Branching formula for \( q \)-Toda functions of type B

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Abstract
We present a proof of the explicit formula for the asymptotically free eigenfunctions of the \( B_N \) \( q \)-Toda operator which was conjectured by the first and third authors. This formula can be regarded as a branching formula from the \( B_N \) \( q \)-Toda eigenfunction restricted to the \( A_{N-1} \) \( q \)-Toda eigenfunctions. The proof is given by a contiguity relation of the \( A_{N-1} \) \( q \)-Toda eigenfunctions and a recursion relation of the branching coefficients.

Keywords
Toda system · Macdonald symmetric function · Quantum group

Mathematics Subject Classification
33D52 · 37K10 · 81R50

1 Introduction
Let \( f^{A_{N-1}}_{\text{Toda}}(x|s|q) \) and \( f^{B_N}_{\text{Toda}}(x|s|q) \) be the asymptotically free eigenfunctions of the \( A_{N-1} \) and \( B_N \) \( q \)-Toda operators, respectively (Definition 2.2, Definition 2.6). Here, \( q \) is a generic parameter, and \( x = (x_1, \ldots, x_N) \) is an \( N \)-tuple of variables. We introduce an \( N \)-tuple of continuous parameters (or indeterminates) \( s = (s_1, \ldots, s_N) \), while the ordinary \( q \)-Toda functions contain a weight as a set of discrete parameters. A combinatorial explicit formula is known for the asymptotically free eigenfunctions of Macdonald’s difference operator of type A [2,10,12], and the one of the \( A_{N-1} \)
q-Toda functions $f^A_{N-1}Toda(x|s;q)$ can be given by taking a certain limit ($t \to 0$) of that formula. The aim of this paper is to prove the following explicit formula for $f^B_{N}Toda(x|s;q)$ in terms of $f^A_{N-1}Toda(x|s;q)$ that was conjectured in [9].

**Theorem.** 2.7. The $B_N$ q-Toda function $f^B_{N}Toda(x|s;q)$ is of the form

$$f^B_{N}Toda(x_1, \ldots, x_N|s_1, \ldots, s_N;q) = \sum_{\theta=(\theta_1, \ldots, \theta_N) \in \mathbb{Z}^N_{\geq 0}} e^B_{\theta/A_{N-1}}(s;q) \cdot \prod_{i=1}^N x_i^{\theta_i} \cdot f^A_{N-1}Toda(x_1, \ldots, x_N|q^{-\theta_1}s_1, \ldots, q^{-\theta_N}s_N;q).$$

(1.1)

where we have set

$$e^B_{\theta/A_{N-1}}(s;q) := \prod_{k=1}^N q^{(N-k+1)\theta_k} \frac{(q; q)_{\theta_k} (q/s_k^2; q)_{\theta_k}}{(q/s_j s_j; q)_{\theta_i} (q/s_j s_i; q)_{\theta_j}} \prod_{1 \leq i < j \leq N} \frac{1}{(q/s_j s_i; q)_{\theta_i} (q/s_i s_j; q)_{\theta_j}} (q/s_j s_i; q)_{\theta_i} (q/s_i s_j; q)_{\theta_j},$$

(1.2)

$(a; q)_n := (a; q)_{\infty} / (q^n a; q)_{\infty}$, and $(a; q)_{\infty} := \prod_{k=1}^\infty (1 - q^{k-1}a)$.

The q-Toda system has been studied in the connection with representation theory of the quantum groups. In particular, the eigenfunctions of the q-Toda operators can be constructed by Whittaker functions in the Verma module [3, 11] and expressed via fermionic formulas [4]. Moreover, the q-Toda functions are closely related to characters of Demazure modules [5–7] and the equivariant K-theory of Laumon spaces [1, 8].

The main result (1.1) can be regarded as a branching rule for the $B_N$ q-Toda function restricted to the $A_{N-1}$ q-Toda eigenfunctions. The proof is given by direct calculation, in which we give a contiguity relation of $f^A_{N-1}Toda(x|s;q)$ (Proposition 3.1). It is an interesting problem to find similar branching formulas for q-Toda functions of other types.

This paper is organized as follows. In Sect. 2, we recall the definitions of the q-Toda functions and state the main theorem. The proof is given in Sect. 3.

