THE SECOND MAIN THEOREM IN THE HYPERBOLIC CASE

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Abstract. We develop Nevanlinna’s theory for a class of holomorphic maps when the source is a disc. Such maps appear in the theory of foliations by Riemann Surfaces.

1. Introduction

In 1929, Nevanlinna [15] established the Second Main Theorem for meromorphic functions on the complex plane \( \mathbb{C} \). Later, S. S. Chern [5] extended the result to holomorphic mappings from the complex plane into compact Riemann surfaces. In 1933, H. Cartan [4] developed the theory for holomorphic mappings from the complex plane to \( \mathbb{P}^n(\mathbb{C}) \) and studied the intersection with hyperplanes in general position. At the same time, it was observed (first by Nevanlinna) that the results also hold for meromorphic functions from the unit disc \( \triangle(1) \) to \( \mathbb{P}^1(\mathbb{C}) \), under the condition that

\[
\lim_{r \to 1} \frac{T_f(r)}{\log \frac{1}{1-r}} = \infty.
\]

Tsuji [26] gives an exposition of this theory. In this paper, we introduce a new class of maps from the disc of radius \( R \) with \( 0 < R \leq \infty \), for which we obtain a Second Main Theorem. Let \( \triangle(R) \) denote the disc of radius \( R \) with the convention that \( \triangle(\infty) = \mathbb{C} \). Let \( M \) be a Hermitian manifold and \( \omega \) be a positive \((1,1)\) form of finite mass on \( M \). Recall that, for a non-constant holomorphic map \( f : \triangle(R) \to M \), the characteristic (or height) function of \( f \) with respect to \( \omega \) is defined, for \( 0 < r < R \), as

\[
T_{f,\omega}(r) = \int_0^r \frac{dt}{t} \int_{|z|<t} f^* \omega.
\]

For each \( c < \infty \), let

\[
\mathcal{E}_c = \left\{ f \mid \int_0^R \exp(cT_{f,\omega}(r)) dr = \infty \right\},
\]

(1)

\[
\mathcal{E} = \bigcup_{c<\infty} \mathcal{E}_c \quad \text{and} \quad \mathcal{E}_0 = \cap_{c>0} \mathcal{E}_c.
\]

(2)
Observe that the set $E_c$ contains the maps from the unit disc to $M$ which satisfy, for $r$ close to 1,
\[
\frac{T_{f,\omega}(r)}{\log \frac{1}{1-r}} \geq \frac{1}{c}.
\]
This is an important class of maps. They occur, for example, as the universal covering maps of leaves in foliation by Riemann surfaces. This is our main motivation.

Generic foliations in $\mathbb{P}^n(\mathbb{C})$ are “Brody hyperbolic”, i.e. they do not admit a non-constant image of $\mathbb{C}$ tangent to the foliation out of the singular points (see [8] and [14]). So leaves are uniformized by the unit disc. It turns out that frequently the uniformizing map is in $E_c$. When the foliation is “Brody hyperbolic”, we get also the inequality $\frac{T_{f,\omega}(r)}{\log \frac{1}{1-r}} \leq A$ for a positive constant $A$.

The value distribution of these maps is related to the complexity of the dynamics. It is conjectured that, for generic foliations, the leaves are dense. So their distribution is far from trivial. The extension of Cartan’s theorem which we obtained can be applied. In the next section, we will list some examples. The space $E_0$ is the space of maps of fast growth.

**Definition 1.1.** Let $M$ be a complex manifold and $\omega$ be a positive $(1,1)$ form of finite volume on $M$. Let $0 < R \leq \infty$ and $f : \Delta(R) \to M$ be a holomorphic map. We define the growth index of $f$ with respect to $\omega$ as
\[
c_{f,\omega} := \inf \left\{ c > 0 \mid \int_0^R \exp(cT_{f,\omega}(r))dr = \infty \right\}.
\]
The critical constant of $M$ with respect to $\omega$, denoted by $c_{\text{cri},M}^\omega$, is defined as
\[
c_{\text{cri},M}^\omega = \inf \{ c \mid \exists a \text{ non-constant holomorphic map } f : \Delta(1) \to M,
\text{ and } \int_0^1 \exp(cT_{f,\omega}(r))dr = \infty \}.
\]

In this paper, whenever $c_{f,\omega}$ is involved, we always assume that the set $\left\{ c > 0 \mid \int_0^R \exp(cT_{f,\omega}(r))dr = \infty \right\}$ is non-empty. If $f$ is of bounded characteristic (hence $R < \infty$), then $c_{f,\omega} = \infty$. In the case where $R = \infty$, noticing that $\int_0^R \exp(cT_{f,\omega}(r))dr = \infty$ for any arbitrary small $\epsilon$ if $f$ is not constant, $c_{f,\omega} = 0$ and $f$ is in $E_0$. Thus our results also include the classical results for mappings on the whole complex plane $f : \mathbb{C} \to M$.

When $M$ is compact the spaces $E$ and $E_0$ are independent of the form $\omega$, so they are intrinsic objects. Indeed we can characterize the Kobayashi hyperbolicity by using $E_0$ (see Theorem 2.1 below) as follows: Let $M$ be a...
compact complex manifold. \( M \) is hyperbolic if and only if the class \( \mathcal{E}_0(\triangle(1)) \) is an empty set.

The Second Main Theorems will be derived for maps \( f : \triangle(R) \to M \) with \( c_{f,\omega} < \infty \). In particular, we derive the defect for \( f \) in \( M \) in terms of \( c_{f,\omega} \).

In the case where \( M \) is hyperbolic, for example \( M \) is a Riemann surface of genus \( \geq 2 \), there is no non-constant holomorphic map \( f : \mathbb{C} \to M \). However, there are many non-constant holomorphic maps \( f : \triangle(1) \to M \) which are in \( \mathcal{E} \). Our result (see Theorem 1.5) shows that if \( c_{f,\omega} < \infty \), then \( \sum_{j=1}^{q} \delta_{f,\omega}(a_j) \leq c_{f,\omega} - 1 \) for any distinct points \( a_1, \ldots, a_q \in M \), where \( \omega \) is the Poincaré form on \( M \) and \( \delta_{f,\omega}(a) \) is the defect properly measured. This is a new phenomenon. We also get a similar result for a compact Riemann surface with finitely many points removed.

The theory here can be regarded as a new illustration of Bloch’s principle: *Nihil est in infinito quod non prius fuerit in finito*. This is explained as: every proposition with a statement on the actual infinity can be always considered a consequence of a proposition in finite terms.

We introduce some notations. For a complex variable \( z \), let

\[
\partial u = \frac{\partial u}{\partial z} dz, \quad \bar{\partial} u = \frac{\partial u}{\partial \bar{z}} d\bar{z}.
\]

Let \( d = \partial + \bar{\partial}, d^c = \frac{\sqrt{-1}}{4\pi} (\bar{\partial} - \partial) \). We have \( dd^c = \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \). Let \( M \) be a Riemann surface. Let \( \omega = a(z) \frac{\sqrt{-1}}{2\pi} dz \land d\bar{z} \) be a non-negative \((1,1)\) form on \( M \). Let \( \text{Ric}(\omega) := dd^c \log a \). Then we have

\[
\text{Ric}(\omega) = -K\omega,
\]

where \( K \) is the Gauss curvature of the metric form \( \omega \). For example, on the unit disc \( \triangle(1) \), the Poincaré metric form \( \omega = \frac{2}{(1-|z|^2)^2} \frac{\sqrt{-1}}{2\pi} dz \land d\bar{z} \) has Gauss curvature \(-1\).

We state our results. For notations, see Section 2.

**Theorem 1.2** (The Second Main Theorem). Let \( M \) be a compact Riemann surface. Let \( \omega \) be a positive \((1,1)\) form on \( M \). Let \( f : \triangle(R) \to M \) be a holomorphic map such that \( c_{f,\omega} < +\infty \), where \( 0 < R \leq \infty \). Let \( a_1, \ldots, a_q \) be distinct points on \( M \). Then, for every \( \epsilon > 0 \), the inequality

\[
\sum_{j=1}^{q} m_{f,\omega}(r, a_j) + T_{f,\text{Ric}(\omega)}(r) + N_{f,\text{ram}}(r) 
\leq (1 + \epsilon)(c_{f,\omega} + \epsilon)T_{f,\omega}(r) + O(\log T_{f,\omega}(r)) + \epsilon \log r
\]

holds for all \( r \in (0, R) \) outside a set \( E \) with \( \int_{E} \exp((c_{f,\omega} + \epsilon)T_{f,\omega}(r)) dr < \infty \), where \( N_{f,\text{ram}}(r) \) is the counting function for the ramification divisor of \( f \).
Remarks. (a) We note that in the case where $R = \infty$ we have $c_{f,\omega} = 0$, so we recover the usual Second Main Theorem for $f : \mathbb{C} \to M$ (due to Chern) with a better error term $\epsilon \log r$. Since the error term is $\epsilon \log r$ rather than $O(\log r)$ in the Second Main Theorem, we don’t need anymore to assume that $f$ is transcendental.

(b) The above theorem also works if we replace the compact Riemann surface $M$ by an open Riemann surface $U$ in a compact Riemann surface $M$ such that $M \setminus U$ is a set of finite number of points. However, to get positive results, we need to consider a metric defined only in $U$. See the remark after Theorem 1.3.

(c) Note that we can also let $c_{f,\omega}$ depends on $r$, i.e., we can consider the $c(r) > 0$ with

$$\int_0^R \exp(c(r)T_{f,\omega}(r))dr = \infty.$$ 

We then get similar results.

In the case where $M = \mathbb{P}^1(\mathbb{C})$, since

$$\omega_{FS} = \frac{1}{(1 + |w|^2)^2} \sqrt{-1} \frac{\sqrt{-1}}{2\pi} dw \wedge d\bar{w} = dd^c \log(1 + |w|^2),$$

we get that

$$\text{Ric}(\omega_{FS}) = -2\omega_{FS}.$$ 

Hence Theorem 1.2 gives

**Theorem 1.3.** Let $f : \Delta(R) \to \mathbb{P}^1(\mathbb{C})$ be a holomorphic map such that $c_f < +\infty$, where $c_f := c_{f,\omega_{FS}}$ and $0 < R \leq \infty$. Let $a_1, \ldots, a_q$ be distinct points on $\mathbb{P}^1(\mathbb{C})$. Then

$$\sum_{j=1}^q \delta_f(a_j) \leq 2 + c_f.$$ 

In particular, $f$ cannot omit more than $[2+c_f]$ points in $\mathbb{P}^1(\mathbb{C})$ if $c_f$ is finite.

**Remark.** Let $U$ be an open subset of $\mathbb{P}^1(\mathbb{C})$ such that $\mathbb{P}^1(\mathbb{C}) \setminus U$ is an infinite set. Let $\phi$ denote the universal covering map $\phi : \Delta(1) \to U$. From the fact that the image of $\phi$ omits infinitely many points in $\mathbb{P}^1(\mathbb{C})$, Theorem 1.3 tells us that $c_f = \infty$. If $\mathbb{P}^1(\mathbb{C}) \setminus U$ is finite, then Theorem 1.3 implies that $c_f \geq (q - 2)$ where $q = \#(\mathbb{P}^1(\mathbb{C}) \setminus U)$.

In the elliptic case, the canonical metric is flat, i.e. there exists a positive $(1,1)$ form $\omega$ whose curvature is 0, so $\text{Ric}(\omega) = 0$. As a consequence of Theorem 1.2, we get
Theorem 1.4. Let $M$ be a compact Riemann surface of genus 1 and $\omega$ be the positive $(1,1)$ form with $\text{Ric}(\omega) = 0$. Let $f : \triangle(R) \to M$ be a holomorphic map with $c_{f,\omega} < \infty$, where $0 < R \leq \infty$. Then
\[
\sum_{j=1}^{q} \delta_{f,\omega}(a_j) \leq c_{f,\omega}.
\]
In particular, $f$ cannot omit more than $\lfloor c_{f,\omega} \rfloor$ points in $M$ if $c_{f,\omega}$ is finite.

In the case where the compact Riemann surface is of genus $\geq 2$, there is a positive $(1,1)$ form $\omega$ whose curvature $-1$ so $\text{Ric}(\omega) = -\omega$. We get the following result using a variation of the proof of Theorem 1.2.

Theorem 1.5. Let $U$ be either a compact Riemann surface or a Riemann surface in a compact Riemann surface $M$ such that $M \setminus U$ consists of a finite number of points. Let $\omega$ be a positive $(1,1)$ form of finite volume on $U$ whose Gauss curvature is bounded from above by $-\lambda$ with $\lambda > 0$, i.e., $\text{Ric}(\omega) \geq \lambda \omega$. Let $f : \triangle(R) \to U$ be a holomorphic map with $c_{f,\omega} < \infty$, where $0 < R \leq \infty$. Then $c_{f,\omega} \geq \lambda$. Furthermore, let $a_1, \ldots , a_q$ be distinct points on $U$, then, for every $\epsilon > 0$, the inequality
\[
\sum_{j=1}^{q} m_{f,\omega}(r, a_j) + N_{f,\text{ram}}(r) 
\leq ((1 + \epsilon)(c_{f,\omega} + \epsilon) - \lambda)T_{f,\omega}(r) + O(\log T_{f,\omega}(r)) + \epsilon \log r
\]
holds for all $r \in (0, R)$ outside a set $E$ with $\int_{E} \exp((c_{f,\omega} + \epsilon)T_{f,\omega}(r))dr < \infty$. In particular, we have
\[
\sum_{j=1}^{q} \delta_{f,\omega}(a_j) \leq c_{f,\omega} - \lambda.
\]

Note that in the above case, i.e. $U$ is hyperbolic, there is no non-constant holomorphic map $f : \triangle(1) \to U$. However, there are many non-constant maps from the unit-disk into $U$, for example, the universal covering map $\phi : \triangle(1) \to U$. If we take the Poincaré metric form $\omega_P$ (i.e., whose Gauss curvature is $-1$), then it is easy to compute that $c_{\phi,\omega_P} = 1$ since $\phi^*\omega_P$ is the Poincaré metric on $\triangle(1)$. On the other hand, from Theorem 1.5 above, we know that for any non-constant holomorphic map $f : \triangle(1) \to U$ we have $c_{f,\omega_P} \geq 1$. So the universal covering map $\phi : \triangle(1) \to U$ is the (non-constant) map whose growth index achieves the lower bound 1.

Part of the above theorem can be extended to higher dimension. Theorem 5.7.2 in [28], corresponds to the case $c_f = 0, R = \infty$.

