ABSTRACT

The Gauss law constraint in the Hamiltonian form of the $SU(2)$ gauge theory of gluons is satisfied by any functional of the gauge invariant tensor variable $\phi^{ij} = B^{ia}B^{ja}$. Arguments are given that the tensor $G_{ij} = (\phi^{-1})_{ij} \det B$ is a more appropriate variable. When the Hamiltonian is expressed in terms of $\phi$ or $G$, the quantity $\Gamma^{i}_{jk}$ appears. The gauge field Bianchi and Ricci identities yield a set of partial differential equations for $\Gamma$ in terms of $G$. One can show that $\Gamma$ is a metric-compatible connection for $G$ with torsion, and that the curvature tensor of $\Gamma$ is that of an Einstein space. A curious 3-dimensional spatial geometry thus underlies the gauge-invariant configuration space of the theory, although the Hamiltonian is not invariant under spatial coordinate transformations. Spatial derivative terms in the energy density are singular when $\det G = \det B = 0$. These singularities are the analogue of the centrifugal barrier of quantum mechanics, and physical wave-functionals are forced to vanish in a certain manner near $\det B = 0$. It is argued that such barriers are an inevitable result of the projection on the gauge-invariant subspace of the Hilbert space, and that the barriers are a conspicuous way in which non-abelian gauge theories differ from scalar field theories.

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1. Introduction

The implementation of the Gauss law constraint on physical states in the Hamiltonian form of non-abelian gauge theories is a major obstacle for non-perturbative studies. Since the difficulty in treating Gauss’ law stems from the non-covariant gauge transformation of the vector potential $A^a_i$, one can attempt to solve the problem by formulating the theory in terms of variables which transform covariantly. For example, Goldstone and Jackiw [1] suggested the use of the electric field as the fundamental variable. Their approach led to an exact implementation of Gauss’ law in the $SU(2)$ gauge theory of gluons, but to a complicated Hamiltonian which has not to our knowledge been used in concrete calculations.

One may also consider the problem in magnetic variables. Here Simonov [2] has applied a “polar representation” of the vector potential which allows the removal of gauge degrees of freedom, but gives a Hamiltonian and functional measure which are non-local. A year ago, one of us [3] proposed a canonical transformation to the magnetic field $B_{ia}$ and a conjugate $C_{ai}$, both of which transform homogeneously. The Gauss law constraint is then satisfied by state functionals $\psi[\phi^{ij}]$ which depend on the gauge invariant tensor variable $\phi^{ij} = B_{ia} B^{ja}$. (See [4] for an earlier, less complete proposal to use the magnetic field).

In this paper we review briefly the proposal of [3] and then discuss further ideas related to the use of gauge-invariant magnetic variables in the $SU(2)$ gluon theory. For most of the considerations it is not necessary to make a complete canonical transformation. Rather we observe that even if $A^a_i$ is taken as the fundamental variable, states $\psi[\phi^{ij}]$ satisfy the Gauss law constraint, and it is sensible to study the form of the Hamiltonian for such states. There are several reasons why the variable $G_{ij} = \phi_{ij} \det B$, where $\phi_{ik} \phi^{kj} = \delta^j_i$, is more appropriate than $\phi^{ij}$, and we also obtain the form of the Hamiltonian for functionals $\psi[G_{ij}]$.

In both cases a gauge invariant quantity $\Gamma^{i\ell}_{jk}$ appears in the electric energy density, and the gauge field Bianchi and Ricci identities lead to partial differential equations from which one can, in principle, determine $\Gamma^{i\ell}_{jk}$ in terms of $G_{ij}$. An unexpected geometrical structure then emerges. If $G_{ij}$ is viewed as an (indefinite) metric tensor on $\mathbb{R}^3$, then $\Gamma^{i\ell}_{jk}$ is a metric-compatible affine connection with torsion, and the Riemann $R^i_{\ell jk\ell}$ and Ricci $R_{ij}$ tensors of $\Gamma$ are those of a 3-dimensional Einstein space. The Hamiltonian is always expressed in Cartesian coordinates on $\mathbb{R}^3$. It cannot be and is not diffeomorphism invariant because it involves the Cartesian metric $\delta_{ij}$ as well as $G_{ij}$. It turns out that the energy density transforms in a specific tensor representation of $GL(3)$.

It can be argued that $\phi^{ij}$ or, somewhat more precisely, $G_{ij}$ are symmetric tensors whose six independent components describe the local gauge invariant degrees of freedom of the $SU(2)$ gauge theory. It is then curious that the gauge invariant phase space admits a fairly natural spatial geometry while the Hamiltonian is not invariant. Thus, two configurations of $G_{ij}$ mathematically related by a “diffeomorphism” describe physically distinct gauge field configurations, typically with different energies.

To clarify things it should be stated that given $A^a_i(x)$, one can calculate $B^{ia}(x)$, $G_{ij}(x)$ and $\Gamma^{k}_{ij}(x)$ by straightforward local formulas, and it then turns out that $G_{ij}(x)$ and $\Gamma^{k}_{ij}(x)$ are the metric and connection of a 3-dimensional Einstein geometry with torsion. We refer
to this as the forward map. It is then an interesting question whether this transformation can be inverted. For SU(2) it is easy to show that given $G_{ij}(x)$ and $\Gamma^k_{ij}(x)$, one can reconstruct $B^{ia}(x)$ and $A^i_a(x)$, up to a gauge transformation, by a local construction. Reasonable but non-rigorous arguments are given that for a given configuration $G_{ij}(x)$ the Einstein space condition

$$R_{ij}(\Gamma) = -2G_{ij} \tag{1.1}$$

can be solved to obtain the contortion tensor $K^i_{jk}$ which is related to $\Gamma$ by

$$\Gamma^i_{jk} = \Gamma^i_{jk} - K^i_{jk} \tag{1.2}$$

where $\Gamma$ is the standard Christoffel symbol. In general, $K^i_{jk}$ is nonlocally related to $G_{ij}$, and we expect two solutions for $K^i_{jk}$ for each configuration of $G_{ij}$. Thus the geometry appears to generate a double-valued map from $G_{ij}$ to a magnetic field $B^{ia}$, unique up to a gauge, and a pair of potentials $A^i_a$. This structure seems to be consistent with the Wu-Yang ambiguity [5]. Some examples of specific geometries and the gauge field configurations related to them are studied.

Our motivation in formulating the theory in terms of gauge-invariant local variables was to implement Gauss’ law exactly, so that the resulting Hamiltonian in the physical subspace could be used for dynamical calculations of the vacuum structure and glueball spectrum. It is the slowly-varying modes (compared to $\frac{1}{\Lambda_{QCD}}$) of the system for which non-perturbative treatment is most urgently required, and the use of gauge-invariant variables allows a gauge-invariant definition of slow variation. It is not fully clear how to implement dynamical calculations or whether the geometrical structure discussed above will be useful for this purpose.

