Continuous Patrolling and Hiding Games

Tristan Garrec*

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Abstract

We present two zero-sum games modeling situations where one player attacks (or hides in) a finite dimensional nonempty compact set, and the other tries to prevent the attack (or find him). The first game, called patrolling game, corresponds to a dynamic formulation of this situation in the sense that the attacker chooses a time and a point to attack and the patroller chooses a continuous trajectory to maximize the probability of finding the attack point in a given time. Whereas the second game, called hiding game, corresponds to a static formulation since both the searcher and the hider choose simultaneously a point and the searcher maximizes the probability of being at distance less than a given threshold of the hider.

1 Introduction

To ensure the security of vulnerable facilities, a planner may deploy either dynamic or static security devices. The dynamic case includes, but is not limited to: soldiers or police officers patrolling the streets of a city, robots patrolling a shopping mall, drones flying above a forest to detect fires, or naval radar systems signaling the detection of an enemy ship. On the other hand, security guards positioned in the rooms of a museum, security cameras scrutinizing subway corridors, or motion detectors placed in a house are some examples of static security devices. Note that these real-world situations include both human and electronic agents. The paradigm we adopt is the one of an adversarial threat, hence we propose a game theoretical approach to these security problems. Some game theoretic security systems are already in use, for example in the Los Angeles international airport (Pita et al. (2008)) and in some ports of the United States (Shieh et al. (2012)).

Motivated by the examples given above, we study two zero-sum games in which a player (the patroller or searcher) aims to detect another player (the attacker or hider). Patrolling games model dynamic security devices.

* Toulouse School of Economics, Université Toulouse 1 Capitole. E-mail address: tristan.garrec@ut-capitole.fr
In these games, the patroller moves continuously in a search space with bounded speed. The attacker chooses a point in the search space and a time to attack it. The attack takes a certain duration to be successful (think of a terrorist needing time to set off a bomb). The patroller wins if and only if he detects the attack before it succeeds. The case of static security devices is modeled by *hiding games*, in which both the searcher and the hider simultaneously deploy at a particular point in a search space, the searcher wins if and only if the hider lies within his detection radius. We provide links between patrolling and hiding games, and show how patrolling games reduce to hiding games if the attack duration is zero, that is the patroller has to detect the attack at the exact time it occurs, or if the patroller is alerted of the attack point when the attack begins.

### 1.1 Contribution

For patrolling games, we prove that the value always exists and obtain a general upper bound. We then study patrolling games on networks. In particular we compute the value as well as optimal strategies for the class of Eulerian networks. The special network composed of two nodes linked by three parallel arcs is also examined and bounds on the value are computed. Lastly, we study patrolling games on $\mathbb{R}^2$ and obtain an asymptotic expression for the value as the detection radius of the patroller goes to 0.

For hiding games, we first focus on a particular class of strategies for both the searcher and the hider called “equalizing”. These strategies have the property that if one exists, then it is optimal for both players. Our main result regarding hiding games is an asymptotic formula for the value of hiding games on a compact set with positive Lebesgue measure. A counterexample based on a Cantor-type set showing that this last result cannot extend to compact sets with zero Lebesgue measure is also presented.

Finally, we discuss some basic properties of monotonicity and continuity of the value function of continuous patrolling and hiding games.

### 1.2 Related literature

Continuous patrolling and hiding games belong to the literature of search and security games, consult [Holzaki (2016)](#) for a survey. These games have their source in search theory, a field of operations research whose origins can be found in the works of [Koopman (1956a,b, 1957)](#). The use of game theory in a search and security context goes back to the famous book of [Morse and Kimball (1951)](#).

#### 1.2.1 Patrolling games

Patrolling games were introduced by [Alpern et al. (2011)](#) in a discrete setting, that is the patroller visits nodes of a graph, where the attacker can
strike, at discrete times. A companion article Alpern et al. (2016a) is dedicated to the resolution of patrolling a discrete line. The idea of investigating a continuous version of patrolling games is suggested in Alpern et al. (2011). In Alpern et al. (2016b), the authors solve the continuous patrolling game played on the unit interval. Several papers in the field of search games have dealt with the transcription of discrete models to a continuous ones, consult Ruckle and Kikuta (2000) and Ruckle (1981).

Other models of games involving a patroller and an attacker can be found in Basilico et al. (2012, 2015), in which the authors design algorithms to solve large instances of Stackelberg patrolling security games on graphs. Lin et al. (2013, 2014) use linear programming and heuristics to study a large class of patrolling problems on graphs, with nodes having different values. Zoroa et al. (2012) study a patrolling game with a mobile attacker on a perimeter.

Continuous patrolling games are closely related to search games with an immobile hider, introduced in the seminal book of Isaac (1999) and developed in the monographs of Gal (1980) and Alpern and Gal (2003). In these games, a searcher intends to minimize the time necessary to find a hider. Search games have been extensively studied, let us mention Gal (1979); Alpern et al. (2008); Dagan and Gal (2008) for search games on a network. In particular, see Bostock (1984); Pavlovic (1993), for the special network consisting of two nodes linked by three parallel arcs, for which the solution is surprisingly complicated.

1.2.2 Hiding games

The first published example of a hiding game goes back to Gale and Glassey (1974). The proposers gave a solution to the problem of hiding in a disc when the detection radius is $r = 1/2$. Later, Ruckle (1983) considered in his book several examples of hiding games (hiding on a sphere, hiding in a disc, among others). Computing the value of hiding games is in general a very difficult task. Danskin (1990) improved substantially the resolution of the hiding game played on a disc, he called the cookie-cutter game. However the solution is not complete for small values of $r$ and no progress has been made since then, see also Alpern et al. (2013) and Washburn (2014). Hiding games in a discrete setting, when the search space is a graph, have also been studied, see Bishop and Evans (2013).

Games in which the payoff is the distance between the two points selected by the players, introduced by Karlin (1953), have been extensively studied, consult Ibragimov and Satimov (2012) and references therein. Although these games resemble hiding games to a certain extent, the lack of continuity of the payoff function in hiding games makes their analysis much more involved.

Finally, ambush games can be seen as hiding games in which the players
1.3 Organization of the paper

The paper is organized as follows. In section 2 the models of patrolling and hiding games are formally presented. Section 3 is dedicated to patrolling games, and section 4 is dedicated to hiding games. Finally in section 5 we give some basic properties of the value function of continuous patrolling and hiding games. The proofs that are not included in the body of the paper are postponed to the appendix section 6.

1.4 Notations

In all the article, $\mathbb{R}^n$ is endowed with a norm denoted $\| \cdot \|$, which induces a metric $d$. For all $x \in \mathbb{R}^n$ and $r > 0$, the closed ball of center $x$ with radius $r$ is denoted $B_r(x) = \{ y \in \mathbb{R}^n \mid \| x - y \| \leq r \}$. For all Lebesgue measurable set $B \subset \mathbb{R}^n$, $\lambda(B)$ denotes its Lebesgue measure. Finally, $\lambda(B_r)$ denotes the Lebesgue measure of any ball of radius $r$.

Let $X$ be a topological space, the set of Borel probability measures on $X$ is denoted $\Delta(X)$, and the set of probability measures on $Y$ with finite support is denoted $\Delta_f(Y)$.

2 The models

2.1 Patrolling games

In a patrolling game two players, an attacker and a patroller, act on a set $Q$ called the search space, which is assumed to be a nonempty compact subset of $\mathbb{R}^n$. An example of this could be a metric network, as defined in section 3.3 below. The attacker chooses an attack point $y$ in $Q$ and a time to attack $t$ in $\mathbb{R}_+$. The patroller walks continuously in $Q$ with speed at most 1. When the attack occurs at time $t$ and point $y$, the patroller has a time limit $m \in \mathbb{R}_+$ to be at distance at most $r \in \mathbb{R}_+$ of the attack point $y$. In this case he detects the attack and wins, and otherwise he does not. Thus, $m$ represents the time needed for an attack to be successful, and $r$ represents the detection radius of the patroller.

