Mathematical analysis/Harmonic analysis

Sharp estimates of integral functionals on classes of functions with small mean oscillation

Estimations précises de certaines fonctionnelles intégrales sur des classes de fonctions avec une petite oscillation moyenne

Paata Ivanisvili\textsuperscript{a}, Nikolay N. Osipov\textsuperscript{b,c}, Dmitriy M. Stolyarov\textsuperscript{b,d}, Vasily I. Vasyunin\textsuperscript{b}, Pavel B. Zatitskiy\textsuperscript{b,d}

\textsuperscript{a} Department of Mathematics, Michigan State University, East Lansing, MI 48823, USA
\textsuperscript{b} St. Petersburg Department of Steklov Mathematical Institute RAS, Fontanka 27, St. Petersburg, Russia
\textsuperscript{c} Norwegian University of Science and Technology (NTNU), IME Faculty, Dep. of Math. Sci, Alfred Getz' vei 1, Trondheim, Norway
\textsuperscript{d} Chebyshev Laboratory, St. Petersburg State University, 14th Line, 29b, St. Petersburg, 199178 Russia

\textbf{A B S T R A C T}

We unify several Bellman function problems treated in [1,2,4–6,9–12,14–16,18–25]. For that purpose, we define a class of functions that have, in a sense, small mean oscillation (this class depends on two convex sets in $\mathbb{R}^2$). We show how the unit ball in the BMO space, or a Muckenhoupt class, or a Gehring class can be described in such a fashion. Finally, we consider a Bellman function problem on these classes, discuss its solution and related questions.

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\textbf{R É S U M É}

Nous unifions plusieurs problèmes concernant la fonction de Bellman traités dans [1,2,4–6,9–12,14–16,18–25]. Dans ce but, nous introduisons une classe de fonctions dont l’oscillation moyenne est petite dans un certain sens (cette classe depend de deux sous-ensembles convexes de $\mathbb{R}^2$). Nous démontrons que la boule unité de l’espace BMO, ou de la classe de Muckenhoupt, ou de la classe de Gehring, peut être décrite de cette façon. Finalement, nous considérons un problème de fonction de Bellman sur chacune de ces classes et discutons sa résolution ainsi que des questions voisines.

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Since Slavin [12] and Vasyunin [18] proved the sharp form of the John–Nirenberg inequality (see [15]), there have been many papers where similar principles were used to prove sharp estimates of this kind. However, there is no theory or even a unifying approach; moreover, the class of problems to which the method can be applied has not been described yet.
There is a portion of heuristics in the folklore that is each time applied to a new problem in a very similar manner. The first attempt to build a theory (at least for BMO) was made in [16], then the theory was developed in the paper [4] (see the short report [5] also). We would also like to draw the reader’s attention to the preprint [6], which can be considered as a description of the theory for the BMO space in a sufficient generality. Problems of this kind were considered not only in BMO, but in Muckenhoupt classes, Gehring classes, etc. (see [1,2,11,13,19,20]). In this short note, we define a class of functions and an extremal problem on it that includes all the problems discussed above.

1. Setting

Let $\Omega_0$ be a non-empty open strictly convex subset of $\mathbb{R}^2$ and let $\Omega_1$ be an open strictly convex subset of $\Omega_0$. We define the domain $\Omega$ as $\text{cl}(\Omega_0 \setminus \Omega_1)$ (the word “domain” comes from “domain of a function”; the symbol $\text{cl}$ denotes the closure) and the class $A_{\Omega}$ of summable $\mathbb{R}^2$-valued functions on an interval $I \subseteq \mathbb{R}$ as follows:

$$
A_{\Omega} = \{ \varphi \in L^1(I, \mathbb{R}^2) \mid \varphi(I) \subset \partial \Omega_0 \text{ and } \forall \text{ subinterval } J \subset I \langle \varphi \rangle_J \notin \Omega_1 \}. 
$$

(1)

Here $\langle \varphi \rangle_J = \frac{1}{|J|} \int_J \varphi(s) \, ds$ is the average of $\varphi$ over $J$. In Section 2 we show how the unit ball in BMO as well as the “unit balls” in Muckenhoupt and Gehring classes can be represented in the form (1). Let $f: \partial \Omega_0 \to \mathbb{R}$ be a bounded from below Borel measurable locally bounded function. We are interested in sharp bounds for the expressions of the form $\langle f(\varphi) \rangle_J$, where $\varphi \in A_{\Omega}$.

