Centrally symmetric Cohen–Macaulay complexes and a conjecture of Stanley

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Abstract

In 1987, Stanley conjectured that if a centrally symmetric Cohen–Macaulay simplicial complex $\Delta$ of dimension $d - 1$ satisfies $h_i(\Delta) = \binom{d}{i}$ for some $1 \leq i \leq d - 1$, then $h_j(\Delta) = \binom{d}{j}$ for all $j \geq i$. This note proves Stanley’s conjecture.

1 Introduction

This paper is devoted to Stanley’s conjecture on the face numbers of centrally symmetric Cohen–Macaulay complexes.

In the seventies, Stanley and Hochster (independently from each other) introduced the notion of Stanley–Reisner rings and started developing their theory, see [3, 7, 8, 9]. In the fifty years since, this theory has become a major tool in the study of face numbers of simplicial complexes. One of its first applications was a complete characterization of face numbers of Cohen–Macaulay (CM, for short) simplicial complexes [9]: it asserted that a vector is the $h$-vector of a CM complex if and only if its entries are non-negative integers satisfying Macaulay-type inequalities. Another application was a complete characterization of flag face numbers of balanced CM complexes, [10, 2].

A simplicial complex $\Delta$ is called centrally symmetric (or cs) if its vertex set is endowed with a free involution $\alpha : V \rightarrow V$ that induces a free involution on the set of all non-empty faces of $\Delta$. Motivated by the desire to understand face numbers of cs simplicial polytopes as well as to find a complete characterization of face numbers of cs CM complexes, Stanley [11] proved the following Lower Bound Theorem:

Theorem 1.1. Let $\Delta$ be a $(d - 1)$-dimensional cs CM simplicial complex. Then $h_i(\Delta) \geq \binom{d}{i}$ for all $0 \leq i \leq d$.

These inequalities are sharp: indeed, the boundary complex of the $d$-cross-polytope has $h_i = \binom{d}{i}$ for all $i$. Stanley also proposed the following conjecture:

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Conjecture 1.2. Let $\Delta$ be a $(d - 1)$-dimensional cs CM simplicial complex. Suppose $h_i(\Delta) = \binom{d}{i}$ for some $i \geq 1$. Then $h_j = \binom{d}{j}$ for all $j \geq i$.

Stanley verified this conjecture in the case that $j$ is even or $j - i$ is even. In this note we prove Stanley’s conjecture in full generality. Along the way, we show that any such complex $\Delta$ contains the boundary complex of a $d$-cross-polytope as a subcomplex — the fact that might be of independent interest. Our proof utilizes the theory of stress spaces developed by Lee [5]. Specifically, the $h_i$-number of a Cohen–Macaulay complex $\Delta$ can be viewed as the dimension of a certain space of $i$-stresses on $\Delta$. A key observation is that if $\Delta$ is a $(d - 1)$-dimensional cs CM complex with $h_i = \binom{d}{i}$, then all $i$-stresses on $\Delta$ are symmetric, see the discussion in Section 2. (A similar idea was utilized in [4] to characterize cs simplicial $d$-polytopes with $g_2 = \binom{d}{2} - d$.)

While some strengthenings of Macaulay-type inequalities for cs CM complexes were established in [6], a complete characterization of face numbers of cs CM complexes remains elusive.

2 Setting the stage

We review several definitions and results on simplicial complexes, Stanley–Reisner rings, stress spaces, and Cohen–Macaulayness, as well as prepare ground for the proofs. The proof of Conjecture 1.2 is given in Section 3. For all undefined terminology we refer the reader to [5, 12].

A simplicial complex $\Delta$ on the ground set $V$ is a collection of subsets of $V$ that is closed under inclusion; $v$ is a vertex of $\Delta$ if $\{v\} \in \Delta$, but not all elements of $V$ are required to be vertices. The elements of $\Delta$ are called faces. The dimension of a face $\tau \in \Delta$ is $\dim \tau := |\tau| - 1$. The dimension of $\Delta$, $\dim \Delta$, is the maximum dimension of its faces. A face of a simplicial complex $\Delta$ is a facet if it is maximal w.r.t. inclusion. We say that $\Delta$ is pure if all facets of $\Delta$ have the same dimension. If $\tau$ is a face of $\Delta$, then the star of $\tau$ and the link of $\tau$ in $\Delta$ are the following subcomplexes of $\Delta$:

$\text{st}_\Delta(\tau) = \text{st}(\tau) := \{\sigma \in \Delta : \sigma \cup \tau \in \Delta\}$ and $\text{lk}_\Delta(\tau) = \text{lk}(\tau) := \{\sigma \in \text{st}_\Delta(\tau) : \sigma \cap \tau = \emptyset\}$.

