CONSTRUCTION OF DIRECTED STRONGLY REGULAR GRAPHS USING FINITE INCIDENCE STRUCTURES

OKTAY OLMEZ AND SUNG Y. SONG

Abstract. We use finite incident structures to construct new infinite families of directed strongly regular graphs with parameters

\[(l(q-1)q^l, l(q-1)q^{l-1}, (lq-l+1)q^{l-2}, (l-1)(q-1)q^{l-2}, (lq-l+1)q^{l-2})\]

for integers \(q\), and \(l \geq 2\), and

\[(lq^2(q-1), lq(q-1), lq-l+1, (l-1)(q-1), lq-l+1)\]

for all prime powers \(q\) and \(l \in \{1, 2, \ldots, q\}\). The new graphs given by these constructions have parameters \((36, 12, 2, 5, 2), (54, 18, 7, 4, 7), (72, 24, 10, 4, 10), (96, 24, 7, 3, 7), (108, 36, 14, 8, 14)\) and \((108, 36, 15, 6, 15)\) listed as feasible parameters on “Parameters of directed strongly regular graphs,” at http://homepages.cwi.nl/~aeb/math/dsrg/dsrg.html by S. Hobart and A. E. Brouwer. We review these constructions and show how our methods may be used to construct other infinite families of directed strongly regular graphs.

Keywords: Affine planes; group-divisible designs; partial geometries.

1. Introduction and preliminaries

Directed strongly regular graphs were introduced by Duval \[5\] in 1988 as directed versions of strongly regular graphs. There are numerous sources for these graphs. Some of known constructions of these graphs use combinatorial block designs \[9\], coherent algebras \[9, 14\], finite geometries \[8, 9, 10, 15\], matrices \[5, 7, 10\], and regular tournaments \[7, 12\]. Some infinite families of these graphs also appear as Cayley graphs of groups \[6, 11, 12, 13, 14\]. (For the current state-of-the-art on known constructions and the existence and non-existence results of directed strongly regular graphs are found in the Hobart and Brouwer’s web \[3\] and the Handbook of Combinatorial Designs \[4\] Chap. VII.12.)

In this paper, we construct some new infinite families of directed strongly regular graphs by using certain finite incidence structures, such as, non-incident point-block pairs of a divisible design and anti-flags of a partial geometry.

In Section 2, we describe a construction of a directed strongly regular graph with parameters \((v, k, t, \lambda, \mu)\) given by

\[(l(q-1)q^l, l(q-1)q^{l-1}, (lq-l+1)q^{l-2}, (l-1)(q-1)q^{l-2}, (lq-l+1)q^{l-2})\]

for integers \(l \geq 2\) and \(q \geq 2\). The graph is defined on the set of non-incident point-block pairs of group divisible design GD\((l, q^{l-2}, q; q^l)\). Among the feasible parameters listed in \[3\], our construction realizes the feasibility of the parameter sets \((36, 12, 5, 2, 5)\) and \((96, 24, 7, 3, 7)\); and then \((72, 24, 10, 4, 10)\) and \((108, 36, 15, 6, 15)\) by applying for a construction of Duval \[5\].
In section 3, we construct a directed strongly regular graph on the set of anti-flags of a partial geometry. In particular, if we use the partial geometry obtained from the affine plane of order $q$ by considering all $q^2$ points and taking the $ql$ lines from exactly $l$ pencils (parallel classes) of the plane, we obtain a directed strongly regular graph with parameters $(lq^2(q - 1), lq(q - 1), lq - l + 1, (l - 1)(q - 1), lq - l + 1)$. Thus, for example, it follows the feasibility of the parameter sets $(54, 18, 7, 4, 7)$ and $(108, 36, 14, 8, 14)$ among those listed in [3].

In section 4, we construct two families of directed strongly regular graphs with parameters $(ql(l - 1), q(l - 1), q, 0, q)$ and $(ql(l - 1), 2ql(l - 1) - 1, ql - 1, ql - 2, 2q)$ for integers $q \geq 2$ and $l \geq 3$. These graphs are not new. Jørgensen [13] showed that the graph with parameters $(ql(l - 1), q(l - 1), q, 0, q)$ is unique for every integers $l \geq 2$ and $q \geq 2$. Godsil, Hobart and Martin constructed the second family of graphs in [10, Corollary 6.3]. Our construction for the second family is essentially similar to the one in [10]. However, both families of the graphs are constructed by using the almost trivial incidence structure obtained from a partition of a set of $ql$ elements into $l$ mutually disjoint $q$-element subsets.

While there is a number of families of directed strongly regular graphs constructed by using flags of finite geometries (cf. [3, 9, 10, 14, 15]), there are few results for using anti-flags or partial geometries. The results presented here can only begin to illustrate some constructions using anti-flags of partial geometries or divisible designs. More work is needed to explore more general incidence structures and address the existence and non-existence problems of directed strongly regular graphs and their related incidence structures.

In the remainder of this section, we recall the definition and some properties of directed strongly regular graphs.

A loopless directed graph $D$ with $v$ vertices is called a directed strongly regular graph with parameters $(v, k, t, \lambda, \mu)$ if and only if $D$ satisfies the following conditions:

i) Every vertex has in-degree and out-degree $k$.

ii) Every vertex $x$ has $t$ out-neighbors, all of which are also in-neighbors of $x$.

iii) The number of directed paths of length two from a vertex $x$ to another vertex $y$ is $\lambda$ if there is an edge from $x$ to $y$, and is $\mu$ if there is no edge from $x$ to $y$.

Another definition of a directed strongly regular graph, in terms of its adjacency matrix, is often conveniently used. Let $D$ be a directed graph with $v$ vertices. Let $A$ denote the adjacency matrix of $D$, and let $I = I_v$ and $J = J_v$ denote the $v \times v$ identity matrix and all-ones matrix, respectively. Then $D$ is a directed strongly regular graph with parameters $(v, k, t, \lambda, \mu)$ if and only if (i) $JA = AJ = kJ$ and (ii) $A^2 = tI + \lambda A + \mu(J - I - A)$.

Duval observed that if $t = \mu$ and $A$ satisfies above equations (i) and (ii), then so does $A \otimes J_m$ for every positive integer $m$; and so, we have:
Definition 2.1. Let \((P, B)\) be the incidence structure defined as above. Let \(D = D(P, B)\) be the directed graph with its vertex set

\[ V(D) = \{(p, B) \in P \times B : p \notin B\}, \]

and directed edges given by \((p, B) \to (p', B')\) if and only if \(p \in B'\).

Theorem 2.1. Let \(D\) be the graph \(D(P, B)\) defined in Definition 2.1. Then \(D\) is a directed strongly regular graph with parameters

\[
\begin{align*}
v &= lq^2(q-1), \\
k &= lq^{l-1}(q-1), \\
t &= \mu = q^{l-2}(lq-l+1), \\
\lambda &= q^{l-2}(l-1)(q-1).
\end{align*}
\]

Proof: It is easy to verify the values of \(v\) and \(k\). To find the value of \(t\), let \((p, B) \in V(D)\) with \(B = \{b_1, b_2, \ldots, b_j\}\), and let \(N^+(p, B)\) and \(N^-(p, B)\) denote the set of out-neighbors and that of in-neighbors of \((p, B)\), respectively. Then for \(t\) we need to count the elements of

\[ N^+(p, B) \cap N^-(p, B) = \{(p', B') \in V(D) : p \in B^*, p' \in B^*\}. \]

Without loss of generality, suppose that \(p \in S_1\). Then condition \(p \in B^*\) requires that \(b_1^*\) must be \(p\). The second condition \(p^* \in B^*\) requires that \(p^* = b_i\) for some \(i\). With the choice of \(p^* = b_1\), there are \(q^{l-1}\) blocks \(B^* = \{b_1^*, b_2^*, \ldots, b_j^*\}\) with \(b_j^* = p\) can be paired with \(p^* = b_1\). On the other hand, with \(p^* = b_j^*\) for \(j \neq 1\), we have \((q-1)q^{l-2}\) choices for \(B^*\) with \(b_j^* = p\), \(b_j^* \in S_j \setminus \{b_j\}\) and \(b_i^* \in S_i\) for all \(i \in \{2, 3, \ldots, l\} \setminus \{j\}\). Hence \(t = q^{l-1} + (l-1)(q-1)q^{l-2}\).
We now claim that the number of directed paths of length two from a vertex \((p, B)\) to another vertex \((p', B')\) depends only on whether there is an edge from \((p, B)\) to \((p', B')\) or not.

