DERIVED EQUIVALENCES BETWEEN ASSOCIATIVE DEFORMATIONS

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Abstract. We prove that if two associative deformations (parameterized by the same complete local ring) are derived Morita equivalent, then they are Morita equivalent (in the classical sense).

0. Introduction

Let $K$ be a commutative ring, and let $A$ and $B$ be associative unital $K$-algebras. We denote by $\text{Mod} A$ and $\text{Mod} B$ the corresponding categories of left modules. One says that $A$ and $B$ are Morita equivalent relative to $K$ (in the classical sense) if there is a $K$-linear equivalence of categories $\text{Mod} A \rightarrow \text{Mod} B$.

Let $D^b(\text{Mod} A)$ denote the bounded derived category of complexes of left $A$-modules. This is a $K$-linear triangulated category. If there is a $K$-linear equivalence of triangulated categories $D^b(\text{Mod} A) \rightarrow D^b(\text{Mod} B)$, then one says that $A$ and $B$ are derived Morita equivalent relative to $K$.

There are plenty of examples of pairs of algebras that are derived Morita equivalent, but are not Morita equivalent in the classical sense.

Now suppose $K$ is a complete noetherian local ring, with maximal ideal $m$ and residue field $k$. Let $A$ be a flat $m$-adically complete $K$-algebra, such that the $k$-algebra $\bar{A} := k \otimes_K A$ is commutative. We then say that $A$ is an associative $K$-deformation of $\bar{A}$; see [Ye3].

The most important example of an associative deformation is when $k = \mathbb{R}$; $\bar{A} = C^\infty(X)$, the $\mathbb{R}$-algebra of smooth functions on a differentiable manifold $X$; $K = \mathbb{R}[[\hbar]]$, the ring of formal power series in the variable $\hbar$; and $A = \bar{A}[[\hbar]]$. In this case the multiplication in $A$ is called a star product.

Let us assume that $A$ and $B$ are associative $K$-deformations, and moreover the commutative rings $\bar{A}$ and $\bar{B}$ have connected prime spectra (i.e. they have no non-trivial idempotents). The main result of the paper (Theorem 2.7) says that if $T$ is a two-sided tilting complex over $B-A$ relative to $K$, then $T \cong P[n]$ for some invertible bimodule $P$ and integer $n$. (Tilting complexes and their properties are recalled in Section 1.) A direct consequence (Corollary 2.8) is that if $A$ and $B$ are derived Morita equivalent, then they are Morita equivalent in the classical sense.

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1. Base Change for Tilting Complexes

In this section we recall some facts about two-sided tilting complexes, and also prove one new theorem. Throughout this section \(K\) is a commutative ring. By “\(K\)-algebra” we mean an associative unital algebra; i.e. a ring \(A\), with center \(\mathbb{Z}(A)\), together with a ring homomorphism \(\mathbb{K} \to \mathbb{Z}(A)\).

For a \(K\)-algebra \(A\) we denote by \(A^{\text{op}}\) the opposite algebra, namely with reverse multiplication. We view right \(A\)-modules as left \(A^{\text{op}}\)-modules. Let \(B\) be some other \(K\)-algebra. By \(B\)-\(A\)-bimodule relative to \(K\) we mean a \(K\)-central \(B\)-\(A\)-bimodule. We view \(B\)-\(A\)-bimodules relative to \(K\) as left \(B \otimes_K A^{\text{op}}\)-modules.

The category of left \(A\)-modules is denoted by \(\text{Mod} A\). This is a \(K\)-linear abelian category. Classical Morita theory says that any \(K\)-linear equivalence \(\text{Mod} A \to \text{Mod} B\) is of the form \(P \otimes_A \_\), where \(P\) is some invertible \(B\)-\(A\)-bimodule relative to \(K\).

The derived category of \(\text{Mod} A\) is \(D(\text{Mod} A)\). This is a \(K\)-linear triangulated category. We follow the conventions of [RD] on derived categories. For instance, \(D^b(\text{Mod} A)\) is the full subcategory of \(D(\text{Mod} A)\) consisting of bounded complexes.

Here is a definition from Rickard’s paper [Ri1].

**Definition 1.1.** Let \(A\) and \(B\) be \(K\)-algebras. If there exists a \(K\)-linear equivalence of triangulated categories \(D^b(\text{Mod} A) \to D^b(\text{Mod} B)\) then we say that \(A\) and \(B\) are derived Morita equivalent relative to \(K\).

Now assume that \(A\) is flat over \(K\). Since \(A \otimes_K B\) is flat over \(B\), it follows that the forgetful functor \(\text{Mod} A \otimes_K B \to \text{Mod} B\) sends flat modules to flat modules.

