On the number of classes of conjugate Hall subgroups in finite simple groups

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Abstract

In this paper we find the number of classes of conjugate \( \pi \)-Hall subgroups in all finite almost simple groups. We also complete the classification of \( \pi \)-Hall subgroups in finite simple groups and correct some mistakes from our previous paper.

Key words: \( \pi \)-Hall subgroup, Hall property, subgroup of odd index, classical group, group of Lie type

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1 Introduction

Let \( \pi \) be a set of primes. We denote by \( \pi' \) the set of all primes not in \( \pi \), by \( \pi(n) \) the set of all prime divisors of a positive integer \( n \), for a finite group \( G \) we denote \( \pi(|G|) \) by \( \pi(G) \). A positive integer \( n \) with \( \pi(n) \subseteq \pi \) is called a \( \pi \)-number, a group \( G \) with \( \pi(G) \subseteq \pi \) is called a \( \pi \)-group. Given positive integer \( n \) denote by \( n_\pi \) the maximal divisor \( t \) of \( n \) with \( \pi(t) \subseteq \pi \). A subgroup \( H \) of \( G \) is called a \( \pi \)-Hall subgroup, if \( \pi(H) \subseteq \pi \) and \( \pi(|G : H|) \subseteq \pi'. \) According to [12] we say that \( G \) satisfies \( E_\pi \) (or briefly \( G \in E_\pi \)), if \( G \) possesses a \( \pi \)-Hall subgroup. If \( G \in E_\pi \) and every two \( \pi \)-Hall

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subgroups are conjugate, then we say that $G$ satisfies $C_{\pi}$ ($G \in C_{\pi}$). If $G \in C_{\pi}$ and each $\pi$-subgroup of $G$ is included in a $\pi$-Hall subgroup of $G$, then we say that $G$ satisfies $D_{\pi}$ ($G \in D_{\pi}$). Thus $G \in D_{\pi}$ means that a complete analogue of the Sylow theorem for $\pi$-Hall subgroups of $G$ holds. A group satisfying $E_{\pi}$ ($C_{\pi}$, $D_{\pi}$) is also called an $E_{\pi}$-group (respectively $C_{\pi}$-group, $D_{\pi}$-group).

In [28, Theorem 7.7] the authors proved that a finite group $G$ satisfies $D_{\pi}$ if and only if each composition factor of $G$ satisfies $D_{\pi}$. In the next series of papers [24–27] for every finite simple group $S$ and for every set $\pi$ of primes pure arithmetic necessary and sufficient condition for $S$ to satisfy $D_{\pi}$ were found.

The authors intend to write a series of papers, where arithmetic criteria for $E_{\pi}$ and $C_{\pi}$ will be obtained. The present paper is the first one in this series. Since, in contrast with $D_{\pi}$-groups, the class of $E_{\pi}$-group is not closed under extensions, while the class of $C_{\pi}$-groups is not closed under normal subgroups, the general theory for $E_{\pi}$ and $C_{\pi}$ is more complicated than the theory for $D_{\pi}$. In particular, the answer to the question, whether given group $G$ satisfies $E_{\pi}$ or $C_{\pi}$, cannot be obtained in terms of composition factors of $G$, i.e., this question cannot be reduced to similar questions about simple groups. We intend to reduce the question to similar questions for almost simple groups. Recall that a finite group $G$ is called almost simple, if its generalized Fitting subgroup $F^{*}(G)$ is a nonabelian simple group $S$, or, equivalently, if $\text{Inn}(S) \leq G \leq \text{Aut}(S)$ for a nonabelian finite simple group $S$. An important step in this direction was made by F.Gross in [7].

In this paper we prove (by using the classification of finite simple groups) an important theorem on the number of classes of conjugate $\pi$-Hall subgroups in finite simple groups. We show that this number is bounded and is a $\pi$-number. This result (in a more general form, which will be used in future research) can be found in Theorem 1.1 below. First we need to introduce some notations.

We denote by Hall$_{\pi}(G)$ the set of all $\pi$-Hall subgroups of $G$ (notice that this set can be empty). Assume that $A \subseteq G$. Then define Hall$_{\pi}^{G}(A) = \{ H \cap A \mid H \in \text{Hall}_{\pi}(G) \}$. By Lemma 2.1(a) (see below), Hall$_{\pi}^{G}(A) \subseteq \text{Hall}_{\pi}(A)$. The elements of Hall$_{\pi}^{G}(A)$ are called $G$-induced $\pi$-Hall subgroups of $A$. Clearly $A$ acts by conjugation on both Hall$_{\pi}(A)$ and Hall$_{\pi}^{G}(A)$. Denote by $k_{\pi}(A)$ and $k_{\pi}^{G}(A)$ the number of orbits under this action, respectively. Thus $k_{\pi}(A)$ is the number of classes of conjugate $\pi$-Hall subgroups of $A$, $k_{\pi}^{G}(A)$ is the number of classes of conjugate $G$-induced $\pi$-Hall subgroups of $A$, and $k_{\pi}^{G}(A) \leq k_{\pi}(A)$.

**Theorem 1.1** Let $\pi$ be a set of primes, $G$ a finite almost simple group with nonabelian simple socle $S$. Then the following statements hold:

(a) if $2 \notin \pi$, then $k_{\pi}^{G}(S) \in \{0, 1\}$;
(b) if $3 \notin \pi$, then $k_{\pi}^{G}(S) \in \{0, 1, 2\}$;
(c) if $2, 3 \in \pi$, then $k_{\pi}^{G}(S) \in \{0, 1, 2, 3, 4, 9\}$. 
In particular, $k^G_\pi(S)$ is bounded and, if $G \in E_\pi$, then $k^G_\pi(S)$ is a $\pi$-number.

Setting $G = S$ we obtain that $k^G_\pi(S) = k_\pi(S)$, so the same statement on the number of classes of conjugate $\pi$-Hall subgroups is true for every simple group $S$.

Theorem 1.1 generalizes a result by F.Gross [9, Theorem B], which states that for every set $\pi$ of odd primes every finite simple $E_\pi$-group satisfies $C_\pi$ (equivalently $k_\pi(S) = 1$). Since by Chunikhin’s theorem the class of $C_\pi$-groups is closed under extension (see Lemma 2.1(f) below), it follows that $k_\pi(G) = 1$ for every finite group $G$, see [9, Theorem A].

In contrast with the case $2 \notin \pi$, Example 1.2 shows that in case $2 \in \pi$ Theorem 1.1 cannot be generalized to arbitrary (nonsimple) groups.

Example 1.2 Assume that $X \in E_\pi$ is such that $k = k_\pi(X) > 1$ (in particular, $\pi$ is not equal to the set of all primes). Suppose $p \in \pi'$. Denote a cyclic subgroup of order $p$ of $\text{Sym}_p$ by $Y$. Consider $G = X \wr Y$ and let

$$M \cong \underbrace{X \times \cdots \times X}_{p \text{ times}}$$

be the base of the wreath product. It is clear that $k_\pi(M) = k^p$. Since $M$ is a normal subgroup of $G$ and $|G : M| = p$ is a $\pi'$-number, we have $\text{Hall}_\pi(G) = \text{Hall}_\pi(M)$. The subgroup $Y$ acts on the set of classes of conjugate $\pi$-Hall subgroups of $M$. Applying the Burnside formula to this action it is easy to show that

$$k_\pi(G) = \frac{k^p + (p - 1)k}{p}.$$ 

Now assume that $\pi = \{2, 3\}$ and $X = \text{SL}_3(2)$. Then [22, Theorem 1.2] implies that $k_\pi(X) = 2$. Since $p \in \pi'$ can be taken arbitrarily large and $(2^p - 2)/p + 2$ tends to infinity as $p$ tends to infinity, we obtain that for a nonsimple group $G$ the number $k_\pi(G)$ is not bounded in general. Moreover, if we take $p = 7$, then $k_\pi(G) = 20$, whence it is possible that $k_\pi(G)$ is not a $\pi$-number.

Although Theorem 1.1 is not true for arbitrary groups, it can be used in order to obtain important results for finite groups. As an example of this using we give a short solution to Problem 13.33 from “Kourovka notebook” [34]. Earlier this problem was solved by the authors in [23, 28] by using other arguments.

Corollary 1.3 Let $\pi$ be a set of primes, $A$ a normal subgroup of a finite $D_\pi$-group $G$. Then $A \in D_\pi$.

Theorem 1.1 follows from the classification of Hall subgroups in finite simple groups. We briefly outline the history.

Hall subgroups in finite groups close to simple were investigated by many authors.
In [12] P.Hall found solvable $\pi$-Hall subgroups in symmetric groups. Together with the famous Odd Order Theorem (see [8]) this result implies the classification of Hall subgroups of odd order in alternating groups. Later J.Thompson in [29] found non-solvable $\pi$-Hall subgroups in symmetric groups. The problem of classification of Hall subgroups of even order in alternating groups remained open for quite a long period. This classification was completed in [28]. The classification of Hall subgroups in the alternating and symmetric groups is given in Lemma 2.3 below.

The Hall subgroups of odd order in sporadic simple groups are classified by F.Gross in [8]. The classification of Hall subgroups in the sporadic groups was completed by the first author in [23].

The classification of $\pi$-Hall subgroups in groups of Lie type in characteristic $p$ with $p \in \pi$ was obtained by F.Gross (in case $2 \notin \pi$, [8] and [10]) and by the first author (in case $2 \in \pi$, [22]). The classification of $\pi$-Hall subgroups in groups of Lie type with $2, p \notin \pi$ was obtained by F.Gross in classical groups [11], and by the authors in exceptional groups [31]. The classification of $\pi$-Hall subgroups with $2 \in \pi$ and $3, p \notin \pi$ was obtained by F.Gross in linear and symplectic groups [14] and by the authors in the remaining cases [28].

In [28] the authors announced the classification of $\pi$-Hall subgroups in finite simple groups. Unfortunately, Lemma 3.14 in [28] contains a wrong statement. Namely, it states in item (a) that if $G \simeq PSL_2(q)$ and $Alt_4 \simeq H$ is a $\{2, 3\}$-Hall subgroup of $G$, then $PGL_2(q)$ does not possesses a $\{2, 3\}$-Hall subgroup $H_1$ such that $H_1 \cap G = H$, but this is not true. Due to this gap we lost several series of $\pi$-Hall subgroups in groups of Lie type with $2, 3 \in \pi$. In the present paper we correct this mistake and complete the classification of $\pi$-Hall subgroups in finite simple groups (see Lemmas 3.1, 4.3, 4.4, 6.7, 7.1–7.6). We correct also other known minor mistakes in proofs and statements from [28] and make some proofs more elementary. The description of $\pi$-Hall subgroups in the groups of Lie type over a field of characteristic $p$ in case $2, 3 \in \pi$ and $p \notin \pi$ takes a significant part of the paper. We also prove that in this case the groups of Lie type do not satisfy $D_\pi$. This fact means that all results from [28] concerning $D_\pi$ (in particular, the above-mentioned [28, Theorem 7.7]), and the results from [24–27] remain valid and their proofs need no corrections.

2 Notation and preliminary results

Our notation for finite groups agrees with that of [4]. For groups $A$ and $B$ symbols $A \times B$ and $A \circ B$ denote direct and central products, respectively. Recall that $A \circ B$ is a group possessing normal subgroups $A_1$ and $B_1$ isomorphic to $A$ and $B$ respectively such that $G = \langle A_1, B_1 \rangle$ and $[A_1, B_1] = 1$. By $A : B, A ^\prime B,$ and $A \cdot B$ we denote a split, a nonsplit, and an arbitrary extension of a group $A$ by a group $B$. For a group $G$ and a subgroup $S$ of $\text{Sym}_n$ by $G \wr S$ we always denote the permutation wreath product. If
If $A$ is a $\pi$-subgroup of $G$, then $O_\pi(G)$ denotes the $\pi$-radical of $G$, i.e., the largest normal $\pi$-subgroup of $G$, while $O^{\pi'}(G)$ denotes the subgroup of $G$ generated by all $\pi$-element.

We write $m \leq n$ if a real number $m$ is not greater than $n$, while we use notations $H \leq G$ and $H \unlhd G$ instead of “$H$ is a subgroup of $G$” and “$H$ is a normal subgroup of $G$”, respectively. If $r$ is an odd prime and $q$ is a positive integer coprime to $r$, then $e(q, r)$ denotes a multiplicative order of $q$ modulo $r$, that is a minimal natural number $m$ with $q^m \equiv 1 \pmod{r}$. If $q$ is an odd positive integer, then

$$e(q, 2) = \begin{cases} 
1 & \text{if } q \equiv 1 \pmod{4}, \\
2 & \text{if } q \equiv -1 \pmod{4}.
\end{cases}$$

For $M \subseteq G$ we set $M^G = \{Mg | g \in G\}$. For a group $G$ we denote by Aut($G$), Inn($G$), and Out($G$) the groups of all, inner, and outer automorphisms of $G$, respectively.

A finite group $G$ is called $\pi$-separable, if $G$ possesses a subnormal series such that each factor of the series is either a $\pi$- or a $\pi'$- group. It is clear that every $\pi$-separable group possesses a normal series $1 = G_0 \triangleleft G_1 \triangleleft \ldots \triangleleft G_k = G$ such that each factor of the series is either a $\pi$- or a $\pi'$- group. Clearly we may assume that $G_i$ is invariant under Aut($G$) for all $i$.

In Lemma 2.1 we collect some known facts about Hall subgroups in finite groups. Most of the results mentioned in Lemma 2.1 are known and we just give hints of their proofs.

**Lemma 2.1** Let $G$ be a finite group, $A$ a normal subgroup of $G$.

(a) For $H \in \text{Hall}_\pi(G)$, we have $H \cap A \in \text{Hall}_\pi(A)$ and $HA/A \in \text{Hall}_\pi(G/A)$, in particular $\text{Hall}^G_\pi(A) \subseteq \text{Hall}_\pi(A)$.

(b) If $M/A$ is a $\pi$-subgroup of $G/A$, then there exists a $\pi$-subgroup $H$ of $G$ with $M/A = HA/A$.

(c) If $G \in E_\pi$ (resp. $G \in D_\pi$), then $G/A \in E_\pi$ (resp. $G/A \in D_\pi$).

(d) If $A$ is $\pi$-separable and $G/A \in E_\pi$ (resp. $G/A \in C_\pi$, $G/A \in D_\pi$), then $G \in E_\pi$ (resp. $G \in C_\pi$, $G \in D_\pi$).

(e) Assume that $A \in E_\pi$ and $\pi(G/A) \subseteq \pi$. Then a $\pi$-Hall subgroup $H$ of $A$ lies in $\text{Hall}^G_\pi(A)$ if and only if $H^A = H^G$.

(f) Assume that $A$ satisfies $C_\pi$, $G/A$ satisfies $E_\pi$, and $M \leq G$ is chosen so that $A \leq M$ and $M/A \in \text{Hall}_\pi(G/A)$. Then $\emptyset \neq \text{Hall}_\pi(M) \subseteq \text{Hall}_\pi(G)$, and every $H, K \in \text{Hall}_\pi(G)$ are conjugate in $G$ if and only if $HA/A$ and $KA/A$ are conjugate in $G/A$. In particular, $G \in E_\pi$ and if $G/A \in C_\pi$, then $G \in C_\pi$ (this statement generalizes Chunikhin’s results [12, Theorems C1,C2]).

**Proof.** (a) See [12, Lemma 1].

(b) Let $H$ be a minimal subgroup of $G$ such that $HA/A = M/A$. Then $H \cap A$ is contained in the Frattini subgroup of $H$. Indeed, if there exists a maximal subgroup $L$ of $H$, not containing $H \cap A$, then clearly $LA = M$, which contradicts the minimality.
of \(H\). Thus the group \(H/(H \cap A) \cong HA/A\) is a \(\pi\)-group, while \(H \cap A\) is a nilpotent normal subgroup of \(H\). Therefore a \(\pi'\)-Hall subgroup \(K\) of \(H \cap A\) is a normal \(\pi'\)-Hall subgroup of \(H\). In view of the Schur-Zassenhaus theorem [33, Chapter IV, Satz 27] (see also [12, Theorems D6, D7]) we obtain that \(H\) possesses a \(\pi\)-Hall subgroup \(H_1\). Clearly \(H_1A = HA\), hence \(H_1 = H\) and \(H\) is a \(\pi\)-group.

(c) Follows from (a) and (b).

(d) Follows from the Schur-Zassenhaus theorem [33, Chapter IV, Satz 27] (see also [12, Theorems D6, D7]) and induction on the order of \(A\).

(e) The “only if” part is evident, since \(G = H_1A\) for a \(\pi\)-Hall subgroup \(H_1\) of \(G\), containing \(H\). Now we prove the “if” part. Since \(G\) leaves the set \(\{H^a \mid a \in A\}\) invariant, Frattini argument implies that \(G = AN_G(H)\). Now \(N_G(H)\) possesses a normal series \(1 < H < N_A(H) < N_G(H)\) and all section in this series are either \(\pi\)- or \(\pi'\)-groups, so \(N_G(H)\) is \(\pi\)-separable. Statement (d) of the lemma implies that \(N_G(H)\) is a total ordering of \(\{\pi\}\). Hence \(N_G(H)\) is a \(\pi'\)-number. Hence \(H_1\) is a \(\pi\)-Hall subgroup of \(G\).

(f) Assume that \(M/A \in \text{Hall}_\pi(G)\) and \(M\) is the complete preimage of \(M/A\) in \(G\) under the natural homomorphism \(G \to G/A\). Point (e) of the lemma implies that \(M \in E_\pi\), in particular there exists \(L \in \text{Hall}_\pi(M)\). Since \(|G : L| = |G : M| \cdot |M : L|\) is a \(\pi'\)-number, we obtain that \(L \in \text{Hall}_\pi(G)\). Assume that \(H, K \in \text{Hall}_\pi(G)\). If \(H\) and \(K\) are conjugate in \(G\), then it is clear that \(HA/A\) and \(KA/A\) are conjugate in \(G/A\). Assume that \(HA/A\) and \(KA/A\) are conjugate in \(G/A\). Then, up to conjugation in \(G\), we have that \(HA = KA\). Since \(A \in C_\pi\) we may assume that \(H \cap A = K \cap A\), i.e., \(H, K \leq N_{HA}(H \cap A)\). As in the proof of item (e) we obtain that \(N_{HA}(H \cap A) \in D_\pi\), hence \(H, K\) are conjugate.

If \(<\) is a total ordering of \(\pi(G)\), then \(G\) has a Sylow tower of complexion \(<\) provided \(G\) has a normal series \(G = G_0 > G_1 > \ldots > G_n = 1\), where \(G_{i-1}/G_i\) is isomorphic to a Sylow \(r_i\)-subgroup of \(G\) and \(r_1 < r_2 < \ldots < r_n\).

**Lemma 2.2** [12, Theorem A1] Assume that \(H_1, H_2\) are two \(\pi\)-Hall subgroups of a finite group \(G\) such that both \(H_1, H_2\) have a Sylow tower of the same complexion. Then \(H_1\) and \(H_2\) are conjugate in \(G\).

**Lemma 2.3** ([12, Theorem A4 and the note after it], [23, Theorem 4.3 and Corollary 4.4], [29]) Let \(\pi\) be a set of primes. Then the following statements hold:

(A) If \(\text{Sym}_n \in E_\pi\) and \(H\) is a \(\pi\)-Hall subgroup of \(\text{Sym}_n\), then \(n, \pi, \) and \(H\) satisfy exactly one of the following statements:

(a) \(|\pi \cap \pi(\text{Sym}_n)| \leq 1\). In this case a \(\pi\)-Hall subgroup of \(\text{Sym}_n\) is either its Sylow \(p\)-subgroup (if \(\pi \cap \pi(\text{Sym}_n) = \{p\}\)) or trivial (if \(\pi \cap \pi(\text{Sym}_n) = \emptyset\)).

(b) \(n = p \geq 7\) is a prime and \(\pi \cap \pi(\text{Sym}_p) = \pi((p-1)!))\). In this case a \(\pi\)-Hall
subgroup of \( \text{Sym}_n \) is isomorphic to \( \text{Sym}_{p-1} \).

(c) \( \pi(\text{Sym}_n) \subseteq \pi \) and \( n \geq 5 \). In this case \( \text{Sym}_n \) is a \( \pi \)-Hall subgroup of \( \text{Sym}_n \).

(d) \( \pi \cap \pi(\text{Sym}_n) = \{2, 3\} \) and \( n \in \{3, 4, 5, 7, 8\} \). In this case for a \( \pi \)-Hall subgroup \( H \) of \( \text{Sym}_n \) we have that \( H = \text{Sym}_3 \) if \( n = 3 \), \( H \cong \text{Sym}_4 \) if \( n \in \{4, 5\} \), \( H \cong \text{Sym}_3 \times \text{Sym}_4 \) if \( n = 7 \), and \( H \cong \text{Sym}_4 \wr \text{Sym}_2 \) if \( n = 8 \).

(B) Conversely, if \( n \) and \( \pi \) satisfy one of statements (a)–(d), then \( \text{Sym}_n \in E_\pi \).

(C) \( \text{Hall}^{\text{Sym}_n}(\text{Alt}_n) = \text{Hall}_\pi(\text{Alt}_n) \) and \( k_\pi(\text{Sym}_n) = k_\pi^\text{Sym}(\text{Alt}_n) = k_\pi(\text{Alt}_n) \in \{0, 1\} \).

(D) If \( 2, 3 \in \pi \) and \( \pi(n!) \not\subseteq \pi \), then both \( \text{Alt}_n \) and \( \text{Sym}_n \) do not satisfy \( D_\pi \).

**Corollary 2.4** Assume that \( \pi \) is a set of primes such that \( 2, 3 \in \pi \). Suppose that \( \text{Sym}_n \in E_\pi \) and \( H \in \text{Hall}_\pi(\text{Sym}_n) \). Then \( N_{\text{Sym}_n}(H) = H \).

**Lemma 2.5** Let \( M \) be a \( \pi \)-subgroup of \( \text{Sym}_n \), \( L \) be a finite group, and \[
L \wr M = \langle L_1 \times \ldots \times L_n \rangle : M
\]
be a permutation wreath product and \( L \cong L_i \) for \( i = 1, \ldots, n \). Assume that \( G \) is a normal subgroup of \( L \wr M \) and \( A = G \cap (L_1 \times \ldots \times L_n) \). Suppose also that \( A \cong L_1 \circ \ldots \circ L_n \) and \( G/A \cong M \). Denote by \( t \) the number of orbits of \( M \), and by \( k \) the number \( k_\pi(L) \). Then \( k_\pi(G) = k_\pi(L \wr M) = k^t \). Moreover, if \( 2, 3 \in \pi \) and \( M \) is a \( \pi \)-Hall subgroup of \( \text{Sym}_n \), then \( k_\pi(G) = k_\pi(L \wr M) = k_\pi(L \wr \text{Sym}_n) \).

**Proof.** By Lemmas 2.1(f) and 2.3(C) we may assume \( k_\pi(L) \geq 1 \), i.e., \( L \in E_\pi \). In view of Lemma 2.1(f) we may assume that \( Z(A) = 1 \), i.e., \( A = L_1 \times \ldots \times L_n \) and \( G = L \wr M \). Assume that \( H, K \in \text{Hall}_\pi(G) \). First we prove

\[ H, K \text{ are conjugate in } G \text{ if and only if } H \cap L_i, K \cap L_i \text{ are conjugate in } L_i \text{ for every } i = 1, \ldots, n. \quad (1) \]

Assume that \( H = K^g \) for some \( g \in G \). Condition \( \pi(M) \subseteq \pi \) implies \( HA = KA = G \). So we may suppose \( g \in A \) and \( g = g_1 \ldots g_n \), where \( g_i \in L_i \). Since \( [L_i, L_j] = 1 \) for \( i \neq j \), we obtain

\[ H \cap L_i = K^g \cap L_i = (K \cap L_i)^{g_i} = (K \cap L_i)^{g_i} \]

for every \( i = 1, \ldots, n \). Conversely suppose that \( H \cap L_i \) and \( K \cap L_i \) are conjugate in \( L_i \) for every \( i = 1, \ldots, n \). It follows that \( H \cap A \) and \( K \cap A \) are conjugate in \( A \). Therefore we may assume that \( H \cap A = K \cap A \), so \( H, K \cong N_G(H \cap A) \). As in the proof of item (e) of Lemma 2.1, we obtain \( N_G(H \cap A) \in D_\pi \), hence \( H, K \) are conjugate.

Let \( \Omega_1, \ldots, \Omega_t \) be the orbits of \( M \). Since, for every \( H \in \text{Hall}_\pi(G) \) we have \( G = HA \) and \( M \cong G/A \), it follows that \( \Omega_1, \ldots, \Omega_t \) are the orbits of \( H \) on \( \{L_1, \ldots, L_n\} \). Now for every \( h \in H \) and \( i = 1, \ldots, n \) the identity \( H \cap L_i^h = (H \cap L_i)^h \) holds, so for every \( j = 1, \ldots, t \) the subgroup \( \prod_{L_i \in \Omega_j} L_i \) possesses at most \( k \) classes of conjugate \( \pi \)-Hall subgroup invariant under \( G \). Thus Lemma 2.1(e) implies the inequality \( k_\pi(G) \leq k^t \).

Conversely assume that \( H \in \text{Hall}_\pi(L \wr M) \) (as we noted at the beginning of the
proof, we may assume \( G = L \triangle M \). If we take \( L_{j_i} \in \Omega_i \) and \( H_{j_i} \in \text{Hall}_\pi(L_{j_i}) \) for some \( i = 1, \ldots, t \), then for every \( g_1, g_2 \in M \) with \( L_{j_i}^{g_1} = L_{j_i}^{g_2} \) we have \((H_{j_i}^{L_{j_i}})^{g_1} = (H_{j_i}^{L_{j_i}})^{g_2}\). So for every \( i = 1, \ldots, t \) in the subgroup \( \prod_{L_i \in \Omega_i} L_i \) we can take at least \( k \) classes of conjugate \( \pi \)-Hall subgroups invariant under \( L \triangle M \). Hence \( A \) possesses at least \( k^t \) classes of conjugate \( \pi \)-Hall subgroups invariant under \( L \triangle M \). By Lemma 2.3(e) and statement (i) we obtain that \( L \triangle M \) possesses at least \( k' \) classes of conjugate \( \pi \)-Hall subgroups, whence \( k' \geq k^t \).

