Branching Rules for Supercuspidal Representations of \( \text{SL}_2(k) \), for \( k \) a \( p \)-adic field

Monica Nevins

Department of Mathematics and Statistics, University of Ottawa, Ottawa, Canada K1N 6N5

Abstract

The restriction of a supercuspidal representation of \( \text{SL}_2(k) \) to a maximal compact subgroup decomposes as a multiplicity-free direct sum of irreducible representations. We explicitly describe this decomposition, and determine how the spectrum of this decomposition varies as a function of the parameters describing the supercuspidal representation.

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1. Introduction

The study of branching rules is that of considering the decomposition of an irreducible representation to an interesting subgroup, with the goal of revealing additional internal symmetries of the original representation, and exposing commonalities in families of representations. In the case of Lie groups, the most interesting subgroup to consider is a maximal compact subgroup (which is unique up to conjugacy) and the resulting theory of minimal \( K \)-types \cite{3} is a significant milestone in that representation theory.

When we turn to the case of a \( p \)-adic reductive algebraic group \( G \), there are several immediate differences. Maximal compact subgroups are open, and not themselves \( p \)-adic algebraic groups; moreover, there exist in general several conjugacy classes of maximal compact subgroups. Nevertheless, through the work of A. Moy, G. Prasad, C. Bushnell, P. Kutzko and others \cite{21,4}, a kind of analogue of the theory of minimal \( K \)-types has emerged, now called the theory of types. The various compact subgroups arising in this theory are open but generally not maximal.

It remains an open question to characterize the representations of a given maximal compact subgroup \( K \) which occur in the decomposition of an irreducible representation of \( G \). Aspects of this question were considered for the group \( \text{PGL}_2(k) \) by A. Silberger \cite{30,31} and for \( \text{GL}_2(k) \) by W. Casselman \cite{8} and K. Hansen \cite{12}. Branching rules for the Weil representation of \( \text{Sp}_{2n}(k) \) were considered for the restriction to one maximal compact subgroup by D. Prasad in \cite{28} and a more explicit decomposition, relative to all conjugacy classes of maximal compact subgroups, was given by K. Maktouf and P. Torasso in \cite{17}. The author considered the case of principal series of \( \text{SL}_2(k) \) in \cite{24} and, together with P. Campbell, addressed principal series of \( \text{GL}_3(k) \) in \cite{5,6}. Recently, U. Onn and P. Singla completed the study of branching rules of unramified principal series of \( \text{GL}_3(k) \) by
giving an explicit description of the decomposition into irreducible representations of $K$ [27]. The work with $GL_3(k)$, along with related calculations by U. Onn, A. Prasad and L. Vaserstein on sizes of double coset spaces in [26], makes it clear that the question of decomposing principal series in the general case will be highly nontrivial.

The work to date suggests that the presentation of a satisfactory answer is tied to the development of the representation theory of Lie groups over finite local rings. This theory is of high current interest but as yet far from complete; as only one example note the work of A.-M. Aubert, U. Onn, A. Prasad and A. Stansinski [2].

We would also like to signal the closely related work of J. Lansky and A. Raghuram [16], whose focus is the determination of newforms for $SL_2(k)$; this amounts to identifying and characterizing the irreducible component of least depth (and least degree) occurring in the restriction of a representation of $SL_2(k)$ to a maximal compact subgroup.

Constructions of supercuspidal representations, phrased in terms of compact open subgroups which arise from the Bruhat-Tits building of $G$, offer the possibility of a geometric, or at least building-theoretic, approach to the description of the branching rules of the corresponding representations. These include works by J. Adler [1], L. Morris [18, 19] and more recently J. K. Yu [32], whose construction of tame supercuspidal representations was shown to be exhaustive by J. Kim [13].

The present work is the first step of a longer term project of describing branching rules for tame supercuspidal representations. In this paper, we consider the restriction of supercuspidal representations of $SL_2(k)$, where $k$ is a local nonarchimedean field of residual characteristic different from 2, to the maximal compact subgroup $K = SL_2(R)$, where $R$ denotes the integer ring of $k$. The main results on branching rules are given in Theorems 5.3 and 6.2. An explicit description of all irreducible representations of $K$ was given by J. Shalika [29], so our answer is given as a sum of known representations. We then analyse the intertwining over $K$ of different supercuspidal representations of $G$, culminating in Corollary 7.7 and Theorem 7.10.

This work, together with [24], completes the branching rules for $SL_2(k)$. Some consequences of these results, based on an early version of the present paper, are given in [25].

The methods and approach used here are greatly inspired by those in [12]. For instance, the idea of using character calculations in Section 5 to resolve the branching rules in the depth-zero case comes from the similar approach of K. Hansen. In fact, although our presentation makes no use of the corresponding results for $GL_2(k)$, the description of the decomposition could be accomplished via the following alternate route, which was used in the author’s first draft. Each tame supercuspidal representation of $SL_2(k)$ appears in the restriction to $SL_2(k)$ of some supercuspidal representation of $GL_2(k)$ [28]. The decomposition of the restriction to $GL_2(R)$ of a supercuspidal representation of $GL_2(k)$ is given in [12]. Consequently, it suffices to describe the restriction of each of these representations of $GL_2(R)$ to $SL_2(R)$ and identify how the various pieces are apportioned between the supercuspidal representations of $SL_2(k)$.

Conversely, from the present work and the literature one may deduce that, given a supercuspidal representation of $GL_2(k)$ of depth $r$, all of the irreducible representations of $GL_2(R)$ of depth greater than $r$ which occur in its restriction decompose, upon further restriction, to a direct sum of two irreducible representations of $SL_2(R)$. Furthermore, the methods of Section 7 could easily be used to extend the work of K. Hansen to determine the intertwining over $GL_2(R)$ of different supercuspidal representations. Note that the $GL_2$ analogue of Corollary 7.7 was previously known [8] and is much simpler.

The advantage of the current approach to the branching rules for $SL_2(k)$ is two-fold. For one, the present classification of supercuspidal representations of $SL_2(k)$ is tight: work by J. Hakim and
F. Murnaghan\cite{11} has answered the question of equivalence of supercuspidal representations in J. K. Yu’s construction as a function of the tamely ramified cuspidal $G$-data of their construction; whereas this is less clear from the description of supercuspidal representations of $\mathrm{GL}_2(k)$ used in \cite{12}. For another, this building-theoretic approach seems to be a most promising language for considering tame supercuspidal representations of general connected reductive $p$-adic groups.

In this paper we consider only one conjugacy class of maximal compact subgroup, namely $K = \mathrm{SL}_2(\mathbb{R})$. There is another conjugacy class, represented by $K^\eta$, where $\eta \in \mathrm{GL}_2(k)$ is as in \cite{2}. As was shown explicitly in \cite{24} for the principal series of $\mathrm{SL}_2(k)$, the conjugacy of these two groups under $\mathrm{GL}_2(k)$ implies that their branching rules are completely analogous.

Generalizing the present work to higher rank cases poses significant challenges since, for example, in our arguments we do at several points use explicit descriptions of the anisotropic tori in $G$. Also, to parametrize certain double cosets, and in some calculations, we represent elements of the group in matrix form. Some of these difficulties are surmountable, using for example the work of S. Debacker \cite{9} to describe the tori, and accepting a less strongly explicit description of the representations which arise. That said, it is expected that any success towards the general case would advance the representation theory of the maximal compact subgroups, and so is intrinsically interesting.

The paper is organized as follows. In Section 2 we establish our notation. We also describe some Moy-Prasad filtration subgroups and give a list of representatives of the anisotropic tori in $\mathrm{SL}_2(k)$. In Section 3 we give the known parametrization of all supercuspidal representations of $\mathrm{SL}_2(k)$, which is naturally divided into two cases: zero depth and positive depth. We also present a subset of J. Shalika’s parametrization of representations of $\mathrm{SL}_2(\mathbb{R})$. We describe the necessary tools for restricting representations in Section 4, including a statement of Mackey theory applicable to the present setting, and then we present a first decomposition of the supercuspidal representations upon restriction to $\mathrm{SL}_2(\mathbb{R})$.

Section 5 is devoted to the depth-zero case. We decompose into irreducible representations the components obtained through the Mackey decomposition, and identify their simple uniform description relative to Shalika’s parametrization. For the case of positive depth the corresponding results are proven in Section 6. We conclude in Section 7 with an analysis of the intertwining over $\mathrm{SL}_2(\mathbb{R})$ of distinct supercuspidal representations of $\mathrm{SL}_2(k)$.

2. Notation and Background

2.1. General notational conventions

Let $k$ be a local nonarchimedean field of residual characteristic $p \neq 2$. Its characteristic may be 0 or $p$. The results of this paper are all valid in this general setting, but for the sake of brevity we will refer to our field as a $p$-adic field and our group as a $p$-adic group.

Denote the residue field of $k$ by $\kappa$, a finite field of order $q$. Let the integer ring of $k$ be $\mathcal{O}$ and its maximal ideal $\mathfrak{P}$. Let $\varpi$ be a uniformizer, and normalize the valuation on $k$ so that $\mathrm{val}(\varpi) = 1$. Then $k^\times/(k^\times)^2$ can be represented by $\{1, \varepsilon, \varpi, \varepsilon \varpi\}$ where $\varepsilon$ is a fixed nonsquare in $\mathcal{O}^\times$ (which is chosen to be $-1$ when $-1 \notin (k^\times)^2$). We shall use $k' = k[\gamma]$ to denote a quadratic extension field of $k$ by an element $\gamma$ such that $\gamma^2 \in \{\varepsilon, \varpi, \varepsilon \varpi\}$; these give all quadratic extensions of $k$ (up to isomorphism). The units of $\mathcal{O}$ admit a filtration by subgroups $\mathcal{U}_n$ where we set

$$\mathcal{U}_n = \begin{cases} 
\mathcal{O}^\times & \text{if } n = 0, \\
1 + \mathfrak{P}^n & \text{if } n > 0.
\end{cases}$$
We fix an additive quasi-character $\Psi$ of $k$ which is trivial on $\mathcal{P}$ but nontrivial on $\mathcal{R}$.

Given a subgroup $H$ of a group $G$ we write $H^g$ for the group $gHg^{-1}$. If $\sigma$ is a representation of $H$ we write $\sigma^g$ for the corresponding representation of $H^g$ given on elements $h$ of $H^g$ by $\sigma^g(h) = \sigma(g^{-1}hg)$. Thus unfortunately we have $\sigma^{gh} = (\sigma^h)^g$.

Given a closed subgroup $K$ of a connected reductive $p$-adic group $G$, and a representation $(\pi, V)$ of $K$, the compactly induced representation $\text{c-Ind}^G_K \pi$ is given by right action by $G$ on the space of functions

$$\left\{ f : G \to V \mid \forall k \in K, g \in G, f(kg) = \pi(k)f(g), \text{ f is smooth and is compactly supported mod } K \right\}.$$

By Mautner's theorem, if $K$ is open and compact mod the center $Z$ of $G$, and if $\text{c-Ind}_K^G \pi$ is irreducible, then it is supercuspidal. It is a lasting conjecture, proven now in many cases \cite{35, 13, 32} that all supercuspidal representations of $G$ arise in this way, for some choice of $K$ and $\pi$.

Now let $G = \text{SL}_2$ denote the algebraic group of $2 \times 2$ matrices of determinant one and set $\mathcal{G} = G(k)$. Its center $Z$ is the two-element group $\{\pm I\}$. Let $\mathfrak{g}$ denote its Lie algebra and $\mathfrak{g}^*$ its algebraic dual. Throughout, we abuse notation by writing $\mathfrak{g} = \mathfrak{g}(k)$ and $\mathfrak{g}^* = \mathfrak{g}^*(k)$. Given subsets $S_i$ of $k$ or $k^\times$, we define a corresponding set of matrices with the notation

$$\left\{ \begin{bmatrix} S_1 & S_2 \\ S_3 & S_4 \end{bmatrix} \right\} = \left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \mid a_i \in S_i \right\}.$$  

We also write $\text{diag}(a, b)$ for a diagonal matrix with diagonal entries $a, b \in k$, and will write $X(u, v)$ for an antidiagonal matrix starting in Section \ref{section:antidiagonal}.

For any $Y \in \mathfrak{g}^*$, the map $X \mapsto \Psi((Y, X))$ defines a smooth quasi-character of $\mathfrak{g}$, and all smooth quasi-characters of $\mathfrak{g}$ arise in this way. In the presence of a nondegenerate bilinear form on $\mathfrak{g}$ (taken without loss of generality to be the trace form $\text{Tr}$ in this case) we can and do simply parametrize these quasi-characters by elements of $\mathfrak{g}$. Thus for $Y \in \mathfrak{g}$ we have a quasi-character $X \mapsto \Psi(\text{Tr}(YX))$. While convenient for us, this shortcut is not used in \cite{22}, where the use of the dual Lie algebra throughout permits a uniform general construction.

2.2. Filtration subgroups in $\text{SL}_2(k)$

Let $\mathcal{S}$ denote a maximal torus of $G$, split over $k$, with associated root system $\Phi$. Let $\mathcal{A} = \mathcal{A}(G, \mathcal{S}, k)$ denote the corresponding apartment (as in \cite{22, 1.2}). We may think of $\mathcal{A}$ as the affine space under $X_*(\mathcal{S}) \otimes \mathbb{R}$, where $X_*(\mathcal{S})$ is the group of $k$-rational cocharacters of $\mathcal{S}$. Let $\mathcal{B} = B(G, k)$ denote the reduced Bruhat-Tits building for $G$ over $k$ and let $y \in \mathcal{B}$. When, as here, $\mathcal{G}$ is semisimple, simply connected and split over $k$, the stabilizer in $\mathcal{G}$ of $y$ is equal \cite{22, \S3.1} to $G_{y,0}$, the connected parahoric subgroup associated with $y$, as described by A. Moy and G. Prasad in \cite{21}. It is thus unambiguous to write $G_y$ for $G_{y,0}$.

Define $\tilde{\mathbb{R}} = \mathbb{R} \cup (\mathbb{R}^+ \cup \{\infty\}$ as in \cite{22, 6.4.1] and associate to each $y \in \mathcal{A}, a \in \Phi$ and $r \in \tilde{\mathbb{R}}$ a subgroup $\mathcal{G}_a(k)_{y,r}$ of the corresponding root subgroup as well as the associated $\mathcal{R}$-submodule $\mathfrak{g}_{a,y,r} \subset \mathfrak{g}$. For $r \geq 0$, one can similarly define filtration subgroups on the torus, $\mathcal{S}(k)_{y,r}$. Then the Moy-Prasad filtration subgroup $\mathcal{G}_{y,r}$ is the group generated by $\{ \mathcal{S}(k)_{y,r} \cup \mathcal{G}_a(k)_{y,r} \mid a \in \Phi \}$. Given a subset $I \subseteq \mathcal{A}$, one sets $\mathcal{G}_{I,r} = \cap_{y \in I} \mathcal{G}_{y,r}$. The lattices $\mathfrak{g}_{y,r}$ in $\mathfrak{g}$, for $y \in \mathcal{A}$ and $r \in \tilde{\mathbb{R}}$, are defined similarly.

