Equivalent description of Hom-Lie algebroids

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Abstract

In this paper, we study representations of Hom-Lie algebroids, give some properties of Hom-Lie algebroids and discuss equivalent statements of Hom-Lie algebroids. Then, we prove that two known definitions of Hom-Lie algebroids can be transformed into each other under some conditions.

1 Introduction

The notion of Hom-Lie algebras was introduced by Hartwig, Larsson, and Silvestrov in [3] as a part of a study of deformations of the Witt and the Virasoro algebras. In a Hom-Lie algebra, the Jacobi identity is twisted by a linear map, called the Hom-Jacobi identity. Some $q$-deformations of the Witt and the Virasoro algebras have the structure of a Hom-Lie algebra [3, 4]. Because of close relations to discrete and deformed vector fields and differential calculus [3, 5, 6], more people pay special attention to this algebraic structure. For a party of $k$-cochains on Hom-Lie algebras, name $k$-Hom-cochains, there is a series of coboundary operators [11]; for regular Hom-Lie algebras, [12] gives a new coboundary operator on $k$-cochains, and there are many works have been done by the special coboundary operator [12, 13]. In [15], there is a series of coboundary operators, and the author generalizes the result * If $\mathfrak{g}$ is a Lie algebra, $\rho : \mathfrak{g} \rightarrow \mathfrak{gl}(V)$ is a representation if and only if there is a degree-1 operator $D$ on $\Lambda^* \otimes V$ satisfying $D^2 = 0$, and

$$D(\xi \wedge \eta \otimes u) = d_\xi \xi \wedge \eta \otimes u + (-1)^\lambda \xi \wedge D(\eta \otimes u), \quad \forall \xi \in \Lambda^k \mathfrak{g}^*, \eta \in \Lambda^l \mathfrak{g}^*, u \in V,$$

where $d_\xi : \Lambda^k \mathfrak{g}^* \rightarrow \Lambda^{k+1} \mathfrak{g}^*$ is the coboundary operator associated to the trivial representation.*

Geometric generalizations of Hom-Lie algebras are given in [7, 13]. In [7], C. Laurent-Gengoux and J. Teles proved that there is a one-to-one correspondence between Hom-Gerstenhaber algebras and Hom-Lie algebroids; in [14], base on Hom-Lie algebroids from [7], the authors study representation of Hom-Lie algebroids. In [15], the authors make small modifications to the definition of Hom-Lie algebroids, and give a new definition of Hom-Lie algebroids, base on the new definition of Hom-Lie algebroids, definitions of Hom-Lie bialgebroids and Hom-Courant algebroids are given.

In this article, we first study representations of Hom-Lie algebroids, give equivalent statements of Hom-Lie algebroids and prove that different definitions of Hom-Lie algebroids are given by the same Hom-Lie algebras and their representations.

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*Keyword: Hom-Lie algebroids; Hom-Lie algebras; representations

**MSC: 17B99,58H05

Supported by the NSF of China (No.11771382) and the Science and Technology Project(GJJ161029)of Department of Education, Jiangxi Province.
The notion of a Hom-Lie algebra was introduced in [3], see also [2, 8] for more information.

We study representations of Hom-Lie algebroids, and some properties of Hom-Lie algebroids.

(1) Definition 2.1. A Hom-Lie algebra is a triple \((\mathfrak{g}, [\cdot , \cdot ], \alpha)\) consisting of a vector space \(\mathfrak{g}\), a skew-symmetric bilinear map (bracket) \([\cdot , \cdot ] : \wedge^2 \mathfrak{g} \rightarrow \mathfrak{g}\) and a linear transformation \(\alpha : \mathfrak{g} \rightarrow \mathfrak{g}\) satisfying \([\alpha(x), [y, z]] + [\alpha(y), [z, x]] + [\alpha(z), [x, y]] = 0, \ \forall x, y, z \in \mathfrak{g}\). (1)

A Hom-Lie algebra is called a regular Hom-Lie algebra if \(\alpha = \text{id}\).

(2) A subspace \(\mathfrak{h} \subset \mathfrak{g}\) is a Hom-Lie sub-algebra of \((\mathfrak{g}, [\cdot , \cdot ], \alpha)\) if \(\alpha(\mathfrak{h}) \subset \mathfrak{h}\) and \(\mathfrak{h}\) is closed under the bracket operation \([\cdot , \cdot ]\), i.e. for all \(x, y \in \mathfrak{h}\), \([x, y] \in \mathfrak{h}\).

(3) A morphism from the Hom-Lie algebra \((\mathfrak{g}, [\cdot , \cdot ], \alpha)\) to the Hom-Lie algebra \((\mathfrak{h}, [\cdot , \cdot ], \gamma)\) is a linear map \(\psi : \mathfrak{g} \rightarrow \mathfrak{h}\) such that \(\psi([x, y]_\mathfrak{g}) = [\psi(x), \psi(y)]_\mathfrak{h}\) and \(\psi \circ \alpha = \gamma \circ \psi\).

Representation and cohomology theories of Hom-Lie algebra are systematically introduced in [1, 9]. See [10] for homology theories of Hom-Lie algebras.

