Abstract. In this paper, we give a complete classification of \( \kappa \)-solutions of Kähler-Ricci flow on compact complex manifolds. Namely, they must be quotients of products of irreducible compact Hermitian symmetric manifolds.

1. Introduction

As an important problem in the analysis of the singularities of Ricci flow, the classification of ancient solutions to Ricci flow has attracted many attentions. 2-dimensional \( \kappa \)-solutions was classified by Perelman in [17]. 2-dimensional ancient Ricci flows with bounded curvature has also been classified even without the \( \kappa \)-noncollapsed condition by Daskalopoulos-Hamilton-Sesum (cf. [7],[8]). Very recently, Brendle classfied 3-dimensioanl noncompact \( \kappa \)-solutions [2]. It remains an open problem for higher dimensional \( \kappa \)-solutions and 3-dimensional compact \( \kappa \)-solutions. It is also interesting to ask the same question for \( \kappa \)-solutions to Kähler-Ricci flow. For eternal Kähler-Ricci flows with uniformly bounded and nonnegative holomorphic bi-sectional curvature, the authors have shown that they must be flat [9],[10]. In this paper, we give a complete classification of compact \( \kappa \)-solutions to Kähler-Ricci flow by using a non-existence result of steady Kähler-Ricci soliton in [10]. Namely, we prove

**Theorem 1.1.** Any \( \kappa \)-solution of Kähler-Ricci flow must be a quotient of product of some irreducible compact Hermitian symmetric manifolds.

By the definition of compact \( \kappa \)-solution of Kähler-Ricci flow \(( M, g(t) )\) (cf. Definition [21]), \(( M, g(t) )\) is a compact Kähler manifold with nonnegative holomorphic bi-sectional curvature. The classification of such manifolds is known as the generalized Frankel conjecture. The conjecture was proved by

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There are also many works related to the conjecture such as [5], [11], etc. The generalized Frankel conjecture is stated as follows.

**Theorem 1.2.** Let \((M, g)\) be an \(n\)-dimensional compact Kähler manifold of nonnegative holomorphic bisectional curvature and let \((\tilde{M}, \tilde{g})\) be its universal cover. Then, there exist nonnegative integers \(k, N_1, \ldots, N_l\) and irreducible compact Hermitian symmetric spaces \(M_1, \ldots, M_r\) of rank \(\geq 2\) such that such that \((\tilde{M}, \tilde{g})\) is isometrically biholomorphic to

\[
(C^k, g_0) \times (\mathbb{CP}^{n_1}, g_1) \times \cdots \times (\mathbb{CP}^{n_l}, g_l) \times (M_1, h_1) \times \cdots \times (M_r, h_r)
\]

where \(g_0\) is the Euclidean metric on \(C^k\), \(h_1, \ldots, h_r\) are canonical metrics on \(M_1, \ldots, M_r\) and \(g_i(1 \leq i \leq l)\) are Kähler metrics on \(\mathbb{CP}^{n_i}\) with nonnegative holomorphic bisectional curvature.

With the help of the generalized Frankel conjecture, we can classify compact backward type I \(\kappa\)-solutions similar to a result of Ni for \(\kappa\)-solutions with positive curvature operator in [15] (cf. Proposition 2.8). On the other hand, based on our former work [10], we will show that any compact \(\kappa\)-solution of Kähler-Ricci flow must be of Type I (cf. Lemma 2.5). Thus Theorem 1.1 follows from Proposition 2.8 and Lemma 2.5.

### 2. Proof of Theorem 1.1

**Definition 2.1.** A complete Kähler-Ricci flow \((M, g(t))\) on \(t \in (-\infty, 0]\) is called ancient if the bisectional curvature of \(g(t)\) is bounded and nonnegative for any \(t \in (-\infty, 0]\). A complete Kähler-Ricci flow \((M, g(t))\) is called a \(\kappa\)-solution to Kähler-Ricci flow if it is a \(\kappa\)-noncollapsed, non-flat ancient solution.

We first recall a theorem on the character of Type II limit solutions by Cao in [3], which is

**Theorem 2.2.** Any simply connected Type II limit solution of Kähler-Ricci flow on a simply connected complex manifold must be a gradient steady Kähler-Ricci soliton.

The Type II limit solution in Theorem 2.2 is an eternal solution of Kähler-Ricci flow with uniformly bounded and nonnegative bisectional curvature and positive Ricci curvature where the scalar curvature assumes its maximum in space-time. The assumption of scalar curvature is only used to make sure that there is a point \((x_0, t_0)\) such that

\[
\frac{\partial R}{\partial t}(x_0, t_0) = 0.
\]

Moreover, the ancient condition is sufficient in Cao’s argument. Hence, Cao’s argument in [3] implies the following corollary.
Corollary 2.3. Let \((M, g(t))\) be an ancient solution of Kähler-Ricci flow with uniformly bounded nonnegative bisectional curvature and positive Ricci curvature. If there is a point \((x_0, t_0)\) such that (2.1) holds, then it is a gradient steady Kähler-Ricci soliton.

Now, we introduce the notation of backward type I Ricci flow.