Theorem 1.6. Let $\omega$ be a positive $(1,1)$-form on a complex manifold $V$ whose holomorphic sectional curvatures are bounded from above by $-\lambda$ with
$\lambda > 0$, i.e. for any holomorphic map $g : U \to V$ ($U \subset \mathbb{C}$ is an open subset), \( \text{Ric}(g^* \omega) \geq \lambda g^* \omega \). Let $f : \triangle(R) \to V$ be a holomorphic map with \( c_f, \omega < \infty \), where $0 < R \leq \infty$. Then, for every $\epsilon > 0$, the inequality

\[
(\lambda - (1 + \epsilon)(c_f, \omega + \epsilon))T_{f, \omega}(r) + N_{f, \text{ram}}(r) \leq O(\log T_{f, \omega}(r)) + \epsilon \log r
\]

holds for all $r \in (0, R)$ outside a set $E$ with $\int_E \exp((c_f, \omega + \epsilon)T_{f, \omega}(r))dr < \infty$. In particular, we have \( c_{f, \omega} \geq \lambda \).

From Theorem 1.6, if $M$ is a Hermitian manifold and $\omega_P$ is a positive $(1,1)$ form on $M$ whose holomorphic sectional curvature is bounded from above by $-1$ on $M$, then $c_{\omega_P} \geq 1$.

We now turn to the Second Main Theorem for holomorphic curves in $\mathbb{P}^n(\mathbb{C})$. We prove the following theorem which generalizes (by taking $R = \infty$) the result of Nochka.

**Theorem 1.7.** Let $f : \triangle(R) \to \mathbb{P}^n(\mathbb{C})$ be a holomorphic map with $c_f < \infty$, where $c_f = c_{f, \omega_F S}$ and $0 < R \leq \infty$. Assume that the image of $f$ is contained in some $k$-dimensional subspace of $\mathbb{P}^n(\mathbb{C})$ but not in any subspace of dimension lower than $k$. Let $H_j, 1 \leq j \leq q$, be hyperplanes in $\mathbb{P}^n(\mathbb{C})$ in general position. Assume that $f(\triangle(R)) \not\subset H_j$ for $1 \leq j \leq q$. Then, for any $\epsilon > 0$, the inequality,

\[
\sum_{j=1}^{q} m_{f, H_j}(r) + \left( \frac{n+1}{k+1} \right) N_{f, \text{ram}}(r) \leq (2n - k + 1)T_f(r)
\]

\[
+ \frac{(2n - k + 1)k}{2} ((1 + \epsilon)(c_f + \epsilon)T_f(r) + \epsilon \log r) + O(\log T_f(r))
\]

holds for all $r \in (0, R)$ outside a set $E$ with $\int_E \exp((c_f + \epsilon)T_f(r))dr < \infty$. Here $N_{f, \text{ram}}(r)$ is the counting function for the ramification divisor of $f$.

When $k = n$, this gives an extension of H. Cartan’s result.

**Corollary 1.8.** Let $H_1, \ldots, H_q$ be hyperplanes in $\mathbb{P}^n(\mathbb{C})$ in general position. Let $f : \triangle(R) \to \mathbb{P}^n(\mathbb{C})$ be a linearly non-degenerate holomorphic curve (i.e. its image is not contained in any proper subspace of $\mathbb{P}^n(\mathbb{C})$) with $c_f < \infty$, where $c_f = c_{f, \omega_F S}$ and $0 < R \leq \infty$. Then, for any $\epsilon > 0$, the inequality

\[
\sum_{j=1}^{q} m_f(r, H_j) + N_W(r, 0) \leq (n + 1)T_f(r) + \frac{n(n+1)}{2} (1 + \epsilon)(c_f + \epsilon)T_f(r)
\]

\[+ O(\log T_f(r)) + \frac{n(n+1)}{2} \epsilon \log r\]

holds for all $r \in (0, R)$ outside a set $E$ with $\int_E \exp((c_f + \epsilon)T_f(r))dr < \infty$. Here $W$ denotes the Wronskian of $f$. 
As a consequence of Theorem 1.7, we get

**Corollary 1.9.** Let $H_1, \ldots, H_q$ be hyperplanes in $\mathbb{P}^n(\mathbb{C})$ in general position. Let $f : \Delta(\mathbb{C}) \rightarrow \mathbb{P}^n(\mathbb{C})$ be a non-constant holomorphic curve with $c_f < \infty$, where $c_f = c_{f,\omega_F}$ and $0 < R \leq \infty$. Assume that $f(\Delta(\mathbb{C})) \not\subset H_j$ for $1 \leq j \leq q$. Then, for any $\epsilon > 0$, the inequality

$$\sum_{j=1}^{q} m_f(r, H_j) + N_{f,\text{ram}}(r) \leq 2nT_f(r) + \frac{(2n + 1)^3}{8}((1 + \epsilon)(c_f + \epsilon)T_f(r) + \epsilon \log r) + O(\log T_f(r))$$

holds for all $r \in (0, R)$ outside a set $E$ with $\int_{E} \exp((c_f + \epsilon)T_f(r))dr < \infty$.

It turns out that our treatment of the error term in Nevanlinna’s theory permits to extend many of the classical results, using the known strategy. Since the new results seem of interest, we repeat the literature in places. We give in particular a version of Bloch’s theorem for maps with values in a complex torus which belong to the space $E_0(\Delta(1))$. We also prove a defect relation for the intersection of the image of a map in $E_0(\Delta(1))$ with an ample divisor in an abelian variety extending results by Siu-Yeung [22].

2. Some examples and applications

In this section, we provide some examples of holomorphic maps on the unit disc which are in the class we study.

**Example 1.** Let $N$ be a compact Riemann surface of genus $\geq 2$. Then $N$ has a smooth metric form $\omega_p$ whose Gauss curvature is $-1$. We take $\phi : \Delta(1) \rightarrow N$ as the uniformizing map. Then

$$T_{\phi,\omega_p}(r) = \log \frac{1}{1 - r} + O(1).$$

Hence $c_{\phi,\omega_p} = 1$, and thus $\phi \in E_1$. Note that not only we know that $\phi$ is onto but also we get, from Theorem 1.5, that $\delta_{\phi,\omega_p}(a) = 0$ for every $a \in N$.

**Example 2.** Let $M$ be a compact Kobayashi hyperbolic manifold and let $\omega$ be metric form. Then, by Brody’s theorem (see [13] or [20]), there is a constant $C > 0$ such that for any holomorphic map $f : \Delta(1) \rightarrow M$, we have $|f'(0)|_\omega \leq C$. Hence $|f'(z)|_\omega \leq \frac{C}{1-|z|}$ on $\Delta(1)$. Consequently, we have $T_{f,\omega}(r) \leq C \log \frac{1}{1 - r}$. So the space $E_0$ is empty. However, $c_{f,\omega}$ is not necessarily finite since it requires an estimate on the lower bound on $T_{f,\omega}(r)$. The following two examples give the lower bound on $T_{f,\omega}(r)$ in terms of $\log \frac{1}{1 - r}$.  


Example 3. Let \((X, \mathcal{L})\) be a compact, 1-dimensional lamination in a compact Hermitian manifold \((M, \omega)\) (see [8], [9] and the references therein). Assume that \((X, \mathcal{L})\) is Brody hyperbolic, which means that there is no non-constant image of \(\mathbb{C}\) directed by the lamination \(\mathcal{L}\). So for every leave \(L\), we have the universal covering map \(f : \triangle(1) \to L\). It is known (see [9]) that there are two positive constants \(C, C'\) (which do not depend on the leave) such that
\[
\frac{C}{1 - |\zeta|} \leq |f'(|\zeta|)|_\omega \leq \frac{C'}{1 - |\zeta|}.
\]
Therefore
\[
T_{f, \omega}(r) \sim \log \frac{1}{1 - r},
\]
so \(f \in \mathcal{E}\).

Example 4. Let \((M, \omega)\) be a compact Hermitian manifold and \(F\) be a Brody hyperbolic foliation with a finite number of singularities which are linearizable. According to a result of Dinh-Nguyen-Sibony (See [8]), for any extremal positive \(\partial \bar{\partial}\)-closed current \(T\) directed by the foliation which gives full mass to hyperbolic leaves, there are two positive constants \(C, C'\) (which do not depend on the leaves) such that
\[
C \log \frac{1}{1 - r} \leq T_{\phi, \omega}(r) \leq C' \log \frac{1}{1 - r}
\]
for \(T\)-almost every leave \(L\) (in terms of the measure \(T \wedge \omega\)). So \(\phi \in \mathcal{E}\). Here \(\phi : \triangle \to L \subset M\) is the universal covering map of \(L\).

In the case where \(F\) is a foliation in \(\mathbb{P}^2(\mathbb{C})\), our Theorem 1.7 implies that, for any line \(\Lambda \subset \mathbb{P}^2\), except for countably many lines, there are cluster points of the sequence of the measures
\[
\frac{1}{T_{\phi}(r)} \sum_{\phi(a) \in \Lambda, |a| < r} \delta_a \log^+ \frac{r}{|a|}
\]
which are probability measures on the unit circle.

We end this section with the following theorem which characterizes the Kobayashi hyperbolicity of \(M\).

**Theorem 2.1.** Let \(M\) be a compact complex manifold. Then the following are equivalent.

(a) \(M\) is Kobayashi hyperbolic;

(b) For any given positive \((1,1)\)-form \(\omega\) on \(M\), there are positive constants \(c_0\) and \(A\) such that for every holomorphic map \(f : \triangle(1) \to M\),
\[
f_0^1 \exp(cT_{f, \omega}(r))dr \leq A
\]
for every \(c < c_0\);

(c) The class \(\mathcal{E}_0(\triangle(1))\) is empty.
Proof. We first prove (a) ⇒ (b). Indeed, since $M$ is Kobayashi hyperbolic, there is a constant $C > 0$ such that for any holomorphic map $f : \triangle(1) \to M$, we have $|f'(0)|_\omega \leq C$. Hence $|f'(z)|_\omega \leq \frac{C}{1 - |z|}$. Consequently we have $T_{f,\omega}(r) \leq C \log \frac{1}{1 - r}$. We take $c_0 = \frac{1}{2C}$, then it is easy to see that
\[
\int_0^1 \exp(cT_{f,\omega}(r))dr \leq \int_0^1 \frac{1}{(1 - r)^{1/2}}dr = A
\]
for every $c < c_0$.

The fact that (b) implies (c) is obvious. So we only need to prove that (c) implies (a). It suffices to prove that if $M$ is not Kobayashi hyperbolic then $E_0(\triangle(1))$ is not empty. We first construct a holomorphic map $g : \triangle(1) \to \mathbb{C}$, such that for most $a$’s,
\[
\lim_{r \to 1} \frac{N_g(r, a)}{\log \frac{1}{1 - r}} = \infty.
\]
Indeed such a holomorphic map $g_1 : \triangle(1) \to \mathbb{P}^1(\mathbb{C})$ exists (see [26]). Let $E$ denote the preimage of the point at infinity in $\mathbb{P}^1(\mathbb{C})$. We can assume that the point 0 is not in $E$. Let $h : \triangle(1) \to \triangle(1) \setminus E$ denote the universal covering map from with $h(0) = 0$. Then the map $g = g_1(h)$ satisfies our condition.

Since $M$ is not Kobayashi hyperbolic there is a non-constant holomorphic map $f : \mathbb{C} \to M$. The map $F = f(g_1(h))$ satisfies that for most $a$’s
\[
\lim_{r \to 1} \frac{N_F(r, a)}{\log \frac{1}{1 - r}} = \infty.
\]
Then a similar growth is valid for $T_F(r)$. Indeed we have:
\[
N_F(r, a) = \int \log^+ \frac{r}{|z|} F^*(\delta_a).
\]
Similarly for any positive measure $\mu$ we have
\[
\int N_F(r, a)d\mu(a) = \int \log^+ \frac{r}{|z|} F^*(\mu).
\]
It suffices to apply this to the form $\omega$ considered as a measure on $F(\triangle(1))$. It follows that if $N(r, a)$ grows fast for most $a$’s, the same is true for $T(F, r)$. Hence $F \in E_0(\triangle(1))$ and thus $E_0(\triangle(1))$ is not empty. □

3. Holomorphic mappings into compact Riemann surfaces

Lemma 3.1 (Calculus Lemma). Let $0 < R \leq \infty$ and let $\gamma(r)$ be a non-negative function defined on $(0, R)$ with $\int_0^R \gamma(r)dr = \infty$. Let $h$ be a non-decreasing function of class $C^1$ defined on $(0, R)$. Assume that $\lim_{r \to R} h(r) = \infty$. Let $\omega$ be a given positive measure on $\mathbb{C}$, then
\[
\int \log^+ \frac{r}{|z|} F^*(\delta_a) = \int \log^+ \frac{r}{|z|} F^*(\mu).
\]
\[ \infty \text{ and } h(r_0) \geq c > 0. \] Then, for every \( 0 < \delta < 1 \), the inequality
\[
h'(r) \leq h^{1+\delta}(r)\gamma(r)
\]
holds for all \( r \in (0, R) \) outside a set \( E \) with \( \int_E \gamma(r)dr < \infty \).

**Proof.** Let \( E \subset (r_0, R) \) be the set of \( r \) such that \( h'(r) \geq h^{1+\delta}(r)\gamma(r) \). Then
\[
\int_E \gamma(r)dr \leq \int_{r_0}^R \frac{h'(r)}{h^{1+\delta}(r)}dr = \int_c^\infty \frac{dt}{t^{1+\delta}} < \infty
\]
which proves the lemma. \( \square \)

**Lemma 3.2.** Let \( 0 < R \leq \infty \) and let \( \gamma(r) \) be a function defined on \( (0, R) \) with \( \int_0^R \gamma(r)dr = \infty \). Let \( h \) be a function of class \( C^2 \) defined on \( (0, R) \) such that \( rh' \) is a nondecreasing function. Assume that \( \lim_{r \to R} h(r) = \infty \). Then
\[
\frac{1}{r} \frac{d}{dr} \left( r \frac{dh}{dr} \right) \leq r^{\delta} \cdot \gamma^{2+\delta}(r) \cdot h^{(1+\delta)^2}(r)
\]
holds outside a set \( E \subset (0, R) \) with \( \int_E \gamma(r)dr < \infty \).

