On the physics of propagating Bessel modes in cylindrical waveguides

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J. E. Gómez-Correa
Facultad de Ingeniería Mecánica y Eléctrica,
Universidad Autónoma de Nuevo León,
C.P. 66451, Nuevo León, México and Centro de Investigación Científica y de Educación Superior de Ensenada,
Unidad Monterrey, C.P. 66629, Nuevo León, México

S. E. Balderas-Mata
Departamento de Electrónica, Universidad de Guadalajara,
C.P. 44840, Guadalajara, Jalisco, México

V. Coello
Centro de Investigación Científica y de Educación Superior de Ensenada,
Unidad Monterrey, C.P. 66629, Nuevo León, México

N. P. Puente
Facultad de Ingeniería Mecánica y Eléctrica,
Universidad Autónoma de Nuevo León, C.P. 66451, Nuevo León, México

J. Rogel-Salazar
Science and Technology Research Institute,
School of Physics Astronomy and Mathematics,
University of Hertfordshire, Hatfield, Herts., AL10 9AB, UK and Blackett Laboratory, Department of Physics, Imperial College London,
Prince Consort Road, London, SW7 2BZ, UK

S. Chávez-Cerda
Departamento de Óptica, Instituto Nacional de Astrofísica,
Óptica y Electrónica, Apdo. Postal 51/216, Puebla, México and
Abstract

In this paper we demonstrate that using a mathematical physics approach (focusing the attention to the physics and using mathematics as a tool) it is possible to visualize the formation of the transverse modes inside a cylindrical waveguide. In opposition, the physical mathematics solutions (looking at the mathematical problem and then trying to impose a physical interpretation), when studying cylindrical waveguides yields to the Bessel differential equation and then it is argued that in the core are only the Bessel functions of the first kind those who describe the transverse modes. And the Neumann functions are deemed non physical due to its singularity at the origin and eliminated from the final description of the solution. In this paper we show, using a geometrical-wave optics approach, that the inclusion of this function is physically necessary to describe fully and properly the formation of the propagating transverse modes. Also, the field in the outside of a dielectric waveguide arises in a natural way.
I. INTRODUCTION

When working on the model of a physical phenomenon, it is sometimes easy to concentrate on understanding the intricacies of the mathematical methods employed to obtain the solutions that attempt to describe the problem at hand, distracting the attention on the significance of the actual physical process in question. Once we have been successful in getting the mathematical solution, the physical process itself may not be properly addressed and emphasized, and in the worst cases it may even get lost.

In this work we deal with the description of the modes in cylindrical waveguides, where typically the physical phenomena involved are treated in such a way that the balance between the mathematical methods used and the physical constraints of the problem is tipped towards the former, leading to a somewhat unsatisfactory physical description of the formation of the modes in the waveguides. In this regard, it is worth remembering the words of Sommerfeld in the preface of his book on Partial Differential Equations, which although dealt with some physics, its main subject was actually mathematics\(^\text{1}\): “We do not really deal with mathematical physics, but physical mathematics; not with the mathematical formulation of physical facts, but with the physical motivation of mathematical methods. The oft-mentioned prestabilized harmony between what is mathematically interesting and what is physically important is met at each step and lends an esthetic - I should like to say metaphysical - attraction to our subject.” Of course, the last statement can be interpreted in both ways.

In order to provide a framework for the work presented here, let us exemplify the difference between the physical mathematics (p-mathematics) and the mathematical physics (m-physics) approaches alluded to by Sommerfeld and which we will use in the rest of this paper. Consider the problem of finding the transverse modes in a planar dielectric waveguide, on the one hand, the p-mathematics approach is used when presenting the mathematical solutions and boundary conditions in different regions of the slab. These solutions and boundary conditions are matched at each interface between regions and the description is said to be obtained. On the other hand, the m-physics approach will look instead at the physical wave as it travels through each of the regions of the slab. We can then observe how the wave is reflected and transmitted as it reaches an interface between the media in question and then set the equations of the mathematical model according to the bound-
ary conditions. We would like to note that both approaches attempt to describe the same phenomenon, however they may emphasize different aspects of the description.

