ON MAXIMAL COMMUTATIVE SUBALGEBRAS OF POISSON ALGEBRAS ASSOCIATED WITH INVOLUTIONS OF SEMISIMPLE LIE ALGEBRAS

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ABSTRACT. For any involution \( \sigma \) of a semisimple Lie algebra \( g \), one constructs a non-reductive Lie algebra \( k \), which is called a \( \mathbb{Z}_2 \)-contraction of \( g \). In this paper, we attack the problem of describing maximal commutative subalgebras of the Poisson algebra \( S(k) \). This is closely related to the study of the coadjoint representation of \( k \) and the set, \( k^*_{\text{reg}} \), of the regular elements of \( k^* \). By our previous results, in the context of \( \mathbb{Z}_2 \)-contractions, the argument shift method provides maximal commutative subalgebras of \( S(k) \) whenever 
\[
\text{codim}(k^* \setminus k^*_{\text{reg}}) \geq 3.
\]
Our main result here is that 
\[
\text{codim}(k^* \setminus k^*_{\text{reg}}) \geq 3
\]
if and only if the Satake diagram of \( \sigma \) has no trivial nodes. (A node is trivial, if it is white, has no arrows attached, and all adjacent nodes are also white.) The list of suitable involutions is provided. We also describe certain maximal commutative subalgebras of \( S(k) \) if the \((-1)\)-eigenspace of \( \sigma \) in \( g \) contains regular elements.

INTRODUCTION

Let \( Q \) be a connected algebraic group, with Lie algebra \( q \), over an algebraically closed field \( k \) of characteristic zero. The symmetric algebra \( S(q) \simeq k[q^*] \) is equipped with the standard Lie-Poisson bracket \( \{ \cdot, \cdot \} \), and the algebra of invariants \( S(q)^Q \) is the centre of \( (S(q), \{ \cdot, \cdot \}) \). We say that a subalgebra \( A \subset S(q) \) is commutative if the bracket \( \{ \cdot, \cdot \} \) vanishes on \( A \). As is well known, a commutative subalgebra cannot have the transcendence degree larger than \((\dim q + \text{ind } q)/2\), where \( \text{ind } q \) is the index of \( q \). If this bound is attained, then \( A \) is said to be of maximal dimension. A commutative subalgebra \( A \) is said to be maximal, if it is not contained in a larger commutative subalgebra of \( S(q) \). It is shown in [9] that natural commutative subalgebras of \( S(q) \) can be constructed through the use of \( S(q)^Q \) and any \( \xi \in q^* \). This procedure is known as the “argument shift method”, see Section 1.2 for details. We write \( \mathcal{F}_\xi(S(q)^Q) \) for the resulting commutative subalgebra of \( S(q) \).

It was proved in [9] that if \( q = g \) is semisimple and \( \xi \in g^* \simeq g \) is regular semisimple, then \( \mathcal{F}_\xi(S(g)^G) \) is of maximal dimension. (Later on, it was realised that these subalgebras are also maximal [23].) Let \( q_{\text{reg}}^* \) be the set of \( Q \)-regular elements of \( q^* \). By [2], if \( \text{trdeg}(S(q)^Q) = \text{ind } q \) and \( \text{codim}(q^* \setminus q^*_{\text{reg}}) \geq 2 \), then \( \mathcal{F}_\xi(S(q)^Q) \) is of maximal dimension for

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\end{itemize}
any $\xi \in q^*_\text{reg}$. In [18], we extended this result by proving that the maximality of $F_\xi(S(q)^Q)$ is related to the property that $\text{codim} (q^* \setminus q^*_\text{reg}) \geq 3$, see also Theorem 1.4 for the precise statement.

An important class of non-reductive Lie algebras consists of $\mathbb{Z}_2$-contractions of semisimple Lie algebras $\mathfrak{g}$. A $\mathbb{Z}_2$-contraction of $\mathfrak{g}$ is the semi-direct product $\mathfrak{k} = \mathfrak{g}_0 \ltimes \mathfrak{g}_1$, where $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ is a $\mathbb{Z}_2$-grading of $\mathfrak{g}$ and $\mathfrak{g}_1$ becomes an abelian ideal in $\mathfrak{k}$. All $\mathbb{Z}_2$-contractions $\mathfrak{k}$ satisfy the property that $\text{trdeg} (S(\mathfrak{k})^\mathfrak{k}) = \text{ind} \mathfrak{k}$ and $\text{codim} (\mathfrak{k}^* \setminus \mathfrak{k}^*_\text{reg}) \geq 2$ [14]. In this paper, we attack the problem of describing maximal commutative subalgebras in $S(\mathfrak{k})$. Our first approach is to verify when Theorem 1.4 applies to $\mathfrak{k}$. To this end, it suffices to check that $\mathfrak{k}^* \setminus \mathfrak{k}^*_\text{reg} \geq 3$. Our main result is that such “codim–3 property” can be characterised in terms of the Satake diagram of the corresponding involution of $\mathfrak{g}$. Namely, $\mathfrak{k} = \mathfrak{g}_0 \ltimes \mathfrak{g}_1$ has the codim–3 property if and only if each white node of the Satake diagram has either a black adjacent node or an arrow attached (Theorem 4.1). See also Table 1 for the list of relevant involutions and Satake diagrams. Thus, for those $\mathbb{Z}_2$-contractions, the commutative subalgebras $F_\xi(S(\mathfrak{k})^\mathfrak{k})$, $\xi \in \mathfrak{k}^*_\text{reg}$, are maximal. Quite a different approach works if $\mathfrak{g}_1$ contains a regular nilpotent element of $\mathfrak{g}$. Here we prove that $S(\mathfrak{k})^{\mathfrak{g}_1}$ is a maximal commutative subalgebra (Theorem 3.3). An interesting feature is that, for all cases, both constructions provide maximal commutative subalgebras of $S(\mathfrak{k})$ that are polynomial. Unfortunately, these results do not cover all $\mathbb{Z}_2$-contractions. On the other hand, there is an involution of $\mathfrak{g} = \mathfrak{sl}_{2n+1}$, with $\mathfrak{g}_0 = \mathfrak{sl}_n + \mathfrak{sl}_{n+1} + \mathfrak{t}_1$, where both approaches apply and the resulting commutative subalgebras appear to be rather different.

In Section 1, we gather basic facts on coadjoint representations and commutative subalgebras of $S(q)$, including our sufficient condition for the maximality of subalgebras of the form $F_\xi(S(q)^Q)$. Necessary background on the isotropy representations of symmetric spaces and Satake diagrams is presented in Section 2. In Section 3, we recall basic properties of $\mathbb{Z}_2$-contractions and prove that the subalgebra $S(\mathfrak{k})^{\mathfrak{g}_1}$ is maximal and commutative if and only if $\mathfrak{g}_1$ contains a regular nilpotent element of $\mathfrak{g}$. Section 4 is devoted to our characterisation of the involutions of semisimple Lie algebras having the property that $\text{codim} (\mathfrak{k}^* \setminus \mathfrak{k}^*_\text{reg}) \geq 3$. Finally, in Section 5, we summarise our knowledge on maximal commutative subalgebras of $\mathbb{Z}_2$-contractions and pose open problems.

**Notation.** If $Q$ acts on an irreducible affine variety $X$, then $\mathbb{k}[X]^Q$ is the algebra of $Q$-invariant regular functions on $X$ and $\mathbb{k}(X)^Q$ is the field of $Q$-invariant rational functions. If $\mathbb{k}[X]^Q$ is finitely generated, then $X//Q := \text{Spec} \mathbb{k}[X]^Q$, and the quotient morphism $\pi : X \to X//Q$ is the mapping associated with the embedding $\mathbb{k}[X]^Q \hookrightarrow \mathbb{k}[X]$.

If $V$ is a $Q$-module and $v \in V$, then $q_v$ is the stabiliser of $v$ in $q$. We write $t_n$ for the Lie algebra of an $n$-dimensional torus and use ‘$+$’ to denote the direct sum of Lie algebras.
1. Preliminaries on the coadjoint representation and commutative subalgebras

Let $Q$ be an affine algebraic group with Lie algebra $q$. If $Q$ acts regularly on an irreducible algebraic variety $X$, then we also write $(Q : X)$ for this. Let $X_{\text{reg}}$ be the set of $Q$-regular (= $q$-regular) elements of $X$. That is,

$$X_{\text{reg}} := \{ x \in X \mid \dim Q \cdot x \geq \dim Q \cdot x' \text{ for all } x' \in X \} = \{ x \in X \mid \dim q_x \leq \dim q_{x'} \text{ for all } x' \in X \}.$$

As is well-known, $X_{\text{reg}}$ is a dense open subset of $X$, hence $\text{codim}_X (X \setminus X_{\text{reg}}) \geq 1$.

**Definition 1.** We say that $(Q : X)$ has the codim–$n$ property if $\text{codim}_X (X \setminus X_{\text{reg}}) \geq n$.

1.1. The coadjoint representation. Henceforth, we assume that $Q$ is connected. There are two natural representations ($Q$-modules) associated with $Q$, the adjoint and coadjoint ones. Accordingly, we write $q^*_{\text{reg}}$ for the set of $Q$-regular elements of $q^*$, with respect to the coadjoint representation.

**Definition 2.** We say that $q$ has the codim–$n$ property if $\text{codim} (q^* \setminus q^*_{\text{reg}}) \geq n$, i.e., if the coadjoint representation (action) of $Q$ has the codim–$n$ property.

