Computing the lines of a smooth cubic surface

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Abstract

We give an explicit formula for the 27 lines of a smooth cubic surface near the Fermat surface. Our formula involves convergent power series with coefficients in the extension of rational numbers with the sixth root of unity. Our main tool is the Artinian Gorenstein ring of socle two attached to such lines.

1 Introduction

One of the most well-known and beautiful objects in classical algebraic geometry is the 27 lines of a smooth cubic surface $X$. This has been partially justified by plasters of cubic surfaces with their lines made by mathematicians in the 19th century. For historical account on this and the visualization of cubic surface with their lines see [vSL03]. Algorithms to compute such lines are mainly based on brute force substitution of equations of lines in the equation of $X$. As the Hodge decomposition of the second cohomology of $X$ consists only of the middle piece, the study of lines of $X$ using Hodge theory might seem hopeless. However, it turns out that the Artinian Gorenstein ring attached to Hodge cycles originated from the works of P. Griffiths in 1970’s and further elaborated in [Voi89, Otw03, Dan17, MV18, MV21] can be useful in order to write down the equation of simple algebraic cycles like lines, using two dimensional periods of $X$. This idea has been explained in [MS21] which uses approximation of periods with a high precision in [Ser19]. Near to the Fermat variety the author in [Mov21] has written down explicit formulas for the Taylor series of such periods which leads us to the main result Theorem 1 of the present paper.

We write down a cubic surface in the format

\[
X_t : F_t := x_0^3 + x_1^3 + x_2^3 + x_3^3 - \sum_{i \in I} t_i x_i^3 = 0, \quad t := (t_i, i \in I) \in T := \mathbb{C}^{20}\{\Delta = 0\},
\]

where $I$ is the set of exponents of monomials of degree 3 in four variables $x_0, x_1, x_2, x_3$ and $\Delta = 0$ is the loci of singular cubic surfaces. We have written this as perturbation of the Fermat surface $X_0$, as our main computations is done in a neighborhood of Fermat. For $\beta \in \mathbb{N}_0^4$ we denote by $\beta_i$ its $(i + 1)$-th coordinate, that is, $\beta = (\beta_0, \beta_1, \beta_2, \beta_3)$, and for $n \in \mathbb{Z}$, $\bar{n} \in \mathbb{N}_0$ is defined by the rules $0 \leq \bar{n} \leq 2$, $n \equiv_3 \bar{n}$. For a positive rational number $r$, $[r]$ is the integer part of $r$, that is $[r] \leq r < [r] + 1$, $\{r\} := r - [r]$ and $\langle r \rangle = (r - 1)(r - 2) \cdots (r - \lfloor r \rfloor)$ (and hence $\langle r \rangle = 0$ if $r \in \mathbb{N}$).

We consider the set of $(m, n, 1, \zeta_1, \zeta_2)$, where $(m, n, 1) = (1, 2, 3), (2, 1, 3), (3, 1, 2)$ and $\zeta_1, \zeta_2$ are roots of $-1$, that is, $\zeta_1^3 = \zeta_2^3 = -1$. This set consists of 27 elements. In this article we prove the following theorem:

**Theorem 1.** For the twenty seven choice of $k = (m, n, 1, \zeta_1, \zeta_2)$ as above we have the following rational curve inside $X_t$:

\[
\mathbb{P}^1_{k,t} : \left\{ \begin{array}{l}
c_{0212} \cdot x_0 - c_{0202} \cdot x_1 + c_{0201} \cdot x_2 + 0 \cdot x_3 = 0 \\
c_{0223} \cdot x_0 + 0 \cdot x_1 - c_{0203} \cdot x_2 + c_{0202} \cdot x_3 = 0
\end{array} \right.,
\]

where

\[
c_{i_1 i_2 j_1 j_2} = \det \begin{bmatrix}
p_{i_1 j_1} & p_{i_1 j_2} \\
p_{i_2 j_1} & p_{i_2 j_2}
\end{bmatrix},
\]

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Let $X \subset \mathbb{P}^3$ be a smooth cubic surface given by the homogeneous polynomial $F$ of degree 3. For a homogeneous polynomial $P$ of degree 2 in $x_0, x_1, x_2, x_3$ define
\[
\omega_P := \text{Res}_i \left( \frac{P \cdot \sum_{i=0}^{3} (-1)^i x_i \, dx_i}{f_t^2} \right) \in H^2_{\text{dR}}(X),
\]
where $\text{Res}_i : H^3(\mathbb{P}^3 \setminus X) \to H^2_{\text{dR}}(X)$ is the Griffiths residue map, see for instance [MV21 Chapter 7]. The following theorem tells us how to recover the equations of $\mathbb{P}^1$ using its periods.

