WEAK SATURATION PROPERTIES AND SIDE CONDITIONS

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Abstract. Towards combining “compactness” and “hugeness” properties at \( \omega_2 \), we investigate the relevance of side-conditions forcing. We reduce the upper bound on the consistency strength of the weak Chang’s Conjecture at \( \omega_2 \) using Neeman’s forcing. But we find a barrier to the applicability of these methods to our problem and give a counterexample to a claim of Neeman about the effects of iterating such forcing.

1. Introduction

Can “compactness” and “hugeness” properties coexist at small cardinals? More specifically, can \( \omega_2 \) satisfy the tree property and also carry a saturated ideal? This question is natural for anyone concerned with “large cardinal properties” that can hold of small cardinals. It turns out to be surprisingly difficult to answer. Cox and the author showed in [4] that if this situation is consistent, then (a) it requires the continuum to be at least \( \omega_3 \), and (b) it cannot be forced via a “Kunen-style” generic lifting of an almost-huge embedding, which is presently the only known method to produce models with saturated ideals on \( \omega_2 \). This means that, if these properties of \( \omega_2 \) are mutually consistent, a proof will require rather novel methods.

A promising idea is to look towards the newer techniques of forcing with sequences of models. The basic idea of using models as side-conditions to control a forcing construction was invented by Todorčević [31]. Work of Friedman [12] and Mitchell [21] expanded upon this idea, and influential work of Neeman [23] made further advances, providing an elegant framework using two types of models. Mitchell [21] and Neeman [23] also suggested iterating such forcing to obtain the tree property on successive cardinals. The author realized that, when combined with very large cardinals, this could also simultaneously force an approximation of saturation at \( \omega_2 \), namely the weak Chang’s Conjecture \( \text{wCC}(\omega_2, \text{cof}(\omega_1)) \).

However, we ultimately found some combinatorial constraints that make the Neeman technique unsuitable for combining compactness and hugeness properties at successor cardinals. Thus the present work can be seen as adding to the limitative results of [4]. In particular, Theorem 5 says that if the continuum is at most \( \omega_2 \) and \( \text{wCC}(\omega_2, \text{cof}(\omega_1)) \) holds, then there is a special \( \omega_2 \)-Aronszajn tree. This implies that an iteration of side-conditions forcing cannot work as claimed in [23] in full generality.

Another method to investigate would be that developed by Mohammadpour and Velicković [22], which obtains the tree property at \( \omega_2 \) and \( \omega_3 \) while strengthening the celebrated consistency result of Mitchell [21]. Since their model satisfies \( 2^{\omega_2} = \omega_3 \), it evades the constraint imposed by Theorem 5. However, Mohammadpour showed that their poset always forces \( \neg \text{wCC}(\omega_2, \text{cof}(\omega_1)) \), and thus that there is no saturated ideal on \( \omega_2 \). We reproduce his argument here with his kind permission.

Our results here are not entirely negative. We uncover some new combinatorial facts that add to the tension between “compactness” and “hugeness” properties. We isolate a new large cardinal notion, “ambitious” cardinals, and show several

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equivalent characterizations. We show how Neeman forcing may introduce the weak Chang’s Conjecture and derive a forcing characterization of ambitiousness. We reduce the known upper bound on the consistency strength of $\text{wCC}(\omega_2, \text{cof}(\omega_1))$.

The structure of the paper is as follows. In §2 we discuss saturation properties of ideals, versions of Chang’s Conjecture, and equivalences in terms of properties of collections of elementary submodels. In §3 we show some new combinatorial results connecting versions of Chang’s Conjecture and square principles. In §4 we introduce a modified version of Neeman forcing and lay out its important properties. In §5 we discuss a large cardinal above $0^+$ but below measurable that can be used with Neeman forcing to obtain weak presaturation at $\omega_1$. In §6 we obtain a new upper bound on $\text{wCC}(\omega_2, \text{cof}(\omega_1))$. In §7 we introduce ambitious cardinals and discuss their connection with so-called Magidor models. Finally, in §8 we give a forcing characterization of ambitiousness, discuss the implications for iterated Neeman forcing, and sketch Mohammadpour’s argument that the forcing of §2 does not allow hugeness properties of $\omega_2$.

We assume familiarity with the basics of stationary sets, large cardinals, forcing, and the notions of properness and strong properness for a class of models. We also assume a strong familiarity with [22], as we will rely on it.

To fix some notation, if $\kappa$ is a cardinal and $X$ is a set, we use $[X]^{<\kappa}$ to denote $\{z \subseteq X : |z| < \kappa\}$, and if $\kappa \subseteq X$, we use $P_\kappa(X)$ to denote $\{z \subseteq X : |z| < \kappa \land z \cap \kappa \in \kappa\}$. For a set of ordinals $x$, we let $\text{ot}(x)$ denote its order-type. $H_\kappa$ denotes the collection of all sets of hereditary cardinality less than $\kappa$. For a structure $\mathfrak{A}$ on $X$ carrying a well-order and $Y \subseteq X$, we write $\text{Sk}^\mathfrak{A}(Y)$ for the Skolem hull of $Y$, i.e. the closure of $Y$ under the definable Skolem functions.

2. Weak saturation properties

Foreman [10] proved an equivalence between a normal fine ideal being saturated and a statement about elementary submodels. Inspired by this, we identify some weakenings of saturation that also have equivalences in terms of elementary submodels, in the hopes that a forcing such as Neeman’s, which is built from sequences of models, may be able to achieve desired results by directly manipulating properties of collections of elementary submodels. Let us first recall some basic notions.

$I$ is an ideal over a set $Z$ when $I \subseteq P(Z)$, and $I$ is closed under subsets and pairwise unions. For a cardinal $\kappa$, $I$ is $\kappa$-complete when it is closed under unions of size $<\kappa$. We say a set $A \subseteq Z$ is $I$-positive when it is not in $I$. We denote the collection of all such sets as $I^+$, and we let $I^d$ denote the filter dual to $I$. If $Z \subseteq P(X)$, we say that $I$ is fine when $\{z \in Z : x \in z\} \in I^d$ for all $z \in X$, and normal when for every $I$-positive $A$ and every function $f : A \to X$ such that $f(z) \in z$ for all $z \in A$, $f$ is constant on an $I$-positive set. A normal fine ideal $I$ on $Z \subseteq P(X)$ is said to be saturated when for every collection $\mathcal{A}$ of $I$-positive sets that have pairwise intersection in $I$, $|\mathcal{A}| \leq |X|$. In other words, the Boolean algebra $P(Z)/I$ has the $|X|^+$-chain condition.

A set $Z \subseteq P(X)$ is stationary when for all $F : X^{<\omega} \to X$, there is $z \in Z$ closed under $F$. The set of all $z \in P(X)$ closed under such an $F$ is called a closed-unbounded set or a club. It is well-known that for all stationary sets $Z$, the collection of all nonstationary subsets of $Z$ forms the smallest nonstationary ideal over $Z$. The ideal of nonstationary subsets of $Z$ will be denoted by $\text{NS}_Z$, or just NS when $Z$ is clear from context. If $Z$ is a regular cardinal $\kappa$, this notion of club coincides with the usual order-topological notion.

Suppose $\pi : Z_1 \to Z_0$ and $J$ is an ideal on $Z_1$. The map $\pi$ induces an ideal $\mathfrak{I}$ on $Z_0$, where we put $A \in \mathfrak{I}$ iff $\pi^{-1}[A] \in J$. If $J$ is $\kappa$-complete, then so is $\mathfrak{I}$. If $Z_i \subseteq P(X_i)$ for $i < 2$, $X_0 \subseteq X_1$, and $\pi$ is the map $z \mapsto z \cap X_0$, then we say that
\( I \) is the canonical projection of \( J \). In this case, if \( J \) is normal and fine, then so is \( I \). For a normal fine ideal \( I \) on \( Z \subseteq P(X) \) and \( \theta > 2^{2^{|\omega|}} \), Foreman [10] defined a model \( M \prec H_\theta \) to be \( I \)-good when \( I \in M \) and \( M \cap X \in A \) for every \( A \in I^2 \cap M \). He showed:

**Theorem 1** (Foreman). Suppose \( I \) is a normal fine ideal on \( Z \subseteq P(X) \), and \( \theta > 2^{2^{|\omega|}} \) is regular.

1. If \( \mathcal{G} \) is the set of \( I \)-good \( M \prec H_\theta \), then \( \mathcal{G} \) is stationary, and \( I \) is the canonical projection of \( \text{NS} \upharpoonright \mathcal{G} \).
2. If \( |X| = |Z| \), then \( I \) is saturated if and only if for every \( I \)-good \( M \prec H_\theta \) and every maximal antichain \( A \subseteq P(Z) / I \) in \( M \), there is \( a \in A \cap M \) such that \( M \cap X \in a \).

It is well-known that saturated ideals produce well-founded generic ultrapowers, which are moreover highly closed from the perspective of the generic extension, and that these properties can be obtained from the following weaker notion. A normal ideal \( I \) on a regular cardinal \( \kappa \) is called presaturated when for all \( I \)-positive \( B \), and all collections of antichains \( \{ A_i : i < \kappa \} \), there is an \( I \)-positive \( C \subseteq B \) such that \( \{|a \in A_i : a \cap C \subseteq I \} : i < \kappa \} \subseteq \kappa \) for all \( i < \kappa \). It is not hard to show that forcing with a presaturated ideal on \( \kappa \) preserves \( \kappa^+ \). Thus if \( \kappa \) is a successor cardinal, any generic embedding arising from such an ideal must send \( \kappa \) to \( (\kappa^+)^V \).

Woodin [33] calls a normal ideal \( I \) on a successor cardinal \( \kappa \) weakly presaturated just when the above property occurs, i.e., \( \text{NS} / I \upharpoonright \sigma_\kappa = (\kappa^+)^V \). This can be explicated in terms of canonical functions. Recall that the canonical functions on a regular cardinal \( \kappa \) represent ordinals \( \alpha < \kappa^+ \) in any generic ultrapower coming from a normal ideal on \( \kappa \), and they can be defined (up to equivalence modulo clubs) as follows: for \( \alpha < \kappa^+ \), pick any surjection \( \sigma_\alpha : \kappa \rightarrow \alpha \), and define the \( \alpha \)th canonical function \( \psi_\alpha \) by \( \psi_\alpha(\beta) = \text{ot}(\sigma_\alpha[\beta]) \) for all \( \beta < \kappa \).

**Lemma 2.** Suppose \( I \) is a normal ideal on a successor cardinal \( \kappa \). The following are equivalent:

1. \( I \) is weakly presaturated.
2. For all \( I \)-positive \( A \subseteq \kappa \) and all \( f : \kappa \rightarrow \kappa \), there is \( \alpha < \kappa^+ \) and an \( I \)-positive \( B \subseteq A \) such that \( f(\beta) < \psi_\alpha(\beta) \) for all \( \beta \in B \).
3. There is \( \theta > \kappa \) and a stationary set \( S \subseteq \{ M \prec H_\theta : M \cap \kappa \in \kappa \} \) such that:
   a. \( I \) is the canonical projection of \( \text{NS} \upharpoonright S \).
   b. For all \( A \in I^+ \) and all \( f : \kappa \rightarrow \kappa \), the set
      \[ \{ M \in S : M \cap \kappa \in A \land f(M \cap \kappa) < \text{ot}(M \cap \kappa^+) \} \]
   is stationary.

**Proof.** Suppose (1). Let \( A \) be \( I \)-positive and \( f : \kappa \rightarrow \kappa \). \( f \) represents some ordinal \( < j(\kappa) \), and thus there is some \( I \)-positive \( B \subseteq A \) deciding that for some canonical function \( \psi_\alpha, [f]_G < [\psi_\alpha]_G \). This implies that for \( I \)-almost-all \( \beta \in B \), \( f(\beta) < \psi_\alpha(\beta) \), establishing (2). On the other hand, if (2) holds, we have that every ordinal of the generic ultrapower \( < j(\kappa) \) is represented by a function \( f : \kappa \rightarrow \kappa \), and the statement says that for such \( f \), there is a dense set of conditions in the forcing \( \text{NS} \upharpoonright I \) deciding that \( f \) represents an ordinal below \( (\kappa^+)^V \), showing (1).

Suppose (2). By Foreman’s theorem, \( I \) is the canonical projection of \( \text{NS} \upharpoonright G \), where \( G \) is the set of \( I \)-good \( M \prec H_{\theta^+} \). Suppose \( f : \kappa \rightarrow \kappa \) and \( A \) is \( I \)-positive. By hypothesis, there is an \( I \)-positive \( B \subseteq A \) and an \( \alpha < \kappa^+ \) such that \( \psi_\alpha \) dominates \( f \) on \( B \). There are stationary-many \( M \in G \) such that \( M \cap \kappa \in B \). Let \( \sigma_\alpha : \kappa \rightarrow \alpha \) be a surjection defining \( \psi_\alpha \). If \( M \) is any such model with \( \sigma_\alpha \in M \), then
\[
\psi_\alpha(M \cap \kappa) = \text{ot}(\sigma_\alpha[M \cap \kappa]) = \text{ot}(M \cap \alpha) < \text{ot}(M \cap \kappa^+).
\]
Thus there are stationary-many $M \in \mathcal{G}$ with $M \cap \kappa \in A$ and $f(M \cap \kappa) < \psi_\alpha(M \cap \kappa) < \text{ot}(M \cap \kappa^+)$, establishing (3).

Suppose (3). Let $A$ be $I$-positive and $f : \kappa \rightarrow \kappa$. There are stationary-many $M \in \mathcal{S}$ such that $M \cap \kappa \in A$ and $f(M \cap \kappa) < \text{ot}(M \cap \kappa^+)$. By Fodor’s Lemma, there is an $\alpha < \kappa^+$ and a stationary $T \subseteq \mathcal{S}$ such that for all $M \in T$, $\alpha \in M$, $M \cap \kappa \in A$, and $f(M \cap \kappa) < \text{ot}(M \cap \alpha)$. We may assume that there is a surjection $\sigma_\alpha : \kappa \rightarrow \alpha$ that belongs to each $M \in T$. For all $M \in T$, $f(M \cap \kappa) < \text{ot}(M \cap \alpha) = \text{ot}(\sigma_\alpha[M \cap \kappa]) = \psi_\alpha(M \cap \kappa)$. Let $B = \{M \cap \kappa : M \in T\}$. Then $B$ is an $I$-positive subset of $A$, and $\psi_\alpha$ dominates $f$ on $B$. This shows (2).

Chang’s Conjecture $(\kappa_1, \kappa_0) \rightarrow (\mu_1, \mu_0)$ states that for every function $F : \kappa_1^{< \omega} \rightarrow \kappa_1$, there is a $z \subseteq \kappa_1$ closed under $F$ such that $|z| = \mu_1$ and $|z \cap \kappa_0| = \mu_0$. If $\kappa = \mu^+$, then we abbreviate $(\kappa^+, \kappa) \rightarrow (\mu^+, \mu)$ by $\text{CC}(\kappa)$, which is equivalent to the statement that for all $F : (\kappa^+)^{< \omega} \rightarrow \kappa^+$, there is a $z \subseteq \kappa^+$ such that $z \cap \kappa \in \kappa$ and $\text{ot}(z) = \kappa$ (see [10]). According to [8], Magidor showed that if $\kappa = \mu^+$ and $\text{CC}(\kappa)$ holds, then there is a weakly presaturated ideal on $\kappa$. (Note that the dual of a weakly presaturated ideal is called a “filter” there.) In fact, such an ideal is obtained as the canonical projection of the nonstationary ideal on $\mathcal{P}(\kappa^+)$ restricted to the set $\{z : z \cap \kappa \in \kappa \text{ and } \text{ot}(z) = \kappa\}$. A result of Foreman and Magidor [11] shows that all but nonstationary-many $z$ in this set satisfy $\text{cf}(z \cap \kappa) = \text{cf}(\mu)$.

The weak Chang’s Conjecture at a regular cardinal $\kappa$ says: for all functions $F : (\kappa^+)^{< \omega} \rightarrow \kappa^+$, there is $\alpha < \kappa$ such that for all $\beta < \kappa$, there is $z \subseteq \kappa^+$ closed under $F$ with $z \cap \kappa = \alpha$ and $\text{ot}(z) > \beta$. To further specify this kind of property, let us write $w\text{CC}(\kappa, A)$ for $A \subseteq \kappa$ to assert that such $\alpha$ can be found in $A$. When $\kappa = \mu^+$, the result of Foreman and Magidor mentioned above tells us that $\text{CC}(\kappa)$ implies $w\text{CC}(\kappa, \text{cof}(\mu))$. The next result shows that $w\text{CC}(\kappa, A)$ follows from the existence of a weakly presaturated normal ideal $I$ on $\kappa$ such that $A \subseteq I^\kappa$.

**Lemma 3.** Suppose $\kappa$ is a regular cardinal and $A \subseteq \kappa$. The following are equivalent:

1. $w\text{CC}(\kappa, A)$.
2. For all $f : \kappa \rightarrow \kappa$, there are stationary-many $M \in \mathcal{P}_\kappa(H_{\kappa^+})$ such that $M \cap \kappa \in A$ and $f(M \cap \kappa) < \text{ot}(M \cap \kappa^+)$. 
3. For all $f : \kappa \rightarrow \kappa$, there is $\alpha < \kappa^+$ such that $\{\beta \in A : f(\beta) < \psi_\alpha(\beta)\}$ is stationary.

**Proof.** Suppose $w\text{CC}(\kappa, A)$ holds and $f : \kappa \rightarrow \kappa$. Let $\mathfrak{A}$ be any structure on $H_{\kappa^+}$ in a countable language with a well-order. If $f : (\kappa^+)^{< \omega} \rightarrow \kappa^+$ is the result of restricting inputs and outputs of Skolem functions for $\mathfrak{A}$ to ordinals, then every $z \cap \kappa$ closed under $F$ has $\text{Sk}\mathfrak{A}(z) \cap \kappa^+ = z$. By $\text{CC}(\kappa, A)$, there is $\mathfrak{A} \prec H_{\kappa^+}$ such that $M \cap \kappa \in A$ and $f(M \cap \kappa) < \text{ot}(M \cap \kappa^+)$. Thus (1) implies (2).

Suppose towards a contradiction that (2) holds and (3) fails. Let $\phi$ be a well-order of $H_{\kappa^+}$, and assume that each canonical function $\psi_\alpha$ on $\kappa$ is defined using the $\langle \cdot \rangle$-least surjection $\sigma : \kappa \rightarrow \alpha$. Let $f : \kappa \rightarrow \kappa$ be such that for all $\alpha < \kappa^+$, $\psi_\alpha(\beta) < f(\beta)$ for all but nonstationary-many $\beta \in A$. Let $\mathfrak{A} = (H_{\kappa^+}, \in, \langle \cdot \rangle, f, A)$. Let $M \prec H_{\kappa^+}$ be such that $M \cap \kappa \in A$ and $f(M \cap \kappa) < \text{ot}(M \cap \kappa^+)$. Let $\alpha \in M$ be such that $f(M \cap \kappa) < \text{ot}(M \cap \alpha)$. Let $\sigma : \kappa \rightarrow \alpha$ be the $\langle \cdot \rangle$-least surjection, so $\sigma \in M$ and $\psi_\alpha(\beta) = \text{ot}(\sigma(\beta))$ for all $\beta < \kappa$. By elementarity, there is a club $C \subseteq \kappa$ in $M$ such that $\psi_\alpha(\beta) < f(\beta)$ for all $\beta \in C \cap A$. But clearly $M \cap \kappa \in C \cap A$ and $\psi_\alpha(M \cap \kappa) = \text{ot}(M \cap \alpha) > f(M \cap \kappa)$, a contradiction.

