Domain wall solution in $F(R)$ gravity and variation of the fine structure constant

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Abstract

We construct a domain wall solution in $F(R)$ gravity. We reconstruct a static domain wall solution in a scalar field theory. We also reconstruct an explicit $F(R)$ gravity model in which a static domain wall solution can be realized. Moreover, we show that there could exist an effective (gravitational) domain wall in the framework of $F(R)$ gravity. In addition, it is demonstrated that a logarithmic non-minimal gravitational coupling of the electromagnetic theory in $F(R)$ gravity may produce time-variation of the fine structure constant which may increase with decrease of the curvature, and that this model would be ruled out by the constraints on the time variation of the fine structure constant from quasar absorption lines. We also present cosmological consequences of the coupling of the electromagnetic field to a scalar field as well as the scalar curvature and discuss the relation between variation of the fine structure constant and the breaking of the conformal invariance of the electromagnetic field.

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I. INTRODUCTION

According to recent cosmological observations, e.g., Supernovae Ia (SNe Ia) [1], cosmic microwave background (CMB) radiation [2, 3], large scale structure (LSS) [4], baryon acoustic oscillations (BAO) [5], and weak lensing [6], it has been implied that the current expansion of the universe is accelerating. Studies on the late time cosmic acceleration are classified into the representative two categories. One is to introduce dark energy such as cosmological constant in the framework of general relativity (for a recent review, see, e.g., [7]). The other is to modify the gravitational theory, for instance, $F(R)$ gravity, where $F(R)$ is an arbitrary function of the scalar curvature $R$ (for recent reviews on $F(R)$ gravity, see, e.g., [8–10]).

Recently, not only temporal [11, 12] but also spatial [13] variations of the fine structure constant $\alpha_{\text{EM}}$ have been suggested. To account for the spatial variation of $\alpha_{\text{EM}}$, the signature of a domain wall produced in the spontaneous symmetry breaking involving a dilaton-like scalar field coupled to electromagnetism has been considered in Ref. [14]. Furthermore, in Ref. [15] it has been shown that a runaway domain wall, which is formed by a runaway type potential of a scalar field [16], can explain both the time variation by its potential and the spatial one by its formation simultaneously. It is interesting to note that in Ref. [17] time and spatial variations of coupling constant have been studied and that when the chameleon field is introduced, variations of coupling constant is related to the chameleon mechanism [18].

On the other hand, a domain wall solution in the framework of $F(R)$ gravity has not been investigated in detail yet. In particular, it is interesting to reconstruct an $F(R)$ gravity model in which a domain wall solution can be realized. It is known that $F(R)$ gravity can be written as a corresponding scalar field theory through a conformal transformation to the Einstein frame. In this paper, we reconstruct an explicit $F(R)$ gravity model in which a static domain wall solution can be realized. First, by using a procedure proposed in Ref. [19], we reconstruct a static domain wall solution in a scalar field theory. Next, in a similar configuration, we reconstruct an explicit form of $F(R)$ with forming a static domain wall solution. Moreover, by applying the method of reconstruction of $F(R)$ gravity in Ref. [20], we show that there could exist an effective (gravitational) domain wall in the framework of $F(R)$ gravity. In addition, we discuss an issue of a connection between $F(R)$ gravity and variation of the fine structure constant by exploring non-minimal Maxwell-$F(R)$ gravity. Furthermore, we present cosmological consequences of the coupling of the electromagnetic...
field to a scalar field as well as the scalar curvature. We also study the relation between variation of the fine structure constant and the breaking of the conformal invariance of the electromagnetic field. We use units of $k_B = c = \hbar = 1$ and denote the gravitational constant $8\pi G$ by $\kappa^2 \equiv 8\pi/M_{\text{Pl}}^2$ with the Planck mass of $M_{\text{Pl}} = G^{-1/2} = 1.2 \times 10^{19}\text{GeV}$. Moreover, in terms of electromagnetism we adopt Heaviside-Lorentz units.

The paper is organized as follows. In Sec. II, we describe $F(R)$ gravity and a corresponding scalar field theory by using a conformal transformation of $F(R)$ gravity to the Einstein frame. In Sec. III, we reconstruct a static domain wall solution in a scalar field theory. In Sec. IV, we also reconstruct an explicit $F(R)$ gravity model in which a static domain wall solution can be realized. In Sec. V, we demonstrate that there could exist an effective (gravitational) domain wall in $F(R)$ gravity. In Sec. VI, we consider non-minimal Maxwell-$F(R)$ gravity and examine a relation between $F(R)$ gravity and variation of the fine structure constant. In addition, we investigate cosmological consequences of the coupling of the electromagnetic field to a scalar field as well as the scalar curvature in Sec. VII. Finally, conclusions are given in Sec. VIII.

II. COMPARISON OF $F(R)$ GRAVITY WITH A SCALAR FIELD THEORY HAVING A RUNAWAY TYPE POTENTIAL

A. $F(R)$ gravity and a corresponding scalar field theory

The action of $F(R)$ gravity with matter is written as

$$S = \int d^4x \sqrt{-g} \frac{F(R)}{2\kappa^2} + \int d^4x \mathcal{L}_M (g_{\mu\nu}, \Psi_M), \quad (2.1)$$

where $g$ is the determinant of the metric tensor $g_{\mu\nu}$ and $\mathcal{L}_M$ is the matter Lagrangian.

We make a conformal transformation to the Einstein frame:

$$\bar{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}, \quad (2.2)$$

where

$$\Omega^2 \equiv F_R, \quad (2.3)$$

$$F_R \equiv \frac{dF(R)}{dR}. \quad (2.4)$$
Here, a tilde represents quantities in the Einstein frame. We introduce a new scalar field $\phi$, defined by

$$\phi \equiv \sqrt{\frac{3}{2}i} \ln F_{,R}.$$  (2.5)\)

The relation between $R$ and $\tilde{R}$ is expressed as

$$R = e^{1/\sqrt{3}i\phi} \left[ \tilde{R} + \sqrt{3} \tilde{\Box} (\kappa \phi) - \frac{1}{2} \tilde{g}^{\mu \nu} \partial_\mu (\kappa \phi) \partial_\nu (\kappa \phi) \right],$$  (2.6)\)

where

$$\tilde{\Box} (\kappa \phi) = \frac{1}{\sqrt{-\tilde{g}}} \partial_\mu \left[ \sqrt{-\tilde{g}} \tilde{g}^{\mu \nu} \partial_\nu (\kappa \phi) \right].$$  (2.7)\)

The action in the Einstein frame is given by

$$S_E = \int d^4x \sqrt{-\tilde{g}} \left( \frac{\tilde{R}}{2\kappa^2} - \frac{1}{2} \tilde{g}^{\mu \nu} \partial_\mu \phi \partial_\nu \phi - V(\phi) \right) + \int d^4x \mathcal{L}_M \left( (F_{,R})^{-1} (\phi) \tilde{g}_{\mu \nu}, \Psi_M \right),$$  (2.8)\)

where

$$V(\phi) = \frac{F_{,R} \tilde{R} - F}{2\kappa^2 (F_{,R})^2}.$$  (2.9)\)

### B. Runaway domain wall and a varying fine structure constant $\alpha_{EM}$

In Ref. [15], the following action describing a runaway domain wall and a space-time varying fine structure constant $\alpha_{EM}$ has been proposed:

$$S_E = \int d^4x \sqrt{-\tilde{g}} \left( \frac{\tilde{R}}{2\kappa^2} - \frac{1}{2} \tilde{g}^{\mu \nu} \partial_\mu \phi \partial_\nu \phi - V(\phi) \right) + \int d^4x \sqrt{-\tilde{g}} \left( -\frac{1}{4} B(\phi) \tilde{g}^{\alpha \beta} \tilde{g}^{\mu \nu} F_{\mu \nu} F_{\alpha \beta} \right)$$

$$+ S_{\text{matter}},$$  (2.10)\)

where

$$V(\phi) = \frac{M^{2p+4}}{(\phi^2 + \sigma^2)^p},$$  (2.11)\)

$$B(\phi) = e^{-\xi \phi}.$$  (2.12)\)

$$F_{\mu \nu} = \partial_\mu A_\nu - \partial_\nu A_\mu.$$  (2.13)\)

Here, $F_{\mu \nu}$ is the electromagnetic field-strength tensor and $A_\mu$ is the $U(1)$ gauge field. $S_{\text{matter}}$ is the action for other ordinary matters. Moreover, $V(\phi)$ is a scalar field potential of runaway type, $M$ is a mass scale, $p(>1)$ is a constant assumed to be larger than unity, $\sigma(\leq \phi)$ is a constant assumed to be smaller than the value of $\phi$. It is known that although there is no
minima in the potential $V(\phi)$, the discrete symmetry $\phi \leftrightarrow -\phi$ can be broken dynamically and consequently a domain wall can be formed. Furthermore, $B(\phi)$ is a coupling function of $\phi$ to the electromagnetic kinetic term and $\xi$ is a constant. The spatio-temporal variations of $\alpha_{\text{EM}}$ come from the variation of $B(\phi)$ in terms of space and time because $\alpha_{\text{EM}}(\phi) = \alpha_{\text{EM}}^{(0)}/B(\phi)$, where $\alpha_{\text{EM}}^{(0)} = e^2 / (4\pi)$ with $e$ being the charge of the electron [22], is the bare fine structure constant, and $\xi$ is a constant. We note that since the electromagnetic fields have the conformal invariance, the conformal transformation in Eq. (2.2) does not generate the non-trivial coupling of the scalar filed $\phi$ with the electromagnetic fields.