Suppose that \( 2,3 \in \pi, \text{Sym}_n \in E_\pi, \) and \( M \in \text{Hall}_\pi(\text{Sym}_n) \). Suppose \( H, K \in \text{Hall}_\pi(L \triangle \text{Sym}_n) \). Since \( \text{Sym}_n \in C_\pi \), we may assume that \( H, K \in \text{Hall}_\pi(L \triangle M) \). So we need to show that \( H, K \) are conjugate in \( L \triangle M \) if and only if \( H, K \) are conjugate in \( L \triangle \text{Sym}_n \). Suppose there exists \( x \in L \triangle \text{Sym}_n \) such that \( H = K^x \). Then \( L \triangle M = (L_1 \times \ldots \times L_n)H = (L_1 \times \ldots \times L_n)K \), hence the image of \( x \) under the natural homomorphism \( L \triangle M \rightarrow M \) is in \( N_{\text{Sym}_n}(M) \), which is equal to \( M \) in virtue of Corollary 2.4. Therefore \( x \in L \triangle M \). \( \square \)

**Lemma 2.6** (\([\text{I}], \text{Lemma } 2.2\), \([\text{II}], \text{Lemmas } 2.4 \text{ and } 2.5\), and \([\text{III}]\)) Let \( q \) be a rational integer and \( r \) a prime such that \((q,r) = 1\). Denote \( e(q,r) \) by \( e \) and

\[
e^* = \begin{cases} 
2e, & \text{if } e \equiv 1 \pmod 2, \\
e, & \text{if } e \equiv 0 \pmod 4, \\
e/2, & \text{if } e \equiv 2 \pmod 4.
\end{cases}
\]

Then the following identities hold:

\[
(q^n - 1)_r = \begin{cases} 
(q^n - 1)_r(n/e)_r, & \text{if } n \text{ is divisible by } e, \\
(r, 2) & \text{otherwise};
\end{cases}
\]

\[
(q^e - (-1)^e)_r = \begin{cases} 
(q^e - (-1)^e)_r(n/e^*)_r, & \text{if } n \text{ is divisible by } e^*, \\
(r, 2) & \text{otherwise};
\end{cases}
\]

\[
\prod_{i=1}^n(q^{r_i} - 1)_r = (q^e - 1)^{e/2}((n/e)!)_r.
\]

If \( r \in \{2,3\} \), then Lemma 2.6 implies the following corollaries.

**Corollary 2.7** Let \( q \) be a rational integer such that \((q,3) = 1\). Then the following
identities hold:

\[
(q^n - \eta^n)_3 = \begin{cases} 
(q - \eta)_n, & \text{if } q \equiv \eta \pmod{3}, \\
(q + \eta)(n/2), & \text{if } q \equiv -\eta \pmod{3} \text{ and } n \text{ is even}, \\
1, & \text{if } q \equiv -\eta \pmod{3} \text{ and } n \text{ is odd};
\end{cases}
\]

\[
\prod_{i=1}^{n}(q^i - \eta^i)_3 = \begin{cases} 
(q - \eta)^{\eta(n!)}_3, & \text{if } q \equiv \eta \pmod{3}, \\
(q + \eta)^{[n/2]([n/2]!)}_3, & \text{if } q \equiv -\eta \pmod{3}.
\end{cases}
\]

**Corollary 2.8** Let \( q \) be an odd rational integer. Then the following identities hold:

\[
(q^n - \eta^n)_{2^t} = (q - \eta)_{2^t}, \quad \text{where } t = \begin{cases} 
(q + \eta)(n/2), & \text{if } n \text{ is even}, \\
1, & \text{if } n \text{ is odd};
\end{cases}
\]

\[
\prod_{i=1}^{n}(q^i - \eta^i)_{2^t} = (q - \eta)^{\eta(n)_{2^t}}(q + \eta)^{[n/2]([n/2]!)}_{2^t}.
\]

**Lemma 2.9** Assume that \( r, p \) are distinct odd primes, \( q = p^\alpha \) and \( m \geq \frac{r+1}{2} \). Then the inequality \(((q^2 - 1)(q^4 - 1)\ldots(q^{2(m-1)} - 1))_r > (m!)_r \) holds.

**Proof.** Note that

\[
(m!)_r = \prod_{k=1}^{[m/r]} r \cdot k_r.
\]

To every number \( r \cdot k \) we put into correspondence the number \((q^{2k(r-1)} - 1)_r = (q^{2(r-1)} - 1)_r \cdot k_r \geq r \cdot k_r\).

So the following inequalities

\[
((q^2 - 1)(q^4 - 1)\ldots(q^{2(m-1)} - 1))_r \geq (q^{2((r-1)/2)} - 1)_r \cdot \prod_{k=1}^{[m/r]} (q^{2(k(r-1))} - 1)_r \geq r \cdot \prod_{k=1}^{[m/r]} r \cdot k_r > (m!)_r,
\]

hold, whence the lemma follows. \(\square\)
For linear algebraic groups our notation agrees with that of [13]. For finite groups of Lie type we use the notation from [2]. If $\overline{G}$ is a simple connected linear algebraic group over the algebraic closure $\overline{F}_p$ of a finite field of characteristic $p$, then a surjective endomorphism $\sigma : \overline{G} \to \overline{G}$ is called a Frobenius map, if the set of $\sigma$-stable points $\overline{G}_\sigma$ is finite. Every group $G$ such that $O^\sigma(\overline{G}_\sigma) \leq G \leq \overline{G}_\sigma$ is called a finite group of Lie type. Notation for classical groups agrees with [19]. In order to make uniform statements and arguments we use the following notations $GL^n(q) = GL_n(q)$, $GL^{-}_n(q) = GU_n(q)$, $SL^n(q) = SL_n(q)$, $SL^{-}_n(q) = SU_n(q)$. In this paper we consider groups of Lie type with the base field $F_q$ of odd characteristic $p$ and order $q = p^\alpha$, and we fix the symbols $p$ and $q$ for this purposes. We always choose $\epsilon(q) \in \{+1, -1\}$ (usually we write just $\epsilon$, since $q$ is fixed by the choice of $G$) so that $q \equiv \epsilon(q) \pmod{4}$, i.e. $\epsilon(q) = (-1)^{(q-1)/2}$. The same symbol $\epsilon(q)$ is used to denote the sign of $\epsilon(q)$. Following [19], by $O^\eta_n(q)$ we denote the general orthogonal group of degree $n$ and of sign $\eta \in \{\circ, +, -\}$ over $F_q$, while the symbol $GO^\eta_n(q)$ denotes the group of similarities. Here $\circ$ is an empty symbol, and we use it only if $n$ is odd. By $\eta$ we always mean an element from the set $\{\circ, +, -\}$, and if $\eta \in \{+,-\}$, then we use $\eta$ instead of $\eta 1$ as well. In classical groups the symbol $P$ will also denote the reduction modulo scalars. Thus for every subgroup $H$ of $GL_n(q)$ the image of $H$ in $PGL_n(q)$ is denoted by $PH$.

Let $I \in \{GL_n(q), GU_n(q), Sp_n(q), O^\eta_n(q)\}$ be a general classical group over a field of odd characteristic $p$. Let $V$ be the natural module for $I$ of dimension $n$ over a field either of order $q$ or of order $q^2$ if $I = GU_n(q)$. In all cases we say that $F_q$ is the base field for $V$. Assume that $V$ is equipped with the corresponding form (trivial for $GL_n(q)$, unitary for $GU_n(q)$, skew-symmetric for $Sp_n(q)$, and symmetric for $O^\eta_n(q)$). Then $I$ can be identified in a natural way with the (general) group of isometries of $V$. We set also

$$S = S(V) = I(V) \cap SL(V) \in \{SL(V), SU(V), Sp(V), SO^\eta(V)\}$$

$$\Omega = \Omega(V) = O^\sigma(S(V)).$$

Given subspaces $U, W \leq V$, we write $U + W = U \perp W$ if $U \cap W = \{0\}$ and $U, W$ are mutually orthogonal. Following [19], we say that a subgroup $H$ of $G$, where $\Omega(V) \leq G \leq I(V)$, is of type $I(U) \perp I(W)$, if $H$ is the stabilizer in $G$ of the decomposition $U \perp W$ and $H$ stabilizes both $U, W$, while $H$ is isomorphic to $(I(U) \times I(W)) \cap G$ as an abstract group. If $V$ is an orthogonal space of even dimension, then we denote the sign of the corresponding quadratic form by $\eta(V)$, and the discriminant of the form by $D(V)$. We write $D(V) = \square$ if the discriminant of the form is a square in $F_q$, and $D(V) = \diamond$ if the discriminant of the form is a non-square in $F_q$. If $\eta = \eta(V)$ and dim($V$) = $2m$, then [19, Proposition 2.5.10] implies that $D(V) = \diamond$ if and only if $\eta = \epsilon(q)^m$.

In our arguments we use the classification of subgroups of odd index in finite simple groups obtained by M.W.Liebeck and J.Saxl [20] and independently by W.M.Kantor [16]. A more detailed description of subgroups of odd index in the
finite simple classical groups is obtained by N.V. Maslova in [21, Theorem 1] and for classical groups we refer to this description. Since [21] is published in Russian, we cite the main theorem from this paper here.

Assume that $n$ is a positive integer and $\alpha_0 \cdot 2^0 + \alpha_1 \cdot 2^1 + \ldots$, where $\alpha_i \in \{0, 1\}$, is the 2-adic expansion of $n$ (for our purposes we assume that this expansion is infinite, but only finitely many coefficients are not equal to 0). Define $\psi(n) = (\alpha_0, \alpha_1, \ldots)$. Let $\triangleright$ be a linear order on $\{0, 1\}$ such that $1 \triangleright 0$. We say that $\psi(n) = (\alpha_0, \alpha_1, \ldots) \triangleright \psi(m) = (\beta_0, \beta_1, \ldots)$ if $\alpha_i \triangleright \beta_i$ for all $i$. Notice that $\triangleright$ is a partial order.

**Theorem 2.10** [21, Theorem 1] Let $G$ be one of the finite classical groups: $\text{SL}_n(q)$ with $n \geq 2$, $\text{SU}_n(q)$ with $n \geq 3$, $\text{Sp}_n(q)$ with $n \geq 4$ and $n$ even, $\Omega_n(q)$ with $n \geq 7$ and $n$ odd, and $\Omega^+_n(q)$ with $n \geq 8$ and $n$ even. Assume that the base field of $G$ has odd order $q$, and $V$ is the natural module for $G$. Then $M$ is a maximal subgroup of odd index of $G$ if one of the following statements holds:

(a) $M = N_G(C_G(\sigma))$, where $\sigma$ is a field automorphism of odd prime order of $G$.
(b) $G = \text{SL}_n(q)$, $M$ is the stabilizer of a subspace of dimension $m$ of $V$ and $\psi(n) \triangleright \psi(m)$.
(c) $G = \text{SU}_n(q)$ or $G = \text{Sp}_n(q)$, $M$ is the stabilizer of a nondegenerate subspace of dimension $m$ of $V$ and $\psi(n) \triangleright \psi(m)$.
(d) $G = \Omega_n(q)$, $n$ is odd, $M$ is the stabilizer of a nondegenerate subspace $U$ of even dimension $m$ of $V$, $D(U) = \varnothing$, $\psi(n) \triangleright \psi(m)$ and $(q, m, \eta(U)) \neq (3, 2, +)$.
(e) $G = \Omega^+_n(q)$, $n$ is even, $M$ is the stabilizer of a nondegenerate subspace $U$ of dimension $m$ of $V$, and one of the following holds:
   (e.1) $m$ is odd, $D(V) = \varnothing$, and $\psi(n-2) \triangleright \psi(m-1)$, except for the case $m = n/2$ and subspaces $U$ and $U^\perp$ are nonisometric;
   (e.2) $m$ is even, $(q, m, \eta(U)) \neq (3, 2, +), (3, n-2, +)$, $D(U) = D(V) = \varnothing$, and $\psi(n-2) \triangleright \psi(m-2)$;
   (e.3) $m$ is even, $(q, m, \eta(U)) \neq (3, 2, +), (3, n-2, +)$, $D(U) = D(V) = \varnothing$, and $\psi(n) \triangleright \psi(m)$.
(f) $G = \text{SL}_n(q)$, $M$ is the stabilizer of a decomposition $V = \bigoplus V_i$ into a direct sum of subspaces of the same dimension $m$ and either $m = 2^u \geq 2$ and $(n, m, q) \neq (4, 2, 3)$, or $m = 1$, $q \equiv 1 \pmod{4}$ and $q \geq 13$ if $n = 2$.
(g) $G = \text{SU}_n(q)$, $M$ is the stabilizer of an orthogonal decomposition $V = \bigoplus V_i$ into a direct sum of isometric subspaces $V_i$ of dimension $m$, and either $m = 2^u \geq 2$, or $m = 1$, $q \equiv 3 \pmod{4}$ and $(n, q) \neq (4, 3)$.
(h) $G = \text{Sp}_n(q)$, $H$ is a stabilizer of an orthogonal decomposition $V = \bigoplus V_i$ into a direct sum of isometric subspaces $V_i$ of dimension $m$ and $m = 2^u \geq 2$.
(i) $G = \Omega_n(q)$, $n$ is odd, $M$ is the stabilizer of an orthogonal decomposition $V = \bigoplus V_i$ into a direct sum of isometric subspaces $V_i$ of dimension 1, $q$ is a prime.

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2 In the original paper the theorem is proven in both directions, i.e., it has “if and only if” form, however the proof of “only if” part does use the unpublished PhD thesis of P. Kleidman, that known to have inaccuracies.
and \( q \equiv \pm 3 \pmod{8} \).

(j) \( G = \Omega_n^0(q) \), \( n \) is even, \( M \) is the stabilizer of an orthogonal decomposition \( V = \bigoplus V_i \) into a direct sum of isometric subspaces \( V_i \) of dimension \( m \), and either \( m = 1 \), \( q \) is a prime, \( q \equiv \pm 3 \pmod{8} \) and \((n, \eta) \neq (8,+)\); or \( m = 2^w \geq 2 \), \( D(V) = D(V_i) = \emptyset \), and \((m, q, \eta(V_i)) \neq (2, 3, \pm), (2, 5, +)\).

(k) \( PG = \text{PSL}_2(q) \) and \( PM = \text{PGL}_2(q_0) \), where \( q = q_0^2 \).

(l) \( PG = \text{PSL}_2(q) \) and \( PM \cong \text{Alt}_4 \), where \( q \) is a prime, \( q \equiv 3, 5, 13, 27, 37 \pmod{40} \).

(m) \( PG = \text{PSL}_2(q) \) and \( PM \cong \text{Sym}_4 \), where \( q \) is a prime, \( q \equiv 7 \pmod{16} \).

(n) \( PG = \text{PSL}_2(q) \) and \( PM \cong \text{Alt}_5 \), where \( q \) is a prime, \( q \equiv 11, 19, 21, 29 \pmod{40} \).

(o) \( PG = \text{PSL}_2(q) \) and \( PM \) is the dihedral group of order \( q + 1 \), where \( 7 < q \equiv 3 \pmod{4} \).

(p) \( PG = \text{PSU}_3(5) \) and \( PM \cong M_{10} \cong \text{Alt}_6 \cdot 2 \).

(q) \( PG = \text{PSL}_4(q) \) and \( PM \cong 2^d \cdot \text{Alt}_6 \), where \( q \) is a prime, \( q \equiv 5 \pmod{8} \).

(r) \( PG = \text{PSL}_4(q) \) and \( PM \cong \text{PSp}_4(q) \cdot 2 \), where \( q \equiv 3 \pmod{4} \).

(s) \( PG = \text{PSU}_4(q) \) and \( PM \cong 2^d \cdot \text{Alt}_6 \), where \( q \) is a prime, \( q \equiv 3 \pmod{8} \).

(t) \( PG = \text{PSU}_4(q) \) and \( PM \cong \text{PSp}_4(q) \cdot 2 \), where \( q \equiv 1 \pmod{4} \).

(u) \( PG = \text{PSp}_4(q) \) and \( PM \cong 2^d \cdot \text{Alt}_5 \), where \( q \) is a prime, \( q \equiv 3 \pmod{8} \).

(v) \( PG = \text{PSL}_7(q) \) and \( PM \cong \Omega_7(2) \), where \( q \) is a prime, \( q \equiv \pm 3 \pmod{8} \).

(w) \( PG = \text{PSO}_8(q) \) and \( PM \cong \Omega_8^+(2) \), where \( q \) is a prime, \( q \equiv \pm 3 \pmod{8} \).

**Lemma 2.11** Let \( V \) be a vector space of dimension \( n \) over a finite field \( \mathbb{F}_q \) of odd characteristic \( p \), equipped with a trivial, unitary, symmetric, or skew-symmetric form. Assume that \( V = V_1 \perp \cdots \perp V_k \) is a decomposition of \( V \) into a direct orthogonal sum of nondegenerate (arbitrary if the form is trivial) subspaces. Let \( L = \Omega(V), L_0 = (I(V_1) \times \cdots \times I(V_k)) \cap L, \) and \( \rho_i : I(V_1) \times \cdots \times I(V_k) \to I(V_i) \) be the natural projection. Then \( L_0^{\rho_i} = I(V_i) \) for every \( i = 1, \ldots, k \), except for the case, when \( V \) is an orthogonal space, \( k = 2 \) and, up to renumbering, \( \dim(V_1) = n - 1, \) \( \dim(V_2) = 1, \) and \( i = 1 \). In this exceptional case one of the following statements holds:

(a) \( n \) is odd, \( L_0^{\rho_i} \) is equal to \( \Omega(V_i) \), extended by a graph automorphism.

(b) \( n \) is even, \( D(V) = \emptyset \), \( L_0^{\rho_i} = \Omega(V_i) \).

(c) \( n \) is even, \( D(V) = \emptyset \), \( L_0^{\rho_i} = \text{SO}(V_i) \).

**Proof.** Assume that \( L_0^{\rho_i} \neq I(V_i) \). Then \[19, \text{Lemma 4.1.1} \] implies that \( V \) is an orthogonal space, \( k = 2 \), up to renumbering, \( \dim(V_1) = n - 1, \) \( \dim(V_2) = 1, \) and \( i = 1 \). Now for \( n \) even the statement of the lemma follows from \[19, \text{Proposition 4.1.6} \] and the fact that \( \text{O}_{n-1}(q) \times \text{O}_1(q) \) does not induce graph automorphisms on \( \Omega_{n-1}(q) \).

Assume that \( n \) is odd. Let \( K \) be an algebraic closure of \( \mathbb{F}_q, \) \( \overline{L} = \text{SO}_n(K) \) a simple linear algebraic group (the group of orthogonal matrices of determinant 1), \( \Omega_n(K) \) a group of all orthogonal matrices. Assume that \( \sigma \) is a Frobenius map of \( \overline{L}, \) i.e., a surjective endomorphism such that the set of \( \sigma \)-stable points \( \overline{L}_{\sigma} \) is finite. Then \( \overline{L}_{\sigma} = \text{SO}_n(q) \) and \( L = O''(\overline{L}_{\sigma}) = \Omega_n(q), \) the index \( [\overline{L}_{\sigma} : G] \) is equal to 2 (recall that \( q \) is odd) and \( \overline{L}_{\sigma} = \text{SO}_n(q) \) is generated by \( L \) and a diagonal automorphism.
Let $V$ be the natural module for $L$. Then $\overrightarrow{V} = K \otimes_{\mathbb{F}_q} V$ is the natural module for $\overline{L}$. Moreover, if $V_1 \perp V_2$ is a decomposition of $V$ into an orthogonal direct sum, then $(K \otimes_{\mathbb{F}_q} V_1) \perp (K \otimes_{\mathbb{F}_q} V_2)$ is a decomposition of $\overrightarrow{V}$ into an orthogonal direct sum. So, for every subgroup $L_0$ of $L$, stabilizing the decomposition $V_1 \perp V_2$, there corresponds a unique subgroup $\overline{L}_0$ of $\overline{L}$, stabilizing the decomposition $(K \otimes_{\mathbb{F}_q} V_1) \perp (K \otimes_{\mathbb{F}_q} V_2)$, and $L_0 = \overline{L}_0 \cap \overline{L}$. The subgroup $\overline{L}_0$ is a reductive subgroup of maximal rank of $\overline{L}$. By [30, Theorem 2] it follows that $\text{Aut}_L(L_0^\pi)$ does not contain diagonal automorphisms, hence by using [19, Proposition 4.1.6], we obtain the statement of the lemma for $n$ odd. □

**Lemma 2.12** Assume that a simple classical group $G$ and its subgroup $H$ of satisfy one of the following statements:

(a) $G \cong \text{PSL}_2(q), (q, 6) = 1, H \cong \text{Sym}_4$;
(b) $G \cong \text{PSL}_2(q), (q, 30) = 1, H \cong \text{SL}_2(4)$;
(c) $G \cong \text{PΩ}_7(q), (q, 210) = 1, H \cong \Omega_7(2)$;
(d) $G = \text{PΩ}_8^+(q), (q, 210) = 1, H \cong \Omega_8^+(2)$.

Suppose $K$ is chosen so that $G < K \leq \hat{G}$, where $\hat{G}$ is the group of inner-diagonal automorphisms of $G$. Then there is no subgroup $H_1$ of $K$ such that $H_1 \cap G = H$ and $|H_1 : H| = |K : G|$.

**Proof.** If $H$ satisfies either (a) or (b), then the lemma follows from [14, Chapter II, § 8]. If $H$ satisfies statement (d), then the lemma follows from [18, Proposition 2.3.8]. Assume that $H$ satisfies (c). By using [3], we obtain that the minimal nontrivial irreducible representation of $\Omega_7(2)$ has degree 7. Since $|H|, q = 1$, all ordinary characters of this group have rational values, and in view of [15, Theorem 9.14 and Corollary 15.12], the same property holds for the representations over $\mathbb{F}_q$. In view of [3] it also follows that $\text{Out}(\Omega_7(2))$ is trivial, while the universal central extension $2 \cdot \Omega_7(2)$ of $\Omega_7(2)$ has no faithful irreducible representations of degree 7 over $\mathbb{F}_q$. Therefore $N_G(\Omega_7(2)) = N_G(\Omega_7(2)) = \Omega_7(2)$. □

The next lemma follows from Lemmas 2.1(e), 2.11 and 2.12.

**Lemma 2.13** Assume $\pi \cap \pi(G) = \{2, 3, 5, 7\}$, $G$ is isomorphic to either $\Omega_7(q)$, or $\Omega_8^+(q)$, or $\Omega_9(q)$, and a $\pi$-Hall subgroup $H$ of $G$ is isomorphic to either $\Omega_7(2)$, or $2 \cdot \Omega_8^+(2)$, or $\left(2 \cdot \Omega_8^+(2)\right)^\cdot 2$, respectively. Denote by $G_1$ either $\text{SO}_7(q)$, or $\text{SO}_8^+(q)$, or $\text{SO}_9(q)$, respectively. Then $G_1$ does not possesses a $\pi$-Hall subgroup $H_1$ such that $H_1 \cap G = H$. 

13
3 Maximal subgroups of odd index in classical groups of small dimension

In this section we classify \( \pi \)-Hall subgroups in groups \( \text{SL}_2^n(q) \) and \( \text{GL}_2^n(q) \), and give a complete list of maximal subgroups of odd index in classical groups of small dimension.

**Lemma 3.1** Let \( \pi \) be a set of primes with \( 2, 3 \in \pi \). Assume that \( G \cong \text{SL}_2(q) \cong \text{PSL}_2(q) \), where \( q \) is a power of an odd prime \( p \notin \pi \), and \( \varepsilon = \varepsilon(q) \). Recall that for a subgroup \( A \) of \( G \) we denote by \( \text{PA} \) the reduction modulo scalars. Then the following statements hold:

(A) If \( G \in E_\pi \) and \( H \in \text{Hall}_\pi(G) \), then one of the following statements holds:

(a) \( \pi \cap \pi(G) \subseteq \pi(q - \varepsilon) \), \( PH \) is a \( \pi \)-Hall subgroup in the dihedral subgroup \( D_{q - \varepsilon} \) of order \( q - \varepsilon \) of \( PG \). All \( \pi \)-Hall subgroups of this type are conjugate in \( G \).

(b) \( \pi \cap \pi(G) = \{2, 3\} \), \( (q^2 - 1)_{2,3} = 24 \), \( PH \cong \text{Alt}_4 \). All \( \pi \)-Hall subgroups of this type are conjugate in \( G \).

(c) \( \pi \cap \pi(G) = \{2, 3\} \), \( (q^2 - 1)_{2,3} = 48 \), \( PH \cong \text{Sym}_4 \). There exist exactly two classes of conjugate subgroups of this type, and \( \text{PGL}_2^n(q) \) interchanges these classes.

(d) \( \pi \cap \pi(G) = \{2, 3, 5\} \), \( (q^2 - 1)_{2,3,5} = 120 \), \( PH \cong \text{Alt}_5 \). There exist exactly two classes of conjugate subgroups of this type, and \( \text{PGL}_2^n(q) \) interchanges these classes.

(B) Conversely, if \( \pi \) and \( (q^2 - 1)_\pi \) satisfy one of statements (a)–(d), then \( G \in E_\pi \).

(C) If \( G \in E_\pi \), then \( k_\pi(G) \in \{1, 2, 3\} \).

(D) If \( G \in E_\pi \) and \( H \in \text{Hall}_\pi(G) \), then there exists a \( \pi \)-subgroup \( K \trianglelefteq G \), nonconjugate in \( G \) to a subgroup of \( H \), in particular \( G \notin D_\pi \).

(E) Every \( \pi \)-Hall subgroup of \( PG \) can be obtained as \( PH \) for some \( H \in \text{Hall}_\pi(G) \).

Conversely, if \( PH \in \text{Hall}_\pi(PG) \) and \( H \) is a complete preimage of \( PH \) in \( G \), then \( H \in \text{Hall}_\pi(G) \).

*Proof.* (E) Follows from Lemma 2.1(a), (d) and from \(|Z(G)| = 2\).

(A) Assume that \( PH \) is a \( \pi \)-Hall subgroup of \( \text{PSL}_2(q) \). Then \( PH \) is included in a maximal subgroup \( M \) of odd index of \( \text{PSL}_2(q) \). By using Theorem 2.10, we obtain the following list of maximal subgroups of odd index in \( \text{PSL}_2(q) \):

1. \( M = N_G(C_G(\sigma)) \), where \( \sigma \) is a field automorphism of odd prime order of \( G \). In view of [19, Proposition 4.5.3] we obtain that \( M \cong \text{PSL}_2(q^2) \), where \( q = p^\alpha \) and \( r \) is an odd prime divisor of \( \alpha \), and all subgroups of this type are conjugate in \( \text{PSL}_2(q) \).
2. \( q = q_0^2, M \cong \text{PGL}_2(q_0) \).
3. \( q \equiv 1 \pmod{4}, q \geq 13, M \) is the dihedral group \( D_{q - 1} \) of order \( q - 1 \).
4. \( q \equiv -1 \pmod{4}, q \geq 11, M \) is the dihedral group \( D_{q + 1} \) of order \( q + 1 \).
5. \( q \equiv 3, 5, 13, 27, 37 \pmod{40}, M \cong \text{Alt}_4 \).
6. \( q \equiv \pm 7 \pmod{16}, M \cong \text{Sym}_4 \).
7. \( q \equiv 11, 19, 21, 29 \pmod{40}, M \cong \text{Alt}_5 \).
Consider all these cases separately.