Suppose towards a contradiction that (3) holds but $w\text{CC}(\kappa, A)$ fails. Let $F : (\kappa^+)^{< \omega} \rightarrow \kappa^+$ be such that for all $\alpha \in A$, the set $\{\text{ot}(z) : z \cap \kappa = \alpha \cap F[z^{< \omega}] \subseteq \z\}$ is bounded below $\kappa$. Let $f : A \rightarrow \kappa$ return such a bound. Let $\mathfrak{A}$ be a structure on $H_{\kappa^+}$ in a countable language such that if $M \prec \mathfrak{A}$, then $M \cap \kappa^+$ is closed under $F$. By (3), let $\alpha < \kappa^+$ be such that $B = \{\beta \in A : f(\beta) < \psi_\alpha(\beta)\}$ is stationary. Let $M \prec \mathfrak{A}$
be such that \( M \cap \kappa \in B \) and \( \alpha, f \in M \). Then \( f(M \cap \kappa) < \psi_\kappa(M \cap \kappa) = \text{ot}(M \cap \alpha) \), contrary to the definition of \( f \).

**Corollary 4.** Suppose \( \kappa \) is a regular cardinal and \( A \subseteq \kappa \) is stationary. The following are equivalent.

1. \( \text{NS}_\kappa \upharpoonright A \) is weakly presaturated.
2. \( \text{wCC}(\kappa, B) \) holds for every stationary \( B \subseteq A \).

3. Chang’s Conjecture and coherent sequences

In this section, we discuss several ways in which versions of Chang’s Conjecture can lead to non-compact structures, expanding on the theme of \([\textbf{4}]\). Let us first recall some basic definitions and facts. For a cardinal \( \kappa \), Jensen’s weak square principle \( \square^*_\kappa \) says that there is a sequence \( \langle C_\alpha : \alpha < \kappa^+ \rangle \) such that each \( C_\alpha \) is a set of at most \( \kappa \)-many club subsets of \( \alpha \), each of order-type at most \( \kappa \), and if \( C \in C_\alpha \) and \( \beta \) is a limit point of \( C \), then \( C \cap \beta \in C_\beta \).

These principles are closely connected to properties of trees. If \( \kappa \) is a regular cardinal, \( T \) is a \( \kappa \)-tree when it is a partially ordered set, well-ordered below any given node, such that the rank of any given node is \( < \kappa \), and if \( T_\alpha \) is the set of nodes of rank \( \alpha \), then \( |T_\alpha| < \kappa \). A \( \kappa \)-tree is Aronszajn if it has no cofinal branch, i.e. a linearly ordered subset of order-type \( \kappa \). If \( \kappa = \mu^+ \), a \( \kappa \)-tree \( T \) is special if there is a function \( f : T \to \mu \) such that if \( a <_T b \), then \( f(a) \neq f(b) \). Clearly, special trees are Aronszajn. Jensen [\textbf{14}] showed that \( \square^*_\kappa \) is equivalent to the existence of a special \( \kappa^+ \)-tree.

**Theorem 5.** Suppose \( \mu \) is a regular cardinal, \( 2^{<\mu} \leq \mu^+ \), and \( \text{wCC}(\mu^+, \text{cof}(\mu)) \). Then \( \square^*_\mu \) holds.

**Proof.** We can assume that \( \mu \) is uncountable, since \( \square^*_\kappa \) always holds. Let \( \kappa = \mu^+ \), let \( < \) be a well-order of \( H_{\kappa^+} \), and let \( \mathfrak{A} = (H_{\kappa^+}, \in, <) \). By \( \text{wCC}(\kappa, \text{cof}(\mu)) \), let \( \delta < \kappa \) be such that \( \text{cf}(\delta) = \mu \) and for every \( \alpha < \kappa \), there is \( M \prec \mathfrak{A} \) such that \( M \cap \kappa = \delta \) and \( \text{ot}(M \cap \kappa^+) > \alpha \). Let \( \langle x_\alpha : \alpha < \kappa \rangle \) be the \( < \)-least enumeration of \( [\kappa]^{<\mu} \). Let \( Q_0 = \{ x_\alpha : \alpha < \delta \} \). For each \( M \prec \mathfrak{A} \) with \( M \cap \kappa = \delta \), \( M \cap [\kappa]^{<\mu} = Q_0 \subseteq [\delta]^{<\mu} \).

**Claim 6.** \( Q_0 \) is \( \subseteq \)-cofinal in \( [\delta]^{<\mu} \).

**Proof of claim.** For \( \alpha < \kappa \) let \( \sigma_\alpha \) be the \( < \)-least surjection from \( \mu \) to \( \alpha \). Then for each \( \alpha < \delta \) and each \( M \prec \mathfrak{A} \) with \( M \cap \kappa = \delta \), \( \sigma_\alpha \in M \). If \( x \in [\delta]^{<\mu} \), then since \( \text{cf}(\delta) = \mu \) and \( |x| < \mu \), there is \( \alpha < \delta \) such that \( x \subseteq \alpha \), and there is \( \beta < \mu \) such that \( x \subseteq \sigma_\alpha(\beta) \in Q_0 \).

Let \( \mathcal{E} \) be the set of transitive structures \( N \in H_\kappa \) such that \( N \) is elementarily equivalent to \( H_{\kappa^+} \). \( N \) is correct about cardinals \( \leq \mu \), \( \delta = (\mu^+)^N \), \( \delta \) is the largest cardinal in \( N \), and \( N \cap [\delta]^{<\mu} = Q_0 \). By taking transitive collapses of \( M \prec \mathfrak{A} \) with \( M \cap \kappa = \delta \), we have that for every \( \alpha < \kappa \), there is a \( N \in \mathcal{E} \) with \( \alpha \in N \).

Suppose \( N \in \mathcal{E} \) and \( \alpha \in N \cap \kappa \). Then there is a surjection \( f : \delta \to \alpha \) in \( N \), and \( f \) is coded by a set \( X \subseteq \delta \) in \( N \). More specifically, \( X \) codes, via Gödel pairing, a prewellordering of \( \delta \) of length \( \alpha \), which is inter-constructible with the surjection \( f \). \( X \) has the property that whenever \( z \in Q_0 \), then \( X \cap z \in Q_0 \), since \( X \cap z \in ([\delta]^{<\mu})^N = Q_0 \). Following the terminology of [\textbf{4}], we say that \( X \) is approximated by \( Q_0 \).

Let \( N^* = \text{Sk}^N(\delta + 1) \). Let \( Q_1 = [\delta]^{<\mu} \cap N^* \supseteq Q_0 \). Let \( C^* \in N^* \) be a club in \( \delta \) of order-type \( \mu \). We may assume that only limit ordinals are in \( C^* \).

**Claim 7.** Suppose \( \alpha < \kappa \), and \( X, Y \) are two subsets of \( \delta \) that are approximated by \( Q_0 \) and code prewellorderings of \( \delta \) of length \( \alpha \). Then \( f_X, f_Y \) be the corresponding
surjections from $\delta$ to $\alpha$. Then:

$$\{f_X[z] : z \in Q_1\} = \{f_Y[z] : z \in Q_1\}$$

Proof of claim. Let $r \in Q_1$ be arbitrary. We must find $s \in Q_1$ such that $f_X[r] = f_Y[s]$. Let $\sigma : \mu \rightarrow \delta$ be a surjection in $N^*$. There is a club $C \subseteq \mu$ such that for all $\beta \in C$, $f_X \circ \sigma[\beta] = f_Y \circ \sigma[\beta]$ and $\sigma[\beta]$ is closed under Gödel pairing. Let $\gamma \in C$ be such that $r \subseteq \sigma[\gamma] = z$. Let $z' \in Q_0$ be such that $z \subseteq z'$. Then $X \cap z'$ and $Y \cap z'$ are in $Q_0$, and thus $X \cap z = (X \cap z') \cap z$ and $Y \cap z = (Y \cap z') \cap z$ are in $Q_1$. $X \cap z$ and $Y \cap z$ code prewellorderings of $z$ of length $\eta = \text{ot}(f_X[z]) = \text{ot}(f_Y[z]) < \mu$. Let $g_X$ and $g_Y$ be the corresponding surjections from $z$ to $\eta$.

Let $r' = g_X[r]$. Note that if $\pi : \eta \rightarrow f_X[z]$ is the unique order-preserving map, then $\pi[r'] = f_X[r]$. Let $s = g_Y^{-1}[r']$. Then $s \in Q_1$, and $g_Y[s] = r'$. Moreover, $f_Y[s] = \pi \circ g_Y[s] = f_X[r]$. \qed

For $\alpha < \kappa$ of cofinality $\mu$, let

$$\mathcal{C}_\alpha = \{f[z] : z \in Q_1 \land f[z] \text{ is club in } \alpha\},$$

where $f$ is a surjection from $\delta$ to $\alpha$ in some $N \in \mathcal{E}$. By Claim \[1\] this set is the same no matter which such $f$ we choose. Note that $\mathcal{C}_\alpha$ has size at most $|Q_1| = \mu$. Furthermore, this set is nonempty for each such $\alpha$. To see this, it suffices to show that whenever $\alpha < N \in \mathcal{E}$, then $N \models \text{cf}(\alpha) < \mu$. If this were to fail, then such an $N$ would have an increasing cofinal sequence in $\alpha$ of length either $\mu$ or $\delta$. But thus is impossible, since $\mu$ is regular and $\text{cf}(\delta) = \mu$.

For $\alpha < \kappa$ of cofinality $\mu$, there are two cases. In the first case, there is some $N \in \mathcal{E}$ with $\alpha \in N$ such that $N \models \text{cf}(\alpha) = \mu$. Choose such a model $N_\alpha$, and a club $C_\alpha \subseteq \alpha$ in $N_\alpha$ with $\text{ot}(C_\alpha) = \mu$, and let $\mathcal{C}_\alpha = \{C_\alpha\}$. In the second case, all $N \in \mathcal{E}$ with $\alpha \in N$ satisfy that $\text{cf}(\alpha) = \delta$. Choose such an $N_\alpha$, and choose a club $D_\alpha \subseteq \alpha$ in $N_\alpha$ such that $\text{ot}(D_\alpha) = \delta$. Let $\langle \xi_i : i < \delta \rangle$ be its increasing enumeration. Then let $\mathcal{C}_\alpha = \{\xi_i : i \in C^*\}$, and let $\mathcal{C}_\alpha = \{C_\alpha\}$.

We want to show that the sequence $\langle \mathcal{C}_\alpha : \alpha < \kappa \rangle$ is coherent. Suppose $\text{cf}(\alpha) < \mu$, $C \in \mathcal{C}_\alpha$, and $\beta$ is a limit point of $C$. Let $N \in \mathcal{E}$ be such that $\alpha \in N$, and let $f : \delta \rightarrow \alpha$ be a surjection in $N$ with the property that $C = f[z]$ for some $z \in Q_1$. Let $g : \delta \rightarrow \beta$ be the surjection in $N$ defined by putting $g(\gamma) = f(\gamma)$ when $f(\gamma) < \beta$, and otherwise letting $g(\gamma)$ be the least point of $C$. Then $C \cap \beta = g[z] \in \mathcal{C}_\beta$.

Suppose that $\alpha < \kappa$ and some $N \in \mathcal{E}$ satisfies that $\text{cf}(\alpha) = \mu$. Let $N_\alpha, C_\alpha$ be as in the construction of $\mathcal{C}_\alpha$. Let $f : \delta \rightarrow \alpha$ be a surjection in $N_\alpha$ with the property that $f \upharpoonright \mu$ enumerates $C_\alpha$ in increasing order. Then for any limit point $\beta$ of $C_\alpha$, there is $\gamma < \mu$ such that $C_\alpha \cap \beta = f[\gamma]$. If $g : \delta \rightarrow \beta$ is a surjection in $N_\alpha$ defined from $f$ as in the previous case, then $g[\gamma] = C_\alpha \cap \beta$, and of course $\gamma \in Q_1$, so $C_\alpha \cap \beta \in \mathcal{C}_\beta$.

Suppose that $\alpha < \kappa$ and all $N \in \mathcal{E}$ with $\alpha \in N$ satisfy that $\text{cf}(\alpha) = \delta$. Let $N_\alpha, C_\alpha$ be as in the construction of $\mathcal{C}_\alpha$. Let $f : \delta \rightarrow \alpha$ be a surjection in $N_\alpha$ with the property that $f$ restricted to the limit ordinals enumerates $D_\alpha$ in increasing order. Let $\beta$ be a limit point of $C_\alpha$. Then there is some $\gamma < \delta$ such that $f[C^* \cap \gamma] = C_\alpha \cap \beta$. If $g : \delta \rightarrow \beta$ is a surjection in $N_\alpha$ defined from $f$ as in the previous cases, then $g[C^* \cap \gamma] = C_\alpha \cap \beta$. Since $C^* \cap \gamma \in Q_1$, $C_\alpha \cap \beta \in \mathcal{C}_\beta$. \qed

Suppose $\mu$ is a regular cardinal, $\kappa = \mu^+$, $M \prec H_{\kappa^+}$, $M \cap \kappa = \delta < \kappa$, and $\text{cf}(\delta) = \mu$. For any $\alpha \in M \cap \kappa^+$, if $\alpha \not= \sup(M \cap \alpha)$, then $\alpha$ must have cofinality $\kappa$, since otherwise $\text{cf}(\alpha) \subseteq M$, and so there would be a cofinal subset of $\alpha$ of order-type $\text{cf}(\alpha)$ that is both an element and a subset of $M$. If $f : \kappa \rightarrow \alpha$ is an increasing cofinal function in $M$, then $f \upharpoonright \delta$ is cofinal in $M \cap \alpha$, and so $\text{cf}(M \cap \alpha) = \mu$. Thus if $g : \text{ot}(M \cap \kappa^+) \rightarrow M \cap \kappa^+$ is the increasing enumeration, then the discontinuity points of $g$ all have cofinality $\mu$. \qed
If CC($\kappa$) holds, then stationary-many $M \prec H_{\kappa+}$ have $M \cap \kappa = \delta < \kappa$ with cf($\delta$) = $\mu$, and ot($M \cap \kappa^+$) = $\kappa$. Thus $M \cap \kappa^+$ is a $<\mu$-closed set of ordinals. Such $M$ will have some discontinuity points of cofinality $\mu$, but it is natural to ask whether there might be a club in sup($M \cap \kappa^+$) that avoids them. Let clubCC($\kappa$) stand for the assertion that there are stationary-many such models.

**Proposition 8.** For regular $\mu$, clubCC($\mu^+$) implies $\square^{*}_{\mu}$.

**Proof.** Let $\kappa = \mu^+$. It follows from a result of Shelah [27] that $\kappa^+$ carries a “partial weak square,” i.e. a sequence $\langle C_\alpha : \alpha \in \kappa^+ \cap \text{cof}(\langle <\kappa\rangle) \rangle$ such that each $C_\alpha$ is a set of at most $\kappa$-many clubs in $\alpha$, each of order-type $\leq \mu$, and if $C \in C_\alpha$ and $\beta \in \text{lim}(C)$, then $C \cap \beta \in C_\beta$. Let $\mathbb{A} = \langle H_{\kappa+}, \in, \mathcal{C} \rangle$. Let $M \prec \mathbb{A}$ be such that ot($M \cap \kappa^+$) = $\kappa$, $M \cap \kappa = \delta < \kappa$, and $M \cap \kappa^+$ contains a club $D \subseteq \text{sup}(M \cap \kappa^+)$. We may assume that $D$ possesses only ordinals of cofinality $<\kappa$. Let $\pi : M \rightarrow N$ be the transitive collapse. Let $x = \pi(x)$ for each $x \in M$, and let $D = \pi(D)$.

For each $\alpha \in D$, $C_\alpha$ is defined, and $\pi(C) = \pi[C]$ for each $C \in C_\alpha$. Thus $C_\alpha$ is a set of at most $\mu$-many clubs in $\alpha$, each of order-type at most $\mu$. To complete this to a $\square^*_\mu$-sequence, we only need to fill in the gaps at ordinals in $\kappa \setminus D$. But it is a standard fact, easy to show by induction, that for each $\eta < \kappa$, there exists a “short square sequence” of length $\eta$, i.e. a sequence $\langle C_\alpha : \alpha < \eta \rangle$ such that each $C_\alpha$ is a club in $\alpha$ of order-type at most $\mu$, and if $\beta \in \text{lim}(C_\alpha)$, then $C_\alpha \cap \beta = C_\beta$. Thus for $\alpha \in D$, choose a short square sequence $\langle C_\beta : \alpha < \beta < \alpha' \rangle$, where $\alpha'$ is the next point of $D$ above $\alpha$, and each $C_\beta$ is contained in $[\alpha, \beta)$. Then putting $D_\alpha = C_\alpha$ for $\alpha \in D$ and $D_\alpha = \{C_\alpha\}$ for $\alpha \in \kappa \setminus D$, we have that $(D_\alpha : \alpha < \kappa)$ is a $\square^*_\mu$-sequence.

A weakening of the principle $\square^*_\mu$ is the assertion that $\kappa^+$ is approachable, a notion due to Shelah [29]. More generally, we say that an ordinal $\alpha < \kappa^+$ is approachable with respect to a sequence $\langle x_\beta : \beta < \kappa^+ \rangle$ if there is a cofinal set $y \subseteq \alpha$ of order-type $\text{cf}(\alpha)$ such that all initial segments are in $\{x_\beta : \beta < \alpha\}$, and we say that a set $S \subseteq \kappa^+$ is approachable when there is a sequence $\langle x_\beta : \beta < \kappa^+ \rangle$ and a club $C \subseteq \kappa$ such that all $\alpha \in S \cap C$ are approachable with respect to $\langle x_\beta : \beta < \kappa^+ \rangle$. Shelah’s partial square result [27] shows that if $\kappa$ is regular, then $\kappa^+ \cap \text{cof}(\kappa^+) < \kappa$ is approachable. Moreover, the collection of approachable subsets of $\kappa^+$ forms a normal ideal denoted by $I[\kappa^+]$. We say that $I[\kappa^+]$ is trivial when it is as small as possible under the constraint imposed by Shelah’s partial square result, namely $I[\kappa^+] = NS_{\kappa^+} \cap \text{cof}(\kappa^+).$ If $2^{<\kappa} \leq \kappa^+$, then it is not hard to show using an enumeration of $[\kappa^+]^{<\kappa}$ that $I[\kappa^+]$ is nontrivial, so the triviality of $I[\kappa^+]$ requires $2^{<\kappa} > \kappa^+$. Mitchell [21] showed that it is consistent relative to a greatly Mahlo cardinal that $I[\omega_2]$ is trivial.

