SAMPLINGS AND OBSERVABLES. INVARIANTS OF METRIC MEASURE SPACES.

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Abstract. In the paper we are dealing with metric measure spaces of diameter at most one and of total measure one. Gromov introduced the sampling compactification of the set of these spaces. He asked whether the metric measure space invariants extend to the compactification. Using ideas of the newly developed theory of graph limits we identify the elements of the compactification with certain geometric objects and show how to extend various invariants to this space. We will introduce the notion of ultralimit of metric measure spaces, that will be the main technical tool of our paper.

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1. Introduction and preliminaries

1.1. Metric measure spaces. Let $\chi$ denote the set (up to isomorphisms) of Polish spaces $X$ of diameter at most one equipped with a Borel probability measure $\mu_X$ of full support. In the course of the paper, we refer to these objects as mm-spaces. We say that $(X, \mu_X)$ and $(Y, \mu_Y)$ are equivalent if there exists an isometry $\Phi : X \to Y$ such that $\Phi_* (\mu_X) = \mu_Y$. The space $\chi$ has a Polish space structure. Note that we will often use the notation $X$ instead of $(X, \mu_X)$. The distance of $X, Y \in \chi$ is defined the following way [5]. Consider the set $MPM(X)$ of measure-preserving maps $\Psi([0,1], \lambda) \to (X, \mu_X)$, where $\lambda$ is the Lebesgue measure. Let $\Psi^{-1}(d_X)$ be the pull-back of the distance function of $X^2$. Then

$$\square_1 (X, Y) = \inf_{\Psi_1 \in MPM(X), \Psi_2 \in MPM(Y)} \square_1 (\Psi_1^{-1}(d_X), \Psi_2^{-1}(d_Y)),$$

where the $\square_1$-distance of the measurable functions $f, g : [0,1]^2 \to [0,1]$ is defined as the supremal $\epsilon$ such that $f$ and $g$ are $\epsilon$-close outside a subset $X_\epsilon \subset [0,1]$ of measure at most $\epsilon$, that is,

$$|f(x,y) - g(x,y)| \leq \epsilon$$

for $(x,y) \in ([0,1] \setminus X_\epsilon) \times ([0,1] \setminus X_\epsilon)$.

1.2. Samplings. Gromov introduced another notion of convergence in $\chi$, convergence in samplings [5] (see also [10]). Let $M_\infty$ be the convex compact metric space of infinite matrices $\{d_{ij}\}_{i,j \geq 1}$ with $0 \leq d_{ij} \leq 1$, $d_{ii} = d_{ji} = 0$. We have a continuous map $\rho : X^N \to M_\infty$ that assigns $\{d_{ij} = d_X(x_i, x_j)\}$ to the sequence $(x_1, x_2, \ldots) \in X^N$. We denote by $\mu^X_\infty$ the push-forward of the measure $\mu_X \times \mu_X \ldots$. According to the Reconstruction Theorem (3.75. [5]), the map $\tau : \chi \to \mathcal{P}(M_\infty)$ is a continuous injective map (where $\mathcal{P}(M_\infty)$ is the space of probability measures on $M_\infty$). Thus the closure of $\tau(\chi)$ gives us a compactification $\overline{\chi}$ of $\chi$. We say that $\{X_n\}_{n=1}^\infty$ is convergent in sampling if $\{\tau(X_n)\}_{n=1}^\infty$ is a weakly convergent sequence. We use the notation $X_n \xrightarrow{s} X$, if $\{\tau(X_n)\}_{n=1}^\infty \to \tau(X)$ weakly. One of the main motivations for writing this paper was a remark of Gromov in Section 3.1 of his book [5]. He asked whether the invariants of mm-spaces can be extended to $\overline{\chi}$. In our paper we argue that the answer is yes, the elements of $\overline{\chi}$ can be regarded as geometric objects.

Let $G$ be a finite simple graph. We can associate an element $X_G \in \chi$ the following way: $X_G = V(G)$, $d_X(a,b) = 1/2$ if $a$ and $b$ are connected and $d_X(a,b) = 1$ otherwise. It is important to note that $\{G_n\}_{n=1}^\infty$ is convergent as a dense graph sequence (see e.g. [2]) if and only if $\{X_{G_n}\}_{n=1}^\infty$ is convergent in sampling. Our paper uses the ideas and methods of graph limit theory in a substantial way.

1.3. Quantum metric spaces. Let $(X, \mu)$ be standard Borel space with a probability measure and $d^* : X \times X \to \mathcal{P}[0,1]$ be a probability measure space valued Borel-measurable function, such that $d^*(x,x) = \delta_0$ and $d^*(x,y) = d^*(y,x)$. Then for a triple of different points $a, b, c$ points we can pick independent random lengths $l(a,b), l(b,c), l(a,c)$ using the probability measures $d^*(a,b), d^*(b,c), d^*(a,c)$. If for $\mu$-almost triples the random lengths
satisfy the triangle inequality, we call \((X, \mu, d^*)\) a quantum metric measure space, a qmm-space. Then, for any qmm-space \(X\) we can associate an \(S_\infty\)-invariant measure \(\tau(X)\) the following way. First consider \(\hat{M}_\infty\) the convex compact metric space of infinite matrices with coefficients \(d^*\) in \(\mathcal{P}[0,1]\) such that \(d^*_{i,j} = d^*_{i,j}, d^*_{i,j} = \delta_0\). An element \(\nu \in \hat{M}_\infty\) defines a probability measure on \(M_\infty\). By the definition of qmm-spaces, we have a continuous map \(X^N \to \hat{M}_\infty\) and we consider the push-forward of \(\mu_X \times \mu_X \times \ldots\). This is a probability measure on a convex compact space (i.e., the space of probability measures on \(M_\infty\)) so one can consider its barycenter. This will be the associated probability measure \(\tau((X, \mu, d))\). We will prove that qmm’s completely represent \(\chi\).

**Theorem 1.** If \(\kappa \in \chi\) then there exists a qmm-space \(X\) such that \(\tau(X) = \kappa\).

We can extend the metric \(\Box_1\) to qmm-spaces as well. If \(f\) and \(g\) are \(\mathcal{P}[0,1]\)-valued weak-\(\ast\)-measurable functions on \([0,1]^2\) then their \(\Box_1\)-distances can be defined again as the supremal supremal \(\epsilon\) such that \(f\) and \(g\) are \(\epsilon\)-close in the \(d_{\text{ext}}\)-distance, outside a subset \(X, Y \subset [0,1]\) of measure at most \(\epsilon\). Here \(d_{\text{ext}}\) is the metric extension of the usual distance on \([0,1]\) onto \(\mathcal{P}[0,1]\). That is

\[
d_{\text{ext}}(\mu, \nu) = \sup_{f \in \text{Lip}_1[0,1]} \left| \int_0^1 f \, d\mu - \int_0^1 f \, d\nu \right|
\]

Thus

\[
\Box_1(X, Y) = \inf_{\Psi_1 \in \text{MPM}(X), \Psi_2 \in \text{MPM}(Y)} \Box_1((\Psi_1)^{-1}(d_X), (\Psi_2)^{-1}(d_Y)).
\]

Note that two qmm-spaces with zero \(\Box_1\)-distance are not necessarily isomorphic. However, we will prove the following reconstruction theorem.

**Theorem 2.** If \(X, Y\) are qmm-spaces then \(\tau(X) = \tau(Y)\) if and only if \(\Box_1(X, Y) = 0\). In fact, if \(\tau(X) = \tau(Y)\), then there exist \(\Psi_1 \in \text{MPM}(X)\) and \(\Psi_2 \in \text{MPM}(Y)\) such that \((\Psi_1)^{-1}(d_X) = (\Psi_2)^{-1}(d_Y)\).

It is important to note that the idea of qmm-spaces is already implicit in the work of Lovász and Szegedy [7]. Also, Theorem 2 is an analogue of the uniqueness theorem of [1], proved originally for graphons (which can be regarded as special kind of qmm-spaces).

### 1.4. Observables and mm-invariants

Let us recall some of the most important mm-invariants from [3]. Let \(Y\) be a compact metric space with \(\text{diam}(Y) \leq 1\) and \(X \in \chi\). Denote by \(\text{Lip}_1(X, Y)\) the set of 1-Lipschitz functions from \(X\) to \(Y\). We can associate to \(X\) a compact subset \(\mathcal{M}_Y(X)\) of \(\mathcal{P}(Y)\) by

\[
\mathcal{M}_Y(X) = \{f_* (\mu_X) \mid f \in \text{Lip}_1(X, Y)\}.
\]

One can think about \(\mathcal{M}_Y(X)\) as the information \(Y\) can see by screening \(X\). We will extend the notion of Lipschitz maps to qmm-spaces and prove that \(\mathcal{M}_Y(\zeta)\) is well-defined on \(\chi\).

Let \(\nu \in \mathcal{P}(Y)\) and \(0 < \kappa < 1\). Then \(\text{diam}(\nu, \kappa)\) is the infimal \(D\) such that \(D\) contains a subset \(Y_0\) such that \(\text{diam}(Y_0) \leq D\) and \(\nu(Y_0) \geq 1 - \kappa\). The **observational diameter** is defined as

\[
\text{ObsDiam}_Y(X, \kappa) = \sup_{f \in \text{Lip}_1(X, Y)} \text{diam}(f_* (\mu_X), \kappa).
\]
Example 1 Let \( \{K_n\}_{n=1}^\infty \) be the sequence of complete graphs on \( n \) vertices. Then \( \{\tau(X_{K_n})\}_{n=1}^\infty \) tends to a point measure \( \alpha \) on \( M_\infty \). Also, \( \text{ObsDiam}_Y(X_{K_n}, \kappa) \to \text{diam}(Y, \kappa) \) for any \( Y, \text{diam}(Y) \leq 1/2 \) and \( 0 < \kappa < 1 \). This follows from the fact that any map \( K_n \) to \( Y \) is \( 1 \)-Lipschitz. Now, consider the sequence \( \{S^n\}_{n=1}^\infty \) of Riemannian spheres of dimension \( n \) with diameter 1 equipped with the normalized volume measure. Then \( \{\tau(S^n)\}_{n=1}^\infty \) tends to \( \alpha \) as well. However, \( \text{ObsDiam}_Y(S^n, \kappa) \to 0 \) for any \( Y \) and \( \kappa \) (by the Lévy Concentration Phenomenon). Thus the samplings, in general, do not capture the observational diameter. However, we have the following proposition.