Given three \(K\)-algebras \(A, B, C\), and complexes \(M \in D^-(\text{Mod} A \otimes_K B^{\text{op}})\) and \(N \in D^-(\text{Mod} B \otimes_K C^{\text{op}})\), and assuming \(A\) is flat over \(K\), the derived tensor product

\[
M \otimes^L_B N \in D^-(\text{Mod} A \otimes_K C^{\text{op}})
\]

can be defined as follows: choose a quasi-isomorphism \(P \to M\) with \(P\) a bounded above complex of projective \(A \otimes_K B^{\text{op}}\)-modules. Then \(P\) is a bounded above complex of flat \(B^{\text{op}}\)-modules, and we take

\[
M \otimes^L_B N := P \otimes_B N.
\]

This operation is functorial in \(M\) and \(N\). As usual the requirements can be relaxed: it is enough to resolve \(M\) by a bounded above complex \(P\) of bimodules that are flat over \(B^{\text{op}}\). If \(C\) is flat over \(K\) then we can resolve \(N\) instead of \(M\). The derived tensor product \(M \otimes^L_B N\) is “indifferent” to the algebras \(A\) and \(C\); we can forget them before or after calculating \(M \otimes^L_B N\), and get the same answer in \(D^-(\text{Mod} K)\).

We record the following useful technical results.

**Lemma 1.2** (Projective truncation trick). Let \(M \in D(\text{Mod} A)\) and let \(i_0\) be an integer. Suppose that \(H^i M = 0\) for all \(i > i_0\), and \(P := H^{i_0} M\) is a projective \(A\)-module. Then there is an isomorphism \(M \cong P[-i_0] \oplus N\) in \(D(\text{Mod} A)\), where \(N\) is a complex satisfying \(N^i = 0\) for all \(i \geq i_0\).

**Proof.** By the usual truncation trick (cf. [RD] Section I.7) we can assume that \(M^i = 0\) for all \(i > i_0\). Hence we get an exact sequence \(M^{i_0-1} \xrightarrow{d} M^{i_0} \to P \to 0\). But \(P\) is projective, and therefore \(M^{i_0} \cong P \oplus \text{Ker}(d) \subseteq M^{i_0-1}\). Define \(N^{i_0} := \text{Ker}(d) \subseteq M^{i_0-1}\) and \(N^i := M^i\) for \(i < i_0\) and \(i = i_0 - 1\). \(\square\)

Recall that a complex \(M \in D(\text{Mod} A)\) is called perfect if it is isomorphic to bounded complex of finitely generated projective modules. We denote by \(D(\text{Mod} A)_{\text{perf}}\) the full subcategory of perfect complexes.

**Lemma 1.3.** Let \(M \in D(\text{Mod} A)_{\text{perf}}\) and let \(i_0\) be an integer. If \(H^i M = 0\) for all \(i > i_0\), then the \(A\)-module \(N := H^{i_0} M\) is finitely presented.
Proof. This is a bit stronger then [Ye1 Lemma 1.1(2)]. By truncation reasons we can assume that $M \cong P$, where $P$ is a bounded complex of finitely generated projective $A$-modules, and $P^i = 0$ for $i > i_0$. So we get an exact sequence $P^{i_0 - 1} \to P^{i_0} \to N \to 0$. Suppose $P^{i_0}$ is a direct summand of $A^r$ (the free module of rank $r$), and $P^{i_0 - 1}$ is a direct summand of $A^s$. Then be rearranging terms we get an exact sequence $A^{r+s} \to A^r \to N \to 0$. □

**Lemma 1.4** (Küneth trick). Let $A$ be a $\mathbb{K}$-algebra, let $M \in D^{-}(\text{Mod } A^{\text{op}})$ and let $N \in D^{-}(\text{Mod } A)$. Let $i_0, j_0 \in \mathbb{Z}$ be such that $H^i M = 0$ and $H^j N = 0$ for all $i > i_0$ and $j > j_0$. Then

$$(H^i_0 M) \otimes_A (H^{j_0} N) \cong H^{i_0+j_0}(M \otimes^L_A N)$$

as $\mathbb{K}$-modules.

**Proof.** See [Ye1 Lemma 2.1]. □

The next definition is from [Ri2].

**Definition 1.5.** Let $A$ and $B$ be flat $\mathbb{K}$-algebras. A two-sided tilting complex over $B-A$ relative to $\mathbb{K}$ is a complex $T \in D^b(\text{Mod } B \otimes_{\mathbb{K}} A^{\text{op}})$ with the following property:

$(\ast)$ there exists a complex $S \in D^b(\text{Mod } A \otimes_{\mathbb{K}} B^{\text{op}})$, and isomorphisms $S \otimes^L_B T \cong_A A$ and $T \otimes^L_A S \cong_B B$ in $D^b(\text{Mod } A \otimes_{\mathbb{K}} A^{\text{op}})$ and $D^b(\text{Mod } B \otimes_{\mathbb{K}} B^{\text{op}})$ respectively.

The complex $S$ is called an inverse of $T$.

In case $B = A$ we say that $T$ is a two-sided tilting complex over $A$ relative to $\mathbb{K}$.

The inverse $S$ in the definition is unique up to isomorphism in $D^b(\text{Mod } A \otimes_{\mathbb{K}} B^{\text{op}})$. Of course $S$ is a two-sided tilting complex over $A-B$ relative to $\mathbb{K}$.