### 2 $A_{N-1}$ and $B_N$ q-Toda functions

First, we recall the asymptotically free eigenfunctions for $A_{N-1}$ q-Toda operator. Let $q$ be a generic parameter and let $s = (s_1, \ldots, s_N)$ be an $N$-tuple of indeterminates. Set

$$\Lambda^A_{Q(s,q)} = Q(s,q)([x_2/x_1, \ldots, x_N/x_{N-1}]).$$

(2.1)
Definition 2.1 Let \( x = (x_1, \ldots, x_N) \). The \( q \)-Toda operator \( D^{A_{N-1}}_{\text{Toda}}(x|s|q) \) of type A acting on \( \Lambda^2_{Q(s,q)} \), is defined to be

\[
D^{A_{N-1}}_{\text{Toda}}(x|s|q) = \sum_{i=1}^{N-1} s_i (1 - x_{i+1}/x_i) T_{q,x_i} + s_N T_{q,x_N}.
\] (2.2)

Here, \( T_{q,x_i} \) is the difference operator defined by

\[
T_{q,x_i} f(x_1, \ldots, x_N) = f(x_1, \ldots, q x_i, \ldots, x_N).
\] (2.3)

The eigenfunctions of \( D^{A_{N-1}}_{\text{Toda}}(x|s|q) \) are given as follows. We use the notation in [9].

Definition 2.2 ([5]). Set

\[
f^{A_{N-1}}_{\text{Toda}}(x|s|q) = \sum_{\theta \in M^{(N)}} c^\text{Toda}_N(\theta; s; q) \prod_{1 \leq i < j \leq N} (x_j/x_i)^{\theta_{i,j}}.
\] (2.4)

Here, \( M^{(N)} = \{ \theta = (\theta_{ij})_{i,j=1}^N | \theta_{ij} \in \mathbb{Z}_{\geq 0}, \theta_{kl} = 0 \text{ if } k \geq l \} \) is the set of \( N \times N \) strictly upper triangular matrices with nonnegative integer entries, and the coefficients \( c^\text{Toda}_N(\theta; s; q) \) are defined by

\[
c^\text{Toda}_N(\theta; s; q) = \prod_{k=2}^{N} \prod_{1 \leq i \leq j \leq k-1} \frac{1}{(q \sum_{a=k+1}^{N} (\theta_{i,a} - \theta_{j+1,a}) q s_{j+1}/s_i; q)_{\theta_{i,k}} (q^{\theta_{i,k}-\theta_{i,k}-\sum_{a=k+1}^{N} (\theta_{i,a} - \theta_{j,a})} q s_i/s_j; q)_{\theta_{i,k}}}. \] (2.5)

Fact 2.3 ([5,9]) We have

\[
D^{A_{N-1}}_{\text{Toda}}(x|s|q) f^{A_{N-1}}_{\text{Toda}}(x|s|q) = \sum_{i=1}^{N} s_i f^{A_{N-1}}_{\text{Toda}}(x|s|q).
\] (2.6)

This formula was originally proved in [5]. A combinatorial explicit formula was given for the asymptotically free eigenfunctions of the Macdonald operator in [2,10,12], and the formula \( f^{A_{N-1}}_{\text{Toda}} \) can also be directly obtained by taking a certain limit \( t \to 0 \) of that combinatorial formula [9]. As for the limit from the Macdonald functions to the Toda functions, see also [5–7].

Notation 2.4 We introduce

\[
a^\text{Toda}_N((\theta_{i,n})_{1 \leq i \leq N-1}; (s_i)_{1 \leq i \leq N}; q) := \frac{c^\text{Toda}_N((\theta_{i,j})_{1 \leq i < j \leq N}; (s_i)_{1 \leq i \leq N}|q)}{c^\text{Toda}_{N-1}((\theta_{i,j})_{1 \leq i < j \leq N-1}; (q^{-\theta_{i,n}} s_i)_{1 \leq i \leq N-1}|q)} \quad (N \geq 2).
\] (2.7)
Then, $d_N^{\text{Toda}}$ is of the form

$$
d_N^{\text{Toda}} \left( (\theta_i)_{1 \leq i \leq N-1}; (s_i)_{1 \leq i \leq N}; q \right) = \prod_{i=1}^{N-1} \frac{1}{(q; q)^{\theta_i}} \prod_{1 \leq i < j \leq N-1} \frac{1}{(q s_j / s_i; q)^{\theta_i}} \frac{q^{\theta_j}}{(q^{\theta_j - \theta_i + 1} s_i / s_j; q)^{\theta_i}},
$$

and the $A_{N-1}$ $q$-Toda function can be expressed as

$$
f_{A_{N-1}}^{\text{Toda}}(x | s | q) = \sum_{\theta = (\theta_1, \ldots, \theta_{N-1}) \in \mathbb{Z}^{N-1} \geq 0} d_N^{\text{Toda}}(\theta; s; q) \prod_{i=1}^{N-1} \left( x_N / x_i \right)^{\theta_i} \cdot f_{A_{N-2}}^{\text{Toda}}(x | (q^{\theta_i} s_i)_{1 \leq i \leq N-1} | q).
$$

(2.9)

Although (2.9) follows from the case of the Macdonald functions, we can also prove (2.9) in a similar manner to Sect. 3.

