NEARBY CYCLES AND COMPOSITION WITH A NON-DEGENERATE POLYNOMIAL

by

Gil Guibert, François Loeser & Michel Merle

1. Introduction

Let $X_j$ be smooth varieties over a field $k$ of characteristic zero, for $1 \leq j \leq p$. Consider a family $f$ of $p$ functions $f_j : X_j \to \mathbb{A}^1_k$. We shall denote also by $f_j$ the function on the product $X = \prod_j X_j$ obtained by composition with the projection. We denote by $X_0(f)$ the set of common zeroes in $X$ of the functions $f_j$. Let $P \in k[y_1, \ldots, y_p]$ be a polynomial, which we assume to be non-degenerate with respect to its Newton polyhedron. In the present note we shall compute the motivic nearby cycles $S_P(f)$ of the composed function $P(f)$ on $X_0(f)$ as a sum over the set of compact faces $\delta$ of the Newton polyhedron of $P$. For every such $\delta$, let us denote by $P_\delta$ the corresponding quasi-homogeneous polynomial. We associate to such a quasi-homogeneous polynomial a convolution operator $\Psi_P_\delta$, which in the special case where $P_\delta$ is the polynomial $\Sigma = y_1 + y_2$ is nothing but the operator $\Psi_\Sigma$ considered in [9]. For such a compact face $\delta$, one may also define generalized nearby cycles $S^\sigma_\delta$, constructed as the limit, as $T \to \infty$, of certain truncated motivic zeta functions.

Our main result, Theorem 3.4, follows from additivity from the following statement, Theorem 3.5:

\[ i^* S_{P(f), U} = \sum_{\delta \in \Gamma^\emptyset} \Psi_{P_\delta}(S^\sigma_\delta). \]

Here $U$ denotes the complement of the locus where at least one function $f_j$ vanishes, $\Gamma^\emptyset$ denotes the set of compact faces of the Newton polyhedron of $P$ not contained in any coordinate hyperplane, $S_{P(f), U}$ refers to the extension of $S_{P(f)}$ constructed in [11] and [9], and $i^*$ denotes restriction to $X_0(f)$.

When $p = 2$ and $P = \Sigma$, one recovers the motivic Thom-Sebastiani, cf. [5], [10], and [6], in the way stated in [9]. When $f$ is the set of coordinate functions on the affine space $\mathbb{A}^p_k$, our result is equivalent to recovers a result obtained by Guibert in [8].
This paper is a natural continuation of [9], from which part of the notation and several results are borrowed.

2. Preliminaries

2.1. Grothendieck rings. — Throughout the paper \( k \) will be a field of characteristic zero. By a variety over \( k \), we mean a separated and reduced scheme of finite type over \( k \). If a linear algebraic group \( G \) acts on a variety \( X \), we say the action is good if every \( G \)-orbit is contained in an affine open subset of \( X \). We denote by \( \text{Var}^{G,\text{eq}} \) the category of varieties with good \( G \)-action, morphisms being \( G \)-equivariant morphisms. If \( S \) is a variety with good \( G \)-action, we denote by \( \text{Var}^{G,\text{eq}}_S \) the category of objects over \( S \), that is the category whose objects are morphisms \( Y \to S \) in \( \text{Var}^{G,\text{eq}} \), morphisms in \( \text{Var}^{G,\text{eq}} \) being defined in the standard way. Let \( Y \) be a variety over \( k \) and let \( p : A \to Y \) be an affine bundle for the Zariski topology (the fibers of \( p \) are affine spaces and the transition morphisms between trivializing charts are affine). In particular the fibers of \( p \) have the structure of affine spaces. Let \( G \) be a linear algebraic group. A good action of \( G \) on \( A \) is said to be affine if it is a lifting of a good-action on \( Y \) and its restriction to all fibers is affine.

One defines \( K_0(\text{Var}^{G,\text{eq}}_S) \) as the free abelian group on isomorphism classes of objects \( Y \to S \) in \( \text{Var}^{G,\text{eq}}_S \), modulo the relations

\[
(2.1.1) \quad [Y \to S] = [Y' \to S] + [Y \setminus Y' \to S]
\]

for \( Y' \) closed \( G \)-invariant in \( Y \) and, for \( f : Y \to S \) in \( \text{Var}^{G,\text{eq}}_S \),

\[
(2.1.2) \quad [Y \times A^n_k \to S, \sigma] = [Y \times A^n_k \to S, \sigma']
\]

if \( \sigma \) and \( \sigma' \) are two liftings of the same \( G \)-action on \( Y \) to an affine action, the morphism \( Y \times A^n_k \to S \) being composition of \( f \) with projection on the first factor.

Fiber product over \( S \) induces a product in the category \( \text{Var}^{G,\text{eq}}_S \), which allows to endow \( K_0(\text{Var}^{G,\text{eq}}_S) \) with a natural ring structure. Note that the unit \( 1_S \) for the product is the class of the identity morphism \( S \to S \).

2.2. — Let \( s \) denote a positive integer and let \( S \) be a \( k \)-variety. From now on, we will consider only \( G_m^s \)-actions on \( S \times G'_m \) which are trivial on the first factor.

Let us consider the category \( \mathcal{C} \) whose objects are finite morphisms \( \varphi : G_m^s \to G_m^r \), a morphism between \( \varphi : G_m^s \to G_m^r \) and \( \varphi' : G_m^s \to G_m^r \) being a finite morphism \( \varphi : G_m^s \to G_m^s \) such that \( \varphi \circ \varphi = \varphi' \).

We consider also the full subcategory \( \mathcal{C}' \) of \( \mathcal{C} \) the objects of which are finite morphisms \( \varphi : G_m^s \to G_m^s \). The subcategory \( \mathcal{C}' \) is final in \( \mathcal{C} \) in the language of [11].

A morphism \( \varphi : G_m^s \to G_m^r \) induces a natural functor

\[
(2.2.1) \quad \Phi : \text{Var}^{G_m^r,\text{eq}}_{S \times G_m^r} \to \text{Var}^{G_m^s,\text{eq}}_{S \times G_m^s}
\]

where an object \( Y \to S \times G_m^r \) with a good \( G_m^r \)-action is sent on the same underlying object of \( \text{Var}^{G_m^s,\text{eq}}_{S \times G_m^s} \) with the \( G_m^s \)-action induced via \( \varphi \).
The functor $\Phi$ induces a morphism

\[(2.2.2) \quad K_0(\varphi) : K_0(\text{Var}_{S \times G_m}^{G_{m}', \text{eq}}) \to K_0(\text{Var}_{S \times G_m}^{G_{m}'})\]

We will denote by $K_0(\text{Var}_{S \times G_m}^{G_{m}', \text{eq}})$ the image of the morphism $K_0(\varphi)$.

