\textbf{L}^2 \textit{estimates of trilinear oscillatory integrals of convolution type on }\mathbb{R}^2\

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\textbf{Abstract} \hspace{1em} This paper is devoted to \(L^2\) estimates for trilinear oscillatory integrals of convolution type on \(\mathbb{R}^2\). The phases in the oscillatory factors include smooth functions and polynomials. We shall establish sharp \(L^2\) decay estimates of trilinear oscillatory integrals with smooth phases, and then give \(L^2\) uniform estimates for these integrals with polynomial phases.

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\section{Introduction}

Consider the following trilinear oscillatory integrals of convolution type,

\begin{equation}
\Lambda_{\lambda}(f_1, f_2, f_3) = \int_{\mathbb{R}^2} e^{i\lambda S(x, y)} f_1(x) f_2(y) f_3(x + y) \varphi(x, y) \, dx \, dy,
\end{equation}

where \(\lambda \in \mathbb{R}\) is a parameter, \(S\) is a real-valued smooth phase defined in a neighborhood of the origin, and \(\varphi \in C_0^\infty(\mathbb{R}^2)\) is a cut-off function near the origin.

If \(S(x, y) \equiv 0\) and \(\varphi\) is removed, then \(\Lambda_{\lambda}(f_1, f_2, f_3)\) is equal to \(\int_{\mathbb{R}} f_1(x) (\tilde{f}_2 * f_3(x)) \, dx\), where \(\tilde{f}_2(x) = f_2(-x)\) and \(\tilde{f}_2 * f_3\) is the convolution of \(\tilde{f}_2\) and \(f_3\). The boundedness of \(\Lambda_{\lambda}\) in Lebesgue spaces is a consequence of Young’s inequality.

Our purpose is to establish \(L^2\) decay estimates of \(\Lambda_{\lambda}\) when the phase function \(S\) is non-degenerate. In other words, we are going to prove decay estimates of the following type:

\begin{equation}
|\Lambda_{\lambda}(f_1, f_2, f_3)| \leq C|\lambda|^{-\epsilon} \|f_1\|_2 \|f_2\|_2 \|f_3\|_2,
\end{equation}

where \(\epsilon\) is a positive exponent, and \(\|f_i\|_2\) is the \(L^2\) norm of \(f_i\).

A more general framework of multi-linear oscillatory integrals was studied in the seminal

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work by Christ, Li, Tao, and Thiele [6]. It takes the form

\[ \Lambda_{\lambda}(f_1, f_2, \cdots, f_n) = \int_{\mathbb{R}^m} e^{i\lambda P(x)} \prod_{j=1}^{n} f_j(\pi_j(x)) \eta(x) \, dx, \quad (1.3) \]

where \( \lambda \in \mathbb{R} \) is a parameter, \( P \in \mathbb{R}[x_1, \cdots, x_m] \) is a real polynomial, \( \pi_j : \mathbb{R}^m \rightarrow V_j \) is a surjective linear mapping from \( \mathbb{R}^n \) onto some subspaces \( V_j \) of \( \mathbb{R}^m \), and \( \eta \in C_c^\infty(\mathbb{R}^m) \) is a smooth cut-off function. In [6], all subspaces \( V_j \) are assumed to have the same dimension. One of the main results in [6] is the following theorem.

**Theorem 1.1** ( [6] ) Assume \( n < 2m, \dim V_j = 1 \), and all \( V_j \) lie in general position. If \( P \in \mathbb{R}[x_1, \cdots, x_m] \) is a real polynomial which is non-degenerate with respect to \( \{\pi_j\}_{j=1}^{n} \), then

\[ |\Lambda_{\lambda}(f_1, f_2, \cdots, f_n)| \leq C|\lambda|^{-\epsilon} \prod_{j=1}^{n} \|f_j\|, \]

where \( \epsilon > 0 \) depends only on \( n, m, \deg P \) and \( \{\pi_j\}_{j=1}^{n} \). Moreover, this estimate is uniform if \( \deg P \) is bounded above by a fixed number \( d \), and the non-degenerate norm of \( P \) has a uniform positive lower bound.

The trilinear oscillatory integral \( \Lambda_{\lambda}(f_1, f_2, f_3) \) in (1.1) is a special case of Theorem 1.1. The proof of Theorem 1.1 in [6] invokes the concept of \( \lambda \)-uniformity and reduces the multi-linear oscillatory integral to the trilinear one by induction on \( n \).

Although a uniform decay estimate of \( \Lambda_{\lambda}(f_1, f_2, f_3) \) has been established in Theorem 1.1, it seems that the sharp (uniform) \( L^2 \) decay estimate of \( \Lambda_{\lambda}(f_1, f_2, f_3) \) may be of independent interest. In this paper, we shall prove the sharp \( L^2 \) decay estimates of trilinear \( \Lambda_{\lambda} \) when the phase function \( S \) is a real-valued smooth function, and then for polynomial phases satisfying certain non-degenerate assumptions, we will establish the uniform (also sharp) \( L^2 \) decay estimate for \( \Lambda_{\lambda} \).

Assume \( S(x, y) \) is a real-analytic function near the origin. Let \( H(x, y) = \partial_x \partial_y (\partial_x - \partial_y) S(x, y) \). By Taylor’s expansion, we have \( H(x, y) = \sum_{\alpha, \beta \geq 0} c_{\alpha, \beta} x^\alpha y^\beta \) in a small neighborhood of the origin. For simplicity, assume \( H(x, y) \) is not identical to zero. Otherwise, if \( H(x, y) \equiv 0 \), then \( S(x, y) \) is degenerate in the sense that \( S(x, y) = p(x) + q(y) + r(x + y) \) for some smooth functions \( p, q \) and \( r \), and hence there is no decay for \( \Lambda_{\lambda} \). Let \( d \) be the order of \( H \) at \((0, 0)\), which is defined by

\[ d = \min\{\alpha + \beta | \alpha, \beta \in \mathbb{N}, c_{\alpha, \beta} \neq 0\}. \quad (1.4) \]

If \( S(x, y) \) is smooth and \( H(x, y) = \partial_x \partial_y (\partial_x - \partial_y) S(x, y) \neq 0 \) on the support of \( \varphi \), Li [14] proved the following theorem.

**Theorem 1.2** ( [14] ) Assume \( S(x, y) \) is a real-valued smooth function near the origin. If \( H(x, y) = \partial_x \partial_y (\partial_x - \partial_y) S(x, y) \neq 0 \) on the support of \( \varphi \), then the trilinear function \( \Lambda_{\lambda} \) satisfies

\[ |\Lambda_{\lambda}(f_1, f_2, f_3)| \leq C|\lambda|^{-\frac{1}{d}} \|f_1\|_2 \|f_2\|_2 \|f_3\|_2. \]

Moreover, if \( \varphi(0, 0) \neq 0 \) then the above estimate is sharp.
In [23], Xiao proved the following sharp decay estimate with real-analytic phase functions.

**Theorem 1.3** ([23]) Assume $S(x, y)$ is a real-valued real-analytic phase function, and $H(x, y) = \partial_x \partial_y (\partial_x - \partial_y) S(x, y)$ is a non-zero function. Then the trilinear functional $\Lambda_\lambda$ satisfies

$$|\Lambda_\lambda(f_1, f_2, f_3)| \leq C|\lambda|^{-\frac{1}{2}(\alpha + \beta)} ||f_1||_2 ||f_2||_2 ||f_3||_2,$$

where the order $d$ is given by (1.4). Moreover, this estimate is sharp provided that $\varphi(0, 0) \neq 0$.

A question arises naturally whether Theorem 1.3 is still valid if the phase function $S$ is merely smooth. One of our purposes is to give a positive answer to this question. Assume $S(x, y)$ is smooth and $H(x, y) = \partial_x \partial_y (\partial_x - \partial_y) S(x, y)$ has a non-zero Taylor’s expansion $H(x, y) \sim \sum_{\alpha, \beta \geq 0} c_{\alpha, \beta} x^\alpha y^\beta$. It should be pointed out that there is no decay for $\Lambda_\lambda$ if $\partial_x^\alpha \partial_y^\beta H(0, 0) = 0$ for all $\alpha, \beta \geq 0$. We say that $H$ is of finite type at $(0, 0)$ if $\partial_x^\alpha \partial_y^\beta H(0, 0) \neq 0$ for some non-negative integers $\alpha$ and $\beta$. Let $d$ be the order of $H$ at $(0, 0)$, defined as in (1.4), if $H$ is of finite type. Then one of our main results is

**Theorem 1.4** Assume $S$ is a real-valued smooth function, and $H(x, y) = \partial_x \partial_y (\partial_x - \partial_y) S(x, y)$ is of finite type at the origin. Then the trilinear functional $\Lambda_\lambda$ satisfies

$$|\Lambda_\lambda(f_1, f_2, f_3)| \leq C|\lambda|^{-\frac{1}{2}(\alpha + \beta)} ||f_1||_2 ||f_2||_2 ||f_3||_2.$$  

Moreover, this decay estimate is also sharp if $\varphi(0, 0) \neq 0$.

Our proof of Theorem 1.4 is quite different from that of Theorem 1.3. The difference lies in the resolution of singularities for $H(x, y)$. In the real-analytic case, resolution of singularities is a consequence of the Newton-Puiseux algorithm. For its applications to related topics, we refer the reader to Phong and Stein [15], Greenblatt [8], and Xiao [23]. This method does not apply to a smooth phase function. Our resolution of singularities in this paper is due to Greenblatt [9].

Away from the coordinate axes, we need resolution of singularities near $H(x, y) = 0$ as in [9]. By the van der Corput Lemma and an almost orthogonality argument, the sharp $L^2$ decay estimate for one-dimensional oscillatory integral operators near the coordinate axes in [9] can be proved without resolution of singularities. However, we find the resolution of singularities for $H$ is also necessary to establish sharp $L^2$ decay estimate for $\Lambda_\lambda$ near the coordinate axes. In fact, this estimate cannot be proved by the same argument as in [9].

On the other hand, we also consider uniform estimates of $\Lambda_\lambda$ when the phase function $S(x, y)$ is a polynomial. If some partial derivatives of $H(x, y) = \partial_x \partial_y (\partial_x - \partial_y) S(x, y)$ is bounded away from 0, then we are able to establish uniform (also sharp) estimates for trilinear oscillatory integrals.

**Theorem 1.5** Assume $S(x, y) \in \mathbb{R}[x, y]$ is a real polynomial. Let $H(x, y) = \partial_x \partial_y (\partial_x - \partial_y) S(x, y)$. If $H(x, y)$ satisfies

$$|\partial_x^{(i)} H(x, y)| \geq 1, \quad 1 \leq i \leq N, \quad (x, y) \in [0, 1]^2,$$

where $\alpha^{(1)}, \alpha^{(2)}, \ldots, \alpha^{(N)}$ belong to $\mathbb{N}^2$, and $\partial_x^{(i)} H(x, y) = \partial_x^{(i)} \partial_y^{(i)} H(x, y)$ with $\alpha^{(i)} = (\alpha_1^{(i)}, \alpha_2^{(i)})$, then for each cut-off function $\varphi \in C_0^\infty(\mathbb{R}^2)$ with $\text{supp}(\varphi) \subseteq [0, 1]^2$, there exists a constant $C$,
depending only on $\deg S$ and $\varphi$, such that
\[
|A_\lambda(f_1, f_2, f_3)| \leq C|\lambda|^{-\frac{5}{25+\eta}}\|f_1\|_2\|f_2\|_2\|f_3\|_2,
\]
where $d = \min\{|\alpha^{(i)}| : 1 \leq i \leq N\}$.