More concretely, for the case $G = \text{SL}_2(k)$ we choose $\mathcal{S}$ to be the diagonal torus, with $\Phi = \{\pm \alpha\}$. Then the root subgroup $\mathcal{G}_\alpha$ (respectively $\mathcal{G}_{-\alpha}$) is the group of unit upper triangular (respectively lower triangular) matrices in $\mathcal{G}$. By identifying an origin $y = 0$ in $\mathcal{A}$, and choosing coordinates so
that \( \alpha^\vee = 2 \), the vertices of the simplicial complex representing the action of the affine Weyl group on \( \mathcal{A} \) occur for integer coordinates. We have \( G_0 = SL_2(\mathbb{R}) \); this is a maximal compact subgroup of \( G \) which we denote by \( K \) throughout. On the other hand, at \( y = 1 \) we obtain

\[
G_1 = \left\{ \begin{bmatrix} R & P^{-1} \\ P & R \end{bmatrix} \right\} \cap G = K^\eta
\]

where \( \eta \in GL_2(k) \) is given by

\[
\eta = \begin{bmatrix} 1 & 0 \\ 0 & \bar{x} \end{bmatrix}.
\]

Then \( K \) and \( K^\eta \) are representatives of the two conjugacy classes of maximal compact subgroups of \( G \). If we define, for finite \( r \in \mathbb{R} \),

\[
[r] = \begin{cases} 
\min\{ n \in \mathbb{Z} | n \geq r \} & \text{if } r \in \mathbb{R}, \text{ and} \\
\min\{ n \in \mathbb{Z} | n > r \} & \text{if } r \in \mathbb{R}^+,
\end{cases}
\]

then for any \( y \in \mathcal{A} \) and \( r \geq 0 \) we have

\[
G_\alpha(k)_{y,r} = \left\{ \begin{bmatrix} 1 & P^{[r-y]} \\ 0 & 1 \end{bmatrix} \right\}, \quad G_{-\alpha}(k)_{y,r} = \left\{ \begin{bmatrix} 1 & 0 \\ P^{[y-r]} & 1 \end{bmatrix} \right\}
\]

and \( S(k)_{y,r} \) consists of those diagonal matrices in \( G \) with entries in \( U_{[r]} \). Thus for finite positive \( r \in \mathbb{R} \) we have

\[
G_{y,r} = \left\{ \begin{bmatrix} U_{[r]} & P^{[r-y]} \\ P^{[y-r]} & U_{[r]} \end{bmatrix} \right\} \cap G.
\]

Note that the Moy-Prasad filtration subgroups \( G_{0,r} \) of \( K \) and \( G_{1,r} \) of \( K^\eta \), for \( r \geq 0 \), are just the standard congruence subgroups \( K_r \) and \( K^\eta_r \), with distinct ones indexed by \( r \in \mathbb{N} \).

Given an irreducible smooth representation \( (\pi, V) \) of \( G \), the depth of \( \pi \) is defined as the least \( r \in \mathbb{R}_{\geq 0} \) such that there exists \( x \in B(G, k) \) for which \( V \) contains vectors invariant under \( G_{x,r+} \). We also refer to the depth of a representation of \( G_x \), for fixed \( x \).

### 2.3. Anisotropic tori in \( G \)

The construction of supercuspidal representations of \( G \) proceeds from its non-split tori. In \( SL_2(k) \) all non-split tori are totally anisotropic and split over a quadratic extension \( k' \) of \( k \). An anisotropic torus is called **unramified** if \( k' \) is unramified, and **ramified** otherwise. We give a list of representatives of conjugacy classes of anisotropic tori in Table 1; these are well-known. There are six when \( -1 \in (k^\times)^2 \); otherwise, we have \( T_1, \infty \cong T_{-1, \infty} \) and \( T_1, -\infty \cong T_{-1, \infty} \) and there are only four conjugacy classes.

The last column of Table 1 is needed for specifying, among other things, the Moy-Prasad filtration on the torus and its Lie algebra, and is determined as follows.

Suppose \( T \) is an anisotropic torus of \( G \) over \( k \). Since \( p \neq 2 \), the extension \( k' \) over which it splits is tamely ramified and so by [33, 2.6.1] one can view \( B \) as the subset of \( B(G, k') \) fixed by \( \text{Gal}(k'/k) \) and identify \( \mathcal{A}(G, T, k') = \mathcal{A}(G, T, k') \cap B(G, k) \). Since \( T(k) \) is totally anisotropic, this intersection consists of a single point (which one also sometimes has cause to identify with \( B(T, k) \)).

To determine this intersection, we use the identification \( B(G, k') = (G(k') \times \mathcal{A})/\sim \), where \( (g, x) \sim (h, y) \) if and only if there exists \( n \in N \), the normalizer of \( S(k') \), so that \( n \cdot x = y \) and \( g^{-1}hn \in G(k')_x \) [3, (7.4.1)]. Since \( T \) splits over \( k' \), there is a \( g \in G(k') \) such that \( T = S^g \), and \( \mathcal{A}(G, T, k') = g \cdot \mathcal{A} \) is identified with the image of \( \{ g \} \times \mathcal{A} \) in \( B(G, k') \). Write \( [g, x] \) for the point in
Table 1: Representatives of equivalence classes of anisotropic tori \( \mathbb{T}(k) \) in \( G \) (with \( a, b \) symbols denoting elements in \( R \), and where the torus consists of those matrices \( t(a, b) \) of determinant one); \( \eta \) was defined in [4]. Listed are: defining parameters \((\gamma_1, \gamma_2)\), the splitting field \( k' \) of the torus and the point \( y \in A \) such that \( \{y\} = A(G, \mathbb{T}, k') \cap B \). If \(-1 \not\in (k^\times)^2\) then \( T_{1, \varepsilon} \cong T_{e, \varepsilon^{-1} \varepsilon} \) and \( T_{1, \varepsilon} \cong T_{e, \varepsilon} \).

| Anisotropic torus \( \mathbb{T}_{\gamma_1, \gamma_2} \) = \( \mathbb{T}(k) \) | \( t(a, b) = \begin{bmatrix} a & b \gamma_1 \\ b & a \end{bmatrix} \) | Splitting field \( k' \) | \( A(G, \mathbb{T}, k) \) |
|---|---|---|---|
| \( T_{1, \varepsilon} \) \( = T_{\varepsilon^{-1}, \varepsilon} \) | \( \begin{bmatrix} a & b \\ b & a \end{bmatrix} \) | \( k(\sqrt{\varepsilon}) \) | \( y = 0 \) |
| \( T_{\varepsilon^{-1}, \varepsilon} \) | \( \begin{bmatrix} a & b \gamma_1 \\ b \gamma_1^{-1} & a \end{bmatrix} \) | \( k(\sqrt{\varepsilon}) \) | \( y = 1 \) |
| \( T_{1, \varepsilon} \) | \( \begin{bmatrix} a & b \gamma_1 \\ b & a \end{bmatrix} \) | \( k(\sqrt{\varepsilon}) \) | \( y = \frac{1}{2} \) |
| \( T_{e, \varepsilon^{-1} \varepsilon} \) | \( \begin{bmatrix} a & b \gamma_1 \\ b \gamma_1^{-1} & a \end{bmatrix} \) | \( k(\sqrt{\varepsilon}) \) | \( y = \frac{1}{2} \) |
| \( T_{e, \varepsilon} \) | \( \begin{bmatrix} a & b \gamma_1 \\ b \gamma_1^{-1} & a \end{bmatrix} \) | \( k(\sqrt{\varepsilon}) \) | \( y = \frac{1}{2} \) |

the building corresponding to the equivalence class of the pair \((g, x)\); then for example \([1, x] = x\) in our previous notation. The Galois group action on \( B(G, k') \) is given by \( \sigma([g, x]) = [\sigma(g), x] \) for each \( \sigma \in \text{Gal}(k'/k) \). It follows that the Galois-fixed points of \( B(G, k') \), that is, the elements of \( B(G, k) \), are those \([g, x]\) for which \( \sigma(g^{-1})g \in \mathbb{G}(k')_1 \) for all \( \sigma \in \text{Gal}(k'/k) \).

In our case, the Galois group has order two and we write \( \text{Gal}(k'/k) = \{1, \sigma\} \). Given \( T \) such that \( \mathbb{T}(k) = \mathbb{T}_{\gamma_1, \gamma_2} \) for some \((\gamma_1, \gamma_2) \in k^2\) as in Table 1 the element

\[
g = \begin{bmatrix} 1 & -\frac{1}{2} \sqrt{\gamma_1 \gamma_2^{-1}} \\ \sqrt{\gamma_1^{-1} \gamma_2} & \frac{1}{2} \end{bmatrix} \in \mathbb{G}(k')
\]

satisfies \( T = \mathbb{S}^g \). For these judicious choices of \((T, g)\), we see that both \( g \) and \( \sigma(g^{-1}) \) lie in \( \mathbb{G}(k')_y \) for the given point \( y \). It follows that the unique Galois-fixed point of \( A(G, \mathbb{T}, k') \) is \([g, y] = [1, y] = y \in A(G, \mathbb{T}, k') \cap A \).

The filtration on the split torus \( \mathbb{S}(k') \) defines one on \( \mathbb{T}(k') \) by conjugation [21, Section 2.6], and hence on \( T = \mathbb{T}(k) \) by restriction to the set of Galois-fixed points. Since \( g \in \mathbb{G}(k')_y \) and \( T \subseteq \mathbb{G}_y \), this filtration is simply the intersection with \( T \) of the Moy-Prasad filtration of \( \mathbb{G}_y \). In particular, \( T_0 = T \) and for each positive \( r \in \bar{R} \) we have

\[
T_r = \{t(a, b) \in T \mid a \in U_{[r]}, b \gamma_1 \in \mathcal{P}^{[r-y]}\}
\]

when \( T = \mathbb{T}_{\gamma_1, \gamma_2} \). The Lie algebra \( t \) of \( T \) is the one-dimensional subalgebra of \( g \) spanned by

\[
X_T = \begin{bmatrix} 0 & \gamma_1 \\ \gamma_2 & 0 \end{bmatrix}
\]

For any \( r \in \bar{R} \), the corresponding filtration subring of \( t \) is

\[
t_r = \{aX_T \mid a \in k, a \gamma_1 \in \mathcal{P}^{[r-y]}\}.
\]

For any \( r \geq 0 \), elements \( X = aX_T \in t_{-r} \) satisfying \( \text{val}(a \gamma_1) = -r - y \) are called \( \mathcal{G} \)-generic of depth \( r \).
Remark 2.1. An element $X \in t_{-r}$ (or more precisely, the element of $t^*$ that it represents via our identification $t \cong t^*$) is $G$-generic of depth $r$ if it satisfies the two conditions (GE1) and (GE2) of [35, §8]. Since $p \neq 2$, (GE2) follows from (GE1) by [35, Lemma 8.1]. Condition (GE1) is that $\text{val}(\text{Tr}(XH_\alpha)) = -r$, where $H_\alpha = d\alpha^\vee(1)$ corresponds to the coroot $\alpha^\vee$ of the unique positive root $\alpha$ of $(G, T)$. Explicitly, we have $H_\alpha = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}^g$, where $g$ is as defined in (4), and so we verify directly that (GE1) is equivalent to our stated condition.

Let $r > 0$. Then there is a natural group isomorphism $e: t_r/t_{r+} \to T_r/T_{r+}$. If $\phi$ is a character of $T$ of depth $r$, we say that $\phi$ (or Res$_T$ $\phi$) is $G$-generic if there exists a $G$-generic element $\Gamma \in t_{-r}$ of depth $r$ such that for all $X \in t_r$, we have $\phi(e_r(X)) = \Psi(\text{Tr}(\Gamma X))$. In our case all positive-depth characters of $T$ are $G$-generic.

3. Supercuspidal representations of $SL_2(k)$

The representation theory of $SL_2(k)$ has been known since the 1960s. In this section we present the classification of supercuspidal representations of $SL_2(k)$ following the work of J. K. Yu [35]. This allows us to exploit the modern language of buildings and to describe this elegant theory in the simplest possible case. Additional sources for the material in this section include the survey paper of J. Kim [14] and the exposition in the paper by J. Hakim and F. Murnaghan [11]. The supercuspidal representations of so-called degree 1 (which are the only supercuspidal representations of positive depth to occur in $SL_2(k)$) are also called toral supercuspidal representations; these were first described in this way by J. Adler in [1].

3.1. Depth-zero supercuspidal representations

Depth-zero supercuspidal representations are induced from cuspidal representations of $G(\kappa) = SL_2(\kappa)$, which are well-known; see, for example, [10]. Briefly: let $\kappa'$ denote the unique quadratic extension field of $\kappa$, and $N: \kappa' \to \kappa$ the norm map. To each nontrivial character $\omega$ of $\ker(N)$, Deligne and Lusztig associate a representation $\sigma = \sigma(\omega)$ of $SL_2(\kappa)$ of degree $q-1$. When $\omega^2 \neq 1$, this representation is cuspidal (a so-called Deligne-Lusztig representation). When $\omega = \omega_0$, the nontrivial quadratic character, the representation $\sigma_0 \cong \sigma(\omega_0)$ decomposes instead as $\sigma_0 = \sigma_0^\vee \oplus \sigma_0^\wedge$, a direct sum of inequivalent cuspidal representations of half the degree. The choice of label $\pm$ is related to a choice of nontrivial additive character on $\kappa$; without loss of generality we may assume the inflation to $R$ of this character coincides with the restriction of $\Psi$ to $R$, where $\Psi$ was fixed in Section 2.1.

Let $\sigma$ be a cuspidal representation of $G(\kappa)$. Inflate $\sigma$ to a representation (also denoted $\sigma$) of $G(R) = K$, and let $\sigma^\eta$ denote the corresponding representation of $K^\eta$. All depth-zero cuspidal representations of $K$ and $K^\eta$ arise in this way and it is known that the compactly induced representations

$$c\text{-Ind}_{K}^{G}\sigma \quad \text{and} \quad c\text{-Ind}_{K^\eta}^{G}\sigma^\eta$$

are irreducible, hence supercuspidal, and have depth zero. Note that their central characters coincide with the restriction of the character $\omega$ to the $\{\pm 1\}$ subgroup.

As a special case of the general result for depth-zero supercuspidals found in [22, 20], one has the following well-known fact.

Proposition 3.1. Up to equivalence, any depth-zero supercuspidal representation of $G$ arises as in (5), for a unique pair $(\sigma, G_0)$ of cuspidal representation $\sigma$ of $SL_2(\kappa)$ and maximal compact subgroup $G_0$ with $y \in \{0, 1\} \subset A$. 

3.2. Positive-depth supercuspidal representations

To construct positive-depth supercuspidal representations, Yu introduced the notion of a generic tamely ramified cuspidal $G$-datum [35, §3, §15]. For $G = \text{SL}_2(k)$, these data fall into one of two kinds. The first consists of data of so-called degree 0, and each is specified by: a choice of anisotropic torus $T$, the corresponding point $y \in \mathcal{A}$ as in Table 1, a positive real number $r$, and a $G$-generic character $\phi$ of $T$ of depth $r$. We note that if $T$ is unramified then $r \in \mathbb{Z}$ whereas if $T$ is ramified then $r \in \frac{1}{2} + \mathbb{Z}$. We set $s = r/2$.

**Remark 3.2.** Let us relate the abbreviated data given here to the definition of a cuspidal $G$-datum from [35, §3, §15] or [11]. This is given as a sequence of five axioms (D1) to (D5); we further justify statements (D3) to (D5) below.

**D1** The tamely ramified twisted Levi sequence, which is of degree 1, is $\mathcal{G} = (\mathcal{G}^0, \mathcal{G}^1) = (\mathbb{T}, G)$, where the quotient $\mathbb{Z}(T)/\mathbb{Z}(G)$ is anisotropic since $\mathbb{T}$ is;

**D2** The point $y$ lies in (is the only point in) $\mathcal{A}(G, T, k)$;

**D3** The sequence of real numbers $0 < r_0 \leq r_1$ may be taken to be $0 < r \leq r$;

**D4** The irreducible representation $\rho$ of $T$, such that $T_{y,0+}$ is 1-isotypic and $\text{c-Ind}_T^G \rho$ is supercuspidal, may without loss of generality be taken to be trivial;

**D5** The sequence of quasi-characters $(\phi_0, \phi_1)$ may be taken to be $(\phi, 1)$ where $\phi$ is a $G$-generic character of $T$ of depth $r$.

In (D3), the value $r_0$ defines the depth of the character $\phi_0$ of $T$ required in (D5), and as such will be either an integer or a half-integer, depending on the type of torus. One is permitted to choose some $r_1 > r_0$, but the purpose of the parameter $r_1$ is to allow a twisting by a quasi-character $\phi_1$ of $\mathcal{G}$ of depth $r_1$. Since $\mathcal{G}$ admits no nontrivial quasi-characters, no such $\phi_1$ exists, so the convention is to set $r_1 = r_0$ and take $\phi_1 = 1$ in (D5). The resulting representation has depth $r = r_0$, so we exclude $r = 0$ here.

