Reality conditions for (2+1)-dimensional gravity coupled with the Dirac field

Manabu Sawaguchi and Chol-Bu Kim

Physics Department, Saitama University
Urawa, Saitama 338, Japan

Abstract

The canonical formalism of three dimensional gravity coupled with the Dirac field is considered. We introduce complex variables to simplify the Dirac brackets of canonical variables and examine the canonical structure of the theory. We discuss the reality conditions which guarantee the equivalence between the complex and real theory.

1e-mail address: sawa@th.phy.saitama-u.ac.jp
2e-mail address: cbkim@th.phy.saitama-u.ac.jp
1 Introduction

In the canonical formalism of general relativity, the spatial metric and its canonical momentum have been considered as canonical variables. However, since constraints are non-polynomial form, it is difficult to solve the constraint equations. To avoid this difficulty, Ashtekar has introduced new variables which consist of a complexified connection and a densitized triad as canonical variables, and has shown that all constraints become of a polynomial form [1]. Inclusion of matter fields has been treated in the Ashtekar formalism, and consequences similar to those for pure gravity were obtained [2]. In Ashtekar formalism, to recover real general relativity, the reality conditions must be imposed on the canonical variables [3, 4].

In 2+1 dimensions, the canonical formalism which is an extension of Ashtekar formalism has been proposed in the pure gravity [5] and the $\mathcal{N} = 2$ supergravity [6]. In the case of including the Dirac field, Kim et al [7] have shown that the constraint structure is similar to the 3+1 dimensional gravity [2], and have found a new physical observable and its eigenstate. In [8], the solution of all quantum constraints based on the loop representation has been found. In this theory, since the spinor action consists of derivative of $\psi$ only, the total action becomes non-Hermitian. Vergeles [9] has also carried out the quantization on the basis of the dynamic quantization method, in which the spinor action contains derivatives of both $\psi$ and $\bar{\psi}$. Thus in contrast with the non-Hermitian case [7], the real total action is used. In this formalism, the Dirac field modes with gauge invariant creation and annihilation operators were selected, and the gauge invariant states were constructed by using the gauge invariant fermion creation operators similarly to the usual construction of states in any Fock space.

In this paper we consider 2+1 dimensional gravity coupled with the Dirac field. We start with a real action. First, we carry out a Hamiltonian formulation. By solving the second class constraints, we obtain the Dirac brackets of canonical variables, which are not a simple form. We also present the algebra of the constraints, in which some quadratic terms of the Gauss-law constraint appear. Next, we introduce new variables which obey simpler Dirac brackets. We show that the canonical structure of the theory is similar to the case of the non-Hermitian action in 2+1 dimensions [7], and also to the case of matter coupled gravity in 3+1 dimensions [3]. Since the new variables are complex, the reality conditions must be imposed like in the 3+1 dimensional gravity. In section 4, we describe the reality conditions in
the form of functionals of the canonical variables and their complex conjugate. It is shown that, by imposing the reality conditions, the correct number of degrees of freedom remains. The last section is devoted to conclusions and comments.

2 Canonical formalism

We start with the first-order gravity in 2+1 dimensions coupled with the Dirac field. The fundamental variables we use are the triad field $e^i_\mu$ and the dual spin connection

$$A^i_\mu = -\frac{1}{2} \epsilon^{ijk} \omega_{ijk},$$

(1)

instead of the triad field and the usual spin connection $\omega_{\mu ij}$.

The action is written as

$$I = \int d^3x \left[ \frac{1}{2} \epsilon^{\mu
u\rho} e^i_\mu F_{\nu\rho i} - \frac{1}{2} e(\bar{\psi}\gamma^\mu D_\mu \psi - D_\mu \bar{\psi}\gamma^\mu \psi) - me\bar{\psi}\psi \right],$$

(2)

where $F_{\nu\rho i} = \partial_\nu A^i_\rho - \partial_\rho A^i_\nu - \epsilon_{ijk} A^j_\rho A^k_\nu$ is the curvature tensor of the spin connection, $\epsilon^{\mu\nu\rho}$ is the Levi-Civita antisymmetric tensor density and $D_\mu \psi = (\partial_\mu + \frac{1}{2} A^i_\mu \gamma_i) \psi$. The matrices $\gamma_i$ generate the SO(2,1) Lorentz group. The Dirac conjugation is defined by $\bar{\psi} := i\psi^\dagger \gamma_0$. We use Greek letters $\mu, \nu, \cdots$ for curved indices in three dimensions and Latin letters $i, j, \cdots$ for tangent space indices. The spacetime metric has a signature $(-, +, +)$ and we use the convention $\epsilon^{012} = 1 = -\epsilon_{012}$.

