A NEW TYPE OF UNIQUE RANGE SET WITH DEFICIENT VALUES

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Abstract. In the paper, we introduce a new type of unique range set for meromorphic function having deficient values which will improve all the previous result in this aspect.

1. Introduction Definitions and Results

In this paper by meromorphic functions we will always mean meromorphic functions in the complex plane. It will be convenient to let $E$ denote any set of positive real numbers of finite linear measure, not necessarily the same at each occurrence. For any non-constant meromorphic function $h(z)$ we denote by $S(r, h)$ any quantity satisfying

$$S(r, h) = o(T(r, h)) \quad (r \to \infty, r \not\in E).$$

We denote by $T(r)$ the maximum of $T(r, f)$ and $T(r, g)$. The notation $S(r)$ denotes any quantity satisfying $S(r) = o(T(r))$ as $r \to \infty, r \not\in E$. Throughout this paper, we denote

$$\Theta(a, f) = 1 - \limsup_{r \to \infty} \frac{N(r, a; f)}{T(r, f)},$$

where $a$ is a value in the extended complex plane.

We adopt the standard notations of the Nevanlinna theory of meromorphic functions as explained in [12].

Let $f$ and $g$ be two non-constant meromorphic functions and let $a$ be a finite complex number. We say that $f$ and $g$ share the value $a$ CM (counting multiplicities), provided that $f - a$ and $g - a$ have the same zeros with the same multiplicities. Similarly, we say that $f$ and $g$ share the value $a$-IM (ignoring multiplicities), provided that $f - a$ and $g - a$ have the same set of zeros, where the multiplicities are not taken into account. In addition we say that $f$ and $g$ share $\infty$ CM (IM), if $1/f$ and $1/g$ share $0$ CM (IM).

Let $S$ be a set of distinct elements of $\mathbb{C} \cup \{\infty\}$ and $E_f(S) = \bigcup_{a \in S} \{z : f(z) = a\}$, where each zero is counted according to its multiplicity. If we do not count the multiplicity, then the set $\bigcup_{a \in S} \{z : f(z) = a\}$ is denoted by $E_f'(S)$. If $E_f(S) = E_g(S)$ we say that $f$ and $g$ share the set $S$ CM. On the other hand, if $E_f'(S) = E_g'(S)$, we say that $f$ and $g$ share the set $S$ IM. Evidently, if $S$ contains only one element, then it coincides with the usual definition of CM (respectively, IM) sharing of values.

In continuation with the famous question proposed in [10], in 1982, F.Gross and C.C.Yang [11] first introduced the novel idea of unique range set for entire function. The analogous definition for meromorphic functions can also be given in similar fashion. Below we are recalling the same.

Let a set $S \subset \mathbb{C}$ and $f$ and $g$ be two non-constant meromorphic (entire) functions. If $E_f(S) = E_g(S)$ implies $f \equiv g$ then $S$ is called a unique range set for meromorphic (entire) functions or in brief URSM (URSE).

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In 1997, Yi [23] introduced the analogous definition for reduced unique range sets. We shall
call any set $S \subset \mathbb{C}$ a unique range set for meromorphic (entire) functions ignoring multiplicity
(URSM-IM) (URSE-IM) or a reduced unique range set for meromorphic (entire) functions
(RURSM) (RURSE) if $\sum f(S) = \sum g(S)$ implies $f \equiv g$ for any pair of non-constant meromorphic
(entire) functions.

During the last few years the notion of unique as well as reduced unique range sets have been
generating an increasing interest among the researchers and naturally a new area of research
have been developed under the aegis of uniqueness theory. The prime concern of the researchers
is to find new unique range sets or to make the cardinalities of the existing range sets as small as
possible imposing sum restrictions on the deficiencies of the generating meromorphic functions.
To see the remarkable progress in this regard readers can make a glance to [1], [4]-[6], [9],
[10]-[17], [20], [21]-[22].

In 1994, H.X.Yi [21] exhibited a URSE with 15 elements and in 1995 P.Li and C.C.Yang [20]
exhibited a URS with 15 elements and a URSE with 7 elements. Till date the URSM with
11 elements and R-URSM with 17 elements are the smallest available URSM and R-URSM
obtained by Frank-Reinders [9] and Bartels [10] respectively. This URSM by Frank of Reinders
is highlighted by a number of researchers.

In 1995, Li and Yang [20] first elucidated the fact that the finite URSM’s are nothing
but the set of distinct zeros of some suitable polynomials and subsequently the study of the
characteristics of these underlying polynomials should also be given utmost priority.

