Monotone Iterative Method for $\psi$-Caputo Fractional Differential Equation with Nonlinear Boundary Conditions

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Abstract: The main contribution of this paper is to prove the existence of extremal solutions for a novel class of $\psi$-Caputo fractional differential equation with nonlinear boundary conditions. For this purpose, we utilize the well-known monotone iterative technique together with the method of upper and lower solutions. Finally, we provide an example along with graphical representations to confirm the validity of our main results.

Keywords: $\psi$-Caputo operator; extremal solutions; monotone iterative style; upper (lower) solutions; boundary conditions

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1. Introduction

In the mathematical modeling of real life phenomena, the study of fractional differential equations has gained notable importance among interested researchers. It is realized that the use of fractional calculus methods is quite prominent in modeling various processes. The main reason for the widespread use (applications) of fractional operators is the fact that, unlike “integer” operators, these operators possess non-local behavior which enables us to trace the past effects of the involved phenomena [1–5]. Based on some classical approaches, such as Riemann-Liouville, Caputo, and Hadamard fractional operators, there are many new definitions which have attempted to provide a general platform that includes these classical operators [6–8]. For the sake of consolidating these different definitions under one single fractional operator, the $\psi$-fractional operator has been introduced [9,10]. The main feature of the the $\psi$-fractional operator is that the function in its integral kernel an be adapted to accommodate other definitions when replacing it by specific functions. Other significant features include non-local behavior and the semigroup property which are clearly preserved. It has been recognized that these types of operators have been successfully used to describe and model many real life phenomena; therefore, several related research works have been produced [11–13]. Along with the recent developments in fractional differential equations, researchers have contributed many research studies that discus the solutions’ behavior in terms of different types of fractional differential equations [14–27].
There are different types of fractional differential equations involving different fractional operators, which are associated with the various types of initial and boundary conditions that have been investigated by many researchers in various research works [28-30]. By exploring the literature, one can figure out that the existence of solutions has been the main target of investigations. In order to prove this, researchers often utilize some fixed point hypothesis along with certain mathematical inequalities. To the best of our observations, the monotone iterative technique combined with the method of upper and lower solutions has not been used to study the existence of solutions for \( \psi \)-Caputo fractional differential equation with nonlinear boundary conditions. For more expository details on monotone iterative method, the readers can consult some interesting research works [31–35].

Oriented by the above discussion, we study the following \( \psi \)-Caputo fractional differential equation (CpFDE) with nonlinear boundary conditions:

\[
\begin{cases}
\epsilon D_{a+}^{\tau \psi} \left( \epsilon D_{a+}^{\lambda \psi} - \sigma \right) \hat{z}(\theta) = F \left( \theta, \hat{z}(\theta), \epsilon D_{a+}^{\lambda \psi} (\theta) \right), \\
G \left( \hat{z}(a), \hat{z}(b) \right) = 0,
\end{cases}
\]  

(1)

for \( \theta \in \Omega := [a, b] \), where \( \epsilon D_{a+}^{\tau \psi} \) and \( \epsilon D_{a+}^{\lambda \psi} \) denote the \( \psi \)-Caputo fractional derivatives of order \( \tau \) and \( \lambda \), respectively, such that \( \tau, \lambda \in (0,1), \sigma > 0, F \in C(\Omega \times \mathbb{R}^2, \mathbb{R}), G, G \in C(\mathbb{R}^2, \mathbb{R}). \) It is worth mentioning here that, unlike the above mentioned relevant works, the CpFDE (1) is subject to nonlinear boundary conditions. The above equation is the deterministic fractional differential equation, therefore, a fractional differential equation with its deterministic solution is only considered in this work without the involvement of any random processes.

The remaining part of the paper is organized as follows: In Section 2, we present some definitions and lemmas that will be used to prove our results. In Section 3, we prove our main results, which conveys the existence of extremal solutions for CpFDE (1). For this purpose, we use the monotone iterative method together with the technique of upper and lower solutions. In Section 4, we apply our results by providing an example and illustrate the solutions of behavior graphically.

2. Relevant Preliminaries

In the current section, we state some basic concepts of fractional calculus that are related to our work. Let \( \Omega = [a, b], 0 \leq a < b < \infty \) be a finite interval and \( \psi : \Omega \rightarrow \mathbb{R} \) be an increasing differentiable function such that \( \psi'(\theta) \neq 0, \) for all \( \theta \in \Omega. \)

**Definition 1 ([9]).** The Riemann–Lebesgue (RL) fractional integral of order \( \tau > 0 \) for an integrable function \( \hat{z} : \Omega \rightarrow \mathbb{R} \) with respect to \( \psi \) is described by the following:

\[
\Gamma_{a+}^{\tau \psi} \hat{z}(\theta) = \frac{1}{\Gamma(\tau)} \int_a^\theta \psi'(\eta)(\psi(\theta) - \psi(\eta))^{\tau-1} \hat{z}(\eta) \, d\eta,
\]

(2)

where \( \Gamma(\tau) = \int_0^{+\infty} \theta^{\tau-1} e^{-\theta} \, d\theta, \tau > 0 \) is the Gamma function.

**Definition 2 ([9]).** Let \( \psi, \hat{z} \in C^n(\Omega, \mathbb{R}) \). The RL fractional derivative of a function \( \hat{z} \) of order \( n - 1 < \tau < n \) with respect to \( \psi \) is given by the following:

\[
\Gamma_{a+}^{\tau \psi} \hat{z}(\theta) = \left( \frac{D_\theta}{\psi'(\theta)} \right)^n \Gamma_{a+}^{\tau \psi} \hat{z}(\theta) = \frac{1}{\Gamma(n-\tau)} \left( \frac{D_\theta}{\psi'(\theta)} \right)^n \int_a^\theta \psi'(\eta)(\psi(\theta) - \psi(\eta))^{n-\tau-1} \hat{z}(\eta) \, d\eta,
\]

where \( n = \lceil \tau \rceil + 1, n \in \mathbb{N} \) and \( D_\theta = \frac{d}{d\theta}. \)
Definition 3 ([9]). Let \( \psi, z \in C^0(\Omega, \mathbb{R}) \). The Caputo fractional derivative of \( z \) of order \( n-1 < \tau < n \) with respect to \( \psi \) is defined by the following

\[
c_{D}\tau;\psi D_{a+}z(\theta) = \sum_{n=\lceil \tau \rceil}^{n-1} \frac{D^n_{a+}z(\theta)}{\Gamma(n-\tau)}(\psi(\theta) - \psi(\eta)) - \frac{D^n_{a+}z(\theta)}{\Gamma(n-\tau)}(\psi(\eta) - \psi(\theta)) d\eta,
\]

where \( n = \lceil \tau \rceil + 1 \) for \( \tau \notin \mathbb{N} \), \( n = \tau \) for \( \tau \in \mathbb{N} \) and \( D^n_{a+}z(\theta) = \left( \frac{D^\theta}{\psi(\theta)} \right)^n D^\theta z(\theta) \). From the definition, it is clear that the following is the case.

\[
c_{D}\tau;\psi D_{a+}z(\theta) = \begin{cases} \int_0^\theta \frac{\psi'(|\tau|)(\psi(\theta)|\tau|^{-\tau} - \psi(\eta))}{\Gamma(n-\tau)} \frac{D^n_{a+}z(\theta)}{\psi(\eta)} d\eta, & \tau \notin \mathbb{N}, \\ \frac{D^n_{a+}z(\theta)}{\psi(\eta)}(\psi(\eta) - \psi(\theta)), & \tau \in \mathbb{N}. \end{cases}
\]

Some basic properties of the \( \psi \)-fractional operators are listed in the following Lemma.

