Category analog of sup-measurability problem

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Abstract

A function \( F: \mathbb{R}^2 \to \mathbb{R} \) is sup-measurable if \( F_f: \mathbb{R} \to \mathbb{R} \) given by \( F_f(x) = F(x, f(x)) \), \( x \in \mathbb{R} \), is measurable for each measurable function \( f: \mathbb{R} \to \mathbb{R} \). It is known that under different set theoretical assumptions, including CH, there are sup-measurable non-measurable functions, as well as their category analog. In this paper we will show that the existence of category analog of sup-measurable non-measurable functions is independent of ZFC. A similar result for the original measurable case is a subject of a work in prepartion by Roslanowski and Shelah.

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1 Introduction

Our terminology is standard and follows that from [3], [4], [9], or [10].

The study of sup-measurable functions comes from the theory of differential equations. More precisely it comes from a question for which functions \( F : \mathbb{R}^2 \rightarrow \mathbb{R} \) the Cauchy problem

\[
y' = F(x, y), \quad y(x_0) = y_0
\]

has a (unique) a.e.-solution in the class of locally absolutely continuous functions on \( \mathbb{R} \) in a sense that \( y(x_0) = y_0 \) and \( y'(x) = G(x, y(x)) \) for almost all \( x \in \mathbb{R} \). (For more on this motivation see [8] or [2].) It is not hard to find measurable functions which are not sup-measurable. (See [11] or [1, Cor. 1.4].)

Under the continuum hypothesis CH or some weaker set-theoretical assumptions nonmeasurable sup-measurable functions were constructed in [6], [7], [1], and [8]. An independence from ZFC of the existence of such example is a subject of a work in preparation by Rosłanowski and Shelah.

A function \( F : \mathbb{R}^2 \rightarrow \mathbb{R} \) is a category analog of sup-measurable function (or Baire sup-measurable) provided \( F_f : \mathbb{R} \rightarrow \mathbb{R} \) given by \( F_f(x) = F(x, f(x)) \), \( x \in \mathbb{R} \), has the Baire property for each function \( f : \mathbb{R} \rightarrow \mathbb{R} \) with the Baire property. Baire sup-measurable function without the Baire property has been constructed under CH in [4]. (See also [1] and [2].) The main goal of this paper is to show that the existence of such functions cannot be proved in ZFC. For this we need the following easy fact. (See [1, Prop. 1.5].)

**Proposition 1** The following conditions are equivalent.

(i) There is a Baire sup-measurable function \( F : \mathbb{R}^2 \rightarrow \mathbb{R} \) without the Baire property.

(ii) There is a function \( F : \mathbb{R}^2 \rightarrow \mathbb{R} \) without the Baire property such that \( F_f \) has the Baire property for every Borel function \( f : \mathbb{R} \rightarrow \mathbb{R} \).

(iii) There is a set \( A \subset \mathbb{R}^2 \) without the Baire property such that the projection \( \text{pr}(A \cap f) = \{ x \in \mathbb{R} : (x, f(x)) \in A \} \) has the Baire property for each Borel function \( f : \mathbb{R} \rightarrow \mathbb{R} \).

(iv) There is a Baire sup-measurable function \( F : \mathbb{R}^2 \rightarrow \{0, 1\} \) without the Baire property.
The equivalence of (i) and (ii) follows from the fact that the function $F: \mathbb{R}^2 \to \mathbb{R}$ is Baire sup-measurable if and only if $F_f$ has a Baire property for every Borel function $f: \mathbb{R} \to \mathbb{R}$. (It is also true that $F: \mathbb{R}^2 \to \mathbb{R}$ is Baire sup-measurable provided $F_f$ has a Baire property for every Baire class one function $f: \mathbb{R} \to \mathbb{R}$, and that $F: \mathbb{R}^2 \to \mathbb{R}$ is sup-measurable provided $F_f$ is measurable for every continuous function $f: \mathbb{R} \to \mathbb{R}$. See for example [2, Lem. 1 and Rem. 1].)

The main theorem of the paper is the following.

**Theorem 2** It is consistent with the set theory ZFC that for every $A \subset 2^\omega \times 2^\omega$ for which the sets $A$ and $A^c = (2^\omega \times 2^\omega) \setminus A$ are nowhere meager in $2^\omega \times 2^\omega$ there is a homeomorphism $f$ from $2^\omega$ onto $2^\omega$ such that the set $\text{pr}(A \cap f)$ does not have the Baire property in $2^\omega$.

Before proving this theorem let us notice that it implies easily the following corollary.

**Corollary 3** The existence of Baire sup-measurable function $F: \mathbb{R}^2 \to \mathbb{R}$ without the Baire property is independent from the set theory ZFC.

**Proof.** Since under CH there are Baire sup-measurable functions without the Baire property it is enough to show that Theorem 2 implies consistency with ZFC that there are no such functions. For this we will work in the model from Theorem 2.

