Killing superalgebra deformations of ten-dimensional supergravity backgrounds

José Figueroa-O’Farrill\(^1\) and Bert Vercnocke\(^2\)

\(^1\) Maxwell Institute and School of Mathematics, The University of Edinburgh, Edinburgh, UK
\(^2\) Instituut voor Theoretische Fysica, Katholieke Universiteit Leuven, Leuven, Belgium

E-mail: \texttt{J.M.Figueroa@ed.ac.uk} and \texttt{Bert.Vercnocke@fys.kuleuven.be}

Received 7 September 2007
Published 21 November 2007
Online at stacks.iop.org/CQG/24/6041

Abstract

We explore Lie superalgebra deformations of the Killing superalgebras of some ten-dimensional supergravity backgrounds. We prove the rigidity of the Poincaré superalgebras in types I, IIA and IIB, as well as of the Killing superalgebra of the Freund–Rubin vacuum of type IIB supergravity. We also prove rigidity of the Killing superalgebras of the NS5-, D0-, D3-, D4- and D5-branes, whereas we exhibit the possible deformations of the D1-, D2-, D6- and D7-brane Killing superalgebras, as well as of that of the type II fundamental string solutions. We relate the superalgebra deformations of the D2- and D6-branes to those of the (delocalized) M2-brane and the Kaluza–Klein monopole, respectively. The good behaviour under Kaluza–Klein reduction suggests that the deformed superalgebras ought to have a geometric interpretation.

PACS numbers: 11.25.Yb, 04.65.+e, 02.20.Sv, 02.40. – k

1. Introduction

This paper continues the study initiated in [1] of the Lie superalgebra deformations of the Killing superalgebras of supergravity backgrounds [2–11]. The focus in [1] was on 11-dimensional supergravity backgrounds: the Minkowski and Freund–Rubin backgrounds, whose Killing superalgebras were shown to be rigid, as well as the elementary branes and the elementary purely gravitational backgrounds. Of these, the M5-brane Killing superalgebra is rigid, but all the others admit deformations. The physical interpretation, if any, of these deformations was not explored in [1]: they could be due perhaps to quantum corrections or perhaps to geometric limits within classical supergravity. This latter possibility is explored further in [12]; although in the present paper we shall give indirect evidence for the geometric origin of such deformations. In this paper, we treat the case of types I and II ten-dimensional supergravity backgrounds. We discuss the Minkowski vacua in all three theories as well as the elementary brane backgrounds. As in the analysis of Kaluza–Klein reductions in [13], the asymptotic flatness of the brane backgrounds allows us to rephrase questions about the
symmetries of these backgrounds in terms of the symmetries of the asymptotic Minkowski vacuum. In particular, their Killing superalgebras are subsuperalgebras of the relevant Poincaré superalgebra and the computation of deformations will borrow much from the case of the Poincaré superalgebras.

We will not explain the methodology in this paper. It is explained in [1, section 2], which the reader should consult for the details. In a nutshell, the tangent space to the moduli space of deformations of a Lie superalgebra \( \mathfrak{g} \) is given by the cohomology group \( H^2(\mathfrak{g}; \mathfrak{g}) \), which we calculate for the Killing superalgebras of these supergravity backgrounds by using the superalgebra version of the factorization theorem of Hochschild and Serre. If \( H^2(\mathfrak{g}; \mathfrak{g}) = 0 \) we say that \( \mathfrak{g} \) is rigid. Otherwise, every line in \( H^2(\mathfrak{g}; \mathfrak{g}) \) defines an infinitesimal deformation of \( \mathfrak{g} \) and one can investigate whether it integrates to a one-parameter deformation. This requires the vanishing of a potentially infinite number of obstructions in \( H^3(\mathfrak{g}; \mathfrak{g}) \), but in practice we will not have to go beyond second order in any of the deformations found here.

The complexity of the calculations increases as we move from type I to type IIB and then to type IIA supergravities, and we have decided to organize the paper in increasing complexity. Within each theory, however, we have ordered the sections in such a way that we first treat the Minkowski and Freund–Rubin vacua and then the brane-like backgrounds in increasing brane dimension. We now give a summary of the results.

In section 2 we discuss type I backgrounds. In section 2.1 we prove the rigidity of the Poincaré superalgebra and in section 2.3 we prove that of the D5-brane superalgebra, whereas in section 2.2 we exhibit a one-parameter deformation of the D1-brane superalgebra. In section 3 we discuss type IIB backgrounds. The rigidity of the Poincaré superalgebra is demonstrated in section 3.1, whereas in section 3.2 we discuss other maximally supersymmetric backgrounds. We sketch a proof that the superalgebra of the Freund–Rubin background is rigid, whereas the existence of the plane-wave limit shows that the superalgebra of the maximally supersymmetric wave admits at least one deformation. In section 3.3 we exhibit a one-parameter deformation of the D1-brane superalgebra, whereas in sections 3.5 and 3.6 we prove the rigidity of the D3- and D5-brane superalgebras, respectively. The rigidity of the D3-brane superalgebra may come as a surprise in view of the deformation of the four-dimensional Poincaré superalgebra. For completeness and because there seems to be some confusion in the literature on this topic, we work out this deformation in section 3.5.1. In sections 3.4 and 3.8, respectively, we exhibit one-parameter deformations of the superalgebras of the D7-brane and of the fundamental string, whereas in section 3.7 we prove the rigidity of the superalgebra of the NS5 brane. In section 4 we discuss type IIA supergravity backgrounds. The rigidity of the Poincaré superalgebra is shown in section 4.1, whereas the rigidity of the superalgebras of the D0-, D4- and NS5-branes is shown in sections 4.2, 4.5 and 4.6, respectively. We exhibit deformations of the fundamental string, D2- and D6-brane superalgebras in sections 4.3, 4.4 and 4.7, respectively. The latter two deformations have their origin in the deformations of the superalgebras of the delocalized M2-brane and the Kaluza–Klein monopole in 11-dimensional supergravity. The latter deformation was found in [1], whereas the former is described in section 4.4.1. Finally, in section 5, we summarize our results and speculate on the geometric origin of these deformations. The paper ends with the appendix which lists our spinor conventions and records some useful formulae.

2. Type I backgrounds

In this section, we study the Lie superalgebra deformations of the Killing superalgebra [11] of some type I supergravity backgrounds: Minkowski space, which is the unique maximally supersymmetric background, and the half-BPS D1- and D5-brane backgrounds.
2.1. Rigidity of the Poincaré superalgebra

The Killing superalgebra of the Minkowski vacuum of type I supergravity is the type I Poincaré superalgebra. Let \( V \) denote a ten-dimensional Lorentzian vector space and \( \mathfrak{so}(V) \) denote the corresponding Lorentz Lie algebra. The Poincaré Lie algebra is \( \mathfrak{so}(V) \oplus V \). The type I spinors are chiral, and we take them to have positive chirality without loss of generality. Let \( \Delta_+ \) denote their representation space. As a vector superspace, the type I Poincaré superalgebra is \( \mathfrak{I} \). We see that there is a one-dimensional space of 2-coboundaries, spanned by \( \mathfrak{I} \) and \( Q_\alpha Q_\beta \). Let \( \mathfrak{I} \) be the type I Poincaré superalgebra. Let \( \mathfrak{I} \) be denoted by \( \mathfrak{I} \). Let \( \mathfrak{I} \) be denoted by \( \mathfrak{I} \). The supertranslation ideal \( \mathfrak{I} \) is spanned by \( P_\mu \), \( L_{\mu \nu} \) and \( Q_\alpha \). The corresponding basis for \( \mathfrak{I} \) will be denoted \( P_\mu \), \( L_{\mu \nu} \) and \( Q_\alpha \). The Lie brackets are those of the Lorentz subalgebra and in addition

\[
\begin{align*}
[L_{\mu \nu}, Q_\alpha] &= \frac{1}{2} \Gamma_{\mu \nu}^\beta Q_\alpha, \\
[L_{\mu \nu}, P_\rho] &= \eta_{\mu \rho} P_\nu - \eta_{\nu \rho} P_\mu, \\
[Q_\alpha, Q_\beta] &= \Gamma_{\alpha \beta}^\gamma P_\gamma,
\end{align*}
\]

with \( \eta_{\mu \nu} \) the Minkowski metric relative to this orthonormal frame, and where

\[
\Gamma_{\mu \nu} \cdot Q_\alpha = Q_\beta (\Gamma_{\mu \nu}^\beta)_{\alpha},
\]

and similarly for the action of any other element in the Clifford algebra \( \mathcal{C}(V) \), and

\[
\Gamma_{\alpha \beta}^\gamma := (\varepsilon_\alpha, \Gamma^\gamma_{\alpha \beta}),
\]

where \((-,-)\) is the \( s \)-invariant symplectic structure on \( \Delta_+ \).

We are interested in the cohomology group \( H^2(\mathfrak{h}; \mathfrak{g}) \) which can be computed from the complex \( C^* := C^*(\mathfrak{h}; \mathfrak{g}) \) of \( s \)-equivariant linear maps \( \Lambda^* \mathfrak{h} \rightarrow \mathfrak{g} \). We write these maps tensorially as invariant elements in \( \Lambda^* \mathfrak{h} \). It should be pointed out that this way of writing them incurs in some signs. Indeed, whereas the natural isomorphism \( \text{Hom}(\Lambda^* \mathfrak{h}, \mathfrak{g}) \cong \mathfrak{h} \otimes \Lambda^* \mathfrak{h} \) carries no sign, the isomorphism \( \mathfrak{h} \otimes \Lambda^* \mathfrak{h} \cong \Lambda^* \mathfrak{h} \otimes \mathfrak{h} \) does carry signs whenever we are interchanging odd objects. Let \( P^\mu \) and \( Q^\alpha \) denote the canonical dual basis for \( \mathfrak{I}^* \). The differential \( \text{d} \) of the complex \( C^* \) is defined uniquely by the following action on \( I^* \) and on \( \mathfrak{h} \) as an \( I \)-module:

\[
\begin{align*}
\text{d} P^\mu &= \frac{1}{2} \Gamma_{\alpha \beta}^\gamma P^\alpha \wedge Q^\beta, \\
\text{d} Q^\alpha &= 0, \\
\text{d} P_\mu &= 0, \\
\text{d} Q_\alpha &= -\Gamma_{\alpha \beta}^\gamma P_\beta \wedge P_\mu, \\
\text{d} L_{\mu \nu} &= \eta_{\mu \rho} P_\rho \wedge P_\nu - \eta_{\nu \rho} P_\rho \wedge P_\mu + \frac{1}{2} Q^\alpha \wedge P_\mu.
\end{align*}
\]

As there are no Lorentz scalars in \( \mathfrak{h} \), \( C^0 = 0 \). There are also no 1-coboundaries. The space \( C^1 \) of \( I \)-cochains is spanned by the cochains corresponding to the identity maps \( V \rightarrow V \) and \( \Delta_+ \rightarrow \Delta_+ ; \), that is, \( P^\mu \otimes P_\mu \) and \( Q^\alpha \otimes Q_\alpha \). Computing the differential \( \text{d} : C^1 \rightarrow C^2 \), we find

\[
\begin{align*}
\text{d}(P^\mu \otimes P_\mu) &= \frac{1}{2} \Gamma_{\alpha \beta}^\gamma P^\alpha \wedge Q^\beta \otimes P_\mu, \\
\text{d}(Q^\alpha \otimes Q_\alpha) &= \Gamma_{\alpha \beta}^\gamma Q^\alpha \wedge Q^\beta \otimes P_\mu,
\end{align*}
\]

whence we see that there is one cocycle \( 2 P^\mu \otimes P_\mu - Q^\alpha \otimes Q_\alpha \). We conclude that \( H^2(\mathfrak{h}; \mathfrak{g}) \cong \mathbb{R} \), corresponding to the outer derivation which gives \( Q_\alpha \) weight 1, \( P_\mu \) weight 2 and \( L_{\mu \nu} \) weight 0. We also see that there is a one-dimensional space of 2-coboundaries, spanned by \( \Gamma_{\alpha \beta}^\gamma Q^\alpha \wedge Q^\beta \otimes P_\mu \).
The space of 2-cochains consists of $\mathfrak{s}$-equivariant maps $\Lambda^2 I \to \mathfrak{k}$. As there are no such maps $V \otimes \Delta_2 \to \Delta_4$ and $\Delta_4 \otimes \Delta_4 \to \Lambda^2 V$, we find only the following 2-cochains: $P^{\mu} \wedge P^{\nu} \otimes L_{\mu\nu}$ and $I_{\alpha\beta} Q^{\mu} \wedge Q^{\beta} \otimes P_{\mu}$, corresponding to the natural isomorphism $\Lambda^2 V \to \mathfrak{s}\mathfrak{o}(V)$ and the projection $\delta^2 \Delta_4 \to V$. A simple calculation shows that
\begin{equation}
\text{d}(P^{\mu} \wedge P^{\nu} \otimes L_{\mu\nu}) = \Gamma^{\mu\nu}_{\alpha\beta} P^{\alpha} \wedge Q^{\beta} \otimes L_{\mu\nu} + \frac{1}{2} P^{\mu} \wedge P^{\nu} \wedge Q^{\alpha} \otimes \Gamma_{\mu\nu} \cdot Q_{\alpha} \neq 0,
\end{equation}
whence the only cocycle is also a coboundary and hence $H^2(\mathfrak{k}; \mathfrak{k}) = 0$, showing the rigidity of the type I Poincaré superalgebra.

### 2.2. A deformation of the D1-brane superalgebra

To describe the type I D1-brane superalgebra, we split the ten-dimensional space as $V = W \oplus \mathbb{R}$. The space of 1-cochains consists of those maps $V \to \mathfrak{h}$, which is now spanned by $P_{\mu}, Q_{\alpha}$, and $Q_{\beta}$. Since $\mathfrak{so}(W)$ is Abelian, it is not part of the semisimple factor and it must be included in the semisimple subalgebra $\mathfrak{s}$ of $\mathfrak{so}(W \otimes \mathbb{R})$. Indeed, the cohomology group $H^2(\mathfrak{t}; \mathfrak{k})$ can be computed from the complex $C^* := C^*(I; \mathfrak{k})^\mathfrak{t}$ of $\mathfrak{t}$-equivariant linear maps $\Lambda^* I \to \mathfrak{k}$. Letting $L^*, P^{\mu}, Q^{\alpha}$ denote the canonical dual basis for $I^*$, the differential $\text{d}$ of this complex is defined uniquely by the following relations:

\begin{equation}
\begin{aligned}
\text{d}P^{\mu} & = \frac{1}{2} \Gamma^{\mu}_{\alpha\beta} Q^{\alpha} \wedge Q^{\beta} + \epsilon^{\mu\nu}, L^* \wedge P^{\nu}, \\
\text{d}Q^{\alpha} & = \frac{1}{2} L^* \wedge Q^{\alpha}, \\
\text{d}L^* & = 0, \\
\text{d}P_{\mu} & = L^* \otimes \epsilon_{\mu} P_{\nu}, \\
\text{d}Q_{\alpha} & = -\frac{1}{2} L^* \otimes Q_{\alpha} - \Gamma^{\mu}_{\alpha\beta} Q^{\beta} \otimes P_{\mu}, \\
\text{d}L & = -P^{\mu} \otimes \epsilon_{\mu} P_{\nu} - \frac{3}{2} Q^{\alpha} \otimes Q_{\alpha}, \\
\text{d}L_{\mu\nu} & = \frac{1}{2} Q^{\alpha} \otimes \Gamma_{\mu\nu} \cdot Q_{\alpha}.
\end{aligned}
\end{equation}

In this case, $C^0 = \mathfrak{t}$ is spanned by $L$, but since $\text{d}L \neq 0$, $H^1(\mathfrak{t}; \mathfrak{k}) = 0$ and $\text{dim} B^1 = 1$. The space of 1-cochains is four-dimensional, with basis $L^* \otimes L, P^{\mu} \otimes P_{\mu}, P^{\mu} \otimes \epsilon_{\mu\nu} P_{\nu}$, and $Q^{\alpha} \otimes Q_{\alpha}$. Finally, the space of 2-cochains is five-dimensional, spanned by $P^{\mu} \wedge P^{\nu} \otimes \epsilon_{\mu\nu} L, L^* \wedge P^{\mu} \otimes P_{\mu}, L^* \otimes P^{\mu} \otimes \epsilon_{\mu\nu} P_{\nu}, L^* \wedge Q^{\alpha} \otimes Q_{\alpha}$, and $Q^{\alpha} \wedge Q^{\beta} \otimes \Gamma^{\mu}_{\alpha\beta} P_{\mu}$. Computing the differentials $d : C^1 \to C^2$ and $d : C^2 \to C^3$, we find that $H^1(\mathfrak{t}; \mathbb{R})$, with representative cocycle $\phi := 2 P^{\mu} \otimes P_{\mu} - Q^{\alpha} \otimes Q_{\alpha}$, and $H^2(\mathfrak{t}; \mathfrak{k}) \cong \mathbb{R}$, with representative cocycle $L^* \wedge \phi$. This infinitesimal deformation integrates to a one-parameter family of Lie superalgebras with brackets

\begin{equation}
\begin{aligned}
[L, Q_{\alpha}] & = (t - \frac{1}{2}) Q_{\alpha}, \\
[L, P_{\mu}] & = 2t P_{\mu} + \epsilon_{\mu\nu} P_{\nu}, \\
[Q_{\alpha}, Q_{\beta}] & = \Gamma^{\mu}_{\alpha\beta} P_{\mu}.
\end{aligned}
\end{equation}
in addition to those involving $\mathfrak{so}(W^\perp)$, which remain undeformed. The reader may be forgiven for suspecting a discrepancy from the sign of the $t$-dependent terms in the brackets above and the relative sign in the cocycle $\varphi$. As explained above, this is due to the signs in the isomorphism $W \otimes V^* \cong V^* \otimes W$ whenever $V$ and $W$ are both odd subspaces. The signs can be read off from the formulae in [1, section 2].

2.3. Rigidity of the D5-brane superalgebra

The type I D5-brane superalgebra is the subsuperalgebra of the Poincaré superalgebra defined as follows. We first split the ten-dimensional Lorentzian vector space $V = W \oplus W^\perp$, where $W$ is a six-dimensional Lorentzian subspace. Then the D5-brane superalgebra $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$, where

\begin{align*}
[\mathfrak{g}_0, \mathfrak{g}_0] &= \mathfrak{so}(W) \oplus W \oplus \mathfrak{so}(W^\perp), \\
[\mathfrak{g}_0, \mathfrak{g}_1] &= \mathfrak{so}^*(6,1), \\
[\mathfrak{g}_1, \mathfrak{g}_1] &= \mathfrak{m}_1 = \{ \epsilon \in \Delta_1 | \nu_W \cdot \epsilon = \epsilon \},
\end{align*}

where $\nu_W$ is the Clifford algebra element which represents the volume form of $W$. It is skewsymmetric relative to the invariant symplectic form on spinors and satisfies $\nu_W^2 = +1$. Let $\epsilon_\alpha$ be a basis for $\Delta_1$ and $e_\mu$ and $e_\nu$ bases for $W$ and $W^\perp$, respectively. Let $Q_\alpha$ and $P_\mu$ denote the corresponding basis for the ideal $\mathfrak{i}_k$. The semisimple subalgebra $\mathfrak{s}$ is spanned by $L_{\mu\nu}$ and $L_{ab}$. Relative to this basis, the Lie brackets of $\mathfrak{g}$ are given by those of $\mathfrak{so}(W) \oplus \mathfrak{so}(W^\perp)$ and

\begin{align*}
[L_{\mu\nu}, Q_\alpha] &= \frac{1}{2} \Gamma_{\mu\nu} \cdot Q_\alpha, \\
[L_{ab}, Q_\alpha] &= \frac{1}{2} \Gamma_{ab} \cdot Q_\alpha, \\
[L_{\mu\nu}, P_\rho] &= \eta_{\rho\nu} P_\mu - \eta_{\rho\mu} P_\nu, \\
[Q_\alpha, Q_\beta] &= \Gamma_{\mu\nu} \cdot Q_\alpha \\
&\quad \times \Gamma_{\mu\nu} \cdot Q_\beta,
\end{align*}

(10) where again

\begin{align*}
\Gamma_{\mu\nu} \cdot Q_\alpha &= Q_\beta (\Gamma_{\mu\nu})^\beta_{\alpha},
\end{align*}

and

\begin{align*}
\Gamma_{\mu\nu} := (\epsilon_\alpha, \Gamma_{\mu\nu} \epsilon_\beta).
\end{align*}

(11)

Let $P_\mu$ and $Q_\alpha$ denote the canonical dual basis for $\mathfrak{i}^*$. The differential ‘$d$’ of the complex $C^* = C^*(I; \mathfrak{g})$ is defined uniquely by

\begin{align*}
d P_\mu &= \frac{1}{2} \Gamma_{\mu\nu}^{ab} Q_\alpha \wedge Q_\beta, \\
d Q_\alpha &= 0, \\
d P_\mu &= 0, \\
d Q_\alpha &= -\Gamma_{\mu\nu} \cdot Q_\beta \otimes P_\mu, \\
d L_{\mu\nu} &= \eta_{\mu\nu} P_\rho \otimes P_\rho - \eta_{\mu\nu} P_\rho \otimes P_\rho + \frac{1}{2} Q_\alpha \otimes \Gamma_{\mu\nu} \cdot Q_\beta, \\
d L_{ab} &= \frac{1}{2} Q_\alpha \otimes \Gamma_{ab} \cdot Q_\alpha.
\end{align*}

(13)

Again we see that $C^0 = \mathfrak{t}^* = 0$ and that $C^1$ is spanned by $P_\mu \otimes P_\mu$ and $Q_\alpha \otimes Q_\alpha$. Similarly, $C^2$ is spanned by $P_\mu \wedge P_\nu \otimes L_{\mu\nu}$ and $Q_\alpha \wedge Q_\beta \otimes \Gamma_{\mu\nu} \cdot P_\rho$. It is easy to compute the differentials $d : C^1 \to C^2$ and $d : C^2 \to C^3$ and we see that $H^1(\mathfrak{t}, \mathfrak{g}) \cong \mathbb{R}$, with representative cocycle $2 P_\mu \otimes P_\mu - Q_\alpha \otimes Q_\alpha$, and that $H^2(\mathfrak{t}; \mathfrak{g}) = 0$, whence the type I D5-brane superalgebra is rigid.

