Probing nonlocality of Majorana fermions in Josephson junctions of Kitaev chains connected to normal metal leads

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Kitaev chain is a prototypical model for the study of Majorana fermions (MFs). In the topological phase, a Kitaev chain hosts two MFs at its ends. Being separated in space, these two MFs are nonlocal. When the Kitaev chain is connected to two normal metal leads, the nonlocal transport is mediated by electron tunneling (ET) and crossed Andreev reflection (CAR). ET contributes positively while CAR contributes negatively to the nonlocal conductance. Enhanced CAR and hence a negative nonlocal conductance is a definite signature of nonlocality of MFs. But simple conductance measurements in the above setup cannot probe the nonlocality of MFs due to the almost cancellation of currents from ET and CAR. On the other hand, a Josephson junction between two Kitaev chains hosts two Andreev bound states (ABSs) at the junction formed by a recombination of Majorana fermions of the individual Kitaev chains. The energies of the ABSs are away from zero and can be changed by altering the superconducting phase difference. A Josephson junction between two finitely long Kitaev chains hosts two MFs at the two ends and two ABSs at the junction. We show that when normal metal leads are connected to two ends of such a Josephson junction, the nonlocal conductance of the setup can be negative for bias values equal to the energies of the ABSs and thus the nonlocal conductance of this setup can be used as a probe of the nonlocality of the constituent MFs.

I. INTRODUCTION

In 2001, Kitaev proposed a one-dimensional lattice model with p-wave superconductivity in which isolated Majorana fermions (MFs) can exist. This lattice model known as Kitaev chain attracted huge interest because MFs could be used as building blocks of a topological quantum computer. An advance in modeling put forward the idea that isolated MFs can be realized in spin orbit coupled quantum wires placed in proximity to a singlet superconductor in presence of a Zeeman field. In the years that followed, several experiments convincingly observed zero bias conductance peak in normal metal leads connected to such quantum wires as predicted by the theory upholding the realization of isolated MF. However, there are two MFs in a Kitaev chain and they are spatially separated. This nonlocal aspect of MFs has not been studied experimentally. In the limit of infinite length of the Kitaev chain, the two MFs are decoupled and are exactly at zero energy, but for a finite length of the Kitaev chain the two MF wavefunctions overlap leading to the formation of two nonlocal Dirac fermions (this happens generically except for a special choice of parameters). If two normal metal leads are connected to two ends of the Kitaev chain, the nonlocal transport in the setup is mediated by electron tunneling (ET) and crossed Andreev reflection (CAR). In the former (latter), an electron incident from one normal metal tunneling through the Kitaev chain and exits onto the other normal metal as an electron (as a hole). The local transport is mediated by electron reflection (ER) and Andreev reflection (AR). In the former (latter), the electron incident from one normal metal results in a reflected electron (hole) in the same normal metal. Local conductance is the ratio of differential current in the first normal metal to the differential bias applied to the first normal metal maintaining the Kitaev chain and the second normal metal grounded. Nonlocal conductance or transconductance is the ratio of differential current in the second normal metal to the differential bias applied to the first normal metal maintaining the Kitaev chain and the second normal metal grounded. AR is a definite signature of isolated MFs and a local conductance of $2e^2/h$ at zero bias owing to perfect AR is obtained when the two MFs in the Kitaev chain are decoupled. AR is mediated by the single MF present at the end of the Kitaev chain which is connected to the first normal metal. CAR on the other hand is a nonlocal process mediated by the nonlocal Dirac fermion formed by the hybridization of the two MFs. Though ET is also mediated by the two MFs, it does not require MFs to be present. Therefore, an enhanced CAR over ET is a definite signature of nonlocality of the MFs. However, the currents due to CAR and ET almost cancel out in a simple setup consisting of two normal metals connected to a Kitaev chain. This has motivated proposals to probe nonlocality of MFs by noise measurements. Also, we recently proposed to employ a Kitaev ladder in series with Kitaev chain to form a setup which can probe the nonlocality of MFs by measurement of nonlocal conductance. Another setup that can enhance CAR over ET will not only probe the nonlocality of MFs but will also add to an assortment of many existing proposals to enhance CAR over ET.

