Symmetry Algebra of IIB Superstring Scattering

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Abstract

The graviton scattering in IIB superstring theory is examined in the context of S-duality and symmetry. There is an algebra that generates all of the terms in the four-point function to any order in derivatives. A map from the algebra to the scattering is given; it suggests the correctness of the full four-point function with the S-duality. The higher point functions are expected to follow a similar pattern.
Introduction

Derivative corrections to the IIB superstring low-energy effective action at the four-point function have been investigated in many works \[5\]-\[13\]. Complete formal perturbative results are known up to the genus two level, due to the complicated nature of the integrals involved \[1\]-\[2\].

There are also several conjectures for the full four-point function, including instantons \[3\]-\[5\]. The original conjecture of \[4\] based on the Eisenstein functions failed to agree with genus one perturbation theory. The conjecture in \[3\] was ambiguous up to relative coefficients of the pairings of non-holomorphic Eisenstein functions; this seems to be straightened out by imposing a differential condition on the modular construction, in \[5\].

In this work, the systematics of the full four-point function in the currently accepted conjecture is examined. The organization of the perturbative corrections is given in an organized manner, and illustrates some symmetry that is unknown in the superstring. The four-point function can be found by expanding the function,

\[
\prod_{n=1}^{\infty} \frac{1}{(1 - 2x^{2n+1})},
\]

which is also known to be very close to certain vertex algebras. Similar functions are conjectured to generate the higher-point functions. The fact that the conjectured form of the amplitude can be found by expanding such a simple partition function appears to support its validity. The form of the modular ansatz is reviewed only briefly in this work.

Brief Review

The low-energy effective action at four-point consists of an infinite number of terms,

\[
S = \int d^{10}x \sqrt{g} \left[ R + \frac{1}{16} R^4 + \sum_{k=0}^{\infty} g_k(\tau, \bar{\tau}) \Box^k R^4 \right],
\]

with \( \tau = i/g_s + \theta/2\pi \), the string coupling constant, and \( \alpha' \) has been suppressed. The action in (2) does not include the terms which contain the unitarity cuts. The coefficients \( g_k \) can be computed in string perturbation theory, and is difficult to obtain beyond genus two.
Imposing modular invariance, in the Einstein frame of the metric, requires that the functions $g_k$ be invariant under fractional linear transformations. Finding the functions $g_k$ at the four-point level has been investigated in many papers, with consistency checks. The refined conjecture is investigated in this work, with the coefficients $g_k$ proportional to $Z_s$ functions. The latter functions obey a specific differential equation, and are roughly proportional to products of Eisenstein functions. This ansatz is investigated here, and with an emphasis on extracting further symmetry.

**Tree Amplitudes**

The four-point graviton tree amplitude has the form,

$$A_4 = \frac{64}{stu} \prod_{n=1}^{\infty} e^{2\zeta(2n+1)/2n/4^{2n+1+1}(s^{2n+1}+t^{2n+1}+u^{2n+1})}$$

and has the expansion,

$$\frac{64}{stu} \prod_i \frac{2\zeta(2n_i+1)}{(2n_i+1)4^{2n_i+1}}(s^{2n_i+1}+t^{2n_i+1}+u^{2n_i+1})^{m_i} \prod \frac{1}{(m_i)!}.$$  \hspace{1cm} (4)

The zeta functions take values in the odd integers excluding unity. In order to find the kinematic structure of the individual terms, an identity is required that expands the Mandelstam invariants in an appropriate basis. For example,

$$s^3 + t^3 + u^3 = 3stu.$$  \hspace{1cm} (5)

The further higher moments are expanded as

$$s^{2n+1} + t^{2n+1} + u^{2n+1} = c^n_s \times (stu)^{n_s},$$  \hspace{1cm} (6)

pertaining to the tensors with the maximal number of $s = (k_1 + k_2)^2$ invariants. In general the expansion takes the form,

$$(s^{2n+1} + t^{2n+1} + u^{2n+1})^m \rightarrow c^{n_s n_t n_u}_{n m} \times (stu)^{n_s} t^{n_t} u^{n_u}. \hspace{1cm} (7)$$