3. Type IIB backgrounds

In this section, we explore the Lie superalgebra deformations of the Killing superalgebras of certain type IIB backgrounds. We start with the Minkowski vacuum, treat briefly the other maximally supersymmetric backgrounds and then go on to the elementary brane elementary brane.
3.1. Rigidity of the Poincaré superalgebra

The Killing superalgebra of the Minkowski vacuum is the type IIB Poincaré superalgebra \( \mathfrak{t} = \mathfrak{t}_0 \oplus \mathfrak{t}_1 \), with \( \mathfrak{t}_0 = \mathfrak{so}(V) \oplus V \) the Poincaré algebra and \( \mathfrak{t}_1 \) isomorphic to two copies of the positive chirality spinor representation \( \Delta_+ \) of \( \mathfrak{so}(V) \). We will denote these two copies by \( \Delta_+^I \), where \( I = 1, 2 \). Let \( \varepsilon_a \) be a basis for \( \Delta_+ \), and \( \varepsilon_a^I \) and \( Q_I^a \) denote the corresponding bases for \( \Delta_+^I \) and \( \mathfrak{t}_1 \), respectively. Then, \( \mathfrak{t} \) is spanned by \( P_\mu, L_{\mu\nu} \) and \( Q_I^a \) subject to the following brackets, in addition to the ones of the Poincaré subalgebra, in addition to the ones of the Poincaré subalgebra, in addition to the ones of the Poincaré subalgebra,

\[
[L_{\mu\nu}, Q_I^a] = \frac{1}{2} \varepsilon_{[\mu} Q_{\nu]}^a, \quad [Q_I^a, Q_J^b] = \delta^{IJ} \varepsilon_{[a} P_{b]}.
\]

where

\[
\Gamma_{\mu\nu} = Q_I^a (\Gamma_{\mu\nu}^a)^I, \quad (14)
\]

and

\[
\varepsilon_{ab} := (\varepsilon_a, \varepsilon^b), \quad (15)
\]

In other words, \( \mathcal{C}(V) \) and the spinor inner product act independently in each of the two copies of the spinor representation.

The supertranslation ideal \( I \subset \mathfrak{t} \) is now spanned by \( Q_I^a \) and \( P_\mu \), whereas the semisimple factor \( s = \mathfrak{so}(V) \). Let \( Q_I^a \) and \( P^\mu \) denote the canonical dual basis for \( I^* \).

Let \( C^* = C^*(I; \mathfrak{t})^a \) denote the complex of \( s \)-invariant maps \( \Lambda^* I \to \mathfrak{t} \), with differential \('d'\) defined by the following relations:

\[
dP_\mu = \frac{1}{2} \varepsilon_{ab} Q^a \wedge Q^b, \\
dQ_I^a = 0, \\
dP_\mu = 0, \\
dQ_I^a = -\delta^{IJ} \varepsilon_{ab} Q_J^b \otimes P_\mu, \\
dL_{\mu\nu} = \varepsilon_{\rho\mu} P_\rho \otimes P_\nu - \varepsilon_{\rho\nu} P_\rho \otimes P_\mu + \frac{1}{2} Q_I^a \otimes \Gamma_{\mu\nu} \cdot Q_I^a.
\]

As there are no Lorentz scalars in \( \mathfrak{t}, C^0 = \mathfrak{t}^d = 0 \). The space \( C^1 \) of 1-cochains is five-dimensional, spanned by the cochains corresponding to the identity maps \( V \to V \) and \( \Delta_+^I \to \Delta_+^I \), namely \( P^\mu \otimes P_\nu \) and \( Q_I^a \otimes Q_I^a \). The space \( C^2 \) is four-dimensional, spanned by the natural isomorphism \( \Lambda^2 V \cong \mathfrak{so}(V) \) and the projections \( \Delta_+^I \otimes \Delta_+^I \to V \) which are symmetric in \( I \leftrightarrow J \). Evaluating the differential \( d : C^1 \to C^2 \) we find

\[
d(P^\mu \otimes P_\mu) = \frac{1}{2} \delta^{IJ} \varepsilon_{ab} Q_I^a \wedge Q_J^b \otimes P_\mu, \\
d(Q_I^a \otimes Q_J^a) = \delta^{ik} \varepsilon_{ab} Q_k^b \wedge Q_k^b \otimes P_\mu. 
\]

from where we see that \( H^1(\mathfrak{t}; \mathfrak{t}) \cong \mathbb{R}^2 \), with representative cocycles \( 2 P^\mu \otimes P_\mu - Q_I^a \otimes Q_I^a \) and \( \varepsilon_I Q_I^a \otimes Q_I^a \). This means that \( \dim B^2 = 3 \) and since

\[
d(P^\mu \otimes P^\nu \otimes L_{\mu\nu}) = \delta^{IJ} \varepsilon_{ab} Q_I^a \wedge Q_J^b \wedge P_\nu \otimes L_{\mu\nu} + \frac{1}{2} P^\mu \wedge P^\nu \otimes Q_I^a \otimes \Gamma_{\mu\nu} \cdot Q_I^a \neq 0, \quad (19)
\]

we see that \( H^2(\mathfrak{t}; \mathfrak{t}) = 0 \) and the IIB Poincaré superalgebra is rigid.

3.2. Other maximally supersymmetric backgrounds

As shown in [14], there are only two other maximally supersymmetric IIB backgrounds: the Freund–Rubin background [15] with geometry AdS_5 \( \times S^5 \), and the maximally supersymmetric wave [8]. The Killing superalgebra of the Freund–Rubin background is the simple Lie superalgebra \( \mathfrak{su}(2, 2|4) \), whereas that of the maximally supersymmetric wave is the contraction [16–18] induced by the plane-wave limit [19, 20]. This observation implies that the Killing
superalgebra of the maximally supersymmetric wave is not rigid, and it admits at least a one-parameter family of deformations, isomorphic to $\mathfrak{su}(2, 2|4)$ for nonzero values of the parameter. We will not compute the space of deformations in this paper, but as in the similar situation in 11 dimensions [1], we would be surprised if there were any other deformations.

As for $\mathfrak{su}(2, 2|4)$ itself, the fact that it is simple does not immediately imply that it is rigid. A closer look at the rigidity results for simple Lie superalgebras [21] shows that the crucial condition used in the proof is the nondegeneracy of the Killing form; whereas Cartan’s criterion guarantees that this is the case for semisimple Lie algebras, this is not the case for superalgebras. Indeed, in Kac’s list [22] there are simple Lie superalgebras with degenerate (or even zero) Killing form and indeed, the Lie superalgebra of type $D(2, 1)$ has zero Killing form and admits a one-parameter deformation $D(2, 1; \alpha)$ which remains simple for all values of $\alpha$. Curiously, as shown in [3], the Killing superalgebra of the near-horizon geometry of a $\frac{1}{8}$-BPS configuration of rotating intersecting branes in 11-dimensional supergravity is isomorphic to two copies of $D(2, 1; \alpha)$—the parameter $\alpha$ having a geometric interpretation as the ratio of the radii of the two 3-spheres in the near-horizon geometry $AdS_3 \times S^3 \times S^3 \times \mathbb{R}^2$.

The Lie superalgebra $\mathfrak{su}(2, 2|4)$ too has zero Killing form, hence the result of [21] does not apply, and moreover since the algebra is simple, there is no Hochschild–Serre factorization. However, the cohomology $H^2(\mathfrak{k}; \mathfrak{k})$ may be calculated from the subcomplex of cochains which are invariant under the semisimple even subalgebra $\mathfrak{k}_0 = \mathfrak{su}(2, 2) \oplus \mathfrak{su}(4)$. The space of 2-cochains breaks up into three, which are the $\mathfrak{k}_0$-invariant subspaces of $\Lambda^2 \mathfrak{k}_0^* \otimes \mathfrak{k}_0$, $\mathfrak{k}_0^* \otimes \mathfrak{k}_0^* \otimes \mathfrak{k}_0$ and $\Lambda^2 \mathfrak{k}_0^* \otimes \mathfrak{k}_0$. As an $\mathfrak{k}_0$-module, $\mathfrak{k}_0 = (\Lambda^2 \mathfrak{V}^{(2,4)} \otimes \mathbb{R}) \oplus (\mathbb{R} \otimes \Lambda^2 \mathfrak{V}^{(6)})$, where $\mathfrak{V}^{(2,4)}$ and $\mathfrak{V}^{(6)}$ are the vector representations of $\mathfrak{so}(2, 4) \cong \mathfrak{su}(2, 2)$ and $\mathfrak{so}(6) \cong \mathfrak{su}(4)$, respectively. Similarly, $\mathfrak{k}_1 = [\Lambda^2 \mathfrak{k}_0^* \otimes \Lambda^6 \mathfrak{k}_0^*]$, where $\Lambda^2 \mathfrak{k}_0^*$ and $\Lambda^6 \mathfrak{k}_0^*$ are the positive-chirality spinor representations of $\mathfrak{so}(2, 4)$ and $\mathfrak{so}(6)$, respectively, which are complex four-dimensional, and where if $W$ is a complex vector space, $[W]$ is a real vector space defined by $[W] \otimes_{\mathbb{R}} \mathbb{C} = W \oplus \bar{W}$. In other words, it is the vector space spanned by the real and imaginary parts of the vectors in $W$, whence $\dim_{\mathbb{R}}[W] = 2 \dim_{\mathbb{C}} W$. In this case, $\Lambda^2 \mathfrak{k}_0^* \otimes \Lambda^6 \mathfrak{k}_0^*$ is complex and 16-dimensional, whence $\mathfrak{k}_1$ is real and 32-dimensional, as expected. Since $\mathfrak{k}_0$ is semisimple, it is rigid as a Lie algebra, we can assume that its Lie brackets remain undeformed, hence we can assume that a cocycle defining an infinitesimal deformation of $\mathfrak{su}(2, 2|4)$ has no components in $\Lambda^2 \mathfrak{k}_0^* \otimes \mathfrak{k}_0$. By the same token, the rigidity of $\mathfrak{k}_1$ as an $\mathfrak{k}_0$-module says that the putative cocycle cannot have components in $\Lambda^2 \mathfrak{k}_0^* \otimes \Lambda^6 \mathfrak{k}_0^* \otimes \mathfrak{k}_0$, whence the cocycle, if it exists, must belong to the $\mathfrak{k}_0$-invariant subspace of $\Lambda^2 \mathfrak{k}_0^* \otimes \mathfrak{k}_0$. A simple roots-and-weights calculation shows that this space is two-dimensional made out of the natural maps

$$\Delta^{(2,4)} \otimes \Delta^{(2,4)} \to \mathbb{R} \quad \text{and} \quad \Delta^{(6)} \otimes \tilde{\Delta}^{(6)} \to \Lambda^2 \mathfrak{V}^{(6)}; \quad (20)$$

$$\Delta^{(6)} \otimes \tilde{\Delta}^{(6)} \to \mathbb{R} \quad \text{and} \quad \Delta^{(2,4)} \otimes \tilde{\Delta}^{(2,4)} \to \Lambda^2 \mathfrak{V}^{(2,4)}; \quad (21)$$

which means that the $[\mathfrak{k}_1, \mathfrak{k}_1]$ bracket has two parameters, which we can choose to take the value 1 in the undeformed superalgebra $\mathfrak{su}(2, 2|4)$. The $(\mathfrak{k}_1, \mathfrak{k}_1)$ Jacobi identity fixes the ratio of these two parameters to be 1 and we can further set them to be equal to 1 by rescaling the odd generators, hence proving the rigidity of the superalgebra.

3.3. A deformation of the D1-brane superalgebra

The Killing superalgebra of the type IIB D1-brane is the subsuperalgebra $\mathfrak{t}$ of the IIB Poincaré superalgebra with $\mathfrak{t}_0 = \mathfrak{so}(W) \oplus \mathfrak{w} \oplus \mathfrak{w}(W^\perp)$, where $V = \mathfrak{w} \oplus W^\perp$ is the decomposition of the ten-dimensional Lorentzian vector space into a two-dimensional Lorentzian subspace $\mathfrak{w}$, corresponding to the brane worldvolume and its eight-dimensional perpendicular complement.
The odd subspace \( \mathfrak{k} \) is isomorphic to the graph \( \Delta_{D1} \subset \Delta_+ \oplus \Delta_+ \) of the endomorphism \( \nu_W : \Delta_+ \to \Delta_+ \) corresponding to the volume form of \( W \). As in the type I D1-brane, the extension of \( \nu_W \) to the Clifford module is skew symmetric relative to the spinor inner product and obeys \( \nu_W^2 = +1 \).

Let \( e_a \) and \( e_a \) span \( W \) and \( W^\perp \), respectively, and \( e_a \) span \( \Delta_+ \). Let \( \psi_a = \frac{1}{\sqrt{2}} \left( e_a + e_a \right) \) be a basis for \( \Delta_{D1} \). The corresponding basis of \( \mathfrak{t} \) is given by \( P_{\mu}, P_{\mu
u} = \epsilon_{\mu\nu}L, L_{ab} \) and \( Q_a \). The Lie brackets are inherited from those in equation (14) and are given explicitly, in addition to those involving \( L_{ab} \), by

\[
[L, Q_a] = -\frac{i}{2} \nu_W \cdot Q_a, \quad [L, P_\mu] = \epsilon_{\mu\nu} P_\nu, \quad [Q_a, Q_\beta] = \Gamma_{a\beta}^{\mu} P_\mu, \quad (22)
\]

where

\[
\Gamma_{a\beta}^{\mu} := \langle \psi_a, \Gamma^\mu \psi_\beta \rangle = \langle e_a, \Gamma^\mu e_\beta \rangle. \quad (23)
\]

As in the case of the type I D1-brane calculation, the ideal \( I \subset \mathfrak{k} \) includes the generator \( L \), but we may work with cochains which are invariant under the reductive subalgebra \( \tau := \mathfrak{so}(W) \oplus \mathfrak{so}(W^\perp) \). Letting \( L^+, P^\mu \) and \( Q^\mu \) be a basis for \( \Gamma^* \), the differential in the complex \( C^* = C^*(I, \mathfrak{k}) \) is determined by the following relations:

\[
\begin{align*}
&dP^\mu = \frac{i}{2} \epsilon_{\alpha\beta} P^\alpha \gamma^\beta + \epsilon_{\mu\nu} L^\nu \wedge P^\nu, \\
&dQ^\mu = \frac{i}{2} L^\nu \wedge (\nu_W)^\beta \gamma^\beta, \\
&dL^\nu = 0, \\
&dP_{\mu} = L^\nu \otimes \epsilon_{\mu\nu} P_\nu, \\
&dQ_a = -\frac{i}{2} L^\nu \otimes \nu_W \cdot Q_a - \Gamma_{a\beta}^{\mu} Q^\beta \otimes P_\mu, \\
&dL = -P^\mu \otimes \epsilon_{\mu\nu} P_\nu - \frac{i}{2} Q^\alpha \otimes \nu_W \cdot Q_a, \\
&dL_{ab} = \frac{i}{2} Q^\alpha \otimes \Gamma_{a\beta}^{\mu} P_\mu. \quad (24)
\end{align*}
\]

The 0-cochains \( C^0 = \mathfrak{k} \) are spanned by \( L \), but \( dL \neq 0 \), hence \( H^0(I; \mathfrak{k}) = 0 \) and \( \dim B^1 = 1 \). The space of 1-cochains is five-dimensional, spanned by \( P^\mu \otimes P_\mu, P^\mu \otimes \epsilon_{\mu\nu} P_\nu, L^\nu \otimes L^\nu, Q^\mu \otimes Q_a \) and \( Q^\mu \otimes \nu_W \cdot Q_a \). Another way to understand this is to note that the ideal \( I \) is graded by the action of \( 2L \) with \( L \) having degree 0, \( Q_a \) having pieces of degrees \( \pm1 \) and \( P_\mu \) having pieces of degrees \( \pm2 \), corresponding to a Witt basis for \( W \). The five-dimensional space of cochains can be thought of as spanned by the cochains corresponding to the identity maps of each of the five graded subspaces. The space of 2-cochains is seven-dimensional, spanned by \( P^\mu \otimes P^\nu \otimes \epsilon_{\mu\nu} L^\nu, L^\nu \otimes P^\mu \otimes P_\mu, L^\nu \otimes P^\mu \otimes \epsilon_{\mu\nu} P_\nu, L^\nu \otimes Q^\mu \otimes Q_a, L^\nu \otimes Q^\mu \otimes \nu_W \cdot Q_a, Q^\mu \otimes Q^\nu \otimes \Gamma_{a\beta}^{\mu} P_\mu \) and \( Q^\mu \otimes Q^\nu \otimes (\Gamma^\mu \nu_W)_{a\beta} P_\mu \).

Computing the differential \( d : C^1 \to C^2 \), we find that \( H^1(I; \mathfrak{k}) \cong \mathbb{R} \), with representative cocycle \( \varphi := 2P^\mu \otimes P_\mu - Q^\mu \otimes Q_a \). Similarly, computing \( d : C^2 \to C^3 \) we find that \( H^2(I; \mathfrak{k}) \cong \mathbb{R} \), with representative cocycle \( L^\nu \otimes \varphi \). This infinitesimal deformation integrates to a one-parameter family of \( \mathfrak{k} \)-

\[
\begin{align*}
[L, Q_a] &= tQ_a - \frac{i}{2} \nu_W \cdot Q_a, \\
[L, P_\mu] &= 2tP_\mu + \epsilon_{\mu\nu} P_\nu, \\
[Q_a, Q_\beta] &= \Gamma_{a\beta}^{\mu} P_\mu. \quad (25)
\end{align*}
\]

in addition to those involving \( \mathfrak{so}(W^\perp) \), which remain undeformed. This deformation consists of changing the \( L \)-weight of the generators in the \( \mathfrak{k} \) superalgebra in such a way that the \( QQ \) bracket remains invariant. This deformation is familiar with the twisting construction of two-dimensional topological conformal field theories.
3.4. A deformation of the fundamental string superalgebra

The Killing superalgebra of the type IIB fundamental string is the subsuperalgebra \( \mathfrak{t} = \mathfrak{t}_0 \oplus \mathfrak{t}_1 \) of the IIB Poincaré superalgebra, where \( \mathfrak{t}_0 = \mathfrak{so}(W) \oplus \mathfrak{W} \oplus \mathfrak{w}(W^\perp) \), corresponding to a decomposition \( V = W \oplus W^\perp \), where \( W \) is Lorentzian and two-dimensional, corresponding to the string worldsheet. The odd subspace \( \mathfrak{t}_1 \) is isomorphic to the subspace \( \Delta_F \subset \Delta_\epsilon + \Delta_\gamma \) given by

\[
\Delta_F = \left\{ \left( \varepsilon_1, \varepsilon_2 \right) \in \Delta_\epsilon + \Delta_\gamma \bar{\nu}_w \left( \varepsilon_1, \varepsilon_2 \right) = \left( -\varepsilon_1, \varepsilon_2 \right) \right\},
\]

where the Clifford endomorphism \( \nu_w \) corresponding to the volume form of the string worldsheet is skewsymmetric relative to the spinor inner product and obeys \( \nu_w^2 = +1 \).