We decompose the spacetime metric following the ADM formalism. We assume that the spacetime manifold $M$ has a topology $M = \Sigma \otimes R$, where $\Sigma$ is a compact two dimensional manifold. We choose a time coordinate $t$ on the manifold $M$ so that $M$ is foliated by two dimensional spacelike surfaces $\Sigma_t$ each with the topology of $\Sigma$. One can define a timelike unit vector $n^\mu$ with $n^\mu n^\nu g_{\mu\nu} = -1$ which is normal to the $\Sigma_t$ and a smooth time vector field $t^\mu$ which is chosen such that $t^\mu \nabla_\mu t = 1$. We then define the spatial metric $q_{\mu\nu}$ by $q_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu; n^\mu q_{\mu\nu} = 0$, the timelike part $n^i$ of $e^i_\mu$ by $n^i = n^\mu e^i_\mu$ and the projected part of $e^i_\mu$ into $\Sigma_t$ by $E^i_\mu \equiv e^i_\mu (g^\nu_\mu + n_\mu n^\nu); n^\mu E^i_\mu = 0, q_{\mu\nu} = E^i_\mu E^j_\nu \eta_{ij}$. The Levi-Civita density $\epsilon^{\mu\nu\rho}$ is related to a density on $\Sigma_t$, $e^{\mu\nu}$ by $\epsilon^{\mu\nu\rho} = 3Nn^\mu e^{\nu\rho}$. The decomposed action takes the form

$$I = \int d^3x [E^{\mu i}(\mathcal{L}_i A^i_\mu) + \frac{1}{2} E\bar{\psi}(n \cdot \gamma)(\mathcal{L}_i \psi) - \frac{1}{2} E(\mathcal{L}_i \bar{\psi})(n \cdot \gamma)\psi$$

(3)
\begin{align*}
+ N \{ \frac{1}{2} \epsilon^{ijk} \tilde{E}_i^\mu \tilde{E}_j^\nu F_{\mu \nu k} + \frac{1}{2} E \epsilon^{ijk} \tilde{E}_i^\mu n_j \Psi_{\mu k} - m E^2 \bar{\psi} \psi \} \\
+ N^\mu \{ - \tilde{E}^\nu F_{\mu \nu i} - \frac{1}{2} E n_i \Psi_{\mu}^i \} \\
+ A_0^i \{ D_{\mu} \tilde{E}_i^\mu + \frac{1}{2} En_i \bar{\psi} \psi \}\right] (3)
\end{align*}

where \( \tilde{E}_i^\mu, A_0^i, E, N, \mathcal{L}_t \) and \( \Psi_{\mu}^i \) are a vector density on \( \Sigma_t \) with \( \tilde{E}_i^\mu \equiv e^{i\mu} E_{\nu i} \), a time component of \( A_i^\mu \) with \( A_0^i \equiv t^{\mu} A_i^\mu \), the determinant of \( E_{\mu i} \), a lapse with density weight \(-1\), Lie derivative by \( t^{\mu} \) and function defined as \( \Psi_{\mu}^i := \bar{\psi} \gamma^i D_{\mu} \psi - (D_{\mu} \bar{\psi}) \gamma^i \psi \) respectively.