Uniform estimates under assumptions of form (1.6) appeared in Carbery, Christ, and Wright [11], Phong, Stein, and Sturm [16], Carbery, and Wright [2], and Gressman [10]. Our proof of Theorem 3.4 is inspired by the work of Phong, Stein, and Sturm [16]. For related works on this topic in this article, we refer the reader to [4], [5], [7], [12].

2 Sharp estimates with smooth phases

2.1 Some lemmas

In this section, we shall present some basic lemmas which will be frequently used in our analysis. Almost all of these results have appeared in literature cited below.

We begin with a simple size estimate for trilinear integrals of convolution type; see also [23].

Lemma 2.1 Assume $R$ is a rectangle in $\mathbb{R}^2$ with sides parallel to the axes, and it is of dimensions $\delta_1 \times \delta_2$. If $a \in L^\infty(\mathbb{R}^2)$ is supported on $R$, then we have
\[
\left| \int_{\mathbb{R}^2} f_1(x)f_2(y)f_3(x+y)a(x,y) \, dx \, dy \right| \leq \min\{\delta_1^{\frac{1}{2}}, \delta_2^{\frac{1}{2}}\}\|a\|_\infty\|f_1\|_2\|f_2\|_2\|f_3\|_2.
\]
The following lemma is an almost orthogonality principle for trilinear integrals. This lemma has been used in [23].

Lemma 2.2 Suppose $T = \sum_l T_l$ is a trilinear functional, such that each $T_l$ can be written as
\[
T_l(f_1, f_2, f_3) = \int \int K_l(x,y)f_1(x)f_2(y)f_3(x+y) \, dx \, dy,
\]
where $T_l(x,y)$ is supported on a product of intervals $I_l \times J_l$. Suppose there is a positive integer $L$ such that for a fixed $l$, the number of $m$ with $I_l \cap I_m \neq \emptyset$ is bounded by $L$, and the number of $m$ with $J_l \cap J_m \neq \emptyset$ is also bounded by $L$. Then $\|T\| \leq L \sup_l \|T_l\|.$

Proof. By the assumption $T = \sum_l T_l$, we see that $|T(f_1, f_2, f_3)|$ is bounded by
\[
\sum_l \int \int |K_l(x,y)f_1(x)f_2(y)f_3(x+y)| \, dx \, dy \leq \sum_l \|T_l\|\|\chi_{I_l}f_1\|_2\|\chi_{J_l}f_2\|_2\|f_3\|_2
\leq \sup_l \|T_l\|\|f_3\|_2\sum_l \|\chi_{I_l}f_1\|_2\|\chi_{J_l}f_2\|_2.
\]

By Cauchy-Schwarz’s inequality, it follows that
\[
\sum_l \|\chi_{I_l}f_1\|_2\|\chi_{J_l}f_2\|_2 \leq \left(\sum_l \|\chi_{I_l}f_1\|_2^2\right)^{1/2}\left(\sum_l \|\chi_{J_l}f_2\|_2^2\right)^{1/2} \leq L\|f_1\|_2\|f_2\|_2.
\]
This implies \( \|T\| \leq L \sup_l \|T_l\| \). \(\square\)

The following lemma is contained in Christ [3].

**Lemma 2.3** Assume \( f \) is of class \( C^n \) on some interval \( I \) satisfying \( |f^{(n)}(x)| \geq 1 \) for all \( x \in I \). Then there exists a constant \( C = C(n) \) such that

\[
\sup_{x \in I} |f(x)| \geq C_n |I|^n.
\]

The following Bernstein lemma is due to Greenblatt [9].

**Lemma 2.4** (9) Let \( f(x,y) \) be a smooth function in a neighborhood of the origin. Assume \( (k,l) \) is a multiindex such that

\[
\partial^k \partial^l f(0,0) \neq 0. \tag{2.1}
\]

Suppose that \( \delta, N > 0 \). There is a neighborhood \( U \) of the origin such that if \( R \subset U \) is an \( r_1 \) by \( r_2 \) rectangle with \( r_1 \leq Nr_2^\delta, r_2 \leq Nr_1^\delta \), then for any multiindex \( (\alpha,\beta) \) satisfying \( 0 \leq \alpha,\beta \leq 2 \) there exists a constant \( C \) such that

\[
\sup_{(x,y) \in R} \left| \partial^{\alpha+\beta} f(x,y) \right| < C r_1^{-\alpha} r_2^{-\beta} \sup_{(x,y) \in R} |f(x,y)|. \tag{2.2}
\]

Furthermore, if \( R' \) is a subrectangle of \( R \) of dimensions \( r_1^{\frac{1}{2}} \) by \( r_2^{\frac{1}{2}} \), then we have

\[
\sup_{(x,y) \in R} |f(x,y)| < C \sup_{(x,y) \in R'} |f(x,y)|. \tag{2.3}
\]

To give a uniform \( L^2 \) decay estimate for trilinear oscillatory integrals of convolution type, we need an operator van der Corput lemma. For its proof, see also Lemma 3.6 in §3.

**Lemma 2.5** (23) Let \( \Lambda_\lambda \) be defined as in (1.1). Assume the cut-off function \( \varphi \) is supported in a rectangle \( R \) of dimensions \( \delta_1 \times \delta_2 \) with \( \delta_1 > 0, \delta_2 > 0 \), and there exists a constant \( A > 0 \) such that

\[
\delta_2 \sup_R |\partial_y a(x,y)| + \delta_2^2 \sup_R |\partial_y^2 a(x,y)| \leq A.
\]

Let \( S(x,y) \) be a real-valued smooth function, and let \( H(x,y) = \partial_x \partial_y (\partial_x - \partial_y) S(x,y) \). If there exist two constants \( \mu, B > 0 \) such that

\[
|H(x,y)| \geq \mu, \quad \delta_2 \sup_R |\partial_y H(x,y)| + \delta_2^2 \sup_R |\partial_y^2 H(x,y)| \leq B \mu,
\]

for all \( (x,y) \in R \), then we have

\[
|\Lambda_\lambda(f_1,f_2,f_3)| \leq C |\lambda \mu|^{-\frac{1}{2}} \|f_1\|_2 \|f_2\|_2 \|f_3\|_2,
\]

where the constant \( C \) depends only on \( A \) and \( B \).
2.2 Resolution of singularities

In this section, we are going to review the useful resolution of singularities due to Greenblatt [9]. It will play an important role in our proof of Theorem 1.4.

Assume $H$ is a real-valued smooth function near the origin in $\mathbb{R}^2$. Suppose $H$ is of finite type, i.e., there exist nonnegative integers $\alpha, \beta, (\alpha, \beta) \neq (0, 0)$, such that $\partial_x^\alpha \partial_y^\beta H(0,0) \neq 0$. For simplicity, we also assume $H(0,0) = 0$.

Now we shall present resolution of singularities of $H$ away from the coordinate axes. Let $H(x,y) \sim \sum_{\alpha,\beta \geq 0} C_{\alpha,\beta} x^\alpha y^\beta$ be the formal Taylor’s expansion of $H$. Then the Newton polyhedron $N(H)$ of $H$ is the convex hull of $\bigcup_{\alpha,\beta \in \mathbb{N}} \{(x,y): x \geq \alpha, y \geq \beta\}$, where the union is taken over all $\alpha, \beta$ such that $C_{\alpha, \beta} \neq 0$. The vertices of $N(H)$ are denoted by $(A_1, B_1), \ldots, (A_n, B_n)$, where $A_1 < A_2 < \cdots < A_n$ and $B_1 > B_2 > \cdots > B_n$. The case $n \geq 2$ will be considered since resolution of singularities is not necessary for $n = 1$. This means that we assume $N(H)$ has at least two vertices throughout this section.

Let $-\frac{1}{M_i}$ be the slope of the line joining $(A_i, B_i)$ and $(A_{i+1}, B_{i+1})$. For any real number $c$, it is easy to see that

$$H(x, cx^{M_i}) = p_i(c)x^{d_i} + o(x^{d_i}), x \to 0.$$  

Here $p_i \in \mathbb{R}[t]$ is a real polynomial. If $\Gamma_i$ denotes the compact face of $N(H)$ joining $(A_i, B_i)$ and $(A_{i+1}, B_{i+1})$, then $p_i$ is given by

$$p_i(t) = \sum_{(\alpha, \beta) \in \Gamma_i} C_{\alpha, \beta} t^\beta.$$  

One has $H(x, cx^{M_i}) = Ax^d + o(x^d)$ for some $A \neq 0$ unless $p_i(c) = 0$. In contrast with the real analytic setting, we cannot obtain an algebraic curve $\gamma(x)$ by the Newton-Puiseux algorithm such that $\gamma(x) = Cx^{M_i} + o(x^{M_i})$ and $H(x, \gamma(x)) \equiv 0$. For a smooth phase, we shall follow Greenblatt’s resolution of singularities.

Without loss of generality, we assume $p_i(c) = 0$ and $c > 0$. If $c$ is a complex root with $\text{Im}(c) \neq 0$, then there $\gamma(x) = cx^{M_i} + o(x^{M_i})$ will not occur in the plane $\mathbb{R}^2$. We consider $c > 0$ since the treatment of $c < 0$ is similar to that of $c > 0$.

For a sufficiently small $\epsilon > 0$, we shall restrict our attention to the following curved triangle:

$$U = \{(x,y): x > 0, (c-\epsilon)x^{M_i} < y < (c+\epsilon)x^{M_i}\}.$$  

If $j \in \mathbb{Z}, j < 0$ and $|j|$ is sufficiently large, we set

$$U_j = \{(x,y): x > 2^j, (c-\epsilon)x^{M_i} < y < (c+\epsilon)x^{M_i}\},$$  

where $x \sim 2^j$ denotes $2^{j-1} \leq x \leq 2^{j+1}$. For a small number $\mu > 0$, we can cover $U_j$ by finitely many dyadic squares with side length $\mu 2^{M_i}$, $M = \max\{M_i, 1\}$, such that if $R$ is a dyadic square of dimension $\mu 2^{M_j} \times \mu 2^{M_j}$ and $R \cap U_j \neq \emptyset$, then its double $R^* \subseteq U_j^*$. Here $R^*$ has the same center as $R$, and its side length is twice as that of $R$. The domain $U_j^*$ is defined by

$$U_j^* = \{(x,y): x \sim 2^j, (c-2\epsilon)x^{M_i} < y < (c+2\epsilon)x^{M_i}\}.$$
The positive parameter $\epsilon$ is so small that the polynomial $p_i$ has no root other than $c$ in the interval $(c-2\epsilon, c+2\epsilon)$.