The current value of the Hubble parameter is given by $H_0 = 2.1h \times 10^{-42}\text{GeV}$ [22] with $h = 0.7$ [3, 23]. We assume the flat Friedmann-Lemaître-Robertson-Walker (FLRW) metric

$$ds^2 = -dt^2 + a^2(t) \sum_{i=1,2,3} (dx^i)^2.$$  (2.14)

In this background, $R = 6\dot{H} + 12H^2$, where $H = \dot{a}/a$ is the Hubble parameter and the dot denotes the time derivative of $\partial/\partial t$. Hence, the current curvature $R_0$ is $R_0 \approx 12H_0^2$.

III. RECONSTRUCTION OF A STATIC DOMAIN WALL SOLUTION IN A SCALAR FIELD THEORY

In this section, we reconstruct a static domain wall solution in a scalar field theory by using a procedure in Ref. [19].

We consider the following action:

$$S = \int d^3x \sqrt{-g} \left( \frac{R}{2\kappa^2} - \frac{\omega(\varphi)}{2} \partial_\mu \varphi \partial^\mu \varphi - V(\varphi) \right).$$  (3.1)

We also assume the following $D = d + 1$ dimensional warped metric

$$ds^2 = dy^2 + e^{u(y)} \sum_{\mu,\nu=0}^{d-1} \hat{g}_{\mu\nu} dx^\mu dx^\nu,$$  (3.2)

and we also assume the scalar field only depends on $y$. In (3.2), $\hat{g}_{\mu\nu}$ is the metric of the $d$-dimensional Einstein manifold defined by $\hat{R}_{\mu\nu} = \frac{d-1}{l^2} \hat{g}_{\mu\nu}$. The de Sitter space corresponds to $1/l^2 > 0$, the anti-de Sitter space to $1/l^2 < 0$, and the flat space $1/l^2 = 0$. Then the $(y,y)$ component and $(\mu,\nu)$ component of the Einstein equation are given by

$$-\frac{d(d-1)}{2l^2} e^{-u} + \frac{d(d-1)}{8} (u')^2 = \frac{1}{2} \omega(\varphi) (\varphi')^2 - V(\varphi),$$  (3.3)

$$-\frac{(d-1)(d-2)}{2l^2} e^{-u} + \frac{d-1}{2} u'' + \frac{d(d-1)}{8} (u')^2 = \frac{1}{2} \omega(\varphi) (\varphi')^2 - V(\varphi).$$  (3.4)
where the prime denotes the derivative with respect to y. Now we may choose \( \varphi = y \). In this case, we also take \( \kappa^2 = 1 \). Then Eqs. (\ref{eq:3.3}) and (\ref{eq:3.4}) give

\[
\omega(\varphi) = -\frac{d-1}{2} u'' - \frac{d-1}{l^2} e^{-u},
\]

(3.5)

\[
\mathcal{V}(\varphi) = -\frac{d-1}{4} u'' - \frac{d(d-1)}{8} (u')^2 + \frac{(d-1)^2}{2l^2} e^{-u}.
\]

(3.6)

The energy density \( \rho \) is now given by

\[
\rho = \frac{\omega(\varphi)}{2} (\varphi')^2 + \mathcal{V}(\varphi) = -\frac{d-1}{2} u'' - \frac{d(d-1)}{8} (u')^2 + \frac{(d-1)(d-2)}{2l^2} e^{-u}.
\]

(3.7)

When we assume the \( D \) dimensional space is flat, we find \( u \to 0 \) when \( |y| \to \infty \), the second term dominates in (3.5), \( \omega(\varphi) \sim -(d-1)/l^2 \). When \( \omega(\varphi) \) is negative, which corresponds to \( 1/l^2 > 0 \), the scalar field \( \varphi \) becomes a ghost. In case of \( 1/l^2 = 0 \), we find \( \omega(\varphi) = -(d-1)u''/2 \). Then if we assume \( Z_2 \) symmetry of the metric, which is the invariance under the transformation \( y \to -y \), there must be a region where \( \omega(\varphi) \) becomes negative and therefore \( \varphi \) becomes a ghost. We should note that the energy density often becomes negative. Anyway if we admit the ghost and negative energy density, for arbitrary \( u \), we find a model which admits that \( u \) as a solution of the Einstein equation. For example, we may consider

\[
u = u_0 e^{-y^2/y_0^2},
\]

(3.8)

with constants \( u_0 \) and \( y_0 \). Then if we consider the model

\[
\omega(\varphi) = -(d-1) \left( \frac{2\varphi^2}{y_0^4} - \frac{1}{y_0^2} \right) e^{-\varphi^2/y_0^2} - \frac{(d-1)}{l^2} e^{-u_0 e^{-\varphi^2/y_0^2}},
\]

\[
\mathcal{V}(\varphi) = -\frac{d-1}{2} \left( \frac{2\varphi^2}{y_0^4} - \frac{1}{y_0^2} \right) e^{-\varphi^2/y_0^2} + \frac{(d-1)^2}{l^2} e^{-u_0 e^{-\varphi^2/y_0^2}},
\]

(3.9)

we obtain \( u \) in (3.8) as a solution of the Einstein equation. For the model, the distribution of the energy density is given by

\[
\rho(y) = -\frac{d-1}{2} \left( \frac{2y^2}{y_0^4} - \frac{1}{y_0^2} \right) e^{-y^2/y_0^2} + \frac{(d-1)^2}{l^2} e^{-u_0 e^{-y^2/y_0^2}},
\]

(3.10)

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1 The reason why we may choose \( \varphi = y \) is as follows. We here examine the case in which the scalar field \( \varphi \) only depends on \( y \). As a simplest choice, we take \( \varphi = y \). Even if we choose other form such as \( \varphi = \varphi(y) \), by using a variable transformation, we can rewrite the action to the one represented with \( \varphi = y \). Hence, all the consequences, e.g., when \( y \) goes to infinity, \( u \) becomes zero, would not be depend on the choice of the form of \( \varphi \) qualitatively.
which is localized at $y \sim 0$ and makes a domain wall.

As a consequence, we have reconstructed the forms of $\omega(\varphi)$ and $V(\varphi)$ in (3.9) so that the metric in Eq. (3.2) with Eq. (3.8) can be a solution of the Einstein equation. In this model, we have illustrated that the distribution of the energy density given by Eq. (3.10) is localized at $y \sim 0$ and hence it can be regarded as a domain wall. A condition for the localization of the energy density is that in the limit of $|y| \to \infty$, the asymptotic behavior of $u \to 0$ is satisfied.

In this work, as a first step to demonstrate whether the configuration that 1-dimensional domain wall and $d$-dimensional Einstein manifold (e.g., for an ordinary 4-dimensional space-time, $d = 3$) can be realized in the framework of a scalar field theory, we explore a static domain wall, provided a $D = d + 1$ dimensional warped metric in Eq. (3.2). In order to analyze the stability of these domain walls against small perturbations, i.e., the time evolution of the above configuration, it is necessary to consider a different metric including a time component from that in Eq. (3.2). In this work, the existence of domain wall solutions (as an assumption) is only studied, while leaving the stability to future work.

Furthermore, we should caution that since there exist the situation in which the scalar field $\varphi$ becomes a ghost, in the sense of quantum theory a static domain wall solution in a scalar field theory reconstructed in this section is not physically viable. Nevertheless, the motivation to carry on and make the subsequent analysis on this model is as follows. It would be important to explore whether the distribution of the energy density is localized so that such a configuration could be a domain wall, which might correspond to so-called a brane in the literature.

