ON THE ASYMPTOTIC BEHAVIOUR OF THE EIGENVALUES OF A ROBIN PROBLEM

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ABSTRACT. We prove that every eigenvalue of a Robin problem with boundary parameter \( \alpha \) on a sufficiently smooth domain behaves asymptotically like \(-\alpha^2\) as \( \alpha \to \infty \). This generalises an existing result for the first eigenvalue.

1. Introduction and Main Results

We are interested in the eigenvalue problem

\[
-\Delta u = \lambda u \quad \text{in } \Omega,
\]
\[
\frac{\partial u}{\partial \nu} = \alpha u \quad \text{on } \partial \Omega
\]  

(1.1)

where we assume \( \Omega \subset \mathbb{R}^N \) is a bounded domain, that is, a bounded open set, without loss of generality connected, and \( \alpha > 0 \). The problem (1.1) is usually referred to as a Robin problem (in comparison with the case \( \alpha < 0 \)) or sometimes as a generalised Neumann problem. This problem has received considerable attention in the last few years; see for example [1, 4, 5, 6, 8, 9, 10] and the references therein. It is well-known that if \( \Omega \) is Lipschitz then there is a sequence of eigenvalues \( \lambda_1 < \lambda_2 \leq \ldots \to \infty \), which we repeat according to their multiplicities, where \( \lambda_1 < 0 \) is simple and is the unique eigenvalue with a positive eigenfunction \( \psi_1 \). Our main result is as follows.

**Theorem 1.1.** Suppose \( \Omega \subset \mathbb{R}^N \) is a bounded domain of class \( C^1 \). Then for every \( n \geq 1 \) we have

\[
\lim_{\alpha \to \infty} \frac{\lambda_n(\alpha)}{-\alpha^2} = 1.
\]  

(1.2)

It was shown in [8] that for \( \Omega \) piecewise-\( C^1 \) the first eigenvalue \( \lambda_1 \) has the asymptotic behaviour \( \liminf_{\alpha \to \infty} -\lambda_1(\alpha)/\alpha^2 \geq 1 \), with equality if
∂Ω is equivalent in some sense to a sphere. It was also observed in [8] that when Ω is a ball of radius 1, there are ⌊α⌋ + 1 negative eigenvalues of (1.1), and they satisfy \( \sqrt{-\lambda_n(\alpha)} \sim \alpha + O(1) \) as \( \alpha \to \infty \). It was subsequently shown in [10] that in fact
\[
\lim_{\alpha \to \infty} \frac{\lambda_1(\alpha)}{-\alpha^2} = 1
\]
(1.3)
for every bounded and \( C^1 \) domain Ω. Related results have been obtained in [5, 6]. The \( C^1 \) assumption in (1.3) is optimal: the authors in [8] constructed examples of domains with “corners” for which the limit in (1.3) is a constant larger than one. Such results were generalised and further studied in [9].

Remark 1.2. One can also consider the same problem with the boundary condition \( \frac{\partial u}{\partial \nu} = \alpha bu \), where \( b \in C(\partial \Omega) \) is a weight function which is positive somewhere. In this case, if Ω is bounded and \( C^1 \), then
\[
\lim_{\alpha \to \infty} \frac{\lambda_1(\alpha)}{-\alpha^2(\max_{\partial \Omega} b)^2} = 1
\]
(see [10] Remark 1.1). It seems the same should be true for \( \lambda_n, n \geq 1 \). However all we can say at present is that Theorem 1.1 together with the monotonic behaviour of \( \lambda_n \) with respect to changes in \( b \) imply that
\[
\lim_{\alpha \to \infty} \sup \frac{\lambda_n(\alpha)}{-\alpha^2(\max_{\partial \Omega} b)^2} \leq 1.
\]

We will also prove the following result on the eigenfunctions of (1.1).