**Proof.** We apply the Calculus lemma twice, first to the function \( rh'(r) \) and then to the function \( h(r) \). \( \square \)

The typical use of the calculus lemma is as follows. Let \( \Gamma \) be a non-negative function on \( \Delta(R) \) with \( 0 < R \leq \infty \). Define
\[
T_\Gamma(r) := \int_0^r dt \int_{|z| < t} \frac{\sqrt{-1}}{2\pi} dz \wedge d\bar{z}
\]
and
\[
\lambda(r) := \int_0^{2\pi} \Gamma(re^{i\theta}) \frac{d\theta}{2\pi}.
\]
Using the polar coordinates,
\[
\frac{\sqrt{-1}}{2\pi} dz \wedge d\bar{z} = 2r dr \wedge \frac{d\theta}{2\pi}.
\]
Hence
\[
r \frac{dT_\Gamma}{dr} = 2 \int_0^{2\pi} \left( \int_0^r \Gamma(te^{i\theta})t dt \right) \frac{d\theta}{2\pi},
\]
\[
\frac{d}{dr} \left( r \frac{dT_\Gamma}{dr} \right) = 2r \int_0^{2\pi} \Gamma(re^{i\theta}) \frac{d\theta}{2\pi} = 2r \lambda(r).
\]
Thus, from Lemma 3.2 we have
\[
(6) \quad \lambda(r) \leq \frac{1}{2} r^{\delta} \cdot \gamma^{2+\delta}(r) \cdot T_\Gamma^{(1+\delta)^2}(r)
\]
holds for all \( r \in (0, R) \) outside a set \( E \) with \( \int_E \gamma(r)dr < \infty \). Throughout the paper, we will use the inequality \( (6) \) with a proper chosen \( \gamma(r) \).
Theorem 3.3 (Green-Jensen formula, see [19]). Let \( g \) be a function on \( \Delta(r) \) such that \( dd^c[g] \) is of order zero and \( g(0) \) is finite. Then
\[
\int_0^r \frac{dt}{t} \int_{|\zeta|<t} dd^c[g] = \frac{1}{2} \left( \int_0^{2\pi} g(re^{i\theta}) \frac{d\theta}{2\pi} - g(0) \right).
\]

Let \( M \) be a compact Riemann surface and let \( \omega \) be a positive \((1,1)\) form of class \( C^1 \) on \( M \) such that \( \int_M \omega = 1 \). Consider the equation, in the sense of currents,
\[
(7) \quad dd^c u = \omega - \delta_a,
\]
where \( \delta_a \) is the Dirac measure at \( a \).

Theorem 3.4. Let \( U \) be an open set in a compact Riemann surface \( M \) such that \( M \setminus U \) consists of at most a finite number of points.

(a) Let \( \omega \) be a positive smooth \((1,1)\) form of volume 1 on \( M \). Let \( a \in M \). Then equation (7) admits a positive solution \( u_a \), smooth in \( M \setminus \{a\} \), with a log singularity at the point \( a \).

(b) If \( M \setminus U \) is non-empty and \( \omega \) is proportional to the Poincaré form of \( M \) so that it is of volume 1, then equation (7) admits a positive solution \( u_a \), smooth in \( U \setminus \{a\} \), with a log singularity at the point \( a \).

Proof. (a) Since the cohomology class of the right hand side is zero, equation (7) always has a solution. The regularity in the complement of \( a \) and the behavior at \( a \) imply that \( u_a \) is smooth in \( M \setminus \{a\} \), with a log singularity at the point \( a \). By adding a constant if necessary, it gives the positivity of \( u_a \). This proves the case (a).

The proof of case (b) is similar. Note that the Poincaré metric at the points in \( M \setminus U \) behaves like \( \omega_z \wedge \omega_{\bar{z}} / (|z|^2 \log |z|^2) \), which has finite volume. Using that the Poincaré metric of the pointed disc has curvature \(-1\) we can by comparison establish that the solution \( u_a \) goes to \( +\infty \) when approaching the points at the boundary. This gives the positivity of \( u_a \).

Let \( a \in U \) and \( u_a \) be the solution of the equation (7). We define the proximity function
\[
(8) \quad m_{f,\omega}(r, a) = \int_0^{2\pi} u_a(f(re^{i\theta})) \frac{d\theta}{2\pi}
\]
and the counting function
\[
(9) \quad N_f(r, a) = \int_0^r \frac{n_f(t, a)}{t} dt
\]
where \( n(r, a) \) is the number of the elements of \( f^{-1}(a) \) inside \( |z| < r \), counting multiplicities (for simplicity we assume 0 is not in \( f^{-1}(a) \)).
By applying the integral operator
\[ \int_0^r \frac{dt}{t} \int_{|\zeta| \leq t}. \]
to the equation (7) and using the Green-Jensen’s formula, we get

**Theorem 3.5 (First Main Theorem).**

\[ m_{f,\omega}(r, a) + N_f(r, a) = T_{f,\omega}(r) + O(1). \]

The defect for \( f \) with \( c_{f,\omega} < \infty \), is given by,
\[ \delta_{f,\omega}(a) := \liminf_{r \to R} \frac{m_{f,\omega}(r, a)}{T_{f,\omega}(r)} = 1 - \limsup_{r \to R} \frac{N_f(r, a)}{T_{f,\omega}(r)}, \quad \delta_f(a) := \delta_{f,\omega_{FS}}(a). \]

**Proof of Theorem 1.2.** Consider
\[ \Psi = C \left( \prod_{j=1}^q \left( u_{a_j}^2 \exp(u_{a_j}) \right) \right) \omega \]
where \( C \) is chosen such that \( \int_M \Psi = 1 \). Write
\[ f^*\Psi = \Gamma \frac{\sqrt{-1}}{2\pi} d\zeta \wedge d\bar{\zeta}. \]

Then, by the Poincaré-Lelong formula,
\[ dd^c[\log \Gamma] = \sum_{j=1}^q dd^c[u_{a_j} \circ f] + [f^*\text{Ric}(\omega)] + D_{f,\text{ram}} + 2 \sum_{j=1}^q dd^c[\log u_{a_j} \circ f]. \]

Applying the integral operator
\[ \int_0^r \frac{dt}{t} \int_{|\zeta| \leq t}. \]
to the above identity and using the Green-Jensen’s formula, we get
\[ \frac{1}{2} \int_0^{2\pi} \log \Gamma(re^{i\theta}) \frac{d\theta}{2\pi} + O(1) = \sum_{j=1}^q m_f(r, a_j) + T_{f,\text{Ric}(\omega)}(r) + N_{f,\text{ram}}(r) \]
\[ + 2 \sum_{j=1}^q \int_0^r \frac{dt}{t} \int_{|\zeta| \leq t} dd^c[\log u_{a_j} \circ f]. \]

Using the Green-Jensen formula, the concavity of log and the First Main Theorem, we get
\[ 2 \int_0^r \frac{dt}{t} \int_{|\zeta| \leq t} dd^c[\log u_{a_j} \circ f] = \int_0^{2\pi} \log u_{a_j}(f(re^{i\theta})) \frac{d\theta}{2\pi} \]
\[ \leq \log \int_0^{2\pi} u_{a_j}(f(re^{i\theta})) \frac{d\theta}{2\pi} + O(1) = \log m_{f,\omega}(r, a_j) + O(1) \]
\[ \leq \log T_{f_\omega}(r) + O(1). \]

Using the concavity of \( \log \) and (6) by taking \( \gamma(r) := \exp((c_{f_\omega} + \epsilon)T_{f_\omega}(r)) \) and \( \delta = 2\epsilon \), we have,

\[ \frac{1}{2} \int_0^{2\pi} \log \Gamma(re^{i\theta}) \frac{d\theta}{2\pi} \leq \frac{1}{2} \int_0^{2\pi} \Gamma(re^{i\theta}) \frac{d\theta}{2\pi} \]

\[ \leq \frac{1}{2} \left( (2 + 2\epsilon)(c_{f_\omega} + \epsilon)T_{f_\omega}(r) + (1 + 2\epsilon)^2 \log^+ T_{f_\omega}(r) + 2\epsilon \log r \right) \]

holds for all \( r \in (0, R) \) outside a set \( E \) with \( \int_E \exp((c_{f_\omega} + \epsilon)T_{f_\omega}(r))dr < \infty \).

It remains to estimate

\[ T_{f_\omega}(r) = \int_0^r \frac{dt}{t} \int_{|\zeta| \leq t} \frac{1}{2\pi} d\zeta \wedge d\bar{\zeta} = \int_0^r \frac{dt}{t} \int_{|\zeta| \leq t} f^* \Psi. \]

We follow the approach by Ahlfors-Chern. The change of variable formula gives,

\[ \int_M n_f(r, a) \Psi(a) = \int_{|\zeta| \leq r} f^* \Psi. \]

So, using the First Main Theorem,

\[ \int_0^r \frac{dt}{t} \int_{|\zeta| \leq t} f^* \Psi = \int_M N_f(r, a) \Psi(a) \leq \int_M T_{f_\omega}(r) \Psi(a) + O(1) = T_{f_\omega}(r) + O(1). \]

This finishes the proof of Theorem 1.2.

A similar idea can be carried out to prove Theorem 1.5; we have just to use Theorem 3.4 (b).

**Proof of Theorem 1.6**. Write \( f^* \omega = h \frac{1}{2\pi} d\zeta \wedge d\bar{\zeta} \). Then, by the Poincaré-Lelong formula,

\[ dd^c \log h = f^* \text{Ric}(\omega) + D_{f,\text{ram}} = \text{Ric}(f^* \omega) + D_{f,\text{ram}}. \]

where \( D_{f,\text{ram}} \) is the ramification divisor of \( f \). The curvature assumption implies that

\[ dd^c \log h \geq D_{f,\text{ram}} + \lambda f^* \omega. \]

Applying the integral operator

\[ \int_0^r \frac{dt}{t} \int_{|\zeta| \leq t} \]

to the above identity and using the Green-Jensen’s formula, we get

\[ \frac{1}{2} \int_0^{2\pi} \log h(re^{i\theta}) \frac{d\theta}{2\pi} + O(1) \geq \lambda T_{f_\omega}(r) + N_{f,\text{ram}}(r). \]
On the other hand, using the concavity of log and (6) by taking $\gamma(r) := \exp((c f + \epsilon)T_f(r))$ and $\delta = 2\epsilon$, it follows

$$\frac{1}{2} \int_0^{2\pi} \log h(r e^{i\theta}) \frac{d\theta}{2\pi} \leq \frac{1}{2} \log \frac{1}{2\pi} \int_0^{2\pi} h(r e^{i\theta}) \frac{d\theta}{2\pi} + O(1)$$

$$\leq \frac{1}{2} ((2 + 2\epsilon)(c f + \epsilon)T_f(r) + (1 + 2\epsilon)^2 \log^+ T_f(r) + 2\epsilon \log r)$$

holds for all $r \in (0, R)$ outside a set $E$ with $\int_E \exp((c f + \epsilon)T_f(r)) dr < \infty$. This finishes the proof.

4. Holomorphic mappings into $\mathbb{P}^n(\mathbb{C})$.

In this section, we prove Theorem 1.7. We follow Ahlfors’ method with some simplifications (see [6], [25], [20] or [24]). However we treat differently the error term. The key is to use (6) by letting $\gamma(r) := \exp((c f + \epsilon)T_f(r))$ for a given $\epsilon$, where $T_f(r) := T_{f,\omega F}(r)$. In the following we use the notation “$\leq$” to denote the inequality holds for all $r \in (0, R)$ except for a set $E$ with $\int_E \exp((c f + \epsilon)T_f(r)) dr < \infty$. We always assume that the holomorphic map $f : \Delta(R) \to \mathbb{P}^n(\mathbb{C})$ is linearly non-degenerate (except in the last section $E$) with $c_f < \infty$.

A. Associated curves and the Plücker’s formula. Let $f : \Delta(R) \to \mathbb{C}^{n+1} - \{0\}$ be a reduced representation of $f$. Consider the holomorphic map $F_k$ defined by

$$F_k = f \wedge f' \wedge \cdots \wedge f^{(k)} : \Delta(R) \to \bigwedge^{k+1} \mathbb{C}^{n+1}.$$  

Evidently $F_{n+1} \equiv 0$. Since $f$ is linearly non-degenerate, $F_k \neq 0$ for $0 \leq k \leq n$. The map $F_k = \mathbb{P}(F_k) : \Delta(R) \to \mathbb{P}(\bigwedge^{k+1} \mathbb{C}^{n+1}) = \mathbb{P}^{N_k}(\mathbb{C})$, where $N_k = \frac{(n+1)!}{(k+1)!(n-k)!} - 1$ and $\mathbb{P}$ is the natural projection, is called the $k$-th associated map. Let $\omega_k = dd^c \log \|Z\|^2$ be the Fubini-Study form on $\mathbb{P}^{N_k}(\mathbb{C})$, where $Z = [x_0 : \cdots : x_{N_k}] \in \mathbb{P}^{N_k}(\mathbb{C})$. Let

$$\Omega_k = F_k^* \omega_k = \frac{\sqrt{-1}}{2\pi} h_k dz \wedge d\bar{z}, \ 0 \leq k \leq n,$$

be the pull-back via the $k$-th associated curve. Observe that since $F_k$ has no indeterminacy points, $\Omega_k = F_k^* \omega_k$ is smooth and $h_k$ is non-negative.

We recall the following lemma (see [11], [25], [20] or [24]).

**Lemma 4.1.**

$$h_k = \frac{\|F_{k-1}\|^2\|F_{k+1}\|^2}{\|F_k\|^4}.$$
We now turn to the Plücker Formula. By Lemma 4.1 and the Poincaré-Lelong formula, we get
\[ \ddc \log h_k = \Omega_{k-1} + \Omega_{k+1} - 2\Omega_k + [h_k = 0]. \]
where \([h_k = 0]\) is the zero divisor of \(h_k\). We recall a few facts on the geometric meaning of this divisor (see [11], [25]). We consider the point \(z_0\) with \(F_k(z_0) = 0\). Without loss of generality, we assume that \(z_0 = 0\) and \(f(z_0) = [1 : 0 : \cdots : 0]\) and that the reduced representation \(f\) of \(f\) in a neighborhood of 0 has the form
\[ f(z) = (1 + \cdots, z^{\nu_1} + \cdots, z^{\nu_n} + \cdots), \]
with \(1 \leq \nu_1 \leq \cdots \leq \nu_n\). Then it is easy to get that
\[ F_k(z) = z^{m_k}(1 + \cdots, z^{\nu_{k+1} - \nu_k} + \cdots), \]
where \(m_k = \nu_1 + \cdots + \nu_k - \frac{k(k+1)}{2}\). On the other hand, if we write in a neighborhood of 0, \(h_k(z) = z^{2\mu_k}b(z)\) with \(b(0) > 0\), then, it is easy to get \(\mu_k = m_{k+1} - 2m_k + m_{k-1}\) (see [11]).

