A note: graviton spin, gravitomagnetic fields and self-interaction of non-inertial frame of reference

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June 4, 2018

Three weak gravitational effects associated with the gravitomagnetic fields are taken into account in this paper: (i) we discuss the background Lorentz transformation and gauge transformation in a linearized gravity theory, and obtain the expression for the spin of gravitational field by using the canonical procedure and Noether theorem; (ii) we point out that by using the coordinate transformation from the fixed frame to the rotating frame, it is found that the nature of Mashhoon’s spin-rotation coupling is in fact an interaction between the gravitomagnetic moment of a spinning particle and the gravitomagnetic fields. The fact that the rotational angular velocity of a rotating frame can be viewed as a gravitomagnetic field is demonstrated; (iii) a purely gravitational generalization of Mashhoon’s spin-rotation coupling, i.e., the interaction of the graviton spin with the gravitomagnetic fields is actually a self-interaction of the spacetime (gravitational fields). In the present paper, we will show that this self-interaction will also arise in a non-inertial frame of reference itself: specifically, a rotating frame that experiences a fluctuation of its rotational frequency (i.e., the change in the rotational angular frequency) will undergo a weak self-interaction. The self-interaction of the rotating frame, which can also be called the self-interaction of the spacetime of the rotating frame, is just the non-inertial generalization of the interaction of the graviton spin with the gravitomagnetic fields.

Keywords: graviton spin, gravitomagnetic fields, self-interaction of non-inertial frame of reference

I. INTRODUCTION

Historically, the weak gravitational effects and the magnetic-type gravitational fields (referred to as the gravitomagnetic fields) have captured much attention of a number of investigators [1–11]. These effects include the Aharonov-Carmi effect [1] (gravitational Aharonov-Bohm effect) [2], Sagnac-type effect [3,4], spin-rotation coupling [5,6], and the effects and phenomena arising from the interaction between gravity (including the inertial force) and superconductors [7–10]. It is now well known that many analogies can be drawn between gravity and electromagnetic force in some aspects, for example, in a rotating reference frame, a moving particle is acted upon by both the inertial centrifugal force and the Coriolis force, which are analogous to the electric force and the Lorentz magnetic force, respectively, in electrodynamics. For this reason, Aharonov and Carmi proposed a geometric effect related to the vector potential of inertial force [1], and Anandan [12] and Dresden et al. [13] (independently) proposed a quantum-interferometry effect associated with the gravity. In fact, these two effects are the gravitational analog to the well-known Aharonov-Bohm effect, namely, the matter wave propagating along a closed path in a rotating frame will acquire a nonintegral phase factor (geometric phase factor). Thus, this effect can also be called the gravitational Aharonov-Bohm effect. Overhauser et al. [14] and Werner et al. [15] (independently) have shown experimentally the existence of the above effects by means of the neutron-gravity interferometry experiments.

It should be noted that the gravitational Aharonov-Bohm effect (Aharonov-Carmi effect) results from the interaction between the momentum of a particle and the rotating frame, the rotational frequency of which can be considered as a gravitomagnetic field: specifically, the Coriolis force $2m\vec{v} \times \vec{\omega}$ experienced by a moving particle inside a rotating frame is just the gravitational analog to the Lorentz magnetic force $q\vec{v} \times \vec{B}$ acting upon a charged particle in a magnetic field. By analogy with the interaction of the spinning magnetic moment with the magnetic field, one may propose a gravitational counterpart in gravity theory. Historically, Mashhoon et al. considered such a weak gravitational effect, i.e., the so-called spin-rotation coupling, which means that a spinning particle inside a rotating frame of reference will

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*It will be submitted nowhere else for publication, just uploaded at the e-print archives. It is a supplement to a brief report (submitted to PRD) entitled “The purely gravitational generalization of spin-rotation couplings” (by J.Q. Shen).

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1Note that here the fixed frame of reference is also viewed as a rest frame. The rotating frame rotates at an angular velocity, $\vec{\omega}$, relative to the rest frame.
undergo a coupling of its spin to the rotational frequency [5,6,16–18]. A remarkable feature of this interaction lies in that the spin-rotation coupling leads to an inertial effects of the intrinsic spin of the particle, for instance, although the equivalence principle still holds for a spinning particle, the universality of Galileo’s law of freely falling particles is violated, provided that the spin is polarized vertically up or down in the non-inertial frame: specifically, a spinning particle (with spin \( \vec{s} \)) in a rotating frame (with the rotating angular velocity being \( \vec{\omega} \)) will undergo a force \(-\nabla(\vec{\omega} \cdot \vec{s})\). The signs of the two forces \(-\nabla(\vec{\omega} \cdot \vec{s})\) acting upon the two spinning particles polarized vertically up and down are just opposite. For this reason, Mashhoon concluded that the spin possesses an inertial property and the universality of Galileo’s law of freely falling particles may therefore be violated [17].