In this paper, we propose to connect normal metal leads to a Josephson junction of two Kitaev chains as shown in Fig. 1. Such a Josephson junction has already been realized experimentally and connecting normal metal leads to such a Josephson junction appears to be an easy task. Josephson junction made out of p-wave superconductors exhibits fractional Josephson effect. A superconducting phase difference drives a Josephson current between the two superconductors, but we are inter-
are inserted in grounding the two KCs and normal metal (N_R). A bias voltage V is applied to N_L, while grounding the two KCs and N_R. Current meters denoted by I are inserted in N_L and N_R to measure the currents I_L and I_R respectively.

II. CALCULATIONS

The Hamiltonian for the proposed system is
\[ H = H_L + H_{JJ} + H_R + H_{JJL} + H_{JRR}, \]
where \( H_L \) describes the normal metal on left \( N_L \), \( H_{JJ} \) describes the Josephson junction made out of the Kiteev chains, \( H_R \) describes the normal metal on right \( N_R \), \( H_{JJL} \) describes the coupling between \( N_L \) and the Josephson junction and \( H_{JRR} \) describes the coupling between the Josephson junction and \( N_R \). Various terms in the Hamiltonian are
\[
\begin{align*}
H_L &= -t \sum_{n=-\infty}^{-1} (c_{n-1}^\dagger c_n + c_n^\dagger c_{n-1}) \\
H_{JJ} &= -t(c_{1}^\dagger c_1 + \text{h.c.}) + \Delta(e^{i\phi_1}c_0^\dagger c_1^\dagger + \text{h.c.}) - t_{MK}(c_1^\dagger c_2 + \text{h.c.}) - t(c_1^\dagger c_3 + \text{h.c.}) + \Delta(e^{i\phi_2}c_2^\dagger c_3^\dagger + \text{h.c.}) \\
H_R &= -t \sum_{n=4}^{\infty} (c_{n+1}^\dagger c_n + c_n^\dagger c_{n+1}) \\
H_{JJL} &= -t_{LL}(c_{-1}^\dagger c_0 + \text{h.c.}) \\
H_{JRR} &= -t_{RR}(c_{3}^\dagger c_4 + \text{h.c.}).
\end{align*}
\]

The Josephson junction is formed by two Kiteev chains having superconducting phases \( \phi_1 \) and \( \phi_2 \). We have chosen a minimal model for the Kiteev chain. The Kiteev chains here consist of only two sites. They host isolated MFs in the limits \( \Delta = \pm t \). For the choice \( \Delta \) close to \( t \), but \( \Delta \neq t \), the two MFs of the Kiteev chain are coupled. The two MFs at the junction hybridize and form Andreev bound states (ABSs) as shown in Fig. 1. In Fig. 2, energy levels of the isolated Josephson junction made of two Kiteev chains described by the Hamiltonian \( H_{JJ} \) are plotted as a function of the superconducting phase difference \( \phi = (\phi_1 - \phi_2) \) for the choice of parameters \( \Delta = 0.9t \) and \( t_{MK} = 0.4t \). The energy levels close to zero energy are those of the fermions formed by the hybridization of the MFs that live at the ends (away from the junction). The energy levels further away from zero energy but in the range \(-0.5\Delta, 0.5\Delta\) are ABSs that live at the junction as depicted in Fig. 1. These ABSs are formed by the hybridization of the MFs at the junction. The energy levels close to \( \pm 2\Delta \) are those of the quasiparticles that belong to the bulk of the Kiteev chains. Hence, a conductance spectroscopy in the energy range \(-\Delta, \Delta\) will probe the constituent MFs of the Kiteev chain that form the Josephson junction.

The wavefunction \( [\psi_n^e, \psi_n^h]^T \) at site \( n \), for an electron incident from \( N_L \) with energy \( E \) has the form
\[
\begin{align*}
\psi_n^e &= e^{ik_n a} + re^{-ik_n a} & \text{for } n \leq -1 \\
&= te^{ik_n a} & \text{for } n \geq 4 \\
\psi_n^h &= rh e^{-ik_n a} & \text{for } n \leq -1 \\
&= t_h e^{-ik_n a} & \text{for } n \geq 4,
\end{align*}
\]
where \( k_n a = \cos^{-1}[-(E+\mu)/2t] \), \( k_n a = \cos^{-1}[(E-\mu)/2t] \) and \( a \) is the lattice spacing. The scattering coefficients \( r_e, t_e, r_h, t_h \) can be determined by solving the equation \( H \Psi = E \Psi \) using the full Hamiltonian (eq. (1)) where \( \Psi \) is the full wavefunction. An electron incident from \( N_L \) at energy \( E \) contributes to the local differential conductance \( G_{LL} \) and the differential transconductance \( G_{RL} \) at a voltage bias \( V = E/e \) (where \( e \) is the electron charge). \( G_{LL} \ (G_{RL}) \) is the ratio of the change in current \( dI_L \ (dI_R) \)
in \( N_L (N_R) \) to the change in bias in \( N_L \) when the bias is changed from \( V \) to \( V + dV \). The two conductances are given by the Landauer-Buttiker formula\(^{16,17} \):