A patrolling game is thus a zero-sum game given by a triplet $(Q, m, r)$. The attacker’s set of pure strategies is $\mathcal{A} = Q \times \mathbb{R}_+$. An element of $\mathcal{A}$ is called an attack. The patroller’s set of pure strategies is $\mathcal{W} = \{ w : \mathbb{R}_+ \rightarrow Q \mid w$ is 1-Lipschitz continuous$\}$. An element of $\mathcal{W}$ is called a walk. $\mathcal{W}$ is endowed with the topology of compact convergence (consult the proof of proposition 1 in the appendix section 6 and Munkres (2000) for details).
The payoff to the patroller is given by

\[ g_{m,r}(w, (y, t)) = \begin{cases} 
1 & \text{if } d(y, w([t, t + m])) \leq r \\
0 & \text{otherwise},
\end{cases} \]

where \( w([t, t + m]) = \{w(\tau) \mid \tau \in [t, t + m]\} \).

### 2.2 Hiding games

In a hiding game two players, a searcher and a hider, act on a set \( Q \), which is again assumed to be a nonempty compact subset of \( \mathbb{R}^n \). An example of this could be the unit interval, as considered in example 5, or a Cantor-type set, as in example 3. Both players choose a point in \( Q \). The searcher has a detection radius \( r \in \mathbb{R}_+ \). He finds the hider if and only if the two points are at distance at most \( r \).

Hence, a hiding game is a zero-sum game given by a couple \( H = (Q, r) \). The set of pure strategies of both players, the searcher and hider, is \( Q \). The payoff to the searcher is given by

\[ h_r(x, y) = \begin{cases} 
1 & \text{if } \|x - y\| \leq r \\
0 & \text{otherwise}.
\end{cases} \]

### 2.3 Links between patrolling and hiding games

Hiding games can be interpreted as two possible variants of patrolling games. In the first variant, hiding games are considered as a particular class of patrolling games in which the attack duration \( m \) is taken equal to 0. Indeed, consider a hiding game \( H = (Q, r) \) and a patrolling \( P = (Q, 0, r) \). In \( P \), for all \( w \in W \) and \( (y, t) \in A \) the payoff to the patroller is

\[ g_{0,r}(w, (y, t)) = \begin{cases} 
1 & \text{if } \|w(t) - y\| \leq r \\
0 & \text{otherwise}.
\end{cases} \]

A strategy \( x \in Q \) of the searcher in the hiding game \( H \) is mapped in the patrolling game \( P \) to the constant strategy \( w \in W \) equal to \( x \). Similarly, a strategy \( y \in Q \) of the hider in \( H \) is mapped to the strategy \( (y, 0) \in A \) in \( P \). Any quantity guaranteed by the searcher in \( H \) is thus guaranteed by the patroller in \( P \). Conversely, any quantity guaranteed by the hider in \( H \) is guaranteed by the attacker in \( P \). Thus, since \( H \) and \( P \) have a value (see propositions 1 and 3), the values of these two games are the same.

The second interpretation is as follows. Alpern et al. (2011) suggest the study of patrolling games in which the patroller may be informed of the presence of the attacker. Suppose that the patroller is informed of the attack point when the attack occurs. Suppose also the search space \( Q \) is
convex. The detection radius \( r \) is taken equal to 0 for simplicity. The payoff of this game is

\[
g_{m,0}(w,(y,t)) = \begin{cases} 
1 & \text{if } y \in w([t,t+m]) \\
0 & \text{otherwise.} 
\end{cases}
\]

This patrolling game with signal is denoted \( P' \). In \( P' \), if the patroller’s strategy is to choose a point and not move until the attack, then go to the attack point in straight line when he is alerted, the attacker is time-indifferent. In particular, the attacker has a best reply in the set of attacks occurring at time 0. Symmetrically, if the attack occurs at time 0, the patroller has a best reply consisting in choosing a starting point in \( Q \) and going directly to the attack point when he is informed of the attack.

Thus, with the same mappings of strategies in the hiding game \( H' = (Q,m) \) to strategies in \( P' \) as before, any quantity guaranteed by the searcher in \( H' \) is guaranteed by the patroller in \( P' \). Conversely, any quantity guaranteed by the hider in \( H' \) is guaranteed by the attacker in \( P' \). Thus the values of these two games are the same.

3 Patrolling games

3.1 The value of patrolling games

The first result is the existence of the value of a patrolling games. We denote it \( V_Q(m,r) \). In addition, we prove that the patroller has an optimal strategy and the attacker has an \( \varepsilon \)-optimal strategy with finite support. The fact that the patroller has an optimal strategy means that he can guarantee that the probability of detecting the attack is at least \( V_Q(m,r) \), no matter what the attacker does. Similarly, the attacker can guarantee that the probability of being caught is at most \( V_Q(m,r) \), up to \( \varepsilon \), no matter what the patroller does. Hence, in patrolling games, the value represents the probability (up to \( \varepsilon \)) of the attack being intercepted when both the patroller and the attacker play (\( \varepsilon \))-optimally.

**Proposition 1.** The patrolling game \( (Q,m,r) \) played with mixed strategies has a value denoted \( V_Q(m,r) \).

Moreover the patroller has an optimal strategy and the attacker has an \( \varepsilon \)-optimal strategy with finite support, i.e., there exists \( \mu \in \Delta(W) \) such that for any \((y,t) \in A \)

\[
\int_W g_{m,r}(w,(y,t))d\mu(w) \geq V_Q(m,r),
\]

and for every \( \varepsilon > 0 \) there exists \( \nu \in \Delta_f(A) \) such that for any \( w \in W \)

\[
\int_A g_{m,r}(w,(y,t))d\nu(y,t) \leq V_Q(m,r) + \varepsilon.
\]
3.2 A general upper bound

Our goal is now to obtain a general upper bound for the value of patrolling games. As in Alpern and Gal (2003), let us introduce the maximum rate at which the patroller can discover new points of $Q$.

**Definition 1.** The maximum discovery rate is given by

$$\rho = \sup_{w \in W, t > 0} \frac{\lambda(w([0, t]) + B_r(0)) - \lambda(B_r)}{t}.$$ 

The next remark shows that in $\mathbb{R}^2$ endowed with the Euclidean norm, the maximum discovery rate is $2r$, that is the sweep width of the patroller. Similarly, in $\mathbb{R}$ and $\mathbb{R}^3$ endowed with the Euclidean norm, the maximum discovery rate is respectively $1$ and $\pi r^2$.

**Remark 1.** Let $(Q, m, r)$ be a patrolling game. If $Q$ has nonempty interior in $\mathbb{R}^2$ endowed with the Euclidean norm, then $\rho = 2r$.

Indeed, since $Q$ has nonempty interior let $x \in Q$ and $s > 0$ be such that $B_s(x) \subset Q$. Define $w(t) = x + t \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ for $t \in [0, s]$ (and arbitrarily such that $w \in W$ for $t > s$). Then

$$\frac{\lambda(w([0, s]) + B_r(0)) - \lambda(B_r)}{s} = 2rs = 2r, \quad \text{and it is clear that this is the maximum.}$$

Let us now give an upper bound for patrolling games whose search space have nonzero Lebesgue measure. This upper bound is rather powerful and will be extensively used in the remaining of the paper. It is in particular the upper bound used to prove theorems 1, 2 and 3.

**Proposition 2.** Let $Q$ be a search space such that $\lambda(Q) > 0$. Then

$$V_Q(m, r) \leq m \rho + \lambda(B_r) \frac{\lambda(B_r)}{\lambda(Q)}.$$ 

To prove proposition 2 we define a strategy for the attacker called uniform. It corresponds to the strategy for which the attacker uniformly chooses an attack point in $Q$, and attacks this point at time $0$. Intuitively, a best reply of the patroller is to cover as much points in $Q$ as possible between time $0$ and time $m$.

**Definition 2.** Let $Q$ be a search space such that $\lambda(Q) > 0$. The attacker’s uniform strategy on $Q$, denoted $a_\lambda$, is a random choice of the attack point $a$ at time $0$ such that for all measurable sets $B \subset Q$,

$$a_\lambda(B) = \frac{\lambda(B)}{\lambda(Q)}.$$
Proof of proposition For all \( w \in \mathcal{W} \), the payoff to the patroller when the attacker plays \( a_\lambda \) is
\[
\int_A g_{m,r}(w, (y,t))da_\lambda(y,t) = \frac{\lambda(w([0,m]) + B_r(0))}{\lambda(Q)} \leq m\rho + \frac{\lambda(B_r)}{\lambda(Q)}.
\]

3.3 Patrolling a network

We now investigate the particular case of patrolling games on a network. As in a network the patroller can manage to have a nonzero probability of walking through the exact attack point, his detection radius \( r \) is set to 0.