In (D4), any $\rho$ satisfying these conditions must be a depth-zero character of $T$ (and any depth-zero character would do). However, for any such $\rho$ one may obtain the same supercuspidal representation in this construction by replacing $\rho$ with the trivial character and replacing $\phi$ by $\rho \phi$. This fact (which can also be seen directly) follows from the equivalence of cuspidal $G$-data given in [11, Theorem 6.7].

The idea of the construction of a supercuspidal representation of depth $r$ from such a quadruple $(T, y, r, \phi)$ is to extend $\phi$ to a (uniquely determined) depth-$r$ representation $\rho$ of the compact open subgroup $T \mathcal{G}_{y,s}$, whose compact induction to $G$ is irreducible, and hence supercuspidal. It proceeds as follows.

We denote by $e$ the natural isomorphisms of the abelian groups

$$t_{s+}/t_+ \cong T_{s+}/T_+ \quad \text{and} \quad g_{y,s+}/g_{y,r+} \cong g_{y,s+}/g_{y,r+}.$$  

In matrix form one can approximate $e(X)$ by $I + X$, in the sense that these are congruent modulo the appropriate matrix groups.

Since the character $\phi$ of $T$ has depth $r$, its restriction to $T_{s+}$ is a character which factors through $T_{s+}/T_+$. All such characters are represented by elements of $t_{-r}$, in the sense that there exists $\Gamma \in t_{-r} \subset g$ for which

$$\phi(t) = \Psi(\text{Tr}(\Gamma e^{-1}(t))) \quad \text{for all } t \in T_{s+}. \quad (6)$$
The image of $\Gamma$ in $t_{-r}/t_s$ is uniquely defined by this relation. The genericity of $\phi$ implies that there exists $s \in k$ such that $\Gamma = aX_T$ and $\val(\alpha_{T'}) = -r - y$. This element $\Gamma$ also defines a character $\Psi_\Gamma$ of $G_{y,s+}/G_{y,r+}$ via

$$\Psi_\Gamma(g) = \Psi(\Tr(e^{-1}(g))) \quad \text{for all } g \in G_{y,s+}.$$ 

Since $\phi$ and $\Psi_\Gamma$ agree on the intersection $T_{s+}$ of their domains, together they give a unique well-defined character $\phi$ of $T G_{y,s+}$, by setting $\phi(tg) = \phi(t)\Psi_\Gamma(g)$ for any $t \in T$, $g \in G_{y,s+}$.

In case $G_{y,s+} = G_{y,s}$, we are done. This occurs both when $T$ is ramified, and when $T$ is unramified and $r$ is an odd integer, since in these cases $s$ is a fraction for which $[s] = [s+]$ and $[s \pm y] = [(s+) \pm y]$. We thus set $\rho = \tilde{\phi}$, which is a representation of $T G_{y,s}$ (of degree 1 and depth $r$).

Otherwise, that is, when $T$ is unramified and $r$ is even, we have the following lemma.

**Lemma 3.3 (Yu).** With the notation above, suppose $T$ is unramified and $r$ is even. Then there exists a canonical construction of an irreducible representation $\rho$ of $T G_{y,s}$ (of degree $q$) extending $\tilde{\phi}$, satisfying:

1. $\Res_{Z T_{0+}} \rho$ is $\phi$-isotypic;
2. $\Res_{G_{y,s+}} \rho$ is $\Psi_\Gamma$-isotypic.

The construction of $\rho$, via the Weil representation, is central to [35], and we do not repeat it here. Our lemma is a very slight generalization in that it includes the behaviour of the center.

**Sketch of proof.** Fix $T$ and define $g$ as in [41]. Yu defines the subgroup $J^1$ (respectively, $J^1_\pm$) as the intersection with $G$ of the subgroup of $G(k')$ generated by $T(k')_r$ and $(G_{y,s}^\pm(k')_{y,s})^q$ (respectively, $(G_{y,s}^\pm(k')_{y,s+})^q$). Then one has the identities $T J^1 = T G_{y,s}$ and $T J^1_\pm = T G_{y,s+}$.

In [35, §11], Yu gives a canonical construction of a representation $\phi$ of $T \times J^1$, which depends only on $\Res T \phi$, as a pullback of a symplectic action such that

- $\Res_{T_{0+} \times 1} \phi$ is 1-isotypic, and
- $\Res_{1 \times J^1_\pm} \phi$ is $\Psi_\Gamma$-isotypic.

By [35, Prop 11.4], this symplectic action is given by conjugation, so the center $Z$ acts trivially. Hence $\Res_{Z T_{0+} \times 1} \phi$ is also 1-isotypic. Using that $T \cap J^1 \subseteq T$, one sees directly that the formula $\rho(tj) = \phi(t)\phi(t,j)$, for $(t,j) \in T \times J^1$, is well-defined. It follows from the above that this representation $\rho$ of $T G_{y,s}$ has the desired properties.

The following result is well-known for $G = SL_2(k)$. It is a special case of the general results about positive-depth tamely ramified supercuspidal representations given by the combination of [35, Prop 4.6], [12] Thm 19.1] and [11, Thm 6.7].

**Proposition 3.4.** Let $\rho = \rho(T, y, r, \phi)$ be as above. The compactly induced representation

$$c\text{-Ind}_{T G_{y,s}}^G \rho$$

is a supercuspidal representation of $G$ of depth $r$, and all positive-depth supercuspidal representations of $G$ arise in this way. Moreover, two such representations are equivalent if and only if the pairs $(T, \phi)$ occurring in their defining data are $G$-conjugate.
3.3. Representations of \( \text{SL}_2(\mathbb{R}) \)

In his thesis \([29]\), J. Shalika constructed all the irreducible representations of \( K = \text{SL}_2(\mathbb{R}) \). The ones we wish to recall here were those he called “ramified representations”, and are constructed via Clifford theory. Our notation here diverges from \([29]\) in that: we use the depth, rather than the conductor, as the index (so our indices are off by one); and we use the single fixed quasi-character \( \Psi \) and elements of the Lie algebra of negative degree, rather than a collection of additive characters \( \eta_k, k \geq 0 \), and elements of depth zero, in the parametrization.

Let \( d \in \mathbb{Z}_{>0} \) and let \( u, v \in k \) be such that \( \text{val}(v) > \text{val}(u) = -d \). Set

\[
X = X(u, v) = \begin{bmatrix} 0 & u \\ v & 0 \end{bmatrix} \in \mathfrak{g}_0, -d. \tag{8}
\]

Then the function \( g \mapsto \Psi_X(g) = \Psi(\text{Tr}(X(g - I))) \) defines a character of the group

\[
\mathcal{G}_{[0, \frac{1}{2}], d/2} = \mathcal{G}_{0, d/2} \cap \mathcal{G}_{\frac{1}{2}, d/2} = \left\{ \begin{bmatrix} \mathcal{U}_{[d/2]} & \mathcal{P}_{[-d/2]} \\ \mathcal{P}_{[d/2]} & \mathcal{U}_{[d/2]} \end{bmatrix} \right\} \cap \mathcal{G}.
\]

Note that \( u \) and \( v \) are uniquely determined by the character \( \Psi_X \) only modulo \( \mathcal{P}^{[d/2]} \) and \( \mathcal{P}^{[-d+1/2]} \), respectively. The normalizer of \( \Psi_X \) in \( K \) is \( T(X) \mathcal{G}_{[0, \frac{1}{2}], d/2} \), where \( T(X) \) is the centralizer of \( X \) in \( K \), namely

\[
T(X) = \left\{ t(a, b) = \begin{bmatrix} a & b \\ bu^{-1}v & a \end{bmatrix} | a, b \in \mathcal{R}, a^2 - b^2u^{-1}v = 1 \right\} = T_{1, u^{-1}v}.
\]

Then Shalika proved the following \([29\text{, Theorems 4.2.1 and 4.25]}\).

**Theorem 3.5 (J. Shalika).** With the notation above, let \( \theta \) be a character of \( T(X) \) which coincides with \( \Psi_X \) on the intersection \( T(X) \cap \mathcal{G}_{[0, \frac{1}{2}], d/2} \). Write \( \Psi_{\theta, X} \) for the unique character of \( T(X) \mathcal{G}_{[0, \frac{1}{2}], d/2} \) which extends \( \theta \) and \( \Psi_X \). Then

\[
S_d(\theta, X) := \text{Ind}_{T(X)\mathcal{G}_{[0, \frac{1}{2}], d/2}}^{\mathcal{G}_{0, d}} \Psi_{\theta, X}
\]

is an irreducible representation of \( K \) of depth \( d \) and of degree \( \frac{1}{2}q^{d-1}(q^2 - 1) \). For varying \( X \) and \( \theta \) as above, these representations exhaust all irreducible representations of \( K \) of this depth and degree.

**Comments on the proof.** Since \( \Psi_{\theta, X} \) is an extension of the character \( \Psi_X \) to its normalizer, by an argument in Clifford theory, one deduces that \( S_d(\theta, X) \) is irreducible. Since \( \Psi_X \) is trivial on \( \mathcal{G}_{0, d} \) and nontrivial on \( \mathcal{G}_{0, d} \), we deduce that the depth of \( S_d(\theta, X) \) as a representation of \( K = \mathcal{G}_0 \) is \( d \). Its degree is calculated directly.

For the last statement, note that for all \( q \in K \) we have \( S_d(\theta, X) \cong S_d(\theta^q, X^q) \). Shalika lists representatives of all \( K \)-orbits in \( \mathfrak{g}_{0,0} \) in \([29\text{, Lemma 4.2.2}]\). Our set of elements \( \{ X(u, v) \mid u \in \mathcal{P}^{d} \setminus \mathcal{P}^{d+1}, v \in \mathcal{P}^{-d+1} \} \) was chosen to meet all those orbits corresponding to Shalika’s ramified representations. Shalika shows that these exhaust all irreducible representations of \( K \) of the given depth and degree in \([29\text{, \S4.3}]\). \( \square \)

4. Restriction to \( K \)

One of the main tools for decomposing restrictions of induced representations is Mackey theory. Applying versions of Frobenius reciprocity \([7\text{, 1.5(33)}]\) and Mackey theory for compactly induced representations \([12]\) yields the following result.
Lemma 4.1. Let $G$ be the $k$-points of a linear algebraic group defined over $k$. Let $H$ be a compact-mod-center subgroup of $G$ and $\rho$ a smooth representation of $H$ such that the compactly induced representation $\pi = c{\text{Ind}}_H^G \rho$ is admissible. Let $K$ be a compact open subgroup of $G$. Then the restriction of $\pi$ to $K$ decomposes into a direct sum of (not necessarily irreducible) representations induced from subgroups of $K$ as

$$\text{Res}_K c{\text{Ind}}_H^G \rho \cong \bigoplus_{\alpha \in K \backslash G/H} \text{Ind}^K_{K \cap H^\alpha} \rho^\alpha.$$  \hspace{1cm} (9)

Let us find representatives of these double cosets and the depths and degrees of these Mackey components in each of the depth-zero and positive depth cases.

4.1. Depth-zero case: double coset representatives, depth and degree

From the Cartan decomposition of $G$, and a short calculation, we see that a set of double coset representatives for either $K \backslash G / K$ or $K \backslash G / \text{ker}(\pi)$ is

$$\left\{ \alpha^t \right\} = \left\{ \begin{pmatrix} \omega^{-t} & 0 \\ 0 & \omega^t \end{pmatrix} \right\} | t \geq 0 \}.$$  \hspace{1cm} (10)

Thus by Lemma 4.1. the depth-zero supercuspidal representations decompose as

$$\text{Res}_K c{\text{Ind}}_K^G \sigma \cong \bigoplus_{t \geq 0} \text{Ind}^K_{K \cap K^{2t}} \sigma^{2t}$$

and

$$\text{Res}_K c{\text{Ind}}_K^G \sigma \cong \bigoplus_{t \geq 0} \text{Ind}^K_{K \cap K^{2t+1}} \sigma^{2t+1}.$$  \hspace{1cm} (11)

Lemma 4.2. Let $\sigma$ be a depth-zero irreducible cuspidal representation of $K$. Then for any $d \geq 0$, the maximum depth of any irreducible component of $\text{Ind}^K_{BK_d} \sigma^d$ is $d$.

Proof. When $d = 0$ the summand is simply $\sigma$, which has zero depth by hypothesis. Let $d > 0$. Note that the maximum depth of an irreducible component of an induced representation $V$ of $K$ is the least integer $n \geq 0$ such that $V = V^{K_{n+1}}$, equivalently, such that the intersection of $K_{n+1}$ with the inducing subgroup lies in the kernel of the representation being induced. Let $B^{op}$ denote the subgroup of lower triangular matrices in $K$.

Given $\sigma$ a nontrivial irreducible representation of $K$ of depth zero, we note that the value of $\sigma^{\alpha^d}$ on elements of $BK_d$ is determined by that of $\sigma$ on $B^{op}$. That is, for $a = (a_{ij}) \in BK_d$, we have

$$\sigma^{\alpha^d}(a) = \sigma \left( a^{-\alpha^d} \right) = \sigma \left( \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \right) = \sigma \left( \begin{pmatrix} a_{11} & 0 \\ a_{21} & a_{22} \end{pmatrix} \right)$$  \hspace{1cm} (12)

since $a_{12} \omega^d \in P$ and $\sigma$ is trivial on $K_1$. If $n > d$ then $K_n \cap BK_d \subseteq \ker(\sigma^{\alpha^d})$ since in this case $a \equiv I$ mod $P^n$ and so $a^{-\alpha^d} \equiv I$ mod $P$. Conversely, since the subgroup $U^{op} = \{ \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix} | c \in \mathbb{R^*} \}$ is contained in no proper normal subgroup of $K$ but $(U^{op})^{\alpha^d} \subseteq K_d \cap BK_d$, we conclude that $K_d \cap BK_d \not\subseteq \ker(\sigma^{\alpha^d})$. Thus the maximal depth is $d$.  \hfill $\Box$
Finally, note that for any \(d > 0\), \(BK_d\) has index \((q + 1)q^{d-1}\) in \(K\). Hence

\[
\deg \left( \text{Ind}_{BK_d}^K \sigma^\nu \right) = (q + 1)q^{d-1} \deg(\sigma),
\]

where if \(\sigma \in \{\sigma_0^\pm\}\) then \(\deg(\sigma) = (q - 1)/2\), but for all other \(\sigma\), we have \(\deg(\sigma) = q - 1\).

In Section 5, we will use this as a starting point to give the complete decomposition of depth-zero supercuspidal representations into irreducible \(K\)-representations.

### 4.2. Positive depth case: double coset representatives

Let us now turn to the positive-depth case. To determine double coset representatives for \(K\backslash G/TG_{y,s}\), for the various triples \((T, y, s)\) that occur in the construction, we begin with the following observation.

**Lemma 4.3.** Let \(T\) be one of the tori in Table 1 such that \(T \subset K\). Choose \(x, y \in \mathbb{R}\) satisfying \(x^2 - y^2 \varepsilon = \varepsilon\). Set

\[
d = \begin{cases} 
    e = \begin{bmatrix} x & y \\ y & \varepsilon^{-1}x \end{bmatrix} & \text{if } T \text{ is unramified, and} \\
    w = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} & \text{if } T \text{ is ramified.}
\end{cases}
\]

Then \(\Lambda(T) = \{I, d\}\) is a set of representatives of \(B^{op}\backslash K/T\). Similarly, \(\Lambda(T^{12}) = \{I, e^0\}\) is a set of representatives of \(B^{op}\backslash K^{12}/T^{12}\).

Note that if \(-1 = t^2\), then one can take \((x, y) = (0, t)\); otherwise, we solve \(x^2 + y^2 = -1 = \varepsilon\).