A two-sided tilting complex $T$ induces a $\mathbb{K}$-linear equivalence of triangulated categories

$$T \otimes^L_A - : D(\text{Mod } A) \to D(\text{Mod } B).$$

This functor restricts to equivalences

$$D^*(\text{Mod } A) \to D^*(\text{Mod } B),$$

where $\ast$ is either $+$, $-$ or $b$; and also to an equivalence

$$D(\text{Mod } A)_{\text{perf}} \to D(\text{Mod } B)_{\text{perf}}.$$

See [Ri2 or Ye1 Corollary 1.6(4)].

Conversely we have the next important result, due to Rickard [Ri2]. For alternative proofs see [Ke1 or Ye1 Corollary 1.9].

**Theorem 1.6** (Rickard). Let $A$ and $B$ be flat $\mathbb{K}$-algebras that are derived Morita equivalent relative to $\mathbb{K}$. Then there exists a two-sided tilting complex over $B-A$ relative to $\mathbb{K}$.

**Remark 1.7.** Suppose $F : D(\text{Mod } A) \to D(\text{Mod } B)$ is a $\mathbb{K}$-linear equivalence of triangulated categories. Then $F$ restricts to an equivalence between the subcategories of perfect complexes (cf. [Ke2]). This implies that $F$ has finite cohomological dimension (bounded by the amplitude of $HF(A)$). Hence $F$ restricts to an equivalence between the bounded derived categories – i.e. a derived Morita equivalence.

**Remark 1.8.** In our paper [Ye1] the base ring $\mathbb{K}$ is taken to be a field. However the results in Sections 1-3 of that paper hold for any commutative base ring $\mathbb{K}$, as long as the $\mathbb{K}$-algebras in question are flat.

It is possible to remove even the flatness condition, at the price of working with DG algebras. Here is how to do it: choose a DG $\mathbb{K}$-algebra $\tilde{A}$ such that $\tilde{A}^i = 0$ for $i > 0$ and every $\tilde{A}^i$ flat as $\mathbb{K}$-module, with a DG algebra quasi-isomorphism $\tilde{A} \to A$. We call $\tilde{A} \to A$ a flat DG algebra resolution of $A$ relative to $\mathbb{K}$. This can
be done (cf. [YZ, Section 1] for commutative $\mathbb{K}$-algebras). Likewise choose a flat DG algebra resolution $\tilde{B} \to B$.

Let $\mathcal{D}(\text{DGMod } A)^b$ be the derived category of DG $A$-modules with bounded cohomologies. It is known (cf. [YZ, Proposition 1.4]) that the restriction of scalars functor $\mathcal{D}^b(\text{Mod } A) \to \mathcal{D}(\text{DGMod } A)^b$ is an equivalence. Therefore a $\mathbb{K}$-linear equivalence $\mathcal{D}^b(\text{Mod } A) \to \mathcal{D}^b(\text{Mod } B)$ is the same as a $\mathbb{K}$-linear equivalence $\mathcal{D}(\text{DGMod } A)^b \to \mathcal{D}(\text{DGMod } B)^b$. Now the proof of [Ye1, Theorem 1.8] shows that there is a complex $T \in \mathcal{D}(\text{DGMod } B \otimes_\mathbb{K} A^{op})^b$ which is two-sided tilting.

A different choice of flat DG algebra resolutions $\tilde{A} \to A$ and $\tilde{B} \to B$ will give rise to an equivalent triangulated category $\mathcal{D}(\text{DGMod } B \otimes_\mathbb{K} A^{op})^b$. In this sense two-sided tilting complexes are independent of the resolutions.

See Remark 1.11 for the history of the next theorem.

**Theorem 1.9.** Let $A$ and $B$ be flat $\mathbb{K}$-algebras. Assume $A$ is commutative with connected spectrum. Let $T$ be a two-sided tilting complex over $B\text{-}A$ relative to $\mathbb{K}$. Then there is an isomorphism

$$T \cong P[n]$$

in $\mathcal{D}^b(\text{Mod } B \otimes_\mathbb{K} A^{op})$ for some invertible $B\text{-}A$-bimodule $P$ and integer $n$.

**Proof.** We may assume that $A \neq 0$, so that $T \neq 0$. The complex $T$ is perfect over $B$ and over $A^{op}$ (cf. [Ye1, Theorem 1.6]). As in [Ye1, Proposition 2.4] the complex $T$ induces a $\mathbb{K}$-algebra isomorphism $A \cong \mathbb{Z}(B)$.

Let

$$n := \sup \{ i \mid H^iT \neq 0 \},$$

and let $P := H^{-n}T$. This is a $B$-$A$-bimodule. By Lemma 1.3 $P$ is finitely presented as right $A$-module.

For a prime $p \in \text{Spec } A$, with corresponding local ring $A_p$, we write $P_p := P \otimes_A A_p$. Define $Y \subset \text{Spec } A$ to be the support of $P$, i.e.

$$Y := \{ p \in \text{Spec } A \mid P_p \neq 0 \}.$$

Since $P$ is finitely generated it follows that $Y$ is a closed subset of $\text{Spec } A$.