For a vertex $v$, we write $v, \text{st}_\Delta(v)$, and $\text{lk}_\Delta(v)$ instead of $\{v\}, \text{st}_\Delta(\{v\})$, and $\text{lk}_\Delta(\{v\})$, resp.

Let $\Delta$ be a $(d - 1)$-dimensional simplicial complex. For $-1 \leq i \leq d - 1$, the $f$-number of $\Delta$, $f_i = f_i(\Delta)$, denotes the number of $i$-dimensional faces of $\Delta$. The $h$-numbers of $\Delta$, $h_i = h_i(\Delta)$ for $0 \leq i \leq d$, are defined by the relation $\sum_{j=0}^{d} h_j \lambda^{d-j} = \sum_{i=0}^{d} f_i (\lambda - 1)^{d-i}$.

Let $\Delta$ be a simplicial complex on the ground set $V$. Let $X = \{x_v : v \in V\}$ be the set of variables and let $\mathbb{R}[X]$ be the polynomial ring over the real numbers $\mathbb{R}$ in variables $X$. The Stanley–Reisner ideal of $\Delta$ is defined as

$I_\Delta = (x_{v_1}x_{v_2} \ldots x_{v_i} : \{v_1, v_2, \ldots, v_i\} \notin \Delta),$

i.e., it is the ideal generated by the squarefree monomials corresponding to non-faces of $\Delta$. The Stanley–Reisner ring of $\Delta$ is $\mathbb{R}[\Delta] := \mathbb{R}[X]/I_\Delta$. The ring $\mathbb{R}[\Delta]$ has a $\mathbb{Z}$-grading: $\mathbb{R}[\Delta] = \bigoplus_{i=0}^{\infty} \mathbb{R}[\Delta]_i$, where the $i$th graded component $\mathbb{R}[\Delta]_i$ is the space of homogeneous polynomials of degree $i$ in $\mathbb{R}[\Delta]$. In general, for a $\mathbb{Z}$-graded vector space $M$, denote by $M_i$ the $i$th graded component of $M$.

Let $\Delta$ be a $(d - 1)$-dimensional simplicial complex. A sequence $\Theta = \theta_1, \theta_2, \ldots, \theta_d$ of linear forms in $\mathbb{R}[X]$ is a linear system of parameters (or l.s.o.p., for short) if $\mathbb{R}[\Delta]/\Theta \mathbb{R}[\Delta]$ is a finite-dimensional $\mathbb{R}$-vector space. We denote the quotient $\mathbb{R}[\Delta]/\Theta \mathbb{R}[\Delta]$ by $\mathbb{R}(\Delta, \Theta)$, or by $\mathbb{R}(\Delta)$ if $\Theta$ is fixed or understood from context.
We say that $\Delta$ is \textit{Cohen–Macaulay} (or CM, for short) if for some (equivalently, every) l.s.o.p. $\Theta = \theta_1, \theta_2, \ldots, \theta_d$, 
\[
\dim \mathbb{R}(\Delta, \Theta)_i = h_i(\Delta), \quad \forall \ 0 \leq i \leq d.
\]
There are other equivalent definitions of CM complexes. The most standard one is that $\Delta$ is CM if some (equivalently, every) l.s.o.p. of $\mathbb{R}[\Delta]$ is a regular sequence. It is also worth mentioning that CM complexes have a topological characterization due to Reisner \cite{7}. This characterization implies, for instance, that CM complexes are pure and that links and stars of CM complexes are also CM.

For our proofs, we will work in the dual setting of stress spaces developed by Lee \cite{5}, see also \cite{11} Section 3. Observe that a variable $x_v$ acts on $\mathbb{R}[X]$ by $\frac{\partial}{\partial x_v}$; for brevity, we will denote this operator by $\partial_{x_v}$. More generally, if $c(X) = \sum_{v \in V} c_v x_v$ is a linear form in $\mathbb{R}[X]$, then we define
\[
\partial_{c(X)} : \mathbb{R}[X] \to \mathbb{R}[X],
\]
\[
w \mapsto \sum_{v \in V} c_v \cdot \partial_{x_v} w = \sum_{v \in V} c_v \frac{\partial w}{\partial x_v}.
\]

For a monomial $\mu \in \mathbb{R}[X]$, the \textit{support} of $\mu$ is $\text{supp}(\mu) = \{v \in V : x_v | \mu\}$. A homogeneous polynomial $w \in \mathbb{R}[X]$ of degree $i$ is called an \textit{i-stress} on $\Delta$ w.r.t. an l.s.o.p. $\Theta$ if it satisfies the following conditions:

- Every term $\mu$ of $w$ is supported on a face of $\Delta$: $\text{supp}(\mu) \subseteq \Delta$, and
- $\partial_{\theta_k} w = 0$ for all $k = 1, \ldots, d$.

