Arithmetic Spacetime Geometry from String Theory

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ABSTRACT:

An arithmetic framework to string compactification is described. The approach is exemplified by formulating a strategy that allows to construct geometric compactifications from exactly solvable theories at $c = 3$. It is shown that the conformal field theoretic characters can be derived from the geometry of spacetime, and that the geometry is uniquely determined by the two-dimensional field theory on the world sheet. The modular forms that appear in these constructions admit complex multiplication, and allow an interpretation as generalized McKay-Thompson series associated to the Mathieu and Conway groups. This leads to a string motivated notion of arithmetic moonshine.

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1 Introduction

One of the long-standing questions in string theory has been the problem to understand the nature of spacetime, in particular how spacetime can emerge from a theory of 1-dimensional objects. Phrased this way, the question is perhaps too vague to lend itself to a direct approach, but it can be made more precise in the context of exactly solvable string models\cite{1, 2}. In these models the string input is described in terms of an extension of the Virasoro algebra defined by affine Kac-Moody algebras, and it is believed that such theories are closely related to Calabi-Yau varieties.

It is the purpose of this paper to outline a strategy that allows the construction of varieties from string theoretic objects, and to show how this works in the simplest context of toroidal compactifications, more precisely the class of Gepner models at $c = 3$. The main virtue of this type of string models is that their structure is irreducible in a certain sense, in contrast to the complicated motivic structure of higher dimensional varieties. This can be expressed more concretely by noting that cohomologically the only motivic invariant available is the Jacobian of the curve. In higher dimensions the relation between string theory and varieties is more complicated because this irreducibility is lost. The strategy described here is general enough to be applied in any dimension, but the rules that apply to these different situations will be more involved.

The class of Gepner models with $c = 3$ is a small set of three models, simple enough to allow for a concise and coherent discussion. Geometrically it concerns the elliptic curves defined by Brieskorn-Pham polynomials

\[
E_3 = \{(z_0 : z_1 : z_2) \in \mathbb{P}_2 \mid z_0^3 + z_1^3 + z_2^3 = 0\}
\]
\[
E_4 = \{(z_0 : z_1 : z_2) \in \mathbb{P}_{(1,1,2)} \mid z_0^4 + z_1^4 + z_2^2 = 0\}
\]
\[
E_6 = \{(z_0 : z_1 : z_2) \in \mathbb{P}_{(1,2,3)} \mid z_0^6 + z_1^3 + z_2^2 = 0\},
\]

which are expected to correspond to tensor models of $A_1^{(1)}$ theories at various levels, equipped with the diagonal affine invariant. Denote by $A_{1,k}^{(1)}$ the affine Lie algebra $A_1^{(1)}$ at affine level $k$. 

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It is conjectured that there exist relations

\[ E_i \sim \left( \bigotimes_{k_{i,j}} A^{(1)}_{1,k_{i,j}} \right)_{\text{GSO}}, \tag{2} \]

where the subscript GSO indicates the supersymmetry projection, the affine levels are given by

\[ (k_{i,1}, k_{i,2}, k_{i,3}) \in \{(1, 1, 1), (2, 2, 0), (4, 1, 0)\}, \tag{3} \]

and 0 denotes a trivial factor.

The first part of this paper completes the modular analysis of the class of elliptic Brieskorn-Pham models, initiated in [3], and continued as part of a discussion with a different focus in [4]. More precisely, the following string theoretic interpretation of the L-function of the degree six elliptic Brieskorn-Pham curve is shown to hold. Denote by \( \Gamma_0(N) \) the congruence subgroup of \( \text{SL}(2, \mathbb{Z}) \) defined by matrices that are upper triangular mod some integer \( N \),

\[ \Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbb{Z}) \mid \begin{pmatrix} a & b \\ c & d \end{pmatrix} \sim \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \pmod{N} \right\}, \tag{4} \]

and denote by

\[ \eta(q) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n) \tag{5} \]

the Dedekind eta function, where \( q = e^{2\pi i \tau} \). Let furthermore \( c_{i,m}^{k}(\tau) \) be the Kac-Peterson string functions of \( A^{(1)}_{1,k} \), defined in Section 3, and denote by \( f \otimes \chi \) the twisted modular form defined as

\[ f(q) \otimes \chi = \sum_n a_n \chi(n) q^n. \tag{6} \]

**Theorem 1.** The Mellin transform of the elliptic curve \( E_6 \) is a modular form in \( S_2(\Gamma_0(144)) \) which is determined by the Hecke indefinite modular form \( \Theta_{1,1}^{1}(\tau) = \eta^3(\tau)c_{1,1}^1(\tau) \) of the string function \( c_{1,1}^1(\tau) \) at conformal level \( k = 1 \) as \( f_{\text{HW}}(E_6, q) = \Theta_{1,1}^{1}(q^6)^2 \otimes \chi_3 \), where \( \chi_3(p) = \left( \frac{3}{p} \right) \) is the Legendre symbol.

Combining this result with the results of [3] and [4] leads to the following complete identification of the elliptic Brieskorn-Pham curve L-functions in terms of string theoretic modular forms.
Theorem 2. The Mellin transforms $f_{HW}(E_i, q)$ of the Hasse-Weil $L$-functions $L_{HW}(E_i, s)$ of the curves $E_i, i = 3, 4, 6$ are modular forms $f_{HW}(E_i, q) \in S_2(\Gamma_0(N))$, with $N \in \{27, 64, 144\}$ respectively. These forms factor into products of Hecke indefinite modular forms $\Theta^k_{\ell,m}(\tau) = \eta^3(\tau)c^k_{\ell,m}(\tau)$, determined by the string functions $c^k_{\ell,m}(\tau)$, as follows

\begin{align*}
  f_{HW}(E_3, q) &= \Theta^{1,1}_{1,1}(q^3)\Theta^{1,1}_{1,1}(q^9) \\
  f_{HW}(E_4, q) &= \Theta^{2,1}_{1,1}(q^4)^2 \otimes \chi_2 \\
  f_{HW}(E_6, q) &= \Theta^{1,1}_{1,1}(q^6)^2 \otimes \chi_3, \tag{7}
\end{align*}

where $\chi_n(p) = \left(\frac{p}{n}\right)$ denotes the Legendre symbol.

The appearance of the Legendre symbols $\chi_n$ in the above theorems can be explained physically. Their origin can be found in the algebraic number field that is determined by the conformal weights of the underlying conformal field theory via the Rogers dilogarithm.

Corollary. The twist characters $\chi_n$ are the quadratic characters of the fields of quantum dimensions of the underlying affine Lie algebra $A^{(1)}_1$.

More concretely, these quadratic characters determine the factorization behavior of the rational primes in the field extensions determined by quantum dimensions of the conformal field theory. In the discussion of the cubic curve $E_3$ in [3] no such character appeared because the field of quantum dimensions in that case does not define an extension of the field of rational numbers, but is the field $\mathbb{Q}$ itself.

Theorem 2 shows that certain conformal field theoretic characters that enter the string partition functions of these models are determined by the arithmetic properties of the topological part of spacetime, and vice versa. Mathematically it establishes a relation between the arithmetic of elliptic curves and the theory of affine Kac-Moody algebras.

In the second part of this paper it is shown that it is possible to reverse the above logic and to formulate a set of assumptions which uniquely determine the conformal field theoretic modular forms of the string which appear in Theorem 2, without any a priori notions from geometry. Given this identification of potentially geometric forms derived from the string, one can ask whether it is possible to construct the geometry of spacetime purely from string
theoretic ingredients. In the context of abelian varieties this can be achieved by the theory initiated by Klein, Hurwitz and Hecke, and further developed by Eichler and Shimura.

The relation described here, between the arithmetic geometry of elliptic curves on the one hand, and the theory of affine Lie algebras on the other, can also be used to establish a new link between Kac-Moody algebras and modular moonshine for some finite sporadic groups, such as the Mathieu group and the Conway group. The theory of generalized Kac-Moody algebras has been used by Borcherds [6] to prove the monstrous moonshine conjecture of Conway and Norton [7], providing an interpretation of hauptmoduln as McKay-Thompson series. The idea to interpret modular forms of higher weight in a similar way has been discussed by Mason [8] and Martin [9] for the largest Mathieu group $M_{24}$ as well as the Conway group Co. Combining their work with the results described here leads to an interpretation of string theoretic modular forms as generalized McKay-Thompson series in the context of affine Lie algebras.

