CLOSED SETS OF REAL ROOTS IN KAC–MOODY ROOT SYSTEMS

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Abstract. In this note, we provide a complete description of the closed sets of real roots in a Kac–Moody root system.

1. Introduction

Let \( A = (a_{ij})_{i,j \in I} \) be a generalised Cartan matrix, and let \( \mathfrak{g}(A) \) be the associated Kac–Moody algebra (see [Kac90]). Like (finite-dimensional) semisimple complex Lie algebras, \( \mathfrak{g}(A) \) possesses a root space decomposition

\[
\mathfrak{g}(A) = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_{\alpha}
\]

with respect to the adjoint action of a Cartan subalgebra \( \mathfrak{h} \), with associated set of roots \( \Delta \subseteq \mathfrak{h}^* \) contained in the \( \mathbb{Z} \)-span of the set \( \Pi = \{ \alpha_i \mid i \in I \} \) of simple roots, as well as a Weyl group \( \mathcal{W} \subseteq \mathrm{GL}(\mathfrak{h}^*) \) stabilising \( \Delta \). However, as soon as \( A \) is not a Cartan matrix (i.e. as soon as \( \mathfrak{g}(A) \) is infinite-dimensional), the set \( \Delta_{re} := \mathcal{W}.\Pi \) of real roots is properly contained in \( \Delta \). In some sense, the real roots of \( \Delta \) are those that behave as the roots of a semisimple Lie algebra; in particular, \( \dim \mathfrak{g}_{\alpha} = 1 \) for all \( \alpha \in \Delta_{re} \).

A subset \( \Psi \subseteq \Delta \) is closed if \( \alpha + \beta \in \Psi \) whenever \( \alpha, \beta \in \Psi \) and \( \alpha + \beta \in \Delta_{re} \). Note that, denoting by \( \mathfrak{g}_\Psi \) the subspace \( \mathfrak{g}_\Psi := \bigoplus_{\alpha \in \Psi} \mathfrak{g}_\alpha \) of \( \mathfrak{g}(A) \), a subset \( \Psi \subseteq \Delta_{re} \) is closed if and only if \( \mathfrak{h} \oplus \mathfrak{g}_\Psi \) is a subalgebra of \( \mathfrak{g}(A) \); in particular, if \( \Psi \subseteq \Delta_{re} \) is closed, the subalgebra generated by \( \mathfrak{g}_\Psi \) is contained in \( \mathfrak{h} \oplus \mathfrak{g}_\Psi \). Closed sets of real roots in Kac–Moody root systems thus arise naturally, and the purpose of this note is to provide a complete description of these sets. Our main theorem is as follows. For each \( \alpha \in \Delta_{re} \), let \( \alpha^\vee \in \mathfrak{h} \) be the coroot of \( \alpha \), i.e. the unique element of \( [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \) with \( \langle \alpha, \alpha^\vee \rangle = 2 \).

Main Theorem. Let \( \Psi \subseteq \Delta_{re} \) be a closed set of real roots and let \( \mathfrak{g} \) be the subalgebra of \( \mathfrak{g}(A) \) generated by \( \mathfrak{g}_\Psi \). Set \( \Psi_s := \{ \alpha \in \Psi \mid -\alpha \in \Psi \} \) and \( \Psi_n := \Psi \setminus \Psi_s \). Set also \( \mathfrak{h}_s := \sum_{\gamma \in \Psi_s} C_{\gamma^\vee}, \mathfrak{g}_s := \mathfrak{h}_s \oplus \mathfrak{g}_\Psi, \) and \( \mathfrak{g}_n := \mathfrak{g}_{\Psi_n} \). Then

1. \( \mathfrak{g}_s \) is a subalgebra and \( \mathfrak{g}_n \) is an ideal of \( \mathfrak{g} \). In particular, \( \mathfrak{g} = \mathfrak{h}_s \oplus \mathfrak{g}_\Psi = \mathfrak{g}_s \ltimes \mathfrak{g}_n \).
2. \( \mathfrak{g}_n \) is nilpotent; it is the largest nilpotent ideal of \( \mathfrak{g} \).
3. \( \mathfrak{g}_s \) is a semisimple finite-dimensional Lie algebra with Cartan subalgebra \( \mathfrak{h}_s \) and set of roots \( \Psi_s \).