Let \( \varepsilon_\mu \) and \( \varepsilon_\alpha \) span \( W \) and \( W^\perp \), respectively, and \( P_\mu, L_{\mu\nu} = \varepsilon_\mu, L \) and \( L_{ab} \) be the generators of \( \mathfrak{t}_0 \). Let \( \varepsilon_\alpha \) and \( \bar{\varepsilon}_\bar{\alpha} \) be basis elements for the subspaces of \( \Delta_\epsilon \) satisfying \( \nu_w \varepsilon_\alpha = -\varepsilon_\alpha \) and \( \nu_w \bar{\varepsilon}_\bar{\alpha} = \bar{\varepsilon}_\bar{\alpha} \), respectively. Then, \( \{ \varepsilon_\alpha \} \) and \( \{ \bar{\varepsilon}_\bar{\alpha} \} \) span \( \Delta_F \). Let \( Q_\alpha \) and \( \bar{Q}_\bar{\alpha} \) denote the corresponding basis for \( \mathfrak{t}_1 \). The nonzero Lie brackets in this basis are given, in addition to those of \( \mathfrak{so}(W^\perp) \), by

\[
\begin{align*}
[L, Q_\alpha] &= \frac{1}{2} Q_\alpha, & [L, \bar{Q}_\bar{\alpha}] &= -\frac{1}{2} \bar{Q}_\bar{\alpha}, & [L, P_\mu] &= \varepsilon_\mu, P_\nu, \\
\{Q_\alpha, Q_\beta\} &= \Gamma^\mu_{ab} P_\mu, & \{\bar{Q}_\bar{\alpha}, \bar{Q}_\bar{\beta}\} &= \Gamma^\mu_{\bar{\alpha}\bar{\beta}} P_\mu,
\end{align*}
\]

where

\[
\Gamma^\mu_{ab} := \langle \varepsilon_a, \Gamma^\mu \varepsilon_b \rangle \quad \text{and} \quad \Gamma^\mu_{\bar{\alpha}\bar{\beta}} := \langle \bar{\varepsilon}_{\bar{\alpha}}, \Gamma^\mu \bar{\varepsilon}_{\bar{\beta}} \rangle.
\]

The ideal \( I \subset \mathfrak{t} \) contains the supertranslation ideal and the generator \( L \), but \( H^2(\mathfrak{t}; \mathfrak{t}) \) can be computed from the complex \( C^* := C^*(I; \mathfrak{t})^t \) of cochains which are invariant under the reductive subalgebra \( \mathfrak{r} := \mathfrak{so}(W) \oplus \mathfrak{so}(W^\perp) \). Letting \( L^* \), \( P^\mu \), \( Q^\alpha \) and \( \bar{Q}^{\bar{\alpha}} \) denote the canonical dual basis for \( I^* \), the differential ‘\( d \)’ in \( C^* \) is determined uniquely by the following relations:

\[
\begin{align*}
dP^\mu &= \frac{1}{2} \Gamma^\mu_{ab} Q^a \wedge Q^b + \frac{1}{2} \Gamma^\mu_{\bar{\alpha}\bar{\beta}} \bar{Q}^{\bar{\alpha}} \wedge \bar{Q}^{\bar{\beta}} + \varepsilon_\mu, L^* \wedge P^\nu, \\
dQ^\alpha &= -\frac{1}{2} L^* \wedge Q^\alpha, \\
d\bar{Q}^{\bar{\alpha}} &= \frac{1}{2} L^* \wedge \bar{Q}^{\bar{\alpha}}, \\
dl^* &= 0, \\
dP_\mu &= L^* \otimes \varepsilon_\mu, P_\nu, \\
dQ_\alpha &= \frac{1}{2} L^* \otimes Q_\alpha - \Gamma^\mu_{ab} Q^a \otimes P_\mu, \\
d\bar{Q}_{\bar{\alpha}} &= -\frac{1}{2} L^* \otimes \bar{Q}_{\bar{\alpha}} - \Gamma^\mu_{\bar{\alpha}\bar{\beta}} \bar{Q}^{\bar{\beta}} \otimes P_\mu, \\
dl &= -P^\mu \otimes \varepsilon_\mu, P_\nu + \frac{1}{2} Q^\alpha \otimes Q_\alpha - \frac{1}{2} \bar{Q}^{\bar{\alpha}} \otimes \bar{Q}_{\bar{\alpha}}, \\
dl_{ab} &= \frac{1}{2} Q^a \otimes \Gamma_{ab} \cdot Q_\alpha + \frac{1}{2} \bar{Q}^{\bar{\alpha}} \otimes \Gamma_{ab} \cdot \bar{Q}_{\bar{\alpha}}.
\end{align*}
\]

The space of 0-cochains is one-dimensional and spanned by \( L \), but since \( dL \neq 0 \), \( H^2(\mathfrak{t}; \mathfrak{t}) = 0 \) and \( \dim B^1 = 1 \). The space \( C^1 \) is five-dimensional and is spanned by \( P^\mu \otimes P_\mu, P^\mu \otimes \varepsilon_\mu, P_\nu, L^* \otimes L, Q^\alpha \otimes Q_\alpha \) and \( \bar{Q}^{\bar{\alpha}} \otimes \bar{Q}_{\bar{\alpha}} \). As in the case of the D-string, this can be understood by the fact that the ideal \( I \) is graded by the action of \( 2L \) with \( L \) having degree 0, \( Q_\alpha \) and \( \bar{Q}_{\bar{\alpha}} \) having degrees \( \pm 1 \), respectively, and \( P_\mu \) having pieces of degrees \( \pm 2 \), corresponding to a Witt basis for \( W \). The five-dimensional space of cochains can be

\footnote{Here and in the sequel we use the bars on the spinors to distinguish them from the unbarred spinors and \textit{not} to denote the Dirac conjugate.}
though of as spanned by the cochains corresponding to the identity maps of each of the five graded subspaces. The space $C^2$ of 2-cochains is seven-dimensional, spanned by $P^\mu \wedge P^\nu \otimes \epsilon_{\mu\nu} L, L^* \wedge P^\mu \otimes P_\mu, L^* \wedge P^\mu \otimes \epsilon_{\mu\nu} P_\nu, L^* \wedge Q^\nu \otimes Q_\nu, L^* \wedge \bar{Q}^\beta \otimes \bar{Q}_\beta, Q^\nu \wedge Q^\beta \otimes \Gamma^\mu_{\beta\gamma} P_\mu$ and $\bar{Q}^\alpha \wedge \bar{Q}^\beta \otimes \Gamma^\mu_{\alpha\beta} P_\mu$.

Computing the differential $d : C^1 \to C^2$, we find that $H^1(t; \mathfrak{t}) \cong \mathbb{R}$, with representative cocycle $\psi := 2P^\mu \otimes P_\mu - Q^\nu \otimes Q_\nu - \bar{Q}^\alpha \otimes \bar{Q}_\alpha$. Similarly, computing $d : C^2 \to C^3$ we find that $H^2(t; \mathfrak{t}) \cong \mathbb{R}$, with representative cocycle $L^* \wedge \psi$. This infinitesimal deformation integrates to a one-parameter family of Lie superalgebras with brackets

$$ [L, Q_a] = (t + \frac{1}{2}) Q_a, \quad [L, \bar{Q}_a] = (t - \frac{1}{2}) \bar{Q}_a, \quad [L, P_\mu] = 2t P_\mu + \epsilon_{\mu\nu} P_\nu, \quad [Q_a, Q_\beta] = \Gamma^\mu_{\alpha\beta} P_\mu, \quad [\bar{Q}_a, \bar{Q}_\beta] = \Gamma^\mu_{\alpha\beta} P_\mu, $$

in addition to those involving $\mathfrak{so}(W^\perp)$ which remain undeformed. This deformation consists of changing the $L$-weight of the generators in the Lie superalgebra in such a way that the $Q Q$ and $\bar{Q} \bar{Q}$ brackets remain invariant. As in the case of the D-string, this deformation is reminiscent of the construction of two-dimensional topological conformal field theories via twisting.

### 3.5. Rigidity of the D3-brane superalgebra

The Killing superalgebra of the D3-brane background is the subalgebra $\mathfrak{t}$ of the type IIB Poincaré superalgebra with $t_\mathfrak{t} = \mathfrak{so}(W) \oplus W \oplus \mathfrak{so}(W^\perp)$, where $V = W \oplus W^\perp$, with $W$ a four-dimensional Lorentzian subspace. The odd subspace $\mathfrak{t}_\mathfrak{o}$ is isomorphic to the subspace $\Delta_{D3} \subset \Delta_+ \oplus \Delta_-$, defined by the graph of the endomorphism $\nu_w : \Delta_+ \to \Delta_-$ corresponding to the volume form of $W$. This endomorphism obeys $\nu_w^2 = -1$ and, when extended to the irreducible Clifford module, is symmetric with respect to the spinor inner product.

Let $e_\mu$ and $e_a$ span $W$ and $W^\perp$, respectively, and $e_a$ span $\Delta_-$. Let $\psi_a = \frac{1}{\sqrt{2}}(e_a \epsilon_a)$ be a basis for $\Delta_{D3}$. The corresponding basis of $\mathfrak{t}$ is given by $P_\mu, L_{\mu\nu}, L_{ab}$ and $Q_a$. The Lie brackets are inherited from those in equation (14) and are given explicitly, in addition to those involving $L_{\mu\nu}$ and $L_{ab}$, by

$$ [Q_a, Q_\beta] = \Gamma^\mu_{\alpha\beta} P_\mu, $$

where

$$ \Gamma^\mu_{\alpha\beta} := \langle \psi_a, \Gamma^\mu \psi_b \rangle = \langle e_a, \Gamma^\mu e_b \rangle. $$

The ideal $I < \mathfrak{t}$ is spanned by $P_\mu$ and $Q_a$ and the semisimple factor $s$ by $L_{\mu\nu}$ and $L_{ab}$. Letting $P^\mu$ and $Q^a$ denote the canonical dual basis for $I^*$, the differential ‘$d$’ on the complex $C^* = C^*(I; \mathfrak{t})^*$ is determined by the following relations:

$$ dP^\mu = \frac{1}{t} \Gamma^\mu_{\alpha\beta} Q^\alpha \wedge Q^\beta, $$

$$ dQ^a = 0, $$

$$ dP_\mu = 0, $$

$$ dQ_a = -\Gamma^\mu_{\alpha\beta} Q^\beta \otimes P_\mu, $$

$$ dL_{\mu\nu} = \eta_{\mu\nu} P^\rho \otimes P_\rho - \eta_{\mu\nu} P^\rho \otimes P_\rho + \frac{1}{2} Q^\rho \otimes \Gamma_{\mu\nu} \cdot Q_a, $$

$$ dL_{ab} = \frac{1}{2} Q^\rho \otimes \Gamma_{ab} \cdot Q_a. $$

There are no nonzero 0-cochains, since $C^0 = \mathfrak{t}^\perp$ and there are no scalars in the superalgebra. There is a three-dimensional space of 1-cochains, spanned by $P^\mu \otimes P_\mu, Q^a \otimes Q_a$ and $Q^a \otimes \nu_w \cdot Q_a$. The first cochain is the identity map $W \to W$, whereas the other two are
linear combinations involving the identity maps of the two irreducible complex representations of $s$ into which the complexification of $\Delta_{D3}$ decomposes. In terms of real maps, we have the identity and the complex structure $n_W$. The space of 2-cochains is also three-dimensional, spanned by the cochains corresponding to the natural isomorphism $\Lambda^2 W \to \mathfrak{so}(W)$ and to its precomposition with the Hodge star $\ast : \Lambda^2 W \to \Lambda^2 W$, as well as to the projection $S^2 \Delta_{D3} \to W$. The corresponding cochains are $P^\mu \wedge P^\nu \otimes L_{\mu\nu}$, $P^\mu \wedge P^\nu \otimes \epsilon_{\mu\nu\rho\sigma} L_{\rho\sigma}$, and $Q^a \wedge Q^b \otimes 1_{ab} P_\mu$.

Computing the differentials $d : C^1 \to C^2$ and $d : C^2 \to C^3$, we find that $H^1(\mathfrak{g}; \mathbb{R}) \cong \mathbb{R}^2$ with representative cocycles $2P^\mu \otimes P_\mu - Q^a \otimes Q_a$ and $Q^a \otimes n_W \cdot Q_a$. The fact that this latter cocycle is a cocycle rests on the skewsymmetry of $(\Gamma^\mu n_W)_{a\beta}$ in $\alpha \leftrightarrow \beta$. Similarly, we find that $H^2(f ; \mathbb{R}) = 0$, whence the D3-brane superalgebra is rigid.

This rigidity might seem a little unexpected due to the fact that the four-dimensional Poincaré superalgebra admits a deformation [23, 24]. The calculation that the anti-de Sitter superalgebra is the unique deformation of the four-dimensional Poincaré superalgebra had been announced in [25], but the expression of the deformed algebra in that paper is incorrect. The calculation in [24] is correct, but we find that the way of writing the algebra is perhaps not the most transparent. For this reason, we present this calculation in the following section.

### 3.5.1. A deformation of the four-dimensional Poincaré superalgebra

We choose to do the calculation using two-component spinor language, which simplifies many of the calculations. Our conventions are taken from [26, appendix B]. The generators of the Poincaré superalgebra $\mathfrak{p} = Q_a, \bar{Q}_a, P_{\alpha\beta}, L_{\alpha\beta}$, with Lie brackets

\begin{align}
[L_{\alpha\beta}, Q_\gamma] &= \epsilon_{\beta\gamma} Q_\alpha + \epsilon_{\alpha\gamma} Q_\beta, \\
[L_{\alpha\beta}, P_{\gamma\alpha}] &= \epsilon_{\beta\gamma} P_{\alpha\alpha} + \epsilon_{\alpha\gamma} P_{\beta\alpha}, \\
[Q_\alpha, \bar{Q}_\beta] &= P_{\alpha\beta},
\end{align}

(34)

together with the conjugate versions of the first two brackets, obtained from those by the replacements $\epsilon_{\alpha\beta} \mapsto \bar{\epsilon}_{\alpha\beta}, L_{\alpha\beta} \mapsto \bar{L}_{\alpha\beta}, P_{\alpha\beta} \mapsto \bar{P}_{\alpha\beta}$ and $Q_\alpha \mapsto \bar{Q}_\alpha$.

Let $I$ be the ideal spanned by $P_{\alpha\alpha}, Q_\alpha$ and $\bar{Q}_\alpha$ and $s$ the semisimple factor spanned by $L_{\alpha\beta}$ and $\bar{L}_{\alpha\beta}$. We denote by $P^{\alpha\alpha}$, $Q^\alpha$ and $\bar{Q}^\alpha$ the canonical dual basis for $I^*$.

The differential in the complex $C^* := C^*(I; \mathfrak{p})^s$ is determined by the following relations:

\begin{align}
d P^{\alpha\alpha} &= Q^\alpha \wedge \bar{Q}^\alpha, \\
d Q^\alpha &= 0, \\
d P_{\alpha\alpha} &= 0, \\
d Q_a &= -\bar{Q}_a \otimes P_{\alpha\alpha}, \\
d L_{\alpha\beta} &= -\epsilon_{\alpha\beta} P_{\gamma\gamma} \otimes P_{\rho\rho} - \epsilon_{\beta\gamma} P_{\gamma\rho} \otimes P_{\alpha\gamma} + \epsilon_{\alpha\gamma} Q^\gamma \otimes Q_\beta + \epsilon_{\beta\gamma} Q^\beta \otimes Q_\alpha,
\end{align}

(35)

and their conjugates.

There are no 0-cochains since there are no Lorentz scalars in the algebra. The space of 1-cochains is three-dimensional, spanned by $Q^\alpha \otimes Q_a, \bar{Q}^\alpha \otimes \bar{Q}_a$ and $P^{\alpha\alpha} \otimes P_{\alpha\alpha}$. The space of 2-cochains is seven-dimensional and spanned by $P^{\alpha\alpha} \otimes P_{\beta\beta} \otimes \epsilon_{\alpha\beta} \bar{L}_{\alpha\beta}, P^{\alpha\alpha} \otimes P_{\beta\beta} \otimes \bar{L}_{\alpha\beta}, P^{\alpha\alpha} \otimes \bar{Q}^\alpha \otimes \epsilon_{\alpha\beta} \bar{Q}_a, P^{\alpha\alpha} \otimes \bar{Q}^\alpha \otimes P_{\alpha\beta}, Q^\alpha \otimes \bar{Q}_a, Q^\alpha \otimes Q_a, \bar{Q}^\alpha \otimes L_{\alpha\beta}$ and $\bar{Q}^\alpha \otimes \bar{Q}_a \otimes L_{\alpha\beta}$.

Computing the differential $d : C^1 \to C^2$ we find a two-dimensional space of cocycles, spanned by $2P^{\alpha\alpha} \otimes P_{\alpha\alpha} - Q^\alpha \otimes Q_a - \bar{Q}^\alpha \otimes \bar{Q}_a$ and $Q^\alpha \otimes Q_a - \bar{Q}^\alpha \otimes \bar{Q}_a$. Strictly speaking the latter cocycle is not real, so we would have to multiply by ‘$i$’ in order to make it real. It corresponds to the map on spinors induced by multiplication with the volume form; that is, $\gamma_5$ in old money. Computing the differential $d : C^2 \to C^3$ we also find a two-dimensional space
of cocycles, spanned by $2P^\alpha_\beta \wedge Q^\delta \otimes \epsilon a_\beta \dot{Q}_a + Q^\alpha \wedge Q^\beta \otimes L_{a\beta}$ and its conjugate. Since we are interested in deformations of the real form of the Lie superalgebra, we choose the real part of the cocycle; that is,

$$2P^\alpha_\beta \wedge Q^\delta \otimes \epsilon a_\beta \dot{Q}_a + 2P^\alpha_\beta \wedge Q^\delta \otimes \epsilon a_\beta Q_a + Q^\alpha \wedge Q^\beta \otimes L_{a\beta} + Q^a \wedge \dot{Q}^\delta \otimes L_{a\beta},$$

(36) corresponding to the following Lie brackets to first order in the deformation parameter $t$:

$$[P_{aa}, Q_\beta] = -t\epsilon a_\beta \dot{Q}_a, \quad [Q_a, Q_\beta] = -tL_{a\beta}, \quad [Q_a, \dot{Q}_\beta] = P_{a\beta},$$

(37) and their conjugates. There is an obstruction to integrating this deformation at the next order, which requires introducing the bracket

$$[P_{aa}, P_{\beta\beta}] = t^2\epsilon a_\beta L_{a\beta} + t^2\epsilon a_\beta L_{a\beta}.$$

(38) The above brackets, together with the ones involving the Lorentz generators, define a one-parameter family of deformations, first written down in [23], corresponding to the $\text{AdS}_4$ superalgebra. The algebraic reason why this deformation of the four-dimensional Poincaré superalgebra does not lift to a deformation of the $\text{D3-brane}$ superalgebra is that the ten-dimensional chirality of the IIB spinors forbids the necessary extra terms in the $QQ$ bracket.

