POLYNOMIAL GROWTH OF SUBHARMONIC FUNCTIONS IN A STRONGLY SYMMETRIC RIEMANNIAN MANIFOLD

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Abstract. In this article we have studied some properties of subharmonic functions in a strongly symmetric Riemannian manifold with a pole. As a generalization of polynomial growth of a function we have introduced the notion of polynomial growth of some degree of a function with respect to a real function and proved that any non-negative twice differentiable subharmonic functions in an \(n\)-dimensional manifold always admit polynomial growth of degree 1 with respect to a non-negative real valued subharmonic function on real line. We have also given a lower bound of the integration of a convex function in a geodesic ball.

1. INTRODUCTION

A real valued function \(f\) is said to have polynomial growth of order \(q \in \mathbb{R}\) if there is a constant \(C > 0\) such that \(|f|(x) \leq Cr^q(x)\) for all \(x \in M\), where \(r(x)\) is the distance of \(x \in M\) from a fixed point \(x_0 \in M\) and is denoted by \(|f(x)| = O(r^q)\). The growth of harmonic and subharmonic functions have been studied by many authors \([9]\), \([14]\). The growth of a function depends on the geometrical structure of the manifold. The first significant work about the harmonic function has been done by Yau \([14]\) in 1975. A function \(f\) is said to have sublinear growth if \(|f(x)| = o(r)|\). Cheng \([1]\) proved that a complete Riemannian manifold with non-negative Ricci curvature does not admit any non-constant harmonic function with sublinear growth. This led Yau to formulate the following conjecture.

Conjecture 1.1. For each integer \(q\), the space of harmonic functions on a manifold \(M\) with non-negative Ricci curvature satisfying \(|f(x)| = O(r^q(x))\), is finite dimensional.

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Li and Tam [9] proved that if the volume function of geodesic balls has polynomial growth of order \( q > 0 \), then the space of harmonic functions with polynomial growth of degree 1 is finite dimensional. In this article we have generalized the notion of polynomial growth of a real valued function in a Riemannian manifold. We have defined polynomial growth of a function with respect to a real function (If a function is from \( \mathbb{R} \) to \( \mathbb{R} \), then we call it a real function) and showed that under suitable conditions all non-negative twice differentiable subharmonic functions have polynomial growth of degree 1 with respect to a real subharmonic function.

We organize this paper as follows: Section 2 deals with some preliminaries of Riemannian manifold. In this section we have introduced the notion of polynomial growth with respect to a function, which is the natural generalization of polynomial growth of a function in a Riemannian manifold. In section 3 we have proved that every non-negative \( C^2 \) subharmonic function in a strongly symmetric Riemannian manifold with a pole possesses polynomial growth of degree 1 with respect to a non-negative real subharmonic function. In this section we have also given a lower bound for the integration of a convex function in terms of volume, distance function and a real subharmonic function.

2. Preliminaries

In this section we have discussed some basic facts of a Riemannian manifold \((M, g)\), which will be used throughout this paper (for reference see [10]). Throughout this paper by \( M \) we mean a complete Riemannian manifold of dimension \( n \) endowed with some positive definite metric \( g \) unless otherwise stated. The tangent space at the point \( p \in M \) is denoted by \( T_p M \) and the tangent bundle is defined by \( TM = \bigcup_{p \in M} T_p M \). The length \( l(\gamma) \) of the curve \( \gamma : [a, b] \rightarrow M \) is given by

\[
l(\gamma) = \int_{a}^{b} \sqrt{g_{\gamma(t)}(\dot{\gamma}(t), \dot{\gamma}(t))} \, dt \]

\[
= \int_{a}^{b} \|\dot{\gamma}(t)\| \, dt.
\]

