CENTERPOLE SETS FOR COLORINGS OF ABELIAN GROUPS

TARAS BANAKH AND OSTAP CHERVAK

ABSTRACT. A subset $C \subset G$ of a topological group $G$ is called $k$-centerpole if for each $k$-coloring of $G$ there is an unbounded monochromatic subset $C$ which is symmetric with respect to a point $c \in C$ in the sense that $S = cS^{-1}c$. By $c_k(G)$ we denote the smallest cardinality $c_k(G)$ of a $k$-centerpole subset in $G$. We prove that $c_k(G) = c_k(Z^n)$ if $G$ is a discrete abelian group of free rank $m \geq k$. Also we prove that $c_1(Z^{n+1}) = 1$, $c_2(Z^{n+2}) = 3$, $c_3(Z^{n+3}) = 6$, $8 \leq c_4(Z^{n+4}) \leq c_4(Z^4) = 12$ for all $n \in \omega$, and $\frac{1}{3}(k^2 + 3k - 4) \leq c_k(Z^n) \leq 2^k - 1 - \max_{s \leq k-2} \frac{k-s-1}{s-k}$ for all $n \geq k \geq 4$.

1. Introduction

In [3] T. Banakh and I. Protasov proved that for any $k$-coloring $\chi : Z^k \to k = \{0, \ldots, k-1\}$ of the abelian group $Z^k$ there is an infinite monochromatic subset $S \subset Z^k$ such that $S - c = c - S$ for some point $c \in \{0, 1\}^k$. The equality $S - c = c - S$ means that the set $S$ is symmetric with respect to the point $c$. On the other hand, a suitable partition of $R^k$ into $k+1$ convex cones determines a Borel $(k+1)$-coloring of $R^k$ without unbounded monochromatic symmetric subsets. These two results motivate the following definition, cf. [1], [3].

Definition 1. A subset $C$ of a topological group $G$ is called $k$-centerpole for (Borel) colorings of $G$ if for any (Borel) $k$-coloring $\chi : G \to k$ of $G$ there is an unbounded monochromatic subset $C \subset G$, symmetric with respect to some point $c \in C$ in the sense that $Sc^{-1} = cS^{-1}$.

The smallest cardinality $|C|$ of such a $k$-centerpole set $C \subset G$ is denoted by $c_k(G)$ (resp. $c_k^B(G)$). If no $k$-centerpole set $C \subset G$ exists then we write $c_k(G) = \infty$ (resp. $c_k^B(G) = \infty$) and assume that $\infty$ is greater than any cardinal that appears in our considerations.

Now we explain some terminology that appears in this definition. A subset $B$ of a topological group $G$ is called totally bounded if $B$ can be covered by finitely many left shifts of any neighborhood of the neutral element of $G$. In the opposite case $B$ is called unbounded.

A cardinal number $k$ is identified with the set $\{\alpha : |\alpha| < \kappa\}$ of ordinals of smaller cardinality and endowed with the discrete topology.

By a (Borel) $k$-coloring of a topological space $X$ we mean a (Borel) function $\chi : X \to k$. A function $\chi : X \to k$ is Borel if for every color $i \in k$ the set $\chi^{-1}(i)$ of points of color $i$ in $X$ is Borel.

The definition of the numbers $c_k(G)$ and $c_k^B(G)$ implies that $c_k^B(G) \leq c_k(G)$ for any topological group $G$ and any cardinal number $k$. If the topological group $G$ is discrete, then each coloring of $G$ is Borel, so $c_k^B(G) = c_k(G)$ for all $k$. In general, the cardinal numbers $c_k(G)$ and $c_k^B(G)$ are different. For example, $c_\omega^B(R^\omega) = \omega$ while $c_\omega(R^\omega) = \infty$, see Theorem [2].

It follows from the definition that $c_k(G)$ and $c_k^B(G)$ considered as functions of $k$ and $G$ are non-decreasing with respect to $k$ and non-increasing with respect to $G$. More precisely, for a number $k \in \mathbb{N}$, a topological group $G$ and its subgroup $H$ we have the inequalities $c_k(H) \geq c_k(G)$, $c_k(G) \leq c_{k+1}(G)$ and $c_k^B(H) \geq c_k^B(G)$, $c_k^B(G) \leq c_{k+1}^B(G)$.

In the sequel we shall use these monotonicity properties of $c_k(G)$ and $c_k^B(G)$ without any special reference.

In this paper we investigate the problem of calculating the numbers $c_k(G)$ and $c_k^B(G)$ for an abelian topological group $G$ and show that in many cases this problem reduces to calculating the numbers $c_k(R^n \times Z^{m-n})$ and $c_k^B(R^n \times Z^{m-n})$ where $n = r_2(G)$ is the $R$-rank and $m = r_2(G)$ is the $Z$-rank of the group $G$.

For topological groups $G$ and $H$ the $H$-rank $r_H(G)$ of $G$ is defined as

$$r_H(G) = \sup\{k \in \omega : H^k \hookrightarrow G\}$$

where $H^k \hookrightarrow G$ means that $H^k$ is topologically isomorphic to a subgroup of the topological group $G$. It is clear that $r_R(G) \leq r_2(G)$ for each topological group $G$. 

1991 Mathematics Subject Classification. 05E15 and 22B99.
Key words and phrases. Abelian topological group and centerpole set and coloring and symmetric subset and monochromatic subset. 
1 So, a centerpole set can be thought as a set of poles of central symmetries that detect unbounded monochromatic symmetric subsets.
It is interesting to remark that the $\mathbb{Z}$-rank appears in the formula for calculating the value of the function

$$\nu(G) = \min \{ \kappa : c_{\kappa}(G) = \infty \}$$

introduced and studied in [11] and [4]. By [4], for any discrete abelian group $G$

$$\nu(G) = \begin{cases} 
\max\{|G[2]|, \log |G|\} & \text{if } G \text{ is uncountable or } G[2] \text{ is infinite,} \\
r_2(G) + 1 & \text{if } G \text{ is finitely generated,} \\
r_2(G) + 2 & \text{otherwise.}
\end{cases}$$

Here $G[2] = \{ x \in G : 2x = 0 \}$ is the Boolean subgroup of $G$ and $\log |G| = \min \{ \kappa : |G| \leq 2^\kappa \}$.

A topological group $G$ is called inductively locally compact (briefly, an ILC-group) if each finitely generated subgroup $H \subset G$ has locally compact closure in $G$. The class of ILC-groups includes all locally compact groups and all closed subgroups of topological vector spaces.

Our aim is to calculate the numbers $c_k(G)$ and $c_k^B(G)$ for an abelian ILC-group. First, let us exclude two cases in which these numbers can be found in a trivial way.

One of them happens if the number of colors is 1. In this case

$$c_1^B(G) = c_1(G) = \begin{cases} 
1 & \text{if } G \text{ is not totally bounded,} \\
\infty & \text{if } G \text{ is totally bounded.}
\end{cases}$$

The other trivial case happens if the Boolean subgroup $G[2] = \{ x \in G : 2x = 0 \} \subset G$ is unbounded in $G$. In this case, for each finite coloring $\chi : G \to k$ there is a color $i \in k$ such that the set $S = G[2] \cap \chi^{-1}(i)$ is unbounded. Since $S = -S$, we conclude that $S$ is a unbounded monochromatic symmetric subset with respect to 0, which means that the singleton $\{ 0 \}$ is $k$-centerpole in $G$ and thus

$$c_k(G) = c_k^B(G) = 1 \text{ for all } k \in \mathbb{N}.$$
Theorem 3. For any numbers \( k \in \mathbb{N} \) and \( n, m \in \mathbb{N} \cup \{\omega\} \), we get:

1. \( c_k(\mathbb{Z}^n) \leq k^k - 1 - \max_{s \leq k - 2} \binom{k}{s} \) if \( k \leq m \),
2. \( c_k^B(\mathbb{R}^k) \geq \frac{1}{2}(k^2 + 3k - 4) \) if \( k \geq 4 \),
3. \( c_k^B(\mathbb{R}^m) \geq k + 4 \) if \( m \geq k \geq 4 \),
4. \( c_k^B(\mathbb{R}^n) < c_k^B(\mathbb{R}^{n+1}) \) and \( c_k(\mathbb{R}^n) < c_{k+1}(\mathbb{R}^{n+1}) \) if \( k \leq n \);
5. \( c_k^B(\mathbb{R}^n \times \mathbb{Z}^m) \leq c_k^B(\mathbb{R}^{n+1} \times \mathbb{Z}^{m+1}) \) and \( c_k(\mathbb{R}^n \times \mathbb{Z}^m) \leq c_{k+1}(\mathbb{R}^{n+1} \times \mathbb{Z}^{m+1}) \) if \( k \leq n + m \).

The binomial coefficient \( \binom{k}{i} \) in statement (1) equals \( \frac{k^i}{i!(k-i)!} \) if \( i \in \{0, \ldots, k\} \) and zero otherwise. The upper bound from this statement improves the previously known upper bound \( c_k(\mathbb{Z}^n) \leq 2^k - 1 \) proved in \([1]\). For \( k = m \leq 4 \) it yields the upper bounds which coincide with the values of \( c_k(\mathbb{Z}^n) \) given in Theorem 2.

The lower bound \( c_k^B(\mathbb{R}^n) \geq \frac{1}{2}(n^2 + 3n - 4) \) from the item (2) improves the previously known lower bound \( c_n^B(\mathbb{R}^n) \geq \frac{1}{2}(n^2 + n) \), proved in \([1]\). For \( n = 4 \) it gives the lower bound \( 12 \leq c_4^B(\mathbb{R}^4) \), which coincides with the value of \( c_4^B(\mathbb{R}^4) = 14 \).

The statement (5) implies that the sequence \( (c_k(\mathbb{Z}^n))_{k=1}^{\infty} \) is strictly increasing, which answers Question 2 posed in \([1]\). Theorem 3 will be proved in Section 8 and some preparatory work done in Section 2.

For every \( k \in \mathbb{N} \) the sequence \( (c_k(\mathbb{Z}^n))_{n=1}^{\infty} \) is non-increasing and thus it stabilizes starting from some \( n \). The value of this number \( n \) is upper bounded by the cardinal number \( rc_k^B(\mathbb{Z}^n) \) defined as follows.

For a topological group \( G \) and a number \( k \in \mathbb{N} \) let \( rc_k^B(G) \) be the minimal possible \( \mathbb{Z}\)-rank \( r_\mathbb{Z}(\langle C \rangle) \) of a subgroup \( \langle C \rangle \) of \( G \) generated a \( k \)-centerpole subset \( C \subset G \) of cardinality \( |C| = c_k^B(G) \). If such a set \( C \) does not exist (which happens if \( c_k^B(G) = \infty \)), then we put \( rc_k^B(G) = \infty \).

Theorem 4 (Stabilization). Let \( k \geq 2 \) be an integer and \( G \) be an abelian ILC-group with totally bounded Boolean subgroup \( G[2] \) and \( \mathbb{R} \)-rank \( n = r_\mathbb{R}(G) \). Then:

1. \( c_k(G) = c_k^B(\mathbb{Z}^n) \) if \( r_\mathbb{Z}(G) \geq c_k^B(\mathbb{Z}^n) \);
2. \( c_k^B(G) = c_k^B(\mathbb{R}^n \times \mathbb{Z}^m) \) if \( G \) is metrizable and \( r_\mathbb{Z}(G) \geq c_k^B(\mathbb{R}^n \times \mathbb{Z}^m) \);
3. \( c_k^B(G) = c_k^B(\mathbb{R}^n) \) if \( G \) is metrizable and \( r_\mathbb{R}(\mathbb{R}) \geq c_k^B(\mathbb{R}^n) \).

In light of Theorem 4 it is important to have lower and upper bounds for the numbers \( rc_k(G) \).

Proposition 1. For any metrizable abelian ILC-group \( G \) with totally bounded Boolean subgroup \( G[2] \), and a natural number \( 2 \leq k \leq r_\mathbb{R}(G) \) we get:

1. \( rc_k^B(G) = k \) if \( k \leq 3 \) and
2. \( k \leq rc_k^B(G) \leq c_k^B(G) - 3 \) if \( k \geq 3 \).

Finally, let us present the \((k+1)\)-centerpole subset \( \Xi_s \) of \( \mathbb{R}^{1+k} \) that contains \( 2^k - 1 - \binom{k}{s} \) elements and gives the upper bound from Theorem 3.1. This \((k+1)\)-centerpole set \( \Xi_s \) is called the \( \binom{k}{s} \)-sandwich.

Definition 2. Let \( k \) be a non-negative integer and \( s \) be a real number. The subsets

\[ 2^k_{<s} = \{(x_i) \in 2^k : \sum_{i=1}^{k} x_i < s\} \quad \text{and} \quad 2^k_{>s} = \{(x_i) \in 2^k : \sum_{i=1}^{k} x_i > s\} \]

are called the \( s \)-slices of the \( k \)-cube \( 2^k \) where \( 2 = \{0, 1\} \) is the doubleton. For \( s \in \{0, \ldots, k\} \) the union of such slices has cardinality

\[ |2^k_{<s} \cup 2^k_{>s}| = 2^k - \binom{k}{s} = 2^k - \frac{k!}{s!(k-s)!}. \]

The subset

\[ \Xi_s = (-1)^s \times 2^k_{<s} \cup \{0\} \times 2^k_{<k} \cup \{1\} \times 2^k_{>s} \]

of the group \( \mathbb{Z} \times 2^k \) is called the \( \binom{k}{s} \)-sandwich. For \( s \in \{0, \ldots, k\} \) it has cardinality

\[ |\Xi_s| = |2^k_{<k}| + |2^k_{<s} \cup 2^k_{>s}| = 2^{k+1} - 1 - \binom{k}{s}. \]

The following theorem implies the upper bound in Theorem 3. The proof of this theorem (given in Section 3) is not trivial and uses some elements of Algebraic Topology.
Theorem 5. For every \( k \in \mathbb{N} \) and \( s \leq k - 2 \) the \( \binom{k}{s} \)-sandwich \( \Xi^k_s \) is a \((k + 1)\)-centerpole set in the group \( \mathbb{Z} \times \mathbb{Z}^k \).

In light of this theorem it is important to known the geometric structure of \( \binom{k}{s} \)-sandwiches \( \Xi^k_s \) for \( s \leq k - 2 \). For \( k \leq 3 \) those sandwiches are written below:

- \( \Xi^0_2 = \{(1, 0)\} \) is a singleton in \( \mathbb{Z} \times \mathbb{Z}^0 = \mathbb{Z} \times \{0\} \);
- \( \Xi^1_1 = \{(0, 1), (1, 0), (1, 1)\} \) the unit square without a vertex in \( \mathbb{Z}^2 \);
- \( \Xi^2_0 = \{(0, 0, 0), (0, 0, 1), (0, 1, 0), (1, 0, 1), (1, 1, 0), (1, 1, 1)\} \) is the unit cube without two opposite vertices in \( \mathbb{Z}^3 \);
- \( \Xi^3_1 \) is the unit cube without two opposite vertices in \( \mathbb{Z}^4 \), so \( |\Xi^3_0| = 14 \);
- \( \Xi^4_2 \) is a 12-element subset in \( \mathbb{Z}^5 \) whose slices \( \{-1\} \times \mathbb{Z}^2_{<1}, \{0\} \times \mathbb{Z}^2_{\leq 3}, \) and \( \{1\} \times \mathbb{Z}^2_{\geq 1} \) have 1, 7, and 4 points, respectively.

By a triangle (centered at the origin) we shall understand any affinely independent subset \( \{a, b, c\} \in \mathbb{R}^n \) (such that \( a + b + c = 0 \)). A tetrahedron (centered at the origin) is any affinely independent subset \( \{a, b, c, d\} \subset \mathbb{R}^n \) (with \( a + b + c + d = 0 \)).

Let us observe that the sandwich

- \( \Xi^0_2 \) has cardinality \( c_1(\mathbb{R}^1) = 1 \) and is affinely equivalent to any singleton \( \{a\} \) in \( \mathbb{R}^1 \);
- \( \Xi^1_1 \) has cardinality \( c_2(\mathbb{R}^2) = 3 \) and is affinely equivalent to any triangle \( \Delta = \{a, b, c\} \) in \( \mathbb{R}^2 \);
- \( \Xi^2_0 \) has cardinality \( c_3(\mathbb{R}^3) = 6 \) and is affinely equivalent to \( \Delta \cup (x - \Delta) \) where \( \Delta \subset \mathbb{R}^3 \) is a triangle centered at zero and \( x \in \mathbb{R}^3 \) does not belong to the linear span of \( \Delta \);
- \( \Xi^3_1 \) has cardinality \( c_4(\mathbb{R}^4) = 12 \) and is affinely equivalent to \( (x - \Delta) \cup \Delta \cup (\Delta - x) \) where \( \Delta \subset \mathbb{R}^4 \) is a tetrahedron centered at zero and \( x \in \mathbb{R}^4 \) does not belong to the linear span of \( \Delta \).

