An Element $\phi$-$\delta$-Primary to another Element in Multiplicative Lattices

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Abstract

In this paper, we introduce an element $\phi$-$\delta$-primary to another element in a compactly generated multiplicative lattice $L$ and obtain its characterizations. We prove many of its properties and investigate the relations between these structures. By a counter example, it is shown that if an element $b \in L$ is $\phi$-$\delta$-primary to a proper element $p \in L$ then $b$ need not be $\delta$-primary to $p$ and found conditions under which an element $b \in L$ is $\delta$-primary to a proper element $p \in L$ if $b$ is $\phi$-$\delta$-primary to $p$.

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1 Introduction

The notion of an element prime to another element in a multiplicative lattice $L$ is introduced by F. Alarcon et. al. in [3]. Further, the notion of an element primary to another element in a multiplicative lattice $L$ is introduced by C. S. Manjarekar and Nitin S. Chavan in [4]. In an attempt to unify these notions of an element prime to another element and an element primary to another element in a multiplicative lattice $L$ under one frame, an element $\delta$-primary to another element in a multiplicative lattice $L$ is introduced by Ashok V. Bingi in [1].

Further, the concept of an element weakly prime to another element and an element weakly primary to another element in a multiplicative lattice $L$ is introduced by C. S. Manjarekar and U. N. Kandale in [5]. To generalise these concepts, the study of an element $\phi$-prime to another element and an element $\phi$-primary to another element in a multiplicative lattice $L$ is done by Ashok V. Bingi in [2]. In this paper, we introduce and study, the notion of an element $\phi$-$\delta$-primary to another element in a multiplicative lattice $L$ as a generalization of an element $\delta$-primary to another element in $L$ and unify an element $\phi$-prime to another element and an element $\phi$-primary to another element in $L$, under one frame.

A multiplicative lattice $L$ is a complete lattice provided with commutative, associative and join distributive multiplication in which the largest element 1 acts as a multiplicative identity. An element $e \in L$ is called meet principal if $a \land be = ((a : e) \land b)e$ for all $a$, $b \in L$. An element $e \in L$ is called join principal if $(ae \lor b) : e = (b : e) \lor a$ for all $a$, $b \in L$. An element $e \in L$ is called principal if $e$ is both meet principal and join principal. A multiplicative lattice $L$ is said to be
principally generated (PG) if every element of \( L \) is a join of principal elements of \( L \).

An element \( a \in L \) is called compact if for \( X \subseteq L, \ a \leq \bigvee X \) implies the existence of a finite number of elements \( a_1, a_2, \ldots, a_n \) in \( X \) such that \( a \leq a_1 \lor a_2 \lor \cdots \lor a_n \). The set of compact elements of \( L \) will be denoted by \( L_c \). If each element of \( L \) is a join of compact elements of \( L \) then \( L \) is called a compactly generated lattice or simply a CG-lattice.

An element \( a \in L \) is said to be proper if \( a < 1 \). The radical of \( a \in L \) is denoted by \( \sqrt{a} \) and is defined as \( \bigvee \{ x \in L_+ \mid x^n \leq a, \text{ for some } n \in \mathbb{Z}_+ \} \). A proper element \( m \in L \) is said to be maximal if for every element \( x \in L \) such that \( m < x \leq 1 \) implies \( x = 1 \). A proper element \( p \in L \) is called a prime element if \( ab \leq p \) implies \( a \leq p \) or \( b \leq p \) where \( a, b \in L \) and is called a primary element if \( ab \leq p \) implies \( a \leq p \) or \( b \leq \sqrt{p} \) where \( a, b \in L_\star \). For \( a, b \in L_\star \) \( (a : b) = \bigvee \{ x \in L \mid xb \leq a \} \). A multiplicative lattice is called as a Noether lattice if it is modular, principally generated and satisfies ascending chain condition. An element \( a \in L \) is called a zero divisor if \( ab = 0 \) for some \( 0 \neq b \in L \) and is called idempotent if \( a = a^2 \). A multiplicative lattice is said to be a domain if it is without zero divisors and is said to be quasi-local if it contains a unique maximal element. A quasi-local multiplicative lattice \( L \) with maximal element \( m \) is denoted by \( (L, m) \). A Noether lattice \( L \) is local if it contains precisely one maximal prime. In a Noether lattice \( L \), an element \( a \in L \) is said to satisfy restricted cancellation law if for all \( b, c \in L, ab = ac \neq 0 \) implies \( b = c \) (see [8]). According to [6], an expansion function on \( L \) is a function \( \delta : L \rightarrow L \) which satisfies the following two conditions: \( 1 \). \( a \leq \delta(a) \) for all \( a \in L \), \( 2 \). \( a \leq b \) implies \( \delta(a) \leq \delta(b) \) for all \( a, b \in L \). The reader is referred to [3] for general background and terminology in multiplicative lattices.