**Theorem 3.** The function \( \text{ObsDiam}_Y(., \kappa) \) can be extended to \( \overline{X} \) in an essentially upper semi-continuous way. That is, for any \( Y \) and \( 0 < \kappa' < \kappa \), if \( \zeta_m \to \zeta \) weakly then

\[
\limsup_{n \to \infty} \text{ObsDiam}_Y(\zeta_n, \kappa) \leq \text{ObsDiam}_Y(\zeta, \kappa').
\]

Let \( \kappa_1, \kappa_2, \ldots, \kappa_N \) be positive numbers such that \( \sum_{i=1}^N \kappa_i < 1 \). The separation distance \( \text{Sep}(X, \kappa_1, \kappa_2, \ldots, \kappa_N) \) is the supremal \( \delta \) such that there exist Borel sets \( X_i \subset X, \mu(X_i) \geq \kappa_i \) such that \( \text{dist}_X(X_i, X_j) \geq \delta \). Obviously,

\[
\lim_{n \to \infty} \text{Sep}(X_{K_n}, \kappa_1, \kappa_2, \ldots, \kappa_N) = 1/2, \quad \text{Sep}(X_{K_n}, \kappa_1, \kappa_2, \ldots, \kappa_N) = 0.
\]

**Theorem 4.** For any \( \kappa_1, \kappa_2, \ldots, \kappa_N \), the function \( \text{Sep}(., \kappa_1, \kappa_2, \ldots, \kappa_N) \) extends to \( \overline{X} \) as an upper semi-continuous function.

2. **On the Radon-Nikodym-Dunford-Pettis Theorem**

Let \( (X, \mu, \mathcal{A}) \) be a probability measure space with a \( \sigma \)-algebra \( \mathcal{A} \). Let \( f : X \to \mathbb{R} \) be a bounded measurable function and \( \mathcal{B} \subset \mathcal{A} \) be a sub-\( \sigma \)-algebra. According to the Radon-Nikodym Theorem there exists a unique measurable function \( E(f | \mathcal{B}) \in L^\infty(M, \mu, \mathcal{B}) \) such that for any \( B \in \mathcal{B} \)

\[
\int_B E(f | \mathcal{B}) \, d\mu = \int_B f \, d\mu.
\]

For the next paragraph our reference is [Chapter 3.][9]. In our paper we use sometimes Banach valued measurable functions. In this category, there are several notion of measurability and integral. What we need is the Gelfand-Dunford integral of weak-*measurable functions. So, let \( L \) be a Banach space, and \( L^* \) be its dual. An essentially bounded function \( f : (X, \mu, \mathcal{A}) \to L^* \) is called weak-*measurable, if for any \( v \in L \), the function \( x \to \langle f(x), v \rangle \) is \( \mathcal{A} \)-measurable. Note that this is equivalent to say that as a map \( f \) is measurable with respect to the weak-*topology of \( L^* \). We denote the space of these functions by \( L^*_w(X, \mu, \mathcal{A}, L^*) \). The Gelfand-Dunford integral \( \int f \, d\mu \) of such a function is the unique element of \( L^* \) such that

\[
\langle \int f \, d\mu \rangle(v) = \int \langle f, v \rangle \, d\mu.
\]

holds for any \( v \in L \). Then we still have a Radon-Nikodym type theorem, based on the theorem of Dunford and Pettis. The following result was explained to us by Nicolas Monod.
**Proposition 2.1** (Radon-Nikodym-Dunford-Pettis Theorem). Let \((X, \mu, \mathcal{A})\) and \(f\) be as above. Then there exists an essentially unique function \(E(f \mid \mathcal{B})\) which is weak-\(*\)-measurable with respect to the \(\sigma\)-algebra \(\mathcal{B}\), such that for all \(v \in L\) and \(B \in \mathcal{B}\)

\[
\int_{B} \langle E(f \mid \mathcal{B})(x), v \rangle d\mu = \int_{B} \langle f(x), v \rangle d\mu.
\]

**Proof.** By the original Radon-Nikodym Theorem for all \(v \in L\) there exists a bounded \(\mathcal{B}\)-measurable function \(f_{v}\) such that for any \(B \in \mathcal{B}\)

\[
\int_{B} \langle E(f \mid \mathcal{B})(x), v \rangle = \int_{B} f_{v} d\mu.
\]

Observe that the map \(v \mapsto f_{v}\) is a continuous linear operator, that is an element of the space \(\text{Hom}(L, l^{\infty}(X, \mu, \mathcal{B}))\). Note that \(\text{Hom}(L, l^{\infty}(X, \mu, \mathcal{B})) \sim \text{Hom}(l^{1}(X, \mu, \mathcal{B}), L^{*})\). On the other hand, by Proposition 2.3.1 \([9]\)

\[
l^{\infty}_{0}(X, \mu, \mathcal{B}, L^{*}) \sim \text{Hom}(L, l^{\infty}(X, \mu, \mathcal{B}))
\]

Hence the function \(v \mapsto f_{v}\) is represented by a weak \(*\)-measurable function. \(\square\)

Now let \(f : (X, \mathcal{A}, \mu) \rightarrow [-k, k]\) be a measurable function. Notice that \(f\) can be viewed as a function \(\bar{f} : (X, \mathcal{A}, \mu) \rightarrow C[-k, k]^{*}\), where \(C[-k, k]^{*}\) is the dual space of the Banach space \(C[-k, k]\). Here \(\bar{f}(x) = \delta_{f(x)}\), the point measure concentrated in \(f(x)\). Then one can consider both \(E(\bar{f} \mid \mathcal{B})\) and \(E(f \mid \mathcal{B})\). In case of \(f = id : ([0, 1], \lambda) \rightarrow [0, 1]\), \(E(f \mid \{0, 1\}) = 1/2\) and \(E(\bar{f} \mid \{0, 1\})\) is the Lebesgue measure. Now let \(Y\) be a compact metric space and \(f : (X, \mathcal{A}, \mu) \rightarrow Y\) be a measurable map. Then \(E(\bar{f} \mid \mathcal{B})\) is a well-defined \(\mathcal{B}\)-measurable \(C(Y)^{*}\)-valued function on \(X\). On the other hand, in general \(E(f \mid \mathcal{B})\) does not have a meaning. If \(G : Y \rightarrow \mathbb{R}\) is a continuous function then \(\langle E(\bar{f} \mid \mathcal{B}), G \rangle\) is a measurable function on \(X\) and it is the Radon-Nikodym derivative of \(G \circ f\). This also shows that \(E(f \mid \mathcal{B})\) is a probability measure valued function.

Finally, let us recall the notion of an ultralimit. Let \(Y\) be a compact metric space and \(\omega\) be a nonprincipal ultrafilter. Let \(\{y_{n}\}_{n=1}^{\infty} \subset Y\) be a sequence of points. Then the ultralimit \(\lim_{\omega} y_{n}\) is the unique element \(y \in Y\) such that for any \(\epsilon > 0\)

\[
\{n \mid d_{Y}(y_{n}, y) < \epsilon\} \in \omega
\]

Note however that we can define an ultralimit \(\hat{\lim}_{\omega}\) that is valued in \(\mathcal{P}(Y)\), the space of probability metric spaces on \(Y\). Consider the natural embedding \(i : Y \rightarrow \mathcal{P}(Y)\). Then we have a sequence \(\{i(y_{n})\}_{n=1}^{\infty} \subset \mathcal{P}(Y)\) and the ultralimit \(\hat{\lim}_{\omega} y_{n} = \lim_{\omega} i(y_{n}) \in \mathcal{P}(Y)\). Clearly, if \(G \in C(Y)\) then

\[
\langle \hat{\lim}_{\omega} y_{n}, G \rangle = G(\lim_{\omega} y_{n})
\]

We will use the following lemma later.

**Lemma 2.1.** Let \(\mathcal{B} \subset \mathcal{A}\) two \(\sigma\)-algebras on a set \(X\) and let \(f : (X, \mathcal{A}, \mu) \rightarrow [0, 1]\) be an \(\mathcal{A}\)-measurable function. We denote by \(\downarrow E(\bar{f} \mid \mathcal{B})(x)\) resp. by \(\uparrow E(\bar{f} \mid \mathcal{B})(x)\) the infimum resp.
supremum of the support of the measure \( E(f | B)(x) \). Then if \( f \geq \epsilon \mu \)-almost everywhere on a set \( B \in \mathcal{B} \), then \( E(f | B)(x) \geq \epsilon \mu \)-almost everywhere as well. Similarly, if \( E(f | B)(x) \leq \epsilon \mu \)-almost everywhere, then \( E(f | B)(x) \geq \epsilon \mu \)-almost everywhere as well.

**Proof.** Let \( 0 < \delta < \epsilon \) and \( g \in C[0,1] \) such that \( g(t) = 1 \) if \( t \leq \frac{\delta + \epsilon}{2} \), \( g(t) = 0 \) if \( t \geq \epsilon \). Let \( C \in \mathcal{B} \) the subset of \( B \) on which \( E(f | B) \leq \delta \). Then if \( \mu(C) > 0 \),

\[
0 < \int_C (E(f | B), g) d\mu = \int_C f \circ g d\mu = 0 ,
\]

leading to a contradiction.

### 3. Ultralimits of measured metric spaces

Let \( \{X_n\}_{n=1}^{\infty} \) be a sequence of Polish spaces and let \( \omega \) be a nonprincipal ultrafilter. The ultralimit of Polish spaces was defined in 3.22 [5] the following way. We say that \( \{x_i\}_{i=1}^{\infty}, \{x'_i\}_{i=1}^{\infty} \subset \prod_{i=1}^{\infty} X_i \) are equivalent if \( \lim_{\omega} d_{X_i}(x_i, x'_i) = 0 \), where \( \lim_{\omega} \) is the associated ultralimit. The elements of the ultralimit space \( X \) are the equivalence classes \( \{\{x_i\}\}_{i=1}^{\infty} \).