Li and Yang [20], called a polynomial $P$ in $\mathbb{C}$, as uniqueness polynomial for meromorphic
(entire) functions, if for any two non-constant meromorphic (entire) functions $f$ and $g$, $P(f) \equiv
P(g)$ implies $f \equiv g$. We say $P$ is a UPM (UPE) in brief.

On the other hand, T. T. H. An, J. T. Wang and P. Wong [2] called a polynomial $P$ in
$\mathbb{C}$ as strong uniqueness polynomial for meromorphic (entire) functions if for any non-constant
meromorphic (entire) functions $f$ and $g$, $P(f) \equiv cP(g)$ implies $f \equiv g$, where $c$ is a suitable
nonzero constant. In this case we say $P$ is SUPM (SUPE) in brief.

In 2000, H. Fujimoto [7] first discovered a special property of a polynomial, which was
recently termed as critical injection property in [3]. Critical injection property of a polynomial
may be stated as follows : A polynomial $P$ is said to satisfy critical injection property if
$P(\alpha) \neq P(\beta)$ for any two distinct zeros $\alpha, \beta$ of the derivative $P'$.

Clearly the inner meaning of critical injection property is that the polynomial $P$ is injective
on the set of distinct zeros of $P'$, which are known as critical points of $P$. Naturally a polyno-
mial with this property may be called a critically injective polynomial. Let $P(z)$ be a monic
polynomial without multiple zero whose derivatives has mutually distinct k zeros given by
d_1, d_2, \ldots, d_k with multiplicities q_1, q_2, \ldots, q_k respectively. The following theorem of Fujimoto
helps us to find many uniqueness polynomials.

**Theorem A.** [3] Suppose that $P(z)$ is critically injective. Then $P(z)$ will be a uniqueness
polynomial if and only if

$$\sum_{1 \leq i < m \leq k} q_i q_m > \sum_{l=1}^{k} q_l.$$ 

In particular the above inequality is always satisfied whenever $k \geq 4$. When $k = 3$ and
$\max\{q_1, q_2, q_3\} \geq 2$ or when $k = 2$, $\min\{q_1, q_2\} \geq 2$ and $q_1 + q_2 \geq 5$ then also the above
inequality holds.

For $k = 1$, taking $P(z) = (z - a)^q - b$ for some constants $a$ and $b$ with $b \neq 0$ and an integer
$q \geq 2$, it is easy to verify that for an arbitrary non-constant meromorphic function $g$ and a
constant $c(\neq 1)$ with $c^q = 1$, the function $g := cf + (1-c)a(\neq f)$ satisfies the condition
$P(f) = P(g)$. 

A recent development in the uniqueness theory has been to consider the notion of weighted sharing of values and sets \([13, 15]\) which is a scaling between CM sharing and IM sharing and measures a gradual increment from IM sharing to CM sharing.

Let \(k\) be a non-negative integer or infinity. For \(a \in \mathbb{C} \cup \{\infty\}\) we denote by \(E_k(a; f)\) the set of all \(a\)-points of \(f\), where an \(a\)-point of multiplicity \(m\) is counted \(m\) times if \(m \leq k\) and \(k + 1\) times if \(m > k\).

If for two meromorphic functions \(f\) and \(g\) we have \(E_k(a; f) = E_k(a; g)\), then we say that \(f\) and \(g\) share the value \(a\) with weight \(k\).

The IM and CM sharing respectively correspond to weight 0 and \(\infty\).

Following theorem is the main result of the paper.

**Theorem 1.1.** Let \(n(\geq 3), m(\geq 3)\) be two positive integers. Suppose that \(S = \{z : P(z) = 0\}\). Let \(f\) and \(g\) be two non-constant meromorphic functions such that \(E_f(S, l) = E_g(S, l)\). Now if one of the following conditions holds:

(a) \(l \geq 2\) and \(\Theta_f + \Theta_g > (9 - (n + m))\);  
(b) \(l = 1\) and \(\Theta_f + \Theta_g > (10 - (n + m))\);  
(c) \(l = 0\) and \(\Theta_f + \Theta_g > (15 - (n + m))\);  

then \(f \equiv g\), where \(\Theta_f = 2\Theta(0; f) + 2\Theta(\infty; f) + \Theta(1; f) + \frac{1}{2}\min\{\Theta(1; f), \Theta(1; g)\}\) and \(\Theta_g\) is similarly defined.

