ORIENTED COBORDISM OF REAL AND COMPLEX PROJECTIVE SPACES

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Abstract. It is known that the oriented cobordism class of $\mathbb{C}P^{2k-1}$ and $\mathbb{R}P^{2k+1}$ are zero for each $k > 0$. We construct oriented manifolds having the boundary either $\mathbb{C}P^{2k-1}$ or $\mathbb{R}P^{2k+1}$ for each $k > 0$. The construction is different from the previous one. The main tool is the theory of quasitoric manifolds and small covers.

1. Introduction

Cobordism was explicitly introduced by Lev Pontryagin in geometric work on manifolds. In the early 1950’s René Thom showed that cobordism groups could be computed by results of homotopy theory. Thom showed that the cobordism classes of $G$-manifolds, for a Lie group $G$, are in one to one correspondence with the elements of the homotopy group of the Thom space of the group $G \subseteq O_n$. R. Thom in [6] and M. Adachi in [1] proved that the oriented cobordism class of $\mathbb{C}P^{2k-1}$ and $\mathbb{R}P^{2m-1}$ to be zero if $k = 1, 2; m = 1, 2, 3, 4$ and $k = 3; m = 5, 6$ respectively. In this article we construct oriented manifolds having the boundary either $\mathbb{C}P^{2k-1}$ or $\mathbb{R}P^{4k+1}$ for each $k > 0$. The main tool is the theory of quasitoric manifolds and small covers.

Quasitoric manifolds and small covers were introduced by Davis and Januskiewicz in [3]. A manifold with quasitoric (small cover) boundary is a manifold with boundary where the boundary is a disjoint union of some quasitoric manifolds (respectively small covers). The strategy of our proof is to first construct some compact orientable manifolds with quasitoric or small cover boundary. Then identifying suitable boundary components using certain equivariant homeomorphisms we obtain oriented manifolds with the boundary either $\mathbb{C}P^{2k-1}$ or $\mathbb{R}P^{4k+1}$ for each $k > 0$, see theorem 3.3 and theorem 4.2.

2. Preliminaries

An $k$-dimensional simple convex polytope is a convex polytope where exactly $k$ bounding hyperplanes meet at each vertex. The codimension one faces of a convex polytope are called facets. Let $F(P)$ be the set of facets of a $k$-dimensional simple convex polytope $P$.

2.1. Quasitoric manifolds. Following [4] we give definition of quasitoric manifold, characteristic function and classification.

Definition 2.1. An action of $\mathbb{T}^k$ on an $2k$-dimensional manifold $M$ is said to be locally standard if every point $y \in M$ has a $\mathbb{T}^k$-stable open neighborhood $U_y$ and a homeomorphism $\psi : U_y \to V$, where $V$ is a $\mathbb{T}^k$-stable open subset of $\mathbb{C}^k$, and an isomorphism $\delta_y : \mathbb{T}^k \to \mathbb{T}^k$ such that $\psi(t \cdot x) = \delta_y(t) \cdot \psi(x)$ for all $(t, x) \in \mathbb{T}^k \times U_a$.

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Definition 2.2. A \( \mathbb{T}^k \)-manifold \( M \) is called a quasitoric manifold over \( P \) if the following conditions are satisfied:

1. the \( \mathbb{T}^k \) action is locally standard,
2. there is a projection map \( p : M \to P \) constant on \( \mathbb{T}^k \) orbits which maps every \( l \)-dimensional orbit to a point in the interior of a codimension-\( l \) face of \( P \).

All complex projective spaces \( \mathbb{C}P^k \) and their equivariant connected sums, products are quasitoric manifolds.

Lemma 2.1 ([\text{3}], Lemma 1.4). Let \( p : M \to P \) be a 2\( k \)-dimensional quasitoric manifold over \( P \). There is a projection map \( f : \mathbb{T}^k \times P \to M \) so that for each \( q \in P, f \) maps \( \mathbb{T}^k \times q \) onto \( p^{-1}(q) \).

A quasitoric manifold \( M \) over \( P \) is simply connected. So \( M \) is orientable. A choice of orientation on \( \mathbb{T}^k \) and \( P \) gives an orientation on \( M \).

Define an equivalence relation \( \sim_q \) on \( \mathbb{Z}^k \) by \( x \sim_q y \) if and only if \( y = -x \). Denote the equivalence class of \( x \) in the quotient space \( \mathbb{Z}^k / \mathbb{Z}_2 \) by \( [x] \).

Definition 2.3. The map \( \lambda : \mathcal{F}(P) \to \mathbb{Z}^k / \mathbb{Z}_2 \) is called characteristic function if the submodule generated by \( \{\lambda(F_{j_1}), \ldots, \lambda(F_{j_l})\} \) is an \( l \)-dimensional direct summand of \( \mathbb{Z}^k \) whenever the intersection of the facets \( F_{j_1}, \ldots, F_{j_l} \) is nonempty.

The vectors \( [\lambda(F_j)] \) are called characteristic vectors and the pair \( (P, \lambda) \) is called a characteristic pair.

In [\text{3}] the authors show that we can construct a quasitoric manifold from the pair \( (P, \alpha) \).

Also given quasitoric manifold we can associate a characteristic pair to it up to choice of signs of characteristic vectors.

Definition 2.4. Let \( \delta : \mathbb{T}^k \to \mathbb{T}^k \) be an an automorphism. Two quasitoric manifolds \( M_1 \) and \( M_2 \) over the same polytope \( P \) are called \( \delta \)-equivariantly homeomorphic if there is a homeomorphism \( f : M_1 \to M_2 \) such that \( f(t \cdot x) = \delta(t) \cdot f(x) \) for all \( (t, x) \in \mathbb{T}^k \times M_1 \).

The automorphism \( \delta \) induces an automorphism \( \delta_* \) of the poset of subtori of \( \mathbb{T}^k \). This automorphism descends to a \( \delta \)-translation of characteristic pairs, in which the two characteristic maps differ by \( \delta_* \).

Proposition 2.2 ([\text{3}], Proposition 5.14). There is a bijection between \( \delta \)-equivariant homeomorphism classes of quasitoric manifolds and \( \delta \)-translations of pairs \( (P, \lambda) \).