The set \( X \) is a metric space

\[
d_X([x],[x']) = \lim_{\omega} d_{X_i}(x_i, x'_i) .
\]

Then \( X \) is a complete metric space, but usually it is not separable. In order to define a measure on \( X \) we need some preparation. Consider again the spaces \( \{X_i\}_{i=1}^{\infty} \subset \chi \) and their set-theoretical ultraproduct. From now on, we will use the phrase set-theoretical ultralimit, since from our point of view, it is very much like the classical ultralimit. Now, two sequences \( \{x_i\}_{i=1}^{\infty} \) and \( \{x'_i\}_{i=1}^{\infty} \) are equivalent if

\[
\{i : x_i = x'_i\} \in \omega .
\]

The set-theoretical ultralimit is denoted by \( X \). Then we can define a pseudo-metric on \( X \), by

\[
d_X([x],[x']) = \lim_{\omega} d_{X_i}(x_i, x'_i) .
\]

Here, \([x]\) denotes the new equivalence class. Then we have a natural map \( \pi : X \rightarrow X \). In nonstandard analysis, the inverse images of an element \([x]\) in \( X \) are called monads \( M([x]) \).

The elements of \( M([x]) \) are infinitesimally close to each other. Now we define the ultralimit of subsets the following way: \( \{\{a_i\}\} \in A \) if and only if

\[
\{i : a_i \in A_i\} \in \omega .
\]

The ultralimit sets \( A \) form a Boolean algebra \( \mathcal{P} \). We have a finitely additive measure \( \mu_X \) on \( \mathcal{P} \) and this finitely additive measure can be extended to a \( \sigma \)-algebra containing \( \mathcal{P} \) (see [4] and [6]) the following way. Let \( N \subset X \) be a nullset if for any \( \epsilon > 0 \) there exists an element \( A \in \mathcal{P} \) such that \( N \subset A \) and \( \mu_X(A) < \epsilon \). A set \( M \subset X \) is measurable if there exists \( P \in \mathcal{P} \) such that \( P \triangle M \) is a nullset. The set of measurable sets \( \mathcal{M} \) is a \( \sigma \)-algebra with a probability measure, where we define \( \mu_X(M) = \mu_X(P) \). We call a set \( A \in X \) admissible if \( \pi^{-1}(A) \in \mathcal{M} \). The admissible sets form a \( \sigma \)-algebra \( \mathcal{M} \). The measure \( \mu_X \) is
defined on \( A \in \mathcal{M} \) by \( \mu_X(A) = \mu_X(\pi^{-1}(A)) \). We do not claim that \( \mathcal{M} \) always contains all the Borel-sets, in the light of the following example.

**Example 2.** Let \( X_i = K_i \), where \( |K_i| = i, d(x, y) = 1 \) if \( x \neq y \in K_i \). Then \( \mathbf{X} \) is an uncountable discrete set therefore all of its subsets are Borel-sets.

However, we prove that balls are always in \( \mathcal{M} \), so \( \mathcal{M} \) contains the Borel sets, if for some reason \( \mathbf{X} \) is a separable metric space.

**Lemma 3.1.** If \( x \in \mathbf{X} \), \( \epsilon \geq 0 \), then \( B_\epsilon(x) \in \mathcal{M} \). Here \( B_\epsilon(x) = \{ y \in \mathbf{X} \mid d_X(x, y) \leq \epsilon \} \).

**Proof.** We need to prove that \( \pi^{-1}(B_\epsilon(x)) \in \mathcal{M} \). Let \( x = [\{x_i\}] \). Consider the sets \( \lim_\omega B_{\epsilon + \frac{1}{k}}(x_i) = B_k \in \mathcal{M} \).

If \( y \in \pi^{-1}(B_\epsilon(x)) \), then \( y \in \cap_{k=1}^\infty B_k \). On the other hand, if \( y \in \cap_{k=1}^\infty B_k \) then \( d_X(x, y) \leq \epsilon \).

Therefore, \( \pi^{-1}(B_\epsilon(x)) = \cap_{k=1}^\infty B_k \in \mathcal{M} \). \( \square \)

In the course of this paper, we use bold letters for objects in the set-theoretic ultralimit and underlined bold letters for the objects in the metric ultralimit. If \( \mathbf{A} \) is the ultralimit of \( \{A_n\}_{n=1}^\infty \), then we use the notation \( \lim_\omega A_n = \mathbf{A} \).

**Example 3.** For an mm-space \( X \), the distance function is Borel on \( X \times X \). This is not always the case for the ultralimit spaces. It is possible that \( d \) is not even \( \mathcal{M} \times \mathcal{M} \)-measurable on \( \mathbf{X} \times \mathbf{X} \). Note that if one considers the set-theoretic ultralimit of the spaces \( X_i \times X_i \), then it is the same space as \( \mathbf{X} \times \mathbf{X} \), however its algebra of measurable functions \( \mathcal{M}_2 \) can be much bigger than \( \mathcal{M} \times \mathcal{M} \). This phenomenon can be observed if \( X_i = X_{G_i} \), where \( G_i \) is a random graph (each edge is chosen with probability \( 1/2 \)). Then with probability 1, the distance function on the ultralimit will not be \( \mathcal{M} \times \mathcal{M} \)-measurable (see [4]). However, as we shall see soon, the distance function on \( \mathbf{X} \times \mathbf{X} \) is always \( \mathcal{M}_2 \)-measurable.

Let \( (X, \mu) \) be an mm-space. Recall that the support of \( \mu \) is defined the following way. The point \( p \in X \) is not in the support of \( \mu \) if \( \mu(B_\epsilon(p)) = 0 \) for some \( \epsilon > 0 \). Clearly, the support is a closed set with \( \mu(\text{Supp}(\mu)) = 1 \). Note however that for some ultralimit spaces such as in Example 2 the support of the measure can be empty.

### 4. Analysis on the ultralimit

In this section we fix a sequence \( \{X_i\}_{i=1}^\infty \subset \chi \). As in the previous section \( \mathbf{X} \) denotes their set-theoretic ultralimit and \( (\mathcal{M}, \mu_X) \) stands for the algebra of measurable sets in \( \mathbf{X} \) with the ultralimit measure. The results in this section are known in the finite graph setting (see Section 5 of [4]). Let \( \{f_i : X_i \to [a, b]\}_{i=1}^\infty \) be measurable functions. Their ultralimits are defined by

\[
\mathbf{f}([x]) = \lim_\omega f_i(x_i).
\]
Proposition 4.1. The ultralimit function $f$ is $\mathcal{M}$-measurable and

$$\int_X f d\mu_X = \lim_{\omega} \int_{X_i} f_i d\mu_{X_i}.$$ 

Conversely, if $g : X \to [a, b]$ is a $\mathcal{M}$-measurable function, then there exists a sequence of functions $\{f_i : X_i \to [a, b]\}_{i=1}^{\infty}$ such that their ultralimit $\mu_X$-almost everywhere equals to $g$.

Proof. In order to prove that $f$ is measurable, it is enough to see that

$$f_{[c, d]} = \{p \in X \mid c \leq f(p) \leq d\} \in \mathcal{M},$$

for any $[c, d] \subset [a, b]$. Let $P_n = \{\{f_i^n_{[c-\frac{1}{n}, d+\frac{1}{n}]}\}_{i=1}^{\infty}\}$. Clearly, $f_{[c, d]} = \cap_{n=1}^{\infty} P_n$, thus $f_{[c, d]} \in \mathcal{M}$.

Now fix $k \geq 1$ and let $h_i : X_i \to \mathbb{R}$ be a measurable stepfunction such that $h_i(x) = \frac{i}{2^k}$, when $\frac{i}{2^k} \leq f_i(x) < \frac{i+1}{2^k}$. Clearly, $|h - f| \leq \frac{1}{2^k}$ on $X$. That is

$$\left|\int_X h d\mu_X - \int_X f d\mu_X\right| \leq \frac{1}{2^k}.$$ 

Also, $|\int_{X_i} h_i d\mu_{X_i} - \int_{X_i} f_i d\mu_{X_i}| \leq \frac{1}{2^k}$. Observe that the ultralimit function $h$ can be written as $\sum \frac{1}{2^k} \chi_{C_{i,j}}$, where $C_{i,j}$ is the ultralimit set of $\{C_{i,j}^n\}_{i=1}^{\infty}$, $C_{i,j} = \{x \in X_i \mid h_i(x) = \frac{i}{2^k}\}$.

Therefore $\int_X h d\mu_X = \lim_{\omega} \int_{X_i} h_i d\mu_{X_i}$. Consequently, for any $k \geq 1$,

$$\left|\int_X h d\mu_X - \lim_{\omega} \int_{X_i} f_i d\mu_{X_i}\right| \leq \frac{1}{2^{k-1}}.$$ 

Thus

$$\int_X f d\mu_X = \lim_{\omega} \int_{X_i} f_i d\mu_{X_i}.$$ 

Now let us prove the converse statement. Let $g_k : X \to [a-1, b+1]$ be the step function approximation of $g$, that is, $g_k = \sum \frac{1}{2^k} \chi_{C_{j,k}}$, where $C_{j,k} = \{x \in X \mid \frac{j}{2^k} \leq g(x) < \frac{j+1}{2^k}\}$.

By modifying $g$ on a set of $\mu_X$-measure zero, we can suppose that $C_{j,k} \in \mathcal{P}$. Note that $|g(x) - g_k(x)| \leq \frac{1}{2^k}$ on $X$, and $|g_k(x) - g_j(x)| \leq \frac{1}{2^k}$ if $k \geq j$. Let $T_{j,k}^i \subset X_i$ be Borel sets such that $C_{j,k} = \lim_{\omega} T_{j,k}^i$. Let $f_{i, k} = \sum \frac{1}{2^k} \chi_{T_{j,k}^i}$. We define $E_{i, k} \subset X_i$ in the following way

$$E_{i, k} = \{x \in X_i \mid |f_{i, k}(x) - f_{j, k}(x)| < \frac{1}{2^{j-1}} \text{ for all } j \leq k\}.$$ 

Then

$$S_k = \{i \mid \mu_X(E_{i, k}) > 1 - \frac{1}{2^k}\} \in \omega.$$ 

Let $h(i) = \min\{i, \sup\{k \mid i \in S_k\}\}$. Thus for any $k \geq 1$

(1) \hspace{1cm} $\{i \mid h(i) \geq k\} \in \omega$.