3.3.1 Definition of a network

We follow the construction of a network of Fournier (2016). Let \((V, E, l)\) be a weighted undirected graph, \( V \) is the finite set of nodes and \( E \) the finite set of edges whose elements \( e \in E \) are associated to a length \( l(e) \in \mathbb{R}_+ \). An edge \( e \in E \) linking the two nodes \( s \) and \( t \) is also denoted \((s, t)\).

We identify the elements of \( V \) with the vectors of the canonical basis of \( \mathbb{R}^{|V|} \). The network generated by \((V, E)\) is the set of points
\[
\mathcal{N} = \{(s, t, \alpha) \mid \alpha \in [0, 1] \text{ and } (s, t) \in E\},
\]
where \((s, t, \alpha) = \alpha s + (1 - \alpha)t\).

A network \( \mathcal{N} \) is endowed with a natural metric \( d \) as follows. Let \( u_1 \) and \( u_2 \) be two points of the same edge \((s, t)\). There exist \( \alpha_1, \alpha_2 \in [0, 1] \) such that \( u_1 = (s, t, \alpha_1) \) and \( u_2 = (s, t, \alpha_2) \). The distance \( d(u_1, u_2) \) is given by
\[
d(u_1, u_2) = l(s, t) \times |\alpha_1 - \alpha_2|.
\]

If \( u \) and \( v \) are not in the same edge, consider the set of paths \( P(u, v) \) between \( u \) and \( v \) as the set of all sequences \((u_1, \ldots, u_n)\), \( n \in \mathbb{N}^* \) such that \( u_1 = u \), \( u_n = v \) and such that for all \( i \in \{1, \ldots, n - 1\} \), \( u_i \) and \( u_{i+1} \) belong to the same edge. The distance \( d(u, v) \) is then defined as:
\[
d(u, v) = \inf_{(u_1, \ldots, u_n) \in P(u, v)} \sum_{i=1}^{n-1} d(u_i, u_{i+1}).
\]

Finally, we define the Lebesgue measure on \( \mathcal{N} \). Let \( u_1 = (s, t, \alpha_1) \) and \( u_2 = (s, t, \alpha_2) \), suppose \( \alpha_1 < \alpha_2 \). The set
\[
[u_1, u_2] = \{(s, t, \alpha) \mid \alpha \in [\alpha_1, \alpha_2]\}
\]
is called an interval. An interval \([u_1, u_2]\) can be isometrically identified with the real interval \([\alpha_1 l(s, t), \alpha_2 l(s, t)]\). As a subset of \( \mathcal{N} \) can be identified with a finite union of subsets of intervals, the Lebesgue measure on \( \mathcal{N} \) is defined as a natural extension of the Lebesgue measure on a real interval.
3.3.2 Eulerian networks

For a particular class of networks called Eulerian, it is possible to compute the value and optimal strategies of the game. Note that we use for networks a similar vocabulary to the one used in graph theory, see for example Bondy and Murty (1982).

As stated in the next definition, an Eulerian tour is a closed path in \( \mathcal{N} \) visiting all points and having length \( \lambda(\mathcal{N}) \).

**Definition 3.** Let \( u \in \mathcal{N} \) and \( \pi = (u_1, u_2, \ldots, u_n) \in P(u, u) \). If \( \bigcup_{k=1}^{n-1} [u_k, u_{k+1}] = \mathcal{N} \) then \( \pi \) is called a tour.

Moreover, if \( \sum_{k=1}^{n-1} \lambda([u_k, u_{k+1}]) = \lambda(\mathcal{N}) \), then \( \pi \) is called an Eulerian tour.

A network \( \mathcal{N} \) is said to be Eulerian if there exists an Eulerian tour in \( \mathcal{N} \).

**Example 1.** Figure 1 and figure 2 give two examples of networks. \( \mathcal{N}_1 \) is an Eulerian network with Eulerian tour \( \pi_1 = (u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_1) \). In contrast, \( \mathcal{N}_2 \) is not an Eulerian network.

![Figure 1: The network \( \mathcal{N}_1 \) is Eulerian](image1)

![Figure 2: The network \( \mathcal{N}_2 \) is not Eulerian](image2)

Our objective is now to define the uniform strategy of the patroller for Eulerian networks. This strategy is optimal for Eulerian networks. First, we need to define a parametrization of the network.

**Definition 4.** Let \( \mathcal{N} \) be an Eulerian network. A continuous function \( w \) from \( [0, \lambda(\mathcal{N})] \) to \( \mathcal{N} \) such that

i) \( w(0) = w(\lambda(\mathcal{N})) \),

ii) \( w \) is surjective.

iii) \( \forall t_1, t_2 \in [0, \lambda(\mathcal{N})] \) \( \lambda(w([t_1, t_2])) = |t_1 - t_2| \) (the speed of \( w \) is 1).

is called a parametrization of \( \mathcal{N} \).

Moreover such a \( w \) can be extended to a \( \lambda(\mathcal{N}) \)-periodic function on \( \mathbb{R} \) which is still denoted \( w \).
Lemma 1. Let $\mathcal{N}$ be an Eulerian network, then there exists a parametrization of $\mathcal{N}$.

It is now possible to define the uniform strategy of the patroller. The idea behind this strategy is that the patroller uniformly chooses of a starting point in $\mathcal{N}$, and then follows a parametrization as defined in definition 4 above.

Definition 5. Suppose $\mathcal{N}$ is an Eulerian network. Let $w$ be a parametrization of $\mathcal{N}$. Denote $(w_t)_{t \in \left[0, \lambda(\mathcal{N})\right]}$ the family of $\lambda(\mathcal{N})$-periodic walks such that $w_t(\cdot) = w(t_0 + \cdot)$. The patroller’s uniform strategy is given by the uniform choice of $t_0 \in \left[0, \lambda(\mathcal{N})\right]$.

The next theorem is the main result on patrolling games for networks. It gives a simple expression of the value of a patrolling game played on any Eulerian network. The result relies on the fact that for such networks, the patroller can achieve the upper bound of proposition 2 using his uniform strategy.

Theorem 1. If $\mathcal{N}$ is an Eulerian network, then

$$V_{\mathcal{N}}(m, 0) = \min \left( \frac{m}{\lambda(\mathcal{N})}, 1 \right).$$

Moreover the attacker’s and the patroller’s uniform strategies are optimal.

3.3.3 A non Eulerian network

For non Eulerian networks, it may be difficult to compute the value of the corresponding patrolling game. In the next example, we compute bounds on the value of a patrolling game played over the network with three parallel arcs, which is not Eulerian. For some values of the attack duration $m$, these bounds are not tight.

Example 2. We consider again the network $\mathcal{N}_2$ represented in figure 2. In this example, we take $l(u_1, u_2) = l(u_2, u_3) = l(u_1, u_4) = l(u_4, u_3) = 1/2$ and $l(u_1, u_3) = 1$. Notice that $\lambda(\mathcal{N}_2) = 3$. We compute the following bounds on the value of $(\mathcal{N}_2, m, 0)$:

$$V_{\mathcal{N}_2}(m, 0) = \begin{cases} \frac{m}{2} & \text{if } m \leq 2 \\ \frac{5m-2}{3(m+2)}, 1 - \frac{1}{3} \left( \frac{4-m}{2} \right)^2 & \text{if } m \in \left[ \frac{2, 10}{3} \right] \\ \frac{14-2m}{3(6-m)}, 1 - \frac{1}{3} \left( \frac{4-m}{2} \right)^2 & \text{if } m \in \left[ \frac{10}{3}, 4 \right] \\ 1 & \text{if } m \geq 4. \end{cases}$$

These bounds are plotted on figure 3 below.
Intercepts any attack on \( u \) and \( w \) noted respectively. That is, \( \pi^1 \) and \( \pi^2 \) naturally induce two walks on \([0,3]\) at speed 1, respectively denoted \( w^1 \) and \( w^2 \). For all \( u \in \mathcal{N}_2 \setminus \{u_1, u_3\} \) and all \( i \in \{1, 2\} \) there exists a unique \( t^i_u \in [0,3] \) such that \( w^i(t^i_u) = u \). Now for all \( u \in \mathcal{N}_2 \setminus \{u_1, u_3\} \) and all \( t \in \mathbb{R}_+ \), define

\[
w^1_u(t) = \begin{cases} w^1(t + t^1_u) & \text{if } t \in [0,3 - t^1_u] \\ w^2(t - (3(2k + 1) - t^1_u)) & \text{if } t \in (3(2k + 1) - t^1_u, 3(2k + 2) - t^1_u) \\ w^1(t - (3(2k + 2) - t^1_u)) & \text{if } t \in (3(2k + 2) - t^1_u, 3(2k + 3) - t^1_u) \end{cases}
\]

for all \( k \in \mathbb{N} \). The walk \( w^1_u \) starts at \( t^1_u \) and alternates between following \( w^1 \) and \( w^2 \). The walk \( w^2_u \) is defined analogously: switch the superscripts 1 and 2 in the definition above. Denote \( \mu^0 \) the uniform choice of a walk in \( \{w^1_u \}_{u \in \mathcal{N}_2 \setminus \{u_1, u_3\}} \).

