ON GEOMETRIC PROBLEMS RELATED TO BROWN-YORK AND LIU-YAU QUASILOCAL MASS

PENZI MIAO\textsuperscript{1}, YUGUANG SHI\textsuperscript{2} AND LUEN-FAI TAM\textsuperscript{3}

Abstract. We discuss some geometric problems related to the definitions of quasilocal mass proposed by Brown-York\textsuperscript{5,6} and Liu-Yau\textsuperscript{13,14}. Our discussion consists of three parts. In the first part, we propose a new variational problem on compact manifolds with boundary, which is motivated by the study of Brown-York mass. We prove that critical points of this variation problem are exactly static metrics. In the second part, we derive a derivative formula for the Brown-York mass of a smooth family of closed 2 dimensional surfaces evolving in an ambient three dimensional manifold. As an interesting by-product, we are able to write the ADM mass\textsuperscript{1} of an asymptotically flat 3-manifold as the sum of the Brown-York mass of a coordinate sphere $S_r$ and an integral of the scalar curvature plus a geometrically constructed function $\Phi(x)$ in the asymptotic region outside $S_r$. In the third part, we prove that for any closed, spacelike, 2-surface $\Sigma$ in the Minkowski space $\mathbb{R}^{3,1}$ for which the Liu-Yau mass is defined, if $\Sigma$ bounds a compact spacelike hypersurface in $\mathbb{R}^{3,1}$, then the Liu-Yau mass of $\Sigma$ is strictly positive unless $\Sigma$ lies on a hyperplane. We also show that the examples given by ´O Murchadha, Szabados and Tod\textsuperscript{18} are special cases of this result.

1. Introduction

In this work, we will discuss some geometric problems related to the definitions of quasilocal mass proposed by Brown-York\textsuperscript{5,6} and Liu-Yau\textsuperscript{13,14}. In general, there are certain properties that a reasonable definition of quasilocal mass should satisfy, see\textsuperscript{21} for example. The

\textsuperscript{5} Research partially supported by Australian Research Council Discovery Grant #DP0987650.
\textsuperscript{6} Research partially supported by NSF grant of China and Fok YingTong Education Foundation.
\textsuperscript{1} Research partially supported by Hong Kong RGC General Research Fund #GRF 2160357.
most important property is the positivity. There are results on positivity of Brown-York mass and Liu-Yau mass in \cite{19, 14, 22, 20, 23}. In particular, the following is a consequence on the positivity of Brown-York mass proved by the last two authors in \cite{19}. Let $g_e$ be the standard Euclidean metric on $\mathbb{R}^3$. Let $\Omega$ be a bounded strictly convex domain in $\mathbb{R}^3$ with smooth boundary $\Sigma$ which has mean curvature $H_0$. Then $\int_{\Sigma} H_0 d\sigma$ is a maximum of the functional $\int_{\Sigma} H d\sigma$ on the class of smooth metrics with nonnegative scalar curvature on $\Omega$ which agree with $g_e$ tangentially on $\Sigma$ and have positive boundary mean curvature $H$. It is interesting to see if this is still true for general domains in $\mathbb{R}^3$.

In \cite{20}, a similar result was proved for domains in $\mathbb{H}^3$, the hyperbolic 3-space. Namely, it was proved that if $g_h$ is the standard hyperbolic metric on $\mathbb{H}^3$ and $\Omega$ is a bounded domain with strictly convex smooth boundary $\Sigma$ which is a topological sphere and has mean curvature $H_0$, then $\int_{\Sigma} H_0 \cosh r d\sigma$ is a maximum of the functional $\int_{\Sigma} H \cosh r d\sigma$ on the class of smooth metrics with scalar curvature bounded below by $-6$ which agree with $g_h$ tangentially on $\Sigma$ and have positive boundary mean curvature $H$. Here $r$ is the distance function on $\mathbb{H}^3$ from a fixed point in $\Omega$. Again it is interesting to see if this is still true for general domains in $\mathbb{H}^3$.

The results and questions above motivate us to study the functional

$$F_\phi(g) = \int_{\Sigma} H \phi \, d\sigma,$$

where $\Sigma$ is the boundary of an $n$ dimensional compact manifold $\Omega$, $\phi$ is a given smooth nontrivial function (that is $\phi \not\equiv 0$) on $\Sigma$, and $d\sigma$ is the volume form of a fixed metric $\gamma$ on $\Sigma$. The class of metrics $g$ we are interested is the space of metrics with constant scalar curvature $K$ which induce the metric $\gamma$ on $\Sigma$. In Theorem 2.1, we will prove the following: $g$ is a critical point of $F_\phi(\cdot)$ if and only if $g$ is a static metric with a static potential $N$ that equals $\phi$ on $\Sigma$. That is to say:

$$\left\{ \begin{array}{c} -(\Delta_g N)g + \nabla_g^2 N - N \text{Ric}(g) = 0, \quad \text{on } \Omega \\ N = \phi, \quad \text{at } \Sigma. \end{array} \right.$$ 

In the theorem, for $K > 0$, we also assume that the first Dirichlet eigenvalue of $(n-1)\Delta_g + K$ is positive.

In particular, if $\phi = 1$, $K = 0$ and $n = 3$, we can conclude that $g$ is a critical point of $\int_{\Sigma} H \, d\sigma$ if and only if $g$ is a flat metric.

Another important question on quasilocal mass is whether it has some monotonicity property. In \cite{19}, it was shown that the Brown-York mass of the boundaries of certain domains in a space with some quasispherical metric is monotonically decreasing rather than increasing as
the domains become larger. In Theorem 3.1 we will derive a more general formula for the derivative of the Brown-York mass of a smooth family of surfaces with positive Gaussian curvature which evolve in an ambient manifold. The formula gives a generalization of the monotonicity formula in [19] which plays a key role in the proof of the positivity of Brown-York mass. As an interesting by-product of this derivative formula, in Corollary 3.5 we are able to write the ADM mass [1] of an asymptotically flat 3-manifold as the sum of the Brown-York mass of a coordinate sphere $S_r$ and an integral of the scalar curvature plus a geometrically constructed function $\Phi(x)$ in the asymptotic region outside $S_r$.

The Minkowski space $\mathbb{R}^{3,1}$ represents the zero energy state in general relativity. Thus, a reasonable notion of quasilocal mass should be such that its value of a spacelike 2-surface in $\mathbb{R}^{3,1}$ equals zero. In [13, 14], the Liu-Yau mass was introduced and its positivity was proved. In the time symmetric case, this coincides with the Brown-York mass. However, Ó. Murchadha, Szabados and Tod [18] constructed spacelike 2-surfaces $\Sigma$ with spacelike mean curvature vector $\vec{H}$ in $\mathbb{R}^{3,1}$ and with positive Gaussian curvature such that the Liu-Yau mass of $\Sigma$ given by

$$m_{LY}(\Sigma) = \frac{1}{8\pi} \int_{\Sigma} (H_0 - |\vec{H}|) d\sigma$$

is strictly positive. Here $H_0$ is the mean curvature of $\Sigma$ when isometrically embedded in $\mathbb{R}^3$ and $|\vec{H}|$ is the Lorentzian norm of $\vec{H}$ in $\mathbb{R}^{3,1}$. Recently, Wang and Yau [22, 23] introduce another definition of quasilocal mass to address this question. In Theorem 4.1 in this paper, we will prove the following: Let $\Sigma$ be a closed, connected, spacelike 2-surface in the Minkowski space $\mathbb{R}^{3,1}$ with spacelike mean curvature vector and with positive Gaussian curvature. Suppose $\Sigma$ spans a compact, spacelike hypersurface in $\mathbb{R}^{3,1}$, then the Liu-Yau mass of $\Sigma$ is strictly positive, unless $\Sigma$ lies on a hyperplane. The results give some properties on isometric embeddings of compact surfaces with positive Gaussian curvature in the Minkowski space. We will also show that all the examples in [18] satisfy the conditions in Theorem 4.1.

This paper is organized as follows. In section 2, we will prove that static metrics are the only critical points of the functional $F_\phi(\cdot)$. In section 3, a formula for the derivative of the Brown-York mass will be derived and some applications will be given. In section 4, we will prove that for “most” spacelike 2-surfaces in $\mathbb{R}^{3,1}$ for which the Liu-Yau mass is defined, their Liu-Yau mass is strictly positive. In the appendix, we prove some results on the differentiability of a 1-parameter family of
isometric embeddings in $\mathbb{R}^3$, following the arguments of Nirenberg [17]. The results will be used in section 3.

We would like to thank Robert Bartnik for useful discussions on the existence of maximal surfaces.

2. Static Metrics and Brown-York Type Integral

Throughout this section, we let $\Omega$ be an $n$-dimensional ($n \geq 3$) compact manifold with smooth boundary $\Sigma$. Let $\gamma$ be a smooth Riemannian metric on $\Sigma$. As in [16], for a constant $K$ and any integer $k > \frac{n}{2} + 2$, we let $\mathcal{M}^K_\gamma$ be the set of $W^{k,2}$ metrics $g$ on $\Omega$ with constant scalar curvature $K$ such that $g|_{T(\Sigma)} = \gamma$. If $g \in \mathcal{M}^K_\gamma$ and the first Dirichlet eigenvalue of $(n - 1)\Delta_g + K$ is positive, where $\Delta_g$ is the usual Laplacian operator of $g$, then $\mathcal{M}^K_\gamma$ is a manifold near $g$ (see [16] for detail). Let $\phi$ be a given smooth function on $\Sigma$, we define the following functional on $\mathcal{M}^K_\gamma$:

$$F_\phi(g) = \int_\Sigma H_g \phi \, d\sigma,$$

where $H_g$ is the mean curvature of $\Sigma$ in $(\Omega, g)$ with respect to the outward unit normal and $d\sigma$ is the volume form of $\gamma$. Motivated by the results in [19, 20] on the positivity of Brown-York mass and some generalization, we want to determine the critical points of $F_\phi(\cdot)$ on $\mathcal{M}^K_\gamma$.

Before we state the main result, we recall the following definition from [7]:

**Definition 2.1.** A metric $g$ on an open set $U$ is called a static metric on $U$ if there exists a nontrivial function $N$ (called the static potential) on $U$ such that

$$-(\Delta_g N)g + \nabla^2_g N - NRic(g) = 0.$$  

Here $\Delta_g, \nabla^2_g$ are the usual Laplacian, Hessian operator of $g$ and $Ric(g)$ is the Ricci curvature of $g$.

A basic property of static metrics is that they are necessarily metrics of constant scalar curvature [7, Proposition 2.3].

In the following, we obtain a characterization of static metrics in $\mathcal{M}^K_\gamma$ using the function $F_\phi(\cdot)$.

**Theorem 2.1.** With the above notations, let $\phi$ be a nontrivial smooth function on $\Sigma$. Suppose $g \in \mathcal{M}^K_\gamma$ such that the first Dirichlet eigenvalue of $(n - 1)\Delta_g + K$ is positive. Then $g$ is a critical point of $F_\phi(\cdot)$ defined in (2.1) if and only if $g$ is a static metric with a static potential $N$ such that $N = \phi$ on $\Sigma$. 