Lemma 1 ([9]). Let \( \tau, \lambda > 0 \), and \( z \in C(\Omega, \mathbb{R}) \). Then for each \( \theta \in \Omega \), we have the following.

1. \( c_{D}\tau;\psi D_{a+}z(\theta) = z(\theta) \),
2. \( \|D_{a+}z(\theta)\| = \|z(a)\| \) for \( 0 < \tau \leq 1 \),
3. \( \|D_{a+}(\psi(\theta) - \psi(a))\|^{\lambda-1} = \frac{\Gamma(\lambda)}{\Gamma(\lambda+\tau)}(\psi(\theta) - \psi(a))^{\lambda+\tau-1} \),
4. \( \|D_{a+}z(\theta) - \psi(a)\|^{\lambda-1} = \frac{\Gamma(\lambda)}{\Gamma(\lambda+\tau)}(\psi(\theta) - \psi(a))^{\lambda+\tau-1} \),
5. \( \|D_{a+}z(\theta) - \psi(a)\|^{k} = 0 \) for all \( k \in \{0, \ldots, n-1\} \), \( n \geq 1 \).

Definition 4 ([36]). The Mittag–Leffler functions (MLFs) of one and two parameters are given by:

\[
E_\nu(\omega) = \sum_{k=0}^{\infty} \frac{\omega^k}{\Gamma(k+1)}, \quad (\omega \in \mathbb{R}, \nu > 0),
\]

and

\[
E_{\nu,\lambda}(\omega) = \sum_{k=0}^{\infty} \frac{\omega^k}{\Gamma(k+\lambda)}, \quad (\nu, \lambda > 0, \omega \in \mathbb{R}),
\]

respectively. It is obvious that \( E_{1,1}(\omega) = E_1(\omega) = e^\omega \).

We denote the set \( \mathcal{X} \) by the following.

\(\mathcal{X} = C^1(\Omega) = \left\{ x : c_{D}\lambda;\psi D_{a+}x(\xi) \in C(\Omega) \right\} \).

Equipped with the norm, we have the following:

\[
\|x\|_{\mathcal{X}} = \|x\|_{\infty} + \left\| c_{D}\lambda;\psi D_{a+}x \right\|_{\infty},
\]

where \( \|x\|_{\infty} = \max_{\xi \in \Omega} |x(\xi)| \) and one can conclude that \( (\mathcal{X}, \| \cdot \|_{\mathcal{X}}) \) is a Banach space.

Lemma 2. For a given \( \ell \in C(\Omega, \mathbb{R}) \), \( \lambda, \tau \in (0, 1] \) and \( \sigma > 0 \), the linear fractional initial value problem is as follows:

\[
\begin{align*}
c_{D}\tau;\psi D_{a+}z(\theta) &= \ell(\theta), \\
c_{D}\lambda;\psi D_{a+}z(\theta) &= z(\theta) - \sigma z(a) = a,
\end{align*}
\]
for \( \vartheta \in \Omega \) is equivalent to the following Volterra integral equation.

\[
\vartheta(\vartheta) = \vartheta + \frac{3 \lambda - \sigma^3 \vartheta}{\Gamma(\vartheta + 1)} (\vartheta(\vartheta) - \vartheta(a))^\lambda + \sigma^{\lambda + \psi} \vartheta(\vartheta) + \Gamma^{\lambda + \psi}(\vartheta).
\] (7)

Moreover, the explicit solution of the Volterra integral Equation (7) can be represented by the following:

\[
\vartheta(\vartheta) = \vartheta + \frac{3 \lambda - \sigma^3 \vartheta}{\Gamma(\vartheta + 1)} (\vartheta(\vartheta) - \vartheta(a))^\lambda + \sigma^{\lambda + \psi} \vartheta(\vartheta) + \Gamma^{\lambda + \psi}(\vartheta).
\] (8)

Proof. Applying the \( \psi \)-Riemann-Liouville fractional integral of order \( \vartheta \) to both sides of (6) and by using Lemma 1, we obtain the following.

\[
\Gamma^{\lambda + \psi}(\vartheta) = \vartheta(\vartheta) + \sigma^3 \vartheta(\vartheta) + \Gamma^{\lambda + \psi}(\vartheta).
\] (9)

Hence, we have the following.

\[
\vartheta(\vartheta) = \vartheta + \frac{3 \lambda - \sigma^3 \vartheta}{\Gamma(\vartheta + 1)} (\vartheta(\vartheta) - \vartheta(a))^\lambda + \sigma^{\lambda + \psi} \vartheta(\vartheta) + \Gamma^{\lambda + \psi}(\vartheta).
\] (10)

The converse can be proven by direct computation. Now, we apply the method of successive approximations in order to prove that the integral Equation (7) can be expressed by the following.

\[
\vartheta(\vartheta) = \vartheta + \frac{3 \lambda - \sigma^3 \vartheta}{\Gamma(\vartheta + 1)} (\vartheta(\vartheta) - \vartheta(a))^\lambda + \sigma^{\lambda + \psi} \vartheta(\vartheta) + \Gamma^{\lambda + \psi}(\vartheta).
\] (11)

For this, we set the following.

\[
\begin{cases}
3_0(\vartheta) = 3_0 + \frac{3 \lambda - \sigma^3 \vartheta}{\Gamma(\vartheta + 1)} (\vartheta(\vartheta) - \vartheta(a))^\lambda, \\
3_m(\vartheta) = 3_0(\vartheta) + \sigma^{\lambda + \psi} 3_m(\vartheta) + \Gamma^{\lambda + \psi}(\vartheta).
\end{cases}
\] (12)

It follows from Equation (11) and Lemma 1 that the following is the case.

\[
\vartheta(\vartheta) = \vartheta + \frac{3 \lambda - \sigma^3 \vartheta}{\Gamma(\vartheta + 1)} (\vartheta(\vartheta) - \vartheta(a))^\lambda + \sigma^{\lambda + \psi} \vartheta(\vartheta) + \Gamma^{\lambda + \psi}(\vartheta).
\] (13)

Similarly, Equations (11) and (13) and Lemma 1 yield the following.

\[
\vartheta(\vartheta) = \vartheta + \frac{3 \lambda - \sigma^3 \vartheta}{\Gamma(\vartheta + 1)} (\vartheta(\vartheta) - \vartheta(a))^\lambda + \sigma^{\lambda + \psi} \vartheta(\vartheta) + \Gamma^{\lambda + \psi}(\vartheta).
\] (14)
\[ + \sigma \frac{3a}{\Gamma(\lambda + 1)} (\psi(\theta) - \psi(a))^{k\lambda} + \sigma \frac{3\lambda - \sigma 3a}{\Gamma(2\lambda + 1)} (\psi(\theta) - \psi(a))^{k\lambda} + \psi(a)^{2\lambda} + \frac{\psi(\theta)}{\Gamma(\lambda + 1)} + \frac{\lambda + \tau}{\Gamma(\lambda + 1)} \psi(\theta) \]

Taking the limit as \( m \to \infty \), we derive the following relation.

\[ z_m(\theta) = z_a + \frac{\sigma^k}{\Gamma(k\lambda + 1)} (\psi(\theta) - \psi(a))^{k\lambda} \]

\[ + \frac{3\lambda - \sigma 3a}{\Gamma(k\lambda + 1 + \lambda)} (\psi(\theta) - \psi(a))^{k\lambda + \lambda} \]

\[ + \int_a^\theta \psi'(\eta) \frac{\sigma^{k-1}(\psi(\theta) - \psi(\eta))^{k\lambda + \lambda}}{\Gamma(k\lambda + \tau)} \ell(\eta) \, d\eta \]

In continuing this process, we derive the following relation.

