Generalized hierarchy of matrix Burgers type and $n$-wave equations

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Abstract
This article concerns the dressing method for solving of multidimensional nonlinear Partial Differential Equations. In particular, we join hierarchy of matrix Burgers type equation with hierarchies of equations integrable by the Inverse Spectral Transform (IST). Example of resonance interaction of wave packets in (3+1)-dimensions is given.

1 Introduction

It is well known that different versions of the dressing method [1, 2, 3, 4, 5] are very successful tools for solving of so called completely integrable nonlinear Partial Differential Equations (PDE). These equations, in turn, have wide application area in different branches of physics, such as hydrodynamics, plasma physics, superconductivity, nonlinear optics. Although till recently dressing methods have been used only for PDE integrable by the Inverse Spectral Transform (IST), it has been shown in [6] that there is another type of equations (maybe not integrable by IST) which admit properly modified dressing procedure for construction of large manifold of their solutions. But the technique proposed there left opened many questions. For instance, it is not clear whether derived nonlinear PDE can be linearized by some substitution [7, 8]. Also it was difficult to characterize the manifold of available solutions.

In this paper we replace algebraic operator with integral one, generalize system of equations introducing the set of additional parameters (independent variables of nonlinear PDE) and modify significantly the algorithm given in [6]. This allows us to simplify description of PDE’s properties, exhibit more information about solution manifold as well as relations to the classical solvable (both linearizable and integrable by IST) PDE. Although all statements following hereafter can be proved, we omit most of the proofs for the sake of brevity. They will be given in different paper.

Thus, the basic object is the following $N \times N$ matrix integral equation

$$\Phi \equiv \Phi(\lambda, \mu; t) = \int_{D_\nu} \Psi(\lambda, \nu; t)U(\nu, \mu; t) d\nu \equiv \Psi * U,$$

(1)
where $\lambda = (\lambda_1, \ldots, \lambda_N), \mu = (\mu_1, \ldots, \mu_N), \nu = (\nu_1, \ldots, \nu_N)$ are vector variables with different length in general; integration is over whole space $D_\nu$ of the appropriate vector parameter; $\Phi$, $\Psi$ and $U$ are $N \times N$ matrix functions of arguments. Star means integration over space of inner variable: $f \ast g \equiv \int f(\lambda, \nu)g(\nu, \mu)d\nu$. We require that $\Psi$ is invertible operator, i.e. equation (1) can be solved uniquely for the function $U$. By definition, operator $A(\lambda, \mu)$ is invertible, if there are operators $A_L^{-1}(\lambda, \mu)$ and $A_R^{-1}(\lambda, \mu)$ such that $\int_{D_\nu} A(\lambda, \nu)A_R^{-1}(\nu, \mu)d\nu = \int_{D_\nu} A_L^{-1}(\lambda, \nu)A(\nu, \mu)d\nu = \delta(\lambda - \mu)$. The functions $\Phi$ and $\Psi$ are related by means of the compatible system of linear integral-differential equations, which, on the other hand, introduce (infinite) set of additional parameters $t = (t_1, t_2, \ldots)$ (independent variables of nonlinear PDE):

$$M_i \ast \Psi = \sum_k L_{ik} \ast \Phi \ast C_{ki}, \quad C_{ki} = C_{ki}(\lambda, \mu; t), \quad i = 1, 2, \ldots \quad (2)$$

where $M_j = M_j(\lambda, \mu; \partial_{t_1}, \partial_{t_2}, \ldots)$ are first order and $L_{jk} = L_{jk}(\lambda, \mu; \partial_{t_1}, \partial_{t_2}, \ldots)$ are arbitrary order linear differential operators with matrix coefficients depending on $\lambda$ and $\mu$. This overdetermined system together with its compatibility condition defines $\Psi$ and $\Phi$. Finally, the same compatibility condition with substitution $\Phi$ from the eq.(1) results in nonlinear PDE whose solution is expressed in terms of $U$. After this preliminary discussion we derive some general equations using the following simplified version of the system (2):