For every morphism $\vartheta$ between $\varphi$ and $\varphi'$ in $\mathcal{C}$, we get a morphism

\[(2.2.3) \quad K_0(\vartheta) : K_0(\text{Var}_{S \times G_m}^{G_{m}', \text{eq}}) \to K_0(\text{Var}_{S \times G_m}^{G_{m}', \text{eq}})\]

where a class of a good $G_{m}'$-action induced by a $G_{m}''$-action via $\varphi'$ on an object of $\text{Var}_{S \times G_m}^{G_{m}'}$ is sent on the class of the same $G_{m}'$-action as induced by a $G_{m}''$-action via $\varphi$. As a particular case, taking $\varphi = \text{Id}$, we get the natural inclusion of $K_0(\text{Var}_{S \times G_m}^{G_{m}', \text{eq}})$ into $K_0(\text{Var}_{S \times G_m}^{G_{m}'}).$

We define the Grothendieck ring $K_0(\text{Var}_{S \times G_m}^{G_{m}'})$ as the colimit along $\mathcal{C}$ (or along $\mathcal{C}'$, which amounts to the same) of the rings $K_0(\text{Var}_{S \times G_m}^{G_{m}', \text{eq}})$. Note that we could also have defined the rings $K_0(\text{Var}_{S \times G_m}^{G_{m}', \text{eq}})$ and $K_0(\text{Var}_{S \times G_m}^{G_{m}'})$ as suitable Grothendieck rings of the essential image $\text{Var}_{S \times G_m}^{G_{m}', \text{eq}}$ of $\Phi$ and of the colimit $\text{Var}_{S \times G_m}^{G_{m}''}$ along $\mathcal{C}$ (or $\mathcal{C}'$) of the categories $\text{Var}_{S \times G_m}^{G_{m}', \text{eq}}$, respectively.

There is a natural structure of $K_0(\text{Var}_k)$-module on $K_0(\text{Var}_{S \times G_m}^{G_{m}'})$. We denote by $L_{S \times G_m} = L$ the element $L \cdot 1_{S \times G_m}$ in this module, and we set

\[(2.2.4) \quad \mathcal{M}_{S \times G_m}^{G_{m}'} := K_0(\text{Var}_{S \times G_m}^{G_{m}'})[L^{-1}]\]

Note that when $s = r$ the above definitions $K_0(\text{Var}_{S \times G_m}^{G_{m}'})$ and $\mathcal{M}_{S \times G_m}^{G_{m}'}$ coincide with that of 9 by 2.7.

A morphism $\vartheta : G_{m}' \to G_{m}''$ induces a morphism from $\mathcal{M}_{S \times G_m}^{G_{m}'}$ to $\mathcal{M}_{S \times G_m}^{G_{m}'}$. For example, the diagonal morphism $G_{m} \to G_{m}'$ yields a canonical morphism

\[(2.2.5) \quad \Delta : \mathcal{M}_{S \times G_m}^{G_{m}'} \to \mathcal{M}_{S \times G_m}^{G_{m}'}\]

Through this morphism, the class of a $G_{m}'$-action $\alpha$ on an object of $\text{Var}_{S \times G_m}^{G_{m}'}$ is sent on the class of $G_{m}'$-actions induced by $\alpha$ via a finite group morphism from $G_{m}$ to $G_{m}''$. If $f : S \to S'$ is a morphism of varieties, composition with $f$ leads to a push-forward morphism $f_! : \mathcal{M}_{S \times G_m}^{G_{m}'} \to \mathcal{M}_{S' \times G_m}^{G_{m}'}$, while fiber product leads to a pull-back morphism $f^* : \mathcal{M}_{S' \times G_m}^{G_{m}'} \to \mathcal{M}_{S \times G_m}^{G_{m}'}$.

2.3. — Let $A$ be one of the rings $\mathbb{Z}[L, L^{-1}]$, $\mathbb{Z}[L, L^{-1}, (\frac{1}{1-L^i})_{i>0}]$, $\mathcal{M}_{S \times G_m}^{G_{m}'}$, etc. We denote by $A[[T]]_{sr}$ the $A$-submodule of $A[[T]]$ generated by $1$ and by finite sums of products of terms $p_{e,i}(T) = \frac{L^{T^e}}{1-L^i}$, with $e$ in $\mathbb{Z}$ and $i$ in $\mathbb{N}_{>0}$. There is a unique $A$-linear morphism

\[(2.3.1) \quad \lim_{T \to \infty} : A[[T]]_{sr} \to A\]
such that
\[ (2.3.2) \lim_{T \to \infty} \left( \prod_{i \in I} p_{e_i j_i}(T) \right) = (-1)^{|I|}, \]
for every family \( \{(e_i, j_i)\}_{i \in I} \) in \( \mathbb{Z} \times \mathbb{N}_{>0} \), with \( I \) finite, maybe empty.

2.4. — We denote as usual by \( \mathcal{L}_n(X) \) the space of arcs of order \( n \), also known as the \( n \)-th jet space on \( X \). It is a \( k \)-scheme whose \( K \)-points, for \( K \) a field containing \( k \), is the set of morphisms \( \varphi : \text{Spec} \, K[t]/t^{n+1} \to X \). There are canonical morphisms \( \mathcal{L}_{n+1}(X) \to \mathcal{L}(X) \) and the arc space \( \mathcal{L}(X) \) is defined as the projective limit of this system. We denote by \( \pi_n : \mathcal{L}(X) \to \mathcal{L}_n(X) \) the canonical morphism. There is a canonical \( \mathbb{G}_m \)-action on \( \mathcal{L}_n(X) \) and on \( \mathcal{L}(X) \) given by \( a \cdot \varphi(t) = \varphi(at) \).