The \textit{support} of an $i$-stress $w$, $\text{supp}(w)$, is the subcomplex of $\Delta$ generated by the support of all terms of $w$. We say that a face $F \in \Delta$ \textit{participates in a stress} $w$ if $F \in \text{supp}(w)$. We also say that a stress $w$ \textit{lives on a subcomplex} $\Gamma$ of $\Delta$ if $\text{supp}(w) \subseteq \Gamma$.

Denote the set of all $i$-stresses on $\Delta$ w.r.t $\Theta$ by $S(\Delta, \Theta)_i$ or by $S(\Delta)_i$ if an l.s.o.p. $\Theta$ is fixed or understood from context. This set is a vector space \cite{11} \cite{5}; it is a subspace of $\mathbb{R}[X]$. In fact, $S(\Delta, \Theta)_i$ is the orthogonal complement of $(I_\Delta + (\Theta))_i$ in $\mathbb{R}[X]$ w.r.t. a certain inner product on $\mathbb{R}[X]_i$, see \cite{5} Section 3. Thus, as a vector space, $S(\Delta, \Theta)_i$ is canonically isomorphic to $\mathbb{R}(\Delta, \Theta)_i$. (For an alternative approach using the Weil duality, see \cite{11} Section 3.) In particular, if $\Delta$ is CM, then $S(\Delta)_i$ has dimension $h_i(\Delta)$. Another useful and easy fact is that for every linear form $c(X) \in \mathbb{R}[X]$, the operator $\partial_{c(X)}$ maps $S(\Delta)_i$ into $S(\Delta)_{i-1}$. This follows from the fact that $\partial_{\theta_k}$ and $\partial_{c(X)}$ commute, and that a subset of a face of $\Delta$ is a face of $\Delta$.

Stresses are convenient to work with for the following reason: if $\Gamma$ is a subcomplex of $\Delta$ (considered as a complex on the same ground set $V$ as $\Delta$), then there is a natural surjective homomorphism $\rho : \mathbb{R}[\Delta] \to \mathbb{R}[\Gamma]$. Moreover if $\Gamma$ is a full-dimensional subcomplex of $\Delta$ and $\Theta \subseteq \mathbb{R}[X]$ is a fixed l.s.o.p. of $\mathbb{R}[\Delta]$, then it is also an l.s.o.p. of $\mathbb{R}[\Gamma]$. Hence $\rho$ induces a surjective homomorphism $\mathbb{R}(\Delta) \to \mathbb{R}(\Gamma)$. On the level of stress spaces, the situation is much easier to describe: $S(\Gamma)_i$ is a subspace of $S(\Delta)_i$.

A simplicial complex $\Delta$ is \textit{centrally symmetric} or \textit{cs} if its ground set is endowed with a \textit{free involution} $\alpha : V \to V$ that induces a free involution on the set of all non-empty faces of $\Delta$.

\footnote{For any field $k$, one may analogously define the rings $k[\Delta]$ and $k(\Delta, \Theta)$ as well as the notion of $\Delta$ being CM over $k$. However, it follows from Reisner’s criterion along with the universal coefficient theorem that if $\Delta$ is CM over some field $k$, then $\Delta$ is CM over $\mathbb{R}$, i.e., $\Delta$ satisfies the definition given above. In other words, no generality is lost by working over $\mathbb{R}$.}
In more detail, for all non-empty faces $\tau \in \Delta$, the following holds: $\alpha(\tau) \in \Delta$, $\alpha(\tau) \neq \tau$, and $\alpha(\alpha(\tau)) = \tau$. To simplify notation, we write $\alpha(\tau) = -\tau$ and refer to $\tau$ and $-\tau$ as antipodal faces of $\Delta$. One example of a cs simplicial complex is $\partial C^*_d$ — the boundary complex of a $d$-cross-polytope $C^*_d := \text{conv}(\pm p_1, \pm p_2, \ldots, \pm p_d)$, where $p_1, \ldots, p_d$ are affinely independent points in $\mathbb{R}^d \setminus \{0\}$. Here we consider $\partial C^*_d$ as an abstract simplicial complex; as such $\partial C^*_d$ is the $d$-fold suspension of $\{0\}$. It is easy to check that $\partial C^*_d$ is CM and that $h_j(\partial C^*_d) = \binom{d}{j}$ for all $0 \leq j \leq d$.

