Generalized Neutrosophic Set and Generalized Neutrosophic Topological Spaces

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Abstract In this paper we introduce definitions of generalized neutrosophic sets. After given the fundamental definitions of generalized neutrosophic set operations, we obtain several properties, and discussed the relationship between generalized neutrosophic sets and others. Finally, we extend the concepts of neutrosophic topological space [9], intuitionistic fuzzy topological space [5, 6], and fuzzy topological space [4] to the case of generalized neutrosophic sets. Possible application to GIS topology rules are touched upon.

Keywords Neutrosophic Set, Generalized Neutrosophic Set, Neutrosophic Topology

1. Introduction

Neutrosophy has laid the foundation for a whole family of new mathematical theories generalizing both their classical and fuzzy counterparts, such as a neutrosophic set theory. The fuzzy set was introduced by Zadeh [10] in 1965, where each element had a degree of membership. The intuitionistic fuzzy set (IFS for short) on a universe X was introduced by K. Atanassov [1, 2, 3] in 1983 as a generalization of fuzzy set, where besides the degree of membership and the degree of non-membership of each element. After the introduction of the neutrosophic set concept [7, 8, 9]. In this paper we introduce definitions of generalized neutrosophic sets. After given the fundamental definitions of generalized neutrosophic set operations, we obtain several properties, and discussed the relationship between generalized neutrosophic sets and others. Finally, we extend the concepts of neutrosophic topological space [9].

2. Terminologies

We recollect some relevant basic preliminaries, and in particular, the work of Smarandache in [7, 8], Atanassov in [1, 2, 3] and Salama [9]. Smarandache introduced the neutrosophic components T, I, F which represent the membership, indeterminacy, and non-membership values respectively, where \([0, 1]^3\) is nonstandard unit interval.

Definition [7, 8]

Let T, I, F be real standard or nonstandard subsets of \([0, 1]^3\) with

\[
\begin{align*}
\text{Sup}_T &= t_{\text{sup}}, \text{inf}_T = t_{\text{inf}} \\
\text{Sup}_I &= i_{\text{sup}}, \text{inf}_I = i_{\text{inf}} \\
\text{Sup}_F &= f_{\text{sup}}, \text{inf}_F = f_{\text{inf}} \\
n_{\text{sup}} &= t_{\text{sup}} + i_{\text{sup}} + f_{\text{sup}} \\
n_{\text{inf}} &= t_{\text{inf}} + i_{\text{inf}} + f_{\text{inf}},
\end{align*}
\]

T, I, F are called neutrosophic components

Definition [9]

Let X be a non-empty fixed set. A neutrosophic set (NS for short) A is an object having the form

\[
A = \left\{ (x, \mu_A(x), \sigma_A(x), \gamma_A(x)) : x \in X \right\}
\]

Where

\[
\mu_A(x), \sigma_A(x) \text{ and } \gamma_A(x)
\]

which represent the degree of membership function (namely \(\mu_A(x)\)), the degree of indeterminacy (namely \(\sigma_A(x)\)), and the degree of non-membership (namely \(\gamma_A(x)\)) respectively of each element \(x \in X\) to the set \(A\).

Definition [9]

The NSS \(0_N\) and \(1_N\) in \(X\) as follows:

\[
0_N \text{ may be defined as:}
\]

\[
\begin{align*}
(0_0) & : x \in X \\
(0_1) & : x \in X \\
(0_2) & : x \in X \\
(0_3) & : x \in X
\end{align*}
\]

\[
1_N \text{ may be defined as:}
\]

\[
\begin{align*}
(1_0) & : x \in X \\
(1_1) & : x \in X \\
(1_2) & : x \in X \\
(1_3) & : x \in X
\end{align*}
\]
3. Generalized Neutrosophic Sets

We shall now consider some possible definitions for basic concepts of the generalized neutrosophic set.

**Definition**

Let $X$ be a non-empty fixed set. A generalized neutrosophic set ($\textit{GNS}$ for short) $A$ is an object having the form $A = \{ (x, \mu_A(x), \sigma_A(x), \gamma_A(x)) : x \in X \}$ where $\mu_A(x), \sigma_A(x)$ and $\gamma_A(x)$ represent the degree of membership function (namely $\mu_A(x)$), the degree of indeterminacy (namely $\sigma_A(x)$), and the degree of non-membership (namely $\gamma_A(x)$) respectively of each element $x \in X$ to the set $A$ where the functions satisfy the condition $\mu_A(x) \land \sigma_A(x) \land \nu_A(x) \leq 0.5$.