IV. RECONSTRUCTION OF AN EXPPLICIT $F(R)$ GRAVITY MODEL REALIZING A STATIC DOMAIN WALL SOLUTION

In this section, in a similar configuration to that in Sec. III, we reconstruct an explicit $F(R)$ gravity model realizing a static domain wall solution.
A. Gravitational field equations

Varying the action in Eq. (2.1) with respect to \( g_{\alpha\beta} \), we obtain

\[
- \frac{1}{2} F g_{\alpha\beta} + (R_{\alpha\beta} - \nabla_{\alpha} \nabla_{\beta} + g_{\alpha\beta} \Box) F_{,R} = \kappa^2 T^{(M)}_{\alpha\beta} .
\] (4.1)

where \( \nabla_{\alpha} \) is the covariant derivative operator associated with \( g_{\alpha\beta} \), \( \Box \equiv g^{\alpha\beta} \nabla_{\alpha} \nabla_{\beta} \) is the covariant d’Alembertian for a scalar field, and \( T^{(M)}_{\alpha\beta} = - (2/\sqrt{-g}) (\delta L_M / \delta g^{\alpha\beta}) \) is the energy-momentum tensor of matter and given by \( T^{(M)}_{\alpha\beta} = \text{diag}(-\rho_M, P_M, P_M, P_M) \) with \( \rho_M \) and \( P_M \) being the energy density and pressure of matter, respectively.

Equation (4.1) can be described as

\[
G_{\alpha\beta} = \kappa^2 \left( T^{(M)}_{\alpha\beta} + T^{(D)}_{\alpha\beta} \right) ,
\] (4.2)

where

\[
\kappa^2 T^{(D)}_{\alpha\beta} \equiv \frac{1}{2} \left( F - R \right) g_{\alpha\beta} + \left( 1 - F_{,R} \right) R_{,\alpha\beta} + \left( \nabla_{\alpha} \nabla_{\beta} - g_{\alpha\beta} \Box \right) F_{,R} .
\] (4.3)

Here, \( G_{\alpha\beta} \equiv R_{\alpha\beta} - (1/2) g_{\alpha\beta} R \) is the Einstein tensor and \( \kappa^2 T^{(D)}_{\alpha\beta} \) can be regarded as the contribution to the energy-momentum tensor from the deviation of \( F(R) \) gravity from general relativity.

We take the \( D = d + 1 \) dimensional warped metric in Eq. (3.2), in which \( g_{yy} = 1 \) and \( g_{\mu\nu} = e^u \hat{g}_{\mu\nu} \). The \((y, y)\) component and the trace of \((\mu, \nu)\) component of the gravitational field equation (4.1) are given by

\[
- \frac{1}{2} u'' (F_{,R})' - \frac{1}{2} \left[ u'' + \frac{1}{2} (u')^2 \right] F_{,R} - \frac{1}{2} F = \kappa^2 T_{yy}^{(M)} ,
\] (4.4)

\[
d (F_{,R})'' + \frac{d (d - 2)}{2} u' (F_{,R})' + \left\{ - \frac{1}{2} \left[ u'' + \frac{d}{2} (u')^2 \right] + \frac{d (d - 1)}{l^2} e^{-u} \right\} F_{,R} - \frac{d}{2} F
\]

\[
= \kappa^2 \sum_{\mu, \nu = 0}^{d-1} g^{\mu\nu} T^{(M)}_{\mu\nu} .
\] (4.5)

where \( (F_{,R})' \equiv dF_{,R} / dy \) and \( (F_{,R})'' \equiv d^2 F_{,R} / dy^2 \).

Moreover, in the background described by Eq. (3.2), \( R \) is expressed as

\[
R = - \frac{d}{2} \left[ 2 u'' + \frac{1 + d}{2} (u')^2 \right] + \frac{d (d - 1)}{l^2} e^{-u} .
\] (4.6)

We rewrite Eqs. (4.4) and (4.5) in the form of Eq. (4.2) as follows:

\[
- \frac{d}{2} \left[ u'' + \frac{1}{2} (u')^2 \right] - \frac{R}{2} = \kappa^2 \left( T_{yy}^{(M)} + T_{yy}^{(D)} \right) ,
\] (4.7)

\[
- \frac{d}{2} \left[ u'' + \frac{d}{2} (u')^2 \right] + \frac{d (d - 1)}{l^2} e^{-u} - \frac{d}{2} R = \kappa^2 \left( \sum_{\mu, \nu = 0}^{d-1} g^{\mu\nu} T_{\mu\nu}^{(M)} + \sum_{\mu, \nu = 0}^{d-1} g^{\mu\nu} T_{\mu\nu}^{(D)} \right) ,
\] (4.8)
where
\[
\kappa^2 T_{yy}^{(D)} \equiv -\frac{d-1}{2} u' (F'_R)' + \frac{d}{2} \left[ u'' + \frac{1}{2} (u')^2 \right] (F'_R - 1) + \frac{1}{2} (F - R) , \tag{4.9}
\]
\[
\kappa^2 \sum_{\mu,\nu=0}^{d-1} g^{\mu\nu} T_{\mu\nu}^{(D)} \equiv -d (F'_R)'' - \frac{d (d-2)}{2} u' (F'_R)'
+ \left\{ \frac{d}{2} \left[ u'' + \frac{d}{2} (u')^2 \right] - \frac{d (d-1)}{l^2} e^{-u} \right\} (F'_R - 1) + \frac{d}{2} (F - R) . \tag{4.10}
\]
By substituting Eq. (4.6) into Eqs. (4.7) and (4.8), we see that the left-hand side (l.h.s.) of Eq. (4.7) is equal to that of Eq. (3.3) and the l.h.s. of Eq. (4.8) divided by \(d\) is equal to that of Eq. (3.4). We note that Eqs. (4.7) and (4.8) are exactly equivalent to Eqs. (4.4) and (4.5), respectively. By comparing these equations with Eqs. (3.3) and (3.4), we see the difference of the gravitational field equations in \(F(R)\) gravity from those in general relativity.

**B. Explicit form of \(F(R)\)**

We derive an explicit form of \(F(R)\) realizing a domain wall solution. For simplicity, we consider the case in which there is no matter.

We assume that \(u\) is given by a function of \(y\), \(u = u(y)\). For example, we take Eq. (3.8), for which a domain wall can be realized at \(y \sim 0\) as shown in Sec. III. By using Eq. (4.6), we can solve \(y\) as a function of \(R\), \(y = y(R)\), and therefore we have \(u = u(y(R))\). Substituting this expression into Eqs. (4.4) and (4.5) and eliminating \(y\), Eqs. (4.4) and (4.5) can be rewritten as differential equations for \(F(R)\) as a function of \(R\). Since Eqs. (4.4) and (4.5) are not independent with each other, we examine Eq. (4.4). As a consequence, Eq. (4.4) can be expressed as
\[
\Xi_1(R) \frac{d^2 F(R)}{dR^2} + \Xi_2(R) \frac{dF(R)}{dR} - F(R) = 0 , \tag{4.11}
\]
where
\[
\Xi_1(R) \equiv (d-1) u \frac{dR}{dy} = (d-1) \left( \frac{dR}{dy} \right)^2 \frac{du(y(R))}{dR} , \tag{4.12}
\]
\[
\Xi_2(R) \equiv (-d) \left[ u'' + \frac{1}{2} (u')^2 \right] = (-d) \left[ \frac{d^2 R}{dy^2} \frac{du(y(R))}{dR} + \left( \frac{dR}{dy} \right)^2 \frac{d^2 u(y(R))}{dR^2} \right.
+ \frac{1}{2} \left( \frac{dR}{dy} \right)^2 \left( \frac{du(y(R))}{dR} \right)^2 \right] . \tag{4.13}
\]
We solve Eq. (4.16) in terms of $y$. We define $Y \equiv y^2/y_0^2$. For $Y = y^2/y_0^2 \ll 1$, we expand exponential terms and take the first leading terms in terms of $Y$. As a result, we obtain

$$Y \equiv \frac{y^2}{y_0^2} \approx \frac{R - \gamma_1}{\gamma_2},$$

(4.14)

$$\gamma_1 \equiv 2d\frac{u_0}{y_0} + \frac{d(d-1)}{l^2},$$

(4.15)

$$\gamma_2 \equiv -\frac{d u_0}{y_0} \left[6 + (1 + d)u_0 \right] + \frac{d(d-1)}{l^2} u_0,$$

(4.16)

where $\gamma_1$ and $\gamma_2$ are constants.

By using Eq. (4.16) and the similar procedure, we find

$$\frac{dR}{dy} \approx \zeta_1 + \zeta_2 \frac{y^2}{y_0^2},$$

(4.17)

$$\zeta_1 \equiv d\frac{u_0}{y_0} \left(1 + \frac{1 + d}{2}u_0\right) + \frac{d(d-1)u_0}{l^2} \frac{y_0}{y_0},$$

(4.18)

$$\zeta_2 \equiv -d\frac{u_0}{y_0} \left[1 + (1 + d)u_0 \right] - \frac{d(d-1)u_0}{l^2} \frac{u_0 + 1}{y_0},$$

(4.19)

$$\frac{d^2 R}{dy^2} \approx \eta_1 + \eta_2 \frac{y^2}{y_0^2},$$

(4.20)

$$\eta_1 \equiv -d\frac{u_0}{y_0} \left[1 + (1 + d)u_0 \right] - \frac{d(d-1)u_0}{l^2} \frac{1 - u_0}{y_0},$$

(4.21)

$$\eta_2 \equiv d\frac{u_0}{y_0} \left[1 + 2(1 + d)u_0 \right] + \frac{d(d-1)}{l^2} \frac{u_0^2 - 3u_0 + 1}{y_0^2}.$$ 

(4.22)

Here, $\zeta_1$, $\zeta_2$, $\eta_1$ and $\eta_2$ are constants.