Definition 2.4. An ancient solution \((M, g(t))\) defined on \(M \times (-\infty, A)\) is called of backward type I if there exists a constant \(C > 0\) such that

\[
|\text{Rm}|(x, t) \leq \frac{C}{|t|}, \quad \forall \; t \leq t_0,
\]

where \(t_0 < 0\) is a constant.

Lemma 2.5. Suppose that \((M, g(t))\) is a \(\kappa\)-solution of Kähler-Ricci flow on compact complex manifold \(M\). Then, \((M, g(t))\) is a backward type I \(\kappa\)-solution.

Proof. We prove by contradiction. Let \(M(t) = \sup_{x \in M} R(x, t)\). If the lemma is not true, then

\[
\limsup_{t \to -\infty} (-t)M(t) = \infty.
\]

Claim 2.6. Under the assumption of (2.3), there exist a time sequence \(t_i \to -\infty\) such that

\[
\lim_{i \to \infty} \frac{\partial}{\partial t} R(p_i, t_i) \cdot R^{-2}(p_i, t_i) = 0,
\]

where \(p_i\) assume the maximum of \(R(x, t)\) at time slice \(t = t_i\), i.e., \(R(p_i, t_i) = M(t_i)\).

We will use a trick in [2] to prove the claim. Note that \(\frac{\partial}{\partial t} R \geq 0\) by Harnack inequality. If the claim is not true, then there exists some constant \(\varepsilon > 0\) such that

\[
\frac{\partial}{\partial t} \left( - \frac{1}{R(p, t)} \right) \geq \varepsilon > 0, \quad \forall \; t \leq 0,
\]

for all \(p \in M\) that satifies \(R(p, t) = M(t)\). Let \(F(t) = M(t)^{-1}\) and \(G(p, t) = R(p, t)^{-1}\). For \(\Delta t > 0\), we have

\[
\frac{d}{dt} F(t) = \lim_{\Delta t \to 0} \frac{F(t + \Delta t) - F(t)}{\Delta t} \leq \lim_{\Delta t \to 0} \frac{G(p, t + \Delta t) - G(p, t)}{\Delta t} \leq -\varepsilon.
\]
It follows that

\[ \limsup_{t \to -\infty} (-t)M(t) \leq \frac{1}{\varepsilon}. \]

(2.6)

It contradicts to (2.3). We complete the proof of the claim.

Let \((p_i, t_i)\) be a sequence of time-space points such that \(R(p_i, t_i) = M(t_i)\) and \(t_i \to -\infty\). By the claim, we also assume that \((p_i, t_i)\) satisfying the condition (2.4). Now, we consider rescaled Kähler-Ricci flows \((M, g_i(t), p_i)\), where \(g_i(t) = M(t_i)g(M(t_i)^{-1}t + t_i)\). Since \(g_i(t)\) is uniformly bounded for \(t \leq 0\) and is \(\kappa\)-noncollapsed, \((M, g_i(t), p_i)\) converges to an ancient \(\kappa\)-solution \((M_\infty, g_\infty(t), p_\infty)\) in Cheeger-Gromov topology. Moreover, by (2.4), we have

\[ \frac{\partial}{\partial t} R_\infty(p_\infty, 0) = \lim_{i \to \infty} \frac{\partial [M(t_i)^{-1}R(p_i, M(t_i)^{-1}t + t_i)]}{\partial t} \bigg|_{t=0} \]

\[ = \lim_{i \to \infty} M(t_i)^{-2} \frac{\partial [R(p_i, M(t_i)^{-1}t + t_i)]}{\partial (M(t_i)^{-1}t)} \bigg|_{t=0} \]

\[ = \lim_{i \to \infty} \frac{\partial}{\partial t} R(p_i, t_i) \cdot R^{-2}(p_i, t_i) \]

\[ = 0, \]

and

\[ R_\infty(p_\infty, 0) = 1. \]

(2.7)

Note that by Theorem 2.1 in [4], we may assume that \((M_\infty, g_\infty(t), p_\infty)\) has positive Ricci curvature. Thus, by Corollary 2.3 together with (2.8) and (2.7), the universal cover of \((M_\infty, g_\infty(t))\) must be a \(\kappa\)-noncollapsed and non-trivial steady Kähler-Ricci soliton. However, such a steady Ricci soliton doesn’t exist by [10]. Therefore, (2.3) could not hold and we prove the lemma.

We need the following uniqueness result of shrinking Kähler-Ricci solitons on \(\mathbb{C}P^n\).

**Lemma 2.7.** If \((\mathbb{C}P^n, g)\) is a shrinking Kähler-Ricci soliton with nonnegative holomorphic bisectional curvature, then \(g\) has constant holomorphic bisectional curvature.

