Spherical and Planar Ball Bearings —
a Study of Integrable Cases

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Abstract—We consider the nonholonomic systems of \(n\) homogeneous balls \(B_1, \ldots, B_n\) with the same radius \(r\) that are rolling without slipping about a fixed sphere \(S_0\) with center \(O\) and radius \(R\). In addition, it is assumed that a dynamically nonsymmetric sphere \(S\) with the center that coincides with the center \(O\) of the fixed sphere \(S_0\) rolls without slipping in contact with the moving balls \(B_1, \ldots, B_n\). The problem is considered in four different configurations, three of which are new. We derive the equations of motion and find an invariant measure for these systems. As the main result, for \(n = 1\) we find two cases that are integrable by quadratures according to the Euler–Jacobi theorem. The obtained integrable nonholonomic models are natural extensions of the well-known Chaplygin ball integrable problems. Further, we explicitly integrate the planar problem consisting of \(n\) homogeneous balls of the same radius, but with different masses, which roll without slipping over a fixed plane \(\Sigma_0\) with a plane \(\Sigma\) that moves without slipping over these balls.

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1. INTRODUCTION

We continue our study of the spherical and planar ball bearing problems, which we introduced in [11]. Here we focus on four different configurations of the spherical ball bearing problem. In [11] we dealt with the first configuration: \(n\) homogeneous balls \(B_1, \ldots, B_n\) with centers \(O_1, \ldots, O_n\) and the same radius \(r\) roll without slipping around a fixed sphere \(S_0\) with center \(O\) and radius \(R\). A dynamically nonsymmetric sphere \(S\) of radius \(\rho = R + 2r\) with the center that coincides with the center \(O\) of the fixed sphere \(S_0\) rolls without slipping over the moving balls \(B_1, \ldots, B_n\) (case I, Fig. 1).

As the second configuration (case II), we consider homogeneous balls of radius \(r\) within a fixed sphere \(S_0\) of radius \(R\). The balls support a moving, dynamically nonsymmetric sphere \(S\) of radius \(\rho = R - 2r\) (see Fig. 1).

Proposition 1 implies that the centers \(O_1, \ldots, O_n\) of the balls are at rest relative to each other. Thus, there are no collisions of the balls \(B_1, \ldots, B_n\). For \(n \geq 4\) there are initial positions of the balls \(B_1, \ldots, B_n\) that imply the condition that the center of the moving sphere \(S\) coincides with the center \(O\) of the fixed sphere \(S_0\). In order to include all possible initial positions for arbitrary \(n\), the condition that \(O\) coincides with the center of the sphere \(S\) is assumed to be a holonomic constraint.

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For \( n = 1 \) we introduce two additional configurations assuming that \( B_1 \) is not a homogeneous ball but a sphere (spherical shell). The first one is when the sphere \( B_1 \) is within the moving sphere \( S \) and the fixed sphere \( S_0 \) is within \( B_1 \) (case III, \( \rho = 2r - R, \rho > R \), Fig. 2). The second one is when the sphere \( B_1 \) is within the fixed sphere \( S_0 \) and the moving sphere \( S_0 \) is within \( B_1 \) (case IV, \( \rho = 2r - R, \rho < R \), Fig. 2).

In Section 2 we present the equations of motion of the spherical ball bearing systems for all four configurations. The kinetic energy and the distribution are invariant with respect to an appropriate action of the Lie group \( SO(3)^{n+1} \), and the system can be reduced to \( M = D/\text{SO}(3)^{n+1} \), where \( D \subset TQ \) is the nonholonomic distribution and \( Q = \text{SO}(3)^{n+1} \times S^n \) is the configuration space of the problem. In addition, by fixing values of first integrals, the system can be also reduced to a second reduced space \( N = \mathbb{R}^3 \times (S^2)^n \), see Theorem 1.

The system also has an invariant measure, see Theorem 2, Section 3. The proofs of Theorems 1 and 2 are similar to the proofs of the corresponding statements given for configuration I in [11]. Thus, they are omitted.

In this paper we consider the integrability of the spherical balls bearing problem in the case of \( n = 1 \). The system can be reduced to \( \tilde{N} = \mathbb{R}^3 \{ \vec{\Omega} \} \times S^2 \{ \vec{\Gamma} \} \) and takes the form (see Section 3)

\[
\dot{\vec{M}} = \tilde{\vec{M}} \times \vec{\Omega}, \quad \dot{\vec{\Gamma}} = \varepsilon \vec{\Gamma} \times \vec{\Omega},
\]

(1.1)

where \( \tilde{\vec{M}} = I\vec{\Omega} + d\vec{\Gamma} \) and

\[
I = \mathbb{I} + DE - D\vec{\Gamma} \otimes \vec{\Gamma}, \quad E = \text{diag}(1, 1, 1).
\]

Here \( \vec{\Omega} \) is the angular velocity of the sphere \( S \), \( \mathbb{I} = \text{diag}(A, B, C) \) is its inertia tensor, \( \vec{\Gamma} \) is the unit vector determining the position of the homogeneous ball \( B_1 \) and \( \varepsilon, d, D \) are parameters of the problem that are described in Sections 2 and 3.

According to Theorem 2, the flow of (1.1) in variables \( \{ \vec{\Omega}, \vec{\Gamma} \} \) preserves the measure with density \( \sqrt{\det(I)} \). Also, it always has the first integrals \( F_1 = \frac{1}{2} \langle \vec{\Omega}, \vec{\Omega} \rangle \) and \( F_2 = \langle \tilde{\vec{M}}, \tilde{\vec{M}} \rangle \) (see Proposition 2). Since \( \tilde{N} \) is five-dimensional, for the integrability, according to the Euler–Jacobi theorem, one additional first integral is needed.

As the main results of the paper, in Section 3 we prove

**Main result 1.** The spherical ball bearing problem (1.1) in configuration III, when \( 2r = 3R \), i.e., \( \varepsilon = -1 \), is integrable. The third first integral is

\[
F_3 = (B + C - A + D)M_1\Gamma_1 + (A + C - B + D)M_2\Gamma_2 + (A + B - C + D)M_3\Gamma_3.
\]

**Main result 2.** The spherical ball bearing problem (1.1) for \( B = C \) is integrable for any \( \varepsilon \) in all configurations. Along with \( F_1 \) and \( F_2 \), the system has two additional nonalgebraic first integrals.
The exact formulae for these additional nonalgebraic first integrals, indicated in the above Theorem, are given in (3.10).

For \( d = 0 \), Eqs. (1.1) coincide with the equations of motion of a Chaplygin ball with the inertia tensor \( I \) on a sphere (\( \varepsilon \neq 1 \)) and the plane (\( \varepsilon = 1 \)) with slightly different definitions of parameters \( \varepsilon \) and \( D \). Thus, the above results can be seen as natural extensions of well-known integrable Chaplygin ball problems.

In [11] we also considered the associated planar problem. This is case I in the notation of the present paper, when the radius of the fixed sphere \( S_0 \) tends to infinity. We found an invariant measure and proved the integrability by means of the Euler–Jacobi theorem [11]. Here, in Section 4, we perform an explicit integration of the reduced problem.

