On Lie algebra modules which are modules over semisimple group schemes

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Abstract. Let $p$ be a prime. Given a split semisimple group scheme $G$ over a normal integral domain $R$ which is a faithfully flat $\mathbb{Z}_p$-algebra, we classify all finite dimensional representations $V$ of the fiber $G_K$ of $G$ over $K := \text{Frac}(R)$ with the property that the set of lattices of $V$ with respect to $R$ which are $G$-modules is as well the set of lattices of $V$ with respect to $R$ which are Lie($G$)-modules. We apply this classification to get a general criterion of extensions of homomorphisms between reductive group schemes over Spec $K$ to homomorphisms between reductive group schemes over Spec $R$. We also show that for a simply connected semisimple group scheme over a reduced $\mathbb{Q}$-algebra, the category of its representations is equivalent to the category of representations of its Lie algebra.

Key words: category, lattice, Lie algebra, representation, ring, semisimple group scheme

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1 Introduction

Let $R$ be a commutative ring with $1 \neq 0$.

Let $r \in \mathbb{N}$. Let $G$ be a semisimple group scheme over Spec $R$ of rank $r$: it is an affine smooth group scheme over Spec $R$ whose geometric fibers are semisimple groups over algebraically closed fields which admit maximal tori of dimension $r$. We recall that $G$ is called split if it has a maximal torus
isomorphic to $G^m_{t_m, R}$. If $R$ is connected, let $d \in \mathbb{N}$ be the relative dimension of $G$ over $\text{Spec} \, R$.

By a $G$-module we mean a finitely generated projective $R$-module $M$ endowed with a homomorphism $\rho_M : G \to \mathbf{Aut}_M$, where $\mathbf{Aut}_M$ is the affine smooth group scheme over $\text{Spec} \, R$ of linear automorphisms of $M$: if $S$ is an $R$-algebra, then

$$\mathbf{Aut}_M(S) := \{ f : S \otimes_R M \to S \otimes_R M | f \text{ is a bijective } S\text{-linear map} \}.$$ 

Thus, if $M = R^n$ for some $n \in \mathbb{N} \cup \{0\}$, then $\mathbf{Aut}_M = \mathbf{GL}_{n, R}$ is a general linear group scheme over $\text{Spec} \, R$. If $P$ is another $G$-module, then by a $G$-module map between $M$ and $P$ we mean an $R$-linear map $f : M \to P$ such that for each $R$-algebra $S$ (equivalently, for each smooth $R$-algebra $S$) and every $g \in G(S)$, we have an identity

$$1_S \otimes f \circ \rho_M(S)(g) = \rho_P(S)(g) \circ 1_S \otimes f : S \otimes_R M \to S \otimes_R P.$$ (1)

Let $\text{Rep}(G)$ be the category of $G$-modules.

By a $g$-module we mean a finitely generated projective $R$-module $L$ equipped with a Lie algebra homomorphism $\varrho_L : g \to \mathfrak{gl}_R(L)$, where

$$\mathfrak{gl}_R(L) := \{ e : L \to L | e \text{ is an } R\text{-linear map} \}$$

is equipped with the usual Lie bracket $[,]$: if $e_1, e_2 \in \mathfrak{gl}_R(L)$, then we have $[e_1, e_2] := e_1 \circ e_2 - e_2 \circ e_1$. Thus $\mathfrak{gl}_R(L)$ is the Lie algebra over $R$ which is associated to the $R$-algebra $\text{End}_R(L)$ and is identified with $\text{Lie}(\mathbf{Aut}_L)$. If $J$ is another $g$-module, then by a $g$-module map between $L$ and $J$ we mean an $R$-linear map $f : L \to J$ such that for all $a \in g$ we have an identity $f \circ \varrho_L(a) = \varrho_J(a) \circ f : L \to J$. Let $\text{Rep}(g)$ be the category of $g$-modules.

We have a natural functor

$$\text{Lie} = \text{Lie}_G : \text{Rep}(G) \to \text{Rep}(g)$$ (2)

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that maps a $G$-module $M$ defined by the representation $\rho_M : G \to \text{Aut}_M$ to

$$\varrho_M := \text{Lie}(\rho_M) : \mathfrak{g} = \text{Lie}(G) \to \text{Lie}(\text{Aut}_M) = \mathfrak{g}_M.$$  

Note the typesetting difference: $\text{Lie}(G) = \mathfrak{g}$ is a Lie algebra over $R$, $\text{Lie} = \text{Lie}_G$ is a functor, and $\text{Lie}(M) = \text{Lie}_G(M)$ is a representation of $\text{Lie}(G) = \mathfrak{g}$.

If $R$ is a field and $M$ is a simple $G$-module, then the irreducible representation $\rho_M : G \to \text{Aut}_M$ is called \textit{infinitesimally irreducible} if the $\mathfrak{g}$-module $M$ is simple as well, see [1], Sect. 6; one also calls $M$ an infinitesimally simple $G$-module.

One would like first to classify all the $\mathfrak{g}$-modules which are $G$-modules, i.e., are isomorphic to objects in the image of the functor (2), and second to apply such a classification to obtain extension results from $\text{Spec } K$ to $\text{Spec } R$ for homomorphisms between reductive group schemes that are in line with the extension results obtained in [21], Subsect. 4.3, [22] and [23].

Let $K := N^{-1}_R R$ be the total quotient ring of $R$, where $N_R$ is the multiplicative set of nonzero divisors of $R$. If $R$ is an integral domain, then $K$ is a field and we will denote its characteristic by $\text{char}(K)$.

Let $p$ be a prime. We are mainly interested in the following two situations:

(i) The ring $R$ is a $\mathbb{Q}$–algebra.

(ii) The ring $R$ is a faithfully flat $\mathbb{Z}(p)$-algebra (i.e., $K$ is a $\mathbb{Q}$–algebra and for each point $z \in \text{Spec } R$, its residue field $k_z$ has characteristic either 0 or $p$, and there exist such points $z$ with $\text{char}(k_z) = p$).

In the situation (i) we have the following classical result which in essence is well-known (for instance, when $R = K$ is a field see [15] and in the general case see [18], Exp. XXIV, Prop. 7.3.1 which implies the surjectivity of the functor (2) on objects without the reduced assumption):

\textbf{Theorem 1.} We assume that $R$ is a reduced $\mathbb{Q}$–algebra and $G$ is simply connected. Then the functor (2) is an equivalence of categories.

The goal of this paper is to obtain variants of Theorem 1 for the situation (ii). As Theorem 1 fails in the situation (ii) (see Theorem 2 below), one is led to consider a fixed nonzero $G_K$-module $V$ (so, if $R$ is an integral domain, $V$ is a finite dimensional $K$-vector space) and to study the natural map

$$\text{Lie} = \text{Lie}_G : \text{Lat}_G(V) \to \text{Lat}_\mathfrak{g}(V)$$ (3)
induced by the functor \( \text{Lie} \) and denoted in the same way, where \( \text{Lat}_{G}(V) \) (resp. \( \text{Lat}_{\mathfrak{g}}(V) \)) is the set of \textit{lattices} of \( V \) with respect to \( R \) which are \( G \)-modules (resp. \( \mathfrak{g} \)-modules). Here and in what follows, by a lattice of \( V \) with respect to \( R \) we mean a \( \mathcal{R} \)-submodule \( L \) of \( V \) which is a finitely generated \( \mathcal{R} \)-module and for which the injective \( \mathbb{K} \)-linear map \( \mathbb{K} \otimes_{\mathcal{R}} L \to V \) is a bijection. If \( L \) is a \( G \)-module, then \( \text{Lie}(L) \) is \( L \) but viewed as a \( \mathfrak{g} \)-module via the functor \( \text{Lie} \).

Let \( G_{\text{sc}} \) be the simply connected semisimple group scheme cover of \( G \); so \( V \) is also a \( G_{\text{sc}} \)-module.

To study the map (3) we will assume that \( R \) is an integral domain and that char(\( K \)) = 0. Let \( K \) be an algebraic closure of \( K \). We recall from [18], Exp. XXV, Thm. 1.1 that there exists a unique (up to ordering) product decomposition \( G_{\text{sc}} \approx \prod_{i=1}^{n} G_{i} \) such that each \( G_{i} \) has a simple adjoint group scheme \( G_{i}^{\text{ad}} := G_{i}/Z(G_{i}) \) over Spec \( K \), where \( Z(G_{i}) \) is the center of \( G_{i} \); here \( n \in \mathbb{N} \).

For references to the standard facts recalled in this paragraph see Subsection 2.1. As char(\( K \)) = 0, it is well-known that the \( G_{\text{sc}} \)-module \( \mathbb{K} \otimes_{\mathcal{R}} V \) is semisimple and hence we write it as a direct sum \( \mathbb{K} \otimes_{\mathcal{R}} V = \bigoplus_{j=1}^{m} \mathbb{V}_{j} \) of simple \( G_{\text{sc}} \)-modules; here \( m \in \mathbb{N} \). Each \( \mathbb{V}_{j} \) admits a tensor product decomposition \( \mathbb{V}_{j} = \bigotimes_{i=1}^{n} \mathbb{V}_{ij} \), where every \( \mathbb{V}_{ij} \) is a simple \( G_{i} \)-module and where every element \( (g_{1}, \ldots, g_{n}) \in G_{\text{sc}}(\mathbb{K}) = \prod_{i=1}^{n} G_{i}(\mathbb{K}) \) acts on \( \mathbb{V}_{j} \) in the usual tensorial way: for all \( v_{1j}, \ldots, v_{nj} \in \mathbb{V}_{nj} \), it maps \( v_{1j} \otimes v_{2j} \otimes \cdots \otimes v_{nj} \) to \( g_{1}(v_{1j}) \otimes g_{2}(v_{2j}) \otimes \cdots \otimes g_{n}(v_{nj}) \). Moreover, for all \( i \in \{1, \ldots, n\} \), if a maximal torus \( T_{i} \) of a Borel subgroup \( B_{i} \) of \( G_{\text{sc}} \) is given and if \( r_{i} \in \mathbb{N} \) is the dimension of \( T_{i} \), then to \( B_{i} \) corresponds a basis \( \omega_{i,1}, \ldots, \omega_{i,r_{i}} \) of dominant weights of the group of characters \( X^{*}(T_{i}) := \text{Hom}(T_{i}, \mathbb{G}_{m}) \simeq \mathbb{Z}^{r_{i}} \).
and the representation $\overline{V}_{ij}$ is uniquely determined by its highest weight
\[
w_{ij} = \sum_{l=1}^{r_i} c_{ijl} \omega_{i,l},
\]
where each $c_{ijl} \in \mathbb{Z}_{\geq 0}$. We have $w_{ij} = 0$, i.e., $c_{ij1} = \cdots = c_{ijr_i} = 0$, if and only if $\overline{V}_{ij}$ is a trivial simple $G_{i,K}^{sc}$-module (equivalently, $\dim_K(\overline{V}_{ij}) = 1$).

**Definition 1.** We say that the nonzero $G_K$-module (or $G_{i,K}^{sc}$-module) $V$ is $p$-latticed if for each $(i, j) \in \{1, \ldots, n\} \times \{1, \ldots, m\}$, for every $l \in \{1, \ldots, r_i\}$ we have $c_{ijl} \in \{0, \ldots, p-1\}$.

