Generalized Variational Source Condition
Associated with the Bregman Distance-I:
Verification of the Variational Source Condition and
Stability of the Total Error Estimation

Erdem Altuntac
Institute for Numerical and Applied Mathematics, University of Göttingen, Lotzestr. 16-18,
D-37083, Göttingen, Germany
E-mail: e.altuntac@math.uni-goettingen.de

Abstract.
A general deterministic analysis to state the necessary conditions with a coefficient determination
for the variational source condition to hold is provided. Of particular interest in terms of the choice of
the regularization parameter, Morozov’s discrepancy principle enables one to determine new stable
lower and upper bounds for the regularization parameter. With these bounds, it is also possible
to establish quantitative estimations for the index function as well as for the different definitions
of the Bregman distance. Inclusion of the variational source condition into the stability analysis
enables one to re-establish convergence and convergence rate results in terms of the index function.
The coefficient in the variational source condition is explicitly defined as a multivariable function of
constants in Morozov’s discrepancy principle. As expected, the results here are applicable when any
strictly convex, smooth/non-smooth objective functional is considered.

1. Introduction

Variational regularization has commenced by introducing a new image denoising
method named as total variation, [35]. Application and analysis of the method have
been widely carried out in the communities of inverse problem and optimization,
[1, 5, 6, 11, 12, 13, 16, 17, 37, 38]. In variational regularization the usage of
Bregman distance as a tool for the convergence and convergence rate has been well
established over the last decade, [10, 22, 23, 24, 26, 27, 28, 29, 32]. As alternative
to well known regularization theory for minimizing the quadratic Tikhonov functional,
[33, 34], studying convex variational regularization with some general penalty term $J$
has recently become important. In a recent work by Hohage et al. 2015, [28, Eq.
(2)] and references therein, a conventional variational source condition (VSC) with
a logarithmic index function $\Psi$ has been derived for an inverse scattering problem.
This work is followed up by another research wherein the coefficient verification of
the VSC has been carried out in [29]. Authors in [29] have verified the existence of
some necessary coefficient in the VSC for quadratic Tikhonov functionals under some conditions. In this work, we study general type Tikhonov functional. We explore under which conditions the VSC hold and the tight convergence rate results explicitly. The mathematical development of this work entails the specific rule for the choice of the regularization parameter which is Morozov’s discrepancy principle.

Hofmann and Mathé et al. 2012, [26], a priori and a posteriori strategies for the choice of the regularization parameter in Banach spaces under the variational source condition to determine the total error estimation

\[ E(\varphi_{\alpha(\delta,f^{\delta})}, \varphi^\dagger) := ||\varphi_{\alpha(\delta,f^{\delta})} - \varphi^\dagger||_V, \]  

have been studied extensively. This work does not necessarily convey any specific solution space \( V \) since the penalty term of our objective functional is not specified. By establishing some quantitative analysis for the Bregman distance \( D_J \), the total error estimation will also be stabilized owing to the consideration of this work below,

\[ E(\varphi_{\alpha(\delta,f^{\delta})}, \varphi^\dagger) \leq D_J(\varphi_{\alpha(\delta,f^{\delta})}, \varphi^\dagger). \]  

Therefore, the objective of this work is to investigate the stable bounds for \( D_J \) in terms of an increasing and positive definite index function \( \Psi \) depending on the noise amount \( \delta \) such that

\[ \alpha(\delta,f^{\delta}) \to 0 \quad \text{and} \quad \frac{\delta^2}{\alpha(\delta,f^{\delta})} \to 0, \quad \text{as} \quad \delta \to 0. \]  

Organization of this work is as follows; In the following section, we give the necessary preliminaries that are the base of entire mathematical development. In Section 3, we review the fundamentals of Morozov’s discrepancy principle. We then move on to the study of verification of the generalized variational source condition in conjunction with coefficient determination. Those conditions will be related to the rule for the choice of the regularization parameter. In Section 4, we, in light of our a posteriori choice of the regularization parameter, introduce a new interval for the value of the regularization parameter, i.e. stable upper and lower bounds. It is with these bounds that we will be able to stabilize the total error estimation. Final scientific development will be given in Section 5. We also propose a different form the VSC in the Appendix A by comparing the reverse form of the usual Bregman distance to the positive definite index function.

### 2. Notations and Prerequisite Knowledge

#### 2.1. Assumptions about the forward operator and the penalty term

Denote by \( \mathcal{V} \) and \( \mathcal{H} \) some reflexive/non-reflexive Banach and Hilbert spaces respectively. For the given linear, injective and compact forward operator \( T : \mathcal{D}(T) = \mathcal{V} \to \mathcal{H} \), we consider solving a linear ill-posed operator equation formulated by

\[ T\varphi = f^\dagger. \]  

(2.1)
The usual inverse problem is that of reconstruction of the approximate solution $\varphi^\delta$ by minimizing a general Tikhonov type objective functional

$$F_\alpha : \mathcal{V} \times \mathcal{H} \rightarrow \mathbb{R}_+, \quad (\varphi, f^\delta) \mapsto F_\alpha(\varphi, f^\delta) := \frac{1}{2} \| T\varphi - f^\delta \|^2_H + \alpha J(\varphi),$$

from the given data $f^\delta$ of the exact right-hand side $f^\dagger \in \mathcal{H}$ with $f^\delta \in B_\delta(f^\dagger)$, i.e. $\| f^\dagger - f^\delta \|_H \leq \delta$.

In (2.2), the nonsmooth $J : \mathcal{V} \rightarrow \mathbb{R}_+$ is the convex regularizer with the regularization parameter $\alpha > 0$ before it. It is assumed that the any non-zero constant function under the image of the forward operator does not vanish, and this fact can be formulated as follows,

$$T1 \neq 0.$$  

The real valued solution function $\varphi$ is defined on a compact domain $\Omega$.

### 2.2. The existence and the uniqueness of the minimizer

Our argument on the existence and the uniqueness of the minimizer is rather pre-assumptional since this work aims to provide some general analysis. Throughout the available literature, e.g., [26, p. 2-3], [27, 3rd of Assumption 2.1], [36, 4th of Assumption 3.13], the sublevel sets of the objective functional $F_\alpha$, or of the penalty term $J$, have been assumed to be sequentially pre-compact. However, if one considers the penalty term as

$$J^{TV}_\beta(\varphi) := \int_\Omega \sqrt{|\nabla \varphi(x)|^2 + \beta} dx,$$

then it can be shown in a counterexample that the sublevel sets for $J^{TV}_\beta$ are not sequentially pre-compact.

