A formula for the conductor of a semimodule of a numerical semigroup with two generators

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Abstract
We provide an expression for the conductor $c(\Delta)$ of a semimodule $\Delta$ of a numerical semigroup $\Gamma$ with two generators in terms of the syzygy module of $\Delta$ and the generators of the semigroup. In particular, we deduce that the difference between the conductor of the semimodule and the conductor of the semigroup is an element of $\Gamma$, as well as a formula for $c(\Delta)$ in terms of the dual semimodule of $\Delta$.

Keywords  Numerical semigroup · Frobenius problem · $\Gamma$-semimodule · Syzygy

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1 Introduction

Consider $\mathbb{N} = \{x \in \mathbb{Z} : x \geq 0\}$. A numerical semigroup $\Gamma$ is an additive sub-monoid of the monoid $(\mathbb{N}, +)$ such that the greatest common divisor of all its elements is equal...
to 1. The complement $\mathbb{N} \setminus \Gamma$ is therefore finite, and its elements are called gaps of $\Gamma$. Moreover, $\Gamma$ is finitely generated and it is not difficult to find a minimal system of generators of $\Gamma$, see, e.g., Rosales and García Sanchez [7].

The number $c(\Gamma) = \max(\mathbb{N} \setminus \Gamma) + 1$ is called the conductor of $\Gamma$; in particular $c(\Gamma) - 1$ is the Frobenius number of $\Gamma$. The computation of $c(\Gamma)$ for an arbitrary number of minimal generators of $\Gamma$ is NP-hard (see Ramírez Alfonsín [6] for a good account of this), but there are some special cases in which a closed formula is available. For example, if $\Gamma = \alpha \mathbb{N} + \beta \mathbb{N} := \langle \alpha, \beta \rangle$, then $c(\Gamma) = \alpha \beta - \alpha - \beta + 1$. However, for a numerical semigroup with more than two generators it is not possible in general to obtain a closed polynomial formula for its conductor in terms of the minimal set of generators (see Curtis [2]).

We are interested in subsets of $\mathbb{N}$ which have an additive structure over $\Gamma$ (in analogy with the structure of module over a ring): a $\Gamma$-semimodule is a non-empty subset $\Delta$ of $\mathbb{N}$ with $\Delta + \Gamma \subseteq \Delta$. A system of generators of $\Delta$ is a subset $\mathcal{E}$ of $\Delta$ such that $\Delta = \bigcup_{x \in \mathcal{E}} (x + \Gamma)$; it is called minimal if no proper subset of $\mathcal{E}$ generates $\Delta$. Notice that, since $\Delta \setminus \Gamma$ is finite, every $\Gamma$-semimodule is finitely generated and has a conductor

$$c(\Delta) = \max(\mathbb{N} \setminus \Delta) + 1.$$ 

Motivated by the Frobenius problem, it is natural to ask for a closed formula for the conductor of a $\Gamma$-semimodule. The purpose of this note is to give a formula for $c(\Delta)$ in the case $\Gamma = \langle \alpha, \beta \rangle$ in terms of the generators of the semimodule of syzygies of $\Delta$, see [4], as well as in terms of the generators of the dual of this semimodule, see [5]. These are the contents of our two main results, namely Theorem 1 resp. Corollary 2.

2 Semimodules over a numerical semigroup

Let $\Gamma$ be a numerical semigroup. This section is devoted to collect the main properties concerning $\Gamma$-semimodules. The reader is referred to [7] or [6] for specific material about numerical semigroups.

Every $\Gamma$-semimodule $\Delta$ has a unique minimal system of generators (see e.g. [4, Lemma 2.1]). Two $\Gamma$-semimodules $\Delta$ and $\Delta'$ are called isomorphic if there is an integer $n$ such that $x \mapsto x + n$ is a bijection from $\Delta$ to $\Delta'$; we write then $\Delta \cong \Delta'$. For every $\Gamma$-semimodule $\Delta$ there is a unique semimodule $\Delta' \cong \Delta$ containing 0; such a semimodule is called normalized. Moreover, the minimal system of generators $\{x_0 = 0, \ldots, x_n\}$ of a normalized $\Gamma$-semimodule is a $\Gamma$-lean set, i.e. it satisfies that

