ON THURSTON’S PARAMETERIZATION OF 
\(\mathbb{C}P^1\)-STRUCTURES

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Abstract. Thurston related \(\mathbb{C}P^1\)-structures (complex projective structures) and equivariant pleated surfaces in the hyperbolic-three space \(\mathbb{H}^3\), in order to give a parameterization of the deformation space of \(\mathbb{C}P^1\)-structures. In this note, we summarize Thurston’s parametrization of \(\mathbb{C}P^1\)-structures, based on \([KT92]\) and \([KP94]\). We, in addition, give alternative proofs for the following well-known theorems on \(\mathbb{C}P^1\)-structures by means of pleated surfaces given by the parameterization. (1) Goldman’s Theorem on \(\mathbb{C}P^1\)-structures with quasi-Fuchsian holonomy. (2) The path lifting property of developing maps in domain of discontinuities in \(\mathbb{C}P^1\).

Contents

1. Introduction 1
2. \(\mathbb{C}P^1\)-structures on surfaces 3
3. Grafting 5
4. The construction of Thurston’s parameters 11
4.1. Measured laminations on hyperbolic surfaces to projective structures. 5
4.2. \(\mathbb{C}P^1\)-structures to measured laminations on hyperbolic surfaces 6
5. Goldman’s theorem on projective structure with Fuchsian holonomy 11
6. The path lifting property in the domain of discontinuity 12
References 13

1. INTRODUCTION

Let \(P\) be the space of all (marked) \(\mathbb{C}P^1\)-structures on a closed oriented surface \(S\) of genus at two (\(\geq 2\)). Thurston give the following parameterization of \(P\), using pleated surfaces in the hyperbolic three-space \(\mathbb{H}^3\).
Theorem A. (Thurston, [KT92] [KP94])

\[ P \cong \text{ML} \times T, \]

where \( \text{ML} \) is the space of measured laminations on \( S \) and \( T \) is the space of all (marked) hyperbolic structures on \( S \).

In §4, we outline this correspondence, in part, giving more details, following the work of Kulkarni and Pinkall [KP94].

A hyperbolic structure on \( S \) is in particular a \( \mathbb{C}P^1 \) structure, and its holonomy is a discrete and faithful representation into \( \text{PSL}(2, \mathbb{R}) \), called a Fuchsian representation. One holonomy representation of a \( \mathbb{C}P^1 \)-structure on \( S \) corresponds to countably many different \( \mathbb{C}P^1 \)-structures on \( S \). Indeed, there is an operation called \( 2\pi \)-grafting (or simply grafting) which transforms a \( \mathbb{C}P^1 \)-structure to a new \( \mathbb{C}P^1 \)-structure, preserving its holonomy representations. The following theorem of Goldman characterizes all \( \mathbb{C}P^1 \)-structures with fixed Fuchsian holonomy.

Theorem B ([Gol87]). Every \( \mathbb{C}P^1 \)-structure \( C \) on \( S \) with Fuchsian holonomy \( \rho \) is obtained by grafting the hyperbolic structure \( \tau \) along a unique multiloop \( M \).

Goldman actually proved the theorem for more general quasi-Fuchsian groups, although the proof is immediately reduced to the case of Fuchsian representations by a quasiconformal map if \( \mathbb{C}P^1 \). Let \( C \) be a \( \mathbb{C}P^1 \)-structures with Fuchsian holonomy \( \pi_1(S) \to \text{PSL}(2, \mathbb{C}) \). Then, by Theorem B, \( C \) corresponds to \((\tau, M)\), where \( \tau \) is the hyperbolic structure \( \mathbb{H}^2 / \text{Im}\rho \) and each loop of \( M \) has a \( 2\pi \)-multiple weight.

For a subgroup \( \Gamma \subset \text{PSL}(2, \mathbb{C}) \), the limit set of \( \Gamma \) is the set of the accumulation points of a \( \Gamma \)-orbit in \( \mathbb{C}P^1 \). In §5 we give an alternative proof of Theorem B directory using pleated surfaces given by Thurston parameters.

The following Theorem is a technical part of the proof of Theorem B which was originally missing.

Theorem C ([CL97], see also §14.4.1. in [Gol]). Let \((f, \rho)\) be a developing pair of a \( \mathbb{C}P^1 \)-structure on \( S \). Let \( \Gamma \) be the limit set of \( \text{Im}\rho \). Then, for each connected component \( U \) of \( f^{-1}(\Omega) \), the restriction of \( f \) to \( U \) is a covering map onto its image.

Note that, as developing maps are local homeomorphisms, Theorem C is equivalent to saying that \( f \) is has the path lifting property in the domain of discontinuity of \( \text{Im}\rho \).

We also give an alternative proof of Theorem C in §6 using Thurston’s parametrization.
Theorem \[\text{given two } \mathbb{C}P^1\text{-structures } C_1 \text{ and } C_2 \text{ with Fuchsian holonomy, } C_1 \text{ can be transformed to } C_2, \text{ via the hyperbolic structure, by a composition of an inverse-grafting and a grafting (where an inverse grafting is the opposite of grafting which remove a cylinder for } 2\pi\text{-grafting). The following question due to Gallo, Kapovich, and Marden remains open.}

**Conjecture 1.1** (§12.1 in [GKM00]). *Give two \( \mathbb{C}P^1 \)-structures \( C_1, C_2 \) on \( S \) with fixed holonomy \( \pi_1(S) \to \text{PSL}(2, \mathbb{C}) \), there is a composition of grafts and inverses of grafts which transforms \( C_1 \) to \( C_2 \).*

Although they stated it in the form of a question, we would like to state more positively since it has been solved affirmatively for genetic type of holonomy representations, namely, for purely loxodromic representations [Bab15, Bab17]. (For Schottky representations, see [Bab12].) There is also a version of this question for branched \( \mathbb{C}P^1 \)-structures (Problem 12.1:2 in [GKM00]); see [CDF14] [Ruf] for some progress in the case of branched \( \mathbb{C}P^1 \)-structures.