The interaction between the weak gravity and the moving superconductors is of physical interest [7–10]². During the past two decades, many researchers investigated such an interaction both theoretically and experimentally. Peng et al. suggested a unified phenomenological theory to treat the interaction between arbitrarily moving superconductors and gravitational fields including the Newtonian gravity, gravitational waves, vector transverse gravitoelectric fields, and vector gravitomagnetic fields [8]. In the limit of weak field and low velocity, they considered the properties of induced electromagnetic and gravitational fields in the interior of a moving superconductor. It was shown that many physically interesting effects, phenomena and properties, including the Meissner effect, London moment, DeWitt effect, effects of gravitational wave on a superconductor, and induced electric fields in the interior of a freely vibrating superconductor, could be recovered from the theoretical expressions for the induced electromagnetic and gravitational fields in their unified phenomenological theory [8]. By further analysis one can demonstrate that the weak equivalence principle is valid in superconductivity, that Newtonian gravity and gravitational waves will penetrate either a moving superconductor or a superconductor at rest, and that a superconductor at rest cannot shield either vector gravitomagnetic fields or vector transverse gravitoelectric fields [8]³. It is worth stating that Ciubotariu et al. investigated the gravitomagnetic effects on a superconductor in the framework of the weak stationary gravitational field and low velocity, and obtained the similar conclusions [19]. Li et al. reported the investigation of the effects of a pure superconductor on the external gravitomagnetic and magnetic fields in a weak-gravity and low-velocity system [7]. It was found that a small residual uniform magnetic field will pervade the superconductor and that the external fields mutually “induce” additional small internal perturbation fields. Li et al. demonstrated that their obtained results might differ from the previous London theory and Meissner effect: the magnetic field inside a superconductor although very small no longer vanishes, and this nonzero internal magnetic field raises an interesting consequence regarding the internal gravitomagnetic field, i.e., for a regular superconducting material, the magnetically produced gravitomagnetic field is order of about \(10^{11}\) times the internal magnetic field [7].

Historically, the concept of spatial gravitational forces modelled after the electromagnetic Lorentz force has a long time and many names associated with it [20–25]. Born in the Newtonian context of centrifugal and Coriolis forces introduced by a rigidly rotating coordinate system in a flat Euclidean space, it has found a number of closely related but distinct generalizations within the context of general relativity and its linearized approximation. Since the frequent reference to “gravitoelectromagnetism” occurs in recent literature, Jantzen et al. placed all of these notions of “non-inertial forces” into a single framework which in turn might be used to infer relationships among them [26]. In addition, other interesting study of the weak gravitational field includes the optical-mechanical analogy in general relativity, i.e., the gravitational field can be represented as an optical medium with an effective optical index of refraction [27]. Thus, it was shown that the orbits of both massive and massless particles are governed by a variational principle which involves the refractive index and which assumes the form of Fermat’s principle or of Maupertuis’s principle [28].

In this paper, we discuss the related topics on the graviton spin and gravitomagnetic fields. In Sec. II, we obtain the expression for the spin of the weak gravitational field by using the canonical procedure and Noether theorem. In Sec. III, we discuss the connection between the rotational frequency of a non-inertial frame and the gravitomagnetic field. In Sec. IV, we predict that there exists a self-interaction of the rotating frame of reference due to the fluctuation

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²If Podkletnov’s experimental result (so-called weak gravitational shielding effect) [9] is true, it can be interpreted by using the concept of “oscillatorily varying Meissner effect”: specifically, the gravity in the superconductor is governed by the equation \(\nabla^2 g + \lambda g = 0 (\lambda > 0)\), where \(\lambda\) is a parameter associated with the cosmological constant and the self-induced mass current. In the superconducting medium, the cosmological constant is small compared with the contribution of the self-induced mass current, and can therefore be ignored. The solution of this equation is \(g = g_0 \cos(\sqrt{\lambda}z)\), which is approximately equal to \(g_0 (1 - \lambda z^2/2)\). The small term \(-g_0 (\lambda z^2/2)\) may be considered a weak gravitational shielding effect of the superconductors.

³This means that the traditional “exponentially decay Meissner effect” is absent. However, the so-called “oscillatorily varying Meissner effect” exists. Also see “On some weak-gravitational effects” (J.Q. Shen, arXiv: gr-qc/0305094).
II. THE SPIN OF GRAVITATIONAL FIELDS

In this paper, we deal with the weak gravitational fields only, which is described by the linearized gravity theory. One speaks of a linearized theory when the metric deviates only slightly from that of flat space. A weak gravitational field (in which spacetime is nearly flat) is defined as a manifold on which coordinates exist, where the metric has components $g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}$ with $|h_{\alpha\beta}| \ll 1$. Such coordinates are called nearly Lorentz coordinates, where the indices of tensors are raised and lowered with the flat-Minkowski metric $\eta_{\mu\nu}$ and $g^{\mu\nu}$ rather than with $g_{\mu\nu}$ and $g^{\mu\nu}$.