**Example 2.1.** According to [26, p. 2], the sublevel sets for $J^{TV}_\beta$ are defined below,

$$M^{J^{TV}_\beta}_R := \{ \varphi \in W^{1,2}(\Omega) : J^{TV}_\beta(\varphi) \leq R \}, \text{ for } R > 0.$$  

Obviously $M^{J^{TV}_\beta}_R \subset BV(\Omega)$. To ensure that the sublevel sets are weakly sequentially compact, one must show that every sequence $\varphi_n \in M^{J^{TV}_\beta}_R$ has a weakly convergent subsequence with the limit in $M^{J^{TV}_\beta}_R$, i.e. the sequence $\varphi_n \in M^{J^{TV}_\beta}_R$ has a subsequence $\{\varphi_{n_k}\}_{k=1}^\infty \subset \{\varphi_n\}_{n=1}^\infty$ such that $\varphi_{n_k} \rightharpoonup \varphi^*$ as $k \rightarrow \infty$ where $\varphi^* \in M^{J^{TV}_\beta}_R$. However, it can be shown that the sublevel sets $M^{J^{TV}_\beta}_R$ contain sequence which does not have weakly convergent subsequence. To do so, for some real function $\bar{\varphi} \in M^{J^{TV}_\beta}_R$, consider the sequence $\varphi_n(x) = \bar{\varphi}(x) + n1$, where $x \in \Omega$. Although, for the defined sequence $J^{TV}_\beta(\varphi_n) = J^{TV}_\beta(\varphi) \leq R$ holds for any $n \in \mathbb{N}$ the sequence $\varphi_n$ cannot have weakly convergent subsequence in $BV(\Omega)$ since
\[
\|\varphi_n - \varphi_m\|_{BV(\Omega)} \geq \|\varphi_n - \varphi_m\|_{L^1(\Omega)} \\
\geq |n - m| |\Omega| \rightarrow \infty \text{ as } n \rightarrow \infty.
\] (2.4)

According to \[1, \text{Theorem 3.1}\], in order to ensure the existence of the regularized solution, one must be able to ensure the \(BV\)-coercivity of the objective functional \(F_\alpha\). Usually general type of Tikhonov functionals are not strictly convex since the forward operator \(T\) may not necessarily be injective. In our case, uniqueness of the regularized solution is a result of the strict convexity of the objective functional since the forward operator \(T\) is assumed to be injective.

2.3. Bregman distance

Conventional thorough feedback for the following terminology can be found in \[9, 23, 36\].

**Definition 2.2. [Subdifferential]** Let \(J : V \rightarrow \mathbb{R}_+ \cup \{\infty\}\) be defined on an appropriate Banach space and be some convex functional. Then subdifferential \(\partial J(u) \subset V^*\) of \(J\) at \(u \in V\) is defined as the set of all \(p \in V^*\) satisfying the inequality

\[
J(v) - J(u) - \langle p, v - u \rangle \geq 0, \text{ for all } v \in V.
\] (2.5)

Here the element \(p \in \partial J(u)\) is called the subgradient.

Note that when \(\partial J(u)\) is a singleton, Gâteaux differentiability and subdifferentiability of \(J\) are equal to each other.

**Definition 2.3. [The Generalized Bregman Distances]** Let \(J : V \rightarrow \mathbb{R}_+ \cup \{\infty\}\) be a convex functional with the subgradient \(p \in \partial J(u^*)\). Then, for \(u, u^* \in V\), Bregman distance associated with the functional \(J\) is defined by

\[
D_J : V \times V \longrightarrow \mathbb{R}_+ \\
(u, u^*) \longmapsto D_J(u, u^*) := J(u) - J(u^*) - \langle p, u - u^* \rangle.
\] (2.6)

In addition to the traditional definition of Bregman distance in (2.6), the symmetric Bregman distance is also given below, (cf. \[23, \text{Definition 2.1}\]),

\[
D_J^{\text{sym}}(u, u^*) := D_J(u, u^*) + D_J(u^*, u).
\] (2.7)

From here, one can easily observe that

\[
D_J^{\text{sym}}(u, u^*) \geq D_J(u, u^*).
\] (2.8)

Same also holds if one replaces the right hand side by the reverse Bregman distance \(D_J(u^*, u)\).
2.4. Minimization problem

The regularized solution \( \varphi^\delta_\alpha \) is constructed by employing an appropriate regularization strategy for the following convex variational minimization problem,

\[
\varphi^\delta_\alpha \in \arg \min_{\varphi \in \mathcal{V}} F_\alpha(\varphi, f^\delta).
\]  

(2.9)

Inherently, this solution satisfies the following first order optimality condition, (cf. [9, Eq. (3.4)]),

\[
\frac{1}{\alpha} \mathcal{T}^*(f^\delta - \mathcal{T}\varphi^\delta_\alpha) \in \partial J(\varphi^\delta_\alpha).
\]  

(2.10)

**Definition 2.4. [J-minimizing Solution]** Let \( \mathcal{V} \) be an appropriate Banach space and \( \mathcal{H} \) be some Hilbert space. For some given linear, injective and compact forward operator \( \mathcal{T} : \mathcal{V} \to \mathcal{H} \), the \( J \)-minimizing solution is a solution to the linear operator equation

\[
\mathcal{T}\varphi = f^\dagger
\]  

(2.11)

if

\[
J(\varphi^\dagger) := \min \{ J(\varphi) : \varphi \in \mathcal{V}, \mathcal{T}\varphi = f^\dagger \}.
\]  

(2.12)

Although our work rather focuses on determining the stable upper bounds for the Bregman distance \( D_J \), it is still worthwhile to review some norm convergence rates both in the image and in the pre-image spaces. Owing to the \textit{a posteriori} strategy for the choice of regularization parameter \( \alpha = \alpha(\delta, f^\delta) \), see subsection 3.1 for the details, with the deterministic noise model \( f^\delta \in \mathcal{B}_\delta(f^\dagger) \) in the measurement space, the following rates can be quantified;

(i) \( \mathcal{T}\varphi^\delta_\alpha(\delta, f^\delta) \in \mathcal{B}_{\mathcal{O}(\delta)}(\mathcal{T}\varphi^\dagger) \); norm of the discrepancy between \( \mathcal{T}\varphi^\delta_\alpha(\delta, f^\delta) \) and \( \mathcal{T}\varphi^\dagger \) by the rate of \( \mathcal{O}(\delta) \), i.e. \( \| \mathcal{T}\varphi^\delta_\alpha(\delta, f^\delta) - \mathcal{T}\varphi^\dagger \|_\mathcal{H} = \mathcal{O}(\delta) \).

(ii) \( D_J(\varphi^\delta_\alpha(\delta, f^\delta), \varphi^\dagger) = \mathcal{O}(\Psi(\delta)) \); upper bound for the Bregman distance \( D_J \).

(iii) \( \varphi^\delta_\alpha(\delta, f^\delta) \in \mathcal{B}_{\mathcal{O}(\Psi(\delta))}(\varphi^\dagger) \); convergence of the regularized solution \( \varphi^\delta_\alpha(\delta, f^\delta) \) to the true solution \( \varphi^\dagger \) by the rate of the noise amount \( \mathcal{O}(\Psi(\delta)) \), i.e., \( \| \varphi^\delta_\alpha(\delta, f^\delta) - \varphi^\dagger \|_\mathcal{V} = \mathcal{O}(\Psi(\delta)) \).