Proof. Since the Dirichlet eigenvalue of \((n-1)\Delta_g + K\) is positive, we know \(\mathcal{M}_K^\gamma\) is a manifold near \(g\) by the result in [16].

First, we suppose \(g\) is a static metric with a potential \(N\) such that \(N = \phi\) on \(\Sigma\). Let \(g(t)\) be a smooth curve in \(\mathcal{M}_K^\gamma\) with \(g(0) = g\). Let

\[
F(t) = \int_\Sigma H(t)\phi \, d\sigma,
\]

where \(H(t)\) is the mean curvature of \(\Sigma\) in \((\Omega, g(t))\) with respect to the outward unit normal \(\nu\). Let \(\omega_n\) denote the outward unit normal part of a 1-form \(\omega\), i.e. \(\omega_n = \omega(\nu)\), let \(\Pi\) be the second fundamental form of \(\Sigma\) in \((\Omega, g(t))\) with respect to \(\nu\), let \(X\) be the vector field on \(\Sigma\) that is dual to the 1-form \(h(\nu, \cdot)|_{T(\Sigma)}\) on \((\Sigma, \gamma)\) and let \(\text{div}_\gamma X\) be the divergence of \(X\) on \((\Sigma, \gamma)\). For convenience, we often omit writing the volume form in an integral. As in [16, (34)], we have

(2.3)

\[
2F'(0) = \int_\Sigma 2H'(0)N
\]

\[
= \int_\Sigma N(d(\text{tr}_g h) - \text{div}_g h)_n - \text{div}_\gamma X - \langle \Pi, h \rangle_\gamma
\]

\[
= \int_\Sigma N[d(\text{tr}_g h) - \text{div}_g h]_n - \int_\Sigma N\text{div}_\gamma X \quad \text{(because } h|_{T\Sigma} = 0)\]

\[
= \int_\Omega N[\Delta_g(\text{tr}_g h) - \text{div}_g(\text{div}_g h)] - \int_\Omega \text{tr}_g h \Delta_N + \int_\Sigma \text{tr}_g h(dN)_n - \int_\Omega \langle dN, \text{div}_g h \rangle_g - \int_\Sigma N\text{div}_\gamma X
\]

\[
= -\int_\Omega N\langle h, \text{Ric}(g) \rangle_g - \int_\Omega \text{tr}_g h \Delta_N + \int_\Sigma \text{tr}_g h(dN)_n + \int_\Omega \langle \nabla^2 g N, h \rangle_g - \int_\Sigma h(\nu, \nabla N) - \int_\Sigma N\text{div}_\gamma X \quad \text{(using (2.4) and (2.5))}
\]

\[
= \int_\Omega \langle h, -N\text{Ric}(g) - (\Delta_g N)g + \nabla^2 g N \rangle_g
\]

\[
+ \int_\Sigma \text{tr}_g h(dN)_n - \int_\Sigma h(\nu, \nabla N) - \int_\Sigma N\text{div}_\gamma X,
\]

where we have used the facts

(2.4) \[ \int_\Omega \langle dN, \text{div}_g h \rangle_g = -\int_\Omega \langle \nabla^2 g N, h \rangle_g + \int_\Sigma h(\nu, \nabla N) \]
DR_g(h) = \frac{d}{dt} R(t)|_{t=0} = -\Delta_g(\text{tr}_g h) + \text{div}_g(\text{div}_g h) - \langle h, \text{Ric}(g) \rangle_g = 0.

Here $R(t)$ is the scalar curvature of $g(t)$. Let $\nabla^\Sigma N$ denote the gradient of $N$ on $(\Sigma, \sigma)$. Integrating by parts on $\Sigma$, we have

$$
\int_{\Sigma} N \text{div}_\gamma X = -\int_{\Sigma} \langle \nabla^\Sigma N, X \rangle_{\gamma} = -\int_{\Sigma} h(\nu, \nabla N) + \int_{\Sigma} h(\nu, \nu)(dN)_n.
$$

On the other hand, $\text{tr}_g h = h(\nu, \nu)$ at $\Sigma$. Hence, $F'(0) = 0$ by (2.2), (2.3), and (2.6).

To prove the converse, suppose $g$ is a critical point of $F_\phi(\cdot)$. Let \{\{g(t)\}\} be any smooth path in $M^K_\gamma$ passing $g = g(0)$. Let $h = g'(0)$ and $F(t) = F_\phi(g(t))$. As before, we have

$$
2F'(0) = \int_{\Sigma} 2H'(0)\phi
= \int_{\Sigma} \phi[d(\text{tr}_g h) - \text{div}_g h]_n - \int_{\Sigma} \phi \text{div}_\gamma X.
$$

Since the first Dirichlet eigenvalue of $(n - 1)\Delta_g + K$ is positive and $\phi$ is not identically zero, there exists a unique smooth function $N = N_\phi$ which is not identically zero on $\Omega$ such that

$$(n - 1)\Delta_g N + KN = 0, \text{ on } \Omega, \quad N = \phi, \text{ at } \Sigma.
$$

With such an $N$ given, we have

$$
\int_{\Sigma} \phi[d(\text{tr}_g h) - \text{div}_g h]_n - \int_{\Sigma} \phi \text{div}_\gamma X
= \int_{\Omega} N [\Delta_g(\text{tr}_g h) - \text{div}_g(\text{div}_g h)] - \int_{\Omega} \text{tr}_g h \Delta N + \int_{\Sigma} \text{tr}_g h (dN)_n
- \int_{\Omega} \langle dN, \text{div}_g h \rangle_g - \int_{\Sigma} N \text{div}_\gamma X
= \int_{\Omega} N(-1)\langle h, \text{Ric}(g) \rangle_g - \int_{\Omega} \text{tr}_g h \Delta N + \int_{\Omega} \langle \nabla^2_{\gamma} N, h \rangle_g,
$$

where we used the fact $DR_g(h) = 0$ (the boundary terms canceled as before and we have not used (2.8) yet).

Now let $\tilde{h}$ be any smooth symmetric (0,2) tensor with compact support in $\Omega$. For each $t$ sufficiently small, we can find a smooth positive
function \(u(t)\) on \(\Omega\) such that \(u(t) = 1\) at \(\Sigma\) and
\[
g(t) = u(t) \frac{1}{n-2} (g + \hat{h}) \in \mathcal{M}^K_{\gamma}.
\]
Moreover, \(u(t)\) is differentiable at \(t = 0\) and \(u(0) \equiv 1\) on \(\Omega\). See the proof of [16, Theorem 5] for details on the existence of such a \(u(t)\). Now \(g'(0) = \frac{1}{n-2} u'(0) g + \hat{h}\). Hence, by (2.7) and (2.9) we have
\[
2F'(0) = \int_\Omega \langle \frac{4}{n-2} u'(0) g + \hat{h}, -\text{NRic}(g) - (\Delta g N) g + \nabla^2 g N \rangle_g
\]
\[
= \int_\Omega \langle \hat{h}, -\text{NRic}(g) - (\Delta g N) g + \nabla^2 g N \rangle_g
\]
\[
+ \int_\Omega \frac{4}{n-2} u'(0) [-KN - n(\Delta g N) + \Delta g N].
\]
By (2.8), the second integral in the above equation is zero. Hence, we have
\[
2F'(0) = \int_\Omega \langle \hat{h}, -\text{NRic}(g) - (\Delta g N) g + \nabla^2 g N \rangle_g.
\]
Since \(\hat{h}\) can be arbitrary, we conclude that \(g\) and \(N\) satisfy (2.2). \(\square\)

Remark 2.1. If \(K \leq 0\), then the condition that the first Dirichlet eigenvalue of \((n-1)\Delta g + \hat{K}\) is positive holds automatically for \(g \in \mathcal{M}^K_{\gamma}\).

As a direct corollary of Theorem 2.1, we have

Corollary 2.1. With the notations given as in Theorem 2.1, suppose \(K = 0\) and \(\phi = 1\). Then \(g \in \mathcal{M}_0^0\) is a critical point of \(\int_\Sigma H \, d\sigma\) if and only if \(g\) is a Ricci flat metric. In particular, if \(n = 3\), then \(g \in \mathcal{M}_0^0\) is a critical point of \(\int_\Sigma H \, d\sigma\) if and only if \(g\) is a flat metric.

If \(\phi\) does not change sign on the boundary, we further have:

Corollary 2.2. With the notations given as in Theorem 2.1, suppose \(\phi \geq 0\) or \(\phi \leq 0\) on \(\Sigma\). Suppose \(g \in \mathcal{M}^K_{\gamma}\) is a static metric. If \(K > 0\), we also assume that the first Dirichlet eigenvalue of \((n-1)\Delta g + \hat{K}\) is positive. Let \(g(t)\) be a smooth family of smooth metrics on \(\Omega\) with \(g(0) = g\) such that
\[
\begin{align*}
(i) & \quad \text{the scalar curvature of } g(t) \text{ is at least } K, \\
(ii) & \quad g(t) \text{ induces } \gamma \text{ on } \Sigma.
\end{align*}
\]
Then
\[
\frac{d}{dt} F_{\phi}(g(t))|_{t=0} = 0.
\]
Proof. We prove the case that $\phi \geq 0$ on $\Sigma$. The case that $\phi \leq 0$ on $\Sigma$ is similar.

By the assumption of $g$, for $t$ small, we can find smooth positive functions $u(t)$ on $\Omega$ with $u(t) = 1$ on $\Sigma$ such that $\hat{g}(t) = u^4(t)g(t) \in \mathcal{M}^K$, $u(t)$ is differentiable at $t = 0$ and $u(0) \equiv 1$ (see the proof of Proposition 1 in [14]). The mean curvature $\hat{H}(t)$ of $\Sigma$ in $(\Omega, \hat{g}(t))$ is given by

$$\hat{H}(t) = H(t) + \frac{2(n-1)}{n-2} \frac{\partial u}{\partial \nu_t},$$

where $H(t)$ and $\nu_t$ are the mean curvature and the unit outward normal of $\Sigma$ in $(\Omega, g(t))$. Note that $u$ satisfies:

$$\begin{cases}
\frac{4(n-1)}{n-2} \Delta_g(t)u - K(t)u = -K u^{\frac{n+2}{n-2}}, & \text{in } \Omega \\
u = 1, & \text{on } \Sigma,
\end{cases}$$

where $K(t)$ is the scalar curvature of $g(t)$. Since $K(t) \geq K$, by the maximum principle, we have

$$\frac{\partial u}{\partial \nu_t} \geq 0.$$

Hence, $\hat{H}(t) \geq H(t)$ and consequently $F_\phi(\hat{g}(t)) \geq F_\phi(g(t))$ by the assumption $\phi \geq 0$ on $\Sigma$. By Theorem 2.1, we have

$$\frac{d}{dt} F_\phi(\hat{g}(t))|_{t=0} = 0.$$

Since $\hat{g}(0) = g(0)$, we conclude

$$\frac{d}{dt} F_\phi(g(t))|_{t=0} = 0.$$

□

Here are some examples provided by Theorem 2.1:

Example 1: Let $\Omega$ be a bounded domain in $\mathbb{R}^n$ with smooth boundary $\Sigma$. Then the standard Euclidean metric is a critical point of $F_\phi(\cdot)$ with $\phi \equiv 1$. If $\Sigma$ is strictly convex, then this follows also from the result in [19].

