Intertwined isospectral potentials in an arbitrary dimension

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The method of intertwining with n-dimensional (nD) linear intertwining operator \( \mathcal{L} \) is used to construct nD isospectral, stationary potentials. It has been proven that differential part of \( \mathcal{L} \) is a series in Euclidean algebra generators. Integrability conditions of the consistency equations are investigated and the general form of a class of potentials respecting all these conditions have been specified for each \( n = 2, 3, 4, 5 \). The most general forms of 2D and 3D isospectral potentials are considered in detail and construction of their hierarchies is exhibited. The followed approach provides coordinate systems which make it possible to perform separation of variables and to apply the known methods of supersymmetric quantum mechanics for 1D systems. It has been shown that in choice of coordinates and \( \mathcal{L} \) there are a number of alternatives increasing with \( n \) that enlarge the set of available potentials. Some salient features of higher dimensional extension as well as some applications of the results are presented.

PACS:03.65.Fd, 03.65.Ge, 02.30.Ik

I. INTRODUCTION

The method of intertwining provides a unified approach to constructing exactly solvable linear and nonlinear problems and their hierarchies in various fields of physics and mathematics [1-3]. This is closely connected with the supersymmetric (SUSY) methods such as Darboux’s transformation, Schrödinger’s factorization, and shape invariant potential concept which deal with pairs of Hamiltonians having the same energy spectra but different eigenstates [4]. In general, the object of the intertwining is to construct the so called intertwining operator \( \mathcal{L} \) which performs an intertwining between two given operators of the same type (differential, integral, matrix, or, operator-valued matrix operator, etc.). In the context of quantum mechanics \( \mathcal{L} \) is taken to be a linear differential operator which intertwines two Hamiltonian operators \( H_0 \) and \( H_1 \) such that

\[
\mathcal{L} H_0 = H_1 \mathcal{L}.
\]

(1)

Two simple and important facts that are at the heart of the usefulness of this method can be stated as follows; (i) If \( \psi^0 \) is an eigenfunction of \( H_0 \) with eigenvalue of \( E^0 \) then \( \psi^1 = \mathcal{L} \psi^0 \) is an (unnormalized) eigenfunction of \( H_1 \) with the same eigenvalue \( E^0 \). Hence \( \mathcal{L} \) transforms one solvable problem into another. (ii) When \( H_0 \) and \( H_1 \) are Hermitian (on some common function space) \( \mathcal{L}^\dagger \) intertwines in the other direction \( H_0 \mathcal{L} = \mathcal{L}^\dagger H_1 \) and this in turn implies that \( [H_0, \mathcal{L}^\dagger \mathcal{L}] = 0 = [\mathcal{L} \mathcal{L}^\dagger, H_1] \), where \( ^\dagger \) and \( [\cdot,\cdot] \) stand for Hermitian conjugation and commutator. Therefore, two hidden dynamical symmetry operators of \( H_0 \) and \( H_1 \) are immediately constructed in terms of \( \mathcal{L} \). These are dimension and form independent general properties of this method [3]. Despite this fact, like the above mentioned SUSY methods, the intertwining method is mostly studied in the context of one dimensional (1D) systems where \( \mathcal{L} \) is taken to be first order differential operator and Hamiltonians are in the standard potential forms. Two additional properties that arise in that case are that [4]; (i) Every eigenfunction of \( H_0 \) (without regard to boundary conditions or normalizability) can be used to generate a transformation to a new solvable problem (see Eq. (20) below). (ii) A direct connection to a SUSY algebra can be established by constructing a diagonal matrix Hamiltonian \( H = \text{diag}(H_0, H_1) \) and two nilpotent supercharges \( Q^+ = (Q^-)^\dagger \) such that the only nonvanishing element of \( Q^+ \) matrix is \( Q_{21}^+ = \mathcal{L} \). These obey the defining relations of the simplest SUSY algebra

\[
\{Q^+, Q^-\} = H, \quad (Q^+)^2 = (Q^-)^2 = 0,
\]

which imply \( [H, Q^\pm] = 0 \) and emphasize in a compact algebraic form of the spectral equivalence of two 1D systems. In the nomenclature of the SUSY quantum mechanics \( \mathcal{L} \) is known as the supercharge operator and its zeroth-order (in derivatives) term as the super-potential.

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There are important studies in the literature which aim to generalize the SUSY methods beyond 1D problems. These can be classified as (i) Curved-space approach \(1,2,3\) (for recent studies see \(4,14\)), and (ii) Matrix-Hamiltonian approach \(4,5,11,13\). Both are based on the intertwining method and they mostly concentrate on extension to two dimensions.

The application of the first approach to quantum mechanics was motivated by Ref. \(1\) which deals with free particle propagation on a Lie group manifold (see also \(14\)). Later on, this has been advanced to find the propagator of a free particle moving on an \(nD\) sphere \(1\) as well as to solve both ordinary and partial differential equations with applications to symmetric spaces \(2\). These approaches are expected to produce solvable 1D \(n\)-particle problems from an \(nD\) free motion via some projection methods like that used in Refs. \(13,16\). The second approach, appeared for the first time in Ref. \(11\), performs the extension by preserving the connection with a SUSY algebra \(12,13\). This inevitably restricts the consideration to two matrix Hamiltonian such that one of them has off-diagonal entries. Accordingly, a matrix with elements having higher order derivative terms participates as the intertwining operator. This approach establishes the equivalence of two matrix systems but does not establish spectral equivalence between two scalar Hamiltonians. To improve it in this regard, an algorithm called the polynomial SUSY in which the classification given above is by no means exhaustive; for instance one may find a nice method based on integral intertwining operator in Ref. \(3\) (section 2.8) to generate a hierarchy of 2D problems. We should also note that recently the intertwining method has been used for the non-stationary Schrödinger operator \(3,17,18\).

The main purpose of this paper is to extend the intertwining method to an arbitrary dimension by applying it to a pair of \(nD\) systems characterized by Hamiltonian operators of potential form

\[
H_i = -\nabla^2 + V_i, \quad i = 0, 1,
\]

where the potentials \(V_i\) and eigenvalues of \(H_i\) are expressed in terms of \(2m/\hbar^2\) and \(\nabla^2 = \sum_{j=1}^{n} \partial_j^2\) is the Laplace’s operator of \(R^n\). We shall use the Cartesian coordinates \(\{x_k; k = 1, \ldots, n\}\), the convention \(\partial_k = \partial/\partial x_k\) and the abbreviation \(V_i \equiv V_i(x_1, \ldots, x_n)\) throughout the paper. We purpose the ansatz that \(\mathcal{L}\) is the most general first-order linear operator

\[
\mathcal{L} = L_0 + L_d = L_0 + \sum_{k=1}^{n} L_k \partial_k
\]

where \(L_0, L_k\) are some functions of \(\{x_k; k = 1, \ldots, n\}\) which together with \(V_i\) are to be determined from consistency equations of Eq. (1). In terms of the vector field \(\vec{L} = (L_1, \ldots, L_n)\) and \(nD\) gradient operator \(\vec{\nabla}\) the operator \(L_d = \sum_{k=1}^{n} L_k \partial_k\) will be usually written as \(L_d = \vec{L} \cdot \vec{\nabla}\), where \(\cdot\) denotes the usual Euclidean inner product.