Nevertheless, there are some physical implications because the energy density is singular when $\det \phi^{ij} = (\det G_{ij})^2 = (\det B^{ia})^2 = 0$. These conditions correspond to coordinate singularities of the gauge-invariant configuration space and a singularity of the transformation $A^i_a \rightarrow B^{ia}$, just as $r = 0$ is a singular point of the spherical coordinate system and of the transformation from Cartesian coordinates. Thus, one can interpret the singularities as the gauge theory analogue of the centrifugal barrier of quantum mechanics. Such barriers also occur in the electric formulation [1] of non-abelian gauge theory and appear to be an inevitable result of the “projection” onto the gauge-invariant physical subspace. There are no such barriers in scalar field theories. Finite energy eigenfunctionals or variational trial functionals must satisfy certain vanishing conditions near the singularity, and these conditions involve a complicated combination of functional and spatial derivatives. It is then suggested that a better understanding of these barriers may provide qualitative insight into the dynamics of non-abelian gluons.

2. Gauss’ Law and the Variables $\phi^{ij}$ and $G_{ij}$

We begin with the SU(2) Yang-Mills Hamiltonian in the $A^a_0(x) = 0$ gauge:

$$H = \frac{1}{2} \int d^3x \left[ g^2 \left( E^{ia}(x) \right)^2 + \frac{1}{g^2} \left( B^{ia}(x) \right)^2 \right], \tag{2.1}$$
where the electric and magnetic fields are

\[ E^{ia} = \frac{1}{g^2} \dot{A}_i^a, \quad B^{ia} = \epsilon^{ijk} \left( \partial_j A_k^a + \frac{1}{2} \epsilon^{abc} A_j^b A_k^c \right). \] (2.2)

The standard pair of canonical variables are the gauge potential and the electric field, with equal-time commutators:

\[ [A_i^a(x), E^{jb}(y)] = i \delta^{ab} \delta_i^j \delta^{(3)}(x - y). \] (2.3)

The generator of the gauge transformation with parameter \( \theta^a(x) \) is

\[ G[\theta] = \int d^3 x \theta^a(x) G^a(x) \]

\[ G^a(x) = D_k E^{ka}(x) = \partial_k E^{ka} + \epsilon^{abc} A_k^b E^{kc}, \] (2.4)

and the quantum transformation rules of the local fields and the Hamiltonian are:

\[ \delta A_i^a(x) = i \left[ G[\theta], A_i^a(x) \right] = D_i \theta^a(x) = \partial_i \theta^a(x) + \epsilon^{abc} A_i^b(x) \theta^c(x) \]

\[ \delta E^{ia}(x) = i \left[ G[\theta], E^{ia}(x) \right] = \epsilon^{abc} E^{ib}(x) \theta^c(x) \]

\[ \delta B^{ia}(x) = i \left[ G[\theta], B^{ia}(x) \right] = \epsilon^{abc} B^{ib}(x) \theta^c(x) \]

\[ [G[\theta], H] = 0. \] (2.5)

Gauss’ law, \( G^a(x) = 0 \), is one of the classical equations of motion obtained from the Lagrangian of the theory before gauge fixing. In the gauge-fixed quantum theory one must impose it as the constraint

\[ G^a(x) | \psi_{\text{phys}} \rangle = 0 \] (2.6)

on physical states in the Hilbert space. It is the implementation of this constraint that motivates the transformation [3] which we now discuss.

Let us consider the possibility of describing the configuration space of the system using \( B^{ia}(x) \) rather than \( A_i^a(x) \). In three spatial dimensions (and only three), they have the same number of spatial components, but this is certainly not the only consideration. The magnetic field satisfies the Bianchi identity

\[ \partial_i B^i(x) = 0 \] (2.7)

in the abelian case, and

\[ D_i B^{ia} = \partial_i B^{ia}(x) + \epsilon^{abc} A_i^b(x) B^{ic}(x) = 0 \] (2.8)

for the non-abelian theory. Thus, \( B^i \) is constrained to only two independent components in the abelian case, and cannot be used as a variable. The non-abelian Bianchi identity, on the other hand, is not a constraint on \( B^{ia} \), but rather a relation between \( B^{ia} \) and \( A_i^a \), which is compatible with (2.2), and so presents no immediate obstruction to our goal.
In [3] a formal canonical transformation from the conjugate variables \((A^a_i, E^{ja})\) to a new set \((B^{ia}, C^a_j)\) was presented. Now we take the simpler viewpoint that \(B^{ia}\) is a useful, dependent variable on the original configuration space. The action of the electric field on functionals \(\psi[B]\) is determined by the chain rule

\[
E^{ia} = -i \frac{\delta}{\delta A^a_i(x)} = -i \int d^3y \frac{\delta B^{jb}(y)}{\delta A^a_i(x)} \frac{\delta}{\delta B^{jb}(y)}
\]

(2.9)

The generator of gauge transformations then acts as

\[
G^a(x) \psi[B] = D_i E^{ia} \psi[B]
= -i \varepsilon^{ijk} D_j \frac{\delta}{\delta B^{ka}} \psi[B]
= -i \varepsilon^{ijk} B^{ib}(x) \frac{\delta}{\delta B^{ic}(x)} \psi[B].
\]

(2.10)

This is of the form of an “angular momentum” because \(B\) and its canonical conjugate \(C = -i \delta / \delta B\) transform homogeneously.

Next we note that the positive symmetric tensor \(\phi^{ij} = B^{ia} B^{ja}\) is gauge invariant, so that any functional \(\psi[\phi]\) satisfies the physical state constraint (2.6). Although \(\phi^{ij}\) has 6 independent components, which is the correct number necessary to describe the gauge invariant content of a field configuration, it is not quite satisfactory because it does not give a complete description of the gauge-invariant subspace of the configuration space. Naturally, \(\det B\) is an independent invariant, but \((\det B)^2\) can be expressed in terms of \(\phi\), so only the sign of \(\det B\) is independent. Thus, to use \(\phi\) one must introduce a discrete label \(\alpha = \pm\) and consider \(\psi[\phi, \alpha]\).

Alternatively we avoid this problem by introducing another gauge invariant variable which has the same dimension as \(B^{ia}\), namely the tensor

\[
G_{ij} = B^a_i B^a_j \det B,
\]

(2.11)

where \(B^{ia} B^a_j = \delta^i_j\). One finds that \(\det G = \det B\), and that \(G_{ij}\) is either positive- or negative-definite. The relation between \(G\) and \(\phi\) is

\[
\phi^{ij} = \frac{1}{2} \varepsilon^{ik\ell} \varepsilon^{jmn} G_{km} G_{\ell n} = G^{ij} \det G,
\]

(2.12)

and one may show that

\[
F_{ij}^a F_{k\ell}^a = G_{ik} G_{j\ell} - G_{jk} G_{i\ell}.
\]

(2.13)

The magnetic energy density is

\[
\frac{1}{2g^2} \delta_{ii} B^{ia} B^{ja} \frac{\partial}{\partial \psi} = \frac{1}{2g^2} \delta_{ii} \phi^{ii} \frac{\partial}{\partial \psi} = \frac{1}{2g^2} (\delta^{ij} \delta^{kk} - \delta^{ik} \delta^{jk}) G_{jj} G_{kk} \frac{\partial}{\partial \psi}^2
\]

(2.14)

The variable \(G_{ij}\) was used previously to describe constant gauge fields on the torus by Lüscher [6] and others [7]. Indeed for constant potentials \(A^a_i\), (2.11) reduces to

\[
G_{ij} = A^a_i A^a_j
\]