Proposition 1.3. Suppose \( \Omega \subset \mathbb{R}^N \) is bounded and \( C^1 \). Fix \( 2 \leq p < \infty \) and let \( \psi_n \) be any eigenfunction associated with \( \lambda_n \), normalised so that \( \| \psi_n \|_{L^p(\Omega)} = 1 \). Then
(i) \( \psi_n \to 0 \) in \( L^p_{\text{loc}}(\Omega) \) as \( \alpha \to \infty \);
(ii) \( \| \psi_n \|_{L^q(\Omega)} \to 0 \) as \( \alpha \to \infty \) for \( 1 \leq q < p \);
(iii) \( \| \psi_n \|_{L^r(\Omega)} \to \infty \) as \( \alpha \to \infty \) for \( r > p \).

We will prove Theorem 1.1 in the next section and defer the proof of Proposition 1.3 until Section 3. We will use the result of Theorem 1.1 to obtain Proposition 1.3, however, the former is only needed to show that \( \lambda_n(\alpha) \to -\infty \) as \( \alpha \to \infty \). Proposition 1.3 is valid for Lipschitz domains whenever we have this more general asymptotic behaviour.

2. PROOF OF THEOREM 1.1

We first discuss the theory related to (1.1) that will be needed to prove Theorem 1.1. The form associated with (1.1) is given by
\[
a(u,v) = \int_{\Omega} \nabla u \cdot \nabla v \, dx - \int_{\partial \Omega} \alpha uv \, d\sigma,
\]
where \( u,v \in H^1(\Omega) \). We understand eigenvalues \( \lambda \) and associated eigenfunctions \( \psi \) of (1.1) in the weak sense, as satisfying \( a(\psi, v) = \)
\[ \lambda_n(\alpha) = \inf_{0 \neq v \in M_n} \frac{a(v, v)}{\|v\|^2_{L^2(\Omega)}}, \]  

where \( M_n \) is the subspace of \( H^1(\Omega) \) of codimension \( n - 1 \) obtained by taking the orthogonal complement of the \( L^2 \)-span of the first \( n - 1 \) eigenfunctions \( \psi_1, \ldots, \psi_{n-1} \) (see [3, Section VI.1]). If we set \( v_n := v - \sum_{i=1}^{n-1} \langle \psi_i, v \rangle \psi_i \), then \( v_n \in M_n \) and so provided \( v_n \neq 0 \), that is, provided \( v \) is not in the \( L^2 \)-span of \( \psi_1, \ldots, \psi_{n-1} \), we may use \( v_n \) as a test function in (2.1) to estimate \( \lambda_n \) from above.

We will use this representation, together with an appropriate choice of \( v \) and an induction argument on \( n \), to prove Theorem 1.1. Our choice of test function is due to an argument in [5, Theorem 2.3], though also cf. [9, Example 2.4]. We will assume throughout that \( \Omega \subset \mathbb{R}^N \) is bounded and \( C^1 \), although some of the results, including the next lemma, are valid for Lipschitz domains with the same proof.

**Lemma 2.1.** Let \( d \in \mathbb{R}^N \), \( \|d\| = 1 \) be any unit vector. Set \( u_d(x, \alpha) := ce^{\alpha x \cdot d} \in C^\infty(\mathbb{R}^N) \cap H^1(\Omega) \), where \( c = c(d, \alpha) \) is a constant chosen so that \( \|u_d\|_{L^2(\Omega)} = 1 \). Then \( a(u_d, u_d) \leq -\alpha^2 \) for all \( \alpha > 0 \).

**Proof.** For \( x \in \mathbb{R}^N \) writing \( x = (x_1, \ldots, x_N) \), we may without loss of generality rotate our coordinate system if necessary so that \( d = (0, \ldots, 0, 1) \). In this case \( u_d = ce^{\alpha x N} \) and \( \nabla u_d = (0, \ldots, 0, c e^{\alpha x_N}) \).

