Blow-up for self-interacting fractional Ginzburg-Landau equation

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Abstract. The blow-up of solutions for the Cauchy problem of fractional Ginzburg-Landau equation with non-positive nonlinearity is shown by an ODE argument. Moreover, in one dimensional case, the optimal lifespan estimate for size of initial data is obtained.

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1. Introduction

The classical complex Ginzburg-Landau (CGL) equation takes the form

$\partial_t \psi = -(\alpha + i\beta)\Delta \psi + F(\psi, \overline{\psi})$,

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where $\alpha, \beta$ are real parameters. The standard CGL equation has a self-interaction term $F$ of the form
\[
F(\psi, \overline{\psi}) = -\sum_{j=1}^{K} (\alpha_j + i\gamma_j) \psi |\psi|^{p_j - 1},
\]
where $\alpha_j, \beta_j$ are real parameters. We refer to [5] for a review on this subject. Using the representation $\psi(t, x) = u_1(t, x) + iu_2(t, x)$, where $u_1, u_2$ are real-valued functions, we see that the equation (1.1) can be rewritten in the form of a system of reaction diffusion equations
\[
\partial_t U = A\Delta U = F(U),
\]
where
\[
U(t, x) = \begin{pmatrix} u_1(t, x) \\ u_2(t, x) \end{pmatrix}, \quad A = \begin{pmatrix} -\alpha & \beta \\ -\beta & -\alpha \end{pmatrix}.
\]
The limiting case $\alpha \to 0, \alpha_j \to 0$ leads to the nonlinear Schrödinger equation (NLS)
\[
(1.2) \quad \partial_t \psi = -i\beta \Delta \psi - \sum_{j=1}^{N} i\gamma_j \psi |\psi|^{p_j - 1}.
\]
The oscillation synchronization of phenomena modeled by Kuramoto equations (see [4]) lead to a system of ODE having a similar qualitative behavior
\[
(1.3) \quad \partial_t \psi_k = -iH_k \psi_k + F_k(\Psi, \overline{\Psi}), \quad k = 1, \ldots, N.
\]
The nonlinear terms $F_k$ in the system obey the property
\[
\text{Im} \left( F_k(\Psi, \overline{\Psi}) \overline{\psi_k} \right) = 0, \quad k = 1, \ldots, N.
\]
This system simulates the behavior of $N$ oscillators, so that $\Psi = (\psi_1, \ldots, \psi_N)$, with $\psi_j$ being complex-valued functions. The nonlinearities in (1.3) are chosen so that the evolution flow associated to the Kuramoto system leaves the manifold
\[
\mathcal{M} = S^1 \times \cdots \times S^1, \quad \text{N times}
\]
invariant.

The derivation of the Kuramoto system in [4] is based on complex Landau-Ginzburg equation (see equation (2.4.15) in [4])
\[
\partial_t \Psi = i\mathcal{H}\Psi - (\alpha + i\beta) \Delta \Psi - (\alpha_1 + i\beta_1) \Psi |\Psi|^2,
\]
where $\Psi = (\psi_1, \ldots, \psi_N)^t$, $\mathcal{H}$ is a diagonal matrix with real entries. If $\beta$ and $\beta_1$ become very large, then we have an equation very close to Schrödinger self-interacting system (1.2). As it was pointed out (p. 20, [4]), a chemical turbulence of a diffusion-induced type are possible only for regions intermediate between the two extreme cases, where $\beta$ and $\beta_1$ are very small or very large.