To see that \( \Xi^3_1 \) is of this form, observe that \( c = (\frac{1}{4}, \frac{1}{2}, \frac{1}{2}) \) is the barycenter of \( \Xi^3_1 \) and \( \Xi^3_1 - c = (x - \Delta) \cup \Delta \cup (\Delta - x) \) for the tetrahedron

\[
\Delta = \{(0, 0, 0, 1), (0, 0, 1, 0), (0, 1, 0, 0), (1, 1, 1, 1)\} - c
\]

and the point \( x = (\frac{1}{2}, 0, 0, 0) \).

Now briefly describe the structure of this paper. In Section 2 we establish a covering property of sandwiches, which will be essentially used in the proof of Theorem 5 given in Section 3. Section 3 is devoted to T-shaped sets which will give us lower bounds for the numbers \( c_k^B(\mathbb{R}^k) \). In Section 5 we prove some lemmas that will help us to analyze the geometric structure of centerpole sets in Euclidean spaces. In Section 6 we study the interplay between centerpole properties of subsets in a group and those of its subgroups. In Section 7 we prove a particular case of the Stability Theorem 4 for the groups \( \mathbb{R}^n \times \mathbb{Z}^{m-n} \). In Sections 8, 9, and 10 we give the proofs of Theorems 5, 2, and 1 respectively. Sections 11 and 12 are devoted to the proofs of Proposition 1 and Theorem 1. The final Section 13 contains selected open problems.

2. Covering \( \Sigma_o \)-sets by shifts of the sandwich \( \Xi^k_s \)

In this section we shall prove a crucial covering property of the \( \binom{k}{s} \)-sandwich \( \Xi^k_s \). In the next section this property will be used in the proof of Theorem 5. We assume that \( k \in \omega \) and \( s \leq k - 2 \) is integer.

First we introduce the notion of a \( \Sigma_o \)-subset of the cube \( 2^{k+1} = \{0, 1\}^{k+1} \). For \( i \in \{0, \ldots, k\} \) consider the \( i \)-th coordinate projection

\[
pr_i : \mathbb{R}^{k+1} \to \mathbb{R}, \quad pr_i : (x_j)_{j=0}^k \mapsto x_i.
\]

The subsets of the form \( 2^{k+1} \cap pr_i^{-1}(l) \) for \( l \in \{0, 1\} \) are called the facets of the cube \( 2^{k+1} \).

Next, consider the function

\[
\Sigma : \mathbb{R}^{k+1} \to \mathbb{R}, \quad \Sigma : (x_i)_{i=0}^k \mapsto \sum_{i=1}^k x_i,
\]

and observe that \( \Sigma(2^{k+1}) = \{0, \ldots, k\} \).

Taking the diagonal product of the functions \( pr_0 \) and \( \Sigma \), we obtain the linear operator

\[
\Sigma_0 : \mathbb{R}^{k+1} \to \mathbb{R}^2, \quad \Sigma_0 : (x_i)_{i=0}^k \mapsto (x_0, \sum_{i=1}^k x_i).
\]

Definition 3. A subset \( \tau \subset 2^{k+1} \) will be called a \( \Sigma_o \)-set if

- \( \tau \) lies in a facet of \( 2^{k+1} \);
- there exists \( a \in \{0, \ldots, k - 1\} \) such that \( \Sigma_0(\tau) \subset \{(0, a), (0, a + 1), (1, a + 1)\} \) or \( \Sigma_0(\tau) \subset \{(0, a), (1, a), (1, a + 1)\} \).

Lemma 1. Each \( \Sigma_o \)-set \( \tau \subset 2^{k+1} \) is covered by a suitable shift \( x + \Xi^k_s \) of the \( \binom{k}{s} \)-sandwich \( \Xi^k_s \).
Proof. Decompose the \( \Sigma_0 \)-set \( \tau \) into the union \( \tau = \tau_0 \cup \tau_1 \) where \( \tau_i = \tau \cap \text{pr}_i^{-1}(i) \) for \( i \in \{0, 1\} \). By our hypothesis \( \tau \) lies in a facet of the cube \( 2^{k+1} \). Consequently, there are numbers \( \gamma \in \{0, \ldots, k\} \) and \( l \in \{0, 1\} \) such that \( \tau \subset \text{pr}_\gamma^{-1}(l) \).

If \( \tau_0 \) or \( \tau_1 \) is empty, then we can change the facet and assume that \( \gamma = 0 \).

Since \( \tau \) is a \( \Sigma_0 \)-set, the image \( \Sigma_0(\tau) \) lies in one of the triangles: \( \{(0, a), (0, a+1), (1, a+1)\} \) or \( \{(0, a), (1, a), (1, a+1)\} \) for some \( a \in \{0, \ldots, k-1\} \). This implies that \( \Sigma(\tau) \subset \{a, a+1\} \).

Identify the cube \( 2^k \) with the subcube \( \{0\} \times 2^k \) of \( \Xi^k \) and let \( e_0 = (1, 0, \ldots, 0) \in 2^{k+1} \). Then
\[
\Xi^k = 2_{<k}^k \cup (e_0 + 2_{<k}^k) \cup (-e_0 + 2_{<k}^k).
\]
Depending on the value of \( \gamma \), two cases are possible.

0. \( \gamma = 0 \). This case has 4 subcases.

0.1. If \( l = 0 \) and \( a < k - 1 \) then \( \Sigma_0(\tau) \subset \{(0, a), (0, a+1)\} \subset \{0, \ldots, k-1\} \) and \( \tau \subset 2_{<k}^k \subset \Xi^k \).

0.2. If \( l = 0 \) and \( a \geq k - 1 \), then \( a > k - 2 \geq s \) and \( \tau \subset 2_{>s}^k \subset -e_0 + \Xi^k \).

0.3. If \( l = 1 \) and \( a < k - 1 \), then \( \Sigma_0(\tau) \subset \{(1, a), (1, a+1)\} \subset \{0, \ldots, k-1\} \) and hence \( \tau \subset e_0 + 2_{<k}^k \subset e_0 + \Xi^k \).

0.4. If \( l = 1 \) and \( a \geq k - 1 \), then \( a > k - 2 \geq s \) and then \( \tau \subset e_0 + 2_{<k}^k \subset \Xi^k \).

I. \( \gamma \neq 0 \). In this case \( \tau_0 \) and \( \tau_1 \) are not empty. Let \( e_\tau \) be the basic vector whose \( \gamma \)-th coordinate is 1 and the others are zero. By our assumption, \( \Sigma_0(\tau) \subset \{(0, a), (1, a), (1, a+1)\} \) or \( \Sigma_0(\tau) \subset \{(0, a), (0, a+1), (1, a+1)\} \) for some \( a \in \{0, \ldots, k-1\} \). So, we consider two subcases.

I.1. \( \Sigma_0(\tau) \subset \{(0, a), (1, a), (1, a+1)\} \). This case has two subcases.

I.1.0. \( l = 0 \). In this subcase \( \Sigma(\tau) = \Sigma(\tau_0) \cup \Sigma(\tau_1) = \{a, a+1\} \subset \{0, \ldots, k-1\} \) and hence \( a \leq k - 2 \). Depending on the value of \( a \), we have three possibilities.

If \( a > s \), then \( \tau = \tau_0 \cup \tau_1 \subset 2_{<k}^k \subset (e_0 + 2_{<k}^k) \subset \Xi^k \).

If \( a = s \), then for the shifted set \( e_\tau + \tau \) we get
\[
\Sigma_0(e_\gamma + \tau) \subset \{(0, a+1), (1, a+1), (1, a+2)\}.
\]
Since \( a = s \leq k - 2 \), we conclude that \( e_\gamma + \tau_0 \subset 2_{<k}^k \subset \Xi^k \). On the other hand, \( e_\gamma + \tau_1 \subset e_1 + 2_{<k}^k \subset \Xi^k \). Then \( \tau \subset -e_\gamma + \Xi^k \).

I.1.1. \( l = 1 \). In this subcase three possibilities can occur:

If \( a > s \), then \( \tau = \tau_0 \cup \tau_1 \subset 2_{<k}^k \subset (e_0 + 2_{<k}^k) \subset \Xi^k \).

If \( a < s \), then \( a + 1 < s \leq k - 2 \) and then \( \tau = \tau_0 \cup \tau_1 \subset 2_{<k}^k \subset (e_0 + 2_{<k}^k) \subset e_0 + \Xi^k \).

If \( a = s \), then for the shift \( -e_\gamma + \tau \) we get
\[
\Sigma_0(-e_\gamma + \tau) \subset \{(0, a-1), (1, a-1), (1, a)\} \text{ and } -e_\gamma + \tau \subset 2_{<k}^k \subset (e_0 + 2_{<k}^k) \subset e_0 + \Xi^k \).
\]

1. \( \Sigma_0(\tau) \subset \{(0, a), (0, a+1), (1, a+1)\} \). Depending on the value of \( l \in \{0, 1\} \), consider two subcases.

I.2.0. \( l = 0 \). In this case \( \{0, \ldots, k-1\} \supset \Sigma(\tau) = \Sigma(\tau_0) \cup \Sigma(\tau_1) = \{a, a+1\} \cup \{a+1\} \) and consequently, \( a + 1 \leq k - 1 \).

If \( a \geq s \), then \( \tau = \tau_0 \cup \tau_1 \subset 2_{<k}^k \subset (e_0 + 2_{<k}^k) \subset e_0 + \Xi^k \).

If \( a < s \), then we can consider the shift \( e_\gamma + \tau \) and repeating the preceding argument, show that \( e_\gamma + \tau \subset \Xi^k \).

Consequently, \( \tau \subset -e_\gamma + \Xi^k \).

I.2.1. \( l = 1 \). In this case we have four subcases.

If \( a = k - 1 \), then for the shifted set \( -e_\gamma + \tau \) we get
\[
\Sigma_0(-e_\gamma + \tau) \subset \{(0, a-1), (0, a), (1, a)\} \text{ and } -e_\gamma + \tau \subset 2_{<k}^k \subset (e_0 + 2_{<k}^k) = \Xi^k \text{. Then } \tau \subset e_\gamma + \Xi^k \).
\]

If \( s \leq a < k - 1 \), then \( \tau = \tau_0 \cup \tau_1 \subset 2_{<k}^k \cup (e_0 + 2_{<k}^k) = \Xi^k \).

If \( a = s - 1 \), then for the shifted set \( -e_\gamma + \tau \) we get
\[
\Sigma_0(-e_\gamma + \tau) \subset \{(0, a-1), (0, a), (1, a)\} \text{ and then } -e_\gamma + \tau \subset 2_{<s}^k \subset (e_0 + 2_{<k}^k) = e_0 + \Xi^k \text{ and } \tau \subset e_\gamma + e_0 + \Xi^k \).
\]

This was the last of the 17 cases we have considered.

\[ \square \]

3. \textbf{Proof of Theorem 5}

The proof of Theorem 5 uses the idea of the proof of Lemma 6 in [1] (which established the upper bound \( c_3(\Xi^2) \leq 6 \)).

We need to prove that for every \( k \leq n \) and \( s \leq k - 2 \) the \( (\Delta_s) \)-sandwich \( \Xi^k \) is \((k+1)\)-centerpole in \( \mathbb{Z} \times \mathbb{Z}^k = \mathbb{Z}^{1+k} \).

Assuming that this is not true, find a coloring \( \chi : \mathbb{Z}^{1+k} \to k+1 = \{0, \ldots, k\} \) such that \( \mathbb{Z}^{1+k} \) contains no unbounded monochromatic subset, symmetric with respect to some point \( c \in \Xi^k \). Observe that for each color \( i \in \{0, \ldots, k\} \) the
Intersection $A_i \cap (2c - A_i)$ is the largest subset of $A_i$, symmetric with respect to the point $c$. By our assumption, the (maximal $i$-colored $c$-symmetric) set $A_i \cap (2c - A_i)$ is bounded and so is the union

$$B = \bigcup_{i=0}^{k} \bigcup_{c \in \Xi^k_s} A_i \cap (2c - A_i)$$

of all such maximal symmetric monochromatic subsets.

**Claim 1.** $\chi(x) \notin \chi(-x + 2 \Xi^k_s)$ for any $x \notin B$.

**Proof.** Assuming conversely that $\chi(x) = \chi(-x + 2c)$ for some $c \in \Xi^k$, we get $\frac{1}{2} (x + (-x + 2c)) = c$ and hence $x$ and $-x + 2c$ are two points symmetric with respect to the center $c \in \Xi^k$ and colored by the same color. Consequently, $x \in B$ by the definition of $B$. \hfill $\square$

Fix a number $n \in \mathbb{N}$ so big that the cube $K = [-2n, 2n]^{1+k} \subset \mathbb{R}^{1+k}$ contains the bounded set $B$ in its interior and let $\partial K$ be the topological boundary $\partial K$ of the cube $K$ in $\mathbb{R}^{1+k}$. Observe that Claim 1 implies:

**Claim 2.** $\chi(-x) \notin \chi(x + 2 \Xi^k_s)$ for each point $x \in \mathbb{Z}^{1+k} \cap \partial K$.

We recall that for every $i \in k + 1 = \{0, \ldots, k\}$

$$pr_i : \mathbb{R}^{1+k} \to \mathbb{R}, \quad pr_i : (x_j)_{j=0}^{k} \mapsto x_i,$$

denotes the $i$th coordinate projection and $e_i$ is the unit vector along the $i$-th coordinate axis, that is, $pr_j(e_i) = 1$ if $i = j$, and 0 otherwise.

For a subset $J \subset \{0, \ldots, k\}$ let $e_J = \sum_{j \in J} e_j \in \mathbb{R}^{1+k}$ be the vector of the principal diagonal of the cube $2^J = \{(x_i)_{i=0}^{k} \in 2^{1+k} : \forall i \notin J \implies (x_i = 0)\} \subset 2^{1+k}$.

For a point $x \in \mathbb{R}^{1+k}$ let $J_x = \{i \in k + 1 : x_i \notin 2\mathbb{Z}\}$ and let $\lfloor x \rfloor$ be the unique point in $(2\mathbb{Z})^{1+k}$ such that $x \in [\lfloor x \rfloor] + 2 \cdot 2^J$. So, $|x| \leq |\lfloor x \rfloor| + 2 \cdot 2^J$.

Consider the function $\Sigma : \mathbb{R}^{1+k} \to \mathbb{R}$ assigning to each sequence $x = (x_i)_{i=0}^{k} = 0$ the sum $\Sigma(x) = \sum_{i=1}^{k} x_i$. The map $\Sigma$ combined with the 0th coordinate projection $pr_0$ compose the linear operator

$$\Sigma_0 : \mathbb{R}^{1+k} \to \mathbb{R}^2, \quad \Sigma_0 : (x_i)_{i=0}^{k} \mapsto (x_0, \Sigma(x)) = (x_0, \sum_{i=1}^{k} x_i).$$

Choose a triangulation $T$ of the boundary $\partial K$ of the cube $K = [-2n, 2n]^{1+k}$ such that for each simplex $\tau$ of the triangulation there is a point $\hat{\tau} \in (2\mathbb{Z})^{1+k}$ such that $\frac{1}{2} (\tau - \hat{\tau})$ is a $\Sigma_0$-subset of $2^{1+k}$. The reader can easily check that such a triangulation $T$ always exists. The choice of the triangulation $T$ combined with Lemma 1 implies:

**Claim 3.** Each simplex $\tau$ of the triangulation $T$ is covered by a suitable shift $x + 2 \Xi^k_s$ of the homothetic copy $2 \Xi^k_s$ of the $(k)\text{-sandwich } \Xi^k_s$.

Let $\Delta$ be (the geometric realization of) a simplex in $\mathbb{R}^k$ with vertices $w_0, \ldots, w_k$ such that $w_0 + \cdots + w_k = 0$. The latter equality means that $\Delta$ is centered at the origin (which lies in the interior of $\Delta$). By $\Delta^{(0)} = \{w_0, \ldots, w_k\}$ we denote the set of vertices of the simplex $\Delta$.

Each point $y \in \Delta$ can be uniquely written as the convex combination $y = \sum_{i=0}^{k} y_i w_i$ for some non-negative real numbers $y_0, \ldots, y_k$ with $\sum_{i=0}^{k} y_i = 1$. The set

$$\text{supp}(y) = \{i \in \{0, \ldots, k\} : y_i \neq 0\}$$

is called the *support* of $y$. It is clear that $\text{supp}(y)$ is the smallest subset of $\Delta^{(0)}$ whose convex hull contains the point $y$.

Identifying each number $i \in \{0, \ldots, k\}$ with the vertex $w_i$ of $\Delta$, we can think of the coloring $\chi : \mathbb{Z}^{1+k} \to \{0, \ldots, k\}$ as a function $\chi : \mathbb{Z}^{1+k} \to \Delta^{(0)} = \{w_0, \ldots, w_k\}$.