In the next section by solving the first \(n(n+1)/2\) consistency equations we will show that the operator \(L_d\) is a series in generators of the Euclidean algebra in \(n\)-dimension. There remain \(n + 1\) consistency equations which consist of \(n\) linear and \(1\) non-linear partial differential equations. Some particular solutions of these equations for an arbitrary \(n\) are presented in section III. In section IV we take up the integrability conditions of the remaining \(n\) linear equations in the context of the Frobenius integrability theory \(19\). General forms of the potentials respecting all integrability conditions for \(n = 2, 3, 4, 5\) are obtained in section V. A detailed investigation of 2D and 3D isospectral potentials are given in sections VI, VII where we also exhibit how to generate hierarchies of potentials.

II. INTERTWINING IN \(n\) DIMENSION: EUCLIDEAN ALGEBRA

In view of (2) and (3) the intertwining relation (1) can be written as

\[
[\nabla^2, L_d] = -[\nabla^2, L_0] + [V_0, L_d] + P\mathcal{L},
\]

where \(P = V_1 - V_0\). At a glimpse of the right hand side of Eq. (4) and

\[
[\nabla^2, L_d] = \sum_{j,k} (\partial_j^2 L_k) \partial_k + 2 \sum_j (\partial_j L_j) \partial_j^2 + 2 \sum_{j<k} (\partial_j L_k + \partial_k L_j) \partial_j \partial_k,
\]

\[
[\nabla^2, L_0] = (\nabla^2 L_0) + 2 \sum_j (\partial_j L_0) \partial_j,
\]

\[
[V_0, L_d] = -(L_d V_0) = - \sum_j L_j (\partial_j V_0),
\]
we see that the second order derivatives in Eq. (4) come, together with some first order derivatives, only from $[\nabla^2, L_d]$. Therefore by setting their coefficients to zero we obtain two sets of consistency equations:

$$\partial_j L_j = 0, \quad j = 1, \ldots, n; \quad \partial_j L_k + \partial_k L_j = 0, \quad j < k = 2, \ldots, n.$$  \hfill (8)

The first set gives $L_j = a_j + f_j(x)$, where $a_j$’s are constants and $f_j(x)$ depends on all of $x_k$’s except $x_j$. The second set determines $f_j$ as $f_j = \sum_k c_{jk} x_k$ where $c_{jk}$’s are all constants and antisymmetric in $j$ and $k$ : $c_{jk} + c_{kj} = 0$. Hence

$$L_j = a_j + \sum_k c_{jk} x_k.$$  \hfill (9)

These solutions make the first order derivative terms at the right hand side of (5) vanish so that $[\nabla^2, L_d] = 0$. As a result of this the intertwining relation (4) simplifies to

$$[\nabla^2, L_0] = [V_0, L_d] + P(L_0 + L_d).$$  \hfill (10)

From (6), (7) and (10) we get, by equating the coefficients of the first and zeroth powers of derivatives

$$2\partial_j L_0 = PL_j; \quad j = 1, 2, \ldots, n,$$ \hfill (11)

$$(-\nabla^2 + P)L_0 = (L_d V_0).$$  \hfill (12)

These $n + 1$ equations constitute a reduced form of the consistency conditions for three unknown functions $L_0, V_0$ and $V_1$. While Eq. (12) is non-linear, Eqs. (11) are linear since all components of $L_d$ have been found.

Eq. (12) can be considered in the following way. By virtue of

$$\partial_j L_k = c_{kj},$$  \hfill (13)

Eqs. (11) imply that

$$\nabla^2 L_0 = \frac{1}{2}(L_d P).$$  \hfill (14)

Combining this with (12) we arrive at

$$L_0 P = \frac{1}{2} L_d (V_1 + V_0).$$  \hfill (15)

which can be used instead of Eq. (12).

By defining

$$T_j = \partial_j, \quad L_{jk} = x_k \partial_j - x_j \partial_k,$$  \hfill (16)

and using (9) $L_d$ can be written as

$$L_d = \sum_j a_j T_j + \sum_{j<k} c_{jk} L_{jk}.$$  \hfill (17)

The generators $T_j$’s and $L_{jk}$’s obey the following commutation relations

$$[T_j, T_k] = 0,$$

$$[T_j, L_{km}] = \delta_{jm} T_k - \delta_{jk} T_m,$$

$$[L_{jk}, L_{\ell m}] = \delta_{jm} L_{\ell k} - \delta_{j\ell} L_{mk} + \delta_{k\ell} L_{mj} - \delta_{km} L_{\ell j}.$$  \hfill (18)

These are the defining relations of $n(n+1)/2$ dimensional Euclidean algebra $e(n)$, also known as the algebra of rigid motion denoted by $iso(n)$ \cite{1,2}. n translational generators $T_j$’s form the invariant subalgebra $t(n)$ and $n(n-1)/2$ rotational generators $L_{jk}$’s form the semisimple subalgebra $so(n)$. As is well known $e(n)$ is semi-direct sum of $t(n)$ and $so(n)$ and $\sum T_j^2 = \nabla^2$ is a Casimir operator of $e(n)$.

Now, we shall show that the above analysis includes and naturally generalizes the well known 1D case. It is evident that for $n = 1$ we have $L = L_0 + \partial_x$ and $P = 2L_0(x)$, where we take $x = x_1, a_1 = 1$ and we use the prime(s) to denote differentiation(s) with respect to the argument (when there is no risk of confusion the argument will be suppressed).
In that case Eq. (15) yields \( \partial_x(V_0 + L'_0 - L_0^2) = 0 \) from which we recover the well-known forms of the 1D partner potentials:

\[
V_0 = L_0^2 + L'_0 + b, \quad V_1 = L_0^2 + L'_0 + b.
\] (19)

It is a standard procedure of 1D SUSY quantum mechanics to take the constant \( b \) and \( L_0 \) as \( b = \lambda_1 \) and \( L_0(x) = -\partial_x[\ln \phi_1(x)] \). When these are substituted into the first equation of (19) we obtain: 

\[ -\partial_x' (x) + V_0 \phi_1(x) = \lambda_1 \phi_1(x), \]

that is, \( \phi_1(x) \) is the eigenfunction of the Schrödinger’s equation \( -\partial^2/\partial x^2 + V_0 \phi(x) = \lambda \phi(x) \) corresponding to the eigenvalue \( \lambda = \lambda_1 \). Therefore the Schrödinger’s equation remains covariant under the Darboux’s transformations

\[
(\phi, V_0) \rightarrow (L \phi = \phi' - [\ln \phi_1]' \phi, \ V_1 = V_0 - 2[\ln \phi_1]'').
\] (20)

Obviously, instead of \( \phi_1 \), any other fixed eigenfunction can be used to generate a transformation to another new potential \( V_1 \). It is this fact which allows us to apply the Darboux’s transformations successively and to construct a hierarchy of potentials for a given \( V_0 \).

We conclude this section by saying that for \( n \geq 1 \) the differential part of the intertwining operator is a series in generators of \( \mathfrak{e}(n) \). In saying that we have identified the algebra generated by \( \partial_x \) with \( \mathfrak{e}(1) \). A related result is that intertwined potentials have symmetry generators differential part of which are quadratic in the generator of \( \mathfrak{e}(n) \), that is, they belong to universal enveloping algebra of \( \mathfrak{e}(n) \). From now on we assume that \( a_j \)'s and \( c_{jk} \)'s are real constants.

### III. Applications

Before proceeding further we consider some particular cases of Eqs. (11) and (12).