3.6. Rigidity of the $D5$-brane superalgebra

The Killing superalgebra of the type IIB $D5$-brane is the subsuperalgebra $\mathfrak{k} = \mathfrak{t}_0 \oplus \mathfrak{t}_1$ of the type IIB Poincaré superalgebra with $\mathfrak{t}_0 = \mathfrak{so}(W) \oplus \mathfrak{W} \oplus \mathfrak{so}(W^\perp)$, where $V = W \oplus W^\perp$ and $W$ is a six-dimensional Lorentzian subspace, and $\mathfrak{t}_1$ is isomorphic to the subspace $\Delta_{D5} \subset \Delta_6 \oplus \Delta_s$, defined as the graph of the volume form $\nu_W : \Delta_6 \oplus \Delta_s$, which obey $\nu_W^2 = +1$ and is skew-symmetric relative to the spinor inner product when extended to a Clifford endomorphism.

Let $e_a$ and $e_a$ span $W$ and $W^\perp$, respectively, and $e_a$ span $\Delta_s$. Let $\psi_a = \frac{1}{\sqrt{2}}(\epsilon_a e_a)$ be a basis for $\Delta_{D5}$. The corresponding basis of $\mathfrak{k}$ is given by $P_\mu, L_{\mu\nu}, L_{ab}$ and $Q_a$. The Lie brackets are inherited from those in equation (14) and are given explicitly, in addition to those involving $L_{\mu\nu}$ and $L_{ab}$, by

$$[Q_a, Q_\beta] = \Gamma^\mu_{a\beta} P_\mu,$$

(39) where

$$\Gamma^\mu_{a\beta} := \langle \psi_a, \Gamma^\mu \psi_\beta \rangle = \langle e_a, \Gamma^\mu e_\beta \rangle.$$

(40) The ideal $I < \mathfrak{k}$ is spanned by $P_\mu$ and $Q_a$ and the semisimple factor $s$ by $L_{\mu\nu}$ and $L_{ab}$. Letting $P^\mu$ and $Q^\alpha$ denote the canonical dual basis for $I^*$, the differential ‘$d$’ on the complex $C^* = C^*(I; \mathfrak{k})$ is determined by the following relations:

$$dP^\mu = \frac{1}{2}\Gamma^\mu_{a\beta} Q^a \wedge Q^\beta,$$

$$dQ^\alpha = 0,$$

$$dP_\mu = 0,$$

$$dQ_{\alpha} = -\Gamma^\mu_{a\beta} Q^\beta \otimes P_\mu,$$

$$dL_{\mu\nu} = \eta_{\mu\nu} P^\alpha \otimes P_\mu - \eta_{\mu\nu} P^\alpha \otimes P_\mu + \frac{1}{2} Q^\alpha \otimes \Gamma^\mu_{a\beta} Q_a,$$

$$dL_{ab} = \frac{1}{2} Q^\alpha \otimes \Gamma^\mu_{a\beta} Q_a.$$

There are no nonzero 0-cochains, since $C^0 = \mathfrak{k}^*$ and there are no scalars in the superalgebra. There is a three-dimensional space of 1-cochains, spanned by $P^\mu \otimes P_\mu, Q^\alpha \otimes Q_a$ and $Q^\alpha \otimes \nu_W, Q_a$. The space of 2-cochains is also three-dimensional, spanned by the following cochains: $P^\mu \wedge P^\nu \otimes L_{\mu\nu}, Q^\alpha \wedge Q^\beta \otimes \Gamma^\mu_{a\beta} P_\mu$ and $Q^\alpha \wedge Q^\beta \otimes (\nu_W)_{a\beta} P_\mu$. 
Computing the differentials $d : C^1 \to C^2$ and $d : C^2 \to C^3$, we find that $H^1(\mathfrak{t}; \mathfrak{t}) \cong \mathbb{R}$ with representative cocycle $2P^\mu \otimes P_\mu - Q^\alpha \otimes Q_\alpha$, whence dim $B^2 = 2$. Since $d : C^2 \to C^3$ is not identically zero, we conclude that $H^2(\mathfrak{t}; \mathfrak{t}) = 0$, whence the D5-brane superalgebra is rigid.

### 3.7. Rigidity of the NS5-brane superalgebra

The Killing superalgebra of the type IIB NS5-brane is the subsuperalgebra $\mathfrak{k} = \mathfrak{h}_0 \oplus \mathfrak{h}_1$ of the IIB Poincaré superalgebra where $\mathfrak{h}_0 = \mathfrak{so}(W) \oplus W \oplus \mathfrak{so}(W^\perp)$, corresponding to a decomposition $V = W \oplus W^\perp$ where $W$ is Lorentzian and six-dimensional, corresponding to the brane worldvolume. The odd subspace $\mathfrak{h}_1$ is isomorphic to the subspace $\Delta_{NS5} \subset \Delta_+ \oplus \Delta_-$, given by

$$\Delta_{NS5} = \left\{ \left( \frac{\epsilon_1}{\epsilon_2} \right) \in \Delta_+ \oplus \Delta_- \mid \nu_W \left( \frac{\epsilon_1}{\epsilon_2} \right) = \left( -\frac{\epsilon_1}{\epsilon_2} \right) \right\},$$

where the Clifford endomorphism $\nu_W$ corresponding to the volume form of the brane worldvolume is skew-symmetric relative to the spinor inner product and obeys $\nu_W^2 = +1$.

Let $e_a$ and $e_\alpha$ span $W$ and $W^\perp$, respectively, and $P_\mu$, $L_\mu\nu$, and $L_{ab}$ be the generators of $\mathfrak{h}_0$. Let $e_a$ and $\bar{e}_a$ be basis elements for the subspaces of $\Delta_+$ satisfying $\nu_W e_a = -\bar{e}_a$ and $\nu_W \bar{e}_a = e_a$, respectively, so that $(e_a)$ and $(\bar{e}_a)$ span $\Delta_{F1}$. Let $Q_a$ and $\bar{Q}_a$ denote the corresponding basis for $\mathfrak{k}_1$. The nonzero Lie brackets in this basis are given, in addition to those of $\mathfrak{h}_0$, by

$$[L_\mu\nu, Q_a] = \frac{1}{2} \Gamma_{\mu\nu} \cdot Q_a, \quad [L_\mu\nu, \bar{Q}_a] = -\frac{1}{2} \Gamma_{\mu\nu} \cdot \bar{Q}_a,$$

$$[L_{ab}, Q_a] = \frac{1}{2} \Gamma_{ab} \cdot Q_a, \quad [L_{ab}, \bar{Q}_a] = -\frac{1}{2} \Gamma_{ab} \cdot \bar{Q}_a,$$

$$[Q_a, Q_\beta] = \Gamma^\mu_{a\beta} P_\mu, \quad [\bar{Q}_a, \bar{Q}_\beta] = -\Gamma^\mu_{a\beta} P_\mu,$$

where, as before,

$$\Gamma^\mu_{a\beta} := \langle e_a, \Gamma^\mu e_\beta \rangle \quad \text{and} \quad \Gamma^\mu_{\bar{a}\bar{b}} := \langle \bar{e}_a, \Gamma^\mu \bar{e}_\beta \rangle.$$

The ideal $I < \mathfrak{k}$ is spanned by $P_\mu, Q_a$ and $\bar{Q}_a$ and the differential in the complex $C^\ast := C^\ast(I; \mathfrak{k})$ of cochains which are invariant under the semisimple subalgebra $\mathfrak{s} := \mathfrak{so}(W) \oplus \mathfrak{so}(W^\perp)$ is determined uniquely by the following relations:

$$dP^\mu = \frac{1}{2} \Gamma^\mu_{a\beta} Q^a \wedge Q^\beta + \frac{1}{2} \Gamma^\mu_{\bar{a}\bar{b}} \bar{Q}^\bar{a} \wedge \bar{Q}^{\bar{b}},$$

$$dQ_a = 0,$$

$$d\bar{Q}_a = 0,$$

$$dP_\mu = 0,$$

$$dQ_a = -\Gamma^\mu_{a\beta} Q^\beta \otimes P_\mu,$$

$$d\bar{Q}_a = -\Gamma^\mu_{\bar{a}\bar{b}} \bar{Q}^{\bar{b}} \otimes P_\mu,$$

$$dL_\mu\nu = \eta_{\mu\rho} P^\rho \otimes P_\nu - \eta_{\nu\rho} P^\rho \otimes P_\mu + \frac{1}{2} Q^\alpha \otimes \Gamma_{\mu\nu} \cdot Q_\alpha + \frac{1}{2} \bar{Q}^\bar{\alpha} \otimes \Gamma_{\mu\nu} \cdot \bar{Q}_{\bar{\alpha}},$$

$$dL_{ab} = \frac{1}{2} Q^\alpha \otimes \Gamma_{ab} \cdot Q_\alpha + \frac{1}{2} \bar{Q}^\bar{\alpha} \otimes \Gamma_{ab} \cdot \bar{Q}_{\bar{\alpha}}.$$

There are no 0-cochains. The space $C^1$ is three-dimensional and is spanned by the cochains corresponding to the identity maps $P^\mu \otimes P_\mu, Q^\alpha \otimes Q_\alpha$ and $\bar{Q}^{\bar{\alpha}} \otimes \bar{Q}_{\bar{\alpha}}$. The space $C^2$ of 2-cochains is three-dimensional, spanned by $P^\mu \wedge P^\nu \otimes L_{\mu\nu}, Q^\alpha \wedge Q^\beta \otimes \Gamma^\mu_{\alpha\beta} P_\mu$ and $\bar{Q}^{\bar{\alpha}} \wedge \bar{Q}^{\bar{\beta}} \otimes \Gamma^\mu_{\bar{a}\bar{b}} P_\mu$. 

Computing the differential \( d : C^1 \to C^2 \), we find that \( H^1(\mathfrak{t}; \mathfrak{t}) \cong \mathbb{R} \), with representative cocycle \( \varphi := 2P^\mu \otimes P_\mu - Q^a \otimes Q_a - \bar{Q}^\mu \otimes \bar{Q}_a \). Since \( d : C^2 \to C^3 \) is not the zero map, we conclude that \( H^2(\mathfrak{t}; \mathfrak{t}) = 0 \), proving the rigidity of the IIB NS5 superalgebra.

### 3.8. A deformation of the D7-brane superalgebra

The Killing superalgebra of the D7-brane is the subsuperalgebra \( \mathfrak{t} \) of the IIB Poincaré superalgebra with \( \mathfrak{t}_0 = \mathfrak{so}(W) \oplus \mathfrak{so}(W^\perp) \), where \( V = W \oplus W^\perp \) is the decomposition of the ten-dimensional Lorentzian vector space into an eight-dimensional Lorentzian subspace \( W \), corresponding to the brane worldvolume and its two-dimensional perpendicular complement. The odd subspace \( \mathfrak{t}_1 \) is isomorphic to the graph \( \Delta_{D7} \subset \Delta_+ \oplus \Delta_+ \) of the endomorphism \( \nu_W : \Delta_+ \to \Delta_+ \), corresponding to the volume form of \( W \). As in the D3-brane, the extension of \( \nu_W \) to the Clifford module is symmetric relative to the spinor inner product and obeys \( \nu_W^2 = -1 \).

Let \( e_\mu \) and \( e_a \) span \( W \) and \( W^\perp \), respectively, and \( \varepsilon_a \) span \( \Delta_+ \). Let \( \psi_{\alpha} = \frac{1}{\sqrt{2}}(\varepsilon_n \varepsilon_n) \) be a basis for \( \Delta_{D7} \). The corresponding basis of \( \mathfrak{t} \) is given by \( P_\mu, L_{\mu\nu}, L_{ab} = \varepsilon_a L_b \) and \( Q_a \). The Lie brackets are inherited from those in equation (14) and are given explicitly, in addition to those involving the Lorentz subalgebra of the brane worldvolume, by

\[
[L, Q_a] = -\frac{1}{2} \nu_W \cdot Q_a, \quad [Q_a, Q_\beta] = \Gamma^{\mu}_{\alpha\beta} P_\mu,
\]

where

\[
\Gamma^{\mu}_{\alpha\beta} := \langle \psi_{\alpha}, \Gamma^\mu \psi_{\beta} \rangle = (e_\alpha, \Gamma^\mu e_\beta).
\]

The ideal \( I < \mathfrak{t} \) includes the generator \( L \), but we may work with cochains which are invariant under the reductive subalgebra \( \mathfrak{r} := \mathfrak{so}(W) \oplus \mathfrak{so}(W^\perp) \). Letting \( L^*, P^a \) and \( Q^a \) be a basis for \( \mathfrak{r}^* \), the differential in the complex \( C^* = C^*(I, \mathfrak{r})^* \) is determined by the following relations:

\[
\begin{align*}
\text{d}P^\mu &= \frac{1}{2} \Gamma^\mu_{\alpha\beta} Q^\alpha \wedge Q^\beta, \\
\text{d}Q^a &= \frac{1}{2} L^* \wedge (\nu_W)^\alpha \beta Q^\beta, \\
\text{d}L^* &= 0, \\
\text{d}P_\mu &= 0, \\
\text{d}Q_a &= -\frac{1}{2} L^* \otimes \nu_W \cdot Q_a - \Gamma^\mu_{\alpha\beta} Q^\beta \otimes P_\mu, \\
\text{d}L &= -\frac{1}{2} Q^a \otimes \nu_W \cdot Q_a, \\
\text{d}L_{\mu\nu} &= \eta_{\mu\nu} P^\rho \otimes P_\rho - \eta_{\nu\rho} P^\rho \otimes P_\mu + \frac{1}{2} Q^a \otimes \Gamma^\mu_{\alpha\beta} P_\beta.
\end{align*}
\]

The 0-cochains \( C^0 = \mathfrak{t}^* \) are spanned by \( L \), but \( dL \neq 0 \), hence \( H^0(\mathfrak{t}; \mathfrak{t}) \cong \mathbb{R} \) and \( \text{dim} B^1 = 1 \). The space of 1-cochains is four-dimensional, spanned by \( P^\mu \otimes P_\mu, L^* \otimes L_\mu, Q^a \otimes Q_a \) and \( Q^a \otimes \nu_W \cdot Q_a \). The space of 2-cochains is five-dimensional, spanned by \( P^\mu \wedge P^\nu \otimes L_{\mu\nu}, L^* \wedge P^a \otimes P_\mu, Q^a \wedge Q^\mu \otimes Q_a, Q^a \wedge Q^\mu \wedge \nu_W \otimes Q_a \) and \( Q^a \wedge Q^\mu \wedge \nu_W \otimes \Gamma^\mu_{\alpha\beta} P_\beta \).

Computing the differential \( d : C^1 \to C^2 \), we find that \( H^1(\mathfrak{t}; \mathfrak{t}) \cong \mathbb{R} \), with representative cocycle \( \varphi := 2P^\mu \otimes P_\mu - Q^a \otimes Q_a - \bar{Q}^\mu \otimes \bar{Q}_a \). Similarly, computing \( d : C^2 \to C^3 \) we find that \( H^2(\mathfrak{t}; \mathfrak{t}) \cong \mathbb{R} \), with representative cocycle \( L^* \wedge \varphi \). This infinitesimal deformation integrates to a one-parameter family of Lie superalgebras with brackets

\[
[L, Q_a] = \iota Q_a - \frac{1}{2} \nu_W \cdot Q_a, \quad [L, P_\mu] = 2 \iota P_\mu, \quad [Q_a, Q_\beta] = \Gamma^\mu_{\alpha\beta} P_\mu,
\]

in addition to those involving \( \mathfrak{so}(W) \), which remain undeformed.
4. Type IIA backgrounds

In this section, we explore the Lie superalgebra deformations of the Killing superalgebras of certain type IIA backgrounds. We start with the Minkowski vacuum and then go on to the elementary brane backgrounds.

4.1. Rigidity of the Poincaré superalgebra

The Killing superalgebra $\mathfrak{P}$ of the unique maximally supersymmetric solution of type IIA supergravity is the IIA Poincaré superalgebra, which extends the ten-dimensional Poincaré algebra $\mathfrak{so}(V) \oplus V$ by supercharges transforming in the spinorial representation $\Delta_+ \oplus \Delta_-$ of $\mathfrak{so}(V)$. Let $\varepsilon_\mu$ denote an orthonormal basis for $V$ and $P_\mu$ and $L_{\mu\nu}$ the corresponding basis for the Poincaré algebra. Let $\varphi_\alpha$ and $\bar{\varphi}_\dot{\alpha}$ be a basis for $\Delta_+$, respectively, and $Q_\alpha$ and $\bar{Q}_{\dot{\alpha}}$ the corresponding basis for the odd subspace of the Poincaré superalgebra. In this basis, the nonzero Lie brackets are, in addition to those of the Poincaré algebra, the following:

$$[L_{\mu\nu}, Q_\alpha] = \frac{1}{2} \Gamma_{\mu\nu} \cdot Q_\alpha, \quad [L_{\mu\nu}, \bar{Q}_{\dot{\alpha}}] = \frac{1}{2} \Gamma_{\mu\nu} \cdot \bar{Q}_{\dot{\alpha}},$$

$$[Q_\alpha, Q_\beta] = \Gamma^\mu_{\alpha\beta} P_\mu, \quad [Q_\alpha, \bar{Q}_{\dot{\beta}}] = \Gamma^\mu_{\alpha\dot{\beta}} P_\mu,$$  \hfill (50)

where, as usual,

$$\Gamma^\mu_{\alpha\beta} := \langle \varepsilon_\alpha, \Gamma^\mu \varepsilon_\beta \rangle \quad \text{and} \quad \Gamma^\mu_{\alpha\dot{\beta}} := \langle \bar{\varepsilon}_{\dot{\alpha}}, \Gamma^\mu \bar{\varepsilon}_{\dot{\beta}} \rangle.$$  \hfill (51)

The supertranslation ideal is spanned by $P_\mu$, $Q_\alpha$, and $\bar{Q}_{\dot{\alpha}}$ and the semisimple algebra is the Lorentz subalgebra $\mathfrak{g} = \mathfrak{so}(V)$. The complex computing the deformations is $C^* = C^*(I; \mathfrak{P})$ and its differential is determined uniquely by its action on the above basis for $\mathfrak{P}$ and the canonical dual basis $P^\mu$, $Q^\alpha$ and $\bar{Q}^{\dot{\alpha}}$ for $I^*$:

$$dP^\mu = \frac{1}{2} \Gamma^\mu_{\alpha\beta} Q^\alpha \wedge Q^\beta + \frac{1}{2} \Gamma^\mu_{\dot{\alpha}\dot{\beta}} \bar{Q}^{\dot{\alpha}} \wedge \bar{Q}^{\dot{\beta}},$$

$$dQ^\alpha = 0,$$

$$d\bar{Q}^{\dot{\alpha}} = 0,$$

$$dP_\mu = 0,$$

$$dQ_\alpha = -\Gamma^\mu_{\alpha\beta} Q^\beta \otimes P_\mu,$$

$$d\bar{Q}_{\dot{\alpha}} = -\Gamma^\mu_{\alpha\dot{\beta}} \bar{Q}^{\dot{\beta}} \otimes P_\mu,$$

$$dL_{\mu\nu} = \eta_{\mu\nu} P^P \otimes P_\nu - \eta_{\mu\nu} P^P \otimes P_\mu + \frac{1}{2} Q^\alpha \otimes \Gamma_{\mu\nu} \cdot Q_\alpha + \frac{1}{2} \bar{Q}^{\dot{\alpha}} \otimes \Gamma_{\mu\nu} \cdot \bar{Q}_{\dot{\alpha}}.$$  \hfill (52)

There are no 0-cochains and the space of 1-cochains is three-dimensional, spanned by $P^\mu \otimes P_\mu$, $Q^\alpha \otimes Q_\alpha$ and $\bar{Q}^{\dot{\alpha}} \otimes \bar{Q}_{\dot{\alpha}}$, corresponding to the identity maps $V \to V$ and $\Delta_+ \to \Delta_+$. The space $C^2$ of 2-cochains is six-dimensional, spanned by $P^\mu \wedge P^\nu \otimes L_{\mu\nu}$, $P^\mu \wedge Q^\alpha \otimes \Gamma_{\mu\nu} \cdot Q_\alpha$, $P^\mu \wedge \bar{Q}^{\dot{\alpha}} \otimes \Gamma_{\mu\nu} \cdot \bar{Q}_{\dot{\alpha}}$, $Q^\alpha \wedge Q^\beta \otimes \Gamma^\mu_{\alpha\beta} P_\mu$, $\bar{Q}^{\dot{\alpha}} \wedge \bar{Q}^{\dot{\beta}} \otimes \Gamma^\mu_{\alpha\dot{\beta}} P_\mu$ and $Q^\alpha \wedge \bar{Q}^{\dot{\beta}} \otimes \Gamma^\mu_{\alpha\dot{\beta}} L_{\mu\nu}$. These cochains correspond to the isomorphism $\Lambda^2 V \to \mathfrak{so}(V)$, Clifford multiplication $V \otimes \Delta_+ \to \Delta_+$ and the projections $S^2 \Delta_+ \to V$ and $\Delta_+ \otimes \Delta_- \to \Lambda^2 V$.

