Integrable couplings of a generalized D-Kaup-Newell soliton hierarchy and their Hamiltonian structures

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Abstract

We present an enlarged spectral problem starting from a generalization of the D-Kaup-Newell (D-KN) spectral problem. Then we solve the enlarged zero-curvature equations to produce a series of Lax pairs and corresponding evolution equations formulating the desired integrable couplings. A reduction is made of the original enlarged spectral problem and the related zero-curvature equations are solved generating a second integrable coupling system. Next, we discuss how to compute bilinear forms that are symmetric, ad-invariant, and non-degenerate on the given non-semisimple matrix Lie algebra to employ the variational identity. The variational identity is applied to the original enlarged spectral problem of a generalized D-KN hierarchy to furnish Hamiltonian structures. Then we apply the variational identity to the reduced problem to see its bi-Hamiltonian structures. Both hierarchies have infinitely many commuting symmetries and conserved densities, i.e., are Liouville integrable.

Keywords: Integrable coupling, matrix loop algebra, Liouville integrable, Hamiltonian structure

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1. Introduction

The quest for finding new integrable couplings has produced many new ideas and become an important area of research in mathematical physics [1]-[23]. Studying integrable couplings will facilitate with the complete classification of multiple component integrable systems. Originally, integrable couplings were found in the study of centerless Virasoro symmetry algebras of integrable systems [1, 2]. Given an integrable system \( u_t = K(u) \), an integrable coupling of it is a triangular system of the form

\[
\begin{align*}
  u_t &= K(u), \\
  v_t &= S(u, v),
\end{align*}
\]

where potentials \( u \) and \( v \) are scalar functions or vector functions with dependent variables \( \bar{x} = (t, x_1, x_2, ...) \). The non-triviality condition is \( \frac{\partial S}{\partial u} \neq 0 \), where \( [u] \) denotes a vector consisting of all derivatives of \( u \) with respect to the space variable. This condition guarantees that the new differential equations in the bigger system (1) involve the dependent variables of the original system. Integrable couplings were first constructed through perturbations [1, 2, 3] taking the form

\[
\begin{align*}
  u_t &= K(u), \\
  v_t &= K'(u)[v],
\end{align*}
\]

where \( K'(u)[v] = \frac{\partial}{\partial v}|_{v=0} K(u + \epsilon v, u_x + \epsilon v_x, ...) \) is the Gateaux derivative. Then the spectral matrices were enlarged [4, 6]. In 2006, the connection between integrable couplings and semi-direct sums of Lie algebras was realized [10, 11]. Very recently, a novel kind of AKNS integrable couplings was analyzed [23]; the enlarged spectral problem has an additional matrix block depending on the spectral parameter \( \lambda \). This paper uses the same technique on a generalized D-KN soliton hierarchy to enlarge the spectral problem producing integrable couplings.

A generalized D-KN hierarchy is derived from the following isospectral problem:

\[
\phi_x = U\phi = \begin{bmatrix}
  \lambda^2 - r_1 & \lambda p_1 + s_1 \\
  \lambda q_1 + v_1 & -\lambda^2 + r_1
\end{bmatrix} \phi, U \in sl(2, \mathbb{R}), u = \begin{bmatrix}
  p_1 \\
  q_1 \\
  r_1 \\
  s_1 \\
  v_1
\end{bmatrix}, \phi = \begin{bmatrix}
  \phi_1 \\
  \phi_2
\end{bmatrix},
\]

where \( p_1, q_1, r_1, s_1, \) and \( v_1 \) are potentials. A generalized D-KN hierarchy was analyzed and found to be integrable in the Liouville sense [24].

Recall, the D-KN spectral problem is known [25, 28] to be

\[
\phi_x = U\phi = \begin{bmatrix}
  \lambda^2 + r_1 & \lambda p_1 \\
  \lambda q_1 & -\lambda^2 - r_1
\end{bmatrix} \phi, U \in sl(2, \mathbb{R}), u = \begin{bmatrix}
  p_1 \\
  q_1 \\
  r_1
\end{bmatrix}, \phi = \begin{bmatrix}
  \phi_1 \\
  \phi_2
\end{bmatrix},
\]
which depends on three potentials: $p_1, q_1, \text{ and } r_1$. The new spectral problem (3) is a generalization of the D-KN spectral problem adding two new potentials $s_1$ and $v_1$. Previously, the cases where $r_1 = \alpha$ and $r_1 = \alpha pq$ have been shown to generate integrable hierarchies [26, 27] for the D-KN spectral problem (4). It is clear that (3) is a generalization of the Kaup-Newell [32] spectral matrix, as well. We will note that AKNS hierarchy [33] may be found from (3) by letting $p_1 = q_1 = r_1 = 0$ and choosing a suitable Laurent expansion.

This paper has been divided into two major sections: integrable couplings and Hamiltonian structures. In the integrable couplings section, we begin by enlarging the spectral matrix (3) and solving the corresponding zero-curvature equations. We also prove its localness and show the soliton hierarchy of integrable couplings. Next, we reduce the enlarged spectral matrix and follow the same procedure to find the second integrable coupling system. The section of Hamiltonian structures follows. Here, we find a non-degenerate, ad-invariant, symmetric bilinear form. The bilinear form is used in the variational identity to produce Hamiltonian structures of a generalized D-KN integrable couplings and then in the reduced integrable couplings. We find that the reduced integrable couplings have bi-Hamiltonian structures. We discover infinitely many commuting symmetries and conserved functionals for both hierarchies and, thus, their Liouville integrability.

2. Integrable couplings

We construct integrable couplings for a soliton hierarchy by using a loop matrix Lie algebra. Define a triangular block matrix as follows:

$$M(A_1, A_2) = \begin{bmatrix} A_1 & A_2 \\ 0 & A_1 \end{bmatrix}. \quad (5)$$

It can easily be shown that matrices of this form are closed under matrix multiplication, i.e., constitute a Lie algebra. The associated matrix loop algebra $\tilde{g}(\lambda)$ is formed by all block matrices of the type:

$$\tilde{g}(\lambda) = \{M(A_1, A_2) \mid M \text{ defined by } (5), \text{ entries of } A_i \text{ are Laurent series in } \lambda\}. \quad (6)$$

2.1. Generalized D-KN integrable couplings

A spectral matrix is chosen from $\tilde{g}(\lambda)$ as

$$\bar{U} = \bar{U}(\bar{u}, \lambda) = \begin{bmatrix} U & U_1 \\ 0 & U \end{bmatrix} = \begin{bmatrix} \lambda^2 - r_1 & \lambda p_1 + s_1 \\ \lambda q_1 + v_1 & -\lambda^2 + r_1 \end{bmatrix} \begin{bmatrix} \lambda^2 - r_2 & \lambda p_2 + s_2 \\ \lambda q_2 + v_2 & -\lambda^2 + r_2 \end{bmatrix}, \quad (7)$$