Suppose that \((p, B) \to (p', B')\): that is, \(p \in B'\). Then, without loss of generality, we may assume that \(p = b'_i\), and have
\[
\lambda = |N^+( (p, B) ) \cap N^- ( (p', B') ) |
\]
\[
= |\{ (p^*, B^*) \in V(D) : p = b'_i \in B^*, p^* \in B' \} |
\]
\[
= |\{ (p^*, \{ p, b'_2, \ldots, b'_l \}) : p^* \in B' \setminus \{ b'_i \}, b'_i \in S_i \setminus \{ p^* \} \text{ for } i = 2, 3, \ldots, l \} |
\]
\[
= (l - 1)(q - 1)q^{l-2}. 
\]

For \(\mu\), suppose \((p, B) \not\to (p', B')\). Then \(\mu = |\{ (p^*, B^*) : p \in B^*, p^* \in B' \} |\) with \(p \notin B'\). Without loss of generality, we assume that \(p \in S_1\), and have
\[
\mu = |\{ (b'_i, \{ p, b'_2, \ldots, b'_l \}) : b'_i \in S_i \text{ for } i = 2, 3, \ldots, l \} |
\]
\[
+ \sum_{i=2}^{l} |\{ (b'_i, \{ p, b'_2^*, \ldots, b'_l^* \}) : b'_i \in S_i \setminus \{ b'_i^* \}, b'_j^* \in S_j \text{ for } j \in \{ 2, 3, \ldots, l \} \setminus \{ i \} \} |
\]
\[
= q^{l-1} + (l - 1)(q - 1)q^{l-2}. 
\]

This completes the proof. □

**Remark 2.1.** By Proposition [1], there are directed strongly regular graphs with parameters
\(mlq^l(q - 1), mlq^{l-1}(q - 1), mq^{l-2}(lq - l + 1), mq^{l-2}(l - 1)(q - 1), mq^{l-2}(lq - l + 1)\)
for all positive integers \(m\). These graphs can be constructed directly by replacing all edges by multiple edges with multiplicity \(m\) in the above construction.

**Remark 2.2.** In the case of \(l = 2 (m = 1)\) in the above construction, we obtain the graphs with parameters
\((2q^2(q - 1), 2q(q - 1), 2q - 1, q - 1, 2q - 1)\).

These graphs are constructed on the sets of non-incident vertex-edge pairs of complete bipartite graphs \(K_{q,q}\). If we use complete bipartite multigraph with multiplicity \(m\) for each edge as in the above remark, we obtain directed strongly regular graphs with parameters,
\(2mq^2(q - 1), 2mq(q - 1), m(2q - 1), m(q - 1), m(2q - 1)\).

With several combinations of small \(m\) and \(q\), we obtain new directed strongly regular graphs with parameter sets \((36, 12, 5, 2, 5)\), \((96, 24, 7, 3, 7)\), \((72, 24, 10, 4, 10)\) and \((108, 36, 15, 6, 15)\). So the feasibility of these parameter sets which are listed in [3] has been realized.

### 3. Construction of Graphs Using Partial Geometries

In this section, we use partial geometries to construct a family of directed strongly regular graphs. The concept of a partial geometry was introduced by R. C. Bose in connection with his study of large cliques of more general strongly regular graphs in [1].

A partial geometry \(pg(\kappa, \rho, \tau)\) is a set of points \(P\), a set of lines \(\mathcal{L}\), and an incidence relation between \(P\) and \(\mathcal{L}\) with the following properties:
(i) Every line is incident with \( \kappa \) points \((\kappa \geq 2)\), and every point is incident with \( \rho \) lines \((\rho \geq 2)\).

(ii) Any two points are incident with at most one line.

(iii) If a point \( p \) and a line \( L \) are not incident, then there exists exactly \( \tau \) lines that are incident with \( p \) and incident with \( L \).

Here we use parameters \((\kappa, \rho, \tau)\) instead of more traditional notations, \((K, R, T)\) or \((1 + t, 1 + s, \alpha)\) used in [11] or [16]. In what follows, we often identify a line \( L \) as the set of \( \kappa \) points that are incident with \( L \); so, write “\( p \in L \)” as well as “\( L \) passes through \( p \)” when \( p \) and \( L \) are incident.

**Definition 3.1.** Let \( D = D(pg(\kappa, \rho, \tau)) \) be the directed graph with its vertex set

\[
V(D) = \{(p, L) \in P \times L : p \notin L\},
\]

and directed edges given by \((p, L) \rightarrow (p', L')\) if and only if \( p \in L' \).

**Theorem 3.1.** Let \( D \) be the graph \( D(pg(\kappa, \rho, \tau)) \) defined as above. Then \( D \) is a directed strongly regular graph with parameters

\[
\begin{align*}
&v = \frac{\kappa \rho (\kappa - 1)(\rho - 1)}{\tau} \left(1 + \frac{\kappa - 1)(\rho - 1)}{\tau}\right), \\
&k = \frac{\kappa \rho (\kappa - 1)(\rho - 1)}{\tau}, \\
&t = \mu = \kappa \rho - \tau, \\
&\lambda = (\kappa - 1)(\rho - 1).
\end{align*}
\]

Proof: The value \( v \) is clear as it counts the number of anti-flags of \( pg(\kappa, \rho, \tau) \) which has \( \kappa \left(1 + \frac{\kappa - 1)(\rho - 1)}{\tau}\right) \) points and \( \rho \left(1 + \frac{\kappa - 1)(\rho - 1)}{\tau}\right) \) lines. Also it is clear that

\[
k = |\{(p', L') : p \in L'\}| = \rho(v - \kappa) = \kappa \rho (\kappa - 1)(\rho - 1)/\tau.
\]

For \( t \), given a vertex \((p, L) \in V(D)\), we count the cardinality of

\[
N^+(\{(p, L)\}) \cap N^-(\{(p, L)\}) = \{(p', L') \in V(D) : p \in L', p' \in L\}.
\]

Among the \( \rho \) lines passing through \( p \), \( \rho - \tau \) lines are parallel to \( L \). If \( L' \) is parallel to \( L \), then all \( \kappa \) points on \( L \) can make legitimate non-incident point-line pairs \((p', L')\) with given \( L' \). In the case when \( L' \) is not parallel to \( L \), all points on \( L \) except for the common incident point of \( L \) and \( L' \), can form desired pairs. Hence we have

\[
t = (\rho - \tau)\kappa + \tau(\kappa - 1) = \kappa \rho - \tau.
\]

For \( \lambda \), suppose \((p, L) \rightarrow (p', L')\). It is clear that

\[
\lambda = |\{(p^*, L^*) : p \in L^*, p^* \in L'\}| = (\rho - 1)(\kappa - 1),
\]

because each of \( \rho - 1 \) lines passing through \( p \) (excluding \( L' \)) can be paired with any of \( \kappa - 1 \) points on \( L' \setminus \{p\} \).

For given \((p, L)\) and \((p', L')\) with \( p \notin L' \),

\[
\mu = |\{(p^*, L^*) : p \in L^*, p^* \in L'\}| = (\rho - \tau)\kappa + \tau(\kappa - 1) = \kappa \rho - \tau,
\]

since among the \( \rho \) lines passing through \( p \), the ones that are parallel to \( L' \) can form desired pairs with any of \( \kappa \) points on \( L' \), while each of the remaining \( \tau \) lines can be paired with \( \kappa - 1 \) points on \( L' \). \( \square \)
Partial geometries are ubiquitous. For example, partial geometries with \( \tau = 1 \) are generalized quadrangles, those with \( \tau = \kappa - 1 \) are transversal designs, and those with \( \tau = \rho - 1 \) are known as nets. In order to have some concrete examples of new directed strongly regular graphs of small order, we consider a special class of partial geometries that are obtained from finite affine planes.

Let \( \text{AP}(q) \) denote the affine plane of order \( q \). Let \( \overrightarrow{\text{AP}}(q) \) denote the partial geometry obtained from \( \text{AP}(q) \) by considering all \( q^2 \) points and taking the lines of \( l \) parallel classes of the plane. Then \( \overrightarrow{\text{AP}}(q) \) inherits the following properties from \( \text{AP}(q) \): (i) every point is incident with \( l \) lines, and every line is incident with \( q \) points, (ii) any two points are incident with at most one line, (iii) if \( p \) and \( L \) are non-incident point-line pair, there are exactly \( l - 1 \) lines containing \( p \) which meet \( L \). That is, \( \overrightarrow{\text{AP}}(q) = \text{pg}(q, l, l - 1) \).