So, take an arbitrary $A \subset \mathbb{R}^2$ without the Baire property. By (iii) from Proposition 1 it is enough to show there exists a Baire class one function $f: \mathbb{R} \to \mathbb{R}$ for which the set $\text{pr}(A \cap f)$ does not have the Baire property.

We will first show this under the additional assumption that the sets $A$ and $\mathbb{R}^2 \setminus A$ are nowhere meager in $\mathbb{R}^2$. But then the set $A_0 = A \cap (\mathbb{R} \setminus \mathbb{Q})^2$ and its complement are nowhere meager in $(\mathbb{R} \setminus \mathbb{Q})^2$. Moreover, since $\mathbb{R} \setminus \mathbb{Q}$ is homeomorphic to $2^\omega \setminus E$ for some countable set $E$ (the set of all eventually constant functions in $2^\omega$) we can consider $A_0$ as a subset of $(2^\omega \setminus E)^2 \subset 2^\omega \times 2^\omega$. Then $A_0$ and its complement are still nowhere meager in $2^\omega \times 2^\omega$. Therefore, there exists an autohomeomorphism $f$ of $2^\omega$ such that the set $\text{pr}(A_0 \cap f) = \{x \in 2^\omega \setminus E : \langle x, f(x) \rangle \in A_0\}$ does not have the Baire property in $2^\omega$. Now, as before, $f \upharpoonright (2^\omega \setminus E)$ can be considered as defined on $\mathbb{R} \setminus \mathbb{Q}$. So if $\bar{f}: \mathbb{R} \to \mathbb{R}$ is an extension of $f \upharpoonright (2^\omega \setminus E)$ (under such identification) to $\mathbb{R}$ as a constant
on $\mathbb{Q}$ then $\tilde{f}$ is Borel and the set $\text{pr}(A_0 \cap \tilde{f})$ does not have the Baire property in $\mathbb{R}$.

Now, if $A$ is an arbitrary subset of $\mathbb{R}^2$ without the Baire property we can find non-empty open intervals $U$ and $W$ in $\mathbb{R}$ such that $A$ and $(U \times W) \setminus A$ are nowhere meager in $U \times W$. Since $U$ and $W$ are homeomorphic with $\mathbb{R}$ the above case implies the existence of Borel function $f_0: U \to W$ such that $\text{pr}(A \cap f_0)$ does not have the Baire property in $U$. So any Borel extension $f: \mathbb{R} \to \mathbb{R}$ of $f_0$ works.

2 Reduction of the proof of Theorem 2 to the main lemma

The idea of the proof is quite simple. For every nowhere meager $A \subset 2^\omega \times 2^\omega$ for which $A^c = (2^\omega \times 2^\omega) \setminus A$ is also nowhere meager we will find a natural ccc forcing notion $Q_A$ which adds the required homeomorphism $f$. Then we will start with the constructible universe $V = L$ and iterate with finite support these notions of forcing in such a way that every nowhere meager set $A^* \subset 2^\omega \times 2^\omega$, with $(2^\omega \times 2^\omega) \setminus A^*$ nowhere meager, will be taken care of by some $Q_A$ at an appropriate step of iteration.

There are two technical problems with carrying through this idea. First is that we cannot possible list in our iteration all nowhere meager subsets of $2^\omega \times 2^\omega$ with nowhere meager complements since the iteration can be of length at most continuum $\mathfrak{c}$ and there are $2^\mathfrak{c}$ many of such sets. This problem will be solved by defining our iteration as $P_{\omega_2} = \langle \langle P_\alpha, Q_\alpha \rangle : \alpha < \omega_2 \rangle$ such that the generic extension $V[G]$ of $V$ with respect to $P_{\omega_2}$ will satisfy $2^\omega = 2^{\omega_1} = \omega_2$ and have the property that

(m) every non-Baire subset $A^*$ of $2^\omega$ contains a non-Baire subset $A$ of cardinality $\omega_1$.

Thus in the iteration we will use only the forcing notions $Q_\alpha = Q_A$ for the sets $A$ of cardinality $\omega_1$, whose number is equal to $\omega_2$, the length of iteration. Condition (m) will guarantee that this will give us enough control of all nowhere meager subsets $A^*$ of $2^\omega \times 2^\omega$.

The second problem is that even if at some stage $\alpha < \omega_2$ of our iteration we will add a homeomorphism $f$ appropriate for a given set $A \subset 2^\omega \times 2^\omega$,
that is such that
\[ V[G_\alpha] \models \text{“pr}(A \cap f) \text{ is not Baire in } 2^\omega, \]
where \( G_\alpha = G \cap P_\alpha \), then in general there is no guarantee that the set \( \text{pr}(A \cap f) \) will remain non-Baire at the final model \( V[G] \). The preservation of non-Baireness of each appropriate set \( \text{pr}(A \cap f) \) will be achieved by careful crafting our iteration following a method known as the oracle-cc forcing iteration.