2.2. Small covers. Following [\text{3}] we give definition of small cover, \( \mathbb{Z}_2 \)-characteristic function and classification of small covers.

Definition 2.5. A closed \( k \)-dimensional manifold \( N \) is said to be small cover if there is an effective action of \( \mathbb{Z}_2^k \) on \( N \) such that:

1. the \( \mathbb{Z}_2^k \) action is locally isomorphic to the standard action of \( \mathbb{Z}_2^k \) on \( \mathbb{R}^k \),
2. the orbit space is a simple convex polytope.

All real projective spaces \( \mathbb{R}P^k \) and their equivariant connected sums are small covers.

Definition 2.6. The map \( \beta : \mathcal{F}(P) \to \mathbb{Z}_2^k \) is called \( \mathbb{Z}_2 \)-characteristic function on \( P \) if the span of \( \{\beta(F_{j_1}), \ldots, \beta(F_{j_l})\} \) is an \( l \)-dimensional subspace of \( \mathbb{Z}_2^k \) whenever the intersection of the facets \( F_{j_1}, \ldots, F_{j_l} \) is nonempty.

The vectors \( \beta(F_j) \) are called \( \mathbb{Z}_2 \)-characteristic vectors and the pair \( (P, \beta) \) is called \( \mathbb{Z}_2 \)-characteristic pair.
In the authors construct a small cover $N(P, \beta)$ from the pair $(P, \beta)$. Given a small cover they associate a $\mathbb{Z}_2$-characteristic pair to it.

**Lemma 2.3** ([9], Lemma 1.8). Let $Z(P)$ be the set of all $\mathbb{Z}_2$-characteristic functions on $P$. Then the collections of small covers over $P$ is given by \{ $N(P, \beta) : \beta \in Z(P)$ \}.

In [9] the authors give conditions for orientability of small cover. For a basis $\gamma = \{\gamma_1, \ldots, \gamma_k\}$ of $\mathbb{Z}_2^k$, define a homomorphism $\epsilon_{\gamma} : \mathbb{Z}_2^k \rightarrow \mathbb{Z}_2$ by $\epsilon_{\gamma}(\gamma_i) = 1$.

**Theorem 2.4** ([9], Theorem 1.7). A small cover $N(P, \beta)$ is orientable if and only if there exists a basis $\{\gamma_1, \ldots, \gamma_k\}$ of $\mathbb{Z}_2^k$ such that the image of $\epsilon_{\gamma} \circ \beta$ is \{1\}.

### 3. Torus cobordism class of $\mathbb{C}P^{2k-1}$

**3.1. Manifolds with quasitoric boundary.** Set $n = 2k$. Corresponding to each even $n \geq 4$ we construct a manifold with quasitoric boundary. Let $\{A_j : j = 0, \ldots, n\}$ be the standard basis of $\mathbb{R}^{n+1}$. Let

$$\Delta^n = \{(x_0, x_1, \ldots, x_n) \in \mathbb{R}^{n+1} : x_j \geq 0 \text{ and } \Sigma_n x_j = 1\}.$$  

Then $\Delta^n$ is an $n$-dimensional simplex with vertices $\{A_j : j = 0, \ldots, n\}$ in $\mathbb{R}^{n+1}$. Define

$$\Delta_j^{-1} = \{(x_0, x_1, \ldots, x_n) \in \Delta^n : x_j = 0\}.$$  

So $\Delta_j^{-1}$ is the facet of $\Delta^n$ not containing the vertex $A_j$. Let $F$ be the largest face of $\Delta^n$ not containing the vertices $A_{j_1}, \ldots, A_{j_l}$. Then

$$F = \cap_{i=1}^l \Delta_j^{-1} = \{(x_0, x_1, \ldots, x_n) \in \Delta^n : x_{j_i} = 0, \ i = 1, \ldots, l\}.$$  

Define a function $\xi : \{\Delta_j^{-1} : j = 0, \ldots, n\} \rightarrow \mathbb{Z}^{n-1}$ as follows.

$$\xi(\Delta_{n-j}^{-1}) = \begin{cases} 
(0, 0, 1, 0, \ldots, 0) & \text{if } 0 \leq j < \frac{n}{2} - 1, \text{ here } 1 \text{ is in the } (j+1)-\text{th place} \\
(1, 1, 0, \ldots, 0) & \text{if } j = \frac{n}{2} - 1, \text{ here } 1 \text{ occurs up to } \frac{n}{2}-\text{th place} \\
(0, 0, 1, 0, \ldots, 0) & \text{if } \frac{n}{2} \leq j < n, \text{ here } 1 \text{ is in the } j-\text{th place} \\
(0, 0, 1, 1, \ldots, 1) & \text{if } j = n, \text{ here } 0 \text{ occurs up to } (\frac{n}{2} - 1)-\text{th place}
\end{cases}$$

Define

$$\xi_j := \xi(\Delta_{n-j}^{-1}), \text{ for all } j = 0, 1, \ldots, n.$$  

**Example 3.1.** For $n = 4$, let $\Delta^4$ be the 4-simplex in $\mathbb{R}^5$ with vertices $A_0, A_1, A_2, A_3, A_4$ (see figure 4). Define a function $\xi$ from the set of facets of $\Delta^4$ to $\mathbb{Z}^3$ by,

$$\xi(\Delta^4_{n-j}) = \begin{cases} 
(1, 0, 0) & \text{if } j = 0 \\
(1, 1, 0) & \text{if } j = 1 \\
(0, 1, 0) & \text{if } j = 2 \\
(0, 0, 1) & \text{if } j = 3 \\
(0, 1, 1) & \text{if } j = 4
\end{cases}$$