**Proposition 1.** Let $\mathbb{P}^1$ be a line inside $X$. The $4 \times 4$ matrix
\[
A = \left[ \int_{\mathbb{P}^1} \omega_{x_i x_j} \right]_{0 \leq i, j \leq 3}
\]
is of rank two and its kernel is generated by $a_i := (a_{1,i}, a_{2,i}, a_{3,i}, a_{4,i}), \ i = 1, 2$, where $a_{1,i}x_0 + a_{2,i}x_1 + a_{3,i}x_2 + a_{4,i}x_3 = 0, \ i = 1, 2$ are two linear equations of $\mathbb{P}^1$.

**Proof.** The homology $H_2(X, \mathbb{Z})$ is of rank 7 and all cycles $\delta \in H_2(X, \mathbb{Z})$ are Hodge cycles. Let $[Z_\infty]$ be the homology class of the hyperplane section. For every Hodge cycle $\delta \in H_2(X, \mathbb{Z})/\mathbb{Z}[Z_\infty]$ we define its associated Artinian Gorenstein ideal $I(\delta) \subset \mathbb{C}[x]$ and the corresponding Artinian Gorenstein algebra $R(\delta) := \mathbb{C}[x]/I(\delta)$ which is of socle 2. We have $I(\delta)_0 = 0, I(\delta)_a = \mathbb{C}[x]_a, \ a \geq 3$ and for $a = 1, 2$:
\[
I(\delta)_a := \left\{ Q \in \mathbb{C}[x]_a \left| \int_{\delta} \omega_{PQ} = 0, \ \forall P \in \mathbb{C}[x]_{2-a} \right. \right\}.
\]
If $\delta = [\mathbb{P}^1]$, $\mathbb{P}^1 := \{f_1 = f_2 = 0\} \subset X$ and $f_1, f_2$ are homogeneous degree one polynomials then we have $F = f_1 g_1 + f_2 g_2$ for some degree two homogeneous polynomials $g_1, g_2$ and it turns out that

$$\langle f_1, f_2, g_1, g_2 \rangle = I(\delta)$$

as we have the inclusion $\subset$ and both ideals are of the same socle 2. This idea comes originally from [Dan17] and has been further elaborated in [MV21, Chapter 11]. Therefore, $I(\delta)_{1} = \mathbb{C} f_1 + \mathbb{C} f_2$, that is, we can recover the ideal of $\mathbb{P}^1$ from its Artinian Gorenstein ideal $I(\delta)_{1}$. It follows that the matrix $A$ is of rank two and the two linearly independent equations of $\mathbb{P}^1 \subset \mathbb{P}^3$ are give by $a_i x = 0$, where $a_i, i = 1, 2$ are two linearly independent vectors in the kernel of $A$.

**Proof of Theorem**\textsuperscript{a}. We consider the family (2) and the the matrix $A = A(t)$ in (6) has entries which are holomorphic functions in $t \in (T, 0)$. The matrix $A(t)$ evaluated at the Fermat point $t = 0$ is

$$A(0) = \frac{2 \pi \sqrt{-1} \zeta_1 \zeta_2}{9} \begin{bmatrix} 0 & 0 & \zeta_1 \zeta_2 & \zeta_1 \\ 0 & 0 & \zeta_2 & 1 \\ \zeta_1 \zeta_2 & \zeta_2 & 0 & 0 \\ \zeta_1 & 1 & 0 & 0 \end{bmatrix}.$$\textsuperscript{b}

This has been calculated on [MV21, Theorem 1] and it implies that the first and third row of $A(t)$ are linear independent for $t \in (T, 0)$. We have to find two vectors perpendicular to these two vectors. For this we use the fact that the determinant of the $3 \times 3$ minors of $A(t)$ formed by rows 0, 2, 1 and columns 0, 1, 2 (resp. rows 0, 2, 3 and columns 0, 2, 3) are zero. We get two vectors corresponding to coefficients of (2). Note that $c_{0202}(0) \neq 0$.