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2. \( \mathbb{C}P^1 \)-structures on surfaces

(General references for \( \mathbb{C}P^1 \)-structures are, for example, [Dum09, Kap01].)

A \( \mathbb{C}P^1 \)-structure on \( S \) is a \( (\mathbb{C}P^1, \text{PSL}(2, \mathbb{C})) \)-structure, i.e. a maximal atlas of chars embeddings open subsets of \( S \) onto open subsets of \( \mathbb{C}P^1 \) such that their transition maps are in \( \text{PSL}(2, \mathbb{C}) \). Let \( \tilde{S} \) be the universal cover \( \tilde{C} \) of \( S \), which is topologically an open disk. Equivalently, a \( \mathbb{C}P^1 \)-structure on \( S \) is defined as a pair \((f, \rho)\) of

- a local homeomorphism \( f: \tilde{S} \to \mathbb{C}P^1 \) (developing map) and
- a homomorphism \( \rho: \pi_1(S) \to \text{PSL}(2, \mathbb{C}) \) (holonomy representation)

such that \( f \) is \( \rho \)-equivariant (i.e. \( f\alpha = \rho(\alpha)f \) for all \( \gamma \in \pi_1(S) \)). This pair \( (f, \rho) \) is called the developing pair of \( \tilde{C} \), and \( (f, \rho) \) is, by definition, equivalent to \( (\gamma f, \gamma \rho\gamma^{-1}) \) for all \( \gamma \in \text{PSL}(2, \mathbb{C}) \). Due to the equivariant condition, we do not usually need to distinguish an element of \( \pi_1(S) \) and its free homotopy class. Then let \( \mathcal{P} \) be the deformation space of all \( \mathbb{C}P^1 \)-structures on \( S \); then \( \mathcal{P} \) has a natural topology, given by the open-compact topology on developing maps \( f: \tilde{S} \to \mathbb{C}P^1 \).
Notice that hyperbolic structures are, in particular, \( \mathbb{CP}^1 \)-structures, as \( \mathbb{H}^2 \) is the upper half plane in \( \mathbb{C} \) and the orientation-preserving isometry group \( \text{Isom} \mathbb{H}^2 \) is \( \text{PSL}(2, \mathbb{R}) \) in \( \text{PSL}(2, \mathbb{C}) \).

3. Grafting

A grafting is a cut-and-paste operation of a \( \mathbb{CP}^1 \)-structure inserting some structure along a loop, an arc or more generally a lamination, originally due to [STS83, Hej75, Mas69]. There are a slightly different versions of grafting, yet they all yield new \( \mathbb{CP}^1 \)-structures without changing the topological types of the base surfaces.

A \textit{round circle} in \( \mathbb{CP}^1 = \mathbb{C} \cup \{ \infty \} \) is a round circle in \( \mathbb{C} \) or a straight line in \( \mathbb{C} \) plus \( \infty \). A \textit{round disk} in \( \mathbb{CP}^1 \) is a disk bounded by a round circle. An arc \( \alpha \) on a \( \mathbb{CP}^1 \)-structure is \textit{circular} if \( \alpha \) immerses into a round circle on \( \mathbb{CP}^1 \) by the developing map. Similarly, a loop \( \alpha \) on a \( \mathbb{CP}^1 \)-structure \( C \) is \textit{circular} if its lift \( \tilde{\alpha} \) to the universal cover immerses into a circular arc \( \mathbb{CP}^1 \) by the developing map.

We first define a grafting along a circular arc. For \( \theta > 0 \), consider the horizontal biinfinite strip \( \mathbb{R} \times [0, \theta i] \) in \( \mathbb{C} \) of height \( \theta \). Then \( \mathbb{R} \theta \) be the \( \mathbb{CP}^1 \)-structure on the strip whose developing map is the restriction of the exponential map \( \exp: \mathbb{C} \to \mathbb{C} \setminus \{0\} \). This \( \mathbb{CP}^1 \)-structure is called the \textit{crescent} of angle \( \theta \) or simply \( \theta \)-crescent.

Let \( \ell \) be a (biinfinite) circular arc property embedded in a \( \mathbb{CP}^1 \)-surface \( C \). Then the \textit{grafting} of \( C \) along \( \ell \) by \( \theta \) inserts \( \mathbb{R} \theta \) along \( \ell \) (\( \theta \)-grafting), to be precise, as follows: Notice that \( C \setminus \ell \) has two boundary components isomorphic to \( \ell \). Then we take a union of \( C \setminus \ell \) and \( \mathbb{R} \times [0, \theta i] \) by an isomorphism between \( \partial(C \setminus \ell) \) and \( \partial(\mathbb{R} \times [0, \theta i]) \) so that there is “no shearing”, i.e. for each \( r \in \mathbb{R} \), the vertical arc \( r \times [0, \theta i] \) connects the points on of the different boundary components of \( C \setminus \ell \) corresponding to the same point of \( \ell \).

Let \( \ell \) be a circular loop on a projective surface \( C \). We can similarly define a grafting along \( \ell \) by grafting the universal cover \( \tilde{C} \) of \( C \) in an equivariant manner: Letting \( \phi: \tilde{C} \to C \) be the universal covering map, \( \phi^{-1}(\ell) \) is a union of disjoint circular arcs property embedded in \( \tilde{C} \) which is invariant under \( \pi_1(S) \).

