Non-proper value set and the Jacobian condition

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Abstract

The non-proper value set of a nonsingular polynomial map from $\mathbb{C}^2$ into itself, if non-empty, must be a curve with one point at infinity.

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1. Let $f = (P, Q) : \mathbb{C}^2_{(x,y)} \rightarrow \mathbb{C}^2_{(u,v)}$ be a dominant polynomial map, $P, Q \in \mathbb{C}[x, y]$ and denote $J(P, Q) := P_x Q_y - P_y Q_x$. Recall that the so-called non-proper value set $A_f$ of $f$ is the set consists of all point $a \in \mathbb{C}^2$ such that the inverse $f^{-1}(K)$ is not compact for compact neighbourhoods $K \subset \mathbb{C}^2$ of $a$. This set $A_f$, if non-empty, must be a plane curve that each of it’s irreducible components can be parameterized by a non-constant polynomial map from $\mathbb{C}$ into $\mathbb{C}^2$ (See [J]). The mysterious Jacobian conjecture (See [BMW] and [E]), posed first by Keller in 1939 and still open, asserts that a polynomial map $f = (P, Q)$ of $\mathbb{C}^2$ with $J(P, Q) \equiv const. \neq 0$ must have a polynomial inverse. This conjecture can be reduced to prove that the non-proper value set $A_f$ is empty. In any way one may think that in a counterexample to the Jacobian conjecture, if exists, the non-proper value set must has a very special configure. Such knowledges may be useful in pursuit of this conjecture. In [C] it was observed that the irreducible components of $A_f$ in such a counterexample can be parameterized by polynomial maps $\xi \mapsto (p(\xi), q(\xi))$ with $\deg p/\deg q = \deg P/\deg Q$. In this paper we would like to notice that the non-proper value set of a nonsingular polynomial map from $\mathbb{C}^2$ into itself, if non-empty, must be a curve with one point at infinity.

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Theorem 1. Suppose \( f = (P, Q) \) is a polynomial map of \( \mathbb{C}^2 \) with \( J(P, Q) \equiv \text{const.} \neq 0 \), \( \deg P = Kd \) and \( \deg Q = Ke \), \( \gcd(d, e) = 1 \), \( P \) and \( Q \) are monic in \( y \),

\[
P(x, y) = Ay^{Kd} + \ldots + a_1(x)y + a_0(x), \quad A \neq 0
\]

\[
Q(x, y) = By^{Ke} + \ldots + b_1(x)y + b_0(x), \quad B \neq 0.
\]

If the non-proper value set \( A_f \) is not empty, then every irreducible component of \( A_f \) can be parameterized by polynomial maps of the form

\[
\xi \mapsto (A\xi^{md} + \text{lower terms in } \xi, B\xi^{me} + \text{lower terms in } \xi), \quad m \in \mathbb{N}.
\]

By definition \( A_f \) is the set of all value \( a \in \mathbb{C}^2 \) such that the number of solutions counted with multiplicities of the equation \( f(x, y) = a \) is different from those for generic values in \( \mathbb{C}^2 \). Then, considering the components \( P(x, y) \) and \( Q(x, y) \) as elements of \( \mathbb{C}[x][y] \) we can define the resultant

\[
\text{Res}_y(P - u, Q - v) = R_0(u, v)x^N + \ldots + R_N(u, v),
\]

where \( R_i \in \mathbb{C}[u, v] \). \( R_0 \neq 0 \). From the basic properties of the resultant function we know that \( N \) is the geometric degree of \( f \) and \( A_f = \{(u, v) \in \mathbb{C}^2 : R_0(u, v) = 0\} \). Note that a curve given by a polynomial parameter of the form (1) can be defined by a polynomial of the form \( (A^e u^e - B^d v^d)^m + \sum_{0 \leq id+je < Mde} c_{ij}u^iv^j \) and its branch at infinity has a Newton-Puiseux series of the form \( u = cv^d + \text{lower terms in } v \), where \( c \) is a \( d- \) radicals of \( B^d/A^e \).

Thus, Theorem 1 leads to

Corollary 1. Let \( f \) be as in Theorem 1. Then

\[
R_0(u, v) = C(A^e u^e - B^d v^d)^M + \sum_{0 \leq id+je < Mde} c_{ij}u^iv^j
\]

with \( 0 \neq C \in \mathbb{C} \) and \( M \geq 0 \).