Example 2: Let $\Omega$ be a bounded domain in $\mathbb{H}^n$ with smooth boundary $\Sigma$. Then the standard Hyperbolic metric is a critical point of $F_\phi(\cdot)$ with $\phi = \cosh r$, where $r$ is the distance function on $\mathbb{H}^n$ from a fixed point. If $\Sigma$ is strictly convex and $n = 3$, then this follows also from the result in [20].

Example 3: Let $\Omega$ be a domain in $\mathbb{S}^n$ with smooth boundary $\Sigma$ such that the volume of $\Omega$ is less than $2\pi$. Then the standard metric on $\mathbb{S}^n$ is
a critical point of $F_\phi(\cdot)$ with $\phi = \cos r$, where $r$ is the distance function on $\mathbb{S}^n$ from a fixed point.

**Example 4:** Let $\Omega$ be a bounded domain with smooth boundary in the Schwarzschild manifold $\mathbb{R}^3 \setminus \{0\}$ with metric

$$g_{ij} = \left(1 + \frac{m}{2r}\right)^4 \delta_{ij}$$

with $m > 0$ and $r = |x|$. Then on $\Omega$ $g$ is a critical point for $F_\phi(\cdot)$ with $\phi = (1 - \frac{m^2}{2r})/(1 + \frac{m^2}{2r})$.

**Example 5:** Complete conformally flat Riemannian manifolds with static metrics have been classified by Kobayashi in [12, Theorem 3.1]. In addition to the manifolds in the previous examples, there are other kind of static metrics with $N$ being explicitly constructed. Domains in these manifolds will be critical points of $F_\phi(\cdot)$ where $\phi$ is the restriction of $N$ to the boundary. See [12] for more details.

### 3. Derivative of the Brown-York Mass

In this section, we give a derivative formula that describes how the Brown-York mass of a surface changes if the surface is evolving in an ambient Riemannian manifold. Our main result is:

**Theorem 3.1.** Let $\mathbb{S}^2$ be the 2-dimensional sphere. Let $(M, g)$ be a 3-dimensional Riemannian manifold. Let $I$ be an open interval in $\mathbb{R}^1$. Suppose

$$F : \mathbb{S}^2 \times I \longrightarrow M$$

is a smooth map such that, for $t \in I$,

(i) $\Sigma_t = F(\mathbb{S}^2, t)$ is an embedded surface in $M$ and $\Sigma_t$ has positive Gaussian curvature.

(ii) The velocity vector $\frac{\partial F}{\partial t}$ is always perpendicular to $\Sigma_t$, i.e

$$\frac{\partial F}{\partial t} = \eta \nu,$$

where $\nu$ is a given unit vector field normal to $\Sigma_t$ and $\eta = \langle \frac{\partial F}{\partial t}, \nu \rangle$ denotes the speed of $\Sigma_t$ with respect to $\nu$.

Consider $m_{by}(\Sigma_t)$, the Brown-York mass of $\Sigma_t$ in $(M, g)$, defined by

$$m_{by}(\Sigma_t) = \frac{1}{8\pi} \int_{\Sigma_t} (H_0 - H) \, d\sigma_t,$$

where $H_0$ is the mean curvature of $\Sigma_t$ with respect to the outward normal when isometrically embedded in $\mathbb{R}^3$, $H$ is the mean curvature of $\Sigma_t$ with respect to $\nu$ in $(M, g)$, and $d\sigma_t$ is the volume form of the induced metric on $\Sigma_t$. 

We have
\[
\frac{d}{dt} m_{\text{H}}(\Sigma_t) = \frac{1}{16\pi} \int_{\Sigma_t} \left( |A_0 - A|^2 - |H_0 - H|^2 + R \right) \eta \, d\sigma_t,
\]
where \(A_0\) is the second fundamental form of \(\Sigma_t\) with respect to the outward normal when isometrically embedded in \(\mathbb{R}^3\), \(A\) is the second fundamental form of \(\Sigma_t\) with respect to \(\nu\) in \((M, g)\), and \(R\) is the scalar curvature of \((M, g)\).

Our proof of Theorem 3.1 makes use of a recent formula of Wang and Yau (Proposition 6.1 in [23]):

**Proposition 3.1.** Let \(\Sigma\) be an orientable closed embedded hypersurface in \(\mathbb{R}^{n+1}\). Let \(\{\Sigma_t\}_{|t|<\delta}\) be a smooth variation of \(\Sigma\) in \(\mathbb{R}^{n+1}\). Then
\[
\frac{d}{dt} \left( \int_{\Sigma_t} H_0 \, d\sigma_t \right) |_{t=0} = \frac{1}{2} \int_{\Sigma} (H_0 \text{tr}_{\Sigma} h - \langle A_0, h \rangle) \, d\sigma,
\]
where \(H_0\) and \(A_0\) are the mean curvature and the second fundamental form of \(\Sigma\) with respect to the outward normal in \(\mathbb{R}^{n+1}\), \(h\) is the variation of the induced metric \(\sigma\) on \(\Sigma\), \(\text{tr}_{\Sigma} h = \langle \sigma, h \rangle\) denotes the trace of \(h\) with respect to \(\sigma\), and \(d\sigma, d\sigma_t\) denote the volume form on \(\Sigma_t, \Sigma\).

In order to apply Proposition 3.1, we will need to show that, on a closed convex surface \(\Sigma\) in \(\mathbb{R}^3\), an abstract metric variation on \(\Sigma\) indeed arises from a surface variation \(\{\Sigma_t\}\) of \(\Sigma\) in \(\mathbb{R}^3\). Precisely, we have:

**Proposition 3.2.** Given an integer \(k \geq 6\) and a number \(0 < \alpha < 1\), let \(\{\sigma(t)\}_{|t|<\delta}\) be a path of \(C^{k,\alpha}\) metrics on \(S^2\) such that \(\{\sigma(t)\}\) is differentiable at \(t = 0\) in the space of \(C^{k,\alpha}\) metrics. Suppose \(\sigma(0)\) has positive Gaussian curvature. Then there exists a small number \(\delta > 0\) and a path of \(C^{k,\alpha}\) embeddings \(\{f(t)\}_{|t|<\delta}\) of \(S^2\) in \(\mathbb{R}^3\) such that \(f(t)\) is an isometric embedding of \((S^2, \sigma(t))\) for \(|t| < \delta\) and \(\{f(t)\}\) is differentiable at \(t = 0\) in the space of \(C^{2,\alpha}\) embeddings.

The proposition above follows from the arguments by Nirenberg in [17]. For completeness, we include its proof here.

**Proof.** Given \(\{\sigma(t)\}_{|t|<1}\), a path of \(C^{k,\alpha}\) metrics on \(S^2\), let \(h = \sigma'(0)\). Then \(h\) is a \(C^{k,\alpha}\) symmetric \((0,2)\) tensor. Since \(\sigma(0)\) has positive Gaussian curvature, by the result in [17], there exists a \(C^{k,\alpha}\) isometric embedding of \((S^2, \sigma(0))\) in \(\mathbb{R}^3\), which we denote by \(X\). Given such an \(X\), let \(Y : S^2 \to \mathbb{R}^3\) be a \(C^{2,\alpha}\) solution to the linear equation
\[
2dX \cdot dY = h,
\]
where “·” denotes the Euclidean dot product in \( \mathbb{R}^3 \) and (3.4) is understood as
\[
dX(e_1) \cdot dY(e_2) + dX(e_2) \cdot dY(e_1) = h(e_1, e_2)
\]
for any tangent vectors \( e_1, e_2 \) to \( S^2 \). The existence of such a \( Y \) is provided by Theorem 2’ in [17]. Let \( d\bar{\sigma}^2 = h \) and let \( \phi, p_1, p_2 \) be given as in (6.5), (6.6) in [17], then \( \phi \) satisfies (6.15) in [17]. Using the fact that \( X \) is in \( C^{k,\alpha} \) and \( d\bar{\sigma}^2 \) is in \( C^{k,\alpha} \), we check that the coefficients of (6.15) in [17] (when written in a non-divergence form) is in \( C^{k-3,\alpha} \).

Thus, it follows from (6.15) in [17] that \( \phi \in C^{k-1,\alpha} \), from which we conclude \( Y \in C^{k-1,\alpha} \) by (6.11)-(6.13) in [17].

Now consider the \( C^{k-1,\alpha} \) path of embeddings \( \{G(t)\}_{|t|<t_0} \), where
\[
G(t) = X + tY
\]
and \( t_0 \) is chosen so that \( G(t) \) is an embedding. Let \( g_e \) be the Euclidean metric on \( \mathbb{R}^3 \). The pull back metric \( \tau(t) = G(t)^*(g_e) \) which is in \( C^{k-2,\alpha} \) satisfies
\[
\tau(0) = \sigma(0), \quad \tau'(0) = \sigma'(0),
\]
which implies
\[
||\tau(t) - \sigma(t)||_{C^{2,\alpha}} = O(t^2).
\]

Apply Lemma 5.3 in the Appendix to \( \sigma^0 = \sigma(0) = \tau(0) \), for each \( t \) sufficiently small, we can find a \( C^{2,\alpha} \) isometric embedding \( X(t) \) of \( (S^2, \sigma(t)) \) in \( \mathbb{R}^3 \) such that
\[
||G(t) - X(t)||_{C^{2,\alpha}} \leq C||\tau(t) - \sigma(t)||_{C^{2,\alpha}} = O(t^2).
\]
(By Lemma 1’ in [17], \( X(t) \) indeed lies in \( C^{k,\alpha} \)\.) It follows from (3.8) that \( \{X(t)\} \), when viewed as a path in the space of \( C^{2,\alpha} \) embeddings, is differentiable at \( t = 0 \). Proposition 3.2 is therefore proved.

Proposition 3.1 and Proposition 3.2 together imply:

**Proposition 3.3.** Given an integer \( k \geq 6 \) and a number \( 0 < \alpha < 1 \), suppose \( \{\sigma(t)\}_{|t|<1} \) is a differentiable path in the space of \( C^{k,\alpha} \) metrics on \( S^2 \). Suppose \( \sigma(t) \) has positive Gaussian curvature for each \( t \). Let \( H_0 \) be the mean curvature of the isometric embedding of \( (S^2, \sigma(t)) \) in \( \mathbb{R}^3 \) with respect to the outward normal. Let \( d\sigma_t \) be the volume form of \( \sigma(t) \). Then
\[
\int_{S^2} H_0 \ d\sigma_t
\]
is a differentiable function of $t$, and

$$
\frac{d}{dt} \left( \int_{S^2} H_0 \, d\sigma_t \right) = \frac{1}{2} \int_{S^2} \langle H_0 \sigma(t) - A_0, h \rangle \, d\sigma_t,
$$

where $A_0$ is the second fundamental form of the isometric embedding of $(S^2, \sigma(t))$ in $\mathbb{R}^3$ with respect to the outward normal, $h = \sigma'(t)$, “$\langle \cdot, \cdot \rangle$” denotes the metric product with respect to $\sigma(t)$ on the space of symmetric $(0,2)$ tensors.

