A Taxonomy of Recurrent Learning Rules

Guillermo Martín-Sánchez¹, Sander Bohte³, and Sebastian Otte¹,²

¹ Neuro-Cognitive Modeling, University of Tübingen
Sand 14, 72076 Tübingen, Germany
guilemartinsan@gmail.com

² Adaptive AI Lab, University of Lübeck
Ratzeburger Allee 160, 23562 Lübeck, Germany
sebastian.otte@uni-luebeck.de

³ Machine Learning group, CWI
Science Park 123, NL-1098XG Amsterdam, The Netherlands
S.M.Bohte@cwi.nl

Abstract. Backpropagation through time (BPTT) is the de facto standard for training recurrent neural networks (RNNs), but it is non-causal and non-local. Real-time recurrent learning is a causal alternative, but it is highly inefficient. Recently, e-prop was proposed as a causal, local, and efficient practical alternative to these algorithms, providing an approximation of the exact gradient by radically pruning the recurrent dependencies carried over time. Here, we derive RTRL from BPTT using a detailed notation bringing intuition and clarification to how they are connected. Furthermore, we frame e-prop within in the picture, formalising what it approximates. Finally, we derive a family of algorithms of which e-prop is a special case.

Keywords: recurrent neural networks · backpropagation through time · real-time recurrent learning · forward propagation · e-prop.

1 Introduction

Backpropagation through time (BPTT) [7] is currently the most used algorithm for training recurrent neural networks (RNNs) and is derived from applying the chain rule (backpropagation) to the computational graph of the RNN unrolled in time. It suffers however from undesired characteristics both in terms of biological plausibility and large scale applicability: (i) it is non-causal, since at each time step it requires future activity to compute the current gradient of the loss with respect to the parameters; and (ii) it is non-local, since it requires reverse error signal propagating across all neurons and all synapses. An equivalent algorithm is real-time recurrent learning (RTRL) [9]. It uses eligibility traces that are computed at each time step recursively in order to be causal, and can therefore be computed online. However, this comes at the cost of very high computational and memory complexity, since all temporal forward dependencies have to be maintained over time. RTRL is, hence, also non-local. Recently, a new online learning algorithm, called e-prop [3] has been proposed, which is tailored for
training recurrent spiking neural networks (RSNNs) with local neural dynamics. The aim was to find an alternative to BPTT (and RTRL) that is causal, local, but also computational and memory efficient.

In this paper, we look in depth into the formalisation of BPTT and RTRL and formalise e-prop into the picture. To do so, we use the computational graph and notation of the architecture in the e-prop paper [3] to understand how these three algorithms relate to each other. Furthermore, in a posterior paper [10], it was shown that e-prop was an approximation of RTRL. Here, by formalising also RTRL in the same framework we indeed confirm the connection and make it more explicit (cf. Fig. 1). In the process, we uncover a family of algorithms determined by the level of approximation allowed to benefit from causality and locality. The main focus of this paper is to give intuition and understanding of all of these gradient computation rules.

1.1 Background

The most common way to train a model in supervised learning is to compute the gradient of a given loss \( \mathcal{L} \) with respect to the parameters \( \theta \), \( \frac{d\mathcal{L}}{d\theta} \), and use this gradient in some gradient descent scheme of the form \( \theta(\tau + 1) = \theta(\tau) - f(\frac{d\mathcal{L}}{d\theta}) \), where \( \tau \) refers to the current update iteration and \( f \) is some gradient postprocessing. Therefore, we here focus on the algorithms for the computation (or approximation) of this gradient.

In particular, we focus on a general class of RNN models where we have \( n \) computational units. These units have hidden states at each time step \( c^t_i \) that influence the hidden state at the next time step \( c^{t+1}_i \) (implicit recurrence) as well as the output of the unit at the current time step \( h^t_i \). The output \( h^t_i \) of a unit at
A Taxonomy of Recurrent Learning Rules

Fig. 2. Simple example of computational graph and distinction between total and partial derivative of \( f \) with respect to \( x \).

A given time step influences the hidden state of the same and other units at the following time step \( c_{t+1} \) (explicit recurrence) through a weighted connection \( w_{i,j} \). Finally, these outputs also account for the model’s computation (either directly or through some other computations, e.g. a linear readout) and therefore are subject to evaluation by a loss function \( L \). The formalization here is agnostic to the particular dimensionality and computational relation between the variables and therefore apply for different RNNs, such as LSTMs [5] or RSNNs [1].

For a function \( f(x, y(x)) \) we distinguish the notation of the total derivative \( \frac{df}{dx} \) and the partial derivative \( \frac{\partial f}{\partial x} \) because the first one represents the whole gradient through all paths, while the second one expresses only the direct relation between the variables. To illustrate: using the chain rule (cf. Fig. 2) and with the example \( y = 2x \) and \( f(x, y(x)) = xy \), the total derivative is calculated as:

\[
\frac{df}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial x} = y + x \cdot 2 = 4x
\]

2 Backpropagation Through Time

In RNNs, since previous states affect the current state, the trick to applying the chain rule is to unroll the RNN in time, obtain a virtual feed-forward architecture and apply to this computational graph error-backpropagation [8]. The resulting algorithm is BPTT [7] and it is the currently most used algorithm to compute \( \frac{dL}{dw_{ij}} \) since it reuses many previous computations to be highly efficient. Here, we focus our attention on the role of the recurrences dividing the algorithm into the following steps:

Explicit Recurrences: Compute \( \frac{dL}{dh_{t,j}} \) using the recursive definition given by the explicit recurrences (cf. Fig. 3A):

\[
\frac{dL}{dh_{t,j}} = \frac{\partial L}{\partial h_{t,j}} + \frac{dL}{dc_{t+1,j}} \frac{\partial c_{t+1,j}}{\partial h_{t,j}} + \sum_{k \neq j} \frac{dL}{dc_{t+1,k}} \frac{\partial c_{t+1,k}}{\partial h_{t,j}}
\]

(2)

Implicit Recurrences: Compute \( \frac{dL}{dc_{t,j}} \) using the value of the previous step and the recursive definition given by the implicit recurrence (cf. Fig. 3B):

\[
\frac{dL}{dc_{t,j}} = \frac{dL}{dh_{t,j}} \frac{\partial h_{t,j}}{dc_{t,j}} + \frac{dL}{dc_{t+1,j}} \frac{\partial c_{t+1,j}}{dc_{t,j}}
\]

(3)
Finally, compute $\frac{dL}{dw_{ij}}$ using the values obtained in the previous two steps for all time steps (cf. Fig. 3C):

$$
\frac{dL}{dw_{ij}} = \sum_t \frac{dL}{dc_{k}^{t}} \frac{\partial c_{k}^{t} - 1}{\partial w_{ij}}
$$

(4)

We use explicit and implicit recurrences from the maximum time $T$ backwards and for all $t \leq T$, and finally, sum all the results from Eq. (3). The existence of these recurrences makes BPTT present the following problems [6,3]:

**Non-locality:** Due to the explicit recurrences, we need to take into account how the current synaptic strength $w_{ij}$ between the neurons $i$ and $j$ affects the future value of the postsynaptic neuron: $\partial c_{k}^{t+1}/\partial h_{ij}^{t}$ for all $k \neq j$ (cf. Eq. 2). This means that to compute the weight change for synapse magnitude $w_{ij}$ we need information of the hidden variables $c_{k}^{t+1}$ for all $k$. Moreover, this chain of dependencies continues at each time step, such that at the next time step we need information of the variables $c_{k}^{t+2}$ for all $q$ (including $q \neq j$) and so forth. The contraposition would be a **local** algorithm that does not require messages passing from every neuron to every synapse to compute the gradients, but rather only need information close to the given synapse.
Non-causality: Due to the three kinds of recurrences shown before we need to take into account all the gradients in the future (the same way as current layer computations need to use the gradients of posterior layers in feed-forward architectures), leading to two main problems. First, we need to compute the values of the variables (update locking) and the gradients (backwards locking) across all future time steps before computing the current gradient. Secondly, all the values of all the variables across time have to be saved during the inference phase to be used while computing the gradients, requiring a memory overhead \(O(nT)\) with \(n\) neurons and \(T\) time steps. The contraposition would be a causal algorithm, that at each time step would only need information from previous and current activity to compute the current gradient. Therefore, it could do it at each time step (online) and while the inference is running.

3 Real-Time Recurrent Learning

RTRL is a causal learning algorithm that can be implemented as an online learning algorithm and that computes the same gradient as BPTT, at the cost of being more computationally expensive. We derive the equation for RTRL starting with BPTT (cf. Eq. 4) via re-expressing the gradients that connect the computation with future gradients to obtain a causal algorithm. These gradients correspond to the implicit and explicit recurrences.

3.1 Re-expressing Implicit Recurrence

First, we re-express the implicit recurrence gradient \(\frac{\partial c_{t+1}^j}{\partial c_t^j}\).

Unrolling the recursion: To unroll, we plug the equation of implicit recurrence Eq. 3 into Eq. 4:

\[
\frac{dL}{dw_{ij}} = \sum_{t'} \left( \frac{dL}{dh_{t'}^j} \frac{\partial h_{t'}^j}{\partial c_t^j} + \frac{dL}{dc_{t+1}^j} \frac{\partial c_{t+1}^j}{\partial c_t^j} \right) \frac{\partial c_{t'}^j}{\partial w_{ij}}
\]

\[
= \sum_{t'} \left( \frac{dL}{dh_{t'}^j} \frac{\partial h_{t'}^j}{\partial c_t^j} + \left( \frac{dL}{dh_{t+1}^j} \frac{\partial h_{t+1}^j}{\partial c_t^j} + \left( \cdots \frac{dL}{dc_{t+T}^j} \frac{\partial c_{t+T}^j}{\partial c_t^j} \right) \frac{\partial c_{t'}^j}{\partial w_{ij}} \right) \right)
\]

Flip time indices: The derived formula is non-causal since it requires future gradients (for each \(t'\) we sum products of gradients with factors starting from \(t \geq t'\)). To make it causal, we change the indices as follows:

\[
\frac{dL}{dw_{ij}} = \sum_t \frac{dL}{dh_{t}^j} \frac{\partial h_{t}^j}{\partial c_t^j} \sum_{t' \leq t} \frac{\partial c_{t'}^j}{\partial c_{t-1}^j} \cdots \frac{\partial c_{t+T}^j}{\partial c_t^j} \frac{\partial c_{t'}^j}{\partial w_{ij}}
\]
Definition (Implicit variable) We define the implicit variable $\epsilon_{t ij}$ as:

$$
\epsilon_{t ij} = \sum_{t' \leq t} \frac{\partial c_{ij}^{t'}}{\partial c_{ij}^{t'-1}} \frac{\partial c_{ij}^{t}}{\partial c_{ij}^{t'}} \frac{\partial c_{ij}^{t+1}}{\partial w_{ij}} \tag{7}
$$

Backwards interpretation: Starting at $c_{ij}^t$, the implicit variable represents the sum over all the paths going backwards through the implicit recurrence until $c_{ij}^{t'}$ and from there to the synaptic weight $w_{ij}$ (cf. Fig. 4).

Forwards interpretation: The implicit variable represents how the hidden variable of neuron $j$ has been affected by the synapse weight $w_{ij}$ through time, i.e. taking into account also how the hidden variables at previous time steps have affected the variables at the current time step through the implicit recurrence.

Incremental computation: Importantly, there is a recursive relation to this variable that allows it to be updated at each time step:

$$
\epsilon_{t ij} = \frac{\partial c_{ij}^{t}}{\partial c_{ij}^{t-1}} \epsilon_{t-1 ij} + \frac{\partial c_{ij}^{t}}{\partial w_{ij}} \tag{8}
$$

Definition (Implicit eligibility trace) Given the implicit variable $\epsilon_{t ij}$, we define the implicit eligibility trace $e_{t ij}$ as:

$$
e_{t ij} := \frac{\partial h_{ij}^{t}}{\partial c_{ij}^{t}} \tag{9}
$$

Since $\partial h_{ij}^{t}/\partial c_{ij}^{t}$ is causal and local, and so is the implicit variable $\epsilon_{t ij}$ (can be computed at each time step and is specific for each synapse), then the implicit eligibility trace $e_{t ij}$ is also causal and local.
Final equation with re-expressed implicit recurrence: With all of this combined, BPTT (cf. Eq. 4) has become (substituting Eq. 7 in Eq. 6) the following (cf. Fig. 5A):

$$\frac{dL}{dw_{ij}} = \sum_t \frac{dL}{dh_j^t} \frac{\partial h_j^t}{\partial c_{ij}^t} \epsilon_{ij}^t = \sum_t \frac{dL}{dh_j^t} \epsilon_{ij}^t \quad (10)$$

Even though $\epsilon_{ij}^t$ is causal and local, this equation as a whole is not, since the factor $dL/dh_j^t$ still includes explicit recurrences. E-prop will simply ignore these recurrences to solve this problem (cf. Fig. 5B, Section 4).

3.2 Re-expressing Explicit Recurrences of Order 1

Now we re-express the explicit recurrences’ gradient $\partial c_{ij}^{t+1}/\partial h_j^t$ analogously to the implicit recurrence in the previous section. First, we plug Eq. 2 (explicit recurrences) into Eq. 10 (re-expressed implicit recurrence):

$$\frac{dL}{dw_{ij}} = \sum_{t'} \left( \frac{\partial L}{\partial h_j^{t'}} + \sum_k \frac{dL}{d_c^{t+1}} \frac{\partial c_k^{t+1}}{\partial h_j^t} \right) \epsilon_{ij}^{t'} = \sum_{t'} \frac{\partial L}{\partial h_j^{t'}} \epsilon_{ij}^{t'} + \sum_k \sum_{t'} \frac{dL}{d_c^{t+1}} \frac{\partial c_k^{t+1}}{\partial h_j^t} \epsilon_{ij}^{t'} \quad (11)$$

