A NOTE ON NON-ORDINARY PRIMES

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Abstract. Suppose that $O_L$ is the ring of integers of a number field $L$, and suppose that

$$f(z) = \sum_{n=1}^{\infty} a_f(n)q^n \in S_k \cap O_L[[q]]$$

(note: $q := e^{2\pi i z}$) is a normalized Hecke eigenform for $SL_2(\mathbb{Z})$. We say that $f$ is non-ordinary at a prime $p$ if there is a prime ideal $\mathfrak{p} \subset O_L$ above $p$ for which

$$a_f(\mathfrak{p}) \equiv 0 \pmod{p}.$$ 

For any finite set of primes $S$, we prove that there are normalized Hecke eigenforms which are non-ordinary for each $p \in S$. The proof is elementary and follows from a generalization of work of Choie, Kohnen and the third author.

1. Introduction and statement of results

If $k \geq 4$ is even, then let $M_k$ (resp. $S_k$) denote the finite dimensional $\mathbb{C}$-vector space of weight $k$ holomorphic modular forms (resp. cusp forms) on $SL_2(\mathbb{Z})$. Furthermore, let $M'_k$ denote the infinite dimensional space of weakly holomorphic modular forms of weight $k$ with respect to $SL_2(\mathbb{Z})$. Recall that a meromorphic modular form is weakly holomorphic if its poles (if any) are supported at cusps. We shall identify a modular form on $SL_2(\mathbb{Z})$ by its Fourier expansion at infinity

$$f(z) = \sum_{n \gg -\infty} a_f(n)q^n,$$

where $q := e^{2\pi i z}$.

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Very little is known about the distribution of non-ordinary primes. We recall the following well-known open problem (see Gouvêa’s expository article [2]).

Problem. Are there infinitely many non-ordinary primes for a generic normalized Hecke eigenform $f(z)$?

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We do not solve this problem here. It remains open. However, we establish the following related result.

**Theorem 1.1.** If $S$ is a finite set of primes, then there are infinitely many normalized Hecke eigenforms for $\text{SL}_2(\mathbb{Z})$ which are non-ordinary for each $p \in S$.

**Remark.** The proof of Theorem 1.1 relies on a general theorem about the Fourier coefficients of weakly holomorphic modular forms modulo $p$ (see Theorem 2.5). For normalized Hecke eigenforms, this general result incorporates classical results of Hatada [3] (in the case where $p = 2$ and 3) and Hida [4–6] (for primes $p \geq 5$) on non-ordinary primes.

**Remark.** The proof of Theorem 1.1 is constructive. Suppose that $S = \{p_1, p_2, \ldots, p_m\}$ is a finite set of primes. Suppose that $k \geq 12$ is an even integer. If for each $p \in S$ there is a choice of $t \in A = \{4, 6, 8, 10, 14\}$ for which $(p - 1)|(k - t)$, then every prime in $S$ is non-ordinary for every normalized Hecke eigenform $f \in S_k$. The earlier work of Choie, Kohnen and the third author [1] is eclipsed by this result thanks to the flexibility in the choice of $t$ above.

In Section 2 we recall certain facts about modular forms and we prove Theorem 2.5. The proof is elementary. In Section 3 we obtain Theorem 1.1 as a simple consequence when $p \geq 5$, combining with the known result on $p = 2, 3$, and in Section 4 we offer some numerical examples.

2. Preliminaries

2.1. Nuts and bolts. As usual, let $\Delta(z) \in S_{12}$ be the cusp form

$$\Delta(z) := q \prod_{n=1}^{\infty} (1 - q^n)^{24} = q - 24q^2 + \ldots,$$

and, for even $k \geq 4$, let $E_k(z) \in M_k$ be the normalized Eisenstein series

$$E_k(z) := 1 - \frac{2k}{B_k} \sum_{n=1}^{\infty} \left( \sum_{1 \leq d|n} d^{k-1} \right) q^n,$$

where the rational numbers $B_k$ are the usual Bernoulli numbers given by the generating function

$$\sum_{k=0}^{\infty} B_k \cdot \frac{t^k}{k!} = \frac{t}{e^t - 1} = 1 - \frac{1}{2} t + \frac{1}{12} t^2 - \ldots.$$

For convenience, we let $E_0(z) := 1$. Finally, we let $j(z)$ be the usual modular function

$$j(z) := \frac{E_4(z)^3}{\Delta(z)} = q^{-1} + 744 + 196884q + \ldots.$$

Finally, for convenience, if $k \in 2\mathbb{Z}$, then throughout we define $\delta(k) \in \{0, 4, 6, 8, 10, 14\}$ so that

$$\delta(k) \equiv k \pmod{12}.$$

In the proof, we need the following propositions.