When \( P = 0 \) Eqs. (11) and (15) give \( L_0 = \text{constant} \) and \( (L_0 V_0) = 0 \). In view of Eq. (1) these imply, as an expected result, that \( \mathcal{L} \) is a symmetry generator of \( H_0 = H_1 : [H_0, \mathcal{L}] = 0 \).

Next we take \( P \) to be a constant such that \( P = p_0 \neq 0 \). In that case the integrability conditions \( \partial_j \partial_k L_0 = \partial_k \partial_j L_0 \) of Eqs. (11) require that \( c_{jk} = 0 \) for all \( j, k \) which lead to \( 2L_0 = p_0 \vec{a} \cdot \vec{r} + 2b \), where \( \vec{r} = (x_1, \ldots, x_n) \) is the position vector and \( \vec{a} \) represents the constant vector \( \vec{a} = (a_1, \ldots, a_n) \). Taking the constant \( b \) as \( b = \vec{a} \cdot \vec{b} \) we get from (15)

\[
\vec{a} \cdot (p_0^2 \vec{r}^2 + 2p_0 \vec{b} \cdot \vec{r} + 2 \vec{V} V_0) = 0,
\]

which is solved by

\[
V_0 = \frac{1}{4} p_0^2 \vec{r}^2 + p_0 \vec{b} \cdot \vec{r} + g(x),
\]

(21)

where \( \vec{b} \) is a constant vector, \( r^2 = \sum_j x_j^2 \) and \( g(x) \equiv g(x_1, \ldots, x_n) \) is any differentiable function subjected to the constraint \( \vec{a} \cdot \vec{V} g(x) = 0 \). One may take

\[
g(x) = g(\vec{b}_{(1)} \cdot \vec{r}, \ldots, \vec{b}_{(n-1)} \cdot \vec{r}),
\]

(22)

such that \( \vec{b}_{(j)} \)'s are linearly independent vectors perpendicular to \( \vec{a} \). Different choices of \( g \) define different systems which accept

\[
\mathcal{L}^\dagger \mathcal{L} = -(\vec{a} \cdot \vec{V})^2 + |\vec{a} \cdot (\frac{1}{2} p_0 \vec{r} + \vec{b})|^2 - \frac{1}{2} p_0 a^2,
\]

(23)

as a common symmetry generator. Accordingly

\[
\mathcal{L} \mathcal{L}^\dagger = -(\vec{a} \cdot \vec{V})^2 + |\vec{a} \cdot (\frac{1}{2} p_0 \vec{r} + \vec{b})|^2 + \frac{1}{2} p_0 a^2,
\]

(24)

is a common symmetry generator for \( V_1 = V_0 + p_0 \). These also imply that \( \mathcal{L}/a \) and \( \mathcal{L}^\dagger/a \) are a pair of ladder operators for \( H_0 \):

\[
[H_0, \mathcal{L}] = -p_0 \mathcal{L}, \quad [H_0, \mathcal{L}^\dagger] = p_0 \mathcal{L}^\dagger, \quad [\mathcal{L}, \mathcal{L}^\dagger] = p_0 a^2.
\]

As a result of these we recover the existence of harmonic oscillator like spectrum in the spectrum of a class of \( nD \) systems described by \( H_0 \) which contains many parameters and an arbitrary function.
Now we set all of $c_{jk}$’s to zero. From (11) and (15) we get $L_0 = f(\zeta)$ and $P = f'(\zeta)$ where $f$ is an arbitrary differentiable function of $\zeta = \vec{a} \cdot \vec{r}/2$. Defining

$$V_{\pm} = \frac{1}{a^2} f^2(\zeta) \pm \frac{1}{2} f'(\zeta)$$

(25)

we obtain, by virtue of (11) and (15)

$$V_0 = \frac{1}{2} g(x) + V_-; \quad V_1 = \frac{1}{2} g(x) + V_+,$$

(26)

where $g(x)$ may be taken as in (22). Observing that $V_{\pm}$ are form equivalent to (19) we can say that all the known techniques of 1D SUSY quantum mechanics can equally well be used in this case. For this application the intertwining operator is $L = f(\zeta) + \vec{a} \cdot \vec{\nabla}$ and the symmetry generators are

$$L L^\dagger = a^2 V_+ - (\vec{a} \cdot \vec{\nabla})^2, \quad L^\dagger L = a^2 V_- - (\vec{a} \cdot \vec{\nabla})^2.$$

(27)

IV. INTEGRABILITY CONDITIONS

In this section we concentrate on the integrability conditions of $n$ linear equations given by (11). It turns out that once these conditions are well understood all the consistency equations can be tackled more easily.

By considering $L_0$ as the $(n+1)$-th coordinate $x_{n+1} \equiv L_0$ of $R^{n+1}$ and $P$ as a function defined on it we introduce the 1-form

$$\Omega = dL_0 - \frac{1}{2} P \Gamma,$$

(28)

on $R^{n+1}$. Here $d$ stands for the exterior derivative and $\Gamma$ denotes the 1-form

$$\Gamma = \sum_{j=1}^{n} L_j dx_j,$$

(29)

on $R^n$. Now $n$ linear equations given by (11) can be expressed as a single Pfaffian equation $\Omega = 0$. In the Frobenius theory, integrability of this Pfaffian equation amounts to being able to find a positive valued integrating factor $f$ and a function $g$ such that $\Omega = f dg$ [13]. If this is possible then $\Omega = 0$ and $dg = 0$ are equivalent Pfaffian equations and the solution (integral surface) of $\Omega = 0$ is the hypersurface $g =$constant. According to the Frobenius theorem a necessary and sufficient condition for the existence of functions $g$ and $f$ is the fulfillment of the so called Frobenius condition:

$$\Omega \wedge d\Omega = 0,$$

(30)

where $\wedge$ denotes the usual exterior product.

From (28) and (29) we have

$$d\Omega = -\frac{1}{2} [(\partial_{n+1} P) dL_0 \wedge \Gamma + \sum_{j=1}^{n} (\partial_j P) dx_j \wedge \Gamma + P d\Gamma],$$

and therefore

$$\Omega \wedge d\Omega = -\frac{1}{2} [dL_0 \wedge d(P \Gamma) - \frac{1}{2} P^2 \Gamma \wedge d\Gamma],$$

(31)

where $d$ in $d(P \Gamma)$ and $d\Gamma$ stands for the exterior derivative of $R^n$. The Frobenius conditions (30) is therefore equivalent to the following two conditions

$$d(P \Gamma) = 0,$$

$$\Gamma \wedge d\Gamma = 0,$$

(32)

(33)

provided that $P \neq 0$. Since both of these conditions are valid in $R^n$, $P$ is defined on $R^n$. 

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The condition (32) gives \( n(n-1)/2 \) equations

\[ K_{jk}P = -2c_{jk}P, \quad (34) \]

where

\[ K_{jk} = L_j \partial_k - L_k \partial_j. \quad (35) \]

Observe that Eq. (34) can also be obtained from \( \partial_j \partial_k L_0 = \partial_k \partial_j L_0 \) and in deriving it we have used Eq. (13). The condition (33) could also be inferred from Eq. (32) upon exterior multiplication of \( dP \wedge \Gamma + Pd\Gamma = 0 \) by \( \Gamma \). It leads to \( n(n-1)(n-2)/6 \) equations:

\[ L_j c_{k\ell} = 0, \quad (36) \]

where \( j < k < \ell \leq n \) and the square bracket \([\quad]\) enclosing the subindexes means anti-symmetrization. Eq. (36) shows that any three of \( L_j \)'s are linearly dependent, that is

\[ L_j c_{k\ell} + L_k c_{\ell j} + L_\ell c_{jk} = 0. \quad (37) \]

Making use of (9) this can be written as

\[ a_{[j}c_{k\ell]} = \sum_m x_m c_{ml}[j k \ell]. \quad (38) \]

This gives nothing in the case of \( n = 2 \) because \( \Gamma \wedge d\Gamma \) is a 3-form and therefore identically vanishes on \( R^2 \).

