Adaptively Pruning Features for Boosted Decision Trees

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Abstract

Boosted decision trees enjoy popularity in a variety of applications; however, for large-scale datasets, the cost of training a decision tree in each round can be prohibitively expensive. Inspired by ideas from the multi-arm bandit literature, we develop a highly efficient algorithm for computing exact greedy-optimal decision trees, outperforming the state-of-the-art Quick Boost method. We further develop a framework for deriving lower bounds on the problem that applies to a wide family of conceivable algorithms for the task (including our algorithm and Quick Boost), and we demonstrate empirically on a wide variety of data sets that our algorithm is near-optimal within this family of algorithms. We also derive a lower bound applicable to any algorithm solving the task, and we demonstrate that our algorithm empirically achieves performance close to this best-achievable lower bound.

1 Introduction

Boosting algorithms are among the most popular classification algorithms in use today, e.g., in computer vision, learning-to-rank, and text classification. Boosting, originally introduced by Schapire [1990], Freund [1995], Freund and Schapire [1996], is a family of machine learning algorithms in which an accurate classification strategy is learned by combining many “weak” hypotheses, each trained with respect to a different weighted distribution over the training data. These hypotheses are learned sequentially, and at each iteration of boosting the learner is biased towards correctly classifying the examples which were most difficult to classify by the preceding weak hypotheses.

Decision trees [Quinlan 1993], due to their simplicity and representation power, are among the most popular weak learners used in Boosting algorithms [Freund and Schapire 1996, Quinlan 1996]. However, for large-scale data sets, training decision trees across potentially hundreds of rounds of boosting can be prohibitively expensive. Two approaches to ameliorate this cost include (1) approximate decision tree training, which aims to identify a subset of the features and/or a subset of the training examples such that exact training on this subset yields a high-quality decision tree, and (2) efficient exact decision tree training, which aims to compute the greedy optimal decision tree over the entire data set and feature space as efficiently as possible. These two approaches complement each other: approximate training often devolves to exact training on a subset of the data. We consider the task of efficient exact decision tree learning in the context of boosting where our primary objective is to minimize the number of examples that must be examined for any feature in order to perform greedy-optimal decision tree training. Our method is simple to implement, and gains in feature-example efficiency directly corresponds to improvements in computation time.

The main contributions of the paper are as follows:

• We develop a highly efficient algorithm for computing exact greedy-optimal decision trees, Adaptive-Pruning Boost, and we demonstrate through extensive experiments that our method outperforms the state-of-the-art Quick Boost method.
We develop a constrained-oracle framework for deriving feature-example lower bounds on the problem that applies to a wide family of conceivable algorithms for the task, including our algorithm and Quick Boost, and we demonstrate that our algorithm is near-optimal within this family of algorithms through extensive experiments.

Within the constrained-oracle framework, we also derive a feature-example lower bound applicable to any algorithm solving the task, and we demonstrate that our algorithm empirically achieves performance close to this lower bound as well.

We will next expand on the ideas that underlie our three main results above and discuss related work.

The Multi-Armed Bandit (MAB) Inspiration. Our approach to efficiently splitting decision tree nodes is based on identifying intervals which contain the score (e.g. classifier’s training accuracy) of each possible split and tightening those intervals by observing training examples incrementally. We can eventually exclude entire features from further consideration because their intervals do not overlap the intervals of the best splits. Under this paradigm, the optimal strategy would be to assess all examples for the best feature, reducing its interval to an exact value, and only then to assess examples for the remaining features to rule them out. Of course, we do not know in advance which feature is best. Instead, we wish to spend our assessments optimally to identify the best feature with the fewest assessments spent on the other features. This corresponds well to the best arm identification problem studied in the MAB literature. This insight inspired our training algorithm.

A “Pure Exploration” MAB algorithm in the “Fixed-Confidence” setting [Kalyanakrishnan et al., 2012, Gabillon et al., 2012, Kaufmann and Kalyanakrishnan, 2013] is given a set of arms (probability distributions over rewards) and returns the arm with highest expected reward with high probability (subsequently, WHP) while minimizing the number of samples drawn from each arm. Such confidence interval algorithms are generally categorized as LUCB (Lower Upper Confidence Bounds) algorithms, because at each round they “prune” sub-optimal arms whose confidence intervals do not overlap with the most promising arm’s interval until it is confident that WHP it has found the best arm.

In contrast to the MAB setting where one estimates the expected reward of an arm WHP, in the Boosting setting one can calculate the exact (training) accuracy of a feature (expected reward of an arm) if one is willing to assess that feature on all training examples. When only a subset of examples are assessed, one can also calculate a non-probabilistic “uncertainty interval” which is guaranteed to contain the feature’s true accuracy. This interval shrinks in proportion to the boosting weight of the assessed examples. We specialize the generic LUCB-style MAB algorithm of the best arm identification to assess examples in decreasing order of boosting weights, and to use uncertainty intervals in place of the more typical probabilistic confidence intervals.

Our Lower Bounds. We introduce two empirical lower bounds on the total number of examples needed to be assessed in order to identify the exact greedy-optimal node for a given set of boosting weights. Our first lower bound is for the class of algorithms which assess feature accuracy by testing the feature on examples in order of decreasing Boosting weights (we call this the assessment complexity of the problem). We show empirically that our algorithm’s performance is consistently nearly identical to this lower bound. Our second lower bound permits examples to be assessed in any order. It requires a feature to be assessed with the minimal set of examples necessary to prove that its training accuracy is not optimal. This minimal set depends on the boosting weights in a given round, from which the best possible (weighted) accuracy across all weak hypotheses is calculated. For non-optimal features, the minimal set is then identified using Integer Linear Programming.

1.1 Related Work

Much effort has gone to reducing the overall computational complexity of training Boosting models. In the spirit of [Appel et al., 2013], which has the state-of-the-art exact optimal-greedy boosted decision tree training algorithm Quick Boost (our main competitor), we divide these attempts into three categories and provide examples of the literature from each category: reducing 1) the set of features to focus on; 2) the set of examples to focus on; and/or 3) the training time of decision trees. Note that these categories are independent of and parallel to each other. For instance, 3), the focus of this work, can build a decision tree from any subset of features or examples. We show improvements compared to state-of-the-art algorithm both on subsets of the training data and on the full training
matrix. Popular approximate algorithms such as XGBoost \cite{Chen2016} typically focus on 1) and 2) and could benefit from using our algorithm for their training step.

Various works \cite{Dollar2007,Paul2009,Busa-Fekete2010} focus on reducing the set of features. Busa-Fekete and Kégel \cite{Busa-Fekete2010} divides features into subsets and at each round of boosting uses adversarial bandit models to find the most promising subset for boosting. LazyBoost \cite{Escudero2001} samples a subset of features uniformly at random to focus on at a given boosting round.

