LOFTY MODELS OF PEANO ARITHMETIC

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Abstract. We give an answer to a question from 1984: For every model $\mathcal{M}$ of Peano Arithmetic, $\mathcal{M}$ is lofty iff $\mathcal{M}$ has a simple extension that is recursively saturated.

Lofty models of Peano Arithmetic (PA) were introduced in [1] and extensively studied in [1] and [2]. These models are significant because a countable model $\mathcal{M}$ of PA is lofty iff it has a simple extension $\mathcal{N} \succ \mathcal{M}$ that is recursively saturated [1, Th. 3.1]. The right-to-left half of this equivalence holds even if $\mathcal{M}$ is uncountable [1, Th. 3.1]. The question of whether or not this equivalence extends to all models of PA is implicit in [1], explicit in [2] and repeated in [3, Chap. 12, Question 16]. It was proved ([1, §7] and [2, Th. 1.11]) that every lofty $\mathcal{M}$ that fails to have a recursively saturated, simple extension is $\kappa$-like for some regular, uncountable cardinal $\kappa$ and is not recursively saturated. Furthermore, if $\mathcal{M}$ is such a $\kappa$-like counterexample, then there is an $\omega_1$-like $\mathcal{N} \equiv \mathcal{M}$ that is also a counterexample [2, §1]. The purpose of this note is to prove Theorem 1, which asserts that no such $\mathcal{N}$ exists.

Theorem 1: If $\mathcal{M} \models \text{PA}$ is $\omega_1$-like and lofty, then $\mathcal{M}$ is recursively saturated.

As indicated in the first paragraph, Theorem 1 has the following corollary.

Corollary 2: For any $\mathcal{M} \models \text{PA}$, $\mathcal{M}$ is lofty iff $\mathcal{M}$ has a simple elementary extension that is recursively saturated.

The language appropriate for PA is $\mathcal{L}_{\text{PA}} = \{+, \times, \leq, 0, 1\}$. Script letters such as $\mathcal{M}, \mathcal{N}, \mathcal{M}_0, \ldots$ always denote models of PA with universes $M, N, M_0, \ldots$, respectively. For any $\mathcal{M}$, if $A \subseteq M$, then $\mathcal{L}_{\text{PA}}(A)$ is $\mathcal{L}_{\text{PA}}$ augmented with constant symbols denoting elements of $A$. If $k < \omega$ and $R \subseteq M^k$, then $R$ is $A$-definable if it definable in $\mathcal{M}$ by an $\mathcal{L}_{\text{PA}}(A)$-formula. The set of $M$-definable subsets of $M$ is $\text{Def}(\mathcal{M})$. If $e \in M$, then $[0, e) = \{i \in M : \mathcal{M} \models 0 \leq i < e\} \in \text{Def}(\mathcal{M})$. If

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\(a \in M\) and \(A \subseteq M\), then \(\text{tp}(a/A)\), the type of \(a\) over \(A\), is the set of 1-ary \(\mathcal{L}_{PA}(A)\)-formulas \(\varphi(x)\) such that \(M \models \varphi(a)\). If \(f : [0, e) \rightarrow M\) is an \(M\)-definable function, then there is \(c \in M\) that codes \(f\); that is, for every \(i < e\), \((c)_i = f(i)\). We assume that codings have been defined so that there is a unique \(c\) that codes \(f\). If \(k < \omega\), then we sometimes identify \(c\) and the \(k\)-tuple \(\langle f(0), f(1), \ldots, f(k-1) \rangle \in M^k\).

If \(X \subseteq M\), then \(\text{Scl}(X)\), the Skolem closure of \(X\), is that \(N \subseteq M\), where \(N \trianglelefteq M\) is the smallest for which \(X \subseteq N\). If \(X = \{a\}\), we write \(\text{Scl}(a)\) instead of \(\text{Scl}(\{a\})\). If \(M \trianglelefteq N\) and there is \(a \in N\) such that \(\text{Scl}(M \cup \{a\}) = N\), then \(N\) is a simple (or finitely generated) extension of \(M\), and \(a\) generates \(N\) over \(M\).