Then, we insert a \( \theta \)-crescent along each arc of \( \phi^{-1}(\ell) \) as above. Then, by quotienting out the resulting structure by \( \pi_1(S) \), we obtain a new \( \mathbb{CP}^1 \)-structure homeomorphic to \( C \), since a cylinder is inserted to \( C \) along \( \ell \). Indeed, the stabilizer of an arc \( \tilde{\ell} \) of \( \phi^{-1}(\ell) \) is an infinite cyclic group generated by an element \( \gamma \in \pi_1(S) \) whose free homotopy class is \( \ell \), and the cyclic group \( \langle \gamma \rangle \) acts on \( \mathbb{R} \theta \) so that the quotient is the inserted cylinder (grafting cylinder of height \( \theta \)).
Note that $R_\theta$ is foliated by horizontal lines $\mathbb{R} \times \{y\}$, $y \in [0, \theta]$, it has a natural transversal measure given by the difference of the second coordinates. This measured foliation descends to a measured foliation on the grafting cylinder. In addition, there is a natural projection $R_\theta \to \mathbb{R}$ to the first coordinate (collapsing map). Then this projection descends to a collapsing map of a grafting cylinder to a circle.

Let $\text{Gr}_{\ell, \theta}(C)$ denote the resulting $\mathbb{C}P^1$-structure homeomorphic to $C$. Notice that, the holonomy along the circular loop $\ell$ is hyperbolic, as it has exactly two fixed points on $\mathbb{C}P^1$ which are the endpoints of the developments of $\ell$.

In the case that $\theta$ is an integer multiple of $2\pi$, the holonomy $C$ does not change by the $\theta$-grafting, since the developing map does not change in $\phi^{-1}(C \setminus \ell)$. In particular, the $2\pi$-grafting along a circular loop $\ell$ inserts a copy of $\mathbb{C}P^1$-minus a circular arc along each lift of $\ell$.

In fact, a $2\pi$-grafting is still well-defined along a more general loop. A loop $\ell$ on $C = (f, \rho)$ is admissible if $\rho(\gamma)$ is hyperbolic and a (or every) lift $\tilde{\ell}$ of $\ell$ embeds into $\mathbb{C}P^1$ by $f$. Then, we insert a copy of $\mathbb{C}P^1 \setminus (f(\tilde{\ell}) \cup \text{Fix}(\rho(\gamma)))$ along $\tilde{\ell}$, where $\text{Fix}(\rho(\gamma))$ denotes the fixed points of $\rho(\gamma)$. Note that, the quotient of $\mathbb{C}P^1 \setminus \text{Fix}(\rho(\gamma))$ by the infinite cyclic group generated by $\rho(\gamma)$ is a projective structure $T$ on a torus, and the development $f(\tilde{\ell})$ covers to a simple loop on $T$ isomorphic to $\ell$; by abuse of notation, by $\ell$, we also denote the loop on $T$. Then the $2\pi$-grafting of $C$ along $\ell$ is given by identifying the boundary loops of $C \setminus \ell$ and $T \setminus \ell$ by the isomorphism. Let $\text{Gr}_{\ell}(C)$ denote the $2\pi$-grafting of $C$ along an admissible loop $\ell$.

A multiloop is a union of (locally) finite disjoint simple closed curves. Note that if there is a multiloop $M$ on a projective surface consisting of admissible loops, then a grafting can be done along $M$ simultaneously.

4. The construction of Thurston’s parameters

In this section, we explain the correspondence stated in Theorem A in both directions, following [KP94].

4.1. Measured laminations on hyperbolic surfaces to projective structures. Let $(\tau, L) \in T \times \text{ML}$, where $\tau$ is a hyperbolic structure on $S$, and $L$ is a measured geodesic lamination on $\tau$. Then $(\tau, L)$ corresponds to the $\mathbb{C}P^1$-structure on $S$ obtained by grafting $\tau$ along $L$ as follows.

Suppose first that $L$ consists of periodic leaves. Then, for each leaf $\ell$ of $L$, letting $w$ be its weight, we insert a grafting cylinder of height $w$, and obtain a projective structure $C = (f, \rho)$ on $S$. Let $\tilde{L}$ be the pull back of $L$ by the universal covering map. Then, moreover, there
is \( \rho \)-equivariant pleated surface \( \beta : \mathbb{H}^2 \to \mathbb{H}^3 \), obtained by bending \( \mathbb{H}^2 \) along \( \tilde{L} \) by the angles given by the weights.

Let \( \kappa : C \to \tau \) be the collapsing map obtained by collapsing all grafting cylinders in \( C \) in \( \mathbb{H}^3 \). Then \( \beta \circ \tilde{\kappa} = f \), where \( \tilde{\kappa} : \tilde{C} \to \mathbb{H}^2 \) be the lift of \( \kappa \).

Suppose next that \( L \) contains an irrational sublamination. Then, pick a sequence of measured laminations \( L_i \) consisting of closed leaves, such that \( L_i \) converges to \( L \) as \( i \to \infty \). Then, for each \( i \), as above there is a \( \mathbb{CP}^1 \)-structure \( C_i = \text{Gr}_{L_i}(\tau) \) and a \( \rho_i \)-equivariant pleated surface \( \beta_i : \mathbb{H}^2 \to \mathbb{H}^3 \). As \( L_i \) converges to \( L \), then \( \beta_i \) converges to a pleated surface \( \beta : \mathbb{H}^2 \to \mathbb{H}^3 \) uniformly on compact, and therefore \( C_i \) converges to a \( \mathbb{CP}^1 \)-structure on \( S \). (See [CEG87].)