According to [3], an element \( b \in L \) is said to be prime to a proper element \( p \in L \) if \( xb \leq p \) implies \( x \leq p \) where \( x \in L \). According to [4], an element \( b \in L \) is said to be primary to a proper element \( p \in L \) if \( xb \leq p \) implies \( x \leq \sqrt{p} \) where \( x \in L_\star \). According to [5], an element \( b \in L \) is said to be weakly prime to a proper element \( p \in L \) if \( 0 \neq xb \leq p \) implies \( x \leq p \) where \( x \in L \) and an element \( b \in L \) is said to be weakly primary to a proper element \( p \in L \) if \( 0 \neq xb \leq p \) implies \( x \leq \sqrt{p} \) where \( x \in L_\star \).

Further, according to [1], given an expansion function \( \delta \) on \( L \), an element \( b \in L \) is said to be \( \delta \)-primary to a proper element \( p \in L \) if for all \( x \in L, xb \leq p \) implies \( x \leq \delta(p) \). According to [2], given a function \( \phi : L \rightarrow L \), an element \( b \in L \) is said to be \( \phi \)-prime to a proper element \( p \in L \) if for all \( x \in L, xb \leq p \) and \( xb \notin \phi(p) \) implies \( x \leq p \) and an element \( b \in L \) is said to be \( \phi \)-primary to a proper element \( p \in L \) if for all \( x \in L, xb \leq p \) and \( xb \notin \phi(p) \) implies \( x \leq \sqrt{p} \).

In this paper, we define an element \( \phi \cdot \delta \)-primary to another element in \( L \) and obtain their characterizations. The notion of an element \( \phi \cdot \alpha \)-primary to another element in \( L \) is introduced and relations among them are obtained. By counter examples, it is shown that if \( b \in L \) is \( \phi \cdot \delta \)-primary to a proper element of \( p \in L \) then \( b \) need not be \( \phi \)-prime to \( p \), \( b \) need not be prime to \( p \) and \( b \) need not be \( \delta \)-primary to \( p \). In 6 different ways, we have proved if an element \( b \in L \) is \( \phi \cdot \delta \)-primary to a proper element \( p \) then \( b \) is \( \delta \)-primary to \( p \) under certain conditions. We define an element \( 2 \)-potent \( \delta \)-primary to another element of \( L \) and an element \( n \)-potent
δ-primary to another element of \( L \). Finally, we show that for an idempotent element \( p \in L, b \in L \) is \( \phi_2-\delta \)-primary to \( p \) but if \( b \in L \) is \( \phi_2-\delta \)-primary to a proper element \( p \in L \) then \( p \) need not be idempotent. Throughout this paper, (1). \( L \) denotes a compactly generated multiplicative lattice with 1 compact in which every finite product of compact elements is compact, (2). \( \delta \) denotes an expansion function on \( L \) and (3). \( \phi \) denotes a function defined on \( L \).

2 An element \( \phi-\delta \)-primary to another element in \( L \)

We begin with introducing the notion of an element of \( L \) to be \( \phi-\delta \)-primary to another element of \( L \) which is the generalization of the concept of an element to be \( \delta \)-primary to another element of \( L \).

**Definition 2.1.** Given an expansion function \( \delta : L \rightarrow L \) and a function \( \phi : L \rightarrow L \), an element \( b \in L \) is said to be \( \phi-\delta \)-primary to a proper element \( p \in L \) if for all \( x \in L, xb \leq p \) and \( xb \nleq \phi(p) \) implies \( x \leq \delta(p) \).

For the special functions \( \phi_\alpha : L \rightarrow L \), an element “\( \phi_\alpha-\delta \)-primary to” another element in \( L \) is defined by following settings in the definition 2.1 of an element \( \phi-\delta \)-primary to another element in \( L \). For any proper element \( p \in L \) in the definition 2.1 in place of \( \phi(p) \), set

- \( \phi_0(p) = 0 \). Then \( b \in L \) is called weakly \( \delta \)-primary to \( p \).
- \( \phi_2(p) = p^2 \). Then \( b \in L \) is called 2-almost \( \delta \)-primary to \( p \) or \( \phi_2-\delta \)-primary to \( p \) or simply almost \( \delta \)-primary to \( p \).
- \( \phi_n(p) = p^n \). Then \( b \in L \) is called \( n \)-almost \( \delta \)-primary to \( p \) or \( \phi_n-\delta \)-primary to \( p \) \( (n > 2) \).
- \( \phi_\omega(p) = \bigwedge_{n=1}^{\infty} p^n \). Then \( b \in L \) is called \( \omega \)-almost \( \delta \)-primary to \( p \) or \( \phi_\omega-\delta \)-primary to \( p \).

Since for an element \( a \in L \) with \( a \leq q \) but \( a \ngeq \phi(q) \) implies that \( a \ngeq q \land \phi(q) \), there is no loss generality in assuming that \( \phi(q) \leq q \). We henceforth make this assumption.

**Definition 2.2.** Given any two functions \( \gamma_1, \gamma_2 : L \rightarrow L \), we define \( \gamma_1 \leq \gamma_2 \) if \( \gamma_1(a) \leq \gamma_2(a) \) for each \( a \in L \).