Computing the differential $d : C^1 \to C^2$, we find

$$d(P^\mu \otimes P_\mu) = \frac{1}{2} \Gamma^\mu_{\alpha\beta} Q^\alpha \wedge Q^\beta \otimes P_\mu + \frac{1}{2} \Gamma^\mu_{\dot{\alpha}\dot{\beta}} \bar{Q}^{\dot{\alpha}} \wedge \bar{Q}^{\dot{\beta}} \otimes P_\mu,$$

$$d(Q^\alpha \otimes Q_\alpha) = \Gamma^\mu_{\alpha\beta} Q^\alpha \wedge Q^\beta \otimes P_\mu,$$

$$d(\bar{Q}^{\dot{\alpha}} \otimes \bar{Q}_{\dot{\alpha}}) = \Gamma^\mu_{\alpha\dot{\beta}} \bar{Q}^{\dot{\alpha}} \wedge \bar{Q}^{\dot{\beta}} \otimes P_\mu,$$  \hfill (53)

so that $H^1(\mathfrak{P}; \mathfrak{P}) \cong \mathbb{R}$, with representative cocycle $2P^\mu \otimes P_\mu - Q^\alpha \otimes Q_\alpha - \bar{Q}^{\dot{\alpha}} \otimes \bar{Q}_{\dot{\alpha}}$. This implies that dim $B^2 = 2$, spanned by $d(Q^\alpha \otimes Q_\alpha)$ and $d(\bar{Q}^{\dot{\alpha}} \otimes \bar{Q}_{\dot{\alpha}})$, say.
Computing the differential \( d : C^2 \to C^3 \), we find, in addition to the coboundaries,
\[
d(P^\mu \wedge P^\nu \otimes L_{\mu\nu}) = \Gamma^\mu_{\alpha\beta} P^\nu \wedge Q^\alpha \otimes L_{\mu\nu} + \Gamma^\mu_{\alpha\beta} P^\nu \wedge \bar{Q}^\beta \otimes L_{\mu\nu} \]
\[
+ \frac{1}{2} P^\mu \wedge P^\nu \wedge \bar{Q}^\beta \otimes \Gamma_{\mu\nu} \cdot Q_a + \frac{1}{2} P^\mu \wedge P^\nu \wedge \bar{Q}^\beta \otimes \Gamma_{\mu\nu} \cdot \bar{Q}_a.
\]
\[
d(P^\mu \wedge Q^\nu \otimes \Gamma_{\mu} \cdot Q_a) = \frac{1}{2} \Gamma^\mu_{\alpha\beta} Q^\beta \wedge Q^\gamma \otimes \Gamma_{\mu} \cdot Q_a + \frac{1}{2} \frac{1}{2} \Gamma^\mu_{\alpha\beta} \bar{Q}^\beta \wedge \bar{Q}^\gamma \otimes \Gamma_{\mu} \cdot Q_a
\]
\[
- (\Gamma_{\mu})^\beta_{\alpha} \Gamma^\mu_{\beta\gamma} P^\alpha \wedge Q^\gamma \otimes P_v,
\]
\[
d(P^\mu \wedge \bar{Q}^\beta \otimes \Gamma_{\mu} \cdot \bar{Q}_a) = \frac{1}{2} \Gamma^\mu_{\alpha\beta} Q^\beta \wedge Q^\gamma \otimes \bar{Q}_a - \frac{1}{2} \frac{1}{2} \Gamma^\mu_{\alpha\beta} \bar{Q}^\beta \wedge \bar{Q}^\gamma \otimes \bar{Q}_a
\]
\[
- (\Gamma_{\mu})^\beta_{\alpha} \Gamma^\mu_{\beta\gamma} P^\alpha \wedge \bar{Q}^\gamma \otimes P_v,
\]
\[
d(Q^\alpha \wedge \bar{Q}^\beta \otimes \Gamma_{\mu\nu} L_{\mu\nu}) = 2 \eta_{\mu\nu} \Gamma^\alpha_{\beta} Q^\alpha \wedge \bar{Q}^\beta \otimes P_v + \frac{1}{2} \Gamma^\nu_{\alpha\beta} Q^\alpha \wedge \bar{Q}^\beta \wedge \bar{Q}_a \otimes \Gamma_{\mu\nu} \cdot Q_v
\]
\[
+ \frac{1}{2} \Gamma^\nu_{\alpha\beta} Q^\alpha \wedge \bar{Q}^\beta \wedge \bar{Q}_a \otimes \Gamma_{\mu\nu} \cdot \bar{Q}_v.
\]
(54)

It is clear that any possible cocycle must be a linear combination of the last three cochains, since the differential of the first cochain is the only one having explicit dependence on \( L_{\mu\nu} \). Therefore, let
\[
\Theta = a_1 P^\mu \wedge Q^\nu \otimes \Gamma_{\mu} \cdot Q_a + a_2 P^\mu \wedge \bar{Q}^\beta \otimes \Gamma_{\mu} \cdot \bar{Q}_a + a_3 Q^\alpha \wedge \bar{Q}^\beta \otimes \Gamma_{\mu\nu} L_{\mu\nu},
\]
(55)

and consider the equation \( d\Theta = 0 \). The coefficient of the monomial \( P^\mu \wedge Q^\nu \wedge \bar{Q}^\beta \otimes P_v \) is given by
\[
-a_1 (\Gamma_{\mu})^\beta_{\alpha} a \Gamma^\nu_{\beta\gamma} - a_2 (\Gamma_{\mu})^\beta_{\alpha} a \Gamma^\nu_{\beta\gamma} + 2 a_3 (\Gamma_{\mu})^\nu_{\beta\gamma}.
\]
(56)

Now,
\[
(\Gamma_{\mu})^\beta_{\alpha} a \Gamma^\nu_{\beta\gamma} = (\bar{e}_\beta, \Gamma^\nu_{\beta\gamma} e_a)
\]
\[
= -(e_a, \Gamma^\nu_{\beta\gamma} \bar{e}_\gamma)
\]
\[
= -(\Gamma^\nu_{\beta\gamma})_{\alpha} a \gamma_{\beta} (e_a, \bar{e}_\gamma),
\]
(57)

and similarly
\[
(\Gamma_{\mu})^\nu_{\beta\gamma} (\Gamma_{\mu})^\nu_{\beta\gamma} = -(\Gamma^\nu_{\beta\gamma})_{\alpha} a \gamma_{\beta} (e_a, \bar{e}_\gamma),
\]
(58)

whence the expression in equation (56) becomes
\[
(a_1 + a_2 + 2 a_3) (\Gamma_{\mu})^\nu_{\beta\gamma} a \gamma_{\beta} (e_a, \bar{e}_\gamma) + (a_1 - a_2) \delta_{\mu} (e_a, \bar{e}_\gamma),
\]
(59)

which vanishes if and only if \( a_1 = a_2 = a_3 \). In other words, the only possible (nontrivial) cocycle must be proportional to
\[
\Theta' = P^\mu \wedge Q^\nu \otimes \Gamma_{\mu} \cdot Q_a + P^\mu \wedge \bar{Q}^\beta \otimes \Gamma_{\mu} \cdot \bar{Q}_a - Q^\nu \wedge \bar{Q}^\beta \otimes \Gamma_{\mu\nu} L_{\mu\nu},
\]
(60)

whose differential \( d\Theta' \) has terms of four types: \( QQ \otimes \bar{Q}, \bar{Q} \bar{Q} Q \otimes Q, Q \bar{Q} \bar{Q} \otimes \bar{Q} \) and \( QQ \bar{Q} \otimes Q \). As we now show, the first two terms vanish. It will suffice to see this for the first term, which has the form
\[
\frac{1}{2} Q^\alpha \wedge Q^\beta \otimes \bar{Q}^\gamma \otimes \Gamma_{\mu\nu} L_{\mu\nu}.
\]
(61)

By the usual polarization identity, this will vanish if and only if for all \( \varepsilon \in \Delta_+ \),
\[
(\varepsilon, \Gamma^\mu \varepsilon) \Gamma_{\mu} \varepsilon = 0,
\]
(62)

which says that the Clifford multiplication by the Dirac current of a chiral spinor, which is null in ten dimensions, annihilates the spinor. This is known to be true, as proved, for instance in [11, appendix A]. The second term vanishes for precisely the same reasons, except that
Deformations of ten-dimensional Killing superalgebras

Poincaré superalgebra

The Killing superalgebra of the D0-brane background is the subsuperalgebra of the IIA

massless limit to Minkowski spacetime.

massive IIA theory of [27], but our rigidity results imply that none exists which tends in the

demonstrated in [14] for the Romans theory. We are not aware of a similar result for the

non-existence of maximally supersymmetric backgrounds for massive supergravities, which

in that limit and hence we would expect to find the massive supergravity background among

whence any background of such a massive supergravity tends to a IIA supergravity background

deforations are such that as the mass parameter tends to zero one recovers IIA supergravity,

ε ∈ Δ− now. It remains to investigate the \( Q \bar{Q} \bar{Q} \otimes \bar{Q} \) and \( Q \bar{Q} Q \otimes Q \) terms. It will suffice to

analyse the former term, say, which is given by

\[
\frac{1}{2} Q^\alpha \wedge \bar{Q}^\beta \wedge Q^\delta \otimes (\Gamma_{\mu}^\alpha (\Gamma_{\mu}^\beta )^0, - \Gamma_{\mu\alpha} (\Gamma_{\mu\beta})^0 ).
\]

(63)

Again, using a polarization identity, this term will vanish if and only if the following identity holds:

\[
(\bar{e}, \Gamma^\mu e) \Gamma^\mu \bar{e} = (\bar{e}, \Gamma^{\mu\nu} e) \Gamma_{\mu\nu} \bar{e} = 0 \quad \forall \bar{e} \in \Delta_+, \ e \in \Delta_-.
\]

(64)

It is not hard to check that this is not true in general, thus proving the rigidity of the IIA

Poincaré superalgebra.

The rigidity of the IIA Poincaré superalgebra might come as a surprise due to the existence

of massive supergravities [27, 28] which deform IIA supergravity by a mass parameter. These

deforations are such that as the mass parameter tends to zero one recovers IIA supergravity,

whence any background of such a massive supergravity tends to a IIA supergravity background

in that limit and hence we would expect to find the massive supergravity background among

the deformations of the IIA background, with a similar situation reflecting itself in their

superalgebras. The fact that the IIA Poincaré superalgebra is rigid is consistent with the

non-existence of maximally supersymmetric backgrounds for massive supergravities, which

is demonstrated in [14] for the Romans theory. We are not aware of a similar result for the

massive IIA theory of [27], but our rigidity results imply that none exists which tends in the

massless limit to Minkowski spacetime.

4.2. Rigidity of the D0-brane superalgebra

The Killing superalgebra of the D0-brane background is the subsuperalgebra of the IIA

Poincaré superalgebra \( \mathfrak{t} = \mathfrak{t}_0 \oplus \mathfrak{t}_1 \), with \( \mathfrak{t}_0 = \mathfrak{w} \oplus \mathfrak{so}(W^\perp) \), with \( \mathfrak{w} \) the Lorentzian line

spanned by \( e_0 \), and \( \mathfrak{t}_1 \) isomorphic to the subspace \( \Delta_{D0} \subset \Delta_+ \oplus \Delta_- \) defined as the graph of

the linear map \( \Gamma^0 : \Delta_+ \rightarrow \Delta_- \). Let \( e_a \) span \( W^\perp \). The corresponding basis for \( \mathfrak{t}_0 \) are \( P \) and

\( L_{ab} \). Let \( \varepsilon_a \) denote a basis for \( \Delta_+ \) and \( \psi_a := \frac{1}{\sqrt{2}} \varepsilon_a \) be a basis for \( \Delta_{D0} \). Let \( Q_a \) denote the

corresponding basis for \( \mathfrak{t}_1 \). The nonzero Lie brackets are those of \( \mathfrak{so}(W^\perp) \) and

\[
[L_{ab}, \ Q_a] = \frac{1}{2} \Gamma_{ab} \cdot Q_a, \quad [Q_a, \ Q_b] = \Gamma^0_{ab} P.
\]

(65)

where

\[
\Gamma^0_{ab} := (\psi_a, \Gamma^0 \psi_b) = (\varepsilon_a, \Gamma^0 \varepsilon_b).
\]

(66)

The ideal \( I \subset \mathfrak{t} \) is spanned by \( P \) and \( Q_a \), whereas the semisimple factor \( \mathfrak{s} = \mathfrak{so}(W^\perp) \) is

spanned by the \( L_{ab} \). Letting \( P^* \) and \( Q^\beta \) denote the canonical dual basis for \( I^* \), the differential

in the deformation complex \( C^* = C^*(I; \mathfrak{t})^2 \) is determined uniquely by the following relations:

\[
dP^* = \frac{1}{2} \Gamma^0_{\alpha\beta} Q^\alpha \wedge Q^\beta , \quad dQ^\alpha = 0, \quad dP = 0 ,
\]

\[
dQ_a = -\Gamma^0_{\alpha\beta} Q^\beta \otimes P, \quad dL_{ab} = \frac{1}{2} Q^\alpha \otimes \Gamma_{ab} \cdot Q_a.
\]

(67)

The space of 0-cochains is spanned by \( P \), whence \( H^0(I; \mathfrak{t}) \cong \mathbb{R} \). There is a two-dimensional

space of 1-cochains, spanned by \( P^* \otimes P \) and \( Q^\beta \otimes Q_{\alpha} \), and a two-dimensional

space of 2-cochains, spanned by \( P^* \wedge Q^\alpha \otimes Q_{\beta} \) and \( Q^\beta \wedge Q^\alpha \otimes \Gamma_{\alpha\beta} P \). Computing

the differential \( d : C^1 \rightarrow C^2 \), we find that \( H^1(I; \mathfrak{t}) \cong \mathbb{R} \), with representative cocycle
$2P^* \otimes P - Q^\alpha \otimes Q_\alpha$. This means that $\dim B^2 = 1$ and since the differential $d : C^2 \to C^3$ is not identically zero, one concludes that $H^3(\mathfrak{t}; \mathfrak{t}) = 0$, thus proving the rigidity of the D0-brane superalgebra.

4.3. A deformation of the fundamental string superalgebra

The Killing superalgebra of the IIA fundamental string solution is the subsuperalgebra $\mathfrak{t} = \mathfrak{t}_0 \oplus \mathfrak{t}_1$ of the IIA Poincaré superalgebra associated with the decomposition of the ten-dimensional Lorentzian vector space $V = W \oplus W^\perp$, where $W$ is a two-dimensional Lorentzian subspace corresponding to the string worldsheet. This means that $\mathfrak{t}_0 = \mathfrak{so}(W) \oplus W \oplus \mathfrak{so}(W^\perp)$ and $\mathfrak{t}_1$ is isomorphic to the subspace $\Delta_{\mathfrak{FI}} \subset \Delta_+ \oplus \Delta_-$ defined by

$$\Delta_{\mathfrak{FI}} = \left\{ (\ell_+ , \ell_- ) \in \Delta_+ \oplus \Delta_- ; |\nu_W \ell_{\pm} = \pm \ell_{\pm} \right\}, \quad (68)$$

where the volume element $\nu_W$ obeys $\nu_W^2 = +1$ and is skewsymmetric relative to the spinor inner product. Alternatively, we may think of chirality under $\mathfrak{so}(2)$ fundamental strings, this can be understood by the fact that the ideal $I$ having degree 0, and $\dim H^0(\mathfrak{t}; \mathfrak{t}) = 1$. Let $e_\alpha$ and $\bar{e}_\alpha$ be a basis for $W$ and $W^\perp$, respectively, and let $e_\alpha$ and $\bar{e}_\alpha$ span $\Delta_{\mathfrak{FI}} \cap \Delta_\pm$, respectively. The corresponding basis for $\mathfrak{t}$ is then $P_\mu$, $L_{\mu \nu} = \epsilon_{\mu \nu \lambda} L_\lambda$, $Q_\alpha$ and $\bar{Q}_\alpha$, with nonzero Lie brackets

$$[L, Q_\alpha] = -\frac{1}{2} Q_\alpha, \quad [L, \bar{Q}_\alpha] = \frac{1}{2} \bar{Q}_\alpha, \quad [L, P_\mu] = \epsilon_\mu^{\nu \lambda} P_\nu, \quad [Q_\alpha, Q_\beta] = \Gamma^\mu_{\alpha \beta} P_\mu, \quad \bar{Q}_\alpha, \bar{Q}_\beta] = \Gamma^\mu_{\alpha \beta} P_\mu, \quad (69)$$

in addition to those of $\mathfrak{so}(W^\perp)$, where

$$\Gamma^\mu_{\alpha \beta} := (\epsilon_\alpha, \Gamma^\mu \epsilon_\beta) \quad \text{and} \quad \Gamma^\mu_{\alpha \beta} := (\bar{e}_\alpha, \Gamma^\mu \bar{e}_\beta). \quad (70)$$

The ideal $I < \mathfrak{t}$ is spanned by $L$, $P_\mu$, $Q_\alpha$ and $\bar{Q}_\alpha$ and the semisimple factor $\mathfrak{s} = \mathfrak{so}(W^\perp)$ by $L_{ab}$. The subalgebra $\mathfrak{t} = \mathfrak{so}(W) \oplus \mathfrak{so}(W^\perp)$ spanned by $L$ and $L_{ab}$ is reductive and the deformation complex $C^* := C^*(I; \mathfrak{t})$ consists of $\mathfrak{t}$-invariant cochains. Letting $L^*$, $P^*$, $Q^*$ and $\bar{Q}^*$ denote the canonical dual basis for $I^*$, the differential in the deformation complex is defined by the following relations:

$$dP_\mu = \frac{1}{2} \Gamma^\mu_{\alpha \beta} Q^\alpha \wedge Q^\beta + \frac{1}{2} \Gamma^\mu_{\alpha \beta} \bar{Q}^\alpha \wedge \bar{Q}^\beta + \epsilon_\mu^{\nu \lambda} L^* \wedge P^\nu, \quad$$
$$dQ_\alpha = \frac{1}{2} L^* \wedge Q^\alpha, \quad$$
$$d\bar{Q}_\alpha = -\frac{1}{2} L^* \wedge \bar{Q}_\alpha, \quad$$
$$dL^* = 0, \quad$$
$$dP_\mu = L^* \wedge \epsilon_\mu^{\nu \lambda} P_\nu, \quad$$
$$dQ_\alpha = -\frac{1}{2} L^* \wedge Q_\alpha - \Gamma^\mu_{\alpha \beta} Q^\beta \wedge P_\mu, \quad$$
$$d\bar{Q}_\alpha = \frac{1}{2} L^* \wedge \bar{Q}_\alpha - \Gamma^\mu_{\alpha \beta} \bar{Q}^\beta \wedge P_\mu, \quad$$
$$dL = -P^\mu \wedge \epsilon_\mu^{\nu \lambda} P_\nu - \frac{1}{2} Q^\alpha \wedge Q_\alpha + \frac{1}{2} \bar{Q}^\alpha \wedge \bar{Q}_\alpha, \quad$$
$$dL_{ab} = \frac{1}{2} Q^\alpha \wedge \Gamma_{ab} \cdot Q_\alpha + \frac{1}{2} \bar{Q}^\alpha \wedge \Gamma_{ab} \cdot \bar{Q}_\alpha. \quad (71)$$

The space of 0-cochains is one-dimensional and spanned by $L$, but since $dL \neq 0$, $H^0(\mathfrak{t}; \mathfrak{t}) = 0$ and $\dim H^1 = 1$. The space $C^1$ is five-dimensional and is spanned by $P^\mu \wedge P_\mu$, $P^\mu \wedge \epsilon_\mu^{\nu \lambda} P_\nu$, $L^* \wedge L$, $Q^\alpha \wedge Q_\alpha$ and $\bar{Q}^\alpha \wedge \bar{Q}_\alpha$. As in the case of the IIB D- and fundamental strings, this can be understood by the fact that the ideal $I$ is graded by the action of $2L$, with $L$ having degree 0, and $Q_\alpha$ and $\bar{Q}_\alpha$ having degrees $\mp 1$, respectively, and $P_\mu$ having
Deformations of ten-dimensional Killing superalgebras 6059

pieces of degrees ±2, corresponding to a Witt basis for \( W \). The five-dimensional space of cochains can be thought of as spanned by the cochains corresponding to the identity maps of each of the five graded subspaces. The space \( C^2 \) of 2-cochains is 11-dimensional, spanned by \( L^* \otimes P^\mu \otimes P_{\mu}, L^* \otimes P^\mu \otimes \epsilon^\alpha_{\beta} P_{\nu}, L^* \otimes \tilde{Q}^\alpha \otimes Q_{\alpha}, L^* \otimes \tilde{Q}^\beta \otimes \tilde{Q}_\beta, P^\mu \otimes P^\nu \otimes \epsilon_{\mu\nu} L, P^\mu \otimes Q^\alpha \otimes \Gamma_{\mu} \cdot Q_{\alpha}, P^\mu \otimes \tilde{Q}^\beta \otimes \Gamma_{\mu} \cdot \tilde{Q}_\beta, Q^\alpha \otimes Q^\beta \otimes \Gamma_{\alpha\beta} P_{\mu}, \tilde{Q}^\beta \otimes \tilde{Q}^\alpha \otimes (\nu_{W})_{\alpha\beta} L \) and \( Q^\alpha \otimes \tilde{Q}^\beta \otimes \Gamma_{\alpha\beta} L_{ab} \).