Let \( X \) be a smooth variety over \( k \) of pure dimension \( d \) and \( g : X \to A^1_k \). Set \( X_0(g) \) for the zero locus of \( g \), and define, for \( n \geq 1 \), the variety
\[ (2.4.1) \mathcal{X}_n(g) := \left\{ \varphi \in \mathcal{L}_n(X) \mid \text{ord}_g(\varphi) = n \right\}. \]

Note that \( \mathcal{X}_n(g) \) is invariant by the \( \mathbb{G}_m \)-action on \( \mathcal{L}_n(X) \) and that furthermore \( g \) induces a morphism \( g_n : \mathcal{X}_n(g) \to \mathbb{G}_m \), assigning to a point \( \varphi \) in \( \mathcal{L}_n(X) \) the coefficient of \( t^n \) in \( g(\varphi) \), which we shall denote by \( \text{ac}(g)(\varphi) \). We have \( g_n(a \cdot \varphi) = a^n g_n(\varphi) \), hence with the terminology of \([9] \), \( g_n \) is diagonally monomial of weight \( n \) with respect to the \( \mathbb{G}_m \)-action on \( \mathcal{X}_n(g) \). In particular, we may consider the class \([\mathcal{X}_n(g)]\) of \( \mathcal{X}_n(g) \) in \( \mathcal{M}^\mathbb{G}_m \times \mathbb{G}_m \) and the motivic zeta function
\[ (2.4.2) Z_g(T) := \sum_{n \geq 1} [\mathcal{X}_n(g)] \mathbf{L}^{-nd} T^n \]
in \( \mathcal{M}^\mathbb{G}_m \times \mathbb{G}_m \[[T]]\).

Denef and Loeser showed in \([3] \) and \([6] \), see also \([10] \) and \([9] \), that \( Z_g(T) \) is a rational series in \( \mathcal{M}^\mathbb{G}_m \times \mathbb{G}_m \[[T]\]_{\text{sa}} \) by giving a formula for \( Z_g(T) \) in terms of a resolution of \( f \) we shall recall in \([2.5] \).

2.5. Resolutions. — Let us introduce some notation and terminology. Let \( X \) be a smooth variety of pure dimension \( d \) and let \( F \) a closed subset of \( X \) of codimension everywhere \( \geq 1 \). By a log-resolution \( h : Y \to X \) of \((X,F)\), we mean a proper morphism \( h : Y \to X \) with \( Y \) smooth such that the restriction of \( h : Y \setminus h^{-1}(F) \to X \setminus F \) is an isomorphism, and \( h^{-1}(F) \) is a divisor with normal crossings. We denote by \( E_i, i \) in \( A \), the set of irreducible components of the divisor \( h^{-1}(F) \). For \( I \subset A \), we set
\[ (2.5.1) E_I := \bigcap_{i \in I} E_i \]
and
\[ (2.5.2) E_I^c := E_I \setminus \bigcup_{j \notin I} E_j. \]
We denote by \( \nu_{E_i} \) the normal bundle of \( E_i \) in \( Y \) and by \( \nu_{E_I} \) the fiber product of the restrictions to \( E_I \) of the bundles \( \nu_{E_i}, i \in I \). We will denote by \( U_{E_i} \) the complement of the zero section in \( \nu_{E_i} \) and by \( U_I \) the fiber product of the restrictions of the spaces \( U_{E_i}, i \in I \), to \( E_I \).

If \( \mathcal{I} \) is an ideal sheaf defining a closed subscheme of \( X \) and \( h^*(\mathcal{I}) \) is locally principal, we define \( N_i(\mathcal{I}) \), the multiplicity of \( \mathcal{I} \) along \( E_i \), by the equality of divisors

\[
(2.5.3) \quad h^{-1}(F) = \sum_{i \in A} N_i(\mathcal{I})E_i.
\]

If \( \mathcal{I} \) is principal generated by a function \( g \) we write \( N_i(g) \) for \( N_i(\mathcal{I}) \). Similarly, we define integers \( \nu_i \) by the equality of divisors

\[
(2.5.4) \quad K_Y = h^*K_X + \sum_{i \in A} (\nu_i - 1)E_i.
\]

2.6. — Assume again \( g \) is a function on a smooth variety \( X \) of pure dimension \( d \). Let \( F \) a reduced divisor containing \( X_0(g) \) and let \( h : Y \to X \) be a log-resolution of \((X, F)\). Let us explain how \( g \) induces a morphism \( g_I : U_I \to G_m \). Note that the function \( g \circ h \) induces a function

\[
(2.6.1) \quad \bigotimes_{i \in I} \nu_{E_i}^{\otimes N_i(g)}|_{E_i} \to \mathbf{A}_k^1,
\]

vanishing only on the zero section. We define \( g_I : \nu_{E_i} \to \mathbf{A}_k^1 \) as the composition of this last function with the natural morphism \( \nu_{E_I} \to \bigotimes_{i \in I} \nu_{E_i}^{\otimes N_i(g)}|_{E_i} \), sending \((u_i)\) to \( \otimes u_i^{\otimes N_i(g)} \). We still denote by \( g_I \) the induced morphism from \( U_I \) (resp. \( U_{E_i} \)) to \( G_m \).

We view \( U_I \) as a variety over \( X_0(g) \times G_m \) via the morphism \((h \circ \pi_I, g_I)\). The group \( G_m \) has a natural action on each \( U_{E_i} \), so the diagonal action induces a \( G_m \)-action on \( U_I \). Furthermore, the morphism \( g_I \) is monomial, in the terminology of \([9]\), hence \( U_I \to X_0(g) \times G_m \) has a class in \( \mathcal{M}_{X_0(g) \times G_m}^{G_m} \) which we will denote by \([U_I]\).

2.7. — We now assume that \( F = X_0(g) \), that is \( h : Y \to X \) is a log-resolution of \((X, X_0(g))\). In this case, \( h \) induces a bijection between \( \mathcal{L}(Y) \setminus \mathcal{L}(h^{-1}(X_0(g))) \) and \( \mathcal{L}(X) \setminus \mathcal{L}(X_0(g)) \).