For convenience, we introduce some notations. If $R$ is a rectangle with sides parallel to the axes in $\mathbb{R}^2$, we use $l_h(R)$ and $l_v(R)$ to denote the length of the horizontal side and the vertical side, respectively. For a square $R$, its side length is denoted by $l(R)$.

Let $\mathcal{D}$ be the family of all dyadic cubes in $\mathbb{R}^2$. Now define

$$\mathcal{F} = \{ R \in \mathcal{D} \mid l(R) = \mu 2^{M_j}, R \cap U_j \neq \emptyset \}.$$

With the above preliminaries, we are going to describe the resolution of singularities for $H$ in $U_j$. This stopping time argument depends on whether the following inequality is true or not,

$$\sup_R |\nabla H(x,y)| l(R) < \frac{1}{4} \sup_R |H(x,y)|,$$

where $R$ is a dyadic square contained in some $\tilde{R} \in \mathcal{F}$. For any $R \in \mathcal{F}$, if (2.6) holds for $R$, then the process stops. Otherwise, we dyadically divide $R$ into four equal dyadic squares. Then we will obtain a collection of dyadic squares, denoted by $\mathcal{F}_1$. Each member $R$ of $\mathcal{F}_1$ has side length either $l(R) = \mu 2^{M_j}$ or $l(R) = \mu 2^{M_j-1}$. For any $R \in \mathcal{F}_1$, if the inequality (2.6) is true for $R$, then our procedure stops. Otherwise, we divide $R$ into four dyadic squares of side length $l(R) = \mu 2^{M_j-2}$. Let $\mathcal{F}_2$ be the family of all dyadic squares obtained in this way.

By induction, if the family $\mathcal{F}_n$ is given, we define $\mathcal{F}_{n+1}$ be all dyadic squares $R$ satisfying one of the following conditions:

(i) $R \in \mathcal{F}_n$ and the inequality (2.6) holds for $R$;
(ii) (2.6) does not holds for some $R' \in \mathcal{F}_n$, and $R$ is a dyadic $\frac{1}{4}$ subsquare in $R'$.

Then a sequence $\{ \mathcal{F}_n \}$ of families of dyadic squares are obtained. Let $\mathcal{F}_\infty$ be

$$\mathcal{F}_\infty = \bigcap_{n=1}^{\infty} \mathcal{F}_n.$$

By the above stopping time procedure for resolution of singularities, we have

$$\left( \bigcup_{R \in \mathcal{F}_\infty} R \right) \cap U_j = \{ (x, y) \in U_j \mid H(x,y) \neq 0 \}.$$

By the Bernstein inequality, i.e., Lemma 2.4, we have

$$l(R) \sup_R |\nabla H(x,y)| > \delta_0 \sup_R |H|, R \in \mathcal{F}_\infty,$$

for some constant $\delta_0 > 0$ independent of $R \in \mathcal{F}_\infty$. Hence either

$$l(R) \sup_R \left| \frac{\partial H}{\partial x}(x,y) \right| > \frac{\delta}{2} \sup_R |H| \quad (2.7)$$
or
\[
l(R) \sup_R \left| \frac{\partial H}{\partial y}(x,y) \right| > \frac{\delta}{2} \sup_R |H| \tag{2.8}
\]
is true for all \(0 < \delta \leq \delta_0\). Write \(R = I \times J\) for \(R \in \mathcal{F}_\infty\). Take \(\delta > 0\) to be sufficiently small. We shall expand each member \(R \in \mathcal{F}_\infty\) as large as possible such that \(H\) stays within a factor of 2 of a fixed constant on a larger rectangle \(\tilde{R}\). For each \(R \in \mathcal{F}_\infty\), if both (2.7) and (2.8) are true for \(R\), then set \(\tilde{R} = R\), otherwise either (2.7) or (2.8) does not hold, say (2.7) for example, let \(\tilde{R} = \tilde{I} \times J\), where \(\tilde{I}\) is the largest dyadic interval such that \(\tilde{I} \supseteq I\), \(\tilde{R} \subseteq U_j\), and
\[
|\tilde{I}| \sup_{\tilde{R}} \left| \frac{\partial H}{\partial x}(x,y) \right| \leq \delta \sup_{\tilde{R}} |H(x,y)|. \tag{2.9}
\]
Similarly, if (2.8) fails, then choose the largest dyadic interval \(\tilde{J} \supseteq J\), \(\tilde{R} = I \times \tilde{J} \subseteq U_j\) and
\[
|\tilde{J}| \sup_{\tilde{R}} \left| \frac{\partial H}{\partial y}(x,y) \right| \leq \delta \sup_{\tilde{R}} |H(x,y)|. \tag{2.10}
\]
Then we can define a mapping from \(\mathcal{F}_\infty\) into the collection of coordinate rectangles \(\mathcal{R}\),
\[
\alpha : \mathcal{F}_\infty \rightarrow \mathcal{R} \\
R \mapsto \tilde{R}.
\]
Let \(\mathcal{G}\) be the range of \(\alpha\), i.e.
\[
\mathcal{G} = \left\{ \tilde{R} = \alpha(R) | R \in \mathcal{F}_\infty \right\}.
\]
Now we list some important properties of rectangles in \(\mathcal{G}\).

**Theorem 2.6** ([9]) There exists an exponent \(\sigma > 0\) and a constant \(C > 0\) such that for all \(R = I \times J \in \mathcal{G}\), we have
\[
|I| \leq C|J|^\sigma, \quad |J| \leq C|I|^\sigma.
\]

For \(\epsilon > 0\), let \(R^{*,1+\epsilon}\) be the rectangle with the same center as \(R\) and expanded by the factor \(1 + \epsilon\).

**Theorem 2.7** ([9]) There exists a small number \(\epsilon_0 > 0\), such that \(\exists C_1, C_2 > 0\),
\[
C_1 \sup_{R^{*,1+\epsilon}} |H(x,y)| \leq \inf_{R^{*,1+\epsilon}} |H(x,y)|,
\]
\[
\sup_{R^{*,1+\epsilon}} |H(x,y)| \leq C_2 \sup_R |H(x,y)|
\]
for all \(R \in \mathcal{G}\) and \(0 < \epsilon < \epsilon_0\).

To obtain a suitable partition of unity on \(\{(x,y) \in U_j : H(x,y) \neq 0\}\), one expects that appropriate dilations of rectangles in the collection \(\mathcal{G}\) have bounded overlap property. This is true even though members in \(\mathcal{G}\) come from various dyadic squares by performing a complicated procedure.
Theorem 2.8 \([\text{[9]}]\) There exists an \(\epsilon_0 > 0\) such that for \(0 < \epsilon < \epsilon_0\), the collection \(\{R^{s,1+\epsilon} | R \in G\}\) has bounded overlap property in \(U_j\), i.e., there exists an integer \(N \geq 1\) such that each point in \(U_j\) is contained in at most \(N\) rectangles \(R^{s,1+\epsilon}\).

Moreover, the collection of rectangles \(\{R^{s,1+\epsilon} | R \in G\}\) has an appropriate mutual orthogonality. Let \(G(k,m,n) \subseteq G\) consist of all \(R \in G\) such that \(R = I \times J\) satisfies

(i) \(l_b(R) = |I| = \mu 2^{m+j}, l_c(R) = |J| = \mu 2^{n+j};\)

(ii) \(2^{k+d_{j-1}} \leq \sup_R |H(x,y)| < 2^{k+d_{j}}.\)

Then \(G(k,m,n)\) has the following almost orthogonality property.

Theorem 2.9 \([\text{[9]}]\) There exists a positive integer \(N\) such that for all \(k,m,n \in \mathbb{Z}\) and any horizontal line or vertical line \(L\), there are at most \(N\) members in \(\{R^{s,1+\epsilon} | R \in G(k,m,n)\}\) having nonempty intersection with \(L\).

2.3 Estimates away from singularities and coordinate axes

For simplicity, we begin with our decay estimates in a curved triangle domain away from singularities and coordinate axes. Let \(H(x,y) = \partial_x \partial_y(\partial_x - \partial_y)S(x,y)\). Assume \((A_i,B_i)\) is the intersection of two compact edges of \(N(H)\), denoted by \(\Gamma_{i-1}\) and \(\Gamma_i\). Let \(p_{i-1}\) and \(p_i\) be the polynomials associated with \(\Gamma_{i-1}\) and \(\Gamma_i\), respectively. It is more convenient to focus our attention in the first quadrant. The argument is similar in other quadrants.

Let \(\delta_0 > 0\) be a small number. Assume \(\Omega_i = \{(x,y) \in (0,\delta_0)^2 | x^{M_i} \lesssim y \lesssim x^{M_{i-1}}\}\) is a curved triangle domain such that both \(p_{i-1}\) and \(p_i\) have no real roots in \(\Omega_i\). Of course, the implicit constants, appearing in the definition of \(\Omega_i\), are appropriately chosen.

By the symmetry of \(x\) and \(y\) in \(\Lambda_\lambda\), we can further assume that \(M_{i-1} \geq 1\). In fact, by a suitable partition of unity, we can restrict the integration of \(\Lambda_\lambda\) in the domain

\[
\Omega = \left\{(x,y) : x > 0, \ 0 < y < Cx\right\}
\]

with some positive constant \(C\).