The coefficients $c^{n_s n_t n_u}_{n m}$ are found by kinematic identities and are in the basis in which the symmetry of $s \leftrightarrow t$, etc, is manifest.
The term with \( n_s \neq 0 \) and \( n_t = n_u = 0 \) is examined. In this kinematical configuration, the contributions are,

\[
A_n^{s,0,0} = 64 \prod_i \frac{1}{(m_i)!} \left[ \frac{2 \zeta(2n_i + 1)}{(2n_i + 1)4^{2n_i+1}} \right] c_n^{n_s} s^{n_s},
\]

which at a particular order in \( s \) is

\[
= 64 c_{n,m}^{n_s} s^{n_s} \prod_i \frac{1}{(m_i)!} \left[ \frac{2}{(2n_i + 1)4^{2n_i+1}} \right]^{m_i} \zeta(2n_i + 1)^{m_i}.
\]

The individual kinematic contributions follow from 1) taking a term \( stu \times s^{n_s} \) for an integer \( n_s \), then 2) partitioning the number \( n_s + 3 = N \) into odd numbers \((2n_j + 1)m_j\) with \( m_i = 1, \ldots \) so that \( \sum m_i = m \). The numbers \( m_i \) count the duplicates of the numbers \( 2n_i + 1 \), in which case there are \( m_i \) of the identical numbers for any \( i \).

The coefficient of the

\[
(stu)^m s^{n_s} \quad m = \sum m_i \quad n_s + 3 = \sum (2n_i + 1)m_i
\]

is found by collecting the coefficient

\[
64 \frac{1}{(m_i)!} \frac{2}{(2n_i + 1)4^{2n_i+1}} \prod_i \zeta(2n_i + 1),
\]

and multiplying by the group theory factor \( c_{n,m}^{n_s} \). The only constraint in this kinematic configuration is that \( n_s + 3 = \sum (2n_i + 1)m_i \). All possible combinations of \( n_i \) an integer and \( m_i \) an integer are allowed.

The ansatz indicates that the zeta functions are to be replaced in the manner,

\[
\prod_j \zeta(2p_j + 1) \rightarrow Z_{\{p_j + 1/2\}},
\]

with the \( Z \) functions described in the next section.

**Perturbative Modular Contributions**

The functions \( Z_{\{p_j + 1/2\}} \) are described by the modular invariant differential equation on the torus,
\[
\frac{1}{4} \Delta Z_{\{q_j\}} = AZ_{\{q_j\}} + B \prod_j Z_{q_j}, \quad (13)
\]

with the simplest case being the Eisenstein functions,

\[
Z_s = E_s \quad s(s - 1)Z_s = \Delta Z_s. \quad (14)
\]

The Laplacian takes the form, when restricted to the perturbative sector, that is, without the \( \tau_1 \) dependence,

\[
\Delta = 4 \tau_2^2 \delta_\tau \delta_{\bar{\tau}}. \quad (15)
\]

The condition on \( A \) and \( B \) could in principle be determined generically by the tree and one-loop contributions of the usual perturbative string amplitude; however, their numbers are left unknown for the moment.

One possible set of values is

\[
B = -(\sum p_j + \frac{1}{2})(-1 + \sum p_j + \frac{1}{2}) \quad (16)
\]

\[
A = 4 * f * \prod (p_j + \frac{1}{2}) \quad f^{-1} = \frac{1}{4} \sum_i p_j + \frac{1}{2} \quad \text{or} \quad 4 = 2^i. \quad (17)
\]

Their form is found by computing tree and one-loop amplitudes generically, and there is no real reason to believe this ansatz for \( A \) and \( B \) is correct.