Computing the differential \( d : C^1 \rightarrow C^2 \), we find that \( H^1(\mathfrak{t}, \mathfrak{f}) \cong \mathbb{R} \), with representative cocycle \( \varphi := 2P^\mu \otimes P_{\mu} - Q^\alpha \otimes Q_{\alpha} - \tilde{Q}^\alpha \otimes \tilde{Q}_\alpha \), whence \( \dim B^2 = 3 \), spanned by \( d(Q^\alpha \otimes Q_{\alpha}), d(\tilde{Q}^\alpha \otimes \tilde{Q}_\alpha) \) and \( d(L^* \otimes L) \), say. Similarly, computing \( d : C^2 \rightarrow C^3 \) we find that \( H^2(\mathfrak{t}, \mathfrak{f}) \cong \mathbb{R} \), with representative cocycle \( L^* \otimes \varphi \). This infinitesimal deformation integrates to a one-parameter family of Lie superalgebras with brackets

\[
\begin{align*}
[L, Q_\alpha] &= (t - \frac{1}{2}) Q_\alpha, \\
[L, \tilde{Q}_\alpha] &= (t + \frac{1}{2}) \tilde{Q}_\alpha, \\
[L, P_{\mu}] &= 2t P_{\mu} + \epsilon_{\mu\nu} P_{\nu}, \\
[Q_\alpha, Q_\beta] &= \Gamma_{\alpha\beta}^\mu P_{\mu}, \\
[\tilde{Q}_\alpha, \tilde{Q}_\beta] &= \Gamma_{\alpha\beta}^\mu P_{\mu},
\end{align*}
\]

in addition to those involving \( \mathfrak{so}(W^\perp) \) which remain undeformed. This deformation consists of changing the \( L \)-weight of the generators in the Lie superalgebra in such a way that the \( \mathcal{Q} \mathcal{Q} \) and \( \tilde{Q} \tilde{Q} \) brackets remain invariant, just as in the IIB fundamental string. As in the cases of the IIB D- and fundamental strings, this deformation is reminiscent of the construction of two-dimensional topological conformal field theories via twisting.

The fundamental string solution arises as the Kaluza–Klein reduction of the M2-brane along a translational symmetry of the brane worldvolume. In view of this, one might expect that the deformation of the M2 superalgebra found in [1] might induce a deformation of the fundamental string superalgebra, yet the deformation found in (72) is not the reduction of the one for the M2-brane superalgebra. If we take the geometric origin of the deformed M2 superalgebra at face value, the worldvolume of the putative deformed M2-brane is now AdS3 and it follows from the results of [29] that no quotient (regular or singular) of AdS3 preserves all the supersymmetries, whence the superalgebra of the Kaluza–Klein reduction of such a deformed M2-brane would be of strictly smaller dimension than that of the fundamental string superalgebra and hence would not appear among its deformations.

4.4. A deformation of the D2-brane superalgebra

The D2-brane Killing superalgebra \( \mathfrak{f} \) is the subsuperalgebra of the IIA Poincaré superalgebra corresponding to the split \( V = W \oplus W^\perp \), with \( W \) a three-dimensional Lorentzian subspace. The even subsuperalgebra \( \mathfrak{e}_0 = \mathfrak{so}(W) \oplus W \oplus \mathfrak{so}(W^\perp) \) and the odd subspace \( \mathfrak{e}_1 \) is isomorphic to the subspace \( \Delta_{D2} \subset \Delta_+ \oplus \Delta_- \) defined as the graph of \( \nu_W : \Delta_+ \rightarrow \Delta_- \). The linear map \( \nu_W \) is symmetric relative to the spinor inner product and obeys \( \nu_W^2 = +1 \). Let \( e_{\mu} \) and \( e_\alpha \) span \( W \) and \( W^\perp \), respectively. Let \( e_\alpha \) span \( \Delta_+ \) so that \( \psi_\alpha := \frac{1}{\sqrt{2}} (e_\alpha, e_\alpha) \) span \( \Delta_{D2} \). The corresponding basis for \( \mathfrak{f} \) is \( P_{\mu}, L_{\mu\nu}, L_{ab} \) and \( Q_\alpha \), with nonzero Lie brackets given, in addition to those of \( \mathfrak{e}_0 \), by

\[
\begin{align*}
[L_{\mu\nu}, Q_\alpha] &= \frac{1}{2} \Gamma_{\mu\nu} \cdot Q_\alpha, \\
[L_{ab}, Q_\alpha] &= \frac{1}{2} \Gamma_{ab} \cdot Q_\alpha, \\
[Q_\alpha, Q_\beta] &= \Gamma_{\alpha\beta}^\mu P_{\mu},
\end{align*}
\]

where

\[
\Gamma_{\alpha\beta}^\mu := (\psi_\alpha, \Gamma_\mu \psi_\beta) = (e_\alpha, \Gamma_\mu e_\beta).
\]
We observe that
\[ \langle \psi_\mu, \Gamma^{\mu \nu} \psi_\rho \rangle = \langle \epsilon_\mu, \Gamma^{\mu \nu} \nu_W \epsilon_\rho \rangle = : (\Gamma^{\mu \nu} \nu_W)_{\alpha \beta} \] (75)
and, similarly,
\[ \langle \psi_\mu, \Gamma^{\alpha \beta} \psi_\rho \rangle = \langle \epsilon_\mu, \Gamma^{\alpha \beta} \nu_W \epsilon_\rho \rangle = : (\Gamma^{\alpha \beta} \nu_W)_{\alpha \beta}, \] (76)
wheras
\[ \langle \psi_\mu, \Gamma^{\alpha} \psi_\rho \rangle = 0 = \langle \psi_\mu, \Gamma^{\alpha \mu} \psi_\rho \rangle. \] (77)

The ideal \( I < \mathfrak{t} \) is spanned by \( P_\mu \) and \( Q_\alpha \), whereas the semisimple factor \( s = \mathfrak{so}(W) \oplus \mathfrak{so}(W^*) \) is spanned by \( L_{\mu \nu} \) and \( I_{ab} \). The canonical dual basis for \( I^* \) is \( P_\mu \) and \( Q_\alpha \), relative to which the differential in the deformation complex \( C^* := C^*(I; \mathfrak{t})^s \) is defined by the following relations:
\[ dP_\mu = \frac{1}{2} \Gamma^{\mu \nu \rho} Q_\nu \wedge Q_\rho, \]
\[ dQ_\alpha = 0, \]
\[ dP_\mu = 0, \]
\[ dQ_\alpha = -\Gamma^{\mu \nu \rho} P_\mu \wedge Q_\alpha, \]
\[ dL_{\mu \nu} = \eta_{\mu \nu} P_\rho \wedge P_\rho - \eta_{\nu \mu} P_\rho \wedge P_\rho + \frac{1}{2} Q_\alpha \wedge \Gamma_{\mu \nu} \cdot Q_\alpha, \]
\[ dI_{ab} = \frac{1}{2} Q_\alpha \wedge \Gamma_{ab} \cdot Q_\alpha. \] (80)

There are no 0-cochains since \( \mathfrak{t}^0 = 0 \). The space of 1-cochains is spanned by \( P_\mu \otimes Q_\nu \otimes Q_\alpha \) and \( P_\mu \otimes \epsilon_\mu \nu \nu_W \otimes L_{\mu \nu} \), corresponding to the identity maps \( W \rightarrow W, \Delta_{D2} \rightarrow \Delta_{D2} \) and the composition \( W \rightarrow \Lambda^2 W \cong \mathfrak{so}(W) \) where the first map is induced by the Hodge star. The space of 2-cochains is six-dimensional and is spanned by \( P_\mu \wedge P_\nu \otimes Q_\alpha, Q_\alpha \wedge Q_\nu \otimes \Gamma_{\mu \nu} \cdot Q_\alpha, Q_\alpha \wedge Q_\nu \wedge \epsilon_\mu \nu \nu_W \otimes L_{\mu \nu} \), and \( Q_\nu \wedge Q_\rho \otimes \Gamma_{ab} \nu_W \otimes I_{ab} \). Note that the cochain \( P_\rho \otimes Q_\alpha \otimes \epsilon_\rho \mu \nu \nu_W \cdot Q_\alpha \) is already contained in the above span, since
\[ \epsilon_{\mu \nu \rho} \Gamma_{\rho} \cdot Q_\alpha = -\Gamma_{\mu \nu} \cdot Q_\alpha \quad \text{and} \quad \frac{1}{2} \epsilon_{\rho \mu \nu} \Gamma^{\alpha \beta} \cdot Q_\alpha = \Gamma_{\rho} \cdot Q_\alpha. \] (79)

Computing the differential \( d : C^1 \rightarrow C^2 \) we find that \( H^1(\mathfrak{t}; \mathfrak{t}) \cong \mathbb{R} \), with representative cocycle \( 2P_\mu \otimes P_\nu - Q_\nu \otimes Q_\alpha \). This means that \( \dim B^2 = 2 \), spanned by \( Q_\alpha \wedge Q_\nu \otimes \Gamma_{\mu \nu} \cdot P_\mu \) and
\[ d (P_\rho \otimes \epsilon_\rho \nu \nu_W \otimes L_{\mu \nu}) = -\frac{1}{2} (\Gamma^{\mu \nu \rho} \nu_W)_{\alpha \beta} Q_\alpha \wedge Q_\beta \otimes L_{\mu \nu} \]
\[ -2 \epsilon_{\mu \nu \rho} P_\rho \wedge P_\nu \otimes P_\mu - P_\mu \wedge Q_\nu \otimes \Gamma_{\mu \nu} \cdot Q_\alpha, \] (81)

Computing the differential \( d : C^2 \rightarrow C^3 \) we find, in addition to the coboundaries,
\[ d(P_\mu \wedge P_\nu \otimes L_{\mu \nu}) = \Gamma_{\alpha \beta} Q_\alpha \wedge Q_\beta \wedge P_\nu \otimes L_{\mu \nu} + \frac{1}{2} P_\mu \wedge P_\nu \wedge Q_\alpha \otimes \Gamma_{\mu \nu} \cdot Q_\alpha, \]
\[ d (P_\mu \wedge P_\nu \otimes \epsilon_\mu \nu \nu_W P_\mu) = - (\Gamma_{\nu \mu} \nu_W)_{\alpha \beta} Q_\alpha \wedge Q_\beta \wedge P_\mu \otimes P_\nu, \]
\[ d(P_\mu \wedge Q_\alpha \otimes \Gamma_{\mu} \cdot Q_\alpha) = \frac{1}{2} \Gamma_{\alpha \beta} Q_\alpha \wedge Q_\beta \wedge Q_\gamma \otimes \Gamma_{\mu} \cdot Q_\gamma \]
\[ + (\Gamma_{\nu} \nu_W)_{\alpha \beta} Q_\alpha \wedge Q_\beta \wedge P_\mu \otimes P_\nu, \]
\[ d(Q_\alpha \wedge Q_\beta \otimes (\Gamma^{\mu \nu \rho} \nu_W)_{\alpha \beta} L_{\mu \nu}) = 2 (\Gamma_{\nu} \nu_W)_{\alpha \beta} Q_\alpha \wedge Q_\beta \wedge P_\mu \otimes P_\nu \]
\[ + \frac{1}{2} (\Gamma^{\mu \nu \rho} \nu_W)_{\alpha \beta} Q_\alpha \wedge Q_\beta \wedge Q_\gamma \otimes \Gamma_{\mu \nu} \cdot Q_\gamma, \]
\[ d(Q_\alpha \wedge Q_\beta \otimes (\Gamma^{ab} \nu_W)_{\alpha \beta} I_{ab}) = \frac{1}{2} (\Gamma^{ab} \nu_W)_{\alpha \beta} Q_\alpha \wedge Q_\beta \wedge Q_\gamma \otimes \Gamma_{ab} \cdot Q_\gamma. \]
The first term is the only one depending on \( L_{\mu\nu} \), whence any cocycle must be a linear combination of the other cochains
\[
\Theta = a_1 P^\mu \wedge P^\nu \otimes \epsilon_{\mu\nu} P_\rho + a_2 P^\mu \wedge Q^a \otimes \Gamma_\mu \cdot Q_a + a_3 Q^a \wedge Q^b \otimes (\Gamma^{ab} \nu_W)_{ab} L_{\mu\nu} + a_4 Q^a \wedge Q^b \otimes (\Gamma^{ab} \nu_W)_{ab} L_{\mu\nu}.
\] (82)
Computing its differential, we find two types of terms: a \( P Q Q \otimes P \) term proportional to
\[
(\Gamma_\mu \cdot Q_W)_{ab} Q^a \wedge Q^b \otimes P_\mu \wedge P_\nu (-a_1 + a_2 + 2a_3),
\] (83)
and a \( QQ Q \otimes Q \) term proportional to
\[
Q^a \wedge Q^b \otimes Q^c (a_2 \Gamma^{ab}_{\mu\nu} (\Gamma_\mu \cdot Q_W)^3_\gamma + a_3 (\Gamma^{ab} \nu_W)_{ab} (\Gamma_\mu \cdot Q_W)^3_\gamma + a_4 (\Gamma^{ab} \nu_W)_{ab} (\Gamma_\mu \cdot Q_W)^3_\gamma) ,
\] (84)
The vanishing of the first of the above terms forces \( a_1 = a_2 + 2a_3 \), whereas the vanishing of the second term is equivalent to
\[
a_2 \langle \epsilon, \Gamma_\mu \epsilon \rangle \Gamma_\mu \nu \omega \epsilon + a_3 \langle \epsilon, \Gamma^{\mu\nu} \nu_W \epsilon \rangle \Gamma_\mu \nu \epsilon + a_4 \langle \epsilon, \Gamma^{ab} \nu_W \epsilon \rangle \Gamma_\mu \nu \epsilon = 0
\] for all \( \epsilon \in \Delta \). Using equation (A.8), we can rewrite this as
\[
(a_2 - 8a_4) \langle \epsilon, \Gamma_\mu \epsilon \rangle \Gamma_\mu \nu \omega \epsilon + (a_3 - a_4) \langle \epsilon, \Gamma^{\mu\nu} \nu_W \epsilon \rangle \Gamma_\mu \nu \epsilon = 0,
\] (86)
whereas using (79) we can finally rewrite this as
\[
(a_2 - 2a_3 - 6a_4) \langle \epsilon, \Gamma_\mu \epsilon \rangle \Gamma_\mu \nu \omega \epsilon = 0,
\] (87)
which vanishes if and only if \( a_2 = 2a_3 + 6a_4 \). Therefore, we see that there is a two-dimensional space of such cocycles, labelled by \( a_3 \) and \( a_4 \). The line \( a_4 = 0 \) is spanned by the coboundary in equation (80), whence we conclude that \( H^2(t; \ell) \cong \mathbb{R} \), with representative cocycle
\[
6P^\mu \wedge P^\nu \otimes \epsilon_{\mu\nu} P_\rho + 6P^\mu \wedge Q^a \otimes \Gamma_\mu \cdot Q_a + Q^a \wedge Q^b \otimes (\Gamma^{ab} \nu_W)_{ab} L_{\mu\nu},
\] (88)
which spans the line \( a_3 = 0 \). This infinitesimal deformation integrates to a one-parameter family of Lie superalgebras which, in addition to the undeformed brackets involving \( s \), has the following nonzero brackets:
\[
[P_\mu, P_\nu] = 12t \epsilon_{\mu\nu} P_\rho, \\
[P_\mu, Q_a] = -6t \Gamma_\mu \cdot Q_a, \\
[Q_a, Q_b] = \Gamma^{ab}_{\mu\nu} P_\mu - 2t (\Gamma^{ab} \nu_W)_{ab} L_{\mu\nu}.
\] (89)
It is not hard to show that the Jacobi identity is satisfied: only the \( O(t^3) \) terms need to be checked, since, by construction, the others are satisfied. The Jacobi identity breaks up into several types of terms, \( P PP \), \( PPP \), \( PQQ \) and \( QQQ \), the first three of which vanish trivially and the last one vanishes by virtue of the Fierz identity (A.8). For \( t \neq 0 \), we may rescale the generators \( P \) and \( Q \) in order to bring the above Lie superalgebra to the following form:
\[
[P_\mu, P_\nu] = -2t \epsilon_{\mu\nu} P_\rho, \\
[P_\mu, Q_a] = \Gamma_\mu \cdot Q_a, \\
[Q_a, Q_b] = \pm \Gamma_{\mu\nu} (\Gamma^{ab} \nu_W)_{ab} L_{\mu\nu}.
\] (90)
As in the case of the M2-brane superalgebra deformation in [1], the choice of sign corresponds to a duality relating two real forms of the same complex superalgebra, corresponding to multiplying the odd generators by \( i \). The even subalgebra \( \ell_0 \cong so(2, 2) \oplus so(7) \) with \( t_1 \), transforming as \( \Delta^{(2, 2)}_{
u} \otimes \Delta^{(7)}_{\gamma} \), with \( \Delta^{(2, 2)}_{\nu} \) the real two-dimensional positive-chirality spinor representation of \( so(2, 2) \) and \( \Delta^{(7)}_{\gamma} \) the real eight-dimensional spinor representation of \( so(7) \). Let us change the basis from \( L_{\mu\nu} \) to \( L'_{\mu\nu} := L_{\mu\nu} + \frac{1}{3} \epsilon_{\mu\nu} P_\rho \). We note that \( L'_{\mu\nu} \) commute with
the supercharges and hence with $P_\mu$ and that they span an $\mathfrak{so}(2,1)$ subalgebra. The remaining generators $L_{ab}$ and $Q_\alpha$ span a simple subsuperalgebra of $\mathfrak{t}$ isomorphic to the exceptional Lie superalgebra $f(4)$ [22]. Therefore, as an abstract Lie superalgebra, the deformed Lie superalgebra in (90) is isomorphic to $\mathfrak{so}(2,1) \oplus f(4)$.