**Proof.** Write \(T = T_{11,12}\) and for each \(g = (g_{ij}) \in K\) set \(N(g) = g_{11}^2 - \gamma_{11}^{-1}g_{12}^2\). We assume \(T \subseteq K\); the remaining case \(T_{11,12}^{12} \subset K^{12}\) is similar. Given \(g \in B^{op}T\), write \(g = bt\) with \(b = \begin{bmatrix} u & 0 \\ v & u \end{bmatrix}\) and \(t = t(a, b)\); then \(N(g) = u^2\). Conversely, given \(g \in K\) such that \(N(g) = u^2\) for some \(u \in \mathbb{R}^\times\), we see directly that \(t = t(g_{11}u^{-1}, g_{12}^{-1}u^{-1}) \in T\) and that \(gt^{-1} \in B^{op}\). Thus the identity double coset \(B^{op}T\) consists of all \(g \in K\) with \(N(g) \in (\mathbb{R}^\times)^2\).

In the ramified case, \(\text{val}(\gamma_{11}^{-1}) = 1\) so \(N(g) \in (\mathbb{R}^\times)^2\) if and only if \(g_{11} \in \mathbb{R}^\times\). Thus if \(g\) lies in the complement \(K \backslash B^{op}T\), then \(g_{11} \in \mathbb{P} \text{ and } g_{12} \in \mathbb{R}^\times\). Since \(p \neq 2\) we can choose \(u \in \mathbb{R}^\times\) such that \(N(g) = -\gamma_{11}^{-1}u^2\). With this choice of \(u\), \(t = t(g_{11}u^{-1}, g_{11}^{-1}, g_{12}^{-1}u^{-1}) \in T\) and \(gt^{-1}w^{-1} \in B^{op}\). Thus \(K \backslash B^{op}T = B^{op}wT\).

In the unramified case, \(\gamma_{11}^{-1} = \varepsilon\). Let \(g \in K \backslash B^{op}T\); then necessarily \(N(g) = u^2\varepsilon\) for some \(u \in \mathbb{R}^\times\). To show that \(g \in B^{op}eT\), we solve \(g = bt\) for the unknown \(t = t(c, d)\) by setting \(b_{11} = u\) and \(b_{12} = 0\). The first row of this matrix equality gives a linear system with solution

\[
\begin{bmatrix} c \\ d \end{bmatrix} = \frac{1}{u} \begin{bmatrix} x & -\varepsilon y \\ -y & x \end{bmatrix} \begin{bmatrix} g_{11} \\ g_{12} \end{bmatrix}.
\]

Since this yields \(\det(t(c, d)) = 1\), \(t \in T\) and we conclude that \(g \in B^{op}eT\). \(\square\)

**Proposition 4.4.** A set of representatives for the double coset space \(K\backslash G/TG_{y,s}\) is

\[
M(T) = \{I, \alpha \cdot \lambda \mid t > 0, \lambda \in \Lambda(T)\}. 
\]
Thus $K\backslash G/K$ is normalized when $m \neq 0$, with $t \geq 0$ and $\beta$ a representative of $(K \cap K^{\alpha^{-1}})\backslash K/TG_{y,s}$. When $t = 0$, $(K \cap K^{\alpha^{-1}}) = K$ and so we take $\beta = 1$. For any $t > 0$, the group $K \cap K^{\alpha^{-1}}$ contains the group $B^\op$; also $T G_{y,s}$ contains $T$. So each double coset is a union of $B^\op \backslash K / T$ double cosets, and Lemma 4.5 applies. Since $s, t > 0$, we can verify that $B^\op T \equiv (K \cap K^{\alpha^{-1}}) T G_{y,s}$ modulo $P$, so $d \notin (K \cap K^{\alpha^{-1}}) \backslash T G_{y,s}$. Hence $(K \cap K^{\alpha^{-1}}) \backslash K / T G_{y,s}$ is also represented by $\{I, d\}$, as required.

We conclude that with $M(\mathcal{T})$ as in (14) we have
$$\text{Res}_{\mathcal{K}c} \text{-Ind}_{T G_{y,s},p}^{\mathcal{K}} \rho \cong \bigoplus_{\mu \in M(\mathcal{T})} \text{Ind}_{K \cap (T G_{y,s})}^{\mathcal{K}} \rho^\mu. \quad (15)$$

We work towards a more explicit description of these inducing subgroups. Let $\mathcal{T}$ be a torus from Table 1 and $\mu = \alpha^t \lambda \in M(\mathcal{T})$. Set
$$\delta(\mu) = \begin{cases} 2t - y & \text{if } y = \frac{1}{2}, \lambda = w \\ 2t + y & \text{otherwise.} \end{cases}$$

**Lemma 4.5.** Given $\mathcal{T}$ as in Table 1, $\Gamma = t \setminus \{0\}$ and $\mu \in M(\mathcal{T})$, we have
$$T(\Gamma^\mu) = K \cap \mathcal{T}^\mu = Z T_{\delta(\mu)}^\mu$$
where $T(X)$ denotes the centralizer of $X$ in $K$, as in Theorem 5.5. Furthermore, for any $m \in \mathbb{Z}_{>0}$ we have $K_m \cap \mathcal{T}^\mu = (T_{\delta(\mu) + m})^\mu$.

**Proof.** That $K \cap \mathcal{T}^\mu = T(\Gamma^\mu)$ follows directly, since $T$ is the centralizer in $G$ of $\Gamma$. For $m \geq 0$ we have $K_m \cap \mathcal{T}^\mu = G_0^{-1}_m = G_{\mu^{-1} \cdot 0, m}$. If $\mu = \alpha^t$ then this group is $G_{\mu^t \mu^{-1} \cdot 0, m}$ whereas if $\mu = \alpha^t w$ then it is $G_{\mu^t \mu^{-1} \cdot 0, m}$. Comparing these matrix groups reveals that the intersection $K_m^{-1} \cap \mathcal{T}$ is, up to centre when $m = 0$, the filtration subgroup $T_{\delta(\mu) + m}$. In the remaining cases, $\mu = \alpha^t \lambda$ does not necessarily normalize the apartment $A$, but $d \in G_0$ normalizes $\mathcal{T}$ and also its Moy-Prasad filtration subgroups. Thus $K_m^{-1} \cap \mathcal{T} = K_m^{-1} \cdot \mathcal{T} = T_{\delta(\mu) + m} = T_{\delta(\mu) + m}$. The result follows.

For reference we identify in Table 2 the action of each $\lambda \in \Lambda(\mathcal{T})$ on a basis element $X_{\mathcal{T}}$ of $t$. Note that $\mathcal{T}^\mu = \{ aI + bX_{\mathcal{T}}^\mu | a, b \in R \} \cap G$. Thus if $\mu = \alpha^t \lambda$ and $X_{\mathcal{T}}^\mu = X(u, v)$ then
$$K \cap \mathcal{T}^\mu = \left\{ \begin{bmatrix} a & b \\ bu^{-1}v^{-4t} & a \end{bmatrix} | a, b \in R \right\} \cap G = T_{1, u^{-1} v^{-4t}}.$$

Similarly, we can describe $K \cap G_{y,s}^{\mu}$ succinctly: when $y \in \{0, 1\}$, $\Lambda(\mathcal{T}) \subset G_y$, so for any $\mu = \alpha^t \lambda \in M(\mathcal{T})$, we have $G_{y,s}^{\mu} = G_{y,s}^{\alpha^t \lambda}$. As in the proof of Lemma 4.5, we can easily write down $G_{y,s}^{\alpha^t \lambda}$ and $G_{y,s}^{\alpha^t \lambda \cdot w}$. In general, setting $M = \max\{0, [s - \delta(\mu)]\}$, we have
$$K \cap G_{y,s}^{\mu} = \left\{ \begin{bmatrix} U_{\delta(\mu)}^M \\ P_{[s + \delta(\mu)]} \end{bmatrix} | \begin{bmatrix} U_{[s]} \\ P_{[s + \delta(\mu)]} \end{bmatrix} \right\} \cap G. \quad (17)$$

**Proposition 4.6.** Let $\mu \in M(\mathcal{T})$. Then $K \cap (T G_{y,s}^\mu) = (K \cap T^\mu)(K \cap G_{y,s}^{\mu})$. Furthermore, if $\delta(\mu) > s$, then $K \cap (T G_{y,s}^\mu) = Z(K \cap G_{y,s}^{\mu})$. 


Proof. Note that since \( T \) normalizes \( G_{y,s}, K \cap T^\mu \) normalizes \( K \cap G_{y,s}^\mu \). Let \( \mu = \alpha^t \lambda \in M(T) \). If \( t = 0 \) then since \( s > 0 \) we see that for all \( y \in \{0, \frac{1}{2}, 1\}, G_{y,s} \subseteq K \), and there is nothing to show. If \( t > 0 \) and \( \lambda \neq w \), then \( G_{y,s} = G_{y,s}^\lambda \), so it suffices to prove that \( K \cap (T^\lambda G_{y,s})^\alpha^t = (K \cap (T^\lambda)^\alpha^t)(K \cap (G_{y,s})^\alpha^t) \). We use the explicit matrix forms, above.

Factor \( g \in T^\lambda G_{y,s} \) as \( g = uh \) with \( u = (u_{ij}) \in T^\lambda \) and \( h = (h_{ij}) \in G_{y,s} \). If \([s - y] \geq 2t\) then \( h_{12} \in R^{2t} \) and consequently \( h^\alpha^t \in K \); thus \( g^\alpha^t \in K \) if and only if \( w^{\alpha^t} \in K \), and our factorization holds. Otherwise, note that if \( g^{\alpha^t} = (uh)^{\alpha^t} \in K \) then its \((1, 2)\) matrix entry satisfies

\[
u_{11}h_{12}w^{-2t} + u_{12}h_{22}w^{-2t} \in R.
\]

As \( u_{11}, h_{22} \in R^{\times} \), we deduce \( \text{val}(u_{12}) = \text{val}(h_{12}) \), which by definition of \( h \) is at least \([s - y]\). It follows that \( u \in T^\lambda \cap ZG_{y,s} \). We can thus refactor \( g \) as \( ih' \) with \( i \in Z \subset T^\lambda \) and \( h' \in G_{y,s} \). Since \( i = i^{\alpha^t} \in K \) we deduce that \( (h')^{\alpha^t} \in K \), as required.

The case \( \lambda = w \) follows by replacing \( G_{y,s} \) with \( G_{y,s}^w \) and \([s - y]\) with \([s + y]\). \( \square \)

Note that this proposition does not hold if we replace the factorization \( T G_{y,s} \) with the factorization \( T^\lambda J^1 \) referred to in the proof of Lemma 3.3.

4.3. Positive depth case: depths and degrees of the Mackey components

**Proposition 4.7.** Let \( \rho = \rho(T, y, r, \phi) \) and \( \mu \in M(T) \). Then the maximum depth \( d \) of any irreducible \( K \)-component of the representation \( \text{Ind}_{K \cap (T \cap G_{y,s})}^{K \cap (T \cap G_{y,s})^\mu} \rho^\mu \) is \( d = r + \delta(\mu) \).

**Proof.** By Lemma 3.3 we have that \( \ker(\rho) \supseteq G_{y,r^+}, \) and that \( \text{Res}_{ZT_\mu, \phi} \rho \) is \( \phi \)-isotypic, where \( \phi \) has depth \( r \). (In fact, one can explicitly describe \( \ker(\rho) \) as the subgroup generated by \( T_+ \) and \( J_k \) but this is more than is needed here.) It follows that \( K \cap G_{y,r^+} \subseteq \ker(\rho^\mu) \); from (17) we see it contains \( K_n \) for all \( n > r + \delta(\mu) \), the maximal depth is at most \( r + \delta(\mu) \).

To show that \( K_{r+\delta(\mu)} \nsubseteq \ker(\rho^\mu) \), write \( \mu = \alpha^t \lambda \) and let \( \Gamma = aX_T \in t, r \) represent \( \phi \), which is \( G \)-generic of depth \( r \). Then \( \text{val}(aw_1) = -r - y \). First suppose \( \lambda = 1 \). For any \( c \in R^{\times} \), set \( gc = [e^{\frac{1}{c}x+y+1}] \). Then \( \rho^{\alpha^t}(g_{c}) = \rho(g_c) = \Psi_T(g_c) J_{\text{deg}(\rho)} = \Psi(a_{\gamma_1} e^{x^{-r+y}}) J_{\text{deg}(\rho)} \) is nontrivial. Since \( g_{c}^{\alpha^t} \in K_{r+2t} \), we conclude that \( \rho^{\alpha^t} \) has a component of depth \( r + \delta(\alpha^t) \).

Now suppose \( \lambda \neq 1 \). When \( y = 0 \), \( \lambda = e \in K \) normalizes \( K_r \) so \( \rho^\mu((g_c^{\alpha^{-1}})^\mu) = \rho(g_c^{-1}) \) is nontrivial and \( (g_c^{-1})^{\alpha^t} = g_{c}^{\alpha^t} \in K_{r+2t} \). When \( y = 1 \), the element to consider is instead \( g_{c}^{(e^{r+1})} \). When \( y = \frac{1}{2} \) and \( \lambda = w \), replace \( g_c \) by \( g_{c} = [e^{\frac{1}{2}x^{-r+y}}] \). We verify that \( \rho^{\mu}(g_{c}^{\alpha^t}) = \rho(g_{c}^{\alpha^t}) \) is nontrivial and \( g_{c}^{\alpha^t} \in K_{r+2t+y} \). In all cases we conclude that \( \rho^\mu \) has a component of depth \( r + \delta(\mu) \). \( \square \)

Now we determine the degree of each \( K \)-representation occurring in (13).
Thus \( \sigma = \alpha^t \lambda \in M(\mathcal{T}) \). If \( t = 0 \) and \( y = 0 \) then 
\[
\deg \left( \text{Ind}_T^G \rho \right) = (q-1)q^r.
\]
In all other cases, setting \( d = r + \delta(\mu) \) we have 
\[
\deg \left( \text{Ind}_T^G \rho \right) = \frac{q^2-1}{2}q^{d-1}.
\]

**Proof.** Let \( \mu = \alpha^t \lambda \in M(\mathcal{T}) \). The degree of the induced representation is given by \( \deg(\rho) \) times the index of \( \mathcal{K} \cap (T \mathcal{G}_{y,s})^\mu \) in \( \mathcal{K} \). Using Proposition 4.6 and that \( (\mathcal{K} \cap \mathcal{T}^\mu)(\mathcal{K} \cap \mathcal{G}_{y,s}^\mu) \supset \mathcal{G}_{y,s}^\mu \), we find by the second isomorphism theorem 
\[
[\mathcal{K} : \mathcal{K} \cap (T \mathcal{G}_{y,s})^\mu] = \frac{[\mathcal{K} : \mathcal{G}_{y,s}^{+2t+1}]}{[\mathcal{K} \cap \mathcal{G}_{y,s}^\mu : \mathcal{G}_{y,s}^{+2t+1}][\mathcal{K} \cap \mathcal{T}^\mu : (\mathcal{K} \cap \mathcal{T}^\mu \cap \mathcal{G}_{y,s}^\mu)]}.
\]

From our explicit descriptions in (16) and (17) (and noting that \( \mathcal{G}_{y,s}^{+2t+1} \subseteq \mathcal{K} = \mathcal{G}_{0,1} \)), we determine 
\[
[\mathcal{K} : \mathcal{G}_{y,s}^{+2t+1}] = (q^2-1)q^{6t+[s]+[s+y]+[s-y]+1}.
\]
\[
[\mathcal{K} \cap \mathcal{G}_{y,s}^\mu : \mathcal{G}_{y,s}^{+2t+1}] = q^{6t+[s-y]+[s+y]-M-[s+\delta(\mu)]+3}.
\]

When \( t = 0 \) and \( y = 0 \), we have \([\mathcal{K} \cap \mathcal{T}^\mu : (\mathcal{K} \cap \mathcal{T}^\mu \cap \mathcal{G}_{y,s}^\mu)] = [\mathcal{T} : \mathcal{T} \cap \mathcal{G}_{0,s}] = |\mathcal{T}(\kappa)||\mathcal{T}_1 : \mathcal{T}_s| = (q+1)q^{|s|-1}. \)
Since \( \deg(\rho) = q \) exactly when \( r = 2\lfloor s \rfloor \), and is 1 when \( r = 2\lfloor s \rfloor - 1 \), we deduce the total degree is \( (q-1)q^{2\lfloor s \rfloor - 1} \) as required.

