SUB-SOLUTIONS AND A POINT-WISE HOPF’S LEMMA FOR FRACTIONAL $p$-LAPLACIAN

ZAIZHENG LI*

School of Mathematical Sciences, Hebei Normal University
Shijiazhuang, 050024, China
Department of Mathematical Sciences, Yeshiva University
New York, NY 10033, USA

QIDI ZHANG

Department of Mathematics, University of British Columbia
Vancouver, B.C. V6T 1Z2, Canada

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**Abstract.** We prove a Hopf’s lemma in the point-wise sense for fractional $p$-Laplacian. The essential technique is to prove $(-\Delta)^s_p u(x)$ is uniformly bounded in the unit ball $B_1 \subset \mathbb{R}^n$, where $u(x) = (1 - |x|^2)^s_p$. Also we study the global Hölder continuity of bounded positive solutions for $(-\Delta)^s_p u(x) = f(x, u)$.

1. **Introduction and main results.** The fractional $p$-Laplacian operator $(-\Delta)^s_p$ is a non-local operator, which is of the form

$$(-\Delta)^s_p u(x) = C_{n,s,p} \lim_{\epsilon \to 0} \int_{\mathbb{R}^n \setminus B_\epsilon(x)} \frac{|u(x) - u(y)|^{p-2}[u(x) - u(y)]}{|x - y|^{n+sp}} dy,$$

where $p \geq 2$, $s \in (0, 1)$, $C_{n,s,p}$ is a constant and P.V. is the Cauchy’s principal value. The fractional $p$-Laplacian operator is an extension version of fractional Laplacian ($p = 2$). In order that the integral on the right hand is well defined, we require

$$u \in C^{1,1}_{loc} \cap \mathcal{L}_{sp},$$

where

$$\mathcal{L}_{sp} := \left\{ u \in L^{p-1}_{loc} \left( \mathbb{R}^n \right) \left| \int_{\mathbb{R}^n} \frac{1 + u(x)^{p-1}}{1 + |x|^{n+sp}} < \infty \right. \right\}.$$

In recent years, the non-local operators arise from many fields, such as game theory, finance, Lévy processes, and optimization, see [1, 23, 4, 3, 18, 26, 12, 24] and references therein. In the special case $p = 2$, Luis A. Caffarelli and Luis Silvestre in [6] introduce an extension method which turns the non-local operator into a local one in higher dimensions. Luis Silvestre in [22], Xavier Ros-Oton and Joaquim Serra

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* Corresponding author.
in [21] discuss about the regularity of solutions for equations involving the fractional Laplacian. Wenxiong Chen, Congming Li and Yan Li in [8], Wenxiong Chen, Yan Li and Ruobing Zhang in [11] develop direct methods of moving planes and moving spheres. Antonio Greco and Raffaella Servadei in [16] prove a Hopf’s lemma for the fractional Laplacian. There are some explicit solutions for $p = 2$, for example, one can find $(-\Delta)^s u(x) = \text{const}$ for $x \in B_1$, where

$$u(x) = (1 - |x|^2)^s_+ = \begin{cases} (1 - |x|^2)^s, & |x| < 1; \\ 0, & |x| \geq 1. \end{cases}$$

in [15] and $(-\Delta)^s (x_n)^+_s = 0$,

$$(x_n)^s_+ = \begin{cases} x_n^s, & x_n > 0; \\ 0, & x_n \leq 0. \end{cases}$$

in the upper half space $\mathbb{R}^n_+$ in [17]. We refer the reader to [10, 14] and references therein for more related results.

When $p > 2$, $(-\Delta)^s_p$ is a non-local and nonlinear operator. Félix del Teso, David Gómez-Castro and Juan Luis Vázquez in [25] introduce the extension method for $(-\Delta)^s_p$. For $p > 2$, $(-\Delta)^s_p (x_n)^+_s = 0$ in the upper half space $\mathbb{R}^n_+$ still holds, see [17]. Our first result is the following sub-solutions result.

**Theorem 1.1** (Sub-solutions). Let $s \in (0, 1)$, $p > 2$, $n \in \mathbb{N}^*$, $u(x) = (1 - |x|^2)^s_+$, then $(-\Delta)^s_p u(x)$ is uniformly bounded in the unit ball $B_1 \subset \mathbb{R}^n$.

This is really the first time we are able to prove the uniform boundedness of $(-\Delta)^s_p u(x)$, where $u(x) = (1 - |x|^2)^s_+$, and this property plays an essential role in the proof of Hopf’s lemma. It is well known that the Hopf’s lemma is one of the most useful tool in the theory of partial differential equations. For $p = 2$, the proof is based on Fourier transform and hypergeometric functions. But Fourier transform does not work anymore due to the nonlinearity when $p > 2$, and we could not find out any hypergeometric function to exploit in this case. In fact, before this theorem, people even do not know whether $(-\Delta)^s_p u(x)$ is uniformly bounded. Compared with the case $p = 2$, $(-\Delta)^s_p u(x)$ is not constant anymore when $p > 2$ by numerical calculation (see Proposition 2). Our proof is based on rigorous analysis on the singular term, then we figure out the exact coefficient of the singular term and we prove the coefficient is 0. Our second result is the following point-wise Hopf’s lemma.

**Theorem 1.2** (Hopf). Let $\Omega$ be a bounded domain with the uniform interior ball condition. For $u(x) \in C^{1,1}_{loc}(\Omega) \cap L^p(\mathbb{R}^n)$, $p > 2$, $s \in (0, 1)$. Assume $u$ is lower semicontinuous in $\overline{\Omega}$ and pointwisely satisfies

$$\begin{cases} (-\Delta)^s_p u + c(x) u \geq 0, & x \in \Omega, \\ u > 0, & x \in \mathbb{R}^n \setminus \Omega, \\ u = 0, & x \in \mathbb{R}^n \setminus \Omega, \end{cases} \quad (1.1)$$

where $c(x) \geq 0$ is bounded. Then there exists a constant $C = C(\Omega, u) > 0$, such that

$$\liminf_{x \to \partial \Omega} \frac{u(x)}{\text{dist}(x, \partial \Omega)^s} \geq C.$$ 

This is the first time we are able to prove a point-wise Hopf’s lemma for the fractional $p$-Laplacian. There are some previous results about the Hopf’s lemma.
Wenxiong Chen and Congming Li in [7] prove a boundary estimate for \((-\Delta)^s_p\), which is a key part in the moving plane method, and the boundary estimate plays the role of Hopf’s lemma to some degree. Leandro M. Del Pezzo and Alexander Quaas in [13], Wenxiong Chen, Congming Li and Shijie Qi in [9] prove a Hopf’s lemma for \(u \in \mathcal{W}^{s,p}(\Omega)\), where

\[
\mathcal{W}^{s,p}(\Omega) := \left\{ u \in L^p_{loc}(\mathbb{R}^n) \mid \exists U \supset \supset \Omega, \text{suchthat} \int_U \int_U \frac{|u(x) - u(y)|^p}{|x-y|^{n+sp}} \, dx \, dy < +\infty \right\}.
\]

But \(\mathcal{W}^{s,p}(\Omega)\) is different from \(C^{1,1}_1(\Omega) \cap L^s_{sp}(\mathbb{R}^n)\), see Example 4.1.

The third result is the Hölder regularity of positive solutions for \((-\Delta)^s_p u(x) = f(x, u)\) in any domain (bounded or not).

**Theorem 1.3.** Let \(\Omega\) be any domain (bounded or not) with the uniform two-sided ball condition, \(s \in (0, 1), p \geq 2, \) and \(u \in C^{1,1}_1(\Omega) \cap L^s_{sp}(\mathbb{R}^n)\) is a bounded positive solution of

\[
\begin{cases}
(-\Delta)^s_p u = f(x, u), & x \in \Omega, \\
u = 0, & x \in \mathbb{R}^n \setminus \Omega,
\end{cases}
\]

where \(f(x, u)\) is bounded. Then there exists a constant \(\nu_0 \in (0, s)\), such that \(u \in C^{\nu_0}(\mathbb{R}^n)\). Moreover,

\[
[u]_{C^{\nu_0}(\mathbb{R}^n)} \leq C(\nu_0) \left[ 1 + \|u\|_{L^\infty(\Omega)} + C\|f\|_{L^\infty(\Omega)}^{\frac{1}{p-1}} \right].
\]

**Remark 1.** See [2] for more information about the ball condition. We extend the regularity results from bounded domains to unbounded domains. About the Hölder regularity of solutions for \((-\Delta)^s_p u(x) = f(x, u)\), Antonio Iannizzotto, Sunra Mosconi, and Marco Squassina in [17] prove the global Hölder regularity of solutions in \(\mathcal{W}^{s,p}_0(\Omega)\) in bounded domains. Lorenzo Brasco, Erik Lindgren, and Armin Schikorra in [5] consider the higher Hölder regularity of local weak solutions in bounded domains, and they first give an explicit Hölder exponent. Yan Li and Lingyu Jin in [19] prove certain Hölder continuity up to the boundary in bounded domains.

This paper is organized as follows. In section 2, we give some preliminary properties. Section 3 is devoted to showing Theorem 1.1. Section 4 is contributed to proving Hopf’s Theorem 1.2. Section 5 is contributed to proving the global Hölder regularity for bounded positive solutions. In section 6, we list our numerical calculation results. The constant \(C\) may vary from line to line or even in the same line.

2. Preliminaries. Let us start by introducing some notations and properties we will use in this article.

**Lemma 2.1.** Set \(u(x) = (1 - |x|^2)_+^s\), then \((-\Delta)^s_p u(x)\) is radially symmetric in the unit ball.

**Proof.** Let \(G(t) := |t|^{p-2}t\), and \(x \in B_1\) given. Set \(A\) is any orthogonal transformation, then

\[
(-\Delta)^s_p u(x) = C_{n,s,p} P.V. \int_{\mathbb{R}^n} G[(1 - |y|^2)^s - (1 - |y|^2)^s] \frac{1}{|x-y|^{n+sp}} \, dy.
\]
\[= C_{n,s,p} P.V. \int_{\mathbb{R}^n} \frac{G[(1 - |Ax|^2)^s - (1 - |Ay|^2)^s]}{|A(x - y)|^{n+p}} dy \]
\[= C_{n,s,p} P.V. \int_{\mathbb{R}^n} \frac{G[(1 - |Ax|^2)^s - (1 - |z|^2)^s]}{|Ax - z|^{n+p}} dz = (-\Delta)^s u(Ax). \]

which implies the radial symmetry of \((-\Delta)^s u(x)\). \qed

Lemma 2.2. Set \(G(t) := |t|^{p-1} = |t|^{p-2} t, p \geq 2\), then \(G(t)\) is strictly increasing and \(G(t) - G(s) \leq 2^{2-p} G(t - s), \quad \forall t < s\).

Proof. 1) For \(t \neq 0\), by \(G'(t) = (p - 1)|t|^{p-2} > 0\), so \(G(t)\) is strictly increasing.

2) We only need to prove
\[\frac{G(s) - G(t)}{G(s - t)} \geq 2^{2-p}, \quad \forall t < s.\]

If \(s = 0\), it is obviously true. In the following we assume \(s \neq 0\), then by direct calculation,
\[\frac{G(s) - G(t)}{G(s - t)} = \frac{|s|^{p-2} s - |t|^{p-2} t}{s - t|^{p-2} (s - t)} = \frac{1 - \frac{1}{s} |s|^{p-2} \frac{t}{s}}{1 - \frac{1}{s} |s|^{p-2} (1 - \frac{1}{s})}.\]

Set
\[F(\rho) = \frac{1 - |\rho|^{p-2} \rho}{1 - |\rho|^{p-2} (1 - \rho)}, \quad \rho \neq 1.\]

It is readily to have
\[\lim_{\rho \to \pm \infty} F(\rho) = 1, \quad F'(\rho) = \frac{(p - 1)(1 - |\rho|^{p-2})}{|1 - \rho|^p}.\]

Hence for \(p \geq 2\), \(F(\rho)\) attains the minimum value at \(\rho = -1\), i.e. \(F(\rho) \geq F(-1) = 2^{2-p}\). \qed

Proposition 1 (Comparison principle). Let \(\Omega\) be a bounded domain, \(u, v \in C^{1,1}_{\text{loc}}(\Omega) \cap L^p(\mathbb{R}^n)\), \(u\) is lower semi-continuous in \(\overline{\Omega}\) and \(v\) is upper semi-continuous in \(\overline{\Omega}\). \(u, v\) satisfy
\[
\begin{cases}
(-\Delta)^s u(x) + c(x) u(x) \geq (-\Delta)^s v(x) + c(x) v(x), & x \in \Omega, \\
u(x) \geq v(x), & x \in \mathbb{R}^n \setminus \Omega.
\end{cases}
\] (2.1)

where \(c(x) \geq 0\). Then \(u(x) \geq v(x)\) in \(\Omega\).

Proof. We prove the desired result by contradiction. If there is a point \(x_0 \in \Omega\), such that \(u(x_0) < v(x_0)\). Since \(u(x)\) is lower semi-continuous in \(\overline{\Omega}\) and \(v(x)\) is upper semi-continuous in \(\overline{\Omega}\), \(u - v\) attains the minimum value in \(\overline{\Omega}\). Without loss of generality, we assume
\[0 > u(x_0) - v(x_0) = \min_{\Omega} (u - v) = \min_{\mathbb{R}^n} (u - v).\]

Then
\[u(x_0) - v(x_0) \leq u(y) - v(y), \quad \forall y \in \mathbb{R}^n.\]

That is
\[u(x_0) - u(y) \leq v(x_0) - v(y), \quad \forall y \in \mathbb{R}^n.\]

By Lemma 2.2,
\[
G(u(x_0) - u(y)) - G(v(x_0) - v(y)) \leq 0, \quad \forall y \in \mathbb{R}^n, \\
G(u(x_0) - u(y)) - G(v(x_0) - v(y)) < 0, \quad \forall y \in \mathbb{R}^n \setminus \Omega. \] (2.2)
By the equation (2.1) and (2.2), we have

\[ 0 \leq (-\Delta)_p^s u(x_0) - (-\Delta)_p^s v(x_0) + c(x_0)u(x_0) - c(x_0)v(x_0) \]

\[ = C_{n,s,p} \text{P.V.} \int_{\mathbb{R}^n} \frac{G[u(x_0) - u(y)]}{|x_0 - y|^{n+sp}} dy - C_{n,s,p} \text{P.V.} \int_{\mathbb{R}^n} \frac{G[v(x_0) - v(y)]}{|x_0 - y|^{n+sp}} dy + c(x_0) [u(x_0) - v(x_0)] \]

\[ = C_{n,s,p} \text{P.V.} \int_{\mathbb{R}^n} \frac{G[u(x_0) - u(y)] - G[v(x_0) - v(y)]}{|x_0 - y|^{n+sp}} dy + c(x_0) [u(x_0) - v(x_0)] < 0. \]

Which is a contradiction. Thus there is no such \( x_0 \) and \( u(x) \geq v(x) \) in \( \Omega \). \( \square \)

3. **Proof of Theorem 1.1.** In fact, for some fixed \( \delta \in (0, 1) \), when \( |x| < 1 - \delta \), by [7, Lemma 5.2],

\[ |(-\Delta)_p^s u(x)| \leq C \int_{\mathbb{R}^n \setminus B_\delta(x)} \frac{1}{|x - y|^{n+sp}} dy + C |\nabla u(x)|^{p-2} \int_{B_\delta(x)} \frac{|x - y|^p}{|x - y|^{n+sp}} dy \leq C. \]

That is, \( (-\Delta)_p^s u(x) \) is bounded for \( |x| < 1 - \delta \). Therefore in the following we only need to consider the case when \( |x| \) is close to 1.