For derivation of these rates, we refer reader to [26] and references therein.

3. Convex Variational Regularization with the Choice of the Regularization Parameter

It is in this section that we explicitly formulate the necessary condition for the VSC to hold and deliver a coefficient determination.

3.1. Choice of the regularization parameter: Morozov’s discrepancy principle

We are concerned with asymptotic properties of the regularization parameter \( \alpha \) for the Tikhonov-regularized solution obtained by Morozov’s discrepancy principle (MDP). MDP serves as an \textit{a posteriori} parameter choice rule for the Tikhonov type objective
functionals (2.2) and has certain impact on stabilizing the total error functional
\[ E : \mathcal{V} \times \mathcal{V} \rightarrow \mathbb{R}_+ \]
having the assumed relation
\[ E(\varphi_{\alpha(\delta, f^\delta)}, \varphi^\dagger) \leq D_J(\varphi_{\alpha(\delta, f^\delta)}, \varphi^\dagger). \]
As has been introduced in [2, Theorem 3.10] and [3], we use the following set notations
in the theorem formulations that are necessary to establish the error estimation between the
regularized solution \( \varphi_{\alpha(\delta, f^\delta)} \) and the \( J \)-minimizing solution \( \varphi^\dagger \) respectively for the
operator equation (2.1) and for the minimization problem (2.9),
\[
\overline{S} := \left\{ \alpha : \| T \varphi_{\alpha(\delta, f^\delta)} - f^\delta \|_{\mathcal{H}} \leq \overline{\tau} \delta \text{ for some } \varphi_{\alpha} \in \arg \min_{\varphi \in \mathcal{V}} \{ F_\alpha(\varphi, f^\delta) \} \right\}, \quad (3.1)
\]
\[
\underline{S} := \left\{ \alpha : \tau \delta \leq \| T \varphi_{\alpha(\delta, f^\delta)} - f^\delta \|_{\mathcal{H}} \text{ for some } \varphi_{\alpha} \in \arg \min_{\varphi \in \mathcal{V}} \{ F_\alpha(\varphi, f^\delta) \} \right\}, \quad (3.2)
\]
where the discrepancy set radii \( 1 < \tau \leq \overline{\tau} < \infty \) are fixed. Analogously, also as
well known from [19, Eq. (4.57) and (4.58)] and [31, Definition 2.3], we are interested in such a regularization parameter \( \alpha(\delta, f^\delta) \), with some fixed discrepancy set
radii \( 1 < \tau \leq \overline{\tau} < \infty \), that
\[
\alpha(\delta, f^\delta) \in \{ \alpha > 0 \mid \tau \delta \leq \| T \varphi_{\alpha(\delta, f^\delta)} - f^\delta \|_{\mathcal{H}} \leq \overline{\tau} \delta \} &= \overline{S} \cap \underline{S} \text{ for the given } (\delta, f^\delta). \quad (3.3)
\]
It is also the immediate consequences of MDP that the following estimations
\[
\| T \varphi_{\alpha(\delta, f^\delta)} - T \varphi_{\alpha(\delta, f^\dagger)} \|_{\mathcal{H}} \leq (\overline{\tau} + 1) \delta, \quad (3.4)
\]
\[
(\tau - 1) \delta \leq \| T \varphi_{\alpha(\delta, f^\delta)} - T \varphi_{\alpha(\delta, f^\dagger)} \|_{\mathcal{H}}, \quad (3.5)
\]
hold true. Furthermore, according to [26, Corollary 2], the regularization parameter
\( \alpha(\delta, f^\delta) \in \underline{S} \) can be bounded below by,
\[
\alpha(\delta, f^\delta) \geq \frac{1 - \tau^2 - \delta^2}{4 \tau^2 + 1 \Psi((\tau - 1) \delta)}, \quad (3.6)
\]
where \( \Psi \) is a concave, positive definite index function. A new lower bound depending
on this index function for the regularization parameter will be developed. With a stable
lower bound for \( \alpha(\delta, f^\delta) \), possible singularity is avoided as \( \alpha \to 0 \), e.g. see Lemma 5.1
and Lemma 5.2.

3.2. Generalized variational source condition verification

Convergence rates results for some general operator \( T \) can be obtained by formulating
variational inequality which uses the concept of index functions. A function \( \Psi : [0, \infty) \rightarrow [0, \infty) \) is called index function if it is continuous, monotonically increasing
and \( \Psi(0) = 0 \). VSC plays an important role in the development of convergence and
convergence rate results for convex variational regularization strategies. Verification of
this source condition has recently become popular, see [28, 29]. We rather associate the
conventional VSC with the generalized Bregman distance since the objective functional
(2.2) can involve any non-smooth and convex functional \( J \).

Assumption 3.1. [Variational Source Condition] There exists some constant
\( \sigma \in (0, 1) \) and a concave index function \( \Psi \) such that
\[
\frac{\sigma}{2} D_J(\varphi, \varphi_{\alpha(\delta, f^\delta)}) \leq J(\varphi) - J(\varphi_{\alpha(\delta, f^\dagger)}) + \Psi(\| T \varphi - T \varphi_{\alpha(\delta, f^\dagger) \|_{\mathcal{H}}}), \text{ for all } \varphi \in \mathcal{V}. \quad (3.7)
\]
Below the necessary condition for the VSC to hold and a coefficient determination will be formulated. The result is applicable for any convex and smooth/non-smooth penalty term $J$ only in conjunction with MDP.

**Theorem 3.2.** Consider the choice of the regularization parameter a posteriori $\alpha(\delta, f^\delta) \in S \cap S$, with the given data $f^\delta \in B_\delta(f^\dagger)$, for the regularized solution $\varphi^\delta_{\alpha(\delta, f^\delta)} \in D(F_\alpha)$ to the problem (2.9). If, for the positive definite, monotonically increasing and concave index function $\Psi : [0, \infty) \to [0, \infty)$, the following condition holds true

$$\langle p, \varphi^\dagger - \varphi^\delta_{\alpha(\delta, f^\delta)} \rangle \leq C(\tilde{\tau}, \tau) \Psi(\delta),$$

where $p \in \partial J(\varphi^\dagger)$ and $C(\tilde{\tau}, \tau) : (1, \infty) \times [\tilde{\tau}, \infty) \to \mathbb{R}_+$ then the $J$-minimizing solution $\varphi^\dagger$, for some $\tilde{\sigma}(\tilde{\tau}, \tau) : (1, \infty) \times [\tilde{\tau}, \infty) \to (0, 1)$, satisfies the VSC as below,

$$\tilde{\sigma} D_J(\varphi^\delta_{\alpha(\delta, f^\delta)}, \varphi^\dagger) \leq J(\varphi^\delta_{\alpha(\delta, f^\delta)}) - J(\varphi^\dagger) + \Psi \left( ||T \varphi^\delta_{\alpha(\delta, f^\delta)} - T \varphi^\dagger||_{L^2(Z)} \right).$$