4.4.1. Deformations of the delocalized M2-brane superalgebra. The D2-brane solution of type IIA supergravity arises via Kaluza–Klein reduction from an M2-brane which has been delocalized along one transverse direction. In this section, we will show that the deformation of the D2-brane superalgebra (90) has its origin in a deformation of the delocalized M2-brane superalgebra. As Kaluza–Klein reduction is a geometric procedure, this result lends support to the hypothesis that these deformations have a geometric origin.

Let $V$ be here an 11-dimensional Lorentzian vector space and we decompose it as $V = W \oplus U \oplus \mathbb{R}e_1$, where $W$ is a three-dimensional Lorentzian subspace, corresponding to the membrane worldvolume, and $W^\perp = U \oplus \mathbb{R}e_1$ is the transverse space, where $e_1$ denotes the delocalized 11th direction. Delocalization means that the metric and 4-form do not depend on the 11th coordinate. The symmetric delocalized M2-brane has superalgebra $\mathfrak{t} = e_0 \oplus \mathfrak{e}_1$, where $e_0 = \mathfrak{so}(W) \oplus \mathfrak{so}(U) \oplus \mathbb{R}$, with the $\mathbb{R}$ subalgebra corresponds to translations along the 11th direction, and where $e_1$ is isomorphic to the subspace $\Delta_{dM2} \subset \Delta$ of the spinor representation consisting of spinors $\epsilon \in \Delta$ obeying $\nu_W \epsilon = \epsilon$, where $\nu_W$ is the volume element associated with $W$. It is convenient, in order to compare with the IIA results, to break up $\Delta = \Delta_1 \oplus \Delta_\infty$ into eigenspaces of $\Gamma_\infty$, which is in fact the same split as the one induced by chirality in ten dimensions. The subspace $\Delta_{dM2}$ defined above is then the graph of $\nu_\infty : \Delta_1 \to \Delta_\infty$, so that if $\epsilon_\alpha$ is a basis for $\Delta_1$, then $\psi_\alpha = \frac{1}{\sqrt{2}} (\epsilon_\alpha, \epsilon_\beta)$ is a basis for $\Delta_{dM2}$. Let $P_\mu, P = P_\mu, L_{\mu\nu}, L_{\mu\nu}$ and $Q_\alpha$ denote a basis for the $\mathfrak{t}$, whose nonzero Lie brackets are given, in addition to those of $e_0$, by

$$[L_{\mu\nu}, Q_\alpha] = \frac{1}{2} \Gamma_{\mu\nu} \cdot Q_\alpha, \quad [L_{ab}, Q_\alpha] = \frac{1}{2} \Gamma_{ab} \cdot Q_\alpha, \quad [Q_\alpha, Q_\beta] = \Gamma_{\alpha\beta}^\mu P_\mu,$$

where

$$\Gamma_{\alpha\beta}^\mu := \langle \psi_\alpha, \Gamma^\mu \psi_\beta \rangle = \langle \epsilon_\alpha, \Gamma^\mu \epsilon_\beta \rangle.$$  \hspace{1cm} (91)

We observe that

$$\langle \psi_\alpha, \Gamma^{\mu\nu} \psi_\beta \rangle = \langle \epsilon_\alpha, \Gamma^{\mu\nu} \nu_\infty \epsilon_\beta \rangle =: (\Gamma^{\mu\nu} \nu_\infty)_{\alpha\beta}$$

and, similarly,

$$\langle \psi_\alpha, \Gamma^{ab} \psi_\beta \rangle = \langle \epsilon_\alpha, \Gamma^{ab} \nu_\infty \epsilon_\beta \rangle =: (\Gamma^{ab} \nu_\infty)_{\alpha\beta}.$$  \hspace{1cm} (92)

The ideal $I < \mathfrak{t}$ is spanned by $P, P_\mu$ and $Q_\alpha$, whereas the semisimple factor $\mathfrak{s} = \mathfrak{so}(W) \oplus \mathfrak{so}(U)$ is spanned by $L_{\mu\nu}$ and $L_{ab}$. The canonical dual basis for $I^*$ is $P^*, P^\mu$ and $Q^\alpha$, relative to which the differential in the deformation complex $C^* := C^*(I; \mathfrak{t})^\ast$ is defined by the following relations:

\begin{align*}
\text{d}P^* &= 0, \\
\text{d}P^\mu &= \frac{1}{2} \Gamma_{\alpha\beta}^\mu Q^\alpha \wedge Q^\beta, \\
\text{d}Q^\alpha &= 0, \\
\text{d}P &= 0, \\
\text{d}P_\mu &= 0, \\
\text{d}Q_\alpha &= -\Gamma_{\alpha\beta}^\mu Q^\beta \otimes P_\mu, \\
\text{d}L_{\mu\nu} &= \eta_{\mu\rho} P_\rho \otimes P_\nu - \eta_{\nu\rho} P_\rho \otimes P_\mu + \frac{1}{2} Q^\alpha \otimes \Gamma_{\mu\nu} \cdot Q_\alpha, \\
\text{d}L_{ab} &= \frac{1}{2} Q^\alpha \otimes \Gamma_{ab} \cdot Q_\alpha.
\end{align*}

\hspace{1cm} (93)
The space of 0-cochains is spanned by $P$, which is a cocycle, whence $H^0(\mathfrak{t}; t) \cong \mathbb{R}$. The space of 1-cochains is four-dimensional, spanned by $P^* \otimes P$, $P^* \otimes P_\mu$, $Q^a \otimes Q_\alpha$ and $P^* \otimes \epsilon_\mu^{\alpha\beta} L_{\mu\nu}$. The space of 2-cochains is nine-dimensional, spanned by $P^* \wedge P^* \otimes P_\mu$, $P^* \wedge Q^a \otimes Q_\alpha$, $P^* \wedge P^* \otimes L_{\mu\nu}$, $P^* \wedge P^* \otimes \epsilon_{\mu\nu}^\beta P_\beta$, $P^* \wedge Q^a \otimes \Gamma_\mu \cdot Q_\alpha$, $Q^a \wedge \Gamma_\mu \cdot P_\mu$, $Q^a \wedge \Gamma_\mu \cdot P_\mu$. The brackets, which can be read off from (90), denote the corresponding basis for $k$.

4.5. Rigidity of the D4-brane superalgebra

The D4-brane Killing superalgebra $\mathfrak{t} = \mathfrak{t}_0 \oplus \mathfrak{t}_1$ is the subsuperalgebra of the IIA Poincaré superalgebra corresponding to the split $V = W \oplus W^\perp$, with $W$ the five-dimensional Lorentzian. This means that $\mathfrak{t}_0 = \mathfrak{s}\mathfrak{o}(W) \oplus W \oplus \mathfrak{s}\mathfrak{o}(W^\perp)$ and $\mathfrak{t}_1$ is isomorphic to the subspace $\Delta_{D4} \subset \Delta_+ \oplus \Delta_-$ defined as the graph of the linear map $\nu_W : \Delta_+ \to \Delta_-$ defined by the Clifford action of the volume element of $W$, which is skew-symmetric under the spinor inner product and obeys $\nu_W^2 = -1$.

Let $e_\mu$ and $e_a$ span $W$ and $W^\perp$, respectively, and $P_\mu$, $L_{\mu\nu}$ and $L_{ab}$ denote the corresponding basis for $\mathfrak{t}_0$. Let $e_a$ span $\Delta_+$, and $\psi_\alpha := \frac{1}{\sqrt{2}} (\frac{\epsilon_\alpha}{\sqrt{2}})$ be a basis for $\Delta_{D4}$. Let $Q_\alpha$ denote the corresponding basis for $\mathfrak{t}_1$. The nonzero Lie brackets are those of $\mathfrak{s}\mathfrak{o}(W) \oplus \mathfrak{s}\mathfrak{o}(W^\perp)$ and in addition the following:

$$[Q_\alpha, Q_\beta] = \Gamma_{\alpha\beta}^\mu P_\mu, \quad [L_{\mu\nu}, Q_\alpha] = \frac{1}{4} \Gamma_{\mu\nu} \cdot Q_\alpha, \quad [L_{ab}, Q_\alpha] = \frac{1}{2} \Gamma_{ab} \cdot Q_\alpha, \quad (97)$$

where, as usual,

$$\Gamma_{\alpha\beta} := (\psi_\alpha, \Gamma^\mu \psi_\beta) = (e_a, \Gamma^\mu e_a). \quad (98)$$

The ideal $I < \mathfrak{t}$ is spanned by $P_\mu$ and $Q_\alpha$ and the semisimple factor $s$ by $L_{\mu\nu}$ and $L_{ab}$. Let $P^*$ and $Q^a$ be the canonical dual basis for $I^*$. Relative to these, the differential in the deformation complex $C^* := C^*(I; \mathfrak{t})^2$ is defined uniquely by the following relations:

$$d P^\mu = \Gamma_{\alpha\beta}^\mu Q^a \wedge Q^\beta,$$

$$d Q^a = 0,$$

$$d P_\mu = 0,$$

$$d Q_\alpha = -\Gamma_{\alpha\beta}^\mu Q^\beta \otimes P_\mu,$$

$$d L_{\mu\nu} = \eta_{\mu\nu} P^* \otimes P_\rho - \eta_{\mu\nu} P^* \otimes P_\rho + \frac{1}{2} Q^a \otimes \Gamma_{\mu\nu} \cdot Q_\alpha,$$

$$d L_{ab} = \frac{1}{2} Q^a \otimes \Gamma_{ab} \cdot Q_\alpha. \quad (99)$$
There are no 0-cochains, whereas the space of 1-cochains is two-dimensional spanned by the cochains corresponding to the identity maps $W \to W$ and $\Delta_{D4} \to \Delta_{D4}$, namely $P^\mu \otimes P_\mu$ and $Q^a \otimes Q_a$. The space of 2-cochains is now three-dimensional, spanned by $P^\mu \otimes P^\nu \otimes L_{\mu\nu}$, $P^\mu \wedge Q^a \otimes \Gamma_\mu \cdot Q_a$ and $Q^a \wedge Q^b \otimes \Gamma_{ab} \cdot P_\mu$. Computing the differential $d : C^0 \to C^1$ we find that $H^1(\mathfrak{t} ; \mathfrak{t}) \cong \mathbb{R}$, with representative 2-cocycle $2P^\mu \otimes P - Q^a \otimes Q_a$.

This means that $\dim B^2 = 1$, spanned by $d(Q^a \otimes Q_a)$, say. The only possible 2-cocycle is a linear combination

$$\Theta = a_1 P^\mu \wedge P^\nu \otimes L_{\mu\nu} + a_2 P^\mu \wedge Q^a \otimes \Gamma_\mu \cdot Q_a. \quad (100)$$

The first term on the right-hand side is the only one whose differential contains $L_{\mu\nu}$, whence $d\Theta = 0$ forces $a_1 = 0$. Computing the differential of the second term, we find

$$d(P^\mu \wedge Q^a \otimes \Gamma_\mu \cdot Q_a) = \frac{1}{2} \Gamma_{ab}^\mu Q^a \wedge Q^b \wedge \Gamma_\mu \cdot Q_a$$

$$= P^\mu \wedge Q^a \wedge (\Gamma_\mu \nu W) \Gamma_\nu^\alpha \Gamma_{\beta\gamma} P_\alpha. \quad (101)$$

The first term vanishes because of identity (62) and the fact that $\langle \epsilon, \Gamma^\alpha \epsilon \rangle = 0$, but the vanishing of the second term requires

$$\langle \Gamma_\mu \nu W \epsilon, \Gamma_\nu^\alpha \epsilon \rangle \equiv 0 \quad (102)$$

which, by the usual polarization identity, is equivalent to

$$\langle \Gamma_\mu \nu W \epsilon, \Gamma_\nu^\alpha \epsilon \rangle \equiv 0 \quad (103)$$

or, equivalently,

$$\langle \epsilon, \Gamma_\mu \nu W \epsilon \rangle \equiv 0 \quad (104)$$

Using the Clifford algebra and the fact that $\langle \epsilon, \Gamma_\mu \nu W \epsilon \rangle = 0$ for all $\epsilon \in \Delta_+$, we are left with

$$\langle \epsilon, \nu W \epsilon \rangle \equiv 0 \quad (105)$$

which is patently false, thus proving the rigidity of the D4-brane superalgebra.

4.6. Rigidity of the NS5-brane superalgebra

The IIA NS5-brane Killing superalgebra $\mathfrak{k} = \mathfrak{t}_0 \oplus \mathfrak{t}_1$ is the subsuperalgebra of the IIA Poincaré superalgebra corresponding to a decomposition $V = W \oplus W^\perp$, where $W$ is a six-dimensional Lorentzian subspace corresponding to the brane worldvolume. In other words, $\mathfrak{t}_0 = so(W) \oplus W \oplus so(W^\perp)$ and $\mathfrak{t}_1$ is isomorphic to the subspace $\Delta_{NS5} \subset \Delta_+ \supset \Delta_-$, defined as the $+1$ eigenspace of the Clifford action of the volume element $\nu W$ of $W$, which obeys $\nu^\mu W = +1$ and is skewsymmetric relative to the spinor (symplectic) inner product.

Let $e_\mu$ and $e_a$ span $W$ and $W^\perp$, respectively, and $\bar{e}_a$ and $\tilde{e}_a$ denote bases for $\Delta_{NS5} \cap \Delta_{\pm}$, respectively. The corresponding basis for $\mathfrak{k}$ is $P_\mu, L_{\mu\nu}, L_{ab}, Q_a$ and $\bar{Q}_a$, and the Lie brackets are, in addition to those of $\mathfrak{t}_0$, the following:

$$[L_{\mu\nu}, Q_a] = \frac{1}{2} \Gamma_{\mu\nu} \cdot Q_a,$$

$$[L_{ab}, Q_a] = \frac{1}{2} \Gamma_{ab} \cdot Q_a,$$

$$[Q_a, Q_b] = \Gamma_{ab} P_\mu,$$

$$[Q_a, \bar{Q}_b] = \Gamma_{\alpha\beta} P^\mu, \quad [\bar{Q}_a, \tilde{Q}_b] = \Gamma^\mu_{\alpha\beta} P_\mu. \quad (106)$$

where, as usual,

$$\Gamma^\mu_{\alpha\beta} := \langle e_\alpha, \Gamma^\mu \epsilon_\beta \rangle \quad \text{and} \quad \Gamma^\mu_{\bar{a}\bar{b}} := \langle \bar{e}_a, \Gamma^\mu \bar{e}_b \rangle. \quad (107)$$

The ideal $I < \mathfrak{k}$ is spanned by $P_\mu, Q_a$ and $\bar{Q}_a$, whereas the semisimple factor $s = so(W) \oplus so(W^\perp)$ is spanned by $L_{\mu\nu}$ and $L_{ab}$. Letting $P^\mu, Q^a$ and $\bar{Q}^a$ be the canonical dual
basis for $I^*$, the relations defining the differential in the deformation complex $C^* := C^*(I; t)^\delta$ are the following:
\[
d P^\mu = \frac{1}{2} \Gamma^\mu_{ab \delta} P^a \wedge P^b + \frac{1}{2} \Gamma^\mu_{ab \delta} \tilde{Q}^a \wedge \tilde{Q}^b,
\]
\[
d Q^a = 0,
\]
\[
d \tilde{Q}^a = 0,
\]
\[
d P_\mu = 0,
\]
\[
d Q_\alpha = -\Gamma^\alpha_{a \delta} Q^a \otimes P_\mu,
\]
\[
d \tilde{Q}_\alpha = -\Gamma^\alpha_{a \delta} \tilde{Q}^a \otimes P_\mu,
\]
\[
d L_{\mu\nu} = \eta_{\mu\nu} P^\rho \otimes P_\nu - \eta_{\nu\rho} P^\rho \otimes P_\mu + \frac{1}{2} Q^a \otimes \Gamma_\mu \nu \cdot Q_\alpha + \frac{1}{2} \tilde{Q}^a \otimes \Gamma_\mu \nu \cdot \tilde{Q}_\alpha,
\]
\[
d L_{ab} = \frac{1}{2} Q^\nu \otimes \Gamma_{ab \cdot Q_\alpha} + \frac{1}{2} \tilde{Q}^\nu \otimes \Gamma_{ab \cdot \tilde{Q}_\alpha}.
\]

There are no 0-cochains, whereas the space of 1-cochains is three-dimensional, spanned by $P^\mu \otimes P_\mu, Q^a \otimes Q_\alpha$ and $\tilde{Q}^a \otimes \tilde{Q}_\alpha$, corresponding to the identity maps $W \rightarrow W$ and $\Delta_{\text{NSS}} \cap \Delta_\pm \rightarrow \Delta_{\text{NSS}} \cap \Delta_\pm$. The space of 2-cochains is five-dimensional, spanned by $P^\mu \wedge P_\mu \otimes L_{\mu\nu}, P^\mu \wedge Q^a \otimes \Gamma^\alpha\nu \cdot Q_\alpha, P^\mu \wedge \tilde{Q}^a \otimes \Gamma^\alpha\nu \cdot \tilde{Q}_\alpha, Q^a \wedge Q^b \otimes \Gamma_\mu \nu \cdot Q_\alpha$ and $Q^a \wedge \tilde{Q}^b \otimes \Gamma_\mu \nu \cdot \tilde{Q}_\alpha$. Note that $\Gamma^\alpha_{a \delta} = 0 = \Gamma^\alpha_{a \delta}$, whence there are no cochains of the form $Q \tilde{Q} \otimes L$.