**Proof.** Take any $t_0 \in (-1,1)$. By Proposition 3.2, there exists a small positive number $\delta$ (depending on $t_0$) and a path of $C^{k,\alpha}$ embeddings $\{f(t)\}_{|t-t_0|<\delta}$ of $S^2$ in $\mathbb{R}^3$, such that $f(t)$ is an isometric embedding of $(S^2, \sigma(t))$ and $\{f(t)\}$ is differentiable at $t = t_0$ in the space of $C^{2,\alpha}$ embeddings.

Let $\Sigma_t = f(t)(S^2)$, let $H_0(t)$ be the mean curvature of $\Sigma_t$ with respect to the outward normal in $\mathbb{R}^3$, by definition we have

$$
\int_{S^2} H_0 \, d\sigma_t = \int_{\Sigma_t} H_0(t) \, d\sigma_t, \quad \forall |t - t_0| < \delta.
$$

Apply the fact that $\{f(t)\}$ is differentiable at $t = t_0$ in the space of $C^{2,\alpha}$ embeddings and note that $H_0$ only involves derivatives of $f(t)$ up to the second order, we conclude that $\int_{\Sigma_t} H_0(t) \, d\sigma_t$ is differentiable at $t_0$. By (3.10), $\int_{S^2} H_0 \, d\sigma_t$ is differentiable at $t_0$ as well. This shows $\int_{S^2} H_0 \, d\sigma_t$ is a differentiable function of $t$. Equation (3.9) then follows directly from (3.10) and Proposition 3.1.

We are now ready to prove Theorem 3.1 using Proposition 3.3.

**Proof of Theorem 3.1.** By Proposition 3.3 the function $m_{av}(\Sigma_t)$ is a differentiable function of $t$. We have

$$
\frac{d}{dt} m_{av}(\Sigma_t) = \frac{1}{8\pi} \frac{d}{dt} \left( \int_{\Sigma_t} H_0 \, d\sigma_t \right) - \frac{1}{8\pi} \frac{d}{dt} \left( \int_{\Sigma_t} H \, d\sigma_t \right).
$$

Let $\sigma = \sigma(t)$ be the induced metric on $\Sigma_t$. By (3.9) in Proposition 3.3 we have

$$
\frac{d}{dt} \left( \int_{\Sigma_t} H_0 \, d\sigma_t \right) = \frac{1}{2} \int_{\Sigma_t} \langle H_0 \sigma(t) - A_0, \frac{\partial \sigma}{\partial t} \rangle \, d\sigma_t.
$$

Now, applying the fact that $\{\Sigma_t\}$ evolves in $(M,g)$ according to

$$
\frac{\partial F}{\partial t} = \eta \nu,
$$

we have

$$
\frac{\partial \sigma}{\partial t} = 2\eta A,
$$

and

$$
\frac{d}{dt} m_{av}(\Sigma_t) = \frac{1}{8\pi} \frac{d}{dt} \left( \int_{\Sigma_t} H_0 \, d\sigma_t \right) - \frac{1}{8\pi} \frac{d}{dt} \left( \int_{\Sigma_t} H \, d\sigma_t \right).
$$
and

\[ \frac{\partial H}{\partial t} = -\Delta \eta - (|A|^2 + \text{Ric}(\nu, \nu)) \eta, \]

where \( \text{Ric}(\nu, \nu) \) is the Ricci curvature of \((M, g)\) along \( \nu \). Thus,

\[ \frac{d}{dt} \left( \int_{\Sigma_t} H_0 \ d\sigma_t \right) = \frac{1}{2} \int_{\Sigma_t} \langle H_0 \sigma(t) - A_0, 2\eta A \rangle \ d\sigma_t \]

(3.16)

\[ = \int_{\Sigma_t} H_0 H \eta - \langle A_0, A \rangle \eta \ d\sigma_t \]

and

\[ \frac{d}{dt} \left( \int_{\Sigma_t} H \ d\sigma_t \right) = \int_{\Sigma_t} \frac{\partial H}{\partial t} + H^2 \eta \ d\sigma_t \]

(3.17)

\[ = \int_{\Sigma_t} -(|A|^2 + \text{Ric}(\nu, \nu)) \eta + H^2 \eta \ d\sigma_t. \]

Hence, it follows from (3.11), (3.16) and (3.17) that

\[ \frac{d}{dt} m_{BY}(\Sigma_t) = \frac{1}{8\pi} \int_{\Sigma_t} \left[ H_0 H - \langle A_0, A \rangle + (|A|^2 + \text{Ric}(\nu, \nu)) - H^2 \right] \eta \ d\sigma_t. \]

(3.18)

Apply the Gauss equation to \( \Sigma_t \) in \((M, g)\) and to the isometric embedding of \( \Sigma_t \) in \( \mathbb{R}^3 \) respectively, we have

\[ 2K = R - 2\text{Ric}(\nu, \nu) + H^2 - |A|^2, \]

(3.19)

and

\[ 2K = (H_0)^2 - |A_0|^2, \]

(3.20)

where \( K \) is the Gaussian curvature of \( \Sigma_t \). Hence, (3.18)-(3.20) imply that

\[ \frac{d}{dt} m_{BY}(\Sigma_t) = \frac{1}{16\pi} \int_{\Sigma_t} \left[ |A_0 - A|^2 - (H_0 - H)^2 + R \right] \eta \ d\sigma_t. \]

(3.21)

Therefore, (3.2) is proved. \( \square \)

Next, we want to discuss some applications of Theorem 3.1. The first two applications below put the monotonicity property of the Brown-York mass in the construction in [19] into a more general context.

**Corollary 3.1.** Let \((M, g), I, F, \{\Sigma_t\}, \eta, A, A_0, H \) and \( H_0 \) be given as in Theorem 3.1 with \( \eta > 0 \). Suppose at each point \( x \in \Sigma_t, t \in I, \)

\[ A_0 - A \] is either positive semi-definite or negative semi-definite, and \( R \leq 0, \) then \( m_{BY}(\Sigma_t) \) is nonincreasing in \( t \). If in addition, \( A = \alpha A_0 \)
for some number $\alpha$ depending on $x \in \Sigma_t$, then $m_{BY}(\Sigma_t)$ is constant in $I$ if and only if $(S^2 \times I, F^*(g))$ is a domain in $\mathbb{R}^3$.

**Proof.** Let $\lambda_1, \lambda_2$ be the eigenvalues of $A_0 - A$. Suppose $A_0 - A$ is either positive semi-definite or negative semi-definite, then $\lambda_1 \lambda_2 \geq 0$ and hence $|A_0 - A|^2 - |H_0 - H|^2 = -2\lambda_1 \lambda_2 \leq 0$. Since $R \leq 0$, by Theorem 3.1, we have:

$$\frac{d}{dt} m_{BY}(\Sigma_t) \leq 0$$

because $\eta > 0$. This proves the first assertion.

Suppose $(S^2 \times I, F^*(g))$ is a domain in $\mathbb{R}^3$, then by definition we have $m_{BY}(\Sigma_t) = 0$, $\forall t$. Hence,

$$\frac{d}{dt} m_{BY}(\Sigma_t) = 0.$$

Conversely, suppose $A = \alpha A_0$ and

$$\frac{d}{dt} m_{BY}(\Sigma_t) = 0.$$

Then $R = 0$ and $A = A_0$. In particular, $H = H_0$. For any $(t_1, t_2) \subset I$, let $\Omega = S^2 \times (t_1, t_2)$ with the pull back metric $F^*(g)$. Let $D$ be the interior of $\Sigma_{t_1} = F(S^2 \times \{t_1\})$ when it is isometrically embedded in $\mathbb{R}^3$ and $E$ be the exterior of $\Sigma_{t_2} = F(S^2 \times \{t_2\})$ when it is isometrically embedded in $\mathbb{R}^3$. By gluing $\Omega$ with $D$ along $S^2 \times \{t_1\}$, which is identified with $\Sigma_{t_1}$ through $F$, and gluing $\Omega$ with $E$ along $S^2 \times \{t_2\}$, which is identified with $\Sigma_{t_2}$ through $F$, we have an asymptotically flat and scalar flat manifold with corners and with zero mass, and it must be flat by [15] [19]. Hence, $\Omega$ is flat. Since it is simply connected, $\Omega$ can be isometrically embedded in $\mathbb{R}^3$.

\[ \square \]

**Corollary 3.2.** Let $(M, g)$, $I$, $F$, $\{\Sigma_t\}$, $\eta$, $A$, $A_0$, $H$ and $H_0$ be given as in Theorem 3.1. Let $g_e$ be the Euclidean metric on $\mathbb{R}^3$. Suppose there exists another smooth map

$$F^0 : S^2 \times I \longrightarrow \mathbb{R}^3$$

such that

(i) $\Sigma^0_t = F^0(S^2, t)$ is an embedded closed convex surface in $\mathbb{R}^3$ and

$$(F^0_t)^*(g_e) = F^*_t(g),$$

where $F^0_t(\cdot) = F^0(\cdot, t)$ and $F_t(\cdot) = F(\cdot, t)$.
(ii) The velocity vector $\frac{\partial F^0}{\partial t}$ is always perpendicular to $\Sigma_t^0$, i.e.

$$\frac{\partial F^0}{\partial t} = \eta^0 \nu^0,$$

where $\nu^0$ is the outward unit normal to $\Sigma_t^0$ in $\mathbb{R}^3$ and $\eta^0$ denotes the speed of $\Sigma_t^0$ with respect to $\nu^0$.

Suppose $\eta^0 > 0$, $\eta > 0$ and $(M, g)$ has zero scalar curvature, then the Brown-York mass $m_{BY}(\Sigma_t^0)$ is monotonically non-increasing, and $m_{BY}(\Sigma_t^0)$ is a constant if and only if $(S^2 \times I, F^*(g))$ is a domain in $\mathbb{R}^3$.

Proof. Since $\eta^0 > 0$ and $\eta > 0$, we can write ($F^0)^*(g_e)$ and $F^*(g)$ as

$$(3.22) \quad F^*(g_e) = (\eta^0)^2 dt^2 + g_t \quad \text{and} \quad F^*(g) = \eta^2 dt^2 + g_t,$$

where $g_t$ denotes the same induced metric on both $\Sigma_0^t$ and $\Sigma_t$. Now it follows from (3.22) that

$$(3.23) \quad A = \frac{\eta^0}{\eta} A_0.$$ 

Since $A_0$ is positive definite, the results follow from Corollary 3.1. □

Remark 3.1. We note that

(i) Quasi-spherical metrics constructed in [19] satisfy all the assumptions of Corollary 3.2.

(ii) In case $\eta^0 = 1$, one recovers the monotonicity formula in [19].

