A REMARK ON "STUDY OF A LESLIE-GOWER-TYPE TRITROPHIC POPULATION MODEL" [CHAOS, SOLITONS AND FRACTALS 14 (2002) 1275-1293]

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ABSTRACT. In [1] a three species ODE model, based on a modified Leslie-Gower scheme is investigated. It is shown that under certain restrictions on the parameter space, the model has bounded solutions for all positive initial conditions, which eventually enter an invariant attracting set. We show that this is not true. To the contrary, solutions to the model can blow up in finite time, even under the restrictions derived in [1], if the initial data is large enough. We also prove similar results for the spatially extended system. We validate all of our results via numerical simulations.

1. INTRODUCTION

The purpose of this letter is to remark on the well cited research article [1], where the following tri-trophic population model originally proposed in [7, 8] is considered,

(1) \[ \frac{du}{dt} = a_1 u - b_1 u^2 - w_0 \left( \frac{uv}{u + D_0} \right), \]

(2) \[ \frac{dv}{dt} = -a_2 v + w_1 \left( \frac{uv}{u + D_1} \right) - w_2 \left( \frac{vr}{v + D_2} \right), \]

(3) \[ \frac{dr}{dt} = cr^2 - w_3 \frac{r^2}{v + D_3}. \]

This model is based on the Leslie-Gower formulation [3], and considers the interactions between a generalist top predator \( r \), specialist middle predator \( v \), and prey \( u \), where \( (u, v, r) \) are solutions to the above system (1)-(3). The model is very rich dynamically, and has led to a number of works in the literature [2, 4, 5, 6, 9, 10, 11, 12, 13]. In [1] various theorems on the boundedness of the system (1)-(3) are...
proved, and the existence of an invariant attracting set is established. In particular, we recall the following result (Theorem 3 (2i), (3i) [1])

**Theorem 1.1.** Consider the model (1)-(3). Under the assumption that

\[
c < \left( \frac{w_0 b_1 D_3}{w_1 \left( a_1 + \frac{(a_1)^2}{4a_2} \right) + w_0 b_1 D_3} \right) \frac{w_3}{D_3},
\]

all solutions to (1)-(3) are uniformly bounded forward in time, for any initial data in \( \mathbb{R}_+^3 \), and they eventually enter a bounded attracting set \( A \). Furthermore system (1)-(3) is dissipative.

Note, \( A \) is explicitly defined in [1]. Also condition (4) is equation (7) in [1]. The form of (4) is different from equation (7) in [1], as in [1] different constants have been used, than what we are currently using. However it is a matter of simple algebra to convert equation (7) from [1] to our setting.

Our aim in the current letter is to show that (Theorem 3 (2i), (3i) [1]) is incorrect. In particular we show,

1) Solutions to (1)-(3) are not bounded uniformly in time, even if condition (4) from Theorem 1.1 is met. Furthermore, solutions to (1)-(3) can even blow-up in finite time for large initial data. Thus there is no absorbing set \( A \) for all initial conditions in \( \mathbb{R}_+^3 \), and system (1)-(3) is not dissipative in \( \mathbb{R}_+^3 \), even under condition (4).

2) Similar results hold for the spatially explicit model.

3) The above results can be validated numerically. We choose parameters satisfying (4), and show that numerical simulations of (1)-(3), and its spatially extended form, still lead to finite time blow-up.

2. **Finite time blow-up in the ODE model**

We state the following theorem

**Theorem 2.1.** Consider the three species food chain model (1)-(3). For

\[
c < \left( \frac{w_0 b_1 D_3}{w_1 \left( a_1 + \frac{(a_1)^2}{4a_2} \right) + w_0 b_1 D_3} \right) \frac{w_3}{D_3},
\]

\( r(t) \) blows up in finite time, that is

\[
\lim_{t \to T^* < \infty} |r(t)| \to \infty,
\]

as long as the initial data \( v_0, r_0 \) are large enough.