By a simple reasoning making use of the anti-symmetry of \( c_{jk} \)'s we see that for \( n = 3 \) the right hand side of Eq. (38) vanishes identically and a single condition

\[ \bar{L} \cdot \bar{c} = \bar{a} \cdot \bar{c} = 0 \quad (39) \]

results. Here we have made use of the fact that in the case of \( n = 3 \) we have

\[ \bar{L} = \bar{a} + \bar{r} \times \bar{c} \quad (40) \]

where \( \bar{c} = (c_1, c_2, c_3) = (c_{23}, c_{31}, c_{12}) \) and “\( \times \)” stands for the usual cross product of \( R^3 \). For \( n \geq 4 \) more care is needed. It is not hard to check that \( c_{m[l}c_{k\ell]} = 0 \) for any \( n \) and hence for \( n = 4 \) the terms \( c_{i[j}c_{k\ell]}, ..., c_{i[l}c_{k\ell]} \) are equal to each other up to a sign “\( - \)”. These imply that in the case of \( n = 4 \), Eqs. (38) restrict all the coordinates to some constant values. But, as is evident from Eqs. (38), at the expense of constraining the form of \( L \) we can get rid of all these coordinate restrictions by imposing the conditions

\[ a_{[j}c_{k\ell]} = 0 \quad ; \quad j < k < \ell, \quad (41) \]

\[ c_{m[l}c_{k\ell]} = 0 \quad ; \quad m = 1, ..., n. \quad (42) \]

As is mentioned above in the case of \( n = 4 \) Eqs. (42) give only one condition

\[ c_{12}c_{34} + c_{14}c_{32} + c_{13}c_{42} = 0, \]

and Eqs. (41) give conditions which reduce the total number of parameters. To see this more concretely we define the following four vectors

\[ \bar{c}_{(1)} = (0, c_{34}, -c_{24}, c_{23}), \quad \bar{c}_{(2)} = (c_{34}, 0, c_{41}, -c_{31}), \]

\[ \bar{c}_{(3)} = (c_{24}, c_{41}, 0, c_{12}), \quad \bar{c}_{(4)} = (c_{23}, c_{31}, c_{12}, 0). \]

Now Eqs. (41) can be rewritten as \( \bar{a} \cdot \bar{c}_{(j)} = 0, j = 1, 2, 3, 4 \). It is easy to check that, in view of Eq. (42), the determinant of the matrix formed by the components of the vectors \( \bar{c}_{(j)} \)'s has rank two. Therefore Eqs. (41) provide two of \( a_{ij} \)'s as free parameters, or, for a given \( \bar{a} \) two constraints for \( c_{jk} \)'s. By taking into account also (42) we get seven free parameters: five \( c_{jk} \)'s and two \( a_{ij} \)'s, or, three \( c_{jk} \)'s and four \( a_{ij} \)'s. These can be chosen in many different ways. Moreover, one can also chose a lesser number of parameters without destroying the integrability conditions.

For the number of conditions implied by (41-42) exceeds the number of parameters the investigation is getting harder and harder for \( n \geq 5 \). But, in the case of \( n = 5 \) one can keep again 5 of \( c_{jk} \)'s as free parameters by setting all \( a_{ij} \)'s to zero. In that case Eqs. (41) disappear and Eqs. (42) give 5 constraints which reduce the number of \( c_{jk} \)'s from 10 to 5. Note also that, as has been done in section III, for \( n \geq 2 \) one can always set all \( c_{jk} \)'s to zero and keep \( n a_{ij} \)'s as parameters. In such a case the condition (36) completely disappears.

These remarks imply an important property of the intertwining method in higher dimensions; due to integrability conditions there are a number of choices in specifying \( L \). Evidently this fact enriches the set of intertwined potentials (see the Table I in the case of \( n = 3 \)). In the next section by taking \( c_{jk} \neq 0 \) for at least a pair of \( j,k \), we carry out an investigation which will enable us to find out the general forms of a class of potentials for \( n = 2, 3, 4, 5 \) endowed with mentioned richness for \( n \geq 3 \).
V. GENERAL FORM OF POTENTIALS

By making use of Eqs. (13), (35) and (37) one can easily verify the following relations

\[ \partial_m \left( \frac{L_i}{L_j} \right) = c_{ij} \frac{L^2}{L_j}, \]  
\[ (43) \]

\[ K_{mn} \left( \frac{L_i}{L_j} \right) = 0, \]  
\[ (44) \]

\[ K_{mn} \left( \frac{1}{L_j} \right) = -k c_{mn} \frac{L^2}{L_j}, \]  
\[ (45) \]

\[ \bar{L} \cdot \nabla (\frac{L_i}{L_j}) = c_{ij} \frac{L^2}{L_j}, \]  
\[ (46) \]

\[ \bar{L} \cdot \nabla (L^2) = (2 \sum_{ij} L_i L_j c_{ij}) g'(L^2) = 0. \]  
\[ (47) \]

In Eq. (47) \( g \) is an arbitrary function of \( L^2 = \sum_j L_j^2 \). Comparing Eqs. (43) and (11) we see that the general form of \( L_0 \) is

\[ L_0 = f \left( \frac{L_i}{L_j} \right), \]  
\[ (48) \]

provided that \( c_{ij} \neq 0 \). Then from any of Eqs. (11) \( P \) is found to be

\[ P = \frac{2c_{ij}}{L_j^2} f'(\eta), \]  
\[ (49) \]

where \( \eta = L_i/L_j \). Fortunately, Eqs. (44) and (45) imply that the solution (49) respects all the integrability conditions given by (34).

The only equation that remained unsolved is Eq. (15) which is now as follows

\[ \bar{L} \cdot \nabla (V_1 + V_0) = \frac{2c_{ij}}{L_j^2} \partial_\eta [f^2(\eta)]. \]  
\[ (50) \]

From (46) and (47) it is evident that the general solution of this equation is of the form

\[ V_1 + V_0 = h + 2 \frac{f^2(\eta)}{L^2}, \]  
\[ (51) \]

where \( 2f^2(\eta)/L^2 \) accounts for the right hand side of (50) and \( h \) is the general solution of the homogeneous equation

\[ \bar{L} \cdot \nabla h = 0. \]  
\[ (52) \]

Hence, the general forms of \( V_0 \) and \( V_1 \) are, by combining (49) and (51)

\[ V_0 = \frac{1}{2} h + \frac{V_1}{L^2}, \]  
\[ (53) \]

\[ V_1 = \frac{1}{2} h + \frac{V_1}{L^2}, \]  
\[ (54) \]

where

\[ V_\pm = f^2(\eta) \pm c_{ij} \frac{L^2}{L_j^2} f'(\eta). \]  
\[ (55) \]

As a result, the number of consistency equations has been reduced from \( (n+1)(n+2)/2 \) (the sum of the number of Eqs. (8),(11) and (12)) to 1, i.e., to Eq. (52). Geometrically, Eq. (52) means that at each point of the surface \( h = \text{constant} \), \( \bar{L} \) always lies on the local tangent space. Equivalently, \( \bar{L} \) is always perpendicular to the (classical) force field determined by \( \nabla h \). On the other hand, from group theoretical point of view Eq. (52) means that the common
part of the intertwined potentials is invariant under the action of the Euclidean group $E(n)$, i.e., $e^{t\xi}h = h$. For all these statements and the integrability conditions are dimension-dependent $h$ must be determined in each case separately. The rest of the paper is devoted to a detailed investigation of $n = 2$ and $n = 3$ cases.