When \( t > 0 \) or \( y \neq 0 \), then \( \mathcal{K} \cap \mathcal{T}^\mu \) is contained in the standard Iwahori subgroup. With notation as in (16), we see that for every \( b \in \mathcal{R} \), there exist exactly two choices for \( a \) such that \( t(a,b) \in \mathcal{K} \cap \mathcal{T}^\mu \). Furthermore, if \( b \in \mathcal{P}^M \) then \( a \in \pm \mathcal{U}_{\delta(\mu)+2M} \subseteq \mathcal{U}_{\lfloor s \rfloor} \), so that \( t(a,b) \in \mathcal{Z} \mathcal{G}_{y,s}^\mu \) if and only if \( b \in \mathcal{P}^M \). Thus 
\[
[\mathcal{K} \cap \mathcal{T}^\mu : (\mathcal{K} \cap \mathcal{T}^\mu \cap \mathcal{G}_{y,s}^\mu)] = 2q^M.
\]
Consequently 
\[
\deg \left( \text{Ind}_T^G \rho \right) = \frac{1}{2} \deg(\rho)(q^2-1)q^{\lfloor s \rfloor + [s+\delta(\mu)]-2}.
\]

When \( r \) is an even integer, \( \deg(\rho) = q \) and \( y \) and \( s \) are integers, so the expression simplifies to \( \frac{1}{2}(q^2-1)q^r+y+2t-1 \), as required. Otherwise, we have \( \deg(\rho) = 1 \), and either \( r \) is an odd integer and \( y \) is an integer, or else \( r \) and \( y \) are half-integers. In either case, \( \lfloor s \rfloor + \lfloor s \pm y \rfloor = 2s \pm y + 1 \). Thus we again recover the desired formula. \( \square \)

5. Branching rules: depth-zero case

We reprise the notation for depth-zero supercuspidal representations. In particular \( \omega \) denotes a nontrivial character of \( \ker(N) \), the kernel of the norm map of a quadratic extension of \( \kappa \), and \( \sigma = \sigma(\omega) \) denotes both the corresponding representation of \( \text{SL}_2(\kappa) \), and its inflation to \( \mathcal{K} \). We write \( \omega_0 \) for the unique character of order 2, for which \( \sigma_0 = \sigma(\omega_0) \) decomposes into the two cuspidal representations \( \sigma_0^\pm \).

We begin by showing that for each \( d > 0 \) and each \( \sigma = \sigma(\omega) \) the representation \( \text{Ind}_{\mathcal{K}^d}^G \sigma \) is independent of the choice of \( \sigma \) up to its central character, by showing that this is true of the restriction to \( \mathcal{B} \mathcal{K}_d \) of \( \sigma \).
Lemma 5.1. Let $d > 0$. Let $\omega_1, \omega_2$ be two nontrivial characters of $\ker(N)$ and $\sigma_i = \sigma(\omega_i)$, $i \in \{1, 2\}$, the corresponding representations of $K$. Let $\tau$ denote the trivial extension to $BK_d$ of a character of the diagonal torus of $K$ such that $\tau(-1) = -1$. Then we have

$$\text{Res}_{BK_d} \sigma_i^d \cong \begin{cases} \text{Res}_{BK_d} \sigma_2^d & \text{if } \omega_1(-1) = \omega_2(-1); \\ \tau \text{Res}_{BK_d} \sigma_2^d & \text{otherwise.} \end{cases}$$

(18)

Furthermore, this restriction decomposes as a direct sum of two inequivalent irreducible subrepresentations.

Proof. We saw in (12) that the restriction of $\sigma_i^d$ to $BK_d$ is determined by the restriction of $\sigma_i$ to $B^{op}$. It thus suffices to show that

$$\text{Res}_{B^{op}} \sigma_1 \cong \begin{cases} \text{Res}_{B^{op}} \sigma_2 & \text{if } \omega_1(-1) = \omega_2(-1) \text{ and} \\ \tau \text{Res}_{B^{op}} \sigma_2 & \text{otherwise,} \end{cases}$$

(19)

and that these each decompose as a direct sum of two inequivalent irreducible representations. Since these representations factor through the finite group quotient $\text{SL}_2(\mathbb{F}_q) \cong \mathbb{C}/K_1$, it suffices to compare their characters. Write $B^{op}$ also for its image in $\text{SL}_2(\mathbb{F}_q)$. The character $\chi_i$ of $\text{Res}_{B^{op}} \sigma_i$ is given on elements of $B^{op}$ by [10, §15, Table 2]

$$\chi_i \left( \begin{bmatrix} a & 0 \\ c & a^{-1} \end{bmatrix} \right) = \begin{cases} (q - 1)\omega_i(a) & \text{if } a = \pm 1, c = 0; \\ -\omega_i(a) & \text{if } a = \pm 1, c \neq 0; \\ 0 & \text{otherwise.} \end{cases}$$

The character of $\tau \text{Res}_{B^{op}} \sigma_2$ is $\tau \chi_2$.

Setting $g_c = \begin{bmatrix} 1 & 0 \\ c & 1 \end{bmatrix}$ and noting that $\chi_i$ is real-valued, we calculate the intertwining number between $\chi_1$ and $\chi_2$ to be

$$I(\chi_1, \chi_2) = \frac{1}{|B^{op}|} \sum_{g \in B^{op}} \overline{\chi_1(g)} \chi_2(g)$$

$$= \frac{1}{q(q - 1)} \left( \sum_{a \in \{\pm 1\}, c = 0} \chi_1(a)\chi_2(a) + \sum_{a \in \{\pm 1\}, c \neq 0} \chi_1(aga_c)\chi_2(aga_c) \right)$$

$$= \frac{1}{q(q - 1)} (1 + \omega_1(-1)\omega_2(-1)) \left( (q - 1)^2 + (q - 1) \right)$$

$$= \begin{cases} 2 & \text{if } \omega_1(-1) = \omega_2(-1); \\ 0 & \text{otherwise.} \end{cases}$$

It follows that for each $i$, $\text{Res}_{B^{op}} \sigma_i$ decomposes as a direct sum of two inequivalent irreducible representations of $B^{op}$, and that $\text{Res}_{B^{op}} \sigma_1$ is equivalent to $\text{Res}_{B^{op}} \sigma_2$ exactly when their central characters coincide. On the other hand, when $\sigma_1$ and $\sigma_2$ have opposite central character, we may argue as above that $I(\chi_1, \tau \chi_2) = 2$, which completes the proof.

When $\omega = \omega_0$, the unique nontrivial quadratic character of $\ker(N)$, we know that $\sigma_0 = \sigma_+^\tau \oplus \sigma_0^-$. Applying Lemma 5.1 with $\omega_2 = \omega_0$ therefore yields an explicit description of the decomposition of
Res_{BK_d} \sigma_1^{\tau^d} into irreducible subrepresentations. Consequently, for \( \theta \) a character of \( Z \) and \( d > 0 \), we define

\[
\pi_d^\pm(\theta) = \begin{cases} 
\text{Ind}_{BK_d}^K(\sigma_0^{\pm} \eta^d) & \text{if } \theta \text{ coincides with the central character of } \sigma_0; \\
\text{Ind}_{BK_d}^K(\tau(\sigma_0^{+}) \eta^d) & \text{otherwise.}
\end{cases}
\]

Then for any \( \sigma = \sigma(\omega) \) with central character \( \theta \) we have

\[\text{Ind}_{BK_d}^K \sigma^\eta^d \cong \pi_d^+(\theta) \oplus \pi_d^-(\theta)\]

where each \( \pi_d^\pm(\theta) \) has degree \( \frac{1}{2}(q^2 - 1)q^{d-1} \). We now offer an alternative description of these representations as per Theorem 3.3 and as a consequence deduce their irreducibility.

**Proposition 5.2.** Let \( \theta \) be a character of \( Z \). Write \( X(a,0) \) as usual for \([0 \ z \ 0] \). Then for each \( d > 0 \), we have \( \pi_d^+(\theta) \cong S_d(\theta, X(-\infty,0)) \) and \( \pi_d^-(\theta) \cong S_d(\theta, X(-\infty,0)) \). Consequently, each \( K \)-representation \( \pi_d(\theta) \) is irreducible.

**Proof.** Fix \( x \in \{1, \varepsilon\} \) and set \( X_x = X(-x\infty,0) \). Let \( \zeta \) be the nontrivial character of \( \kappa^x \) of order 2, inflated to a character of \( R^x \). Its kernel is \((R^x)^2\). For any \( s \in R^x \), let \( \text{sgn}(s) \in \{+,-\} \) denote the sign of \( \zeta(s) \in \{\pm\} \). In these terms, we need to show that

\[\pi_d^{\text{sgn}(x)}(\theta) \cong S_d(\theta, X_x).\] (21)

We henceforth write \( X \) for \( X_x \). We first show that \( S_d(\theta, X) \) is well-defined. The centralizer of \( X \) in \( K \) is \( T(X) = ZU \) where \( U = G_{\infty}(R) \) is the unipotent upper triangular subgroup. We extend \( \theta \) trivially over \( U \) to a character, also denoted \( \theta \), of \( T(X) \). We easily verify that the characters \( \Psi_X \) and \( \theta \) are both trivial on the intersection \( T(X) \cap G_{[0, t]}d/2 \), so we may apply Theorem 3.3 to conclude that \( S_d(\theta, X) \) is well-defined and an irreducible representation of \( K \).

The representations in (21) have the same degree so it suffices to show that they admit nonzero intertwining. By Frobenius reciprocity and Mackey theory, it suffices to show that

\[\text{Ind}_{BK_d}^K \Psi_{\theta, X}\]

intertwines on \( BK_d \) with either \((\sigma_0^{\text{sgn}(x)})\eta^d\) or \(\tau(\sigma_0^{\text{sgn}(x)})\eta^d\), according to central character. Since these representations have (maximal) depth \( d \), they factor through the finite group quotient \( BK_d/K_{d+1} \). Thus our approach is to evaluate the characters of these representations of finite groups and calculate their intertwining number \( I \).

First, we need some additional notation. Let \( S \) be a set of representatives of \((\kappa^x)^2\) in \( R^x \). For any \( u \in R^x \) (or, since \( \Psi \) factors to a character of the quotient \( \kappa \), in \( \kappa^x \)) define

\[\xi_u = \sum_{y \in S} \Psi(uy).\]

Since \( \Psi \) is trivial on \( P \), \( \xi_u \) takes on only one of two values, \( \xi_1 \) and \( \xi_\varepsilon \). We have \( \overline{\xi_u} = \xi_{-u} \) and we compute directly that \( \sum_{u \in \kappa^x} \xi_u = (q^2 - 1)/4 \).

Let \( \theta_0 \) denote the central character of \( \sigma_0^{\pm} \). Referring again to (10), the character of \( \sigma_0^{\text{sgn}(x)} \) is nonzero only on classes of elements the form \([\varepsilon \varepsilon]\), where it is given by \( \frac{1}{2}(q - 1)\theta_0(\varepsilon) \) if \( c = 0 \) and

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\[ \xi_{-xzg_2\psi_0}(z) \] if \( c \neq 0 \). Therefore the character \( \chi_d^\sigma \) of \((\sigma_{0}^{\text{sgn}(x)})^{g^d}\) on an element \( g = (g_{ij}) \in BK_d \)

is given by \( \chi_d^\sigma(g) = \text{Tr} \left( \sigma_{0}^{\text{sgn}(x)} \left( \begin{array}{cc} g_{11} & g_{12} \\ g_{21} & g_{22} \end{array} \right) \right) \) with \( g_{11}, g_{22} \in \mathcal{R}_x \), \( g_{12} \in \mathcal{R} \) and \( g_{21} \in \mathcal{P}^d \)

\[ = \text{Tr} \left( \sigma_{0}^{\text{sgn}(x)} \left( \begin{array}{cc} g_{11} & g_{12} \omega^{-d} \\ g_{21} \omega^{-d} & g_{22} \end{array} \right) \right) \]

\[ = \begin{cases} \frac{1}{2}(q-1)\psi_0(z) & \text{if } g_{11} \in z + \mathcal{P} \text{ for some } z = \pm 1 \text{ and } g_{21} \in \mathcal{P}^{d+1}, \\ \xi_{-xzg_2\psi_0}(z) & \text{if } g_{11} \in z + \mathcal{P} \text{ for some } z = \pm 1 \text{ and } g_{21} \in \mathcal{P}^d \setminus \mathcal{P}^{d+1}, \\ 0 & \text{otherwise.} \end{cases} \]

The character of \( \tau(\sigma_{0}^{\text{sgn}(x)})^{g^d} \) is given by \( \tau \chi_d^\sigma \).

On the other hand, to calculate the character \( \psi^\tau \) of \( \text{Ind}_{BK_d \cap T(X)G_{0,\frac{d}{2}}/d}^{BK_d \cap T(X)G_{0,\frac{d}{2}}/d} \Psi_{\theta,X} \) we use the Frobenius formula. Since \( BK_d \cap T(X)G_{0,\frac{d}{2}} \) is normal in \( BK_d \), \( \psi^\tau(g) = 0 \) for \( g \not\in BK_d \cap T(X)G_{0,\frac{d}{2}} \).

For the remaining values of \( g \), we need a set of coset representatives \( \Delta \) of \( BK_d/(BK_d \cap T(X)G_{0,\frac{d}{2}}) \); we choose

\[ \Delta = \left\{ \delta_a = \begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix} \mid a \in \mathcal{R}_x/\pm \mathcal{U}_{[d/2]} \right\}. \]

For each \( g \in BK_d \cap T(X)G_{0,\frac{d}{2}} \) choose a factorization \( g = tu \) with \( t = [\delta_a] \in T(X) \) and \( u = (u_{ij}) \in G_{0,\frac{d}{2}} \cap BK_d \). Then \( z \in \{ \pm 1 \} \) satisfies \( z \equiv g_{11} \mod \mathcal{P}^{[d/2]} \). We compute \( \Psi_{\theta,X}(tu) = \theta(t)\Psi(\text{Tr}(Xu)) = \theta(z)\Psi(-x\pi^{-d}u_{21}). \) Since \( u_{21} \equiv zg_{21} \mod \mathcal{P}^{[d/2]} \), this simplifies to \( \Psi_{\theta,X}(g) = \theta(z)\Psi(-xzg_{21}\omega^{-d}). \) Hence

\[ \psi^\tau(g) = \sum_{\delta_a \in \Delta} \Psi_{\theta,X}(\delta_a^{-1}g\delta_a) \]

\[ = \sum_{a \in \mathcal{R}_x/\pm \mathcal{U}_{[d/2]}} \theta(z)\Psi(-xzg_{21}\omega^{-d}a^2), \]

\[ = \begin{cases} \frac{1}{2}(q-1)q^{[d/2]-1} & \text{if } g_{21} \in \mathcal{P}^{d+1}, \\ \theta(z)\xi_{-xzg_2\psi_0}(z)q^{[d/2]-1} & \text{if } g_{21} \in \mathcal{P}^d \setminus \mathcal{P}^{d+1}, \\ 0 & \text{if } g \not\in BK_d \cap T(X)G_{0,\frac{d}{2}}. \end{cases} \]
Thus

\[
I(\chi_d^\tau, \psi_d^\tau) = \frac{1}{|BK_d/K_{d+1}|} \sum_{g \in BK_d/K_{d+1}} \chi_d^\tau(g) \overline{\psi_d^\tau(g)}
\]

\[
= \frac{1}{(q-1)q^{2d+2}} \sum_{g \in (BK_d)^{\tau G} \cap \mathcal{O}(X) \mathcal{G}_{g_1, g_{12}} / K_{d+1}} \chi_d^\tau(g) \overline{\psi_d^\tau(g)}
\]

\[
= \frac{1}{(q-1)q^{2d+2}} \left( \sum_{z \in \pm 1 \in g_{11}, g_{12}} \theta_0(z) \theta(z) \frac{(q-1)^2}{4} q^{[d/2]-1} \right) + \sum_{z \in \pm 1 \in g_{11}, g_{12}} \theta_0(z) \theta(z) \xi_{-x z c} \xi_{x z c} q^{[d/2]-1},
\]

where the sums are over all \(g_{11} \in z + \mathcal{P}^{[d/2]}\) and \(g_{12} \in \mathcal{R}\), taken modulo \(\mathcal{P}^{d+1}\), and where to simplify the expression we have set \(c = g_{21} \omega^{-d}\) in the second sum, so with \(g_{21}\) taken modulo \(\mathcal{P}^{d+1}\) this is effectively a sum over \(\kappa\).