**Proof.** First set \( \left( \frac{w_0 b_1 D_3}{w_1 \left( a_1 + \frac{(a_1)^2}{4a_2} \right) + w_0 b_1 D_3} \right) = k < 1 \).

Consider the following modification to system (1)-(3), with solution \( (u_1, v_1, r_1) \),
THREE SPECIES FOOD CHAIN MODEL

\[
\frac{du_1}{dt} = a_1u_1 - b_1(u_1)^2 - w_0 \left( \frac{u_1v_1}{u_1 + D_0} \right),
\]

(7)

\[
\frac{dv_1}{dt} = -a_2v_1 - w_2v_1r_1,
\]

(8)

\[
\frac{dr_1}{dt} = \delta r_1^2.
\]

(9)

Recall that \(r_1\), the solution to (9), blows up in finite time, at 
\(T^{**} = \frac{1}{\delta |r_1(0)|} \), and we have an exact solution for \(r_1\), for 
\(t \in \left[0, \frac{1}{\delta |r_1(0)|} \right)\), given by,

\[
r_1 = \frac{1}{r_1(0)} - \delta t.
\]

(10)

However, using this exact solution for \(r_1\), one can find an exact solution to (8) 
via separation of variables. Thus

\[
v_1 = v_1(0)e^{-a_2t} (1 - r_1(0)\delta t)^{\frac{w_2}{r_1}}
\]

(11)

for \(t \in \left[0, \frac{1}{\delta |r_1(0)|} \right)\). Next for a given \(c, k, w_3, D_3\) we choose \(\delta\) s.t we can enforce

\[
\frac{w_3}{v_1 + D_3} + \frac{\delta}{2} \leq c < \delta < k \frac{w_3}{D_3}
\]

(12)

to hold \(\forall t\) s.t. \(t \in \left[0, \frac{1}{\delta |r_1(0)|} \right]\). (12) implies

\[
\left( \frac{1}{D_3} - \frac{\delta}{2w_3} - D_3 \right) < v_1.
\]

(13)

(Here we assume \(\left( \frac{1}{D_3} - \frac{\delta}{2w_3} - D_3 \right) > 0\), else it is an uninteresting case)

Equivalently we have

\[
\left( \frac{1}{D_3} - \frac{\delta}{2w_3} - D_3 \right) < v_1(0)e^{-a_2t} (1 - r_1(0)\delta t)^{\frac{w_2}{r_1}}
\]

(14)

This of course is always possible for \(v_1(0), r_1(0)\) large enough.

However, note that \(v_1\) is a subsolution to (2). If \(D_2 \geq 1\), this is immediate as then \(w_2 \left( \frac{w_2}{v_1 + D_2} \right) < w_2vr\). If \(D_2 < 1\), then we can assume \(w_2 \left( \frac{w_2}{v_1 + D_2} \right) < w_4vr\), where we select \(w_4\) s.t \(w_4(v + D_2) > w_2\), and then choose \(w_4\) in place of \(w_2\) in (8).

Also \(r_1\) (with \(\frac{\delta}{2}\) in place of \(\delta\) in (9)) is a subsolution to (3), as long as (12) holds. Thus via direct comparison, \(v > v_1\) and \(r > r_1\). Since (12) implies \(c - \frac{w_3}{v_1 + D_3} \geq \frac{\delta}{2}\), it is immediate that the solution \(r\) to (3) will also blow-up, via direct comparison
with \(r_1\) solving (9) with \(\frac{\delta}{2}\) in place of \(\delta\).

See figure 2, for a simple graphical representation of this idea. Thus we have
ascertained the blow-up of system (1)-(3), via direct comparison to the modified
system (7)-(9). This proves the Theorem.
We next state the following Theorem

**Theorem 2.2.** The three species food chain model (1)-(3), even under condition (5) is not dissipative in all of $\mathbb{R}^3_+$.  