We now explain some definitions and notations which are used in the paper.

**Definition 1.1.** \([13]\) For \(a \in \mathbb{C} \cup \{\infty\}\) we denote by \(N(r, a; f \mid = 1)\) the counting function of simple \(a\)-points of \(f\). For a positive integer \(m\) we denote by \(N(r, a; f \mid \leq m)\) \((N(r, a; f \mid \geq m)\) by the counting function of those \(a\)-points of \(f\) whose multiplicities are not greater\(\leq\) than \(m\) where each \(a\)-point is counted according to its multiplicity. \(\overline{N}(r, a; f \mid \leq m)\) and \(\overline{N}(r, a; f \mid \geq m)\) are the reduced counting function of \(N(r, a; f \mid \leq m)\) and \(N(r, a; f \mid \geq m)\) respectively.

Also \(N(r, a; f \mid < m), N(r, a; f \mid > m), \overline{N}(r, a; f \mid < m)\) and \(\overline{N}(r, a; f \mid > m)\) are defined analogously.

**Definition 1.2.** \([24]\) Let \(f\) and \(g\) be two non-constant meromorphic functions such that \(f\) and \(g\) share \((a, 0)\). Let \(z_0\) be an \(a\)-point of \(f\) with multiplicity \(p\), an \(a\)-point of \(g\) with multiplicity \(q\). We denote by \(\overline{N}_E(r, a; f)\) the reduced counting function of those \(a\)-points of \(f\) and \(g\) where \(p > q\), by \(\overline{N}_E^1(r, a; f)\) the counting function of those \(a\)-points of \(f\) and \(g\) where \(p = q = 1\), by \(\overline{N}_E^2(r, a; f)\) the reduced counting function of those \(a\)-points of \(f\) and \(g\) where \(p = q \geq 2\). In
the same way we can define \( N_L(r, a; g) \), \( N^1_E(r, a; g) \), \( N^2_E(r, a; g) \). In a similar manner we can define \( N_L(r, a; f) \) and \( N_L(r, a; g) \) for \( a \in \mathbb{C} \cup \{ \infty \} \).

When \( f \) and \( g \) share \( (a, m) \), \( m \geq 1 \) then \( N^1_E(r, a; f) = N(r, a; f \mid 1) \).

**Definition 1.3.** We denote by \( \mathcal{N}(r, a; f \mid 1) \) the reduced counting function of those \( a \)-points of \( f \) whose multiplicities is exactly \( k \), where \( k \geq 2 \) is an integer.

**Definition 1.4.** \([14, 15]\) Let \( f, g \) share a value a IM. We denote by \( \mathcal{N}_{*}(r, a; f, g) \) the reduced counting function of those \( a \)-points of \( f \) whose multiplicities differ from the multiplicities of the corresponding \( a \)-points of \( g \).

Clearly \( \mathcal{N}_{*}(r, a; f, g) = \mathcal{N}_{*}(r, a; g, f) \) and \( \mathcal{N}_{*}(r, a; f, g) = \mathcal{N}_{L}(r, a; f) + \mathcal{N}_{L}(r, a; g) \)

2. **LEMMAS**

In this section we present some lemmas which will be needed in the sequel. Let, unless otherwise stated \( F \) and \( G \) be two non-constant meromorphic functions given by \( F = P(f) \) and \( G = P(g) \). Henceforth we shall denote by \( H \) the following function

\[
H = \left( \frac{F''}{F'} - \frac{2F'}{F} \right) - \left( \frac{G''}{G'} - \frac{2G'}{G} \right).
\]

**Lemma 2.1.** \([19]\) Let \( f \) be a non-constant meromorphic function and let

\[
R(f) = \frac{\sum_{k=0}^{n} a_k f^k}{\sum_{j=0}^{m} b_j f^j}
\]

be an irreducible rational function in \( f \) with constant coefficients \( \{a_k\} \) and \( \{b_j\} \) where \( a_n \neq 0 \) and \( b_m \neq 0 \) Then

\[
T(r, R(f)) = dT(r, f) + S(r, f),
\]

where \( d = \max\{n, m\} \).