2. ROLLING OF A DYNAMICALLY NONSYMMETRIC SPHERE OVER \( n \) MOVING HOMOGENEOUS BALLS AND A FIXED SPHERE

2.1. Kinematics

Let \( O\bar{e}_1, \bar{e}_2, \bar{e}_3, O\bar{e}_1', \bar{e}_2', \bar{e}_3' \) be positively oriented reference frames rigidly attached to the spheres \( S_0, S \), and the balls \( B_i, i = 1, \ldots, n \), respectively. By \( \mathbf{g}, \mathbf{g}_i \in SO(3) \) we denote the matrices that map the moving frames \( O\bar{e}_1, \bar{e}_2, \bar{e}_3 \) and \( O\bar{e}_1', \bar{e}_2', \bar{e}_3' \) to the fixed frame \( O\bar{e}_1^0, \bar{e}_2^0, \bar{e}_3^0 \).

Using the standard isomorphism between the Lie algebras \((so(3), [\cdot, \cdot])\) and \((\mathbb{R}^3, \times)\) given by

\[
a_{ij} = -\varepsilon_{ijk} \omega_k, \quad i,j,k = 1,2,3, \tag{2.1}
\]

the skew-symmetric matrices \( \omega = \mathbf{g}^{-1} \mathbf{g}_i \), \( \omega_i = \mathbf{g}_i^{-1} \) correspond to the angular velocities \( \vec{\omega}, \vec{\omega}_i \) of the sphere \( S \) and the \( i \)th ball \( B_i \) in the fixed reference frame \( O\bar{e}_1^0, \bar{e}_2^0, \bar{e}_3^0 \) attached to the sphere \( S_0 \).

The matrices \( \Omega = \mathbf{g}^{-1} \bar{\mathbf{g}} = \mathbf{g}^{-1} \omega \mathbf{g}_i, W_i = \mathbf{g}_i^{-1} \omega_i \mathbf{g}_i \) correspond to the angular velocities \( \vec{\Omega}, \vec{W}_i \) of \( S \) and \( B_i \) in the frames \( O\bar{e}_1, \bar{e}_2, \bar{e}_3 \) and \( O\bar{e}_1', \bar{e}_2', \bar{e}_3' \) attached to the sphere \( S \) and the balls \( B_i \), respectively. We have \( \vec{\omega} = \mathbf{g} \vec{\Omega}, \vec{\omega}_i = \mathbf{g}_i \vec{W}_i \).

Then the configuration space of the problem is

\[
Q = SO(3)^{n+1} \times (S^2)^n \{ \mathbf{g}, \mathbf{g}_1, \ldots, \mathbf{g}_n, \vec{\gamma}_1, \ldots, \vec{\gamma}_n \},
\]

where \( \vec{\gamma}_i \) is the unit vector

\[
\vec{\gamma}_i = \frac{\overrightarrow{O\bar{O}_i}}{|O\bar{O}_i|}.
\]

determining the position of the center of the \( i \)th ball \( B_i, i = 1, \ldots, n \). In cases I and II, the velocity of the center of the \( i \)th ball is \( \vec{v}_{\bar{O}_i} = (R \pm r) \vec{\gamma}_i \), while for cases III and IV (\( n = 1 \)) we
have $\vec{v}_{O_i} = \pm(r - R)\vec{\gamma}_i$. It follows from Proposition 1 that, if the initial conditions are chosen such that the distances between $O_i$ and $O_j$ are all greater than $2r$, $1 \leq i < j \leq n$, then the balls will not have collisions along the course of motion. This is the reason why we do not assume additional one-side constraints

$$|\vec{\gamma}_i - \vec{\gamma}_j| \geq \frac{2r}{R \pm r}, \quad 1 \leq i < j \leq n \quad \text{(cases I and II, n} \geq 2).$$

Let $A_1, \ldots, A_n$ and $B_1, B_2, \ldots, B_n$ be the contact points of the balls $B_1, \ldots, B_n$ with the spheres $S_0$ and $S$, respectively. The condition that the balls $B_1, \ldots, B_n$ and the sphere $S$ roll without slipping leads to the nonholonomic constraints:

$$\vec{v}_{O_i} + \vec{\omega}_i \times \overrightarrow{O_iA_i} = 0, \quad \vec{v}_{O_i} + \vec{\omega}_i \times \overrightarrow{O_iB_i} = \vec{\omega} \times \overrightarrow{OB_i}, \quad i = 1, \ldots, n,$$

that is,

$$\vec{v}_{O_i} = \pm r\vec{\omega}_i \times \vec{\gamma}_i, \quad \vec{v}_{O_i} = (R \pm 2r)\vec{\omega} \times \vec{\gamma}_i \pm r\vec{\gamma}_i \times \vec{\omega}_i \quad \text{(cases I and II)} \quad (2.2)$$

and

$$\vec{v}_{O_i} = \pm r\vec{\omega}_i \times \vec{\gamma}_i, \quad \vec{v}_{O_i} = \pm (2r - R)\vec{\omega} \times \vec{\gamma}_i \pm r\vec{\gamma}_i \times \vec{\omega}_i \quad \text{(cases III and IV)}. \quad (2.3)$$

The dimension of the configuration space $Q$ is $5n + 3$. There are $4n$ independent constraints in (2.2), defining a nonintegrable distribution $\mathcal{D} \subset TQ$ of rank $n + 3$. The phase space of the system, $\mathcal{D}$ considered as a submanifold of $TQ$, has the dimension $6n + 6$.

Note that there are two nonholonomic systems which are close to the spherical ball bearings. One is the so-called spherical support system introduced by Fedorov in [12]. It describes the rolling without slipping of a dynamically nonsymmetric sphere $S$ over $n$ homogeneous balls $B_1, \ldots, B_n$ of possibly different radii, but with fixed centers. The second one is the rolling of a homogeneous ball $B$ over a dynamically asymmetric sphere $S$, introduced by Borisov, Kilin, and Mamaev in [5].

### 2.2. Symmetries

Let $\mathbb{I}$ be the inertia operator of the sphere $S$. We choose the moving frame $O\vec{e}_1, \vec{e}_2, \vec{e}_3$, such that $O\vec{e}_1, O\vec{e}_2, O\vec{e}_3$ are the principal axes of inertia: $\mathbb{I} = \text{diag}(A, B, C)$. Let $\mathbb{I}_i = \text{diag}(I_i, I_i, I_i)$ and $m_i$ be the inertia operator and the mass of the $i$th ball $B_i$. Then $\langle \mathbb{I}_i \vec{W}_i, \vec{W}_i \rangle = I_i \langle \vec{\omega}_i, \vec{\omega}_i \rangle$ and the kinetic energy, which plays the role of the Lagrangian, is given by

$$T = \frac{1}{2}(\mathbb{I} \vec{\Omega}, \vec{\Omega}) + \frac{1}{2} \sum_{i=1}^{n} I_i \langle \vec{\omega}_i, \vec{\omega}_i \rangle + \frac{1}{2} \sum_{i=1}^{n} m_i \langle \vec{v}_{O_i}, \vec{v}_{O_i} \rangle.$$

The equations of motion of the problem are given by the Lagrange–d’Alembert equations [1, 2, 7]

$$\delta T = \left(\frac{\partial T}{\partial \dot{q}} - \frac{d}{dt} \frac{\partial T}{\partial q}, \delta q \right) = 0, \quad \text{for all virtual displacements } \delta q \in \mathcal{D}_q. \quad (2.4)$$

The kinetic energy and the constraints are invariant with respect to the $SO(3)^{n+1}$–action defined by

$$(\mathbf{g}, \mathbf{g}_1, \ldots, \mathbf{g}_n, \vec{\gamma}_1, \ldots, \vec{\gamma}_n) \mapsto (ag, ag_1a_1^{-1}, \ldots, ag_na_n^{-1}, a\vec{\gamma}_1, \ldots, a\vec{\gamma}_n), \quad (2.5)$$

$a, a_1, \ldots, a_n \in SO(3)$, representing a freedom in the choice of the reference frames $O\vec{e}_1, \vec{e}_2, \vec{e}_3$, $O_i\vec{e}_1, \vec{e}_2, \vec{e}_3$, $i = 1, \ldots, n$. Also, note that (2.5) does not change the vectors

$$\tilde{O}_i = \mathbf{g}^{-1}\vec{\omega}_i, \quad \tilde{\Gamma}_i = \mathbf{g}^{-1}\vec{\gamma}_i, \quad \tilde{V}_{O_i} = \mathbf{g}^{-1}\vec{v}_{O_i}$$

of the moving frame $O\vec{e}_1, \vec{e}_2, \vec{e}_3$.

