The $\Delta S = 1$ Effective Hamiltonian Including Next-to-Leading Order QCD and QED Corrections

M. Ciuchini$^{a,b}$, E. Franco$^b$, G. Martinelli$^{b,c}$ and L. Reina$^d$

$^a$ INFN, Sezione Sanità, V.le Regina Elena 299, 00161 Roma, Italy.
$^b$ Dip. di Fisica, Università degli Studi di Roma “La Sapienza” and INFN, Sezione di Roma, P.le A. Moro 2, 00185 Roma, Italy.
$^c$ Laboratoire de Physique Théorique de l’École Normale Supérieure
24 rue Lhomond, 75231 Paris CEDEX 05, France.
$^d$ Service de Physique Théorique, Université Libre de Bruxelles, Boulevard du Triomphe, CP 225 B-1050 Brussels, Belgium.

Abstract

In this paper we present a calculation of the $\Delta S = 1$ effective weak Hamiltonian including next-to-leading order QCD and QED corrections. At a scale $\mu$ of the order of few GeV, the Wilson coefficients of the operators are given in terms of the renormalization group evolution matrix and of the coefficients computed at a large scale $\sim M_W$. The expression of the evolution matrix is derived from the two-loop anomalous dimension matrix which governs the mixing of the relevant current-current and penguin operators, renormalized in some given regularization scheme. We have computed the anomalous dimension matrix up to and including order $\alpha_s^2$ and $\alpha_e\alpha_s$ in two different renormalization schemes, NDR and HV, with consistent results. We give many details on the calculation of the anomalous dimension matrix at two loops, on the determination of the Wilson coefficients at the scale $M_W$ and of their evolution from $M_W$ to $\mu$. We also discuss the dependence of the Wilson coefficients/operators on the regularization scheme.
1 Introduction

In this paper we present a calculation of the two-loop anomalous dimension matrix relevant for $\Delta S = 1$ decays. The anomalous dimension matrix includes leading and sub-leading corrections at orders $\alpha_s$, $\alpha_s^2$, $\alpha_e$ and $\alpha_s \alpha_e$. The calculation has been performed in two different regularization schemes: naive dimensional regularization (NDR) and ’t Hooft-Veltman regularization (HV)\cite{1}. We verify that the results obtained in the two schemes are compatible both in the strong and electro-magnetic case. We give many details on the calculation itself, on the definition of the renormalized operators, on the relation between different regularization schemes, on the role of counter-terms for operators which vanish by the equations of motion, on the gauge invariance of the final result etc. We also report a list containing the double and single pole contribution of all the diagrams, counter-terms and “effervescent” operator counter-terms in both the schemes used in this calculation. The list of all the diagrams may be useful to check our results and for further applications.

A comparison of our results with a parallel calculation reported in refs.\cite{2}-\cite{4} is also presented. We agree with the authors of these references in the NDR scheme. For some diagrams computed in the HV regularization scheme we however disagree and we explain the origin of the difference. In the electromagnetic case we show that, by using the values of ref.\cite{3} for the diagrams computed in HV, it is not possible to satisfy the expected relation between the two-loop anomalous dimension matrix computed in NDR and HV. On the contrary our results satisfy the expected relations for both terms of order $\alpha_s^2$ and $\alpha_s \alpha_e$. We have also checked the consistency of our calculation at order $\alpha_e^2$.

The authors of ref.\cite{4} have only computed the anomalous dimension in NDR and derived the result in HV by using the one-loop anomalous dimension and coefficient matrix. We thus agree with their final result for the two-loop anomalous dimension matrix in HV too, in spite of the different results for some diagrams between this work and ref.\cite{3}.

In this paper we have preferred to report only the calculation of the two-loop anomalous dimension matrix with as many details as possible and postpone a discussion of the numerical calculation and uncertainties for the
The paper is organized as follows. In sec. 2 we introduce the general formalism. The calculation of the coefficient functions at the \( W \) scale is summarized in sec. 3. A detailed discussion of the scheme dependence of the anomalous dimension matrix and of the relation between different renormalization schemes is given in sec. 4. We also present a convenient definition of the renormalized operators which makes the evolution matrix scheme independent. This definition may be useful to predict weak amplitudes by combining the Wilson coefficients with the matrix elements of the corresponding operators computed with a non-perturbative method, as for example lattice QCD \([5, 7]\). In the same section we also recall some basic features of the HV and NDR regularizations. Sec. 5 is the main section of this paper. There we describe the calculation of one- and two-loop diagrams, discuss the role of the so called “effervescent” operators and comment on the subtraction of counter-terms corresponding to operators which vanish by the equations of motion. Double and single pole contributions of all the relevant Feynman diagrams, in the NDR and HV regularizations are given in the Appendix. From the calculation of the Feynman diagrams, the two-loop anomalous dimension matrix is derived and given in the NDR and HV schemes in sec. 6.

2 General Formalism

Effective Hamiltonians for non-leptonic decays of hadrons composed by light quarks (\( K, D \) and \( B \) mesons for example) are defined by Wilson operator expansions of products of weak currents \([8-9]\):

\[
\langle F | H_{\text{eff}} | I \rangle = g_W^2/8 \int d^4 x D_W(x^2, M_W^2) \langle F | T \left( J_{\mu}(x), J_{\mu}^\dagger(0) \right) | I \rangle \\
\to \sum_i \langle F | Q_i(\mu) | I \rangle C_i(\mu)
\]

\( g_W \)

For kaon decays, in the limit in which we neglect quark masses, only four-quark operators appear on the r.h.s. of eq. (1). The \( \Delta S = 1 \) effective hamil-

\[2 \text{ Dependence of the coefficient functions on the renormalization scale } \mu, \text{ on } \Lambda_{QCD} \text{ and on the renormalization prescription for example.} \]
tonian can then be written as:

$$
\mathcal{H}_{eff}^{\Delta S = 1} = \lambda_u \frac{G_F}{\sqrt{2}} \left[ (1 - \tau) (C_1(\mu) (Q_1(\mu) - Q_1^c(\mu)) + C_2(\mu) (Q_2(\mu) - Q_2^c(\mu)) \right] + \tau \sum_{i=1} \lambda_i C_i(\mu) 
$$

(2)

where \( \lambda_u = V_{ud}V_{us}^* \) and similarly we can define \( \lambda_c \) and \( \lambda_t \). \( \tau = -\lambda_t/\lambda_u \) and \( V_{ij} \) is one of the elements of the CKM\[10, 11\] mixing matrix. The operator basis is given by:

- \( Q_1 = (\bar{s}_\alpha u_\beta)(V - A)(\bar{u}_\beta d_\alpha)(V - A) \)
- \( Q_2 = (\bar{s}_\alpha u_\alpha)(V - A)(\bar{u}_\beta d_\beta)(V - A) \)
- \( Q_{3,5} = (\bar{s}_\alpha d_\alpha)(V - A) \sum_{q=u,d,s,\cdots} (\bar{q}_\beta q_\beta)(V \mp A) \)
- \( Q_{4,6} = (\bar{s}_\alpha d_\beta)(V - A) \sum_{q=u,d,s,\cdots} (\bar{q}_\beta q_\alpha)(V \mp A) \)
- \( Q_{7,9} = \frac{3}{2} (\bar{s}_\alpha d_\alpha)(V - A) \sum_{q=u,d,s,\cdots} e_q(\bar{q}_\beta q_\beta)(V \pm A) \)
- \( Q_{8,10} = \frac{3}{2} (\bar{s}_\alpha d_\beta)(V - A) \sum_{q=u,d,s,\cdots} e_q(\bar{q}_\beta q_\alpha)(V \pm A) \)
- \( Q_1^c = (\bar{s}_\alpha c_\beta)(V - A)(\bar{c}_\beta d_\alpha)(V - A) \)
- \( Q_2^c = (\bar{s}_\alpha c_\alpha)(V - A)(\bar{c}_\beta d_\beta)(V - A) \)

(3)

when QCD and QED corrections are taken into account\[8, 9, 12\]-\[19\]. In (3) the subscript \((V \pm A)\) indicates the chiral structure and \( \alpha \) and \( \beta \) are colour indices. The sum is intended over those flavours which are active at the scale \( \mu \). We have completely ignored the effects due to the operators \( Q_{10} \) and \( Q_{12} \) which are the analog of \( Q_1^c \) and \( Q_2^c \) with the charm quark replaced by the bottom quark. In ref.\[12\] it was indeed shown that these operators have a negligible effect on the evolution of the Wilson coefficients of the operators (3). Their inclusion, once that the anomalous dimension matrix is known, is in any case elementary. For \( m_c < \mu < m_b \) we have used the relation:

$$
Q_{10} = Q_9 + Q_4 - Q_3
$$

(4)

to eliminate \( Q_{10} \) from the evolution equations. We think that expression (4) is the most transparent for \( \mu > m_c \). It shows that we can find all the Wilson
coefficients by evolving $C_{1,10}$ ($C_{1,9}$) via a $10 \times 10$ ($9 \times 9$) evolution matrix down to $\mu = m_b$ ($m_c < \mu < m_b$). The generalization to $\mu < m_c$ is straightforward and can be found, for example, in a recent paper on $\epsilon'/\epsilon$ [13].

The operators $Q_i(\mu)$ are renormalized at the scale $\mu$ in some given scheme. The corresponding coefficients $C_i(\mu)$ are scheme dependent. The dependence on the regularization scheme appears at one loop, when we express the original current-current product in terms of the Wilson operator product expansion (OPE), see eq.(I).

To obtain the coefficients $C_i(\mu)$ at next-to-leading order (NLO) two steps are necessary:

1) The calculation of the coefficients at a given scale, for example $M_W$ or $m_t$, including corrections of order $\alpha_s$ and $\alpha_e$.

2) The calculation of the two-loop anomalous dimension up to $O(\alpha_s^2)$ and $O(\alpha_s \alpha_e)$.

The results of steps 1) and 2) depend on the regularization scheme and on the normalization conditions imposed on the renormalized operators, as will be discussed below. We have done our calculations in two popular regularization schemes, i.e. the t’Hooft-Veltman (HV) and the naive (NDR) dimensional regularization schemes. In both cases we have obtained the renormalized operators via the standard modified minimal subtraction procedure $\overline{MS}$. We will also discuss other renormalization prescriptions which can make the renormalization group evolution matrix scheme independent.

In presence of $\gamma_5$ and in certain regularization schemes, the axial vector current may develop an anomalous dimension at the two-loop level. In defining the evolution matrix one has to take into account this effect. Alternatively one can impose to the current a certain one-loop renormalization condition such that its two-loop anomalous dimension is zero. We prefer this second solution and discuss this point in sec.3.

To make easier a comparison with previous calculations on the same subject, [2]-[4] and [18], we follow as close as possible the notation introduced in ref.[2] and write:

$$H_{\Delta S=1}^{\text{eff}} \sim \tilde{Q}^T(\mu) \tilde{C}(\mu)$$

$^{3}$ Through this paper we neglect terms of order $\alpha_s^2$.  

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where $\vec{Q}^T(\mu)$ is a row vector whose components are the operators $Q_{1,10}$ of the basis (3) and $\vec{C}(\mu)$ is a column vector, whose components are the corresponding Wilson coefficients. $\vec{C}(\mu)$ are expressed in terms of $\vec{C}(M_W)$ through the renormalization group evolution matrix $\hat{W}[\mu, M_W]$:

$$\vec{C}(\mu) = \hat{W}[\mu, M_W]\vec{C}(M_W)$$

(6)

The coefficients $\vec{C}(\mu)$ obey the renormalization group equations:

$$\left(-\frac{\partial}{\partial t} + \beta(\alpha_s)\frac{\partial}{\partial \alpha_s} - \frac{\hat{\gamma}(\alpha_s, \alpha_e)}{2}\right)\vec{C}(t, \alpha_s(t), \alpha_e) = 0$$

(7)

where $t = \ln(M_W^2/\mu^2)$ and we ignore the running of $\alpha_e$. The factor of 2 in eq.(7) normalizes the anomalous dimension matrix as in refs.[2]-[4].

$$\hat{\gamma} = \hat{\gamma}_Q - 2\gamma J \hat{1}$$

(8)

is the anomalous dimension matrix of the operators minus twice the anomalous dimension of the weak current in a given renormalization scheme. In eq.(8) $\hat{1}$ is the identity matrix.

To simplify the discussion, we first consider the case where there is no crossing of a quark threshold when going from $M_W$ to $\mu$. The relevant formulae for the general case will be given at the end of this section. At the next-to-leading order, by expanding $\hat{W}[\mu, M_W]$, we can write:

$$\hat{W}[\mu, M_W] = \hat{M}[\mu]\hat{U}[\mu, M_W]\hat{M}'[M_W]$$

(9)

with:

$$\hat{M}[\mu] = \left(\hat{1} + \frac{\alpha_e}{4\pi}\hat{K}\right)\left(\hat{1} + \frac{\alpha_s(\mu)}{4\pi}\hat{J}\right)\left(\hat{1} + \frac{\alpha_e}{\alpha_s(\mu)}\hat{P}\right)$$

(10)

and

$$\hat{M}'[M_W] = \left(\hat{1} - \frac{\alpha_e}{\alpha_s(M_W)}\hat{P}\right)\left(\hat{1} - \frac{\alpha_s(M_W)}{4\pi}\hat{J}\right)\left(\hat{1} - \frac{\alpha_e}{4\pi}\hat{K}\right)$$

(11)

We substitute the expression of $\vec{C}(\mu)$ given in eq.(3) in the renormalization group equations (3), using $\hat{W}[\mu, M_W]$ written as in eqs.(9-11). By expanding the anomalous dimension matrix, which includes gluon and photon corrections, up to order $\alpha_s^2$ and $\alpha_e\alpha_s$:

$$\hat{\gamma} = \frac{\alpha_s}{4\pi}\hat{\gamma}_s^{(0)} + \frac{\alpha_e}{4\pi}\hat{\gamma}_e^{(0)} + \left(\frac{\alpha_s}{4\pi}\right)^2\hat{\gamma}_s^{(1)} + \frac{\alpha_s\alpha_e}{4\pi}\hat{\gamma}_e^{(1)}$$

(12)
we obtain the expression for $\hat{U}$:

$$
\hat{U}[\mu, M_W] = T_{\alpha_s} \exp\left(- \int_{\alpha_s(M_W)}^{\alpha_s(\mu)} \frac{d \alpha_s}{\alpha_s} \frac{\hat{\gamma}_s(0)^T}{2\beta_0}\right)
\rightarrow \left[ \frac{\alpha_s(M_W)}{\alpha_s(\mu)} \right]^{\hat{\gamma}_s(0)^T/2\beta_0}
$$

(13)

in the basis where $\hat{\gamma}_s(0)$ is diagonal. $T_{\alpha_s}$ is the ordered product, with increasing couplings from right to left. The matrices $\hat{P}$, $\hat{J}$ and $\hat{K}$ are solutions of the equations:

$$
\hat{P} + \left[ \hat{P}, \frac{\hat{\gamma}_s(0)^T}{2\beta_0} \right] = \frac{\hat{\gamma}_s(0)^T}{2\beta_0}
$$

(14)

$$
\hat{J} - \left[ \hat{J}, \frac{\hat{\gamma}_s(0)^T}{2\beta_0} \right] = \frac{\beta_1}{2\beta_0^2} \hat{\gamma}_s(0)^T - \frac{\hat{\gamma}_e(1)^T}{2\beta_0}
$$

(15)

$$
\left[ \hat{K}, \hat{\gamma}_s(0)^T \right] = \hat{\gamma}_e(1)^T + \hat{\gamma}_e(0)^T \hat{J} + \hat{\gamma}_s(1)^T \hat{P} + \left[ \hat{\gamma}_s(0)^T, \hat{J} \hat{P} \right] - 2\beta_1 \dot{\hat{P}} - \frac{\beta_1}{\beta_0} \hat{P} \hat{\gamma}_s(0)^T
$$

(16)

In eqs. (13-16), $\beta_0$ and $\beta_1$ are the first two coefficients of the $\beta$-function of $\alpha_s$. $\hat{U}$ and $\hat{P}$ are determined by the leading logarithmic (LO) anomalous dimension matrices $\hat{\gamma}_s(0)$ and $\hat{\gamma}_e(0)$ and are regularization scheme independent. On the other hand the two-loop anomalous dimensions $\hat{\gamma}_s(1)$ and $\hat{\gamma}_e(1)$, and consequently $\hat{J}$, $\hat{K}$ and $\hat{W}[\mu, M_W]$, are regularization scheme dependent. Eqs. (14-16) can be easily solved in the basis where $\hat{\gamma}_s(0)$ is diagonal. The solutions develop singularities which however cancel in the final expression of $\hat{W}[\mu, M_W]$. It is indeed possible to find an explicit form of $\hat{W}[\mu, M_W]$ which is not singular. This form was used in the numerical calculation of ref. [6].