One deduces from Lemma 3.4 in \([4]\), in a way completely similar to \([3]\) and \([6]\), the equality

\[
(2.7.1) \quad Z_g(T) = \sum_{\emptyset \neq I \subseteq A} [U_I] \prod_{i \in I} \frac{1}{T^{N_i(g)} - 1} \in \mathcal{M}_{X_0(g) \times G_m}^{G_m}[[T]].
\]

In particular, the function \( Z_g(T) \) is rational and belongs to \( \mathcal{M}_{X_0(g) \times G_m}^{G_m}[[T]]_{ad} \), with the notation of \(2.5\), hence we can consider \( \lim_{T \to \infty} Z_g(T) \) in \( \mathcal{M}_{X_0(g) \times G_m}^{G_m} \) and
set
\[(2.7.2) \quad S_g := - \lim_{T \to \infty} Z_{g(T)},\]
which by \((2.7.1)\) may be expressed on a resolution \(h\) as
\[(2.7.3) \quad S_g = - \sum_{\emptyset \neq I \subset A} (-1)^{|I|}[U],\]
in \(M_{X_{\phi(g)} \times G_m}^G\). The element \(S_g\) is called the motivic Milnor fiber or the motivic nearby fiber of \(f\). It was first considered by Denef and Loeser, cf. \([8], [6]\) and \([7]\). For recent results concerning \(S_g\), we refer the reader to \([8], [1]\) and \([9]\).

2.8. — Consider a family \(f\) of \(p\) functions \(f_j : X \to A^1_k, 1 \leq j \leq p\). We denote by \(X_0(f)\) the set of common zeros of the functions \(f_j, 1 \leq j \leq p\) and by \(F\) the product function \(f_1 \ldots f_p\).

Let us fix a rational polyhedral convex cone \(C\) in \(R^p_{>0}\) and an integral linear form \(\ell\) on \(Z^p\) which is positive on \(C \setminus \{0\}\), where \(\overline{C}\) denotes the closure of \(C\) in \(R^p\).

We will consider the modified zeta function \(Z^{C,\ell}_f\) defined as follows: for a vector \(n\) in \(N^p_{>0}\), we denote by \(s(n)\) the sum of its components and we consider, similarly as in \((2.4.1)\), the variety
\[(2.8.1) \quad X_n(f) := \left\{ \varphi \in L_{s(n)}(X) \mid \text{ord}(f_j(\varphi)) = n_j, 1 \leq j \leq p \right\}.\]
Note that \(X_n(f)\) is stable under the \(G_m\)-action on \(L_n(X)\) and that \(f\) induces a morphism
\[(2.8.2) \quad f_n : X_n(f) \to G_m^p,\]
whose components are \(ac(f_j), 1 \leq j \leq p\) defined similarly as in \((2.4.1)\). Since \(f_n(a \cdot \varphi) = a^n f_n(\varphi)\), we may consider the class \([X_n(f)]\) of \(X_n(f) \to X_0(f) \times G_m\) in \(M_{X_0(f) \times G_m}^{G_m}\). We set
\[(2.8.3) \quad Z^{C,\ell}_f(T) := \sum_{n \in C} [X_n(f)] L^{-s(n)d} T^{\ell(n)}\]
in \(M_{X_0(f) \times G_m}^{G_m}[[T]]\).

2.9. — Let \(h : Y \to X\) be a log-resolution of the set \(X_0(F)\). We keep the notations of \(2.5\). In particular we denote by \(A\) the set of irreducible components of \(h^{-1}(X_0(F))\). For \(i\) in \(A\) we will denote by \(N_i(f_j)\) the integral vector of the orders \(N_i(f_j)\) of the functions \(f_j, 1 \leq j \leq p\), along the divisor \(E_i\), and by \(N_I\) the linear map
\[(2.9.1) \quad N_I : \begin{cases} \mathbb{R}^p_{>0} \to \mathbb{R}^p_{>0} \\ k \to \sum_{i \in I} k_i N_i. \end{cases}\]

Similarly the set of integers \(v_i\) defines a linear integral form \(\nu : k \to \sum_{i \in I} k_i v_i\) on \(\mathbb{R}^p_{>0}\).
Using Lemma 3.4 in [4] similarly as for the proof of (2.7.1), see for example [6, 10], one gets the following formula for the zeta function $Z_{f}^{C,\ell}(T)$ in terms of the resolution:

\[(2.9.2)\]

\[Z_{f}^{C,\ell}(T) = \sum_{\emptyset \neq I \subset A} [U_I] \sum_{\{k \in \mathbb{N}_{>0} \mid N_I(k) \in C\}} \prod_{i \in I} (T^{\ell(N_i)} L^{-\nu_i})^{ki}.\]

Here $[U_I]$ stands for the class in $\mathcal{M}_{X_0(f) \times G_{m}^p}$ of the morphism $(h, f_I) : U_I \to X_0(f) \times G_{m}^p$.

It follows that $Z_{f}^{C,\ell}(T)$ belongs to $\mathcal{M}_{X_0(f) \times G_{m}^p}[[T]]_{sr}$, hence we may set

\[(2.9.3)\]

\[S_{f}^{C,\ell} := \lim_{T \to \infty} Z_{f}^{C,\ell}(T)\]

in $\mathcal{M}_{X_0(f) \times G_{m}^p}$. By section 2.9 of [9] we have:

\[(2.9.4)\]

\[S_{f}^{C,\ell} = \sum_{I} \chi(N^{-1}(C))[U_I],\]

where $\chi$ denotes Euler characteristic with compact supports. Note that this is independent of $\ell$, so we may write $S_{f}^{C}$ instead of $S_{f}^{C,\ell}$.

### 3. Composition with a non-degenerate polynomial

#### 3.1. The generalized convolution $\Psi_{P}$.

Let $P$ be a quasi-homogeneous polynomial function on $G_{m}^p$, that is $P$ is homogeneous for a $G_{m}$-action $\alpha$ on $G_{m}^p$ monomial of weight $w = (w_1, \ldots, w_p)$.

Let $X$ be a smooth variety. We will denote by $pr_1$ the projection of $X \times G_{m}^p \times G_{m}$ on $X \times G_{m}$ (forgetting the $G_{m}^p$ factor) and by $i$ the inclusion of the complement of $X \times P^{-1}(0)$ into $X \times G_{m}^p$.