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1) Throughout the paper, the sign $\pm$ denotes $+$ for cases I and III, and $-$ for cases II and IV.
Thus, for the coordinates in the space \((TQ)/SO(3)^{n+1}\) we can take the angular velocities and the unit position vectors in the reference frame attached to the sphere \(S\):

\[
(TQ)/SO(3)^{n+1} \cong \mathbb{R}^{3(n+1)} \times (TS^2)\mathbb{P}\{\vec{\Omega}, \vec{\Omega}_1, \ldots, \vec{\Omega}_n, \vec{\Gamma}_1, \ldots, \vec{\Gamma}_n, \vec{\Gamma}_{n+1}, \ldots, \vec{\Gamma}_n\}.
\]

In the moving reference frame \(O\vec{e}_1, \vec{e}_2, \vec{e}_3\), the constraints become

\[
\vec{V}_{O_i} = (R \pm 2r)\vec{\Omega} \times \vec{\Gamma}_i \pm r\vec{\Gamma}_i \times \vec{\Omega}, \quad \text{(cases I and II)} \tag{2.6}
\]

\[
\vec{V}_{O_i} = \pm(2r - R)\vec{\Omega} \times \vec{\Gamma}_i \pm r\vec{\Gamma}_i \times \vec{\Omega}, \quad \text{(cases III and IV)} \tag{2.7}
\]

\[
\vec{V}_{O_i} = \pm r\vec{\Omega}_i \times \vec{\Gamma}_i, \quad i = 1, \ldots, n, \tag{2.8}
\]

defining the reduced phase space \(\mathcal{M} = \mathcal{D}/SO(3)^{n+1} \subset (TQ)/SO(3)^{n+1}\) of dimension \(3n + 3\).

Since both the kinetic energy and the constraints are invariant with respect to the \(SO(3)^{n+1}\)-action (2.5), the equations of motion (2.4) are also \(SO(3)^{n+1}\)-invariant. Thus, they induce a well-defined system on the reduced phase space \(\mathcal{M}\).

To simplify the constraints and the equations below, we introduce parameters

\[
\varepsilon = \frac{R}{2R \pm 2r} \quad \text{and} \quad \delta = \pm \frac{R \pm 2r}{2r} \quad \text{(cases I and II)}, \tag{2.9}
\]

\[
\varepsilon = \frac{R}{2R - 2r} \quad \text{and} \quad \delta = \frac{2r - R}{2r} \quad \text{(cases III and IV)}. \tag{2.10}
\]

In particular, the constraints (2.6), (2.7), (2.8) are equivalent to

\[
\vec{V}_{O_i} = \pm r\vec{\Omega}_i \times \vec{\Gamma}_i, \quad \vec{\Omega}_i \times \vec{\Gamma}_i = \delta\vec{\Omega} \times \vec{\Gamma}_i, \quad i = 1, \ldots, n. \tag{2.11}
\]

### 2.3. Equations of Motion

The statements below are derived in [11] for configuration I. The inclusion of configurations II, III, and IV can be obtained similarly and we will omit the proofs.

Let \(\vec{F}_{B_i}\) and \(\vec{F}_{A_i}\) be the reaction forces that act on the ball \(B_i\) at the points \(B_i\) and \(A_i\), respectively. The reaction force at the point \(B_i\) on the sphere \(S\) is then \(-\vec{F}_{B_i}\). By using the laws of change of angular momentum and momentum of a rigid body in the moving reference frame for the balls \(B_i\) and the sphere \(S\), we get

\[
I_i \dot{\vec{\Omega}}_i = I_i \vec{\Omega}_i \times \vec{\Omega} \pm r\vec{\Gamma}_i \times (\vec{F}_{B_i} - \vec{F}_{A_i}), \quad \text{(2.12)}
\]

\[
m_i \dot{\vec{V}}_{O_i} = m_i \vec{V}_{O_i} \times \vec{\Omega} + \vec{\Omega}_i \times \vec{\Gamma}_i + \vec{F}_{B_i} + \vec{F}_{A_i}, \quad i = 1, \ldots, n \tag{2.13}
\]

\[
\vec{\Omega} = \vec{\Omega} \times \vec{\Omega} = 2r \sum_{i=1}^{n} \delta\vec{\Gamma}_i \times \vec{F}_{B_i}. \tag{2.14}
\]

On the other hand, from the constraint, we obtain the following kinematic equations for the unit position vectors \(\vec{\Gamma}_i\):

\[
\dot{\vec{\Gamma}}_i = \varepsilon\vec{\Gamma}_i \times \vec{\Omega}, \quad i = 1, \ldots, n. \tag{2.15}
\]

As a direct consequence of Eqs. (2.12) and (2.15), we get

**Proposition 1.** The following functions are the first integrals of the equations of motion (2.12), (2.14), and (2.15):

\[
\langle \vec{\Gamma}_i, \vec{\Gamma}_j \rangle = \gamma_{ij} = \text{const}, \quad 1 \leq i < j \leq n, \tag{2.16}
\]

\[
\langle \vec{\Omega}_i, \vec{\Gamma}_i \rangle = c_i = \text{const}, \quad i = 1, \ldots, n. \tag{2.17}
\]
Equations (2.16) are a consequence of the kinematic equations (2.15) only and they imply that the centers \( O_i \) of the homogeneous balls \( B_i \) are at rest with respect to each other. After fixing the values of \( \gamma_{ij} \), we can consider the relations \( (\vec{\gamma}_i, \vec{\gamma}_j) = (\vec{\Gamma}_i, \vec{\Gamma}_j) = \gamma_{ij} \) as holonomic constraints that have no influence on the motion of the system.