\[ z_m(\theta) = z_a + m \frac{\sigma^k}{\Gamma(k\lambda + 1)} (\psi(\theta) - \psi(a))^{k\lambda} \]

\[ + (3\lambda - \sigma 3a) m \frac{\sigma^k}{\Gamma(k\lambda + 1 + \lambda)} (\psi(\theta) - \psi(a))^{k\lambda + \lambda} \]

\[ + \int_a^\theta \psi'(\eta) \frac{\sigma^{k-1}(\psi(\theta) - \psi(\eta))^{k\lambda + \lambda - 1}}{\Gamma(k\lambda + \tau)} \ell(\eta) \, d\eta \]

Taking the limit as \( m \to \infty \), we obtain the following explicit solution \( z(\theta) \) of the integral Equation (7).

\[ z(\theta) = z_a + \frac{\sigma^k}{\Gamma(k\lambda + 1)} (\psi(\theta) - \psi(a))^{k\lambda} \]

\[ + (3\lambda - \sigma 3a) \frac{\sigma^k}{\Gamma(k\lambda + 1 + \lambda)} (\psi(\theta) - \psi(a))^{k\lambda + \lambda} \]

\[ + \int_a^\theta \psi'(\eta) (\psi(\theta) - \psi(\eta))^{k\lambda + \lambda - 1} \frac{\sigma^{k-1}}{\Gamma(k\lambda + \tau)} \ell(\eta) \, d\eta \]
\[ + \int_{\theta}^{\psi} (\psi(\theta) - \psi(\eta)) \Delta(\theta) \geq 0, \quad \theta \in (a, b], \]
\[ \Delta(a) \geq 0, \quad cD_{a}^{\psi} \Delta(a) \geq 0, \]

Then, the proof is completed. \[ \square \]

**Lemma 3** (Comparison Result). Let \( \lambda, \tau \in (0, 1) \), and \( \sigma > 0 \). If \( \Delta \in C(\Omega, \mathbb{R}) \) fulfills the following inequalities:

\[
\left\{ \begin{array}{ll}
\int cD_{a}^{\psi} \left( cD_{a}^{\lambda, \psi} - \sigma \right) \Delta(\theta) \geq 0, & \theta \in (a, b], \\
\Delta(a) \geq 0, & cD_{a}^{\lambda, \psi} \Delta(a) \geq 0,
\end{array} \right.
\]

then \( \Delta(\theta) \geq 0 \) and \( cD_{a}^{\lambda, \psi} \Delta(\theta) \geq 0 \) for all \( \theta \in \Omega \). 

**Proof.** Since \( E_{\rho_{1}, \rho_{2}}(x) \geq 0 \) for \( \rho_{1}, \rho_{2} \in (0, 1) \), we allow the following:

\[ \ell(\theta) = cD_{a}^{\psi} \left( cD_{a}^{\lambda, \psi} - \sigma \right) \Delta(\theta) \geq 0, \]

\( \Delta(a) = \lambda \geq 0 \) and \( cD_{a}^{\lambda, \psi} \Delta(a) = \lambda \geq 0 \) in Lemma 2. Then, it follows by Equations (8) and (9) that the conclusion of Lemma 3 holds. \[ \square \]

Let \( (X, \mathcal{T}) \) be a topological Hausdorff space and \( g_{1}, g_{2} : X \to \mathbb{R} \) be a lower semi-continuous function and an upper semi-continuous function, respectively. This means that for every \( r \in \mathbb{R} \), the subsets of the following:

\[ \{ g_{1} > r \} := \{ x \in X : g_{1}(x) > r \}, \quad \{ g_{2} < r \} := \{ x \in X : g_{2}(x) < r \}, \]

are open in \( X \). Suppose that \( g_{1}(x) \leq g_{2}(x) \) for all \( x \in X \) and we allow the interval \( [g_{1}, g_{2}] \) consist of those upper or lower semi-continuous functions \( h : X \to \mathbb{R} \) such that \( g_{1}(x) \leq h(x) \leq g_{2}(x) \) for all \( x \in X \). Let \( \Omega : [g_{1}, g_{2}] \to [g_{1}, g_{2}] \) be a monotone mapping in the sense that \( g_{1} \leq h_{1} \leq h_{2} \leq g_{2} \) implies \( g_{1} \leq \Omega(h_{1}) \leq \Omega(h_{2}) \leq g_{2} \). In addition, suppose that the sequence \( \{ \Omega(h_{n}) \}_{n \in \mathbb{N}} \subset [g_{1}, g_{2}] \) consist of lower semi-continuous functions that increase pointwise to \( h \) whenever the sequence \( \{ h_{n} \}_{n \in \mathbb{N}} \subset [g_{1}, g_{2}] \) consist of lower semi-continuous functions that decrease pointwise to \( h \). A similar assumption is made when the sequence \( \{ h_{n} \}_{n \in \mathbb{N}} \subset [g_{1}, g_{2}] \) consist of upper semi-continuous functions, which decreases pointwise to \( h \in [g_{1}, g_{2}] \). In particular, assume that \( \Omega(h) \) is lower semi-continuous whenever \( h \) is lower semi-continuous and that \( \Omega(h) \) is upper semi-continuous whenever \( h \) is so. Then for every \( n \in \mathbb{N} \), we have the following:

\[ g_{1} \leq \Omega^{n}(g_{1}) \leq \Omega^{n+1}(g_{1}) \leq \Omega^{n+1}(g_{2}) \leq \Omega^{n}(g_{2}) \leq g_{2}. \]

Substitute \( \omega_{1} = \sup_{n \in \mathbb{N}} \Omega^{n}(g_{1}) \) and \( \omega_{2} = \sup_{n \in \mathbb{N}} \Omega^{n}(g_{2}) \). Then, \( \omega_{1} \) and \( \omega_{2} \) belong to the interval \( [g_{1}, g_{2}] \), the function \( \omega_{1} \) is lower semi-continuous, the function \( \omega_{2} \) is upper semi-continuous, and the equalities \( \Omega(\omega_{1}) = \omega_{1} \) and \( \Omega(\omega_{2}) = \omega_{2} \) are valid. If the monotone mapping \( \Omega \) has at most one fixed point, then \( \omega_{1} = \omega_{2} = \Omega(\omega_{1}) = \Omega(\omega_{2}) \) is a continuous function. When the mapping \( \Omega : [g_{1}, g_{2}] \to [g_{1}, g_{2}] \) does not posses this sequential continuity property, then one needs a more subtle version of the Tarski-Knaster fixed point theorem. More precisely, define the functions \( h_{\text{prefix}} \) and \( h_{\text{postfix}} \), respectively, by the following:

\[ h_{\text{prefix}} = \inf \{ h \in [g_{1}, g_{2}] : \Omega(h) \leq h, \ h \text{ is upper semi-continuous} \}, \]

and the following.

\[ h_{\text{postfix}} = \sup \{ h \in [g_{1}, g_{2}] : \Omega(h) \geq h, \ h \text{ is lower semi-continuous} \}. \]
Then, we obtain the following.

\[ \Omega(h_{\text{prefix}}) = h_{\text{prefix}} \leq \inf_{n \in \mathbb{N}} \Omega^n(g_2), \quad \Omega(h_{\text{postfix}}) = h_{\text{postfix}} \geq \sup_{n \in \mathbb{N}} \Omega^n(g_1). \]

Moreover, the function \( h_{\text{prefix}} \) is upper semi-continuous and the function \( h_{\text{postfix}} \) is lower semi-continuous. Consequently, if \( \Omega \) has at most one fixed point, then

\[ \Omega(h_{\text{prefix}}) = h_{\text{prefix}} = \Omega(h_{\text{postfix}}) = h_{\text{postfix}}, \]

and, therefore, this unique fixed point is a continuous function.