**Theorem 2.** We assume that $R$ is a normal integral domain which is a faithfully flat $\mathbb{Z}(p)$-algebra. We consider the following two statements on the fixed nonzero $G_K$-module $V$:

1. The map $\text{Lie}: \text{Lat}_{G}(V) \to \text{Lat}_{q}(V)$ is a bijection.
2. The $G_K$-module is $p$-latticed.

Then the following three properties hold:

(a) The implication 2 $\Rightarrow$ 1 always holds.

(b) We assume that there exists a discrete valuation ring $D$ of mixed characteristic $(0, p)$ which is a subring of $R$ such that $G$ is the pullback of a semisimple group scheme $G_D$ over $\text{Spec} D$ and the $G_K$-module $V$ is the pullback of a $G_{\text{Frac}(D)}$-module $V_{\text{Frac}(D)}$, where $\text{Frac}(D) = D[1/p]$ is the subfield of $K$ which is the field of fractions of $D$ (for instance, this holds if $G$ is split). Then the implication 1 $\Rightarrow$ 2 holds.

(c) If $G$ is split, then we have an equivalence 1 $\Leftrightarrow$ 2.

**Example 1.** We assume that $G = \text{SL}_2, R$ and $R$ is as in Theorem 2. Then the map (3) is a bijection if and only if the $G_K$-module $V$ is a direct sum of simple $G_K$-modules of dimension at most $p$.

For instance, suppose $R = \mathbb{Z}(p)$ and $V$ is simple of dimension $p+1$, so it is the $p$-th symmetric power $V = \mathbb{Q}x^p \oplus \mathbb{Q}x^{p-1}y \oplus \cdots \oplus \mathbb{Q}y^p$ of the standard $G_{\mathbb{Q}}$-module $\mathbb{Q}x \oplus \mathbb{Q}y$ of rank 2 (here $x$ and $y$ are viewed as indeterminates). Then the map (3) is not surjective: consider the lattice
\[
L := \mathbb{Z}(p)x^p \oplus \mathbb{Z}(p)x^{p-1}y \oplus \cdots \oplus \mathbb{Z}(p)xy^{p-1} \oplus \frac{1}{p}\mathbb{Z}(p)(x^p + y^p)
\]
of $V$ with respect to $\mathbb{Z}_p$. Let $T$ be the split torus of $G$ which normalizes both $\mathbb{Z}_p \cdot x$ and $\mathbb{Z}_p \cdot y$; it has rank 1, i.e., $T \simeq \mathbb{G}_m \cdot \mathbb{Z}_p$. The elements of the standard $\mathbb{Z}_p$-basis of $\mathfrak{g}$ map $(x, y)$ to $(y, 0)$ or $(0, x)$ or $(x, -y)$ (respectively) and thus map $\frac{1}{p}(x^p + y^p)$ to elements of $L$. This implies that $L$ is a $\mathfrak{g}$-module. But $L$ is not a $T$-module and thus it is also not a $G$-module.

The highest weights of Definition 1 show up in the works of Curtis and Borel (see [1], Sects. 6 and 7; see also [6] and [7] for original results under certain restrictions such as $p \geq 7$): they are precisely all the highest weights which in characteristic $p$ define infinitesimally irreducible representations (see [1], Thms. 6.4 and 7.5 (iii)).

Theorem 1 is proved in Section 3 based on the review of Section 2 that recalls classical properties of roots and of closed subgroup schemes of semisimple group schemes over Spec $\mathbb{R}$. Theorem 2 is proved in Section 5 based on the proof of Theorem 1, on the mentioned works of Curtis and Borel, and on the following general result proved in Section 4.

**Theorem 3.** Let $H$ be a semisimple group over an algebraically closed field $\kappa$. Let $P$ be an $H$-module such that the $\text{Lie}(H)$-module $P$ is semisimple of the same length as the $H$-module $P$. Then the $H$-module $P$ is itself semisimple.

The following example shows that the “same length” assumption of Theorem 3 is always necessary in positive characteristic.

**Example 2.** We assume $\text{char}(\kappa) = p$. Let $0 \rightarrow L_1 \rightarrow Q \rightarrow L_2 \rightarrow 0$ be a nonsplit short exact sequence of $H$-modules with $L_1$ and $L_2$ simple and $\dim_\kappa(Q) > 2$ (see [12], Part 2, Ch. 7, Sects. 7.1 and 7.2 for general examples). For an $H$-module $V$, we consider the $H$-module $V^{(p)}$ defined by the representation which is the composite of the surjective Frobenius homomorphism $H \rightarrow H^{(p)}$ and the pullback $H^{(p)} \rightarrow \text{GL}_V^{(p)} = \text{GL}_{V^{(p)}}$ via the Frobenius endomorphism $\text{Frob}$ of Spec $\kappa$ of the representation defining $V$; here $V^{(p)} := \kappa \otimes_{\text{Frob}, \kappa} V$. Let $P := Q^{(p)}$. Then $P$ is a trivial (hence semisimple) $\text{Lie}(H)$-module of length $\dim_\kappa(Q) > 2$. But $P$ is an indecomposable $H$-module of length 2; this is so as $L_1^{(p)}$ and $L_2^{(p)}$ are simple $H$-modules and the short exact sequence $0 \rightarrow L_1^{(p)} \rightarrow P \rightarrow L_2^{(p)} \rightarrow 0$ of $H$-modules is nonsplit (see [12], Part 2, Ch. 10, Prop. 10.16 for the injectivity of the natural pullback map $\text{Ext}_H^1(L_1, L_2) \rightarrow \text{Ext}_H^1(L_1^{(p)}, L_2^{(p)})$).

By combining Theorem 2 with [23], in Section 6 we prove the following theorem which is an application used in [24] to simplify the arguments of [21].
Subsect. 4.3 on extending homomorphisms between reductive group schemes in contexts related to integral models of Shimura varieties of Hodge type.

**Theorem 4.** We assume that $R$ is a normal integral domain and a faithfully flat $\mathbb{Z}_p$-algebra. Let $G_K$ be a simply connected semisimple group over $\text{Spec } K$. Let $V$ be a $G_K$-module which is $p$-latticed and let $H_K := \text{Im}(G_K \to \text{Aut}_V)$. Let $M$ be a lattice of $V$ with respect to $R$ such that there exists a perfect symmetric bilinear form $B : M \times M \to R$ which, over $K$, it is fixed by $H_K$ and whose restriction to $\text{Lie}(H_K) \cap \text{End}_R(M)$ is a unit of $R$ times the Killing form (thus we have $p > 2$, see [23], Prop. 3.5 (a)). Then the schematic closure of the image $H_K$ in $\text{Aut}_M$ is a semisimple group scheme $H$ over $\text{Spec } R$ whose simply connected semisimple group scheme cover $G$ extends $G_K$ and has the same Lie algebra $\text{Lie}(H_K) \cap \mathfrak{g}l_M$ as either $H$ or its adjoint $G^{\text{ad}}$ (i.e., the isogenies $G \to H \to H^{\text{ad}} = G^{\text{ad}}$ are étale).

Theorem 2 (c) was first obtained by the first author in the case when $R$ is noetherian while he was a graduate student.

## 2 A review

In this section we assume that $G = G^{\text{sc}}$ is simply connected and split and that $R$ is connected.

### 2.1 The split context

In this subsection we assume that $K$ is a field with $\text{char}(K) = 0$. Thus a group scheme $\triangle$ of finite type over $\text{Spec } K$ is smooth (for Cartier’s Theorem, for instance, see [3], Ch. II, Sect. 6, Thm. of Subsect. 1.1). Moreover, $\triangle$ is a (not necessarily connected) reductive group if and only if $\triangle$ is linearly reductive, i.e., each $\triangle$-module is semisimple (completely reducible), see [3], p. 178. Similarly, if $\triangle$ is semisimple, then its Lie algebra $\text{Lie}(\triangle)$ is semisimple (this can be easily checked over $\overline{K}$), and Weyl’s complete reducibility theorem implies that each $\text{Lie}(\triangle)$-module is semisimple (see [4], Ch. I, Subsect. 6.2, Thm. 2). Thus the categories $\text{Rep}(G_K)$ and $\text{Rep}(K \otimes_R \mathfrak{g}) = \text{Rep}(\text{Lie}(G_K))$ are semisimple abelian categories.

Let $T_K$ be a maximal torus of $G_K$ which is split. Let $B_K$ be a Borel subgroup of $G_K$ that contains $T_K$. The Lie algebra $\text{Lie}(T_K)$ is a split Cartan subalgebra of $K \otimes_R \mathfrak{g}$ and thus $K \otimes_R \mathfrak{g}$ is also split. Moreover, $\text{Lie}(B_K)$ is
a Borel subalgebra of $K \otimes_R g$. The simple $G_K$-modules are classified by the dominant weights of $T_K$ with respect to $B_K$ (see [12], Part 2, Ch. 2, Cor. 2.7) and the simple $K \otimes_R g$-modules are classified by the dominant weights of $\text{Lie}(T_K)$ with respect to $\text{Lie}(B_K)$ (see [5], Ch. VIII, Sect. 7, Cor. 2).

2.2 Roots

For centers of semisimple group schemes see [18], Exp. XXII, Cor. 4.1.7.

As $R$ is connected, from [18], Exp. XXV, Thm. 1.1 we get that:

- There exists a unique (up to isomorphism) simply connected split semisimple group $G_Z$ over $\text{Spec } \mathbb{Z}$ such that $G = \text{Spec } R \times_{\text{Spec } \mathbb{Z}} G_Z$.
- There exists a unique direct sum decomposition $G_Z = \prod_{i=1}^{n} G_{i,Z}$, such that each $G_{i,Z}$ has an adjoint group scheme $G_{i,\text{ad},Z} := G_{i,Z}/Z(G_{i,Z})$ whose geometric fibers are simple, where $Z(G_{i,Z})$ is the center of $G_{i,Z}$.

Defining $G_i := \text{Spec } R \times_{\text{Spec } \mathbb{Z}} G_{i,Z}$, we get a product decomposition

$$G = \prod_{i=1}^{n} G_i$$

over $\text{Spec } R$ and a product decomposition $G_K = \prod_{i=1}^{n} G_{i,K}$ over $\text{Spec } K$.

For $i \in \{1, \ldots, n\}$ let $T_{i,Z}$ be a (split) maximal torus of a Borel subgroup scheme $B_{i,Z}$ of $G_{i,Z}$, let $T_i := \text{Spec } R \times_{\text{Spec } \mathbb{Z}} T_{i,Z}$ and $B_i := \text{Spec } R \times_{\text{Spec } \mathbb{Z}} B_{i,Z}$. Therefore, $T := \prod_{i=1}^{n} T_i$ is a maximal torus of the Borel subgroup scheme $B := \prod_{i=1}^{n} B_i$ of $G$. Let $B_{\text{op}}$ be the Borel subgroup scheme of $G$ which is the opposite of $B$ with respect to $T$.