**Corollary 3.2.** Let \( D = D(\overrightarrow{\text{AP}}(q)) \) be the directed graph \( D(\text{pg}(q, l, l - 1)) \) defined as in Definition 3.1. Then \( D \) is a directed strongly regular graph with parameters \((v, k, t, \lambda, \mu) = (lq^2(q - 1), lq(q - 1), lq - l + 1, (l - 1)(q - 1), lq - l + 1)\).

Proof: It immediately follows from Theorem 3.1. \( \square \)

In particular, if \( l = q \), \( \overrightarrow{\text{AP}}(q) = \text{pg}(q, q, q - 1) \) is a transversal design \( \text{TD}(q, q) = (P, G, L) \) of order \( q \), block size \( q \), and index 1 in the following sense.

(i) \( P \) is the set of \( q^2 \) points of \( \overrightarrow{\text{AP}}(q) \);
(ii) \( G \) is the partition of \( P \) into \( q \) classes (groups) such that each class consists of \( q \) points that were collinear in \( \text{AP}(q) \) but not in \( \overrightarrow{\text{AP}}(q) \);
(iii) \( L \) is the set of \( q^2 \) lines (blocks);
(iv) every unordered pair of points in \( P \) is contained in either exactly one group or in exactly one block, but not both.

**Corollary 3.3.** Let \( D \) be the graph \( D(\overrightarrow{\text{AP}}(q)) \) defined as above. Then \( D \) is a directed strongly regular graph with parameters
\[(v, k, t, \lambda, \mu) = (q^3(q - 1), q^2(q - 1), q^2 - q + 1, (q - 1)^2, q^2 - q + 1).\]

**Example 3.1.** This method produces new directed strongly regular graphs with parameters \((54, 18, 7, 4, 7)\) when \( q = 3 \); and so \((108, 36, 14, 8, 14)\) by Proposition 1.1.

4. **Construction of graphs using partitioned sets**

In this section we construct directed strongly regular graphs for two parameter sets,
\[(ql(l - 1), q(l - 1), q, 0, q)\]
and
\[(ql(l - 1), 2q(l - 1) - 1, ql - 1, ql - 2, 2q)\]
for all positive integers \( q \geq 1 \) and \( l \geq 3 \). The incidence structure which will be used here may be viewed as a degenerate case of those used in earlier
The graphs produced by this construction are not new. The latter family of graphs have been constructed by Godsil, Hobart and Martin in [10, Corollary 6.3]. Nevertheless, we introduce our construction to illustrate a variation of constructions which may be applied to produce new families of directed strongly regular graphs.

Let $P$ be a set of $ql$ elements (‘points’), and let $S_1, S_2, \ldots, S_l$ be $l$ mutually disjoint $q$-element subsets of $P$ (‘blocks’). We denote the family of blocks by $S = \{S_1, S_2, \ldots, S_l\}$. We will say that point $x \in P$ and block $S \in S$ is a non-incident point-block pair if and only if $x \not\in S$.

For the directed strongly regular graph with the first parameter set, let $D = D(P, S)$ be the directed graph with its vertex set

$$V(D) = \{(x, S) \in P \times S : x \not\in S\},$$

and directed edges defined by $(x, S) \to (x', S')$ if and only if $x \in S'$.

**Theorem 4.1.** Let $P$, $S$ and $D(P, S)$ be as the above. Then $D(P, S)$ is a directed strongly regular graph with parameters

$$(ql(l - 1), q(l - 1), q, 0, q).$$

**Proof:** Easy counting arguments give the values for parameters. □

**Remark 4.1.** Jørgensen [13] showed that the directed strongly regular graph with parameters $(ql(l-1), q(l-1), q, 0, q)$ is unique for all positive integers $l$ and $q$.

For the directed strongly regular graph with the second parameter set, let $G = G(P, S)$ be the directed graph with its vertex set

$$V(G) = \{(x, S) \in P \times S : x \not\in S\},$$

where $P$ and $S$ are as the above, and let edges on $V(G)$ be defined by: $(x, S) \to (x', S')$ if and only if

1. $x \in S'$; or
2. $S = S'$ and $x \neq x'$.

**Theorem 4.2.** Let $G$ be the graph $G(P, S)$ defined as above. Then $G$ is a directed strongly regular graph with parameters

$$(ql(l - 1), 2q(l - 1) - 1, ql - 1, ql - 2, 2q).$$

**Proof:** Clearly $v = ql(l - 1)$ as $|V(G)| = |V| \cdot (|S| - 1)$.

Given a vertex $(x, S)$, let

- $N_1((x, S)) := \{(x', S') : x \in S' \text{ and } x' \in S\}$
- $N_2((x, S)) := \{(x', S') : S = S' \text{ and } x \neq x'\}$
- $N_3((x, S)) := \{(x', S') : x \in S' \text{ and } x' \not\in S\}$

Then by simple counting, we have

$$|N_1((x, S))| = q$$
$$|N_2((x, S))| = q(l - 2) + q - 1$$
$$|N_3((x, S))| = ql - 2.$$
Hence, $t = |N_1((x, S))| + |N_2((x, S))| = ql - 1$, and $k = t + |N_3((x, S))| = 2ql - 2q - 1$.

Finally, we count the number of vertices $(x^*, S^*)$ that belongs to $N^+((x, S)) \cap N^-(x^*, S^*)$ for $\lambda$ and $\mu$.

For $\lambda$, given an edge $(x, S) \rightarrow (x', S')$, we shall need to consider the following two cases separately: (Case 1) when $x \in S'$, and (Case 2) when $S = S'$ and $x \neq x'$.

Case (1). Suppose $x \in S'$. Then unless $S^* = S$, $S^*$ must contain $x$ which forces $S^* = S'$. With $S^* = S'$, all points that are not belong to $S' \cup \{x^*\}$ can be chosen to be $x^*$; and so, there are $q(l - 1) - 1$ choices for $x^*$. With $S^* = S$, the $q - 1$ points of the set $S' \setminus \{x\}$ are possible for $x^*$. Together there are $ql - 2$ vertices $(x^*, S^*)$ belonging to $N^+((x, S)) \cap N^-(x^*, S^*)$.

Case (2). Let $S = S'$. Then with the choice of $S^* = S = S'$, $x^*$ can be chosen from $P \setminus (S \cup \{x, x^*\})$; and so, there are $q(l - 1) - 2$ choices for $x^*$. Also with the choice of $S^*$ being the block containing $x$, any point in $S$ can be chosen as $x^*$; and so, there are $q$ possible choices for $x^*$. Together we have $ql - 2$ choices for $(x^*, S^*)$ as well. Hence we have $\lambda = ql - 2$.

For $\mu$, suppose $S \neq S'$ and $x \notin S'$. Then it is clear that

$$\mu = |\{(x^*, S^*) : x \in S^*, x^* \in S'\} \cup \{(x^*, S^*) : S^* = S, x \in S'\}| = q + q.$$

This completes the proof. $\square$

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Department of Mathematics, Iowa State University, Ames, Iowa, 50011, U. S. A.

E-mail address, O. Olmez: oolmez@iastate.edu

Department of Mathematics, Iowa State University, Ames, Iowa, 50011, U. S. A.

E-mail address, S. Y. Song: sysong@iastate.edu
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Abstract. We use finite incident structures to construct new infinite families of directed strongly regular graphs with parameters

\((lq^l(q-1), (lq^{l-1}(q-1), q^{l-2}(q-1), (l-1)q^{l-2}(q-1), q^{l-2}(q-l+1))\),

for integers \(q\) and \(l\) \((q, l \geq 2)\), and

\((lq^2(q-1), lq(q-1), lq-l+1, (l-1)(q-1), lq-l+1)\)

for all prime powers \(q\) and \(l \in \{1, 2, \ldots, q\}\). The new graphs given by these constructions include those with parameter sets \((36, 12, 5, 2, 5)\), \((54, 18, 7, 4, 7)\), \((72, 24, 10, 4, 10)\), \((96, 24, 7, 3, 7)\), \((108, 36, 14, 8, 14)\) and \((108, 36, 15, 6, 15)\) listed as feasible parameters on “Parameters of directed strongly regular graphs” by S. Hobart and A. E. Brouwer at http://homepages.cwi.nl/~aeb/math/dsrg/dsrg.html. We also review these constructions and show how our methods may be used to construct other infinite families of directed strongly regular graphs. We then characterize finite incidence structures from which we can produce directed strongly regular graphs.

