MATRIX INVERSION IS AS EASY AS EXPONENTIATION

SUSHANT SACHDEVA AND NISHEETH K. VISHNOI

ABSTRACT. We prove that the inverse of a positive-definite matrix can be approximated by a weighted-sum of a small number of matrix exponentials. Combining this with a previous result [6], we establish an equivalence between matrix inversion and exponentiation up to polylogarithmic factors. In particular, this connection justifies the use of Laplacian solvers for designing fast semi-definite programming based algorithms for certain graph problems. The proof relies on the Euler-Maclaurin formula and certain bounds derived from the Riemann zeta function.

1. Matrix Inversion vs. Exponentiation

Given a symmetric \( n \times n \) matrix \( A \), its matrix exponential is defined to be \( e^A = \sum_{i \geq 0} \frac{A^i}{i!} \). This operator is of fundamental interest in several areas of mathematics, physics, and engineering, and has recently found important applications in algorithms, optimization and quantum complexity. Roughly, these latter applications are manifestations of the matrix-multiplicative weight update method and recently found important applications in algorithms, optimization and quantum complexity. Roughly, the operator is of fundamental interest in several areas of mathematics, physics, and engineering, and has found important applications in algorithms, optimization and quantum complexity. Roughly, the operator is of fundamental interest in several areas of mathematics, physics, and engineering, and has found important applications in algorithms, optimization and quantum complexity.

Theorem 1.1. Given \( \varepsilon, \delta \in (0, 1] \), there exist \( \text{poly}(\log \frac{1}{\delta \varepsilon}) \) numbers \( 0 < w_j, t_j = O(\text{poly}(\frac{1}{\delta \varepsilon})) \), such that for all symmetric matrices \( A \) satisfying \( \delta I \preceq A \preceq I \), \( (1 - \varepsilon)A^{-1} \preceq \sum_j w_je^{-t_jA} \preceq (1 + \varepsilon)A^{-1} \).

This proves that the problems of matrix exponentiation and matrix inversion are equivalent up to polylogarithmic factors. This result justifies the somewhat surprising use of Laplacian solvers for matrix-exponential based methods for designing fast semi-definite programming based algorithms for certain graph problems. Note that this equivalence does not require the matrix \( A \) to be a Laplacian, but only that it be a symmetric positive-definite matrix. It would be interesting to investigate if this result can be used to construct fast solvers for linear systems more general than those arising from graph Laplacians. Finally, note that the numbers \( w_j, t_j \) in the above theorem are independent of the matrix \( A \), and are given explicitly in the proof.

The proof of Theorem 1.1 follows from the lemma below, which gives such an approximation in the scalar world.

Lemma 1.2. Given \( \varepsilon, \delta \in (0, 1] \), there exist \( \text{poly}(\log \frac{1}{\delta \varepsilon}) \) numbers \( 0 < w_j, t_j = O(\text{poly}(\frac{1}{\delta \varepsilon})) \), such that for all \( x \in [\delta, 1] \), \( (1 - \varepsilon)x^{-1} \leq \sum_j w_je^{-t_jx} \leq (1 + \varepsilon)x^{-1} \).

Note that as \( x \) approaches 0 from the right, \( x^{-1} \) is unbounded, where as \( e^{-t x} \) is bounded by 1 for any \( t > 0 \). This justifies the assumption that \( x \in [\delta, 1] \). Versions of this lemma were proved in [2, 3].
Proof of Theorem 1.1. Let \( \{\lambda_i\} \) be the eigenvalues of \( A \) with corresponding eigenvectors \( \{u_i\} \). Since \( A \) is symmetric and \( I \leq A \leq I \), we have \( \lambda_i \in [\delta, 1] \), for all \( i \). Let \( w_j, t_j > 0 \) denote the numbers given by Lemma 1.2 for parameters \( \epsilon \) and \( \delta \). Thus, Lemma 1.2 implies that all \( i \), \( (1 - \epsilon)\lambda_i^{-1} \leq \sum_j w_j e^{-t_i \lambda_i} \leq (1 + \epsilon)\lambda_i^{-1} \). Note that if \( \lambda_i \) is an eigenvalue of \( A \), then \( \lambda_i^{-1} \) is the corresponding eigenvalue of \( A^{-1} \) and \( e^{-t_i \lambda_i} \) is that of \( e^{t_i A} \) with the same eigenvector. Thus, multiplying the scalar inequalities by \( u_i u_i^\top \) and summing up, we obtain the matrix inequality \( (1 - \epsilon) \sum_i \lambda_i^{-1} u_i u_i^\top \leq \sum_j w_j \sum_i e^{-t_i \lambda_i} u_i u_i^\top \leq (1 + \epsilon) \sum_i \lambda_i^{-1} u_i u_i^\top \). Hence, \( (1 - \epsilon)A^{-1} \leq \sum_j w_j e^{-t_i A} \leq (1 + \epsilon)A^{-1} \).