Clearly, we have the following order:

\[
\phi_0 \leq \phi_\omega \leq \cdots \leq \phi_{n+1} \leq \phi_n \leq \cdots \leq \phi_2 \leq \phi_1
\]

Further as \( \phi(p) \leq p \) and \( p \leq \delta(p) \) for each \( p \in L \), the relation between the functions \( \delta \) and \( \phi \) is \( \phi \leq \delta \).

According to [6], \( \delta_0 \) is an expansion function on \( L \) defined as \( \delta_0(p) = p \) for each \( p \in L \) and \( \delta_1 \) is an expansion function on \( L \) defined as \( \delta_1(p) = \sqrt{p} \) for each \( p \in L \).

The following 2 results relate an element \( \phi \)-prime to another element and an element \( \phi-\delta \)-primary to another element with some element \( \phi-\delta \)-primary to another element in \( L \).
Theorem 2.3. An element \( b \in L \) is \( \phi-\delta_0 \)-primary to a proper element \( p \in L \) if and only if \( b \) is \( \phi \)-prime to \( p \).

Proof. The proof is obvious. \( \square \)

Theorem 2.4. An element \( b \in L \) is \( \phi-\delta_1 \)-primary to a proper element \( p \in L \) if and only if \( b \) is \( \phi \)-primary to \( p \).

Proof. The proof is obvious. \( \square \)

Theorem 2.5. Let \( \delta, \gamma : L \to L \) be expansion functions on \( L \) such that \( \delta \leq \gamma \). Let \( p \in L \) be a proper element and \( b \in L \). If \( b \) is \( \phi-\delta \)-primary to \( p \) then \( b \) is \( \phi-\gamma \)-primary to \( p \). In particular, for every expansion function \( \delta \) on \( L \), if \( b \) is \( \phi \)-prime to \( p \) then \( b \) is \( \phi-\delta_0 \)-primary to \( p \).

Proof. Assume that \( b \in L \) is \( \phi-\delta \)-primary to a proper element \( p \in L \). Suppose \( xb \leq p \) and \( xb \notin \phi(p) \) for \( x \in L \). Then \( x \leq \delta(p) \leq \gamma(p) \) and so \( b \) is \( \phi-\gamma \)-primary to \( p \). Next, for any expansion function \( \delta \) on \( L \), we have \( \delta_0 \leq \delta \). So if \( b \) is \( \phi-\delta_0 \)-primary to \( p \) then \( b \) is \( \phi-\delta \)-primary to \( p \) and we are done because if \( b \) is \( \phi \)-prime to \( p \) then \( b \) is \( \phi-\delta_0 \)-primary to \( p \). \( \square \)

Corollary 2.6. For every expansion function \( \delta \) on \( L \), if an element \( b \in L \) is prime to a proper element \( p \in L \) then \( b \) is \( \phi-\delta \)-primary to \( p \).

Proof. The proof follows by using Theorem 2.3 to the fact that if an element \( b \in L \) is prime to a proper element \( p \in L \) then \( b \) is \( \phi \)-prime to \( p \). \( \square \)

The following example shows that (by taking \( \phi \) as \( \phi_2 \) and \( \delta \) as \( \delta_1 \) for convenience)

- If \( b \in L \) is \( \phi-\delta \)-primary to a proper element \( p \in L \) then \( b \) need not be \( \phi \)-prime to \( p \).
- If \( b \in L \) is \( \phi-\delta \)-primary to a proper element \( p \in L \) then \( b \) need not be prime to \( p \).

Example 2.7. Consider the lattice \( L \) of ideals of the ring \( R = \langle \mathbb{Z}_{24}, +, \cdot \rangle \). Then the only ideals of \( R \) are the principal ideals \( (0), (2), (3), (4), (6), (8), (12), (1) \). Clearly, \( L = \{ (0), (2), (3), (4), (6), (8), (12), (1) \} \) is a compactly generated multiplicative lattice. It is easy to see that the element \( (2) \in L \) is \( \phi_2-\delta_1 \)-primary to \( (4) \in L \) while \( (2) \) is not \( \phi_2 \)-prime to \( (4) \). Also \( (2) \) is not prime to \( (4) \).

Now before obtaining the characterizations of an element \( \phi-\delta \)-primary to another element of \( L \), we state the following essential lemma which is outcome of Lemma 23.13 from [7].

Lemma 2.8. Let \( a_1, a_2 \in L \). Suppose \( b \in L \) satisfies the following property:

(\( \ast \)). If \( h \in L \) with \( h \leq b \) then either \( h \leq a_1 \) or \( h \leq a_2 \).

Then either \( b \leq a_1 \) or \( b \leq a_2 \).
Theorem 2.9. Let $p$ be a proper element of $L$ and $b \in L$. Then the following statements are equivalent:

1. $b$ is $\phi$-$\delta$-primary to $p$.
2. either $(p : b) \leq \delta(p)$ or $(p : b) = (\phi(p) : b)$.
3. for every $r \in L_*$, $rb \leq p$ and $rb \notin \phi(p)$ implies $r \leq \delta(p)$.