In the linearized gravity theory, there are two fundamental types of coordinate transformations, which take one nearly Lorentz coordinate system into another. The two coordinate transformations are the background Lorentz transformation (the background metric takes the form of simple diagonal $(+1, -1, -1, -1)$) and the gauge transformation (for $h_{\mu\nu}$). For the weak gravitational field, we can see that, under a background Lorentz transformation, $h_{\mu\nu}$ transforms as if it were a tensor in SR all by itself. Of course it is not a tensor. But this tells us that we can think of a slightly curved spacetime as a flat one with a “tensor” $h_{\mu\nu}$ defined on it, and moreover, all physical fields such as $R_{\alpha\beta\mu\nu}$ can be defined in terms of $h_{\mu\nu}$, and they will look like physical fields on a flat background spacetime [29].

The gauge transformation is such that a very small change in coordinates, which is of the form $R_{\alpha\beta\mu\nu}$, is generated by a vector $\zeta^\alpha$, the components of which are functions of position. If we demand that the “vector” $\zeta^\alpha$ be small in the sense that the relation $|\zeta^{\alpha,\beta}| \ll 1$ is satisfied, then we have [29]

$$L_\beta = \frac{\partial x^\alpha}{\partial x^\beta} = \delta^\alpha_\beta + \zeta^{\alpha,\beta}, \quad \Lambda_\alpha^\beta = \frac{\partial x^\alpha}{\partial x^\beta} = \delta^\alpha_\beta - \zeta^{\alpha,\beta} + \mathcal{O}\left(|\zeta^{\alpha,\beta}|^2\right), \quad (2.1)$$

By using the background Lorentz transformation $g_{\alpha\beta} = \Lambda_\alpha^\gamma \Lambda_\beta^\delta g_{\gamma\delta}$, keeping in mind Eq.(2.1), one can easily verify that, to the first order in small quantities,

$$g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta} - \zeta_\alpha,\beta - \zeta_\beta,\alpha, \quad (2.2)$$

where we define $\zeta_\alpha = \eta_{\alpha\beta} \zeta^\beta$. This means that the effect of the coordinate change is to re-define $h_{\alpha\beta}$:

$$h_{\alpha\beta} \rightarrow h_{\alpha\beta} - \zeta_\alpha,\beta - \zeta_\beta,\alpha, \quad (2.3)$$

which can be considered as a gauge transformation. Such a gauge transformation enables us to choose an appropriate gauge condition so as to simplify the expressions for the physical quantities (including the canonical momentum, spin) of gravitational fields.

In the following, we will obtain the expression for the spin of gravitational fields by making use of the Noether theorem and canonical procedure with the linearized gravity theory.

In the linearized gravity theory, to the first order in $h_{\mu\nu}$, the Riemann tensor is written

$$R_{\alpha\beta\mu\nu} = \frac{1}{2} (h_{\alpha\nu,\beta\mu} + h_{\beta\mu,\alpha\nu} - h_{\alpha\mu,\beta\nu} - h_{\beta\nu,\alpha\mu}) + \mathcal{O}(h^2), \quad (2.4)$$

and consequently the curvature scalar is of the form

$$R = g^{\alpha\mu} g^{\beta\nu} R_{\alpha\beta\mu\nu} = 2 h^{\beta\nu} \left(h_{\nu,\beta\mu} - \frac{1}{2} h_{\beta,\mu,\nu} - \frac{1}{2} h_{\mu,\beta,\nu} \right) + \mathcal{O}(h^3). \quad (2.5)$$

Thus, the action of the gravitational field may be written in the form

$$I = \int_{\Omega} d\Omega \sqrt{-g} R \simeq \int_{\Omega} d\Omega \left[-2 \left(h_{\beta,\mu,\nu} h_{\nu,\beta} - \frac{1}{2} h_{\beta,\mu,\nu} h_{\beta,\mu} - \frac{1}{2} h_{\mu,\beta,\nu} h_{\beta,\mu} \right) \right] + \text{S.T.}, \quad (2.6)$$

where S.T. denotes the surface term. It follows that, in the nearly Lorentz coordinate system, the Lagrangian density of the gravitational field takes the form

$$\mathcal{L} = -2 \left(h_{\beta,\mu,\nu} h_{\mu,\nu,\beta} - \frac{1}{2} h_{\beta,\mu,\nu} h_{\beta,\nu,\mu} - \frac{1}{2} h_{\beta,\nu,\mu} h_{\mu,\beta,\nu} \right). \quad (2.7)$$
According to the canonical procedure, the canonical momentum of the linearized gravitational field is

$$\pi^{\mu\nu} = \frac{\partial L}{\partial h_{\mu\nu}} = -4 \left( h^{0\nu, \mu} - \frac{1}{2} h_{\mu\nu,0} \right) + \eta^{\mu\nu} h^{0\lambda, \lambda} + \eta^{\nu\lambda} h_{\lambda, \mu},$$