$$\Psi_{t_i} = S_i \ast \Phi \ast C_i, \quad i = 1, 2, \ldots \quad (3)$$

where $S_i(\lambda, \mu; t)$ and $C_i(\lambda, \mu; t)$ are known functions of $t$, which will be seen later. The compatibility condition for the system (3) has the form

$$S_{it_i} \ast \Phi \ast C_i - S_{j_{t_i}} \ast \Phi \ast C_j + S_i \ast \Phi \ast C_{it_j} - S_j \ast \Phi \ast C_{jt_i} + S_i \ast \Phi_{t_j} \ast C_i - S_j \ast \Phi_{t_i} \ast C_j = 0 \quad (4)$$

which is linear system of compatible integral-differential equations for the function $\Phi$. Solving this equation, substituting result in (3) and integrating it, we obtain the expression for $\Psi$: $\Psi(\lambda, \mu; t) = \partial_{t_i}^{-1}(S_i \ast \Phi \ast C_1)(\lambda, \mu; t) + E(\lambda, \mu) + F(\lambda, \mu; t_2, t_3, \ldots)$. Here $E$ is invertible operator, function $F$ provides compatibility of the system (3). Being invertible, operator $\Psi$ provides unique solution to the eq.(1).

On the other hand, eq.(4) may be given another form after substitution eq.(1) for $\Phi$ and (3) for $\Psi_{t_i}$

$$S_{it_j} \ast \Psi \ast U \ast C_i - S_{j_{t_i}} \ast \Psi \ast U \ast C_j + S_i \ast \Psi \ast U \ast C_{it_j} - S_j \ast \Psi \ast U \ast C_{jt_i} + S_i \ast (S_j \ast \Psi \ast U \ast C_j \ast U + \Psi \ast U_{t_j}) \ast C_i - S_j \ast (S_i \ast \Psi \ast U \ast C_i \ast U + \Psi \ast U_{t_i}) \ast C_j = 0 \quad (5)$$

which is nonlocal equation quadratic in $U$. It may result in nonlinear PDE for the dependent variables, expressed in terms of $U$, $S_i$ and $C_i$. To provide this possibility we must impose specific dependence of the functions $S_i$ and $C_i$ on their arguments, for instance, in accordance
to the following set of relations:

\[ S_i(\lambda, \mu; t) = S(\lambda, \mu), \quad \Phi = S \ast \Phi + \chi, \quad \chi = \chi(\lambda, \mu; t), \quad (6) \]

\[ C_i(\lambda, \mu; t) = \int_{D_\nu} A_i(\lambda, \nu)p_1(\nu; t)p_2(\mu)d\nu + c_1(\lambda) B_i c_2(\mu; t) \equiv A_i \ast p_1(t)p_2 + c_1 B_i c_2(t), \quad (7) \]

\[ A_i \ast c_1 = c_1 B_i, \quad \ast (8) \]

where \( A \) is invertible operator, \( A_i \ast A_j = A_j \ast A_i, \) \([B_i, B_j] = 0\) and \([\ast, \ast]\) means commutator of two matrices. Eqs.(6-8) split eq.(4) into the following set of three integral-differential equations for \( \Phi, \) \( p_1 \) and \( c_2 \):

\[ S \ast \Phi_{t_j} \ast A_1 - S \ast \Phi_{t_1} \ast A_j = 0, \quad (9) \]

\[ A_1 \ast (p_1)_{t_j} - A_j \ast (p_1)_{t_1} = 0, \quad (10) \]

\[ B_1 c_{2t_j} - B_j c_{2t_1} = 0. \quad (11) \]

Then one has the following nonlinear equation instead of (5)

\[ \Psi \ast (U_{t_j} \ast A_1 - U_{t_1} \ast A_j + U \ast C_j \ast U \ast A_1 - U \ast C_1 \ast U \ast A_j) + \chi_{t_1} \ast A_j - \chi_{t_j} \ast A_1 + \chi \ast (C_1 \ast U \ast A_j - C_j \ast U \ast A_1) = 0. \quad (12) \]