It is not difficult to check that \( \mu^0 \) guarantees \( m/3 \) to the patroller (moreover \( \mu^0 \) yields a payoff of \( m/3 \) for every \((y, t) \in \mathcal{A}\)). Hence \( V_{\mathcal{N}_2}(m, 0) = \frac{m}{3} \).

**Second case: \( 2 < m < 4 \).** We detail the computation for \( m = 3 \). The walks \( w^3, w^4 \) and \( w^5 \) hereafter can be adapted and similar strategies can be used to derive the bounds for all \( m \in (2, 4) \).

Let us define three paths \( \pi^3, \pi^4 \) and \( \pi^5 \) as in figure 4, 5, and 6 respectively. That is, \( \pi^3 = (u_1, u_2, u_3, u_5, u_3, u_1, u_6, u_1) \), \( \pi^4 = (u_1, u_7, u_1, u_3, u_8, u_3, u_4, u_1) \) and \( \pi^5 = (u_1, u_2, u_3, u_{10}, u_3, u_4, u_1, u_9, u_1) \). Where \( u_5 = (u_3, u_4, 1/2), u_6 = (u_1, u_4, 1/2), u_7 = (u_1, u_2, 1/2), u_8 = (u_2, u_3, 1/2), u_9 = (u_1, u_3, 1/4) \) and \( u_{10} = (u_1, u_3, 3/4) \).

\( \pi^3, \pi^4 \) and \( \pi^5 \) naturally induce three 3-periodic walks at speed 1, denoted respectively \( w^3, w^4 \) and \( w^5 \). These are such that for \( i \in \{3, 4, 5\} \), \( w^i \) intercepts any attack on \( w^i([0,3]) \) with probability 1.
With a slight abuse of notation, for $y \in [0, 1/2]$, denote $y$ the point $(u_1, u_3, y)$ and $1 - y$ the point $(u_1, u_3, 1 - y)$. By symmetry it is enough to consider attacks occurring at $y$. Moreover, $\mu^0$, $w^3$, $w^4$ and $w^5$ make the patroller time indifferent, hence we only consider attacks at time 0.

$\mu^0$ intercepts the attack $(y, 0)$ with probability $1 - \frac{2y}{6} = \frac{3}{2} + \frac{y}{3}$. Indeed, only the walks $w_u^1$, such that $u$ belongs to the open interval $\{(u_1, u_3, \alpha) \mid \alpha \in (y, 1 - y)\}$ do not intercept the attack. Finally, define $\tilde{\mu} = \frac{1}{15}(\delta_{w^3} + \delta_{w^4} + \delta_{w^5}) + \frac{4}{5}\mu^0$, where $\delta_{w^i}$ is the Dirac measure at $w \in W$.

At any time, an attack at $y \leq 1/4$ is intercepted by $\tilde{\mu}$ with probability

$$\frac{1}{15} \cdot 3 + \frac{4}{5} \left(\frac{5}{6} + \frac{y}{3}\right) \geq \frac{3}{15} + \frac{4}{5} \cdot \frac{5}{6} = \frac{13}{15}.$$  

An attack at $y > 1/4$ is intercepted by $\tilde{\mu}$ with probability

$$\frac{1}{15} \cdot 2 + \frac{4}{5} \left(\frac{5}{6} + \frac{y}{3}\right) \geq \frac{2}{15} + \frac{4}{5} \left(\frac{5}{6} + \frac{1}{12}\right) = \frac{13}{15}.$$  

Hence $V_{N_2}(3, 0) \geq \frac{13}{15}$.

Define the following attack $\tilde{a}$: choose uniformly a point in $N_2 \times [0, 3]$. The tour $(u_1, u_2, u_3, u_4, u_3, u_3, u_1, u_3, u_4, u_1)$ induces a 6-periodic walk $w^6$ which is a best reply for the patroller. Moreover $g_{3,0}(w^6, \tilde{a}) = 11/12$. Hence $V_{N_2}(3, 0) \leq \frac{11}{12}$.

**Third case:** $m \geq 4$. The tour $(u_1, u_2, u_3, u_1, u_4, u_3, u_4, u_1)$ induces a 4-periodic walk which guarantees 1 to the patroller. Hence $V_{N_2}(m, 0) = 1$.

### 3.4 Patrolling a simple search space in $\mathbb{R}^2$

In this section, we are interested in patrolling games in $\mathbb{R}^2$ for a large class of search spaces called simple. To introduce this class of search spaces, we first need to recall the notion of bounded variation of a function $f$. 

![Figure 4: The path $\pi^3$](image1)

![Figure 5: The path $\pi^4$](image2)

![Figure 6: The path $\pi^5$](image3)
Definition 6. Let $a > 0$. Let $f : [0, a] \to \mathbb{R}^n$ be a continuous function. Then the total variation of $f$ is the quantity:

$$TV(f) = \sup \left\{ \sum_{i=1}^{n} \| f(t_i) - f(t_{i-1}) \|_2 \mid n \in \mathbb{N}^*, 0 = t_0 < t_1 < \cdots < t_n = a \right\}.$$ 

If $TV(f) < +\infty$, then $f$ is said to have bounded variation.

The next definition introduces a classical assumption on the boundary of a search space in $\mathbb{R}^2$. This is a weak assumption already made in Gal (1980) and Alpern and Gal (2003).

Definition 7. Let $a > 0$, let $f_1$ and $f_2$ be two continuous functions from $[0, a]$ to $\mathbb{R}$ such that $f_1 \geq f_2$, $f_1 \neq f_2$, and $f_1$ and $f_2$ have bounded variation. Then the nonempty compact set $\{(x, t) \in [0, a] \times \mathbb{R} \mid f_2(x) \leq t \leq f_1(x)\}$ is called an elementary search space.

Let $Q$ be the finite union of elementary search spaces such that any two have disjoint interiors. If $Q$ is path-connected, then it is called a simple search space.

The next theorem is the main result on patrolling games on simple search spaces. It gives a simple asymptotic expression of the value as the detection radius $r$ goes to 0. The result relies on the fact that the patroller can use a uniform strategy in the spirit of what has been done in the previous section for Eulerian networks. This strategy yields a lower bound that asymptotically matches the upper bound of proposition 2.

As one would expect, the value goes to 0 as $r$ goes to 0. It is interesting to note that due to the movement of the patroller the convergence is linear in $r$ and not quadratic. Indeed, the relevant parameter is the sweep width of the patroller and not the area of detection.

Theorem 2. If $Q$ is a simple search space endowed with the Euclidean norm, then

$$V_Q(m, r) \sim \frac{2rm}{\lambda(Q)},$$

as $r$ goes to 0.

4 Hiding games

Recall that the value of a hiding game is equal to the value of a patrolling game with time limit $m$ equal to 0. Hiding games have a value which represents the probability (up to $\varepsilon$) that the searcher and the hider are at distance less that $r$ when they play ($\varepsilon$-)optimally.

Proposition 3. The hiding game $(Q, r)$ played in mixed strategies has a value denoted $V_Q(r)$. Moreover the searcher has an optimal strategy and the hider has an $\varepsilon$-optimal strategy with finite support.
4.1 Equalizing strategies

We now study particular strategies called equalizing, these have been introduced in Bishop and Evans (2013) when the search space is a graph (see definition 7.3 and proposition 7.3 therein). We adapt those considerations to our compact setting.

