Honda formal group as Galois module in unramified extensions of local fields

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Abstract

For given rational prime number $p$ consider the tower of finite extensions of fields $K_0/\mathbb{Q}_p$, $K/K_0, L/K, M/L$, where $K/K_0$ is unramified and $M/L$ is a Galois extension with Galois group $G$. Suppose one dimensional Honda formal group over the ring $\mathcal{O}_K$, relative to the extension $K/K_0$ and uniformizer $\pi \in K_0$ is given. The operation $x + y = F(x, y)$ sets a new structure of abelian group on the maximal ideal $p_M$ of the ring $\mathcal{O}_M$ which we will denote by $F(p_M)$. In this paper the structure of $F(p_M)$ as $\mathcal{O}_K[G]$-module is studied for specific unramified $p$-extensions $M/L$.

0.1 Introduction

Let $p$ be a rational prime, $K/\mathbb{Q}_p, L/K, M/L$ be a tower of finite extensions of local fields, $M/L$ be a Galois extension with Galois group $G$ and $F$ be a one dimensional formal group law over the ring $\mathcal{O}_K$. The operation $x + y = F(x, y)$ sets a new structure of abelian group on the maximal ideal $p_M$ of the ring $\mathcal{O}_M$ which we will denote by $F(p_M)$. Taking into account the natural action of the group $G$ on $F(p_M)$, one may consider it as an $\text{End}_{\mathcal{O}_K}(F)[G]$-module, in which the multiplication by scalars from $\text{End}_{\mathcal{O}_K}(F)$ is performed by the rule $f \ast x = f(x)$. We refer the reader to [1, Chapter 6, §3], [2, Chapter 3, §6], [3, Chapter 4] and [4, Chapter 4] for more details concerning formal groups and the group $F(p_M)$.

If $F$ is a Lubin-Tate formal group law, then there is an injection $\mathcal{O}_K \hookrightarrow \text{End}_{\mathcal{O}_K}(F)$ (see [1, Chapter 6, Prop. 3.3]), which enables us to regard $F(p_M)$ as an $\mathcal{O}_K[G]$-module. The structure of this module in case of multiplicative formal group $F = G_m$ and $K = \mathbb{Q}_p$ is studied in sufficient detail in [5–7]. The starting point of the current study is the following theorem of Borevich in [7].

Theorem (Borevich, 1965). Suppose $M/L$ is an unramified $p$-extension and $K = \mathbb{Q}_p$. If the fields $M$ and $L$ have the same irregularity degree $^1$ then for the $\mathcal{O}_K[G]$-module $U_M$ there exists a system of generating elements $\theta_1, \ldots, \theta_{n-1}, \xi, \omega$ with the unique defining relation $\xi^{p^n} = \omega^{p-1}$, where $n = [L : \mathbb{Q}_p]$ and $\sigma$ is a generating element of the Galois group $G = \text{Gal}(M/L)$.

It may seem that the group of principal units $E_M$ has nothing to do with formal groups, but in fact it is easy to show that for the multiplicative formal group $F = G_m$ there is an isomorphism

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$^1$This means that the field $L$ contains a $p^s$-th primitive root of unity, while $M$ does not contain a primitive $p^{s+1}$-th root of unity for some $s \geq 1$
The joint work of S.V. Vostokov and I.I. Nekrasov [8], where they generalized the aforementioned theorem to the case of Lubin-Tate formal groups. More precisely, they managed to prove the following

**Theorem (Vostokov-Nekrasov, 2014).** Suppose $M/L$ is an unramified $p$-extension and $F$ is a Lubin-Tate formal group for the prime element $\pi \in K$. Assume moreover that the fields $M$ and $L$ have the same irregularity degree, namely they contain a generator of $\ker[\pi^s]_F$ and do not contain a generator of $\ker[\pi^{s+1}]_F$ for some $s \geq 1$ \footnote{In fact, $\ker[\pi^s]_F$ is a cyclic $O_K$-module, whenever $F$ is a Lubin-Tate formal group (See [2, Chapter 3, Prop. 7.2])}. Then for the $O_K[G]$-module $F(p_M)$ there exists a system of generating elements $\theta_1, \ldots, \theta_{n-1}, \xi, \omega$ with the unique defining relation $[\pi^s]_F(\xi) = \omega^s - \omega$, where $n = [L : K]$ and $\sigma$ is a generating element of the Galois group $G = \text{Gal}(M/L)$.