3.1. \( n = 1 \). In this part, for \( u(x) = (1 - x^2)_+^s \), we will prove that \( (-\Delta)_p^s u(x) \) is uniformly bounded in \( (-1, 1) \). Due to Lemma 2.1, we only need to consider \( x \in (0, 1) \). The proof is divided into 3 steps.

Step 1. Firstly we give a general estimate for \( (-\Delta)_p^s u(x) \) when \( x \) is close to 1.

For simplicity, we omit the constant \( C_{n,s,p} \).

\[ (-\Delta)_p^s u(x) \]

\[ = \int_{-\infty}^{1-x} \frac{(1-x^2)^{s(p-1)}}{(x-y)^{1+sp}} dy + \int_{1-x}^{1} \frac{(1-x^2)^{s(1-y)^{1+sp}}}{{(y-x)^{1+sp}}} dy + \int_{1}^\infty \frac{(1-x^2)^{s(1-y)^{1+sp}}}{{(y-x)^{1+sp}}} dy \]

\[ + \lim_{\delta \to 0} \left\{ \int_{-\infty}^{1-x} \frac{ -(1-x^2)^{s(1-y)^{1+sp}}}{{(y-x)^{1+sp}}} dy + \int_{1-x}^{1} \frac{ -(1-x^2)^{s(1-y)^{1+sp}}}{{(y-x)^{1+sp}}} dy \right\} \]

\[ = \int_{-\infty}^{1-x} \frac{(1-x^2)^{s(p-1)}}{z^{1+sp}} dz + \int_{1-x}^{1} \frac{(1-x^2)^{s(1-(x-z)^2)^{1+sp}}}{{z^{1+sp}}} dz + \int_{1}^{\infty} \frac{(1-x^2)^{s(1-y)^{1+sp}}}{{z^{1+sp}}} dz \]

\[ + \lim_{\delta \to 0} \left\{ \int_{-\infty}^{1-x} \frac{ -(1-x^2)^{s(1-y)^{1+sp}}}{{z^{1+sp}}} dz + \int_{1-x}^{1} \frac{ -(1-x^2)^{s(1-y)^{1+sp}}}{{z^{1+sp}}} dz \right\} \]

\[ = I_1 + I_2 + I_3 + I_4 + I_5 + I_6. \]

Where

\[ I_1 + I_6 = (1-x^2)^{s(p-1)} \left[ \int_{1-x}^{\infty} \frac{1}{z^{1+sp}} dz + \int_{1}^{\infty} \frac{1}{z^{1+sp}} dz \right] = (1-x^2)^{s(p-1)} \left[ \frac{1}{sp(1+x)^{sp}} + \frac{1}{sp} \right], \]
and
\[ |I_2| + |I_5| \leq \int_{-1}^{1-x} \frac{1}{z^{1+s} p} \, dz + \int_{1}^{2-x} \frac{1}{z^{1+s} p} \, dz = \int_{1}^{1-x} \frac{1}{z^{1+s} p} \, dz = \frac{1}{sp} \left[ 1 - \frac{1}{(1+x)^{sp}} \right]. \]

So \( I_1, I_2, I_5, I_6 \) are uniformly bounded. Then we only need to consider \( I_3, I_4 \).

\[ I_3 = \lim_{\epsilon \to 0} \int_{-1}^{1-x} \frac{1}{z^{1+s} p} \left[ (1-x)^s - (1-(x+z)^s) \right]^{ p-1} - \left[ (1-(x-z)^s) - (1-x^s) \right]^{ p-1} \, dz \]
\[ = (1-x^2)^{(p-1)} \lim_{\epsilon \to 0} \int_{x}^{2-x} \frac{1}{z^{1+s} p} \left[ (1-k - \frac{1-x^2}{4x^2} k^2)^s - 1 \right]^{ p-1} \, dz \]
\[ = \frac{(2x)^{sp}}{(1-x)^{sp}} \lim_{\epsilon \to 0} \int_{x}^{2-x} \frac{1}{z^{1+s} p} \left[ (1-k - \frac{1-x^2}{4x^2} k^2)^s - 1 \right]^{ p-1} \, dk \]
\[ = \frac{(2x)^{sp}}{(1-x)^{sp}} I_3', \]

where we use the substitution
\[ k = \frac{2x}{1-x^2} z. \]

Similarly, we deal with \( I_4 \).

\[ I_4 = \int_{-1}^{1} \frac{1}{z^{1+s} p} \frac{(1-x^2)^s - (1-x^2)^s}{(1-x)^s} \, dz \]
\[ = (1-x^2)^{(p-1)} \frac{1}{sp(1-x)^{sp}} - 1 \frac{x}{sp} \int_{1-x}^{1} \frac{1}{z^{1+s} p} \left[ (1-(x-z)^s) - (1-x^s) \right]^{ p-1} \, dz \]
\[ = (1-x^2)^{(p-1)} \frac{1}{sp(1-x)^{sp}} - 1 \frac{x}{sp} \int_{1-x}^{1} \frac{1}{z^{1+s} p} \left[ (1-k - \frac{1-x^2}{4x^2} k^2)^s - 1 \right]^{ p-1} \, dz \]
\[ = \frac{1}{(1-x)^s} \left[ \frac{1+x}{sp} - \frac{(2x)^{sp}}{(1+x)^{sp}} \int_{1-x}^{1} \frac{1}{z^{1+s} p} \left[ (1+k - \frac{1-x^2}{4x^2} k^2)^s - 1 \right]^{ p-1} \, dk \right] \]
\[ = \frac{1}{(1-x)^s} \left[ \frac{1+x}{sp} - \frac{(2x)^{sp}}{(1+x)^{sp}} I_4' \right] - \frac{(1-x^2)^{(p-1)}}{sp}. \]

Step 2. The aim of this part is to simplify \((1-x)^{-s}I_3'\) and \((1-x)^{-s}I_4'\). To be precise, we will prove

\[ -C \leq (1-x)^{-s}I_3' - (1-x)^{-s} \int_{0}^{1} \frac{1}{k^{1+s} p} \left[ (1-k)^s - (1+k)^s - 1 \right]^{ p-1} \, dk \leq C, \quad (3.1) \]

and

\[ -C \leq (1-x)^{-s}I_4' - (1-x)^{-s} \int_{1}^{\infty} \frac{1}{k^{1+s} p} \left[ (1+k)^s - 1 \right]^{ p-1} \, dk \leq C. \quad (3.2) \]

Firstly, we cope with the term \((1-x)^{-s}I_3'\).

\[ (1-x)^{-s}I_3' = (1-x)^{-s} \int_{0}^{2-x} \frac{1}{z^{1+s} p} \left[ (1-k - \frac{1-x^2}{4x^2} k^2)^s - 1 \right]^{ p-1} \, dk \]
\[ \leq (1-x)^{-s} \int_{0}^{2-x} \frac{1}{z^{1+s} p} \left[ (1-k - \frac{1-x^2}{4x^2} k^2)^s - 1 \right]^{ p-1} \, dk + C. \]
This is because
\[
\limsup_{x \to 1} (1-x)^s \int_0^1 \left[ 1 - \left( 1 - k - \frac{1-x^2}{4x^2} k^2 \right) \right]^{p-1} \frac{1}{k^{1+sp}} \left[ 1 + k - \left( \frac{1-x^2}{4x^2} k^2 \right)^s \right]^{-1} \, dk
\]
\[
\leq \limsup_{x \to 1} (1-x)^s \frac{(1+x)^{sp} (1-x)}{1+x} = 0.
\]

Next we are going to prove there exists a constant \( C > 0 \), such that
\[
-C \leq (1-x)^s \int_0^1 \left[ 1 - \left( 1 - k - \frac{1-x^2}{4x^2} k^2 \right) \right]^{p-1} \frac{1}{k^{1+sp}} \left[ 1 + k - \left( \frac{1-x^2}{4x^2} k^2 \right)^s \right]^{-1} \, dk
\]
\[
-(1-x)^s \int_0^{1/2} \left[ 1 - \left( 1 - k - \frac{1-x^2}{4x^2} k^2 \right) \right]^{p-1} \frac{1}{k^{1+sp}} \left[ 1 + k - \left( \frac{1-x^2}{4x^2} k^2 \right)^s \right]^{-1} \, dk \leq C.
\]

Just consider the difference above,
\[
(1-x)^s \int_0^1 \left[ 1 - \left( 1 - k - \frac{1-x^2}{4x^2} k^2 \right) \right]^{p-1} \frac{1}{k^{1+sp}} \, dk
\]
\[
= (1-x)^s \int_0^{1/2} \left[ 1 - \left( 1 - k - \frac{1-x^2}{4x^2} k^2 \right) \right]^{p-1} \frac{1}{k^{1+sp}} \, dk
\]
\[
+ (1-x)^s \int_{1/2}^1 \left[ 1 - \left( 1 - k - \frac{1-x^2}{4x^2} k^2 \right) \right]^{p-1} \frac{1}{k^{1+sp}} \, dk
\]
\[
\leq \frac{1}{2} C k^{sp-2} \left[ (1-k)^s - \left( 1 - k - \frac{1-x^2}{4x^2} k^2 \right)^s \right] \, dk
\]
\[
+ (1-x)^s \int_0^{1/2} \left[ 1 - \left( 1 - k - \frac{1-x^2}{4x^2} k^2 \right) \right]^{s} \frac{k^{1+sp}}{k^{1+sp}} \, dk
\]
\[
\leq (1-x)^s \int_0^{1/2} C k^{sp-2} (1-k)^s \left[ 1 - \left( 1 - \frac{1-x^2}{4x^2} k^2 \right) \right] \, dk
\]
\[
+ (1-x)^s \int_{1/2}^1 C k^{sp-2} \left( 1-x^2 \right)^s k^{2s} \frac{1+sp}{k^{1+sp}} \, dk
\]
\[
\leq (1-x)^s \int_0^{1/2} \left[ 1 - \left( 1 - \frac{1-x^2}{4x^2} k^2 \right) \right]^{s} \int_0^{1/2} C k^{sp-2} (1-k)^s \left[ 1 - \left( 1 - \frac{1-x^2}{4x^2} k^2 \right) \right] \, dk + C \leq C.
\]

And similarly,
\[
(1-x)^s \int_0^1 \left[ 1 - \left( 1 + k - \frac{1-x^2}{4x^2} k^2 \right) \right]^{p-1} \frac{1}{k^{1+sp}} \left[ 1 + k - \left( \frac{1-x^2}{4x^2} k^2 \right)^s \right]^{-1} \, dk
\]
\[
\leq (1-x)^s \int_0^{1/2} \left[ 1 - \left( 1 + k - \frac{1-x^2}{4x^2} k^2 \right) \right]^{p-1} \frac{1}{k^{1+sp}} \left[ 1 + k - \left( \frac{1-x^2}{4x^2} k^2 \right)^s \right]^{-1} \, dk + C \leq C.
\]
Secondly, we estimate the term \((1 - x)^{-s} I'_4\),

\[
I'_4 = \int_{1+\varepsilon}^{2x} \left( \frac{(1 + k - \frac{1-x^2}{4x^2} k^2)^s - 1}{k^{1+s}} \right)^{p-1} \, dk.
\]

Next we will prove

\[
(1 - x)^{-s} I'_4 = (1 - x)^{-s} \int_{1+\varepsilon}^{2x} \left( \frac{(1 + k - \frac{1-x^2}{4x^2} k^2)^s - 1}{k^{1+s}} \right)^{p-1} \, dk
\]

\[
\leq (1 - x)^{-s} \int_{1}^{1+\varepsilon} \left( \frac{(1 + k - \frac{1-x^2}{4x^2} k^2)^s - 1}{k^{1+s}} \right)^{p-1} \, dk + C.
\]

This is due to

\[
\limsup_{x \to 1} (1 - x)^{-s} \int_{1+\varepsilon}^{2x} \left( \frac{(1 + k - \frac{1-x^2}{4x^2} k^2)^s - 1}{k^{1+s}} \right)^{p-1} \, dk
\]

\[
\leq \limsup_{x \to 1} C (1 - x)^{-s} \frac{1-x}{1+x} = 0,
\]

and

\[
\limsup_{x \to 1} (1 - x)^{-s} \left| \int_{1}^{1+\varepsilon} \left( \frac{(1 + k - \frac{1-x^2}{4x^2} k^2)^s - 1}{k^{1+s}} \right)^{p-1} \, dk \right|
\]

\[
\leq \limsup_{x \to 1} C (1 - x)^{-s} \int_{1+\varepsilon}^{2x} \frac{k^s(p-1)}{k^{1+s}} \, dk
\]

\[
= \limsup_{x \to 1} C (1 - x)^{-s} (1 - x^2)^{1+s} \left( \frac{1}{1-x} - \frac{2x}{1-x^2} \right) = 0.
\]

Now we claim that there exists a constant \(C > 0\), such that

\[
\left| (1 - x)^{-s} \int_{1}^{1+\varepsilon} \left( \frac{(1 + k - \frac{1-x^2}{4x^2} k^2)^s - 1}{k^{1+s}} \right)^{p-1} \, dk - (1 - x)^{-s} \int_{1}^{1+\varepsilon} \left( \frac{(1 + k)^s - 1}{k^{1+s}} \right)^{p-1} \, dk \right| \leq C.
\]