**Proof.** Via Theorem 2.1, there exists initial data in $\mathbb{R}^3_+$, for which solutions blow-up in finite time, and thus do not enter any bounded attracting set $A$. Thus system (1)-(3) is not dissipative. \)

**Remark 1.** The essential error made in the proof in [1] is in equation (12) in [1]. The derived bound for $v$ is inserted in an estimate for the sum of $u, v$ and $r$. Although it is true that $v$ is bounded, and enters an attracting set eventually, there is some transition time before this happens. If $v_0$ is chosen arbitrarily large, then this transition time can be made arbitrarily long. The key is for $r_0, v_0$ chosen large enough, during this transition time, we can enforce $c - \frac{w_0}{u + D_0} > \delta_1 > 0$, for as long as it takes $\frac{dr}{dt} = \delta_1 r^2$ to bow up. In this case $r$ will also blow-up in finite time, (by comparison to $\frac{dr}{dt} = \delta_1 r^2$), and hence never enter any bounded attracting set.

3. Finite time blow-up in the PDE model

3.1. Preliminaries. We now consider the following spatially extended version of (1)-(3)

\[ \partial_t u - d_1 \Delta u = f(u, v, r) = a_1 u - b_1 u^2 - w_0 \frac{uv}{u + D_0}, \]

\[ \partial_t v - d_2 \Delta v = g(u, v, r) = -a_2 v + w_1 \frac{uv}{u + D_1} - w_2 \frac{vr}{v + D_2}, \]

\[ \partial_t r - d_3 \Delta r = h(u, v, r) = cr^2 - \omega_3 \frac{vr^2}{v + D_3}, \]

defined on $\mathbb{R}^+ \times \Omega$. Here $\Omega \subset \mathbb{R}^N$ and where $a_1, a_2, b_1, c, D_0, D_1, D_2, D_3, w_0, w_1, w_2$ and $w_3$, the parameters in the problem as earlier, are positive constants. We can prescribe either Dirichlet or Neumann boundary conditions.

$\Omega$ is an open bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial \Omega$. $d_1, d_2$ and $d_3$ are the positive diffusion coefficients.

The initial data

\[ u(0, x) = u_0(x), \quad v(0, x) = v_0(x), \quad r(0, x) = r_0(x) \quad \text{in} \quad \Omega \]

are assumed to be nonnegative and uniformly bounded on $\Omega$.

The nonnegativity of the solutions is preserved by application of classical results on invariant regions ([18]), since the reaction terms are quasi-positive, i.e.

\[ f(0, v, r) \geq 0, \quad g(u, 0, r) \geq 0, \quad h(u, v, 0) \geq 0, \quad \text{for all} \quad u, v, r \geq 0. \]

The usual norms in the spaces $L^p(\Omega), L^\infty(\Omega)$ and $C(\overline{\Omega})$ are respectively denoted by

\[ \|u\|_p = \frac{1}{|\Omega|} \int_\Omega |u(x)|^p \, dx, \]

\[ \|u\|_\infty = \max_{x \in \Omega} |u(x)|. \]
Since the reaction terms are continuously differentiable on $\mathbb{R}^{+3}$, then for any initial data in $C(\bar{\Omega})$ or $L^p(\Omega)$, $p \in (1, +\infty)$, it is easy to check directly their Lipschitz continuity on bounded subsets of the domain of a fractional power of the operator $I_3 (d_1, d_2, d_3)^t \Delta$, where $I_3$ the three dimensional identity matrix, $\Delta$ is the Laplacian operator and $(\cdot)^t$ denotes the transposition. Under these assumptions, the following local existence result is well known (see [14, 15, 18, 16, 17]).

**Proposition 1.** The system (15)-(17) admits a unique, classical solution $(u,v,r)$ on $[0, T_{\text{max}}] \times \Omega$. If $T_{\text{max}} < \infty$ then

$$\lim_{t \nearrow T_{\text{max}}} \{\|u(t,.)\|_{\infty} + \|v(t,.)\|_{\infty} + \|r(t,.)\|_{\infty}\} = \infty.$$ 

3.2. **Blow-up.** Here we will show that (15)-(17), blows up in finite time. We will do this by looking back at the blow-up for $r$ in (1), and then using a standard comparison method. Consider (1)-(3), with initial conditions $u_0^-, v_0^-$ and $r_0^-$ strictly positive.