Let $g_i' = f_{h(i)}$. We claim that $g' = \lim_{\omega} g_i' = g$ $\mu_X$-almost everywhere. Let

$$V_i = \{x \in X \mid |g'(x) - g(x)| > \frac{1}{2^{i-2}}\}.$$
What we need to prove is that for all \( l \geq 1 \), \( \mu_X(V_l) = 0 \). Since \( g_l = \lim_\omega f_l \), it is enough to show that for any \( k > 0 \)

\[
\{ i \mid \mu_Xi(x) \mid g_l'(x) - f_l'(x) \mid < \frac{1}{2^{l-1}} > 1 - \frac{1}{2^k} \} \in \omega
\]

However, (2) follows from (1). \( \square \)

5. Sampling qmm-spaces

5.1. q-samplings. In this subsection, we show how one can extend \( \tau \) onto qmm’s. This is described in [7] in a slightly different situation using somewhat different terminology. Let \( M_n \) be the convex compact space of \( n \times n \) real matrices satisfying \( 0 \leq d_{i,j} \leq 1 \), \( d_{i,j} = d_{j,i} \) and \( d_{i,i} = 0 \). Also, let \( \hat{M}_n \) be the convex compact space of \( \mathcal{P}[0,1] \)-valued matrices satisfying \( d_{ij}^* = d_{ji}^*, d_{ii}^* = \delta_0 \). An element of \( \hat{M}_n \) can be viewed as a probability measure on \( M_n \). For a probability measure \( \nu \) on \( \hat{M}_n \) one can consider its barycenter \( b(\nu) \). Let \( (X, \mu, d^*) \) be a qmm-space. Then the push-forward of \( (X, \mu, d^*) \) by the definition of the barycenter \( \rho_n : X^n \to \hat{M}_n \) is a probability measure \( \nu_n \) on \( \hat{M}_n \). Its barycenter \( b(\nu_n) = \tau_n((X, \mu, d^*)) \) is the \( n \)-sampling measure of \( (X, \mu, d^*) \). Note that \( M_\infty \) is the inverse limit of the spaces \( M_n \) and \( b(\nu_n) \) is the push-forward of \( b(\nu) = \tau((X, \mu, d^*)) \) constructed in the Introduction. One can also look at the measures \( b(\nu_n) \) by taking moments. Let \( g = \{ g_{ij} : [0,1] \to \mathbb{R} \}_{1 \leq i \leq j \leq n} \) be a system of continuous functions. They define a continuous function \( q_g = \prod_{1 \leq i \leq j \leq n} g_{ij} \) on \( M_n \).

Lemma 5.1.

\[
\int_{X^n} \prod_{1 \leq i \leq j \leq n} \langle d(x_i, x_j), g_{ij} \rangle d\mu(x_1)d\mu(x_2) \ldots d\mu(x_n) = \int_{M_n} q_g db(\nu_n).
\]

Proof. We have a continuous map \( i_g : \hat{M}_n \to \mathbb{R} \) defined by \( i_g(s) = \prod_{1 \leq i \leq j \leq n} \langle s_{ij}, g_{ij} \rangle \), where \( s = \{ s_{ij} \}_{1 \leq i \leq j \leq n} \in \hat{M}_n \). By the definition of the push-forward,

\[
\int_{X^n} \prod_{1 \leq i \leq j \leq n} \langle d(x_i, x_j), g_{ij} \rangle d\mu(x_1)d\mu(x_2) \ldots d\mu(x_n) = \int_{M_n} i_g(s) d\nu_n(s).
\]

On the other hand, by the definition of the barycenter

\[
\int_{M_n} i_g(s) d\nu_n(s) = \int_{M_n} \prod_{1 \leq i \leq j \leq n} \langle s_{ij}, g_{ij} \rangle d\nu_n(s) = \int_{M_n} \left( \prod_{1 \leq i \leq j \leq n} g_{ij} \right) db(\nu_n) \quad \square
\]

Following [7], we denote \( \int_{X_r} \prod_{1 \leq i \leq j \leq r} \langle d(x_i, x_j), g_{ij} \rangle d\mu^r \) by \( t(g, x) \). Observe that the linear combinations of the functions \( q_g \) are dense in \( C(M_r) \). Therefore, \( \tau((X_1, \mu_1, d_1^*)) = \tau((X_2, \mu_2, d_2^*)) \) if and only if \( t(g, X_1) = t(g, X_2) \) for all \( r \geq 1 \) and system \( g \).

Let \( \{ X_n \}_{n=1}^\infty \subset \chi \) be mm-spaces. Let \( \{ X^k_n \}_{n=1}^\infty \subset \chi \) be their \( k \)-fold product spaces with the metric \( d_k(x, y) = \max_{1 \leq i \leq k} d(x_i, y_i) \). We consider the ultralimits \( X, X^2, \ldots \). As we noted before, the \( \sigma \)-algebras of measurable sets \( M_k \) in \( X^k \) are in general much bigger than the
Lemma 5.2. \[ E \]

By definition, \[ \frac{\partial}{\partial x} \] 

Proof. The ultralimit distance function \( d \) is measurable function in \( (X^2, \mu^2_X, M_2) \), hence we can consider the Radon-Nikodym-Dunford-Pettis derivative \( d^* = E(d \mid M \times \mathcal{M}) : X \times X \to \mathcal{P}[0, 1] \).

A separable realization of \( d^* \) is a measurable map \( \Psi : (X, \mu_X, d^*) \to ([0, 1], \mu_X, d^*) \), where the Borel measure \( \mu_X \) is the push-forward of \( \mu_X \), \( d^* \) is the pull-back of the Borel function \( d^* \). Now we show that \( ([0, 1], \mu_X, d^*_X) \) is always a qmm-space. In fact, we have the following proposition.

Proposition 5.1. Let \( \{X_n\}_{n=1}^{\infty} \subset \chi \) be mm-spaces and \( ([0, 1], \mu_X, d^*_X) \) be a separable realization of their ultralimits then \( \lim_{\omega} \tau(X_n) = \tau([0, 1], \mu_X, d^*_X) \).

Proof. First, let us fix an \( r \geq 1 \) and a system of continuous functions as in Subsection 5.1. By Proposition 4.1,

\[ \int_{X^n} \prod_{1 \leq i,j \leq r} g_{ij}(d_{X_n}(x_i, x_j))d\mu^r_{X_n} = \int_{X^r} \prod_{1 \leq i,j \leq r} g_{ij}(d(x_i, x_j))d\mu^r_X, \]

where \( d \) is the ultralimit of the functions \( d_{X_n} \).

Lemma 5.2.

\[ \int_{X^r} \prod_{1 \leq i,j \leq r} g_{ij}(d(x_i, x_j))d\mu^r_X = \int_{X^r} \prod_{1 \leq i,j \leq r} \langle d^*(x_i, x_j), g_{ij} \rangle d\mu^r_X \]

Proof. By definition,

\[ E(g_{ij}(d(x_i, x_j)) \mid M_i \times M_j) = \langle d^*(x_i, x_j), g_{ij} \rangle \]

By the Integration Rule [Proposition 5.3][4], if \( h_{ij} : X^r \to \mathbb{R} \) are bounded \( \mathcal{M}_{i,j} \)-measurable functions then

\[ \int_{X^r} \prod_{1 \leq i,j \leq r} h_{ij}d\mu^r_X = \int_{X^r} \prod_{1 \leq i,j \leq r} E(h_{ij} \mid M \times \mathcal{M} \times \ldots \mathcal{M})d\mu^r_X. \]

Hence by [4] the lemma follows. □

Since \( d^* \) is measurable on \( \mathcal{L} \times \mathcal{L} \), we immediately obtain from the lemma that for any separable realization \( ([0, 1], \mu_X, d^*) \):

\[ \lim_{\omega} \int_{X^n} \prod_{1 \leq i,j \leq r} g_{ij}(d_{X_n}(x_i, x_j))d\mu^r_{X_n} = \int_{[0,1]^r} \prod_{1 \leq i,j \leq r} \langle g_{ij}, d^*(x_i, x_j) \rangle d\mu^r_X. \]

Note that \( \lim_{\omega} \tau(X_n) \) is well-defined since \( \mathcal{P}(M_\infty) \) is a compact metric space. By the definition of the ultralimit, \( \lim_{\omega} \tau(X_n) = \kappa \) if for any continuous function \( g \in C(M_\infty) \)

\[ \lim_{\omega} \langle \tau(X_n), g \rangle = \langle \kappa, g \rangle. \]

Hence our proposition follows. □
Note that we never actually checked that the triangle inequality condition holds for the limit qmm-space \([0, 1], \mu_X, d^*\). However, let \(g_{i,j,k}^* : M_\infty \to \mathbb{R}\) be a continuous function, that equals to 1 if \(d_{ij} + d_{jk} - d_{ik} < \epsilon\). Since \(0 = \lim_\omega (\tau(X_n), g_{i,j,k}^*) = \langle \tau([0, 1], \mu_X, d^*), g_{i,j,k}^* \rangle\) one can immediately see that the measure of “bad” triangles is zero. That is \(([0, 1], \mu, d^*)\) is a qmm. Thus we proved that for any \(\kappa \in \mathcal{X}\), there exists a qmm-space \(X\) such that \(\tau(X) = \kappa\).