Definition 2.2. A representation of the Hom-Lie algebra \((\mathfrak{g}, [\cdot , \cdot ], \alpha)\) on a vector space \(V\) with respect to \(\beta \in \mathfrak{gl}(V)\) is a linear map \(\rho : \mathfrak{g} \rightarrow \mathfrak{gl}(V)\), such that for all \(x, y \in \mathfrak{g}\), the following equalities are satisfied:

\[
\rho([x, y]) = [\rho(x), \rho(y)] + [\rho(y), \rho(x)].
\]

Let \((\mathfrak{g}, [\cdot , \cdot ], \alpha)\) be a Hom-Lie algebra, \(V\) be a vector space, \(\rho : \mathfrak{g} \rightarrow \mathfrak{gl}(V)\) be a representation of \((\mathfrak{g}, [\cdot , \cdot ], \alpha)\) on the vector space \(V\) with respect to \(\beta \in \mathfrak{gl}(V)\).

The set of \(k\)-cochains on \(\mathfrak{g}\) with values in \(V\), which we denote by \(C^k(\mathfrak{g}; V)\), is the set of skew-symmetric \(k\)-linear maps from \(\mathfrak{g} \times \cdots \times \mathfrak{g}(k\text{-times})\) to \(V\):

\[
C^k(\mathfrak{g}; V) := \{ \eta : \wedge^k \mathfrak{g} \rightarrow V \text{ is a linear map} \}.
\]

In [15], when \(\beta \in GL(V)\), there is a series operators \(d^k : C^k(\mathfrak{g}; V) \rightarrow C^{k+1}(\mathfrak{g}; V)\) is given by

\[
d^k \eta(x_1, \ldots, x_{k+1}) = \sum_{i=1}^{k+1} (-1)^{i+1} \beta^{k+1+s} \rho(x_i) \beta^{-2-s} \eta(\alpha(x_1), \ldots, \widehat{x_i}, \ldots, \alpha(x_{k+1}))
\]

\[
+ \sum_{i<j} (-1)^{i+j} \eta([x_i, x_j], \alpha(x_1), \ldots, \widehat{x_i}, \ldots, \widehat{x_j}, \ldots, \alpha(x_{k+1})),
\]

where \(\beta^{-1}\) is the inverse of \(\beta\), \(\eta \in C^k(\mathfrak{g}; V)\), and the author have the results: \(d^k \circ d^k = 0\).
2.2 Hom-Lie algebroids

Now, we introduce two kinds of definitions of Hom-Lie algebroids, they are from [7] and [13]. More about Hom-Lie algebroids, please see [7] and [13].

**Definition 2.3.** [7] A Hom-Lie algebroid is a quintuple \((A,\varphi,\cdot,\cdot,\rho_A,\alpha_A)\), where \(A\) is a vector bundle over a manifold \(M, \varphi : M \rightarrow M\) is a smooth map, \([\cdot,\cdot] : \Gamma(A) \otimes \Gamma(A) \rightarrow \Gamma(A)\) is a bilinear map, called bracket, \(\rho_A : \varphi^*A \rightarrow \varphi^*TM\) is a vector bundle morphism, called anchor and \(\alpha_A : \Gamma(A) \rightarrow \Gamma(A)\) is a linear endomorphism of \(\Gamma(A)\), for \(X, Y \in \Gamma(A), f \in C^\infty(M)\) such that:

1. \(\alpha_A(fX) = \varphi^*(f)\alpha_A(X)\);
2. the triple \((\Gamma(A),\cdot,\cdot,\alpha_A)\) is a Hom-Lie algebra;
3. the following Hom-Leibniz identity holds:
   \[ [X,fY] = \varphi^*(f)[X,Y] + \rho_A(X)(f)\alpha_A(Y); \]
4. \(\rho_A\) is a representation of Hom-Lie algebra \((\Gamma(A),\cdot,\cdot,\alpha_A)\) on \(C^\infty(M)\) with respect to \(\varphi^*\).

In fact, according to Definition 2.3 for \(X, Y \in \Gamma(A), f, g \in C^\infty(M)\), we have the following properties:

(a) \(\alpha_A([X,Y]) = [\alpha_A(X),\alpha_A(Y)];\)
(b) \(\varphi^* : C^\infty(M) \rightarrow C^\infty(M)\), defined by \(\varphi^*(f) = f \circ \varphi, \varphi^*\) is a morphism of \(C^\infty(M)\);
(c) \(\rho_A(X)(fg) = \varphi^*(f)\rho_A(X)(g) + \varphi^*(g)\rho_A(X)(f)\). According to (3) in Definition:
   \[ \rho_A(X)(fg) = [[X,f]g] = [[X,f]]\alpha_A(g) + [[X,g]]\alpha_A(f) = \alpha_A(g)\rho_A(X)(f) + \alpha_A(f)\rho_A(X)(g) \]
   where \([\cdot,\cdot]\) is defined in Definition 3.1 of [7].
(d) \(\alpha_A(f) = \varphi^*(f)\), when \(\alpha = \text{id}\), then \(\varphi^* = \text{id}\), Hom-Lie algebroid \((A,\varphi,\cdot,\cdot,\rho_A,\alpha_A)\) is just a Lie algebroid;
(e) \(\rho_A(fX) = \varphi^*(f)\rho_A(X)\). It follows from:
   \[ \rho_A(fX)(g) = [[fX,g]] = \alpha_A(f)g = \varphi^*(f)\rho_A(X)(g). \]