Turning back to CGL equation and comparing (1.1) with Kuramoto system, we see that it is natural to take $\alpha \to 0, \beta_j \to 0$ so that we have the following simplified CGL equation
\[
\partial_t \psi = -i\beta \Delta \psi - \alpha_1 \psi |\psi|^{p-1}.
\]
A similar system was discussed in [1] with nonlinearity typical for the Kuramoto system.
The fractional dynamics seems more adapted to synchronization models due to the considerations in [6], therefore we can consider the following fractional Ginzburg-Landau equations

$$\partial_t \psi = -i\sqrt{-\Delta}\psi \pm |\psi|^{p-1}.$$  

The study of the attractive case

$$\partial_t \psi = -i|D|\psi - |\psi|^{p-1}, \quad |D| = \sqrt{-\Delta},$$

is initiated in [2], where the well-posedness is established for the cases $1 \leq n \leq 3$.

In this article, we study the repulsive case

$$\partial_t u = -i|D|u + |u|^{p-1}u, \quad t \in [0, T), \quad x \in \mathbb{R}^n,$$

where $n \geq 1$, and $p > 1$. Our main goal is to obtain a blow-up result under the assumption that initial data are in $H^s(\mathbb{R}^n)$ with $s > n/2$, where $H^s(\mathbb{R}^n)$ is the usual Sobolev space defined by $(1 - \Delta)^{-s/2}L^2(\mathbb{R}^n)$.

We denote $\langle x \rangle = (1 + |x|^2)^{1/2}$. We abbreviate $L^q(\mathbb{R}^n)$ to $L^q$ and $\| \cdot \|_{L^q(\mathbb{R}^n)}$ to $\| \cdot \|_q$ for any $q$. We also denote by $\|T\|$ the operator norm of bounded operator $T : L^2 \to L^2$.

The following statements are the main results of this article.

**Proposition 1.1.** Let $h$ be a function satisfying $\frac{1}{h} \in L^\infty \cap L^2$ and

$$\left\| \frac{1}{h} (-\Delta)^{1/2}, h \right\|$$

Let $u_0 \in hL^2$ satisfy

$$\|\frac{1}{h} u_0\|_2 \geq \|\frac{1}{h} [D, h]\|^{-1} \|\frac{1}{h}\|.$$  

If there is a solution $u \in C([0, T); hL^2)$ for (1.4), then

$$\|\frac{1}{h} u(t)\|_2 
\geq e^{-2\|\frac{1}{h} [D, h]\| t} \left(\|\frac{1}{h} u_0\|_2^{-p+1} + \|\frac{1}{h} [D, h]\|^{-1} \|\frac{1}{h}\|_2^{-p+1} \{e^{-\|\frac{1}{h} [D, h]\|(p-1) t} - 1\}\right)^{-\frac{1}{p-1}}.$$  

Therefore, the lifespan is estimated by

$$T \leq -\frac{2}{p-1} \|\frac{1}{h} [D, h]\|^{-1} \log \left(1 - \|\frac{1}{h} [D, h]\| \|\frac{1}{h}\|_2^{-p+1} \|\frac{1}{h} u_0\|_2^{-p+1}\right).$$

We remark that for $n = 1$, we can take $h = \langle \cdot \rangle$ for Proposition 1.1. Proposition 1.1 is a blow-up result for a kind of large data of $hL^2$. However, in a subcritical case where $p < p_F = 3$, solutions blow up even for small $L^2$ initial data.

**Corollary 1.2.** Let $u_0 \in L^2(\mathbb{R}) \setminus \{0\}$ and $1 < p < p_F$. Then the corresponding solution in $C([0, T); L^2(\mathbb{R}))$ blows up at a finite positive time.

**Remark 1.3.** If we choose $h(x) = \langle x \rangle$, the statement of our main result guarantees the blow-up of the momentum