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \lambda^2 - r_1 & \lambda p_1 + s_1 \\ \lambda q_1 + v_1 & -\lambda^2 + r_1 \end{bmatrix}.$$
where \{p_i, q_i, r_i, s_i, v_i, i = 1, 2\} are potentials and \( \bar{u} = (u, v)^T \),

\[ u = (p_1, q_1, r_1, s_1, v_1)^T, \quad v = (p_2, q_2, r_2, s_2, v_2)^T. \]

The isospectral problem is

\[
\bar{\phi}_x = \bar{U} \bar{\phi} = \begin{bmatrix}
\lambda^2 - r_1 & \lambda p_1 + s_1 & \lambda^2 - r_2 & \lambda p_2 + s_2 \\
\lambda q_1 + v_1 & -\lambda^2 + r_1 & \lambda q_2 + v_2 & -\lambda^2 + r_2 \\
0 & 0 & \lambda^2 - r_1 & \lambda p_1 + s_1 \\
0 & 0 & \lambda q_1 + v_1 & -\lambda^2 + r_1
\end{bmatrix} \bar{\phi}, \quad \bar{\phi} = \begin{bmatrix} \psi \\ \phi \end{bmatrix}, \quad (8)
\]

where \( \psi = (\psi_1, \psi_2)^T \) and \( \phi = (\phi_1, \phi_2)^T \). Note that \( U \) is the same matrix as \( \bar{U} \).

Assume that the solution to the stationary zero-curvature equation, \( \bar{W}_x = [\bar{U}, \bar{W}] \), is of the form

\[
\bar{W} = \begin{bmatrix} W & W_1 \\ 0 & W \end{bmatrix} = \begin{bmatrix} a & b & e & f \\ c & -a & g & -e \\ 0 & 0 & a & b \\ 0 & 0 & c & -a \end{bmatrix} \in \tilde{g}(\lambda), \quad (9)
\]

then we get the following matrix formulas:

\[
\begin{align*}
\bar{W}_x &= UW - WU, \\
W_{1,x} &= U_1W - WU_1 + UW_1 - W_1U.
\end{align*} \quad (10)
\]

Solving these two formulas, we get the differential equations:

\[
\begin{align*}
a_x &= -q_1b\lambda + p_1c\lambda - v_1b + s_1c, \\
b_x &= -2p_1a\lambda + 2b\lambda^2 - 2s_1a - 2r_1b, \\
c_x &= 2q_1a\lambda - 2c\lambda^2 + 2v_1a + 2r_1c, \\
e_x &= p_1g\lambda + p_2c\lambda - q_2b\lambda - q_1f\lambda + s_1g + s_2c - v_1f - v_2b, \\
f_x &= 2b\lambda^2 + 2f\lambda^2 - 2p_1e\lambda - 2p_2a\lambda - 2r_1f - 2r_2b - 2s_1e - 2s_2a, \\
g_x &= -2c\lambda^2 - 2g\lambda^2 + 2q_1e\lambda + 2q_2a\lambda + 2r_1g + 2r_2c + 2v_1e + 2v_2a.
\end{align*} \quad (11)
\]

By assuming \( a, b, c, e, f, g \), have the following Laurent series expansions:

\[
\begin{align*}
a &= \sum_{i=0}^{\infty} a_i\lambda^{-i}, \\
b &= \sum_{i=0}^{\infty} b_i\lambda^{-i}, \\
c &= \sum_{i=0}^{\infty} c_i\lambda^{-i}, \\
e &= \sum_{i=0}^{\infty} e_i\lambda^{-i}, \\
f &= \sum_{i=0}^{\infty} f_i\lambda^{-i}, \\
g &= \sum_{i=0}^{\infty} g_i\lambda^{-i}.
\end{align*} \quad (12)
\]
and substituting (12) into (11), we have the recursion relations

\[
\begin{align*}
\begin{cases}
b_{i+1} & = \frac{b_{i-1}}{2} + p_1 a_i + s_1 a_{i-1} + r_1 b_{i-1}, \\
c_{i+1} & = -\frac{c_{i-1}}{2} + q_1 a_i + v_1 a_{i-1} + r_1 c_{i-1}, \\
a_{i+1,x} & = -q_1 \frac{b_{i-1}}{2} - p_1 \frac{c_{i-1}}{2} + (p_1 v_1 - q_1 s_1) a_i - q_1 r_1 b_i + p_1 r_1 c_i + s_1 c_{i+1} \\
& \quad - v_1 b_{i+1}, \\
f_{i+1} & = \frac{f_{i-1}}{2} - b_{i+1} + p_2 a_i + p_1 e_i + s_2 a_{i-1} + s_1 e_{i-1} + r_2 b_{i-1} + r_1 f_{i-1}, \\
g_{i+1} & = -\frac{g_{i-1,x}}{2} - c_{i+1} + q_2 a_i + q_1 e_i + v_2 a_{i-1} + v_1 e_{i-1} + r_2 c_{i-1} + r_1 g_{i-1}, \\
e_{i+1,x} & = -\frac{e_{i-1,x}}{2} p_1 - \frac{e_{i-1}}{2} q_1 + (p_2 - p_1)\left[-\frac{e_{i-1}}{2} + v_1 a_{i-1} + r_1 c_{i-1}\right] \\
& \quad + (q_1 - q_2)\left[\frac{b_{i-1}}{2} + s_1 a_{i-1} + r_1 b_{i-1} + s_1 g_{i+1} + s_2 c_{i-1} - v_1 f_{i+1} - v_2 b_{i+1} \\
& \quad + (p_1 v_2 - q_1 s_2) a_i + (p_1 v_1 - q_1 s_1) e_i + p_1 r_2 c_i - q_1 r_2 b_i \\
& \quad + p_1 r_1 g_i - q_1 r_1 f_i, \\
\end{cases}
\end{align*}
\]

for all \( i \geq 1 \) with initial values

\[
\begin{align*}
a_0 & = \alpha, \quad b_0 = c_0 = 0, \quad a_1 = 0, \quad b_1 = \alpha p_1, \quad c_1 = \alpha q_1, \\
e_0 & = \beta, \quad f_0 = g_0 = 0, \quad e_1 = 0, \quad f_1 = (\beta - \alpha) p_1 + p_2 \alpha, \quad g_1 = (\beta - \alpha) q_1 + q_2 \alpha, \\
\end{align*}
\]

and the conditions for integration

\[
\begin{align*}
\begin{cases}
a_i|_{u=0} = b_i|_{u=0} = c_i|_{u=0} = 0, \quad i \geq 1, \\
e_i|_{u=0} = f_i|_{u=0} = g_i|_{u=0} = 0, \quad i \geq 1, \\
\end{cases}
\end{align*}
\]

which determine the sequence of \( \{a_i, b_i, c_i, e_i, f_i, g_i | i \geq 0\} \) uniquely. For \( i = 2, 3 \), we have the following:

\[
\begin{align*}
b_2 & = \alpha s_1, \quad c_2 = \alpha v_1, \quad a_2 = -\frac{1}{2} p_1 q_1, \\
f_2 & = (\beta - \alpha) s_1 + \alpha s_2, \quad e_2 = (\alpha q_1 + \frac{1}{2} \beta q_1 - \frac{1}{2} q_2) p_1 - \frac{1}{2} \alpha p_2 q_1, \\
g_2 & = (\beta - \alpha) v_1 + \alpha v_2; \\
b_3 & = \frac{1}{2}(p_1 v_1 + q_1 s_1), \quad c_3 = -\frac{1}{2} (q_1^2 p_1 - 2 q_1 r_1 + q_1 x), \\
a_3 & = -\frac{1}{2} (p_1 v_1 + q_1 s_1), \\
f_3 & = \frac{1}{2} ((\beta - 2 \alpha) p_1 v_1 + 2 p_2 r_2 + (\beta + 3 \alpha) p_1^2 q_1 + (2 \beta - 4 \alpha) r_1 p_1 \\
& \quad - \alpha (p_2 q_1 p_1 - 2 r_2 p_1 + q_2 p_1^2) + 2 \alpha p_2 r_1), \\
g_3 & = -\frac{1}{2} ((\beta - 2 \alpha) q_1 v_1 + \alpha q_2 v_2 - (\beta + 3 \alpha) q_2^2 p_1 \alpha - (2 \beta - 4 \alpha) r_1 q_1 \\
& \quad + \alpha (q_2 p_1 q_1 - 2 r_2 q_1 + p_2 q_1^2) - 2 \alpha q_2 r_1), \\
e_3 & = (v_1 \alpha - \frac{1}{2} v_1 \beta - \frac{1}{2} v_2 \alpha) p_1 + (s_1 \alpha - \frac{1}{2} s_1 \beta - \frac{1}{2} s_2 \alpha) q_1 - \frac{1}{2} p_2 v_1 \alpha - \frac{1}{2} q_2 s_1 \alpha.
\end{align*}
\]
All \( \{a_i, b_i, c_i, e_i, f_i, g_i | i \geq 0 \} \) can be proven as differential polynomials of \( \bar{u} \) with respect to \( x \).

**Proposition 2.1.** Let \( \{a_i, b_i, c_i, e_i, f_i, g_i | i = 0, 1 \} \) be given by equations (14). Then all functions \( \{a_i, b_i, c_i, e_i, f_i, g_i | i \geq 0 \} \) determined by equation (13) with the conditions (15) are differential polynomials in \( \bar{u} \) with respect to \( x \), and thus, are local.

Proof. We compute from the enlarged stationary zero-curvature equation, \( \bar{W}_x = [\bar{U}, \bar{W}] \),

\[
\frac{d}{dx}\text{tr}(\bar{W}^2) = 2\text{tr}(\bar{W}\bar{W}_x) = 2\text{tr}(\bar{W}[\bar{U}, \bar{W}]) = 2(\text{tr}(\bar{W}^2 \bar{U}) - \text{tr}(\bar{W}^2 \bar{U})) = 0, \tag{16}
\]

and seeing that the \( \text{tr}(\bar{W}^2) = 4(a^2 + bc) \), we have

\[
a^2 + bc = (a^2 + bc)|_{u=0} = a^2, \tag{17}
\]

following from the initial data (14). Now, we use (12), the Laurent expansions of \( a, b, c \), to give

\[
a_i = \frac{\alpha}{2} - \frac{1}{2\alpha} \sum_{k+l=i} a_k a_l - \frac{1}{2\alpha} \sum_{k+l=i} b_k c_l, \quad i \geq 1. \tag{18}
\]

Based on the recursion relation above (18) and the previous (13), we use mathematical induction to see that all functions \( \{a_i, b_i, c_i, i \geq 0 \} \) are differential polynomials in \( u \) with respect to \( x \), and therefore, are local.

Now, we have

\[
\frac{d}{dx}(2ae + fc + gb) = 2ae_x + 2ae_x + fxc + fcx + gxb + gbx
= 2e(-q_1 b\lambda + p_1 c\lambda - v_1 b + s_1 c) + 2e(p_1 g\lambda + p_2 c\lambda
- q_2 b\lambda - q_1 f\lambda + s_2 g + s_2 c - v_1 f - v_2 b) + c(2b\lambda^2
+ 2f\lambda^2 - 2q_1 e\lambda - 2p_2 a\lambda - 2r_1 f - 2r_2 b - 2s_1 e - 2s_2 a)
+ f(2q_1 a\lambda - 2c\lambda^2 + 2v_1 a + 2r_1 c) + b(-2c\lambda^2 - 2g\lambda^2
+ 2q_1 c\lambda + 2q_2 a\lambda + 2r_1 g + 2r_2 c + 2v_1 c + 2v_2 a)
+ g(-2p_1 a\lambda + 2b\lambda^2 - 2s_1 a - 2r_1 b) = 0.
\]

Similarly, we get

\[
2ae + fc + gb = (2ae + fc + gb)|_{\bar{u}=0} = \alpha \beta.
\]

Therefore, using the Laurent expansions of \( a, b, c, e, f, \) and \( g \) in (12), we have

\[
e_i = \beta - \frac{\beta}{\alpha} a_i - \frac{1}{2\alpha} \sum_{k+l=i, k,l \geq 0} f_k c_l - \frac{1}{2\alpha} \sum_{k+l=i, k,l \geq 0} g_k b_l - \frac{1}{\alpha} \sum_{k+l=i, k,l \geq 1} a_k c_l, \tag{19}
\]

for all \( i \geq 1 \). Using the localness of \( \{a_i, b_i, c_i | i \geq 0 \} \) and the recursive relations (13) and (19), we may see through mathematical induction that all functions \( \{e_i, f_i, g_i | i \geq 0 \} \) are differential polynomials in \( \bar{u} \) with respect to \( x \). This completes the proof.
Now, we need to solve the zero-curvature equations,
\[ \bar{U}_t - \bar{V}^{[m]} + [\bar{U}, \bar{V}^{[m]}] = 0, \quad m \geq 0, \] (20)
which are the compatibility conditions between \[ \Phi \] and the temporal problems,
\[ \bar{\phi}_t = \bar{V}^{[m]} \bar{\phi} = \bar{V}^{[m]}(\bar{u}, \lambda) \bar{\phi}, \quad m \geq 0. \] (21)

In order to do this, we introduce a series of Lax operators
\[ \bar{V}^{[m]}(\bar{u}, \lambda) = (\lambda^m \bar{W})_. \] (22)

After solving (20), we generate a hierarchy of soliton equations, for all \( m \geq 0, \)
\[ \bar{u}_t = \bar{K}_m = \begin{bmatrix} 2b_{m+1} \\ -2c_{m+1} \\ q_1 b_{m+1} - p_1 c_{m+1} \\ -2p_1 a_{m+1} + 2b_{m+2} \\ 2q_1 a_{m+1} - 2c_{m+2} \\ 2f_{m+1} + 2b_{m+1} \\ -2g_{m+1} - 2c_{m+1} \\ q_1 f_{m+1} + q_2 b_{m+1} - p_1 g_{m+1} - p_2 c_{m+1} \\ -2p_1 e_{m+1} - 2p_2 a_{m+1} + 2b_{m+2} + 2f_{m+2} \\ 2q_1 e_{m+1} + 2q_2 a_{m+1} - 2c_{m+2} - 2g_{m+2} \end{bmatrix}. \] (23)