**Proposition 9.** If $\mu$ is regular and uncountable and CC($\mu^+$) holds, then $I[\mu^+]$ is nontrivial.

**Proof.** Let $\kappa = \mu^+$. Let $\mathbb{A} = \langle H_{\kappa+}, \in, \mathcal{C} \rangle$, where $\mathcal{C}$ is a well-order of $H_{\kappa+}$. Suppose $M$ and $N$ are elementary in $\mathbb{A}$, $M \cap \kappa = N \cap \kappa = \delta < \kappa$, $\text{sup}(M \cap \kappa^+) = \eta$, and $\text{cf}(\delta), \text{cf}(\eta) \geq \mu$. By the observations preceding Proposition 8, both $M \cap \kappa^+$ and $N \cap \kappa^+$ are $<\mu$-closed. Then $M \cap N \cap \eta$ is an unbounded subset of $\eta$. If $\alpha \in M \cap N \cap \eta$, then there is a surjection $f : \kappa \rightarrow \alpha$ in $M \cap N$. We have that $f[\delta] = M \cap \alpha = N \cap \alpha$. As $\alpha$ was arbitrary, it follows that $M \cap \kappa^+ = N \cap \kappa^+$. Thus for each $\delta < \kappa$ and $\eta < \kappa^+$, each of cofinality $\geq \mu$, there is at most one set $X(\delta, \eta)$ that equals $M \cap \kappa^+$ for some $M \prec \mathbb{A}$ with $M \cap \kappa = \delta$ and $\text{sup}(M \cap \kappa^+) = \eta$. Let $\langle x_\alpha : \alpha < \kappa^+ \rangle$ enumerate all initial segments of such $X(\delta, \eta)$.

Now let $C \subseteq \kappa^+$ be a club. Let $C' \subseteq C$ be a club such that whenever $\alpha \in C'$, $\delta, \eta < \alpha$, and $X(\delta, \eta)$ is defined, then $X(\delta, \eta)$ and all its initial segments are in
$$\{x_\beta : \beta < \alpha\}$$ Expand $\mathfrak{A}$ to $\mathfrak{A}'$ to include a constant for $C'$. By $\text{CC}(\kappa)$, let $M \prec \mathfrak{A}'$ be such that $M \cap \kappa = \delta < \kappa$ and $\text{ot}(M \cap \kappa^+) = \kappa$. Then $\text{sup}(M \cap \kappa^+) \in C'$. By Foreman-Magidor [11], $\text{cf}(\delta) = \mu$. For all $\eta \in M \cap \kappa^+$ of cofinality $\geq \mu$ such that $\text{SK}^\mathfrak{A}(M \cap \eta) \cap \eta = M \cap \eta$, we have $M \cap \eta = X(\delta, \eta)$. Thus $M \cap \kappa^+$ has the property that all initial segments are in $\{x_\alpha : \alpha < \text{sup}(M \cap \kappa^+)\}$. This shows that there is an ordinal in $C$ that is approachable with respect to the sequence $\langle x_\alpha : \alpha < \kappa^+\rangle$. □

The above result cannot be strengthened much further, since in general, $\text{CC}(\kappa)$ does not imply that the full $\kappa^+$ is approachable when $\kappa$ is the successor of a regular cardinal. We give a sketch of the consistency proof for the reader who is familiar with both Mitchell’s forcing for the tree property [20] and the Kunen-style forcings for saturated ideals and Chang’s Conjecture [16].

**Proposition 10.** If $\kappa$ is huge and $\mu < \kappa$ is regular, then there is a forcing extension in which $\kappa = \mu^+$, $\text{CC}(\kappa)$ holds, and $\kappa^+$ is not approachable.

**Proof (sketch).** We first define a variant of Mitchell’s forcing using Easton supports. If $\mu$ is a regular cardinal and $\kappa > \mu$ is inaccessible, then the standard Mitchell forcing to make $\kappa = \mu^+$ consists of pairs $\langle p, q \rangle$, where:

1. $p \in \text{Add}(\mu, \kappa)$.
2. $q$ is a function with domain in $[\kappa]^\kappa$.
3. For each $\alpha \in \text{dom}(q)$, $q(\alpha)$ is a name for a condition in $\text{Col}(\mu^+, \alpha)$, as defined in $V_{\text{Add}(\mu, \kappa)}$.

We put $\langle p_1, q_1 \rangle \leq \langle p_0, q_0 \rangle$ when $p_1 \leq p_0$ and for each $\alpha \in \text{dom}(q)$, $p_1 \upharpoonright q_1(\alpha) \leq q_0(\alpha)$. Variations on this forcing fix some $A \subseteq \kappa$ and require the domain of the second coordinate to be in $[A]^\kappa$. This can change the combinatorial effects of the forcing, such as whether $\kappa$ is forced to be approachable or not (see [6]). In particular, if $\kappa$ is Mahlo and $A$ is the set of inaccessibles below $\kappa$ which are not limits of inaccessibles, then $\kappa \setminus A$ is forced to be non-approachable.

Now, our modification is simply to require that the second coordinate is function with domain an Easton subset of $A$, rather than a $\leq \mu$-sized subset of $A$. Recall that a set of ordinals $X$ is Easton when $\text{sup}(X \cap \alpha) < \alpha$ whenever $\alpha$ is regular. It is not hard to check that the same arguments of [6] work to show that this modification still forces that $\kappa$ is not approachable. Let us call this forcing $\mathcal{P}(\mu, \kappa)$.

It is a standard fact, owing in part to the supports of the functions in the second coordinate, that Mitchell’s forcing is a projection of $\text{Add}(\mu, \kappa) \times \text{Col}(\mu^+, \kappa)$. The same analysis yields that our modified poset is a projection of $\text{Add}(\mu, \kappa) \times \mathcal{E}(\mu^+, \kappa)$, where $\mathcal{E}(\mu^+, \kappa)$ is the Easton-support product of $\text{Col}(\mu^+, \alpha)$ over $\alpha < \kappa$, or the Easton collapse, introduced by Shioya [29].

By the work of the author and Hayut [9], if $\kappa$ is huge with target $\theta$ and $\mu < \kappa$ is regular, then the two-step iteration of Easton collapses, $\mathcal{E}(\mu, \kappa) * \mathcal{E}(\kappa, \theta)$, forces $(\theta, \kappa) \rightarrow (\kappa, \mu)$. The reason is that, if $G * H \subseteq \mathcal{E}(\mu, \kappa) * \mathcal{E}(\kappa, \theta)$ is generic, then a hugeness embedding $j : V \rightarrow M$, with $\text{crit}(j) = \kappa$, $j(\kappa) = \theta$, and $j[\theta] \in M$, can be lifted by a further forcing, yielding an embedding $j : V[G][H] \rightarrow M[G][H']$. Then for any $F : [\theta]^{< \omega} \rightarrow \theta$ in $V[G][H]$, $j(F)$ is a set closed under $j(F)$ when $j(\theta) \cap j(\kappa) = \kappa < j(\kappa)$ and $\text{ot}(j(\theta)) = \theta = j(\kappa)$. By elementarity, there is a set $X \in V[G][H]$ closed under $F$ with $X \cap \kappa \in \kappa$ and $\text{ot}(X) = \kappa$. Since $V[G][H] \models \kappa^{< \kappa} = \mu$, $\text{Add}(\mu, \kappa)$ is $\kappa$-c.c. in this model. Thus by standard arguments, we can lift the embedding further through any $K \subseteq \text{Add}(\mu, \kappa)$ that is generic over $V[G][H]$, and $V[G][H][K]$ will satisfy $(\theta, \kappa) \rightarrow (\kappa, \mu)$ for the same reason.

Let $G \ast Q \subseteq \mathcal{E}(\mu, \kappa) * \mathcal{P}(\mu, \theta)$ be generic. In $V[G][Q]$, $\kappa = \mu^+$, $\theta = \kappa^+$, and $\theta$ is not approachable. Then force with the quotient $(\text{Add}(\mu, \theta) \times \mathcal{E}(\kappa, \theta))/Q$, where the product is as defined in $V[G]$, yielding a generic $G \ast (H \times K)$ for $\mathcal{E}(\mu, \kappa)$.
\[(\mathbb{P}(\kappa, \theta) \times \text{Add}(\mu, \theta)).\] A further forcing yields a lifted embedding \(j : V[G][H][K] \to M[G'][H'][K']\) with \(\text{crit}(j) = \kappa, j(\kappa) = \theta, \text{and } j[\theta] \in M.\) The generic \(H' \times K'\) projects to a generic \(Q'\) for \(j(\mathbb{P}(\mu, \theta))\), and restricting the map yields an elementary \(j : V[G][Q] \to M[G'][Q'][\kappa].\) The same reflection argument as above then gives that \(V[G][Q] \models (\theta, \kappa) \Rightarrow (\kappa, \mu).\)

\[\square\]

4. Modified Neeman Forcing

Let us recall the definition of Neeman’s model sequence poset \([23]\). We fix some transitive set \(K\) satisfying a sufficient amount of ZFC. In our applications, \(K\) will always be \(H_\theta\) for some regular \(\theta\). We fix two classes of elementary submodels of \(K\), the small models \(S\) and the transitive models \(T\), and a cardinal \(\kappa\) with the following properties:

- \(S \cup T \subseteq K.\)
- \(\kappa + 1 \subseteq M\) for all \(M \in S \cup T.\)
- If \(M, N \in S\) and \(M \in N\), then \(M \subseteq N.\)
- If \(W \in T, M \in S,\) and \(W \in M,\) then \(M \cap W \in S \cap W.\)
- Each \(W \in T\) is transitive and \(<\kappa\)-closed.

A pair \(\langle S, T \rangle\) satisfying these conditions is called appropriate for \(\kappa\) and \(K\). Usually these conditions are implied by defining \(S\) and \(T\) such that for some regular cardinal \(\lambda \geq \kappa\), every \(M \in S\) has \(M \cap \lambda \in \lambda\) and \(|M| < \lambda,\) while every \(W \in T\) is \(<\lambda\)-closed. The poset \(\mathbb{P}_{\kappa, S, T}\) consists of sets of models \(s \in [S \cup T]^{\kappa}\) such that the rank function is injective on \(s,\) and if \(\langle M_i : i < \alpha \rangle\) enumerates \(s\) by order of rank, then for each \(\beta, \gamma < \alpha: \)

- \(\{i < \beta : M_i \in M_\beta\}\) is cofinal in \(\beta.\)
- \(s \cap M_\beta \in M_\beta.\)
- \(M_\beta \cap M_\gamma = M_i\) for some \(i < \alpha.\)

A condition \(t\) is stronger than a condition \(s\) when \(t \supseteq s.\) Neeman also introduced a decorated version of this poset, which enforces some continuity of the generic object added. We will slightly modify these decorations as follows. We define \(\mathbb{P}^{\text{dec}^+}_{\kappa, S, T}\) to consist of pairs \(\langle s, f \rangle\) where:

1. \(s \in \mathbb{P}_{\kappa, S, T}.\)
2. \(f\) is a function with \(\text{dom}(f) \subseteq [s \times \kappa]^{\kappa}.\) If \(\langle M, \alpha \rangle \in \text{dom}(f),\) then \(f(M, \alpha) \in M^* \cap W,\) where \(M^*\) is the successor of \(M\) in \(s\) if it exists, \(M^* = K\) if \(M\) is the largest model in \(s,\) and \(W\) is the smallest member of \(T\) such that \(M \in W.\)
3. \(f \in K,\) and for each \(M \in s, f \upharpoonright M \in M.\)

In order for (2) to make sense, we assume that \(T\) is linearly ordered by \(\subseteq\) and \(\subseteq\)-cofinal in \(K.\) We note that by Claim 2.34 of \([23]\), if \(\langle s, f \rangle\) is a condition and \(M \in s,\) then \(\text{ran}(f \upharpoonright M) \subseteq M,\) so (3) is equivalent to saying that \(f \cap M \in M\) for all \(M \in s.\) We put \(\langle t, g \rangle \leq \langle s, f \rangle\) when \(s \subseteq t\) and \(f \subseteq g.\) For a model \(M \in S \cup T\) and a condition \(p = \langle s, f \rangle\), we sometimes write \(p \upharpoonright M\) for \(\langle s \cap M, f \cap M \rangle.\)

Essentially, the only differences between our decorated poset and that of \([23]\) are that we consider a smaller class of decorations, namely those \(<\kappa\)-sized sets that are partial functions on \(\kappa,\) and we don’t allow the decorations attached to a model \(M\) to jump past the next transitive model above \(M.\) One can check that the proof of strong properness for \(S \cup T,\) in particular Claim 2.38 of \([23]\), still holds for this modified poset. (Almost no change to the argument is needed to accommodate our modified decorations.) Let us state this result:

**Lemma 11.** Suppose \(\langle s, f \rangle \in \mathbb{P}^{\text{dec}^+}_{\kappa, S, T}\) and \(M \in s.\) Then \(\langle s \cap M, f \cap M \rangle\) is a condition, and if \(\langle t, g \rangle \in M\) and \(\langle t, g \rangle \leq \langle s \cap M, f \cap M \rangle,\) then \(\langle t, g \rangle\) is compatible.
with \((s,f)\). Furthermore, these conditions have a greatest lower bound \((r,h)\), where \(r\) is the closure of \(s \cup t\) under intersections, \(r \cap M = t\), and \(h = f \cup g\).

When we take a generic filter \(G\) for \(P_{\kappa,S,T}\) or \(P_{\kappa,S,T}^{\text{dec}}\), we will say that a model \(M\) "appears in \(G\)" to mean, if we are using the undecorated poset, that there is some \(s \in G\) with \(M \in s\), and if we are using the decorated poset, that there is some \((s,f) \in G\) with \(M \in s\).

As noted in \([23]\), if each \(M \in S\) is \(<\kappa\)-closed then the whole forcing is \(<\kappa\)-closed. The key points of these modifications are as follows:

\(1) \) The union of the decorations appearing in a generic that are attached to a model \(M\) will be a surjection from \(\kappa\) to the successor model \(M'\), which in typical situations must be a small model.

\(2) \) If \(K = H_\theta\), \(\theta^{<\alpha} = \theta\), and \(T\) is sufficiently rich, then the poset satisfies the \(\theta\)-c.c. This does not hold for the version of the decorated poset appearing in \([23]\), since, for \(\kappa = \omega\), it adds a club subset of \(\theta\) that contains no infinite ground model set.

\(3) \) In typical situations, the sequence of small models appearing between two consecutive transitive models will be continuous at limits of cofinality \(\kappa\).

Suppose \(K = H_\theta\) for some regular \(\theta\). Then \(\theta^{<\alpha} = \theta\) if and only if there is a continuous \(\epsilon\)-increasing sequence of transitive sets \(\dot{W} = \{W_\alpha : \alpha < \theta\}\) such that each \(W_\alpha \in H_\theta\) and \(\bigcup_{\alpha < \theta} W_\alpha = H_\theta\). Let us call such a sequence a filtration of \(H_\theta\).

In case \(\theta\) is an inaccessible cardinal, then we can simply take \(W_\alpha = V_\alpha\). Otherwise, it will be useful to take \(\dot{W}\) as a predicate and consider collections of elementary submodels of \((H_\theta, \epsilon, \dot{W})\).

When \(\alpha^{<\kappa} < \theta\) for each \(\alpha < \theta\), each \(W_\alpha < (H_\theta, \epsilon, \dot{W})\) with \(\text{cf}(\alpha) \geq \kappa\) is \(<\kappa\)-closed. This is because for any \(x \in [W_\alpha]^{<\kappa}\), there is some \(\beta < \alpha\) such that \(x \subseteq W_\beta\), and \([W_\beta]^{<\kappa} \subseteq W_\alpha\) by elementarity. Thus the set of such \(W_\alpha\)'s can be part of an appropriate pair for \(\kappa\) and \(H_\theta\).

**Lemma 12.** Suppose \(\theta\) is regular, \(\dot{W} = \{W_\alpha : \alpha < \theta\}\) is a filtration of \(H_\theta\), \(\mathfrak{A} = (H_\theta, \epsilon, \dot{W})\), \(M < \mathfrak{A}\), and \(\alpha \in M\). Then \(M \cap W_\alpha \prec \mathfrak{A}\) if and only if \(W_\alpha \prec \mathfrak{A}\). Furthermore, if \(W_\beta \prec \mathfrak{A}\) and \(\alpha = \min(M \setminus \beta)\), then \(W_\alpha \prec \mathfrak{A}\).

**Proof.** We use the Tarski-Vaught criterion. Suppose \(W_\alpha \prec \mathfrak{A}\). Let \(a \in M \cap W_\alpha\) and suppose \(\mathfrak{A} \models \exists x \varphi(a, x)\). Then \(W_\alpha \models \exists x \varphi(a, x)\) and \(M \models (\exists x \in W_\alpha) \varphi(a, x)\).

Thus there is \(b \in M \cap W_\alpha\) such that \(\mathfrak{A} \models \varphi(b, a)\).

Now suppose \(M \cap W_\alpha \prec \mathfrak{A}\). If \(W_\alpha \not\prec \mathfrak{A}\), then there is some \(a \in W_\alpha\) and a formula \(\varphi(x, y)\) such that \(\mathfrak{A} \models \exists x \varphi(x, a)\), but a witness cannot be found in \(W_\alpha\). Then

\[ M \models (\exists y \in W_\alpha)(\exists x \varphi(x, y) \land (\forall z \in W_\alpha) \neg \varphi(z, y)) \]

Let \(b \in M \cap W_\alpha\) witness the outermost quantifier. By our suppositions, there is \(c \in M \cap W_\alpha\) such that \(\mathfrak{A} \models \varphi(c, b)\). This contradicts that \(\mathfrak{A} \models (\forall z \in W_\alpha) \neg \varphi(z, b)\).

Now suppose \(W_\beta \prec \mathfrak{A}\) and \(\alpha = \min(M \setminus \beta)\). If \(W_\alpha \not\prec \mathfrak{A}\), there is \(a \in W_\alpha\) and a formula \(\varphi(x, y)\) such that \(\mathfrak{A} \models \exists y \varphi(a, y) \land (\forall y \in W_\alpha) \neg \varphi(a, y)\).

By elementarity, there is \(b \in M \cap W_\alpha = M \cap W_\beta\) such that \(M \models (\exists y \in W_\alpha) \neg \varphi(a, y)\). Since \(W_\beta \prec \mathfrak{A}\), there is \(c \in W_\beta \subseteq W_\alpha\) such that \(\mathfrak{A} \models \varphi(b, c)\). But this contradicts that \(\mathfrak{A} \models (\forall y \in W_\alpha) \neg \varphi(b, y)\).