Note that the possible closed root subsystems \( \Psi_s \) in the statement of the Main Theorem were explicitly determined in [KV18] when \( A \) is an affine GCM.

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2. Preliminaries

2.1. Kac–Moody algebras. The general reference for this section is [Kac90, Chapters 1–5].

Let \( A = (a_{ij})_{i,j \in I} \) be a generalised Cartan matrix with indexing set \( I \), and let \((\frak h, \Pi, \Pi^\vee)\) be a realisation of \( A \) in the sense of [Kac90, §1.1], with set of simple roots \( \Pi = \{ \alpha_i \mid i \in I \} \) and set of simple coroots \( \Pi^\vee = \{ \alpha_i^\vee \mid i \in I \} \). Let \( \frak g(A) \) be the corresponding Kac–Moody algebra (see [Kac90, §1.2-1.3]). Then \( \frak g(A) \) admits a root space decomposition

\[
\frak g(A) = \frak h \oplus \bigoplus_{\alpha \in \Delta} \frak g_\alpha
\]

with respect to the adjoint action of the Cartan subalgebra \( \frak h \), with associated set of roots \( \Delta \subseteq \frak h^* \). Set \( Q := \bigoplus_{i \in I} \mathbb{Z}\alpha_i \subseteq \frak h^* \) and \( Q_+ := \bigoplus_{i \in I} \mathbb{N}\alpha_i \subseteq Q \). Then \( \Delta \subseteq Q_+ \cup -Q_+ \), and we let \( \Delta_+ := \Delta \cap Q_+ \) (resp. \( \Delta_- := \Delta \setminus \Delta_+ = -\Delta_+ \)) denote the set of positive (resp. negative) roots. The height of a root \( \alpha = \sum_{i \in I} n_i \alpha_i \in Q \) is the integer \( \text{ht}(\alpha) := \sum_{i \in I} n_i \). Thus a root is positive if and only if it has positive height. We also introduce a partial order \( \leq \) on \( Q \) defined by

\[
\alpha \leq \beta \iff \beta - \alpha \in Q_+.
\]

The Weyl group of \( \frak g(A) \) is the subgroup \( \mathcal{W} \) of \( \text{GL}(\frak h^*) \) generated by the fundamental reflections

\[
r_i = r_{\alpha_i} : \frak h^* \to \frak h^* : \alpha \mapsto \alpha - \langle \alpha, \alpha_i^\vee \rangle \alpha_i \quad \text{for } i \in I.
\]

Alternatively, \( \mathcal{W} \) can be identified with the subgroup of \( \text{GL}(\frak h) \) generated by the reflections \( r_i : \frak h \to \frak h : h \mapsto h - \langle \alpha_i, h \rangle \alpha_i^\vee \). Then \( \mathcal{W} \) stabilises \( \Delta \), and we let \( \Delta^r := \mathcal{W}\Pi \) (resp. \( \Delta^r_+ := \Delta^r \cap \Delta_+ \)) denote the set of (resp. positive) real roots. For each \( \alpha \in \Delta^r \), say \( \alpha = w_0 \alpha \), for some \( w \in \mathcal{W} \) and \( i \in I \), the element \( \alpha^\vee := w_0 \alpha^\vee \) only depends on \( \alpha \), and is called the coroot of \( \alpha \). One can then also define the reflection \( r_\alpha \in \mathcal{W} \) associated to \( \alpha \) as

\[
r_\alpha = wr_iw^{-1} : \frak h^* \to \frak h^* : \beta \mapsto \beta - \langle \beta, \alpha^\vee \rangle \alpha.
\]

Let \( \Psi \subseteq \Delta \) be a subset of roots. We call \( \Psi \) closed if \( \alpha + \beta \in \Psi \) whenever \( \alpha, \beta \in \Psi \) and \( \alpha + \beta \in \Delta \). The closure \( \overline{\Psi} \) of \( \Psi \) is the smallest closed subset of \( \Delta \) containing \( \Psi \). A subset \( \Psi' \subseteq \Psi \) is an ideal in \( \Psi \) if \( \alpha + \beta \in \Psi' \) whenever \( \alpha \in \Psi, \beta \in \Psi' \) and \( \alpha + \beta \in \Delta \). The set \( \Psi \) is prenilpotent if there exist some \( w, w' \in \mathcal{W} \) such that \( w\Psi \subseteq \Delta_+ \) and \( w'\Psi \subseteq \Delta_- \); in that case, \( \Psi \) is finite and contained in \( \Delta^r \). If \( \Psi \) is both prenilpotent and closed, it is called nilpotent. We further call \( \Psi \) pro-nilpotent if it is a directed union of nilpotent subsets.