The first factor of this sum is already causal since it only requires the direct derivative and the implicit eligibility trace introduced in Eq. 7. Focusing on the second factor, this term represents the gradient until $c_k^{t+1}$, the jump to $h_j^{t'}$ and the implicit eligibility trace $\epsilon_{ij}^{t'}$ stored there that represents the sum over all of the paths from there to $w_{ij}$.
Unrolling the recursion: We can now unroll the recursion by plugging the equation of explicit recurrences Eq. (2) into the second term of Eq. (11)

\[
\sum_{t'} \frac{\partial \beta_{ij}^{t+1}}{\partial c_k^{t+1}} \frac{\partial c_k^{t+1}}{\partial h_j^{t'} \partial c_k^{t+1}} = \sum_{t'} \left( \frac{\partial \beta_{ij}^{t+1}}{\partial h_k^{t+1}} \frac{\partial h_k^{t+1}}{\partial c_k^{t+1}} + \frac{\partial \beta_{ij}^{t+1}}{\partial c_k^{t+2}} \frac{\partial c_k^{t+2}}{\partial h_j^{t'} \partial c_k^{t+1}} \right) \frac{\partial c_k^{t+1}}{\partial h_j^{t'}} \epsilon_{ij}^{t'}
\]

\[
= \sum_{t'} \left( \frac{\partial \beta_{ij}^{t+1}}{\partial h_k^{t+1}} \frac{\partial h_k^{t+1}}{\partial c_k^{t+1}} + \left( \frac{\partial \beta_{ij}^{t+1}}{\partial c_k^{t+2}} \frac{\partial c_k^{t+2}}{\partial h_j^{t'} \partial c_k^{t+1}} \right) \frac{\partial c_k^{t+1}}{\partial h_j^{t'}} \epsilon_{ij}^{t'}
\right)

\[
= \sum_{t'} \sum_{t \geq t'} \frac{\partial \beta_{ij}^{t+1}}{\partial h_k^{t+1}} \frac{\partial c_k^{t+1}}{\partial c_k^{t+2}} \frac{\partial c_k^{t+2}}{\partial h_j^{t'}} \epsilon_{ij}^{t'}
\]

Flip time indices: We flip the indices again to have a causal formula:

\[
\sum_{t'} \frac{\partial \beta_{ij}^{t+1}}{\partial c_k^{t+1}} \frac{\partial c_k^{t+1}}{\partial h_j^{t'}} \epsilon_{ij}^{t'} = \sum_{t} \frac{dL}{dh_k^{t+1}} \frac{\partial h_k^{t+1}}{\partial c_k^{t+1}} \sum_{t'} \frac{\partial \beta_{ij}^{t+1}}{\partial c_k^{t+2}} \frac{\partial c_k^{t+2}}{\partial h_j^{t'}} \epsilon_{ij}^{t'}
\]

Definition (Explicit variable) We define the explicit variable \( \beta_{ij}^{t}(k, k', ..., j) \) as:

\[
\beta_{ij}^{t}(k, k', ..., j) := \sum_{t' \leq t-1} \frac{\partial c_k^{t'}}{\partial c_k^{t}} \frac{\partial c_k^{t+1}}{\partial h_k^{t'}} \frac{\partial h_k^{t'+2}}{\partial h_k^{t'} \partial c_k^{t+1}} \frac{\partial h_k^{t'}}{\partial c_k^{t'}} \beta_{ij}^{t}(k', k'', ..., j)
\]

with \( \beta_{ij}^{t}(j) = \epsilon_{ij}^{t} \).

Backwards interpretation: The explicit variable represents the idea of starting at \( c_k^{t} \), moving an arbitrary number of steps through the implicit recurrence \( h_k^{t'} \) until at a certain \( t' \) you jump to the output variable of another neuron \( h_k^{t''} \), down to its hidden variable \( c_k^{t''} \) and then start again, with a path, now starting at
A Taxonomy of Recurrent Learning Rules

c\_t^{'}_k. In total, it considers all possible paths , with arbitrary length, spending an arbitrary number of steps in each of the neurons (through implicit recurrences) from c\_t^{'}_k to c\_t^{'}_j through the neurons k', k'', ... and then times the implicit variable e\_t^{ij} (cf. Fig. 6).

Forwards interpretation: The explicit variable accounts for the influence of the activity of neuron j at any previous time step c\_t^{'}_j to neuron k' at a future time step c\_t^{'}_k through the neurons k', k'', ....

Incremental computation: The recursive relation to this variable that allows it to be updated at each time step is:

\[ \beta^{t}_{ij}(k, k', ..., j) = \frac{\partial c^{t+1}_{k'}}{\partial c^{t}_{k'}} \beta^{t-1}_{ij}(k, k', ..., j) + \frac{\partial c^{t+1}_{k'}}{\partial h^{t-1}_{k'}} \frac{\partial h^{t-1}_{k'}}{\partial c^{t}_{k'}} \beta^{t-1}_{ij}(k', ..., j) \]  

(15)

Definition (Explicit eligibility trace) Given the explicit variable \( \beta^{t}_{ij}(k, k', ..., j) \), we define the explicit eligibility trace \( b^{t}_{ij}(k, k', ..., j) \) as:

\[ b^{t}_{ij}(k, k', ..., j) := \frac{\partial h^{t}_{k}}{\partial c^{t}_{k'}} \beta^{t}_{ij}(k, k', ..., j) \]  

(16)

with \( b^{t}_{ij}(j) = e^{t}_{ij} \)

Since \( \frac{\partial h^{t}_{k}}{\partial c^{t}_{k'}} \) is causal and local, and the explicit variable \( \beta^{t}_{ij}(k, k', ..., j) \) is causal but only partially local (it requires message passing from the presynaptic neuron k' to the postsynaptic neuron k), then the explicit eligibility trace \( b^{t}_{ij}(k, k', ..., j) \) is also causal but only partially local.

Final equation with re-expressed explicit recurrence of order 1: Substituting the explicit variable Eq. [14] in Eq. [13] yields:

\[ \sum_{k} \sum_{t} \frac{dL}{dk_{k}} \frac{\partial c^{t+1}_{k'}}{\partial h^{t}_{j}} e^{t}_{ij} = \sum_{k} \sum_{t} \frac{dL}{dh^{t+1}_{k}} b^{t+1}_{ij}(k, j) \]  

(17)

And substituting this back to the original equation (cf. Eq. [11]):

\[ \frac{dL}{dw_{ij}} = \sum_{t} \frac{dL}{dh^{t}_{j}} e^{t}_{ij} + \sum_{k} \sum_{t} \frac{dL}{dk^{t+1}_{k}} e^{t}_{ij} + \sum_{k} \sum_{t} \frac{dL}{dh^{t+1}_{k}} b^{t+1}_{ij}(k, j) \]  

(18)

Here it becomes clear how setting this second factor to 0 is what gives us e-prop (cf. the right arrow in Fig. 1), since we forcefully ignore the influence of a neuron on other neurons (and itself) through the explicit recurrences.
3.3 Re-expressing Explicit Recurrences of Order > 1

Now that we have seen how the explicit eligibility connects the activity of neuron \(j\) with other neurons through explicit recurrences, we can use it to re-express higher-order explicit recurrences.