In the case of the example used above, the p-mathematics approach will give the sine and cosine functions as solutions inside the waveguide, and the arguments of these functions are set to satisfy the boundary conditions. In a different manner, the m-physics approach provides a more extensive physical picture, i.e. that within the waveguide there will be traveling waves suffering reflections at both interfaces that in turn will have transverse components with the same frequency but traveling in opposite directions. In this way, for some particular conditions, the transverse modes in waveguides happen to be transverse standing waves also referred to as stationary waves.

The m-physics approach is sometimes discussed in the study of planar waveguides, mostly for the cosine function solution but, to the best of our knowledge, it has never been used for cylindrical waveguides and optical fibers. For the latter two the p-mathematics approach is the standard one presenting the solutions in terms of the Bessel functions of the first and second kind, then discarding the latter arguing that they are unphysical or not the proper ones since they are singular at the axis of the waveguide. With such arguments, based on the mathematical properties of the functions without further thinking on what these properties might physically represent, unfortunately, the physics of what really might be happening in the physical system of the cylindrical waveguides can be lost.

In this work we endeavor to use the mathematical formulation of the physical facts, i.e. the mathematical physics approach, to present a detailed physical analysis of the formation of modes in cylindrical waveguides. We demonstrate physically, with the help of propagating wave analysis, that the well-known Bessel modes are in fact the result of the interference of the counter-propagating transverse components of traveling conical waves. Contrary to the prevailing approach in the literature we show that to fully describe the propagating nature of the wave field in the core and the cladding, it is necessary to use both Hankel functions; constructed by the complex superposition of the Bessel function of the first kind $J_m$ and the Bessel function of the second kind, or Neumann function, $N_m$. Moreover, the singular behavior of these solutions at the origin, in particular of the Neumann functions, is easily explained in clear physical terms. Finally, our mathematical physics traveling-wave approach shows how the traveling wave described by the Hankel function within the waveguide becomes, in a natural way, an evanescent wave at the interface between the core
and the cladding of dielectric cylindrical waveguides as it occurs for the planar waveguides.

II. MATHEMATICAL PHYSICS OF CYLINDRICAL WAVEGUIDES

In the literature on electromagnetic theory studying wave propagation in cylindrical waveguides one encounters that the transverse field is described by the Bessel differential equation. The set of independent solutions of this equation are the Bessel functions of the first kind, \( J_m \), as well as the Bessel functions of the second kind, also known as Neumann functions, \( N_m \).

Following the literature related to cylindrical waveguides, the treatment usually follows what we are referring to as the p-mathematics approach. In other words, inclined more towards the mathematical approach. It is commonly argued that a general solution can be constructed by a linear combination of these functions, namely \( E(\rho) = AJ_m + BN_m \), where \( A \) and \( B \) are constants. This superposition is mathematically correct but unfortunately it does not necessarily give a complete physical insight to the problem at hand. The argument used is along the lines of “looking for the physical solution” inside the cylindrical waveguide, which leads to the conclusion that the constant \( B \) in the superposition above needs to be zero. This is because the Neumann function has to be discarded due to the fact that, for any \( m \), it has a singularity at the origin \( \rho = 0 \), diverging to minus infinity. This argument considers that such behavior of the solution is inconsistent with physical fields within the core of the waveguide and that, the only “proper” and physically allowed solution that is bounded is \( J_m \), i.e. the Bessel function of the first kind.