**Proof.** Firstly, observe that for the fixed $1 < \tilde{\tau} \leq \tau < \infty$ discrepancy radii,

$$1 < \frac{1}{\tilde{\tau} - 1} = \frac{\tau}{\tau - 1} \leq \frac{\tau}{\tau - 1},$$

which implies

$$\frac{\tau - 1}{\tau} < 1.$$  

This will be beneficial to the coefficient estimation. Now, for the monotonically increasing and concave index function $\Psi : [0, \infty) \to [0, \infty)$, we can estimate

$$\Psi(\delta) \leq \ell \Psi \left( \frac{1}{\tilde{\tau} - 1} ||T \varphi^\delta_{\alpha(\delta, f^\delta)} - T \varphi^\dagger||_H \right) \leq \ell \Psi \left( \frac{\tau}{\tau - 1} ||T \varphi^\delta_{\alpha(\delta, f^\delta)} - T \varphi^\dagger||_H \right) \leq \ell \frac{\tau}{\tau - 1} \Psi \left( ||T \varphi^\delta_{\alpha(\delta, f^\delta)} - T \varphi^\dagger||_H \right),$$

holds true. On the other hand, convexity of the penalty term $J$ implies

$$J(\varphi^\dagger) - J(\varphi^\delta_{\alpha(\delta, f^\delta)}) \leq \langle p, \varphi^\dagger - \varphi^\delta_{\alpha(\delta, f^\delta)} \rangle \leq \frac{\tau}{\tau - 1} \Psi \left( ||T \varphi^\delta_{\alpha(\delta, f^\delta)} - T \varphi^\dagger||_H \right),$$

where $p \in \partial J(\varphi^\dagger)$ and $C(\tilde{\tau}, \tau) := \frac{\tau}{\tau - 1}$. Adding the $J$-difference $J(\varphi^\delta_{\alpha(\delta, f^\delta)}) - J(\varphi^\dagger)$ and taking into consideration the Bregman distance definition (2.6) provides

$$D_J(\varphi^\delta_{\alpha(\delta, f^\delta)}, \varphi^\dagger) \leq J(\varphi^\delta_{\alpha(\delta, f^\delta)}) - J(\varphi^\dagger) + \frac{\tau}{\tau - 1} \Psi \left( ||T \varphi^\delta_{\alpha(\delta, f^\delta)} - T \varphi^\dagger||_H \right),$$

where $1 < \tilde{\tau} \leq \tau < \infty$ are fixed. By defining the coefficient

$$\tilde{\sigma} := \frac{\tau - 1}{\tau} < 1,$$

the VSC has been verified. \[\Box\]

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$\S$ by (3.5)

$\S$ by concavity of $\Psi$

$\S$ by monotone increasing
Remark 3.3. Note that the coefficient defined by (3.12) does not violate the conventional coefficient condition in the VSC presented in (3.7), i.e., the VSC holds for $\sigma \in (0, 1]$. By defining the coefficient $\tilde{\sigma}$ as a multivariable function of discrepancy radii, we only intended to give a general definition for the coefficient. If one sets $\tau = T = \tau^*$, then the coefficient $\tilde{\sigma} \equiv \tilde{\sigma}(\tau^*) : (1, \infty) \to (0, 1)$ boils down to a single variable function.

4. New Bounds for the Regularization Parameter $\alpha(\delta, f^\delta)$

MDP brings new stable lower and upper bounds for the regularization parameter. Let us consider the solution space as a reflexive Banach space $V = L^2(\Omega)$. Having motivated by the condition stated in (3.8), a global estimation for the coefficient $\sigma$ as a result of the Bregman distance definition (2.6) and of the variational source condition can be derived,

$$\frac{\sigma}{2} \langle p, \varphi^\dagger - \varphi \rangle \leq \Psi \left( ||T\varphi - T\varphi^\dagger||_H \right), \text{ for } p \in \partial J(\varphi^\dagger) \text{ and } \varphi \in V. \quad (4.1)$$

Furthermore, since the penalty term $J : L^2(\Omega) \to \mathbb{R}_+$ is convex then a lower bound, which follows from (4.1), for the index function $\Psi$ can be given in terms of $J$-difference as such

$$\frac{\sigma}{2} (J(\varphi^\dagger) - J(\varphi)) \leq \Psi \left( ||T\varphi - T\varphi^\dagger||_H \right) \quad (4.2)$$

On the other hand $\varphi^\delta_\alpha$ is the minimizer of the objective functional (2.2). Thus the estimation (4.2) reads,

$$\frac{\sigma}{4\alpha} ||T\varphi^\delta_\alpha - f^\delta||_H^2 - \frac{\sigma \delta^2}{4\alpha} \leq \frac{\sigma}{2} (J(\varphi^\dagger) - J(\varphi^\delta_\alpha)) \leq \Psi \left( ||T\varphi^\delta_\alpha - T\varphi^\dagger||_H \right), \quad (4.3)$$

which implies

$$\frac{\sigma}{4\alpha} ||T\varphi^\delta_\alpha - f^\delta||_H^2 - \frac{\sigma \delta^2}{4\alpha} \leq \Psi \left( ||T\varphi^\delta_\alpha - T\varphi^\dagger||_H \right). \quad (4.4)$$

A new lower bound for the regularization parameter will rise from this global estimation. As also well known by the literature (e.g [26]) lower bound is crucial to control the trade off between $\delta$ and $\alpha$. Unlike in the aforementioned literature, our lower bound contains the coefficient $\sigma$ from (3.7) and has a simpler form.

Theorem 4.1. Let the regularization parameter $\alpha = \alpha(\delta, f^\delta)$, for the minimizer $\varphi^\delta_{\alpha(\delta, f^\delta)} \in \mathcal{D}(F_\alpha)$ of the objective functional $F_\alpha$ in (2.2), be chosen according to the discrepancy principle $\alpha(\delta, f^\delta) \in \tilde{S} \cap \underline{S}$ where the given data $f^\delta \in B_{\delta}(f^\dagger)$. Then this choice of regularization parameter, for $\sigma \in (0, 1]$, and for the fixed $1 < \underline{\tau} \leq \tau < \infty$ coefficients, implies the following lower bound for the regularization parameter $\alpha(\delta, f^\delta)$,

$$\frac{\sigma}{4} (\tau - 1) \frac{\delta^2}{\Psi(\delta)} \leq \alpha(\delta, f^\delta). \quad (4.5)$$

Proof. If the regularization parameter is chosen according to the discrepancy principle $\alpha(\delta, f^\delta) \in \tilde{S} \cap \underline{S}$ where the given data $f^\delta \in B_{\delta}(f^\dagger)$, then it follows from (4.4) that

$$\frac{\sigma \tau \delta^2}{4\alpha(\delta, f^\delta)} - \frac{\sigma \delta^2}{4\alpha(\delta, f^\delta)} \leq \Psi \left( ||T\varphi^\delta_{\alpha(\delta, f^\delta)} - T\varphi^\dagger||_H \right). \quad (4.6)$$
Recall that the index function $\Psi$ is a concave function. We then, from (3.4), conclude that
\[ \frac{\sigma (\tau^2 - 1)}{4} \frac{\delta^2}{(\tau + 1) \alpha(\delta, f^\delta)} = \frac{\sigma (\tau - 1)}{4} \frac{\delta^2}{\alpha(\delta, f^\delta)} \leq \Psi(\delta), \text{ for } 1 < \tau \leq \tau < \infty. \] (4.7)

From the assertion above, a stable lower bound for the index function can also be obtained. However, this rises the question of a stable maximum value of the regularization parameter. Regardless of the choice of regularization parameter, there exists some $\delta_{\text{max}} > \delta$ such that $\alpha < \alpha_{\text{max}} = \alpha(\delta_{\text{max}})$. Analogous to [27, Eq (3.2) of Proposition 3.1], we will estimate an improvised form of this maximum value in consideration of the introduced lower bound (4.5).