5.2. Martingales. Let \((X, \mu, d^*)\) be a qmm-space. The way we obtained \(\tau_n(X)\) can be described by the following sampling process. First pick \(n\) points in \(X\) \(\mu\)-randomly, independently and then for any \(i < j\) choose \(d(i,j)\) randomly according to the probability measure \(d^*(x_i, x_j)\). If \(\mu\) is atomless, then by probability one we obtain a finite mm-space. For measures with atoms, we get pseudometric spaces. Now let us fix \(r \geq 1\) and a system \(g = \{g_{ij} : [0, 1] \to \mathbb{R}\}_{1 \leq i \leq j \leq r}\), as in the previous section. Then we have the following martingale \(\{B_m, M_m, \mathcal{B}_m\}_{m=0}^n\). The \(\sigma\)-algebras are the standard Borel algebras \(\mathcal{B}_m\) on \(M_m\), for \(1 \leq m \leq n\). \(\mathcal{B}_0\) is the trivial algebra. The measures are \(\tau_m(X), 1 \leq m \leq n\). The function \(B_n : M_n \to \mathbb{R}\) is defined by

\[
B_n(Y) = \frac{1}{n^r} \sum_\phi t_\phi(g, Y),
\]

where the summation is taken for all maps \(\phi : [r] \to [n]\) and

\[
t_\phi(g, Y) = \prod_{1 \leq i \leq j \leq r} g_{ij}(d_Y(i, j)).
\]

Recall that \(t(g, Y) = \prod_{1 \leq i \leq j \leq r} g_{ij}(d_Y(p_i, p_j))d\mu^r(p)\). Since \(Y\) as a measure space is just \([n]\) with the uniform measure, \(t(g, Y) = B_n(Y)\). Then let

\[
B_m = \frac{1}{n^r} \sum_\phi E(t_\phi(g) \mid M_m)(Y),
\]

Note that this is a standard use of martingales in graph limit theory see e.g [7]. Let \(B_0 = \int_{M_n} B_n \, d\tau_n = t(g, X)\). Observe that if \(\phi\) does not take the value \(m\), then

\[
E(t_\phi(g) \mid M_m) = E(t_\phi(g) \mid M_{m-1})
\]

Then we have the inequality

\[
|B_m - B_{m-1}| \leq \frac{1}{n^r} \sum_\phi |E(t_\phi(g) \mid M_m) - E(t_\phi(g) \mid M_{m-1})|.
\]

Otherwise, \(|E(t_\phi(g) \mid M_m) - E(t_\phi(g) \mid M_{m-1})| \leq \prod_{1 \leq i < j \leq r} \|g_{ij}\| = c_g\). Hence, \(|B_m - B_{m-1}| \leq \frac{1}{n} c_g\). Therefore, using the Azuma Inequality we obtain the following proposition.

**Proposition 5.2.**

\[
Prob(Y \in M_n \mid t(g, Y) - t(g, X) \geq \epsilon) \leq 2 \exp\left(-\frac{\epsilon^2 n}{2c_g}\right).
\]

By the Borel-Cantelli Lemma we immediately obtain the following corollary.
Proposition 5.3. For any fixed \( r \geq 1 \) and system \( g \), for \( \tau(X) \)-almost all \( \zeta \in M_\infty \)
\[ t(g, X) = \lim_{n \to \infty} t(g, \zeta_n), \]
where \( \zeta_n \) is the pseudometric space on the first \( n \) coordinate on \( \zeta \).

So, if \( X \) is atomless, then \( \zeta_m \in \chi \) and \( \tau(X) = \lim_{n \to \infty} \tau(\zeta_n) \). If \( X \) has atoms, then \( \tau(X) = \lim_{n \to \infty} \tau(\zeta'_n) \), where \( \zeta'_n \) is the \( m \)-mm-space associated to \( \zeta_n \). Therefore, \( \tau(X) \in \overline{\chi} \) for any \( \text{qmm-space} \). This finishes the proof of Theorem \( [1] \)

6. The Reconstruction Theorem

6.1. The compactification and \( \text{qmm-spaces} \). In this section, we prove Theorem \( [2] \)

Proposition 6.1. Let \( \chi_Q \) be the metric space of equivalence classes (under the pseudometric \( \square_1 \)) of \( \text{qmm-spaces} \). Then \( \tau : \chi_Q \to \overline{\chi} \) is a continuous bijection. In particular, if \( \square_1(X, Y) = 0 \) then \( \tau(x) = \tau(y) \).

Proof. Let \( \Psi : ([0, 1], \lambda) \to X \) be a measure preserving map. Then clearly, \( t(g, X) = t(g, \Psi^{-1}(X)) \), where \( \Psi^{-1}(X) \) is the induced \( \text{qmm-structure} \). Since Lipschitz-functions are dense in \( C(Z) \) for any compact metric space \( Z \), the proposition follows from the lemma below.

Lemma 6.1. For any \( r \geq 1 \), system of \( K \)-Lipschitz functions \( g = \{g_{ij}\}_{1 \leq i, j \leq r} \) and \( \epsilon > 0 \) there exists \( \delta = \delta_{r,g,\epsilon} > 0 \) such that if \( \square_1(f_1, f_2) < \delta \) for two weak \( \ast \)-measurable \( \mathcal{P}[0, 1] \)-valued function on \( [0, 1]^2 \), then \( |t(g, ([0, 1], \lambda, f_1)) - t(g, ([0, 1], \lambda, f_2))| < \epsilon \).

Proof.
\[
|t(g, ([0, 1], \lambda, f_1)) - t(g, ([0, 1], \lambda, f_2))| =
\[
= \left| \int_{[0,1]^r} \left( \prod_{1 \leq i \leq j \leq r} \langle f_1(x_i, x_j), g_{ij} \rangle - \prod_{1 \leq i \leq j \leq r} \langle f_2(x_i, x_j), g_{ij} \rangle \right) d\lambda^r \right| \leq
\[
\leq \left| \int_{x \in [0,1]^r, (x_i, x_j) \in N(f_1, f_2, \epsilon)} \left( \prod_{1 \leq i \leq j \leq r} \langle f_1(x_i, x_j), g_{ij} \rangle - \prod_{1 \leq i \leq j \leq r} \langle f_2(x_i, x_j), g_{ij} \rangle \right) d\lambda^r \right| +
\[
+ \left| \int_{x \in [0,1]^r, (x_i, x_j) \notin N(f_1, f_2, \epsilon)} \left( \prod_{1 \leq i \leq j \leq r} \langle f_1(x_i, x_j), g_{ij} \rangle - \prod_{1 \leq i \leq j \leq r} \langle f_2(x_i, x_j), g_{ij} \rangle \right) d\lambda^r \right|
\]
where \( N(f_1, f_2, \epsilon) = \{(x, y) \in X \times X \mid d_{\text{ext}}(f_1(x, y), f_2(x, y) \leq \epsilon) \} \). For the second term, we have the upper bound \( 2c_g \lambda^r (x \in [0,1]^r, (x_i, x_j) \in N(f_1, f_2, \epsilon)) < 2c_g (r+1)^2 \lambda^2(N(f_1, f_2, \epsilon)) \). To estimate the first term of the right hand side of inequality above, observe that by the definition of the extended metric \( d_{\text{ext}} \)
\[
|\langle f_1(x_i, x_j) - f_2(x_i, x_j), g_{ij} \rangle| \leq Kd_{\text{ext}}(f_1(x_i, x_j) - f_2(x_i, x_j))
\]
Note that for positive numbers $c_i, d_i, 1 \leq i \leq n$ and $T$:

$$|\prod_{i=1}^{n}(c_i + T) - \prod_{i=1}^{n}c_i| \leq (2T)^n \sup_{1\leq i \leq n} |c_i|^n.$$  

This gives us the upper bound

$$\left| \int_{x \in [0,1]^r, (x_i,x_j) \in N(f_1,f_2,\epsilon)} \left( \prod_{1\leq i \leq j \leq r} \langle f_1(x_i, x_j), g_{ij} \rangle - \prod_{1\leq i \leq j \leq r} \langle f_2(x_i, x_j), g_{ij} \rangle \right) d\lambda^r \right| \leq (2K \epsilon \sup |g_{ij}|)^{\binom{r+1}{2}}.$$  

This immediately shows that if $d(f_1, f_2)$ is small enough then

$$|t(g, ([0,1], \lambda, f_1)) - t(g, ([0,1], \lambda, f_2))| < \epsilon.$$  

\[\square\]

6.2. Random maps. Let $(X, \mu, d^*)$ be a qmm-space. We can suppose that $X = [0,1]$. Let us pick a sequence $\{x_n\}_{n=1}^{\infty}$ of independent $\mu$-random points. For each pair $(i, j), i < j$ we pick a real number $d(i,j)$ independently according to the probability measure $d^*(x_i, x_j)$. That is we pick a $\tau(X, \mu, d^*)$-random element $x$ of $M_{\infty}$. Let $X_n$ be the restriction of $x$ on $[n]$. Then we have a natural map $\pi_n : X_n \to [0,1]$ defined by $\pi_n(i) = x_i$. We denote by $\pi$ the ultralimits of the maps $\pi_n$.

**Theorem 5.** The map $\pi : X \to [0,1]$ is a separable realization with probability 1. That is $\pi_* (\mu_X) = \mu$ and $(\pi \times \pi)^{-1}(d^*) = d^X_\pi$ almost everywhere.

**Proof.** We have two kind of randomness in our construction. First, the random choice of $\{x_n\}$, then the choice of $d(i,j)$ according to the law $d^*(x_i, x_j)$. The following lemma is about the second kind of randomness.

**Lemma 6.2.** For any choice of $\{x_n\}_{n=1}^{\infty}$ and $k > 0$, with probability one

$$\lim_{n \to \infty} \frac{\sum_{i \in A_n, j \in B_n} d(i,j)^k}{|A_n||B_n|} = \frac{\sum_{i \in A_n, j \in B_n} \langle d^*(x_i, x_j), t^k \rangle}{|A_n||B_n|}$$

for all sequences $\{A_n, B_n \subset [n]\}_{n=1}^{\infty}$, where $|A_n|, |B_n| \geq \epsilon n$ for some $\epsilon > 0$. Here $t^k$ denotes the $k$-th power of the identity function on $[0,1]$.

**Proof.** First, let us recall the Chernoff inequality. If $X_1, X_2, \ldots X_m$ are independent random variables, taking values in $[0,1]$ and $\delta > 0$ then

$$\text{Prob} \left( \left| \frac{\sum_{i=1}^{m} X_i}{m} - \frac{\sum_{i=1}^{m} E(X_i)}{m} \right| > \delta \right) \leq 2 \exp \left( -\frac{-\delta^2 m}{2} \right).$$  

Let us apply (5) for a fixed pair $A_n, B_n \subseteq [n], |A_n|, |B_n| \geq \epsilon n$, and the random variables $\{d(i,j)^k\}_{i \in A_n, j \in B_n}$. We get that

$$\text{Prob} \left( \left| \frac{\sum_{i \in A_n, j \in B_n} d(i,j)^k}{|A_n||B_n|} - \frac{\sum_{i \in A_n, j \in B_n} \langle d^*(x_i, x_j), t^k \rangle}{|A_n||B_n|} \right| > \delta \right) \leq 2 \exp \left( -\frac{-\delta^2 \epsilon^2 n^2}{2} \right).$$
Therefore the probability that
\[
\frac{\sum_{i \in A_n, j \in B_n} d(i,j)^k}{|A_n||B_n|} - \frac{\sum_{i \in A_n, j \in B_n} \langle d^*(x_i, x_j), t \rangle}{|A_n||B_n|}
\]
is larger than \(\delta\) for at least one such pair is less than \(4^n \exp\left(-\frac{\delta^2 s^2 n^2}{2}\right)\). Hence our lemma follows from the Borel-Cantelli lemma. \(\square\)

Taking ultralimits we immediately obtain the following proposition.