As our investigation for an arbitrary dimension is completed two remarks are in order. (i) The above analysis enables us to write down a class of $nD$ isospectral potentials provided that at least one of $c_{jk}$’s is different from zero. For instance, if only $c_{jk} \neq 0$ then Eqs. (37) imply that $L_m = 0$ for $m \neq j,k$ and Eqs. (11) require $L_0$ to depend only on $x_j$ and $x_k$. In such a case, after defining $\eta = L_j/L_k$ it remains to solve Eq. (52) to find suitable $n - 1$ coordinate functions. (ii) When the number of non-zero $c_{jk}$’s is greater than one there are a number of choices (at most $n(n - 1)/2$) for $\eta$. But, from Eq. (37) we see that these are all functionally dependent to each other. For example, in the case of $n = 3$ we have three choices $\eta = L_1/L_2, \eta_2 = L_1/L_3, \eta_3 = L_2/L_3$ which obey the following relations

$$\eta_3 = \eta_2/\eta, \quad \eta_2 c_{23} + \eta_3 c_{31} = -c_{12}.$$  

Instead of $\eta$, one may choose one of the variables $\alpha_i = L_i/\tilde{c} \cdot \tilde{a} = L_i/\tilde{f} \cdot \tilde{a}$, or for $n = 3, \sigma_i = L_i/\tilde{c} \cdot \tilde{L}_i$. It is easy to verify that each of these satisfies relations similar to Eqs. (43-44) and (46) and enables us to express $L_0, P, V_\pm$ in terms of them. This freedom in the choice of coordinates once again manifests the largeness of the set of intertwined potentials. But, we should emphasize that these are all functionally dependent since the differential of any variable obeying (43) is proportional to $\Gamma = \tilde{L} \cdot \tilde{c} \tilde{\Delta}$ and therefore $d\eta \wedge da_i = 0$, etc. This also proves that as long as first order intertwining is concerned $V_\pm$ depend only on one variable.

VI. 2D ISOSPECTRAL POTENTIALS

In two dimension we have $L_1 = (a_1 + cy)$ and $L_2 = (a_2 - cx)$, where $c = c_{12}, x = x_1, y = x_2$. From Eq. (47) we see that, in terms of

$$\kappa = [L_1^2 + L_2^2]^{1/2} = [(a_1 + cy)^2 + (a_2 - cx)^2]^{1/2}$$  

the general solution of Eq. (52) is $h = h(\kappa)$, where $h$ is an arbitrary differentiable function. Taking $\eta = L_1/L_2$ and noting that $L_2/L_2^2 = 1 + \eta^2$, by Eqs. (53-55) the general forms of the 2D isospectral potentials are found to be

$$V_0 = \frac{1}{2} h(\kappa) + \frac{V}{\kappa^2}, \quad V_1 = \frac{1}{2} h(\kappa) + \frac{V}{\kappa^2},$$  

where

$$V_\pm = f^2(\eta) \pm c(1 + \eta^2)f'(\eta).$$  

In that case the intertwining operator is

$$L = f(\eta) + (a_1 + cy)\partial_x + (a_2 - cx)\partial_y = f(\eta) + c(1 + \eta^2)\partial_\eta.$$  

As is well known, for a 2D stationary system the existence of a symmetry generator means that the system is completely integrable in the Liouville sense. Recalling that $L^1L$ and $LL^1$ are symmetry generators of $H_0$ and $H_1$, the potentials given by (57) are the most general forms of 2D integrable potentials which can be intertwined by a first order operator.

We shall now present some examples in which for some simple forms of $V_\pm$ we consider the Riccati’s equation (58) for dependent variable $f$ and by solving it we construct the corresponding potentials. As the simplest case we take $V_0 = 0$. This may happen in two different cases; (i) $h = 0, V_\pm = 0$, and (ii) $h = -2b/\kappa^2, V_\pm = b$, where $b$ is a constant. In these cases (58) is a separable equation of the form

$$f^2 - c(1 + \eta^2)f' = b,$$  

which has the general solution

$$f = (-b)^{1/2} \tan\left[\frac{(-b)^{1/2}}{c}(\tan^{-1} \eta - b_1)\right]$$  

for $b < 0$. This should be read as $f = b^{1/2}\tanh[|b|^{1/2}/c(1 - \tan^{-1} \eta)]$ for $b > 0$ and as $f = c(b_1 - \tan^{-1} \eta)^{-1}$ for $b = 0$, where $b_1$ is an integration constant. From (58) we have
for the case (i) and
\[ V_1 = -2b\{\kappa \cos\left[\left(-\frac{b^{1/2}}{e}(\tan^{-1} \eta - b_1)\right)\right]\}^{-2}; \]
\[ V_1 = -2b\{\kappa \cosh\left[\left(-\frac{b^{1/2}}{e}(\tan^{-1} \eta - b_1)\right)\right]\}^{-2}, \]  
(63)
for the case (ii) corresponding to \( b < 0 \) and \( b > 0 \) respectively. As a result we have found a two parameter family of 2D potentials that are intertwined to 2D free motion. Note that for \( b = -c^2, b_1 = 0 \) we have \( f = c\eta \) and
\[ V_1 = 2c^2\eta^2 + \frac{1}{\kappa^2} = \frac{2c^2}{(a_2 - cx)^2}. \]  
(64)
As another example, taking \( V_0 = b = -c^2 \) and \( h = \left(2c^2/\kappa^2\right) + 2g(\kappa) \) in (57) leads us to the partner potentials
\[ V_0 = g(\kappa), \quad V_1 = g(\kappa) + 2c^2\frac{1 + \eta^2}{\kappa^2} \]  
for \( f = c\eta \). In particular, for \( g(\kappa) = \kappa^2 \), \( H_0 \) represents a 2D isotropic displaced harmonic oscillator and \( H_1 \) a 2D Calogero’s type system for which
\[ V_1 = \frac{2c^2}{(a_2 - cx)^2} + (a_2 - cx)^2 + (a_1 + cy)^2. \]  
(66)
In that case for any choice of \( g(\kappa) \) we have \( \mathcal{L} = c[\eta + (\eta^2 + 1)\partial_\eta] \). This explicitly shows that two different families of potentials, such as that given by (65) can be intertwined by the same \( \mathcal{L} \). This is an important property that we do not have in one dimension. It is evident that this arises from the separability of the problem that we shall analyze in the next section. It is also worth mentioning that after a simple affine transformation of the coordinates and a restriction on \( c^2 \) one can easily recognize (66) as one of the four superintegrable the Smorodinsky-Winternitz 2D potentials \[21\]. The above particular example shows that this potential is intertwined to the harmonic oscillator and one of its symmetry generators is immediately obtained as \( \mathcal{L}\mathcal{L}^\dagger \).

