APOLOGY OF EUCLID

S. S. KUTATELADZE

April 21, 2005

Abstract. This is a short apology of the style of the Elements by Euclid and Bourbaki.

A somewhat derogatory term “bourbakism” proliferates in many public discussions about the teaching of mathematics. We hear many funny anecdotes about commutativity as a method of calculation as well as separate addition of nominators and denominators. Professional mathematicians and teachers divide into the hostile groups that discuss with alienation and indignation of the medieval scholastics the “problem of the naturalness of zero” as well as the priority rights between the concepts “greater than,” “greater than or equal to,” and “strictly greater than.” All these stories and philippics are nice and true to some extent but rest upon a clear-cut misunderstanding.

It stands to reason to recall that there was no teacher whose name was Bourbaki. It is also reasonable to bear in mind that the treatise of Bourbaki is written as imitation of Euclid’s Elements. The style of Bourbaki’s Elements of Mathematics is exactly the style of Euclid.

Any serious criticism of the books by Bourbaki bases on pretensions to their content rather than style. Bourbaki’s treatise is evidently incomplete. Many important mathematical theories are absent or covered inadequately. A few volumes present the dead ends of exuberant theories. All these shortcomings are connected with the major capital distinction between the books by Euclid and Bourbaki. In his Elements Euclid set forth the theory that was almost complete in his times, the so-called “Euclidean” plane and space geometry. Most of this section of science was made clear once and forever in the epoch of Euclid.

The Bourbaki project was implemented in the period of very rapid progress in mathematics. Many books of the treatise became obsolete at the exact moment of publication. In particular, functional analysis had been developing contrary to what one might imagine reading the book Topological Vector Spaces. But to a failure was doomed the heroic and ambitions plan of Bourbaki to present the elements of the whole mathematics of the twentieth century in a single treatise along the methodological lines of Euclid. Mathematics renews and enriches itself with outstanding brilliant achievements much faster than the books of Bourbaki’s treatise were compiled. There is no wonder that the mathematical heroes who create the twentieth century mathematics have distinctly and immediately scented the shortcomings of Bourbaki. The treatise encountered severe criticism and even condemnation since it omits many important topics. As usual, this serious criticism convened all sorts of educationists, would-be specialists in “propaedeutics” and
“methodology” who are hardly aware of what is going on in the real mathematics. Everyone knows that to criticize a book for incompleteness is a weak argument since it is strange to judge an article for what is absent in this article. Grudges against the content of the treatise transform by necessity to the criticism of its form. The terseness, conciseness, and lapidary of the style of exposition fall victim to criticism and even ostracism by the adversaries of the malicious “bourbakism” in education.

Salomon Bochner, one of the famous mathematicians of the past, observed with a witty smile:

Also, if examined “objectively,” Euclid’s work ought to have been any educationist’s nightmare. The work presumes to begin from a beginning; that is, it presupposes a certain level of readiness, but makes no other prerequisites. Yet it never offers any “motivations,” it has no illuminating “asides,” it does not attempt to make anything “intuitive,” and it avoids “applications” to a fault. It is so “humorless” in its mathematical purism that, although it is a book about “Elements,” it nevertheless does not unbend long enough in its singlemindedness to make the remark, however incidentally, that if a rectangle has a base of 3 inches and a height of 4 inches then it has an area of 12 square inches. Euclid’s work never mentions the name of a person; it never makes a statement about, or even an (intended) allusion to, genetic developments of mathematics; it makes no cross references, except once, the exception being in proposition 2 of Book 13, where the text refers to, and repeats the content of, the “first theorem of the tenth book,” which, as it happens, is Euclid’s “substitute” for the later axiom of Archimedes. Euclid has a fixed pattern for the enunciation of a proposition, and, through the whole length of 13 books, he is never tempted to deviate from it. In short, it is almost impossible to refute an assertion that the Elements is the work of an unsufferable pedant and martinet... Euclid’s work became one of the all-time best sellers. According to “objective” Pestalozzi criteria, it should have been spurned by students and “progressive” teachers in every generation. But it nevertheless survived intact all the turmoil, ravages, and illiteracies of the dissolving Roman Empire, of the early Dark Ages, of the Crusades, and of the plagues and famines of the later Middle Ages. And, since printing began, Euclid has been printed in as many editions, and in as many languages, as perhaps no other book outside the Bible.1

Euclid’s book is a totally appalling, terse and formal presentation of axioms, definitions, lemmas and theorems without any motivation and digression, lacking any illuminating examples from physics, economics, social or spiritual life. However, it is the book that lives about two and a half millennia and shows no indication of dying. In contract, the textbooks fail to survive the gerontological tests that define the area of a figure by sowing it with some grain or cutting it off a sheet of paper.