### 3. Main Results

In this section, we prove the existence of extremal solutions for problem (1). Before proceeding, we provide the definitions of lower and upper solutions of the problem (1).

**Definition 5.** A function \( 0 \in \mathbb{X} \) is called a lower solution of (1), if it satisfies the following:

\[
\begin{align*}
\left\{ \begin{array}{l}
\mathring{c}D^{\lambda, \psi}_{a^+} \left( cD^{\lambda, \psi}_{a^+} - \sigma \right) \bar{z}_0(\theta) & \leq \mathcal{F} \left( \theta, \bar{z}_0(\theta), cD^{\lambda, \psi}_{a^+} \bar{z}_0(\theta) \right), \\
\mathcal{H} \left( cD^{\lambda, \psi}_{a^+} \bar{z}_0(a), cD^{\lambda, \psi}_{a^+} \bar{z}_0(b) \right) & \leq 0, \quad \mathcal{G}(\bar{z}_0(a), \bar{z}_0(b)) \leq 0,
\end{array} \right.
\end{align*}
\]

for \( \theta \in \Omega \).

**Definition 6.** A function \( \bar{z}_0 \in \mathbb{X} \) is called an upper solution of (1), if it satisfies the following:

\[
\begin{align*}
\left\{ \begin{array}{l}
\mathring{c}D^{\lambda, \psi}_{a^+} \left( cD^{\lambda, \psi}_{a^+} - \sigma \right) \tilde{z}_0(\theta) & \geq \mathcal{F} \left( \theta, \tilde{z}_0(\theta), cD^{\lambda, \psi}_{a^+} \tilde{z}_0(\theta) \right), \\
\mathcal{H} \left( cD^{\lambda, \psi}_{a^+} \tilde{z}_0(a), cD^{\lambda, \psi}_{a^+} \tilde{z}_0(b) \right) & \geq 0, \quad \mathcal{G}(\tilde{z}_0(a), \tilde{z}_0(b)) \geq 0,
\end{array} \right.
\end{align*}
\]

for each \( \theta \in \Omega \).

**Theorem 1.** Let \( \mathcal{F} : \Omega \times \mathbb{R}^2 \to \mathbb{R} \) be a continuous function such that the following assumptions hold:

1. **(H1)** There exist \( \bar{z}_0 \) and \( \tilde{z}_0 \) as lower and upper solutions of (1) in \( \mathbb{X} \), respectively, with \( \bar{z}_0(\theta) \leq \tilde{z}_0(\theta) \) and:

\[ cD^{\lambda, \psi}_{a^+} \bar{z}_0(\theta) \leq cD^{\lambda, \psi}_{a^+} \tilde{z}_0(\theta), \quad \theta \in \Omega. \]

2. **(H2)** \( \mathcal{F} \) satisfies the following condition:

\[ \mathcal{F}(\theta, y(\theta), cD^{\lambda, \psi}_{a^+} y(\theta)) \leq \mathcal{F}(\theta, \bar{z}(\theta), cD^{\lambda, \psi}_{a^+} \bar{z}(\theta)), \]

for \( y_0(\theta) \leq y(\theta) \leq \bar{z}(\theta) \) and the following:

\[ cD^{\lambda, \psi}_{a^+} y_0(\theta) \leq cD^{\lambda, \psi}_{a^+} y(\theta) \leq cD^{\lambda, \psi}_{a^+} \bar{z}(\theta) \leq cD^{\lambda, \psi}_{a^+} \tilde{z}(\theta), \]

for each \( \theta \in \Omega \).

3. **(H3)** There exist constants \( c > 0 \) and \( d \geq 0 \), such that for \( \bar{z}_0(a) \leq \bar{z}_1 \leq \bar{z}_2 \leq \tilde{z}_0(a) \) and \( \bar{z}_0(b) \leq \bar{z}_1 \leq \bar{z}_2 \leq \tilde{z}_0(b) \),

\[ \mathcal{G}(\bar{z}_2, \bar{z}_1) - \mathcal{G}(\bar{z}_1, \bar{z}_2) \leq c(\bar{z}_2 - \bar{z}_1) - d(\bar{z}_2 - \bar{z}_1). \]

4. **(H4)** There exist constants \( c > 0 \) and \( f \geq 0 \), such that for the following:

\[ cD^{\lambda, \psi}_{a^+} \bar{z}_0(a) \leq \bar{z}_1 \leq \bar{z}_2 \leq cD^{\lambda, \psi}_{a^+} \tilde{z}_0(a), \]
\[ c\mathbb{D}_{a+}^{\lambda, \Psi} z_0(b) \leq \zeta_1 \leq \zeta_2 \leq c\mathbb{D}_{a+}^{\lambda, \Psi} z_0(b), \]

and the following obtains.

\[ \mathbb{H}(\zeta_2, \zeta_2) - \mathbb{H}(\zeta_1, \zeta_1) \leq c(\zeta_2 - \zeta_1) - f(\zeta_2 - \zeta_1). \]

Then, there exist monotone iterative sequences \( \{z_n\} \) and \( \{\bar{z}_n\} \), which converge uniformly on \( \Omega \) to the extremal solutions of (1) in the sector \([z_0, \bar{z}_0] \), where

\[ [z_0, \bar{z}_0] = \left\{ z \in \mathbb{X} : z_0(\theta) \leq z(\theta) \leq \bar{z}_0(\theta), \quad \theta \in \Omega \right\}. \]

**Proof.** For any \( z_0, \bar{z}_0 \in \mathbb{X} \), we define the following:

\[
\begin{cases}
\mathbb{D}_{a^+}^{\tau, \Psi} \left( c\mathbb{D}_{a^+}^{\lambda, \Psi} - \sigma \right) z_{n+1}(\theta) = \mathbb{F}(\theta, z_n(\theta), c\mathbb{D}_{a^+}^{\lambda, \Psi} z_n(\theta)), & \theta \in \Omega, \\
\mathbb{D}_{a^+}^{\lambda, \Psi} z_{n+1}(a) = c\mathbb{D}_{a^+}^{\lambda, \Psi} z_n(a) - \frac{1}{\varepsilon} \mathbb{H} \left( c\mathbb{D}_{a^+}^{\lambda, \Psi} z_n(a), c\mathbb{D}_{a^+}^{\lambda, \Psi} z_n(b) \right), & \end{cases}
\]

and the following as well.

\[
\begin{cases}
\mathbb{D}_{a^+}^{\tau, \Psi} \left( c\mathbb{D}_{a^+}^{\lambda, \Psi} - \sigma \right) \bar{z}_{n+1}(\theta) = \mathbb{F}(\theta, \bar{z}_n(\theta), c\mathbb{D}_{a^+}^{\lambda, \Psi} \bar{z}_n(\theta)), & \theta \in \Omega, \\
\mathbb{D}_{a^+}^{\lambda, \Psi} \bar{z}_{n+1}(a) = c\mathbb{D}_{a^+}^{\lambda, \Psi} \bar{z}_n(a) - \frac{1}{\varepsilon} \mathbb{H} \left( c\mathbb{D}_{a^+}^{\lambda, \Psi} \bar{z}_n(a), c\mathbb{D}_{a^+}^{\lambda, \Psi} \bar{z}_n(b) \right), & \end{cases}
\]

By Lemma 2, we know that (13) and (14) have unique solutions in \( \mathbb{X} \) that are the following.