The key point in this work was the proof of the triviality of the cohomology groups $H^i(G, F(p_M))$ with $i = 0, -1$ for unramified extensions $M/L$. In its turn, our work is devoted to the generalization of the last result to the case of Honda formal groups. Namely, let $K_0/\mathbb{Q}_p$ be a finite extension such that $K/K_0$ is unramified, $\pi \in K_0$ be a uniformizer, $F$ be a Honda formal group over $O_K$ relative to the extension $K/K_0$ of type $u \in O_{K,F}[T]$. We refer the reader to [9, §2.3] and [10] for more information concerning Honda formal groups. Suppose $K^{\text{alg}}$ is a fixed algebraic closure of the field $K$, $p_{K^{\text{alg}}}$ is the valuation ideal, i.e. the set of all points in $K^{\text{alg}}$ with positive valuation. Define $W^n_F = \ker[\pi^n]_F \subset F(p_{K^{\text{alg}}})$ to be the $\pi^n$-torsion submodule. More precisely, let $W^n_F = \{ x \in p_{K^{\text{alg}}} \mid [\pi^n]_F(x) = 0 \}$, where $[\pi^n]_F \in \text{End}_{O_K}(F)$, and let $W_F = \bigcup_{n=1}^\infty W^n_F$.

It is known (See [9, §2, Thm. 3]) that there is a ring embedding $O_{K_0} \rightarrow \text{End}_{O_K}(F)$, which allows us to regard $F(p_M)$ as an $O_{K_0}[G]$-module. In this paper, using generators and defining relations we describe the structure of this module in the case of unramified $p$-extension $M/L$, provided that $W_F \cap F(p_L) = W_F \cap F(p_M) = W^s_F$, for certain $s \geq 1$. It is known that any finite unramified extension of a local field is a cyclic extension, so that $G$ is a cyclic $p$-group.

We agree in the following notation

- $n$—the degree of the field $L$ over $K_0$;
- $h$—the height of the type $u = \pi + \sum_{i \geq 1} a_i T^i$ of the formal group $F$, i.e. the minimal $h$, for which $a_h$ is invertible.
- $f$—the logarithm of $F$;
- $p^m$—the order of the group $G = \text{Gal}(M/L)$;
- $\sigma$—a generating element of $G$;
- $\zeta_i$, $1 \leq i \leq h$—a fixed basis of the $O_{K_0}/\pi^s O_{K_0}$-module $W^n_F$;
- $k_0, l$—the residue fields of $K_0$ and $L$ respectively;
- $q$—the order of $k_0$;
- $x + y := F(x, y)$;
- $\sum_{F,i=1}^k x_i := x_1 + x_2 + \ldots + x_k$. 
0.2 Auxiliary lemmas

Lemma 1. The $\mathcal{O}_{K_0}$-module $W_F^n$ is isomorphic to $(\mathcal{O}_{K_0}/\pi^n\mathcal{O}_{K_0})^h$.

Proof. See [10, Prop. 1].

Lemma 2. In the case of an unramified extension $M/L$, the groups $H^i(G, F(\mathfrak{p}_M))$ are trivial for $i = 0, -1$.

Lemma 3. If the elements $x_1, x_2, \ldots, x_k$ from $F(\mathfrak{p}_M)$ are such that the system
\[\{N_{F(\mathfrak{p}_M)}(x_i), 1 \leq i \leq k\}\] is linearly independent in the $k_0$-vector space $F(\mathfrak{p}_M)/[\pi]_F(F(\mathfrak{p}_M)$, then so is the system $\{x_1^{\sigma^i}, 1 \leq i \leq k, 0 \leq j \leq p^m - 1\}$.