We identify $G_{m,Z} = \text{Spec } \mathbb{Z}[x, x^{-1}]$ and $\text{Lie}(G_{m,Z}) = \mathbb{Z}$ in such a way that $1 \in \mathbb{Z} = \text{Lie}(G_{m,Z})$ gets identified with the $\mathbb{Z}$-linear map

$$\Omega^1_{\mathbb{Z}[x,x^{-1}]/\mathbb{Z}} = \mathbb{Z} \frac{dx}{x} \to \mathbb{Z}$$

that maps $\frac{dx}{x}$ to 1. We also identify $\text{Lie}(G_{m,R}) = R \otimes_{\mathbb{Z}} \text{Lie}(G_{m,Z}) = R$.

We can assume that, if $K$ is a field, then the choices made in Section 1 and Subsection 2.1 are compatible with our notation, i.e., for each $i \in \{1, \ldots, n\}$ the maximal torus $T_{i,K}$ and the Borel subgroup $B_{i,K}$ of $G_{i,K}$ are indeed the extensions to $\text{Spec } K$ of $T_i$ and $B_i$ (respectively) and $T_K = \text{Spec } K \times_{\text{Spec } \mathbb{Z}} T_Z$ and $B_K = \text{Spec } K \times_{\text{Spec } \mathbb{Z}} B_Z$. As such, we identify

$$\mathcal{W}_{G_i} := X^*(T_{i,K}) = X^*(T_{i,Z}) = X^*(T_i) := \text{Hom}(T_i, G_{m,R}) \simeq \mathbb{Z}^{r_t},$$

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\[ W_G := X^*(T_R) = X^*(T_K) = X^*(T) = \text{Hom}(T, \mathbb{G}_{m,R}) = \bigoplus_{i=1}^{n} W_{G_i} \]
and we speak about the monoid of dominant weights
\[ W_{G}^{\geq 0} := \bigoplus_{i=1}^{n} (\bigoplus_{j=1}^{r_i} \mathbb{Z}_{\geq 0} \omega_{ij}) \subset \bigoplus_{i=1}^{n} W_{G_i} = W_G \]
of \( G \) with respect to the split maximal torus \( T \) of the Borel subgroup scheme \( B \) of \( G \).

We recall that, if \( K \) is a field, then the (integral) weights of \( K \otimes \mathbb{R} \mathfrak{g} \) are the elements of
\[ \mathcal{W}_g := \text{Hom}_{\mathbb{Z}}(\text{Lie}(T_\mathbb{Z}), \text{Lie}(\mathbb{G}_{m,\mathbb{Z}})) = \text{Hom}_{\mathbb{Z}}(\text{Lie}(T_\mathbb{Z}), \mathbb{Z}) \]
inside
\[ \text{Hom}_K(\text{Lie}(T_K), \text{Lie}(\mathbb{G}_{m,K})) = \text{Hom}_K(\text{Lie}(T_K), K), \]
with the dominant weights \( \mathcal{W}_g^{\geq 0} \) being those with respect to \( B_K \). We have a bijection
\[ \mathcal{L} : \mathcal{W}_G \to \mathcal{W}_g \]
given by the rule \( w \to \text{Lie}_{T_K}(w) \) which induces a bijection \( \mathcal{L} : \mathcal{W}_G^{\geq 0} \to \mathcal{W}_g^{\geq 0} \) denoted in the same way.

Let \( G_\mathbb{Q} := \text{Spec} \mathbb{Q} \times_{\text{Spec} \mathbb{Z}} G_\mathbb{Z} \). If \( K \) is a field, as the abelian categories \( \text{Rep}(G_K) \) and \( \text{Rep}(G_\mathbb{Q}) \) are semisimple, from the classification of simple \( G_K \)-modules or \( G_\mathbb{Q} \)-modules in terms of dominant weights (see [12], Part II, Ch. 2, Cor. 2.7), we get that these simple modules are absolutely simple (see [12], Part II, Ch. 2, Cor. 2.9). Thus, if \( K \) is a field, then the pullback functor
\[ K \otimes \mathbb{Q} \text{Rep}(G_\mathbb{Q}) \to \text{Rep}(G_K) \]
is an equivalence between \( K \)-linear semisimple abelian categories. In particular, for each \( G_K \)-module \( V \), there exists a unique (up to isomorphism) \( G_\mathbb{Q} \)-module \( V_\mathbb{Q} \) such that the \( G_K \)-modules \( V \) and \( K \otimes \mathbb{Q} V_\mathbb{Q} \) are isomorphic.

### 2.3 Closed subgroups

In this subsection we also assume that \( \text{Spec} \mathbb{R} \) is reduced. Let \( \Phi(G, T) \) be the root system of \( G \) with respect to \( T \) and let \( \Phi^+(G, T) \) be the set of positive roots of \( \Phi(G, T) \) with respect to \( B \). We have a disjoint union
\[ \Phi(G, T) = \Phi^+(G, T) \sqcup -\Phi^+(G, T) \]
as well as direct sum decompositions of $R$-modules

$$g = \text{Lie}(T) \oplus_{\alpha \in \Phi(G,T)} g_\alpha \quad \text{and} \quad \text{Lie}(B) = \text{Lie}(T) \oplus_{\alpha \in \Phi^+(G,T)} g_\alpha,$$

where each $g_\alpha$ is the weight space of the $T$-module $g$ corresponding to the weight $\alpha$, with $g$ being viewed as a $G$-module (hence also as a $T$-module) via the adjoint representation $\text{Ad} : G \to \text{Aut}_g$. For each $\alpha \in \Phi(G,T)$ there exists a unique $G_{a,R}$ closed subgroup scheme $U_\alpha$ of $G$ which is normalized by $T$ and whose Lie algebra is $g_\alpha$ (see [18], Exp. XII, Sect. 1, Thm. 1.1 or [12], Part II, Ch. 1, Sects. 1.1 and 1.2); as $R$ is a reduced $\mathbb{Q}$-algebra, $\text{Lie}(U_\alpha) = g_\alpha$ implies that $U_\alpha$ is normalized by $T$ as one can easily check based on [2], Ch. II, Sect. 7, Subsect. 7.1. As $U_\alpha \cong G_{a,R}$, $g_\alpha$ is a free $R$-module of rank 1.

We recall from [18], Exp. XII, Sect. 4, Prop. 4.1.2 that the product morphism

$$\iota : \left( \prod_{\alpha \in -\Phi^+(G,T)} U_\alpha \right) \times_{\text{Spec} \ R} T \times_{\text{Spec} \ R} \left( \prod_{\alpha \in \Phi^+(G,T)} U_\alpha \right) \to G$$

is an open embedding whose image $U := \text{Im}(\iota)$ does not depend on the orderings of the first and third factor of the source of $\iota$, being in fact equal to the image $B^\text{op} B$ of the product morphism $B^\text{op} \times_{\text{Spec} \ R} B \to G$. In particular, $\iota$ induces an isomorphism

$$j : \left( \prod_{\alpha \in -\Phi^+(G,T)} U_\alpha \right) \times_{\text{Spec} \ R} T \times_{\text{Spec} \ R} \left( \prod_{\alpha \in \Phi^+(G,T)} U_\alpha \right) \to U. \quad (4)$$

### 3 Proof of Theorem [1]

In this section we assume that $R$ is a reduced $\mathbb{Q}$-algebra and $G = G^{\text{sc}}$ is simply connected.

We write $R = \lim \text{ind}_{\lambda \in \Lambda} R_{\lambda}$ as an inductive limit of finitely generated $\mathbb{Z}$-subalgebras of $R$ where $\Lambda$ is the set of finite subsets of $R$ and $R_{\lambda}$ is the $\mathbb{Z}$-subalgebra of $R$ generated by the elements of $\lambda$.

In this paragraph we recall the essentially well-known property that there exists $\lambda_0 \in \Lambda$ such that $G$ is the pullback of a simply connected semisimple group scheme $G_{\lambda_0}$ over $\text{Spec} R_{\lambda_0}$. As group objects in a category are defined in terms of commutative diagrams, from [11], Thm. (8.8.2) we get that there exists $\lambda_0 \in \Lambda$ such that $G$ is the pullback of an affine group scheme $G_{\lambda_0}$ over $\text{Spec} R_{\lambda_0}$ of finite type. Based on [16], Exp. VII, Subsect. 10.9 and...
Prop. 3.9 we can assume that there exists an open subgroup scheme $G^0_{\lambda_0}$ of $G_{\lambda_0}$ whose fibers over Spec $R_{\lambda_0}$ are connected. From this and the affineness part of [11], Thm. (8.10.5) we get that we can assume that $G^0_{\lambda_0}$ is affine and hence we can also assume that $G_{\lambda_0} = G^0_{\lambda_0}$. Based on [11], Thm. (11.2.6) we can assume that $G_{\lambda_0}$ is flat over Spec $R_{\lambda_0}$. From [18], Exp. XIX, Thm. 2.5 we get that there exist a largest open subscheme $S_{\lambda_0}$ of Spec $R_{\lambda_0}$ such that $S_{\lambda_0} \times_{\text{Spec } R_{\lambda_0}} G_{\lambda_0}$ is a semisimple group scheme over $S_{\lambda_0}$. From this and the isomorphism part of [11], Thm. (8.10.5) we get first that we can assume that $S_{\lambda_0} = \text{Spec } R_{\lambda_0}$ and second that we can assume $G_{\lambda_0}$ is simply connected.

For $\lambda_0 \subset \lambda \in \Lambda$, let $G_{\lambda} := \text{Spec } R_{\lambda} \times_{\text{Spec } R_{\lambda_0}} G_{\lambda_0}$ and let $g_{\lambda} := \text{Lie}(G_{\lambda})$.

As $\mathbb{Z}$ is a universally Japanese ring (see [10], Cor. (7.7.4)), each finitely generated $\mathbb{Z}$-algebra which is an integral domain has a normalization which is a finitely generated $\mathbb{Z}$-algebra and hence noetherian. Thus, if $R$ is a normal integral domain, the normalization of each $R_{\lambda}$ is noetherian, and hence $R$ is the inductive limit of such normal noetherian integral domains.

From [11], Thm. (8.5.2) we get that the categories $\text{Rep}(G)$ and $\text{Rep}(g)$ are the inductive limits of the categories $\text{Rep}(G_{\lambda})$ and $\text{Rep}(g_{\lambda})$ (respectively) indexed by $\lambda \in \Lambda$, $\lambda \supset \lambda_0$. Thus to prove Theorem [11] by replacing $R$ with $R_{\lambda}$ for some $\lambda \in \Lambda$, $\lambda \supset \lambda_0$, we can assume that $R$ is a finitely generated $\mathbb{Z}$-algebra, hence noetherian; hence, as $R$ is reduced, $K$ is a finite product of fields.

To check that the functor (2) is faithful, by replacing $R$ by a direct factor of $K$ which is a field, we can assume that $R = K$ is a field of characteristic zero and this case is well-known (for instance, using graphs, this follows from [2], Ch. II, Sect. 7, Subsect. 7.1).

Thus to prove Theorem [11] it suffices to show that the functor (2) is surjective on objects and on morphisms. To check this, we can work locally in the étale topology of Spec $R$ (cf. Equation (1)) and hence we can also assume that $R$ is connected and that (see [18], Exp. XIX, Prop. 6.1) $G$ has a maximal torus which is split.