(2.8)

where dot denotes the derivative with respect to time, and $h = \eta^{\mu\nu} h_{\mu\nu}$. In accordance with the Noether theorem, the spin of a field that is characterized by the Lagrangian density (2.7) takes the form

$$S^0^\tau = \int d^3x \left( \pi^{\mu\nu} \frac{\partial}{\partial h_{\mu\nu}} + \eta^{\mu\nu} \frac{\partial L}{\partial h^{0\lambda, \lambda}} + \eta^{\nu\lambda} \frac{\partial L}{\partial h_{\lambda, \mu}} \right) \left( \delta^\theta_\mu \delta^\tau_\eta - \delta^\theta_\eta \delta^\tau_\mu \right).$$

(2.9)

Now we calculate the integrand $S^0^\tau$ of the above integral. With the help of (2.8), one can arrive at

$$S^0^\tau = \pi^{\mu\nu} \sum_{\nu\eta} h^\eta_\mu = \left[ -4 \left( h^{0\nu, \mu} - \frac{1}{2} h_{\mu\nu,0} \right) + \eta^{\mu\nu} h^{0\lambda, \lambda} + \eta^{\nu\lambda} h_{\lambda, \mu} \right] \left( \delta^\theta_\mu \delta^\tau_\eta - \delta^\theta_\eta \delta^\tau_\mu \right).$$

(2.10)

Note that here the indices $\theta, \tau$ take values from 1 to 3. Further calculation yields

$$S^0^\tau = -4 \left( h^{0\nu, \mu} h^\tau_{\mu} - h^{0\nu, \mu} h^\theta_{\mu} \right) + 2 \left( h^{0\nu, \theta} h^\tau_{\mu} - h^{0\nu, \mu} h^\theta_{\tau} \right) + \left( h^{0\lambda, \lambda} h^\theta_\tau - h^{0\lambda, \lambda} h^{\tau\theta} \right) + h_{\mu\nu} \left( \eta^{\rho\theta} h^\tau_{\mu} - \eta^{\tau\rho} h^\theta_{\mu} \right).$$

(2.11)

It is apparently seen that the third term on the right-handed side of the expression (2.11) vanishes, i.e., $h^{0\lambda, \lambda} h^\theta_\tau - h^{0\lambda, \lambda} h^{\tau\theta} = 0$, since $h^{0\nu, \mu}$ is symmetric in indices. Additionally, here we will choose a gauge condition in which the relation $h_{\mu\nu} = h_{11} = h_{22} = h_{33} = 0$ is satisfied. According to the gauge transformation (2.3), this condition (containing four relations) can be easily realized only by introducing an appropriate “gauge vector” (vector parameters) $\zeta_\alpha$ ($\alpha$ runs from 0 to 3). This, therefore, means that $h = \eta^{\mu\nu} h_{\mu\nu} = 0$ and, in consequence, $h^\nu_\mu = 0$. Thus, the fourth term on the right-handed side of the expression (2.11) is also vanishing. Hence, it follows from (2.11) that the density of the spin of the weak gravitational field is rewritten as

$$S^0^\tau = -4 \left( h^{0\nu, \mu} h^\tau_{\mu} - h^{0\nu, \mu} h^\theta_{\mu} \right) + 2 \left( h^{0\nu, \theta} h^\tau_{\mu} - h^{0\nu, \mu} h^\theta_{\tau} \right).$$

(2.12)

In what follows we will let the derivatives of the “gauge vector” (vector parameters) $\zeta_\alpha$ agree with the following three conditions $h_{i\j} \rightarrow h_{i\j} - \zeta_{i\j} \rightarrow 0$ ($i, j = 1, 2, 3$), which means that the off-diagonal spacial-part field components $h_{12}, h_{23}, h_{31}$ approach 0, and the only retained components are the gravitomagnetic vector potentials $h_{0i}$ ($i = 1, 2, 3$). It should be noted that this gauge condition can be readily realized in the linearized gravity theory, for example, for the rotating gravitating body, the gravitomagnetic components $h_{0i}$ ($i = 1, 2, 3$) are in general much greater than the other off-diagonal field components $h_{i\j}$ ($i, j = 1, 2, 3$), that is, there exists a coordinate system in which the off-diagonal field components $h_{i\j}$ ($i, j = 1, 2, 3$) vanishes (or nearly vanishes). Hence, in this gauge condition, the spin of gravitational field can be simplified as the following form $S^0^\tau = -2 \left( h^{0\nu, \theta} h^{\tau\theta} - h^{0\nu, \theta} h^{\theta\tau} \right)$. If the gravitomagnetic vector potentials are represented by a three-dimensional “vector” $g^\theta$, i.e., $h^{0\theta} = g^\theta$, then the above expression for $S^0^\tau$ can be rewritten as

$$S^0^\tau = -2 \left( g^\theta g^\tau - g^\tau g^\theta \right),$$

(2.13)

where dot denotes the time derivative. Note that in the electrodynamics, the spin of the electromagnetic field is expressed by $s^\theta = - \left( \dot{A}^\theta - \dot{A}^\theta \right)$ [30]. It follows from the comparison of the expression for the spin of electromagnetic field with (2.13) that the factor 2 in (2.13) may imply that the spin of a weak gravitational field is just two times that of the electromagnetic field.