This notion will mostly be used with $n = 2$ or 3.

**Examples 1.1.** 1) If $q$ is semisimple, then $\text{ad} \simeq \text{ad}^*$ and $\text{codim} (q \setminus q^*_{\text{reg}}) = 3$, see [6]. If $q$ is toral (= Lie algebra of a torus), then $q^*_{\text{reg}} = q^*$. This also implies that all reductive Lie algebras have the codim–3 property. (We assume that $\text{codim} (\emptyset) = -\infty$.)

2) If $q$ is abelian, then $q^*_{\text{reg}} = q^*$ and thereby $q$ has the codim–$n$ property for all $n$.

3) For each $n \in \mathbb{N}$ there exist non-commutative Lie algebras with codim–$n$ property [18, Example 1.1].

If $\xi \in q^*_{\text{reg}}$, then $\dim q_\xi$ is called the index of $q$, denoted $\text{ind} q$. In other words, $\text{ind} q$ is the minimal codimension of $Q$-orbits in $q^*$. By Rosenlicht’s theorem, we have $\text{trdeg} k(q^*)^Q = \text{ind} q$. In particular, $\text{trdeg} k[q^*]^Q \leq \text{ind} q$. Set $b(q) = (\dim q + \text{ind} q)/2$. If $q$ is semisimple, then $b(q)$ is the dimension of a Borel subalgebra.

Let $S(q) \simeq k[q^*]$ be the symmetric algebra of $q$ (= the algebra of polynomial functions on $q^*$). For $f \in S(q)$, the differential of $f$, $df$, is a polynomial mapping from $q^*$ to $q$, i.e., an element of $\text{Mor}(q^*, q) \simeq S(q) \otimes q$. More precisely, if $\deg f = d$ (i.e., $f \in S^d(q)$), then $df \in S^{d-1}(q) \otimes q$. We write $(df)_\xi$ for the value of $df$ at $\xi \in q^*$, and the element $(df)_\xi \in q$ is defined as follows. If $\nu \in q^*$ and $\langle , \rangle$ is the natural pairing between $q$ and $q^*$, then

$$\langle (df)_\xi, \nu \rangle := \text{the coefficient of } t \text{ in the Taylor expansion of } f(\xi + t\nu).$$
The Lie-Poisson bracket in \( S(q) \) is defined by \( \{ f_1, f_2 \}(\xi) = \langle (df_1)_\xi, (df_2)_\xi, \xi \rangle \) for \( \xi \in q^* \). Since \( Q \) is connected, the algebra of invariants \( S(q)^Q = S(q)^q \) is the centre of \( (S(q), \{ , \}) \). We also write \( Z(q) \) for the Poisson centre of \( S(q) \).

**Warning.** The symbols \( S(q)^Q, Z(q) \), and \( k[q^*]^Q \) refer to one and the same algebra. But we prefer to use \( S(q)^Q \) and \( Z(q) \) (resp. \( k[q^*]^Q \)) in the Poisson-related (resp. invariant-theoretic) context.

From the invariant-theoretic point of view, the usefulness of the \( \text{codim}-2 \) property is clarified by the following result, see [14, Theorem 1.2].

**Theorem 1.2.** Suppose that \( q \) has the \( \text{codim}-2 \) property and \( \text{trdeg} k[q^*]^Q = \text{ind} q \). Let \( f_1, \ldots, f_l \in k[q^*]^Q \) be arbitrary homogeneous algebraically independent polynomials. Then

1. \( \sum_{i=1}^l \deg f_i \geq b(q) \);
2. If \( \sum_{i=1}^l \deg f_i = b(q) \), then
   - \( k[q^*]^Q \) is freely generated by \( f_1, \ldots, f_l \) and
   - \( \xi \in q_{reg}^* \) if and only if \( (df_1)_\xi, \ldots, (df_l)_\xi \) are linearly independent.

The second assertion in (ii) is a generalisation of Kostant’s result for reductive Lie algebras [6, (4.8.2)].

### 1.2. Commutative subalgebras of \( S(q) \)

Let \( A \) be a subalgebra of the symmetric algebra \( S(q) \). Then \( A \) is said to be commutative if the restriction of \( \{ , \} \) to \( A \) is zero.

By definition, the transcendence degree of \( A \) is that of the quotient field of \( A \). It is well-known that if \( A \) is commutative, then \( \text{trdeg} A \leq b(q) \). Indeed, if \( f_1, \ldots, f_n \in A \) are algebraically independent, then for a generic \( \xi \in q_{reg}^* \), the linear span of \( (df_1)_\xi, \ldots, (df_n)_\xi \) is \( n \)-dimensional, and it is an isotropic space with respect to the Kirillov-Kostant form \( K_\xi \) on \( q \). (Recall that \( K_\xi(x, y) := \langle \xi, [x, y] \rangle \) and hence \( \dim(\ker K_\xi) = \dim q_\xi = \text{ind} q_\xi \).

A commutative subalgebra of \( S(q) \) is said to be of maximal dimension, if its transcendence degree equals \( b(q) \). A commutative subalgebra of \( S(q) \) is maximal, if it is maximal with respect to inclusion among all commutative subalgebras. It is known that commutative subalgebras of maximal dimension always exist, see [21]. It is plausible that if \( q \) is algebraic and \( \text{trdeg} Z(q) = \text{ind} q \), then any maximal commutative subalgebra of \( S(q) \) is of maximal dimension. (In [11, 2.5], Ooms provides an example of a maximal commutative subalgebra of \( S(q) \) that is not of maximal dimension. In his example, \( q \) is not algebraic, which can easily be mended. However, even after that modification one still has \( \text{trdeg} Z(q) < \text{ind} q \).)

Let \( f \in S(q) \) be a polynomial of degree \( d \). For \( \xi \in q^* \), consider a shift of \( f \) in direction \( \xi \): \( f_{a,\xi}(\mu) = f(\mu + a\xi) \), where \( a \in k \). Expanding the right hand side as polynomial in \( a \), we obtain the expression \( f_{a,\xi}(\mu) = \sum_{j=0}^d f_j^\xi(\mu)a^j \) and the family of polynomials \( f_j^\xi \), where
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\[ j = 0, 1, \ldots, d - 1. \] Since \( \deg f^j_\xi = d - j \), the value \( j = d \) is not needed.) Notice that \( f^0_\xi = f \)
and \( f^{d-1}_\xi \) is a linear form on \( q^* \), i.e., an element of \( q \). Actually, \( f^{d-1}_\xi = (df)_\xi \). There is also an obvious symmetry with respect to \( \xi \) and \( \mu \): \( f^j_\xi(\mu) = f^{d-j}_\mu(\xi) \).

The following observation is due to Mishchenko–Fomenko [9].

**Lemma 1.3.** Suppose that \( h_1, \ldots, h_m \in Z(q) \) are homogeneous. Then for any \( \xi \in q^* \), all the polynomials \( \{ (h_i)^j_\xi \mid i = 1, \ldots, m; j = 0, 1, \ldots, \deg h_i - 1 \} \) pairwise commute with respect to the Lie–Poisson bracket.

This procedure has been used for constructing commutative subalgebras of maximal dimension in \( S(q) \). Given \( \xi \in q^* \) and an arbitrary subset \( B \subset Z(q) \), let \( F_\xi(B) \) denote the subalgebra of \( S(q) \) generated by the \( \xi \)-shifts of all elements of \( B \). If \( B \) is the subalgebra generated by \( B \), then \( F_\xi(B) = F_\xi(B) \). By Lemma 1.3, any algebra \( F_\xi(B) \) is commutative. In particular, algebras \( F_\xi(Z(q)) \) are natural candidates on the role of commutative subalgebras of maximal dimension.

For semisimple \( g \), it is proved in [9] that \( F_\xi(Z(g)) \) is of maximal dimension whenever \( \xi \in g^* \simeq g \) is regular semisimple. A general sufficient condition for \( F_\xi(Z(q)) \) to be of maximal dimension is found by Bolsinov [2, Theorem 3.1]. In [18], we have generalised these results and obtained a sufficient condition for \( F_\xi(Z(q)) \) to be maximal:

**Theorem 1.4** (see [18, Theorem 3.2]). Let \( q \) be an algebraic Lie algebra.

(i) Suppose that \( q \) has the codim–2 property and \( Z(q) \) contains algebraically independent polynomials \( f_1, \ldots, f_l \), where \( l = \text{ind} \ q \), such that \( \sum_{i=1}^l \deg f_i = b(q) \). Then \( F_\xi(Z(q)) \) is a polynomial algebra of Krull dimension \( b(q) \) for any \( \xi \in q^*_\text{reg} \). (Hence \( F_\xi(Z(q)) \) is a polynomial commutative subalgebra of maximal dimension.)

(ii) Moreover, if \( q \) has the codim–3 property, then \( F_\xi(Z(q)) \) is a maximal commutative subalgebra of \( S(q) \).

In the rest of the paper, we consider a special class of non-reductive Lie algebras, the so-called \( \mathbb{Z}_2 \)-contractions of semisimple algebras. These algebras always satisfy the conditions stated in Theorem 1.4(i), see Theorem 3.1 below. Therefore, one of our objectives is to study the codim–3 property for them.