Substituting Eqs. (4.17) and (4.20) into Eqs. (4.12) and (4.13), expanding exponential terms, and taking the first leading terms in terms of $Y$, we acquire

$$\Xi_i(R) = \Xi_i^{(0)} + \Xi_i^{(1)} Y = \Xi_i^{(0)} - \frac{\gamma_1}{\gamma_2} \Xi_i^{(1)} + \Xi_i^{(1)} R,$$

(4.23)

with

$$\Xi_i^{(0)} \equiv (d-1) \left( -\frac{u_0}{\gamma_2} \zeta_1^{2} \right),$$

(4.24)

$$\Xi_i^{(1)} \equiv (d-1) \frac{u_0}{\gamma_2} \zeta_1 \left( \zeta_1 - 2\zeta_2 \right),$$

(4.25)

$$\Xi_2^{(0)} \equiv (-d) \left[ -\frac{u_0}{\gamma_2} \eta_1 + \frac{u_0}{\gamma_2} \zeta_1 \left( 1 + \frac{u_0}{2} \right) \right],$$

(4.26)

$$\Xi_2^{(1)} \equiv (-d) \left\{ \frac{u_0}{\gamma_2} (\eta_1 - \eta_2) - \zeta_1 \frac{u_0}{\gamma_2} \left[ \zeta_1 (1 + u_0) - 2\zeta_2 \left( 1 + \frac{u_0}{2} \right) \right] \right\}.$$ 

(4.27)
In deriving the second equality in Eq. (4.23), we have used Eq. (4.14). Here, $i, j = 1, 2$, and the superscriptions $(0)$ and $(1)$ denotes the terms proportional to the zeroth power of $Y (Y^0 = 1)$ and the first one of $Y (Y^1 = Y)$, respectively.

For $Y = y^2/y_0^2 \ll 1$, when $\Xi^{(1)}/\Xi^{(0)}_i \lesssim \mathcal{O}(1)$, we can consider that $\Xi_i \approx \Xi^{(0)}_i (= \text{constant})$. In such a case, Eq. (4.11) can be regarded as

$$
\frac{d^2 F(R)}{dR^2} + \mathcal{C} \frac{dF(R)}{dR} + \mathcal{D} F(R) = 0,
$$

where

$$
\mathcal{C} \equiv \frac{\Xi^{(0)}_2}{\Xi^{(0)}_1},
$$

$$
\mathcal{D} \equiv -\frac{1}{\Xi^{(0)}_1}.
$$

A general solution of Eq. (4.28) is given by

$$
F(R) = F_+ e^{\lambda_+ R} + F_- e^{\lambda_- R},
$$

with

$$
\lambda_\pm \equiv \frac{1}{2} \left( -\mathcal{C} \pm \sqrt{\mathcal{C}^2 - 4\mathcal{D}} \right).
$$

Here, $F_\pm$ are arbitrary constants, and the subscriptions $\pm$ of $\lambda_\pm$ correspond to the sign ($\pm$) on the right-hand side (r.h.s.) of Eq. (4.32).

It follows from the considerations in Sec. III that if we take Eq. (3.8), the distribution of the energy density is localized at $y \sim 0$ and hence a domain wall can be made. Thus, it is interpreted that in an exponential model of $F(R)$ gravity given by Eq. (4.31), a domain wall can be realized at $y \sim 0$. In Ref. [24], such an exponential model of $F(R)$ gravity has been studied.

We should also note that the metric ansatz Eq. (3.8) will lead to the same ghost and negative energy problem which make the model physically not viable as in Sec. III. This is because in this subsection, we examine an explicit form of $F(R)$ realizing a domain wall solution constructed in Sec. III.

C. Form of the potential in a corresponding scalar field theory

We examine the form of the potential $V(\phi)$ in Eq. (2.3) in a corresponding scalar field theory in the Einstein frame to which an exponential model of $F(R)$ gravity in Eq. (4.31)
is transformed through a conformal transformation in Eq. (2.2). As an exponential model, for example, by choosing $F_+ \neq 0$ and $F_- = 0$, we take $F(R) = F_+ e^{\lambda_+ R}$. In this case, from Eq. (2.5) we have the following relation between $R$ and $\phi$:

$$R = \frac{1}{\lambda_+} \left[ \ln \left( \frac{1}{F_+ \lambda_+} \right) + \sqrt{\frac{2}{3}} \kappa \phi \right].$$

(4.33)

By using Eqs. (2.5), (2.9) and (4.33), we find

$$V(\phi) = \frac{1}{2\kappa^2 \lambda_+} e^{-\sqrt{2/3} \kappa \phi} \left[ \sqrt{\frac{2}{3}} \kappa \phi + \ln \left( \frac{1}{F_+ \lambda_+} \right) - 1 \right].$$

(4.34)

By defining $\bar{\phi} \equiv \sqrt{2/3} \kappa \phi$, $\bar{\phi}_0 \equiv \ln [1 / (F_+ \lambda_+)] - 1$, and $V_0 \equiv 1 / (2\kappa^2 \lambda_+)$, $V(\phi)$ in Eq. (4.34) is expressed as $V(\bar{\phi}) = V_0 e^{-\bar{\phi}} (\bar{\phi} + \bar{\phi}_0)$. We note that $\bar{\phi}$ is a dimensionless quantity.

We show $V/V_0$ as a function of $\bar{\phi}$ in Fig. 1 for $\bar{\phi}_0 = 1$, i.e., $F_+ \lambda_+ = 1/e^2$. From Fig. 1 we see that the potential energy is localized at $\bar{\phi} \equiv \sqrt{2/3} \kappa \phi \sim 0$. However, it should again be cautioned that in the Einstein frame with the potential $V(\phi)$ in Eq. (4.34), a domain wall is not formed. In other words, it is considered that the form of the potential $V(\phi)$ in Eq. (4.34) drawn in Fig. 1 is just a corresponding form to realize an $F(R)$ gravity model of $F(R) = F_+ e^{\lambda_+ R}$ with realizing a static domain wall solution in the Jordan frame. The analyses and considerations in this subsection correspond to those in Sec. II B, and the direction of the conformal transformation (i.e., from the Jordan frame to the Einstein frame) is the opposite to that (i.e., from the Einstein frame to the Jordan frame) in Sec. II B.

V. EFFECTIVE (GRAVITATIONAL) DOMAIN WALL

In this section, we demonstrate that there could exist an effective (gravitational) domain wall in the framework of $F(R)$ gravity by using the reconstruction method of $F(R)$ gravity in Ref. [20].

A. Reconstruction method

We now consider $F(R)$ model whose action is given by

$$S_{F(R)} = \int d^4x \sqrt{-g} \left( \frac{F(R)}{2\kappa^2} + L_{\text{matter}} \right).$$

(5.1)
Here $F(R)$ is an appropriate function of the scalar curvature $R$. The action (5.1) is equivalently rewritten as

$$S = \int d^4x \sqrt{-g} \left[ \frac{1}{2\kappa^2} (P(\psi)R + Q(\psi)) + L_{\text{matter}} \right].$$  \hspace{1cm} (5.2)

Here, $P$ and $Q$ are proper functions of the auxiliary scalar $\psi$. By the variation over $\psi$, it follows that $0 = P'(\psi)R + Q'(\psi)$, which may be solved with respect to $\psi$ as $\psi = \psi(R)$. Here, the prime denotes the derivative with respect to the argument of the function as $P'(\psi) = dP(\psi)/d\psi$. By substituting the obtained expression of $\psi(R)$ into (5.2), one arrives again at the $F(R)$-gravity:

$$S = \int d^4x \sqrt{-g} \left( \frac{F(R)}{2\kappa^2} + L_{\text{matter}} \right), \quad F(R) \equiv P(\psi(R))R + Q(\psi(R)).$$  \hspace{1cm} (5.3)

For the action (5.2), the variation of the metric gives

$$0 = \frac{1}{2} g_{\mu\nu} (P(\psi)R + Q(\psi)) - P(\psi)R_{\mu\nu} + \nabla_\mu \nabla_\nu P(\psi) - g_{\mu\nu} \Box P(\psi).$$  \hspace{1cm} (5.4)

Here we have neglected the contribution from the matter.