Define the \(k\)th characteristic function
\[ T_{F_k}(r) = \int_0^r \frac{dt}{t} \int_{|z| \leq t} F_k^* \omega, \]
Denote by
\[ N_{d_k}(r) = \int_0^r n_{d_k}(t) \frac{dt}{t} \]
where \(n_{d_k}(t)\) is the number of zeros of the \(h_k\) in \(|z| < t\), counting multiplicities. Note that \(N_{d_k}(r, s)\) does not depend on the choice of the reduced representation. Define
\[ S_k(r) = \frac{1}{2} \int_0^{2\pi} \log h_k(re^{i\theta}) \frac{d\theta}{2\pi}. \]
Then, by applying the integral operator
\[ \int_0^r \frac{dt}{t} \int_{|z| \leq t} \]
to (11) and using the Green-Jensen’s formula, we get the following lemma.

**Lemma 4.2 (Plücker Formula).** For any integers \(k\) with \(0 \leq k \leq n\),
\[ N_{d_k}(r) + T_{F_{k-1}}(r) - 2T_{F_k}(r) + T_{F_{k+1}}(r) = S_k(r) + O(1) \]
where \(T_{F_{-1}}(r) \equiv 0\) and \(T_{F_0}(r) = T_f(r)\).

The Plücker formula implies the following lemma which gives the estimates of \(T_{F_k}(r)\) in terms of \(T_f(r)\). We use our estimate of the error term.
Lemma 4.3.  For $0 \leq k \leq n - 1$ and every $\delta > 0$,

$$T_{F_k}(r) \leq (n + 2)^3(1 + (2 + \delta)c_f)T_f(r) + n(n + 1)^2\delta \log r + O(1) \|.$$  

Proof. Write $T(r) = \sum_{k=0}^{n-1} T_{F_k}(r)$. Observe that

$$\frac{1}{r} \frac{d}{dr} \left( \frac{d}{dr} T_{F_k}(r) \right) = 2 \int_0^{2\pi} h_k(re^{i\theta}) \frac{d\theta}{2\pi}.$$  

Applying the Calculus Lemma (see (6)) with $\gamma(r) = \exp((c_f + \delta)T_f(r))$, we get

$$\int_{|z|=r} h_k(re^{i\theta}) \frac{d\theta}{2\pi} \leq r^2e^{c_f(4+2\delta)T_f(r)}T_{F_k}^{(1+2\delta)^2}(r) \|.$$  

This implies

$$S_k(r) = \frac{1}{2} \int_0^{2\pi} \log h_k(re^{i\theta}) \frac{d\theta}{2\pi} \leq \frac{1}{2} \log \int_0^{2\pi} h_k(re^{i\theta}) \frac{d\theta}{2\pi} + O(1)$$  

$$(13) \quad \leq (2 + \delta)c_fT_f(r) + \frac{1}{2}(1 + 2\delta)^2 \log T(r) + \delta \log r \|.$$  

From Lemma 4.2 we claim that, for $0 \leq q \leq p$,

$$T_{F_q}(r) + (p - q)T_{F_{q-1}}(r) \leq (p - q + 1)T_{F_q}(r) + \sum_{j=q}^{p-1} (p - j)S_j(r) + O(1).$$  

In fact, the claim is true for $p = q$. Assume that the claim is true for $q, q + 1, \ldots, p$. If $p = n$, the proof is done. If $p < n$, we proceed, by using Lemma 4.2:

$$T_{F_{q-1}}(r) - T_{F_q}(r) + T_{F_{p+1}}(r) - T_{F_p}(r)$$  

$$= \sum_{j=q}^{p} (T_{F_{j-1}}(r) - 2T_{F_j}(r) + T_{F_{j+1}}(r)) = \sum_{j=q}^{p} S_j(r) - \sum_{j=q}^{p} N_{d_j}(r) + O(1)$$  

$$\leq \sum_{j=q}^{p} S_j(r) + O(1).$$  

So

$$T_{F_{p+1}}(r) + T_{F_{q-1}}(r) \leq T_{F_p}(r) + T_{F_q}(r) + \sum_{j=q}^{p} S_j(r) + O(1).$$  

Thus

$$T_{F_{p+1}}(r) + (p + 1 - q)T_{F_{q-1}}(r) = T_{F_{p+1}}(r) + T_{F_{q-1}}(r) + (p - q)T_{F_{q-1}}(r)$$  

$$\leq T_{F_p}(r) + T_{F_q}(r) + (p - q)T_{F_{q-1}}(r) + \sum_{j=q}^{p} S_j(r) + O(1).$$
On the other hand, from Lemma \[4.2\] again, we have
\[
T_{F_p}(r) - (p - q + 1)T_{F_q}(r) + (p - q)T_{F_{q-1}}(r) = \sum_{j=q}^{p} (p-j) \left( T_{F_{j-1}}(r) - 2T_F(r) + T_{F_{j+1}}(r) \right) \leq \sum_{j=q}^{p} (p-j)S_j(r) + O(1).
\]
Hence
\[
T_{F_p}(r) + T_{F_q}(r) + (p-q)T_{F_{q-1}}(r) \leq (p-q+2)T_F(r) + \sum_{j=q}^{p} (p-j)S_j(r) + O(1).
\]
Therefore
\[
T_{F_{p+1}}(r) + (p+1-q)T_{F_{q-1}}(r) \leq (p-q+2)T_F(r) + \sum_{j=q}^{p} (p+1-j)S_j(r) + O(1).
\]
This proves our claim. Now take \( q = 0 \) and \( p = k \) and notice that \( T_{F_{-1}}(r) \equiv 0 \), then
\[
T_{F_k}(r) \leq (k+1)T_F(r) + \sum_{j=0}^{k-1} (k-j)S_j(r) + O(1).
\]
This, together with \([13]\) gives, for \( 0 \leq k \leq n \),
\[
T_{F_k}(r) \leq (k+1)T_F(r) + \frac{1}{2}k(k+1) \left( (2+\delta)c_FT_F(r) + (1+2\delta)^2 \log T(r) + \delta \log r + O(1) \right). \|
\]
Therefore,
\[
T(r) \leq (n+1)^2T_F(r) + \frac{1}{2}n(n+1)^2 \left( (2+\delta)c_FT_F(r) + \frac{1}{2}(1+2\delta)^2 \log T(r) + \delta \log r + O(1) \right) \|. \]
Because \( \frac{1}{2}n(n+1)^2(1+2\delta)^2 \log T(r) \leq \frac{1}{2}T_F(r) \) where \( r \) is close enough to \( R \), we have
\[
T(r) \leq (n+2)^3(1+(2+\delta)c_FT_F(r) + n(n+1)^2\delta \log r + O(1) \|. \]

\[\Box\]

**B. The projective distance.** For integers \( 1 \leq q \leq p \leq n+1 \), the interior product \( \xi(\alpha) \in \Lambda^{p-q}(\mathbb{C}^{n+1}) \) of vectors \( \xi \in \Lambda^{p+1}(\mathbb{C}^{n+1}) \) and \( \alpha \in \Lambda^{q+1}(\mathbb{C}^{n+1})^* \) is defined by
\[
\beta(\xi|\alpha) = (\alpha \wedge \beta)(\xi)
\]
for any \( \beta \in \Lambda^{p-q}(\mathbb{C}^{n+1})^* \). Let
\[
H = \{ [x_0 : \cdots : x_n] \mid a_0x_0 + \cdots + a_nx_n = 0 \}
\]
be a hyperplane in \( \mathbb{P}^n(\mathbb{C}) \) with unit normal vector \( \mathbf{a} = (a_0, \cdots, a_n) \). In the rest of this section, we regard \( \mathbf{a} \) as a vector in \( (\mathbb{C}^{n+1})^* \) which is defined by
$a(x) = a_0x_0 + \cdots + a_nx_n$ for each $x = (x_0, \cdots, x_n) \in \mathbb{C}^{n+1}$, where $(\mathbb{C}^{n+1})^*$ is the dual space of $\mathbb{C}^{n+1}$. Let $x \in \mathbb{P}(\wedge^{k+1}\mathbb{C}^{n+1})$, the **projective distance** is defined by

\[(14) \quad \|x; H\| = \frac{\|\xi \cdot a\|}{\|\xi\|\|a\|}\]

where $\xi \in \wedge^{k+1}\mathbb{C}^{n+1}$ with $\mathbb{P}(\xi) = x$. Define

\[(15) \quad m_{F_k}(r, H) = \int_{0}^{2\pi} \frac{1}{\|F_k(re^{i\theta}); H\|} \frac{d\theta}{2\pi}.\]

We have the following weak form of the First Main Theorem for $F_k$.

**Theorem 4.4 (Weak First Main Theorem).**

\[
m_{F_k}(r, H) \leq T_{F_k}(r) + O(1).
\]

**Proof.** Let $f_k : \triangle(R) \to \wedge^{k+1}\mathbb{C}^{n+1}$ be a reduced representation of $F_k$, and we consider the holomorphic map

\[
F_k\{a\} : \triangle(R) \to \mathbb{P}(\wedge^{k}\mathbb{C}^{n+1})
\]

which is given by $F_k\{a\} := \mathbb{P}(G)$ where $G = f_k\{a\}$. Note that $G$ is a representation of the holomorphic map $F_k\{a\}$, but is not reduced. We denote by $\nu_G$ the divisor of $G$ on $\triangle(R)$, and $N_G(r, 0)$ the counting function associated to $\nu_G$ (which is independent of the choices of the reduced representation of $F_k$). We have

\[
(F_k\{a\})^* \omega_k + \nu_G = dd^c \log \|G\|^2.
\]

Applying the integral operator

\[
\int_{0}^{r} dt \int_{|\zeta| \leq t}
\]

to the above identity and using the Green-Jensen’s formula, we get

\[
T_{F_k}\{a\}(r) + N_G(r, 0) = \int_{0}^{2\pi} \log \|G(re^{i\theta})\| \frac{d\theta}{2\pi} + O(1)
\]

\[
= \int_{0}^{2\pi} \log \|f_k\{a\}(re^{i\theta})\| \frac{d\theta}{2\pi} + O(1).
\]

On the other hand, from the definition (notice that $f_k$ is a reduced representation of $F_k$),

\[
T_{F_k}(r) = \int_{0}^{2\pi} \log \|f_k\|(re^{i\theta}) \frac{d\theta}{2\pi} + O(1).
\]

Hence, from the definition of $m_{F_k}(r, H)$,

\[
T_{F_k}\{a\}(r) + N_G(r, 0) + m_{F_k}(r, H)
\]
\[= \int_0^{2\pi} \log \|f_k(a)(re^{i\theta})\| \frac{d\theta}{2\pi} + O(1) + \int_0^{2\pi} \log \frac{||f_k(a)||}{||f_k||} (re^{i\theta}) \frac{d\theta}{2\pi} + O(1) = T_F(r) + O(1). \]

We shall need the following product to sum estimate. It is an extension of the estimate of the geometric mean by the arithmetic mean.

**Lemma 4.5** (See Theorem 3.5.7 in [20]). Let \(H_1, \ldots, H_q\) (or \(a_1, \ldots, a_q\)) be hyperplanes in \(\mathbb{P}^n(\mathbb{C})\) in general position. Let \(k \in \mathbb{Z}[0, n-1]\) with \(n-k \leq q\).

Then there exists a constant \(c_k > 0\) such that for every \(0 < \lambda < 1\) and \(x \in \mathbb{P}(\bigwedge^k \mathbb{C}^{n+1})\) with \(x \not\subset H_j, 1 \leq j \leq q\) and \(y \in \mathbb{P}(\bigwedge^{k+1} \mathbb{C}^{n+1})\) we have

\[
\prod_{j=1}^q \frac{\|y; H_j\|^2}{\|x; H_j\|^2} \leq c_k \left( \sum_{j=1}^q \frac{\|y; H_j\|^2}{\|x; H_j\|^2} \right)^{n-k}. 
\]

**C. The Ahlfors’ estimate.** Let \(\phi_k(H) = \|F_k; H\|^2\). Define

\[
h_k(H) = \frac{\phi_k(H)\phi_{k+1}(H)}{\phi_k^2(H)} \Omega_k. 
\]

The function \(\phi_k(H)\) is defined out of the stationary points, however the analysis near those points shows that \(\phi_k(H)\) can be extended smoothly at those points [25]. The key of this Ahlfors’ approach is the following so-called Ahlfors’ estimate. We include a proof here.

**Theorem 4.6** (Ahlfors’ estimate ([20] or [24]). Let \(H\) be a hyperplane in \(\mathbb{P}^n(\mathbb{C})\). Then for any \(0 < \lambda < 1\), we have

\[
\int_s^r \int_{|z| < t} \frac{\phi_{k+1}(H)}{\phi_k(H)^{1-\lambda}} \Omega_k \frac{dt}{t} \leq \frac{1}{\lambda^2} (8 T_F(r) + O(1)).
\]

To prove Ahlfors’ estimate, the following lemma plays a crucial role (see [25], [20] or [24]). The proof of the lemma is based on a standard but lengthy computation. For the details of the proof, see Lemma A3.5.10 in [20].