2. **Symmetric pairs, isotropy representations, and Satake diagrams**

Let \( g \) be a semisimple Lie algebra and \( \sigma \) an involutory automorphism of \( g \). Let \( g_i \) denote the \((-1)^i\)-eigenspace of \( \sigma \). Then \( g_0 \) is a reductive subalgebra and \( g_1 \) is an orthogonal \( g_0 \)-module. We also say that \((g, g_0)\) is a symmetric pair and \( g = g_0 \oplus g_1 \) is a \( \mathbb{Z}_2 \)-grading of \( g \). Let \( G \) be the adjoint group of \( g \) and \( G_0 \) the connected subgroup of \( G \) with \( \text{Lie} \ G_0 = g_0 \). The representation \((G_0 : g_1)\) is the isotropy representation of the symmetric pair \((g, g_0)\).
Below we introduce some notation and recall basic invariant-theoretic properties of the representation \((G_0 : g_1)\). The standard reference for this is \([7]\). Let \(N\) denote the variety of nilpotent elements of \(g\).

(†1) For any \(v \in g_1\) and the induced \(\mathbb{Z}_2\)-grading \(g_v = g_{0,v} \oplus g_{1,v}\), one has

\[
\dim g_0 - \dim g_{0,v} = \dim g_1 - \dim g_{1,v}.
\]

The closure of \(G_0 \cdot v\) contains the origin if and only if \(v \in N\); and \(G_0 \cdot v\) is closed if and only if \(v\) is semisimple. Write \(G_{0,v}\) for the stabiliser of \(v\) in \(G_0\), which is not necessarily connected.

(‡2) Let \(c \subset g_1\) be a maximal subspace consisting of pairwise commuting semisimple elements. Any such subspace is called a Cartan subspace. All Cartan subspaces are \(G_0\)-conjugate and \(G_0 \cdot c\) is dense in \(g_1\); \(\dim c\) is called the rank of the \(\mathbb{Z}_2\)-grading or pair \((g, g_0)\), denoted \(\text{rk} (g, g_0)\). If \(h \in c\) is \(G_0\)-regular in \(g_1\), then \(g_{1,h} = c\) and \(g_{0,h}\) is the centraliser of \(c\) in \(g_0\). We also write \(\tau = \delta(c)_0\) for this centraliser.

(‡3) The algebra \(k[g_1]^{G_0}\) is polynomial and \(\dim g_1 / G_0 = \text{rk} (g, g_0)\). The quotient map \(\pi : g_1 \to g_1 / G_0\) is equidimensional, i.e., the irreducible components of all fibres of \(\pi\) are of dimension \(\dim g_1 - \dim g_1 / G_0\). Any fibre of \(\pi\) contains finitely many \(G_0\)-orbits and each closed \(G_0\)-orbit in \(g_1\) meets \(c\). We write \(N(g_1)\) for \(\pi^{-1}(\pi(0)) = N \cap g_1\).

If \(v \in g_1\) is semisimple, then both \(g_v\) and \(g_{0,v}\) are reductive, and \(G_0 \cdot v\) is the unique closed orbit in \(\pi^{-1}(\pi(v))\). By Luna’s slice theorem \([8]\), there is an isomorphism

\[
G_0 \times_{G_{0,v}} N(g_{1,v}) \xrightarrow{\sim} \pi^{-1}(\pi(v)),
\]

which takes \((y, y) \in G_0 \times_{G_{0,v}} N(g_{1,v})\) to \(g \cdot (v + y)\). This implies that \(g_{0,v+y} = g_{0,v} \cap g_{0,y}\) and \(y \in N(g_{1,v})\) is \(G_{0,v}\)-regular if and only if \(v + y\) is \(G_0\)-regular in \(\pi^{-1}(\pi(v))\) and hence in \(g_1\).

Let \(h = [g_v, g_v]\) and let \(z\) be the centre of \(g_v\). Then \(g_{0,v} = h_0 + z_0\) and \(g_{1,v} = h_1 + z_1\). Write \(H_0\) for the connected subgroup of \(G_0\) with \(\text{Lie} H_0 = h_0\). The \(H_0\)-orbits in \(g_{1,v}\) coincide with the orbits of the identity component of \(G_{0,v}\), and since \(z\) consists of semisimple elements, \(N(g_{1,v}) = N(h_1)\). Furthermore,

\[
g_{0,v+y} = (h_0 + z_0) \cap g_{0,y} = z_0 + (h_0 \cap g_{0,y}) = z_0 + h_{0,y}.
\]

In particular, \(g_{0,v+y}\) has the \(\text{codim} - n\) property if and only if \(h_{0,y}\) has.

Thus, the \(G_0\)-orbits in \(\pi^{-1}(\pi(v))\) and the corresponding centralisers in \(g_0\) can be studied via the isotropy representation \((H_0 : h_1)\) related to the “smaller” symmetric pair \((h, h_0)\). The latter is called a reduced sub-symmetric pair of \((g, g_0)\).

An explicit description of all reduced sub-symmetric pairs associated with \((g, g_0)\) can be given via Satake diagrams. One usually associates the Satake diagram to a real form of \(g\) (see e.g. \([24, \text{Ch. 4, } \S 4.3]\)). But, in view of a one-to-one correspondence between the real forms of \(g\) and the \(\mathbb{Z}_2\)-gradings of \(g\), one obtains Satake diagrams for the symmetric pairs as well. A direct construction goes as follows. The Satake diagram \(\text{Sat}(g, g_0)\) associated
with \((\mathfrak{g}, \mathfrak{g}_0)\) is the Dynkin diagram of \(\mathfrak{g}\), where each node is either black or white, and some pairs of white nodes are joined by a new arrow. More precisely, choose a Cartan subspace \(\mathfrak{c} \subset \mathfrak{g}_1\). Let \(t\) be a \(\sigma\)-stable Cartan subalgebra of \(\mathfrak{g}\) containing \(\mathfrak{c}\) and let \(\Delta\) be the root system of \((\mathfrak{g}, t)\). Since \(t\) is \(\sigma\)-stable, \(\sigma\) acts on \(\Delta\). It is possible to choose the set of positive roots, \(\Delta^+\), such that if \(\beta \in \Delta^+\) and \(\beta|_\mathfrak{c} \neq 0\), then \(\sigma(\beta) \in -\Delta^+\). Let \(\Pi \subset \Delta^+\) be the corresponding set of simple roots. We identify the simple roots and the nodes of the Dynkin diagram. For \(\alpha \in \Pi\), there are the following possibilities:

- If \(\alpha|_\mathfrak{c} = 0\), then the root space \(\mathfrak{g}^\alpha \subset \mathfrak{g}\) belongs to \(\mathfrak{g}_0\) and \(\alpha\) is black in \(\text{Sat}(\mathfrak{g}, \mathfrak{g}_0)\);
- if \(\alpha|_\mathfrak{c} \neq 0\) and \(\sigma(\alpha) = -\alpha\), then \(\alpha\) is white, without arrows attached;
- if \(\alpha|_\mathfrak{c} \neq 0\) and \(\sigma(\alpha) = -\beta \neq -\alpha\), then \(\beta\) is another simple root, and the corresponding white nodes are joined by an arrow. (In this case, we have \(\alpha|_\mathfrak{c} = \beta|_\mathfrak{c}\)).

**Examples 2.1.** (1) Suppose that \(\mathfrak{g} = \mathfrak{h} + \mathfrak{h}\) and \(\sigma\) is the permutation. Then \(\mathfrak{g}_0 = \Delta_\mathfrak{h} \simeq \mathfrak{h}\) and \(\text{Sat}(\mathfrak{h} + \mathfrak{h}, \Delta_\mathfrak{h})\) is the union of two copies of the Dynkin diagram for \(\mathfrak{h}\), where the corresponding nodes are joined by arrows. This diagram has no black nodes at all.

(2) If \(\mathfrak{c} \subset \mathfrak{g}_1\) is a Cartan subalgebra, then \(\text{Sat}(\mathfrak{g}, \mathfrak{g}_0)\) has neither arrows nor black nodes. The corresponding symmetric pair (or, involution) is said to be of maximal rank. Recall that every simple Lie algebra has a unique, up to conjugation, involution of maximal rank.

**Remark 2.2.** We say that \(\text{Sat}(\mathfrak{g}, \mathfrak{g}_0)\) is connected, if it is connected as a graph, where the new arrows are also taken into account. For instance, if \(\mathfrak{h}\) is simple in Example 2.1(1), then \(\text{Sat}(\mathfrak{h} + \mathfrak{h}, \Delta_\mathfrak{h})\) is connected. Recall that \((\mathfrak{g}, \mathfrak{g}_0)\) is indecomposable if \(\mathfrak{g}\) cannot be presented as a direct sum of two nonzero \(\sigma\)-stable ideals. It is easily seen that \(\text{Sat}(\mathfrak{g}, \mathfrak{g}_0)\) is connected if and only if \((\mathfrak{g}, \mathfrak{g}_0)\) is indecomposable if and only if either \(\mathfrak{g}\) is simple, or \(\mathfrak{h}\) is simple in Example 2.1(1). In general, \((\mathfrak{g}, \mathfrak{g}_0)\) is a direct sum of indecomposable symmetric pairs that correspond to the minimal \(\sigma\)-stable ideals of \(\mathfrak{g}\).