The initial conditions for the evolution equations, $\hat{C}(M_W)$ are obtained by matching the full theory, which includes propagating $W$, $Z^0$ and six quarks, to the effective theory, where the $W$, $Z^0$ and top quark have been removed simultaneously. In general, $\hat{C}(M_W)$ depend on the definition of the renormalized operators in a given regularization scheme. A scheme independent way of defining $\hat{C}(M_W)$ and the evolution matrix will be discussed in

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4 The last term in eq.(15) of ref. [6], corresponding to eq. (16) of this paper, contains a misprint. The expression which is reported here has however been used in all the numerical calculations of the Wilson coefficients of ref. [6], which are consequently correct. We thank A. Buras for finding the misprint.
In the full theory, all the coefficients coming from current-current and penguin operators have been computed in refs.\cite{12},\cite{20}-\cite{22}. In this work we have only computed, in the effective theory for the HV and NDR regularization schemes, the \(O(\alpha_s)\) and \(O(\alpha_e)\) one-loop corrections necessary to impose the matching conditions on \(\vec{C}(M_W)\).

We now discuss the modifications to the evolution matrix in presence of a heavy quark threshold. These modifications are necessary when \(\mu < m_b\). The extension to the case when \(\mu < m_c\) is straightforward. The matrix \(\hat{W}[\mu_1, \mu_2]\) depends on the number of active flavours \(f\) in the interval \([\mu_1, \mu_2]\). We denote it by \(\hat{W}_f[\mu_1, \mu_2]\). On the other hand when we cross a threshold, the renormalization conditions of the operators are in general changed\footnote{This however does not happen with scheme independent renormalization conditions.}. At threshold it is thus necessary to introduce a suitable matrix \(\hat{T}\)\footnote{The matrix \(\hat{T}\) is different for different thresholds, depending on the quark electric charge.}. This matrix allows for the matching of the evolution between scales larger and smaller than the threshold. Thus for example, to evolve the coefficients from \(M_W\) to \(\mu < m_b\), one has to use the following expression\cite{23}:

\[
\hat{W}[\mu, M_W] = \hat{W}_4[\mu, m_b] \hat{T} \hat{W}_5[m_b, M_W] \tag{17}
\]

where

\[
\hat{T} = \hat{1} + \frac{\alpha_s(m_b)}{4\pi} \delta \hat{r} + \frac{\alpha_e}{4\pi} \delta \hat{s} \tag{18}
\]

The matrices \(\delta \hat{r}\) and \(\delta \hat{s}\) relevant for eq.\(\tag{17}\) are given in sec.6.1 together with all other one-loop results.

### 3 Calculation of the Coefficients \(\vec{C}(M_W)\)

In order to compute the Wilson coefficients of the OPE at a scale \(\mu \sim M_W\) (\(m_t\)), we have to consider the full set of current-current, box and strong, electro-magnetic and \(Z^0\) penguin diagrams up to and including \(O(G_F\alpha_s)\) and \(O(G_F\alpha_e)\)\cite{12},\cite{20}-\cite{22}. In the current-current case we compute the diagrams with external momenta \(|p_i^2| \sim \mu_0^2 \ll M_W^2\) (i.e. we neglect terms of order \(\mu_0^2/M_W^2\)) with massless external quark states. The dependence on the...
external momenta only appears in logarithms, proportional to the anomalous dimension of the operators, and in the operator matrix elements. Strong and electro-magnetic penguin diagrams are also computed with external momenta $|p_i^2| \ll M_W^2$, $|p_i^2| \ll m_t^2$ and massless external states. In the case of penguin diagrams, the logarithmic dependence on the external states appears as a dependence on the momentum transferred through the gluon or photon propagators, $q^2 \sim \mu^2$. Z$^0$-penguin and box diagrams can be computed with zero external momenta and including only the top quark contribution, since they are infrared finite, cf. $B(x_t)$ and $C(x_t)$ in table I.

We now introduce the notation necessary for the calculation of the coefficients $\vec{C}(M_W)$. In the full theory, the direct calculation of the current-current, box and penguin diagrams at one loop (including order $\alpha_s$ and $\alpha_e$ corrections) has the form:

$$<JJ> \sim <\vec{Q}^{(0)}T> \left[ \vec{T}^{(0)} + \frac{\alpha_s}{4\pi} \vec{T}^{(1)} + \frac{\alpha_e}{4\pi} \vec{D}^{(1)} \right]$$

$$= <\vec{Q}^{(0)}(M_W)T> \vec{C}(M_W)$$

where $<\vec{Q}^{(0)}T>$ are the tree-level matrix elements and $\vec{T}^{(1)}$ and $\vec{D}^{(1)}$ depend on the regularization scheme and on the external quark states. Indeed all the $W$-$g$ and $W$-$\gamma$ box diagrams, being finite, are regularization scheme independent. The axial vector vertex diagram however does depend on the regularization scheme. The diagrams necessary to obtain $\vec{T}^{(0),(1)}$ and $\vec{D}^{(1)}$ are shown in figs.1,2 and 4-8. By inserting the renormalized operators of the effective hamiltonian (3) in the diagrams reported in figs.3 and 9, we then compute the one-loop current-current, strong and electromagnetic penguin diagrams between the same external states, using the same regularization scheme. In this case we obtain:

$$<\vec{Q}(M_W)> = \left( 1 + \frac{\alpha_s}{4\pi} \hat{r} + \frac{\alpha_e}{4\pi} \hat{s} \right) <\vec{Q}^{(0)}>$$

$\vec{T}^{(1)}$ and $\vec{D}^{(1)}$ contain logarithms of the external momenta, whose coefficients are proportional to the one-loop anomalous dimensions of the relevant operators (the $W$ and top masses acting as an effective ultraviolet cutoff). Thus for example $\vec{T}^{(1)}$ contains terms $\sim ln(M_W^2/\mu_0^2)$ ($\sim ln(m_t^2/\mu_0^2)$). On the other hand the insertion of the renormalized operators $\vec{Q}(\mu)$ between the same external states goes like $ln(\mu^2/\mu_0^2)$, where $\mu$ is the renormalization scale.
If we choose the renormalization scale $\mu = M_W$, the logarithms disappear when we compute the coefficients $\vec{C}(M_W)$, which are obtained by comparing eq.(19) with eq.(20):

$$\vec{C}(M_W) = \vec{T}(0) + \frac{\alpha_s}{4\pi} \left( \vec{T}(1) - \vec{T}(0) \right) + \frac{\alpha_e}{4\pi} \left( \vec{D}(1) - \vec{D}(0) \right)$$  \hspace{1cm} (21)

We take $\alpha_s$ as a fixed coupling constant and $\alpha_s$ in eq.(21) has to be interpreted as $\alpha_s(M_W)$. $\vec{T}(1)$ and $\vec{D}(1)$ depend on the external states and on the regularization scheme. However their difference depends only on the regularization scheme. We will give the results for both the HV and NDR regularizations.

We now describe separately the strong and electro-magnetic current-current, penguin and box diagrams.

### 3.1 Current-Current $O(\alpha_s)$ Diagrams

The coefficients $\vec{C}(M_W)$ depend on the combination $\vec{T}(1) - \vec{T}(0)$ which is independent of the external states. Thus we can choose to compute the diagrams in figs. 2a-2c and 3a-3c with external states different from diagram to diagram (but equal for corresponding diagrams in the full and effective theory, 2a and 3a for example). We have chosen the external momenta as shown in the figures. At one loop in $\alpha_s$, $\langle JJ \rangle$ can only mix with the operators $Q_{1,2}$ and $Q_{c,1,2}$ of the list given above. We only discuss the mixing of $Q_{1,2}$, since the case $Q_{c,1,2}$ proceeds in the same way.

When we compute the vertex corrections to the weak charged current, fig.2a, and combine it with the renormalization of the external quark lines, fig.4, the axial current is in general subject to a finite renormalization which depends on the regularization scheme. A finite correction to the axial current at one loop implies that the current have a non-zero two-loop anomalous dimension. The current anomalous dimension must be subtracted from the anomalous dimension of the operators of the effective Hamiltonian, see eq.(8). Alternatively we can apply a finite renormalization to the current in such a way that its two–loop anomalous dimension is zero. This procedure modifies

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7 The same choice of external momenta for the current-current diagrams has been adopted for the $O(\alpha_e)$ corrections described in sec.3.2.
the coefficients $\bar{C}(M_W)$. It is equivalent to impose that the current obeys the Ward identity which states that, in perturbation theory, the axial current must be conserved in the limit in which the external quark masses vanish. The two definitions of the axial current will give, at the NLO, the same physical result. We prefer to choose the second alternative, i.e. $\gamma_J = 0$ at two-loops, and avoid the subtraction of the current anomalous dimension.

With our choice of the renormalized weak current, in the HV scheme, we find:

$$C^{(1)}_{HV -ccg}(M_W) = \frac{\alpha_s(M_W)}{4\pi} \frac{7}{2}$$
$$C^{(2)}_{HV -ccg}(M_W) = -\frac{\alpha_s(M_W)}{4\pi} \frac{7}{6}$$

(22)

where the subscript $ccg$ indicates that this is the contribution from current-current diagrams, "cc", from the exchange of a gluon, "g". In the calculation of the coefficient functions $\bar{C}(M_W)$, $\alpha_s(M_W)$ has to be understood as the running coupling constant, computed in the 5-flavour theory at the scale $M_W$. This is the correct procedure when $m_t$ is larger than $M_W$, as suggested by the mass lower bound obtained for a top quark which decays in the standard modes. In the NDR scheme, we find instead:

$$C^{(1)}_{NDR -ccg}(M_W) = \frac{\alpha_s(M_W)}{4\pi} \frac{11}{2}$$
$$C^{(2)}_{NDR -ccg}(M_W) = -\frac{\alpha_s(M_W)}{4\pi} \frac{11}{6}$$

(23)

### 3.2 Current-Current $O(\alpha_e)$ Diagrams

The electro-magnetic corrections are more complicated due to the presence of the non-abelian diagrams of figs.5d and 6b and the $W-Z^0$ box diagrams. The sum of the vertex and self-energy diagrams, figs.5a and 6, contrary to the $O(\alpha_s)$ case, is neither finite nor gauge invariant. This is related to the fact that the sum of all the diagrams in figs.5 and 6 contributes both to the renormalization of the Fermi constant $G_F$, which is universal for quarks and leptons, and to the electromagnetic corrections to the effective Hamiltonian at order $\alpha_e$\[24, 25\]. This means that we can reabsorb a part of the $O(\alpha_e)$ corrections by a suitable redefinition of $G_F$. We proceed following ref.\[24\].
In the vertex and self-energy diagrams in figs.5a and 6a, we write the photon contribution using the identity:

\[ k^{-2} = k^{-2}M_W^2(M_W^2 - k^2)^{-1} + (k^2 - M_W^2)^{-1} \]  \hspace{1cm} (24)

The contributions coming from the second term in eq.(24) are divergent. The divergent part of these diagrams, combined with the divergent terms from diagrams in figs.5d and 6b, gives a universal contribution (equal for quarks and leptons) to the renormalization of the weak coupling constant \( g_W \) and \( W \) mass. Thus this term can be reabsorbed in the definition of the physical \( G_F \) as measured in \( \mu \)-decays [25]. The first term in eq.(24) is cut-off by the additional convergent factor \( M_W^2(M_W^2 - k^2)^{-1} \). For this reason it can only give a finite contribution \( \sim \ln(M_W^2/\mu_0^2) \), similar to the contribution of the \( \gamma-W \) box diagrams of figs.5b-c. The sum of the terms \( \sim \ln(M_W^2/\mu_0^2) \) gives indeed the current-current contribution to \( \gamma^{(0)}_e \). The remaining genuine \( O(G_F\alpha_e) \) corrections (i.e. not containing \( \ln(M_W^2/\mu_0^2) \) ) are all proportional to \( Q_2 \) both in the effective and in the full theory [24]-[25]. They give thus a correction of order \( \alpha_e/4\pi \sim 10^{-3} \) to a coefficient of \( O(1) \) and can be safely ignored. In ref.[13] they have instead computed the finite \( O(\alpha_e) \) corrections by analogy with the \( O(\alpha_s) \) case, i.e. by considering the differences between the diagrams of fig.3a-c and 5a-c with a photon exchanged. In this case one would have found:

\[ C^{(2)}_{HV-cc\gamma}(M_W) = -\frac{\alpha_e}{4\pi} \frac{13}{6} \]

\[ C^{(2)}_{NDR-cc\gamma}(M_W) = -\frac{\alpha_e}{4\pi} \frac{35}{18} \]  \hspace{1cm} (25)

We observe that the above coefficients do not exaust the \( O(\alpha_e) \) corrections. For example they do not contain the terms coming from the \( Z^0-W \) box diagrams which exist only in the full theory.

### 3.3 QCD Penguin Diagrams

Penguin diagrams have been extensively studied in the literature [22]-[12]. In the full theory, for \( \mu_0 \gg m_c \), using the unitarity of the CKM mixing matrix, we have only to consider the contribution coming from the difference between the top-penguin and the up-penguin diagrams. The analogous contribution
\[
B(x_t) = \frac{1}{4} \left[ x_t/(1 - x_t) + x_t \ln x_t/(x_t - 1)^2 \right]
\]
\[
C(x_t) = \frac{1}{8} x_t [(x_t - 6)/(x_t - 1) + (3x_t + 2)/(x_t - 1)^2 \ln x_t]
\]
\[
D(x_t) = -\frac{3}{4} \ln x_t + (-19x_t^2 + 25x_t^2)/[36(x_t - 1)^3] + [x_t^2(5x_t^2 - 2x_t - 6)]/[18(x_t - 1)^4] \ln x_t
\]
\[
E(x_t) = -\frac{2}{3} \ln x_t + [x_t^2(15 - 16x_t + 4x_t^2)]/[6(1 - x_t)^4] \ln x_t + [x_t(18 - 11x_t - x_t^2)]/[12(1 - x_t)^3]
\]

Table 1: Basic functions governing the \(m_t\)-dependence of various weak amplitudes

The modified Inami-Lim function \(E(x_t)\) is given in table 1. In the NDR scheme one finds:

\[
C^{(3)}_{\text{NDR\-pg}}(M_W) = C^{(5)}_{\text{NDR\-pg}}(M_W) = -\frac{\alpha_s(M_W)}{24\pi} \left( E(x_t) - \frac{2}{3} \right)
\]

where \(x_t = m_t^2/M_W^2\). The modified Inami-Lim function \(E(x_t)\) is given in table 1. In the NDR scheme one finds:

\[
C^{(3)}_{\text{NDR\-pg}}(M_W) = C^{(5)}_{\text{NDR\-pg}}(M_W) = -\frac{\alpha_s(M_W)}{24\pi} \left( E(x_t) - \frac{2}{3} \right)
\]

These results were already presented in ref.\[1\].

3.4 QED Penguin Diagrams

In the case of electro-magnetic penguins we have also to consider, besides the diagram of fig.8a, the non-abelian diagram given in fig.8b. In HV, we find
the following global contribution for QED-penguin diagrams:

\[
C^{(7)}_{HV-p\gamma}(M_W) = \frac{\alpha_e}{6\pi} D(x_t) \\
C^{(9)}_{HV-p\gamma}(M_W) = \frac{\alpha_e}{6\pi} D(x_t)
\]

where the Inami-Lim function \(D(x_t)\) is given in table 1. In NDR one finds:

\[
C^{(7)}_{NDR-p\gamma}(M_W) = \frac{\alpha_e}{6\pi} \left( D(x_t) - \frac{4}{9} \right) \\
C^{(9)}_{NDR-p\gamma}(M_W) = \frac{\alpha_e}{6\pi} \left( D(x_t) - \frac{4}{9} \right)
\]

### 3.5 \(Z^0\) Penguin Diagrams

\(Z^0\) penguin diagrams give \(O(\alpha_e)\) contributions\(^2\) as it was the case for photon-penguins. The corresponding modified Inami-Lim function \(^2\) vanishes as \(x_t \to 0\). We can then reasonably neglect the up and charm quark contributions, even though the GIM mechanism is not active in this case. One gets:

\[
C^{(3)}_{HV-pZ^0}(M_W) = \frac{\alpha_e}{6\pi} \frac{1}{\sin^2 \vartheta_W} C(x_t) \\
C^{(7)}_{HV-pZ^0}(M_W) = \frac{\alpha_e}{6\pi} 4C(x_t) \\
C^{(9)}_{HV-pZ^0}(M_W) = \frac{\alpha_e}{6\pi} \left\{ 4C(x_t) - \frac{1}{\sin^2 \vartheta_W} 4C(x_t) \right\}
\]

and the same in NDR. \(C(x_t)\) is given in table 1.