Let us say that a tuple \((\kappa, \lambda, \theta, \dot{W}, S, T)\) is nice when:

\(1) \) \(\kappa < \lambda < \theta\) are regular cardinals;

\(2) \) \(\alpha^{<\kappa} < \theta\) for each \(\alpha < \theta\);

\(3) \) \(\dot{W} = \{W_\alpha : \alpha < \theta\}\) is a filtration of \(H_\theta\);

\(4) \) \(S\) is a set of \(M \prec \mathfrak{A} = (H_\theta, \epsilon, \dot{W})\) such that each \(M \in S\) satisfies \(|M| < \kappa\) and \(M \cap \lambda \in \lambda\);

\(5) \) \(T = \{W_\alpha \prec \mathfrak{A} : \text{cf}(\alpha) \geq \lambda\}\);
(6) $\langle S, T \rangle$ is appropriate for $\kappa$ and $\frak A$.

The following is an extension of Claim 5.7 of \cite{23}:

**Lemma 13.** Suppose $\langle \kappa, \lambda, \theta, \bar{W}, S, T \rangle$ is nice. Let $\mathbb{P}$ be either $\mathbb{P}_{\kappa, S, T}$ or $\mathbb{P}^{\text{dec}}_{\kappa, S, T}$. Then for any $p \in \mathbb{P}$ and $W \in T$, $p$ can be extended to include the model $W$.

**Proof.** We give the proof for the decorated poset; the other case is just slightly simpler. Let $\langle s, f \rangle \in \mathbb{P}$ and let $W_{\alpha} \in T$. First suppose $s \subseteq W_{\alpha}$. Since for each $\langle M, \alpha \rangle \in \text{dom}(f)$, $f(M, \alpha)$ is required to be in the smallest $W \in T$ such that $M \in W$, $f \subseteq W_{\alpha}$. Since $W_{\alpha}$ is $<\kappa$-closed, $\langle s, f \rangle \in W_{\alpha}$. Thus $\langle s \cup \{W_{\alpha}\}, f \rangle$ is a condition witnessing the claim. If the claim fails for $W_{\alpha}$, then it must be witnessed by $\langle s, f \rangle$ with $s \not\subseteq W_{\alpha}$. Let us assume that $\langle s, f \rangle$ is a witness to failure with $\min(\text{rank}(M) : M \in s \setminus W_{\alpha})$ as small as possible. If $M$ is of minimal rank $\geq \alpha$ in $s$, then $\text{rank}(M) > \alpha$. If $\alpha \in M$, then $\langle s \cap M \cup \{W_{\alpha}\}, f \cap M \rangle$ is a condition in $M$ below $\langle s \cap M, f \cap M \rangle$, since $f \cap M \subseteq W_{\alpha}$. Thus by Lemma \[11\] it is compatible with $\langle s, f \rangle$.

Therefore, we must have $\alpha \notin M$. Thus $M$ must be in $S$. Let $\beta = \min(M \cap \theta \setminus \alpha)$. Since $M$ is not cofinal in $\beta$, $\text{cf}(\beta) \geq \lambda$. By Lemma \[12\] $W_{\beta} \in T$. Thus $\langle s \cap M, f \cap M \rangle$ can be extended in $M$ to include $W_{\beta}$. By Lemma \[11\] there is a condition $p \leq \langle s, f \rangle$ that includes $W_{\beta}$. By the minimality assumption on $\langle s, f \rangle$, there is $q \leq p$ that includes $W_{\alpha}$. This contradicts that $\langle s, f \rangle$ witnesses the failure of the claim. \[\square\]

**Corollary 14.** Under the hypotheses of Lemma \[13\] $\mathbb{P}$ is $\theta$-c.c.

**Proof.** Suppose that $\mathcal{A} \subseteq \mathbb{P}$ is a maximal antichain. Let $W \in T$ be such that $\mathcal{A} \cap W$ is maximal in $\mathbb{P} \cap W$. Let $p \in \mathbb{P}$ be arbitrary, and extend it to $q$ that includes $W$. There is $a \in \mathcal{A} \cap W$ that is compatible with $q \mid W$. By Lemma \[11\] $a$ is compatible with $q$. Thus $\mathcal{A} \subseteq W$. \[\square\]

**Lemma 15.** Suppose $\langle \kappa, \lambda, \theta, \bar{W}, S, T \rangle$ is nice and $S$ is stationary. Let $G \subseteq \mathbb{P}^{\text{dec}}_{\kappa, S, T}$ be generic, let $W_0 \in W_1$ be two consecutive transitive models of $T$, and let $\langle W_0, W_1 \rangle_G$ be the set of models appearing in $G$ between $W_0$ and $W_1$. Then $\langle W_0, W_1 \rangle_G$ is a subset of $S$ that is:

1. linearly ordered by both $\subseteq$ and $\subseteq$;
2. $\subseteq$-cofinal in $W_1$; and
3. continuous at limit points of $V$-cofinality at least $\kappa$.

**Proof.** The linearity of $\langle W_0, W_1 \rangle_G$ follows from \cite{23} Claim 2.10. To show that it is cofinal in $W_1$, let $\langle s, f \rangle$ be a condition such that $W_0, W_1 \in s$ and let $x \in W_1$. By the stationarity of $S$, there is $M \in S$ such that $\{\langle s, f \rangle, x \} \in M$. Applying Lemma \[11\] there is $\langle t, g \rangle \leq \langle s, f \rangle$ with $M \in t$. We must have $x \in M \cap W_1 \in t$. Thus genercity implies that the union of $\langle W_0, W_1 \rangle_G$ is $W_1$.

Now suppose that $\langle s, f \rangle$ forces that $M$ is a model in the interval $\langle W_0, W_1 \rangle_G$ whose index in the increasing enumeration following its $\subseteq$-ordering is a limit ordinal $\eta$ such that $\text{cf}^V(\eta) \geq \kappa$. Let $x \in M$ be arbitrary. Since there are no transitive models between $W_0$ and $M$, $s \cap M$ includes the interval $\langle W_0, M \rangle$. Since $|s| < \kappa$, there is $\langle t, g \rangle \leq \langle s, f \rangle$ such that $\langle W_0, M \rangle$ is contained in a strict initial segment of $\langle W_0, M \rangle$. Let $N$ be the first model of $t$ above $\langle W_0, M \rangle$. Let $r \in M$ be the initial segment of $t \cap M$ up to and including $N$. Then $r \in \mathbb{P}_{\kappa, S, T} \cap M$. Let $h$ be the function $g \upharpoonright (r \times \kappa) \cup \{\langle (M, 0), x \rangle \}$. Then $\langle r, h \rangle$ is a condition in $M$ below $\langle s \cap M, f \cap M \rangle$. By Lemma \[11\] it is compatible with $\langle s, f \rangle$. For any condition $\langle s', f' \rangle$ below both $\langle r, h \rangle$ and $\langle s, f \rangle$ in which $N$ is not the largest model below $M$, $x$ is a member of the successor of $N$ in $s'$. Since $\langle s, f \rangle$ was an arbitrary condition forcing $M$ to be at place $\eta$ in $\langle W_0, W_1 \rangle_G$, it follows that models appearing at such places must be the union of the models in $\langle W_0, W_1 \rangle_G$ appearing below them. \[\square\]
Lemma 16. If \( (\kappa, \lambda, \theta, W, S, T) \) is nice and \( S \) is stationary, then \( P_{\kappa, S, T}^{\text{dec}} \) forces that \( \lambda = \kappa^+ \) and \( \theta = \lambda^+ \).

Proof. Since \( P = P_{\kappa, S, T}^{\text{dec}} \) is strongly proper for \( S \cup T \), the regularity of \( \lambda \) and \( \theta \) are preserved. Since \( S \) is stationary, for every \( p \in P \), and every \( \alpha < \lambda \), there is \( M \in S \) such that \( \alpha, p \in M \). Let \( q \leq p \) be such that \( M \) appears in \( q \), \( M \) has a successor \( N \) in \( q \), and \( q \) forces that \( N \) is the next model above \( M \) appearing in the generic. Then a density argument shows that \( q \) forces that the union of the decorations attached to \( M \) will be a surjection from \( \kappa \) to \( N \). Thus \( P \) forces that \( |\alpha| \leq \kappa \). Now let \( \alpha < \theta \) and \( p \in P \) be arbitrary. Let \( W \in T \) be such that \( \alpha, p \in W \). Let \( W' \) be the next transitive model above \( W \). Then by Lemma 15 \( P \) forces that \( W' \) is the union of a \( \subseteq \)-increasing chain of sets of size \( \lambda \), so it forces that \( |\alpha| \leq \lambda \).

The remainder of this section borrows ideas from [32]. Suppose \( (\kappa, \lambda, \theta, W, S, T) \) is nice, and let \( P \) be either \( P_{\kappa, S, T} \) or \( P_{\kappa, S, T}^{\text{dec}} \). Suppose \( G \subseteq P \) is generic over \( V \). For notational convenience, let \( W_0 = \emptyset \) and \( W_\theta = H_\theta \). Let us say \( \delta \leq \theta \) is relevant if either \( \delta = 0 \), \( \delta = \theta \), or \( \delta < \theta \). We define a decreasing sequence of sets \( S^\delta \) for relevant \( \delta \leq \theta \): \( S^\delta = \{ M \in S : M \cap W_\delta \text{ appears in } G \cap W_\delta \} \). Note that for \( \delta < \theta \), \( G \cap W_\delta \) is \( (\mathbb{P} \upharpoonright W_\delta) \)-generic over \( V \) by strong properness, and \( S^\delta \subseteq G \cap W_\delta \).

Lemma 17. Suppose \( G \subseteq P \) is generic. If \( \delta \) is relevant and \( S' \subseteq S^\delta \) is stationary in \( V[G \cap W_\delta] \), then \( S' \cap S^\delta \) is stationary in \( V[G] \).

Proof. Suppose \( G, \delta, \) and \( S' \) are as hypothesized. Let \( F : W_\delta^{\text{c}}_\theta \to W_\theta \) be a function in \( V[G] \). Work in \( V[G \cap W_\delta] \). Let \( F \) be a \((\mathbb{P} \upharpoonright (G \cap W_\delta))\)-name for \( F \), and let \( p_0 \in \mathbb{P} \upharpoonright (G \cap W_\delta) \) be arbitrary. Let \( \theta^* > \theta \) be regular and let \( A \) be the structure \((H_{\theta^*}, \epsilon, P, G \cap W_\delta, p_0, F) \) (as defined in \( V[G \cap W_\delta] \)). Let \( N < A \) be such that \( N \cap W_\theta = M \in S' \). Now go back to \( V \). Let \( s \) be the set of models of \( p_0 \). By Lemma 11 (or [23] Corollary 2.32), there is \( p_1 \leq p_0 \in \mathbb{P} \) in which \( M \) appears, and if \( t \) is the set of models of \( p_1 \), then \( t \) is the closure of \( s \cup \{ M \} \) under intersections, and if we are using the decorated poset, then the decorating function is unchanged. By hypothesis, \( M \cap W_\delta \) appears in \( G \cap W_\delta \), so \( p_1 \upharpoonright W_\delta \in G \cap W_\delta \), since \( p_1 \upharpoonright W_\delta \) is the weakest condition extending \( p_0 \upharpoonright W_\delta \) in which \( M \cap W_\delta \) appears. \( p_1 \) is a strong master condition for \( M \), and thus it forces that \( M \) is closed under \( F \). By the arbitrariness of \( p_0 \), it is forced by \( \mathbb{P} \upharpoonright (G \cap W_\delta) \) that there is a model in \( S' \cap S^\delta \) that is closed under \( F \).

Lemma 18. Suppose \( (\kappa, \lambda, \theta, W, S, T) \) is nice and \( S \) is stationary. Assume also that \( P_{\kappa, S, T}^{\text{dec}} \) preserves the regularity of \( \kappa \). If \( G \subseteq P_{\kappa, S, T}^{\text{dec}} \) is generic over \( V \), then in \( V[G] \), \( \text{NS}_\lambda \land \text{cof}(\kappa) \) is the canonical projection of \( \text{NS} \upharpoonright S^\delta \). Equivalently, for every stationary \( A \subseteq \lambda \land \text{cof}(\kappa) \) in \( V[G] \), \( \{ M \in S^\delta : M \cap \lambda \in A \} \) is stationary.

Proof. Suppose \( A \subseteq \lambda \land \text{cof}(\kappa) \) is a stationary set in \( V[G] \). Let \( \delta \) be such that \( W_\delta \in T \) and \( A \in V[G \cap W_\delta] \). Let \( S^\delta(A) = \{ M \in S^\delta : M \cap \lambda \in A \} \). By Lemma 17, it suffices to show that \( S^\delta(A) \) is stationary in \( V[G \cap W_\delta] \). Note that for all \( \eta \) such that \( \delta < \eta < \theta \) and \( W_\eta \in T \), \( \{ M \cap W_\eta : M \in S^\delta(A) \} = S^\delta(A) \cap W_\eta \). If \( \delta < \eta < \eta' < \theta \) and \( S^\delta(A) \cap W_{\eta'} \) is a stationary subset of \( P_\lambda(W_{\eta'}) \), then \( S^\delta(A) \cap W_\eta \) is a stationary subset of \( P_\lambda(W_\eta) \).

Suppose towards a contradiction that \( S^\delta(A) \) is nonstationary in \( V[G \cap W_\delta] \), and let \( F : W_{\delta}^{\omega} \to W_\theta \) be a function in \( V[G \cap W_\delta] \) such that no \( M \in S_{\delta}^G \) is closed under \( F \). There is \( \eta \) such that \( \delta < \eta < \theta \) and \( W_\eta \) is closed under \( F \), and thus \( F \upharpoonright W_\eta \) witnesses that \( S^\delta(A) \cap W_\eta \) is nonstationary in \( P(W_\eta) \). Hence \( S^\delta(A) \cap W_{\eta'} \) is nonstationary for \( \eta \leq \eta' < \theta \). Now let \( W_{\eta'} \) be the successor of \( W_\eta \) in \( T \). By Lemma 15, the models in \( (W_\eta, W_{\eta'}) \cap \) form a chain of length \( \lambda \) that is \( \subseteq \) and \( \subseteq \)-increasing and continuous at points of cofinality \( \kappa \). In \( V[G] \), let \( \{ M_\alpha : \alpha < \lambda \} \)
enumerate this chain. Since \( A \) is stationary in \( V[G] \), \( \{ \alpha \in A : M_\alpha \cap \lambda = \alpha \} \) is also stationary in \( V[G] \). But this means that \( S^G_\delta(A) \cap W_\nu \) is stationary in \( V[G] \), a contradiction. \( \square \)

5. END-EXTENDING CARDINALS AND \( \text{NS}_{\omega_1} \)

In this section, we investigate a relatively weak large cardinal notion and its forcing applications for the nonstationary ideal on \( \omega_1 \).

For two sets \( M, N \) we say that \( N \) is an end-extension of \( M \), or \( M \subseteq N \), when \( N \cap \text{sup}(M \cap \text{Ord}) = M \cap \text{Ord} \). Suppose \( \kappa \) is a cardinal and \( S \subseteq [\kappa]^{<\kappa} \). We say that \( S \) admits gap end-extensions when for every structure \( \mathfrak{A} \) on \( \kappa \) in a countable language, there is an expansion \( \mathfrak{A}^* \) of \( \mathfrak{A} \), also in a countable language, such that for every \( M \in S \) elementary in \( \mathfrak{A}^* \), there are cofinally many \( \alpha < \kappa \) such that for some \( N < \mathfrak{A}^*, M \cup \{ \alpha \} \subseteq N \) and \( N \cap \alpha = M \). For a cardinal \( \mu \leq \kappa \), we say that \( \kappa \) is \( \mu \)-end-extending when \( [\kappa]^{<\mu} \) admits gap end-extensions, and we say \( \kappa \) is end-extending when it is \( \kappa \)-end-extending.

Lemma 19. Suppose \( \mu < \kappa \) and \( S \subseteq [\kappa]^{<\kappa} \) admits gap end-extensions. Then for all structures \( \mathfrak{A} \) on \( \kappa \) and all clubs \( C \subseteq \kappa \), there is an expansion \( \mathfrak{A}^* \) such that for all \( M < \mathfrak{A}^* \) with \( M \in S \) and \( M \cap \mu \in \mu \), there are cofinally many \( \alpha \in C \) such that \( \text{cf}(\alpha) \geq \mu \) and \( \text{Sk}^\mathfrak{A}^*(M \cup \{ \alpha \}) \cap \alpha = M \).

Proof. Let \( f : \kappa^2 \rightarrow \kappa \) be such that if \( \text{cf}(\alpha) \leq \beta \), then \( \{ f(\alpha, \gamma) : \gamma < \beta \} \) is cofinal in \( \alpha \). Let \( \mathfrak{A} \) be any structure on \( \kappa \) with \( f, C \), and \( \mu \) in its language. Then for \( \alpha < \kappa \), \( \text{cf}(\alpha) \) is definable in \( \mathfrak{A} \) as the least \( \beta \) such that for all \( \gamma < \alpha \), there is \( \delta < \beta \) with \( f(\alpha, \delta) \geq \gamma \). Let \( \mathfrak{A}^* \) be an expansion witnessing that \( S \) admits gap end-extensions.

Suppose \( M < \mathfrak{A}^* \) is in \( S \) and \( M \cap \mu \in \mu \). For \( \alpha < \kappa \), let \( N_\alpha = \text{Sk}^\mathfrak{A}^*(M \cup \{ \alpha \}) \). If \( \alpha > \text{sup}(M) + 1 \) is such that \( N_\alpha \cap \alpha = M \), then \( \text{cf}(\alpha) \geq \mu \), because otherwise, \( \text{cf}(\alpha) \in N_\alpha \cap \mu \) and \( \{ f(\alpha, \delta) : \delta < \text{cf}(\alpha) \} \) is cofinal in \( \alpha \) and contained in \( N_\alpha \). Also, if \( N_\alpha \cap \alpha = M \), then \( N_\alpha \models "C \text{ is unbounded in } \alpha," \) so \( \alpha \in C \) by elementarity. \( \square \)

Proposition 20. If \( \mu \) is regular, \( \mu < \kappa \), and \( \kappa \) is \( \mu \)-end-extending, then \( \text{cf}(\kappa) > \mu \).

If \( \kappa \) is end-extending, then \( \kappa \) is weakly inaccessible.

Proof. Suppose \( \mu \) is regular, \( \mu < \kappa \), and \( \kappa \) is \( \mu \)-end-extending. Towards a contradiction, suppose that \( \text{cf}(\kappa) \leq \mu \). Let \( C \subseteq \kappa \) be a club consisting of ordinals of cofinality \( < \mu \). By the previous lemma, there is a set \( M \in \mathcal{P}_\mu(\kappa) \) and an \( \alpha \in C \) above \( \text{sup}(M) \) such that \( \text{cf}(\alpha) \geq \mu \). This is a contradiction.