From (2.17), we also find that the reduced phase space \( \mathcal{M} = \mathcal{D}/SO(3)^{n+1} \) is foliated by \( 2n + 3 \)-dimensional invariant manifolds

\[
\mathcal{M}_c : \quad (\vec{\Omega}_i, \vec{\Gamma}_i) = c_i = \text{const}, \quad i = 1, \ldots, n.
\]

By using the constraints, the vector functions \( \vec{\Omega}_i \) can be uniquely expressed as functions of \( \vec{\Omega}, \vec{\Gamma}_i \) on the invariant manifold \( \mathcal{M}_c \):

\[
\vec{\Omega}_i = c_i \vec{\Gamma}_i + \delta \vec{\Omega} - \delta (\vec{\Gamma}_i, \vec{\Omega}) \vec{\Gamma}_i.
\]

Hence, \( \vec{\Omega} \) determines all velocities of the system on \( \mathcal{M}_c \) and \( \mathcal{M}_c \) is diffeomorphic to the second reduced phase space

\[
\mathcal{N} = \mathbb{R}^3 \times (S^2)^n \{ \Omega, \vec{\Gamma}_1, \ldots, \vec{\Gamma}_n \}.
\]

This can be seen as follows. Consider the natural projection

\[
\pi : (TQ)/SO(3)^{n+1} \cong \mathbb{R}^{3(n+1)} \times (TS^2)^n \to \mathcal{N},
\]

\[
\pi (\vec{\Omega}, \vec{\Gamma}_1, \ldots, \vec{\Omega}_n, \vec{\Gamma}_1, \ldots, \vec{\Gamma}_n) = (\vec{\Omega}, \vec{\Gamma}_1, \ldots, \vec{\Gamma}_n),
\]

and let \( \pi_c \) be the restriction to \( \mathcal{M}_c \subset \mathcal{M} \subset (TQ)/SO(3)^{n+1} \) of \( \pi \). Then the projection

\[
\pi_c : \mathcal{M}_c \to \mathcal{N}
\]

is a bijection. Further, since \( (\vec{\omega}_i, \vec{\gamma}_i) = (\vec{\Omega}_i, \vec{\Gamma}_i) \), we have that \( \mathcal{D} \) is foliated by invariant manifolds

\[
\mathcal{D}_c : \quad (\vec{\omega}_i, \vec{\gamma}_i) = c_i, \quad i = 1, \ldots, n, \quad \dim \mathcal{D}_c = 5n + 6
\]

and \( \mathcal{M}_c = \mathcal{D}_c/\text{SO}(3)^{n+1} \). As a result, we obtain the following diagram:

\[
\begin{array}{ccc}
\mathcal{D}_c & \longrightarrow & \mathcal{D} \\
/\text{SO}(3)^{n+1} & /\text{SO}(3)^{n+1} & /\text{SO}(3)^{n+1} \\
\mathcal{M}_c & \longrightarrow & \mathcal{M} \\
\pi_c & \cong & \pi \\
\mathcal{N} = \mathbb{R}^3 \times (S^2)^n, & & \\
\end{array}
\]

which implies that \( \mathcal{D}_c \) and \( \mathbb{R}^3 \times \text{SO}(3)^{n+1} \times (S^2)^n \{ \Omega, g, g_1, \ldots, g_n, \vec{\Gamma}_1, \ldots, \vec{\Gamma}_n \} \) are diffeomorphic:

\[
\mathcal{D}_c \cong \mathbb{R}^3 \times \text{SO}(3)^{n+1} \times (S^2)^n \{ \Omega, g, g_1, \ldots, g_n, \vec{\Gamma}_1, \ldots, \vec{\Gamma}_n \}.
\]

We define the modified operator of inertia \( \mathbf{I} \) as

\[
\mathbf{I} = I + \delta^2 \sum_{i=1}^{n} (I_i + m_i r_i^2) \mathbf{pr}_i,
\]

where \( \mathbf{pr}_i : \mathbb{R}^3 \to \vec{\Gamma}_i^\perp \) is the orthogonal projection to the plane orthogonal to \( \vec{\Gamma}_i \), and set

\[
\mathbf{M} = \mathbf{I} \vec{\Omega} = \mathbf{I} \vec{\Omega} + \delta^2 \sum_{i=1}^{n} (I_i + m_i r_i^2) \vec{\Omega} - \delta^2 \sum_{i=1}^{n} (I_i + m_i r_i^2) (\vec{\Gamma}_i, \vec{\Omega}) \vec{\Gamma}_i,
\]

\[
\mathbf{N} = \delta \sum_{i=1}^{n} I_i c_i \vec{\Gamma}_i.
\]
Theorem 1. (i) The complete equations of motion of the sphere $S$ and the balls $B_1, \ldots, B_n$ of the spherical ball bearing problem on the invariant manifold $D_c$ are given by

$$
\dot{\vec{M}} = \vec{M} \times \vec{\Omega} + (1 - \varepsilon)\vec{N} \times \vec{\Omega},
$$

$$
\dot{\vec{\Gamma}_i} = \varepsilon \vec{\Gamma}_i \times \vec{\Omega}, \quad i = 1, \ldots, n,
$$

$$
\dot{g} = g\vec{\Omega},
$$

$$
\dot{g}_i = g\vec{\Gamma}_i (\vec{\Omega}, \vec{\Gamma}_i, c_i)g_i,
$$

where $\vec{M}, \vec{N}$, are given by (2.20) and (2.21). Here $\Omega$ and $\Omega_i(\vec{\Omega}, \vec{\Gamma}_i, c_i)$ are skew-symmetric matrices related to $\vec{\Omega}$ and $\vec{\Omega}_i$ after the identification (2.1); $\vec{\Omega}_i = \vec{\Omega}_i(\vec{\Omega}, \vec{\Gamma}_i, c_i)$ as in Eq. (2.18).

(ii) The reduced system on $M_c \cong N$ is described by the closed system (2.22)–(2.23).

Remark 1. If we formally set $\varepsilon = 1$ in the system (2.22)–(2.23), we obtain the equation of the spherical support system introduced by Fedorov in [12]. The system describes the rolling without slipping of a dynamically nonsymmetric sphere $S$ over $n$ homogeneous balls $B_1, \ldots, B_n$ of possibly different radii, but with fixed centers. It is an example of a class of non-Hamiltonian L+R systems on Lie groups with an invariant measure (see [14, 15, 19]). On the other hand, if we set $\vec{N} = 0$, we obtain an example of the $\varepsilon$-modified L+R system studied in [20].

3. INVARIANT MEASURE AND INTEGRABLE CASES

3.1. Invariant Measure

The modified inertia operator $I$ (2.19) can be rewritten as

$$
I = \mathbb{I} - \Pi, \quad \Pi = \delta^2 \sum_{i=1}^{n} (I_i + m_ir^2)(\vec{\Gamma}_i \otimes \vec{\Gamma}_i - \mathbb{E}).
$$

Along the flow of the system, $\Pi$ satisfies the matrix equation

$$
\frac{d}{dt} \Pi = \varepsilon [\Pi, \Omega],
$$

where $\Omega$ is the skew-symmetric matrix that corresponds to the angular velocity $\vec{\Omega}$ via isomorphism (2.1).

Theorem 2. For arbitrary values of parameters $c_i$, the reduced system (2.22)–(2.23) has the invariant measure

$$
\mu(\vec{\Gamma}_1, \ldots, \vec{\Gamma}_n) d\Omega \wedge \sigma_1 \wedge \cdots \wedge \sigma_n, \quad \mu = \sqrt{\det(I)} = \sqrt{\det(\mathbb{I} - \Pi)},
$$

where $d\Omega$ and $\sigma_i$ are the standard measures on $\mathbb{R}^3\{\vec{\Omega}\}$ and $S^2\{\vec{\Gamma}_i\}, i = 1, \ldots, n$.

Note that the existence of an invariant measure for nonholonomic problems is well studied in many classical problems [4, 6]. After Kozlov’s theorem on obstruction to the existence of an invariant measure for the variant of the classical Suslov problem (see, e.g., [8, 15]) on Lie algebras [22], general existence statements for nonholonomic systems with symmetries are obtained in [25] and [16].