Note, that reduction leading to the equations introduced in [6] will be discussed in different paper. Here we consider two other examples of multidimensional systems. First of them (Sec.2.) represents combination of linearizable (Burgers type) and completely integrable (n-wave) \((3 + 1)-dimensional\) systems, having solutions depending on arbitrary functions of three variables. Second example (Sec.3) is another generalization of the matrix \(n\)-wave system [9]. Properly introduced multiple scales expansion of this system results in the multidimensional \((3+1)-dimensional\) equation describing resonance interaction of wave packets. Its solutions may depend on arbitrary functions of two variables. Both examples have extension into \((n + 1)\)-dimensions with arbitrary \(n\).

2 Generalized hierarchy of linearizable (Burgers type) and integrable by IST \((n\)-wave\) systems

In this section \(S(\lambda, \mu) = \delta(\lambda - \mu), \chi = 0, \) \(A_j = A \ast \cdots \ast A \equiv A^j, B_j = B^j, \) where \(A(\lambda, \mu)\) is invertible operator and \(B\) is nondegenerate constant matrix. Thus \(\Psi_{t_j} = \Phi \ast C_i\) with \(C_i\) given by (7). After applying operator \(\Psi^{-1}\) from the left to the eq.(12) one results in

\[ E_j = U_{t_j} \ast A + U \ast A^j \ast p_1p_2 \ast U \ast A + U \ast c_1 B^j c_2 \ast U \ast A - (U_{t_1} \ast A^j + U \ast A \ast p_1p_2 \ast U \ast A^j + U \ast c_1 B c_2 \ast U \ast A^j) = 0 \quad (13) \]

We may derive nonlinear system for the functions

\[ u = p_2 \ast U \ast c_1, \quad q_n = p_2 \ast U \ast A^n \ast p_1, \quad v_n = \partial^n c_2 \ast U \ast c_1, \quad w_{nm} = \partial^n c_2 \ast U \ast A^m \ast p_1, \quad (14) \]
which has the following ”short” form

\[ p_2 * E_j * c_1 = 0, \quad p_2 * E_j * A^n * p_1 = 0, \quad \partial^n c_2 * E_j * c_1 = 0, \quad \partial^n c_2 * E_j * A^n * p_1 = 0, \]  \tag{15} 

or, extended form

\[
\begin{align*}
    u_{t_j} - u_{t_1}B^{j-1} + q_j u - q_1 uB^{j-1} + uB^j v_0 - uBv_0B^{j-1} &= 0, \quad (16) \\
    q_{nt_j} - q_{n+j-1}u + q_1 q_n - q_1 q_{n+j-1} + uB^j w_{0n} - uBw_{0(n+j-1)} &= 0, \\
    v_{nt_j} - v_{nt_1}B^{j-1} + w_{nj} u - w_{n1} uB^{j-1} - B^{j-1} v_{n+1} + v_{n+1} B^{j-1} + v_n B^j v_0 - v_n Bv_0 B^{j-1} &= 0, \quad (17) \\
    w_{mnt_j} - w_{m(n+j-1)t_1} + w_{mj} q_n - w_{m1} q_{n+j-1} - B^{j-1} w_{(m+1)n} + v_{(m+1)(n+j-1)} + v_m B^j w_0 - v_m Bw_0 B^{j-1} &= 0. \quad (18)
\end{align*}
\]

The complete system of pure PDE is represented by the following set: eq. (16) with \( j = 2 \), eqs. (17) and (18) with \( j = 2, 3 \), eq. (19) with \( j = 2, 3, 4 \). Thus, this system is \((3+1)\)-dimensional. It may be given the compact form if one introduces column of matrices \( u, q_n, v_n, w_{nm} \): \( \chi = [u, q_1, q_2, \ldots, v_1, v_2, \ldots, w_{00}, w_{10}, w_{01}, \ldots]^T \):

\[
    \sum_{l=1}^{4} \sum_{mn} V_{lijmn} \partial_l \chi_{mn} + \sum_{klmn} T_{ijklmn} \chi_{kl} \chi_{mn} = 0, \quad (20)
\]

where \( V_{lijmn} \) and \( T_{ijklmn} \) are constants expressed in terms of the elements of the matrix \( B \).