**Proposition 2.1.** A normalized Hecke eigenform is non-ordinary at $p$ if there is an $m \geq 1$ such that $a_f(p^m) \equiv 0 \pmod{p}$. 

Proof. This follows from the fact that $T_p f(z) = a_f(p) f(z)$ for every prime $p$ when $f(z)$ is a normalized Hecke eigenform of weight $k$. Here $T_p$ is the $p$-th Hecke operator. In particular, on prime power exponents, we have

$$a_f(p)a_f(p^n) = a_f(p^{n+1}) + p^{k-1}a_f(p^{n-1}) \equiv a_f(p^{n+1}) \pmod{p}$$

for every non-negative integer $n$. By induction, we find that

$$a_f(p^n) \equiv a_f(p)^n \pmod{p}.$$ 

This proves the proposition. \[\square\]

The following well-known propositions play a central role in the proof of Theorem 2.5.

Proposition 2.2. If $p \geq 5$ is prime, then as a $q$-series, $E_{p-1}(z) \equiv 1 \pmod{p}$.

Proof. This can be found on page 38 of \[7\]. \[\square\]

Proposition 2.3. If $f(z) = \sum_{n \gg -\infty} a_f(n) q^n \in M_k^1$, then $a_f(0) = 0$.

Proof. By a simple generalization of Lemma 2.34 of \[7\], it is known that every weakly holomorphic modular form $h(z)$ of weight 2 may be represented as $P(j(z))E_{14}(z)\Delta(z)^{-1}$, where $P(x)$ is a polynomial of $x$. Dropping the dependence on $z$ for convenience, we have the following well-known identities:

$$-\frac{1}{2\pi i} \frac{d}{dz} j = \frac{E_{14}}{\Delta},$$

$$j^w \frac{d}{dz} j = \frac{1}{w+1} \frac{d}{dz} j^{w+1},$$

where $w \in \mathbb{Z}_{\geq 0}$. Therefore, it follows that $h$ is the derivative of a polynomial in $j$, and so its constant term in the Fourier expansion is zero. \[\square\]

Remark. For more standard facts about modular forms the reader may see \[7\].

2.2. Our main technical result. In 2005 Choie, Kohlen and the third author proved the following (see Corollary 1.3 of \[1\]). This result recovered earlier aforementioned results of Hatada and Hida.

Theorem 2.4. Let $p$ be a prime, and suppose that $f(z) = \sum_{n=1}^{\infty} a_f(n) q^n \in S_k$ is a normalized Hecke eigenform. Let $L_f$ be the number field generated by the coefficients of $f(z)$, and let $p \in O_{L_f}$ be any prime ideal above $p$.

(1) If $p = 2, 3$, then

$$a_f(p) \equiv 0 \pmod{p}.$$ 

(2) If $p \geq 5$, $\delta(k) \in \{4, 6, 8, 10, 14\}$ and $k \equiv \delta(k) \pmod{p-1}$, then

$$a_f(p) \equiv 0 \pmod{p}.$$ 

Here we strengthen this result for primes $p \geq 5$ by extending it to all $k$ without any condition on $\delta(k)$.

Theorem 2.5. Let $p \geq 5$ be prime, and suppose that $f(z) = \sum_{n=1}^{\infty} a_f(n) q^n \in M_k^1 \cap O_L[[q]]$, where $k \in 2\mathbb{Z}$ and $O_L$ is the ring of algebraic integers of a number field $L$. 