**Theorem 4.2.** Provided that the regularization parameter $\alpha(\delta, f^\delta) \in \mathcal{S} \cap \mathcal{S}$ for the regularized solution $\varphi^\delta(\alpha(\delta, f^\delta))$ of the problem (2.9) and $\varphi^\dagger$ is the $J$-minimizing solution of (2.4), then there can be defined a maximum value for the regularization parameter depending on the positive definite and concave index function $\Psi : [0, \infty) \rightarrow [0, \infty)$,
\[ \alpha_{\text{max}} := (O(\Psi(\delta)) + J(\varphi^\dagger))^{-1} \] (4.8)
such that $F_{\alpha_{\text{max}}} < \infty$.

**Proof.** Let us consider the following form of the objective functional
\[ F_{\alpha_{\text{max}}} (\varphi^\delta, f^\delta) = \frac{1}{2} \| T \varphi^\delta - f^\delta \|_H^2 + \alpha_{\text{max}} J(\varphi^\delta), \text{ for some } \alpha > 0. \] (4.9)
It follows from here, for some $\alpha > 0$, that
\[ F_{\alpha_{\text{max}}} (\varphi^\delta, f^\delta) \leq \frac{1}{2} \| T \varphi^\delta - f^\delta \|_H^2 + \delta^2 + \alpha_{\text{max}} J(\varphi^\delta) \]
\[ = F_{\alpha} (\varphi^\delta, f^\delta) + (\alpha_{\text{max}} - \alpha) J(\varphi^\delta) + \delta^2 \]
\[ \leq F_{\alpha} (\varphi^\dagger, f^\delta) + (\alpha_{\text{max}} - \alpha) J(\varphi^\delta) + \delta^2 \]
\[ \leq \frac{\delta^2}{2} + \alpha J(\varphi^\dagger) + (\alpha_{\text{max}} - \alpha) J(\varphi^\delta) + \delta^2 \]

Since $\varphi^\delta \in \mathcal{D}(F_{\alpha})$, then $\alpha J(\varphi^\delta) \leq \frac{\alpha \delta^2}{2} + \alpha J(\varphi^\dagger)$ for some $\alpha > 0$. Furthermore $\frac{\alpha_{\text{max}}}{\alpha} \geq 1$.

Thus, these facts yield that
\[ F_{\alpha_{\text{max}}} (\varphi^\delta, f^\delta) \leq \frac{\delta^2}{2} + \alpha J(\varphi^\dagger) + (\alpha_{\text{max}} - \alpha) J(\varphi^\dagger) + \frac{3\delta^2}{2} \]
\[ = 2\delta^2 + \alpha_{\text{max}} J(\varphi^\dagger) \]
\[ \leq 2\delta^2 \frac{\alpha_{\text{max}}}{\alpha} + \alpha_{\text{max}} J(\varphi^\dagger) \]
\[ = \alpha_{\text{max}} \left( \frac{2\delta^2}{\alpha} + J(\varphi^\dagger) \right), \text{ for some } \alpha > 0. \]

We proceed with this estimation by making use of the lower bound estimated in (4.5) as such,
\[ F_{\alpha_{\text{max}}} (\varphi^\delta, f^\delta) \leq \alpha_{\text{max}} \left( \frac{8}{\sigma^2} (\tau - 1) \Psi(\delta) + J(\varphi^\dagger) \right). \]
The result is hence obtained by defining
\[
\alpha_{\text{max}} := \left( \frac{8}{\sigma} (\tau - 1) \Psi(\delta) + J(\varphi^\dagger) \right)^{-1}.
\tag{4.10}
\]

As has been mentioned above, a stable maximum value for the regularization parameter \( \alpha(\delta, f^\delta) \) yields a stable lower bound for the index function \( \Psi \) owing to (4.11). We, then, close this section with providing the following corollary.

**Corollary 4.3.** If one plugs the maximum value for the regularization parameter explicitly defined by (4.10) into (4.5), then one obtains
\[
\frac{\sigma}{4} \left( \frac{8}{\sigma} (\tau - 1) \Psi(\delta) + J(\varphi^\dagger) \right) \delta^2 \leq \Psi(\delta).
\tag{4.11}
\]

5. Contribution of the VSC to Stabilize the Bregman Distance

As has been motivated above in the subsection 3.1, our choice of regularization parameter must fulfill (1.3). Moving on from here and together with (4.1), we will obtain stable upper bounds for the Bregman distance \( D_J \), or for the total error value functional \( E \), see (1.2). We will also see that it is also possible to bound the reverse Bregman distance \( D_J(\varphi^\dagger, \varphi) \). With this upper bound, we will eventually arrive at the quantitative estimation for the symmetric Bregman distance \( D_{\text{sym}}^J \).

Therefore, the important question to be answered is how to control the trade-off between the noise amount \( \delta \) and the regularization parameter \( \alpha \). It will be observed that this controllability is only possible when the choice of the regularization parameter is specified which is Morozov’s discrepancy principle in our case. As a result of this choice and of the inclusion of the VSC, the quantitative estimations for the Bregmans distance depend on the discrepancy set radii and the coefficient in the VSC. In this section, the function space of the measured data will be taken as \( L^2(\mathbb{Z}) \) where \( \mathbb{Z} = \mathcal{D}(f^\delta) \).