The above terminology is motivated by its Lie algebra counterpart: consider the subspace \( \frak g_\Psi := \bigoplus_{\alpha \in \Psi} \frak g_\alpha \) for each \( \Psi \subseteq \Delta \). If \( \Psi \) is closed, then \( \frak h \oplus \frak g_\Psi \) is a subalgebra. If, moreover, \( \Psi \cap -\Psi = \emptyset \) and \( \Psi' \) is an ideal in \( \Psi \), then \( \frak g_{\Psi'} \) is an ideal in \( \frak g_\Psi \). If \( \Psi \) is nilpotent, then \( \frak g_\Psi \) is a nilpotent subalgebra. This remains valid for pro-nilpotent sets of roots.

**Lemma 1.** Let \( \Psi \subseteq \Delta \) be a pro-nilpotent set of roots. Then \( \frak g_\Psi \) is a nilpotent subalgebra.
Proof. By definition, \( g_\Psi \) is a directed union of nilpotent subalgebras \( g_{\Psi_n} \), associated to nilpotent sets of roots \( \Psi_n \subseteq \Psi \). Since there is a uniform bound on the nilpotency class of these subalgebras by \([\text{Cap}07, \text{Theorem 1.1}]\), the claim follows. \( \square \)

2.2. Davis complexes. The general reference for this section is \([\text{AB}08]\) (see also \([\text{Nos}11]\)).

The pair \((W, S := \{r_i \mid i \in I\})\) is a Coxeter system, and we let \( \Sigma = \Sigma(W, S) \) denote the corresponding \textbf{Davis complex}. Thus \( \Sigma \) is a CAT(0) cell complex whose underlying 1-skeleton \( \Sigma^{(1)} \) is the Cayley graph of \((W, S)\). Moreover, the \( W \)-action on \( \Sigma \) is by cellular isometries, is proper, and induces on \( \Sigma^{(1)} \) the canonical \( W \)-action on its Cayley graph. The vertices of \( \Sigma^{(1)} \) (which we identify with the elements of \( W \)) are called the \textbf{chambers} of \( \Sigma \), and their set is denoted \( \text{Ch}(\Sigma) \). The vertex \( 1_W \) is called the fundamental chamber of \( \Sigma \), and is denoted \( C_0 \). Two chambers of \( \Sigma \) are \textbf{adjacent} if they are adjacent in the Cayley graph of \((W, S)\). A \textbf{gallery} is a sequence \( \Gamma = (x_0, x_1, \ldots, x_d) \) of chambers such that \( x_{i-1} \) and \( x_i \) are distinct adjacent chambers for each \( i = 1, \ldots, d \). The integer \( d \in \mathbb{N} \) is the length of \( \Gamma \), and \( \Gamma \) is called \textbf{minimal} if it is a gallery of minimal length between \( x_0 \) and \( x_d \). In that case, \( d \) is called the chamber \textbf{distance} between \( x_0 \) and \( x_d \), denoted \( d_{\text{ch}}(x_0, x_d) \). If \( X \) is a nonempty subset of \( \text{Ch}(\Sigma) \) and \( x \in \text{Ch}(\Sigma) \), we also set \( d_{\text{ch}}(x, X) := \min_{x' \in X} d_{\text{ch}}(x, x') \).