4.2. \( \mathbb{CP}^1 \)-structures to measured laminations on hyperbolic surfaces. Let \( C = (f, \rho) \) be a projective structure on \( S \) given by a developing pair. Let \( \tilde{C} \) be the universal cover of \( C \).

Identify \( \mathbb{CP}^1 \) conformally with a unite sphere \( S^2 \) in \( \mathbb{R}^3 \). Then, each round circle on \( \mathbb{CP}^1 \) is the intersection of \( S^2 \) with some (affine) hyperplane \( \mathbb{R}^2 \) in \( \mathbb{R}^3 \). A (open) round disk \( D \) in \( \tilde{C} \) is an open subset of \( \tilde{C} \) homeomorphic to an open disk, such that \( f \) embeds \( D \) onto an open round disk in \( \mathbb{CP}^1 \). A maximal disk \( D \) in \( \tilde{C} \) is a round disk, such that there is no round disk in \( \tilde{C} \) strictly containing \( D \). Let \( D \) be a maximal disk in \( \tilde{C} \). Then the closure of its image, \( f(D) \), is a closed round disk in \( \mathbb{CP}^1 \).

We first see a basic example illustrating the pleated surface corresponding a \( \mathbb{CP}^1 \)-structure. Let \( U \) be a region of \( \mathbb{CP}^1 \) homeomorphic to an open disk such that \( \mathbb{CP}^1 \setminus U \) contains more than one point (i.e. \( U \not\equiv \mathbb{CP}^1, \mathbb{C} \)). Regard \( \mathbb{CP}^1 \) as the ideal boundary of hyperbolic three space \( \mathbb{H}^3 \). Then consider the convex full of \( \mathbb{CP}^1 \setminus U \) in \( \mathbb{H}^3 \). Then its boundary in \( \mathbb{H}^3 \) is a hyperbolic plane \( \mathbb{H}^2 \) bent along a measured lamination \( L_U \) [EM87]. This lamination corresponds to the lamination in the Thurston coordinates.

There is an orthogonal projection \( \Psi_U \) from \( U \) to \( \partial \text{Conv}(\mathbb{CP}^1 \setminus U) \), which yields to a continuous map from \( U \) to \( \mathbb{H}^2 \). A stratum of \( (\mathbb{H}^2, L) \) is either leaf with the closure of the complementary region, a leaf atomic measure, a leaf not contained in the boundary of a complementary region. Then, for each stratum \( \sigma \) of \( (\mathbb{H}^2, L_U) \), there is a maximal disk \( D \) in \( U \) such that, letting \( H \) be the hyperbolic plane in \( \mathbb{H}^3 \) bounded by \( \partial D \), \( H \) intersects \( \partial \text{Conv}(\mathbb{CP}^1 \setminus U) \) in \( \sigma \). The transversal measure of \( L_U \) is, infinitesimally, given by the angles between such hyperbolic planes.
Moreover there is a natural measured lamination \( L_U \) on \( U \) which maps to \( L_U \) by \( \Psi_U \). If a leaf \( \ell \) has a positive atomic measure \( w > 0 \), then \( \Psi_U^{-1}(\ell) \) is a crescent region \( R_w \) of angle \( w \), and \( R_w \) is foliated by circular arcs \( \ell' \) which project to \( \ell \). Then \( \Psi_U \) takes \( L_U \) to \( L_U \) isomorphically except that such collapsing of foliated crescent regions to leaves with positive atomic measured. The transversal measure of \( L \) is given by infinitesimal angles between "very close" maximal disks.

As developing maps of \( \mathbb{CP}^1 \)-structures are, in general, not embedding, we need to find such projections somewhat more "locally" using maximal disks.

Let \( D \) be a maximal disk in the universal cover \( \tilde{C} \). Then, let \( \overline{D} \) be the closure of \( D \) in \( \tilde{C} \). In other words, \( \overline{D} \) is the connected component of \( f^{-1}(\overline{f(D)}) \) containing \( D \). Then \( \overline{f(D)} \setminus f(D) \) is a subset of the boundary circle of the round disk \( f(D) \), and the points in this subset are called the ideal points of \( D \). (Given a point \( p \) of the boundary circle \( f(D) \), pick a path \( \alpha : [0, 1) \to f(D) \) limiting to \( p \) as the parameter goes to 1. Then \( p \) is a ideal point of \( D \) if and only if the lift of \( \alpha \) to \( \tilde{C} \) leaves every compact subset of \( \tilde{C} \).)

Let \( \partial_\infty D \subset \mathbb{CP}^1 \) denote the set of all ideal points of \( D \). As \( f|D \) is an embedding onto a round disk, we regard \( \partial_\infty D \) as a subset of the boundary circle of \( D \) abstractly (not as a subset of \( \mathbb{CP}^1 \)). Then \( \partial_\infty D \) is a closed subset of \( S^1 \), since its complement is open. Identifying \( D \) with a hyperbolic disk conformally, we let \( \text{Core}(D) \) be the convex hull of \( \partial_\infty D \).

For each point \( p \) of \( \tilde{C} \), there is a round disk containing \( p \), and moreover, as \( C \) is not \( \mathbb{CP}^1 \) or \( \mathbb{C} \), there is a maximal disk containing \( p \). The canonical neighborhood \( U_p \) of \( C \) is the union of all maximal disks \( D_j \) (\( j \in J \)) in \( \tilde{C} \) which contains \( p \).