$$|x_i - x_j| \notin \Gamma \quad \text{for any} \quad 0 \leq i < j \leq n,$$

and conversely, every $\Gamma$-lean set of $\mathbb{N}$ minimally generates a normalized $\Gamma$-semimodule. Hence there is a bijection between the set of isomorphism classes of $\Gamma$-semimodules and the set of $\Gamma$-lean sets of $\mathbb{N}$. See Sect. 2 in [4] for the proofs of those statements.
There is another kind of system of generators—not minimal—for a semimodule \( \Delta \) of \( \Gamma \) relative to \( s \in \Gamma \setminus \{0\} \): this is the set of the \( s \) smallest elements in \( \Delta \) in each of the \( s \) classes modulo \( s \), namely the set \( \Delta \setminus (s + \Delta) \), and is called the Apéry set of \( \Delta \) with respect to \( s \); we write \( \text{Ap}(\Delta, s) \).

A formula for the conductor in terms of \( \text{Ap}(\Delta, s) \) for \( s \in \Gamma \setminus \{0\} \) is easily deduced.

**Proposition 1** Let \( \Delta \) be a \( \Gamma \)-semimodule. For any \( s \in \Gamma \setminus \{0\} \) we have that

\[ c(\Delta) - 1 = \max_{\leq \mathbb{N}} \text{Ap}(\Delta, s) - s. \]

**Proof** The equality follows as in the case \( \Delta = \Gamma \), see e.g., Lemma 3 in Brauer and Shockley [1]. \( \square \)

In this paper we will consider numerical semigroups with two generators, say \( \Gamma = \langle \alpha, \beta \rangle \), with \( \alpha, \beta \in \mathbb{N} \) with \( \alpha < \beta \) and \( \gcd(\alpha, \beta) = 1 \). As mentioned above, the conductor of \( \Gamma \) can be expressed as \( c = c(\langle \alpha, \beta \rangle) = (\alpha - 1)(\beta - 1) \). The gaps of \( \langle \alpha, \beta \rangle \) are also easy to describe: they admit a unique representation \( a\beta - a\alpha - b\beta \), where \( a \in [0, \beta - 1] \cap \mathbb{N} \) and \( b \in [0, \alpha - 1] \cap \mathbb{N} \). This writing yields a map from the set of gaps of \( \langle \alpha, \beta \rangle \) to \( \mathbb{N}^2 \) given by

\[ a\beta - a\alpha - b\beta \mapsto (a, b), \]

which allows us to identify a gap with a lattice point in the lattice \( L = \mathbb{N}^2 \); since the gaps are positive numbers, the point lies inside the triangle with vertices \((0, 0)\), \((0, \alpha)\), \((\beta, 0)\).

In the following we will use the notation

\[ e = \alpha\beta - a(e)\alpha - b(e)\beta \]

for a gap \( e \) of the semigroup \( \langle \alpha, \beta \rangle \); if the gap is subscripted as \( e_i \) then we write \( a_i = a(e_i) \) and \( b_i = b(e_i) \).

Let us denote by \( \leq \) the total ordering in \( \mathbb{N} \); sometimes we will write \( \leq \mathbb{N} \) to emphasize that it is the natural ordering. In addition, we define the following partial ordering \( \preceq \) on the set of gaps:

**Definition 1** Given two gaps \( e_1, e_2 \) of \( \langle \alpha, \beta \rangle \), we define

\[ e_1 \preceq e_2 : \iff a_1 \leq a_2 \ \& \ b_1 \geq b_2 \]

and

\[ e_1 < e_2 : \iff a_1 < a_2 \ \& \ b_1 > b_2. \]

Observe that the ordering \( \preceq \) differs from the one used by the second author and Uliczka in [3–5]: there the gaps \( e_i \) are ordered by decreasing sequence of the corresponding \( a_i \).
Let $\mathcal{E} = \{0, e_1, \ldots, e_n\} \subseteq \mathbb{N}$ with gaps $e_i = \alpha \beta - a_i \alpha - b_i \beta$ of $\langle \alpha, \beta \rangle$ for every $i = 1, \ldots, n$ such that $a_1 < a_2 < \cdots < a_n$. Corollary 3.3 in [4] ensures that $\mathcal{E}$ is $\langle \alpha, \beta \rangle$-lean if and only if $b_1 > b_2 > \cdots > b_n$.