Take any prime $p \in Y$, and let $B_p := B \otimes_A A_p$. Then, by [Ye1, Lemma 2.5], the complex

$$T_p := B_p \otimes_B T \otimes_A A_p \in \mathcal{D}^b(\text{Mod } B_p \otimes_\mathbb{K} A_p^{op})$$

is a two-sided tilting complex over $B_p\text{-}A_p$. Since

$$H^{-n}T_p \cong P_p \neq 0,$$

[Ye1, Theorem 2.3] implies that

$$T_p \cong P_p[n] \in \mathcal{D}^b(\text{Mod } B_p \otimes_\mathbb{K} A_p^{op}).$$

Thus $P_p$ is an invertible $B_p\text{-}A_p$-bimodule. This implies that $P_p$ is a free $A_p$-module, of rank $r > 0$. According to [CA, Section II.5.1, Corollary] there is an open neighborhood $U$ of $p$ in $\text{Spec } A$ on which $P$ is free of rank $r$. In particular $P_q \neq 0$ for all $q \in U$. Therefore $U \subset Y$.

The conclusion is that $Y$ is also open in $\text{Spec } A$. Since $\text{Spec } A$ is connected it follows that $Y = \text{Spec } A$. Another conclusion is that $P$ is projective as $A$-module – see [CA, Section II.5.2, Theorem 1].

Going back to equation (1.10) we see that $(H^iT)_p \cong H^iT_p = 0$ for all $i \neq -n$. Therefore $H^iT = 0$ for $i \neq -n$. By truncation we get an isomorphism $T \cong P[n]$ in $\mathcal{D}^b(\text{Mod } B \otimes_\mathbb{K} A^{op})$. Finally by [Ye1] Proposition 2.2 the $B\text{-}A$-bimodule $P$ is invertible. \hfill \square
Remark 1.11. Theorem [K9] (for a field $\mathbb{K}$) is [Ye1, Theorem 2.6]. However the proof there is only correct when $A$ is noetherian (the hidden assumption is that $\text{Spec } A$ is a noetherian topological space).

The same result was proved independently (and pretty much simultaneously, i.e. circa 1997) by Rouquier and Zimmermann [RZ].

Corollary 1.12. Let $A$ and $B$ be flat $\mathbb{K}$-algebras with $A$ commutative. If $A$ and $B$ are derived Morita equivalent relative to $\mathbb{K}$, then they are Morita equivalent relative to $\mathbb{K}$.

Proof. Use the first paragraph in the proof of [Ye1, Theorem 2.6] to pass to the case when $\text{Spec } A$ is connected, and then apply Theorem [K9]. \square

We denote by $\text{Pic}_\mathbb{K}(A)$ the noncommutative Picard group of $A$, consisting of isomorphism classes of invertible $A$-$A$-bimodules relative to $\mathbb{K}$. The operation is $- \otimes_A -$. Here is a definition from [Ye1] extending this notion to the derived setting:

Definition 1.13. Let $A$ be a flat $\mathbb{K}$-algebra. The derived Picard group of $A$ relative to $\mathbb{K}$ is

$$\text{DPic}_\mathbb{K}(A) := \frac{\text{two-sided tilting complexes over } A \text{ relative to } \mathbb{K}}{\text{isomorphism}},$$

where isomorphism is in $\text{D}^b(\text{Mod } A \otimes_{\mathbb{K}} A^{\text{op}})$. The operation is $- \otimes^L_A -$ , and the unit element is the bimodule $A$.

There is a canonical injective group homomorphism

$$\text{Pic}_\mathbb{K}(A) \times \mathbb{Z} \to \text{DPic}_\mathbb{K}(A).$$

It formula is $(P, n) \mapsto P[n]$.

Remark 1.14. When $A$ is either local, or commutative with connected spectrum, the homomorphism above is in fact bijective. On the other hand, if $A$ is the algebra of upper triangular $n \times n$ matrices over $\mathbb{K}$, then the bimodule $A^* := \text{Hom}_\mathbb{K}(A, \mathbb{K})$ is a two-sided tilting complex that does not belong to $\text{Pic}_\mathbb{K}(A) \times \mathbb{Z}$. This is a sort of “Calabi-Yau” phenomenon. See [Ye1] for details.

Let $A$ and $B$ be $\mathbb{K}$-algebras, and let $P$ be an invertible $B$-$A$-bimodule relative to $\mathbb{K}$. Let $\mathbb{K}'$ be any commutative $\mathbb{K}$-algebra, and define $A' := \mathbb{K}' \otimes_{\mathbb{K}} A$, $B' := \mathbb{K}' \otimes_{\mathbb{K}} B$, and $P' := \mathbb{K}' \otimes_{\mathbb{K}} P$. Then $P'$ is an invertible $B'$-$A'$-bimodule relative to $\mathbb{K}'$. When we take $B = A$ this fact gives rise to a group homomorphism

$$\text{Pic}_\mathbb{K}(A) \to \text{Pic}_{\mathbb{K}'}(A').$$

For the derived version we need flatness. The next theorem is the only new result in this section of the paper.