We have
\[ \bar{K}_m = \bar{\Phi} \bar{K}_{m-1} = \bar{\Phi}^m \bar{K}_0, \quad m \geq 0, \] (24)
where
\[ \bar{\Phi} = \begin{bmatrix} \Phi \\ \Phi_1 - \Phi \\ \Phi \end{bmatrix}. \] (25)
where $\Phi$ is the recursion operator of the original system $u_i = K(u)$ equal to

$$
\begin{bmatrix}
-p_1 \partial^{-1} v_1 & -p_1 \partial^{-1} s_1 & 2s_1 \partial^{-1} & 1 - p_1 \partial^{-1} q_1 & -p_1 \partial^{-1} p_1 \\
-s_1 \partial^{-1} q_1 & -s_1 \partial^{-1} p_1 & & & \\
q_1 \partial^{-1} v_1 & q_1 \partial^{-1} s_1 & -2v_1 \partial^{-1} & q_1 \partial^{-1} q_1 & 1 + q_1 \partial^{-1} p_1 \\
+p_1 \partial^{-1} q_1 & +p_1 \partial^{-1} p_1 & & & \\
-p_1 \partial^{-1} \frac{q_1}{2} & -p_1 \partial^{-1} \frac{p_1}{2} & & & \\
-q_1 s_1 \partial^{-1} \frac{q_1}{2} & -q_1 s_1 \partial^{-1} \frac{p_1}{2} & & & \\
\frac{1}{2} \partial v_1 + r_1 - s_1 \partial^{-1} v_1 & -s_1 \partial^{-1} s_1 & 2p_1 r_1 \partial^{-1} & -s_1 \partial^{-1} q_1 & -s_1 \partial^{-1} p_1 \\
-p_1 r_1 \partial^{-1} q_1 & -p_1 r_1 \partial^{-1} \frac{p_1}{2} & +p_1 \partial^{-1} & & \\
-\partial p_1 \partial^{-1} \frac{q_1}{2} & -p_1 r_1 \partial^{-1} p_1 & & & \\
v_1 \partial^{-1} q_1 & -\frac{1}{2} \partial v_1 + r_1 + v_1 \partial^{-1} s_1 & -2q_1 r_1 \partial^{-1} & v_1 \partial^{-1} q_1 & v_1 \partial^{-1} p_1 \\
+\partial q_1 \partial^{-1} \frac{q_1}{2} & +q_1 r_1 \partial^{-1} p_1 & -\partial q_1 \partial^{-1} & & \\
+q_1 r_1 \partial^{-1} q_1 & +\partial q_1 \partial^{-1} \frac{p_1}{2} & & & \\
\end{bmatrix}
$$

and $\Phi_1$ is the supplemental matrix differential operator with entries

$$
[\Phi_1]_{11} = -p_1 \partial^{-1} v_2 - p_2 \partial^{-1} v_1 - s_2 \partial^{-1} q_1 - s_1 \partial^{-1} q_2, \\
[\Phi_1]_{12} = -p_1 \partial^{-1} s_2 - p_2 \partial^{-1} s_1 - s_2 \partial^{-1} p_1 - s_1 \partial^{-1} p_2, \\
[\Phi_1]_{13} = 2s_2 \partial^{-1} + 2s_1 \partial^{-1}, [\Phi_1]_{14} = 1 - p_1 \partial^{-1} q_2 - p_2 \partial^{-1} q_1, \\
[\Phi_1]_{15} = -p_1 \partial^{-1} p_2 - p_2 \partial^{-1} p_1, \\
[\Phi_1]_{21} = q_1 \partial^{-1} v_2 + q_2 \partial^{-1} v_1 + v_1 \partial^{-1} q_2 + v_2 \partial^{-1} q_1, \\
[\Phi_1]_{22} = q_1 \partial^{-1} s_2 + q_2 \partial^{-1} s_1 + v_1 \partial^{-1} p_2 + v_2 \partial^{-1} p_1, \\
[\Phi_1]_{23} = -2v_1 \partial^{-1} - 2v_2 \partial^{-1}, \\
[\Phi_1]_{24} = q_1 \partial^{-1} q_2 + q_2 \partial^{-1} q_1, [\Phi_1]_{25} = 1 + q_1 \partial^{-1} p_2 + q_2 \partial^{-1} p_1, \\
[\Phi_1]_{31} = \frac{(v_1 p_2 - q_2 s_1)}{2} \partial^{-1} q_1 + \frac{(p_1 v_2 - q_1 s_2)}{2} \partial^{-1} q_1 - \frac{(p_1 v_1 - q_1 s_1)}{2} \partial^{-1} q_1 \\
+ \frac{(p_1 v_1 - q_1 s_1)}{2} \partial^{-1} q_2
$$

(26)
\[
[\Phi_{1}]_{32} = \left(\frac{v_1 p_2 - q_2 s_1}{2}\right) \partial^{-1} p_1 + \left(\frac{p_1 v_2 - q_1 s_2}{2}\right) \partial^{-1} p_1 - \left(\frac{p_1 v_1 - q_1 s_1}{2}\right) \partial^{-1} p_1
\]
\[
+ \left(\frac{p_1 v_1 - q_1 s_1}{2}\right) \partial^{-1} p_2
\]
\[
[\Phi_{1}]_{33} = -(p_1 v_2 - q_1 s_2) \partial^{-1} - (p_2 v_1 - q_2 s_1) \partial^{-1}
\]
\[
[\Phi_{1}]_{34} = \frac{q_2}{2}, \quad [\Phi_{1}]_{35} = \frac{p_2}{2}
\]
\[
[\Phi_{1}]_{41} = r_2 - s_1 \partial^{-1} v_2 - s_2 \partial^{-1} v_1 - \partial \left(\frac{p_2}{2} - p_1\right) \partial^{-1} q_1 - (r_1 p_2 + p_1 r_2) \partial^{-1} q_1
\]
\[
+ p_1 r_1 \partial^{-1} q_1 - r_1 p_1 \partial^{-1} q_2 - \partial \left(\frac{p_1}{2}\right) \partial^{-1} q_2,
\]
\[
[\Phi_{1}]_{42} = -s_1 \partial^{-1} s_2 - s_2 \partial^{-1} s_1 - \partial \left(\frac{p_2}{2} - p_1\right) \partial^{-1} p_1 - (r_1 p_2 + p_1 r_2) \partial^{-1} p_1
\]
\[
+ p_1 r_1 \partial^{-1} p_1 - r_1 q_1 \partial^{-1} p_2 - \partial \left(\frac{p_1}{2}\right) \partial^{-1} p_2,
\]
\[
[\Phi_{1}]_{43} = \partial (p_2 - p_1) \partial^{-1} + 2(r_1 p_2 + p_1 r_2) \partial^{-1} + \partial p_1 \partial^{-1}
\]
\[
[\Phi_{1}]_{44} = -s_1 \partial^{-1} q_2 - s_2 \partial^{-1} q_1, \quad [\Phi_{1}]_{45} = -s_1 \partial^{-1} p_2 - s_2 \partial^{-1} p_1
\]
\[
[\Phi_{1}]_{51} = v_1 \partial^{-1} v_2 + v_2 \partial^{-1} v_1 + \partial \left(\frac{q_1 - q_2}{2}\right) \partial^{-1} q_1 + (r_1 q_2 + q_1 r_2) \partial^{-1} q_1
\]
\[
- q_1 r_1 \partial^{-1} q_1 + r_1 q_1 \partial^{-1} q_2 - \partial \left(\frac{q_1}{2}\right) \partial^{-1} q_2,
\]
\[
[\Phi_{1}]_{52} = r_2 + v_1 \partial^{-1} s_2 + v_2 \partial^{-1} s_1 + \partial \left(\frac{q_1 - q_2}{2}\right) \partial^{-1} p_1 + (r_1 q_2 + q_1 r_2) \partial^{-1} p_1
\]
\[
- q_1 r_1 \partial^{-1} p_1 + r_1 q_1 \partial^{-1} p_2 - \partial \left(\frac{q_1}{2}\right) \partial^{-1} p_2
\]
\[
[\Phi_{1}]_{53} = -\partial (q_1 - q_2) \partial^{-1} - 2(r_1 q_2 + q_1 r_2) \partial^{-1} + \partial q_1 \partial^{-1},
\]
\[
[\Phi_{1}]_{54} = v_1 \partial^{-1} q_2 + v_2 \partial^{-1} q_1, \quad [\Phi_{1}]_{55} = v_1 \partial^{-1} p_2 + v_2 \partial^{-1} p_1
\]
\]
with \(\partial = \frac{\partial}{\partial x}\) and \(\partial^{-1}\) as the inverse operator of \(\partial\).