A closely related problem is the integrability of nonholonomic systems [1].

Note that the kinetic energy of the system takes the form

$$
T = \frac{1}{2} \langle \vec{M}, \vec{\Omega} \rangle + \frac{1}{2} \sum_{i=1}^{n} I_i c_i^2.
$$

Also, since

$$
\frac{d}{dt} \vec{N} = \varepsilon \vec{N} \times \vec{\Omega},
$$
Eq. (2.22) is equivalent to
\[ \frac{d}{dt}(\dot{M} + \dot{N}) = (\dot{M} + \dot{N}) \times \ddot{\Omega}. \]  

(3.3)

From the above considerations we get:

**Proposition 2.** The system (2.22)–(2.23) always has the following first integrals:

\[ F_1 = \frac{1}{2} \langle \dot{M}, \ddot{\Omega} \rangle, \quad F_2 = \langle \dot{M} + \dot{N}, \dot{M} + \dot{N} \rangle, \quad F_{ij} = \langle \Gamma_i, \Gamma_j \rangle, \quad 1 \leq i < j \leq n. \]

Thus, in the special case \( n = 1 \), we have the 5-dimensional phase space \( N = \mathbb{R}^3 \times S^2(\Omega, \Gamma_1) \), and the system has two first integrals \( F_1, F_2 \) and an invariant measure. For the integrability, one needs to find a third independent first integral.

### 3.2. System with One Homogeneous Ball

We proceed with the case \( n = 1 \). To simplify notation, we denote \( \Gamma_1 \) by \( \Gamma \) and set

\[ D = \delta^2(I_1 + m_1 r^2), \quad d = \delta I_1 c_1, \quad L = \langle \ddot{\Omega}, \Gamma \rangle, \]
\[ \ddot{M} = \dot{M} + \dot{N} = \ddot{M} + d \Gamma = I \ddot{\Omega} + D \ddot{\Omega} + (d - DL) \Gamma. \]

The system (3.3), (2.23), the operator \( \mathbf{I} \), and its determinant now read

\[ \dot{\mathbf{M}} = \mathbf{M} \times \ddot{\Omega}, \quad \dot{\Gamma} = \varepsilon \Gamma \times \ddot{\Omega}, \]

\[ \mathbf{I} = \mathbf{I} + \mathbf{DE} - D \Gamma \otimes \Gamma = \begin{pmatrix} A + D - D \Gamma_1^2 & -D \Gamma_1 \Gamma_2 & -D \Gamma_1 \Gamma_3 \\ -D \Gamma_1 \Gamma_2 & B + D - D \Gamma_2^2 & -D \Gamma_2 \Gamma_3 \\ -D \Gamma_1 \Gamma_3 & -D \Gamma_2 \Gamma_3 & C + D - D \Gamma_3^2 \end{pmatrix}, \]

\[ \det(\mathbf{I}) = (A + D)(B + D)(C + D) \left( 1 - D \left( \frac{\Gamma_1^2}{A + D} + \frac{\Gamma_2^2}{B + D} + \frac{\Gamma_3^2}{C + D} \right) \right). \]

From (2.22) we get

\[ \ddot{\mathbf{\Omega}} = (\mathbf{I} \ddot{\mathbf{\Omega}} + D \ddot{\mathbf{\Gamma}} - D L \ddot{\mathbf{\Gamma}}) \times \ddot{\mathbf{\Omega}} + (1 - \varepsilon) d \Gamma \times \ddot{\Omega} + D \frac{d}{dt} \left( \Gamma \otimes \ddot{\mathbf{\Gamma}} \right) \ddot{\mathbf{\Omega}} \]
\[ = (\mathbf{I} \ddot{\mathbf{\Omega}} - D L \ddot{\mathbf{\Gamma}}) \times \ddot{\mathbf{\Omega}} + (1 - \varepsilon) d \Gamma \times \ddot{\Omega} + \varepsilon D L \ddot{\mathbf{\Gamma}} \times \ddot{\mathbf{\Omega}} \]
\[ = \mathbf{I} \ddot{\mathbf{\Omega}} + \ddot{\mathbf{\Omega}} + (\varepsilon - 1)(D L - d) \Gamma \times \ddot{\mathbf{\Omega}}, \]

implying the explicit from of Eqs. (3.4)

\[ \ddot{\mathbf{\Omega}} = \mathbf{I}^{-1} (\mathbf{I} \ddot{\mathbf{\Omega}} + (\varepsilon - 1)(D L - d) \Gamma \times \ddot{\mathbf{\Omega}}), \quad \dot{\Gamma} = \varepsilon \Gamma \times \ddot{\mathbf{\Omega}}. \]

(3.5)

Thus, according to Theorem 2, the flow of (3.5) preserves the measure \( \sqrt{\det(\mathbf{I})} d \Omega \wedge d \sigma \) on \( \mathbb{R}^3 \times S^2(\Omega, \Gamma) \).

For \( d = 0 \), the equations coincide with the equations of a Chaplygin ball with inertia tensor \( \mathbf{I} \) on a sphere (\( \varepsilon \neq 1 \)) and the plane (\( \varepsilon = 1 \)) with slightly different definitions of parameters \( \varepsilon \) and \( D \). The equations have a similar structure as the Euler–Poisson equations of the Euler case of rigid body motion about a fixed point.
3.3. The First Integrable Case (Generic $\mathbb{I}$, $\varepsilon = -1$)

It is well known that the rolling of a Chaplygin ball over a plane ($\varepsilon = 1$), for an arbitrary inertia operator $\mathbb{I}$, has the third integral $\langle \mathbf{\hat{M}}, \mathbf{\hat{F}} \rangle$. That is why for $\varepsilon \neq 1$ we are looking for an integral of the form

$$F_3 = x_1 M_1 \Gamma_1 + x_2 M_2 \Gamma_2 + x_3 M_3 \Gamma_3.$$

Along the flow of the system (3.4) we have

$$\dot{F}_3 = \Gamma_1 \Omega_2 \Omega_3 ((B - C)x_1 - \varepsilon (B + D)x_2 + \varepsilon (C + D)x_3)$$
$$+ \Omega_1 \Gamma_2 \Omega_3 (\varepsilon (A + D)x_1 + (C - A)x_2 - \varepsilon (C + D)x_3)$$
$$+ \Omega_1 \Gamma_2 \Omega_3 (\varepsilon (A + D)x_1 + \varepsilon (B + D)x_2 + (A - B)x_3)$$
$$+ \Gamma_1 \Omega_2 \Omega_3 ((d + \varepsilon d)x_1 + (-d - \varepsilon d)x_2)$$
$$+ \Gamma_1 \Omega_2 \Omega_3 ((-d - \varepsilon d)x_1 + (d + \varepsilon d)x_3)$$
$$+ \Omega_1 \Gamma_2 \Gamma_3 ((d + \varepsilon d)x_2 + (-d - \varepsilon d)x_3)$$
$$+ L \Gamma_1 \Omega_2 \Omega_3 ((-D - \varepsilon D)x_1 + (D + \varepsilon D)x_2)$$
$$+ L \Gamma_1 \Omega_2 \Gamma_3 ((D + \varepsilon D)x_1 + (-D - \varepsilon D)x_3)$$
$$+ L \Omega_1 \Gamma_2 \Gamma_3 ((-D - \varepsilon D)x_2 + (D + \varepsilon D)x_3).$$