**Lemma 2.2.** If \( F, G \) are two non-constant meromorphic functions such that they share \( (0, 0) \) and \( H \neq 0 \) then

\[
N^1_{E}(r, 0; F \mid 1) = N^1_{E}(r, 0; G \mid 1) \leq N(r, H) + S(r, f) + S(r, g).
\]

**Proof.** By the Lemma of Logarithmic derivative we obtain

\[
m(r, H) = S(r, f) + S(r, g) \quad (:= S(r)).
\]

By Laurent expansion of \( H \) we can easily verify that each simple zero of \( F \) (and so of \( G \)) is a zero of \( H \). Hence

\[
N^1_{E}(r, 0; F \mid 1) = N^1_{E}(r, 0; G \mid 1)
\]

\[
\leq N(r, 0; H)
\]

\[
\leq T(r, H) + O(1)
\]

\[
= N(r, \infty; H) + S(r, f) + S(r, g).
\]

\[\square\]

**Lemma 2.3.** Let \( S \) be the set of zeros of \( P \). If for two non-constant meromorphic functions \( f \) and \( g \), \( E_f(S, 0) = E_g(S, 0) \) and \( H \neq 0 \) then

\[
\mathcal{N}(r, \infty; H) \leq \mathcal{N}(r, 0; f) + \mathcal{N}(r, 1; f) + \mathcal{N}(r, 0; g) + \mathcal{N}(r, 1; g) + \mathcal{N}_{*}(r, 0; F, G)
\]

\[
+ \mathcal{N}(r, \infty; f) + \mathcal{N}(r, \infty; g) + \mathcal{N}_{0}(r, 0; f') + \mathcal{N}_{0}(r, 0; g'),
\]

\[\square\]
where by $N_0(r, 0; f')$ we mean the reduced counting function of those zeros of $f'$ which are not the zeros of $Ff(f−1)$ and $N_0(r, 0; g')$ is similarly defined.

**Proof.** Since $E_f(S, 0) = E_g(S, 0)$ it follows that $F$ and $G$ share $(0, 0)$. Also we observe that $F' = f^n(f−1)^m f'$. It can be easily verified that possible poles of $H$ occur at (i) poles of $f$ and $g$, (ii) those 0-points of $F$ and $G$ with different multiplicities, (iii) zeros of $f'$ which are not the zeros of $Ff(f−1)$, (iv) zeros of $g'$ which are not zeros of $Gg(g−1)$, (v) 0 and 1 points of $f$ and $g$. Since $H$ has only simple poles, the lemma follows from above. This proves the lemma. □

**Lemma 2.4.** $Q(1)$ is not an integer. In particular, $P(1) \neq -1$, where $n \geq 3$, $m \geq 3$ are integers.

**Proof.** We claim that

$$S_n(m) = \sum_{i=0}^{m} \binom{m}{i} \frac{(-1)^i}{n + m + 1 - i} = \frac{\binom{m}{0}}{n + m + 1} - \frac{\binom{m}{1}}{n + m + 1 - 1} + \ldots + (-1)^m \frac{\binom{m}{m}}{n + 1} = \frac{(-1)^m m!}{(n + m + 1)(n + m) \ldots (n + 1)}.$$

We prove the claim by method of induction on $m$.

At first for $m = 3$ we get

$$S_n(3) = \frac{1}{n + 4} - \frac{3}{n + 3} + \frac{3}{n + 2} - \frac{1}{n + 1} = \frac{(−1)^3 \cdot 3!}{(n + 4)(n + 3)(n + 2)(n + 1)}.$$

So, $S_n(m)$ is true for $m = 3$. Now we assume that $S_n(m)$ is true for $m = k$, where $k$ is any given positive integer such that $k \geq 3$. Now we will show that $S_n(m)$ is true for $m = k + 1$. i.e.,

$$\frac{\binom{k+1}{0}}{n + k + 2} - \frac{\binom{k+1}{1}}{n + k + 1} + \ldots + (-1)^{k+1} \frac{\binom{k+1}{k+1}}{n + 1} = \frac{(-1)^{(k+1)}(k + 1)!}{(n + k + 2)(n + k + 1) \ldots (n + 1)}.$$
Using induction hypothesis we have
\[
S_n(k + 1) = \frac{1}{n + k + 2} - \frac{k + 1}{n + k + 1} + \frac{(k + 1)k}{2(n + k)} - \ldots + \frac{(-1)^{k+1}}{n + 1}
\]
\[
= \left[ \frac{1}{n + k + 2} - \frac{k}{n + k + 1} + \frac{k(k - 1)}{2(n + k)} - \ldots + \frac{(-1)^k}{n + 2} \right]
\]
\[
- \left[ \frac{1}{n + k + 1} - \frac{2k}{2(n + k)} + \frac{3k(k - 1)}{2(3(n + k - 1))} - \ldots + \frac{(-1)^k}{n + 1} \right]
\]
\[
= \left[ \frac{(k)}{(n + 1) + k + 1} - \frac{(k)}{(n + 1) + k} + \frac{(k)}{(n + 1) + k - 1} - \ldots + \frac{(-1)^k}{n + 1} \right]
\]
\[
= S_{n+1}(k) - S_n(k)
\]
\[
= \frac{(-1)^{k+1}}{n + k + 2} - \frac{(-1)^k}{n + k + 1}(n + 2) - \ldots - \frac{(-1)^k}{(n + k + 1)(n + k + 1)}.
\]