Since \(x\) is close to 1, we have

\[
(1 - x)^{-s} \int_{1}^{1+\varepsilon} \left[ (1 + k)^s - 1 \right]^{p-1} \left( \frac{1}{k^{1+s}} \right) \, dk
\]

\[
\leq (1 - x)^{-s} \int_{1}^{1+\varepsilon} \frac{C k^{s(p-2)}}{k^{1+s}} \left( (1 + k)^s - \left( 1 + k - \frac{1-x^2}{4x^2} k^2 \right)^s \right) \, dk
\]

\[
\leq (1 - x)^{-s} \int_{1}^{1+\varepsilon} \frac{C k^{s(p-2)} (1 + k)^s}{k^{1+s}} \left( 1 - \left( 1 - \frac{1-x^2}{4x^2} k^2 \right)^s \right) \, dk
\]

\[
\leq (1 - x)^{-s} \int_{1}^{1+\varepsilon} \frac{C k^{s(p-2)} (1 + k)^s}{k^{1+s}} \left( 1 + \frac{1-x^2}{4x^2} k^2 + o \left( \frac{1-x^2}{4x^2} \right) \frac{k^2}{1+k} \right) \, dk
\]

\[
\leq (1 - x)^{-s} \int_{1}^{1+\varepsilon} \frac{C k^{s(p-2)} (1 - x) k^{1+s}}{k^{1+s}} \, dk
\]
Furthermore, there is a constant where we use the estimate

\[ \epsilon \]

Now we begin to prove the above identity. For any \( C \), the singular term of \( \Delta \) is

\[ \int_0^1 \left( \frac{(1 + k)^s - 1}{k^{1 + sp}} \right) dk - (1 - x)^{-s} \int_1^\infty \left[ \frac{(1 + k)^s - 1}{k^{1 + sp}} \right] dk \leq C. \]

This is because when \( x \) is close to 1, we have

\begin{align*}
(1 - x)^{-s} & \int_1^\infty \left[ \frac{(1 + k)^s - 1}{k^{1 + sp}} \right] dk \\
& \leq (1 - x)^{-s} \int_1^\infty \frac{k^{s(p-1)}}{k^{1 + sp}} dk = (1 - x)^{-s} \int_1^\infty k^{-s-1} dk \leq C.
\end{align*}

**Step 3.** We will prove all terms are bounded uniformly. By the above simplification, the singular term of \( (-\Delta)^s u(x) \) is

\[ (1 - x)^{-s} \left( \frac{2x}{1 + x} \right)^p \left\{ \int_0^1 \left[ \frac{1 - (1 - k)^s - 1}{k^{1 + sp}} \right] dk + \frac{(1 + x)^{sp}}{sp(2x)^{sp}} - \int_1^\infty \left[ \frac{(1 + k)^s - 1}{k^{1 + sp}} \right] dk \right\}. \]

Furthermore, there is a constant \( C > 0 \),

\[ -C \leq (1 - x)^{-s} \frac{(1 + x)^{sp}}{sp(2x)^{sp}} - (1 - x)^{-s} \frac{1}{sp} \leq C. \]

Because

\[ (1 - x)^{-s} \frac{1}{sp} \left[ \frac{(1 + x)^{sp}}{sp(2x)^{sp} - 1} \right] = (1 - x)^{-s} \frac{1}{sp} \left[ \left( 1 + \frac{1 - x}{2x} \right)^{sp} - 1 \right] \]

\[ = (1 - x)^{-s} \frac{1}{sp} \left[ \frac{1 - x}{2x} + o\left( \frac{1 - x}{2x} \right) \right] \leq C. \]

Hence we just need to prove the following identity:

\[ \frac{1}{sp} + \int_0^1 \left[ \frac{1 - (1 - k)^s - 1}{k^{1 + sp}} \right] dk - \int_1^\infty \left[ \frac{(1 + k)^s - 1}{k^{1 + sp}} \right] dk = 0. \]  (3.3)

Now we begin to prove the above identity. For any \( \epsilon \in (0, 1) \) fixed, we have

\[ \int_0^1 \left[ \frac{1 - (1 - k)^s - 1}{k^{1 + sp}} \right] dk - \int_1^\infty \left[ \frac{(1 + k)^s - 1}{k^{1 + sp}} \right] dk = \int_0^\epsilon \left[ \frac{1 - (1 - k)^s - 1}{k^{1 + sp}} \right] dk + \int_1^\epsilon \left[ \frac{1 - (1 - k)^s - 1}{k^{1 + sp}} \right] dk + \int_\epsilon^1 \left[ \frac{1 - (1 - k)^s - 1}{k^{1 + sp}} \right] dk + \int_1^\infty \left[ \frac{1 - (1 - k)^s - 1}{k^{1 + sp}} \right] dk.
\]
Notice that
\[
\int_\epsilon^1 \frac{[1-(1-k)^s]^{p-1}}{k^{1+sp}} \, dk = \int_\epsilon^\infty \frac{(1+t)^{s-1} [(1+t)^s-1]^{p-1}}{t^{1+sp}} \, dt,
\]
where we use the change of variable
\[
t = \frac{k}{1-k}.
\]
Then
\[
\int_0^1 \frac{[1-(1-k)^s]^{p-1} - [(1+k)^s - 1]^{p-1}}{k^{1+sp}} \, dk = \int_0^\infty \frac{[(1+k)^s - 1]^{p-1}}{k^{1+sp}} \, dk
\]
\[
- \int_\epsilon^\infty \frac{[(1+t)^s - 1]^{p-1}}{t^{1+sp}} \, dt - \int_\epsilon^\infty \frac{[(1+k)^s - 1]^{p-1}}{k^{1+sp}} \, dk
\]
\[
= \int_0^\epsilon \frac{[1-(1-k)^s]^{p-1} - [(1+k)^s - 1]^{p-1}}{k^{1+sp}} \, dk - \int_\epsilon^\infty \frac{[(1+k)^s - 1]^{p-1}}{k^{1+sp}} \, dk
\]
\[
- \int_\epsilon^\infty \frac{[(1+t)^s - 1]^{p-1} [1-(1+t)^{s-1}]}{t^{1+sp}} \, dt
\]
\[
= \int_0^\epsilon \frac{[1-(1-k)^s]^{p-1} - [(1+k)^s - 1]^{p-1}}{k^{1+sp}} \, dk - \int_\epsilon^\infty \frac{[(1+k)^s - 1]^{p-1}}{k^{1+sp}} \, dk
\]
\[
- \frac{1}{sp} \int_\epsilon^\infty \frac{[(1+t)^s - 1]^{p-1}}{t^{sp}} \, dt \bigg|_{t=\epsilon^{-1}} =: H_1 + H_2 + H_3.
\]
Now we estimate term by term and let \( \epsilon \to 0 \).

\[H_1 = \int_0^\epsilon \frac{[1-(1-k)^s]^{p-1} - [(1+k)^s - 1]^{p-1}}{k^{1+sp}} \, dk\]
\[\leq \int_0^\epsilon \frac{Ck^{s(p-2)} \left[ 2 - (1-k)^s - (1+k)^s \right]}{k^{1+sp}} \, dk\]
\[\leq \int_0^\epsilon \frac{Ck^{s(p-2)} \left[ 2 - \left( 1 - sk - \frac{s(1-s)}{2} k^2 + o(k^2) \right) - \left( 1 + sk - \frac{s(1-s)}{2} k^2 + o(k^2) \right) \right]}{k^{1+sp}} \, dk\]
\[\leq \int_0^\epsilon \frac{Ck^{s(p-2)} \left[ k^2 + o(k^2) \right]}{k^{1+sp}} \, dk \to 0 \quad \text{as} \quad \epsilon \to 0.\]

\[H_2 = \int_\epsilon^\infty \frac{[(1+k)^s - 1]^{p-1}}{k^{1+sp}} \, dk = \int_\epsilon^\infty \frac{[(1+k) + o(k)] - 1]^{p-1}}{k^{1+sp}} \, dk\]
\[\leq \int_\epsilon^\infty \frac{Ck^{p-1}}{k^{1+sp}} \, dk\]
\[= \begin{cases} \int_\epsilon^\infty \frac{Ck^{p-1-s} \, dk}{k^{1+sp}} \leq C \left( \frac{\epsilon}{1-\epsilon} \right)^{p-sp-1}, & \text{if} \quad p-sp > 1; \\ \int_\epsilon^\infty Ck^{p-1} \, dk \leq C \epsilon^{2(p-1)} \left( \frac{\epsilon}{1-\epsilon} - \epsilon \right) = C \epsilon^{p-sp-1} \left( \frac{\epsilon}{1-\epsilon} - \epsilon \right), & \text{if} \quad p-sp = 1; \\ \int_\epsilon^\infty C \frac{1}{k^{2-(p-sp)}} \, dk \leq C \epsilon^{2-[(p-sp)]} \left( \frac{\epsilon}{1-\epsilon} - \epsilon \right) = C \epsilon^{p-sp} \left( \frac{\epsilon}{1-\epsilon} - \epsilon \right), & \text{if} \quad p-sp < 1, \end{cases}\]
\[\to 0 \quad \text{as} \quad \epsilon \to 0.\]
Now we roughly analyze the integration term.

\[
H_3 = - \frac{1}{sp} \int \frac{[(1 + t)^s - 1]^p}{t^{sp}} \left[ \int_{r=\epsilon}^{\infty} \frac{1}{sp} \frac{[1 - (1 - \epsilon)^s]^p}{\epsilon^{sp}} \right] \, dt = - \frac{1}{sp} + \frac{1}{sp} \frac{[1 - (1 - s\epsilon - o(\epsilon))]^p}{\epsilon^{sp}} \rightarrow - \frac{1}{sp} \text{ as } \epsilon \rightarrow 0.
\]

Therefore we have proved the identity (3.3). Hence we have completed the proof for \( n = 1 \).

3.2. \( n \geq 2 \). The purpose of this section is to prove that for \( u(x) = (1 - |x|^2)^s \), \((-\Delta)_p^su(x)\) is uniformly bounded in \( B_1 \) for higher dimensions. There are totally 14 steps. In this subsection, we will often use \( G[t] \) to represent \( |t|^{p-2}t \) for convenience.

Step 1. Due to Lemma 2.1, without loss of generality, we assume a fixed \( 0 \leq (y_1, y_2, \ldots, y_n) =: (y_1, \bar{y}) \). We omit the constant \( C_{n,s,p} \) for simplicity. Then

\[
(-\Delta)_p^s u(x) = \lim_{\epsilon \to 0} \int_{\mathbb{R}^n \setminus B_\epsilon(x)} \frac{|u(x) - u(y)|^{p-2}[u(x) - u(y)]}{|x - y|^{n+sp}} \, dy
\]

\[
= \lim_{\epsilon \to 0} \int_{\{y \in \mathbb{R}^n: |x - y| \geq \epsilon\}} \frac{|(1 - x^2)^s - (1 - |y|^2)^s|^p}{|x - y|^{n+sp}} \left[ \int_{\mathbb{R}^n \setminus B_\epsilon(x)} \frac{2^{sp} \left[ (1 - x^2)^s - (1 - |y|^2)^s \right]}{|x - y|^2 + |y|^2} \, dy \right]
\]

\[
= \lim_{\epsilon \to 0} \int_{\{y \in \mathbb{R}^n: |x - y| \geq \epsilon\}} \frac{G[(1 - x^2)^s - (1 - |y|^2)^s]}{|x - y|^2 + |y|^2} \, dy,
\]

where \( G[t] := |t|^{p-2}t \). Set \( z = (z_1, \bar{z}) \), where \( z_1 = x - y_1, \bar{z} = \bar{y} \). Then

\[
(-\Delta)_p^s u(x) = \lim_{\epsilon \to 0} \int_{\{z \in \mathbb{R}^n: |z| \geq \epsilon\}} \frac{G[(1 - x^2)^s - (1 - (x - z_1)^2 - |\bar{z}|^2)^s]}{\left( z_1^2 + |\bar{z}|^2 \right)^{\frac{n+sp}{2}}} \, dz
\]

\[
= (1 - x^2)^s(p-1) \lim_{\epsilon \to 0} \int_{\{z \in \mathbb{R}^n: |z| \geq \epsilon\}} \frac{G[1 - \left( 1 + \frac{2x_1z_1}{1 - x^2} - \frac{|\bar{z}|^2}{1 - x^2} \right)^s]}{\left| z \right|^{n+sp}} \, dz.
\]

Let \( w = 2x_1z_1 \), where \( w_1 = \frac{2x_1z_1}{1 - x^2} \) and \( \bar{w} = \frac{2x_1z_1}{1 - x^2} \), then

\[
(-\Delta)_p^s u(x) = \frac{(2x_1)^p}{(1 - x^2)^s} \lim_{\epsilon \to 0} \int_{\{w \in \mathbb{R}^n: |w| \geq \frac{2x_1}{1 - x^2}\}} \frac{G\left[ 1 - \left( 1 + w_1 - \frac{1 - x^2}{4x_1^2} |w|^2 \right)^s \right]}{|w|^{n+sp}} \, dw.
\]

Now we roughly analyze the integration term.

\[
\int_{|w| \geq \frac{2x_1}{1 - x^2}} \frac{G\left[ 1 - \left( 1 + w_1 - \frac{1 - x^2}{4x_1^2} |w|^2 \right)^s \right]}{|w|^{n+sp}} \, dw
\]
Firstly, we can readily estimate 

\[ J = \int_{-\infty}^{\infty} dw_1 \int_{|w|^{2} \geq \frac{1}{(1+\epsilon^2)^2}} \frac{G \left[ 1 - \left( 1 + w_1 - \frac{1}{4\epsilon^2} w_1^2 - \frac{1}{4\epsilon^2} |w|^2 \right)_+ \right]}{(w_1^2 + |w|^2)^{\frac{n+np}{2}}} dw \]

In the sequel, we set \( f(w, \rho) = 1 + w - \frac{1}{4\epsilon^2} w^2 - \frac{1}{4\epsilon^2} \rho^2 \) for the fixed \( x \).