Theorem 1.15. Let $A, B, C$ be flat $\mathbb{K}$-algebras, and let $\mathbb{K}'$ be a commutative $\mathbb{K}$-algebra. Define $A' := \mathbb{K}' \otimes_{\mathbb{K}} A$, $B' := \mathbb{K}' \otimes_{\mathbb{K}} B$ and $C' := \mathbb{K}' \otimes_{\mathbb{K}} C$. Given complexes $M \in \text{D}^-(\text{Mod } A \otimes_{\mathbb{K}} B^{\text{op}})$ and $N \in \text{D}^-(\text{Mod } B \otimes_{\mathbb{K}} C^{\text{op}})$, let us define

$$M' := \mathbb{K}' \otimes^L_{\mathbb{K}} M \in \text{D}^-(\text{Mod } A' \otimes_{\mathbb{K}'} B'^{\text{op}})$$

and

$$N' := \mathbb{K}' \otimes^L_{\mathbb{K}} N \in \text{D}^-(\text{Mod } B' \otimes_{\mathbb{K}'} C'^{\text{op}}).$$

Then there is an isomorphism

$$M' \otimes^L_{B'} N' \cong \mathbb{K}' \otimes^L_{\mathbb{K}} (M \otimes^L_{B} N)$$

in $\text{D}^-(\text{Mod } A \otimes_{\mathbb{K}} C^{\text{op}})$, functorial in $M$ and $N$. 

Proof. First let us observe that $A \otimes_\mathbb{K} B^{\text{op}}$ is a flat $\mathbb{K}$-algebra, and
\[ A' \otimes_\mathbb{K} B'^{\text{op}} \cong \mathbb{K}' \otimes_\mathbb{K} (A \otimes_\mathbb{K} B^{\text{op}}) \]
as $\mathbb{K}'$-algebras.
Choose an isomorphism $M \cong P$ in $D^- (\text{Mod} \ A \otimes_\mathbb{K} B^{\text{op}})$, where each $P^i$ is projective over $A \otimes_\mathbb{K} B^{\text{op}}$. Then
\[ M' \cong \mathbb{K}' \otimes_\mathbb{K} P \in D^- (\text{Mod} \ A' \otimes_\mathbb{K} B'^{\text{op}}), \]
and each $\mathbb{K}' \otimes_\mathbb{K} P^i$ is flat over $A'$ and over $B'^{\text{op}}$.
Similarly let us choose an isomorphism $N \cong Q$ in $D^- (\text{Mod} \ B \otimes_\mathbb{K} C^{\text{op}})$; so $N' \cong \mathbb{K}' \otimes_\mathbb{K} Q$.
Now
\[ M' \otimes_B' N' \cong (\mathbb{K}' \otimes_\mathbb{K} P) \otimes_B' (\mathbb{K}' \otimes_\mathbb{K} Q) \]
in $D^- (\text{Mod} \ A' \otimes_\mathbb{K} C'^{\text{op}})$. There is a canonical isomorphism
\[ (\mathbb{K}' \otimes_\mathbb{K} P) \otimes_B' (\mathbb{K}' \otimes_\mathbb{K} Q) \cong \mathbb{K}' \otimes_\mathbb{K} (P \otimes_B Q) \]
as complexes of $A' \otimes_\mathbb{K} C'^{\text{op}}$-modules; and therefore this is also an isomorphism in $D^- (\text{Mod} \ A' \otimes_\mathbb{K} C'^{\text{op}})$.
Next we have
\[ M \otimes_B^L N \cong P \otimes_B Q \]
in $D^- (\text{Mod} \ A \otimes_\mathbb{K} C^{\text{op}})$. But since $P \otimes_B Q$ is a complex of flat $\mathbb{K}$-modules, we also have
\[ \mathbb{K}' \otimes_\mathbb{K} (M \otimes_B^L N) \cong \mathbb{K}' \otimes_\mathbb{K} (P \otimes_B Q) \]
in $D^- (\text{Mod} \ A' \otimes_\mathbb{K} C'^{\text{op}})$. \[ \square \]

Corollary 1.16. Let $A$ and $B$ be flat $\mathbb{K}$-algebras, and let $\mathbb{K}'$ be a commutative $\mathbb{K}$-algebra. Define $A' := \mathbb{K}' \otimes_\mathbb{K} A$ and $B' := \mathbb{K}' \otimes_\mathbb{K} B$. Suppose $T$ is a two-sided tilting complex over $B-A$ relative to $\mathbb{K}$, with inverse $S$. Define
\[ T' := \mathbb{K}' \otimes_\mathbb{K} T \in D^b (\text{Mod} \ B' \otimes_\mathbb{K} A'^{\text{op}}) \]
and
\[ S' := \mathbb{K}' \otimes_\mathbb{K} S \in D^b (\text{Mod} \ A' \otimes_\mathbb{K} B'^{\text{op}}). \]
Then $T'$ is a is a two-sided tilting complex over $B'-A'$ relative to $\mathbb{K}'$, with inverse $S'$.