Hence

\[ a(u_d, u_d) = c^2 \alpha^2 \int_{\Omega} e^{2\alpha x_N} dx - c^2 \alpha \int_{\partial \Omega} e^{2\alpha x_N} d\sigma. \]

We will now use the divergence theorem on \( V := (0, \ldots, 0, e^{2\alpha x_N}) \in C^\infty(\mathbb{R}^N, \mathbb{R}^N) \) and the domain \( \Omega \) (see for example [11, Théorème 3.1.1]). Denoting the outer unit normal to \( \Omega \) by \( \nu_\Omega(x) = (\nu_1(x), \ldots, \nu_N(x)) \), \( x \in \partial \Omega \), we have

\[ \int_{\partial \Omega} e^{2\alpha x_N} d\sigma \geq \int_{\partial \Omega} e^{2\alpha x_N} \nu_N d\sigma = \int_{\partial \Omega} V \cdot \nu_\Omega d\sigma \]

\[ = \int_{\Omega} \text{div} V \, dx = 2\alpha \int_{\Omega} e^{2\alpha x_N} dx. \]
Proof. Since bilinearity of the form $a$ we now estimate $a$ shows that
A simple calculation using the orthonormality of the eigenfunctions establishing (2.2). □

Remark 2.2. An easy calculation shows that the function $u(x) := e^{\alpha x_N}$ is a positive eigenfunction, with eigenvalue $-\alpha^2$, of (1.1) on the half-space $T = \{x \in \mathbb{R}^N : x_N < 0\}$.

For $d \in \mathbb{R}^N$ a fixed unit vector and $n \geq 1$ also fixed, set $u_{n+1} := u_d - \sum_{i=1}^n \langle u_d, \psi_i \rangle \psi_i \in M_{n+1}$. We will use $u_{n+1}$ as a test function in the variational characterisation in order to establish (2.2). To that end, we estimate $\lambda_{n+1}$ in terms of the previous $n$ eigenvalues and functions.

Lemma 2.3. Suppose $u_d \not\in \text{span}\{\psi_1, \ldots, \psi_n\}$. Then

$$\lambda_{n+1}(\alpha) \leq \frac{-\alpha^2 - \sum_{i=1}^n \lambda_i (u_d, \psi_i)^2}{1 - \sum_{i=1}^n (u_d, \psi_i)^2}. \quad (2.2)$$

Proof. Since $u_d$ is not a linear combination of the first $n$ eigenfunctions, we can use $u_{n+1} = u_d - \sum_{i=1}^n \langle u_d, \psi_i \rangle \psi_i \not= 0$ as a test function in (2.1). A simple calculation using the orthonormality of the eigenfunctions shows that

$$0 < \langle u_{n+1}, u_{n+1} \rangle = 1 - \sum_{i=1}^n (u_d, \psi_i)^2.$$  

We now estimate $a(u_{n+1}, u_{n+1})$. Using the definition of $u_{n+1}$ and the bilinearity of the form $a$, we see that $a(u_{n+1}, u_{n+1})$ is given by

$$a(u_d, u_d) - 2 \sum_{i=1}^n \langle u_d, \psi_i \rangle a(u_d, \psi_i) + \sum_{i=1}^n \sum_{j=1}^n \langle u_d, \psi_i \rangle^2 a(\psi_i, \psi_j).$$

Since $a(u_d, \psi_i) = \lambda_i (u_d, \psi_i)$, and since $a(\psi_i, \psi_j) = \lambda_i$ if $i = j$ and 0 otherwise, we obtain

$$a(u_{n+1}, u_{n+1}) = a(u_d, u_d) - \sum_{i=1}^n \lambda_i (u_d, \psi_i)^2.$$

(Cf. the abstract theory in [7, Section I.6.10].) Using the estimate of $a(u_d, u_d)$ from Lemma (2.1) and putting everything together yields

$$\lambda_{n+1}(\alpha) \leq \frac{a(u_{n+1}, u_{n+1})}{\|u_{n+1}\|_{L^2(\Omega)}^2} \leq \frac{-\alpha^2 - \sum_{i=1}^n \lambda_i (u_d, \psi_i)^2}{1 - \sum_{i=1}^n (u_d, \psi_i)^2},$$

establishing (2.2).