VII. SEPARATION OF VARIABLES AND HIERARCHY OF 2D POTENTIALS

The above analysis suggests the variables \((\kappa, \eta)\) as a new coordinate system. This is a kind of the orthogonal polar coordinate system with displaced center in which we have
\[ \nabla^2 = \frac{c^2}{\kappa^2} \{\kappa \partial_{\kappa}(\kappa \partial_{\kappa}) + (1 + \eta^2)\partial_\eta(1 + \eta^2)\partial_\eta\}. \]  
(67)
This implies that the eigenvalue equations of \( H_i \) accept the separation of variables in terms of \((\kappa, \eta)\). In fact, this can be carried out in an easier way by introducing the coordinates
\[ \rho = \frac{1}{c} \ln \kappa, \quad \xi = \frac{1}{c} \tan^{-1} \eta. \]  
(68)
From (59) and (67) we get
\[ \mathcal{L} = f(\xi) + \partial_\xi, \]  
(69)
and \( \nabla^2 = e^{-2c\rho}(\partial_\rho^2 + \partial_\xi^2) \). By defining
\[ H_\rho = -\partial_\rho^2 + \frac{1}{2}e^{2c\rho}h(\rho), \quad H_\pm = -\partial_\xi^2 + V_\pm(\xi), \]
and
\[ V_\pm = f^2(\xi) \pm f'(\xi). \]  
(70)
the Hamiltonians can be written as
\[ H_0 = e^{-2cp}(H_\rho + H_-), \quad H_1 = e^{-2cp}(H_\rho + H_+). \]  

(71)

If we take \( \psi^0(\rho, \xi) = R(\rho)U^0(\xi) \) the eigenvalue equation \( H_0\psi^0(\rho, \xi) = E^0\psi^0(\rho, \xi) \) separates as follow

\[ H_-U^0(\xi) = MU^0(\xi), \]  

(72)

\[ (H_\rho - E^0e^{2cp})R(\rho) = -MR(\rho), \]  

(73)

where \( M \) is the separation constant. For given \( E^0 \) \( \rho \)-equation for \( H_1 \) is the same as Eq. (73), but \( \xi \)-equation is \( H_-U^1(\xi) = MU^1(\xi) \). \( L \) given by (69) intertwines only solutions of \( H_- \) to that of \( H_+ \) by \( U^1(\xi) = LU^0(\xi) \).

We shall now briefly describe how to generate a hierarchy of 2D isospectral potentials.

Taking \( f(\xi) = -\phi'(\xi)/\phi(\xi) \) in Eq. (70) yields \( V_- (\xi) = \phi'(\xi)/\phi(\xi) \). This is the same as Eq. (72) for \( M = 0 \). Therefore, each solution of (72) with \( M = 0 \) can be used to generate a transformation to a new problem with potential \( V_1 \). In fact, by keeping analogy with 1D SUSY methods we can do more than that. For this purpose let us take

\[ V_-(\xi) = \mathcal{V}(\xi) - \mathcal{E}_n, \quad f(\xi) = -\frac{\phi'_n(\xi)}{\phi_n(\xi)}, \]  

(74)

in Eq. (70) and suppose that the resulting stationary Schrödinger’s equation

\[ [-\partial^2_\xi + \mathcal{V}(\xi)]\phi_n(\xi) = \mathcal{E}_n\phi_n(\xi) \]  

(75)

is exactly solvable, where \( n \) is a quantum number labelling the eigenvalues and eigenfunctions. If together with (74) we take

\[ h(\rho) = 2e^{-2cp}[\mathcal{H}(\rho) + \mathcal{E}_n], \]  

(76)

then from Eq. (57) \( V_1 \) are found to be

\[ V_0 = e^{-2cp}[\mathcal{V}(\xi) + \mathcal{H}(\rho)], \]
\[ V_1 = e^{-2cp}[2(\frac{\phi'_n(\xi)}{\phi_n(\xi)})^2 + 2\mathcal{E}_n - \mathcal{V}(\xi) + \mathcal{H}(\rho)]. \]

In that case the separated equations of \( H_0 \) are

\[ [-\partial^2_\xi + \mathcal{V}(\xi)]U^0_n(\xi) = (\mathcal{E}_n + M)U^0_n(\xi), \]  

(77)

\[ [-\partial^2_\rho + \mathcal{H}(\rho) - e^{2cp}E^0]R_n(\rho) = -(\mathcal{E}_n + M)R_n(\rho). \]  

(78)

Let us choose \( M \) such that

\[ \mathcal{E}_{n_\pm} = \mathcal{E}_n \pm M \]  

(79)

This amounts to the fact that \( \xi \)-equation of \( H_0 \) is the same as Eq. (75). Therefore, \( U^0_n(\xi) = \phi_{n_+}(\xi) \) and \( E^0, R_n(\rho) \) must be labelled by \( n_+ \). Accordingly Eq. (78) must be rewritten as

\[ [-\partial^2_\rho + \mathcal{H}(\rho) - e^{2cp}E^0_{n_+}]R_{n_+}(\rho) = -\mathcal{E}_{n_+}R_{n_+}(\rho), \]  

(80)

The eigenvalue equation of \( H_1 \) corresponding to the same \( E^0_{n_+} \) can be separated such that the \( \rho \)-equation is the same as Eq. (80) and \( \xi \)-equation reads

\[ [-\partial^2_\xi + 2(\frac{\phi'_n(\xi)}{\phi_n(\xi)})^2 - \mathcal{V}(\xi)]U^1_{n_+}(\xi) = -\mathcal{E}_{n_+}U^1_{n_+}(\xi), \]  

(81)

where

\[ U^1_{n_+}(\xi) = \mathcal{L}U^0_{n_+}(\xi) = [-\frac{\phi'_n(\xi)}{\phi_n(\xi)} + \partial_\xi]\phi_{n_+}(\xi). \]  

(82)

The function \( \phi_n(\xi) \) that generates the transformation is annihilated by the action of \( \mathcal{L} \), i.e., \( \mathcal{L}\phi_n(\xi) = [(\phi'_n(\xi)/\phi_n(\xi)] - \partial_\xi)\phi_n(\xi) = 0 \). Hence, in the case of \( M = 0 \) the function \( U^1_{n_+}(\xi) \) corresponding to \( \phi_n(\xi) \) can
not be found in this way. But, by referring to a well-known theorem of the theory of ordinary differential equations $U_n^1$ can be constructed. This theorem says that if $y_0(x)$ is a particular non-trivial solution of the equation $a_0(x)y'' + a_1(x)y' + a_2(x)y = 0$ then the second solution $y_1$ linearly independent from $y_0$ is given by

$$y_1 = y_0 \int \frac{\exp[- \int \frac{a_0(x)}{y_0} dx]}{y_0^2} dx. \quad (83)$$

Adopting this theorem to Eq. (75) where $a_0 = -1$ and $a_1 = 0$ the second solution linearly independent from $\phi_n(\xi)$ is found to be $Y(\xi) = \phi_n(\xi) \int d\xi/\phi_n^2(\xi)$. $L$ generated by $\phi_n(\xi)$ applied to $Y(\xi)$ gives $Y_n(\xi) = L Y(\xi) = -1/\phi_n(\xi)$. Inserting this (as $y_0$) into (83) yields the desired eigenfunction corresponding to $\phi_n(\xi)$

$$U_n^1(\xi) = - \frac{1}{\phi_n(\xi)} \int \phi_n^2(\xi) d\xi. \quad (84)$$

As a result, changing the eigenfunction of Eq. (75) used to generate the transformation will lead us to a new eigenvalue problem given by Eq. (81). In that way a hierarchy of $2D$ isospectral potentials can be constructed.