**Definition 2.4.** [13] A Hom-Lie algebroid is a quintuple \((B,\varphi,\cdot,\cdot,\rho_B,\alpha_B)\), where \(B\) is a vector bundle over a manifold \(M, \varphi : M \rightarrow M\) is a smooth map, \([\cdot,\cdot] : \Gamma(B) \otimes \Gamma(B) \rightarrow \Gamma(B)\) is a bilinear map, called bracket, \(\rho_B : \varphi^*B \rightarrow \varphi^*TM\) is a bundle morphism, called anchor and \(\alpha_B : \Gamma(B) \rightarrow \Gamma(B)\) is a linear endomorphism of \(\Gamma(B)\), for \(X, Y \in \Gamma(B), f \in C^\infty(M)\) such that:

1. \(\alpha_B(fX) = \varphi^*(f)\alpha_B(X);\)
2. the triple \((\Gamma(B),\cdot,\cdot,\alpha_B)\) is a Hom-Lie algebra;
3. the following Hom-Leibniz identity holds:
   \[ [X,fY] = \varphi^*(f)[X,Y] + \rho_B(\alpha_B(X))(f)\alpha_B(Y); \]
4) \( \rho_B \) is a representation of Hom-Lie algebra \( (\Gamma(B), [\cdot, \cdot], \alpha_B) \) on \( C^\infty(M) \) with respect to \( \varphi^* \).

From Definition 2.4, for \( X, Y \in \Gamma(A) \), \( f, g \in C^\infty(M) \), we have:

a) \( \alpha_B([X, Y]) = [\alpha_B(X), \alpha_B(Y)] \);

b) \( \varphi^* : C^\infty(M) \to C^\infty(M) \), defined by \( \varphi^*(f) = f \circ \varphi \), is a morphism of \( C^\infty(M) \);

c) \( \rho_B(X)(fg) = \varphi^*(f)\rho_B(X)(g) + \varphi^*(g)\rho_B(X)(f) \).

d) \( \alpha_B(f) = \varphi^*(f) \).

e) \( \rho_B(fX) = f\rho_B(X) \).

When \( \alpha_B \) and \( \varphi \) are invertible, Hom-Lie bialgebroids and Hom-Courant algebroids are given in [13].

3 Representations of Hom-Lie algebroids

In this section, we assume that map \( \varphi : M \to M \) is an involution, i.e. \( \varphi^2 = \text{id} \).

Let \( (E, \varphi, [\cdot, \cdot], \rho_E, \alpha_E) \) be a Hom-Lie algebroid. Whence \( (\rho_E, \varphi^*) \) is a representation of \( (\Gamma(E), [\cdot, \cdot], \alpha_E) \) on \( C^\infty(M) \). Where \( E \) is \( A \) or \( B \). We define \( d^s : C^k(\Gamma(E); C^\infty(M)) \to C^{k+1}(\Gamma(E); C^\infty(M)) \), \( s = 0, 1, \ldots \), by setting

\[
d^s\eta(X_1, \cdots, X_{k+1}) = \sum_{i=1}^{k+1}(-1)^{i+1}\varphi^*\rho_E(X_i)\varphi^{*k-s-1}\eta(\alpha_E(X_1), \cdots, \hat{X}_i, \cdots, \alpha_E(X_{k+1})) + \sum_{i<j}(-1)^{i+j}\eta([X_i, X_j], \alpha_E(X_1), \cdots, \hat{X}_i, \cdots, \hat{X}_j, \cdots, \alpha_E(X_{k+1})),
\]

where \( C^k(\Gamma(E); C^\infty(M)) = \{ \eta : \wedge^k \Gamma(E) \to C^\infty(M) \} \) is a \( \mathbb{R} \) linear map, \( X_i \in \Gamma(E) \).

We define map \( \alpha_E^* : C^k(\Gamma(E); C^\infty(M)) \to C^k(\Gamma(E); C^\infty(M)) \) by

\[
\alpha_E^*(\eta)(X_1, \cdots, X_k) = \varphi^* \circ \eta(\alpha_E(X_1), \cdots, \alpha_E(X_k)), \quad X_i \in \Gamma(E).
\]

When \( f \in C^0(\Gamma(E); C^\infty(M)) = C^\infty(M) \), we have \( \alpha_E^*(f) = \varphi^*(f) \).

Let \( \Gamma(\wedge^k E^*) = \{ \eta \in C^k(\Gamma(E); C^\infty(M)) \mid f \in C^\infty(M), \quad \eta(fX_1, \cdots, X_k) = f\eta(X_1, \cdots, X_k) \} \).