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Keywords: Affine planes, affine resolvable designs, group-divisible designs, partial geometries.

1. INTRODUCTION AND PRELIMINARIES

Directed strongly regular graphs were introduced by Duval [7] in 1988 as directed versions of strongly regular graphs. There are numerous sources for these graphs. Some of known constructions of these graphs use combinatorial block designs [11], coherent algebras [11, 18], finite geometries [10, 11, 13, 19],...
matrices \([7, 9, 13]\), and regular tournaments \([9, 16]\). Some infinite families of these graphs also appear as Cayley graphs of groups \([8, 14, 16, 17, 18]\).

In this paper, we construct some new infinite families of directed strongly regular graphs by using certain finite incidence structures, such as, non-incident point-block pairs of a divisible design and anti-flags of a partial geometry.

In Section 2, we describe a construction of a directed strongly regular graph with parameters \((v, k, t, \lambda, \mu)\) given by

\[
(lq(q - 1), \, lq^{-1}(q - 1), \, q^{l-2}l(q - l + 1), \, (l - 1)q^{l-2}(q - 1), \, q^{l-2}(lq - l + 1)),
\]

for integers \(l \geq 2\) and \(q \geq 2\). The graph is defined on the set of non-incident point-block pairs of group divisible design \(GD(l, q^l - 2, q^l; q^l)\). Among the feasible parameters listed in \([4]\), our construction realizes the feasibility of the parameter sets \((36, 12, 5, 2, 5)\) and \((96, 24, 7, 3, 7)\); and then \((72, 24, 10, 4, 10)\) and \((108, 36, 15, 6, 15)\) by applying for a construction of Duval \([7]\).

In section 3, we construct directed strongly regular graphs on the sets of anti-flags of partial geometries. In particular, if we use the partial geometry obtained from the affine plane of order \(q\) by considering all \(q^2\) points and taking the \(ql\) lines from exactly \(l\) pencils (parallel classes) of the plane, we obtain a directed strongly regular graph with parameters

\[
(lq^2(q - 1), \, lq(q - 1), \, lq - l + 1, \, (l - 1)(q - 1), \, lq - l + 1).
\]

Thus, for example, it follows the feasibility of the parameter sets \((54, 18, 7, 4, 7)\) and \((108, 36, 14, 8, 14)\) among those listed in \([4]\).

In section 4, we construct two families of directed strongly regular graphs with parameters

\[
(q(l - 1), \, q(l - 1), \, q, \, 0, \, q)
\]

and

\[
(q(l - 1), \, 2q(l - 1) - 1, \, ql - 1, \, ql - 2, \, 2q)
\]

for integers \(q \geq 2\) and \(l \geq 3\). These graphs are not new. Jørgensen \([17]\) showed that the graph with parameters \((ql(l - 1), \, q(l - 1), \, q, \, 0, \, q)\) is unique for every integers \(l \geq 2\) and \(q \geq 2\). Godsil, Hobart and Martin constructed the second family of graphs in \([13, Corollary 6.3]\). Our construction for the second family is essentially similar to the one in \([13]\). However, both families of the graphs are constructed by using the almost trivial incidence structure obtained from a partition of a set of \(ql\) elements into \(l\) mutually disjoint \(q\)-element subsets.

In Section 5 and Section 6 we introduce how to construct directed strongly regular graphs by using the anti-flags over the affine resolvable designs and arbitrary 2-designs, respectively. In Section 7, we introduce a prolific construction method for directed strongly regular graphs by using sets of anti-flags over certain tactical configurations.

While there is a number of families of directed strongly regular graphs constructed by using flags of finite geometries (cf. \([4, 11, 13, 18, 19]\)), there
are few results for using anti-flags or partial geometries. The results presented here adds some constructions using sets of anti-flags of various finite incidence structures including partial geometries, combinatorial designs, and certain tactical configurations. More work is needed to explore other incidence structures and address the existence and non-existence problems of directed strongly regular graphs and their related incidence structures.

In the remainder of this section, we recall the definition and some properties of directed strongly regular graphs.

A loopless directed graph $D$ with $v$ vertices is called directed strongly regular graph with parameters $(v, k, t, \lambda, \mu)$ if and only if $D$ satisfies the following conditions:

i) Every vertex has in-degree and out-degree $k$.

ii) Every vertex $x$ has $t$ out-neighbors, all of which are also in-neighbors of $x$.

iii) The number of directed paths of length two from a vertex $x$ to another vertex $y$ is $\lambda$ if there is an edge from $x$ to $y$, and is $\mu$ if there is no edge from $x$ to $y$.

Another definition of a directed strongly regular graph, in terms of its adjacency matrix, is often conveniently used. Let $D$ be a directed graph with $v$ vertices. Let $A$ denote the adjacency matrix of $D$, and let $I = I_v$ and $J = J_v$ denote the $v \times v$ identity matrix and all-ones matrix, respectively. Then $D$ is a directed strongly regular graph with parameters $(v, k, t, \lambda, \mu)$ if and only if (i) $JA = AJ = kJ$ and (ii) $A^2 = tI + \lambda A + \mu(J - I - A)$.

Duval observed that if $t = \mu$ and $A$ satisfies above equations (i) and (ii), then so does $A \otimes J_m$ for every positive integer $m$; and so, we have:

**Proposition 1.1.** [7] If there exists a directed strongly regular graph with parameters $(v, k, t, \lambda, \mu)$ and $t = \mu$, then for each positive integer $m$ there exists a directed strongly regular graph with parameters $(mv, mk, mt, m\lambda, m\mu)$.

**Proposition 1.2.**[7] A directed strongly regular graph with parameters $(v, k, t, \lambda, \mu)$ has three distinct integer eigenvalues

$$
\theta_0 = k, \quad \theta_1 = \frac{1}{2}(\lambda - \mu + \delta), \quad \theta_2 = \frac{1}{2}(\lambda - \mu - \delta)
$$

with multiplicities

$$
m_0 = 1, \quad m_1 = \frac{k + \theta_2(v - 1)}{\theta_1 - \theta_2}, \quad m_2 = \frac{k + \theta_1(v - 1)}{\theta_1 - \theta_2},
$$

respectively, where $\delta = \sqrt{(\mu - \lambda)^2 + 4(t - \mu)}$ is a positive integer.

Throughout the paper, we will write $x \rightarrow y$ if there is an edge from a vertex $x$ to another vertex $y$, and $x \nrightarrow y$ if there is no edge from $x$ to $y$. We will also write $x \leftrightarrow y$ if and only if both $x \rightarrow y$ and $y \rightarrow x$.

2. **Graphs obtained from group divisible designs**

The first family of directed strongly regular graphs, which we shall describe in this section, use non-incident point-block pairs of group divisible designs $GD(l, q^{l-2}, q; ql)$ for integers $q \geq 2$ and $l \geq 2$. 
Let $P$ be a $ql$-element set with a partition $\mathcal{P}$ of $P$ into $l$ parts (‘groups’) of size $q$. Let $\mathcal{P} = \{S_1, S_2, \ldots, S_l\}$. Let

$$B = \{B \subset P : |B \cap S_i| = 1 \text{ for all } i = 1, 2, \ldots, l\}.$$ 

Then $B$ consists of $q^l$ subsets of size $l$ called blocks. The elements of $P$ will be called points of the incidence structure $(P, B)$ with the natural point-block incidence relation $\in$. This structure has property that any two points from the same group never occur together in a block while any two points from different groups occur together in $q^{l-2}$ blocks. It is known as a group-divisible design $GD(l, q^{l-2}, q; ql)$.

**Definition 2.1.** Let $(P, B)$ be the incidence structure defined as above. Let $D = D(P, B)$ be the directed graph with its vertex set

$$V(D) = \{(p, B) \in P \times B : p \notin B\},$$

and directed edges given by $(p, B) \to (p', B')$ if and only if $p \in B'$.