By applying the co-area formula directly to (3.2), we also obtain

**Corollary 3.3.** Let $(M, g), F, \{\Sigma_t\}, \eta, A_0, H$ and $H_0$ be given as in Theorem 3.1. Suppose $\eta > 0$. For any $t_1 < t_2$, let $\Omega_{[t_1, t_2]}$ be the region bounded by $\Sigma_{t_1}$ and $\Sigma_{t_2}$. Then

$$(3.24) \quad m_{BY}(\Sigma_{t_2}) - m_{BY}(\Sigma_{t_1}) = \frac{1}{16\pi} \left( \int_{\Omega_{[t_1, t_2]}} R \, dV + \int_{\Omega_{[t_1, t_2]}} \Phi \, dV \right),$$

where $R$ is the scalar curvature of $(M, g)$, $dV$ is the volume form of $g$ on $M$, and $\Phi$ is the function on $\Omega_{[t_1, t_2]}$, depending on $\{\Sigma_t\}$, defined by

$$(3.25) \quad \Phi(x) = |A_0 - A|^2 - (H_0 - H)^2, \quad x \in \Sigma_t.$$

The function $\Phi(x)$ defined above clearly depends on the foliation $\{\Sigma_t\}$ connecting $\Sigma_{t_1}$ to $\Sigma_{t_2}$. However, it is interesting to note that the integral $\int_{\Omega_{[t_1, t_2]}} \Phi \, dV$ turns out to be $\{\Sigma_t\}$ independent by (3.24).

We can apply formula (3.24) to small geodesic balls in a general 3-manifold and to asymptotically flat regions in an asymptotically flat 3-manifold.
Corollary 3.4. Let \((M,g)\) be a 3-dimensional Riemannian manifold. Let \(p \in M\) and \(B_\delta(p)\) be a geodesic ball centered at \(p\) with geodesic radius \(\delta\). Suppose \(\delta\) is small enough such that

1. \(\delta < i_p(M)\), where \(i_p(M)\) is the injectivity radius of \((M,g)\) at \(p\).
2. For any \(0 < r \leq \delta\), the geodesic sphere \(S_r(p)\), centered at \(p\) with geodesic radius \(r\), has positive Gaussian curvature.

Then the Brow-York mass of \(S_\delta(p)\) can be written as

\[
\text{m}_{\text{BY}}(S_\delta(p)) = \frac{1}{16\pi} \left( \int_{B_\delta(p)} R \, dV + \int_{B_\delta(p) \setminus \{p\}} \Phi \, dV \right),
\]

where \(R\) is the scalar curvature of \(M\), \(dV\) is the volume form on \(M\), and \(\Phi\) is the function on \(B_\delta(p) \setminus \{p\}\), defined by

\[
\Phi(x) = |A_0 - A|^2 - (H_0 - H)^2, \quad x \in S_r.
\]

Here \(A, H\) are the second fundamental form, the mean curvature of \(S_r\) in \(M\) with respect to the outward normal; and \(A_0, H_0\) are the second fundamental form, the mean curvature of the isometric embedding of \(S_r\) in \(\mathbb{R}^3\) with respect to the outward normal.

Proof. Let \((r, \omega)\) be the geodesic polar coordinate of \(x \in B_\delta(p) \setminus \{p\}\), where \(r\) denotes the distance from \(x\) to \(p\). Since \(\frac{\partial}{\partial r} \perp S_r\), we can choose the foliation \(\{\Sigma_t\}\) in Corollary 3.1 to be \(\{S_r\}\) with \(t = r\). By (3.24), we have

\[
\text{m}_{\text{BY}}(S_\delta(p)) - \text{m}_{\text{BY}}(S_r(p)) = \frac{1}{16\pi} \int_{B_\delta(p) \setminus B_r(p)} (R + \Phi) \, dV.
\]

By [8], we have

\[
\lim_{r \to 0^+} \text{m}_{\text{BY}}(S_r(p)) = 0.
\]

Hence, (3.26) follows from (3.28) and (3.29). \(\square\)

Next, we express the ADM mass \([1]\) as the sum of the Brown-York mass of a coordinate sphere and an integral involving the scalar curvature and the function \(\Phi(x)\).

Corollary 3.5. Let \((M,g)\) be an asymptotically flat 3-manifold with a given end. Let \(\{x^i \mid i = 1, 2, 3\}\) be a coordinate system at \(\infty\) defining the asymptotic structure of \((M,g)\). Let \(S_r = \{x \in M \mid |x| = r\}\) be the coordinate sphere, where \(|x|\) denotes the coordinate length. Suppose \(r_0 \gg 1\) is a constant such that \(S_r\) has positive Gaussian curvature for each \(r \geq r_0\). Then

\[
\text{m}_{\text{ADM}} = \text{m}_{\text{BY}}(S_{r_0}) + \frac{1}{16\pi} \int_{M \setminus D_{r_0}} R \, dV + \frac{1}{16\pi} \int_{M \setminus D_{r_0}} \Phi \, dV,
\]
where $m_{\text{ADM}}$ is the ADM mass of $(M, g)$, $R$ is the scalar curvature of $(M, g)$, $D_{r_0}$ is the bounded open set in $M$ enclosed by $S_{r_0}$, and $\Phi$ is the function on $M \setminus D_{r_0}$ defined by

\begin{equation}
\Phi(x) = |A_0 - A|^2 - (H_0 - H)^2, \quad x \in S_r.
\end{equation}

Here $A, H$ are the second fundamental form, the mean curvature of $S_r$ in $M$; and $A_0, H_0$ are the second fundamental form, the mean curvature of $S_r$ when isometrically embedded in $\mathbb{R}^3$.

Proof. \{S_r\}_{r \geq r_0}$ consists of level sets of the function $r$ on $M \setminus D_{r_0}$, hence can be reparameterized to evolve in a way that its velocity vector is perpendicular to the surface at each time. To be precise, we can define the vector field $X = \nabla r / |\nabla r|^2$ on $M \setminus D_{r_0}$ and let $\gamma_p(t)$ be the integral curve of $X$ starting at $p \in S_{r_0}$. For any $t \geq 0$, let $\Sigma_t = \{\gamma_p(t) | p \in S_{r_0}\}$, then $\Sigma_t = S_{r_0} + t$. For any $T > 0$, apply (3.24) to $\{\Sigma_t\}_{0 \leq t \leq T}$, we have

\begin{equation}
m_{\text{BY}}(S_{r_0} + t) - m_{\text{BY}}(S_{r_0}) = \frac{1}{16\pi} \left( \int_{\Omega_{[0,T]}} R \, dV + \int_{\Omega_{[0,T]}} \Phi \, dV \right),
\end{equation}

where $\Omega_{[0,T]}$ is the region in $M$ bounded by $\Sigma_0 = S_{r_0}$ and $\Sigma_T = S_{r_0} + T$. Letting $T \to +\infty$, by [8] we have

\begin{equation}
\lim_{T \to +\infty} m_{\text{BY}}(S_T) = m_{\text{ADM}}.
\end{equation}

Hence, (3.30) follows from (3.32) and (3.33). $\square$

4. Liu-Yau mass of spacelike two-surfaces in $\mathbb{R}^{3,1}$

Let $\Sigma$ be a closed, connected, 2-dimensional spacelike surface in a spacetime $N$. Suppose $\Sigma$ has positive Gaussian curvature and has spacelike mean curvature vector $\vec{H}$ in $N$. Let $H_0$ be the mean curvature of $\Sigma$ with respect to the outward unit normal when it is isometrically embedded in $\mathbb{R}^3$. The Liu-Yau mass of $\Sigma$ is then defined as (see [13, 14]):

\begin{equation}
m_{\text{LY}}(\Sigma) = \frac{1}{8\pi} \int_{\Sigma} (H_0 - |\vec{H}|) \, d\sigma,
\end{equation}

where $|\vec{H}|$ is Lorentzian norm of $\vec{H}$ in $N$ and $d\sigma$ is the volume form of the induced metric on $\Sigma$.

In [14], the following positivity result was proved: Let $\Omega$ be a compact, spacelike hypersurface in a spacetime $N$ satisfying the dominant energy conditions. Suppose the boundary $\partial \Omega$ has finitely many components $\Sigma_i$, $1 \leq i \leq l$, each of which has positive Gaussian curvature and has spacelike mean curvature vector in $N$. Then $m_{\text{LY}}(\Sigma_i) \geq 0$ for all $i$;
moreover if $m_{LY}(\Sigma_i) = 0$ for some $i$, then $\partial \Omega$ is connected and $N$ is a flat spacetime along $\Omega$.

We remark that in their proof of the above result, it is assumed implicitly that the mean curvature of $\partial \Omega$ in $\Omega$ with respect to the outward unit normal is positive. See the statement [22, Theorem 1.1]. The condition is necessary as can be seen by the following example in the time symmetric case:

Let $g_e$ be the Euclidean metric on $\mathbb{R}^3$ and let $m > 0$ be a constant. Consider the Schwarzschild metric (with negative mass)

$$g = \left(1 - \frac{m}{2|x|}\right)^4 g_e,$$

defined on $\{0 < |x| < \frac{m}{2}\}$. Given any $0 < r_1 < r_2 < \frac{m}{2}$, consider the domain

$$\Omega = \{r_1 < |x| < r_2\}.$$

For any constant $r$, the mean curvature $H$ of the sphere $S_r = \{|x| = r\}$ with respect to the unit normal in the direction of $\partial/\partial r$ is

$$H = \frac{1}{(1 - \frac{m}{2r})^2} \left(\frac{2}{r} - \frac{4}{1 - \frac{m}{2r} r^2}\right).$$

The mean curvature of $S_r$ when it is embedded in $\mathbb{R}^3$ is

$$H_0 = \frac{1}{(1 - \frac{m}{2r})^2} \frac{2}{r}.$$

Suppose $r < \frac{m}{2}$, then $H < 0$ and

$$|H| - H_0 = -\frac{4}{r(1 - \frac{m}{2r})^3} > 0.$$

Hence, $m_{LY}(S_{r_1}) < 0$ and $m_{LY}(S_{r_2}) < 0$ where $\partial \Omega = S_{r_1} \cup S_{r_2}$.

In [13], Ó Murchadha, Szabados and Tod gave some examples of a spacelike 2-surface, lying on the light cone of the Minkowski space $\mathbb{R}^{3,1}$, whose Liu-Yau mass is strictly positive. Motivated by their result, we want to understand the Liu-Yau mass of more general spacelike 2-surfaces in $\mathbb{R}^{3,1}$. In the sequel, we always regard $\mathbb{R}^3$ as the $t = 0$ slice in $\mathbb{R}^{3,1}$. We have the following:

**Theorem 4.1.** Let $\Sigma$ be a closed, connected, smooth, spacelike 2-surface in $\mathbb{R}^{3,1}$. Suppose $\Sigma$ spans a compact spacelike hypersurface in $\mathbb{R}^{3,1}$. If $\Sigma$ has positive Gaussian curvature and has spacelike mean curvature vector, then $m_{LY}(\Sigma) \geq 0$; moreover $m_{LY}(\Sigma) = 0$ if and only if $\Sigma$ lies on a hyperplane in $\mathbb{R}^{3,1}$. 

In order to prove this theorem, we need the following result which can be proved by the method of Bartnik and Simon \cite{BartnikSimon} and by an idea from Bartnik \cite{Bartnik1}. In fact, it is just a special case of the results by Bartnik \cite{Bartnik2}.