The curve \( \gamma \) is said to be a geodesic if \( \nabla_{\dot{\gamma}(t)} \dot{\gamma}(t) = 0 \ \forall t \in [a, b] \), where \( \nabla \) is the Riemannian connection of \( g \). For any point \( p \in M \), the exponential map \( exp_p : V_p \rightarrow M \) is defined by

\[exp_p(u) = \sigma_u(1),\]
where \( \sigma_u \) is a geodesic with \( \sigma(0) = p \) and \( \dot{\sigma}_u(0) = u \) and \( V_p \) is a collection of vectors of \( T_pM \) such that for each element \( u \in V_p \), the geodesic with initial tangent vector \( u \) is defined on \([0,1]\). It can be easily seen that for a geodesic \( \sigma \), the norm of a tangent vector is constant, i.e., \( \|\dot{\gamma}(t)\| \) is constant. If the tangent vector of a geodesic is of unit norm, then the geodesic is called normal. If the exponential map \( \exp \) is defined at all points of \( T_pM \) for each \( p \in M \), then \( M \) is called complete. Hopf-Rinow theorem provides some equivalent cases for the completeness of \( M \). Let \( x, y \in M \). The distance between \( p \) and \( q \) is defined by

\[
d(x,y) = \inf \{l(\gamma) : \gamma \text{ be a curve joining } x \text{ and } y \}.
\]

A geodesic \( \sigma \) joining \( x \) and \( y \) is called minimal if \( l(\sigma) = d(x,y) \). Hopf-Rinow theorem guarantees the existence of minimal geodesic between two points of \( M \). A smooth vector field is a smooth function \( X : M \to TM \) such that \( \pi \circ X = id_M \), where \( \pi : TM \to M \) is the projection map.

A pole in \( M \) is such a point where the tangent space is diffeomorphic to the whole manifold. Gromoll and Meyer [5] introduced the notion of pole in \( M \). A point \( o \in M \) is called a pole of \( M \) if the exponential map at \( o \) is a global diffeomorphism and a manifold \( M \) with a pole \( o \) is denoted by \((M,o)\). If a manifold possesses a pole then the manifold is complete. Simply connected complete Riemannian manifold with non-positive sectional curvature and a paraboloid of revolution are the examples of Riemannian manifolds with a pole. A manifold with a pole is diffeomorphic to the Euclidean space but the converse is not always true, see [6].

The gradient of a smooth function \( f : M \to \mathbb{R} \) at the point \( p \in M \) is defined by \( \nabla f(p) = g^{ij} \frac{\partial f}{\partial x_j} \frac{\partial}{\partial x_i} \big|_p \). It is the unique vector field such that \( g(\nabla f, X) = X(f) \) for all smooth vector field \( X \) in \( M \). The Hessian \( Hess(f) \) is the \((0,2)\)-tensor field, defined by \( Hess(f)(X,Y) = g(\nabla_X \nabla f, Y) \) for all smooth vector fields \( X, Y \) of \( M \). For a vector field \( X \), the divergence of \( X \) is defined by

\[
div(X) = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^j} \sqrt{g} X^j,
\]

where \( g = det(g_{ij}) \) and \( X = X^j \frac{\partial}{\partial x^j} \). The Laplacian of \( f \) is defined by \( \Delta f = div(\nabla f) \).

**Definition 2.1.** A \( C^2 \)-function \( u : M \to \mathbb{R} \) is said to be harmonic if \( \Delta u = 0 \). The function \( u \) is called subharmonic (superharmonic) if \( \Delta \geq (\leq)0 \).
Definition 2.2. [3][11] A real valued function \( f \) on \( M \) is called convex if for every geodesic \( \gamma : [a, b] \to M \), the following inequality holds

\[
f \circ \gamma((1 - t)a + tb) \leq (1 - t)f \circ \gamma(a) + tf \circ \gamma(b) \quad \forall t \in [0, 1],
\]

or if \( f \) is differentiable, then

\[
g(\nabla f, X)_x \leq f(exp_x \nabla f) - f(x), \quad \forall X \in T_x M.
\]

If the function \( f \) is \( C^2 \), then the convexity of \( f \) is equivalent to \( Hess(f) \geq 0 \). Zhang and Xu [15] introduced the notion of strongly symmetric manifold. And they studied some properties of subharmonic function in strongly symmetric manifold with a pole. Some discussion about strongly symmetric manifold can also be found in [3], where they use the term “model” instead of “strongly symmetric manifold”.

Definition 2.3. [15] A manifold \((M, o)\) with a pole \(o\) is strongly symmetric around \(o\) if and only if every linear isometry \(\phi_o : T_o M \to T_o M\) can be realized as the differential of an isometry \(\phi : M \to M\), i.e., \(\phi(o) = o\) and \(d\phi(o) = \phi_o\).