So our claim has been established. We note that \(S_n(m) = (-1)^m \prod_{i=1}^{m} \frac{i}{(n + m + 1)}\) and hence it can not be an integer. In particular we have proved that \(Q(1) \neq -2\) i.e., \(P(1) \neq -1\). □

**Lemma 2.5.** [15] If \(N(r, 0; f^{(k)} | f \neq 0)\) denotes the counting function of those zeros of \(f^{(k)}\) which are not the zeros of \(f\), where a zero of \(f^{(k)}\) is counted according to its multiplicity then
\[
N(r, 0; f^{(k)} | f \neq 0) \leq kN(r, \infty; f) + N(r, 0; f < k) + kN(r, 0; f \geq k) + S(r, f).
\]

### 3. Proof of the theorem

**Proof of Theorem [7]**. First we observe that since \(P(0) = 1 \neq P(1) = Q(1) + 1, P(z)\) is critically injective polynomial. Also \(P(z) - 1\) and \(P(z) - P(1)\) have a zero of multiplicity \(n + 1\) and \(m + 1\) respectively at 0 and 1, it follows that the zeros of \(P(z)\) are simple. Let the zeros be given by \(\alpha_j, j = 1, 2, \ldots, n + m + 1\). Since \(E_f(S, l) = E_g(S, l)\) it follows that \(F, G\) share \((0, l)\).

**Case 1.** If possible let us suppose that \(H \neq 0\).

**Subcase 1.1.** \(l \geq 1\). While \(l \geq 2\), using **Lemma 2.3** we note that
\[
(3.1) \quad \overline{N}_0(r, 0; g') + \overline{N}(r, 0; G \geq 2) + \overline{N}_s(r, 0; F, G) 
\leq \overline{N}_0(r, 0; g') + \overline{N}(r, 0; G \geq 2) + \overline{N}(r, 0; G \geq 3) 
\leq N(r, 0; g' | g \neq 0) + S(r, g) 
\leq \overline{N}(r, 0; g) + \overline{N}(r, \infty; g) + S(r, g).
\]
Similarly we can obtain Lemma 2.2 and 2.3 we get from second fundamental theorem for \( \varepsilon > 0 \) that

\[
(3.2) \quad (n + m + 2)T(r, f) \\
\leq \ \mathcal{N}(r, \infty; f) + \mathcal{N}(r, 0; f) + \mathcal{N}(r, 1; f) + N(r, 0; F \geq 1) + \mathcal{N}(r, 0; F \geq 2) \\
- \mathcal{N}_0(r, 0; f') + S(r, f) \\
\leq \ 2\mathcal{N}(r, 0; f) + 2\mathcal{N}(r, 1; f) + \mathcal{N}(r, 0; g) + \mathcal{N}(r, 1; g) + 2\mathcal{N}(r, \infty; f) + \mathcal{N}(r, \infty; g) \\
+ \mathcal{N}_0(r, 0; g') + \mathcal{N}(r, 0; G \geq 2) + \mathcal{N}_*(r, 0; F, G) + S(r) \\
\leq \ 2\{\mathcal{N}(r, 0; f) + \mathcal{N}(r, 1; f) + \mathcal{N}(r, \infty; f) + \mathcal{N}(r, 0; g) + \mathcal{N}(r, \infty; g)\} + \mathcal{N}(r, 1; g) + S(r). \\
\leq \ (11 - 2\Theta(0; f) - 2\Theta(\infty; f) - 2\Theta(1; f) - 2\Theta(0; g) - 2\Theta(\infty; g) - \Theta(1; g) + \varepsilon)T(r) + S(r).
\]