**Lemma 5.1.** Let the regularization parameter \( \alpha = \alpha(\delta, f^\delta) \), for the minimizer \( \varphi_{\alpha(\delta, f^\delta)}^\dagger \in \mathcal{D}(F_\alpha) \) of the objective functional \( F_\alpha \) in (2.2), be chosen according to the discrepancy principle \( \alpha(\delta, f^\delta) \in \mathfrak{S} \cap \mathfrak{S} \) where the given data \( f^\delta \in \mathcal{B}_\delta(f^\dagger) \). Furthermore, suppose that the \( J \)-minimizing solution obeys the VSC (3.7). Then, this a posteriori rule for the choice of the regularization parameter stabilises the following \( J \)-difference
\[
J(\varphi_{\alpha(\delta, f^\delta)}^\dagger) - J(\varphi^\dagger) = \mathcal{O}(\Psi(\delta)).
\tag{5.1}
\]

**Proof.** Since \( \varphi_{\alpha}^\dagger \in \mathcal{D}(F_\alpha) \) is the minimizer of the objective functional \( F_\alpha \), for some \( \alpha > 0 \) and for any \( \varphi \in \mathcal{D}(F_\alpha) \) it holds that \( F_\alpha(\varphi_{\alpha}^\dagger) \leq F_\alpha(\varphi) \). This implies the following,
\[
J(\varphi_{\alpha}^\dagger) - J(\varphi^\dagger) \leq \frac{\delta^2}{2\alpha}.
\tag{5.2}
\]
Here, the decrease in $\alpha$ will cause a blow-up on the right hand side. This is controlled by the choice of the regularization parameter $\alpha = \alpha(\delta, f^\delta) \in S$. Thus, we make use of the lower bound for the regularization parameter given in (3.6) to have a stable upper bound by using the facts that $\Psi$ is a concave and increasing function,

$$\frac{\delta^2}{2\alpha(\delta, f^\delta)} \leq 2\frac{\tau^2 + 1}{\tau - 1} \Psi((\tau - 1)\delta) \leq \frac{2\tau^2 + 1}{\tau - 1} \Psi((\tau + 1)\delta) \leq \frac{2\tau^2 + 1}{\tau - 1} \Psi(\delta)$$  \hspace{1cm} (5.3)

Hence, this control over the trade-off between $\delta^2$ and $\alpha$ yields the desired result. \hfill $\square$

The lemma above is comparable to its counterparts in the literature, [2, Corollary 4.2], [3, Lemma 2.8], [4, Eq. (2.17)], [23, Theorem 4.4] [26, Lemma 1].

We, below, reformulate the result with a new proof since a new lower bound for the regularization parameter that has been stated in Theorem 4.1 will be included.

Lemma 5.2. Let the regularization parameter $\alpha = \alpha(\delta, f^\delta)$, for the minimizer $\varphi^\delta_{(\delta, f^\delta)} \in D(F_\alpha)$ of the objective functional $F_\alpha$ in (2.2), be chosen according to the discrepancy principle $\alpha(\delta, f^\delta) \in S \cap S$ where the given data $f^\delta \in B_\delta(f^\dagger)$. Then this a posteriori rule for the choice of the regularization parameter stabilises the following $J$ difference

$$J(\varphi^\delta_{(\delta, f^\delta)}) - J(\varphi^\dagger) = \mathcal{O}(\Psi(\delta)).$$ \hspace{1cm} (5.4)

Proof. Likewise before, since $F_\alpha(\varphi^\delta_{(\delta, f^\delta)}) \leq F_\alpha(\varphi)$,

$$J(\varphi^\delta_{(\delta, f^\delta)}) - J(\varphi^\dagger) \leq \frac{\delta^2}{2\alpha},$$ \hspace{1cm} (5.5)

holds true for some $\alpha > 0$. The choice of regularization parameter, as we have seen in Theorem 4.1, provides the stable lower bound (4.5). Plugging that lower bound into (5.5) stabilizes the $J$-difference as such,

$$\frac{\delta^2}{2\alpha(\delta, f^\delta)} \leq \frac{1}{\sigma(\tau - 1)} \Psi(\delta).$$ \hspace{1cm} (5.6)

$\square$

Now tight rates for the total error estimation can be established. We will present two results, one of which is for the usual Bregman distance and the other one is for its reverse form. These results will inherently lead to the stable upper bound for the symmetric Bregman distance that has been defined by (2.7).

Theorem 5.3. Let the $J$-minimizing solution $\varphi^\dagger \in V$ for the operator equation (2.1) satisfy Assumption 3.1. Under the same conditions in Lemma 5.2, we then have

$$D_J(\varphi^\delta_{(\delta, f^\delta)}, \varphi^\dagger) = \mathcal{O}(\Psi(\delta)),$$ \hspace{1cm} (5.7)

as $\delta \to 0$.

$\|\text{Since } \Psi \text{ is an increasing function, then } \Psi((\tau - 1)\delta) \leq \Psi((\tau + 1)\delta)$.  
$\|\text{Due to the concavity of } \Psi, \Psi((\tau + 1)\delta) \leq (\tau + 1)\Psi(\delta) \text{ holds, see [26, Eq. 2.3 of Proposition 1].}$
Variational Source Condition with the Bregman Distance

Proof. Since the true solution $\varphi^\dagger$ satisfies Assumption 3.1,

$$D_J(\varphi^\delta_{\alpha(\delta,f^\delta)}, \varphi^\dagger) \leq J(\varphi^\delta_{\alpha(\delta,f^\delta)}) - J(\varphi^\dagger) + \Psi\left(||T \varphi^\delta_{\alpha(\delta,f^\delta)} - T \varphi^\dagger||_{L^2(Z)}\right).$$

$$\leq \frac{2}{\sigma(\tau - 1)} \Psi(\delta) + \Psi((\tau + 1)\delta)$$

$$\leq \frac{2}{\sigma(\tau - 1)} \Psi(\delta) + (\tau + 1)\Psi(\delta).$$

(5.8)

The first term on the right hand side, the bound for the $J$ difference, comes from Lemma 5.2. As in the estimation (5.6) of Lemma 5.2, $\Psi$ is concave function, thus $\Psi((\tau + 1)\delta) \leq (\tau + 1)\Psi(\delta)$. □

Theorem 5.4. Let the regularization parameter $\alpha = \alpha(\delta, f^\delta)$, for the minimizer $\varphi^\delta_{\alpha(\delta,f^\delta)} \in D(F_\alpha)$ of the objective functional $F_\alpha$ in (2.2), be chosen according to the discrepancy principle $\alpha(\delta, f^\delta) \in \overline{S} \cap \underline{S}$ where the given data $f^\delta \in B_\delta(f^\dagger)$. Suppose that the $J$–minimizing solution $\varphi^\dagger \in \mathcal{V}$, where $T \varphi^\dagger = f^\dagger$, satisfies Assumption 3.1 with the concave and monotonically increasing index function $\Psi : [0, \infty) \to [0, \infty)$. Then, this a posteriori rule for the choice of regularization parameter yields the following rate,

$$D_J(\varphi^\dagger, \varphi^\delta_{\alpha(\delta,f^\delta)}) = O(\Psi(\delta)), \quad (5.9)$$
as $\delta \to 0$.