$$Q_{-1}(t) = \int_{\mathbb{R}} \langle x \rangle^{-1} |u(t, x)|^2 dx.$$
for the solution to the fractional CGL equation
\[ \partial_t u = -i|D|u + |u|^{p-1}u \]
in (1.4). The blow-up mechanism is based on the differential inequality
\[ Q_{-1}'(t) \geq C_0 (Q_{-1}(t))^{(p+1)/2} - C_1 Q_{-1}(t), \quad C_0, C_1 > 0. \]
Comparing the fractional CGL equation with the classical NLS
\[ \partial_t u = i\Delta u + i|u|^{p-1}u, \]
we see that introducing the momentum
\[ Q_2(t) = \int_{\mathbb{R}^n} |x|^2 |u(t, x)|^2 dx, \]
and using a Virial identity one can show that
\[ Q_2''(t) \sim E(u)(t), \]
where \( E(u)(t) = \|Du(t)\|_{L^2}^2 - c\|u(t)\|_{L^{p+1}}^{p+1} \) and \( c > 0 \) is an appropriate constant. Therefore, the blow-up mechanism for NLS is based on the estimate
\[ E(u)(t) \leq -\delta, \quad \delta > 0, \]
that implies differential inequality
\[ Q_2''(t) \leq -\delta \]
and the last inequality cannot be satisfied for the whole interval \( t \in (0, \infty) \) since \( Q_2(t) \) is a positive quantity.

Moreover, for large \( R \), if \( u_0 \) is given by \( Rf \) with \( f \in hL^2(\mathbb{R}) \) and \( h \) satisfying (1.5), then (1.8) means \( T \leq CR^{-p+1} \). In one dimensional case, this upper bound is shown to be sharp for \( f \in (hL^2 \cap H^1)(\mathbb{R}) \).

**Proposition 1.4.** Let \( u_0 = Rf \) with \( R > 0 \) sufficiently large and \( f \in H^1(\mathbb{R}). \) Then there exists an \( H^1(\mathbb{R}) \) solution for \( u_0 \) for which its lifespan is estimated by \( T \geq CR^{-p+1} \) with some positive constant \( C \).

### 2. Preliminary

In this section, we recall the blow-up solutions for an ODE which gives the mechanism of blow-up for weighted \( L^2 \) norm of solutions. We also study the condition for weight functions of Corollary 1.2.

#### 2.1. Blow-up solutions for an ODE.

**Lemma 2.1.** Let \( C_1, C_2 > 0 \) and \( q > 1 \). If \( f \in C^1([0, T); \mathbb{R}) \) satisfies \( f(0) > 0 \) and
\[ f' + C_1 f = C_2 f^q \quad \text{on } [0, T) \]
for some \( T > 0 \), then
\[ f(t) = e^{-C_1 t} \left( f(0)^{-(q-1)} + C_1^{-1} C_2 e^{-C_1 (q-1) t} - C_1^{-1} C_2 \right)^{-\frac{1}{q-1}}. \]
Moreover, if \( f(0) > C_1^{-1} C_2^{-\frac{1}{q-1}} \), then \( T < \frac{1}{C_1 (q-1)} \log(1 - C_1 C_2^{-1} f(0)^{-q+1}). \)
Proof. Let \( f = e^{-C_1 t}g \). Then
\[
g'(t) = C_2 e^{-C_1(q-1)t} g'.
\]
Therefore,
\[
\frac{1}{1-q} \left( g^{1-q}(t) - g^{1-q}(0) \right) = \frac{C_2}{C_1(1-q)} (e^{-C_1(q-1)t} - 1).
\]

□

2.2. Condition for weight function.

Lemma 2.2 (Coifman - Meyer). Let \( p \in C^\infty(\mathbb{R}^{2n}) \) satisfy the estimates
\[
|D_x^\alpha D_\xi^\beta p(x, \xi)| \leq C_{\alpha, \beta} |\xi|^{1-|\alpha|}
\]
for all multi-indices \( \alpha \) and \( \beta \). Then for any Lipschitz function \( h \),
\[
||p(x, D), h||_2 \leq C ||h||_{\text{Lip}} ||f||_2.
\]