Proof. (1)$\implies$(2). Suppose (1) holds. Let $h \in L_*$ be such that $h \leq (p : b)$. Then $hb \leq p$. If $hb \leq \phi(p)$ then $h \leq (\phi(p) : b)$. If $hb \notin \phi(p)$ then since $b$ is $\phi$-$\delta$-primary to $p$, $hb \leq p$ and $hb \notin \phi(p)$, it follows that $h \leq \delta(p)$. Hence by Lemma 2.8, either $(p : b) \leq (\phi(p) : b)$ or $(p : b) \leq \delta(p)$. Consequently, either $(p : b) = (\phi(p) : b)$ or $(p : b) \leq \delta(p)$.

(2)$\implies$(3). Suppose (2) holds. Let $rb \leq p$ and $rb \notin \phi(p)$ for $r \in L_*$. By (2) if $(p : b) = (\phi(p) : b)$ then as $r \leq (p : b)$, it follows that $r \leq (\phi(p) : b)$ which contradicts $rb \notin \phi(p)$ and so we must have $(p : b) \leq \delta(p)$. Therefore $r \leq (p : b)$ gives $r \leq \delta(p)$.

(3)$\implies$(1). Suppose (3) holds. Let $xb \leq p$ and $xb \notin \phi(p)$ for $x \in L$. Then as $L$ is compactly generated, there exist $y' \in L_*$ such that $y' \leq x$ and $y'b \notin \phi(p)$. Let $y \leq x$ be any compact element of $L$. Then $(y \lor y') \in L_*$ such that $(y \lor y')b \leq p$ and $(y \lor y')b \notin \phi(p)$. So by (3), it follows that $(y \lor y') \leq \delta(p)$ which implies $x \leq \delta(p)$ and therefore $b$ is $\phi$-$\delta$-primary to $p$.

Theorem 2.10. Let $(L, m)$ be a quasi-local Noether lattice. If a proper element $p \in L$ is such that $p^2 = m^2 \leq p \leq m$ and $b \in L$ then $b$ is either $\phi_2$-$\delta_1$-primary to $p$ or $b \leq p$.

Proof. Let $xb \leq p$ and $xb \notin \phi_2(p)$ for $x \in L$. If $x \notin m$ then $x = 1$. So $xb \leq p$ gives $b \leq p$. Now if $x \leq m$ then $x^2 \leq m^2 = p^2 \leq p$ and hence $x \leq \delta_1(p)$ which implies $b$ is $\phi_2$-$\delta_1$-primary to $p$. Thus $b$ is either $\phi_2$-$\delta_1$-primary to $p$ or $b \leq p$.

To obtain the relation among an element $\phi_\alpha$-$\delta$-primary to another element in $L$, we prove the following lemma.

Lemma 2.11. Let $\gamma_1, \gamma_2 : L \to L$ be functions on $L$ such that $\gamma_1 \leq \gamma_2$ and $p$ be proper element of $L$. If an element $b \in L$ is $\gamma_1$-$\delta$-primary to $p$ then $b \in L$ is $\gamma_2$-$\delta$-primary to $p$.

Proof. Let an element $b \in L$ be $\gamma_1$-$\delta$-primary to $p$. Suppose $xb \leq p$ and $xb \notin \gamma_2(p)$ for $x \in L$. Then as $\gamma_1 \leq \gamma_2$, we have $xb \leq p$ and $xb \notin \gamma_1(p)$. Since $b$ is $\gamma_1$-$\delta$-primary to $p$, it follows that $x \leq \delta(p)$ and hence $b$ is $\gamma_2$-$\delta$-primary to $p$.

Theorem 2.12. For an element $b \in L$ and a proper element $p \in L$, consider the following statements:

(a) $b$ is $\delta$-primary to $p$.
(b) $b$ is $\phi_0$-$\delta$-primary to $p$.
(c) $b$ is $\phi_\omega$-$\delta$-primary to $p$.
(d) $b$ is $\phi_{(n+1)}$-$\delta$-primary to $p$. 
(e) \( b \) is \( \phi_n-\delta \)-primary to \( p \) where \( n \geq 2 \).

(f) \( b \) is \( \phi_2-\delta \)-primary to \( p \).

Then \( (a) \implies (b) \implies (c) \implies (d) \implies (e) \implies (f) \).

**Proof.** Obviously, if \( b \) is \( \delta \)-primary to \( p \) then \( b \) is weakly \( \delta \)-primary to \( p \) and hence \( (a) \implies (b) \). The remaining implications follow by using Lemma 2.11 to the fact that \( \phi_0 \leq \phi_\omega \leq \cdots \leq \phi_{n+1} \leq \phi_n \leq \cdots \leq \phi_2 \). □

**Corollary 2.13.** Let \( p \in L \) be a proper element and \( b \in L \). Then \( b \) is \( \phi_\omega-\delta \)-primary to \( p \) if and only if \( b \) is \( \phi_n-\delta \)-primary to \( p \) for every \( n \geq 2 \).