The outline of the paper is as follows. In Section 2 the Hasse-Weil L-function is computed and the geometries are analyzed to the extent necessary to compute the geometric conductors of the curves. These conductors determine the levels of the modular forms expected from the modularity theorem of Wiles, Taylor, and Breuil et.al. In Section 3 the basic building blocks of the affine Kac-Moody algebra are determined at the levels $k = 1, 2, 4$. Section 4 completes the proof of the string theoretic interpretation of the Hasse-Weil L-functions for the Brieskorn-Pham type elliptic curves. In Section 5 a set of criteria is formulated which geometric modular forms should satisfy. These constraints uniquely determine the Hasse-Weil modular forms of the elliptic Brieskorn-Pham curves considered in this paper. Section 6 describes how the arithmetic geometry of these curves can be recovered from the modular forms. In Section 7 it is shown that the modular forms encountered here admit complex multiplication in the sense of Shimura-Taniyama, in agreement with the conjecture of [10]. Section 8 points out that the Hasse-Weil modular forms can be interpreted as generalized McKay-Thompson series of either the largest Mathieu group $M_{24}$, or the Conway group Co determined as the automorphism group of the Leech lattice, thereby leading to sporadic modular moonshine.
2 From geometry to modularity

2.1 Hasse-Weil L-functions

The strategy in this paper is to use the arithmetic of elliptic curves to produce modular forms and to show that these modular forms admit a conformal field theoretic interpretation. More details, physical motivation, historical remarks, and further references can be found in [3]. The idea is to use the congruent zeta function of Artin, Schmidt, and Weil

$$Z(X/\mathbb{F}_p, t) = \exp \left( \sum_{r \geq 1} \frac{N_{r,p}(X)}{r} t^r \right),$$  \hfill (8)

where $N_{r,p}(X)$ denotes the number of solutions of the variety $X$ over the finite field extension $\mathbb{F}_p$ and $t$ is a formal variable. Schmidt has shown that the zeta function of curves $C$ of genus $g$ factorizes into the rational form

$$Z(C, t) = P(p)(t)/(1 - t)(1 - pt),$$  \hfill (9)

where $P(p)(t)$ is a polynomial of degree $2g$. It is this polynomial which is used to define the Hasse-Weil L-function. The Hasse-Weil L-function globalizes the local congruent functions by combining the information at all primes

$$L_{HW}(C, s) = \prod_{p \text{ prime}} \frac{1}{P(p)(p^{-s})}.$$  \hfill (10)

For elliptic curves the polynomials $P(p)(t)$ can be written as

$$P(p)(t) = 1 + \beta_1(p)t + \delta(p)pt^2,$$  \hfill (11)

where $\beta_1(p)$ and $\delta(p)$ depend on the properties of the reduction over the curve of the finite field $\mathbb{F}_p$. At the good primes $\beta_1(p)$ is given by

$$\beta_1(p) = N_{1,p}(E) - (p + 1),$$  \hfill (12)

while at the bad primes its structure depends on the type of the singularity

$$\beta_1(p) = \begin{cases} 
\pm 1 & \text{if the singularity at } p \text{ is a node} \\
0 & \text{if the singularity at } p \text{ is a cusp}
\end{cases}.$$  \hfill (13)
Here the sign in the first case depends on whether the node is split or non-split. \( \delta(p) \) finally is given as
\[
\delta(p) = \begin{cases} 
0 & \text{if } p \text{ is a bad prime} \\
1 & \text{if } p \text{ is a good prime}
\end{cases}.
\] (14)

With these ingredients the Hasse-Weil L-function can then be defined as
\[
L_{HW}(X, s) = \prod_{p \in S} \frac{1}{1 + \beta_1(p)p^{-s}} \prod_{p \notin S} \frac{1}{1 + \beta_1(p)p^{-s} + p^{2-2s}},
\] (15)
where \( S \) denotes the set of bad primes.

The Hasse-Weil L-functions of \( E_3 \) and \( E_4 \) can be found in [3] and [4] respectively. Here the analysis is completed by computing the L-functions for the degree six elliptic curve. Table 1 contains the cardinalities and the resulting coefficients \( \beta_1(p) \) for the lower primes.

| Prime \( p \) | 2 3 5 7 11 13 17 19 23 29 31 37 41 43 |
|---------------|-----------------------------------|
| \( N_{1,p} \) | 3 4 6 4 12 12 18 28 30 28 48 42 52 |
| \( \beta_1(p) \) | 0 0 0 −4 0 −2 0 8 0 0 −4 10 0 8 |

**Table 1.** Cardinalities \( N_{1,p}(E_6) \) for the elliptic curve of degree six \( E_6 \subset \mathbb{P}_{(1,2,3)} \).

By taking into account the fact that the bad primes for \( E_6 \) are given by \( p = 2, 3 \) one finds that its L-function takes the form
\[
L_{HW}(E_6, s) = 1 + \frac{4}{7^s} + \frac{2}{13^s} - \frac{8}{19^s} - \frac{5}{25^s} + \frac{4}{31^s} + \cdots
\] (16)

To any \( L \)-series \( L = \sum_n a_n n^{-s} \) one can associate a \( q \)-series by replacing \( n^{-s} \mapsto q^n \). Doing so leads to what might be called a Hasse-Weil \( q \)-expansion. Table 2 summarizes the results of the expansions for the curves discussed here and in [3, 4].

| Curve | Hasse-Weil \( q \)-expansion |
|-------|-----------------------------|
| \( E_3 \) | \( q - 2q^4 - q^7 + 5q^{13} + 4q^{16} - 7q^{19} - 5q^{25} + 2q^{28} - 4q^{31} + 11q^{37} + \cdots \) |
| \( E_4 \) | \( q + 2q^5 - 3q^9 - 6q^{13} + 2q^{17} - q^{25} + 10q^{29} + 2q^{37} + 10q^{41} + \cdots \) |
| \( E_6 \) | \( q + 4q^7 + 2q^{13} - 8q^{19} - 5q^{25} + 4q^{31} - 10q^{37} - 8q^{43} + \cdots \) |

**Table 2.** The Hasse-Weil \( q \)-expansions \( f_{HW}(E_i, q) \) for the curves \( E_3, E_4, E_6 \).
2.2 Modular forms from the Hasse-Weil L-function

The proof of the Shimura-Taniyama conjecture in the modular curve theorem says that the \( q \)-expansion associated to \( L_{\text{HW}}(E, s) \) is a modular form of weight two and some modular level \( N \). More concretely, the weight of a modular form which is a Hecke eigenfunction can be determined from its multiplicative structure, while the level can be obtained from the conductor of the curve and checked via the Fricke operator. Both of these ingredients are provided by the Hecke algebra of operators that act on modular forms. The basic ingredients in the present context can be summarized as follows.

One possible way to define Hecke operators \( T^w_n \), \( n \in \mathbb{N} \), on modular forms \( f(\tau) \) of weight \( w \) with character \( \chi \) is by considering the following linear combination of shifts

\[
T^w_n f(\tau) = \frac{1}{n} \sum_{a \equiv n} \chi(a) a^w \sum_{0 \leq b < d} \left( \frac{a\tau + b}{d} \right). \tag{17}
\]

If \( f(q) = \sum_{n=0}^{\infty} a_n q^n \) is the \( q \)-expansion of the modular form, the expansion image \( T^w_n f(q) = \sum_{n=0}^{\infty} b_m q^m \) is given by

\[
b_m = \sum_{d \mid (m,n)} \chi(d) d^{w-1} a_{(mn/d^2)}. \tag{18}\]

The operators \( T^w_n \) map the space of cusp forms \( S_w(\Gamma_0(N), \chi) \) into itself, and for prime \( p \) the operator simplifies on cusp forms to

\[
T^w_p f(q) = \sum_{n=1}^{\infty} a_{np} q^n + \chi(p) p^{w-1} \sum_{n=1}^{\infty} a_n q^{np}. \tag{19}\]

Then

\[
T^w_{mn} = T^w_m T^w_n, \quad m, n \text{ coprime,}
\]

\[
T^w_{p^{e+1}} = T^w_p T^w_p - p^{w-1} T^w_{p^e}. \tag{20}
\]

A special operator has to be considered at primes \( p \) which divide the level \( N \) of the form. This is the Atkin-Lehner operator [12], often denoted by \( U_k(p) \)

\[
U_p f(q) = \sum a_{pn} q^n. \tag{21}\]
For eigenforms of these operators the operator structure translates into identical relation
between their coefficients $a_n$

$$a_{mn} = a_m a_n \quad (m, n) = 1$$
$$a_{p^{n+1}} = a_p a_p - p^{k-1} a_{p^{n-1}}, \quad \text{for } p \nmid N$$
$$a_{p^n} = (a_p)^n, \quad \text{for } p | N. \quad (22)$$

The Hasse-Weil forms $f_{HW}(E_i, q)$ of the elliptic curves of Brieskorn-Pham type satisfy these
relations with $w = 2$, hence define normalized cusp Hecke eigenforms of weight two.