Now, we turn to the case of type B. Set

$$
\Lambda_{Q(s, q)}^{B_N} = Q(s, q)[[x_2/x_1, \ldots, x_N/x_N-1, 1/x_N]].
$$

(2.10)

**Definition 2.5** Define the $B_N$ $q$-Toda operator $D_{B_N}^{\text{Toda}}(x | s | q)$ acting on $\Lambda_{Q(s, q)}^{B_N}$ by

$$
D_{B_N}^{\text{Toda}}(x | s | q) = \sum_{i=1}^{N-1} s_i (1 - x_{i+1} / x_i) T_{q,x_i} + s_N (1 - 1 / x_N) T_{q,x_N}
$$

$$
+ s_1^{-1} T_{q,x_1}^{-1} + \sum_{i=2}^{N} s_i^{-1} (1 - x_i / x_{i-1}) T_{q,x_i}^{-1}.
$$

(2.11)

This operator can be obtained by the limit of the $B_N$ Macdonald operator [9]. As for the description of the $q$-Toda operators by the quantum groups, see [3,4,11].

**Definition 2.6** The asymptotically free eigenfunction $f_{B_N}^{\text{Toda}}(x | s | q) \in \Lambda_{Q(s, q)}^{B_N}$ of the $B_N$ $q$-Toda operator is defined by

$$
D_{B_N}^{\text{Toda}}(x | s | q) f_{B_N}^{\text{Toda}}(x | s | q) = \sum_{i=1}^{N} (s_i + s_i^{-1}) f_{B_N}^{\text{Toda}}(x | s | q),
$$

(2.12)

$$
\left[ f_{B_N}^{\text{Toda}}(x | s | q) \right]_{x,1} = 1.
$$

(2.13)

Here, $[ \quad ]_{x,1}$ means the constant term with respect to $x_i$’s.
Note that $f_{BN}^{Toda}(x|s|q)$ is uniquely determined. We obtain an explicit formula for the $B_N$ $q$-Toda function $f_{BN}^{Toda}(x|s|q)$ in terms of the $A_{N-1}$ $q$-Toda functions $f_{A_{N-1}}^{Toda}(x|s|q)$.

**Theorem 2.7** The $B_N$ $q$-Toda function $f_{BN}^{Toda}(x|s|q)$ satisfies the branching formula

\[
f_{BN}^{Toda}(x_1, \ldots, x_N|s_1, \ldots, s_N|q) = \sum_{\theta=(\theta_1, \ldots, \theta_N) \in \mathbb{Z}^N_{\geq 0}} e_{\frac{B_N}{A_{N-1}}}(s|q) \cdot \prod_{i=1}^{N} x_i^{-\theta_i} \cdot f_{A_{N-1}}^{Toda}(x_1, \ldots, x_N|q^{-\theta_1} s_1, \ldots, q^{-\theta_N} s_N|q),
\]

(2.14)

where we have set

\[
e_{\frac{B_N}{A_{N-1}}}(s|q) := \prod_{k=1}^{N} \frac{q^{(N-k+1)\theta_k}}{(q;q)_{\theta_k} (q/s_k^2; q)_{\theta_k}} \times \prod_{1 \leq i < j \leq N} \frac{1}{(qs_j/s_i; q)_{\theta_i} (q^{\theta_j-\theta_i} qs_i/s_j; q)_{\theta_i} (qs_i/s_j; q)_{\theta_i} (qs_i/s_j; q)_{\theta_i} (q/s_i s_j; q)_{\theta_i}},
\]

(2.15)

$(\alpha; q)_n := \frac{(\alpha; q)_\infty}{(q^n \alpha; q)_\infty}$, and $(\alpha; q)_\infty := \prod_{k=1}^{\infty} (1 - q^{k-1} \alpha)$. Note that the constant term of $f_{A_{N-1}}^{Toda}$ is 1. Hence, the constant term of (2.14) is also 1.