\[
G_{LL} = \frac{e^2}{h} \left[ 1 - |r_L|^2 + |r_L|^2 \frac{\sin k_h a}{\sin k_e a} \right]
\]
\[
G_{RL} = \frac{e^2}{h} \left[ |r_L|^2 - |t_h|^2 \frac{\sin k_h a}{\sin k_e a} \right]
\]  \hspace{1cm} (4)

III. RESULTS AND ANALYSIS

We calculate the local conductance \( G_{LL} \) and the transconductance \( G_{RL} \) as a function of the bias \( eV \) and the superconducting phase difference \( \phi = \phi_1 - \phi_2 \) for the choice of parameters: \( \Delta_0 = 0.9t \), \( t_{MK} = 0.4t \), \( t_L = 0.2t \) and \( t_R = 0.6t \).

![Figure 3](image3.png)

**FIG. 3.** \( G_{LL} \) (left panel) and \( G_{RL} \) (right panel) in units of \( e^2/h \) versus bias \( eV \) and the phase difference \( \phi = \phi_1 - \phi_2 \) for the choice of parameters: \( \Delta_0 = 0.9t \), \( t_{MK} = 0.4t \), \( t_L = 0.2t \) and \( t_R = 0.6t \).

Further, we find that an asymmetry in the hopping amplitudes from the normal metal leads to the Josephson junction leads to rich features in the conductance results. We find that \( 0 < t_L < t_R \leq t \) enhances the nonlocal transport by the following mechanism. An electron enters the Josephson junction at the resonant energies by resonant tunneling from \( N_L \) despite a weak coupling \( t_L \). The electron gets converted into BdG quasiparticle in the Josephson junction. Now, having \( t_R > t_L \) will increase the transparency of the BdG quasiparticle into \( N_R \) that into \( N_L \). So, the transmission (either as an electron or as a hole) onto \( N_R \) is enhanced, there by aiding nonlocal transport.

There are four MFs in the Josephson junction that is made of two Kitaev chains. The two MFs in single Kitaev chain are coupled due to a finite \( \Delta \). Further, the MFs in the two Kitaev chains are coupled due to a finite \( t_{MK} \). So, the four Dirac fermion states in the energy range \((-0.5\Delta, 0.5\Delta)\) in Fig. 2 are formed by the hybridization of four MFs that are nonlocal. Hence, the negative transconductance at energies of ABSs is a signature of nonlocality of constituent MFs.

IV. CONCLUSION

We have seen that the Josephson junction of Kitaev chains connected to normal metal leads can be used to enhance crossed Andreev reflection over electron tunneling. This means the setup can be used to probe the nonlocality of the constituent Majorana fermions. The peaks in local conductance and the valleys and peaks in the transconductance match with the energy levels of the iso-

![Figure 4](image4.png)

**FIG. 4.** \( G_{LL} \) (left panel) and \( G_{RL} \) (right panel) in units of \( e^2/h \) versus bias \( eV \) and the Josephson coupling \( t_{MK} \) for the choice of parameters: \( \Delta_0 = 0.9t \), \( \phi_1 - \phi_2 = 0.7\pi \), \( t_L = 0.2t \) and \( t_R = 0.6t \).
lated Josephson junction. Further, we see that asymmetry in couplings of the Josephson junction to the normal metal can enhance nonlocal transport. Since the Josephson junctions between p-wave superconductors hosting Majorana fermions have already been fabricated, we envisage that our proposal can be tested experimentally with present technology.

ACKNOWLEDGMENTS

The author thanks DST-INSPIRE Faculty Award (Faculty Reg. No.: IFA17-PH190) for financial support.

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