We have studied a purely gravitational generalization of spin-rotation couplings, the Lagrangian density of which is [31].

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4The introduction of $\zeta_\alpha$ means that the four gauge conditions are introduced and can fix the four gravitational field components $h_{\mu\nu}$, say, $h_{00}, h_{11}, h_{22}, h_{33}$. According to the gauge transformation (2.3), the four gauge conditions are $h_{i\j} \rightarrow h_{i\j} - 2\zeta_{i\j} = 0$ with $i = 0, 1, 2, 3$. Thus, in this gauge condition, all the diagonal components $h_{i\i} = 0$ ($i = 0, 1, 2, 3$).

5For the detailed derivation, see, for example, “The interaction between graviton spin and gravitomagnetic fields” (J.Q. Shen, arXiv: gr-qc/0312116).


\[ \mathcal{L}_{\text{v}} = \frac{1}{2} K^2 (g_i \dot{g}_j - g_j \dot{g}_i) (\partial_i g_j - \partial_j g_i), \tag{2.14} \]

where \( i, j = 1, 2, 3, \) and the coefficient \( K \) and the gravitomagnetic vector potential \( g_i \) are so defined that the relation \( \sqrt{-g_{00}} \simeq K g_i \) is satisfied. Since the expression \( (g_i \dot{g}_j - g_j \dot{g}_i) \) in \( \tag{2.14} \) can be thought of as a term associated with the spin of gravitational field, and \( \partial_i g_j - \partial_j g_i \) is an expression for the gravitomagnetic field, the nature of such a purely gravitational generalization is an interaction between the gravitational spinning moment (gravitomagnetic moment) and the gravitomagnetic fields. In the next section, we will show that for the spacetime in the rotating frame of reference, the expression \( \partial_i g_j - \partial_j g_i \) contains the angular velocity of the rotating frame, i.e., the rotational frequency is only a piece of the gravitomagnetic field. Hence, it is shown that the interaction in \( \tag{2.14} \) is truly a purely gravitational generalization of Mashhoon’s spin-rotation couplings.

### III. Mashhoon’s Spin-Rotation Coupling and Gravitomagnetic Fields

Mashhoon showed that a spinning particle in a rotating frame of reference will experience an interaction between its spin and the angular frequency of the rotating frame. This interaction is referred to as the spin-rotation coupling \([5,6,16–18]\). It is clearly seen that this coupling is in fact an inertial effect of spinning particles.

Further analysis may show that here the spin of the spinning particle can be regarded as the gravitomagnetic moment, and the angular velocity of the rotating frame can be thought of as a gravitomagnetic field\(^6\), since it can be verified that, in a rotating frame of reference, the spin-rotation coupling is involved in the interaction of the gravitomagnetic moment and the gravitomagnetic field.

In the following, by taking account of the coordinate transformation from the rest frame to the rotating frame, we will show that, for a moving particle, the rotating angular frequency of a rotating frame of reference can truly be viewed as a gravitomagnetic field. Hence, in order to deal with this problem, we consider the Kerr metric of the exterior gravitational field of the rotating spherically symmetric body, which is of the form

\[
\begin{align*}
\text{ds}^2 &= \left[ 1 - \frac{2\text{GM}r}{c^2 (r^2 + a^2 \cos^2 \theta)} \right] c^2 dt^2 - \frac{r^2 + a^2 \cos^2 \theta}{r^2 + a^2 - \frac{2\text{GM}r}{c^2}} dr^2 - (r^2 + a^2 \cos^2 \theta) d\theta^2 \\
&\quad - \sin^2 \theta \left( \frac{2a \sin^2 \theta}{r^2 + a^2 \cos^2 \theta} \frac{\text{GM}r}{c^2} + r^2 + a^2 \right) d\phi^2 + \frac{2a \sin^2 \theta}{r^2 + a^2 \cos^2 \theta} \frac{\text{GM}r}{c} dt d\phi,
\end{align*}
\tag{3.1}
\]

where \( r, \theta, \phi \) are the displacements of spherical coordinate. Here \( a \) is so defined that \( ac \) is the angular momentum of unit mass of the rotating gravitating body, and \( M \) denotes the mass of this gravitating body. Note that the spacetime coordinate of Kerr metric \( \tag{3.1} \) is in the rest (fixed) reference frame. We can transform the above Kerr metric into the form in the rotating reference frame. Because of the smallness of the Earth’s rotating velocity, one can apply the following Galileo transformation to the coordinates of a test particle moving radially in the rotating frame