1.1. Integral Representation, Discretization and Smoothness. The starting point of the proof of Lemma 1.2 is the easy integral identity \( x^{-1} = \int_0^\infty e^{-xt} \, dt \). Thus, by discretizing this integral to a sum, the fact that one can approximate \( x^{-1} \) as a weighted sum of exponentials as claimed Lemma 1.2 is not surprising. The crux is to prove that this can be achieved using a sparse sum of exponentials. One way to discretize an integral to a sum is the so-called trapezoidal rule. If \( g \) is the integrand, and \([a, b]\) is the interval of integration, this rule approximates the integral \( \int_a^b g(t) \, dt \) by covering the area under \( g \) in the interval \([a, b]\) using trapezoids of small width, say \( h \), as follows:

\[
\int_a^b g(t) \, dt \approx T_{[a, b], h} \overset{\text{def}}{=} \frac{h}{2} \sum_{j=0}^{K-1} (g(a + jh) + g(a + (j + 1)h)),
\]

where \( K \overset{\text{def}}{=} \frac{b-a}{h} \) is an integer. The choice of \( h \) determines the discretization of the interval \([a, b]\), and hence \( K \), which is essentially the sparsity of the approximating sum. To apply this to the integral representation for \( x^{-1} \), we have to first truncate the infinite integral \( \int_0^\infty e^{-xt} \, dt \) to a large enough interval \([0, b]\), and then bound the error in the trapezoidal rule. Recall that the error needs to be of the form

\[
\left| x^{-1} - \frac{h}{2} \sum_j (e^{-xjh} + e^{-x(j+1)h}) \right| \leq \epsilon x^{-1}.
\]

For such an error guarantee to hold, \( xh \leq O(1) \). Thus, if we want the approximation to hold for all \( 0 < x \leq 1 \), we require \( h \leq O(1) \), which in turn implies that \( K \geq \Omega(x) \). Also, if we restrict the interval to \([0, b]\), the truncation error is \( \int_0^b e^{-xt} \, dt = x^{-1} e^{-bx} \), forcing \( b \geq \delta^{-1} \log \frac{1}{\epsilon} \) for this error to be at most \( \epsilon x^{-1} \) for all \( x \in [\delta, 1] \). Thus, this way of discretizing can only give us a sum which uses \( \text{poly}(1/\delta) \) exponentials, which does not suffice for our application.

This suggests that we should pick a discretization such that \( t \), instead of increasing linearly with \( h \), increases much more rapidly. Thus, a natural idea is to allow \( t \) to grow geometrically. This can be achieved by substituting \( t = e^s \) in the above integral to obtain the identity \( x^{-1} = \int_0^\infty e^{-xe^{s}} \, ds \). We show that discretizing this integral using the trapezoidal rule does indeed give us the lemma.