For canonical treatment, we compute the canonical momenta. They are
\begin{align*}
\Pi_{\mu}^i &= \frac{\delta \mathcal{L}}{\delta \dot{E}_i^\mu} = 0, \quad P_{\mu}^i = \frac{\delta \mathcal{L}}{\delta \dot{A}_i^\mu} = \tilde{E}_i^\mu \\
\pi &= \frac{\delta \mathcal{L}}{\delta \dot{\psi}} = -\frac{1}{2} E \bar{\psi} (n \cdot \gamma), \quad \bar{\pi} = \frac{\delta \mathcal{L}}{\delta \dot{\bar{\psi}}} = -\frac{1}{2} E (n \cdot \gamma) \psi \\
\Pi_N &= \frac{\delta \mathcal{L}}{\delta \dot{N}} = 0, \quad \Pi_N^\mu = \frac{\delta \mathcal{L}}{\delta \dot{N}^\mu} = 0, \quad \Pi_{Ai} = \frac{\delta \mathcal{L}}{\delta \dot{A}_0^i} = 0, (4c)
\end{align*}

where we use the convention that \( \dot{q} \equiv \mathcal{L}_t q \) and the canonical momenta of \( \psi \) and \( \bar{\psi} \) are defined by the left derivative. Because these momenta do not depend on the velocities, these result in the primary constraints:
\begin{align*}
\Pi_{\mu}^i &\approx 0, \quad P_{\mu}^{i \mu} := P_{\mu}^i - \tilde{E}_i^\mu \approx 0 \quad (5a) \\
\pi' := \pi + \frac{1}{2} E \bar{\psi} (n \cdot \gamma) &\approx 0, \quad \bar{\pi}' := \bar{\pi} + \frac{1}{2} E (n \cdot \gamma) \psi \approx 0 \quad (5b) \\
\Pi_N &\approx 0, \quad \Pi_N^\mu \approx 0, \quad \Pi_{Ai} \approx 0. \quad (5c)
\end{align*}

The consistency condition that the primary constraints are conserved requires secondary constraints,
\begin{align*}
H &= \frac{1}{2} \epsilon^{ijk} \tilde{E}_i^\mu \tilde{E}_j^\nu F_{\mu \nu k} + \frac{1}{2} E \epsilon^{ijk} \tilde{E}_i^\mu n_j \Psi_{\mu k} - m E^2 \bar{\psi} \psi \approx 0 \quad (6a) \\
H_{\mu} &= -\tilde{E}^\nu F_{\mu \nu i} - \frac{1}{2} E n_i \Psi_{\mu}^i \approx 0 \quad (6b) \\
G_i &= D_{\mu} \tilde{E}_i^\mu + \frac{1}{2} E n_i \bar{\psi} \psi \approx 0 \quad (6c)
\end{align*}

and the velocity conditions which determine \( \tilde{E}_i^\mu, A_0^i, \psi \) and \( \bar{\psi} \). The consistency conditions of the secondary constraints become the combination of the secondary
constraints, from which it follows that there are no tertiary constraints. From some algebra, it follows that the constraints (5a) and (5b) are second class. The constraints (6) do not commute with constraints (5a,b). But when adding some linear combination of constraints (5a,b) to the constraints (6), we find that these are first class. The remaining constraints (5c) tell us that the variables \( N_{\sim}, N_{\mu}, A_{i}^0 \) and the corresponding momenta (4c) play a non-essential role in the Hamiltonian dynamics and become unphysical variables. Hereafter, we regard \( N_{\sim}, N_{\mu} \) and \( A_{i}^0 \) as Lagrange multipliers and the constraints (5c) as zero strongly.

Now we have to calculate the Dirac brackets to eliminate second class constraints. The non-vanishing brackets of the canonical variables are