Now extend the restriction $\chi|_{\partial K \cap (2\mathbb{Z})^{1+k}}$ of $\chi$ to a simplicial map $f : \partial K \to \Delta$ (which is affine on the convex hull of each simplex $\tau \in T$). The simpliciality of $f$ implies:

**Claim 4.** For each simplex $\tau \in T$ and a point $x \in \text{conv}(\tau)$

$$\text{supp}(f(x)) \subset \chi(\tau) \subset \chi([x] + 2 \cdot 2^J).$$

This Claim has the following corollary.

**Claim 5.** $f(\partial K) \subset \partial \Delta$. 
It follows from Claim 7 that the geometric simplex $\Delta$ is a homeomorphism.

Consequently, identified with the space $R$, Claim 8.

Proof. Given any point $x \in \partial K$, find a simplex $\tau \in T$ whose convex hull contains $x$. By the choice of the triangulation $T$ and Lemma 1, $\tau \subset -y + 2\mathbb{Z}^k$ for some point $y \in \mathbb{Z}^{1+k}$. By Claim 2 $\chi(-y) \notin \chi(\tau)$ and thus $f(x) \in \text{conv}(f(\tau)) = \text{conv}(\chi(\tau)) \subset \text{conv}(\Delta(0) \setminus \chi(-y)) \subset \partial \Delta$.

\[\square\]

Now consider the intersection $K_0 = \{0\} \times [-2n, 2n]^k$ of the cube $K$ with the hyperplane $\{0\} \times \mathbb{R}^k$, which will be identified with the space $\mathbb{R}^k$, and let $\partial K_0 = \partial K \cap \mathbb{R}^k$ be the boundary of $K_0$.

For each subset $J \subset k+1 = \{0, \ldots, k\}$ consider the map

$$p_J : \mathbb{R}^{1+k} \to \mathbb{R}, \quad p_J : (x_i)_{i=0}^k \mapsto 1 \cdot \prod_{j \in J} x_j.$$

Here we assume that $p_0(x) = 1$. It follows that $\sum_{\substack{j \subseteq k+1 \quad p_j(x) > 0 \, \text{for all} \quad x \in [0, 2]^k \cup 2\mathbb{Z}^k \cup \Xi_k}}$.

We remind that for a point $x \in \mathbb{R}^{1+k}$, $J_x = \{ j \in \{0, \ldots, k\} : x_i \notin 2\mathbb{Z} \}$ has cardinality $|J_x| < k$ and thus $2^{J_x} \subset \{0\} \times 2^k \subset \Xi_k$. Consequently, $|x| + 2 \cdot 2^{J_x} \subset |x| + 2\Xi_k$ and $\chi(|x| + 2 \cdot 2^{J_x}) \subset \chi(|x| + 2\Xi_k)$.

Claim 7. $\varphi(x) \neq \varphi(-x)$ for all $x \in \partial K_0$.

Proof. Observe that $J_x = J_{-x}$ and $[-x] = [-x] - 2e_{J_x}$.

On the other hand, Claim 6 guarantees that $\chi(-[x]) = \chi([-x] + 2 \cdot e_{-J_x}) \in \chi([-x] + 2 \cdot 2^{J_x}) = \text{supp}(\varphi(-x))$.

Finally, consider the homotopy $$(f_t) : \partial K_0 \times [0, 1] \to \Delta, \quad f_t : x \mapsto t \varphi(x) + (1-t)f(x),$$

connecting the map $f = f_0$ with the map $\varphi = f_1$.

Claim 8. $\text{supp}(f_t(x)) \subset \chi(|x| + 2 \cdot 2^{J_x}) \subset \partial \Delta$ for all $x \in \partial K_0$ and $t \in [0, 1]$.

Proof. The inclusion $\text{supp}(f_t(x)) \subset \chi(|x| + 2 \cdot 2^{J_x})$ follows from Claims 4 and 6.

The inclusion $x \in \partial K_0$ implies that the set $J_x = \{ j \in \{0, \ldots, k\} : p_j(x) \notin 2\mathbb{Z} \}$ has cardinality $|J_x| < k$ and thus $2^{J_x} \subset \{0\} \times 2^k \subset \Xi_k$. By Claim 1 $\chi(-[x]) \notin \chi([-x] + 2\Xi_k)$ and then $f_t(x) \in \text{conv}((\text{supp}(f_t(x)) \subset \text{conv}(\chi(|x| + 2 \cdot 2^{J_x})) \subset \text{conv}(\chi(|x| + 2\Xi_k)) \subset \text{conv}(\Delta(0) \setminus \chi([-x])) \subset \partial \Delta$.

Let $S^{k-1} = \{ x \in \mathbb{R}^k : ||x|| = 1 \}$ be the unit sphere in $\mathbb{R}^k$ with respect to the Euclidean norm $|| \cdot ||$ and $r : \mathbb{R}^k \setminus \{0\} \to S^{k-1}, \quad r : x \mapsto x/||x||$, the radial retraction. Observe that its restriction $r|\partial \Delta$ to the boundary of the geometric simplex $\Delta$ is a homeomorphism.

By Claim 5 $f(\partial K) \subset \partial \Delta \subset \mathbb{R}^k \setminus \{0\}$, so we can consider the map $g_0 : \partial K \to S^{k-1}$ defined by $g_0(x) \mapsto r \circ f(x) = f(x)/||f(x)||$. By Claim 5 the map $g_0|\partial K_0$ is homotopic to the map $g_1 : \partial K_0 \to S^{k-1}, \quad g_1(x) \mapsto r \circ f_1(x) = r \circ \varphi(x)$.

It follows from Claim 7 that $g_1(x) \neq g_1(-x)$ for all $x \in \partial K_0$. This implies that the formula $h_t(x) = \frac{g_1(x) - tg_1(-x)}{||g_1(x) - tg_1(-x)||}, \quad x \in \partial K_0, \quad t \in [0, 1]$. 

\[\square\]
determines a well-defined homotopy \( h_t : \partial K_0 \to S^{k-1} \) connecting the map \( g_1 \) with the map
\[
h_1(x) = \frac{g_1(x) - g_1(-x)}{\|g_1(x) - g_1(-x)\|},
\]
which is antipodal in the sense that \( h_1(-x) = -h_1(x) \). By [12] Chap.4,§7.10, each antipodal map between spheres of the same dimension is not homotopically trivial. Consequently, the antipodal map \( h_1 : \partial K_0 \to S^{k-1} \) is not homotopically trivial. On the other hand, \( h_1 \) is homotopic to the map \( h_0 = g_1 \), which is homotopic to \( g_0 | \partial K_0 \) and the latter map is homotopically trivial since the boundary \( \partial K_0 \) of the cube \( K_0 \) is contractible in the boundary \( \partial K \) of \( K \). This contradiction completes the proof of Theorem [3].

4. T-shaped sets in \( \mathbb{R}^n \)

Theorem [5] proved in the preceding section, yields an upper bound for the numbers \( c_k(\mathbb{Z}^k) \). A lower bound for the numbers \( c_k^B(\mathbb{R}^k) \) will be obtained by the technique of T-shaped sets created in [1].

Let \( \mathbb{R}^+ = [0, \infty) \) be the closed half-line. For every \( n \geq 0 \) consider the subset \( T_0 \subset \mathbb{R}^0 \) defined inductively:
\[
T_0 = \emptyset \subset \mathbb{R}^0 = \{0\}, \quad T_1 = \{0\} \subset \mathbb{R}^1, \quad \text{and} \quad T_n = (\mathbb{R}^{n-1} \times \{0\}) \cup (T_{n-1} \times \mathbb{R}^+) \subset \mathbb{R}^n
\]
for \( n > 1 \).

**Definition 4.** A subset \( C \subset \mathbb{R}^n \) is called **T-shaped** if \( f(C) \subset \mathbb{R} \times T_{n-1} \) for some affine transformation \( f : \mathbb{R}^n \to \mathbb{R} \times \mathbb{R}^{n-1} \). The smallest cardinality of a subset \( A \subset \mathbb{R}^n \), which is not T-shaped is denoted by \( t(\mathbb{R}^n) \).

Let us describe the geometric structure of T-shaped sets.

We say that for \( k \leq n \), hyperplanes \( H_1, \ldots, H_k \) in \( \mathbb{R}^n \) are in *general position* if they are pairwise distinct and their normal vectors are linearly independent. This happens if and only if there is an affine transformation \( f : \mathbb{R}^n \to \mathbb{R}^n \) that maps the \( i \)-th hyperplane onto the hyperplane \( \mathbb{R}^{i-1} \times \{0\} \times \mathbb{R}^{n-i} \) for all \( i \in \{1, \ldots, k\} \).

We shall say that a hyperplane \( H \subset \mathbb{R}^n \) does not separate a subset \( S \subset \mathbb{R}^{n+1} \) if \( S \) lies in one of two closed half-spaces bounded by the hyperplane \( H \). Such a hyperplane \( H \) will be called *non-separating* for \( S \). A hyperplane \( H \) is called a **support hyperplane** for \( S \) if \( H \cap S \neq \emptyset \) and \( H \) does not separate \( S \).

**Proposition 2.** Let \( n \in \mathbb{N} \). A subset \( S \subset \mathbb{R}^{n+1} \) is T-shaped if and only if
\[
S \subset H_1 \cup \cdots \cup H_n
\]
for some hyperplanes \( H_1, \ldots, H_n \) in general position such that each hyperplane \( H_i, 1 \leq i \leq n \), does not separate the set \( S \setminus (H_1 \cup \cdots \cup H_{i-1}) \).

**Proof.** This proposition can be easily derived from the equality
\[
\mathbb{R} \times T_n = \bigcup_{i=0}^{n-1} \mathbb{R}^{n-i} \times \{0\} \times \mathbb{R}_+^i
\]
that can be easily proved by induction on \( n \). \( \square \)

By Lemma 7 of [1], T-shaped subsets of Euclidean spaces \( \mathbb{R}^k \) are \( k \)-centerpole for Borel colorings. Consequently, \( t(\mathbb{R}^n) \leq c_n^B(\mathbb{R}^n) \). This gives us a lower bound for the numbers \( c_k^B(\mathbb{R}^n) \) and \( c_k(\mathbb{R}^n) \):

**Proposition 3.** \( t(\mathbb{R}^k) \leq c_k^B(\mathbb{R}^k) \leq c_k^B(\mathbb{R}^{n}) \leq c_k(\mathbb{R}^{n}) \) for any finite \( k \leq n \).

In the following theorem we collect all the available information on the numbers \( t(\mathbb{R}^n) \).

**Theorem 6.** (1) \( t(\mathbb{R}^1) = 1 \),
(2) \( t(\mathbb{R}^2) = 3 \),
(3) \( t(\mathbb{R}^3) = 6 \),
(4) \( t(\mathbb{R}^4) = 12 \),
(5) \( t(\mathbb{R}^n) \leq n^2 - n + 1 \) for every \( n \geq 1 \);
(6) \( t(\mathbb{R}^n) \geq t(\mathbb{R}^{n-1}) + n + 1 \) for any \( n \geq 4 \),
(7) \( t(\mathbb{R}^n) \geq \frac{1}{2}(n^3 + 3n - 4) \) for any \( n \geq 4 \).

**Proof.** 1. Since \( T_0 = \emptyset \), a subset of \( \mathbb{R}^1 \) is T-shaped if and only if it is empty. Consequently, \( t(\mathbb{R}^1) = 1 \).

2. Since \( T_1 = \{0\} \subset \mathbb{R}^1 \), a subset \( C \subset \mathbb{R}^2 \) is T-shaped if and only if \( C \) lies in an affine line. Consequently, \( t(\mathbb{R}^2) = 3 \).

3. By Theorem [6] the 6-element \( \binom{2}{0} \)-sandwich \( \Xi_0^2 \) is 3-centerpole in \( \mathbb{R}^3 \). Consequently, \( c_3(\mathbb{R}^3) \leq 6 \). By Proposition [3] \( t(\mathbb{R}^3) \leq c_3(\mathbb{R}^3) \leq 6 \). To see that \( t(\mathbb{R}^3) \geq 6 \), we need to check that a subset \( C \subset \mathbb{R}^3 \) of cardinality \( |C| \leq 5 \) is
T-shaped, which means that after a suitable affine transformation of $\mathbb{R}^3$, $C$ can be embedded into $\mathbb{R} \times T_2$. By the definition, $T_2 = \mathbb{R} \times \{0\} \cup \{0\} \times \mathbb{R}$.

Consider the convex hull $\text{conv}(C)$ of $C$ in $\mathbb{R}^3$. If $C$ lies in an affine plane $H$, then applying to $\mathbb{R}^3$ a suitable affine transformation, we can assume that $C \subset H = \mathbb{R} \times \mathbb{R} \times \{0\} \subset \mathbb{R} \times T_2$. If $C$ does not lie in a plane, then the convex polyhedron $\text{conv}(C)$ has a supporting plane $H_1$ such that $|H_1 \cap C| \geq 3$. So, $C \setminus H_1$ lies in one of the closed half-spaces with respect to the plane $H_1$. Denote this subspace by $H_1^+$. The set $C \setminus H_1$ has cardinality $|C \setminus H_1| \leq 2$ and hence it lies in an affine plane $H_2 \subset \mathbb{R}^3$ that meets $H_1$. Find an affine transformation $f : \mathbb{R}^3 \to \mathbb{R}^3$ such that $f(H_1) = \mathbb{R} \times \mathbb{R} \times \{0\}$, $f(H_1^+) = \mathbb{R} \times \mathbb{R} \times \mathbb{R}$ and $f(H_2) = \{0\} \times \{0\} \times \{0\}$. Then

$$f(C) \subset \mathbb{R} \times \mathbb{R} \times \{0\} \cup \mathbb{R} \times \{0\} \times \mathbb{R} = \mathbb{R} \times T_2$$

and hence $C$ is T-shaped.

4. By Theorem 5 the $(\mathbb{Z}^4)$-sandwich $\overline{X}$ is 4-centerpole in $\mathbb{Z}^4$. Consequently,

$$t(\mathbb{R}^4) \leq \min \{2t(\mathbb{R}^{n-1}), t(\mathbb{R}^{n-1}) + n + 1\}$$

for every $n \geq 2$. Take any subset $C \subset \mathbb{R}^n$ of cardinality $|C| < \min \{2t(\mathbb{R}^{n-1}), t(\mathbb{R}^{n-1}) + n + 1\}$. We need to show that $C$ is T-shaped.

Consider the convex hull $\text{conv}(C)$ of $C$ in $\mathbb{R}^n$. If $\text{conv}(C)$ lies in some hyperplane, then $C$ is T-shaped by the definition. So, we assume that $\text{conv}(C)$ does not lie in a hyperplane and then $\text{conv}(C)$ is a compact convex body in $\mathbb{R}^n$. Let $H$ be a supporting hyperplane of $\text{conv}(C)$ having maximal possible cardinality of the intersection $C \cap H$. It is clear that $|C \cap H| \geq n$.

Now two cases are possible:

a) The set $C \setminus H$ lies in a hyperplane $H_1$, parallel to $H$. Then $H_1$ is a supporting hyperplane of $\text{conv}(C)$ and then $|C \cap H_1| \leq |C \cap H|$ by the choice of $H$. Now we see that $|C \cap H_1| \leq \frac{1}{2}|C| < t(\mathbb{R}^{n-1})$.

Applying to $\mathbb{R}^n = \mathbb{R}^{n-1} \times \mathbb{R}$ a suitable affine transformation, we can assume that $H = \mathbb{R}^{n-1} \times \{0\}$ and $C \setminus H \subset \mathbb{R}^{n-1} \times \mathbb{R}_+$. Let $\text{pr} : \mathbb{R}^n \to \mathbb{R}^{n-1}$ be the coordinate projection. Since $|\text{pr}(C \setminus H)| < t(\mathbb{R}^{n-1})$, the set $C' = \text{pr}(C \setminus H)$ is T-shaped. This means that there is an affine transformation $f : \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$ such that $f(C') \subset \mathbb{R} \times T_{n-2}$. This affine transformation $f$ induces the affine transformation

$$\Phi : \mathbb{R}^{n-1} \times \mathbb{R} \to \mathbb{R}^{n-1} \times \mathbb{R}, \quad \Phi(x, y) = (f(x), y),$$

such that

$$\Phi(C) = \Phi(C \cap H) \cup \Phi(C \setminus H) \subset (\mathbb{R} \times \mathbb{R}^{n-2} \times \{0\}) \cup (\mathbb{R} \times T_{n-2} \times \mathbb{R}_+) = \mathbb{R} \times T_{n-1}.$$ 

The affine transformation $\Phi$ witnesses that the set $C$ is T-shaped.

b) The set $C \setminus H$ does not lie in a hyperplane parallel to $H$. Then $C \setminus H$ contains two distinct points $x, y$ such that the vector $\bar{x}y$ is not parallel to $H$. Applying to $\mathbb{R}^n = \mathbb{R}^{n-1} \times \mathbb{R}$ a suitable affine transformation, we can assume that $H = \mathbb{R}^{n-1} \times \{0\}$, $C \setminus H \subset \mathbb{R}^{n-1} \times \mathbb{R}_+$, and under the projection $\text{pr} : \mathbb{R}^{n-1} \times \mathbb{R} \to \mathbb{R}^{n-1}$ the images of the points $x$ and $y$ coincide. Then the projection $C' = \text{pr}(C \setminus H)$ has cardinality $|C'| \leq |C \setminus H| - 1 < |C| - |C \cap H| - 1 < t(\mathbb{R}^{n-1}) + n + 1 - n - 1 = t(\mathbb{R}^{n-1})$. Continuing as in the preceding case, we can find an affine transformation $\Phi$, witnessing that $C$ is a T-shaped set in $\mathbb{R}^n$.