Proof. Firstly, by Assumption 3.1, it can easily be observed that,

$$J(\varphi^\dagger) - J(\varphi^\delta_{\alpha(\delta,f^\delta)}) \leq \Psi\left(||T \varphi^\delta_{\alpha(\delta,f^\delta)} - T \varphi^\dagger||_{L^2(Z)}\right).$$

From the early observation (3.4) and since $\Psi$ is a monotonically increasing, concave function, we obtain,

$$J(\varphi^\dagger) - J(\varphi^\delta_{\alpha(\delta,f^\delta)}) \leq \Psi((\tau + 1)\delta) \leq (\tau + 1)\Psi(\delta). \quad (5.10)$$

Regarding the aimed upper bound for the Bregman distance, use the estimation (5.10) for $\alpha(\delta, f^\delta) \in \overline{S}$ and observe the following for $p \in \partial J(\varphi^\delta_{\alpha(\delta,f^\delta)})$,

$$D_J(\varphi^\dagger, \varphi^\delta_{\alpha(\delta,f^\delta)}) = J(\varphi^\dagger) - J(\varphi^\delta_{\alpha(\delta,f^\delta)}) - \langle p, \varphi^\dagger - \varphi^\delta_{\alpha(\delta,f^\delta)}\rangle$$

$$= J(\varphi^\dagger) - J(\varphi^\delta_{\alpha(\delta,f^\delta)}) + \langle p, \varphi^\delta_{\alpha(\delta,f^\delta)} - \varphi^\dagger\rangle$$

$$\leq (\tau + 1)\Psi(\delta) + \langle p, \varphi^\delta_{\alpha(\delta,f^\delta)} - \varphi^\dagger\rangle.$$ \hspace{1cm} (5.11)

Since the regularized solution $\varphi^\delta_{\alpha(\delta,f^\delta)}$ satisfies the first order optimality condition (2.10), we then have,

$$D_J(\varphi^\dagger, \varphi^\delta_{\alpha(\delta,f^\delta)}) \leq (\tau + 1)\Psi(\delta) + \frac{1}{\alpha(\delta, f^\delta)}\langle \mathcal{T}^\star(f^\delta - T \varphi^\delta_{\alpha(\delta,f^\delta)}), \varphi^\delta_{\alpha(\delta,f^\delta)} - \varphi^\dagger\rangle.$$

Now apply the Cauchy-Schwarz inequality and take into account the estimation (3.4), $||T \varphi^\delta_{\alpha(\delta,f^\delta)} - f^\delta||_{L^2(Z)} \leq \tau \delta$ for the choice of regularization parameter $\alpha(\delta, f^\delta) \in \overline{S}$, to
arrive at
\[
D_J(\varphi^\dagger, \varphi^\delta_{\alpha(\delta,f)})(\varphi^\dagger, \varphi^\delta_{\alpha(\delta,f)}) \leq (\tau + 1)\Psi(\delta) + \frac{1}{\alpha(\delta, f^\delta)}\|
\nT \varphi^\delta_{\alpha(\delta,f)} - f^\delta\|_{L^2(Z)}\|\mathcal{T} \varphi^\delta_{\alpha(\delta,f)} - T \varphi^\dagger\|_{L^2(Z)}
\leq (\tau + 1)\Psi(\delta) + \frac{1}{\alpha(\delta, f^\delta)}\tau \delta\|\mathcal{T} \varphi^\delta_{\alpha(\delta,f)} - T \varphi^\dagger\|_{L^2(Z)}
\leq (\tau + 1)\Psi(\delta) + \frac{1}{\alpha(\delta, f^\delta)}\tau(\tau + 1)\delta^2
\leq (\tau + 1)\Psi(\delta) + \frac{4\tau(\tau + 1)}{\sigma(\tau - 1)}\Psi(\delta).
\]

Here, again, the lower bound for the regularization parameter \(\alpha(\delta, f^\delta)\) given in (4.5) has controlled the trade-off between \(\delta^2\) and \(\alpha\). Hence, this yields the stable upper bound (5.9).

Upper bounds obtained in the theorems 5.3 and 5.4 provide upper bound for the symmetric Bregman distance defined in (2.7). The finalizing result of this section can be compared to [23, Proof of Theorem 4.4].

**Corollary 5.5.** From the theorems 5.3 and 5.4, and by the definition given in (2.7), it is concluded that
\[
D^\text{sym}_J(\varphi^\delta_{\alpha(\delta,f)}, \varphi^\dagger) = \mathcal{O}(\Psi(\delta)), \text{ as } \delta \to 0.
\]

6. Conclusion and Future Prospects

The goal of this work has been providing a general analysis for the verification of the generalized variational source condition given by (3.7). Without specification of the rule for the choice of the regularization parameter, the results above would not have been obtained. Certainly, further necessary tool is a stable lower bound for \(\alpha(\delta, f^\delta)\) stated by Theorem 4.1. The condition given in (3.8) has been mentioned in [23, Theorem 4.4] but only for the quadratic Tikhonov functional.

Further generalization of this work would be possible by considering the following form of the Tikhonov functional,
\[
F^\alpha(\varphi, f^\delta) := \frac{1}{q}\||\mathcal{T} \varphi - f^\delta\|_{H}^q + \alpha J(\varphi).
\]

The order of norm \(q\) will change the rates of the error estimation.

Interpretation of this work will be introduced with considering different penalty terms \(J\). From the early assumption (1.2), a lower bound which is a function of corresponding norm, say \(\Phi(||\varphi^\delta_{\alpha(\delta,f)} - \varphi^\dagger||_V)\) per different \(J\) will permit one to obtain norm convergence result. With the involvement of any \(J\) in (2.2), or equivalently in (6.1), defining regularity properties for the solution function \(\varphi\) will be broadened. To be more specific, different norm convergence and convergence rates results will also follow from which function space \(V\) is considered.
APPENDIX

A. A Symmetric Form of the Variational Source Condition

Here, we question whether it is plausible to state different form of VSC which is rather associated with the reverse Bregman distance.

**Lemma A.1.** Denote by \( \varphi^\alpha_{(\delta, f^\delta)} \) the regularized solution for the problem (2.9) where the regularization parameter \( \alpha(\delta, f^\delta) \in \mathcal{S} \cap \mathcal{S}^c \) is chosen a posteriori. Let \( \varphi^\dagger \) be the \( J \)-minimizing solution introduced in (2.12). Then, by this choice of regularization parameter, the following \( J \)-difference

\[
J(\varphi^\alpha_{(\delta, f^\delta)}) - J(\varphi^\dagger) = \mathcal{O}(\delta^2),
\]

holds true.