Given two distinct adjacent chambers \( x, x' \in \text{Ch}(\Sigma) \), any chamber of \( \Sigma \) is either closer to \( x \) or to \( x' \) (for the chamber distance). This yields a partition of \( \text{Ch}(\Sigma) \) into two subsets, which are the underlying chamber sets of two closed convex subcomplexes of \( \Sigma \), called \textbf{half-spaces}. If \( x = C_0 \) and \( x' = sC_0 \) for some \( s \in S \), we let \( H(\alpha_s) \) denote the corresponding half-space containing \( C_0 \) and \( H(-\alpha_s) \) the other half-space. If \( x = wC_0 \) and \( x' = wsC_0 \) (\( w \in W \) and \( s \in S \)), the corresponding half-spaces \( H(\pm \alpha) = wH(\pm \alpha_s) \) depend only on \( \alpha = w\alpha_s \). The map

\[
\Delta^{re} \rightarrow \{\text{half-spaces of } \Sigma \} : \alpha \mapsto H(\alpha)
\]

is a \( W \)-equivariant bijection mapping \( \Delta^{re} \) onto the set of half-spaces of \( \Sigma \) containing \( C_0 \). The \textbf{wall} associated to \( \alpha \in \Delta^{re} \) is the closed convex subset \( \partial \alpha := H(\alpha) \cap H(-\alpha) \) of \( \Sigma \), and coincides with the fixed-point set of the reflection \( r_\alpha \in W \). A \textbf{wall} \( \partial \alpha \) \textbf{separates} two chambers \( x, x' \in \text{Ch}(\Sigma) \) if \( x \in H(\epsilon \alpha) \) and \( x' \in H(-\epsilon \alpha) \) for some \( \epsilon \in \{\pm 1\} \).

We conclude this preliminary section with a short dictionary between roots and half-spaces. Call two roots \( \alpha, \beta \in \Delta^{re} \) \textbf{nested} if either \( H(\alpha) \subseteq H(\beta) \) or \( H(\beta) \subseteq H(\alpha) \).

\textbf{Lemma 2.} Let \( \alpha, \beta \in \Delta^{re} \) with \( \alpha \neq \pm \beta \), and set \( m := \langle \alpha, \beta^\vee \rangle \) and \( n := \langle \beta, \alpha^\vee \rangle \).

(1) \( \{\alpha, \beta\} \) is not prenilpotent \( \iff \) \( \{\alpha, -\beta\} \) is nested \( \iff \) \( m, n < 0 \) and \( mn \geq 4 \)

(2) \( \langle r_\alpha, r_\beta \rangle \subseteq W \) is finite \( \iff \) \( \partial \alpha \cap \partial \beta \neq \emptyset \iff \{\pm \alpha, \pm \beta\} \) does not contain any nested pair \( \iff \) \( \{\pm \alpha, \pm \beta\} \) is a root system of type \( A_1 \times A_1, A_2, B_2 \) or \( G_2 \).

(3) A subset \( \Psi \subseteq \Delta^{re} \) of roots is prenilpotent if and only if the subcomplexes \( \bigcap_{\alpha \in \Psi} H(\alpha) \) and \( \bigcap_{\alpha \in \Psi} H(-\alpha) \) of \( \Sigma \) both contain at least one chamber.

\textbf{Proof.} (1) The first equivalence is \([\text{AB}08, \text{Lemma 8.42}(3)]\), and the second and third equivalences follow from \([\text{Mar}13, \text{Proposition 4.31}(3)]\) (see also \([\text{Mar}18, \text{Exercise 7.42}]\).
(2) For the first equivalence, note that \( \langle r_\alpha, r_\beta \rangle \) fixes \( \partial \alpha \cap \partial \beta \). Hence if \( \partial \alpha \cap \partial \beta \neq \emptyset \), then \( \langle r_\alpha, r_\beta \rangle \) is finite (because the \( W \)-action on \( \Sigma \) is proper, hence has finite point stabilisers). Conversely, if \( \partial \alpha \cap \partial \beta = \emptyset \), then \( r_\alpha, r_\beta \) generate an infinite dihedral group.

For the second equivalence, note that since half-spaces are convex, they are in particular arc-connected, and hence if \( \partial \alpha \cap \partial \beta = \emptyset \) then either \( \{ \alpha, \beta \} \) or \( \{ \alpha, -\beta \} \) is nested. Conversely, if \( \{ \pm \alpha, \pm \beta \} \) contains a nested pair, then \( \langle r_\alpha, r_\beta \rangle \subseteq W \) is an infinite dihedral group.