**Proposition 6.2.** For any \(A, B \in \mathcal{M}_1\)
\[
\int_A \int_B \langle \hat{d}_X(x,y), t^k \rangle d\mu^2_X = \int_A \int_B d_X(x,y)^k d\mu_X^2,
\]
where \(\hat{d}_X = (\pi \times \pi)^{-1}(d^*)\). Consequently, \((\pi \times \pi)^{-1}(d^*) = d_X^*\) almost everywhere.

**Lemma 6.3.** With probability one, \(\pi_*(\mu_X) = \mu\).

*Proof.* Fix a Borel-set \(A \subseteq [0,1]\). By the Law of Large Numbers, with probability one,
\[
\lim_{n \to \infty} \frac{|\{i : x_i \in A\}|}{n} = \mu(A).
\]
That is, with probability one \(\pi_*(\mu_X)\) and \(\mu\) coincide on all dyadic intervals. Therefore, by Caratheodory’s Theorem the two measures are equal. \(\square\)

Now Theorem 5 follows from Proposition 6.2 and Lemma 6.3. \(\square\)

6.3. **The Maharam Lemma.** Let \(([0,1], \mathcal{A}, \mu) \subset ([0,1], \mathcal{B}, \mu)\) be two separable \(\sigma\)-algebras. We say that \(\mathcal{A}\) is complemented in \(\mathcal{B}\) if there exists a \(\sigma\)-algebra \(\mathcal{C} \subset \mathcal{B}\) such that the generated algebra \((\mathcal{A}, \mathcal{C})\) is dense in \(\mathcal{B}\) and the elements of \(\mathcal{C}\) are independent from \(\mathcal{A}\). Note that it means that there exists a \(\sigma\)-algebra \(([0,1], \mathcal{C}', \mu')\) and the measure preserving bijection
\[
\Phi : ([0,1] \times [0,1], \mathcal{A} \times \mathcal{C}', \mu \times \mu') \to ([0,1], \mathcal{B}, \mu)
\]
such that \(\Phi^{-1}(\mathcal{A}) = \mathcal{A} \times [0,1]\). Maharam ([8], see also [4]) gave a necessary and sufficient condition for having a complement. Namely, for any \(k > 0\) there exists a partition \(S_1 \cup S_2 \cup \cdots \cup S_k = [0,1]\), such that \(S_i \in \mathcal{B}\) and \(S_i\) is independent of \(\mathcal{A}\).

**Lemma 6.4.** Let \(\pi : (X, \mu_X) \to (X, \mu)\) as above a separable realization. Then there exists a Maharam partition for any \(k \geq 1\) in \(\mathcal{M}_1^X\).

*Proof.* Let \((x_1, x_2, \ldots)\) be a random sequence as above. For each \(i \geq 1\), pick an element \(s(x_i) \subset \{1,2,\ldots,k\}\) randomly with uniform distribution. Then by the Law of Large Numbers, for each dyadic set \(A\) and \(1 \leq j \leq k\)
\[
\lim_{n \to \infty} \frac{|\{1 \leq i \leq n \mid x_i \in A, s(x_i) = j\}|}{n} = \frac{1}{k} \mu(A),
\]
with probability 1. Hence the ultralimits of \(S^n_j = \{s^{-1}(j) \cap [n]\}\), \(\{S_1, S_2, \ldots, S_k\}\) form a Maharam partition. \(\square\)
6.4. The proof of Theorem 2. Let \([0,1], \mu_X, d_X^*\) be two qnm-spaces such that \(\tau(([0,1], \mu_X, d_X^*)) = \tau(([0,1], \mu_Y, d_Y^*))\). Let us consider a \(\tau(([0,1], \mu_X, d_X^*))\)-random element of \([0,1]\). With probability one, both \(\pi_X : (X, \mu_X, d_X^*) \to ([0,1], \mu_X, d_X^*)\) and \(\pi_Y : (X, \mu_X, d_X^*) \to ([0,1], \mu_Y, d_Y^*)\) are separable realizations. By Lemma 5.3, we have a separable \(\sigma\)-algebra in \(\mathcal{M}_I^X\) that contains Maharam-partitions for any \(k > 0\) with respect to both \((\pi_X)^{-1}(\mathcal{B}_{[0,1]})\) and \((\pi_Y)^{-1}(\mathcal{B}_{[0,1]})\), where \(\mathcal{B}_{[0,1]}\) is the Borel algebra. Therefore there exist measure preserving maps \(\rho_X : ([0,1], \mu_Z) \to ([0,1], \mu_X), \rho_Y : ([0,1], \mu_Z) \to ([0,1], \mu_Y)\) such that \((\rho_X \times \rho_Y)^{-1}(d_X^*)\) and \((\rho_Y \times \rho_Y)^{-1}(d_Y^*)\) coincide. Note that for any Borel probability measure \(([0,1], \mu_Z)\) there exists a measure preserving map \(\rho : ([0,1], \lambda) \to ([0,1], \mu_Z)\). Hence Theorem 2 follows.

\[\Box\]

7. LIMITS VS. ULTRALIMTS

Proposition 7.1. Let \(\{X_n\}_{n=1}^\infty \subseteq \chi\) converge to \(X \in \chi\) in the \(\Box\)-metric. Then the support of their metric ultralimit \(\Box X\) is isometric to \(X\). Conversely, if for some sequence \(\{X_n\}_{n=1}^\infty \subseteq \chi\) the support of the ultralimit is isometric to \(X \in \chi\), then \(X = \lim_\omega X_n\). That is for any \(\epsilon > 0\)

\[\{n \mid \Box_1(X_n, X) < \epsilon\} \in \omega.\]

Proof. We start with two lemmas.

Lemma 7.1. Let \((Y, d)\) be a complete metric space equipped with a probability measure \(\mu\), such that \(\text{supp}(\mu) = Y\). Then \(Y\) is separable.

Proof. Let us consider the set \(S_{\epsilon/2}\) of \(\epsilon/2\)-balls in \(Y\). Take a well-ordering \(B_1, B_2, \ldots\) of \(S_{\epsilon/2}\) and for each ordinal \(\alpha\), let \(f(\alpha) = \mu(\cup_{\beta \leq \alpha} B_\beta)\). Then there is a countable ordinal \(\alpha\) such that \(f(\alpha) = f(\gamma)\) for any \(\gamma \geq \alpha\). Therefore, we have countable many balls of radius \(\epsilon/2\) such that the measure of their union is greater or equal than the measure of the union of any countable subset of \(S_{\epsilon/2}\). However, it means that the \(\epsilon\)-balls around the centers of these balls cover the whole set \(Y\).

Now the first part of the proposition easily follows. By (3),

\[\tau(\text{supp}(\mu_X)) = \tau(X).\]

Hence by the reconstruction theorem of Gromov, \(X\) and \(\text{supp}(\mu_X)\) are isomorphic.

Now let us turn to the converse statement. Let \((X, \mu, d)\) be an mm-space. with \(k\) given points \(x_1, x_2, \ldots, x_k\) such that for some \(\epsilon > \delta > 0\), \(\mu(\cup_{i=1}^k B_{x_i}(\delta)) \geq 1 - \epsilon\), where \(B_{x_i}(\delta) = \{y \in X \mid d(x_i, y) \leq \delta\}\). For \(1 \leq j \leq k\), let \(C_j = B_{x_j}(\delta) \setminus \cup_{i=1}^{j-1} B_{x_i}(\delta)\). Then consider the following discrete mm-space \((Y, \mu)\): \(Y = \{y_1, y_2, \ldots, y_k, z\}, \nu(y_i) = \nu(C_i), \nu(z) = 1 - \sum_{i=1}^k \mu(C_i)\), \(d'(y_i, y_j) = d(x_i, x_j), d'(y_i, z) = 1\).

Lemma 7.2. Let \((X, \mu), (Y, \nu)\) as above. Then \(\Box_1(X, Y) \leq 3\epsilon\).

Proof. Let \(\Psi : X \to Y\) be defined the following way. \(\Psi(C_i) = y_i, \Psi(X \setminus \cup_{i=1}^k C_i) = z\). Then \(\Box_1(\Psi^{-1}(d'), d) \leq 3\epsilon\). Indeed, \(|\Psi^{-1}(d') - d| \leq 3\epsilon\) on \(C_i \times C_j\) and \(\mu(\cup_{i=1}^k C_i) > 1 - \epsilon\).
The following lemma is trivial.

**Lemma 7.3.** For each \( m > 0 \) and \( \epsilon > 0 \), there exists some \( \delta > 0 \) such that if \((G, \mu)\) and \((H, \nu)\) are discrete mm-spaces on the same set \( \{a_1, a_2, \ldots, a_m\} \) such that for all \( 1 \leq i \leq m \) \(|\mu(a_i) - \nu(a_i)| \leq \delta\) and for all \( 1 \leq i, j \leq m \) \(|d_G(a_i, a_j) - d_H(a_i, a_j)| \leq \delta\) then \( \overline{\square}(G, H) \leq \epsilon \).