**Lemma 4.1.** Let $\Sigma$ be a closed, connected, smooth, spacelike 2-surface in $\mathbb{R}^{3,1}$. Suppose $\Sigma$ spans a compact spacelike hypersurface in $\mathbb{R}^{3,1}$. Then $\Sigma$ spans a compact, smoothly immersed, maximal spacelike hypersurface in $\mathbb{R}^{3,1}$.

**Proof.** Let $M$ be a compact spacelike hypersurface in $\mathbb{R}^{3,1}$ spanned by $\Sigma$. By extending $M$ a bit, we may assume that there exists a spacelike hypersurface $\tilde{M}$ in $\mathbb{R}^{3,1}$ such that $M \subset \tilde{M}$. Since $\tilde{M}$ is spacelike, $\tilde{M}$ is locally a graph over an open set in $\mathbb{R}^3$. Hence, the projection map $\pi : \tilde{M} \to \mathbb{R}^3$, given by $\pi(x,t) = x$, is a local diffeomorphism. Now consider the map

$$F : \tilde{M} \times \mathbb{R}^1 \longrightarrow \mathbb{R}^{3,1},$$

given by $F(p,s) = (x,s)$ for any $p = (x,t) \in \tilde{M}$, then $F$ is a local diffeomorphism as well. Let $N = \tilde{M} \times \mathbb{R}^1$ equipped with the pull back metric. Let $v$ be the time function on $\tilde{M}$ in $\mathbb{R}^{3,1}$, i.e. $v(x,t) = t$. Since $v$ is a smooth function on $\tilde{M}$, we can consider its graph in $N$. Let $\hat{\Sigma}$ and $\hat{G}$ be the graph of $v$ over $\Sigma$ and $\tilde{M}$ in $N$ respectively. Then $\hat{G}$ is a compact, spacelike hypersurface in $N$ whose boundary is $\hat{\Sigma}$. Moreover, $F|_{\hat{G}} : \hat{G} \to \overline{M} \subset \mathbb{R}^{3,1}$ and $F|_{\hat{\Sigma}} : \hat{\Sigma} \to \Sigma \subset \mathbb{R}^{3,1}$ are both isometries.

Now one can carry over the arguments in section 3 in \cite{BartnikSimon} to prove that there is a smooth solution (defined on $M$) to the maximal surface equation in $N$ such that, if $G$ is its graph in $N$, then $\partial G = \hat{\Sigma}$. For example, Lemma 3.3 in \cite{BartnikSimon} can be rephrased as: For $\theta > 0$, let

$$\mathcal{D} = \{ \phi \in C^{0,1}(\overline{M}) | |D\phi| \leq (1 - \theta) \}$$

and

$$\mathcal{F} = \{ u \in C^2(M) | |Du| < 1 \text{ with maximal graph} $$

and $u = \phi$ on $\Sigma$ for some $\phi \in \mathcal{D}$.

Then there exists $r_0 > 0$ and $\theta_1 > 0$ such that for all $u \in \mathcal{F}$ and for all $p,q \in M$ with $d(p,\Sigma), d(q,\Sigma) < \frac{1}{3}r_0$ (say) and $d(p,q) = r_0$ we have: $|u(p) - u(q)| \leq (1 - \theta_1)r_0$.

One then readily checks that $F(G)$ is a compact, smoothly immersed, maximal hypersurface in $\mathbb{R}^{3,1}$ spanned by $\Sigma = F(\hat{\Sigma})$. $\square$
To prove Theorem 4.1, we also need a technical lemma concerning the boundary mean curvature of a compact spacelike hypersurface in \( \mathbb{R}^{3,1} \), whose boundary has spacelike mean curvature vector.

**Lemma 4.2.** Let \( M \) be a compact 3-manifold with boundary \( \partial M \). Let \( F : M \to \mathbb{R}^{3,1} \) be a smooth, maximal spacelike immersion such that \( F|_{\partial M} : \partial M \to F(\partial M) \subset \mathbb{R}^{3,1} \) has spacelike mean curvature vector. Let \( g \) be the pull back metric on \( M \) and \( k \) be the mean curvature of \( \partial M \) in \( (M,g) \) with respect to the outward unit normal. Then \( k \) must be nonnegative at some point on \( \partial M \).

**Proof.** Suppose \( k < 0 \) everywhere on \( \partial M \). Since \( F(M) \) is a compact subset in \( \mathbb{R}^{3,1} \), without loss of generality, we may assume that \( F(M) \subset \{x_1 \leq 0\} \) and \( F(M) \cap \{x_1 = 0\} \neq \emptyset \). Let \( X_0 = F(q) \in F(M) \cap \{x_1 = 0\} \) for some \( q \in M \). If \( q \) is an interior point of \( M \), then there exists an open neighborhood \( V \) of \( q \) in the interior of \( M \) such that the tangent space of \( F(V) \) at \( X_0 \) is \( \{x_1 = 0\} \). This is impossible, because \( F(V) \) needs to be spacelike. Therefore, \( q \in \partial M \). Using the fact that \( F \) is a spacelike immersion again, we know there exists an open neighborhood \( U \) of \( q \) in \( M \) such that \( F(U) \) is a spacelike graph of some function \( f \) over \( \overline{D} \) for some open set \( D \subset \mathbb{R}^3 \cap \{x_1 \leq 0\} \). Let \( B = F(\overline{U} \cap \partial M) \) and let \( \hat{B} \) be the part of \( \partial D \) such that \( B \) is the graph of \( f \) over \( \hat{B} \). We note that \( X_0 \in B \). Without loss of generality, we may assume that \( X_0 \) is the origin.

To proceed, we let \( T = \frac{\partial}{\partial t} \) and define the following notations:
- \( n \): the future time like unit normal to \( F(\overline{U}) \) in \( \mathbb{R}^{3,1} \);
- \( \nu \): the unit outward normal to \( B \) in \( F(\overline{U}) \);
- \( \hat{\nu} \): the unit outward normal to \( \hat{B} \) in \( \overline{D} \).

We parallel translate \( \nu \), \( \hat{\nu} \) and all the tangent vectors of \( B \), \( \hat{B} \) along the \( T \) direction. Also, we consider \( f \) as a function on \( D \times (-\infty, \infty) \) so that \( f \) is independent of \( t \).

Now \( \hat{\nu} \) is normal to \( B \), so \( \hat{\nu} = u\nu + vn \) for some numbers \( u, v \) satisfying \( u^2 - v^2 = 1 \). At \( X_0 \in B \), we have \( \hat{\nu} = \frac{\partial}{\partial x_1} \). Suppose \( \alpha(s) = (x_1(s), x_2(s), x_3(s), t(s)) \) is a curve in \( F(\overline{U}) \) such that \( \alpha(0) = X_0 \) and \( \alpha'(0) = \nu \). Then, for \( t < 0 \) small, \( \alpha(t) \in F(\overline{U}) \) and so \( x_1(t) < 0 \). Since \( x_1(0) = 0 \), we have \( x_1'(0) \geq 0 \), hence \( u = \langle \hat{\nu}, \nu \rangle \geq 0 \). Since \( u^2 = 1 + v^2 \), we have \( u > |v| \) at \( X_0 \).

Let \( \vec{H} \) be the mean curvature vector of \( B \) in \( \mathbb{R}^{3,1} \). Let \( p = p_{ij} \) be the second fundamental form of \( F(\overline{U}) \) in \( \mathbb{R}^{3,1} \) with respect to \( n \). Then

\[
(4.3) \quad \vec{H} = -k \nu + (\text{tr}_B p)n,
\]
where \( \text{tr}_B p \) denotes the trace of \( p \) restricted to \( B \). Hence,
\[
- \langle \vec{H}, \hat{\nu} \rangle = u k + v (\text{tr}_B p).
\]
At \( X_0 \), we have shown \( u > |v| \). On the other hand, we know \( |k| > |\text{tr}_B p| \)
(because \( \vec{H} \) is spacelike) and \( k < 0 \) (by the assumption), therefore we have \(- \langle \vec{H}, \hat{\nu} \rangle < 0 \) at \( X_0 \). Recall that
\[
\langle \vec{H}, \hat{\nu} \rangle = \left( \sum_{i=1}^{2} \nabla e_i e_i, \nu \right),
\]
where \( \{e_1, e_2\} \) is an orthonormal frame in \( T_{X_0} B \) and \( \nabla \) is the covariant derivative in \( \mathbb{R}^{3,1} \). Hence there exists a unit vector \( e \in T_{X_0} B \) such that
\[
- \langle \nabla e, \hat{\nu} \rangle < 0.
\]
Suppose \( e \) is the tangent of a curve \( \gamma(s) \subset B \) at \( s = 0 \). Let \( \hat{\gamma}(s) \subset \hat{B} \) be the projection of \( \gamma(s) \) in \( \mathbb{R}^3 \). Then
\[
\gamma'(s) = \hat{\gamma}'(s) + \frac{d}{ds} f(\hat{\gamma}(s)) T.
\]
Hence,
\[
- \langle \nabla \gamma'(s) \hat{\gamma}'(s), \hat{\nu} \rangle = - \langle \nabla \gamma'(s) \hat{\gamma}'(s), \hat{\nu} \rangle = - \langle \nabla \gamma'(s) \hat{\gamma}'(s), \hat{\nu} \rangle,
\]
where we have used the facts that \( T \) is parallel, \( T \perp \hat{\nu} \) and \( \hat{\gamma}'(s) \) is parallel translated along \( T \). Thus, it follows from \( (4.6), (4.8) \) and the fact \( e = \gamma'(0) \) that
\[
- \langle \nabla \gamma'(0) \hat{\gamma}'(0), \hat{\nu} \rangle < 0.
\]
But this is impossible because \( \hat{\gamma}(s) \subset \hat{B} \subset \{ x_1 \leq 0 \} \cap \mathbb{R}^3 \) and \( \hat{\nu} = \frac{\partial}{\partial x_1} \) at \( \hat{\gamma}(0) \). Therefore, we have proved that \( k \) can not be negative everywhere on \( \partial M \). \( \square \)

**Proof of Theorem 4.1.** By Lemma 4.1, we know that \( \Sigma \) indeed bounds a compact, smoothly immersed, maximal spacelike hypersurface in \( \mathbb{R}^{3,1} \). Precisely, this means that there exists a compact 3-manifold \( M \) with boundary \( \partial M \) and a smooth, maximal spacelike immersion \( F : M \to \mathbb{R}^{3,1} \) such that \( F : \partial M \to \Sigma \) is a diffeomorphism.

Let \( g = g_{ij} \) be the pull back metric on \( M \). Let \( p = p_{ij} \) be the second fundamental form of the immersion \( F : M \to \mathbb{R}^{3,1} \). Let \( R \) be the scalar curvature of \( (M, g) \). Since \( F \) is a maximal immersion, it follows from the constraint equations (or simply the Gauss equation) that
\[
R = |p|^2 \geq 0,
\]
where \(| \cdot |\) is taken with respect to \(g\). On the other hand, let \(k\) be the mean curvature of \(\partial M\) in \((M, g)\) with respect to the outward unit normal and let \(\vec{H}\) be the mean curvature vector of \(\Sigma = F(\partial M)\) in \(\mathbb{R}^{3,1}\), it is known that
\[
|\vec{H}|^2 = k^2 - (\text{tr}_\Sigma p)^2, \tag{4.11}
\]
where \(\text{tr}_\Sigma p\) is the trace of \(p\) restricted to \(\Sigma\). Since \(\vec{H}\) is spacelike, (4.11) implies that either \(k > 0\) or \(k < 0\) on \(\partial M\) because \(\partial M\) is connected.