This simple fact leads to an identification (cf. [4, Lemma 3.4]) between an $\langle \alpha, \beta \rangle$-lean set and a lattice path with steps downwards and to the right from $(0, \alpha)$ to $(\beta, 0)$ not crossing the line joining these two points, where the lattice points identified with the gaps in $\mathcal{E}$ mark the turns from the $x$-direction to the $y$-direction; these turns will be called ES-turns for abbreviation. Figure 1 shows the lattice path corresponding to the $\langle 5, 7 \rangle$-lean set $\{0, 9, 11, 8\}$.

Let $g_0 = 0, g_1, \ldots, g_n$ be the minimal system of generators of a $\langle \alpha, \beta \rangle$-semimodule $\Delta$. From now on, we will assume that the indexing in the minimal set of generators of $\Delta$ is such that $g_0 = 0 \preceq g_1 \preceq \cdots \preceq g_n$; accordingly we will use the notation $[g_0, \ldots, g_n]$ rather than $\{g_0, \ldots, g_n\}$. In [4] it was introduced the notion of syzygy of $\Delta$ as the $\langle \alpha, \beta \rangle$-semimodule

$$\text{Syz}(\Delta) := \bigcup_{i,j \in \{0, \ldots, n\}, i \neq j} \left( (\Gamma + g_i) \cap (\Gamma + g_j) \right).$$

The semimodule of syzygies of the semimodule $\Delta$ minimally generated by $[g_0 = 0, g_1, \ldots, g_n]$ can be characterized as follows (see [4, Theorem 4.2]; since Definition 1 differs from the corresponding in [3–5]—as mentioned above, Definition 2 must be conveniently adapted here):

**Definition 2**

$$\text{Syz}(\Delta) = \bigcup_{0 \leq k < j \leq n} \left( (\Gamma + g_k) \cap (\Gamma + g_j) \right) = \bigcup_{k=0}^{n} (\Gamma + h_k),$$

where $h_1, \ldots, h_{n-1}$ are gaps of $\Gamma$, $h_0, h_n \preceq \alpha \beta$, and The next formulas must be aligned at $\equiv$ as shown (also in the pdf file!)

$$h_k \equiv g_k \mod \beta, \; h_k > g_k \; \text{for} \; k = 0, \ldots, n$$

$$h_k \equiv g_{k+1} \mod \alpha, \; h_k > g_{k+1} \; \text{for} \; k = 0, \ldots, n - 1$$

$$h_n \equiv 0 \mod \alpha, \; \text{and} \; h_n \geq 0$$
In particular, \( J = [h_0, \ldots, h_n] \) is a minimal system of generators of the semimodule \( \text{Syz}(\Delta) \), hence \( h_0 \leq h_1 \leq \cdots \leq h_n \). Therefore it is easily seen that the SE-turns of the lattice path associated to \( \Delta \) can be identified with the minimal set of generators of the syzygy module (we call SE-turns to the turns from the \( y \)-direction to the \( x \)-direction). After that, we can associate to any \( \Gamma \)-semimodule \( \Delta \) a lean set \([I, J]\), where \( I \) is a minimal set of generators of \( \Delta \) and \( J \) a minimal set of generators of \( \text{Syz}(\Delta) \); or, equivalently, a lattice path. An easy consequence of this fact is the following lemma.

**Lemma 1** Let \( \Delta \) be a \( \Gamma \)-semimodule with associated \( \Gamma \)-lean set \([I, J]\) for \( I = [g_0 = 0, g_1, \ldots, g_n] \) and \( J = [h_0, \ldots, h_n] \). Then, for any \( h \in J \) we have \( h - \alpha - \beta \notin \Delta \).