We see that the first term vanishes if \(\theta_0(1) \neq \theta(-1)\); otherwise it has sum

\[
2 \left( \frac{1}{4} (q-1)^2 q^{[d/2]-1} \right) |\mathcal{P}^{[d/2]}| / |\mathcal{R}| = \frac{1}{2} (q-1)^2 q^{d+1}.
\]

Similarly, the second sum vanishes unless \(\theta_0(-1) = \theta(-1)\), in which case it gives

\[
q^{2d+1} \sum_{\kappa} (\xi_{-x c} \xi_{x c} + \xi_{x c} \xi_{-x c}) = q^{2d+1} \left( 2 \sum_{\kappa} \xi_{u \xi_{-u}} \right) = \frac{q^2 - 1}{2} q^{2d+1}.
\]

Thus these representations intertwine only when \(\theta_0 = \theta\), in which case

\[
I(\chi_d^\tau, \psi_d^\tau) = \frac{1}{(q-1)q^{2d+2}} \left( \frac{1}{2} (q-1)^2 q^{d+1} + \frac{q^2 - 1}{2} q^{2d+1} \right) = 1.
\]

On the other hand, when \(\theta = \tau \theta_0\) we have instead that \(I(\tau \chi_d^\tau, \psi_d^\tau) = 1\). 

We summarize the conclusions of this section in the following theorem.

**Theorem 5.3** (Branching Rules for Depth-Zero Supercuspidal Representations). Let \(\sigma = \sigma(\omega)\) with \(\omega^2 \neq 1\), and let \(\theta\) denote its central character. Then the decomposition into irreducible \(K\)-representations of the restrictions to \(K\) of the corresponding depth-zero supercuspidal representations are given by

\[
\text{Res}_K c\text{-Ind}_K^G \sigma \cong \sigma \oplus \bigoplus_{t \geq 1} \left( \pi_{2t}^+(\theta) \oplus \pi_{2t}^- (\theta) \right)
\]

and

\[
\text{Res}_K c\text{-Ind}_K^G \sigma^n \cong \bigoplus_{t \geq 1} \left( \pi_{2t-1}^+(\theta) \oplus \pi_{2t-1}^- (\theta) \right).
\]
On the other hand, for $\sigma = \sigma_0^{\pm}$, which each have central character $\theta_0$, the decompositions are given by

$$\text{Res}_K c-\text{Ind}_K^G \sigma_0^{\pm} \cong \sigma_0^{\pm} \oplus \bigoplus_{t \geq 1} \pi_{2t}^{\pm}(\theta_0)$$

and

$$\text{Res}_K c-\text{Ind}_K^G (\sigma_0^{\pm})^n \cong \bigoplus_{t \geq 1} \pi_{2t-1}^{\pm}(\theta_0).$$

6. Positive depth case

We reprise our notation for positive depth supercuspidal representations.

**Theorem 6.1.** Let $(\mathcal{T}, y, r, \phi)$ be a generic tamely ramified cuspidal $G$-datum with $r > 0$. Let $\rho = \rho(\mathcal{T}, y, r, \phi)$ and set $s = r/2$. Choose $\Gamma \in \mathcal{T}$, representing the restriction of $\phi$ to $\mathcal{T}_{s+}$ as in $(23)$ and $(24)$. Let $\mu = \alpha^t \lambda \in M(\mathcal{T})$ and set $d = r + \delta(\mu)$. Then

$$\text{Ind}_{K}^{\mathcal{T} \cap (\mathcal{T} G_{0+})} \rho^\mu \cong S_d(\phi^\mu, \Gamma^\mu),$$

(22)

unless $y = 0$ and $t = 0$, in which case $\text{Ind}_{K}^{\mathcal{T} G_{0+}} \rho$ is not equivalent to any $S_*(\phi, \Gamma)$.

**Proof.** The final statement of the theorem is evident by comparison of degrees and depth; in fact $\text{Ind}_{K}^{\mathcal{T} G_{0+}} \rho$ is explicitly an example of what Shalika classified as an (irreducible) unramified representation in (29).

So assume $t > 0$ or $y \neq 0$. By Proposition 4.6 we have that

$$\text{Ind}_{K}^{\mathcal{T} \cap (\mathcal{T} G_{0+})} \rho^\mu = \text{Ind}_{K}^{(\mathcal{T} \cap \mathcal{T} G_{0+})} \rho^\mu$$

(23)

whereas by definition,

$$S_d(\phi^\mu, \Gamma^\mu) = \text{Ind}_{T(\Gamma^\mu)}^{(K \cap \mathcal{T} G_{0+})} \Psi_{\phi^\mu, \Gamma^\mu}.$$  

(24)

These two representations have the same depth and degree, by Theorem 3.5 and Propositions 4.7 and 4.8. Since $S_d(\phi^\mu, \Gamma^\mu)$ is irreducible, it suffices to prove that they intertwine; in particular, it suffices to show that $\rho^\mu$ and $\Psi_{\phi^\mu, \Gamma^\mu}$ intertwine on the intersection

$$((K \cap \mathcal{T}^\mu)(K \cap G_{y,s}^\mu)) \cap \left(T(\Gamma^\mu)G_{[0, 1/2], d/2}\right).$$

(25)

Noting that $T(\Gamma^\mu) = K \cap \mathcal{T}^\mu$, this intersection is simply

$$T(\Gamma^\mu)(G_{y,s}^\mu \cap G_{[0, 1/2], d/2}).$$

When $G_{y,s} = G_{y,s+}$, $\rho$ is a character, and the restriction of $\rho^\mu$ to an element $tu$ of this group, with $t \in T(\Gamma^\mu)$ and $u \in G_{y,s}^\mu \cap G_{[0, 1/2], d/2}$, is given by $\rho^\mu(tu) = \phi^\mu(t)\Psi_{\phi^\mu, \Gamma^\mu}(u) = \phi^\mu(t)\Psi_{\phi^\mu, \Gamma^\mu}(tu)$, so (23) and (24) clearly intertwine.

When $G_{y,s} \neq G_{y,s+}$, that is, when $s$ is an integer and $\mathcal{T}$ is unramified so $y \in \{0, 1\}$, the inducing representation $\rho^\mu$ has degree $q$. In what follows we require the additional hypothesis that $\delta(\mu) > 0$, which excludes exactly the case $t = 0$, $y = 0$.

By Lemma 14.6 we have $T(\Gamma^\mu) = Z \mathcal{T}^\mu_{\delta(\mu)} \subseteq Z \mathcal{T}^{\mu}_{0+}$, so by Part (1) of Lemma 3.3 we know the restriction of $\rho^\mu$ to $T(\Gamma^\mu)$ is $\phi^\mu$-isotypic. However, as

$$G_{y,s}^\mu \cap G_{[0, 1/2], d/2} = \left\{\left[\begin{array}{c|c}
U_{[d/2]} & P_{[d/2]} \\
\hline
P_{s+\delta(\mu)} & U_{[d/2]} \\
\end{array}\right]\right\} \cap \left[\begin{array}{c}
G \not
\subseteq G_{y,s+}^\mu,
\end{array}\right]$$

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Part (2) of the lemma does not apply. In fact, although $\Gamma$ is uniquely determined by $\phi$ in $t_r$, only modulo $t_{-s}$, we can see that the restriction of $\Psi_{\phi^\mu,\Gamma}$ to the above subgroup depends on the choice of $\Gamma$ modulo $t_{-s+1}$. We claim that for any of these $q$ choices of $\Gamma$ modulo $t_{-s+1}$, $\Psi_{\phi^\mu,\Gamma}$ intertwines with $\rho^\mu$, and hence that the isomorphism is independent of the choice.

To see this, note that the restriction of $\rho^\mu$ factors through the quotient

$$H = (G^\mu_{y,s} \cap G_{[0,\frac{d}{2}]}) / (G^\mu_{y,r+} \cap G_{[0,\frac{d}{2}]}) .$$

The group $G^\mu_{y,r+} \cap G_{[0,\frac{d}{2}]}$ is given as follows. Set $A = \max\{r + 1, \lfloor d/2 \rfloor\}$, $B = \max\{r + 1 - \delta(\mu), \lfloor d/2 \rfloor\}$ and $C = r + 1 + \delta(\mu)$. Then

$$G^\mu_{y,r+} \cap G_{[0,\frac{d}{2}]} = \left\{ \left[ \frac{A}{C} \frac{B}{U} \right] \cap G \right\} .$$

Since $2([y/2] + s + t) > r$ and $(\lfloor y/2 \rfloor + s + t) + (y + s + 2t) > r + y + 2t$, the quotient group $H$ is abelian. Thus the restriction of $\rho^\mu$ to $H$ decomposes as a sum of $q$ characters. The distinct characters of $H$ are given by $\Psi_Y$, where $Y$ is chosen from the dual lattice quotient; more precisely, the distinct characters correspond to elements $Y$ of the set

$$\tilde{H} = \left\{ \left[ \frac{f}{g} \frac{h}{-f} \right] \ | \ f \in \mathcal{P}^{-A}/\mathcal{P}^{-\lfloor d/2 \rfloor}, h \in \mathcal{P}^{-B}/\mathcal{P}^{-\lfloor d/2 \rfloor}, g \in \mathcal{P}^{-C}/\mathcal{P}^{-s-\delta(\mu)} \right\} .$$

We thus have $\text{Res}_{G^\mu_{y,s} \cap G_{[0,\frac{d}{2}]}} \rho^\mu = \bigoplus_{\gamma \in \tilde{H}} \Psi_Y(\gamma)$ for some $Y \in \tilde{H}$. By Part (2) of Lemma 3.3, $\rho^\mu$ is $\Psi_{\Gamma^\mu}$-isotypic on $G^\mu_{y,s+1}$. Thus the characters $\Psi_Y$ and $\Psi_{\Gamma^\mu}$ must coincide on their restriction to $G^\mu_{y,s+1}$. The elements of $\tilde{H}$ satisfying this condition are exactly those of the form

$$Y(x) = x \left[ \begin{array}{c|c} 0 & -C \\ \hline 0 & 0 \end{array} \right] + \Gamma^\mu,$$

for some $x \in R$. We conclude that $\text{Res}_{G^\mu_{y,s} \cap G_{[0,\frac{d}{2}]}} \rho^\mu = \bigoplus_{\gamma \in \tilde{H}} \Psi_Y(\gamma)$. In particular, since $Y(0) = \Gamma^\mu$, it follows that the representation $\rho^\mu$ and the character $\Psi_{\phi^\mu,\Gamma^\mu}$ intertwine on the intersection.

We summarize the result in the following theorem.

**Theorem 6.2 (Branching Rules for Positive-Depth Supercuspidal Representations).** Let $\rho = \rho(\mathcal{H}, y, r, \phi)$ for a generic tamely ramified cuspidal $\mathcal{G}$-datum of positive depth $r$, and let $\Gamma$ be a $\mathcal{G}$-generic element of depth $r$ representing $\phi$. Then the decomposition into irreducible $K$-representations of the restriction to $K$ of the corresponding supercuspidal representation of $\mathcal{G}$ is, for $y = 0, \frac{d}{2}$ and 1, respectively:

$$\text{Res}_K \text{Ind}_{\mathcal{G}_{y,s}} \mathcal{G} \rho \cong \text{Ind}_{\mathcal{G}_{y,s}} \mathcal{G} \rho \oplus \bigoplus_{t > 0} \left( S_{r+2t}(\phi_{\alpha^t}, \Gamma_{\alpha^t}) \oplus S_{r+2t}(\phi_{\alpha^t}^\epsilon, \Gamma_{\alpha^t}^\epsilon) \right),$$

$$\text{Res}_K \text{Ind}_{\mathcal{G}_{y,s}} \mathcal{G} \rho \cong S_{r+\frac{d}{2}}(\phi, \Gamma) \oplus \bigoplus_{t > 0} \left( S_{r+\frac{d}{2}+2t}(\phi_{\alpha^t}, \Gamma_{\alpha^t}) \oplus S_{r-\frac{d}{2}+2t}(\phi_{\alpha^t}^w, \Gamma_{\alpha^t}^w) \right),$$

$$\text{Res}_K \text{Ind}_{\mathcal{G}_{y,s}} \mathcal{G} \rho \cong S_{r+1}(\phi, \Gamma) \oplus \bigoplus_{t > 0} \left( S_{r+2t+1}(\phi_{\alpha^t}, \Gamma_{\alpha^t}) \oplus S_{r+2t+1}(\phi_{\alpha^t}^\epsilon, \Gamma_{\alpha^t}^\epsilon) \right).$$
7. Intertwining results

In this section, we answer the question of when, and to what extent, two supercuspidal representations of $G$ intertwine as representations of $K$. By the results in the preceding sections, this can be reduced to determining the equivalences between Shalika’s representations.

7.1. Equivalences among Shalika’s ramified representations

Using the notation of Section 3.3, the following theorem is deduced from [29].

**Theorem 7.1** (Shalika). Suppose $X_1 = X(u_1, v_1)$ and $X_2 = X(u_2, v_2)$. For each $i \in \{1, 2\}$ suppose $-d = \text{val}(u_i) < \text{val}(v_i)$ and let $\theta_i$ be a character of $T(X_i)$ which agrees with $\Psi_{X_i}$ on $T(X_i) \cap G_{0, \frac{1}{2}, d/2}$. Then

1. If $X_1 = X_2$ then $S_d(\theta_1, X_1) \cong S_d(\theta_2, X_2)$ if and only if $\theta_1 = \theta_2$.
2. If $S_d(\theta_1, X_1) \cong S_d(\theta_2, X_2)$ then there exists a diagonal matrix $g \in K$ such that $\Psi_{X_1} = \Psi_{X_2}$.

Recall that $\Psi_{X_1} = \Psi_{X_2}$ as characters of $G_{0, \frac{1}{2}, d/2}$ only if $u_1 \equiv u_2$ modulo $\mathcal{P}^{-d/2}$ and $v_1 \equiv v_2$ modulo $\mathcal{P}^{(1-d+1/2)}$.

**Comments on proof.** The first statement comes from [29, Thm 4.2.1, Thm 4.2.5]. It follows from Clifford theory, since $T(X)G_{0, \frac{1}{2}, d/2}$ is the normalizer of $\Psi_{X}$ in $K$ and thus $\text{Ind}_{G_{0, \frac{1}{2}, d/2}}^{T(X)G_{0, \frac{1}{2}, d/2}} \Psi_{X}$ decomposes as a multiplicity-free direct sum of characters of the form $\Psi_{X_0}$. If $d$ is odd, then the second property is stated explicitly in [29, Thm 4.2.1] as $\Psi_{X_1} = \Psi_{X_2}^g = \Psi_{X_2}$ for some $g \in K$; by our choice of representatives of the orbits we may without loss of generality assume $g$ is diagonal.