This leaves the question what the level $N$ is of the modular forms listed in Table 2. The answer
to this problem involves two ingredients, Weil’s conductor conjecture, and an involution $w_N$
that preserves the space of cusp forms on $\Gamma_0(N)$ and is defined by

$$w_N(f)(\tau) = N\tau^2 f\left(\frac{-1}{N\tau}\right). \quad (23)$$

Weil observed that the geometric conductor of an elliptic curve is in fact the level of the
modular form [13], hence it provides guidance which can then be used for the Fricke involution
$w_N$.

The first step therefore is to compute the conductors of the elliptic curves $E_i$. For an elliptic
curve this is a quantity which is determined both by the rational primes for which the reduced
curve degenerates, i.e. the bad primes, as well as the degeneration type. The conductor of the
curve can be computed by first transforming the Fermat cubic into a Weierstrass form and
then applying Tate’s algorithm [14]. Since it is necessary to consider fields of characteristic 2
and 3, the usual (small) Weierstrass form $y^2 = 4x^3 + Ax + B$ is not appropriate. Instead one
has to consider the generalized Weierstrass form given by

$$E : \quad y^2 + a_1 xy + a_3 y = x^3 + a_2 x^2 + a_4 x + a_6, \quad (24)$$

where the unusual index structure indicates the weight of the coefficients under admissible
transformations which preserve this form

$$(x, y) \mapsto (u^2 x + r, u^3 y + u^2 s x + t), \quad (25)$$
with \( r, s, t \in K \) and \( u \in K^\times \) if \( E \) is defined over the field \( K \). Curves of this type can acquire certain types of singularities when reduced over finite prime fields \( \mathbb{F}_p \). The quantity which detects such singularities is the discriminant

\[
\Delta = \frac{c_4^3 - c_6^2}{1728},
\]

where

\[
c_4 = b_2^2 - 24b_4
\]

\[
c_6 = -b_2^2 + 36b_2b_4 - 216b_6,
\]

with

\[
b_2 = a_1^2 + 4a_2
\]

\[
b_4 = a_1a_3 + 2a_4
\]

\[
b_6 = a_3^2 + 4a_6.
\]

The curve \( E \) acquires singularities at those primes \( p \) for which \( p \mid \Delta \). The singularity types that can appear have been classified by Kodaira and Néron [15], and are indicated by Kodaira’s symbols. \( I_0 \) describes the smooth case, \( I_n \), \((n > 0)\) involve bad multiplicative reduction, and \( I_n^*, II^*, III^*, IV^* \) denote bad additive reduction.

The conductor itself depends on the detailed structure of the bad fiber and the discriminant. Conceptually, it is defined as an integral ideal of the field \( K \) over which the elliptic curve \( E \) is defined. By a result of Ogg [16] this ideal is determined by the number \( s_p \) of irreducible components of the singular fiber at \( p \) as well as the order \( \text{ord}_p\Delta_{E/K} \) of the discriminant \( \Delta_{E/K} \) at \( p \). In the present cases the curves are defined over the field \( K = \mathbb{Q} \), hence the ring of integers is a principal domain. The conductors can therefore be viewed as numbers defined by

\[
N_{E/Q} = \prod_{\text{bad } p} p^{f_p},
\]

where the exponent \( f_p \) is given by

\[
f_p = \text{ord}_p\Delta_{E/Q} + 1 - s_p.
\]

\( E_3 \): The Fermat cubic can be transformed into the form \( v^2 - 9v = u^3 - 27 \), which in turn can be transformed further by completing the square and introducing the variables \( x = u \) and
$y = v - 5$, leading to the affine curve

$$y^2 + y = x^3 - 7.$$ (31)

This curve has discriminant $\Delta(E_3) = -3^9$ and $j$-invariant $j = 0$, while the singular fiber resulting from Tate’s algorithm is of Kodaira type $IV^*$ with $s_3 = 7$ components. Using the fact that $\text{ord}_3(\Delta(E_3)) = 9$ leads to the conductor $N = 27$.

$E_4$: The Brieskorn-Pham quartic can be transformed into the Weierstrass form

$$v^2 = u^3 + u,$$ (32)

which has discriminant $\Delta(E_4) = -2^6$. It follows that $\text{ord}_2(\Delta(E_4)) = 6$, and since the singular fiber resulting from Tate’s algorithm is of Kodaira type $II$ with $s_2 = 1$ component, this leads to the conductor $N = 64$.

$E_6$: The Weierstrass form of this curve is given by

$$v^2 = u^3 - 1.$$ (33)

The discriminant of this curve is $\Delta(E_6) = -2^4 \cdot 3^3$, hence both 2 and 3 are bad primes, as expected, and we have $\text{ord}_2(\Delta(E_6)) = 4$ and $\text{ord}_3(\Delta(E_6)) = 3$. The singularity types are given by the fibers of the type

$$p = 2 : \text{II}, \quad p = 3 : \text{III}.$$ (34)

Hence Ogg’s result leads to the conductor $N = 144$.

### 2.3 Dimension of spaces of modular forms

The conductor computations above show that the modular forms derived from the Hasse-Weil $L$-functions considered here are cusp forms of conductors

$$N \in \{27, 64, 144\}.$$ (35)

It is useful to compute the dimensions of the corresponding spaces $S_2(\Gamma_0(N))$. The dimension of the space of cusp forms at fixed level $N$ can be found in many places, e.g. [11]. It turns
out to be given by the genus of an algebraic curve $X(\Gamma_0(N))$ constructed from the congruence group $\Gamma_0(N)$, for reasons that will become clear further below.

**Proposition.** The dimension of the space $S_2(\Gamma_0(N))$ is given by

$$\dim S_2(\Gamma_0(N)) = g(X(\Gamma_0(N))),$$ (36)

where the genus of $X(\Gamma_0(N))$ is determined by

$$g(X(\Gamma_0(N))) = 1 + \frac{\mu(N)}{12} - \frac{\nu_2(N)}{4} - \frac{\nu_3(N)}{3} - \frac{\nu_\infty(N)}{2},$$ (37)

where $\mu(N), \nu_2(N), \nu_3(N), \nu_\infty(N)$ denote the index of $\Gamma_0(N)$, the number of elliptic points of order 2,3, and the number of $\Gamma_0(N)$–inequivalent cusps respectively.

The ingredients of this dimension result only depend on the modular level $N$ and its prime divisors, and closed formulae are available.

**Proposition.** The index $\mu(N)$ of $\Gamma_0(N)$ is given by

$$\mu(N) = N \prod_{p \mid N} \left(1 + \frac{1}{p}\right),$$ (38)

where the product is over all prime divisors of $N$. $\Gamma_0(N)$ has elliptic elements (i.e. $\gamma$ with $|\text{Tr}(\gamma)| < 2$). Their numbers are given by

$$\nu_2(N) = \begin{cases} 0 & \text{if } 4 \mid N \\ \prod_{p \mid N} \left(1 + \left(\frac{-4}{p}\right)\right) & \text{otherwise} \end{cases},$$

$$\nu_3(N) = \begin{cases} 0 & \text{if } 2 \mid N \text{ or } 9 \mid N \\ \prod_{p \mid N} \left(1 + \left(\frac{-3}{p}\right)\right) & \text{otherwise} \end{cases},$$ (39)

where $\left(\frac{-}{p}\right)$ denotes the Legendre symbol. Finally, the number of cusps is given by

$$\nu_\infty(N) = \sum_{0 < d \mid N} \phi((d, N/d)),$$ (40)

where $(d, N/d)$ denotes the greatest common divisor and $\phi(n)$ is the Euler totient function.

The results for $N = 27$ and $N = 64$ have been described e.g. in [3] and [4]. For the space $S_2(\Gamma_0(144))$ one obtains $\mu(144) = 288$. There are no elliptic points of order 2, $\nu_2(144) = 0,$
because 144 is divisible by 4. There are also no elliptic points of order 3, because 9|144, hence \( \nu_3(144) = 0 \). Finally, the number of \( \Gamma_0(144) \) independent cusps is \( \nu_\infty(144) = 24 \). Plugging these ingredients into the genus formula gives \( g(X(\Gamma_0(144))) = 13 \). Collecting everything one finds that the curves \( E_i \) lead to the results collected in Table 3.

| Curve | \( \Delta \) | Type | \( s_p \) | \( N \) | \( \dim S_2(\Gamma_0(N)) \) |
|-------|-------------|------|--------|-------|-----------------|
| \( E_3 \) | \(-3^9\) | IV* | 7 | 27 | 1 |
| \( E_4 \) | \(-2^6\) | II | 1 | 64 | 3 |
| \( E_6 \) | \(-2^4 \cdot 3^4\) | II, III | (1,2) | 144 | 13 |

Table 3. Relevant characteristics for the elliptic curves \( E_i \).