### 3.1 Surjectivity of the functor (2) on objects

Though the surjectivity of the functor (2) on objects follows from [18], Exp. XXIV, Prop. 7.3.1 without even assuming that the $\mathbb{Q}$-algebra $R$ is reduced, several parts of the proof included here are used in the proof of Theorem 2.

Let $1_\ast$ be the identity automorphism of a (group) scheme $\ast$. 

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Let $L$ be a $g$-module. To check that there exists a $G$-module $M$ such that we have $M = L$ as $g$-modules, we consider four disjoint cases in order to include several proofs, including simpler ones in the easier cases such as when $R$ is a field or a discrete valuation ring or a normal integral domain.

**Case 1: $R = K$ is a field.** We include two proofs in this case.

The first proof, slightly sketched, is well-known and relies on the classification of simple $G$-modules and simple $g$-modules. As the abelian category Rep($g$) is semisimple, we can assume that $L$ is a simple $g$-module. Let $w \in W_{\geq 0}$ be such that $L(w) \in W_{\geq 0}$ is the dominant weight with the property that $L$, up to isomorphism, is the simple $g$-module of highest weight $L(w)$. Let $M$ be the simple $G$-module of highest weight $w \in G_{\geq 0}$. It is an easy exercise to check that $\text{Lie}(M)$ and $L$ are isomorphic $g$-modules.

For the second proof we consider a $G$-module $V$ such that the representation $\rho_v : G \rightarrow \text{Aut}_A$ is faithful, to be viewed as a closed embedding. Thus the $g$-module $N := \text{Lie}(V) \oplus L = V \oplus L$ is such that the representation $\varphi_N : g \rightarrow \text{End}(N)$ is also faithful, to be viewed as an inclusion. Let $\mathcal{T}(N) := \oplus_{a,b \geq 0} N^a \otimes_K (N^*)^b$, where $N^* := \text{Hom}_K(N,K)$ is the dual of $N$ and $\text{Lie}_c$, with $c \in \mathbb{Z}_{\geq 0}$, is the tensor product over $K$ of $c$-copies of the $K$-vector space $\mathbf{1}$ ($\mathbf{1}^0 := K$). From [19], Ch. VI, Sect. 5, Thm. 5.2 we get that there exists a finite subset $\mathcal{F} \subset \mathcal{T}(N)$ with the property that $g$ is the Lie subalgebra of $\text{End}_K(N)$ that annihilates every tensor of $\mathcal{F}$. We can assume that the projection $\pi \in \text{End}_K(N) = N \otimes_K N^*$ of $N$ on $L$ along $V$ belongs to $\mathcal{F}$. Let $E$ be the subgroup of $\text{Aut}_N$ which fixes each tensor of $\mathcal{F}$. Let $E^0$ be the connected component of the identity element of $E$. We have $\text{Lie}(E) = g$, thus also $\text{Lie}(E^0) = g$. As $E$ fixes $\pi$, both $L$ and $V$ are $E^0$-modules. In particular, we have a representation $\sigma : E^0 \rightarrow \text{Aut}_V$ with the property that $\text{Lie}(\sigma)$ is injective (due to the identity $\text{Lie}(E^0) = g$). This implies that $\sigma$ induces an étale isogeny $\sigma : E^0 \rightarrow \text{Im}(\sigma)$ and moreover we have $\text{Lie(Im(\sigma))} = \text{Lie}(G) = g \subset \text{End}_K(V)$. From this and [2], Ch. II, Sect. 7, Subsect. 7.1 we get that $\text{Im}(\sigma) = G$ and hence we have an étale isogeny $E^0 \rightarrow G$. As $G$ is simply connected, we conclude that $E^0 \rightarrow G$ is an isomorphism. As $L$ is an $E^0$-module, we conclude that it is as well a $G$-module in such a way that $\text{Lie}(L)$ is the $g$-module $L$.

**Case 2: $R$ is a discrete valuation ring.** Let $k$ be the residue field of $R$; we have char($k$) = 0.

Let $V$ be the $G_K$-module such that $\text{Lie}(V) = K \otimes_R L$, see Case 1. We identify $V = K \otimes_R L$ as $K$-vector spaces and let $M$ be the lattice of $V$ with
respect to $R$ which, under the mentioned identification, gets identified with $L$. It suffices to show that $M$ is a $G$-module.

Based on [16], Exp. VIB, Rm. 11.11.1 we get the existence of a $G$-module $P$ such that the representation $\rho_P : G \to \text{Aut}_P$ is a closed embedding. From [12], Part I, Ch. X, Lem. of Sect. 10.4 we get that there exists a lattice $M'$ of $V$ which is a $G$-module. As $G$ is split and simply connected and as $R$ is connected, the existence of $P$ (respectively $M'$) also follows by pullback to $\text{Spec} R$ from the mentioned references applied over $\mathbb{Z}_{(p)}$ to $G_{\mathbb{Z}_{(p)}} := \text{Spec} \mathbb{Z}_{(p)} \times_{\text{Spec} \mathbb{Z}} G_{\mathbb{Z}}$ (respectively to $G_{\mathbb{Z}_{(p)}}'$ and a $V^*_Q$ as in Subsection 222). Let $Q := P \oplus M$ and $Q' := P \oplus M'$; we have $K \otimes_R Q = K \otimes_R Q'$.

The representation $\rho_{Q'} : G \to \text{Aut}_{Q'}$ is a closed embedding and thus, with the notation of Subsection 223 for each $q \in \Phi(G, T)$, $U_\alpha$ is also a closed subgroup scheme of $\text{Aut}_{Q'}$ whose Lie algebra is identified with $\mathfrak{g}_\alpha$ via the faithful representation $\rho_{Q'} = \text{Lie}(\rho_{Q'}) : \mathfrak{g} \to \text{End}_R(Q')$.

For $\alpha \in \Phi(G, T)$ let $\mathbb{V}_\alpha$ be the vector group scheme over $\text{Spec} R$ whose group of $S$-valued points is $S \otimes_R \mathfrak{g}_\alpha$ for each $R$-algebra $S$. As $R$ is a $\mathbb{Q}$-algebra and $Q$ and $Q'$ are $\mathfrak{g}$-modules, we have homomorphisms

$$\eta_\alpha : \mathbb{V}_\alpha \to \text{Aut}_Q \quad \text{and} \quad \eta'_\alpha : \mathbb{V}_\alpha \to \text{Aut}_{Q'}$$

which for each $R$-algebra $S$ map $x \in \mathbb{V}_\alpha(S) = S \otimes_R \mathfrak{g}_\alpha$ to the sums $\sum_{q=0}^\infty \frac{\rho_Q(x)^q}{q!}$ and $\sum_{q=0}^\infty \frac{\rho_{Q'}(x)^q}{q!}$ (respectively). These sums coincide as elements of $\text{End}_K(V)$ and are finite sums as each $x$ acts nilpotently on $S \otimes_R Q$ and $S \otimes_R Q'$. The image of $\eta_{\alpha,K} = \eta_{\alpha,K} : U_{\alpha,K}$ have the same Lie algebras and thus they coincide, see [2], Ch. II, Sect. 7, Subsect. 7.1. This implies that $\eta'_\alpha$ factors through a homomorphism $\zeta_\alpha : \mathbb{V}_\alpha \to U_\alpha$ which induces an isomorphism at the level of Lie algebras, and hence is étale. As $\mathbb{G}_m$ over each field of characteristic 0 has no finite nontrivial subgroup, we deduce that the fibers of $\zeta_\alpha$ are isomorphisms, based on which we easily see that $\zeta_\alpha$ itself is an isomorphism.

We get a homomorphism $\eta_\alpha \circ \zeta_\alpha^{-1} : U_\alpha \to \text{Aut}_Q$, hence $Q$ is a $U_\alpha$-module.

We fix an identification $T = \mathbb{G}_m^r$ and with respect to it we speak about the $\mathbb{G}_{m,R}$ factors of $T$ (there exist $r$ such factors). If $F = \mathbb{G}_{m,R}$ is such a factor of $T$, then we have a direct sum decomposition $K \otimes_R Q = \oplus_{q \in \mathbb{Z}} W_q$ such that $F_K$ acts on $W_q$ via the $q$-th power of the identity character of $F_K$. The standard generator $x$ of $\text{Lie}(F)$ acts on $W_q$ as the multiplication by $q$. As $Q$ is a $\mathfrak{g}$-module, we have $x(Q) \subset Q$. As $x(Q) \subset Q$ and as for distinct integers $q_1, q_2$ which are eigenvalues of $x$ acting on the $K$-vector space $K \otimes_R Q$, the
difference \( q_1 - q_2 \) is invertible in \( k \), it is an easy exercise to check that we have a direct sum decomposition \( Q = \oplus_{q \in Z} Q \cap W_q \). This implies that \( Q \) is an \( F \)-module. The resulting homomorphism \( \eta_F : F \to \text{Aut}_Q \) is a closed embedding as this is so over \( \text{Spec} \ K \), see [23], Lem. 2.3.2 (b) and (c). The images \( \text{Im}(\eta_F) \) indexed by such factors \( F \) of \( T \) commute as this is so over \( \text{Spec} \ K \) and hence we get a product homomorphism \( \eta_T : T \to \text{Aut}_Q \) which over \( \text{Spec} \ K \) is a closed embedding. Again from [23], Lem. 2.3.2 (b) and (c) we get that \( \eta_T \) is a closed embedding. In particular, \( Q \) is a \( T \)-module.

From the last two paragraphs we get a product morphism

\[
\eta : ( \prod_{\alpha \in \Phi^+ (G,T)} U_\alpha ) \times_{\text{Spec} \ R} T \times_{\text{Spec} \ R} ( \prod_{\alpha \in \Phi^- (G,T)} U_\alpha ) \to \text{Aut}_Q
\]

which is the product of the \( \eta_\alpha \)'s and \( \eta_T \) and which is compatible with the representation \( \rho_{K \otimes R} : G_K \to \text{Aut}_{K \otimes R} Q \), in the sense that \( \rho_{K \otimes R} \) restricted to \( U_K \) is \( \eta_K \circ f^{-1} \).

The union \( U_+ := G_K \cup U \) is an open subscheme of \( G \) whose complement \( C := G \setminus U_+ \), when endowed with the reduced structure, is a reduced closed subscheme of \( G_k \) of dimension less than \( d = \dim(G_k) \). Thus, as we have \( \dim(G) = d + \dim(R) = d + 1 \), we get that \( \text{codim}_G(C) \geq 2 \).

From the last two paragraphs we get the existence of a morphism

\[
\rho_{Q,U_+} : U_+ \to \text{Aut}_Q
\]

which extends both \( \rho_{K \otimes R} \) and the composite

\[
\rho_{Q,U} := \eta \circ f^{-1} : U = \text{Im}(\iota) \to \text{Aut}_Q.
\]

As \( \text{codim}_G(C) \geq 2 \) and the scheme \( \text{Spec} \ R \) is normal noetherian, from [3], Ch. 4, Sect. 4.4, Thm. 1 we get that the morphism \( \rho_{Q,U_+} \) extends to a morphism \( \rho_Q : G \to \text{Aut}_Q \) which, as it extends \( \rho_{K \otimes R} \), is a homomorphism. So \( Q \) is a \( G \)-module.