Proof. By the theorem we have
\[ T' \otimes_B^{L} S' \cong \mathbb{K}' \otimes_\mathbb{K} (T \otimes_B^L S) \cong \mathbb{K}' \otimes_\mathbb{K} B \cong B' \]
in $D^b (\text{Mod} \ B' \otimes_\mathbb{K} B'^{\text{op}})$; and similarly $S' \otimes_B^{L} T' \cong A'$.
\[ \square \]

Corollary 1.17. Let $A$ be a flat $\mathbb{K}$-algebra, and let $\mathbb{K}'$ be a commutative $\mathbb{K}$-algebra. Define $A' := \mathbb{K}' \otimes_\mathbb{K} A$. Then the formula $T \mapsto \mathbb{K}' \otimes_\mathbb{K} T$ defines a group homomorphism
\[ \text{DPic}_\mathbb{K} (A) \rightarrow \text{DPic}_\mathbb{K} (A'). \]

Proof. Immediate from the previous corollary. \[ \square \]
2. Associative Deformations

In this section we keep the following setup:

**Setup 2.1.** \( K \) is a complete local noetherian commutative ring, with maximal ideal \( m \) and residue field \( k = K / m \).

Let \( M \) be a \( K \)-module. Its \( m \)-adic completion is the \( K \)-module

\[
\hat{M} := \lim_{\longrightarrow i} M / m^i M.
\]

Recall that \( M \) is called \textit{\( m \)-adically complete} (some texts, e.g. [CA], use the term “separated and complete”) if the canonical homomorphism \( M \to \hat{M} \) is bijective. Every finitely generated \( K \)-module is complete; but this is not true for infinitely generated modules. For instance, if \( N \) is a free \( K \)-module of infinite rank, and if the ideal \( m \) is not nilpotent, then the canonical homomorphism \( N \to \hat{N} \) in injective but not surjective. Still in this instance the induced homomorphism \( k \otimes_k \hat{N} \to k \otimes_k \hat{N} \) is bijective. See [Ye2, Theorem 1.12].

In [Ye2] Corollary 2.12] we prove that a \( K \)-module \( M \) is flat and \( m \)-adically complete if and only if \( M \cong \hat{N} \) for some free \( K \)-module \( N \).

Sometimes one is given a ring homomorphism \( k \to \hat{k} \) lifting the canonical surjection \( K \to k \); and then \( K \) becomes a \( k \)-algebra. In this case the free \( K \)-module \( N \) can be expresses as \( N = K \otimes_k V \) for some \( k \)-module \( V \); and its completion is \( M = \hat{N} = K \otimes_k \hat{V} \). Moreover \( V \cong k \otimes_k N \cong k \otimes_k M \) as \( k \)-modules.

**Example 2.2.** Take \( K := \langle k[\hbar] \rangle \), the power series ring in the variable \( \hbar \) over the field \( k \). The maximal ideal \( m \) is generated by \( \hbar \). For a \( k \)-module \( V \) we have a canonical isomorphism \( k[\hbar] \otimes_k \hat{V} \cong V[[\hbar]] \), the latter being set of formal power series with coefficients in \( V \).

The next definition is used in [Ye3]:

**Definition 2.3.** Let \( A \) be a flat \( \mathfrak{m} \)-adically complete \( K \)-algebra, such that the \( k \)-algebra \( \hat{A} := k \otimes_K A \) is commutative. Then we call \( A \) an \textit{associative \( K \)-deformation of} \( \hat{A} \).

If \( K \) is a \( k \)-algebra then we can find a (noncanonical) isomorphism of \( K \)-modules \( A \cong K \otimes_k \hat{A} \). The multiplication induced on \( K \otimes_k \hat{A} \) by such an isomorphism is called a \textit{star product}.

**Example 2.4.** Suppose \( \hat{A} \) is some commutative \( k \)-algebra, and \( K = k[\hbar] \). Then a star product \( \ast \) on the \( k[\hbar] \)-module \( A := \hat{A}[\hbar] \) is expressed by a series \( \{ \beta_i \}_{i \geq 1} \) of \( k \)-bilinear functions \( \beta_i : \hat{A} \times \hat{A} \to \hat{A} \), as follows:

\[
c_1 \ast c_2 = c_1 c_2 + \sum_{i \geq 1} \beta_i (c_1, c_2) \hbar^i
\]

for \( c_1, c_2 \in \hat{A} \).

We shall need this version of the Nakayama Lemma:

**Lemma 2.5.** Let \( K \) be as in Setup 2.1 let \( A \) be an \( \mathfrak{m} \)-adically complete \( K \)-algebra, and let \( M \) be a finitely generated left \( A \)-module. If \( k \otimes_K M = 0 \) then \( M = 0 \).