3 Affine Kac-Moody Algebras

3.1 \( N = 2 \) supersymmetric models

Supersymmetric string models can be constructed in terms of conformal field theories with \( N=2 \) supersymmetry. The simplest class of \( N=2 \) supersymmetric exactly solvable theories is built in terms of the affine \( SU(2)_k \) algebra at level \( k \) as a coset model

\[
\frac{SU(2)_k \otimes U(1)_2}{U(1)_{k+2, \text{diag}}}. \tag{41}
\]

Coset theories \( G/H \) lead to central charges of the form \( c_G - c_H \), hence the supersymmetric affine theory at level \( k \) still has central charge \( c_k = 3k/(k+2) \). The spectrum of conformal weights \( \Delta_{q,s}^\ell \) and U(1)—charges \( Q^\ell \) of the primary fields \( \Phi_{q,s}^\ell \) at level \( k \) is given by

\[
\Delta_{q,s}^\ell = \frac{\ell(\ell + 2) - q^2}{4(k+2)} + \frac{s^2}{8},
\]

\[
Q_{q,s}^\ell = -\frac{q}{k+2} + \frac{s}{2}, \tag{42}
\]

where \( \ell \in \{0, 1, \ldots, k\} \), \( \ell + q + s \in 2\mathbb{Z} \), and \( |q - s| \leq \ell \). Associated to the primary fields are characters defined as

\[
\chi_{\ell,q,s}^k(\tau, z, u) = e^{-2\pi i u} \text{tr}_{H_{\ell,q,s}} e^{2\pi i (L_0 - \frac{c}{24})} e^{2\pi i J_0}, \tag{43}
\]

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where the trace is to be taken over a projection $\mathcal{H}^{\ell}_{q,s}$ to a definite fermion number $(\text{mod } 2)$ of a highest weight representation of the (right-moving) $N = 2$ algebra with highest weight vector determined by the primary field. It is of advantage to express these maps in terms of the string functions and theta functions, leading to the form

$$\chi^k_{\ell,q,s}(\tau, z, u) = \sum c^k_{\ell,q+4j-s}(\tau)\theta_{2q+(4j-s)(k+2),2k(k+2)}(\tau, z, u) \quad (44)$$

because it follows from this representation that the modular behavior of the $N = 2$ characters decomposes into a product of the affine SU(2) structure in the $\ell$ index and into $\Theta$-function behavior in the charge and sector index. The string functions $c^k_{\ell,m}$ are given by

$$c^k_{\ell,m}(\tau) = \frac{1}{\eta^3(\tau)} \sum_{|x| < |y| \leq |x| \atop (x,y) \text{ or } (\frac{x+1}{2}, \frac{y}{2}) \in \mathbb{Z}^2 + (\frac{k+1}{2}, 0)} \text{sign}(x)e^{2\pi i((k+2)x^2-ky^2)} \quad (45)$$

while the classical theta functions $\theta_{n,m}$ are defined as

$$\theta_{n,m}(\tau, z, u) = e^{-2\pi imu} \sum_{\ell \in \mathbb{Z} + \frac{d}{2m}} e^{2\pi im\ell^2\tau + 2\pi iz}. \quad (46)$$

It follows from the coset construction that the essential ingredient in the conformal field theory is the SU(2) affine theory.

### 3.2 Hecke indefinite modular forms

This section records the Hecke indefinite forms $\Theta^k_{\ell,m}(\tau) = \eta^3(\tau)c^k_{\ell,m}$ that are associated to the independent string functions at the level $k = 1, 2, 4$. The number of string functions defined in (45) at level $k$ is restricted by the level $k$ via the constraints

$$0 \leq \ell \leq k, \quad -\ell \leq m \leq 2k - \ell, \quad \ell = m \text{ mod } 2. \quad (47)$$

Not all the resulting string functions are independent; there are relations which can be encoded as follows [1]

$$c^k_{\ell,m} = c^k_{\ell,-m} = c^k_{\ell,m+2k} = c^k_{k-\ell,m+k}. \quad (48)$$

These identifications lead to a unique modular form at level 1, three independent forms at level 2, and seven independent forms at level 4. The expansions of these theta forms are collected in Table 3. These are important in particular for the inverse problem of finding 'geometric' forms at their various levels, considered in the second part of this paper.
Hecke modular form $\Theta$

A sign difference in two modular forms suggests that they are related via twists. For a modular form $f(q) = \sum_n a_n q^n$ and a Dirichlet character

$$\chi : \mathbb{Z} \rightarrow K^\times$$

(49)

### Table 4

| Level $k$ | Hecke modular form $\Theta_k^m$ |
|-----------|----------------------------------|
| 1         | $\Theta_{1,1}^1(\tau) = q^{1/12} (1 - 2q - q^2 + 2q^3 + q^4 + 2q^5 - 2q^6 - 2q^8 + \cdots)$ |
| 2         | $\Theta_{0,0}^2(\tau) = q^{1/16} (1 - 2q + q^2 + q^3 + q^5 - 2q^6 + 2q^7 - 2q^{12} + \cdots)$  
$\Theta_{1,1}^2(\tau) = q^{1/8} (1 - q - 2q^2 + q^3 + 2q^5 + q^6 - 2q^9 + \cdots)$  
$\Theta_{2,0}^2(\tau) = q^{9/16} (1 - q - 2q^2 + 2q^4 + 2q^5 + q^7 - 2q^8 + \cdots)$ |
| 4         | $\Theta_{0,0}^4(\tau) = q^{1/24} (1 - 2q + 2q^3 - q^5 - 2q^6 + q^7 + 2q^{10} + \cdots)$  
$\Theta_{0,2}^4(\tau) = q^{19/24} (1 - q - q^2 + q^5 + q^6 - q^8 + \cdots)$  
$\Theta_{0,4}^4(\tau) = q^{25/24} (1 - q - 2q^3 + 2q^5 + 2q^7 - 2q^{10} + \cdots)$  
$\Theta_{1,1}^4(\tau) = q^{5/48} (1 - q - q^2 + q^3 - q^4 + q^5 - q^6 + q^8 + \cdots)$  
$\Theta_{1,3}^4(\tau) = q^{29/48} (1 - 2q^2 - q^3 + q^5 + q^6 + 2q^7 - q^9 + \cdots)$  
$\Theta_{2,0}^4(\tau) = q^{3/8} (1 - q - q^2 - q^3 + 2q^5 + 2q^8 - q^9 + \cdots)$  
$\Theta_{2,2}^4(\tau) = q^{1/8} (1 - q - q^3 - q^6 + 2q^7 + 2q^9 + \cdots)$ |

### 4 From Geometry to Conformal Field Theory

At this point all the ingredients are in place to complete the proof of the theorems formulated in the introduction. It follows from Weil’s conjecture that the modular forms associated to the curves $E_i$ have levels that agree with the conductors $N_i = 27, 64, 144$ determined above. These conductors are divisible by the bad primes, which gives guidance as to what kind of theta products $\Theta_{\ell,m}(q^a)\Theta_{\ell',m'}(q^b)$ should be considered. The bad primes of $E_6$ are $p = 2, 3$, which shows that the form $\Theta_{1,1}^1(q^6)$ is a plausible starting point. This leaves the proof of the identity in Theorem 1 to all orders, and the explanation of the twist characters. The proof that the third relation in (7) holds to all orders of the $q-$expansion follows from a theorem of Sturm quoted in [3].

A sign difference in two modular forms suggests that they are related via twists. For a modular form $f(q) = \sum_n a_n q^n$ and a Dirichlet character

$$\chi : \mathbb{Z} \rightarrow K^\times$$

(49)
with values in a field $K$, the twisted form $f(q) \otimes \chi$ is defined as in (6). An important class of characters is provided by Legendre symbols. These are defined on rational primes as

$$\chi_n(p) = \left( \frac{n}{p} \right) = \begin{cases} 1 & n \text{ is a square in } \mathbb{F}_p \\ -1 & n \text{ is not a square in } \mathbb{F}_p \end{cases}$$

(50)

The conductor of $\chi_n(\cdot)$ is given by $n$ if $n = 1 \pmod{4}$ and $4n$ for $n = 2, 3 \pmod{4}$. For non-prime numbers the generalized Legendre symbol is defined by using the prime decomposition. Every natural number $m$ can be decomposed into primes as $m = p_1 \cdots p_r$ and the generalized symbol is defined as

$$\chi_n(m) = \prod_{i=1}^{r} \left( \frac{n}{p_i} \right).$$

(51)

Computing the character $\chi_3$ of Theorems 1 and 2 shows that it produces the claimed sign changes.

The physical interpretation of the twist characters derives from the fact that the Legendre character $\chi_n$ determines the factorization behavior of rational primes $p$ in the number fields $\mathbb{Q}(\sqrt{n})$. The following result holds (see e.g. [19]).