The main aim of this paper is to show that in a full description and analysis of a cylindrical waveguide, the Neumann functions become a natural part of the solution and furthermore, their presence provide a whole physical picture of how the modes are formed in cylindrical waveguides. In order to show this, we first solve the wave equation or the Helmholtz equation in cylindrical coordinates, \( \nabla^2 E(\rho, \varphi, z) + k^2 E(\rho, \varphi, z) = 0 \), by applying separation of variables and using \( E(\rho, \varphi, z) = R(\rho) \Phi(\varphi) Z(z) \) as an ansatz, we get three differential equations. Those for \( Z(z) \) and \( \Phi(\varphi) \) are:

\[
\frac{d^2 Z(z)}{dz^2} + k_z^2 Z(z) = 0, \tag{1}
\]

\[
\frac{d^2 \Phi(\varphi)}{d\varphi^2} + m^2 \Phi(\varphi) = 0, \tag{2}
\]
whose solutions, in complex form, are respectively

$$Z(z) = e^{\pm ik_z z},$$  \hspace{1cm} (3)

and

$$\Phi(\varphi) = e^{\pm im\varphi}.$$  \hspace{1cm} (4)

The third differential equation, corresponding to the radial part \(R(\rho)\) of the ansatz, is the Bessel differential equation:

$$\frac{d^2}{d\rho^2}R(\rho) + \frac{1}{\rho}\frac{d}{d\rho}R(\rho) + \left[(k^2 - k_z^2) - \frac{m^2}{\rho^2}\right]R(\rho) = 0,$$  \hspace{1cm} (5)

where we can define \(k_\rho = k^2 - k_z^2\) and the solutions in complex form are given by

$$H_m^{(1)}(k_\rho \rho) = J_m(k_\rho \rho) + i N_m(k_\rho \rho),$$  \hspace{1cm} (6)

and

$$H_m^{(2)}(k_\rho \rho) = J_m(k_\rho \rho) - i N_m(k_\rho \rho),$$  \hspace{1cm} (7)

which are known as the Hankel functions of the first and second kind, respectively. The Hankel functions are singular due to the presence of the singularity of the Neumann function. However, we will show below that this singularity has an actual physical meaning. Notice that Equation (6), represents the Green’s function of the Helmholtz equation in cylindrical coordinates: it implies that we have a source of light that emanates energy radially.

We can now focus our attention on the physics of these solutions. To simplify the description and the visualization let us take \(m = 0\); the observations below still apply for any \(m\). Incorporating the \(z\)-dependence we have \(H_0^{(1)}(k_\rho \rho) e^{ik_z z}\) and \(H_0^{(2)}(k_\rho \rho) e^{ik_z z}\). These equations represent conical waves with a total wavevector \(\vec{k} = k_\rho \hat{\rho} + k_z \hat{z}\), see Fig. 1 a). The zero-th order Hankel function of the first kind \(H_0^{(1)}\), describes radially symmetric cylindrical waves traveling away from the axis (outgoing waves). The radial component of the outgoing conical wave on reflection at the surface of the waveguide becomes an incoming conical wave described by \(H_0^{(2)}\), i.e., the zero-th order Hankel function of the second kind \(H_0^{(2)}\), represents waves traveling towards the axis (incoming waves). This is easy to visualize if the propagation of the conical wavefronts are known. The evolution of the propagation of a transverse section of the conical wavefronts are shown in Fig. 2, where the red dots represent outgoing conical wave while the black dots incoming conical wave. It is possible to observe that a
reflected incoming conical wave is generated when the outgoing one has reached the border and that the incoming conical wave is transformed into an outgoing wave when the former passes through the axis of the waveguide. It is important to say that the transverse section of the conical wavefronts is generated by the points ACDE from Fig. 1, and the transverse section of the wavefronts is divided in representative sections, solid circles in Fig. 2.

FIG. 1. a) Representation of conical counter-propagating waves which are solutions to the Helmholtz equation in a cylindrical waveguide. They generate Hankel functions. b) The sum of conical counter-propagating waves generate a Bessel function of order zero.

FIG. 2. The propagation evolution of the conical wavefronts.