Step 2. In this part, we will prove \( \lim_{\epsilon \to 0} J_1 = 0 \).

\[ J_1 = \int_{\frac{2\pi}{1+\epsilon^2}}^{\frac{2\pi}{1-\epsilon^2}} dw_1 \int_{\frac{2\pi}{1-\epsilon^2}}^{\infty} \frac{G \left[ 1 - \left( 1 + w_1 - \frac{1}{4\epsilon^2} w_1^2 - \frac{1}{4\epsilon^2} \rho^2 \right)_+ \right]}{(w_1^2 + \rho^2)^{\frac{n+np}{2}}} \rho^{n-2} d\rho \]

Firstly, we can readily estimate \( J_{12} \).

|\( J_{12} \)| \( \leq \int_{\frac{2\pi}{1-\epsilon^2}}^{\frac{2\pi}{1+\epsilon^2}} dw_1 \int_{\frac{2\pi}{1-\epsilon^2}}^{\infty} \frac{G \left[ 1 - \left( 1 + w_1 - \frac{1}{4\epsilon^2} w_1^2 - \frac{1}{4\epsilon^2} \rho^2 \right)_+ \right]}{(w_1^2 + \rho^2)^{\frac{n+np}{2}}} \rho^{n-2} d\rho \)

\( \leq \int_{\frac{2\pi}{1-\epsilon^2}}^{\frac{2\pi}{1+\epsilon^2}} dw_1 \int_{\frac{2\pi}{1-\epsilon^2}}^{\infty} \frac{\rho^{n-2}}{(w_1^2 + \rho^2)^{\frac{n+np}{2}}} d\rho \)

\( \leq \int_{\frac{2\pi}{1-\epsilon^2}}^{\frac{2\pi}{1+\epsilon^2}} dw_1 \int_{\frac{2\pi}{1-\epsilon^2}}^{\infty} \frac{1}{\rho^{2+np}} d\rho \leq C(x) \int_{\frac{2\pi}{1-\epsilon^2}}^{\frac{2\pi}{1+\epsilon^2}} dw_1 \to 0 \) as \( \epsilon \to 0 \).

Secondly, we estimate \( J_{11} \) by Taylor expansion,

\[ J_{11} = \int_{0}^{\frac{2\pi}{1+\epsilon^2}} dw_1 \int_{\frac{2\pi}{1+\epsilon^2}}^{\infty} \frac{G \left( 1 - f (-w_1, \rho)^n \right) - G \left( f (w_1, \rho)^n - 1 \right)}{(w_1^2 + \rho^2)^{\frac{n+np}{2}}} \rho^{n-2} d\rho \]
Moreover, we have

\[ \int_0^{2 \epsilon} d \omega_1 \int \frac{\sqrt{\frac{4 \omega_1^2}{(1 - \omega_1^2)^2} w_1 - w_1^2}}{w_1^2 - w_1^2} \left[ 1 - \left( 1 + w_1 - \frac{1 - \omega_1^2}{4 \omega_1^2} w_1^2 - \frac{1 - \omega_1^2}{4 \omega_1^2} \rho^2 \right)^{\frac{1}{2}} \frac{\rho}{w_1^2 + \rho^2} \right]^{\frac{n-1}{2}} \rho^{-2} d \rho \]

\[ + \int_0^{2 \epsilon} d \omega_1 \int \frac{\sqrt{\frac{4 \omega_1^2}{(1 - \omega_1^2)^2} w_1 - w_1^2}}{w_1^2 - w_1^2} \rho^{-2} \left[ 1 - \left( 1 - w_1 + \frac{1 - \omega_1^2}{4 \omega_1^2} w_1^2 + \frac{1 - \omega_1^2}{4 \omega_1^2} \rho^2 \right)^{\frac{1}{2}} \frac{\rho}{w_1^2 + \rho^2} \right]^{\frac{n-1}{2}} \rho^{-2} d \rho \]

\[ \leq \int_0^{2 \epsilon} d \omega_1 \int \frac{\sqrt{\frac{4 \omega_1^2}{(1 - \omega_1^2)^2} w_1 - w_1^2}}{w_1^2 - w_1^2} \left[ C w_1^{p-2} \left[ 2 - f (-w_1, \rho) - f (w_1, \rho) \right] \right]^{\frac{n-1}{2}} \rho^{-2} d \rho \]

Moreover, we have

\[ \int_0^{2 \epsilon} d \omega_1 \int \sqrt{\frac{4 \omega_1^2}{(1 - \omega_1^2)^2} w_1 - w_1^2} \left[ \frac{C w_1^{p-2} \left( 1 - \frac{1 - \omega_1^2}{2 \omega_1^2} w_1^2 + O(w_1^2) \right) \rho^{-2} \left( w_1^2 + \rho^2 \right)^{\frac{n-1}{2}}}{\rho^{-2} \left( w_1^2 + \rho^2 \right)^{\frac{n-1}{2}}} \right] d \rho \]

\[ \leq \int_0^{2 \epsilon} d \omega_1 \int \sqrt{\frac{4 \omega_1^2}{(1 - \omega_1^2)^2} w_1 - w_1^2} \left( w_1^2 + \rho^2 \right)^{\frac{n-1}{2}} \rho^{-2} d \rho \]

\[ \leq \int_0^{2 \epsilon} d \omega_1 \int \sqrt{\frac{4 \omega_1^2}{(1 - \omega_1^2)^2} w_1 - w_1^2} \left( w_1^2 + \rho^2 \right)^{\frac{n-1}{2}} \rho^{-2} d \rho \]

\[ = C \int_0^{2 \epsilon} d \omega_1 \left\{ \arctan \frac{\rho}{w_1} \right\} \sqrt{\frac{4 \omega_1^2}{(1 - \omega_1^2)^2} w_1 - w_1^2} \left( w_1^2 + \rho^2 \right)^{\frac{n-1}{2}} \rho^{-2} d \rho \]

\[ \leq C \int_0^{2 \epsilon} d \omega_1 w_1^{p-1} \rightarrow 0 \quad \text{as} \quad \epsilon \rightarrow 0. \]

and

\[ \int_0^{2 \epsilon} d \omega_1 \int \sqrt{\frac{4 \omega_1^2}{(1 - \omega_1^2)^2} w_1 - w_1^2} \left( w_1^2 + \rho^2 \right)^{\frac{n-1}{2}} \rho^{-2} d \rho \]
\[
\leq C \int_0^{2\epsilon} dw_1 \int \sqrt{\frac{2\epsilon}{1-x^2}} \frac{\rho^{2p+n-4}}{w^2 + \rho^2} \, d\rho \\
\leq C \int_0^{2\epsilon} dw_1 \int \sqrt{\frac{2\epsilon}{1-x^2}} \frac{\rho^{2p+n-4}}{w^2 + \rho^2} \, d\rho \\
\leq C \int_0^{2\epsilon} dw_1 \int \sqrt{\frac{4\epsilon^2}{1-x^2}} \frac{\rho^{2p+n-4}}{(w_1^2 + \rho^2)^{\frac{1}{2}}} \, d\rho \\
\leq C \int_0^{2\epsilon} dw_1 \frac{\sqrt{\frac{4\epsilon^2}{1-x^2}}}{(w_1^2 + \rho^2)^{\frac{1}{2}}} \leq C \int_0^{2\epsilon} w_1^{2p-4} \, dw_1, \quad \text{if } 2p - sp - 3 < 0; \\
\leq C \int_0^{2\epsilon} dw_1 \frac{\sqrt{\frac{4\epsilon^2}{1-x^2}}}{(w_1^2 + \rho^2)^{\frac{1}{2}}} \leq C \int_0^{2\epsilon} w_1^{2p-4} \, dw_1, \quad \text{if } 2p - sp - 3 \leq 0; \\
\rightarrow 0 \quad \text{as } \epsilon \rightarrow 0.
\]

Hence

\[J_1 \rightarrow 0 \quad \text{as } \epsilon \rightarrow 0.\]

Step 3. We work on \(J_2\) partly,

\[J_2 = \int_{|w_1| \geq \frac{2\epsilon}{1-x^2}} dw_1 \int_0^\infty G \left[ 1 - \left( 1 + w_1 - \frac{1-x^2}{4\epsilon^2} w_1^2 - \frac{1-x^2}{4\epsilon^2} \rho^2 \right)^s \right] \frac{\rho^{n-2}}{(w_1^2 + \rho^2)^{\frac{n+2}{2}}} \, d\rho \\
= \int_0^\infty dw_1 \int_0^\infty G \left[ 1 - \left( 1 + w_1 - \frac{1-x^2}{4\epsilon^2} w_1^2 - \frac{1-x^2}{4\epsilon^2} \rho^2 \right)^s \right] \frac{\rho^{n-2}}{(w_1^2 + \rho^2)^{\frac{n+2}{2}}} \, d\rho \\
+ \int_0^\infty dw_1 \int_0^\infty G \left[ 1 - \left( 1 - w_1 - \frac{1-x^2}{4\epsilon^2} w_1^2 - \frac{1-x^2}{4\epsilon^2} \rho^2 \right)^s \right] \frac{\rho^{n-2}}{(w_1^2 + \rho^2)^{\frac{n+2}{2}}} \, d\rho.
\]

Set \(y = \frac{\rho}{w_1}\), then

\[J_2 = \int_{\frac{2\epsilon}{1-x^2}}^{\infty} \frac{1}{w_1^{1+sp}} dw_1 \int_0^\infty G \left[ 1 - \left( 1 + w_1 - \frac{1-x^2}{4\epsilon^2} (1+y^2) w_1^2 \right)^s \right] \frac{y^{n-2}}{(1+y^2)^{\frac{n+2}{2}}} \, dy \\
+ \int_{\frac{2\epsilon}{1-x^2}}^{\infty} \frac{1}{w_1^{1+sp}} dw_1 \int_0^\infty G \left[ 1 - \left( 1 - w_1 - \frac{1-x^2}{4\epsilon^2} (1+y^2) w_1^2 \right)^s \right] \frac{y^{n-2}}{(1+y^2)^{\frac{n+2}{2}}} \, dy \\
=: J_{21} + J_{22}.
\]

In the following, we set \(h(w, y) := 1 + w - \frac{1-x^2}{4\epsilon^2} (1+y^2) w^2\), then

\[J_{22} = \int_{\frac{2\epsilon}{1-x^2}}^{\infty} \frac{1}{w_1^{1+sp}} dw_1 \int_0^\infty G \left[ 1 - \left( 1 - w_1 - \frac{1-x^2}{4\epsilon^2} (1+y^2) w_1^2 \right)^s \right] \frac{y^{n-2}}{(1+y^2)^{\frac{n+2}{2}}} \, dy \\
= \int_{\frac{2\epsilon}{1-x^2}}^{\infty} \frac{1}{w_1^{1+sp}} dw_1 \int_0^\infty \sqrt{\frac{4\epsilon^2}{1-x^2} \frac{1-x^2}{w_1^2}} \frac{1 - h(-w_1, y)^{sp-1}}{(1+y^2)^{\frac{n+2}{2}}} \, y^{n-2} \, dy.
\]
Step 4. In this step, we will figure out \(1 + \int 1 + \int \) and

\[
J_{21} = \int_{\frac{x}{1+x}}^{1} \frac{1}{w_1^{1+sp}} dw_1 \int_{0}^{\infty} G \left[ 1 - \left( 1 + w_1 - \frac{1-x^2}{4\pi} (1 + y^2)w_1^2 \right)^s \right] \frac{y^{n-2}}{(1 + y^2)^{\frac{n+sp}{x}}} dy =: (1) + (2) + (3).
\]

and

\[
\begin{align*}
&+ \int_{\frac{x}{1+x}}^{1} \frac{1}{w_1^{1+sp}} dw_1 \int_{0}^{\infty} \frac{y^{n-2}}{(1 + y^2)^{\frac{n+sp}{x}}} dy =: 1 + (2) + (3) + (4) + (5) + (6).
\end{align*}
\]

Step 4. In this step, we will figure out \(1 + 1^\prime\),

\[
1 + 1^\prime = \int_{\frac{x}{1+x}}^{1} \frac{1}{w_1^{1+sp}} \int_{0}^{\infty} G \left[ 1 - \left( 1 + w_1 - \frac{1-x^2}{4\pi} (1 + y^2)w_1^2 \right)^s \right] \frac{y^{n-2}}{(1 + y^2)^{\frac{n+sp}{x}}} dy dw_1
\]

\[
+ \int_{\frac{x}{1+x}}^{1} \frac{1}{w_1^{1+sp}} \int_{0}^{\infty} \frac{y^{n-2}}{(1 + y^2)^{\frac{n+sp}{x}}} dy dw_1 =: (I) + (II) + (III) - (IV).
\]
At present, there are 11 terms: \(\mathcal{I}, \mathcal{J}, \mathcal{K}, \mathcal{L}, \mathcal{M}, \mathcal{N}, \mathcal{O}, \mathcal{P}, \mathcal{Q}, \mathcal{R}, \mathcal{S}\).