We now assume the $D = d + 1$ dimensional warped metric in Eq. (3.2) and we also assume the scalar field only depends on $y$. Then $(y,y)$ and $(i,j)$ components of (5.4) give

\[ \text{FIG. 1: } V/V_0 \text{ as a function of } \bar{\phi} \text{ for } \bar{\phi}_0 = 1 \text{ (} F_+ \lambda_+ = 1/e^2 \text{).} \]
the following equations:

\[
0 = \frac{1}{2} \left\{ P(\psi) \left[ -du'' - \frac{d(d+1)}{4} (u')^2 + \frac{d(d-1)e^{-u}}{l^2} \right] + Q(\psi) \right\} - P(\psi) \left[ -\frac{d}{2} u'' - \frac{d}{4} (u')^2 \right] - \frac{d-1}{2} u'u'P'(\psi),
\]

\begin{align*}
0 &= \frac{1}{2} e^u \left\{ P(\psi) \left[ -du'' - \frac{d(d+1)}{4} (u')^2 + \frac{d(d-1)e^{-u}}{l^2} \right] + Q(\psi) \right\} \\
&\quad - P(\psi) \left\{ \frac{d-1}{l^2} + e^u \left[ -\frac{1}{2} u'' - \frac{d}{4} (u')^2 \right] \right\} \\
&\quad + \frac{1}{2} e^u u'u'P'(\psi) - e^u \left[ \psi''P'(\psi) + (\psi')^2 P''(\psi) + \frac{d-1}{2} u'u'P'(\psi) \right],
\end{align*}

(5.6)

where \( u' = du(y)/dy \) and \( u'' = d^2u(y)/dy^2 \). By choosing \( \psi = y \), in case \( 1/l^2 = 0 \), by rewriting Eqs. (5.5) and (5.6), we obtain,

\begin{align*}
0 &= P''(\psi) - \frac{u'(\psi)}{2} P'(\psi) + \frac{(d-1)u''(\psi)}{2} P(\psi), \\
Q(\psi) &= \frac{d(d-1)(u'(\psi))^2}{4} P(\psi) + (d-1)u'(\psi)P'(\psi).
\end{align*}

(5.7)

(5.8)

Equation (5.7) can be further rewritten as

\[
u'(\psi) = -\frac{2}{d-1} P(\psi)^{\frac{4}{d-1}} \int d\psi P(\psi)^{-\frac{d}{d-1}} P''(\psi) \\
= -\frac{2}{d-1} \left[ \frac{P'(\psi)}{P(\psi)} + \frac{d}{d-1} P(\psi)^{\frac{1}{d-1}} \int d\psi P(\psi)^{-\frac{2d-1}{d-1}} (P'('\psi))^2 \right].
\]

(5.9)

In the second equality in (5.9), we have used partial integration. Furthermore by writing

\[
P(\psi) = U(\psi)^{-2(d-1)},
\]

(5.10)

we find

\[
u'(\psi) = \frac{4U''(\psi)}{U(\psi)} - \frac{8d}{U(\psi)^2} \int d\psi U'(\psi)^2.
\]

(5.11)

As an example, we consider a model

\[
U(\psi) = U_0 \left( \psi^2 + \psi_0^2 \right)^\chi.
\]

(5.12)

Here \( U_0, \psi_0, \) and \( \chi \) are constants. Then we find

\[
u'(\psi) = \frac{2\chi \psi}{\psi^2 + \psi_0^2} - \frac{32d\chi^2\psi^4\chi^{-1}}{(\psi^2 + \psi_0^2)^{2\chi}} \sum_{k=0}^{\infty} \frac{\Gamma (2\chi - 1)}{\Gamma (2\chi - 1 - k) \Gamma (2\chi - 1 - k)} \left( \frac{\psi^2}{\psi_0^2} \right)^k.
\]

(5.13)
When $\psi = y$ is large, $u'(\psi)$ behaves as

$$u'(\psi) = \left(2\chi - \frac{32d\chi}{4\chi - 1}\right) \frac{1}{\psi} + \left[-2\chi + \frac{64d\chi^2}{4\chi - 1} - \frac{64d\chi^2(\chi - 1)}{4\chi - 3}\right] \frac{\psi_0}{\psi^2} + \mathcal{O}\left(\frac{(\psi_0^2)^2}{\psi^2}\right). \tag{5.14}$$

Therefore if we choose

$$\chi = -\frac{1}{4(4d - 1)}, \tag{5.15}$$

we find

$$u'(\psi) = \frac{1}{4(6d - 1)} \frac{\psi_0}{\psi^2} + \mathcal{O}\left(\frac{(\psi_0^2)^2}{\psi^2}\right). \tag{5.16}$$

Then by imposing the boundary condition that the universe becomes flat ($u \to 0$) when $|y| = |\psi| \to \infty$, we find

$$u(\psi) = -\frac{1}{4(6d - 1)} \frac{\psi_0}{\psi} + \mathcal{O}\left(\frac{(\psi_0^3)}{\psi}\right). \tag{5.17}$$

Since $u(\psi)$ behaves non-trivially when $\psi = y \sim 0$, we may regard that there could be an effective (gravitational) domain wall at $y = 0$.

For a model in Eq. (5.12), by using Eq. (5.11), $u(\psi)$ can be described as

$$u(\psi) = 8\chi \int_{-\infty}^{\psi} d\psi' \left(\frac{\psi}{\psi^2 + \psi_0^2} - 32d\chi^2 \int_{-\infty}^{\psi'} d\psi'' \left(\frac{1}{(\psi''^2 + \psi_0^2)^{2\chi}} \int_0^{\psi'} d\tilde{\psi} \left(\tilde{\psi}^2 + \psi_0^2\right)^{2(\chi - 1)} \tilde{\psi}^2\right)\right). \tag{5.18}$$

In Fig. 2, we illustrate $u(\psi)$ in Eq. (5.18) as a function of $\psi$ for $d = 3$, $\chi = 2$, and $\psi_0 = 1$. From Fig. 2, we see that $u(\psi)$ has a local maximum around $\psi = y \sim 0$, and hence it is considered that there could exist an effective (gravitational) domain wall at $y = 0$. In the numerical analysis of Eq. (5.18) in Fig. 2, we have substituted the minimum of $\psi$ in the integral range $\psi_{\text{min}} = -10^8$ for $-\infty$. We have also checked that the qualitative behavior of $u(\psi)$ as a function of $\psi$ does not depend on these values of parameters sensitively.

Furthermore, for $\chi = 1/2$ in a model in Eq. (5.12), it is possible to acquire an analytic solution as follows.

$$u(\psi) = 2 \left(1 - 2d\right) \ln \left(\psi^2 + \psi_0^2\right) + 4d \left(\arctan \left(\frac{\psi}{\sqrt{\psi_0^2}}\right)\right)^2 + c_0, \tag{5.19}$$

where $c_0$ is an integration constant. We illustrate the behavior of $u(\psi)$ in Eq. (5.19) for $d = 3$ (i.e., $D = 4$ dimension), $\psi_0 = 1$ and $c_0 = 0$ in Fig. 3. From Fig. 3, we see that $u(\psi)$ has a local maximum around $\psi \sim 0.8$, although $u(\psi)$ does not asymptotically approaches 0 in the limit of $|\psi| \to \infty$. Thus, it might be interpreted that in the range of $|\psi| \lesssim 1.4$, i.e., a small amplitude of $\psi$, the distribution of the energy density is localized and hence such a configuration could be regarded as an effective (gravitational) domain wall.
FIG. 2: $u(\psi)$ as a function of $\psi$ for $d = 3$, $\chi = 2$, and $\psi_0 = 1$.

FIG. 3: $u(\psi)$ in Eq. (5.19) as a function of $\psi$ for $d = 3$, $\psi_0 = 1$ and $c_0 = 0$.

B. Reconstruction of an explicit form of $F(R)$

For $u(\psi)$ in Eq. (5.19), we explicitly derive a form of $F(R)$. We note that as a possible analytic solution, we here consider $u(\psi)$ in Eq. (5.19), even though only in a small amplitude of $\psi$, the distribution of the energy density might correspond to an effective (gravitational)
domain wall.

By using \(0 = P'(\psi)R + Q'(\psi)\) and Eq. (5.8), we find
\[
R = -\frac{Q'(\psi)}{P'(\psi)} = -(d - 1) \left\{ \frac{d}{2} \left( \frac{u'(\psi) u''(\psi)}{P'(\psi)} + u''(\psi) + \frac{P''(\psi)}{P'(\psi)} \right) \right\}.
\] (5.20)

From Eq. (5.20), we derive an analytic relation \(\psi = \psi(R)\). By substituting this relation into the second equation in Eq. (5.3), we can obtain an explicit form of \(F\).