By integrating the third equation of the ODE system, we have

$$\frac{1}{r} + \frac{1}{r_0} = ct - w_3 \int_0^t \frac{ds}{v + D_3},$$

which gives

$$r = \frac{1}{\frac{1}{r_0} - ct + w_3 \int_0^t \frac{ds}{v + D_3}}.$$

If we prove that the function: $t \mapsto \psi(t) = \frac{1}{r_0} - ct + w_3 \int_0^t \frac{ds}{v + D_3}$ vanishes at a time $T > 0$ and since $\psi(0) > 0$, then the solution will blow-up in finite time.

Since the reaction terms are continuous functions, then the solutions are classical and continuous and

$$\lim_{t \to 0} \left(1 - t \int_0^t \frac{ds}{v + D_3}\right) = 1 \frac{1}{v_0^- + D_3}.$$

If $v_0^-$ is sufficiently large, then there exists $\delta > 0$ such that

$$\frac{1}{t} \int_0^t \frac{ds}{v + D_3} < \frac{c}{2w_3}, \quad \text{for all } t \in (0, \delta).$$

Then

$$\frac{1}{r_0} - ct + w_3 \int_0^t \frac{ds}{v + D_3} = \frac{1}{r_0} - \left[-c + \frac{w_3}{t} \int_0^t \frac{ds}{v + D_3}\right] t < \frac{1}{r_0} - \frac{c}{2}, \quad \text{for all } t \in (0, \delta).$$

If $r_0^-$ is sufficiently large, then we can find $T^* \in (0, \delta)$ such that

$$\frac{1}{r_0^*} - \frac{c}{2} T^* = 0.$$

This entails
\( \psi(T^*) = \frac{1}{r_0} - cT^* + w_3 \int_0^{T^*} \frac{ds}{v + D_3} < \frac{1}{r_0} - \frac{c}{2}T^* = 0. \)

Thus one has \( \psi(T^*) < 0 \), but \( \psi(0) > 0 \), and by application of the mean value theorem, we obtain the existence of some \( T \in (0, \delta) \), \( T < T^* \), s.t. \( \psi(T) = 0 \). This implies the solution of (1)-(3) blows up in finite time, at \( t = T \), and by a standard comparison argument [18], the solution of the corresponding PDE system (15)-(17), also blows up in finite time. We can thus state the following theorem

**Theorem 3.1.** Consider the spatially explicit three species food chain model (15)-(17). For \( c < \frac{w_3}{D_3} \), \( r \) blows up in finite time, that is

\[
\lim_{t \to T^{**}} ||r(t)||_{\infty} \to \infty,
\]

as long as the initial data \( v_0, r_0 \) are large enough. Here \( T^{**} < T < \infty \).

Note the above argument easily generalises to the case \( c < k \left( \frac{w_3}{D_3} \right) \), where

\[
0 < k = \left( \frac{w_0b_1D_3}{w_1 \left( a_1 + \frac{(a_1)^2}{4a_2} \right) + w_0b_1D_3} \right) < 1.
\]

Thus one can also state the following corollary

**Corollary 1.** Consider the three species food chain model (15)-(17). Even if

\[
c < \left( \frac{w_0b_1D_3}{w_1 \left( a_1 + \frac{(a_1)^2}{4a_2} \right) + w_0b_1D_3} \right) \frac{w_3}{D_3},
\]

solutions to (15)-(17) with certain initial data are not bounded forward in time. In fact the solution \( r \) to (17) can blow-up in finite time, that is

\[
\lim_{t \to T^{***}} ||r(t)||_{\infty} \to \infty,
\]

as long as the initial data \( v_0, r_0 \) are large enough. Here \( T^{***} < T < \infty \).