Suppose the faces $F'$ and $F''$ of $\Delta^n$ are the intersection of facets $\{\Delta^{n-1}_n, \Delta^{n-1}_{n-1}, \ldots, \Delta^{n-1}_0\}$ and $\{\Delta^{n-1}_0, \Delta^{n-1}_{0-1}, \ldots, \Delta^{n-1}_0\}$ respectively. Then

$$F' = \{(x_0, x_1, \ldots, x_n) \in \Delta^n : x_{\frac{n}{2}} = 0, \ldots, x_n = 0\},$$

$$F'' = \{(x_0, x_1, \ldots, x_n) \in \Delta^n : x_0 = 0, \ldots, x_{\frac{n}{2}} = 0\}.$$
Hence \( \dim(F') = \dim(F'') = \frac{n}{2} - 1 \geq 1 \). The set of vectors \( \{\xi_0, \ldots, \xi_{\frac{n}{2}}\} \) and \( \{\xi_{\frac{n}{2}}, \ldots, \xi_n\} \) are linearly dependent sets in \( \mathbb{Z}^{n-1} \). But the submodules generated by the vectors \( \{\xi_0, \ldots, \hat{\xi}_j, \ldots, \xi_{\frac{n}{2}}\} \) and \( \{\xi_{\frac{n}{2}}, \xi_{\frac{n}{2}+1}, \ldots, \hat{\xi}_l, \ldots, \xi_n\} \) are \( \frac{n}{2} \)-dimensional direct summands of \( \mathbb{Z}^{n-1} \) for each \( j = 0, \ldots, \frac{n}{2} \) and \( l = \frac{n}{2}, \ldots, n \) respectively. Here the symbol \( \hat{} \) represents the omission of the corresponding entry.

Suppose \( e \) is an edge of \( \Delta^n \) not contained in \( F' \cup F'' \). Then \( e = \cap_{j=1}^{n-1} \Delta_{l_j}^{n-1} \) for some \( \{l_j : j = 1, \ldots, n-1\} \subset \{0, 1, \ldots, n\} \). Observe that \( \{\xi_0, \ldots, \xi_{\frac{n}{2}}\} \not\subset \{\xi_{l_1}, \ldots, \xi_{l_{n-1}}\} \) and \( \{\xi_{\frac{n}{2}}, \xi_{\frac{n}{2}+1}, \ldots, \xi_n\} \not\subset \{\xi_{l_1}, \ldots, \xi_{l_{n-1}}\} \). Hence the set of vectors \( \{\xi_{l_1}, \ldots, \xi_{l_{n-1}}\} \) form a basis of \( \mathbb{Z}^{n-1} \).

Let \( r_1, r_2 \) be two positive real numbers such that \( r_1 < r_2 \) and \( r_1 + 2r_2 < 1 \). Consider the following hyperplanes in \( \mathbb{R}^{n+1} \).

\[
\begin{align*}
H_1 &= \{(x_0, x_1, \ldots, x_n) \in \mathbb{R}^{n+1} : x_{\frac{n}{2}} + \cdots + x_n = r_2\}, \\
H_2 &= \{(x_0, x_1, \ldots, x_n) \in \mathbb{R}^{n+1} : x_0 + \cdots + x_{\frac{n}{2}} = r_2\}.
\end{align*}
\]
\[(3.11)\quad H_3 = \{(x_0, x_1, \ldots, x_n) \in \mathbb{R}^{n+1} : x_\frac{n}{2} = 1 - r_1\}.
\]

\[(3.12)\quad H = \{(x_0, x_1, \ldots, x_n) \in \mathbb{R}^{n+1} : x_0 + \cdots + x_n = 1\}.
\]

Then the intersections $\triangle^n \cap H_1 \cap H_2$, $\triangle^n \cap H_1 \cap H_3$, $\triangle^n \cap H_3 \cap H_2$ are empty. We cut off an open neighborhood of faces $F'$, $F''$ and $\{A_\frac{n}{2}\}$ by affine hyperplanes $H_1 \cap H$, $H_2 \cap H$ and $H_3 \cap H$ respectively in $H$. Let $H_j'$ be the closed half space associated to the affine hyperplane $H_j$ such that the interior of half spaces $H_1', H_2', H_3'$ do not contain the faces $F', F'', \{A_\frac{n}{2}\}$ respectively. We illustrate such hyperplanes for the case $n = 4$ in figure 4.

Define

\[(3.13)\quad \triangle^n_Q = \triangle^n \cap H_1' \cap H_2' \cap H_3', \quad P_1 = \triangle^n \cap H_1, \quad P_2 = \triangle^n \cap H_2 \quad \text{and} \quad P_3 = \triangle^n \cap H_3.
\]

The convex polytope $\triangle^n_Q$ is a simple convex polytope of dimension $n$ and $P_1$, $P_2$ and $P_3$ are also facets of $\triangle^n_Q$. The polytopes $P_1$, $P_2$ and $P_3$ are given by the following equations.

\[
\begin{align*}
(3.14) \quad P_1 &= \{(x_0, x_1, \ldots, x_n) \in \triangle^n : x_0 + \cdots + x_{\frac{n}{2}-1} = 1 - r_2 \quad \text{and} \quad x_{\frac{n}{2}} + \cdots + x_n = r_2\} \\
(3.15) \quad P_2 &= \{(x_0, x_1, \ldots, x_n) \in \triangle^n : x_0 + \cdots + x_{\frac{n}{2}} = r_2 \quad \text{and} \quad x_{\frac{n}{2} + 1} + \cdots + x_n = 1 - r_2\} \\
(3.16) \quad P_3 &= \{(x_0, x_1, \ldots, x_n) \in \triangle^n : x_{\frac{n}{2}} = 1 - r_1 \quad \text{and} \quad x_0 + \cdots + x_{\frac{n}{2}} + \cdots + x_n = r_1\}
\end{align*}
\]

By equations (3.14) and (3.15), the convex polytopes $P_1$ and $P_2$ are diffeomorphic to $\triangle^{(\frac{n}{2}-1)} \times \triangle^{\frac{n}{2}}$. From equation (3.16), $P_3$ is diffeomorphic to $\triangle^{n-1}$. The facets of $P_1$, $P_2$ and $P_3$ are given by the following equations respectively.

\[
\begin{align*}
(3.17) \quad \triangle^{n-1} \cap P_1 &= \{(x_0, x_1, \ldots, x_n) \in P_1 : x_j = 0\} \quad \text{for all} \ j \in \{0, \ldots, n\} \\
(3.18) \quad \triangle^{n-1} \cap P_2 &= \{(x_0, x_1, \ldots, x_n) \in P_2 : x_j = 0\} \quad \text{for all} \ j \in \{0, \ldots, n\} \\
(3.19) \quad \triangle^{n-1} \cap P_3 &= \{(x_0, x_1, \ldots, x_n) \in P_3 : x_j = 0\} \quad \text{for all} \ j \in \{0, \ldots, \frac{n}{2}, \ldots, n\}
\end{align*}
\]