This formula was conjectured in [9]. The proof is given in the next subsection.

**Remark 2.8** The region of convergence of $f_{A_{N-1}}^{Toda}$ can be derived from the case of the Macdonald functions (Proposition 6.1 in [10]) by taking the limit $t \to 0$. It is an interesting problem to consider the convergence of formula (2.14).

### 3 Proof of Theorem 2.7

In this section, we prove Theorem 2.7. First we give the following relation of the $q$-Toda functions of type A.

**Proposition 3.1** The $q$-Toda functions of type A satisfy the contiguity relation

\[
f_{A_{N-1}}^{Toda}(x_1, \ldots, x_{N-1}, q x_N|s|q) = \sum_{k=1}^{N} (-1)^{N-k} q^{N-k} \prod_{i=k+1}^{N-1} x_i/s_i \prod_{i=k+1}^{N} (1 - x_i/s_i) (1 - q x_i/s_i) (x_N/x_k) f_{A_{N-1}}^{Toda}(x_1, \ldots, x_N|q^{-\varepsilon_k} s|q).
\]

(3.1)

Here, we used the notation

\[
q^{\pm \varepsilon_i} \cdot s = (s_1, \ldots, s_{i-1}, q^{\pm 1} s_i, s_{i+1}, \ldots, s_N).
\]

(3.2)
**Proof** First, we show the following equation of the rational functions of \(a_i\) and \(s_i\):

\[
\prod_{i=1}^{N-1} a_i = \sum_{k=1}^{N} (s_k/s_N) \prod_{i=1}^{N-1} \left( 1 - a_i s_k/s_i \right) \prod_{1 \leq i < N, i \neq k} (1 - s_k/s_i). \tag{3.3}
\]

Regarding \(s_i\)'s in the RHS as complex variables, we set

\[
F(s) := \sum_{k=1}^{N} (s_k/s_N) \prod_{1 \leq i < N, i \neq k} (1 - s_k/s_i) = \sum_{k=1}^{N} \prod_{i=1}^{N-1} \left( s_i - a_i s_k \right) \prod_{1 \leq i < N, i \neq k} (s_i - s_k). \tag{3.4}
\]

For any \(\ell = 1, \ldots, N\), the residue at \(s_\ell = s_{\ell'} (\ell' \neq \ell)\) is

\[
\text{Res}_{s_\ell = s_{\ell'}} F(s) = \lim_{s_\ell \to s_{\ell'}} F(s)(s_\ell - s_{\ell'}) = \lim_{s_\ell \to s_{\ell'}} \left( - \prod_{i=1}^{N-1} (s_i - a_\ell s_i) + \prod_{i \neq \ell, \ell'} (s_i - s_{\ell'}) \right) = 0. \tag{3.5}
\]

Hence \(F(s)\) is regular on the whole complex plane with respect to each \(s_\ell\), and it is clear that \(F(s)\) is bounded. This indicates that \(F(s)\) is a constant function. By the specialization \(s_i = a_i^{-1} a_{i-1}^{-1} \cdots a_2^{-1} s_1 (i = 2, \ldots, N)\), we have

\[
F(s) = \sum_{k=1}^{N} (A_{N-1} A_{N-2} \cdots a_k) \prod_{i=1}^{N-1} \left( 1 - \frac{a_i a_{i-1} \cdots a_1}{a_{k-1} a_{k-2} \cdots a_1} \right) \prod_{1 \leq i < N, i \neq k} (1 - \frac{a_i a_{i-1} \cdots a_1}{a_{k-1} a_{k-2} \cdots a_1}) \tag{3.6}
\]

This gives (3.3).

Substituting \(a_i = q^{b_i}\) into (3.3) yields

\[
\prod_{i=1}^{N-1} q^{b_i} = \sum_{k=1}^{N-1} (-1)^{N-k} q^{N-k} \prod_{i=k+1}^{N} s_i/s_k \prod_{i=k+1}^{N} (1 - s_i/s_k) (1 - q s_i/s_k) \frac{d_{Toda}^{N} (\theta_1, \ldots, \theta_{k-1}, 1, \ldots, \theta_{N-1} | q^{-\varepsilon} s \cdot s)}{d_{Toda}^{N} (\theta_1, \ldots, \theta_{N-1} | s)} + \frac{d_{Toda}^{N} (\theta_1, \ldots, \theta_{N-1} | q^{-\varepsilon} N \cdot s)}{d_{Toda}^{N} (\theta_1, \ldots, \theta_{N-1} | s)}. \tag{3.7}
\]