Now let \( \{X_n\}_{n=1}^\infty \subset \chi \) be mm-spaces as in the statement of the proposition. and let \( Z \) be the support of \( \overline{\chi} \). By Lemma 3.1 and Lemma 7.1 \( Z \in \chi \). Pick \( \epsilon > 0 \). Let \( x_1, x_2, \ldots, x_k \in Z \) and \( 0 < \delta < \epsilon \) such that

- \( \mu(\bigcup_{k=1}^\infty B_{\overline{X}}(\delta)) > 1 - \epsilon \).
- For each \( 1 \leq i \leq k \), \( \mu(B_{\overline{X}_i}(\delta) \setminus B'_{\overline{X}_i}(\delta)) = 0 \), where \( B'_{\overline{X}_i}(\delta) = \{y \mid d_{\overline{X}}(\overline{x}_i, y) < \delta\} \)

Now let us consider \( \{x_1^n, x_2^n, \ldots, x_k^n\} \subset X_n \) such that \( \overline{x}_i = [\{x_i^n\}_{n=1}^\infty] \). For any fix \( q > 0 \)

\[
\{n \mid \mu(X_n(B_{x_i^n}(\delta)) - \mu_{\overline{X}}(B_{x_i}(\delta))) < q\} \in \omega
\]

Indeed, for any \( \delta'' < \delta < \delta' \)

\[
\lim_{n} \mu_{X_n}(B_{x_i^n}(\delta)) \leq \mu_{\overline{X}}(B_{x_i}(\delta'')) , \lim_{n} \mu_{X_n}(B_{x_i^n}(\delta)) \geq \mu_{\overline{X}}(B_{x_i}(\delta'')).
\]

So (6) follows from our condition on negligibility of the boundary of the balls. Similarily, for any \( j \),

\[
\{n \mid \mu_{X_n}(C^n_j) - \mu_{\overline{X}}(C_j) < q\} \in \omega,
\]

where \( C^n_j = B_{x_j^n}(\delta) \cup \bigcup_{i=1}^{n-1} B_{x_i^n}(\delta) \). By Lemmas 7.2 and 7.3

\[
\{n \mid \square(X_n, \overline{Z}) \leq 4\epsilon\} \in \omega.
\]

Hence the proposition follows. \( \square \)

According the proposition, any class of \( \chi \) closed under taking ultralimits is precompact in \( (\chi, \overline{\square}) \).

**Example 4.** Let \( \{C(n) > 0\}_{n=1}^\infty \), \( \{D(n) > 0\}_{n=1}^\infty \) be two sequences of integers such that \( \lim_{n \to \infty} D(n) = 0 \). Then \( \chi_{C,D} \subset \chi \) is defined as the set of mm-spaces for which

\[
\mu\{x \in X \mid \mu(B_{1/n}(x)) \leq C(n)\} \leq D(n).
\]

By definition, the support of any ultralimit of a sequence in \( \chi_{C,D} \) has full measure, so the ultralimit in also in the class \( \chi_{C,D} \) (see [5] Section 31/2.14)

8. Lipschitz functions and their ultralimits

8.1. Lipschitz maps and qmm-spaces. Let \((X, \mu, d_X)\) be a qmm-space and \( Y \) be a compact metric space. We say that a measurable map \( f : X \to Y \) is a 1-Lipschitz, if for \( \mu \times \mu \)-almost all pairs \((x_1, x_2) \in X \times X\), \( d_Y(f(x_1), f(x_2)) \leq d_X(x_1, x_2) \). Note that the metric space structure is still well-defined on \( Lip_1(X, Y) \). Also, we can define the sets \( \mathcal{M}(X, Y) \subset \mathcal{P}(Y) \).
Lemma 8.1. If $X_1 \sim X_2$ ($\square_1(X_1, X_2) = 0$), then $\text{Lip}_1(X_1, Y)$ and $\text{Lip}_1(X_2, Y)$ are isomorphic. Also, the sets $\mathcal{M}(X_1, Y)$ and $\mathcal{M}(X_2, Y)$ are equal.

Proof. Let $\Phi : [0, 1] \to X_1$ be a measurable map. Then $\text{Lip}_1(\Phi^{-1}(X_1), Y)$ is isomorphic to $\text{Lip}_1(X_1, Y)$. Hence our lemma immediately follows from Theorem 2. \qed

Now let $\{X_n\}_{n=1}^\infty \subset \chi$ and $Y$ be a compact metric space. Let $f_n : X_n \to Y$ be a sequence of 1-Lipschitz functions.

Lemma 8.2. The ultralimit of $\{f_n\}_\infty$, $f : X \to Y$ is 1-Lipschitz and $\lim_\omega (f_n)_*(\mu_n) = f_*(\mu_X)$.

Proof. Let $x_1 = \lim_\omega x_n^1$, $x_2 = \lim_\omega x_n^2$. Then

$$d_X(x_1, x_2) = \lim_\omega d_{X_n}(x_n^1, x_n^2) \geq \lim_\omega d_Y(f_n(x_n^1), f_n(x_n^2)) = d_Y(f(x_1), f(x_2)).$$

Now let $g : Y \to \mathbb{R}$ be a continuous function. Then $\langle (f_n)_*(\mu_n), g \rangle = \langle \mu_n, g \circ f_n \rangle$ and $\langle f_*(\mu_X), g \rangle = \langle \mu_X, g \circ f \rangle$. Since $\lim_\omega g \circ f_n = g \circ f$, by Proposition 4.1

$$\lim_\omega \langle (f_n)_*(\mu_n), g \rangle = \langle f_*(\mu_X), g \rangle.$$

That is $\lim_\omega (f_n)_*(\mu_n) = f_*(\mu_X)$. \qed

Let us consider the space $(X, \mu_X, d^*)$, where $X$ and $d$ are as above.

Lemma 8.3. $f$ is still 1-Lipschitz on $(X, \mu_X, d^*)$.

Proof. Let us apply Lemma 2.1 for the $\mathcal{M}_2$-measurable function $d_f(x, y) = d(x, y) - |f(x) - f(y)|$. Since $d_f$ is $\mu_X \times \mu_X$-almost positive, the function $\downarrow d^* - |f(x) - f(y)|$ is $\mu_X \times \mu_X$-almost positive as well. Therefore, $f : (X, \mu_X, d^*) \to Y$ is 1-Lipschitz. \qed

Lemma 8.4. Let $(X, \mu, d^*)$ be a separable realization of $(X, \mu_X, d^*)$. Then there exists $\hat{f} \in \text{Lip}_1(X, Y)$ such that $\hat{f}_*(\mu) = f_*(\mu_X)$.

Proof. Let $(X', \mu', (d^*)')$ be a separable realization of $(X, \mu_X, d^*)$ for which $f$ is measurable. Note that such separable realization exists, since any separable extension of a separable realization is a separable realization. Then for this particular realization the function $\hat{f}$ must exist. Since $\text{Lip}_1(X, Y)$ depends only on the $\square_1$-equivalence class of $X$, the lemma follows. \qed

We can summarize the previous lemmas in a proposition.

Proposition 8.1. Let $\{X_n\}_{n=1}^\infty \subset \chi$ converging to a qmm-space $(X, \mu, d^*)$ in sampling. Also let $f_n \in \text{Lip}_1(X_n, Y)$ such that $(f_n)_*(\mu_n) \to \nu$. Then there exists $f \in \text{Lip}_1(X, Y)$ such that $(f)_*(\mu) = \nu$.

Proposition 8.2. Let $(X, \mu_X, d^*)$ be a qmm-space, $Z$ be a compact metric space and $f \in \text{Lip}_1(X, Z)$. Then there exists $\{X_n\}_{n=1}^\infty \subset \chi$ and $\{f_n \in \text{Lip}_1(X_n, Z)\}_{n=1}^\infty$ such that $(f_n)_*(\mu_n) \to (f)_*(\mu)$. 

Proof. Let \( \{X_n\}_{n=1}^\infty \) be the discrete spaces as in Theorem 8.2. We define \( f_n \) by \( f_n(i) = f(x_i) \). That is \( f_n = f \circ \pi_n \), where \( \pi_n : X_n \to X \) is the natural map. Then
\[
\lim_{n \to \infty} (f_n)_*(\mu_n) = f_*(\mu_X) = (f \circ \pi)_*(\mu_X) = f_*(\mu).
\]
\( \square \)

8.2. The proof of Theorem 8.2

Proposition 8.3. Let \( Y \) be a compact metric space and \( 0 < \kappa' < \kappa < 1 \). Let \( \{\nu_n\}_{n=1}^\infty, \nu \in \mathcal{P}(Y) \) such that \( \nu_n \to \nu \). Then
\[
\limsup_{n \to \infty} \operatorname{diam}(\nu_n, \kappa) \leq \operatorname{diam}(\nu, \kappa').
\]
\[
\liminf_{n \to \infty} \operatorname{diam}(\nu_n, \kappa) \geq \operatorname{diam}(\nu, \kappa).
\]

Proof. First, let \( Y_0 \subset Y \) be a closed set such that \( \operatorname{diam}(Y_0) = t \) and \( \nu(Y_0) \geq 1 - \kappa' \). Let \( Y_\epsilon = \{ y \in Y \mid d_Y(y, Y_0) \leq \epsilon \} \). Pick a continuous function \( g : Y \to [0, 1] \) such that \( g(Y_{\epsilon/2}) = 1 \), \( g(Y^c_\epsilon) = 0 \). Then
\[
\limsup_{n \to \infty} \nu_n(Y_\epsilon) \geq \lim_{n \to \infty} \int_{Y_{\epsilon}} g d\nu_n = \int_{Y_{\epsilon}} g d\nu \geq 1 - \kappa'.
\]
Hence, for large enough \( n \),
\[
\operatorname{diam}(\nu_n, \kappa) \leq t + 2\epsilon.
\]

Therefore, \( \limsup_{n \to \infty} \operatorname{diam}(\nu_n, \kappa) \leq \operatorname{diam}(\nu, \kappa') \). One should note that \( \limsup_{n \to \infty} \operatorname{diam}(\nu_n, \kappa) \leq \operatorname{diam}(\nu, \kappa) \) does not always hold.