**Remark**

A generalized neutrosophic set $A = \{ (x, \mu_A(x), \sigma_A(x), \gamma_A(x)) : x \in X \}$ can be identified to an ordered triple $< \mu_A, \sigma_A, \gamma_A >$ in $\prod_{0.0}^{1.0}$ on $X$, where the triple functions satisfy the condition $\mu_A(x) \land \sigma_A(x) \land \nu_A(x) \leq 0.5$.

**Remark**

For the sake of simplicity, we shall use the symbol $A = \{ < x, \mu_A(x), \sigma_A(x), \gamma_A(x) > : x \in X \}$ for the $\textit{GNS}$ set.

**Example**

Every $\textit{GIFS}$ $A$ a non-empty set $X$ is obviously on $\textit{GNS}$ having the form 

$A = \{ < x, \mu_A(x), 1 - (\mu_A(x) + \gamma_A(x)), \gamma_A(x) : x \in X \}$

**Definition**

Let $A = \{ \mu_A, \sigma_A, \gamma_A \}$ a $\textit{GNSS}$ on $X$, then the complement of the set $A$ ($\overline{A}$) for short) may be defined as three kinds of complements

$(C_1)$ $C(A) = \{ (x, 1 - \mu_A(x), \sigma_A(x), 1 - \nu_A(x)) : x \in X \}$

$(C_2)$ $C(A) = \{ (x, \nu_A(x), \sigma_A(x), \mu_A(x)) : x \in X \}$

$(C_3)$ $C(A) = \{ (x, \nu_A(x), 1 - \sigma_A(x), \mu_A(x)) : x \in X \}$

One can define several relations and operations between $\textit{GNS}$ as follows:

**Definition**

Let $X$ be a non-empty set, and $\textit{GNS}$ $A$ and $B$ in the form $A = \{ x, \mu_A(x), \sigma_A(x), \gamma_A(x) \}$, $B = \{ x, \mu_B(x), \sigma_B(x), \gamma_B(x) \}$, then we may consider two possible definitions for subsets ($A \subseteq B$)

$(A \subseteq B)$ may be defined as $A \subseteq B \Leftrightarrow \mu_A(x) \leq \mu_B(x), \gamma_A(x) \geq \gamma$ and $\sigma_A(x) \leq \sigma_B(x)$

$\forall x \in X$

$A \subseteq B \Leftrightarrow \mu_A(x) \leq \mu_B(x), \gamma_A(x) \geq \gamma_B(x)$ and $\sigma_A(x) \geq \sigma_B(x)$

**Proposition**

For any $\textit{GNS}$ set $A$ the following are holds

$0_N \subseteq A$, $0_N \subseteq 0_N$

$A \subseteq 1_N$, $1_N \subseteq 1_N$

**Definition**

Let $X$ be a non-empty set, and $A = \{ x, \mu_A(x), \gamma_A(x), \sigma_A(x) \}$, $B = \{ x, \mu_B(x), \gamma_B(x), \sigma_B(x) \}$ are $\textit{GNSS}$ then $A \cap B$ may be defined as:

$(I_1)$ $A \cap B = \{ x, \mu_A(x) \land \mu_B(x), \gamma_A(x) \lor \gamma_B(x), \sigma_A(x) \lor \sigma_B(x) \}$

$(I_2)$ $A \cap B = \{ x, \mu_A(x) \land \mu_B(x), \gamma_A(x) \lor \gamma_B(x), \sigma_A(x) \lor \sigma_B(x) \}$

$(I_3)$ $A \cap B = \{ x, \mu_A(x) \land \mu_B(x), \gamma_A(x) \lor \gamma_B(x), \sigma_A(x) \lor \sigma_B(x) \}$

$A \cup B$ may be defined as:

$(U_1)$ $A \cup B = \{ x, \mu_A(x) \lor \mu_B(x), \gamma_A(x) \lor \gamma_B(x), \sigma_A(x) \lor \sigma_B(x) \}$

$(U_2)$ $A \cup B = \{ x, \mu_A(x) \lor \mu_B(x), \gamma_A(x) \lor \gamma_B(x), \sigma_A(x) \lor \sigma_B(x) \}$

$A \Delta B$ may be defined as:

$(\Delta_1)$ $A \Delta B = \{ x, \mu_A(x) \lor \mu_B(x), \gamma_A(x) \lor \gamma_B(x), \sigma_A(x) \lor \sigma_B(x) \}$

$(\Delta_2)$ $A \Delta B = \{ x, \mu_A(x) \lor \mu_B(x), \gamma_A(x) \lor \gamma_B(x), \sigma_A(x) \lor \sigma_B(x) \}$

$(\Delta_3)$ $A \Delta B = \{ x, \mu_A(x) \lor \mu_B(x), \gamma_A(x) \lor \gamma_B(x), \sigma_A(x) \lor \sigma_B(x) \}$

