On the Modes of Polynomials Derived from Nondecreasing Sequences

Donna Q. J. Dou¹, Arthur L. B. Yang²
¹School of Mathematics
Jilin University, Changchun 130012, P. R. China
²Center for Combinatorics, LPMC-TJKLC
Nankai University, Tianjin 300071, P. R. China
Email: ¹qjdou@jlu.edu.cn, ²yang@nankai.edu.cn

Abstract. Wang and Yeh proved that if \( P(x) \) is a polynomial with non-negative and nondecreasing coefficients, then \( P(x + d) \) is unimodal for any \( d > 0 \). A mode of a unimodal polynomial \( f(x) = a_0 + a_1x + \cdots + a_mx^m \) is an index \( k \) such that \( a_k \) is the maximum coefficient. Suppose that \( M_*(P, d) \) is the smallest mode of \( P(x + d) \), and \( M^*(P, d) \) the greatest mode. Wang and Yeh conjectured that if \( d_2 > d_1 > 0 \), then \( M_*(P, d_1) \geq M_*(P, d_2) \) and \( M^*(P, d_1) \geq M^*(P, d_2) \). We give a proof of this conjecture.

Keywords: Unimodal polynomials; The smallest mode; The greatest mode.

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1 Introduction

This paper is concerned with the modes of unimodal polynomials constructed from nonnegative and nondecreasing sequences. Recall that a sequence \( \{a_i\}_{0 \leq i \leq m} \) is unimodal if there exists an index \( 0 \leq k \leq m \) such that
\[
a_0 \leq \cdots \leq a_{k-1} \leq a_k \geq a_{k+1} \geq \cdots \geq a_m.
\]
Such an index \( k \) is called a mode of the sequence. Note that a mode of a sequence may not be unique. It is said to be spiral if
\[
a_m \leq a_0 \leq a_{m-1} \leq a_1 \leq \cdots \leq a_{\lfloor \frac{m-1}{2} \rfloor},
\]
where \( \lfloor \cdot \rfloor \) stands for the greatest integer less than \( \frac{m-1}{2} \). Clearly, the spiral property implies unimodality. We say that a sequence \( \{a_i\}_{0 \leq i \leq m} \) is log-concave if for \( 1 \leq k \leq m-1 \),
\[
a_k^2 \geq a_{k+1}a_{k-1},
\]
and it is ratio monotone if
\[ \frac{a_m}{a_0} \leq \frac{a_{m-1}}{a_1} \leq \cdots \leq \frac{a_{m-i}}{a_i} \leq \cdots \leq \frac{a_{m-[\frac{m-1}{2}]}^{(m-1)}}{a_{m-[\frac{m-1}{2}]]}} \leq 1 \]  
(1.2)

and
\[ \frac{a_0}{a_{m-1}} \leq \frac{a_1}{a_{m-2}} \leq \cdots \leq \frac{a_{i-1}}{a_i} \leq \cdots \leq \frac{a_{m-[\frac{m}{2}]-1}}{a_{m-[m/2]}} \leq 1. \]  
(1.3)

It is easily checked that the ratio monotonicity implies both log-concavity and the spiral property.

Let \( P(x) = a_0 + a_1 x + \cdots + a_m x^m \) be a polynomial with nonnegative coefficients. We say that \( P(x) \) is unimodal if the sequence \( \{a_i\}_{0 \leq i \leq m} \) is unimodal. A mode of \( \{a_i\}_{0 \leq i \leq m} \) is also called a mode of \( P(x) \). Similarly, we say that \( P(x) \) is log-concave or ratio monotone if the sequence \( \{a_i\}_{0 \leq i \leq m} \) is log-concave or ratio monotone.