Since each wave is the complex conjugate of the other, the incoming and outgoing waves have the same frequency and amplitude, and consequently, their transverse radial components move in opposite directions, and in superposition the singular Neumann functions cancel out. An alternative way to explain this fact is by means of mathematical physics: the
cylindrical waveguide supports cylindrical incoming waves traveling towards the longitudinal axis ($\rho = 0$). As they get closer and closer to the axis, the waves “collapse” into a line, and it is actually this line which simultaneously acts as the source from which the outgoing cylindrical waves emanate. This is the physical explanation of the singularity of the Hankel functions, within the waveguide there are simultaneously a sink and a source that cancel out resulting in the $J_0$ non-singular stationary wave solution\textsuperscript{16}. In the core of the cylindrical waveguide both waves exist simultaneously, this is, the solution must be given by

$$E(\rho, z) = \left[ H_0^{(1)}(k_\rho \rho) + H_0^{(2)}(k_\rho \rho) \right] e^{ik_z z} = 2J_0(k_\rho \rho)e^{ik_z z}. \quad (8)$$

In order to expose the propagating wave behavior described by the Hankel functions we introduce its asymptotic approximation and the harmonic temporal dependence $\exp(-i\omega t)$ of the wave equation is reincorporated\textsuperscript{17}:

$$H_m^{(1,2)}(k_\rho \rho) e^{ik_z z} \approx \frac{A}{\sqrt{k_\rho \rho}} e^{-i(\omega t \mp k_\rho \rho + ik_z z) - i\frac{\pi}{2}(m+\frac{1}{2})}. \quad (9)$$

From this expression it is clear the traveling wave behavior and the conical nature of the wavefronts.

One may wonder how big has to be the argument of the Hankel functions to the asymptotic expression to be valid. One finds in the literature (see e.g. Ref.\textsuperscript{17}) that this expression can be used when $k_\rho \rho \gg (4m^2 - 1)/8$ with $m \geq 1$. However, giving a quantitative value, if we require that the error be of the order of $10^{-2}$ or less we have to increase $k_\rho \rho$ by at least a factor of five:

$$k_\rho \rho \geq 5\frac{4m^2 - 1}{8}, \quad (10)$$

this approximation is now valid for representing the Bessel functions of the first kind as well as for the Neumann functions, including the zero order ones. This shown in Fig. 3 for $m = 0$ where in the top plot the Bessel and Neumann functions together with their asymptotic are superposed and the simple difference is shown in in bottom plot.

In the slab waveguides there exist incident and reflected waves inside it, as described in Refs.\textsuperscript{4} and\textsuperscript{5}. In analogy to the latter, we notice that inside the cylindrical waveguide there also exist incident as well as a reflected waves at and from the cylindrical wall of the waveguide. In both cases, when the waves are added together they form stationary waves inside the waveguide that are discussed in the literature for slab waveguides and we introduce
FIG. 3. a) The magenta continuous line is the First Kind Bessel functions and the blue dashed-line is the Neumann functions. The black asterisks and the black circles represent the asymptotic approximation of the First Kind Bessel functions and the asymptotic approximation of the Neumann functions, respectively. b) The red horizontal line represents an error of $10^{-2}$. The magenta continuous line represents the error difference between the asymptotic approximation and the First Kind Bessel functions. The blue dashed-line represents the error difference between the asymptotic approximation of the Neumann functions and the Neumann functions.

for the first time for cylindrical waveguides. From equation (8), the solution can be defined as the sum of conical counter-propagating waves.