### VIII. 3D ISOSPECTRAL POTENTIALS

In order to find the general solution of Eq. (52) for $n = 3$ we firstly recall the integrability condition (39) and the relation (40). Secondly we observe that the set $\{\vec{L}, \vec{c}, \vec{c} \times \vec{c}\}$ forms a right-handed (unnormalized) orthogonal moving frame which “moves” about fixed direction of $\vec{c} = (c_1, c_2, c_3) = (c_{23}, c_{31}, c_{12})$. By using $x = x_1, y = x_2, z = x_3$ we now introduce the variables

$$\beta = \vec{r} \cdot \vec{c},$$

$$\gamma = \frac{1}{2} \vec{r} \cdot (\vec{a} + \vec{L} \times \vec{c}) = \vec{r} \cdot (\vec{a} \times \vec{c}) + \frac{1}{2}[(\vec{r} \cdot \vec{c})^2 - c^2]$$

$$\eta = \frac{L_1}{L_2} = \frac{a_1 + c_3 y - c_2 z}{a_2 - c_3 x + c_1 z}. \quad (85)$$

These obey the following relations

$$\vec{c} \cdot \vec{L} \beta = \vec{c} \cdot \vec{L} \gamma = 0, \quad (86)$$

$$\vec{c} \cdot \vec{L} \eta = p(\eta) \quad (87)$$

where $p(\eta)$ is a quadratic polynomial in $\eta$:

$$p(\eta) = c_3 \frac{L^2}{L_2^2} = \frac{1}{c_3} [2 c_1^2 \eta^2 + 2 c_1 c_2 \eta + (c^2 - c_1^2)]. \quad (88)$$

and $c^2 = c_1^2 + c_2^2 + c_3^2$. In deriving this we assumed $c_3 \neq 0$ and made use of Eq. (37).

It is now easy to see that, in view of (85) and (86), the general solution of (52) is $h = h(\beta, \gamma)$ where $h : R^2 \to R$ is an arbitrary differentiable function. On the other hand, from (53-55) the general forms of the potentials are

$$V_0 = \frac{1}{2} h(\beta, \gamma) + \frac{V_-}{L^2}; \quad V_1 = \frac{1}{2} h(\beta, \gamma) + \frac{V_+}{L^2}, \quad (89)$$

where

$$V_\pm = f^2(\eta) \pm p(\eta) f'(\eta), \quad (90)$$

$$L^2 = a^2 - 2 \gamma. \quad (91)$$

Making use of Eqs. (84-87) $L$ is found to be

$$L = f(\eta) + (\vec{a} + \vec{r} \times \vec{c}) \cdot \vec{\nabla} = f(\eta) + p(\eta) \partial_\eta.$$
If instead of \( \eta = L_1/L_2 \) we had taken one of the variables
\[
\eta_2 = \frac{L_1}{L_3} = \frac{a_1 + c_3 y - c_2 z}{a_3 + c_2 x - c_1 y}, \quad \eta_3 = \frac{L_2}{L_3} = \frac{a_2 - c_3 x + c_1 z}{a_3 + c_2 x - c_1 y},
\]
we would have obtained \((\vec{L} \cdot \nabla) \eta_j = p_j(\eta_j), j = 2, 3 \) and \( V_1 = f_j^2(\eta_j) \pm p_j(\eta_j)f_j'(\eta_j) \), where
\[
p_2(\eta_2) = -c_2 \frac{L_2}{L_3} = -\frac{1}{c_2}[(c^2 - c_3^2)\eta_2^2 + 2c_1c_3\eta_2 + (c^2 - c_1^2)],
\]
\[
p_3(\eta_3) = c_1 \frac{L_2}{L_3} = \frac{1}{c_1}[(c^2 - c_3^2)\eta_3^2 + 2c_2c_3\eta_3 + (c^2 - c_2^2)].
\]
Without any change in the \( \beta, \gamma \) dependence merely \( L \) would have been changed as \( L = f_j(\eta_j) + p_j(\eta_j)\partial_{\eta_j} \).

As an application we again consider the simplest case \( V_0 = 0 \). Following an analysis similar to that made in section VI one can easily verify that the following 3 different potentials:
\[
V_1^{(1)} = 2\frac{f_1^2}{L}, \quad V_1^{(2)} = 2\frac{f_2}{L}^2 \left[ c^2 + \frac{1}{4}p^2(\eta) \right], \quad V_1^{(3)} = 2\frac{f_3}{L}^2 \left[ c^2 + \frac{1}{2}(\frac{p}{b_1 - \eta})^2 \right],
\]
are intertwined to 3D free motion respectively by
\[
\mathcal{L}^{(1)} = f_1 + p(\eta)\partial_\eta, \quad \mathcal{L}^{(2)} = \frac{1}{2}p'(\eta) + p(\eta)\partial_\eta, \quad \mathcal{L}^{(3)} = [\frac{1}{2}p'(\eta) + \frac{p(\eta)}{b_1 - \eta}] + p(\eta)\partial_\eta,
\]
where \( f_1 = [b_1 - (c_3^2)^{-1}\tan^{-1}(p'(\eta)/2c)]^{-1} \) and \( b_1 \) is an integration constant. More generally, a two parameter family of potentials can be constructed by means of \( f = (-b)^{1/2}\tan[(-b)^{1/2}(b + \int d\eta/p(\eta))] \).

We shall now show that in terms of \( (\beta, \gamma, \eta) \) the eigenvalue equations of \( H_i \)'s accept separation of variables. Starting with
\[
d\beta = \vec{c} \cdot d\vec{r}, \quad d\gamma = (\vec{L} \times \vec{c}) \cdot d\vec{r}, \quad d\eta = \frac{c_3}{L_3} \vec{L} \cdot d\vec{r},
\]
onceq one can easily write the differentials \( dx, dy, dz \) in terms of \( d\beta, d\gamma, d\eta \). These are as follow
\[
d\vec{r} = \frac{1}{c^2}\vec{c} d\beta + \frac{1}{c^2L^2} (\vec{L} \times \vec{c}) d\gamma + \frac{L_2}{c_3L^2} \vec{L} d\eta.
\]
With the help of these relations the volume form \( dV = dx \wedge dy \wedge dz \), the metric \( ds^2 = d\vec{r} \cdot d\vec{r} \), and \( \nabla^2 \) are found to be
\[
dV = \frac{1}{c^2p(\eta)} d\beta \wedge d\gamma \wedge d\eta,
\]
\[
ds^2 = \frac{1}{c^2} (d\beta)^2 + \frac{1}{c^2L^2} (d\gamma)^2 + \frac{L_2^4}{c_3^2L^2} (d\eta)^2,
\]
\[
\nabla^2 = c^2[\partial_\beta^2 + \partial_\gamma(\partial_\gamma \partial_\eta)] + \frac{p(\eta)}{L^2} \partial_\eta [p(\eta)\partial_\eta].
\]

From Eq. (94) we infer that the Jacobian determinant of the transformation \( (x, y, z) \rightarrow (\beta, \gamma, \eta) \) is \( 1/c^2p(\eta) \). On the other hand Eq. (95) manifestly shows that the coordinate system \( (\beta, \gamma, \eta) \) is orthogonal. In virtue of (96) the eigenvalue equation \( H_0 \psi^0(\beta, \gamma, \eta) = E^0 \psi^0(\beta, \gamma, \eta) \) separates, by taking \( \psi^0(\beta, \gamma, \eta) = U^0(\beta, \gamma)R^0(\eta) \), as
\[
H_{\beta,\gamma}U^0(\beta, \gamma) = MU^0(\beta, \gamma), \quad H_\eta R^0(\eta) = -MR^0(\eta),
\]
where \( M \) is a separation constant and
\[
H_{\beta,\gamma} = -c^2L^2[\partial_\beta^2 + \partial_\gamma(\partial_\gamma \partial_\eta)] + L^2[\frac{1}{2}h(\beta, \gamma) - E^0],
\]
\[
H_\eta = -p(\eta)\partial_\eta [p(\eta)\partial_\eta] + f^2(\eta) - p(\eta)f'(\eta).
\]
At this point we will be content with saying that by following the similar steps as for section VII one can construct hierarchy of 3D isospectral potentials.