**Theorem 2.1.** Let $D$ be the graph $D(P, B)$ defined in Definition 2.1. Then $D$ is a directed strongly regular graph with parameters

$$v = lq^l(q - 1),$$

$$k = lq^{l-1}(q - 1),$$

$$t = \mu = q^{l-2}(lq - l + 1),$$

$$\lambda = q^{l-2}(l - 1)(q - 1).$$

Proof: It is easy to verify the values of $v$ and $k$. To find the value of $t$, let $(p, B) \in V(D)$ with $B = \{b_1, b_2, \ldots, b_l\}$, and let $N^+((p, B))$ and $N^-((p, B))$ denote the set of out-neighbors and that of in-neighbors of $(p, B)$, respectively. Then for $t$ we need to count the elements of

$$N^+((p, B)) \cap N^-((p, B)) = \{(p^*, B^*) \in V(D) : p \in B^*, p^* \in B\}.$$ 

Without loss of generality, suppose that $p \in S_1$. Then condition $p \in B^*$ requires that $b_1^*$ must be $p$. The second condition $p^* \in B$ requires that $p^* = b_i$ for some $i$. With the choice of $p^* = b_1$, there are $q^{l-1}$ blocks $B^* = \{b_1^*, b_2^*, \ldots, b_l^*\}$ with $b_1^* = p$ can be paired with $p^* = b_1$. On the other hand, with $p^* = b_j$ for $j \neq 1$, we have $(q - 1)q^{l-2}$ choices for $B^*$ with $b_1^* = p$, $b_j^* \in S_j \setminus \{b_j\}$ and $b_i^* \in S_i$ for all $i \in \{2, 3, \ldots, l\} \setminus \{j\}$. Hence $t = q^{l-1} + (l - 1)(q - 1)q^{l-2}$.

We now claim that the number of directed paths of length two from a vertex $(p, B)$ to another vertex $(p', B')$ depends only on whether there is an edge from $(p, B)$ to $(p', B')$.

Suppose that $(p, B) \to (p', B')$; that is, $p \in B'$. Then, without loss of generality, we may assume that $p = b_1^*$, and have

$$\lambda = |N^+((p, B)) \cap N^-((p', B'))|$$

$$= |\{(p^*, B^*) \in V(D) : p = b_1^* \in B^*, p^* \in B'\}|$$

$$= |\{(p^*, \{p, b_2^*, \ldots, b_l^*\}) : p^* \in B' \setminus \{b_1^*\}, b_i^* \in S_i \setminus \{p^*\} \text{ for } i = 2, 3, \ldots, l\}|$$

$$= (l - 1)(q - 1)q^{l-2}.$$
For $\mu$, suppose $(p, B) \not\rightarrow (p', B')$. Then $\mu = |\{(p^*, B^*) : p \in B^*, p^* \in B'\}|$ with $p \not\in B'$. Without loss of generality, we assume that $p \in S_1$, and have

$$
\begin{align*}
\mu &= |\{(b'_i, \{p, b'_2, \ldots, b'_l\}) : b'_i \in S_i \text{ for } i = 2, 3, \ldots, l}\}|
+ \sum_{i=2}^{l} |\{(b'_i, \{p, b'_2, \ldots, b'_l\}) : b'_i \in S_i \setminus \{b'_i\}, b'_j \in S_j \text{ for } j \in \{2, 3, \ldots, l\} \setminus \{i\}\}|
= q^{l-1} + (l - 1)(q - 1)q^{l-2}.
\end{align*}
$$

This completes the proof. $\square$

**Remark 2.1.** By Proposition 1.1, there are directed strongly regular graphs with parameters $(mlq^2(q-1), mlq^2(q-1), mq^2(lq-l+1), mlq^2(l-1)(q-1), mlq^2(lq-l+1))$ for all positive integers $m$. These graphs can be constructed directly by replacing all edges by multiple edges with multiplicity $m$ in the above construction.

**Remark 2.2.** In the case of $l = 2$ ($m = 1$) in the above construction, we obtain the graphs with parameters

$$(2q^2(q-1), 2q(q-1), 2q - 1, q - 1, 2q - 1).$$

These graphs are constructed on the sets of non-incident vertex-edge pairs of complete bipartite graphs $K_{q,q}$. If we use complete bipartite multigraph with multiplicity $m$ for each edge as in the above remark, we obtain directed strongly regular graphs with parameters,

$$(2mq^2(q-1), 2mq(q-1), m(2q-1), m(q-1), m(2q-1)).$$

With several combinations of small $m$ and $q$, we obtain new directed strongly regular graphs with parameter sets $(36, 12, 5, 2, 5)$, $(96, 24, 7, 3, 7)$, $(72, 24, 10, 4, 10)$ and $(108, 36, 15, 6, 15)$. So the feasibility of these parameter sets which are listed in [1] has been realized.

By Proposition 1.2, we can easily compute the eigenvalues for $D(P,B)$.

**Lemma 2.2.** If a directed strongly regular graph with parameters $(v, k, t, \lambda, \mu)$ satisfies $t = \mu = \lambda + q^{l-1}$ for positive integers $q$ and $l \ge 2$, then its eigenvalues are $\theta_0 = k$, $\theta_1 = 0$, $\theta_2 = -q^{l-1}$ with multiplicities $m_0 = 1$, $m_1 = v - 1 - \frac{k}{q^{l-1}}$, $m_2 = \frac{k}{q^{l-1}}$, respectively.
Example 2.3. (Parameter sets for small DSRGs (orders up to 110) constructed.)

\begin{align*}
\begin{array}{ll}
(v, k, t, \lambda, \mu) & l, q, m \\
(8, 4, 3, 1, 3) & l = q = 2 \\
(16, 8, 6, 2, 6) & l = q = 2; m = 2 \\
(24, 12, 9, 3, 9) & l = q = 2; m = 3 \\
(24, 12, 8, 4, 8) & l = 3, q = 2 \\
(32, 16, 12, 4, 12) & l = q = 2; m = 4 \\
(36, 12, 5, 2, 5) & l = 2, q = 3 \\
(40, 20, 15, 5, 15) & l = q = 2; m = 5 \\
(48, 24, 16, 8, 16) & l = 3, q = 2; m = 2 \\
(48, 24, 18, 6, 18) & l = q = 2; m = 6 \\
(56, 28, 21, 7, 21) & l = q = 2; m = 7 \\
(64, 32, 20, 4, 20) & l = 4, q = 2 \\
(64, 32, 24, 8, 24) & l = q = 2; m = 8 \\
(72, 24, 10, 4, 10) & l = 2, q = 3; m = 2 \\
(72, 36, 24, 12, 24) & l = 3, q = 2; m = 2 \\
(72, 36, 27, 9, 27) & l = q = 2; m = 9 \\
(80, 40, 30, 10, 30) & l = q = 2; m = 10 \\
(88, 44, 33, 11, 33) & l = q = 2; m = 11 \\
(96, 24, 7, 3, 7) & l = 2, q = 4 \\
(96, 48, 32, 16, 32) & l = 3, q = 2; m = 4 \\
(96, 48, 36, 12, 36) & l = q = 2; m = 12 \\
(104, 52, 39, 13, 39) & l = q = 2; m = 13 \\
(108, 36, 15, 6, 15) & l = 2, q = 3; m = 3 \\
\end{array}
\end{align*}

3. Graphs obtained from partial geometries

In this section, we use partial geometries to construct a family of directed strongly regular graphs. The concept of a partial geometry was introduced by R. C. Bose in connection with his study of large cliques of more general strongly regular graphs in [2].

A partial geometry \( pg(\kappa, \rho, \tau) \) is a set of points \( P \), a set of lines \( L \), and an incidence relation between \( P \) and \( L \) with the following properties:

(i) Every line is incident with \( \kappa \) points \( (\kappa \geq 2) \), and every point is incident with \( \rho \) lines \( (\rho \geq 2) \).

(ii) Any two points are incident with at most one line.

(iii) If a point \( p \) and a line \( L \) are not incident, then there exists exactly \( \tau \) \( (\tau \geq 1) \) lines that are incident with \( p \) and incident with \( L \).

Here we use parameters \( (\kappa, \rho, \tau) \) instead of more traditional notations, \( (K, R, T) \) or \( (1 + t, 1 + s, \alpha) \) used in [21, 3] or [22]. In what follows, we often identify a line \( L \) as the set of \( \kappa \) points that are incident with \( L \); so, when \( p \) and \( L \) are incident, we write “\( p \in L \)” as well as “\( p \) is on \( L \)” and “\( L \) passes through \( p \)”.