Suppose \( \kappa \) is end-extending. Then it is \( \mu \)-end-extending for all \( \mu < \kappa \), so by the previous paragraph, \( \kappa \) is regular. Note that \( \kappa > \omega_1 \), since we can take \( \mathfrak{A} \) on \( \omega_1 \) such that all substructures of \( \mathfrak{A} \) are transitive, while admitting gap end-extensions requires non-transitive substructures. If \( \kappa = \nu^+ \), let \( f(x, y) \) be a function such that for all \( \alpha < \kappa \), \( f(\alpha, \cdot) \) is an injection of \( \alpha \) into \( \nu \). Let \( \mathfrak{A} \) incorporate \( f \) in its language, and let \( \mathfrak{A}^* \) be given by the hypothesis. Let \( M < \mathfrak{A}^* \) be such that \( \nu \in M \) and \( |M \cap \nu| < \nu \). We can inductively build \( N < \mathfrak{A}^* \) such that \( N \) is cofinal in \( \kappa \) and \( M \subseteq N \), by applying end-extendibility at successor stages and taking unions at limits. Let \( \alpha \in N \) be such that \( |N \cap \alpha| \geq \nu \). Then \( f(\alpha, \cdot) \) injects \( N \cap \alpha \) into \( N \cap \nu \), which is a contradiction since \( N \cap \nu = M \cap \nu \in [\nu]^{<\nu} \). \( \square \)

Proposition 21. Suppose \( \kappa \) is \( \mu \)-end-extending.

1. For all infinite cardinals \( \nu < \delta < \kappa \), \( (\kappa, \delta) \rightarrow (\mu, \nu) \).
2. \( 0^\mu \) exists.
3. If \( \mu = \kappa \), then \( \kappa \) is Rowbottom and weakly \( \kappa \)-Mahlo.

Proof. For the first item, let \( \mathfrak{A} \) be any structure on \( \kappa \) and let \( \mathfrak{A}^* \) be an expansion witnessing that \( \kappa \) is \( \mu \)-end-extending. Let \( M < \mathfrak{A}^* \) be such that \( \delta \in M \) and
$|M \cap \delta| = \nu$. Build a strictly $\subseteq$-increasing continuous sequence $\langle M_i : i \leq \mu \rangle$ with $M_0 = M$. Then $M_\mu \prec \mathfrak{A}$, $|M_\mu \cap \kappa| = \mu$, and $|M_\mu \cap \delta| = \nu$.

The second item follows from $(\kappa, \delta) \to (\mu, \nu)$, using Theorem 18.27 and the argument for Corollary 18.29 in [13].

For the third item, let $c$ be any coloring of the finite subsets of $\kappa$ in $\lambda$-many colors, $\lambda < \kappa$, and let $\mathfrak{A}$ be a structure incorporating $c$ into its language, such that every submodel of size $< \kappa$ can be end-extended. By repeatedly end-extending a countable $M \prec \mathfrak{A}$ with $\lambda \in M$, we obtain a subset of $\kappa$ of size $\kappa$ on which $c$ takes only countably many values. This implies that $\kappa$ is a regular Rowbottom cardinal, and by Shelah [23], that it is weakly $\kappa$-Mahlo. 

The following is inspired by arguments in [3] and [25].

**Proposition 22.** Suppose $\mathcal{I}$ is a $\kappa$-complete ideal on $\kappa$, $S^* \subseteq [H(2^{2^{\kappa}})]^{< \kappa}$ is stationary, $\mathcal{P}(\kappa)/\mathcal{I}$ is $S^*$-proper, and $\mathcal{S} = \{ M \cap \kappa : M \in S^* \}$. Then $\mathcal{S}$ admits gap end-extensions.

**Proof.** First we claim that $\mathcal{I}$ is precipitous. Let $X$ be an $\mathcal{I}$-positive set, and suppose $X \models \langle \tau_n : n \in \omega \rangle$ is a descending sequence of ordinals in the generic ultrapower. Let $D_0$ be the dense set of conditions deciding $\tau_n = \bar{f}$ for some function $f$ on $\kappa$. Let $\mathfrak{A} = \langle H(2^{2^\kappa})^+, \in, \prec, \mathcal{I}, X, \bar{f} \rangle$, where $\prec$ is a well-order, and let $M \prec \mathfrak{A}$ be in $S^*$. Let $Y \subseteq X$ be $(M, \mathcal{P}(\kappa)/\mathcal{I})$-generic.

Note that for each dense set $D \subseteq \mathcal{P}(\kappa)/\mathcal{I}$ in $M$, $Y \setminus \bigcup (M \cap D) \in \mathcal{I}$. Otherwise, there is a $\mathcal{I}$-positive $Z \subseteq Y$ such that $Z \cap A = \emptyset$ for each $A \in D \cap M$. But this contradicts that $Z$ is $(M, \mathcal{P}(\kappa)/\mathcal{I})$-generic. By $\kappa$-completeness, there is $\alpha \in Y$ such that for each dense $D \in M$, there is $A \in D \cap M$ with $\alpha \in A$. Let $\mathcal{U} = \{ A \in \mathcal{P}(\kappa) : \alpha \in A \}$. Then $\mathcal{U}$ is an $(M, \mathcal{P}(\kappa)/\mathcal{I})$-generic ultrapower containing $X$. For each $n$, let $f_n$ be a function such that some $A \in D_n \cap \mathcal{U}$ decides $\tau_n = f_n$. By the statement forced by $X$, $\{ \beta : f_{n+1}(\beta) \in f_n(\beta) \} \in \mathcal{U}$ for each $n$. So $\langle f_n(\alpha) : n \in \omega \rangle$ is a descending sequence of ordinals, a contradiction.

Now let $\mathfrak{A}$ be a structure on $\kappa$ let $\xi < \kappa$. Let $\mathfrak{B} = \langle H(2^{2^\kappa})^+, \in, \prec, \mathcal{I}, \mathfrak{A} \rangle$, where $\prec$ is a well-order. Let $\mathfrak{A}^*$ be the result of restricting the Skolem functions of $\mathfrak{B}$ to $\kappa$. Let $N^* \in S^*$ be elementary in $\mathfrak{B}$, and let $G$ be a $\mathcal{P}(\kappa)/\mathcal{I}$-generic filter with a master condition for $N^*$. Let $N = N^* \cap \kappa < \mathfrak{A}^*$.

Let $j : V \to M$ be the generic ultrapower embedding. Since $|N^*| < \kappa$, $j(N^*) = j[\mathfrak{B}]$. Suppose $\alpha$ is an ordinal in $Q = \text{Sk}^{\mathfrak{B}}(j(N^*) \cup \{ \kappa \})$. Then there is a function $f : \kappa \to \text{Ord}$ in $N^*$ such that $\alpha = j(f)(\kappa)$. Let $\tau \in N^*$ be a $\mathcal{P}(\kappa)/\mathcal{I}$-name for $j_G(f)(\kappa)$. Since $G$ has a master condition for $N^*$, $\tau^G \in N^*$. Since $j(N^*) \cap \kappa = N^* \cap \kappa = N$, we have $Q \cap \kappa = N$. By elementarity, there is $\beta$ such that $\xi < \beta < \kappa$ and $\text{Sk}^{\mathfrak{A}^*}(N \cup \{ \beta \}) \cap \beta = N$.

It follows that measurable cardinals are end-extending. But since $\kappa$ being end-extending depends only on $\mathcal{P}(\kappa)$, reflection shows that every normal ultrafilter on $\kappa$ has a measure-one set of end-extending cardinals. Furthermore, since we can consistently have weakly inaccessible cardinals carrying $\omega_1$-saturated ideals, end-extending cardinals are not necessarily strong limit. Even successor cardinals can be end-extending in limited degrees. For example, the statement that $\omega_2$ is $\omega_1$-end-extending is easily seen to be equivalent to the notion $\text{SCC}^\text{gap}_{\omega_1}$ defined in [3]. It follows from the above proposition that if $\kappa$ carries a $\kappa$-complete ideal $\mathcal{I}$ such that $\mathcal{P}(\kappa)/\mathcal{I}$ is a proper forcing, then $\kappa$ is $\omega_1$-end-extending. If $\kappa$ is measurable and $\mu < \kappa$ is regular and uncountable, then this is forced by the Lévy collapse $\text{Col}(\mu, < \kappa)$ (see [10]).

Let us now see how this notion can be used with Neeman’s forcing. Suppose $\theta^{< \theta} = \theta$ and $\alpha^{< \alpha} < \theta$ for each $\alpha < \theta$. Let $\mathcal{W}$ be a filtration of $H_\theta$, and let
\[ A = \langle H_\theta, \epsilon, \vec{W} \rangle. \] Let \( S \) be the set of countable \( M \prec A \), and let \( T \) be the set of \( W_\alpha \prec A \) such that \( \text{cf}(\alpha) \) is uncountable. Then according to the terminology of the previous section, \( (\omega_1, \omega_1, \theta, \vec{W}, S, T) \) is nice. Let us set \( C_{\vec{W}} = \mathbb{P}^{\text{dec*}}_{\omega, S, T} \) under these hypotheses. If \( \theta \) is inaccessible, let \( C_\theta = C_{\vec{W}} \), for \( \vec{W} = \langle V_\alpha : \alpha < \theta \rangle \).

**Theorem 23.** If \( \theta \) is an inaccessible \( \omega_1 \)-end-extending cardinal, or a stationary limit of such cardinals, \( \text{then} \ C_\theta \text{forces that NS}_{\omega_1} \) is weakly presaturated. If \( \theta \in \text{Ord} \), \( \alpha^+ < \theta \) for all \( \alpha < \theta \), and \( \theta \) is \( \omega_1 \)-end-extending, then the same conclusion is forced by \( C_{\vec{W}} \), where \( \vec{W} \) is any filtration of \( H_\theta \).

**Proof.** Let us first show the latter statement. Let \( G \subseteq C_{\vec{W}} = \mathbb{P}^{\text{dec*}}_{\omega, S, T} \) be generic, and let \( S^G_\omega \) be the collection of \( M \in S \) that appear in \( G \). By Lemma 16, \( P \) forces \( \theta \) to become \( \omega_2 \). By Lemma 18, \( V[G] \) satisfies that \( \text{NS} \upharpoonright S^G_\omega \) is nice. By Lemma 2 it suffices to show that for all stationary \( A \subseteq \omega_1 \) and \( f : \omega_1 \to \omega_1 \), \( \{ M \in S^G_\omega : M \cap \omega_1 \in A \wedge f(M \cap \omega_1) < \text{ot}(M \cap \theta) \} \) is stationary.

Let \( \dot{A} \) be a name for a stationary subset of \( \omega_1 \), \( \dot{f} \) a name for a function from \( (H^Y_\theta)^{<\omega} \) to \( H^Y_\theta \). Let \( p_0 \in C_{\vec{W}} \) be arbitrary. Let \( \lambda > \theta \) be regular and let \( \mathcal{B} = \langle H_{\lambda, \epsilon}, \mathcal{A}, p_0, \dot{f}, \dot{\omega} \rangle \), where \( \dot{\omega} \) is a well-order. Let \( \mathcal{A}^* \) be the result of restricting the Skolem functions for \( \mathcal{B} \) to \( H_{\theta} \).

Let \( \mathcal{A}^{**} \) be an expansion of \( \mathcal{A}^* \) given by the \( \omega_1 \)-end-extending hypothesis.

In \( V[G] \), there is some \( M \prec \mathcal{A}^{**} \) such that \( M \in S^G_\omega \) and \( M \cap \omega_1 \in A \). Let \( p_1 \leq p_0 \) decide the value of such \( M \), force that \( M \cap \omega_1 \in A \), and decide \( f(M \cap \omega_1) = \xi \) for some ordinal \( \xi < \omega_1 \). We may assume that \( p_1 \) takes the form \( (s_1, g) \) with \( M \in s_1 \).

Let \( \alpha \) be least such that \( p_1 \in W_\alpha \in T \). By end-extensibility, there is \( M_1 \prec \mathcal{A}^{**} \) such that \( M \subseteq M_1 \), and if \( \alpha_0 \) is the least ordinal in \( M_1 \setminus M \), then \( \alpha < \alpha_0 \). By Lemma 12, \( W_{\alpha_0} \in T \). Now repeat this \( (\xi + 1) \)-many times to obtain a continuous \( \subseteq \)-increasing sequence of models \( \langle M_i : i \leq \xi + 1 \rangle \), all elementary in \( \mathcal{A}^{**} \), with \( M_0 = M_1 = M \), and a corresponding sequence of ordinals \( \langle \alpha_i : i \leq \xi \rangle \) such that \( \alpha_i \in M_{i+1} \), \( M_{i+1} \cap W_{\alpha_i} = M_i \). Let \( M^* = M_\xi+1 \). Then \( M^* \cap W_{\alpha_0} = M \), and \( \text{ot}(M^* \cap \theta) > \xi \).

Let \( s_2 = s_1 \cup \{ W_{\alpha_0}, M^* \} \). Clearly, \( s_2 \) is an \( \epsilon \)-chain. To see that it is closed under intersections, note that \( W_{\alpha_0} \cap M^* = M \in s_1 \), and for any \( Q \in s_1 \), \( Q \cap W_{\alpha_0} = Q \) and \( M^* \cap Q = M^* \cap W_{\alpha_0} \cap Q = M \cap Q \in s_1 \). Then \( p_2 = (s_2, g) \) is a condition below \( p_1 \).

Let \( N^* = \text{Sk}^\mathcal{B}(M^*) \). Since \( p_2 \) is a master condition for \( N^* \) and \( \dot{F} \in \dot{N}^* \), \( p_2 \) forces \( N^*[G] \upharpoonright H^V_\theta = M^* \), so it forces \( M^* \) to be closed under \( \dot{F} \). Also, \( p_2 \) forces \( M^* \) to be in \( S^G_\omega \) and \( M \cap \omega_1 = M^* \cap \omega_1 \in A \). Since \( p_0 \) and \( \dot{F} \) were arbitrary, it is forced that the set \( \{ Q \in S^G_\omega : Q \cap \omega_1 \in A \wedge f(Q \cap \omega_1) < \text{ot}(Q \cap \theta) \} \) is stationary.

Now suppose that \( \theta \) is an inaccessible stationary limit of regular \( \omega_1 \)-end-extending cardinals. Let \( A, f, F, p_0 \) be as above. By the chain condition, we may assume that \( A, f, p_0 \in V_\theta \), and on a club of \( \alpha < \theta \), \( F \cap V_\alpha \) is forced to be a name for \( \dot{F} \upharpoonright V_\alpha \).

We can find an inaccessible \( \kappa < \theta \) such that \( \kappa \) is \( \omega_1 \)-end-extending, \( A, f, p_0 \in V_\kappa \in T \), and \( F \cap V_\kappa \) is a \( C_\kappa \)-name for a function from \( V_\kappa^{<\omega} \) to \( V_\kappa \). By the arguments of the previous paragraphs, there is \( q \leq p_0 \in V_\kappa \) that forces some \( M \in S \cap V_\kappa \) to appear in the generic \( G \cap V_\kappa \), be closed under \( \dot{F} \), and have the properties that \( M \cap \omega_1 \in \dot{A} \) and \( f(M \cap \omega_1) < \text{ot}(M \cap \kappa) \). Since \( V_\kappa \in T \), \( C_\kappa \) is a regular suborder of \( C_\theta \), and any generic \( G \subseteq C_\theta \) possessing \( q \) will yield an extension satisfying these statements. By the arbitrariness of \( p_0 \) and \( \dot{F} \), the collection of such models \( M \) is forced to be stationary.

We remark that above forcing does not necessarily render \( \text{NS}_{\omega_1} \) precipitous. Claverie and Schindler [3] showed that a precipitous weakly presaturated ideal (a.k.a. “strong ideal”) is equiconsistent with a Woodin cardinal.
Another strengthening of weak presaturation for $\text{NS}_{\omega_2}$ is the statement that for every $f : \omega_1 \to \omega_1$, there is a club $C \subseteq \omega_1$ and a canonical function $\psi_\alpha$ such that $f(\beta) < \psi_\alpha(\beta)$ for all $\beta \in C$. Deiser-Donder [11] and Larson-Shelah [17] showed that this property, which the latter call “Bounding”, is equiconsistent with an inaccessible limit of measurable cardinals.

Besides consistency strength considerations, there is a combinatorial reason why $\mathbb{C}_\theta$ does not necessarily force Bounding. Suppose $\theta > \omega_1$ is a club sequence. Define a function $f : \omega_1 \to \omega_1$ as follows. If $x_\alpha$ codes, via the Gödel pairing function, a countable transitive set $M$, let $f(\alpha) = \text{ot}(N \cap \omega_1)$. Otherwise, let $f(\alpha) = 0$.

Let $\theta > \omega_1$ be regular and let $\mathfrak{A}$ be a structure on $H_\theta$ in a countable language. Let $\langle M_i : i < \omega_1 \rangle$ be a continuous, $\varepsilon$-increasing sequence of countable elementary substructures of $\mathfrak{A}$. Let $C \subseteq \omega_1$ be the club of $\alpha$ such that $M_\alpha \cap \omega_1 = \alpha$. Let $M$ be the union of the $M_\alpha$, and let $\kappa$ be the transitive collapse of $M$. Let $A \subseteq \omega_1$ code the $\varepsilon$-structure of $M$ via the pairing function. There is a club $D \subseteq C$ such that for all $\alpha \in D$, $A \cap \alpha$ codes a structure isomorphic to $M_\alpha$. There is some $\alpha \in D$ such that $x_\alpha = A \cap \alpha$. We have $f(\alpha) = f(M_\alpha \cap \omega_1) = \text{ot}(M_\alpha \cap \theta)$. Thus there are stationary-many $z \in P_{\omega_1}(\theta)$ such that $f(z \cap \omega_1) = \text{ot}(z \cap \theta)$.

If $\mathbb{P}$ is a proper forcing, then this continues to hold in $V^\mathbb{P}$ with respect to the same $f$. In particular, it holds in any extension by $\mathbb{C}_\theta$. Let $G \subseteq \mathbb{P}$ be generic. There is some $M \prec H_{\omega_1}[G]$ such that $f \in M$, and $f(M \cap \omega_1) = \text{ot}(M \cap \omega_2)$. If Bounding were to hold in $V[G]$, then by elementarity, there would be an ordinal $\gamma \in M$ and a club $C \subseteq M$ such that $M \models f(\beta) < \psi_\gamma(\beta)$ for all $\beta \in C$. But $\alpha = M \cap \omega_1 \in C$ and $\psi_\gamma(\alpha) = \text{ot}(M \cap \gamma) < \text{ot}(M \cap \omega_2) = f(\alpha)$, a contradiction.