The differential solution to the perturbative sector of the modular construction is generated as follows. The Eisenstein functions have the expansion,

\[
Z_s \big|_{\text{pert}} = 2\zeta(2s)\tau^s + \frac{2\sqrt{\pi}\Gamma(s - 1/2)\zeta(2s - 1)}{\Gamma(s)}\tau_2^{1-s}. \quad (18)
\]

Using their form in the differential equation,

\[
(\tau_2^2 \delta_\tau^2 - A)Z_{\{p_j + 1/2\}} \quad (19)
\]
\[
= (\tau_2^2 \partial_{\tau_2}^2 - A) (a_0 \tau_2^{3/2+k} + \ldots + a_{\text{gmax}} \tau_2^{3/2+k-2\text{gmax}})
\]
(20)

\[
= a_0 [(3/2 + k)(3/2 + k - 1) - A] \tau_2^{3/2+k} + \ldots
\]
(21)

\[+ a_{\text{gmax}} [(3/2 + k - 2\text{gmax})(3/2 + k - 2\text{gmax} - 1) - A] \tau_2^{3/2+k-2\text{gmax}}
\]
(22)

\[
= b_0 \tau_2^{3/2+k} + \ldots + b_{\text{gmax}} \tau_2^{3/2+k-2\text{gmax}}
\]
(23)

and ignores two additional terms on the left hand side, one of which has the \(\tau_2\) dependence to be not physical. The number \(k\) is defined by \(\sum 2p_j + 1 = 3/2 + k\) and corresponds to the \(\Box^k R^4\) term in the low-energy effective action. The other one is ignored for unphysical reasons. The solution to the terms is,

\[
a_i = \frac{b_i}{[(3/2 + k - 2i)(3/2 + k - 2i - 1) - A]}
\]
(24)

Then, the coefficients \(b_i\) are found by expanding the product \(B \times \prod_j Z_{p_j+1/2},\)

\[
B \prod_j Z_{p_j+1/2} = B \prod_j (a_{p_j+1/2} \tau_2^{2p_j+1} + a_{-p_j+1/2} \tau_2^{-2p_j}).
\]
(25)

Examples are listed below. For \(\sum 2p_j + 1 = 3/2 + k,\)

\[
b_0 = B \prod_j a_{p_j+1/2}
\]
(26)

\[
a_0 = B \prod_j \frac{a_{p_j+1/2}}{[(3/2 + k)(3/2 + k - 1) - A]}
\]
(27)

For \(\sum 2p_j + 1 = 3/2 + k - 2,\)

\[
b_1 = B \sum_{j \neq i} a_{p_i-1/2} \prod_j a_{p_j+1/2}
\]
(28)

\[
a_1 = B \sum_{j \neq i} a_{p_i-1/2} \prod_j \frac{a_{p_j+1/2}}{[(3/2 + k - 2)(3/2 + k - 2 - 1) - A]}
\]
(29)
The sum extends until $3/2 + k - 2g_{\text{max}}$, in which case we have $\sum 2p_j + 1 = 3/2 + k - 2g_{\text{max}}$,

$$b_{\text{max}} = B \prod_j a_{-p_j + 1/2}$$  \hspace{1cm} (30)

$$a_{\text{max}} = B \prod_j \frac{a_{-p_j + 1/2}}{[(3/2 + k - 2g_{\text{max}})(3/2 + k - 2g_{\text{max}} - 1) - A]}.$$  \hspace{1cm} (31)

The genus number $g_{\text{max}} = \frac{1}{2}(2k + 1)$ or $\frac{1}{2}(2k + 2)$ for $k = n/2$ with $n$ either odd or even, and for the $\Box^k R^4$ term; all of the products of the $Z_{p_j+1/2}$ have a perturbative truncation due to the individual expansion of the $Z_{p_j+1/2}$ functions. The examples describe the perturbative contributions from the modular functions $Z_{\{2p_j+1\}}$.