Lemma 4. If the elements $x_1, x_2, \ldots, x_k$ from $F(\mathfrak{p}_M)$ generate the $k_0$-vector space $F(\mathfrak{p}_M)/[\pi]_F(F(\mathfrak{p}_M)$, then they generate $F(\mathfrak{p}_M)$ as an $\mathcal{O}_{K_0}$-module.

The proofs of lemmas 2.8-2.10 can be found in the article [8], as well as in [7, §3].

Lemma 5. The natural linear map
\[\varphi : F(\mathfrak{p}_L)/[\pi]_F(F(\mathfrak{p}_L)) \to F(\mathfrak{p}_M)/[\pi]_F(F(\mathfrak{p}_M))\]
of $k_0$-vector spaces, induced by inclusion, has kernel of dimension $h$.

Proof. Consider the elements $\eta_i = [\pi^{x_{i-1}}]_{F\mathfrak{c}_i}, 1 \leq i \leq h$. They form a basis of $W_F^1$ as an $\mathcal{O}_{K_0}/\pi\mathcal{O}_{K_0}$-module. Since $N_{F(\mathfrak{p}_M)}\eta_i = [p^m]_{F\mathfrak{c}_i} = 0$, then by Lemma 2 we get that $\eta_i = t_i^\sigma - t_i$ for some elements $t_i \in F(\mathfrak{p}_M)$. Suppose that $x \in F(\mathfrak{p}_L)$ and $x = [\pi]_F(y)$ for some $y \in F(\mathfrak{p}_M)$. Then
\[[\pi]_F(y^\sigma - y) = x^\sigma - x = 0,\]
from which it follows that
\[y^\sigma - y = \sum_{F_i=1}^{h} [a_i]_F(\eta_i) = \sum_{F_i=1}^{h} \left( ([a_i]_F(t_i))^\sigma - [a_i]_F(t_i) \right),\]
for certain elements $a_i \in \mathcal{O}_{K_0}$, uniquely determined modulo $\pi$. The last relationship indicates the existence of $z \in F(\mathfrak{p}_L)$, for which $y = \sum_{F_i=1}^{h} [a_i]_F(t_i) + z$. Therefore,
\[x = [\pi]_F(y) = \sum_{F_i=1}^{h} [a_i]_F([\pi]_F(t_i)) + [\pi]_F(z).

Hence the elements $[\pi]_F(t_i), 1 \leq i \leq h$ constitute a basis of ker $\varphi$. The lemma is proved.

Lemma 6. The dimension of the $k_0$-vector space $F(\mathfrak{p}_L)/[\pi]_F(F(\mathfrak{p}_L))$ is equal to $n + h$.

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3 Here $N_{F(\mathfrak{p}_M)}$ is the $G$-module norm.
Proof. According to [4, Chapter 4, Thm. 6.4] for \( i > \frac{e(L/Q_p)}{p-1} \) there is an isomorphism of groups 
\( f : F(p^i_L) \cong \mathfrak{p}_L \), which is in fact an isomorphism of \( \mathcal{O}_{K_0} \)-modules due to the relation 
\( f \circ [a]_F = af \) which holds for all \( a \in \mathcal{O}_{K_0} \). Consequently, \( F(p^i_L) \) is a free \( \mathcal{O}_{K_0} \)-module of rank \( n \).

From the exactness of sequences of \( \mathcal{O}_{K_0} \)-modules:
\[
0 \rightarrow F(p^{i+1}_L) \rightarrow F(p^i_L) \rightarrow \mathfrak{l} \rightarrow 0, \quad i \geq 1
\]
it follows that \( F(p^i_L) \) is an \( \mathcal{O}_{K_0} \)-submodule of finite index in \( F(p_L) \). Therefore \( F(p_L) \) is a finitely generated \( \mathcal{O}_{K_0} \)-module of rank \( n \). The theory of finitely generated modules over a PID yields \( F(p^i_L) = T \oplus A \), where \( T \) is the torsion submodule, which in our case coincides with \( W^p_{K_0} \), while \( A \) is a free \( \mathcal{O}_{K_0} \)-module of rank \( n \). In the long run, we get
\[
\left| F(p^i_L) / [\pi]_F(F(p^i_L)) \right| = \left| T / [\pi]_F T \right| \cdot \left| A / [\pi]_F A \right| = q^h \cdot q^n = q^{n+h},
\]
completing the proof of the lemma.

\[ \square \]

Remark 1. Likewise we get that \( \dim_{k_0} (F(p^i_M) / [\pi]_F(F(p^i_M))) = np^m + h \).