(2.15)

which is exactly the variable used in [6].
3. Geometry

The plan now is to work out the form taken by the electric energy density in gauge-invariant variables. As an intermediate step we use the $\phi$ variable, but later return to $G$. The functional chain rule gives

$$\frac{\delta}{\delta A_i^a} \psi[\phi] = \frac{\delta B^{kb}}{\delta A_i^a} \frac{\delta \phi^{mn}}{\delta B^{kb}} \frac{\delta \psi[\phi]}{\delta \phi^{mn}}$$

$$= 2 \epsilon^{ijk} D_j^{ab} \left( B^{\ell b} \frac{\delta \psi}{\delta \phi^{k \ell}} \right)$$

$$= 2 \epsilon^{ijk} \left[ B^{\ell a} \partial_j \frac{\delta \psi}{\delta \phi^{k \ell}} + D_j B^{\ell a} \frac{\delta \psi}{\delta \phi^{k \ell}} \right].$$

We define a connection-like quantity $\Gamma'^m_{j \ell}$ by

$$B^{\ell a} \Gamma'^m_{j \ell} \equiv -D_j B^{ma}$$

so that (3.1) can be rewritten as

$$\frac{\delta}{\delta A_i^a} \psi[\phi] = 2 \epsilon^{ijk} B^{\ell a} \left[ \delta^m_j \partial_j - \Gamma'^m_{j \ell} \right] \frac{\delta \psi}{\delta \phi^{km}}.$$  

Let us analyze the properties of $\Gamma'$, first multiplying (3.2) by $B^{ia}$ to get

$$\phi^{i \ell} \Gamma'^m_{j \ell} \equiv -B^{ia} D_j B^{ma}.$$  

The $im$ symmetric part of this equation can be written as

$$\partial_j \phi^{im} + \Gamma'^i_{j \ell} \phi^{\ell m} + \Gamma'^m_{j \ell} \phi^{i \ell} = 0$$

which looks like a metric-compatible condition for the (inverse) metric $\phi^{im}$. Next differentiate (3.2) and manipulate as follows

$$D_i \left( B^{at} \Gamma'^m_{j \ell} \right) \equiv -D_i D_j B^{am}$$

$$B^{ak} \left( \partial_i \Gamma'^m_{jk} - \Gamma'^i_{ik} \Gamma'^m_{j \ell} \right) = -D_i D_j B^{am}$$

With the help of the gauge theory Ricci identity, the $ij$ anti-symmetric part of (3.6) can be written as

$$\partial_i \Gamma'^m_{jk} - \partial_j \Gamma'^m_{ik} - \Gamma'^i_{ik} \Gamma'^m_{j \ell} + \Gamma'^j_{jk} \Gamma'^m_{i \ell} = -B^a_k \left[ D_i, D_j \right] B^{ma}$$

$$= -\epsilon^{abc} \epsilon_{ij \ell} B^{a}_k B^{b} B^{mc}$$

$$= -\epsilon_{ij \ell} \epsilon^{\ell mn} B^{a}_k B^{a}_n \det B$$

$$= \delta^m_j G_{ik} - \delta^m_i G_{jk}.$$
A geometrical interpretation is not yet at hand, because (2.12) shows that both $\phi$ and $G$ cannot transform as tensors, which would be necessary for (3.5) and the curvature-like (3.7) to be compatible. Since (3.7) signals that $G$ is tensorial, we insert (2.12) in (3.5), and substitute

$$\Gamma'_{j\ell} = -\frac{1}{2}\delta^i_j \partial \ln \det G + \Gamma_{j\ell}^i,$$  \hspace{1cm} (3.8)

or equivalently,

$$\frac{B_{\ell a}}{\sqrt{\det G}} \Gamma'_{j\ell} = -D_j \left( \frac{B^{ia}}{\sqrt{\det G}} \right).$$  \hspace{1cm} (3.9)

Then, (3.5) becomes

$$\partial_j G^{im} + \Gamma_{j\ell}^i G_{\ell m} + \Gamma_{m j\ell}^i G_{i \ell} = 0.$$  \hspace{1cm} (3.10)

The $\partial_j \ln \det G$ term drops out on the left side of (3.7), and we find that the curvature tensor of the $\Gamma$ connection satisfies

$$R^\ell_{\kappa ij} \equiv \partial_i \Gamma^\ell_{jk} - \partial_j \Gamma^\ell_{ik} - \Gamma_{ik}^m \Gamma^\ell_{jm} + \Gamma_{jk}^m \Gamma^\ell_{im} = \delta^\ell_j G_{ik} - \delta^\ell_i G_{jk},$$  \hspace{1cm} (3.11)

while the Ricci tensor obeys

$$R_{kj} = -2G_{kj}.$$  \hspace{1cm} (3.12)

One more contraction gives $R = -6$ for the Ricci scalar. One may now interpret (3.10) as a metric compatibility condition for $\Gamma$ with respect to $G$, so that $\Gamma$ must take the form of a connection with torsion, namely

$$\Gamma^i_{jk} = \tilde{\Gamma}^i_{jk} - K^i_{jk},$$

$$\tilde{\Gamma}^i_{jk} = \frac{1}{2} G^{i\ell} (\partial_j G_{\ell k} + \partial_k G_{j\ell} - \partial_\ell G_{jk})$$

$$K^j_{jk} = -K^j_{jk}.$$  \hspace{1cm} (3.13)

Clearly (3.12) is the Einstein condition for the (in principle non-symmetric) Ricci tensor. For a metric-compatible connection,

$$R_{\ell kij} = -R_{k\ell ij}$$  \hspace{1cm} (3.14)

in addition to the manifest antisymmetry in $ij$. One can then easily show (using $\epsilon$-tricks) that, even with torsion, the Riemann tensor for $D = 3$ is fully determined by its contractions and takes the form

$$R_{ij\kappa \ell} = G_{ik} R_{j\ell} - G_{i\ell} R_{jk} - G_{jk} R_{i\ell} + G_{j\ell} R_{ik} - \frac{R}{2} (G_{ik} G_{j\ell} - G_{i\ell} G_{jk}).$$  \hspace{1cm} (3.15)

Thus (3.11) and the simpler (3.12) have the same content.

An apparent further constraint on the geometry follows from the gauge theory Bianchi identity (2.8). When applied to (3.4) one finds that $\Gamma'_{i\kappa} = 0$. Using (3.8) and (3.13) we see that this is equivalent to the additional trace condition on the contortion

$$K^j_{jk} = 0.$$  \hspace{1cm} (3.16)
which implies that $K$ can be represented as

$$K^{ij}_{jk} = \epsilon_{jkn} S^{ni} \frac{1}{\sqrt{\det G}},$$

(3.17)

where $S^{ni}$ is a symmetric tensor and $\epsilon_{jkn}$ is defined at the end of this Section.

It turns out that (3.16) can also be derived directly from geometry without reference to gauge theory. To do this one starts with a contracted form of the second Bianchi identity of a curvature tensor with torsion [8], and uses (3.11) and (3.12) to replace the Riemann and Ricci tensors by metrics. This quickly gives (3.16) which can thus be viewed as an integrability condition for an Einstein space with torsion.

Equations (3.10 - 17) completely define the spatial geometry associated with the gauge-invariant subspace of the configuration space of $SU(2)$ gauge theory. It is worthwhile to emphasize that given a potential $A^a_i$, the magnetic field $B^{ia}$, the metric $G_{ij}$, and the connection $\Gamma^i_{jk}$ can be calculated directly from the formulas (2.2), (2.11), and (3.9) respectively. $K$ and $R$ can then be calculated through (3.13), and all geometrical conditions are then satisfied. Later we will begin to address the converse question, namely given a symmetric tensor $G_{ij}$, can one find a contortion tensor $K_{ijk}$ satisfying (3.16) and such that the Einstein condition (3.12) is satisfied. This is essentially the question whether the change of variables $A^a_i \to B^{ia} \to G_{ij}$ is invertible.