2.2. A specific reduction with two less potentials

A spectral matrix, \(\tilde{U}\), chosen from \(\tilde{g}(\lambda)\), is of the form:

\[
\tilde{U} = \begin{bmatrix}
\lambda^2 - \tilde{r}_1 & \lambda p_1 + s_1 & \lambda^2 - \tilde{r}_2 & \lambda p_2 + s_2 \\
\lambda q_1 + v_1 & -\lambda^2 + \tilde{r}_1 & \lambda q_2 + v_2 & -\lambda^2 + \tilde{r}_2 \\
0 & 0 & \lambda^2 - r_1 & \lambda p_1 + s_1 \\
0 & 0 & \lambda q_1 + v_1 & -\lambda^2 + \tilde{r}_1
\end{bmatrix},
\]

where \(\tilde{r}_1 = \frac{1}{2} p_1 q_1, \tilde{r}_2 = \frac{1}{2} (p_1 q_2 + p_2 q_1 - p_1 q_1), p_i, q_i, s_i, v_i, i = 1, 2\) are potentials, and \(\tilde{u} = (u, v)^T, \ u = (p_1, q_1, s_1, v_1)^T, v = (p_2, q_2, s_2, v_2)^T\). The corresponding spacial spectral problem is

\[
\tilde{\phi}_x = \tilde{U}(\tilde{u}, \lambda) \tilde{\phi}, \quad \tilde{\phi} = \begin{bmatrix}
\psi \\
\phi
\end{bmatrix},
\]

(30)
where \( \psi = (\psi_1, \psi_2)^T \) and \( \phi = (\phi_1, \phi_2)^T \).

Again, we assume that the solution to the stationary zero-curvature equation, \( W_x = [\hat{U}, W] \), is of the form

\[
\hat{W} = \begin{bmatrix} W_1 & W_1 \end{bmatrix} = \begin{bmatrix} a & b & e & f \\ c & -a & g & -e \\ 0 & 0 & a & b \\ 0 & 0 & c & -a \end{bmatrix} \in \mathfrak{g}(\lambda). \tag{31}
\]

Solving the stationary zero-curvature equation \( (30) \), we have the following differential equations:

\[
\begin{aligned}
a_x &= -q_1 b \lambda + p_1 c \lambda - v_1 b + s_1 c, \\
b_x &= -2 p_1 a \lambda + 2 b \lambda^2 - 2 s_1 a - p_1 q_1 b, \\
c_x &= 2 q_1 a \lambda - 2 c \lambda^2 + 2 v_1 a + p_1 q_1 c, \\
e_x &= p_1 g \lambda + p_2 c \lambda - q_2 b \lambda - q_1 f \lambda + s_1 g + s_2 c - v_1 f - v_2 b, \\
f_x &= 2 b \lambda^2 + 2 f \lambda^2 - 2 p_1 e \lambda - 2 p_2 a \lambda - p_1 q_1 f - (p_1 q_2 + p_2 q_1 - p_1 q_1)b, \\
g_x &= -2 c \lambda^2 - 2 g \lambda^2 + 2 q_1 e \lambda + 2 q_2 a \lambda + p_1 q_1 g + (p_1 q_2 + p_2 q_1 - p_1 q_1)c, \\
&+ 2 v_1 e + 2 v_2 a.
\end{aligned} \tag{32}
\]

By assuming \( a, b, c, e, f, g \), have the Laurent expansions \( (12) \), we have the recursion relations

\[
\begin{aligned}
b_{i+1} &= b_{i-1,x} + p_1 a_i + s_1 a_{i-1} + \frac{1}{2} p_1 q_1 b_{i-1}, \\
c_{i+1} &= -c_{i-1,x} + q_1 a_i + v_1 a_{i-1} + \frac{1}{2} p_1 q_1 c_{i-1}, \\
a_{i+1,x} &= -q_1 b_{i,x} + p_1 a_i + s_1 a_{i-1} + \frac{1}{2} p_1 q_1 b_{i-1}, \\
f_{i+1} &= f_{i-1,x} - b_{i+1} + p_2 a_i + p_1 e_i + s_2 a_{i-1} + s_1 e_{i-1}, \\
g_{i+1} &= -g_{i-1,x} - c_{i+1} + q_2 a_i + q_1 e_i + v_2 a_{i-1} + v_1 e_{i-1}, \\
e_{i+1,x} &= -g_{i-1,x} - q_2 a_i + q_1 e_i + v_2 a_{i-1} + v_1 e_{i-1} + \frac{1}{2} (p_1 q_2 + p_2 q_1 - p_1 q_1) b_{i-1}, \\
&+ \frac{1}{2} (p_1 q_2 + p_2 q_1 - p_1 q_1) e_{i-1}, \\
f_i &= f_{i-1} + \frac{1}{2} (p_1 q_2 + p_2 q_1 - p_1 q_1) c_{i-1} + \frac{1}{2} p_1 q_1 g_{i-1}, \\
g_i &= g_{i-1} + \frac{1}{2} (p_1 q_2 + p_2 q_1 - p_1 q_1) e_{i-1} + v_2 a_{i-1} + v_1 e_{i-1} + \frac{1}{2} (p_1 q_2 + p_2 q_1 - p_1 q_1) c_{i-1}, \\
e_{i+1} &= e_{i-1} + \frac{1}{2} (p_1 q_2 + p_2 q_1 - p_1 q_1) e_{i-1} + v_2 a_{i-1} + v_1 e_{i-1} + \frac{1}{2} (p_1 q_2 + p_2 q_1 - p_1 q_1) c_{i-1},
\end{aligned} \tag{33}
\]