In a similar way we can obtain

\[
(3.3) \quad (n + m + 2)T(r, g) \leq \ (11 - 2\Theta(0; f) - 2\Theta(\infty; f) - \Theta(1; f) - 2\Theta(0; g) \\
- 2\Theta(\infty; g) - \Theta(1; g) + \varepsilon)T(r) + S(r).
\]

Combining (3.2) and (3.3) we see that

\[
(3.4) \quad (n + m - 9 + 2\Theta(0; f) + 2\Theta(\infty; f) + \Theta(1; f) + 2\Theta(0; g) \\
+ 2\Theta(\infty; g) + \Theta(1; g) + \min\{\Theta(1; f), \Theta(1; g)\} - \varepsilon)T(r) \leq S(r).
\]

Since \( \varepsilon > 0 \) is arbitrary (3.4) leads to a contradiction.

While \( l = 1 \), using Lemma 2.4 (5.1) can be changed to

\[
(3.5) \quad \mathcal{N}_0(r, 0; g') + \mathcal{N}(r, 0; G \geq 2) + \mathcal{N}(r, 0; F, G) \\
\leq \ \mathcal{N}_0(r, 0; g') + \mathcal{N}(r, 0; G \geq 2) + \mathcal{N}_*(r, 0; G) + \mathcal{N}(r, 0; F \geq 3) \\
\leq \ N(r, 0; g' \neq 0) + \sum_{j=1}^{n+m+1} \mathcal{N}(r, \alpha_j; f \geq 3) \\
\leq \ \mathcal{N}(r, 0; g) + \mathcal{N}(r, \infty; g) + \frac{1}{2} \sum_{j=1}^{n+m+1} \{N(r, \alpha_j; f) - \mathcal{N}(r, \alpha_j; f)\} + S(r, g) \\
\leq \ \mathcal{N}(r, 0; g) + \mathcal{N}(r, \infty; g) + \frac{1}{2} \mathcal{N}(r, 0; f' \neq 0) + S(r, g) \\
\leq \ \mathcal{N}(r, 0; g) + \mathcal{N}(r, \infty; g) + \frac{1}{2} \{\mathcal{N}(r, 0; f) + \mathcal{N}(r, \infty; f)\} + S(r, f) + S(r, g),
\]

So using (3.5), Lemmas 2.2 and 2.3 and proceeding as in (3.2) we get from second fundamental theorem for \( \varepsilon > 0 \) that

\[
(3.6) \quad (n + m + 2)T(r, f) \leq \ (12 - 2\Theta(0; f) - 2\Theta(\infty; f) - 2\Theta(1; f) \\
- 2\Theta(0; g) - 2\Theta(\infty; g) - \Theta(1; g) + \varepsilon)T(r) + S(r).
\]

Similarly we can obtain

\[
(3.7) \quad (n + m + 2)T(r, g) \leq \ (12 - 2\Theta(0; f) - 2\Theta(\infty; f) - \Theta(1; f) - 2\Theta(0; g) \\
- 2\Theta(\infty; g) - \Theta(1; g) + \varepsilon)T(r) + S(r).
\]

Combining (3.6) and (3.7) we see that

\[
(3.8) \quad (n + m - 10 + 2\Theta(0; f) + 2\Theta(\infty; f) + \Theta(1; f) + 2\Theta(0; g) \\
+ 2\Theta(\infty; g) + \Theta(1; g) + \min\{\Theta(1; f), \Theta(1; g)\} - \varepsilon)T(r) \leq S(r).
\]
Clearly, (3.8) leads to a contradiction for \( \varepsilon > 0 \).  

**Subcase 1.2.** \( l = 0 \). Using Lemma 2.7 we note that  
\begin{align*}
(3.9) & \quad \mathcal{N}_0(r, 0; g') + \mathcal{N}_0^2(r, 0; F) + 2\mathcal{N}_L(r, 0; G) + 2\mathcal{N}_L(r, 0; F) \\
\leq & \quad \mathcal{N}_0(r, 0; g') + \mathcal{N}_0^2(r, 0; G) + \mathcal{N}_L(r, 0; G) + 2\mathcal{N}_L(r, 0; F) \\
\leq & \quad N(r, 0; g') + N(r, 0; G) | g \geq 2 \) + 2\mathcal{N}_L(r, 0; F) \\
\leq & \quad N(r, 0; g') + \mathcal{N}(r, 0; g) + \mathcal{N}(r, 0; g) + \mathcal{N}(r, 0; g) \\
+ & \quad 2\mathcal{N}(r, 0; f) + 2\mathcal{N}(r, 0; f) + S(r, f) + S(r, g).
\end{align*}