Roughly speaking, to prove Theorem (1.1) using the estimate of $\lambda_{n+1}$ in Lemma (2.3) we have to prove that we can find a direction $d$ such that $\langle u_d, \psi_i \rangle$ stays small as $\alpha \to \infty$ for all $1 \leq i \leq n$. To that end we will study the functions $u_d$ more carefully. We start by observing that,
for any given \( \alpha > 0 \), the upper level sets of \( u_d \) are restrictions to \( \Omega \) of half-planes of the form \( \{ x \in \mathbb{R}^N : x \cdot d > \kappa \} \), where \( \kappa \in \mathbb{R} \). The key place where we will use the assumption that \( \Omega \) has \( C^1 \) boundary is in parts (iii) and (iv) of the next lemma.

**Lemma 2.4.** Let \( d \in \mathbb{R}^N \), \( \|d\| = 1 \). For \( \kappa \in \mathbb{R} \) set

\[
U_d(\kappa) := \{ x \in \Omega : x \cdot d > \kappa \},
\]

\[
\kappa_d := \sup\{ \kappa \in \mathbb{R} : U_d(\kappa) \neq \emptyset \},
\]

\[
K_d := \{ x \in \overline{\Omega} : x \cdot d = \kappa_d \}.
\]

Then

(i) the \( U_d(\kappa) \) are open, nested (i.e. \( U_d(\kappa) \subset U_d(\kappa') \) if \( \kappa > \kappa' \)), nonempty if and only if \( \kappa < \kappa_d \), and \( 0 \neq |U_d(\kappa)| \to 0 \) as \( \kappa \to \kappa_d \) from below;

(ii) \( \emptyset \neq K_d \subset \partial \Omega \);

(iii) if \( z \in K_d \), then \( d = \nu_\Omega(z) \), the outer unit normal to \( \Omega \) at \( z \);

(iv) if \( d \neq e \in \mathbb{R}^N \), \( \|e\| = 1 \) is another unit vector with \( U_e(\kappa) \) and \( \kappa_e \) defined as in (2.3), then there exists \( \varepsilon > 0 \) such that \( U_d(\kappa) \cap U_e(\kappa) = \emptyset \) for all \( \kappa \in (\kappa_d - \varepsilon, \kappa_d) \) and all \( \kappa \in (\kappa_e - \varepsilon, \kappa_e) \).

**Proof.** (i) is obvious. For (ii), to show \( K_d \neq \emptyset \) we note that \( K_d = \cap_{\kappa < \kappa_d} U_d(\kappa) \), that is, \( K_d \) is the intersection of nested, compact and nonempty sets. That \( K_d \subset \partial \Omega \) is immediate from the definitions and the fact that the \( U_d \) are open. For (iii), we assume as in the proof of Lemma 2.1 that \( d = (0, \ldots, 0, 1) \), so that \( U_d(\kappa) = \{ x \in \Omega : x_N > \kappa \} \). Then \( z = (z_1, \ldots, z_N) \in K_d \) means \( z_N = \kappa_d \), that is, \( z_N = \max\{ x_N : x \in \overline{\Omega} \} \). Since \( \Omega \) is \( C^1 \), this means the tangent plane to \( \Omega \) at \( z \in K_d \) must be horizontal. Thus \( \nu_\Omega(z) \) points in the direction \( x_N \). That is, \( \nu_\Omega(z) = (0, \ldots, 1) \). For (iv), suppose for a contradiction that there exist \( \kappa_j \neq \kappa_d \) and \( \kappa_j \neq \kappa_e \) such that, for each \( j \geq 1 \), there exists \( x_j \in U_d(\kappa_j) \cap U_e(\kappa_j) \). Since \( \overline{\Omega} \) is compact, a subsequence of the \( x_j \) converges to some \( z \in \overline{\Omega} \). Since \( x_j \in U_d(\kappa_j) \) and \( \cap_{j \geq 1} U_d(\kappa_j) = K_d \), we see \( z \in K_d \). By a similar argument, \( z \in K_e \). This contradicts (iii) since \( d \neq e \). \( \square \)

We now show that for \( d \) fixed, all the mass of \( u_d \) becomes concentrated in an arbitrarily small region of \( \Omega \) as \( \alpha \to \infty \).