### 3.6 Box Diagrams

In the approximation of massless light quarks, the box diagrams of fig.7 give:

\[
C^{(3)}_{(\Box)}(M_W) = \frac{\alpha_e}{6\pi \sin^2 \vartheta_W} B(x_t) \\
C^{(9)}_{(\Box)}(M_W) = \frac{\alpha_e}{6\pi \sin^2 \vartheta_W} 10B(x_t)
\]

14
both in HV and in NDR.

A summary of the results discussed in this section for the HV and NDR regularizations is reported below. We have also included the tree-level contribution from the diagram of fig. 1 and neglected terms of \(O(\alpha_e/4\pi)\) in \(C_2\).

\(i)\) HV

\[
C_1(M_W) = \frac{\alpha_s^{(5)}(M_W)}{4\pi} \frac{7}{2}
\]

\[
C_2(M_W) = 1 - \frac{\alpha_s^{(5)}(M_W)}{4\pi} \frac{7}{6}
\]

\[
C_3(M_W) = -\frac{\alpha_s^{(5)}(M_W)}{24\pi} E(x_t) + \frac{\alpha_e}{6\pi \sin^2 \vartheta_W} \left[ 2B(x_t) + C(x_t) \right]
\]

\[
C_4(M_W) = \frac{\alpha_s^{(5)}(M_W)}{8\pi} E(x_t)
\]

\[
C_5(M_W) = -\frac{\alpha_s^{(5)}(M_W)}{24\pi} E(x_t)
\]

\[
C_6(M_W) = \frac{\alpha_s^{(5)}(M_W)}{8\pi} E(x_t)
\]

\[
C_7(M_W) = \frac{\alpha_e}{6\pi} \left[ 4C(x_t) + D(x_t) \right]
\]

\[
C_8(M_W) = 0
\]

\[
C_9(M_W) = \frac{\alpha_e}{6\pi} \left[ 4C(x_t) + D(x_t) + \frac{1}{\sin^2 \vartheta_W} \left( 10B(x_t) - 4C(x_t) \right) \right]
\]

\[
C_{10}(M_W) = 0
\]

and:

\(ii)\) NDR

\[
C_1(M_W) = \frac{\alpha_s^{(5)}(M_W)}{4\pi} \frac{11}{2}
\]

\[
C_2(M_W) = 1 - \frac{\alpha_s^{(5)}(M_W)}{4\pi} \frac{11}{6}
\]

\[
C_3(M_W) = -\frac{\alpha_s^{(5)}(M_W)}{24\pi} \left( E(x_t) - \frac{2}{3} \right) + \frac{\alpha_e}{6\pi \sin^2 \vartheta_W} \left[ 2B(x_t) + C(x_t) \right]
\]

\[
C_4(M_W) = \frac{\alpha_s^{(5)}(M_W)}{8\pi} \left( E(x_t) - \frac{2}{3} \right)
\]
\[
C_5(M_W) = -\frac{\alpha_s(5)(M_W)}{24\pi} \left( E(x_t) - \frac{2}{3} \right) \\
C_6(M_W) = \frac{\alpha_s(5)(M_W)}{8\pi} \left( E(x_t) - \frac{2}{3} \right) \\
C_7(M_W) = \frac{\alpha_e}{6\pi} \left[ 4C(x_t) + \left( D(x_t) - \frac{4}{9} \right) \right] \\
C_8(M_W) = 0 \\
C_9(M_W) = \frac{\alpha_e}{6\pi} \left[ 4C(x_t) + \left( D(x_t) - \frac{4}{9} \right) + \frac{1}{\sin^2 \vartheta_W} (10B(x_t) - 4C(x_t)) \right] \\
C_{10}(M_W) = 0
\]  

4 Anomalous Dimensions at One and Two Loops

In this section we introduce the notation necessary for the calculation of the anomalous dimension matrix \( \hat{\gamma}(\alpha_s, \alpha_e) \) in dimensional regularization, eq. (8), and recall the rules for the HV and NDR schemes.

4.1 General Definitions and Scheme Dependence

For simplicity, we start by considering only one- and two-loop corrections due to strong interactions. The modifications, necessary to include the electromagnetic corrections will be given in sec. 6.

The anomalous dimension matrix for the operators appearing in the effective Hamiltonian is defined from the operator renormalization matrix:

\[
\hat{\gamma}_Q(\alpha_s) = 2 \hat{Z}^{-1} \mu^2 \frac{d}{d\mu^2} \hat{Z} 
\]

where \( \hat{Z} \) is defined by the relation:

\[
\bar{Q} = \hat{Z}^{-1} \bar{Q}^B
\]

which gives the renormalized operators in terms of the bare ones.
In a dimensional regularization, as in the HV and NDR schemes, from eq.(35), we obtain:

\[ \hat{\gamma}_Q = 2 \hat{Z}^{-1} (-\epsilon \alpha_s + \beta(\alpha_s)) \frac{\partial}{\partial \alpha_s} \hat{Z} \]  

(37)

By writing \( \hat{\gamma}_Q \) and \( \hat{Z} \) as series in the strong coupling constant\[^8\]:

\[ \hat{\gamma}_Q = \frac{\alpha_s}{4\pi} \hat{\gamma}_s^{(0)} + \frac{\alpha_s^2}{(4\pi)^2} \hat{\gamma}_s^{(1)} + \cdots \]  

(38)

\[ \hat{Z} = 1 + \frac{\alpha_s}{4\pi} \hat{Z}^{(1)} + \frac{\alpha_s^2}{(4\pi)^2} \hat{Z}^{(2)} + \cdots \]  

(39)

we derive the following relations:

\[ \hat{\gamma}_s^{(0)} = -2\epsilon \hat{Z}^{(1)} \]  

(40)

and

\[ \hat{\gamma}_s^{(1)} = -4\epsilon \hat{Z}^{(2)} - 2\beta_0 \hat{Z}^{(1)} + 2\epsilon \hat{Z}^{(1)} \hat{Z}^{(1)} \]  

(41)

where \( \epsilon = (4 - D)/2 \). \( \beta_0 \) is the one-loop coefficient of the \( \beta \)-function \( \beta(\alpha_s) \) which governs the evolution of the effective coupling constant:

\[ \mu^2 \frac{d\alpha_s}{d\mu^2} = \beta(\alpha_s) \]  

(42)

and:

\[ \beta(\alpha_s) = -\beta_0 \frac{\alpha_s^2}{4\pi} - \beta_1 \frac{\alpha_s^3}{(4\pi)^2} + O(\alpha_s^4) \]  

(43)

\( \beta_0 \) and \( \beta_1 \) are given by:

\[ \beta_0 = \frac{(11N - 2f)}{3} \]

\[ \beta_1 = \frac{34}{3} N^2 - \frac{10}{3} Nf - \frac{(N^2 - 1)}{N} f \]  

(44)

\[ ^8 \text{Since we normalize the weak current in such a way that } \gamma_J = 0, \text{ see below, we have } \hat{\gamma} = \hat{\gamma}_Q. \]
where \( f \) is the number of flavours. The running coupling constant, solution of eq.(42), is:

\[
\frac{\alpha_s(\mu^2)}{4\pi} = \frac{1}{\beta_0 \ln(\mu^2/\Lambda_{QCD}^2)} \left( 1 - \frac{\beta_1 \ln[\ln(\mu^2/\Lambda_{QCD}^2)]}{\beta_0^2 \ln(\mu^2/\Lambda_{QCD}^2)} \right) + \cdots
\]

(45)

The above equation defines \( \Lambda_{QCD} \) at the NLO.

We can expand \( \hat{Z}^{(i)} \) in eqs.(40) and (41) in inverse powers of \( \epsilon \):

\[
\hat{Z}^{(i)} = \sum_{j=0}^{i} \left( \frac{1}{\epsilon} \right)^j \hat{Z}^{(i)}_j
\]

(46)

The anomalous dimension is finite as \( \epsilon \to 0 \). This implies a relation between the one- and two-loop coefficients of \( \hat{Z} \):

\[
4 \hat{Z}^{(2)}_2 + 2\beta_0 \hat{Z}^{(1)}_1 - 2 \hat{Z}^{(1)}_1 \hat{Z}^{(1)}_1 = 0
\]

(47)

From the above equations we finally obtain:

\[
\hat{\gamma}^{(0)}_s = -2 \hat{Z}^{(1)}_1
\]

(48)

and

\[
\hat{\gamma}^{(1)}_s = -4 \hat{Z}^{(2)}_1 - 2\beta_0 \hat{Z}^{(1)}_0 + 2(\hat{Z}^{(1)}_1 \hat{Z}^{(1)}_1 + \hat{Z}^{(1)}_0 \hat{Z}^{(1)}_1)
\]

(49)

We thus conclude that it is sufficient to compute the pole and finite part of \( \hat{Z}^{(1)} \) and the single pole of \( \hat{Z}^{(2)} \) in order to obtain the two-loop anomalous dimension. Eq.(43) tells how to derive \( \hat{\gamma}^{(1)} \). In dimensional regularizations, such as HV, NDR or DRED (dimensional reduction) however, the calculation is complicated by the presence of the so called "effervescent" operators (EO), which appear in the intermediate steps of the calculation [18, 19]. The EO are independent operators which are present in D-dimensions but disappear in the physical basis of operators in 4-dimensions. Because of the presence of the EO, the products of the matrices \( \hat{Z}^{(i)}_j \) in eq.(49) have to be done by summming indices over the full set of operators, including the EO. Only at the end of the calculation we can restrict the set of operators to the operators of the physical 4-dimensional basis [3].

Different renormalization prescriptions will define different renormalized operators. This happens for example if we adopt the \( \overline{\text{MS}} \) subtraction procedure in the HV and NDR schemes. Let us denote as \( \hat{Z}_{HV} \) (\( \hat{Z}_{NDR} \)) the
renormalization matrices in the two cases and \( \hat{r}_{HV} (\hat{r}_{NDR} \) ), cf. eq.(20), the one-loop matrices of the effective theory in the two regularizations. Then we must have:

\[
\hat{Z}_{HV} = \hat{Z}_{NDR} \left( 1 + \frac{\alpha_s}{4\pi} \Delta \hat{r} \right)
\]

with \( \Delta \hat{r} = \hat{r}_{NDR} - \hat{r}_{HV} \). Using eq.(35), expanded in powers of \( \alpha_s \) as in eqs.(38)-(39), we then find:

\[
\Delta \hat{\gamma}_s^{(1)} = \left[ \Delta \hat{r}, \hat{\gamma}_s^{(0)} \right] + 2 \beta_0 \Delta \hat{r}
\]

Eq.(51) implies that the combination:

\[
\hat{G} = \hat{\gamma}_s^{(1)} - \left[ \hat{r}, \hat{\gamma}_s^{(0)} \right] - 2 \beta_0 \hat{r}
\]

is regularization independent. It can be shown that a consequence of eq.(52) is the scheme independence of the combination \( \hat{R} = \hat{r}_{T} + \hat{J} \) (\( \hat{S} = \hat{s}_{T} + \hat{K} \) in the case of electromagnetic corrections) on the regularization scheme, see for example ref.[2]. \( \hat{R} \) and \( \hat{S} \) are precisely the combinations which appear in the final expressions of the coefficient functions.

Using eqs.(9)-(11) and (21), we obtain:

\[
\vec{C}(\mu) = \hat{M}[\mu] \hat{U}[\mu, M_W] \hat{N}'[M_W] \hat{C}'(M_W)
\]

where \( \hat{M}[\mu] \) has been defined in eq.(10) and:

\[
\hat{N}'[M_W] = \left( \hat{1} - \frac{\alpha_e}{\alpha_s(M_W)} \hat{P} \right) \left( \hat{1} - \frac{\alpha_s(M_W)}{4\pi} [\hat{r}_{T} + \hat{J}] \right) \left( \hat{1} - \frac{\alpha_e}{4\pi} [\hat{s}_{T} + \hat{K}] \right)
\]

\[
\vec{C}'(M_W) = \vec{T}^{(0)} + \frac{\alpha_s}{4\pi} \vec{T}^{(1)} + \frac{\alpha_e}{4\pi} \vec{D}^{(1)}
\]

In the above equation we have neglected higher order terms in \( \alpha_s \) or \( \alpha_e \). From eq.(54), we conclude that the matrix \( \hat{N}'[M_W] \) is independent of the regularization. We can obtain scheme independent coefficients \( \vec{C}(\mu) \) by a suitable redefinition of the renormalized operators \( \vec{Q}^T(\mu) \):

\[
\vec{V}^T(\mu) = \vec{Q}^T(\mu) \left( 1 - \frac{\alpha_s(\mu)}{4\pi} \hat{r}_{T} - \frac{\alpha_e}{4\pi} \hat{s}_{T} \right)
\]

9 The corresponding formula for \( \hat{\gamma}_e^{(1)} \) is \( \Delta \hat{\gamma}_e^{(1)} = \left[ \Delta \hat{r}, \hat{\gamma}_e^{(0)} \right] + \left[ \Delta \hat{s}, \hat{\gamma}_s^{(0)} \right] \).

10 This is strictly true if we include the terms of \( \hat{O}(\alpha_e) \) of eq.(23).
With the above redefinition, one has:

\[
\hat{M}[\mu] \rightarrow \hat{N}[\mu] = \left(1 + \frac{\alpha_e}{4\pi}(\hat{K} + \hat{s}^T)\right)\left(1 + \frac{\alpha_s(\mu)}{4\pi}(\hat{J} + \hat{r}^T)\right)\left(1 + \frac{\alpha_e}{\alpha_s(\mu)}\hat{P}\right)
\]

so that \(\vec{C}(\mu) = \hat{N}[\mu]\hat{U}[\mu,M_W]\hat{N}'[M_W]\vec{C}'(M_W)\) is regularization scheme independent.

It is not difficult to understand how it is possible to find renormalization conditions which do not depend on the scheme. Let us fix the renormalization conditions by imposing that the matrix elements of the renormalized operators have a given value for a certain set of external quark (gluon) states:

\[
<\beta|\vec{V}|\alpha> = 1
\]

This procedure defines the same renormalized operators, i.e. the same coefficients \(\vec{C}(\mu)\), in all the regularization schemes\(^{11}\). Notice however that the coefficient functions depend now on the external states chosen to fix the renormalization conditions, \(|\alpha>\) and \(|\beta>\) in eq.(58). The external states have thus to be specified if one wants to use renormalization scheme independent coefficients. The values of the coefficients can change in a substantial way by going from the \(\overline{MS}\) HV or NDR prescription to the scheme independent one. For example, using \(\overline{MS}\) HV we found that \(C_6\) is reduced by the inclusion of the NLO corrections\(^{1}\) while it is enhanced in the scheme independent case\(^{2,3}\). We remark that a consistent treatment of the Wilson coefficients and renormalized operators is necessary in order to get, up to higher order corrections, the physical result. Such a treatment is possible, at least in principle, in lattice QCD but not in other approaches as for example the \(1/N\) expansion.

Expression (49), which allows us to compute the two-loop anomalous dimension in terms of the one- and two-loop renormalization matrices, is indeed valid diagram by diagram. This means that we can define the contribution of any given two-loop diagram to the anomalous dimension by combining:

\(^{11}\) If the renormalized strong coupling constant differs in two different renormalization schemes, as it is the case for HV and DRED for example, the coefficients will have the same expression only when given in terms of the same renormalized \(\alpha_s\).
i) the single pole obtained from the diagram where we insert a given bare operator;

ii) the single pole of the diagram obtained by substituting to any divergent sub-diagram the appropriate counter-terms, including those proportional to “effervescent” operators;

iii) the single pole coming from the substitution, in any divergent sub-diagram, of an appropriate combination of effervescent operators to take into account the term $2(\hat{Z}_1^{(1)}\hat{Z}_0^{(1)} + \hat{Z}_0^{(1)}\hat{Z}_1^{(1)})$ in eq.(49).