Now let \( Y_n \subset Y \) be closed subsets such that \( \operatorname{diam}(Y_n) \leq t \) and \( \nu_n(Y_n) \geq 1 - \kappa \). We can suppose, by taking a subsequence, that \( \{Y_n\}_{n=1}^\infty \) converges to a closed set \( Y_0 \subset Y \) in the Hausdorff topology. Then, \( \operatorname{diam}(Y_0) \leq t \). Let \( g : Y \to [0, 1] \) be a continuous function such that \( g(Y_{\epsilon/2}) = 1 \) and \( g(Y^c_\epsilon) = 0 \). Then
\[
\liminf_{n \to \infty} \nu_n(Y_n) \leq \lim_{n \to \infty} \int_{Y_{\epsilon}} g d\nu_n = \int_{Y_{\epsilon}} g d\nu \leq \nu(Y_\epsilon).
\]
Hence \( \nu(Y_\epsilon) \geq 1 - \kappa \) and \( \operatorname{diam}(Y_\epsilon) \leq t + 2\epsilon \). That is \( \liminf_{n \to \infty} \operatorname{diam}(\nu_n, \kappa) \geq \operatorname{diam}(\nu, \kappa) \).
\( \square \)

Lemma 8.5. Let \( (X, \mu, d^*) \) be a qmm-space and \( 0 < \kappa' < \kappa \). Then
\[
\begin{itemize}
  \item for all \( \{X_n\}_{n=1}^\infty \subset \chi \) such that \( \{X_n\}_{n=1}^\infty \overset{\delta}{\to} X \)
    \[
    \limsup_{n \to \infty} \text{ObsDiam}_Y(X_n, \kappa) \leq \text{ObsDiam}_Y(X, \kappa').
    \]
  \item There exists \( \{X_n\}_{n=1}^\infty \overset{\delta}{\to} X \) such that
    \[
    \liminf_{n \to \infty} \text{ObsDiam}_Y(X_n, \kappa) \geq \text{ObsDiam}_Y(X, \kappa).
    \]
\end{itemize}
\]
Proof. Let \( \{X_n\}_{n=1}^{\infty} \xrightarrow{\delta} X \). Pick \( f_n \in \text{Lip}_1(X_n, Y) \) such that 
\[ \text{diam}((f_n)_*(\mu_n), \kappa) \geq \text{ObsDiam}_Y(X_n, \kappa) - \frac{1}{n} \].
Let \( \{(f_n)_*(\mu_n)\}_{k=1}^{\infty} \in \mathcal{P}(Y) \) be an arbitrary convergence subsequence such that
\[
\lim_{k \to \infty} \text{diam}((f_n)_*(\mu_n), \kappa) = \limsup_{n \to \infty} \text{ObsDiam}_Y(X_n, \kappa).
\]
Then by Proposition 8.1, there exists \( f \in \text{Lip}_1(X, \mu) \) such that \( (f_n)_*(\mu_n) \to f_*(\mu) \). By the previous proposition,
\[
\limsup_{n \to \infty} \text{ObsDiam}_Y(X_n, \kappa) \leq \text{ObsDiam}_Y(X, \kappa').
\]
Now let \( f \in \text{Lip}_1(X, Y) \) such that
\[
\text{diam}(f_*(\mu), \kappa) \geq \text{ObsDiam}_Y(X, \kappa) - \epsilon.
\]
By Proposition 8.2, there exists a sequence \( \{X_n\}_{n=1}^{\infty} \subset \chi \) and \( f_n \in \text{Lip}_1(X_n, Y) \) such that
- \( \{X_n\}_{n=1}^{\infty} \xrightarrow{\delta} X \).
- \( (f_n)_*(\mu_n) \to f_*(\mu) \).
Therefore
\[
\liminf_{n \to \infty} \text{ObsDiam}_Y(X_n, \kappa) \geq \liminf_{n \to \infty} \text{diam}((f_n)_*(\mu_n), \kappa) \geq \text{ObsDiam}_Y(X, \kappa) - \epsilon.
\]
Since (7) holds for all \( \epsilon > 0 \), the lemma follows.

Now let us finish the proof of our theorem. Let \( 0 < \kappa' < \kappa'' < \kappa \), and \( \{(X_n, \mu_n, d_n^*)\}_{n=1}^{\infty} \) be a sequence of qmm-spaces converging to \( (X, \mu, d^*) \). Pick a sequence \( \{Z_n\}_{n=1}^{\infty} \subset \chi \) such that
- \( \text{ObsDiam}_Y(Z_n, \kappa'') + \frac{1}{n} \geq \text{ObsDiam}_Y(X_n, \kappa) \).
- \( \{Z_n\}_{n=1}^{\infty} \xrightarrow{\delta} X \).

By the previous lemma,
\[
\limsup_{n \to \infty} \text{ObsDiam}_Y(X_n, \kappa) \leq \limsup_{n \to \infty} \text{ObsDiam}_Y(Z_n, \kappa'') \leq \text{ObsDiam}_Y(X, \kappa')
\]

9. The Proof of Theorem [4]

Let \( (X, \mu, d^*) \) be a qmm-space, \( 0 < \kappa_i < 1 \), \( \sum_{i=1}^{m} \kappa_i \leq 1 \). Then \( \text{Sep}(X, \kappa_1, \kappa_2, \ldots, \kappa_m) \) is defined as the supremum of \( \delta \)'s such that there exist disjoint measurable subsets \( \{A_i\}_{i=1}^{m} \), \( \mu(A_i) \geq \kappa_i \), with
\[
\text{ess inf}_{x \in A_i, y \in A_j} d^*(x, y) \geq \delta.
\]
Recall that
\[
\text{ess inf}_{U \downarrow d^*} = \inf\{t \mid \mu(p \in U \downarrow d^*(p) \geq t) > 0\}.
\]

Lemma 9.1. If \( X_1 \sim X_2 \) then \( \text{Sep}(X_1, \kappa_1, \kappa_2, \ldots, \kappa_m) = \text{Sep}(X_2, \kappa_1, \kappa_2, \ldots, \kappa_m) \).

Proof. Let \( \Psi : [0, 1] \to X_1 \) be a measure preserving map. Then it is easy to see that
\[
\text{Sep}(\Psi^{-1}(X_1), \kappa_1, \kappa_2, \ldots, \kappa_m) = \text{Sep}(X_1, \kappa_1, \kappa_2, \ldots, \kappa_m).
\]
Hence the lemma follows from Theorem [2]
Lemma 9.2. Let \( \{X_n\}_{n=1}^{\infty} \subset \chi \) converging in sampling to a qmm-space \( X \). Then
\[
\limsup_{n \to \infty} \text{Sep}(X_n, \kappa_1, \kappa_2, \ldots, \kappa_m) \leq \text{Sep}(X, \kappa_1, \kappa_2, \ldots, \kappa_m).
\]

Proof. By taking a subsequence, we can suppose that \( \lim_{n \to \infty} \text{Sep}(X_n, \kappa_1, \kappa_2, \ldots, \kappa_m) \) exists. Let \( \{A^1_n, A^2_n, \ldots, A^m_n\} \) be disjoint subsets such that \( \mu_n(A^i_n) \geq \kappa_i \) and
\[
\inf_{i,j} d_{X_n}(A^i_n, A^j_n) \geq \text{Sep}(X_n, \kappa_1, \kappa_2, \ldots, \kappa_m) - \frac{1}{n}.
\]
Let \( A^i = \lim \omega A^i_n \). Then \( \mu_X(A^i) \geq \kappa_i \) and
\[
\inf_{i,j} d(A^i, A^j) \geq \lim_{n \to \infty} \text{Sep}(X_n, \kappa_1, \kappa_2, \ldots, \kappa_m).
\]
Let \( \mathcal{L} \subset \mathcal{M} \) be a separable subalgebra such that \( A^i \subset \mathcal{L} \) for all \( 1 \leq i \leq m \). Then by Lemma 9.1
\[
\text{ess} \inf \downarrow d^*(A^i, A^j) \geq \lim_{n \to \infty} \text{Sep}(X_n, \kappa_1, \kappa_2, \ldots, \kappa_m).
\]
Therefore we have a separable realization \( Y \) of \( X \) such that
\[
\text{Sep}(Y, \kappa_1, \kappa_2, \ldots, \kappa_m) \geq \lim_{n \to \infty} \text{Sep}(X_n, \kappa_1, \kappa_2, \ldots, \kappa_m).
\]
Thus the lemma follows from Lemma 9.1. \( \square \)

Lemma 9.3. Let \( (X, \mu, d^*) \) be a qmm-space. Then there exist \( \{Y_n\}_{n=1}^{\infty} \subset \chi \) such that \( \{Y_n\}_{n=1}^{\infty} \) converges to \( X \) in sampling and
\[
\lim_{n \to \infty} \text{Sep}(Y_n, \kappa_1, \kappa_2, \ldots, \kappa_m) = \text{Sep}(X, \kappa_1, \kappa_2, \ldots, \kappa_m).
\]

Let \( A_1, A_2, \ldots, A_m \subset X \) such that for any \( 1 \leq i < j \leq m \)
\[
\text{ess} \inf_{x_1 \in A_i, x_2 \in A_j} \downarrow d^*(x_1, x_2) \geq t.
\]
Choose a random sequence \( y_1, y_2, \ldots \) such that
\[
\lim_{n \to \infty} \frac{|(1 \leq j \leq n \mid y(j) \in A_i)|}{n} = \mu(A_i),
\]
and pick \( d(i, j) \) randomly according to the law \( d^*(y(i), y(j)) \). Let us normalize the measure on \( Y_n \) in such a way that \( \mu_n(B^n_i) = \mu(A_i) \), where \( B^n_i = \{i \mid y_i \in A_i\} \). Then with probability one, \( (Y_n, \mu_n) \) converges to \( X \) in sampling and for any \( 1 \leq i < j \leq n \),
\[
\inf_{i,j} d_{Y_n}(B^n_i, B^n_j) \geq t.
\]
Now we finish the proof of Theorem 11. Let \( \{X_n, \mu_n, d^n_\ast\}_{n=1}^{\infty} \) be a sequence of qmm-spaces converging to \( (X, \mu, d^*) \) in sampling. Let us pick a sequence \( \{Z_n\}_{n=1}^{\infty} \subset \chi \) such that
- \( \text{Sep}(Z_n, \kappa_1, \kappa_2, \ldots, \kappa_m) + \frac{1}{n} \geq \text{Sep}(X_n, \kappa_1, \kappa_2, \ldots, \kappa_m) \).
- \( \{Z_n\}_{n=1}^{\infty} \overset{\text{d}}{\to} X. \)

By the previous lemma,
\[
\limsup_{n \to \infty} \text{Sep}(X_n, \kappa_1, \kappa_2, \ldots, \kappa_m) \leq \limsup_{n \to \infty} \text{Sep}(Z_n, \kappa_1, \kappa_2, \ldots, \kappa_m) \leq \text{Sep}(X, \kappa_1, \kappa_2, \ldots, \kappa_m) \]
\( \square \)
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