Definition 2.4. A function \(u : M \to \mathbb{R}\) is said to have polynomial growth of degree \(q \in \mathbb{R}\) with respect to \(v\), where \(v\) is a real function, if \(|f(x)| = O(r^q v(r))\), where \(r(x)\) is the distance of \(x\) from a fixed point in \(M\).

We say that a function has \((v, p)\)-polynomial growth if it has polynomial growth of degree \(p\) with respect to \(v\). If \(v \equiv 1\), then we get the definition of polynomial growth. Then using this generalized notion of polynomial growth, we can generalized the Conjecture 1.1 and ask the following:

Conjecture 2.1. For a fixed integer \(p\) and real function \(v\), the space of harmonic functions with \((v, p)\)-polynomial growth on a Riemannian manifold having non-negative Ricci curvature is finite dimensional?
3. POLYNOMIAL GROWTH AND SUBHARMONIC FUNCTIONS

Theorem 1. Let $M$ be an $n$-dimensional complete manifold with non-negative Ricci curvature. Then any non-negative subharmonic function $u : M \to \mathbb{R}$ satisfies

\[
\sup_{B_{r/2}} u \leq \frac{C}{\text{Vol}(B_r)} \int_{B_r} u,
\]

where the constant $C$ depends only on $n$.

Theorem 2. Let $(M, o)$ be an $n$-dimensional manifold with a pole $o$ and $\text{Ric}_M \geq 0$. If $M$ is strongly symmetric around $o$, then for every non-negative subharmonic function $u \in C^2(M)$, there exists a non-negative subharmonic function $v$ on $\mathbb{R}$ such that

\[
\sup_{B_{r/2}} u \leq C v(r) \quad \text{for every } r > 0,
\]

where $C$ is the constant, depends only on $n$. In particular, $u(x) = O(r \omega(r))$, for some non-negative subharmonic function $\omega$ in $\mathbb{R}$.

Proof. Since $M$ is diffeomorphic to the euclidean space so by taking the polar coordinates of $\mathbb{R}^n$ as $(r, \theta^1, \cdots, \theta^{n-1})$, the metric of $M$ can be expressed in polar from, i.e.,

\[
ds^2 = dr^2 + \sum_{i,j} g_{ij} d\theta^i d\theta^j = dr^2 + h(r)^2 d\Theta^2
\]

on $M - \{o\}$, where $g_{ij} = g\left(\frac{\partial}{\partial \theta^i}, \frac{\partial}{\partial \theta^j}\right)$ and $d\Theta^2$ is the canonical metric on the unit sphere of $T_oM$. Since $M$ is strongly symmetric around $o$ so $h$ depends only on $r$. Hence $r(x)$ denotes the geodesic distance from $o$ to $x$ for any $x \in M$. The Riemannian volume element of $S_r$ can be expressed as $dS_r = \sqrt{D(r, \Theta)} d\theta^1 \cdots d\theta^{n-1}$, where $D = \text{det}(g_{ij})$. Since $u$ is subharmonic so from (1), we get

\[
\sup_{B_{r/2}} u \leq \frac{C}{\text{Vol}(B_r)} \int_{B_r} u dV.
\]

Since for $r > 0$, $\text{Vol}(\partial B_r) \leq \text{Vol}(B_r)$, which implies that $\frac{1}{\text{Vol}(B_r)} \geq \int_{\epsilon}^{r} \frac{1}{\text{Vol}(\partial B_1)} dt$, for some arbitrary small $\epsilon > 0$ and hence we get

\[
\sup_{B_{r/2}} u \leq \frac{C}{\text{Vol}(B_r)} \int_{0}^{r} \int_{\partial B_t} u dS_t dt \\
\leq \liminf_{\epsilon \to 0} C \int_{\epsilon}^{r} \frac{1}{\text{Vol}(\partial B_1)} \int_{\partial B_t} u dS_t dt
\]
Now for \( r > 0 \) define

\[
(4) \quad v(r) = \frac{1}{Vol(\partial B_r)} \int_{\partial B_r} udS_r.
\]

Then we get

\[
(5) \quad \sup_{B_{r/2}} u \leq C \liminf_{\epsilon \to 0} \int_0^r v(t)dt.
\]

Using the coordinate system \( (3) \), the equation \( (4) \) can be represented as

\[
v(r) = \frac{1}{Vol(\partial B_1)} \int_{\partial B_1} u(r\xi)dS_1, \; \text{for} \; \xi \in \partial B_1.
\]