**Theorem.** The splitting behavior of rational primes $p$ into prime ideals $p_i \subset \mathcal{O}_{\mathbb{Q}(\sqrt{n})}$ in the ring of algebraic integers $\mathcal{O}_{\mathbb{Q}(\sqrt{n})}$ is determined by the Legendre symbol $\chi_n$ as follows:

$$\chi_n(p) = \begin{cases} 1 & \text{if } (p) = p_1p_2 \\ -1 & \text{if } (p) \text{ is prime} \\ 0 & \text{if } (p) = p^2 \end{cases}.$$  

(52)

The interpretation of these characters arises from the structure of the quantum dimensions of the underlying conformal field theory. The modular behavior of the $N = 2$ characters follows from the modular behavior of the string functions $c^k_{\ell,m}$ and the theta functions $\theta_{m,k}$: the $N = 2$ characters transform like SU(2)-affine characters in their $\ell$ index and as theta functions in their $q$ and $s$ index. More precisely, the modular behavior of these characters is given according to Gepner by

$$\chi^k_{\ell,q,s}(\tau + 1, z, u) = e^{2\pi i (\Delta^k_{\ell,q,s} - c_k)} \chi^k_{\ell,q,s}(\tau, z, u)$$

(53)
for the modular shift $\tau \mapsto \tau + 1$, where $\Delta_{\ell,q,s}^k$ are the conformal weights, which, together with the $U(1)$-charges, are given by

$$\Delta_{\ell,q,s}^k = \frac{\ell(\ell + 2) - q^2}{4(k + 2)} + \frac{s^2}{8}$$  \hspace{1cm} (54)$$

$$Q_{\ell,q,s}^k = \frac{q}{k + 2} - \frac{s}{2}$$  \hspace{1cm} (55)$$

and $c_k$ is the central charge at level $k$.

For the other generator of $SL(2, \mathbb{Z})$ one finds

$$\chi_{\ell,q,s}^k \left( -\frac{1}{\tau}, z, u + \frac{z^2}{2\tau} \right) = f'(k) \sum_{\ell',q',s'} S_{qq',ss'}^{\ell\ell'} \chi_{\ell',q',s'}^k (\tau, z, u),$$  \hspace{1cm} (56)$$

where $f'(k)$ is a constant, and

$$S_{qq',ss'}^{\ell\ell'} = e^{\pi i \frac{q q'}{2}} e^{-\pi i \frac{s s'}{2}} \sin \left( \frac{\pi (\ell + 1)(\ell' + 1)}{k + 2} \right).$$  \hspace{1cm} (57)$$

Defining the generalized quantum dimensions of the $N = 2$ theory as

$$Q_{qp, st}^\ell = \frac{S_{qp, st}^{\ell m}}{S_{q0, st}^{m0}},$$  \hspace{1cm} (58)$$

it follows that the quantum dimensions $Q_{q,s}^\ell := Q_{q0,s0}^{\ell 0}$ are independent of the charge and spin quantum numbers, and agree with the $SU(2)$ quantum dimensions $Q_{\ell} = S_{\ell 0}/S_{00}$, where $S$ is the $S$-matrix of the affine theory $A_1^{(1)}$ determined by the transformation behavior of the characters

$$\chi_{\ell} \left( -\frac{1}{\tau}, u \right) = e^{\pi i k u^2/2} \sum_{\ell'} S_{\ell \ell'} \chi_{\ell'} (\tau, u)$$  \hspace{1cm} (59)$$

with

$$S_{\ell \ell'} = \sqrt{\frac{2}{k + 2}} \sin \left( \frac{(\ell + 1)(\ell' + 1)\pi}{k + 2} \right), \quad 0 \leq \ell, \ell' \leq k.$$  \hspace{1cm} (60)$$

These numbers are important because even though they do not directly provide the scaling behavior of the correlation functions, they do contain information about the conformal weights as well as the central charge via the relations

$$\frac{1}{L(1)} \sum_{i=1}^{k} L \left( \frac{1}{Q_{ij}^2} \right) = \frac{3k}{k + 2} - 24\Delta_j^{(k)} + 6j,$$  \hspace{1cm} (61)$$

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where \( L \) is Rogers’ dilogarithm

\[
L(z) = Li_2(z) + \frac{1}{2} \log(z) \log(1 - z)
\]

(62)

and \( Li_2 \) is Euler’s classical dilogarithm

\[
Li_2(z) = \sum_{n \in \mathbb{N}} \frac{z^n}{n^2}.
\]

(63)

It follows that the quantum dimensions contain the essential information about the spectrum of the conformal field theory and Rogers’ dilogarithm provides, via Euler’s dilogarithm, the map from the quantum dimensions to the central charge and the conformal weights. A review of these results and references to the original literature can be found in [18].

In the present context the fields of quantum dimensions that appear at the levels \( k = 1, 2, 4 \) will be of interest. The result,

\[
\begin{array}{c|ccc}
\text{Level } k & 1 & 2 & 4 \\
\text{QD Field} & \mathbb{Q} & \mathbb{Q}(\sqrt{2}) & \mathbb{Q}(\sqrt{3}) \\
\end{array}
\]

(64)

will explain the twisting behavior that emerges in Theorem 2.

5 From Conformal Field Theory to Geometry

At this point modular forms have been identified for all Gepner models at \( c = 3 \). It is therefore of interest to reverse the logic of providing string theoretic interpretations of geometric quantities by asking which properties string theoretic modular forms should have in order to lead to reasonable geometries. Once a list of properties has been established that determine such forms one can further address the problem of constructing the geometry of spacetime from the string theoretic conformal field theory on the world sheet. These two problems are discussed in this section. The goal is to identify a well-motivated set of conformal field theoretic quantities, and to see whether spacetime can be derived from these objects. Concretely, the question becomes to what extent some of the considerations presented above can be turned around, and how powerful the properties described so far are when one aims to identify ‘geometrically induced’ quantities on the world sheet.
5.1 The constraints

I. Conformal field theoretic modular forms

Given the importance of modular invariance for string theory, it is natural to test whether modularity holds the key to the construction of spacetime. Starting from modular forms on the world sheet, one encounters a problem of riches — even after having recognized the modular forms associated to the characters of the partition function as the key ingredients, there are a great many of them. The focus in this paper is on Hecke indefinite modular forms as the basic building blocks. Only at level \( k = 1 \) is there a single such modular form. At level two there are three independent forms, and at level four there are seven. Combining these forms leads to a large number of forms, not all of which are useful for geometric purposes.

II. Dimension and weight

The fact that there are a large number of potentially useful modular forms that arise from a given exactly solvable model is useful, because the conformal field theoretic models that provide the building blocks of Gepner and Kazama-Suzuki models should geometrically be thought of as motives, i.e. pieces of the variety. The same motive can form a building block for different varieties, as indicated in [3, 4]. This suggests that in order to make progress some assumptions are needed about the varieties to be constructed. One such piece of information is the dimension, and the question arises what property of the modular forms is related to the dimension of the corresponding geometry.

A statement about the dimension of the target space translates into a statement about the Hodge type of the variety. If this Hodge type is \( (0, r) \) then it is conjectured that if such Hodge structures are modular they correspond to modular forms of weight \( (r + 1) \) [20]. In the case of elliptic curves this is known via the general proof of the Shimura-Taniyama conjecture [21, 22]. Hence the assumption of a one-dimensional target space translates into the requirement that the modular forms of interest are of weight two.
III. Cusp forms

It appears that forms that are of cusp type are the ones that are most useful in the context of a geometric interpretation. Hence the focus should be on such forms, that is forms for which $a_0 = 0$. Normalizing these cusp forms means that $a_1 = 1$. This criterion can be viewed as an argument to consider the SU(2) theta functions of the conformal field theory, instead of other forms. Another argument is based on the fact that characters that enter the partition functions by themselves are not useful quantities in the present context because their coefficients count multiplicities, hence are always positive. This explains why neither the parafermionic characters $\eta(\tau)c_{k,\ell,m}(\tau)$, nor the SU(2) characters $\chi_{k,\ell,m}(\tau)$, nor the N=2 characters $\chi_{k,\ell,q,s}(\tau)$ themselves are useful.

IV. Hecke eigenforms

At this point the focus has narrowed to modular cusp forms of weight 2. Not all such forms are expected to allow a geometric interpretation. The Hasse-Weil modular form of any variety is defined via an L-function that is a product of factors defined at each prime. This suggests to focus on conformal field theoretic modular forms which admit such a product representation. This means that the relevant forms should be eigenforms of the Hecke operators. This condition turns out to be a very useful and strong restriction for elliptic geometry.