Corollary 2. Let \( f \) be as in Theorem 1. If \( A_f \neq \emptyset \), then \( A_f \) is a curve with one point at infinity and the irreducible branches at infinity of \( A_f \) have Newton-Puiseux series of the form

\[
u = cv^d + \text{lower terms in } v
\]

with coefficients \( c \) to be \( d- \) radicals of \( B^d/A^e \).

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As seen later, the monic representation in (1) of $P$ and $Q$ is only used to visualize the coefficient $B^d/A^e$. In fact, when $A_f \neq \emptyset$ the numbers $d$, $e$, $B^d/A^e$ and the polynomial $R_0(u,v)$ are invariant of $f$ under right actions of automorphisms of $\mathbb{C}^2$, since the set $A_f$ does not depend on the coordinate $(x,y)$. Furthermore, the coefficient $B^d/A^e$ is uniquely determined in the relation

$$P_+(x,y) = (B^d/A^e)Q_+(x,y),$$

which is dominated by the Jacobian condition when $\deg P > 1$ and $\deg Q > 1$. Here, $P_+$ and $Q_+$ are leading homogenous components of $P$ and $Q$, respectively.

Theorem 1 will be proved in sections 2 - 5 by an elementary way using Newton-Puiseux expansions and the Newton Theorem. It is worth to determine the form of $R_0(u,v)$ by examining directly the resultant function $\text{Res}_y(P - u, Q - v)$.

2. Dicritical series of $f$. In order to prove Theorem 1, we need to setup our hypothesis. From now on, $f = (P,Q) : \mathbb{C}_2^2 \rightarrow \mathbb{C}_2^2(x,y)$ is a given polynomial map with $J(P,Q) \equiv \text{const.} \neq 0$, $\deg P = Kd > 0$ and $\deg Q = Ke > 0$, $\gcd(d,e) = 1$. The Jacobian condition will be used really in Lemma 3 and the proof of Theorem 1. Since $A_f$ does not depend on the coordinate $(x,y)$, to examine it we can assume that $P$ and $Q$ are monic in $y$,

$$P(x,y) = Ay^{Kd} + \ldots + a_1(x)y + a_0(x), \ A \neq 0$$
$$Q(x,y) = By^{Ke} + \ldots + b_1(x)y + b_0(x), \ B \neq 0. \quad (5)$$

With this representation the Newton-Puiseux roots at infinity $y(x)$ of each equations $P(x,y) = 0$ and $Q(x,y) = 0$ are fractional power series of the form

$$y(x) = \sum_{k=0}^{\infty} c_k x^{1 - \frac{k}{m}}, \ m \in \mathbb{N}, \ \gcd\{k : c_k \neq 0\} = 1,$$

for which the map $\tau \mapsto (\tau^m, y(\tau^m))$ is meromorphic and injective for $\tau$ larger enough. In view of the Newton theorem we can represent

$$P(x,y) = A \prod_{i=1}^{\deg P} (y - u_i(x)), \ Q(x,y) = B \prod_{j=1}^{\deg Q} (y - v_i(x)), \quad (6)$$

where $u_i(x)$ and $v_j(x)$ are the Newton-Puiseux roots at infinity of the equations $P = 0$ and $Q = 0$, respectively. We refer the readers to [A] and [BK] for the Newton theorem and the Newton-Puiseux roots.
We begin with the description of the non-proper value set $A_f$ of $f$ via Newton-Puiseux expansion. We will work with finite fractional power series $\varphi(x, \xi)$ of the form

$$\varphi(x, \xi) = \sum_{k=1}^{K-1} a_k x^{1 - \frac{k}{m}} + \xi x^{1 - \frac{K}{m}}, m \in \mathbb{N}, \text{gcd}\{k : a_k \neq 0\} = 1,$$

(7)

where $\xi$ is a parameter. For convenience, we denote $\text{mult}(\varphi) := m$. A such series is called dicritical series of $f$ if

$$f(x, \varphi(x, \xi)) = f_\varphi(\xi) + \text{lower terms in } x, \quad \deg f_\varphi > 0.$$

The following description of $A_f$ was presented in [C].