For the Fermat variety $X_0$, the twenty seven lines are given by

$$\mathbb{P}^1_{k,0} := \begin{cases} x_0 - \zeta_1 x_m = 0, \\ x_n - \zeta_2 x_1 = 0, \end{cases} \quad \zeta_1 = \zeta_2 = -1, \{0, m, n, 1\} = \{0, 1, 2, 3\}, \quad n < 1.$$\textsuperscript{c}

For $t \in T$ near to the Fermat point 0, there is a unique rational curve $\mathbb{P}^1_{k,t}$ which is obtained by deformation of $\mathbb{P}^1_{k,0}$. We want to compute its equations. Its homology class $\delta_t = [\mathbb{P}^1_{k,t}] \in H_2(X_t, \mathbb{Z})$ is the monodromy (parallel transport) of the homology class of $\mathbb{P}^1_{k,0}$. The Taylor series of integration of $\omega_{x_i x_j}$ over $\delta_t$ at $t = 0$ is computed in [Mov21, §18.3] and we have

$$\frac{-6}{2 \pi \sqrt{-1}} \int_{\delta_t} \omega_{x_i x_j} = p_{ij}, \quad i, j = 0, 1, 2, 3,$$

where $p_{ij}$ is the power series (3). This together with Proposition 6 finishes the proof.\textsuperscript{d}

**Remark 1.** For the family (2) define

$$P_{ij}(y) := \prod_{k=1}^{27} (y - \int_{\mathbb{P}^1_{k,t}} \omega_{ij}) = y^{27} + \sum_{i=1}^{27} \tilde{p}_i(t) y^{27-i},$$

where $\mathbb{P}^1_{k,t}$, $k = 1, 2, \ldots, 27$ are lines of $X_t$. The coefficients $\tilde{p}_{i}$ of this polynomial in $y$ are rational functions in $t$ with coefficients in $\mathbb{Q}$ and poles along the discriminant. This follows from the finite growth of integrals near degeneration points $\Delta = 0$, see [AGZV88, Chapter 13], and simple Galois action argument. The computations of these coefficients seems to be out of reach even for the following family written in the affine chart $x_0 = 1$:

$$X_t : \quad f_i := x_1^3 + x_2^3 + x_3^3 - \sum_{i \in I} t_i x^i = 0, \quad t := (t_i, i \in I) \in T := \mathbb{C}^{10} \setminus \{\Delta = 0\},$$

3
where $x^i$'s are monomials of degree $\leq 2$ in $x_1, x_2, x_3$. This is a tame polynomial in the sense of [Mov21, Chapter 10]. We consider the $\mathbb{C}^*$-action

$$
\mathbb{C}^3 \times \mathbb{C}^* \to \mathbb{C}^3, \ ((x_1, x_2, x_3), a) \to (a^{-1}x_1, a^{-1}x_2, a^{-1}x_3)
$$

which induces an action in $T$:

$$(t_\alpha, \alpha \in I) \cdot a = (t_\alpha ^{\deg(x^\alpha)} a, \alpha \in I)$$

and an isomorphism $h : X_{t\bullet a} \to X_t$ such that for a monomial $x^\beta$ with $\deg(x^\beta) \leq 2$, we have $h^* \frac{x^\beta dx}{f_t^2} = a^{3-\deg(x^\beta)} \frac{x^\beta dx}{f_t^2}$. We conclude that

$$(9) \int_{\delta_{t\bullet a}} \frac{x^\beta dx}{f_t^2} = a^{\deg(x^\beta) - 3} \int_{\delta_t} \frac{x^\beta dx}{f_t^2}$$

which implies that this integral is homogeneous of negative degree $\deg(x^\beta) - 3$ in variables $t_\alpha, \deg(t_\alpha) = 3 - \deg(x^\alpha)$. The discriminant $\Delta$ is a homogeneous polynomial of degree $(d - 1)^{n+1}d = 2^33 = 24$, see [Mov21, Section 10.9]. The pole order of $\tilde{p}_i$ along $\Delta$ is at most $2i$, see [Mov21, Equality 10.25 and Proposition 11.2]. If this is the case then $\tilde{p}_i = \frac{p_i}{\Delta^2}$, $\tilde{p}_i$ a homogeneous polynomial, and $\deg(p_i) - 2i \deg(\Delta) = i(\deg(x^\beta) - 3)$. Therefore, $p_i$ is of degree $(45 + \deg(x^\beta))i$. In theory, one can use the formal power series $p_{ij}$ in (3) and compute the polynomials $p_i$, however, in practice this is out of the capacity of the author’s simple computer code written in Singular. This can be found in the tex file of the present text in arxiv.

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