Now we want to construct $(2n - 1)$-dimensional manifold with quasitoric boundary. Let $F$ be a face of $\triangle^n$ of codimension $l$. Then $F = \triangle^{n-1}_{j_1} \cap \ldots \cap \triangle^{n-1}_{j_l}$ for a unique $\{j_1, \ldots, j_l\} \subset \{0, 1, \ldots, n\}$. Suppose $T_F$ be the torus subgroup of $\mathbb{T}^{n-1}$ determined by the submodule generated by $\{\xi_{j_1}, \ldots, \xi_{j_l}\}$ in $\mathbb{Z}^{n-1}$. Assume $T_{\triangle^n} = \{1\}$. We define an equivalence relation $\sim$ on the product $\mathbb{T}^{n-1} \times \triangle^n$ as follows.

\[(3.20)\quad (s, p) \sim (t, q) \quad \text{if and only if} \quad p = q \quad \text{and} \quad ts^{-1} \in T_F
\]

where $F$ is the unique face of $\triangle^n$ containing the point $p$ in its relative interior.

Restrict the equivalence relation $\sim$ on $\mathbb{T}^{n-1} \times \triangle^n_Q$. Define $W(\triangle^n_Q, \xi) := (\mathbb{T}^{n-1} \times \triangle^n_Q)/\sim$ to be the quotient space. So $W(\triangle^n_Q, \xi)$ is a $\mathbb{T}^{n-1}$-space. Let $p : W(\triangle^n_Q, \xi) \to \triangle^n_Q$, defined by $p([s, p]) = p$, be the corresponding orbit map.

Let $\xi^1$, $\xi^2$ and $\xi^3$ be the restriction of the function $\xi$ on the set of facets of $P_1$, $P_2$ and $P_3$ respectively. Define

\[
(3.21) \quad \xi^i_j := \xi^i(\triangle^{n-1}_j \cap P_i) = \begin{cases} 
\xi_j & \text{if} \ i = 1, 2 \quad \text{and} \quad j \in \{0, 1, \ldots, n\} \\
\xi_j & \text{if} \ i = 3 \quad \text{and} \quad j \in \{0, 1, \ldots, \frac{n}{2}, \ldots, n\}
\end{cases}
\]

Let $v$ be a vertex of $P_i$. So $v$ belongs to the relative interior of a unique edge $e_v$ of $\triangle^n$ not contained in $F' \cup F''$. If $e_v = e^{n-1}_{i_j} \triangle^{n-1}_{i_j}$ for some $\{i_j : j = 1, \ldots, n-1\} \subset \{0, 1, \ldots, n\}$,
the vectors \( \{ \xi_{i_1}, \ldots, \xi_{i_{n-1}} \} \) form a basis of \( \mathbb{Z}^{n-1} \). So \( v = \cap_{i=1}^{n-1} (\triangle_{i_j}^{n-1} \cap P_i) \) and the vectors \( \{ \xi_{i_1}, \ldots, \xi_{i_{n-1}} \} \) form a basis of \( \mathbb{Z}^{n-1} \). So \( \xi^i \) defines the characteristic function of a quasitoric manifold \( M_i \) over \( P_i \). Hence from the definition of equivalence relation \( \sim \) we get that

\[
M_i = (\mathbb{T}^{n-1} \times P_i)/ \sim \quad \text{for } i = 1, 2, 3.
\]

Let \( U_i \) be the open subset of \( \triangle_Q^n \) obtained by deleting all facets \( F \) of \( \triangle_Q^n \) such that the intersection \( F \cap P_i \) is empty. Then \( \triangle_Q^n = U_1 \cup U_2 \cup U_3 \). The space \( U_i \) is diffeomorphic as manifold with corners to \([0, 1) \times P_i \). Let \( \phi_i : U_i \to [0, 1) \times P_i \) be a diffeomorphism. From the definition of \( \xi \) and \( \sim \) we get the following homeomorphisms

\[
(\mathbb{T}^{n-1} \times \phi_i^{-1}([a] \times P_i))/ \sim \cong [a] \times M_i \quad \text{for all } a \in [0, 1).
\]

Hence the space \( p^{-1}(U_i) \) is homeomorphic to \((\mathbb{T}^{n-1} \times \phi_i^{-1}([0, 1) \times P_i))/ \sim \cong [0, 1) \times M_i \). Since \( W(\triangle_Q^n, \xi) = p^{-1}(U_1) \cup p^{-1}(U_2) \cup p^{-1}(U_3) \), the space \( W(\triangle_Q^n, \xi) \) is a manifold with quasitoric boundary. The intersections \( P_1 \cap P_2, P_2 \cap P_3 \) and \( P_1 \cap P_3 \) are empty. Hence the boundary \( \partial W(\triangle_Q^n, \xi) = M_1 \sqcup M_2 \sqcup M_3 \).

3.2. Orientability of \( W(\triangle_Q^n, \xi) \). Fix the standard orientation on \( \mathbb{T}^{n-1} \). Then the boundary orientations on \( P_1, P_2 \) and \( P_3 \) induced from the orientation of \( \triangle_Q^n \) give the orientations of \( M_1, M_2 \) and \( M_3 \) respectively.