**Remark 2.** We remark that the methods of this section can be directly applied to prove blow up in the ODE case as well. However the earlier proof via Theorem 2.1 has the advantage, that we can explicitly give a sufficient condition on the largeness of the data, required for blow-up. Also, not just the \( L^\infty(\Omega) \) norm, but every \( L^p(\Omega) \) norm, \( p \geq 1 \), blows up. This is easily seen in analogy with the equation \( r_t = d_3\Delta r + \delta r^2 \), and an application of the first eigenvalue method. Also note the blow-up times \( T^*, T^{**} \) for the PDE case are not to be confused with the blow-up times \( T^*, T^{**} \) for the ODE case.
4. Numerical validation

In this section we numerically simulate the ODE system (1)-(3), as well as the
PDE system (15)-(17), (in 1d and 2d), in order to validate our results Theorem
2.1, Theorem 2.2, Theorem 3.1 and Corollary 1. To this end we select the following
parameter range,

\[ a_1 = 1, b_1 = 0.5, D_0 = 10, a_2 = 1, w_1 = 0.1, D_1 = 13, w_2 = 0.25, D_2 = 10, \]
\[ c = 0.055, w_3 = 1.2, D_3 = 20. \]

These parameters satisfy condition (4), from [1].

\[ c = 0.055 < 0.0587 = \left( \frac{w_0 b_1 D_3}{w_1 \left( a_1 + \frac{(a_1)^2}{4a_2} \right) + w_0 b_1 D_3} \right) \frac{w_3}{D_3}. \]

Despite this, we see finite time blow-up. The systems are simulated in MATLB
R2011a. For simulation of the ODE systems we have used the standard
ode45 routine which uses a variable time step Runge Kutta method. To explore the
spatiotemporal dynamics of the PDE system in one and two dimensional spatial
domain, the system of partial differential equations is numerically solved using a
finite difference method. A central difference scheme is used for the one dimensional
diffusion term, whereas standard five point explicit finite difference scheme is used
for the two dimensional diffusion terms. The system is studied with positive initial
condition and Neumann boundary condition in the spatial domain \( 0 \leq x \leq L_x, \)
\( 0 \leq y \leq L_y, \) where \( L_x = L_y = \pi. \) Note, our proof of blow-up, allows for Dirichlet,
Neumann or Robin type boundary conditions. Simulations are done over this square
domain with spatial resolution \( \Delta x = \Delta y = 0.01, \) and time step size \( \Delta t = 0.01. \) We
next present the results of our simulations.

5. Conclusion

In the current letter we have shown that the solutions to the system (1)-(3),
modeling a tri-trophic food chain can exhibit finite time blow-up under the condition
(4) from theorem 1.1, as long as the initial data is large enough. This is also true in
the case of the spatially explicit model. Thus the basin of attraction of the invariant
set \( A, \) explicitly constructed in [1], is not all of \( \mathbb{R}^3_+, \) as claimed in [1]. Furthermore
system (1)-(3) is not dissipative in all of \( \mathbb{R}^3_+, \) also as claimed in [1]. For a numerical
validation of these results please see figures 2, 3, 4.

However, the model posesses very rich dynamics, in the parameter region

\[ \frac{w_3}{v + D_3} < c < \frac{w_3}{D_3}. \]

Thus an extremely interesting open question is, what is the basin of attraction
for an appropriately defined and constructed \( A? \) This is tantamount to asking,
which sorts of initial data lead to globally existing solutions, under the dynamics
of (1)-(3), and the parameter range (27)? The same questions can be asked, in the
case of the spatially explicit model.
Figure 1. These figures show blow up in the ODE case.

Figure 2. We also compare the original system with the modified system to illustrate our idea, to show blowup.
Figure 3. These figures show blow up in the 1d PDE case.

Figure 4. These figures show a surface plot of blowup in the 2d PDE case.
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