\[
\text{dr} + v dt = \text{dr}', \quad \text{d}\theta = \text{d}\theta', \quad \text{d}\phi = \text{d}\phi + \omega dt, \quad \text{dt}' = \text{dt}
\tag{3.2}
\]

with \( v \) being the radial velocity of the test particle relative to the rotating reference frame, \( (r', \theta', \phi', t') \) and \( (r, \theta, \phi, t) \) the spacetime coordinates of the rotating frame and fixed frame, respectively. \( \omega \) denotes the rotational frequency of the rotating frame with respect to the fixed reference frame. For the simplicity of calculation, the radial velocity \( v \) is taken to be much less than \( \omega r \). The substitution of Eq.\( \tag{3.2} \) into Eq.\( \tag{3.1} \) yields

\[
\begin{align*}
\text{ds}^2 &= \left[ 1 - \frac{2\text{GM}r}{c^2 (r^2 + a^2 \cos^2 \theta)} - \frac{(r^2 + a^2 \cos^2 \theta) v^2}{r^2 + a^2 - \frac{2\text{GM}r}{c^2}} - \sin^2 \theta \left( \frac{r^2 + a^2}{r^2 + a^2 - \frac{2\text{GM}r}{c^2}} \right) \frac{\text{GM}r}{c^2} - \frac{2a}{r^2 + a^2 \cos^2 \theta} \frac{\text{GM}r}{c^2} \right) \omega^2 \\
&\quad \times c^2 dt^2 - \frac{r^2 + a^2 \cos^2 \theta}{r^2 + a^2 - \frac{2\text{GM}r}{c^2}} dr^2 - (r^2 + a^2 \cos^2 \theta) d\theta^2 - \sin^2 \theta \left( \frac{2a \sin^2 \theta}{r^2 + a^2 \cos^2 \theta} \frac{\text{GM}r}{c^2} + r^2 + a^2 \right) d\phi^2 \\
&\quad - \frac{2(r^2 + a^2 \cos^2 \theta) v}{r^2 + a^2 - \frac{2\text{GM}r}{c^2}} c dt d\phi + \left[ \frac{2a \sin^2 \theta}{r^2 + a^2 \cos^2 \theta} \frac{\text{GM}r}{c} + 2 \sin^2 \theta \left( \frac{r^2 + a^2 + \frac{2a \sin^2 \theta}{r^2 + a^2 \cos^2 \theta} \frac{\text{GM}r}{c^2}}{r^2 + a^2} \right) \omega \right] dt' d\phi'.
\end{align*}
\tag{3.3}
\]

\(^6\)These viewpoints are adopted from Ref.\([32]\).

where the term $\frac{2a^2}{\epsilon r^2} \sin^2 \theta$ in $g_{\theta \theta}$ results in the inertial centrifugal force written as $\vec{F} = m\vec{\omega} \times (\vec{\omega} \times \vec{r})$. Ignoring the small terms associated with the relation $\frac{a^2}{\epsilon r^2} \ll 1$ in $g_{\theta \theta}$, one can arrive at

$$g_{\theta \theta}' d\varphi' \, dt' = \left( \frac{2aGM \sin^2 \theta}{c r^2} + 2 \omega r^2 \sin^2 \theta \right) dt' \, d\varphi'$$

$$= \left( \frac{2aGM \sin \theta}{c r^2} + 2 \omega r \sin \theta \right) dt' \sin \theta \, d\varphi'.$$

Thus, the gravitomagnetic potentials can be written as

$$g_{\phi} = \frac{2aGM \sin \theta}{c r^2} + 2 \omega r \sin \theta, \quad g_r = -2v, \quad g_{\theta} = 0.$$  \hspace{1cm} (3.5)

It follows that the first term, $\frac{2aGM \sin \theta}{c r^2}$, on the right-hand side of (3.5) is exactly analogous to the magnetic potential $\frac{\mu_0}{4\pi} \frac{ca}{r^2} \sin \theta$ produced by a rotating charged spherical shell in the electrodynamics. So, the first term $\frac{2aGM \sin \theta}{c r^2}$ of $g_{\phi}$ shows the existence of a gravitomagnetic field of the rotating gravitating body, the exterior gravitomagnetic strength of which is [33]

$$\vec{B}_g = \frac{2G}{c} \left[ \frac{\vec{a}}{r^3} - \frac{3 (\vec{a} \cdot \vec{r}) \vec{r}}{r^5} \right].$$  \hspace{1cm} (3.6)