Throughout this paper \( P(x) \) is assumed to be a polynomial with nonnegative and nondecreasing coefficients. Boros and Moll [2] proved that \( P(x+1) \), as a polynomial of \( x \), is unimodal. Alvarez et al. [1] showed that \( P(x+n) \) is also unimodal for any positive integer \( n \), and conjectured that \( P(x+d) \) is unimodal for any \( d > 0 \). Wang and Yeh [6] confirmed this conjecture and studied the modes of \( P(x+d) \). Llamas and Martínez-Bernal [5] obtained the log-concavity of \( P(x+c) \) for \( c \geq 1 \). Chen, Yang and Zhou [4] showed that \( P(x+1) \) is ratio monotone, which leads to an alternative proof of the ratio monotonicity of the Boros-Moll polynomials [3].

Let \( M_*(P,d) \) and \( M^*(P,d) \) denote the smallest and the greatest mode of \( P(x+d) \) respectively. Our main result is the following theorem, which was conjectured by Wang and Yeh [6].

**Theorem 1.1** Suppose that \( P(x) \) is a monic polynomial of degree \( m \geq 1 \) with nonnegative and nondecreasing coefficients. Then for \( 0 < d_1 < d_2 \), we have \( M_*(P,d_1) \geq M_*(P,d_2) \) and \( M^*(P,d_1) \geq M^*(P,d_2) \).

From now on, we further assume that \( P(x) \) is monic, that is \( a_m = 1 \). For \( 0 \leq k \leq m \), let
\[ b_k(x) = \sum_{j=k}^{m} \binom{j}{k} a_j x^{j-k}. \]  
(1.4)

Therefore, \( b_k(x) \) is of degree \( m - k \) and \( b_k(0) = a_k \). For \( 1 \leq k \leq m \), let
\[ f_k(x) = b_{k-1}(x) - b_k(x), \]  
(1.5)

which is of degree \( m - k + 1 \). Let \( f_k^{(n)}(x) \) denote the \( n \)-th derivative of \( f_k(x) \).
Our proof of Theorem 1.1 relies on the fact that \( f_k(x) \) has only one real zero on \((0, +\infty)\). In fact, the derivative \( f_k^{(n)}(x) \) of order \( n \leq m - k \) has the same property. We establish this property by induction on \( n \).

2 Proof of Theorem 1.1

To prove Theorem 1.1, we need the following three lemmas.

Lemma 2.1 For any \( 0 \leq k \leq m \), we have \( b_k'(x) = (k + 1)b_{k+1}(x) \).

Proof. It can be checked that

\[
\begin{align*}
b_k'(x) &= \sum_{j=k}^{m} \binom{j}{k} a_j (x^{j-k})' \\
&= \sum_{j=k+1}^{m} (j - k) \binom{j}{k} a_j x^{j-k-1} \\
&= \sum_{j=k+1}^{m} (j - k) \frac{j!}{(j-k)!} a_j x^{j-(k+1)} \\
&= \sum_{j=k+1}^{m} \frac{j!}{(j-(k+1))!} a_j x^{j-(k)} \\
&= (k + 1)b_{k+1}(x),
\end{align*}
\]

as required. \( \blacksquare \)

Lemma 2.2 For \( n \geq 1 \) and \( 1 \leq k \leq m \), we have

\[
f_k^{(n)}(x) = (k + n - 1)b_{k+n-1}(x) - (k + n)b_{k+n}(x),
\] (2.6)

where \( (m)_j = m(m - 1) \cdots (m - j + 1) \).

Proof. Use induction on \( n \). For \( n = 1 \), we have

\[
f_k^{(1)}(x) = f'(x) = kb_k - (k + 1)b_{k+1}.
\]

Assume that the lemma holds for \( n = j \), namely,

\[
f_k^{(j)}(x) = (k + j - 1)b_{k+j-1}(x) - (k + j)b_{k+j}(x).
\]
Thus, \( f(x) = n \)

Proof of Theorem 1.1.

In view of (1.4), we have

**Lemma 2.3** For \( 1 \leq k \leq m \) and \( 0 \leq n \leq m - k \), the polynomial \( f_k^{(n)}(x) \) has only one real zero on the interval \((0, +\infty)\). In particular, \( f_k(x) \) has only one real zero on the interval \((0, +\infty)\).