By (2.9) and (3.7), we obtain formula (3.1). \(\square\)

**Proposition 3.2** The branching coefficients \(e_0^{B_N/A_{N-1}} (s | q)\) satisfy the recursion relation

\[
\sum_{i=1}^{N} \left( (1 - q^{-b_i}) s_i + (1 - q^{b_i}) s_i^{-1} \right) e_0^{B_N/A_{N-1}} (s | q) = 0. \tag{3.8}
\]
\[ \sum_{k=1}^{N} s_N (-1)^{N-k+1} q^{-\theta_N + \delta_k} q^{N-k} \prod_{i=k+1}^{N} (1 - q^{-\theta_i + \theta_k - 1} s_i/s_k) e^{(\theta_1, \ldots, \theta_N)/(s|q)}. \]

**Proof** By substituting (2.15) into (3.8), it can be shown that (3.8) is equivalent to

\[ \sum_{i=1}^{N} (1 - q^{-\theta_i}) s_i + (1 - q^{\theta_i}) s_i^{-1} = -\sum_{k=1}^{N} q^{-\theta_k} s_k \prod_{i \neq k}^{N} (1 - q^{\theta_k - \theta_i} s_i/s_k)(1 - q^{\theta_k + \theta_i}/s_i s_k) \]

(3.9)

By replacing \( q^{\theta_i} \) with generic parameters \( Q_i \) and shifting \( s_i \) to \( Q_i s_i \), Eq. (3.9) becomes

\[ \sum_{i=1}^{N} (1 - Q_i) s_i + (1 - Q_i^{-1}) s_i^{-1} = \sum_{k=1}^{N} s_k \prod_{i \neq k}^{N} (1 - s_i/s_k)(1 - 1/s_i s_k) \]

(3.10)

The proof is completed by showing this equation. Regarding \( s_i \)'s as complex variables, we define the function

\[ F(s) := \sum_{k=1}^{N} s_k \prod_{i \neq k}^{N} (1 - Q_i s_i/s_k)(1 - Q_i^{-1}/s_i s_k) \]

(3.11)

A direct calculation shows that the residue at \( s_\ell = s_{\ell'}^{\pm 1} (\ell \neq \ell') \) is

\[ \text{Res}_{s_\ell = s_{\ell'}^{\pm 1}} F(s) = \lim_{s_\ell \to s_{\ell'}^{\pm 1}} F(s)(s_\ell - s_{\ell'}) = 0 \]

(3.12)

and these singularities are removable. Hence, \( F(s) \) is a regular with respect to each variable \( s_\ell (\ell = 1, \ldots, n) \) on the complex plane except for the origin 0 (and \( \infty \)). Therefore, for arbitrary \( \ell \), the function \( F(s) \) can be given by the Laurent series on \( 0 < |s_\ell| < \infty \)

\[ F(s) = \sum_{i \in \mathbb{Z}} C_i s_\ell^i, \]

(3.13)

where \( C_i \) is a function of \( s_1, \ldots, s_{\ell-1}, s_{\ell+1}, \ldots, s_N \). Since the orders of the poles at \( s_\ell = 0 \) and \( s_\ell = \infty \) are at most 1, we have \( C_i = 0 \) (\( i < -1 \) or \( i > 1 \)). It can be shown that the residues at \( s_\ell = 0 \) and \( s_\ell = \infty \) are

\[ C_{-1} = \text{Res}_{s_\ell = 0} F(s) = 1 - Q_\ell^{-1}, \]

(3.14)

\[ C_1 = \text{Res}_{s_\ell = \infty} F(s) = 1 - Q_\ell. \]

(3.15)
Therefore, with a constant $\widetilde{C}_0$ independent of $s_i$’s, we can write
\[
F(s) = \sum_{i=1}^{N} \left( (1 - Q_i)s_i + (1 - Q_i^{-1})s_i^{-1} \right) + \widetilde{C}_0. \tag{3.16}
\]

Furthermore, we obtain
\[
\widetilde{C}_0 = F \left( \sqrt{Q_1^{-1}}, \sqrt{Q_2^{-1}}, \ldots, \sqrt{Q_N^{-1}} \right)
= \sum_{k=1}^{N} \sqrt{Q_k^{-1}} \prod_{i=1}^{N} (1 - \sqrt{Q_i/Q_k}) (1 - \sqrt{Q_k/Q_i}) (1 - \sqrt{Q_i/Q_k})
= 0. \tag{3.17}
\]