A strategy is equalizing if the induced payoff does not depend on the strategy of the other player. The interest of equalizing strategies lies in the fact that if such a strategy exists, then it is optimal for both players.

Definition 8. Let $Q$ be a search space. A strategy $\mu \in \Delta(Q)$ is said to be equalizing if there exists $c \in \mathbb{R}^+$ such that $\mu(B_r(y) \cap Q) = c$ for all $y \in Q$.

Proposition 4. Let $\mu \in \Delta(Q)$. Then $\mu$ is an equalizing strategy (with constant payoff $c$) if and only if $\mu$ is optimal for both players (and in that case $V_Q(r) = c$).

The following game is an example of a hiding game with finite search space without equalizing strategies.

Example 3. Let $r = 1$ and $Q = \{x_1, x_2, x_3, x_4, x_5\}$ be the finite subset of $\mathbb{R}^2$ such that $x_1 = (0,0)$, $x_2 = (0,1)$, $x_3 = (1,1)$, $x_4 = (1,0)$ and $x_5 = (1/2,0)$. Denote for $i \in \{1, \ldots, 5\}$ $Q_i = \{j \in \{1, \ldots, 5\} \mid \|x_i - x_j\|_2 \leq r\}$. That is $Q_1 = \{1,2,4,5\}$, $Q_2 = \{1,2,3\}$, $Q_3 = \{2,3,4\}$, $Q_4 = \{1,3,4,5\}$ and $Q_5 = \{1,4,5\}$.

The game $(Q, r)$ admits an equalizing strategy if and only if the following system of equations admits a solution $p = (p_i)_{1 \leq i \leq 5}$:

\[
\begin{align*}
& p_i \geq 0 \text{ for all } i \in \{1, \ldots, 5\} \\
& \sum_{i=1}^5 p_i = 1 \\
& \sum_{i \in Q_j} p_i = \sum_{i \in Q_j} p_i \text{ for all } j \in \{2, \ldots, 5\}.
\end{align*}
\]

It is easy to verify that this system does not admit a solution, hence the game $(Q, r)$ does not have an equalizing strategy.

4.2 An asymptotic result for hiding games

The next theorem is the main result on hiding games. For any search space $Q \subset \mathbb{R}^n$ with positive Lebesgue measure, it gives a simple asymptotic expression of the value when the detection radius goes to 0. In this static setting the value is equivalent, as $r$ goes to 0, to the ratio of the volume of the ball of radius $r$ over the volume of $Q$. This result relies on the fact that the searcher has a strategy that yields a lower bound which asymptotically matches the upper bound of proposition 2.
Theorem 3. Let $Q$ be a compact subset of $\mathbb{R}^n$. Suppose $\lambda(Q) > 0$. Then

$$V_Q(r) \sim \frac{\lambda(B_{r})}{\lambda(Q)}$$

as $r$ goes to 0.

A consequence of theorem 3 is that for a compact set $Q$ included in $\mathbb{R}^n$ such that $\lambda(Q) > 0$, $V_Q(r) \sim r^n \frac{\lambda(B_{r})}{\lambda(Q)}$ as $r$ goes to 0. When $\lambda(Q) = 0$, it is not always the case that $V_Q$ admits an equivalent of the form $Mr^\alpha$, with $\alpha$ and $M$ positive, as $r$ goes to 0, as it is shown in example 4.

Example 4. Let $Q \subset [0,1]$ be the following Cantor-type set. Define $C_0 = [0,1]$, and for all $n \in \mathbb{N}^*$ $C_n = \frac{1}{4} C_{n-1} \cup \left( \frac{3}{4} + \frac{1}{4} C_{n-1} \right)$. Finally, let $Q = \bigcap_{n \in \mathbb{N}} C_n$. $Q$ is compact and $\lambda(Q) = 0$.

The value of the hiding game played on $Q$ is given by the following formula:

$$V_Q(r) = \begin{cases} \frac{2}{3^n} & \text{if } r \in \left[ \frac{1}{2^{2n}}, \frac{3}{2^{2n}} \right), \\ \frac{1}{3^n} & \text{if } r \in \left[ \frac{3}{2^{2n}}, \frac{1}{2^{n-1}} \right), n \in \mathbb{N}^*. \end{cases}$$

Indeed, let $\Sigma_1 = \{0, 1\}$ and for all $n \in \mathbb{N}^* \setminus \{1\}$ let $\Sigma_n = \frac{1}{4} \Sigma_{n-1} \cup \left( \frac{3}{4} + \frac{1}{4} \Sigma_{n-1} \right)$. For $n \in \mathbb{N}^*$, consider the following strategy $\sigma_n$: choose uniformly a point in $\Sigma_n$, that is with probability $|\Sigma_n|^{-1} = \frac{1}{2^n}$. Let $n \in \mathbb{N}^*$ suppose $r \in \left[ \frac{1}{2^{2n}}, \frac{3}{2^{2n}} \right)$. Then for all $q \in Q$ there is exactly one point $s$ in $\Sigma_n$ such that $|q - s| \leq r$. Hence $\sigma_n$ is an equalizing strategy which guarantees $\frac{1}{2^n}$ to both players.

Let $\Sigma'_1 = \{\frac{1}{2}\}$ and for all $n \in \mathbb{N}^* \setminus \{1\}$ let $\Sigma'_n = \frac{1}{4} \Sigma'_{n-1} \cup \left( 1 - \frac{1}{4} \Sigma'_{n-1} \right)$. For $n \in \mathbb{N}^*$ consider the following strategy $\sigma'_n$: choose uniformly a point in $\Sigma'_n$, that is with probability $|\Sigma'_n|^{-1} = \frac{1}{2^n}$. Suppose now that $r \in \left[ \frac{3}{2^{2n}}, \frac{1}{2^{n-1}} \right)$. Then for all $q \in Q$ there is exactly one point $s$ in $\Sigma'_n$ such that $|q - s| \leq r$. Hence $\sigma'_n$ is an equalizing strategy which guarantees $\frac{1}{2^n}$ to both players.

In particular, for all $n \in \mathbb{N}^*$

$$V_Q \left( \frac{1}{2^{2n-1}} \right) = V_Q \left( \frac{1}{2^{2n}} \right) = \frac{1}{2^n}.$$ 

Let $(r_n)_{n \in \mathbb{N}^*} = \left( \frac{1}{2^n} \right)_{n \in \mathbb{N}^*}$ and let $\alpha > 0$. Then for all $n \in \mathbb{N}^*$

$$V_Q(r_{2n-1})^{\alpha} = \frac{1}{2^n} 2^{(2n-1)\alpha} \text{ and } V_Q(r_{2n})^{\alpha} = 2^{(2n-1)\alpha}.$$ 

Thus we have

$$\lim_{n \to +\infty} \frac{V_Q(r_{2n-1})}{(r_{2n-1})^\alpha} = \begin{cases} +\infty & \text{if } \alpha > 1/2 \\
 \frac{1}{\sqrt{2}} & \text{if } \alpha = 1/2 \text{ and } \lim_{n \to +\infty} \frac{V_Q(r_{2n})}{(r_{2n})^\alpha} = \begin{cases} +\infty & \text{if } \alpha > 1/2 \\
 1 & \text{if } \alpha = 1/2 \\
 0 & \text{if } \alpha < 1/2. \end{cases} \end{cases}$$

Hence $r \mapsto V_Q(r)$ does not admit an equivalent of the form $r \mapsto Mr^\alpha$, with $\alpha$ and $M$ positive numbers, as $r$ goes to 0.
5 Properties of the value function of patrolling and hiding games

In this section we give some elementary properties of the function $V_Q$ for patrolling and hiding games.

5.1 The value function of patrolling games

Proposition 5. Let $Q$ be a search space. The function

$$V_Q : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow [0, 1]$$

$(m, r) \mapsto V_Q(m, r)$

is

i) non decreasing in $m$ and $r$,

ii) upper semi-continuous in $r$ for all $m$,

iii) upper semi-continuous in $m$ for all $r$.

Example 5 in the next section shows that in general, for fixed $m$, $V_Q(\cdot, m)$ is not lower semi-continuous.

Remark 2. Let $m, r \geq 0$, and $Q_1, Q_2$ be two search spaces, it is clear that if $Q_1 \subset Q_2$ then the attacker is better off in $Q_2$ hence $V_{Q_1}(m, r) \geq V_{Q_2}(m, r)$.