Then, \( \Gamma(\wedge^k E^*) \) is a subset of \( C^k(\Gamma(E); C^\infty(M)) \). Let \( \alpha_E^* \) acts on \( \Gamma(\wedge^k E^*) \), we have:

\[
\alpha_E^* : \Gamma(\wedge^k E^*) \to \Gamma(\wedge^k E^*).
\]

Actually, when \( \xi \in \Gamma(E^*) \), for \( f \in C^\infty(M) \), we have

\[
\alpha_E^*(\xi)(fX) = \varphi^* \circ \xi(\alpha_E(fX)) = \varphi^* \circ \xi(\varphi^*(f)\alpha_E(X)) = \varphi^2(f)\varphi^* \circ \xi(\alpha_E(X)) = f\alpha_E^*(\xi)(X),
\]

so, \( \alpha_E^*(\xi) \in \Gamma(E^*) \), and we have

\[
\alpha_E^*(\xi_1 \wedge \cdots \wedge \xi_k) = \alpha_E^*(\xi_1) \wedge \cdots \wedge \alpha_E^*(\xi_k), \quad \xi_1, \cdots, \xi_k \in \Gamma(E^*).
\]

Let \( \Gamma(\wedge^* E^*) = \bigoplus_k \Gamma(\wedge^k E^*) \), \( C^* \Gamma(E); C^\infty(M) = \bigoplus_k C^k(\Gamma(E); C^\infty(M)) \), we have: \( \Gamma(\wedge^* E^*) \) is a subset of \( C^* \Gamma(E); C^\infty(M) \). So, if \( \xi \in \Gamma(\wedge^* E^*) \), we have \( d^s\xi \in C^* \Gamma(E); C^\infty(M) \).
At the same time, \( \varphi^* \) can induce a map \( \overline{\varphi^*} : C^k(\Gamma(E); C^\infty(M)) \to C^k(\Gamma(E); C^\infty(M)) \), which define by

\[
\overline{\varphi^*}(\eta)(X_1, \cdots, X_k) = \varphi^* \circ \eta(X_1, \cdots, X_k), \quad X_i \in \Gamma(E).
\]

Then, we have:

\[
\overline{\varphi^*}(\eta_1 \wedge \cdots \wedge \eta_k) = \overline{\varphi^*}(\eta_1) \wedge \cdots \wedge \overline{\varphi^*}(\eta_k).
\]

**Proposition 3.1.** With above notations, for \( \eta_1 \in C^k(\Gamma(E); C^\infty(M)), \eta_2 \in C^l(\Gamma(E); C^\infty(M)), \) we have:

\[
d^s(\eta_1 \wedge \eta_2) = d^{s+1}\eta_1 \wedge \overline{\varphi^*} \circ \alpha^*_E(\eta_2) + (-1)^l \overline{\varphi^*} \circ \alpha^*_E(\eta_1) \wedge d^{s+1} \eta_2.
\]

**Proof.** First let \( k = 1, \eta_1 \wedge \eta_2 \in C^{d+1}(\Gamma(E); C^\infty(M)) \), we have:

\[
d^s(\eta_1 \wedge \eta_2)(X_1, \cdots, X_{l+2})
= \sum_{i=1}^l (-1)^{i+1} \varphi_i \rho_{E}(X_1) \varphi_i^{-l-3-s} \eta_1 \wedge \eta_2, \quad \alpha_E(X_1), \cdots, \alpha_E(X_{l+2})
+ \sum_{i<j} (-1)^{i+j} \eta_1 \wedge \eta_2([X_i, X_j], \alpha_E(X_1), \cdots, \alpha_E(X_{l+2}))
= \sum_{i=1}^l (-1)^{i+1} \eta_1 \wedge \eta_2(\alpha_E(X_1), \cdots, \alpha_E(X_{l+2}))(\varphi_i^{l+2+s} \rho_{E}(X_1) \varphi_i^{-l-3-s} \eta_1 \wedge \eta_2)
+ \sum_{i<j} (-1)^{i+j} \eta_1([X_i, X_j], \alpha_E(X_1), \cdots, \alpha_E(X_{l+2}))
= \sum_{i=1}^l (-1)^{i+1} \eta_1 \wedge \eta_2(\alpha_E(X_1), \cdots, \alpha_E(X_{l+2}))(\varphi_i^{l+2+s} \rho_{E}(X_1) \varphi_i^{-l-3-s} \eta_1 \wedge \eta_2)
+ \sum_{i<j} (-1)^{i+j} \eta_1([X_i, X_j], \alpha_E(X_1), \cdots, \alpha_E(X_{l+2}))

So, when \( k = 1, \) we have:

\[
d^s(\eta_1 \wedge \eta_2) = d^{s+1}\eta_1 \wedge \overline{\varphi^*} \circ \alpha^*_E(\eta_2) + (-1)^l \overline{\varphi^*} \circ \alpha^*_E(\eta_1) \wedge d^{s+1} \eta_2.
\]