**Proof.** The regularized solution \( \varphi^\alpha \in \mathcal{D}(F_\alpha) \) surely implies

\[
\alpha \left( J(\varphi^\alpha) - J(\varphi^\dagger) \right) \leq \frac{1}{2} \left( \| \mathcal{T} \varphi^\dagger - f^\delta \|_{L^2(\mathcal{Z})}^2 - \| \mathcal{T} \varphi^\alpha - f^\delta \|_{L^2(\mathcal{Z})}^2 \right).
\]

The right hand side of this inequality can recalculated in the following way,

\[
\| \mathcal{T} \varphi^\dagger - f^\delta \|_{L^2(\mathcal{Z})}^2 - \| \mathcal{T} \varphi^\alpha - f^\delta \|_{L^2(\mathcal{Z})}^2 = \langle \mathcal{T} \varphi^\dagger - f^\delta, \mathcal{T} \varphi^\dagger - f^\delta \rangle - \langle \mathcal{T} \varphi^\alpha - f^\delta, \mathcal{T} \varphi^\alpha - f^\delta \rangle
\]

\[
= \langle \mathcal{T} \varphi^\dagger - f^\delta, \mathcal{T} \varphi^\dagger - f^\delta \rangle - \langle \mathcal{T} \varphi^\alpha - f^\delta, \mathcal{T} \varphi^\alpha - f^\delta \rangle - \langle \mathcal{T} \varphi^\alpha - f^\delta, \mathcal{T} \varphi^\dagger - f^\delta \rangle
\]

\[
= \langle \mathcal{T} \varphi^\dagger - f^\delta, \mathcal{T} \varphi^\dagger - f^\delta \rangle - \langle \mathcal{T} \varphi^\alpha - f^\delta, \mathcal{T} \varphi^\alpha - f^\delta \rangle
\]

We now apply Cauchy-Schwarz inequality,

\[
\| \mathcal{T} \varphi^\dagger - f^\delta \|_{L^2(\mathcal{Z})}^2 - \| \mathcal{T} \varphi^\alpha - f^\delta \|_{L^2(\mathcal{Z})}^2 \leq \delta \| \mathcal{T} \varphi^\dagger - f^\delta \|_{L^2(\mathcal{Z})} \| \mathcal{T} \varphi^\alpha - f^\delta \|_{L^2(\mathcal{Z})} + \| \mathcal{T} \varphi^\alpha - f^\delta \|_{L^2(\mathcal{Z})} \| \mathcal{T} \varphi^\dagger - f^\delta \|_{L^2(\mathcal{Z})}.
\]

By the choice of regularization parameter \( \alpha(\delta, f^\delta) \in \mathcal{S} \cap \mathcal{S}^c \), we obtain

\[
\| \mathcal{T} \varphi^\dagger - f^\delta \|_{L^2(\mathcal{Z})}^2 - \| \mathcal{T} \varphi^\alpha_{(\delta, f^\delta)} - f^\delta \|_{L^2(\mathcal{Z})}^2 \leq \delta \| \mathcal{T} \varphi^\alpha_{(\delta, f^\delta)} - \mathcal{T} \varphi^\dagger \|_{L^2(\mathcal{Z})} + \| \mathcal{T} \varphi^\alpha_{(\delta, f^\delta)} - f^\delta \|_{L^2(\mathcal{Z})} \| \mathcal{T} \varphi^\dagger - f^\delta \|_{L^2(\mathcal{Z})}
\]

\[
= (1 + \delta) \| \mathcal{T} \varphi^\alpha_{(\delta, f^\delta)} - \mathcal{T} \varphi^\dagger \|_{L^2(\mathcal{Z})}
\]

Further step from here can be estimated by taking into account (3.4)

\[
\| \mathcal{T} \varphi^\dagger - f^\delta \|_{L^2(\mathcal{Z})} - \| \mathcal{T} \varphi^\alpha_{(\delta, f^\delta)} - f^\delta \|_{L^2(\mathcal{Z})} \leq (1 + \delta) \| \mathcal{T} \varphi^\dagger - f^\delta \|_{L^2(\mathcal{Z})}
\]

Thus, from (1.2), we arrive at

\[
J(\varphi^\alpha_{(\delta, f^\delta)}) - J(\varphi^\dagger) \leq (1 + \delta) \frac{\delta^2}{\alpha(\delta, f^\delta)}.
\]

We make use of the estimation (1.4) to compare the well-known index function against generalized reverse Bregman distance in the following theorem without avoiding possible singularity as \( \alpha \to 0 \). This comparison may give birth to a new form of the VSC.
Theorem A.2. Denote by \( \varphi^\delta_{\alpha(\delta, f^\delta)} \) the regularized solution for the problem (2.9) where the regularization parameter \( \alpha(\delta, f^\delta) \in \overline{S} \cap \underline{S} \) is chosen a posteriori. Let \( \varphi^\dagger \) be the \( J \)-minimizing solution introduced in (2.12). Then, for some index function \( \Psi : [0, \infty) \to [0, \infty) \), the following
\[
D_J(\varphi^\dagger, \varphi^\delta_{\alpha(\delta, f^\delta)}) \leq \Psi(t),
\]
holds true.

Proof. Let us begin the proof with assuming the opposite, i.e. let us assume that
\[
D_J(\varphi^\dagger, \varphi^\delta_{\alpha(\delta, f^\delta)}) > \Psi(t),
\]
holds for any \( t \in [0, \infty) \). Then, by the definition of Bregman distance in (2.6), for \( p \in \partial J(\varphi^\delta_{\alpha(\delta, f^\delta)}) \) (1.6) reads
\[
J(\varphi^\dagger) - J(\varphi^\delta_{\alpha(\delta, f^\delta)}) - \langle p, \varphi^\dagger - \varphi^\delta_{\alpha(\delta, f^\delta)} \rangle > \Psi(t).
\]
Let us include the \( J \)-difference (1.4) and rewrite the inner product
\[
\langle p, \varphi^\delta_{\alpha(\delta, f^\delta)} - \varphi^\dagger \rangle - (1 + \tau)^2 \frac{\delta^2}{\alpha(\delta, f^\delta)} > \Psi(t).
\]
The regularized solution \( \varphi^\delta_{\alpha(\delta, f^\delta)} \) must satisfy the first order optimality condition (2.10). Thus,
\[
\frac{1}{\alpha(\delta, f^\delta)} \langle \mathcal{T}^* (f^\delta - \mathcal{T} \varphi^\delta_{\alpha(\delta, f^\delta)}), \varphi^\delta_{\alpha(\delta, f^\delta)} - \varphi^\dagger \rangle - (1 + \tau)^2 \frac{\delta^2}{\alpha(\delta, f^\delta)} > \Psi(t).
\]
By the Cauchy-Schwarz inequality, and by the fact that \( \alpha(\delta, f^\delta) \in \overline{S} \cap \underline{S} \) provides the immediate estimation in (3.4), we eventually arrive at the following estimation
\[
\Psi(t) < \frac{1}{\alpha(\delta, f^\delta)} \tau \delta \| \mathcal{T} \varphi^\delta_{\alpha(\delta, f^\delta)} - \mathcal{T} \varphi^\dagger \|_{L^2(z)} - (1 + \tau)^2 \frac{\delta^2}{\alpha(\delta, f^\delta)}
\]
\[
< \frac{1}{\alpha(\delta, f^\delta)} \tau (1 + \tau) \delta^2 - (1 + \tau)^2 \frac{\delta^2}{\alpha(\delta, f^\delta)}
\]
which is a direct contradiction to the positive definiteness of the index function \( \Psi \). Hence, the assumption in (1.6) is wrong. \( \Box \)
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