It is well accepted that the stationary modes inside a slab waveguide are generated by the sum of counter-propagating waves bouncing back and forth from the plane walls of the slab. With this in mind we introduce another approach to describe the stationary waves inside a cylindrical waveguide from the picture of the modes of the slab waveguide. The latter is possible by rotating the slab waveguide $\pi$ radians, as it is shown in Ref. [18]. Since a plane wave in a polar coordinates can be represent by $e^{i k \rho (x \cos \varphi + y \sin \varphi) + ik_z z}$ the continuous interference of these counter-propagating plane waves in the $\pi$ rotation can be represent by

$$
\frac{1}{\pi} \int_0^\pi \cos k \rho (x \cos \varphi + y \sin \varphi) e^{i k_z z} d\varphi = \frac{1}{2\pi} \int_0^{2\pi} e^{i k \rho (x \cos \varphi + y \sin \varphi + ik_z z)} d\varphi \equiv J_0(k \rho \rho) e^{i k_z z}.
$$

(11)

In a similar way, under a change of variables it can be demonstrated that if the plane
waves have an azimuthal phase shift of the form \( e^{im\varphi} \) as they rotate the resulting field is described by

\[
J_m (k \rho) e^{-im\theta + ik_z z} = \frac{1}{2\pi} \int_0^{2\pi} e^{ik \rho (\cos \alpha \sin \alpha - ima + ik_z z)} d\alpha.
\] (12)

where \( \theta \) is now the azimuthal variable.

For \( m = 0 \), the wavefront generated by the rotation of the plane waves forms two conical surfaces; cone A, B, C represents the outgoing conical wave incident to the inside of the waveguide, and cone D, B, E represents the reflected conical incoming wavefront. This is easy to visualize due to the change in sign of the \( k_\rho \) component, see Fig. 1.

From Fig. 1 we can see that the vector \( \vec{k} \) and its components \( k_\rho \) and \( k_z \) are located in the sagital plane of the cylinder, i.e. the vectors are coplanar vectors. Notice that the integral in (11) creates a the cone of wavevectors.

Also observe that using the rotation of the planar slab description, the singular Neumann function cannot be constructed and this solution is discarded in a natural way. Physically, we can also deduce this from equations (6) and (7) since in order to get the Neumann solution we would require to subtract the waves, what would imply that there was a relative phase of \( \pi \) between the incoming and outgoing conical waves along the whole waveguide, something that does no occur. In this manner the approach presented here contributes to making the modes inside the cylindrical waveguide more physically understandable.

Now, we will demonstrate how the traveling waves inside the core become, in a natural way, the evanescent wave at the cladding. We have said that in the core \( H_0^{(1)} (k \rho \rho) e^{ik z} \) represents the outgoing wave. At the cylindrical waveguide surface \( \rho = a \) the incident wave, by total reflection, must give rise to an evanescent field outside the core. In the same way as occurs with the slab waveguide, the transverse wave number becomes purely imaginary, i.e. \( k_\rho t = i\kappa_\rho t \), resulting in having the Hankel function of the first type to be transmitted to the outer part of the cylindrical waveguide. In this case the solution in this region is given by

\[
E (\rho > a, z) = H_0^{(1)} (i\kappa_\rho \rho) e^{ik z z}.
\] (13)

This equation can be rewritten using the modified Bessel function of the second kind, i.e., \( K_m (\kappa_\rho \rho) = \frac{\pi}{2} l^{m+1} H_m^{(1)} (i\kappa_\rho \rho) \) and for \( m = 0 \) we have

\[
E (\rho > a, z) = \frac{2}{\pi} K_0 (\kappa_\rho \rho) e^{i(k z z + \frac{\pi}{2})}.
\] (14)
This result demonstrates how the outgoing traveling wave transforms in a natural way in an evanescent wave described by the modified Bessel function $K_0(\kappa \rho)$. 

Now, the transmitted wave just at the cladding is

$$J_0(\kappa \rho a) = \left| \frac{e^{i\frac{\pi}{2}}}{\pi} K_0(\kappa \rho a) \right|,$$

where we can observe that the amplitude coefficient of the transmitted wave is given by $e^{i\frac{\pi}{2}}/\pi$. Moreover, this $\pi/2$ phase shift can be interpreted as the rotation of the radial component of the wavevector to create the surface waves.