Lemma 1. (Lemma 4, [C])

$$A_f = \bigcup_{\varphi \text{ is a dicritical series of } f} f_\varphi(C).$$

To see it, note that by definitions the non-proper value set $A_f$ consists of all values $a \in \mathbb{C}^2$ such that there exists a sequence $\mathbb{C}^2 \ni p_i \to \infty$ with $f(p_i) \to a$. If $\varphi$ is a dicritical series of $f$ of the form (7), we can define the map $\Phi(t, \xi) := (t^{-m}, \varphi(t^{-m}, \xi))$. Then, $\Phi$ sends $\mathbb{C}^* \times \mathbb{C}$ to $\mathbb{C}^2$ and the line $\{0\} \times \mathbb{C}$ to the line at infinity of $\mathbb{C} \mathbb{P}^2$. The polynomial map $F_\varphi(t, \xi) := f \circ \Phi(t, \xi)$ sends the line $\{0\} \times \mathbb{C}$ into $A_f \subset \mathbb{C}^2$. Therefore, $f_\varphi(C)$ is an irreducible component of $A_f$, since $\deg f_\varphi > 0$. Conversely, if $\ell$ is an irreducible component of $A_f$, one can choose a smooth point $(u_0, v_0)$ of $A_f$, $(u_0, v_0) \in \ell$, and an irreducible branch at infinity $\gamma$ of the curve $P = u_0$ (or the curve $Q = v_0$) such that the image $f(\gamma)$ is a branch curve intersecting transversally $\ell$ at $(u_0, v_0)$. Let $u(x)$ be a Newton-Puiseux expansion at infinity of $\gamma$. Then, we can construct an unique dicritical series $\varphi(x, \xi)$ such that $u(x) = \varphi(x, \xi_0 + \text{lower term in } x)$. For this dicritical series $\varphi$ we have $f_\varphi(C) = \ell$.

4. Associated sequence of dicritical series. Let $\varphi$ be a given dicritical series of $f$. Let us to represent

$$\varphi(x, \xi) = \sum_{k=0}^{K-1} c_k x^{1 - \frac{nk}{m_k}} + \xi x^{1 - \frac{K}{m_K}},$$

(8)
where $0 \leq \frac{m_0}{m_0} < \frac{m_1}{m_1} < \ldots < \frac{m_{K-1}}{m_{K-1}} < \frac{m_K}{m_K} = \frac{n_0}{m_0}$ and $c_i \in \mathbb{C}$ may be the zero, so that the sequence of series $\{\varphi_i\}_{i=0,1,\ldots,K}$ defined by

$$
\varphi_i(x,\xi) := \sum_{k=0}^{i-1} c_k x^{1-\frac{n_k}{m_k}} + \xi x^{1-\frac{n_i}{m_i}}, i = 0, 1, \ldots, K - 1,
$$

(9)

and $\varphi_K := \varphi$ satisfies the following properties:

S1) $\text{mult}(\varphi_i) = m_i$.

S2) For every $i < K$ at least one of polynomials $p_{\varphi_i}$ and $q_{\varphi_i}$ has a zero point different from the zero.

S3) For every $\psi(x,\xi) = \varphi_i(x,c_i) + \xi x^{1-\alpha}, \frac{m_i}{m_i} < \alpha < \frac{m_{i+1}}{m_{i+1}}$, each of the polynomials $p_\psi$ and $q_\psi$ is either constant or a monomial of $\xi$.

The representation (8) of $\varphi$ is thus the longest representation such that for each index $i$ there is a Newton-Puiseux root $y(x)$ of $P = 0$ or $Q = 0$ such that $y(x) = \varphi_i(x,c + \text{lower terms in } x), c \neq 0$ if $c_i = 0$. This representation and the associated sequence $\{\varphi_i\}_{i=0,1,\ldots,K}$ are well defined and unique. Further, $\varphi_0(x,\xi) = \xi x$.

We will use the associated sequence $\{\varphi_i\}$ to determine the form of the polynomials $f_\varphi(\xi)$. For simplicity in notations, in below we shall use lower indeces “$i$” instead of the lower indeces “$\varphi_i$”.

For each associated series $\varphi_i, i = 0,\ldots,K$, let us represent

$$
P(x,\varphi_i(x,\xi)) = p_i(\xi)x^{\frac{a_i}{m_i}} + \text{lower terms in } x
$$

$$
Q(x,\varphi_i(x,\xi)) = q_i(\xi)x^{\frac{b_i}{m_i}} + \text{lower terms in } x,
$$

(10)

where $p_i, q_i \in \mathbb{C}[\xi] - \{0\}, a_i, b_i \in \mathbb{Z}$ and $m_i := \text{mult}(\varphi_i)$.