Therefore, $\dot{F}_3 = 0$ if and only if the parameters $x_1, x_2, x_3$ satisfy the system of 9 homogeneous linear equations corresponding to the 9 terms given above. Since $D \neq 0$, if $\varepsilon \neq -1$, from the last three equations we get $x_1 = x_2 = x_3 = x$. We can take $x = 1$. The 4th, 5th and the 6th equations are then also satisfied, while from the first 3 equations we obtain the conditions on the parameters $A, B, C$:

$$(1 - \varepsilon)B + (\varepsilon - 1)C = 0$$
$$(\varepsilon - 1)A + (1 - \varepsilon)C = 0$$
$$(1 - \varepsilon)A + (\varepsilon - 1)B = 0.$$ 

Thus, if $\varepsilon = 1$, the function $\langle \mathbf{\hat{M}}, \mathbf{\hat{F}} \rangle$ is the integral of equation (3.4) (for any $d \in \mathbb{R}$), while for $\varepsilon \neq 1$, we find that $F_3 = \langle \mathbf{\hat{M}}, \mathbf{\hat{F}} \rangle$ is the integral in the totally symmetric case $A = B = C$. However, then $F_1, F_2, F_3$ are functionally dependent.

On the other hand, for $\varepsilon = -1$, the last 6 equations become trivial, while the first 3 have a nontrivial solution

$$x_1 = B + C - A + D, \quad x_2 = A + C - B + D, \quad x_3 = A + B - C + D.$$ 

As a result, since $\varepsilon = -1$ in configuration III for $2r = 3R$, we get the following statement.

**Theorem 3.** The spherical ball bearing problem (3.4) in configuration III, when $2r = 3R$, i.e., the radius of the moving sphere $S$ is twice the radius of the fixed sphere $S_0$, is integrable. The third integral is

$$F_3 = (B + C - A + D)M_1 \Gamma_1 + (A + C - B + D)M_2 \Gamma_2 + (A + B - C + D)M_3 \Gamma_3.$$ 

Thus, for $d = 0$, the integral $F_3$ reduces to the one found by Borisov and Fedorov for the rolling of a Chaplygin ball over a sphere [3].
3.4. The Second Integrable Case \((B = C, \text{Generic } \varepsilon)\)

Further, note that, for \(B = C\), the density of the measure becomes the function of \(\Gamma_1\) only:

\[
\rho = \rho(\Gamma_1) = \frac{\sqrt{\det(I)}}{\sqrt{C + D}} \sqrt{(A + D)(C + D)(1 - \frac{D\Gamma_1^2}{A + D} - \frac{D - D\Gamma_1^2}{C + D})}
\]

\[
= \sqrt{C(A + D) + D(A - C)\Gamma_1^2}.
\]

Also, for the motion of a symmetric Chaplygin ball \((B = C)\) over a plane, we have an integral of the form (up to multiplication by a constant, see [4, 9, 23])

\[
f = \rho^2(\Gamma_1)\Omega_1^2 = C(A + D)\Omega_1^2 + D(A - C)\Omega_1^2\Gamma_1^2.
\]

It appears that in the study of the rolling of the symmetric Chaplygin ball over a sphere, it is convenient to use variables \(F\) and \(G\) defined by (see [4])

\[
F = \rho(\Gamma_1)\Omega_1, \quad (f = F^2)
\]

\[
G = A\Omega_1\Gamma_1 + C(\Omega_2\Gamma_2 + \Omega_3\Gamma_3) = (A - C)\Omega_1\Gamma_1 + CL \quad \left( L = \frac{G + (C - A)\Omega_1\Gamma_1}{C} \right).
\]

We are going to determine the time derivatives of \(F\) and \(G\) along the flow (3.4), i.e., (3.5).

Note that \(G = \langle \tilde{M}, \tilde{\Gamma} \rangle - d\). Therefore, from Eq. (3.6), where we set \(x_1 = x_2 = x_3 = 1, B = C\), we get

\[
\dot{G} = (\varepsilon - 1)(A - C)\Omega_1\Gamma_2\Omega_3 - (\varepsilon - 1)(A - C)\Omega_1\Omega_2\Gamma_3
\]

\[
= (\varepsilon - 1)(A - C)\Omega_1(\Gamma_2\Omega_3 - \Omega_2\Gamma_3)
\]

\[
= \frac{\varepsilon - 1}{\varepsilon}(A - C)\Omega_1\dot{\Gamma}_1 = (\varepsilon - 1)(A - C)F\frac{\dot{\Gamma}_1}{\varepsilon\rho}.
\]

In order to find \(\dot{F}\), we need to use Eqs. (3.5). We have

\[
\begin{pmatrix}
\dot{\Omega}_1 \\
\dot{\Omega}_2 \\
\dot{\Omega}_3
\end{pmatrix} = \frac{1}{\det(I)} \begin{pmatrix}
\Delta_1 & (C + D)D\Gamma_1\Gamma_2 & (C + D)D\Gamma_1\Gamma_3 \\
(C + D)D\Gamma_1\Gamma_2 & \Delta_2 & (A + D)D\Gamma_2\Gamma_3 \\
(C + D)D\Gamma_1\Gamma_3 & (A + D)D\Gamma_2\Gamma_3 & \Delta_3
\end{pmatrix} \cdot
\begin{pmatrix}
(\varepsilon - 1)(DL - d)(\Gamma_2\Omega_3 - \Gamma_3\Omega_2) \\
(\varepsilon - 1)(DL - d)(\Gamma_3\Omega_1 - \Gamma_1\Omega_3) + (C - A)\Omega_1\Omega_3 \\
(\varepsilon - 1)(DL - d)(\Gamma_1\Omega_2 - \Gamma_2\Omega_1) + (A - C)\Omega_1\Omega_2
\end{pmatrix},
\]

where

\[
\Delta_1 = (C + D)^2 - (C + D)D(\Gamma_2^2 + \Gamma_3^2),
\]

\[
\Delta_2 = (A + D)(C + D) - (C + D)D\Gamma_1^2 - (A + D)D\Gamma_3^2,
\]

\[
\Delta_3 = (A + D)(C + D) - (C + D)D\Gamma_1^2 - (A + D)D\Gamma_2^2.
\]