Again, in Section 2 we explain how the John–Nirenberg inequality or other inequalities of harmonic analysis can be rewritten as estimations of such an expression. The said estimates are delivered by the corresponding Bellman function

$$
B_{\Omega,f}(x) = \sup \{ \langle f(\varphi) \rangle_J \mid \langle \varphi \rangle_J = x, \varphi \in A_{\Omega} \}.
$$

(2)

**Problem 1.1.** Given a domain $\Omega$ and a function $f$, calculate the function $B_{\Omega,f}$.

The particular cases of this problem were treated in the papers [1,2,4–6,9–12,14–16,18–25] (see Section 2 for a detailed explanation). The main reason for Problem 1.1 to be solvable (and it is been heavily used in all the preceeding work) is that the function $B$ enjoys good properties.

**Definition 1.2.** Let $\omega$ be a subset of $\mathbb{R}^d$. We call a function $G: \omega \to \mathbb{R} \cup \{+\infty\}$ locally concave on $\omega$ if for every segment $\ell \subset \omega$ the restriction $G|_{\ell}$ is concave.

Define the class of functions on $\Omega$:

$$
\Lambda_{\Omega,f} = \left\{ G: \Omega \to \mathbb{R} \cup \{+\infty\} \mid G \text{ is locally concave on } \Omega, \forall x \in \partial \Omega_0 \ G(x) \geq f(x) \right\}.
$$

(3)

The function $\mathfrak{B}_{\Omega,f}$ is given as follows:

$$
\mathfrak{B}_{\Omega,f}(x) = \inf_{G \in \Lambda_{\Omega,f}} G(x).
$$

(4)

**Conjecture 1.3.** $B_{\Omega,f} = \mathfrak{B}_{\Omega,f}$.

In particular, the conjecture states that the Bellman function is locally concave (because the function $\mathfrak{B}_{\Omega,f}$ is).

**Problem 1.4.** Prove Conjecture 1.3 in adequate generality.

Though it may seem that one should solve Problem 1.4 before turning to Problem 1.1, it is not really the case. All the preceding papers used Conjecture 1.3 as an assumption that allowed the authors to guess $B$, then to prove that this function was the Bellman function indeed, and only then verify Conjecture 1.3 for $\Omega$ and $f$ chosen. However, to treat Problem 1.4 in itself, one has to invent a different approach, see Section 3.

2. Examples

From now on, we follow the agreement: if $g: \mathbb{R} \to \mathbb{R}^2$ is some fixed parameterization of $\partial \Omega_0$, then the function $f(g): \mathbb{R} \to \mathbb{R}$ is denoted by $f$.

**The BMO space.** We consider the BMO space with the quadratic seminorm. Let $\varepsilon$ be a positive number. Set $\Omega_0 = \{ x \in \mathbb{R}^2 \mid x_1^2 < x_2 \}$ and $\Omega_1 = \{ x \in \mathbb{R}^2 \mid x_1^2 + \varepsilon^2 < x_2 \}$. A function

$$
\varphi = (\varphi_1, \varphi_2): I \to \partial \Omega_0
$$

where $I$ is a subinterval of $\mathbb{R}$, then the function $\langle \varphi \rangle_I$ can be considered as a function $\mathbb{R} \to \mathbb{R}$.
belongs to the class $A_\Omega$ if and only if its first coordinate $\varphi_1$ belongs to $\text{BMO}_\varepsilon$ (the ball of radius $\varepsilon$ in BMO). Indeed, for any $t \in I$, we have $\varphi_2(t) = \varphi_2^2(t)$; therefore, the condition $(\varphi)_I \notin \Omega_1$ can be rewritten as

$$\langle \varphi_1^2 \rangle_j \leq \langle \varphi_1 \rangle_j^2 + \varepsilon^2,$$

which is the same as

$$\langle (\varphi_1 - \langle \varphi_1 \rangle_j) \rangle^2_j \leq \varepsilon^2. \quad (5)$$