**Lemma 4.7** (Lemma A3.5.10 in [20]). Let \(H\) be a hyperplane \(\mathbb{P}^n(\mathbb{C})\) and \(\lambda\) be a constant with \(0 < \lambda < 1\). Then, for \(0 \leq k \leq n\), the following inequality holds on \(\Delta(R) - \{z \mid \phi_k(H)(z) = 0\}\)

\[
\frac{\lambda^2 \phi_{k+1}(H)}{4 \phi_k^{1-\lambda}(H)} \Omega_k - \lambda(1+\lambda) \Omega_k \leq dd^c \log (1 + \phi_k(H)^\lambda). 
\]

We now prove Theorem 4.6 (Ahlfors’ Estimate).
Proof. By Lemma 4.7,
\[ dd^c \log(1 + \phi_k(H)^\lambda) \geq \frac{\lambda^2 \phi_{k+1}(H)}{4 \phi_k^{1-\lambda}(H)} \Omega_k - \lambda(1 + \lambda)\Omega_k. \]

Thus
\[ \frac{\lambda^2 \phi_{k+1}(H)}{4 \phi_k^{1-\lambda}(H)} \Omega_k \leq dd^c \log(1 + \phi_k(H)^\lambda) + \lambda(1 + \lambda)\Omega_k. \]

By the Green-Jensen’s formula,
\[ \int_s^t \frac{dt}{t} \int_{|z| \leq t} dd^c \log(1 + \phi_k(H)^\lambda) = \frac{1}{2} \int_0^{2\pi} \log(1 + \phi_k(H)^\lambda) \frac{d\theta}{2\pi} + O(1) \]

This, together with (17) implies that
\[ \frac{\lambda^2}{4} \int_0^r \frac{dt}{t} \int_{|z| \leq t} \frac{\phi_{k+1}(H)}{\phi_k^{1-\lambda}(H)} \Omega_k \leq \int_0^r \frac{dt}{t} \int_{|z| \leq t} dd^c \log(1 + \phi_k(H)^\lambda) + \lambda(1 + \lambda)T_{F_k}(r) \]
\[ = \frac{1}{2} \int_0^{2\pi} \log(1 + \phi_k(H)^\lambda) \frac{d\theta}{2\pi} + \lambda(1 + \lambda)T_{F_k}(r) + O(1) \]
\[ \leq \lambda(1 + \lambda)T_{F_k}(r) + \frac{1}{2} \log 2 + O(1) \leq 2T_{F_k}(r) + O(1), \]
using \( 0 \leq \phi_k(H) \leq 1. \)

\[ \square \]

**D. A general theorem.** We prove the following general version of H. Cartan’s theorem.

**Theorem 4.8 (A General Form of the SMT).** \( f : \Delta(R) \rightarrow \mathbb{P}^n(\mathbb{C}) \) be a linearly non-degenerate holomorphic curve (i.e. its image is not contained in any proper subspace of \( \mathbb{P}^n(\mathbb{C}) \)) with \( c_f < \infty \), where \( c_f = c_{f,\omega_F} \) and \( 0 < R \leq \infty \). Let \( H_1, \ldots, H_q \) (or linear forms \( a_1, \ldots, a_q \)) be arbitrary hyperplanes in \( \mathbb{P}^n(\mathbb{C}) \). Then, for any \( \epsilon > 0 \), the inequality
\[ \int_0^{2\pi} \max_K \sum_{j \in K} \log \frac{1}{\|f(\epsilon e^{i\theta}); H_j\|} \frac{d\theta}{2\pi} + N_W(r, 0) \]
\[ \leq (n + 1)T_f(r) + \frac{n(n + 1)}{2} (1 + \epsilon)(c_f + \epsilon)T_f(r) \]
\[ + O(\log T_f(r)) + \frac{n(n + 1)}{2} \epsilon \log r, \]
where the max is taken over all subsets \( K \) of \( \{1, \ldots, q\} \) such that the linear forms \( a_j, j \in K \), are linearly independent.
Proof. Without loss of generality, we may assume $q \geq n + 1$ and that $\#K = n + 1$. Let $T$ be the set of all the injective maps $\mu : \{0, 1, \ldots, n\} \to \{1, \ldots, q\}$ such that $a_{\mu(0)}, \ldots, a_{\mu(n)}$ are linearly independent. Take

(18) \[ \lambda := \Lambda(r) = \min_k \left\{ \lambda_k \right\} \]

For any $\mu \in T$, by Lemma 4.5 with $\lambda = \Lambda(r)$ and notice that $\phi_k(H) = \|F_k, H\|$, it gives, for $0 \leq k \leq n - 1$,

\[ \prod_{j=0}^{n} \phi_{k+1}(H_{\mu(j)}) \leq c_k \left( \sum_{j=0}^{n} \frac{\phi_{k+1}(H_{\mu(j)})}{\phi_k(H_{\mu(j)})^{2-2\Lambda(r)}} \right)^{n-k} \]

for some constant $c_k > 0$. Since $\phi_n(H_{\mu(j)})$ is a constant for any $0 \leq j \leq n$ and $F_0 = f$, the above inequality implies that

\[ \prod_{j=0}^{n} \frac{1}{\|f; H_{\mu(j)}\|^2} \leq c \prod_{k=0}^{n-1} \left( \sum_{j=0}^{n} \frac{\phi_{k+1}(H_{\mu(j)})}{\phi_k(H_{\mu(j)})^{2-2\Lambda(r)}} \right)^{n-k} \cdot \prod_{k=0}^{n-1} \frac{1}{\phi_k(H_{\mu(j)})^{2\Lambda(r)}} + O(1) \]

for some constant $c > 0$. Therefore

\[ \int_0^{2\pi} \max K \sum_{j \in K} \log \frac{1}{\|f(re^{i\theta})\|} \frac{d\theta}{2\pi} = \frac{1}{\max K} \int_0^{2\pi} \max \mu \in T \prod_{j=0}^{n} \frac{1}{\|f(re^{i\theta})\|^2} \frac{d\theta}{2\pi} \]

\[ \leq \sum_{k=0}^{n-1} \int_0^{2\pi} \max \mu \in T \log \left( \sum_{j=0}^{n} \frac{\phi_{k+1}(H_{\mu(j)})}{\phi_k(H_{\mu(j)})^{2-2\Lambda(r)}} \right)^{n-k} \frac{d\theta}{2\pi} \]

\[ + \sum_{k=0}^{n-1} \sum_{j=0}^{n} \int_0^{2\pi} \max \mu \in T \log \phi_k(H_{\mu(j)})^{2\Lambda(r)} \frac{d\theta}{2\pi} + O(1) \]

\[ = \sum_{k=0}^{n-1} (n-k) \int_0^{2\pi} \max \mu \in T \log \left( \sum_{j=0}^{n} \frac{\phi_{k+1}(H_{\mu(j)})}{\phi_k(H_{\mu(j)})^{2-2\Lambda(r)}} \right)^{n-k} \frac{d\theta}{2\pi} \]

\[ - 2 \sum_{k=0}^{n-1} (n-k)S_k(r) + \sum_{k=0}^{n-1} \sum_{j=0}^{n} \int_0^{2\pi} \max \mu \in T \log \phi_k(H_{\mu(j)})^{2\Lambda(r)} \frac{d\theta}{2\pi} + O(1), \]

where $h_k$ is defined in (10). By Lemma 4.2 noticing that $N_W(r, 0) = N_{d_n}(r)$, we have

\[ \sum_{k=0}^{n-1} (n-k)S_k(r) = \sum_{k=0}^{n-1} (n-k)N_{d_k}(r) \]

\[ + \sum_{k=0}^{n-1} (n-k)(T_{F_{k-1}}(r) - 2T_{F_k}(r) + T_{F_{k+1}}(r)) + O(1) \]
Also, by Theorem 4.4 (the weak First Main Theorem) and (18),

\[ G \]

where

\[ \frac{1}{2\pi} \int_0^{2\pi} \max_{\mu \in T} \log \| f(re^{i\theta}) \| \frac{d\theta}{2\pi} \leq (n + 1)T_f(r) - N_W(r, 0) + G(r), \]

where

\[ G(r) = \frac{1}{2} \sum_{k=0}^{n-1} (n-k) \int_0^{2\pi} \max_{\mu \in T} \log \left( \sum_{j=0}^{n-1} \frac{\phi_{k+1}(H_{\mu(j)})}{\phi_k(H_{\mu(j)})} \right) \frac{d\theta}{2\pi}. \]

We now estimate \( G(r) \). Let

\[ \hat{T}(r) := \int_0^r \left( \int_{|z|<t} \frac{\phi_{k+1}(H)}{\phi_k(H)^{2-2\Lambda(r)}} h_k(re^{i\theta}) \frac{dz \wedge d\bar{z}}{2\pi} \right) dt. \]

Then, from Lemma 4.6 (18) and Lemma 4.3, we get

(20) \[ \hat{T}(r) \leq O(T_{F_k}(r)) = O(T_f^3(r)). \]

Then, by (3) with \( \gamma(r) = e^{(c_f+\epsilon)T_f(r)} \), for every hyperplane \( H \),

\[ \int_0^{2\pi} \frac{\phi_{k+1}(H)(re^{i\theta})}{\phi_k(H)^{2-2\Lambda(r)}} h_k(re^{i\theta}) \frac{d\theta}{2\pi} \leq 2\epsilon e^{(c_f+\epsilon)(2+2\epsilon)T_f(r)} \cdot \hat{T}(1+2\epsilon)^2(r) \]

This, together with the concavity of \( \log \) and (20), gives

\[ G(r) = \frac{1}{2} \sum_{k=0}^{n-1} (n-k) \int_0^{2\pi} \log \max_{\mu \in T} \sum_{j=0}^{n-1} \frac{\phi_{k+1}(H_{\mu(j)})}{\phi_k(H_{\mu(j)})} h_k(re^{i\theta}) \frac{d\theta}{2\pi} \]

\[ \leq \sum_{k=0}^{n-1} \frac{n-k}{2} \log \int_0^{2\pi} \sum_{j=1}^{q} \frac{\phi_{k+1}(H_j)(re^{i\theta})}{\phi_k(H_j)^{2-2\Lambda(r)}} h_k(re^{i\theta}) \frac{d\theta}{2\pi} \]

\[ \leq ((c_f + \epsilon)(2 + 2\epsilon)T_f(r) + 2\epsilon \log r) \sum_{k=0}^{n-1} \frac{n-k}{2} + O(\log T(r)) \]
\[
= \frac{n(n+1)}{2} ((1 + \varepsilon)(c_f + \varepsilon)T_f(r) + \varepsilon \log r) + O(\log T_f(r)) \parallel.
\]
Combining this with (19) proves Theorem 4.8. \(\square\)

E. The proof of Theorem 1.7

We first consider the case when \(k = n\), i.e. \(f\) is linearly non-degenerate. We need the following lemma.

**Lemma 4.9** (see Lemma A3.1.6 in [20]). Let \(H_1, \ldots, H_q\) be hyperplanes in \(\mathbb{P}^n(\mathbb{C})\) in general position. Denote by \(T\) the set of all injective maps \(\mu: \{0, 1, \ldots, n\} \rightarrow \{1, \ldots, q\}\). Then

\[
\sum_{j=1}^{q} m_f(r, H_j) \leq \int_{0}^{2\pi} \max_{\mu \in T} \sum_{i=0}^{n} \log \frac{1}{\|f(re^{i\theta}); H_{\mu(i)}\|} \frac{d\theta}{2\pi} + O(1).
\]

Theorem 4.8 together with the above Lemma, proves Theorem 1.7 is this case.

We now deal with the case when \(f\) is degenerate. By the assumption, we can assume that \(f(\triangle(R)) \subset \mathbb{P}^k(\mathbb{C})\) with \(0 \leq k < n\) and \(f\) becomes linearly non-degenerate. We also assume that \(q \geq 2n - k + 1\). Denote by \(\bar{H}_j = H_j \cap \mathbb{P}^k(\mathbb{C})\). Then \(\bar{H}_j\) are hyperplanes in \(\mathbb{P}^k(\mathbb{C})\) located in \(n\)-subgeneral position. Here hyperplanes \(H_1, \ldots, H_q\) (or \(a_1, \ldots, a_q\)) in \(\mathbb{P}^k(\mathbb{C})\) are said to be in \(n\)-subgeneral position if, for every \(1 \leq i_0 < \cdots < i_n \leq q\), the linear span of \(a_{i_0}, \ldots, a_{i_n}\) is \(\mathbb{C}^{k+1}\). We recall the following result due to Nochka.

**Lemma 4.10** (See Theorem A3.4.3 in [20]). Let \(H_1, \ldots, H_q\) (or \(a_1, \ldots, a_q\)) be hyperplanes in \(\mathbb{P}^k(\mathbb{C})\) in \(n\)-subgeneral positions with \(2n - k + 1 \leq q\). Then there exists a function \(\omega: \{1, \ldots, q\} \rightarrow (0, 1]\) called a Nochka weight and a real number \(\theta \geq 1\) called Nochka constant satisfying the following properties:

(i) \(\text{If } j \in \{1, \ldots, q\}, \text{ then } 0 \leq \omega(j) \theta \leq 1.\)

(ii) \(q - 2n + k - 1 = \theta(\sum_{j=1}^{q} \omega(j) - k - 1).\)

(iii) \(\text{If } 0 \neq B \subset \{1, \ldots, q\} \text{ with } \#B \leq n + 1, \text{ then } \sum_{j \in B} \omega(j) \leq \dim L(B), \text{ where } L(B) \text{ is the linear space generated by } \{a_j | j \in B\}.\)

(iv) \(1 \leq (n + 1)/(k + 1) \leq \theta \leq (2n - k + 1)/(k + 1).\)

(v) \(\text{Given real numbers } E_1, \ldots, E_q \text{ with } E_j \geq 1 \text{ for } 1 \leq j \leq q, \text{ and given any } Y \subset \{1, \ldots, q\} \text{ with } 0 < \#Y \leq n + 1, \text{ there exists a subset } M \text{ of } Y \text{ with } \#M = \dim L(Y) \text{ such that } \{a_j | j \in M\} \text{ is a basis for } L(Y) \text{ where } L(Y) \text{ is the linear space generated by } \{a_j | j \in Y\}, \text{ and}\)

\[
\prod_{j \in Y} E_{\omega(j)} \leq \prod_{j \in M} E_j.
\]
We now continue our proof. Since $H_1, \ldots, H_q$ (or $a_1, \ldots, a_q$) are hyperplanes in $n$-subgeneral position, for each $z \in \triangle(R)$, there are (see the proof of Lemma B3.4.4 in [20] for detail) indices $i(z,0), \ldots, i(z,n) \in \{1, \ldots, q\}$ such that

$$\prod_{j=1}^{q} \frac{1}{\|f(z); H_j\|\omega(j)} \leq C \prod_{l=0}^{n} \frac{1}{\|f(z); H_{i(z,l)}\|\omega(i(z,l))}$$

where $\omega(j)$ is the Nochka weight corresponding to $\hat{H}_j$ and $C > 0$ is a constant. Applying Lemma 4.10 with

$$E_l = \frac{1}{\|f(z); H_{i(z,l)}\|}, \quad 0 \leq l \leq n,$$

there is a subset $M$ of $Y = \{i(z,0), \ldots, i(z,n)\}$ with $\#M = k + 1$ such that $\{\hat{H}_{i(z,j)}| i(z,j) \in M\}$ is linearly independent, and

$$\prod_{l=0}^{n} \frac{1}{\|f(z); H_{i(z,l)}\|\omega(i(z,l))} \leq \prod_{i(z,j) \in M} \frac{1}{\|f(z); \hat{H}_{i(z,l)}\|}$$