Assume that either $PH \leqslant \operatorname{PSL}_2(q^{1/2}) = \operatorname{PSL}_2(q_0)$, or $PH \leqslant \operatorname{PGL}_2(q^{1/2}) = \operatorname{PGL}_2(q_0)$. The condition $p \notin \pi$ implies that $\operatorname{PSL}_2(q_0) \nsubseteq PH$. By Lemma 2.1(a) we obtain that $PH \cap \operatorname{PSL}_2(q_0)$ is a $\pi$-Hall subgroup of $\operatorname{PSL}_2(q_0)$. Induction on $q$ implies that $PH \cap \operatorname{PSL}_2(q_0)$ satisfies one of statements (a)–(d) of the lemma. So we obtain the statement of the lemma by induction if $PH \subseteq \operatorname{PSL}_2(q_0)$. If $PH \nsubseteq \operatorname{PSL}_2(q_0)$, then Lemma 2.1(e) implies that $PH \cap \operatorname{PSL}_2(q_0)$ satisfies either (a) or (b) of the lemma, hence $PH$ satisfies either (a) or (c) of the lemma.

If $PH$ is included in a dihedral subgroup $D_{q-\epsilon}$ of order $q - \epsilon$, then statement (a) of the lemma holds.

Assume that $PH$ is included in $\operatorname{Alt}_4$. Since $2, 3 \in \pi$, we obtain that $H = \operatorname{Alt}_4$. So statement (b) of the lemma holds in this case.

Assume that $PH$ is included in $\operatorname{Sym}_4$. Since $2, 3 \in \pi$, we obtain that $H = \operatorname{Sym}_4$. So statement (c) of the lemma holds in this case.

Assume, finally, that $PH$ is included in $\operatorname{Alt}_5$. Since $2, 3 \in \pi$, by Lemma 2.5 we obtain that either $PH = \operatorname{Alt}_5$, or $PH \cong \operatorname{Alt}_4$. The case $PH \cong \operatorname{Alt}_4$ is considered above, so we may assume that $PH = \operatorname{Alt}_5$. Thus statement (d) of the lemma holds in this case.

(B) Now we prove that every subgroup $PH$, satisfying one of statements (a)–(d) of the lemma, is a $\pi$-Hall subgroup of $PG$. This fact is evident for statements (b)–(d), since $|\operatorname{PGL}_2| = \frac{1}{2}(q^2 - 1)_\pi = |PH|$. If statement (a) holds, i.e., $PH$ is a $\pi$-Hall subgroup in a dihedral subgroup $M$ of order $q - \epsilon$ of $G$, then $|PG : M| = \frac{1}{2}q(q + \epsilon)$ is a $\pi'$-number, whence $PH \in \operatorname{Hall}_{\pi'}(PG)$.

(C) If a $\pi$-Hall subgroup $PH$ satisfies statement (a) of the lemma, then $PH$ has a Sylow tower of complexion $<$, where $<$ is the natural order, so all subgroups of this type are conjugate by Lemma 2.2. If a $\pi$-Hall subgroup $PH$ satisfies statement (b) of the lemma, then $PH$ has a Sylow tower of complexion $<$, where $3 < 2$. Hence all subgroups of this type are conjugate by Lemma 2.2. In view of [2] Chapter XII, if either (c) or (d) holds, then $PG$ possesses precisely two classes of conjugate subgroups isomorphic respectively to either $\operatorname{Sym}_4$ or $\operatorname{Alt}_5$, and these classes are interchanged by $\operatorname{PGL}_2(q)$. Notice that there can be more than one type of $\pi$-Hall subgroups in $G$, namely, there can exist subgroups, satisfying either (a) and (b), or (a) and (c), or (a) and (d). Hence, if $G \in E_\pi$, then $k_\pi(G) \in \{1, 2, 3\}$.

(D) If $G$ possesses more than one class of conjugate $\pi$-Hall subgroups, then we have nothing to prove. So we may assume that $G$ possesses one class of conjugate $\pi$-Hall subgroups, i.e., either statement (a) or statement (b) of the lemma holds. Assume that statement (a) of the lemma holds. Then $H$ is included in a dihedral subgroup $D_{2(q-\epsilon)}$ of order $2(q - \epsilon)$, in particular $H$ possesses a normal abelian subgroup of
index 2, while its Sylow 2-subgroup is not normal. On the other hand \( G \) possesses a \( \pi \)-subgroup \( K \cong SL_2(3) \cong 2 \cdot 2^2 \cdot 3 \). Clearly \( K \) is not isomorphic to a subgroup of \( H \). Assume that statement (c) of the lemma holds, i.e., \( H \cong SL_2(3) \). Assume that \( q \equiv \nu \pmod{3} \), where \( \nu = \pm 1 \), then \( G \) possesses a dihedral \( \pi \)-subgroup \( K = D_{2(q-\nu); 2,3} \). Clearly \( K \) is not isomorphic to a subgroup of \( H \). □

As a corollary to Lemmas 2.1(e) and 3.1 we obtain the following lemma.

**Lemma 3.2** Let \( G = GL_2^n(q) \), \( PG = G/Z(G) = PGL_2^n(q) \), where \( q \) is a power of a prime \( p \), and \( \varepsilon = \varepsilon(q) \). Let \( \pi \) be a set of primes such that \( 2,3 \in \pi \) and \( p \notin \pi \). A subgroup \( H \) of \( G \) is a \( \pi \)-Hall subgroup if and only if \( H \cap SL_2(q) \) is a \( \pi \)-Hall subgroup of \( SL_2(q) \), \( |H : H \cap SL_2(q)|_\pi = (q-\eta)_\pi \), and either statement (a), or statement (b) of Lemma 3.1 holds. More precisely, one of the following statements holds:

(a) \( \pi \cap \pi(G) \subseteq \pi(q-\varepsilon) \), where \( \varepsilon = \varepsilon(q) \), \( PH \) is a \( \pi \)-Hall subgroup in the dihedral group \( D_{2(q-\varepsilon)} \) of order \( 2(q-\varepsilon) \) of \( PG \). All \( \pi \)-Hall subgroups of this type are conjugate in \( G \).

(b) \( \pi \cap \pi(G) = \{2,3\} \), \( (q^2-1)_{2,3} = 24 \), \( PH \cong Sym_4 \). All \( \pi \)-Hall subgroups of this type are conjugate in \( G \).

**Lemma 3.3** Let \( G = Sp_4(q) \), \( \varepsilon = \varepsilon(q) \), and \( M \) be a maximal subgroup of odd index of \( G \). Then one of the following statements hold:

(a) \( M \cong Sp_4(q^{1/2}) \), where \( q = p^r \) and \( r \) is an odd prime divisor of \( \alpha \), and all subgroups of this type are conjugate in \( G \);

(b) \( M \cong Sp_2(q) \cap Sym_2 \cong SL_2(q) \cap Sym_2 \) and all subgroups of this type are conjugate in \( G \);

(c) \( q \equiv \pm 3 \pmod{8} \), \( M \cong 2^{1+4} \cdot \Omega^+_4(2) \cong 2^{1+4} \cdot SL_2(4) \) and all subgroups of this type are conjugate in \( G \).

**Proof** In view of [20] and Theorem 2.10(a), (c), (h), (u) we obtain that either \( M = N_G(C_G(\sigma)) \); or \( M \) is the stabilizer of a nondegenerate subspace \( U \) of dimension \( m \) of \( V \) and \( \psi(4) \gg \psi(m) \); or \( M \) is the stabilizer of an orthogonal decomposition \( V = \bigoplus V_i \) into a direct sum of isometric subspaces \( V_i \) of dimension \( m = 2^\mu \geq 2 \); or \( q \) is a prime, \( q \equiv \pm 3 \pmod{8} \), and \( M \cong 2^{1+4} \cdot \Omega^+_4(2) \cong 2^{1+4} \cdot SL_2(4) \). In the first case [19] Proposition 4.5.8] implies statement (a) of the lemma. It is easy to see that the second case is impossible. In the third case we obtain that \( m = 2 \) and \( M \cong Sp_2(q) \cap Sym_2 \). So [19], Table 3.5.C implies that all subgroups of this type are conjugate in \( G \). In the fourth case we obtain that \( M \) satisfies (c) of the lemma and [19], Table 3.5.C implies that all subgroups of this type are conjugate in \( G \). □

**Corollary 3.4** Suppose \( G = \Omega_5(q) \), \( \varepsilon = \varepsilon(q) \), and let \( M \) be a maximal odd-index subgroup of \( G \). Then one of the following statements holds:

(a) \( M \cong \Omega_5(q^{1/2}) \), where \( q = p^r \) and \( r \) is an odd prime divisor of \( \alpha \), and all subgroups of this type are conjugate in \( G \);
(b) \( M \simeq \Omega^+_4(q) \cdot 2 \) and all subgroups of this type are conjugate in \( G \);
(c) \( q \equiv \pm 3 \pmod{8} \), \( M \simeq 2^4 \cdot \text{Alt}_5 \) and all subgroups of this type are conjugate in \( G \).

Proof. The corollary follows from known isomorphisms \( \text{SL}_2(q) \simeq \text{Sp}_2(q) \), \( \text{PSp}_4(q) \simeq \Omega^+_5(q) \), \( \Omega^-_4(q) \simeq \text{Sp}_2(q) \circ \text{Sp}_2(q) \), and Lemma 3.3. \( \square \)

Lemma 3.5  Suppose \( G \simeq \text{SL}^\eta_4(q) \), \( \varepsilon = \varepsilon(q) \), and \( M \) is a maximal subgroup of odd index of \( G \). Then one of the following statements holds:

(a) \( M \simeq \text{SL}^\eta_4(q^2) \), where \( q = p^r \) and \( r \) is an odd prime divisor of \( \alpha \), and all subgroups of this type are conjugate in \( G \);
(b) \( \eta = -\varepsilon \), \( M \simeq \text{Sp}_4(q) \cdot 2 \simeq 2 \cdot \text{SO}_5(q) \), and there exist two classes of subgroup of this type; interchanged by \( \text{GL}^\eta_4(q) \);
(c) \( M \simeq (\text{GL}^\eta_4(q) : \text{Sym}_3) \cap \text{SL}^\eta_4(q) \), and all subgroups of this type are conjugate in \( G \);
(d) \( \eta = \varepsilon \), \( M \simeq (\text{GL}^\eta_4(q) : \text{Sym}_4) \cap \text{SL}^\eta_4(q) \) and all subgroups of this type are conjugate in \( G \);
(e) \( \eta = \varepsilon \), \( q \equiv 5\varepsilon \pmod{8} \), \( M \simeq 4 \cdot 2^4 \cdot \text{Alt}_6 \), and there exist two classes of subgroup of this type, interchanged by \( \text{GL}^\eta_4(q) \).

Proof. In view of Theorem 2.10(a), (b), (c), (f), (g), (q), (r), (s), (t) we obtain that \( M \) satisfies one of the following statements:

1. \( M = N_G(C_G(\sigma)) \), where \( \sigma \) is a field automorphism of odd prime order of \( G \).
2. \( M \) is the stabilizer of a nondegenerate (arbitrary if \( \eta = + \)) subspace \( U \) of dimension \( m \) of \( V \), and \( \psi(4) \gg \psi(m) \).
3. \( M \) is the stabilizer of an orthogonal (arbitrary if \( \eta = + \)) decomposition \( V = \bigoplus V_i \) into a direct sum of isometric subspaces \( V_i \) of dimension \( m = 2^w \geq 2 \).
4. \( M \simeq \text{Sp}_4(q) \cdot 2 \simeq 2 \cdot \text{SO}_5(q) \), \( q \equiv -\eta \pmod{4} \).
5. \( M \) is the stabilizer of an orthogonal (arbitrary if \( \eta = + \)) decomposition \( V = \bigoplus V_i \) into a direct sum of isometric subspaces \( V_i \) of dimension 1, and \( q \equiv \eta \pmod{4} \).
6. \( M \simeq 4 \cdot 2^4 \cdot \text{Alt}_6 \), \( q \) is prime, and \( q \equiv 5\eta \pmod{8} \).

If \( M \) satisfies the first statement, then [19, Proposition 4.5.3] implies statement (a) of the lemma. The second statement is impossible. If \( M \) satisfies the third statement, then by using [19, Proposition 4.2.9] we obtain statement (c) of the lemma. If \( M \) satisfies the fourth statement, then statement (b) of the lemma follows from [19, Propositions 4.5.6 and 4.8.3]. If \( M \) satisfies the fifth statement, then by using [19, Proposition 4.2.9] we obtain statement (d) of the lemma. Assume that \( M \) satisfies the sixth statement. Then [19, Proposition 4.6.6] and the condition \( q \equiv 5\eta \pmod{8} \) imply that \( \eta = \varepsilon \). Now statement (e) of the lemma follows from [19, Proposition 4.6.6 and Tables 3.5.A and 3.5.B]. \( \square \)

Corollary 3.6  Suppose \( G = \text{PO}_4^\eta(q) \), \( \varepsilon = \varepsilon(q) \), and \( M \) is a maximal subgroup of

\[ 3 \] In this case \( |Z(\text{SL}^\eta_4(q))| = 2 \]
odd index of \( G \). Then one of the following statements holds:

(a) \( M \simeq \PO_6^\eta(q^k) \), where \( q = p^\alpha \) and \( r \) is an odd prime divisor of \( \alpha \), and all subgroups of this type are conjugate in \( G \);

(b) \( \eta = -\varepsilon \), \( M \simeq \Omega_5(q) \cdot 2 \simeq \text{SO}_5(q) \), there exist two classes of subgroups of this type, and \( \PO_6^\eta(q) \) interchanges these classes;

(c) \( \eta = -\varepsilon \), \( M = (\Omega_2^\eta(q) \times \Omega_2^\eta(q)) \cdot [4] \) and all subgroups of this type are conjugate in \( G \);

(d) \( \eta = \varepsilon \), \( M = 2 \cdot (\PO_2^\eta(q) \times \PO_4^\eta(q)) \cdot [4] \), and all subgroups of this type are conjugate in \( G \);

(e) \( \eta = \varepsilon \), \( q \equiv 5 \varepsilon \pmod{8} \), \( M \simeq 2^4 \cdot \text{Alt}_6 \), there exist two classes of subgroups of this type, these classes are invariant under \( \PO_6^\eta(q) \), and \( \PO_6^\eta(q) \) interchanges these classes;

(f) \( \eta = \varepsilon \), \( M \simeq 2^2 \cdot \PO_2^\eta(q)^3 \cdot 2^4 \cdot \text{Sym}_3 \), and all subgroups of this type are conjugate in \( G \).

**Proof.** Follows from known isomorphism \( \PSL_4^\eta(q) \simeq \PO_6^\eta(q) \) and Lemma 3.5. \( \square \)

## 4 Hall subgroups in linear, unitary, and symplectic groups

**Lemma 4.1** Let \( \pi \) be a set of primes such that \( 2, 3 \in \pi \). Suppose \( V \) is a linear, unitary, or symplectic space of dimension \( n \) with the base field \( \overline{\mathbb{Q}} \) of characteristic \( p \not\in \pi \). Assume that \( G \) is chosen so that \( \Omega(V) \leq G \leq I(V) \), and \( G \) possesses a \( \pi \)-Hall subgroup \( H \). Then one of the following statements holds:

(a) \( H \) stabilizes a decomposition \( V = V_1 \perp \cdots \perp V_k \) into a direct sum of pairwise orthogonal nondegenerate (arbitrary if \( V \) is linear) subspaces \( V_i \), and \( \dim(V_i) \leq 2 \) for \( i = 1, \ldots, k \).

(b) \( V \) is a linear or a unitary space, \( \dim(V) = 4 \), \( I(V) = \text{GL}^\eta(V) \), \( |PG : \PSL^\eta(V)| \leq 2 \), \( \pi \cap \pi(G) \nsubseteq \{2, 3, 5\} \), \( q \equiv 5 \eta \pmod{8} \) (in particular \( |\text{PGL}^\eta(V) : \PSL^\eta(V)| = 4 \) and \( \text{PG} \neq \text{PGL}^\eta(V) \)), \( (q + \eta)_3 = 3 \), \( (q^2 + 1)_3 = 5 \). Moreover \( H \simeq 4 \cdot 2^4 \cdot \text{Sym}_6 \), if \( |PG : \PSL^\eta(V)| = 2 \), and \( H \simeq 4 \cdot 2^4 \cdot \text{Alt}_6 \), if \( \text{PG} = \PSL^\eta(V) \).

**Proof.** We proceed by induction on \( \dim(V) \). If \( \dim(V) \leq 2 \) we have nothing to prove.

Assume that \( \dim(V) > 2 \). Since \( p \neq 2 \), it follows that \( G/Z(G) \) has a simple socle, and \( G \) induces inner-diagonal automorphisms on this socle. In view of the main theorem from [20] (see also Theorem 2.10) we obtain one of the following cases:

(1) \( V \) is unitary, \( q = 5 \), \( n = 3 \), and \( H \cap \Omega(V) \) is included in \( 3 \cdot M_{10} \simeq 3 \cdot \text{Alt}_6 \cdot 2 \).

---

4 Here and below, following [19], by \( \PO_\eta^\eta(q) \) we always mean a cyclic group of order \((q - \eta)/(4 - q - \eta)\), while by \( \Omega_2^\eta(q) \) we always mean a cyclic group of order \((q - \eta)/(2 - q - \eta)\).
(2) \( V \) is symplectic, \( n = 4 \), and \( H \leq M \), where \( M \) is a maximal subgroup of odd index, satisfying Lemma 5.3(c).

(3) \( V \) is a linear or a unitary space, \( n = 4 \), and \( H \cap \Omega(V) \leq M \), where \( M \) is a maximal subgroup of odd index in \( \Omega(V) \) satisfying Lemma 5.5(e).

(4) \( H \leq M < I(V) \) for a group \( M \) such that \( \Omega(V_0) \leq M \leq I(V_0) \), where \( I(V_0) \) is a group of the same type as \( I(V) \), \( \dim(V_0) = \dim(V) \) and the base field \( \mathbb{F}_{q_0} \) for \( V_0 \) is a proper subfield of \( \mathbb{F}_q \).

(5) \( V \) possesses a proper \( H \)-invariant nondegenerate (arbitrary if \( V \) is linear) subspace \( U \).

(6) \( H \) stabilizes a proper decomposition \( V = U_1 \perp \ldots \perp U_m \) of \( V \) into an orthogonal direct sum of pairwise isometric subspaces \( U_i \).

Now we proceed case by case.

(1) In this case \( H \cap \Omega(V) \) is a \( \pi \)-Hall subgroup of \( M \cong \mathrm{Alt}_6 \cdot 2 \). By Lemma 2.1(a) we obtain that \( \mathrm{Alt}_6 \) possesses a \( \pi \)-Hall subgroup. Lemma 2.3 implies that \( \mathrm{Alt}_6 \) does not possess a proper \( \pi \)-Hall subgroups with \( 2, 3 \in \pi \), hence \( p = 5 \in \pi \), a contradiction with \( p \notin \pi \).

(2) In this case \( I(V) = \Omega(V) \) and \( |\Omega(V) : M|_i \geq 3 \), a contradiction with \( 2, 3 \in \pi \).

(3) In this case \( H \cap \Omega(V) \) is a \( \pi \)-Hall subgroup of \( M \cong \mathrm{SL}_2(q) \cdot \mathrm{PGL}_2(q) \), i.e., \( (q - \eta)_i \geq 4 \). By using Lemmas 2.1(a) and 2.3, as in case (1) we obtain that \( \pi \cap \pi(H) = \{2, 3, 5\} \) and \( H \cap \Omega(V) = M \). Now \( H \) is a \( \pi \)-Hall subgroup of \( G \) if and only if \( \pi(|G : H|) \subseteq \pi' \). Since \( \Omega(V) : M \) divides \( |G : H| \), it follows that \( H \) is a \( \pi \)-Hall subgroup of \( G \) only if \( \pi(\Omega(V) : M) \) is not divisible by \( 2, 3, \) and \( 5 \), or, equivalently, only if \( |\mathrm{SL}_2(q)|_{\{2, 3, 5\}} = |H| = 2^9 \cdot 3^2 \cdot 5 \). The condition \( p \notin \pi \) implies that \( p \neq 2, 3, 5 \). So \( |\mathrm{SL}_2(q)|_{\{2, 3, 5\}} = \left( (q^2 - 1)(q^3 - \eta)(q^4 - 1) \right)_{\{2, 3, 5\}} \). By Lemma 3.5(e), we have \( |\mathrm{SL}_2(q)|_{2} = |M|_{2} \) if and only if \( q \equiv 5 \eta \pmod{8} \). Clearly, \( \left( (q^2 - 1)(q^3 - \eta)(q^4 - 1) \right)_{2} = 3^2 \) if and only if \( (q + \eta)_3 = 3 \). Finally, \( \left( (q^2 - 1)(q^3 - \eta)(q^4 - 1) \right)_{3} = 5 \) if and only if \( (q^2 + 1)_3 = 5 \). Condition \( \eta = \varepsilon(q) \) implies equality \( |\mathrm{PGL}_2(q) : \mathrm{PSL}_2(q)| = 4 \). By Lemma 3.5(e), it follows that \( \mathrm{PSL}_2(q) \) possesses two classes of conjugate subgroups isomorphic to \( L \) and \( \mathrm{PGL}_2(q) \) interchanges these classes. Since \( \mathrm{PGL}_2(q)/\mathrm{PSL}_2(q) \) is cyclic, Lemma 2.1(e) implies that \( \mathrm{PGL}_2(q) \) does not possesses a \( \pi \)-Hall subgroup \( H \) such that \( H \cap \mathrm{PGL}_2(q) \cong M \), while every subgroup \( G \) such that \( \mathrm{PSL}_2(q) \leq G < \mathrm{PGL}_2(q) \) possesses a \( \pi \)-Hall subgroup \( H \) such that \( H \cap \mathrm{PGL}_2(q) \cong M \). Thus statement (b) of the lemma holds in this case.

(4) We may assume that \( q_0 \) is the minimal possible number with \( H \leq I(V_0) \). Since \( |H| \) is coprime to \( p \), then \( H \) is a proper subgroup of \( M \cap G \). Hence for \( H, V_0 \) either case (5) or case (6) holds. By using natural embeddings \( \mathbb{F}_{q_0} \leq \mathbb{F}_q \) and \( V_0 \leq V \) we obtain that for \( H, V \) either case (5) or case (6) holds.
(5) In this case there exists a subspace $W \leq V$ such that $V = U \perp W$ and $W$ is $H$-invariant (if $I(V) = \text{GL}(V)$, then the existence follows from Maschke Theorem, while in the remaining cases we can take $W = U^\perp = \{w \in W \mid (u, w) = 0, \forall u \in U\}$). Thus $H$ is included in $G_0 = G \cap (I(U) \times I(W))$ and $H$ is a $\pi$-Hall subgroup of $G_0$. Denote by $\rho_U$ and $\rho_W$ the projections from $I(U) \times I(W)$ onto $I(U)$ and $I(W)$, respectively. Then $H \leq H^{\rho_U} \times H^{\rho_W}$, $\Omega(U) \leq G_0^{\rho_U} \leq I(U)$, and $\Omega(W) \leq G_0^{\rho_W} \leq I(W)$. Lemma 2.11 implies that $H^{\rho_U}$ and $H^{\rho_W}$ are $\pi$-Hall subgroups of $G_0^{\rho_U}$ and $G_0^{\rho_W}$, respectively. Furthermore, Lemma 2.11 implies that $G_0^{\rho_U} = I(U)$ and $G_0^{\rho_W} = I(W)$. Hence $H^{\rho_U}$ in $G_0^{\rho_U}$ and $H^{\rho_W}$ in $G_0^{\rho_W}$ cannot satisfy statement (b) of the lemma. By induction both $U$ and $W$ have a decomposition into a direct sum of pairwise orthogonal nondegenerate (arbitrary if $I(V)$ is linear) subspaces of dimensions at most 2, whence statement (a) of the lemma holds.

(6) Since we have already considered case (5), we may assume that $H$ is irreducible. In this case $H$ is included in a subgroup of type $I(U_1) \times \text{Sym}_m$ of $I(V)$. In particular, $H$ normalizes the subgroup $G_0 = G \cap (I(U_1) \times \ldots \times I(U_m))$ of $G$ and Lemma 2.11 implies that $H_0 = H \cap (I(U_1) \times \ldots \times I(U_m))$ is a $\pi$-Hall subgroup of $G_0$. Let $N_1 = \{x \in H \mid U_1x = U_1\}$ be the stabilizer of $U_1$ in $H$. Clearly $H_0 \leq N_1$. Denote by $\sigma$ the natural representation $N_1 \rightarrow I(U_1)$ of $N_1$. Assume also that $\rho : I(U_1) \times \ldots \times I(U_m) \rightarrow I(U_1)$ is the natural projection. We obtain from definition that the restrictions of $\rho$ and $\sigma$ on $H_0$ coincide. Denote $G_0^\rho$ by $G_1$ and $H_0^\rho$ by $H_1$. Lemma 2.11 implies that $H_1$ is a $\pi$-Hall subgroup of $G_1$. By using Lemma 2.11, we also obtain that $G_1 = I(U_1)$. Thus $H_1$ in $G_1$ does not satisfy statement (b) of the lemma. Now $H_1 = H_0^\rho = H_0^\sigma \leq N_1^\sigma$. So $N_1^\sigma$ is a $\pi$-Hall subgroup of $G_1$ and $N_1^\sigma = H_1$. By induction there exists an $N_1^\sigma$-invariant decomposition $U_1 = W_{11} \perp \ldots \perp W_{1k}$ of $U_1$ into an orthogonal direct sum of nondegenerate (arbitrary if $V$ is linear) subspaces of dimensions at most 2. By definition this decomposition is $N_1$-invariant. Let $g_1, g_2, \ldots, g_m$ be a right transversal for the cosets of $N_1$ in $H$. Since $H$ is irreducible, without lost of generality we may assume that $U_i = U_ig_i$. Now we set $W_{ij} = W_{1j}g_i$. Clearly $H$ stabilizes the decomposition $W_{11} \perp \ldots \perp W_{1k} \perp \ldots \perp W_{m1} \perp \ldots \perp W_{mk}$. □

Lemma 4.2 Let $\pi$ be a set of primes with $2, 3 \in \pi$. Assume that $V$ is a linear or a unitary space with the base field $\mathbb{F}_q$ of characteristic $p \notin \pi$ and $G$ is chosen so that $\text{SL}^\pi(V) \leq G \leq \text{GL}^\pi(V)$. Suppose also that $H$ is a $\pi$-Hall subgroup of $G$, and $H$ stabilizes a decomposition $V = V_1 \perp \cdot \cdot \cdot \perp V_m \perp U_1 \perp \cdot \cdot \cdot \perp U_k$ into a direct sum of pairwise orthogonal nondegenerate (arbitrary if $V$ is linear) subspaces such that $\dim(V_i) = 2$ for $i = 1, \ldots, m$ and $\dim(U_j) = 1$ for $j = 1, \ldots, k$, and the decomposition cannot be refined. Then one of the following statements holds:

(a) $\dim(V) = 2$;
(b) $q \equiv \eta \pmod{12}$ and $m = 0$;
(c) $q \equiv -\eta \pmod{3}$ and $k \leq 1$;
(d) $q \equiv -\eta \pmod{3}$, $q \equiv \eta \pmod{4}$, $m$ is even, $k = 3$, $(q + \eta)_1 = 3$, and $m \not\equiv -1 \pmod{3}$;
(e) \( q \equiv \eta \pmod{12}, m = 1, k \equiv 0 \pmod{3}, k(k - 1) \equiv 0 \pmod{4}.