**Proof** Consider \( h \in J \) such that that \( g_i < h < g_{i+1} \). Let us denote \((a_j, b_j)\) resp. \((a_{j+1}, b_{j+1})\) the coordinates of \( g_j \) resp. \( g_{j+1} \) in the lattice \( \mathcal{L} \); then the element \( h \) is represented in the lattice path as \((a_j, b_{j+1})\), see Definition 2. By contradiction, assume that \( h - \alpha - \beta \in \Delta \); then there exists a gap \( g \in I \) together with two integers \( v_1, v_2 \in \mathbb{N} \) such that

\[
h - \alpha - \beta = v_1 \alpha + v_2 \beta + g.
\]

Since \( h - \alpha - \beta \notin \Gamma \), we may write

\[
h - \alpha - \beta = a\beta - (a_j + 1)\alpha - (b_{j+1} + 1)\beta.
\]

The writing of \( g \) as \( g = a\beta - a\alpha - b\beta \) is unique whenever \((a, b) \in \mathcal{L}\), therefore

\[
a_j + 1 = a - v_1, \quad b_{j+1} + 1 = b - v_2.
\]

These equalities yield the conditions \( a_j < a \) and \( b_{j+1} < b \). But the unique minimal generator which fulfills these conditions is \( g_{j+1} \); however, \( h \) cannot be expressed as \( h = g_{j+1} + v + \alpha + \beta \) since \( h \) is represented in the lattice path as \((a_j, b_{j+1})\), a contradiction. \( \square \)

**Example 1** For \( \Gamma = \langle 5, 7 \rangle \) and the \( \Gamma \)-semimodule \( \Delta_I \) minimally generated by \( I = [0, 9, 11, 8] \), it is easily deduced that the syzygy module \( \text{Syz}(\Delta_I) \) is minimally generated by \( J = [14, 16, 18, 15] \), cf. Fig. 1; there we have extended the labelling beyond the axis in the natural way in order to have also an interpretation of \( J \) in terms of the lattice path. Observe that by Lemma 1 we have \( 14 - 7 - 5 = 2 \notin \Delta \), \( 16 - 7 - 5 = 4 \notin \Delta \), \( 18 - 7 - 5 = 6 \notin \Delta \) and \( 15 - 7 - 5 = 3 \notin \Delta \); this can be read off from Fig. 1 as well.

### 3 A formula for the conductor of an \( \langle \alpha, \beta \rangle \)-semimodule

In this section we are going to provide a formula for the conductor of a \( \Gamma \)-semimodule with any number of generators in terms of the generators of \( \Gamma \) and a special syzygy of the \( \Gamma \)-semimodule. In particular, we will obtain some relations between the conductor of \( \Gamma \) and the conductor of the \( \Gamma \)-semimodule. Finally, we will provide a formula for the conductor of the \( \Gamma \)-semimodule in terms of its dual.
Theorem 1 Let $\Delta$ be a $\Gamma$-semimodule with associated lean set $[I, J]$ as above, and let $M := \max_{\leq \mathbb{N}} \{ h \in J \}$ denote the biggest (with respect to the total ordering of the natural numbers) minimal generator of $\text{Syz}(\Delta)$. Then

$$c(\Delta) = M - \alpha - \beta + 1.$$  

In particular, if $(m_1, m_2)$ are the coordinates of the point representing $M$ in the lattice $L$, then we have

$$c(\Delta) = c(\Gamma) - m_1\alpha - m_2\beta.$$  

Proof Since $c(\Delta) - 1$ is the Frobenius number of the $\Gamma$-semimodule $\Delta$, it is enough to check that (i) $M - \alpha - \beta \notin \Delta$, and (ii) if $\ell \notin \Delta$, then $\ell \leq M - \alpha - \beta$. The statement (i) is clear by Lemma 1, since $M \in J$. To see (ii), consider an element $\ell \notin \Delta$, which in particular means $\ell \notin \Gamma$. So we can associate to $\ell$ a point $(a, b)$ in the lattice $L$. Moreover, $\ell$ is upon and not contained in the lattice path associated to $I$. This means that there exists some $j \in J$ with coordinates $(j_1, j_2)$ in the lattice path such that $a > j_1$ and $b > j_2$, otherwise $\ell$ would be an element of $\Delta$, since the elements represented by lattice points on and under the lattice path belong to $\Delta$. Therefore, $a \geq j_1 + 1$ and $b \geq j_2 + 1$. Thus, from the representation of $\ell$ and $j$ as gaps we can check that

$$\ell = \alpha\beta - a\alpha - b\beta \leq \mathbb{N} \alpha\beta - (j_1 + 1)\alpha - (j_2 + 1)\beta = j - \alpha - \beta.$$  

Hence, since $M = \max_{\leq \mathbb{N}} \{ h \in J \}$ and $M \in J$, we have that $M - \alpha - \beta \geq \mathbb{N} \ell$ for any $\ell \notin \Delta$, which proves (ii).