6. $w\text{CC}(\omega_2, \text{cof}(\omega_1))$ from $\text{(+2)-subcompactness}$

As far as we are able to ascertain, until the time of this writing, the best known upper bounds for the consistency strength of $w\text{CC}(\mu^+, \text{cof}(\mu))$, for $\mu$ regular uncountable, remained what was discovered in the late 1970s, namely an almost-huge cardinal. Kunen [15] developed a forcing strategy that collapses a huge cardinal $\kappa$ to become the successor of a chosen regular cardinal $\mu < \kappa$, and forces Chang’s Conjecture $(\kappa^+; \kappa) \rightarrow (\mu^+; \mu)$ and the existence of a saturated ideal on $\kappa$. Shortly thereafter, Magidor showed that the saturated ideal, and consequently $w\text{CC}(\kappa, \text{cof}(\mu))$, can be obtained from an almost-huge $\kappa$ (see [10]). In [1], the author and Hayut showed by a reflection argument that the strength of $(\omega_3, \omega_2) \rightarrow (\omega_2, \omega_1)$ is weaker than a huge cardinal, and that a single huge cardinal can be used to get Chang’s Conjecture between many pairs of cardinals simultaneously, but the reflection did not take us down to the level of almost-huge or lower. On the other hand, we showed that if we allow some distance between the pairs of cardinals, the strength can be shown to be much lower by a very different forcing argument. In particular, we got a model of the generalized continuum hypothesis (GCH) plus $(\omega_4, \omega_3) \rightarrow (\omega_2, \omega_1)$ from a model of GCH with a $(+2)$-subcompact cardinal. We get some more information using an observation of Adolf [1]:

**Lemma 24** (Adolf). Suppose $2^{\kappa^+} = \kappa^{++}$ and $(\kappa^{++}, \kappa^{+++}) \rightarrow (\kappa^+, \kappa)$. Then for every regular $\mu \leq \kappa$ the set of $M \prec H_{\kappa+3}$ such that $M \cap \kappa^+ \in \kappa^+$, $\text{cf}(M \cap \kappa^+) = \mu$, $\text{cf}(M \cap \kappa^{++}) = \kappa$, and $\text{ot}(M \cap \kappa^{+++}) = \kappa^+$, is stationary.

Let us show that the above result yields many witnesses to Chang’s Conjecture that are countably closed. Suppose GCH holds and $M \prec H_{\omega_1}$ is such that $\omega_1 \subseteq M$ and $\text{cf}(M \cap \kappa) > \omega$ for each uncountable cardinal $\kappa \in M$. By CH, $M$ contains all reals. Since $\text{cf}(M \cap \omega_2)$ is uncountable, $[\omega_2]^{<\omega_1} \cap M = \bigcup\{[\alpha]^{<\omega_1} : \alpha \in M \cap \omega_2\} = [M \cap \omega_2]^{<\omega_1}$. Similarly, $[\omega_3]^{<\omega_1} \cap M = \bigcup\{[\alpha]^{<\omega_1} \cap M : \alpha \in M \cap \omega_3\}$, and for any
κ models behave with end-extending cardinals. That $\dot{v}$ and for any $x, y\in M$, a bijection $f : \omega_2 \to \alpha$ in $M$ yields that $[\alpha]^{<\omega_1} \cap M = \{ f(z) : z \in [M \cap \omega_2]^{<\omega_1}\} = [M \cap \alpha]^{<\omega_1}$. Thus $[\alpha]^{<\omega_1} \cap M = [M \cap \omega_3]^{<\omega_1}$. Similarly, $[\omega_2]^{<\omega_1} \cap M = [M \cap \omega_2]^{<\omega_1}$. Thus $M$ is countably closed.

Now assume GCH and $(\omega_2, \omega_3) \to (\omega_2, \omega_1)$. Let us introduce a Neeman forcing. Let $\tilde{W}$ be a filtration of $H_{\omega_4}$, and let $\mathfrak{A} = (H_{\omega_4}, \in, \tilde{W})$. Let $\mathcal{S}$ be the collection of all countably closed $\mathcal{N} \prec \mathfrak{A}$ of size $\omega_1$ such that $N \subseteq M$ for some countably closed $M \prec \mathfrak{A}$ with $\ot(M \cap \omega_4) = \omega_2$. By taking countably closed elementary initial segments of witnesses to Chang's Conjecture, we see that $\mathcal{S}$ is stationary. Let $\mathcal{T}$ be the collection of all $W_\alpha \prec \mathfrak{A}$ with $\cf(\alpha) \geq \omega_2$. Each $W \in \mathcal{T}$ is closed under $\omega_1$-sequences. For any $M \in \mathcal{S}$ and $W \in \mathcal{T}$, $M \cap W$ in $\mathcal{S}$, since $W \cap M \subseteq M$, and it is a countably closed elementary substructure of $\mathfrak{A}$. Thus $(\mathcal{S}, \mathcal{T})$ is appropriate for $\omega_1$ and $\mathfrak{A}$.

Let $\mathcal{P} = \mathcal{P}^{\text{dec}}_{\omega_1, \mathcal{S}, \mathcal{T}}$. Then $\mathcal{P}$ is countably closed, preserves $\omega_2$ and $\omega_4$, and collapses $\omega_3$ so that $\omega_3^{\mathcal{P}} = \omega_1^{\mathcal{P}}$. Let $\dot{F}$ be a $\mathcal{P}$-name for a function from $(H^{\omega_4}_{\omega_4})^{<\omega}$ to $H^{\omega_4}_{\omega_1}$. Let $p_0 \in \mathcal{P}$ be arbitrary, and let $\check{f}$ be a $\mathcal{P}$-name for a function from $\omega_2$ to $\omega_2$. Let $\mathfrak{B} = (H_{\omega_4}, \in, \check{W}, \check{F}, \check{p_0}, \check{f})$, where $\prec$ is a well-order. Let $\mathfrak{B}^*$ be the result of restricting the Skolem functions for $\mathfrak{B}$ to $H_{\omega_4}$.

By $(\omega_2, \omega_3) \to (\omega_2, \omega_1)$ and the above lemma, there are stationary-many countably closed $M \prec \mathfrak{B}^*$ such that $\ot(M \cap \omega_4) = \omega_2$ and $[M \cap \omega_3] = \omega_1$. For each $\alpha < \omega_1$, let $M_\alpha \prec \mathfrak{B}^*$ be such a model with $\alpha \in M_\alpha$. For each $\alpha < \omega_4$ of cofinality at least $\omega_2$, $M_\alpha \cap \alpha$ is bounded below $\alpha$. Using Fodor's Lemma, there is a stationary $A \subseteq \omega_4 \cap \cf(\omega_2)$ and $\gamma < \omega_4$ such that $M_\alpha \cap \alpha \subseteq \gamma$ for all $\alpha \in A$. By GCH, there is a stationary $B \subseteq A$ and a set $N$ such that $M_\alpha \cap W_\alpha = N$ for all $\alpha \in B$. We may assume that $W_\alpha \prec \mathfrak{B}^*$ for all $\alpha \in B$, which implies $N \prec \mathfrak{B}^*$. Note that $N \in \mathcal{S}$.

Let $p_1 = \langle s_1, h \rangle$ be a condition below $p_0$ with $N \in s_1$, such that $p_1$ decides the value of $\check{f}(N \cap \omega_2)$, say as $\xi < \omega_2$. Let $\alpha \in B$ be such that $p_1 \in W_\alpha$. Let $N^* \subseteq M_\alpha$ be such that $N^* \prec M_\alpha$, $N^*$ is countably closed, and $\ot(N^* \cap \omega_4) > \xi$. Let $s_2 = s_1 \cup \{ W_\alpha, N^* \}$. Then $s_2$ is closed under intersections, since $N^* \cap W_\alpha = N \in s_1$, and for any $Q \in s_1$, $W_\alpha \cap Q = Q$ and $N^* \cap Q = N^* \cap W_\alpha \cap Q = N \cap Q \in s_1$. Then $p_2 = \langle s_2, h \rangle$ is a condition below $p_1$, and it forces that $N^*$ is closed under $\check{F}$ and that $\check{f}(N^* \cap \omega_2) < \ot(N^* \cap \omega_2)$. Note also that $\cf(N^* \cap \omega_2) = \omega_1$. As $p_0, \check{f}$, and $\check{F}$ were arbitrary, we have:

**Theorem 25.** If ZFC$+$GCH is consistent with a $(+2)$-subcompact cardinal, then ZFC is consistent with $\text{wCO}(\omega_2, \cf(\omega_1))$.

7. Magidor models and ambitious cardinals

Suppose $\kappa$ is a regular cardinal. Following [22], we call a set $M$ $\kappa$-Magidor if $\kappa \in M$, $M \cap \kappa \in M$, $M$ is extensional, $M$ has size $< \kappa$, and the transitive collapse of $M$ is equal to $V_\alpha$ for some $\alpha < \kappa$. Magidor [19] showed that $\kappa$ is supercompact if and only if for every $\theta > \kappa$, the set of $\kappa$-Magidor $M \in \mathcal{P}_\kappa(V_\theta)$ is stationary.

**Lemma 26.** Suppose $\mu < \kappa$ is a regular cardinal, $M \prec V_\theta$ is a $\kappa$-Magidor model, $\mu \subseteq M \cap \kappa$, and $\cf(\sup(M \cap \theta)) \geq \mu$. Then $M$ is $< \mu$-closed.

**Proof.** Let $i : V_\theta \to M$ be the inverse of the transitive collapse of $M$. Suppose $x \in [M]^{<\mu}$. Since $\cf(M \cap \theta) \geq \mu$, there is $\alpha < \eta$ such that $x \subseteq i(V_\eta)$. Then $y = i^{-1}[x] \in V_\eta$, and $x = i(y) \in M$ since $|y| < M \cap \kappa = \text{crit}(i)$. \qed

In this section, we identify a species of supercompactness that has a characterization in terms of Magidor models that is somewhat analogous to how countable models behave with end-extending cardinals.
Definition. A cardinal $\kappa$ is ambitious when for all $\lambda \geq \kappa$, there is $\delta_0 \geq \lambda$ such that for all $\delta \geq \delta_0$, there is a $\lambda$-closed transitive $M$, a $\delta$-closed transitive $N$, and elementary embeddings $j : V \to M$ and $k : M \to N$ such that:

- $\text{crit}(j) = \kappa$ and $\lambda < j(\kappa)$.
- $\text{crit}(k) > \lambda$, $k[j(\kappa)] \subseteq \delta$, and $k(j(\kappa)) > \delta$.

Recall that $\kappa$ is almost-huge if there is an elementary $j : V \to M$ with critical point $\kappa$ and $M$ is a transitive class closed under $\langle j(\kappa) \rangle$-sequences. The name ambitious is chosen because it is as if such $\kappa$ are trying hard to be almost-huge.

The first embedding $j$ sends $\kappa$ to a highly closed model, but $j(\kappa)$ overshoots the closure. The next embedding $k$ tries to make up for this by sending $j(\kappa)$ into a model whose closure is above $k(\alpha)$ for each $\alpha$ at which $j$ may have fallen short of being an almost-huge embedding. Now $k(j(\kappa))$ may overshoot the new closure, but it looks like progress. Indeed, this is a local property of systems of measures that characterize almost-huge cardinals (see [15, Theorem 24.11]).

Proposition 27. The following are equivalent:

1. $\kappa$ is ambitious.
2. (High-jump property) For all $\lambda \geq \kappa$, there is $\delta_0 \geq \lambda$ such that for all $\delta \geq \delta_0$, there is a normal ultrafilter $\mathcal{U}$ on $\mathcal{P}_\kappa(\delta)$ such that for all $f : \mathcal{P}_\kappa(\lambda) \to \kappa$, $\{z \in \mathcal{P}_\kappa(\delta) : f(z \cap \lambda) < \text{ot}(z)\} \in \mathcal{U}$.
3. (Shelah property) For all $\lambda \geq \kappa$ and all $f : \mathcal{P}_\kappa(\lambda) \to \kappa$, there is $\delta_0 \geq \lambda$ such that for all $\delta \geq \delta_0$, there is a normal ultrafilter $\mathcal{U}$ on $\mathcal{P}_\kappa(\delta)$ such that $\{z \in \mathcal{P}_\kappa(\delta) : f(z \cap \lambda) < \text{ot}(z)\} \in \mathcal{U}$.

Proof. (1) $\Rightarrow$ (2): Let $\lambda \geq \kappa$, let $\delta_0 \geq \lambda$ be given by (1), let $\delta \geq \delta_0$, and let $j : V \to M$ and $k : M \to N$ also be given by (1). Let $i = k \circ j$. Let $\mathcal{U} = \{X \subseteq \mathcal{P}_\kappa(\lambda) : j[|X|] \in j(X)\}$, and let $W = \{X \subseteq \mathcal{P}_\kappa(\delta) : i[|\delta|] \in i(X)\}$. Let $\pi : \mathcal{P}_\kappa(\delta) \to \mathcal{P}_\kappa(\lambda)$ be $z \mapsto z \cap \lambda$. Then:

$$X \in \mathcal{U} \iff j[|X|] \in j(X) \iff k[|j|] = i[|X|] \in i(X) \quad \text{(since} \quad \text{crit}(k) > \lambda) \iff i(\pi)(i[|\delta|]) \in i(X) \iff \pi^{-1}[X] \in W$$

Let $\bar{k} : \text{Ult}(V, \mathcal{U}) \to \text{Ult}(V, W)$ be the map $[f]_\mathcal{U} \mapsto [f \circ \pi]_W$. Let $\bar{f}_\mathcal{U} : \text{Ult}(V, \mathcal{U}) \to M$ be $[f]_\mathcal{U} \mapsto j(f)(|X|)$, and let $f_W : \text{Ult}(V, W) \to N$ be $[f]_W \mapsto i(f)(i[|\delta|])$. With $j_\mathcal{U}, j_W$ denoting the usual ultrapower embeddings, we have the following commutative diagram:

\[ V \xrightarrow{j} M \xrightarrow{k} N \xrightarrow{j_W} N \xrightarrow{\bar{k}} \text{Ult}(V, \mathcal{U}) \xrightarrow{\bar{f}_\mathcal{U}} \text{Ult}(V, W) \]

Since $j_W[|\delta|]$ is represented by $[\text{id}]_W$, the claim that for all $f : \mathcal{P}_\kappa(\lambda) \to \kappa$, $\{z \in \mathcal{P}_\kappa(\delta) : f(z \cap \lambda) < \text{ot}(z)\} \in W$ is equivalent to the claim that $\sup \bar{k}(j_\mathcal{U}(\kappa)) \leq \delta$. To
show this, note:
\[
\begin{align*}
\delta & \geq \sup \{ k(\alpha) : \alpha < j(\kappa) \} \\
& \geq \sup \{ k(\ell_U(\alpha)) : \alpha < j_U(\kappa) \} \\
& = \sup \{ \ell_{\pi_U}(k(\alpha)) : \alpha < j_U(\kappa) \} \\
& \geq \sup \{ k(\alpha) : \alpha < j_U(\kappa) \}
\end{align*}
\]

2) \implies 1): Let \( \lambda \geq \kappa \), let \( \delta_0 \geq \lambda \) be given by (2), let \( \delta \geq \delta_0 \) and let \( W \) be a normal ultrafilter on \( P_\kappa(\delta) \) given by (2). Let \( \pi : P_\kappa(\delta) \to P_\kappa(\lambda) \) be \( z \mapsto z \cap \lambda \), and let \( U \) be the projection of \( W \) to \( P_\kappa(\lambda) \) via \( \pi \). Let \( j : V \to M = \text{Ult}(V, U) \) and \( i : V \to N = \text{Ult}(V, W) \) be the ultrapower embeddings, and let \( k : M \to N \) be \( [j]\delta \mapsto [j \circ \pi]_W \). Ordinals \( \alpha \leq \lambda \) are represented in \( \text{Ult}(V, U) \) by the function \( f_\alpha : z \mapsto \text{ot}(z \cap \alpha) \), and \( k(\alpha) \) is the \( W \)-equivalence class of the function \( z \mapsto \text{ot}(z \cap \lambda \cap \alpha) \), which represents \( \alpha \) in \( \text{Ult}(V, W) \). Thus \( \text{crit}(k) > \lambda \). For all \( \alpha < j(\kappa) \), there is a function \( f : P_\kappa(\lambda) \to \kappa \) representing \( \alpha \). Since \( \{ z \in P_\kappa(\delta) : f \circ \pi(z) < \text{ot}(z) \} \in W \), \( k(\alpha) < \delta \).

The implication 2) \implies 3) is trivial. For the reverse direction, let \( \lambda \geq \kappa \). Let \( \lambda' = 2^\lambda < \kappa \), and let \( \delta_0 \geq \lambda' \) witness 3) for \( \lambda' \). Enumerate all functions from \( P_\kappa(\lambda) \) to \( \kappa \) as \( \{ f_\alpha : \alpha < \lambda' \} \). Let \( g : P_\kappa(\lambda') \to \kappa \) be \( z \mapsto \sup \{ f_\alpha(z \cap \lambda) : \alpha < \lambda \} \). Let \( \delta \geq \delta_0 \) and let \( U \) be a normal ultrafilter on \( P_\kappa(\delta) \) such that \( \{ z \in P_\kappa(\delta) : g(z \cap \lambda) < \text{ot}(z) \} \in U \).

By the fineness of \( U \), for all \( \alpha < \lambda' \), \( \{ z \in P_\kappa(\delta) : f_\alpha(z \cap \lambda) < \text{ot}(z) \} \in U \). \( \square \)

**Theorem 28.** Suppose \( \kappa < \theta \) are inaccessible. The following are equivalent:

1. \( V_\theta \models \kappa \) is ambitious.
2. For all structures \( A \) on \( V_\theta \) in a countable language, there is a \( \kappa \)-Magidor \( M < A \) such that for all \( \alpha < \kappa \), there is a \( \kappa \)-Magidor \( N < A \) such that \( M \subseteq N \) and \( \text{ot}(N \cap \theta) > \alpha \).

Furthermore, in (2), we can take \( M \) and \( N \) to be \( \langle M \cap \kappa \rangle \)-closed.

**Proof.** Suppose 1). If 2) fails, then there is a structure \( A \) on \( V_\theta \) in a countable language extending (\( V_\theta, \in, \kappa \)) such that for every \( \kappa \)-Magidor \( M < A \), there is a bound \( b_\kappa(M) < \kappa \) on the order-type of \( N \cap \theta \) whenever \( N < A \) is a \( \kappa \)-Magidor end-extension of \( M \).

Let \( \lambda < \theta \) be such that \( V_\lambda < A \) and \( \text{cf}(\lambda) \geq \kappa \). Let \( \delta_0 \geq \lambda \) witness ambitousness. Let \( \delta \geq \delta_0 \) be such that \( V_\delta < A \) and \( \text{cf}(\delta) \geq \kappa \). Let \( W \) be a normal ultrafilter on \( P_\kappa(V_\delta) \) such that for all \( f : P_\kappa(V_\delta) \to \kappa \), \( \{ z \in P_\kappa(V_\delta) : f(z \cap V_\lambda) < \text{ot}(z) \} \in W \).

Let \( U \) be the projection of \( W \) to \( P_\kappa(V_\delta) \), and let \( j : V \to M = \text{Ult}(V_\delta, U) \) be the ultrapower embedding via \( U \). Then in \( M \), \( j(V_\lambda) \) is a \( (j(\kappa)) \)-Magidor elementary submodel of \( j(V_\lambda) \). It is \( \langle \kappa \rangle \)-closed by Lemma 26. Let \( b = b_{V_\delta} \upharpoonright P_\kappa(V_\lambda) \), and let \( \xi : j(b)(j(V_\lambda)) \). Let \( i : V \to N \) be the ultrapower map via \( W \), and let \( k : M \to N \) be the factor map. Then \( \delta > k(\xi) \). In \( N \), \( k(\xi) = i(b_\kappa)(i(V_\lambda)) \) is a bound on the order-types of \( i(\kappa) \)-Magidor end-extensions of \( i(V_\lambda) \) that are elementary in \( i(\kappa) \).