Lemma 2.3. Let \( \phi \in C^\infty_0([0, \infty); \mathbb{R}) \) satisfy
\[
\phi(\rho) = \begin{cases} 1 & \text{if } 0 \leq \rho \leq 1, \\ 0 & \text{if } \rho \geq 2. \end{cases}
\]
Then
\[
\left| \int_{\mathbb{R}^n} \phi(|\xi|)|\xi|e^{ix \cdot \xi} d\xi \right| \leq C(x)^{-n-1}.
\]

Proof. It suffices to consider the case where \( |x| \) is sufficiently large. Let \( \psi \in C^\infty_0([0, \infty); \mathbb{R}) \) satisfy
\[
\psi(\rho) = \begin{cases} 1 & \text{if } 0 \leq \rho \leq 1, \\ 0 & \text{if } \rho \geq 2. \end{cases}
\]
Let \( e_1 = (1, 0, \cdots, 0) \). Let \( \xi_1 = \xi \cdot e_1 \) and \( \xi' = \xi - \xi_1 e_1 \). Assume \( x = |x|e_1 \). Then
\[
\int_{\mathbb{R}^n} \phi(|\xi|)|\xi|e^{ix \cdot \xi} d\xi = \int_{\mathbb{R}^n} \phi(|\xi|)|\xi|e^{ix \cdot \xi_1} d\xi.
\]
By integrating by parts \( k \) times,
\[
\int_{\mathbb{R}^n} \phi(|\xi|)|\xi|e^{ix \cdot \xi_1} d\xi = (-i|x|)^{-k} \int_{\mathbb{R}^n} \partial_1^k (\phi(|\xi|)|\xi|)e^{ix \cdot \xi_1} d\xi
\]
\[
= (-i|x|)^{-k} \int_{\mathbb{R}^n} \partial_1^k (|\xi|)\phi(|\xi|)e^{ix \cdot \xi_1} d\xi + R_k(\xi)d\xi,
\]
where \( R_{n+1} \in L^1(\mathbb{R}^n) \). Here \( \partial_1^k |\xi| \) is estimated by \( C|\xi|^{1-k} \). Moreover,
\[
\left| \int_{\mathbb{R}^n} \partial_1^k (\xi)|\xi\phi(|\xi|)e^{ix \cdot \xi_1} d\xi \right|
\]
\[
= |x|^{-1} \left| \int_{\mathbb{R}^n} \int_{\mathbb{R}} \partial_1 \{ \partial_1^n (|\xi|)\phi(|\xi|)\} (e^{ix \cdot \xi_1} - 1) d\xi d\xi' \right|
\]
\[
\leq C \left| \int_{\mathbb{R}^n} \psi(|x||\xi|)|\xi|^{1-n} |\phi(|\xi|)| + |\partial_1 \phi(|\xi|)| d\xi \right|
\]
\[
+ C|x|^{-1} \left| \int_{\mathbb{R}^n} (1 - \psi(|x||\xi|))|\xi|^{-n} |\phi(|\xi|)| + |\partial_1 \phi(|\xi|)| d\xi \right|
\]
since $|e^{i|x|\xi_1} - 1| \leq |x||\xi|$. The first integral is estimated by

$$C \int_0^2 |x|^{-1} |\phi(\rho)| + |\phi'(\rho)|d\rho \leq C\|\phi\|_{C^1(0,2)}|x|^{-1}.$$ 

By letting $\Psi(\rho) = \int_0^\rho |1 - \psi(\rho')|d\rho'$ and integrating by parts once again, the second integral is estimated by

$$C|x|^{-2} \int_{|x|^{-1}}^2 \rho^{-2}\|\phi\|_{C^2(0,2)}|x|d\rho \leq C\|\phi\|_{C^2(0,2)}|x|^{-1}.$$ 

This proves the lemma. □

**Lemma 2.4.** Let $h$ be a Lipschitz function on $\mathbb{R}^n$ satisfying the estimate

$$\left\| \frac{1}{h(\cdot)} \int_{\mathbb{R}^n} \langle \cdot - y \rangle^{-n-1} h(y)f(y)dy \right\|_2 \leq C\|f\|_2$$

for any $f \in L^2$. Then $\frac{1}{h}[D,h]$ is a bounded operator from $L^2$ to $L^2$.