**Definition 3.1.** Let \( D = D(pg(\kappa, \rho, \tau)) \) be the directed graph with its vertex set

\[ V(D) = \{(p, L) \in P \times L : p \notin L\}, \]

and directed edges given by \((p, L) \rightarrow (p', L')\) if and only if \( p \in L' \).
Theorem 3.1. Let $D$ be the graph $D(pg(\kappa, \rho, \tau))$ defined as above. Then $D$ is a directed strongly regular graph with parameters

$$v = \frac{\kappa \rho (\kappa - 1)(\rho - 1)}{\tau},$$

$$k = \frac{\kappa \rho (\kappa - 1)(\rho - 1)}{\tau},$$

$$t = \mu = \kappa \rho - \tau,$$

$$\lambda = (\kappa - 1)(\rho - 1).$$

Proof: The value $v$ is clear as it counts the number of anti-flags of $pg(\kappa, \rho, \tau)$ which has $\kappa \left(1 + \frac{(\kappa - 1)(\rho - 1)}{\tau}\right)$ points and $\rho \left(1 + \frac{(\kappa - 1)(\rho - 1)}{\tau}\right)$ lines. Also it is clear that

$$k = \left|\{(p', L') : p \in L'\}\right| = \rho(v - \kappa) = \kappa \rho (\kappa - 1)(\rho - 1)/\tau.$$  

For $t$, given a vertex $(p, L) \in V(D)$, we count the cardinality of

$$N^+(pg, L) \cap N^-(pg, L) = \{(p', L') \in V(D) : p \in L', p' \in L\}.$$  

Among the $\rho$ lines passing through $p$, $\rho - \tau$ lines are parallel to $L$. If $L'$ is parallel to $L$, then all $\kappa$ points on $L$ can make legitimate non-incident point-line pairs $(p', L')$ with given $L'$. In the case when $L'$ is not parallel to $L$, all points on $L$ except for the common incident point of $L$ and $L'$, can form desired pairs. Hence we have

$$t = (\rho - \tau)k + \tau(\kappa - 1) = \kappa \rho - \tau.$$  

For $\lambda$, suppose $(p, L) \to (p', L')$. It is clear that

$$\lambda = \left|\{(p^*, L^*) : p \in L^*, p^* \in L'\}\right| = (\rho - 1)(\kappa - 1),$$

because each of $\rho - 1$ lines passing through $p$ (excluding $L'$) can be paired with any of $\kappa - 1$ points on $L' \setminus \{p\}$.

For given $(p, L)$ and $(p', L')$ with $p \not\in L'$,

$$\mu = \left|\{(p^*, L^*) : p \in L^*, p^* \in L'\}\right| = (\rho - \tau)k + \tau(\kappa - 1) = \kappa \rho - \tau,$$

since among the $\rho$ lines passing through $p$, the ones that are parallel to $L'$ can form desired pairs with any of $\kappa$ points on $L'$, while each of the remaining $\tau$ lines can be paired with $\kappa - 1$ points on $L'$.

Partial geometries are ubiquitous. For example, partial geometries with $\tau = 1$ are generalized quadrangles, those with $\tau = \kappa - 1$ are transversal designs, and those with $\tau = \rho - 1$ are known as nets. In order to have some concrete examples of new directed strongly regular graphs of small order, we consider a special class of partial geometries that are obtained from finite affine planes.

Let $AP(q)$ denote the affine plane of order $q$. Let $\overline{AP}(q)$ denote the partial geometry obtained from $AP(q)$ by considering all $q^2$ points and taking the lines of $l$ parallel classes of the plane. Then $\overline{AP}(q)$ inherits the following properties from $AP(q)$: (i) every point is incident with $l$ lines, and every line is incident with $q$ points, (ii) any two points are incident with at most one line, (iii) if $p$ and $L$ are non-incident point-line pair, there are exactly $l - 1$ lines containing $p$ which meet $L$. That is, $\overline{AP}(q) = pg(q,l,l-1)$. 

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Corollary 3.2. Let \( D = D(\overline{AP}^l(q)) \) be the directed graph \( D(pg(q,l,l-1)) \) defined as in Definition 3.1. Then \( D \) is a directed strongly regular graph with parameters
\[
(v, k, \lambda, \mu) = (lq^2(q-1), \ lq(q-1), \ lq-l+1, \ (l-1)(q-1), \ lq-l+1).
\]

Proof: It immediately follows from Theorem 3.1. \( \square \)

In particular, if \( l = q, \overline{AP}^l(q) = pg(q,q,q-1) \) is a transversal design \( TD(q,q) = (P, G, L) \) of order \( q \), block size \( q \), and index 1 in the following sense.

(i) \( P \) is the set of \( q^2 \) points of \( \overline{AP}^l(q) \);
(ii) \( G \) is the partition of \( P \) into \( q \) classes (groups) such that each class consists of \( q \) points that were collinear in \( AP(q) \) but not in \( \overline{AP}^l(q) \);
(iii) \( L \) is the set of \( q^2 \) lines (blocks);
(iv) every unordered pair of points in \( P \) is contained in either exactly one group or in exactly one block, but not both.

Corollary 3.3. Let \( D \) be the graph \( D(\overline{AP}^l(q)) \) defined as above. Then \( D \) is a directed strongly regular graph with parameters
\[
(v, k, \lambda, \mu) = (q^3(q-1), \ q^2(q-1), \ q^2 - q + 1, \ (q-1)^2, \ q^2 - q + 1).
\]
The eigenvalues of this graph are \( q^2(q-1), \ -q \) with multiplicities \( 1, \ q^2 - q^2 + q - 1, \ q(q-1) \), respectively.

Example 3.1. This method produces the DSRGs with the following parameter sets (with order up to 110 and \( m = 1 \) only):

\[
\begin{array}{c|cc}
(v, k, \lambda, \mu) & l, q, m \\
\hline
(8, 4, 3, 1, 3) & l = q = 2 \\
(12, 6, 4, 2, 4) & l = 3, q = 2 \\
(16, 8, 5, 3, 5) & l = 4, q = 2 \\
(20, 10, 7, 4, 7) & l = 5, q = 2 \\
(24, 12, 7, 5, 7) & l = 6, q = 2 \\
(28, 14, 8, 6, 8) & l = 7, q = 2 \\
(32, 16, 9, 7, 9) & l = 8, q = 2 \\
(36, 12, 5, 2, 5) & l = 2, q = 3 \\
(54, 18, 7, 4, 7) & l = q = 3 \\
(72, 24, 9, 6, 9) & l = 4, q = 3 \\
(96, 24, 7, 3, 7) & l = 2, q = 4
\end{array}
\]

Among these, the parameter sets \((54, 18, 7, 4, 7)\): and so \((108, 36, 14, 8, 14), (162, 54, 21, 12, 21) \cdots \) confirm the feasibility of the putative parameter sets listed in \([4]\). These graphs share the same automorphism group which is isomorphic to \(((Z_3 \times Z_3) \times Z_3) \rtimes Z_2) \rtimes Z_2 \).