Computing the differential $d : C^1 \rightarrow C^2$, we see that $H^1(t; t) \cong \mathbb{R}$, with representative cocycle $2P^\mu \otimes P_\mu - Q^a \otimes Q_\alpha - \tilde{Q}^a \otimes \tilde{Q}_\alpha$. This implies that $\dim B^2 = 2$, spanned by $d(\tilde{Q}^a \otimes Q_\alpha)$ and $d(Q^a \otimes \tilde{Q}_\alpha)$, say. Therefore, any cohomology in dimension 2 must be represented by a cocycle of the form
\[
\Theta = a_1 P^\mu \wedge P_\nu \otimes L_{\mu\nu} + a_2 Q^a \otimes Q_\alpha + a_3 P^\mu \wedge Q^b \otimes \Gamma_\mu \nu \cdot Q_\alpha.
\]

As usual, the cocycle condition implies $a_1 = 0$ because that term is the only one whose differential involves $L_{\mu\nu}$. The differential of the remaining terms have the form $P \tilde{Q} \otimes \tilde{Q} \otimes \tilde{Q} \otimes \tilde{Q} \otimes Q$, $\tilde{Q} \otimes \tilde{Q} \otimes \tilde{Q} \otimes \tilde{Q} \otimes Q$ and $Q \otimes Q \otimes Q \otimes Q \otimes Q$. The $P \tilde{Q} \otimes \tilde{Q} \otimes Q$ terms are $-P^\mu \wedge Q^a \otimes \tilde{Q}^b \otimes P_\alpha (a_1 (\Gamma_\mu)^{\gamma}_{a \beta} + a_2 (\Gamma_\mu)^{\gamma}_{a \beta} \Gamma^\nu_{\gamma a})$
\[
= P^\mu \wedge Q^a \wedge \tilde{Q}^b \otimes P_\alpha (a_1 (\Gamma_\mu)^{\gamma}_{a \beta} + a_2 (\Gamma_\mu)^{\gamma}_{a \beta} \Gamma^\nu_{\gamma a}),
\]
which vanishes because
\[
(\Gamma_\mu \Gamma^\nu)_{a \beta} = (\epsilon_a, \Gamma_\mu \Gamma^\nu \tilde{e}_\beta)
\]
\[
= (\nu \omega \epsilon_a, \Gamma_\mu \Gamma^\nu \tilde{e}_\beta)
\]
\[
= - (\epsilon_a, \nu \omega \Gamma_\mu \Gamma^\nu \tilde{e}_\beta)
\]
\[
= - (\epsilon_a, \Gamma_\mu \Gamma^\nu \nu \omega \tilde{e}_\beta)
\]
\[
= - (\epsilon_a, \Gamma_\mu \Gamma^\nu \tilde{e}_\beta)
\]
\[
= - \Gamma_\mu \Gamma^\nu \epsilon_{a \beta},
\]

whence $(\Gamma_\mu \Gamma^\nu)_{a \beta} = 0$ and, similarly, $(\Gamma_\mu \Gamma^\nu)_{\delta a} = 0$. The $QQ \otimes \tilde{Q}$ term also vanishes by polarizing identity (62) and the fact that $\Gamma^\alpha_{a \delta} = 0$. By a similar argument we see that the $\tilde{Q} \tilde{Q} \otimes Q$ term also vanishes. The remaining terms would vanish if and only if
\[
(\epsilon_{\pm}, \Gamma_\mu \epsilon_{\pm})_{a \beta} \equiv 0 \quad \forall \epsilon_{\pm} \in \Delta_{\text{NSS}} \cap \Delta_\pm.
\]
It is not hard to show that this is false, proving the rigidity of the NS5-brane superalgebra.
4.7. A deformation of the D6-brane superalgebra

The D6-brane Killing superalgebra $\mathfrak{t}$ is the subsuperalgebra of the IIA Poincaré superalgebra associated with a split $V = W \oplus W^\perp$, with $W$ a seven-dimensional Lorentzian subspace corresponding to the brane worldvolume. This means that $\mathfrak{t}_0 = \mathfrak{so}(W) \oplus \mathfrak{so}(W^\perp)$ and $\mathfrak{t}_1$ is isomorphic to the subspace $\Delta_{D6} \subset \Delta_\ast \oplus \Delta_\ast$ defined as the graph of the linear map $\nu_W : \Delta_\ast \to \Delta_\ast$ corresponding to the volume element of $W$, which is symmetric relative to the spinor inner product and satisfies $\nu_W^2 = +1$. Let $e_\mu$ and $e_a$ span $W$ and $W^\perp$, respectively, and $P_\mu$, $L_\mu$, and $L_{\mu\nu}$ be the corresponding basis for $\mathfrak{t}_0$. If $e_a$ is a basis for $\Delta_\ast$, then $\psi_a := \frac{1}{\sqrt{2}}(\Gamma_0^a e_a)$ span $\Delta_{D6}$. The corresponding basis for $\mathfrak{t}_1$ is $Q_\alpha$. The nonzero Lie brackets of $\mathfrak{t}$ are given, in addition to those of $\mathfrak{t}_0$, by

\[ [L_{\mu\nu}, Q_a] = \frac{i}{2} \Gamma_{\mu\nu} Q_a, \quad [L_{ab}, Q_\alpha] = \frac{i}{2} \Gamma_{ab} Q_\alpha, \quad [Q_\alpha, Q_\beta] = \Gamma_{\alpha\beta}^\mu P_\mu, \] (113)

where

\[ \Gamma_{ab}^\mu := (\psi_a, \Gamma^\mu \psi_b) = (e_a, \Gamma^\mu e_b). \] (114)

The ideal $I < \mathfrak{t}$ is spanned by $P_\mu$ and $Q_\alpha$, whereas the semisimple factor $s = \mathfrak{so}(W) \oplus \mathfrak{so}(W^\perp)$ is spanned by $L_\mu$ and $L_{\mu\nu}$. The canonical dual basis for $I^\ast$ is $P^\mu$ and $Q^a$, relative to which the differential in the deformation complex $C^\ast := C^\ast(I; \mathfrak{t})^s$ is defined by the following relations:

\[ dP^\mu = \frac{i}{2} \Gamma_{ab}^\mu Q^a \wedge Q^b, \]
\[ dQ^a = 0, \]
\[ dP_\mu = 0, \]
\[ dQ_a = -\Gamma_{ab}^\mu Q^b \otimes P_\mu, \]
\[ dL_{\mu\nu} = \eta_{\mu\nu} P^\rho \otimes P_\rho - \eta_{\nu\rho} P^\mu \otimes P_\rho + \frac{i}{2} Q^\rho \otimes \Gamma_{\mu\nu} Q_\rho, \]
\[ dL_{ab} = \frac{i}{2} Q^\rho \otimes \Gamma_{ab} Q_\rho. \] (115)

There are no 0-cochains since $\mathfrak{t}^0 = 0$. The space of 1-cochains is two-dimensional, spanned by $P^\mu \otimes P_\mu$ and $Q^a \otimes Q_a$, corresponding to the identity maps $W \to W$ and $\Delta_{D6} \to \Delta_{D6}$. The space of 2-cochains is five-dimensional and is spanned by $P^\mu \wedge P^\nu \otimes L_{\mu\nu}$, $P^\mu \wedge Q^a \otimes \Gamma_{\mu} Q_a$, $Q^a \wedge Q^b \otimes \Gamma_{ab}^\mu P_\mu$, $Q^a \wedge Q^b \otimes (\Gamma_{\mu}^\mu \nu_W)_{ab} L_{\mu\nu}$, and $Q^a \wedge Q^b \otimes (\Gamma_{ab}^\mu \nu_W)_{ab} L_{ab}$.

Computing the differential $d : C^1 \to C^2$, we find that $H^1(\mathfrak{t}; \mathfrak{t}) \cong \mathbb{R}$, with representative cocycle $2P^\mu \otimes P_\mu - Q^a \otimes Q_a$, which means that $\dim B^2 = 1$, spanned by $d(Q^a \otimes Q_a)$, say. Computing the differential $d : C^2 \to C^3$ and employing the usual arguments, we see that any nonzero cocycle must be of the form

\[ \Theta = a_1 P^\mu \wedge Q^a \otimes (\Gamma_{ab} \nu_W)_{ab} L_{\mu\nu} + a_2 Q^a \wedge Q^b \otimes (\Gamma_{\mu}^\mu \nu_W)_{ab} L_{\mu\nu} + a_3 Q^a \wedge Q^b \otimes (\Gamma_{ab}^\mu \nu_W)_{ab} L_{ab}. \] (116)

Computing its differential, we find two types of terms which must vanish separately for $\Theta$ to be a cocycle. The first term takes the form

\[ P^\mu \wedge Q^a \wedge Q^b \otimes P_\chi (-a_1 (\Gamma_{\mu} \nu_W)^\chi_{\alpha\beta} \Gamma_{\alpha\beta}^\mu + 2a_2 (\Gamma_{\alpha\beta} \nu_W)_{\alpha\beta}^\mu). \] (117)

Using that

\[ (\Gamma_{\mu} \nu_W)^{\alpha\beta}_{\alpha\beta} = (\Gamma_{\mu} \nu_W e_{\alpha\beta}, \Gamma_{\alpha\beta} e_{\beta}), \]
\[ = - (\nu_W e_{\alpha\beta}, \Gamma_{\alpha\beta} e_{\beta}), \]
\[ = - (e_{\alpha\beta}, \Gamma_{\alpha\beta} \nu_W e_{\alpha\beta}) \]
\[ = - (\Gamma_{\alpha\beta} \nu_W)_{\alpha\beta} - \delta_{\mu}^\nu (e_{\alpha\beta}, \nu_W e_{\alpha\beta}), \] (118)
and that the second term is skewsymmetric in $\alpha \leftrightarrow \beta$, the above term in $d\Theta$ becomes
\begin{equation}
(a_1 + 2a_2)(\Gamma^\mu_{\nu, \rho})_{a\beta} P^\mu \wedge Q^\nu \wedge Q^\rho \otimes P_\gamma,
\end{equation}
whence $d\Theta = 0$ forces $a_1 = -2a_2$. The second type of term in $d\Theta$ is given by
\begin{equation}
\frac{1}{2} Q^\mu \wedge Q^\beta \wedge Q^\gamma \otimes (a_1 \Gamma^\mu_{\alpha\beta} (\Gamma_{\mu, \nu})_{a\beta} P^\mu + a_2 (\Gamma^\mu_{\nu, \rho})_{a\beta} (\Gamma_{\mu, \nu})_{a\beta} P^\mu + a_3 (\Gamma^\mu_{\nu, \rho})_{a\beta} (\Gamma_{\mu, \nu})_{a\beta} P^\mu).
\end{equation}
whose vanishing is equivalent, via a polarization identity, to the vanishing of
\begin{equation}
a_1 \langle \epsilon, \Gamma^\mu_{\nu} \nu \rangle \mu \nu + a_2 \langle \epsilon, \Gamma^\mu_{\nu} \nu \rangle \mu \nu + a_3 \langle \epsilon, \Gamma^\mu_{\nu} \nu \rangle \mu \nu = 0
\end{equation}
for all $\epsilon \in \Delta_\ast$. Using the Clifford identities
\begin{equation}
\Gamma^a_{\nu} \nu \epsilon = \epsilon a_b \Gamma^a_{\nu} \epsilon \quad \text{and} \quad \frac{1}{2} \epsilon a_b c = -\epsilon a_b c,
\end{equation}
using equation (62). Plugging the above into the Fierz identity (A.8), we find
\begin{equation}
\langle \epsilon, \Gamma^\mu_{\nu} \nu \rangle \mu \nu = -10 \langle \epsilon, \Gamma^\mu_{\nu} \nu \rangle \mu \nu \epsilon,
\end{equation}
whence the second part of the cocycle condition $d\Theta = 0$ becomes
\begin{equation}
(a_1 - 10a_2 + 2a_3) \langle \epsilon, \Gamma^\mu_{\nu} \nu \rangle \mu \nu \epsilon = 0
\end{equation}
for all $\epsilon \in \Delta_\ast$, which implies $a_1 = 10a_2 - 2a_3$. Putting both conditions together, we find that the nontrivial cocycle is a multiple of
\begin{equation}
\Theta = -2 P^\mu \wedge Q^\alpha \otimes \Gamma^\mu_{\nu} \nu \epsilon \mu_a + 2 Q^\mu \wedge Q^\nu \otimes (\Gamma^\mu_{\nu} \nu \epsilon) \rho \beta L^\beta_{\rho \mu} + 6 Q^\mu \wedge Q^\rho \otimes (\Gamma^\mu_{\nu} \nu \epsilon) \rho \beta L^\beta_{\rho \mu},
\end{equation}
which shows that $H^2(\mathfrak{g}; \mathfrak{t}) \cong \mathbb{R}$.

To first order in the deformation parameter $t$, the nonvanishing Lie brackets corresponding to the above cocycle, in addition to those of $a$ which do not deform, are given by
\begin{equation}
[P_\mu, Q_\alpha] = 2 \Gamma_{\mu} \cdot Q_\alpha, \quad [Q_\alpha, Q_\beta] = \Gamma^\beta_{\alpha\beta} P_\mu - 2 t (\Gamma^\mu_{\nu, \rho})_{a\beta} L^\beta_{\rho \mu} - 12 t (\Gamma^\mu_{\nu, \rho})_{a\beta} L^\beta_{\rho \mu}.
\end{equation}
There is an obstruction to integrating this deformation at order $t^2$, which can be overcome by introducing the bracket
\begin{equation}
[P_\mu, P_\rho] = 16 t^2 L_{\mu \rho},
\end{equation}
One can check that the above Lie brackets now satisfy the Jacobi for all $t$. When $t \neq 0$, one can rescale $P_\rho$ and $Q_\alpha$ in order to get rid of $t$ and bring the Lie algebra to the following form:
\begin{equation}
[P_\mu, Q_\alpha] = L_{\mu \alpha}, \quad [P_\mu, Q_\beta] = \frac{1}{2} \Gamma_{\mu} \cdot Q_\beta, \quad [Q_\alpha, Q_\beta] = \frac{1}{2} (\Gamma^\mu_{\nu, \rho})_{a\beta} L^\beta_{\rho \mu} - 3 (\Gamma^\mu_{\nu, \rho})_{a\beta} L^\beta_{\rho \mu}.
\end{equation}
The bosonic subalgebra is now $\mathfrak{so}(2, 6) \oplus \mathfrak{so}(3)$ and the Lie superalgebras above are real forms of the simple Lie superalgebra of type $D(4, 1)$ in Kac’s classification [22]. In fact, it is isomorphic to $\mathfrak{osp}(6, 2|2)$, which is the conformal superalgebra of six-dimensional Minkowski spacetime, which suggests that the worldvolume of the $D6$-brane curves to $\text{AdS}_7$.

This deformation is related via Kaluza–Klein reduction to the similar deformation of the Kaluza–Klein monopole superalgebra in equation (95) of [1, section 5.2].
5. Conclusions

In this paper, we have explored the Lie superalgebra deformations of the Killing superalgebras of ten-dimensional supergravity backgrounds. We have concentrated largely on the flat vacua, which have been shown to be rigid, and the elementary asymptotically flat branes. All have been found to be rigid except for the following:

- Types I and IIB D1-brane superalgebra, with deformed superalgebras given by (9) and (25), respectively.
- Types IIA and IIB fundamental string superalgebras, with deformations (72) and (30), respectively.
- Type IIA D2-brane superalgebra, with deformation (90), isomorphic to $\mathfrak{so}(2, 1) \oplus \mathfrak{f}(4)$.
- Type IIA D6-brane superalgebra, with deformation (129), isomorphic to $\mathfrak{osp}(6, 2|2)$.
- Type IIB D7-brane superalgebra, with deformation (49).

In particular, these two latter deformations are related to the deformation of the delocalized M2-brane found in section 4.4.1 and that of the Kaluza–Klein monopole background of 11-dimensional supergravity in equation (95) of [1, section 5.2].

These results seem to suggest a geometric origin for the deformations found in this paper, simply because it would otherwise be difficult to justify their good behaviour under a geometric process such as Kaluza–Klein reduction. If a background has a symmetry superalgebra $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ and we consider its Kaluza–Klein reduction along the one-parameter subgroup generated by some element $X \in \mathfrak{g}_0$, then the symmetry superalgebra $\mathfrak{k}$ of the quotient is isomorphic to $\mathfrak{n}/h_X$, where $h_X < \mathfrak{g}_0$ is the one-dimensional Lie subalgebra spanned by $X$ and $n < \mathfrak{g}$ is the normalizer of $h_X$ in $\mathfrak{g}$. As a Lie superalgebra, $\mathfrak{k} = \mathfrak{k}_0 \oplus \mathfrak{k}_1$, where $\mathfrak{k}_1$ are those elements of $\mathfrak{g}_1$ which commute with $X$ and $\mathfrak{k}_0$ is the normalizer of $X$ in $\mathfrak{g}_0$ modulo the span of $X$. It is easy to check that for the delocalized M2 brane and the D2, which is its reduction along the delocalized direction, and for the Kaluza–Klein monopole and the D6, which is its reduction along the central element, the Killing superalgebras do indeed behave in the way just stated and moreover do their deformations. It is precisely this coherence under Kaluza–Klein reduction which suggests that the deformations do have a geometric construction. Furthermore, as explained in section 4.3, the deformation of the M2 brane does not induce a deformation of the fundamental string because the dimension of the superalgebras are different. Although it has not been the purpose of this paper to elucidate the geometric interpretation of the deformations found here and in [1]—this will be reported on in [12]—we nevertheless believe that we have given evidence that such an interpretation ought to exist.

Acknowledgments

We have enjoyed discussions about this topic with Patricia Ritter and Joan Simón. A part of this work was done while BV was visiting the School of Mathematics of the University of Edinburgh, with support from the Marie Curie Research Training Network Grant ‘ForcesUniverse’ (contract no MRTN-CT-2004-005104) from the European Community’s Sixth Framework Programme.

Appendix. Spinorial conventions

We work with a mostly plus metric and with a plus sign in the Clifford algebra. Let $V$ denote an 11-dimensional Lorentzian vector space with signature $(10, 1)$ and $C\ell(V)$ denote its Clifford algebra. It is well known that $C\ell(V) \cong \text{End}(\Delta) \oplus \text{End}(\Delta')$, where the irreducible Clifford
modules $\Delta$ and $\Delta'$ are real and 32-dimensional and are distinguished by the action of the central element $\nu$ corresponding to the volume form on $V$. We will work with $\Delta$, say. Introduce an orthonormal frame $e_0, e_1, \ldots, e_5$ for $V$ and denote the corresponding elements in $\text{Cl}(V)$ by $\Gamma_0, \ldots, \Gamma_5$. Let $W = (e_i)^\perp$ be the ten-dimensional Lorentzian subspace perpendicular to $e_i$. Although $\Delta$ is irreducible under $\mathfrak{so}(V)$, it decomposes under $\mathfrak{so}(W)$ as $\Delta = \Delta_+ \oplus \Delta_-$, each summand being an eigenspace of $\Gamma_5$. The invariant symplectic form $\langle - , - \rangle$ on $\Delta$ is such that $\Delta_\pm$ are Lagrangian subspaces. If $\psi \in \Delta$, then there is an 11-dimensional Fierz-type identity which reads

$$\frac{1}{2} \langle \psi, \Gamma^{MN} \psi \rangle \Gamma_M \Gamma_N \psi = -5 \langle \psi, \Gamma^M \psi \rangle \Gamma_M \psi,$$

and a similar identity involving the natural 5-form:

$$\frac{1}{2} \langle \psi, \Gamma^{MN} \psi \rangle \Gamma_{M_1} \ldots \Gamma_{M_5} \psi = -6 \langle \psi, \Gamma^M \psi \rangle \Gamma_M \psi,$$

which is consistent with the Fierz identity

$$32 \lambda_\psi = \langle \psi, \Gamma^M \psi \rangle \Gamma_M = \frac{1}{2} \langle \psi, \Gamma^{MN} \psi \rangle \Gamma_{M_1} \ldots \Gamma_{M_5} + \frac{1}{2} \langle \psi, \Gamma^{M_1} \ldots \Gamma^{M_5} \psi \rangle \Gamma_{M_1} \ldots \Gamma_{M_5},$$

with $\lambda_\psi \in \text{End}(\Delta)$ the rank-1 endomorphism defined by $\lambda_\psi(\chi) = \langle \psi, \chi \rangle \psi$.

We may reinterpret identity (A.1) in ten dimensions as follows. Let $\psi = (\psi_\ell)$, with $\psi_\pm \in \Delta_\pm$, using that for any $\varepsilon \in \Delta_\pm$,

$$\langle \varepsilon, \Gamma^A \varepsilon \rangle \Gamma_A \varepsilon = 0,$$

where $A = 0, 1, \ldots, 9$, we may unpack equation (A.1) as

$$4 \langle \varepsilon_\pm, \Gamma^A \varepsilon_\pm \rangle \Gamma_A \varepsilon_\pm = 0.$$

In this paper, we will need the restriction of this identity to various subspaces of $\Delta$ or $\Delta_\pm$ or of $2\Delta_\pm$, depending on the supergravity theory in question. For type II D-branes, we will be interested in the graphs of linear maps $\nu_W : \Delta_\pm \rightarrow \Delta_\pm$ inside $\Delta_\pm \oplus \Delta_\mp$, with $\nu_W$ the volume form of a Lorentzian subspace of $V$ associated with a brane worldvolume.

In the case of IIB D-branes, $W$ is even-dimensional, and hence $\nu_W : \Delta_+ \rightarrow \Delta_+$. The Fierz identity of relevance is the restriction of equation (A.3), which now says

$$\langle \varepsilon, \Gamma^A \varepsilon \rangle \Gamma_A \varepsilon = 0,$$

for $A = 0, \ldots, \text{dim } W$.

In the case of IIA D-branes, $W$ is odd-dimensional, and hence $\nu_W : \Delta_+ \rightarrow \Delta_-$. In this case, the two identities (A.4) become equivalent to this one:

$$4 \langle \varepsilon, \Gamma^A \varepsilon \rangle \Gamma_A \nu_W \varepsilon = -10 \langle \varepsilon, \nu_W \varepsilon \rangle \varepsilon + \langle \varepsilon, \Gamma^{AB} \nu_W \varepsilon \rangle \Gamma_{AB} \varepsilon = 0,$$

for all $\varepsilon \in \Delta_\pm$. We can refine this identity further. Let $\sigma_W := (-1)^{p/2}$, where $\text{dim } W = p+1$ with $p$ even. Then it follows that $\langle \nu_W \varepsilon_1, \varepsilon_2 \rangle = \sigma_W \langle \varepsilon_1, \nu_W \varepsilon_2 \rangle$ and similarly that $\nu_W^\dagger = \sigma_W I$. Letting $e_\mu$ and $e_\nu$ denote orthonormal frames for $W$ and $W^\perp$, respectively, we have that

$$\nu_W ^\dagger = \Gamma^a \nu_W \quad \text{wheras} \quad \nu_W \Gamma_a = -\Gamma_a \nu_W.$$