Unroll the recursion: Starting from the equation with one order of explicit recurrence already re-expressed (cf. Eq. 18), and alternatively using the definition of explicit recurrences (cf. Eq. 2) and the action of the explicit eligibility trace (cf. Eq. 17), we can repeat the previous steps for higher orders:

\[
\frac{dL}{dw_{ij}} = \sum_{t} \sum_{k_{1}} \frac{\partial L}{\partial h_{k_{1}}^{t}} e_{t}^{ij} + \sum_{k_{1}} \left( \frac{\partial L}{\partial h_{k_{1}}^{t+1}} + \sum_{k_{2}} \frac{dL}{dc_{k_{2}}^{t+2}} \frac{\partial c_{k_{2}}^{t+2}}{\partial h_{k_{1}}^{t+1}} b_{ij}^{t+1}(k_{1}, j) \right) b_{ij}^{t+1}(k_{1}, j)
\]

\[
= \sum_{t} \frac{\partial L}{\partial h_{j}^{t}} e_{t}^{ij} + \sum_{t} \sum_{k_{1}} \frac{\partial L}{\partial h_{k_{1}}^{t+1}} b_{ij}^{t+1}(k_{1}, j) + \sum_{t} \sum_{k_{1}, k_{2}} \frac{dL}{dc_{k_{2}}^{t+2}} \frac{\partial c_{k_{2}}^{t+2}}{\partial h_{k_{1}}^{t+1}} b_{ij}^{t+1}(k_{1}, j)
\]

\[
= \sum_{t} \sum_{k_{1}} \frac{\partial L}{\partial h_{k_{1}}^{t}} e_{t}^{ij} + \sum_{t} \sum_{k_{1}} \frac{\partial L}{\partial h_{k_{1}}^{t+1}} b_{ij}^{t+1}(k_{1}, j) + \sum_{t} \sum_{k_{1}, k_{2}} \frac{dL}{dc_{k_{2}}^{t+2}} b_{ij}^{t+2}(k_{2}, k_{1}, j)
\]

\[
= \sum_{t} \sum_{t' \geq t \atop k_{0}=j, k_{1}, \ldots, k_{t}} \frac{\partial L}{\partial h_{k_{t}}^{t'}} b_{ij}^{t'}(k_{t}, \ldots, k_{1}, k_{0} = j)
\]

This gives us a high overview of separating the different levels of explicit recurrences which will lead to the definition of the m-order e-prop (Section 4).

Flip time indices: As before we change the time indices and reorganise to allow for causality,

\[
\frac{dL}{dw_{ij}} = \sum_{t} \sum_{k} \frac{\partial L}{\partial h_{k}^{t}} \frac{\partial h_{k}^{t}}{\partial h_{k_{0}}^{t}} \sum_{t' \leq t \atop k_{0}=j, k_{1}, \ldots, k_{t'-1}} \beta_{ij}^{t}(k, k_{t'-1}, \ldots, k_{1}, k_{0} = j)
\]

Definition (Recurrence variable) We define the recurrence variable \(\alpha_{ij}^{t,r}\) as:

\[
\alpha_{ij}^{t,r} = \sum_{t' \leq t \atop k_{0}=j, k_{1}, \ldots, k_{t'-1}} \beta_{ij}^{t}(r, k_{t'-1}, \ldots, k_{1}, k_{0} = j)
\]

Backwards interpretation: Starting at current time \(t\) in neuron \(r\), the recurrence variable represents all combinations of paths through any combination of neurons \(k_{t'-1}, \ldots, k_{1}\) ending in neuron \(j\).

Forwards interpretation: The recurrence variable accounts for the influence of the activity of neuron \(j\) at any previous time step to neuron \(r\) at the current timestep \(t\) through all possible paths through all neurons.
Incremental computation: Once again, importantly, we have a recursive equation for computing the recurrence variable:

$$
\alpha_{t,r}^{i,j} = \frac{\partial c_{t,r}^{i,j}}{\partial c_{t-1,r}^{i,j}} + \sum_{k} \frac{\partial c_{t,r}^{i,j}}{\partial h_{t-1,k}^{i,j}} \frac{\partial h_{t-1,k}^{i,j}}{\partial c_{t-1,k}^{i,j}} + \frac{\partial c_{t,r}^{i,j}}{\partial w_{ij}}
$$

(22)

Observe $\forall r \neq j$, $\frac{\partial c_{t,r}^{i,j}}{\partial w_{ij}} = 0$.

**Definition (Recurrence eligibility trace)** Given the recurrence variable $\alpha_{t,r}^{i,j}$, we define the recurrence eligibility trace $a_{t,r}^{i,j}$ as:

$$
a_{t,r}^{i,j} := \frac{\partial h_{t,r}^{i,j}}{\partial c_{t,r}^{i,j}} \alpha_{t,r}^{i,j}
$$

(23)

Since $\frac{\partial h_{t,r}^{i,j}}{\partial c_{t,r}^{i,j}}$ is causal and local, but the recurrence variable $\alpha_{t,r}^{i,j}$ is causal but non-local, the recurrence eligibility trace $a_{t,r}^{i,j}$ is also causal but non-local. It is non-local in an equivalent way as BPTT is not: each synapse $ij$ requires to store a variable representing how the activation in the past of any other neuron $r$ would affect its computation in the present, even if $r \neq i, j$ and through all possible paths of synapses. The recursive computation of $a_{t,r}^{i,j}$ requires of the summation of the recurrence variables of all the neurons requiring non-local communication.

**Final equation of RTRL**: Eq. 20 transforms into (by substituting Eq. 21 into Eq. 20) the final equation for RTRL (cf. Fig. 7):

$$
\frac{dL}{dw_{ij}} = \sum_{t} \sum_{k} \frac{\partial L}{\partial h_{t,k}^{i,j}} \frac{\partial h_{t,k}^{i,j}}{\partial c_{t-1,k}^{i,j}} \alpha_{t,k}^{i,j} = \sum_{t} \sum_{k} \frac{\partial L}{\partial h_{t,k}^{i,j}} a_{t,k}^{i,j}
$$

(24)

We now have a causal but still non-local gradient computation algorithm.
4 E-prop

The e-prop algorithm approximates the gradient by not considering the explicit recurrences in RNNs. E-prop was originally formulated for RSNNs, since it is considered more biologically plausible than BPTT and RTRL due to its characteristics of being causal and local [3]. The approximation of the gradient that defines e-prop is:

\[
\frac{d\mathcal{L}}{dw_{ij}} \approx \sum_t \frac{\partial \mathcal{L}}{\partial h^t_{ij}} e^t_{ij} \tag{25}
\]

Through the derivation of RTRL from BPTT, e-prop has arisen naturally in three different places. This allows us for equivalent interpretations of the approximation, each more detailed than the previous one.