We will call the contribution to the anomalous dimension of a given diagram plus the counter-terms and the insertion of the effervescent operators, i)-iii), the “complete” contribution. The advantage of combining different terms (bare diagram, counter-terms and insertion of effervescent operators) diagram by diagram is that this allows several checks on the contribution of any given two-loop diagram to the anomalous dimension. Thus for example, in HV, two diagrams which go one into another via a Fierz rearrangement, diagrams $V_{10}$ and $V_{12}$ in fig.10 or $P_2$ and $F_2$ in fig.11 for instance, give the same “complete” contribution to $\hat{\gamma}^{(1)}$ for left-left operators {19}. 

Similarly, relation (51) can be shown to be true for the “complete” contribution of any single two-loop diagram. It is then possible to use it as a further check of the calculation in two different regularization schemes. Eq.(51), used diagram by diagram, is a relation which connects single poles and finite terms at one loop with single poles at two loops. When satisfied, it automatically ensures that the relation holds both for $\alpha_s^2$ and $\alpha_e\alpha_s$ corrections. A more extensive discussion of the checks done on the calculation of the single diagrams will be given in sections 5.4 and 5.5. Before this we recall some basic facts about the HV and NDR regularization schemes, sec.4.2, and introduce the ”effervescent” operators by considering the one-loop diagrams in these two regularizations, secs.5.1-5.3. Then we explain the construction of “complete” diagrams by two specific examples, one taken from the set of current-current diagrams, fig.10, and the other from the set of penguin diagrams, fig.11, sec.5.4 and 5.5.
4.2 HV and NDR Regularization Schemes

In this subsection we report for completeness the computational rules of the HV and NDR regularization schemes, which have been used in the present work. In both the schemes Feynman diagrams are regularized by performing the integrals over the loop momenta in $D = 4 - 2\epsilon$ dimensions. The two schemes differ in the definition of $\gamma_5$, which, in the case of the NDR regularization, can lead to inconsistencies and has to be treated with particular care.

1. Naive Dimensional Regularization (NDR)

In NDR $\gamma$-matrices are $D$-dimensional and only the $D$-dimensional metric tensor is introduced, following the convention:

$$g_{\mu\nu} = g_{\nu\mu}, \quad g^\rho_{\mu}g^\nu_{\rho} = g^\nu_{\mu}, \quad g_{\mu}^\mu = D, \quad Tr(\mathbb{I}) = 4 \quad (59)$$

The $D$-dimensional Dirac matrices obey the usual algebra:

$$\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu} \quad (60)$$

and $\gamma_5$ anticommutes with all the $\gamma$'s:

$$\{\gamma_5, \gamma_\mu\} = 0 \quad (61)$$

The above definition of $\gamma_5$ may give problems in the evaluation of closed odd parity fermion loops, as for example $Tr(\gamma_5\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma)$, which are not unambiguously defined. For this reason it is not guaranteed that it is possible to obtain the correct results by using the NDR scheme. In ref. [3] and in the present calculation it is shown that, up to two loops, by fixing the ambiguity in the closed odd parity fermion loops, it is possible to obtain results which are in agreement with those obtained in the HV regularization scheme. In the present work we have fixed the ambiguity following the prescription of ref. [26]. We have verified that the prescription of ref. [26] gives the correct result for the triangle anomaly, as it is also the case for the HV scheme.

2. ’t Hooft-Veltman Regularization (HV)

The HV regularization scheme has been proved to be unaffected by ambiguities or inconsistencies in the algebra of the $\gamma$ matrices [1, 27, 28].

22
The $D$-dimensional $\gamma$-matrices $\gamma^\mu$ are decomposed in two parts, a 4-dimensional part $\tilde{\gamma}^\mu$ and a $(D-4)$-dimensional part $\hat{\gamma}^\mu$, i.e.:

\[ \gamma^\mu = \tilde{\gamma}^\mu + \hat{\gamma}^\mu \quad (62) \]

with different anticommutation relations with respect to $\gamma_5$. The 4-dimensional $\gamma$'s anticommute with $\gamma_5$, while the $\hat{\gamma}$'s commute with $\gamma_5$:

\[ \{ \tilde{\gamma}_\mu, \gamma_5 \} = 0 \quad , \quad [ \hat{\gamma}_\mu, \gamma_5 ] = 0 \quad (63) \]

Besides the $D$-dimensional tensor of eq.(59), two other metric tensors are introduced, namely the 4-dimensional tensor $\tilde{g}_{\mu\nu}$ and the $(-2\epsilon)$-dimensional one $\hat{g}_{\mu\nu}$:

\[ \{ \tilde{\gamma}_\mu, \tilde{\gamma}_\nu \} = 2\tilde{g}_{\mu\nu} \quad , \quad \{ \hat{\gamma}_\mu, \hat{\gamma}_\nu \} = 2\hat{g}_{\mu\nu} \quad , \quad \{ \hat{\gamma}_\mu, \tilde{\gamma}_\nu \} = 0 \quad (64) \]

with the following “mixed” contraction properties:

\[ \tilde{g}_\mu^\mu = 4 \quad , \quad \hat{g}_\mu^\mu = -2\epsilon \quad , \quad \tilde{g}_\rho^\rho \hat{g}_\nu^\nu = 0 \quad (65) \]

and

\[ \tilde{g}_\mu^\rho \hat{g}_\rho^\nu = \hat{g}_\mu^\nu \quad , \quad \hat{g}_\mu^\rho \hat{g}_\rho^\nu = \hat{g}_\mu^\nu \quad (66) \]

Using HV for computing Feynman diagrams, it is convenient to take the external momenta in four dimensions (the loop momenta being integrated in $D$-dimension).

The bare operators inserted in the one- and two-loop diagrams are defined by using only 4-dimensional gamma matrices, i.e.:

\[ \tilde{\gamma}^\mu (1-\gamma_5) \otimes \tilde{\gamma}_\mu (1-\gamma_5) \quad \text{or} \quad \hat{\gamma}^\mu (1-\gamma_5) \otimes \hat{\gamma}_\mu (1+\gamma_5) \quad (67) \]

In the algebraic programs, written for the calculation of the diagrams, it is convenient to write the weak four-fermion operators in $\tilde{g}_\mu^\rho \hat{g}_\rho^\nu$ using $(1 \pm \gamma_5)$ as projectors:

\[ \frac{1}{4} (1+\gamma_5) \gamma^\mu (1-\gamma_5) \otimes (1+\gamma_5) \gamma_\mu (1-\gamma_5) \quad (68) \]

and analogously for the $\gamma_L^\mu \otimes \gamma_{\mu R}$ case.
In 4-dimensions, in the massless theory, all operators appearing in the weak effective Hamiltonian are suitable flavour combinations of four-fermion left-left and left-right operators. This happens because long strings of $\gamma$ matrices, appearing at one- or two- loops, can be reduced to left-left or left-right four-fermion operators, using the completeness of the Dirac algebra. This completeness is lost however using a dimensional regularization. In this case we can define an infinite number of independent four-fermion operators which, by quantum numbers, can mix with the original basis, at different orders in perturbation theory. The extra operators are those called previously “effervescent” operators. They do not exist in four dimensions and are artefact of the regularization. It would be not correct however to compute the renormalization of the operators without taking into account the mixing of EO with the 4-dimensional operators. The reduction to the four dimensional basis can only be done at the end of the calculation of the two-loop anomalous dimension, as explained in the next section.
5 Diagrams and Counter-terms

In this section we report the calculation of the one-loop diagrams in HV and NDR. This allows us to introduce the EO induced by the regularization and give the elements necessary to compute the coefficient functions $\tilde{C}(M_W)$ as explained in sec.3. We also give some specific examples of the calculation of two-loop “complete” diagrams, both in the current-current and in the penguin case, in order to explain the method used in this work and the relations between the HV and NDR regularizations. A complete list of the contribution of all the diagrams is reported in Appendix.

5.1 Current-Current Diagrams at One Loop

Let us consider the diagrams in figs.3a-c, where the wavy lines can be due to a gluon or a photon exchange. The Dirac structure inserted in the vertex can be of the form $\gamma^\mu_L \otimes \gamma^\mu_L$ or $\gamma^\mu_L \otimes \gamma^\mu_R$ and we consider both cases separately. After the loop integration, but before the simplification of the Dirac algebra, the pole terms of the diagrams have the following structure (for the two-loop anomalous dimension we only need to consider the pole contribution of 4-dimensional and effervescent operators):

\[
G_{LL,LR}^{(1)} = C^{(1)} \frac{1}{4\epsilon} (\gamma^\nu \gamma^\rho \Gamma^\mu_L \gamma^\rho \gamma^\nu \otimes \Gamma_{\mu L,R} + \Gamma^\mu_L \otimes \gamma^\nu \gamma^\rho \Gamma_{\mu L,R} \gamma^\rho \gamma^\nu )
\]

\[
G_{LL,LR}^{(2)} = -C^{(2)} \frac{1}{4\epsilon} (\gamma^\nu \gamma^\rho \Gamma^\mu_L \otimes \gamma^\nu \gamma^\rho \Gamma_{\mu L,R} + \Gamma^\mu_L \otimes \gamma^\nu \gamma^\rho \Gamma_{\mu L,R} \gamma^\rho \gamma^\nu )
\]

\[
G_{LL,LR}^{(3)} = C^{(3)} \frac{1}{4\epsilon} (\gamma^\nu \gamma^\rho \Gamma^\mu_L \otimes \Gamma_{\mu L,R} \gamma^\nu \gamma^\rho + \Gamma^\mu_L \gamma^\rho \gamma^\nu \otimes \gamma^\nu \gamma^\rho \Gamma_{\mu L,R} )
\]

where $C^{(1)}$, $C^{(2)}$ and $C^{(3)}$ are coefficients which summarize the colour/charge dependence of each diagram. $\Gamma^\mu_L$ and $\Gamma^\mu_R$ denote the weak vertex structure, given in NDR and in HV respectively by:

\[
\Gamma^\mu_L = \begin{cases} \gamma^\mu (1-\gamma_5), & \text{NDR} \\ \bar{\gamma}^\mu (1-\gamma_5), & \text{HV} \end{cases} \quad \Gamma^\mu_R = \begin{cases} \gamma^\mu (1+\gamma_5), & \text{NDR} \\ \bar{\gamma}^\mu (1+\gamma_5), & \text{HV} \end{cases}
\]

By reducing the $\gamma$-algebra in (63), we can separate the contributions of 4-dimensional operators from those of EO. By definition, the contributions of
EO correspond to those terms which vanish under a suitable projection on the four dimensional basis. There are several possible choices of the projection operators, which in general will define different renormalized operators, i.e. they will give different two-loop anomalous dimension matrices. The results can be easily related, by computing at one-loop the renormalized operators obtained by different projections and we have checked the consistency of the contribution of several two-loop diagrams obtained with different projections. For the sake of comparison, in the following we shall use the same projection as in refs. 3, 19:

- for $\gamma^\mu_L \otimes \gamma_\mu L \rightarrow P_{LL} = \gamma^\nu_R \otimes \gamma_\nu R$
- for $\gamma^\mu_L \otimes \gamma_\mu R \rightarrow P_{LR} = (1 - \gamma_5) \otimes (1 + \gamma_5)$

Then, in order to project on $\gamma^\mu_L \otimes \gamma_\mu L$, we take the following trace:

$$P_{LL} (\gamma^\mu_L \otimes \gamma_\mu L) = Tr [\gamma^\mu (1 - \gamma_5) \gamma^\nu (1 + \gamma_5) \gamma_\mu (1 - \gamma_5) \gamma_\nu (1 + \gamma_5)]$$ (71)

while, in order to project on $\gamma^\mu_L \otimes \gamma_\mu R$, we use:

$$P_{LR} (\gamma^\mu_L \otimes \gamma_\mu R) = Tr [\gamma^\mu (1 - \gamma_5)(1 - \gamma_5) \gamma_\mu (1 + \gamma_5)(1 + \gamma_5)]$$ (72)

In eq.(71) the sum over $\nu$ is intended in 4-dimension in HV and in D-dimensions in NDR. When projecting a string of $\gamma$-matrices, the values of the traces introduced in eq.(71) and (72) are taken as normalization factors. With the projectors introduced in eq.(71) and eq.(72), we obtain the following decomposition for the one-loop vertex diagrams in (69):

$$G^{(1)}_{LL,LR} = C^{(1)} \frac{1}{2\epsilon} \left( F^{(1)}_{LL,LR}(\epsilon) \gamma^\mu_L \otimes \gamma_\mu L,R + E^{(1)}_{LL,LR} \right)$$

$$G^{(2)}_{LL,LR} = C^{(2)} \frac{1}{2\epsilon} \left( F^{(2)}_{LL,LR}(\epsilon) \gamma^\mu_L \otimes \gamma_\mu L,R + E^{(2)}_{LL,LR} \right)$$

$$G^{(3)}_{LL,LR} = C^{(3)} \frac{1}{2\epsilon} \left( F^{(3)}_{LL,LR}(\epsilon) \gamma^\mu_L \otimes \gamma_\mu L,R + E^{(3)}_{LL,LR} \right)$$ (73)

where the $\epsilon$-dependent coefficients are given, up to $O(\epsilon)$, by:

$$F^{(1)}_{LL}(\epsilon) = F^{(3)}_{LL}(\epsilon) = \begin{cases} 4(1 - 2\epsilon) & \text{NDR} \\ 4 & \text{HV} \end{cases}$$

$$F^{(2)}_{LL}(\epsilon) = \begin{cases} -4(4 - \epsilon) & \text{NDR} \\ -4(4 - \epsilon) & \text{HV} \end{cases}$$ (74)
and:

\[
F_{LR}^{(1)}(\epsilon) = \begin{cases} 
4(1 - 2\epsilon) & \text{NDR} \\
4 & \text{HV}
\end{cases}
\]

\[
F_{LR}^{(2)}(\epsilon) = \begin{cases} 
-4(1 + \epsilon) & \text{NDR} \\
-4(1 - \epsilon) & \text{HV}
\end{cases}
\]

\[
F_{LR}^{(3)}(\epsilon) = \begin{cases} 
16(1 - \epsilon) & \text{NDR} \\
16 & \text{HV}
\end{cases}
\]

(76)

(77)

(78)

\[E_{LL,LR}^{(i)}\] are the contributions due to the EO.

From eq.(74), we obtain the counter-terms to left-left and left-right operators, including those proportional to EO:

\[
\hat{G}_{LL}^{(1)} = C^{(1)} \frac{1}{2\epsilon} \left( 4 \gamma_\mu \otimes \gamma_{\mu L} + E_{LL}^{(1)} \right)
\]

\[
\hat{G}_{LL}^{(2)} = C^{(2)} \frac{1}{2\epsilon} \left( -16 \gamma_\mu \otimes \gamma_{\mu L} + E_{LL}^{(2)} \right)
\]

\[
\hat{G}_{LL}^{(3)} = C^{(3)} \frac{1}{2\epsilon} \left( 4 \gamma_\mu \otimes \gamma_{\mu L} + E_{LL}^{(3)} \right)
\]

(79)

and similarly for the LR case. By subtracting the counter-terms in eq.(79) from the result of the calculation of the diagrams in figs.3a-c, we thus obtain the renormalized operator matrix elements expressed in terms of the tree-level matrix elements and of the matrix \( \hat{r} \) (\( \hat{s} \)), see eq.(20). The pole and finite terms coming from the calculation of diagrams \( V_{1,3} \) are reported in table 2 both in HV and NDR, for the insertion of a \( \gamma_\mu \otimes \gamma_{\mu L} \) and \( \gamma_\mu \otimes \gamma_{\mu R} \) operator. Only the terms proportional to the original operators are shown.