The projection of \( Q \) on \( M \) along \( P \) is fixed by \( G \) (as it is fixed by \( G_K \)). This implies that \( \rho_Q \) induces a homomorphism \( \rho_M : G \to \text{Aut}_M \) that extends \( \rho_V \), hence \( M \) is a \( G \)-module.

**Case 3:** \( R \) is normal but neither a field nor a discrete valuation ring. Let \( \mathcal{D} \) be the set of all local rings of \( R \) which are discrete valuation rings; we recall (for instance, see [14], Thm. 11.5) that, as \( R \) is noetherian, \( \mathcal{D} \) is nonempty and in fact we have \( R = \cap_{O \in \mathcal{D}} O \).
Let $M := L$. From Case 2 we get the existence of an open subscheme $Y$ of Spec $R$ which contains all points of Spec $R$ of codimension in Spec $R$ at most 1 (i.e., the closed subscheme Spec $R \setminus Y$ has codimension in Spec $R$ at least 2) and for which we have a homomorphism $\rho_{M,Y} : G_Y \to Y \times_{\text{Spec } R} \text{Aut}_M$ between reductive group schemes over $Y$ with the property that for each $O \in \mathcal{D}$, the $O \otimes_R g$-module $O \otimes_R M$ is $O \otimes_R L$. Considering the closed embedding 

$$(\rho_{M,Y}, 1_{G_Y}) : G_Y \to (Y \times_{\text{Spec } R} \text{Aut}_M) \times_Y G_Y,$$

from [24], Prop. 5.1 we get that it extends uniquely to a closed embedding homomorphism $(\rho_M, 1_G) : G \to \text{Aut}_M \times_{\text{Spec } R} G$. The resulting homomorphism $\rho_M : G \to \text{Aut}_M$ endows $M$ with the structure of a $G$-module. The fact that the $g$-module structure on $M$ is the same one as the one given by $M = L$ follows from the fact that this is so over $O$ for one (hence all) $O \in \mathcal{D}$.

**Case 4:** $R$ is not normal. Let $V$ be the $G_K$-module such that we have $\text{Lie}(V) = K \otimes_R L$, see Case 1 applied to the direct factors of $K$ which are fields. Let $G_{\mathbb{Z}(p)}$ and $G_{\mathbb{Q}}$ be as in Subsection 2.2 and let $M$ be as in Case 2. As $R$ is connected and $G = G^{\text{sc}}$ is split, there exists a $G_{\mathbb{Q}}$-module $V_{\mathbb{Q}}$ such that the $G_K$-module $V$ is isomorphic to $K \otimes_{\mathbb{Q}} V_{\mathbb{Q}}$ (see end of Subsection 2.2). This implies that there exist $G$-modules $P$ and $M'$ such that $\rho_P : G \to \text{Aut}_P$ is a closed embedding and $V = K \otimes_R M'$ (they are obtained, to be compared with Case 2, by pullback from Spec $\mathbb{Z}(p)$ to Spec $R$). Based on this, as in Case 2 we argue the existence of a product morphism $\eta$ as in Equation (5), and hence we get a morphism $\rho_{Q,U} : U = \text{Im}(i) \to \text{Aut}_Q$ as in Equation (6). The pullback of $\rho_{Q,U}$ to Spec $K$ coincides with the restriction to $U_K$ of the homomorphism $\rho_{K \otimes_{\mathbb{R}} \mathbb{Q}} : G_K \to \text{Aut}_{K \otimes_{\mathbb{R}} \mathbb{Q}}$.

The product morphism

$$\Theta : U \times_{\text{Spec } R} U \to G$$

is surjective and smooth, in particular it is a faithfully flat morphism between affine schemes. We will use affine faithfully flat descent with respect to $\Theta$ to show that the morphism $\rho_{Q,U}$ extends to a morphism $\rho_Q : G \to \text{Aut}_Q$ that extends $\rho_{K \otimes_{\mathbb{R}} \mathbb{Q}}$. We consider the two projections

$$\Pi_1, \Pi_2 : (U \times_{\text{Spec } R} U) \times_G (U \times_{\text{Spec } R} U) \to U \times_{\text{Spec } R} U$$

defined by $\Theta$. The two composite morphisms

$$\rho_{Q,U} \circ \Pi_1, \rho_{Q,U} \circ \Pi_2 : (U \times_{\text{Spec } R} U) \times_G (U \times_{\text{Spec } R} U) \to \text{Aut}_Q$$

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coincide as this is so after pullback to Spec $K$. This implies the existence of $ho_Q$ with the desired property.

As $\rho_{K \otimes_R Q}$ is homomorphism we get that $\rho_Q$ is a homomorphism. As in the last paragraph of Case 2 we argue that $M$ is a $G$-module. The fact that the $\mathfrak{g}$-module structure on $M$ is the same one as the one given by $M = L$ follows from the fact that this is so over $K$.

### 3.2 Surjectivity of the functor (2) on morphisms

Let $f : L \to J$ be a morphism of $\text{Rep}(\mathfrak{g})$. Based on Subsection 3.1 we know that there exist $G$-modules $M$ and $P$ such that $\text{Lie}(M) = L$ and $\text{Lie}(P) = J$, i.e., we have $M = L$ and $P = J$ as $R$-modules but, in connection to $f : L \to J$, we view them as $\mathfrak{g}$-modules. We denote also by $f : M \to P$ the $R$-linear map defined by $f$ and the identifications $M = L$ and $P = J$, and to end the proof of Theorem 1 it suffices to show that $f : M \to P$ is a morphism of $G$-modules. To check this, recall (see beginning of Section 3) that we are assuming that Spec $R$ is connected and $R$ is a reduced finitely generated $\mathbb{Z}$-algebra. As each smooth $R$-algebra $S$ is still a reduced finitely generated $\mathbb{Z}$-algebra, by replacing $R$ with smooth $R$-algebras whose spectra are connected, it suffices to show that for every $g \in G(R)$ we have (cf. Equation (1))

$$f \circ \rho_M(R)(g) = \rho_P(R)(g) \circ f : M \to P.$$  \hspace{1cm} (7)

To check this, by giving up on the second recalled assumption on $R$, we can assume that $R = K = \overline{K}$ is an algebraically closed field and we will only use $K$.

Let $A$ be the subgroup of $\text{Aut}_M \times_{\text{Spec} K} \text{Aut}_P$ defined by the identity

$$A(K) = \{(g_1, g_2) \in \text{Aut}_M(K) \times \text{Aut}_P(K) | f \circ g_1 = g_2 \circ f \}.$$ 

Let $I$ be the image of the homomorphism

$$(\rho_M, \rho_P) : G \to \text{Aut}_M \times_{\text{Spec} K} \text{Aut}_P.$$ 

Considering the short exact sequence $1 \to \text{Ker}(G \to I) \to G \to I \to 1$, as $\text{Ker}(G \to I)$ is smooth over Spec $K$ (due to Cartier’s theorem), we get a short exact sequence of Lie algebras

$$0 \to \text{Lie(}\text{Ker}(G \to I)) \to \mathfrak{g} \to \text{Lie}(I) \to 0.$$
As \( f : L \to J \) is a morphism of \( g \)-modules and as \( g \to \text{Lie}(I) \) is surjective, we get that we have an inclusion

\[
\text{Lie}(I) \subset \text{Lie}(A).
\]

From this and [2], Ch. II, Sect. 7, Subsect. 7.1 we get that \( I \) is a subgroup of \( A \) which implies that Equation (7) holds. Thus the functor (2) is surjective on morphisms. We conclude that Theorem 1 holds. \( \Box \)

4 Proof of Theorem 3

We will first prove the following basic lemma:

**Lemma 1.** Let \( I \) be a subgroup of a semisimple group \( H \) of adjoint type over the spectrum of an algebraically closed field \( \kappa \) such that \( \dim(H/I) = 1 \). Then there exists an isomorphism \( H \cong \text{PGL}_{2, \kappa} \times_{\text{Spec} \, \kappa} H' \) which induces via restriction an isomorphism \( I \cong B_{2, \kappa} \times_{\text{Spec} \, \kappa} H' \), where \( B_{2, \kappa} \) is a Borel subgroup of \( \text{PGL}_{2, \kappa} \) and \( H' \) is an arbitrary adjoint group over \( \text{Spec} \, \kappa \).

**Proof:** We consider the connected smooth projective curve \( C \) having \( H/I \) as an open subscheme and let \( g(C) \) be its genus.

For simplicity, we define \( \text{Aut}(H/I) \) to be the reduced (smooth) subgroup of the group of automorphisms \( \text{Aut}(C) \) of \( C \) that leave invariant the complement \( C_0 := C \setminus (H/I) \). As for each field extension \( \mu \) of \( \kappa \), every automorphism of \((H/I)_\mu \) extends to an automorphism of \( C_\mu \), the left multiplication action of \( H(\mu) \) on \((H/I)_\mu \) induces an abstract homomorphism \( H(\mu) \to \text{Aut}(H/I)(\mu) \). Taking \( \mu \) to be the field of fractions of \( H \), we obtain a natural rational morphism from \( H \) to \( \text{Aut}(H/I) \) and using translates it follows that it is defined everywhere. The reduced kernel of the resulting homomorphism \( H \to \text{Aut}(H/I) \) has a connected component \( H' \) of the identity element which is the largest connected normal smooth subgroup of \( H \) contained in \( I \). Thus, as \( H/H' \) is semisimple, we have inequalities

\[
3 \leq \dim(H/H') \leq \dim(\text{Aut}(H/I)). \tag{8}
\]

\[\text{Another approach to prove this lemma due to Gabber is to use induction on the number of simple factors of } H. \text{ One is reduced to the base of the induction case, so } H \text{ is simple, and it would suffice to prove that } \dim(H/I) \text{ is at least equal to the rank of } H; \text{ such an inequality is well-known in characteristic 0 but we could not find a reference for it in positive characteristic (however see [13]).}\]
As $\text{Aut}(H/I) \subset \text{Aut}(C)$ and as the connected component $\text{Aut}(C)^0$ of the identity element of $\text{Aut}(C)$ is trivial if $g(C) \geq 2$, is an elliptic curve (thus abelian) if $g(C) = 1$, and it is $\text{PGL}_{2,\kappa}$ if $g(C) = 0$, we conclude that $g(C) = 0$ and we have a finite homomorphism $H/H' \to \text{PGL}_{2,\kappa}$. As $H/H'$ is semisimple, by reasons of dimensions or by the classification of adjoint groups over $\kappa$, we get that $H/H'$ is isomorphic to either $\text{PGL}_{2,\kappa}$ or $\text{SL}_{2,\kappa}$. As $H$ is adjoint, the short exact sequence $1 \to H' \to H \to H/H' \to 1$ splits. Thus $H \simeq H' \times_{\text{Spec} \kappa} H/H'$ and we conclude that $H/H' \simeq \text{PGL}_{2,\kappa}$.