Remark 2. Since \( F(p^i_M) \) is a finitely generated \( \mathcal{O}_{K_0} \)-module, then by Nakayama’s lemma we obtain a new proof of the assertion of Lemma 4.

Lemma 7. The elements \( \zeta_i, 1 \leq i \leq h \) are linearly independent modulo \( \ker \varphi \).

Proof. Suppose the relation \( \sum_{F; i=1}^{h} [a_i]_F \zeta_i = [\pi]_F(y) \) holds for some \( a_i \in \mathcal{O}_{K_0}, \ y \in F(p^i_M) \). Applying the endomorphism \( [\pi^s]_F \), we get that \( [\pi^{s+1}]_F(y) = 0 \), which gives \( [\pi^s]_F(y) = 0 \). The latter means that \( \sum_{F; i=1}^{h} [\pi^{s-1} a_i]_F \zeta_i = 0 \), which is equivalent to the condition \( a_i : \pi, 1 \leq i \leq h \). The lemma is proved.

\[ \square \]

Corollary 1. \( h \leq n \)

Proof. In view of the lemmas proved, it follows that the maximal number of linearly independent vectors modulo \( \ker \varphi \) in \( F(m_L) / [\pi]_F(F(p_L)) \) is equal to
\[
\dim \text{Im} \varphi = \dim_{k_0}(F(p^i_L) / [\pi]_F(F(p^i_L))) - \dim \ker \varphi = (n + h) - h = n.
\]

By Lemma 7 we already have \( h \) linearly independent vectors modulo \( \ker \varphi \), from which the desired result follows.

\[ \square \]
0.3 The main theorem

Theorem. If the extension \( M/L \) is unramified and \( W_F \cap F(p_L) = W_F \cap F(p_M) = W_F^s \), for some \( s \geq 1 \), then \( h \leq n \) and for the \( \mathcal{O}_{K_0}[G] \)-module \( F(p_M) \) there exist a system of generating elements \( \theta_j, \xi_i, 1 \leq j \leq n - h, 1 \leq i \leq h \) with the only defining relations \([\pi^s]_F(\xi_i) = \omega^s_i - \omega_i, 1 \leq i \leq h\).

Proof. From the triviality of the group \( H^0(G, F(p_M)) \) follows the existence of elements \( \xi_i \in F(p_M), 1 \leq i \leq h \), such that \( N_{F(p_M)}(\xi_i) = \xi_i \). Since \( N_{F(p_M)}([\pi^s]_F(\xi_i)) = [\pi^s]_F(\xi_i) = 0 \) and the group \( H^{-1}(G, F(p_M)) \) is trivial, there exist elements \( \omega_i \in F(p_M), 1 \leq i \leq h \), satisfying the relations \([\pi^s]_F \xi_i = \omega_i^s - \omega_i \). In view of Corollary 1, the system \( \xi_i, 1 \leq i \leq h \) can be supplemented to a basis modulo \( \ker \varphi \) via elements \( \varepsilon_j \in F(m_L), 1 \leq j \leq n - h \). For \( 1 \leq j \leq n - h \) we select elements \( \theta_j \in F(p_M) \), so that \( N_{F(p_M)}(\theta_j) = \varepsilon_j \) for all \( j \) and we prove that the system \( \mathcal{E} = \{\omega_i, \xi_i^a, \theta_j^a | 1 \leq i \leq h, 1 \leq j \leq n - h, 0 \leq k \leq p^m - 1\} \) is linearly independent modulo \([\pi]_F(F(p_M))\). Assume the contrary that there exist elements \( a_i, a_{i,k}, b_{j,k} \in \mathcal{O}_{K_0} \) and \( \beta \in F(p_M) \) such that

\[
\sum_{F;i} [a_i]_F \omega_i + \sum_{F;i,k} [a_{i,k}]_F (\xi_i^a) + \sum_{F;j,k} [b_{j,k}]_F (\theta_j^a) + [\pi]_F(\beta) = 0.
\]