Remark 3.2. When \( l = 2 \), we obtain the directed strongly regular graph with parameters \((4q^2, 4q, 2q-1, q-1, 2q-1)\) in both Section 2 and Section 3. The two graphs are shown to be isomorphic although the construction methods are different. We demonstrate the isomorphism through an example; namely, for the case when \( q = 3 \), the directed strongly regular graph with parameters...
by S-disjoint \( q \times, S_D = \{1, 2, \ldots, 9\} \) may be applied to produce new families [13, Corollary 6.3]. Nevertheless, we introduce our construction to illustrate a family of graphs have been constructed by Godsil, Hobart and Martin in sections. The graphs produced by this construction are not new. The latter be used here may be viewed as a degenerate case of those used in earlier sections. The graphs obtained from partitioned sets

\[ (h, ij) \to (h', i'j') \text{ iff } h \in i'j' \]

while that of \( D_2 \) is defined by

\[ (hij, l) \to (h'i'j', l') \text{ iff } l' \in \{hij\} \]

| 1,24 ↔ 123,4 | 2,14 ↔ 456,1 | 3,14 ↔ 789,1 |
| 1,25 ↔ 123,5 | 2,15 ↔ 456,2 | 3,15 ↔ 789,2 |
| 1,26 ↔ 123,6 | 2,16 ↔ 456,3 | 3,16 ↔ 789,3 |
| 1,34 ↔ 123,7 | 2,34 ↔ 456,7 | 3,24 ↔ 789,4 |
| 1,35 ↔ 123,8 | 2,35 ↔ 456,8 | 3,25 ↔ 789,5 |
| 1,36 ↔ 123,9 | 2,36 ↔ 456,9 | 3,26 ↔ 789,6 |
| 4,15 ↔ 147,2 | 5,14 ↔ 258,1 | 6,14 ↔ 369,1 |
| 4,16 ↔ 147,3 | 5,16 ↔ 258,3 | 6,15 ↔ 369,2 |
| 4,25 ↔ 147,5 | 5,24 ↔ 258,4 | 6,24 ↔ 369,4 |
| 4,26 ↔ 147,6 | 5,26 ↔ 258,6 | 6,25 ↔ 369,5 |
| 4,35 ↔ 147,8 | 5,34 ↔ 258,7 | 6,34 ↔ 369,7 |
| 4,36 ↔ 147,9 | 5,36 ↔ 258,9 | 6,35 ↔ 369,8 |

4. Graphs obtained from partitioned sets

In this section we construct directed strongly regular graphs for two parameter sets,

\[ (ql(l - 1), q(l - 1), q, 0, q) \]

and

\[ (ql(l - 1), 2q(l - 1) - 1, ql - 1, ql - 2, 2q) \]

for all positive integers \( q \geq 1 \) and \( l \geq 3 \). The incidence structure which will be used here may be viewed as a degenerate case of those used in earlier sections. The graphs produced by this construction are not new. The latter family of graphs have been constructed by Godsil, Hobart and Martin in [13] Corollary 6.3]. Nevertheless, we introduce our construction to illustrate a variation of constructions which may be applied to produce new families of directed strongly regular graphs.

Let \( P \) be a set of \( ql \) elements (‘points’), and let \( S_1, S_2, \ldots, S_l \) be \( l \) mutually disjoint \( q \)-element subsets of \( P \) (‘blocks’). We denote the family of blocks by \( S = \{S_1, S_2, \ldots, S_l\} \). We will say that point \( x \in P \) and block \( S \in S \) is a non-incident point-block pair if and only if \( x \notin S \).

For the directed strongly regular graph with the first parameter set, let \( D = D(P, S) \) be the directed graph with its vertex set

\[ V(D) = \{(x, S) \in P \times S : x \notin S\}, \]

and directed edges defined by \( (x, S) \to (x', S') \) if and only if \( x \in S' \).
Theorem 4.1. Let $P, S$ and $D(P, S)$ be as the above. Then $D(P, S)$ is a directed strongly regular graph with parameters

$$(ql(l - 1), q(l - 1), q, 0, q).$$

Proof: Easy counting arguments give the values for parameters. □

Remark 4.1. Jørgensen [17] showed that the directed strongly regular graph with parameters $(ql(l - 1), q(l - 1), q, 0, q)$ is unique for all positive integers $l$ and $q$.

For the directed strongly regular graph with the second parameter set, let $G = G(P, S)$ be the directed graph with its vertex set

$$V(G) = \{(x, S) \in P \times S : x \notin S\},$$

where $P$ and $S$ are as the above, and let edges on $V(G)$ be defined by:

$$\{(x, S) \rightarrow (x', S')\} \text{ if and only if}$$

$$\left\{\begin{array}{ll}
(1) & x \in S'; \\
(2) & S = S' \text{ and } x \neq x'.
\end{array}\right.$$

Theorem 4.2. Let $G$ be the graph $G(P, S)$ defined as above. Then $G$ is a directed strongly regular graph with parameters

$$(ql(l - 1), 2q(l - 1) - 1, ql - 1, ql - 2, 2q).$$

Proof: Clearly $v = ql(l - 1)$ as $|V(G)| = |V| \cdot (|S| - 1)$.

Given a vertex $(x, S)$, let

$$N_1((x, S)) := \{(x', S') : x \in S' \text{ and } x' \in S\},$$

$$N_2((x, S)) := \{(x', S') : S = S' \text{ and } x \neq x'\},$$

$$N_3((x, S)) := \{(x', S') : x \in S' \text{ and } x' \notin S\}.$$

Then by simple counting, we have

$$|N_1((x, S))| = q,$$

$$|N_2((x, S))| = q(l - 2) + q - 1,$$

$$|N_3((x, S))| = q(l - 2).$$

Hence, $t = |N_1((x, S))| + |N_2((x, S))| = ql - 1$, and $k = t + |N_3((x, S))| = 2ql - 2q - 1$.

Finally, we count the number of vertices $(x^*, S^*)$ that belongs to $N^+(x, S) \cap N^-(x', S')$ for $\lambda$ and $\mu$.

For $\lambda$, given an edge $(x, S) \rightarrow (x', S')$, we shall need to consider the following two cases separately: (Case 1) when $x \in S'$, and (Case 2) when $S = S'$ and $x \neq x'$.

Case (1). Suppose $x \in S'$. Then unless $S^* = S$, $S^*$ must contain $x$ which forces $S^* = S'$. With $S^* = S'$, all points that are not belong to $S' \cup \{x\}$ can be chosen to be $x^*$; and so, there are $q(l - 1) - 1$ choices for $x^*$. With $S^* = S$, the $q - 1$ points of the set $S' \setminus \{x\}$ are possible for $x^*$. Together there are $ql - 2$ vertices $(x^*, S^*)$ belonging to $N^+(x, S) \cap N^-(x', S')$.

Case (2). Let $S = S'$. Then with the choice of $S^* = S = S'$, $x^*$ can be chosen from $P \setminus (S \cup \{x, x'\})$; and so, there are $q(l - 1) - 2$ choices for $x^*$. Also with the choice of $S^*$ being the block containing $x$, any point in $S$ can
be chosen as \(x^*\); and so, there are \(q\) possible choices for \(x^*\). Together we have \(ql - 2\) choices for \((x^*, S^*)\) as well. Hence we have \(\lambda = ql - 2\).

For \(\mu\), suppose \(S \neq S'\) and \(x \notin S'\). Then it is clear that
\[
\mu = \left| \{(x^*, S^*) : x \in S^*, x^* \in S'\} \cup \{(x^*, S^*) : S^* = S, x \in S'\} \right| = q + q.
\]
This completes the proof. \(\square\)

5. Graphs obtained from affine resolvable designs

Let \(v, k\) and \(\Lambda\) be positive integers such that \(v > k > \Lambda\). Let \(D = (P, B)\) be a \(2 - (v, k, \Lambda)\) design such that \(|P| = v\), each block consists of \(k\) points, and each pair of distinct points occurs in exactly \(\Lambda\) blocks. Every point belongs to \(r = \frac{A(v-1)}{k-1}\) blocks, and the block set \(B\) consists of \(b = \frac{A(v-1)}{k(k-1)}\) blocks. A parallel class is a set of pairwise disjoint blocks that is a partition of \(P\). Obviously a parallel class can exist only if \(v\) is divisible by \(k\) or equivalently, \(b\) is divisible by \(r\).

A \(2 - (v, k, \Lambda)\) design is resolvable if the block set \(B\) can be partitioned into \(r\) disjoint parallel classes. A resolvable design \(D = (P, B)\) is called affine (or affine resolvable) if there exists a positive integer \(m\) such that any two non-parallel blocks intersect in exactly \(m\) points. All parameters of an affine design may be expressed in terms of \(m\) and another parameter \(s\) where \(s = (v/k) = (b/r)\), the number of blocks in a parallel class, as follows (cf. [1]):

\[
v = ms^2, \quad b = \frac{s(ms^2 - 1)}{s - 1}, \quad r = \frac{ms^2 - 1}{s - 1}, \quad k = ms, \quad \Lambda = \frac{ms - 1}{s - 1}.
\]

Let \((P, B)\) be an affine resolvable \(2 - (v, b, k, r, \Lambda)\) design with \(m = k^2/v\) and \(s = v/k\). We denote this design by \(AP_m(s)\). Let \(AP_m^l(s)\) denote the incidence structure obtained from \(AP_m(s)\) by considering all \(v\) points in \(P\) and taking the blocks of any \(l\) parallel classes of the design \((P, B)\) for \(1 < l < r\). Let \(\overline{B}\) denote the set of blocks in the \(l\) parallel classes. Then \(AP_m^l(s) = (P, \overline{B})\) inherits the following properties from \(AP_m(s)\): (i) every point is incident with \(l\) blocks, and every block is incident with \(k\) points, (ii) any two points are incident with at most \(\Lambda\) blocks, (iii) if \(p\) and \(B\) are non-incidence point-block pair, there are exactly \(l - 1\) blocks containing \(p\) which intersect with \(B\) in \(m\) point(s).