Another attempt at computational complexity reduction involve sampling a set of examples. Given a fixed budget of examples, Laminating \cite{Dubout2014} attempts to find the best among a set of hypotheses by testing each surviving hypothesis on a increasingly larger set of sampled examples while pruning the worst performing half and doubling the number of examples, until it is left with one hypothesis. It returns this hypothesis to boosting as the best one with probability $1 - \delta$. The hypothesis identification part of Laminating is fairly identical to the best arm identification algorithm Sequential Halving \cite{Karnin2013}, Stochastic Gradient Boost \cite{Friedman2002}, and the weight trimming approach of Friedman et al. \cite{Friedman1998} are a few other instances of reducing the set of examples. FilterBoost \cite{Bradley2008} uses an oracle to sample a set of examples from a very large dataset and uses this set to train a weak learner.

Another line of research focuses on reducing the training time of decision trees \cite{Sharp2008,Wu2008}. More recently, Appel et al. \cite{Appel2013} proposed Quick Boost, which trains decision tree as weak learners while pruning underperforming features earlier than a classic Boosting algorithm would. They build their algorithm on the insight that the (weighted) error rate of a feature when trained on a subset of examples can be used to bound its error rate on all examples. This is because the error rate is simply the normalized sum of the weights of the misclassified examples; if one supposes that all unseen examples may be correctly classified, that yields a lower bound on the error rate. If this lower bound is above the best observed error rate of a feature trained on all examples, the underperforming feature may be pruned and no more effort spent on it.

Our Adaptive-Pruning Boost algorithm carries forward the ideas introduced by Quick Boost. In contrast to Quick Boost, our algorithm is parameter-free and adaptive. Our algorithm uses fewer training examples and thus faster training CPU time than Quick Boost. It works by gradually adding weight to the “winning” feature with the smallest upper bound on, e.g., its error rate and the “challenger” feature with smallest lower bound, until all challengers are pruned. We demonstrate consistent improvement over Quick Boost on a variety of datasets, and show that when speed improvements are more modest this is due to Quick Boost approaching the lower bound more tightly than due to our algorithm using more examples than are necessary. Our algorithm is consistently nearly-optimal in terms of the lower bound for algorithms which assess examples in weight order, and this lower bound in turn is close to the global lower bound. Experimentally, we show that the reduction in total assessed examples also reduces the CPU time.

2 Setup and Notation

We adopt the setup, description and notation of \cite{Appel2013} for ease of comparison.

A Generic Boosting Algorithm. Boosting algorithms train a linear combination of classifiers $H_T(x) = \sum_{t=1}^{T} \alpha_i h_t(x)$ such that an error function $E$ is minimized by optimizing scalar $\alpha_i$ and the weak learner $h_t(x)$ at round $t$. Examples $x_i$ misclassified by $h_t(x)$ are assigned “heavy” weights $w_i$ so that the algorithm focuses on these heavy weight examples when training weak learner $h_{t+1}(x)$ in round $t + 1$. Decision trees, defined formally below, are often used as weak learners.

Decision Tree. A binary decision tree $h_{tree}(x)$ is a tree-based classifier where every non-leaf node is a decision stump $h(x)$. A decision stump can be viewed as a tuple $(p, k, \tau)$ of a polarity (either $+1$ or $-1$), the feature column index, and threshold, respectively, which predicts a binary label from the set $\{-1, +1\}$ for any input $x \in \mathbb{R}^K$ using the function $h(x) = \text{sign}(x[k] - \tau)$.

A decision tree $h_{tree}(x)$ is trained, top to bottom, by “splitting” a node, i.e. selecting a stump $h(x)$ that optimizes some function such as error rate, information gain, or GINI impurity. While this paper focuses on selecting stumps based on error rate, we provide bounds for information gain in the supplementary material which can be used to split nodes on information gain. Our algorithm Adaptive-Pruning Stump (Algorithm 1), a subroutine of Adaptive-Pruning Boost.
We define the uncertainty intervals (which we call uncertainty intervals) and permits us to train a decision stump $h$. To describe how $h$ prunes features based on exact intervals, we need to take the following steps:

1. **Calculate Error Rate**: The weighted error rate for stump $h$ on the first $m$ examples is then $E_m^j := \epsilon_m^j / Z_m$.

2. **Algorithm**: Adaptive-Pruning Stump prunes features based on exact intervals. Our lower bound assumes that all unseen examples are classified correctly and our upper bound assumes that all unseen examples are classified incorrectly.

3. **Extension**: We use $L_m^j$ as the lower bound on the error rate for stump $h$ on all $n$ examples, when computed on the first $m$ examples, and $U_m^j$ as the corresponding upper bound. For any $1 \leq m \leq n$, we define, using $c_m^j := \mathbb{I}\{h_j(x_i) \neq y_i\}$ to indicate whether stump $h_j$ incorrectly classifies example $i$,

$$\begin{align*}
L_m^j := \frac{1}{Z_m} \sum_{i=1}^{m} w_i c_m^j \leq \frac{1}{Z_n} \sum_{i=1}^{n} w_i c_m^j \leq \frac{1}{Z_n} (\epsilon_m^j + \sum_{i=m+1}^{n} w_i) = \frac{1}{Z_n} (\epsilon_m^j + (Z_n - Z_m)) =: U_m^j.
\end{align*}$$

For any two stumps $i$ and $j$ when numbers $m$ and $m'$ exist such that $L_m^i > U_{m'}^j$, then we can safely discard stump $i$, as it cannot have the lowest error rate. This extension of the pruning rule used by Appel et al. [2013] permits each feature to have its own interval of possible error rates, and permits us to compare features for pruning without first needing to assess all $n$ examples for any feature (Quick Boost’s subroutine requires the current-best feature to be tested on all $n$ examples).

Now we describe our algorithm in detail; see the listing in Algorithm 1. We use $f_k$ to denote an object which stores all decision stumps $h(x)$ for feature $x[k]$. Recall that $x \in \mathbb{R}^K$ and that $x[k]$ is the $k_{th}$ feature of $x$, for $k \in \{1, \ldots, K\}$. $f_k$ has method $\text{assess}(\text{batch})$, when given a “batch” of examples, updates $L_m$, $E_m$, $U_m$ (defined above) for all decision stumps of feature $x[k]$ based on the examples in the batch. It also has methods $\text{LB}()$ and $\text{UB}()$, which report the $L_m$ and $U_m$ for the single hypothesis with smallest error $E_m$ on the $m$ examples seen so far, and $\text{bestStump}()$, which returns the hypothesis with smallest error $E_m$.

Adaptive-Pruning Stump proceeds until there is some feature $k^*$ whose upper bound is below the lower bounds for all other features. We then know that the best hypothesis uses feature $k^*$. We assess any remaining unseen examples for feature $k^*$ in order to identify the best threshold and polarity and to calculate $E_n^{k^*}$. Thus, our algorithm always finds the exact greedy-optimal hypothesis.