Looking at \(\text{Sat}(\mathfrak{g}, \mathfrak{g}_0)\), one immediately reads off many properties of a symmetric pair under consideration. Recall that \(\tau = \mathfrak{g}_{0,h}\) for generic \(h \in \mathfrak{c}\). Then \([\tau, \tau]\) is the semisimple subalgebra of \(\mathfrak{g}\) corresponding to the set of black nodes in \(\text{Sat}(\mathfrak{g}, \mathfrak{g}_0)\) and the dimension of the centre of \(\tau\) equals the number of arrows. Therefore,

\[ \dim \mathfrak{c} = \text{rk} (\mathfrak{g}, \mathfrak{g}_0) = (\text{number of white nodes}) - (\text{number of arrows}). \]

We say that \(S'\) is a subdiagram of \(\text{Sat}(\mathfrak{g}, \mathfrak{g}_0)\) if \(S'\) is obtained from \(\text{Sat}(\mathfrak{g}, \mathfrak{g}_0)\) by iterating the following steps: one can either remove one white node, if it does not have an arrow attached; or one can remove a pair of nodes connected by an arrow. The geometric meaning of this notion is the following.

**Proposition 2.3.** (see [17, Prop. 1.5]) A symmetric pair \((\mathfrak{h}, \mathfrak{h}_0)\) occurs as a reduced subsymmetric pair of \((\mathfrak{g}, \mathfrak{g}_0)\) if and only if \(\text{Sat}(\mathfrak{h}, \mathfrak{h}_0)\) is a subdiagram of \(\text{Sat}(\mathfrak{g}, \mathfrak{g}_0)\).
Remark 2.4. The connected components of $\text{Sat}(\mathfrak{h}, \mathfrak{h}_0)$ that consist only of black nodes correspond to the simple factors of $\mathfrak{h}$ that lie entirely in $\mathfrak{g}_0$. Therefore they do not affect $\mathcal{N}(\mathfrak{h}_1)$ and the structure of the corresponding fibre of $\pi : \mathfrak{g}_1 \to \mathfrak{g}_1 / \mathcal{G}_0$.

Proposition 2.5. Given a $\mathbb{Z}_2$-grading $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$, let $z$ be a $G_0$-regular element of $\mathfrak{g}_1$. Then $\dim \mathfrak{g}_{1,z} = \text{rk} (\mathfrak{g}, \mathfrak{g}_0)$ and $\text{ind} \mathfrak{g}_{0,z} = \text{rk} \mathfrak{g} - \text{rk} (\mathfrak{g}, \mathfrak{g}_0)$.

Proof. Consider the induced $\mathbb{Z}_2$-grading $\mathfrak{g}_z = \mathfrak{g}_{0,z} \oplus \mathfrak{g}_{1,z}$ for $z \in \mathfrak{g}_1$. If $z \in c$ is $G_0$-regular, then $\mathfrak{g}_{1,z} = c$, i.e., it is a (abelian) toral subalgebra of dimension $\text{rk} (\mathfrak{g}, \mathfrak{g}_0)$. Next, $\mathfrak{g}_{0,z} = \mathfrak{r}$ is reductive, with $\text{ind} \mathfrak{r} = \text{rk} \mathfrak{r} = \text{rk} \mathfrak{g} - \text{rk} (\mathfrak{g}, \mathfrak{g}_0)$, and the assertion holds in this case.

For an arbitrary $G_0$-regular $z$, we still have $\dim \mathfrak{g}_{1,z} = \text{rk} (\mathfrak{g}, \mathfrak{g}_0)$ in view of Eq. (2.1). Since the $G_0$-regular semisimple elements are dense in the set of all $G_0$-regular elements, $\mathfrak{g}_{1,z}$ is an abelian subalgebra. By [16, Prop. 2.6(1)], the Lie bracket $\mathfrak{g}_{0,z} \times \mathfrak{g}_{1,z} \to \mathfrak{g}_{1,z}$ is trivial for the $G_0$-regular elements in $\mathfrak{g}_1$. Hence $\mathfrak{g}_z = \mathfrak{g}_{0,z} \oplus \mathfrak{g}_{1,z}$ is a direct sum of Lie algebras. Consequently, $\text{ind} \mathfrak{g}_{0,z} = \text{ind} \mathfrak{g}_z - \text{rk} (\mathfrak{g}, \mathfrak{g}_0)$. Finally, by the ”Elashvili conjecture”, we have $\text{ind} \mathfrak{g}_z = \text{rk} \mathfrak{g}$ for all $z \in \mathfrak{g}$, see [3].

3. Generalities on $\mathbb{Z}_2$-contractions of semisimple Lie algebras

For a $\mathbb{Z}_2$-grading $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$, the semi-direct product $\mathfrak{t} = \mathfrak{g}_0 \ltimes \mathfrak{g}_1$, where $\mathfrak{g}_1$ is an abelian ideal, is called a $\mathbb{Z}_2$-contraction of $\mathfrak{g}$. This is a particular case of the general concept of “contractions of Lie algebras”, see [24, Ch. 7, §2]. The corresponding connected algebraic group is the semi-direct product $K = G_0 \ltimes \mathfrak{g}_1$. Then $G_0$ is a Levi subgroup of $K$ and the unipotent radical of $K$, $K^u$, is commutative. Of course, $K^u$ is isomorphic to $\exp(\mathfrak{g}_1)$. The study of $\mathbb{Z}_2$-contraction was initiated in [13],[14].

The vector space $\mathfrak{t}^*$ is isomorphic to $\mathfrak{g}_0^\ast \oplus \mathfrak{g}_1^\ast$. Since the Killing form is non-degenerate on $\mathfrak{g}_i$, we obtain a fixed isomorphism of $G_0$-modules $\mathfrak{g}_i \simeq \mathfrak{g}_i^\ast$. Consequently, $\mathfrak{t}$ and $\mathfrak{t}^*$ are also identified as $G_0$-modules, and one can speak about Cartan subspaces of $\mathfrak{g}_1^\ast$ and apply all invariant-theoretic results stated in (†1)-(†3) to the action $(G_0 : \mathfrak{g}_1^\ast)$. We write $\mathfrak{g}_i^\ast$, if we wish to stress that $\mathfrak{g}_i$ is regarded as a subspace of $\mathfrak{t}^\ast$. Upon these identifications, the coadjoint representation of $\mathfrak{t}$ is given by the following formula. If $(x_0, x_1) \in \mathfrak{t}$ and $(\xi_0, \xi_1) \in \mathfrak{t}^\ast$, then $(x_0, x_1) \ast (\xi_0, \xi_1) = ([x_0, \xi_0] + [x_1, \xi_1], [x_0, \xi_1])$. It follows that $\mathfrak{g}_0^\ast$ is a $K$-stable subspace of $\mathfrak{t}^\ast$ and $\mathfrak{t}^\ast / \mathfrak{g}_0^\ast$ is a $K$-module with trivial action of $K^u$. That is, $\mathfrak{t}^\ast / \mathfrak{g}_0^\ast$ is identified with the $G_0$-module $\mathfrak{g}_1^\ast$.

We summarise below some fundamental results on the coadjoint representation of $\mathfrak{t}$.

Theorem 3.1. For any $\mathbb{Z}_2$-contraction $\mathfrak{t}$ of a semisimple Lie algebra $\mathfrak{g}$, we have

(i) $\text{ind} \mathfrak{t} = \text{ind} \mathfrak{g} = \text{rk} \mathfrak{g}$ [13, Prop. 9.3]. Therefore $b(\mathfrak{t}) = b(\mathfrak{g})$.

(ii) $\mathfrak{t}$ has the codim–2 property [14, Theorem 3.3].
(iii) The algebra \( k[t^*]^K \) is polynomial, of Krull dimension \( l = \text{rk} \, g \). If \( f_1, \ldots, f_i \) are algebraically independent homogeneous generators of \( k[t^*]^K \), then \( \sum_{i=1}^{l} \deg f_i = b(t) \) [14, 25, 26].

Remark 3.2. The first assertion in (iii) is achieved via case-by-case considerations. Namely, results of [14, 25] together cover all but four cases related to simple algebras of type \( E_n \). The remaining involutions will be handled in [26]. In each case, a certain set of homogeneous generators of \( k[t^*]^K \) can be constructed, and the equality \( \sum_{i=1}^{l} \deg f_i = b(t) \) arises a posteriori. However, once one knows somehow that the algebra \( k[t^*]^K \) is polynomial, there is also a conceptual way to establish the equality for the sum of degrees:

By [5, Theorem 2.2], which generalises a sum rule obtained in [12, Theorem 1.1], if a Lie algebra \( q \) is unimodular, \( k[q^*]^Q \) is polynomial, and the fundamental semi-invariant \( p_q \) of \( q \) is an invariant, then the sum of degrees of generators of \( k[q^*]^Q \) equals \( b(q) - \deg p_q \). One easily proves that all \( Z_2 \)-contractions are unimodular, and since they have the \( \text{codim} \)-2 property, \( p_e = 1 \) for all of them.