The free involution $\alpha$ on $\Delta$ induces the free involution on $X$ via $\alpha(x_v) = x_{-v}$, which in turn induces a $\mathbb{Z}/2\mathbb{Z}$-action on $\mathbb{R}[X]$ and $\mathbb{R}[\Delta]$. For any $\mathbb{R}$-vector space $W$ endowed with such an action $\alpha$, one has $W = W^+ \oplus W^-$, where $W^+ := \{w \in W : w = \alpha(w)\}$ and $W^- := \{w \in W : w = -\alpha(w)\}$.

Thus, $\mathbb{R}[\Delta]_i = \mathbb{R}[\Delta]_{i}^+ \oplus \mathbb{R}[\Delta]_i^-$. As $\mathbb{R}[\Delta]_{i}^+ \cdot \mathbb{R}[\Delta]_{j}^- \subseteq \mathbb{R}[\Delta]_{i+j}^-$, and similar inclusions hold for all choices of plus and minus signs, it follows that $\mathbb{R}[\Delta]$ has a $(\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z})$-grading.

Stanley \cite{Stanley Lectures} showed that if $\Delta$ is cs, then there exists an l.s.o.p. $\Theta = \theta_1, \ldots, \theta_d$ of $\mathbb{R}[\Delta]$ with the property that each $\theta_k$ lies in $\mathbb{R}[X]_1^-$. Throughout the paper, we will use this special l.s.o.p. For such a choice of l.s.o.p., $\mathbb{R}(\Delta, \Theta)$ also inherits a $(\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z})$-grading. So if $\Delta$ is cs and CM, we can consider $\mathbb{R}(\Delta, \Theta)^+_i$ and $\mathbb{R}(\Delta, \Theta)^-_i$. Utilizing the $(\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z})$-grading of $\mathbb{R}[\Delta]$ and $\mathbb{R}(\Delta, \Theta)$, Stanley proved the following Lower Bound Theorem for cs CM complexes \cite[Theorem 3.2]{Stanley Lectures}.

**Theorem 2.1.** Let $\Delta$ be a $(d-1)$-dimensional cs CM simplicial complex, and let $\Theta$ be Stanley’s special l.s.o.p. Then $\dim_{\mathbb{R}} \mathbb{R}(\Delta, \Theta)^+_i = \frac{1}{2}(h_i(\Delta) - \binom{d}{i})$ for all $0 \leq i \leq d$. In particular, $h_i(\Delta) \geq \binom{d}{i}$ for all $0 \leq i \leq d$.

Let $\Delta$ be a cs simplicial complex with an involution $\alpha$, and let $\Theta$ be Stanley’s special l.s.o.p. of $\mathbb{R}[\Delta]$. Since $\alpha(I_\Delta + (\Theta)) = I_\Delta + (\Theta)$ and since for any $w, w' \in \mathbb{R}[X]_i$, $\langle \alpha(w), \alpha(w') \rangle = \langle w, w' \rangle$, where $\langle -, - \rangle$ is the inner product from \cite[Section 3]{Stanley Lectures} used to define the isomorphism $\Phi_i$ between $\mathbb{R}(\Delta)_i$ and $\mathbb{R}(\Delta)^+_i$, it follows that $\alpha$ also acts on $S(\Delta)_i$ and that this action commutes with $\Phi_i$. Hence, $S(\Delta)_i = S(\Delta)^+_i \oplus S(\Delta)^-_i$, where the subspaces $S(\Delta)^+_i$ and $S(\Delta)^-_i$ of $S(\Delta)_i$ are isomorphic (as vector spaces) to $\mathbb{R}(\Delta)^+_i$ and $\mathbb{R}(\Delta)^-_i$, resp. We refer to the elements of $S(\Delta)^+_i$ as symmetric $i$-stresses. This discussion along with Theorem 2.1 leads to the following:

**Corollary 2.2.** Let $\Delta$ be a $(d-1)$-dimensional cs CM simplicial complex, let $\Theta$ be a special l.s.o.p., and let $1 \leq \ell \leq d$ be an integer. Then $h_\ell(\Delta) = \binom{d}{\ell}$ if and only if all $\ell$-stresses on $\Delta$ are symmetric, i.e., $S(\Delta)_\ell = S(\Delta)^+_\ell$.