For a variety $A$ of dimension $e$ in $\text{Var}_{X \times G_{m}^p}$, the function $P$ induces by composition with the second projection a function on $A$ we still denote by $P$

\[(3.1.1)\]

\[P : A \to A_{k}.\]

We now define the (augmented) zeta function $Z_{f}^{0}(T)$ as

\[(3.1.2)\]

\[Z_{f}^{0}(T) = \sum_{n \geq 0} [X_n(P)] L^{-n} T^{n} = [X_0(P)] + Z_{f}(T),\]

where $X_n(P)$ is

\[(3.1.3)\]

\[X_n(P) := \left\{ \varphi \in L_n(A) \mid \text{ord}_{P}(\varphi) = n \right\},\]

for $n \geq 0$. It belongs to $\mathcal{M}_{X \times G_{m}^p \times G_{m}}[[T]]_{sr}$. We define $\Psi_{P}(A)$ as the limit, as $T \to \infty$, of the opposite $-Z_{f}^{0}(T)$. Thus, with the notations of [9], it is nothing but

\[(3.1.4)\]

\[-\lim_{T \to \infty} Z_{f}^{0}(T) = -[A \setminus P^{-1}(0)] + S_{P}([A]).\]
It is an object in $\mathcal{M}_{X \times G_m \times G_m}^G$, the $G_m$-action and the morphism to $G_m$ being the usual ones. On $A \setminus P^{-1}(0)$ the $G_m$-action is trivial and the morphism to $G_m$ is the restriction of $P$ to $A \setminus P^{-1}(0)$. Taking the direct image by the projection $\text{pr}_1$ we get the following object in $\mathcal{M}_{X \times G_m}^{G_m}$:

\begin{equation}
\Psi_0^0(A) := \text{pr}_1([A \setminus P^{-1}(0)] + S_p(A)).
\end{equation}

One may then extend uniquely this construction to a $\mathcal{M}_k$-linear group morphism

\begin{equation}
\Psi_P : \mathcal{M}_{X \times G_m}^G \rightarrow \mathcal{M}_{X \times G_m}^{G_m}.
\end{equation}

If $A$ is endowed with a $G_m$-action $\alpha$ for which the morphism to $G_m^p$ is monomial of weight $w$, $A \setminus P^{-1}(0)$ is endowed with an additional action which is homogeneous with respect to the composed morphism to $G_m$. Hence we may attach to $A \setminus P^{-1}(0)$ a class $[A \setminus P^{-1}(0)]$ in $\mathcal{M}_{X \times G_m^p \times G_m}^{G_m^2}$. In [9], §3.10, we attached to such an $A$ with the action $\alpha$ an element $S_p(A)$ in $\mathcal{M}_{X \times G_m \times G_m}^{G_m^2}$. Hence we can consider $\text{pr}_1([A \setminus P^{-1}(0)] + S_p(A))$ as an element of $\mathcal{M}_{X \times G_m}^{G_m^2}$.

Composing with the canonical morphism $\mathcal{M}_{X \times G_m}^{G_m^2} \rightarrow \mathcal{M}_{X \times G_m}^{G_m}$ induced by the diagonal action, we get an element of $\mathcal{M}_{X \times G_m}^{G_m}$ we shall denote by $\Psi_P(A)$. This construction extends uniquely to a $\mathcal{M}_k$-linear group morphism

\begin{equation}
\Psi_P : \mathcal{M}_{X \times G_m}^{G_m} \rightarrow \mathcal{M}_{X \times G_m}^{G_m}.
\end{equation}

3.2. **Remark.** — When $P$ is the sum of coordinates $\Sigma$ on $G_m^2$, then $\Psi_\Sigma$ is nothing but the convolution product from [9]. More precisely, the convolution product $\Psi_\Sigma$ defined in [9] is equal to the composition of the morphism $\Psi_\Sigma$ defined in this paper with the morphism $\Delta$ defined in (2.2.5).

3.3. **Composed maps.** — For $1 \leq j \leq p$, let $f_j : X_j \rightarrow A^1_k$ be a function on a smooth $k$-variety $X_j$. By composition with the projection, $f_j$ becomes a function on the product $X = \prod X_j$. We write $d$ for the dimension of $X$. Define $f$ as the family of the $f_j$ on $X$, $1 \leq j \leq p$. The product of the log-resolutions of the $X_{j,0}(f_j)$ is a log-resolution $h : Y \rightarrow X$ of $X_0(F)$ (recall that $F = f_1 \ldots f_p$).

Let $P = \sum_{a \in \mathbb{N}^p} a_\alpha y^a$ be a polynomial in $k[y_1, \ldots, y_p]$. We denote by $\text{supp}(P)$ the set of exponents $\alpha$ in $\mathbb{N}^p$ with $a_\alpha \neq 0$. The Newton polyhedron $\Gamma$ of $P$ is the convex hull of $\text{supp}(P) + \mathbb{R}_+^p$. For a compact face $\delta$ of $\Gamma$ we denote by $P_\delta$ the sum of the monomials of $P$ supported in $\delta$:

\begin{equation}
P_\delta = \sum_{\alpha \in \delta} a_\alpha y^a.
\end{equation}

We say $P$ is non-degenerate with respect to its Newton polyhedron $\Gamma$, if, for every compact face $\delta$ of $\Gamma$, the function $P_\delta$ is smooth on $G_m^p$. To the Newton polyhedron $\Gamma$ one may associate a fan of rational polyhedral cones subdividing $\mathbb{R}_+^p$ as follows. We consider the function $\ell_\Gamma$ assigning to a vector $a$ in
\( R^p_\gamma \) the value \( \inf_{b \in \Gamma} \langle a, b \rangle \), with \( \langle , \rangle \) the standard inner product. For any \( a \) in \( R^p_\gamma \) we may consider the compact face

\[
\delta_a = \{ b \in \Gamma_c | \langle a, b \rangle = \ell_\gamma(b) \},
\]

with \( \Gamma_c \) the union of all compact faces of \( \Gamma \).

For a compact face \( \delta \) of the Newton polyhedron \( \Gamma \), we denote by \( \sigma(\delta) \) its dual cone \( \{ a \in R^n_\gamma | \delta_a = \delta \} \). The cones \( \sigma(\delta) \), for \( \delta \) running over the compact faces of \( \Gamma \), form a fan partitioning \( R^n_\gamma \) by rational polyhedral cones. The function \( \ell_\gamma \) is linear on each cone \( \sigma(\delta) \).

We write \( \Gamma_c \) for the set of compact faces of \( \Gamma \). For \( J \) a subset of \( \{1, \ldots, p\} \), we denote by \( \Gamma^J \) the set of compact faces of \( \Gamma \) contained in the coordinate hyperplanes \( x_i = 0 \) for \( i \) in \( J \), and in no other coordinate hyperplane, so that \( \Gamma_c \) is the disjoint union of the subsets \( \Gamma^J \). Note that \( \ell_i \) is positive on \( \sigma(\delta) \setminus \{0\} \) if and only if \( \delta \) is in \( \Gamma^0 \).