Step 5. From now on, we will estimate item by item. Given (3.4), in this step, we claim that \(\mathcal{I}, \mathcal{K}, \mathcal{L}, \mathcal{O}\) are uniformly bounded when multiplied by \((1 - x)^{-s}\) when \(x\) is close to 1.

\[
(1 - x)^{-s} \left( \mathcal{I} + \mathcal{L} \right) \leq C (1 - x)^{-s} \int_{\frac{2x}{1-x}}^{\frac{2x}{1+x}} \int_0^\infty \frac{1}{w_1^{1+sp}} \int_0^\infty \frac{y^{n-2}}{(1 + y^2)^{\frac{n+sp}{2}}} \, dy \, dw_1
\]
\[
\leq C (1 - x)^{-s} \int_{\frac{2x}{1-x}}^{\frac{2x}{1+x}} \int_0^\infty \frac{1}{w_1^{1+sp}} \, dw_1
\]
\[
\leq C (1 - x)^{-s} \left( \frac{1 - x^2}{4x^2} \right)^{sp} \leq C (1 - x)^{(p-1)}.
\]

and

\[
(1 - x)^{-s} \left( \mathcal{K} \right) \leq C (1 - x)^{-s} \int_{\frac{2x}{1-x}}^{\frac{2x}{1+x}} \int_0^\infty \frac{1}{w_1^{1+sp}} \int_0^\infty \frac{y^{n-2}}{(1 + y^2)^{\frac{n+sp}{2}}} \, dy \, dw_1
\]
\[
\leq C (1 - x)^{-s} \int_{\frac{2x}{1-x}}^{\frac{2x}{1+x}} \int_0^\infty \frac{1}{w_1^{1+sp}} \, dw_1
\]
\[
\leq C (1 - x)^{-s} \left( \frac{1 - x}{2x} \right)^{sp} \leq C (1 - x)^{(p-1)}.
\]

Step 6. We assert that \(\mathcal{J}\) can be replaced by

\[
\int_1^\infty \frac{1}{w_1^{1+sp}} \, dw_1 \int_0^\infty \frac{y^{n-2}}{(1 + y^2)^{\frac{n+sp}{2}}} \, dy = \frac{1}{sp} \int_0^\infty \frac{y^{n-2}}{(1 + y^2)^{\frac{n+sp}{2}}} \, dy. \tag{3.5}
\]

This is because

\[
(1 - x)^{-s} \int_{\frac{2x}{1-x}}^{\frac{2x}{1+x}} \int_0^1 \frac{1}{w_1^{1+sp}} \, dw_1 \int_0^\infty \frac{y^{n-2}}{(1 + y^2)^{\frac{n+sp}{2}}} \, dy
\]
\[
\leq C (1 - x)^{-s} \int_{\frac{2x}{1-x}}^{\frac{2x}{1+x}} \int_0^1 \frac{1}{w_1^{1+sp}} \, dw_1
\]
\[
= C (1 - x)^{-s} \left( \frac{1 + x}{2x} \right)^{sp} \left[ 1 - \left( \frac{1 - x}{1 + x} \right)^{sp} \right]
\]
\[
\leq C (1 - x)^{-s} \left( \frac{1 + x}{2x} \right)^{sp} \left[ C \cdot sp \frac{1 - x}{1 + x} \right] \leq C (1 - x)^{-s}.
\]

Step 7. We will reformulate (III) as

\[
\int_{\frac{1}{2}}^1 \frac{[1 - (1 - w_1)^s]^{p-1}}{w_1^{1+sp}} \, dw_1 \int_0^\infty \frac{y^{n-2}}{(1 + y^2)^{\frac{n+sp}{2}}} \, dy.
\]

Firstly, we will substitute (III) by

\[
\int_{\frac{1}{2}}^1 \frac{1}{w_1^{1+sp}} \int_0^{\frac{2x}{1-x}} \frac{1 - (1 - w_1)^s]^{p-1}}{w_1^{1+sp}} y^{n-2} \, dy \, dw_1.
\]
Since
\[(1-x)^{-s} \int_{1/2}^{2x} \frac{1}{w_1^{1+sp}} \sqrt{\frac{4x}{1-x} - w_1^{-1}} \left[ 1 - h(-w_1, y)^s \right]_{p-1} \frac{[1 - (1 - w_1)^s]_{p-1}}{(1 + y^2)^{n+sp/2}} y^{-n-2} dy dw_1 \leq C(1-x)^{-s} \int_{1/2}^{2x} \frac{1}{w_1^{1+sp}} \sqrt{\frac{4x}{1-x} - w_1^{-1}} \frac{1}{w_1^{s(p-2)}} \left[ 1 - h(-w_1, y)^s \right]_{p-1} \frac{[1 - (1 - w_1)^s]_{p-1}}{(1 + y^2)^{n+sp/2}} y^{-n-2} dy dw_1 \leq C(1-x)^{-s} \int_{1/2}^{2x} \frac{1}{w_1^{1+sp}} \sqrt{\frac{4x}{1-x} - w_1^{-1}} \left(1 + y^2 \right)^{s(p-2)} \frac{1}{w_1^{s}} \frac{1}{y^{-n-2}} dy dw_1 \leq C \int_{1/2}^{2x} \frac{1}{w_1^{1+sp}} \frac{1}{y^{-n}} dy dw_1 \leq C.
\]

Moreover, (III) can be replaced by
\[\int_{1/2}^{2x} \frac{1}{w_1^{1+sp}} \int_0^\infty \frac{[1 - (1 - w_1)^s]_{p-1}}{(1 + y^2)^{n+sp/2}} y^{-n-2} dy dw_1.
\]

Due to the fact that
\[(1-x)^{-s} \int_{1/2}^{2x} \frac{1}{w_1^{1+sp}} \sqrt{\frac{4x}{1-x} - w_1^{-1}} \left[ 1 - (1 - w_1)^s \right]_{p-1} \frac{[1 - (1 - w_1)^s]_{p-1}}{(1 + y^2)^{n+sp/2}} y^{-n-2} dy dw_1 \leq C(1-x)^{-s} \int_{1/2}^{2x} \frac{1}{w_1^{1+sp}} \sqrt{\frac{4x}{1-x} - w_1^{-1}} \frac{1}{y^{-n-2}} dy dw_1 \leq C(1-x)^{-s} \int_{1/2}^{2x} \frac{1}{w_1^{1+sp}} \sqrt{\frac{4x}{1-x} - w_1^{-1}} \frac{1}{y^{1+2s}} \frac{1}{y^{-n-2}} dy dw_1 \leq C(1-x)^{-s} \int_{1/2}^{2x} \frac{1}{w_1^{1+sp}} \sqrt{\frac{4x}{1-x} - w_1^{-1}} \frac{1}{y^{1+2s}} dy dw_1 \leq C(1-x)^{-s} \int_{1/2}^{2x} \frac{1}{w_1^{1+sp}} \frac{1}{y^{1+2s}} \left( \frac{2x}{1+x} \right)^s dw_1 \leq C \int_{1/2}^{2x} \left( \frac{2x}{1+x} \right)^{-s} dw_1 \leq C.
\]

Furthermore, we transform (III) into
\[\int_{1/2}^{2x} \frac{1}{w_1^{1+sp}} \int_0^\infty \frac{[1 - (1 - w_1)^s]_{p-1}}{(1 + y^2)^{n+sp/2}} y^{-n-2} dy dw_1.
\]

This is because
\[(1-x)^{-s} \int_{1/2}^{2x} \frac{1}{w_1^{1+sp}} \int_0^\infty \frac{[1 - (1 - w_1)^s]_{p-1}}{(1 + y^2)^{n+sp/2}} y^{-n-2} dy dw_1.
\]
\[
\leq C(1-x)^{-s} \int_0^1 \frac{1}{w_1^{1+sp}} \, dw_1 \\
\leq C(1-x)^{-s} \left( \frac{1 + x}{2x} \right)^{sp} \left[ 1 - \left( \frac{1 - x}{1 + x} \right)^{sp} \right] \\
\leq C(1-x)^{-s} \left( \frac{1 + x}{2x} \right)^{sp} \left[ C \cdot sp \frac{1 - x}{1 + x} \right] \leq C(1-x)^{1-s}.
\]

Step 8. In this part, we are going to replace (I) by
\[
\int_0^1 \left[ 1 - (1-w_1)^s \right]^{p-1} - \left[ (1+w_1)^s - 1 \right]^{p-1} \, dw_1 \int_0^\infty \frac{y^{n-2}}{(1+y^2)^{n+sp}} \, dy. \quad (3.6)
\]

Firstly, we will reduce (I) to
\[
\int_0^1 \frac{1}{w_1^{1+sp}} \left[ \int_0^{\frac{1-x}{1+w_1}} \frac{x^2}{w_1^{1+sp}} \, dx \right] \left[ 1 - (1-w_1)^s \right]^{p-1} - \left[ (1+w_1)^s - 1 \right]^{p-1} \, dy \, dw_1.
\]

On the one hand,
\[
(1-x)^{-s} \int_0^1 \frac{1}{w_1^{1+sp}} \left[ \int_0^{\frac{1-x}{1+w_1}} \frac{x^2}{w_1^{1+sp}} \, dx \right] \left[ 1 - h(-w_1,y)^s \right]^{p-1} - \left[ (1-w_1)^s - 1 \right]^{p-1} \, dy \, dw_1
\]
\[
\leq (1-x)^{-s} \int_0^1 \frac{1}{w_1^{1+sp}} \left[ \int_0^{\frac{1-x}{1+w_1}} \frac{x^2}{w_1^{1+sp}} \, dx \right] \left[ 1 - (1-w_1)^s \right]^{p-1} - \left[ (1+w_1)^s - 1 \right]^{p-1} \, dy \, dw_1
\]
\[
\leq C(1-x)^{-s} \int_0^1 \frac{1}{w_1^{1+sp}} \left[ \int_0^{\frac{1-x}{1+w_1}} \frac{x^2}{w_1^{1+sp}} \, dx \right] \left[ 1 - (1-w_1)^s \right]^{p-1} - \left[ (1+w_1)^s - 1 \right]^{p-1} \, dy \, dw_1
\]
\[
\leq C(1-x)^{-s} \int_0^1 \frac{1}{w_1^{1+sp}} \left[ \int_0^{\frac{1-x}{1+w_1}} \frac{x^2}{w_1^{1+sp}} \, dx \right] \left[ 1 - (1-w_1)^s \right]^{p-1} - \left[ (1+w_1)^s - 1 \right]^{p-1} \, dy \, dw_1
\]

Further,
\[
(1-x)^{-s} \int_0^1 \frac{1}{w_1^{1+sp}} \left[ \int_0^{\frac{1-x}{1+w_1}} \frac{x^2}{w_1^{1+sp}} \, dx \right] \left[ 1 - (1-w_1)^s \right]^{p-1} - \left[ (1+w_1)^s - 1 \right]^{p-1} \, dy \, dw_1
\]
\[
\leq C(1-x)^{-s} \int_0^1 \frac{1}{w_1^{1+sp}} \left[ \int_0^{\frac{1-x}{1+w_1}} \frac{x^2}{w_1^{1+sp}} \, dx \right] \left[ 1 - (1-w_1)^s \right]^{p-1} - \left[ (1+w_1)^s - 1 \right]^{p-1} \, dy \, dw_1
\]
\[
\leq C(1-x)^{1-s} \int_0^1 \frac{1}{w_1^{p-1+sp-1}} \int_0^{\frac{1-x}{1+w_1}} \frac{1}{dy} \, dw_1
\]
Furthermore, we conclude this step by the following estimate,

$$
\leq \begin{cases} 
C(1-x)^{1-s} \int_0^{\infty} \frac{1}{w_1^{1+sp}} \, dw_1 \leq C(1-x)^{1-s}, & \text{if } sp > 2; \\
C(1-x)^{1-s} \int_0^{\infty} \frac{1}{w_1^{1+sp}} \left( \frac{4x^2}{1-x^2} \right) \, dw_1 \\
\leq C(1-x)^{1-s} \int_0^{\infty} \frac{1}{w_1^{1+sp/2}} \, dw_1 \leq C(1-x)^{\frac{s}{2(p-2)}}, & \text{if } sp \leq 2.
\end{cases}
$$

and

$$(1-x)^{-s} \int_0^{\frac{1}{2}} \frac{1}{w_1^{1+sp}} \int_0^{\frac{1}{2}} \frac{1}{w_1^{1+sp}} \, dw_1 \leq (1-x)^{-s} \int_0^{\frac{1}{2}} \frac{1}{w_1^{1+sp}} \, dw_1 \leq C.$$

On the other hand, we have

$$(1-x)^{-s} \int_0^{\frac{1}{2}} \frac{1}{w_1^{1+sp}} \int_0^{\frac{1}{2}} \frac{1}{w_1^{1+sp}} \, dw_1 \leq (1-x)^{-s} \int_0^{\frac{1}{2}} \frac{1}{w_1^{1+sp}} \, dw_1 \leq C.$$

Moreover, we derive from the above arguments that (I) can be replaced by

$$\int_0^{\frac{1}{2}} \frac{1}{w_1^{1+sp}} \int_0^{\frac{1}{2}} \frac{1}{w_1^{1+sp}} \, dw_1 \leq (1-x)^{-s} \int_0^{\frac{1}{2}} \frac{1}{w_1^{1+sp}} \, dw_1.$$

Furthermore, we conclude this step by the following estimate,

$$(1-x)^{-s} \int_0^{\frac{1}{2}} \frac{1}{w_1^{1+sp}} \int_0^{\frac{1}{2}} \frac{1}{w_1^{1+sp}} \, dw_1 \leq (1-x)^{-s} \int_0^{\frac{1}{2}} \frac{1}{w_1^{1+sp}} \, dw_1 \leq C(1-x)^{-s} \int_0^{\frac{1}{2}} \frac{1}{w_1^{1+sp}} \, dw_1.$$
\[
\begin{align*}
\leq & C(1-x)^{-s} \int_0^1 \frac{1}{w_1^{1+p-2}} \left[ 2 - (1 - w_1)^s - (1 + w_1)^s \right] \frac{1}{w_1^{1+sp}} \left[ \frac{4x^2}{1-x^2} \right]^{1+sp} \frac{1}{w_1^{1+sp}} \, dw_1 \\
\leq & C(1-x)^{\frac{1+s(p-2)}{2}} \int_0^1 \frac{2^{p-1}}{w_1^{1+sp}} \, dw_1 \leq C(1-x)^{\frac{1+s(p-2)}{2}}.
\end{align*}
\]

Step 9. We demonstrate that \((1 - x)^{-s}\) is uniformly bounded when \(x\) is close to 1.

\[
(1-x)^{-s} \int_0^{\frac{2x}{1-x}} \frac{1}{w_1^{1+sp}} \int_0^{\infty} \frac{y^{n-2}}{(1 + y^2)^{n+sp}} \, dy \, dw_1 = (1-x)^{-s} \int_0^{\frac{2x}{1-x}} \frac{1}{w_1^{1+sp}} \int_0^{\infty} \frac{y^{n-2}}{(1 + y^2)^{n+sp}} \, dy \, dw_1.
\]

For one thing,

\[
(1-x)^{-s} \int_0^{\frac{2x}{1-x}} \frac{1}{w_1^{1+sp}} \int_0^{\infty} \frac{1}{(1 + y^2)^{n+sp}} \, dy \, dw_1 \leq C(1-x)^{\frac{1+s(p-2)}{2}}.
\]

For another thing,

\[
C(1-x)^{\frac{1+s(p-2)}{2}} \int_0^{\frac{2x}{1-x}} \frac{1}{w_1^{1+sp}} \int_0^{\infty} \frac{1}{(1 + y^2)^{n+sp}} \, dy \, dw_1 \leq C.
\]
Step 10. We will prove that \((1 - x)^{-s}\mathcal{O}\) is uniformly bounded when \(x\) is close to 1.