The theory of the oracle-cc forcings is well described in [10, Ch. IV] and here we will recall only the fragments that are relevant to our specific situation. In particular if \( \Gamma \) stands for the set of all limit ordinals less than \( \omega_1 \) then

- an \( \omega_1 \)-oracle is any sequence \( \mathcal{M} = \langle M_\delta : \delta \in \Gamma \rangle \) where \( M_\delta \) is a countable transitive model of ZFC\(^-\) (i.e., ZFC without the power set axiom) such that \( \delta \subset M_\delta, M_\delta \models \text{“}\delta \text{ is countable,”} \) and the set \( \{ \delta \in \Gamma : A \cap \delta \in M_\delta \} \) is stationary in \( \omega_1 \) for every \( A \subset \omega_1 \).

The existence of an \( \omega_1 \)-oracle is equivalent to the diamond principle \( \diamond \).

We will also need the following fact which, for our purposes, can be viewed as a definition of \( \mathcal{M} \)-cc property.

**Fact 4** If \( P \) is a forcing notion of cardinality \( \leq \omega_1 \), \( e : P \to \omega_1 \) is one-to-one, \( \mathcal{M} = \langle M_\delta : \delta \in \Gamma \rangle \) is an \( \omega_1 \)-oracle, and \( C \) is a closed unbounded subset of \( \omega_1 \) such that for every \( \delta \in \Gamma \cap C \)

\[
ed^{-1}(E) \text{ is predense in } P \text{ for every set } E \in M_\delta \cap \mathcal{P}(\delta) \text{ for which } \]
\[
ed^{-1}(E) \text{ is predense in } e^{-1}(\{ \gamma : \gamma < \delta \})
\]

then \( P \) has the \( \mathcal{M} \)-cc property.

This follows immediately from the definition of \( \mathcal{M} \)-cc property [11, Definition 1.5, p. 119] and [11, Claim 1.4(3), p. 118].

Our proof will rely on the following main lemma.

**Lemma 5** For every \( A \subset 2^\omega \times 2^\omega \) for which \( A \) and \( A^c = (2^\omega \times 2^\omega) \setminus A \) are nowhere meager in \( 2^\omega \times 2^\omega \) and for every \( \omega_1 \)-oracle \( \mathcal{M} \) there exists an \( \mathcal{M} \)-cc forcing notion \( Q_A \) of cardinality \( \omega_1 \) such that \( Q_A \) forces

\[ \text{there exists an autohomeomorphism } f \text{ of } 2^\omega \text{ such that the sets } \]
\[ \text{pr}(f \cap A) \text{ and } \text{pr}(f \setminus A) \text{ are nowhere meager in } 2^\omega. \]
The proof of Lemma 5 represents the core of our argument and will be presented in the next section. In the reminder of this section we will sketch how Lemma 5 implies Theorem 2. Since this follows the standard path, as described in [10], the experts familiar with this treatment may proceed directly to the next section.

Now, the iteration \( P_{\omega_2} \) is defined by choosing by induction the sequence \( \langle \langle P_\alpha, \dot{A}_\alpha, \dot{M}_\alpha, \dot{Q}_\alpha, \dot{f}_\alpha \rangle : \alpha < \omega_2 \rangle \) such that for every \( \alpha < \omega_2 \)

- \( P_\alpha = \langle \langle P_\beta, \dot{Q}_\beta \rangle : \beta < \alpha \rangle \) is a finite support iteration,
- \( \dot{A}_\alpha \) is a \( P_\alpha \)-name and for every \( \beta \leq \alpha \)
  \[ P_\alpha \models \text{``} \dot{A}_\beta \text{ and } (\dot{A}_\beta)^c \text{ are nowhere meager subsets of } 2^\omega \times 2^\omega,\text{''} \]
- \( \dot{M}_\alpha \) is a \( P_\alpha \)-name such that \( P_\alpha \) forces
  \[ \dot{M}_\alpha \text{ is an } \omega_1\text{-oracle and for every } \beta < \alpha \text{ if } Q \text{ satisfies } \dot{M}_\alpha\text{-cc then} \]
  \[ Q \models \text{``} \text{pr}(\dot{f}_\beta \cap \dot{A}_\beta), \text{pr}(\dot{f}_\beta \setminus \dot{A}_\beta) \subset 2^\omega \text{ are nowhere meager in } 2^\omega,\text{''} \]
- \( \dot{Q}_\alpha \) is a \( P_\alpha \)-name for a forcing such that \( P_\alpha \) forces
  \[ \dot{Q}_\alpha \text{ is an } \dot{M}_\alpha\text{-cc forcing } Q_{\dot{A}_\alpha} \text{ from Lemma 5} \]
- \( \dot{f}_\alpha \) is a \( P_{\alpha+1} \)-name for which \( P_{\alpha+1} \) forces that
  \( \dot{f}_\alpha \) is a \( \dot{Q}_\alpha \)-name for the function \( f \) from Lemma 5.