Let $\{u_i(x), i = 1,\ldots,\deg P\}$ and $\{v_j(x), j = 1,\ldots,\deg Q\}$ be the collections of the Newton-Puiseux roots of $P = 0$ and $Q = 0$, respectively. As shown in Section 2, by the Newton theorem the polynomials $P(x,y)$ and $Q(x,y)$ can be factorized in the form

$$
P(x,y) = A \prod_{i=1}^{\deg P} (y - u_i(x)), Q(x,y) = B \prod_{j=1}^{\deg Q} (y - v_j(x)).
$$

(11)

For each $i = 0,\ldots,K$, let us define

$- \quad S_i := \{k : 1 \leq k \leq \deg P : u_k(x) = \varphi_i(x,a_{ik} + \text{lower terms in } x)a_{ik} \in \mathbb{C}\}$;
- $T_i := \{ k : 1 \leq k \leq \deg Q : v_k(x) = \varphi_i(x, b_{ik} + \text{lower terms in } x), b_{ik} \in \mathbb{C} \};$
- $S_i^0 := \{ k \in S_i : a_{ik} = c_i \};$
- $T_i^0 := \{ k \in T_i : b_{ik} = c_i \}.$

Represent

\[ p_i(\xi) = A_i \bar{p}_i(\xi)(\xi - c_i)^\#_{S_i^0}, \bar{p}_i(\xi) := \prod_{k \in S_i \setminus S_i^0} (\xi - a_{ik}), \]

and

\[ q_i(\xi) = B_i \bar{q}_i(\xi)(\xi - c_i)^\#_{T_i^0}, \bar{q}_i(\xi) := \prod_{k \in T_i \setminus T_i^0} (\xi - b_{ik}). \]

**Lemma 2.**

i) $n_0 = 0, m_0 = 1$ and

\[ A_0 = A, \deg p_0 = a_0 = Kd \]

\[ B_0 = B, \deg q_0 = b_0 = Ke. \]

ii) For $i = 1, \ldots, K$

\[ A_i = A_{i-1} \bar{p}_{i-1}(c_{i-1}), \deg p_i = \# S_i = \# S_i^0 \]

\[ \frac{a_i}{m_i} = \frac{a_{i-1}}{m_{i-1}} + \# S_{i-1}^0 \left( \frac{n_{i-1}}{m_{i-1}} - \frac{n_i}{m_i} \right) \]

\[ B_i = B_{i-1} \bar{q}_{i-1}(c_{i-1}), \deg q_i = \# T_i = \# T_i^0, \]

\[ \frac{b_i}{m_i} = \frac{b_{i-1}}{m_{i-1}} + \# T_{i-1}^0 \left( \frac{n_{i-1}}{m_{i-1}} - \frac{n_i}{m_i} \right). \]

**Proof.** Note that $\varphi_0(x, \xi) = \xi x$ and $\varphi_i(x, \xi) = \varphi_{i-1}(x, c_{i-1}) + \xi x^1 - \frac{n_i}{m_i}$ for $i > 0$. Then, substituting $y = \varphi_i(x, \xi), i = 0, 1, \ldots, K$, into the Newton factorizations of $P(x, y)$ and $Q(x, y)$ in (11) one can easy verify the conclusions.

**4. The Jacobian condition.** Let $\varphi$ be a dicritical series of $f$ and $\{ \varphi_i \}$ be it’s associated series. Denote

\[ J_i(\xi) := a_i \dot{p}_i(\xi) \dot{q}_i(\xi) - b_i \ddot{p}_i(\xi) q_i(\xi). \]

The Jacobian condition will be considered in the following meaning.
Lemma 3: Let $0 \leq i < K$. If $a_i > 0$ and $b_i > 0$, then

$$J_i(\xi) \equiv \begin{cases} -m_i J(P, Q) & \text{if } a_i + b_i = 2m_i - n_i, \\ 0 & \text{if } a_i + b_i > 2m_i - n_i. \end{cases}$$

Further, $J_i(\xi) \equiv 0$ if and only if $p_i(\xi)$ and $q_i(\xi)$ have a common zero point. In this case

$$p_i(\xi)^{b_i} = Cq_i(\xi)^{a_i}, \ C \in \mathbb{C}^*.$$

Proof. Since $a_i > 0$ and $b_i > 0$, taking differentiation of $Df(t^{-m_i}, \varphi_i(t^{-m_i}, \xi))$, we have that

$$m_i J(P, Q)t^{n_i-2m_i-1} + \text{higher terms in } t = -J_i(\xi)t^{-a_i-b_i-1} + \text{higher terms in } t.$$

Comparing two sides of it we can get the first conclusion. The remains are left to the readers as an elementary exercise. 