**Proof.** If \( \dim(V) = 2 \) then statement (a) holds. Assume that \( \dim(V) = 2m + k > 2 \).

Then \( H \) is included in a subgroup \( M \) of type \((\text{GL}_2^n(q) \rtimes \text{Sym}_m) \subseteq (\text{GL}_2^n(q) \rtimes \text{Sym}_k) \) of \( \text{GL}^n(V) \). Denote by \( L \) the intersection \( M \cap G \). Since \( H \) contains a Sylow 2-subgroup and a Sylow 3-subgroup of \( G \), the identities \( |G : L|_2 = |G : L|_3 = 1 \) hold.

Since \( H \cap \text{SL}^n(V) \) is a \( \pi \)-Hall subgroup of \( \text{SL}^n(V) \), it is enough to prove statements (b)–(e) in case \( G = \text{SL}^n(V) \). In this case \( |M : L| = q - \eta \) and, by using Corollaries 2.7 and 2.8, we obtain the following identities

\[
|G|_3 = \prod_{i=2}^{2m+k} (q^i - \eta^i)_3 = \begin{cases} 
(q - \eta)_3^{2m+k-1}((2m + k)!)_3, & \text{if } q \equiv \eta \pmod{3}, \\
(q + \eta)_3^{m+k/2}((m + [k/2])!)_3, & \text{if } q \equiv -\eta \pmod{3};
\end{cases}
\]

\[
|L|_3 = \frac{1}{(q - \eta)_3} (q - \eta)_3^m(q^2 - \eta^2)_3^m(m!)(q - \eta)_3^k(k!)_3 = \begin{cases} 
(q - \eta)_3^{2m+k-1}(m!)(k!)_3, & \text{if } q \equiv \eta \pmod{3}, \\
(q + \eta)_3^m(m!)(k!)_3, & \text{if } q \equiv -\eta \pmod{3};
\end{cases}
\]

\[
|G|_2 = \prod_{i=2}^{2m+k} (q^i - \eta^i)_2 = (q - \eta)_2^{2m+k-1}(q + \eta)_2^{m+k/2}((m + [k/2])!)_2;
\]

\[
|L|_2 = \frac{1}{(q - \eta)_2} (q - \eta)_2^m(q^2 - \eta^2)_2^m(m!)(q - \eta)_2^k(k!)_2 = (q - \eta)_2^{2m+k-1}(q + \eta)_2^m(m!)(k!)_2.
\]

Assume first that \( q \equiv \eta \pmod{3} \). We have

\[
|G : L|_3 = \frac{(2m + k)!_3}{(m!)(k!)_3} \geq \frac{(2m)!_3}{(m!)_3} = (m + 1)_3(m + 2)_3 \ldots (2m)_3.
\]

Hence \( m \leq 1 \) in this case. If \( m = 1 \), then

\[
|G : L|_3 = \frac{(k + 2)!_3}{(k!)_3},
\]

so \( |G : L|_3 \) equals 1 if and only if \((k + 1)_3(k + 2)_3\) equals 1. Thus, if \( q \equiv \eta \pmod{3} \), then the equality \( |G : L|_3 = 1 \) can be true only if either \( m = 0 \), or \( m = 1 \) and \( k \equiv 0 \).
(mod 3). Furthermore
\[
|G : L_2| = (q + \eta_2^{\lfloor k/2 \rfloor}) \frac{(m + \lfloor k/2 \rfloor)!}{(k!)^2} \geq (q + \eta_2^{\lfloor k/2 \rfloor}) \frac{(\lfloor k/2 \rfloor)!}{(k!)^2}.
\]

Now
\[
(k!)^2 = 2^{[k/2] + [k/2]^2 + \ldots} < 2^{k/2 + k/2^2 + \ldots} = 2^k.
\]

Since we have a strict inequality, it follows that $(k!)^2 \leq 2^{k-1}$. Suppose $q \equiv -\eta \pmod{3}$ (mod 4). Then $(q + \eta)^2 \geq 4$ and
\[
1 = |G : L_2| \geq 2^{[k/2]} \frac{([k/2]!)}{2^{k-1}},
\]
whence $k \leq 1$. The case $m = 1$ and $k = 1$ is not possible, since $k \equiv 0 \pmod{3}$ for $m = 1$. Therefore, if $q \equiv \eta \pmod{3}$, then either statement (a) of the lemma holds or $q \equiv \eta \pmod{3}$ and $m \leq 1$. If $q \equiv \eta \pmod{4}$ and $m = 0$, then we obtain statement (b) of the lemma. Assume that $q \equiv \eta \pmod{4}$ and $m = 1$. Then Corollary 2.8 implies that $|G_1| = (q - \eta_1^{k+2} \cdot (k + 2)!)^2$, and $|L_2| = (q - \eta_2^{k+2} \cdot 2 \cdot (k!)^2$. Since $|G : L_2| = 1$, it follows that $(k + 1)! \equiv 1 \pmod{2}$, hence $k(k - 1) \equiv 0 \pmod{4}$, and statement (e) of the lemma holds.

Now assume that $q \equiv -\eta \pmod{3}$. Then
\[
|G : L_3| = (q + \eta_3^{\lfloor k/2 \rfloor}) \frac{(m + \lfloor k/2 \rfloor)!}{(m!)^3 (k!)^3}.
\]

We have that
\[
(k!)^3 = 3^{[k/3] + [k/3^2] + \ldots} < 3^{k/3 + k/3^2 + \ldots} = 3^{k/2},
\]
hence $(k!)^3 \leq 3^{[k/2]}$. Now $(q + \eta_3^{\lfloor k/2 \rfloor}) \geq 3^{[k/2]}$, so
\[
|G : L_3| \geq \frac{(m + \lfloor k/2 \rfloor)!}{(m!)^3}.
\]

Since $|G : L|$ is not divisible by 3, we have $\lfloor k/2 \rfloor \leq 2$, i.e., $k \leq 5$. If $k \leq 1$, then statement (c) of the lemma holds. Suppose $k \in \{2, 3, 4, 5\}$. For $k = 2$ we have
\[
|G : L_3| = (q + \eta_3) \frac{(m + 1)!}{(m!)^3} \geq 3,
\]
for $k = 4, 5$ we have
\[
|G : L_3| = (q + \eta_3)^2 \frac{(m + 2)!}{3(m!)^3} \geq 3,
\]
which is impossible. If $k = 3$, then
\[
|G : L_3| = \frac{(q + \eta_3)(m + 1)!}{3(m!)^3},
\]

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so \( m \not\equiv -1 \pmod{3} \) and \( (q + \eta)_3 = 3 \). Moreover

\[
|G : L|_2 = (q + \eta)_2 \frac{(m + 1)!}{2(m!)_2} = \frac{(q + \eta)_2}{2}(m + 1)_2,
\]

so \( m \) is even, \( q \equiv \eta \pmod{4} \) and statement (d) holds. \( \square \)

**Lemma 4.3** Assume \( G = \text{SL}_n^\eta(q) \) is a special linear or unitary group with the base field \( \mathbb{F}_q \) of characteristic \( p \) and \( n \geq 2 \). Let \( \pi \) be a set of primes such that \( 2, 3 \in \pi \) and \( p \not\in \pi \). Then the following statements hold:

(A) Suppose \( G \in E_\pi \), and \( H \) is a \( \pi \)-Hall subgroup of \( G \). Then for \( G, H, \) and \( \pi \) one of the following statements holds:

(a) \( n = 2 \) and one of the statements (a)–(d) of Lemma 3.1 holds.

(b) either \( q \equiv \eta \pmod{12} \), or \( n = 3 \) and \( q \equiv \eta \pmod{4} \), \( \text{Sym}_n \) satisfies \( E_\pi \), \( \pi \cap \text{Sym}(G) \subseteq \pi(q - \eta) \cup \pi(n!) \) and if \( r \in (\pi \cap \pi(n!)) \setminus \pi(q - \eta) \), then \( |G|_r = |\text{Sym}_n|_r \).

The subgroup \( H \) is included in

\[
M = L \cap G \approx Z^{n-1} \cdot \text{Sym}_n,
\]

where \( L = Z \cap \text{Sym}_n \leq \text{GL}_n^\eta(q) \) and \( Z = \text{GL}_n^\eta(q) \) is a cyclic group of order \( q - \eta \). All \( \pi \)-Hall subgroups of this type are conjugate in \( G \).

(c) \( n = 2m + k \), where \( k \in \{0, 1\} \), \( m \geq 1 \), \( q \equiv -\eta \pmod{3} \), \( \pi \cap \text{Sym}(G) \subseteq \pi(q^2 - 1) \), the groups \( \text{Sym}_m \) and \( \text{GL}_2^\eta(q) \) satisfy \( E_\pi \).\(^5\) The subgroup \( H \) is included in

\[
M = L \cap G \approx (\text{GL}_2^\eta(q) \circ \cdots \circ \text{GL}_2^\eta(q)) \cdot \text{Sym}_m \circ Z,
\]

where \( L = \text{GL}_2^\eta(q) \circ \text{Sym}_m \times Z \leq \text{GL}_n^\eta(q) \) and \( Z \) is a cyclic group of order \( q - \eta \) for \( k = 1 \) and \( Z \) is trivial for \( k = 0 \). The subgroup \( H \) acting by conjugation on the set of factors in the central product

\[
\text{GL}_2^\eta(q) \circ \cdots \circ \text{GL}_2^\eta(q)
\]

\( m \) times

has at most two orbits. The intersection of \( H \) with each factor \( \text{GL}_2^\eta(q) \) in (2) is a \( \pi \)-Hall subgroup in \( \text{GL}_n^\eta(q) \). The intersections with the factors from the same orbit all satisfy the same statement (a) or (b) of Lemma 3.2. Two \( \pi \)-Hall subgroups of \( M \) are conjugate in \( G \) if and only if they are conjugate in \( M \). Moreover \( M \) possesses one, two, or four classes of conjugate \( \pi \)-Hall subgroups, while all subgroups \( M \) are conjugate in \( G \).

(d) \( n = 4 \), \( \pi \cap \pi(G) = \{2, 3, 5\} \), \( q \equiv 5\eta \pmod{8} \), \( (q + \eta)_3 = 3 \), and \( (q^2 + 1)_3 = 5 \). The subgroup \( H \) is isomorphic to \( 4 \cdot 2^4 \cdot \text{Alt}_5 \). \( G \) possesses exactly two classes of conjugate \( \pi \)-Hall subgroups of this type and \( \text{GL}_n^\eta(q) \) interchanges these classes.

---

\(^5\) Notice that, in view of Lemma 3.1, the conditions \( \text{GL}_2^\eta(q) \in E_\pi \) and \( q \equiv -\eta \pmod{3} \) imply that \( q \equiv -\eta \pmod{r} \) for every odd \( r \in \pi(q^2 - 1) \cap \pi \).
(e) $n = 11$, $\pi \cap \pi(G) = \{2, 3\}$, $(q^2 - 1)_{(2,3)} = 24$, $q \equiv -\eta \pmod{3}$, and $q \equiv \eta \pmod{4}$. The subgroup $H$ is included in a subgroup $M = L \cap G$, where $L$ is a subgroup of $G$ of type $\left((GL^2_n(q) \wr \text{Sym}_4) \perp (GL^4_n(q) \wr \text{Sym}_3)\right)$, and
\[
H = (((Z \circ 2 \cdot \text{Sym}_4) \wr \text{Sym}_4) \times (Z \wr \text{Sym}_3)) \cap G,
\]
where $Z$ is a Sylow 2-subgroup of a cyclic group of order $q - \eta$. All $\pi$-Hall subgroups of this type are conjugate in $G$.

(B) Conversely, if the conditions on $\pi$, $n$, $\eta$, and $q$ in one of statements (a)–(e) are satisfied, then $G \in E_\pi$.

(C) If $G \in E_\pi$, then $k_\pi(G) \in \{1, 2, 3, 4\}$.

(D) If $G \in E_\pi$ and $H \in \text{Hall}_4(G)$, then $G$ possesses a $\pi$-subgroup nonconjugate to a subgroup of $H$, in particular, $G$ does not satisfy $D_\pi$.

Proof. (A) Assume that $G \in E_\pi$. In view of Lemma 3.1, we may assume that $n > 2$. By Lemmas 4.1 and 4.2, it follows that either statement (d) of the lemma holds, or every $\pi$-Hall subgroup $H$ of $G$ is included in $M$, where $M$ is the stabilizer of an $H$-invariant decomposition $V = V_1 \perp \cdots \perp V_m \perp U_1 \perp \cdots \perp U_k$ into a direct sum of pairwise orthogonal subspaces such that $\dim(V_i) = 2$ for $i = 1, \ldots, m$ and $\dim(U_j) = 1$ for $j = 1, \ldots, k$, and the decomposition cannot be refined. Below we denote by $T$ the intersection
\[
(GL^2_n(V_1) \times \cdots \times GL^2_n(V_m) \times GL^2_n(U_1) \times \cdots \times GL^2_n(U_k)) \cap G
\]
and by $\rho_i$ the natural projection $T \to GL^2_n(V_i)$. Notice that Lemma 2.11 implies the equality $T^\rho_i = GL^2_n(V_i)$ for all $i$, since $n > 2$. Moreover one of the statements (b)–(e) of Lemma 4.2 holds. Now we consider these statements separately.

Suppose statement (b) of Lemma 4.2 holds, i.e., $m = 0$ (so $k = n$) and $q \equiv \eta \pmod{12}$. In this case, $M = T : \text{Sym}_n$, and
\[
T = (q - \eta) \times \cdots \times (q - \eta).
\]

Since $M$ includes a $\pi$-Hall subgroup $H$ of $G$, we have $|G : M|_r = 1$. If $r \in \pi(q - \eta)$, then Lemma 2.3 (Corollary 2.8, if $r = 2$) implies that $|G : M|_r = 1$. If $r \notin \pi(q - \eta)$, then $r \in \pi(\text{Sym}_n)$ and the identity $|G : M|_r = 1$ holds if and only if $|G|_r = |\text{Sym}_n|_r$. Thus for every $r \in \pi \cap \pi(G) \setminus \pi(q - \eta)$ the identity $|G|_r = |\text{Sym}_n|_r$ holds. Lemma 2.11(a) implies that $HT/T$ is a $\pi$-Hall subgroup of $M/T \cong \text{Sym}_n$, hence $\text{Sym}_n \in E_\pi$. Moreover by [19], Tables 3.5A and 3.5B it follows that all subgroups of $G$ stabilizing a decomposition into direct orthogonal sum of 1-dimension subspaces are conjugate in $G$. By Lemma 2.3, we have $\text{Sym}_n \in C_\pi$, hence, by Lemma 2.11(f), $M \in C_\pi$ and all $\pi$-Hall subgroups of this type are conjugate in $G$. Thus statement (b) of Lemma 4.3 holds.

Suppose statement (c) of Lemma 4.2 holds, i.e., $k \leq 1$ (so $m > 0$) and $q \equiv -\eta \pmod{3}$. In this case, $M = T : \text{Sym}_m$. Since $M$ includes a $\pi$-Hall subgroup $H$ of
By Lemma 2.3, it follows that \( k \) satisfies statement (b) of Lemma 3.2. If \((\pi \cap (\pi (q^2 - 1)) \cap \pi (m!)) \neq 1\), then \( |G : M|_r > 1 \). Hence \( \pi \cap (\pi (G) \subseteq \pi (q^2 - 1) \cup \pi (m!)) \). Lemma 2.5 implies that if \( \pi \cap (\pi (G) \subseteq \pi (q^2 - 1) \cup \pi (m!)) \), then \( |G : M|_r > 1 \). Hence \( \pi \cap \pi (G) \subseteq \pi (q^2 - 1) \cup \pi (m!)) \). By Lemma 2.1(a), we obtain that \( HT/T \) is a \( \pi \)-Hall subgroup of \( M/T \simeq \text{Sym}_m \), while \( H \cap T \) is a \( \pi \)-Hall subgroup of \( T \). Hence \((H \cap T)^{\pi_i} \) is a \( \pi \)-Hall subgroup of \( \text{GL}^\pi (V_i) \approx \text{GL}_2^\pi (q) \) and \( \text{GL}_2^\pi (q) \in \mathcal{E}_\pi \). By Lemma 2.7(e), it follows that if \( i, j \) are in the same \((HT/T)\)-orbit, then \((H \cap T)^{\pi_i} \) in \( \text{GL}^\pi (V_i) \) and \((H \cap T)^{\pi_j} \) in \( \text{GL}^\pi (V_j) \) satisfy the same statement of Lemma 3.2. In view of Lemma 3.3, \( HT/T \) has at most two orbits. Lemma 3.2 implies that \( \text{GL}_2^\pi (q) \) possesses at most two classes of conjugate \( \pi \)-Hall subgroups. By Lemma 2.5, it follows that \( M \) possesses one, two or four classes of conjugate \( \pi \)-Hall subgroups. Now we show that if, for some \( g \in G \), the \( \pi \)-Hall subgroups \( H \) and \((H \cap T)^{\pi_i} \) are in \( M \), then \( g \in M \). Assume \( \Delta \) is the set of subgroups \( \text{SL}^\pi (V_i) \). By [I, Theorem 2] it follows that for every Sylow 2-subgroup \( S \) of \( M \) (and, so of \( G \) ) the identity \( \Delta = \text{Fun}(S) \) (in the notations of [I]) holds. It follows that \( g \in N_G(\Delta) = M \).

Suppose statement (d) of Lemma 4.2 holds, i.e., \( q \equiv -\eta \) (mod 3), \( q \equiv \eta \) (mod 4), \( m \) is even, \( k = 3 \), \((q + \eta), = 3 \), and \( m \neq -1 \) (mod 3). Assume first that \( m = 0 \). Then it is easy to see, that statement (b) of Lemma 4.3 holds in this case. Now assume that \( m > 0 \). Then \( M = T \). (\( \text{Sym}_m \times \text{Sym}_3 \)), and by Lemma 2.1(a), \( HT/T \) is a \( \pi \)-Hall subgroup of \( M/T \), while \( H \cap T \) is a \( \pi \)-Hall subgroup of \( T \), and \((H \cap T)^{\pi_i} \) is a \( \pi \)-Hall subgroup of \( \text{GL}^\pi (V_i) \approx \text{GL}_2^\pi (q) \). In particular, \( \text{GL}_2^\pi (q) \in \mathcal{E}_\pi \). Since \( q \equiv \eta \) (mod 4) and \( q \equiv -\eta \) (mod 3), Lemma 3.2 implies that \( \pi \cap \pi (\text{GL}_2^\pi (q)) = \{2, 3\} \) and \( \text{GL}_2^\pi (q) \in C_\pi \). By Lemma 2.5, it follows that for every \( r \in \pi (m!) \setminus \pi (q^2 - 1) \) the inequality \( |G : M|_r > 1 \) holds. So \( \pi \cap \pi (G) = \{2, 3\} \). Since \( HT/T \) is a \( \pi \)-Hall subgroup of \( \text{Sym}_m \times \text{Sym}_3 \), we obtain that \( \text{Sym}_m \in \mathcal{E}_{\{2, 3\}} \). Lemma 2.3 and the conditions \( m \) is even and \( m \neq -1 \) (mod 3) imply that \( m = 4 \), in particular, \( \text{Sym}_m = \text{Sym}_4 \) is a \( \{2, 3\} \)-group. Thus, by Lemma 2.1(f), we obtain that \( M \in C_{\{2, 3\},} \), and statement (e) of the lemma holds.

Suppose statement (e) of Lemma 4.2 holds, i.e., \( q \equiv \eta \) (mod 12), \( m = 1 \), \( k = 0 \) (mod 3), \( k(k - 1) \equiv 0 \) (mod 4). Again, by Lemma 2.1(a), we obtain that \( HT/T \) is a \( \pi \)-Hall subgroup of \( M/T \simeq \text{Sym}_k \), \( H \cap T \) is a \( \pi \)-Hall subgroup of \( T \), and \((H \cap T)^{\pi_i} \) is a \( \pi \)-Hall subgroup of \( \text{GL}^\pi (V_i) \approx \text{GL}_2^\pi (q) \). So \((H \cap T)^{\pi_i} \) satisfies either statement (a), or statement (b) of Lemma 3.2. If \((H \cap T)^{\pi_i} \) satisfies statement (a) of Lemma 3.2, then \( V_1 \) possesses a decomposition \( V_1 = V_1'' \) into an orthogonal sum of 1-dimension subspaces and \((H \cap T)^{\pi_i} \) stabilizes this decomposition. Hence \( H \) stabilizes a decomposition \( V_1 = V_1'' \subseteq U_1 \subseteq \ldots \subseteq U_k \) into an orthogonal sum of 1-dimension subspaces, i.e., \( H \) satisfies statement (b) of Lemma 4.2. Since this case is already considered, we may assume that \((H \cap T)^{\pi_i} \) satisfies statement (b) of Lemma 3.2. In particular, \( \pi \cap \pi (q^2 - 1) = \{2, 3\} \) and \((q^2 - 1)_{\{2, 3\}} = 24 \). Since \( \pi (M) = \pi (q^2 - 1) \cup \pi (m!) \) and \( M \) contains a \( \pi \)-Hall subgroup of \( G \), we obtain that for every \( r \in \pi (\pi (G) \setminus \{2, 3\} \) the identity \( |G|_r = |\text{Sym}_k|_r \) holds. By Lemma 2.1(a), \( HT/T \) is a \( \pi \)-Hall subgroup of \( \text{Sym}_k \). The conditions \( k \equiv 0 \) (mod 3) and \( k(k - 1) \equiv 0 \) (mod 4) imply that \( k > 9 \). By Lemma 2.3, it follows that \( 5 \in \pi (\pi (G) \setminus \{5 \notin \pi (q^2 - 1) \}) \). By Lemma 2.6, we
Suppose statement (b) of the lemma holds. In this case, $|G| = (q^2 + 1)^{\frac{k+2}{4} \cdot \binom{k+2}{4}}
$. Since $|\text{Sym}_k| = 5 \cdot \left( \frac{k}{2} \right)^{k-1} < 5^{\frac{k}{2} + \frac{k}{2} + \cdots} = 5^k \leq (q^2 + 1)^{\frac{k+2}{4}}$
 (the last inequality is true, since $k(k - 1) \equiv 0 \pmod{4}$), we obtain a contradiction with $|G| = |\text{Sym}_k|$. Thus there does not exist a $\pi$-Hall subgroup satisfying statement (e) of Lemma 4.2.

(B) In view of Lemma 3.1, we may assume that $n > 2$. If statement (d) holds, then it is easy to check that $H$ is a $\pi$-Hall subgroup of $G$.

By [19], it follows that there always exists a subgroup $M$ with the structure described in statements (b), (c), and (e) of the lemma. Namely, $M$ is the stabilizer in $G$ of a decomposition

$$V = V_1 \perp \cdots \perp V_m \perp U_1 \perp \cdots \perp U_k$$

of the natural module $V$ of $G$ into a direct orthogonal sum of $m$ nondegenerate subspaces $V_i$ of dimension 2 and $k$ nondegenerate subspaces $U_j$ of dimension 1. Moreover $m = 0$ in case (b), $k = 1$ in case (c), $m = 4$ and $k = 3$ in case (e). By [19, Tables 3.5A, 3.5B, and 3.5C], it also follows that the stabilizers of decompositions with the same numbers $m$ and $k$ are conjugate in $G$.

Assume that one of statements (b), (c), and (e) of the lemma holds. The subgroup $M$ of $G$ is obtained as the intersection $L \cap G$, where $L$ is specified in the corresponding statements. In order to prove that $G \in E_\pi$, it is sufficient to show that $|\text{GL}_n^\pi(q) : L| = 1$ and $L \in E_\pi$. Indeed, the identity $|\text{GL}_n^\pi(q) : L| = 1$ implies that every $H_1 \in \text{Hall}_\pi(L)$ is a $\pi$-Hall subgroup of $\text{GL}_n^\pi(q)$, i.e., $\emptyset \neq \text{Hall}_\pi(L) \subseteq \text{Hall}_\pi(\text{GL}_n^\pi(q))$. Since $G$ is normal in $\text{GL}_n^\pi(q)$, Lemma 2.3(a) implies that $H = H_1 \cap G$ is a $\pi$-Hall subgroup of $G$. Notice also that the identity $|\text{GL}_n^\pi(q) : L| = 1$ and the condition $L \in E_\pi$ imply $|G : M| = 1$ and $M \in E_\pi$. Now we consider statements (b), (c), and (e) separately.

Suppose statement (b) of the lemma holds. In this case, $L = \text{GL}_n^\pi(q) \cdot \text{Sym}_\pi = (q^2 - 1)^{\frac{k+2}{4} \cdot \binom{k+2}{4}}$. Assume first that $q \equiv \eta \pmod{12}$. Then, for every $r \in \pi(q - \eta)$, Lemma 2.6 (Corollary 2.8 if $r = 2$) implies that $|\text{GL}_n^\pi(q) : L| = 1$. If $r \in (\pi \cap \pi(G)) \setminus \pi(q - \eta)$, then the condition $|G| = |\text{Sym}_\pi| \implies |\text{GL}_n^\pi(q) : L| = 1$. So, for every $r \in \pi$, the identity $|\text{GL}_n^\pi(q) : L| = 1$ holds. Hence $|\text{GL}_n^\pi(q) : L| = 1$. Now assume that $n = 3$, $q \equiv -\eta \pmod{3}$, and $q \equiv \eta \pmod{4}$. In this case, $3 \notin \pi(q - \eta)$ and the condition $|G| = |\text{Sym}_3| \implies |\text{GL}_3^\pi(q) : L| = 1$. For the remaining primes $r \in \pi \cap \pi(G)$, we have $r \in \pi(q - \eta)$ and Lemma 2.6 (Corollary 2.8 if $r = 2$) implies $|\text{GL}_3^\pi(q) : L| = 1$. Hence $|\text{GL}_3^\pi(q) : L| = 1$. Now we show that $L \in E_\pi$. Consider a $\pi$-Hall subgroup $\overline{H}_2$ of $\text{Sym}_n$ (this subgroup exists, since $\text{Sym}_n \in E_\pi$), let $H_2$ be its complete preimage under the natural homomorphism $\varphi : L \rightarrow \text{Sym}_n$. The kernel of $\varphi$ is an abelian
By Corollaries 2.7 and 2.8, we obtain that

$$T = \prod_{i=1}^{n} \GL_n^\eta(q) = (q - \eta) \times \prod_{i=1}^{n} (q - \eta),$$

and $H_2/T \cong \overline{H}_2$ is a $\pi$-group. By Lemma 2.1(e) we obtain that $H_2 \in E_\pi$. Since $|L : H_2|_\pi = |\Sym_n : \overline{H}_2|_\pi = 1$, it follows that $L \in E_\pi$.

Suppose statement (c) of the lemma holds. In this case, $L = (\GL_2^\eta(q) \cdot \Sym_m) \times Z$.