The existence of appropriate \( \omega_1 \)-oracles and the fact that each \( P_\alpha \) obtained that way preserves nowhere meagerness (i.e., non-meagerness of their traces on all basic open sets) of all the projections \( \text{pr}(\dot{f}_\beta \cap \dot{A}_\beta) \) and \( \text{pr}(\dot{f}_\beta \setminus \dot{A}_\beta) \) for \( \beta < \alpha \) follows from Example 2.2 and results from section 3 of [10, ch. IV]. Also, Claim 3.2 from [10, ch. IV] implies that all sets \( \text{pr}(\dot{f}_\alpha \cap \dot{A}_\alpha) \) and \( \text{pr}(\dot{f}_\alpha \setminus \dot{A}_\alpha) \) remain nowhere meager in \( 2^\omega \) in the final model \( V[G] \). Thus it is enough to ensure that each nowhere meager subset \( A^* \) of \( 2^\omega \times 2^\omega \) from \( V[G] \) with nowhere meager complement contains an interpretation of some \( \dot{A}_\alpha \). However, a choice of \( \dot{A}_\alpha \)’s which guarantee this can be made with a help of the diamond principle \( \diamondsuit_{\omega_2} \) and the fact that the set

\[ \{ \alpha < \omega_2 : \text{cf}(\alpha) \neq \omega_1 \text{ or } V[G_\alpha] \models A \cap V[G_\alpha] \text{ is nowhere meager in } 2^\omega \times 2^\omega \} \]

contains a closed unbounded set. (Compare [10, Claim 4.4, p. 130].)
3 Proof of Lemma 5

Let $\mathcal{K}$ be the family of all sequences $\bar{h} = \langle h_\xi : \xi \in \Gamma \rangle$ such that each $h_\xi$ is a function from a countable set $D_\xi \subset 2^\omega$ onto $R_\xi \subset 2^\omega$ and that

$$D_\xi \cap D_\eta = R_\xi \cap R_\eta = \emptyset \text{ for every distinct } \xi, \eta \in \Gamma.$$ 

For each $\bar{h} \in \mathcal{K}$ we will define a forcing notion $Q_{\bar{h}}$. Forcing $Q_A$ satisfying Lemma 5 will be chosen as $Q_{\bar{h}}$ for some $\bar{h} \in \mathcal{K}$.

So fix an $\bar{h} \in \mathcal{K}$. Then $Q_{\bar{h}}$ is defined as the set of all triples $p = \langle n, \pi, h \rangle$ for which

(A) $h$ is a function from a finite subset $D$ of $\bigcup_{\xi \in \Gamma} D_\xi$ into $2^\omega$;

(B) $n < \omega$ and $\pi$ is a permutation of $2^n$;

(C) $|D \cap D_\xi| \leq 1$ for every $\xi \in \Gamma$;

(D) if $x \in D \cap D_\xi$ then $h(x) = h_\xi(x)$ and $h(x) \upharpoonright n = \pi(x \upharpoonright n)$.

Forcing $Q_{\bar{h}}$ is ordered as follows. Condition $p' = \langle n', \pi', h' \rangle$ is stronger than $p = \langle n, \pi, h \rangle$, $p' \preceq p$, provided

$$n \leq n', \ h \subset h', \text{ and } \pi'(\eta) \upharpoonright n = \pi(\eta \upharpoonright n) \text{ for every } \eta \in 2^{n'}.$$  \hspace{1cm} (2)

In the reminder of this paper we will write $[\eta]$ for the basic open neighborhood in $2^\omega$ generated by $\eta \in 2^{<\omega}$, that is,

$$[\eta] = \{x \in 2^\omega : \eta \subset x\}.$$ 

Note that using this notation the second part of the condition (D) says that for every $x \in D$ and $\eta \in 2^n$

$$x \in [\eta] \text{ if and only if } h(x) \in [\pi(\eta)].$$  \hspace{1cm} (3)

Also, if $n < \omega$ we will write $[\eta] \upharpoonright 2^n$ for $\{x \upharpoonright 2^n : x \in [\eta]\}$. Note that in this notation the part of (4) concerning permutations says that $\pi'$ expands $\pi$ in a sense that $\pi'$ maps $[\zeta] \upharpoonright 2^{n'}$ onto $[\pi(\zeta)] \upharpoonright 2^{n'}$ for every $\zeta \in 2^n$.

In what follows we will use the following basic property of $Q_{\bar{h}}$.