For convenience, we define \( f_x(s) \overset{\text{def}}{=} e^{-xe^s} + s \). First, observe that \( f_x(s) = x^{-1} f_1(s + \ln x) \). Since we also allow the error to scale as \( x^{-1} \), as \( x \) varies over \([\delta, 1]\), \( s \) needs to change only by an additive \( \log \frac{1}{\delta} \) to compensate for \( x \). Roughly, this suggests that when approximating this integral by the trapezoidal rule, the dependence on \( 1/\delta \) is likely logarithmic, instead of polynomial. The proof formalizes this intuition and uses the fact that the error in the approximation by the trapezoidal rule can be expressed using the Euler-Maclaurin formula (see Section 2.1) which involves higher order derivatives of \( f_x \). We establish the following properties about the derivatives of \( f_x \) which, when combined with known estimates on Bernoulli numbers obtained from the Riemann zeta function, allow us to bound this error with relative ease (see Section 2.2): (1) All the derivatives of \( f_x \) up to any fixed order, vanish at the end points of the integration interval (in the limit). (2) The derivatives of \( f_x \) are reasonably smooth; the \( L_3 \) norm of the \( k \)-th derivative is bounded roughly by \( x^{-k} k^k \) (see Lemma 1.4). In summary, this allows us to approximate \( x^{-1} \) as an infinite sum of exponentials. In this sum, the contribution beyond about \( \text{poly}(\log 1/\delta) \) terms turns out to be negligible, and hence we can truncate the infinite sum to obtain our final approximation (see Section 2.3).
We now present simple properties of the derivatives of $f_x$, alluded to above, which underlie the technical intuition as to why an approximation of the kind claimed in Lemma 1.2 should exist. Let $f^{(k)}_x(s)$ denote the $k$th derivative of the function $f_x$ with respect to $s$. The first fact relates $f^{(k)}_x(s)$ to $f_x(s)$.

**Fact 1.3.** For any non-negative integer $k$, $f^{(k)}_x(s) = f_x(s) \sum_{j=0}^{k} c_{k,j}(-xe^s)^j$, where $c_{k,j}$ are some non-negative integers satisfying $\sum_{j=0}^{k} c_{k,j} \leq (k+1)^{k+1}$.

**Proof.** We prove this lemma by induction on $k$. For $k = 0$, we have $f^{(0)}_x(s) = f_x(s)$. Hence, $f^{(0)}_x$ is of the required form, with $c_{0,0} = 1$, and $\sum_{j=0}^{0} c_{0,j} = 1$. Hence, the claim holds for $k = 0$. Suppose the claim holds for $k$. Hence, $f^{(k)}_x(s) = f_x(s) \sum_{j=0}^{k} c_{k,j}(-xe^s)^j$, where $c_{k,j}$ are non-negative integers satisfying $\sum_{j=0}^{k} c_{k,j} \leq (k+1)^{k+1}$. We can compute $f^{(k+1)}_x(s)$ as follows,

$$
\frac{d}{ds} \left( \sum_{j=0}^{k} c_{k,j}(-xe^s)^j f_x(s) \right) = \sum_{j=0}^{k} c_{k,j} (j - xe^s + 1)(-xe^s)^j f_x(s)
$$

$$
= f_x(s) \sum_{j=0}^{k} ((j+1)c_{k,j} + c_{k,j-1})(-xe^s)^j,
$$

where we define $c_{k+1,0} \overset{\text{def}}{=} 0$, and $c_{k,-1} \overset{\text{def}}{=} 0$. Thus, if we define $c_{k+1,j} \overset{\text{def}}{=} (j+1)c_{k,j} + c_{k,j-1}$, we get that $c_{k+1,j} \geq 0$, and that $f^{(k+1)}_x(s)$ is of the required form. Moreover, we get, $\sum_{j=0}^{k+1} c_{k+1,j} \leq (k+2)(k+1)^{k+1} + (k+1)^{k+1} = (k+3)(k+1)(k+1)^k \leq (k+2)^2(k+1)^k \leq (k+2)^{k+2}$. This proves the claim for $k+1$ and, hence, the fact follows by induction.

The next lemma uses the fact above to bound the $L_1$ norm of $f^{(k)}_x$.

**Lemma 1.4.** For every non-negative integer $k$, $\int_{-\infty}^{\infty} \left| \frac{d}{ds} f^{(k)}_x(s) \right| ds \leq \frac{2}{x} \cdot e^k(k+1)^{2k}$.