Let \( W := W(\triangle_Q^n, \xi) \). The boundary \( \partial W \) has a collar neighborhood in \( W \). Hence by the proposition 2.22 of [1], \( H_i(W, \partial W) = \tilde{H}_i(W/\partial W) \) for all \( i \). We show the space \( W/\partial W \) has a CW-structure. We choose a linear functional \( L : \mathbb{R}^n \to \mathbb{R} \) which distinguishes the vertices of \( \triangle_Q^n \), as in the proof of Theorem 3.1 in [3]. The vertices are linearly ordered according to ascending value of \( L \). We make the 1-skeleton of \( \triangle_Q^n \) into a directed graph by orienting each edge such that \( L \) increases along edges. For each vertex \( v \) of \( \triangle_Q^n \) define its index \( \text{ind}(v) \) as the number of incident edges that point towards \( v \). Suppose \( V(\triangle_Q^n) \) is the set of all vertices and \( E(\triangle_Q^n) \) is the set of all edges of \( \triangle_Q^n \). For each \( j \in \{1, 2, \ldots, n\} \), let

\[
I_j = \{(v, e_v) \in V(\triangle_Q^n) \times E(\triangle_Q^n) : \text{ind}(v) = j \text{ and } e_v \text{ is the incident edge that points towards } v \text{ such that } e_v = e \cap \triangle_Q^n \text{ for an edge } e \text{ of } \triangle_Q^n \}.
\]

Suppose \( (v, e_v) \in I_j \) and \( F_v \) is the unique face of \( \triangle_Q^n \) containing \( e_v \) such that \( \text{ind}(v) = \dim(F_v) \). Let \( U_{e_v} \) be the open subset of \( F_v \) obtained by deleting all faces of \( F_v \) not containing the edge \( e_v \). The restriction of the equivalence relation \( \sim \) on \((\mathbb{T}^{n-1} \times U_{e_v})\) gives that the quotient space \((\mathbb{T}^{n-1} \times U_{e_v})/ \sim \) is homeomorphic to the open disk \( B^{2j-1} \). Hence the quotient space \((W/\partial W) \) has a CW-complex structure with odd dimensional cells and one zero dimensional cell only. The number of \((2j - 1)\)-dimensional cell is \(|I_j|\), the cardinality of \( I_j \) for \( j = 1, 2, \ldots, n \). So we get the following theorem.

**Theorem 3.2.** \( H_i(W, \partial W) = \begin{cases} \mathbb{Z} & \text{if } i = 2j - 1 \text{ and } j \in \{1, \ldots, n\} \\ |I_j| & \text{if } i = 0 \\ 0 & \text{otherwise} \end{cases} \)

When \( j = n \) the cardinality of \( I_j \) is one. So \( H_{2n-1}(W, \partial W) = \mathbb{Z} \). Hence \( W \) is an oriented manifold with quasitoric boundary. From the definition [3,2] we get that the
boundary orientation on $M_i$ is same as the orientation on $M_i$ as the quasitoric manifold, for all $i = 1, 2, 3$.

3.3. Identification of $M_1$ and $M_2$. We show the quasitoric manifolds $M_1$ and $M_2$ are equivariantly homeomorphic up to an automorphism of $\mathbb{T}^{n-1}$. Consider the permutation $\rho: \{0, 1, \ldots, n\} \to \{0, 1, \ldots, n\}$ defined by

$$\rho(j) = \begin{cases} 
  n - 1 - j & \text{if } 0 \leq j < \frac{n}{2} 	ext{ and } \frac{n}{2} < j < n \\
  n & \text{if } j = \frac{n}{2} - 1 \\
  \frac{n}{2} & \text{if } j = \frac{n}{2} \\
  \frac{n}{2} - 1 & \text{if } j = n 
\end{cases}$$

(3.24)

Define a linear automorphism $\Phi$ on $\mathbb{R}^{n+1}$ by

$$\Phi(x_0, \ldots, x_j, \ldots, x_n) = (x_{\rho(0)}, \ldots, x_{\rho(j)}, \ldots, x_{\rho(n)})$$

(3.25)

Since $n$ is even $\rho$ is an even permutation. So $\Phi$ is an orientation preserving diffeomorphism. From equations 3.14 and 3.15 it is clear that $\Phi$ maps $P_1$ diffeomorphically onto $P_2$. We denote the restriction of $\Phi$ on the faces of $P_1$ by $\Phi$. Also from the equations 3.17 and 3.18 we get that $\Phi$ maps the facet $(\Delta_j^{n-1} \cap P_1)$ of $P_1$ diffeomorphically onto the facet $(\Delta_j^{n-1} \cap P_2)$ of $P_2$. So

$$\Phi(\Delta_j^{n-1} \cap P_1) = (\Delta_j^{n-1} \cap P_2), \text{ for all } j = 0, \ldots, n.$$  

(3.26)

Let $\alpha_1, \ldots, \alpha_{n-1}$ be the standard basis of $\mathbb{Z}^{n-1}$ over $\mathbb{Z}$. Let $h$ be the linear automorphism of $\mathbb{Z}^{n-1}$ defined by $h(\alpha_i) = \alpha_{n-i}$ for all $i = 1, \ldots, (n-1)$. Hence

$$h(\xi_i) = \xi_{\rho(i)} \text{ and } h(\xi_{\rho(i)}) = \xi_i \text{ for } i = 0, 1, \ldots, n.$$  

(3.27)

Let $h'$ be the automorphism of $\mathbb{T}^{n-1}$ induced by $h$. Hence the automorphism $h'$ is orientation reversing if $n$ is divisible by 4, otherwise it is orientation preserving. From the equations 3.21, 3.26 and 3.27 we get that the following commutative diagram.

$$\begin{array}{ccc}
\mathcal{F}(P_1) & \xrightarrow{\Phi} & \mathcal{F}(P_2) \\
\xi^1 \downarrow & & \xi^2 \downarrow \\
\mathbb{Z}^{n-1} & \xrightarrow{h} & \mathbb{Z}^{n-1}
\end{array}$$

So the diffeomorphism $h' \times \Phi: \mathbb{T}^{n-1} \times P_1 \to \mathbb{T}^{n-1} \times P_2$ induces a $h'$-equivariant orientation reversing homeomorphism $g_n: M_1 \to M_2$ if $n$ is divisible by 4, otherwise it is orientation preserving. From the definition 3.23 of characteristic function $\xi^3$ we get that the quasitoric manifold $M_3$ is equivariantly homeomorphic to $\mathbb{CP}^{n-1}$ if $n = 4l + 2$ and $\overline{\mathbb{CP}}^{n-1}$ if $n = 4l$.

Suppose $n = 4l$. Define an equivalence relation $\sim_n$ on $W(\Delta_Q^n, \xi)$ by

$$x \sim_n y \text{ if and only if } x \in M_1 \text{ and } y = g_n(x).$$  

(3.28)

So the quotient space $W(\Delta_Q^n, \xi)/ \sim_n$ is an oriented manifold with boundary and the boundary is $\overline{\mathbb{CP}}^{n-1}$.