for all \( i \geq 1 \) with the same initial values \( (1) \) and conditions for integration \( (15) \) which determine the sequence of \( \{a_i, b_i, c_i, e_i, f_i, g_i \mid i \geq 0\} \) uniquely. All \( \{a_i, b_i, c_i, e_i, f_i, g_i \} \) can be proven as differential polynomials of \( \tilde{u} \) with respect to \( x \).
Proposition 2.2. Let \( \{a_i, b_i, c_i, e_i, f_i, g_i | i = 0, 1\} \) be given by equation (14). Then all functions \( \{a_i, b_i, c_i, e_i, f_i, g_i | i \geq 0\} \) determined by equations (35) with the conditions (17) are differential polynomials in \( \bar{u} \) with respect to \( x \), and thus, are local.

Proof. For brevity, we leave the proof out. It is similar to Proposition 2.1.

We solve the zero-curvature equations (20) with the Lax matrices (22) to generate a hierarchy of soliton equations for all \( m \geq 0 \),

\[
\ddot{u}_m = \ddot{K}_m = \begin{bmatrix}
2h_{m+1} & -2c_{m+1} & -2p_1a_{m+1} + 2b_{m+2} & 2q_1a_{m+1} - 2c_{m+2} & 2f_{m+1} + 2b_{m+1} & -2g_{m+1} - 2c_{m+1} \\
-2p_1c_{m+1} & 2c_{m+1} & 2f_{m+1} + 2b_{m+1} & -2g_{m+1} - 2c_{m+1} & -2p_1e_{m+1} + 2b_{m+2} + 2f_{m+2} & 2q_1e_{m+1} + 2q_2a_{m+1} - 2c_{m+2} - 2g_{m+2}
\end{bmatrix}.
\] (34)

We have

\[
\ddot{K}_m = \ddot{\Phi}K_{m-1} = \ddot{\Phi}^m \ddot{K}_0, \quad m \geq 0,
\] (35)

where \( \dddot{\Phi} \) is a recursion operator determined from (34) and given by

\[
\dddot{\Phi} = \begin{bmatrix}
\Phi & 0 \\
\Phi_1 - \Phi & \dddot{\Phi}
\end{bmatrix}.
\] (36)

The matrix blocks of \( \dddot{\Phi} \) are defined by

\[
\Phi = \begin{bmatrix}
-p_1\partial^{-1}v_1 & -p_1\partial^{-1}s_1 & 1 - p_1\partial^{-1}q_1 & -p_1\partial^{-1}p_1 \\
q_1\partial^{-1}v_1 & q_1\partial^{-1}s_1 & q_1\partial^{-1}q_1 & 1 + q_1\partial^{-1}p_1 \\
\frac{1}{2}\partial + \bar{r}_1 - s_1\partial^{-1}v_1 & -s_1\partial^{-1}s_1 & -s_1\partial^{-1}q_1 & -s_1\partial^{-1}p_1 \\
v_1\partial^{-1}v_1 & -\frac{1}{2}\partial + \bar{r}_1 + v_1\partial^{-1}s_1 & v_1\partial^{-1}q_1 & v_1\partial^{-1}p_1
\end{bmatrix}.
\] (37)

and

\[
\Phi_1 = \begin{bmatrix}
-p_1\partial^{-1}v_2 & -p_1\partial^{-1}s_2 & 1 - p_1\partial^{-1}q_2 & -p_1\partial^{-1}p_2 \\
-q_1\partial^{-1}v_2 & -q_1\partial^{-1}s_2 & q_1\partial^{-1}q_2 & 1 + q_1\partial^{-1}p_2 \\
+q_2\partial^{-1}v_1 & +q_2\partial^{-1}s_1 & +q_2\partial^{-1}q_1 & +q_2\partial^{-1}p_1 \\
-v_1\partial^{-1}v_2 & \bar{r}_2 - s_1\partial^{-1}v_2 & -s_1\partial^{-1}s_1 & -s_1\partial^{-1}q_2 & -s_1\partial^{-1}p_2 \\
-v_2\partial^{-1}v_1 & \bar{r}_2 + v_2\partial^{-1}s_1 & v_1\partial^{-1}q_2 & v_1\partial^{-1}p_2 & v_2\partial^{-1}p_2
\end{bmatrix}.
\] (38)

where \( \bar{r}_1 = \frac{1}{2}p_1q_1 \) and \( \bar{r}_2 = \frac{1}{2}(p_1q_2 + p_2q_1 - p_1q_1) \).
A specific example can be found from the reduced hierarchy of integrable couplings \((\text{34})\) when \(m = 6\) by setting the eight potentials and \(\alpha\) and \(\beta\) to be the following: \(\{p_1 = q_1 = 0, s_1 = u, v_1 = -u, p_2 = q_2 = v, s_2 = w, v_2 = r, \alpha = -4, \beta = -8\}\). We find a coupled mKdV \((\text{14, 15})\) system of equations:

\[
\begin{align*}
    u_t &= -u_{xxx} - 6u^2u_x, \\
    v_t &= -v_{xxx} - 4uu_x - 2u^2v_x + (4r - 4w)u^2, \\
    w_t &= -w_{xxx} + u_{xxx} + (6u^2 + (4v - 4w)u)u_x - 4u^2v_x - 2u^2w_x, \\
    r_t &= -r_{xxx} - u_{xxx} + (-6u^2 + (-4r + 4v)u)u_x - 2u^2r_x - 2u^2v_x.
\end{align*}
\]  

(39)

3. Hamiltonian structures

3.1. Constructing bilinear forms over a non-semisimple Lie algebra

There is a systematic approach for generating Hamiltonian structures for the integrable couplings in \((\text{23})\) and \((\text{34})\) using the variational identity over the enlarged matrix loop algebra \(\tilde{\mathfrak{g}}(\lambda)\) \((\text{12, 16, 17})\). As seen in \((\text{12})\), there is a convenient method to constructing non-degenerate, symmetric, and ad-invariant bilinear forms on \(\tilde{\mathfrak{g}}(\lambda)\) by rewriting \(\tilde{\mathfrak{g}}(\lambda)\) into a vector form. The following four steps have been suggested in \((\text{12})\) to produce the required bilinear forms on \(\tilde{\mathfrak{g}}(\lambda)\):

(1) Construct an isomorphism between the loop algebra \(\tilde{\mathfrak{g}}(\lambda)\) and a vector Lie algebra;
(2) Derive the commutator on the vector Lie algebra;
(3) Compute the required non-degenerate, symmetric, and ad-invariant bilinear forms on the vector Lie algebra;
(4) Establish the corresponding bilinear forms on the original Lie algebra \(\tilde{\mathfrak{g}}(\lambda)\).