We apply \( \sigma - 1 \) to both parts of the latter relation and use the relations

\[
\sum_{F;i,k} [a_{i,k} - a_{i,k-1}]_F (\xi_i^a) + \sum_{F;j,k} [b_{j,k} - b_{j,k-1}]_F (\theta_j^a) + \sum_{F;i} [a_i]_F [\pi^s]_F \xi_i + [\pi]_F(\beta^s - \beta) = 0.
\]

From lemmas 3 and 7 it follows that the system

\[ \mathcal{E}_0 = \{\xi_i^a, \theta_j^a | 1 \leq i \leq h, 1 \leq j \leq n - h, 0 \leq k \leq p^m - 1\} \]

is linearly independent modulo \([\pi]_F(F(p_M))\), so that \( a_{i,k} \) (mod \( \pi \)) and \( b_{j,k} \) (mod \( \pi \)) are independent of \( k \). Therefore, without loss of generality we may assume that

\[
\sum_{F;i} [a_i]_F [\pi^s]_F \xi_i + [\pi]_F(\beta^s - \beta) = 0,
\]

changing if needed \( \beta \). From the obtained follows the existence of \( b_i \in \mathcal{O}_{K_0}, 1 \leq i \leq h \) such that

\[
\sum_{F;i} [a_i]_F [\pi^{s-1}]_F (\xi_i) + \beta^s - \beta = \sum_{F;i} [b_i]_F \eta_i
\]

Taking norms \( N_{F(p_M)} \), the obtained relation leads to the equality \( \sum_{F;i} [a_i]_F [\pi^{s-1}]_F (\xi_i) = 0 \) which implies that \( a_i : \pi, 1 \leq i \leq h \). From the linear independence of the system \( \mathcal{E}_0 \) it follows that \( a_{i,k} : \pi \) and \( b_{j,k} : \pi \) for all \( i, j \) and \( k \). This completes the proof of the linear independence of the system \( \mathcal{E} \). The number of vectors in it is \( np^m + h = \dim_{K_0}(F(p_M)/[\pi]_F(F(p_M))) \), so that they
generate the space \( F(\mathfrak{p}_M)/[\pi]_F(F(\mathfrak{p}_M)) \). From lemma 4 it follows that they generate \( F(\mathfrak{p}_M) \) as an \( \mathcal{O}_{K_0} \)-module, and consequently the elements \( \theta_j, \xi_i, \omega_i, 1 \leq j \leq n - h, 1 \leq i \leq h \) generate \( F(\mathfrak{p}_M) \) as an \( \mathcal{O}_{K_0}[G] \)-module. It remains only to prove the assertion concerning defining relations. Let us further agree to write multiplication by elements of the ring \( \mathcal{O}_{K_0}[G] \) through exponentiation.

Suppose that the relation

\[
\sum_{b} \xi_{i}^{\alpha_{i}} + \sum_{F;i} \omega_{i}^{\beta_{i}} + \sum_{F;j} \theta_{j}^{\delta_{j}} = 0,
\]

holds for some elements \( \alpha_{i}, \beta_{i}, \delta_{j} \in \mathcal{O}_{K_0}[G] \). Our goal is to prove the existence of elements \( \gamma_{i} \in \mathcal{O}_{K_0}[G] \) for which \( \alpha_{i} = \pi^{*}\gamma_{i}, \beta_{i} = (1 - \sigma)\gamma_{i} \) and \( \delta_{j} = 0 \). Indeed, let \( \beta_{i} = b_{i} + (1 - \sigma)\gamma_{i} \) for certain elements \( b_{i} \in \mathcal{O}_{K_0} \) and \( \gamma_{i} \in \mathcal{O}_{K_0}[G] \). Taking into account the relations \( [\pi^{s}]_F \xi_{i} = \omega_{i}^{\sigma} - \omega_{i} \) for \( 1 \leq i \leq h \) we get