3.1. \( Z_2 \)-contractions associated with \( N \)-regular symmetric pairs. A symmetric pair \((g, g_0)\) is called \( N \)-regular if \( g_1 \) contains a regular nilpotent element of \( g \). By a result of Antonyan [1], a symmetric pair is \( N \)-regular if and only if \( g_1 \) contains a regular semisimple element of \( g \). Therefore, \( N \)-regularity is equivalent to that \( \text{Sat}(g, g_0) \) contains no black nodes. For the \( N \)-regular symmetric pairs, \( m := \text{rk} \, g - \text{rk} \, (g, g_0) \) is equal to the number of arrows in \( \text{Sat}(g, g_0) \). The list of all indecomposable \( N \)-regular symmetric pairs includes the symmetric pairs of maximal rank (with \( m = 0 \)) and also the following pairs:

\begin{enumerate}
\item \((sl_{n+k}, sl_n + sl_k + t_1), |n-k| \leq 1, m = \min\{n, k\}\);
\item \((so_{2n+2}, so_n + so_{n+2}), m = 1\);
\item \((E_6, sl_6 + sl_2), m = 2\);
\item \((\mathfrak{h} + \mathfrak{h}, \Delta h), \) where \( \mathfrak{h} \) is simple, \( m = \text{rk} \, \mathfrak{h} \).
\end{enumerate}

The following result is a straightforward consequence of the theory developed in [14].

**Theorem 3.3.** Suppose that \( g = g_0 \oplus g_1 \) is \( N \)-regular. Then \( k[t^*]^K = k[t^*]^0 \) is a maximal commutative subalgebra of \( k[t^*] \).

**Proof.** For the \( N \)-regular \( Z_2 \)-gradings, \( k[t^*]^K \) is a polynomial algebra of Krull dimension \( \dim g_1 + \text{rk} \, g - \text{rk} \, (g, g_0) = b(g) \). More precisely, let \( e_1, \ldots, e_n \) be a basis for \( g_1 \). We regard the \( e_i \)'s as linear function on \( g_1^* \) and hence on \( t^* \). Then \( k[t^*]^K \) is freely generated by \( e_1, \ldots, e_n, \hat{F_1}, \ldots, \hat{F_m} \), where \( m = \text{rk} \, g - \text{rk} \, (g, g_0) \) and \( \hat{F_1}, \ldots, \hat{F_m} \) are explicitly described polynomials that are even \( K \)-invariant [14, Theorem 5.2]. Since \( g_1 \) is an Abelian ideal in \( t \) and the \( \hat{F}_j \)'s belong to the centre of the Poisson algebra \( k[t^*] \), the algebra \( k[t^*]^K \) is commutative. On the other hand, if \( A \supseteq k[t^*]^K \), then \( A \) contains the whole space \( g_1 \). Hence the commutativity of \( A \) implies that \( A \subset k[t^*]^0 \). \( \square \)
Since all elements of $S(\mathfrak{t})^K$ can naturally be lifted to the centre of $\mathcal{U}(\mathfrak{t})$ and there is no problem with lifting elements of degree 1, the above description of free generators shows that $S(\mathfrak{t})_{g_1}^0 = \mathbb{k}[\mathfrak{t}^*]_{g_1}^0$ can be lifted to the enveloping algebra $\mathcal{U}(\mathfrak{t})$; that is, there exists a commutative subalgebra $\tilde{A} \subset \mathcal{U}(\mathfrak{t})$ such that $\text{gr}(\tilde{A}) = S(\mathfrak{t})_{g_1}^0$. In particular, if $m = 0$, i.e., $(\mathfrak{g}, \mathfrak{g}_0)$ is of maximal rank, then just $S(\mathfrak{g}_1) = \mathbb{k}[\mathfrak{g}_1^*]$ is a maximal commutative subalgebra of $\mathcal{U}(\mathfrak{t})$.

Remark 3.4. The algebra $\mathbb{k}[\mathfrak{t}^*]_{g_1}^0$ can be considered for any $\mathbb{Z}_2$-contraction. However, we can prove that

the algebra $\mathbb{k}[\mathfrak{t}^*]_{g_1}^0$ is commutative $\iff (\mathfrak{g}, \mathfrak{g}_0)$ is $\mathcal{N}$-regular.

Recall that $\mathfrak{t} = \mathfrak{z}(\mathfrak{c})_0 = \mathfrak{g}_0, \xi$ for a Cartan subspace $\mathfrak{c} \in \mathfrak{g}_1 \simeq \mathfrak{g}_1^*$ and generic $\xi \in \mathfrak{c}$. Then

$$\text{trdeg} \mathbb{k}[\mathfrak{t}^*]_{g_1}^0 = \dim \mathfrak{t} - \max_{\xi \in \mathfrak{t}^*} \dim K^u, \xi = \dim \mathfrak{t} - \dim \mathfrak{g}_1 + \min_{\xi \in \mathfrak{g}_1^*} \dim \mathfrak{g}_1, \xi = \dim \mathfrak{t} - \dim \mathfrak{g}_0 + \min_{\xi \in \mathfrak{g}_1^*} \dim \mathfrak{g}_0, \xi = \dim \mathfrak{g}_1 + \dim \mathfrak{r}.$$ 

On the other hand, Rais’ formula for the index of semi-direct product [19] shows that $\text{ind} \mathfrak{t} = \dim \mathfrak{g}_1 - \dim \mathfrak{g}_0 + \dim \mathfrak{r} + \text{ind} \mathfrak{r}$ and therefore

$$b(\mathfrak{t}) = (\dim \mathfrak{t} + \text{ind} \mathfrak{t})/2 = \dim \mathfrak{g}_1 + b(\mathfrak{r}).$$

It follows that $\text{trdeg} \mathbb{k}[\mathfrak{t}^*]_{g_1}^0 \geq b(\mathfrak{t})$ and the equality occurs if and only if $\mathfrak{r}$ is toral. The latter is equivalent to that $\mathfrak{c}$ contains regular semisimple elements of $\mathfrak{g}$, i.e., $(\mathfrak{g}, \mathfrak{g}_0)$ is $\mathcal{N}$-regular.

For involutions (symmetric pairs) of maximal rank, $\dim \mathfrak{g}_1 = b(\mathfrak{g})$. Therefore, the commutative Lie subalgebra $\mathfrak{g}_1$ is a commutative polarisation of $\mathfrak{t}$ [10, Sect. 5]. Conversely, using [10, Prop. 20], one can prove that if $\mathfrak{t}$ admits a commutative polarisation, then $(\mathfrak{g}, \mathfrak{g}_0)$ is of maximal rank.

4. $\mathbb{Z}_2$-CONTRACTIONS WITH CODIMENSION–3 PROPERTY

The $\text{codim}$–3 property does not hold for all $\mathbb{Z}_2$-contractions. In [18, Example 4.1], we noticed that, for the involutions of maximal rank and $\xi \in \mathfrak{t}_{\text{reg}}^* \cap \mathfrak{g}_1^*$, the commutative subalgebras $\mathcal{F}_\xi(\mathbb{Z}(\mathfrak{t}))$ fail to be maximal, and thereby $\mathfrak{t}$ does not have the codim–3 property. In this section, we obtain a characterisation of $\mathbb{Z}_2$-contractions with the codim–3 property.

Definition 3. A node of the Satake diagram $\text{Sat}(\mathfrak{g}, \mathfrak{g}_0)$ is said to be trivial, if it is white, does not have an arrow attached, and all adjacent nodes are also white.

Theorem 4.1. A $\mathbb{Z}_2$-contraction $\mathfrak{t} = \mathfrak{g}_0 \times \mathfrak{g}_1$ of a semisimple algebra $\mathfrak{g}$ has the codim–3 property if and only if $\text{Sat}(\mathfrak{g}, \mathfrak{g}_0)$ has no trivial nodes.
**Remark 4.2.** A description of the reduced sub-symmetric pairs via $\text{Sat}(g, g_0)$ (see Prop. 2.3) shows that this theorem can be restated as follows. A $\mathbb{Z}_2$-contraction $\mathfrak{t} = g_0 \ltimes g_1$ has the codim–3 property if and only if there is no reduced sub-symmetric pairs of rank 1 associated with $(g, g_0)$ of the form $(\mathfrak{h}, h_0) = (\mathfrak{sl}_2, \mathfrak{so}_2) + (l, l)$, where $l$ is a semisimple Lie algebra. (That is, $\mathfrak{sl}_2$ is the only simple ideal of $\mathfrak{h}$ with non-trivial restriction of $\sigma$.) It is also easily verified that the $\mathbb{Z}_2$-contraction $\mathfrak{t}$ arising from the symmetric pair $(\mathfrak{sl}_2, \mathfrak{so}_2)$ does not have the codim–3 property.

Before dwelling on the proof, we provide an explicit description of the corresponding symmetric pairs. If $(g, g_0)$ is not indecomposable, then $\mathfrak{t}$ is the direct sum of the $\mathbb{Z}_2$-contractions corresponding to the minimal $\sigma$-stable ideals of $g$ (cf. Remark 2.2). Therefore, it suffices to point out the admissible indecomposable symmetric pairs.