**Example 3.2.** Let $X = \{ a, b, c, d, e \}$ and $A = \{ x, \mu_A(x), \sigma_A(x), \nu_A(x) \}$ given by:

| $x$ | $\mu_A(x)$ | $\sigma_A(x)$ | $\nu_A(x)$ | $\mu_A(x) \land \sigma_A(x) \land \nu_A(x)$ |
|-----|-------------|---------------|-------------|------------------------------------------|
| a   | 0.6         | 0.3           | 0.5         | 0.3                                      |
| b   | 0.5         | 0.3           | 0.6         | 0.3                                      |
| c   | 0.4         | 0.5           | 0.4         | 0.4                                      |
| d   | 0.3         | 0.6           | 0.3         | 0.3                                      |
| e   | 0.3         | 0.6           | 0.4         | 0.3                                      |

Then the family $G = \{ O_x, A \}$ is an $\textit{GNSS}$ on $X$.

We can easily generalize the operations of $\textit{GNS}$ intersection and union in definition 3.4 to arbitrary family of $\textit{GNSS}$ as follow:

**Definition**

Let $\{ A_j : j \in J \}$ be an arbitrary family of $\textit{GNS}$ in $X$, then

$\bigcap_{j \in J} A_j$ may be defined as:

$(I_1)$ $\bigcap_{j \in J} A_j = \{ x, \land \mu_{A_j}(x), \land \sigma_{A_j}(x), \lor \nu_{A_j}(x) \}$
2) \[ \bigcap A_j = \left\{ x, \wedge \mu_{A_j}(x), \vee \sigma_{A_j}(x), \vee \gamma_{A_j}(x) \right\} \]

\[ \bigcup A_j \]

may be defined as:

1) \[ \bigcup A_j = \left\{ x, \vee \mu_{A_j} \wedge \sigma_{A_j} \wedge \gamma_{A_j} \right\} \]

2) \[ \bigcup A_j = \left\{ x, \vee \mu_{A_j} \wedge \gamma_{A_j} \wedge \gamma_{A_j} \right\} \]

**Definition**

Let \( A \) and \( B \) are generalized neutrosophic sets then \( A \subseteq B \) may be defined as:

\[ A \subseteq B = \left\{ x, \mu_A \wedge \gamma_B, \sigma_A(x) \wedge B(x), \sigma_A \vee B(x) \right\} \]

**Proposition**

For all \( A, B \) two generalized neutrosophic sets then the following are true

i) \[ C(A \cap B) = C(A) \cap C(B) \]

ii) \[ C(A \cup B) = C(A) \cup C(B) \]

### 4. Generalized Neutrosophic Topological Spaces

Here we extend the concepts of and intuitionistic fuzzy topological space \([5, 7]\), and neutrosophic topological Space \([9]\) to the concept of generalized neutrosophic sets.

**Definition**

A generalized neutrosophic topology \((G\, NT\, S)\) for short) an a non empty set \( X \) is a family \( \tau \) of generalized neutrosophic subsets in \( X \) satisfying the following axioms

\[
\begin{align*}
(GNT_1) \quad & \forall \tau, l, v \in \tau , \\
(GNT_2) \quad & \forall G_1, G_2 \in \tau , \\
(GNT_3) \quad & \forall G_i, \forall J, J \subseteq \tau
\end{align*}
\]

In this case the pair \((X, \tau)\) is called a generalized neutrosophic topological space \((G\, NT\, S)\) for short) and any neutrosophic set in \( \tau \) is known as neutrosophic open set \((N\, O\, S)\) for short) in \( X \). The elements of \( \tau \) are called open generalized neutrosophic sets, A generalized neutrosophic set \( F \) is closed if and only if it \( C(F) \) is generalized neutrosophic open.

**Remark** A generalized neutrosophic topological spaces are very natural generalizations of intuitionistic fuzzy topological spaces allow more general functions to be members of intuitionistic fuzzy topology.

**Example**

Let \( X = \{ x \} \) and

\[ A = \{ x, 0.5, 0.5, 0.4 : x \in X \} \]

\[ B = \{ x, 0.4, 0.6, 0.8 : x \in X \} \]

\[ D = \{ x, 0.5, 0.6, 0.4 : x \in X \} \]

\[ C = \{ x, 0.4, 0.5, 0.8 : x \in X \} \]

Then the family \( \tau = \{ O, 1, A, B, C, D \} \) of G\, N\, S in \( X \) is generalized neutrosophic topology on \( X \)

**Example**

Let \((X, \tau_0)\) be a fuzzy topological space in Changes [4] sense such that \( \tau_0 \) is not indiscrete suppose now that

\[ \tau_0 = \{ 0, l, j \} \] \( \tau_0 \) can be construct two \( G\, NT\, S\) on \( X \) as follows

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\[ \tau_0 = \{ 0, l, j \} \] \( \tau_0 \) can be construct two \( G\, NT\, S\) on \( X \) as follows

**Proposition**

Let \((X, \tau)\) be a G\, NT\, S on \( X \), then we can also construct several G\, NT\, S on \( X \) in the following way:

a) \( \tau_{0,1} = \{ T : G \in \tau \} \),

b) \( \tau_{0,2} = \{ G : G \in \tau \} \),

d) \( \tau_{0,3} = \{ G : G \in \tau \} \),

e) \( \tau_{0,4} = \{ G : G \in \tau \} \)

**Proof**

\( (GNT_1) \) and \( (GNT_2) \) are easy.