Thus, together with (21),

$$\prod_{j=1}^{q} \frac{1}{\|f(z); H_j\|\omega(j)} \leq C \max_{\gamma \in \Gamma} \prod_{l=0}^{k} \frac{1}{\|f(z); H_{\gamma(l)}\|}$$

where $\Gamma$ is the set of all maps $\gamma : \{0, \ldots, k\} \to \{1, \ldots, q\}$ such that $\hat{H}_{\gamma(0)}, \ldots, \hat{H}_{\gamma(k)}$ are linearly independent. Hence, by applying the integration, we get, together with Theorem 1.8,

$$\sum_{j=1}^{q} \omega(j) \mu_f(H_j, r) \leq \int_{0}^{2\pi} \max_{\gamma \in \Gamma} \sum_{l=0}^{k} \log \frac{1}{\|f(re^{i\theta}); H_{\gamma(l)}\|} \frac{d\theta}{2\pi} + O(1)$$

$$\leq (k + 1)T_f(r) - N_{f,\text{ram}}(r) + \frac{k(k + 1)}{2} (1 + \epsilon)(c_f + \epsilon)T_f(r)$$

$$+ O(\log T_f(r)) + \frac{k(k + 1)}{2} \epsilon \log r \|.$$

By Lemma 4.10 and recalling that $\mu_f(r, H_j) \leq T_f(r) + O(1)$, it gives

$$\sum_{j=1}^{q} \mu_f(r, H_j) = \sum_{j=1}^{q} (1 - \theta \omega(j)) \mu_f(r, H_j) + \sum_{j=1}^{q} \theta \omega(j) \mu_f(r, H_j)$$

$$\leq \sum_{j=1}^{q} (1 - \theta \omega(j)) \mu_f(r, H_j) + \theta (k + 1)T_f(r) - \theta N_{f,\text{ram}}(r)$$

$$+ \theta \frac{k(k + 1)}{2} ((1 + \epsilon)(c_f + \epsilon)T_f(r) + \epsilon \log r) + O(\log T_f(r))$$
\[ \sum_{j=1}^{q} (1 - \theta \omega(j)) T_f(r) + \theta (k + 1) T_f(r) - \left( \frac{n + 1}{k + 1} \right) N_{f,\text{ram}}(r) \]

\[ + \frac{(2n - k + 1)k}{2} (1 + \epsilon)(c_f + \epsilon)T_f(r) + \epsilon \log r + O(\log T_f(r)) \]

\[ = \left\{ q - \theta \left( \sum_{1 \leq j \leq q} \omega(j) - k - 1 \right) \right\} T_f(r) - \left( \frac{n + 1}{k + 1} \right) N_{f,\text{ram}}(r) \]

\[ + \frac{(2n - k + 1)k}{2} (1 + \epsilon)(c_f + \epsilon)T_f(r) + \epsilon \log r + O(\log T_f(r)) \]

\[ = (2n - k + 1)T_f(r) - \left( \frac{n + 1}{k + 1} \right) N_{f,\text{ram}}(r) \]

\[ + \frac{(2n - k + 1)k}{2} (1 + \epsilon)(c_f + \epsilon)T_f(r) + \epsilon \log r + O(\log T_f(r)), \]

where the inequality holds for all \( r \in (0, R) \) outside a set \( E \subset (0, R) \) with \( \int_E \exp((c_f + \epsilon)T_f(r))dr < \infty \). This proves Theorem 1.7.

5. The Logarithmic Derivative Lemma and the Fundamental Vanishing Theorem

We begin with the following Logarithmic Derivative Lemma for meromorphic functions.

**Theorem 5.1** (Logarithmic Derivative Lemma). Let \( 0 < R \leq \infty \) and let \( \gamma(r) \) be a function defined on \( (0, R) \) with \( \int_0^R \gamma(r)dr = \infty \). Let \( f(z) \) be a meromorphic function on \( \Delta(R) \). Then, for \( \delta > 0 \), the inequality

\[ \int_0^{2\pi} \log^+ \left| \frac{f'(re^{i\theta})}{f(re^{i\theta})} \right| \frac{d\theta}{2\pi} \leq (1 + \delta) \log \gamma(r) + \delta \log r + O(\log T_f(r)) \]

holds outside a set \( E \subset (0, R) \) with \( \int_E \gamma(r)dr < \infty \).

**Proof.** For \( w \in \mathbb{C} \), we define the \((1, 1)\) form on \( \mathbb{C} \) with singularities at \( w = 0, \infty \):

\[ \Phi = \frac{1}{(1 + \log^2|w||w|^2} \frac{\sqrt{-1}}{4\pi^2} dw \wedge d\bar{w}. \]

The form \( \Phi \) is of integral 1. By the change of variable formula,

\[ \int_{\Delta(t)} f^*\Phi = \int_{w \in \mathbb{C}} n_f(t, w)\Phi(w). \]
Thus, defining $\mu(r) := \int_0^r \frac{dt}{t} \int_{\triangle(t)} f^* \Phi$, we have

$$
\mu(r) = \int_0^r \frac{dt}{t} \int_{\triangle(t)} \frac{|f'|^2}{(1 + \log^2 |f|)|f|^2} \sqrt{-1} \frac{dz}{4\pi^2} d\bar{z}
$$

$$
= \int_{\mathbb{C}} \int_0^r \frac{dt}{t} n_f(t,w) \Phi(w) = \int_{\mathbb{C}} N_f(r,w) \Phi(w) \leq T_f(r) + O(1)
$$

where the last inequality holds as a consequence of the First Main Theorem.

Using the observation (6) (or Lemma 3.2) we get

$$
\frac{1}{2\pi} \int_{|z|=r} \frac{|f'|^2}{(1 + \log^2 |f|)|f|^2} d\theta \leq \frac{1}{2} r^{2\delta} \cdot \gamma^{2+2\delta}(r) \cdot T_f^{(1+2\delta)^2}(r)
$$

outside a set $E \subset (0,1)$ with $\int_E \gamma(r) dr < \infty$. By making use of this, the Calculus lemma and the concavity of the logarithm function, we carry out the following classical computations, except for the error term:

$$
\int_0^{2\pi} \log^+ \left| \frac{f'}{f}(re^{i\theta}) \right| \frac{d\theta}{2\pi}
$$

$$
= \frac{1}{4\pi} \int_{|z|=r} \log^+ \left( \frac{|f'|^2}{(1 + \log^2 |f|)|f|^2} \right)\left(1 + \log^2 |f| \right) d\theta
$$

$$
\leq \frac{1}{4\pi} \int_{|z|=r} \log^+ \left( \frac{|f'|^2}{(1 + \log^2 |f|)|f|^2} \right) d\theta
$$

$$
+ \frac{1}{4\pi} \int_{|z|=r} \log^+ (1 + \log^+ |f| + \log^+ (1/|f|))^2) d\theta
$$

$$
\leq \frac{1}{4\pi} \int_{|z|=r} \log \left( \frac{1 + \log^2 |f|}{(1 + \log^2 |f|)|f|^2} \right) d\theta
$$

$$
+ \frac{1}{2\pi} \int_{|z|=r} \log^+ (|f| + \log^+ (1/|f|)) d\theta + \frac{1}{2} \log 2
$$

$$
\leq \frac{1}{2} \log \left( \frac{1 + \frac{1}{2\pi} \int_{|z|=r} \frac{|f'|^2}{(1 + \log^2 |f|)|f|^2} d\theta}{2\pi} \right)
$$

$$
+ \frac{1}{2\pi} \int_{|z|=r} \log (1 + \log^+ |f| + \log^+ (1/|f|)) d\theta + \frac{1}{2} \log 2
$$

$$
\leq \frac{1}{2} \log \left( 1 + \frac{r^{2\delta} \cdot \gamma^{2+2\delta}(r) \cdot T_f^{(1+2\delta)^2}(r)}{2\pi} \right)
$$

$$
+ \log (1 + m_f(r, \infty) + m_f(r,0) + \frac{1}{2} \log 2
$$

$$
\leq \frac{1}{2} \log \left( 1 + r^{2\delta} \cdot \gamma^{2+2\delta}(r) \cdot T_f^{(1+2\delta)^2}(r) \right) + \log^+ T_f(r) + O(1)
$$

$$
\leq (1 + \delta) \log \gamma(r) + \delta \log r + O(\log T_f(r))
$$
holds outside a set $E \subset (0, R)$ with $\int_E \gamma(r)dr < \infty$. This proves the theorem.

We actually need to estimate the higher order derivatives.

**Theorem 5.2.** Let $0 < R \leq \infty$ and let $\gamma(r)$ be a function defined on $(0, R)$ with $\int_0^R \gamma(r)dr = \infty$. Let $f(z)$ be a meromorphic function on $\triangle(R)$. Then for $k \geq 1$ and $\delta > 0$ (small enough), the inequality

$$\int_0^{2\pi} \log^+ \left| \frac{f^{(k)}}{f}(re^{i\theta}) \right| \frac{d\theta}{2\pi} \leq (1 + \delta)k \log \gamma(r) + \delta k \log r + O(\log T_f(r) + \log \log \gamma(r) + \log \log r)$$

holds outside a set $E \subset (0, R)$ with $\int_E \gamma(r)dr < \infty$.

**Proof.** Note that

$$\frac{f^{(k)}}{f} = \frac{f^{(k)}}{f^{(k-1)}} \frac{f^{(k-1)}}{f^{(k-2)}} \cdots \frac{f'}{f}$$

hence, by using Theorem 5.1,

$$\int_0^{2\pi} \log^+ \left| \frac{f^{(k)}}{f}(re^{i\theta}) \right| \frac{d\theta}{2\pi} \leq \sum_{j=1}^{k} \log^+ \left| \frac{f^{(j)}}{f^{(j-1)}}(re^{i\theta}) \right| \frac{d\theta}{2\pi}$$

holds outside a set $E \subset (0, R)$ with $\int_E \gamma(r)dr < \infty$. On the other hand,

$$T_f^{(j-1)}(r) = m_f^{(j-1)}(r, \infty) + N_f^{(j-1)}(r, \infty)$$

$$\leq m_f^{(j-1)}f_f^{(j-2)}(r, \infty) + m_f^{(j-2)}(r, \infty) + N_f^{(j-2)}(r, \infty) + T_f^{(j-2)}(r) + O(1)$$

$$\leq \int_0^{2\pi} \log^+ \left| \frac{f^{(j-2)}}{f^{(j-3)}}(re^{i\theta}) \right| \frac{d\theta}{2\pi} + 2T_f^{(j-2)}(r) + O(1)$$

$$\leq (1 + \delta) \log \gamma(r) + \delta \log r + O(\log T_f^{(j-2)}(r))$$

holds outside a set $E \subset (0, R)$ with $\int_E \gamma(r)dr < \infty$. The theorem is proved by induction.

We now extend the above theorem to jet differentials. Jet bundles are generalizations of tangent bundles. Kobayashi attributes the introduction of the concept of jets and jet bundles to Ehresmann. We refer to [10], Kobayashi’s book [12] and Demailly’s survey paper [7]. See also [19]. Let $X$ be a complex manifold with dim $X = n$. Let $x \in X$ and consider the germs of holomorphic mappings $\phi : \triangle(1) \to X$ with $\phi(0) = x$. Two germs $\phi, \tilde{\phi}$...
osculate to order \( k \) (denote it as \( \phi \sim^k \tilde{\phi} \)) if \( \phi^{(i)}(0) = \tilde{\phi}^{(i)}(0) \), for \( 0 \leq i \leq k \).
Let \( j_k(\phi) \) denote the equivalence class of \( \phi \) and set
\[
J_k(X)_x = \{ j_k(\phi) \mid \phi : (\Delta, 0) \to (X, x) \}.
\]
Clearly \( J_k(X)_x = \mathbb{C}^{nk} \), i.e. every element \( v \in J_k(X)_x \) is represented by \( (\frac{d}{d\zeta}(z^i \circ \phi))(0)_{1 \leq j \leq k, 1 \leq i \leq n} \) for some holomorphic map \( \phi \) from an open neighborhood \( U \) of 0 in \( \mathbb{C} \) to \( M \) such that \( \phi(0) = x \). Of course this isomorphism depends on the choice of local coordinates \( z^1, \ldots, z^n \). Let \( J_k(X) = \bigcup_{x \in U} J_k(X)_x \). Locally \( J_k(U) = U \times \mathbb{C}^{kn} \), so \( J_k(X) \) is a complex manifold of dimension \( n + nk \). For a holomorphic map \( f : \Delta(R) \to X \), at each point \( z \in \Delta \), the map \( f \) has a jet in \( J_k(X)_f(z) \), denoted by \( j_k(f(z)) \).

The notation \( j_k(f) : \Delta(R) \to J_k(X) \) will be used to denote the natural lifting of \( f \) to \( k \)-jet. The 1-jet bundle \( J_1(X) \) is simply the tangent bundle of \( M \). For \( k > 1 \), \( J_k(X) \) is no longer a vector bundle, just a holomorphic fiber bundle, i.e. \( J_k(X) \) is a complex analytic space with a natural projection \( p : J_k(X) \to X \) with \( p^{-1}(U) = U \times \mathbb{C}^{nk} \).

When \( X \) is an analytic set, we can consider the space \( J_k(\text{Reg}X) \). Let \( G_k \) denote the group of \( k \)-jets of biholomorphisms of \( (\mathbb{C}, 0) \). One can consider the space \( J_k(\text{Reg}X)/G_k \) following [7], one can construct a compactification \( X_k \) of this space. There is a natural projection \( \pi_k : X_k \to X \), the fiber at a non-singular point is a rational manifold. See [7], for more details.

Let \( x \in X \) and let \( z^1, \ldots, z^n \) be a local coordinate of \( X \) centered at \( x \). We consider the symbols
\[
 dz^1, \ldots, dz^n, d^2z^1, \ldots, d^2z^n, \ldots, d^kz^1, \ldots, d^kz^n
\]
and we say that the weight of the symbol \( d^p z^i \) is equal to \( p \), for any \( i = 1, \ldots, n \). A (Green-Griffiths) jet differential of order \( k \) and degree \( m \) at \( x \) is a homogeneous polynomial of degree \( m \) in \( \{ d^p z^i \}_{p=1,\ldots,k,i=1,\ldots,n} \). We denote \( E_{k,m}^{\text{GG}}(X) \) the set of (Green-Griffiths) jet differentials of weight \( m \) and order \( k \).

Let \( D = Y_1 + \cdots + Y_l \) be an effective divisor, such that the pair \( (X, D) \) is log-smooth (this last condition means that the hypersurfaces \( Y_j \) are non-singular, and that they have transverse intersections). A jet differential of order \( k \) and degree \( m \) with possible log-pole along \( D \) is locally a homogeneous polynomial of degree \( m \) in \( d^p \log z^1, \ldots, d^p \log z^d, d^p z^{d+1}, \ldots, d^p z^n \) where \( p = 1, \ldots, k \) and \( z^1 \cdots z^d = 0 \) is a local defining equation of the divisor \( D \). We denote \( E_{k,m}^{\text{GG}}(\log D) \) the set of jet differential of order \( k \) and degree \( m \) with possible log-pole along \( D \).