First, and as originally proposed [3], we can understand e-prop from the equation that arises after re-expressing the implicit eligibility trace (cf. Eq. 10):

\[
\frac{d\mathcal{L}}{dw_{ij}} = \sum_t \frac{d\mathcal{L}}{dh^t_{ij}} e^t_{ij} = \sum_t \frac{d\mathcal{L}}{dh^t_{ij}} e^t_{ij} + \sum_k \sum_t \frac{d\mathcal{L}}{dh^t_{k}b^t_{k+1}} e^t_{ij} (k, j)
\]

Here we approximate the non-causal and non-local total derivative by the causal and local partial derivative, i.e. \(d\mathcal{L}/dh^t_{ij} \approx \partial \mathcal{L}/\partial h^t_{ij}\) (cf. Fig. 5).

Second, we can understand it from the equation after re-expressing the explicit recurrences of order 1 (cf. Eq. 18):

\[
\frac{d\mathcal{L}}{dw_{ij}} = \sum_t \frac{d\mathcal{L}}{dh^t_{ij}} e^t_{ij} + \sum_k \sum_t \frac{d\mathcal{L}}{dh^t_{k}b^t_{k+1}} e^t_{ij} (k, j) + \sum_k \sum_{t_1} \frac{d\mathcal{L}}{dh^t_{k_1}b^t_{k_2}} e^{t+1}_{ij} (k_2, k_1, j) + \sum_{t_1, t_2} \frac{d\mathcal{L}}{dh^t_{k_1}b^t_{k_2}} e^{t+2}_{ij} (k_2, k_1, j) + \cdots
\]

Here we define the \(m\)-order e-prop as the approximation resulting from setting in the above equation \(\sum_t \sum_{k_0=j, k_1, \ldots, k_m} \frac{d\mathcal{L}}{dh^t_{k_m}} e^{t+m}_{ij} (k_m, \ldots, k_1, k_0 = j) = 0\).
By increasing the order $m$ we better approximate the gradient at the cost of needing the activities of other neurons $m$ time steps ahead to compute the current gradient of the loss. Under this scope, standard e-prop is just the 1-order e-prop (fully causal and local but the most inaccurate approximation). On the other extreme, the T-order e-prop (nothing is approximated or set to 0) corresponds to the full gradient computation, in a middle form between BPTT and RTRL (the exact computation of the gradient but completely non-causal and non-local). Moreover, synapses arriving into neurons connected to the readout through up to $m$ synapses will be modified by the m-order e-prop (m-order e-prop computes through up to $m - 1$ additional feed-forward layers).

5 Conclusion

In this paper, we formally explored how BPTT, RTRL, and e-prop relate to each other. We extended the general scheme for re-expressing recurrences as eligibility traces from and applied it iteratively to go from BPTT to RTRL. In the process, we found intermediate expressions that allow for better intuition of these algorithms. Moreover, we showed how e-prop can be seen as an extreme case of a series of approximation algorithms, which we coin m-order e-prop.

References

1. Bellec, G., Salaj, D., Subramoney, A., Legenstein, R., Maass, W.: Long short-term memory and learning-to-learn in networks of spiking neurons (2018)
2. Bellec, G., Scherr, F., Hajek, E., Salaj, D., Legenstein, R., Maass, W.: Biologically inspired alternatives to backpropagation through time for learning in recurrent neural nets (2019)
3. Bellec, G., Scherr, F., Subramoney, A., Hajek, E., Salaj, D., Legenstein, R., Maass, W.: A solution to the learning dilemma for recurrent networks of spiking neurons. Nature Communications 11 (07 2020)
4. Czarnecki, W.M., Świrszcz, G., Jaderberg, M., Osindero, S., Vinyals, O., Kavukcuoglu, K.: Understanding synthetic gradients and decoupled neural interfaces (2017)
5. Hochreiter, S., Schmidhuber, J.: Long short-term memory. Neural computation 9(8), 1735–1780 (1997)
6. Marblestone, A., Wayne, G., Kording, K.: Toward an integration of deep learning and neuroscience. Frontiers in Computational Neuroscience 10 (06 2016)
7. Werbos, P.: Backpropagation through time: what it does and how to do it. Proceedings of the IEEE 78, 1550 – 1560 (11 1990)
8. Werbos, P., John, P.: Beyond regression: new tools for prediction and analysis in the behavioral sciences / (01 1974)
9. Williams, R.J., Zipser, D.: A learning algorithm for continually running fully recurrent neural networks. Neural Computation 1, 270–280 (1989)
10. Zenke, F., Nefci, E.O.: Brain-inspired learning on neuromorphic substrates. CoRR abs/2010.11931 (2020)
Appendix

A Online Computation vs Online Update

In the main text, we have used $w_{ij}$ as a constant variable over the $T$ time steps. This is based on the usual usage of BPTT where we compute \( \frac{dL}{dw_{ij}} \) after the $T$ time steps of execution and only then do the update $w_{ij}^{T+1} = w_{ij}^0 + f(\frac{dL}{dw_{ij}})$. 

Looking at the RTRL equation Eq. 24:

\[
\frac{dL}{dw_{ij}} = \sum_{t} \sum_{k} \frac{\partial L}{\partial h_t^k} a_{ij}^{t,k}
\]

(26)

we observe that since the factors for a given $t$ are causal, we can compute the quantity $C_t = \sum_k \frac{\partial L}{\partial h_t^k} a_{ij}^{t,k}$ at each time step $t$ (while doing the forward pass) and add them all after $T$ time-steps to exactly compute $\frac{dL}{dw_{ij}} = \sum_t C_t$. Equivalently with e-prop and computing the value $C_t = \frac{\partial L}{\partial y_t^k} e_{ij}^t$ to approximate $\frac{dL}{dw_{ij}} \approx \sum_t C_t$. This is why we consider these algorithms as online gradient computing algorithms.

However, the exact equivalence between BPTT and these causal algorithms is only held if the weights are not updated with these $C_t$ values, since the derivations of these algorithms from BPTT depend on the weights being constant for the $T$ time steps. That is why in the computational graph we use $w_{ij}$ and not different $w_{ij}^t \in 0, 1, ..., T$ (cf. Fig. 3).

In general, we can define an online update algorithm from each online gradient computation algorithm with the updating scheme $w_{ij}^{t+1} = w_{ij}^t + f(C^t)$ with $C^t$ being the causal values described before. In general, however, this is computationally different from BPTT, RTRL, and e-prop as described in this paper.

B E-prop with Read-Out Neurons

When the RNN has a readout module, this has to be considered in e-prop in the computation of the gradient of the loss with respect to the output of the neuron, i.e. $\frac{\partial L}{\partial y_t^k}$. Note that now, focusing on the read-out subnetwork, this derivative is not necessarily direct, so we will use a medium notation to refer to a derivative that has no paths into the main RNN but that is not completely direct. So $\frac{\partial L}{\partial y_t^k}$ in the main text becomes $\frac{d^r L}{d^r h_t^j}$ in this point of view.

We find ourselves in a subnetwork where we are computing $\frac{d^r L}{d^r h_t^j}$. The BPTT equation (cf. Eq. 4) becomes in this case:

\[
\frac{d^r L}{d^r h_t^j} = \sum_k \frac{d^r L}{d^r y_t^k} \frac{\partial y_t^k}{d^r h_t^j}
\]

(27)

where $y_t^k$ is the activity of the readout neuron $k$ at time $t$. 

Fig. 8. Computational graph for A) implicit recurrence gradients in read-out neurons B) computation of $\frac{dL}{dh_j}$.