**Corollary 4.3.** Suppose that $(g, g_0)$ is indecomposable and $\mathfrak{t} = g_0 \ltimes g_1$ has the codim–3 property. Then either $g = \mathfrak{h} + \mathfrak{h}$, where $\mathfrak{h}$ is simple and $g_0 = \Delta_0$, or $g$ is simple and $(g, g_0)$ occurs in Table 1.

| $(g, g_0)$ | $\text{Sat}(g, g_0)$ | $\text{rk}(g, g_0)$ | $r = \delta(c)_0$ |
|------------|----------------------|-------------------|-----------------|
| $(\mathfrak{sl}_n, \mathfrak{sl}_k + \mathfrak{sl}_{n-k} + t_1)$, $0 < k < n - k$ | ![Diagram](chart1.png) | $k$ | $\mathfrak{sl}_{n-2k} + t_k$ |
| $(\mathfrak{sl}_{2n}, \mathfrak{sp}_{2n})$, $n \geq 2$ | ![Diagram](chart2.png) | $n - 1$ | $(\mathfrak{sl}_2)^n$ |
| $(\mathfrak{so}_{4n+2}, \mathfrak{gl}_{2n+1})$, $n \geq 2$ | ![Diagram](chart3.png) | $n$ | $(\mathfrak{sl}_2)^n + t_1$ |
| $(\mathfrak{so}_n, \mathfrak{so}_{n-1})$, $n \geq 5$ | ![Diagram](chart4.png) | $1$ | $\mathfrak{so}_{n-2}$ |
| $(\mathfrak{sp}_{2n}, \mathfrak{sp}_{2k} + \mathfrak{sp}_{2n-2k})$, $1 \leq k \leq n - k$ | ![Diagram](chart5.png) | $k$ | $(\mathfrak{sl}_2)^k + \mathfrak{sp}_{2n-4k}$ |
| $(\mathfrak{e}_6, \mathfrak{f}_4)$ | ![Diagram](chart6.png) | $2$ | $\mathfrak{so}_8$ |
| $(\mathfrak{e}_6, \mathfrak{so}_{10} + t_1)$ | ![Diagram](chart7.png) | $2$ | $\mathfrak{sl}_4 + t_1$ |
| $(\mathfrak{f}_4, \mathfrak{so}_9)$ | ![Diagram](chart8.png) | $1$ | $\mathfrak{so}_7$ |

**Table 1.** The symmetric pairs with $g$ simple and the codim–3 property for $\mathfrak{t}$.

**Remarks on Table 1.** (i) The number of black nodes in item 1) equals $n - 1 - 2k$ and the number of arrows equals $k$. If $k + 1 = n - k$, then $\text{Sat}(g, g_0)$ has no black nodes at all.

(ii) The right-hand end of $\text{Sat}(g, g_0)$ in item 4) depends on the parity of $n$ (type B or D).
Proof of Theorem 4.1. We begin with describing certain $K$-regular elements of $\mathfrak{t}^* \simeq g_0^* \oplus g_1^*$. Consider the mappings

$$\mathfrak{t}^* \xrightarrow{\psi} g_1^* \xrightarrow{\pi} g_1^*/G_0,$$

where $\psi$ is the projection with kernel $g_0^*$ and $\pi$ is the quotient morphism. Recall that $g_0^*$ is a $K$-submodule of $\mathfrak{t}^*$, hence $\psi$ is a surjective homomorphism of $K$-modules (the unipotent radical $K^u$ acts trivially on $g_1^*$). Let $\eta = (\alpha, \beta) \in \mathfrak{t}^*$ be an arbitrary point, where $\alpha \in g_0^*$ and $\beta \in g_1^*$. Write $g_{0, \beta}$ for the stabiliser of $\beta$ in $g_0$. Then $g_1 \ast \beta = \text{Ann}(g_{0, \beta}) \subset g_0^*$ and therefore $g_0^*/(g_1 \ast \beta) \simeq g_{0, \beta}^*$. Using the last isomorphism, we let $\hat{\alpha}$ denote the image of $\alpha$ in $g_{0, \beta}^*$. By [13, Prop. 5.5],

$$\dim \mathfrak{t}_\eta = \text{codim}_{g_1^*}(G_0; \beta) + \text{dim}(g_{0, \beta}) \hat{\alpha},$$

where the last summand refers to the stabiliser of $\hat{\alpha}$ with respect to the coadjoint representation of $g_{0, \beta}$. Since $\psi^{-1}(G_0; \beta) = g_0^* \times G_0; \beta$ is $K$-stable, it follows from Eq. (4.1) that

$$\min\{\text{codim}_\mathfrak{t}, \{K\text{-orbits in } \psi^{-1}(G_0; \beta))\} = \text{codim}_{g_1^*}(G_0; \beta) + \text{ind}(g_{0, \beta}).$$

If $\beta \in (g_1^*)_{\text{reg}}$, then $\text{ind}(g_{0, \beta}) = \text{rk} g - \text{rk}(g, g_0)$ (see Proposition 2.5) and $\text{codim}_{g_1^*}(G_0; \beta) = \text{rk}(g, g_0)$. Consequently,

$$\text{if } \beta \in (g_1^*)_{\text{reg}}, \text{ then } \psi^{-1}(G_0; \beta) = g_0^* \times G_0; \beta \text{ contains } K\text{-regular elements.}$$

Consider the Luna stratification of the quotient variety $g_1^*//G_0$, see [8, III.2]. By definition, $\bar{\xi}, \bar{\xi}' \in g_1^*/G_0$ belong to the same stratum, if the closed $G_0$-orbits in $\pi^{-1}(\bar{\xi})$ and $\pi^{-1}(\bar{\xi}')$ are isomorphic as $G_0$-varieties. Each stratum is locally closed, and there are finitely many of them. (An exposition of Luna’s theory can also be found in [22].) Write $\bar{\Omega}_i$ for the union of all strata of codimension $i$. In particular, $\bar{\Omega}_0$ is the unique open stratum.

Set $\Omega_i = \pi^{-1}(\bar{\Omega}_i)$ and $\Xi_i = \psi^{-1}(\Omega_i) = g_0^* \times \Omega_i$. Since both $\pi$ and $\psi$ are equidimensional, $\text{codim}_\mathfrak{t} \Xi_i = \text{codim}_{g_1^*} \Omega_i = i$. Therefore, $\Xi_0 \cup \Xi_1 \cup \Xi_2$ has the complement of codimension $\geq 3$ in $\mathfrak{t}^*$, $\Omega_0 \cup \Omega_1 \cup \Omega_2$ has the complement of codimension $\geq 3$ in $g_1^*$, and we may not care about the strata of codimension $\geq 3$. Our ultimate goal is to characterise the symmetric pairs such that $(\Xi_0 \cup \Xi_1 \cup \Xi_2) \cap \mathfrak{t}^*_{\text{reg}}$ still has the complement of codimension $\geq 3$ in $\mathfrak{t}^*$. More precisely, we are going to find out whether $(\Xi_i)_{\text{sg}} := \Xi_i \setminus (\Xi_i \cap \mathfrak{t}^*_{\text{reg}})$ is of codimension $\geq (3 - i)$ in $\Xi_i$. It appears to be that for $i = 0, 2$, this condition is satisfied for all $\mathbb{Z}_2$-contractions, and non-trivial constraints occur only for $i = 1$.

$(\Xi_0)$-case. If $\bar{\xi} \in \bar{\Omega}_0$, then $\pi^{-1}(\bar{\xi}) = G_0; \xi$ (a sole closed and $G_0$-regular orbit!). Here $g_{0, \xi}$ is reductive and

$$\Xi_0 = \bigsqcup_{\xi \in \bar{\Omega}_0} (g_0^* \times G_0; \xi).$$

It follows from Eq. (4.1) that $(\alpha, \xi) \in \mathfrak{t}^*_{\text{reg}}$ if and only if $\hat{\alpha}$ is $g_{0, \xi}$-regular. Since $g_{0, \xi}$ has the codim–3 property (see Example 1.1), we conclude that $(\Xi_0)_{\text{sg}}$ is of codimension $\geq 3$. 
(\Xi_1)-case. If \( \xi \in \tilde{\Omega}_1 \), then \( \pi^{-1}(\xi) \) is not a sole orbit. Below, we use the notation that
- \( \xi \in \pi^{-1}(\xi) \) is semisimple and, without loss of generality, we assume that \( \xi \in \mathfrak{c} \);
- \( \zeta \in \pi^{-1}(\xi) \) is \( G_0 \)-regular and hence \( G_0 \cdot \zeta \) is open in \( \pi^{-1}(\xi) \);
- \( \pi^{-1}(\xi)_{sg} \) is the complement of set of \( G_0 \)-regular elements of \( \pi^{-1}(\xi) \).

Let \( \Omega_1^{(j)} \) be a Luna stratum of codimension 1 and \( \Omega_1^{(j)} := \pi^{-1}(\Omega_1^{(j)}) \), \( \Xi_1^{(j)} := \psi^{-1}(\Omega_1^{(j)}) \) the corresponding strata in \( g_1^* \) and \( \mathfrak{t}^* \). Then
\[
\Xi_1^{(j)} = g_0^* \times \Omega_1^{(j)} = \bigcup_{\xi \in \Omega_1^{(j)}} (g_0^* \times \pi^{-1}(\xi)) = \\
\left( \bigcup_{\zeta} (g_0^* \times G_0 \cdot \zeta) \right) \cup \left( \bigcup_{\xi \in \Omega_1^{(j)}} (g_0^* \times \pi^{-1}(\xi)_{sg}) \right) =: \mathcal{Y}^{(j)} \cup \mathcal{Z}^{(j)},
\]
where \( \zeta \) ranges over the set of representative of all \( G_0 \)-regular orbits in \( \Omega_1^{(j)} \).