We now need some notation to cope with the original fiber-bundle geometry of the gauge theory and the new spatial geometry. We use $D_i$ to denote the gauge theory covariant derivative, as implicitly defined in (2.4 - 5), which "sees" only gauge indices. Then $\nabla_i$ and $\hat{\nabla}_i$ are used to denote spatial derivatives with and without torsion, e.g., on a covariant vector

$$\nabla_i V_j \equiv \partial_i V_j - \Gamma^k_{ij} V_k$$

$$\hat{\nabla}_i V_j \equiv \partial_i V_j - \hat{\Gamma}^k_{ij} V_k .$$

(3.18)

Later we will use $R$ and $\hat{R}$ to denote curvatures with and without torsion. The Levi-Civita density $\epsilon^{ijk}$ takes the usual values $\pm 1, 0$, and one applies 3 factors of $G_{il}$, etc. to obtain $\epsilon_{lmn}$ so that $\epsilon_{lmn}/\sqrt{\det G}$ is a tensor. Indeed, indices of all quantities are raised and lowered from their initially defined form with $G$, while all contractions with the Cartesian metric $\delta_{ij}$ are indicated explicitly. (The exceptions to these conventions are that $B^{ia}$, $B^a_i$ and $\phi^{ij}$, $\phi_{ij}$ are matrix inverses, and $\epsilon_{ijk}$ in (3.7) = $\{\pm 1, 0\}$. Throughout, $\sqrt{\det G}$ means $\sqrt{|\det G|}$.)

4. The Energy Density in Geometric Variables

Let us return to a more physical question, namely, the form of the Hamiltonian. With gravity neglected, this cannot be invariant under spatial diffeomorphisms. This is already clear from the magnetic energy density (2.14) which involves contractions of the true metric $\delta_{ij}$ and the gauge-invariant tensor $G_{ij}$.

For the electric energy density we might expect noncovariance both due to the $\delta_{ij}$ contraction, and because the derivative appearing in (3.3) does not appear to be fully covariant. It turns out, however, that this is not the case, and in fact we find that the
unique source of noncovariance in the energy density lies in the $\delta_{ij}$ contraction needed to square the electric and magnetic fields in the Hamiltonian.

To find the electric field, we use (3.3), (3.8), and the torsion tensor

$$ T_{jk}^m = \Gamma_j^m_{jk} - \Gamma_k^m_{jk} = -K_{jk}^m + K_{kj}^m $$

and obtain

$$ \frac{\delta \psi}{\delta A_i^a} = 2\epsilon^{ijk} B^{\ell a} \left( \tilde{\nabla}_j \frac{\delta \psi}{\delta \phi^{k\ell}} + \frac{1}{2} T_{jk}^m \frac{\delta \psi}{\delta \phi^{m\ell}} \right) $$

$$ \equiv 2\epsilon^{ijk} B^{\ell a} D_j \frac{\delta \psi}{\delta \phi^{k\ell}} , $$

where

$$ \tilde{\nabla}_j \frac{\delta \psi}{\delta \phi^{k\ell}} \equiv \partial_j \frac{\delta \psi}{\delta \phi^{k\ell}} - \Gamma_j^m \frac{\delta \psi}{\delta \phi^{m\ell}} - \Gamma_j^m \frac{\delta \psi}{\delta \phi^{m\ell}} + \frac{1}{2} \partial_j (\ln \det G) \frac{\delta \psi}{\delta \phi^{k\ell}} . $$

This expression is precisely the covariant derivative of a rank 2 covariant tensor density of weight $-1$. To see that $\frac{\delta \psi}{\delta \phi^{k\ell}}$ really is a density of weight $-1$, we note that in order for $G_{ij}$ to be of weight 0, $B^{ai}$ must be a density of weight +1, from which it follows that $\phi_i^j$ has weight +2, and thus if $\psi[\phi]$ is an invariant functional, $\frac{\delta \psi}{\delta \phi^{k\ell}}$ will have weight $-2 + 1 = -1$. From (4.2) it then follows that the electric field is, like the magnetic field, a vector density of weight +1. It can be checked that this weight assignment to $B^{ai}$, together with the fact that the Levi-Civita symbol $\epsilon^{ijk}$ has weight +1, leads to consistent tensor densities throughout.

We can now easily write down the electric energy density:

$$ \frac{1}{2} g^2 \delta_{ii} \frac{\delta \psi^*}{\delta A_i^a} \frac{\delta \psi}{\delta A_i^a} = 2 g^2 \delta_{ii} \epsilon^{ijk} \epsilon^{\ell\ell} \phi^{k\ell} \left( D_j \frac{\delta \psi^*}{\delta \phi^{k\ell}} \right) \left( D_j \frac{\delta \psi}{\delta \phi^{k\ell}} \right) . $$

(4.3)

If we consider wavefunctionals $\psi[G]$ rather than $\psi[\phi]$, then we should rewrite the electric field in terms of derivatives w.r.t. $G_{ij}$. This is done by simply taking (4.2) together with the chain rule giving

$$ \frac{\delta}{\delta \phi^{k\ell}} = \frac{\delta G_{pq}}{\delta \phi^{k\ell}} \frac{\delta}{\delta G_{pq}} = \frac{1}{2 \det G} \left( G_{pq} G_{k\ell} - 2 G_{pk} G_{q\ell} \right) \frac{\delta}{\delta G_{pq}} . $$

(4.4)

Due to metric compatibility, it is possible to push the covariant derivative through this prefactor, and we find

$$ \frac{\delta \psi}{\delta A_i^a} = \frac{\epsilon^{ijk} B^{\ell a}}{\det G} \left( G_{pq} G_{m\ell} - 2 G_{pm} G_{q\ell} \right) \left( \delta_k^m \tilde{\nabla}_j - K_{jk}^m \right) \frac{\delta \psi}{\delta G_{pq}} , $$

(4.5)

with the caveat that this covariant derivative is now acting on a contravariant density of weight +1, so that connections and the density term are changed in sign from (4.2).

We note also that the electric energy density,

$$ \frac{1}{2} g^2 \delta_{ii} \frac{\delta \psi^*}{\delta A_i^a} \frac{\delta \psi}{\delta A_i^a} \equiv \delta_{ii} \mathcal{E}^{ii} $$

(4.6)
is given by the contraction of a fixed Cartesian tensor, \( \delta_{\bar{ij}} \), with a rank 2 contravariant tensor of \( GL(3) \) with weight 2. Alternatively, one can use (4.5) and form the contraction

\[
\delta_{\bar{ij}} \epsilon^{ijk} \epsilon^{\bar{k}\bar{q}} = \delta^{i\bar{j}} \delta^{k\bar{q}} - \delta^{i\bar{k}} \delta^{\bar{j}q}
\]

which is another fixed Cartesian tensor which then multiplies a fourth rank \( GL(3) \) tensor of definite symmetry. One can see from (2.14) that the \( GL(3) \) properties of the magnetic energy density are exactly the same. In fact, this \( GL(3) \) behavior also holds in the original variable \( A^a_i \) if it is taken to transform as a covariant vector. Because we are now dealing with the explicit geometric variable \( G_{ij} \), one may hope that the definite \( GL(3) \) transformation property of the Hamiltonian might lead to a group theoretic approach to gauge field dynamics.