This proves the inequality (1). By analogy we can prove that $t(\mathbb{R}^n) \geq t(\mathbb{R}^{n-1}) + n$. Since $t(\mathbb{R}^1) = 1$, by induction we can show that $t(\mathbb{R}^n) \geq \frac{1}{n}(n + 1)$. In particular, $t(\mathbb{R}^{n-1}) \geq \frac{1}{n}(n - 1) \geq n + 1$ for all $n \geq 4$. In this case

$$t(\mathbb{R}^n) \geq \min \{2t(\mathbb{R}^{n-1}), t(\mathbb{R}^{n-1}) + n + 1\} = t(\mathbb{R}^{n-1}) + n + 1.$$ 

7. The lower bound $t(\mathbb{R}^n) \geq \frac{1}{n}(n^2 + 3n - 4)$, $n \geq 4$, will be proved by induction. For $n = 4$ it is true according to the statement (4). Assuming that it is true for some $n > 4$ and applying the lower bound (6), we get

$$t(\mathbb{R}^{n+1}) \geq t(\mathbb{R}^n) + (n + 1) + 1 \geq \frac{1}{2}(n^2 + 3n - 4) + n + 2 = \frac{1}{2}((n + 1)^2 + 3(n + 1) - 4).$$

To finish the proof of Theorem 6 it remains to prove the promised:
Lemma 2. Each subset $C \subset \mathbb{R}^4$ of cardinality $|C| < 12$ is $T$-shaped.

Proof. Assume that some subset $C \subset \mathbb{R}^4$ of cardinality $|C| < 12$ is not $T$-shaped. Without loss of generality, $|C| = 11$.

We recall that a hyperplane $H \subset \mathbb{R}^4$ is called a support hyperplane for $C$ if $C \cap H \neq \emptyset$ and $H$ does not separate $C$ (which means that $C$ lies in a closed half-space $H^+$ bounded by the hyperplane).

Claim 9. Each support hyperplane $H \subset \mathbb{R}^4$ for $C$ has at most 5 common points with $C$.

Proof. Assume that $H$ is a support hyperplane for $C$ with $|H \cap C| > 5$. After a suitable affine transformation of $\mathbb{R}^4$, we can assume that $H = \mathbb{R}^3 \times \{0\}$ and $C \subset \mathbb{R}^3 \times \mathbb{R}_+$. Let $\text{pr} : \mathbb{R}^4 \to \mathbb{R}^3$ be the coordinate projection. Since $|C \setminus H| = |C| - |C \cap H| < 11 - 5 = 6$ and $t(\mathbb{R}^3) = 6$ (by Theorem 5), $\text{pr}(C \setminus H)$ is $T$-shaped in $H$ and so $C$ is $T$-shaped $\mathbb{R}^4$.

Claim 10. For any two parallel hyperplanes $H_1$ and $H_2$ in $\mathbb{R}^4$ the set $C \setminus (H_1 \cup H_2)$ is non-empty.

Proof. Otherwise one of these hyperplanes contains more than 6 points, which contradicts Claim 9.

Claim 11. Each support hyperplane $H$ for the set $C$ has less than 5 common points with $C$.

Proof. Previous claim guarantees the existence of two distinct points $a, b \in C$ that lie in an affine line $L$ that meets $H$. After a suitable affine transformation of $\mathbb{R}^4$, we can assume that $H = \mathbb{R}^3 \times \{0\}$, $C \subset \mathbb{R}^3 \times \mathbb{R}_+$, and $L = \{0\}^3 \times \mathbb{R}$. Let $\text{pr} : \mathbb{R}^4 \to \mathbb{R}^3$ be the coordinate projection. Assuming that $|H \cap C| \geq 5$ and taking into account that $\text{pr}(a) = \text{pr}(b)$, we conclude that

$$|\text{pr}(C \setminus H)| \leq |C \setminus H| - 1 = |C| - |C \cap H| - 1 \leq 5 < 6 = t(\mathbb{R}^3).$$

It follows that $\text{pr}(C \setminus H)$ is $T$-shaped in $\mathbb{R}^3$ and then $C$ is $T$-shaped in $\mathbb{R}^4$.

The characterization of $T$-shaped sets given in Proposition 2 implies:

Claim 12. If $H_1$ is a support hyperplane for $C$, $H_2$ is a support hyperplane for $C \setminus H_1$ and $H_1, H_2$ are not parallel, then $|C \setminus (H_1 \cup H_2)| \geq 3$ and if $|C \setminus (H_1 \cup H_2)| = 3$, then the set $C \setminus (H_1 \cup H_2)$ does not lie in a line but lies in a plane, parallel to $H_1 \cap H_2$.

Claim 13. If $H_1$ and $P_2$ are parallel support hyperplanes for $C$ and $|H_1 \cap C| = 4$, then $|P_2 \cap C| = 1$.

Proof. By Claim 11 $C \setminus H_1$ does not lie in a hyperplane. Now consider 4 cases.

1) $|P_2 \cap C| > 4$. In this case $C$ is $T$-shaped by Claim 11.

2) $|P_2 \cap C| = 4$. We claim that the set $P_2 \cap C$ does not lie in a plane $P$. Otherwise $P$ can be enlarged to a support hyperplane that contains $\geq 5$ points of $C$, which is forbidden by Claim 11. Therefore, the convex hull of $P_2 \cap C$ is a convex body in $P_2$ and we can find a support hyperplane $H_2$ for $C \setminus H_1$ that meets $H_1$, has at least 4 common points with $C \setminus H_1$ and exactly three common points with the set $P_2$. In this case the unique point $c_2$ of the set $C \cap P_2 \setminus H_2$ lies in $C \setminus (H_1 \cup H_2)$. By Proposition 2, the set $C \setminus (H_1 \cup H_2)$ contains exactly 3 points that lie in a plane parallel to $H_1 \cap H_2$. Since this set contains the point $c_2 \in C \cap P_2$, we conclude that $C \setminus (H_1 \cup H_2) \subset P_2$ and hence $|C \cap P_2| = 6$, which is a contradiction.

3) $|P_2 \cap C| = 3$. Let $Pl$ be a plane which contains $P_2 \cap C$ and lies in the hyperplane $P_2$. We claim that the set $C \setminus (H_1 \cup Pl)$ lies in a plane $Pl_1$ that is parallel to $Pl$. Let $S$ be the set of all points $x \in C \setminus (H_1 \cup Pl)$ that belong to a support hyperplane $H_x$ to $C \setminus H_1$ that has at least 4 common points with $C \setminus H_1$ and contains the plane $Pl$. Claim 12 guarantees that the set $C \setminus (H_1 \cup H_x)$ contains exactly 3 elements and lies in a plane that is parallel to the intersection $H_1 \cap H_x$ (which is parallel to $Pl$). Since the set $C \setminus H_1$ does not lie in a hyperplane, the set $S$ contains more than one point, which implies that the set $C \setminus (H_1 \cup Pl) = \bigcup_{x \in S} C \setminus (H_1 \cup H_x)$ lies in a plane $Pl_1$ that is parallel to the plane $Pl$. Let $H_2$ be the hyperplane that contains the parallel planes $Pl$ and $Pl_1$. Since $H_2$ meets $H_1$, we see that $C \subset H_1 \cup H_2$ is $T$-shaped by Proposition 2 and this is a contradiction.

4) $|P_2 \cap C| = 2$. Since $C \setminus H_1$ does not lie in a hyperplane, there is a support hyperplane $H_2$ to $C \setminus H_1$ such that $|H_2 \cap (C \setminus H_1)| \geq 4$ and $|H_2 \cap P_2 \cap C| = 1$. It follows that the hyperplane $H_2$ does not coincide with $P_2$ and hence meets the hyperplane $H_1$. By Claim 12 the complement $C \setminus (H_1 \cup H_2)$ contains exactly 3 points that lie in a plane, parallel to $H_1 \cap H_2$. Since $C \setminus (H_1 \cup H_2)$ meets the hyperplane $P_2$ we conclude that $C \setminus (H_1 \cup H_2) \subset P_2$ and $|C \cap P_2| \geq 4$, which is a contradiction.

Claim 14. If $P_1$ and $P_2$ are parallel support hyperplanes for $C$ and $|P_1 \cap C| = 4$, then the set $C \setminus (P_1 \cup P_2)$ lies in a hyperplane $P_3$ that is parallel to $P_1$ and $P_2$. 

□
Proof. By Claim 13, \(|P_2 \cap C| = 1\) and hence \(|C \setminus (P_1 \cup P_2)| = 6\). Let \(x\) be the unique point of \(P_2 \cap C\). Take any support hyperplane \(H \ni x\) for the set \(C \setminus P_1\) such that \(|H \cap C| \geq 4\). Since \(H\) meets \(P_1\), Proposition 2 guarantees that the set \(C' = C \setminus (P_1 \cup H)\) contains exactly 3 points that lie in a plane parallel to the intersection \(P_1 \cap H\) and hence parallel to \(P_1\). The hyperplane \(H'\) containing the set \(C' \cup \{x\}\) is a support hyperplane for the set \(C \setminus P_1\). Applying Proposition 2, we conclude that the set \(C'' = C \setminus (P_1 \cup H') = C \cap H \setminus P_2\) contains exactly 3 points lying in a plane parallel to \(P_1 \cap H'\). Thus \(C \setminus (P_1 \cup P_2)\) lies in two planes parallel to \(P_1\) and hence it lies in a hyperplane \(P_3\). Proposition 2 implies that the hyperplane \(P_3\) is parallel to \(P_1\).

By an octahedron in a linear space \(L\) we understand a set of the form
\[c + \{e_i, -e_i : 1 \leq i \leq 3\}\]
where \(e_1, e_2, e_3\) are linearly independent vectors in \(L\) and \(c \in L\) is the center of the octahedron. Up to an affine equivalence an octahedron is a unique 6-element set \(X\) with 3-dimensional affine hull \(A\) such that for each support plane \(P \subset X\) with \(|P \cap X| \geq 3\) the set \(X \setminus P\) contains exactly 3 points and lies in a plane \(P'\), parallel to \(P\).

Claim 15. If \(P_1\) and \(P_2\) are parallel support hyperplanes for \(X\) and \(|P_1 \cap C| = 4\), then the set \(C \setminus (P_1 \cup P_2)\) is an octahedron that lies in a hyperplane \(P_3\), parallel to \(P_1\).

Proof. By the preceding Claim, the set \(K = C \setminus (P_1 \cup P_2)\) lies in a hyperplane \(P_3\), parallel to \(P_1\). Let us show that \(K\) does not lie in a plane. In the opposite case, we could find a hyperplane \(H_2\) that contains the set \(K\) and meets the hyperplane \(P_1\). Then for each hyperplane \(H_3\) that contains the unique point \(C \cap P_2\) and has one-dimensional intersection with \(P_1 \cap H_2\), we get \(C \subset P_1 \cup H_2 \cup H_3\) witnessing that \(C\) is \(T\)-shaped.

Thus the affine hull of \(K\) is 3-dimensional. To see that \(K\) is an octahedron, it suffices to check that for each support plane \(P \subset P_3\) of \(K\) with \(|P \cap K| \geq 3\) the set \(K \setminus P\) contains exactly 3 points and lies in a plane parallel to \(P\).

Let \(x\) be the unique point of the set \(C \cap P_2\) and \(H_2\) be the hyperplane containing the plane \(P\) and passing through \(x\). It follows that \(H_2\) is a support hyperplane for the set \(C \setminus P_1\). By Claim 12 the set \(C \setminus (P_1 \cup H_2) = K \setminus P\) contains exactly 3 elements and lies in a plane \(P'\) parallel to the intersection \(H_1 \cap H_2\).

Now let \(H_2'\) be the hyperplane that contains the support plane \(P'\) and passes through the point \(x\). Since \(P'\) is a support plane for \(K\) in the hyperplane \(P_3\), \(H_3\) is a support hyperplane for \(K \cup \{x\} = C \setminus P_1\) in \(\mathbb{R}^3\). Since \(H_3\) intersects \(P_1\), Claim 12 guarantees that the set \(C \setminus (P_1 \cup H_2') = K \setminus P'\) contains exactly 3 points and the plane \(P\) containing these 3 points is parallel to \(P_1 \cap H'\) which is parallel to the plane \(P'\).

After this preparatory work we are ready to finish the proof of Lemma 2. As \(C\) is not \(T\)-shaped, it does not lie in a hyperplane. So, we can find a support hyperplane \(P_1\) for \(C\) such that \(|P_1 \cap C| \geq 4\). Let \(P_2\) be a support hyperplane for \(C\), which is parallel to \(P_1\). By Claim 13, \(|P_2 \cap C| = 4\) and \(|P_2 \cap C| = 1\). Let \(p_2\) be the unique point of the set \(P_2 \cap C\). By Claim 14, \(C \setminus (P_1 \cup P_2)\) is an octahedron that lies in a hyperplane \(P_3\), parallel to the hyperplanes \(P_1\) and \(P_2\). Let \(c\) be the center of this octahedron and \(2c - p_2\) be the point, symmetric to \(p_2\) with respect to \(c\).

Fix any 3-element subset \(F\) of \(P_1 \cap C\) such that \(2c - p_2 \in F\) if \(2c - p_2 \in C \setminus P_1\). Next, find a hyperplane \(H_1\) for \(C\) that contains \(F\) and meets \(C \setminus H_1\) at some point \(a\). If \(a = p_2\), then the set \(C \subset H_1 \cup P_3 \cup (C \setminus P_1)\) is \(T\)-shaped by Proposition 2.

Consequently, \(a\) is a point of the octahedron \(C \cap P_3\) with center \(c\). Let \(H_2\) be a support hyperplane for \(C\) that is parallel to the hyperplane \(H_1\). By Claims 13 and 15, \(|C \cap H_1| = 4\), \(|C \cap H_2| = 1\) and \(C \setminus (H_1 \cup H_2)\) is an octahedron that lies in a hyperplane \(H_3\), parallel to \(H_1\) and \(H_2\). If \(H_3\) does not meet the octahedron \(C \cap P_3\), then \(C \cap P_3\) and \(C \cap H_3\) have 5 common points and hence lie in the same hyperplane \(P_3\), which is not possible. So, the support hyperplane \(H_3\) meets the octahedron \(C \cap P_3 \setminus \text{at a single point and this point is } 2c - a\). In this case the octahedra \(C \cap P_3\) and \(C \cap H_3\) have 4 common points which belong to the set \(C \cap P_3 \setminus \{a, 2c - a\}\) and lie in the 2-dimensional plane \(P_3 \cap H_3\). This implies that the octahedra \(C \cap P_3\) and \(C \cap H_3\) have the common center \(c\). Since \(p_2 \in C \cap H_3\), the point \(2c - p_2\) belongs to the octahedron \(C \cap H_3 \subset C\). It follows from \(p_2 \in P_2\) and \(c \in P_3\) that \(2c - p_2 \in C \setminus (P_2 \cup P_3) = C \setminus P_1\) and hence \(2c - p_2 \in F \subset H_1\) by the choice of the set \(F\). On the other hand, \(2c - p_2\) belongs to the hyperplane \(H_3\), which is disjoint with \(H_1\) and this is a desired contradiction.

5. ENLARGING NON-CENTERPOLE SETS

In this section we prove several lemmas on enlarging non-centerpole subsets. Namely, we show that under certain conditions, a non-\(k\)-centerpole subset \(C\) of a topological group \(X\) (possibly enlarged by one or two points) remains not \(k\)-centerpole in the direct sum \(X \oplus \mathbb{R}\). The group \(X \oplus \mathbb{R}\) can be identified with the direct product \(X \times \mathbb{R}\), so that \(X\) is identified with the subgroup \(X \times \{0\} \subset X \times \mathbb{R}\) while the real line \(\mathbb{R}\) is identified with the subgroup \(\{e\} \times \mathbb{R} \subset X \times \mathbb{R}\) where \(e\) is the neutral element of the group \(X\).
Lemma 3. If for \( k \geq 2 \) a subset \( C \subset X \) of a topological group \( X \) is not \( k \)-centerpole (for Borel colorings), then set \( C \) is not \( k \)-centerpole in \( X \oplus \mathbb{R} \).