Notice that the insertion of an operator \( \gamma_\mu \otimes \gamma_{\mu L} \) does not produce "effervescent" operators when we compute the vertex renormalization of the vector current \( \gamma_{\mu L} \), diagram \( V_1 \), since \( \gamma_5 \) is not involved. As a consequence, in HV, using the \( \overline{MS} \) subtraction procedure, the finite term of the operator \( \gamma_\mu \otimes \gamma_{\mu L} \) is different from the finite term of the operator \( 1/2 \left( \gamma_\mu \otimes \gamma_{\mu L} + \gamma_\mu \otimes \gamma_{\mu R} \right) \). In particular for the vertex renormalization of \( \gamma_\mu \) one obtains:

\[
\frac{\alpha_s}{4\pi} \left( \frac{1}{\epsilon} + \frac{1}{2} \right) \times \gamma_\mu \otimes \gamma_{\mu L}
\]

instead of:

\[
\frac{\alpha_s}{4\pi} \left( \frac{1}{\epsilon} + \frac{5}{2} \right) \times \frac{1}{2} \left( \gamma_\mu \otimes \gamma_{\mu L} + \gamma_\mu \otimes \gamma_{\mu R} \right)
\]

(80)

(81)
Table 2: Singular and finite terms for diagrams in figs.3a-c and 9, with a \( \gamma^\mu_L \otimes \gamma^\mu_L \) or a \( \gamma^\mu_L \otimes \gamma^\mu_R \) Dirac structure. The multiplicity of the diagrams is also reported in the table. Colour-charge factors and the factor \( \alpha_s/4\pi \) (or \( \alpha_e/4\pi \)) are omitted.

as can be read in table 2. This difference has important consequences as will be discussed in the following.

### 5.2 One-Gluon/Photon Penguin Diagrams at One Loop.

The one-loop penguin diagrams are much simpler, due to the absence of EO at the one-loop level. The computation of the diagrams in fig.9 gives, both in the HV and in the NDR schemes, the following Dirac structure (we omit the colour-charge factors):

\[
\text{(Penguin)} = -\frac{4}{3} F_P^{HV, NDR}(\epsilon) \left( q^2 \Gamma^\mu_L - q^\mu q_L \right) \tag{82}
\]

where \( q^\mu \) is the momentum of the gluon/photon (see fig.9) and \( \Gamma^\mu_L \) is 4-dimensional or D-dimensional in the HV or in the NDR case respectively. We find:

\[
F_P^{HV}(\epsilon) = 1 + \frac{5}{3} \epsilon
\]

\[
F_P^{NDR}(\epsilon) = 1 + \frac{2}{3} \epsilon \tag{83}
\]
We conclude that both in HV and NDR the counter-term has the following form:
\[
\hat{G}_{1g} = \frac{-41}{3} \frac{1}{\epsilon} \left( q^2 \Gamma_{L}^\mu - q^\mu \hat{q} \right)
\]  
(84)

5.3 Two-Gluon or One-Gluon+One-Photon Penguin One Loop Diagrams

The presence of two gluon contributions is a typical feature of the two-loop calculation. The two-gluon (one gluon+one photon) diagrams, fig.12c, only enter as counter-terms at the two-loop level, see for example fig.15. These counter-terms exist only if, in any two-loop penguin-like diagram, we subtract completely the divergent part of the internal sub-diagram, without making use of the equations of motion. This is also true for the longitudinal component \( \sim q^\mu \hat{q} \) of the one gluon (photon) counter-term, which vanishes by the equations of motion. However one may choose to subtract only counter-terms which do not vanish by the equations of motion. The contribution of single diagrams will be modified, but the final result will be the same. Thus, for example, consider the diagrams \( P_2 \) and \( P_3 \). They can be computed with or without the two gluon counter-terms, \( \hat{G}_{2g}^a \) and \( \hat{G}_{2g}^b \) respectively, whose sum is equal to zero, fig.16. The abelian part, i.e. the sum of the two "complete" diagrams, remains the same as schematically indicated in fig.16, even though each of them is modified. Something similar happens to the longitudinal contribution of the penguin counter-term, eq.(84), for diagrams \( P_{14} \) and \( P_{15} \). If we subtract all the terms \( \sim 1/\epsilon \) in the penguin internal sub-diagram, we are also subtracting the longitudinal term \( q^\mu \hat{q} \). We may however make use of the equations of motion and subtract only the term \( \sim q^2 \gamma_{L}^\mu \). The sum of the two diagrams (corresponding to the abelian case) remains the same because of the cancellation of the contributions of the counter-terms due to the longitudinal components, fig.17. The same happens in the non-abelian case, but in a more complicated way which involves the difference of the two diagrams and also non-abelian diagrams. We have explicitly checked that, if we subtract only those counter-terms which do not vanish by the equations of motion, we obtain the same anomalous dimension matrix. However we will give the results by subtracting all the pole parts of the internal sub-diagrams

\[\text{12 That is their sum is the same with or without the subtraction of } \hat{G}_{2g}^{a,b}.\]
because in this way eq.(51) remains valid diagram by diagram, see below.

The cancellation of the sum of the counter-terms proportional to the longitudinal contribution in eq.(84) and of the two-gluon (one gluon-one photon) counter-terms corresponds to the substitution [29]:

\[
\bar{\psi}_1 \gamma^\mu_L D^\nu F_{\mu,\nu} \psi_2 \rightarrow (\bar{\psi}_1 \gamma^\mu_L \psi_2) \sum_q \bar{q} \gamma_\mu q
\]

i.e. to the subtraction done using only four-fermion operators. Notice however that minimal subtraction of the pole term proportional to the operator \(\bar{\psi}_1 \gamma^\mu_L D^\nu F_{\mu,\nu} \psi_2\) in HV, is non-minimal in the basis where we use the \(\gamma^\mu_L \otimes \gamma_\mu\) and \(\gamma^\mu_L \otimes \gamma_\mu\) operators. This happens because, as explained above, the finite term of the renormalized operator \(\gamma^\mu_L \otimes \gamma_\mu\) is different from the finite term of the operator \(\frac{1}{2}(\gamma^\mu_L \otimes \gamma_\mu_L + \gamma^\mu_L \otimes \gamma_\mu_R)\). It is this last operator that we have indeed to subtract for consistency with the current-current diagrams. The difference between \(\gamma^\mu_L \otimes \gamma_\mu\) and \(\frac{1}{2}(\gamma^\mu_L \otimes \gamma_\mu_L + \gamma^\mu_L \otimes \gamma_\mu_R)\) is important only for the counter-terms of diagrams \(P_{16}\) and \(F_{16}\). Since we have to apply a non-minimal subtraction to the lower vertex, the contribution of \(P_{16}\) and \(F_{16}\) is non-zero, but it is given by one half of the pole of the one-loop penguin diagram times the difference of the finite terms of the two operators \(\gamma^\mu_L \otimes \gamma_\mu\) and \(\frac{1}{2}(\gamma^\mu_L \otimes \gamma_\mu_L + \gamma^\mu_L \otimes \gamma_\mu_R)\):

\[
\frac{1}{2} \times \left( -\frac{4}{3\epsilon} \right) \times (-2) = \frac{4}{3} \frac{1}{\epsilon}
\]

in agreement with eq.(86). This point was overlooked in ref. [2].

The results in Appendix have been obtained by subtracting the pole term of the internal divergent sub-diagrams, without making use of the equations of motion. With this choice, the relation between the anomalous dimension in NDR and HV remains valid diagram by diagram, eq.(51). We have checked that our results satisfy eq.(51) diagram by diagram, as the reader can verify by himself using the results from tables 2 and 3. Below we will discuss how this works in three specific examples.

The fact that the sum of all the counter-terms due to operators which vanish by the equations of motion cancel, makes the relation (51) valid also for the complete matrix.

For completeness we report also the explicit form of the two-gluon counter-term. The pole part, obtained from the first diagram in figs.12c, is given by
(colour-charge factors omitted):

\[
\frac{4}{3} F_P^{HV,NDR}(\epsilon) Q^{\mu\nu}
\]

(87)

\[Q^{\mu\nu} = (g^{\mu\nu}(k_1 - k_2)_\perp - (2k_1 + k_2)\gamma^\mu_L + (k_1 + 2k_2)\gamma^\nu_L)\]  \hspace{1cm} (88)

and the finite part, i.e. \(F_P^{HV,NDR}(\epsilon)\) is (as expected) equal to the finite part obtained from the diagram of fig.12b, cf. eq.(83). We thus find that the counter-term is in this case:

\[
\hat{G}_{2g}^a = -\hat{G}_{2g}^b = \frac{4}{3} \frac{1}{\epsilon} Q^{\mu\nu}
\]

(89)

A diagramatic representation of the one-loop diagrams and the corresponding counter-terms is given in figs.12a-c.

### 5.4 Two-Loop Current-Current Diagrams.

After the detailed study of the one loop diagrams, we are ready to show the construction of a “complete” two-loop diagram, including all necessary counter-terms. We will consider as an example the diagram of fig.10 denoted as \(V_{17}\), where we insert a \(\gamma^\mu_L \otimes \gamma^\mu_L\) vertex. We denote by \(D_{17}^{LL}\) the double and single pole contribution from this diagram. We then substitute to the internal loop, including the string of Dirac matrices, the suitable counter-term, \(\hat{G}^{(1)}_{LL}\). We denote by \(C_{17}^{LL}\) the double and single pole contribution from this counter-term. We have finally to correct for the mixing between 4-dimensional operators and EO which occurs at one loop. This corresponds to the last term in eq.(49), the formula which gives the two-loop anomalous dimension\(^{13}\). To obtain this term we substitute to the internal loop the combination:

\[
\hat{E}^{(17)}_{LL} = -\frac{1}{2} \frac{C^{(1)}}{\epsilon} \left( \frac{\gamma^\nu \gamma^\rho \gamma^\mu_L \gamma^\rho_L \gamma^\nu_L}{4\epsilon} \frac{E^{(1)}_{LL}(0)}{F^{(1)}_{LL}(\epsilon)} \left( -\frac{\Gamma^\mu_L}{\epsilon} \right) \right)
\]

\[
\sim -\frac{C^{(1)}}{8\epsilon} E^{(1)}_{LL}
\]

(90)

\(^{13}\) The last term in eq.(49) receives a contribution also from non ”effervescent” operators in the case of a non minimal subtraction. In this respect there is not much difference between EO and finite subtractions of 4-dimensional operators.
The above equality is valid up to terms which do not contribute to the two-loop anomalous dimension because are of higher order in $\epsilon$, cf. eq. (79). We denote by $E_{C,LL}^{(17)}$ the contribution from this insertion. The contribution proportional to the first coefficient of the beta function $\beta_0$ in eq. (49) is absent for the diagram $V_{17}$. It will only give a contribution for those diagrams, which contain an internal loop corresponding to the renormalization of $\alpha_s$, as for example $V_{12}$. The term $\sim \beta_0$ is automatically taken into account by subtracting the counter-term corresponding to the renormalization of the strong vertex.

The result for the “complete” diagram is thus given by the sum of the three different contributions, see fig.13:

$$\bar{D}_{LL}^{(17)} = D_{LL}^{(17)} - C_{C,LL}^{(17)} + E_{C,LL}^{(17)}$$  \hspace{1cm} (91)

This ends our discussion of current-current counter-terms. In tables 6 and 7 we report in units of $(\alpha_s/4\pi)^2$, $(\alpha_s\alpha_e/16\pi^2)$, the double and single pole contribution of all the current-current diagrams and of the corresponding counter-terms, specifying the contribution of EO counter-terms whenever present.

From table 6 we notice that, in HV, diagrams which can be changed one into the other by Fierz rearrangement, like $V_{17}$ and $V_{20}$ for example, give exactly the same contribution to the anomalous dimension (double and single poles). This is however true only for the “complete” diagrams, $\bar{D}_{LL}^{(17)}$ and $\bar{D}_{LL}^{(20)}$, while there is no relation between the corresponding bare diagram contributions, $D_{LL}^{(17)}$ and $D_{LL}^{(20)}$. This happens because the HV regularization scheme preserves the Fierz properties of the four dimensional basis of the renormalized operators defined via $\overline{MS}$ subtraction. The renormalized operators are however obtained only in combination with all the possible counter-terms. On the other hand, $\overline{MS}$ NDR does not respect the Fierz properties of the renormalized operators, as it can be seen from the results reported in tables 8-10. It would be possible however to obtain a two-loop anomalous dimension which respects the Fierz symmetry by a suitable finite one-loop redefinition of the renormalized operators [18, 19]. Since Fierz

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14 This is strictly true with the projection operators (71) and (72). Also in HV one can choose projection operators such that Fierz rearrangement relations are not satisfied at two loops.
rearrangement is not valid in NDR, $\gamma_L^\mu \otimes \gamma_\mu R$ and $-2(1 - \gamma_5) \otimes (1 + \gamma_5)$ Fierz rearranged operators renormalize differently. Notice that in the case of $-2(1 - \gamma_5) \otimes (1 + \gamma_5)$ operators, diagrams denoted by $P_i$ in fig.11 do not vanish. We have also computed the renormalization of the effective Hamiltonian in the Fierz rearranged basis and verified that we obtain consistent results, i.e. that the anomalous dimension matrices in the basis (3) and in the rearranged one satisfy eq.(51).

As already anticipated, the relation which connects the two-loop anomalous dimension in two different regularization schemes (51) holds diagram by diagram. This means that in eq.(51) (or in eq.(41)), we have to interpret the two-loop anomalous dimension $\hat{\gamma}^{(1)}$ (or $\hat{Z}_1^{(2)}$) as the contribution of the particular diagram under consideration. We have also to interpret the one-loop anomalous dimension $\hat{\gamma}^{(0)}$ (or $\hat{Z}_1^{(1)}$) and one-loop matrix $\hat{r}$ (or $\hat{Z}_0^{(1)}$) as due to all the possible diagrams obtained by eliminating one of the gluon propagators in the two-loop diagram.

We illustrate further this point by discussing again the current-current diagram $V_{17}$ of fig.10. Eq.(51), for a single diagram, has to be interpreted as follows. The commutator $[\Delta r, \gamma_s^{(0)}]$ corresponds to the difference of the finite parts of the internal diagram $\Delta r_{V_1}$, corresponding to diagram $V_1$, times the pole term of the external diagram $p_{V_2}$ minus the difference of the finite parts of the external diagram $\Delta r_{V_2}$ times $p_{V_1}$. Omitting colour-charge factors, in the case at hand we have:

$$\Delta p^{(1)} = p_{NDR}^{(1)} - p_{HV}^{(1)} = \frac{1}{2} \left( \Delta r_{V_1} p_{V_2}^{(0)} - p_{V_1}^{(0)} \Delta r_{V_2} \right) \to \frac{7}{2} - \left( -\frac{1}{2} \right) = \frac{1}{2} \left( \left(-2\right) \times \left(-4\right) - 1 \times 0 \right)$$

which coincides with the difference of the $1/\epsilon$ term reported in table 3 for $V_{17}$.

In general the appropriate formula is:

$$\Delta p^{(1)} = p_{NDR}^{(1)} - p_{HV}^{(1)} = \frac{1}{2} \sum_{a,b} \left[ \Delta r_a \cdot p_b^{(0)} - p_a^{(0)} \cdot \Delta r_b \right]$$

$a$ and $b$ indicate all the possible sub-diagrams which appear in the two-loop diagram. $p_{NDR,HV}^{(1)}$, which is the single pole of the two-loop diagram, has a
subscript because it depends on the regularization. \( p_a^{(0)} \) is the one-loop pole term for the diagram \( a \). \( \Delta r_a \) is the difference of the finite terms between NDR and HV for the sub-diagram \( a \). If \( p_a^{(0)} \) includes the poles corresponding to the renormalization of the strong coupling constant, eq. (93) incorporates automatically the last term of eq. (51). We have verified diagram by diagram that all our results satisfy eq. (13).

5.5 Two-Loop Penguin Diagrams.

We will consider, as an example of the two-loop penguin diagrams, the diagram of fig.11 denoted as \( P_3 \), where we insert a \( \gamma_\mu^L \otimes \gamma_\mu^L \) vertex. We denote by \( P_{LL}^{(3)} \) the double and single pole contribution from this diagram. We then substitute to the internal loop, including the string of Dirac matrices, the suitable counter-term, \( \hat{G}_{LL}^{(2)} \). In this case however we have also to subtract the two-gluon counter-term since the first of the two diagrams in fig.12c is also contained in \( P_3 \) as a divergent sub-diagram. We denote by \( C^{(3)}_{P,LL} \) the double and single pole contribution from all the counter-terms. We have finally to correct for the mixing between 4-dimensional operators and EO which occurs at one loop. This corresponds to the last term in eq. (49), the formula which gives the two-loop anomalous dimension. As was the case for \( V_{17} \), to obtain this term, we substitute to the internal loop the combination:

\[
\hat{E}_{LL}^{(3)} = -\frac{1}{2} C^{(2)} \left( \frac{\Gamma^\mu_\nu \gamma^\rho \otimes \Gamma_{\mu L} \gamma_\nu \gamma_\rho}{4\epsilon} \frac{F_{LL}^{(2)}(0)}{F_{LL}^{(2)}(\epsilon)} - \frac{(-4\Gamma^\mu_\mu)}{\epsilon} \right)
\]

\[
\sim -C^{(2)} \times \frac{1}{8\epsilon} E_{LL}^{(2)}
\]

and denote by \( E_{P,LL}^{(3)} \) the double and single pole contribution from this insertion. We thus obtain:

\[
\hat{P}_{LL}^{(3)} = P_{LL}^{(3)} - C_{P,LL}^{(3)} + E_{P,LL}^{(3)}
\]

A diagramatic representation of the subtraction procedure is reported in fig.14. In HV two diagrams which go one into the other by Fierz rearrangement give the same contribution to the anomalous dimension. All the results
for the “complete” penguin diagrams are given in tables 8-10 for HV and for NDR.