**Proof.** Assume that \( b \) is \( \phi_n-\delta \)-primary to \( p \) for every \( n \geq 2 \). Let \( xb \leq p \) and \( xb \notin \bigwedge_{n=1}^\infty p^n \) for \( x \in L \). Then \( xb \leq p \) and \( xb \notin p^n \) for some \( n \geq 2 \). Since \( b \) is \( \phi_n-\delta \)-primary to \( p \), we have \( x \leq \delta(p) \) and hence \( b \) is \( \phi_\omega-\delta \)-primary to \( p \). The converse follows from Theorem 2.12. □

Now we show that under a certain condition, if an element \( b \in L \) is \( \phi_n-\delta \)-primary \((n \geq 2)\) to a proper element \( p \in L \) then \( b \) is \( \delta \)-primary to \( p \).

**Theorem 2.14.** Let \( L \) be a local Noetherian domain. Let \( p \in L \) be a proper element and \( 0 \neq b \in L \). Then \( b \) is \( \phi_n-\delta \)-primary to \( p \) for every \( n \geq 2 \) if and only if \( b \) is \( \delta \)-primary to \( p \).

**Proof.** Assume that \( b \) is \( \phi_n-\delta \)-primary to \( p \) for every \( n \geq 2 \). Let \( xb \leq p \) for \( x \in L \). If \( xb \notin \phi_n(p) \) for \( n \geq 2 \) then as \( b \) is \( \phi_n-\delta \)-primary to \( p \), we have \( x \leq \delta(p) \). If \( xb \leq \phi_n(p) = p^n \) for all \( n \geq 1 \) then as \( L \) is local Noetherian, by Corollary 3.3 of [?], it follows that \( xb \leq \bigwedge_{n=1}^\infty p^n = 0 \) and so \( x = 0 \). Since \( L \) is domain and \( 0 \neq b \), we have \( x = 0 \) which implies \( x \leq \delta(p) \). Hence, in any case, \( b \) is \( \delta \)-primary to \( p \). Converse follows from Theorem 2.12. □

**Corollary 2.15.** Let \( L \) be a local Noetherian domain. Let \( p \in L \) be a proper element and \( 0 \neq b \in L \). Then \( b \) is \( \phi_\omega-\delta \)-primary to \( p \) if and only if \( b \) is \( \delta \)-primary to \( p \).

**Proof.** The proof follows from Theorem 2.14 and Corollary 2.13. □

Clearly, if an element \( b \in L \) is \( \delta \)-primary to a proper element \( p \in L \) then \( b \) is \( \phi-\delta \)-primary to \( p \). The following example shows that its converse is not true (by taking \( \phi \) as \( \phi_2 \) and \( \delta \) as \( \delta_1 \) for convenience).

**Example 2.16.** Consider the lattice \( L \) of ideals of the ring \( R = \langle Z_{30}, +, \cdot \rangle \). Then the only ideals of \( R \) are the principal ideals \( \langle 0 \rangle, \langle 2 \rangle, \langle 3 \rangle, \langle 5 \rangle, \langle 6 \rangle, \langle 10 \rangle, \langle 15 \rangle, \langle 1 \rangle \). Clearly \( L = \{ \langle 0 \rangle, \langle 2 \rangle, \langle 3 \rangle, \langle 5 \rangle, \langle 6 \rangle, \langle 10 \rangle, \langle 15 \rangle, \langle 1 \rangle \} \) is a compactly generated multiplicative lattice. It is easy to see that the element \( \langle 2 \rangle \in L \) is \( \phi_2-\delta_1 \)-primary to \( \langle 6 \rangle \in L \) but \( \langle 2 \rangle \) is not \( \delta_1 \)-primary to \( \langle 6 \rangle \).

In the following successive six theorems, we show conditions under which if an element \( b \in L \) is \( \phi-\delta \)-primary to a proper element \( p \) then \( b \) is \( \delta \)-primary to \( p \).
Theorem 2.17. Let \( L \) be a Noether lattice. Let \( 0 \neq p \in L \) be a non-nilpotent proper element satisfying the restricted cancellation law. Let \( b \in L \) be such that \( p < b \). Then \( b \) is \( \phi \)-\( \delta \)-primary to \( p \) for some \( \phi \leq \phi_2 \) if and only if \( b \) is \( \delta \)-primary to \( p \).

Proof. Assume that \( b \in L \) is \( \delta \)-primary to \( p \in L \). Then obviously, \( b \) is \( \phi \)-\( \delta \)-primary to \( p \) for every \( \phi \) and hence for some \( \phi \leq \phi_2 \). Conversely, assume that \( b \) is \( \phi \)-\( \delta \)-primary to \( p \) for some \( \phi \leq \phi_2 \). Then by Lemma 2.11, \( b \) is \( \phi_2 \)-\( \delta \)-primary (almost \( \delta \)-primary) to \( p \). Let \( xb \leq p \) for \( x \in L \). If \( xb \notin \phi_2(p) \) then as \( b \) is \( \phi_2 \)-\( \delta \)-primary to \( p \), we have \( x \leq \delta(p) \). If \( xb \leq \phi_2(p) = p^2 \) then \( xp \leq p^2 \neq 0 \) as \( p < b \). Hence \( x \leq p \leq \delta(p) \) by Lemma 1.11 of [8] and thus \( b \) is \( \delta \)-primary to \( p \).