In order to efficiently compare two features $i$ and $j$ to decide whether to prune feature $i$, we want to “add” the minimum weight to these arms to possibly obtain that $L_m^i > U_{m'}^j$. The most efficient way to do this is to test each feature against a batch of the heaviest unseen examples whose weight is at least the gap $U_{m'}^j - L_m^i$. This permits us to choose batch sizes adaptively, based on the minimum weight needed to prune a feature given the current boosting weights and the current uncertainty intervals for each arm. We note that our “weight order” lower bound on the sample complexity of the problem in the next section is also calculated based on this insight. This is in contrast to Quick Boost, which...
We compare Adaptive-Pruning Boost against two lower bounds, defined empirically based on the boosting weights in a given round. In our weight order lower bound, we consider the minimum number of examples required to determine that a given feature is underperforming with the assumption that examples will be assessed in order of decreasing boosting weight. Our exact lower bound permits examples to be assessed in any order, and so bounds any possible algorithm which finds the best-performing feature.

Weight Order Lower Bound. For this bound, we first require that Adaptive-Pruning Stump selects the feature with minimal error. In the case of ties, an optimal feature may be chosen arbitrarily. Adaptive-Pruning Stump need to assess every example for the returned feature in order for Adaptive-Pruning Boost to calculate α and update weights w, so the lower bound for the returned feature is simply the total number of examples n.

Let k∗ be the returned feature, and E∗ its error rate when assessed on all n examples. For any feature k ≠ k∗ which is not returned, we need to prove that it is underperforming (or tied with the best feature). Let Jk be the set of decision stumps which use feature k; then we need to find the smallest value m such that for all stumps j ∈ Jk, we have Lm_j ≥ E*. Our lower bound is simply LBw := n + \sum_{k≠k^*} \min \{ m : \forall j \in J_k, L_{m_j} \geq E^* \}. We present results in Figure 2 showing that Adaptive-Pruning Boost achieves this bound on a variety of datasets. Quick Boosting, in contrast, sometimes approaches this bound but often uses more examples than necessary.

Algorithm 1 Adaptive-Pruning Stump

| Input: Examples \{x_1, \ldots, x_n\}, Labels \{y_1, \ldots, y_n\}, Weights \{w_1, \ldots, w_n\} |
| Output: h(x) |
| m ← min. index s.t. Z_m ≥ 0.5 |
| for k = 1 to K do |
| f_k.assess([x_1, \ldots, x_m]); m_k ← m |
| end for |
| a ← k with min f_k.UB() |
| b ← k ≠ a with min f_k.LB() |
| while f_a.UB() > f_b.LB() do |
| gap ← f_a.UB() - f_b.LB() |
| m ← min index s.t. Z_m ≥ Z_m_a + gap |
| f_a.assess([x_{m_a+1}, \ldots, x_m]); m_a ← m |
| gap ← f_a.UB() - f_b.LB() |
| if gap > 0 then |
| m ← min index s.t. Z_m ≥ Z_m_a + gap |
| f_a.assess([x_{m_a+1}, \ldots, x_m]); m_b ← m |
| end if |
| if f_a.UB() < f_b.UB() then |
| a ← b |
| end if |
| b ← k ≠ a with min f_k.LB() |
| end while |
| return h(x) := f_a.bestStump() |

Algorithm 2 Adaptive-Pruning Boost

| Input: Instances \{x_1, \ldots, x_n\}, Labels \{y_1, \ldots, y_n\} |
| Output: \mathcal{H}_T(x) |
| Initialize Weights: \{w_1, \ldots, w_n\} |
| for t = 1 to T do |
| Train Decision Tree \mathcal{H}_t(x) one node at a time by calling Adaptive-Pruning Stump |
| Choose \alpha_t and update \mathcal{H}_t(x) |
| Update and Sort (in descending order) w |
| end for |
**Exact Lower Bound.** In order to test the idea that adding examples in weight order is nearly optimal, and to provide a lower bound on any algorithm which finds the optimal stump, we also present an exact lower bound on the problem. Like the weight order lower bound, this bound is defined in terms of the boosting weights in a given round; unlike it, examples may be assessed in any order. It is not clear how one might achieve the exact lower bound without incurring an additional cost in time. We leave such a solution to future work. However, we show in Figure 1 that this bound is, in fact, very close to the weight order lower bound.

For the exact lower bound, we still require the selected feature \( k^* \) to be assessed against all examples; this is imposed by the boosting algorithm. For any other feature \( k \neq k^* \), we simply need the size of the smallest set of examples which would prune the feature (or prove it is tied with \( k^* \)). We will use \( M \subseteq \{1, \ldots, n\} \) to denote a set of indexes of examples assessed for a given feature, and \( L^j_M \) to denote the lower bound of stump \( j \) when assessed on the examples in subset \( M \). This bound, then, is:

\[
LB_{\text{exact}} := n + \sum_{k \neq k^*} \min_{M, |L^j_M| \geq E^*} |M|.
\]

We identify the examples included in the smallest subset \( M \) for a given feature \( k \neq k^* \) using integer linear programming. We define binary variables \( c_1, \ldots, c_n \), where \( c_i \) indicates whether example \( i \) is included in the set \( M \). We then create a constraint for each stump \( j \in J_k \) defined for feature \( k \) which requires that the stump be proven underperforming. Our program, then, is:

Minimize \( \sum_{i=1}^n c_i \) s.t. \( c_i \in \{0, 1\} \) \( \forall i \), and \( \sum_{i=1}^n c_i w_i \mathbb{I}\{h_j(x_i) \neq y_i\} \geq E^* \) \( \forall j \in J_k \).

**Discussion.** Figure 1 shows a non-cumulative comparison of our weight order lower bound to the global lower bound. Minimizing the global lower bound function mentioned above is computationally expensive. For this reason we used binary class datasets of moderate size and trees of depth 1 as weak leaners, but we have no reason to believe that the technique would not work for deeper trees and multi-class datasets. Refer to Table 1 for details of datasets. The weight order lower bound and Adaptive-Pruning Boost are within 10-20% of the exact lower bound, but Quick Boost often uses half to all of the unnecessary training examples in a given round.

### 5 Experiments

We experimented with shallow trees on various binary and multi-class datasets. We report both assessment complexity and CPU time complexity for each dataset. Though Adaptive-Pruning Boost is a general Boosting algorithm, we experimented with the following class of algorithms (1) Boosting exact greedy-optimal decision trees and (2) Boosting approximate decision trees.