Physical application of the eqs. (16-19) is not found yet. In particular, it reduces into the following \((2+1)\)-dimensional systems:

1. **Matrix Burgers type system** (i.e. linearizable) for the function \( q_0 \), if \( c_1 = 0 \) or \( N = 1 \).

2. **Matrix n-wave equation** \( (n = N(N - 1)/2) \) for the function \( v_0 \), if \( p_1 = 0 \).

### 2.1 Construction of solutions

First, one needs to solve the system (8-10) for the functions \( c_1, \Phi, p_1, c_2 \):

\[
\begin{align*}
    A * c_1 &= c_1 B, \quad (21) \\
    \Phi(\lambda, \nu; t) &= \int_{\Omega_k} \int_{D_{\nu}} \Phi_0(\lambda, \nu; k) e^{\eta_i(v; k)} \phi_0(\nu; \mu; k) dk d\nu, \quad (22) \\
    p_1(\lambda; t) &= \int_{\Omega_k} \int_{D_{\nu}} p_0(\lambda, \nu; k) e^{\eta_i_2(v; k)} p_{10}(\nu; k) dk d\nu, \quad (23) \\
    c_2(\lambda; t) &= \int_{\Omega_k} e^{\kappa \sum_i B^{(i)} c_{20}(\lambda; k)} dk \quad (24)
\end{align*}
\]

where \( \eta_i(\mu; k) = \sum_{j=1}^{4} \eta_{ij}(\mu; k)t_j, i = 1, 2, [\eta_{ij}, \eta_{ik}] = 0, \det(\eta_{ij}) \neq 0 \). Parameter \( k \) is complex in general, integration is over whole complex plane \( \Omega_k \) of this parameter. Function \( c_{20} \) is
arbitrary and functions $\phi_0$ and $p_0$ solve the following system:

$$
\eta_{1j}(\nu,k)\phi_0(\nu,\mu;k) = \int_{D\nu_1} \eta_{1(j-1)}(\nu,k)\phi_0(\nu,\nu_1;k)A(\nu_1,\mu)d\nu_1, \quad (25)
$$

$$
p_0(\lambda,\nu;k)\eta_{2j}(\nu,k) = \int_{D\nu_1} A(\lambda,\nu_1)p_0(\nu_1,\nu;k)\eta_{2(j-1)}(\nu,k)d\nu_1, \quad j = 2,3,\ldots. \quad (26)
$$

Functions $\eta_{1j}$ $(j = 1,2)$ are arbitrary, while $\eta_{jn}$ with $n > 1$ provide compatibility of eqs. (25) and (26).

Now one can integrate (3) to get $\Psi(j = 1)$:

$$
\Psi(t) = E + \partial^{-1}_t [\Phi(t) * A * p_1(t)p_2 + \Phi(t) * c_1c_2(t)], \quad (27)
$$

where $E = E(\lambda,\mu)$ is invertible operator independent on $t$. For instance, $E(\lambda,\mu) = \delta(\lambda - \mu)$. Next, find $U$ from (1): $U = \Psi^{-1} * \Phi$. In general, operator $\Psi^{-1}$ can be constructed only numerically, unless $\Phi_0$ is degenerate ($\Phi_0(\lambda,\mu;k) = \sum\Phi_{0j}(\lambda)\Phi_{0j}(\mu;k)$) and explicit form for $E_L^-1$ is known. In this case $\Psi^{-1}$ may be found analytically, following the procedure proposed, for instance, in [10], where $\partial$-problem with degenerate kernel has been solved. Similarly, eqs. (25) and (26) can be solved numerically, unless $A$ has the following structure: $A(\lambda,\mu) = A_0(\lambda,\mu) + \sum_j A_{j1}(\lambda)A_{j2}(\mu)$, where operator $A_0$ is invertible with known analytical form for $A_0R_1^-$. For instance, $A_0(\lambda,\mu) = \delta(\lambda - \mu)$. Then compatibility condition of the system (25) and (26) produces dispersion relations in the form $\eta_{1n}(\mu;k) = \eta_{11}(\mu;k)F_1[(\phi_0 * A_{11})(\mu;k), \ i = 1,2,\ldots], \eta_{2n}(\mu;k) = F_2[(A_{21} * p_0)(\mu;k), \ i = 1,2,\ldots] \eta_{21}(\mu;k)$, where $F_i$ are given matrix functions of matrix arguments.