B.1 No Recurrences

When the readout module is not recurrent (e.g. linear readout), we have that $\frac{dL}{dy_k} = \frac{\partial L}{\partial y_k}$ and therefore Eq. 27 becomes the usual equation for Backpropagation for one layer:

$$\frac{dL}{dh_j} = \sum_k \frac{\partial L}{\partial y_k} \frac{\partial y_k}{\partial h_j} \tag{28}$$

Final e-prop equation with non-recurrent read-out: Substituting Eq. 28 in the equation for e-prop Eq. 25 we get the final equation for RNNs with a non-recurrent readout module:

$$\frac{dL}{dw_{ij}} \approx \sum_t \sum_k \frac{\partial L}{\partial y_k} \frac{\partial y_k}{\partial h_j} \frac{\partial h_j}{\partial w_{ij}} \tag{29}$$

B.2 With Implicit Recurrences

When the readout module has implicit recurrences (e.g. non-spiking readout neurons), we have to use an implicit variable as in the main text (cf. Eq. 7)

Looking at the computational graph for $\frac{dL}{dy_k}$ in this case (cf. Fig 8B) we see we can do an equivalent trick as we did with the implicit recurrence in Section 3.1. To do this, we first write a recursive definition of the gradient $\frac{dL}{dy_k}$ (cf. Fig. 8A):

$$\frac{dL}{dy_k} = \frac{\partial L}{\partial y_k} + \frac{dL}{dy_k} \frac{\partial y_k}{\partial y_k} + \frac{dL}{dy_k} \frac{\partial y_k}{\partial y_k}$$

$$\frac{dL}{dh_j} = \sum_k \frac{\partial L}{\partial y_k} \frac{\partial y_k}{\partial h_j} \tag{30}$$
Fig. 9. Computational graph for the read-out implicit variable $\gamma_{t,k}^{k}$ with $t' = t - 2$.

**Unrolling the recursion:** To unroll, we plug Eq. 30 into Eq. 27:

$$
\frac{d' \mathcal{L}}{dh_{j}^{t'}} = \sum_{k} \left( \frac{\partial \mathcal{L}}{\partial h_{k}^{t'}} + \frac{d' \mathcal{L}}{dh_{k}^{t+1'}} \frac{\partial y_{k}^{t+1'}}{\partial h_{k}^{t'}} \right) \frac{\partial y_{k}^{t'}}{\partial h_{j}^{t'}} \\
= \sum_{k} \left( \frac{\partial \mathcal{L}}{\partial h_{k}^{t'}} + \left( \frac{\partial \mathcal{L}}{\partial y_{k}^{t+1'}} + \cdots \frac{\partial y_{k}^{t+2}}{\partial y_{k}^{t+1'}} \frac{\partial y_{k}^{t+1}}{\partial y_{k}^{t'}} \right) \frac{\partial y_{k}^{t'}}{\partial h_{j}^{t'}} \right) \\
= \sum_{k} \sum_{t' \geq t'} \frac{\partial \mathcal{L}}{\partial y_{k}^{t'}} \frac{\partial y_{k}^{t'}}{\partial y_{k}^{t'-1}} \cdots \frac{\partial y_{k}^{t+1}}{\partial y_{k}^{t'}} \\
= \sum_{k,t} \frac{\partial \mathcal{L}}{\partial y_{k}^{t}} \sum_{t' \geq t} \frac{\partial y_{k}^{t}}{\partial y_{k}^{t'-1}} \cdots \frac{\partial y_{k}^{t+1}}{\partial y_{k}^{t'}} \\
(31)
$$

**Flip time indices:** Contrary to Section 3.1 with $w_{ij}$, in this case, we have $h_{ij}^{t'}$ which changes over time, and so there is no summation over $t$ that would allow us for an exchange of time indices. However, we can introduce Eq. 31 in the equation of symmetric e-prop Eq. 25 which has a confluence of paths at $w_{ij}$ to do the time index flip:

$$
\frac{d \mathcal{L}}{d w_{ij}} = \sum_{t} \frac{\partial \mathcal{L}}{\partial h_{j}^{t'}} e_{ij}^{t'} = \sum_{t} \frac{d' \mathcal{L}}{dh_{j}^{t'}} e_{ij}^{t'} \\
= \sum_{k} \sum_{t' \geq t} \frac{\partial \mathcal{L}}{\partial y_{k}^{t}} \frac{\partial y_{k}^{t}}{\partial y_{k}^{t'-1}} \cdots \frac{\partial y_{k}^{t+1}}{\partial y_{k}^{t'}} \\
= \sum_{k,t} \frac{\partial \mathcal{L}}{\partial y_{k}^{t}} \sum_{t' \geq t} \frac{\partial y_{k}^{t}}{\partial y_{k}^{t'-1}} \cdots \frac{\partial y_{k}^{t+1}}{\partial y_{k}^{t'}} \\
(32)
$$

**Definition (Read-out implicit variable)** We define the read-out implicit variable $\gamma_{ij}^{t,k}$ as:

$$
\gamma_{ij}^{t,k} := \sum_{t' \geq t} \frac{\partial y_{k}^{t'}}{\partial y_{k}^{t'-1}} \cdots \frac{\partial y_{k}^{t+1}}{\partial y_{k}^{t'}} \\
(33)
$$
Backwards interpretation: Starting at $y_k^t$, the read-out implicit variable represents the sum over all the paths going backwards through the implicit recurrence until $y_k^t$ and the connection to all other paths in the RNN (without explicit recurrences since we are using e-prop) stored there in $e_{ij}^t$ (cf. Fig. 9).

Forwards interpretation: The read-out implicit variable represents how the read-out neuron’s value $y_k^t$ has been affected by the states of the RNN through time, i.e. taking into account also how the values of the read-out neuron at previous time steps have affected the value at the current time step through the implicit recurrence.

Incremental computation: The recursive relation:

$$\gamma_{ij}^{t,k} = \frac{\partial y_k^t}{\partial y_k^{t-1}} \gamma_{ij}^{t-1,k} + \frac{\partial y_k^t}{\partial h_j^t} e_{ij}^t$$

(34)

Definition (Read-out implicit eligibility trace) Given the read-out implicit variable $\gamma_{ij}^{t,k}$, we define the read-out implicit eligibility trace $g_{ij}^{t,k}$ as:

$$g_{ij}^{t,k} := \frac{\partial L}{\partial y_k^t} \gamma_{ij}^{t,k}$$

(35)

Since $\partial L / \partial y_k^t$ is causal, and so is the read-out implicit variable $\gamma_{ij}^{t,k}$ (can be computed at each time step), then the read-out implicit eligibility trace $g_{ij}^{t,k}$ is also causal. However, it is semi-local since it requires information both from the connection $jk$ and $ij$.