**Lemma 2.5.** Let \( d \in \mathbb{R}^N \) and \( u_d(x) = ce^{\alpha x \cdot d} \) be as in Lemma 2.1 and let \( U_d(\kappa) \) and \( \kappa_d \) be as in Lemma 2.4. For every \( \varepsilon > 0 \) and \( \kappa' < \kappa_d \) there exists \( \alpha_\varepsilon := \alpha(\varepsilon, \kappa') > 0 \) such that

\[
\|u_d\|^2_{L^2(\Omega \setminus U_d(\kappa'))} < \varepsilon
\]

(2.4)

for all \( \alpha > \alpha_\varepsilon \).

**Proof.** Since \( u_d(x) \leq ce^{\alpha \kappa'} \) for all \( x \in \Omega \setminus U_d(\kappa') \), we have

\[
\|u_d\|^2_{L^2(\Omega \setminus U_d(\kappa'))} \leq ce^{2\alpha \kappa'} |\Omega|.
\]
Choose \( \kappa'' \in (\kappa', \kappa_d) \). Then \( U_d(\kappa'') \subset U_d(\kappa') \) with \( |U_d(\kappa'')| \neq 0 \) and
\[
1 = \| u_d \|_{L^2(\Omega)}^2 \geq \| u_d \|_{L^2(U_d(\kappa''))}^2 \geq c e^{2\alpha \kappa''} |U_d(\kappa'')|.
\]
For \( \varepsilon > 0 \) fixed, choose \( \alpha_\varepsilon > 0 \) such that
\[
e^{2\alpha \kappa'} |\Omega| < \varepsilon e^{2\alpha \kappa''} |U_d(\kappa'')|,
\]
which we can do since \( \kappa' < \kappa'' \). Then (2.5) will hold uniformly in \( \alpha > \alpha_\varepsilon \) and so
\[
\| u_d \|_{L^2(\Omega \setminus U_d(\kappa'))}^2 < c e^{2\alpha \kappa'} |\Omega| < \varepsilon e^{2\alpha \kappa''} |U_d(\kappa'')| < \varepsilon
\]
for all \( \alpha > \alpha_\varepsilon \).

Lemma 2.5 implies that for fixed \( d, u_d \to 0 \) weakly in \( L^2(\Omega) \) as \( \alpha \to \infty \); it turns out that the same is true of the \( \psi_i \) (see Proposition 1.3). But this is not enough to show directly that \( \langle u_d, \psi_i \rangle \) is uniformly small, since both \( u_d \) and \( \psi_i \) vary with \( \alpha \). Instead, we will use the following rather technical result concerning the \( u_d \). Since this does not use any specific properties of the \( \psi_i \), we set it up so it works for arbitrary \( L^2 \)-functions.

**Lemma 2.6.** Fix \( n \geq 1 \) and \( \delta > 0 \). Suppose we have a sequence \( \alpha_k \to \infty \) and for each \( k \in \mathbb{N} \) a family of \( n \) functions \( \varphi_i(k) \in L^2(\Omega) \), \( 1 \leq i \leq n \), such that \( \| \varphi_i(k) \|_{L^2(\Omega)} = 1 \) for all \( 1 \leq i \leq n \) and \( k \in \mathbb{N} \). Then there exists a unit vector \( d \in \mathbb{R}^N \) and a subsequence \( \alpha_{k_l} \to \infty \) of the \( (\alpha_k) \) such that
\[
\sum_{i=1}^{n} \langle u_d(k_l), \varphi_i(k_l) \rangle^2 \leq \delta,
\]
for all \( l \in \mathbb{N} \), where \( u_d(k_l) = u_d(x, \alpha_{k_l}) \) is as in Lemma 2.7.