Now taking derivative with respect to \( r \) we get

\[
v'(r) = \frac{1}{Vol(\partial B_1)} \int_{\partial B_1} \partial_r u(r\xi)dS_1 = \frac{1}{Vol(\partial B_r)} \int_{\partial B_r} \partial_r udS_r.
\]

Hence by using divergence theorem, we obtain

\[
(6) \quad v'(r)Vol(\partial B_r) = \int_{\partial B_r} \partial_r udS_r = \int_{B_r} \Delta u dV,
\]

where \( dV \) is the volume element of \( M \). Now it can be easily seen that \( v \) is radially symmetric function. Hence

\[
(7) \quad \Delta v = v'' + (\Delta r)v'.
\]

Again we have the following equation

\[
\int_{B_r} \Delta u dV = \int_0^r \int_{\partial B_t} \Delta u dS_t dt.
\]

Also in \( [15] \) it is proved that \( \Delta r = \frac{Vol'(\partial B_r)}{Vol(\partial B_r)} \), hence using this relation and \( (4) \) and \( (7) \) we get

\[
\int_{\partial B_r} \Delta u dS_r = \frac{d}{dr} \int_{B_r} \Delta u dV = \frac{d}{dr} [v'(r)Vol(\partial B_r)]
\]

\[
= Vol(\partial B_r) \left[ v''(r) + \frac{Vol'(\partial B_r)}{Vol(\partial B_r)} v'(r) \right]
\]

\[
= Vol(\partial B_r) \left[ v''(r) + (\Delta r)v'(r) \right]
\]

\[
= Vol(\partial B_r) \Delta v(r).
\]
Thus we obtain

\[ \Delta v(r) = \frac{1}{Vol(\partial B_r)} \int_{\partial B_r} \Delta u dS. \]

Since \( M \) is strongly symmetric, hence \([15, Lemma 3.1]\) \( \lim_{r \to 0} r\Delta r = n - 1 \). Then for \( r = 0 \), we have \( v'(0) = 0, \ v''(0) = \frac{1}{n}\Delta u(0), \ \lim_{r \to 0} \Delta v(r) = \Delta u(0) \). Then \( v \in C^2(\mathbb{R}) \) \([15]\). Now \( \Delta u \geq 0 \), hence for any \( r \geq 0 \), the above inequality implies that \( \Delta v \geq 0 \), i.e., \( v \) is subharmonic. Then \([5]\) implies that

\[ \sup_{B_{r/2}} u \leq C \int_0^r v(t) \quad \text{(since } v \text{ is continuous at } 0) \]

\[ \leq rC \sup_{[0,r]} v \text{ for any } r > 0. \]

Now \( v \) is subharmonic in \([0, R]\), so using maximum principle \( \sup_{[0,R]} v = v(R) \), since \( v \) is non-decreasing \([15]\). Hence we get

\[ \sup_{B_{r/2}} u \leq rCv(r) \text{ for any } r > 0. \]

The second part is proved trivially from the first part and by taking \( \omega(r) = v(2r) \).

\[ \square \]

**Corollary 2.1.** Let \( u \in C^2(\mathbb{R}^2) \) be a subharmonic function. Then \( u(x) = O(r \omega(r)) \), for some non-negative subharmonic function \( \omega \) in \( \mathbb{R} \).

**Proof.** Since there is no non-constant negative subharmonic function in \( \mathbb{R}^2 \), so the proof easily follows from the above Theorem.

\[ \square \]

**Theorem 3.** \([4]\) Let \( M \) be a complete noncompact Riemannian manifold of positive sectional curvature. If \( u \) is a continuous nonnegative subharmonic function on \( M \), then for any \( p > 1 \) there exist positive constants \( C \) and \( r_0 \) such that

\[ \int_{B_r} u^p dV \geq C(r - r_0) \quad \text{for all } r \geq r_0. \]