5. Proof of Theorem 1.

i) Assume that $A_f \neq \emptyset$. Then, $A_f$ is a plane curve in $\mathbb{C}^2$. Let $\ell$ be an irreducible component of $A_f$. By Lemma 1 there is a dicritical series $\varphi$ of $f$ such that $\ell$ can be parameterized by the polynomial map $f_\varphi(\xi) = (p_\varphi(\xi), q_\varphi(\xi))$, i.e. $\ell = f_\varphi(\mathbb{C})$. We will show that

$$f_\varphi(\xi) = (AC^{d\xi}_\varphi D^{d\xi}_\varphi + \ldots, BC^{e\xi}_\varphi D^{e\xi}_\varphi + \ldots), \ C_\varphi \neq 0, \ D_\varphi \in \mathbb{N}. \quad (12)$$

Then, by changing variable $\xi \mapsto C^{-1}_\varphi \xi$ we get the desired parameterization $\xi \mapsto (A\xi^{D\xi_\varphi} + \ldots, B\xi^{D\xi_\varphi} + \ldots)$ of $\ell$.

ii) Consider the associated sequence $\{\varphi_i\}_{i=1}^K$ of $\varphi$. Since $A_f \neq \emptyset$ as assumed,

$$\deg P > 1, \deg Q > 1.$$

Otherwise, $f$ is bijective and $A_f = \emptyset$. Since $\varphi$ is a dicritical series of $f$, without loss of generality we can assume that

$$\deg p_K > 0, \ a_K = 0 \text{ and } b_K \leq 0.$$

Then, from the constructing of the sequence $\varphi_i$ it follows that

$$\begin{cases} p_i(c_i) = 0 \text{ and } a_i > 0, & i = 0, 1, \ldots, K - 1 \\ q_i(c_i) = 0 & \text{if } b_i > 0 \end{cases} \quad (13)$$
This allows us to use the Jacobian condition in the meaning of Lemma 3. Then, by induction using Lemma 2, Lemma 3 and (13) we can obtain without difficult the following.

**Assertion:** For \( i = 0, 1, \ldots, K - 1 \) we have

\[
a_i > 0, b_i > 0, \quad (a)
\]

\[
\frac{a_i}{b_i} = \frac{\#S_i}{\#T_i} = \frac{d}{e} \quad (b)
\]

and

\[
\frac{\#S_i^0}{\#T_i^0} = \frac{d}{e}, \quad \bar{p}_i(\xi)^e = \bar{q}_i(\xi)^d. \quad (c)
\]

iii) Now, we prove (12). By Lemma 2 (iii) and (b-c) we have

\[
\frac{b}{m_K} = \frac{b_{K-1}}{m_{K-1}} + \frac{T_{K-1}^0(n_{K-1} - n_K)}{m_{K-1}}
\]

\[
= \frac{d}{e} \left[ \bar{p}_{K-1}(c_{K-1}) + \#S_{K-1}^0 \right]
\]

\[
= \frac{d}{e} \frac{a_K}{m_K} = 0,
\]

as \( a_K = 0 \). Hence, \( f_\varphi(\xi) = (p_K(\xi), q_K(\xi)) \) by definition and (a). Using Lemma 2 (ii-iii) to compute the coefficient \( A_K \) and \( B_K \) we can get

\[
A_K = A\left( \prod_{k \leq K-1} \bar{p}_k(c_k) \right), \quad B_K = B\left( \prod_{k \leq K-1} \bar{q}_k(c_k) \right).
\]

Let \( C_\varphi \) be a \( d \)-radical of \( (\prod_{k \leq K-1} \bar{p}_k(c_k)) \) and \( D_\varphi := \gcd(\#S_{K-1}^0, \#T_{K-1}^0) \). Then, by Lemma 2 (ii) and (b-c) we have that \( A_K = AC_\varphi^{d}, \quad B_K = BC_\varphi^e, \)

\[ \deg p_K = \#S_{K-1}^0 = D_\varphi d \]

\[ \deg q_K = \#T_{K-1}^0 = D_\varphi e. \]

Thus,

\[
f_\varphi(\xi) = (AC_\varphi^{d} D_\varphi^{d} + \ldots, BC_\varphi^{e} D_\varphi^{e} + \ldots).
\]

\[ \blacksquare \]

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