(1) Suppose that \( a \geq 0 \) and \( m \in A = \{4,6,8,10,14\} \) are integers for which
\[ k - 2 \leq (m - 2)p^a. \]
If \( \text{ord}_\infty(f) > -p^a \) and \((p - 1)|(k - m)\), then for any integer \( b \geq a \), we have
\[ a_f(p^b) \equiv -\frac{2m}{B_m}a_f(0) \pmod{p}. \]

(2) Suppose that \( k \leq 2 \), \( r, s \in \mathbb{Z}_{\geq 0} \) and \( t, u \in \mathbb{Z}_{>0} \) are integers for which
\[ 2 - k = r(p - 1) + sp^t, \]
where \( s \neq 2 \). If \( \text{ord}_\infty(f) > -p^v, u \leq t \), then for any integer \( v \) such that \( u \leq v \leq t \), we have
\[ a_f(p^v) \equiv a_f(0) \equiv 0 \pmod{p}. \]

**Proof.** The proofs in both cases begin with the construction of suitable weakly holomorphic modular forms of weight \( 2 - k \). The product of such forms with \( f \) have weight \( 2 \), and so Proposition 2.3 implies that their constant terms vanish.

For case (1), first note that \((k - 2) - (m - 2)p^b \equiv k - m \pmod{p - 1}\). As we have \((p - 1)|(k - m)\) and \( k - 2 \leq (m - 2)p^b \), we may find a non-negative integer \( c \) such that
\[ 2 - k = c(p - 1) - (m - 2)p^b. \]

Let \( g_m \) be the function
\[
g_m := j \frac{E_6^{(1+i^m)/2}}{E_4^{(m+1+3i^m)/4}} = \begin{cases} 
  j \frac{E_6}{E_4} & \text{for } m = 4 \\
  j \frac{1}{E_4} & \text{for } m = 6 \\
  j \frac{E_6}{E_4} & \text{for } m = 8 \in M_{2-m}^1, \\
  j \frac{1}{E_4} & \text{for } m = 10 \\
  j \frac{1}{E_4} & \text{for } m = 14 
\end{cases} \]

Then we have
\[ g_m^b E_{p-1}^c \in M_{2-k}. \]
That is to say, the constant term of \( g_m^b E_{p-1}^c f \) is zero. From Proposition 2.2, we know that
\[ E_{p-1} \equiv 1 \pmod{p}. \]

Then we have that the constant term of \( g_m^b f \) is zero modulo \( p \). By using Fermat’s little theorem to compute the multinomials, we get
\[
g_m^b f = (q^{-1} + 744 + O(q))p^b (1 - 504q + O(q^2))^{\frac{x^b(1+i^m)}{2}} (1 + (-240)q + O(q^2))^{x^b(m+1+3i^m)} f \equiv (q^{-p^b} + 744 + O(q^{p^b}))(1 - 252(1 + i^m)q^{p^b} + O(q^{2p^b})) (1 - 60(m + 1 + 3i^m)q^{p^b} + O(q^{2p^b})) \sum_{n \gg -\infty} a_f(n)q^n \equiv (q^{-p^b} + 432 - 60m - 432i^m + O(p^b)) \sum_{n \gg -\infty} a_f(n)q^n \pmod{p}. \]

We already know that \( \text{ord}_\infty(f) > -p^a \geq -p^b \), so we know that the constant term \( c_{m,p} \) of \( g_m^b f \) must satisfy the congruence
\[ c_{m,p} \equiv a_f(p^b) + (432 - 60m - 432i^m)a_f(0) \pmod{p}. \]
As $c_{m,p}$ is known to be zero modulo $p$ and for $m \in A$,
\[
\frac{2m}{B_m} = 432 - 60m - 432i^m,
\]
we get the conclusion.

For case (2), as we have $2 - k = r(p - 1) + sp^t$ and $sp^t - u \neq 2$, we can find $c_1, c_2 \in \mathbb{Z}_{\geq 0}$ such that $4c_1 + 6c_2 = sp^t - u$. Then we have
\[
(E_4^c E_6^c)^p^n E_p^{r-1} f \in M_1^j.
\]
Hence we have that the constant term of $(E_4^c E_6^c)^p^n E_p^{r-1} f$ is zero. As
\[
(E_4^c E_6^c)^p^n E_p^{r-1} f \equiv (1 + O(q^{p^n})) f \pmod{p}
\]
and $\text{ord}_\infty(f) > -p^a$, we know $a_f(0) \equiv 0 \pmod{p}$. To prove the case of $a_f(p^u)$ for $u \leq v \leq t$, we may find $c_1', c_2' \in \mathbb{Z}_{\geq 0}$ such that $4c_1' + 6c_2' = sp^t - v$. Then we have
\[
j^{p^u} (E_4^{c_1'} E_6^{c_2'})^p E_p^{r-1} f \in M_1^j.
\]
Hence the constant term of $j^{p^u} (E_4^{c_1'} E_6^{c_2'})^p E_p^{r-1} f$ is zero. As
\[
(j E_4^{c_1'} E_6^{c_2'})^p E_p^{r-1} f \equiv (q^{-p^u} + 744 + 240c_1' - 504c_2' + O(q^{p^u})) f \pmod{p}
\]
and $\text{ord}_\infty(f) > -p^a \geq -p^u$, we get
\[
a_f(p^u) + (744 + 240c_1' - 504c_2')a_f(0) \equiv 0 \pmod{p}.
\]
Knowing that $a_f(0) \equiv 0 \pmod{p}$, we get the conclusion. \qed