Finally we would like to emphasize that the 3D potentials we have found depend on six parameters such that a large number of potentials can be generated by setting some of them to zero, or, to some particular values. Possible choices of parameters are represented in the Table I. The corresponding potentials can be read off from the expressions in the main text.

IX. CONCLUDING REMARKS

Main results of this study can be summarized as follows. We have studied a pair of \( nD \) Hamiltonians of potential forms that intertwine by first order operator \( \mathcal{L} \) and proved that the differential part of \( \mathcal{L} \) is an element of the Euclidean algebra \( e(n) \). These imply that so-intertwined systems have symmetry operators whose differential parts belong to enveloping algebra of \( e(n) \). The integrability conditions of consistency equations are dimension dependent and therefore have been considered for each case separately. The general form of potentials have been specified for \( n = 2, 3, 4, 5 \) where only one linear partial differential equation which determines the common part of the potential remains unsolved. We have found the general solution of this equation in cases of \( n = 2 \) and \( n = 3 \).

Three distinctive features of the higher dimensional extension of the intertwining method are that: (i) The method suggests coordinate systems which allows us to do the separation of variables and to utilize, in one of the variable, all the methods of the 1D SUSY quantum mechanics. (ii) In the choice of this variable and \( \mathcal{L} \) itself one has a number of alternatives increasing with \( n \). This fact enlarges the set of available potentials for each \( n \geq 3 \). (iii) There exist families of potentials accepting the same intertwining operator.

2D and 3D isospectral potentials we have obtained involve two arbitrary functions. The former constitute the most general integrable potentials which admit first order intertwining. Particular forms of these potentials may be of special interest for various purposes. Having in mind the projection techniques which produce exactly solvable lower dimensional problems from the higher dimensional one-particle problems our analysis in section VI-VIII must be continued also for \( n = 4 \) and \( n = 5 \). As is implied by the last example of section VI, it seems to be possible to investigate connections among the superintegrable potentials as well as to construct related potentials by repeated Darboux’s transformations in the context of intertwining method. Velocity dependent, stationary and non-stationary problems can as well be considered within our approach. Work on 2D and 3D isospectral potentials which are at the same time superintegrable is in progress.

ACKNOWLEDGMENTS

We thank M. Önder for a critical reading of this manuscript and for illuminating discussions. Special thanks are due to A. U. Yılmazer and B. Demircioğlu for useful conversations. This work was supported in part by the Scientific and Technical Research Council of Turkey (TÜBİTAK).

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In 3-dimension the special choices of parameters and corresponding coordinates. Note that in each case further choices are possible. For example, in the first three cases $c_j$ which does not appear in the first column can be set to zero and instead of $\eta_1$ one can also use $\eta_2$, or, $\eta_3$. As a completely different case in which all $c_{jk}$’s are zero has been presented in section III for any $n > 1$.

| $a_1$ | $a_2c_2 + a_3c_1 = 0$ | $a_2 = 0; a_1c_1 + a_3c_1 = 0$ | $a_3 = 0; a_1c_1 + a_2c_2 = 0$ | $2\gamma = a^2 - L^2$ | $\eta$ | $p(\eta)$ |
|---|---|---|---|---|---|---|
| $a_1 = 0; a_2c_2 + a_3c_1 = 0$ | $\vec{c} \cdot \vec{a}$ | $2\vec{r} \cdot (\vec{a} \times \vec{c}) + (\vec{r} \cdot \vec{c})^2 - c^2 r^2$ | $\frac{c_{1y} - c_{2z}}{a_{1y} - a_{2z}}$ | $p(\eta)$ |
| $a_2 = 0; a_1c_1 + a_3c_1 = 0$ | $\vec{r} \cdot \vec{c}$ | $2\vec{r} \cdot (\vec{a} \times \vec{c}) + (\vec{r} \cdot \vec{c})^2 - c^2 r^2$ | $\frac{c_{1y} - c_{2z}}{a_{1y} - a_{2z}}$ | $p(\eta)$ |
| $a_3 = 0; a_1c_1 + a_2c_2 = 0$ | $\vec{r} \cdot \vec{c}$ | $2\vec{r} \cdot (\vec{a} \times \vec{c}) + (\vec{r} \cdot \vec{c})^2 - c^2 r^2$ | $\frac{c_{1y} - c_{2z}}{a_{1y} - a_{2z}}$ | $p(\eta)$ |
| $a_1 = 0; a_2c_2 + a_3c_1 = 0$ | $c_1x + c_2y$ | $2a_3(c_1y - c_2x) + (c_1x + c_2y)^2 - (c_1^2 + c_2^2)r^2$ | $\frac{c_{1y} - c_{2z}}{a_{1y} - a_{2z}}$ | $p(\eta)$ |
| $a_2 = 0; a_1c_1 + a_3c_1 = 0$ | $c_1x + c_2z$ | $2a_2(c_3x - c_1z) + (c_1x + c_3z)^2 - (c_1^2 + c_3^2)r^2$ | $\frac{c_{1y} - c_{2z}}{a_{1y} - a_{2z}}$ | $p(\eta)$ |
| $a_3 = 0; a_1c_1 + a_2c_2 = 0$ | $c_2y + c_3z$ | $2a_1(c_2z - c_3y) + (c_2y + c_3z)^2 - (c_2^2 + c_3^2)r^2$ | $\frac{c_{1y} - c_{2z}}{a_{1y} - a_{2z}}$ | $p(\eta)$ |
| $a_1 = 0; a_2c_2 + a_3c_1 = 0$ | $c_1x$ | $2c_1(a_2y - a_2z) - c_1(y^2 + z^2)$ | $\frac{c_{1y} - c_{2z}}{a_{1y} - a_{2z}}$ | $p(\eta)$ |
| $a_2 = 0; a_1c_1 + a_3c_1 = 0$ | $c_2y$ | $2c_2(-a_3x + a_1z) - c_2(x^2 + z^2)$ | $\frac{c_{1y} - c_{2z}}{a_{1y} - a_{2z}}$ | $p(\eta)$ |
| $a_3 = 0; a_1c_1 + a_2c_2 = 0$ | $c_3z$ | $2c_3(a_2x - a_1y) - c_3(x^2 + y^2)$ | $\frac{c_{1y} - c_{2z}}{a_{1y} - a_{2z}}$ | $p(\eta)$ |
| $a_1 = a_2 = a_3 = 0$ | $\vec{r} \cdot \vec{c}$ | $(\vec{r} \cdot \vec{c})^2 - c^2 r^2$ | $\frac{c_{1y} - c_{2z}}{a_{1y} - a_{2z}}$ | $p(\eta)$ |