V. Integral exponents

The Mellin transform of the Hasse-Weil L-function produces $q-$expansions which have integral exponents. The basic building blocks of the string partition function, however, involve $q-$expansions that have rational exponents. In particular the string functions, and their associated Hecke indefinite modular forms, have leading orders that are essentially determined by the (rational) conformal weights of the associated fields. Aiming at modular forms of weight two with integral exponents leads to the consideration of products of the form

$$\Theta_{\ell,m}(a\tau)\Theta_{\ell',m'}(b\tau)$$ (65)
such that

\[ at^k_{\ell,m} + bt'^{k'}_{\ell',m'} \in \mathbb{Z}, \tag{66} \]

where \( t^k_{\ell,m} \) and \( t'^{k'}_{\ell',m'} \) are the leading orders of the theta functions \( \Theta^k_{\ell,m} \) and \( \Theta'^{k'}_{\ell',m'} \).

VI. Level of the modular form

Geometric modular forms are usually modular only with respect to some congruent subgroup of the full modular group \( \text{SL}(2, \mathbb{Z}) \), determined by the level \( N \). It was first observed by Weil, on the basis of experimental computations, that the modular level \( N \) is determined by the degeneration behavior of the curve at those primes \( p \) for which the reduced variety \( X/\mathbb{F}_p \) is singular, i.e. the bad primes [13]. It was this notion of relating the geometric conductor to the modular level which helped to make the Shimura-Taniyama conjecture computationally attractive by making the relation between elliptic curves and modular forms more concrete.

A priori there are a great many choices for the coefficients \( a \) and \( b \) in (65), leading to modular forms of the same weight but different modular level \( N \). What is needed is a constraint for the level \( N \) that can be formulated in terms of the conformal field theory. More precisely, one wants to have a criterion which prescribes at which primes the curve should have bad reduction, i.e. provides some of the divisors of \( N \). This leads to the consideration of products of the type

\[ \Theta^k_{\ell,m}(ap\tau)\Theta'^{k'}_{\ell',m'}(a'p'\tau), \tag{67} \]

where \( k, k' \) are the levels, and \( p, p' \) are bad primes, or powers thereof, and \( a, a' \) are integers that need to be determined.

In the present context a suggestive constraint is to require that the prime factorization of the quantity \( (k + g) \), where \( g \) is the dual Coxeter number, involves only the primes of bad reduction

\[ k + g = \prod_{p \text{ bad prime}} p. \tag{68} \]

In the case of \( A_1^{(1)} \) the dual Coxeter number is \( g = 2 \) and the assumption means that the bad primes are only those that appear in the factorization of \( k + 2 \).
VII. Twisting and quantum dimensions

At this point enough constraints have been identified to select a very small set of conformal field theoretic forms. It turns out, however, that some of them describe the 'wrong' geometry, i.e. not the geometry of either $E_3, E_4$ or $E_6$. This is not necessarily unexpected because there is additional information in the conformal field theory that at this point has not been encoded geometrically. What is missing is the input of the quantum dimensions. The correct geometric form emerges only after multiplying the forms constructed at this point by the Legendre character associated to the field generated by the quantum dimensions. This means that for the modular forms $f(q) = \sum a_n q^n$ identified by the conditions formulated above, its twisted version $f(q) \otimes \chi$ defined in (6) should be considered. In Section 3 it was shown that the fields generated by the quantum dimensions at the levels $k = 1, 2, 4$ are given by $\mathbb{Q}$, $\mathbb{Q}(\sqrt{2})$, and $\mathbb{Q}(\sqrt{3})$, respectively. Associated to these fields are the quadratic characters $\chi_n$, defined by the Legendre symbols (50), and these characters provide the necessary twists.

5.2 Applications

5.2.1 $(1)^{\otimes 3}$

It was shown above that at conformal level $k = 1$ there is only one Hecke indefinite modular form $\Theta_{1,1}(\tau)$. The assumptions that the forms should be cusp forms and of weight two lead to the set of forms of the type $\Theta_{1,1}(a\tau)\Theta_{1,1}(b\tau)$ with $a, b \in \mathbb{N}$. The next step is to restrict these cusp forms to those with integral exponents. This turns out to be a strong requirement, leading to a set of six forms

$$(a, b) \in \{(1, 11), (2, 10), (3, 9), (4, 8), (5, 7), (6, 6)\}. \quad (69)$$

The Hecke criterion is not very strong at this point, excluding only the form with $(a, b) = (5, 7)$. This leaves five cusp Hecke eigenforms with integral exponents.

The modular level condition requires that the form should have a conductor which is divisible only by the prime $k + 2 = p = 3$. This final constraint uniquely determines the form $\Theta_{1,1}(q^3)\Theta_{1,1}(q^9)$. The field of quantum dimensions at $k = 1$ is given by the rational num-
bers, hence this form should not be twisted. It follows that the conditions formulated above uniquely determine the modular form identified in [3] as the Mellin transform of the Hasse-Weil L-function of the cubic Fermat curve. The construction of the geometry is described below.

5.2.2 $(2)^{\otimes 2}$

At conformal level $k = 2$ there are three independent Hecke indefinite modular forms and the resulting number of a priori possible forms $\Theta_{l,m}^2(a\tau)\Theta_{l',m'}^2(b\tau)$ is much larger than at $k = 1$. The requirements of having forms of weight two, cusp type, and integral exponents leaves a large number of forms, most of which, however, are not Hecke eigenforms. Multiplicativity reduces the possible forms to three of type $\Theta_{1,1}^2(a\tau)\Theta_{1,1}^2(b\tau)$ with

$$(a, b) \in \{(1, 7), (2, 6), (4, 4)\}.$$  

(70)

The level constraint at $k = 2$ implies that the only prime of bad reduction is $p = 2$. This uniquely identifies the form as $\Theta_{1,1}^2(4\tau)^2$.

The quantum dimension argument finally determines the unique cusp Hecke eigenform of weight two with integral exponents to be given by $\Theta_{1,1}^2(4\tau)^2 \otimes \chi_2$, which is the form determined in [4] as the Mellin transform of the quartic elliptic Brieskorn-Pham curve.

5.2.3 $(4 \otimes 1)$

For the last of the Gepner models at $c = 3$ it is useful to employ a trick, because the number of a priori possible forms is rather large. The idea is to use the fact that all Calabi-Yau varieties are projective and to use inhomogeneous coordinates to analyze the variety. This leads to the conclusion that the only nontrivial modular ingredient of the model is the theta function at level $k = 1$. Hence the analysis presented earlier can be applied, except that in the present case the level constraint must be modified to require that the modular level should be divisible by the primes $p = 2, 3$. This uniquely singles out the form $\Theta_{1,1}^1(6\tau)^2$. The quantum dimension twist finally leads to the geometric form $\Theta_{1,1}^1(6\tau)^2 \otimes \chi_3$. 

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Summary

The results of this section concerning the conformal field theoretic identification of the target space geometry are summarized in Table 5.

| Curve | $N$  | $\bar{k}$ | QD Field | Character | CFT form |
|-------|------|-----------|-----------|-----------|----------|
| $E_3$ | 27   | (1, 1, 1) | $\mathbb{Q}$ | $\chi_1 = 1$ | $\Theta_{1,1}^1(q^3)\Theta_{1,1}^1(q^9)$ |
| $E_4$ | 64   | (2, 2, 0) | $\mathbb{Q}(\sqrt{2})$ | $\chi_2(\cdot) = (\frac{2}{2})$ | $\Theta_{1,1}^2(q^4)^2 \otimes \chi_2$ |
| $E_6$ | 144  | (4, 1, 0) | $\mathbb{Q}(\sqrt{3})$ | $\chi_3(\cdot) = (\frac{2}{2})$ | $\Theta_{1,1}^1(q^6)^2 \otimes \chi_3$ |

Table 5. Modular forms associated to the curves $E_i$, $i = 3, 4, 6$.

6 Modular Geometry

In the previous sections a set of constraints was formulated which uniquely identify particular Hecke cusp eigenforms in certain conformal field theories. One can therefore reverse the process described earlier, and ask what kind of geometry can be constructed from such modular forms. This question can be asked for cusp forms of arbitrary weight $w$ and level $N$, i.e. for any form $f \in S_w(\Gamma_0(N))$, and it is known how to derive a geometric object from such forms via a construction by Deligne and others. The obstacle that arises for the general case is that the geometry associated to cusp forms is given by a Kuga-Sato variety, which is different from the Calabi-Yau geometry. It is not obvious what the relation is in general between these Kuga-Sato varieties and Calabi-Yau spaces, if any. One of the virtues of the present focus on elliptic curves is not only that the theory of Eichler-Shimura which applies here is simpler, but that it is possible to apply some theorems of the arithmetic theory of curves to see how the abelian varieties associated to cusp forms of weight 2 are related to the elliptic curve itself.