**Proof:** Theorem 2.1 implies that $S(\Delta)^-_\ell = (0)$ if and only if $h_\ell(\Delta) = \binom{d}{\ell}$. \hfill $\Box$

Let $\Delta$ be a $(d-1)$-dimensional cs complex, and $\Theta$ a special l.s.o.p. of $\mathbb{R}[\Delta]$. Since $\theta_1, \ldots, \theta_d \in \mathbb{R}[X]_1^-$, it follows that for every vertex $v$ of $\Delta$ and $1 \leq k \leq d$, $\theta_{\theta_k}(x_v + x_{-v}) = 0$. We conclude that for every vertex $v$ of $\Delta$, $x_v + x_{-v}$ is a symmetric 1-stress. Furthermore, if $\Delta$ is the boundary complex of the $d$-cross-polytope on the vertex set $V$, then every $i$-subset $W$ of $V$ that contains at most one vertex from each pair of antipodal vertices in $V$ forms a face. That $\theta_k \in \mathbb{R}[X]_1^-$ for all $k$ then implies that $\prod_{v \in W}(x_v + x_{-v})$ is an $i$-stress on $\Delta$. Conversely, if $\prod_{v \in W}(x_v + x_{-v})$ is a stress on a cs CM complex $\Delta$, then, by definition of stresses, $\Delta$ contains as a subcomplex the boundary complex of the cross-polytope on the vertex set $W \cup (-W)$. These stresses play a prominent role in our proofs.

3 Proof of Stanley’s conjecture

The goal of this section is to prove Conjecture \cite{Conjecture} We start with the following simple lemma.
Lemma 3.1. Let $\Delta$ be a cs complex, $\Theta$ a special l.s.o.p., and $\tau$ a face of $\Delta$. If $w$ is a symmetric stress on $\Delta$ that lives on $\text{st}(\tau)$, then, in fact, $w$ lives on $\text{lk}(\tau) \cap \text{lk}(-\tau)$.

Proof: By definition of cs complexes, no vertex of $-\tau$ belongs to $\text{st}(\tau)$. Thus the assumption that $w$ is symmetric and lives on $\text{st}(\tau)$ implies that $w$ lives on $\text{lk}(\tau)$. Now, since $w$ is symmetric, a face $F$ participates in $w$ if and only if $-F$ does. This together with the symmetry of $\Delta$ yields that $w$ lives on $\text{lk}(\tau) \cap \text{lk}(-\tau)$.

Throughout the rest of this section, we fix integers $d \geq 2$ and $0 < i < d$. The next lemma and corollaries culminate in a characterization of $\ell$-stresses on $\Delta$ for all $\ell > i$. As the first step, we show that any $i$-stress on $\Delta$ that lives on the star of a vertex is the sum of $i$-stresses whose supports are the boundary complexes of $i$-cross-polytopes, see Corollary 3.3.

Lemma 3.2. Let $\Delta$ be a cs CM complex of dimension $d-1$ with a special l.s.o.p. $\Theta$ and with $h_i(\Delta) = \binom{d}{i}$. Let $u_0$ be a vertex of $\Delta$. If $w$ is a non-zero $i$-stress that lives on $\text{st}(u_0)$, then for any vertex $u_1$ that participates in $w$, $w$ can be expressed as the sum $w = (x_{u_1} + x_{-u_1}) \cdot \partial_{x_{u_1}} w + \bar{w}$. In this sum, both $(x_{u_1} + x_{-u_1}) \cdot \partial_{x_{u_1}} w$ and $\bar{w}$ are $i$-stresses, $\bar{w}$ has the property that neither $u_1$ nor $-u_1$ participate in it, and $\partial_{x_{u_1}} w$ is a symmetric $(i-1)$-stress that lives on $\text{lk}(u_1) \cap \text{lk}(-u_1) \cap \text{lk}(u_0) \cap \text{lk}(-u_0)$.