\[
\{ \tilde{E}^{\mu}_{i}(x), A_{\nu}^{\mu}(y) \} = q_{\mu}^{\nu} \eta_{ij} \delta^{2}(x,y) \\
\{ \bar{\psi}_{\alpha}(x), \tilde{\psi}_{\beta}(y) \} = E^{-1}(n \cdot \gamma)_{\alpha\beta} \delta^{2}(x,y) \\
\{ \psi_{\alpha}(x), A_{i}^{\mu}(y) \} = \frac{1}{2} E^{-1} \epsilon_{\mu\nu}\epsilon^{ijk} \tilde{E}_{j}^{\nu}((n \cdot \gamma)\gamma_{k}\psi)_{\alpha} \delta^{2}(x,y) \\
\{ \bar{\psi}_{\alpha}(x), A_{i}^{\mu}(y) \} = \frac{1}{2} E^{-1} \epsilon_{\mu\nu}\epsilon^{ijk} \tilde{E}_{j}^{\nu}((\bar{\psi}\gamma_{k}(n \cdot \gamma))_{\alpha} \delta^{2}(x,y) \\
\{ A_{i}^{\mu}(x), A_{\nu}^{\mu}(y) \} = \frac{1}{2} \epsilon_{\mu\nu} n^{i} n^{j} \bar{\psi}\psi \delta^{2}(x,y) \tag{7}
\]

Unfortunately, \( A_{i}^{\mu} \) does not commute with \( \psi, \bar{\psi} \) and itself. In this respect the present theory differs from the case that starts with non-Hermitian action \[7\], and the Dirac brackets are complicated at first sight.

After eliminating all the second class constraints, remaining first class constraints are

\[
H \approx 0, \ H_{\mu} \approx 0, \ G_{i} \approx 0. \tag{8}
\]

These constraints are called the Hamiltonian, the vector and the Gauss-law constraints respectively. The total Hamiltonian in the reduced phase space is described by these constraints as follows,

\[
\mathcal{H} = -(NH + N^{\mu}H_{\mu} + A_{0}^{i}G_{i}). \tag{9}
\]

Note that, since constraints and multipliers contained in (8) are real, this Hamiltonian is also real. So we do not have to consider the reality conditions.

Now we discuss the algebra of the constraints. Instead of the constraints (8), we
use the following constraints smeared with suitable well-defined fields on $\Sigma_t$,

$$H[N] = \int d^2x N \left\{ \frac{1}{2} \epsilon^{ijk} \tilde{E}_i^\mu \tilde{E}_j^\nu F_{\mu \nu k} + \frac{1}{2} E \epsilon^{ijk} \tilde{E}_i^\mu n_j \Psi_{\mu k} - m E^2 \bar{\psi} \psi \right\}$$  \hspace{1cm} (10a)

$$H_\mu[N^\mu] = \int d^2x N^\mu \left\{ - \tilde{E}^{\nu i} F_{\mu \nu i} - \frac{1}{2} E n_i \Psi^i \right\}$$  \hspace{1cm} (10b)

$$G_i[\Lambda^i] = \int d^2x \Lambda^i \left\{ D_\mu \tilde{E}_i^\mu + \frac{1}{2} E n_i \bar{\psi} \psi \right\}.$$  \hspace{1cm} (10c)

where $N, N^\mu$ and $\Lambda^i$ are smearing fields.

The constraint algebra are

$$\{ G_i[\Lambda^i], G_j[\Gamma^j] \} = G_i[\epsilon^{ijk} \Lambda_j \Gamma_k]$$

$$\{ G_i[\Lambda^i], H[N] \} = 0$$

$$\{ G_i[\Lambda^i], H_\mu[N^\mu] \} = 0$$

$$\{ H_\mu[N^\mu], H_\nu[M^\nu] \} = H_\mu[\mathcal{L}_N M^\mu] + G_i[N^\mu M^\nu (F_{i \mu}^i + \frac{1}{2} \epsilon_{\mu \nu \lambda} n^i \bar{\psi} \psi (n \cdot G))]$$

$$\{ H_\mu[N^\mu], H[N] \} = H[\mathcal{L}_N N^\mu] + G_i[N N^\mu \epsilon^{ijk} (\tilde{E}_j^\nu F_{\mu \nu k} + \frac{1}{2} E n_j \Psi_{\mu k}$$

$$+ m E \epsilon_{\mu \nu \lambda} \tilde{E}_j^\nu n_k \bar{\psi} \psi - \frac{1}{2} \epsilon_{\mu \nu \lambda} \tilde{E}_j^\nu n_k \bar{\psi} \psi (n \cdot G))]$$

$$\{ H[N], H[M] \} = H_\mu[\tilde{E}_j^\nu \tilde{E}_i^\nu (N \partial_\nu M - M \partial_\nu N)]$$  \hspace{1cm} (11)

Note that in these algebra, some quadratic terms of constraint $G_i$ appear in the right hand side, which do not appear in the case of the gravity including matter in 3+1 dimensions [4] and the non-Hermitian theory in 2+1 dimensions [7]. In classical level since these terms weakly vanish, the constraints are actually first class. So we think there is no problem. But we do not know whether these terms have an effect in quantum theory.