\[
\bar{z}_{n+1}(\theta) = \bar{z}_n(a) - \frac{1}{\varepsilon} \mathbb{G}(\bar{z}_n(a), \bar{z}_n(b))
\]

\[
+ \left( c\mathbb{D}_{a^+}^{\lambda, \Psi} \bar{z}_n(a) - \frac{1}{\varepsilon} \mathbb{H} \left( c\mathbb{D}_{a^+}^{\lambda, \Psi} \bar{z}_n(a), c\mathbb{D}_{a^+}^{\lambda, \Psi} \bar{z}_n(b) \right) \right) \times (\mathbb{F}(\theta, \bar{z}_n(a), c\mathbb{D}_{a^+}^{\lambda, \Psi} \bar{z}_n(b)) (\sigma(\mathbb{F}(\theta, \bar{z}_n(a), c\mathbb{D}_{a^+}^{\lambda, \Psi} \bar{z}_n(b)))^\lambda)
\]

\[
+ \int_\theta^\sigma \psi^\prime(\eta) (\mathbb{F}(\theta, \bar{z}_n(a), c\mathbb{D}_{a^+}^{\lambda, \Psi} \bar{z}_n(b)) (\sigma(\mathbb{F}(\theta, \bar{z}_n(a), c\mathbb{D}_{a^+}^{\lambda, \Psi} \bar{z}_n(b)))^\lambda)
\]

\[
\times E_{\lambda, \delta} \left( \bar{z}_{n+1}(\theta) - \psi(\eta) \left( c\mathbb{D}_{a^+}^{\lambda, \Psi} \bar{z}_n(\eta) \right) \right) d\eta
\]

\[
\bar{z}_{n+1}(\theta) = \bar{z}_n(a) - \frac{1}{\varepsilon} \mathbb{G}(\bar{z}_n(a), \bar{z}_n(b))
\]

\[
+ \left( c\mathbb{D}_{a^+}^{\lambda, \Psi} \bar{z}_n(a) - \frac{1}{\varepsilon} \mathbb{H} \left( c\mathbb{D}_{a^+}^{\lambda, \Psi} \bar{z}_n(a), c\mathbb{D}_{a^+}^{\lambda, \Psi} \bar{z}_n(b) \right) \right) \times (\mathbb{F}(\theta, \bar{z}_n(a), c\mathbb{D}_{a^+}^{\lambda, \Psi} \bar{z}_n(b)) (\sigma(\mathbb{F}(\theta, \bar{z}_n(a), c\mathbb{D}_{a^+}^{\lambda, \Psi} \bar{z}_n(b)))^\lambda)
\]

\[
+ \int_\theta^\sigma \psi^\prime(\eta) (\mathbb{F}(\theta, \bar{z}_n(a), c\mathbb{D}_{a^+}^{\lambda, \Psi} \bar{z}_n(b)) (\sigma(\mathbb{F}(\theta, \bar{z}_n(a), c\mathbb{D}_{a^+}^{\lambda, \Psi} \bar{z}_n(b)))^\lambda)
\]

\[
\times E_{\lambda, \delta} \left( \bar{z}_{n+1}(\theta) - \psi(\eta) \left( c\mathbb{D}_{a^+}^{\lambda, \Psi} \bar{z}_n(\eta) \right) \right) d\eta
\]

First, we show that the sequences \( z_n(\theta) \) and \( \bar{z}_n(\theta) \) (\( n \geq 1 \)) are lower and upper solutions of (1), respectively, and \( z_n(\theta) \) and \( \bar{z}_n(\theta) \) (\( n \geq 1 \)) satisfy the following relations:

\[ z_0(\theta) \leq z_1(\theta) \leq \cdots \leq z_n(\theta) \leq \cdots \leq \bar{z}_n(\theta) \leq \cdots \leq \bar{z}_1(\theta) \leq \bar{z}_0(\theta), \quad (15) \]
for \( \vartheta \in \Omega \) and the following is the case:

\[
\cd_{a^+}^{\lambda \psi} 3_0(\vartheta) \leq \cd_{a^+}^{\lambda \psi} 3_1(\vartheta) \leq \cd_{a^+}^{\lambda \psi} 3_{n}(\vartheta) \leq \cd_{a^+}^{\lambda \psi} 3_0(\vartheta),
\]

for each \( \vartheta \in \Omega \). Now, we show that \( 3_0(\vartheta) \leq 3_1(\vartheta) \leq 3_0(\vartheta) \), for \( \vartheta \in \Omega \) and

\[
\cd_{a^+}^{\lambda \psi} 3_0(\vartheta) \leq \cd_{a^+}^{\lambda \psi} 3_1(\vartheta) \leq \cd_{a^+}^{\lambda \psi} 3_0(\vartheta),
\]

for this end, set \( \Delta(\vartheta) = 3_1(\vartheta) - 3_0(\vartheta) \). From (13) and Definition 5, we obtain the following.

\[
\cd_{a^+}^{\lambda \psi} \left( \cd_{a^+}^{\lambda \psi} - \sigma \right) \Delta(\vartheta) = \cd_{a^+}^{\lambda \psi} \left( \cd_{a^+}^{\lambda \psi} - \sigma \right) 3_1(\vartheta) - \cd_{a^+}^{\lambda \psi} \left( \cd_{a^+}^{\lambda \psi} - \sigma \right) 3_0(\vartheta)
\]

Again, this is because the following is the case.

\[
\Delta(a) = -\frac{1}{c} G(3_0(a), 3_0(b)) \geq 0,
\]

\[
\cd_{a^+}^{\lambda \psi} \Delta(a) = -\frac{1}{c} H(\cd_{a^+}^{\lambda \psi} 3_0(a), \cd_{a^+}^{\lambda \psi} 3_0(b)) \geq 0.
\]

Invoking Lemma 3, we obtain \( \Delta(\vartheta) \geq 0 \) and \( \cd_{a^+}^{\lambda \psi} \Delta(\vartheta) \geq 0 \) for \( \vartheta \in \Omega \). Thus, \( 3_0(\vartheta) \leq 3_1(\vartheta) \) and the following is the case:

\[
\cd_{a^+}^{\lambda \psi} 3_0(\vartheta) \leq \cd_{a^+}^{\lambda \psi} 3_1(\vartheta),
\]

\( \vartheta \in \Omega \). In a similar manner, we can find that \( 3_1(\vartheta) \leq 3_0(\vartheta) \) and the following:

\[
\cd_{a^+}^{\lambda \psi} 3_1(\vartheta) \leq \cd_{a^+}^{\lambda \psi} 3_0(\vartheta),
\]

\( \vartheta \in \Omega \). Now, let \( \Delta(\vartheta) = \bar{3}_1(\vartheta) - 3_1(\vartheta) \). Using (13) and (14) together with assumptions (H1)–(H3) we obtain the following.

\[
\cd_{a^+}^{\lambda \psi} \left( \cd_{a^+}^{\lambda \psi} - \sigma \right) \Delta(\vartheta) = F(\vartheta, 3_0(\vartheta), \cd_{a^+}^{\lambda \psi} 3_0(\vartheta)) - F(\vartheta, 3_0(\vartheta), \cd_{a^+}^{\lambda \psi} 3_0(\vartheta)) \geq 0.
\]

Notice the following inequalities:

\[
\Delta(a) = 3_0(a) - 3_0(a) - \frac{1}{c} [f(G(3_0(a), 3_0(b)) - G(3_0(a), 3_0(b)))]
\]

\[
\geq \frac{d}{c} (3_0(b) - 3_0(b)) \geq 0,
\]

and the following is the case.

\[
\cd_{a^+}^{\lambda \psi} \Delta(a) = \cd_{a^+}^{\lambda \psi} 3_0(a) - \cd_{a^+}^{\lambda \psi} 3_0(a)
\]

\[
- \frac{1}{c} \left[ f\left( \cd_{a^+}^{\lambda \psi} 3_0(a), \cd_{a^+}^{\lambda \psi} 3_0(b) \right) - f\left( \cd_{a^+}^{\lambda \psi} 3_0(a), \cd_{a^+}^{\lambda \psi} 3_0(b) \right) \right]
\]

\[
\geq \frac{f}{c} \left( \cd_{a^+}^{\lambda \psi} 3_0(b) - \cd_{a^+}^{\lambda \psi} 3_0(b) \right) \geq 0.
\]