In essence, the theory of Eichler [23] and Shimura [24] establishes a correspondence between cusp forms of weight 2 and level $N$ and the cohomology for a particular type of curve $X_0(N)$, constructed from the modular group $\Gamma_0(N)$ as the compactification

$$X_0(N) = \overline{H}/\Gamma_0(N),$$

(71)
of the quotient curve $Y_0(N) = \mathcal{H}/\Gamma_0(N)$. More precisely, there exists a map

$$S_2(\Gamma_0(N)) \rightarrow H^0(X_0(N), \Omega^1)$$

(72)

between cusp forms of weight 2, and level $N$, and the differential forms of the modular curve $X_0(N)$, defined as

$$f \mapsto 2\pi i f(z)dz.$$  

(73)

Given an element $f(q) = \sum_n a_n q^n \in S_2(\Gamma_0(N))$ and its coefficient field $K = \mathbb{Q}(\{a_n\})$ Eichler and Shimura construct an abelian variety $A_f$ of dimension

$$\dim_\mathbb{C} A_f = [K: \mathbb{Q}],$$

(74)

where $[K: \mathbb{Q}]$ denotes the degree of $K$. Roughly, the abelian variety is obtained as a subvariety, or quotient, of the Jacobian $J(X_0(N))$ of the modular curve $X_0(N)$ determined by the set $\{f^\sigma\}_\sigma$ of conjugate forms obtained from the set of embeddings $\sigma : K \rightarrow \mathbb{C}$ of the field $K$.

If $f$ has rational coefficients then $A_f$ is an elliptic curve, and given any of the string theoretic modular forms derived in section 4, it is possible to associate an elliptic geometry to it. The only remaining question is what happens when one starts from an elliptic curve $E$, computes its Hasse-Weil L-function, considers the Mellin transform $f_{\text{HW}}(E, q)$ of this L-function, and then determines from this modular form the elliptic curve via the Eichler-Shimura construction. It turns out that the resulting curve is isogenous to the original curve

$$E \xleftrightarrow{L_{\text{HW}}(E, s)} L_{\text{HW}}(E, s)$$

$$E_{f_{\text{HW}}} \xleftrightarrow{f_{\text{HW}}(E, q)} f_{\text{HW}}(E, q).$$

(75)

It was furthermore shown in a sequence of papers by Eichler, Shimura, Igusa and Carayol that the Dirichlet series $L(f, s)$ of the modular form is identical to the Hasse-Weil L-series $L_{\text{HW}}(E, s)$. More precisely one has the following result.

**Theorem.**[26] Let $f \in S_2(\Gamma_0(N))$ be a normalized newform with coefficients in $\mathbb{Z}$ and let $E_f$ be the elliptic curve associated to $f$ via the Eichler-Shimura construction. Then

$$L(E_f, s) = L(f, s)$$

(76)
and \( N \) is the conductor of \( E_f \).

This theorem is of importance because it relieves us from an explicit construction of the elliptic curve \( E_f \). Such a construction can be obtained in a straightforward, if tedious, way if the generators of the group \( \Gamma_0(N) \) are known. The simplest example is provided by the curve of conductor eleven, which can be found in [27].

7 Complex multiplication

It was suggested in [10] that exactly solvable Calabi-Yau varieties are distinguished by the fact that they admit complex multiplication (CM) in the sense formulated in [28]. For complex dimension one the notion of Calabi-Yau CM coincides with the usual notion of complex multiplication of elliptic curves, and it is of interest to ask whether the exactly solvable elliptic curves discussed here admit complex multiplication in the sense of Shimura-Taniyama. This question will be addressed in the first subsection, where the CM type of these curves will be described. This question can also be turned around, and one can ask whether the modular forms derived from the Mellin transform of the Hasse-Weil L-function of elliptic curves with complex multiplication have special properties. If so, then this opens up the possibility of using this special property as an additional criterion to select conformal field theoretical modular forms for which a geometric interpretation might be attempted. This circle of ideas will be addressed in the second subsection.

7.1 Geometric complex multiplication

The easiest way to see that the elliptic curves \( E_i, i = 3, 4, 6 \) admit complex multiplication is by finding the lattices \( \Lambda_i \subset \mathbb{C} \) that define the curves as \( E_i = \mathbb{C}/\Lambda_i \). Every elliptic curve can be constructed in terms of a lattice \( \Lambda \subset \mathbb{C} \), generated by two periods \( \omega_i \) such that \( \tau = \omega_2/\omega_1 \) has a positive imaginary part. The notion of complex multiplication arises from the observation that for some lattices there are homomorphisms that are more interesting than the obvious multiplication by an integer. Denote by \( \text{End}(E) \) the group of endomorphisms of the elliptic
Definition. An elliptic curve \( E = \mathbb{C}/\Lambda \) is said to admit complex multiplication, or to be a CM curve, if \( \text{End}(E) \neq \mathbb{Z} \).

It follows from the basic notion of the endomorphism algebra (77) that both the modulus \( \tau \) of the curve, and the endomorphisms, have to be elements of a quadratic imaginary field. Define an order \( \mathcal{O} \) of a number field \( K \) to be a subgroup of maximal rank.

Theorem. If the elliptic curve \( E_\tau = \mathbb{C}/\Lambda(\tau) \) is defined via the lattice \( \Lambda(\tau) = \mathbb{Z} + \tau\mathbb{Z} \), \( \text{Im}(\tau) > 0 \), then either
1) \( \text{End}(E_\tau) = \mathbb{Z} \), or
2) \( \mathbb{Q}(\tau) \) is a quadratic imaginary field and \( \text{End}(E_\tau) \) is isomorphic to an order in \( \mathbb{Q}(\tau) \).

As a consequence of this result the notion of CM can be viewed more conceptually as a map which embeds a CM field \( F \) into the endomorphism algebra tensored with the rational field

\[
\theta : F \longrightarrow \text{End}(E) \otimes \mathbb{Q}.
\]

This is the notion of CM used in [28].

The type of lattices \( \Lambda_i \) which lead to the curves \( E_i \) can be identified from the affine embedding of \( \mathbb{C}/\Lambda \), given by the Weierstrass function \( \wp_\Lambda(z) \) defined on the complex plane

\[
\begin{align*}
\mathbb{C}/\Lambda & \longrightarrow \mathbb{C}^2 \\
z & \mapsto (\wp_\Lambda(z), \wp'_\Lambda(z)),
\end{align*}
\]

where \( \wp'_\Lambda \) denotes the derivative. With \( (x, y) = (\wp_\Lambda(z), \wp'_\Lambda(z)) \) this leads to the Weierstrass equation

\[
y^2 = 4x^3 - g_2(\Lambda)x - g_3(\Lambda),
\]

where \( g_k(\Lambda) = \sum_{\omega \in \Lambda \setminus 0} 1/\omega^{2k} \) are the Eisenstein series. It follows from these series that a lattice that admits the Gaussian ring of integers as a symmetry must have the general form

\[
E_{\mathbb{Q}(\sqrt{-1})} : y^2 = x^3 + ax,
\]
while a lattice that admits the Eisenstein integers in \( \mathbb{Q}(\sqrt{-3}) \) as a symmetry leads to elliptic curves of the form

\[
E_{\mathbb{Q}(\sqrt{-3})} : \quad y^2 = x^3 + b. 
\] (82)

Comparing these types of curves with the Weierstrass forms of the Brieskorn-Pham curves discussed above shows that the curves \( E_3 \) and \( E_6 \) admit complex multiplication by the ring of integers \( \mathcal{O}_K \) for \( K = \mathbb{Q}(\sqrt{-3}) \), and that \( E_4 \) admits CM by the Gaussian ring of integers \( \mathcal{O}_K \) for \( K = \mathbb{Q}(\sqrt{-1}) \).

### 7.2 Modular complex multiplication

In this subsection it will be shown that the modular forms \( f_{HW}(E_i, q) \) \( i = 3, 4, 6 \) considered above admit complex multiplication as defined by Ribet [29] (which can be motivated by a result concerning the geometry of CM elliptic curves described in [30]).

**Definition.**

A modular cusp form \( f(q) = \sum_n a_n q^n \) is said to admit complex multiplication if there exists a Dirichlet character \( \varphi \) such that

\[
\varphi(p) a_p = a_p \quad (83)
\]

for all primes \( p \) in a set of density 1.

This means, essentially, that the expansion of a CM modular form is ‘sparse’, i.e. many coefficients are zero. This can be made more precise by describing the vanishing behavior more concretely. A modular form is CM in the sense of Ribet if the coefficients \( a_p \) vanish at all primes that are inert in some quadratic extension of the rational field \( \mathbb{Q} \) [31]. This is a formulation that is practical enough for concrete tests. In the remainder of this subsection the modular forms \( f_{HW}(E_i, q) \) are shown to admit complex multiplication in the above sense. The proof given here combines the arithmetic geometric origin of the modular forms with certain number theoretic results. The first step is to use the fact that the modular forms under consideration are derived from the Hasse-Weil L-series of the curve \( E_i \). In the previous section the L-series was explicitly computed for low primes. This was enough because a result of Sturm implies that the modular form is determined uniquely by a small number of coefficients. In the present context it is more useful to determine the coefficients of this
L-series in a more systematic way. This can be achieved by using the Weierstrass form of the curve.