We must avoid mixing together the full-time and extramural forms of training, the transfer and saving of knowledge. The Babylonian texts on mathematics are in fact problem-books with solutions. This style of teaching is still alive. However, no problem-book of any sort can compare with Euclid’s Elements in its long-term impact on mathematics and culture as a whole. Any student’s notes of a mathematical course still remind us of Euclid’s Elements and its successor in style, Bourbaki’s Elements of Mathematics.

In common parlance, bourbakism stands for “formalistic structural mathematics,” whatever the bizarre term means. In fact, this vogue word rarely implies something more than a simple reference to the century-old tradition of shortening and saving mathematical theories in axiomatic form. This marvelous and noble tradition stems from the writings of Euclid. Elimination of extravagancy and pursuit of consistency, clarity, terseness, and rationality in exposition stimulate, organize,

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1Bochner S. The Role of Mathematics in the Rise of Science. Princeton University Press, 1981, pp. 35–36
and discipline mind and thought, revealing the intrinsic beauty and harmony of mathematics. It is exactly the impersonal style of Euclid’s Elements, lacking any temporal inklings, that makes them especially valuable and allows anybody to understand what they tell us when centuries have elapsed.

The “verbal” problems, practical motivations and emphasis on a person’s creativity as well as the subjective coloring of exposition and present-day allusions are absolutely obligatory gadgets in the tool-kit for training. However, the particular products of these immortal teaching tools are rather volatile, momentary, and fragile; they often die at the spur of the moment of their enunciation.

Science must preserve old knowledge as well as meet the challenges of nowadays by solving the new and pending problems. Therefore, teaching has the twofold task of preserving and transferring knowledge, “filling the mind” in combination with “igniting a fire,” i.e., the initiation and stimulation of creative search into new knowledge. There is no reason to oppose the transfer and preservation of knowledge and the training of creativity and practical skills in raising and solving the problems of today. Preservation of mathematical knowledge in the impersonal and dry style of textbooks never excludes the possibility of creative search of the teacher. On the contrary, the style of Euclid presupposes perpetual creativity, calling the teacher for finding and using subtle personal adjustments, subjective keys and even mysteries for igniting students’ interest in mathematics, the understanding of its place and role in science, industry, and other areas of public life as well as for training skills of application of mathematics in practical problems.

The everlasting duty of the teacher is to destroy the obstacles to the understanding of mathematics, reveal the liberating essence of its free thinking, and explain that MATHEMATICS IS THE MOST HUMAN OF ALL HUMAN SCIENCES. There is no math without a man or a woman. The physical world still prevails but math vanishes without men and women. We people do math. We do it, thinking about everyone and we do it for everybody. The purpose and essence of mathematics reside in the freedom it brings to us.

Mathematics welcomes everyone, combining free access, democracy, and openness with the indisputable prohibition of any prejudice, subjectiveness, and arbitrariness of judgements.

One of the most personalized sciences which requires everybody’s personal effort for solving a however simple arithmetical problem, mathematics has learned to make the complex the simple and comprehensible to each of us.

The most human among sciences, mathematics has elaborated its beautiful “unhuman” form of the objective transfer of knowledge in writing—the classic style of the Hellenistic “Elements.”

There are no King’s ways to mathematics; the road to mathematics was charted by Euclid. The style of Euclid not only lives in the books by Bourbaki but also proliferates in hundreds of thousands of students’ notes throughout the world. This style is an achievement and article of pride of our ancient science.

Sobolev Institute of Mathematics
Novosibirsk State University
E-mail address: sskut@member.ams.org