Therefore,

\[
\dot{\Omega}_1 = \frac{1}{\rho^2}((C + D) - D(1 - \Gamma_1^2))(\varepsilon - 1)(DL - d)(\Gamma_2\Omega_3 - \Gamma_3\Omega_2)
\]

\[
+ \frac{1}{\rho^2}D\Gamma_1\Gamma_2((\varepsilon - 1)(DL - d)(\Gamma_3\Omega_1 - \Gamma_1\Omega_3) + (C - A)\Omega_1\Omega_3)
\]
\[ + \frac{1}{\rho^2} D \Gamma_1 \Gamma_3 ((\varepsilon - 1)(DL - d)(\Gamma_1 \Omega_2 - \Gamma_2 \Omega_1) + (A - C) \Omega_1 \Omega_2) \]

\[ = \frac{1}{\rho^2} (C(\varepsilon - 1)(DL - d)(\Gamma_2 \Omega_3 - \Gamma_3 \Omega_2) + D \Gamma_1 \Omega_1 (C - A)(\Gamma_2 \Omega_3 - \Gamma_3 \Omega_2)) \]

\[ = \frac{1}{\varepsilon \rho^2} (C(\varepsilon - 1)(DL - d) + D \Gamma_1 \Omega_1 (C - A)) \frac{\dot{\Gamma}_1}{\dot{\Gamma}_1} \]

\[ = \frac{1}{\varepsilon \rho^2} (D(\varepsilon - 1)G + (C - A)\Omega_1 \Gamma_1) + D \Gamma_1 \Omega_1 (C - A) - Cd(\varepsilon - 1)) \frac{\dot{\Gamma}_1}{\dot{\Gamma}_1} \]

and

\[ \dot{F} = \dot{\rho} \Omega_1 + \rho \dot{\Omega}_1 \]

\[ = \frac{1}{\rho} D (A - C) \Gamma_1 \frac{\dot{\Gamma}_1}{\dot{\Gamma}_1} \Omega_1 \]

\[ = \frac{1}{\varepsilon \rho^2} (D(\varepsilon - 1)G + (C - A)\Omega_1 \Gamma_1) + D \Gamma_1 \Omega_1 (C - A) - Cd(\varepsilon - 1)) \frac{\dot{\Gamma}_1}{\dot{\Gamma}_1}. \]

Let \( \Phi(\Gamma_1) \) be a primitive function of \( 1/\varepsilon \rho \):

\[ \frac{d\Phi}{d\Gamma_1} = \frac{1}{\varepsilon \rho} \left( \frac{\dot{\Phi}}{\dot{\Gamma}_1} \right). \]

Due to the structure of the expressions for \( \dot{F} \) and \( \dot{G} \), we are looking for a third first integral in the form

\[ F_3 = (y_1 F + y_2 G + y_3) \exp(y_4 \Phi). \]

We have

\[ \dot{F}_3 = (y_1 \dot{F} + y_2 \dot{G}) \exp(y_4 \Phi) + y_4 (y_1 F + y_2 G + y_3) \dot{\Phi} \exp(y_4 \Phi) \]

\[ = (y_1 (D(\varepsilon - 1)G - Cd(\varepsilon - 1)) + y_2 (\varepsilon - 1)(A - C)F) \dot{\Phi} \exp(y_4 \Phi) \]

\[ + y_4 (y_1 F + y_2 G + y_3) \dot{\Phi} \exp(y_4 \Phi). \]

Therefore, if \( y_1, y_2, y_3, y_4 \) are solutions of the system

\[ (\varepsilon - 1)(A - C)y_2 + y_1 y_4 = 0, \]

\[ D(\varepsilon - 1)y_1 + y_2 y_4 = 0, \]

\[ -Cd(\varepsilon - 1)y_1 + y_3 y_4 = 0, \]

the function \( F_3 \) is a first integral of the system (3.4) for \( B = C \).

By dividing Eqs. (3.7) and (3.8) we get

\[ \frac{A - C}{D} \frac{y_2}{y_1} = \frac{y_1}{y_2}, \]

whence

\[ y_1 = \pm \sqrt{\frac{A - C}{D}} y_2 \]

and

\[ y_4 = \pm (1 - \varepsilon) \sqrt{\frac{A - C}{D}} y_2. \]

On the other hand, by dividing Eqs. (3.8) and (3.9) we obtain

\[ y_3 = -y_2 \frac{Cd}{D}. \]

Here \( y_2 \neq 0 \) is arbitrary. By taking \( y_2 = D \) we get
Theorem 4. The spherical ball bearing problem (3.4) for $B = C$ is integrable for all $\varepsilon$. Along with $F_1$ and $F_2$, the system has two additional, nonalgebraic first integrals $F_3$ and $F_4$:

$$F_{3,4} = (\pm \sqrt{D(A-C)F + DG - dC}) \exp(\pm(1 - \varepsilon)\sqrt{D(A-C)\Phi}).$$ (3.10)

Note that the product of the two nonalgebraic first integrals

$$F_3 F_4 = (DG - dC)^2 - D(A-C)F^2$$

is an affine combination of $F_1$ and $F_2$:

$$F_3 F_4 = CD(C + D)\langle \vec{M}, \vec{\Omega} \rangle - CD\langle \vec{M}, \vec{M} \rangle - C(C + D)d^2.$$

4. INTEGRATION OF THE PLANAR BALL BEARINGS

In [11], we also introduced the planar ball bearing problem. It was proved there that this system is integrable in quadratures. Here, we are going to present the procedure of its integration. In [11] the equations of motion were derived for the case of three homogeneous balls. Nevertheless, it was pointed out in [11] that all considerations could be adopted for the general case of $n$ balls. Here we consider this general case of $n$ balls.

The planar $n$ balls bearing problem is a system that consists of $n$ homogeneous balls $B_1, \ldots, B_n$ of radius $r$ rolling without slipping over the fixed plane $\Sigma_0$. In addition, it is assumed that the moving plane $\Sigma$ of mass $m$ is placed over the balls, such that there is no slipping between the balls and the moving plane. Let $O_0$ be a fixed point of the plane $\Sigma_0$ and $O, O_1, \ldots, O_n$ be the centers of masses of the plane $\Sigma$ and the balls $B_1, \ldots, B_n$, respectively. In the fixed reference frame, the positions of the points $O, O_i$ are given by $O(x, y, 2r), O_i(x_i, y_i, r), i = 1, \ldots, n$.

Let $A_1, \ldots, A_n$ and $B_1, B_2, \ldots, B_n$ be the contact points of the balls $B_1, \ldots, B_n$ with the planes $\Sigma_0$ and $\Sigma$, respectively, see Fig. 3 for the special case $n = 3$. It appears that the moving configuration of the centers $(O_1(t), \ldots, O_n(t))$, i.e., of the contact points $(B_1(t), \ldots, B_n(t))$, is congruent to the configuration $(O_1, \ldots, O_n)$ formed under the initial condition [11]. In other words, as in the spherical case, the centers $O_i$ of the balls $B_i$ are at rest relative to each other along the motion.

Fig. 3. Planar ball bearings for $n = 3$.

We denote $v_\phi = \dot{\varphi}, v_x = \dot{x}, v_y = \dot{y}$ and introduce

$$\vec{N} = (N_1, N_2, 0) = \sum_{i=1}^{n} \delta_i \vec{OB}_i, \quad M = \sum_{i=1}^{n} \delta_i \langle \vec{OB}_i, \vec{OB}_i \rangle,$$

$$\delta_i = \frac{m_i r^2 + I_i}{4r^2}, \quad i = 1, \ldots, n, \quad \delta = \delta_1 + \cdots + \delta_n.$$ (4.1)
where \( \text{diag}(I_1, I_2, I_3) \) is the inertia operator and \( m_i \) is the mass of the \( i \)th ball \( B_i \).