**Proof.** Let \( \mathfrak{a} := mA \), which is a two-sided ideal of \( A \), and \( \mathfrak{m}^i A = \mathfrak{a}^i \) for every \( i \). It follows that \( A \) is \( \mathfrak{a} \)-adically complete. According to [CA] Section III.3.1, Lemma 3] the ideal \( \mathfrak{a} \) is inside the Jacobson radical of \( A \). By the usual Nakayama Lemma (which holds also for noncommutative rings, cf. [CA] Section II.3.2, Proposition 4]) we see that \( M / \mathfrak{a} M = 0 \) implies \( M = 0 \). \( \square \)
Note that there is no commutativity or finiteness assumption on the algebra $A$; only its structure as $\mathbb{K}$-module is important.

The next proposition might be of interest.

**Proposition 2.6.** Let $\mathbb{K}$ be as in Setup 2.1, let $A$ be an $m$-adically complete $\mathbb{K}$-algebra, and let $M$ be a perfect complex in $D(\text{Mod } A)$. If $k \otimes_{\mathbb{K}}^L M = 0$ then $M = 0$.

**Proof.** Assume $M \neq 0$, and let $H^i M$ be its highest nonzero cohomology module. By Lemmas 1.3 and 2.5 we see that $k \otimes_{\mathbb{K}}^L H^i M \neq 0$. On the other hand by the Künneth trick (Lemma 1.4) we have

$$k \otimes_{\mathbb{K}}^L H^i M \cong H^i (k \otimes_{\mathbb{K}}^L M).$$

Hence $k \otimes_{\mathbb{K}}^L M \neq 0$. \hfill $\square$

Here is the main result of our paper:

**Theorem 2.7.** Let $\mathbb{K}$ be as in Setup 2.1 and let $A$ and $B$ be a flat $m$-adically complete $\mathbb{K}$-algebras, such that the $k$-algebras $\overline{A} := k \otimes_{\mathbb{K}}^L A$ and $\overline{B} := k \otimes_{\mathbb{K}}^L B$ are commutative with connected spectra. Suppose $T$ is a two-sided tilting complex over $B-A$ relative to $\mathbb{K}$. Then there is an isomorphism

$$T \cong P[n]$$

in $D^b(\text{Mod } B \otimes_{\mathbb{K}}^L A^{op})$, for some invertible $B-A$-bimodule $P$ and integer $n$.

**Proof.** This is very similar to the proof of Theorem 1.9. We may assume that $A \neq 0$. Define

$$n := -\sup\{i \mid H^i T \neq 0\},$$

and let $P := H^{-n}T$. This is a $B$-$A$-bimodule. By Lemma 1.3, $P$ is a nonzero finitely generated right $A$-module. So according to Lemma 2.5, the right $A$-module $\overline{P} := \overline{k} \otimes_{\mathbb{K}}^L P$ is nonzero. By the Künneth trick (Lemma 1.4) there is an isomorphism

$$\overline{P} = k \otimes_{\mathbb{K}}^L H^{-n}T \cong H^{-n}(k \otimes_{\mathbb{K}}^L T).$$

According to Corollary 1.16, the complex $\overline{T} := k \otimes_{\mathbb{K}}^L T$ is a two-sided tilting complex over $B-A$ relative to $\mathbb{K}$. Since $A$ is commutative and Spec $\overline{A}$ is connected, we can apply Theorem 1.9. The conclusion is that $\overline{T}$ has exactly one nonzero cohomology module. But by the calculation above this must be $H^{-n}\overline{T} \cong \overline{P}$. Therefore we get an isomorphism $\overline{T} \cong \overline{P}[n]$ in $D(\text{Mod } \overline{B} \otimes_{\mathbb{K}}^L \overline{A}^{op})$, and $\overline{P}$ is an invertible $\overline{B}$-$\overline{A}$ bimodule relative to $\mathbb{K}$.

Let $S \in D^b(\text{Mod } \overline{A} \otimes_{\mathbb{K}}^L \overline{B}^{op})$ be an inverse of $\overline{T}$. Define

$$m := -\sup\{i \mid H^i S \neq 0\},$$

$$Q := H^{-m}S, \overline{S} := \overline{k} \otimes_{\mathbb{K}}^L S$$

and $\overline{Q} := \overline{k} \otimes_{\mathbb{K}}^L Q$. By the same considerations as above we see that $\overline{S} \cong \overline{Q}[m]$ in $D(\text{Mod } \overline{A} \otimes_{\mathbb{K}}^L \overline{B}^{op})$, and $\overline{Q}$ is an invertible $\overline{A}$-$\overline{B}$ bimodule relative to $\mathbb{K}$.

From Corollary 1.16 it follows that

$$\overline{P}[n] \otimes_{\overline{A}} \overline{Q}[m] \cong \overline{T} \otimes_{\overline{A}} \overline{S} \cong \overline{B}.$$

Therefore $n = -m$. Using the Künneth trick we see that

$$B \cong H^0(T \otimes_{\overline{A}} S) \cong (H^{-n}T) \otimes_{\overline{A}} (H^nQ) = P \otimes_{\overline{A}} Q.$$

Similarly we get

$$A \cong Q \otimes_{\overline{B}} P.$$

So $P$ is an invertible $B$-$A$-bimodule relative to $\mathbb{K}$. 