Each algorithm was run with either the state-of-the-art method (Quick Boost) or our decision tree training method (Adaptive-Pruning Boost), apart from the case of Figure 2 that also uses the brute-force decision tree search method (Classic AdaBoost). The details of our datasets are in Table 1. For datasets SATIMAGE, W4A, A6A, and RCV1 tree depth of three was used and for MNIST Digits tree depth of four was used (as in Appel et al. [2013]). Train and test error results are provided as supplementary material.
Table 1: The datasets used in our experiments.

| Dataset    | Source          | Train / Test Size | Total Features | Classes |
|------------|-----------------|-------------------|----------------|---------|
| A6A        | Platt [1999]    | 11220 / 21341     | 123            | 2       |
| MNIST Digits | Lecun et al. [1988] | 60000 / 10000   | 780            | 10      |
| RCV1 (Binary) | Lewis et al. [2004] | 20242 / 67799   | 47236          | 2       |
| SATIMAGE   | Hu and Lin [2002] | 4435 / 2000      | 36             | 6       |
| W4A        | Platt [1999]    | 7366 / 42383      | 300            | 2       |

Boosting Exact Greedy-Optimal Decision Trees. We used AdaBoost for exact decision tree training. Figure 2 shows the total number of example assessments used by AdaBoost when it uses three different decision tree building methods described above. In all of these experiments, our algorithm, Adaptive-Pruning Boost, not only consistently beats Quick Boost but it also almost matches the weight order lower bound. The Classic AdaBoost can be seen as the upper bound on the total number of example assessments.

Table 2 shows that CPU time improvements correspond to example-assessments improvements for Adaptive-Pruning Boost for all our datasets, except for RCV1. This could be explained by Figure 2 wherein Quick Boost is seen approaching the lower bound for this particular dataset. While Adaptive-Pruning Boost is closer to the lower bound, its example-assessments improvements are not enough to translate to CPU time improvements.

![Figure 2](image_url)

Figure 2: We report the total number of assessments at various boosting rounds used by the algorithms, as well as the weight order lower bound. In all of these experiments, our algorithm, AP Boost, not only consistently beats Quick Boost but it also almost matches the lower bound.

Table 2: Computational Complexity for AdaBoost. All results are for 500 rounds of boosting except MNIST (300 rounds) and RCV1 (400 rounds).

| Dataset  | Boosting | CPU Time in Seconds | # Example Assessments |
|----------|----------|----------------------|-----------------------|
|          |          | AP-B | QB | IMPROV. | AP-B | QB | IMPROV. |
| A6A      | AdaBoost | 4.49E+02 | 4.46E+02 | 5.3% | 1.69E+09 | 1.83E+09 | 7.8%                 |
| MNIST    | AdaBoost | 6.32E+05 | 6.60E+05 | 4.2% | 3.52E+11 | 3.96E+11 | 11.1%                |
| RCV1     | AdaBoost | 1.58E+05 | 1.58E+05 | -0.5% | 6.15E+11 | 6.58E+11 | 6.5%                 |
| SATIMAGE | AdaBoost | 9.21E+02 | 1.19E+03 | 18.9% | 8.64E+08 | 1.11E+09 | 22.5%                |
| W4A      | AdaBoost | 3.03E+02 | 3.96E+02 | 27.1% | 1.69E+09 | 2.41E+09 | 29.8%                |
| MEAN     |          | 11% | 15.54% |
**Boosting Approximate Decision Trees.** We used two approximate boosting algorithms. We experimented with Boosting with Weight-Trimming 90% and 99% [Friedman et al., 1998], wherein the weak hypothesis is trained only on 90% or 99% of the weights, and LazyBoost 90% and 50% [Escudero et al., 2001] wherein the weak hypothesis is trained only on 90% or 50% randomly selected features. Table 3 shows that the CPU time improvements correspond to assessment improvements. Note that approximate algorithms like XGBoost of Chen and Guestrin [2016] are not competitors to Adaptive-Pruning Boost but rather potential “clients” because such algorithms train on a subset of the data. Therefore, they are not appropriate baselines to our method.

Table 3: Computational Complexity for LazyBoost and Boosting with Weight Trimming. All results are for 500 rounds of boosting except MNIST (300 rounds) and RCV1 (400 rounds).

| Dataset | Boosting        | CPU Time in Seconds | # Example Assessments |
|---------|----------------|---------------------|-----------------------|
|         |                | AP-B    | QB    | IMPROV. | AP-B    | QB    | IMPROV. |
| A6A     | LazyBoost (0.5)| 1.86e+02 | 1.95e+02 | 4.8% | 8.48e+08 | 9.22e+08 | 8.1% |
| MNIST   | LazyBoost (0.5)| 3.46e+05 | 3.52e+05 | 1.8% | 1.87e+11 | 2.07e+11 | 9.7% |
| RCV1    | LazyBoost (0.5)| 7.86e+04 | 7.54e+04 | -4.2% | 3.18e+11 | 3.29e+11 | 3.4% |
| SATIMAGE| LazyBoost (0.5)| 4.70e+02 | 5.48e+02 | 14.2% | 5.17e+08 | 6.11e+08 | 15.4% |
| W4A     | LazyBoost (0.5)| 1.15e+02 | 1.38e+02 | 26.8% | 8.61e+08 | 1.22e+09 | 29.3% |
| MEAN    |                | 8.68%   |       |        | 13.18%   |        |        |
| A6A     | Wt. Trim (0.9) | 3.28e+02 | 3.48e+02 | 5.6% | 1.51e+09 | 1.64e+09 | 7.7% |
| MNIST   | Wt. Trim (0.9) | 5.89e+05 | 6.09e+05 | 3.3% | 3.20e+11 | 3.59e+11 | 10.9% |
| RCV1    | Wt. Trim (0.9) | 1.38e+05 | 1.37e+05 | -1.0% | 5.60e+11 | 5.93e+11 | 5.6% |
| SATIMAGE| Wt. Trim (0.9) | 7.37e+02 | 8.89e+02 | 17.1% | 8.05e+08 | 1.01e+09 | 20% |
| W4A     | Wt. Trim (0.9) | 2.04e+02 | 2.82e+02 | 27.7% | 1.52e+09 | 2.19e+09 | 30.5% |
| MEAN    |                | 10.54%  |       |        | 14.94%   |        |        |
| A6A     | Wt. Trim (0.99)| 3.34e+02 | 3.38e+02 | 1.3% | 1.54e+09 | 1.58e+09 | 2.6% |
| MNIST   | Wt. Trim (0.99)| 5.80e+05 | 5.69e+05 | -1.8% | 3.16e+11 | 3.37e+11 | 6.1% |
| RCV1    | Wt. Trim (0.99)| 1.38e+05 | 1.37e+05 | -1.0% | 5.61e+11 | 5.86e+11 | 4.4% |
| SATIMAGE| Wt. Trim (0.99)| 6.49e+02 | 6.68e+02 | 2.9% | 7.01e+08 | 7.39e+08 | 5.1% |
| W4A     | Wt. Trim (0.99)| 1.91e+02 | 2.03e+02 | 6.0% | 1.44e+09 | 1.52e+09 | 5.3% |
| MEAN    |                | 1.48%   |       |        | 4.7%     |        |        |

6 Conclusion

In this paper, we introduced an efficient exact greedy-optimal algorithm, Adaptive-Pruning Boost, for boosted decision trees. Our experiments on various datasets show that our algorithm use fewer total example assessments compared to the state-of-the-art algorithm Quick Boost. We further showed that Adaptive-Pruning Boost almost matches the lower bound for its class of algorithms and the global lower bound for any algorithm.
References

Ron Appel, Thomas Fuchs, Piotr Dollar, and Pietro Perona. Quickly boosting decision trees – pruning underachieving features early. In *Proceedings of the 30th International Conference on Machine Learning (ICML)*, 2013.