Finally, for the third equivalence, note that \( \langle r_\alpha, r_\beta \rangle \) stabilises \( R := \{ \pm \alpha, \pm \beta \} \), and that \( \langle r_\alpha, r_\beta \rangle \) is finite if and only if \( R \) is finite. In this case, \( R \) is a (reduced) root system of rank 2 in the sense of [Bou68, VI, §1.1], and hence of one of the types \( A_1 \times A_1 \), \( A_2 \), \( B_2 \) or \( G_2 \) by [Bou68, VI, §4.2 Théorème 3], as desired.

(3) \( \Psi \) is prenilpotent if and only if there exist \( v, w \in W \) such that \( v\Psi \subseteq \Delta^<_r \) (i.e. \( v^{-1}C_0 \in \bigcap_{\alpha \in \Psi} H(\alpha) \)) and \( w\Psi \subseteq \Delta^>_r \) (i.e. \( w^{-1}C_0 \in \bigcap_{\alpha \in \Psi} H(-\alpha) \)). \( \square \)

3. Closed and prenilpotent sets of roots

We shall need the following lemma, which is a slight variation of [Cap06, Lemma 12].

**Lemma 3.** Let \( x \in \text{Ch}(\Sigma) \) be a chamber and \( \alpha \in \Delta^r \) be a root such that \( x \not\in H(\alpha) \). Let \( y \in \text{Ch}(\Sigma) \) be a chamber contained in \( H(\alpha) \) and at minimal distance from \( x \), and let \( \beta \in \Delta^r \) be a root such that \( H(\beta) \) contains \( x \) but not \( y \), and that \( H(-\beta) \) contains a chamber adjacent to \( x \). Assume that \( \beta \neq -\alpha \). Then:

(i) \( \langle \alpha, \beta^\vee \rangle < 0 \).

(ii) \( x \not\in H(\beta r(\alpha)) \).

**Proof.** For the first assertion, we invoke [Cap06, Lemma 12] that we apply to the roots \( \phi := \alpha \) and \( \psi := \beta \). We deduce that \( r_\alpha(\beta) \neq \beta \) and that \( H(r_\alpha(\beta)) \supseteq H(\alpha) \cap H(\beta) \). If \( \langle r_\alpha, r_\beta \rangle \) is infinite, then \( \partial \alpha \cap \partial \beta = \emptyset \), and since \( \{ \alpha, \beta \} \) is not nested by construction, the pair \( \{ \alpha, -\beta \} \) is nested and \( \langle \alpha, \beta^\vee \rangle < 0 \) (see Lemma 2(1)). If \( \langle r_\alpha, r_\beta \rangle \) is finite, on the other hand, the root system generated by \( \alpha \) and \( \beta \) is an irreducible root system of rank 2 (see Lemma 2(2)). A quick case-by-case inspection of the rank 2 root systems of type \( A_2 \), \( B_2 \) and \( G_2 \) (see [Hum78, §9.3]) reveals that the condition \( H(r_\alpha(\beta)) \supseteq H(\alpha) \cap H(\beta) \) implies that the angle between the walls of \( \alpha \) and \( \beta \) (in the Euclidean plane spanned by \( \alpha, \beta \) and with scalar product \( \langle \cdot, \cdot \rangle \) such that \( \langle \alpha, \beta^\vee \rangle = 2(\alpha|\beta)/(\alpha|\alpha) \)), see [Hum78, §9.1]) is acute. Thus the angle between the corresponding roots is obtuse, hence \( \langle \alpha, \beta^\vee \rangle < 0 \). This proves (i).