**Proof.** Fix \( n \geq 1 \), \( \delta > 0 \) and a sequence \( \alpha_k \to \infty \). Choose \( m \geq 1 \) and \( \varepsilon > 0 \), to be specified precisely later on. Now choose any \( m \) distinct unit vectors \( d_j \in \mathbb{R}^N \), \( 1 \leq j \leq m \), and for each \( j \) let \( u_j := u_{d_j}(x, \alpha_k) \) be as in Lemma 2.4. By making an appropriate choice of \( \kappa_j \) we may assume the \( U_j \) are pairwise disjoint. Using Lemma 2.5 we find an \( \alpha_\varepsilon > 0 \) such that
\[
\| u_j \|_{L^2(\Omega \setminus U_j)}^2 < \varepsilon
\]
for all \( \alpha > \alpha_\varepsilon \) and all \( 1 \leq j \leq m \). By discarding at most finitely many \( k \), we may assume \( \alpha_k > \alpha_\varepsilon \) for all \( k \in \mathbb{N} \). Now for each \( k \in \mathbb{N} \), we have
\[
\int_{\Omega} \sum_{i=1}^{n} \| \varphi_i(k) \|_{L^2(\Omega)}^2 dx = \sum_{i=1}^{n} \| \varphi_i(k) \|_{L^2(\Omega)}^2 = n.
\]
Since the \( U_j \) are disjoint, it follows that for each \( k \in \mathbb{N} \), there exists at least one \( j = j_k \) such that
\[
\int_{U_{j_k}} \sum_{i=1}^{n} \| \varphi_i(k) \|_{L^2(\Omega)}^2 dx \leq \frac{n}{m}.
\]
For this $j_k$, using Hölder’s inequality, for each $1 \leq i \leq n$ we have
\[
|\langle u_{jk}, \varphi_i(k) \rangle| \leq \int_{\Omega \setminus U_{jk}} |u_j \varphi_i| \, dx + \int_{U_{jk}} |u_j \varphi_i| \, dx
\]
\[
\leq \|u_j\|_{L^2(\Omega)} \left( \frac{n}{m} \right)^{\frac{1}{2}} + \varepsilon \frac{1}{\sqrt{m}} \|u_j\|_{L^2(\Omega)} \|\varphi_i\|_{L^2(\Omega)}
\]
\[
= \left( \frac{n}{m} \right)^{\frac{1}{2}} + \varepsilon \frac{1}{\sqrt{m}},
\]
where we have used the bound $\int_{U_j} |\varphi_i|^2 \, dx \leq n/m$. We now specify $m \geq 1$ and $\varepsilon > 0$ to be such that
\[
n \left( \left( \frac{n}{m} \right)^{\frac{1}{2}} + \varepsilon \frac{1}{\sqrt{m}} \right)^2 \leq \delta,
\]
noting that this depends only on $n$ and $\delta$. Squaring the above estimate for $|\langle u_{jk}, \varphi_i(k) \rangle|$ and summing over $i$, this implies that for all but finitely many $k \in \mathbb{N}$, (2.6) holds for at least one of the $m$ fixed $u_j$.

By a simple counting argument, there must exist at least one $j^*$ between 1 and $m$ such that (2.6) holds for this fixed $u_{j^*}$ and infinitely many $\alpha_k$. This gives us our $u_d$ and $(\alpha_{k_i})$.

**Proof of Theorem 1.1.** The proof is by induction on $n$. The step when $n = 1$ is given by [10, Theorem 1.1]. Now fix $n \geq 1$ and suppose we know that for all $1 \leq i \leq n$, $-\lambda_i(\alpha_k)/\alpha_k^2 \to 1$ as $k \to \infty$ for every sequence $\alpha_k \to \infty$. It suffices to prove that for every such sequence $\alpha_k \to \infty$, there exists a subsequence $\alpha_{k_l} \to \infty$ such that $-\lambda_{n+1}(\alpha_{k_l})/\alpha_{k_l}^2 \to 1$ as $l \to \infty$.