We now wish to give a preliminary discussion about the energy barriers which appear because we have reexpressed the theory in terms of gauge invariant variables. It is clear that the transformation involves both \( G_{ij} \) and its inverse \( G^{ij} \), so that there is a singularity when \( \det G = \det B = 0 \). Indeed, there are explicit singular factors of \( (1/\det G) \) in (4.5), and more singularities in the connection which enters both (4.3) and (4.5). We will be more specific about the nature of these energy barriers in Section 7, where we restrict to submanifolds of the function space where we can find explicit expressions for \( K_{ij}^k \). However, it is clear that wavefunctionals of finite energy must vanish in a certain manner for field configurations \( B^{ia} \) or \( G_{ij} \) whose determinant vanishes somewhere in space.

5. Inversion of the Transformation \( A^a_i \to B^{ai} \to G_{ij} \)

A basic assertion of our approach to \( SU(2) \) gauge theory is that any locally gauge invariant variable, such as the contortion \( K_{ij}^k \), can be expressed in terms of the tensor \( G_{ij} \). As we will see, we must expect these expressions to be non-local, and they seem to be bi-unique; e.g. there are two configurations \( K_{ij}^k \) for each configuration of \( G_{ij} \). It is correct that any functional \( \psi[G_{ij}] \) is gauge invariant, but it is certainly not convenient to express all gauge-invariant functionals in this form. One can also construct gauge-invariant functionals using, e.g., Wilson loops and Chern-Simons terms. For this we would like to be able to reconstruct \( B^{ia} \) and \( A^a_i \), up to an \( SU(2) \) gauge transformation, from \( G_{ij} \). The forward map \( A^a_i \to B^{ia} \to G_{ij} \) automatically satisfies the geometrical conditions (3.10, .12, .16), but we would like to know how big is the image of the space of vector potentials \( A^a_i \) within the space of symmetric tensors \( G_{ij} \). This is the type of question we discuss in this section. We will give reasonable arguments that the situation is favorable but there is more to be done.

Let us first discuss the reconstruction of \( A^a_i \) given \( G_{ij} \) and \( \Gamma^k_{ij} \). This is elementary, but specific to the gauge group \( SU(2) \), which is locally isomorphic to the tangent space group of a 3-manifold. We consider the quantities

\[
\begin{align*}
    b^{ia}(x) &= \frac{1}{|\det B(x)|^{1/2}} B^{ia}(x) \\
    b^a_i(x) &= |\det B(x)|^{1/2} B^a_i(x)
\end{align*}
\]

(5.1)
which are matrix inverses. For \( \det B > 0 \), we see from (2.11) that \( b^a_i \) is an orthonormal frame for \( G_{ij} \), and it is well-defined when \( \det B < 0 \). Thus we start the reconstruction by diagonalizing \( G_{ij}(x) \), writing

\[
G_{ij}(x) = \pm R^a_i(x) \lambda_a(x) R^a_j(x) \tag{5.2}
\]

where the upper(lower) sign refers to the the case \( \det G > 0(<0) \), and \( R^a_i(x) \) is a special orthogonal matrix. The eigenvalues satisfy \( \lambda_a > 0 \). We then take

\[
b^a_i = \pm \sqrt{\lambda_a} R^a_i \quad \text{no sum on } a. \tag{5.3}
\]

Any other “frame” is related to this by application of an orthogonal matrix on the left. The magnetic field associated with the “metric” \( G_{ij} \) is then defined by

\[
B^{ia} = \sqrt{\det G} b^{ia}. \tag{5.4}
\]

We then rewrite (3.9) as

\[
\Gamma^i_{jk} = b^{ia} D_j b^a_k = b^{ia} (\partial_j b^a_k + \epsilon^{abc} A^b_j b^c_k), \tag{5.5}
\]

which can be rearranged as a “dreibein postulate”

\[
\partial_j b^a_k - \Gamma^i_{jk} b^a_i + \epsilon^{abc} A^b_j b^c_k = 0 \tag{5.6}
\]

from which we can identify the vector potential as the spin connection, viz.,

\[
\epsilon^{abc} A^c_j = -\omega^a_j = \frac{1}{2} \epsilon^{abc} b^{ka}(\partial_j b^b_k - \Gamma^i_{jk} b^b_i). \tag{5.7}
\]

From the standpoint of the inverse map, it may not be clear that \( B^{ia} \) and \( A^a_i \) now defined, respectively, as the frame and spin connection for the geometry, satisfy the gauge theory relation (2.2). However, (2.2) is a direct consequence of the Einstein space condition (3.11). Contracting this with \( b^n_i b^b_b \) one finds (for both signs of \( \det B !)\),

\[
R^{ab}_{ij} = \partial_i \omega^a_j - \partial_j \omega^a_i + \omega^{ac}_{ij} \omega^c_j - \omega^{ac}_{ji} \omega^c_i \\
= -\det B (B^a_i B^b_j - B^a_j B^b_i) \\
= -\frac{1}{\det B} \epsilon^{abc} \epsilon_{ijk} B^{kc}. \tag{5.8}
\]

This is equivalent to (2.2) if (5.7) is used to relate \( \omega \) and \( A! \).

Since both \( G_{ij} \) and \( \Gamma^k_{ij} \) are needed to reconstruct \( A^a_i \), we must ask how to find \( \Gamma \), given \( G \). This means that one must be able to solve (3.12) to find a contortion tensor which
satisfies (3.16). To see what this involves, we expand out (3.12) by splitting \( \Gamma = \hat{\Gamma} - K \), finding, with the help of (3.16),

\[
\hat{R}_{ij} - \hat{\nabla}_\ell K_{j\ell} - K_{jm} K_{i\ell}^m = -2G_{ij} .
\] (5.9)

This constitutes 9 equations for the 6 independent components of \( K \), but it turns out that there is a Bianchi identity which imposes 3 relations among these equations, thus one expects that there is a solution for \( K_{ij}^k(x) \) for any given configuration \( G_{ij}(x) \).

To analyze (5.9) we first insert the representation (3.17) which gives

\[
\hat{\nabla}_j (S^{kj} - G^{kj} S^p_p) = 0 .
\] (5.10)

while the \( \delta^j_i \) contraction is the purely algebraic condition

\[
\hat{R} + \text{sgn}(\det G)(S^p_i s^j_p - S^p_p S^j_i) = -6 .
\] (5.12)

It is natural to try to generalize the known Bianchi identity

\[
\hat{\nabla}_j (\hat{R}^j_i - \frac{1}{2} \delta^j_i \hat{R}) = 0 .
\] (5.13)

by applying \( \hat{\nabla}_j \) to the difference between (5.10) and \( \frac{1}{2} \delta^j_i \) times the trace (5.12). The use of (5.11) helps to cancel many \( \hat{\nabla}_j S \) terms leading to

\[
0 = -\frac{\varepsilon^{\ell m}}{\sqrt{\det G}} \hat{\nabla}_j \hat{\nabla}_\ell S^m_i + \text{sgn}(\det G)(\hat{\nabla}_j S^m_i - \hat{\nabla}_i S^m_j) S^j_m .
\] (5.14)