Final e-prop equation with read-out re-expressed implicit recurrence: With all of this combined, substituting Eq. 33 in Eq. 32 we get the following:

$$\frac{dL}{dw_{ij}} \approx \sum_{k,t} \frac{\partial L}{\partial y_k^t} \gamma_{ij}^{t,k} = \sum_{k,t} g_{ij}^{t,k}$$

(36)

B.3 LSNNs

For completeness, we repeat here the reformulation of read-out neurons with implicit recurrence in the original e-prop paper [2]. Particularly, they use as read-out non-spiking neurons with update equation:

$$y_k^t = \kappa y_k^{t-1} + \sum_j w_{jk}^{out} h_j^t + b_{k}^{out}$$

(37)

where $\kappa$ is the read-out neurons’ time constant, $w_{jk}^{out}$ is the $jk$ connection’s strength and $b_{k}^{out}$ is a bias.

From the definition of read-out implicit variable Eq. 33 and using $\partial y_k^t / \partial y_k^{t-1} = \kappa$ and $\partial y_k^t / \partial h_j^t = w_{jk}^{out}$ (cf. Eq. 37) we conclude:
that can be rewritten as:

\[ \gamma_{ij}^{t,k} = w_{jk}^{out} \mathcal{F}_\kappa(e_{ij}^t) \tag{39} \]

where \( \mathcal{F}_\alpha(x^t) = \alpha \mathcal{F}_\alpha(x^{t-1}) + x^t \) is a low-pass filter operator for a time series \( x \).

Finally, if we plug this in the final e-prop equation for read-outs with implicit recurrence Eq. \[36\] and update gradient:

\[
\frac{d\mathcal{L}}{d\delta w_{ij}} \approx \sum_{k,t} \frac{\partial \mathcal{L}}{\partial y_{kj}^t} \gamma_{ij}^{t,k} = \sum_{k,t} \frac{\partial \mathcal{L}}{\partial y_{kj}^t} w_{jk}^{out} \mathcal{F}_\kappa(e_{ij}^t) = \sum_{t} L_j^t \mathcal{F}_\kappa(e_{ij}^t) \tag{40}
\]

where \( L_j^t = \sum_k w_{jk}^{out} \frac{\partial \mathcal{L}}{\partial y_{kj}^t} \) is called a learning signal.

## C Complexity

E-prop solves the non-locality problem of RTRL (except for the learning signal) while being causal. Also, it is less computational and memory expensive than RTRL. Given \( n \) neurons simulated \( T \) time steps with \( p \) synapses, we see the complexities of the different algorithms in Table 1.

Notice that in general in dense architectures \( p = n^2 \). RTRL has a computational complexity of \( O(p^2 T) \) in total (\( O(p^2) \) per time step). This complexity is prohibitive in comparison to BPTT which belongs to \( O(pT) \) in total (\( O(p) \) per time step, even though since it is not causal it cannot be implemented online and so there is no per time step computation). E-prop has a computational complexity of \( O(pT) \) in total (and indeed now \( O(p) \) per time step), so is as efficient as BPTT.

With regards to memory complexity, BPTT is \( O(nT) \) (the variables’ values of each neuron at each time step, since it is not local), usually much less than the memory requirements of RTRL which is \( O(np) \) (the recurrent eligibility variable of each neuron for each synapse \( \alpha_{ij}^{t,r} \)). E-prop has a memory complexity of \( O(p) \) (the implicit eligibility variable of each synapse \( \epsilon_{ij}^t \)). This is not as good as BPTT but way better than RTRL.

### Table 1. Computational and memory complexity of BPTT, RTRL and e-prop.

| Algorithm | Computation | Memory |
|-----------|-------------|--------|
| BPTT      | \( O(pT) \) | \( O(nT) \) |
| RTRL      | \( O(p^2T) \) | \( O(np) \) |
| e-prop    | \( O(pT) \) | \( O(p) \) |
The main drawback of e-prop is that it is an approximation of the gradient and, therefore, in theory, it should take more iterations to converge to an optimal value. In practical examples the increase in the number of iterations is not prohibitive [23].

D Proofs of incremental computations

D.1 Proof of the Incremental Computation of the Implicit Variable

Incremental computation of the implicit variable Eq. 8:

\[ \epsilon_{ij}^t = \frac{\partial c_{ij}^t}{\partial c_{ij}^{t-1}} \epsilon_{ij}^{t-1} + \frac{\partial c_{ij}^t}{\partial w_{ij}} \]

**Proof:** Using the definition of the implicit variable Eq. 7

\[ \epsilon_{ij}^t := \sum_{t' \leq t} \frac{\partial c_{ij}^{t'}}{\partial c_{ij}^{t'-1}} \cdots \frac{\partial c_{ij}^{t'+1}}{\partial w_{ij}} \]

we get

\[ \epsilon_{ij}^t = \frac{\partial c_{ij}^t}{\partial c_{ij}^{t-1}} \sum_{t' \leq t-1} \frac{\partial c_{ij}^{t'-1}}{\partial c_{ij}^{t-2}} \cdots \frac{\partial c_{ij}^{t'+1}}{\partial w_{ij}} + \frac{\partial c_{ij}^t}{\partial w_{ij}} \]

\[ = \sum_{t' \leq t-1} \frac{\partial c_{ij}^t}{\partial c_{ij}^{t'-1}} \frac{\partial c_{ij}^{t'-1}}{\partial c_{ij}^{t-2}} \cdots \frac{\partial c_{ij}^{t'+1}}{\partial w_{ij}} + \frac{\partial c_{ij}^t}{\partial w_{ij}} \]

\[ = \sum_{t' \leq t} \frac{\partial c_{ij}^t}{\partial c_{ij}^{t'-1}} \cdots \frac{\partial c_{ij}^{t'+1}}{\partial w_{ij}} \]

\[ \square \]

Equivalent proof for the incremental computation of the read-out implicit variable Eq. 34.

D.2 Proof of the Incremental Computation of the Explicit Variable

Incremental computation of the explicit variable Eq. 15

\[ \beta_{ij}^t(k, k', ..., j) = \frac{\partial c_{k}^t}{\partial c_{k}^{t-1}} \beta_{ij}^{t-1}(k, k', ..., j) + \frac{\partial h_{k'}^{t-1}}{\partial h_{k'}^{t-1}} \beta_{ij}^{t-1}(k', ..., j) \]
Proof: Using the definition of the explicit variable Eq. 14

\[
\beta_{ij}^{t}(k, k', ..., j) := \sum_{t' \leq t-1} \frac{\partial c_{k'}^{t}}{\partial c_{k}^{t-1}} \cdot \frac{\partial c_{k}^{t+1}}{\partial c_{k'}^{t+1}} \cdot \frac{\partial h_{k'}^{t}}{\partial c_{k'}^{t}} \cdot \beta_{ij}^{t-1}(k', k'', ..., j)
\]

we get

\[
\beta_{ij}^{t}(k, k', ..., j) = \frac{\partial c_{k}^{t}}{\partial c_{k}^{t-1}} \sum_{t' \leq t-2} \frac{\partial c_{k'}^{t-2}}{\partial c_{k}^{t-2}} \cdot \frac{\partial c_{k}^{t+1}}{\partial c_{k'}^{t+1}} \cdot \frac{\partial h_{k'}^{t}}{\partial c_{k'}^{t}} \cdot \beta_{ij}^{t-1}(k', k'', ..., j)
\]