Finally, one can show that solutions of our (3+1)-dimensional system (16-19) constructed in accordance with definitions (14) may depend on arbitrary functions of three real parameters, for instance, $t_1, t_2$ and $t_3$. This is owing to the factor $\Psi * A * p_1$.

3 Resonance wave interaction in (3+1)-dimensions

In this section we consider eqs.(6-11) with $S(\lambda,\mu) \neq \delta(\lambda-\mu), \chi \neq 0, \ p_1 = 0$ and $\Psi_{ti} = S * \Phi * C_i$.

It is convenient to apply operator $c_1$ to the eqs.(1) and (9) from the right, giving them the form:

$$
S * \Phi + \chi = \Psi * \tilde{U}, \ \tilde{U} = U * c_1, \ \tilde{\chi} = \chi * c_1, \quad (28)
$$

$$
(S * \Phi)_{tj} = (S * \Phi)_{ti}B_j, \ B_1 = I, \quad (29)
$$

$I$ is identity matrix, $B_i$ are diagonal matrices and $N \geq 4$. Let $\tilde{\chi}_{ti}(\lambda,t) = \tilde{\chi}(\lambda,t)a_j$, where $a_j$ are constant matrices. We will need the following notations: $b_j = a_1B_j - a_j, \ V_0 = c_2 * \tilde{U}$ and $V_1 = c_{21} * \tilde{U}$.

Nonlinear eq.(12) gets the following form after applying operator $c_1$ from the right:

$$
\Psi * (\partial_\nu \tilde{U} - \partial_t \tilde{U}B_2 + \tilde{U}[B_2, V_0]) + \tilde{\chi}(b_j - [B_j, V_0]) = 0. \quad (30)
$$

Now assume that det($b_j - [B_j, V_0]$) $\neq 0$ for all $j$ and use two equations (30) with indexes $j$ and $k, j \neq k$ to eliminate function $\tilde{\chi}$. After applying operator $c_2 * \Psi^{-1}$ from the left to the resulting equation, we receive:

$$
(\partial_\nu V_0 - \partial_t V_0B_k + [V_1, B_k] + V_0[B_k, V_0])(b_k - [B_k, V_0])^{-1} = \quad (31)
$$

$$
(\partial_\nu V_0 - \partial_t V_0B_j + [V_1, B_j] + V_0[B_j, V_0])(b_j - [B_j, V_0])^{-1}. \quad (31)
$$
Next, let us introduce different scales for variables $t_k$, $V_0$, $V_1$: $\partial_{t_k} \rightarrow \epsilon \partial_{t_k}$, $V_0 = \epsilon v$, $V_1 = \epsilon^2 v$.

Keeping only leading terms, we get from the eq. (31):

$$E_k = v_1 (B_j b_{j}^{-1} - B_k b_{k}^{-1}) + v_t b_k^{-1} - v_j b_j^{-1} + [v_1, B_k] b_k^{-1} - [v_1, B_j] b_j^{-1} + v[v, B_j] b_j^{-1} - v[v, B_k] b_k^{-1} = 0.$$  \hspace{1cm} (32)

Thus the complete system is represented by the pair of equations (32), $E_k$ and $E_n$, $k \neq n$. One can see that the following combination of these equations has no function $v_1$ and contains only off-diagonal elements of $v$:

$$E_k (B_n b_{n}^{-1} - B_j b_{j}^{-1}) - E_n (B_k b_{k}^{-1} - B_j b_{j}^{-1}) + B_j (E_k - E_n) b_j^{-1} - B_n E_k b_n^{-1} + B_k E_n b_k^{-1} = 0.$$  \hspace{1cm} (33)