Suppose $n = 4l + 2$. Let $f$ be an automorphism of $\mathbb{Z}^{n-1}$ defined by $f(\alpha_1) = -\alpha_1$ and $f(\alpha_j) = \alpha_j$ for all $j = 2, \ldots, n - 1$. The automorphism $f$ induces an orientation reversing homeomorphism $\tilde{f}: M_2 \to M_2$. The composition map $\tilde{f} \circ g_n: M_1 \to M_2$ is an orientation reversing homeomorphism. Define an equivalence relation $\sim_n$ on $W(\Delta_Q^n, \xi)$ by

$$x \sim_n y \text{ if and only if } x \in M_1 \text{ and } y = \tilde{f} \circ g_n(x).$$  

(3.29)
So the quotient space $W(\triangle^n, \xi) / \sim_n$ is an oriented manifold with boundary and the boundary is $\mathbb{C}P^{n-1}$. Hence we prove the following theorem.

**Theorem 3.3.** The complex projective space $\mathbb{C}P^{2k-1}$ is boundary of an oriented manifold, for all $k > 0$.

**3.4. Examples.** We adhere to definition and notations given in the example 3.1. The faces $A_0A_1$ and $A_3A_4$ are the intersection of facets $\{\triangle^3, \triangle^3, \triangle^3\}$ and $\{\triangle^3, \triangle^3, \triangle^3\}$ respectively.

Here the polytopes $P_1, P_2$ are prism and $P_3$ is 3-simplex, see the figure 3. The restriction of $\xi$ (namely $\xi^1, \xi^2$ and $\xi^3$) on the facets of $P_1, P_2$ and $P_3$ are given in following figure 4.

![Figure 3](image)

**Figure 3.** The simple convex polytope $\triangle^4_Q$ with the facets $P_1, P_2$ and $P_3$.

![Figure 4](image)

**Figure 4.** The characteristic functions $\xi^1, \xi^2$ and $\xi^3$ of $P_1, P_2$ and $P_3$ respectively.

Let $h$ be the automorphism of $\mathbb{Z}^3$ defined by $h(1,0,0) = (0,0,1)$, $h(0,1,0) = (0,1,0)$ and $h(0,0,1) = (1,0,0)$. Clearly the combinatorial pairs $(P_1, \xi^1)$ and $(P_2, \xi^2)$ give two $h'$-equivariantly homeomorphic quasitoric manifolds, namely $M_4^4$ and $M_2^4$ respectively. The combinatorial pair $(P_3, \xi^3)$ gives the quasitoric manifold $\mathbb{C}P^3$ over $P_3$. So the boundary of $W(\triangle^4_Q, \xi)$ is $M_4^4 \sqcup M_2^4 \sqcup \mathbb{C}P^3$. Hence after identifying $M_4^4$ and $M_2^4$ via an orientation preserving homeomorphism, we get an oriented manifold with boundary $\mathbb{C}P^3$.

**4. Oriented cobordism class of $\mathbb{R}P^{4k+1}$**

**4.1. Manifolds with small cover boundary.** We follow the notations of subsection 3.3. Let $q : \mathbb{Z} \to \mathbb{Z}_2$ be the quotient homomorphism. Let $q^{n-1} : \mathbb{Z}^{n-1} \to \mathbb{Z}_2^{n-1}$ be
homomorphism defined by \( q \) componentwise. Define a function \( \mu \) from the set of facets of \( \Delta^n \) to \( \mathbb{Z}_2^{n-1} \) by
\[
\mu_j = \mu((\Delta_{n-j}^{n-1}) := q^{n-1}(\xi_j), \text{ for } j = 0, 1, \ldots, n.
\]
where \( \xi_j \) is defined in equation (3.1). Suppose \( F \) is a codimension \( l \) face of \( \Delta^n \). Then \( F = \Delta^{j_1}_{n-1} \cap \ldots \cap \Delta^{j_l}_{n-1} \) for a unique \( \{j_1, \ldots, j_l\} \subset \{0, 1, \ldots, n\} \). Let \( G_F \) be the subgroup of \( \mathbb{Z}_2^{n-1} \) determined by the vectors \( \{\mu_{j_1}, \ldots, \mu_{j_l}\} \) in \( \mathbb{Z}_2^{n-1} \). Define an equivalence relation \( \sim_s \) on the product \( (\mathbb{Z}_2^{n-1} \times \Delta^n) \) as follows,
\[
(s, p) \sim (t, q) \text{ if and only if } p = q \text{ and } t - s \in G_F,
\]
where \( F \) is the unique face of \( \Delta^n \) containing \( p \) in its relative interior.

Let \( S(\Delta^n, \mu) = (\mathbb{Z}_2^{n-1} \times \Delta^n)/\sim_s \) be the quotient space. Let \( p_s : S(\Delta^n, \mu) \to \Delta^n \) be the corresponding orbit map. Resist the equivalence relation \( \sim_s \) on \( \mathbb{Z}_2^{n-1} \times \Delta^n \). Let \( S(\Delta^n_Q, \mu) = (\mathbb{Z}_2^{n-1} \times \Delta^n_Q)/\sim_s \) be the quotient space. So \( S(\Delta^n_Q, \mu) \) is a \( \mathbb{Z}_2^{n-1} \)-space. Denote the restriction of \( p_s \) on \( S(\Delta^n_Q, \mu) \) again by \( p_s \).