\[
+ \frac{\partial c_{k}^{t}}{\partial h_{k'}^{t}} \frac{\partial h_{k'}^{t}}{\partial c_{k'}^{t}} \beta_{ij}^{t-1}(k', ..., j)
\]

\[
= \sum_{t' \leq t-2} \frac{\partial c_{k}^{t}}{\partial c_{k}^{t-1}} \cdot \frac{\partial c_{k'}^{t-1}}{\partial c_{k}^{t-1}} \cdot \frac{\partial c_{k}^{t+1}}{\partial c_{k'}^{t+1}} \cdot \frac{\partial h_{k'}^{t}}{\partial c_{k'}^{t}} \cdot \beta_{ij}^{t-1}(k', k'', ..., j)
\]

\[
+ \frac{\partial c_{k}^{t}}{\partial h_{k'}^{t}} \frac{\partial h_{k'}^{t}}{\partial c_{k'}^{t}} \beta_{ij}^{t-1}(k', ..., j)
\]

\[
= \sum_{t' \leq t-1} \frac{\partial c_{k}^{t}}{\partial c_{k}^{t-1}} \cdot \frac{\partial c_{k'}^{t+1}}{\partial c_{k}^{t+1}} \cdot \frac{\partial h_{k'}^{t}}{\partial h_{k'}^{t}} \cdot \beta_{ij}^{t-1}(k', k'', ..., j)
\]

\[
\square
\]

D.3 Proof of the Incremental Computation of the Recurrence Variable

Incremental computation of the recurrence variable Eq. 22

\[
\alpha_{ij}^{t,r} = \frac{\partial c_{r}^{t}}{\partial c_{r}^{t-1}} \alpha_{ij}^{t-1,r} + \sum_{k} \frac{\partial c_{r}^{t}}{\partial h_{k'}^{t-1}} \frac{\partial h_{k'}^{t-1}}{\partial c_{k}^{t-1}} \alpha_{ij}^{t-1,k} + \frac{\partial c_{r}^{t}}{\partial w_{ij}}
\]

Proof: Using the definition of the explicit variable Eq. 14

\[
\beta_{ij}^{t}(k, k', ..., j) = \frac{\partial c_{k}^{t}}{\partial c_{k}^{t-1}} \beta_{ij}^{t-1}(k, k', ..., j)
\]

\[
+ \frac{\partial c_{k}^{t}}{\partial h_{k'}^{t-1}} \frac{\partial h_{k'}^{t-1}}{\partial c_{k'}^{t-1}} \beta_{ij}^{t-1}(k', ..., j)
\]

and the definition of the recurrence variable Eq. 21

\[
\alpha_{ij}^{t,r} = \sum_{t' \leq t} \sum_{k_{0}=j}^{j} \sum_{k_{1}, k_{2}, ..., k_{t-1}} \beta_{ij}^{t}(r, k_{t-1}, ..., k_{1}, k_{0})
\]
we get

\[ \begin{align*}
\alpha_{t,r}^{t,r} &= \frac{\partial x}{\partial c_{t-1}} \sum_{t' \leq t-1} \sum_{k_0 = j}^{k_{t'-1}} \beta^{t-1}_{ij}(r, k_{t'-1}, \cdots, k_0 = j) \\
+ \sum_{k} \frac{\partial x}{\partial h_{t-1}^{k-1}} \frac{\partial h_{t-1}^{k-1}}{\partial c_{t-1}} \sum_{t' \leq t-1} \sum_{k_0 = j}^{k_{t'-1}} \beta^{t-1}_{ij}(k, k_{t'-1}, \cdots, k_0 = j) \\
+ \frac{\partial x}{\partial w_{ij}} \\
= \frac{\partial x}{\partial c_{t-1}} \sum_{0 < t' \leq t-1} \sum_{k_0 = j}^{k_{t'-1}} \beta^{t-1}_{ij}(r, k_{t'-1}, \cdots, k_1, k_0 = j) \\
+ \sum_{k} \frac{\partial x}{\partial h_{t-1}^{k-1}} \frac{\partial h_{t-1}^{k-1}}{\partial c_{t-1}} \sum_{t' < t-1} \sum_{k_0 = j}^{k_{t'-1}} \beta^{t-1}_{ij}(k, k_{t'-1}, \cdots, k_0 = j) \\
+ \left( \frac{\partial x}{\partial c_{t-1}} \beta^{t-1}_{ij}(r) + \frac{\partial x}{\partial w_{ij}} \right) \\
+ \sum_{k} \frac{\partial x}{\partial h_{t-1}^{k-1}} \frac{\partial h_{t-1}^{k-1}}{\partial c_{t-1}} \sum_{k_0 = j}^{k_{t-2}} \beta^{t-1}_{ij}(k, k_{t-2}, \cdots, k_0 = j) \\
= \sum_{t' \leq t-1} \sum_{k_0 = j}^{k_{t'-1}} \beta^{t-1}_{ij}(r, k_{t'-1}, \cdots, k_1, k_0 = j) \\
+ \sum_{k} \frac{\partial x}{\partial h_{t-1}^{k-1}} \frac{\partial h_{t-1}^{k-1}}{\partial c_{t-1}} \sum_{k_0 = j}^{k_{t-2}} \beta^{t-1}_{ij}(k, k_{t-2}, \cdots, k_0 = j) \\
\end{align*} \]

Now, since \( \beta^{t-1}_{ij}(r, k, k_{t-2}, \cdots, k_1, k_0 = j) = 0 \) \( \forall r, k, k_{t-2}, \cdots, k_1 \) (since it considers \( t \) explicit jumps in \( t-1 \) steps), we can add it to the equation:

\[ \begin{align*}
\alpha_{t,r}^{t,r} &= \sum_{t' \leq t-1} \sum_{k_0 = j}^{k_{t'-1}} \beta^{t-1}_{ij}(r, k_{t'-1}, \cdots, k_1, k_0 = j) \\
+ \frac{\partial x}{\partial c_{t-1}} \sum_{k_0 = j}^{k_{t-2}} \beta^{t-1}_{ij}(r, k, k_{t-2}, \cdots, k_1, k_0 = j) \\
+ \sum_{k} \frac{\partial x}{\partial h_{t-1}^{k-1}} \frac{\partial h_{t-1}^{k-1}}{\partial c_{t-1}} \sum_{k_0 = j}^{k_{t-2}} \beta^{t-1}_{ij}(k, k_{t-2}, \cdots, k_1, k_0 = j) \\
= \sum_{t' \leq t-1} \sum_{k_0 = j}^{k_{t'-1}} \beta^{t-1}_{ij}(r, k_{t'-1}, \cdots, k_1, k_0 = j) \\
+ \sum_{k=0}^{k_{t-1}} \sum_{k_0 = j}^{k_{t-1}} \beta^{t-1}_{ij}(r, k_{t-1}, k_{t-2}, \cdots, k_1, k_0 = j) \\
= \sum_{t' \leq t} \sum_{k_0 = j}^{k_{t'-1}} \beta^{t-1}_{ij}(r, k_{t'-1}, \cdots, k_1, k_0 = j) \\
\end{align*} \]

\( \square \)