By using Lemma 2.6 (Corollary 2.8, if $r = 2$), we obtain that, for every $r \in \pi$, the identity $|\GL_2^\eta(q) : L_r| = 1$ holds. Hence $|\GL_2^\eta(q) : L|_\pi = 1$. Now we show that $L \in E_\pi$. Consider a $\pi$-Hall subgroup $\overline{H}_2$ of $\Sym_m$ (this subgroup exists, since $\Sym_m \in E_\pi$), let $H_2$ be its complete preimage in $L$ under the natural homomorphism $\varphi : L \rightarrow \Sym_m$. The kernel of $\varphi$ is the subgroup

$$T = \prod_{i=1}^{m} \GL_2^\eta(q) \times \prod_{i=1}^{N} \GL_2^\eta(q) \times Z,$$

and $H_2/T \cong \overline{H}_2$ is a $\pi$-group. Let $H_3$ be a $\pi$-Hall subgroup of $\GL_2^\eta(q)$ (this subgroup exists, since $\GL_2^\eta(q) \in E_\pi$) and let $H_4$ be a $\pi$-Hall subgroup of $Z$. Then

$$H_5 = H_3 \times \ldots \times H_3 \times H_4$$

is a $\pi$-Hall subgroup of $T$ and $H_5^T = H_5^{H_2}$. By Lemma 2.1(e), we obtain that $H_2 \in E_\pi$. Since $|L : H_2|_\pi = |\Sym_m : \overline{H}_2|_\pi = 1$, it follows that $L \in E_\pi$.

Suppose finally that statement (e) of the lemma holds. Then

$$L = (\GL_2^\eta(q) \cdot \Sym_4) \times (\GL_1^\eta(q) \cdot \Sym_3),$$

$q \equiv -\eta \pmod{3}$, $q \equiv \eta \pmod{4}$, $\pi \cap \pi(G) = \{2, 3\}$, and $n = 11$. We have

$$|L|_{\{2, 3\}} = (q - \eta)^2 \cdot (q + \eta)^4 \cdot 2^4 \cdot 24 \cdot (q - \eta)^2 \cdot 2 \cdot 3 = (q - \eta)^{11} \cdot (q + \eta)^{8} \cdot 2^8 \cdot 3^2.$$

By Corollaries 2.7 and 2.8, we obtain that

$$|\GL_1^\eta(q)|_{\{2, 3\}} = (q - \eta)^{11} \cdot (q + \eta)^{8} \cdot 2^8 \cdot 3.$$

The condition $(q^2 - 1)_{\{2, 3\}} = 24$ implies that $(q + \eta)_3 = 3$. Hence $|\GL_1^\eta(q) : L|_{\{2, 3\}} = 1$. Now we show that $L \in E_\pi$. The group $L$ includes a subgroup

$$H = \left(\left((q - \eta)_{\{2, 3\}} \cdot \Sym_4 \cdot \Sym_3 \right) \times \left((q - \eta)_{\{2, 3\}} \cdot \Sym_3 \right) \right),$$

which is a $\pi$-group. Since $(q^2 - 1)_{\{2, 3\}} = 24$ and $q \equiv -\eta \pmod{3}$, we have $|H|_{\{2, 3\}} = (4 \cdot 24)^4 \cdot 24 \cdot 3^6$, and $|L|_{\{2, 3\}} = (4 \cdot 24)^4 \cdot 24 \cdot 3^6 = 2^{30} \cdot 3^6$. Hence $H$ is a $\pi$-Hall subgroup of $L$.

Thus we proved that if one of the statements (a)–(e) of the lemma holds, then $G \in E_\pi$. 

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(C) Note that if statement (e) of the lemma holds, then there also exist two classes of conjugate π-Hall subgroups of G satisfying statement (c), while the remaining statements cannot be satisfied simultaneously. Therefore if G ∈ Eπ, then kπ(G) ∈ {1, 2, 3, 4}.

(D) Given a π-Hall subgroup H of G, it remains to prove that there exists a π-subgroup K of G such that K is not conjugate to a subgroup of H. If n = 2, then this statement is proven in Lemma 5.1(D). If statement (d) of the lemma holds, then G possesses two classes of π-Hall subgroups. If statement (e) of the lemma holds, then, as we noted above, kπ(G) = 3. So we may assume that either statement (b), or statement (c) of the lemma holds. If statement (b) holds then, in the proof of Lemma 6.1 from [28], it is shown that G possesses a π-subgroup K such that K is not isomorphic to a subgroup of H. Assume that statement (c) of the lemma holds. Sym4 is known to have an irreducible representation of degree 3 over a field of characteristic not equal to 2 and 3, while Alt4 is included in SL3(q) and Alt4 does not stabilize a decomposition into a sum of a 2-dimensional and a 1-dimensional subspaces. Hence if statement (c) holds, then we can take K to be equal to Alt4 acting irreducibly on a 3-dimensional subspace of the natural module of G. □

Lemma 4.4 Let G = Sp2n(q) be a symplectic group over a field Fp of characteristic p. Assume that π is a set of primes such that 2, 3 ∈ π and p ∉ π. Then the following statements hold:

(A) Suppose G ∈ Eπ and H ∈ Hallπ(G). Then both Symπn and SL2(q) satisfy Eπ and π ∩ π(G) ⊆ π(q2 − 1). Moreover, H is a π-Hall subgroup of

\[ M = \text{Sp}_2(q) \wr \text{Sym}_n \cong \left( \frac{\text{SL}_2(q) \times \cdots \times \text{SL}_2(q)}{n \text{ times}} \right) : \text{Sym}_n \subseteq G. \]

(B) Conversely, if both Symπn and SL2(q) satisfy Eπ and π ∩ π(G) ⊆ π(q2 − 1), then M ∈ Eπ and every π-Hall subgroup H of M is a π-Hall subgroup of G.

(C) π-Hall subgroups of M are conjugate in G if and only if they are conjugate in M, while all such subgroups M are conjugate in G. In particular, if G ∈ Eπ, then kπ(G) ∈ {1, 2, 3, 4, 9}.

(D) If G ∈ Eπ and H ∈ Hallπ(G), then G possesses a π-subgroup nonconjugate to a subgroup of H, in particular, G ∉ Dπ.

Proof. (A) Assume that G ∈ Eπ and H is a π-Hall subgroup of G. The statement that H is included in M follows from Lemma 4.1. By Lemma 2.4(a) we obtain that both Symπn and SL2(q) satisfy Eπ. Lemma 2.9 implies that for every r ∈ π(n!) \ π(q2 − 1) the inequality |G : M|r > 1 holds. So π ∩ π(G) ⊆ π(q2 − 1).

(B) Conversely, assume that both Symπn and SL2(q) satisfy Eπ and π ∩ π(G) ⊆ π(q2 − 1). In view of [19, Table 3.5C] we obtain that G possesses a subgroup M ∼ Sp2(q) \ Symπn and all such subgroups are conjugate in G. The condition π ∩ π(G) ⊆ π(q2 − 1) and Corollaries 2.6 and 2.8 imply that for every r ∈ π ∩ π(G) the equality |G : M|r =
1 holds. Let \( \overline{H}_1 \) be a \( \pi \)-Hall subgroup of \( \text{Sym}_n \) and \( H_1 \) be its complete preimage under the natural homomorphism \( M \to \text{Sym}_n \). Let \( H_2 \) be a \( \pi \)-Hall subgroup of \( \text{SL}_2(q) \) and

\[
H_3 = H_2 \times \ldots \times H_2 \leq \text{SL}_2(q) \times \ldots \times \text{SL}_2(q) = G_1 \times \ldots \times G_n = T \subseteq M.
\]

Then \( H_3^T = H_3^{H_1} \), so Lemma 2.1(a) implies that \( H_1 \) possesses a \( \pi \)-Hall subgroup \( H \). Since \( |M : H_1|_\pi = |\text{Sym}_n : \overline{H}_1|_\pi = 1 \) and \( |G : M|_\pi = 1 \), we obtain that \( H \) is a \( \pi \)-Hall subgroup of both \( G \) and \( M \).

(C) Lemma 2.3 (in the notations introduced in the proof of statement (B)) implies that \( H \) has \( t \leq 2 \) orbits on the set of factors \( \{G_1, \ldots, G_n\} \). By Lemma 3.1 we obtain that \( k_\pi(\text{SL}_2(q)) \in \{1, 2, 3\} \). Thus Lemma 2.5(b) implies that \( k_\pi(M) = k_\pi(\text{SL}_2(q))^t \in \{1, 2, 3, 4, 9\} \). We show that if for some \( g \in G \) the \( \pi \)-Hall subgroups \( H \) and \( H^g \) are in \( M \), then \( g \in M \), in particular, \( k_\pi(G) = k_\pi(M) \). Assume \( \Delta \) is the set of subgroups \( \text{SL}_2(q) \) in \( T \). By [1], Theorem 2, it follows that for every Sylow 2-subgroup \( S \) of \( M \) (and, hence of \( G \)) the identity \( \Delta = \text{Fun}(S) \) (in the notations of [1]) holds. It follows that \( g \in N_G(\Delta) = M \).

(D) Given a \( \pi \)-Hall subgroup \( H \) of \( G \) it remains to prove that there exists a \( \pi \)-subgroup \( K \) of \( G \) such that \( K \) is not conjugate to a subgroup of \( H \). If \( k_\pi(G) > 1 \) we have nothing to prove, so assume that \( k_\pi(G) = 1 \). Lemma 2.5 implies that \( k_\pi(\text{SL}_2(q)) = 1 \). In the proof of Lemma 3.1(D), we have shown that if \( k_\pi(\text{SL}_2(q)) = 1 \) then \( SL_2(q) \) possesses a \( \pi \)-subgroup \( X \) such that \( X \) is not isomorphic to a subgroup of \( H \cap G_1 \). It is clear that \( X \cap (HT/T) \) is a \( \pi \)-subgroup of \( M \) (hence, of \( G \)) and it is not isomorphic to a subgroup of \( H \).  

\[ \square \]

5 Hall subgroups in orthogonal groups of dimension at most 6

Recall that we denote by \( \text{O}^\eta_n(q) \) the general orthogonal group of degree \( n \) and of sign \( \eta \in \{-, +, -\} \) over \( \mathbb{F}_q \), while the symbol \( \text{GO}^\eta_n(q) \) denotes the group of similarities. Here \( o \) is an empty symbol, and we use it only if \( n \) is odd.

**Lemma 5.1** Let \( \pi \) be a set of primes with \( 2, 3 \in \pi \). Assume that \( G = \Omega^\eta_n(q) \), where \( q \) is a power of a prime \( p \not\in \pi \); and \( H \) is a \( \pi \)-Hall subgroup of \( G \). Suppose also that \( \varepsilon = \varepsilon(q) \). Then one of the following statements holds.

(a) \( n = 2, G \) is cyclic of order \( (q - \eta)/2 \), \( H \) is a unique \( \pi \)-Hall subgroup of \( G \).
(b) \( n = 3, \pi \cap \pi(G) \subseteq \pi(q - \varepsilon), H \) is a \( \pi \)-Hall subgroup in a dihedral subgroup \( D_{q-\varepsilon} \) of order \( q - \varepsilon \) of \( G \). The subgroup \( H \) stabilizes a decomposition \( V_1 \perp V_2 \) of \( V \) into an orthogonal sum of subspaces of dimension 1 and 2, respectively, with \( \eta(V_2) = \varepsilon \). All \( \pi \)-Hall subgroups of this type are conjugate in \( G \).
(c) $n = 3, \pi \cap \pi(G) = \{2, 3\}, (q^2 - 1)_{[2,3]} = 24, H \simeq \text{Alt}_4$. The subgroup $H$ stabilizes
the decomposition $V_1 \perp V_2 \perp V_3$ of $V$ into an orthogonal sum of 1-dimensional
subspaces, and all $\pi$-Hall subgroups of this type are conjugate in $G$.

(d) $n = 3, \pi \cap \pi(G) = \{2, 3\}, (q^2 - 1)_{[2,3]} = 48, H \simeq \text{Sym}_4$. The subgroup $H$ stabilizes
the decomposition $V_1 \perp V_2 \perp V_3$ of $V$ into an orthogonal sum of 1-dimensional
subspaces, there exist two classes of conjugate subgroups of this type, and $SO_3(q)$ interchanges these classes.

(e) $n = 3, \pi \cap \pi(G) = \{2, 3, 5\}, (q^2 - 1)_{[2,3,5]} = 120, H \simeq \text{Alt}_5$. The subgroup $H$ is
irreducible and primitive, there exist two classes of conjugate subgroups of this type, and $SO_3^+(q)$ interchanges these classes.

(f) $n = 4, \eta = +, \pi \cap \pi(G) \subseteq \pi(q - \varepsilon), H$ is a $\pi$-Hall subgroup in the central
product of two subgroups isomorphic to $2 \cdot D_{q-\varepsilon}$. The subgroup $H$ stabilizes a
decomposition $V_1 \perp V_2$ of $V$ into an orthogonal sum of 2-dimensional subspaces
with $\eta(V_1) = \eta(V_2) = \varepsilon$. All $\pi$-Hall subgroups of this type are conjugate in $G$.

(g) $n = 4, \eta = +, \pi \cap \pi(G) = \{2, 3\}, (q^2 - 1)_{[2,3]} = 24, H \simeq \text{SL}_2(3) \circ \text{SL}_2(3)$. All $\pi$-Hall
subgroups of this type are conjugate in $G$.

(h) $n = 4, \eta = +, \pi \cap \pi(G) = \{2, 3\}, (q - \varepsilon)_{[2,3]} = 12, H \simeq 2 \cdot D_{12} \circ \text{SL}_2(3)$. There exist
two classes of conjugate subgroups of this type, $SO_4^+(q)$ stabilizes these classes,
while $O_4^+(q)$ interchanges them.

(i) $n = 4, \eta = +, \pi \cap \pi(G) = \{2, 3\}, (q^2 - 1)_{[2,3]} = 48, H \simeq 2 \cdot \text{Sym}_4 \circ 2 \cdot \text{Sym}_4$.
There exist four classes of conjugate subgroups of this type and every element of
$SO_4^+(q) \setminus \Omega_4^+(q)$ induces an involution with cyclic structure $(ij)(kl)$ on the set of
these classes.

(j) $n = 4, \eta = +, \pi \cap \pi(G) = \{2, 3\}, (q - \varepsilon)_{[2,3]} = 24, H \simeq 2 \cdot D_{24} \circ 2 \cdot \text{Sym}_4$.
There exist four classes of conjugate subgroups of this type and each element of
$O_4^+(q) \setminus \Omega_4^+(q)$ induces an involution with cyclic structure $(ij)(kl)$ on the set of
these classes.

(k) $n = 4, \eta = +, \pi \cap \pi(G) = \{2, 3, 5\}, (q^2 - 1)_{[2,3,5]} = 120, H \simeq \text{SL}_2(5) \circ \text{SL}_2(5)$.
There exist four classes of conjugate subgroups of this type and every element of
$SO_4^+(q) \setminus \Omega_4^+(q)$ induces an involution with cyclic structure $(ij)(kl)$ on the set of
these classes.

(l) $n = 4, \eta = +, \pi \cap \pi(G) = \{2, 3, 5\}, (q - \varepsilon)_{[2,3,5]} = 60, H \simeq 2 \cdot D_{60} \circ \text{SL}_2(5)$.
There exist four classes of conjugate subgroups of this type and each element of
$O_4^+(q) \setminus \Omega_4^+(q)$ induces an involution with cyclic structure $(ij)(kl)$ on the set of
these classes.

(m) $n = 4, \eta = -, \pi \cap \pi(G) \subseteq \pi(q^2 - 1), H$ is a $\pi$-Hall subgroup in a dihedral
group $D_{q^2-1}$. The subgroup $H$ stabilizes a decomposition $V_1 \perp V_2$ of $V$ into an
orthogonal sum of 2-dimensional subspaces with $\eta(V_1) = +$ and $\eta(V_2) = -$. All
$\pi$-Hall subgroups of this type are conjugate in $G$.

(n) $n = 4, \eta = -, \pi \cap \pi(G) = \{2, 3\}, (q^2 - 1)_{[2,3]} = 24, H \simeq \text{Sym}_4$. The subgroup $H$
stabilizes a decomposition $V_1 \perp V_2 \perp V_3 \perp V_4$ of $V$ into an orthogonal sum of
1-dimensional subspaces. There exist two classes of conjugate $\pi$-Hall subgroups
of this type, invariant under $O_4^+(q)$, and $GO_4^+(q)$ interchanges these classes.

(o) $n = 5, \pi \cap \pi(G) \subseteq \pi(q - \varepsilon), H$ is isomorphic to an extension of a $\pi$-Hall subgroup
in the central product of two groups isomorphic to $2 \cdot D_{q-\varepsilon}$ by a group of order 2.
The subgroup $H$ stabilizes a decomposition $V_1 \perp V_2 \perp V_3$ of $V$ with $\dim(V_1) = 1$, $\dim(V_2) = \dim(V_3) = 2$, and $\eta(V_2) = \eta(V_3) = \varepsilon$. All $\pi$-Hall subgroups of this type are conjugate in $G$.

(p) $n = 5$, $\pi \cap \pi(G) = \{2, 3\}$, $(q^2 - 1)_{|2,3|} = 24$, $H \cong (2 \cdot \Alt_4 \circ 2 \cdot \Alt_4) \cdot 2$. The subgroup $H$ stabilizes a decomposition $V_1 \perp V_2$ of $V$ with $\dim(V_1) = 1$, $\dim(V_2) = 4$, $\eta(V_2) = +$, and all $\pi$-Hall subgroups of this type are conjugate in $G$.

(q) $n = 5$, $\pi \cap \pi(G) = \{2, 3\}$, $(q^2 - 1)_{|2,3|} = 48$, $H \cong (2 \cdot \Sym_4 \circ 2 \cdot \Sym_4) \cdot 2$. The subgroup $H$ stabilizes a decomposition $V_1 \perp V_2$ of $V$ with $\dim(V_1) = 1$, $\dim(V_2) = 4$, $\eta(V_2) = +$, there exist two classes of conjugate subgroups of this type, and $\SO_5(q)$ interchanges these classes.

(r) $n = 5$, $\pi \cap \pi(G) = \{2, 3, 5\}$, $(q^2 - 1)_{|2,3,5|} = 120$, $H \cong (\SL_2(5) \circ \SL_2(5)) \cdot 2$. The subgroup $H$ stabilizes a decomposition $V_1 \perp V_2$ of $V$ with $\dim(V_1) = 1$, $\dim(V_2) = 4$, $\eta(V_2) = +$, there exist two classes of conjugate subgroups of this type, and $\SO_5(q)$ interchanges these classes.

(s) $n = 6$, $\eta = \varepsilon$, $\pi \cap \pi(G) \subseteq \pi(q - \varepsilon) \cup \{3\}$ and, if $3 \not\in \pi(q - \varepsilon)$, then $(q + \varepsilon)_3 = 3$, $H$ is a $\pi$-Hall subgroup of a solvable group $\left((q - \varepsilon)^2 \times (q - \varepsilon)/2\right) \cdot \Sym_3$. The subgroup $H$ stabilizes a decomposition $V_1 \perp V_2 \perp V_3$ of $V$ into a sum of isometric 2-dimensional subspaces with $\eta(V_i) = \eta$ for $i = 1, 2, 3$. All $\pi$-Hall subgroups of this type are conjugate in $G$.

(t) $n = 6$, $\eta = -\varepsilon$, $\pi \cap \pi(G) \subseteq \pi(q - \varepsilon)$, $H$ is a $\pi$-Hall subgroup of a solvable group $\left((q - \varepsilon)^2 \times (q - \varepsilon)/2\right) \cdot 2$. The subgroup $H$ stabilizes a decomposition $V_1 \perp V_2 \perp V_3$ of $V$ into a sum of 2-dimensional subspaces with $\eta(V_1) = \eta(V_2) = \varepsilon$, $\eta(V_3) = -\varepsilon$. All $\pi$-Hall subgroups of this type are conjugate in $G$.

(u) $n = 6$, $q \equiv -\eta \pmod{3}$, $\pi \cap \pi(G) = \{2, 3\}$, $(q^2 - 1)_{|2,3|} = 24$, $H$ is isomorphic to $\left((q - \eta) \circ 2 \cdot \Sym_4 \circ 2 \cdot \Sym_4\right) \cdot 2$. The subgroup $H$ stabilizes a decomposition $V_1 \perp V_2$ of $V$ with $\dim(V_1) = 2$, $\dim(V_2) = 4$, $\eta(V_1) = \eta$, $\eta(V_2) = +$. All $\pi$-Hall subgroups of this type are conjugate in $G$.

(v) $n = 6$, $\eta = \varepsilon$, $q \equiv \pm 3 \pmod{8}$, $\pi \cap \pi(G) = \{2, 3, 5\}$, $(q^2 - 1)_{|2,3|} = 24$, $q^2 \equiv -1 \pmod{5}$, $q \equiv -\eta \pmod{3}$, $H \cong 2^3 \cdot \Alt_6$. The subgroup $H$ stabilizes decomposition $V = V_1 \perp V_2 \perp V_3 \perp V_4 \perp V_5 \perp V_6$ of $V$ into a sum of 1-dimensional subspaces. There exist two classes of $\pi$-Hall subgroups of this type, invariant under $\OO_6^+(q)$, and $\GO_6^+(q)$ interchanges these classes.

Proof. In the proof we use the known isomorphisms of orthogonal groups of small dimension and other groups (see [19], Proposition 2.9.1, for example). Without further references we also use the fact that $q$ is odd.

If $n = 2$, then the isomorphism $\Omega_2^+(q) \cong (q - \eta)/2$ implies statement (a) of the lemma.

If $n = 3$, then the isomorphisms $\Omega_3(q) \cong \PSL_3(q)$ and $\SO_3(q) \cong \PGL_2(q)$ together with Lemma [3] imply statements (b)–(e) (except for the decomposition part). Indeed, if $H$ is a $\pi$-Hall subgroup of $\Omega_3(q) \cong \PSL_2(q)$, then one of the following holds:
(1) \( \pi \) and \( H \) satisfy Lemma 3.1(a), so \( \pi \cap \pi(G) \subseteq \pi(q - \varepsilon) \) and \( H \) is a \( \pi \)-Hall subgroup in the dihedral subgroup \( D_{q-\varepsilon} \) of \( G \). Moreover all \( \pi \)-Hall subgroups of this type are conjugate in \( G \). So statement (b) of the lemma holds (except for the decomposition part).

(2) \( \pi \) and \( H \) satisfy Lemma 3.1(b), so \( \pi \cap \pi(G) = \{2, 3\} \), \( (q^2 - 1)_{(2,3)} = 24 \), and \( H \cong \text{Alt}_4 \). Moreover all \( \pi \)-Hall subgroups of this type are conjugate in \( G \). So statement (c) of the lemma holds (except for the decomposition part).

(3) \( \pi \) and \( H \) satisfy Lemma 3.1(c), so \( \pi \cap \pi(G) = \{2, 3\} \), \( (q^2 - 1)_{(2,3)} = 48 \), and \( H \cong \text{Sym}_4 \). Moreover there exist exactly two classes of conjugate subgroups of this type, and \( \text{SO}_3(q) \cong \text{PGL}_2(q) \) interchanges these classes. So statement (d) of the lemma holds (except for the decomposition part).

(4) \( \pi \) and \( H \) satisfy Lemma 3.1(d), so \( \pi \cap \pi(G) = \{2, 3, 5\} \), \( (q^2 - 1)_{(2,3)} = 120 \), and \( H \cong \text{Alt}_5 \). Moreover there exist exactly two classes of conjugate subgroups of this type, and \( \text{SO}_3(q) \cong \text{PGL}_2(q) \) interchanges these classes. So statement (e) of the lemma holds (except for the decomposition part).

Now we show the statements about decomposition in statements (b)–(e) of the lemma.

There exists a subgroup \( L = \left( \text{O}_1(q) \times \text{O}_2(q) \right) \cap \Omega_3(q) \). By Lemma 2.11, the subgroup \( L \) is isomorphic to a dihedral group of order \( q - \varepsilon \). Since all \( \pi \)-Hall subgroups satisfying statement (b) of the lemma are conjugate, it follows that \( H \leq L \), whence the decomposition in statement (b) of the lemma.

In view of [[19], Proposition 4.2.15], there exists a subgroup of type \( \text{O}_1(q) \wr S_3 \) of \( \Omega_3(q) \), and it is isomorphic to \( \text{Alt}_4 \) under the conditions of statement (c) of the lemma (in this case all subgroups of this type are conjugate), and to \( \text{Sym}_4 \) under the conditions of statement (d) of the lemma (in this case there exists two classes of subgroups of this type). It follows that there exist decompositions of \( V \) as in statements (c) and (d).

Suppose that \( H \cong \text{Alt}_5 \). The minimal degree of faithful complex representation of \( \text{Alt}_5 \) equals 3. Since \( (q, |H|) = 1 \), we obtain that the minimal degree of a nontrivial representation of \( \text{Alt}_5 \) over an algebraically closed field of characteristic \( p \) equals 3 as well. Hence the subgroup \( \text{Alt}_5 \) in statement (e) is absolutely irreducible and primitive.

Suppose \( n = 4 \) and \( \eta = + \). Then \( \Omega_4^+(q) \) is isomorphic to \( \text{SL}_2(q) \circ \text{SL}_2(q) \), \( \text{SO}_4^+(q) \) normalizes each factor and induces on it the group of all inner-diagonal automorphisms, while each element of \( \text{O}_4^+(q) \setminus \text{SO}_4^+(q) \) interchanges the factors. Thus statements (f)–(l) of the lemma (except for the decomposition of \( V \) in statement (f)) follows from Lemma 3.1. A \( \pi \)-Hall subgroup \( H \) in statement (f), up to conjugation, is included in \( \left( \text{O}_2^+(q) \times \text{O}_2^+(q) \right) \cap \Omega_4^+(q) \), whence the decomposition in statement (f).
Suppose $n = 4$ and $\eta = 4$. We have that $\Omega_4^4(q)$ is isomorphic to $\text{PSL}_2(q^2)$, while $\text{O}_4^-(q)$ is isomorphic to $2 \times \text{PSL}_2(q^2)$. 2 and induces on $\text{PSL}_2(q^2)$ the subgroup of $\text{Aut}(\text{PSL}_2(q^2))$ generated by the inner automorphisms and a field automorphism of order 2 [19, Proposition 2.9.1(v) and proof]. Since $((q^2)^2 - 1)^{1/3}$ divides 48, it follows that $H \cap \text{PSL}_2(q^2)$ is a $\pi$-Hall subgroup of $\text{PSL}_2(q^2)$ satisfying either (a) or (c) of Lemma 3.1. If $H \cap \text{PSL}_2(q^2)$ satisfies Lemma 3.1(a), i.e., $H \cap \text{PSL}_2(q^2)$ is a $\pi$-Hall subgroup in $D_{q-1}$, then $\text{PSL}_2(q^2) \cong (H \cap \text{PSL}_2(q^2))^{\text{Aut}(\text{PSL}_2(q^2))}$, so by Lemma 2.1(e) we obtain that statement (m) of the lemma (except the decomposition) holds. If $H \cap \text{PSL}_2(q^2)$ satisfies Lemma 3.1(c), i.e., $H \cap \text{PSL}_2(q^2) \cong \text{Sym}_4$, then $\text{PSL}_2(q^2)$ possesses two classes of conjugate subgroups of this type. Moreover $H \cap \text{PSL}_2(q^2)$ is a subgroup of symplectic type (type $C_6$ in terms of [19]) of $\text{PSL}_2(q^2)$ and [19]. Tables 3.5A and 3.5.G imply that a field automorphism leaves these classes invariant. So Lemma 2.1(e) implies statement (n) of the lemma (except the decomposition). A subgroup $H$ in statement (m) is included in $(\text{O}_2^+(q) \times \text{O}_2^-(q)) \cap \Omega_4^4(q)$, whence the decomposition in this statement. A subgroup $H$ in statement (n) is included in $(\text{O}_1(q) \times (\text{O}_1(q) \rtimes \text{Sym}_3)) \cap \Omega_4^4(q)$, whence the decomposition in statement (n).