We can check eq. (51) also in the case of $P_3$. We call $\Delta r_{2g}$, $\Delta r_{2gct}$ and $p_2g$, $p_2g_{ct}$ the finite part and pole term of the two-gluon diagram of fig. 12c and $\Delta r_{2gct}$ and $\Delta r_{2gct}$ the corresponding quantities for the insertion of the operator $Q^{\mu\nu}$ of eq. (88), see fig. 15. One then obtains ($p_2g = -4/3$, $p_2g_{ct} = 1$, $\Delta r_{2g} = 4/3$ and $\Delta r_{2gct} = 0$):

$$\Delta p(1) = \frac{1}{2} \left( \Delta r_{V2} p_{P1}^{(0)} - p_{V2}^{(0)} \Delta r_{P1} + \Delta r_{2g} p_{2g_{ct}} - p_{2g} \Delta r_{2g_{ct}} \right) = \frac{10}{3} \quad (96)$$

in agreement with the results of table 8.$^{15}$.

We notice that our results for the diagrams $P_8$ and $F_8$ in HV satisfy eq. (51), as can be easily verified. For these diagrams, in HV, the authors of ref. [2] found a different result which fails to satisfy eq. (51).

5.6 Terms Which Vanish by Equations of Motion and Gauge Invariance

The results reported in tables 8-10 have been obtained by subtracting the pole term, including the EO, eqs. (79), for all the divergent sub-diagrams. In doing so, we have subtracted also operators which vanish by the equations of motion or non-gauge invariant operators. As already discussed before, we could have subtracted only four-fermion operators and obtained the same final result. However contributions of single diagrams would have been different and the check between HV and NDR, eq. (51), would have not be valid diagram by diagram. Besides this we have of course to take into account the mixing with “effervescent” operators. This mixing is responsible, besides other effects, of the non-vanishing of diagrams $P_{16}$ and $F_{16}$ in HV.

The presence of operators which vanish by the equations of motion and non-gauge invariant operators is signaled by the appearence of various tensor products in the calculation of the different two-loop Feynman diagrams (we follow the notation of ref. [3]):

$$T_1 = \gamma^L_\mu \otimes \gamma^\mu$$

$^{15}$When we take into account the multiplicity.
\[T_2 = \gamma^\mu (1 - \gamma_5) \otimes \gamma^\rho q^2 \]
\[T_3 = T_1 \times \frac{p \cdot r}{q^2} \tag{97}\]
\[T_4 = (\gamma^\mu (1 - \gamma_5) \otimes \gamma^\nu (1 - \gamma_5) \otimes \gamma^\rho q^2) \frac{1}{q^2} \]
\[T_5 = S_{\mu \sigma \nu} p^\mu r^\nu (1 - \gamma_5) \otimes \gamma^\sigma q^2 \]

where \(-p\) is the ingoing momentum, \(r\) the outgoing momentum, \(q = p + r\) and \(S_{\mu \sigma \nu} = \gamma_\mu \gamma_\sigma \gamma_\nu - \gamma_\nu \gamma_\sigma \gamma_\mu\).

\(T_2\) and \(T_4\) vanish by using the equations of motion:
\[\bar{u}(r)f = \dot{p} u(p) = 0 \tag{98}\]

and
\[T_3 \rightarrow \frac{T_1}{2} \quad T_5 \rightarrow -T_1 \tag{99}\]

Non-gauge invariant operators can be eliminated by using the background field Feynman gauge\[30\] for the non-abelian penguin diagrams, for example \(P_3\) and \(P_6\) in fig.11. This allows simple checks of the gauge invariance of the final result, because non-gauge invariant operators cancel when combining together sub-sets of two-loop penguin diagrams.

We give an explicit example of this cancellation by considering diagrams \(P_2, P_3\) and \(P_4\) in HV. The results for these diagrams, using the background field gauge, putting \(p^2 = r^2 = 0\) and reporting only double and single poles are:

\[\hat{P}_2^\mu = \left( \frac{1}{2 \epsilon^2} - \frac{59}{36 \epsilon} \right) q^2 \gamma^\mu_L + \left( -\frac{2}{3 \epsilon^2} + \frac{1}{9 \epsilon} \right) \gamma^\rho q^\mu \]
\[+ \left( -\frac{1}{3 \epsilon^2} + \frac{29}{18 \epsilon} \right) \gamma^\rho q^\mu + \left( -\frac{1}{3 \epsilon^2} + \frac{5}{18 \epsilon} \right) \hat{P}_L q^\mu \]
\[+ \left( \frac{4}{9 \epsilon^2} - \frac{16}{27 \epsilon} \right) \hat{P}_L q^\mu + \left( \frac{1}{2 \epsilon^2} + \frac{1}{2 \epsilon} \right) i \epsilon^{\mu \nu \rho \sigma} q_\nu p_\rho \gamma^\sigma \]
\[+ (p \leftrightarrow r) \]

\[\hat{P}_3^\mu = \left( -\frac{5}{2 \epsilon^2} + \frac{59}{36 \epsilon} \right) q^2 \gamma^\mu_L + \left( \frac{8}{3 \epsilon^2} - \frac{10}{9 \epsilon} \right) \gamma^\rho q^\mu \]
The explicit results are given for a gluon propagator stemming from the upper incoming quark leg, as shown in fig.11. $p$ is replaced by $r$ (and the term proportional to $\epsilon_{\mu\nu\rho\sigma}$ changes sign) when the gluon is attached to the outgoing quark leg. Notice that $q_\mu \bar{P}_\mu^2$ is different from zero and similarly for $\bar{P}_3^\mu$ and $\bar{P}_4^\mu$, indicating the presence of non-gauge invariant operators\textsuperscript{16}. However:

$$q_\mu \left( \bar{P}_2^\mu + \bar{P}_3^\mu \right) = 0$$
$$q_\mu \left( \bar{P}_2^\mu - \bar{P}_3^\mu + 2 \bar{P}_4^\mu \right) = 0$$

if we use the background field gauge for the non-abelian diagram. The sum of the above diagrams eliminates the non-gauge invariant operators. By writing the term $\sim \epsilon_{\mu\nu\rho\sigma}$ as a term proportional to $q^2 \gamma^\mu_L - \hat{q}_L q^\mu$ plus terms which vanish by the equations of motion and by eliminating for the same reason all terms proportional to $\hat{p}$ or $\hat{r}$, we arrive to the results reported in the tables.

We conclude this section by summarizing some checks done to verify the correctness of our calculation.

1) We have verified that the $1/\epsilon^2$ contribution of the counter-term is twice the corresponding term of the original diagram, as imposed by eq.(47).

\textsuperscript{16}We use the on-shell identity $p \cdot q = q^2/2$
2) We have verified the cancellation of all the single poles accompanied by logarithms of the external momenta, i.e. \( \sim 1/\epsilon \times \ln(p^2/\mu^2), 1/\epsilon \times \ln(r^2/\mu^2) \) or \( 1/\epsilon \times \ln(q^2/\mu^2) \). For penguin diagrams, this cancellation does not follow automatically from 1).

3) We have verified the cancellation of all non-gauge invariant terms in the background Feynman gauge and repeated the calculation in the standard Feynman gauge with the same final result.

4) We have verified that we get the same result by subtracting all the pole terms in the internal loops, i.e. by subtracting also operators which vanish by the equations of motion, or by subtracting only those counter-terms which are proportional to four-fermion operators.

5) We have verified that eq.(51), which relates the NDR and HV schemes, is valid diagram by diagram. We have also verified that eq.(51) is valid for the matrices \( \hat{\gamma}^{(1)}_s \) and \( \hat{\gamma}^{(1)}_e \).

In doing so we have found a difference in the result of diagram \( P_8 (F_8) \) between our calculation and the calculation reported in ref.[2], as can be read from tables 8-10. Our results for these diagrams, \( P_8 \) and \( F_8 \), satisfy eq.(51), while the results of ref.[2] do not. Moreover the authors of ref.[2] overlooked the different renormalization of the operator \( \gamma^\mu_L \otimes \gamma^\mu_L + \gamma^\mu_L \otimes \gamma^\mu_R \) with respect to the renormalization of \( 1/2 (\gamma^\mu_L \otimes \gamma_{\mu L} + \gamma^\mu_L \otimes \gamma_{\mu R}) \) and consequently do not consider the diagrams that we call \( P_{16} \) and \( F_{16} \). These differences compensate in the calculation of \( \hat{\gamma}^{(1)}_s \). It is however easy to show that, by using for the diagrams the results of ref.[2], one cannot satisfy the relation (51) for \( \hat{\gamma}^{(1)}_e \). As a further check we also computed the anomalous dimension matrix at \( O(\alpha_s^2) \), since in this case different diagrams enter with different weights with respect to \( \hat{\gamma}^{(1)}_s \) and \( \hat{\gamma}^{(1)}_e \). We found again that only our results satisfy (51), contrary to those of ref.[2]. This gives us further confidence on the results reported here.

We finally notice that in ref.[4], the authors have only computed the anomalous dimension matrix in NDR. They get the matrix in HV using eq.(51). Since we agree in NDR, the final result of ref.[4] is correct, in spite of the difference found for the two diagrams in HV.
5.7 Quark Self-Energy Diagrams

The calculation of the quark self energy diagrams at one and two loops is straightforward and has already been done by several authors, see for example [13]. We only report in table 5 of the Appendix the results of the different diagrams shown in figs.4, 6 and 18.
6 The anomalous dimension matrices

In this section we collect the results of the calculation of the anomalous dimension matrices at one- and two-loop level both in the HV and NDR renormalization schemes and the matrices $\Delta \hat{\gamma}$ and $\Delta \hat{s}$, relevant for the comparison of the results in the two schemes.

6.1 One-loop results

The one-loop anomalous dimension matrices [15, 17] and $\Delta \hat{\gamma}$ are presented here. They can be computed from the pole and finite parts of the one-loop diagrams in figs.3a-c and 9, given in table 2, and from the diagrams of figs.4 and 6a, table 5. We also report the $\delta \hat{\gamma}$ and $\delta \hat{s}$ matrices introduced at the end of sec.2 to take into account the bottom threshold.

The one-loop $O(\alpha_s)$ matrix is given by

$$\hat{\gamma}^{(0)}_s =$$

$$\begin{pmatrix}
-\frac{6}{N} & 6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
6 & -\frac{6}{N} & -\frac{2}{3N} & \frac{2}{3} & -\frac{2}{3N} & \frac{2}{3} & 0 & 0 & 0 & 0 \\
0 & 0 & -\frac{6}{N} & \frac{6}{N} & -\frac{18N^2}{2f} & -\frac{9N^2f}{3N} & 0 & 0 & 0 & 0 \\
0 & 0 & -\frac{2}{3N}f & \frac{2}{3}f & -\frac{2}{3N}f & \frac{2}{3}f & 2f & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{6}{N} & -6 & 0 & 0 & 0 & 0 \\
0 & 0 & -\frac{2}{3N}f & \frac{2}{3}f & -\frac{2}{3N}f & \frac{2}{3}f & \frac{18-18N^2+2Nf}{3N} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{6}{N} & -6 & 0 & 0 & 0 & 0 \\
0 & 0 & -\frac{2u-d}{3N} & \frac{2u-d}{3N} & -\frac{2u-d}{3N} & \frac{2u-d}{3N} & 0 & 0 & \frac{6-6N^2}{N} & 0 \\
0 & 0 & \frac{2u-d}{3N} & \frac{2u-d}{3N} & -\frac{2u-d}{3N} & \frac{2u-d}{3N} & 0 & 0 & -\frac{6}{N} & 6 \\
0 & 0 & -\frac{2u-d}{3N} & \frac{2u-d}{3N} & -\frac{2u-d}{3N} & \frac{2u-d}{3N} & 0 & 0 & 6 & -\frac{6}{N}
\end{pmatrix}$$

where $f$, $u$ and $d$ represent the number of flavours, the number of charge $2/3$ up-like quarks and the number of charge $-1/3$ down-like quarks respectively.

For the electro-magnetic anomalous dimension matrix one finds:
\[ \dot{\gamma}_e^{(0)} = \] (101)

\[
\begin{pmatrix}
-\frac{8}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -\frac{8}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

At one loop, from the finite terms in table 2 we also obtain:

\[ \Delta \hat{r} = \] (102)

\[
\begin{pmatrix}
\frac{4-2N^2}{N} & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-2 & \frac{4-2N^2}{N} & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{10-6N^2}{3N} & \frac{1}{3} & -\frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -2 & \frac{4-2N^2}{N} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{5-2N^2}{N} & -6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -4 & \frac{8-4N^2}{N} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{8-2N^2}{N} & -6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -4 & \frac{8-4N^2}{N} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -4 & \frac{8-2N^2}{N} & -6 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -4 & \frac{8-4N^2}{N} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & \frac{4-2N^2}{N} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & \frac{4-2N^2}{N} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & \frac{4-2N^2}{N} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & \frac{4-2N^2}{N} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & \frac{4-2N^2}{N} & 0
\end{pmatrix}
\]

in the strong case and

\[ \Delta \hat{s} = \] (103)
The matrices $\delta \hat{r}$ and $\delta \hat{s}$ in eq.(18) are given by

$$\delta \hat{r} = \hat{r}(u, d) - \hat{r}(u, d - 1) =$$

$$
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{5}{9N} & -\frac{5}{9} & \frac{5}{9N} & -\frac{5}{9} & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{5}{9N} & -\frac{5}{9} & \frac{5}{9N} & -\frac{5}{9} & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{5}{9N} & -\frac{5}{9} & \frac{5}{9N} & -\frac{5}{9} & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{5}{18N} & \frac{5}{18} & -\frac{5}{18N} & \frac{5}{18} & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{5}{18N} & \frac{5}{18} & -\frac{5}{18N} & \frac{5}{18} & 0 & 0 & 0 & 0 \\
\end{pmatrix}
$$

$$\delta \hat{s} = \hat{s}(u, d) - \hat{s}(u, d - 1) =$$

$$
\begin{pmatrix}
-\frac{2}{9} & 0 & 0 & 0 & 0 & 0 & \frac{8N}{27} & 0 & \frac{8N}{27} & 0 \\
0 & -\frac{2}{9} & 0 & 0 & 0 & 0 & \frac{8N}{27} & 0 & \frac{8N}{27} & 0 \\
0 & 0 & -\frac{2}{3} & 0 & 0 & 0 & -\frac{8}{27} & 0 & \frac{8}{27} & 0 \\
0 & 0 & 0 & -\frac{2}{3} & 0 & 0 & -\frac{8}{27} & 0 & \frac{8}{27} & 0 \\
0 & 0 & 0 & 0 & -\frac{2}{3} & 0 & \frac{20}{9} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -\frac{2}{3} & 0 & \frac{20}{9} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{10}{9} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{10}{9} & 0 & 0 & 0 \\
0 & 0 & \frac{2}{9} & 0 & 0 & 0 & -\frac{8}{27} & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{2}{9} & 0 & 0 & -\frac{8}{27} & 0 & 0 & 0 \\
\end{pmatrix}
$$

in the electromagnetic one.
where \( u (d) \) is the number of active u-type (d-type) quarks. These matrices are equal in the HV and NDR schemes.

### 6.2 Two-loop results in HV and NDR

All the ingredients needed for the calculation of the two loop anomalous dimension have been given in secs. 4 and 5. Our final results for the two-loop anomalous dimension matrix introduced in eq.(7) are given below in the HV and NDR regularization schemes.

Each Feynman diagram can be schematically represented as the product of three factors:

\[
\text{(Colour - Charge)} \times \text{(Dirac)} \times \text{(Numerical result of loop integrations)}
\]

The numerical contribution of each diagram can be found in tables 6-10, for vertex and penguin diagrams. It is given in units of \( \alpha_s/4\pi (\alpha_e/4\pi) \). The Dirac structure can be read directly from the diagrams with few exceptions which are explicitly reported in the tables. The results of the easy but tedious calculation of the colour-charge factors are not given.