**Theorem 4.** Let \((M, o)\) be an \( n \)-dimensional manifold with a pole \( o \) and of positive sectional curvature. Then for every non-negative convex function \( u \) on \( M \), there exist constants \( C > 0 \) and \( r_1 > 0 \) such that

\[ u(o) \geq \frac{2C}{Vol(\partial B_{r_1})} - \sup_{\partial B_{r_1}} u. \]
Proof. Since $u$ is convex, so $u$ is also subharmonic \[2\]. Hence from (11) we get

$$C(r - r_0) \leq \int_{B_r} u^p dV \leq \int_{\partial B_r} \int_0^1 u(\sigma_x(t)) dt dS_r$$

for all $r \geq r_0$, where $\sigma_x : [0, 1] \to M$ is the minimal geodesic such that $\sigma_x(0) = o$ and $\sigma_x(1) = x$. Now using convexity of $u$ and for $r > r_0$, we obtain

$$C(r - r_0) \leq \int_{\partial B_r} \int_0^1 [(1 - t)u(o) + tu(x)] dt dS_r$$

$$\leq \frac{1}{2} \int_{\partial B_r} [u(o) + u(x)] dS_r$$

$$\frac{2C(r - r_0)}{vol(\partial B_r)} \leq u(o) + \sup_{\partial B_r} u.$$

Now taking $r = R_0 + 1$, we get

$$u(o) \geq \frac{2C}{vol(\partial B_{R_0+1})} - \sup_{\partial B_{R_0+1}} u.$$

Let $u$ be a non-negative subharmonic function. Then in $B_{2R}$ we have [12, p. 78]

$$\int_{B_r} |\nabla u|^2 dV \leq \frac{C}{r^2} \int_{B_{2r}} u^2 dV \leq Vol(B_{2r}) \frac{C}{r^2} \sup_{B_{2r}} u^2.$$

Since $u$ is subharmonic so $u^2$ is also subharmonic. Hence by applying the Theorem [2] we obtain

$$\int_{B_r} |\nabla u|^2 dV \leq Vol(B_{2r}) \frac{C}{r^2} rv(4r) = Vol(B_{2r}) \frac{C}{r} v(4r),$$

for some non-negative subharmonic function $v$ in $[0, 4r]$. Hence

$$\frac{r}{Vol(B_{2r})} \int_{B_r} |\nabla u|^2 dV \leq Cv(4r).$$

Now taking limit, we get

$$\limsup_{r \to \infty} \frac{r}{Vol(B_{2r})} \int_{B_r} |\nabla u|^2 dV \leq C \limsup_{r \to \infty} v(4r).$$

Since $Ric_M \geq 0$, so by Bishop volume comparison Theorem [12, p. 11], $Vol(B_r) \leq C_nr^n$. So from the above inequality we get

$$\limsup_{r \to \infty} \frac{1}{r^{n-1}} \int_{B_r} |\nabla u|^2 dV \leq C_1 \limsup_{r \to \infty} v(4r),$$
for some constant $C_1$, depends only on $n$. Hence we state the following Proposition:

**Proposition 5.** Under the assumption of Theorem 2, there exists a non-negative subharmonic function $v$ in $\mathbb{R}$ such that, for all $r > 0$

$$\limsup_{r \to \infty} \frac{1}{r^{n-1}} \int_{B_r} |\nabla u|^2 dV \leq C_1 \limsup_{r \to \infty} v(4r).$$

**Theorem 6.** Let $(M, o)$ be an $n$-dimensional manifold with a pole $o$ and $\text{Ric}_M \geq 0$. If $M$ is strongly symmetric around $o$, then for every convex function $u \in C^2(M)$ with $u \geq 1$ and $|\nabla u| \geq 1$, there exists a positive subharmonic function $v$ in $\mathbb{R}$ such that, for all $r > 0$

$$\int_{B_r} u(e^{x \cdot \nabla u}) dV \geq C_6(n) \frac{(\text{Vol}(B_{4r}))^2}{r^{n+1}v^3(4r)},$$

where $C_6 > 0$ is a constant, depends only on $n$.

**Proof.** The convexity of $u$ and (1) imply that

$$\sup_{\partial B_r/2} u \leq \frac{C}{\text{Vol}(B_r)} \int_{B_r} u dV, \quad \text{for all } r > 0.$$  

Now applying Schwarz’s Inequality, we obtain

$$\sup_{\partial B_r/2} u \leq \frac{C}{\text{Vol}(B_r)} \left( \int_{B_r} \frac{u^2}{|\nabla u|^2} dV \right)^{1/2} \left( \int_{B_r} |\nabla u|^2 dV \right)^{1/2}.  \leq \frac{C_3}{\text{Vol}(B_r)} \left( \int_{B_r} \frac{u^2}{|\nabla u|^2} dV \right)^{1/2} \left( \frac{\text{Vol}(B_{2r})}{r} v(4r) \right)^{1/2},$$