Joseph K Bradley and E Schapire. Filterboost: Regression and classification on large datasets. In J. C. Platt, D. Koller, Y. Singer, and S. T. Roweis, editors, *Advances in Neural Information Processing Systems 20*, pages 185–192. Curran Associates, Inc., 2008.

R. Busa-Fekete and B. Kégl. Fast boosting using adversarial bandits. In *Proceedings of the 27th International Conference on Machine Learning (ICML)*, 2010. http://www.machinelearning.org.

Tianqi Chen and Carlos Guestrin. Xgboost: A scalable tree boosting system. In *Proceedings of the 22Nd ACM SIGKDD International Conference on Knowledge Discovery and Data Mining*, KDD ’16, pages 785–794, New York, NY, USA, 2016. ACM. ISBN 978-1-4503-4232-2. doi: 10.1145/2939672.2939785.

P. Dollar, Zhuowen Tu, H. Tao, and S. Belongie. Feature mining for image classification. In *IEEE Conference on Computer Vision and Pattern Recognition (CVPR ’07)*, pages 1–8, June 2007. doi: 10.1109/CVPR.2007.383046.

Charles Dubout and François Fleuret. Adaptive sampling for large scale boosting. *J. Mach. Learn. Res.*, 15(1):1431–1453, January 2014. ISSN 1532-4435.

G. Escudero, L. Màrquez, and G. Rigau. Using lazyboosting for word sense disambiguation. In *The Proceedings of the Second International Workshop on Evaluating Word Sense Disambiguation Systems*, 2001.

Yoav Freund. Boosting a weak learning algorithm by majority. *Inf. Comput.*, 121(2):256–285, September 1995. ISSN 0890-5401. doi: 10.1006/inco.1995.1136.

Yoav Freund and Robert E. Schapire. Experiments with a new boosting algorithm. In *Proceedings of the Thirteenth International Conference on International Conference on Machine Learning*, ICML’96, pages 148–156, San Francisco, CA, USA, 1996. Morgan Kaufmann Publishers Inc. ISBN 1-55860-419-7.

J. H. Friedman. Stochastic gradient boosting. In *In Computational Statistics & Data Analysis*, 2002., 2002.

Jerome Friedman, Trevor Hastie, and Robert Tibshirani. Additive logistic regression: a statistical view of boosting. *Annals of Statistics*, 28:2000, 1998.

Victor Gabillon, Mohammad Ghavamzadeh, and Alessandro Lazaric. Best arm identification: A unified approach to fixed budget and fixed confidence. In *Advances in Neural Information Processing Systems (NIPS)*, 2012.

Chih-Wei Hsu and Chih-Jen Lin. A comparison of methods for multiclass support vector machines. *IEEE Transactions on Neural Networks*, 13(2):415–425, Mar 2002. ISSN 1045-9227. doi: 10.1109/72.991427.

Shivaram Kalyanakrishnan, Ambuj Tewari, Peter Auer, and Peter Stone. PAC subset selection in stochastic multi-armed bandits. In *Proceedings of the 29th International Conference on Machine Learning (ICML)*, 2012.

Zohar Karnin, Tomer Koren, and Oren Somekh. Almost optimal exploration in multi-armed bandits. In *Proceedings of the 30th International Conference on Machine Learning (ICML-13)*, 2013.

E. Kaufmann and S. Kalyanakrishnan. Information complexity in bandit subset selection. In *Proceeding of the 26th Conference On Learning Theory*, 2013.

Y. Lecun, L. Bottou, Y. Bengio, and P. Haffner. Gradient-based learning applied to document recognition. *Proceedings of the IEEE*, 86(11):2278–2324, Nov 1998. ISSN 0018-9219. doi: 10.1109/5.726791.
David D. Lewis, Yiming Yang, Tony G. Rose, and Fan Li. Rcv1: A new benchmark collection for text categorization research. *J. Mach. Learn. Res.*, 5:361–397, December 2004. ISSN 1532-4435.

Biswajit Paul, G Athithan, and M Murty. Speeding up adaboost classifier with random projection, 03 2009.

John C. Platt. Advances in kernel methods. chapter Fast Training of Support Vector Machines Using Sequential Minimal Optimization, pages 185–208. MIT Press, Cambridge, MA, USA, 1999. ISBN 0-262-19416-3.

J. R. Quinlan. Bagging, boosting, and c4.s. In *Proceedings of the Thirteenth National Conference on Artificial Intelligence - Volume 1*, AAAI’96, pages 725–730. AAAI Press, 1996. ISBN 0-262-51091-X.

J. Ross Quinlan. *C4.5: Programs for Machine Learning*. Morgan Kaufmann Publishers Inc., San Francisco, CA, USA, 1993. ISBN 1-55860-238-0.

Robert E. Schapire. The strength of weak learnability. In *Machine Learning*, 1990.

Toby Sharp. Implementing decision trees and forests on a gpu. In *ECCV (4)*, volume 5305, pages 595–608. Springer, January 2008. ISBN 978-3-540-88692-1.

Jianxin Wu, S Charles Brubaker, Matthew D Mullin, and James Rehg. Fast asymmetric learning for cascade face detection. 30:369–82, 04 2008.
Appendix A  Additional Results

A.1  Train and Test Error for AdaBoost

Table 4 reports test and train errors at various Boosting rounds. Our algorithm achieves the test and train error in fewer total number of example assessments, compared to Quick Boost. Note that both algorithms, except in the case of RCV1, have the same test and train error at a given round, as they should because both train identical decision trees. The case of RCV1 is due to the algorithms picking a weak learner arbitrarily in case of ties, without changing the overall results significantly.