So fix a particular sequence $\alpha_k \to \infty$ and also fix $0 < \delta < 1$. Let $u_d$ satisfy the conclusion of Lemma 2.6 for a subsequence which we will still denote by $(\alpha_k)$, this $\delta > 0$ and the family of $n$ functions $\psi_i(\alpha_k) = \varphi_i(1), 1 \leq i \leq n$. Then by Lemma 2.6 we know that
\[
\sum_{i=1}^n \langle u_d(\alpha_k), \psi_i(\alpha_k) \rangle^2 \leq \delta \tag{2.7}
\]
for all $k \in \mathbb{N}$ and the fixed direction $d$. In particular, (2.7) implies $u_d \not\in \text{span}\{\psi_1(\alpha_k), \ldots, \psi_n(\alpha_k)\}$ for any $k \in \mathbb{N}$, since $\delta < 1$. Applying Lemma 2.3 to $u_d$ for each $k \in \mathbb{N}$, we obtain
\[
\lambda_{n+1}(\alpha_k) \leq \frac{-\alpha_k^2 - \sum_{i=1}^n \lambda_i(\alpha_k) \langle u_d, \psi_i \rangle^2}{1 - \sum_{i=1}^n \langle u_d, \psi_i \rangle^2}
\]
for all $k \in \mathbb{N}$. This implies
\[
\frac{\lambda_i(\alpha_k)}{-\alpha_k^2} \geq \frac{\lambda_{n+1}(\alpha_k)}{-\alpha_k^2} \geq \frac{1 - \sum_{i=1}^n \lambda_i(\alpha_k) \langle u_d, \psi_i \rangle^2}{1 - \sum_{i=1}^n \langle u_d, \psi_i \rangle^2} \tag{2.8}
\]
Using the bound (2.7), which holds independently of $k \in \mathbb{N}$, together with the induction assumption $-\lambda_i(\alpha_k^2)/\alpha_k^2 \to 1$ as $k \to \infty$ for all $i \leq n$ it follows that the term on the right in (2.8) converges to 1 as $k \to \infty$. 

□
This establishes the desired limit for $-\lambda_{n+1}(\alpha_k)/\alpha_k^2$, which completes the proof. □

3. PROOF OF PROPOSITION 1.3

Fix $n \geq 1$ and $p \geq 2$. We first obtain the following interior estimate for $\psi_n$, from which the proof of the proposition will follow easily.

**Lemma 3.1.** Under the assumptions of Proposition 1.3, if $\phi \in C^\infty_c(\Omega)$, then

$$\lambda_n \geq -(p-1)^{-1}\frac{\int_\Omega |\psi_n|^p |\nabla \phi|^2 \, dx}{\int_\Omega |\psi_n|^p \phi^2 \, dx}$$

for all $\alpha > 0$ and all $n \geq 1$.

**Proof.** Given $\phi \in C^\infty_c(\Omega)$, we will use $\phi := \phi^2|\psi_n|^p|\psi_n|^2$ as a test function in the weak form of (1.1) given by

$$\lambda_n \int_\Omega \psi_n v \, dx = a(\psi_n, v) = \int_\Omega \nabla \psi_n \cdot \nabla v \, dx - \int_{\partial \Omega} \alpha \psi_n v \, d\sigma$$

for all $v \in H^1(\Omega)$. We first note that if $p \geq 2$, then since $\psi_n \in C(\Omega)$ (see [4, Corollary 4.2]) we have $\phi \in H^1(\Omega)$ with $\nabla \phi = 2\phi^2|\psi_n|^p|\psi_n|^2\psi_n \nabla \phi + (p-1)\phi^2|\psi_n|^p|\psi_n|^2\nabla \psi_n$. Moreover $\langle \phi, \psi_n \rangle = \int_\Omega \phi^2|\psi_n|^p \, dx \neq 0$, since $\psi_n$ cannot vanish identically on an open set (see [2]). Hence, by completing the square,

$$\int_\Omega \nabla \psi_n \cdot \nabla \phi \, dx = \int_\Omega 2\phi|\psi_n|^p|\psi_n|^2|\nabla \phi| \, dx + (p-1)\phi^2|\psi_n|^p|\nabla \psi_n|^2 \, dx$$

$$= \int_\Omega (p-1)^{-1}|\psi_n|^p|\nabla \phi|^2 \, dx.$$

Substituting this into (3.1), and using that $\phi \equiv 0$ on $\partial \Omega$,

$$\lambda_n \int_\Omega \phi^2|\psi_n|^p \, dx = \int_\Omega \nabla \psi_n \cdot \nabla \phi \, dx \geq - \int_\Omega (p-1)^{-1}|\psi_n|^p|\nabla \phi|^2 \, dx.$$

Rearranging gives the conclusion of the lemma. □

To prove the proposition, part (i) uses the result of Theorem 1.1 that $\lambda_n \to -\infty$ as $\alpha \to \infty$; parts (ii) and (iii) follow directly from (i).