The next step is to compute the product of \( \varepsilon^{j pq} S^i_q \) with (5.10). SSS terms miraculously cancel in the resulting expression:

\[
\varepsilon^{j pq} \hat{R}^j_i S^i_q - \frac{\det G}{\sqrt{\det G}} (\hat{\nabla}^p S^q_i - \hat{\nabla}^q S^p_i) S^i_q = 0 .
\] (5.15)

This is exactly what is needed to convert the last term in (5.14) into an algebraic expression which then gives

\[
0 = \varepsilon^{\ell m} \hat{\nabla}_j \hat{\nabla}_\ell S^m_i + \varepsilon_{ijk} \hat{R}^j_k S^{mk} .
\] (5.16)

Finally this reduces to \( 0 = 0 \) when the Ricci identity and the representation (3.15) of the curvature tensor are used. It is possible to investigate the question of independence of the 9 Einstein equations directly from (3.12) using the known form \( [8] \) of Bianchi identities with torsion, but we prefer the present determination which generalizes the familiar Riemannian identity (5.13) by intricate but straightforward manipulation of torsion terms in (5.10).
6. Sample Geometries

In order to develop a better intuition concerning the new spatial geometry, we consider some particular examples of gauge field geometries and obtain the metrics $G_{ij}$ and contortions $K_{ijk}$ associated with them. In other words, we explore the spatial geometry associated with some submanifolds of the function space of $SU(2)$ gauge theory. Among other things, we will find a fringe benefit of the $GL(3)$ invariance. Namely, we can automatically extend our solutions to orbits of the group of diffeomorphisms.

We first explore the case where the magnetic field $B^i_a$ is constant in some gauge in a finite region $V \subset \mathbb{R}^3$, and the potential $A^a_i$ is also constant there (we exclude abelian configurations with linear potentials). Then

$$B^i_a = \frac{1}{2} \epsilon^{abc} \epsilon^{ijk} A^b_j A^c_k, \quad (6.1)$$

which implies $\det B = (\det A)^2 \geq 0$. Equation (6.1) can be inverted to find two solutions for $A_i^a$ namely

$$A^a_i = \pm \sqrt{\det BB^a_i}. \quad (6.2)$$

Thinking in terms of the forward map, we compute from (2.11) and (6.2)

$$G_{ij} = A_i^a A_j^a \quad (6.3)$$

which is an arbitrary positive constant matrix (see (2.15)).

To obtain the connection, which is pure contortion for constant fields, we use the definition (3.9), and obtain

$$K_{ij}^k = B^a_j D_i B^{ka}$$

$$= \pm \frac{\epsilon_{ij}^k}{\sqrt{\det G}} \quad (6.4)$$

where the last line requires (6.2,3) and some calculation using properties of the inverse of 3 $\times$ 3 matrices. Comparing with (3.17) we see that the tensor $S^{ni} = \pm G^{ni}$.

One can also derive (6.4) directly from the Einstein condition (5.9). For constant fields $\hat{R}_{ij}$ and $\hat{\nabla}_\ell K_{ij}^\ell$ vanish, and (5.9) becomes algebraic, viz.,

$$K_{jm}^\ell K_{ki}^m = -2G_{ij}. \quad (6.5)$$

It is easy to show that the only solutions are given by (6.4). Thus the contortion tensor is totally antisymmetric, and it is minimally constructed from the metric $G_{ij}$, hence covariantly constant,

$$\hat{\nabla}_\ell K_{ij}^\ell = 0. \quad (6.6)$$

Of course, the Christoffel geometry is flat, i.e., $\hat{R}_{ij} = 0$.

We may now generalize this solution by introducing arbitrary “coordinate functions” $y^\alpha(x^i)$, with

$$\det \frac{\partial y^\alpha}{\partial x^i} \neq 0$$
and considering the new metric

\[ G'_{\alpha\beta}(y(x)) = G_{ij} \frac{\partial x^i}{\partial y^\alpha} \frac{\partial x^j}{\partial y^\beta} \]  

(6.7)

which is “diffeomorphic” to \( G_{ij} \). Then the transformed version of (5.9) will have the solution

\[ K'_{\alpha\beta} = \pm \epsilon_{\alpha\beta\gamma} \sqrt{\det G} \]  

(6.8)

and one can then reconstruct gauge fields \( B'^{\alpha\alpha}(y(x)) \) and \( A'^{\alpha}_a(y(x)) \) which are “diffeomorphic” to \( B^{\alpha\alpha}(y(x)) \) and \( A^\alpha_a(y(x)) \). The new fields are not constant, and they are related by

\[ B'^{\alpha\alpha}(y(x)) = \frac{1}{2} \epsilon^{abc} \epsilon^{\alpha\beta\gamma} A'^{\beta}_b A'^{\gamma}_c . \]  

(6.9)

It is in this way that a correspondence between gauge field configurations and geometries on subspaces of the function space of the theory can be extended by action of the group of diffeomorphisms.

There is one other important implication of this sample geometry. Although we do not know the general solution of (5.9), it presumably must reduce to (6.4), whenever \( \tilde{R}_{ij}(x) \) vanishes in any finite subset of \( \mathbb{R}^3 \). It therefore seems correct to infer that in the general case, there are always two solutions \( K_{ij}^k \) for a given metric \( G_{ij} \).

The second class of gauge field configurations to be studied are those where, in some gauge, the potential takes the form

\[ A^a_i(x) = \lambda A^a_i(x) \]  

(6.10)

where \( \lambda \) is any real number and \( A^a_i(x) \) is a pure gauge, i.e.,

\[ \partial_i A^a_j(x) - \partial_j A^a_i(x) + \epsilon^{abc} A^b_j(x) A^c_i(x) = 0 . \]  

(6.11)

The magnetic field is then

\[ B^{ia}(x) = \lambda (\lambda - 1) \frac{1}{2} \epsilon^{ijk} \epsilon^{abc} A^b_j A^c_k \]  

(6.12)

\[ = \lambda (\lambda - 1) A^{ia} \det A , \]

where \( \det A = \det A^a_i \) and \( A^{ia} \) is the matrix inverse of \( A^a_i \). These gauge fields were first studied in [9] as an example of the Wu-Yang ambiguity, in which different potentials, for \( \lambda \) and \( 1 - \lambda \), give the same magnetic field.

The gauge-invariant “metric” variable \( G_{ij} \) is given by

\[ G_{ij}(x) = \lambda (\lambda - 1) A^a_i(x) A^a_j(x) . \]  

(6.13)
One sees that \( \det B = \det G = [\lambda(\lambda - 1)]^3(\det A)^2 \), so we have \( \det G < 0 \) for \( 0 < \lambda < 1 \), and \( \det G > 0 \) otherwise.