If $H/I$ is an affine rational curve, then $C \simeq \mathbb{P}^1_{\kappa}$ and the connected component $\text{Aut}(H/I)^0$ of the identity element of $\text{Aut}(H/I)$ is the subgroup of $\text{Aut}(\mathbb{P}^1_{\kappa}) \simeq \text{PGL}_{2,\kappa}$ that fixes the finite nonempty set $C_0$; it follows that $\dim(\text{Aut}(H/I)) \leq \dim(\text{PGL}_{2,\kappa}) - 1 = 2$ which contradicts Inequality (8).

Thus $H/I$ is projective isomorphic to $\mathbb{P}^1_{\kappa}$ which implies that $I/H'$ is a parabolic subgroup of $H/H'$, hence a Borel subgroup of $H/H' \simeq \text{PGL}_{2,\kappa}$. The lemma follows from the last sentence and the isomorphisms $H/H' \simeq \text{PGL}_{2,\kappa}$ and $H \simeq H' \times_{\text{Spec} \kappa} H/H'$.

To prove Theorem 3 for an $H$-module $\diamond$, let $\ell_H(\diamond)$ be its length and let $\ell_{\text{Lie}(H)}(\diamond)$ be its length as a $\text{Lie}(H)$-module. We have a general inequality

$$\ell_H(\diamond) \leq \ell_{\text{Lie}(H)}(\diamond). \tag{9}$$

As $\ell_H$ and $\ell_{\text{Lie}(H)}$ are additive, from Inequality (9) we get immediately:

**Fact 1.** If the Inequality (9) is an equality for $\diamond$, then it is an equality for each $H$-submodule or quotient of $\diamond$.

We will use Lemma 1 to prove Theorem 3, i.e., that $P$ is a semisimple $H$-module, by induction on $\ell := \ell_H(P) = \ell_{\text{Lie}(H)}(P) \in \mathbb{Z}_{\geq 0}$. The base of the induction for $\ell \in \{0, 1\}$ is trivial. For $\ell \geq 2$ the passage from $\ell - 1$ to $\ell$ goes as follows. Let $Q$ be a simple $H$-submodule of $P$: it is a simple $\text{Lie}(H)$-module (by Fact 1) and the $\text{Lie}(H)$-module $P/Q$ is semisimple of the same length $\ell - 1$ as the $H$-module $P/Q$. By the induction assumption, the $H$-module $P/Q$ is semisimple. Thus, to prove that the short exact sequence

$$0 \to Q \to P \to P/Q \to 0$$

splits, we can assume that $\ell = 2$.

As $\ell = 2$, the $\text{Lie}(H)$-module $P/Q$ is simple and we consider a simple $\text{Lie}(H)$-submodule $N$ of $P$ which maps isomorphically onto $P/Q$. We have a direct sum decomposition $P = Q \oplus N$ of $\text{Lie}(H)$-modules.
We consider two cases as follows.

**Case 1: the Lie\((H)\)-modules \(N\) and \(Q\) are not isomorphic.** Thus \(P\) has only two simple Lie\((H)\)-submodules: \(Q\) and \(N\). As for all \(h \in \text{Lie}(H)\), we have \(h\text{Lie}(H)h^{-1} = \text{Lie}(H)\), we get that \(h(Q)\) and \(h(N)\) are simple Lie\((H)\)-modules for all \(h \in \text{Lie}(H)\). As \(H\) is connected, from the last two sentences we get that for all \(h \in \text{Lie}(H)\) we have \(Q = h(Q)\) and \(N = h(N)\). This implies that both \(Q\) and \(N\) are \(H\)-submodules of \(P\) which, as \(\ell = 2\), are simple. Thus \(P = Q \oplus N\) is a semisimple \(H\)-module in this case.

**Case 2: the Lie\((H)\)-modules \(N\) and \(Q\) are isomorphic.** We fix a Lie\((H)\)-isomorphism \(a: Q \to N\): it is unique up to multiplication by nonzero elements of \(\kappa\). All simple Lie\((H)\)-submodules of \(P\) are of the form \(Q_{[t_0 : t_1]} := \{t_0 x + t_1 a(x) | x \in Q\} \subset P = Q \oplus N\) for a uniquely determined point \([t_0 : t_1] \in \mathbb{P}_\kappa^1\). For instance, \(Q = Q_{[1 : 0]}\) and \(N = Q_{[0 : 1]}\). Similar to Case 1, for each field extension \(\mu\) of \(\kappa\) and for every \(h \in \text{Lie}(\mu)\), \(h(\mu \otimes \kappa Q)\) is a simple Lie\((H)\_\mu\)-module and hence there exists a unique point \(\delta(h) = [v_0 : v_1] \in \mathbb{P}_\kappa^1\) such that \(h(Q) = Q_{\delta(h)} := \{v_0 x + v_1 a(x) | x \in \mu \otimes \kappa Q\}\).

An argument similar to the one involving \(\mu\)s in the proof of Lemma \([\text{III}]\) shows that the association \(h \to \delta(h)\) defines a morphism \(\delta: H \to \mathbb{P}_\kappa^1\).

We show that the assumption that \(\delta\) is nonconstant leads to a contradiction. For the stabilizer \(I\) of \(Q\) in \(H\) we have \(\dim(H/I) = \dim(\text{Im}(\delta)) = 1\) and from Lemma \([\text{III}]\) applied to the adjoint group \(H_{\text{ad}}\) of \(H\) we get that there exists an isomorphism \(H_{\text{ad}} \simeq \text{PGL}_{2,\kappa} \times_{\text{Spec} \kappa} H'\) which induces via restriction an isomorphism \(\text{Im}(I \to H_{\text{ad}}) \simeq B_2 \times_{\text{Spec} \kappa} H'\), where \(B_2\) is a Borel subgroup of \(\text{PGL}_{2,\kappa}\) and where \(H'\) is an adjoint group over \(\kappa\). This implies that \(\delta\) is surjective and thus the simple Lie\((H)\)-submodules of \(P\) are permuted transitively under the natural left multiplication action by \(H(\kappa)\). But a simple \(H\)-submodule of \(P\) is among the simple Lie\((H)\)-submodules of \(P\) (by Fact \([\text{I}]\) and it is fixed by \(H(\kappa)\), hence we reached a contradiction.

Thus \(\delta\) is constant of constant value \([1 : 0]\). Hence for all \(h \in H(\kappa)\) we have \(h(Q) = Q\) which implies that \(Q\) is an \(H\)-submodule of \(P\). The same applies to \(N\). Thus \(P = Q \oplus N\) is a semisimple \(H\)-module even in Case 2.

This ends the induction and the proof of Theorem \([\text{II}]\) \(\square\).
5 Proof of Theorem 2

In this section we assume that $R$ is a faithfully flat $\mathbb{Z}(p)$-algebra and a normal integral domain.

Theorem 2 (a) is proved in Subsection 5.1. Theorem 2 (b) is proved in Subsection 5.2. If $G$ is split, then the hypotheses of Theorem 2 (b) hold: as $D$ we can take $\mathbb{Z}(p)$, see the existence of $G_{\mathbb{Z}}$ and $V_{\mathbb{Z}}$ in Subsection 2.2. Thus, Theorem 2 (c) follows directly from Theorems 2 (a) and (b).

We recall the following well-known fact.

Fact 2. Let $S$ be an affine smooth scheme over the spectrum of a discrete valuation ring $D$ with uniformizer $\varpi$ and residue field $k$. Let $M$ be a free $D$-module which is a $S$-module (i.e., it is equipped with a homomorphism $S \to \text{Aut}_M$). Let $L$ be a $D$-submodule of $\varpi^{-1}M$ which contains $M$. Then $L$ is a $S$-module (resp. a Lie($S$)-module) if and only if $L/M$ is a $S_k$-submodule (resp. is a Lie($S_k$)-submodule) of $\varpi^{-1}M/M$.

Proof: The ‘only if’ parts and the case of Lie algebras are obvious, hence it suffices to check that if $L/M$ is a $S_k$-module, then $L$ is a $S$-module. Writing $S = \text{Spec} \, A$, this is equivalent to checking that the comultiplication $D$-linear map $\nabla : M \to A \otimes_D M$ is such that $\nabla(L) \subset A \otimes_D L$. To check this we can assume that $k$ is algebraically closed (it suffices to be infinite) and $D$ is complete and we will check directly (i.e., without mentioning $\nabla$ again) that $L$ is a $S$-module. As $L/M$ is a $A_k$-submodule of $\varpi^{-1}M/M$, we get that for each $h \in S(D)$ we have $h(L) = L$. From the last two sentences and [21], Prop. 3.1.2.1 a) we get that the homomorphism $S \to \text{Aut}_M$ over the spectrum of the field of fractions of $D$ extends to a homomorphism $S \to \text{Aut}_L$, thus $L$ is a $S$-module.

5.1 Proof of Theorem 2 (a)

Let $Z := \text{Ker}(G^{sc} \to G)$; it is a finite flat group scheme over $\text{Spec} \, R$ of multiplicative type which is contained in the center of $G^{sc}$ and which, if $G$ is split, is the kernel of the induced homomorphisms between split maximal tori. In particular, the Zariski (or the schematic) closure of $Z_K$ in $G^{sc}$ is $Z$ itself. Thus, if $M$ is a $G^{sc}$-module such that we can identify $K \otimes_R M = V$ as $G^{sc}_K$-modules, then the kernel of the homomorphism $\rho_M : G^{sc} \to \text{Aut}_M$ contains $Z_K$ and therefore it contains $Z$; hence $M$ is in fact a $G$-module. Thus we can identify $\text{Lat}_{G^{sc}}(V) = \text{Lat}_G(V)$. Based on this and the inclusions
Lat\(_G(V) \subset \text{Lat}_g(V) \subset \text{Lat}_{\text{Lie}(G^{sc})}(V) \supset \text{Lat}_{G^{sc}}(V)\), it suffices to prove that Theorems 2 (a) and (b) hold in the case when \(G = G^{sc}\) is simply connected.

To prove that Theorem 2 (a) holds, as in the beginning of Section 3, using inductive limits and working in the étale topology of \(\text{Spec} \, R\), we can assume that \(R\) is also noetherian and that \(G\) is split. Based on Case 2 of Subsection 3.1, as in Case 3 of Subsection 3.1 we argue that Theorem 2 (a) holds provided it holds for discrete valuation rings of mixed characteristic \((0, p)\). Thus we can also assume that \(R = D\) is a discrete valuation ring of mixed characteristic \((0, p)\); let \(\pi\) be a uniformizer of it. Let \(k := D/(\pi)\): it is a field of characteristic \(p\). Let \(H := G_k\) and let \(h := g/\pi g = k \otimes_D g\).

Let \(L \in \text{Lat}_g(V)\); we have \(V = K \otimes_D L\). With the notation of Subsections 2.2 and 2.3 for \(G = G^{sc}\), if we have \(L \in \text{Lat}_{G_i}(V)\) for all \(i \in \{1, \ldots, n\}\), then we obtain homomorphisms \(\rho_{L,i} : G_i \to \text{Aut}_L\) whose fibers over \(K\) are restrictions of \(\rho_V : G_K \to \text{Aut}_V\). These homomorphisms over \(\text{Spec} \, D\) commute as they commute over \(K\). This implies that their product defines a homomorphism \(\rho_L : G \to \text{Aut}_L\) which extends \(\rho_V\) and hence we have \(L \in \text{Lat}_G(V)\).