We expand \(P\), \(Q\), and \(R\), where \(Q\) are constants. Moreover, we have
\[
Q = Q_1 \bar{Y} + Q_2 \bar{Y}^2,
\]
\[
Q_1 \equiv \frac{4 (d - 1)}{\psi_0^2} \left[d - 2 (d - 1) (U_0 \psi_0)^{-2(d-1)} \right],
\]
\[
Q_2 \equiv \frac{8 (d - 1)}{\psi_0^2} \left[-d \left(1 + \frac{2d}{3}\right) + (U_0 \psi_0)^{-2(d-1)} (d - 1) \left(1 + \frac{5d}{3}\right) \right],
\]
where \(Q_1\) and \(Q_2\) are constants. By using Eq. (5.21), we express \(\bar{Y}\) as
\[
\bar{Y} = \bar{Y}_0 + \bar{Y}_1 R,
\]
\[
\bar{Y}_0 \equiv -\frac{R_0}{R_1},
\]
\[
\bar{Y}_1 \equiv 1.
\]

We expand \(P(\psi)\) as \(P(\psi) \approx (U_0 \psi_0)^{-2(d-1)} \left\{1 - (d - 1) \bar{Y} + \frac{d (d - 1)}{2} \bar{Y}^2 \right\}\). We substitute this relation and Eq. (5.24) with Eq. (5.27) into the second equation in Eq. (5.3) and take terms which is of order of \(R^2\). As a consequence, we acquire
\[
F(R) = F_0 + F_1 R + F_2 R^2,
\]
\[
F_0 \equiv Q_1 \bar{Y}_0 + Q_2 \bar{Y}_0^2,
\]
\[
F_1 \equiv (U_0 \psi_0)^{-2(d-1)} \left[1 - (d - 1) \bar{Y}_0 + \frac{d (d - 1)}{2} \bar{Y}_0^2 \right] + Q_1 \bar{Y}_1 + 2 Q_2 \bar{Y}_0 \bar{Y}_1,
\]
\[
F_2 \equiv (U_0 \psi_0)^{-2(d-1)} (d - 1) \bar{Y}_1 (-1 + d \bar{Y}_0) + Q_2 \bar{Y}_1^2,
\]
where \( F_0, F_1 \) and \( F_2 \) are constants. Since we have derived an explicit form of \( F(R) \) in Eq. (5.30) for \( Y = \psi^2/\psi_0^2 \ll 1 \), from Eq. (5.21) it can be considered that this \( F(R) \) form in Eq. (5.30) corresponds to the one for \( R \sim \mathcal{O}(1) \) when \( R_0 \sim \mathcal{O}(1) \). If we set \( F_0 = 0 \) and \( F_1 = 1 \), from Eq. (5.33) we find \( F(R) = R + F_2 R^2 \). In the limit of the small curvature regime, \( F(R) \) asymptotically approaches \( R \), i.e., general relativity. Thus, for \( u(\psi) \) in Eq. (5.19) forming an effective (gravitational) domain wall, an explicit form of \( F(R) \) is described by a power-law model.

Finally, we clearly explain the difference between the investigations for \( F(R) \) gravity with a static domain wall solution in Sec. IV and the demonstrations for an effective (gravitational) domain wall in \( F(R) \) gravity in Sec. V. In Sec. IV, we regard the deviation of \( F(R) \) gravity from general relativity as a geometrical contribution to the energy-momentum tensor, which can be described in Eq. (1.2). Since we have a static domain wall solution in a scalar field theory in general relativity in Sec. III, by comparing Eqs. (4.7) and (4.8) with Eqs. (3.3) and (3.4), we find the difference of the gravitational field equations in \( F(R) \) gravity from those in general relativity. Furthermore, in principle we can derive an explicit form of \( F(R) \) realizing a domain wall solution, as discussed in Sec. IV B. In other words, in Sec. IV we first suppose the existence of a static domain wall solution in \( F(R) \) gravity, which is equivalent to that obtained in a scalar field theory in general relativity in Sec. III. Then, through the comparison of gravitational field equations in \( F(R) \) gravity with those in a scalar field theory in general relativity, we reconstruct an explicit form of \( F(R) \). On the other hand, in Sec. V, by using the reconstruction method of \( F(R) \) gravity [20], we directly show that the distribution of the energy density could be localized and hence such a configuration could be regarded as an effective (gravitational) domain wall. Here, the reason why we call “an effective (gravitational) domain wall”, i.e., what is the definition of it, is that a domain wall solution obtained in Sec. V is realized by a pure gravitational effect. This is because in Sec. V we consider the case in which there is no matter, such as a scalar field, whereas the realization of a static domain wall solution explored in Sec. III comes from the existence of a scalar field \( \varphi \) in the action in Eq. (3.1). As a result, the fundamental difference of an effective (gravitational) domain wall investigated in Sec. V from the domain wall solution obtained in Sec. IV is summarized as follows. An effective (gravitational) domain wall in Sec. V is realized by a pure gravitational effect. On the other hand, a static domain wall solution, the existence of which is shown in Sec. III, is made by a scalar field. In Sec. IV, the
deviation of \( F(R) \) gravity from general relativity contributes the energy-momentum tensor geometrically, and eventually it plays an equivalent role of matter, such as a scalar field in Sec. III.

VI. NON-MINIMAL MAXWELL-\( F(R) \) GRAVITY

It is known that a coupling between the scalar curvature and the electromagnetic field arises in curved space-time due to one-loop vacuum-polarization effects in Quantum Electrodynamics (QED) \(^{25}\). Therefore, in this section, as a possible way to examine a connection between \( F(R) \) gravity and variation of the fine structure constant, we investigate non-minimal Maxwell-\( F(R) \) gravity. This might lead to a clue to solve an issue of variation of the fine structure constant. Furthermore, a non-minimal gravitational coupling of the electromagnetic field breaks the conformal invariance of the electromagnetic field.

A. Variation of the fine structure constant

We study a case in which there exists a non-minimal gravitational coupling of the electromagnetic field in \( F(R) \) gravity \(^{26}\). Cosmological consequences of such a non-minimal gravitational coupling of the Maxwell field \(^{27}\) and a non-minimal gravitational coupling of the Yang-Mills field \(^{28}\) have also been studied.

We consider the following action \(^{26}\):

\[
S = \int d^4x \sqrt{-g} \frac{F(R)}{2\kappa^2} + \int d^4x \sqrt{-g} \left( -\frac{1}{4} I(R) g^{\mu\alpha} g^{\nu\beta} F_{\mu\nu} F_{\alpha\beta} \right), \tag{6.1}
\]

where

\[
I(R) = 1 + \mathcal{I}(R). \tag{6.2}
\]

Here, \( \mathcal{I}(R) \) is an arbitrary function of \( R \).

We investigate a situation in which a domain wall as well as the variation of the fine structure constant can be realized in non-minimal Maxwell-\( F(R) \) gravity. As an \( F(R) \) gravity model to form a domain wall, we take \( F(R) = F_+ e^{\lambda R} \) as in Sec. IV C. Moreover, we choose a logarithmic non-minimal gravitational coupling of the electromagnetic field as

\[
I(R) = 1 + \ln \left( \frac{R}{R_0} \right), \tag{6.3}
\]
where \( R_0 \) is the current curvature. (Here, \( \mathcal{I}(R) = \ln (R/R_0) \).) In Ref. [29], it has been found that such a logarithmic-type non-minimal gravitational coupling appears in the effective renormalization-group improved Lagrangian for an \( SU(2) \) gauge theory in matter sector for a de Sitter background. This comes from the running gauge coupling constant with the asymptotic freedom in a non-Abelian gauge theory, which approaches zero in very high energy regime.

Furthermore, from the second part of the action in Eq. (6.1) describing non-minimal electromagnetic field theory we find

\[
\alpha_{\text{EM}}(R) = \frac{\alpha_{\text{EM}}^{(0)}}{\mathcal{I}(R)},
\]

where \( \alpha_{\text{EM}}^{(0)} \) is the bare fine structure constant and hence \( \alpha_{\text{EM}}^{(0)} = \alpha_{\text{EM}}(R_0) \). Since \( R \) is large in the early universe and it decreases in time as the universe evolves, \( \alpha_{\text{EM}} \) varies in time. For a logarithmic-type non-minimal gravitational coupling in Eq. (6.3), we see that \( \alpha_{\text{EM}} \) increases as the universe evolves and approaches the value of the bare fine structure constant at the present time.

It is known that there exist strong constraints on variation of the fine structure constant from the big-bang nucleosynthesis (BBN) at redshift \( z \sim 1 \times 10^3 \) and from the primary CMB signal. Furthermore, there are astronomical constraints from quasar absorption lines. Moreover, the start formation could be affected by a time-varying \( \alpha_{\text{EM}} \) as well.