For the second assertion, let \( x' \in \text{Ch}(\Sigma) \) be the unique chamber adjacent to \( x \) and contained in \( H(-\beta) \). If \( x \in H(\beta r(\alpha)) \), then \( x' = r_\beta(x) \in r_\beta(H(\beta r(\alpha))) = H(\alpha) \). Since \( x \not\in H(\alpha) \) by hypothesis, it follows that the wall \( \partial \alpha \) separates \( x \) from \( x' \). Since \( x \) and \( x' \) are adjacent, we deduce that \( \beta = -\alpha \), contradicting the hypotheses. This proves (ii). \( \square \)

Given a set of roots \( \Phi \subseteq \Delta \) and a subset \( \Psi \subseteq \Phi \), we set

\[ \Phi_{\geq \Psi} := \{ \phi \in \Phi \mid \phi \geq \psi \text{ for some } \psi \in \Psi \} \]
Lemma 4. Let $\Phi \subseteq \Delta^re$ be a closed set of roots and let $\Psi \subseteq \Phi$ be a non-empty finite subset. Assume that $\Phi \cap -\Phi = \emptyset$. Then $\bigcap_{\phi \in \Phi \cap \Psi} H(\phi)$ contains a chamber. In particular, $\bigcap_{\psi \in \Psi} H(\psi)$ contains a chamber.

Proof. For simplicity, we identify each half-space $H(\phi)$ with its underlying set of chambers; we thus have to show that $\bigcap_{\phi \in \Phi \cap \Psi} H(\phi) \neq \emptyset$. For each set of roots $B \subseteq \Delta$, we set $B_x = B \cap \Delta_x$ for $x \in \{+, -, \emptyset\}$.

Set $A = \Phi \cap \Psi$. Observe that the set $A_-$ is finite, since $\Psi$ is finite by hypothesis. We shall proceed by induction on $|A_-|$.

In the base case $|A_-| = 0$, we have $A \subseteq \Delta^re$, and each half-space $H(\Phi)$ with $\Phi \in A$ contains the fundamental chamber $C_0$. Thus $\bigcap_{\Phi \in A} H(\Phi)$ is nonempty in this case.

We assume henceforth that $A_-$ is non-empty. We set $\mathcal{X} = \bigcap_{\Phi \in \Phi_+} H(\Phi)$. In the special case where $\Phi_+$ is empty, we adopt the convention that $\mathcal{X} = \text{Ch}(\Sigma)$. In all cases, we observe that $\mathcal{X}$ is non-empty since it contains $C_0$. We now distinguish two cases.

Assume first that there exists $\psi \in A_-$ such that $\mathcal{X}$ is not entirely contained in the half-space $H(-\psi)$. We may then choose a chamber $C \in \mathcal{X} \cap H(\psi)$. Let $w \in W$ be such that $wC = C_0$. For each $\Phi \in \Phi_+$, we have $C \in \mathcal{X} \subseteq H(\Phi)$, hence $C_0 \in H(w\Phi)$. Therefore $w\Phi_+ \subseteq \Delta_+$, and in particular $wA_+ \subseteq \Delta_+$. Moreover $wA_+ \cap \Delta_+$ contains $w\psi$, since $C \in H(\psi)$. It follows that $|wA_+ \cap \Delta_+| = |wA_- \cap \Delta_-| < |wA_-| = |A_-|$. Set

$$B = wA = w(\Phi \cap \Psi) = w(\Phi)_{w(\Psi)}.$$

We have just seen that $|B_-| < |A_-|$. Hence the induction hypothesis ensures that $\bigcap_{\phi \in B} H(\phi) \neq \emptyset$. Therefore, $\bigcap_{\phi \in A} H(\phi) = w^{-1}\left(\bigcap_{\phi \in B} H(\phi)\right)$ is also non-empty, and we are done in this case.

Assume finally that for all $\psi \in A_-$, we have $\mathcal{X} \subseteq H(-\psi)$. Choose $x \in \mathcal{X}$ and $\alpha \in A_-$ such that $d_{\text{Ch}}(x, H(\alpha))$ is minimal. Choose also $y \in H(\alpha)$ such that $d_{\text{Ch}}(x, y) = d_{\text{Ch}}(x, H(\alpha))$. Let $x = x_0, x_1, \ldots, x_m = y$ be a minimal gallery from $x$ to $y$, and let $\beta \in \Delta^re$ be the unique root such that $H(\beta)$ contains $x_0$ but not $x_1$.

We have $\beta \in \Phi_+$, since otherwise we would have $x_1 \in H(\phi)$ for all $\phi \in \Phi_+$, so that $x_1 \in \mathcal{X}$, contradicting the minimality condition in the definition of $x$. In particular, we have $\alpha \neq -\beta$ since $\Phi \cap -\Phi = \emptyset$.