**Theorem 5.1.** For \(m \geq 1, s \geq 2\) and \(l \geq 2\), let \(AP_m^l(s)\) be the tactical configuration defined as above. Let \(D = D(AP_m^l(s))\) be the directed graph with its vertex set

\[
V(D) = \{(p, B) \in P \times \overline{B} : p \notin B\},
\]
and directed edges defined by
\[
(p, B) \rightarrow (p', B') \text{ if and only if } p \in B'.
\]

Then \(D\) is a directed strongly regular graph with parameters
\[
v = mls^2(s-1), \quad k = mls(s-1), \quad t = \mu = m(ls-l+1), \quad \lambda = m(l-1)(s-1).
\]
Proof: By straightforward counting, we find
\[ v = 1s(v-k), \quad k = l(v-k), \quad t = (l-1)(k-m)+k, \quad \lambda = (l-1)(k-m), \quad \mu = (l-1)(k-m)+k. \]
If we write these parameters in terms of \( l, m \) and \( s \), we get the desired expression. □

**Remark 5.1.** This is a generalization of the construction of the directed strongly regular graphs obtained from the degenerated affine plane described in Corollary 5.2. In fact, the graphs obtained for \( m = 1 \) in Theorem 5.1 are the ones obtained in Corollary 5.2. From the graphs with \( m = 1 \), we can obtain graphs with \( m \geq 2 \) by the ‘\( m \)-multiple construction method’ introduced by Duval [7].

**Remark 5.2.** In regard to the source of affine resolvable designs, we would like to recall the concept of orthogonal arrays. An orthogonal array of strength 2, \( A[n, q, \mu] \) from an \( \mu n \times n \) matrix with entries from an \( n \)-set with the property that any two of its rows contain each of the \( n^2 \) ordered pairs of symbols exactly \( \mu \) times. The parameters of an orthogonal array satisfy the inequality \( q \leq \frac{\mu n^2-1}{n-1} \). An orthogonal array is called complete if if satisfy the equality \( q = \frac{\mu n^2-1}{n-1} \). The existence of an affine resolvable \( 2-(v, b, k, r, \Lambda) \) is equivalent to the existence of a complete orthogonal array with \( n = \frac{k^2}{v} \), \( q = r \), \( \mu = \frac{k^2}{v} \) [21]. Many concrete examples may be obtained from the study of orthogonal arrays.

Here are the parameters of the graphs (with small orders) that are constructed by our method described above. – This table may be deleted.

| \( v \) | \( k \) | \( t \) | \( \lambda \) | \( \mu \) | \( (v,k,\Lambda; s,m,l) \) | \( v \) | \( k \) | \( t \) | \( \lambda \) | \( \mu \) | \( (v,k,\Lambda; s,m,l) \) |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 12 | 6 | 4 | 2 | 4 | (4, 2, 6; 2, 1, 3) | 132 | 66 | 36 | 30 | 36 | (12, 6, 22; 2, 3, 11) |
| 8 | 4 | 3 | 1 | 3 | (4, 2, 6; 2, 1, 2) | 120 | 60 | 33 | 27 | 33 | (12, 6, 22; 2, 3, 10) |
| 56 | 28 | 16 | 12 | 16 | (8, 4, 14; 2, 2, 7) | 108 | 54 | 30 | 24 | 30 | (12, 6, 22; 2, 3, 9) |
| 48 | 24 | 14 | 10 | 14 | (8, 4, 14; 2, 2, 5) | 96 | 48 | 27 | 21 | 27 | (12, 6, 22; 2, 3, 8) |
| 40 | 20 | 12 | 8 | 12 | (8, 4, 14; 2, 2, 5) | 84 | 42 | 24 | 18 | 24 | (12, 6, 22; 2, 3, 7) |
| 32 | 16 | 10 | 6 | 10 | (8, 4, 14; 2, 2, 4) | 72 | 36 | 21 | 15 | 21 | (12, 6, 22; 2, 3, 6) |
| 24 | 12 | 8 | 4 | 8 | (8, 4, 14; 2, 2, 3) | 60 | 30 | 18 | 12 | 18 | (12, 6, 22; 2, 3, 5) |
| 16 | 8 | 6 | 2 | 6 | (8, 4, 14; 2, 2, 2) | 48 | 24 | 15 | 9 | 15 | (12, 6, 22; 2, 3, 4) |
| 72 | 24 | 9 | 6 | 9 | (9, 3, 12; 3, 1, 4) | 36 | 18 | 12 | 6 | 12 | (12, 6, 22; 2, 3, 3) |
| 54 | 18 | 7 | 4 | 7 | (9, 3, 12; 3, 1, 3) | 24 | 12 | 9 | 3 | 9 | (12, 6, 22; 2, 3, 2) |
| 36 | 12 | 5 | 3 | 5 | (9, 3, 12; 3, 1, 2) |   |   |   |   |   |   |

6. **Graphs obtained from \( 2-(v, k, \Lambda) \) designs**

The balanced incomplete block designs have been used to produce directed strongly regular graphs (cf. [4]). In this section, we will see that we can also construct many directed strongly regular graphs on the sets of anti-flags of every \( 2-(v, b, k, \Lambda) \) design (with \( b+\Lambda > 2r \)) in the following two different ways.
Theorem 6.1. Let $D = (P, \mathcal{B})$ be a $2-(v, b, k, r, \Lambda)$ design with $b+\Lambda > 2r$. Let $D = D(P, \mathcal{B})$ be the directed graph with its vertex set

$$V(D) = \{(p, B) \in P \times \mathcal{B} : p \notin B\},$$

and directed edges defined by $(p, B) \rightarrow (p', B')$ if and only if $p' \in B$.

Then $D$ is a directed strongly regular graph with parameters

$$v = v(b - r) = \frac{\Lambda v(v - 1)(v - k)}{k(k - 1)}, \quad k = k(b - r) = \frac{\Lambda(v - 1)(v - k)}{k(k - 1)},$$

$$t = \mu = k(r - \Lambda) = \frac{\Lambda k(v - k)}{k - 1}, \quad \lambda = (k - 1)(r - \Lambda) = \Lambda(v - k).$$

Proof: It is easy to verify the values of the parameters. \hfill \Box

Remark 6.1. This is another way to construct the graphs that may be constructed by the method of [T8] in Brouwer and Hobart’s home page. The construction [T8] uses the complementary designs of 2-designs instead of 2-designs themselves.

The second method of obtaining directed strongly regular graphs from the anti-flags of 2-designs uses a different rule for the adjacency.

Theorem 6.2. Let $D = (P, \mathcal{B})$ be a $2-(v, b, k, r, \Lambda)$ design with $b+\Lambda > 2r$. Let $D = D(P, \mathcal{B})$ be the directed graph with its vertex set

$$V(D) = \{(p, B) \in P \times \mathcal{B} : p \notin B\},$$

and directed edges given by $(p, B) \rightarrow (p', B')$ if and only if either $p' \in B$ or $p = p'$ and $B \neq B'$. Then $D$ is a directed strongly regular graph with parameters

$$v = v(b - r),$$

$$k = k(b - r) + (b - r - 1),$$

$$t = k(r - \Lambda) + (b - r - 1),$$

$$\lambda = k(r - \Lambda) + (b - r - 2),$$

$$\mu = (k + 1)(r - \Lambda).$$

Proof: The values of the parameters can be verified by the similar counting argument. \hfill \Box

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DEPARTMENT OF MATHEMATICS, IOWA STATE UNIVERSITY, AMES, IOWA, 50011, U. S. A.

E-mail address, O. Olmez: oolmez@iastate.edu

DEPARTMENT OF MATHEMATICS, IOWA STATE UNIVERSITY, AMES, IOWA, 50011, U. S. A.

E-mail address, S. Y. Song: sysong@iastate.edu