In accordance with the equation of geodesic line of a particle in the post-Newtonian approximation, the gravitomagnetic strength can be defined by $-\frac{1}{2} \nabla \times \vec{g}$ with $\vec{g} = (g_{01}, g_{02}, g_{03})$ as assumed above. If we set $\beta_{\phi} = 2 \omega \sin \theta, \beta_r = -2v, \beta_{\theta} = 0$, then the gravitomagnetic strength that arises in the rotating reference frames is given as follows

$$-\frac{1}{2} \nabla \times \vec{\beta} = -2 \omega \cos \theta e_r + 2 \omega \sin \theta e_{\theta}$$  \hspace{1cm} (3.7)

with $e_r, e_{\theta}$ being the unit vector. It follows from Eq.(3.7) that such a gravitomagnetic strength is related to the rotation of non-inertial frame and independent of the Newtonian gravitational constant $G$. From the point of view of Newtonian mechanics, it is the inertial force field in essence rather than the field produced by mass current. Since we have assumed that the velocity of a test particle is parallel to $e_r$, i.e., $\vec{v} = v e_r$, the gravitational Lorentz force acting on the test particle in the gravitomagnetic field is thus given by

$$\vec{F} = m\vec{\omega} \times \left( -\frac{1}{2} \nabla \times \vec{\beta} \right) = 2v \omega \sin \theta e_{\phi} = 2m\vec{v} \times \vec{\omega}.$$  \hspace{1cm} (3.8)

We conclude from Eq.(3.8) that the gravitational Lorentz force in a rotating reference frame is just the familiar Coriolis force, and that the rotational frequency $\vec{\omega}$ can be regarded as a gravitomagnetic field strength (or a piece of the gravitomagnetic field strength).

Since the angular frequency of the rotating frame is just a piece of the gravitomagnetic field, Mashhoon’s spin-rotation coupling is actually the interaction of the gravitomagnetic moment with the gravitomagnetic field. In the literature, to the best of our knowledge, the spin-rotation coupling of photon, electron and neutron has been taken into account [17,32,34]. However, the gravitational coupling of graviton spin to the gravitomagnetic fields, which may also be of physical interest, has so far never yet been considered. We think it is necessary to extend Mashhoon’s spin-rotation coupling to a purely gravitational case, where the graviton spin will be coupled to the gravitomagnetic fields.

**IV. THE SELF-INTERACTION OF THE NON-INERTIAL FRAME OF REFERENCE**

For the present, Mashhoon’s spin-rotation coupling could be tested only in microwave experiments [17], since it is relatively weak due to the smallness of the rotational frequencies of various rotating frames on the Earth. We argue that besides the microwave experiments, there is another new method to test for the spin-rotation coupling, i.e., the rotational motion of C$_{60}$ molecules may provide us with an ingenious way to detect this weakly gravitational (gravitomagnetic) effect [34]. It follows that since in the high-temperature phase (orientationally disordered phase) of C$_{60}$ solid, the rotational frequency, $\omega$, of C$_{60}$ molecules may be about $10^{11}$ rad/s [34], the precessional frequency $\Omega$ is therefore compared to $\omega$, i.e.,
\[ \Omega \simeq \frac{|\vec{M}|}{I \omega} \]  

(4.1)  
ranges from \(10^{10}\) to \(10^{12}\) rad/s. Here, \(\vec{M}\) and \(I\) denote the moment of noncentral intermolecular force and the \(C_{60}\) moment of inertia, respectively. Because the rotational angular velocity is much greater than that of any rotating bodies on the Earth, the \(C_{60}\) molecule is an ideal non-inertial frame of reference, where the effects for the valency electrons in the \(C_{60}\) molecule resulting from the electron spin-rotation coupling may be easily observed experimentally.

In addition to the above electron spin-rotation coupling in the rotating \(C_{60}\) molecule, there may exist another novel interaction, which is related closely to the inertial property of the rotating frame of reference itself. The linearized gravity theory indicates that the interaction of the graviton spin with the gravitomagnetic fields (a purely gravitational generalization of Mashhoon’s spin-rotation coupling) is a self-interaction of the spacetime (gravitational fields). Here, we will show that this self-interaction will also arise in a non-inertial frame of reference itself, namely, a rotating frame, the rotating frequency of which fluctuates, will undergo a weak self-interaction described by (2.14). It is believed that such a self-interaction of the rotating frame is just the non-inertial generalization of the interaction of the graviton spin with the gravitomagnetic fields (i.e., the self-interaction of the spacetime). According to Sec. 2, in a rotating frame, the \(\varphi\)-component of the gravitomagnetic vector potential is \(g_\varphi = \frac{2GM\sin\theta}{r^2} + 2\omega r \sin \theta\). Generally speaking, in the weak gravitational field, the contribution of the term \(2\omega r \sin \theta\) is much greater than that of \(\frac{2GM\sin\theta}{r^2}\). So, the \(\varphi\)-component of the gravitomagnetic vector potential (measured by the observer fixed in the \(C_{60}\) rotating frame) is \(g_\varphi \simeq 2\omega r \sin \theta\). It follows that if either the direction or the magnitude of the angular velocity of the rotating frame changes, then the time derivative of the gravitomagnetic vector potentials is nonvanishing, i.e.,

\[ |\dot{g}_\varphi| \simeq |2\omega r \sin \theta| \neq 0, \]

(4.2)
and according to (2.14), this rotating frame will be subjected to a self-interaction, the nature of which is just a nonlinear interaction of the spacetime of the non-inertial frame of reference itself. It is apparent that such a self-interaction of the non-inertial frame of reference is one of the predictions of both Einstein’s field equation and the principle of equivalence, while in the Newtonian mechanics, there exists no such a self-interaction.