Table 4: AdaBoost results, reported at rounds 100, 300 and 500 (400 for RCV1).

| ALG: DATA | # ASSESS. | 100 | TRAIN | TEST | 300 | TRAIN | TEST | 400/500 | TRAIN | TEST |
|-----------|-----------|-----|-------|------|-----|-------|------|---------|-------|------|
| AP-B: A6A | 3.35e+08  | 0.142 | 0.155 | 1.02e+09 | 0.131 | 0.157 | 1.69e+09 | 0.128 | 0.160 |
| QB: A6A   | 3.57e+08  | 0.142 | 0.155 | 1.09e+09 | 0.131 | 0.157 | 1.83e+09 | 0.128 | 0.160 |
| AP-B: MNIST | 1.26e+11 | 0.106 | 0.111 | 3.52e+11 | 0.057 | 0.064 | — | — | — |
| QB: MNIST | 1.36e+11  | 0.106 | 0.111 | 3.96e+11 | 0.057 | 0.064 | — | — | — |
| AP-B: RCV1 | 1.73e+11 | 0.027 | 0.059 | 4.83e+11 | 0.005 | 0.047 | 6.15e+11 | 0.001 | 0.044 |
| QB: RCV1  | 1.85e+11  | 0.029 | 0.061 | 5.13e+11 | 0.004 | 0.047 | 6.58e+11 | 0.001 | 0.046 |
| AP-B: SATIMAGE | 1.98e+08 | 0.113 | 0.150 | 5.46e+08 | 0.070 | 0.121 | 8.64e+08 | 0.049 | 0.109 |
| QB: SATIMAGE | 2.20e+08 | 0.113 | 0.150 | 6.61e+08 | 0.070 | 0.121 | 1.11e+09 | 0.049 | 0.109 |
| AP-B: W4A | 3.92e+08  | 0.011 | 0.019 | 1.07e+09 | 0.006 | 0.018 | 1.69e+09 | 0.006 | 0.018 |
| QB: W4A   | 4.64e+08  | 0.011 | 0.020 | 1.45e+09 | 0.006 | 0.018 | 2.41e+09 | 0.006 | 0.018 |

A.2  Train and Test Error for LazyBoost and Weight Trimming

Table 5: Performance for A6A

| ALG LazyBoost (0.5) | # ASSESS. | 100 | TRAIN | TEST | 300 | TRAIN | TEST | 500 | TRAIN | TEST |
|---------------------|-----------|-----|-------|------|-----|-------|------|-----|-------|------|
| AP LazyBoost (0.5)  | 1.69e+08  | 0.145 | 0.156 | 5.11e+08 | 0.134 | 0.159 | 8.48e+08 | 0.129 | 0.160 |
| QB LazyBoost (0.5)  | 1.80e+08  | 0.145 | 0.157 | 5.50e+08 | 0.137 | 0.158 | 9.22e+08 | 0.132 | 0.160 |
| AP LazyBoost (0.9)  | 2.99e+08  | 0.141 | 0.156 | 9.07e+08 | 0.133 | 0.157 | 1.51e+09 | 0.130 | 0.159 |
| QB LazyBoost (0.9)  | 3.18e+08  | 0.141 | 0.156 | 9.75e+08 | 0.133 | 0.157 | 1.64e+09 | 0.130 | 0.159 |
| AP Wt. Trim (0.9)   | 2.45e+08  | 0.151 | 0.157 | 7.35e+08 | 0.151 | 0.157 | 1.23e+09 | 0.151 | 0.157 |
| QB Wt. Trim (0.9)   | 2.49e+08  | 0.151 | 0.157 | 7.46e+08 | 0.151 | 0.157 | 1.24e+09 | 0.151 | 0.157 |
| AP Wt. Trim (0.99)  | 3.16e+08  | 0.141 | 0.156 | 9.34e+08 | 0.132 | 0.157 | 1.54e+09 | 0.126 | 0.158 |
| QB Wt. Trim (0.99)  | 3.28e+08  | 0.141 | 0.156 | 9.62e+08 | 0.132 | 0.157 | 1.58e+09 | 0.126 | 0.160 |

Table 6: Performance for MNIST Digits

| ALG LazyBoost (0.5) | # ASSESS. | 100 | TRAIN | TEST | 200 | TRAIN | TEST | 300 | TRAIN | TEST |
|---------------------|-----------|-----|-------|------|-----|-------|------|-----|-------|------|
| AP LazyBoost (0.5)  | 6.65e+10  | 0.150 | 0.145 | 1.28e+11 | 0.098 | 0.098 | 1.87e+11 | 0.076 | 0.079 |
| QB LazyBoost (0.5)  | 7.07e+10  | 0.150 | 0.145 | 1.39e+11 | 0.098 | 0.098 | 2.07e+11 | 0.076 | 0.079 |
| AP LazyBoost (0.9)  | 1.17e+11  | 0.117 | 0.118 | 2.22e+11 | 0.079 | 0.085 | 3.20e+11 | 0.061 | 0.069 |
| QB LazyBoost (0.9)  | 1.25e+11  | 0.117 | 0.118 | 2.43e+11 | 0.079 | 0.085 | 3.59e+11 | 0.061 | 0.069 |
| AP Wt. Trim (0.9)   | 1.53e+11  | 0.901 | 0.901 | 3.07e+11 | 0.901 | 0.901 | 4.61e+11 | 0.901 | 0.901 |
| QB Wt. Trim (0.9)   | 1.53e+11  | 0.900 | 0.900 | 3.07e+11 | 0.900 | 0.900 | 4.61e+11 | 0.900 | 0.901 |
| AP Wt. Trim (0.99)  | 1.19e+11  | 0.117 | 0.124 | 2.21e+11 | 0.076 | 0.080 | 3.16e+11 | 0.062 | 0.068 |
| QB Wt. Trim (0.99)  | 1.29e+11  | 0.115 | 0.117 | 2.37e+11 | 0.074 | 0.078 | 3.37e+11 | 0.056 | 0.061 |
Table 7: Performance for RCV1

|                     | # Assess. | Train | Test | # Assess. | Train | Test | # Assess. | Train | Test |
|---------------------|-----------|-------|------|-----------|-------|------|-----------|-------|------|
| AP LazyBoost (0.5)  | 8.93e+10  | 0.029 | 0.061| 2.48e+11  | 0.006 | 0.047| 3.18e+11  | 0.002 | 0.046|
| QB LazyBoost (0.5)  | 9.06e+10  | 0.028 | 0.060| 2.55e+11  | 0.005 | 0.048| 3.29e+11  | 0.002 | 0.046|
| AP LazyBoost (0.9)  | 1.59e+11  | 0.027 | 0.058| 4.35e+11  | 0.005 | 0.047| 5.60e+11  | 0.002 | 0.045|
| QB LazyBoost (0.9)  | 1.64e+11  | 0.027 | 0.058| 4.62e+11  | 0.004 | 0.047| 5.93e+11  | 0.001 | 0.045|
| AP Wt. Trim (0.9)   | 1.19e+11  | 0.022 | 0.059| 2.92e+11  | 0.003 | 0.047| 3.65e+11  | 0.001 | 0.046|
| QB Wt. Trim (0.9)   | 1.22e+11  | 0.025 | 0.058| 3.03e+11  | 0.003 | 0.047| 3.79e+11  | 0.001 | 0.046|
| AP Wt. Trim (0.99)  | 1.62e+11  | 0.027 | 0.059| 4.40e+11  | 0.004 | 0.047| 5.61e+11  | 0.001 | 0.045|
| QB Wt Trim (0.99)   | 1.70e+11  | 0.027 | 0.059| 4.66e+11  | 0.004 | 0.048| 5.86e+11  | 0.001 | 0.046|