Let \( \mu_1, \mu_2 \) and \( \mu_3 \) be the restriction of \( \mu \) on the facets of \( P_1, P_2 \) and \( P_3 \) respectively. So \( \mu_1, \mu_2 \) and \( \mu_3 \) are defined by,
\[
\mu_j^i := \mu^i((\Delta_{n-j}^{n-1}) \cap P_i) = \begin{cases} \mu_j & \text{if } i = 1, 2 \text{ and } j \in \{0, 1, \ldots, n\} \\ \mu_j & \text{if } i = 3 \text{ and } j \in \{0, 1, \ldots, \frac{n}{2}, \ldots, n\} \end{cases}
\]
Similarly as in the subsection (3.2), we can show that \( \mu_1, \mu_2 \) and \( \mu_3 \) are \( \mathbb{Z}_2 \)-characteristic functions of some small covers \( N_1, N_2 \) and \( N_3 \) over \( P_1, P_2 \) and \( P_3 \) respectively. Hence from the definition of \( \sim_s \) we get that
\[
N_i = (\mathbb{Z}_2^{n-1} \times P_i)/\sim_s, \text{ for } i = 1, 2, 3.
\]
So by the definitions of \( \mu \) and \( \sim_s \) we get the following homeomorphisms.
\[
p_s^{-1}(U_i) \cong (\mathbb{Z}_2^{n-1} \times \phi_i^1([0, 1) \times P_i))/\sim_s \cong [0, 1) \times N_i, \text{ for all } i = 1, 2, 3.
\]
Hence the quotient space \( S(\Delta^n_Q, \mu) \) is a manifold with small cover boundary and the boundary \( \partial S(\Delta^n_Q, \mu) = N_1 \cup N_2 \cup N_3 \). From the definition of characteristic function \( \mu^3 \) we get that the small cover \( N_3 \) is equivariantly homeomorphic to \( \mathbb{R}P^{n-1} \).

4.2. Orientability of \( S(\Delta^n_Q, \mu) \). We adhere to notations given in the subsection (3.2). By theorem (2.3) we get that the small covers \( N_1, N_2 \) and \( N_3 \) are orientable.
Using similar decomposition to subsection (3.3), we can give a CW-structure on \( X := S(\Delta^n_Q, \mu)/\partial S(\Delta^n_Q, \mu) \) where the attaching maps are natural quotient maps. For each \((v, e_v) \in I_j \) the quotient space \( X_v := (\mathbb{Z}_2^{n-1} \times U_v)/\sim_s \) is homeomorphic to the \( j \)-dimensional open disc \( B^j \) in \( \mathbb{R}^j \). Corresponding to this CW-structure we get a cellular chain complex. We calculate the boundary maps of this chain complex.

Since \( \Delta^n \) is a simplex, without any loss of generalities we may choose the linear functional \( L \) such that \( I_n = \{(\overline{v}, e_v)\} \) where \( e_v = e \cap \Delta^n_Q \) and the vertices of the edge \( e \) of \( \Delta^n \) are \( \{A_0, A_{\frac{n}{2} + 1}\} \). Then \( e = \cap \{\Delta^{n-1}_Q : i \neq 0, \frac{n}{2} + 1\} \). Hence the \((n-1)\)-dimensional cells of \( X \) come from the facets \( \Delta^{n-1}_0 \) and \( \Delta^{n-1}_{\frac{n}{2} + 1} \). Let \( v_1 \) be the vertex of \( \Delta^{n-1}_0 \) and \( v_2 \) be the vertex of \( \Delta^{n-1}_{\frac{n}{2} + 1} \) such that \( I_{n-1} = \{(v_1, e_{v_1}), (v_2, e_{v_2})\} \). So \( X_\overline{v} \) is the only \( n \)-dimensional cell of \( X \) and \( X_{v_1}, X_{v_2} \) are the only \((n-1)\)-dimensional cells of \( X \).
Now we describe the attaching map of the cell $X_{\mathfrak{f}}$. Define a function $\tilde{\mu}$ from the set of facets of $\Delta^n$ to $\mathbb{Z}_2^{n-1}$ by
\[
(4.6) \quad \tilde{\mu}_j = \tilde{\mu}(\Delta_n^{n-1}) := \left\{ \begin{array}{ll}
0 & \text{if } j = \frac{n}{2} - 1, n \\
\mu_j & \text{otherwise}
\end{array} \right.
\]
where $\mu_j$ is defined in equation (4.1). For a face $F = \Delta_n^{n-1} \cap \ldots \cap \Delta_n^{n-1}$, let $Z_F$ be the subgroup of $\mathbb{Z}_2^{n-1}$ determined by the vectors $\{\tilde{\mu}_j, \ldots, \tilde{\mu}_j\}$ in $\mathbb{Z}_2^{n-1}$. Define an equivalence relation $\sim_b$ on $(\mathbb{Z}_2^{n-1} \times \Delta^n)$ as follows,
\[(4.7) \quad (s, p) \sim_b (t, q) \text{ if and only if } p = q \text{ and } t - s \in Z_F\]
where $F$ is the unique face of $\Delta^n$ containing $p$ in its relative interior. The quotient space $(\mathbb{Z}_2^{n-1} \times \Delta^n)/ \sim_b$ is homeomorphic to the closed disc $\overline{B}^n$ in $\mathbb{R}^n$. Let $U_e$ be the open subset of $\Delta^n$ obtained by deleting all facets of $\Delta^n$ not containing the edge $e$. So $(\mathbb{Z}_2^{n-1} \times (\Delta^n - U_e))/ \sim_b$ is homeomorphic to $S^{n-1}$. Let
\[(4.8) \quad V_i = \{ (\Delta^n - U_e) \cap (\mathbb{R}^{n+1} - H^i_1) \} \cup P_i \text{ for } i = 1, 2, 3.
\]
Let $X_{n-1}$ be the $(n-1)$-th skeleton. So $X_{n-1} = X - X_{\mathfrak{f}}$. Let $S_V = p_s^{-1}(V_1) \cup p_s^{-1}(V_2) \cup p_s^{-1}(V_3)$. Let $\eta$ be the composition of following quotient maps.
\[(4.9) \quad (\mathbb{Z}_2^{n-1} \times (\Delta^n - U_e))/ \sim_b \to (\mathbb{Z}_2^{n-1} \times (\Delta^n - U_e))/ \sim_b \to ((\mathbb{Z}_2^{n-1} \times (\Delta^n - U_e))/ \sim_b)/S_V
\]
Hence $\eta$ is the attaching map of the cell $X_{\mathfrak{f}}$. If $\mathfrak{f} \in P_1$ ($P_2$) then the other vertex $v_0$ of $e_{\mathfrak{f}}$ belongs to $P_2$ (respectively $P_1$). Suppose $\mathfrak{f} \in P_2$. For each $t \in (0, 1)$ define $f(t) = tv + (1 - t)v_0 \in e_{\mathfrak{f}}$. Let $H_t$ be the hyperplane in $\mathbb{R}^{n+1}$ such that
\[(4.10) \quad H_t \text{ is parallel to } H_1 \text{ and } f(t) = e_{\mathfrak{f}} \cap H_t.
\]
Hence there exists a maximum $t_0 \in (0, 1)$ such that
\[(4.11) \quad H_t \cap H_2 \text{ and } H_t \cap H_3 \text{ are empty for all } t \in (0, t_0).
\]
Let
\[(4.12) \quad Z = \Delta_0^n \cap \{ \cup_{t \in (0, t_0)} H_t \}.
\]
Hence $Z$ is an open subset of $\Delta_0^n$ containing $\{ f(t) : t \in (0, t_0) \}$. Let
\[(4.13) \quad f_1(t) = e_{v_1} \cap H_t \text{ and } f_2(t) = e_{v_2} \cap H_t \text{ for all } t \in (0, t_0).
\]
Let
\[(4.14) \quad Z_1 = Z \cap \Delta_0^{n-1} = \cup_{t \in (0, t_0)} (H_t \cap \Delta_0^{n-1}) \quad \text{and} \quad Z_2 = Z \cap \Delta_2^{n-1} = \cup_{t \in (0, t_0)} (H_t \cap \Delta_2^{n-1}).
\]
Hence $Z_1$ and $Z_2$ are open subset of $\Delta_0^{n-1}$ and $\Delta_2^{n-1}$ containing $\{ f_1(t) : t \in (0, t_0) \}$ and $\{ f_2(t) : t \in (0, t_0) \}$ respectively. So $p_s^{-1}(Z_1)$ and $p_s^{-1}(Z_2)$ are open subset of $X_{v_1}$ and $X_{v_2}$ respectively. By the definition (4.1.2) the isotopy groups of $p_s^{-1}(Z_1)$ and $p_s^{-1}(Z_2)$ are the subgroups generated by $\{ \mu_n \}$ and $\{ \mu_{\frac{n}{2}-1} \}$ respectively. Then by equation (4.9)
\[(4.15) \quad \eta^{-1}(p_s^{-1}(Z_i)) = Y_i \cup Y_{\grave{i}} \text{ for } i = 1, 2.
\]
From definition (4.6) and (4.7) we get that the identification of two facets in $\mathbb{Z}_2^{n-1} \times \Delta^n$ containing the edge $A_0A_{\frac{n}{2}+1}$ and corresponding to the vector $\tilde{\mu}_j$ is realized by a reflection, when $j \neq \frac{n}{2}-1$ and $j \neq n$. Note that $\mu_{\frac{n}{2}-1} = \tilde{\mu}_0 + \cdots + \tilde{\mu}_{\frac{n}{2}-2} + \tilde{\mu}_{\frac{n}{2}}$ and $\mu_n = \tilde{\mu}_{\frac{n}{2}} + \cdots + \tilde{\mu}_{n-1}$. So by definition (4.2) the identification of two facets in $\mathbb{Z}_2^{n-1} \times \Delta^n$ not containing the edge $A_0A_{\frac{n}{2}+1}$ and corresponding to the vector $\mu_{\frac{n}{2}-1}$ or $\mu_n$ is realized by the composition of
reflections. Since the facets \( \Delta^{n-1}_{\frac{n}{2}+1} \) and \( \Delta^{n-1}_0 \) correspond to the vectors \( \mu_{\frac{n}{2}-1} \) and \( \mu_n \) respectively, from \( 4.14 \) and \( 4.15 \) we infer that \( Y_i \) is the image of \( Y_{i-1} \) under the composition of \( \frac{n}{2} \) reflections.