\( (GNT_3) \) Let \( \tau_{0,1} = \{ T : G \in \tau \} \),

\( (GNT_4) \) Let \( \tau_{0,1} = \{ T : G \in \tau \} \),

\( (GNT_5) \) Let \( \tau_{0,1} = \{ T : G \in \tau \} \),

\( (GNT_6) \) Let \( \tau_{0,1} = \{ T : G \in \tau \} \),

This similar to (a)

**Definition**

A generalized neutrosophic topology \((G\, NT\, S)\) on \( X \) is called a generalized neutrosophic closed set \((G\, N\, C\, S)\) if and only if \( G \in \tau \),

**Proposition**

Let \( \tau \) be a G\, NT\, S on \( X \) then we can also construct several G\, NT\, S on \( X \) in the following way:

a) \( \tau_{0,1} = \{ T : G \in \tau \} \),

b) \( \tau_{0,2} = \{ G : G \in \tau \} \),

d) \( \tau_{0,3} = \{ G : G \in \tau \} \),

e) \( \tau_{0,4} = \{ G : G \in \tau \} \)

**Proof**

Obvious

**Definition**

The complement of \( A \) \((C(A)\) for short) of \( N\, O\, S \) \( A \) is called a generalized neutrosophic closed set \((G\, N\, C\, S)\) for short) in \( X \).

**Proposition**

Let \( \tau \) be a G\, NT\, S on \( X \) then \( \tau \) is a G\, NT\, S on \( X \) containing all \( \tau \).

**Proof**

Obvious

**Definition**

The complement of \( A \) \((C(A)\) for short) of \( N\, O\, S \) \( A \) is called a generalized neutrosophic closed set \((G\, N\, C\, S)\) for short) in \( X \).

**Proposition**

Let \( \tau \) be a G\, NT\, S on \( X \) then \( \tau \) is a G\, NT\, S on \( X \) containing all \( \tau \).

**Proof**

Obvious
For any generalized neutrosophic set $A$ in $(x, \tau)$ we have
(a) $GNCI(C(A)) = C(\text{GNInt}(A))$,
(b) $\text{NGInt}(C(A)) = C(GNCI(A))$.

**Proof.**
Let $A = \{x, \mu_A, \sigma_A, \nu_A : x \in X\}$ and suppose that the family of generalized neutrosophic subsets contained in $A$ are indexed by the family $A = \{x, \mu_{G_i}, \sigma_{G_i}, \nu_{G_i} : i \in J\}$. Then we see that $\text{GNInt}(A) = \{x, \lor \mu_{G_i}, \lor \sigma_{G_i}, \lor \nu_{G_i} : i \in J\}$ and hence $C(\text{GNInt}(A)) = \{x, \lor \mu_{G_i}, \lor \sigma_{G_i}, \lor \nu_{G_i} : i \in J\}$. Since $C(A)$ and $\mu_{G_i} \leq \mu_A$ and $\nu_{G_i} \geq \nu_A$ for each $i \in J$, we obtaining $C(A)$, i.e.
$GNCI(C(A)) = \{x, \lor \mu_{G_i}, \lor \sigma_{G_i}, \lor \nu_{G_i} : i \in J\}$. Hence $GNCI(C(A)) = C(\text{GNInt}(A))$, follows immediately 
This is analogous to (a).

**Proposition**
Let $(x, \tau)$ be a GNTS and $A, B$ be two neutrosophic sets in $X$. Then the following properties hold:
$\text{GNInt}(A) \subseteq A$,
$A \subseteq B \Rightarrow \text{GNInt}(A) \subseteq \text{GNInt}(B)$,
$A \subseteq B \Rightarrow \text{GNCI}(A) \subseteq \text{GNCI}(B)$,
$\text{GNInt}(\text{GNInt}(A)) = \text{GNInt}(A) \land \text{GNInt}(B)$,
$\text{GNCI}(A \cup B) = \text{GNCI}(A) \lor \text{GNCI}(B)$,
$\text{GNInt}(1_N) = 1_N$,
$\text{GNCI}(O_N) = O_N$,

**Proof** (a), (b) and (e) are obvious (c) follows from (a) and

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