Proof. Since the set \( C \subset X \) is not \( k \)-centerpole (for Borel colorings), there exists a (Borel) coloring \( \chi : X \to k \) such that \( X \) contains no monochromatic unbounded subset, which is symmetric with respect to a point \( c \in C \). Extend \( \chi \) to a (Borel) coloring \( \tilde{\chi} : X \times \mathbb{R} \to k \) letting

\[
\tilde{\chi}(x, t) = \begin{cases} 
\chi(x) & \text{if } t = 0, \\
0 & \text{if } t < 0, \\
1 & \text{if } t > 0.
\end{cases}
\]

This coloring witnesses that \( C \) is not \( k \)-centerpole in \( X \oplus \mathbb{R} \) (for Borel colorings).

Lemma 4. If for \( k \geq 3 \) a subset \( C \subset X \) of a topological group \( X \) with \( c^B_2(X) \geq 2 \) is not \( k \)-centerpole (for Borel colorings), then for each \( x \in X \times (0, \infty) \) the set \( C \cup \{x\} \) is not \( k \)-centerpole for (Borel) colorings of the topological group \( X \oplus \mathbb{R} \).

Proof. Without loss of generality we may assume that \( x = (e, 1) \) where \( e \) is the neutral element of topological group \( X \). Fix a (Borel) coloring \( \chi : X \to k \) witnessing that the subset \( C \subset X \) is not \( k \)-centerpole (for Borel colorings).

This coloring induces a (Borel) 2-coloring \( \chi_2 : X \to 2 \) defined by

\[
\chi_2(x) = \min \left( \{0, 1\} \setminus \chi(x^{-1}) \right) \quad \text{for } x \in X.
\]

Since \( c^B_2(X) \geq 2 \), there exists a Borel coloring \( \chi_1 : X \to 2 \) witnessing that the singleton \( \{e\} \) is not 2-centerpole for Borel colorings of \( X \).

It is easy to see that the (Borel) coloring \( \tilde{\chi} : X \times \mathbb{R} \to k \) defined by

\[
\tilde{\chi}(x, t) = \begin{cases} 
\chi(x), & \text{if } t = 0, \\
\chi_1(x), & \text{if } t = 1, \\
\chi_2(x), & \text{if } t = 2, \\
0, & \text{if } 1 < t \neq 2, \\
1, & \text{if } 0 < t < 1, \\
2, & \text{if } t = 0,
\end{cases}
\]

witnesses that the set \( C \cup \{(e, 1)\} \) fails to be \( k \)-centerpole for (Borel) colorings of the topological group \( X \oplus \mathbb{R} \).

Lemma 5. \( c^B_3(\mathbb{R}^m) \geq 6 \) for all \( m \geq 3 \).

Proof. By Theorem \( \Box \) 3 and Proposition \( \Box \) \( c^B_3(\mathbb{R}^3) \geq t(\mathbb{R}^3) = 6 \).

Next, we check that \( c^B_3(\mathbb{R}^4) \geq 6 \). Assuming that \( c^B_3(\mathbb{R}^4) < 6 \) find a subset \( C \subset \mathbb{R}^4 \) of cardinality \( |C| \leq 5 \), which is 3-centerpole for Borel colorings of \( \mathbb{R}^4 \).

Since \( |C| \leq 5 \), there is a 3-dimensional hyperplane \( H_3 \subset \mathbb{R}^4 \) such that \( |C \setminus H_3| \leq 1 \). Since \( |C \cap H_3| \leq |C| < 6 = c^B_3(\mathbb{R}^3) \), the set \( C \cap H_3 \) is not 3-centerpole for Borel colorings of \( H_3 \). By (the proof of) Proposition 4.1 of \( \Box \), \( c^B_3(\mathbb{R}^3) = 3 \geq 2 \). By Lemma \( \Box \) the set \( C \) is not 3-centerpole for Borel colorings of \( H_3 \oplus \mathbb{R} \) (which can be identified with \( \mathbb{R}^4 \)).

Now assume that the inequality \( c^B_3(\mathbb{R}^{m-1}) \geq 6 \) has been proved for some \( m \geq 4 \). Assuming that \( c^B_3(\mathbb{R}^m) \leq 5 \) find a subset \( C \subset \mathbb{R}^m \) of cardinality \( |C| \leq 5 \), which is 3-centerpole for Borel colorings of \( \mathbb{R}^m \). This set lies in a \( m-1 \) dimensional hyperplane and according to Lemma \( \Box \) is 3-centerpole for Borel colorings of \( \mathbb{R}^{m-1} \). Then \( c^B_3(\mathbb{R}^{m-1}) \leq |C| \leq 5 \), which contradicts the inductive assumption.

Lemma 6. If for \( k \geq 4 \) a subset \( C \subset X \) of a topological group \( X \) with \( c^B_2(X) \geq 3 \) is not \( k \)-centerpole (for Borel colorings), then for any 2-element set \( A \subset X \times (0, \infty) \) the set \( C \cup A \) is not \( k \)-centerpole for (Borel) colorings of the topological group \( X \oplus \mathbb{R} \).

Proof. Let \((a, v)\) and \((b, w)\) be the points of the 2-element set \( A \subset X \times (0, \infty) \). We can assume that \( v \leq w \). Let \( \chi_0 : X \to k \) be a (Borel) coloring witnessing that the set \( C \) is not \( k \)-centerpole for (Borel) colorings of the group \( X \).

Consider the Borel 4-coloring \( \psi : \mathbb{R} \to 4 \) of the real line defined by

\[
\psi(t) = \begin{cases} 
3 & \text{if } t \leq 0, \\
0 & \text{if } 0 < t \leq v, \\
1 & \text{if } v < t \leq w, \\
2 & \text{if } w < t.
\end{cases}
\]

and observe that for each \( c \in \{0, v, w\} \) and \( t \in \mathbb{R} \setminus \{c\} \) we get \( \psi(t) \neq \psi(2c - t) \).
We consider 2 cases.

1) $v = w$. In this case we can assume that $v = w = 1$. Since $c^B_2(X) \geq 3$, there exists a Borel coloring $\chi_1 : X \rightarrow 2$ witnessing that the 2-element set $\{a, b\} \subset X$ is not 2-centerpole for Borel colorings of $X$. The (Borel) coloring $\chi_0$ induces the (Borel) coloring $\chi_2 : X \rightarrow 3$ defined by the formula

$$\chi_2(x) = \min \{\{0, 1, 2\} \setminus \{\chi_0(ax^{-1}a), \chi_0(bx^{-1}b)\}\}.$$

Now we see that the (Borel) coloring $\tilde{\chi} : X \times \mathbb{R} \rightarrow k$ defined by

$$\tilde{\chi}(x, t) = \begin{cases} \chi_2(x), & \text{if } t \in \{0, 1, 2\}, \\ \psi(t), & \text{otherwise} \end{cases}$$

witnesses that the set $C \cup A$ is not $k$-centerpole for (Borel) colorings of the topological group $X \oplus \mathbb{R}$.

2) The second case occurs when $v \neq w$. Without loss of generality, $v < w$ and $w - v = 1$. This case has three subcases.

2a) $v = 1$ and $w = 2$. In this case we can assume that $b = e$ is the neutral element of the group $X$.

Since $c^B_2(X) \geq 3$, there is a Borel 2-coloring $\chi_1 : X \rightarrow 2$ witnessing that the singleton $\{a\}$ is not 2-centerpole in $X$. By the same reason, there is a Borel 2-coloring $\phi : X \rightarrow 2$ witnessing that the singleton $\{b\} = \{e\}$ is not 2-centerpole for Borel colorings of $X$. Using the colorings $\phi$ and $\chi_0$ one can define a (Borel) 3-coloring $\chi_2 : X \rightarrow 3$ such that $\chi_2(x) \neq \chi_0(ax^{-1}a)$ for all $x \in X$ and $\chi_2(x) \neq \chi_2(x^{-1})$ if and only if $\phi(x) \neq \phi(x^{-1})$.

Such a coloring $\chi_2 : X \rightarrow 3$ can be defined by the formula

$$\chi_2(x) = \begin{cases} \min \{3 \setminus \{\chi_0(axa), \chi_0(ax^{-1}a)\}\}, & \text{if } \phi(x) = \phi(x^{-1}); \\ \phi(x), & \text{if } \chi_0(ax^{-1}a) \neq \phi(x) \neq \phi(x^{-1}) \neq \chi_0(axa); \\ \min \{3 \setminus \{\phi(x^{-1}), \chi_0(ax^{-1}a)\}\}, & \text{if } \chi_0(ax^{-1}a) = \phi(x) \neq \phi(x^{-1}) \neq \chi_0(axa); \\ \phi(x), & \text{if } \chi_0(ax^{-1}a) = \phi(x) \neq \phi(x^{-1}) = \chi_0(axa); \\ \phi(x^{-1}), & \text{if } \chi_0(ax^{-1}a) = \phi(x) = \phi(x^{-1}) = \chi_0(axa). \end{cases}$$

Let $\chi_3 : X \rightarrow 2$ be the Borel 2-coloring defined by $\chi_3(x) = 1 - \chi_1(x^{-1})$ for $x \in X$. It is clear that $\chi_3(x^{-1}) \neq \chi_1(x)$ for all $x \in X$. Finally, consider the Borel 2-coloring $\chi_4 : X \rightarrow 2$ defined by

$$\chi_4(x) = \min \{\{0, 1\} \setminus \{\chi_0(x^{-1})\}\} \text{ for } x \in X.$$

The (Borel) colorings $\psi, \chi_0, \chi_1, \chi_2, \chi_3, \chi_4$ compose a (Borel) $k$-coloring $\tilde{\chi} : X \times \mathbb{R} \rightarrow k$,

$$\tilde{\chi}(x, t) = \begin{cases} \chi_1(x), & \text{if } t \in \{0, 1, 2, 3, 4\}, \\ \psi(t), & \text{otherwise}, \end{cases}$$

witnessing that the set $C \cup A$ is not $k$-centerpole for (Borel) colorings of $X \oplus \mathbb{R}$.

2b) $v = 2$ and $w = 3$. Since $c^B_2(X) \geq 3 > 1$, there is a Borel 2-coloring $\chi_2 : X \rightarrow 2$ witnessing that the singleton $\{a\}$ is not 2-centerpole for Borel colorings of $X$. By the same reason, there is a Borel 2-coloring $\chi_3 : X \rightarrow 2$ witnessing that the singleton $\{b\}$ is not 2-centerpole for Borel colorings of $X$.

Next consider the (Borel) colorings $\chi_1 : X \rightarrow 2$, $\chi_4 : X \rightarrow 3$, and $\chi_6 : X \rightarrow 2$ defined by the formulas

$$\chi_1(x) = 1 - \chi_3(ax^{-1}a),$$

$$\chi_4(x) = \min \{3 \setminus \{\chi_0(ax^{-1}a), \chi_2(bx^{-1}b)\}\},$$

$$\chi_6(x) = \min \{2 \setminus \{\chi_0(bx^{-1}b)\}\}.$$

The (Borel) colorings $\psi$ and $\chi_t$, $t \in \{0, 1, 2, 3, 4, 6\}$, compose the (Borel) coloring $\tilde{\chi} : X \times \mathbb{R} \rightarrow k$ defined by

$$\tilde{\chi}(x, t) = \begin{cases} \chi_t(x), & \text{if } t \in \{0, 1, 2, 3, 4, 6\}, \\ \psi(t), & \text{otherwise}. \end{cases}$$

This coloring $\tilde{\chi}$ witnesses that the set $C \cup A$ is not $k$-centerpole for (Borel) colorings of $X \oplus \mathbb{R}$.

2c) $v \notin \{1, 2\}$. Since $c^B_2(X) > 1$ there is a Borel 2-coloring $\chi_v : X \rightarrow 2$ witnessing that the singleton $\{a\}$ is not 2-centerpole for Borel colorings of $X$. By the same reason, there is a Borel 2-coloring $\chi_w : X \rightarrow \{1, 2\}$ witnessing that the singleton $\{b\}$ is not 2-centerpole for Borel colorings of $X$.

Next, define the (Borel) colorings $\chi_{2v}, \chi_{2w} : X \rightarrow 3$ by the formula

$$\chi_{2v}(x) = \min \{3 \setminus \{\chi_0(ax^{-1}a), \psi(2)\}\} \text{ and } \chi_{2w}(x) = \min \{2 \setminus \{\chi_0(bx^{-1}b)\}\}.$$

Here let us note that the points $2v$ and $2$ are symmetric with respect to $w$ in the group $\mathbb{R}$.
Finally, define a (Borel) $k$-coloring $\tilde{\chi} : X \oplus \mathbb{R} \to k$ letting

$$\tilde{\chi}(x,t) = \begin{cases} 
\chi_t(x) & \text{if } t \in \{0, v, 2v, 2w\} \\
\psi(t) & \text{otherwise.}
\end{cases}$$

This coloring witnesses that the set $C \cup A$ is not $k$-centerpole for (Borel) colorings of the topological group $X \oplus \mathbb{R}$. □

Lemma 7. $c_B^k(\mathbb{R}^m) \geq 8$ for all $m \geq 4$.

Proof. This lemma will be proved by induction on $m \geq 4$. For $m = 4$ the inequality $c_B^4(\mathbb{R}^4) \geq t(\mathbb{R}^4) = 12 \geq 8$ follows from Lemma 2. Assume that for some $m \geq 4$ we know that $c_B^k(\mathbb{R}^m) \geq 8$. The inequality $c_B^k(\mathbb{R}^{m+1}) \geq 8$ will follow as soon as we check that each 7-element subset $C \subset \mathbb{R}^{m+1}$ is not 4-centerpole for Borel colorings of $\mathbb{R}^{m+1}$.

Given a 7-element subset $C \subset \mathbb{R}^{m+1}$, find a support $m$-dimensional hyperplane $H \subset \mathbb{R}^{m+1}$ that has at least $\min\{m+1, |C|\} \geq 5$ common points with the set $C$. After a suitable shift, we can assume that the intersection $C \cap H$ contains the origin of $\mathbb{R}^{m+1}$. In this case $H$ is a linear subspace of $\mathbb{R}^{m+1}$ and $\mathbb{R}^{m+1}$ can be written as the direct sum $\mathbb{R}^m \oplus \mathbb{R}$.

Since $|H \cap C| \leq |C| \leq 7$, the inductive assumption guarantees that $H \cap C$ is not 4-centerpole for Borel colorings of $H$. By Lemma 6, $c_B^k(\mathbb{R}^m) \geq 3$. Since $|C \setminus H| \leq 2$, we can apply Lemma 4 and conclude that $C$ is not 4-centerpole for Borel colorings of the topological group $H \oplus \mathbb{R} = \mathbb{R}^{m+1}$. □

6. CENTERPOLE SETS IN SUBGROUPS AND GROUPS

It is clear that each $k$-centerpole subset $C \subset H$ in a subgroup $H$ of a topological group $G$ is $k$-centerpole in $G$. In some cases the converse statement also is true.

Lemma 8. If a subset $C$ of an abelian topological group $G$ is $k$-centerpole in $G$ for some $k \geq 2$, then it is $k$-centerpole in the subgroup $H = \langle C \rangle + G[2]$.

Proof. Observe that for each $x \in G \setminus H$ the cosets $c + 2(C)$ and $-x + 2(C)$ are disjoint. Assuming the opposite, we would conclude that $2x \in 2(C)$ and hence $x \in \langle C \rangle + G[2] = H$, which contradicts the choice of $x$.

Now we are able to prove that the set $C$ is $k$-centerpole in the group $H$. Given any $k$-coloring $\chi : H \to k$, extend $\chi$ to a $k$-coloring $\tilde{\chi} : G \to k$ such that for each $x \in G \setminus H$ the coset $x + 2(C)$ is monochromatic and its color is different from the color of the coset $-x + 2(C)$.

Since $C$ is $k$-centerpole in the group $G$, there is an unbounded monochromatic subset $S \subset G$ such that $S = 2c - S$ for some $c \in C$. We claim that $S \subset H$. Assuming the converse, we would find a point $x \in S \setminus H$ and conclude that the coset $x + 2(C)$ has the same color as the coset $2c - x + 2(C) = -x + 2(C)$, which contradicts the choice of the coloring $\tilde{\chi}$. □

The Borel version of this result is a bit more difficult.

Lemma 9. Let $k \geq 2$ and $H$ be a Borel subgroup of an abelian topological group $G$ such that $G[2] \subset H$. A subset $C \subset H$ is $k$-centerpole for Borel colorings of $H$ if $C$ is $k$-centerpole for Borel colorings of $G$, the subgroup $2H = \{2x : x \in H\}$ is closed in $G$, and the subspace $X = (G/2H) \setminus (H/2H)$ contains a Borel subset $B$ that has one-point intersection with each set $\{x, -x\}, x \in X$. Such a Borel set $B \subset X$ exists if the space $X$ is paracompact.