In [11] it is proved that the equation of motion can be reduced to

\[
Q = \{(v_x, v_y, v_\varphi, N_1, N_2, M) \in \mathbb{R}^6 \mid \delta M > N_1^2 + N_2^2 \}. \tag{4.2}
\]

Let \( I \) be the moment of inertia of the plane \( \Sigma \) with respect to the line perpendicular to \( \Sigma \) at \( O \). If we introduce

\[
v = (v_x, v_y, v_\varphi), \quad n = (N_1, N_2, M),
\]

\[
m = \frac{1}{2}(N_1 v_\varphi^2 - \delta v_\varphi v_y, N_2 v_\varphi^2 + \delta v_\varphi v_x, v_\varphi (N_1 v_x + N_2 v_y)),
\]

\[
I = \begin{pmatrix}
m + \delta & 0 & -N_2 \\
0 & m + \delta & N_1 \\
-N_2 & N_1 & I + M
\end{pmatrix}, \quad J = -\frac{1}{2} \begin{pmatrix}
d \quad 0 \quad N_2 \\
0 \quad \delta \quad -N_1 \\
2N_1 \quad 2N_2 \quad 0
\end{pmatrix},
\]

then the reduced equations of motion on \( Q \) become

\[
\dot{v} = I^{-1} m, \quad \dot{n} = J v. \tag{4.3}
\]

It was proved in [11] that Eqs. (4.3) have the following first integrals:

\[
f_1 = (m + \delta) v_x - v_\varphi N_2,
\]

\[
f_2 = (m + \delta) v_y + v_\varphi N_1,
\]

\[
f_3 = \delta M - (N_1^2 + N_2^2),
\]

\[
f_4 = T = \frac{1}{2} (I + M) v_\varphi^2 + \frac{1}{2} (m + \delta) (v_x^2 + v_y^2) + v_\varphi (N_1 v_x + N_2 v_x).
\]

Also, it was shown in [11] that Eqs. (4.3) possess the invariant measure

\[
\sqrt{\det(I)} \, dv_x \wedge dv_y \wedge dv_\varphi \wedge dN_1 \wedge dN_2 \wedge dM,
\]

where

\[
\det(I) = (m + \delta)((m + \delta) I + m M + (\delta M - (N_1^2 + N_2^2)) > 0|_Q.
\]

The system (4.3) can be solved by quadratures.

On an invariant level set of the three first integrals

\[
Q_d: \quad f_1 = d_1, \quad f_2 = d_2, \quad f_3 = d_3,
\]

where \( d = (d_1, d_2, d_3) \) are given constants, we obtain a closed system in the space \( \mathbb{R}^3 \{ v_\varphi, N_1, N_2 \} \) given by

\[
\dot{v}_\varphi = \frac{mv_\varphi(N_1 d_1 + N_2 d_2)}{2 \det(I)},
\]

\[
\dot{N}_1 = -\frac{m + 2\delta}{2(m + \delta)} N_2 v_\varphi - \frac{\delta d_1}{2(m + \delta)}, \tag{4.5}
\]

\[
\dot{N}_2 = \frac{m + 2\delta}{2(m + \delta)} N_1 v_\varphi - \frac{\delta d_2}{2(m + \delta)},
\]

where

\[
\det(I) = (m + \delta) \left[ (m + \delta) I + \frac{m}{\delta} (N_1^2 + N_2^2) + \frac{md_3}{\delta} + d_3 \right] .
\]

In order to integrate the system (4.5), we introduce the polar coordinates

\[
N_1 = A \cos \theta, \quad N_2 = A \sin \theta.
\]
In the new coordinates Eqs. (4.5) become

\[
\dot{v}_\varphi = \frac{m v_\varphi A_1}{2(d_5 + \frac{m(m+\delta)}{\delta} A_2^2)} (d_1 \cos \theta + d_2 \sin \theta),
\]

\[
\dot{A} = -\frac{\delta}{2(m+\delta)} (d_1 \cos \theta + d_2 \sin \theta),
\]

\[
A \dot{\theta} = \frac{m + 2\delta}{2(m+\delta)} A v_\varphi + \frac{\delta}{2(m+\delta)} (d_1 \sin \theta - d_2 \cos \theta).
\]

From the first two equations, when \(\dot{A} \neq 0\), one gets

\[
\frac{dv_{\varphi}}{dA} = -\frac{m(m+\delta)}{\delta d_5 + \frac{m(m+\delta)}{\delta} A^2} A v_{\varphi},
\]

The integration leads to

\[
v_{\varphi} = v_{\varphi}(A) = \frac{d_6}{\sqrt{d_5 + \frac{m(m+\delta)}{\delta} A^2}},
\]

where \(d_6\) is the constant of integration.

The case when \(\dot{A} = 0\) is considered in [11] (see Remark 4 in [11]).

**Remark 2.** Equation (4.7) gives the first integral

\[
F = \frac{m(m+\delta)}{\delta} v_{\varphi}^2 A^2 + \frac{(I\delta + d_3)(m+\delta)^2}{\delta} v_{\varphi}^2 = d_6 = \text{const},
\]

a modification of the energy integral that in the new coordinates has the form

\[
f_4 = \frac{m}{2\delta(m+\delta)} v_{\varphi}^2 A^2 + \frac{I\delta + d_3}{2\delta} v_{\varphi}^2 + \frac{d_1^2 + d_2^2}{2(m+\delta)}.
\]

Thus, we have \(F = 2(m+\delta)^2 f_4 - 2(m+\delta)(d_1^2 + d_2^2)\).

Let us introduce the constant \(\alpha\) such that

\[
d_1 \cos \theta + d_2 \sin \theta = \sqrt{d_1^2 + d_2^2} \cos(\theta - \alpha),
\]

\[
d_1 \sin \theta - d_2 \cos \theta = \sqrt{d_1^2 + d_2^2} \sin(\theta - \alpha).
\]

By using (4.7), Eqs. (4.6) reduce to

\[
\dot{A} = -k \cos(\theta - \alpha),
\]

\[
A \dot{\theta} = k \sin(\theta - \alpha) + \frac{m + 2\delta}{2(m+\delta)} A v_{\varphi}(A),
\]

where \(k = \frac{\delta}{2(m+\delta)} \sqrt{d_1^2 + d_2^2}\).

As mentioned above, the original system has an invariant measure with the density \(\mu = \sqrt{\det(\mathbb{I})}\). One can check that the density of the invariant measure for the last equations reduces to \(\mu = A\). We will use this observation about the invariant measure to finish the integration.

Equations (4.8) can be rewritten in the form

\[
\frac{dA}{-k \cos(\theta - \alpha)} = A \frac{d\theta}{k \sin(\theta - \alpha) + \frac{m + 2\delta}{2(m+\delta)} A v_{\varphi}(A)},
\]
or equivalently

\[
(k \sin(\theta - \alpha) + \frac{m + 2\delta}{2(m + \delta)} A v_\varphi(A)) dA + k A \cos(\theta - \alpha) d\theta = 0.
\]

We observe that the left-hand side of the last formula presents a total differential. Thus, we finally get

\[
k A \sin(\theta - \alpha) + \frac{(m + 2\delta)\delta}{2m(m + \delta)^2} \sqrt{d_5 + \frac{m(m + \delta)}{\delta}} A^2 = d_7 = \text{const}.
\]

From the last equation we express \(\theta\) as a function of \(A\). By plugging it into the system (4.8) and performing one more integration, we calculate \(A\) as a function of time. From (4.7) we find also \(v_\varphi\) as a function of time.

Finally, we note that it would be interesting to study the above nonholonomic systems with added gyroscopes (see, e.g., [7, 10, 26]), and also to study variations of the problems in \(\mathbb{R}^d\) with an arbitrary dimension \(d > 3\) (e.g., see [13, 15, 17, 18, 21, 24]).

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CONFLICT OF INTEREST

The authors declare that they have no conflicts of interest.

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