To understand this geometry we use the representation

\[
-\frac{i}{2} \tau^a A_i^a = U^{-1} \partial_i U
\]  

(6.14)

where \( U(x) \) is an arbitrary \( SU(2) \) matrix. Then (6.11) is satisfied, and one finds

\[
G_{ij} = 2\lambda(\lambda - 1) \text{Tr} \partial_i U \partial_j U^{-1}.
\]  

(6.15)

One can then write

\[
U(x) = \alpha_4(x) + i\vec{\tau} \cdot \vec{\alpha}(x)
\]

\[
\sum_{s=1}^{4} \alpha_s^2 = 1,
\]

(6.16)

and obtain

\[
G_{ij}(x) = 4\lambda(\lambda - 1) \sum_{s=1}^{4} (\partial_i \alpha_s)(\partial_j \alpha_s).
\]

(6.17)

Since

\[
ds^2 \equiv G_{ij} dx^i dx^j
\]

\[
= 4\lambda(\lambda - 1) \sum_{s=1}^{4} d\alpha_s d\alpha_s
\]

(6.18)

with \( \alpha_s \) a unit 4-vector, one sees that \( G_{ij} \) is proportional to a metric on the round 3-sphere. Globally there may be multiple coverings of the sphere by the map \( \alpha_s(x) \) from \( \mathbb{R}^3 \). In this case a diffeomorphism is implemented by \( \alpha'_s(y(x)) = \alpha_s(x) \) and \( A'^a_s(y(x)) = (\partial x^i/\partial y^\alpha) A^a_i(x) \). This leaves us within the initially defined class of potentials.

To compute the connection one uses the definition (3.9), separated into two terms:

\[
\frac{B^{ka}}{\sqrt{\det G}} \Gamma^i_{jk} = - \left[ \partial_j \left( \frac{B^{ia}}{\sqrt{\det G}} \right) + \epsilon^{abc} A^b_j \frac{B^{ic}}{\sqrt{\det G}} \right].
\]

(6.19)

The second term is similar to that of the constant case,

\[
-\epsilon^{abc} A^b_j \frac{B^{ic}}{\sqrt{\det G}} = \pm \left( \frac{B^{ka}}{\sqrt{\det G}} \right) \epsilon^{ijk} \frac{\lambda A^i_k}{|\lambda(\lambda - 1)|^{1/2} \sqrt{\det G}}.
\]

(6.20)

To treat the first term we use (6.12) and obtain

\[
-\partial_j \left( \frac{B^{ia}}{\sqrt{\det G}} \right) = \frac{B^{ka}}{\sqrt{\det G}} A^{ib} \partial_j A^b_k.
\]

(6.21)
The last factor is split onto symmetric and antisymmetric terms in $jk$. One uses (6.11) for the latter and (6.11) plus the definition of $\hat{\Gamma}_{jk}^i$ in (3.13) for the former. The result is

$$-\partial_j \left( \frac{B^{ia}}{\sqrt{\det G}} \right) = -\frac{B^{ka}}{\sqrt{\det G}} \left[ \hat{\Gamma}_{jk}^i \mp \frac{1}{2} \frac{\epsilon_{ijk}}{\lambda(\lambda - 1)^{1/2} \sqrt{\det G}} \right].$$  

(6.22)

The contortion tensor can be identified from (6.20) and (6.22) as

$$K_{jk}^i = \mp \frac{|\lambda - \frac{1}{2}|}{|\lambda(\lambda - 1)|^{1/2} \sqrt{\det G}} \epsilon_{ijk}.$$  

(6.23)

The sign in (6.23) is the product $\mp = -\text{sgn}[\lambda(\lambda - 1)] \text{sgn det } \mathcal{A}$. So the contortion is again the minimal totally antisymmetric structure.

The two Wu-Yang related potentials $\lambda \mathcal{A}$ and $(1 - \lambda) \mathcal{A}$ give contortion tensors of opposite signs, as in (6.23), although the interpretation of the signs is best stated in terms of the inverse map. Suppose one is given a metric of the form (6.18) and a pair of contortion tensors of the form

$$K_{jk}^i = \mp \frac{|\lambda - \frac{1}{2}|}{|\lambda(\lambda - 1)|^{1/2} \sqrt{\det G}} \epsilon_{ijk}.$$  

(6.24)

Then from $G$ and $K$ with the upper sign, one will reconstruct via (5.4) and (5.7) a potential $A^i_0(x)$ which is in general not proportional to a pure gauge, but is the gauge transform of either $\lambda \mathcal{A}$ or $(1 - \lambda) \mathcal{A}$. Then for $G$ and $K$ with lower sign, the reconstruction leads to another potential which is gauge equivalent to the Wu-Yang related form $(1 - \lambda) \mathcal{A}$ or $\lambda \mathcal{A}$, respectively.

The 3-sphere is an Einstein space, and one can compute by the standard method of the Cartan structure equations that the metric (6.17) satisfies

$$\hat{\mathcal{O}}_{ij} = \frac{1}{2\lambda(\lambda - 1)} G_{ij}.$$  

(6.25)

This suggests that we investigate special solutions of the geometrical equations (5.9) in which $G_{ij}$ is an Einstein space in the Riemannian sense and $K_{jk}^i$ is minimal, viz.,

$$\hat{\mathcal{O}}_{ij} = \Lambda G_{ij}$$

$$K_{ij}^k = c \frac{\epsilon_{ijk}}{\sqrt{\det G}}.$$  

(6.26)

Then (5.9) is satisfied if the parameters are related by

$$c^2 = \text{sgn}(\det G)(\Lambda + 1).$$  

(6.27)

If $\Lambda > 0$, then (6.26) is satisfied by a positive definite $G_{ij}$ which is locally a round metric on $S^3$. Since $\hat{\mathcal{O}}_{ij}(G) = \hat{\mathcal{O}}_{ij}(-G)$, we see that for $\Lambda < 0$, (6.26) is satisfied by a negative-definite $G_{ij}$ which is the negative of a round $S^3$ metric. In this case the torsion is real only
if $\Lambda < -1$. Both ranges $\Lambda > 0$ and $\Lambda < -1$ are covered if we take $\Lambda = 1/(4\lambda(\lambda - 1))$ as in (6.25), and it is easy to see that the magnitude of the contortion $c$ agrees with (6.23). Thus the solutions of (6.26) we are now discussing give Deser-Wilczek potentials under the inverse map. We believe that they exhaust this class of potentials, although we are uncertain of global issues such as multiple coverings of $S^3$.

If $\Lambda < 0$, then there are certainly positive definite metrics $G_{ij}$ which are solutions of (6.26). These should also correspond to gauge field configurations, at least in the range $-1 < \Lambda < 0$ in which the torsion is real.

We also suggest that it may be useful to investigate solutions of (5.9) restricted by the condition of spherical symmetry. This should lead to a system of ordinary differential equations for 4 radial functions, 2 contained in the most general ansatz for a spherically symmetric $G_{ij}$ and 2 more for $S^{ij}$. The corresponding gauge field configurations should include monopole solutions of Yang-Mills-Higgs systems with all but spatial potentials $A_i^a$ and $B^{ia}$ discarded.

7. Energy Barriers

We now write the energy density in a form suitable for our discussion. It is useful to define the quantities

$$V^{impq} = \varepsilon^{ijk} \left( \delta_k^m \tilde{\nabla}_j - K_{jk}^m \right) \frac{\delta \psi}{\delta G_{pq}}$$

$$Q^{im} = G_{pq} (V^{impq} - 2V^{iqp}) .$$

(7.1)

Then by straightforward index-shuffling one can reexpress the energy density (4.6) in the form

$$\delta_{ii} \mathcal{E}^{ii} = \frac{1}{2} g^2 \frac{\delta}{\delta G^{im}} Q^{*im} Q^{im} .$$

(7.2)

Separating out the torsion from the derivative $\tilde{\nabla}$ in (7.1), one can obtain

$$Q^{im} = G_{pq} \left( \varepsilon^{ijm} \tilde{\nabla}_j \frac{\delta \psi}{\delta G_{pq}} - 2 \varepsilon^{ijq} \tilde{\nabla}_j \frac{\delta \psi}{\delta G_{pm}} \right)$$

$$+ \varepsilon^{ijk} \left( K_{jk}^m G_{pq} \frac{\delta \psi}{\delta G_{pq}} - 2K_{jn}^m G_{pk} \frac{\delta \psi}{\delta G_{pm}} \right)$$

(7.3)

where $\tilde{\nabla}_j$ is a torsion-free covariant derivative with density term.