If a set \(\Phi(x)\) of \(\mathcal{L}_{PA}(M)\)-formulas is recursive, then, necessarily, there is a finite \(A \subseteq M\) such that \(\Phi(x)\) is a set of \(\mathcal{L}_{PA}(A)\)-formulas. We say that \(\Phi(x)\) is bounded if, for some \(a \in M\), the formula \(x < a\) in \(\Phi(x)\).

Recall that \(M\) is (boundedly) recursively saturated iff whenever \(\Phi(x)\) is a (bounded) finitely satisfiable, recursive set of 1-ary \(\mathcal{L}_{PA}(M)\) formulas, then there is a singleton subset \(\{a\} \subseteq M\) such that each finite \(\Phi_0(x) \subseteq \Phi(x)\) is satisfied by some element of \(\{a\}\).

1 We have stated these characterizations in this odd way so as to motivate the next definition.

**Definition 3:** ([1, Def. 1.4]) Let \(M\) be a model. If \(e \in M\), then \(M\) is (boundedly) \(e\)-lofty if \(M\) is nonstandard and whenever \(\Phi(x)\) is a (bounded) finitely satisfiable, recursive set of formulas, then there is \(A \in \text{Def}(M)\) such that \(M \models |A| = e\) and each finite \(\Phi_0(x) \subseteq \Phi(x)\) is satisfied by some element of \(A\). If \(M\) is (boundedly) \(e\)-lofty for some \(e \in M\), then \(M\) is (boundedly) lofty.

No \(M\) is (boundedly) 0-lofty. If \(1 \leq e < \omega\), then \(M\) is (boundedly) \(e\)-lofty iff \(M\) is (boundedly) recursively saturated. A fundamental result ([1, Th. 3.1]) is that if \(M\) is countable and \(e \in M\), then \(M\) is \(e\)-lofty iff it has a simple extension \(N \succcurlyeq M\) generated by an element \(a\) such that \(N \models a < e\).

The next lemma gives an alternate characterization of (bounded) loftiness.

**Lemma 4:** ([2, Th. 1.7]) Suppose \(M\) is a model and \(e \in M\) is nonstandard.

1. \(M\) is boundedly \(e\)-lofty iff for every \(a, b \in M\), there is \(c \in M\) such that \(\{(c)_i : i < e\} \supseteq \text{Scl}(a) \cap [0, b)\).
(2) $\mathcal{M}$ is $e$-lofty iff for every $a \in M$, there is $c \in M$ such that
\[ \{(c)_i : i < e\} \supseteq \text{Scl}(a). \]
\[ \square \]

For any $\mathcal{M}$ and $a \in M$, we let $\mathcal{M}(a) \equiv_{\text{end}} \mathcal{M}$ be such that $\text{Scl}(a)$
is a cofinal subset of $M(a)$. For each $a \in M$, this uniquely determines $\mathcal{M}(a)$. If there is no $a \in M$ such that $M(a) = M$, then $\mathcal{M}$ is tall. Clearly, $\mathcal{M}$ is $e$-lofty iff it is boundedly $e$-lofty and tall.

We will prove two lemmas about boundedly lofty models. Even though the first one, Lemma 6, is essentially Lemma 1.9 of [2], we give a proof whose presentation seems more cogent than the one in [2]. The second, Lemma 7, is the key new ingredient in the proof of Theorem 1.

Before stating and proving these lemmas, we review some material that will be used in their proofs.