Now suppose 2). Suppose \( \kappa \leq \lambda < \theta \), and let \( A \) be any structure in a countable language extending (\( V_\theta, \in, \kappa, \lambda \)). Let \( M < A \) be a \( \kappa \)-Magidor model that can be end-extended to other \( \kappa \)-Magidor models elementary in \( A \) of arbitrarily high order-type below \( \kappa \). There is \( \gamma < \kappa \), a structure \( A \) on \( V_\gamma \), and an elementary \( j : A \to A \), with \( j(V_\gamma) = M \). Let \( \langle \kappa, \lambda \rangle = j^{-1}((\kappa, \lambda)) \). Then we can define a normal ultrafilter \( U \) on \( P_\kappa(\lambda) \) by \( X \in \text{Ult}(A, U) \) iff \( j(\lambda) \in j(X) \). Of course, \( U \in V_\lambda \). Let \( j_U : V_\gamma \to P = \text{Ult}(V_\gamma, U) \) be the ultrapower embedding. As usual, there is a factor map \( \ell_0 : P \to V_\delta \) defined by \( \ell_0([j]_U) = j(f)(j(\kappa)) \), with \( f = \ell_0 \circ j_U \). Let \( \xi = \sup \ell_0(\kappa(j_U(\kappa))) \), which is above \( \lambda \) since \( j_U(\kappa) > \lambda \). Since \( \gamma < \kappa \) and \( \kappa \) is regular, \( \xi < \kappa \).
Now let $N \prec \mathbb{A}$ be a $\kappa$-Magidor end-extension of $M$ with $\text{ot}(N \cap \theta) > \xi$. Let $\eta < \kappa$ be such that $N \cong V_\eta$. Let $i : V_\eta \to V_\theta$ be the inverse of the transitive collapse map. Then $i$ is an extension of $j$. Let $\xi \leq \delta < \eta$. Let $W$ be the normal ultrafilter on $\mathcal{P}_\kappa(\delta)$ defined by $X \in W$ if $i[\delta] \in i(X)$. Then $U$ is the projection of $W$ via the map $\pi : z \mapsto z \cap \lambda$. Let $j_W : V_\eta \to Q = \text{Ult}(V_\eta, W)$ be the ultrapower embedding, let $k : P \to Q$ be the factor map $[f]_U \mapsto [f \circ \pi]_W$, and let $\ell_0 : Q \to V_\theta$ be the map $[f]_W \mapsto i(f)(i[\delta])$. We have the following commutative diagram:

$$
\begin{array}{c}
V_\eta \\
| \quad \quad \downarrow j_W \\
V_\theta \\
\end{array}
\begin{array}{c}
P \\
| \quad \quad \searrow k \\
Q \\
\end{array}
\begin{array}{c}
V_\lambda \\
\downarrow \ell_0 \\
V_\theta \\
\end{array}
$$

We have that:

$$\bar{\delta} \geq \sup\{\ell_0(\alpha) : \alpha < j_U(\bar{\kappa})\} = \sup\{\ell_1(\kappa(\alpha)) : \alpha < j_U(\bar{\kappa})\} \geq \sup\{k(\alpha) : \alpha < j_U(\bar{\kappa})\}$$

The relation $\bar{\delta} \geq \sup k[i(j_U(\bar{\kappa}))]$ is equivalent to the statement that for all $f : \mathcal{P}_\kappa(\bar{\lambda}) \to \bar{\kappa}$, $\{ z \in \mathcal{P}_\kappa(\bar{\delta}) : f(z \cap \lambda) < \text{ot}(z) \} \in W$, which can be computed in $V_\eta$. Note that to arrange this property, we only needed to take $\delta \geq \xi$. By the elementarity of $i = \ell_0 \circ j_W$, for every $\delta \geq i(\xi)$ in $V_\theta$, there is a normal ultrafilter $W'$ on $\mathcal{P}_\kappa(\delta)$ such that for all $f : \mathcal{P}_\kappa(\bar{\lambda}) \to \kappa$, $\{ z \in \mathcal{P}_\kappa(\delta) : f(z \cap \lambda) < \text{ot}(z) \} \in W'$.

In contrast to the class of countable models, Magidor models are more constrained in their end-extendibility.

**Proposition 29.** Suppose $\kappa < \theta$ are inaccessible and $V_\theta \models \kappa$ is supercompact. Then there is a set $B \subseteq \kappa$ and a function $f : B \to \kappa$ such that:

1. $M \cap \kappa \in B$ for stationary many $\kappa$-Magidor $M \prec V_\theta$.
2. For all $\kappa$-Magidor $M \prec V_\theta$ such that $M \cap \kappa \in B$, $\text{ot}(M \cap \theta) \leq f(M \cap \kappa)$.

**Proof.** Let $B = \{ \alpha < \kappa : V_\kappa \models \text{“}\alpha \text{ is not supercompact}”\}$. For $\alpha \in B$, let $f(\alpha)$ be the least $\beta$ such that $\alpha$ is not $\beta$-supercompact.

Suppose $\kappa$ is $\lambda$-supercompact Let $\mathcal{U}, W$ be two normal ultrafilters on $\mathcal{P}_\kappa(\lambda)$. If $\mathcal{U} \in \text{Ult}(V, W)$, then $j_U(\kappa) \leq j_W(\kappa)$ since $j_W(\kappa)$ is inaccessible in $\text{Ult}(V, W)$, and $\text{Ult}(V, W)$ can compute $\text{Ult}(\kappa, \mathcal{U})$. Thus if $\mathcal{U}$ has $j_U(\kappa)$ as small as possible, then $\kappa$ is not $\lambda$-supercompact in $\text{Ult}(V, \mathcal{U})$. Thus $\kappa \in j_U(B)$ and $j_U(f(\kappa)) \leq \lambda$.

Let $\mathfrak{A}$ be a structure on $V_\theta$ in a countable language, and let $\lambda$ be such that $\kappa < \lambda < \theta$ and $V_\lambda \prec \mathfrak{A}$. Let $\mathcal{U}$ be a normal ultrafilter on $\mathcal{P}_\kappa(\lambda)$ with $j_U(\kappa)$ as small as possible. In $M = \text{Ult}(V, \mathcal{U})$, $N = j_U[V_\lambda]$ is a $j_U(\kappa)$-Magidor elementary submodel of $j_U(\mathfrak{A})$. By the above observations, $\kappa = N \cap j_U(\kappa) \in j_U(B)$. By elementarity, there is $\kappa$-Magidor $N' \prec \mathfrak{A}$ such that $N' \cap \kappa \in B$.

Now suppose $M \prec V_\theta$ is $\kappa$-Magidor and $M \cap \kappa \in B$. Suppose towards a contradiction that there is a $\kappa$-Magidor $N \prec V_\theta$ such that $M \subseteq N$ and $\text{ot}(N) > f(M \cap \kappa)$. Then there is an elementary $i : V_\theta \to V_\lambda$, with $i[V_\lambda] = N$ and $\eta > f(M \cap \kappa)$. We can define a normal ultrafilter $W$ on $\mathcal{P}_{M \cap \kappa}(f(M \cap \kappa))$ by $X \in W$ if $i[f(M \cap \kappa)] \in i(X)$. This contradicts the fact that $M \cap \kappa$ is not $f(M \cap \kappa)$-supercompact. \qed

To situate ambitiousness among the large cardinal notions, we show that it is strictly below almost-hugeness and strictly above the strongest notion found in the literature below almost-huge. Perlmutter’s paper [23] seems to be the latest word on this. A cardinal $\kappa$ is called *high-jump* when for some $\lambda \geq \kappa$, there is a normal...
ultrafilter $\mathcal{U}$ on $\mathcal{P}_\kappa(\lambda)$ such that $\lambda \geq \sup\{j_\mathcal{U}(f)(\kappa) : f : \kappa \rightarrow \kappa\}$. Such an ultrafilter is called a \textit{high-jump measure}. This notion was first introduced in [90] and given the name “high-jump” by [2]. The strongest notion considered in [24] below almost-huge is \textit{high-jump with unbounded excess closure}. Perlmutter [24 Proposition 4.6] shows that this is equiconsistent with the existence of a cardinal $\kappa$ such that for all sufficiently large $\lambda \geq \kappa$, there is a high-jump measure on $\mathcal{P}_\kappa(\lambda)$.

**Proposition 30.** Suppose $\kappa$ is almost-huge with target $\lambda$. Then $V_\lambda \models \text{“$\kappa$ is ambitious,”}$ and $V_\kappa \models \text{“There are unboundedly many ambitious cardinals.”}$

\begin{proof}
Let $j : V \rightarrow M$ be such that $\text{crit}(j) = \kappa$, $j(\kappa) = \lambda$, and $M^{< \lambda} \subseteq M$. Suppose $\kappa \leq \alpha < \lambda$. Since $\lambda$ is inaccessible, there is $\delta_0 < \lambda$ such that for all $f : \mathcal{P}_\kappa(\alpha) \rightarrow \kappa$, $j(f)(\alpha) < \delta_0$. Let $\delta_0 \leq \delta < \lambda$ and let $\mathcal{U}$ be the normal ultrafilter on $\mathcal{P}_\kappa(\delta)$ derived from $j$. Then for all $f : \mathcal{P}_\kappa(\alpha) \rightarrow \kappa$, $\{z \in \mathcal{P}_\kappa(\delta) : f(z \cap \kappa) < \text{ot}(z)\} \in \mathcal{U}$. The second claim follows from an easy reflection argument.
\end{proof}

**Proposition 31.** If $\kappa$ is ambitious, then for all sufficiently large $\lambda \geq \kappa$, there is a high-jump measure on $\mathcal{P}_\kappa(\lambda)$. Furthermore, the former property has strictly greater consistency strength than the latter.

\begin{proof}
If $\kappa$ is ambitious, then there is $\delta_0 > \kappa$ such that for all $\delta \geq \delta_0$, there is a normal ultrafilter $\mathcal{U}$ on $\mathcal{P}_\kappa(\delta)$ such that for all $f : \mathcal{P}_\kappa(\alpha) \rightarrow \kappa$, $\{z \in \mathcal{P}_\kappa(\delta) : f(z \cap \kappa) < \text{ot}(z)\} \in \mathcal{U}$. This means that if $j : V \rightarrow M$ is the ultrapower embedding via $\mathcal{U}$, then for all $f : \mathcal{P}_\kappa(\alpha) \rightarrow \kappa$, $j(f)(\kappa) < \delta$. Thus $\mathcal{U}$ is a high-jump measure.

To prove the strict consistency comparison, we show that there are unboundedly many inaccessible cardinals above an ambitious cardinal $\kappa$. Then if $\theta$ is any inaccessible above the witness $\delta_0$ for $\kappa$, then $V_\theta \models \text{ZFC + “For all $\delta \geq \delta_0$, there is a high-jump measure on $\mathcal{P}_\kappa(\delta)$.”}$. To this end, let $\lambda \geq \kappa$ be arbitrary. Let $j : V \rightarrow M$, $k : M \rightarrow N$ be as in the definition of ambitiousness. There is $\delta$ such that $\lambda < \theta < j(\kappa)$ and $M \models \theta$ is inaccessible. There is $\delta$ such that $N$ is $\delta$-closed and $k(\theta) < \delta$. Thus $k(\theta)$ is inaccessible in $V$, and $k(\theta) > \lambda$.
\end{proof}

8. \textbf{Forcing with Magidor Models}

In this final section, we end on a negative note. We first describe a variety of circumstances under which forcing with Magidor models as the small type in a Neeman forcing will force the weak Chang’s Conjecture, and we demonstrate a certain reversal, yielding a forcing characterization of ambitious cardinals. Then we address a claim of Neeman [23, \S 5.1] about iterating his forcing and show that ambitious cardinals provide a counterexample. Finally, we present an argument of Mohammapour that his forcing with Veličković [22] does not permit $\text{wCC}({\omega}_2, \text{cof}({\omega}_1))$. This seems to close the door on combining compactness and hugeness at $\omega_2$ with existing side-conditions technology.

8.1. \textbf{A forcing characterization of ambitiousness.} If $M \prec V_\theta$ is a Magidor model and $\mathcal{P} \in M$ is a partial order, then $\mathcal{P} \in M$ is a strong master condition for $M$ if and only if it is an ordinary master condition for $M$. For suppose $\mathcal{P}$ is an ordinary master condition. If we take a dense $D \subseteq \mathcal{P} \cap M$ and let $\pi : M \rightarrow V_\theta$ be the transitive collapse, then $\pi[D] = E \subseteq V_\theta$ is a dense subset of $\pi(\mathcal{P})$. Then $\pi^{-1}(E) \cap M = D$, and since $\mathcal{P}$ is a master condition for $M$, $\mathcal{P}$ forces the generic filter $\mathcal{G}$ to meet $\pi^{-1}(E)$ in $M$, i.e. it forces $\mathcal{G} \cap D \neq \emptyset$.

Let us fix the following assumptions and notation for the remainder of this section:

1. $\mu < \kappa < \theta$ are regular cardinals, and $\theta$ is inaccessible.
2. $\mathcal{P} \in V_\theta$ is a partial order.
3. $\mathcal{A} = \{V_\theta, \mathcal{P}, \kappa, \mathcal{A}\}$. 

Claim 32. Suppose $M, N \in S_0$, $W \in T_0$, and $G \subseteq \mathbb{P}$ is generic.

1. $\mathbb{P}$ is a strong master condition for $W$.
2. $M \cap W \in W \cap S_0$.
3. If $M \subseteq N$, then $M[G] \in S$ iff $N[G] \in S$.
4. If $M[G] \in S$, and $W \in M$, then $M[G] \cap W[G] = (M \cap W)[G] \in S \cap W[G]$.

Proof. For (1), note that $M \cap W$ is $\kappa$-Magidor since it is an initial segment of $M$, elementary in $\mathcal{A}$ by Lemma 12 and a member of $W$ by the $\kappa$-closure of $W$.

For (2), suppose $M \cap W \subseteq M$ are both in $S_0$. Recall that for $X < \langle H_\theta, \in, \mathbb{P} \rangle$, $p$ is a strong master condition for $X$ iff for $G$ generic over $X$ for the partial order $\mathbb{P} \cap X$, let $(p_\alpha: \alpha < |\mathbb{P}|)$ be an enumeration of $\mathbb{P}$ in $M$.

For (3), suppose $M \cap W \subseteq M$, it follows by (2) that $(M \cap W)[G] \in S$.

Since $M \cap W \subseteq W$, $(M \cap W)[G] \in W[G]$. To show $(M \cap W)[G] = M[G] \cap W[G]$, let $x \in M[G] \cap W[G]$. Since $M[G] \models [\tau]: \tau \in W$ is a $\mathbb{P}$-name, there is $\mathbb{P}$-name $\tau \in M \cap W$ such that $x = \tau^G$. Thus $M[G] \cap W[G] \subseteq (M \cap W)[G]$. The reserve inclusion is trivial.

Claim 33. Suppose $S_0$ is stationary in $V$ and $G \subseteq \mathbb{P}$ is generic.

1. $S$ is stationary in $V[G]$.
2. $T = \{V_\alpha^{V[G]} < V_\theta^{V[G]} : \text{cf}(\alpha) \geq \kappa \land \mu, \kappa, \mathbb{P} \in V_\alpha \}$.

Proof. For (1), consider a name for a function $\dot{F}$: $(H_\theta^{V[G]} \times \omega \rightarrow H_\theta^{V[G]})$. Let $p \in \mathbb{P}$ be arbitrary.

For (2), note that by the $S_0$-properness of $\mathbb{P}$, the class of ordinals cof($\kappa$) is the same in $V[G]$. It is a standard fact that if $\mathbb{P} \in V_\alpha$, then $V_\alpha^{V[G]} = V_\alpha[G]$. If $\mathbb{P} \in V_\alpha$ and $V_\alpha \prec V_\theta$, then if $V_\alpha[G] \models \varphi(\tau^G)$, this is forced by some $p \in G$.

For (3), suppose $V_\alpha[G] \prec V_\theta[G]$. By Laver’s result 18 that the ground model is a definable class of a forcing extension, there is a translation of formulas $\varphi(\check{x}) \rightarrow \varphi(\check{x})^{Gr}$ such that for $\check{a} \in V_\alpha^{\omega}$, $\check{V}_\alpha \models \varphi(\check{a})$ if $V_\alpha[G] \models \varphi(\check{a})^{Gr}$.

Lemma 34. Suppose $V_\theta \models \langle \kappa \text{ is ambitious} \rangle$, and $\mathbb{P}$ preserves the regularity of $\mu$.

Let $A = \{M \cap \kappa : M \in S_0 \}$. Then for each $n < 3$, $\mathbb{P} \ast Q_n$ forces $\text{ccc}(\kappa, A)$.

Proof. By Claims 32 and 33, and by the preservation of $\mu$, $\langle S, T \rangle$ is appropriate for $\mu$ and $V_\mu[G]$, whenever $G \subseteq \mathbb{P}$ is generic. By 23, or Lemma 16, $\mathbb{P} \ast Q_n$ preserves the regularity of $\kappa$ and $\theta$ and forces that $\theta = \kappa^+$.

Let $\dot{F}$ be a $\mathbb{P} \ast Q_n$-name for a function from $\theta^{<\omega}$ to $\theta$, let $\check{f}$ be a name for a function from $\kappa$ to $\kappa$, and let
\[ \langle p_0, q_0 \rangle \in \mathbb{P} \times \check{Q}_\kappa \text{ be arbitrary.} \] Let \( \mathfrak{B} = \langle H_{\mathfrak{B}}, \in, \mathfrak{A}, \check{F}, \check{f}, \langle p_0, q_0 \rangle, \rangle \), where \( \langle \) is a well-order of \( H_{\mathfrak{B}} \). Let \( \mathfrak{C} \) be the result of restricting the Skolem functions for \( \mathfrak{B} \) to \( V_\theta \). Let \( F^*: V_{\check{\kappa}} \rightarrow V_\theta \) be another function. By Theorem 25, there is a \( \kappa \)-Magidor \( M < \mathfrak{C} \) that is \( (M \cap \kappa) \)-closed, \( M \) is closed under \( F^* \), and for every \( \alpha < \kappa \), there is a \( \kappa \)-Magidor \( N < \mathfrak{C} \) such that \( M \subseteq N \), \( N \) is \( (N \cap \kappa) \)-closed, and \( \text{ot}(N \cap \theta) > \alpha \).

It follows that there are stationary-many \( \kappa \)-Magidor \( M < \mathfrak{C} \) that are \( (M \cap \kappa) \)-closed and can be end-extended to \( \kappa \)-Magidor \( N < \mathfrak{C} \) with the same closure and such that \( \text{ot}(N \cap \theta) \) is arbitrarily large below \( \kappa \). Let \( \mathcal{E} \) be the set of such \( M \). For each \( \alpha < \theta \) choose \( M_\alpha \in \mathcal{E} \) such that \( \alpha < M_\alpha \). If \( \text{cf}(\alpha) \geq \kappa \), then \( \sup(M_\alpha \cap \alpha) < \alpha \).