**Proof.** Let $\phi$ be a smooth function on $[0, \infty)$ satisfying that $\phi(\xi) = 1$ if $|\xi| \leq 1$ and $\phi(\xi) = 0$ if $|\xi| \geq 2$. Let $\phi(D)f = \hat{\phi}^{-1}\hat{f}$. We divide the proof into the following two estimates: $\|\frac{1}{h}\phi(D)(Dh)f\|_2 \leq \|f\|_2$ and $\|\frac{1}{h}(1 - \phi(D))D,h[f]\|_2 \leq \|f\|_2$.

At first,

$$\|\frac{1}{h}\phi(D)(Dh)f\|_2 \leq C\left\| \frac{1}{h(\cdot)} \int_{\mathbb{R}^n} \langle \cdot - y \rangle^{-n-1} h(y)f(y)dy \right\|_2 \leq C\|f\|_2,$$

since

$$|\hat{\phi}^{-1}(\cdot)\cdot\phi| \leq C(x)^{-n-1}.$$ 

Secondly, $(1 - \phi(|\xi|))|\xi|$ satisfies the condition of Lemma 2.2. So the second estimate follows from Lemma 2.2. □

**Remark 2.5.** $h(x) = \langle x \rangle$ satisfies the condition of Lemma 2.4. Actually $h$ is Lipshitz and by using triangle inequality,

$$\left\| \langle \cdot \rangle^{-1} \int_{\mathbb{R}^n} \langle x - y \rangle^{-n-1} \langle y \rangle f(y)dy \right\|_2 \leq \left\| \int_{\mathbb{R}^n} \langle x - y \rangle^{-n-1} f(y)dy \right\|_2 + \left\| \langle \cdot \rangle^{-1} \int_{\mathbb{R}^n} \langle x - y \rangle^{-n} f(y)dy \right\|_2 \leq (\|\langle \cdot \rangle^{-n-1}\|_1 + \|\langle \cdot \rangle^{-1}\|_q\|\langle \cdot \rangle^{-n}\|_q')\|f\|_2,$$

where $n < q < \infty$.

**Corollary 2.6.** Let $h$ satisfy the condition of Lemma 2.4 and let $h_R = h(\cdot/R)$. Then

$$\left\| \frac{1}{h_R}[D,h_R] \right\| \leq R^{-1}\left\| \frac{1}{h}[D,h] \right\|.$$
Proof.

\[
\frac{1}{h_R(x)}[D, h_R]f(x) = \frac{1}{h(\frac{x}{R})} \int_{\mathbb{R}^n} e^{i(x-y) \cdot \xi} |\{h(\frac{y}{R}) - h(\frac{x}{R})\}| f(y) d\xi dy
\]

\[
= R^n \frac{1}{h(\frac{x}{R})} \int_{\mathbb{R}^n} e^{i(\frac{x}{R} - y) \cdot R \xi} |\{h(y) - h(\frac{x}{R})\}| f(Ry) d\xi dy
\]

\[
= R^{-1} \frac{1}{h(\frac{x}{R})} \int_{\mathbb{R}^n} e^{i(\frac{x}{R} - y) \cdot \xi} |\{h(y) - h(\frac{x}{R})\}| f(Ry) d\xi dy
\]

\[
= R^{-1} \frac{1}{h(\frac{x}{R})}[D, h]f_{R^{-1}}(\frac{x}{R}).
\]

This implies

\[
\| \frac{1}{h_R} [D, h_R]f \|_{L^2} = R^{-1+n/2} \| \frac{1}{h} [D, h]f_{R^{-1}} \|_{L^2} \leq R^{-1} \| \frac{1}{h} [D, h] \| \| f \|_{L^2}.
\]

\[
\square
\]