Hence using (3.9), Lemmas 2.1, 2.2 and 2.3 we get from second fundamental theorem for \( \varepsilon > 0 \) that  
\begin{align*}
(3.10) & \quad (n + m + 2)T(r, f) \\
\leq & \quad \mathcal{N}(r, \infty; f) + \mathcal{N}(r, 0; f) + \mathcal{N}(r, 1; f) + N_1^0(r, 0; F) + \mathcal{N}_L(r, 0; F) + \mathcal{N}_L(r, 0; G) \\
+ & \quad N^2_0(r, 0; F) - N_0(r, 0; f') + S(r, f) \\
\leq & \quad 2\{N(r, 0; f) + \mathcal{N}(r, 1; f)\} + \mathcal{N}(r, 0; g) + \mathcal{N}(r, 1; g) + 2\mathcal{N}(r, \infty; f) + \mathcal{N}(r, \infty; g) \\
+ & \quad N^2_0(r, 0; F) + 2\mathcal{N}_L(r, 0; G) + 2\mathcal{N}_L(r, 0; F) + N_0(r, 0; g') + S(r, f) + S(r, g) \\
\leq & \quad (17 - 2\Theta(0; f) - 2\Theta(\infty; f) - 2\Theta(1; f) - 2\Theta(0; g) - 2\Theta(\infty; g) \\
& \quad - \Theta(1; g) + \varepsilon)T(r) + S(r).
\end{align*}

In a similar manner we can obtain  
\begin{align*}
(3.11) & \quad (n + m + 2)T(r, g) \leq (17 - 2\Theta(0; f) - 2\Theta(\infty; f) - \Theta(1; f) - 2\Theta(0; g) \\
& \quad - 2\Theta(\infty; g) - 2\Theta(1; g) + \varepsilon)T(r) + S(r).
\end{align*}

Combining (3.10) and (3.11) we see that  
\begin{align*}
(3.12) & \quad (n + m - 15 + 2\Theta(0; f) + 2\Theta(\infty; f) + \Theta(1; f) + 2\Theta(0; g) \\
& \quad + 2\Theta(\infty; g) + \Theta(1; g) + \min\{\Theta(1; f), \Theta(1; g)\} - \varepsilon)T(r) \leq S(r).
\end{align*}

Since \( \varepsilon > 0 \), be arbitrary (3.12) leads to a contradiction.  

**Case 2.** \( H \equiv 0 \). On integration we get from (2.1)  
\begin{align*}
(3.13) & \quad \frac{1}{F} \equiv A + B,
\end{align*}

where \( A, B \) are constants and \( A \neq 0 \). From Lemma 2.7 we get  
\begin{align*}
(3.14) & \quad T(r, f) = T(r, g) + S(r, g).
\end{align*}

**Subcase 2.1.** First suppose that \( B \neq 0 \). From (3.13) we have  
\begin{align*}
\overline{N}(r, \infty; f) = \overline{N}(r, -\frac{A}{B}; G).
\end{align*}

**Subcase 2.1.1.** Let \( \frac{A}{B} \neq 1 \). 
If \( \frac{A}{B} \neq Q(1) + 1 \), then in view of (3.14), from the second fundamental theorem we get  
\begin{align*}
(3.15) & \quad (n + 2m + 1)T(r, g) \\
\leq & \quad \mathcal{N}(r, \infty; g) + \mathcal{N}(r, 1; G) + \mathcal{N}(r, -\frac{A}{B}; G) + S(r, g) \\
= & \quad \mathcal{N}(r, \infty; g) + \mathcal{N}(r, 0; g) + mT(r, g) + \mathcal{N}(r, \infty; f) + S(r, g) \\
\leq & \quad (m + 2)T(r, g) + \mathcal{N}(r, \infty; f) + S(r, g) \\
\leq & \quad (m + 3)T(r, g) + S(r, g),
\end{align*}

where \( m = 1 \).
which is a contradiction for $n \geq 3$ and $m \geq 3$.