**Proof.** By Fact 1.3, $\int_{-\infty}^{\infty} \left| \frac{d}{ds} f^{(k)}_x(s) \right| ds$ is at most

$$
\int_{-\infty}^{\infty} \left| \sum_{j=0}^{k} c_{k,j}(-xe^s)^j \right| e^{-xe^s+\frac{s}{x}} \frac{1}{x} \int_{0}^{\infty} \left| \sum_{j=0}^{n} c_{k,j}(-t)^j \right| e^{-t} dt
$$

$$
\overset{\text{Fact 1.3}}{\leq} \frac{1}{x} (k+1)^{k+1} \left( \int_{0}^{1} e^{-t} dt + \int_{1}^{\infty} t^k e^{-t} dt \right) \leq \frac{1}{x} \cdot (k+1)^{k+1} \cdot (1 + k!) \leq \frac{2}{x} \cdot e^k(k+1)^{2k},
$$

where the last inequality uses $k+1 \leq e^k$, and $1 + k! \leq 2(k+1)^k$.

We conclude this section by giving a brief comparison of our proof to that from [2]. While the authors in [2] employ both the trapezoidal rule and the Euler-Maclaurin formula, our proof strategy is different and leads to a shorter and simpler proof. In contrast to the previous proof, we use the Euler-Maclaurin formula in the limit over $[-\infty, \infty]$, and since the derivatives of $f_x$ vanish in the limit, we save considerable effort in bounding the derivatives at the end points of the integral, which is required when using the Euler-Maclaurin formula to bound the error. We manage to use simpler bounds, at the cost of slightly worse parameters. On the way, we obtain an approximation of $x^{-1}$ as an infinite sum of exponentials that holds for all $x > 0$, which we believe is interesting in itself.

2. **Proof of Lemma 1.2**

Before we introduce the Euler-Maclaurin formula which captures the error in the approximation of an integral by the trapezoidal rule, we introduce the Bernoulli numbers and polynomials, bounds on which are derived using a connection to the Riemann zeta function.
2.1. Bernoulli Polynomials and Euler-Maclaurin Formula. The Bernoulli numbers, denoted by \( b_k \) for any integer \( k \geq 0 \), are a sequence of rational numbers which, while discovered in an attempt to compute sums of the form \( \sum_{j=0}^{\infty} j^k \), have deep connections to several areas of mathematics, including number theory and analysis. They can be defined recursively as: \( b_0 = 1 \), and the following equation which is satisfied for all positive integers \( k \geq 2 \), \( \sum_{j=0}^{k-1} \binom{k}{j} b_j = 0 \). This implies that \((e^t - 1) \sum_{k=0}^{\infty} \frac{b_k t^k}{k!} = t \). Further, it can be checked that \( \frac{t}{e^t-1} - \frac{1}{e^t} \) is an even function, thus implying that \( b_k = 0 \) for odd \( k \geq 2 \). Given the Bernoulli numbers, the Bernoulli polynomials are defined as \( B_k(s) = \sum_{j=0}^{k} \binom{k}{j} b_j s^{k-j} \). It follows from the definition that, for all \( k \), \( B_{2k}(s) \equiv 0 \) for odd \( k \geq 2 \).

We also get \( B_0(s) \equiv 1 \), \( B_1(s) \equiv s - \frac{1}{2} \). Moreover, using the definition of Bernoulli numbers, we get that \( B_k(0) = B_k(1) = b_k \) for all \( k \geq 2 \). We also need the following bounds on the Bernoulli polynomials and the Bernoulli numbers.

**Lemma 2.1.** For any non-negative integer \( k \), and for all \( s \in [0, 1] \), \( |B_{2k}(s)| \leq \frac{|b_{2k}|}{(2k)!} \leq \frac{4}{(2\pi)^{2k}} \).

**Proof.** The first inequality follows from a well-known fact that \( |B_{2k}(s)| \leq |b_{2k}| \) for all \( s \in [0, 1] \) (see [1]). For the second inequality, we recall the following connection between Bernoulli numbers and the Riemann zeta function for any even positive integer, proved by Euler (see [3]), \( \zeta(2k) \equiv \sum_{j=1}^{\infty} j^{-2k} = (-1)^{k+1} \frac{b_{2k} (2\pi)^{2k}}{2(2k)!} \). Thus, \( |b_{2k}| = \frac{2}{(2\pi)^{2k}} \sum_{j=1}^{\infty} j^{-2k} \leq 4(2\pi)^{-2k} \).