We conclude this section with comment on the quantization of this theory. Since the Dirac brackets of the canonical variables are complicated, it is difficult to perform quantization. Because of the fact that $A^i_\mu$ does not commute with $\psi, \bar{\psi}$ and itself, we can no longer represent $A^i_\mu$ by a multiplication operator, which is different from the case in [4, 8]. Vergeles [9] also have carried out the canonical formalism of the 2+1 dimensional gravity coupled with the Dirac field and obtained the same result that the Dirac brackets of the canonical variables are complicated. He has also found that the quadratic terms appear in the constraint algebra. In [3] the quantization have been carried out on the basis of the dynamic quantization method. To perform this,
the Dirac field modes with gauge invariant creation and annihilation operators are selected. The gauge invariant states are built by using the gauge invariant fermion creation operators similarly to the usual construction of states in any Fock space. In contrast with this method, we stand to transform the canonical variables for simplifying their Dirac bracket. This procedure is discussed in the next section.

3 New variables

In the last section we found that the Dirac brackets of the canonical variables are complicated. This complication cause some difficulty. For example, we can no longer construct quantum theory in the connection representation.

In order to simplify the Dirac brackets of canonical variables, we introduce the following complex variables:

\[ A_{\mu}^{\prime i} := A_{\mu}^i - \frac{1}{2} \epsilon_{\mu\nu\rho} \epsilon^{ijk} \tilde{E}^\nu_j \tilde{\psi} \gamma_k \psi, \]

(12)

which commutes with itself and \( \psi \). However, it does not commute with \( \bar{\psi} \), which then is not a suitable canonical variable. So we use \( \pi := -E \tilde{\psi} (n \cdot \gamma) \) as the canonical variable which replaces \( \bar{\psi} \). The non-vanishing Dirac brackets of new variables are

\[ \{ \tilde{E}^\mu_i(x), A_{\mu}^{\prime i}(y) \} = q^\mu_i \eta^2 (x, y), \quad \{ \pi_\alpha(x), \psi_\beta(y) \} = \delta_{\alpha\beta} \delta^2(x, y), \]

(13)

which are the same Dirac brackets in the non-Hermitian case. By means of this simplification, we can avoid the difficulty indicated above. So we can represent \( A_{\mu}^{\prime i} \) by a multiplication operator in quantum theory.

Now we recast the constraints in terms of the new variables. The constraints (6) are rewritten as

\[ H = H' + \frac{1}{2} \epsilon_{ijk} n^i \pi (n \cdot \gamma) \gamma^j \psi G'^{tk} \approx 0 \]

(14a)

\[ H_\mu = H'_{\mu} - \frac{1}{2} E^{-1} \epsilon_{\mu\nu\rho} \epsilon^{ijk} \tilde{E}^\nu_i \pi (n \cdot \gamma) \gamma^j \psi G'^{tk} \approx 0 \]

(14b)

\[ G_i = G'_i \approx 0, \]

(14c)

where

\[ H' = \frac{1}{2} \epsilon^{ijk} \tilde{E}^\mu_i \tilde{E}^\nu_j F'_{\mu
u\rho} - \tilde{E}^\mu_i \pi \gamma^j D'_{\mu} \psi - m E \pi (n \cdot \gamma) \psi + \frac{3}{4} (\pi \psi)^2 \]