Now we see that the class $A_\Omega$ corresponds to $\text{BMO}_\varepsilon$. The Bellman function (2) estimates the functional $(\tilde{f}(\varphi_1))_j$. The solution to Problem 1.1 with $f(t) = e^{it^2}$ leads to the John–Nirenberg inequality in its integral form, the case $f(t) = \chi_{[-\infty, -\varepsilon]}(t)$ corresponds to the weak form of the John–Nirenberg inequality, and the case $f(t) = |t|^p$ leads to equivalent definitions of BMO. We refer the reader to the paper [4] for a detailed discussion. This case is the subject of study for the papers [4, 6, 9, 10, 15, 16, 21, 22, 25].

**Classes $A_{p_1, p_2}$.** Let $p_1$ and $p_2$, $p_1 > p_2$, be real numbers and let $Q \geq 1$. Suppose

$$\Omega_0 = \{ x \in \mathbb{R}^2 | x_1, x_2 > 0, \ x_2^\frac{1}{p_2} < x_1^\frac{1}{p_1} \} \quad \text{and} \quad \Omega_1 = \{ x \in \mathbb{R}^2 | x_1 > 0, \ Q x_2^\frac{1}{p_2} < x_1^\frac{1}{p_1} \}.$$ 

If a function $\varphi$ belongs to the class $A_\Omega$, then its first coordinate $\varphi_1$ belongs to the so-called $A_{p_1, p_2}$ class. The “norm” in this class is defined as

$$[\varphi]_{A_{p_1, p_2}} = \sup_{I \subset \Omega} \langle \varphi \rangle^\frac{1}{p_1} \langle \varphi^p \rangle^\frac{1}{p_2},$$

where the supremum is taken over all subintervals of $I$. These classes were introduced in [20]. If $p \in (1, \infty)$, then $A_{1, -\frac{1}{p}} = A_p$, where $A_p$ stands for the classical Muckenhoupt class. When $p_2 = 1$ and $p_1 > 1$, the class $A_{p_1, p_2}$ coincides with the so-called Gehring class (see [7] or [8]). Estimates of integral functionals as provided by the Bellman function (2) lead to various sharp forms of the reverse Hölder inequality, see [20]. These cases were treated in the papers [1, 2, 11, 19, 20].

**Reverse Jensen classes.** These classes were introduced in [7], Let $\Phi: \mathbb{R}_+ \to \mathbb{R}_+$ be a convex function. Let $Q > 1$. Consider the class of functions $\psi: I \to \mathbb{R}_+$ such that

$$\forall J \subset I \quad \langle \Phi(\psi)_j \rangle_j \leq Q \Phi(\langle \psi \rangle_j).$$

Surely, both a Muckenhoupt class and a Gehring class can be described as certain Reverse Jensen classes. The corresponding domain is $\{ x \in \mathbb{R}^2 | x_1, x_2 \geq 0, \ \Phi(x_1) \leq x_2 \leq Q \Phi(x_1) \}$. Consult a very recent paper [13], where the Bellman function on the domain $\{ x \in \mathbb{R}^2 | e^{x_1} \leq x_2 \leq Ce^{x_1} \}, C > 1$, provides sharp constants in the John–Nirenberg inequality for the BMO space equipped with the $L^p$-type seminorm.

### 3. Hints to solutions

First, we note that strict convexity of $\Omega_0$ implies the fact that $B(x) = f(x)$ for $x \in \partial \Omega_0$. Second, we need $\Omega$ to fulfill several assumptions that all the domains listed in Section 2 do satisfy.