($\ast$) For every $q = \langle n, \pi, h \rangle \in Q_{\bar{h}}$ and $m < \omega$ there exist an $n' \geq m$ and a permutation $\pi'$ of $2^{n'}$ such that $q' = \langle n', \pi', h \rangle \in Q_{\bar{h}}$ and $q'$ extends $q$.  \hspace{1cm} (4)
The choice of such \( n' \) and \( \pi' \) is easy. First pick \( n' \geq \max\{m, n\} \) such that \( x \upharpoonright n' \neq y \upharpoonright n' \) for every different \( x \) and \( y \) from either domain \( D \) or range \( R = h[D] \) of \( h \). This implies that that for every \( \zeta \in 2^n \) the set \( D_\zeta = \{ x \upharpoonright n' : x \in D \cap [\zeta] \} \subset [\zeta] \upharpoonright 2^{n'} \) has the same cardinality that \( D \cap [\zeta] \) and \( H_\zeta = \{ x \upharpoonright n' : x \in h[D] \cap [\pi(\zeta)] \} \subset [\pi(\zeta)] \upharpoonright 2^{n'} \) has the same cardinality that \( h[D] \cap [\pi(\zeta)] \). Since, by (3), we have also \( |D \cap [\zeta]| = |h[D] \cap [\pi(\zeta)]| \) we see that \( |D_\zeta| = |H_\zeta| \). Define \( \pi' \) on \( D_\zeta \) by \( \pi'(x \upharpoonright n') = h(x) \upharpoonright n' \) for every \( x \in D_\zeta \). Then \( \pi' \) is a bijection from \( D_\zeta \) onto \( H_\zeta \) and this definition ensures that an appropriate part of the condition (D) for \( h \) and \( \pi' \) is satisfied. Also, if for each \( \zeta \in 2^n \) we extend \( \pi' \) onto \( [\zeta] \upharpoonright 2^{n'} \) as a bijection from \( ([\zeta] \upharpoonright 2^{n'}) \setminus D_\zeta \) onto \( ([\pi(\zeta)] \upharpoonright 2^{n'}) \setminus H_\zeta \), then the condition (3) will be satisfied. Thus such defined \( q' = \langle n', \pi', h \rangle \) belongs to \( Q_h \) and extends \( q \).

Next note that forcing \( Q_h \) has the following properties needed to prove Lemma 6. In what follows we will consider \( 2^\omega \) with the standard distance:

\[
d(r_0, r_1) = 2^{-\min\{\eta < \omega : r_0(\eta) \neq r_1(\eta)\}}
\]

for different \( r_0, r_1 \in 2^\omega \).

**Fact 6** Let \( \tilde{h} = \langle h_\xi : \xi \in \Gamma \rangle \in \mathcal{K} \) and \( f = \bigcup \{ h : \langle n, \pi, h \rangle \in H \} \), where \( H \) is a \( V \)-generic filter over \( Q_{\tilde{h}} \). Then \( f \) is a uniformly continuous one-to-one function from a subset \( D \) of \( 2^\omega \) into \( 2^\omega \). Moreover, if for every \( \xi \in \Gamma \) the graph of \( h_\xi \) is dense in \( 2^\omega \times 2^\omega \) then \( D \) and \( f[D] \) are dense in \( 2^\omega \) and \( f \) can be uniquely extended to an autohomeomorphism \( \tilde{f} \) of \( 2^\omega \).

**Proof.** Clearly \( f \) is a one-to-one function from a subset \( D \) of \( 2^\omega \) into \( 2^\omega \). To see that it is uniformly continuous choose an \( \varepsilon > 0 \). We will find \( \delta > 0 \) such that \( r_0, r_1 \in D \) and \( d(r_0, r_1) < \delta \) imply \( d(f(r_0), f(r_1)) < \varepsilon \). For this note that, by (\ast), the set

\[
S = \{ q = \langle n, \pi, h \rangle \in Q_{\tilde{h}} : 2^{-n} < \varepsilon \}
\]

is dense in \( Q_{\tilde{h}} \). So take \( q = \langle n, \pi, h \rangle \in H \cap S \) and put \( \delta = 2^{-n} \). We claim that this \( \delta \) works.

Indeed, take \( r_0, r_1 \in D \) with \( d(r_0, r_1) < \delta \). Then there is \( q' = \langle n', \pi', h' \rangle \in H \) stronger than \( q \) such that \( r_0 \) and \( r_1 \) are in the domain of \( h' \). Therefore, \( n \leq n' \) and for \( j < 2 \)

\[
f(r_j) \upharpoonright n = h'(r_j) \upharpoonright n = (h'(r_j) \upharpoonright n') \upharpoonotrseq n = \pi'(r_j \upharpoonright n') \upharpoonotrseq n = \pi(r_j \upharpoonright n)
\]
by conditions (D) and $[2]$. Since $d(r_0, r_1) < \delta = 2^{-n}$ implies $r_0 \upharpoonright n = r_1 \upharpoonright n$ we obtain

$$f(r_0) \upharpoonright n = \pi(r_0 \upharpoonright n) = \pi(r_1 \upharpoonright n) = f(r_1) \upharpoonright n$$

that is, $d(f(r_0), f(r_1)) < 2^{-n} < \varepsilon$. So $f$ is uniformly continuous.

Essentially the same argument (with the same values of $\varepsilon$ and $\delta$) shows that $f^{-1}: f[D] \to D$ is uniformly continuous. Thus, if $\bar{f}$ is the unique continuous extension of $f$ into $\text{cl}(D)$ then $\bar{f}$ is a homeomorphism from $\text{cl}(D)$ onto $\text{cl}f[D]$.