(15a)
\[ H'_{\mu} = -\tilde{E}^\nu F'^{i\nu}_{\mu} + \pi D'_{\mu} \psi \]  \hfill (15b)

\[ G'_i = D'_{\mu} \tilde{E}^\mu_i - \frac{1}{2} \pi \gamma_i \psi, \]  \hfill (15c)

where \( F'^{i\nu}_{\mu} \) is the curvature tensor of \( A'^i_{\mu} \) and \( D'_{\mu} \psi \) is the covariant derivative defined by \( D'_{\mu} \psi := (\partial_{\mu} + \frac{1}{2} A'^i_{\mu} \gamma_i) \psi \). In this calculation we used the Fierz transformation, \( (\pi(n \cdot \gamma) \psi)^2 = (\pi \psi)^2 \). In terms of the new variables, \( H \) and \( H'_{\mu} \) become non-polynomial, and non-polynomial terms are proportional to \( G'_i \). When \( G'_i \approx 0 \), \( H' \) and \( H'_{\mu} \) are weakly equal to \( H \) and \( H_{\mu} \) respectively. So in this theory, \( H' \) and \( H'_{\mu} \) which are polynomial in terms of the new variables can be employed as the constraints:

\[ H' \approx 0, \quad H'_{\mu} \approx 0. \]  \hfill (16)

Note that the new constraints (14c) and (16) are almost the same as the ones of the non-Hermitian case in 2+1 dimensions [7] and Ashtekar formalism including the Dirac field in 3+1 dimensions [2]. The only difference is that \( (\pi \psi)^2 \) appears in the Hamiltonian constraint \( H' \). The effect of this term will be discussed in section 5.

Using the new variables, the Hamiltonian is described as follows,

\[ \mathcal{H} = -(\mathcal{N} H' + N^\mu H'_{\mu} + \Lambda^i G'_i), \]  \hfill (17)

where

\[ \Lambda^i \equiv A^i_0 + \frac{1}{2} N \epsilon^{ijk} n_j \pi (n \cdot \gamma) \gamma_k \psi - \frac{1}{2} N^\mu \epsilon^{-1 \mu \nu} \epsilon^{ijk} \tilde{E}^\nu_j \pi (n \cdot \gamma) \gamma_k \psi. \]  \hfill (18)

The Hamiltonian takes a form of linear combination of the constraints with multipliers. Note that, as a result of getting the polynomial constraints, the multiplier of the Gauss-law constraint is different from the \( A^i_0 \) of (13).

Now we discuss the algebra of these constraints. Using the Dirac brackets (13), the constraint algebra are

\[ \{G'_i[\Lambda^j], G'_j[\Gamma^i]\} = G'_i[\epsilon^{ijk} \Lambda_j \Gamma_k] \]

\[ \{G'_i[\Lambda^i], H'_{\mu}[N^\mu]\} = 0 \]

\[ \{G'_i[\Lambda^i], H'_{\mu}[N^\mu]\} = 0 \]

\[ \{H'_{\mu}[N^\nu], H'_{\nu}[M^\mu]\} = H'_{\mu}[\mathcal{L}_N M^\mu] + G'_i[N^\mu M^\nu F'^{i\nu}_{\mu\nu}] \]

\[ \{H'_{\mu}[N^\nu], H'[N]\} = H'[\mathcal{L}_N N] \]
\[ G'_{i}[\mathcal{N}\mathcal{N}_{\mu} (\epsilon_{ijk} \tilde{E}_{\nu} F'_{\mu \nu} - \pi \gamma^i D'_{\mu} \psi + m \epsilon_{\mu \nu} \epsilon_{ijk} \tilde{E}_{\nu} \pi \gamma_k \psi)] \]
\[ \{H'[\mathcal{N}], H'[\mathcal{M}]\} = H'_{\mu} \tilde{E}_{\nu} (\mathcal{N} \partial_{\nu} \mathcal{M} - \mathcal{M} \partial_{\nu} \mathcal{N}). \]  

(19)

Note that, in contrast with the case based on the real variables, no quadratic terms of \( G'_{i} \) appear in the right hand side. This algebra is similar to the non-Hermitian case except for mass term, so that the structure of the constraint algebra is not changed by the extra term \((\pi \psi)^2\) in (19).