\[
\sum_{F;i} \omega_{i}^{b_{i}} + \sum_{F;i} \xi_{i}^{\alpha_{i}'} + \sum_{F;j} \theta_{j}^{\delta_{j}} = 0,
\]

where \( \alpha_{i}' = \alpha_{i} - \pi^{*}\gamma_{i} \). Factoring the latter relation modulo \( [\pi]_F(F(\mathfrak{p}_M)) \) and recalling that the system \( \mathcal{E} \) is a basis modulo \( [\pi]_F(F(\mathfrak{p}_M)) \), we find that there exist elements \( b_{i}^{(1)}, \beta_{i}', \delta_{j}^{(1)} \in \mathcal{O}_{K_0}[G] \) such that \( b_{i} = \pi b_{i}^{(1)}, \alpha_{i}' = \pi \beta_{i}', \delta_{j} = \pi \delta_{j}^{(1)} \). Therefore, for some elements \( a_{i} \in \mathcal{O}_{K_0} \) we must have the equality

\[
\sum_{F;i} \omega_{i}^{b_{i}^{(1)}} + \sum_{F;i} \xi_{i}^{\beta_{i}' - a_{i} \sum_{k} \sigma_{k}} + \sum_{F;j} \theta_{j}^{\delta_{j}^{(1)}} = 0,
\]

due to the fact that \( \xi_{i} = N_{F(\mathfrak{p}_M)}(\xi_{i}) = \xi_{i} \sum_{k} \sigma_{k} \). For the same reasons, all \( b_{i}^{(1)} \) and \( \delta_{j}^{(1)} \) are divisible by \( \pi \). By induction we construct sequences \( (b_{i}^{(v)})_{v \geq 0} \) and \( (\delta_{j}^{(v)})_{v \geq 0} \) satisfying the conditions

\[
b_{i}^{(0)} = b_{i}, \delta_{j}^{(0)} = \delta_{j}, b_{i}^{(v)} = \pi b_{i}^{(v+1)} \quad \text{and} \quad \delta_{j}^{(v)} = \pi \delta_{j}^{(v+1)} \quad \text{for all} \quad v \geq 0, 1 \leq i \leq h, 1 \leq j \leq n - h,
\]

from which it follows that \( b_{i} = 0 \) for all \( i \) and \( \delta_{j} = 0 \) for all \( j \). There remains only the relation

\[
\sum_{F;i} \xi_{i}^{a_{i}'} = 0.
\]

Let now \( \alpha_{i}' = \sum_{k} a_{i,k} \sigma_{k} \), where \( a_{i,k} \in \mathcal{O}_{K_0} \) for all \( i,k \). The factorization modulo \( [\pi]_F(F(\mathfrak{p}_M)) \) yields \( a_{i,k} = \pi b_{i,k} \). Further, we obtain that

\[
\sum_{F;i} \xi_{i}^{b_{i,k} \sigma_{k}} = \sum_{F;i} [\lambda_{i}]_F(\xi_{i}) = \sum_{F;i} \xi_{i}^{\lambda_{i} \sum_{k} \sigma_{k}},
\]

for some elements \( \lambda_{i} \in \mathcal{O}_{K_0} \). Consequently \( b_{i,k} \) (mod \( \pi \)) is the same for all \( k \), and so on. In the end we get that \( a_{i,k} = a_{i} \) and that \( \sum_{F;i} [a_{i}]_F(\xi_{i}) = 0 \), i.e. \( a_{i} = \pi^{*}t_{i} \) for certain \( t_{i} \in \mathcal{O}_{K_0} \) and therefore

\[
\alpha_{i} - \pi^{*}\gamma_{i} = \alpha_{i}' = \pi^{*}t_{i} \sum_{k} \sigma_{k}.
\]

If we denote \( \gamma_{i}' = \gamma_{i} + t_{i} \sum_{k} \sigma_{k} \), then we will have \( \alpha_{i} = \pi^{*}\gamma_{i}' \) and \( \beta_{i} = (1 - \sigma)\gamma_{i} = (1 - \sigma)\gamma_{i}' \), thus completing the proof of the theorem .

\[\square\]
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