The Logarithmic Derivative Lemma is extended to the jet differentials with possible log-pole along \( D \) as follows.
Theorem 5.3 (Logarithmic derivative lemma for jet differentials). Let \( X \) be a complex projective manifold and let \( D \) be a divisor on \( X \) such that the pair \((X,D)\) is log-smooth. Let \( A \) be an ample divisor on \( X \) and \( \omega_A \) be its curvature form. Let \( \mathcal{P} \) be a logarithmic \( k \)-jet differential along \( D \) on \( X \). Let \( f : \Delta(R) \to X \) be a holomorphic map such that \( f(\Delta(R)) \not\subset D \). Let \( \xi(z) := \mathcal{P}(J_k(f))(z) \) which is a meromorphic function on \( \Delta(R) \). Assume that \( c_{f,\omega_A} < \infty \). Then, for \( \epsilon > 0 \), the inequality

\[
\int_0^{2\pi} \log^+ |\xi(re^{i\theta})| \frac{d\theta}{2\pi} \leq C ((c_{f,\omega_A} + \epsilon)T_{f,A}(r) + \epsilon \log r + \log T_{f,A}(r))
\]

holds outside a set \( E \subset (0, R) \) with \( \int_E e^{(c_{f,\omega_A}+\epsilon)T_{f,A}(r)} \, dr \leq C > 0 \) is a constant.

Proof. We follow the argument in [23] (see also [20] Theorem A7.5.4). Since \( X \) is projective, we can embed \( X \) into a projective space \( \mathbb{P}^N \) with homogeneous coordinates \([w_0 : \ldots : w_N]\). Let \( Z = \{ \prod_{i=0}^N w_i = 0 \} \subset \mathbb{P}^N \). Choose elements \( A_t \in GL(N+1, \mathbb{C}) \), \( 0 \leq t \leq N \) such that \( \cap_{i=0}^N A_t(Z) = \emptyset \), where \( A_t : \mathbb{P}^N \to \mathbb{P}^N \) is the map induced by \( A_t \). Let

\[
\{ \mu_{j,\nu} \}_{0 \leq j \leq N, 1 \leq \nu \leq N(N+1)} := \left\{ \left. \frac{w_\lambda}{w_j} \circ A_t \right|_{0 \leq \lambda \leq N, \lambda \neq j, 0 \leq t \leq N} \right\}.
\]

Then for any point \( P_0 \in \mathbb{P}^N \) there exist \( 0 \leq j_1, \ldots, j_N \leq N, 1 \leq \nu_1, \ldots, \nu_N \leq N(N+1) \), such that one can choose local branches \( \log u_{j_1,\nu_1}, \ldots, \log u_{j_N,\nu_N} \) to form a local coordinate system of \( \mathbb{P}^N \) at \( P_0 \). As a consequence there exists a positive constant \( C \) such that

\[
|f^*\mathcal{P}| \leq C \sum_{j=0}^N \sum_{\nu=0}^{N(N+1)} |f^* \prod_{\nu=0}^{N(N+1)} (d^{\alpha_{j,\nu}} \log u_{j,\nu})^{\beta_{j,\nu}}|,
\]

where the second summation \( \sum \) is over the indices \( \{ \alpha_{j,\nu}, \beta_{j,\nu} \}_{1 \leq \nu \leq N(N+1)} \), with \( \sum_{\nu=1}^{N(N+1)} \alpha_{j,\nu}\beta_{j,\nu} = m, \ 0 \leq \alpha_{j,\nu} \leq k, \beta_{j,\nu} \geq 0 \). Since \( f^*\mathcal{P} = \xi(d\zeta)^m \), the above gives

\[
\frac{1}{2\pi} \int_0^{2\pi} \log^+ |\xi(re^{i\theta})| \, d\theta \leq C' \sum_{h \in \mathcal{H}} \sum_{1 \leq s \leq k} \frac{1}{2\pi} \int_0^{2\pi} \log^+ \left| \frac{(h \circ f)^{(s)}}{h \circ f} (re^{i\theta}) \right| \, d\theta,
\]

where \( C' > 0 \) is a constant, and \( \mathcal{H} \) is the set \( \{ u_{j,\nu} \} \). By applying Theorem 5.2 with \( \gamma(r) := \exp((c_{f,\omega_A} + \epsilon)T_{f,A}(r)) \), the inequality

\[
\int_0^{2\pi} \log^+ \left| \frac{(h \circ f)^{(s)}}{h \circ f} (re^{i\theta}) \right| \frac{d\theta}{2\pi} \leq (1 + \epsilon)s(c_{f,\omega_A} + \epsilon)T_{f,A}(r) + \epsilon s \log r + O(\log T_{h\circ f}(r))
\]
holds outside a set $E \subset (0, R)$ with $\int_E e^{(c_f, \omega_A + \epsilon)T_{f,A}(r)} dr < \infty$. Since $h$ is a rational function,

$$\log T_{h_{0f}}(r) \leq O(\log T_{f,A}(r))$$

and we arrive at the estimate

$$\int_0^{2\pi} \log^+|\xi(re^{i\theta})| \frac{d\theta}{2\pi} \leq C((c_f, \omega_A + \epsilon)T_{f,A}(r) + \epsilon \log r + \log T_{f,A}(r)),$$

for some constant $C > 0$, where the inequality holds outside a set $E \subset (0, R)$ with $\int_E e^{(c_f, \omega_A + \epsilon)T_{f,A}(r)} dr < \infty$. \hfill \Box

As a corollary of the above Theorem, we get the following result.

**Corollary 5.4** (Fundamental Vanishing Theorem). Let $X$ be a complex projective manifold. Let $f : \Delta(R) \to X$ be a holomorphic map. Assume that $f \in \mathcal{E}_0$, i.e $\int_0^R \exp(cT_{f,A}(r))dr = \infty$ for any $\epsilon > 0$ for some (hence for any) ample divisor $A$. Let $\mathcal{P}$ be a holomorphic (or log-pole) $k$-jet differential on $X$ which vanishes on an ample divisor $A$ of $X$ (and the image of $f$ is disjoint from the log-pole of $\mathcal{P}$), i.e. $\mathcal{P} \in H^0(X, E_{k,m}^G(X) \otimes \mathcal{O}(-A))$ or $\mathcal{P} \in H^0(X, E_{k,m}^G(\log D) \otimes \mathcal{O}(-A))$. Then $f^*\mathcal{P}$ is identically zero on $\Delta(R)$.

**Remark.** We observe that if $R = \infty$, then $f$ is necessarily in $\mathcal{E}_0$ if $f$ is non-constant. So the above result extends the Fundamental Vanishing Theorem for maps defined in the complex plane $\mathbb{C}$. See Green-Griffiths [10], Siu-Yeung [23] and Demailly’s survey paper [7].

**Proof.** Assume that $f^*\mathcal{P} \neq 0$, we will derive a contradiction. Choose a positive integer $l$ such that $lA$ is very ample. The canonical map $\phi_{lA}$ associated to $lA$ embeds $X$ into the projective space $\mathbb{P}^N(\mathbb{C})$ with homogeneous coordinates $[w_0 : \cdots : w_N]$. By Cartan’s Second Main Theorem, we conclude that for any $1 > \epsilon > 0$, there exists a hyperplane $H = \{[w_0 : \cdots : w_N] | \sum_{i=0}^{N} a_iw_i = 0\}$ such that

$$N_{\phi_{lA^0f}}(r, H) \geq (1 - \epsilon)T_{\phi_{lA^0f}}(r).$$

Let $s_A$ denote the canonical section of of the line bundle associated to $A$ (i.e. $[s_A = 0] = A$). By replacing $\mathcal{P}$ by $\left(\frac{s_A}{s_A}\right)^{N} \phi^*_{lA}(\sum_{i=0}^{N} a_iw_i)$ we can assume without loss of generality that $\ell = 1$ and $A = \phi_{lA}^* H$ so we have

$$N_f(r, A) \geq (1 - \epsilon)T_{f,A}(r).$$

Write $f^*\mathcal{P}(z) = \xi(z)^{\otimes m}$. Since $\mathcal{P}$ vanishes on $A$, by (22), the Jensen formula and Theorem 5.3 (noticing that $c_{f,\omega_A} = 0$ under our assumption),

$$(1 - \epsilon)T_{f,A}(r) \leq N_f(r, A) \leq \int_0^{2\pi} \log |\xi(re^{i\theta})| \frac{d\theta}{2\pi} \leq C(\epsilon T_{f,A}(r) + 2\epsilon \log r + \log T_{f,A}(r))$$
holds outside a set $E \subset (0, R)$ with $\int_E e^{\varepsilon T_f \cdot \lambda(r)} dr < \infty$, which gives a contradiction by taking $\varepsilon$ small enough. \hfill \Box

6. **Bloch’s theorem and the Second Main Theorem for mappings into Abelian varieties**

A. **Bloch Theorem.**

The following is a fundamental theorem in value distribution theory (see Bloch [2], Siu [21], Noguchi-Ochiai [16], and [19], [20]).

**Theorem 6.1** (Bloch). Let $A$ be an Abelian variety and let $f : \mathbb{C} \to A$ be a holomorphic map. Then the Zariski closure of $f(\mathbb{C})$ is a translate of a sub-abelian variety.

We extend the above result to mappings on the disc. We follow the strategy from Siu [21] as carried out in [19] where $\mathbb{C}$ is replaced by a parabolic Riemann Surface. We recall the following result due to Ueno [27].

**Theorem 6.2** (Ueno). Let $X$ be a subvariety of a complex torus $T$. Then there exist a complex torus $T_1 \subset T$, a projective variety $W$ and an abelian variety $A$ such that

1. We have $W \subset A$ and $W$ is a variety of general type;
2. There exists a dominant (reduction) map $R : X \to W$ whose general fiber is isomorphic to $T_1$.

We now prove the following result.

**Theorem 6.3.** Let $T$ be a complex torus and let $f : \triangle(R) \to T$ be a non-constant holomorphic map in the space $\mathcal{E}_0$ (i.e. $\int_0^R \exp(\varepsilon T_f(r)) dr = \infty$ for any $\varepsilon > 0$). Let $X$ be the Zariski closure of $f(\triangle(R))$. Then either $X$ is the translate of a sub-torus of $T$, or there is a variety of general type $W$ and map $R : X \to W$ such that $R \circ f$ dose not belong to the space $\mathcal{E}_0$.

**Remarks.** (1) The characteristic function $T_f(r)$ is defined by $T_f(r) = T_{f,\omega}(r)$ where $\omega = \pi_s(dw_1 + \cdots + dw_m)$ where $\pi : \mathbb{C}^m \to T$ is the projection map. (2) We observe that if $R = \infty$, then $f$ and $R \circ f$ are necessarily in $\mathcal{E}_0$. So the above result extends the classical Bloch’s Theorem.

To prove Theorem 6.3, let $n$ be the complex dimension of $T$. Let $J_k(T) = T \times \mathbb{C}^{kn}$, and $J'_k(T) = T \times \mathbb{P}^{nk-1}$. Let $\mathcal{X}'_k$ be the Zariski closure of $J'_k(f)(\triangle(R))$ in $J'_k(T)$. Let $\pi_k : \mathcal{X}'_k \to \mathbb{P}^{nk-1}$ be the projection on the second factor. The proof relies on the following two Propositions whose idea goes back to Bloch [2] (see also [7] and [19]).
Proposition 6.4. Assume that the Zariski closure of $f$ is $X$. We assume that for each $k \geq 1$ the fibers of $\tau_k$ are positive dimensional. Then the dimension of the subgroup $A_X$ of $T$ defined by

$$A_X := \{ a \in T \mid a + X = X \}$$

is strictly positive.

In the following statement we discuss the other possibility.

Proposition 6.5. Let $k$ be a positive integer such that the map $\tau_k : X_k \to \mathbb{P}^{nk-1}$ has finite generic fibers. Then there exists a jet differential $\mathcal{P}$ of order $k$ with values in the dual of an ample line bundle, and whose restriction to $X_k$ is non-identically zero.

Proof. The hyperplane line bundle $O_{\mathbb{P}^{nk-1}}(1)$ is ample, and since the generic fibers of $\tau_k$ are of dimension zero, the restriction to $X_k$ of the line bundle $O_k(1) := \tau_k^*O_{\mathbb{P}^{nk-1}}(1)$ is big. Hence, for $m >> 0$ large enough, we have

$$H^0(X_k, O_k(m) \otimes A^{-1}) \neq \emptyset,$$

which means that there exists a jet differential $\mathcal{P}$ of order $k$ with values in the dual of an ample line bundle $A$, and whose restriction to $X_k$ is non-identically zero. The proposition is proved. \qed

Proof of Theorem 6.3. Let $X$ be the Zariski closure of $f$. Thanks to Ueno’s result (Theorem 6.2), we can consider the reduction map $R : X \to W$. We claim that, if $X$ is not a translate of a sub-torus, then $R \circ f$ is not in the space $\mathcal{E}_0$. If $W$ is a point, this means that $X$ is the translate of a sub-torus. If this is not the case, then we can assume that $X$ is of general type and $R \circ f$ is in $\mathcal{E}_0$. If the hypothesis in Proposition 6.5 is verified, then $X_k$ is algebraic and Corollary 5.4 gives a contradiction. So the hypothesis of Proposition 6.5 will never be verified for any $k \geq 1$. Hence the hypothesis of the Proposition 6.4 are verified, and so $X$ will be invariant by a positive dimensional sub-torus of $T$. Since $X$ is assumed to be a manifold of general type, its automorphism group is finite, so this cannot happen. This finishes the proof.

B. The Second Main Theorem for Holomorphic Curves Into Abelian Varieties.

We prove the following result which generalizes the result of Siu-Yeung \cite{23} (see also \cite{18}, \cite{17}).

Theorem 6.6. Let $A$ be an Abelian variety, and let $D$ be an ample divisor on $A$. Let $f : \Delta(R) \to A$ be a holomorphic map with Zariski dense image. Assume that $f \in \mathcal{E}_0$. Then there is a positive integer $k_0$ such that, for any $\epsilon > 0$,

$$T_{f,D}(r) \leq N_f^{(k_0)}(r,D) + \epsilon T_{f,D}(r) + O(\log T_{f,D}(r)) + \epsilon \log r$$
holds for \( r \in (0, R) \) except for a set \( E \) with \( \int_E \exp(\epsilon T_{f,D}(r))dr < \infty \).