Notice that $x$ is a chamber contained in $H(\beta)$ but not in $H(\alpha)$, whereas $y$ is a chamber contained in $H(\alpha)$ but not $H(\beta)$. We now invoke Lemma 3. It follows that $\langle \alpha, \beta \rangle < 0$ and that $x \not\in H(r_\beta(\alpha))$. Since $\beta$ is a positive root, it follows that $r_\beta(\alpha) = \alpha - \langle \alpha, \beta \rangle \beta \geq \alpha \in A = \Phi \cap \Psi$. Moreover $r_\beta(\alpha) \in \{\alpha, \beta\} \subseteq \Phi$. Hence $r_\beta(\alpha) \in A$. In particular, $r_\beta(\alpha) \in \Delta_-$, for otherwise $x \in \mathcal{X} \subseteq H(r_\beta(\alpha))$, a contradiction. We conclude that $r_\beta(\alpha) \in A_-$. Now we observe that the gallery $x = x_0 = r_\beta(x_1), r_\beta(x_2), \ldots, r_\beta(x_m) = r_\beta(y)$ is of length $m - 1$ and joins $x$ to a chamber in $H(r_\beta(\alpha))$. Since $r_\beta(\alpha)$ belongs to $A_-$, this contradicts the minimality condition in the definition of $\alpha$. Thus this final case does not occur, and the proof is complete. \qed
4. Nilpotent sets

Lemma 5. A subset $\Phi \subseteq \Delta^{re}$ is nilpotent if and only if it satisfies the following three conditions:

(i) $\Phi$ is closed.
(ii) $\Phi$ is finite.
(iii) $\Phi \cap -\Phi = \emptyset$.

Proof. Assume that $\Phi \subseteq \Delta^{re}$ satisfies the three conditions (i)–(iii). Applying Lemma 4 with $\Phi = \Psi$, we deduce that $\bigcap_{\phi \in \Phi} H(\phi)$ contains a chamber. Similarly, Lemma 4 applied to $-\Phi$ implies that $\bigcap_{\phi \in \Phi} H(-\phi)$ also contains a chamber. Hence Lemma 2(3) implies that $\Phi$ is nilpotent, as desired.

The converse assertion is clear by the definition of a nilpotent set of roots.

5. On the closure of a finite set

Lemma 6. Let $\Phi \subseteq \Delta^{re}$ be a closed set of roots such that $\Phi \cap -\Phi = \emptyset$. For any finite subset $\Psi \subseteq \Phi$, the closure $\overline{\Psi}$ is finite.

Proof. Applying Lemma 4 to $\Psi$ and $-\Psi$, we deduce that the intersections $\bigcap_{\psi \in \Psi} H(\psi)$ and $\bigcap_{\psi \in \Psi} H(-\psi)$ both contain a chamber. Hence $\Psi$ is prenilpotent by Lemma 2(3). The closure of any prenilpotent set of real roots is nilpotent, hence finite by Lemma 5.

6. Pro-nilpotent sets

Proposition 7. Let $\Phi \subseteq \Delta^{re}$ be a closed set of real roots such that $\Phi \cap -\Phi = \emptyset$. Then $\Phi$ is pro-nilpotent.

Proof. Let $n$ be a positive integer and let $\Phi_{\leq n}$ denote the subset of those $\phi \in \Phi$ with $|ht(\phi)| \leq n$. Thus $\Phi_{\leq n}$ is finite for all $n$, and the sets $\Phi_{\leq n}$ are linearly ordered by inclusion. By Lemma 6, the closure $\overline{\Phi_{\leq n}}$ is finite, hence nilpotent by Lemma 5. Thus $\Phi$ is the union of an ascending chain of nilpotent subsets.

7. Levi decomposition

Lemma 8. Let $\Psi \subseteq \Delta^{re}$ be a closed set of real roots. Set $\Psi_s := \{ \alpha \in \Psi \mid -\alpha \in \Psi \}$ and $\Psi_n := \Psi \setminus \Psi_s$. Then $\Psi_s$ is closed and $\Psi_n$ is an ideal in $\Psi$.