**Quantum Extension and Symmetry**

In this section a set of quantum rules is defined that generates the graviton amplitudes. The partitions of numbers are useful in parameterizing these contributions; also these partitions are connected to a fundamental symmetry of the quantum theory.

There have been several proposals for the quantum completion of the S-matrix, and higher derivative terms up to genus two have been computed. The modular invariant completion due to S-duality enforces certain structures on the coupling dependence. A basis for the coupling structure is formed from the Eisenstein functions, the contribution of which have recently been elucidated more completely in [5].

The polynomial system generating the perturbative contributions can be determined from a graphical illustration and also through a 'vertex' algebra. The latter can be found from expanding the function,

$$\prod_{n=1} \frac{1}{(1 - 2 x^{2n+1})},$$  \hspace{1cm} (32)

which is similar to the partition function of a boson on the torus,

$$\prod_{n=0} \frac{1}{(1 - x^{2n+1})}.$$  \hspace{1cm} (33)

The latter is associated to a vertex algebra. The former will be shown to correspond to the perturbative four-point function, without the non-analytic terms required by unitarity.
There are a set of trees as depicted in figure 1. Each tree is found by taking a number $N$, an odd number, and partitioning it into odd numbers $3, 5, 7, \ldots$. At each pair of nodes the numbers in the partition are attached. The partition is labeled by the set $N(\{\zeta\})$.

The nodes of the trees are chosen, one from each pair, and a set of lines could be drawn between these nodes. For each tree there are

$$\frac{a!}{b!(b-a)!}$$

ways of partitioning the tree into the same number of 'up' and 'down' nodes. Each tree set is labeled by $a = N(\{\zeta\})$, and the terms in (34) are spanned by $b = 0, \ldots, N(\{\zeta\})$ for $a = N(\{\zeta\})$. There are $2^{N(\{\zeta\})}$ terms or polynomials in each tree, as found by summing (34).
The derivative terms are labeled by $\square^{2k} R^4$ with $k$ either half integral or integral starting at $k = 0$. At a specific order $\square^{2k}$, the tree system is found by partitioning the number $N = 2k + 3$. The number of partitions a number $N$ can have into odd numbers, excluding unity, is denoted $P_{\text{odd}}(N)$. The perturbative contributions to this derivative order are found as follows,

1) attach weights to the nodes of the tree
2) take the product of the weights of the nodes in the tree
3) add the sums of the products

The sum of the products reintroduces the contribution of the modular ansatz to the four-point scattering.

For reference, the genus truncation property holds at $g_{\text{max}} = \frac{1}{2}(2k + 1), \frac{1}{2}(2k + 2)$ for $k = n/2$ with $n$ either odd or even. There are $P_{\text{odd}}(2k + 3)$ partitions of $2k + 3$ into odd numbers excluding unity, with a maximum number. The genus truncation follows from the fact that there a maximum number of nodes in the tree.

Some examples of the partitions of numbers at a given derivative order $\square^{2k} R^4$ are described in the following table. For example, at $k = 4$, the number 11 can be partitioned into 11 and $3 + 3 + 5$, with counts of $2^1$ and $2^3$.

\[
\begin{pmatrix}
  k = 0 & 2 & k = 4 & 2 + 2^3 = 10 \\
  k = 1 & 2 & k = 5 & 2 + 2^3 + 2^3 = 18 \\
  k = 2 & 2 & k = 6 & 2 + 2^5 + 2^3 + 2^3 = 68 \\
  k = 3 & 2 + 2^3 = 10 & k = 7 & 2 + 2^3 + 2^5 + 2^3 + 2^3 = 58
\end{pmatrix}
\] (35)

One would like to represent the terms group theoretically with the quantum numbers being zeta entries. This can be done using the expansion in (32).