The following proposition give an insight for extending the above construction of the pleated surface and such for \( U \) embedded in \( \mathbb{CP}^1 \) naturally generalizes to a general developing map.

**Proposition 4.1** ([KP94], Proposition 4.1). For every point \( p \) in \( \tilde{C} \), \( f : \tilde{S} \to \mathbb{CP}^1 \) embeds its canonical neighborhood \( U_p \) into \( \mathbb{CP}^1 \). Moreover \( U_p \) is homeomorphic to an open disk.

**Proof.** Set \( U_p = \cup D_j \), where \( D_j \) are maximal disks in \( \tilde{C} \) containing \( p \). Let \( x, y \) be distinct points in \( U_p \); let \( D_x \) and \( D_y \) be maximal disks containing \( \{ p, x \} \) and \( \{ p, y \} \), respectively. Then \( f \) is injective on \( D_x \cup D_y \), and \( f(x) \neq f(y) \). Therefore \( f \) embeds \( U_p \).

The image \( f(U_p) \) is not surjective (as \( S \) is not homomorphic to a sphere). Thus we can normalize \( \mathbb{CP}^1 = \mathbb{C} \cup \{ \infty \} \) so that \( f(p) = 0 \) and
$f(U_p)$ does not contain $\infty$. Then $f(U_p) = \bigcup f(D_j)$ is a union of round open disks containing 0 in $\mathbb{C}$. Therefore it is a star-shaped region, and thus $U_p$ is homeomorphic to an open disk.

**Lemma 4.2.** Every maximal disk in $U_p$ is a maximal disk in $\tilde{\mathcal{C}}$. Every maximal disk $D$ in $U_p$ is also a maximal disk in $\tilde{\mathcal{C}}$.

Moreover, if a maximal disk $D$ in $U_p$ contains $p$, then the ideal points of $D$ in $U_p$ coincides with its ideal points in $\tilde{\mathcal{C}}$.

**Proof.** If $D$ is an (open) maximal disk in $U_p$, then its closure $\overline{D}$ contains $p$.

Suppose first that the boundary circle of $D$ contains $p$. Then, as $D$ is maximal, the boundary of $D$ must contain at least two ideal points of $\tilde{\mathcal{C}}$. Thus $D$ is also maximal in $\tilde{\mathcal{C}}$.

Suppose next that (the interior of) $D$ contains $p$. Then, if $D$ is not maximal in $\tilde{\mathcal{C}}$, then there is a round disk $D'$ in $\tilde{\mathcal{C}}_p$ which strictly contains $D$. Then $U_p$ contains $D'$. This is a contradiction, as $D$ is maximal in $U_p$.

Suppose that $D$ contains $p$ and there is an ideal point $q$ of $D$ in $U_p$ which is not an ideal point of $\tilde{\mathcal{C}}$. Then there is a round disk $D'$ in $\tilde{\mathcal{C}}$ which contains $p$ and $q$. Then as $D'$ is contained in $U_p$, this is a contradiction against the assumption that $q$ is an ideal point of $D$ in $U_p$. $\square$

The following proposition yields a lamination on $\tilde{\mathcal{C}}$ invariant under $\pi_1(S)$.

**Proposition 4.3** ([KP94], Theorem 4.4). The cores $\text{Core}(D)$ of all maximal disks $D$ in $\tilde{\mathcal{C}}$ are all disjoint and their union is $\tilde{\mathcal{C}}$.

**Proof.** We first show that the cores are disjoint. Let $D_1$ and $D_2$ be distinct maximal disks in $\tilde{\mathcal{C}}$. If $D_1 \cap D_2 \neq \emptyset$, then $f(D_1)$ and $f(D_2)$ are round disks intersecting a crescent. Therefore $\text{Core}(D_1)$ and $\text{Core}(D_2)$ are disjoint. (Consider the circular arc in $D_1$ orthogonal to $\partial D_1$; then, indeed, this arc separates $\text{Core}(D_1)$ and $\text{Core}(D_2)$ in $D_1 \cup D_2$.)

**Claim 4.4.** Given a convex subset $V$ of $\mathbb{C}$, there is a unique round disk $D$ in $\mathbb{C}$ of minimal radius containing $V$.

**Proof.** Suppose, to the contrary, that there are two different round disks $D_1, D_2$ containing $V$ which attain the minimal radius. Then, clearly, there is a round disk $D_3$ of strictly smaller radius which contains $V$ (such that $D_3 \supset D_1 \cap D_2$ and $D_3 \subset D_1 \cup D_2$). This is a contradiction. $\square$
Claim 4.5. The convex hull of $\partial D \cap V$ contains the center of $c$, where $D$ is conformally identified with $\mathbb{H}^2$.

Proof. Suppose not; then the closure of $V$ is contained in the interior of a (Euclidean) half disk of $D$. Then one can easily find a round disk of smaller radius containing $V$. \hfill $\square$

Note that, by the inversion of $\mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$ about $\partial D$ which exchanges the center of $D$ and $\infty$.