To finish the argument assume that all functions $h_\xi$ have dense graphs, take an $\eta \in 2^m$ for some $m < \omega$, and notice that the set

$$S_\eta = \{ q = \langle n, \pi, h \rangle \in Q_h : \text{the domain } D' \text{ of } h \text{ intersects } [\eta] \}$$

is dense in $Q_h$. Indeed, if $q = \langle n, \pi, h \rangle \in Q_h$ then, by ($\ast$), strengthening $q$ if necessary, we can assume that $m \leq n$. Then, refining $\eta$ if necessary, we can also assume that $m = n$, that is, that $\eta$ is in the domain of $\pi$. Now, if $[\eta]$ intersects the domain of $h$ then already $q$ belongs to $S_\eta$. Otherwise take $\xi \in \Gamma$ with $D' \cap D_\xi = \emptyset$ and pick $\langle x, h_\xi(x) \rangle \in [\eta] \times [\pi(\eta)]$, which exists by the density of the graph of $h_\xi$. Then $\langle n, \pi, h \cup \{ \langle x, h_\xi(x) \rangle \} \rangle$ belongs to $S_\eta$ and extends $q$.

This shows that $D \cap [\eta] \neq \emptyset$ for every $\eta \in 2^{<\omega}$, that is, $D$ is dense in $2^\omega$.

The similar argument shows that for every $\eta \in 2^{<\omega}$ the set

$$S^\eta = \{ q = \langle n, \pi, h \rangle \in Q_h : \text{the range of } h \text{ intersects } [\eta] \}$$

is dense in $Q_h$, which implies that $h[D]$ is dense in $2^\omega$. Thus $\bar{f}$ is a homeomorphism from $\text{cl}(D) = 2^\omega$ onto $\text{cl}h[D] = 2^\omega$.

Now take $A \subset 2^\omega \times 2^\omega$ for which $A$ and $A^c = (2^\omega \times 2^\omega) \setminus A$ are nowhere meager in $2^\omega \times 2^\omega$ and fix an $\omega_1$-oracle $\mathcal{M} = \langle M_\delta : \delta \in \Gamma \rangle$. By Fact $[3]$ in order to prove Lemma $[3]$ it is enough to find an $\bar{h} = \langle h_\xi : \xi \in \Gamma \rangle \in \mathcal{K}$ such that

$$Q_A = Q_h \text{ is } \mathcal{M}\text{-cc}$$

and $Q_h$ forces that, in $V[H]$,

the sets $\text{pr}(f \cap A)$ and $\text{pr}(f \setminus A)$ are nowhere meager in $2^\omega$.  \hspace{1cm} (5)

(In (5) function $f$ is defined as in Fact $[3]$.)
To define $\bar{h}$ we will construct a sequence $\langle \langle x_\alpha, y_\alpha \rangle \in 2^\omega \times 2^\omega : \alpha \in \omega \rangle$ aiming for $h_\xi = \{ \langle x_{\xi+n}, y_{\xi+n} \rangle : n < \omega \}$, where $\xi \in \Gamma$.

Let $\{ \eta_n, \zeta_n : n < \omega \}$ be an enumeration of $2^{<\omega} \times 2^{<\omega}$. Points $\langle x_{\xi+n}, y_{\xi+n} \rangle$ are chosen inductively in such a way that

(i) $\langle x_{\xi+n}, y_{\xi+n} \rangle$ is a Cohen generic number over $M_\delta[\langle \langle x_\alpha, y_\alpha \rangle : \alpha < \xi + n \rangle]$ for every $\delta \leq \xi, \delta \in \Gamma$, that is, $\langle x_{\xi+n}, y_{\xi+n} \rangle$ is outside of all meager subsets of $2^\omega \times 2^\omega$ which are coded in $M_\delta[\langle \langle x_\alpha, y_\alpha \rangle : \alpha < \xi + n \rangle]$;

(ii) $\langle x_{\xi+n}, y_{\xi+n} \rangle \in A$ if $n$ is even, and $\langle x_{\xi+n}, y_{\xi+n} \rangle \in A^c$ otherwise.

(iii) $\langle x_{\xi+n}, y_{\xi+n} \rangle \in [\eta_n] \times [\zeta_n]$.

The choice of $\langle x_{\xi+n}, y_{\xi+n} \rangle$ is possible since both sets $A$ and $A^c$ are nowhere meager, and we consider each time only countably many meager sets. Condition (iii) guarantees that the graph of each of $h_\xi$ will be dense in $2^\omega \times 2^\omega$.