Let $\mathcal{M}$ be a model. As usual, an $L_{PA}(M)$-formula is $\Delta_0 = \Sigma_0 = \Pi_0$ if all of its quantifiers are bounded. For $k < \omega$, an $L_{PA}(M)$-formula is $\Sigma_{k+1}$ if it has the form $\exists y \varphi$, where $\varphi$ is a $\Pi_k$ formula, and it is $\Pi_{k+1}$ if it has the form $\forall y \varphi$, where $\varphi$ is a $\Sigma_k$ formula. If $d = \langle d_0, d_1, \ldots, d_{k-1} \rangle \in M^k$ and $\sigma$ is a $L_{PA}(M)$-sentence having the form
\[ \sigma = Q_0 y_0 Q_1 y_1 \cdots Q_{k-1} y_{k-1} \varphi(y_0, y_1, \ldots, y_{k-1}, \overline{x}), \]
where $\varphi(y_0, y_1, \ldots, y_{k-1}, \overline{x})$ is $\Delta_0$ and each $Q_i$ is either $\exists$ or $\forall$, then we let
\[ \sigma^{(d)} = Q_0 y_0 \leq d_0 Q_1 y_1 \leq d_1 \cdots Q_{k-1} y_{k-1} \leq d_{k-1} \varphi(y_0, y_1, \ldots, y_{k-1}, \overline{x}), \]
which is a $\Delta_0$ sentence. If $m \in M$, then we say that a $k$-tuple $d = \langle d_0, d_1, \ldots, d_{k-1} \rangle \in M^k$ is $m$-fast if
\[ \mathcal{M} \models \sigma^{(d)} \iff \sigma \]
whenever $\sigma$ is a $\Sigma_k L_{PA}([0, m])$-sentence. We say that $d$ is fast if it is 0-fast.

We are assuming in the previous paragraph that $k < \omega$. However, it still makes sense, if we are working in $\mathcal{M}$, to let $k \in M$ be arbitrary and then refer to the sets $\Sigma_k$, $\Pi_k$ and to $\sigma^{(d)}$. We often identify a formula with its Gödel number. For $\sigma$ a sentence in $\Sigma_k$, where $k \in M$ is nonstandard, there is in general no meaning to $\mathcal{M} \models \sigma$ although $\mathcal{M} \models \sigma^{(d)}$ does have meaning.

For $k < \omega$, we will say that a function $f : M \rightarrow M$ is a $\Sigma_k$-function if it is $\varnothing$-definable in $\mathcal{M}$.

The next proposition is routinely proved by induction on $k$.

**Proposition 5:** Suppose that $\mathcal{M}$ is a model and that $k < \omega$, $m \in M$ and $d \in M^k$. Then, $d$ is $m$-fast iff the following:

- if $k > 0$, then $d_0 > f(m)$ for every $\Sigma_k$-function $f$;
• if $1 \leq i < k$, then $d_i > f(d_{i-1})$ for every $\Sigma_{k-i}$-function $f$. □

A consequence of this proposition is that if $d = \langle d_0, d_1, \ldots, d_{k-1} \rangle$ is an $m$-fast $k$-tuple and $k > 0$, then $\langle d_1, d_2, \ldots, d_{k-1} \rangle$ is a $d_0$-fast $(k-1)$-tuple.

**Lemma 6:** Suppose that $M$ is boundedly e-lofty. Let $a, s, d \in M$ be such that $e \in M(a)$, $s \in M \setminus M(a)$, $d \in M \setminus M(s)$ and $M(a)$ is not boundedly recursively saturated. Then there is $c \in M$ such that $\text{tp}(c/\langle 0, e \rangle)$ is not realized by any element in $M(a)$.

**Proof.** For each $n < \omega$, let $f_n : M \rightarrow M$ be a strictly increasing $\Sigma_{n+1}$ function such that for every $\Sigma_n$ function $f$ and every nonstandard $b \in M$, $M \models f_n(b) > f(b)$. Choose these functions so that there is a recursive sequence of formulas whose $n$-th one is a $\Sigma_{n+2}$ formula that defines $f_n$. Let $g_n : M \rightarrow M$ be such

$$M \models \forall x, y[g_n(x) = y \leftrightarrow (x \leq f_n(y) \land \forall z < y (f_n(z) < x))].$$