The perturbative contribution to the order $\square^{2k}$ can be read off of the tree by associating the weights to the nodes. For the 'up' node there is a factor

\[
\frac{2\zeta(2s)}{[(3/2 + k)(3/2 + k - 1) - A]},
\] (36)

and for the 'down' node there is

\[
\frac{2\sqrt{\pi}\Gamma(s - 1/2)\zeta(2s - 1)}{\Gamma(s)[(3/2 + k - 2g_{\text{max}})(3/2 + k - 2g_{\text{max}} - 1) - A]}.
\] (37)
Each perturbative contribution is found by multiplying the node contributions. There are the various weighted trees that contribute to the perturbative contribution at \(2k + 3\), when partitioned into the various odd numbers. Each contribution has the weighted factor of \(B\) in the product. (The weighted trees resemble a fermionic system with \(g_{\text{max}}\) fermions with a quantum level degeneracy non-identical fermions.) The partitioning of the number \(2k + 3\) into the weighted trees is a convenient way of describing all of the contributions to the particular derivative term.

**Partitions**

Basically the perturbative contributions to the four-point function come about from partitioning an integer \(2k + 3\) into odd numbers \(3, 5, 7, \ldots\), and then subsequently choosing one of two choices for each number in the partition.

The partition of a number into odd numbers can be achieved by expanding the function

\[
O(N) = \prod_{n=0}^{\infty} \frac{1}{(1 - x^{2n+1})} \tag{38}
\]

into the polynomials

\[
x^{n_1}x^{n_2} \ldots x^{n_m}, \tag{39}
\]

for all sets of numbers \(n_i\). The expansion and summation of all terms generates the sum \(\sum P(N)x^N\) with \(P(N)\) the total number of partitions.

The number of partitions of the number \(N\) into all odd numbers, and excluding unity, is achieved by expanding the function

\[
O1(N) = \prod_{n=1}^{\infty} \frac{1}{(1 - x^{2n+1})} \ . \tag{40}
\]

This expansion generates the sum \(\sum P1(N)x^N\) with \(P1(N)\) numbering the total number of partitions, with

\[
P1(N) = \frac{1}{N!} \partial_x^N \prod_{n=1}^{\infty} \frac{1}{(1 - x^{2n+1})} . \tag{41}
\]
The sum $O1(N)$ follows from the same expansion as $O(N)$ but with additional factor of $(1 - x)$ removed from the denominator in the infinite product.

In the CFT language, the function $O(N)$ arises from a scalar on the torus. The $O1(N)$ function has one mode deleted, and represents a non-modular modification.

To complete the count of the number of tree systems the factor of two must be added to each of the nodes. The number of trees is,

$$T(N) = \sum_{\text{tree systems}} 2^{N(\zeta)} ,$$

with the corresponding sum without the $2^{N(\zeta)}$ being the count $P1(N)$. This number $T(N)$ can be found from,

$$P1(N) = \frac{1}{N!} \frac{\partial^N}{\partial x^N} \prod_{n=1}^{\infty} \left( \frac{1}{1 - 2x^{2n+1}} \right) ,$$

and the 2 at each node follows from the number of $x$s in the expansion, in all possible partitions. The relevant function is

$$\prod_{n=1}^{\infty} \frac{1}{(1 - 2x^{2n+1})} ,$$

and in the CFT language requires another modification of the modes by a factor of 2. Termwise the rescaling of $x$ to $2x$ corresponds with $x = e^{-2\pi \tau_2}$, with $\tau_2 \rightarrow \tau_2 - \ln 2^{-1/2\pi}$.

**Higher-Point Functions**

The higher point functions require expanding the function $\sqrt{g} \Box^{2k} R^4$ to higher order in the plane wave expansion, which stems essentially from the $\sqrt{g}$. There are also terms with higher numbers of curvatures, requiring $R^p$ at $p$-point. The tree amplitudes of the higher point functions are not currently available in the literature, in either an expanded or closed form.