Corollary 2.18. Let \( L \) be a Noether lattice. Let \( 0 \neq p \in L \) be a non-nilpotent proper element satisfying the restricted cancellation law. Let \( b \in L \) be such that \( p < b \). If \( b \) is \( \phi_2 \)-\( \delta \)-primary to \( p \) then \( b \) is \( \delta \)-primary to \( p \).

Proof. The proof follows from proof of the Theorem 2.17.

The following result is general form of Theorem 2.17.

Theorem 2.19. Let \( L \) be a Noether lattice. Let \( 0 \neq p \in L \) be a non-nilpotent proper element satisfying the restricted cancellation law. Let \( b \in L \) be such that \( p < b \). Then \( b \) is \( \phi \)-\( \delta \)-primary to \( p \) for some \( \phi \leq \phi_n \) and for all \( n \geq 2 \) if and only if \( b \) is \( \delta \)-primary to \( p \).

Proof. Assume that \( b \) is \( \delta \)-primary to \( p \). Then obviously, \( b \) is \( \phi \)-\( \delta \)-primary to \( p \) for every \( \phi \) and hence for some \( \phi \leq \phi_n \), for all \( n \geq 2 \). Conversely, assume that \( b \) is \( \phi \)-\( \delta \)-primary to \( p \) for some \( \phi \leq \phi_n \) and for all \( n \geq 2 \). Then by Lemma 2.11, \( b \) is \( \phi_n \)-\( \delta \)-primary (almost \( \delta \)-primary) to \( p \) and for all \( n \geq 2 \). Let \( xb \leq p \) for \( x \in L \). If \( xb \notin \phi_n(p) \) for some \( n \geq 2 \) then as \( b \) is \( \phi_n \)-\( \delta \)-primary to \( p \), we have \( x \leq \delta(p) \) and we are done. So let \( xb \leq \phi_n(p) \) for all \( n \geq 2 \). Then \( xb \leq p^n \leq p^2 \) as \( n \geq 2 \). This implies \( xp \leq p^2 \neq 0 \) as \( p < b \). Hence \( x \leq p \leq \delta(p) \) by Lemma 1.11 of [8] and thus \( b \) is \( \delta \)-primary to \( p \).

Corollary 2.20. Let \( L \) be a Noether lattice. Let \( 0 \neq p \in L \) be a non-nilpotent proper element satisfying the restricted cancellation law. Let \( b \in L \) be such that \( p < b \). If \( b \) is \( \phi_n \)-\( \delta \)-primary to \( p \) (\( \forall \ n \geq 2 \)) then \( b \) is \( \delta \)-primary to \( p \).

Proof. The proof follows from proof of the Theorem 2.19.

Definition 2.21. An element \( b \in L \) is said to be \( 2 \)-potent \( \delta \)-primary to a proper element \( p \in L \) if for all \( x \in L \), \( xb \leq p^2 \) implies \( x \leq \delta(p) \).

Theorem 2.22. Let \( b \in L \) be \( 2 \)-potent \( \delta \)-primary to a proper element \( p \in L \). Then \( b \) is \( \phi \)-\( \delta \)-primary to \( p \) for some \( \phi \leq \phi_2 \) if and only if \( b \) is \( \delta \)-primary to \( p \).

Proof. Assume that \( b \) is \( \delta \)-primary to \( p \). Then obviously, \( b \) is \( \phi \)-\( \delta \)-primary to \( p \) for every \( \phi \) and hence for some \( \phi \leq \phi_2 \). Conversely, assume that \( b \) is \( \phi \)-\( \delta \)-primary to \( p \) for some \( \phi \leq \phi_2 \). Then by Lemma 2.11, \( b \) is \( \phi_2 \)-\( \delta \)-primary (almost \( \delta \)-primary) to \( p \). Let \( xb \leq p \) for \( x \in L \). If \( xb \notin \phi_2(p) \) then as \( b \) is \( \phi_2 \)-\( \delta \)-primary to \( p \), we have \( x \leq \delta(p) \). If \( xb \leq \phi_2(p) = p^2 \) then as \( b \) is \( 2 \)-potent \( \delta \)-primary to \( p \), we have \( x \leq \delta(p) \). Hence \( b \) is \( \delta \)-primary to \( p \).
Corollary 2.23. Let \( p \in L \) be a proper element and \( b \in L \). If \( b \) is \( \phi_2 \)-\( \delta \)-primary to \( p \) and \( b \) is 2-potent \( \delta \)-primary to \( p \) then \( b \) is \( \delta \)-primary to \( p \).