Now from (12)

$$\int_{B_r} |\nabla u|^2 \leq C_1 \frac{\text{Vol}(B_{2r})}{r} v(4r), \quad \forall r > 0,$$

where $v$ is a positive subharmonic function in $\mathbb{R}$ and $C_1 > 0$ is a constant depends only on $n$. Hence putting this value in (14), we get

$$\sup_{\partial B_r/2} u \leq \frac{C_3}{\text{Vol}(B_r)} \left( \int_{B_r} \frac{u^2}{|\nabla u|^2} dV \right)^{1/2} \left( \frac{\text{Vol}(B_{2r})}{r} v(4r) \right)^{1/2},$$

for some constant $C_3 > 0$, depends only on $n$. Now by Bishop volume comparison $\text{Vol}(B_{2r}) \leq C_n 2^n r^n$, we get

$$\sup_{\partial B_r/2} u \leq \frac{C_3}{\text{Vol}(B_r)} \left( \int_{B_r} \frac{u^2}{|\nabla u|^2} dV \right)^{1/2} \left( C_n 2^n r^{n-1} v(4r) \right)^{1/2},$$

i.e.,

$$\left( \sup_{\partial B_r/2} u \right)^2 \leq \frac{C_4(n)}{(\text{Vol}(B_r))^2} \left( \int_{B_r} \frac{u^2}{|\nabla u|^2} dV \right) r^{n-1} v(4r),$$
for some constant $C_4 > 0$. From \(2\), we get $rC\alpha(2r) \geq \sup_{B_r} u \geq u(x) \forall x \in B_r$, i.e., $r^2C\alpha^2(2r) \geq u^2(x) \forall x \in B_r$. Hence $\sup_{B_r} u^2 \leq r^2C\alpha^2(2r)$. Now rewriting the above inequality we have

\[
1 \leq \frac{C_4(n)}{(\text{Vol}(B_r))^2} \left( \int_{B_r} \frac{u^2}{|\nabla u|^2} \, dV \right) r^{n-1}\alpha(4r)
\]

\[
\leq \frac{C_4(n)}{(\text{Vol}(B_r))^2} \left( \sup_{B_r} u^2 \int_{B_r} \frac{1}{|\nabla u|^2} \, dV \right) r^{n-1}\alpha(4r)
\]

\[
\leq r^2C\alpha^2(2r) \frac{C_4(n)}{(\text{Vol}(B_r))^2} \left( \int_{B_r} \frac{1}{|\nabla u|^2} \, dV \right) r^{n-1}\alpha(4r).
\]

Since $\alpha$ is non-decreasing so $\alpha(2r) \leq \alpha(4r)$ for $r > 0$. Hence the above inequality implies that

\[
\frac{C_5(n)}{(\text{Vol}(B_r))^2} \left( \int_{B_r} \frac{1}{|\nabla u|^2} \, dV \right) r^{n+1}\alpha^3(4r) \geq 1,
\]

i.e.,

\[
\int_{B_r} \frac{1}{|\nabla u|^2} \, dV \geq \frac{(\text{Vol}(B_r))^2}{C_5(n)r^{n+1}\alpha^3(4r)}
\]

for some constant $C_5 > 0$. Again $|\nabla u| > 1$, so $|\nabla u|^2 \geq \frac{1}{|\nabla u|^2}$. Hence \(15\) implies that

\[
\int_{B_r} |\nabla u|^2 \, dV \geq \frac{(\text{Vol}(B_r))^2}{C_5(n)r^{n+1}\alpha^3(4r)}.
\]

Now $u$ is convex and $u > 0$, so we get

\[
|\nabla u|^2 \leq u(\exp_x \nabla u) - u(x) \leq u(\exp_x \nabla u) \forall x \in B_r.
\]

Hence by taking $C_6 = 1/C_5$, \(16\) implies

\[
\int_{B_r} u(\exp_x \nabla u) \, dV \geq C_6(n) \frac{(\text{Vol}(B_r))^2}{r^{n+1}\alpha^3(4r)}.
\]

And the positivity of $\alpha$ can easily be seen by using \(4\) and $u > 0$.

\[ \square \]

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