\[
(1 - x)^{-s} \int_{\frac{2x^2}{1 - x^2}}^{\frac{4x^2}{1 - x^2}} \frac{1}{w_1^{1+sp}} \int_{\frac{2x^2}{1 - x^2}}^{\frac{4x^2}{1 - x^2}} \frac{1}{w_1^{1+sp}} \frac{y^{n-2}}{(1 + y^2)^{\frac{n+sp}{p}}} dy dw_1
\]

\[
= (1 - x)^{-s} \int_{\frac{2x^2}{1 - x^2}}^{\frac{4x^2}{1 - x^2}} \frac{1}{w_1^{1+sp}} \int_{\frac{2x^2}{1 - x^2}}^{\frac{4x^2}{1 - x^2}} \frac{1}{w_1^{1+sp}} \frac{y^{n-2}}{(1 + y^2)^{\frac{n+sp}{p}}} dy dw_1
\]

\[
= (1 - x)^{-s} \int_{\frac{2x^2}{1 - x^2}}^{\frac{4x^2}{1 - x^2}} \frac{1}{w_1^{1+sp}} \int_{\frac{2x^2}{1 - x^2}}^{\frac{4x^2}{1 - x^2}} \frac{1}{w_1^{1+sp}} \frac{y^{n-2}}{(1 + y^2)^{\frac{n+sp}{p}}} dy dw_1
\]

\[
+ (1 - x)^{-s} \int_{\frac{2x^2}{1 - x^2}}^{\frac{4x^2}{1 - x^2}} \frac{1}{w_1^{1+sp}} \int_{\frac{2x^2}{1 - x^2}}^{\frac{4x^2}{1 - x^2}} \frac{1}{w_1^{1+sp}} \frac{y^{n-2}}{(1 + y^2)^{\frac{n+sp}{p}}} dy dw_1.
\]

On the one hand,

\[
(1 - x)^{-s} \int_{\frac{2x^2}{1 - x^2}}^{\frac{4x^2}{1 - x^2}} \frac{1}{w_1^{1+sp}} \int_{\frac{2x^2}{1 - x^2}}^{\frac{4x^2}{1 - x^2}} \frac{y^{n-2}}{(1 + y^2)^{\frac{n+sp}{p}}} dy dw_1
\]

\[
\leq C(1 - x)^{-s} \int_{\frac{2x^2}{1 - x^2}}^{\frac{4x^2}{1 - x^2}} \frac{1}{w_1^{1+sp}} \int_{\frac{2x^2}{1 - x^2}}^{\frac{4x^2}{1 - x^2}} \frac{y^{n-2}}{(1 + y^2)^{\frac{n+sp}{p}}} dy dw_1
\]

\[
\leq C(1 - x) \frac{1+sp}{1} \int_{\frac{2x^2}{1 - x^2}}^{\frac{4x^2}{1 - x^2}} \frac{1}{w_1^{1+sp}} \left( w_1 + \frac{2x}{1 + x} \right)^{-\frac{1+sp}{1}} dw_1
\]

\[
\leq \begin{cases} 
  C(1 - x) \frac{1+sp}{1}, & \text{if } sp > 1; \\
  C(1 - x) \frac{1+sp}{1} \log \frac{1}{1-x}, & \text{if } sp = 1; \\
  C(1 - x) \frac{1+sp}{1} \frac{1}{1-x} & = C(1 - x)^{s(p-1)}, & \text{if } sp < 1.
\end{cases}
\]

On the other hand,

\[
(1 - x)^{-s} \int_{\frac{2x^2}{1 - x^2}}^{\frac{4x^2}{1 - x^2}} \frac{1}{w_1^{1+sp}} \int_{\frac{2x^2}{1 - x^2}}^{\frac{4x^2}{1 - x^2}} \frac{y^{n-2}}{(1 + y^2)^{\frac{n+sp}{p}}} dy dw_1
\]

\[
\leq C(1 - x)^{-s} \int_{\frac{2x^2}{1 - x^2}}^{\frac{4x^2}{1 - x^2}} \frac{1}{w_1^{1+sp}} dw_1 \leq C(1 - x)^{s(p-1)}.
\]

Step 11. This part is intended to prove \((1 - x)^{-s}\mathcal{O}\)' is uniformly bounded when \(x\) is close to 1.

\[
(1 - x)^{-s} \int_{\frac{2x^2}{1 - x^2}}^{\frac{4x^2}{1 - x^2}} \frac{1}{w_1^{1+sp}} \int_{\frac{2x^2}{1 - x^2}}^{\frac{4x^2}{1 - x^2}} \frac{1}{w_1^{1+sp}} \frac{1}{(1+y)^{n+sp}} dy dw_1
\]

\[
= (1 - x)^{-s} \int_{\frac{2x^2}{1 - x^2}}^{\frac{4x^2}{1 - x^2}} \frac{1}{w_1^{1+sp}} \int_{\frac{2x^2}{1 - x^2}}^{\frac{4x^2}{1 - x^2}} \frac{1}{w_1^{1+sp}} \frac{1}{(1+y)^{n+sp}} dy dw_1
\]
\[ (1 - x)^{-s} \int_{1 - x^{-2}}^{2x^{-2}} \frac{1}{w_1^{1 + sp}} \sqrt{\frac{4x^2 \frac{1}{w_1^{-1}}}{1 - x^{-2} \frac{1}{w_1^{-1}}}} \left[ 1 - h \left( w_1, y \right)^{p - 1} \right] \frac{y^{n - 2}}{(1 + y^2)^{n - sp}} dyw_1. \]

We will estimate the two terms separately.

\[ (1 - x)^{-s} \int_{1 - x^{-2}}^{2x^{-2}} \frac{1}{w_1^{1 + sp}} \sqrt{\frac{4x^2 \frac{1}{w_1^{-1}}}{1 - x^{-2} \frac{1}{w_1^{-1}}}} \left[ 1 - h \left( w_1, y \right)^{p - 1} \right] \frac{y^{n - 2}}{(1 + y^2)^{n - sp}} dyw_1 \]

\[ = (1 - x)^{-s} \int_{1 - x^{-2}}^{2x^{-2}} \frac{1}{w_1^{1 + sp}} \sqrt{\frac{4x^2 \frac{1}{w_1^{-1}}}{1 - x^{-2} \frac{1}{w_1^{-1}}}} \left[ 1 - h \left( w_1, y \right)^{p - 1} \right] \frac{y^{n - 2}}{(1 + y^2)^{n - sp}} dyw_1 \]

\[ + (1 - x)^{-s} \int_{1 - x^{-2}}^{2x^{-2}} \frac{1}{w_1^{1 + sp}} \sqrt{\frac{4x^2 \frac{1}{w_1^{-1}}}{1 - x^{-2} \frac{1}{w_1^{-1}}}} \left[ 1 - h \left( w_1, y \right)^{p - 1} \right] \frac{y^{n - 2}}{(1 + y^2)^{n - sp}} dyw_1 \]

\[ + (1 - x)^{-s} \int_{1 - x^{-2}}^{2x^{-2}} \frac{1}{w_1^{1 + sp}} \sqrt{\frac{4x^2 \frac{1}{w_1^{-1}}}{1 - x^{-2} \frac{1}{w_1^{-1}}}} \left[ 1 - h \left( w_1, y \right)^{p - 1} \right] \frac{y^{n - 2}}{(1 + y^2)^{n - sp}} dyw_1 \]

\[ \leq C(1 - x)^{-s} \int_{1 - x^{-2}}^{2x^{-2}} \frac{1}{w_1^{1 + sp}} \sqrt{\frac{4x^2 \frac{1}{w_1^{-1}}}{1 - x^{-2} \frac{1}{w_1^{-1}}}} \left[ 1 - h \left( w_1, y \right)^{p - 1} \right] \frac{y^{n - 2}}{(1 + y^2)^{n - sp}} dyw_1 \]

\[ + C(1 - x)^{-s} \int_{1 - x^{-2}}^{2x^{-2}} \frac{1}{w_1^{1 + sp}} \sqrt{\frac{4x^2 \frac{1}{w_1^{-1}}}{1 - x^{-2} \frac{1}{w_1^{-1}}}} \left[ 1 - h \left( w_1, y \right)^{p - 1} \right] \frac{y^{n - 2}}{(1 + y^2)^{n - sp}} dyw_1 + C(1 - x)^{(p - 1)}. \]

Moreover,

\[ (1 - x)^{-s} \int_{1 - x^{-2}}^{2x^{-2}} \frac{1}{w_1^{1 + sp}} \sqrt{\frac{4x^2 \frac{1}{w_1^{-1}}}{1 - x^{-2} \frac{1}{w_1^{-1}}}} \left[ 1 - h \left( w_1, y \right)^{p - 1} \right] \frac{y^{n - 2}}{(1 + y^2)^{n - sp}} dyw_1 \]

\[ \leq (1 - x)^{p - s - 1} \int_{1 - x^{-2}}^{2x^{-2}} \frac{1}{w_1^{1 + sp - 3}} \sqrt{\frac{4x^2 \frac{1}{w_1^{-1}}}{1 - x^{-2} \frac{1}{w_1^{-1}}}} \left( 1 + y^2 \right)^{p - 2} dyw_1 \]

\[ + C(1 - x)^{p - s - 1} \int_{1 - x^{-2}}^{2x^{-2}} \frac{1}{w_1^{1 + sp - 3}} \sqrt{\frac{4x^2 \frac{1}{w_1^{-1}}}{1 - x^{-2} \frac{1}{w_1^{-1}}}} \left( 1 + y^2 \right)^{p - 2} dyw_1 \]

\[ \leq \begin{cases} 
C(1 - x)^{p - s - 1} \int_{1 - x^{-2}}^{2x^{-2}} \frac{1}{w_1^{1 + sp - 3}} \sqrt{\frac{4x^2 \frac{1}{w_1^{-1}}}{1 - x^{-2} \frac{1}{w_1^{-1}}}} \left( 1 + y^2 \right)^{p - 2} dyw_1 
\leq C(1 - x)^{\frac{1 + s(p - 2)}{2}}, & \text{if } p - \frac{sp}{2} - 2 \geq 0; \\
C(1 - x)^{p - s - 1} \int_{1 - x^{-2}}^{2x^{-2}} \frac{1}{w_1^{1 + sp - 3}} \sqrt{\frac{4x^2 \frac{1}{w_1^{-1}}}{1 - x^{-2} \frac{1}{w_1^{-1}}}} \left( 1 + y^2 \right)^{p - 2} dyw_1 
\leq C(1 - x)^{\frac{1 + s(p - 2)}{2}}, & \text{if } p - \frac{sp}{2} - 2 < 0.
\end{cases} \]

And

\[ (1 - x)^{-s} \int_{1 - x^{-2}}^{2x^{-2}} \frac{1}{w_1^{1 + sp}} \sqrt{\frac{4x^2 \frac{1}{w_1^{-1}}}{1 - x^{-2} \frac{1}{w_1^{-1}}}} dyw_1 \]
\[
C(1-x) \frac{1+x(p-2)}{2} \int_1^{2x} \frac{1}{w_1^{1+sp}} \, dw_1 \leq \begin{cases} 
C(1-x) \frac{1+x(p-2)}{2}, & \text{if } sp > 1; \\
C(1-x) \frac{1+x(p-2)}{2} \log \frac{1}{1-x}, & \text{if } sp = 1; \\
C(1-x)^{s(p-1)}, & \text{if } sp < 1.
\end{cases}
\]

In addition,
\[
(1-x)^{-s} \int_1^{2x} \frac{1}{w_1^{1+sp}} \sqrt{\frac{4x^2}{1-x^2} \frac{1+w_1}{w_1} - 1} \frac{[1 - h (w_1, y)^s]^{p-1}}{(1 + y^2)^{\frac{n+sp}{2}}} y^{n-2} \, dy \, dw_1
= (1-x)^{-s} \int_1^{2x} \frac{1}{w_1^{1+sp}} \sqrt{\frac{4x^2}{1-x^2} \frac{1+w_1}{w_1} - 1} \frac{[1 - h (w_1, y)^s]^{p-1}}{(1 + y^2)^{\frac{n+sp}{2}}} y^{n-2} \, dy \, dw_1
+ (1-x)^{-s} \int_1^{2x} \frac{1}{w_1^{1+sp}} \sqrt{\frac{4x^2}{1-x^2} \frac{1+w_1}{w_1} - 1} \frac{[1 - h (w_1, y)^s]^{p-1}}{(1 + y^2)^{\frac{n+sp}{2}}} y^{n-2} \, dy \, dw_1
\leq C(1-x)^{-s} \int_1^{2x} \frac{1}{w_1^{1+sp}} \sqrt{\frac{4x^2}{1-x^2} \frac{1+w_1}{w_1} - 1} \frac{1}{(1-x)^{s(p-1)}} \, dw_1 + C(1-x)^{s(p-1)}.
\]

Further,
\[
(1-x)^{-s} \int_1^{2x} \frac{1}{w_1^{1+sp}} \sqrt{\frac{4x^2}{1-x^2} \frac{1+w_1}{w_1} - 1} \frac{1}{w_1} \, dw_1
\leq C(1-x)^{1+s(p-2)} \int_1^{2x} \left( \frac{1}{2} + w_1 \right)^{-\frac{1}{1+sp}} \, dw_1 \leq \begin{cases} 
C(1-x)^{1+s(p-2)} \frac{y}{2}, & \text{if } sp > 1; \\
C(1-x)^{1+s(p-2)} \log \frac{1}{1-x}, & \text{if } sp = 1; \\
(1-x)^{s(p-1)}, & \text{if } sp < 1.
\end{cases}
\]