As a result of introducing the new canonical variables, we constructed a theory which has simple Dirac brackets. However, since the complex variables are introduced, the reality of the theory is not manifest. So we must impose reality conditions.

### 4 Reality conditions

In order to ensure that a complex theory is equivalent to a real one, we must impose the reality conditions like in the 3+1 dimensional gravity. If we ignore the reality conditions, that is, if we consider \( A'^{\mu}_{i} \) and corresponding momentum \( \tilde{E}^{\mu}_{i} \) as complex, we can no longer recover the real theory. So we must impose the reality conditions which restrict the phase space.

In our case, the canonical variables \( \tilde{E}^{\mu}_{i} \) and \( A'^{\mu}_{i} \) are not independent of their complex conjugates but must satisfy the reality conditions

\[ \tilde{E}^{\mu}_{i} = \tilde{E}^{\mu}_{i}, \quad A'^{\mu}_{i} = A'^{\mu}_{i} + E^{-1} \epsilon_{\mu \nu} \epsilon_{ijk} \pi \gamma_{j} \psi. \]  

(20)

The latter condition of \( A'^{\mu}_{i} \) ensure that the original variable \( A^{\mu}_{i} \), which is related to \( A'^{\mu}_{i} \) by (12), is real. Furthermore, when spinor fields are included, additional conditions for the reality of observable currents are needed [2]. In our case the reality conditions for spinor fields are given by

\[ (\pi \tilde{E}^{\mu}_{i} \gamma^{i} \psi)^{\dagger} = -\pi \tilde{E}^{\mu}_{i} \gamma^{i} \psi, \quad (\pi \tilde{E}^{\mu}_{i} \gamma^{i} \pi)^{\dagger} = E^{2} (\psi \tilde{E}^{\mu}_{i} \gamma^{i} \psi). \]  

(21)

Using these conditions, the constraints \( H', H'_{\mu} \) and \( G'_{i} \) obey the following relations

\[ H'^{\dagger} = H' + \epsilon^{ijk} n_{i} \pi (n \cdot \gamma) \gamma_{j} \psi G'_{k} \]
\[ H'^{\dagger}_{\mu} = H'_{\mu} + E^{-1} \epsilon_{\mu \nu} \epsilon^{ijk} \tilde{E}^{\nu}_{i} \pi (n \cdot \gamma) \gamma_{j} \psi G'_{k} \]
\[ G'^{\dagger}_{i} = G'_{i}. \]  

(22)
Note that the constraints $H'$ and $H'_\mu$ are complex. Therefore the Hamiltonian takes a form of linear combination of the complex constraints. In order to satisfy the reality conditions at any time, the Hamiltonian must be real. So the Lagrange multipliers in the Hamiltonian are not totally arbitrary but must satisfy

$$\begin{align*}
N^\dagger &= N, \quad N^{\mu\dagger} = N^\mu \\
\Lambda^\dagger &= \Lambda + N\epsilon^{ijk}n_j \pi (n \cdot \gamma) \gamma_k \psi - N^\mu E^{-1} \epsilon_{\mu\nu} \epsilon^{ijk} \tilde{E}^\nu_j \pi (n \cdot \gamma) \gamma_k \psi.
\end{align*}$$

(23)

If the reality conditions (20) and (21) are satisfied at initial time and the multipliers obey the relations (23), the equivalence between the real theory and the complex one is guaranteed. Note that the reality conditions (20) and the multiplier conditions (23) are similar to the ones in the case of pure gravity in 3+1 dimensions [10]. In both theories, complex connection is introduced to simplify the canonical structure. In our case the aim is to simplify the Dirac brackets of canonical variables. In the case of pure gravity in 3+1 dimensions, on the other hand, the complex connection is used to get the constraints in polynomial form.