When \( R = \infty \) then \( f \in \mathcal{E}_0 \). So the above theorem recovers the result of Siu-Yung [23]. Note that in the case \( R = \infty \), K. Yamanoi [29] showed that one can indeed take \( k_0 = 1 \). The proof here follows from the argument in the book by Noguchi and Winkelmann (see Theorem 6.3.1 in [17]).

**Proof.** For \( k \geq 1 \), let \( X_k(f) \) be the Zariski closure of the image of the \( k \)-jet lifting \( j_k(f) \) of \( f \). Let \( I_k \) denote the restriction to \( X_k(f) \) of the jet projection \( p_k : J_k(A) = A \times \mathbb{C}^{nk} \to \mathbb{C}^{nk} \), where \( n = \dim A \). Let \( x \in D \) and \( \sigma = 0 \) be a local defining equation of \( D \) near \( x \). For a given holomorphic map \( \phi : (\Delta(1), 0) \to (A, x) \), we denote its \( k \)-jet by \( j_k(\phi) \) and write

\[
d^j\sigma(\phi) = \left. \frac{d^j}{d\zeta^j} \right|_{\zeta = 0} \sigma(\phi(\zeta)).
\]

We set \( J_{k,x}(D) = \{j_k(\phi) \in J_k(A) \mid d^j\sigma(\phi) = 0, 1 \leq j \leq k \} \), and \( J_k(D) = \bigcup_{x \in D} J_{k,x}(D) \). To continue the proof, we need the following key lemma.

**Key Lemma.** There is \( k_0 \in \mathbb{N} \) such that for \( k \geq k_0 \)

\[
I_k(X_k(f)) \cap I_k(J_k(D)) \neq I_k(X_k(f)).
\]

**Proof.** It suffices to show that there is \( k \in \mathbb{N} \) such that \( I_k(j_k(f)(0)) \notin I_k(J_k(D)) \). Suppose that \( I_k(j_k(f)(0)) \in I_k(J_k(D)) \) for all integers \( k \geq 0 \). Then we have that

\[
J_k(D) \cap I_k^{-1}(I_k(j_k(f)(0))) \neq \emptyset
\]

for all \( k \geq 0 \). Define

\[
V_k := p_{1,k}(J_k(D) \cap I_k^{-1}(I_k(j_k(f)(0)))) \neq \emptyset,
\]

where \( p_{1,k} \) is the projective \( J_k(A) \to A \). Note that \( V_k \) is Zariski closed (because \( p_{1,k} : J_k(A) \to A \) has a section \( id_A \times \{I_k(j_k(f)(0))\} : A \to J_k(A) \), and \( V_k \) is the pull-back of supp\( J_k(D) \) by this section), and note that \( V_{k+1} \subset V_k \). Thus we have the sequence of Zariski closed set

\[
\cdots \subset V_3 \subset V_2 \subset V_1 \subset D
\]

that eventually stabilizes at the variety \( V \). Since we are assuming that \( V_k \neq \emptyset \), \( V \) is not empty. Let \( a \in V \), and translate \( f \) by \( a - f(0) \), i.e \( \hat{f}(z) = f(z) + a - f(0) \). Then by the construction of \( \hat{f} \), we have \( \hat{f}(0) = a \) and \( j_k(\hat{f})(0) \in J_k(D) \). Considering the Taylor series, we get \( \hat{f}(\Delta(R)) \subset D \), and hence a contradiction since we are assuming that \( f \) is non-degenerate. Thus the lemma is proved.
Write $Y_k := I_k(X_k(f))$. Note that $I_k$ is proper, therefore $Y_k := I_k(X_k(f))$ is an irreducible algebraic subset of $\mathbb{C}^{nk}$. By the key lemma, there is $k = k_0$ for which there is a polynomial $P$ on $\mathbb{C}^{nk}$ satisfying
\[ P|_{Y_k} \neq 0, \quad P|_{J_k(D)} \equiv 0. \]
Let $\{U_\lambda\}$ be an affine covering of $A$ such that $D \cap U_\lambda = \{\sigma_\lambda = 0\}$ for a regular function $\sigma_\lambda$ on $U_\lambda$. The defining functions of $J_k(D)|_{U_\lambda}$ are given by
\[ \sigma_\lambda = d\sigma_\lambda = \cdots = d^k\sigma_\lambda = 0. \]
On each $U_\lambda$ one obtains the following equation:
\[ a_{\lambda 0}\sigma_\lambda + \cdots + a_{\lambda k}d^k\sigma_\lambda = I_k^*P|_{U_\lambda}. \]
Here $a_{\lambda j}$ are polynomials in jet coordinates with coefficients of rational holomorphic functions on $U_\lambda$ restricted on $J_k(A)|_{U_\lambda}$.

Using a Hermitian metric on the line bundle $[D]$ associated to $D$, we have positive functions $\rho_\lambda \in C^\infty(U_\lambda)$ such that $\frac{d\sigma_\lambda}{\sigma_\lambda} = \frac{d\rho_\lambda}{\rho_\lambda}$ on $U_\lambda \cap U_\mu$. Therefore
\[ \rho_\lambda a_{\lambda 0} + \rho_\lambda a_{\lambda 1} \frac{d\sigma_\lambda}{\sigma_\lambda} + \cdots + \rho_\lambda a_{\lambda k} \frac{d^k\sigma_\lambda}{\sigma_\lambda} = \frac{\rho_\lambda}{\sigma_\lambda} I_k^*P|_{U_\lambda}. \]
Substituting $j_k(f)(z)$, $f(z) \in U_\lambda$ in the above equation, we have
\[ \left| \rho_\lambda(f(z)) a_{\lambda 0}(f(z)) + \cdots + \rho_\lambda(f(z)) a_{\lambda k}(f(z)) \frac{d^k\sigma_\lambda(f(z))}{\sigma_\lambda(f(z))} \right| = \frac{\rho_\lambda(f(z))}{|\sigma_\lambda(f(z))|} |P(I_k(J_k(f)(z)))|. \]
Let $\{\tau_\lambda\}$ be a partition of unity subordinated to the covering $\{U_\lambda\}$. Then
\[ \frac{1}{||\sigma(f(z))||} \leq \frac{1}{|P(I_k(J_k(f)(z)))|} \times \sum_\lambda \left\{ \tau_\lambda \rho_\lambda(f(z)) a_{\lambda 0}(f(z)) + \cdots + \tau_\lambda \rho_\lambda(f(z)) a_{\lambda k}(f(z)) \left| \frac{d^k\sigma_\lambda(f(z))}{\sigma_\lambda(f(z))} \right| \right\}. \]
Since $a_{\lambda j}$ are polynomials in jet coordinates with coefficients of holomorphic functions on $U_\lambda$, Theorem 5.3 with $\epsilon$ properly chosen yields that
\[ m_f(r, D) \leq C \left( m_{1/P(I_k(J_k(f)))}(r, \infty) + \sum_{\lambda, 1 \leq j \leq k} m_{(\sigma_\lambda \sigma_j)(r, \infty)} \right) \]
\[ + \epsilon(T_{f, D}(r) + \log r + O(\log T_{f, D}(r)) \]
holds for $r \in (0, R)$ except a set $E$ with $\int_E \exp(\epsilon T_{f, D}(r))dr < \infty$, where $C > 0$ is a constant. Since $\sigma_\lambda$ is a rational function on $A$, $d^j\sigma_\lambda/\sigma_\lambda$ is a logarithmic jet differential carrying logarithmic poles on zeros and poles of
It follows, from Theorem \[5.3\] with \(\epsilon\) properly chosen (notice that \(c_{f,\omega_D}\) is arbitrarily small in our case),
\[
\frac{m(\sigma r, f, j)}{\sigma r} \leq \epsilon(T_f, D(r) + \log r) + O(\log T_f, D(r))
\]
holds for \(r \in (0, R)\) except a set \(E\) with \(\int_E \exp(\epsilon T_f, D(r)) dr < \infty\). Moreover, the First Main Theorem and Theorem \[5.3\] with \(\epsilon\) properly chosen imply that
\[
m_{\lambda/(I_k(J_k(f)))}(r, \infty) \leq T_{P(J_k(J_k(f)))}(r) + O(1) \leq \epsilon(T_f, D(r) + \log r) + O(\log T_f, D(r))
\]
holds for \(r \in (0, R)\) except a set \(E\) with \(\int_E \exp(\epsilon T_f, D(r)) dr < \infty\). Thus
\[
m_f(r, D) \leq \epsilon(T_f, D(r) + \log r) + O(\log T_f, D(r))
\]
holds for \(r \in (0, R)\) except a set \(E\) with \(\int_E \exp(\epsilon T_f, D(r)) dr < \infty\). It is inferred from Theorem \[5.3\] with \(\epsilon\) properly chosen and (24) that
\[
N_f(r, D) - N_f^{(k)}(r, D) \leq T_{P(J_k(J_k(f)))}(r, 0) \leq \epsilon T_f, D(r) + \epsilon \log r + O(\log T_f, D(r))
\]
holds for \(r \in (0, R)\) except a set \(E\) with \(\int_E \exp(\epsilon T_f, D(r)) dr < \infty\). Hence, from the First Main Theorem and (24),
\[
T_f, D(r) = N_f(r, D) + m_f(r, D) \leq N_f^{(k)}(r, D) + \epsilon T_f, D(r) + \epsilon \log r + O(\log T_f, D(r))
\]
holds for \(r \in (0, R)\) except a set \(E\) with \(\int_E \exp(\epsilon T_f, D(r)) dr < \infty\). This finishes the proof. \(\square\)

References

[1] L. Ahlfors. The Theory of meromorphic curves. Acta Soc. Sci. Fenn. Nova, Ser. A, 3(4)(1941), 171-183.
[2] A. Bloch. Sur les systèmes de fonctions holomorphes à variétés linéaires. Ann. Sci. Ecole Norm. Sup. 43(1926), 309-362.
[3] M. Brunella. In existence of invariant measures for generic rational differential equations in the complex domain. Bol. Soc. Mat. Mexicana (3) 12(2006), no. 1, 43-49.
[4] H. Cartan. Sur les zeros des combinaisons lineaires de p fonctions holomorpes don- nées. Mathematica(Cluj), 7(1933), 80-103.
[5] S.S. Chern. Complex Analytic Mappings of Riemann Surfaces I. Amer. J. Math., 82(1960), 323-337.
[6] M. Cowen and Ph. Griffiths. Holomorphic curves and metrics of nonnegative curvature. J. Analyse Math., 29(1976), 93-153.
[7] J.P. Demailly. A Criteria for Kobayashi hyperbolic varieties and jet differentials. Proc. Symp. Pur. Math. Amer. Math. Soc., 62, Part 2(1995), 285-360.
[8] T.C. Dinh, V.A. Nguyen and N. Sibony. Heat equation and ergodic theorems for Riemann surface laminations. Math. Ann., 354(2012), 331-376.
[9] J.E. Fornaess and N. Sibony. Riemann surface laminations with singularities, J. Geom. Analysis, 18(2008), 400-442.
[10] M. Green and Ph. Griffiths. Two applications of algebraic geometry to entire holomorphic mappings. The Chern Symposium 1979, Proc. Internat. Sympos., Berkeley, 1979, Springer-Verlag.
[11] PH. GRIFFITHS AND J. HARRIS. Principle of Algebraic Geometry. Wiley, New York, 1978.
[12] S. KOBAYASHI. Hyperbolic Complex Spaces. Grundlehren der math. Wissenschaften 315. Springer, Berlin (1998).
[13] S. LANG. Introduction to Complex Hyperbolic Spaces. Springer-Verlag, New York-Berlin-Heidelberg, 1987.
[14] LINS NETO. Simultaneous uniformization for the leaves of projective foliations by curves. Bol. Soc. Brasil. Mat. (N.S.) 25(1994), no. 2, 181-206.
[15] R. NEVANLINNA. Zur Theorie der meromorphen Funktionen. Acta Mathematica, 46(1925), 1-99.
[16] J. NOGUCHI AND T. OCHIAI. Geometric function theory in several complex variables. Translated from the Japanese by Noguchi. Translations of Mathematical Monographs, 80. American Mathematical Society, Providence, RI, 1990.
[17] J. NOGUCHI AND J. WINKERLMANN. Nevanlinna theory in several complex variables and Diophantine approximation. Springer-Verlag, New York-Berlin-Heidelberg, 2014.
[18] J. NOGUCHI, J. WINKERLMANN AND K. YAMANOI. The second main theorem for holomorphic curves into semi-abelian varieties. Acta Math. 188 no. 1(2002), 129-161.
[19] M. PAUN AND N. SIBONY. Nevanlinna Theory for parabolic Riemann surfaces. arXiv: 1403.6596V5.
[20] MIN RU. Nevanlinna Theory and Diophantine approximation. World Scientific, 2001.
[21] Y.-T. SIU. Recent techniques in hyperbolicity problems. Several complex variables (Berkeley, CA, 1995–1996), 429–508, Math. Sci. Res. Inst. Publ., 37, Cambridge Univ. Press, Cambridge, 1999.
[22] Y. -T. SIU AND S.K. YEUNG. A generalized Bloch’s theorem and the hyperbolicity of the complement of an ample divisor in an abelian variety. Math. Ann., 306(1996), 743-758.
[23] Y.-T. SIU AND S.K. YEUNG. Defects for ample divisors of abelian varieties, Schwarz lemma, and hyperbolic hypersurfaces of low degree. Amer. J. of Math., 119(1997), 1139-1172.
[24] W. STOLL AND P.M. WONG. Second main theorem of Nevanlinna theory for nonequidimensional meromorphic maps. Amer. J. Math., 116(1994), 1031-1071.
[25] B.V. SHABAT. Distribution of values of holomorphic mappings. Translated from the Russian by J. R. King. Translation edited by Lev J. Leifman. Translations of Mathematical Monographs, 61. American Mathematical Society, Providence, RI, 1985. v+225 pp.
[26] M. TSUJI. Potential theory in modern function theory. Maruzen Co.,Tokyo, 1959.
[27] K. UENO. Classification theory of algebraic varieties and compact complex spaces. Notes written in collaboration with P. Cherenack; Lecture Notes in Mathematics, Vol. 439. Springer-Verlag, Berlin-New York, 1975.
[28] P.VOJTA. Diophantine approximations and value distribution theory. Lecture Notes in Math. 1239, Springer-Verlag, 1987.
[29] K. YAMANOI. Holomorphic curves in abelian varieties and intersections with higher codimensional subvarieties. Forum Math., 16(2004), 749-788.
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