As in Case 2 of Subsection 3.1 based on [12], Part I, Ch. X, Lem. of Sect. 10.4 we get the existence of a lattice \(M\) of \(V\) with respect to \(R\) which is a \(G\)-module. So \(M/\pi M\) is an \(H\)-module.

To prove that \(L\) is a \(G\)-module, we can assume that \(k\) is algebraically closed and we can replace \(L\) by \(\pi^r L\) with \(r \in \mathbb{Z}\). Thus we can assume that \(M \subset L\) but \(\pi^{-1} M \not\subset L\). Let \(s \geq 0\) be the smallest integer such that \(L \subset \pi^{-s} M\). We will prove by induction on \(s \in \mathbb{Z}_{\geq 0}\) that, regardless of what the \(G\)-module \(M\) is, \(L\) is a \(G\)-module, i.e., \(L \in \text{Lat}_G(V)\). The base of the induction is trivial: if \(s = 0\), then \(L = M\) is a \(G\)-module.

For \(s \in \mathbb{N}\) the passage from at most \(s - 1\) to \(s\) goes as follows. If \(s \geq 2\), then \(M \subset \pi L + M \subset \pi^{-s+1} M\) are inclusions between \(g\)-modules and hence by induction applied first with \((L, s)\) replaced by \((\pi L + M, s - 1)\) we get that \(\pi L + M\) is a \(G\)-module and applied second with \((M, s)\) replaced by \((\pi L + M, 1)\) we get that \(L\) is a \(G\)-module. Thus we can assume that \(s = 1\), i.e., we have \(M \subsetneq L \subsetneq \pi^{-1} M\). Let

\[n := L/M \subset m := \pi^{-1} M/M \simeq M/\pi M;\]

it is a nonzero \(h\)-module.

We will prove using a second induction on the length \(t \in \mathbb{N}\) of the \(h\)-module \(n\) that, regardless of what the \(G\)-module \(M\) is, \(L\) is a \(G\)-module. Let
Let $\mathfrak{p}$ be a $\mathfrak{h}$-submodule of $\mathfrak{n}$ of length $t - 1$: thus the $\mathfrak{h}$-module $\mathfrak{n}/\mathfrak{p}$ is simple. Let $M_+$ be the inverse image of $\mathfrak{p}$ via the $D$-linear map $\varpi^{-1}M \to \varpi^{-1}M/M = \mathfrak{m}$. If $t = 1$, then $M_+ = M$ is a $G$-module. Thus, if $t \geq 2$ and the statement is true for $\leq t - 1$, then by the (second) induction assumption we get first that $M_+$ is a $G$-module and second, by replacing $(M, t)$ by $(M_+, 1)$, that $L$ itself is a $G$-module. Hence to end the proof of both inductions we can assume that not only $s = 1$ but we also have $t = 1$. So $\mathfrak{n}$ is a simple $\mathfrak{h}$-module.

For a maximal torus $T$ of $G$ which is split, the weights of the action of $T$ on $\mathfrak{m}$ are the same as the weights of the action of $T_k$ on $\mathfrak{m}$. Based on this, as statement (2) holds, for each composition series of the $H$-module $\mathfrak{m}$, the simple factors are irreducible $H$-modules associated to highest weights $\sum_{l=1}^{r_1} \gamma_{1,l} \omega_{1,l}$ with the property that for all $l \in \{1, \ldots, r_1\}$ we have $\gamma_{1,l} \in \{0, \ldots, p - 1\}$ and hence are simple $\mathfrak{h}$-modules (see [1], Thm. 6.4). This implies that the $H$-module $\mathfrak{m}$ and the $H$-module $\mathfrak{m}$ have the same length, and, by Fact [1] the same holds for each $H$-submodule $\mathfrak{p}$ of $\mathfrak{m}$.

We take $\mathfrak{p}$ to be the $H$-submodule of $\mathfrak{m}$ generated by $\mathfrak{n}$. As $k$ is algebraically closed, we have an identity

$$\mathfrak{p} = \sum_{h \in H(k)} h(\mathfrak{n})$$

of $k$-vector spaces. As $h$ normalizes $\mathfrak{h}$, each $h(\mathfrak{n})$ is a simple $\mathfrak{h}$-module and therefore $\mathfrak{p}$, being a sum of simple $\mathfrak{h}$-modules, is a semisimple $\mathfrak{h}$-submodule of $\mathfrak{m}$. From Theorem 3 we get that $\mathfrak{p}$ is a semisimple $H$-module.

Writing $\mathfrak{p} = \bigoplus_{u=1}^b \mathfrak{p}_u$ as a direct sum of simple $H$-modules, from Fact [1] we get that each $\mathfrak{p}_u$ is a simple $\mathfrak{h}$-module. From this and the fact that $\mathfrak{p}$ is the $H$-submodule of $\mathfrak{m}$ generated by $\mathfrak{n}$, we get that the $\mathfrak{h}$-module $\mathfrak{n}$ projects isomorphically onto each $\mathfrak{p}_u$. Hence the $\mathfrak{h}$-module $\mathfrak{p}$ is isomorphic to $b \mathfrak{n} := \bigoplus_{u=1}^b \mathfrak{n}$. If $\omega_u$ is the highest weight of the $H$-module $\mathfrak{p}_u$, then as the isomorphism class of the $\mathfrak{h}$-module $\mathfrak{p}_u$ does not depend on $u$, we easily get that $\omega_u \in \Omega_1 := \{\sum_{l=1}^{r_1} \gamma_{1,l} \omega_{1,l} | \gamma_{1,1}, \ldots, \gamma_{1,r_1} \in \{0, 1, \ldots, p - 1\}\}$, does not depend on $u$ (this is also proved in [1], Subsect. 6.6). Thus we have $\mathfrak{p} = b \mathfrak{p}_1$ as $H$-modules as well as $\mathfrak{h}$-modules and this implies that a $k$-vector subspace of $\mathfrak{p}$ is an $H$-module if and only if it is a $\mathfrak{h}$-module. Therefore $\mathfrak{n}$ is an $H$-module (and in particular we have $\mathfrak{p} = \mathfrak{n}$ and $b = 1$). This implies that $L$ is a $G$-module (see Fact 2). This ends the proof of both inductions and hence of Theorem 2 (a).
5.2 Proof of Theorem 2 (b)

Using the contrapositive, it suffices to show that if statement $\textcircled{2}$ does not hold, then there exists $L \in \text{Lat}_g(V)$ which is not a $G$-module. Considering pullbacks via $\text{Spec } R \to \text{Spec } D$ (i.e., the tensorization of elements of $\text{Lat}_{\text{Lie}(D)}(V_{\text{Frac } D})$ over $D$ with $R$), to find such an $L$, we can assume that $R = D$ is a discrete valuation ring of mixed characteristic $(0, p)$. Let $\varpi$, $k$ and $M$ be as in Subsection 5.1, we will only use $G$ (as $G = G^{\text{sc}} = G^D$), $D$ (as $R = D$), and $V$ (as $V = V_K$).

The factors of a composition series of the $H := G_k$-module $m \simeq M/\varpi M$ do not depend on the choice of the $G$-module $M$, see [12], Part I, Ch. X, Sects. 10.7 and 10.9. Let $D^h$ be the henselization of $D$.

We consider two disjoint cases as follows.

Case 1: $H$ is split. As $H$ is split, the affine smooth scheme $T_G$ over $\text{Spec } D$ that parametrizes maximal tori of $G$ (see [17], Exp. XII,Cors. 1.10 and 5.4) has a $k$-valued point defining a split maximal torus of $H$ and therefore, due to the smoothness of $T_G$, it lifts to a $D^h$-valued point of $T_G$. Thus $G_{D^h} := \text{Spec } D^h \times_{\text{Spec } D} G$ has a maximal torus whose fiber over $\text{Spec } k$ is split and hence, as $D^h$ is henselian, it is split. Thus $G_{D^h}$ is split. This implies that there exists a $D$-subalgebra $D'$ of $\overline{K}$ which is étale and a discrete valuation ring of residue field $k$ and which is such that $G_{D'}$ is split.

Let $K' := \text{Frac}(D')$. If $T'$ is a maximal torus of $G_{D'}$ which is split, then the weights of the action of $T'_k$ on $K' \otimes_K V = K' \otimes_D M$ and of the action of $T'_k$ on $\mathfrak{m}$ are the same. As statement $\textcircled{2}$ does not hold, we deduce that the composition series of the $H = \prod_{i=1}^n G_{i,k}$-module $\mathfrak{m}$ has a simple factor $n$ which, up to isomorphism, is a tensor product $\otimes_{i=1}^n \mathfrak{n}_i$, where each $\mathfrak{n}_i$ is a simple $G_{i,k}$-module of highest weight $w_i$, and there exists $i_0 \in \{1, \ldots, n\}$ such that we can write

$$w_{i_0} = \sum_{t=1}^{r_{i_0}} c_{i_0,t} \omega_{i_0,t}$$

with all $c_{i_0,t} \in \mathbb{Z}_{\geq 0}$ but there exists $l_0 \in \{1, \ldots, r_{i_0}\}$ such that $c_{i_0,l_0} \geq p$. This implies that we can write

$$w_{i_0} = \sum_{t=0}^q p^t w_{i_0,t}$$
with \( q \in \mathbb{N} \), and \( w_{i_0,q} \neq 0 \), and

\[
w_{i_0,0}, \ldots, w_{i_0,q} \in \Omega_{i_0} := \left\{ \sum_{l=1}^{r_{i_0}} \gamma_{i_0,l} \omega_{i_0,l} \left| \gamma_{i_0,1}, \ldots, \gamma_{i_0,r_{i_0}} \in \{0, 1, \ldots, p - 1\} \right. \right\}.
\]

The key point is (see [20], Thm. 1.1; see also [1], Thm. 7.5 (i)) that we have a tensor product decomposition

\[
n_{i_0} \cong \bigotimes_{t=0}^{q} n_{i_0,t}^{(p^t)}
\]
to be viewed as an identification, where \( n_{i_0,t} \) is the simple \( G_{i_0,k} \)-module associated to the highest weight \( w_{i_0,t} \) and we view the \( G_{i_0,k}^{(p^t)} \)-module \( n_{i_0,t}^{(p^t)} \) as a \( G_{i_0,k} \)-module via the functorial Frobenius homomorphism \( G_{i_0,k} \to G_{i_0,k}^{(p^t)} \). We recall that \( G_{i_0,k}^{(p^t)} \) is the pullback of \( G_{i_0,k} \) via the morphism \( \text{Spec} \ k \to \text{Spec} \ k \) defined by the Frobenius endomorphism \( \text{Fr}_t : k \to k \) that maps \( x \) to \( x^{p^t} \) and that \( n_{i_0,t}^{(p^t)} := k \otimes_{\text{Fr}_t,k} n_{i_0,t} \). For \( t \in \{1, \ldots, q\} \), \( \text{Lie}(G_{i_0,k}) \) acts trivially on \( n_{i_0,t}^{(p^t)} \).