Choose a smooth function \(\phi \in C^\infty(\frac{1}{2},2)\) such that \(\sum_{j \in \mathbb{Z}} \phi(j/2) = 1\) for all \(x > 0\). Let \(A_{j,k}\) be defined as \(\Lambda_\lambda\) in \([11]\), but with insertion of \(\phi(j/2)\phi(j/2)\) into the cut-off function. Since our analysis is restricted to \(\Omega_i\), we must have \(M_j + O(1) \leq k \leq M_{i-1}j + O(1)\), where \(O(1)\) is a bounded constant independent of \(j\) and \(k\). By Lemma 2.5 we obtain

\[
|A_{j,k}(f_1,f_2,f_3)| \leq C \left(|\lambda|2^j A_{j,k}B_j\right)^{-\frac{1}{2}} \|f_1\|_2 \|f_2\|_2 \|f_3\|_2.
\]

The size estimate in Lemma 2.1 gives

\[
|A_{j,k}(f_1,f_2,f_3)| \leq C \min\{2^\frac{j}{4}, 2^\frac{k}{2}\} \|f_1\|_2 \|f_2\|_2 \|f_3\|_2.
\]

It follows from our assumption \(M_j + O(1) \leq k \leq M_{i-1}j + O(1)\) that \(2^j \gtrsim 2^k\). Hence we shall take \(2^\frac{k}{2}\) as our upper bounds for size estimates of \(\Lambda_{j,k}\). For clarity, we divide our proof into three cases.
Case (i) $2^{A_i j + (B_i + 3) M_{i-1,j}} \lesssim |\lambda|^{-1}$

In this case, we choose the size estimate for $\Lambda_{j,k}$. Then

$$\sum_j \sum_{M_j + O(1) \leq k \leq M_{i-1,j} + O(1)} \| \Lambda_{j,k} \| \lesssim \sum_j \sum_{k \leq M_{i-1,j}} 2^\frac{k}{2}$$

$$\lesssim \sum_j 2^{\frac{M_{i-1,j}}{2}}$$

$$\lesssim |\lambda|^{-\frac{1}{2}[A_i + (B_i + 3) M_{i-1}]}.$$

Case (ii) $2^{A_i j + (B_i + 3) M_{i,j}} \gtrsim |\lambda|^{-1}$

The oscillation estimate is better than the size estimate. This implies

$$\sum_j \sum_k \| \Lambda_{j,k} \| \lesssim \sum_j \sum_{M_j \leq k \leq M_{i-1,j}} (|\lambda|^{2^{A_i} 2^k B_i})^{-\frac{1}{6}}$$

$$\lesssim \sum_j (|\lambda|^{2^{A_i} 2^k M_i B_i})^{-\frac{1}{6}}$$

$$\lesssim |\lambda|^{-\frac{M_i}{2[A_i + (B_i + 3) M_{i-1}]}},$$

Case (iii) $2^{A_i j + (B_i + 3) M_{i,j}} \lesssim |\lambda|^{-1} \lesssim 2^{A_i j + (B_i + 3) M_{i-1,j}}$

We denoted by $k_j$ the solution of $2^{A_i j + (B_i + 3) k_j} = |\lambda|^{-1}$. Then it is true that

$$\sum_j \sum_{M_j + O(1) \leq k \leq k_j} \| \Lambda_{j,k} \| \lesssim \sum_j \sum_{M_j + O(1) \leq k \leq k_j} 2^\frac{k}{2} \lesssim \sum_j 2^\frac{k_j}{2}.$$

On the other hand, for $k_j < k \leq M_{i-1,j} + O(1)$, we have

$$\sum_j \sum_{k_j < k \leq M_{i-1,j} + O(1)} \| \Lambda_{j,k} \| \lesssim \sum_j \sum_{M_j + O(1) \leq k \leq k_j} (|\lambda|^{2^{A_i} 2^k B_i})^{-\frac{1}{6}} \lesssim \sum_j 2^\frac{k_j}{2}.$$

Let $j_1$ be the solution of $2^{A_i j_1 + (B_i + 3) M_{i-1,j_1}} = |\lambda|^{-1}$. Then it follows that

$$\sum_{j \geq j_1} 2^\frac{k_j}{2} \lesssim \sum_j (|\lambda|^{2^{A_i} j})^{-\frac{1}{6}} \lesssim |\lambda|^{-\frac{M_{i-1}}{2[A_i + (B_i + 3) M_{i-1}]}},$$

We shall point out that the order $d$ in (1.4) satisfies $d \geq \min\{A_i + M_i B_i, \frac{A_i}{M_i} + B_i\}$. This inequality is also true if $M_i$ is replaced by $M_{i-1}$. Hence the decay estimate in Theorem 1.4 is true if we restrict the integration of $\Lambda_{j,k}$ over $\Omega_i$.

2.4 Estimates near the singularity

In this section, by Greenblatt’s method of resolution of singularities in §2.2, $L^2$ decay estimates near the singularities of $H$ will be proved. Without loss of generality, we focus our attention in the first quadrant.
As in §2.2, let \( \Gamma_i \) denotes the compact face of \( N(H) \) joining \((A_i, B_i)\) and \((A_{i+1}, B_{i+1})\), and \( p_i(t) = \sum_{(a, \beta) \in \Gamma_i} C_{a, \beta} t^\beta \). Recall that \(-1/M_i\) is the slope of the straight line through \( \Gamma_i \). Our argument is quite different depending on whether \( M_i = 1 \) or not. Indeed, in the original coordinates, one is able to prove the sharp decay estimate in Theorem 1.4 only for \( M_i = 1 \). However, the argument breaks down for \( M_i \neq 1 \). The reason is that a direct resolution of singularities does not exploit fully the convolution structure of \( \Lambda_\lambda \), and that a logarithmic term \( \log |\lambda| \) will appear in our estimate. Hence we cannot obtain the desired decay estimate in this way. For \( M_i \neq 1 \), it is necessary to make changes of variables before our application of resolution of singularities.

We begin with the case \( M_i = 1 \). By insertion of a cut-off function, we define a trilinear operator \( T(f_1, f_2, f_3) \) by

\[
T(f_1, f_2, f_3) = \int \int e^{i\lambda S(x, y)} f_1(x) f_2(y) f_3(x + y) \varphi(x, y) \phi \left( \frac{y - \eta x}{\rho x} \right) \chi_{\{x, y > 0\}}(x, y) \, dx \, dy,
\]

where \( \eta \) is a positive real zero of \( p_i(t) \), \( \rho \) is a small positive number, and \( \chi_{\{x, y > 0\}} \) is the characteristic function of the first quadrant. Let

\[
U = \{(x, y) : x > 0, (\eta - \rho/2)x < y < (\eta + 2\rho)x\}.
\]

It is easy to see that the cut-off function in \( T(f_1, f_2, f_3) \) is compactly supported in \( U \).

For a negative integer \( j \), \( |j| \gg 1 \), we define

\[
T_j(f_1, f_2, f_3) = \int \int_{\mathbb{R}^2} e^{i\lambda S(x, y)} f_1(x) f_2(y) f_3(x + y) \varphi(x, y) \psi \left( \frac{y - \eta x}{\rho x} \right) \chi_{\{x, y > 0\}} \psi_j(x) \, dx \, dy,
\]

where \( \psi_j(x) = \phi \left( \frac{x}{2^j} \right) \). Set

\[
U_j = \{(x, y) : x \sim 2^j, (\eta - \rho/2)x < y < (\eta + 2\rho)x\}.
\]

This definition is slightly different from (2.5) in §2.2.

As in §2.2, by resolution of singularities for \( H \) in \( U_j \), we will obtain a collection of rectangles with dimensions \( \mu 2^{m+j} \) by \( \mu 2^{n+j} \), denoted by \( \mathcal{G} \). Here \( \mu > 0 \) is a small fixed dyadic number, and \( m, n \) are negative integers. Choose a nonnegative \( C^\infty \) function \( \tau(x) \) supported on \([-\frac{1}{2} - \frac{\epsilon}{2}, \frac{1}{2} + \frac{\epsilon}{2}] \). Here \( \epsilon = \frac{1}{2\epsilon_0} \), and \( \epsilon_0 \) is chosen as in Lemma 2.7. Assume \( R \) is a \( l_h(R) \) by \( l_v(R) \) rectangle in \( \mathcal{G} \), and \((x_0, y_0)\) is its center. Let \( \tau_R(x, y) \) be defined by

\[
\tau_R(x, y) = \tau \left( \frac{x - x_0}{l_h(R)} \right) \tau \left( \frac{y - y_0}{l_v(R)} \right).
\]

By Lemma 2.8 we see that each \( \xi_R(x, y) \) defined by

\[
\xi_R(x, y) = \frac{\tau_R(x, y)}{\sum_{R \in \mathcal{G}} \tau_R(x, y)}, \quad R \in \mathcal{G},
\]

is a smooth cut-off function in \( C^\infty_0(R^{s,1+\epsilon}) \). Moreover, \( \{\xi_R : R \in \mathcal{G}\} \) forms a partition of unity.
of \( \{(x, y) \in U_j : H(x, y) \neq 0\} \). Define a new cut-off function by
\[
\phi_{j,R} = \varphi(x, y)\psi\left(\frac{y - \eta x}{px}\right) \chi_{x>0}(x, y)\psi_j(x)\xi_R(x, y).
\]

Now we have the decomposition of \( T_j = \sum_{R \in \mathcal{G}} T_j,R \), where
\[
T_j,R(f_1, f_2, f_3) = \int \int e^{i\lambda S(x,y)} f_1(x) f_2(y) f_3(x + y) \phi_{j,R}(x, y) \, dx \, dy.
\]

Using Lemma 2.4, Theorems 2.6, 2.7 and 2.8, we are able to prove the following upper bounds for \( \phi_{j,R} \) and its partial derivatives; see also Lemma 4.8 in [9].

**Lemma 2.10** Suppose \( \phi_{j,R} = \phi(x, y)\psi_j(x)\xi_R(x, y) \) for some \( R \in \mathcal{G} \) which is of dimensions \( r_1 \) by \( r_2 \). For \( \alpha = 0, 1, 2 \), there is a constant \( C > 0 \) such that
\[
\left| \frac{\partial^\alpha \phi_{j,R}}{\partial x^\alpha}(x, y) \right| < C r_1^{-\alpha}, \quad \left| \frac{\partial^\alpha \phi_{j,R}}{\partial y^\alpha}(x, y) \right| < C r_2^{-\alpha}.
\]

In addition, there exist a constant \( W_{j,R} > 0 \) and a constant \( \delta > 0 \) such that on \( R \) we have
\[
\delta W_{j,R} < |H(x, y)| < W_{j,R}.
\]
Also, there exist \( \delta', C' > 0 \) such that we have
\[
\delta' W_{j,R} < r_1 \sup_{(x,y) \in R} \left| \frac{\partial H}{\partial x}(x, y) \right|, \quad r_2 \sup_{(x,y) \in R} \left| \frac{\partial H}{\partial y}(x, y) \right|; \quad \text{ (2.13)}
\]
\[
\sup_{(x,y) \in R} \left| \frac{\partial^2 H}{\partial x^2}(x, y) \right|, \quad \sup_{(x,y) \in R} \left| \frac{\partial^2 H}{\partial y^2}(x, y) \right| < C' W_{j,R};
\]
\[
\sup_{(x,y) \in R} \left| \frac{\partial H}{\partial y}(x, y) \right|, \quad \sup_{(x,y) \in R} \left| \frac{\partial^2 H}{\partial y^2}(x, y) \right| < C' W_{j,R}.
\]
Moreover, all above estimates are also true with \( R^{\epsilon,1+\epsilon} \) in place of \( R \), where \( \epsilon > 0 \) is sufficiently small.