Proof. Given any Borel $k$-coloring $\chi : H \to k$, extend $\chi$ to a Borel $k$-coloring $\tilde{\chi} : G \to k$ defined by

$$\tilde{\chi}(x) = \begin{cases} 
\chi(x), & \text{if } x \in H, \\
0, & \text{if } x \in G \setminus H \text{ and } x + 2H \in B, \\
1, & \text{if } x \in G \setminus H \text{ and } x + 2H \notin B.
\end{cases}$$

Since $C$ is $k$-centerpole for Borel colorings of the group $G$, there is an unbounded monochromatic subset $S \subset G$, symmetric with respect to some point $c \in C$. We claim that $S \subset H$, witnessing that $C$ is $k$-centerpole for Borel colorings of $H$.

Assuming conversely that $S \not\subset H$, find a point $x \in S \setminus H$. It follows that $x$ and $2c - x$ have the same color. If this color is 0, then the cosets $x + 2H$ and $2c - x + 2H = -x + 2H = -(x + 2H)$ both belong to the set $B \subset G/2H$. By our hypothesis $B$ has one-point intersection with the set $\{x + 2H, -(x + 2H)\}$. Consequently, $x + 2H = -(x + 2H)$ and hence $2x \in 2H$ and $x \in H + G[2] = H$, which contradicts the choice of the point $x$. If the color of the cosets $x + 2H$ and $2c - x + 2H = -(x + 2H)$ is 1, then $(x + 2H), -(x + 2H) \notin B$ and then $x + 2H = -(x + 2H)$ because $B$ has one-point intersection with the set $\{x + 2H, -(x + 2H)\}$. This again leads to a contradiction.

Claim 16. If the space $X = (G/2H) \setminus (H/2H)$ is paracompact, then $X$ contains a Borel subset $B \subset X$ that has one-point intersection with each set $\{x, -x\}, x \in X$. 

Consider the action
\[ \alpha : C_2 \times X \to X, \quad \alpha : (\varepsilon, x) \mapsto \varepsilon \cdot x, \]
of the cyclic group \( C_2 = \{1, -1\} \) on the space \( X \) and let \( X/C_2 = \{ \{x, -x\} : x \in X \} \) be the orbit space of this action. It is easy to check that the orbit map \( q : X \to X/C_2 \) is closed and then the orbit space \( X/C_2 \) is paracompact as the image of a paracompact space under a closed map, see Michael Theorem 5.1.33 in [7].

Since \( H \supseteq 2H + G[2] \), for every \( x \in G \setminus H \) the cosets \( x + 2H \) and \( -x + 2H \) are disjoint, which implies that each point \( x \in X \) is distinct from \( -x \). Then each point \( x \in X \) has a neighborhood \( U_x \subset X \) such that \( U_x \cap -U_x = \emptyset \). Replacing \( U_x \) by \( U_x \cap (-U_x) \) we can additionally assume that \( U_x = -U_x \). Now consider the open neighborhood \( U_{\pm x} = q(U_x) = q(U_x - x) \subset X/C_2 \) of the orbit \( \{x, -x\} \in X/C_2 \) of the point \( x \in X \). By the paracompactness of \( X/C_2 \) the open cover \( \{U_{\pm x} : x \in X\} \) of \( X/C_2 \) has a \( \Sigma \)-discrete refinement \( U = \bigcup_{n \in \omega} U_n \). This means that each family \( U_n \), \( n \in \omega \), is discrete in \( X/C_2 \). For each \( U \in U \) find a point \( x_U \in X \) such that \( U \subset U_{x_U} \). For every \( n \in \omega \) consider the open subset \( W_n = \bigcup_{U \in U_n} q^{-1}(U) \cap U_{x_U} \) of the space \( X \) and let \( +W_n = -W_n \cup W_n \). One can check that the Borel subset
\[ B = \bigcup_{n \in \omega} (W_n \setminus \bigcup_{i < n} \pm W_i) \]
of \( X \) has one-point intersection with each orbit \( \{x, -x\}, x \in X \).

The following lemma will be helpful in the proof of the upper bound \( rc_k^B(G) \leq c_k^B(G) - 2 \) from Proposition [1].

**Lemma 10.** Let \( k \geq 4 \) and \( C \subset \mathbb{R}^\omega \) be a finite \( k \)-centerpole subset for Borel colorings of \( \mathbb{R}^\omega \). Then the affine hull of \( C \) in \( \mathbb{R}^\omega \) has dimension \( \leq |C| - 3 \).

**Proof.** This lemma will be proved by induction on the cardinality \( |C| \).

First observe that \( |C| \cdot c_3^B(\mathbb{R}^\omega) \geq c_3^B(\mathbb{R}^\omega) \geq 6 \) by Lemma [5]. So, we start the induction with \( |C| = 6 \).

Suppose that either \( m = 6 \) or \( m > 6 \) and the lemma is true for all \( C \) with \( 6 \leq |C| < m \). Fix a \( k \)-centerpole subset \( C \subset \mathbb{R}^\omega \) for Borel colorings of cardinality \( |C| = m \). We need to show that the affine hull \( A \) of \( C \) has dimension \( \dim A \leq m - 3 \). Assuming the opposite, we can find a support hyperplane \( H \subset A \) for \( C \) such that \( |H \cap C| \geq \dim H + 1 = \dim A \geq |C| - 2 \) and hence \( 0 < |C \setminus H| \leq 2 \). After a suitable shift, we can assume that \( H \) contains the origin of \( \mathbb{R}^\omega \) and hence is a subgroup of \( \mathbb{R}^\omega \). In this case the affine hull \( A \) is a linear subspace in \( \mathbb{R}^\omega \) that can be identified with the direct sum \( H \oplus \mathbb{R} \). It follows that \( \dim H = \dim A - 1 \geq |C| - 2 - 1 \geq |C \cap H| \).

We claim that the set \( H \cap C \) is not \( k \)-centerpole for Borel colorings of the topological group \( H \).

If \( 6 \leq |C \cap H| < |C| = m \), then by the inductive assumption, the set \( C \cap H \) is not \( k \)-centerpole for Borel colorings of \( \mathbb{R}^\omega \) because its affine hull \( H \) has dimension \( \dim H \geq |C \cap H| - 2 \). If \( |C \cap H| < 6 \) (which happens for \( m = 6 \)), then the inequalities \( c_k^B(H) \geq c_k^B(C) \geq 6 = m = |C| > |H \cap C| \) given by Lemma [5] guarantee that \( C \cap H \) is not \( k \)-centerpole for Borel colorings of \( \mathbb{R}^\omega \).

By (the proof) of Proposition 1 in [3], \( c_k^B(H) = 3 \). Since \( H \) is a support hyperplane for \( C \) and \( |C \cap H| \leq 2 \), we can apply Lemma [6] and conclude that \( C \) is not \( k \)-centerpole for Borel colorings of \( H \oplus \mathbb{R} = A \). Since the subgroup \( 2A \) is closed in the metrizable group \( \mathbb{R}^\omega \), by Lemma [10] \( C \) is not \( k \)-centerpole for Borel colorings of \( \mathbb{R}^\omega \) and this is a desired contradiction that completes the proof of the inductive step and base of the induction.

### 7. Stability Properties

In this section we shall prove some particular cases of the Stability Theorem [4].

**Lemma 11.** For any numbers \( k \geq 2 \) and \( n \leq m \)
\[ c_k^B(\mathbb{R}^n \times \mathbb{Z}^{m-n}) = \begin{cases} c_k^B(\mathbb{R}^n \times \mathbb{Z}^{m-n}), & \text{if } m \geq rc_k^B(\mathbb{R}^n \times \mathbb{Z}^{m-n}), \\ c_k^B(\mathbb{R}^n), & \text{if } n \geq rc_k^B(\mathbb{R}^n). \end{cases} \]

**Proof.** First assume that \( m \geq rc_k^B(\mathbb{R}^n \times \mathbb{Z}^{m-n}) \). By the definition of the number \( r = rc_k^B(\mathbb{R}^n \times \mathbb{Z}^{m-n}) \), the topological group \( G = \mathbb{R}^n \times \mathbb{Z}^s \) contains a \( k \)-centerpole subset \( C \subset G \) of cardinality \( |C| = c_k^B(G) \) that generates a subgroup \( \langle C \rangle \subset \mathbb{Z}^s \) of \( \mathbb{Z}^s \)-rank \( r \). It follows that the linear subspace \( L \subset \mathbb{R}^n \times \mathbb{Z}^s \) generated by the set \( C \) has dimension \( r \). Then \( H = L \cap G \), being a closed subgroup of \( \mathbb{Z}^s \) in the \( r \)-dimensional vector space \( L \) is topologically isomorphic to \( \mathbb{R}^s \times \mathbb{Z}^{r-s} \) for some \( s \leq r \leq m \), see Theorem 6 in [10]. Taking into account that \( H \) is a closed subgroup of \( G = \mathbb{R}^n \times \mathbb{Z}^s \), we conclude that \( s \leq n \). By Lemma [9] the set \( C \) is \( k \)-centerpole in \( H \) for Borel colorings. Consequently,
\[ c_k^B(\mathbb{R}^n \times \mathbb{Z}^s) \leq c_k^B(\mathbb{R}^n \times \mathbb{Z}^{m-n}) \leq c_k^B(\mathbb{R}^n \times \mathbb{Z}^{r-s}) \leq c_k^B(\mathbb{R}^n) \leq |C| = c_k^B(G) = c_k^B(\mathbb{R}^n \times \mathbb{Z}^s), \]
implies the desired equality \( c_k^B(\mathbb{R}^n \times \mathbb{Z}^{m-n}) = c_k^B(\mathbb{R}^n \times \mathbb{Z}^{r-s}) \).

Now assume that \( n \geq rc_k^B(\mathbb{R}^n) \). In this case we can repeat the above argument for a set \( C \subset \mathbb{R}^n \) of cardinality \( |C| = c_k^B(\mathbb{R}^n) \) that generates a subgroup \( \langle C \rangle \subset \mathbb{R}^n \) of \( \mathbb{R}^n \)-rank \( r = rc_k^B(\mathbb{R}^n) \). Then the linear subspace \( L \subset \mathbb{R}^n \)
generated by the set $C$ is topologically isomorphic to $\mathbb{R}^r$. By Lemma 9, the set $C$ is $k$-centerpole for Borel colorings of $L$. Since $\mathbb{R}^r \rightarrow \mathbb{R}^n \times \mathbb{Z}^{m-n} \rightarrow \mathbb{R}^r$, we get

$$c_k^B(\mathbb{R}^r) \leq c_k^B(\mathbb{R}^n \times \mathbb{Z}^{m-n}) \leq c_k^B(\mathbb{R}^r) = c_k^B(L) \leq |C| = c_k^B(\mathbb{R}^\omega)$$

and hence $c_k^B(\mathbb{R}^n \times \mathbb{Z}^{m-n}) = c_k^B(\mathbb{R}^\omega)$. \hfill $\square$

Lemma 12. $c_k(\mathbb{R}^n \times \mathbb{Z}^{m-n}) = c_k^B(\mathbb{Z}^\omega)$ for any numbers $k \in \mathbb{N}$ and $n \leq m$ with $m \geq c_k(\mathbb{Z}^\omega)$.

Proof. For $k = 1$ the equality $c_k(\mathbb{R}^n \times \mathbb{Z}^{m-n}) = 1 = c_k^B(\mathbb{Z}^\omega)$ is trivial. So we assume that $k \geq 2$.

We claim that $c_k^B(\mathbb{Z}^\omega) \leq c_k(\mathbb{R}^m)$. Indeed, take any $k$-centerpole subset $C \subset \mathbb{R}^\omega$ of cardinality $|C| = c_k(\mathbb{R}^m)$. By Lemma 8 the set $C$ is $k$-centerpole in the subgroup $\langle C \rangle \subset \mathbb{R}^\omega$ generated by $C$. Being a torsion-free finitely-generated abelian group, $\langle C \rangle$ is algebraically isomorphic to $\mathbb{Z}^r$ for some $r \in \omega$. Then

$$c_k(\mathbb{Z}^r) \leq c_k(\langle C \rangle) \leq |C| = c_k(\mathbb{R}^m).$$

On the other hand, Lemma 11 ensures that

$$c_k(\mathbb{R}^m) \leq c_k(\mathbb{Z}^m) = c_k^B(\mathbb{Z}^m) = c_k^B(\mathbb{Z}^\omega).$$

Unifying these inequalities we get

$$c_k^B(\mathbb{Z}^\omega) \leq c_k^B(\mathbb{Z}^m) = c_k(\mathbb{Z}^m) \leq c_k(\mathbb{R}^m) \leq c_k(\mathbb{Z}^m) = c_k^B(\mathbb{Z}^\omega),$$

which implies the desired equality $c_k(\mathbb{R}^n \times \mathbb{Z}^{m-n}) = c_k^B(\mathbb{Z}^\omega)$. \hfill $\square$

8. Proof of Theorem 3

1. The upper bound $c_k(\mathbb{Z}^n) \leq c_k(\mathbb{Z}^k) \leq 2^k - 1 - \max_{s \leq k-2} \left( \begin{array}{c} k-1 \\ s \end{array} \right)$ for $k \leq n$ follows from Theorem 5.

2. By Proposition 3 and Theorem 7, $c_n(\mathbb{Z}^n) \geq c_n(\mathbb{R}^n) \geq c_n^B(\mathbb{R}^n) \geq t(\mathbb{R}^n) \geq \frac{1}{2}(n^2 + 3n - 4)$.

For technical reasons, first we prove the statement (4) of Theorem 3 and after that return back to the statement (3).

4. Let $1 \leq k \leq m \leq \omega$ be two numbers. We need to prove that $c_k^B(\mathbb{R}^m) < c_k^B(\mathbb{R}^{m+1})$ and $c_k(\mathbb{R}^m) < c_k(\mathbb{R}^{m+1})$.

First we assume that $m$ is finite. The strict inequality $c_k^B(\mathbb{R}^m) < c_k^B(\mathbb{R}^{m+1})$ will follow as soon as we show that any subset $C \subset \mathbb{R}^{m+1}$ of cardinality $|C| \leq c_k^B(\mathbb{R}^m)$ fails to be $(k+1)$-centerpole for Borel colorings of $\mathbb{R}^{m+1}$. If $C$ is a singleton, then it is not $(k+1)$-centerpole since $c_k^B(\mathbb{R}^{m+1}) \geq c_k^B(\mathbb{R}^m) \geq 3$ (by the proof of) Proposition 4.1 in [3]. So, $C$ contains two distinct points $a,b$. Let $L = \mathbb{R} \cdot (a - b) \subset \mathbb{R}^{m+1}$ be the linear subspace generated by the vector $a - b$. Write the space $\mathbb{R}^{m+1}$ as the direct sum $\mathbb{R}^{m+1} = H \oplus L$ where $H$ is a linear $m$-dimensional subspace of $\mathbb{R}^{m+1}$ and consider the projection $pr : \mathbb{R}^{m+1} \rightarrow H$ whose kernel is equal to $L$. Since $pr(a) = pr(b)$, the projection of the set $C$ onto the subspace $H$ has cardinality $|pr(C)| < |C| \leq c_k^B(\mathbb{R}^m) = c_k^B(H)$ and hence $pr_H(C)$ is not $k$-centerpole for Borel $k$-colorings of the group $H$. Consequently, there is a Borel $k$-coloring $\chi : H \rightarrow k$ such no monochromatic unbounded subset of $H$ is symmetric with respect to a point $c \in pr(C)$.

For a real number $\gamma \in \mathbb{R}$, consider the half-line $L_+^{(\gamma)} = \{t(a - b) : t \geq \gamma\}$ of $L$. Since the subset $C \subset \mathbb{R}^{m+1}$ is finite, there is $\gamma \in \mathbb{R}$ such that $C \subset H + L_+^{(\gamma)}$.

Now define a Borel $(k+1)$-coloring $\tilde{\chi} : H \oplus L \rightarrow k+1 = \{0, \ldots, k\}$ by the formula

$$\tilde{\chi}(x) = \begin{cases} \chi(pr(x)), & \text{if } x \in H + L_+^{(\gamma)}; \\ k, & \text{otherwise}. \end{cases}$$

It can be shown that this coloring witnesses that $C$ is not $(k+1)$-centerpole for Borel colorings of $\mathbb{R}^{m+1} = H \oplus L$.

Now assume that the number $m$ is infinite. Then for the finite number $r = \max\{rc_k^B(\mathbb{R}^n), rc_k^B(\mathbb{R}^\omega)\}$ we get $c_k^B(\mathbb{R}^r) = c_k^B(\mathbb{R}^r)$ and $c_k^B(\mathbb{R}^{r+1}) = c_k(\mathbb{R}^\omega)$ by the stabilization Lemma 11. Since $r$ is finite, the case considered above guarantees that

$$c_k^B(\mathbb{R}^m) = c_k^B(\mathbb{R}^n) = c_k^B(\mathbb{R}^r) < c_k^B(\mathbb{R}^{r+1}) = c_k^B(\mathbb{R}^\omega) = c_k(\mathbb{R}^{m+1}).$$

By analogy we can prove the strict inequality $c_k(\mathbb{R}^m) < c_k(\mathbb{R}^{m+1})$.