Obviously, $g_n(x) \leq x$ for every $x \in M$. By Lemma 4(1), let $c \in M$ be such that $(c)_0 = d$ and $\{(c)_i : i < e\} \supseteq \text{Scl}(d) \cap [0, d+1)$. Let $i_n$ be the least $i < e$ such that $(c)_i = g_n(c)$. If $b \in M \setminus M(a)$ and $k < \omega$, then the $(k+1)$-tuple $\langle g_k(b), g_{k-1}(b), \ldots, g_0(b) \rangle$ is $a$-fast. Thus, for each $k < \omega$, the $(k+1)$-tuple $\langle (c)_{k-i} : i \leq n \rangle$ is $a$-fast.

**Claim:** The sequence $\langle i_n : n < \omega \rangle$ is not coded.

To prove the claim by contradiction, assume that $\langle i_n : n < \omega \rangle$ is coded. Then the sequence $\langle g_n(d) : n < \omega \rangle$ is also be coded and, therefore, $M(a)$ (even $M(s)$) is boundedly recursively saturated.

We now show that for no $b \in M(a)$ does $\text{tp}(b/\langle 0, e \rangle) = \text{tp}(c/\langle 0, e \rangle)$. For a contradiction, suppose that $b \in M(a)$ is a counterexample. In particular, for each $k < \omega$, the $(k+1)$-tuple $\langle (b)_{k-i} : i \leq n \rangle$ is $e$-fast.

In $M$, we define a sequence $\langle j_n : n < e \rangle$ so that if $n < e$, then $j_n$ is the least such that

$$M \models (g_n((b)_0) = (b)_{j_n})^{(r_n)},$$

where $r_n$ is the $(n+1)$-tuple $\langle s, (c)_{j_{n-1}}, (c)_{j_{n-2}}, \ldots, (c)_{j_0} \rangle$. If $n < \omega$, then $r_n$ is $b$-fast, so by Proposition 5, $M \models g_{n+1}((b)_0) = (b)_{j_n}$. It is then easily verified by induction on $n < \omega$ that $j_n = i_n$. Thus, $\langle i_n : n < \omega \rangle$ is coded, contradicting the claim. □

In the next lemma, we consider the structure $(M, \text{Th}(M))$. Recall that we are identifying $\mathcal{L}_{\text{PA}}$-sentences with their Gödel numbers, so that $\text{Th}(M) \subseteq \omega$. Where appropriate, we extend terminology that was intended to apply to $M$ to apply to expansions of $M$ as well.
**Lemma 7:** Suppose that $\mathcal{M}$ is boundedly $e$-lofty, $e \in M$ is nonstandard, $a \in M$ and $M(a) \neq M$. Then $\text{tp}(a/[0,e))$ is $[0,e)$-definable in $(\mathcal{M}, \text{Th}(\mathcal{M}))$.

**Proof.** Let $\mathcal{M}$, $e$ and $a$ be as given, and let $T = \text{Th}(\mathcal{M})$. Clearly, $\omega \in \text{Def}(\mathcal{M}, T)$. Since $\mathcal{M}$ is boundedly $e$-lofty and $M(a) \neq M$, we let $c \in M$ be such that $\{(c)_i : i < e\} \supseteq \text{Scl}(a)$ by applying Lemma 4(1) with $b \in M \setminus M(a)$.

Let $D \in \text{Def}(\mathcal{M})$ be the set of all triples $(b, \sigma, \varphi(x))$ such that for some $k < e$,

- $b$ codes a $(k+1)$-tuple $(b_i : i \leq k)$ of elements of $[0,e)$,
- $(b)_0 = 0$,
- $\sigma$ is a $\Sigma_{k+1}$ $L_{PA}([0,e))$-sentence of length at most $e$,
- $\varphi(x)$ is a $\Sigma_k$ 1-ary $L_{PA}([0,e))$-formula of length at most $e$,
- if $d$ is the $(k+1)$-tuple where $(d)_i = (c)(b)_i$, then $\sigma^{(d)}$,
- if $d'$ is the $k$-tuple where $(d')_i = (d)_{i+1}$, then $\varphi(a)^{(d')}$.