This gives (3.10).

\[\square\]

**Proof of Theorem 2.7** The action of $D^{B_N \text{Toda}}(x|s|q)$ on the right-hand side of (2.14) gives
\[
D^{B_N \text{Toda}}(x|s|q) \text{ (RHS of } (2.14))
= \sum_{\theta \in \mathbb{Z}_{\geq 0}^N} e_{B_N/AN-1}^{\theta}(s|q) \prod_{i=1}^{N} x_i^{-\theta_i} \cdot \left\{ D_{AN-1}^{B_N \text{Toda}}(x|s|q) - q^{-\theta_N} s_N/x_N T_{q,x_N} ight\}
+ D_{AN-1}^{B_N \text{Toda}}((x_{N-i+1}^{-1})^{N}_{i=1} | (q^{\theta_{N-i+1}} s_{N-i+1}^{-1})^{N}_{i=1} | q) \right\}
= \sum_{\theta \in \mathbb{Z}_{\geq 0}^N} e_{B_N/AN-1}^{\theta}(s|q) \prod_{i=1}^{N} x_i^{-\theta_i} \cdot \left\{ \sum_{i=1}^{N} q^{-\theta_i} s_i + \sum_{i=1}^{N} q^{\theta_i} s_i^{-1} - q^{-\theta_N} s_N/x_N T_{q,x_N} \right\}
\times f_{AN-1}^{B_N \text{Toda}}(x|q^{-\theta_i} s_i) | q). \tag{3.18}
\]

Here, we used Fact 2.3 and the symmetry
\[
f_{AN-1}^{B_N \text{Toda}}(x|s|q) = f_{AN-1}^{B_N \text{Toda}}(x_{N-i+1}^{-1} | s_{N-i+1}^{-1}) | q). \tag{3.19}
\]

By Proposition 3.1, we have
\[
D^{B_N \text{Toda}}(x|s|q) \text{ (RHS of } (2.14))
= \sum_{\theta \in \mathbb{Z}_{\geq 0}^N} e_{B_N/AN-1}^{\theta}(s|q) \prod_{i=1}^{N} x_i^{-\theta_i} \cdot \left\{ \sum_{i=1}^{N} (q^{-\theta_i} s_i + q^{\theta_i} s_i^{-1}) f_{AN-1}^{B_N \text{Toda}}(x|q^{-\theta_i} s_i) | q)\right\}
- q^{-\theta_N} \sum_{k=1}^{N} (-1)^{N-k} (s_N/x_k) \frac{q^{N-k} \prod_{i=k+1}^{N-1} (q^{-\theta_i+1} s_i/s_k)}{\prod_{i=k+1}^{N} (1 - q^{-\theta_i+1} s_i/s_k)(1 - qq^{-\theta_i+1} s_i/s_k)}, \tag{3.18}
\]

\[\square\] Springer
\begin{align*}
& \times f^{A_{N-1}}_{\text{Toda}}(x | q^{-\varepsilon_k} \cdot \left( q^{-\theta_l s_i} \right)_{1 \leq l \leq N} | q) \\
& = \sum_{\theta \in \mathbb{Z}^N_{\geq 0}} \prod_{i=1}^N x_i^{-\theta_i} \cdot \left\{ \sum_{i=1}^N \left( q^{-\theta_i s_i} + q^{\theta_i s_i^{-1}} \right) e_B^{B_{N/A_{N-1}}}(s | q) \right\} \\
& + \sum_{k=1}^N s_N^k \prod_{i=k+1}^N \left( 1 - q^{-\theta_i + \theta_k - 1} s_i / s_k \right) \left( 1 - q^{\theta_i + \theta_k - 1} s_i / s_k \right) e_B^{B_{N/A_{N-1}}}(s | q) \\
& \times f^{A_{N-1}}_{\text{Toda}}(x | (q^{-\theta_l s_i})_{1 \leq l \leq N} | q),
\end{align*}

where we have used that $e_B^{B_{N/A_{N-1}} \cdot 1} = 0$ if $\theta_j = -1$ for some $j$. Proposition 3.2 shows that this is equal to $\sum_{i=1}^N (s_i + s_i^{-1}) \cdot (\text{RHS of (2.14)})$. This completes the proof. \hfill \Box

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