One of the most significant connections in analysis involving the Bernoulli numbers is the Euler-Maclaurin formula which describes the error in approximating an integral by the trapezoidal rule.

**Lemma 2.2 (Euler-Maclaurin Formula).** Given a function \( g : \mathbb{R} \to \mathbb{R} \), for any \( a < b \), any positive \( h \), and any positive integer \( N \in \mathbb{N} \), we have,

\[
\int_{a}^{b} g(s)ds - T_{g}[a;b]h = h^{2N+1} \int_{0}^{K} \frac{B_{2N}(s-[s])}{(2N)!} g^{(2N)}(a+sh)ds - \sum_{j=1}^{N} \frac{b_{2j}}{(2j)!} h^{2j} \left( g^{(2j-1)}(b) - g^{(2j-1)}(a) \right),
\]

where \( K = \frac{b-a}{h} \) is an integer, and \([\cdot]\) denotes the integer part.

Note that the Euler-Maclaurin formula is really a family of formulae, one each for the choice of \( N \), which we call the order of the formula. Also note that this formula captures the error exactly. This error can be much less than the naive bound obtained by summing up the absolute value of the error due to each trapezoid. The first term in (1), after removing the contribution due to the Bernoulli polynomials via Lemma 2.1 can be bounded by the \( L_1 \) norm of \( g^{(2N)} \). The second term in (1) depends only on \( g^{(2N-1)} \) evaluated at the ends of the interval. The choice of \( N \) is influenced by how well behaved the higher order derivatives of the function are. For example, if \( g(s) \) is a polynomial, when \( 2N > \text{degree}(g) \), we get an exact expression for \( \int_{a}^{b} g(s)ds \) in terms of the values of the derivatives of \( g \) at \( a \) and \( b \).

In the next section, we use the Euler-Maclaurin formula to bound the error in approximating the integral \( \int_{a}^{b} f(s)ds \) using the trapezoidal rule. For our application, we pick \( a \) and \( b \) such that the derivatives up to order \( 2N-1 \) at \( a \) and \( b \) are negligible. Since the sparsity of the approximation is \( \Omega(1/h) \), for the sparsity to depend logarithmically on the error parameter \( \varepsilon \), we need to pick \( N \) to be roughly \( \Omega(\log 1/\varepsilon) \), so that the first error term in (1) is comparable to \( \varepsilon \).

We end this section by giving a proof sketch for the Euler-Maclaurin formula (see also [3]). By a change of variables, it suffices to prove the formula for \( h = 1 \) and for the interval \([0, 1]\). Consider the

\[\text{The story goes that when Charles Babbage designed the Analytical Engine in the 19th century, one of the most important tasks he hoped the Engine would perform was the calculation of Bernoulli numbers.}\]
integral \( \int_0^1 \frac{B_n^{(2N)}(s)}{(2N)!} g(s)ds \), and apply integration by parts\(^3\) to it repeatedly to obtain
\[
\int_0^1 B_{2N}^{(2N)}(s)g(s)ds = \frac{B_{2N-1}^{(2N-1)}(s)}{(2N)!}g(s) \bigg|_0^1 - \frac{B_{2N-2}^{(2N-2)}(s)}{(2N)!}g^{(1)}(s) \bigg|_0^1 + \frac{B_{2N-3}^{(2N-3)}(s)}{(2N)!}g^{(2)}(s) \bigg|_0^1 - \cdots - \frac{B_2^{(2N)}(s)}{(2N)!}g^{(2N-1)}(s) \bigg|_0^1 + \int_0^1 \frac{B_{2N}^{(2N)}(s)}{(2N)!}g^{(2N)}(s)ds.
\]

Using the fact that for all \( k \leq 2N \), \( \frac{B_{2N-k}^{(2N-k)}(s)}{(2N-k)!} = \frac{B_{2N}^{(2N-k)}(s)}{(2N-k)!} \), and rearranging, we get,
\[
\int_0^1 B_0(s)g(s)ds - B_1(s)g(s) \bigg|_0^1 = \sum_{k=2}^{2N} (-1)^{k-1} \frac{B_k(s)}{k!}g^{(k-1)}(s) \bigg|_0^1 + \int_0^1 \frac{B_{2N}^{(2N)}}{(2N)!}g^{(2N)}(s)ds.
\]