Let $j = 2$, $k = 3$, $n = 4$ and write this equation in the following form

$$\sum_{n=1}^{4} s_{nij} \partial_{t_n} v_{ij} + \sum_{k:k \neq i \neq j} T_{ikj} v_{ik} v_{kj} = 0, \hspace{0.5cm} i \neq j,$$  \hspace{1cm} (34)

where $s_{kij}$ and $T_{ikj}$ are constants, expressed in terms of the elements of the matrices $B_j$ and $b_j$. If $v_{ij}$ are real, then this equation describes resonance interaction of wave packets.

Reduction $t_k \rightarrow it_k$, $v_{ij} \rightarrow \tilde{v}_{ji}$, with real $s_{nij}$ and $T_{ikj}$, $s_{nij} = s_{nji}$, $T_{ikj} = T_{jki}$ (bar means complex conjugated value) transforms the (3+1)-dimensional eq. (34) into (2+1)-dimensional $n$-wave equation with independent variables $\tau_k = t_k + t_1$, $k = 2, 3, 4$.

### 3.1 Construction of solutions

In this section we give the algorithm for construction the solution $V_0$ to the eq. (31).

Solutions of the eq. (29) and expression for $\tilde{\chi}$ have the form:

$$S * \Phi(\lambda) = \int_{\Omega_k} \Phi_0(\lambda, k) e^{k \sum_n B_n t_n} dk, \hspace{0.5cm} c_2(\lambda) = \int_{\Omega_k} e^{k \sum_n B_n t_n} c_{20}(\lambda, k) dk,$$  \hspace{1cm} (35)

$$\tilde{\chi}(\lambda) = \chi_0(\lambda) e^{\sum_n a_n t_n}.$$  \hspace{1cm} (36)

To find $\Psi$ we integrate eq. (3) ($j = 1$, remember that $B_1 = I$):

$$\Psi(\lambda, \mu) = \int_{\Omega_k} \int_{\Omega_q} \Phi_0(\lambda, k) e^{(k+q) \sum_n B_n t_n} c_{20}(\mu, q) \frac{dk dq}{k+q} + \delta(\lambda - \mu).$$  \hspace{1cm} (37)

Thus

$$\tilde{U}(\lambda) = S * \Phi(\lambda) - \int_{\Omega_k} \int_{\Omega_q} \Phi_0(\lambda, k) e^{\sum_n B_n t_n (k+q)} \phi(q) \frac{dk dq}{k+q} + \tilde{\chi}(\lambda), \hspace{0.5cm} \phi = c_{20} * \tilde{U}$$  \hspace{1cm} (38)

and

$$V_0 = c_2 * \tilde{U}.$$  \hspace{1cm} (39)

Unknown function $\phi$, related with $\tilde{U}$, can be found only numerically in general case, unless functions $\Phi(\lambda, k)$ is degenerate [10]. Eq. (39) shows that $V_0$ may depend on $N \times N$ matrix function of two real variables, for instance, $t_1$ and $t_2$.

Regarding the multi-scale expansion given by eqs. (32), one should replace $t_n \rightarrow \epsilon t_n$, in formulae (35-39) and take arbitrary functions $\Phi_0$ and $\chi_0$ proportional to $\epsilon$. Thus $V_0 \sim \epsilon$. 


4 Conclusions

Working with dressing methods we underline two directions: (a) increase of dimension of solvable nonlinear PDE and (b) provide rich class of their solutions. Nonlinear PDE derived with our algorithm admit infinite set of commuting flows corresponding to different parameters $t_j$. Since general equations are rather complicated (see (16-19), (31)), the reasonable problem is construction of their reductions, which would exhibit physical application of these systems. Another way is multi-scale expansion of general systems, which in our case reveals (3+1)-dimensional equation describing resonance interaction of wave packets (see eq.(32) and (34)).

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