5.2 The value function of hiding games

Recall that the value of a hiding game is equal to the value of a patrolling game with time limit $m$ equal to 0. Hence, the negative results presented in this section also hold for patrolling games when $m = 0$.

The following simple example of a hiding game on the unit interval was first solved by Ruckle (1983). It shows that in general, $V_Q$ is not lower semi-continuous.

Example 5. Let $Q$ be the $[0, 1]$ interval, then

$$V_Q(r) = \begin{cases} \min \left( \left\lfloor \frac{1}{2r} \right\rfloor^{-1}, 1 \right) & \text{if } r > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Indeed, it is clear when $r$ equals 0 and $r \geq 1/2$. Let $n \in \mathbb{N}^*$ and suppose $r \in \left[ \frac{1}{2(n+1)}, \frac{1}{2n} \right)$, then the patroller guarantees $\frac{1}{n+1}$ by choosing equiprobably a point in \( \left\{ \frac{1+2k}{2(n+1)} \right\}_{0 \leq k \leq n} \). And the attacker, choosing equiprobably a point in \( \left\{ \frac{(2+\varepsilon)k}{2(n+1)} \right\}_{0 \leq k \leq n} \), with $0 < \varepsilon \leq 2/n$, also guarantees $\frac{1}{n+1}$. 

16
The next proposition disproves the somehow intuitive belief that the value of hiding games is continuous with respect to the Hausdorff metric between nonempty compact sets.

**Proposition 6.** Let \( r \geq 0 \). The function which maps any search space \( Q \) to \( V_Q(r) \) is in general not continuous with respect to the Hausdorff metric between nonempty compact sets.

**Proof.** Let \( D_s = \{ x \in \mathbb{R}^2 \mid \|x\|_2 < s \} \) be the Euclidean disc of radius \( s > 0 \) centered at 0. From [Danskin (1990)](https://doi.org/10.1016/0022-0396(90)90064-5), it is known that

\[
V_{D_s}(1) = \begin{cases} 
1 & \text{if } s \in [0, 1] \\
\frac{1}{\pi} \arcsin \left( \frac{1}{s} \right) & \text{if } s \in (1, \sqrt{2}].
\end{cases}
\]

Hence \( \lim_{s \to 1, s > 1} V_{D_s}(1) = \frac{1}{2} < 1 \).

The intuition is the following: it is clear that when \( s \) equals 1 the searcher guarantees 1 by playing \( x = (0, 0) \). Suppose now that \( s \) equals \( 1 + \varepsilon \). Then the searcher covers almost all the area of the disc but less than half of its circumference. Hence the hider guarantees \( 1/2 \) by choosing uniformly a point on the boundary of \( D_s \).

\( \square \)

### 6 Appendix: omitted proofs

#### Subsection 3.1

**Proof of proposition 7**. Let us first define a metric on the set \( \mathcal{W} \) inducing the topology of compact convergence. For \( n \in \mathbb{N} \), define \( K_n = [0, n] \). Then

\[
D : \mathcal{W} \times \mathcal{W} \to \mathbb{R}_+ \\
(f, g) \mapsto \sum_{n=1}^{\infty} \frac{1}{n} \sup_{x \in K_n} \| f(x) - g(x) \|.
\]

is a metric on \( \mathcal{W} \).

We recall the following fact about the topology of compact convergence topology.

**Proposition 7** (Application of theorem 46.2 in [Munkres (2000)](https://doi.org/10.2307/1973983)). Let \( Q \) be a search space. A sequence \( f_n : \mathbb{R}_+ \to Q \) of functions converges to the function \( f \) in the topology of compact convergence if and only if for each compact subspace \( K \) of \( \mathbb{R}_+ \), the sequence \( f_n|_K \) converges uniformly to \( f|_K \).

The following corollary follows from Sion’s theorem, [Sion (1958)](https://doi.org/10.1016/0022-0396(58)90006-8).

**Corollary 1** (Proposition A.10 in [Sorin (2002)](https://transition.dl.sourceforge.net/ Dolphins/citeulike/7608474)). Let \( (X, Y, g) \) be a zero-sum game such that: \( X \) is a compact metric space, for all \( y \in Y \), the function \( g(\cdot, y) \) is upper semi-continuous. Then the game \( (\Delta(X), \Delta_f(Y), g) \) has a value and player 1 has an optimal strategy.
We are now able to complete the proof. By Ascoli’s theorem (application of theorem 47.1 in Munkres (2000)), \( W \) is compact for the topology of compact convergence. Moreover, for all \((y, t) \in A\) the function \( g_{m,r}(\cdot, (y, t)) \) is upper semi-continuous. The conclusion follows from corollary 1.

**Subsection 3.3**

*Proof of lemma* \[7\] Let \( \pi = (u_1, u_2, \ldots, u_{n-1}, u_n), u_1 = u_n = u \in V \) be an Eulerian tour. Without loss of generality, suppose \( l(u_i, u_{i+1}) \neq 0 \) for all \( i \in \{1, \ldots, n-1\} \). The parametrization is constructed in the following way.

If \( t \in [0, l(u_1, u_2)] \) then
\[
 w(t) = \left( u_1, u_2, \frac{t}{l(u_1, u_2)} \right).
\]

Else, suppose \( n \geq 3 \). For all \( k \in \{2, \ldots, n-1\} \) if
\[
 t \in \left[ \sum_{i=1}^{k-1} l(u_i, u_{i+1}), \sum_{i=1}^k l(u_i, u_{i+1}) \right]
\]
then
\[
 w(t) = \left( u_k, u_{k+1}, \frac{t - \sum_{i=1}^{k-1} l(u_i, u_{i+1})}{l(u_k, u_{k+1})} \right).
\]

It is not difficult to verify that such \( w \) is appropriate.

*Proof of theorem* \[7\] If \( m \geq \lambda(N) \), the patroller guarantees 1 by playing a parametrization of \( N \). Suppose that \( m < \lambda(N) \). Let \((y, t) \in N \times \mathbb{R}_+\) be a pure strategy of the attacker and let \( w \) be a in definition \[4\] There exists \( t_y \in [0, \lambda(N)] \) such that \( w(t_y) = y \). Now let \( t_0 \in [t_y - t - m, t_y - t] \). Then \( w(t_0)(t_y - t_0) = w(t_y) = y \). And \( t_y - t_0 \in [t, t + m] \). Thus \( y \in w_{t_0}([t, t + m]) \).

Hence under the patroller’s uniform strategy
\[
 \mathbb{P}(y \in w_{t_0}([t, t + m])) \geq \mathbb{P}(t_0 \in [t_y - t - m, t_y - t]) = \frac{m}{\lambda(N)}.
\]

The other inequality follows from proposition \[2\] since in this case, \( \rho \) equals 1.

**Subsection 3.4**

To prove theorem \[2\] we first need some preliminary definition and lemmas.

**Definition 9.** Let \( Q \) be a search space. A continuous function \( L : [0, 1] \rightarrow Q \) such that \( L(0) = L(1) \) is called an \( r \)-tour if for any \( x \in Q \) there exists \( l \in L([0, 1]) \) such that \( d(x, l) \leq r \).

The next lemma shows that when the radius of detection \( r \) is small, one can find in \( Q \) an \( r \)-tour with length not exceeding \( \lambda(Q)/2r \), up to some \( \varepsilon \).
Lemma 2 (Lemma 3.39 in Alpern and Gal (2003)). Let $Q \subset \mathbb{R}^2$ be a simple search space. Endow $Q$ with the Euclidean norm. Then for any $\varepsilon > 0$ there exists $r_\varepsilon > 0$ such that for any $r < r_\varepsilon$ there exists an $r$-tour $L : [0, 1] \to Q$ such that

$$TV(L) \leq (1 + \varepsilon) \frac{\lambda(Q)}{2r}.$$  

The next lemma gives a parametrization of $L([0, 1])$ in terms of walks.

Lemma 3. Let $L$ be an $r$-tour as in lemma 2. Then for all $\varepsilon' > 0$ there exists $w : [0, TV(L) + \varepsilon'] \to L([0, 1])$ continuous such that:

i) $w(0) = w(TV(L) + \varepsilon')$,

ii) $w$ is surjective,

iii) $w$ is 1-Lipschitz continuous,

iv) $TV(w) = TV(L)$.