We denote by \( X_J \) the closed subset of \( S \) defined by the vanishing of the functions \( f_i, i \in J \), and by \( f_j : X_J \to A^{\{1, \ldots, p\} \setminus J} \) the morphism induced by the functions \( f_j, j \notin J \).

For every variety \( Z \) containing \( X_0(f) \) we denote by \( i^\ast \) the restriction morphisms

\[
\mathcal{M}_{Z \times G_m}^G \longrightarrow \mathcal{M}_{X_0(f) \times G_m}^G
\]

and

\[
\mathcal{M}_{Z \times G_m}^G[[T]] \longrightarrow \mathcal{M}_{X_0(f) \times G_m}^G[[T]].
\]

3.4. Theorem. — With the previous notations and hypotheses, we have the following formula for \( i^\ast \mathcal{S}_P(f) \) in \( \mathcal{M}_{X_0(f) \times G_m}^G \):

\[
i^\ast \mathcal{S}_P(f) = \sum_{J \subset \{1, \ldots, p\}} \sum_{\delta \in \Gamma^J} \Psi_P(\mathcal{S}_f^\delta, \ell_\gamma).
\]

Proof. — Following [9], for \( \gamma \) in \( \mathbb{N}_{>0} \), we consider the constructible set

\[
\mathcal{X}^\gamma_n := \{ \varphi \in L_\gamma (X) \mid \text{ord}_T P(f)(\varphi) = n, \text{ord}_F(\varphi) \leq \gamma n \}
\]

together with the morphism \( ac(P(f)) : \mathcal{X}^\gamma_n \to G_m \), giving rise to a class \( [\mathcal{X}^\gamma_n] \) in \( \mathcal{M}_{X_0(F) \times G_m}^{G_m} \). By Proposition 3.8 in [9], for \( \gamma \gg 0 \), the corresponding zeta function

\[
Z_{P(f),X_\gamma_n}(T) := \sum_{n>0} [\mathcal{X}^\gamma_n] L^{-\gamma nd} T^n
\]

lies in \( \mathcal{M}_{X_0(F) \times G_m}^{G_m}[[T]] \) and its limit as \( T \to \infty \) is independent of \( \gamma \), so we may set

\[
\mathcal{S}_{P(f),X_\gamma_n}(T) := - \lim_{T \to \infty} Z_{P(f),X_\gamma_n}(T).
\]

Furthermore, by additivity of \( \mathcal{S}_{P(f)} \), cf. Theorem 3.11 of [9], we have

\[
\mathcal{S}_{P(f)} = \sum_{J \subset \{1, \ldots, p\}} \mathcal{S}_{P(f),X_J}(T).
\]
with $X_0^\circ$ the largest open in $X_J$, where no $f_j$, $j \notin J$, vanishes. Theorem 3.4 follows now directly from Theorem 3.3.

**3.5. Theorem.** With the previous notation the following holds:

\[ i^* \mathcal{S}_{P(f), X \setminus X_0(F)} = \sum_{\delta \in \Gamma} \Psi_{P_{\delta}}(\mathcal{S}^{\sigma(\delta)}). \]

**Proof.** Let us fix a log-resolution $h : Y \to X$ of $X_0(F)$. We shall keep the notations of (2.5).

Fix a subset of $I$ of $A$ and $k = (k_i)_{i \in I}$ in $\mathbb{N}_{>0}^I$. For $\varphi$ in $\mathcal{L}_{\gamma \eta}(Y)$ with $\varphi(0)$ in $E_i$, we set $\text{ord}_{E_i} \varphi := \text{ord}_{i} z_i(\varphi)$, for $z_i$ any local equation of $E_i$ at $\varphi(0)$. We denote by $X_{n,k}$ the set of arcs $\varphi$ in $\mathcal{L}_{\gamma \eta}(Y)$ such that $\varphi(0)$ is in $E_i^\circ$ and $\text{ord}_{E_i} \varphi = k_i$ for $i \in I$. We also consider the subset $\mathcal{Y}_{n,k}$ of $X_{n,k}$ consisting of arcs $\varphi$ such that $\text{ord}_{i}((P(f)) \circ h)(\varphi) = n$. The variety $\mathcal{Y}_{n,k}$ is stable by the usual $\mathbb{G}_m$-action on $\mathcal{L}_{\gamma \eta}(Y)$ and the morphism $\text{ac}(P(f)) \circ h$ defines a class $[\mathcal{Y}_{n,k}]$ in $\mathcal{M}_{X_0(F) \times \mathbb{G}_m}$. Note that $\mathcal{Y}_{n,k} = \emptyset$ if $n < \ell_{\Gamma}(N_I(k))$.

By a now standard calculation, using Lemma 3.6 in [2], $Z^\gamma_{P(f), X \setminus X_0(F)}$ may be expressed on the log-resolution $Y$, as

\[ Z^\gamma_{P(f), X \setminus X_0(F)} = \sum_{\emptyset \neq I \subset A} \sum_{\substack{N_I(k) \subset \sigma(\delta) \cap (N_J(k), 1) \leq n \in \mathbb{N}_{>0}}} [\mathcal{Y}_{n,k}]_{\mathcal{L}} \sum_{i \in I} (\nu_i - 1) k_i \mathcal{L}^{-\gamma d} T^n. \]

We denote by $B$ the set of all subsets $I$ of $A$ such that $h(E_i^\circ)$ is contained in $X_0(F)$. We fix $I$ in $B$ and $k = (k_i)_{i \in I}$ in $\mathbb{N}_{>0}^I$. Note that there is a unique compact face $\delta$ of $\Gamma$ such that $N_I(k)$ lies in $\sigma(\delta)$.

To go further on, we shall use the following variant of the classical deformation to the normal cone already considered in [2]. We consider the affine line $A_{k_1} = \text{Spec} \ k[u]$ and the subsheaf

\[ A_k := \sum_{n \in \mathbb{N}^I} \mathcal{O}_{Y \times A_{k_1}} \left( - \sum_{i \in I} n_i(E_i \times A_{k_1}^1) \right) \left( u - \sum_{i \in I} k_i n_i \right) \]

of $\mathcal{O}_{Y \times A_{k_1}}[u^{-1}]$. It is a sheaf of rings and we set

\[ CY_k := \text{Spec} \ A_k. \]

The natural inclusion $\mathcal{O}_{Y \times A_{k_1}} \to A_k$ induces a morphism $\pi : CY_k \to Y \times A_{k_1}$, hence a morphism $p : CY_k \to A_{k_1}$. Via the same inclusion, the functions $P(f \circ h)$ and $F \circ h$ are, in $A_k$, divisible by $u^{k_i(N_I(k))}$ and by $u^{(N_I(k), 1)}$, where $1$ denotes the vector with all coordinates equal to 1, and we denote the corresponding quotients by $\tilde{P}(f)_k$ and $\tilde{F}_k$, respectively.