If $d$ is even, then the second property is instead implicit in the proof of [29, Thm 4.2.5]. Namely, the statement of that theorem gives a $g \in K$ conjugating $X_1$ to some $X'_1$ where $X'_1 \equiv X_2$ modulo $G_{0, -d/2}$. Thus $g$ may be assumed to be diagonal and replacing $S_d(\theta_1, X_1)$ with its conjugate by $g$, we may assume that $X_1 = X'_1$. Then $\Psi_{X_1}$ and $\Psi_{X_2}$ are equal upon restriction to $G_{0, (d+1)/2} \subseteq G_{0, \frac{1}{2}, d/2}$. One checks that the normalizer of $\text{Res}_{G_{0, (d+1)/2}} \Psi_{X_i}$ is $T(X_i)G_{0, d/2}$, whence $T(X_1)G_{0, d/2} = T(X_2)G_{0, d/2}$.

By [29, Lemma 4.1.1], we know that if a character $\psi$ of a subgroup $H$ occurs in the restriction to $H$ of $\Xi_1$ and $\Xi_2$, where each $\Xi_i$ is an irreducible representation of the normalizer $N(\psi)$ of $\psi$ in $K$, then $\text{Ind}_{N(\psi)}^K \Xi_1 \cong \text{Ind}_{N(\psi)}^K \Xi_2$ if and only if $\Xi_1 \cong \Xi_2$. In our case, for each $i \in \{1, 2\}$ set

$$\Xi_i = \text{Ind}_{T(X_i)G_{0, d/2}}^{T(X_i)G_{0, \frac{1}{2}, d/2}} \Psi_{\theta_i, X_i};$$

clearly $\text{Res}_{G_{0, (d+1)/2}} \Psi_{X_i}$ occurs in the restriction of each $\Xi_i$. Since $\text{Ind}_{T(X_1)G_{0, d/2}}^K \Xi_1 = S_d(\theta_1, X_1) \cong \text{Ind}_{T(X_1)G_{0, d/2}}^K \Xi_2$ is irreducible, so are the $\Xi_i$, so applying the above result yields $\Xi_1 \cong \Xi_2$.

As noted at the beginning of the proof, $\Xi_i$ occurs as an irreducible component of $\text{Ind}_{G_{0, \frac{1}{2}, d/2}}^{T(X_i)G_{0, d/2}} \Psi_{X_i}$, and so these representations of $T(X_1)G_{0, d/2} = T(X_2)G_{0, d/2}$ intertwine. But $G_{0, \frac{1}{2}, d/2}$ is normal in $T(X_1)G_{0, d/2}$ so by [29, Lemma 4.1.2] the representations are isomorphic, and there exists some $h \in T(X_1)G_{0, d/2}$ such that $\Psi_{X_1} = \Psi_{X_2}^h$. Because both $X_1$ and $X_2$ are already antidiagonal, this forces $\Psi_{X_1} = \Psi_{X_2}$, as required.
Throughout this section, given $X = X(u, v)$ we denote by “$t(a, b) \in T(X)$” the element $\left[\begin{array}{cc} 1 & \frac{ad}{b} \\ -ba & 1 \end{array}\right]$, leaving the dependence on $u$ and $v$ implicit. Recall that for any $u \in k^\times$, if $X = X(u, 0)$ then $T(X) = ZU$ for $U$ the unipotent upper triangular subgroup of $K$.

**Corollary 7.2.** Suppose $X = X(u, 0)$ and $X' = X(u', 0)$, with $\text{val}(u) = \text{val}(u') = -d$. Then $S_d(\theta, X) \cong S_d(\theta', X')$ if and only if there is some $c \in \mathcal{R}^\times$ such that $c^2u = u'$ and for all $t(z, b) \in T(X)$, we have $\theta(t(z, c^{-2}b)) = \theta'(t(z, b))$.

**Proof.** By part (2) of Theorem 7.1 if $S_d(\theta, X) \cong S_d(\theta', X')$ then there exists some $g = \text{diag}(c, c^{-1}) \in K$ such that $\Psi^\theta_X = \Psi_{X'}$. Thus $c^2u \equiv u'$ modulo $\mathcal{P}^{-d/2}$, so replacing $c$ by a suitable multiple if necessary we may assume $c^2u = u'$. Therefore $S_d(\theta, X) \cong S_d(\theta', X') \cong S_d(\theta', X')$ and $X' = X'$, so by part (1), we must have $\theta'^\circ = \theta'$, which is the property sought. The converse is immediate. 

We wish to generalize this Corollary to provide an explicit complement to part (2) of Theorem 7.1. The challenge lies in that in general $T(X_1) \neq T(X_2)$ even though $\Psi_{X_1} = \Psi_{X_2}$ and thus $T(X_1)\mathcal{G}_{[\frac{1}{2}, \frac{1}{2}], d/2} = T(X_2)\mathcal{G}_{[\frac{1}{2}, \frac{1}{2}], d/2}$.

Let $X = X(u, v)$ and $X' = X(u', v')$ be given and set $n = \text{val}(u^{-1}v - u'^{-1}v')$. Suppose $t' = t(a', b) \in T(X')$; so $a'^2 = 1 + b^2a'^{-1}v'$. Let $a \in \mathcal{R}$ be the unique solution to $a^2 = 1 + b^2u^{-1}v$ satisfying $a \equiv a' \mod \mathcal{P}^n$. Let $t$ be the corresponding element $t(a, b) \in T(X)$, which we call the transfer of $t'$ to $T(X)$. The transfer is not a homomorphism but has the property that

$$t^{-1}t' = \left[\begin{array}{cc} 1 & (a - a')a' \\ (a - a')bn - 1 & (a - a') \end{array}\right] \in \mathcal{G}_{0, n}.$$  \hfill (26)

In particular, when $T(X)\mathcal{G}_{[\frac{1}{2}, \frac{1}{2}], d/2} = T(X)\mathcal{G}_{[\frac{1}{2}, \frac{1}{2}], d/2}$ (equivalently, when $n \geq [(d + 1)/2]$, which implies $t^{-1}t' \in \mathcal{G}_{[\frac{1}{2}, \frac{1}{2}], d/2}$) the transfer gives a refactorization

$$T(X)\mathcal{G}_{[\frac{1}{2}, \frac{1}{2}], d/2} \ni t'g' = t((t^{-1}t')g') = tg \in T(X)\mathcal{G}_{[\frac{1}{2}, \frac{1}{2}], d/2}.$$  \hfill (27)

**Proposition 7.3.** Let $X = X(u, v)$ with $\text{val}(v) > \text{val}(u) = -d$, and suppose $\theta$ is a character of $T(X)$ extending $\Psi_X$. Suppose $X' = X(u', v')$ is such that $\Psi_X = \Psi_{X'}$. For any $t' \in T(X')$ let $t \in T(X)$ be its transfer and set

$$\theta(t') = \theta(t)\Psi_{X'}(t^{-1}t').$$  \hfill (28)

Then $\theta'$ is a character of $T(X')$ extending $\Psi_{X'}$, and $S_d(\theta, X) \cong S_d(\theta', X')$. Moreover, if $u^{-1}v \equiv u'^{-1}v'$ modulo $\mathcal{P}^{d+1}$ then $S_d(\theta, X) \cong S_d(\theta', X')$ if and only if $\theta(t) = \theta'(t')$ for all such transfer pairs $(t, t')$.

**Proof.** Since $\Psi_X = \Psi_{X'}$, these characters have the same normalizer $T(X)\mathcal{G}_{[\frac{1}{2}, \frac{1}{2}], d/2} = T(X')\mathcal{G}_{[\frac{1}{2}, \frac{1}{2}], d/2}$ and so $n = \text{val}(u^{-1}v - u'^{-1}v') \geq [(d + 1)/2]$. Since the induced representation $\text{Ind}_{\mathcal{G}_{[\frac{1}{2}, \frac{1}{2}], d/2}}^{T(X)\mathcal{G}_{[\frac{1}{2}, \frac{1}{2}], d/2}} \Psi_X$ decomposes as a multiplicity-free direct sum of characters of the form $\Psi_{\theta, X}$, we conclude that for each character $\theta$ of $T(X)$ extending $\Psi_X$ there is a unique character $\theta'$ of $T(X')$ extending $\Psi_{X'}$ such that $\Psi_{\theta, X} = \Psi_{\theta', X'}$. It suffices to verify that $\theta'$ satisfies the given identity.

Given $t' = t(a', b) \in T(X')$ and its transfer $t = t(a, b) \in T(X)$, we may for any $g' \in \mathcal{G}_{[\frac{1}{2}, \frac{1}{2}], d/2}$ refactorize $t'g' = t(t^{-1}t')g'$ as in (27). Evaluating $\Psi_{\theta, X} = \Psi_{\theta', X'}$ on both sides yields

$$\theta'(t')\Psi_{X'}(g') = \theta(t)\Psi_X(t^{-1}t')\Psi_X(g')$$
whence (28). For the final statement, we use the explicit form (26) to compute

\[ \Psi_X(t^{-1}t') = \Psi(abu(u^{-1}v' - u^{-1}v))\Psi(2bv(a - a')). \]

(29)

Therefore this correction factor disappears if and only if \( n > d \). Note that in this case the transfer realizes an isomorphism \( T(X)/\ker(\theta) \cong T(X')/\ker(\theta') \).

**Remark 7.4.** By applying this proposition to a character \( \phi \) of \( T \) and two choices of good elements of depth \(-r\) representing \( \phi \), say \( \Gamma \) and \( \Gamma' = f\Gamma \), for some \( f \in U \), one obtains explicitly the equivalence \( S_d(\phi^\mu, \Gamma^\nu) \cong S_d(\phi^\mu, \Gamma'^\nu) \) for any \( \mu \); this was implicitly a consequence of Theorem 6.1.

7.2. Equivalences among \( K \)-components of supercuspidal representations of positive depth \( r \): case \( d > 2r \)

From now on, let \( \pi \) be a supercuspidal representation of \( G \) of positive depth arising from a (generic tamely ramified cuspidal \( G \)-) datum \((T, y, r, \phi)\). As usual set \( s = r/2 \). Let \( \mu = \alpha^s\lambda \in M(T) \) with \( t > 0 \). Then \( \mu \) parametrizes an irreducible representation of \( K \) of depth \( d = r + \delta(\mu) \) occurring in \( \text{Res}_K \pi \) which we denote \( \pi_\mu \). We have shown that if \( \Gamma = aX_T \in t_r \) represents \( \text{Res}_{T_r} \phi \), then \( \pi_\mu \cong S_d(\phi^\mu, \Gamma^\nu) \).

**Lemma 4.5** implies \( t \). We thus simply require \( \delta(\mu) + s > d \). Hence \( \delta(\mu) + s > d \) if and only if \( T \in \calT^\mu_m \) for some \( \delta(\mu) > 0 \). We thus simply require \( \delta(\mu) + s > d \).

**Remark 7.5.** A key property we exploit is the following. Let \( m > s \) and \( m \geq \delta(\mu) \). If \( t \in \calT_m^\mu = K_{m-\delta(\mu)} \cap T(\Gamma^\nu) \), then

\[ \phi^\mu(t) = \phi(\nu^{-1}) = \Psi(\nu^{-1}) \]

which is computed simply as \( \Psi(\text{Tr}(\Gamma(\nu^{-1} - I))) = \Psi(\text{Tr}(\Gamma^\nu(t - I))) \). That is, \( \Gamma^\nu \) determines the value of the Shalika inducing character on \( T_m^\mu G^{[0, \frac{1}{2}], d/2} \geq G^{[0, \frac{1}{2}], d/2} \).

**Proposition 7.6.** Suppose \( d > \frac{4}{3}r \). Given \( \Gamma^\nu = X(u, v) \), set \( X' = X(u, 0) \in g_{0, -d} \). Then there exists a character \( \theta' \) of \( ZU = T(X') \), which agrees with \( \phi \) on \( Z \), such that

\[ S_d(\phi^\mu, \Gamma^\nu) \cong S_d(\theta', X'). \]

Explicitly, for each \( t' = t(z, b) \in ZU \), let \( t \in T(\Gamma^\nu) \) be its transfer; then

\[ \theta'(t') = \phi^\mu(t)\Psi(zbv)^{-1}. \]

(30)

If \( d \leq 2r \) then the depth of \( \theta' \) is \( r - \delta(\mu) = 2r - d \). Otherwise, \( \theta' \) is trivial on \( U \).

**Proof.** Note that \( d > \frac{4}{3}r \) implies \( \text{val}(v) \geq (\frac{1}{3} - 1)/2 \), so \( \Psi_{\Gamma^\nu} = \Psi_X \) on \( G^{[0, \frac{1}{2}], d/2} \). The existence of \( \theta' \) then follows from Proposition 7.3. Let \( t' = t(z, b) \in ZU \) have transfer \( t = t(a, b) \in T(\Gamma^\nu) \).

Then \( a \equiv z \mod P^{2d-2r} \), which for \( d > \frac{4}{3}r \) implies \( (a - z)bv \in P \). Therefore (29) yields \( \Psi_{\Gamma^\mu}(t^{-1}t') = \Psi(-abv) = \Psi(zbv)^{-1} \) and so (28) simplifies to (30) in this case.

Evidently \( \theta' \) and \( \phi \) coincide on \( Z \). To determine the depth of \( \theta' \), take \( z = 1 \) and \( b \in P^m \), for \( m = \max\{0, r - \delta(\mu) \} \). Then the transfer of \( t(1, b) \in ZU \) to \( t(a, b) \in T(\Gamma^\nu) \) lies in \( K_m \cap T(\Gamma^\nu) \). Lemma 7.3 implies \( t \in T_{\delta(\mu) + m} \subseteq T_{\delta(\mu)} \subseteq T_{\delta(\mu) + m} \). Therefore by Remark 7.3 \( \phi^\mu(t) = \Psi(\text{Tr}(\Gamma^\nu(t - I))) \).

We thus simply require \( \delta(\mu) - r \), this is trivial for \( b \in P^m + 1 \) and, if \( m > 0 \), is nontrivial for \( b \in P^m \). When \( d > 2r \), we have \( \text{val}(bv) > 0 \) for all \( b \in R \) and so \( \theta'(t(1, b)) = 1 \).

**Corollary 7.7.** Suppose that \( d > 2r \). Then \( \pi_\mu \) depends only on the central character \( \theta \) of \( \phi \) and the class of \( ab^d \) modulo \( R^{\times 2} \), where \( \Gamma^\nu = X(u, v) \). In particular, with notation as in [20], we have \( \pi_\mu \in \{ \pi_\phi^\theta(\theta), \pi_\phi^\theta(\theta) \} \).
Proof. Indeed, when \( d > 2r \), Proposition 7.6 implies that \( \pi_\mu \cong \mathcal{S}_d(\theta', X(u, 0)) \), where in effect \( \theta' \) is the trivial extension of the central character of \( \pi \) to \( ZU \). Corollary 7.2 implies that this representation depends on \( u \) only up to an element of \( \mathbb{R}^\times \), whence either \( \pi_\mu \cong \mathcal{S}_d(\theta, X(-\infty, 0)) \cong \pi_d^-(\theta) \) or \( \pi_\mu \cong \mathcal{S}_d(\theta, X(-\infty, 0)) \cong \pi_d^-(\theta) \).

Thus for any supercuspidal representation \( \pi \) of \( G \), its \( K \)-components of sufficiently large depth \( d > 2r \) occur among the four irreducible representations of \( K \) — corresponding to classes in \( \mathbb{R}^\times/(\mathbb{R}^\times)^2 \) and characters of \( Z \) — which occur as depth-\( d \) components of the decompositions of supercuspidal representations of depth zero. Further implications of this result, also vis-à-vis principal series representations, are explored in [25].

7.3. Equivalences among \( K \)-components of supercuspidal representations of positive depth \( d \): case \( r < d \leq 2r \)

We begin with some immediate examples.

Proposition 7.8. Let \( (T, y, r, \phi) \) be a datum giving rise to a supercuspidal representation of positive depth. Let \( \psi \) be a nontrivial character of \( T \) of depth \( m < r \), which is trivial on \( Z \). Set \( \phi' = \psi \phi \) and construct the supercuspidal representation \( \pi' \) corresponding to \( (T, y, r, \phi') \). Then \( \pi_\mu \cong \pi'_\mu \) if and only if \( \delta(\mu) > m \).