We are now able to prove theorem 2

Proof of theorem 2. Let $L$ and $w$ be as in lemma 2 and lemma 3 respectively. For all $t_0 \in [0, TV(L) + \varepsilon']$ define $w_{t_0}(\cdot)$ as $w(t_0 + \cdot)$.

Let $(l, t) \in L([0, 1]) \times \mathbb{R}_+$. By lemma 3 ii), there exists $t_1 \in [0, TV(L) + \varepsilon']$ such that $w(t_1) = l$. Now let $t_0 \in [t_1 - t - m, t_1 - t]$. Then $w_{t_0}(t_1 - t_0) = w(t_1) = l$. And $t_0 - t_0 \in [t, t + m]$. Hence $l \in w_{t_0}([t, t + m])$.

Suppose $t_0$ is chosen uniformly in $[0, TV(L) + \varepsilon']$. By lemma 3 iii) this is an admissible strategy for the patroller. Let $(y, t) \in A$ be a pure strategy of the attacker. Then if $l \in L([0, 1])$ is such that $d(y, l) \leq r,$

$$P(d(y, w_{t_0}([t, t + m])) \leq r) \geq P(l \in w_{t_0}([t, t + m])),$$

$$\geq P(t_0 \in [t_1 - t - m, t_1 - t])$$

$$= \frac{r}{TV(L) + \varepsilon'}.$$ 

By lemma 2, this last quantity is greater than or equal to $\frac{m}{(1 + \varepsilon) \lambda(Q) + \varepsilon'}$. Hence the patroller guarantees $\frac{m}{1 + \varepsilon \lambda(Q) + \varepsilon'}$ for all $\varepsilon' > 0$, that is

$$V_Q(m, r) \geq \frac{2rm}{(1 + \varepsilon) \lambda(Q)} \sim 2rm \frac{2rm}{\lambda(Q)}$$

19
as $r$ goes to 0.

In this context, proposition 2 yields $V_Q(m, r) \leq \frac{2rm + \pi r^2}{x(Q)} \sim \frac{2r}{x(Q)}$ as $r$ goes to 0 (see remark 1).

\begin{proof}[Proof of proposition 4]
Suppose $\mu \in \Delta(Q)$ is an equalizing strategy. If the searcher plays $\mu$, then for all $y \in Q$ $\mu(B_r(y) \cap Q) = c$, hence $V_Q(r) \geq c$. Symmetrically, if the hider plays $\mu$, then for all $x \in Q$ $\mu(B_r(x) \cap Q) = c$, hence $V_Q(r) \leq c$, and $V_Q(r) = c$.

Conversely, suppose $\mu \in \Delta(Q)$ is optimal for both players. Then the searcher guaranties $V_Q(r)$ that is for all $y \in Q$ $\mu(B_r(y) \cap Q) \geq V_Q(r)$, and the hider guaranties $V_Q(r)$ that is for all $x \in Q$ $\mu(B_r(x) \cap Q) \leq V_Q(r)$. Hence for all $y \in Q$

$$\mu(B_r(y) \cap Q) = V_Q(r).$$

\end{proof}

\begin{proof}[Subsection 4.1]

To prove theorem 3 we first need to introduce a technical lemma.

Denote $B^2_r(x) = \{ y \in \mathbb{R}^n \mid \| x - y \|_2 \leq r \}$ the closed ball of center $x$ with radius $r$ for the Euclidean norm, and $\partial B^2_r(x) = \{ y \in \mathbb{R}^n \mid \| x - y \|_2 = r \}$ the sphere of center $x$ with radius $r$ for the Euclidean norm.

The intuition behind lemma 4 below is the following. We consider the balls $B^2_z(0)$ and $B^2_r(x)$ with $x$ on the boundary of $B^2_z(0)$. When $r$ goes to zero, the ratio between the volume of the ball $B^2_z(x)$ and the ball $B^2_z(0)$ intersected with the ball $B^2_r(0)$ goes to 2. Lemma 4 gives an upper bound to this ratio, as $r$ goes to 0, for a non necessary Euclidean ball $B_r(x)$.

\begin{lemma}
Let $\| \cdot \|$ be a norm on $\mathbb{R}^n$ and $c_1, c_2 > 0$ be such that $c_1 \| \cdot \| \leq \| \cdot \|_2 \leq c_2 \| \cdot \|$. Then for all $x \in \partial B^2_z(0)$

$$\limsup_{r \to 0} \frac{\lambda(B_r(x))}{\lambda(B^2_z(0) \cap B_r(x))} \leq 2 \left( \frac{c_2}{c_1} \right)^n.$$

\end{lemma}

\begin{proof}[Proof of lemma 4]
Let $x \in \partial B^2_z(0)$, let $\varepsilon > 0$. Denote $I$ the regularized incomplete Beta function: for $a, b > 0$ and $0 < z < 1$, $I_z(a, b) = \frac{B(z; a, b)}{B(a, b)}$.

Where $B(z; a, b) = \int_0^z t^{a-1}(1 - t)^{b-1}dt$ and $B(a, b) = B(1; a, b)$ is the Beta function. Then we have, see \cite{Li2011},

$$\lambda \left( B^2_z(0) \cap B^2_r(x) \right) = \frac{\pi^{n/2}}{2\Gamma(\frac{n}{2} + 1)} \left( \frac{r}{z} \right)^{n/2} \left( 1 - \frac{r}{z} \right)^{n/2} \left( 1 - \frac{r}{z} \right)^{n/2} \left( \frac{n + 1}{2} \cdot \frac{1}{2} \right) \left( \frac{n + 1}{2} \cdot \frac{1}{2} \right),$$

$$+ \varepsilon^n \frac{\pi^{n/2}}{2\Gamma(\frac{n}{2} + 1)} \left( \frac{r}{z} \right)^{n/2} \left( 1 - \frac{r}{z} \right)^{n/2} \left( 1 - \frac{r}{z} \right)^{n/2} \left( \frac{n + 1}{2} \cdot \frac{1}{2} \right) \left( \frac{n + 1}{2} \cdot \frac{1}{2} \right).$$

\end{proof}
Since \( t \mapsto t^\frac{n+1}{2}(1-t)^{-1/2} \) is integrable over \([0, 1)\),
\[
I_{1-(\frac{\pi}{2})^2} \left( \frac{n+1}{2}, \frac{1}{2} \right) = \int_0^{1-(\frac{\pi}{2})^2} t^\frac{n+1}{2}(1-t)^{-1/2} dt \to 1,
\]
as \( r \) goes to 0. And,
\[
I_{\left(\frac{\pi}{2}\right)^2 \left(1-(\frac{\pi}{2})^2\right)} \left( \frac{n+1}{2}, \frac{1}{2} \right) = \int_0^{\left(\frac{\pi}{2}\right)^2 \left(1-(\frac{\pi}{2})^2\right)} t^\frac{n+1}{2}(1-t)^{-1/2} dt
\]
which, since \( 1 \leq (1-t)^{-1/2} \) when \( t \in [0, 1) \), is greater than
\[
\frac{2}{\pi^{n+1}} \left( \frac{\pi}{2} \right)^{\frac{n}{2}} \frac{(1-(\frac{\pi}{2})^2)^{\frac{n}{2}}}{B\left(\frac{n-1}{2}, \frac{1}{2}\right)} = \frac{2\pi^{n+1}}{(n+1)\varepsilon^{n+1}B\left(\frac{n-1}{2}, \frac{1}{2}\right)} + o(r^{2n+2})
\]
when \( r \) goes to 0. Hence we have
\[
\lambda \left( B^2_\varepsilon(0) \cap B^2_r(x) \right) \geq \frac{\pi^{n/2}}{2(\frac{\pi}{2}+1)} (r^n + o(r^n))
\]
as \( r \) goes to 0. Moreover since
\[
B^2_\varepsilon(0) \cap B_r(x) = \{ y \in \mathbb{R}^n \mid \|y\|_2 \leq \varepsilon \text{ and } \|x - y\| \leq r \}
\]
and \( B_r(0) \subset B^2_{c_1r}(0) \), we have \( \lambda(B^2_\varepsilon(0) \cap B_r(x)) \geq \lambda(B^2_\varepsilon(0) \cap B^2_{c_1r}(x)) \),
and \( c_2^3 \lambda(B^2_\varepsilon) \geq \lambda(B_r) \). Finally, dividing by \( \lambda(B^2_\varepsilon(0) \cap B_r(x)) \) and taking the lim sup, since \( \lambda(B^2_\varepsilon) = \frac{\pi^{n/2}r^n}{\Gamma(\frac{n}{2}+1)} \)
\[
\limsup_{r \to 0} \frac{\lambda(B_r)}{\lambda(B^2_\varepsilon(0) \cap B_r(x))} \leq \limsup_{r \to 0} \frac{c_2^3 \lambda(B^2_\varepsilon)}{\lambda(B^2_\varepsilon(0) \cap B^2_{c_1r}(x))} \leq 2 \left( \frac{c_2}{c_1} \right)^n.
\]