According to Lemma 3, we obtain \( 3_1(\vartheta) \leq \bar{3}_1(\vartheta) \) and

\[
\cd_{a^+}^{\lambda \psi} 3_1(\vartheta) \leq \cd_{a^+}^{\lambda \psi} 3_1(\vartheta),
\]
\( \theta \in \Omega \). Next, we show that the functions \( \bar{z}_1(\theta), \underline{z}_1(\theta) \) are a lower and an upper solution of the equation in (1), respectively. Since \( \bar{z}_0 \) and \( \underline{z}_0 \) are lower and upper solutions of (1), by (H2) and (H3), it follows that the following is the case:

\[
\begin{align*}
&cD_{\alpha^+}^{\lambda, \psi}(cD_{\alpha^+}^{\lambda, \psi} - c)\bar{z}_1(\theta) = F(\theta, \bar{z}_0(\theta), cD_{\alpha^+}^{\lambda, \psi} \underline{z}_0(\theta)) \\
&\quad \leq F(\theta, \bar{z}_1(\theta), cD_{\alpha^+}^{\lambda, \psi} \underline{z}_1(\theta)) \\
&\quad \leq \mathbb{E}(\bar{z}_1(a), \underline{z}_1(b)) \leq \mathbb{G}(\bar{z}_0(a), \underline{z}_0(b)) + c(\bar{z}_1(a) - \underline{z}_0(a)) - d(\bar{z}_1(b) - \underline{z}_0(b)) \\
&\quad = -d(\bar{z}_1(b) - \underline{z}_0(b)) \leq 0,
\end{align*}
\]

and the following obtains.

\[
\begin{align*}
\mathbb{H}(cD_{\alpha^+}^{\lambda, \psi} \bar{z}_1(a), cD_{\alpha^+}^{\lambda, \psi} \underline{z}_1(b)) &\leq \mathbb{G}(\bar{z}_0(a), \underline{z}_0(b)) \\
&\quad + e(cD_{\alpha^+}^{\lambda, \psi} \bar{z}_1(a) - cD_{\alpha^+}^{\lambda, \psi} \underline{z}_0(a)) \\
&\quad - f(cD_{\alpha^+}^{\lambda, \psi} \bar{z}_1(b) - cD_{\alpha^+}^{\lambda, \psi} \underline{z}_0(b)) \\
&= -f(cD_{\alpha^+}^{\lambda, \psi} \bar{z}_1(b) - cD_{\alpha^+}^{\lambda, \psi} \underline{z}_0(b)) \leq 0.
\end{align*}
\]

Therefore, \( \bar{z}_1(\theta) \) is a lower solution of (1). Analogously, it can be obtained that \( \underline{z}_1(\theta) \) is an upper solution of (1). By the above arguments and mathematical induction, we can show that the sequences \( \bar{z}_n(\theta), \underline{z}_n(\theta), (n \geq 1) \) are lower and upper solutions of (1), respectively, and the relations (15) and (16) are true. On the contrary, by employing the earlier arguments, together with Ascoli–Arzelà’s Theorem, we can show that the following:

\[
\| \bar{z}_n - \bar{z}^* \|_X \to 0, \quad \| \underline{z}_n - \underline{z}^* \|_X \to 0,
\]

when \( n \to \infty \). Finally, it remains to show that \( \bar{z}^* \) and \( \underline{z}^* \) are extremal solutions of (1) in \( [\bar{z}_0, \underline{z}_0] \). To conduct this, let \( \bar{z} \in [\bar{z}_0, \underline{z}_0] \) be any solution of (1). Suppose for some \( n \in \mathbb{N}^+ \) that the following is the case:

\[
\begin{align*}
\bar{z}_n(\theta) &\leq \bar{z}(\theta) \leq \underline{z}_n(\theta), \\
cD_{\alpha^+}^{\lambda, \psi} \bar{z}_n(\theta) &\leq cD_{\alpha^+}^{\lambda, \psi} \bar{z}(\theta) \leq cD_{\alpha^+}^{\lambda, \psi} \underline{z}_n(\theta), \quad (17)
\end{align*}
\]

for \( \theta \in \Omega \). Setting \( \Delta(\theta) = \bar{z}(\theta) - \bar{z}_{n+1}(\theta) \). It follows that the following obtains.

\[
\begin{align*}
&cD_{\alpha^+}^{\lambda, \psi}(cD_{\alpha^+}^{\lambda, \psi} - c)\Delta(\theta) = F(\theta, \bar{z}(\theta), cD_{\alpha^+}^{\lambda, \psi} \bar{z}(\theta)) - F(\theta, \bar{z}_n(\theta), cD_{\alpha^+}^{\lambda, \psi} \bar{z}_n(\theta)) \geq 0.
\end{align*}
\]

Notice the inequalities in the following:

\[
\begin{align*}
\bar{z}_{n+1}(a) &= \bar{z}_n(a) + \frac{1}{c} \left[ \mathbb{G}(\bar{z}(a), \underline{z}(b)) - \mathbb{G}(\bar{z}(a), \underline{z}_n(b)) \right] \\
&\leq \bar{z}(a) - \frac{d}{c}(\underline{z}(b) - \underline{z}_n(b)) \\
&\leq \underline{z}(a),
\end{align*}
\]

and the following.

\[
\begin{align*}
cD_{\alpha^+}^{\lambda, \psi} \bar{z}_{n+1}(a) &= cD_{\alpha^+}^{\lambda, \psi} \bar{z}_n(a) + \frac{1}{c} \left[ \mathbb{H}(\bar{z}(a), \underline{z}(b)) - \mathbb{H}(\bar{z}_n(a), \underline{z}_n(b)) \right] \\
&\leq cD_{\alpha^+}^{\lambda, \psi} \bar{z}(a) - \frac{f}{c} \left( cD_{\alpha^+}^{\lambda, \psi} \bar{z}(b) - cD_{\alpha^+}^{\lambda, \psi} \underline{z}_n(b) \right) \\
&\leq cD_{\alpha^+}^{\lambda, \psi} \bar{z}(a).
\end{align*}
\]

By Lemma 3, we obtain \( \Delta(\theta) \geq 0, \theta \in \Omega \), which implies \( \bar{z}_{n+1}(\theta) \leq \bar{z}(\theta) \) and the following:

\[
\begin{align*}
cD_{\alpha^+}^{\lambda, \psi} \bar{z}_{n+1}(\theta) &\leq cD_{\alpha^+}^{\lambda, \psi} \bar{z}(\theta),
\end{align*}
\]

for almost all \( \theta \in \Omega \).
By using the same method, we can show that $\tilde{z}(\theta) \leq \tilde{z}_{n+1}(\theta)$ and

$$cD_{a+}^{\lambda; \psi} \tilde{z}(\theta) \leq cD_{a+}^{\lambda; \psi} \tilde{z}_{n+1}(\theta),$$

for each $\theta \in \Omega$. Hence, $\tilde{z}_{n+1}(\theta) \leq \tilde{z}(\theta) \leq \tilde{z}_{n+1}(\theta)$, for $\theta \in \Omega$, and the following is the case:

$$cD_{a+}^{\lambda; \psi} \tilde{z}_{n+1}(\theta) \leq cD_{a+}^{\lambda; \psi} \tilde{z}(\theta) \leq cD_{a+}^{\lambda; \psi} \tilde{z}_{n+1}(\theta),$$

for $\theta \in \Omega$. Therefore, (17) holds on $\Omega$ for all $n \in \mathbb{N}$. Taking the limit as $n \to \infty$ on both sides of (17), we obtain $\tilde{z}^*(\theta) \leq \tilde{z}(\theta) \leq \tilde{z}^*(\theta)$ and the following:

$$cD_{a+}^{\lambda; \psi} \tilde{z}^*(\theta) \leq cD_{a+}^{\lambda; \psi} \tilde{z}(\theta) \leq cD_{a+}^{\lambda; \psi} \tilde{z}^*(\theta),$$

for each $\theta \in \Omega$. This means that $\tilde{z}^*, \tilde{z}^*$ are the extremal solutions of (1) in $[\tilde{z}_0, \tilde{z}_0]$. Thus, the proof of Theorem 1 is complete. □

4. Illustration

The theoretical outcomes are verified by a particular example with specific values. All the experiments are carried out in MATLAB Ver. 8.5.0.197613 (R2015a) on a computer equipped with a CPU AMD Athlon(tm) II X2 245 at 2.90 GHz running under the operating system Windows 7.