There is $e_0 \in M(a)$ such that $D \subseteq [0,e_0)^3$. Since $D \in \text{Def}(\mathcal{M})$, we can consider $D$ to be an element of $M(e)$. We claim: $\text{tp}(a/[0,e))$ is $\emptyset$-definable in $(\mathcal{M}, T, D, e)$. More specifically, we claim that $\varphi(x) \in \text{tp}(a/[0,e))$ iff the following:

There is $k < \omega$ such that $\varphi(x)$ is a standard $\Sigma_k$ 1-ary $L_{PA}([0,e))$-formula and there is a $(k+1)$-tuple $b = (b_0, b_1, \ldots, b_k) \in [0,e)^{k+1}$ such that $(b, \sigma, \varphi(x)) \in D$ for all $\Sigma_{k+1} \sigma \in T$.

It is clear that the purported definition of $\text{tp}(a/[0,e))$ can be made in $(\mathcal{M}, \omega, D, e)$. We check that it actually defines $\text{tp}(a/[0,e))$. Let $k < \omega$ and $\varphi(x)$ be a standard 1-ary $\Sigma_k$ $L_{PA}([0,e))$-formula.

**Suppose that $\varphi(x) \in \text{tp}(a/[0,e))$.** Let $d = (d_0, d_1, \ldots, d_k) \in \text{Scl}(a)^{k+1}$ be such that $(d)_0 = a$ and $d$ is fast. Then, $d' = (d_1, d_2, \ldots, d_k) \in \text{Scl}(a)^k$ is $a$-fast. Let $b = (b_0, b_1, \ldots, b_k) \in [0,e_0)^{k+1}$ be such that $(c)_{b_i} = (d)_i$ for $i \leq k$. By Proposition 5, $\mathcal{M} \models \varphi(a)^{(d)}$. If $\sigma \in T$ is $\Sigma_{k+1}$, then by Proposition 5 again, $\mathcal{M} \models \sigma^{(d)}$. Thus, $(b, \sigma, \varphi(x)) \in D$ for all $\Sigma_{k+1} \sigma \in T$.

**Suppose that $\varphi(x) \notin \text{tp}(a/[0,e))$.** Suppose, for a contradiction, that $b = (b_0, b_1, \ldots, b_{k+1}) \in [0,e)^{k+1}$ is such that $(b)_0 = 0$ and that for every $\Sigma_{k+1} \sigma \in T$, $(b, \sigma, \varphi(x)) \notin D$. Let $d = (d_0, d_1, \ldots, d_k)$ and $d' = (d_1, d_2, \ldots, d_k)$ be such that $(c)_{b_i} = d_i$ for $i \leq k$. Then, for every such $\sigma$, $\mathcal{M} \models \sigma^{(d)}$. Hence, by Proposition 5, $d$ is fast, so that $d'$ is $a$-fast. Since $\mathcal{M} \models \varphi(a)^{(d)}$, then by Proposition 5 again, $\mathcal{M} \models \varphi(a)$, contradicting that $\varphi(x) \notin \text{tp}(a/[0,e))$. \qed
Having Lemmas 6 and 7, we show how Theorem 1 follows. Let $\mathcal{M}$ be $\omega_1$-like and $e$-lofty. Lemma 7 implies that $\text{tp}(a/[0, e])$ is $[0, e]$-definable in $(\mathcal{M}, \text{Th}(\mathcal{M}))$ for every $a \in M$. Since $M(e)$ is countable, the set $\{\text{tp}(a/[0, e]) : a \in M\}$ is countable. But Lemma 6 implies that if $\mathcal{M}$ were not recursively saturated, then this set would be uncountable. Therefore, $\mathcal{M}$ is recursively saturated.

Note that this proof also works when $\mathcal{M}$ in Theorem 1 is assumed to be $\kappa$-like for some regular, uncountable $\kappa$ instead of being $\omega_1$-like.

References

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