Table 8: Performance for SATIMAGE

|                     | # Assess. | Train | Test | # Assess. | Train | Test | # Assess. | Train | Test |
|---------------------|-----------|-------|------|-----------|-------|------|-----------|-------|------|
| AP LazyBoost (0.5)  | 1.11e+08  | 0.133 | 0.152| 3.22e+08  | 0.094 | 0.123| 5.17e+08  | 0.073 | 0.115|
| QB LazyBoost (0.5)  | 1.23e+08  | 0.130 | 0.150| 3.68e+08  | 0.090 | 0.129| 6.11e+08  | 0.067 | 0.113|
| AP LazyBoost (0.9)  | 1.88e+08  | 0.114 | 0.128| 5.13e+08  | 0.071 | 0.119| 8.05e+08  | 0.050 | 0.110|
| QB LazyBoost (0.9)  | 2.06e+08  | 0.114 | 0.128| 6.07e+08  | 0.071 | 0.119| 1.01e+09  | 0.050 | 0.110|
| AP Wt. Trim (0.9)   | 2.51e+08  | 0.756 | 0.766| 7.56e+08  | 0.756 | 0.766| 1.26e+09  | 0.756 | 0.766|
| QB Wt. Trim (0.9)   | 2.51e+08  | 0.755 | 0.765| 7.57e+08  | 0.755 | 0.765| 1.26e+09  | 0.755 | 0.765|
| AP Wt. Trim (0.99)  | 1.80e+08  | 0.109 | 0.141| 4.66e+08  | 0.066 | 0.121| 7.01e+08  | 0.045 | 0.113|
| QB Wt. Trim (0.99)  | 1.89e+08  | 0.109 | 0.141| 4.91e+08  | 0.066 | 0.121| 7.39e+08  | 0.045 | 0.113|

Table 9: Performance for W4A

|                     | # Assess. | Train | Test | # Assess. | Train | Test | # Assess. | Train | Test |
|---------------------|-----------|-------|------|-----------|-------|------|-----------|-------|------|
| AP LazyBoost (0.5)  | 2.00e+08  | 0.012 | 0.019| 5.46e+08  | 0.008 | 0.018| 8.61e+08  | 0.006 | 0.018|
| QB LazyBoost (0.5)  | 2.35e+08  | 0.012 | 0.019| 7.35e+08  | 0.008 | 0.018| 1.22e+09  | 0.006 | 0.018|
| AP LazyBoost (0.9)  | 3.48e+08  | 0.012 | 0.020| 9.66e+08  | 0.007 | 0.018| 1.52e+09  | 0.006 | 0.018|
| QB LazyBoost (0.9)  | 4.27e+08  | 0.012 | 0.020| 1.32e+09  | 0.007 | 0.018| 2.19e+09  | 0.006 | 0.018|
| AP Wt. Trim (0.9)   | 2.87e+08  | 0.016 | 0.021| 8.41e+08  | 0.016 | 0.021| 1.40e+09  | 0.016 | 0.021|
| QB Wt. Trim (0.9)   | 2.97e+08  | 0.016 | 0.021| 8.63e+08  | 0.016 | 0.021| 1.43e+09  | 0.016 | 0.021|
| AP Wt. Trim (0.99)  | 3.63e+08  | 0.012 | 0.020| 9.44e+08  | 0.007 | 0.017| 1.44e+09  | 0.006 | 0.018|
| QB Wt. Trim (0.99)  | 3.96e+08  | 0.012 | 0.020| 1.01e+09  | 0.007 | 0.018| 1.52e+09  | 0.006 | 0.018|
A.3 Different Tree Depths

Table 10: Different Tree Depths: Number of Assessments after 500 rounds

|       | 1     | 2     | 3     | 4     | 5     |
|-------|-------|-------|-------|-------|-------|
| A6A   | AP Boost | 6.40E+08 | 1.23E+09 | 1.69E+09 | 2.08E+09 | 2.44E+09 |
| W4A   | AP Boost | 8.71E+08 | 1.38E+09 | 1.69E+09 | 1.90E+09 | 2.12E+09 |

We also experimented with different tree depths, and found that Adaptive-Pruning Boost shows more dramatic gains in terms of total number of assessments when it uses deeper trees as weak learners. We believe this is because of accumulated gains for training more nodes in each tree. We have included an example of this in Table [10], where for two datasets (W4A, and A6A) we show experiments at depth 1 through 5. We report the total number of assessments used by AdaBoost (exact greedy-optimal decision trees) after 500 rounds.
Appendix B  Information Gain

Notation reference:

- $Z_n$ is the total weight of all $n$ training examples
- $Z_p$ is the weight of examples which reached some leaf $\rho$.
- $Z_u$ and $Z_\bar{u}$ are the seen and unseen weight for leaf $\rho$ (where $\rho$ should be clear from context), so $Z_u + Z_\bar{u} = Z_p$.
- $Z_y^u$ is the total weight for leaf $\rho$ with label $y$.
- $Z_y^u$ and $Z_y^\bar{u}$ are the seen and unseen weight for leaf $\rho$ with label $y$, so $Z_y^u + Z_y^\bar{u} = Z_y^\rho$.
- $Z_y^\rho$ is the total weight for leaf $\rho$ with some label other than $y$, so $Z_y^\rho = Z_p - Z_y^u$.
- $Z_y$ is the total weight of all leaves, so $Z_y = \sum_{\rho} Z_\rho^y$.
- $w^u$ and $w^\bar{u}$ are the fraction of total unseen weight with and without label $y$, so $w^u + w^\bar{u} = w$.