**Proof.** Use induction on \( n \) from \( m - k \) to 0. First, we consider the case \( n = m - k \). Recall that

\[
f_k(x) = \sum_{j=k-1}^{m} \binom{j}{k-1} a_j x^{j-k+1} - \sum_{j=k}^{m} \binom{j}{k} a_j x^{j-k}.
\]

Thus \( f_k(x) \) is a polynomial of degree \( m - k + 1 \). Note that

\[
f_k^{(m-k)}(x) = (m-k+1)! \left( \binom{m}{k-1} a_m x + \left[ \binom{m-1}{k-1} a_{m-1} - \binom{m}{k} a_m \right] (m-k) \right).
\]

Clearly, \( f_k^{(m-k)}(x) \) has only one real zero \( x_0 \) on \((0, +\infty)\). So the lemma is true for \( n = m - k \).

Suppose that the lemma holds for \( n = j \), where \( m - k \geq j \geq 1 \). We proceed to show that \( f_k^{(j-1)}(x) \) has only one real zero on \((0, +\infty)\). From the inductive hypothesis it follows that \( f_k^{(j)}(x) \) has only one real zero on \((0, +\infty)\).

In light of (2.6), it is easy to verify that \( f_k^{(j)}(+\infty) > 0 \) and

\[
f_k^{(j)}(0) = (k+j-1)j a_{k+j} - (k+j)j a_{k+j} \leq 0.
\]

It follows that the polynomial \( f_k^{(j-1)}(x) \) is decreasing up to certain point and becomes increasing on the interval \((0, +\infty)\). Again by (2.6) we find \( f_k^{(j-1)}(+\infty) > 0 \) and

\[
f_k^{(j-1)}(0) = (k+j-2)j a_{k+j-2} - (k+j-1)j a_{k+j-1} \leq 0.
\]

So we conclude that \( f_k^{(j-1)}(x) \) has only one real zero on \((0, +\infty)\). This completes the proof.

**Proof of Theorem 1.1.** In view of (1.4), we have

\[
P(x + d) = \sum_{k=0}^{m} a_k (x + d)^k = \sum_{k=0}^{m} b_k d x^k.
\]
Let us first prove that $M^*(P, d_1) \geq M^*(P, d_2)$. Suppose that $M^*(P, d_1) = k$. If $k = m$, then the inequality $M^*(P, d_1) \geq M^*(P, d_2)$ holds. For the case $0 \leq k < m$, it suffices to verify that $b_k(d_2) > b_{k+1}(d_2)$. By Lemma 2.2, $f_{k+1}(x)$ has only one real zero on $(0, +\infty)$. Note that

$$f_{k+1}(0) \leq 0 \quad \text{and} \quad f_{k+1}(+\infty) > 0.$$ 

From $M^*(P, d_1) = k$ it follows that $b_k(d_1) > b_{k+1}(d_1)$, that is $f_{k+1}(d_1) > 0$. Therefore, $f_{k+1}(d_2) > 0$, that is, $b_k(d_2) > b_{k+1}(d_2)$.

Similarly, it can be seen that $M^*(P, d_1) \geq M^*(P, d_2)$. Suppose that $M^*(P, d_2) = k$. If $k = 0$, then we have $M^*(P, d_1) \geq M^*(P, d_2)$. If $0 < k \leq m$, it is necessary to show that $b_{k-1}(d_1) < b_k(d_1)$. Again, by Lemma 2.2, we know that $f_k(x)$ has only one real zero on $(0, +\infty)$. From $M^*(P, d_2) = k$, it follows that $b_{k-1}(d_2) < b_k(d_2)$, that is $f_k(d_2) < 0$. By the boundary conditions

$$f_k(0) \leq 0 \quad \text{and} \quad f_k(+\infty) > 0,$$

we obtain $f_k(d_1) < 0$, that is $b_{k-1}(d_1) < b_k(d_1)$. This completes the proof.

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