Proof. The proof follows from proof of the Theorem 2.22.

Definition 2.24. Let \( n \geq 2 \). An element \( b \in L \) is said to be \( n \)-potent \( \delta \)-primary to a proper element \( p \in L \) if for all \( x \in L \), \( xb \leq p^n \) implies \( x \leq \delta(p) \).

Obviously, if an element \( b \in L \) is \( n \)-potent \( \delta_0 \)-primary to a proper element \( p \in L \) then \( b \) is 2-potent \( \delta \)-primary to \( p \).

The following result is general form of Theorem 2.22.

Theorem 2.25. Let \( p \in L \) be a proper element and \( b \in L \). Then \( b \) is \( \phi \)-\( \delta \)-primary to \( p \) for some \( \phi \leq \phi_n \) where \( n \geq 2 \) if and only if \( b \) is \( \delta \)-primary to \( p \), provided \( b \) is \( k \)-potent \( \delta \)-primary to \( p \) for some \( k \leq n \).

Proof. Assume that \( b \) is \( \delta \)-primary to \( p \). Then obviously, \( b \) is \( \phi \)-\( \delta \)-primary to \( p \) for every \( \phi \) and hence for some \( \phi \leq \phi_n \) where \( n \geq 2 \). Conversely, assume that \( b \) is \( \phi \)-\( \delta \)-primary to \( p \) for some \( \phi \leq \phi_n \) where \( n \geq 2 \). Then by Lemma 2.21, \( b \) is \( \phi_n \)-\( \delta \)-primary (\( n \)-almost \( \delta \)-primary) to \( p \). Let \( xb \leq p \) for \( x \in L \). If \( xb \not\leq \phi_k(p) = p^k \) then \( xb \not\leq \phi_n(p) = p^n \) as \( k \leq n \). Since \( b \) is \( \phi_n \)-\( \delta \)-primary to \( p \), we have \( x \leq \delta(p) \). If \( xb \not\leq \phi_k(p) = p^k \) then as \( b \) is \( k \)-potent \( \delta \)-primary to \( p \), we have \( x \leq \delta(p) \). Hence \( b \) is \( \delta \)-primary to \( p \).

Corollary 2.26. Let \( p \in L \) be a proper element and \( b \in L \). If \( b \) is \( \phi_n \)-\( \delta \)-primary to \( p \) and \( b \) is \( k \)-potent \( \delta \)-primary to \( p \) where \( k \leq n \) then \( b \) is \( \delta \)-primary to \( p \).

Theorem 2.27. Let \( p \in L \) be a proper element and \( b \in L \) be \( \phi \)-\( \delta \)-primary to \( p \). If \( pb \not\leq \phi(p) \) then \( b \) is \( \delta \)-primary to \( p \).

Proof. Let \( xb \leq p \) for \( x \in L \). If \( xb \not\leq \phi(p) \) then as \( b \) is \( \phi \)-\( \delta \)-primary to \( p \), we have \( x \leq \delta(p) \). So assume that \( xb \leq \phi(p) \). Then as \( pb \not\leq \phi(p) \), we have \( db \not\leq \phi(p) \) for some \( d \leq p \) in \( L \). Also \( (x \lor d)b = xb \lor db \leq p \) and \( (x \lor d)b \not\leq \phi(p) \). As \( b \) is \( \phi \)-\( \delta \)-primary to \( p \), we have \( x \leq (x \lor d) \leq \delta(p) \) and hence \( b \) is \( \delta \)-primary to \( p \).

From the Theorem 2.27, it follows that, if an element \( b \in L \) is \( \phi \)-\( \delta \)-primary to a proper element \( p \in L \) but \( b \) is not \( \delta \)-primary to \( p \) then \( pb \leq \phi(p) \) and hence \( pb \leq p \).

Corollary 2.28. If an element \( b \in L \) is \( \phi_0 \)-\( \delta \)-primary to a proper element \( p \in L \) but \( b \) is not \( \delta \)-primary to \( p \) then \( pb = 0 \).

Proof. The proof is obvious.

Theorem 2.29. Let an element \( b \in L \) be \( \phi \)-\( \delta \)-primary to a proper element \( p \in L \). If \( b \) is \( \delta \)-primary to \( \phi(p) \) then \( b \) is \( \delta \)-primary to \( p \).

Proof. Let \( xb \leq p \) for \( x \in L \). If \( xb \not\leq \phi(p) \) then as \( b \) is \( \phi \)-\( \delta \)-primary to \( p \), we have \( x \leq \delta(p) \) and we are done. Now if \( xb \leq \phi(p) \) then as \( b \) is \( \delta \)-primary to \( \phi(p) \), we have \( x \leq \delta(\phi(p)) \). This implies that \( x \leq \delta(p) \) because \( \phi(p) \leq p \) and we are done.
The following theorem shows that a under certain condition, \( b \in L \) is \( \phi \)-\( \delta \)-primary to \( (p : q) \in L \) if \( b \) is \( \phi \)-\( \delta \)-primary to \( p \in L \) where \( q \in L \).