3. Proof of Theorem [1.1]

By Theorem [2.3], $p = 2$ and 3 are non-ordinary for every normalized Hecke eigenform on $\text{SL}_2(\mathbb{Z})$. Therefore, we may assume that $S$ consists only of primes $p \geq 5$.

For the given finite set of primes $S$, let $k_S(j, m) := j \prod_{p \in S} (p - 1) + m$, where $j$ is an arbitrary non-negative integer, $m \in A$. For each $j$ and $m$ let $b_S(j, m)$ be any integer for which
\[
k_S(j, m) - 2 < (m - 2)p^{b_S(j, m)}
\]
for all $p \in S$. Let $f = \sum_{n=1}^{\infty} a_f(n) q^n$ be any Hecke eigenform of weight $k_S(j, m)$. By Theorem [2.5](1), since $a_f(0) = 0$, we have
\[
a_f(p^{b_S(j, m)}) \equiv 0 \pmod{p}
\]
for all $p \in S$. Applying Proposition [2.1], we know that $f$ is non-ordinary for each $p \in S$. As $j$ can be chosen freely, we get the conclusion.

4. Examples

**Example.** Let $S = \{2, 3, 5, 7, 11, 13, 17, 19\}$. In the following table we list some of the weights $k$ for which Hecke eigenforms are non-ordinary at each prime $p$. 
\[
\begin{array}{c|cccccccccccccccc}
p & 12 & 14 & 16 & 18 & 20 & 22 & 24 & 26 & 28 & 30 & 32 & 34 & 36 & 38 & 40 & 42 \\
\hline
2 &  &  &  &  &  &  &  &  &  &  &  &  &  &  &  &  \\
3 &  &  &  &  &  &  &  &  &  &  &  &  &  &  &  &  \\
5 &  &  &  &  &  &  &  &  &  &  &  &  &  &  &  &  \\
7 &  &  &  &  &  &  &  &  &  &  &  &  &  &  &  &  \\
11 &  &  &  &  &  &  &  &  &  &  &  &  &  &  &  &  \\
13 &  &  &  &  &  &  &  &  &  &  &  &  &  &  &  &  \\
17 &  &  &  &  &  &  &  &  &  &  &  &  &  &  &  &  \\
19 &  &  &  &  &  &  &  &  &  &  &  &  &  &  &  &  \\
\end{array}
\]

In particular, we consider the case \( k = 26 \) and check its non-ordinariness. We have the following \( q \)-expansion of the normalized weight 26 Hecke eigenform \( f_{26} = \Delta E_6 E_4 \):

\[
f_{26}(z) = q - 48q^2 - 195804q^3 - 33552128q^4 - 741989850q^5 + 9398592q^6 + 39080597192q^7 \\
+ 322114880q^8 - 808949403027q^9 + 35615512800q^{10} + 8419515299052q^{11} \\
+ 6569640870912q^{12} - 8165104535314q^{13} - 1875868665216q^{14} \\
+ 145284580589400q^{15} + 1125667983917056q^{16} - 2519900028948078q^{17} \\
+ 38829571345296q^{18} - 6082056370308940q^{19} + O(q^{20}).
\]

We can easily check that \( a_{f_{26}}(p) \equiv 0 \pmod{p} \) for each \( p \in S \). Of course we can also choose weights \( k \) of the form \( k = 26 + 720j \), for every \( j \in \mathbb{N} \). Note that \( 720 = [5 - 1, 7 - 1, 11 - 1, 13 - 1, 17 - 1, 19 - 1] \).

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