The isomorphism

\[
\sigma : \tilde{\mathfrak{g}}(\lambda) \to \mathbb{R}^6, A \mapsto (a_1, ..., a_6)^T,
\]

(40)

where

\[
A = M(A_1, A_2) \in \tilde{\mathfrak{g}}(\lambda), \quad A_i = \begin{bmatrix} a_{3i-2} & a_{3i-1} \\ a_{3i} & -a_{3i-2} \end{bmatrix}, i = 1, 2,
\]

(41)

and a constant symmetric matrix,

\[
F = \begin{bmatrix} 2\eta_1 & 0 & 0 & 2\eta_2 & 0 & 0 \\ 0 & 0 & \eta_1 & 0 & 0 & \eta_2 \\ 0 & \eta_1 & 0 & 0 & \eta_2 & 0 \\ 2\eta_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & \eta_2 & 0 & 0 & 0 \\ 0 & \eta_2 & 0 & 0 & 0 & 0 \end{bmatrix},
\]

(42)

with arbitrary constants \(\eta_1\) and \(\eta_2\) furnish the bilinear forms on \(\tilde{\mathfrak{g}}(\lambda)\) defined as

\[
\langle A, B \rangle_{\tilde{\mathfrak{g}}(\lambda)} = \langle \sigma(A), \sigma(B) \rangle_{\mathbb{R}^6} = \langle (a_1, ..., a_6)F(b_1, ..., b_6)^T \rangle = (2a_1b_1 + a_2b_2 + a_3b_2)\eta_1 + (2a_1b_1 + a_2b_2 + a_3b_5 + 2a_4b_1 + a_5b_3 + a_6b_2)\eta_2.
\]

(43)
The bilinear forms \( \text{(43)} \) are symmetric and ad-invariant due to the isomorphism \( \sigma \). The bilinear forms, defined by \( \text{(43)} \), are non-degenerate iff the determinant of \( F \) is not zero, i.e.,

\[
\text{det}(F) = -4\eta_2^6 \neq 0.
\]  

Therefore, we choose \( \eta_2 \neq 0 \) to obtain the required non-degenerate, symmetric, and ad-invariant bilinear forms over the enlarged matrix loop algebra \( \tilde{g}(\lambda) \). For simplicity, we choose \( \eta_1 = 0 \) and \( \eta_2 = 1 \).

### 3.2. Hamiltonian structures of generalized D-KN integrable couplings

Now, we begin with the enlarged spectral matrix of a generalized D-KN hierarchy \( \text{(5)} \) and compute

\[
\langle \bar{W}, \bar{U}_\lambda \rangle_{\tilde{g}(\lambda)} = (4a + 4e)\lambda + f q_1 + bq_2 + cp_2 + gp_1 \tag{45}
\]

and

\[
\langle \bar{W}, \bar{U}_{\bar{u}} \rangle_{\tilde{g}(\lambda)} = [g\lambda, f\lambda, -2e, g, f, c\lambda, b\lambda, -2a, c, b]^T. \tag{46}
\]

Substituting the Laurent series and comparing powers of \( \lambda \), we have

\[
\frac{\delta}{\delta \bar{u}} \int \left( 4a_{m+2} + 4e_{m+2} \right) + f_{m+1}q_1 + b_{m+1}q_2 + c_{m+1}p_2 + g_{m+1}p_1 dx =
\]

\[
[g_{m+1}, f_{m+1}, -2e_m, g_m, f_m, c_{m+1}, b_{m+1}, -2a_m, c_m, b_m]^T, \quad m \geq 1.
\]  

A long calculation involving the recursion relations \( \text{(13)} \) shows that

\[
\frac{\delta \bar{H}_{m+1}}{\delta \bar{u}} = \bar{\Psi} = \Phi_1^+ = \begin{bmatrix} \Phi_1^+ \\ 0 \end{bmatrix}, \tag{49}
\]

where

\[
\Phi = \begin{bmatrix} g_{m+1} \\ f_{m+1} \\ -2e_m \\ g_m \\ f_m \\ c_{m+1} \\ b_{m+1} \\ c_m \\ b_m \end{bmatrix}, \quad \Phi_1 = \begin{bmatrix} \Phi \end{bmatrix}, \quad \Phi_1^+ = \begin{bmatrix} \Phi_1^+ \\ 0 \end{bmatrix},
\]

and

\[
\frac{\delta \bar{H}_m}{\delta \bar{u}} = \bar{\Psi} = \Phi_1^+ = \begin{bmatrix} \Phi_1^+ \\ 0 \end{bmatrix}, \tag{48}
\]

with \( \Phi \) and \( \Phi_1 \) from \( \text{(26)}, \text{(27)} \) and \( \text{(28)}, \) respectively.

We consequently obtain Hamiltonian structures for the hierarchy of integrable couplings \( \text{(23)} \), i.e.,

\[
\bar{u}_t = \int \frac{\delta \bar{H}_m}{\delta \bar{u}}, \quad m \geq 0, \tag{50}
\]
with the Hamiltonian functionals,
\[ \bar{H}_m = \int \left( \frac{4a_{m+2} + 4e_{m+2}}{m} \right) + f_{m+1}q_1 + b_{m+1}q_2 + c_{m+1}p_2 + g_{m+1}p_1 \, dx, \] (51)
for \( m \geq 1 \), and
\[ \bar{H}_0 = \int [\beta - \alpha]q_1 + \alpha(p_1q_2 + p_2q_1) - 2\beta r_1 - 2\alpha r_2 \, dx \] (52)
calculated directly from \([g_1, f_1, -2e_0, g_0, f_0, c_1, b_1, -2a_0, c_0, b_0]^T\). The Hamiltonian operator in (50) is of the form:
\[
\bar{J} = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} \partial & s_1 & -v_1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -s_1 & 0 & \partial + 2r_1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & v_1 & \partial - 2r_1 & 0 & 0 \\
-2 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{2} \partial & s_1 & -v_1 & 0 & 0 & 0 & s_2 & -v_2 \\
0 & 0 & -s_1 & 0 & \partial + 2r_1 & 0 & 0 & -s_2 & 0 & 2r_2 \\
0 & 0 & v_1 & \partial - 2r_1 & 0 & 0 & 0 & v_2 & -2r_2 & 0
\end{bmatrix}
\] (53)
As a direct result of the Hamiltonian structures (50), the recursion structure (24) and (48), and the property \( \bar{J} \bar{\Psi} = \bar{\Psi}^\dagger \bar{J} \), the hierarchy (23) has the following commutativity of flows:
\[ \{ \bar{H}_k, \bar{H}_l \}_J = \int \left( \frac{\delta \bar{H}_k}{\delta \bar{u}} \right)^T \bar{J} \frac{\delta \bar{H}_l}{\delta \bar{u}} \, dx = 0. \] (54)
We also have the commutativity of symmetries for \( \{ \bar{K}_k \} \), i.e.,
\[ [\bar{K}_k, \bar{K}_l] = \bar{K}'_k(\bar{u})[\bar{K}_l] - \bar{K}'_l(\bar{u})[\bar{K}_k] = 0, \quad k, l \geq 0. \] (55)
Therefore, the hierarchy (23) is Liouville integrable, as expected.