**Example 1.** Consider problem (1) with the following:

$$\tau = \lambda = 0.5, \sigma = \frac{\sqrt{\pi}}{2}, a = 0, b = 1, \psi(\theta) = \theta.$$  

(18)

In order to illustrate Theorem 1, we take the following:

$$F(\theta, \tilde{z}(\theta), cD_{0+}^{\lambda; \psi} \tilde{z}(\theta)) = (1 - \sqrt{\theta})$$

$$\times \exp \left( \tilde{z}(\theta) + cD_{0+}^{\lambda; \psi} \tilde{z}(\theta) - \frac{2}{\sqrt{\pi}} - 2 \right),$$

(19)

for $\theta \in [0, 1]$ and obtain.

$$H\left(cD_{0+}^{\lambda; \psi} \tilde{z}(0), cD_{0+}^{\lambda; \psi} \tilde{z}(1) \right) = cD_{0+}^{\lambda; \psi} \tilde{z}(0),$$

(20)

$$G(\tilde{z}(0), \tilde{z}(1)) = \tilde{z}(0) - 1.$$

Obviously, $F, G$ and $H$ are continuous. In addition, we can easily verify that $\tilde{z}_0(\theta) = 1$ and $\tilde{z}_0(\theta) = 1 + \theta$ are lower and upper solutions of (1), respectively. Moreover, we can obtain $\tilde{z}_0(\theta) \leq \tilde{z}_0(\theta)$ and the following:

$$cD_{0+}^{\lambda; \psi} \tilde{z}_0(\theta) \leq cD_{0+}^{\lambda; \psi} \tilde{z}_0(\theta),$$

for all $\theta \in [0, 1]$. On the other hand, one can observe that the assumptions (H2)–(H4) of Theorem 1 are fulfilled. So, An application of Theorem 1 shows that the problem (1) with the data (18) and (19) has extremal solutions in $[\tilde{z}_0, \tilde{z}_0]$, which can be approximated by the following iterative sequences:

$$\tilde{z}_{n+1}(\theta) = 1 + \int_0^\theta E_{0.5} \left( \frac{\sqrt{\pi}(\theta - \eta)}{2} \right) (1 - \sqrt{\eta})$$

$$\times \exp \left( \tilde{z}_n(\eta) + cD_{0+}^{\lambda; \psi} \tilde{z}_n(\eta) - \frac{2}{\sqrt{\pi}} - 2 \right) d\eta,$$

(21)
with \( z_0(\theta) = 1 \) and the following:

\[
\tilde{z}_{n+1}(\theta) = 1 + \int_0^\theta E_{0.5} \left( \frac{\sqrt{\pi(\theta - \eta)}}{2} \right) (1 - \sqrt{\eta}) \times \exp \left( \tilde{z}_n(\eta) + cD^{\lambda, \mu}_{0.5} \tilde{z}_n(\eta) - \frac{2}{\sqrt{\pi}} - 2 \right) \, d\eta, \quad (22)
\]

with \( \tilde{z}_0(\theta) = 1 + \theta \). Tables 1 and 2 show the numerical results of the iterative sequences of \( \tilde{z}_{n+1}(\theta) \) for \( \theta = 0, 0.1, 0.2, 0.3, 0.4 \) and \( 0.5, 0.6, 0.7, 0.8, 0.9 \), respectively. We plot these results in Figure 1a,b.

(a) \( \theta = 0, 0.1, 0.2, 0.3, 0.4 \)  
(b) \( \theta = 0.5, 0.6, 0.7, 0.8, 0.9 \)

**Figure 1.** Graphical representation of \( \tilde{z}_{n+1}(\theta) \) in Example 1.

**Table 1.** Numerical results of \( \tilde{z}_{n+1}(\theta) \) for \( n = 1, 2, \ldots, 20 \) and \( \theta = 0, 0.1, 0.2, 0.3, 0.4 \) by using (21) in Example 1.

|  | 0   | 0.1  | 0.2  | 0.3  | 0.4  |
|---|-----|------|------|------|------|
| 1 | 1.000000 | 1.000000 | 1.000000 | 1.000000 | 1.000000 |
| 2 | 1.000000 | 1.011901 | 1.023786 | 1.035626 | 1.047386 |
| 3 | 1.000000 | 1.012044 | 1.024359 | 1.036918 | 1.049685 |
| 4 | 1.000000 | 1.012045 | 1.024373 | 1.036966 | 1.049799 |
| 5 | 1.000000 | 1.012045 | 1.024373 | 1.036968 | 1.049805 |
| 6 | 1.000000 | 1.012045 | 1.024373 | 1.036968 | 1.049805 |
| 7 | 1.000000 | 1.012045 | 1.024373 | 1.036968 | 1.049805 |
| 8 | 1.000000 | 1.012045 | 1.024373 | 1.036968 | 1.049805 |
| 9 | 1.000000 | 1.012045 | 1.024373 | 1.036968 | 1.049805 |
| 10 | 1.000000 | 1.012045 | 1.024373 | 1.036968 | 1.049805 |
| 11 | 1.000000 | 1.012045 | 1.024373 | 1.036968 | 1.049805 |
| 12 | 1.000000 | 1.012045 | 1.024373 | 1.036968 | 1.049805 |
| 13 | 1.000000 | 1.012045 | 1.024373 | 1.036968 | 1.049805 |
| 14 | 1.000000 | 1.012045 | 1.024373 | 1.036968 | 1.049805 |
| 15 | 1.000000 | 1.012045 | 1.024373 | 1.036968 | 1.049805 |
| 16 | 1.000000 | 1.012045 | 1.024373 | 1.036968 | 1.049805 |
| 17 | 1.000000 | 1.012045 | 1.024373 | 1.036968 | 1.049805 |
| 18 | 1.000000 | 1.012045 | 1.024373 | 1.036968 | 1.049805 |
| 19 | 1.000000 | 1.012045 | 1.024373 | 1.036968 | 1.049805 |
| 20 | 1.000000 | 1.012045 | 1.024373 | 1.036968 | 1.049805 |
Table 2. Numerical results of $z_{n+1}(\theta)$ for $n = 1, 2, \ldots, 20$ and $\theta = 0.5, 0.6, 0.7, 0.8, 0.9$ by using (21) in Example 1.