3. Now we are able to prove the lower bound $c_k^B(\mathbb{R}^\omega) \geq k + 4$ from the statement (3) of Theorem 3. By the preceding item, $c_k^B(\mathbb{R}^\omega) \geq 1 + c_k^B(\mathbb{R}^\omega)$ for all $k \in \mathbb{N}$. By induction, we shall show that $c_k^B(\mathbb{R}^\omega) \geq k + 4$ for all $k \geq 4$. For $k = 4$ the inequality $c_4^B(\mathbb{R}^\omega) \geq 8 \geq 4 + 4$ was proved in Lemma 7. Assuming that $c_k^B(\mathbb{R}^\omega) \geq k + 4$ for some $k \geq 4$, we conclude that $c_{k+1}^B(\mathbb{R}^\omega) \geq c_k^B(\mathbb{R}^\omega) \geq k + 4$ and hence $c_k^{B+1}(\mathbb{R}^\omega) \geq (k + 1) + 4$.

Now we see that for every $n \geq k \geq 4$ we have the desired lower bound:

$$c_k^B(\mathbb{R}^n) \geq c_k^B(\mathbb{R}^\omega) \geq k + 4.$$
5. Let \( k \in \mathbb{N} \) and \( n, m \in \omega \cup \{\omega\} \) be numbers with \( 1 \leq k \leq n + m \). We need to prove that \( c_k^B(\mathbb{R}^n \times \mathbb{Z}^m) < c_{k+1}^B(\mathbb{R}^n \times \mathbb{Z}^{m+1}) \) and \( c_k(\mathbb{R}^n \times \mathbb{Z}^m) < c_{k+1}(\mathbb{R}^n \times \mathbb{Z}^{m+1}) \). According to the Stabilization Lemma \( \text{[11]} \) it suffices to consider the case of finite numbers \( n, m \).

First we prove the inequality \( c_k^B(\mathbb{R}^n \times \mathbb{Z}^m) < c_{k+1}^B(\mathbb{R}^n \times \mathbb{Z}^{m+1}) \). We need to show that each subset \( C \subset \mathbb{R}^n \times \mathbb{Z}^{m+1} \) of cardinality \( |C| \leq c_k^B(\mathbb{R}^n \times \mathbb{Z}^m) \) is not \((k+1)\)-centerpole in \( \mathbb{R}^n \times \mathbb{Z}^{m+1} \) for Borel colorings. We shall identify \( \mathbb{R}^n \times \mathbb{Z}^{m+1} \) with the direct sum \( \mathbb{R}^n \oplus \mathbb{Z}^{m+1} \). Since \( k \leq n + m \), Theorem \( \text{[5]} \) implies that the numbers \( |C| \leq c_k^B(\mathbb{R}^n \times \mathbb{Z}^m) \leq c_k(\mathbb{Z}^{n+m}) \) all are finite.

Three cases are possible:

(i) \( |C| \leq 1 \). In this case we can assume that \( C = \{0\} \) and take any coloring \( \chi : \mathbb{R}^n \oplus \mathbb{Z}^{m+1} \to k + 1 \) such that the color of each non-zero element \( x \in \mathbb{R}^n \times \mathbb{Z}^{m+1} \) differs from the color of \(-x\). This coloring witnessed that \( C \) is not \((k+1)\)-centerpole in \( \mathbb{R}^n \times \mathbb{Z}^{m+1} \).

(ii) \( |C| > 1 \) and \( C \subset z \oplus \mathbb{R}^n \) for some \( z \in \mathbb{Z}^{m+1} \). Without lose of generality, \( z = 0 \) and hence \( C \subset \mathbb{R}^n \). Take two distinct points \( a, b \in C \) and consider the 1-dimensional linear subspace \( L = \mathbb{R} \cdot (a - b) \subset \mathbb{R}^n \) generated by the vector \( a - b \). Write the space \( \mathbb{R}^n \) as the direct sum \( \mathbb{R}^n = L \oplus H \) where \( H \) is a linear \((n-1)\)-dimensional subspace of \( \mathbb{R}^n \) and consider the projection \( \text{pr} : \mathbb{R}^n \oplus \mathbb{Z}^{m+1} \to H \oplus \mathbb{Z}^{m+1} \) whose kernel is equal to \( L \). Since \( \text{pr}(a) = \text{pr}(b) \), the projection of the set \( C \) onto the subgroup \( H \oplus \mathbb{Z}^{m+1} \) of \( \mathbb{R}^n \oplus \mathbb{Z}^{m+1} \) has cardinality
\[
|\text{pr}(C)| < |C| \leq c_k^B(\mathbb{R}^n \times \mathbb{Z}^{m+1}) = c_k^B(\mathbb{Z}^{n+m})
\]
and hence \( \text{pr}_H(C) \) is not \( k \)-centerpole for Borel colorings of the group \( H \oplus \mathbb{Z}^{m+1} \). Consequently, there is a Borel \( k \)-coloring \( \chi : H \oplus \mathbb{Z}^{m+1} \to k \) such no monochromatic unbounded subset of \( H \oplus \mathbb{Z}^{m+1} \) is symmetric with respect to a point \( c \in \text{pr}(C) \).

For a real number \( \gamma \in \mathbb{R} \), consider the half-line \( L_\gamma^+ = \{t(a - b) : t \geq \gamma\} \) of \( L \). Since the subset \( C \subset \mathbb{R}^n \oplus \mathbb{Z}^{m+1} = H \oplus L \oplus \mathbb{Z}^{m+1} \) is finite, there is \( \gamma \in \mathbb{R} \) such that \( C \subset H + L_\gamma^+ \oplus \mathbb{Z}^{m+1} \).

Now define a Borel \((k+1)\)-coloring \( \tilde{\chi} : H \oplus L \oplus \mathbb{Z}^{m+1} \to k + 1 = \{0, \ldots, k\} \) by the formula
\[
\tilde{\chi}(x) = \begin{cases} \chi(\text{pr}(x)), & \text{if } x \in H + L_\gamma^+ + \mathbb{Z}^{m+1}, \\ k, & \text{otherwise.} \end{cases}
\]
It can be shown that this coloring witnesses that \( C \) is not \((k+1)\)-centerpole for Borel colorings of \( \mathbb{R}^n \oplus \mathbb{Z}^{n+m} = H \oplus L \oplus \mathbb{Z}^{m+1} \).

(iii) The set \( C \subset \mathbb{R}^n \oplus \mathbb{Z}^{m+1} \) contains two points \( a, b \) whose projections on the subspace \( \mathbb{Z}^{m+1} \) are distinct. Without loss of generality, the projections of \( a, b \) on the last coordinate are distinct. Then the 1-dimensional subspace \( L = \mathbb{R} \cdot (a - b) \) of \( \mathbb{R}^n \times \mathbb{Z}^{m+1} \) meets the subspace \( \mathbb{R}^n \oplus \mathbb{R}^m \) and hence \( \mathbb{R}^n \oplus \mathbb{R}^m \) can be identified with the direct sum \( \mathbb{R}^n \oplus \mathbb{R}^m \oplus L \). Let \( \text{pr} : \mathbb{R}^n \times \mathbb{R}^{m+1} \to \mathbb{R}^n \times \mathbb{R}^m \) be the projection whose kernel coincides with \( L \). Since \( \text{pr} \) is an open map, the image \( H = \text{pr}(\mathbb{R}^n \times \mathbb{R}^{m+1}) \) is a locally compact (and hence closed) subgroup of \( \mathbb{R}^n \times \mathbb{R}^m \), which can be written as the countable union of shifted copies of the space \( \mathbb{R}^n \). By Theorem 6 of \( \text{[10]} \), \( H \) is topologically isomorphic to \( \mathbb{R}^n \times \mathbb{Z}^m \). It follows from the definition of \( H \) that \( \mathbb{R}^n \oplus \mathbb{Z}^{m+1} \subset H \oplus L \).

Since \( \text{pr}(a) = \text{pr}(b) \), the projection of the set \( C \) has cardinality \( |\text{pr}(C)| \leq |C| \leq c_k^B(\mathbb{R}^n \times \mathbb{Z}^{m+1}) = c_k^B(H) \), which means that \( \text{pr}(C) \) is not \( k \)-centerpole for Borel colorings of \( H \). Consequently, there is a Borel \( k \)-coloring \( \chi : H \to k \) such no monochromatic unbounded subset of \( H \) is symmetric with respect to a point \( c \in \text{pr}(C) \).

For a real number \( \gamma \in \mathbb{R} \), consider the half-line \( L_\gamma^+ = \{t(a - b) : t \geq \gamma\} \) of \( L \). Since the subset \( C \subset H \oplus L \) is finite, there is \( \gamma \in \mathbb{R} \) such that \( C \subset H + L_\gamma^+ \).

Now define a Borel \((k+1)\)-coloring \( \tilde{\chi} : H \oplus L \to k + 1 \) by the formula
\[
\tilde{\chi}(x) = \begin{cases} \chi(\text{pr}(x)), & \text{if } x \in H + L_\gamma^+, \\ k, & \text{otherwise.} \end{cases}
\]
It can be shown that this coloring witnesses that \( C \) is not \((k+1)\)-centerpole for Borel colorings of \( H \oplus L \supset \mathbb{R}^n \oplus \mathbb{Z}^{m+1} \).

After considering these three cases, we can conclude that
\[
c_k^B(\mathbb{R}^n \times \mathbb{Z}^{m+1}) > c_k^B(\mathbb{R}^n \times \mathbb{Z}^m).
\]

Deleting the adjective “Borel” from the above proof, we get the proof of the strict inequality
\[
c_k(\mathbb{R}^n \times \mathbb{Z}^m) < c_{k+1}(\mathbb{R}^n \times \mathbb{Z}^{m+1}).
\]

9. Proof of Theorem 2

In this section we prove Theorem 2. Let \( k, n, m \) be cardinals. We shall use known upper bounds for the numbers \( c_k(\mathbb{Z}^n) \), lower bounds for \( t(\mathbb{R}^n) \) and the inequality
\[
t(\mathbb{R}^{n+m}) \leq c_k^B(\mathbb{R}^{n+m}) \leq c_k^B(\mathbb{R}^n \times \mathbb{Z}^m) \leq c_k(\mathbb{R}^n \times \mathbb{Z}^m) \leq c_k(\mathbb{Z}^m).
\]
established in Proposition 3.

1. Assume that \( n + m \geq 1 \). Since each singleton is 1-centerpole for (Borel) colorings of the group \( \mathbb{R}^n \times \mathbb{Z}^m \), we conclude that \( c_1(\mathbb{R}^n \times \mathbb{Z}^m) = c_2^B(\mathbb{R}^n \times \mathbb{Z}^m) = 1 \).

2. Assume that \( n + m \geq 2 \). The inequalities \( 3 \leq t(\mathbb{R}^2) \leq c_2^B(\mathbb{R}^2) \leq c_2(\mathbb{Z}^2) \leq 3 \) follow from Theorem 5 (2) and Proposition 3.

   We claim that \( c_2^B(\mathbb{R}^\omega) \geq 3 \). Assuming that \( c_2^B(\mathbb{R}^\omega) < 3 \) we conclude that \( rc_k^B(\mathbb{R}^\omega) \leq c_2^B(\mathbb{R}^\omega) - 1 \leq 1 \). Then by the Stabilization Lemma 11 we get that \( c_2(\mathbb{R}^1) = c_2(\mathbb{R}^\omega) \) is finite. On the other hand, the real line has the 2-coloring \( \chi : \mathbb{R} \to 2, \chi^{-1}(1) = (0, \infty) \), without unbounded monochromatic symmetric subsets. This coloring witnesses that \( c_2(\mathbb{R}^1) = \infty \) and this is a contradiction. Therefore,

\[
3 \leq c_2^B(\mathbb{R}^\omega) \leq c_2^B(\mathbb{R}^n \times \mathbb{Z}^{m-n}) \leq c_2(\mathbb{R}^n \times \mathbb{Z}^{m-n}) \leq c_2(\mathbb{Z}^2) = 3.
\]

3. Assume that \( n + m \geq 3 \). Lemma 5 and Theorem 3 imply the inequalities

\[
6 \leq c_3^B(\mathbb{R}^n) \leq c_3^B(\mathbb{R}^n \times \mathbb{Z}^{m-n}) \leq c_3^B(\mathbb{R}^n \times \mathbb{Z}^{m-n}) \leq c_3(\mathbb{Z}^3) = 6
\]

that turn into equalities.

4. Assume that \( n + m = 4 \). Theorem 5 (4) and Proposition 3 imply the inequalities

\[
12 \leq t(\mathbb{R}^2) \leq c_2^B(\mathbb{R}^4) \leq c_2^B(\mathbb{R}^n \times \mathbb{Z}^m) \leq c_4(\mathbb{R}^n \times \mathbb{Z}^m) \leq c_4(\mathbb{Z}^4) \leq 12,
\]

which actually are equalities.

5. We need to prove that \( c_2^B(\mathbb{R}^n \times \mathbb{Z}^m) = \infty \) if \( k \geq n + m + 1 < \omega \). This equality will follow as soon as we check that \( c_k^B(\mathbb{R}^{n+m}) = \infty \). Let \( \Delta \) be a simplex in \( \mathbb{R}^{n+m} \) centered at the origin. Write the boundary \( \partial \Delta \) as the union \( \partial \Delta = \bigcup_{i=0}^{n+m} \Delta_i \) of its facets. Define a Borel \( k \)-coloring \( \chi : \mathbb{R}^n \to \{0, \ldots, n + m\} \subset k \) assigning to each point \( x \in \mathbb{R}^n \setminus \{0\} \) the smallest number \( i \leq n + m \) such that the ray \( \mathbb{R}_x \cdot x \) meets the facet \( \Delta_i \). Also put \( \chi(0) = 0 \). It is easy to check that the coloring \( \chi \) witnesses that the set \( \mathbb{R}^n \times \mathbb{Z}^m \) is not \( k \)-centerpole for Borel colorings of \( \mathbb{R}^{n+m} \) and consequently, \( c_2^B(\mathbb{R}^{n+m}) = \infty \).

6. Assume that \( k \geq n + m + 1 \), we shall show that \( c_k(\mathbb{R}^n \times \mathbb{Z}^m) = \infty \). If \( n + m \) is finite, then this follows from the preceding item. So, we assume that \( n + m \) is infinite. Then the group \( G = \mathbb{R}^n \times \mathbb{Z}^m \) has cardinality \( 2^{n+m} \). By Theorem 4 of [4], for the group \( G \) endowed with the discrete topology, we get \( \nu(G) = \log |G| = \min\{\gamma : 2^\gamma \geq |G|\} \leq n + m \leq k \), which means that \( G \) admits a \( k \)-coloring without infinite monochromatic symmetric subset. This implies that the set \( G \) is not \( k \)-centerpole in \( G \) and thus \( c_k(G) = \infty \).

7. Assume that \( n + m \geq \omega \) and \( \omega \leq k < \text{cov}(\mathcal{M}) \). The lower bound from Theorem 5 (3) implies that \( \omega \leq c_k^B(\mathbb{R}^\omega) \leq c_k^B(\mathbb{Z}^\omega) \). The upper bound \( c_k^B(\mathbb{Z}^\omega) \leq \omega \) will follow as soon as we check that each countable dense subset \( C \subset \mathbb{Z}^\omega \) is \( k \)-centerpole for Borel colorings of \( \mathbb{Z}^\omega \). Let \( \chi : \mathbb{Z}^\omega \to k \) be a Borel \( k \)-coloring of \( \mathbb{Z}^\omega \). Taking into account that \( \mathbb{Z}^\omega = \bigcup_{i \in k} \chi^{-1}(i) \) is homeomorphic to a dense \( G_k \)-subset of the real line, we conclude that for some color \( i \in k \) the preimage \( A = \chi^{-1}(i) \) is not meager in \( \mathbb{Z}^\omega \). Being a Borel subset of \( \mathbb{Z}^\omega \), the set \( A \) has the Baire property, which means that for some open subset \( U \subset \mathbb{Z}^\omega \) the symmetric difference \( \Delta U \) is meager in \( \mathbb{Z}^\omega \). Since \( A \) is not meager, the set \( U \) is not empty. Take any point \( c \in U \cap C \) and observe that \( V = U \cap (2c - U) \) is an open symmetric neighborhood of \( c \). It follows that for the set \( B = A \cap (2c - A) \) the symmetric difference \( B \Delta V \) is meager. Since \( V \) is not meager in \( \mathbb{Z}^\omega \), the set \( B \) is not meager and hence is unbounded in \( \mathbb{Z}^\omega \) (since totally bounded subsets of \( \mathbb{Z}^\omega \) are nowhere dense in \( \mathbb{Z}^\omega \)). Now we see that \( B = A \cap (2c - A) \) is a monochromatic unbounded subset, symmetric with respect to the point \( c \), witnessing that the set \( C = \omega \)-centerpole for Borel coloring of \( \mathbb{Z}^\omega \).