Now we get to estimate \( \|T\| \). By the almost orthogonality principle in Lemma 2.2, it follows from \( T = \sum_j T_j \) that \( \|T\| \lesssim \sup_j \|T_j\| \). Hence it suffices to estimate \( \|T_j\| \) for fixed integer \( j \). By our assumption \( M_i = 1 \), one has \( d_i = A_i + M_i B_i = A_i + B_i \) for each \( (A_i, B_i) \in \Gamma_i \). There is a constant \( C > 0 \) such that
\[
\sup_{R} |H(x, y)| < C 2^{d_i,j}. \quad \text{ (2.14)}
\]
Let \( r \geq 1 \) be the order of the zero \( \eta \) of \( p_i(t) \). This implies that \( p_i(\eta) = \cdots = p_i^{(r-1)}(\eta) = 0 \) and \( p_i^{(r)}(\eta) \neq 0 \). For \( k, m, n \in \mathbb{Z}_{>0} \), we define \( \mathcal{G}(k, m, n) \subseteq \mathcal{G} \) as in Theorem 2.9 in §2.2. Then we have the following lemma; see also [9].
Lemma 2.11 Suppose $T_{j,R} \in \mathcal{G}(k,m,n)$. For some constants $c_1$ and $c_2$ we have

$$c_1 + k < m < c_2 + \frac{k}{r}, \quad c_1 + k < n < c_2 + \frac{k}{r}.$$ 

Proof. We prove only the inequality for $m$. Observe that $H$ satisfies $\sup_R |\partial_x H(x,y)| \lesssim 2^{d_j - j}$. By Lemma 2.10 we have

$$2^{m+j} \sup_R \left| \frac{\partial H}{\partial x}(x,y) \right| \gtrsim \sup_R |H(x,y)| \sim 2^{k + jd_i}.$$ 

Hence there exists a constant $c_1$ such that $m \geq c_1 + k$. On the other hand,

$$\inf_R \left| \frac{\partial H}{\partial x}(x,y) \right| \gtrsim 2^{-jr + jd_i}.$$ 

By Lemma 2.3 we get

$$\sup_R |H(x,y)| \gtrsim 2^{mp + jd_i}.$$ 

Combining with $\sup_R |H(x,y)| \sim 2^{k + jd_i}$, we see that $m \leq c_2 + \frac{k}{r}$ for some constant $c_2$. \hfill \Box

Observe that $T_j = \sum_{k \leq 0} \sum_{c_1 + k < m, n < c_2 + k/r} \sum_R \mathcal{G}(k,m,n) T_{j,R}$. In view of the orthogonality property in Theorem 2.9 we see that

$$\|T_j\| \lesssim \sum_{k \leq 0} \sum_{c_1 + k < m, n < c_2 + \frac{k}{r}} \sup_R \|T_{j,R}\|.$$ 

(2.15)

We use Lemmas 2.1 and 2.5 to bound $\|T_{j,R}\|$. More precisely, by the size estimate in Lemma 2.1 one has

$$\|T_{j,R}\| \lesssim \min(2^{\frac{m+j}{r}}, 2^{\frac{n+j}{r}}).$$ 

On the other hand, the oscillation estimate in Lemma 2.5 gives us the following bound,

$$\|T_{j,R}\| \lesssim |\lambda|^{-\frac{1}{6}} 2^{-\frac{k+jd_i}{6}}.$$ 

By (2.15), it follows immediately that

$$\|T_j\| \lesssim \sum_{k \leq 0} \sum_{c_1 + k < m, n < c_2 + \frac{k}{r}} \min\left(2^{\frac{m+j}{r}}, 2^{\frac{n+j}{r}}, |\lambda|^{-\frac{1}{6}} 2^{-\frac{k+jd_i}{6}}\right).$$ 

(2.16)

We fix a constant $c_3 > \max(1, c_2 - c_1)$. Observe that $c_1$ is a constant, so the right hand of (2.16) is not greater than a constant multiple of

$$\sum_{k \leq 0} \sum_{k<m,n<c_3 + \frac{k}{r}} \min\left(2^{\frac{m+j}{r}}, 2^{\frac{n+j}{r}}, |\lambda|^{-\frac{1}{6}} 2^{-\frac{k+jd_i}{6}}\right).$$ 

(2.17)
Without loss of generality, we assume $|\lambda| \geq 2$. Then we consider the equation
\[
\frac{1}{2} \left( \frac{k'}{r} + j' \right) = -\frac{1}{6} \log_2 |\lambda| - \frac{1}{6} (k' + j'd_i).
\] (2.18)

Let $j_0$ be $j'$ in (2.18) when $k' = 0$. However, this solution $j_0$ is not necessarily an integer. The definition of $j_0$ implies $2^{j_0} = |\lambda|^{-1/(3+d_i)}$. Now we divide our argument into two cases $j \leq j_0$ and $j > j_0$.

If $j \leq j_0$, we take the size estimate in (2.18). It follows that
\[
\|T_{j_0}\| \lesssim \sum_{k \leq 0} \sum_{k < m, n < c_3 + \frac{k}{r}} \min \left( 2^{\frac{m+j}{2}}, 2^{\frac{n+j}{2}} \right) \lesssim |\lambda|^{-\frac{1}{2} \frac{d_i}{\pi(\theta)}}. \tag{2.19}
\]

Now we consider $j > j_0$. Let $k_j$ be the solution of $k'$ in (2.18) when $j' = j$. For simplicity, we shall divide the summation in (2.17) into the following two parts:
\[
I_1 := \sum_{k \geq k_j} \sum_{k < m, n < c_3 + \frac{k}{r}} \min \left( 2^{\frac{m+j}{2}}, 2^{\frac{n+j}{2}}, |\lambda|^{-\frac{1}{2} \frac{k+jd_i}{6}} \right),
\]
\[
I_2 := \sum_{k < k_j} \sum_{k < m, n < c_3 + \frac{k}{r}} \min \left( 2^{\frac{m+j}{2}}, 2^{\frac{n+j}{2}}, |\lambda|^{-\frac{1}{2} \frac{k+jd_i}{6}} \right).
\]

For $I_2$, we apply the size estimate to obtain
\[
I_2 = \sum_{k < k_j} \sum_{k < m, n < c_3 + \frac{k}{r}} 2^{\frac{m+n+2j}{2}} \lesssim 2^{\frac{k_j}{r}}.
\]

Note that $(k', j') = (j, k_j)$ is a solution of the equation (2.18). This implies
\[
\frac{r + 3}{r} k_j + (d_i + 3)j = -\log |\lambda|.
\]

Hence
\[
I_2 \lesssim 2^{\frac{r-d_i}{2(\theta)} j} |\lambda|^{-\frac{1}{2(\theta)}} \lesssim 2^{\frac{r-d_i}{2(\theta)} j} |\lambda|^{-\frac{1}{2(\theta)}} \lesssim 2^{\frac{r-d_i}{2(\theta)} j} |\lambda|^{-\frac{1}{2(\theta)}},
\]

where we have used the facts $r \leq d_i$, $j < j_0$, and $2^{j_0} = |\lambda|^{-\frac{1}{2(\theta)}}$. 

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Now we turn to $I_1$. Consider the following inequality:

\[
\frac{1}{4}(m + n + 2j) \geq -\frac{1}{6}\log_2 |\lambda| - \frac{1}{6}(k + j d_i).
\]

Observe that the equation (2.18) holds for $k' = k_j, j' = j$. By subtracting this equation from the above inequality, we have

\[
(m - \frac{k}{r}) + (n - \frac{k}{r}) \geq (\frac{2}{3} + \frac{2}{r}) (k_j - k).
\]

Let

\[
A = \left\{ (m, n) \in \mathbb{Z}_{<0}^2 : m \leq \frac{k}{r}, n \leq \frac{k}{r} \text{ and } (m - \frac{k}{r}) + (n - \frac{k}{r}) \geq (\frac{2}{3} + \frac{2}{r}) (k_j - k) \right\},
\]

\[
B = \left\{ (m, n) \in \mathbb{Z}_{<0}^2 : m \leq \frac{k}{r}, n \leq \frac{k}{r} \text{ and } (m - \frac{k}{r}) + (n - \frac{k}{r}) < (\frac{2}{3} + \frac{2}{r}) (k_j - k) \right\}.
\]

It is easy to see that the number of elements in $A$, denoted by $|A|$, is comparable to $(k - k_j)^2$.

On the other hand, for $t < 0$ which satisfies $t + \frac{2k}{r} \in \mathbb{Z}_{<0}$, we define

\[
B_t = \left\{ (m, n) \in \mathbb{Z}_{<0}^2 : m \leq \frac{k}{r}, n \leq \frac{k}{r} \text{ and } (m - \frac{k}{r}) + (n - \frac{k}{r}) = t \right\}.
\]

It is clear that $|B_t| \leq |t| + 1$ and

\[
B = \bigcup_t B_t,
\]

where the union is taken over all $t$ satisfying $t + \frac{2k}{r} \in \mathbb{Z}_{<0}$ and $t \leq (\frac{2}{3} + \frac{2}{r})(k_j - k)$. So, by calculation and definition of $k_j$, we have

\[
I_1 = \sum_{k_j \leq k \leq 0} \sum_{k \leq m, n < c_3 + \frac{k}{r}} \min \left(2^{\frac{m+k}{2}}, 2^{\frac{n+k}{2}}, |\lambda|^{-\frac{1}{6}} 2^{-\frac{k+j d_i}{6}} \right)
\]

\[
\leq \sum_{k \geq k_j} \sum_{m, n \leq c_3 + \frac{k}{r}} \max \left(2^{\frac{m+k}{2}}, |\lambda|^{-\frac{1}{6}} 2^{-\frac{k+j d_i}{6}} \right)
\]

\[
\leq \sum_{k \geq k_j} \sum_{(m, n) \in A} |\lambda|^{-\frac{1}{6}} 2^{-\frac{k+j d_i}{6}} + \sum_{k \geq k_j} \sum_{(m, n) \in B} 2^{\frac{m+k}{2}} := I_{1,1} + I_{1,2}.
\]

For $I_{1,1}$, it follows from $|A| \lesssim 1 + (k - k_j)^2$ that

\[
I_{1,1} \lesssim \sum_{k \geq k_j} (1 + (k_j - k)^2)|\lambda|^{-\frac{1}{6}} 2^{-\frac{k+j d_i}{6}}
\]

\[
\lesssim |\lambda|^{-\frac{1}{6}} 2^{-\frac{k_j+j d_i}{6}} = 2^{\frac{k_j+j}{6}}.
\]
where the last equality is true since \((k', j') = (k_j, j)\) is a solution to (2.18). For \(I_{2,2}\), we have
\[
I_{1,2} \lesssim \sum_{k \geq k_j} \sum_{t \in \mathbb{Z} < 0, t \leq 2(k_j - k)} \sum_{(m,n) \in B_t} 2^{\frac{m+2t+n}{4}}
\]
\[
\lesssim \sum_{k \geq k_j} \sum_{t \in \mathbb{Z} < 0, t \leq 2(k_j - k)} \frac{(|t| + 1) 2^{-\frac{t+2k_j}{4}}}{j^{\frac{1}{2}}}
\]
\[
\lesssim \frac{k_j^{\frac{1}{2} j}}{j^{\frac{1}{2}}}.\]
By the same argument as in the estimate of \(I_2\), we see that \(I_1\) is bounded by a constant multiple of \(|\lambda|^{-\frac{1}{\pi (d_1 + 3)}}\).

Now we turn to the case \(M_i > 1\). We have pointed out that the desired decay estimate cannot be obtained by resolution of singularities in the original coordinates \(x,y\). For this reason, we shall make a change of variables in the following domain:
\[
\Omega_\epsilon = \{(x,y) : x > 0, |y| \leq 2\epsilon x\}. \tag{2.20}
\]
where \(\epsilon > 0\) is a suitably small number.