| $n$ | 0.5     | 0.6     | 0.7     | 0.8     | 0.9     |
|-----|---------|---------|---------|---------|---------|
| 1   | 1.00000 | 1.00000 | 1.00000 | 1.00000 | 1.00000 |
| 2   | 1.05902 | 1.07048 | 1.08172 | 1.09266 | 1.10326 |
| 3   | 1.06261 | 1.07563 | 1.08867 | 1.10166 | 1.11449 |
| 4   | 1.06283 | 1.07602 | 1.08929 | 1.10257 | 1.11578 |
| 5   | 1.06285 | 1.07604 | 1.08935 | 1.10267 | 1.11593 |
| 6   | 1.06285 | 1.07605 | 1.08935 | 1.10268 | 1.11595 |
| 7   | 1.06285 | 1.07605 | 1.08935 | 1.10268 | 1.11595 |
| 8   | 1.06285 | 1.07605 | 1.08935 | 1.10268 | 1.11595 |
| 9   | 1.06285 | 1.07605 | 1.08935 | 1.10268 | 1.11595 |
| 10  | 1.06285 | 1.07605 | 1.08935 | 1.10268 | 1.11595 |
| 11  | 1.06285 | 1.07605 | 1.08935 | 1.10268 | 1.11595 |
| 12  | 1.06285 | 1.07605 | 1.08935 | 1.10268 | 1.11595 |
| 13  | 1.06285 | 1.07605 | 1.08935 | 1.10268 | 1.11595 |
| 14  | 1.06285 | 1.07605 | 1.08935 | 1.10268 | 1.11595 |
| 15  | 1.06285 | 1.07605 | 1.08935 | 1.10268 | 1.11595 |
| 16  | 1.06285 | 1.07605 | 1.08935 | 1.10268 | 1.11595 |
| 17  | 1.06285 | 1.07605 | 1.08935 | 1.10268 | 1.11595 |
| 18  | 1.06285 | 1.07605 | 1.08935 | 1.10268 | 1.11595 |
| 19  | 1.06285 | 1.07605 | 1.08935 | 1.10268 | 1.11595 |
| 20  | 1.06285 | 1.07605 | 1.08935 | 1.10268 | 1.11595 |

Tables 3 and 4 show the numerical results of the iterative sequences of $\tilde{z}_{n+1}(\theta)$ for $\theta = 0, 0.1, 0.2, 0.3, 0.4$ and $0.5, 0.6, 0.7, 0.8, 0.9$, respectively. We plot these results in Figure 2a,b.

Table 3. Numerical results of $\tilde{z}_{n+1}(\theta)$ for $n = 1, 2, \ldots, 20$ and $\theta = 0, 0.1, 0.2, 0.3, 0.4$ by using (22) in Example 1.

| $n$ | 0     | 0.1    | 0.2    | 0.3    | 0.4    |
|-----|-------|--------|--------|--------|--------|
| 1   | 1.00000 | 1.10000 | 1.20000 | 1.30000 | 1.40000 |
| 2   | 1.00000 | 1.01879 | 1.04812 | 1.08922 | 1.14431 |
| 3   | 1.00000 | 1.01733 | 1.04134 | 1.07226 | 1.11752 |
| 4   | 1.00000 | 1.01730 | 1.04106 | 1.07105 | 1.10817 |
| 5   | 1.00000 | 1.01730 | 1.04105 | 1.07096 | 1.10785 |
| 6   | 1.00000 | 1.01730 | 1.04104 | 1.07095 | 1.10743 |
| 7   | 1.00000 | 1.01730 | 1.04104 | 1.07095 | 1.10739 |
| 8   | 1.00000 | 1.01730 | 1.04104 | 1.07095 | 1.10738 |
| 9   | 1.00000 | 1.01730 | 1.04104 | 1.07095 | 1.10738 |
| 10  | 1.00000 | 1.01730 | 1.04104 | 1.07095 | 1.10738 |
| 11  | 1.00000 | 1.01730 | 1.04104 | 1.07095 | 1.10738 |
| 12  | 1.00000 | 1.01730 | 1.04104 | 1.07095 | 1.10738 |
| 13  | 1.00000 | 1.01730 | 1.04104 | 1.07095 | 1.10738 |
| 14  | 1.00000 | 1.01730 | 1.04104 | 1.07095 | 1.10738 |
| 15  | 1.00000 | 1.01730 | 1.04104 | 1.07095 | 1.10738 |
| 16  | 1.00000 | 1.01730 | 1.04104 | 1.07095 | 1.10738 |
| 17  | 1.00000 | 1.01730 | 1.04104 | 1.07095 | 1.10738 |
| 18  | 1.00000 | 1.01730 | 1.04104 | 1.07095 | 1.10738 |
| 19  | 1.00000 | 1.01730 | 1.04104 | 1.07095 | 1.10738 |
| 20  | 1.00000 | 1.01730 | 1.04104 | 1.07095 | 1.10738 |
Figure 2. Graphical representation of $\tilde{z}_{n+1}(\theta)$ in Example 1.

Table 4. Numerical results of $\tilde{z}_{n+1}(\theta)$ for $n = 1, 2, \ldots, 20$ and $\theta = 0.5, 0.6, 0.7, 0.8, 0.9, 1.0$ by using (22) in Example 1.

| $n$ | 0.5   | 0.6   | 0.7   | 0.8   | 0.9   | 1.0   |
|-----|-------|-------|-------|-------|-------|-------|
| 1   | 1.500000 | 1.600000 | 1.700000 | 1.800000 | 1.900000 | 2.000000 |
| 2   | 1.216106 | 1.307784 | 1.422976 | 1.565785 | 1.740782 | 1.953002 |
| 3   | 1.162695 | 1.229793 | 1.320631 | 1.447644 | 1.631747 | 1.909249 |
| 4   | 1.154233 | 1.212553 | 1.289440 | 1.397785 | 1.566487 | 1.870324 |
| 5   | 1.152934 | 1.208920 | 1.280551 | 1.378410 | 1.530698 | 1.837098 |
| 6   | 1.152735 | 1.208162 | 1.278068 | 1.371156 | 1.512041 | 1.809741 |
| 7   | 1.152705 | 1.208004 | 1.277379 | 1.368474 | 1.502577 | 1.787890 |
| 8   | 1.152700 | 1.207972 | 1.277187 | 1.367487 | 1.497842 | 1.770860 |
| 9   | 1.152699 | 1.207965 | 1.277134 | 1.367124 | 1.495491 | 1.757843 |
| 10  | 1.152699 | 1.207963 | 1.277120 | 1.366991 | 1.493427 | 1.748043 |
| 11  | 1.152699 | 1.207963 | 1.277116 | 1.366942 | 1.493752 | 1.740747 |
| 12  | 1.152699 | 1.207963 | 1.277115 | 1.366925 | 1.493469 | 1.735363 |
| 13  | 1.152699 | 1.207963 | 1.277114 | 1.366918 | 1.493329 | 1.731414 |
| 14  | 1.152699 | 1.207963 | 1.277114 | 1.366916 | 1.493260 | 1.728532 |
| 15  | 1.152699 | 1.207963 | 1.277114 | 1.366914 | 1.493225 | 1.726435 |
| 16  | 1.152699 | 1.207963 | 1.277114 | 1.366914 | 1.493209 | 1.724913 |
| 17  | 1.152699 | 1.207963 | 1.277114 | 1.366914 | 1.493194 | 1.722437 |
| 18  | 1.152699 | 1.207963 | 1.277114 | 1.366914 | 1.493193 | 1.722021 |

5. Conclusions

This paper is devoted to the study of a new type of $\psi$-Caputo fractional differential equation. The addressed problem is considered in the framework of nonlinear boundary value conditions. Our work is different from the existing results in the literature, which are basically based on fixed point approaches. However, we have proved our main results with the help of monotone iterative techniques along with the method of upper and lower solutions. It is observed that these methods are closely related to the Tarski–Knaster theorem and that, essentially speaking, also results in fixed point results. The main results have been verified and demonstrated by an example with explicit numerical values.
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