Suppose that an elliptic curve is given in the form

$$E : y^2 = f(x)$$

(84)

with a polynomial of degree three or four. The Hasse-Weil L-series

$$L(E, s) = \prod_{p \neq 2} \frac{1}{1 - \frac{a_p}{p^s} + \frac{r}{p^{2s}}} ,$$

(85)

where $a_p = p + 1 - \#(E/F_p)$, can be written as

$$a_p = - \sum_{x \mod p} \left( \frac{f(x)}{p} \right).$$

(86)

These sums can be explicitly computed for concrete polynomials $f(x)$, and allow to systematically determine the behavior of the coefficients $a_p$ at various types of primes, in particular their vanishing behavior.

The second step then consists of trying to match the vanishing behavior found in step one with the prime decomposition of a number field. This can often be achieved because the factorization of rational primes in quadratic imaginary fields is known.

The recipe of this proof can be applied to the curves discussed above. Consider e.g. the curve $E_6$. Denote by $\left( \frac{x^2 - 1}{p} \right)$ the Legendre symbol.

**Theorem.**[32]

For the curve $y^2 = x^3 - 1$ the following holds.

$$\sum_{x \mod p} \left( \frac{x^3 - 1}{p} \right) = \begin{cases} 0, & p \equiv 2 \mod 3 \\ \pm 2c, & p \equiv 1 \mod 3, \end{cases}$$

(87)

where $p = c^2 + 3d^2$.

This result determines the vanishing behavior of the form $f_{HW}(E_6, q)$. It remains to find a number field for which the primes at which the coefficients $a_p$ vanish are precisely the inert
primes. The natural candidate is the field $\mathbb{Q}(\mu_3)$, for which one finds the following result, e.g. in [33].

**Theorem.** Let $\omega = \frac{1}{2}(-1 + i\sqrt{3})$ be the cube root of unity. The Eisenstein primes in the field $\mathbb{Q}(\sqrt{-3})$ are given as follows:
1) The ordinary primes $p \equiv 2 \pmod{3}$.
2) The prime $1 - \omega \in \mathbb{Z}[\omega]$.
3) The primes $p \equiv 1 \pmod{3}$ factor as $p = \pi\pi'$, where $\pi$ and $\pi'$ are primes in $\mathbb{Z}[\omega]$ which are not associates of each other, i.e. equivalent modulo multiplication by the Eisenstein units $\{\pm1, \pm\omega, \pm\omega^2\}$.

This completes the proof for $f_{HW}(E_6, q)$.

The proofs for the two forms $f_{HW}(E_i, q)$ with $i = 3, 4$ is similar. The modular form $f_{HW}(E_3, q)$ also has CM by $\mathbb{Q}(\sqrt{-3})$, while $E_4$ has CM by $\mathbb{Q}(\sqrt{-1})$. The latter can be obtained by combining Gauss's cardinality result

$$\sum_{x \mod p} \left( \frac{x^3 - x}{p} \right) = \begin{cases} 0, & p \equiv 3 \pmod{4} \\ \pm2a, & p \equiv 1 \pmod{4}, \end{cases}$$

with $p = a^2 + 4b^2$ with the prime factorization of the Gauss number field (see [33]).

**Theorem.** Let $p$ be a rational prime. Then $p$ factors in the ring $\mathbb{Z}[i]$ of Gauss integers as follows:
1) If $p = 2$ then $p = -i\pi^2$ where $\pi = 1 + i$ is a Gaussian prime.
2) If $p \equiv 3 \pmod{4}$ then $p$ is inert.
3) The primes $p \equiv 1 \pmod{3}$ factor as $p = \pi\pi'$, where $\pi$ and $\pi'$ are primes in $\mathbb{Z}[\omega]$ and $\pi$ and $\pi'$ are unique up to associates, i.e. equivalent modulo multiplication by the Gaussian units $\{\pm1, \pm i\}$. 

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8 Arithmetic moonshine

The main result of the present paper shows that for the simplest class of exactly solvable Calabi-Yau compactifications it is possible to prove modular identities which establish a link between arithmetic geometry and affine Kac-Moody algebras. Generalized Kac-Moody algebras have played a pivotal role in the proof [6] of an old relation between modular functions and the representation theory of the largest finite simple group, the monstrous moonshine conjecture of Conway and Norton [7]. This conjecture was motivated by the original observation of McKay and Thompson that the first few coefficients of the classical modular function $j$ suggest a relation between $j$ and traces of representations of the monster group.

In the present context the starting point is provided by the arithmetic modular forms of higher weight derived from the Hasse-Weil L-function which were shown above to arise from Kac-Moody algebras. It is of interest to ask whether these forms can be related to some finite simple sporadic group, leading to what might be called arithmetic moonshine. More concretely, the question is whether the coefficients of modular forms $f_{\text{HW}}(E_i, q)$ admit an interpretation as traces over representations of one of the sporadic groups. The purpose of this section is to briefly indicate how this can be achieved by using results obtained by Mason for the largest Mathieu group $M_{24}$ [8], defined as a subgroup of the permutation group of 24 letters, or by Martin for the Conway group, defined as the automorphism group of the Leech lattice [9]. A different application of modular functions in the context of enumerative aspects of mirror symmetry has been discovered by Lian and Yau [34], and further investigated by Verrill and Yui [35] and Doran [36].

Mason’s idea is to associate traces to pairs of elements $(g, h) \in G \times G$, for any finite group $G$. More precisely, assume that the pair $(g, h)$ is a rational commuting pair, i.e. the action of the element $h$ is assumed to be rational on the eigenspaces of the element $g$. This notion was originally introduced by Norton in the context of the Fischer-Griess monster. The basic concept is that of an elliptic system associated to commuting rational pairs $(g, h)$, defined as a map which associates to $h$ a graded infinite dimensional complex vector space $V_h = \oplus n V_{h,n}q^n$ such that every homogeneous component of $V_h$ affords a finite dimensional complex representation for the centralizer of $h$ in $G$, and if $g \in G$ commutes with $h$, then its graded
trace $f(g, h; q) = \text{tr}_{V_h}(g)$ in $V_h$ is a modular function or modular form. In [8] Mason considers the elliptic system of $M_{24}$, and using his results, extended to the Conway group in [9], leads to the interpretation of the modular forms of the three elliptic Brieskorn-Pham curves as forms associated to pairs of group elements $(g, h) \in M_{24} \times M_{24}$

$$f_{HW}(E, q) = \text{tr}_{V_h}(g).$$

These relations are summarized in Table 6, where the group elements are written in terms of the conjugacy classes of the groups. The numerical prefix indicates the order of the elements, and the letter symbol is used to distinguish between classes of the same order [37]. An enumeration of these classes and Frame shapes can be found in [8] and [38].

| Curve $E_i$ | Theta form | Mathieu pair $(g, h)$ |
|-------------|------------|-----------------------|
| $E_3$       | $\Theta^1_{1,1}(q^3)\Theta^1_{1,1}(q^9)$ | $(3B, 3A)$ |
| $E_4$       | $\Theta^2_{1,1}(q^4)^2 \otimes \chi_2$ | $(4A, 4A)$ |
| $E_6$       | $\Theta^1_{1,1}(q^6)^2 \otimes \chi_3$ | $(6B, 2B)$ |

**Table 6.** Mathieu classes for $f_{HW}(E_i, q)$, $i = 3, 4, 6$.

The interpretation of the modular forms of elliptic Brieskorn-Pham curves as Kac-Moody theoretic objects, and as series associated to sporadic finite simple groups, provides a link between affine Lie algebras and moonshine that applies to standard string theory models, without the need of the constructions of [6]. Perhaps the notion of arithmetic moonshine will help to gain a more conceptual understanding of the link between Kac-Moody algebras and arithmetic geometry described here and in [3, 4].

9 Final comment

The analysis of this paper may appear to have succeeded only in recovering part of the geometry of spacetime, namely the arithmetic part that is obtained by reducing the complex variety that is normally considered. The results obtained so far indicate that it may well be that this is all that is needed in the general case to understand the physics of the string from
the geometry of the variety. Put differently, it is possible that all the string needs are the arithmetic properties of spacetime.

In the present simple case it turns out that the discrete structure is enough to recover the Calabi-Yau geometry from the arithmetic properties. The reason for this is a conjecture of Tate from the 1960s to the effect that abelian varieties are determined up to isogeny by the Tate module, i.e. essentially by the torsion points of the abelian variety. This translates into the fact that two abelian varieties defined over the rational numbers must be isogenous if and only if their L-functions coincide. It follows that an equivalence of the arithmetic of two varieties over all finite fields implies an isogeny over the rationals, dense everywhere in the reals.

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