By induction on \( k, \) assume that when \( k = n, \) we have:

\[
d^s(\eta_1 \wedge \eta_2) = d^{s+1}\eta_1 \wedge \overline{\varphi^*} \circ \alpha^*_E(\eta_2) + (-1)^n \overline{\varphi^*} \circ \alpha^*_E(\eta_1) \wedge d^{s+n} \eta_2.
\]
For any \( \eta_3 \in C^1(\Gamma(E); C^\infty(M)) \), then \( \eta_1 \wedge \eta_3 \in C^{n+1}(\Gamma(E); C^\infty(M)) \), we have
\[
d^s(\eta_1 \wedge \eta_3 \wedge \eta_2) = d^s(\eta_1 \wedge (\eta_3 \wedge \eta_2)) = d^{s+1}(\eta_1 \wedge \varphi^s \circ \alpha_E^s(\eta_3) \wedge \eta_2) + (-1)^s \varphi^s \circ \alpha_E^s(\eta_1) \wedge d^{s+1}(\eta_3 \wedge \eta_2) = d^{s+1}(\eta_1 \wedge \varphi^s \circ \alpha_E^s(\eta_3) \wedge \varphi^s \circ \alpha_E^s(\eta_2) + (-1)^s \varphi^s \circ \alpha_E^s(\eta_1) \wedge d^{s+1}(\eta_3 \wedge \eta_2)
\]
\[
\begin{aligned}
&\quad + (-1)^s \varphi^s \circ \alpha_E^s(\eta_1) \wedge \varphi^s \circ \alpha_E^s(\eta_3) \wedge \varphi^s \circ \alpha_E^s(\eta_2) \\
&= d^{s+1}(\eta_1 \wedge \varphi^s \circ \alpha_E^s(\eta_3) \wedge \varphi^s \circ \alpha_E^s(\eta_2) + (-1)^s \varphi^s \circ \alpha_E^s(\eta_1) \wedge d^{s+1}(\eta_3 \wedge \eta_2) \\
&\quad + (-1)^s \varphi^s \circ \alpha_E^s(\eta_1) \wedge \varphi^s \circ \alpha_E^s(\eta_3) \wedge \varphi^s \circ \alpha_E^s(\eta_2) \\
&= d^{s+1}(\eta_1 \wedge \eta_3 \wedge \eta_2) \quad \text{and} \quad d^{s+1}(\eta_1 \wedge \eta_3 \wedge \eta_2).
\end{aligned}
\]
The proof is completed. ■

**Proposition 3.2.** With above notations, we have:
\[
\alpha_E^s \circ d^s = d^s \circ \alpha_E^s; \quad \varphi^s \circ d^s = d^{s+1} \circ \varphi^s.
\]

**Proof.** With straightforward computations, for any \( \eta \in C^k(\Gamma(E); C^\infty(M)) \), we have:
\[
\begin{aligned}
&\alpha_E^s \circ d^s \eta(X_1, \cdots, X_{k+1}) \\
&= \varphi^s \circ d^s \eta(\alpha_E(X_1), \cdots, \alpha_E(X_{k+1})) \\
&= \sum_{i=1}^{n} (-1)^{i+1} \varphi^{s+k+2-s} \rho_E(\alpha_E(X_i)) \varphi^{s-k-2-s} \eta(\alpha_E^s(X_1), \cdots, \hat{X}_i, \cdots, \alpha_E^s(X_{k+1})) \\
&\quad + \sum_{i<j} (-1)^{i+j} \varphi^s \circ \eta([\alpha_E(X_i), \alpha_E(X_j)], \alpha_E^s(X_1), \cdots, \hat{X}_{i,j}, \cdots, \alpha_E^s(X_{k+1})) \\
&= \sum_{i=1}^{n} (-1)^{i+1} \varphi^{s+k+3-s} \rho_E(X_i) \varphi^{s-k-3-s} \alpha_E^s(\eta)(\alpha_E(X_1), \cdots, \hat{X}_i, \cdots, \alpha_E(X_{k+1})) \\
&\quad + \sum_{i<j} (-1)^{i+j} \alpha_E^s(\eta)([X_i, X_j], \alpha_E(X_1), \cdots, \hat{X}_{i,j}, \cdots, \alpha_E(X_{k+1})) \\
&= d^s \circ \alpha_E^s(\eta)(X_1, \cdots, X_{k+1}).
\end{aligned}
\]
At the same time, we have:
\[
\varphi^s \circ d^s \eta(X_1, \cdots, X_{k+1}) \\
= \sum_{i=1}^{n} (-1)^{i+1} \varphi^{s+k+2-s} \rho_E(X_i) \varphi^{s-k-2-s} \eta(\alpha_E(X_1), \cdots, \hat{X}_i, \cdots, \alpha_E(X_{k+1})) \\
= \sum_{i<j} (-1)^{i+j} \alpha_E^s(\eta)([X_i, X_j], \alpha_E(X_1), \cdots, \hat{X}_{i,j}, \cdots, \alpha_E(X_{k+1})) \\
= d^{s+1} \circ \varphi^s(\eta)(X_1, \cdots, X_{k+1}).
\]
We complete this proof. ■

Now, we revisited representations of Hom-Lie algebroids respectively base on Definition 2.3 and Definition 2.4.
Theorem 3.3. Let $A$ be a vector bundle over manifold $M, \varphi : M \to M$ is a smooth map and $\varphi^2 = \text{id}$, $\alpha_A : \Gamma(A) \to \Gamma(A)$ is a linear endomorphism of $\Gamma(A)$ i.e. for $f \in C^\infty(M), X \in \Gamma(A), \alpha_A(fX) = \varphi^*(f)\alpha_A(X)$. Then $(A, \varphi, [,\cdot,\cdot], \rho_A, \alpha_A)$ is a Hom-Lie algebriod define by Definition 2.3 if and only if there is a series operators $d^s : C^k(\Gamma(A); C^\infty(M)) \to C^{k+1}(\Gamma(A); C^\infty(M)), s = 0, 1, \ldots$, and such that:

(i) $d^s \circ d^s = 0$;
(ii) for any $\eta_1 \in C^k(\Gamma(A); C^\infty(M)), \eta_2 \in C^l(\Gamma(A); C^\infty(M))$, we have

$$d^s(\eta_1 \wedge \eta_2) = d^{s+1}\eta_1 \wedge \varphi^* \circ \alpha_A^*(\eta_2) + (-1)^k \varphi^* \circ \alpha_A^*(\eta_1) \wedge d^{s+k}\eta_2.$$ 

(iii) $\alpha_A^* \circ d^s = d^s \circ \alpha_A^*$;
(iv) for $f \in C^\infty(M) = C^0(\Gamma(A); C^\infty(M))$, we have: $d^0 f \in \Gamma(A^*)$.
(v) for $f \in C^\infty(M), \xi \in \Gamma(A^*), \ we $ have:

$$d^0 \xi(fX_1, X_2) = \varphi^*(f)d^0 \xi(X_1, X_2).$$

Proof. For necessity, with above Propositions which we proved, we just need to prove (iv) and (v). For Hom-Lie algebriod $(A, \varphi, [,\cdot,\cdot], \rho_A, \alpha_A)$ and $f, g \in C^\infty(M) = C^0(\Gamma(E); C^\infty(M)), X \in \Gamma(A)$, by the definition of $d^s$, we have:

$$d^s f(gX) = \varphi^* \rho_A(gX)f = g \varphi^* \rho_A(X)f = gd^0 f(X).$$

For $f \in C^\infty(M), \xi \in \Gamma(A^*)$, we have:

$$d^0 \xi(fX_1, X_2) = \rho_A(fX_1) \varphi^* \xi(\alpha_A(X_2)) - \rho_A(X_2) \varphi^* \xi(\alpha_A(fX_1)) - \xi([fX_1, X_2])$$

$$= \varphi^*(f) \rho_A(X_1) \varphi^* \xi(\alpha_A(X_2)) - \varphi^*(f) \rho_A(X_2) \varphi^* \xi(\alpha_A(X_1)) - \xi(\alpha_A(X_1)) \rho_A(X_2) f$$

$$- \xi(\varphi^*(f)[X_1, X_2]) - \rho_A(X_2)f \alpha_A(X_1))$$

$$= \varphi^*(f) d^0 \xi(X_1, X_2).$$

So, we proved the necessity of this Theorem. Now, we prove the adequacy of this theorem. Sept1, we define $\rho_A : \varphi^* A \to \varphi^* TM$ by:

$$\varphi^* \rho_A(X)f = d^0 f(X), \quad X \in \Gamma(A), f \in C^\infty(M).$$

Then, by (iv), for $g \in C^\infty(M), \varphi^* \rho_A(gX)f = d^0 f(gX) = gd^0 f(X) = g \varphi^* \rho_A(X)f$, we have:

$$\rho_A(gX) = \varphi^*(g) \rho_A(X).$$

On the other hand, we have:

$$d^0 (fg)(X) = \varphi^* \rho_A(X)(fg).$$

By (ii), for $f, g \in C^0(\Gamma(A); C^\infty(M)) = C^\infty(M)$, we have:

$$d^0 (fg) = d^0 fg + fd^0 g.$$
Then, we have:
\[ \rho_A(X)(fg) = \varphi^*(g)\rho_A(X)f + \varphi^*(f)\rho_A(X)g. \]  \hspace{1cm} (4)

The definition of \( \rho_A \) is reasonable.

By (iii), we have:
\[ \alpha_A^* \circ d^0 f(X) = \varphi^* \circ d^0 f(\alpha_A(X)) \]
\[ = \varphi^* \circ \varphi^* \rho_A(\alpha_A(X))f \]
\[ = d^0 \alpha_A^*(f)(X) \]
\[ = \varphi^* \circ \rho_A(X) \varphi^*(f). \]

We find the result:
\[ \rho_A(\alpha_A(X)) \varphi^* = \varphi^* \rho_A(X). \]  \hspace{1cm} (5)

Sept2, for any \( \xi \in \Gamma(A^*), X, Y \in \Gamma(A) \), we define \([\cdot, \cdot] : \Gamma(A) \wedge \Gamma(A) \rightarrow \Gamma(A)\) by
\[ \xi([X, Y]) = \rho_A(X) \varphi^* \xi(\alpha_A(Y)) - \rho_A(Y) \varphi^* \xi(\alpha_A(X)) - d^0 \xi(X, Y). \]  \hspace{1cm} (6)