Finally, since $M$ can be represented as a lattice point $(m_1, m_2) \in L$, we have

$$c(\Delta) = M - \alpha - \beta + 1 = \alpha\beta - m_1\alpha - m_2\beta - \alpha - \beta + 1 = c(\Gamma) - m_1\alpha - m_2\beta.$$  

Example 2 Again in the case of $\Gamma = \langle 5, 7 \rangle$ and the $\Gamma$-semimodule minimally generated by $[0, 9, 11, 8]$, Fig. 1 illustrates that the maximal syzygy is $M = 18$, and so the conductor of the semimodule is $c(\Gamma) - 5m_1 - 7m_2 = 24 - 5 \cdot 2 - 7 \cdot 1 = 7$.

Notice that for the particular case of $\Delta = \Gamma$ we have $M = \alpha\beta$, and we recover the well-known formula $c(\Gamma) = \alpha\beta - \alpha - \beta + 1$. The value $M$ can be easily characterized in terms of the Apéry set of $\Delta$ with respect to $\alpha + \beta$:

Proposition 2 Let $M := \max_{\leq \mathbb{N}} \{ h \in J \}$ be the biggest minimal generator of the syzygy module with respect to the natural ordering of $\mathbb{N}$ as above, then

$$M = \max_{\leq \mathbb{N}} \text{Ap}(\Delta, \alpha + \beta).$$  

Proof This is a consequence of Proposition 1 for $s = \alpha + \beta \in \langle \alpha, \beta \rangle$.  

\[\square\]
A straightforward consequence of Theorem 1 is the following.

**Corollary 1** Let $\Delta$ be a $\Gamma$-semimodule. Then

$$c(\Gamma) - c(\Delta) \in \Gamma.$$  

We conclude this paper rewriting the formula of Theorem 1 in terms of the dual $\Gamma$-semimodule of $\Delta$,

$$\Delta^* := \{ z \in \mathbb{Z} \mid z + \Delta \subseteq \Gamma \},$$

see [5]. An important fact about the dual semimodule is that the minimal set of generators of $\text{Syz}(\Delta)$ is in bijection with the minimal set of generators of $\Delta^*$:

**Lemma 2** ([5], Lemma 6.1) The minimal sets of generators of $\Delta^*$ and $\text{Syz}(\Delta)$ are in correspondence via the map $x \mapsto \alpha \beta - x$.

In particular, this bijection together with Theorem 1 allows us to compute the conductor of the semimodule $\Delta$ in terms of the minimal generators of $\Delta^*$ in a natural way:

**Corollary 2** Let $\Delta$ be a $\Gamma$-semimodule, and let $\Delta^*$ be its dual, minimally generated by $x_0, \ldots, x_n$. Then

$$c(\Delta) = \alpha \beta - \min_{\leq N}\{x_0, \ldots, x_n\} - \alpha - \beta + 1.$$  

**Proof** By Theorem 1 we have that $c(\Delta) = \max_{\leq N}\{h \in J\} - \alpha - \beta + 1$, where $J$ is a minimal set of generators of $\text{Syz}(\Delta)$. Lemma 2 yields the equality

$$\min_{\leq N}\{x_0, x_1, \ldots, x_n\} = \alpha \beta - \max_{\leq N}\{h \in J\},$$

which allows us to conclude. $\square$

**Example 3** By [5, Theorem 2.5], the minimal generators of the dual of the $\langle 5, 7 \rangle$-semimodule $\Delta_I$ are given by $[20, 17, 19, 21]$; notice that, for the explicit calculation, the mentioned theorem requires the reverse ordering $\succeq$ instead of the ordering $\preceq$ we use here. The minimum of this set is 17, therefore by Corollary 2 we have $c(\Delta) = 35 - 17 - 12 + 1 = 7$, as computed in Example 2.

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