Suppose $n = 5$, then $\Omega_5^5(q) \cong \text{PSp}_4(q)$. By Lemma 4.4, it follows that $M$ is included in

$$L = (\text{O}_1(q) \times \text{O}_4^+(q)) \cap \Omega_5^5(q) \cong (\text{Sp}_2(q) \circ \text{Sp}_2(q)) \cdot \text{Sym}_2$$

$$\cong (\text{SL}_2(q) \circ \text{SL}_2(q)) \cdot \text{Sym}_2$$

and statements (o)–(r) (except the statements about classes in (q) and (r)) of the lemma follows from statements (f)–(l) of the lemma. It is easy to see, that two classes of conjugate $\pi$-Hall subgroups in statements (i) and (k) are invariant under a graph automorphism, while the remaining two classes are interchanged by this automorphism. Hence the statement on the number of classes follows from Lemmas 2.1(e) and 2.11.

Suppose $n = 6$. Then $\Omega_6^6(q) \cong \text{PSL}_4^6(q)$. Statements (s) and (t) (except the decomposition of $V$) follow from Lemma 4.3(b). Statement (u) (except the decomposition of $V$) follows from Lemma 4.3(c). Statement (v) (except the decomposition of $V$) follows from Lemma 4.3(d). A subgroup $H$, satisfying statement (s), is included in $\left(\text{O}_2^+(q) \rtimes \text{Sym}_3\right) \cap \Omega_6^6(q)$, whence the decomposition of $V$ in statement (s). A subgroup $H$, satisfying statement (t), is included in $\left(\text{O}_2^+(q) \times \text{O}_2^-(q) \times \text{O}_2^6(q)\right) \cap \Omega_6^6(q)$, whence the decomposition of $V$ in statement (t). A subgroup $H$, satisfying (u), is included in $(\text{O}_1(q) \times \text{O}_4^+(q)) \cap \Omega_6^6(q)$, whence the decomposition of $V$ in statement (u). Finally, by [19, Proposition 4.2.15] it follows that $H$ satisfying (v) exists only if $\eta = e$, and $H$ is included in $(\text{O}_1(q) \rtimes S_6) \cap \Omega_6^6(q)$, whence the decomposition of $V$ in statement (v).
6 Hall subgroups in orthogonal groups

In this section we say that a hypothesis ($\star$) is true, if the following statements hold:

1. $V$ is a vector space with a nondegenerate symmetric bilinear form over a field $\mathbb{F}_q$ of odd characteristic $p$;
2. $\varepsilon = \varepsilon(q)$;
3. $\pi$ is a set of primes such that $2, 3 \in \pi$, and $p \not\in \pi$;
4. $\Omega(V) \subseteq G \subseteq I(V)$;
5. $H$ is a $\pi$-Hall subgroup of $G$;
6. $V, G,$ and $H$ do not satisfy the following statements:
   
   (a) $\dim(V) = 7$, $G = \Omega(V)$, $H \cong \Omega_7(2)$;
   (b) $\dim(V) = 8$, $\eta(V) = +$, $G = \Omega(V)$, $H \cong 2 \cdot \Omega_8^+(2)$;
   (c) $\dim(V) = 9$, $G = \Omega(V)$, $H$ stabilizes a decomposition $V = U \perp W$, $\dim(U) = 8$, $\eta(U) = +$, $\dim(W) = 1$, $H \subseteq (I(U) \times I(W)) \cap G$ and the projection of $H$ on $I(U)$ is isomorphic to $2 \cdot \Omega_8^+(2)$.

**Lemma 6.1** Assume that ($\star$) holds. Then there exists an $H$-invariant decomposition

$$V = V_1 \perp \ldots \perp V_l$$

of $V$ into an orthogonal sum of nondegenerate subspaces such that $\dim(V_i) \leq 4$ for all $i = 1, \ldots, l$.

**Proof.** If $n \leq 4$, we have nothing to prove. If $n = 5$ or $n = 6$, then the claim follows from Lemma 5.1. So we may assume $n \geq 7$. Since $p \not\in \pi$, we obtain that $H \neq G$. Hence $H$ is included in a maximal subgroup $M$ of odd index of $G$. In view of [20] we obtain that one of the following statements holds:

1. $\Omega(V_0) \subseteq M \subseteq I(V_0)$, where $I(V_0)$ is a group of the same type as $I(V)$, $\dim(V_0) = \dim(V)$, and the base field $\mathbb{F}_{q_0}$ of $V_0$ is a proper subfield of $\mathbb{F}_q$.
2. $\dim(V) = 7$, $M \cap \Omega(V) = \Omega_7(2)$, $q$ is prime and $q \equiv 3 \pmod{8}$.
3. $\dim(V) = 8$, $\eta = +$, $M \cap \Omega(V) = 2 \cdot \Omega_8^+(2)$, $q$ is prime and $q \equiv \pm 3 \pmod{8}$.
4. $M$ is the stabilizer of a non-singular subspace.
5. $M$ is the stabilizer of a decomposition $V = V_1 \perp \ldots \perp V_k$ with all $V_i$ isometric.

Now we consider all statements separately.

1. As in the proof of Lemma 4.1 we obtain the statement of the lemma by induction on $q$.

2. (2), (3) By [22, Theorem 1.2] it follows that both $\Omega_7(2)$ and $\Omega_8^+(2)$ do not possess proper $\pi$-Hall subgroups with $2, 3 \in \pi$. Hence $H \cap \Omega(V) = M \cap \Omega(V)$, i.e., ($\star$) is not satisfied, a contradiction.
(4) In this case $M$ stabilizes a decomposition $U \perp W$ of $V$, hence $M = (I(U) \times I(W)) \cap G$. Without loss of generality we may assume that $\dim(U) = k \geq \dim(W) = m$. Denote by $\rho_U$ and $\rho_W$ the natural projections of $M$ into $I(U)$ and $I(W)$, respectively. By Lemma 2.1(a) it follows that $H^{\rho_U} \in \text{Hall}_k(M^{\rho_U})$ and $H^{\rho_W} \in \text{Hall}_m(M^{\rho_W})$. If $m$ is greater than 1, then Lemma 2.11 implies that $M^{\rho_U} = I(U)$ and $M^{\rho_W} = I(W)$. Hence Lemma 2.13 implies that for $U$, $M^{\rho_U}$, $H^{\rho_U}$ and $W$, $M^{\rho_W}$, $H^{\rho_W}$ condition ($\star$) holds and we obtain the statement of the lemma by induction on dimension. If $m = 1$ and $k \not\in \{7, 8, 9\}$, then again for $U$, $M^{\rho_U}$, $H^{\rho_U}$ and $W$, $M^{\rho_W}$, $H^{\rho_W}$ hypothesis ($\star$) holds and the statement of the lemma follows by induction. If $m = 1$ and $k \in \{7, 8, 9\}$, then Theorem 2.10(e) implies that $k = 8$ and $n = 9$. In this case either condition ($\star$) is not true, or we obtain the statement of the lemma by induction.

(5) Since we already considered statement (4), we may assume that $M$ is irreducible. Then

$$M = ((I(V_1) \times \ldots \times I(V_k)) : \text{Sym}_k) \cap G$$

and, by Lemma 2.11 we have $M^{\rho_i} = I(V_i)$ for $i = 1, \ldots, k$. By using Lemma 2.13, we obtain the statement of the lemma as in the proof of Lemma 4.4. □

Assume that ($\star$) holds. By Lemma 6.1 there exists an $H$-invariant decomposition

$$V = V_1 \perp \cdots \perp V_s,$$

which cannot be refined. All subspaces $V_i$ in decomposition (3) are nondegenerate and $\dim(V_i) \leq 4$. For $k \in \{1, 2, 3, 4\}$ and $\delta \in \{+, -, 0\}$, we denote by $V(k, \delta)$ the sum of $V_i$ with $\dim(V_i) = k$ and $\eta(V_i) = \delta$, and by $d(k, \delta)$ the number of such subspaces $V_i$. It is clear that $\dim(V(k, \delta)) = kd(k, \delta)$, subspace $V(k, \delta)$ is $H$-invariant for every $k$ and $\delta$, and

$$V = \sum_{(k, \delta)} V(k, \delta) = V(1, 0) \perp V(2, 0) \perp V(2, -\varepsilon) \perp V(3, 0) \perp V(4, 0) \perp V(4, -).$$

For brevity we write $\eta(k, \delta)$ and $D(k, \delta)$ instead of $\eta(V(k, \delta))$ and $D(V(k, \delta))$, respectively. If $k$ is even then $\eta(k, \delta) = \delta^{d(k, \delta)}$. Set also $I(k, \delta) = I(V(k, \delta))$, $\Omega(k, \delta) = \Omega(V(k, \delta))$. Thus $H$ is included in a subgroup $G_0$ of $G$ of type

$$G_0 = (I(1, 0) \times I(2, 0) \times I(2, -\varepsilon) \times I(3, 0) \times I(4, 0) \times I(4, -)) \cap G.$$ 

Denote by $G(k, \delta)$ and $H(k, \delta)$ the projection of $G_0$ and $H$ on $I(k, \delta)$, respectively. Then $\Omega(k, \delta) \leq G(k, \delta)$ and by Lemma 2.1(a) it follows that $H(k, \delta)$ is a $\pi$-Hall subgroup of $G(k, \delta)$.

We also set

$$U = \sum_{(k, \delta) \neq (2, 0)} V(k, \delta),$$

and $W = V(2, 0)$. So $V = U \perp W$. Now we show that the dimension of $U$ is bounded. Clearly it is enough to prove that $d(k, \delta)$ is bounded for $(k, \delta) \neq (2, 0)$. 

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Since the decomposition (3) cannot be refined, Lemma 5.1 implies that 
\( d(4, -) = 0 \). By Lemma 5.1 we obtain that a \( \pi \)-Hall subgroup \( H \) of \( \Omega_3(q) \) is irreducible if 
and only if \( H \cong \text{Alt}_5 \). By using Lemmas 2.1(e), 2.1(f), and 3.1(e), we obtain that 
\( d(3, \circ) = 0 \). Thus we obtain

**Lemma 6.2** In the above notation we have \( d(4, -) = d(3, \circ) = 0 \).

Now we need to consider the possible values for \( d(1, \circ), d(2, -\varepsilon), \) and \( d(4, +) \). First assume that \( d(1, \circ) = t \). Note that [19, Tables 3.5D, 3.5E, and 3.5F] imply that 
if \( t \) is odd or \( D(1, \circ) = \square \) then \( H(1, \circ) \) is included in \( \Omega_1(q) \triangleleft \text{Sym}_t \) (the stabilizer 
of the decomposition of \( V(1, \circ) \) into an orthogonal direct sum of isometric one-
dimensional subspaces), while if \( t \) is even and \( D(1, \circ) = \heartsuit \), then \( H(1, \circ) \) is included 
in a subgroup of type \( \Omega_1(q) \perp \Omega_{t-1}(q) \), and the projection of \( H(1, \circ) \) on \( \Omega_{t-1}(q) \) is 
included in \( \Omega_1(q) \triangleleft \text{Sym}_{t-1}(q) \). In order to distinguish between these cases for \( t \) even, 
we write respectively \( d(1, \circ) = t_{\square} \) and \( d(1, \circ) = t_{\heartsuit} \).

**Lemma 6.3** Suppose (\( \ast \)) holds and the decomposition (3) of \( V \) cannot be refined. 
In the above-introduced notation the following statements hold:

(a) \( d(1, \circ) \in \{0, 1, 2_{\square}, 2_{\heartsuit}, 3, 4_{\heartsuit}, 6_{\square}\} \);
(b) \( d(2, -\varepsilon) \in \{0, 1\} \);
(c) \( d(4, +) \in \{0, 1\} \);
(d) if \( d(1, \circ) = 6_{\square} \), then \( \dim(V) = 6 \), and \( \eta(V) = \varepsilon \);
(e) if \( d(1, \circ) = 4_{\heartsuit} \), then \( \eta(1, \circ) = - \) and \( d(2, -\varepsilon) = d(4, +) = 0 \);
(f) if \( d(1, \circ) = 3 \), then \( d(2, -\varepsilon) = d(4, +) = 0 \);
(g) if \( d(1, \circ) = 2_{\heartsuit} \), then \( \eta(1, \circ) = -\varepsilon \), and \( d(2, -\varepsilon) = 0 \);
(h) if \( d(1, \circ) = 1 \), then \( d(2, -\varepsilon) = 0 \);
(i) if \( d(1, \circ) = 2_{\square} \), then either \( \dim(V) = 2 \) and \( \eta(V) = \varepsilon \), or \( \dim(V) = 4 \) and
\( \eta(V) = -\varepsilon \), or \( \dim(V) = 6 \) and \( \eta(V) = \varepsilon \).

**Proof.** Let \( M \) be the stabilizer in \( G(k, \delta) \) of the decomposition of \( V(k, \delta) \) into the 
\( H \)-invariant sum of \( k \)-dimensional subspaces. Set \( \Omega = \Omega(k, \delta), M_\Omega = \Omega \cap M, \) 
\( L = \Omega/Z(\Omega), \) and \( M_L = M_\Omega/Z(\Omega) \). Since \( M \) includes a \( \pi \)-Hall subgroup \( H(k, \delta) \) of 
\( G(k, \delta), \) Lemma 2.1(a) implies that \( M_L \) includes a \( \pi \)-Hall subgroup of \( L \). In particular, 
\( |L : M_L| \) is not divisible by 2 and 3.

(a) Assume \((k, \delta) = (1, \circ)\). Denote \( d(1, \circ) \) by \( t \).

Consider the case \( t \) odd first. We set \( m = (t - 1)/2 \) for brevity. Then 
\[
|L|_3 = \prod_{i=1}^{m} (q^{2i} − 1)_3 = (q^2 − 1)_3^m (m!)_3 \geq 3^m (m!)_3,
\]
\[
|M_L|_3 = (t!)_3 = 3^{(t!/3)!+t!/3^2!+...} < 3^{(t!/3!+t!/3^2+...)} = 3^{t/2}.
\]
Since \( t \) is odd, we obtain \( |M_L|_3 \leq 3^{(t-1)/2} = 3^m \). So \( |L : M_L| \geq (m!)_3 \), whence \( m < 3 \) 
and \( t < 7 \). If \( t = 5 \), then \( |L|_3 \geq 3^2 \), while \( |M_L|_3 = 3 \), a contradiction. Thus, if \( t \) is odd,
then $t \in \{1, 3\}$.

Now assume that $t$ is even. If $D(1, \circ) = \varnothing$, then case $t$ odd and Lemma [5,1] imply that $t \in \{2_\perp, 4_\perp\}$. Thus it remains to consider the case $D(1, \circ) = \varnothing$. Notice that $\eta(1, \circ) = \epsilon^{t/2}$ in this case. Set $m = t/2 - 1$. We have

$$|L|_3 = (q^{m+1} - \epsilon^{m+1})_3 \prod_{i=1}^{m} (q^{2i} - 1)_3 = (q^{m+1} - \epsilon^{m+1})_3(q^2 - 1)_3(m!)_3 \geq 3^m(m!)_3,$$

$$|M_L|_3 = (t!)_3 = 3^{[t/3]+[t/3^2]+...} < 3^{t/3+t/3^2+...} = 3^{t/2} = 3^{m+1}.$$

Since the last inequality is strict, it follows that $|M_L|_3 \leq 3^m$. Therefore, $|L : M_L|_3 \geq (m!)_3$, whence $m < 3$ and $t < 8$. For $t = 4_\perp$, we get $|L|_3 = (q^2 - 1)_3 \geq 3^2$, while $|M_L|_3 = 3$ that is impossible. Hence, if $t$ is even then $t \in \{2_\perp, 2_\perp, 4_\perp, 6_\perp\}$.

(b) Assume $(k, \delta) = (2, -\epsilon)$ and set $d(2, -\epsilon) = t$. By Theorem [2,10(j)], we obtain that $|L : M_L|$ is even if $t > 1$.

(c) Assume $(k, \delta) = (4, +)$ and set $d(4, +) = t$. By using [19, Proposition 4.2.11], we obtain that

$$|L|_3 = (q^2 - 1)_3(2t!)_3, \quad |M_L|_3 = (q^2 - 1)_3(t!)_3,$$

hence $|L : M_L|$ is divisible by $3$ if $t > 1$.

(d) Assume $d(1, \circ) = 6_\perp$ and $n = \dim(V) > 6$. Since $d(3, \circ) = 0$, it follows that $\dim(V)$ is even, i.e., $\Omega(V) = \Omega^H_{2m}(q)$ for some integer $m$. Since $H(1, \circ)$ is a $\pi$-Hall subgroup of $G(1, \circ)$, then Lemma [5,1(y)] implies that $q \equiv -\epsilon \pmod{3}$. Moreover, $H$ is included in a subgroup $F$ of type $O_n^\perp(q) \perp O_{2m-6}^\perp(q)$. Since $Z(G)$ is a $2$-group, the inclusions $Z(G) \leq H \leq F$ hold. So the index of $F/Z(G)$ in $G/Z(G)$ is a $\pi'$-number. By [19, Proposition 4.1.6], we obtain that either $D = \varnothing$ (i.e., $\eta = -\epsilon^m$) and

$$(F \cap \Omega(V))/Z(G) = \left(O_n^\perp(q) \times O_{2m-6}^\perp(q) \right) \cdot [4],$$

or $D = \varnothing$ (i.e., $\eta = \epsilon^m$) and

$$(F \cap \Omega(V))/Z(G) = 2 \cdot \left(PO_n^\perp(q) \times PO_{2m-6}^\perp(q) \right) \cdot [4].$$

In the first case, we have

$$|G : F|_{\pi'} = \frac{(q^m + \epsilon^m)(q^{2m-1} - 1)(q^{2m-2} - 1)(q^{2m-3} - 1)}{2(q^2 - 1)(q^4 - 1)(q^6 - \epsilon)(q^{m-3} + \epsilon^{m-3})}.$$

Consider the $3$-part of this index:

$$|G : F|_3 = \frac{(q^m + \epsilon^m)_3(q^2 - 1)_3^3(m - 1)_3(m - 2)_3(m - 3)_3}{(q^2 - 1)_3^2(q^{m-3} + \epsilon^{m-3})_3} \geq (q^m + \epsilon^m)_3.$$

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Since \( q \equiv -\varepsilon \pmod{3} \), it follows that \((q^m + \varepsilon^m)_3 > 1\) if and only if \( m \) is odd. Hence if \( m \) is odd, then \( F \) does not include \( \pi \)-Hall subgroups of \( G \) (recall that \( 3 \in \pi \)). If \( m \) is even, then

\[
|G : F|_1 = \frac{(q^2 - 1)^2 \cdot 2 \cdot (m - 2)}{(q^2 - 1)^2 \cdot (q - \varepsilon)^2 \cdot 2 \cdot 2} = \frac{(m - 2)}{2},
\]

i.e., we must have \( m \equiv 2 \pmod{4} \) and \( \eta = -\varepsilon \) in this case. Since \( \eta = -\varepsilon^m \), it follows that \( d(2, -\varepsilon) \neq 0 \), so \( d(2, -\varepsilon) = 1 \) by statement (b) of the lemma. Now Lemma 2.1(a) implies that \((G(1, 0) \times G(2, -\varepsilon)) \cap \Omega_8(q)\) contains a \( \pi \)-Hall subgroup of \( \Omega_8(q) \). Since \( D(V(1, \circ) \perp V(2, -\varepsilon)) = \emptyset \), for the subspace \( V(1, \circ) + V(2, -\varepsilon) \), we can apply the same arguments as for \( V \). In particular, we have \( 4 = \frac{1}{2} \dim(V(1, \circ) + V(2, -\varepsilon)) \equiv 2 \pmod{4} \), a contradiction.

If \( D = \emptyset \), then

\[
|G : F|_1 = \frac{(q^m - \varepsilon^m) \cdot (q^2 - 1)_2 \cdot (m - 2) \cdot (m - 3)}{(q^2 - 1)_2 \cdot (q - \varepsilon)_2 \cdot 2 \cdot 2} \geq (q^2 - 1), \geq 3,
\]

so \( F \) does not include \( \pi \)-Hall subgroups of \( G \) in this case.

(e), (f) Assume \( d(1, \circ) = 4 \emptyset \), then \( \eta(1, \circ) = -\varepsilon \). Suppose \( d(2, -\varepsilon) = 1 \) (resp. \( d(4, +) = 1 \)). Then \( \Omega(V(1, \circ) \perp V(2, -\varepsilon)) \cong \Omega_8(q) \) (resp. \( \Omega(V(1, \circ) \perp V(4, +)) \cong \Omega_8(q) \)). Set \( \Omega = \Omega(V(1, \circ) \perp V(2, -\varepsilon)) \cong \Omega_8(q) \) (resp. \( \Omega = \Omega(V(1, \circ) \perp V(4, +)) \cong \Omega_8(q) \)) and define groups \( M_{\Omega}, L, \) and \( M_L \) as above. Then \( M_L \) includes a \( \pi \)-Hall subgroups of \( L \), therefore \( |L : M_L| \) is not divisible by 2 and 3. If \( L \cong \Pi \Omega_8(q) \), then Corollary 3.6 implies that \( |L : M_L| \) is even, while if \( L \cong \Pi \Omega_8(q) \), then [19, Proposition 4.1.6] implies that \( |L : M_L|_3 = 3 \). Statement (f) can be obtained arguing in the same way.

(g) The identity \( \eta(1, \circ) = -\varepsilon \) is evident. The identity \( d(2, -\varepsilon) = 0 \) follows from the fact that the decomposition (3) cannot be refined.

(h) can be obtained arguing as in (e) and (f) by using the fact that a subgroup of type \( \Omega_1(q) \perp \Omega_2^{-\varepsilon}(q) \) in \( \Pi \Omega_3(q) \) has even index.

(i) If \( d(1, \circ) = 2 \emptyset \), then \( \eta(1, \circ) = \varepsilon \) and, since the decomposition (3) cannot be refined, \( d(2, \varepsilon) = 0 \). Hence, the previous statements imply that one of the following cases occurs:

1. \( d(2, -\varepsilon) = d(4, +) = 0, \dim(V) = 2, \eta(V) = \varepsilon; \)
2. \( d(2, -\varepsilon) = 1, d(4, +) = 0, \dim(V) = 4, \eta(V) = -; \)
3. \( d(2, -\varepsilon) = 0, d(4, +) = 1, \dim(V) = 6, \eta(V) = \varepsilon; \)
4. \( d(2, -\varepsilon) = d(4, +) = 1, \dim(V) = 8, \eta(V) = -; \)

As in the proof of statement (e) one can show that the last case is impossible.  

□
Lemma 6.4 Suppose (⋆) holds, \( \dim(V) \geq 7 \), and the decomposition (3) of \( V \) cannot be refined. Assume also that either \( d(4,+) = 1 \) or \( d(1,\circ) \geq 3 \). Then \( q \equiv \pm 3 \) (mod 8).

Proof. Assume that \((k, \delta) \in \{(1, \circ), (4, +)\}\). Since \( \dim(V) \geq 7 \), Lemmas 2.11 and 6.3 imply that \( G(k, \delta) = I(V(k, \delta)) \supseteq \SO(V(k, \delta)) \). The statement follows from Lemma 6.5(c), (d), (g), (h), (i), (j), (k), (l), (v) for a \( \pi \)-Hall subgroup \( H(k, \delta) \cap \SO(V(k, \delta)) \) of \( \SO(V(k, \delta)) \). \( \square \)

Lemmas 6.3 and 6.4 give us the following:

Lemma 6.5 Suppose (⋆) holds, \( \dim(V) \geq 7 \), and the decomposition (3) of \( V \) cannot be refined. Then \( d(3, \circ) = d(4, -) = 0 \), and one of the following statements holds:

(a) \( \dim(U) = 0 \) and \( d(k, \delta) = 0 \) for \((k, \delta) \neq (2, \varepsilon)\);
(b) \( \dim(U) = 1 \) and \( d(1, \circ) = 1 \) and \( d(2, -\varepsilon) = d(4, +) = 0 \);
(c) \( \dim(U) = 2 \), \( \eta(U) = -\varepsilon \), \( d(1, \circ) = 2\varepsilon \), and \( d(2, -\varepsilon) = d(4, +) = 0 \), or \( d(2, -\varepsilon) = 1 \) and \( d(1, \circ) = d(4, +) = 0 \);
(d) \( \dim(U) = 3 \), \( q \equiv \pm 3 \) (mod 8), \( d(1, \circ) = 3 \), and \( d(2, -\varepsilon) = d(4, +) = 0 \);
(e) \( \dim(U) = 4 \), \( \eta(U) = +, q \equiv \pm 3 \) (mod 8), \( d(4, +) = 1 \), and \( d(1, \circ) = d(2, -\varepsilon) = 0 \);
(f) \( \dim(U) = 4 \), \( \eta(U) = -, q \equiv \pm 3 \) (mod 8), \( d(1, \circ) = 4\varepsilon \), and \( d(2, -\varepsilon) = d(4, +) = 0 \);
(g) \( \dim(U) = 5 \), \( q \equiv \pm 3 \) (mod 8), \( d(4, +) = d(1, \circ) = 1 \), and \( d(2, -\varepsilon) = 0 \);
(h) \( \dim(U) = 6 \), \( \eta(U) = -\varepsilon \), \( q \equiv \pm 3 \) (mod 8), and either \( d(1, \circ) = 2\varepsilon \), \( d(4, +) = 1 \) and \( d(2, -\varepsilon) = 0 \), or \( d(2, -\varepsilon) = d(4, +) = 1 \) and \( d(1, \circ) = 0 \).

In particular, \( \dim(U) \leq 6 \).

Lemma 6.6 Suppose (⋆) holds and \( \dim(V) \geq 7 \). Then \( s \in \pi(q - \varepsilon) \) whenever \( s \in \pi \cap \pi(q^2 - 1) \). In particular, \( q \equiv \varepsilon \) (mod 12).

Proof. Assume to the contrary that \( q \equiv -\varepsilon \) (mod \( s \)), in particular \( s \) is odd. Lemma 6.5 implies that \( \dim(W) \geq 1 \), i.e., \( t = d(2, \varepsilon) \geq 1 \).