Now, using \( B_0(s) \equiv 1 \), we get that the first term on the l.h.s. is \( \int_0^1 g(s)ds \). Also, since \( B_1(1) = 1/2, B_1(0) = -1/2 \), we see that the second term on the l.h.s. is \( 1/2 \cdot (g(0) + g(1)) = T_0^{[0,1]} \). Finally, using \( B_k(0) = B_k(1) = b_k \) for \( k \geq 2 \), and that \( b_k = 0 \) when \( k \geq 2 \) is odd, we get the desired formula.

### 2.2. Approximation Using an Infinite Sum

As mentioned in Section 1.1, we approximate the integral \( \int_{-\infty}^{\infty} f_x(s)ds \) using the trapezoidal rule. We bound the error in this approximation using the Euler-Maclaurin formula. Since the Euler-Maclaurin formula applies to finite intervals, we first fix the step size \( h \), use the Euler-Maclaurin formula to bound the error in the approximation over the interval \([-bh, bh]\) (where \( b \) is some positive integer), and then let \( b \) go to \( \infty \). This allows us to approximate the integral over \([-\infty, \infty]\) by an infinite sum of exponentials. In the next section, we truncate this sum to obtain our final approximation.

We are given \( \varepsilon, \delta \in (0, 1] \). Fix an \( x \in [\delta, 1] \), the step size \( h = \Theta \left((\log 1/x)^{-2}\right) \), and the order of the Euler-Maclaurin formula, \( N = \Theta \left((\log 1/x)\right) \) (exact parameters to be specified later). For any positive integer \( b \), applying the order \( N \) Euler-Maclaurin formula to the integral \( \int_{-bh}^{bh} f_x(s)ds \), and using bounds from Lemma 2.1, we get,
\[
(2) \quad \left| \int_{-bh}^{bh} f_x(s)ds - T_{f_x}^{[-bh,bh],h} \right| \leq 4 \left( \frac{h}{2\pi} \right)^{2N} \int_{-bh}^{bh} \left| f_x^{(2N)}(s) \right|ds + \sum_{j=1}^{N} 4 \left( \frac{h}{2\pi} \right)^{2j} \left( \left| f_x^{(2j-1)}(-bh) \right| + \left| f_x^{(2j-1)}(bh) \right| \right).
\]

Now, we can use Fact 1.3 to bound the derivatives in the last term of (2). Fact 1.3 implies that for any \( s \) and any positive integer \( k \), \( |f^{(k)}(s)| \leq f_x(s)(k + 1)^{k+1} \max\{1, (xe^x)^k\} \). Thus, for \( b \geq -\frac{1}{h} \log \frac{1}{x} \), we have \( xe^{-bh} \leq 1 \) and \( |f^{(k)}(-bh)| \leq e^{-bh}(k + 1)^{k+1} \), and hence \( f^{(k)}(-bh) \) vanishes for any fixed \( k \) and \( h \), as \( b \) goes to \( \infty \). Also, for any \( x > 0 \), and \( b > 1/\varepsilon \log 1/x \), we get, \( |f^{(k)}(bh)| \leq xe^{(k+1)bh-xe^x}(k + 1)^{k+1} \), which again vanishes for any fixed \( k \) and \( h \), as \( b \) goes to \( \infty \). Thus, letting \( b \) go to \( \infty \) and observing that \( T_{f_x}^{[-bh,bh],h} \) converges to \( h \sum_{j \in \mathbb{Z}} f_x(jh) \), (2) implies,
\[
(3) \quad \left| \int_{-\infty}^{\infty} f_x(s)ds - h \sum_{j \in \mathbb{Z}} f_x(jh) \right| \leq 4 \left( \frac{h}{2\pi} \right)^{2N} \int_{-\infty}^{\infty} \left| f_x^{(2N)}(s) \right|ds.
\]

Hence, since the derivatives of the function \( f_x(s) \) vanish as \( s \) goes to \( \pm \infty \), the error in approximating the integral over \([-\infty, \infty]\) is just controlled by its smoothness. Since we already know \( f_x \) is a very