The required information on \( \mathcal{Y}^{(j)} \) and \( \mathcal{Z}^{(j)} \) will be extracted from the coadjoint representation of \( g_{0,\xi} \) and the Satake diagram associated with the closed orbits in \( \Omega_1^{(j)} \), respectively. For \( \xi \in \Omega_1^{(j)} \cap \mathfrak{c} \), the corresponding reduced sub-symmetric pair is of rank 1. As in Section 2, we consider \( \mathfrak{h} = [\mathfrak{g}_0, \mathfrak{g}_0] = \mathfrak{h}_0 \oplus \mathfrak{h}_1 \) and the action \( (H_0 : N(\mathfrak{h}_1)) \).

- By Eq. (4.1) and (4.3), \( \mathcal{Y}^{(j)} \) contains \( K \)-regular elements and the dimension of their complement is determined by the coadjoint representation of \( g_{0,\xi} \). Namely, if \( g_{0,\xi} \) has the \( \text{codim} - n \) property, then \( \text{codim}_{\mathfrak{t}^*} (\mathcal{Y}^{(j)} \setminus (\mathcal{Y}^{(j)} \cap \mathfrak{t}^*_{\text{reg}})) = n + 1 \). Hence we need the \( \text{codim} - 2 \) property for \( g_{0,\xi} \), i.e., for stabilisers of \( G_0 \)-regular elements in \( \Omega_1^{(j)} \).

- For \( \mathcal{Z}^{(j)} \), we have \( \text{codim}_{\mathfrak{t}^*} \mathcal{Z}^{(j)} = 1 + \text{codim} \pi^{-1}(\xi)_{sg} = 1 + \text{codim}_{\mathfrak{h}(\mathfrak{h}_1)} N(\mathfrak{h}_1)_{sg} \).
Hence \( \mathcal{Z}^{(j)} \) is irrelevant for the \( \text{codim} - 3 \) property whenever \( \text{codim}_{\mathfrak{h}(\mathfrak{h}_1)} N(\mathfrak{h}_1)_{sg} \geq 2 \). If \( \text{codim}_{\mathfrak{h}(\mathfrak{h}_1)} N(\mathfrak{h}_1)_{sg} = 1 \), then a more accurate analysis of \( \mathcal{Z}^{(j)} \) is needed.

Because \( \text{rk} (\mathfrak{h}, \mathfrak{h}_0) = 1 \) and \( \text{Sat}(\mathfrak{h}, \mathfrak{h}_0) \) is a sub-diagram of \( \text{Sat}(\mathfrak{g}, \mathfrak{g}_0) \) (Proposition 2.3), \( \text{Sat}(\mathfrak{h}, \mathfrak{h}_0) \) contains all black nodes from \( \text{Sat}(\mathfrak{g}, \mathfrak{g}_0) \) and either a unique white node or a unique pair of white nodes joined by an arrow. We consider all the possibilities in turn.

(1) \( \text{Sat}(\mathfrak{h}, \mathfrak{h}_0) \) contains a unique white node and this node is trivial in \( \text{Sat}(\mathfrak{g}, \mathfrak{g}_0) \).
In other words, \( \text{Sat}(\mathfrak{h}, \mathfrak{h}_0) \) is a disjoint union of one white node and the subdiagram of all black nodes. In this case, \( \mathfrak{h} = \mathfrak{s}(\mathfrak{t}) + [\mathfrak{r}, \mathfrak{r}] \) and \( [\mathfrak{r}, \mathfrak{r}] \subset \mathfrak{h}_0 \). That is, the \( \mathbb{Z}_2 \)-grading of \( \mathfrak{h} \) is determined by the unique non-trivial \( \mathbb{Z}_2 \)-grading of \( \mathfrak{s}(\mathfrak{t}) \). Here \( \dim \mathfrak{h}_1 = 2 \) and \( N(\mathfrak{h}_1) \) is the union of two lines in \( \mathfrak{h}_1 \) (the “coordinate cross”). Using (2.2), we obtain that \( G_0 \cdot \xi \) is of codimension 1 in \( \pi^{-1}(\xi) \) (and \( \pi^{-1}(\xi) \) also contains two \( G_0 \)-orbits of regular elements). Therefore, the union of all closed \( G_0 \)-orbits in \( \Omega_1^{(j)} \) yields a subvariety \( Z^{(j)} \), of codimension 1 in \( \Omega_1^{(j)} \), hence of codimension 2 in \( g_1^* \). Let us prove that \( Z^{(j)} = \psi^{-1}(Z^{(j)}) \) does not contain \( K \)-regular elements. Indeed, in this case \( g_{0,\xi} \) is reductive and \( \text{dim} g_{0,\xi} = \text{dim} \mathfrak{r} + 1 \). Hence \( g_{0,\xi} = \mathfrak{r} + \mathfrak{t}_1 \). It follows that \( \text{ind} (g_{0,\xi}) = \text{ind} \mathfrak{r} + 1 = \text{rk} \mathfrak{g} - \text{rk} (\mathfrak{g}, \mathfrak{g}_0) + 1 \). Now, using Eq. (4.2)
and the fact that \( \text{codim}_{g \mathfrak{t}} (G_0 \xi) = \text{rk} (g, g_0) + 1 \), we obtain that \( \dim \mathfrak{t} \eta = \text{rk} g + 2 = \text{ind} \mathfrak{t} + 2 \) for all \( \eta \in Z^{(j)} \). Thus, here \( \mathfrak{t} \) does not have the \( \text{codim} \)-3 property.

\textbf{(II)} \( \text{Sat}(h, h_0) \) has a unique white node which is adjacent to a black node. To realise the structure of \( \pi^{-1}(\xi) \simeq G_0 \times_{G_0,\xi} N(h_1) \), we may only consider the connected component of \( \text{Sat}(h, h_0) \) that contains the white node. That is, we look at \( N(h_1) \) for the \( \mathbb{Z}_2 \)-gradings of rank one of simple Lie algebras such that the Satake diagram has no arrows. The corresponding list consists of the following symmetric pairs \((h, h_0)\):

\[
I_1: (so_n, so_{n-1}), n \geq 5; \quad I_2: (sp_{2n}, sp_{2n-2} + sp_2), n \geq 2; \quad I_3: (F_4, so_9).
\]

[Note that the case \((I_1, n = 5)\) coincides with \((I_2, n = 2), \) and \((I_1, n = 6)\) is also equal to \((sl_4, sp_4)\).] For all three cases, we have \( \text{codim}_{N(h_1)} N(h_1)_{sg} \geq 2 \). Hence \( Z^{(j)} \) is irrelevant for the \( \text{codim} \)-3 property for \( \mathfrak{t} \) and, as explained above, we only have to verify the \( \text{codim} \)-2 property for \( \mathfrak{g}_{0,\zeta} \). Without loss of generality, we may assume that \( \zeta = \xi + y \), where \( y \in N(h_1) \) is \( H_0 \)-regular. Then \( \mathfrak{g}_{0,\zeta} = \mathfrak{g}_0 + h_{0,y} \), see Eq. (2.3), and it suffices to check the \( \text{codim} \)-2 property for \( h_{0,y} \). Because \( \text{rk} (h, h_0) = 1 \), we have \( \dim h_{1,y} = 1 \). Hence \( h_y = h_{0,y} \langle y \rangle \), and we can work with either \( h_y \) or \( h_{0,y} \) in the above cases \((I_1-I_3)\).

For \( h = sl_n \) or \( sp_{2n} \), all the centralisers \( h_v \) have the \( \text{codim} \)-2 property [15, Sect. 3], which covers cases \((I_1, n \leq 6)\) and \( I_2 \). For \((I_1, n \geq 7)\), an explicit description of centralisers shows that \( h_{0,y} \) is a \( \mathbb{Z}_2 \)-contraction of \( so_{n-2} \), and all \( \mathbb{Z}_2 \)-contractions have the \( \text{codim} \)-2 property. For \( I_3 \), we have \( h_{0,y} = g_2 \ltimes \mathbb{R}^7 \), which is a contraction of \( so_7 \), and the verification is straightforward.

Thus, the strata occurring in part (II) provide no obstacles for the \( \text{codim} \)-3 property.

\textbf{(III)} \( \text{Sat}(h, h_0) \) has a unique pair of nodes joined by an arrow. There are three possibilities for the connected component of \( \text{Sat}(h, h_0) \) containing these two white nodes.

\textbf{(III-a)} There are black nodes between the white ones. The corresponding connected component of \( \text{Sat}(h, h_0) \) looks like item 1) in Table 1 with \( k = 1 \) and \( n \geq 4 \). Here again \( \text{codim}_{N(h_1)} N(h_1)_{sg} \geq 2 \) and the argument goes through as in part (II) of the proof. An essential point is that \( h_{0,y} \) has the \( \text{codim} \)-2 property, because it is a \( \mathbb{Z}_2 \)-contraction of \( sl_{n-2} \).