### 10. Proof of Theorem 11

Let \( k \geq 2 \) be a finite cardinal number and \( G \) be an abelian ILC-group with totally bounded Boolean subgroup \( G[2] \) and ranks \( n = r_B(G) \) and \( m = r_Z(G) \). Let \( \widehat{G} \) be the completion of the group with respect to its (two-sided) uniformity.

First we give a proof the statements (3) and (4) of Theorem 11 holding under the additional assumption of the metrizability of the group \( G \).

Since \( c_k^B(\mathbb{R}^n \times \mathbb{Z}^{m-n}) \leq \omega \) if \( k \leq m \), the Borel version of Theorem 11 will follow as soon as we prove two inequalities:

1. \( c_k^B(\mathbb{R}^n) \leq c_k^B(\mathbb{R}^n \times \mathbb{Z}^{m-n}) \) if \( k \leq m \) and
2. \( c_k^B(\mathbb{R}^n \times \mathbb{Z}^{m-n}) \leq c_k^B(\mathbb{R}^n) \) if \( c_k^B(\mathbb{R}^n) \) is finite.

1. Assume that \( k \leq m \). If the \( \mathbb{Z} \)-rank \( m = r_Z(G) \) is finite, then so is the \( \mathbb{R} \)-rank \( n = r_B(G) \) and we can find copies of the topological groups \( \mathbb{R}^n \) and \( \mathbb{Z}^m \) in \( G \). Now consider the closure \( H \) of the subgroup \( \mathbb{R}^n + \mathbb{Z}^m \) in \( G \). Since \( G \) is an ILC-group and \( \mathbb{R}^n + \mathbb{Z}^m \) contains a dense finitely generated subgroup, the group \( H \) is locally compact. By the structure theorem of locally compact abelian groups [10] (Theorem 25), \( H \) is topologically isomorphic to \( \mathbb{R}^r \oplus \mathbb{Z}^s \) for some \( r \in \omega \) and a closed subgroup \( Z \subset H \) that contains an open compact subgroup \( K \). It follows from the
inclusion $\mathbb{R}^n \subset H$ that $n \leq r$. On the other hand, $r \leq r_\omega(G)$. By the same reason, $r_\omega(H) = m = r_\omega(G)$. In particular, $r_\omega(Z) = m - n$ and hence $H$ contains an isomorphic copy of the group $\mathbb{R}^n \times \mathbb{Z}^{m-n}$. Now we see that $r_\omega^B(G) \leq r_\omega^B(\mathbb{R}^n \times \mathbb{Z}^{m-n})$.

Next, assume that the $\mathbb{Z}$-rank $m = r_\omega(G)$ is infinite but $n = r_\omega(G)$ is finite. By the Stabilization Lemma\[1\]
$c_k^B(\mathbb{R}^n \times \mathbb{Z}^{m-n}) = c_k^B(\mathbb{R}^n \times \mathbb{Z}^n) = c_k^B(\mathbb{R}^n \times \mathbb{Z}^n)$ for $r = r c_k^B(\mathbb{R}^n \times \mathbb{Z}^n) \leq c_k^B(\mathbb{R}^n \times \mathbb{Z}^n) < \infty$. Repeating the above argument we can find a copy of the group $\mathbb{R}^n \times \mathbb{Z}^{s-n}$ in $G$ for some finite $s \geq r$ and conclude that $c_k^B(G) \leq c_k^B(\mathbb{R}^n \times \mathbb{Z}^{s-n}) \leq c_k^B(\mathbb{R}^n \times \mathbb{Z}^{s-n}) = c_k^B(\mathbb{R}^n \times \mathbb{Z}^{m-n})$.

Finally, assume that the $\mathbb{R}$-rank $n = r_\omega(G)$ is infinite. Then $c_k^B(\mathbb{R}^n \times \mathbb{Z}^{m-n}) = c_k^B(\mathbb{R}^n) = c_k^B(\mathbb{R})$ for $r = r c_k^B(\mathbb{R}^n) \leq c_k^B(\mathbb{R}^n) < \omega$. By the definition of the $\mathbb{R}$-rank $r_\omega(G) = n = \omega$, we can find a copy of the group $\mathbb{R}$ in $G$ and conclude that $c_k^B(G) \leq c_k^B(\mathbb{R}^n \times \mathbb{Z}^{m-n})$. This completes the proof of the inequality $c_k^B(G) \leq c_k^B(\mathbb{R}^n \times \mathbb{Z}^{m-n})$.

Deleting the adjective “Borel” from the above proof we get the proof of the inequality $c_k(G) \leq c_k(\mathbb{R}^n \times \mathbb{Z}^{m-n})$ holding for each $k \leq m$.

2. Now assuming that $c_k^B(G)$ is finite and the group $G$ is metrizable, we prove the inequality $c_k^B(\mathbb{R}^n \times \mathbb{Z}^{m-n}) \leq c_k^B(G)$.

Fix a $k$-centerpole subset $C \subset G$ for Borel colorings of $G$ with cardinality $|C| = c_k^B(G)$. The subgroup $G[2]$ is totally bounded and hence has compact closure $K_2$ in the completion $\hat{G}$ of the group $G$. It follows that $K_2 \subset \hat{G}[2]$. Since $G$ is an ILC-group, the finitely-generated subgroup $\langle C \rangle$ has locally compact closure $\langle C \rangle$ in $G$. It follows from the compactness of the subgroup $K_2$ that the sum $H = \langle C \rangle + K_2$ is a locally compact subgroup of $G$. This subgroup is compactly generated because it contains a dense subgroup generated by the compact set $C + K_2$.

By the Structure Theorem for compactly generated locally compact abelian groups\[10\], Theorem 24], $H$ is topologically isomorphic to $\mathbb{R}^r \oplus \mathbb{Z}^{s-r} \oplus K$ for some compact subgroup $K$ that contains all torsion elements of $H$. In particular, $K_2 \subset K$. Now we see that the subgroup $2H = \{2x : x \in H\}$ is closed in $H$ and consequently, the subgroup $2H \cap G$ is closed in $G$. The group $G$ is metrizable and so is the quotient group $G/2H$. Then the subspace $X = (G/2H) \setminus (H/2H)$ is metrizable and thus paracompact. Since $H \supset G[2]$ we can apply Lemma 8 and conclude that the set $C$ is $k$-centerpole for Borel colorings of the subgroup $H \cap G$. Since $H \cap G \subset H$, the set $C$ is $k$-centerpole for Borel colorings of the group $H$.

The compactness of the subgroup $K \subset H$ implies that the image $q(C)$ of $C$ under the quotient map $q : H \to H/K$ is a $k$-centerpole set for Borel colorings of the quotient group $H/K = \mathbb{R}^r \times \mathbb{Z}^{s-r}$. Since $H = \langle C \rangle + K_2$ and $K_2 \subset K$, we conclude that $\langle C \rangle / \langle \langle C \rangle \cap K \rangle \subset q(C) = H/K = \mathbb{R}^r \times \mathbb{Z}^{s-r}$ and hence $r \leq n$ and $s \leq m$. Consequently, $\mathbb{R}^r \times \mathbb{Z}^{s-r} \rightarrow \mathbb{R}^n \times \mathbb{Z}^{m-n}$ and

\[ c_k^B(\mathbb{R}^n \times \mathbb{Z}^{m-n}) \subset c_k^B(\mathbb{R}^r \times \mathbb{Z}^{s-r}) = c_k^B(H/K) \leq |C| = c_k^B(G). \]

Deleting the adjective “Borel” from the above proof and applying Lemma 8 instead of Lemma 8, we get the proof of the inequality $c_k(\mathbb{R}^n \times \mathbb{Z}^{m-n}) \leq c_k(G)$ under the assumption that the number $c_k(G)$ is finite. Since Lemma 8 does not require the metrizability of $G$, this upper bound hold without this assumption.

11. Proof of Proposition 1

Let $G$ be a metrizable abelian ILC-group with totally bounded Boolean subgroup $G[2]$ and $k \in \mathbb{N}$ be such that $2 \leq k \leq r_\omega(G)$. Theorems 1 and 3 guarantee that $c_k^B(G) = c_k^B(\mathbb{R}^n \times \mathbb{Z}^{m-n}) < \infty$ where $n = r_\omega(G)$ and $m = r_\omega(G)$.

Let $r = r c_k(G)$ and $C \subset G$ be a subset of cardinality $|C| = c_k^B(G)$ such that $r \omega(\langle C \rangle) = r$. Without loss of generality, $0 \in C$. Since $G$ is an ILC-group, the finitely generated subgroup $\langle C \rangle$ has locally compact closure in $G$.

The totally bounded Boolean subgroup $G[2]$ has compact closure $K_2$ in the completion $\hat{G}$ of the abelian topological group $G$. It follows that the subgroup $H = \langle C \rangle + K_2$ of $G$ is locally compact and compactly generated. Consequently, it contains a compact subgroup $K \subset G$ such that the quotient group $H/K$ is topologically isomorphic to $\mathbb{R}^s \times \mathbb{Z}^{t-s}$ for some $s \leq r$. It follows from Lemma 8 that the set $C$ is $k$-centerpole for Borel colorings of the group $H$. The compactness of the subgroup $K \subset H$ implies that the image $q(C) \subset H/K$ of $C$ under the quotient homomorphism $q : H \to H/K$ is a $k$-centerpole set for Borel colorings of $H/K$. Consequently,

\[ c_k^B(H/K) \leq c_k^B(\mathbb{R}^s \times \mathbb{Z}^{t-s}) = c_k^B(H/K) \leq |q(C)| \leq |C| = c_k^B(G) < \infty \]

and hence $r \geq k$ by Theorem 3(5).

Now assume that $k \geq 4$. Since the set $q(C)$ is $k$-centerpole for Borel colorings of $H/K = \mathbb{R}^s \times \mathbb{Z}^{t-s} \subset \mathbb{R}^r$, Lemma 10 implies that the affine hull of $q(C)$ in the linear space $\mathbb{R}^r$ has dimension $\leq |q(C)| - 3$. Since $0 \in q(C)$, the affine hull of the set $q(C)$ coincides with its linear hull. Consequently, $r = r_\omega(\langle C \rangle) = r_\omega(\langle q(C) \rangle) \leq |q(C)| - 3 \leq |C| - 3 = c_k^B(G) - 3$. This completes the proof of the lower and upper bounds

\[ k \leq r c_k(G) \leq c_k^B(G) - 3 \]

for all $k \geq 3$. 
Next, we show that $rc_k(G) = k$ for $k \in \{2, 3\}$. In this case $c_k^B(G) = c_k(\mathbb{Z}^k)$ by Theorems 1 and 2. Since $r_2(G) \geq k$, the group $G$ contains an isomorphic copy of the group $\mathbb{Z}^k$. Then each $k$-centerpole subset $C \subset \mathbb{Z}^k \leq G$ with $|C| = c_k(\mathbb{Z}^k)$ is $k$-centerpole for Borel colorings of $G$ and thus $k \leq rc_k^B(G) \leq r_2(|C|) \leq k$, which implies the desired equality $rc_k^B(G) = k$.

12. Proof of Stabilization Theorem 4

Let $k \geq 2$ and $G$ be an abelian $\text{ILC}$-group with totally bounded Boolean subgroup $G[2]$. Let $n = r_k^B(G)$ and $m = r_2(G)$.

1. Assume that $m = r_2(G) \geq rc_k^B(\mathbb{Z}^\omega)$. By Proposition 1 $k \leq rc_k^B(\mathbb{Z}^\omega) \leq r_2(G)$ and then $c_k(G) = c_k(\mathbb{R}^n \times \mathbb{Z}^{m-n})$ by Theorem 1. Since $m = r_2(\mathbb{R}^n \times \mathbb{Z}^{m-n}) \geq rc_k^B(\mathbb{Z}^\omega)$, Lemma 2 guarantees that $c_k(G) = c_k^B(\mathbb{R}^n \times \mathbb{Z}^{m-n}) = c_k^B(\mathbb{Z}^\omega)$.

2. Assume that the group $G$ is metrizable and $r_2(G) \geq rc_k^B(\mathbb{R}^n \times \mathbb{Z}^\omega)$. By Proposition 1 $k \leq rc_k^B(\mathbb{R}^n \times \mathbb{Z}^\omega) \leq r_2(G) = m$ and hence $c_k^B(G) = c_k^B(\mathbb{R}^n \times \mathbb{Z}^{m-n})$ by Theorem 5. Since $m = r_2(\mathbb{R}^n \times \mathbb{Z}^{m-n}) \geq rc_k^B(\mathbb{R}^n \times \mathbb{Z}^\omega)$, Lemma 11 guarantees that $c_k^B(G) = c_k^B(\mathbb{R}^n \times \mathbb{Z}^{m-n}) = c_k^B(\mathbb{R}^n \times \mathbb{Z}^\omega)$.

3. By analogy with the preceding case we can prove that $c_k^B(G) = c_k^B(\mathbb{R}^\omega)$ if $G$ is metrizable and $r_2(G) \geq rc_k^B(\mathbb{R}^\omega)$.

13. Selected Open Problems

By Theorem 2 $c_k^B(\mathbb{Z}^\omega) = c_k(\mathbb{Z}^\omega) = c_k(\mathbb{Z}^k)$ for all $k \leq 4$.

Problem 1. Is $c_k(\mathbb{Z}^\omega) = c_k(\mathbb{Z}^k)$ for all $k \in \mathbb{N}$? In particular, is $c_k(\mathbb{Z}^n) = 12$ for every $n \geq 4$?

Problem 2. Is $c_k^B(\mathbb{R}^n) = c_k(\mathbb{R}^n)$ for every $k \leq n$?

Theorem 5 gives an upper and lower bounds for the numbers $c_k(\mathbb{Z}^k)$ that have exponential and polynomial growths, respectively.

Problem 3. Is the growth of the sequence $(c_n(\mathbb{Z}^n))_{n \in \mathbb{N}}$ exponential?

By [1], for every $k \in \{1, 2, 3\}$ any $k$-centerpole subset $C \subset \mathbb{Z}^k$ of cardinality $|C| = c_k(\mathbb{Z}^k)$ is affinely equivalent to the $(\mathbb{Z}^{k-1})$-sandwich $\Xi_{k-1}$.

Problem 4. Is each 12-element 4-centerpole subset of $\mathbb{Z}^4$ affinely equivalent to the $(\mathbb{Z}^3)$-sandwich $\Xi_3$?

It follows from the proof of Theorem 1 in [5] that the free group $F_2$ with two generators and discrete topology has $c_2(F_2) \leq 13$.

Problem 5. What is the value of the cardinal $c_2(F_2)$? Is $c_3(F_2)$ finite?

The last problem can be posed in a more general context.

Problem 6. Investigate the cardinal characteristics $c_k(G)$ and $c_k^B(G)$ for non-commutative topological groups $G$.

REFERENCES

[1] T. Banakh, On a cardinal group invariant related to partition of abelian groups, Mat. Zametki, 64:3, 341–350 (1998).
[2] T. Banakh, B. Bokalo, I. Guran, T. Radul, M. Zarichnyi, Problems from the Lviv topological seminar, in: E. Pearl (ed.), Open Problems in Topology, II, pp.655–667, Elsevier (2007).
[3] T. Banakh, A. Dudko, R. Repovs, Symmetric monochromatic subsets in colorings of the Lobachevsky plane, Discrete Math. Theor. Comput. Sci. 12:1, 12–20 (2010).
[4] T. Banakh, I. Protasov, Asymmetric partitions of Abelian groups, Mat. Zametki 66:1, 17–30 (1999).
[5] T. Banakh, I. Protasov, Symmetry and colorings: some results and open problems, Izv. Gosel Univ. Voprosy Algebry, Issue 4(17), 5–16 (2001).
[6] T. Banakh, O. Verbitski, Ya. Vorobets, Ramsey treatment of symmetry, Electron. J. Combin. 7:1, R52, 25 p. (2000).
[7] R. Engelking, General topology. Sigma Series in Pure Mathematics, 06. Heldermann Verlag, Berlin (1989).
[8] Yu. Gryshko, Monochromatic symmetry subsets in 2-colorings of groups, Electron. J. Combin. 10, R28, 8 p. (2003).
[9] W. Just, M. Weese, Discovering modern set theory. II. Set-theoretic tools for every mathematician. GSM, 18. Amer. Math. Soc., Providence, RI (1997).
[10] S. Morris, Pontryagin duality and the structure of locally compact abelian groups. London Mathematical Society Lecture Note Series, No. 29. Cambridge University Press, Cambridge-New York-Melbourne (1977).
[11] I.V. Protasov, Asymmetrically decomposable abelian groups, Mat. Zametki 59:3, 468–471 (1996).
[12] E.H. Spanier, Algebraic Topology., McGraw-Hill Book Co., New York (1966).