Proof. By construction \( \phi' \) also has depth \( r \), so there exists \( \Gamma' \in t_r \), a good element of depth \( r \) representing \( \phi' \). We have \( \pi_\mu \cong \mathcal{S}_d(\phi^\mu, \Gamma^\mu) \) and \( \pi'_\mu \cong \mathcal{S}_d(\phi'^\mu, \Gamma'^\mu) \).

If \( m \leq s \), then \( \phi \) and \( \phi' \) agree on \( T_{s+} \), so we may choose \( \Gamma' = \Gamma \). Applying part (1) of Theorem 7.1 yields \( \pi_\mu \cong \pi'_\mu \) if and only if \( \phi'^\mu = \phi^\mu \), which happens if and only if \( \psi^\mu = 1 \) as a character of \( T(\Gamma^\mu) = Z \mathcal{T}_d(\mu) \). Since \( \psi \) is trivial on \( Z \), this happens if and only if \( \delta(\mu) > m \), as required.

Otherwise, we have \( s < m < r \). Write \( \Gamma' = f \Gamma \) for some \( f \in \mathbb{R}^\times \); since \( \phi \) and \( \phi' \) agree precisely on \( T \cap G_{y, m+} \), \( f - 1 \in \mathcal{P}^{r-m} \setminus \mathcal{P}^{r-m+1} \) and so \( f \in (\mathbb{R}^\times)^2 \). Choose \( c \in \mathbb{R}^\times \) such that \( c^2 = f^{-1} \) and set \( g = \text{diag}(c, c^{-1}) \in K \). Writing \( \Gamma^\mu = X(u, v) \) we have \( (\Gamma^\mu)^\gamma = X(f u, f v)^\gamma = X(u, f^2 v) \).

Now \( \text{val}(f^2 v - v) = m - \delta(\mu) \); if this is less than \((d + 1)/2 \) then \( \Psi_{\Gamma^\mu} \) and \( \Psi((\Gamma^\mu)^\gamma) \) are not conjugate by any diagonal matrix and we conclude by part (2) of Theorem 7.1 that \( \pi_\mu \not\cong \pi'_\mu \). Otherwise, we simply have \( \Psi_{\Gamma^\mu} = \Psi((\Gamma^\mu)^\gamma) \). We apply Proposition 7.3 with \( X = (\Gamma^\mu)^\gamma, \theta = (\phi'^\mu)^\gamma \) and \( X' = \Gamma^\mu \) to deduce that \( \pi^\mu_{\Gamma^\mu} \cong \mathcal{S}_d(\theta', \Gamma^\mu) \), where

\[
\theta'(t') = (\phi'^\mu)^\gamma(t') \Psi_{\Gamma^\mu}(t^{-1} t')
\]

for each \( t' \in T(\Gamma^\mu) \) with transfer \( t \in T((\Gamma^\mu)^\gamma) \). Since \( \pi^\mu_\mu \cong \pi^\mu_{\Gamma^\mu} \), it follows from part (1) of Theorem 7.1 that \( \pi_\mu \cong \pi'_\mu \) if and only if \( \phi'^\mu = \theta' \) as characters of \( T(\Gamma^\mu) = Z \mathcal{T}_d(\mu) \).

Let \( n = \max\{\delta(\mu), m\} \). Let \( t' = (t'(a', b') \in T^\mu_n \), with \( a \equiv 1 \mod \mathcal{P} \), and let \( t = t(a, b) \in T((\Gamma^\mu)^\gamma) \) be its transfer. Then \( t^{-1} = t(a, b f) \in T(\Gamma^\mu) \) also lies in \( T^\mu_n \). Since \( m > s \), \( \phi \) and \( \phi' \) are given by \( \Psi_\Gamma \) and \( \Psi_{\Gamma^\mu} \), respectively, on \( T_n \). Using Remark 7.5 we find that \( \phi'^\mu(t') = \Psi((2 b v)^\gamma) \phi'(t'^{-1}) = \Psi((2 b f^2 v)^\gamma) \). Also, noting that \( 2 m > r \) implies under these circumstances that \( a b v \equiv a' b' v \equiv b v \mod \mathcal{P} \), we compute using (29) that \( \Psi_{\Gamma^\mu}(t^{-1} t') = \Psi((1 - f^2) b v) \). Thus \( \theta'(t) = \Psi((1 + f^2) b v) \). Note that \( (1 + f^2) b v - 2 b v \in \mathcal{P}^{n-m} \setminus \mathcal{P}^{n-m+1} \). If \( \delta(\mu) < m \), then \( n = m \) and \( \theta' \neq \phi'^\mu \) on \( T^\mu_n \), whence \( \pi_\mu \not\cong \pi'_\mu \). If \( \delta(\mu) \geq m \), then \( n = \delta(\mu) \) and we conclude that \( \theta' \) and \( \phi'^\mu \) coincide as characters of \( \mathcal{T}_d(\mu) \). Since they also agree on \( Z \) by construction, we have \( \pi_\mu \cong \pi'_\mu \).  

Proposition 7.9. Given \((\pi, \mathcal{T}, y, r, \phi)\) as above, let \( \psi \) be a character of \( T \) of depth \( m < r \) which is trivial on \( Z \). Let \( \phi' = \psi \phi^{-1} \) and construct the supercuspidal representation \( \pi' \) corresponding to
(T, y, r, φ'). Then if \(-1 \in (k^x)^2\), we have \(π_μ \cong π'_μ\) whenever \(δ(μ) > m\), whereas if \(-1 \notin (k^x)^2\), we have \(π_μ \cong π'_μ\) whenever \(δ(μ') > m\) and \(μ ≠ μ'\).

Proof. By Proposition 7.8, it suffices to prove the result for \(ψ = 1\), where \(m = 0\). Since now \(φ' = φ^{-1}\), we may choose \(Γ' = -Γ\).

If \(-1 \in (k^x)^2\), then let \(g = \text{diag}(c, c^{-1}) \in K\) where \(c^2 = -1\). For such \(g\), and any \(μ\), we have \((Γμ)^g = -Γμ = Γμ'\). For any \((a, b) ∈ T(Γμ)\), we have \((t(a, b))g^{-1} = t(a, c^{-2}b) = t(a, b)τ\), so for all \(t ∈ T(Γμ) = T(Γμ')\), we have \((φμ)g(t) = φμ(t) = φμ(t)^{-1} = φμ(t)^{-1}\). We thus conclude

\[π_μ \cong S_d(φμ, Γμ) \cong S_d((φμ)g, (Γμ)^g) = S_d(φμ, Γμ'') \cong π'_μ.\]

Now suppose \(-1 \notin (k^x)^2\), \(δ(μ) = δ(μ') > 0\) and \(μ ≠ μ'\). The latter two conditions can be satisfied only if \(T\) is unramified. We have assumed in this case that \(ε = -1\) (otherwise one must conjugate appropriately) so from \(Γ = -Γ'\) we obtain \(Γμ = Γμ'\). We verify directly that for \(t ∈ T(Γμ)\), \(t(μ^{-1}) = t^{-1}\), so that \(φμ(t) = φμ(t)^{-1}\). The equivalence now follows as above.

Finally, let us show that the cases arising in Propositions 7.8 and 7.9 are in fact exhaustive.

**Theorem 7.10.** Suppose \((T, y, r, φ)\) and \((T', y', r', φ')\) are data defining two supercuspidal representations of positive depth, denoted \(π\) and \(π'\) respectively. Suppose they contain a common \(K\)-component of depth \(d\), with \(r < d ≤ 2r\). Then

- \(r = r'\), that is, \(π\) and \(π'\) have the same depth;
- \(y = y'\) and \(T = T'\), or more generally, their defining tori are conjugate; and
- \(φ\) and \(φ'\) are related as in one of Propositions 7.8 or 7.9.

Proof. Suppose the common irreducible \(K\)-component is \(π_μ \cong π'_μ\). Then \(S_d(φμ, Γμ) \cong S_d(φμ', Γμ')\) and so \(d = d'\). Set \(X(u, v) = Γμ\) and \(X(u', v') = Γμ'\). Then \(\text{val}(u) = \text{val}(u') = -d\) whereas \(a\) \(pr\) \(\text{val}(v) = \text{val}(v') = d - 2r\) and \(\text{val}(v') = d - 2r'\). We first show that \(\text{val}(v) = \text{val}(v')\).

Since \(S_d(φμ, Γμ) \cong S_d(φμ', Γμ')\), Theorem 7.1 implies there exists \(c ∈ R^x\) such that (scaling if necessary) we have

\[c^2u = u' \quad \text{and} \quad c^{-2}v = v' \mod P^{(-1)(d + 1)/2}.\]  

(31)

If \(r < d ≤ 1\), then \(\text{val}(v) = d - 2r ≤ -\frac{1}{2}d < \lfloor (d - 1)/2\rfloor\). Thus \(v\) is nonzero modulo \(P^{(-1)(d - 1)/2}\) and the second congruence implies \(\text{val}(v') = \text{val}(v)\).

Otherwise, we have \(\frac{1}{2}d < d ≤ 2r\). By the preceding we also have \(d > \frac{1}{2}r'\); without loss of generality we may assume \(r ≥ r'\). Then Proposition 7.6 applies, yielding characters \(θ_1\) and \(θ_2\) of \(ZU\) such that

\[π_μ \cong S_d(θ_1, X_1) \quad \text{and} \quad π'_μ \cong S_d(θ_2, X_2)\]

where \(X_1 = X(u, 0)\) and \(X_2 = X(u', 0)\). Since \(c^2u = u'\), Corollary 7.2 implies that for all \(t = t(z, b) ∈ ZU\),

\[θ_1(t(z, c^{-2}b)) = θ_2(t(z, b)).\]

Let \(t(a, c^{-2}b) ∈ T(Γμ)\) be the transfer of \(t = t(z, c^{-2}b) ∈ T(X_1)\) and let \(t(a', b) ∈ T(Γμ')\) be the transfer of \(t(z, b) ∈ T(X_2)\). Expanding the above equality using (30) yields

\[φμ(t(a, c^{-2}b))Ψ(zc^{-2}b)^{-1} = φμ'(t'(a', b))Ψ(zb')^{-1}\]

(32)

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This already implies \( \phi \) and \( \phi' \) agree on \( Z \), so suppose \( z = 1 \) and let \( \text{val}(b) = r - \delta(\mu) = 2r - d \geq 0 \). Then \( t(a, c^{-2}b) \in T^\mu \), so by Remark 7.3 we compute
\[
\phi'^\mu(t(a, c^{-2}b)) = \Psi_\Gamma(t(a, c^{-2}b)^{\mu^{-1}}) = \Psi(2bc^{-2}v).
\]
A similar argument, and our hypothesis \( r \geq r' \), gives \( \phi'^{\mu'}(t(a', b)) = \Psi(2bv') \). Therefore for \( z = 1 \) and \( b \in P^{r-\delta(\mu)} \setminus P^{r-\delta(\mu)+1} \), \( \Psi(2bc^{-2}v) = \Psi(bv') \), and by construction the left side is not identically 1. We conclude that \( c^{-2}v \equiv v' \) modulo \( P^{\text{val}(v)+1} \), whence in particular \( \text{val}(v) = \text{val}(v') \).

Thus in both cases we have that \( \text{val}(v) = \text{val}(v') \), whence \( \delta(\mu) = \delta(\mu') \) and \( r = r' \). From the definition of \( \delta \) we then conclude \( y \equiv y' \) modulo \( 2Z \), whence \( y = y' \).

To prove the final statement, begin by noting that in both cases above we have found \( c \in R^\times \) such that \( c^2u = u' \) and \( c^{-2}v \equiv v' \) modulo \( P^{\text{val}(v)+1} \). If \( \mu \neq \mu' \) then \( \delta(\mu) = \delta(\mu') \) implies \( y \in \{0, 1\} \). Thus \( T = T' \) is an unramified torus and we may assume without loss of generality that \( \mu = \alpha' \) and \( \mu' = \alpha'd \) where \( \Lambda(T) = \{1, d\} \). Since \( \Gamma \) and \( \Gamma' \) are good elements of depth \( -r \), we can write \( \Gamma = aX_T \) and \( \Gamma' = a'X_T \) with in this case \( \text{val}(a) = \text{val}(a') = -r \). Using Table 2, we can write
\[
\Gamma^\mu = X(a\omega^{-\delta(\mu)}, a\omega^{\delta(\mu)}) \quad \text{and} \quad \Gamma'^{\mu'} = X(a'\epsilon\omega^{-\delta(\mu)}, a'\epsilon\omega^{\delta(\mu)}).
\]
Thus \( c^2 = u'/u = a'\epsilon/a \). On the other hand, the congruence \( c^2 \equiv v/v' \) modulo \( P \) yields \( c^2 \equiv a'\epsilon/a' \) modulo \( P \). Thus modulo \( P \), the quotient \( a/a' \) is a self-invertible nonsquare, which exists if and only if \( -1 \notin (k^\times)^2 \). So taking \( \epsilon = -1 \), we have simply \( \Gamma \equiv -\Gamma' \) modulo \( t_{-r+1} \). The character \( \psi = \phi\phi' \) is represented on \( T \), \( \Gamma + \Gamma' \) and so is trivial there, hence is of depth \( m < r \). It is also trivial on \( Z \); we are thus in the setting of Proposition 7.9 and we deduce that \( m < \delta(\mu) \).

So now we may assume that \( \mu = \mu' \).

If \( y \in \{0, 1\} \), then \( T = T' \) and an argument as above yields \( c^2 = a'/a \) and \( c^2 \equiv a/a' \) modulo \( P \). Thus modulo \( P \), the quotient \( a/a' \) is a self-invertible square. A similar argument applies, showing that if \( z = 1 \) then \( \phi' \) can be factored as \( \psi\phi \) whereas if \( z = -1 \), then \( \phi' \) can be factored as \( \psi\phi^{-1} \), where in each case the depth of \( \psi \) is less than \( \delta(\mu) \). Thus we are in the case of Proposition 7.3 or 7.4 respectively.

If \( y = \frac{1}{2} \), then there are up to 4 possible choices for \( T = T_{\gamma_1, \gamma_2} \) and \( T' = T_{\gamma_1', \gamma_2'} \). We suppose that \( \mu = \alpha^2 \); the case that \( \mu = \alpha't \) (so \( \delta(\mu) = 2t - y \)) is accomplished by interchanging \( \gamma_1 \) with \( \gamma_2 \) (and \( \gamma_1' \) with \( \gamma_2' \)) throughout the following argument. Again by Table 2 for some \( a, a' \) of valuation \( r - y \), we have
\[
\Gamma^{a'} = X(a_\gamma_1\omega^{-2t}, a_\gamma_2\omega^{2t}) \quad \text{and} \quad \Gamma'^{a'} = X(a_\gamma'_1\omega^{-2t}, a_\gamma'_2\omega^{2t}).
\]
Thus \( c^2 = \frac{a\gamma_1}{\gamma_1} \) and \( c^{-2} = \frac{a\gamma_2}{\gamma_2} \) modulo \( P \). It follows that the quotients \( \gamma_2'/\gamma_2 \) and \( \gamma_1'/\gamma_1 \) lie in the same class modulo \( (k^\times)^2 \).

We claim it cannot be the case that both are nonsquare. For if they were then Table 1 would imply: that \( T \neq T' \); that \( T \) and \( T' \) correspond to some ramified extension field; and that the quotients are mutually inverse. Repeating the argument of the case \( \mu \neq \mu' \) above, we would conclude that \( -1 \notin (k^\times)^2 \), in which circumstance the two are conjugate, hence equal by our choices, which is a contradiction.

So both quotients are squares, whence from Table 1 we see \( T = T' \) and we have \( \gamma_i = \gamma'_i \) for \( i \in \{1, 2\} \). Then as before there exists a \( z \in \{\pm 1\} \cap (k^\times)^2 \) such that \( \Gamma \equiv z\Gamma' \) modulo \( t_{-r+1} \), and we are done, as above.

\[\square\]
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