Using the above claims, we show that, for every $x \in \tilde{C}$, there is a maximal disk $D$ in $\tilde{C}$ whose core contains $x$. Let $U_x = \cup_{j \in J} D_j$ be the canonical neighborhood of $x$, where $D_j$ are the maximal disks in $\tilde{C}$ which contain $x$. Normalize $\mathbb{CP}^1$ so that $f(x) = \infty$. Let $D^c_j = \mathbb{CP}^1 \setminus f(D_j)$. Then $\mathbb{CP}^1 \setminus f(U_x) = \cap_j D^c_j$. By Lemma 4.2, all maximal disks in $U_x$ are maximal disks in $\tilde{C}$. Let $D$ be the maximal disk in $U_x$, by Lemma 4.3 such that $D^c$ has the minimal radius in $\mathbb{C}$. Then, as $D$ contains $x$, by Lemma 4.2, the ideal points of $D$ in $U_x$ coincides with those in $\tilde{C}$. Then, Core$(D)$ contains $\infty = x$ by Claim 4.5. \hfill ERS

Proposition 4.3, $\tilde{C}$ decomposes into the cores of maximal disks in $\tilde{C}$, which yields a stratification of $\tilde{C}$. Note that this decomposition is invariant under $\pi_1(S)$, as the maximal balls and ideal points are preserved by the action. Moreover, for each maximal disk $D$ in $\tilde{C}$, its Core$(D)$ is properly embedded in $\tilde{C}$. Then the one-dimensional cores and the boundaries components of two-dimensional cores yield a lamination $\tilde{\lambda}$ on $\tilde{C}$, and it descends to a lamination $\lambda$ on $C$.

Next we see that the angles between infinitesimally close maximal disks yield a natural transversal measure of this lamination. Given a point $x \in \tilde{C}$, let $D_x$ be the maximal disk in $\tilde{C}$ whose core contains $x$. If $y \in \tilde{C}$ is sufficiently close to $x$, then $D_y$ intersects $D_x$. Then let $\angle(D_x, D_y)$ denote the angle between the boundary circles of $D_x$ and $D_y$, to be precise the angles of the crescents $D_x \setminus D_y$ and $D_y \setminus D_x$ at the vertices. Then $\angle(D_x, D_y) \to 0$ as $y \to x$.

Let $x$ and $y$ be distinct points in $\tilde{C}$, contained in different strata of $(\tilde{C}, \tilde{\lambda})$. Then pick a path $\alpha : [0, 1] \to \tilde{C}$ connecting $x$ to $y$ such that $\alpha$ is transversal to $\tilde{\lambda}$. Let $\Delta : 0 = t_0 < t_1 < \cdots < t_n = 1$ be a finite division of $[0, 1]$, and let $x_i = \alpha(t_i)$ for each $i = 0, \ldots, n$. Let $|\Delta| = \min_{i=0}^{n-1} (x_{i+1} - x_i)$, the smallest width of the subintervals. Then, let $\Theta(\Delta) = \Sigma_{i=1}^{n-1} \angle(D_{x_i}, D_{x_{i+1}})$ for a subdivision $\Delta$ of $[0, 1]$ with sufficiently small $|\Delta|$. Pick a sequence of subdivisions $\Delta_i$ such that $|\Delta_i| \to 0$ as $i \to \infty$. Then $\lim_{i \to \infty}(\Theta(\Delta_i))$ exists and it is independent on the choice of $\Delta_i$ as $i \to \infty$ (CEG87, II.1)). Then we define the transversal measure
of $\alpha$ to be $\lim_{i \to \infty} (\Theta(\Delta_i))$. Then $\tilde{\lambda}$ with this transversal measure yields a measured lamination $\tilde{\mathcal{L}}$ invariant under $\pi_1(S)$. Thus $\tilde{\mathcal{L}}$ descends to a measured lamination $\mathcal{L}$ on $\tilde{C}$.

By Lemma 1.2, for every $x \in \tilde{C}$, the measured lamination $\mathcal{L}$ near $x$ is determined by the canonical neighborhood $U_x$ of $x$. Let $\mathcal{L}_x$ be the measured lamination on $U_x$, which descends to the measured lamination on the boundary of $\operatorname{Conv}(\mathbb{CP}^1 \setminus U_x)$. Then there is a neighborhood $V$ of $x$ in $U_x$ such that $\mathcal{L}$ and $\mathcal{L}_x$ coincide in $V$ by the inclusion $U_x \subset \tilde{C}$.

For each point $x \in \tilde{C}$, the boundary circle of the maximal disk $D_x$ bounds a hyperbolic plane $H_x$ in $\mathbb{H}^3$. Let $\Psi_x: f(D_x) \to H_x$ be the projection along geodesics in $\mathbb{H}^3$ orthogonal to $H_x$. Then $H_x$ has a canonical normal direction pointing $D_x$. Then, by Lemma 1.2 there is a neighborhood $V$ of $x$ in $U_x$ such that $\Psi_y(y) = \Psi_x(y)$. Moreover, $\Psi_x$ coincides with the projection onto the boundary pleated surface of $\operatorname{Conv}(\mathbb{CP}^1 \setminus U_x)$. Therefore, as in the case of regions in $\mathbb{CP}^1$, we have a pleated surface $\mathbb{H}^2 \to \mathbb{H}^3$ which is $\rho$-equivariant, as in the following paragraph.

We assume that crescents $R$ in $\tilde{C}$ are always foliated by leaves of $\tilde{\mathcal{L}}$ sharing their endpoints at the vertices of $R$. We have a well-defined continuous map $\Psi: \tilde{C} \to \mathbb{H}^3$ defined by $\Psi(x) = \Psi_x(x)$. We shall take an appropriate quotient of $\tilde{C}$ to turn it to a hyperbolic plane. For each crescent $R$ in $\tilde{C}$, $\Psi$ takes each leaf in $R$ to the geodesic in $\mathbb{H}^3$ connecting the vertices of $R$. Identify $x, y \in \tilde{C}$, if $x, y$ are contained in a single crescent in $\tilde{C}$ and $\Psi_x(x) = \Psi_y(y)$; let $\tilde{k}: \tilde{C} \to \tilde{C}/\sim$ be the quotient map by this identification, which collapses each foliated crescent region to a single leaf. Then by the equivalence relation, $\Psi: \tilde{C} \to \mathbb{H}^3$ induces a continuous map $\beta: (\tilde{C}/\sim) \to \mathbb{H}^3$ such that $\Psi_x(x) = \beta \circ \kappa$. Moreover that, $\tilde{C}/\sim$ is $\mathbb{H}^2$ with respect to the path metric in $\mathbb{H}^3$ via $\Psi$, since, for every $x \in \tilde{C}$, $\Psi$ coincides with the projection $U_x \to \partial \operatorname{Conv}(\mathbb{CP}^1 \setminus U_x)$ in a neighborhood of $x$. Thus we have a $\rho$-equivariant pleated surface $\mathbb{H}^2 \to \mathbb{H}^3$.