**Proof.** The lemma is already known by the uniqueness of Kähler-Ricci solitons on compact manifolds (cf. [18, 19]). In the following, we give another proof. We may assume that \((\mathbb{C}P^n, g, f)\) is a shrinking Kähler-Ricci soliton for some smooth function \(f \in C^\infty(\mathbb{C}P^n)\). Let \(g(t) = \phi(t)g \) and \(\phi_t\) is generated by \(-\nabla f\). We first note that the \((\mathbb{C}P^n, g(t))\) has positive Ricci curvature by Theorem 2.1 in [4]. If \((\mathbb{C}P^n, g)\) is symmetric, then it is Kähler-Einstein by the Kähler-Ricci soliton equation. Hence, \(g\) has constant holomorphic bisectional curvature by the uniqueness of Kähler-Einstein metric (cf. [11]).
If \((\mathbb{C}P^n, g)\) is not symmetric, then it must be irreducible and the holonomy group \(\text{Hol}(g(t)) = U(n)\). By the argument in the proof of Theorem 1.2 in \([11]\), we see that \((\mathbb{C}P^n, g(t))\) has positive holomorphic bisectional curvature. By \([6]\), we see that \(g(t)\) converge to a Kähler-Einstein metric with positive holomorphic bisectional curvature as \(t \to \infty\). By the convergence of \(g(t)\) and the definition of \(g(t)\), we conclude that \(g\) has constant holomorphic bisectional curvature. We complete the proof. \(\square\)

By Theorem 1.2 and Lemma 2.7, we prove

**Proposition 2.8.** Any backward Type I \(\kappa\)-solution of Kähler-Ricci flow on compact complex manifolds must be a quotient of products of irreducible compact symmetric manifolds.

**Proof.** Let \((M, g(t))\) be a backward Type I \(\kappa\)-solution of Kähler-Ricci flow on compact complex manifold \(M\). By Theorem 1.2, the universal cover of \((M, g(0))\) is holomorphically isometric to

\[(C^k, g_0) \times (\mathbb{C}P^{n_1}, g_1) \times \cdots \times (\mathbb{C}P^{n_l}, g_l) \times (M_1, h_1) \times \cdots \times (M_r, h_r)\]

We only need to show that \(k = 0\) and \(g_i\) is holomorphically isometric to the Fubini-Study metric on \(\mathbb{C}P^{n_i}\) for \(1 \leq i \leq l\).

We first show that \(k = 0\). Fix \(p \in M\), for any \(\tau_i \to \infty\), let \(g_{\tau_i}(t) = \tau_i^{-1}g(\tau_i t)\). Then, \((M, g_{\tau_i}(t), p)\) will converge to a shrinking Ricci soliton \((M_\infty, g_\infty(t), p_\infty)\) by taking a subsequence if necessary (cf. \([16]\)). By the diameter estimate in \([15]\), we see that \((M_\infty, g_\infty(t))\) is compact for \(t < 0\) and \(M_\infty\) is diffeomorphic to \(M\). Hence, \(M_\infty\) is covered by

\[(C^k, g_0) \times (\mathbb{C}P^{n_1}, g_1) \times \cdots \times (\mathbb{C}P^{n_l}, g_l) \times (M_1, h_1) \times \cdots \times (M_r, h_r)\]

Therefore, the fundamental group of \(M_\infty\) is infinite if \(k \geq 1\). However, the fundamental group of compact shrinking Ricci soliton \(M_\infty\) is finite (cf. \([13]\)). Hence, we get \(k = 0\).

Now, we are left to show that \(g_i\) is holomorphically isometric to the Fubini-Study metric on \(\mathbb{C}P^{n_i}\) for \(1 \leq i \leq l\). In this case, we may assume that \(M = \mathbb{C}P^n\) for convenience. For fixed \(p \in M\) and \(\tau_i \to \infty\), we have \((M, g_{\tau_i}(t), p)\) will converge to a shrinking Kähler-Ricci soliton \((M_\infty, g_\infty(t), p_\infty)\) with non-negative holomorphic bisectional curvature, where \(g_{\tau_i}(t) = \tau_i^{-1}g(\tau_i t)\). We also note that \(M_\infty\) is diffeomorphic to \(M = \mathbb{C}P^n\). Therefore it is biholomorphic to \(\mathbb{C}P^n\) by the generalized Frankel conjecture (cf. Theorem 1.2). By Lemma 2.7, we know that \((M_\infty, g_\infty(t))\) has positive constant holomorphic bisectional curvature for all \(t < 0\). By the convergence of \((M, g_{\tau_i}(t), p)\), we know that \(g_{\tau_i}(t)\) also has positive holomorphic bisectional curvature for large \(\tau_i\) and \(t < 0\). Since the positivity of bisectional curvature is preserved along the flow, we know that \(g(t)\) has positive holomorphic bisectional curvature.
for all $t$. By [6], $g(t)$ blows up at some finite time $T$ and converge to a metric of constant holomorphic bisectional curvature under rescaling as $t \to T$ (also see [20]). Note that the entropy invariant $\nu(M, g(t))$ is monotone along the flow. By the forward and backward convergence of $g(t)$, the entropy invariant $\nu(M, g(t))$ is a constant along $g(t)$. Hence, $(M, g(t))$ is a shrinking Kähler-Ricci soliton with positive holomorphic bisectional curvature. Hence, $(M, g(t))$ has constant positive holomorphic bisectional curvature by Lemma 2.7.

□

Theorem 1.1 follows from Proposition 2.8 and Lemma 2.5 immediately.

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