We remark that it was not necessary to make any mathematical assumptions in order to end up with evanescent waves with Eq. (14) at the surface of the cylindrical waveguide. Contrary to what is done in the physical mathematics approach where it is usually that the modified Bessel function $K_m$ is chosen over the modified Bessel function $I_m$ because the latter grows to infinity while the former one decays. In our analysis the direct use of the Hankel function as a solution inside the waveguide gives rise naturally to the evanescent wave in the form of the modified Bessel function $K_m$ without having to make any further mathematical assumptions. Furthermore, in the core, the solutions we have obtained are also able to describe the phenomena observed in tubular mirrors\textsuperscript{20,21}.

As it has been seen throughout this section, for sake of clarity the order of the Hankel function has been taken as zero, i.e., $m = 0$. In the following section we will be briefly describe the solution of the wavefronts of higher order modes $m \neq 0$ inside of the cylindrical waveguide. These modes are also known as skew modes in the Optical Fibers literature.

III. HIGHER-ORDER MODES IN CYLINDRICAL WAVEGUIDES

We will now turn our attention to the higher-order modes in cylindrical waveguides, which have the solution: a) in the core

$$E_{om}(\rho, \varphi, z) = e^{-i(k_z z + m \varphi)} H^{(1)}_m(k \rho \rho)$$

$$E_{im}(\rho, \varphi, z) = e^{-i(k_z z + m \varphi)} H^{(2)}_m(k \rho \rho),$$

The factor $e^{-im\varphi}$ represents that the phase rotates $m$ times in a period of 0 a $2\pi$, as is shown in Fig. 4.

b) In the cladding

$$E(\rho, \varphi, z) = \frac{2}{\pi} e^{-i(k_z z + m[\varphi - \frac{\pi}{2}] - \frac{\pi}{2})} K_m(k \rho \rho).$$
We have explained that lower order modes in a cylindrical waveguide have a conical wavefront. For the case of higher order modes, the wavefront is a conical helicoid, as shown in Fig. 4.

FIG. 4. High-order mode in cylindrical waveguides with $m = 4$ and its conical helicoidal wavefronts.

As the wavefront is a conical helicoid, the vector of propagation $\vec{k}$ follows a screw-like helical trajectory and its components no longer lie on a plane perpendicular to the tangential plane of the cylinder at any given point.

IV. CONCLUSIONS

We have presented a discussion of the differences between the physical mathematics and mathematical physics approaches used to describe problems in physics. We have demonstrated that using the mathematical physics approach some physical aspects of the propagation of electromagnetic waves in cylindrical waveguides are recovered that otherwise using the physical mathematics would be lost.

Having the traveling wave idea in mind, we demonstrated that the waves inside the cylindrical waveguide are in fact propagating conical waves. From a physical mathematics point of view these conical waves would never be seen, since they are described with the Hankel functions of the first and second kinds, that have a singularity due to the presence of the singular Neumann functions. Nonetheless, with the aid of the mathematical physics picture, we have shown that the Neumann functions are a very important component in the full description of the physics of these conical waves. The standing waves inside the cylindrical waveguide, which have profile given by the Bessel function of the first kind, is the result of the transverse counter-propagating component of the conical waves described with the Hankel functions. The counter-propagation results in the superposition of the incoming and outgoing waves canceling out the term with the Neumann functions in a natural manner, without the need of arbitrarily discarding them. Also, we demonstrated
how the outgoing wave transforms in a simple straightforward manner in the evanescent field when the condition for total internal reflection is satisfied.

In general, the physics oriented method presented in this paper, the mathematical physics approach, gives a more physically rich insight of the modes inside cylindrical waveguide than more prevalent physical mathematics methods in the literature.

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*jesus.gomezcr@uanl.edu.mx; permanent address: Avenida Universidad s/n. Ciudad Universitaria San Nicolás de los Garza, Nuevo León, México
†sandra.balderas@cucei.udg.mx
‡vcoello@cicese.mx
§norma.puenterm@uanl.edu.mx
¶j.rogel@physics.org
**sabino@inaoep.mx

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