\textbf{(III-b)} The two white nodes are not adjacent in the Dynkin diagram of \( g \), and there is no black nodes between them. Such a sub-diagram occurs only for the symmetric pair \((sl_{2n+1}, sl_n + sl_{n+1} + t_1) \) \((n \geq 2)\) and \( \text{Sat}(h, h_0) \) is just \( \circ - \circ \). Here \( h = sl_2 + sl_2 \) and \( h_0 \) is the diagonal in \( h \) (cf. Example 2.1). In this case, we have \( \dim N(h_1) = 2 \) and \( N(h_1)_{sg} = \{0\} \). Hence \( \text{codim}_{N(h_1)} N(h_1)_{sg} = 2 \). If \( y \in N(h_1) \setminus \{0\} \), then \( h_{0,y} \) is 1-dimensional and abelian. Therefore, \( h_{0,y} \) has the \( \text{codim} \)-2 property.
3.3 The two white nodes are adjacent in the Dynkin diagram, i.e., $\text{Sat}(\mathfrak{h}, \mathfrak{h}_0)$ contains a connected component $\mathfrak{g}_0^*$. Again, this means that $(\mathfrak{g}, \mathfrak{g}_0) = (\mathfrak{sl}_{2n+1}, \mathfrak{sl}_n + \mathfrak{sl}_{n+1} + t_1)$, with $\text{rk} (\mathfrak{g}, \mathfrak{g}_0) = n$, and there is no black nodes at all.

Here $\mathfrak{h} = \mathfrak{sl}_3$, $\mathfrak{h}_0 = \mathfrak{gl}_2$, and $\mathcal{N}(\mathfrak{h}_1)$ consists of four $H_0$-orbits of dimension $3, 2, 2, 0$. Hence $\text{codim}_{\mathcal{N}(\mathfrak{h}_1)}\mathcal{N}(\mathfrak{h}_1)_{sg} = 1$, $\text{codim}_{\pi^{-1}(\bar{\xi})}\pi^{-1}(\bar{\xi})_{sg} = 1$, and $\mathfrak{z}^{(j)}$ is of codimension 2 in $\mathfrak{t}^*$, which resembles the bad case of part (I). But, unlike that situation, here $\mathfrak{z}^{(j)}$ does contain $K$-regular elements. Recall that $\xi \in \pi^{-1}(\bar{\xi}) \cap c$, $\mathfrak{h} = [\mathfrak{g}_\xi, \mathfrak{g}_\xi]$, and $\mathfrak{z}$ is the centre of $\mathfrak{g}_\xi$. Since $\dim \mathcal{N}(\mathfrak{h}_1) = 3$, the orbit $G_0(\mathfrak{g}_\xi)$ is of dimension 3 in $\pi^{-1}(\bar{\xi})$. Therefore, $\mathfrak{g}_\xi = \mathfrak{sl}_3 + \mathfrak{t}_{2n-2}$ and $\mathfrak{g}_0,\xi = \mathfrak{gl}_2 + \mathfrak{t}_{n-1}$, i.e., $\mathfrak{z} = \mathfrak{t}_{2n-2}$ and $\mathfrak{z}_0 = \mathfrak{t}_{n-1}$. Let $\nu \in \mathcal{N}(\mathfrak{h}_1)$ belong to a two-dimensional $H_0$-orbit. Then $G_0(\mathfrak{g}_\xi + \nu)$ is of codimension 1 in $\pi^{-1}(\bar{\xi})$, i.e., $\text{codim}_{\mathfrak{g}_0} G_0(\mathfrak{g}_\xi + \nu) = n + 1$. Here $\mathfrak{h}_{0,\nu}$ is the 2-dimensional non-abelian Lie algebra, hence $\text{ind} \mathfrak{h}_{0,\nu} = 0$. Since $\mathfrak{g}_{0,\xi + \nu} = \mathfrak{z}_0 + \mathfrak{h}_{0,\nu} = \mathfrak{t}_{n-1} + \mathfrak{h}_{0,\nu}$, we have $\text{ind} \mathfrak{g}_{0,\xi + \nu} = (n - 1) + \text{ind} \mathfrak{h}_{0,\nu} = n - 1$. Using Eq. (4.2), we conclude that $\psi^{-1}(G_0(\mathfrak{g}_\xi + \nu))$ contains $K$-orbits of codimension $(n + 1) + (n - 1) = 2n = \text{rk} \mathfrak{g} = \text{ind} \mathfrak{t}$.

Finally, we notice that if $y$ belongs to the 3-dimensional $H_0$-orbit in $\mathcal{N}(\mathfrak{h}_1)$, then $\mathfrak{h}_{0,y}$ is 1-dimensional and abelian. Therefore, $\mathfrak{h}_{0,y}$ has the codim–2 property, which guarantees the “good behaviour” of $\mathfrak{z}^{(j)}$.

Thus, the strata occurring in part (III) provides no obstacles for the codim–3 property.

(III)-case. Here we only have to prove that $(\Xi_2)_{sg}$ has smaller dimension than $\Xi_2$. Since

$$\Xi_2 = \mathfrak{g}_0^* \times \Omega_2 = \bigsqcup_{\xi \in \Omega_2} (\mathfrak{g}_0^* \times \pi^{-1}(\bar{\xi}))$$

and each irreducible component of $\pi^{-1}(\bar{\xi})$ contains $G_0$-regular elements, we conclude using Eq. (4.3) that the set of $K$-regular elements is dense in $\Xi_2$. Thus, the codim 2 strata cause no harm with respect to the codim–3 property.

Thus, the codim–3 property for $\mathfrak{t}$ fails if and only if $\text{Sat}(\mathfrak{g}, \mathfrak{g}_0)$ has a trivial node.

Remark 4.4. The symmetric pair $(\mathfrak{g}, \mathfrak{g}_0) = (\mathfrak{sl}_{2n+1}, \mathfrak{sl}_n + \mathfrak{sl}_{n+1} + t_1)$ provides a curious and unique example such that the complement of the set of $G_0$-regular points in $\mathfrak{g}_0^*$ contains a component of codimension two in $\mathfrak{g}_0^*$, but nevertheless $\mathfrak{t}$ possesses the codim–3 property.

5. CONCLUDING REMARKS AND OPEN PROBLEMS

We have given a description of maximal commutative subalgebras of the Poisson algebra $S(\mathfrak{t}) = \mathbb{k}[\mathfrak{t}^*]$ in the following two cases:

1) If $(\mathfrak{g}, \mathfrak{g}_0)$ is $N$-regular, i.e., $\text{Sat}(\mathfrak{g}, \mathfrak{g}_0)$ contains no black nodes, then $S(\mathfrak{t})^{\mathfrak{g}_1} = \mathbb{k}[\mathfrak{t}^*]^{\mathfrak{g}_1}$ is a maximal commutative subalgebra (Theorem 3.3).

2) If $\text{Sat}(\mathfrak{g}, \mathfrak{g}_0)$ has no trivial nodes, then the argument shift method provides maximal commutative subalgebras $\mathcal{F}_\xi(\mathcal{Z}(\mathfrak{t}))$ for $\xi \in \mathfrak{t}_{reg}^*$ (Theorems 1.4(ii) and 4.1).
The list of remaining symmetric pairs with $g$ simple consists of the following items:

1. $(\mathfrak{so}_{4n}, \mathfrak{gl}_{2n}), n \geq 2$;
2. $(\mathfrak{so}_{n+m}, \mathfrak{so}_m \oplus \mathfrak{so}_n), n \geq m > 1, n - m \geq 3$;
3. $(\mathfrak{e}_7, \mathfrak{e}_6 \oplus \mathfrak{t}_1)$;
4. $(\mathfrak{e}_7, \mathfrak{so}_{12} \oplus \mathfrak{sl}_2)$;
5. $(\mathfrak{e}_8, \mathfrak{e}_7 \oplus \mathfrak{sl}_2)$.

For these symmetric pairs, no maximal commutative subalgebras of $S(\mathfrak{t})$ is known. Of course, $\mathcal{F}_\xi(\mathbb{Z}(\mathfrak{t}))$ is always of maximal dimension, since $\mathfrak{t}$ has the codim–2 property. But the maximality can fail; it does fail for the involutions of maximal rank and $\xi \in \mathfrak{g}_1^1 \cap \mathfrak{t}_{\text{reg}}^*$, see [18, Example 4.1].

On the other hand, there are symmetric pairs, where both above constructions apply. For $(\mathfrak{sl}_{2n+1}, \mathfrak{sl}_n + \mathfrak{sl}_{n+1} + \mathfrak{t}_1)$ and $(\mathfrak{h} + \mathfrak{h}, \Delta_0)$, the Satake diagram contains neither black nor trivial nodes. Here the maximal commutative subalgebras $\mathcal{F}_\xi(\mathbb{Z}(\mathfrak{t}))$ and $S(\mathfrak{t})^{\mathfrak{g}_1}$ are quite different. Indeed, both algebras are graded polynomial, but the degrees of free homogeneous generators differ considerably.

For any $\xi \in \mathfrak{g}^{\text{reg}}_1$, the maximal commutative subalgebra $\mathcal{F}_\xi(\mathbb{Z}(\mathfrak{g}))$ can be lifted to $\mathcal{U}(\mathfrak{g})$ [4, Theorem 3.14] (for the regular semisimple elements $\xi$, the result was earlier obtained in [20]).

**Problems.**

1. Is it possible to quantise (=lift to $\mathcal{U}(\mathfrak{t})$) commutative subalgebras of the form $\mathcal{F}_\xi(\mathbb{Z}(\mathfrak{t})), \xi \in \mathfrak{t}_{\text{reg}}^*$, for $\mathfrak{t} = \mathfrak{g}_0 \times \mathfrak{g}_1$?

2. Suppose that $\mathfrak{t}$ has the codim–3 property. Is it true that any maximal commutative subalgebra of maximal dimension in $S(\mathfrak{t})$ is necessarily polynomial?

It might also be interesting to have some structure results on maximal commutative subalgebra of maximal dimension for more general Lie algebras.

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