Proof. If $\alpha, \beta \in \Psi_s$ and $\alpha + \beta \in \Delta$, then $-\alpha, -\beta \in \Psi_s$ and hence $-(\alpha + \beta) \in \Psi$, that is, $\alpha + \beta \in \Psi_s$. This shows that $\Psi_s$ is closed.

If now $\alpha \in \Psi$ and $\beta \in \Psi_n$ are such that $\alpha + \beta \in \Delta$, then $\alpha + \beta \in \Psi_n$. Otherwise, $-(\alpha + \beta) \in \Psi$, and hence $-\beta = -(\alpha + \beta) + \alpha \in \Psi$, contradicting the fact that $\beta \in \Psi_n$. This shows that $\Psi_n$ is an ideal in $\Psi$.

Lemma 9. Let $\Psi \subseteq \Delta^{re}$ be a closed set of real roots such that $\Psi = -\Psi$. Then $\Psi$ is the root system of a finite-dimensional semisimple complex Lie algebra.
Proof. Let $H$ be the reflection subgroup of $\mathcal{W}$ generated by $R_\Psi := \{ r_\alpha \mid \alpha \in \Psi \}$. Then $H$ is a Coxeter group in its own right (see [Deo89]). Moreover, the set $M_\Psi$ of walls of $\Sigma(\mathcal{W}, S)$ corresponding to reflections in $R_\Psi$ is stabilised by $H$. Indeed, if $\alpha, \beta \in \Psi$, we have to check that $r_\alpha(\beta) = \beta - \beta(\alpha^\vee)\alpha \in \Psi$. But as $\pm \alpha \in \Psi$, this follows from the fact that $\Psi$ is closed. In particular, $H$ is finite, for otherwise it would contain two reflections $r_\alpha, r_\beta \in R_\Psi$ generating an infinite dihedral group by [Héé93, Prop. 8.1, p. 309], and hence $\Psi$ would contain a non-prenilpotent pair of roots by Lemma 2(1.2), contradicting the fact that $\Psi \subseteq \Delta^{re}$ (see Lemma 2(1)). This shows that $\Psi$ is a finite (reduced) root system in the sense of [Bou68, VI, §1.1], hence the root system of a finite-dimensional semisimple complex Lie algebra (see [Bou68, VI, §4.2 Théorème 3]), as desired. □

Theorem 10. Let $\Psi \subseteq \Delta^{re}$ be a closed set of real roots and let $\mathfrak{g}$ be the subalgebra of $\mathfrak{g}(A)$ generated by $\mathfrak{g}_\Psi$. Set $\Psi_s := \{ \alpha \in \Psi \mid -\alpha \in \Psi \}$ and $\Psi_n := \Psi \setminus \Psi_s$. Set also $\mathfrak{h}_s := \sum_{\gamma \in \Psi_s} C_\gamma^\vee$, $\mathfrak{g}_s := \mathfrak{h}_s \oplus \mathfrak{g}_s$, and $\mathfrak{g}_n := \mathfrak{g}_\Psi$. Then

1. $\mathfrak{g}_s$ is a subalgebra and $\mathfrak{g}_n$ is an ideal of $\mathfrak{g}$. In particular, $\mathfrak{g} = \mathfrak{g}_s \times \mathfrak{g}_n$.
2. $\mathfrak{g}_n$ is the unique maximal nilpotent ideal of $\mathfrak{g}$.
3. $\mathfrak{g}_s$ is a semisimple finite-dimensional Lie algebra with Cartan subalgebra $\mathfrak{h}_s$ and set of roots $\Psi_s$.

Proof. (1) follows from Lemma 8 and (3) from Lemma 9. For (2), note that $\Psi_n$ is a pro-nilpotent set of roots by Proposition 7, and hence $\mathfrak{g}_n$ is nilpotent by Lemma 1. Finally, since $\mathfrak{g}_s$ is semisimple, the image of any nilpotent ideal $\mathfrak{i}$ of $\mathfrak{g}$ under the quotient map $\mathfrak{g} \to \mathfrak{g}_s$ must be zero, that is, $\mathfrak{i} \subseteq \mathfrak{g}_n$. Therefore, $\mathfrak{g}_n$ is the unique maximal nilpotent ideal of $\mathfrak{g}$, as desired. □

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