The “error” term for Information Gain is the conditional entropy of the leaves, written as follows.

$$\epsilon_n := \sum_{\rho} Z_p \frac{Z_y^u}{Z_n} \left( - \sum_{y} Z_y^u \frac{Z_y^u}{Z_p} \right) \Rightarrow Z_n \epsilon_n = \sum_{\rho} \left( - \sum_{y} Z_y^u \frac{Z_y^u}{Z_p} \right)$$

$$Z_p \epsilon_\rho = - \sum_{y} Z_y^u \frac{Z_y^u}{Z_p} = - \sum_{y} (Z_u^y + Z_\bar{u}^y) \frac{Z_y^u + Z_\bar{u}^u}{Z_u + Z_\bar{u}}$$

$$= \left( - \sum_{y} Z_u^y \frac{Z_y^u}{Z_u} \right) + \left( - \sum_{y} Z_\bar{u}^y \frac{Z_y^\bar{u}}{Z_\bar{u}} \right) + \sum_{y} Z_y^\rho \text{KL} \left( \mathcal{B} \left( \frac{Z_y^u}{Z_p} \right) \| \mathcal{B} \left( \frac{Z_u}{Z_p} \right) \right),$$

where the final equality follows by Lemma 2 proved below. The bounds on information gain thus ultimately depend on $Z_u \epsilon_\rho$ and on the KL divergence term,

$$\sum_{y} Z_y^\rho \text{KL} \left( \mathcal{B} \left( \frac{Z_y^u}{Z_p} \right) \| \mathcal{B} \left( \frac{Z_u}{Z_p} \right) \right)$$

(1)

where KL $(\cdot \| \cdot)$ is the Kullback-Liebler divergence and $\mathcal{B} (\cdot)$ is a Bernoulli probability distribution.

$$\text{KL} \left( \mathcal{B} (p) \| \mathcal{B} (q) \right) = p \log \frac{p}{q} + (1 - p) \log \frac{1 - p}{1 - q}$$

Since $Z_u \epsilon_\rho \geq 0$ and KL divergence are non-negative, a trivial lower bound is

$$Z_p \epsilon_\rho \geq Z_u \epsilon_\rho = - \sum_{y} Z_y^u \frac{Z_y^u}{Z_u}.$$  

(2)

It remains to prove an upper bound. We upper bound the weight $Z_y^\rho$ of KL divergence as $Z_y^\rho \leq Z_y^u + w^u$. Below, we prove the following upper bound on the KL divergence in Eq. 1.

Lemma 1 (KL Upper Bound). For any individual leaf $\rho$ and label $y$, we have

$$\text{KL} \left( \mathcal{B} \left( \frac{Z_y^u}{Z_p} \right) \| \mathcal{B} \left( \frac{Z_y}{Z_p} \right) \right) \leq \log \frac{Z_u^u + w^u}{Z_y^u}.$$
In order to complete our upper bound, we note that \( Z \bar{n} u \) is simply the unassessed weight \( Z \bar{n} u \) times the label entropy for the unassessed weight, and with \(|Y|\) total labels the label entropy is upper bounded as \( \lg |Y| \). This yields the following bounds on the conditional entropy term for Information Gain.

\[
\sum_{\rho} Z_{u} \epsilon_{u} \leq Z_{\rho} \epsilon_{\rho} \leq \sum_{\rho} \left[ Z_{u} \epsilon_{u} + w \lg |Y| + \sum_{y} \left( (Z_{u}^{y} + w^{y}) \lg \frac{Z_{u} + w}{Z_{u}^{y}} \right) \right] \tag{3}
\]

Our proofs follow.

**Proof of Lemma [7]** We bound the KL divergence using the Reyni divergence and by bounding the two Bernoulli probability ratios. Our probabilities are

\[
\left( \frac{Z_{u}^{y}}{Z_{\rho}^{y}}, 1 - \frac{Z_{u}^{y}}{Z_{\rho}^{y}} \right) = \left( \frac{Z_{u}^{y}}{Z_{u}^{y} + Z_{u}}, \frac{Z_{u}^{y}}{Z_{u} + Z_{u}} \right)
\]

and

\[
\left( \frac{Z_{u}}{Z_{\rho}}, 1 - \frac{Z_{u}}{Z_{\rho}} \right) = \left( \frac{Z_{u}}{Z_{u} + Z_{u}}, \frac{Z_{u}}{Z_{u} + Z_{u}} \right)
\]

Our two ratio are upper bounded as follows

\[
\left( \frac{Z_{u}^{y}}{Z_{u}^{y} + Z_{u}} \right) = \frac{Z_{u}^{y}}{Z_{u}^{y} + Z_{u}} \leq \frac{Z_{u}^{y}}{Z_{u}} \leq \frac{Z_{u} + w}{Z_{u}} \leq \frac{Z_{u} + w}{Z_{u}} \tag{6}
\]

and

\[
\left( \frac{Z_{u}}{Z_{u} + Z_{u}} \right) = \frac{Z_{u}}{Z_{u} + Z_{u}} \leq \frac{Z_{u} + Z_{u}}{Z_{u} + Z_{u}} \leq \frac{Z_{u} + Z_{u}}{Z_{u} + Z_{u}} \leq \frac{Z_{u} + Z_{u}}{Z_{u} + Z_{u}} \leq \frac{Z_{u} + Z_{u}}{Z_{u} + Z_{u}} \tag{7}
\]

Since

\[
\frac{Z_{u} + w}{Z_{u}} \leq \frac{Z_{u} + w}{Z_{u}} \tag{8}
\]

by the Reyni Divergence of \( \infty \) order \( D_{\infty}(B(p) \parallel B(q)) = \lg \sup_{p \neq q} \frac{p}{q} \) (i.e. the log of the maximum ratio of probabilities) we conclude that

\[
\text{KL} \left( B \left( \frac{Z_{u}^{y}}{Z_{\rho}^{y}} \right) \parallel B \left( \frac{Z_{u}}{Z_{\rho}} \right) \right) \leq \lg \frac{Z_{u} + w}{Z_{u}}.
\]

\( \Box \)

**Lemma 2.** For \( a, b \geq 0 \) and \( \alpha, \beta > 0 \)

\[
(a + b) \lg \frac{a + b}{\alpha + \beta} = a \lg \frac{a}{\alpha} + b \lg \frac{b}{\beta} - (a + b) \text{KL} \left( B \left( \frac{a}{a + b} \right) \parallel B \left( \frac{\alpha}{\alpha + \beta} \right) \right) \tag{9}
\]
Proof.

\[(a + b) \log \frac{a + b}{\alpha + \beta} = a \log a + b \log b + \beta(a + b)\]

\[= a \log a + b \log b + a + b + (a + b) \left[ \frac{a}{a+b} \log \frac{a+\beta}{a} + \frac{b}{a+b} \log \frac{\beta}{a+\beta} \right] \]

\[= a \log a + b \log b + (a + b) \left[ \frac{a}{a+b} \log \frac{a}{a+(a+b)} + \frac{b}{a+b} \log \frac{\beta}{a+\beta} \right] \]

\[= a \log a + b \log b - (a + b) KL \left( \mathcal{B} \left( \frac{a}{a+b} \right) \| \mathcal{B} \left( \frac{\alpha}{\alpha+\beta} \right) \right) \]

\[= a \log a + b \log b - (a + b) KL \left( \mathcal{B} \left( \frac{a}{a+b} \right) \| \mathcal{B} \left( \frac{\alpha}{\alpha+\beta} \right) \right)\]