The non-zero two-loop anomalous dimension of the weak current has been put to zero by a suitable redefinition of the current at one-loop. The effect of this redefinition has been already taken into account in in the calculation of the coefficient functions. Without this redefinition, using the minimal \( \overline{MS} \) subtraction in the HV scheme, we would have found \( \gamma_J = (N^2 - 1)/2N \times 4\beta_0 \). Both the one-loop and the two-loop anomalous dimension matrices are comprehensive of the self-energy contribution.
Table 3: Elements of the two-loop anomalous dimension matrix \((\hat{\gamma}_{s}^{(1)})_{ij}\). The elements which are not reported are equal to zero. (Continue)
| $(i, j)$ | HV | NDR |
|---|---|---|
| $(7, 3)$ | $(-3N + \frac{8}{3N}) \left( u - \frac{1}{2}d \right)$ | $(-3N + \frac{20}{3N}) \left( u - \frac{1}{2}d \right)$ |
| $(7, 4)$ | $\frac{1}{3} \left( u - \frac{1}{2}d \right)$ | $-\frac{11}{3} \left( u - \frac{1}{2}d \right)$ |
| $(7, 5)$ | $\left( 3N - \frac{10}{3N} \right) \left( u - \frac{1}{2}d \right)$ | $\left( 3N + \frac{2}{3N} \right) \left( u - \frac{1}{2}d \right)$ |
| $(7, 6)$ | $\frac{1}{3} \left( u - \frac{1}{2}d \right)$ | $-\frac{1}{3} \left( u - \frac{1}{2}d \right)$ |
| $(7, 7)$ | $\frac{44N^2}{3} - \frac{71}{6} + \frac{15}{2N^2} - \frac{8N}{3} f + \frac{10}{3N} f$ | $\frac{137}{6} + \frac{15}{2N^2} - \frac{22}{3N} f$ |
| $(7, 8)$ | $-\frac{40N^2}{3} - \frac{\gamma}{3} - \frac{8N}{3} f$ | $-\frac{100N}{3} + \frac{15}{2N^2} + \frac{13}{3} f$ |
| $(8, 3)$ | $\left( \frac{128}{27} - \frac{94}{27N} \right) \left( u - \frac{1}{2}d \right)$ | $\left( -\frac{56}{27} - \frac{178}{27N} \right) \left( u - \frac{1}{2}d \right)$ |
| $(8, 4)$ | $\left( \frac{20N}{27} + \frac{202}{27N} \right) \left( u - \frac{1}{2}d \right)$ | $\left( -\frac{16N}{27} + \frac{250}{27N} \right) \left( u - \frac{1}{2}d \right)$ |
| $(8, 5)$ | $\left( -\frac{38}{27} + \frac{86}{27N} \right) \left( u - \frac{1}{2}d \right)$ | $\left( \frac{70}{27} + \frac{74}{27N} \right) \left( u - \frac{1}{2}d \right)$ |
| $(8, 6)$ | $\left( \frac{110N}{27} - \frac{158}{27N} \right) \left( u - \frac{1}{2}d \right)$ | $\left( \frac{110N}{27} - \frac{254}{27N} \right) \left( u - \frac{1}{2}d \right)$ |
| $(8, 7)$ | $-\frac{9N^2}{2} - \frac{17}{6} + \frac{15}{2N^2} - \frac{18}{N} - \frac{4}{3} f - 2N f + \frac{10}{3N} f$ | $-\frac{203N^2 + 479}{6} + \frac{15}{2N^2} + \frac{10}{3N} f - \frac{22}{3N} f$ |
| $(8, 8)$ | $-\frac{32}{27} - \frac{86}{27N^2} + \frac{3N - \frac{4}{3}}{N} \left( u - \frac{1}{2}d \right)$ | $-\frac{170}{27} + \frac{230}{27N} - \frac{1}{3} \left( u - \frac{1}{2}d \right)$ |
| $(9, 3)$ | $\frac{56}{27} - \frac{86}{27N^2} + \frac{3N - \frac{4}{3}}{N} \left( u - \frac{1}{2}d \right)$ | $\frac{32}{27} - \frac{86}{27N^2} + \frac{3N - \frac{8}{3N}}{3N} \left( u - \frac{1}{2}d \right)$ |
| $(9, 4)$ | $-\frac{110}{27} + \frac{140}{27N} + u - \frac{1}{2}d$ | $-\frac{140}{27} + \frac{94}{27N} + \frac{1}{3} \left( u - \frac{1}{2}d \right)$ |
| $(9, 5)$ | $\frac{128}{27} + \frac{58}{27N^2} + \frac{1}{3} \left( u - \frac{1}{2}d \right)$ | $\frac{122}{27} + \frac{94}{27N^2} + \frac{1}{3} \left( u - \frac{1}{2}d \right)$ |
| $(9, 6)$ | $\left( -\frac{38N}{27} + \frac{2}{N} \right) \left( u - \frac{1}{2}d \right)$ | $\left( -\frac{3N + \frac{10}{3N}}{27N} \right) \left( u - \frac{1}{2}d \right)$ |
| $(9, 7)$ | $-\frac{140}{27} + \frac{94}{27N^2} + \frac{1}{3} \left( u - \frac{1}{2}d \right)$ | $-\frac{86N}{27} - \frac{130}{27N} + \frac{1}{3} \left( u - \frac{1}{2}d \right)$ |
| $(9, 8)$ | $\left( -\frac{38N}{27} + \frac{2}{N} \right) \left( u - \frac{1}{2}d \right)$ | $\left( -\frac{3N + \frac{10}{3N}}{27N} \right) \left( u - \frac{1}{2}d \right)$ |
| $(9, 9)$ | $\frac{44N^2}{3} - \frac{110}{3N} + \frac{57}{2N^2} - \frac{8N}{3} f + \frac{14}{3N} f$ | $-\frac{22}{3N} + \frac{57}{2N^2} - \frac{23}{3N} f$ |
| $(9, 10)$ | $\frac{44N^2}{3} - \frac{110}{3N} - \frac{57}{2N^2} - \frac{8N}{3} f + \frac{14}{3N} f$ | $\frac{19}{3} - \frac{57}{2N^2} + \frac{3}{3N} f$ |

Table 3: (Continued) Elements of the two-loop anomalous dimension matrix $(\tilde{\gamma}_a^{(1)})_{ij}$.  

45
| $(i, j)$ | HV | NDR |
|---------|----|-----|
| (1, 1) | $8N - \frac{22}{3N}$ | $8N - \frac{22}{3N}$ |
| (1, 2) | $-\frac{2}{3}$ | $-\frac{2}{3}$ |
| (1, 3) | $-\frac{8}{243}$ | $-\frac{8}{243}$ |
| (1, 4) | $\frac{268}{243}$ | $\frac{268}{243}$ |
| (1, 5) | $-\frac{8}{243}$ | $-\frac{8}{243}$ |
| (1, 6) | $\frac{8N^2 - 16}{27}$ | $\frac{8N^2 - 16}{27}$ |
| (1, 7) | $-\frac{8N}{9}$ | $-\frac{8N}{9}$ |
| (1, 8) | $\frac{8N^2}{27}$ | $\frac{8N^2}{27}$ |
| (1, 9) | $-\frac{8N}{9}$ | $-\frac{8N}{9}$ |
| (1, 10) | $\frac{124}{27}$ | $\frac{124}{27}$ |
| (2, 1) | $-\frac{22}{3N}$ | $-\frac{22}{3N}$ |
| (2, 2) | $-\frac{268}{243}$ | $-\frac{268}{243}$ |
| (2, 3) | $\frac{268}{243}$ | $\frac{268}{243}$ |
| (2, 4) | $-\frac{268}{243}$ | $-\frac{268}{243}$ |
| (2, 5) | $\frac{268}{243}$ | $\frac{268}{243}$ |
| (2, 6) | $\frac{520N}{243} + \frac{884}{243N}$ | $-\frac{200N}{243} + \frac{1316}{243N}$ |
| (2, 7) | $-\frac{52}{9}$ | $-\frac{52}{9}$ |
| (2, 8) | $\frac{520N}{243} - \frac{1636}{243N}$ | $-\frac{200N}{243} + \frac{1348}{243N}$ |
| (2, 9) | $\frac{124}{27}$ | $\frac{124}{27}$ |
| (2, 10) | $\frac{124}{27}$ | $\frac{124}{27}$ |
| (3, 3) | $\frac{1402}{243N} - \frac{88}{243} \left( u - \frac{1}{2}d \right)$ | $\frac{1690}{243N} - \frac{136}{243} \left( u - \frac{1}{2}d \right)$ |
| (3, 4) | $\frac{1402}{243} + \frac{88}{243} \left( u - \frac{1}{2}d \right)$ | $\frac{1690}{243} + \frac{136}{243} \left( u - \frac{1}{2}d \right)$ |
| (3, 5) | $-\frac{56}{243N} - \frac{88}{243} \left( u - \frac{1}{2}d \right)$ | $-\frac{56}{243N} - \frac{88}{243} \left( u - \frac{1}{2}d \right)$ |
| (3, 6) | $\frac{56}{243} + \frac{88}{243} \left( u - \frac{1}{2}d \right)$ | $\frac{56}{243} + \frac{88}{243} \left( u - \frac{1}{2}d \right)$ |
| (3, 7) | $-\frac{256N}{243} - \frac{1040}{243} + \frac{8N^2 - 16}{27} \left( u - \frac{1}{2}d \right)$ | $\frac{464N}{243} - \frac{1040}{243} + \frac{8N^2 - 16}{27} \left( u - \frac{1}{2}d \right)$ |
| (3, 8) | $\frac{16}{3} - \frac{8N}{27} \left( u - \frac{1}{2}d \right)$ | $\frac{64}{27} + \frac{8N}{27} \left( u - \frac{1}{2}d \right)$ |
| (3, 9) | $\frac{1688N}{243} + \frac{1976}{243N} + \frac{8N^2}{9} \left( u - \frac{1}{2}d \right)$ | $\frac{2480N}{243} - \frac{1400}{243N} + \frac{8N^2}{9} \left( u - \frac{1}{2}d \right)$ |
| (3, 10) | $\frac{32}{27} - \frac{8N}{9} \left( u - \frac{1}{2}d \right)$ | $\frac{112}{27} + \frac{8N}{27} \left( u - \frac{1}{2}d \right)$ |

Table 4: Elements of the two-loop anomalous dimension matrix $(\hat{\gamma}_c^{(1)})_{ij}$. The elements which are not shown are equal to zero. (Continue)
| $(i, j)$ | HV | NDR |
|---------|-----|-----|
| (4, 3) | $-\frac{641}{243} + \frac{8}{27N} f - \frac{340}{243} u - \frac{100}{243} d$ | $-\frac{641}{243} - \frac{388}{243} u + \frac{32}{243} d$ |
| (4, 4) | $-\frac{641}{243} + \frac{8}{27N} f - \frac{340}{243} u + \frac{100}{243} d$ | $-\frac{641}{243} + \frac{388}{243} u - \frac{32}{243} d$ |
| (4, 5) | $\frac{243}{27N} f - \frac{340}{243} u - \frac{100}{243} d$ | $\frac{243}{27N} f - \frac{340}{243} u + \frac{100}{243} d$ |
| (4, 6) | $-\frac{88N^2}{9} + \frac{16}{27} + \frac{88N f}{243} u - \frac{243}{243} f + \frac{104}{9N} (u - \frac{1}{2} d)$ | $-\frac{88N^2}{9} + \frac{104}{9N} (u - \frac{1}{2} d)$ |
| (4, 7) | $-\frac{8N^2 f}{27} - \frac{4}{27} f - \frac{152}{27} (u - \frac{1}{2} d)$ | $-\frac{40N^2 f}{27} - \frac{44}{27} f - \frac{56}{27} (u - \frac{1}{2} d)$ |
| (4, 8) | $32 - \frac{8N^2}{3} - \frac{8N f}{27} + \frac{124}{243} f + \frac{16N}{9 - \frac{56}{9N}} (u - \frac{1}{2} d)$ | $-\frac{8N^2}{9} + \frac{386}{243} f - \frac{100}{243} f + \frac{16N}{27 - \frac{56}{9N}} (u - \frac{1}{2} d)$ |
| (4, 9) | $\frac{26N}{9} - \frac{40}{3N} + \frac{4}{27} f + \frac{40}{9} (u - \frac{1}{2} d)$ | $-\frac{2N}{27} - \frac{40}{3N} - \frac{4}{27} f + \frac{152}{27} (u - \frac{1}{2} d)$ |
| (4, 10) | $-\frac{88}{243} (u - \frac{1}{2} d)$ | $-\frac{136}{243} (u - \frac{1}{2} d)$ |
| (5, 3) | $-\frac{6N}{27} - \frac{88N}{243} (u - \frac{1}{2} d)$ | $-\frac{6N}{27} - \frac{136}{243} (u - \frac{1}{2} d)$ |
| (5, 4) | $6 + \frac{88N}{243} (u - \frac{1}{2} d)$ | $6 + \frac{136}{243} (u - \frac{1}{2} d)$ |
| (5, 5) | $-\frac{8N^2 f}{9} + \frac{16}{27} (u - \frac{1}{2} d)$ | $-\frac{8N^2 f}{9} - \frac{112}{27} (u - \frac{1}{2} d)$ |
| (5, 6) | $\frac{40}{3} - \frac{8N^2}{9} (u - \frac{1}{2} d)$ | $\frac{40}{3} + \frac{88N}{27} (u - \frac{1}{2} d)$ |
| (5, 7) | $\frac{8N^2 f}{9} (u - \frac{1}{2} d)$ | $\frac{8N^2 f}{9} - \frac{32}{27} (u - \frac{1}{2} d)$ |
| (5, 8) | $\frac{3N}{27} f - \frac{12N}{243} u - \frac{64}{243} d$ | $\frac{3N}{27} f - \frac{12N}{243} u + \frac{64}{243} d$ |
| (5, 9) | $3 + \frac{3N}{27} f - \frac{12N}{243} u + \frac{64}{243} d$ | $3 + \frac{3N}{27} f - \frac{12N}{243} u + \frac{64}{243} d$ |
| (5, 10) | $3N - \frac{9}{27} f + \frac{12N}{243} u + \frac{64}{243} d$ | $3N - \frac{9}{27} f + \frac{12N}{243} u + \frac{64}{243} d$ |
| (6, 3) | $-\frac{2N}{27} f - \frac{12N}{243} u - \frac{64}{243} d$ | $-\frac{2N}{27} f + \frac{12N}{243} u + \frac{64}{243} d$ |
| (6, 4) | $27N f - \frac{27N}{243} u + \frac{64}{243} d$ | $27N f - \frac{27N}{243} u + \frac{64}{243} d$ |
| (6, 5) | $3 + \frac{3N}{27} f - \frac{12N}{243} u + \frac{64}{243} d$ | $3 + \frac{3N}{27} f - \frac{12N}{243} u + \frac{64}{243} d$ |
| (6, 6) | $3N - \frac{9}{27} f + \frac{12N}{243} u + \frac{64}{243} d$ | $3N - \frac{9}{27} f + \frac{12N}{243} u + \frac{64}{243} d$ |
| (6, 7) | $-2 + \frac{88N}{243} f + \frac{12N}{243} f + \frac{104}{9N} (u - \frac{1}{2} d)$ | $-2 + \frac{136}{243} f + \frac{200}{243} f + \frac{104}{9N} (u - \frac{1}{2} d)$ |
| (6, 8) | $\frac{22N}{3} - \frac{16}{3N} f - \frac{4}{27} f + \frac{136}{27} (u - \frac{1}{2} d)$ | $\frac{22N}{3} - \frac{16}{3N} f - \frac{4}{27} f + \frac{136}{27} (u - \frac{1}{2} d)$ |
| (6, 9) | $\frac{22N}{3} - \frac{16}{3N} f - \frac{4}{27} f + \frac{136}{27} (u - \frac{1}{2} d)$ | $\frac{22N}{3} - \frac{16}{3N} f - \frac{4}{27} f + \frac{136}{27} (u - \frac{1}{2} d)$ |
| (6, 10) | $\frac{22N}{3} - \frac{16}{3N} f + \frac{4}{27} f + \frac{136}{27} (u - \frac{1}{2} d)$ | $\frac{22N}{3} - \frac{16}{3N} f + \frac{4}{27} f + \frac{136}{27} (u - \frac{1}{2} d)$ |