3.3. Bi-Hamiltonian structures of the reduced integrable couplings

Next, we focus on the reduced spectral matrix (50) and compute
\[ \langle \bar{W}, \bar{U}_\lambda \rangle_{\bar{g}(\lambda)} = (4a + 4e)\lambda + f_1q_1 + bq_2 + cp_2 + gp_1 \] (56)
and
\[ \langle \bar{W}, \bar{U}_{\bar{u}} \rangle_{\bar{g}(\lambda)} = [(a - e)q_1 - aq_2 + g\lambda, (a - e)p_1 - ap_2 + f\lambda, g, f, -aq_1 + c\lambda, -ap_1 + b\lambda, c, b]^T. \] (57)
Again, we compare powers of \( \lambda \) after substituting the Laurent series for \( a, b, c, e, f, g \) to get
\[
\frac{\delta}{\delta \bar{u}} \int \frac{(4a_{m+2} + 4\epsilon_{m+2}) + f_{m+1}q_1 + b_{m+1}q_2 + c_{m+1}p_2 + g_{m+1}p_1}{m} \, dx =
\]
\[
[(a_m - \epsilon_m)q_1 - a_m q_2 + g_{m+1}, (a_m - \epsilon_m)p_1 - a_m p_2 + f_{m+1}, g_m, f_m]
\]
\[
- a_m q_1 + c_{m+1}, - a_m p_1 + b_{m+1}, c_m, b_m]^T, \ m \geq 1.
\]  
(58)

Now using the recursion relations (33), we have
\[
\frac{\delta \bar{H}_{m+1}}{\delta \bar{u}} = \bar{\Psi} \frac{\delta \bar{H}_m}{\delta \bar{u}} \ , \text{ i.e.,}
\]
\[
\begin{pmatrix}
(a_{m+1} - \epsilon_{m+1})q_1 - a_{m+1}q_2 + g_{m+2} \\
(a_{m+1} - \epsilon_{m+1})p_1 - a_{m+1}p_2 + f_{m+2} \\
g_m \\
f_m \\
-a_{m+1}q_1 + c_{m+2} \\
-a_{m+1}p_1 + b_{m+2} \\
c_{m+1} \\
b_{m+1}
\end{pmatrix} = \bar{\Psi} \begin{pmatrix}
(a_m - \epsilon_m)q_1 - a_m q_2 + g_{m+1} \\
(a_m - \epsilon_m)p_1 - a_m p_2 + f_m \\
g_m \\
f_m \\
-a_m q_1 + c_{m+1} \\
-a_m p_1 + b_{m+1} \\
c_m \\
b_m
\end{pmatrix},
\]  
(59)

where
\[
\bar{\Psi} = \Phi \frac{\Phi}{\Phi^\dagger} \Phi \frac{\Phi}{\Phi^\dagger},
\]  
(60)

with \( \Phi \) and \( \Phi^\dagger \) from (57) and (58), respectively.

We finally obtain the bi-Hamiltonian structure for the hierarchy of integrable couplings (54),
\[
\bar{u}_m = \bar{J} \frac{\delta \bar{H}_{m+1}}{\delta \bar{u}} = \bar{M} \frac{\delta \bar{H}_m}{\delta \bar{u}}, \ m \geq 0,
\]  
(61)

with the Hamiltonian functionals
\[
\bar{H}_m = \int \frac{(4a_{m+2} + 4\epsilon_{m+2}) + f_{m+1}q_1 + b_{m+1}q_2 + c_{m+1}p_2 + g_{m+1}p_1}{m} \, dx,
\]  
(62)

for \( m \geq 1 \), and the Hamiltonian operators
\[
\bar{J} = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 2 \\
0 & 0 & 0 & 0 & 0 & -2 & 0 \\
0 & 0 & 0 & 0 & 0 & 2 & 0 \\
0 & 0 & 0 & 0 & -2 & 0 & 0 \\
0 & 0 & 2 & 0 & 0 & 0 & 2 \\
0 & 0 & -2 & 0 & 0 & -2 & 0 \\
0 & 2 & 0 & 0 & 2 & 0 & 0 \\
-2 & 0 & 0 & -2 & 0 & 0 & 0
\end{pmatrix},
\]  
(63)

and \( \bar{M} = \bar{\Phi} \bar{J} \) where \( \bar{J} \) is above (58) and \( \bar{\Phi} \) is the recursion operator (55) for the reduced integrable couplings (54). Recall, a bi-Hamiltonian property means that \( \bar{J} \) and \( \bar{M} \) constitute a Hamiltonian pair, or, \( \bar{N} = \alpha \bar{J} + \beta \bar{M} \), for any \( \alpha, \beta \in \mathbb{R} \), is
a Hamiltonian operator. As a direct result of the bi-Hamiltonian structure (61), we can say that the soliton hierarchy (34) is integrable in the Liouville sense:

\[
\{\bar{H}_k, \bar{H}_l\}_{\bar{J}} = \int \left( \frac{\delta \bar{H}_k}{\delta \bar{u}} \right)^T \bar{J} \left( \frac{\delta \bar{H}_l}{\delta \bar{u}} \right) dx = 0,
\]

\[
\{\bar{H}_k, \bar{H}_l\}_{\bar{M}} = \int \left( \frac{\delta \bar{H}_k}{\delta \bar{u}} \right)^T \bar{M} \left( \frac{\delta \bar{H}_l}{\delta \bar{u}} \right) dx = 0,
\]

and

\[
[\bar{K}_k, \bar{K}_l] = \bar{K}_k'(\bar{u})[\bar{K}_l] - \bar{K}_l'(\bar{u})[\bar{K}_k] = 0, \quad k, l \geq 0. \tag{65}
\]

4. Concluding remarks

We have introduced a new spectral matrix that is a generalization of the D-Kaup-Newell and Kaup-Newell spectral problems. In fact, the AKNS hierarchy can be found through a reduction of this spectral problem as discussed in the introduction. Integrable couplings were generated on an enlarged spectral problem solving zero-curvature equations. Using the variational identity, the Hamiltonian structures of the integrable couplings were constructed. A reduction of the enlarged spectral problem was made and the corresponding integrable couplings were found. Again, the variational identity produced the Hamiltonian structures of the reduced integrable couplings. The reduced hierarchy of integrable couplings was found to be bi-Hamiltonian and hierarchies are Liouville integrable.

This paper uses a relatively new idea of having \( \frac{\partial \bar{U}_1}{\partial \lambda} \neq 0 \) in the enlarged spectral matrix \( \bar{U} \). Although the calculations are more difficult, many new applications may arise from integrable couplings starting from enlarged spectral matrices of this form. One application is the Darboux transformation method for the construction of soliton solutions to integrable couplings. Such a new process of construction creates new integrable systems associated with non-semisimple Lie algebras and brings us new insightful thoughts to classify integrable systems from an algebraic point of view.

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