If there exists some \(M_i = 1\), then we choose a small number \(\epsilon > 0\) such that \(p_i(t)\), defined by (2.4), has no non-zero roots in \((-2\epsilon, 2\epsilon)\). Otherwise let \(\epsilon = 1\) if there is no \(M_i = 1\). Choose a smooth function \(a(x) \in C_0^\infty(-2,2)\) such that \(a(x) = 1\) on \([-1,1]\). First, we insert \(a\left(\frac{y}{\epsilon x}\right)\chi_{\{x > 0\}}\) into the cut-off function of \(\Lambda_\lambda\). We shall make the following changes of variables:
\[
u = x, \quad v = x + y. \tag{2.21}
\]
It is easy to see that \(\widetilde{H}(u,v) = H(x,y)\) has the same generalized order \(\tilde{d}\) as \(H\), i.e., \(\tilde{d} = d\). The trilinear oscillatory integral \(\Lambda_\lambda\) becomes
\[
\tilde{\Lambda}_\lambda(f_1, f_2, f_3) = \int_{\mathbb{R}^2} e^{i\lambda S(u,v-u)} f_1(u) f_2(v-u) f_3(v) \varphi(u,v-u) a \left(\frac{v-u}{\epsilon u}\right) \chi_{\{u > 0\}} du dv. \tag{2.22}
\]
This implies that \(\tilde{\Lambda}_\lambda\) is also a trilinear oscillatory integral of convolution type. On the other hand, the curved box \(\Omega_\epsilon\) in (2.20) becomes
\[
\tilde{\Omega}_\epsilon = \{(u,v) : u > 0, u - 2\epsilon u < v < u + 2\epsilon u\}. \tag{2.23}
\]
If we apply the method of resolution of singularities for \(\widetilde{H}(u,v)\), then it is suffices to consider the corresponding compact face of \(N(\widetilde{H})\) with slope \(-1\). Hence our above argument works.

Combining all above results, we have completed the proof of Theorem 1.4.

**Remark** The decay estimate in Theorem 1.4 and the Newton polyhedron of \(H\) are unrelated. In fact, during our proof, it suffices to treat only the compact face of \(N(H)\) with slope \(-1\).
3 \( L^2 \) uniform estimates for trilinear oscillatory integrals

In this section, we shall establish \( L^2 \) uniform estimates for the trilinear oscillatory integrals \( \Lambda_{\lambda}(f,g,h) \) with polynomial phases. The resulting \( L^2 \) estimate will be a consequence of an operator van der Corput lemma and uniform estimates for a class of sublevel set operators. We begin with a useful decomposition of an algebraic domain, due to Phong, Stein, and Sturm [16], which will play an important role in our argument.

**Definition 3.1** Assume \( a \) and \( b \) are real numbers, \( a < b \). Suppose \( g \) and \( h \) are continuous monotone functions on \([a,b]\), and \( g(x) < h(x) \) for all \( a < x < b \). Then the domain

\[
\Omega = \{(x,y) \in \mathbb{R}^2 : a < x < b, g(x) < y < h(x)\}
\]

is called a curved trapezoid.

**Definition 3.2** We say that \( D \subseteq U = [0,1]^d \) is a simple algebraic domain of type \((r,n)\) if \( D \) can be written as

\[
D = \{(x_1, \ldots, x_d) \in U | P_k(x) \geq \lambda_k, 1 \leq k \leq r'\},
\]

where \( P_k \in \mathbb{R}[x_1, \ldots, x_d] \) are real polynomials, \( \lambda_k \in \mathbb{R}, 1 \leq r' \leq r \), and \( \deg P_k \leq n \).

We say that \( D \) is an algebraic domain of type \((r,n,d,\omega)\) if \( D = \bigcup_{i=1}^{\omega'} D_i \) with \( \omega' \leq \omega \), where each \( D_i \) is a simple algebraic domain of type \((r,n)\).

Phong, Stein, and Sturm proved the following theorem for algebraic domains. Roughly speaking, any algebraic domain can be decomposed into finitely many disjoint curved trapezoids, up to a set of measure zero.

**Lemma 3.3** ([16]) Let \( D \) be an algebraic domain of type \((r,n,2,\omega)\). Then there exists finitely many curved trapezoids \( \Omega_1, \Omega_2, \ldots, \Omega_M \), and a set \( Z \) of measure zero, such that

\[
D = \left( \bigcup_{i=1}^{M} \Omega_i \right) \cup Z.
\]

Moreover, the number \( M \) is bounded in terms of \( r, n, \omega \).

For our application, we shall give an outline of proof of Lemma 3.3. For simplicity, we assume \( D \) is a simple algebraic domain. The treatment for general algebraic domain is essentially the same as that for a simple algebraic domain. Then

\[
D = \{(x,y) \in [0,1]^2 | P_k(x,y) \geq \lambda_k, 1 \leq k \leq r \}.
\]

Let \( Q_k = P_k - \lambda_k \) and decompose \( Q_k \) as

\[
Q_k(x,y) = \prod_{l} (P_{k,l}(x,y))^{m_{i_l}},
\] \hspace{1cm} (3.1)
where \( Q_{k,l} \) are irreducible polynomials in \( \mathbb{R}[x,y] \), \( m_l \geq 1 \), and \( P_{k,l_1}, P_{k,l_2} \) are relatively prime for \( l_1 \neq l_2 \). Let
\[
\Gamma_1 = \bigcup \{ (x,y) \in \mathbb{R}^2 \mid P_{k,l}(x,y) = 0 \},
\]
where the union is taken over all \( P_{k,l} \) such that \( P_{k,l} \) is a polynomial independent of \( x \) or \( y \). In other words, each \( Q_{k,l} \) is a polynomial in only one variable. It is clear that \( \Gamma_1 \) consists of finitely many horizontal (vertical) straight lines. Of course, it is possible that \( \Gamma_1 = \emptyset \).

Define \( \Gamma_2 \) by
\[
\Gamma_2 = \bigcup \{ (x,y) \in \mathbb{R}^2 \mid P_{k,l}(x,y) = 0, \ \partial_x P_{k,l}(x,y) \cdot \partial_y P_{k,l}(x,y) = 0 \},
\]
where the union is taken over all factors appearing in \( \mathcal{L} \) such that both \( \partial_x P_{k,l} \) and \( \partial_y P_{k,l} \) are non-zero polynomials. In other words, \( P_{k,l} \) is not a polynomial in only one variable. The equation \( \partial_x P_{k,l}(x,y) \cdot \partial_y P_{k,l}(x,y) = 0 \) implies that either \( \partial_x P_{k,l}(x,y) = 0 \) or \( \partial_y P_{k,l}(x,y) = 0 \).

We also define \( \Gamma_3 \) by
\[
\Gamma_3 = \bigcup \{ (x,y) \in \mathbb{R}^2 \mid P_{k_1,l_1}(x,y) = P_{k_2,l_2}(x,y) = 0 \},
\]
where the union is taken over all polynomial factors \( P_{k_1,l_1} \) and \( P_{k_2,l_2} \), appearing in the factorization \( \mathcal{L} \), such that \( P_{k_1,l_1} \) and \( P_{k_2,l_2} \) are relatively prime.

It should be pointed out that if some \( P_{k,l} \) has an isolated zero, say \( (x_0,y_0) \), then it follows from the implicit function theorem that \( \partial_x P_{k,l}(x_0,y_0) = 0 \) and \( \partial_y P_{k,l}(x_0,y_0) = 0 \).

By Bézout’s theorem, we see that \( \Gamma_2 \) and \( \Gamma_3 \) have finitely many points, and the number of points in \( \Gamma_2 \) and \( \Gamma_3 \) is bounded in terms of \( r,n \) and \( \omega \). Similarly, by the Fundamental theorem of algebra, it is clear that \( \Gamma_1 \) consists of finitely many straight lines parallel in the axes. The number is also bounded in terms of \( r,n \), and \( \omega \).

Let \( Z(P_{k,l}) \) be the zeros of \( P_{k,l} \), i.e., \( Z(P_{k,l}) = \{ (x,y) \in \mathbb{R}^2 \mid P_{k,l}(x,y) = 0 \} \). Then \( Z(P_{k,l}) \) is a set of measure zero. Let \( \mathcal{L} \) be the collection of all \( x \)-parallel and \( y \)-parallel straight lines, which intersect the union \( \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \), or which contains the intersection of the boundary of \( [0,1]^2 \) and \( (Z(P_{k,l})) \). Then the number of straight lines in \( \mathcal{L} \) is bounded in terms of \( r,n \), and \( \omega \). Then set \( Z \) in Lemma \( \ref{lem:main} \) can be defined by \( Z = \bigcup (Z(P_{k,l})) \cup \mathcal{L} \). Then we can prove that there exists finitely many curved trapezoids \( \Omega_1, \Omega_2, \cdots, \Omega_M \) such that
\[
D = \left( \bigcup_{i=1}^M \Omega_i \right) \bigcup Z.
\]

Consider the following trilinear sublevel set operator
\[
K_\mu(f,g,h) = \int_D \chi_{\{|H(x,y)| \leq \mu\}} f(x)g(y)h(x+y) \, dx \, dy,
\]
where \( D \) is an algebraic domain in \( U \), and \( \chi_{\{|H(x,y)| \leq \mu\}} \) is the characteristic function of the sublevel set \( \{ (x,y) \in \mathbb{R}^2 \mid |H(x,y)| \leq \mu \} \). The function \( H(x,y) \) is a polynomial in \( x \) and \( y \).

Our main result for \( K_\mu \) is the following theorem.
Theorem 3.4 Let $H \in \mathbb{R}[x,y]$ be a polynomial in $x$ and $y$ with real coefficients. Assume $\alpha^{(1)}, \alpha^{(2)}, \ldots, \alpha^{(N)} \in \mathbb{N}^2 \setminus \{(0,0)\}$ and define an algebraic domain by

$$D = \left\{ (x, y) \in [0,1]^2 : |\partial^{\alpha} H(x, y)| \geq 1, 1 \leq i \leq N \right\}.$$  

Then there exists a constant $C > 0$, depending only on $\deg H$, such that

$$|K_\mu(f, g, h)| \leq C\mu^{1/2}\|f\|_2\|g\|_2\|h\|_2, \quad (3.2)$$

where $d = \min\{ |\alpha^{(i)}| : 1 \leq i \leq N \}$.

**Remark** We use the notation $\partial^{\alpha} H(x, y) = \partial^{\alpha_1}_{x_1} \partial^{\alpha_2}_{y_2} H(x, y)$, and $|\alpha^{(i)}| = \alpha^{(i)}_1 + \alpha^{(i)}_2$ is the order of $\alpha^{(i)}$.