By Fodor’s Lemma and the inaccessibility of \( \theta \), there is a stationary \( X \subseteq \theta \cap \text{cof}(\kappa) \) and an \( M^* \) such that for every \( \alpha < X, M_\alpha \cap \alpha = M^* \). We may assume that for every \( \alpha < X, V_\alpha < \mathfrak{C} \). It follows that \( M^* < \mathfrak{C}, M^* \) is \( \kappa \)-Magidor, and \( M^* \) is \( (M^* \cap \kappa) \)-closed.

By \( \mathcal{S}_\kappa \)-properness, let \( p_1 \leq p_0 \) be a master condition for \( M^* \). \( p_1 \) forces that \( M^*[\check{G}] \in \mathcal{S} \) and thus by [23], or Lemma 11 that \( q_0 \) can be extended to include the model \( M^*[\check{G}] \). Let \( \check{q}_1 \) be a name for such an extension. Let \( \langle p_2, q_2 \rangle \leq \langle p_1, \check{q}_1 \rangle \) decide the value of \( f(M^* \cap \kappa) \), say as \( \xi < \kappa \). Let \( \alpha \in X \) be such that \( \langle p_2, q_2 \rangle \in V_\alpha \). Let \( N \supseteq M_\alpha \) be such that \( N \) is \( \kappa \)-Magidor, \( N < \mathfrak{C} \), \( N \) is \( (N \cap \kappa) \)-closed, and \( \text{ot}(N \cap \theta) > \zeta \). Note that by Claim 22 \( p_1 \) is a master condition for \( N \).

Let \( s_2 \) be a name for the set of models appearing in \( q_2 \). Let \( s_3 \) be a name such that \( \forall p \ s_3 = s_2 \cup \{ V_\alpha \check{G}, N[\check{G}] \} \). Then \( p_2 \) forces that \( s_3 \) is closed under intersections, since \( N \cap V_\alpha = M^* \), and thus by Claim 23 that \( N[\check{G}] \cap V_\alpha[\check{G}] = M^*[\check{G}] \in s_2 \), and that if \( R \in s_2 \), then \( V_\alpha[\check{G}] \cap R = R \) and \( N[\check{G}] \cap R \subseteq N[\check{G}] \cap V_\alpha[\check{G}] \cap R = M^*[\check{G}] \cap R \in s_2 \). Furthermore, it is forced that \( s_3 \nsubseteq N[\check{G}] = (s_3 \cap M^*[\check{G}]) \cup \{ V_\alpha[\check{G}] \} \), which is forced to be a member of \( N[\check{G}] \).

If \( n = 0 \), let \( \check{q}_3 = \check{s}_3 \). If \( n = 1 \) and \( \check{h} \) is a name for the decoration of \( \check{q}_2 \), let \( \check{h}' \) be a name for the extension of \( \check{h} \) that assigns the empty set to \( V_\alpha[\check{G}] \) and \( N[\check{G}] \), and let \( \check{q}_3 = \check{s}_3 \). If \( n = 2 \) and \( \check{h} \) is a name for the decoration of \( \check{q}_2 \), let \( \check{q}_3 = \check{s}_3 \). In each case, \( \langle p_2, q_2 \rangle \leq \langle p_2, \check{q}_2 \rangle \), and \( \langle p_2, \check{q}_3 \rangle \) is a master condition for \( \mathcal{S}_\kappa(N) \). Thus \( \langle p_2, \check{q}_3 \rangle \) forces that \( N \) is closed under \( \check{F} \) and that \( f(N \cap \kappa) < \text{ot}(N \cap \theta) \). By the arbitrariness of \( \check{F}, \check{f} \), and \( \langle p_0, q_0 \rangle \), \( \mathbb{P} \times \check{Q}_\kappa \) is forced.

**Theorem 35.** Suppose \( \mathbb{P} \times \check{Q}_\kappa \) preserves the class \( \text{cof}(\geq \mu) \). Then \( \mathbb{P} \times \check{Q}_\kappa \) forces \( \text{wCC}(\mu^+, \text{cof}(\mu)) \) if and only if \( V_0 \models \text{"\( \kappa \) is ambitious."} \)

**Proof.** Suppose \( V_0 \models \text{“\( \kappa \) is ambitious.”} \) By Lemma 44 \( \text{wCC}(\kappa, A) \) holds, where \( A = \{ M \cap \kappa : M \in \mathcal{S}_\kappa \} \). For every \( \kappa \)-Magidor \( M < \mathfrak{C} \), \( V \models \mu \text{ cf}(M \cap \kappa) \), so since all cardinals between \( \mu \) and \( \kappa \) are collapsed, \( \mathbb{P} \times \check{Q}_\kappa \) forces that \( \text{c}(M \cap \kappa) = \mu \).

For the other direction, suppose \( V_0 \models \text{“\( \kappa \) is not ambitious.”} \)

**Case 1:** \( V_0 \models \text{“\( \kappa \) is not supercompact.”} \) Then there is \( \lambda_0 \in [\kappa, \theta) \) such that for all \( \lambda \geq \lambda_0 \), there is no normal fine ultrafilter on \( \mathcal{P}_\kappa(\lambda) \). It follows that for every \( \lambda \in [\lambda_0, \theta) \), there is no \( \kappa \)-Magidor \( M < V_\theta \) with \( \lambda \in M \). This is because if there were such an \( M \), then there would be an elementary \( i : V_\theta \rightarrow V_\theta \) for some \( \eta < \kappa \), with \( i(\text{crit}(i)) = \kappa \). If \( i(\check{\kappa}) = \kappa \) and \( i(\check{\lambda}) = \lambda \), then we could define a normal fine ultrafilter \( U \) on \( \mathcal{P}_\kappa(\lambda) \) by \( X \in U \) iff \( i(\check{\lambda}) \in i(X) \). But then \( U \in V_\eta \), and by elementarity, \( V_0 \models \text{“\( \kappa \) is \( \lambda \)-supercompact,”} \) a contradiction.

Let \( G \subseteq \mathbb{P} \) be generic. In \( V[G] \), for any \( q \in \check{Q}_\kappa \) the only models included in \( p \) of rank \( > \lambda_0 \) are transitive models. Thus if \( V_\alpha \in V_\beta \) are two consecutive transitive models, with \( \alpha > \lambda_0 \), and \( H \subseteq Q_\alpha \) is generic over \( V[G] \), then the union of the decorations attached to \( V_\alpha \) that appear in \( H \) constitutes a surjection from \( \mu \) to \( V_\beta \).

But notice also that Corollary 13 still holds for \( Q_\kappa \), and so \( \theta = \mu^+ \) in \( V[G][H] \). Now in \( V \), let \( f : \theta \rightarrow \theta \) be the function \( \beta \mapsto \beta^+ \). Let \( \alpha < (\theta^+)^V = (\theta^+)^{V[G][H]} \) and
let $\sigma : \theta \to \alpha$ be a surjection in $V$. For all $\beta < \theta$, $\text{ot}(\sigma[\beta]) < (\beta^+)V = f(\beta)$. Thus in $V[G][H]$, $f$ dominates each canonical function pointwise, and so $\text{wCC}(\theta, A)$ fails for every $A \subseteq \theta$.

**Case 2:** $V_0 \models "\kappa is supercompact but not ambitious."$ In this case, $S_0$ is stationary in $V$, so $P * \dot{Q}_2$ forces that $\kappa = \mu^+$ and $\theta = \mu^{++}$. By Theorem [25] there is a structure $\mathfrak{B}$ on $V_0$ in a countable language and a function $b : V_0 \to \kappa$ such that for all $\kappa$-Magidor $M \in \mathfrak{B}$, if $N < \mathfrak{B}$ is $\kappa$-Magidor and $M \subseteq N$, then $\text{ot}(N \cap \theta) < b(M)$. We may assume that $\mathfrak{B}$ has definable Skolem functions.

Let $G * H \subseteq P * \dot{Q}_2$ be generic. $H$ yields a chain of models $\langle M_\alpha : \alpha < \theta \rangle$. Since the set of $M \in S$ that appear in $H$ is stationary, there are stationary-many $M$ that appear in $H$ that are closed under the Skolem functions for $\mathfrak{B}$. For $\alpha < \kappa$ such that $\alpha = M \cap \kappa$ for some model $M$ appearing in $H$ such that $M \cap V \prec \mathfrak{B}$, let $M_\alpha$ be the first such model in the chain. We claim that for all $N$ appearing in $H$ such that $N \cap V \prec \mathfrak{B}$ and $N \cap \kappa = \alpha$, we have $M_\alpha \subseteq N$. If $N$ is such a model, then $M_\alpha \cap N$ also appears in $H$, and since $M_\alpha \cap N \cap V \prec \mathfrak{B}$ and $M_\alpha \cap N \cap \kappa = \alpha$, we must have that $M_\alpha \cap N = M_\alpha$ by the minimality of $M_\alpha$. If $N \neq M_\alpha$, then since $M_\alpha \notin N$, there must be transitive models appearing between them. Let $W$ be the least such transitive model. Then $W \cap N$ also appears in $H$. If $W \cap N \neq M_\alpha$ then $W \cap N$ occurs either before or after $M_\alpha$. If were to occur after, then since there are no transitive models between $M_\alpha$ and $W \cap N$, we would have $M_\alpha \subseteq W \cap N$. But this is impossible, since $M_\alpha \cap \kappa = W \cap N \cap \kappa$. But also $W \cap N$ cannot occur before $M_\alpha$, since $M_\alpha \cap N = M_\alpha$, so $M_\alpha \subseteq W \cap N$ and thus $\text{rank}(M_\alpha) \leq \text{rank}(W \cap N)$. Thus $M_\alpha = W \cap N \subseteq N$.

In $V[G][H]$, define $f : \kappa \to \kappa$ by $f(\alpha) = b(M_\alpha \cap \kappa)$ if $M_\alpha$ is defined, and otherwise $f(\alpha) = 0$. Now let $X \subseteq \kappa \cap \text{cof}(\mu)$ be a stationary set in $V[G][H]$. Since $P$ preserves that $\text{cf}(M \cap \kappa) \geq \mu$ for all $M \in S$, Lemma [13] implies that the set of $M$ appearing in $H$ with $M \cap \kappa \in X$ is stationary. Let $\kappa < \gamma < \theta$, and let $\sigma : \kappa \to \gamma$ be a bijection in $V[G][H]$. Let $M$ be a model appearing in $H$ such that $M \cap V \prec \mathfrak{B}$, $\gamma \in M$, $M$ is closed under $\sigma$ and $\sigma^{-1}$, and $M \cap \kappa = \alpha \in X$. There are $N, N_\alpha \in S_0$ such that $N[G] = M$, $N_\alpha[G] = M_\alpha$, $M \cap V = N$, and $M_\alpha \cap V = N_\alpha$. Since $N_\alpha \subseteq N$, $\text{ot}(M \cap \kappa) < b(N_\alpha) = f(\alpha)$. If $\psi_\gamma$ is the canonical function for $\gamma$ defined using $\sigma$, then $\psi_\gamma(\alpha) = \text{ot}(\sigma[\alpha]) = \text{ot}(M \cap \gamma) < \text{ot}(M \cap \theta) < f(\alpha)$. Since $X$ was an arbitrary stationary subset of $\kappa \cap \text{cof}(\mu)$, the set of $\beta \in \kappa \cap \text{cof}(\mu)$ such that $\psi_\gamma(\beta) \geq f(\beta)$ is nonstationary. Thus $f$ dominates all canonical functions modulo clubs on $\kappa \cap \text{cof}(\mu)$, so $\text{wCC}(\kappa, \text{cof}(\mu))$ fails.

We note that the hypothesis of the above theorem is satisfied when $P$ is the trivial forcing.

**Proposition 36.** Suppose $P * \dot{Q}_2$ preserves $\text{cof}(\geq \mu)$. Then $P * \dot{Q}_2$ forces that $\text{NS}_\kappa \setminus \text{cof}(\mu)$ is not weakly presaturated.

**Proof.** If $V_0 \models "\kappa is not supercompact,"$ then the conclusion follows from the above theorem. So assume $\kappa$ is supercompact in $V_0$. Let $B \subseteq \kappa$ and $f : B \to \kappa$ be as in Proposition [29] so that $B = \{ M \in S_0 : M \cap \kappa \in B \}$ is stationary, and whenever $M \in B$, then $\text{ot}(M \cap \theta) \leq f(M \cap \kappa)$. If $G * H \subseteq P * \dot{Q}_2$ is generic, then there will be stationary-many $M \in B$ such that $M[G]$ appears in $H$.

Using Corollary [3] it suffices to show that $\text{wCC}(\kappa, B)$ fails. Towards a contradiction, suppose that there is $\gamma < \theta$ and a bijection $\sigma : \kappa \to \gamma$ such that $B' = \{ \alpha \in B : f(\alpha) < \text{ot}(\sigma[\alpha]) \}$ is stationary. By Lemma [13] there is $M \in S_0$ such that $M[G]$ appears in $H$, $M \cap \kappa \in B'$, and $M$ is closed under $\sigma$ and $\sigma^{-1}$. But then $\text{ot}(\sigma[M \cap \kappa]) = \text{ot}(M \cap \gamma) < \text{ot}(M \cap \theta) \leq f(M \cap \kappa)$, a contradiction.

$\square$
Question. Under some large cardinal hypothesis, can \( \mathcal{P} \ast \mathcal{Q}_2 \), or some similar poset, force the existence of a weakly presaturated ideal on \( \omega_2? \)

8.2. A claim of Neeman. At the end of Section 5.1 of [22], Neeman gave a brief description of an argument that if \( \kappa \) is supercompact and \( \theta > \kappa \) is weakly compact, then the tree property can be forced simultaneously at \( \omega_2 \) and \( \omega_3 \) by a two-step iteration of the form \( \mathcal{P} \ast \mathcal{Q} \), where \( \mathcal{P} \) is a finite-conditions two-types poset making \( \kappa = \omega_2 \), and \( \mathcal{Q} \) is a countable-conditions two-types poset of size \( \theta \). In private correspondence, he clarified that \( \mathcal{Q} \) should indeed use former \( \kappa \)-Magidor models as the small type. Since \( \mathcal{P} \) forces \( 2^{\omega} = \kappa \), the small models of \( \mathcal{Q} \) will not be countably closed. Thus, as stated in [22], “preservation of \( \omega_1 \) requires a special argument.”

Neeman also clarified that this special argument is that in \( V^\mathcal{Q} \), \( \mathcal{Q} \) is completely proper for a stationary class of countable models. Specifically, if \( \theta^* > \theta \) is regular, \( N < H_{\theta^*} \) is countable model in \( V \), \( \mathcal{P} \ast \mathcal{Q} \in N \), \( G \subseteq \mathcal{P} \) is generic, and \( N \cap V_\kappa \) appears in \( G \), then for any \( q \in N[G] \cap \mathcal{Q} \), there is \( q' \leq q \) such that \( \{ r \in N[G] \cap \mathcal{Q} : q' \leq r \} \) is a \( \mathcal{Q} \)-generic filter over \( N[G] \). In particular, by a standard argument, \( \mathcal{Q} \) will not add \( \omega \)-sequences of ordinals. This complete properness claim also played a key role in the argument for the tree property.

However, Theorem 5 combined with Lemma 34 shows that, setting \( \mu = \omega_1 \) and taking \( \kappa \) and \( \theta \) sufficiently large, if the second stage \( \mathcal{Q} \) is one of our posets \( \mathcal{Q}_n \) for \( n < 3 \), and it preserves \( \omega_1 \) and doesn’t add \( \theta \)-many new reals over \( V^\mathcal{Q} \), than it forces \( \square^*_\omega \), which contradicts the tree property at \( \omega_2 \).

8.3. Failure of \( \text{wCC}(\omega_2, \text{cof}(\omega_1)) \) in the Mohammadpour-Veličković model. We present here, for those familiar with the details of [22], a sketch of an argument due to Mohammadpour that in a forcing extension by the virtual-models poset of [22], there is a function \( g : \omega_2 \to \omega_2 \) that dominates all canonical functions on \( \omega_2 \).

The forcing \( \mathcal{P}^\lambda_\alpha \) uses a supercompact \( \kappa \) and an inaccessible \( \lambda > \kappa \). The poset preserves \( \omega_1 \), turns \( \kappa \) into \( \omega_2 \), and turns \( \lambda \) into \( \omega_3 \). The conditions in their poset are finite sequences of countable and \( \kappa \)-Magidor models, not necessarily elementary in \( V_\lambda \), but possibly sets that look like partial transitive collapses of elementary submodels of \( V_\lambda \). See [22] for details.

Moreover, if \( \alpha < \beta \) are in \( E \), then for all but boundedly-many \( \gamma < \kappa \), \( M^\gamma_\beta \) is active at \( \alpha \). If \( M^\gamma_\alpha, M^\beta_\gamma \) are both defined and active at \( \alpha \), then \( M^\gamma_\alpha \cong M^\beta_\gamma \), since \( M^\gamma_\alpha \) is a partial transitive collapse of \( M^\beta_\gamma \).

In \( V[G] \), we define representatives of cofinally-many of the canonical functions on \( \kappa \) as follows. For \( \alpha \in E \) of cofinality \( < \kappa \) and \( \gamma \in C_\alpha(G) \) of cofinality \( \omega_1 \), let \( f_\alpha(\gamma) = \text{ot}(M^\gamma_\alpha \cap \alpha) \). The set of \( \kappa \)-Magidor models active at \( \alpha \) and appearing in \( C_\alpha \) is a \( \subseteq \)-increasing chain of length \( \kappa \), continuous at limits of uncountable cofinality, whose union covers \( V_\gamma \). If \( \sigma : \kappa \to \alpha \) is any surjection in \( V[G] \), then for club-many \( \gamma < \kappa \) of cofinality \( \omega_1 \), \( \sigma(\gamma) = M^\gamma_\sigma \cap \alpha \). Thus \( f_\sigma \) represents the \( \alpha^{th} \) canonical function on \( \text{cof}(\omega_1) \).

Also, define \( g_\alpha : C_\alpha(G) \to \kappa \) by \( g_\alpha(\gamma) = \text{ot}(M^\gamma_\alpha \cap \lambda) \). By the above remarks, if \( \alpha < \beta \) are in \( E \) and have cofinality \( < \kappa \), then \( g_\alpha(\gamma) = g_\beta(\gamma) \) for all but boundedly-many \( \gamma \in C_\alpha(G) \cap C_\beta(G) \cap \text{cof}(\omega_1) \). Furthermore, it follows from an easy density
argument that for all $\alpha \in E$ of cofinality $< \kappa$ and all but boundedly-many $\gamma < \kappa$, $\alpha \in M^\cup_\gamma$, and thus $M^\cup_\gamma \cap V_\alpha$ is a proper subset of $M^\cup_\gamma$. Thus if $g = g_{\alpha_0}$, where $\alpha_0$ is the minimum point of $E$, then for all $\beta < \lambda$, $g$ is greater than the $\beta^{th}$ canonical function at club-many points of cofinality $\omega_1$.

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