We consider two \( H \)-submodules \( p_0 \subset p_1 \) of \( m \) such that as \( H \)-module \( p_1/p_0 \) is (isomorphic to) such an \( H \)-module \( n \) and the \( H \)-module \( p_0 \) has the smallest length. By replacing the \( G \)-module \( M \) by the inverse image of \( p_0 \) (see Fact [2]) via the composite \( D \)-linear map

\[
\nu : M \to M/\varpi M \cong m,
\]
we can assume that \( p_0 = 0 \). Thus \( p_1 = p_1/p_0 \) is a simple \( H \)-module. Therefore we can assume that \( n = p_1 \) is a simple \( H \)-submodule of \( m \).

As \( w_{i_0,q} \neq 0 \), we have \( \dim_k(n_{i_0,q}) \geq 2 \) and thus there exists a nonzero proper \( k \)-vector subspace \( Q_{i_0,q} \) of \( n_{i_0,q}^{(p^q)} \). It is a trivial \( \text{Lie}(G_{i_0,k}) \)-module which is not a \( G_{i_0,k} \)-module. The \( k \)-vector subspace

\[
q_{i_0} := n_{i_0,0}^{(p^q)} \otimes_k n_{i_0,1}^{(p^1)} \otimes_k \cdots \otimes_k n_{i_0,q-1}^{(p^{q-1})} \otimes_k Q_{i_0,q}
\]
of \( n_{i_0} \) is a \( \text{Lie}(G_{i_0,k}) \)-module which is not a \( G_{i_0,k} \)-module. Defining

\[
q := n_1 \otimes_k \cdots \otimes_k n_{i_0-1} \otimes_k q_{i_0} \otimes_k n_{i_0+1} \otimes_k \cdots \otimes_k n_n,
\]
we get that \( \nu^{-1}(q) \in \text{Lat}_q(V) \) but (see Fact [2]) \( \nu^{-1}(q) \not\in \text{Lat}_G(V) \) as \( q \) is not an \( H \)-module.

**Case 2: \( H \) is not split.** Let \( k' \) be a finite separable field extension of \( k \) such that \( H_{k'} := \text{Spec} \ k' \times_{\text{Spec} \ k} H \) is split. We can assume that the field extension
$k \to k'$ is Galois. The Galois group $\Gamma := \text{Gal}(k'/k)$ acts naturally on the set \{1, \ldots, n\} that indexes the absolutely simple factors of $\text{Spec } k' \times_{\text{Spec } k} G' \cong G' = \prod_{i=1}^n G'_{i,k'}$.

Let $D'$ be a discrete valuation ring which is a finite flat $D$-algebra with the property that we have an identity $D'/\mathcal{O}D' = k'$. Let $K' := \text{Frac}(D')$; we have $K' = D' \otimes_D K = D'[1/\mathcal{O}]$. Let $\mathfrak{g}' := \text{Lie}(G_D') = D' \otimes_D \mathfrak{g}$.

As $\Gamma$ is canonically identified with $\text{Aut}(K'/K)$ and $\text{Aut}(D'/D)$, it acts naturally on $K' \otimes_K V$ and $D' \otimes_D M$.

Based on Case 1 applied to $G_D'$ and the $G_K'$-module $K' \otimes_K V$, we get the existence of a lattice $L' \in \text{Lat}_{\mathfrak{g}'}(K' \otimes_K V)$ which is not a $G_D'$-module. Let $L := L' \cap V$, the intersection being taken inside $K' \otimes_K V$. As $L'$ is a $\mathfrak{g}'$-module, it is also a $\mathfrak{g}$-module and we conclude that $L \in \text{Lat}_\mathfrak{g}(V_K)$. If $L$ is a $G$-module, then $D' \otimes_D L$ is a $G_D'$-module and hence we have $L' \neq D' \otimes_D L$. Thus to end the proof in this case it suffices to show that we can choose $L'$ such that we have $L' = D' \otimes_D L$.

Based on Case 1 applied over $D'$ to the $G_D'$-module $D' \otimes_D M$, we consider $H_{k'}$-submodules $p'_0 \subset p'_1 \subset (D' \otimes_D M)/(\mathcal{O}D' \otimes_D M) = k' \otimes_k m$ such that $n' := p'_1/p'_0$ is a simple $H_{k'}$-module that has the same property as $n$ of Case 1. We can assume that $p'_0$ is such that its length as an $H_{k'}$-module is the smallest. Due to this, by considering $r'_0 := \sum_{\gamma \in \Gamma} \gamma(p_0)$, which, due to Galois descent, is of the form $k' \otimes_k r_0$ with $r_0$ an $H$-submodule of $m$, and $r'_1 := p'_1 + r'_0$, we have an $H_{k'}$-isomorphism $n'_0 \simeq r'_1/r'_0$. Thus, as in Case 1, by replacing $M$ with $\nu^{-1}(r_0)$, we can assume that $p'_0 = 0$ and hence we have a simple $H_{k'}$-module $n'$ of $k' \otimes_k m$ which is the analogue of $n$.

As in Case 1 we get a tensor product decomposition $n' = \prod_{i=1}^n n'_i$, an element $i_0 \in \{1, \ldots, n\}$, and a second product decomposition $n'_{i_0} \simeq \otimes_{i=0}^n (n'_{i_0,i})^{(p')}$.

The stabilizer $\Gamma_{i_0}$ of $i_0$ in $\Gamma$ acts naturally on the set of dominant weights of $G_{i_0,k'}$ and as such let $\Gamma_{i_0}^{-}$ be the subgroup of $\Gamma_{i_0}$ that fixes $w_{i_0,q}$. Thus the $G_{i_0,k'}$-module $n'_{i_0,q}$ is defined over the subfield of $k'$ fixed by $\Gamma_{i_0}$ and as such there exists a nonzero proper $k'$-vector subspace $Q'_{i_0,q}$ of $(n'_{i_0,q})^{(p')}$ left invariant by $\Gamma_{i_0}^-$. We use it to define an element

$$M' := (1_{D'} \otimes_D \nu)^{-1}(q') \in \text{Lat}_{\mathfrak{g}'}(K' \otimes_K V)$$

which (see Fact 2) is not a $G_D'$-module. Then we can take

$$L' := \sum_{\gamma \in \Gamma} \gamma(M').$$
As $L'$ is $\Gamma$-invariant, using Galois descent we get that we have $L' = D' \otimes_D L$. As $g'$ is $\Gamma$-invariant, each $\gamma(M')$ is a $g'$-module. As $\Gamma_i$ leaves $M'$ invariant, we have $L' = \sum_{\gamma \in \Gamma/\Gamma_i} \gamma(M')$ and it follows that $L'$ is not a $G_{D'}$-module.

We conclude that Theorem 2 (b) holds.

6 Proof of Theorem 4

Due to the perfect assumptions we have a direct sum decomposition

$$\text{End}_R(\mathcal{M}) = \text{Lie}(\mathcal{H}_K) \cap \text{End}_R(\mathcal{M}) \oplus [\text{Lie}(\mathcal{H}_K) \cap \text{End}_R(\mathcal{M})]^\perp,$$

where $[\text{Lie}(\mathcal{H}_K) \cap \text{End}_R(\mathcal{M})]^\perp$ is the perpendicular of $\text{Lie}(\mathcal{H}_K) \cap \text{End}_R(\mathcal{M})$ with respect to $\mathcal{B}$. Thus the $R$-module underlying the Lie algebra $\text{Lie}(\mathcal{H}_K) \cap \mathfrak{gl}_M$ is projective. Moreover, the Killing form of $\text{Lie}(\mathcal{H}_K) \cap \mathfrak{gl}_M$ is a perfect bilinear map as it times a unit of $R$ is so. As $\text{char}(K) = 0$, we have $\text{Lie}(\mathcal{H}_K) = \text{Lie}(\mathcal{H}_{Kad})$. Based on the last two sentences, from [23], Cor. 1.3 we get that there exists a unique adjoint group scheme $\mathcal{H}_{ad}$ over Spec $R$ which extends $\mathcal{H}_{Kad}$ and whose Lie algebra is $\text{Lie}(\mathcal{H}_K) \cap \mathfrak{gl}_M$.

Let $G$ be the simply connected semisimple group scheme cover of $\mathcal{H}_{ad}$, we have $G_{ad} = \mathcal{H}_{ad}$ and our notation matches (i.e., the fiber of $G$ over Spec $K$ is the ‘initial’ $G_K$). Let $\mathcal{H}$ be the normalization of $\mathcal{H}_{ad}$ in the field of fractions of $\mathcal{H}_K$. From [23], Lem. 2.3.1 we get that $\mathcal{H}$ has a unique structure of a semisimple group scheme over Spec $R$ which extends $\mathcal{H}_K$ and the morphism $\mathcal{H} \to \mathcal{H}_{ad}$ is in fact a central isogeny. Clearly, $G$ is also the simply connected semisimple group scheme cover of $\mathcal{H}$.

From [23], Prop. 3.5 (b) we get that the homomorphisms $G \to \mathcal{H}$ and $\mathcal{H} \to \mathcal{H}_{ad}$ are étale. Thus we have identifications

$$\text{Lie}(G) = \text{Lie}(\mathcal{H}) = \text{Lie}(\mathcal{H}_{ad}) = \text{Lie}(\mathcal{H}_K) \cap \mathfrak{gl}_M$$

and the kernels $\mathcal{K} := \text{Ker}(G \to \mathcal{H})$ and $\text{Ker}(\mathcal{H} \to \mathcal{H}_{ad})$ are finite étale group schemes over Spec $R$.

As $\mathcal{M}$ is a $\text{Lie}(\mathcal{H}_K) \cap \mathfrak{gl}_M$-module and hence also a $\text{Lie}(G)$-module, from Theorem 2 (a) we get that $\mathcal{M}$ is a $G$-module. The kernel of the resulting homomorphism $G \to \text{Aut}_\mathcal{M}$ contains $\mathcal{K}_K$ and hence contains $\mathcal{K}$. This implies that $G \to \text{Aut}_\mathcal{M}$ factors through a homomorphism $\rho : \mathcal{H} \to \text{Aut}_\mathcal{M}$.

Using a limit argument as in the beginning of Section 3, to check that $\rho$ is a closed embedding we can assume that the normal domain $R$ is also noetherian.
As $\rho_K$ is a closed embedding, from [22], Thm. 1.1 (c) we get that $\rho$ is finite over the spectrum of each local ring of $R$ which is a discrete valuation ring. This implies that there exists an open subscheme $Y$ of $\text{Spec } R$ which contains all points of $\text{Spec } R$ of codimension in $\text{Spec } R$ at most 1 and such that $\rho_Y$ is finite. Based on this, as $p > 2$, from [23], Prop. 5.1 we get that $\rho$ itself is a closed embedding. This implies that the cokernel $\mathfrak{gl}_M/\text{Lie}(\mathcal{H})$ of the inclusion $\text{Lie}(\mathcal{H}) \to \mathfrak{gl}_M$ has constant rank over the points of $\text{Spec } R$, so it is a finitely generated projective module over $R$. Thus $\text{Lie}(\mathcal{H})$ is a direct summand of $\mathfrak{gl}_M$ and hence we have $\text{Lie}(\mathcal{H}) = \text{Lie}({\mathcal{H}}_K) \cap \mathfrak{gl}_M$. Therefore Theorem 4 holds. □

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