We are now able to prove theorem

\section*{Proof of theorem

Let \( \varepsilon > 0 \) and \( r \in (0, \varepsilon) \). We regularize the boundary of \( Q \) by defining \( Q_\varepsilon = Q + B^2_\varepsilon(0) \), and \( I^r(r) = \{ y \in Q_\varepsilon \mid B_r(y) \subset Q_\varepsilon \} \). Define as well \( \lambda^r_{\min}(r) = \min_{y \in Q_\varepsilon} \lambda(B_r(y) \cap Q_\varepsilon) \). Finally define \( \mu \in \Delta(Q_\varepsilon) \) such that for all \( B \subset Q_\varepsilon \) measurable
\[
\mu(B) = \frac{\lambda(B \cap I^r(r)) \lambda^r_{\min}(r) + \lambda(B \cap (Q_\varepsilon \setminus I^r(r))) \lambda(B_r)}{\lambda(I^r(r)) \lambda^r_{\min}(r) + \lambda(B_r) \lambda(Q_\varepsilon \setminus I^r(r))}.
\]
Since by definition $\lambda(B_r) \geq \lambda^e_{\text{min}}(r)$, for all $x \in Q_{\varepsilon}$

$$\mu(B_r(x) \cap Q_{\varepsilon}) \geq \frac{\lambda^e_{\text{min}}(r) \lambda(B_r)}{\lambda(I^e(r)) \lambda^e_{\text{min}}(r) + \lambda(B_r) \lambda(Q_{\varepsilon} \setminus I^e(r))}.$$ 

Because the hider can play in $(Q_{\varepsilon}, r)$ as he would play in $(Q, r)$, $V_{Q_{\varepsilon}}(r) \leq V_{Q}(r)$. By proposition 2,

$$\frac{\lambda^e_{\text{min}}(r) \lambda(B_r)}{\lambda(I^e(r)) \lambda^e_{\text{min}}(r) + \lambda(B_r) \lambda(Q_{\varepsilon} \setminus I^e(r))} \leq V_{Q_{\varepsilon}}(r) \leq V_{Q}(r) \leq \frac{\lambda(B_r)}{\lambda(Q)}.$$

Dividing by $\lambda(B_r)/\lambda(Q)$,

$$\frac{\lambda^e_{\text{min}}(r) \lambda(Q)}{\lambda(I^e(r)) \lambda^e_{\text{min}}(r) + \lambda(B_r) \lambda(Q_{\varepsilon} \setminus I^e(r))} \leq \frac{V_{Q}(r) \lambda(Q)}{\lambda(B_r)} \leq 1. \quad (2)$$

Let us show that for all $\varepsilon > 0$ $\bigcup_{r>0} I^e(r) = \hat{Q}_{\varepsilon}$. Indeed, let $y \in \bigcup_{r>0} I^e(r)$. There exists $r > 0$ such that $y \in I^e(r)$. Thus there exists $r > 0$ such that $B_r(y) \subset Q_{\varepsilon}$. Conversely, let $y \in \hat{Q}_{\varepsilon}$. There exists $r' > 0$ such that $B_{r'}(y) \subset Q_{\varepsilon}$, where $B_{r'}(y) = \{x \in \mathbb{R}^n \mid \|x - y\| < r\}$. Take $0 < r < r'$, then $B_r(y) \subset \hat{Q}_{\varepsilon}$ hence $y \in I^e(r)$.

For all $r_1, r_2 > 0$ such that $r_1 > r_2$ one has $I^e(r_1) \subset I^e(r_2)$. Hence $\lim_{r \to 0} \lambda(I^e(r)) = \lambda(\hat{Q}_{\varepsilon})$. Dividing by $\lambda^e_{\text{min}}(r)$ and letting $r$ go to 0 in equation (2), by lemma 4 one has, since the minimum in $\lambda^e_{\text{min}}(r)$ is reached on the boundary of a Euclidean ball,

$$\frac{\lambda(Q)}{\lambda(\hat{Q}_{\varepsilon}) + 2 \left(\frac{\varepsilon}{\lambda(Q)}\right)^n \lambda(\partial Q_{\varepsilon})} \leq \liminf_{r \to 0} \frac{V_{Q}(r) \lambda(Q)}{\lambda(B_r)} \leq \limsup_{r \to 0} \frac{V_{Q}(r) \lambda(Q)}{\lambda(B_r)} \leq 1. \quad (3)$$

Let us show that $\bigcap_{\varepsilon > 0} \hat{Q}_{\varepsilon} = \bigcap_{\varepsilon > 0} Q_{\varepsilon} = Q$. Indeed, let $y \in \bigcap_{\varepsilon > 0} Q_{\varepsilon}$. For all $\varepsilon > 0 \min_{z \in Q} \|y - z\| \leq \varepsilon$, hence $y \in Q$. Conversely, for all $\varepsilon > 0 \min_{z \in Q} \|y - z\| \leq \varepsilon$, hence $Q \subset \hat{Q}_{\varepsilon}$. Moreover for all $\varepsilon_1, \varepsilon_2 > 0$ such that $\varepsilon_1 < \varepsilon_2$ one has $Q_{\varepsilon_1} \subset Q_{\varepsilon_2}$. Hence $\lim_{\varepsilon \to 0} \lambda(Q_{\varepsilon}) = \lambda(Q)$, $\lim_{\varepsilon \to 0} \lambda(\hat{Q}_{\varepsilon}) = \lambda(Q)$ and $\lambda(\partial Q_{\varepsilon}) = \lambda(\hat{Q}_{\varepsilon}) - \lambda(\hat{Q}_{\varepsilon})$ so $\lim_{\varepsilon \to 0} \lambda(\partial Q_{\varepsilon}) = 0$.

Letting $\varepsilon \to 0$ in equation (3), $1 = \frac{\lambda(Q)}{\lambda(Q)} \leq \lim_{r \to 0} \frac{V_{Q}(r) \lambda(Q)}{\lambda(B_r)} \leq 1. \quad \square$

**Subsection 5.1**

**Proof of proposition 3.** Since i) is direct we only prove ii). For all $(m, r) \in \mathbb{R}^2_+$,

$$V_Q(m, r) = \max_{\mu \in \Delta(W)} \inf_{(y, t) \in A} \int_W g_{m, r}(w, (y, t))d\mu(w) = \inf_{(y, t) \in A} \int_W g_{m, r}(w, (y, t))d\mu^*(w),$$
where \( \mu^* \in \Delta(W) \) is an optimal strategy of the patroller. Let \((y,t) \in A\) and \(m \geq 0\). For all \(w \in W\), the function \(r \mapsto g_{m,r}(w,(y,t))\) is upper semi-continuous, as the indicator function of a closed set. Let \(r_n \rightarrow r\), then by Fatou’s lemma,

\[
\limsup_n \int_W g_{m,r_n}(w,(y,t))d\mu^*(w) \leq \int_W \limsup_n g_{m,r_n}(w,(y,t))d\mu^*(w) \leq \int_W g_{m,r}(w,(y,t))d\mu^*(w).
\]

Thus the function \(r \mapsto \int_W g_{m,r}(w,(y,t))d\mu^*(w)\) is upper semi-continuous. Hence

\[
V_Q(m,\cdot) : r \mapsto \inf_{(y,t) \in A} \int_W g_{m,r}(w,(y,t))d\mu^*(w)
\]

is upper semi-continuous.

Since for all \(w \in W\), the function \(m \mapsto g_{m,r}(w,(y,t))\) is upper semi-continuous, as the indicator function of a closed set, the proof of iii) is strictly analogous.

\[\square\]

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