Consider the case \( t \geq 2 \) first. Let \( \Omega = \Omega(2, \varepsilon) \). Then a \( \pi \)-Hall subgroup \( H(2, \varepsilon) \cap \Omega \) of \( \Omega \) is included in a subgroup \( M_\Omega \) of type \( O_\mathfrak{S}(q) \cap S_t \). We also have that

\[
|\Omega_s| \geq \prod_{i=1}^{t-1} (q^{2i} - 1)_s = (q^2 - 1)^{-1}_s ((t-1)!)_s \geq s^{t-1}_s ((t-1)!)_s,
\]

\[
|M_\Omega|_s = (t!)_s < s^{t-1}_s ((t-1)!)_s \leq |\Omega|_s,
\]

since \( t \geq 2 \). Hence \( |\Omega : M_\Omega| \) is divisible by \( s \), a contradiction.

Consider the case \( t = 1 \). Then \( \dim(W) = 2 \). By Lemma 6.5 it follows that one of the following statements holds.
(1) \( \dim(U) = 5, \dim(W) = 2, \) and \( \dim(V) = 7. \)

(2) \( \dim(U) = 6, \eta(U) = -\epsilon, \dim(W) = 2, \eta(W) = \epsilon, \dim(V) = 8, \) and \( \eta(V) = -\epsilon. \) In this case we also have that \( d(4, +) = 1 \) and either \( d(1, o) = 2\omega, \) or \( d(2, -\epsilon) = 1. \)

Statement (1) is impossible, since by [19, Proposition 4.1.6] the index of a subgroup of type \( O_3(q) \perp O_2^\epsilon(q) \) in \( P\Omega_7(q) \) is divisible by 3. Statement (2) can be eliminated arguing as in the proof of statements (e) and (i) of Lemma 5.3. \( \text{□} \)

By Lemma 6.6 it follows that if \( \dim(V) \geq 7, \) hypothesis \((\star)\) holds, and the decomposition \((\text{3})\) of \( V \) cannot be refined, then a \( \pi\)-Hall subgroup of \( O_2^{\epsilon}(q) \) has order 4 and is equal to the stabilizer of a decomposition of the natural 2-dimensional module into a sum of nondegenerate 1-dimensional subspaces. Hence we also have

\[
d(2, -\epsilon) = 0. \tag{5}
\]

**Lemma 6.7** Assume that \( G = \Omega_n^\delta(q), \eta \in \{+,-,o\}, q \) is a power of a prime \( p, n \geq 7, \epsilon = \epsilon(q). \) Let \( \pi \) be a set of primes such that \( 2, 3 \in \pi, p \notin \pi. \) Then the following statements hold:

(A) If \( G \) possesses a \( \pi\)-Hall subgroup \( H, \) then one of the following statements holds:

(a) \( n = 2m + 1, \pi \cap \pi(G) \subseteq \pi(q - \epsilon), q \equiv \epsilon \pmod{12}, \) and \( \text{Sym}_m \in E_\pi. \) The subgroup \( H \) is a \( \pi\)-Hall subgroup in \( M = (O_2^\epsilon(q) \wr \text{Sym}_m \times O_1(q)) \cap G. \) All \( \pi\)-Hall subgroups of this type are conjugate.

(b) \( n = 2m, \eta = \epsilon^m, \pi \cap \pi(G) \subseteq \pi(q - \epsilon), q \equiv \epsilon \pmod{12}, \) and \( \text{Sym}_m \in E_\pi. \) The subgroup \( H \) is a \( \pi\)-Hall subgroup in \( M = (O_2^\epsilon(q) \wr \text{Sym}_m) \cap G. \) All \( \pi\)-Hall subgroups of this type are conjugate.

(c) \( n = 2m, \eta = -\epsilon^m, \pi \cap \pi(G) \subseteq \pi(q - \epsilon), q \equiv \epsilon \pmod{12}, \) and \( \text{Sym}_{m-1} \in E_\pi. \) The subgroup \( H \) is a \( \pi\)-Hall subgroup of \( M = (O_2^\epsilon(q) \wr \text{Sym}_{m-1} \times O_2^{\epsilon^m}(q)) \cap G. \) All \( \pi\)-Hall subgroups of this type are conjugate.

(d) \( n = 11, \pi \cap \pi(G) = \{2, 3\}, q \equiv \epsilon \pmod{12}, \) and \( (q^2 - 1)_n = 24. \) The subgroup \( H \) is a \( \pi\)-Hall subgroup of \( M = (O_2^\epsilon(q) \wr \text{Sym}_4 \times O_1(q) \wr \text{Sym}_3) \cap G. \) All \( \pi\)-Hall subgroups of this type are conjugate.

(e) \( n = 12, \eta = -\epsilon, \pi \cap \pi(G) = \{2, 3\}, q \equiv \epsilon \pmod{12}, \) and \( (q^2 - 1)_n = 24. \) The subgroup \( H \) is a \( \pi\)-Hall subgroup of \( M = (O_2^\epsilon(q) \wr \text{Sym}_4 \times O_1(q) \wr \text{Sym}_3 \times O_1(q)) \cap G. \) There exist precisely two classes of conjugate subgroups of this type in \( G, \) and the automorphism of order 2 induced by the group of similarities of the natural module interchanges these classes.

(f) \( n = 7, \pi \cap \pi(G) = \{2, 3, 5, 7\}, \) and \( |G|_n = 2^9 \cdot 3^4 \cdot 5 \cdot 7. \) The subgroup \( H \) is isomorphic to \( O_7(2). \) There exist precisely two classes of conjugate subgroups of this type in \( G, \) and \( SO_7(q) \) interchanges these classes.

(g) \( n = 8, \eta = +, \pi \cap \pi(G) = \{2, 3, 5, 7\}, \) and \( |G|_n = 2^{13} \cdot 3^5 \cdot 5^2 \cdot 7. \) The subgroup \( H \) is isomorphic to \( O_8^+(2). \) There exist precisely four classes of conjugate subgroups of this type in \( G. \) The subgroup of \( \text{Out}(G) \) generated by diagonal and graph automorphisms is isomorphic to \( \text{Sym}_4 \) and acts on the set of these classes as \( \text{Sym}_4 \) in its natural permutation representation, and every diagonal
automorphism acts without fixed points.

(h) \( n = 9, \pi \cap \pi(G) = \{2, 3, 5, 7\}, \) and \(|G|_r = 2^{14} \cdot 3^3 \cdot 5^2 \cdot 7. \) The subgroup \( H \simeq 2 \cdot \Omega^+_8(2) \cdot 2. \) There exist precisely two classes of conjugate subgroups of this type in \( G, \) and \( \text{SO}_9(q) \) interchanges these classes.

(B) Conversely, if arithmetic conditions in one of the statements (a)–(h) holds, then \( G \) possesses a \( \pi \)-Hall subgroup with the given structure.

(C) If \( G \in E_n, \) then \( k_n(G) \in \{1, 2, 3, 4\}. \)

(D) If \( H \in \text{Hall}_\pi(G) \neq \emptyset, \) then \( G \) possesses a \( \pi \)-subgroup not conjugate to a subgroup of \( H, \) in particular \( G \notin D_n. \)

(E) All \( \pi \)-Hall subgroups of \( P\Gamma \) have the form \( \text{PH}. \)

Proof. (A) Suppose \( H \) is a \( \pi \)-Hall subgroup of \( G. \) Assume first that \((\ast)\) does not hold. Then \( n = 7, 8, \) or \( 9 \) and \( H \) is isomorphic to \( \Omega_7(2), 2 \cdot \Omega^+_8(2), \) or \( 2 \cdot \Omega^+_8(2) : 2, \) respectively. Thus we obtain statements (f), (g), (h) of the lemma and we only need to prove the statements about the classes of conjugate \( \pi \)-Hall subgroups in this case.

(f) By \([18, \text{Lemma 1.7.1}]\) the number of classes of conjugate subgroups isomorphic to \( \Omega_7(2) \) in \( \text{O}_7(q) \) is not greater than the number of nonequivalent irreducible representation of \( \Omega_7(2) \) of degree 7 over a field of order \( q. \) By using ordinary character tables of \( \Omega_7(2) \) given in \([3] \) or \([35]\) we obtain that it has precisely one irreducible complex representation of degree 7, while \( 2 \cdot \Omega_7(2) \) has no complex representations of degree 7. Since \( p \) does not divide the order of \( \Omega_7(2), \) by \([15, \text{Theorem 15.3 and Corollary 9.7}]\), this statement also holds for the representations over \( \mathbb{F}_q. \) Thus there exist at most two classes of conjugate \( \pi \)-Hall subgroups isomorphic to \( \Omega_7(2). \) By Lemma 2.13, it follows that \( \text{SO}_7(q) \) does not possess a \( \pi \)-Hall subgroup \( H_1 \) such that \( H_1 \cap \Omega_7(q) \simeq \Omega_7(2). \) Thus by Lemma 2.1(e) there exists at least two classes of conjugate \( \pi \)-Hall subgroups isomorphic to \( \Omega_7(2), \) and \( \text{SO}_7(q) \) interchanges these classes.

(g) By \([18, \text{Proposition 2.3.8}]\), we obtain that there exist 4 classes of conjugate subgroups isomorphic to \( \Omega^+_8(2) \) in \( \text{PQ}_8^+(q). \) Moreover \( \text{Aut}(S) \) interchanges these four classes in the natural way. Therefore there exists a homomorphism \( \text{Aut}(S) \rightarrow \text{Sym}_4 \) and, by \([18, \text{Proposition 2.3.8}]\), this homomorphism is surjective.

(h) In this case a \( \pi \)-Hall subgroup \( H \simeq 2 \cdot \Omega^+_8(2) \cdot 2 \) is included in a subgroup of type \( \text{O}_1(q) \leq \Omega^+_8(q) \) which, by Lemma 2.11(a), is isomorphic to \( \Omega^+_8(q) : \langle \tau \rangle, \) where \( \tau \) is a graph automorphism of order 2, and by \([19, \text{Table 3.5D}]\) all such subgroups are conjugate. By \([18, \text{Proposition 2.3.8}]\), there exists 4 classes of conjugate subgroups isomorphic to \( 2 \cdot \Omega^+_8(2) \) in \( \Omega^+_8(q), \) and the graph automorphism \( \tau \) of order 2 stabilizes 2 of these classes and interchanges the remaining 2 classes. By Lemma 2.1(e), \( \Omega^+_8(q) : \langle \tau \rangle \) has 2 classes of conjugate \( \pi \)-Hall subgroups isomorphic to \( 2 \cdot \Omega^+_8(2) \cdot 2. \) By Lemma 2.13, it follows that \( \text{SO}_9(q) \) does not possess a \( \pi \)-Hall subgroup \( H_1 \) such that \( H_1 \cap \Omega_9(q) \simeq 2 \cdot \Omega^+_8(2) \cdot 2. \) Thus \( \text{SO}_9(q) \) interchanges these classes.
Now assume that (⋆) holds. Consider a decomposition of the natural module $V$ of $G$ into $H$-invariant direct orthogonal sum of subspaces of dimension at most 4 such that the decomposition cannot be refined (the existence of this decomposition follows from Lemma 6.5). We preserve the above notations, in particular, the symbols $V(k, δ), U,$ and $W$ are as defined above. Then one of statements (a)–(h) of Lemma 6.5 holds. Moreover, as we noted before Lemma 6.7, the condition that the decomposition cannot be refined implies that $d(2, -ε) = 0.$ Consider statements (a)–(h) of Lemma 6.5 separately.

Suppose statement (a) of Lemma 6.5 holds. Then $V = W = V(2, ε).$ So $n = \dim(V) = 2m,$ where $m = d(2, ε),$ $η = ε^m,$ and $H \leq (O_2(q) \wr Sym_m) \cap G.$ Lemmas 2.9 and 6.6 imply statement (b) of the lemma. By [19, Tables 3.5.E and 3.5.F], we obtain that all subgroups of type $O_{2d}(2m − 1)$ are conjugate in $G.$ By Lemma 2.3 we obtain $Sym_m \subset C_π.$ Since $O_2(q)$ is solvable, Lemma 2.7(f) implies that $(O_{2d}(q) \wr Sym_m) \cap G \subset C_π,$ hence all $π$-Hall subgroups of this type are conjugate.

Suppose statement (b) of Lemma 6.5 holds. Then $n = \dim(V) = 2m + 1,$ where $m = d(2, ε)$ and $H \leq ((O_2(q) \wr Sym_m) \perp O_1(q)) \cap G.$ Using [19, Table 3.5.D] instead of [19, Tables 3.5.E and 3.5.F] we obtain statement (a) of the lemma as in the previous case.

Suppose statement (c) of Lemma 6.5 holds. In this case $n = \dim(V) = 2m, d(2, ε) = m - 1,$ $d(1, o) = 2m$ (in view of (6), the case $d(2, -ε) = 1$ is impossible), $η = -ε \cdot ε^m = -ε^m$ and $H \leq ((O_2(q) \wr Sym_{m-1}) \perp O_{2e}(q)) \cap G.$ As in statement (a) of Lemma 6.5 we obtain that statement (c) of the lemma holds in this case.

Suppose statement (d) of Lemma 6.5 holds, i.e., $\dim(U) = 3, U = V(1, o).$ Set $d(2, ε) = m - 1.$ Then $η(W) = ε^{m-1}, n = \dim(V) = 2m + 1.$ Lemma 2.11 implies that $G(1, o) \simeq O_3(q).$ Now $H(1, o)$ is a $π$-Hall subgroup of $G(1, o),$ stabilizing a decomposition of $U$ into a sum of 1-dimensional subspaces. Lemmas 5.7(b)–(e) and 2.11(e) imply that $|H(1, o)| = 24$ and $H(1, o) = Sym_4,$ hence $(q^2 - 1)_{π} = |O_2(q)|_{π} = 24.$ By [19, Proposition 4.1.6], we obtain that the $p'$-part of index of a subgroup $M$ of type $O_{2d(m-1)}^q(1) \perp O_3(q)$ in $G$ is equal to

$$|L : M|_{p'} = \frac{1}{2} \cdot \frac{q^{2m} - 1}{q^2 - 1}(q^{m-1} + ε^{m-1}).$$

Lemma 6.6 implies that $q \equiv ε$ (mod 3), so we obtain $(q^{m-1} + ε^{m-1})_{π} = 1.$ Since $M$ contains $H,$ it follows that $1 = [L : M], m,$ and $1 = |L : M| = m_{π},$ whence $m$ is not divisible by 2 and by 3, i.e., $m \equiv ±1$ (mod 6). By Lemma 2.9, we obtain that every prime from $π \cap π(Sym_{m-1})$ is contained in $π(q^2 - 1),$ i.e., $Sym_{m-1} \in E_{[2, 3]}.$ Now Lemma 2.3 implies the inequalities $m \leq 9$ and $m \neq 7.$ All restrictions on $m$ give us the only possible value $m = 5,$ whence statement (d) of the lemma follows. The statement about the classes of conjugate $π$-Hall subgroups we obtain as in statement (a) of Lemma 6.5 using [19, Table 3.5.D] instead of [19, Tables 3.5.E and 3.5.F].
Suppose statement (e) of Lemma 6.5 holds, i.e., \(d(4, +) = 1\), \(U = V(4, +)\), \(\dim(U) = 4\), \(\eta(U) = +\). Set \(d(2, \varepsilon) = m - 2\). Then \(\dim(V) = 2m\), \(\eta(V) = \varepsilon^{m-2} = \varepsilon^m\). Lemma 2.11 implies that \(G(4, +) \cong O_4^+(q)\). Now Lemma 2.11(f)–(l) and the fact that the decomposition \((3)\) cannot be refined imply that \(H(4, +) \cap \Omega(4, +)\) is isomorphic to \(2.\text{Alt}_4 \circ 2.\text{Alt}_4\), whence \((q^2 - 1)_p = 24\). By [19, Proposition 4.1.6], the \(p^t\)-part of the index of a subgroup \(M\) of type \(O_{2(m-2)}^+(q) \perp O_4^+(q)\) in \(G\) is equal to

\[
|G : M|_{p^t} = \frac{q^{m-2} + \varepsilon^{m-2}}{2} \cdot \frac{(q^{2(m-1)} - 1)(q^m - \varepsilon^m)}{(q^2 - 1)^2}.
\]

Since, by Lemma 6.6, we have \(q \equiv \varepsilon \pmod{3}\), it follows that

\[
|G : M|_2 = \frac{m_2(m - 1)}{2}, \quad |G : M|_3 = m_3(m - 1),
\]

Moreover \(H\) is included in \(M\), whence \((m - 2)(m - 3)\) is divisible by 4, and \(m - 2\) is divisible by 3. Thus \(m \equiv 2, 11 \pmod{12}\). Lemma 2.9 implies that each prime in \(\pi \cap \pi(\text{Sym}_{m-2})\) is contained in \(\pi(q^2 - 1)\), i.e., \(\text{Sym}_{m-2} \in E_{[2,3]}\). Lemma 2.3 implies that \(m \leq 10\) and \(m \neq 8\). All restrictions on \(m\) and the condition \(n \geq 7\) imply that there does not exist a \(\pi\)-Hall subgroup satisfying statement (e) of Lemma 6.5.

Suppose statement (f) of Lemma 6.5 holds. By definition, the identity \(d(1, \circ) = 4\_\infty\) implies that this case follows from the already considered statement (d) of Lemma 6.5 and statement (e) holds.

Suppose statement (g) or (h) of Lemma 6.5 holds. As in the already considered statement (e) of Lemma 6.5, we obtain that there does not exist a \(\pi\)-Hall subgroup satisfying statement (g) or (h) of Lemma 6.5, respectively.

(B) Clearly a subgroup satisfying statement (f), (g) or (h) of the lemma is a \(\pi\)-Hall subgroup in the corresponding group \(G\). If one of statements (a)–(e) of the lemma holds, then a direct calculation using Lemma 2.6 and Corollaries 2.7, 2.8 implies that \(|G : M|_\pi = 1\), hence \(\emptyset \neq \text{Hall}_\pi(M) \subseteq \text{Hall}_\pi(G)\).

(C) If one of statements (f), (g), (h) of the lemma holds, then \(|G|_7 = 7\), whence \(7 \notin \pi(q - \varepsilon)\) and none of statements (a)–(e) can be fulfilled. Since either statements (a) and (d), or statements (c) and (e) of the lemma can be fulfilled simultaneously, and the remaining cannot, we obtain that in this case \(k_\pi(G) = 3\) and (C) follows.

(D) If one of the statements (e)–(h) of the lemma holds, then \(G\) possesses more than one class of conjugate \(\pi\)-Hall subgroups and we have nothing to prove. Assume that statement (d) of the lemma holds. Then \(d(2, \varepsilon) = 4\), and \(n = \dim(V) = 11\). Then \(L = (((O_2^+(q) \circ \text{Sym}_4) \times O_2^+(q) \times O_3(q)) \cap G\) possesses a \(\pi\)-Hall subgroup \(F\), and \(F\) is not isomorphic to a subgroup of \(H\). Assume that one of statements (a) and (b) of the lemma holds. Then \(G\) possesses a subgroup of type \(O_3(q) \circ S_n\), which contains a subgroup \(2^{n-1}.\text{Alt}_n\). Statements (a) and (b) imply that \(\text{Sym}_m \in E_\pi\), where \(m = [n/2]\). Consider a subgroup \(\text{Alt}_m \times \text{Alt}_m \leq \text{Alt}_n\), let \(R\) be its \(\pi\)-Hall subgroup
and \( R_1 \) the complete preimage in \( 2^{n-1} \cdot \text{Alt}_n \). It is clear that \( R_1 \) is not isomorphic to any subgroup of \( H \). If statement (c) holds, we proceed in the same way considering a subgroup of type \( O_1(q) \perp (O_1(q) \ltimes \text{Sym}_{n-1}) \).

(E) Follows from Lemma 2.1(a), (c), (f). □

7 Hall subgroups in exceptional groups of Lie type

Recall that \( q = p^a \) is a power of a prime \( p \), \( \varepsilon = \varepsilon(q) = (-1)^{(q-1)/2} \), \( \pi \) is a set of primes such that \( 2, 3 \in \pi \), and \( p \not\in \pi \).

Lemma 7.1 Suppose \( G \cong G_2(q) \), \( 2, 3 \in \pi \), \( p \not\in \pi \), and \( \varepsilon = \varepsilon(q) \). Then the following statements hold:

(A) If \( G \) possesses a \( \pi \)-Hall subgroup \( H \), then one of the following statements holds:
   (a) \( \pi \cap \pi(G) = \{2, 3, 7\}, \ (q^2 - 1)_{2,3,7} = 24 \), and \( (q^3 + q^2 + 1)_{7} = 7 \). The subgroup \( H \) is isomorphic to \( G_2(2) \), and all \( \pi \)-Hall subgroups of this type are conjugate in \( G \);
   (b) \( \pi \cap \pi(G) \subseteq \pi(q - \varepsilon) \). The subgroup \( H \) is a \( \pi \)-Hall subgroup of a solvable group \( (q - \varepsilon)^2 \cdot W(G_2) \), and all \( \pi \)-Hall subgroups of this type are conjugate in \( G \).

(B) Conversely, if arithmetic conditions in one of statements (a), (b) holds, then \( G \in E_\pi \).

(C) If \( G \in E_\pi \), then \( k_\pi(G) = 1 \), i.e., \( G \in C_\pi \).

(D) If \( G \in E_\pi \) and \( H \in \text{Hall}_\pi(G) \), then \( G \) possesses a \( \pi \)-subgroup that is not isomorphic to a subgroup of \( H \), in particular, \( G \not\in \ D_\pi \).

Proof. (A) Suppose \( H \) is a \( \pi \)-Hall subgroup of \( G \). Then it is included in a maximal subgroup \( M \) of odd index of \( G \). By [24] one of the following statements holds (recall that \( p \not\in \pi \), while \( 2 \in \pi \), whence \( p \) is odd):

1. \( M \cong G_2(q_0) \), where \( \mathbb{F}_{q_0} \) is a subfield of \( \mathbb{F}_q \).
2. \( M \cong G_2(2) \), \( |M| = 2^6 \cdot 3^3 \cdot 7 \).
3. \( M = N_G(2 \cdot (\text{PSL}_2(q) \times \text{PSL}_2(q))) \), \( |M| = q^2(q^2 - 1)^2 \).
4. \( M \cong 2^3 \cdot \text{SL}_3(2) \), \( |M| = 2^6 \cdot 3 \cdot 7 \).
5. \( M \cong \text{SL}_3^2(q) \cdot 2 \), \( |M| = q^3(q^3 - \varepsilon)(q^2 - 1) \cdot 2 \).
6. \( M \cong (q - \varepsilon)^2 \cdot W(G_2) \), \( |M| = (q - \varepsilon)^2 \cdot 2^2 \cdot 3 \).

Suppose statement (1) holds. Since \( p \not\in \pi \) we obtain that \( H \) is a proper subgroup of \( M \). So \( H \) is included in a maximal subgroup \( M_1 \) of odd index of \( M \cong G_2(q_0) \). In particular, \( M_1 \) satisfies one of statements (1)--(6). If \( M_1 \) satisfies statement (1), we again obtain that \( H \) is a proper subgroup of \( M_1 \) and repeat the arguments. So
we may assume that $M$ satisfies one of statements (2)–(6).ootnote{The case, when a maximal subgroup $M$ is a subgroup of the same type, as $G$, defined over a proper subfield in proofs of Lemmas 7.2–7.6 can be excluded by arguing in the same way, and we do not consider this case.} Assume statement (2) holds. Then [23, Theorem 1.2] implies that $G_2(2)$ does not possess proper $\pi$-Hall subgroups with $2, 3 \in \pi$. Therefore $H = G_2(2)$. Moreover all subgroups isomorphic to $G_2(2)$ are conjugate in view of [17, Theorem A], whence statement (a) of the lemma follows. If either (3) or (4) holds, then $|G : M|_3 > 1$, whence $M$ cannot include a $\pi$-Hall subgroup of $G$ with $3 \in \pi$. Assume statement (5) holds. Then $|G : M|_3 = 1$ if and only if $q \equiv \varepsilon \pmod{3}$. Moreover we obtain that a $\pi$-Hall subgroup $H_1$ of $\text{SL}_3^\varepsilon(q)$ satisfies either statement (b), or statement (c) of Lemma 4.3. Finally, if statement (6) holds, then direct calculations show that the condition $|G : (q - \varepsilon)^2. W(G_2)|_3 = 1$ implies $q \equiv \varepsilon \pmod{3}$. The conjugacy of maximal tori of order $(q - \varepsilon)^2$ is obtained in [28, Lemma 3.10], whence we obtain statement (b) of the lemma.

(B) If statement (a) of the lemma holds, then the index $|G : H|$ is a $\pi'$-number, so $G_2(2)$ is a $\pi$-Hall subgroup of $G$. If statement (b) of the lemma holds, then the index $|G : (q - \varepsilon)^2. W(G_2)|$ is a $\pi'$-number, so $\varnothing \neq \text{Hall}_\pi((q - \varepsilon)^2. W(G_2)) \subseteq \text{Hall}_\pi(G)$.

(C) Notice that statement (a) of the lemma implies that 7 does not divide $(q^2 - 1)$, while statement (b) implies that 7 divides $q^2 - 1$, whence these statements cannot hold simultaneously. Hence $k_\pi(G) = 1$ if $G \in E_\pi$.

(D) In view of [3], there exists a subgroup $L$ of $G$ isomorphic to $2^3 \cdot \text{SL}_3(2)$. If statement (a) holds, then $L$ is a $\pi$-group and $L$ clearly is not isomorphic to a subgroup of $G_2(2)$. In the second case of the proof of Lemma 6.9 in [28], we have constructed a $\{2, 3\}$-subgroup of $L$ which cannot be embedded in a maximal torus, whence if statement (b) holds, then $G$ possesses a $\pi$-subgroup not isomorphic to a subgroup of a $\pi$-Hall subgroup of $G$. \square

**Lemma 7.2** Suppose that $G \simeq F_4(q)$, $2, 3 \in \pi$, $p \notin \pi$, and $\varepsilon = \varepsilon(q)$. Then the following statements hold:

(A) If $G \in E_\pi$ and $H \in \text{Hall}_\pi(G)$, then $\pi \cap \pi(G) \subseteq \pi(q - \varepsilon)$ and $H$ is included in a solvable group $(q - \varepsilon)^4. W(F_4)$.

(B) Conversely, if $\pi \cap \pi(G) \subseteq \pi(q - \varepsilon)$, then $G \in E_\pi$.

(C) If $G \in E_\pi$, then $k_\pi(G) = 1$, i.e., $G \in C_\pi$.

(D) If $H \in \text{Hall}_\pi(G)$, then $G$ possesses a $\pi$-subgroup not isomorphic to a subgroup of $H$. In particular, $G \notin D_\pi$.
Proof. (A) Assume that $G$ possesses a $\pi$-Hall subgroup $H$. Then $H$ is included in a maximal subgroup $M$ of odd index of $G$. As in the proof of Lemma 7.1 by using [28] we may assume that one of the following statements holds:

1. $M = N_G(2, P\Omega_3(q))$, $|M| = q^{16}(q^2 - 1)(q^4 - 1)(q^6 - 1)(q^8 - 1)$,
2. $M = N_G(2^2, P\Omega_8^+(q))$, $|M| = q^{12}(q^2 - 1)(q^4 - 1)^2(q^8 - 1) \cdot 2 \cdot 3$.