So, by (7), we have:
\[ \alpha_A^*(\xi([X, Y])) = \varphi^* \rho_A(X) \alpha_A^*(\xi(\alpha_A(Y))) - \varphi^* \rho_A(Y) \alpha_A^*(\xi(\alpha_A(X))) - d^0 \alpha_A^*(\xi(X, Y)) \]
\[ = \rho_A(X) \alpha_A^*(\xi(\alpha_A(Y))) - \rho_A(Y) \alpha_A^*(\xi(\alpha_A(X))) - d^0 \alpha_A^*(\xi(X, Y)). \]

By \( \alpha_A^*(\xi([X, Y])) = \varphi^* \xi([\alpha_A(X), \alpha_A(Y)]) \), (5) and (iii), (7), we have
\[ \xi([\alpha_A([X, Y])]) = \varphi^* \rho_A(X) \varphi^* \xi(\alpha_A^*(\xi(\alpha_A(Y)))) - \varphi^* \rho_A(Y) \varphi^* \xi(\alpha_A^*(\xi(\alpha_A(X)))) - d^0 \xi(\alpha_A(X), \alpha_A(Y)) \]
\[ = \xi([\alpha_A(X), \alpha_A(Y)]). \]

We have the following:
\[ \alpha_A([X, Y]) = [\alpha_A(X), \alpha_A(Y)]. \]  \hspace{1cm} (7)

For any \( f \in C^\infty(M) \), by (i), (2) and (8), we have:
\[ 0 = d^0 \circ d^0 f(X, Y) \]
\[ = \rho_A(X) \varphi^* d^0 f(\alpha_A(Y)) - \rho_A(Y) \varphi^* d^0 f(\alpha_A(X)) - d^0 f([X, Y]) \]
\[ = \rho_A(X) \rho_A(\alpha_A(Y))f - \rho_A(Y) \rho_A(\alpha_A(X))f - \varphi^* \rho_A([X, Y])f. \]

We get
\[ \rho_A(X) \rho_A(\alpha_A(Y)) - \rho_A(Y) \rho_A(\alpha_A(X)) = \varphi^* \rho_A([X, Y]) \]
\[ \varphi^* \rho_A(X) \rho_A(\alpha_A(Y)) \varphi^* - \varphi^* \rho_A(Y) \rho_A(\alpha_A(X)) \varphi^* = \rho_A([X, Y]) \varphi^* \]
\[ \rho_A(\alpha_A(X)) \rho_A(\alpha_A(Y)) - \rho_A(\alpha_A(Y)) \rho_A(\alpha_A(X)) = \rho_A([X, Y]) \varphi^*. \]  \hspace{1cm} (8)

Sept3, by (6), (3), (4) and (v), we have:
\[ \xi([X, fY]) = \rho_A(X) \varphi^* \xi(\alpha_A(fY)) - \rho_A(fY) \varphi^* \xi(\alpha_A(X)) - d^0 \xi(X, fY) \]
\[ = \rho_A(X) (f \varphi^* \xi(\alpha_A(Y))) - \varphi^* (f) \rho_A(Y) \varphi^* \xi(\alpha_A(X)) - \varphi^* (f) d^0 \xi(X, Y) \]
\[ = \varphi^* (f) \xi([X, Y]) + \xi(\rho_A(X)f \alpha_A(Y)). \]
So, for $f \in C^\infty(M)$, $X, Y \in \Gamma(A)$, we have:

$$[X, fY] = \varphi^*(f)[X, Y] + \rho_A(X)f\alpha_A(Y).$$

(9)

Sept4, by (iii), we have

$$d^1 = \overline{\varphi} \circ d^0 \circ \overline{\varphi}.$$

For $\eta_1, \eta_2 \in C^1(\Gamma(A); C^\infty(M))$, by (ii), we have

$$d^0(\eta_1 \wedge \eta_2) = d^1 \eta_1 \wedge \overline{\varphi} \circ \alpha^*_A(\eta_2) - \overline{\varphi} \circ \alpha^*_A(\eta_1) \wedge d^1 \eta_2$$

$$= \overline{\varphi} \left( d^0(\overline{\varphi}(\eta_1) \wedge \alpha^*_A(\eta_2) - \alpha^*_A(\eta_1) \wedge d^0(\overline{\varphi}(\eta_2)) \right).$$

By (\ref{10}),

$$d^0(\eta_1 \wedge \eta_2)(X, Y, Z)$$

$$= \varphi^*\rho_A(X)\eta_1(\alpha_A(Y)\eta_2(\alpha_A(Z)) - \varphi^*\rho_A(X)\eta_1(\alpha_A(Z)\eta_2(\alpha_A(Y))) + \eta_1(\alpha_A(Y))\varphi^*\rho_A(X)\eta_2(\alpha_A(Z))$$

$$- \eta_1(\alpha_A(Z))\varphi^*\rho_A(X)\eta_2(\alpha_A(Y)) - \varphi^*\rho_A(X)\eta_1(\alpha_A(Z))\eta_2(\alpha_A(Y)) + \varphi^*\rho_A(Y)\eta_1(\alpha_A(Z))\eta_2(\alpha_A(Y))$$

$$- \varphi^*\rho_A(Z)\eta_1(\alpha_A(Y))\eta_2(\alpha_A(X)) + \eta_1(\alpha_A(Y))\varphi^*\rho_A(Z)\eta_2(\alpha_A(X)) - \eta_1(\alpha_A(Y))\varphi^*\rho_A(Z)\eta_2(\alpha_A(X))$$

$$- \eta_1([X, Y])\eta_2(\alpha_A(Z)) + \eta_1(\alpha_A(Z))\eta_2([X, Y]) + \eta_1([X, Z])\eta_2(\alpha_A(Y)) - \eta_1(\alpha_A(Y))\eta_2([X, Z])$$

$$- \eta_1([Y, Z])\eta_2(\alpha_A(X)) + \eta_1(\alpha_A(X))\eta_2([Y, Z]) + \eta_1(\alpha_A(X), Y) - \eta([Y, Z], \alpha_A(X)).$$