By Lemma 4.2, we have \(k > 0\) on \(\partial M\).

Now let \(k_0\) be the mean curvature of \(\Sigma\) with respect to the unit outward normal when it is isometrically embedded in \(\mathbb{R}^3\). It follows from (4.11) that
\[
\int_{\Sigma} (k_0 - |\vec{H}|) \, d\sigma \geq \int_{\Sigma} (k_0 - k) \, d\sigma. \tag{4.12}
\]
On the other hand, by the result of [19], we have
\[
\int_{\Sigma} (k_0 - k) \, d\sigma \geq 0 \tag{4.13}
\]
and equality holds if and only if \((M, g)\) is a domain in \(\mathbb{R}^3\). Thus, we conclude from (4.12) and (4.13) that \(m_{LY}(\Sigma) \geq 0\). Moreover, if \(m_{LY}(\Sigma) = 0\), then \((M, g)\) must be flat, hence \(R = 0\) and consequently \(p = 0\). Therefore, \(F(M)\) and hence \(\Sigma\) lie on a hyperplane in \(\mathbb{R}^{3,1}\). Conversely, if \(\Sigma\) lies on a hyperplane in \(\mathbb{R}^{3,1}\), then obviously \(m_{LY}(\Sigma) = 0\). \(\square\)

In the sequel, we want to show that the examples given in [18] satisfy the assumption in Theorem 4.1. To do that, we need the following definition:

**Definition 4.1.** Two points \(p\) and \(q\) in a Lorentzian manifold \(N\) are said to be *causally related* if \(p\) and \(q\) can be joined by a timelike or null path. A set \(S\) in \(N\) is called *acausal* if no two points in \(S\) are causally related.

We claim that all surfaces in the examples in [18] are acausal. Suppose this claim is true, then by Theorem 3 in [9] (P.4765), we know that those surfaces span spacelike hypersurfaces in \(\mathbb{R}^{3,1}\), hence satisfying the assumption in Theorem 4.1.

To verify the claim, let \(\Sigma\) be an example given in [18], i.e. in terms of the usual spherical coordinates \((t, r, \theta, \phi)\) in \(\mathbb{R}^{3,1}\), \(\Sigma\) is determined by the equation
\[
t = r = F(\theta, \phi), \tag{4.14}
\]
where \( F = F(\theta, \phi) \) is a smooth positive function of \((\theta, \phi) \in S^2\). Suppose \( \Sigma \) is not acausal, then there exists two distinct points \( p, q \) in \( \Sigma \) and a path \( \gamma(\tau) \) in \( \Sigma \) such that \( \gamma(0) = p, \gamma(1) = q \), and

\[
(4.15) \quad (\dot{x})^2 + (\dot{y})^2 + (\dot{z})^2 \leq (\dot{t})^2, \quad \forall \ \tau \in [0, 1],
\]

here we denote \( \gamma(\tau) = (x(\tau), y(\tau), z(\tau), t(\tau)) \) and “\( \cdot \)” denotes the derivative with respect to \( \tau \). Since \( \dot{\gamma} \neq 0 \), without loss of generality, we may assume \( \dot{t} > 0 \). Let \( r = \sqrt{x^2 + y^2 + z^2} \), \( (4.15) \) implies that

\[
(4.16) \quad |\dot{r}| \leq \dot{t}.
\]

Note that \( r(0) = t(0) \) and \( r(1) = t(1) \), we see that

\[
(4.17) \quad |\dot{r}| = \dot{t},
\]

for all \( \tau \in [0, 1] \). By the equality case in the Cauchy-Schwartz inequality, we must have

\[
(4.18) \quad x = k(\tau)\dot{x}, \ y = k(\tau)\dot{y}, \ z = k(\tau)\dot{z}
\]

for some function \( k = k(\tau) \) and for all \( \tau \in [0, 1] \). Clearly, this implies that \( p \) and \( q \) lie on a line which passes through the origin, or equivalently, \( p = aq \) for some positive number \( a \). On the other hand, using the Cartesian coordinates, we may write \( p \) as

\[
(F(\theta_1, \phi_1), F(\theta_1, \phi_1) \sin \theta_1 \cos \phi_1, F(\theta_1, \phi_1) \sin \theta_1 \sin \phi_1, F(\theta_1, \phi_1) \cos \theta_1)
\]

and write \( q \) as

\[
(F(\theta_2, \phi_2), F(\theta_2, \phi_2) \sin \theta_2 \cos \phi_2, F(\theta_2, \phi_2) \sin \theta_2 \sin \phi_2, F(\theta_2, \phi_2) \cos \theta_2)
\]

for some \((\theta_i, \phi_i) \in S^2, i = 1, 2\). The fact \( p = aq \), for some \( a > 0 \), then implies \( p = q \), which is contradiction. Therefore, \( \Sigma \) is acausal.

5. Appendix

In this appendix, we give some Lemmas which are needed to complete the proof of Proposition 3.2. We will follow closely Nirenberg’s argument in [17]. First, we introduce some notations: given an integer \( k \geq 2 \) and a positive number \( \alpha < 1 \), let

\[
\mathcal{E}^{k,\alpha} = \text{the space of } C^{k,\alpha} \text{ embeddings of } S^2 \text{ into } \mathbb{R}^3
\]

\[
\mathcal{A}^{k,\alpha} = \text{the space of } C^{k,\alpha} \mathbb{R}^3\text{-valued vector functions on } S^2
\]

\[
\mathcal{S}^{k,\alpha} = \text{the space of } C^{k,\alpha} \text{ symmetric } (0, 2) \text{ tensors on } S^2
\]

\[
\mathcal{M}^{k,\alpha} = \text{the space of } C^{k,\alpha} \text{ Riemannian metrics on } S^2.
\]

On page 353 in [17], Nirenberg proved
Lemma 5.1. Let $\sigma \in \mathcal{M}^{4,\alpha}$ be a metric with positive Gaussian curvature. Let $X \in \mathcal{E}^{4,\alpha}$ be an isometric embedding of $(S^2, \sigma)$ in $\mathbb{R}^3$. There exists two positive numbers $\epsilon$ and $C$, depending only on $\sigma$, such that if $\tau \in \mathcal{M}^{2,\alpha}$ satisfying

$$||\sigma - \tau||_{C^2,\alpha} < \epsilon,$$

then there is an isometric embedding $Y \in \mathcal{E}^{2,\alpha}$ of $(S^2, \tau)$ in $\mathbb{R}^3$ such that

$$||X - Y||_{C^2,\alpha} \leq C||\sigma - \tau||_{C^2,\alpha}.$$

In what follows, we want to show that the constants $\epsilon$ and $C$ in the above Lemma can be chosen to be independent on $\sigma$, provided $\sigma$ is sufficiently close to some $\sigma_0 \in \mathcal{M}^{5,\alpha}$ (see Lemma 5.3). First, we prove the following:

Lemma 5.2. Let $\sigma_0 \in \mathcal{M}^{5,\alpha}$ be a metric with positive Gaussian curvature. There exists positive numbers $\delta$ and $\tilde{K}$, depending only on $\sigma_0$, such that if $\sigma \in \mathcal{M}^{4,\alpha}$ satisfying

$$||\sigma_0 - \sigma||_{C^2,\alpha} < \delta,$$

then for any $\gamma \in S^{2,\alpha}$ and any $Z \in \mathcal{X}^{2,\alpha}$, there exists a solution $Y \in \mathcal{X}^{2,\alpha}$ to the linear equation

$$2dX^{\sigma} \cdot dY = \gamma - (dZ)^2.$$  \hspace{1cm} (5.1)

Here $X^{\sigma} \in \mathcal{E}^{4,\alpha}$ is any given isometric embedding of $(S^2, \sigma)$. Moreover, for every $Z$ (with $\gamma$ fixed), a particular solution $Y$ denoted by $\Phi(Z)$ may be chosen so that

$$||\Phi(Z)||_{C^2,\alpha} \leq \tilde{K} (||\gamma||_{C^2,\alpha} + ||Z||_{C^2,\alpha}^2),$$

and for any $Z, Z_1 \in \mathcal{E}^{2,\alpha}$,

$$||\Phi(Z) - \Phi(Z_1)||_{C^2,\alpha} \leq \tilde{K} ||Z + Z_1||_{C^2,\alpha} \cdot ||Z - Z_1||_{C^2,\alpha}.  \hspace{1cm} (5.2)$$

$$||\Phi(Z) - \Phi(Z_1)||_{C^2,\alpha} \leq \tilde{K} ||Z + Z_1||_{C^2,\alpha} \cdot ||Z - Z_1||_{C^2,\alpha}.  \hspace{1cm} (5.3)$$

Proof. We proceed exactly as in [17]. For any $\sigma \in \mathcal{M}^{4,\alpha}$, let $X^{\sigma}$ be a given isometric embedding of $(S^2, \sigma)$ in $\mathbb{R}^3$. Let $X_3$ be the unit inner normal to the surface $X^{\sigma}(S^2)$. Let $\{u, v\}$ be a local coordinate chart on $S^2$. Let $\phi, p_1, p_2$ be defined as in (6.5), (6.6) in [17]. Then $\phi, p_1, p_2$ satisfy the system of equations (6.11)-(6.13) in [17] with $c_1, c_2, \Delta$ defined on page 356-357 in [17]. By section 6.3 in [17], the derivatives of $Y$ are completely determined by $\phi$, which satisfies (6.15) in [17]. Let $\phi$ be given by (7.3) in [17], following the first paragraph in section 8.1 in [17], we obtain a unique solution $Y \in \mathcal{E}^{2,\alpha}$ to (5.1), normalized to vanish at a fixed point on $S^2$. We denote such a $Y$ by $Y = \Phi(Z)$. 
To prove estimates (5.2) and (5.3), by the Remark on page 365 in [17] and the proof following it, we know it suffices to show

\[
\|Y\|_{C^{2,\alpha}} \leq C \left[ \|d\bar{\sigma}^2\|_{C^{1,\alpha}} + \|\frac{1}{\Delta}(c_{1v} - c_{2u})\|_{C^{\alpha}} \right],
\]

where \(d\bar{\sigma}^2 = \gamma - (dZ)^2\), \(c_{1v}, c_{2u}\) are derivatives of \(c_1, c_2\) with respect to \(v, u\) respectively. On the other hand, by section 9 in [17], to prove (5.4), it suffices to establish an \(C^{1,\alpha}\) estimate of \(\phi\):

\[
\|\phi\|_{C^{1,\alpha}} \leq C \|d\bar{\sigma}^2\|_{C^{1,\alpha}}.
\]

Therefore, in what follows, we will prove that there are positive numbers \(\delta\) and \(C\), depending only on \(\sigma_0\), such that (5.5) holds for any \(\sigma\) satisfying \(\|\sigma^0 - \sigma\|_{C^{2,\alpha}} < \delta\).