Table 4: (Continued) Elements of the two-loop anomalous dimension matrix $\left(\bar{\gamma}^{(1)}_{e}\right)_{ij}$ (Continue)
| $(i,j)$ | HV | NDR |
|--------|-------------------|-------------------|
| (7, 3) | $-\frac{88}{243}u + \frac{1}{4}d$ | $-\frac{136}{243}u + \frac{1}{4}d$ |
| (7, 4) | $\frac{88N}{243}u + \frac{1}{4}d$ | $\frac{136N}{243}u + \frac{1}{4}d$ |
| (7, 5) | $-4N - \frac{8}{3N} - \frac{88}{243}(u + \frac{1}{4}d)$ | $-4N - \frac{8}{3N} - \frac{136}{243}(u + \frac{1}{4}d)$ |
| (7, 6) | $\frac{20}{3} + \frac{88N}{243}u + \frac{1}{4}d$ | $\frac{20}{3} + \frac{136N}{243}(u + \frac{1}{4}d)$ |
| (7, 7) | $-4N - \frac{36}{3N} + \left(\frac{8N^2}{9} - \frac{16}{27}\right)(u + \frac{1}{4}d)$ | $-4N - \frac{36}{3N} + \left(\frac{8N^2}{9} - \frac{112}{27}\right)(u + \frac{1}{4}d)$ |
| (7, 8) | $\frac{38}{3} - \frac{8N}{27}(u + \frac{1}{4}d)$ | $\frac{38}{3} + \frac{88N}{27}(u + \frac{1}{4}d)$ |
| (7, 9) | $\frac{8N^2}{9}(u + \frac{1}{4}d)$ | $\frac{8N^2}{9}(u + \frac{1}{4}d)$ |
| (7, 10) | $\frac{8N^2}{9}(u + \frac{1}{4}d)$ | $\frac{8N^2}{9}(u + \frac{1}{4}d)$ |
| (8, 3) | $-\frac{340}{243}u - \frac{4}{243}d$ | $-\frac{148}{243}u - \frac{106}{243}d$ |
| (8, 4) | $\frac{240}{243}u + \frac{1}{243}d$ | $\frac{240}{243}u + \frac{106}{243}d$ |
| (8, 5) | $-1 - \frac{340}{243}u - \frac{4}{243}d$ | $-1 - \frac{748}{243}u - \frac{106}{243}d$ |
| (8, 6) | $\frac{11N}{3} - \frac{243}{9}u + \frac{4}{243}d$ | $\frac{11N}{3} - \frac{748}{243}u + \frac{106}{243}d$ |
| (8, 7) | $\frac{240}{243}u - \frac{1564}{243}u + \frac{4}{243}d$ | $\frac{240}{243}u - \frac{1252}{243}u - \frac{500}{243}d$ |
| (8, 8) | $\frac{20N}{3} - \frac{26}{243} + \frac{8}{243}u + \frac{4}{243}d$ | $\frac{20N}{3} - \frac{26}{243} + \frac{188}{243}u + \frac{80}{27}d$ |
| (8, 9) | $\frac{376N}{243}u + \frac{1100}{243}u + \frac{368}{243}d + \frac{368}{243}u - \frac{1252}{243}u - \frac{500}{243}d$ | $\frac{376N}{243}u + \frac{1100}{243}u + \frac{368}{243}d + \frac{368}{243}u - \frac{1252}{243}u - \frac{500}{243}d$ |
| (8, 10) | $-\frac{104}{27}u - \frac{27}{243}d$ | $-\frac{104}{27}u - \frac{38}{27}d$ |
| (9, 3) | $4N - \frac{1592}{243}u - \frac{88}{243}d$ | $4N - \frac{1736}{243}u - \frac{136}{243}d$ |
| (9, 4) | $\frac{620}{243} + \frac{88N}{243}u + \frac{1}{4}d$ | $\frac{764}{243} + \frac{136N}{243}(u + \frac{1}{4}d)$ |
| (9, 5) | $\frac{28}{243}u - \frac{8}{243}d$ | $-\frac{116}{243} + \frac{136}{243}(u + \frac{1}{4}d)$ |
| (9, 6) | $-\frac{28}{243} + \frac{88N}{243}u + \frac{1}{4}d$ | $\frac{116}{243} + \frac{136N}{243}(u + \frac{1}{4}d)$ |
| (9, 7) | $\frac{128N}{243} + \frac{520}{243}d$ | $-\frac{243}{243} + \frac{1082}{243}(u + \frac{1}{4}d)$ |
| (9, 8) | $\left(\frac{8N^2}{9} - \frac{16}{27}\right)(u + \frac{1}{4}d)$ | $\left(\frac{8N^2}{9} - \frac{112}{27}\right)(u + \frac{1}{4}d)$ |
| (9, 9) | $\frac{1100N}{243} - \frac{794}{243}u + \frac{8N^2}{9}(u + \frac{1}{4}d)$ | $\frac{1082}{243} + \frac{8N^2}{9} - \frac{32}{27}(u + \frac{1}{4}d)$ |
| (9, 10) | $-\frac{34}{27} - \frac{8N}{9}(u + \frac{1}{4}d)$ | $\frac{38}{27} + \frac{8N}{27}(u + \frac{1}{4}d)$ |

Table 4: (Continued) Elements of the two-loop anomalous dimension matrix $(\tilde{\gamma}_e^{(1)})_{ij}$ (Continue)
| \((i, j)\) | \(HV\) | \(NDR\) |
|---|---|---|
| (10, 3) | \(\frac{1333}{287N} - \frac{268}{243N} u + \frac{14}{243N} d\) | \(\frac{1333}{287N} - \frac{388}{243N} u + \frac{10}{243N} d\) |
| (10, 4) | \(\frac{243}{3N} + \frac{268}{243N} u - \frac{14}{243N} d\) | \(\frac{243}{3N} + \frac{388}{243N} u + \frac{16}{243N} d\) |
| (10, 5) | \(\frac{44}{243N} u + \frac{14}{243N} d\) | \(\frac{44}{243N} u - \frac{44}{243N} d\) |
| (10, 6) | \(\frac{243}{243N} + \frac{268}{243N} u - \frac{14}{243N} d\) | \(\frac{243}{243N} + \frac{388}{243N} u + \frac{16}{243N} d\) |
| (10, 7) | \(\frac{88}{243N} u + \frac{520}{243N} u + \frac{260}{243N} d\) | \(\frac{88}{243N} u - \frac{32}{243N} d - \frac{40}{243N} d\) |
| (10, 8) | \(-\frac{4N}{27} - \frac{9}{7} u - \frac{3}{7} d\) | \(-\frac{4N}{27} - \frac{9}{7} u + \frac{3}{7} d\) |
| (10, 9) | \(\frac{4N}{27} + \frac{8}{7} + \frac{520}{243N} u - \frac{163}{243N} d - \frac{216}{243N} d\) | \(\frac{4N}{27} + \frac{32}{27} + \frac{260}{243N} u - \frac{161}{243N} d - \frac{328}{243N} d\) |
| (10, 10) | \(-\frac{9}{243N} u + \frac{124}{27} u + \frac{28}{27} d\) | \(-\frac{9}{243N} u - \frac{2}{27} + \frac{148}{243N} u + \frac{10}{27} d\) |

Table 4: (Continued) Elements of the two-loop anomalous dimension matrix \((\hat{\gamma}^{(1)}_{e})_{ij}\)
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Figure Captions

- fig.1: Tree-level current-current diagram.
- fig.2a-c: One-loop $O(\alpha_s)$ current-current diagrams in the full theory.
- fig.3a-c: One-loop $O(\alpha_s)$ current-current diagrams in the effective theory. In the figure the relevant momenta of the external legs are given.
- fig.4: One-loop $O(\alpha_s)$ quark self-energy diagram.
- fig.5a-d: One loop $O(\alpha_e)$ current-current diagrams in the full theory, including the non-abelian diagrams. The $Z^0-W$ diagrams are also shown.
- fig.6a-b: The quark and $W$ self-energy diagrams at $O(\alpha_e)$.
- fig.7: Box diagrams contributing to $B(x_t)$.
- fig.8a-b: Penguin diagrams in the full theory. In the electro-weak case also the non-abelian diagram is shown.
- fig.9: Penguin diagrams in the effective theory. In the figure the relevant momenta of the external legs are given.
- fig.10: Current-current diagrams at two loops.
- fig.11: Penguin diagrams at two loops.
- fig.12: Schematic representation of the counter-terms: current-current (a), one-gluon penguin (b) and two-gluon counterterms (c).
• fig.13: Diagramatic representation of the subtraction procedure for a current-current diagram. The “complete” diagram is obtained by summing the diagram with a bare operator inserted, the counter-terms including those corresponding to effervescent operators and the term defined as $E_{LL}^{(17)}$ in eq.(90).

• fig.14: Diagramatic representation of the subtraction procedure for a penguin diagram. The “complete” diagram is obtained by summing the diagram with a bare operator inserted, the counter-terms including those corresponding to effervescent operators and the term defined as $E_{LL}^{(3)}$ in eq.(94). Also the two-gluon counter-term is shown.

• fig.15: One loop diagram for the $s \rightarrow d + g + g$ ($g + \gamma$) operator. From this diagram one can compute $p_{2gct}$ and $\Delta r_{2gct}$, cf. eq.(96).

• fig.16: This figure shows that the sum of diagrams $P_2$ and $P_3$ is the same with or without the two-gluon counter-terms. This corresponds to the cancellation of counter-terms which vanish by the equations of motion in the abelian case.

• fig.17: The contribution of the longitudinal term $\sim q^\mu \bar{q}$ of the penguin counter-term cancels when we sum the diagrams shown in this figure. This corresponds to the cancellation of counter-terms which vanish by the equations of motion in the abelian case.

• fig.18: Quark self-energy diagrams at two loops.
Appendix

| Diagram | $\frac{1}{\epsilon}$ | $O(1)$ |
|---------|----------------------|--------|
| $S_0$   | -1                   | -1/2   |
| $S_1$   | 1/8                  | -      |
| $S_2$   | $f/2$                | -      |
| $S_3$   | 5/8                  | -      |
| $S_4$   | 1/4                  | -      |
| $S_5$   | 11/4                 | -      |

Table 5: Single pole and finite part for the one-loop and two-loop self-energy diagrams in fig. 4, 6 and 18. For the two-loop case only the pole part is given.
Table 6: Two-loop pole contributions for the $\gamma_L^\mu \otimes \gamma_{\mu L}$ four-quark type diagrams in fig. 10. For $V_{29}, V_{30}$ and $V_{31}$, the results proportional to the number of colour $N (V^N)$ and flavour $f (V^f)$ are separately reported.
Table 7: Two-loop pole contributions for the $\gamma_L^a \otimes \gamma_R$ four-quark type diagrams in fig. 10. For $V_{29}$, $V_{30}$ and $V_{31}$, the results proportional to the number of colour $N$ ($V^N$) and flavour $f$ ($V^f$) are separately reported.
Table 8: Two-loop pole contributions for the $P$-type penguin diagrams in fig. 11 when a $\gamma^\mu_L \otimes \gamma_\mu_L$ structure is inserted in the upper vertex. All penguin diagrams have a $\gamma^\mu_L \otimes \gamma_\mu_L$ structure, except $P_{10}$, $P_{11}$, $P_{14}$ and $P_{15}$, for which we explicitly give the $\gamma^\mu_L \otimes \gamma_\mu \gamma_5$ ($P_{LV}$) and the $\gamma^\mu_L \otimes \gamma_\mu \gamma_5$ ($P_{LA}$) part. For $P_4$ and $P_6$ both the Feynman ($P$) and the background Feynman ($P_{bg}$) gauge results are reported. Diagrams $P_5$, $P_7$ and $P_{12}$ are not included, because their pole parts vanish and they do not contribute to the two-loop anomalous dimension. Some contributions to diagram $P_{13}$ are identical for the bare diagram and the counter-term. This means that one does not need to compute these terms since they cancel in the final result. For this reason they have not been reported in the table.
been reported in the table. For this reason they have not been included.

Some contributions to diagram $F_{13}$ are identical for the bare diagram and the background Feynman ($F_{bg}$) gauge results are reported. Diagrams $F_5$, $F_7$ and $F_{12}$ are not included, because their pole parts vanish and they do not contribute to the two-loop anomalous dimension. Some contributions to diagram $F_{13}$ are identical for the bare diagram and the counter-term. This means that one does not need to compute these terms since they cancel in the final result. For this reason they have not been reported in the table.

Table 9: Two-loop pole contributions for the $F$-type penguin diagrams in fig. 11 when a $\gamma_L^\mu \otimes \gamma_{\mu L}$ structure is inserted in the upper vertex. All penguin diagrams have a $\gamma_{\mu L}^\mu \otimes \gamma_{\mu}$ structure, except $F_{10}$, $F_{12}$, $F_{14}$ and $F_{15}$, for which we explicitly give the $\gamma_{\mu L}^\mu \otimes \gamma_{\mu}$ (FLV) and the $\gamma_{\mu L}^\mu \otimes \gamma_{\mu} \gamma_5$ (FLA) part. For $F_4$ and $F_6$ both the Feynman ($F$) and the background Feynman ($F_{bg}$) gauge results are reported.
been reported in the table. Terms since they cancel in the final result. For this reason they have not been reported. Diagrams $F_{5}$, $F_{7}$ and $F_{12}$ are not included, because their pole parts vanish and they do not contribute to the two-loop anomalous dimension. Some contributions to diagram $F_{13}$ are identical for the bare diagram and the counter-term. This means that one does not need to compute these terms since they cancel in the final result. For this reason they have not been reported in the table.

| $T_{N}$ | $\frac{1}{\zeta}$ | $\frac{1}{\zeta}$ | $\frac{1}{\zeta}$ | $\frac{1}{\zeta}$ | $\frac{1}{\zeta}$ | $\frac{1}{\zeta}$ | $\frac{1}{\zeta}$ | $\frac{1}{\zeta}$ |
|---------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| $F_{2}$ | -4               | -3              | -2              | -1              | -1              | -1              | -1              | -1              |
| $F_{3}$ | -23              | -10             | -5              | -3              | -3              | -3              | -3              | -3              |
| $F_{4}$ | -22              | -22             | -22             | -22             | -22             | -22             | -22             | -22             |
| $F_{5}$ | 8                | 2               | 1               | 1               | 1               | 1               | 1               | 1               |
| $F_{6}$ | 4                | 4               | 4               | 4               | 4               | 4               | 4               | 4               |
| $F_{7}$ | 4                | 4               | 4               | 4               | 4               | 4               | 4               | 4               |
| $F_{8}$ | 4                | 4               | 4               | 4               | 4               | 4               | 4               | 4               |
| $F_{9}$ | 4                | 4               | 4               | 4               | 4               | 4               | 4               | 4               |
| $F_{10}$ | -2               | -2              | -2              | -2              | -2              | -2              | -2              | -2              |
| $F_{11}$ | 2                | 2               | 2               | 2               | 2               | 2               | 2               | 2               |
| $F_{12}$ | -3               | -3              | -3              | -3              | -3              | -3              | -3              | -3              |
| $F_{13}$ | 8                | 8               | 8               | 8               | 8               | 8               | 8               | 8               |
| $F_{14}$ | 2                | 2               | 2               | 2               | 2               | 2               | 2               | 2               |
| $F_{15}$ | -2               | -2              | -2              | -2              | -2              | -2              | -2              | -2              |
| $F_{16}$ | -2               | -2              | -2              | -2              | -2              | -2              | -2              | -2              |

Table 10: Two-loop pole contributions for the $F$-type penguin diagrams in fig. 11 when a $\gamma_{L}^{\mu} \otimes \gamma_{\mu R}$ structure is inserted in the upper vertex. All penguin diagrams have a $\gamma_{L}^{\mu} \otimes \gamma_{\mu}$ structure, except $F_{10}$, $F_{11}$, $F_{14}$ and $F_{15}$, for which we explicitly give the $\gamma_{L}^{\mu} \otimes \gamma_{\mu}$ ($F^{LV}$) and the $\gamma_{L}^{\mu} \otimes \gamma_{\mu} \gamma_{5}$ ($F^{LA}$) part. For $F_{4}$ and $F_{6}$ both the Feynman ($F$) and the background Feynman ($F^{bg}$) gauge results are reported. Some contributions to diagram $F_{13}$ are identical for the bare diagram and the counter-term. This means that one does not need to compute these terms since they cancel in the final result. For this reason they have not been reported in the table.
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