We denote by $\tilde{E}_i$ the strict transform of the divisor $E_i \times A_{k_1}$ by $\pi$, by $D$ the divisor globally defined on $CY_k$ by $u = 0$, and by $CE_i$ the divisors $\tilde{E}_i - k_i D$, $i \in I$. We denote by $CY_k^\circ$ the complement in $CY_k$ of the union of the divisors $CE_i$ in $I$, and we denote by $Y^\circ$ the complement in $Y$ of the union of the $E_i$, $i \in I$. 

As proved in Lemma 5.12 of [9], the scheme $CY_k$ is smooth, the morphism $\pi$ induces an isomorphism over $A_1^k \setminus \{0\}$, the morphism $p$ is a smooth morphism and its fiber $p^{-1}(0)$ may be naturally identified with the bundle $\nu_{E_1}$. Furthermore, when restricted to $CY_k^o$, the fiber of $p$ above 0 is naturally identified with $U_1$ and $\pi$ induces an isomorphism between $CY_k^o \setminus p^{-1}(0)$ and $Y^o \times A_1^k \setminus \{0\}$. The restrictions of $\tilde{P}(f)_k$ and $\tilde{F}_k$ to the fiber $U_1 \subset p^{-1}(0)$ are respectively equal to $P_\delta(f_I)$ and $F_I$.

The ring $A_k$ being a graded subring of the ring $O_Y[u, u^{-1}]$, we may consider the $G_m$-action $\sigma$ on $CY_k$, leaving sections of $O_Y$ invariant and acting by $\lambda \mapsto \lambda^{-1}u$ on $u$. It restricts on $U_1$, to the diagonal action induced by the canonical $G_m^I$-action on $U_1$ via the finite morphism $\lambda \mapsto \lambda^{k_i}$. We have now two different $G_m$-actions on $\mathcal{L}_n(CY_k^o)$: the one induced by the standard $G_m$-action on arc spaces and the one induced by $\sigma$. We denote by $\tilde{\sigma}$ the action given by the composition of these two (commuting) actions.

Let us denote by $\tilde{L}_{\gamma}(CY_k^o)$ the set of arcs $\varphi$ in $\mathcal{L}_{\gamma}(CY_k^o)$ such that $p(\varphi(t)) = t$ (in particular $\varphi(0)$ is in $U_1$). For such an arc $\varphi$, composition with $\pi$ sends $\varphi$ to an arc in $\mathcal{L}_{\gamma}(Y \times A_1^k)$ which is the graph of an arc in $\mathcal{L}_{\gamma}(Y)$ not contained in the union of the divisors $E_i$, $i$ in $I$. Note that $\tilde{L}_n(CY_k^o)$ is stable by $\tilde{\sigma}$.

3.6. Lemma. — Let $I$ be in $B$ and $k$ in $N_{\geq 0}$. Assume $n \geq k_i$ for $i$ in $I$. The morphism $\tilde{\pi} : \tilde{L}_n(CY_k^o) \to \mathcal{X}_{n,k}$ induced by the projection $CY_k^o \to Y$ is an affine bundle with fiber $A_1^\sum_{i \in I} k_i$ induced by $\tilde{\sigma}$ and $\mathcal{X}_{n,k}$ with the standard $G_m$-action. Furthermore if $\tilde{L}_n(CY_k^o)$ is endowed with the $G_m$-action induced by $\tilde{\sigma}$ and $\mathcal{X}_{n,k}$ with the standard $G_m$-action, $\tilde{\pi}$ is $G_m$-equivariant and the action of $G_m$ on the affine bundle is affine. Furthermore, if $n \geq \ell_\Gamma(N_I(k))$, then for every $\varphi$ in $\tilde{L}_{\gamma}(CY_k^o)$

\[(3.6.1) \quad ac(P(f) \circ h)(\tilde{\pi}(\varphi)) = ac(\tilde{P}(f)_k(\varphi)).\]

When $P_\delta(f_I)(\varphi(0)) \neq 0$, hence $(\text{ord}_t(P(f)) \circ h)(\tilde{\pi}(\varphi)) = \ell_\Gamma(N_I(k))$, we have

\[(3.6.2) \quad ac(P(f) \circ h)(\tilde{\pi}(\varphi)) = P_\delta(f_I)(\varphi(0)).\]

Proof. — The first part statement is contained in Lemma 5.13 of [9] and the rest follows from its proof.\[\Box\]

We then define $\tilde{Y}_{n,k}$ as the inverse image of $\tilde{Y}_{n,k}$ by the fibration $\tilde{\pi}$. It is the subset of arcs $\varphi$ in $\mathcal{L}_{\gamma}(CY_k^o)$ such that $\text{ord}_t(\tilde{P}(f)_k(\varphi)) = n - \ell_\Gamma(N_I(f))$. We denote by $[\tilde{Y}_{n,k}]$ the class of $\tilde{Y}_{n,k}$ in $\mathcal{M}_{\mathcal{X}_0(F) \times G_m}$, the morphism $\tilde{\gamma}_{n,k} : G_m$ being $ac(\tilde{P}(f)_k)$ and the $G_m$-action being induced by $\tilde{\sigma}$. We denote by $[U_I \setminus (P_\delta(f_I)^{-1}(0))]$ the class of $U_I \setminus (P_\delta(f_I)^{-1}(0))$ in $\mathcal{M}_{\mathcal{X}_0(F) \times G_m}$, the $G_m$-action being the natural diagonal action of weight $k$ on $U_I \setminus (P_\delta(f_I)^{-1}(0))$ and the morphism to $G_m$ being the restriction of $P_\delta(f_I)$. We also consider the class $[G_m \times F^{-1}_m]$ of $G_m \times P_\delta(f_I)^{-1}(0)$ in $\mathcal{M}_{\mathcal{X}_0(F) \times G_m}$, the $G_m$-action on the second factor being the diagonal one and the morphism to $G_m$ being the first projection.
3.7. Lemma. — Let \( I \) be in \( B \) and \( k \) in \( \mathbb{N}_{>0} \). The following equalities hold in \( \mathcal{M}_{X_0(F) \times G}^m \):

\[
\begin{align*}
(1) \quad \mathcal{Y}_{n,k} &= L^{\gamma nd}[U_I \setminus (P_\delta(f_I)^{-1}(0))], \text{ if } n = \ell_\Gamma(N_I(k)), \\
(2) \quad \widetilde{\mathcal{Y}}_{n,k} &= L^{\gamma nd-m}[G_m \times P_\delta(f_I)^{-1}(0)], \text{ if } n - \ell_\Gamma(N_I(k)) = m > 0.
\end{align*}
\]