1. The domains $\Omega_0$ and $\Omega_1$ are unbounded. \hfill (7)
2. The boundary of $\Omega_1$ is $C^2$-smooth. \hfill (8)
3. Every ray inside $\Omega_0$ can be translated to belong to $\Omega_1$ entirely. \hfill (9)

The first two conditions are technical in a sense, the third one is essential, since (under assumption (7)) it is equivalent to the fact that for any $x \in \Omega$ there exists a function $\varphi \in A_\Omega$ such that $(\varphi)_j = x$ (i.e. the supremum in formula (2) is taken over a non-empty set). Now we are ready to present a solution to Problem 1.4.

**Theorem 3.1.** Let the domain $\Omega$ satisfy the conditions (7), (8), (9). If the function $f$ is bounded from below, then $\partial \Omega, f = B, \partial, f$.

The condition that $f$ is bounded from below is not necessary. However, we note that without this condition the extremal problem in formula (2) is not well posed (the integral of $f(\varphi)$ may be not well defined). In [17] the reader can find the proof of Theorem 3.1 for the case $\partial \Omega_1 \subset \Omega_0$ as well as its analog where $f$ can be unbounded from below.

To solve Problem 1.1, we need to consider even more restrictive conditions, we introduce some notation for that purpose. Choose $g = (g_1, g_2): \mathbb{R} \to \mathbb{R}^2$ to be a continuous parameterization of $\partial \Omega_0$; let the domain $\Omega$ lie on the left of this oriented curve. For any number $u \in \mathbb{R}$ we draw two tangents from the point $g(u)$ to the set $\Omega_1$; by a tangent we mean not a line, but a segment connecting $g(u)$ with the tangency point. We denote the lengths of the left and the right tangents by $\ell_L(u)$ and $\ell_R(u)$ correspondingly (the left tangent lies between the right one and $g'$, see [4] for explanations about this notation).
The boundaries $\partial \Omega_0$ and $\partial \Omega_1$ are $C^3$-smooth curves, the function $f$ is $C^3$-smooth.

2. The curve $\gamma(t) = (g_1(t), g_2(t), \dot{\gamma}(t))$ changes the sign of its torsion a finite number of times.

3. The integrals $\int_{-\infty}^{0} \frac{1}{tR} + \int_{0}^{+\infty} \frac{1}{tL}$ diverge.

In Condition (12), the integration is with respect to the natural parameterization of the curve $\partial \Omega_1$, where the functions $tR$ and $tL$ are considered as the functions of their tangency points lying on $\partial \Omega_1$. The last Condition (12) is more mysterious; we believe that our considerations may work without it.

We also need a summability assumption for the function $f$. Let $\alpha_R(u)$ denote the oriented angle between the right tangent at the point $u$ and the vector $(1,0)$, and let $\alpha_L(u)$ denote the oriented angle between the left tangent at the point $u$ and the vector $(1,0)$. Then, the summability condition requires the bulky integral
\[
\int_{-\infty}^{t} \exp \left( \int_{\tau}^{t} \frac{g_1'(s)}{t \cos(\alpha_R(s))} \right) \frac{\tan(\alpha_R(\tau))}{g_1'(\tau)} \left| \begin{array}{ccc}
\dot{f}'(\tau) & \ddot{f}'(\tau) & \dddot{f}'(\tau) \\
g_1'(\tau) & g_2'(\tau) & g_3'(\tau) \\
g_1''(\tau) & g_2''(\tau) & g_3''(\tau)
\end{array} \right| d\tau
\] (13)
to converge for any $t \in \mathbb{R}$ (and a similar condition with $R$ replaced by $L$ and with $-\infty$ replaced by $+\infty$).

**Claim:** under Conditions (7), (9), (10), (11), (12), and the mentioned convergence conditions for the integrals (13), we can solve Problem 1.1.

As in [4], by “solution” we mean an expression for the function $B$, which may include roots of implicit equations, differentiations, and integrations. Though at the first sight, the benefit of such a “solution” may seem questionable, it occurs to be useful if one has a specific domain $\Omega$ and a function $f$ at hand, see examples in the papers [4,6], the whole paper [22] that treats the cases of functions $f$ extremely difficult from an algebraic point of view, and other papers on the subject.