According to the latest results of Keck/HIRES (High Resolution Echelle Spectrometer) quasi-stellar object (QSO) absorption spectra over the redshift range \( 0.2 < z_{\text{abs}} < 3.7 \) in Ref. [12], \( \alpha_{\text{EM}} \) was smaller in the past and the following weighted mean \( \alpha_{\text{EM}} \) with raw statistical errors has been presented:

\[
\frac{\alpha_{\text{EM}}^{(0)} - \alpha_{\text{EM}}}{\alpha_{\text{EM}}^{(0)}} = (-0.543 \pm 0.116) \times 10^{-5},
\]

representing 4.7\( \sigma \) significance level. For a logarithmic-type non-minimal gravitational coupling in Eq. (6.3), we see that \( \alpha_{\text{EM}} \) was smaller in the past and becomes larger in time.

In addition, in Ref. [13], by analyzing the combined dataset from the Keck telescope and the ESO Very Large Telescope (VLT), the following spatial variation of the fine structure constant has been given:

\[
\frac{\alpha_{\text{EM}}^{(0)} - \alpha_{\text{EM}}}{\alpha_{\text{EM}}^{(0)}} = (1.10 \pm 0.25) \times 10^{-6} r \cos \Theta \text{ Glyr}^{-1},
\]
with a significance of $4.2\sigma$. Here, $r(z) \equiv ct(z)$ with $c$ being the speed of light is the lookback time at redshift $z$ and $\Theta$ is the angle on the sky between sightline and best-fit dipole position. In Ref. [13], by using a new dataset from the ESO VLT, it has also been mentioned that $\alpha_{\text{EM}}$ appears on average to be larger in the past.

It should be cautioned that in our model it is not possible to estimate the spatial variation of $\alpha_{\text{EM}}$ and only the time-variation of alpha could be estimated.

For a logarithmic non-minimal gravitational coupling of the electromagnetic field in Eq. (6.3), in order to compare the theoretical results with the observations on the time variation of the fine structure constant from quasar absorption lines in Eq. (6.5), we estimate the time variation of the fine structure constant from the epoch of the redshift $z = 0.21$ to the present time. In the flat FLRW space-time in Eq. (2.14), from $R/R_0 \approx (1 + z)^3$ we find $R(z = 0.21)/R_0 \approx 1.77$. By using Eqs. (6.3) and (6.4), we obtain

$$\frac{\alpha_{\text{EM}}(R(z = 0.21)) - \alpha_{\text{EM}}^{(0)}}{\alpha_{\text{EM}}^{(0)}} = -0.364.$$  (6.7)

This implies that the naive model of a logarithmic non-minimal gravitational coupling of the electromagnetic field could not satisfy the constraints on the time variation of the fine structure constant from quasar absorption lines in Eq. (6.5) and therefore it would be ruled out.

We remark that the time variation of the fine structure constant in the Jordan frame depends only on a non-minimal gravitational coupling of the electromagnetic field, i.e., the form of $I(R)$, and it does not on the form of $F(R)$, provided that there is no explicit relation between the form of $F(R)$ and that of $I(R)$ in the action in Eq. (6.1). In the next subsections, therefore, we explore the effect of $F(R)$ gravity on variation of the fine structure constant by making a conformal transformation to the Einstein frame.

**B. Relation to a coupling between the electromagnetic field and a scalar field in the Einstein frame**

We study the effect of $F(R)$ gravity with realizing a domain wall on variation of the fine structure constant. By using the same procedure presented in Sec. II A, we make a conformal transformation to the Einstein frame in Eq. (2.2). Consequently, we obtain the
action in the Einstein frame described as
\[ S_E = \int d^4x \sqrt{-\tilde{g}} \left( \frac{\tilde{R}}{2\kappa^2} - \frac{1}{2} \tilde{g}^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi) \right) + \int d^4x \sqrt{-\tilde{g}} \left( -\frac{1}{4} J(\phi) \tilde{g}^{\alpha\beta} \tilde{g}^{\mu\nu} F_{\mu\nu} F_{\alpha\beta} \right), \]  
(6.8)
where
\[ J(\phi) \equiv e^{-2/\sqrt{3}\kappa \phi} \left( I(\tilde{R}) - \frac{dI(\tilde{R})}{d\tilde{R}} \tilde{R} \right) + e^{-1/\sqrt{3}\kappa \phi} \frac{dI(\tilde{R})}{d\tilde{R}} \left[ \tilde{R} + \sqrt{3} \Box (\kappa \phi) - \frac{1}{2} \tilde{g}^{\mu\nu} \partial_\mu (\kappa \phi) \partial_\nu (\kappa \phi) \right]. \]  
(6.9)
Here, the first term on the r.h.s. of Eq. (6.8) is the same as that in Eq. (2.8). We note that if \( \tilde{R} \) can be expressed by \( \phi \), \( J \) can be described as a function of \( \phi \). By comparing the second term on the r.h.s. of Eq. (6.8) with that of Eq. (2.12), we find \( J(\phi) = B(\phi) \). Thus, by using this relation, it might be possible that we obtain the relation between a non-minimal gravitational coupling of the electromagnetic field in the Jordan frame and a coupling of the electromagnetic field to a scalar field in the Einstein frame.

C. Case for an exponential model

First, we take an \( F(R) \) gravity model with realizing a domain wall as \( F(R) = F_+ e^{\lambda_+ R} \) derived in Sec. IV B and a logarithmic non-minimal gravitational coupling of the electromagnetic field in Eq. (6.3). We also assume the \( D = 4 \) (\( d = 3 \)) dimensional warped metric in Eq. (3.2) because such an exponential model \( F(R) = F_+ e^{\lambda_+ R} \) is derived in this metric in Sec. IV B. We consider the case in which \( \phi \) only depends on \( y \). In this case, the effect of \( F(R) \) gravity with realizing a domain wall is involved in \( J(\phi) \) in Eq. (6.9) through the relation between the scalar curvature \( R \) and \( \phi \). From Eq. (6.9), we obtain
\[ J(\phi) = e^{-2/\sqrt{3}\kappa \phi} \ln \left( \frac{R}{R_0} \right) + e^{-1/\sqrt{3}\kappa \phi} \left[ 1 - 3\sqrt{3} \left( \frac{\phi}{y_0} \right) e^{-\phi/(y_0)} \frac{u_0}{y_0 R_0} \left( \frac{d\phi}{dy} \right) - \frac{1}{2} \kappa^2 \left( \frac{d\phi}{dy} \right)^2 \right], \]  
(6.10)
where \( R \) can be described as a function of \( \phi \) by Eq. (4.33). We also take the value of the current curvature \( R_0 = (1/\lambda_+) \left\{ \ln [1/(F_+ \lambda_+)] + \sqrt{2/3} \kappa \phi_p \right\} \) by using Eq. (4.33). Here, \( \phi_p \) is the amplitude of \( \phi \) at the present time. We note that \( R_0 \) is determined by \( \phi_p \) and not \( \phi_0 \).

We may now choose \( \phi = y \) and set \( \kappa^2 = 1 \) as executed in Sec. III. In Fig. 4 we depict \( J(\phi) \) as a function of \( \phi \) for \( F_+ = 1 \), \( \lambda_+ = 1 \), \( u_0 = 1 \), \( y_0 = \phi_0 = 10 \), and \( \phi_p = 1 \). We have
confirmed that the qualitative behavior of $J(\phi)$ as a function of $\phi$ does not depend on these values of parameters sensitively.

Moreover, from the second part of the action in Eq. (6.8) describing electromagnetic field theory we have

$$\alpha_{\text{EM}}(\phi) = \frac{\alpha_{\text{EM}}^{(0)}}{J(\phi)},$$

(6.11)

where $\alpha_{\text{EM}}^{(0)} = \alpha_{\text{EM}}(\phi_p)$. We investigate the time variation of the fine structure constant from the epoch of the redshift $z = 0.21$ to the present time, as executed in Sec. VI A. As an example, we choose $F_+ = 1$, $\lambda_+ = 1$, $u_0 = 1$, $y_0 = \phi_0 = 10$, and $\phi_p = 1$. Since $R = \sqrt{2/3}\phi$ from Eq. (4.33), we have $\phi(z = 0.21) = (R(z = 0.21)/R_0) \phi_p$. From Eq. (6.10), we acquire $J(\phi(z = 0.21)) = 0.420$. By combining this value with Eq. (6.11), we find

$$\frac{\alpha_{\text{EM}}(\phi(z = 0.21)) - \alpha_{\text{EM}}^{(0)}}{\alpha_{\text{EM}}^{(0)}} = \frac{1}{J(\phi(z = 0.21))} - 1 = 1.38.$$ (6.12)

This value is larger than the constraints on the time variation of the fine structure constant from quasar absorption lines in Eq. (6.5) and hence the naive model of a logarithmic non-minimal gravitational coupling of the electromagnetic field could not be consistent with the observations of quasar absorption lines. In Fig. 4 we see that $J(\phi)$ approaches zero as $\phi$ becomes large. It follows from Eq. (6.11) that $\alpha_{\text{EM}}$ decreases as the universe evolves. Such
a behavior of $\alpha_{\text{EM}}$ in the Einstein frame is opposite to that in the Jordan frame examined in Sec. VI A.