Note that if $\Gamma \ni \delta \leq \alpha_0 < \cdots < \alpha_{k-1}$, where $k < \omega$, then (by the product lemma in $M_\delta$)

\[ \langle \langle x_\alpha, y_\alpha \rangle : i < k \rangle \text{ is an } M_\delta\text{-generic Cohen number in } (2^\omega)^k. \] (6)

For $q = \langle n, \pi, h \rangle \in Q_{\bar{h}}$ define

\[ \hat{q} = \bigcup_{\langle \xi, \zeta \rangle \in \pi} [\xi] \times [\zeta]. \]

Clearly $\hat{q}$ is an open subset of $2^\omega \times 2^\omega$ and condition (2) implies that for every $q, r \in Q_{\bar{h}}$ with $r = \langle n', \pi', h' \rangle$

if $q \leq r$ then $\hat{q} \subset \hat{r}$ and $\hat{q} \cap ([\xi] \times [\zeta]) \neq \emptyset$ for every $\langle \xi, \zeta \rangle \in \pi'$.

(7)

Also for $\delta \in \Gamma$ let $(Q_h)^\delta = \{ \langle n, \pi, h \rangle \in Q_h : h \in \bigcup_{\xi \leq \delta} h_\xi \}$. To prove (4) and (5) we will use also the following fact.

**Fact 7** Let $\delta \in \Gamma$ be such that $M_\delta$ contains $(Q_h)^\delta$ and let $E \in M_\delta$ be, in $M_\delta$, a predense subset of $(Q_h)^\delta$. Then for every $k < \omega$ and $p = \langle n, \pi, h \rangle \in (Q_h)^\delta$ the set

\[ B^k_p = \bigcup \{ (\hat{q})^k : q \text{ extends } p \text{ and some } q_0 \in E \} \]

is dense in $(\hat{p})^k \subset (2^\omega \times 2^\omega)^k$. (8)
Proof. By way of contradiction assume that $B_p^k$ is not dense in $(\hat{p})^k$. Then there are $t < \omega$ and $\xi_0, \xi_1, \ldots, \xi_{k-1} \in 2^t$ such that $P = \prod_{i<k}(\{\xi_i\} \times \{\zeta_i\}) \subset (\hat{p})^k$ is disjoint with $B_p^k$. Increasing $t$ and refining $\xi_i$'s and $\zeta_j$'s, if necessary, we may assume that $t \geq n$, all $\xi_i$'s and $\zeta_j$'s are different, $\bigcup_{i<k}\{\xi_i\}$ is disjoint from the domain $D$ of $h$, and $h[D] \cap \bigcup_{i<k}\{\zeta_i\} = \emptyset$. We can also assume that $x \upharpoonright t \neq y \upharpoonright t$ for every different $x$ and $y$ from $D$ and from $h[D]$. Now, refining slightly the argument for (§) we can find $r = (t, \pi', h) \in (Q_h)^{\delta}$ extending $p$ such that $\pi'(\xi_i) = \zeta_i$ for every $i < k$. (Note that $P \subset (\hat{p})^k$.) We will obtain a contradiction with predensity of $E$ in $(Q_h)^{\delta}$ by showing that $r$ is incompatible with every element of $E$.

Indeed if $q$ were an extension of $r \leq p$ and an element $q_0$ of $E$ then we would have $(\hat{q})^k \subset B_p^k$. But then, by (4) and the fact that $\langle \xi_i, \zeta_i \rangle \in \pi'$ for $i < k$, we would also have $(\hat{q})^k \cap P \neq \emptyset$, contradicting $P \cap B_p^m = \emptyset$. This finishes the proof of Fact 4.

Now we are ready to prove (4), that is, that $Q_h$ is $\mathcal{M}$-cc. So, fix a bijection $e: Q_h \to \omega_1$ and let

$$C = \{\delta \in \Gamma: (Q_h)^{\delta} = e^{-1}(\delta)\}.$$

Then $C$ is a closed unbounded subset of $\omega_1$. Take a $\delta \in C$ for which $(Q_h)^{\delta} = e^{-1}(\delta) \in M_\delta$ and fix an $E \subset \delta$, $E \in M_\delta$, for which $e^{-1}(E)$ is predense in $(Q_h)^{\delta}$. By Fact 4 it is enough to show that

$$e^{-1}(E)$$

is predense in $Q_h$.

Take $p_0 = \langle n, \pi, h_0 \rangle$ from $Q_h$, let $h = h_0 \upharpoonright \bigcup_{\eta<\delta} D_\eta$ and $h_1 = h_0 \setminus h$, and notice that the condition $p = \langle n, \pi, h \rangle$ belongs to $(Q_h)^{\delta}$. Assume that $h_1 = \{\langle x_i, y_i \rangle: i < k\}$. Since $s(h_1) = \langle \langle x_i, y_i \rangle: i < k \rangle \in (\hat{p})^k$ and, by Fact 4, $B_p^k$ is dense in $(\hat{p})^k$ condition (4) implies that $s(h_1) \in B_p^k$. So there is $q = \langle n_0, \pi_0, g \rangle \in (Q_h)^{\delta}$ extending $p$ and some $q_0 \in e^{-1}(E)$ for which $s(h_1) \in q^m$. But then $p' = \langle n_0, \pi_0, g \cup h_1 \rangle$ belongs to $Q_h$ and extends $q$. This finishes the proof of (3).