It appears that to solve Problem 1.1, one has to reformulate reasonings from [4] and [6] in geometric terms and observe that in such terms they work for a more general setting of the problem considered. For example, the integral (13) plays the role of the force function coming from $-\infty$ (see [4] for the definition in the case of $BMO$) in the general setting. However, the geometric essence of the matter is even more revealed in the example of the chordal domain (it has already been used in [3]). We remind the reader that a chordal domain is a type of foliation (see [4] for the definition) that consists of chords, i.e. segments that connect two points of $\partial \Omega_0$. In the case of the parabolic strip $g(t) = (t, t^2)$, the chordal domain could match $B_{\frac{5}{2}}$ if and only if it satisfied the cup equation
\[
\frac{\tilde{f}(b) - \tilde{f}(a)}{b - a} = \frac{\tilde{f}'(b) + \tilde{f}'(a)}{2}; \quad (a, a^2) \text{ and } (b, b^2) \text{ are the endpoints of a chord},
\]
and two special differential inequalities (“inequalities for the differentials”) for each of its chord. In the general setting of Problem 1.1, the cup equation turns into
\[
\left| \begin{array}{ccc}
g_1'(a) & g_2'(a) & \tilde{f}'(a) \\
g_1'(b) & g_2'(b) & \tilde{f}'(b) \\
g_1(b) - g_1(a) & g_2(b) - g_2(a) & \tilde{f}(b) - \tilde{f}(a)
\end{array} \right| = 0; \quad g(a) \text{ and } g(b) \text{ are the endpoints of a chord},
\]
which has the following geometrical meaning: the tangent vectors to the curve $\gamma(t) = (g_1(t), g_2(t), \dot{\gamma}(t))$ at the points $a$ and $b$ lie in one two-dimensional plane with the vector $\gamma'(a) - \gamma'(b)$. The special differential inequalities (the so-called inequalities for the differentials) can also be re-stated in purely geometric terms (the triple product of $\gamma'(a), \gamma'(b) - \gamma'(a)$, and $\gamma'(b)$ at the point $a$ should be negative; the same should be fulfilled with $a$ and $b$ interchanged) and then generalized to fit Problem 1.1.

In [4] the roots of $\tilde{f}''$ played the main role. Indeed, the cups sit on the points where $\tilde{f}'''$ changes its sign from + to −. In the general case, the function $\tilde{f}''$ should be replaced by the torsion of the curve $\gamma$. One can see the traces of the torsion in formula (13). Moreover, now we see that Condition (11) is a straightforward generalization of the regularity condition from [4].

We recall that in [4] the problem was treated not in the full generality (we assumed that the roots of $\tilde{f}'''$ were well separated). This narrowed the list of local types of foliations. However, without such an assumption, the collection of figures is wider, see the preprint [6] for the general theory, and the example [22], where almost all figures from the general case appear. The latter paper also highlights the notation that becomes very important when there are lots of different figures (it appeared that a foliation corresponds to a special weighted graph). We only mention that all the figures are transferred to the general setting of Problem 1.1, as well as all the monotonicity lemmas for forces and tails (see [4] for definitions).

**Acknowledgements**

The research of N.O., D.S. and P.Z. is supported by the Chebyshev Laboratory (Department of Mathematics and Mechanics, St. Petersburg State University) under RF Government grant No. 11.G34.31.0026. N.O. and D.S. are supported by RFBR grant
No. 14-01-00198. D.S. and P.Z. are supported by JSC “Gazprom Neft”. N.O. is supported by RFBR grant No. 14-01-31163 and supported by ERCIM “Alain Bensoussan” Fellowship. V.V. is supported by RFBR grant No. 14-01-00748a. P.Z. is supported by the SPbSU grant No. 6.38.223.2014 and by the Dinasty Foundation.

P.I. visited the Hausdorff Research Institute for Mathematics (HIM) in the framework of the Trimester Program “Harmonic Analysis and Partial Differential Equations”. He thanks HIM for the hospitality.

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