Since $P$ is a projective $A^{op}$-module, and it is the highest nonzero cohomology of $T$, by Lemma 1.10 we have an isomorphism $T \cong M \oplus P[n]$ in $D^b(\text{Mod} A^{op})$ for some complex $M$. Suppose, for the sake of contradiction, that $M \neq 0$; and let
\[ l := \sup \{ i \mid H^i M \neq 0 \}. \]
Then $l < -n$, so $l + n < 0$. By the Künneth trick we get
\[ (H^i M) \otimes_A Q \cong (H^i M) \otimes_A (H^n S) \cong H^{i+n}(M \otimes_Q S), \]
which is a direct summand of the $B^{op}$-module
\[ H^{i+n}(T \otimes_A S) \cong H^{i+n}B = 0. \]
But $Q$ is an invertible bimodule, and therefore $H^i M = 0$. This is a contradiction. Hence $T \cong P[n]$ in $D^b(\text{Mod} A^{op})$.

Finally, the last isomorphism implies that $H^i T = 0$ for all $i \neq -n$. By truncation we obtain the isomorphism $T \cong P[n]$ in $D^b(\text{Mod} B \otimes_K A^{op})$. \hfill \Box

The upshot is that associative deformations behave like commutative algebras, as far as derived Morita theory is concerned. Specifically:

**Corollary 2.8.** Let $K$ be as in Setup 2.1, and let $A$ and $B$ be a flat $m$-adically complete $K$-algebras, such that the $k$-algebras $\bar{A} := k \otimes_K A$ and $\bar{B} := k \otimes_K B$ are commutative with connected spectra. Assume that $A$ and $B$ are derived Morita equivalent relative to $K$. Then $A$ and $B$ are Morita equivalent relative to $K$. Moreover the $k$-algebras $\bar{A}$ and $\bar{B}$ are isomorphic.

**Proof.** By Theorem 1.6 there is a two-sided tilting complex $T$ over $B$-$A$-relative to $K$. Therefore by Theorem 2.7 there is an invertible $B$-$A$-bimodule $P$ relative to $K$. So we have classical Morita equivalence between $A$ and $B$.

Now the bimodule $\bar{P} := k \otimes_K P$ is an invertible $\bar{B}$-$\bar{A}$-bimodule relative to $\bar{k}$. Since these are commutative $k$-algebras they must be isomorphic. \hfill \Box

**Corollary 2.9.** Let $K$ be as in Setup 2.1, and let $A$ be a flat $m$-adically complete $K$-algebra, such that the $k$-algebra $\bar{A} := k \otimes_K A$ is commutative with connected spectrum. Then
\[ \text{DPic}_K(A) = \text{Pic}_K(A) \times \mathbb{Z}. \]

**Proof.** As mentioned earlier, there is a canonical inclusion of $\text{Pic}_K(A) \times \mathbb{Z}$ into $\text{DPic}_K(A)$. By Theorem 2.7 this is a bijection. \hfill \Box

**Remark 2.10.** Let $K$ be any commutative ring, and let $A$ be a flat noetherian $K$-algebra. A dualizing complex over $A$ relative to $K$ is a complex $R \in D^b(\text{Mod} A \otimes_K A^{op})$ satisfying a list of conditions; see [Ye1, Definition 4.1]. Presumably [Ye1, Theorem 4.5] holds in this case (it was only proved when $K$ is a field). Then the group $\text{DPic}_K(A)$ classifies isomorphism classes of dualizing complexes (if at least one dualizing complex exists).

Now assume we are in the situation of Corollary 2.9 and that $\bar{A}$ is a finitely generated $k$-algebra. Then $A$ is noetherian. It is reasonable to suppose that $A$ will have some dualizing complex $R$ relative to $K$. What Corollary 2.9 tells us is that any other dualizing complex $R'$ must be isomorphic to $P[n] \otimes_A R$ for some invertible bimodule $P$ and integer $n$.

**Remark 2.11.** In the paper [BW] Bursztyn and Waldmann consider the local ring $\mathbb{k} = \mathbb{k}[[\hbar]]$, and a fixed commutative $\mathbb{k}$-algebra $A$ with connected spectrum. They prove that the Picard group $\text{Pic}_k(\bar{A})$ acts on the set of gauge equivalence classes of associative $K$-deformations $A$ of $\bar{A}$. The orbit of a deformation $A$ under this action is the set of deformations that Morita equivalent to $A$. The stabilizer...
of $A$ in $\text{Pic}_K(A)$ is the image of $\text{Pic}_K(A)$. And the kernel of the homomorphism $\text{Pic}_K(A) \rightarrow \text{Pic}_K(\bar{A})$ is the group of outer gauge equivalences of $A$.

Presumably these results remain true for any complete ring $K$ as in Setup 2.1, not just for $K = k[[\hbar]]$.

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