Step 12. We will prove \((1-x)^{-s} (II)\) is uniformly bounded when \(x\) is close to 1.
\[
(1-x)^{-s} \int_1^{\frac{1}{2}} \frac{1}{w_1^{1+sp}} \sqrt{\frac{4x^2}{1-x^2} \frac{1+w_1}{w_1} - 1} \frac{[1 - h (-w_1, y)^s]^{p-1}}{(1 + y^2)^{\frac{n+sp}{2}}} y^{n-2} \, dy \, dw_1
= (1-x)^{-s} \int_1^{\frac{1}{2}} \frac{1}{w_1^{1+sp}} \sqrt{\frac{4x^2}{1-x^2} \frac{1+w_1}{w_1} - 1} \frac{[1 - h (-w_1, y)^s]^{p-1}}{(1 + y^2)^{\frac{n+sp}{2}}} y^{n-2} \, dy \, dw_1
+ (1-x)^{-s} \int_1^{\frac{1}{2}} \frac{1}{w_1^{1+sp}} \sqrt{\frac{4x^2}{1-x^2} \frac{1+w_1}{w_1} - 1} \frac{[1 - h (-w_1, y)^s]^{p-1}}{(1 + y^2)^{\frac{n+sp}{2}}} y^{n-2} \, dy \, dw_1
= (1-x)^{-s} \int_1^{\frac{1}{4}} \frac{1}{w_1^{1+sp}} \sqrt{\frac{4x^2}{1-x^2} \frac{1+w_1}{w_1} - 1} \frac{[1 - h (-w_1, y)^s]^{p-1}}{(1 + y^2)^{\frac{n+sp}{2}}} y^{n-2} \, dy \, dw_1
+ (1-x)^{-s} \int_1^{\frac{1}{4}} \frac{1}{w_1^{1+sp}} \sqrt{\frac{4x^2}{1-x^2} \frac{1+w_1}{w_1} - 1} \frac{[1 - h (-w_1, y)^s]^{p-1}}{(1 + y^2)^{\frac{n+sp}{2}}} y^{n-2} \, dy \, dw_1
\]
Furthermore,

\[
+ (1 - x)^s \int_0^{\frac{4}{1 - x^2}} \frac{1}{w_{1 + sp}^1} \left( \sqrt{1 - \frac{4x^2}{1 - x^2}} \frac{1 - w_{1 - 1}^1}{w_{1 - 1}^1} \right) \frac{1}{1 + y^2} \left[ 1 - h (-w_{1, y})^{1 - 1 - 1} \right] y^{n - 2} \, dy \, dw_1
\]

\[
\leq C (1 - x)^s \int_0^{\frac{4}{1 - x^2}} \frac{1}{w_{1 + sp}^1} \left( \sqrt{1 - \frac{4x^2}{1 - x^2}} \frac{1 - w_{1 - 1}^1}{w_{1 - 1}^1} \right) \frac{1}{1 + y^2} \left[ 1 - h (-w_{1, y})^{1 - 1 - 1} \right] y^{n - 2} \, dy \, dw_1
\]

\[
+ C (1 - x)^s \int_0^{\frac{4}{1 - x^2}} \frac{1}{w_{1 + sp}^1} \left( \sqrt{1 - \frac{4x^2}{1 - x^2}} \frac{1 - w_{1 - 1}^1}{w_{1 - 1}^1} \right) \frac{1}{1 + y^2} \left[ 1 - h (-w_{1, y})^{1 - 1 - 1} \right] y^{n - 2} \, dy \, dw_1
\]

Step 13. We finally will prove that (IV) can be reduced to

\[
(1 - x)^{p - 1} \int_0^{\frac{4}{1 - x^2}} \frac{1}{w_{1 + sp}^1} \left( \sqrt{1 - \frac{4x^2}{1 - x^2}} \frac{1 - w_{1 - 1}^1}{w_{1 - 1}^1} \right) \frac{1}{1 + y^2} \left[ 1 - h (-w_{1, y})^{1 - 1 - 1} \right] y^{n - 2} \, dy \, dw_1
\]

Furthermore,

\[
(1 - x)^{p - 1} \int_0^{\frac{4}{1 - x^2}} \frac{1}{w_{1 + sp}^1} \left( \sqrt{1 - \frac{4x^2}{1 - x^2}} \frac{1 - w_{1 - 1}^1}{w_{1 - 1}^1} \right) \frac{1}{1 + y^2} \left[ 1 - h (-w_{1, y})^{1 - 1 - 1} \right] y^{n - 2} \, dy \, dw_1
\]

\[
\leq \left\{ \begin{array}{ll}
C (1 - x)^{p - 1} \int_0^{\frac{4}{1 - x^2}} \frac{1}{w_{1 + sp}^1} \left( \sqrt{1 - \frac{4x^2}{1 - x^2}} \frac{1 - w_{1 - 1}^1}{w_{1 - 1}^1} \right) \frac{1}{1 + y^2} \left[ 1 - h (-w_{1, y})^{1 - 1 - 1} \right] y^{n - 2} \, dy \, dw_1
\end{array} \right.
\]

\[
\leq C (1 - x)^{p - 1} \int_0^{\frac{4}{1 - x^2}} \frac{1}{w_{1 + sp}^1} \left( \sqrt{1 - \frac{4x^2}{1 - x^2}} \frac{1 - w_{1 - 1}^1}{w_{1 - 1}^1} \right) \frac{1}{1 + y^2} \left[ 1 - h (-w_{1, y})^{1 - 1 - 1} \right] y^{n - 2} \, dy \, dw_1
\]

Firstly, we prove that (IV) can be replaced by

\[
\int_0^{\frac{4}{1 - x^2}} \frac{1}{w_{1 + sp}^1} \left( \sqrt{1 - \frac{4x^2}{1 - x^2}} \frac{1 - w_{1 - 1}^1}{w_{1 - 1}^1} \right) \frac{1}{1 + y^2} \left[ (1 + w_{1, y})^{1 - 1 - 1} \right] y^{n - 2} \, dy \, dw_1.
\]
Moreover,

\[
\leq C(1-x)^{-s} \int_0^{3 \frac{2}{1-x}} \frac{1}{w_1^{1+sp}} \int_0^{\frac{4x^2 + 1}{1-x}} w_1^{\alpha - 1} \frac{[(1 + w_1)^{\alpha - 1]^{p-1} - [h(w_1,y) \alpha - 1]^{p-1}}{(1 + y^2) \frac{4x^2}{1-x}} y^{n-2} dy dw_1
\]

\[
+ (1-x)^{-s} \int_0^{2 \frac{2}{1-x}} \frac{1}{w_1^{1+sp}} \int_0^{\frac{4x^2 + 1}{1-x}} w_1^{\alpha - 1} [1 - (h(w_1,y) \alpha - 1]^{p-1}}{(1 + y^2) \frac{4x^2}{1-x}} y^{n-2} dy dw_1
\]

\[+ C(1-x)^{(p-1)}.
\]

We begin to evaluate the first two terms separately,

\[
(1-x)^{-s} \int_0^{\frac{3}{2}} \frac{1}{w_1^{1+sp}} \int_0^{\frac{4x^2 + 1}{1-x}} w_1^{\alpha - 1} \frac{[(1 + w_1)^{\alpha - 1]^{p-1} - [h(w_1,y) \alpha - 1]^{p-1}}{(1 + y^2) \frac{4x^2}{1-x}} y^{n-2} dy dw_1
\]

\[
\leq C(1-x)^{-s} \int_0^{\frac{3}{2}} \frac{1}{w_1^{1+sp}} \int_0^{\frac{4x^2 + 1}{1-x}} w_1^{\alpha - 1} [1 - (h(w_1,y) \alpha - 1]^{p-1}}{(1 + y^2) \frac{4x^2}{1-x}} y^{n-2} dy dw_1
\]

\[
\leq C(1-x)^{-s} \int_0^{\frac{3}{2}} \frac{1}{w_1^{1+sp}} \int_0^{\frac{4x^2 + 1}{1-x}} w_1^{\alpha - 1} \frac{[(1 + w_1)^{\alpha - 1]^{p-1} - [h(w_1,y) \alpha - 1]^{p-1}}{(1 + y^2) \frac{4x^2}{1-x}} y^{n-2} dy dw_1
\]

\[
= C(1-x)^{-s} \int_0^{\frac{3}{2}} \frac{1}{w_1^{1+sp}} \int_0^{\frac{4x^2 + 1}{1-x}} \frac{1}{1 + y^2} d y dw_1
\]

\[
\leq \begin{cases} 
C(1-x)^{1-s}, & \text{if } sp \geq 2; \\
C(1-x)^{1-s} \int_0^{\frac{3}{2}} \frac{4x^2 - 1}{w_1} \frac{1}{1 + y^2} d y dw_1 \leq C(1-x)^{(p-2)} & \text{if } sp < 2.
\end{cases}
\]

Moreover,

\[
(1-x)^{-s} \int_0^{\frac{2}{1-x}} \frac{1}{w_1^{1+sp}} \int_0^{\frac{4x^2 + 1}{1-x}} \frac{[(1 + w_1)^{\alpha - 1]^{p-1} - [h(w_1,y) \alpha - 1]^{p-1}}{(1 + y^2) \frac{4x^2}{1-x}} y^{n-2} dy dw_1
\]

\[
= (1-x)^{-s} \int_0^{\frac{2}{1-x}} \frac{1}{w_1^{1+sp}} \int_0^{\frac{4x^2 + 1}{1-x}} \frac{[(1 + w_1)^{\alpha - 1]^{p-1} - [h(w_1,y) \alpha - 1]^{p-1}}{(1 + y^2) \frac{4x^2}{1-x}} y^{n-2} dy dw_1
\]

\[
+ (1-x)^{-s} \int_0^{\frac{2}{1-x}} \frac{1}{w_1^{1+sp}} \int_0^{\frac{4x^2 + 1}{1-x}} \frac{[(1 + w_1)^{\alpha - 1]^{p-1} - [h(w_1,y) \alpha - 1]^{p-1}}{(1 + y^2) \frac{4x^2}{1-x}} y^{n-2} dy dw_1
\]

\[
\leq C(1-x)^{-s} \int_0^{\frac{2}{1-x}} \frac{1}{w_1^{1+sp}} \int_0^{\frac{4x^2 + 1}{1-x}} \frac{[(1 + w_1)^{\alpha - 1]^{p-2}}{(1 + y^2) \frac{4x^2}{1-x}} y^{n-2} dy dw_1
\]

\[
+ C(1-x)^{-s} \int_0^{\frac{2}{1-x}} \frac{1}{w_1^{1+sp}} \int_0^{\frac{4x^2 + 1}{1-x}} \frac{[(1 + w_1)^{\alpha - 1]^{p-2}}{(1 + y^2) \frac{4x^2}{1-x}} y^{n-2} dy dw_1
\]

\[
\leq C(1-x)^{-s} \int_0^{\frac{2}{1-x}} \frac{1}{w_1^{1+sp}} \int_0^{\frac{4x^2 + 1}{1-x}} \frac{[(1 + w_1)^{\alpha - 1]^{p-2}}{(1 + y^2) \frac{4x^2}{1-x}} y^{n-2} dy dw_1
\]

\[
+ C(1-x)^{(p-2)} \int_0^{\frac{2}{1-x}} \frac{1}{w_1^{1+sp}} \int_0^{\frac{4x^2 + 1}{1-x}} \frac{[(1 + w_1)^{\alpha - 1]^{p-2}}{(1 + y^2) \frac{4x^2}{1-x}} y^{n-2} dy dw_1
\]

\[
\leq C(1-x)^{1-s} \int_0^{\frac{2}{1-x}} \frac{w_1^{1+sp}}{(1 + w_1)^{1+s}} \int_0^{\frac{4x^2 + 1}{1-x}} \frac{1}{1 + y^2} dy dw_1 + C(1-x)^{(p-2)}.
\]
Besides,

\[
(1 - x)^{1-s} \int_0^{\frac{2\pi}{1-x^2}} \frac{w_1^{1-2s}}{(1 + w_1)^{1-s}} \int_0^{\frac{2\pi}{1-x^2}} \frac{1}{(1 + y^2)^{\frac{n+sp}{2}}} \, dy \, dw_1 \leq C(1 - x)^{1-s} \int_0^{\frac{2\pi}{1-x^2}} w_1^{-s} \int_0^{\frac{2\pi}{1-x^2}} \frac{1}{(1 + y^2)^{\frac{n+sp}{2}}} \, dy \, dw_1
\]

\[
\leq \begin{cases}
(1 - x)^{1-s} \int_0^{\frac{2\pi}{1-x^2}} w_1^{-s} \, dw_1 \leq C, & \text{if } sp \geq 2; \\
C(1 - x)^{1-s} \int_0^{\frac{2\pi}{1-x^2}} w_1^{-s} \left( \frac{2\pi}{1-x^2} \frac{1}{w_1} \right)^{1- \frac{sp}{2}} \, dw_1 \leq C(1 - x)^{\frac{x(p-2)}{2}} \int_0^{\frac{2\pi}{1-x^2}} w_1^{-(p-2)} \, dw_1 \leq C, & \text{if } sp < 2.
\end{cases}
\]

Secondly, we claim that \((IV)\) can be substituted by

\[
\int_\frac{1}{2}^{\frac{\pi}{2}} \frac{1}{w_1^{1+sp}} \int_0^\infty \frac{[(1 + w_1)^s - 1]^{p-1}}{(1 + y^2)^{\frac{n+sp}{2}}} \, y^{n-2} \, dy \, dw_1.
\]

Since

\[
(1 - x)^{-s} \int_\frac{1}{2}^{\frac{\pi}{2}} \frac{1}{w_1^{1+sp}} \int_0^\infty \frac{[(1 + w_1)^s - 1]^{p-1}}{(1 + y^2)^{\frac{n+sp}{2}}} \, y^{n-2} \, dy \, dw_1 \leq C(1 - x)^{-s} \int_\frac{1}{2}^{\frac{\pi}{2}} \frac{1}{w_1^{1+sp}} \int_0^\infty \frac{[(1 + w_1)^s - 1]^{p-1}}{(1 + y^2)^{\frac{n+sp}{2}}} \, y^{n-2} \, dy \, dw_1
\]

\[
\leq C(1 - x)^{\frac{x(p-2)}{2}} \int_\frac{1}{2}^{\frac{\pi}{2}} \frac{w_1^{s(p-1)}}{w_1^{1+sp}} \, dw_1 \leq C(1 - x)^{\frac{x(p-2)}{2}} \int_\frac{1}{2}^{\frac{\pi}{2}} \frac{w_1^{s(p-1)}}{w_1^{1+sp}} \, dw_1 \leq C.
\]

Thirdly, \((IV)\) can be replaced by

\[
\int_\frac{1}{2}^{\frac{\pi}{2}} \frac{1}{w_1^{1+sp}} \int_0^\infty \frac{[(1 + w_1)^s - 1]^{p-1}}{(1 + y^2)^{\frac{n+sp}{2}}} \, y^{n-2} \, dy \, dw_1.
\]

Due to the fact that

\[
(1 - x)^{-s} \int_\frac{1}{2}^{\frac{\pi}{2}} \frac{1}{w_1^{1+sp}} \int_0^\infty \frac{[(1 + w_1)^s - 1]^{p-1}}{(1 + y^2)^{\frac{n+sp}{2}}} \, y^{n-2} \, dy \, dw_1 \leq C(1 - x)^{-s} \int_\frac{1}{2}^{\frac{\pi}{2}} \frac{w_1^{s(p-1)}}{w_1^{1+sp}} \, dw_1 \leq C.
\]