**Proof of Proposition 1.3.** (i) Fix $p \geq 2$, $n \geq 1$ and $\Omega_0 \subset \subset \Omega$ and assume $\|\psi_n\|_{L^p(\Omega)} = 1$. Let $\phi \in C^\infty_c(\Omega)$ be such that $0 \leq \phi \leq 1$ in $\Omega$ and $\phi \equiv 1$ in $\Omega_0$. Setting $K := (p-1)^{-1}\|\nabla \phi\|^2_{L^\infty(\Omega)} > 0$, which depends only on $p$ and $\Omega_0$, by Lemma 3.1

$$\lambda_n \geq \frac{-K}{\int_{\Omega_0} |\psi_n|^p \, dx}$$
for all $\alpha > 0$. Since $\lambda_n \to -\infty$ as $\alpha \to \infty$ by Theorem 1.1, this forces $\int_{\Omega_0} |\psi_n|^p \, dx \to 0$ as $\alpha \to \infty$.

(ii) Fix $1 \leq q < p$ and $\varepsilon > 0$. Choose $\Omega_\varepsilon \subset \subset \Omega$ such that $|\Omega \setminus \Omega_\varepsilon|^{\frac{p}{p-q}} < \varepsilon/2$, which we may do since $p > q$. Also choose $\alpha_\varepsilon > 0$ such that

$$\|\psi_n\|_{L^q(\Omega_\varepsilon)} < \frac{\varepsilon}{2} |\Omega_\varepsilon|^{\frac{p-q}{p}}$$

for all $\alpha > \alpha_\varepsilon$, which we may do by (i). Noting that $p/q$ and $p/(p-q)$ are dual exponents, Hölder’s inequality implies

$$\|\psi_n\|_{L^q(\Omega)} = \int_{\Omega_\varepsilon} |\psi_n|^q \, dx + \int_{\Omega \setminus \Omega_\varepsilon} |\psi_n|^q \, dx$$

$$\leq \left( \int_{\Omega_\varepsilon} |\psi_n|^p \, dx \right)^{\frac{q}{p}} |\Omega_\varepsilon|^{\frac{p-q}{p}} + \left( \int_{\Omega \setminus \Omega_\varepsilon} |\psi_n|^p \, dx \right)^{\frac{q}{p}} |\Omega \setminus \Omega_\varepsilon|^{\frac{p-q}{p}}$$

$$= \|\psi_n\|_{L^p(\Omega_\varepsilon)}^{\frac{q}{p}} |\Omega_\varepsilon|^{\frac{p-q}{p}} + \|\psi_n\|_{L^p(\Omega \setminus \Omega_\varepsilon)}^{\frac{q}{p}} |\Omega \setminus \Omega_\varepsilon|^{\frac{p-q}{p}} < \varepsilon$$

for all $\alpha > \alpha_\varepsilon$, by choice of $\Omega_\varepsilon$ and $\alpha_\varepsilon$, and since $\|\psi_n\|_{L^p(\Omega \setminus \Omega_\varepsilon)} \leq 1$.

(iii) Fix $r > p$. If we normalise $\psi_n$ so that $\|\psi_n\|_{L^r(\Omega)} = 1$, then (ii) implies $\|\psi_n\|_{L^r(\Omega)} \to 0$, so that

$$\frac{\|\psi_n\|_{L^r(\Omega)}}{\|\psi_n\|_{L^p(\Omega)}} \to \infty$$

as $\alpha \to \infty$. Now re-normalise so that $\|\psi_n\|_{L^p(\Omega)} = 1$. Since this does not affect (3.2), in this case $\|\psi_n\|_{L^r(\Omega)} \to \infty$. □

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