\(^3\) \( \int \frac{dv}{dx}ds = uv - \int u \frac{dv}{dx}ds. \)
smooth function, we are in good shape. Using Lemma 1.4 we get, $(\frac{h}{15})^{2N} \int_{\infty}^{\infty} \left| f_{x}^{(2N)}(s) \right| ds \leq \frac{2}{x} \left( \frac{2(2N+1)^2 e^h}{2\pi} \right)^{2N}$. Thus, if we let $h \equiv \frac{2\pi}{e^{(2N+1)^2}},$ and $N \equiv \left[ \frac{1}{2} \log \frac{24}{\varepsilon} \right], \eqref{4}$ implies that,

\[ |x^{-1} - \sum_{j \in \mathbb{Z}} e^{j} \cdot e^{-x e^j} - h \sum_{j \in \mathbb{Z}} f_{x}(j)h| \leq 8e^{-2N} \cdot \frac{1}{x} \leq \frac{1}{3} \varepsilon \cdot x. \]

Also note that the above approximation holds for all $x > 1$. Thus, in particular, we can approximate the function $x^{-1}$ over $[\delta, 1]$ as an (infinite) sum of exponentials.

2.3. Truncating the Infinite Sum and Proof of Lemma 1.2. Now, we want to truncate the infinite sum of exponentials approximating $x^{-1}$ given by \eqref{4}. Since the function $f_{x}(s) = e^{s} \cdot e^{-x e^s}$ is non-decreasing for $s < \log \frac{1}{x}$, we majorize the lower tail by an integral. For $A \equiv \left[ -\frac{1}{h} \log \frac{3}{\varepsilon} \right] < 0 \leq \frac{1}{h} \log \frac{2}{\varepsilon}$ (since $x \leq 1$),

\[ h \sum_{j < A} e^{j} \cdot e^{-xe^j} \leq h \int_{-\infty}^{A} e^{j} \cdot e^{-xe^j} dj = \int_{0}^{eh} e^{-xt} dt = x^{-1} \left( 1 - e^{-x e^h} \right) \leq \frac{1}{3} \varepsilon \cdot x. \]

Again, for the upper tail, since the function $f_{x}(s) = e^{s} \cdot e^{-x e^s}$ is non-increasing for $s \geq \log \frac{1}{x}$, we majorize by an integral. For $B \equiv \left[ \frac{1}{h} \log \left( \frac{1}{\delta} \log \frac{3}{\varepsilon} \right) \right] \geq \frac{1}{h} \log \frac{1}{x}$ (since $x \geq \delta$ and $\varepsilon \leq 1$),

\[ h \sum_{j > B} e^{j} \cdot e^{-xe^j} \leq h \int_{B}^{\infty} e^{j} \cdot e^{-xe^j} dj = \int_{e^{B h}}^{\infty} e^{-xt} dt = x^{-1} \cdot e^{-x e^B} \leq \frac{1}{3} \varepsilon \cdot x. \]

Before we complete the proof, we list here the setting of all parameters for completeness:

\[ N = \left[ \frac{1}{2} \log \frac{24}{\varepsilon} \right], \quad h = \frac{2\pi}{e^{2(2N+1)^2}}, \quad A = \left[ -\frac{1}{h} \log \frac{3}{\varepsilon} \right], \quad B = \left[ \frac{1}{h} \log \left( \frac{1}{\delta} \log \frac{3}{\varepsilon} \right) \right]. \]

Thus, combining \eqref{4}, \eqref{5} and \eqref{6}, the final error is given by,

\[ \left| \frac{1}{x} - h \sum_{j \geq A} e^{j} \cdot e^{-xe^j} \right| \leq \left| \frac{1}{x} - h \sum_{j \in \mathbb{Z}} e^{j} \cdot e^{-xe^j} \right| + h \sum_{j < A} e^{j} \cdot e^{-xe^j} + h \sum_{j > B} e^{j} \cdot e^{-xe^j} \leq \frac{\varepsilon}{x}. \]

Hence, $(1 - \varepsilon)x^{-1} \leq h e^{j} \cdot e^{-xe^j} \leq (1 + \varepsilon)x^{-1}$, implying the claim of Lemma 1.2.

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