Then, for any $\eta \in C^2(\Gamma(A); C^\infty(M))$, we have

$$d^0\eta(X, Y, Z) = \varphi^*\rho_A(X)\eta(\alpha_A(Y), \alpha_A(Z)) - \varphi^*\rho_A(Y)\eta(\alpha_A(X), \alpha_A(Z)) + \varphi^*\rho_A(Z)\eta(\alpha_A(X), \alpha_A(Y))$$

$$- \eta([X, Y], \alpha_A(Z)) + \eta([X, Z], \alpha_A(Y)) - \eta([Y, Z], \alpha_A(X)).$$

Sept5, for $\xi \in \Gamma(A^*)$, by (\ref{9}), (\ref{8}) and (i), we have:

$$0 = d^0 \circ d^0 \xi(X, Y, Z)$$

$$= \varphi^*\rho_A(X)d^0\xi(\alpha_A(Y), \alpha_A(Z)) - \varphi^*\rho_A(Y)d^0\xi(\alpha_A(X), \alpha_A(Z)) + \varphi^*\rho_A(Z)d^0\xi(\alpha_A(X), \alpha_A(Y))$$

$$- d^0\xi([X, Y], \alpha_A(Z)) + d^0\xi([X, Z], \alpha_A(Y)) - d^0\xi([Y, Z], \alpha_A(X))$$

$$= \xi([X, Y], \alpha_A(Z)) + [Y, Z], \alpha_A(X)) + [Z, X], \alpha_A(Y))].$$

So, we have:

$$[[X, Y], \alpha_A(Z)] + [Y, Z], \alpha_A(X)) + [Z, X], \alpha_A(Y))] = 0.$$ 

(10)

By (\ref{9}) and (\ref{10}), $\Gamma(A), [\cdot, \cdot], \alpha_A)$ is a Hom-Lie algebra.

By (\ref{9}), we have: $[X, fY] = \varphi^*(f)[X, Y] + \rho_A(X)f\alpha_A(Y)$.

By (\ref{7}) and (\ref{8}), $\rho_A$ is a representation of Hom-Lie algebra $(\Gamma(A), [\cdot, \cdot], \alpha_A)$ on $C^\infty(M)$ with respect to $\varphi^*$.

The proof is completed. ■

**Theorem 3.4.** Let $B$ be a vector bundle over manifold $M$, $\varphi : M \to M$ is a smooth map and $\varphi^2 = \text{id}$, $\alpha_B : \Gamma(B) \to \Gamma(B)$ is a linear endomorphism of $\Gamma(B)$ i.e. for $f \in C^\infty(M), X \in \Gamma(B), \alpha_B(fX) = \varphi^*f \alpha_B(X)$. Then $(B, \varphi, [\cdot, \cdot], \rho_B, \alpha_B)$ is a Hom-Lie algebroid define by Definition 2.4 if and only if there is a series operators $d^s : C^\infty(\Gamma(B); C^\infty(M)) \to C^{\infty+1}(\Gamma(B); C^\infty(M)), s = 0, 1, \ldots$, and such that:

-
Theorem 3.3. So, we proved the necessity of this Theorem. The sufficiency of this theorem is similar with

Proof. For necessity, with above Propositions which we proved, we just need to prove 4) and 5).

For Hom-Lie algebrido $(B, \varphi, [\cdot,\cdot], \rho_B, \alpha_B)$ and $f, g \in C^\infty(M) = C^0(\Gamma(B); C^\infty(M)), X \in \Gamma(B)$, by the definition of $d^s$, we have:

$$d^s f(gX) = \rho_B(gX) \varphi^s f = g \rho_B(X) \varphi^s f = g d^sf(X).$$

For Hom-Lie algebrido $(B, \varphi, [\cdot,\cdot], \rho_B, \alpha_B)$ and $f \in C^\infty(M), \xi \in \Gamma(B^*)$, we have:

$$d^\xi (fX_1, X_2) = \varphi^s (f) d^\xi (X_1, X_2).$$

So, we proved the necessity of this Theorem. The sufficiency of this theorem is similar with Theorem 3.3. $lacksquare$

Remark 3.5. For $E$ is a vector bundle over $M$, we hope that we are able to get a pull-back diagram:

$$
\begin{array}{ccc}
E & \xrightarrow{\varphi^1} & E \\
\downarrow & & \downarrow \\
M & \xrightarrow{\varphi} & M.
\end{array}
$$

So, we assume that $\varphi^2 = \text{id}$.

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