We first recall the fact that \(\phi\) is a solution to the second order elliptic equation (6.15) in [17]. For simplicity, we let

\[
L_\sigma(\phi) = \mathcal{L}(\phi_u, \phi_v), \quad F_\sigma(d\bar{\sigma}^2) = \mathcal{L}(c_1, c_2) - T,
\]

where \(\mathcal{L}(\phi_u, \phi_v), \mathcal{L}(c_1, c_2) - T\) are given as in (6.16) and (6.14) in [17], then (6.15) in [17] becomes

\[
L_\sigma(\phi) + H_\sigma \phi = F_\sigma(d\bar{\sigma}^2),
\]

where \(H_\sigma\) is the mean curvature of \(X^\sigma(S^2)\) w.r.t \(X_3\) (note that our \(H\) here equals 2\(H\) in [17]). On the other hand, since we have chosen \(\phi\) to be given by the integral formula (7.3) in [17], we know \(\phi\) is a special solution to (5.7) in the sense that \(\phi\) is \(L^2\)-perpendicular to the kernel of the operator \(L_\sigma(\cdot) + H_\sigma\) (See page 359 in [17]). For any \(\sigma \in \mathcal{M}^{4,\alpha}\), let \(\text{Ker}(\sigma)\) denote the space of solutions \(\psi\) to the the homogeneous equation

\[
L_\sigma(\psi) + H_\sigma \psi = 0.
\]

On page 360 in [17], it was shown that \(\text{Ker}(\sigma)\) is spanned by the coordinate functions of \(X_3\).

Note that the coefficient of (5.7) depends only on the metric \(\sigma\). Therefore, if \(\sigma\) is close to \(\sigma^0\) in \(C^{2,\alpha}\), we know by Theorem 8.32 in [10] that, to prove the \(C^{1,\alpha}\) estimate (5.5), it suffices to prove the following \(C^0\) estimate

\[
\|\phi\|_{C^0} \leq C \|d\bar{\sigma}^2\|_{C^{1,\alpha}},
\]

where \(C\) is some positive constant independent on \(\sigma\), provided \(\sigma\) is sufficiently close to \(\sigma^0\) in \(C^{2,\alpha}\).

Suppose (5.9) is not true, then there exists \(\{\sigma_i\} \subset \mathcal{M}^{4,\alpha}\) which converges to \(\sigma^0\) in \(C^{2,\alpha}\), \(\{d\bar{\sigma}^2_i\} \subset S^{1,\alpha}\) with \(\|d\bar{\sigma}^2_i\|_{C^{1,\alpha}} = 1\), and a sequence
of numbers \( \{C_i\} \) approaching \(+\infty\) so that the corresponding \( \phi_i \) (of \( Y = Y_i \)) satisfies
\[
||\phi_i||_{C^0} \geq C_i.
\]
Consider \( \xi_i = \phi_i/||\phi_i||_{C^0} \), then \( \xi_i \) satisfies
\[
(5.10) \quad L_{\sigma_i}(\xi_i) + 2H_{\sigma_i}\xi_i = ||\phi_i||_{C^0}^{-1}F_{\sigma_i}(d\bar{\sigma}_i^2).
\]
By Theorem 8.32 in [10], we conclude from (5.10) and the facts \( \{\sigma_i\} \) converges to \( \sigma^0 \) in \( C^{2,\alpha} \) and \( ||\xi_i||_{C^0} = 1 \) that
\[
(5.11) \quad ||\xi_i||_{C^{1,\alpha}} \leq C,
\]
where \( C \) is some positive constant independent on \( i \). Now (5.11) implies that \( \xi_i \) converges in \( C^1 \) to some \( \xi \) which is also in \( C^{1,\alpha} \). Moreover, \( ||\xi||_{C^0} = 1 \). By (5.10), \( \xi \) is a weak solution to the equation
\[
(5.12) \quad L_{\sigma^0}\xi + H_{\sigma^0}\xi = 0.
\]
Since \( \sigma^0 \in C^{5,\alpha} \), the coefficients of (5.12) (given by (6.16) in [17]) are then in \( C^{3,\alpha} \), hence in \( C^{2,1} \). By Theorem 8.10 in [10], we know \( \xi \in W^{4,2} \), hence in \( C^2 \). Therefore, \( \xi \) is a classic solution to (5.12), i.e. \( \xi \in Ker(\sigma^0) \). On the other hand, we know \( \phi_i \), hence \( \xi_i \), is \( L^2 \)-perpendicular to \( Ker(\sigma_i) \) for each \( i \). Since \( \{\sigma_i\} \) converges to \( \sigma^0 \) in \( C^{2,\alpha} \) and \( \{\xi_i\} \) converges to \( \xi \) in \( C^1 \), we conclude that \( \xi \) must be \( L^2 \)-perpendicular to \( Ker(\sigma^0) \). Hence, \( \xi \) must be zero. This is a contradiction to the fact \( ||\xi||_{C^0} = 1 \). Therefore, we conclude that (5.9) holds.

As mentioned earlier, once we establish the \( C^0 \) estimate (5.9), we will have the \( C^{1,\alpha} \) estimate (5.5). Then we can proceed as in the rest of section 9 in [17] to prove (5.4), hence prove (5.2) and (5.3).

We note that the constants \( \epsilon \) and \( C \) in Lemma 5.1 indeed can be chosen as \( \epsilon = \frac{1}{4\bar{K}^2} \) and \( C = 2\bar{K} \), where \( \bar{K} \) is the constant in Theorem 2′ on page 352 in [17]. Therefore, by applying the exactly same iteration argument as on page 352-353 in [17], one concludes from Lemma 5.2 that

**Lemma 5.3.** Let \( \sigma^0 \in \mathcal{M}^{5,\alpha} \) be a metric with positive Gaussian curvature. There exists positive numbers \( \delta, \epsilon \) and \( C \), depending only on \( \sigma^0 \), such that for any \( \sigma \in \mathcal{M}^{4,\alpha} \) satisfying
\[
||\sigma^0 - \sigma||_{C^{2,\alpha}} < \delta,
\]
if \( \tau \in \mathcal{M}^{2,\alpha} \) satisfying
\[
||\sigma - \tau||_{C^{2,\alpha}} < \epsilon,
\]
then there is an isometric embedding \( Y \in \mathcal{E}^{2,\alpha} \) of \( (S^2, \tau) \) in \( \mathbb{R}^3 \) such that
\[
||X - Y||_{C^{2,\alpha}} \leq C||\sigma - \tau||_{C^{2,\alpha}}.
\]
Here \( X \in \mathcal{E}^{4,\alpha} \) is any given isometric embedding of \((S^2, \sigma)\).

**References**

1. Arnowitt, R., Deser, S. and Misner, C. W., *Coordinate invariance and energy expressions in general relativity*, Phys. Rev. (2) **122**, (1961), 997–1006.

2. Bartnik, R., *private communications*.

3. Bartnik, R., *Regularity of variational maximal surfaces*, Acta Math. **161** (1988), no. 3-4, 145–181.

4. Bartnik, R. and Simon, L.*Spacelike hypersurfaces with prescribed boundary values and mean curvature* Commun. Math. Phys. 87, 131-152(1982).

5. Brown, J. David and York, Jr., *James W. Quasilocal energy in general relativity*. In *Mathematical aspects of classical field theory (Seattle, WA, 1991)*, volume 132 of *Contemp. Math.*, pages 129–142. Amer. Math. Soc., Providence, RI, 1992.

6. Brown, J. David and York, Jr., *James W. Quasilocal energy and conserved charges derived from the gravitational action*. Phys. Rev. D (3), 47(4):1407–1419, 1993.

7. Corvino, J., *Scalar curvature deformation and a gluing construction for the Einstein constraint equations*, Commun. Math. Phys **214** (1) (2000), 137-189.

8. Fan, X.-Q., Shi, Y.-G. and Tam, L.-F., *Large-sphere and small-sphere limits of the Brown-York mass*, Comm. Anal. Geom., **17** (2009), 3772.

9. F. J. Flatherty. *The boundary value problem for maximal hypersurfaces* Proc. Natl. Acad. Sci. USA. Vol. 76, No. 10, pp. 4765-4767, October 1979

10. Gilbarg, D. and Trudinger, N. S., *Elliptic partial differential equations of second order*, second edition, Springer-Verlag, (1983).

11. Huisken, G. and Ilmanen, T. *The inverse mean curvature flow and the Riemannian Penrose Inequality*, J. Differential Geom. **59** (2001), 353–437.

12. Kobayashi, O., *A differential equation arising from scalar curvature function*, J. Math. Soc. Japan **34** (1982), no. 4, 665–675.

13. Liu, C.-C.M. and Yau, S.-T., *Positivity of quasilocal mass*, Phys. Rev.Lett. 90(2003) No. 23, 231102

14. Liu, C.-C.M. and Yau, S.-T., *Positivity of quasilocal mass II*, J. Amer.Math.Soc. 19(2006) No. 1.181-204.

15. Miao, P., *Positive mass theorem on manifolds admitting corners along a hypersurface*, Adv. Theor. Math. Phys. **6** (2002), no. 6, 1163–1182 (2003).

16. Miao, P. and Tam, L.-F., *On the volume functional of compact manifolds with boundary with constant scalar curvature*, to appear in Calc. Var. Partial Differential Equations, arXiv: 0807.2693

17. Nirenberg, L., *The Weyl and Minkowski problems in differential geometry in the large*, Comm. Pure Appl. Math. 6 (1953), 337-394.

18. N. Ó. Murchadha, L.B. Szabados, and K.P. Tod, *Comment on ” Positivity of quasi-local mass”* Phys. Rev. Letter92(2004), 259001.

19. Shi, Y.-G. and Tam, L.-F., *Positive mass theorem and the boundary behaviors of compact manifolds with nonnegative scalar curvature*, J. Differential Geom. **62** (2002), 79–125.

20. Shi, Y.-G. and Tam, L.-F., *Rigidity of compact manifolds and positivity of quasi-local mass*, Classical Quantum Gravity **24** (2007), no. 9, 2357–2366.
21. Christodoulou, D. and Yau, S.-T., *Some remarks on the quasi-local mass* Mathematics and general relativity (Santa Cruz, CA, 1986), Contemp. Math. 71 (1986), Amer. Math. Soc., p. 9–14.

22. Wang, M.-T. and Yau, S.-T., *A generalization of Liu-Yau’s quasi-local mass* Comm. Anal. Geom. 15 (2007), no. 2, 249–282.

23. Wang, M.-T. and Yau, S.-T., *Isometric embeddings into the Minkowski space and new quasi-local mass*, Comm. Math. Phys. 288(3):919–942, 2009, arXiv: 0805.1370v3

School of Mathematical Sciences, Monash University, Victoria, 3800, Australia.

*E-mail address:* Pengzi.Miao@sci.monash.edu.au

Key Laboratory of Pure and Applied Mathematics, School of Mathematics Science, Peking University, Beijing, 100871, P.R. China.

*E-mail address:* ygshi@math.pku.edu.cn

The Institute of Mathematical Sciences and Department of Mathematics, The Chinese University of Hong Kong, Shatin, Hong Kong, China.

*E-mail address:* lftam@math.cuhk.edu.hk