Next Suppose $\frac{A}{B} = Q(1) + 1$, from (3.14) we have

\begin{equation}
\frac{G}{BF} = G - P(1) = G + \frac{A}{B} = (g - 1)^{m+1}(g - \alpha_1')(g - \alpha_2') \cdots (g - \alpha_n'),
\end{equation}

where $\alpha_i', i = 1, 2, \ldots, n$ are the distinct simple zeros of $P(z) + \frac{A}{B}$. As $B \neq 0$, $f$ and $g$ do not have any common pole. Let $z_0$ be a zero of $g - 1$ of multiplicity $p$ (say) then it must be a pole of $f$ with multiplicity $q \geq 1$ (say). So from (3.15) we have

\[(m + 1)p = (n + m + 1)q \geq m + n + 1.\]

i.e.,

\[p \geq \frac{n + m + 1}{m + 1} > 1.\]

Next suppose $z_i$ be a zero of $g - \alpha_i'$ of multiplicity $p_i$, then in view of (3.15), we have $z_i$ be a pole of $f$ of multiplicity $q_i$, (say) such that

\[p_i = (n + m + 1)q_i \geq n + m + 1.\]

Let $\beta_j, j = 1, 2, \ldots, m$ be the distinct simple zeros of $P(z) - 1$. Now from the second fundamental theorem we get

\[T(r, g) \leq N(r, \infty; g) + N(r, 1; G) + N\left(r, \frac{A}{B}; G\right) + S(r, g)\]

\[\leq N(r, \infty; g) + N(r, 0; g) + \sum_{j=1}^{m} N(r, \beta_j; g) + \sum_{i=1}^{n} N(r, \alpha_i'; g) + S(r, g)\]

\[\leq \left(m + 2 + \frac{1}{2} + \frac{n}{n + m + 1}\right) T(r, g) + S(r, g),\]

which is a contradiction for $n \geq 3, m \geq 3$.

**Subcase 2.1.2.** Next let $\frac{A}{B} = 1$. From (3.13) we have

\[\frac{1}{F} = \frac{B(G - 1)}{G}.\]

Therefore in view of (3.13), second fundamental theorem yields

\[T(r, g) + S(r, g) \geq N(r, \infty; f) = N(r, 1; G) = N(r, 0; g) + \sum_{j=1}^{m} N(r, \beta_j; g)\]

\[\geq (m - 1)T(r, g) + S(r, g),\]

a contradiction as $m \geq 3$.

**Subcase 2.2.** $B = 0$. From (3.13) we get

\[(3.16) \quad AF \equiv G.\]

**Subcase 2.2.1.** Suppose $A \neq 1$.

**Subcase 2.2.1.1.** Let $A = P(1)$, then from (3.10) we have

\[P(1) \left(F - \frac{1}{P(1)}\right) \equiv G - 1.\]

As $P(1) \neq 1$ and Lemma 2.4 implies $P(1) \neq -1$, we have $\frac{1}{P(1)} \neq P(1)$, it follows that $P(z) - \frac{1}{P(z)}$ has simple zeros. Let they be given by $\gamma_i, i = 1, 2, \ldots, n+m+1$. So from the second fundamental
Theorem and (3.13) we get
\[(n + m - 1)T(r, f) \leq \sum_{i=1}^{n+m+1} N(r, \gamma_i; f) + S(r, f)\]
\[\leq N(r, 0; g) + \sum_{j=1}^{m} N(r, \beta_j; f) \leq (m + 1)T(r, f) + S(r, f),\]
a contradiction since \(n \geq 3\).

**Subcase 2.2.1.2.** Let \(A \neq P(1)\).

Then we have from (3.16)
\[A(F - 1) \equiv G - A.\]

Let the distinct zeros of \(P(z) - A\) be given by \(\delta_i, i = 1, 2, \ldots, n + m + 1\). So from the second fundamental theorem and (3.14) we get
\[(n + m - 1)T(r, g) \leq \sum_{i=1}^{n+m+1} N(r, \delta_i; g) + S(r, g)\]
\[= \sum_{j=1}^{m} N(r, \beta_j; f) + N(r, 0; f) + S(r, f)\]
\[\leq (m + 1)T(r, g) + S(r, g),\]
a contradiction since \(n \geq 3\).

**Subcase 2.2.2.** Suppose \(A = 1\). Then from (3.16) we have \(F \equiv G\), i.e., \(P(f) \equiv P(g)\). Here \(k = 2, d_1 = 0, d_2 = 1, q_1 = n, q_2 = m\). Since \(\min\{q_1, q_2\} = \min\{n, m\} \geq 2\) and \(n + m \geq 5\) we see that \(nm > n + m\). So from Theorem B we conclude that \(P(z)\) is an uniqueness polynomial. Therefore \(f \equiv g\). This proves the theorem. \(\square\)

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