So the \( n \)-th boundary map \( d_n \) of the cellular chain complex is given by

\[
d_n = 1 + (-1)^{\frac{n}{2}} = \begin{cases} 2 & \text{if } n = 4l \\ 0 & \text{if } n = 4l + 2 \end{cases}
\]

Hence the manifold \( S(\Delta^4_Q, \mu) \) with small cover boundary is orientable if and only if \( n = 4l + 2 \). From the definition \( 4.3 \) we get that the boundary orientation on \( N_i \) is same as the orientation on \( N_i \) as the small cover, for all \( i = 1, 2, 3 \) whenever \( n = 4l + 2 \).

**Example 4.1.** The simple convex polytope \( \Delta^4_Q \) with index of each vertex is given in figure 5. The vertices \( 0, 1, 2, 3 \) and the edges \( e_0, e_1, e_2 \) are given in this figure.

![Figure 5. The simple convex polytope \( \Delta^4_Q \) with the index of each vertex.](image)

4.3. Identification of \( N_1 \) and \( N_2 \). Suppose \( n = 4l + 2 \). Let \( h_s \) be the automorphism of \( \mathbb{Z}^{n-1}_2 \) defined by \( h_s(q^{n-1}(\alpha_i)) = q^{n-1}(\alpha_{n-i}) \). From the definition \( 4.3 \) of \( \mu^1 \) and \( \mu^2 \) we get the following commutative diagram.

\[
\begin{array}{c}
\mathcal{F}(P_1) \xrightarrow{\Phi} \mathcal{F}(P_2) \\
\mu^1 \downarrow \quad \quad \quad \quad \\
\mathbb{Z}^{n-1}_2 \xrightarrow{h_s} \mathbb{Z}^{n-1}_2
\end{array}
\]

The automorphism \( h_s \) induces an orientation reversing linear automorphism on \( \mathbb{R}^{n-1} \). So the diffeomorphism \( h_s \times \Phi : \mathbb{Z}^{n-1}_2 \times P_1 \to \mathbb{Z}^{n-1}_2 \times P_2 \) induces a \( h_s \)-equivariant orientation.
reversing homeomorphism $g^n_s : N_1 \to N_2$. Define an equivalence relation $\sim^n_s$ on $S(\Delta^n, \mu)$ by

$$x \sim^n_s y \text{ if and only if } x \in N_1 \text{ and } y = g^n_s(x).$$

So the quotient space $S(\Delta^n, \mu)/\sim^n_s$ is an oriented manifold with boundary and the boundary is $\mathbb{R}P^{n-1}$. Hence we have proved the following theorem.

**Theorem 4.2.** The real projective space $\mathbb{R}P^{4k+1}$ is boundary of an oriented manifold, for all $k > 0$.

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