One conjecture for the higher derivative terms at $p$-point would be to take the form,

$$64 c_{n,m,s}^{m_s} \prod_i \frac{1}{(m_i)!} \left[ \frac{2}{(2n_i + 1)4^{2n_i + 1}} \right]^{m_i} \prod_i \zeta(2n_i + 1)^{m_i} ,$$

with the $\sum 2n_i + 1 = 5, 7, \ldots$. The sum indicates that there are this number of vertices in a tree-level $p$-point graph, such as $p = 5$ with $\sum 2n_i + 1 = 5$ on up; likewise, at $p = 7$.
the sum starts at 9. The tensor combination here is for the terms with the maximum number of $s$ invariants, and the tensor function $c^{n,s}_{n,m}$ pertains to the $p$-point. Also, the helicity structure is found from the particular combination in the collection of Weyl tensors.

The origin of the $\zeta(3)$ in the $R^4$ stems from the fact that there are two vertices, with a massive propagator between them. In the low-energy limit of the string amplitude, this results in an infinite sum of the massive modes, with a $1/n$ coming from each vertex and also the propagator. At higher point, the sum of the massive modes requires more vertices, and results in the tree-level $\zeta$ function starting out at $2p - 5$, which is the number of vertices and propagators in a skeleton graph; this graph is a ladder tree diagram with $m - 4$ external gravitons on the internal rung.

Internal gravitons within the tree diagram enforce the entire tree diagram to be built of gravitons. This is in the absence of a massive-graviton-graviton vertex (or massive-graviton ... graviton vertex). Of course, at four-point we know there is a massive-graviton-graviton vertex, with a factor of $1/n$ from the vertex. The ladder diagrams result in the local terms in the amplitudes, that is without any $1/\Box$ terms in the low energy effective action. And each replacement of the massive mode by a graviton eliminates two factors of $1/n$ in the infinite sum; this lowers the $\zeta$ function value by two.

Assuming the $\zeta$ function takes on values in the tree diagrams at $p$-point beginning at $2p - 5$, it is natural to think that the partition function,

$$
\sum_{i=0}^{p+3} \prod_{n=2p-5-2i}^{\pi_2} \frac{1}{1 - 2x^{2n+1}},
$$

would be able to generate these scattering amplitudes. The helicity tensor from the $R^{2p}$ and the tensor function $c^{n,s,...}_{n,m}$ is required. Without more information this is just a partial conjecture for the form of the amplitude, however, the partition function seems appropriate. The terms with $i \neq 1$ correspond to diagrams without the maximal number of internal massive modes; these diagrams produce potential $1/\Box$ terms in the low-energy effective action. (For example, at four-point, there is a contribution $\frac{1}{\Box} R^4$ contribution from the GR graphs.)

Conclusion

The systematics of the four-point amplitude according to the S-duality of the IIB superstring is explored and delimited. The systematics entail the partitioning
of numbers and the construction of weighted trees. The entire four-point function, without the non-analytic terms, can be found through a simple construction.

The procedure of finding the four-point function is found by expanding a particular partition function. This partition function should be related to the S-duality of the IIB superstring. The simplicity of the generating function,

$$\prod_{i=1} \frac{1}{(1 - 2x^{2n+1})},$$

(47)
does seem to suggest the correctness of the four-point function. The higher point functions have similar partition functions.

The form of the amplitude can be used to deduce contributions in the massless sector, that is maximal supergravity. These calculations are typically very complicated, especially at the multi-loop level. The techniques for doing these cancellations are improving, and possible cancellations leading to better ultra-violet behavior beyond those up to five loops have been investigated in [14]-[16].

The origin of the simple partition function that generates the scattering seems not clear. There could be some additional symmetry beyond the naive supersymmetry and S-duality, or an extension as a result of the two. Some indication of this is examined in [17]. Vertex algebras are generated by these partition functions [18].
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