3. Proof

3.1. Proof of Proposition 1.1. Let \( u(t, x) = h(x)v(t, x) \). Then

\[
(3.1) \quad i\partial_t v + Dv + \frac{1}{h} [D, h]v = ih^{p-1} |v|^{p-1} v.
\]

Multiplying both sides of (3.1) by \( \overline{v} \), integrating over \( \mathbb{R}^n \), and taking the imaginary part of the resulting integrals, we obtain

\[
\frac{1}{2} \frac{d}{dt} \| v(t) \|_2^2 = \int_{\mathbb{R}^n} h(x)^{p-1} |v(t, x)|^{p+1} dx - \text{Im} \int_{\mathbb{R}^n} \overline{v(t, x)} \frac{1}{h(x)} [D, h]v(t, x) dx
\]

\[
\geq \int_{\mathbb{R}^n} h(x)^{p-1} |v(t, x)|^{p+1} dx - \frac{1}{h} [D, h] \| v(t) \|_2^2
\]

\[
\geq \left\| \frac{1}{h} \right\|_2^{-p+1} \| v(t) \|_2^{p+1} - \frac{1}{h} [D, h] \| v(t) \|_2^2,
\]

where we used the following estimate:

\[
\| v(t) \|_2 \leq \left\| \frac{1}{h^{\frac{p}{p+1}}} \right\|_2^{2(p+1)} \| h^{\frac{p-1}{p+1}} v(t) \|_{p+1}.
\]

Then (1.7) follows from Lemma 2.1 with \( q = (p + 1)/2 \).

3.2. Proof of Corollary 1.2. Let \( h_R(x) = \langle x/R \rangle \) with \( R > 0 \). \( h_R \) satisfies (1.5) and \( 1/h_R \in (L^\infty \cap L^2)(\mathbb{R}) \), and \( \frac{1}{h_R} u_0 \to u_0 \) in \( L^2 \) as \( R \to \infty \). Moreover, \( \| \frac{1}{h_R} [D, h_R] \| \sim R^{-1} \), and \( \| \frac{1}{h_R} \|_2 \sim R^{1/2} \). Therefore

\[
\text{RHS (1.6)} \sim R^{\frac{1}{2} - \frac{1}{p+1}} \to 0
\]

as \( R \to \infty \) if \( p < 3 \). It means that for any \( u_0 \in L^2(\mathbb{R}) \setminus \{0\} \), there exists \( R_0 \) such that (1.6) is satisfied with \( h(x) = \langle x/R_0 \rangle \).
3.3. Proof of Proposition 1.4. The local well-posedness in $H^1(\mathbb{R})$ is easily obtained by the Sobolev embedding and standard contraction argument. By multiplying (1.4) by $u$ and $(\Delta)u$, integrating over $\mathbb{R}$, we obtain

$$\frac{d}{dt} \| u(t) \|_2^2 = \| u(t) \|_{p+1}^{p+1} \leq C \| u(t) \|_{H^1(\mathbb{R})}^p,$$

$$\frac{d}{dt} \| \nabla u(t) \|_2^2 = \text{Re} \int_{\mathbb{R}} \nabla |u(t,x)|^{p-1} u(t,x) \cdot \nabla u(t,x) dx \leq C \| u(t) \|_{H^1(\mathbb{R})}^p,$$

where $\| f \|_{H^1(\mathbb{R})} = \| f \|_2^2 + \| \nabla f \|_2^2$. By solving the following ordinary differential equality:

$$\frac{d}{dt} U(t) = CU(t)^{\frac{p+1}{2}}$$

we get

$$\| u(t) \|_{H^1(\mathbb{R})} \leq \left( \| u_0 \|_{H^1(\mathbb{R})}^{-(p-1)} - \frac{C(p-1)}{2} t \right)^{-\frac{1}{p-1}}.$$

This proves the Proposition 1.4.

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