We now specialize to the cases discussed in Section 6 where the contortion takes the minimal form (6.26). The torsion term in (7.3) then simplifies, leading to

$$Q^{im} = G_{pq} \left( \varepsilon^{ijm} \tilde{\nabla}_j \frac{\delta \psi}{\delta G_{pq}} - 2 \varepsilon^{ijq} \tilde{\nabla}_j \frac{\delta \psi}{\delta G_{pm}} \right)$$

$$+ 2 c \sgn(\det G) \sqrt{\frac{\det G}{\delta G^{im}}} .$$

(7.4)

When this is substituted in (7.2), one finds an expression for the energy density which can be seen to contain terms with spatial derivatives which are singular when $\det G \to 0$. 
at any point in space and terms without spatial derivatives which are regular. Positive definiteness implies that the singularity does not cancel in any simple way. Instead the gauge invariant state functional $\psi[G]$ must be constrained in a complicated way near the barrier in order to have finite total energy. The energy barriers are certainly present for general configurations $G_{ij}$ for which (7.3) must be used. They are simply clearer when the restriction is made to one of the flat or constant curvature configurations of Section 6 because the contortion tensor is explicitly known. One can also consider the case where a field configuration $G_{ij}$ reduces to a constant curvature or flat metric on some finite subset $V \subset \mathbb{R}^3$. The form (7.4) is then valid for $x \in V$, but the functional $\psi[G]$ and its derivatives depend on the behavior of $G_{ij}$ throughout space.

There are similar energy barriers at $\det \phi = 0$, if the energy density is expressed in terms of the $\phi^{ij}$ variables as in (4.3), and they are known to be present in the electric formulation [1] of nonabelian gauge theories. We believe that they are a general feature of any formulation of the theory in gauge invariant variables, because the transformation to these variables is nonlinear and inevitably has singular points. The perturbative wave functional is not of the form $\psi[G]$, but because it peaks at zero magnetic field where $\det B = 0$, we would expect significant non-perturbative corrections.

Finally we would like to point out that for fields which are constant throughout space, spatial derivative terms in (7.4) can be dropped, and (6.4) tells us that $c = \pm 1$. The electric energy density then becomes simply

$$\delta_{ii} \mathcal{E}^{ii} = \frac{1}{2} g^2 \delta_{ii} G_{m\bar{m}} \frac{\delta \psi^*}{\delta G_{im}} \frac{\delta \psi}{\delta G_{im}}. \quad (7.5)$$

Using (6.3) we see that this is equal to

$$\delta_{ii} \mathcal{E}^{ii} = \frac{1}{2} g^2 \delta_{ii} \frac{\delta \psi^*}{\delta A_{i}^a} \frac{\delta \psi}{\delta A_{i}^a}. \quad (7.6)$$

It is no surprise that this coincides with the electric part of the Hamiltonian used in [6],[7] to describe constant modes of the $SU(2)$ gauge field in a box.

8. Open Questions

The spatial geometry found in the gauge-invariant configuration space of $SU(2)$ gauge theory has not been recognized previously. It is a rather simple geometry, and a natural question is the generalization to larger gauge groups, notably the true color group $SU(3)$. Here the situation is that more independent variables are required to describe the gauge-invariant degrees of freedom of the magnetic field $B^{ia}$. In $SU(3)$ one can take [3] the 6 components of $\phi^{ij} = B^{ia} B^{ja}$ and the 10 components of the symmetric tensor density formed with the $d$-symbol, namely $d^{abc} B^{ia} B^{ib} B^{kc}$. The rectangular matrix $B^{ia}$ has many "right-inverses" and in general no "left-inverse". Partly because of this, careful navigation will be required to find the underlying geometry, but we are optimistic that it can be found.

It is far from clear that the geometry will be helpful in understanding the dynamics of the $SU(2)$ gauge theory. Indeed there are several problems to be overcome if $G_{ij}$ is to be viewed as the fundamental variable. First, one must be able to solve (5.9) for the
contortion to obtain an explicit expression for the Hamiltonian. Second one must represent
the functional Jacobian \( \frac{\delta B^a(x)}{\delta A^b(y)} \) which involves the determinant of a differential
operator with singular symbol. Third one must deal with the fact that a gauge invariant
formulation of the theory, whether magnetic or electric [1], is essentially nonperturbative.
There are \( 1/g \) terms which remain in \( H \) after rescaling the fields either by an integral or
fractional [6] power of \( g \). In the past geometry has provided strong impetus for physicists,
and we hope that it will stimulate new approaches to the important problem of gauge field
dynamics.

**Appendix. Tranformations of the Measure**

As mentioned briefly in Sec. 8 a complete change of variables \( A \rightarrow B \rightarrow G \) requires
consideration of the Jacobian determinant. This will enter in the measure of the trans-
formed functional integral over \( G_{ij}(x) \), and terms from the measure will be generated when
a functional integration by parts is performed in Eq. (7.2) to put the Hamiltonian in the
more standard second derivative form.

The essential part of the measure is the functional determinant of the transformation
operator from \( A \) to \( B \), namely

\[
M^{ij}(x,y) \equiv M^{ia,jb}(x,y) \equiv \frac{\delta B^a(x)}{\delta A^b_j(y)} = \epsilon^{imj} D^{ab}_{mn} \delta^{(3)}(x - y) \tag{A.1}
\]

This operator is gauge covariant, but a manifestly gauge invariant form is useful. This
can be achieved by conjugating with the ultralocal algebraic operator \( B_{jb}^{\ell k}(x,y) \equiv \delta_j^k B^{lb}(x) \delta^{(3)}(x - y) \). Namely one convolutes \( M_{ia,jb}(x,y) \) with this operator from the right
and with its inverse from the left. This gives the new invariant differential operator

\[
N_{mn}^{\ell k}(x,y) \equiv B_{mn}^{-1} \cdot M_{ia,jb} \cdot B_{jb}^{\ell k}(v,y) = \epsilon_m^{ik} (\delta_n^\ell \partial_i - \Gamma_n^\ell_{in}) \delta^{(3)}(x - y), \tag{A.2}
\]

where (3.2) has been used. One nows sees explicitly from (4.2) that \( N \) involves the same
covariant derivative that occurs in the transformed electric field. Formally \( M \) and \( N \) have
the same determinant, so we see that the measure has been brought to geometric form.

The remaining terms in the functional measure are ultralocal factors from the trans-
formation \( \delta G/\delta B \) and the integration over the time-independent \( SU(2) \) gauge group at
each point \( x \) of \( \mathbb{R}^3 \). We simply state the final result:

\[
[dG_{ij}] \frac{\det G}{\det N}. \tag{A.3}
\]

Functional determinants in general require ultraviolet regularization, and this is diffi-
cult in the present case where the highest derivative term of \( N \) has zero modes. We leave
such problems for future investigation because the measure is not of direct concern in this
paper.
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