**Proof.** For each $1 \leq i \leq N$, we shall prove (3.2) with $d$ replaced by $|\alpha^{(i)}|$.

**Case 1.** $D = \{(x, y) \in [0,1]^2 : x^{\alpha^{(1)}_1} y^{\alpha^{(1)}_2} \leq \mu \}$.

In this case, $H(x, y) = x^{\alpha^{(1)}_1} y^{\alpha^{(1)}_2}$ and $\partial^{\alpha^{(i)}}_{x^i} \partial^{\alpha^{(j)}}_{y^j} H(x, y) = \alpha^{(i)}_1 \alpha^{(j)}_2 \geq 1$. It is clear that $D = U \cap \{|H(x, y)| \leq \mu \}$. Here $\mu$ is a positive number.

Let $j_\mu = \sup\{ j \in \mathbb{Z} : 2^j \leq \mu \}$. Then $\mu \in [2^{j_\mu}, 2^{j_\mu+1})$. Hence we have

$$\int_D |f(x)g(y)h(x+y)| \, dx \, dy \leq \sum_{j \leq j_\mu} \int_{D_j} |f(x)g(y)h(x+y)| \, dx \, dy,$$

where $D_j = \{(x, y) \in [0,1]^2 : 2^j \leq x^{\alpha^{(1)}_1} y^{\alpha^{(1)}_2} \leq 2^{j+1} \}$ for $j \in \mathbb{Z}$.

If one component of $\alpha^{(i)}$ is equal to zero, say $\alpha^{(i)}_2 = 0$, then by Cauchy-Schwarz's inequality and Fubini's theorem,

$$\int_{D_j} |f(x)g(y)h(x+y)| \, dx \, dy \leq \left( \int_{D_j} |f(x)|^2 \, dx \, dy \right)^{1/2} \cdot \left( \int_{D_j} |g(y)h(x+y)|^2 \, dx \, dy \right)^{1/2} \leq C2^{2\alpha^{(i)}_1} \|f\|_2 \|g\|_2 \|h\|_2.$$

It follows immediately that

$$\int_D |f(x)g(y)h(x+y)| \, dx \, dy \leq C\mu^{1/2} \|f\|_2 \|g\|_2 \|h\|_2.$$

Now assume that $\alpha^{(i)}_1 > 0$ and $\alpha^{(i)}_2 > 0$. By a dyadic decomposition, we can see that

$$\int_{D_j} |f(x)g(y)h(x+y)| \, dx \, dy \leq \sum \int_{\{(x, y) \in [0,1]^2 : 2^k \leq x^{\alpha^{(i)}_1} \leq 2^{k+1}, 2^l \leq y^{\alpha^{(i)}_2} \leq 2^{l+1} \}} |f(x)g(y)h(x+y)| \, dx \, dy,$$

where the summation is taken over all $k, l \in \mathbb{Z}$ satisfying $k, l \leq 0$ and $j - 1 \leq k + l \leq j + 1$. By
the almost orthogonality principle in Lemma 2.2, we have
\[ \sup_{\|f\|_2,\|g\|_2,\|h\|_2 \leq 1} \int_{D_j} |f(x)g(y)h(x + y)| \, dx \, dy \]
\[ \leq \sup_{\|f\|_2,\|g\|_2,\|h\|_2 \leq 1} \sup_{k,l} \int \{ (x,y) \in [0,1]^2 : 2^k \leq x^{(i)} \leq 2^{k+1}, 2^l \leq y^{(i)} \leq 2^{l+1} \} |f(x)g(y)h(x + y)| \, dx \, dy \]
\[ \leq C \min \{ 2^{2\alpha_1^{(i)}}, 2^{2\alpha_2^{(i)}} \} \]
\[ \leq C 2^{2\alpha^{(i)}}. \]

where the supremum \( \sup_{k,l} \) is taken over all negative integers \( k, l \) satisfying \( j - 1 \leq k + l \leq j + 1 \). Taking summation over all \( j \leq j_{\mu} \), we obtain the desired \( L^2 \) estimate. Since the inequality is true with \( d \) replaced by \( \alpha^{(i)} \) for each \( 1 \leq i \leq N \), it also holds for the growth rate \( \frac{1}{d} \).

**Case 2.** \( D \) is a general algebraic domain.

By Lemma 3.3, we can decompose \( D \) into finitely many curved trapezoids, up to a set of measure zero. Assume
\[ D = \left( \bigcup_{i=1}^{M} \Omega_i \right) \bigcup Z, \]
where each \( \Omega_i \) is a curved trapezoid, and \( Z \) has measure zero. By a similar argument as Phong, Stein, and Sturm [16], one has \( a^{(i)} \leq \mu \) for each coordinate rectangle \( R \in \Omega_j \) of dimensions \( a \) by \( b \). Moreover, as in [16], we can decompose each \( \Omega_j \) into at most countable many curved right triangles, which satisfy the orthogonality assumption in Lemma 2.2. This reduces our estimate in Case 2 to Case 1.

Combining all above results, we have completed the proof of Theorem 3.4. \( \square \)

Now consider
\[ \Lambda(\lambda, f, g, h) = \int \int e^{i\lambda S(x,y)} f(x)g(y)h(x + y)\varphi(x,y) \, dx \, dy, \]
where \( S \in \mathbb{R}[x,y] \) is a polynomial in \( x \) and \( y \), and \( \varphi(x,y) \) is a smooth cut-off near the origin. The problem then is the uniform estimate for \( \Lambda(\lambda, f, g, h) \) as \( \lambda \to \infty \). For convenience, define a linear oscillatory integral operator by
\[ T(\lambda, f) = \int_{-\infty}^{\infty} e^{i\lambda P(x,y)} f(y)\varphi(x,y) \, dy, \]
where \( P \) is a real-valued polynomial, and \( \varphi \in C_0^\infty(\mathbb{R}^2) \) is supported in a neighborhood of the origin. We need the following operator van der Corput lemma for \( T(\lambda, \cdot) \); see [16] for its proof.

**Lemma 3.5** ([16]) Let \( P(x,y) \in \mathbb{R}[x,y] \) be a real polynomial in \( x \) and \( y \), \( \Omega \) be a curved trapezoid in \( \mathbb{R}^2 \), and for each \( x \in (a,b) \), the cut-off function \( \varphi(x,\cdot) \in C_0^\infty([g(x),h(x)]) \). Here \( \Omega \)
is given by
\[ \Omega = \{(x, y) \in \mathbb{R}^2 | a < y < b, g(x) < y < h(x) \}, \]
where \( g(x) < h(x) \) for each \( x \in (a, b) \). Assume the following conditions are true:

(i) there exists two positive constants \( \mu > 0 \) and \( A \geq 1 \) such that
\[ \mu \leq S_{xy}''(x,y) \leq A\mu, \quad (x,y) \in \Omega, \]

(ii) for some \( B > 0 \), it is true that
\[ \sup_{\Omega} \sum_{k=0}^{2} (\tau(x))^k |\partial_y^k \varphi(x,y)| \leq B, \]
where \( \tau(x) \) is the length of the cross section \( \{ y \in \mathbb{R} | (x,y) \in \Omega \} \). Define
\[ T_{\lambda,\Omega}f(x) = \int_{-\infty}^{\infty} e^{i\lambda S(x,y)} \chi_{\Omega}(x,y) \varphi(x,y)f(y) \, dy. \]

Then we have
\[ \|T_{\lambda,\Omega}f\|_2 \leq C|\lambda\mu|^{-\frac{1}{2}}\|f\|_2, \]
where the constant depends only on \( \deg S, A \) and \( B \).

As a consequence of Lemma 3.5, we have the following operator van der Corput lemma for trilinear oscillatory integrals.

**Lemma 3.6** Assume \( S \in \mathbb{R}[x,y] \) is a real polynomial in \( x \) and \( y \), and \( \Omega \) is a curved trapezoid given by
\[ \Omega = \{(x, y) \in \mathbb{R}^2 | a < x < b, \alpha(y) < x < \beta(y) \}, \]
where \( \alpha \) and \( \beta \) are continuous monotone functions on \( [a,b] \) such that \( \alpha(y) < \beta(y) \) for all \( y \in (a,b) \). Let \( H(x,y) = \partial_x \partial_y (\partial_x - \partial_y) S(x,y) \). If the following conditions are true:

(i) for each \( y \in (a,b) \), \( \varphi(\cdot, y) \in C_0^{\infty}([\alpha(y), \beta(y)]) \) and there exists a constant \( B > 0 \) such that
\[ \sup_{\Omega} \sum_{k=0}^{2} (\tau(y))^k |\partial_y^k \varphi(x,y)| \leq B, \]
where \( \tau(y) \) is the length of the cross section \( \{ x \in \mathbb{R} | (x,y) \in \Omega \}; \)

(ii) there exist two positive constants \( \mu > 0 \) and \( A \leq 1 \) such that
\[ \mu \leq |H(x,y)| \leq A\mu, \quad (x,y) \in \Omega, \]
then we have
\[ \|\Lambda_{\lambda,\Omega}(f,g,h)\| \leq C|\lambda\mu|^{-\frac{1}{2}}\|f\|_2\|g\|_2\|h\|_2, \]
where the constant \( C > 0 \) depends only on \( \deg S, A \) and \( B \), and \( \Lambda_{\Omega,f,g,h} \) is defined by

\[
\Lambda_{\Omega,f,g,h} = \int \int e^{i\lambda S(x,y)} f(x)g(y)h(x+y)\chi_{\Omega}(x,y)\varphi(x,y) \, dx \, dy.
\]

**Proof.** Let \( T_\lambda(g,h) \) be defined by

\[
T_\lambda(g,h)(x) = \int_{-\infty}^{\infty} e^{i\lambda S(x,y)} g(y)h(x+y)\chi_{\Omega}(x,y)\varphi(x,y) \, dy.
\]

Then it is clear that \( \Lambda_{\Omega,f,g,h} = \int_{\mathbb{R}} T_\lambda(g,h)(x)f(x) \, dx \).