Now we consider reality conditions which are different from (19) in appearance. First we take $\Lambda^i$ as an arbitrary complex function in the Hamiltonian. Therefore the Hamiltonian is in general complex. As new reality conditions, we impose that the triad $\tilde{E}_i^\mu$ is real,

$$\text{Im} \tilde{E}_i^\mu = 0$$

(24)

and that the time derivative of $\tilde{E}_i^\mu$ is also real,

$$\text{Im} \dot{\tilde{E}}_i^\mu = 0.$$  

(25)

As regards spinor fields, the spinor reality conditions (21) are required. Using the Gauss-law constraint, the spinor reality conditions (21) and the triad reality condition (24), it follows that (25) reduces to

$$\text{Im} A_{\mu}^i = -\frac{1}{2} E^{-1} \epsilon_{\mu\nu} \epsilon^{ijk} \tilde{E}_j^\nu \pi (n \cdot \gamma) \gamma_k \psi,$$

(26)

which is equivalent to the second condition in (20). On the other hand, it is possible to solve (23) with regard to $\Lambda^i$,

$$\text{Im} \Lambda^i = \frac{1}{2} N\epsilon^{ijk}n_j \pi (n \cdot \gamma) \gamma_k \psi + N^\mu \text{Im} A_{\mu}^i.$$  

(27)
This condition restricts a part of gauge freedom, and this situation is similarly to the 3+1 dimensional gravity [4]. Using (26), we see that the condition (27) is the same as the last condition in (23). Thus the reality conditions (24) and (25) are equivalent to the those of (20) and (21).

We count the number of (real) degrees of freedom assuming there is only one Dirac field. The complex canonical variables $A'_i^\mu$ and $\tilde{E}_i^\mu$ have 12 independent components respectively. When we consider that $\psi$ and $\pi$ are independent, there are 8 independent components of spinor fields. The constraints are also complex in general. But due to the reality conditions (20) and (21), all the constraints are not independent of each other. From (22) we see that $G'_i$ is real and that the imaginary parts of $H'_i$ and $H'_\mu$ are proportional to $G'_i$. Thus 6 independent constraints remain. With 12 reality conditions (20) and 4 for the spinor field (21), we find that the remaining number of degrees of freedom is $(24 + 8) - 2 \times 6 - (12 + 4) = 4$. Because there is no graviton in 2+1 dimensions, this corresponds to the degrees of freedom of spinor field $\psi$. So it follows that the reality conditions (20) and (21) are suitable and reproduce the correct number of degrees of freedom.

5 Conclusion

We studied (2+1)-dimensional gravity coupled with the Dirac field. In contrast with the non-Hermitian case [7], the Dirac brackets of the canonical variables are complicated. We found that quadratic terms of the constraint $G_i$ appear in the constraint algebra. No such term appears also in the case of the gravity including matter in 3+1 dimensions [4] and the non-Hermitian theory in 2+1 dimensions [7]. In order to simplify the Dirac brackets of the canonical variables, we introduced the complex variables, and found that quadratic terms of the Gauss-law constraint disappear and that the constraint algebra becomes similar to the case of the gravity including matter in 3+1 dimensions [4] and the non-Hermitian theory in 2+1 dimensions [7]. But being different from both cases, the Hamiltonian constraint of the present theory contains $(\pi \psi)^2$.

Next we considered the reality conditions. By virtue of these conditions, the phase space is restricted, and the original real theory is recovered. In order to retain the reality conditions at any time, it is important that the Hamiltonian is real; accordingly, the Lagrange multipliers are not totally arbitrary but are related to their complex conjugates. We also considered the different reality conditions, which
are imposed only on the triad. We showed that these new conditions are reduced to the original ones after all. We also showed that, owing to the reality conditions, the correct number of degrees of freedom remains.

Finally we give a short comment on the quantum theory. We can define the canonical operators $A^{\mu i}$ and $\psi$ as multiplicative operators, and the corresponding momentum operators $E_i^\mu$ and $\pi$ as the derivative operators. We take the ordering in which momentum operators are placed to the right. In the non-Hermitian case \cite{7}, as a solution including spinor fields for the Hamiltonian constraint, a trivial one $\psi^A \psi_A$ has been found. This solution corresponds to an eigenstate of the fermion number operator $\int d^2 x \psi_A \pi^A$. In our Hermitian case, however, since the Hamiltonian constraint contains $(\psi \pi)^2$, $\psi^A \psi_A$ is no longer a solution. We are now looking for other solutions.

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