The measured lamination $\tilde{\mathcal{L}}$ on $\tilde{C}$ descends to a measured lamination $\tilde{L}$ on $\mathbb{H}^2$ invariant under $\pi_1(S)$. By quotient out, we obtain a desired pair $(\tau, L)$ of a hyperbolic surface $\tau$ and a measured geodesic lamination $\tilde{L}$ on $\tau$.

Then a collapsing map $\tilde{k}: \tilde{C} \to \mathbb{H}^2$ descends to the collapsing map $\kappa: C \to \tau$. Then, for each periodic leaf $\ell$ of $L$, $\kappa^{-1}(\ell)$ is a grafting cylinder foliated by closed leaves of $\mathcal{L}$.

We last note that, as $\beta: \mathbb{H}^2 \to \mathbb{H}^3$ is the obtained by bending $\mathbb{H}^2$ in $\mathbb{H}^3$ along $\tilde{L}$, the pair $(\tau, L)$ corresponds to $C$ by the correspondence in §4.1.
5. Goldman’s theorem on projective structure with Fuchsian holonomy

Let $C$ be a $\mathbb{CP}^1$-structure on $S$ with holonomy $\rho$, and let $(\tau, L) \in \mathbb{T} \times \mathbb{ML}$ be its Thurston parameters. Let $\psi: \mathbb{H}^2 \to \tau$ be the universal covering map, and $\tilde{L}$ be the measured lamination $\psi^{-1}(L)$ on $\mathbb{H}^2$. Let $\Gamma = \text{Im} \rho$, and let $\Lambda$ denote the limit set of $\text{Im} \rho$.

Lemma 5.1. Let $\beta: \mathbb{H}^2 \to \mathbb{H}^3$ be the associated pleated surface, where $\mathbb{H}^2$ is the universal cover of $\tau$. Then, for every leaf $\tilde{\ell}$ of $\tilde{L}$, $\beta|\tilde{\ell}$ is the geodesic connecting different points of $\Lambda$.

Proof. If $\tilde{\ell}$ is a lift of a closed leaf of $L$, then the assertion clearly holds. For every closed curve $\alpha$ on $\tau$, let $\tilde{\alpha}$ be a lift of $\alpha$. Then $\beta|\tilde{\alpha}$ is a quasigeodesic in $\mathbb{H}^3$ whose endpoints are the fixed points of $\rho(\alpha)$, which is a subset of $\Lambda$. Let $\ell$ be a non-periodic leaf of $L$, and let $\tilde{\ell}$ be a lift of $\ell$ to $\mathbb{H}^2$.

There is a sequence of simple closed geodesics $\ell_i$ on $\tau$ such that $\ell_i$ converges to $\ell$ in the Hausdorff topology ([CEG87, I.4.2.14]). For each $i \in \mathbb{N}$, pick a lift $\tilde{\ell}_i$ of $\ell_i$ to $\mathbb{H}^2$ so that $\tilde{\ell}_i \to \ell$ uniformly on compact as $i \to \infty$. Then, $\beta|\tilde{\ell}_i$ converges to $\beta|\tilde{\ell}$ uniformly on compact. Moreover as $\angle_{\tau_i}(\tau_i, L_i) \to 0$, $\beta|\tilde{\ell}_i$ is asymptotically an isometric embedding: To be precise, for large enough $i$, it is a bilipschitz embedding, and it bilipschitz constant converges to 1 as $i \to \infty$ [Bab15, Proposition 4.1].

As $\ell_i$ are closed loops, the endpoints of $\beta|\tilde{\ell}_i$ are in $\Lambda$. Then as the endpoints of $\beta|\tilde{\ell}_i$ converges to the endpoints of $\beta|\tilde{\ell}$ in $\mathbb{CP}^1$, and $\Lambda$ is a closed subset of $\partial \mathbb{H}^3$, the endpoints of $\beta|\ell$ are also contained in $\Lambda$. \qed

We immediately have

Corollary 5.2. For each stratum $\sigma$ of $(\mathbb{H}^2, \tilde{L})$, let $D_\sigma \subset \tilde{C}$ be the maximal disk whose core corresponds to $\sigma$. Then its ideal points $\partial_\infty D_\sigma$ are contained in the limit set $\Lambda$.

We reprove the following theorem by means of pleated surfaces.

Proposition 5.3. (See [Tan88, Theorem 3.7.3.]) Let $C$ be a $\mathbb{CP}^1$-structure with real holonomy $\rho: \pi_1(S) \to \text{PSL}(2, \mathbb{R})$. Let $C \cong (L, \tau)$ in Thurston’s parameters. Then each leaf of $L$ is periodic, and its weight is $\pi$-multiple. If $\rho$ is, in addition, Fuchsian, then each leaf of $L$ is periodic and its weight is a $2\pi$-multiple.