**Theorem 2.30.** Let an element \( b \in L \) be \( \phi \)-\( \delta \)-primary to a proper element \( p \in L \). Then \( b \) is \( \phi \)-\( \delta \)-primary to \( (p : q) \) for all \( q \in L \) if \( (\phi(p) : q) \leq \phi(p : q) \) and \( (\delta(p) : q) \leq \delta(p : q) \).

**Proof.** Let \( x b \leq (p : q) \) and \( x b \not\in \phi(p : q) \) for \( x \in L \). Then \( x q b \leq p \) and \( x q b \not\in \phi(p) \). Now as \( b \) is \( \phi \)-\( \delta \)-primary to \( p \), we have \( x q \leq \delta(p) \) which implies \( x \leq (\delta(p) : q) \leq \delta(p : q) \) and hence \( b \) is \( \phi \)-\( \delta \)-primary to \( (p : q) \).

**Theorem 2.31.** If an element \( b^k \in L \) is \( \phi \)-\( \delta_1 \)-primary to a proper element \( p \in L \) for all \( k \in \mathbb{Z}_+ \) such that \( \delta_1(\phi(p)) = \phi(\delta_1(p)) \) then \( b \) is \( \phi \)-prime to \( \delta_1(p) \) where \( b \in L \).

**Proof.** Assume that \( x b \leq \delta_1(p) \) and \( x b \not\in \phi(\delta_1(p)) \) for \( x \in L \). Then there exists \( n \in \mathbb{Z}_+ \) such that \( x^n \cdot b^n = (x b)^n \leq p \). If \( (x b)^n \leq \phi(p) \) then by hypothesis \( x b \leq \delta_1(\phi(p)) = \phi(\delta_1(p)) \), a contradiction. So we must have \( x^n \cdot b^n = (x b)^n \not\in \phi(p) \). Since \( b^n \) is \( \phi \)-\( \delta_1 \)-primary to \( p \) we have, \( x^n \leq \delta_1(p) \) and hence \( x \leq \delta_1(\delta_1(p)) = \delta_1(p) \). This shows that \( b \) is \( \phi \)-prime to \( \delta_1(p) \).

Now we relate idempotent element of \( L \) with an element \( \phi_n \)-\( \delta \)-primary \( (n \geq 2) \) to another element of \( L \).

**Theorem 2.32.** If \( p \) is an idempotent element of \( L \) then \( b \in L \) is \( \phi_\omega \)-\( \delta \)-primary to \( p \) and hence \( b \) is \( \phi_n \)-\( \delta \)-primary \( (n \geq 2) \) to \( p \).

**Proof.** As \( p \) is an idempotent element of \( L \), we have \( p = p^n \) for all \( n \in \mathbb{Z}_+ \). So \( \phi_\omega(p) = p \). Therefore \( b \) is \( \phi_\omega \)-\( \delta \)-primary to \( p \). Hence \( b \) is \( \phi_n \)-\( \delta \)-primary \( (n \geq 2) \) to \( p \), by Theorem 2.12.

As a consequence of Theorem 2.32 we have following result whose proof is obvious.

**Corollary 2.33.** If \( p \) is an idempotent element of \( L \) then \( b \in L \) is \( \phi_2 \)-\( \delta \)-primary to \( p \).

However, if \( b \in L \) is \( \phi_2 \)-\( \delta \)-primary to \( p \in L \) then \( p \) need not be idempotent as shown in the following example (by taking \( \delta \) as \( \delta_1 \) for convenience).

**Example 2.34.** Consider the lattice \( L \) of ideals of the ring \( R = \langle \mathbb{Z}_8 , + , \cdot , > \). Then the only ideals of \( R \) are the principal ideals \((0),(2),(4),(1)\). Clearly, \( L = \{(0),(2),(4),(1)\} \) is a compactly generated multiplicative lattice. It is easy to see that the element \((2) \in L \) is \( \phi_2 \)-\( \delta_1 \)-primary to \((4) \in L \) but \((4) \) is not idempotent.

We conclude this paper with the following examples, from which it is clear that,

- If \( b \in L \) is \( \phi_2 \)-\( \delta_1 \)-primary to \( p \in L \) then \( b \) need not be 2-potent \( \delta_0 \)-primary to \( p \).
- If \( b \in L \) is 2-potent \( \delta_0 \)-primary to \( p \in L \) and \( b \) is \( \phi_2 \)-\( \delta_1 \)-primary to \( p \) then \( b \) need not be prime to \( p \).
Example 2.35. Consider $L$ as in Example 2.16. Here the element $(3) \in L$ is $\phi_2-\delta_1$-primary to $(6) \in L$ but $(3)$ is not $2$-potent $\delta_0$-primary to $(6)$.

Example 2.36. Consider $L$ as in Example 2.34. Here the element $(2) \in L$ is $\phi_2-\delta_1$-primary to $(4) \in L$ and $(2)$ is $2$-potent $\delta_0$-primary to $(4)$ but $(2)$ is not prime to $(4)$.

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