In what follows, we will consider the change in the angular frequencies of rotating \(C_{60}\) molecules acted upon by a noncentral intermolecular force that causes both the precession and rotation of molecular rotation of \(C_{60}\). Roughly speaking, the moment of noncentral intermolecular force, \(\vec{M}\), is the product of van der Waals force (referring to the noncentral part) and the distance between molecules, the order of magnitude \(|\vec{M}|\) of which is \(10^{-22} \sim 10^{-26}\) N·m. The equation of the rotational motion of \(C_{60}\) molecule coupled to its neighbors is \(\frac{d}{dt}\vec{L} = \vec{M}\) with the angular momentum \(\vec{L} = I\vec{\omega}\), where the \(C_{60}\) moment of inertia is \(I \simeq 1.0 \times 10^{-43}\) Kg·m². Thus the rate of change of the \(C_{60}\) angular velocity \(\vec{\omega}\) is

\[ \frac{d}{dt}\vec{\omega} = \frac{1}{I}\vec{M} = (10^{21} \sim 10^{23}) \text{ s}^{-2}. \]

(4.3)
So, the linear acceleration of valency electrons on the \(C_{60}\) molecular surface due to the above fluctuation of \(\vec{\omega}\) is about \(10^{12} \sim 10^{14}\) m/s², which is the same order of magnitude of the inertial centrifugal acceleration⁸ (~ \(3 \times 10^{12}\) m/s²) of the valency electrons of \(C_{60}\) due to the rotational motion of the \(C_{60}\).

It is well known that the Aharonov-Carni effect [1], neutron Sagnac effect [3,4], spin-rotation coupling [5,6] are the inertial effects of the rotating frame. Most of these inertial effects may have influence on the frequency (energy) shift in atoms, molecules and light wave. For example, in the optical “Foucault pendulum” (the Michelson-Gale light

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⁷Such a large fluctuation of angular velocity (gravitomagnetic field) of frame of reference will generate a kind of space-time ripples, which can be called the non-inertial gravitational wave. Detecting and investigating the gravitational wave is one of the leading areas in both astrophysics and cosmology. Because of the weakness of the gravitational radiation, gravitational wave has not been detected up to now even by means of both resonant-mass detectors and laser-interferometric detectors. Here, however, the density of the non-inertial gravitational wave is no longer restricted to the smallness of the gravitational constant. For the detailed discussion of the non-inertial gravitational wave, see “On some weak-gravitational effects” (J.Q. Shen, arXiv: gr-qc/0305094).

⁸In the high-temperature phase (orientationally disordered phase), the rotational frequency, \(\omega\), of \(C_{60}\) molecules is about \(10^{11}\) rad/s. The \(C_{60}\) radius is \(a = 3.55\) Å. So, the inertial centrifugal acceleration, \(\omega^2a\), experienced by the valency electrons in \(C_{60}\) molecules, is about \(3 \times 10^{12}\) m/s².
interferometer), the rotation of Earth could yield a Sagnac shift of the light waves [3]; the energy of the valency electrons of C$_{60}$ is shifted due to the interaction of the electron spin with the rotation of C$_{60}$ [34]. We think that since both the fluctuation in the angular velocity of C$_{60}$ molecules and the change in the gravitomagnetic vector potential (measured by the observer fixed in the C$_{60}$ rotating frame) is very great, the above-mentioned self-interaction of C$_{60}$ rotating frame may deserve investigation for the treatment of the photoelectron spectroscopy, noncentral intermolecular potential and molecular rotational dynamics of C$_{60}$ solid$^9$.

The consideration of the gravitomagnetic fields, the spin of the weak gravitational field and even the gravitomagnetic dynamo theory can lead to the appearance of many novel gravitomagnetic effects. We hope the above-mentioned nonlinear interaction of the rotating frame of reference itself and its potential effect on the photoelectron spectroscopy of rotating C$_{60}$ molecules would be studied experimentally in the near future.

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$^9$Additionally, maybe such a self-interaction of the non-inertial frame of reference is also worth considering inside the non-inertial frame itself if the change in the angular velocity of the binary pulsar (which can generate the gravitational radiation) and the rotating star due to the gravitational collapse is rather great.