Proof. — As we assume \( P \) is non-degenerate with respect to its Newton polyhedron, \( P_\delta \) is smooth on \( G_m \) and the composed map \( P_\delta(f_I) \) is smooth on \( U_I \). It follows that the morphism \( (\tilde{P}(F)_k, u) : CY_k^\circ \rightarrow A_k^2 \) is smooth on a neighborhood of \( U_I \) in \( CY_k^\circ \), so one can argue similarly as in the proof of Lemma 5.14 of [9].

From Lemma 3.6 and Lemma 3.7, we may rewrite (3.5.2)

\[
i^*Z_{P(f), X \setminus X_0(F)} = \sum_{\delta \in F(T)} Z_{\delta, I}(T),
\]

with

\[
Z_{\delta, I}(T) = ([U_I \setminus (P_\delta(f_I)^{-1}(0))] \Phi_{\delta, I}(T) + [G_m \times P_\delta(f_I)^{-1}(0)] \Psi_{\delta, I}(T),
\]

where

\[
\Phi_{\delta, I}(T) = \sum_{\mathcal{N} \in \sigma(\delta)} T^{\ell_\Gamma(N_I(k))} L^{-\sum_i \nu_i k_i},
\]

and

\[
\Psi_{\delta, I}(T) = \sum_{\mathcal{N} \in \sigma(\delta), n \geq 0} T^{\ell_\Gamma(N_I(k)) + n} L^{-\sum_i \nu_i k_i}.
\]

If \( \delta \) is not contained in a coordinate hyperplane, for \( \gamma \) large enough, the inequality

\[
\langle N_I(k), 1 \rangle \leq \gamma \ell_\Gamma(N_I(k)) + \gamma n
\]

holds for every \( N_I(k) \) in \( \sigma(\delta) \) and every \( n \geq 0 \). It follows that

\[
\lim_{T \rightarrow \infty} \Phi_{\delta, I}(T) = \lim_{T \rightarrow \infty} \Psi_{\delta, I}(T) = \chi(N_I^{-1}(\sigma(\delta))).
\]

If \( \delta \) is contained in some coordinate hyperplane, it follows from Lemma 2.10 of [9] that

\[
\lim_{T \rightarrow \infty} \Phi_{\delta, I}(T) = \lim_{T \rightarrow \infty} \Psi_{\delta, I}(T) = 0.
\]

The result follows now from the definition of \( \Psi_{P_\delta} \) and (2.9.3).

3.8. Example. — When \( p = 2 \) and \( P = \Sigma \), one recovers the motivic Thom-Sebastiani, cf. [5], [10], and [6], in the way stated in [9]. When \( f \) is the family of coordinate functions on the affine space \( A_k^p \), formula 3.5.1 specializes to the one given by Guibert [8], Proposition 2.1.6.
3.9. Remark. — Restricting to a given point $x$ of $X_0(f)$ and applying the Hodge spectrum map $Sp$ of §6 of $[9]$ to $[3.4.1]$, one gets a formula for the Hodge-Steenbrink spectrum (cf. $[12]$, $[13]$, $[15]$) of $P(f)$ at $x$. It is not immediately clear whether this formula coincides with the one obtained by Terasoma ($[14]$, Theorem 3.6.1).

References

[1] F. Bittner, On motivic zeta functions and the motivic Milnor fiber, Math. Z. 249 (2005), 63–83.
[2] J. Denef, On the degree of Igusa’s local zeta function, Amer. J. Math. 109 (1987), 991–1008.
[3] J. Denef, F. Loeser, Motivic Igusa zeta functions, J. Algebraic Geom. 7 (1998), 505–537.
[4] J. Denef, F. Loeser, Germs of arcs on singular algebraic varieties and motivic integration, Invent. Math. 135 (1999), 201–232.
[5] J. Denef, F. Loeser, Motivic exponential integrals and a motivic Thom-Sebastiani Theorem, Duke Math. J. 99 (1999), 285–309.
[6] J. Denef, F. Loeser, Geometry on arc spaces of algebraic varieties, Proceedings of 3rd European Congress of Mathematics, Barcelona 2000, Progress in Mathematics 201 (2001), 327–348, Birkhäuser.
[7] J. Denef, F. Loeser, Lefschetz numbers of iterates of the monodromy and truncated arcs, Topology 41 (2002), 1031–1040.
[8] G. Guibert, Espaces d’arcs et invariants d’Alexander, Comment. Math. Helv. 77 (2002), 783–820.
[9] G. Guibert, F. Loeser, M. Merle, Iterated vanishing cycles, convolution, and a motivic analogue of a conjecture of Steenbrink, math. AG/03012203.
[10] E. Looijenga, Motivic Measures, Astérisque 276 (2002), 267–297, Séminaire Bourbaki exposé 874.
[11] S. MacLane, Categories for the working mathematician, Graduate Texts in Mathematics, Vol. 5. Springer-Verlag, New York-Berlin, 1971.
[12] J. Steenbrink, Mixed Hodge structures on the vanishing cohomology, in Real and Complex Singularities, Sijthoff and Noordhoff, Alphen aan den Rijn, 1977, 525–563.
[13] J. Steenbrink, The spectrum of hypersurface singularities, in Théorie de Hodge, Luminy 1987, Astérisque, 179 -180 (1989), 163–184.
[14] T. Terasoma, Convolution theorem for non-degenerate maps and composite singularities, J. Algebraic Geom. 9 (2000), 265–287.
[15] A. Varchenko, Asymptotic Hodge structure in the vanishing cohomology, Math. USSR Izvestija 18 (1982), 469–512.