It is interesting to emphasize that in the Einstein frame, the differences of $F(R)$ gravity models reflect time-variation of the fine structure constant through $J(\phi)$ in Eq. (6.9) due to the relation (2.5) between $\phi$ and $F_{,R}$.

D. Case for a power-law model

Next, we take an $F(R)$ gravity model with forming an effective (gravitational) domain wall as $F(R) = R + F_2 R^2$ derived in Sec. V B and a logarithmic non-minimal gravitational coupling of the electromagnetic field in Eq. (6.3). We again assume the $D = 4$ $(d = 3)$ dimensional warped metric in Eq. (3.2) because such a power-law model is the one for $u(\psi)$ in Eq. (5.19) derived in this metric in Sec. V A. We consider the case in which $\phi$ only depends on $y$. In this case, the effect of $F(R)$ gravity with forming an effective (gravitational) domain wall is included in $J(\phi)$ in Eq. (6.10) through the following relation between the scalar curvature $R$ and $\phi$:

$$R = \frac{1}{2F_2} \left( e^{\sqrt{2/3}\kappa \phi} - 1 \right),$$

(6.13)

where we have used Eq. (2.5). By using Eq. (6.13), we also take the value of the current curvature $R_0 = \left( e^{\sqrt{2/3}\kappa \phi_p} - 1 \right) / (2F_2)$.

Here, we may choose $\phi = y$ and set $\kappa^2 = 1$ as executed in Sec. III. In Fig. 5 we show $J(\phi)$ as a function of $\phi$ for $F_2 = 1$, $u_0 = 1$, $y_0 = \phi_0 = 10$, and $\phi_p = 1$. We have again verified that the qualitative behavior of $J(\phi)$ as a function of $\phi$ does not depend on these values of parameters sensitively.

We explore the time variation of the fine structure constant from the epoch of the redshift $z = 0.21$ to the present time, similarly to those in Secs. VI A and VI C. As an example, we take $F_2 = 1$, $u_0 = 1$, $y_0 = \phi_0 = 10$, and $\phi_p = 1$, which is the case illustrated in Fig. 5. Since $R = \left( e^{\sqrt{2/3}\phi} - 1 \right) / 2$ from Eq. (6.13), we acquire $\phi(z = 0.21) = \sqrt{3/2} \ln \left( (R(z = 0.21)/R_0) \left( e^{\sqrt{2/3}\phi_p} - 1 \right) + 1 \right)$. From this relation, for $\phi_p = 1$, we obtain $\phi(z = 0.21) = 1.44$. By using Eq. (6.10), we find $J(\phi(z = 0.21)) = 0.632$. By substituting this value into Eq. (6.11), we obtain

$$\frac{\alpha_{\text{EM}}(\phi(z = 0.21)) - \alpha_{\text{EM}}(0)}{\alpha_{\text{EM}}(0)} = \frac{1}{J(\phi(z = 0.21))} - 1 = 0.583.$$  

(6.14)
This value is also larger than the constraints on the time variation of the fine structure constant from quasar absorption lines in Eq. (6.5) and therefore the naive model of a logarithmic non-minimal gravitational coupling of the electromagnetic field would be incompatible with the observations of quasar absorption lines. In Fig. 5, we see that $J(\phi)$ approaches zero as $\phi$ increases. It follows from Eq. (6.11) that $\alpha_{\text{EM}}$ becomes small as the universe evolves, similarly to that for an exponential model in Sec. VI C. Again, such a behavior of $\alpha_{\text{EM}}$ in the Einstein frame is opposite to that in the Jordan frame examined in Sec. VI A.

Finally, we emphasize the main reason why we study three types of non-minimal gravitational couplings between the scalar field (or the Ricci scalar) and the electromagnetic field: logarithmic, exponential and power law, even though these models can be easily ruled out. It would be considered that such a non-minimal coupling of the electromagnetic field to gravity could be one of the most theoretically motivated approaches to investigate a relation between $F(R)$ gravity and variation of the fine structure constant, and thus that cosmological considerations on this model could present an understanding on the origin of variation of the fine structure constant.
VII. COSMOLOGICAL CONSEQUENCES OF THE COUPLING OF THE ELECTROMAGNETIC FIELD TO NOT ONLY A SCALAR FIELD BUT ALSO THE SCALAR CURVATURE

In this section, we consider a scalar field theory with its potential forming a domain wall, e.g., $V(\phi)$ in Eq. (2.11), and its coupling to the electromagnetic field, such as the action in Eq. (2.10). In particular, we extend the coupling of the electromagnetic field not only to a scalar field but also to the scalar curvature as

$$S_{E} = \int d^{4}x \sqrt{-\tilde{g}} \left( \frac{\tilde{R}}{2\kappa^{2}} - \frac{1}{2} \tilde{g}^{\mu \nu} \partial_{\mu} \phi \partial_{\nu} \phi - V(\phi) \right) + \int d^{4}x \sqrt{-\tilde{g}} \left( -\frac{1}{4} \Upsilon(\phi, \tilde{R}) \tilde{g}^{\mu \alpha} \tilde{g}^{\nu \beta} F_{\mu \nu} F_{\alpha \beta} \right) + S_{\text{matter}}, \tag{7.1}$$

where $\Upsilon(\phi, \tilde{R})$ is an arbitrary function of $\phi$ as well as $\tilde{R}$. In this case, the cosmological evolution of the scalar field $\phi$ as well as that of the scalar curvature $\tilde{R}$ can contribute to the variation of the fine structure constant. Hence, a domain wall can be used to account for the spatial variation through a scalar field coupled to electromagnetism as in Ref. [14], whereas the non-minimal gravitational coupling of the electromagnetic field to the scalar curvature can explain the time variation of the fine structure constant. Thus, there exist more choices of the scalar field potential which can make a domain wall.

In addition, it is interesting to remark that the conformal invariance of the electromagnetic field can be broken by the coupling of the electromagnetic field to both a scalar field (or a scalar quantity) [30, 31] and the scalar curvature [30, 32], and therefore large-scale magnetic fields can be generated from inflation even in the FLRW spacetime, which is conformally flat [30, 33] (for a recent review of the generation of primordial magnetic fields, see [35]).

Finally, we mention that we can develop the action in Eq. (7.1) in the framework of $F(R)$ gravity as follows:

$$S = \int d^{4}x \sqrt{-g} \left( \frac{F(R)}{2\kappa^{2}} - \frac{1}{2} g^{\mu \nu} \partial_{\mu} \phi \partial_{\nu} \phi - V(\phi) \right) + \int d^{4}x \sqrt{-g} \left( -\frac{1}{4} \Upsilon(\phi, R) g^{\mu \alpha} g^{\nu \beta} F_{\mu \nu} F_{\alpha \beta} \right) + S_{\text{matter}}. \tag{7.2}$$

\(^2\) In Ref. [34], it has been shown that by assuming an open FLRW background, large-scale magnetic fields with its enough strength to seed the galactic dynamo mechanism can be generated within standard electromagnetism and standard general relativity.
In this model action, power-law inflation can occur due to the non-minimal gravitational coupling of the electromagnetic field as well as the deviation of $F(R)$ gravity from general relativity and the late-time accelerated expansion of the universe can also be realized through the modified part of $F(R)$ gravity in a unified model action \cite{26, 28}. In the scalar-tensor sector of the theory in Eq. (7.2), the domain wall may be created due to combined effect of scalar potential and modified gravity. Then, combined effect of scalar and curvature in the non-minimal electromagnetic sector gives us wider possibility for realizing the time-variation of the fine structure constant in accordance with observational data.

VIII. CONCLUSION

In the present paper, we have studied a domain wall solution in $F(R)$ gravity. We have reconstructed a static domain wall solution in a scalar field theory. We have also reconstructed an explicit $F(R)$ gravity model in which a static domain wall solution can be realized. Furthermore, we have shown that there could exists an effective (gravitational) domain wall in the framework of $F(R)$ gravity. Moreover, it has been illustrated that a logarithmic non-minimal gravitational coupling of the electromagnetic theory in $F(R)$ gravity may produce time-variation of the fine structure constant which may increase as the curvature decreases. In addition, we have described cosmological consequences of the coupling of the electromagnetic field to not only a scalar field but also the scalar curvature and remarked the relation between variation of the fine structure constant and the breaking of the conformal invariance of the electromagnetic field.

The reconstruction technique was applied here to inducing of domain wall solution in modified gravity (cf. the case of black hole reconstruction in Ref. \cite{36}). It is clear that similar methods may be applied to generation of other solutions in modified gravity, like topological defects, cosmic strings, etc. This question will be discussed elsewhere.

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