If statement (1) holds, then $|G : M| \geq 3$, hence $M$ cannot contain a $\pi$-Hall subgroup of $G$. If statement (2) holds, then by Lemma 6.1(a) the image of $H$ in $P\Omega_8^+(q)$ is a $\pi$-Hall subgroup of $P\Omega_8^+(q)$. By Lemma 6.7, we obtain that this image is either equal to $\Omega_8^+(2)$, or included in the normalizer of a maximal torus (i.e., statement (b) of Lemma 6.7 holds). If $H$ is included in the normalizer of a maximal torus, we obtain statement (A) of the lemma. If the image of $H$ in $P\Omega_8^+(q)$ equals $\Omega_8^+(2)$, then $|N_G(2^2, P\Omega_8^+(q))|_q = 7$, and $|G|_q \geq 7^2$. Hence, this case is impossible.

(B), (C) and (D) are proven in [28, Lemma 6.8]. □

Lemma 7.3 Suppose that $G \cong E_6^\eta(q)$, $2, 3 \in \pi$, $p \notin \pi$, and $\varepsilon = \varepsilon(q)$. Then the following statements hold:

(A) If $G \in E_\pi$ and $H \in \text{Hall}_\pi(G)$, then $\pi \cap \pi(G) \subseteq \pi(q - \varepsilon)$ and one of the following statements holds:

(a) $\eta = \varepsilon$, $5 \in \pi$, and $H$ is a $\pi$-Hall subgroup in $T . W(E_6)$, where $T$ is a maximal split torus of order $(q - \eta)^6/(3, q - \eta)$;

(b) $\eta = -\varepsilon$ and $H$ is a $\pi$-Hall subgroup in $(q^2 - 1)^2(q + \eta)^2 \cdot W(F_4)$.

(B) Conversely, if $\pi \cap \pi(G) \subseteq \pi(q - \varepsilon)$, and $5 \in \pi$ for $\eta = \varepsilon$, then $G \in E_\pi$.

(C) If $G \in E_\pi$, then $k_\pi(G) = 1$, i.e., $G \in C_\pi$.

(D) If $H \in \text{Hall}_\pi(G)$, then $G$ possesses a $\pi$-subgroup not isomorphic to a subgroup of $H$. In particular, $G \notin D_\pi$.

Proof. If we show that a $\pi$-Hall subgroup $H$ normalizes a maximal torus, then the claim follows from [28, Lemma 6.5]. Suppose that $H$ is a $\pi$-Hall subgroup of $G$. Then it is included in a maximal subgroup $M$ of odd index of $G$. In view of [28] and arguing as in the proof of Lemma 7.1, we may assume that one of the following statements holds:

1. $\eta = +$, $|M| = \frac{q^{36}(q-1)(q^2-1)(q^4-1)(q^6-1)(q^8-1)}{(3,q-1)}$;
2. $M = N_G((4, q - \eta) \cdot P\Omega_3(q))$, $|M| = \frac{q^{20}(q-\eta)(q^2-1)(q^4-1)(q^6-1)(q^8-1)}{(3,q-\eta)}$;
3. $M = N_G(2^2 \cdot P\Omega_8^+(q))$, $|M| = \frac{q^{12}(q-\eta)^2(q^2-1)(q^4-1)^2(q^6-1)}{2 \cdot 3}$;
4. $M = T \cdot W(E_6)$, where $T$ is a maximal split torus of order $(q-\eta)^6/(3,q-\eta)$, and $|M| = \frac{(q-\eta)^6 \cdot 2 \cdot 3^5}{(3,q-\eta)}$.

If either statement (1) or statement (2) holds, then $|G : M|_q \geq 3$, whence $M$ cannot include a $\pi$-Hall subgroup of $G$. If statement (3) holds, then by Lemma 7.1(a) we
obtain that the image of $H$ in $\Omega^+(\pi)$ is a $\pi$-Hall subgroup of $\Omega^+(\pi)$. Lemma 6.7 implies that either this image satisfies statement (b) of Lemma 6.7 or this image is isomorphic to $\Omega^+(2)$. In the first case it follows that $H$ normalizes a maximal torus, in the second case we obtain that $|H| = 7$, while $|G| \geq 7^2$, a contradiction. If statement (4) holds, then $M$ normalizes a maximal torus. □

**Lemma 7.4** Suppose that $G \cong E_7(q)$, $2, 3 \in \pi$, $p \notin \pi$, $\varepsilon = \varepsilon(q)$ and $T$ is a maximal torus of order $(q - \varepsilon)^7/2$ of $G$. Then the following statements hold:

(A) If $H$ is a $\pi$-Hall subgroup of $G$, then $\pi \cap \pi(G) \subseteq \pi(q - \varepsilon)$, $5, 7 \in \pi$, and $H$, up to conjugation, is included in $T \cdot W(E_7)$.

(B) Conversely, if $\pi \cap \pi(G) \subseteq \pi(q - \varepsilon)$ and $5, 7 \in \pi$, then $G \in E_\pi$.

(C) If $G \in E_\pi$, then $k_\pi(G) = 1$, i.e., $G \in C_\pi$.

(D) If $H \in \text{Hall}_\pi(G)$, then $G$ possesses a $\pi$-subgroup not isomorphic to a subgroup of $H$. In particular, $G \notin D_\pi$.

**Proof.** As in the previous lemma, if we show that a $\pi$-Hall subgroup $H$ normalizes a maximal torus, then the claim follows from [28, Lemma 6.6]. Assume that $H$ is a $\pi$-Hall subgroup of $G$. Then $H$ is included in a maximal subgroup $M$ of odd index of $G$. In view of [20] and arguing as in the proof of Lemma 7.1, we may assume that one of the following statements holds.

1. $|M| = q^7(q^2 - 1)^7 \cdot 2^2 \cdot 3 \cdot 7$.
2. $|M| = 1/2 \cdot q^{31}(q^2 - 1)^2(q^4 - 1)^2(q^6 - 1)^2(q^8 - 1)(q^{10} - 1)$.
3. $M = N_G(2^2 \cdot (\text{PSL}_2(q) \times \Omega^+(\pi)))$, $|M| = q^{15}(q^2 - 1)^4(q^4 - 1)^2(q^6 - 1) \cdot 3$.
4. $M = N(G, T)$, where $T$ is a maximal torus defined in the lemma, and $N(G, T)$ is its algebraic normalizer (defined in [28]).

If one of statements (1)–(3) holds, then $|M : M_\pi| \geq 3$, and if statement (4) holds, then $H$ is in $N(G, T) = T \cdot W(E_7)$. □

**Lemma 7.5** Suppose that $G \cong E_8(q)$, $2, 3 \in \pi$, $p \notin \pi$, $\varepsilon = \varepsilon(q)$ and $T$ is a maximal torus of order $(q - \varepsilon)^8$ of $G$. Then the following statements hold:

(A) If $G \in E_\pi$ and $H \in \text{Hall}_\pi(G)$, then $\pi \cap \pi(G) \subseteq \pi(q - \varepsilon)$, $5, 7 \in \pi$, and $H$, up to conjugation, is included in $T \cdot W(E_8)$.

(B) Conversely, if $\pi \cap \pi(G) \subseteq \pi(q - \varepsilon)$ and $5, 7 \in \pi$, then $G \in E_\pi$.

(C) If $G \in E_\pi$, then $k_\pi(G) = 1$, i.e., $G \in C_\pi$.

(D) If $H \in \text{Hall}_\pi(G)$, then $G$ possesses a $\pi$-subgroup not isomorphic to a subgroup of $H$. In particular, $G \notin D_\pi$.

**Proof.** If we show that a $\pi$-Hall subgroup $H$ normalizes a maximal torus, then the lemma will follow from [28, Lemma 6.7]. In view of [20] and arguing as in the proof of Lemma 7.1, we may assume that a maximal subgroup $M$ of $G$ that includes $H$ satisfies to one of the following statements.

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(1) \(|M| = q^8(q^2 - 1)^8 \cdot 2^6 \cdot 3 \cdot 7,
(2) \(|M| = q^{56}(q^2 - 1)(q^4 - 1)(q^6 - 1)^2(q^{10} - 1)(q^{12} - 1)(q^{14} - 1),
(3) \(M = N_G(2^2 \cdot (P\Omega_8^+(q) \times \Omega_8^+(q)))\), \(|M| = q^{24}(q^2 - 1)^2(q^4 - 1)^4(q^6 - 1)^2 \cdot 2^2 \cdot 3,
(4) \(M = N(G, T)\), where \(T\) is a maximal torus defined in the lemma, and \(N(G, T)\)
is its algebraic normalizer.

If one of statements (1)–(3) holds, then \(|G : M| \geq 3\), and if statement (4) holds, then \(H\) is in \(N(G, T) = T \cdot W(E_8)\). □

**Lemma 7.6** Suppose that \(G \cong 3D_4(q), 2, 3 \in \pi, p \notin \pi, \epsilon = \epsilon(q)\) and \(T\) is a maximal torus of order \((q - \epsilon)(q^3 - \epsilon)\) of \(G\). Then the following statements hold:

(A) If \(G \in E_\pi\) and \(H \in \text{Hall}_e(G)\), then \(\pi \cap \pi(G) \subseteq \pi(q - \epsilon), \epsilon\), and \(H\), up to conjugation, is included in \(T \cdot W(G_2)\).
(B) Conversely, if \(\pi \cap \pi(G) \subseteq \pi(q - \epsilon), \epsilon\), then \(G \in E_\pi\).
(C) If \(G \in E_\pi\), then \(k_\pi(G) = 1\), i.e., \(G \in C_\pi\).
(D) If \(H \in \text{Hall}_e(G)\), then \(G\) possesses a \(\pi\)-subgroup not isomorphic to a subgroup of \(H\). In particular, \(G \notin D_\pi\).

**Proof.** If we show that a \(\pi\)-Hall subgroup \(H\) normalizes a maximal torus, then the claim follows from [28, Lemma 6.10]. In view of [20] and arguing as in the proof of Lemma [7], we may assume that a maximal subgroup \(M\) of \(G\) that includes \(H\), satisfies to the following statements.

(1) \(M \cong G_2(q)\), \(|M| = q^6(q^2 - 1)(q^6 - 1)\),
(2) \(M = N_G(2 \cdot (PSL_2(q) \times PSL_2(q)))\), \(|M| = q^4(q^2 - 1)(q^6 - 1)\),
(3) \(M = N_G(\text{SL}_2(q))\), \(|M| = q^3(q^3 - \epsilon)(q^2 - 1)(q^2 + \epsilon q + 1)/2\).

If one of statements (1), (2) holds, then \(|G : M| \geq 3\). Suppose statement (3) holds. Then the index \(|G : M|\) is not divisible by 3 if and only if \(q \equiv \epsilon \pmod{3}\). By Lemma [4], we obtain that either a \(\pi\)-Hall subgroup normalizes a maximal torus, or is included in \(\text{GL}_2^\epsilon(q)\). In the second case we have \(|M : \text{GL}_2^\epsilon(q)| \geq 1\). □

8 Proof of the Theorem 1.1

We first prove Theorem [1.1] in the particular case \(2, 3 \in \pi\) and \(G = S\).

**Lemma 8.1** Let \(\pi\) be a set of primes such that \(2, 3 \in \pi\) and let \(S\) be a finite simple group. Then one of the following statements holds:

(A) \(k_\pi(S) \in \{0, 1, 2, 3, 4\}\);
(B) \(k_\pi(S) = 9, S \cong \text{PSp}_{2n}(q), q\) is a power of a prime \(p \notin \pi\) and
   (a) either \(n \in \{5, 7\}, \pi \cap \pi(S) = \{2, 3\} \subseteq \pi(q^2 - 1) \text{ and } (q^2 - 1)_\pi = 48;\)
   (b) or \(n = 7, \pi \cap \pi(S) = \{2, 3, 5\} \subseteq \pi(q^2 - 1) \text{ and } (q^2 - 1)_\pi = 120.\)
Table 1
Proper $\pi$-Hall subgroups in sporadic groups, $2, 3 \in \pi$

| $S$     | $\pi$            | Structure $H$          |
|---------|------------------|------------------------|
| $M_{11}$ | $(2, 3)$          | $3^2 : Q_8 \cdot 2$   |
|         | $(2, 3, 5)$       | $\text{Alt}_6 \cdot 2$ |
| $M_{22}$ | $(2, 3, 5)$       | $2^4 : \text{Alt}_6$  |
| $M_{23}$ | $(2, 3)$          | $2^4 : (3 \times \text{Alt}_4) : 2$ |
|         | $(2, 3, 5)$       | $2^4 : \text{Alt}_6$  |
|         | $(2, 3, 5, 7)$    | $2^4 : (3 \times \text{Alt}_5) : 2$ |
|         | $(2, 3, 5, 7)$    | $L_3(4) : 2_2$         |
|         | $(2, 3, 5, 7, 11)$| $2^4 : \text{Alt}_7$  |
|         | $(2, 3, 5, 7, 11)$| $M_{22}$               |
| $M_{24}$ | $(2, 3, 5)$       | $2^6 : 3^2 \cdot \text{Sym}_6$ |
| $J_1$   | $(2, 3)$          | $2 \times \text{Alt}_4$ |
|         | $(2, 7)$          | $2^3 : 7$              |
|         | $(2, 3, 5)$       | $2 \times \text{Alt}_5$ |
|         | $(2, 3, 7)$       | $2^3 : 7 \cdot 3$      |
| $J_4$   | $(2, 3, 5)$       | $2^{11} : (2^6 : 3^2 \cdot \text{Sym}_6)$ |

Proof. We may assume that $S$ is nonabelian and $S \in E_\pi$. Consider all nonabelian finite simple $E_\pi$-groups with $2, 3 \in \pi$.

If $S \cong \text{Alt}_n$, then the claim follows from Lemma 2.3. In sporadic groups all proper $\pi$-Hall subgroups with $2, 3 \in \pi$ are found in [23, Theorem 4.1] and are given in Table 1. By using [4], it is easy to check that $k_\pi(S) \leq 2$ in this case.

If $S$ is a finite group of Lie type with the base field of characteristic $p \notin \pi$, then the claim follows from Lemmas 4.3(C), 4.4(C), 6.7(C), 7.1(C), 7.2(C), 7.3(C), 7.4(C), 7.5(C), 7.6(C), and the proof of Lemma 4.4.

Now assume that $S$ is a finite group of Lie type with the base field $\mathbb{F}_q$ of characteristic $p \in \pi$. Then [8, Theorem 3.2], [10, Theorem 3.1], and [22, Theorem 1.2] imply that one of the following cases 1–4 holds:

Case 1. $\pi(S) \subseteq \pi$. Clearly $k_\pi(S) = 1$.

Case 2. $\pi \cap \pi(S) \subseteq \pi(q - 1) \cup \{p\}$ and each $\pi$-Hall subgroup $H$ of $S$ is included in a Borel subgroup of $S$. In this case the solvability of Borel subgroups and the fact that all Borel subgroups are conjugate imply that $k_\pi(S) = 1$.
Case 3. $S = D^n_\pi(q)$, $p = 2$, $\pi \cap \pi(S) = \pi(D^n_{\pi-1}(q))$ and each $\pi$-Hall subgroup $H$ of $S$ is a parabolic subgroup with the Levi factor isomorphic to $D^n_{\pi-1}(q)$. Since all parabolic subgroups of this type are conjugate, we have $k_\pi(S) = 1$.

Case 4. $S = A_{\pi-1}(q) \cong \text{PSL}(V)$, where $V$ is a vector space of dimension $n$ over $\mathbb{F}_q$; $\pi \cap \pi(S) = \pi(H)$, where $H$ is the image in $\text{PSL}(V)$ of the stabilizer in $\text{SL}(V)$ of a series

$$0 = V_0 < V_1 < \ldots < V_s = V$$

such that $\dim V_i/V_{i-1} = n_i$, $n = \sum n_i$ and one of the following statements holds:

1. $n$ is an odd prime, $s = 2$, $\{n_1, n_2\} = \{1, n-1\}$;
2. $n = 4$, $s = 2$, $n_1 = n_2 = 2$;
3. $n = 5$, $s = 2$, $\{n_1, n_2\} = \{2, 3\}$;
4. $n = 5$, $s = 3$, $\{n_1, n_2, n_3\} = \{1, 2\}$;
5. $n = 7$, $s = 2$, $\{n_1, n_2\} = \{3, 4\}$;
6. $n = 8$, $s = 2$, $n_1 = n_2 = 4$;
7. $n = 11$, $s = 2$, $\{n_1, n_2\} = \{5, 6\}$.

Every $\pi$-Hall subgroup of $S$ is equal to one of such subgroups $H$. It is easy to see that in distinct cases either the dimensions, or the sets $\pi(H)$ are distinct, so distinct statements cannot be fulfilled simultaneously. The number of classes of conjugate $\pi$-Hall subgroups in each statement can be calculated directly. They are as given below.

1. $k_\pi(S) = 2$;
2. $k_\pi(S) = 1$;
3. $k_\pi(S) = 2$;
4. $k_\pi(S) = 3$;
5. $k_\pi(S) = 2$;
6. $k_\pi(S) = 1$;
7. $k_\pi(S) = 2$.

Thus the claim follows. □

**Lemma 8.2** Suppose $S \cong \text{PSp}_{2\pi}(q) \in E_\pi$, where $2, 3 \in \pi$ and $p \notin \pi$. Assume also that $k_\pi(S) = 9$. Choose $G \in E_\pi$ so that $S \leq G \leq \text{Aut}(S)$. Then $k_\pi^G(S) \in \{1, 9\}$.

**Proof.** By Lemma 4.4 each $\pi$-Hall subgroup $H$ of $S$ is included in a subgroup

$$M = (\text{Sp}_{2\pi}(q) \wr \text{Sym}_n)/\text{Z(\text{Sp}_{2\pi}(q))} = L \cdot \text{Sym}_n,$$

where $L = L_1 \circ \ldots \circ L_n \cong \text{Sp}_{2\pi}(q) \circ \ldots \circ \text{Sp}_{2\pi}$). Moreover all such subgroups $M$ are conjugate in $S$, while $H, K \in \text{Hall}_\pi(M)$ are conjugate in $S$ if and only if $H, K$ are conjugate in $M$. Since $p$ is odd, $S$ does not possesses graph automorphisms. So $\text{Out}(S)$ is abelian and it is generated by a diagonal automorphism (of order 2) and a field automorphism. Clearly we may choose representatives $\delta$ of the diagonal
automorphism and $\varphi$ of the field automorphism so that both $\delta$ and $\varphi$ normalize each factor $L_i$, and $\delta$ is a 2-element. Moreover, preserving such a choice, we may also assume that $\delta$ induces a diagonal automorphism on each factor $L_i$, while $\varphi$ induces a field automorphism on each $L_i$.

Now $H = (H_1 \circ \ldots \circ H_n) \cdot \overline{H}$, where $H_i \in \mathrm{Hall}_p(L_i)$ and $\overline{H} \in \mathrm{Hall}_p(\mathrm{Sym}_n)$. Since $\kappa_p(S) = k_p(M) = 9$, Lemma 2.3 and the proof of Lemma 3.4 imply that $\overline{H}$, acting by conjugation on $\{L_1, \ldots, L_n\}$, has 2 orbits and $k_p(L_i) = 3$ for each $i$. By Lemma 3.1, we obtain that $L_i$ possesses a class $K_i$ of conjugate $\pi$-Hall subgroups satisfying Lemma 2.1(a), and classes $\mathcal{L}_i'$ and $\mathcal{L}_i''$ consisting of $\pi$-Hall subgroups satisfying 

If we choose $K_i \in K_i$ and consider $K = K_1 \circ \ldots \circ K_n$, then $K^G = K^{\mathrm{Aut}(S)}$, hence $k_p(G) \geqslant 1$.

If $k^G_p(S) \geqslant 2$, then there exists $F \in \mathrm{Hall}_p(N_G(M)) \subset \mathrm{Hall}_p(G)$ such that either $F \cap L_i \in \mathcal{L}_i'$, or $F \cap L_i \in \mathcal{L}_i''$ for some $i$. Since $\delta$ is a 2-element and normalizes $M$, the Sylow theorem implies that we may assume $\delta \in F$. Moreover, the induced action of both $\delta$ and $\varphi$ on $M/L$ is trivial. Since $FL/L$ is a $\pi$-Hall subgroup of $N_G(M)/L$, we obtain that $F$ contains an element $\psi$ such that $\psi$ normalizes each $L_i$, induces a field automorphism on each $L_i$, and $F = \langle \delta, \psi, H \rangle$, where $H = F \cap M \in \mathrm{Hall}_p(M) \subset \mathrm{Hall}_p(S)$. Consider $K = \langle \delta, \psi, F \cap L \rangle$. Then either $K \cap L_i \in \mathcal{L}_i'$, or $K \cap L_i \in \mathcal{L}_i''$, hence both classes $\mathcal{L}_i'$ and $\mathcal{L}_i''$ are invariant under $K$. By construction, both $\delta$ and $\psi$ induce automorphisms of the same type on each $L_j$, so we obtain that $\mathcal{L}_j'$ and $\mathcal{L}_j''$ are invariant under $K$ for $j = 1, \ldots, n$. In particular, each class of conjugate $\pi$-Hall subgroups of $L$ is invariant under $\delta$ and $\psi$. Lemma 2.1(e) implies that each $\pi$-Hall subgroup of $L$ is embedded into a $\pi$-Hall subgroup of $\langle \delta, \psi, L \rangle$, hence of $\langle \delta, \varphi, L \rangle$. Since both $\delta$ and $\varphi$ induce a trivial automorphism on $M/L \approx \mathrm{Sym}_n$, Lemma 2.1(d), (e) implies that a $\pi$-Hall subgroup $K$ of $L$ is embedded into a $\pi$-Hall subgroup of $\langle \delta, \psi, M \rangle$ if and only if $K$ is embedded into a $\pi$-Hall subgroup of $M$. Thus $\mathrm{Hall}_p(M) = \mathrm{Hall}^{N_G(M)}_p(M)$. Since all subgroups $M$ are conjugate in $S$ and $H_1, H_2 \in \mathrm{Hall}_p(M)$ are conjugate in $S$ if and only if they are conjugate in $M$, we obtain that $9 = k_p(S) = k_p(M) = k^G_p(N_G(M)) \leqslant k^G_p(S) \leqslant 9$. \[\square\]

Now we are able to prove Theorem 1.1.

Without lost of generality we may assume that $G \in E_\pi$, and by Lemma 2.1(a) we have $S \in E_\pi$.

If $2 \notin \pi$, then $\pi$-Hall subgroups of $S$ are conjugate by [9, Theorem B], i.e., $k_p(S) = 1$, whence $k^G_p(S) = 1$.

If $2 \in \pi$ and $3 \notin \pi$, then [28, Lemma 5.1 and Theorem 5.2] implies that $S \in C_\pi$, unless $S = \overline{G}_2(q)$. Therefore $k^G_p(S) = k_p(S) = 1$ if $S \neq \overline{G}_2(q)$. If $S = \overline{G}_2(q)$, $q = 3^{2n+1}$ and either $\pi \cap \pi(S) \neq \{2, 7\}$, or $n \equiv 3 \pmod{7}$, then again $k^G_p(S) = k_p(S) = 1$.

Suppose that $S = \overline{G}_2(q)$, $q = 3^{2n+1}$, $\pi \cap \pi(S) = \{2, 7\}$, $n \not\equiv 3 \pmod{7}$ and $S \in E_\pi$. By [23, Lemma 5.1], we have that $7 \in \pi(q + 1)$ and every $\pi$-Hall subgroup of $S$ is either included in the normalizer of a maximal torus of order $q + 1$, or is a
Frobenius group of order 56. In the first case a $\pi$-Hall subgroup has a Sylow tower of complexion $2 < 7$, and in the second case a $\pi$-Hall subgroup has a Sylow tower of complexion $7 < 2$. Thus $k_\pi(S) = 2$ and, since $k_\pi^G(S) \leq k_\pi(S)$, it follows that $k_\pi^G(S) \in \{1, 2\}$.

Since the inequality $k_\pi(S) \geq k_\pi^G(S)$ is evident, we may assume that $k_\pi(S) > 4$. Lemma [8.1] implies that $k_\pi(S) = 9$ and $S \cong PSp_{2n}(q)$. Now Theorem [17] follows from Lemma [8.2].

9 Proof of Corollary 1.3

Lemma [2.1](a) implies that $A$ satisfies $E_\pi$. Each $\pi$-subgroup $K$ of $A$ is included in a $\pi$-Hall subgroup $H$ of $G$. Hence $K$ is included in a $\pi$-Hall subgroup $H \cap A$ of $A$. So it remains to show that $A \in C_\pi$. We proceed by induction on $|A|$.

Suppose there exists a normal subgroup $N$ of $G$ such that $1 < N < A$. Then Lemma [2.1](c) implies that $G/N$ satisfies $D_\pi$ and $A/N$ is normal in $G/A$. By induction both $A/N$ and $N$ satisfy $C_\pi$, whence $A \in C_\pi$ by Lemma [2.1](f). Thus we may assume that $A$ is a minimal normal subgroup of $G$, so

$$A = S_1 \times \cdots \times S_n,$$

where $S_1, \ldots, S_n$ are conjugate simple subgroups of $G$. In view of the Hall theorem we may also assume that $S_i$ is nonabelian for $i = 1, \ldots, n$.

By Theorem [1.1] it follows that $k = k_\pi(S_i)$ is a $\pi$-number. Clearly each $\pi$-Hall subgroup of $A$ is equal to

$$\langle K_1, \ldots, K_n \rangle = K_1 \times \cdots \times K_n,$$

where $K_i$ is a $\pi$-Hall subgroup of $S_i$ for $i = 1, \ldots, n$, and vice versa. So $k_\pi(A) = k^n$ is a $\pi$-number.

The group $G$ acts by conjugation on the set $\Omega$ of classes of conjugate $\pi$-Hall subgroups of $A$. Moreover this action is transitive. Indeed, since $G$ satisfies $D_\pi$, each $\pi$-Hall subgroup of $A$ has the form $H \cap A$, where $H$ is a $\pi$-Hall subgroup of $G$. Since $G$ acts by conjugation transitively on $HALL_\pi(G)$, it follows that $G$ acts by conjugation transitively on $HALL_\pi(A)$, hence on $\Omega$. We fix $H \in HALL_\pi(G)$, let $\Delta \in \Omega$ be the class of conjugate $\pi$-Hall subgroups of $A$, containing $H \cap A$. Since $H$ normalizes $H \cap A$, it follows that $H$ is included in the stabilizer $G_\Delta$ of this class. So $k_\pi(A) = |\Omega| = |G : G_\Delta|$ divides $|G : H|$. Hence $k_\pi(A)$ is a $\pi'$-number. Thus $k_\pi(A)$ is a $\pi$- and a $\pi'$-number, so $k_\pi(A) = 1$ and $A \in C_\pi$.

Notice that Lemma [2.1](e) implies $k_\pi^G(S) = 2$, since classes of nonisomorphic subgroups cannot be interchanged by an automorphism.
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