The proof of (3) is similar. We will prove only that $\text{pr}(f \setminus A) = \text{pr}(f \cap A^c)$ is nowhere meager in $2^\omega$, the argument for $\text{pr}(f \cap A)$ being essentially the same.

By way of contradiction assume that $\text{pr}(f \setminus A)$ is not nowhere meager in $2^\omega$. So there is an $\eta \in 2^{<\omega}$ such that $\text{pr}(f \setminus A)$ is meager in $[\eta]$. Thus, there is a sequence $\langle \hat{U}_m: m < \omega \rangle$ of $Q_h$-names for which $Q_h$ forces

$$\text{each } \hat{U}_m \text{ is an open dense subset of } [\eta] \text{ and } \text{pr}(f \cap A) \cap \bigcap_{m<\omega} \hat{U}_m = \emptyset.$$
Moreover, since $Q_h$ is ccc (as every $\mathcal{M}$-cc forcing is ccc), we can also assume that there exists a $\delta_0 \in \Gamma$ such that $\langle \hat{U}_m : m < \omega \rangle \in M_{\delta_0}$.

Now, by the definition of $\omega_1$-oracle, the set

$$B_0 = \{ \delta \in \Gamma : \langle \hat{U}_m : m < \omega \rangle \in M_\delta \land M_\delta \cap Q_h = (Q_h)^\delta \in M_\delta \}$$

is stationary in $\omega_1$. Thus, we can find, in $V[H]$, a $\delta \in B_0$ and an odd $j < \omega$ such that $x_{\delta+j} \in [\eta] \cap \text{pr}(f)$. Recall that $\langle x_{\delta+j}, y_{\delta+j} \rangle \in A^c$ for odd $j$’s. Therefore $x_{\delta+j} \in \text{pr}(f \setminus A)$. Let $p_0 = \langle n_0, \pi_0, h_0 \rangle \in H$ be such that $p_0 \forces \langle x_{\delta+j} \in [\eta] \cap \text{pr}(f) \rangle$. We can also assume that $x_{\delta+j}$ belongs to the domain of $h_0$. We will show that

$$p_0 \forces x_{\delta+j} \in \bigcap_{m<\omega} \hat{U}_m,$$

which will finish the proof.

So, assume that this is not the case. Then there exist an $i < \omega$ and $p_1 = \langle n, \pi, h_1 \rangle \in Q_h$ stronger than $p_0$ such that $p_1 \forces \langle x_{\delta+j} \notin \hat{U}_i \rangle$. Let $h = h_1 \upharpoonright \bigcup_{n<\delta} D_n$ and $h_1 \setminus h = \{ \langle a_t, b_t \rangle : t < m \}$. Notice that $p = \langle n, \pi, h \rangle$ belongs to $(Q_h)^\delta$. We can also assume that $\langle x_{\delta+j}, y_{\delta+j} \rangle = \langle a_0, b_0 \rangle$.

Now consider the set

$$Z = \{ q = \langle n^q, \pi^q, h^q \rangle \in (Q_h)^\delta : q \leq p \land (\exists \langle \xi, \zeta \rangle \in \pi^q) q \forces \langle [\xi] \subset \hat{U}_i \rangle \}$$

and note that it belongs to $M_\delta$. Notice also that it is dense below $p$ in $(Q_h)^\delta$. This is the case since any persistent information on $\hat{U}_i$ depends only on conditions from $(Q_h)^\delta$. Thus, by Fact 7 the set

$$B^m_p = \bigcup \{ (\hat{q})^m : q \text{ extends some } q_0 \in Z \} \in M_\delta$$

is dense in $(\hat{p})^m$. So, by 8, $\langle \langle a_t, b_t \rangle : t < m \rangle \in B^m_p$ since $\langle \langle a_t, b_t \rangle : t < m \rangle$ belongs to $(\hat{p})^m$. But this means that there exist $q = \langle n^q, \pi^q, h^q \rangle \in Z$ and $\langle \xi, \zeta \rangle \in \pi^q$ with $q \forces \langle [\xi] \subset \hat{U}_i \rangle$ for which $\langle \langle a_t, b_t \rangle : t < m \rangle \in (\hat{q})^m$. A slight modification of the proof of Fact 7 lets us also to choose $q$ such that $x_{\delta+j} = a_0 \in [\xi]$. But then $p_2 = \langle n^q, \pi^q, h^q \cup \{ \langle a_t, b_t \rangle : t < m \} \rangle$ belongs to $Q_h$ and extends $p_1$. So, $p_2$ forces that $x_{\delta+j} = a_0 \in [\xi] \subset \hat{U}_i$ contradicting our assumption that $p_1 \forces \langle x_{\delta+j} \notin \hat{U}_i \rangle$.

This finishes the proof of 8 and of Lemma 3.
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