Now we compute the \( L^2 \) norm of \( T_\lambda(g,h) \). Set \( \varphi_{\Omega}(x,y) = \varphi(x,y)\chi_{\Omega}(x,y) \). Then we have

\[
\begin{align*}
&\int_{\mathbb{R}} |T_\lambda(g,h)(x)|^2 \, dx \\
&= \int_{\mathbb{R}^3} e^{i\lambda |S(x,y)-S(x,z)|} g(y)h(x+y)\overline{g(z)}\overline{h(x+z)}\varphi_{\Omega}(x,y)\varphi_{\Omega}(x,z) \, dy \, dz \, dx \\
&= \int_{\mathbb{R}^3} e^{i\lambda |S(x,y)-S(x,y+u)|} g(y)g(y+u)h(x+y)\overline{h(x+y+u)}\varphi_{\Omega}(x,y)\overline{\varphi_{\Omega}(x,y+u)} \, dx \, dy \, du \\
&= \int_{\mathbb{R}^2} \left[ \int_{\mathbb{R}} e^{i\lambda S(x,y-v,y-u)} \overline{\varphi_{\Omega}(x,y)} \varphi_{\Omega}(v,y) \, dx \right] \, dv \, dy \quad (3.3)
\end{align*}
\]

where we have made changes of variables \( z = y + u, v = x + y \), and

\[
\begin{align*}
\tilde{S}_u(v,y) &= S(v-y,y) - S(v-y,y+u), \\
\tilde{g}_u(y) &= g(y)\overline{g(y+u)}, \\
\tilde{h}_u(v) &= h(v)\overline{h(v+u)}, \\
\tilde{\varphi}_u(v,y) &= \varphi(v,y)\overline{\varphi(v,y+u)},
\end{align*}
\]

for each \( u \in \mathbb{R} \). Then the inner integral with respect to \( v \) and \( y \) in (3.3) can be written as

\[
\int_{\mathbb{R}} e^{i\lambda \tilde{S}_u(v,y)} \tilde{g}_u(y)\tilde{h}_u(v)\tilde{\varphi}_u(v,y) \chi_{\tilde{\Omega}_u}(v,y) \, dv \, dy. \quad (3.4)
\]

Let \( \tilde{\Omega}_u \) be

\[
\tilde{\Omega}_u = \{(v,y) \in \mathbb{R}^2 | a < y < b, a < y + u < b, \alpha(y) < v-y < \beta(y), \alpha(y+u) < v-y < \beta(y+u) \}
\]

\[
= \left\{ (v,y) \in \mathbb{R}^2 | \tilde{a}_u < y < \tilde{b}_u, \tilde{a}_u(y) < v < \tilde{\beta}_u(y) \right\},
\]

where

\[
\begin{align*}
\tilde{a}_u &= \max\{a, a-u\}, \\
\tilde{b}_u &= \min\{b, b-u\}, \\
\tilde{\alpha}_u(y) &= y + \max\{\alpha(y), \alpha(y+u)\}, \\
\tilde{\beta}_u(y) &= y + \min\{\beta(y), \beta(y+u)\}.
\end{align*}
\]
Assume $\tilde{a}_u < \tilde{b}_u$. Then both $\tilde{a}_u$ and $\tilde{b}_u$ are monotone continuous functions on $[\tilde{a}_u, \tilde{b}_u]$. And for each $y \in (\tilde{a}_u, \tilde{b}_u)$, the cut-off function $\tilde{\varphi}(\cdot, y)$ is a smooth function supported in the horizontal cross section $\{v \in \mathbb{R} | (v, y) \in \tilde{\Omega}_u\}$. Moreover,

$$\sup_{\tilde{\Omega}_u} \sum_{k=0}^{2} (\tau_u(y))^k |\partial_x^k \tilde{\varphi}_u(v, y)| \leq \left( \sup_{\Omega} \sum_{k=0}^{2} (\tau(y))^k |\partial_x^k \varphi(x, y)| \right)^2 \leq B^2.$$ 

Here $\tau_u(y)$ is the length of the interval $\{v \in \mathbb{R} | (v, y) \in \tilde{\Omega}_u\}$.

On the other hand, it is clear that

$$\partial_x \partial_y [S(v - y, y) - S(v - y, y + u)] = u \int_0^1 H(v - y, y + \theta u) \, d\theta.$$ 

Hence, by our assumption (ii),

$$\mu |u| \leq |\partial_x \partial_y [S(v - y, y) - S(v - y, y + u)]| \leq A \mu |u|.$$ 

By Lemma 3.5 the integral in (3.4) is bounded by

$$\left| \int_{\mathbb{R}^2} e^{i\lambda \tilde{S}_u(v, y)} \tilde{g}_u(y) \tilde{h}_u(v) \tilde{\varphi}_u(v, y) \chi_{\tilde{\Omega}_u}(v, y) \, dv \, dy \right| \leq C \left( |\lambda| |\mu| |u| \right)^{-\frac{1}{2}} \|\tilde{g}_u(\cdot)\|_2 \|\tilde{h}_u(\cdot)\|_2,$$

where $C > 0$ depends only on $\deg S, A,$ and $B$. Taking the absolute value into the integral, we also have

$$\left| \int_{\mathbb{R}^2} e^{i\lambda \tilde{S}_u(v, y)} \tilde{g}_u(y) \tilde{h}_u(v) \tilde{\varphi}_u(v, y) \chi_{\tilde{\Omega}_u}(v, y) \, dv \, dy \right| \leq C \|\tilde{g}_u(\cdot)\|_1 \|\tilde{h}_u(\cdot)\|_1,$$

where $C > 0$ depends only on $B$. By the Cauchy-Schwarz inequality, $\|\tilde{g}_u(\cdot)\|_1 \leq \|g\|_2$ and also $\|\tilde{g}_u(\cdot)\|_1 \leq \|h\|_2$. 

Now we are able to estimate $\|T_\lambda(g, h)\|_2^2$. In fact, we make use of the above $L^2$ and $L^1$ estimates together with Cauchy-Schwarz’s inequality to obtain

$$\|T_\lambda(g, h)\|_2^2 \leq \int_{\mathbb{R}} \left| \int_{\mathbb{R}^2} e^{i\lambda \tilde{S}_u(v, y)} \tilde{g}_u(y) \tilde{h}_u(v) \tilde{\varphi}_u(v, y) \chi_{\tilde{\Omega}_u}(v, y) \, dv \, dy \right| \, du$$

$$= \sum_{j \in \mathbb{Z}} \int_{2^j}^{2^{j+1}} \left| \int_{\mathbb{R}^2} e^{i\lambda \tilde{S}_u(v, y)} \tilde{g}_u(y) \tilde{h}_u(v) \tilde{\varphi}_u(v, y) \chi_{\tilde{\Omega}_u}(v, y) \, dv \, dy \right| \, du$$

$$\leq C \sum_{j \in \mathbb{Z}} \min\{(|\lambda| |\mu| 2^j)^{-\frac{1}{2}}, 2^j\} \|g\|_2^2 \|h\|_2^2$$

$$\leq C |\lambda| |\mu|^{-\frac{1}{2}} \|g\|_2 \|h\|_2^2,$$

where $C$ depends only on $\deg S, A$ and $B$. This implies $\|T_\lambda(g, h)\|_2 \leq C |\lambda| |\mu|^{-\frac{1}{2}} \|g\|_2 \|h\|_2$. By
duality, the estimate for $\Lambda_\Omega(f, g, h)$ follows immediately. This completes the proof of Lemma 3.6.

Now we can state our main result in this section.

**Theorem 3.7** Assume $S(x, y) \in \mathbb{R}[x, y]$ is a real polynomial. Let $H(x, y) = \partial_x \partial_y (\partial_x - \partial_y) S(x, y)$, and $\alpha^{(1)}, \alpha^{(2)}, \ldots, \alpha^{(N)} \in \mathbb{N}^2$. Suppose that $\varphi \in C_0^\infty(U)$ and

$$|\partial^{\alpha^{(i)}} H(x, y)| \geq 1, \quad 1 \leq i \leq N,$$

on the support of $\varphi$. Then $\Lambda_\lambda(f, g, h)$ satisfies the inequality

$$|\Lambda_\lambda(f, g, h)| \leq C |\lambda|^{-\frac{1}{2(\deg S)} \cdot \frac{d}{2}} \|f\|_2 \|g\|_2 \|h\|_2$$

with $d = \min\{|\alpha^{(i)}| : 1 \leq i \leq N\}$. Here the constant $C$ depends only on $\deg S$ and the cut-off function $\varphi$.

**Proof.** Choose a smooth function $\phi \in C_0^\infty([\frac{1}{2}, 2])$ such that $\sum_{j \in \mathbb{Z}} \phi\left(\frac{x}{2^j}\right) = 1$ for all $x > 0$. Let $\Lambda_\lambda^{(j)}(f, g, h)$ be defined as $\Lambda_\lambda(f, g, h)$ with insertion of $\phi\left(\frac{|H(x, y)|}{2^j}\right)$ in the cut-off. In other words,

$$\Lambda_\lambda^{(j)}(f, g, h) = \int \int e^{i\lambda S(x, y)} f(x)g(y)h(x + y)\phi\left(\frac{|H(x, y)|}{2^j}\right) \varphi(x, y) \, dx \, dy, \quad j \in \mathbb{Z}.$$

Now consider the simplest case $d = 0$. By removing only the horizontal lines appearing in the proof of Lemma 3.3 we can decompose the following algebraic domain

$$D_j = \{ (x, y) \in U \mid 2^{j-1} \leq |\partial^{\alpha^{(i)}} H(x, y)| \leq 2^j \}$$

into finitely many curved trapezoids described in Lemma 3.6. Moreover, the cut-off function of $\Lambda_\lambda^{(j)}(f, g, h)$, with the vertical variable fixed, is compactly supported in the corresponding horizontal cross section of each curved trapezoid mentioned above. Hence we can apply Lemma 3.6 to obtain $|\Lambda_\lambda^{(j)}(f, g, h)| \leq C (|\lambda|2^j)^{-\frac{1}{2}} \|f\|_2 \|g\|_2 \|h\|_2$ for $j \geq 0$. Taking summation over $j$, we obtain the desired estimate.

Now we focus on the general case $d > 0$. By Theorem 3.4, we obtain

$$|\Lambda_\lambda^{(j)}(f, g, h)| \leq C 2^{-\frac{d}{2}j} \|f\|_2 \|g\|_2 \|h\|_2.$$

On the other hand, we can apply Lemma 3.6 to get

$$|\Lambda_\lambda^{(j)}(f, g, h)| \leq C (|\lambda|2^j)^{-\frac{1}{2}} \|f\|_2 \|g\|_2 \|h\|_2.$$

Hence it follows that

$$|\Lambda_\lambda(f, g, h)| \leq \sum_{j \in \mathbb{Z}} |\Lambda_\lambda^{(j)}(f, g, h)|$$

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\[ \leq C \sum_{j \in \mathbb{Z}} \min \{ 2^j, (|\lambda|2^j)^{-\frac{1}{2}} \} \|f\|_2 \|g\|_2 \|h\|_2 \]

\[ \leq |\lambda|^{-\frac{1}{2(3 + d)}} \cdot \|f\|_2 \|g\|_2 \|h\|_2. \]

This completes the proof of the theorem.

Inspired by an observation in Gressman [11] for bilinear oscillatory integral operators, we can prove the following theorem by a similar argument in this section.

**Theorem 3.8** Assume \( S \in \mathbb{R}[x, y] \) is a real polynomial, and \( H(x, y) = \partial_x \partial_y (\partial_x - \partial_y) S(x, y) \). If \( \phi \in C^\infty_0 ([\frac{1}{12}, 2]) \), then

\[ \left| \int_{[0,1]^2} e^{i\lambda S(x,y)} f(x)g(y)h(x+y)\phi \left( \frac{H(x,y)}{\mu} \right) dx \, dy \right| \leq C |\lambda\mu|^{-\frac{1}{6}} \|f\|_2 \|g\|_2 \|h\|_2 \]

for all \( \mu \in \mathbb{R} \setminus \{0\} \), where the constant \( C \) depends only on \( \deg S \) and the cut-off function \( \phi \).

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