Step 14. Overall, the singular term is

\[
(1 - x)^{-s} \int_0^\infty \frac{y^{n-2}}{(1 + y^2)^{\frac{n+sp}{2}}} \, dy \left\{ \frac{1}{sp} + \int_0^1 \frac{[1 - (1 - w_1)^s]^{p-1} - [(1 + w_1)^s - 1]^{p-1}}{w_1^{1+sp}} \, dw_1 \right\}
\]
Example 4.1. Set

\[ \rho \leq \left| \frac{1}{x^2} \right|, \quad \text{if} \quad x \in B_1^+(0) := \{ x \in B_1(0) \mid x_n > 0 \}; \]
\[ 0, \quad \text{if} \quad x \in \mathbb{R}^n \setminus B_1^+(0). \]  

(4.1)

Then \( u \in C^{1,1}_{loc}(B_1^+(0)) \) and \( u \) is lower semicontinuous,

\[ \int_{\mathbb{R}^n} \frac{\left| 1 + u(x) \right|^{p-1}}{1 + |x|^{n+sp}} \, dx = \int_{B_1^+(0)} \frac{\left| 1 + \frac{1}{x^2} \right|^{p-1}}{1 + |x|^{n+sp}} \, dx + \int_{\mathbb{R}^n \setminus B_1^+(0)} \frac{1}{1 + |x|^{n+sp}} \, dx \]
\[ \leq C(p) \int_{B_1^+(0)} \frac{1}{|x|^{n+sp}} \, dx + C(p) \int_{\mathbb{R}^n \setminus B_1^+(0)} \frac{1}{1 + |x|^{n+sp}} \, dx \]
\[ \leq C(p) \int_{B_1^+(0)} \frac{1}{|x|^{t(p-1)}} \, dx + C(p) \]
\[ \leq C(p, n) \int_0^1 \rho^{n-t(p-1)-1} \, d\rho + C(p, n) < +\infty \quad \text{if} \quad t < \frac{n}{p-1}. \]

So \( u(x) \in C^{1,1}_{loc}(B_1^+(0)) \cap L_{sp}(\mathbb{R}^n) \) when \( t < \frac{n}{p-1}. \) However, when \( t \geq \frac{n}{p} \), \( u(x) \notin L^p(B_1^+(0)). \)

So for \( p \in \left( \frac{n}{p}, \frac{n}{p-1} \right) \), \( u(x) \in \left\{ C^{1,1}_{loc}(B_1^+(0)) \cap L_{sp} \right\} \setminus \left\{ W^{s,p}(B_1^+(0)) \right\}. \)

From the example above, it is meaningful to investigate the Hopf’s lemma for \( u(x) \in C^{1,1}_{loc}(\Omega) \cap L_{sp} \) in the point-wise sense.

**Proof of Theorem 1.2.** Since \( u \) satisfies the uniform interior ball condition, we assume the uniform radius is 10\( p \). Then for every \( z \in \partial \Omega \), there is a ball \( B_{10}(y) \subset \Omega \) centered at \( y \in \Omega \) with \( \{ z \} = \partial B_{10}(y) \cap \partial \Omega \). And \( \delta(x) := \text{dist}(x, \partial \Omega) = |x - z| \). Without loss of generality, we relocate the origin to \( y \) so that \( \delta(x) = \rho - |x| \).

Set

\[ \psi(x) = (1 - |x^2|^p)_+', \quad \psi(y) = \psi \left( \frac{x}{\rho} \right) = \left( \frac{\rho^2 - |x|^2}{\rho^2} \right)^{p}, \quad x \in B_{\rho}. \]

Then by Theorem 1.1, there exists a constant \( C_0 > 0 \), such that

\[ (-\Delta)^p \psi(x) \leq C_0, \quad (-\Delta)^p \psi(y) = \left( \frac{-\Delta \psi \left( \frac{x}{\rho} \right)}{x^{2p}} \right) \leq \frac{C_0 \rho^{2p}}{\rho^{2p}} \]
Since \( u > 0 \) in \( \Omega \), we consider a region \( D := \{ x \in \Omega : \text{dist}(x, \partial \Omega) \geq 3\rho \} \) which has a positive distance with \( B_\rho \). So \( C_D := \inf_{D} u(x) > 0 \). Then we set \( u_-(x) = u(x) - \epsilon \psi_g \), where \( \epsilon < C_D \) is to be specified later. Denote \(|t|^{p-1} := |t|^{p-2} t \). For any \( x \in B_\rho \),

\[
\begin{align*}
(-\Delta)_p u_-(x) &= C_{n,s,p} \text{P.V.} \int_{\mathbb{R}^n} \frac{[\psi_g(x) - u(y)]^{p-1}}{|x-y|^{n+sp}} dy \\
&= C_{n,s,p} \text{P.V.} \int_{B_\rho(y)} \frac{[\psi_g(x) - \psi_g(y)]^{p-1}}{|x-y|^{n+sp}} dy + \int_{D} \frac{[\psi_g(x) - u(y)]^{p-1}}{|x-y|^{n+sp}} dy
\end{align*}
\]

\[
\begin{align*}
&+ \int_{\mathbb{R}^n \setminus (D \cup B_\rho(y))} \frac{[\psi_g(x) - u(y)]^{p-1}}{|x-y|^{n+sp}} dy
\end{align*}
\]

\[
\begin{align*}
&= p^{p-1} (-\Delta)^{\rho}_p \psi_g(x) + C_{n,s,p} \text{P.V.} \int_{D} \frac{[\psi_g(x) - u(y)]^{p-1}}{|x-y|^{n+sp}} dy - \int_{D} \frac{[\psi_g(x)]^{p-1}}{|x-y|^{n+sp}} dy
\end{align*}
\]

\[
\begin{align*}
\leq C_{\rho,p} &+ C_{n,s,p} \text{P.V.} \int_{D} \frac{[\psi_g(x) - u(y)]^{p-1} - [\psi_g(x)]^{p-1}}{|x-y|^{n+sp}} dy
\end{align*}
\]

\[
\begin{align*}
&\leq C_{\rho,p} + C_{n,s,p} \frac{2^{2-p} C_D}{p^{p-1}} \int_{D} \frac{1}{|x-y|^{n+sp}} dy
\end{align*}
\]

\[
\begin{align*}
&\leq C_{\rho,p} - C_{n,s,p} \frac{2^{2-p} C_D}{p^{p-1}} C(p),
\end{align*}
\]

where we use the inequality in Lemma 2.2. So we can choose \( \epsilon \) small such that \((-\Delta)_p u_- (x) + c(x) u_- \leq 0 \) for any \( x \in B_\rho \). Then by Proposition 1, for \( \forall x \in [y, z] \), we have

\[
\begin{align*}
u(x) &\geq u_- (x) = \epsilon \psi_g(x) = \frac{\epsilon (\rho^2 - |x|^2)^s}{\rho^{2s}} = \frac{\epsilon (\rho + |x|)^s (\rho - |x|)^s}{\rho^{2s}} = \frac{\epsilon (\rho + |x|)^s}{\rho^{2s}} \delta(x). \quad \Box
\end{align*}
\]

5. Global Hölder regularity of bounded positive solutions.

**Proof of Theorem 1.2.** First we repeat the last part of the proof in [19, Theorem 2], there exist \( \epsilon_0, \nu_0 \) such that

\[
|u(x)| \leq c [\text{dist}(x, \partial \Omega)]^{\nu_0}, \quad \forall x \in V := \{ x \in \Omega | \text{dist}(x, \partial \Omega) < \epsilon_0 \}.
\]

By the boundedness of \( u \), we have

\[
|u(x)| \leq c [\text{dist}(x, \partial \Omega)]^{\nu_0}, \quad \forall x \in \Omega.
\]

Now we consider \( x \in W := \{ x \in \Omega | \text{dist}(x, \partial \Omega) \geq \epsilon_0 \} \), then \( B_{\frac{\rho}{2}} (x) \subset \subset \Omega, \quad \forall x \in W. \) We claim \( u \in W^{1,p}_{loc}(\Omega) \) if \( u \in C^{1,1}_{loc}(\Omega) \cap L^{sp} \). In fact, for any domain \( \Omega' \subset \subset \Omega, \)

\[
\begin{align*}
\int_{\Omega'} \int_{\Omega'} \frac{|u(x) - u(y)|^p}{|x-y|^{n+sp}} dx \, dy &= \int_{\Omega'} \left( \int_{\Omega'} \frac{|\nabla u(y) \cdot (x-y) + o(|x-y|)|^p}{|x-y|^{n+sp}} dx \right) dy
\end{align*}
\]

\[
\begin{align*}
&\leq C(\Omega') \left( \int_{\Omega'} \frac{|x-y|^p}{|x-y|^{n+sp}} dx \right) dy \leq C(\Omega').
\end{align*}
\]

By [17, Proposition 2.12, Lemma 2.5], \( u \) is a weak solution of \((-\Delta)_p u = f \) in \( B_{\frac{\rho}{2}} (x), \quad \forall x \in W. \) Then by [5, Theorem 1.4], for this \( \nu_0 \), we have

\[
\begin{align*}
u(u)_{C^{\nu_0}(B_{\frac{\rho}{2}} (x))} \leq \frac{C(\nu_0)}{\epsilon_0 \nu_0} \left\| u \right\|_{L^{\infty}(B_{\frac{\rho}{2}} (x))} + \left( \int_{R^n \setminus (B_{\frac{\rho}{2}} (x))} \frac{|u|^{p-1}}{|x-y|^{n+sp}} dy \right)^{\frac{1}{p-1}}.
\end{align*}
\]
Again by [5, Theorem 1.4], we denote \( \delta \). Now for \( x, y \in W \), we have

\[
\frac{|u(x) - u(y)|}{|x - y|^{\nu_0}} \leq C(\nu_0, \epsilon_0) \left( \|u\|_{L^\infty(\Omega)} + \|f\|_{L^\infty(\Omega)}^{\frac{1}{p}} \right).
\]

So for \( \forall x, y \in W \), we have

\[
\frac{|u(x) - u(y)|}{|x - y|^{\nu_0}} \leq C(\nu_0, \epsilon_0) \left( \|u\|_{L^\infty(\Omega)} + \|f\|_{L^\infty(\Omega)}^{\frac{1}{p}} \right).
\]

The next part is similar to the [17, Section 5.2]. Now we consider that \( x, y \in V \). Again by [5, Theorem 1.4], we denote \( \delta_x := \text{dist}(x, \partial\Omega) \),

\[
[u]_{C^{\nu_0}(B_{\delta_x}(x))} \leq C(\nu_0) \frac{\|u\|_{L^\infty(B_{\delta_x}(x))}}{\delta_x^{\nu_0}} \left( \|u\|_{L^\infty(B_{\delta_x}(x))} + \left( \delta_x^{sp} \int_{\mathbb{R}^n \setminus B_{\delta_x}(x)} \frac{|u|^{p-1}}{|x - y|^{n+sp}} dy \right)^{\frac{1}{p-1}} \right).
\]

Finally, we have

\[
[u]_{C^{\nu_0}(B_{\delta_x}(x))} \leq C(\nu_0) \left( 1 + \|f\|_{L^\infty(\Omega)}^{\frac{1}{p}} \right) + C(\nu_0) \frac{\delta_x^{sp} \int_{\mathbb{R}^n \setminus B_{\delta_x}(x)} \frac{C_y^{\nu_0(p-1)}}{|x - y|^{n+sp}} dy}{\delta_x^{\nu_0}} \left( \delta_x^{sp} \int_{\mathbb{R}^n \setminus B_{\delta_x}(x)} \frac{C_y^{\nu_0(p-1)}}{|x - y|^{n+sp}} dy \right)^{\frac{1}{p-1}}.
\]

Now for \( x, y \in V \) and without loss of generality, we assume \( \delta_x \geq \delta_y \).

- When \( |x - y| < \frac{\delta_x}{44} \), i.e. \( y \in B_{\delta_x}(x) \), by (5.1), we have

\[
\frac{|u(x) - u(y)|}{|x - y|^{\nu_0}} \leq C(\nu_0) \left( 1 + \|f\|_{L^\infty(\Omega)}^{\frac{1}{p}} \right).
\]

- When \( |x - y| \geq \frac{\delta_x}{44} \), we have

\[
|u(x) - u(y)| \leq |u(x)| + |u(y)| \leq C(\delta_x^{\nu_0} + \delta_y^{\nu_0}) \leq C|x - y|^{\nu_0}.
\]

Hence our conclusion is \( u \in C^{\nu_0} (\mathbb{R}^n) \).
Appendix. We list the numerical calculation results below. We refer the readers who are interested in the details to [20], which is the Ph.D. thesis of the first author.

**Proposition 2** (Numerical calculation). Let \( n = 1, s = \frac{1}{2}, p = 2, 4, 6, 8 \), \( u(x) = (1 - x^2)^{\frac{1}{r}} \). Then \((-\Delta)^{s}_{p} u(x)\) is uniformly bounded in \((-1, 1)\). More precisely ( omit constant \( C_{n, s, p} \)), we have

\[
(-\Delta)^{\frac{1}{2}} u(x) = 2 \arcsin 1 = \pi;
\]

\[
(-\Delta)^{\frac{1}{4}} u(x) = 3 \sqrt{1 - x^2} \left[ \log(4 - 4x^2) - 1 \right] + 6x \arcsin x;
\]

\[
(-\Delta)^{\frac{1}{6}} u(x) = 20 \sqrt{1 - x^2} \left[ x \log \frac{1 - x}{1 + x} + 2 \right] + \frac{5\pi}{2} (8x^2 - 5);
\]

\[
(-\Delta)^{\frac{1}{8}} u(x) = 7 \sqrt{1 - x^2} \left[ 4(5x^2 - 2) \log(4 - 4x^2) + \frac{(-x^2 + 67)}{6} \right] + 35x(8x^2 - 7) \arcsin x.
\]

In addition, \((-\Delta)^{\frac{1}{4}} u(x), (-\Delta)^{\frac{1}{6}} u(x), (-\Delta)^{\frac{1}{8}} u(x)\) are strictly increasing in \((0, 1)\) and \((-\Delta)^{\frac{1}{4}} u(x) \in [6 \log 2 - 3, 3\pi), (-\Delta)^{\frac{1}{6}} u(x) \in [40 - \frac{25\pi}{2}, \frac{15\pi}{2}), (-\Delta)^{\frac{1}{8}} u(x) \in \left[ \frac{469}{6}, 122 \log 2, \frac{35\pi}{2} \right)\).

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_E-mail address:_ lizaizheng@amss.ac.cn

_E-mail address:_ qidi@math.ubc.ca