Proof. We first show that $L$ consists of periodic leaves. Suppose, to the contrary, that $L$ contains an irrational minimal sublaminations $N$. Then transversal measure is continuous in a neighborhood of $|N|$ in $\tau$ (i.e. no leaf of $N$ has an atomic measure).
Thus there are two-dimensional strata $\sigma, \sigma_1, \sigma_2, \ldots$ of $\mathbb{H}^2 \setminus \bar{L}$, such that $\sigma_i$ converges to an edge of $\sigma$ as $i \to \infty$. Note that, as they are two-dimensional, each $\beta(\sigma_i)$ has at least three ideal points, which lie in a round circle in $\mathbb{C}P^1$. Let $H, H_1, H_2, \ldots$ be the supporting oriented hyperbolic planes in $\mathbb{H}^3$ of $\sigma, \sigma_1, \ldots$. Let $\angle_{\mathbb{H}^3}(H, H_i) \in [0, \pi]$ be the angle between the hyperbolic planes $H$ and $H_i$ with respect to their orientations, if $H$ and $H_i$ intersect. Then, by the continuity, $\angle_{\mathbb{H}^3}(H, H_i) \to 0$ as $i \to \infty$. Thus the ideas points of $\sigma_i$ cannot be contained in a single round circle if $i$ is sufficiently large. This can not happen by Corollary 5.2 as $\Lambda$ is a single round circle.

We first show that the weight of each leaf of $\bar{L}$ is a multiple of $2\pi$. Let $\sigma_1$ and $\sigma_2$ be components of $\mathbb{H}^2 \setminus \bar{L}$ adjacent along a leaf of $\bar{L}$. Let $H_1$ and $H_2$ be their support planes of $\sigma_1$ and $\sigma_2$, respectively. Then the angle between $H_1$ and $H_2$ is the weight of $\ell$. As the ideal points of $\sigma_1$ and $\sigma_2$ must lie in the round circle $\Sigma$, the angle must be a $\pi$-multiple.

Suppose, in addition, that $\rho$ is Fuchsian. Let $\beta_0: \mathbb{H}^2 (= \tau) \to \mathbb{H}^3$ be the $\rho$-equivariant embedding onto the hyperbolic plane $H_\Lambda$ bounded by $\Lambda$. For each $i = 1, 2$, as each boundary component $m$ of $\sigma_i$ covers a periodic leaf of $L$, $\beta = \beta_0$ on $m$. Therefore $H_1 = H_2 = \text{Conv}(\Lambda)$, and $\beta_0 = \beta$ on $\sigma_i$ for each $i = 1, 2$. As the orientation of $H_1$ coincides with that of $H_2$, the weight of $m$ must be a multiple of $2\pi$. \hfill $\Box$

**Proof of Theorem 5.** By Proposition 5.3, $L$ is a union of closed geodesics $\ell$ with $2\pi$-multiple weights. For each (closed) leaf $\ell$ of $L$, let $2\pi n_\ell$ denote the weight of $\ell$, where $n_\ell$ is a positive integer. Let $\kappa: C \to \tau$ be the collapsing map. Then, $\kappa^{-1}(\ell)$ is a grafting cylinder of height $2\pi n_\ell$, which the structure inserted by $2\pi$-grafting $n$ times. Therefore, $C$ is obtained by grafting along a multiloop corresponding to $L$. \hfill $\Box$

### 6. The path lifting property in the domain of discontinuity

Let $C = (f, \rho)$ be a $\mathbb{C}P^1$-structure on $S$. Then, let $\Lambda$ is the limit set of $\text{Im}\rho$, and let $\Omega = \mathbb{C}P^1 \setminus \Lambda$, the domain of discontinuity.

**Proposition 6.1.** For every $x \in \Omega$, there is a neighborhood $V_x$ in $\Omega$ such that, for every $x \in \bar{S}$ with $f(x) \in V_x$, $V_x$ is contained in the maximal disk whose core contains $x$.

**Proof.** The union $\mathbb{H}^3 \cup \partial \mathbb{H}^3$ is a unite ball in the Euclidean space and the visual distance is the restriction of the Euclidean metric.

Suppose, to the contrary, that there is no such a neighborhood $V_x$. Then there is a sequence $x_1, x_2, \cdots \in f^{-1}(x)$ such that, letting
\( H_1, H_2, \ldots \) be the its corresponding hyperbolic support planes, the visual distance from \( H_i \) to \( x \) goes to zero as \( i \to \infty \). Let \( y_i \in \mathbb{H}^3 \) be the nearest point projection of \( f(x_i) \) to \( H_i \). Then, \( y_i \to x \) in the visual metric. Let \( \sigma_i \) be the stratum of \((\mathbb{H}^2, \tilde{L}_i)\) which contains \( \tilde{\kappa}(x_i) \). Then, as the orthogonal projection of \( f(x_i) \) to \( H_i \) is \( y_i \), the visual distance between \( x \) and \( \beta_i(\sigma_i) \) goes to zero as \( i \to \infty \). Therefore, there is an ideal point \( p_i \) of \( \beta(\sigma_i) \) which converges to \( x \) as \( i \to \infty \). As \( \Omega \) is open, this is a contradiction by Corollary 5.2. □

As \( f \) embeds maximal disks of \( \bar{C} \) into \( \mathbb{C}P^1 \), we immediately have

**Corollary 6.2.** For each point \( x \in \Omega \), there is a neighborhood \( V_x \) of \( x \), such that, if \( f(x) \in V_x \) for \( x \in \bar{S} \), then \( f \) embeds a neighborhood \( W_x \) of \( x \) in \( \bar{S} \) homomorphically onto \( V_x \).

Theorem [C] immediately follows from the corollary.

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