Proof: We start by considering $w' = (x_{u_0} + x_{-u_0}) \cdot \partial_{x_{u_1}} w$. We claim that $w'$ is an $i$-stress on $\Delta$. Indeed, according to Corollary 2.2 and Lemma 3.1, $w$ lives on $\text{lk}(u_0) \cap \text{lk}(-u_0)$, and so all terms of $w'$ are supported on faces of $\Delta$. Furthermore, since $\theta_1, \ldots, \theta_d \in \mathbb{R}[X]_{\leq 1}$, it follows that

$$\partial_{\theta_k}(x_{u_0} + x_{-u_0}) = 0 \quad \text{for all } k = 1, 2, \ldots, d,$$

where the last equality follows from the fact that $w$ is an $i$-stress. Thus, $w'$ is indeed an $i$-stress on $\Delta$; as such, it is symmetric (see Corollary 2.2). Consequently, $\partial_{x_{u_1}} w$ is a symmetric $(i-1)$-stress on $\Delta$ that lives on $\text{st}(u_1) \cap \text{lk}(u_0) \cap \text{lk}(-u_0)$. Hence by Lemma 3.1 it lives on $\text{lk}(u_1) \cap \text{lk}(-u_1) \cap \text{lk}(u_0) \cap \text{lk}(-u_0)$. Furthermore, the same argument we used for $w'$, shows that $(x_{u_1} + x_{-u_1}) \cdot \partial_{x_{u_1}} w$ is an $i$-stress, and hence so is the difference $\bar{w} := w - (x_{u_1} + x_{-u_1}) \cdot \partial_{x_{u_1}} w$. Finally, since $x_{u_1} \cdot \partial_{x_{u_1}} w$ contains all the terms of $w$ that involve $x_{u_1}$, and, by symmetry, $x_{-u_1} \cdot \partial_{x_{u_1}} w$ contains all the terms of $w$ that involve $x_{-u_1}$, it follows that neither $u_1$ nor $-u_1$ participate in $\bar{w}$.

Let $w$ be a stress on $\Delta$. We say that $\bar{w}$ is a stress of $w$ if $\bar{w}$ is a stress that is obtained from $w$ by deleting some of the terms of $w$.

Corollary 3.3. Let $\Delta$ be a cs CM complex of dimension $d-1$ with $h_i(\Delta) = \binom{d}{i}$, and let $u_0$ be a vertex of $\Delta$. Then any non-zero $i$-stress $w$ on $\Delta$ that lives on $\text{st}(u_0)$ can be written as a weighted sum of $i$-stresses of the form $\prod_{k=1}^j (x_{u_k} + x_{-u_k})$, where all $\pm v_1, \ldots, \pm v_i \in V \setminus \{\pm u_0\}$ are distinct.

Proof: We claim that for all $1 \leq j \leq i$, there exist distinct vertices $u_1, \ldots, u_j$ such that $w_j := \partial_{x_{u_1}} \partial_{x_{u_2}} \ldots \partial_{x_{u_j}} w$ is a non-zero symmetric $(i-j)$-stress that lives on $\bigcap_{k=0}^j (\text{lk}(u_k) \cap \text{lk}(-u_k))$, $u_j$ participates in $w_{j-1}$ (with $w_0$ defined as $w$), and $\prod_{k=1}^j (x_{u_k} + x_{-u_k}) w_j$ is a substress of $w$. For $j = 1$, this is the content of Lemma 3.2. More generally, the argument of Lemma 3.2 provides the desired inductive step: once the existence of $u_1, \ldots, u_{j-1}$ and $w_{j-1}$ is established, consider a vertex $u_j$ that participates in $w_{j-1}$, and let $w(j) := \prod_{k=0}^{j-1} (x_{u_k} + x_{-u_k}) \cdot \partial_{x_{u_j}} w_{j-1}$. The same
argument as in Lemma 3.2 then shows that \( w(j) \) is an \( i \)-stress, hence symmetric. This, in turn, implies that \( w_j = \partial_{x_{u_j}} w_{j-1} \) is a symmetric \((i-j)\)-stress that lives on \( \bigcap_{k=0}^{d} \{ \text{lk}(u_k) \cap \text{lk}(-u_k) \} \) and that \( \prod_{k=1}^{l} (x_{u_k} + x_{-u_k}) \) is a substress of \( w \). In particular, for \( j = i \), we obtain that \( \prod_{k=1}^{l} (x_{u_k} + x_{-u_k}) \) is a substress of \( w \). Here \( w_i \) is a non-zero 0-stress; hence \( w_i \) is a non-zero constant. The corollary then follows by induction on the number of terms in \( w \).

With Corollary 3.3 in hand, we are ready to characterize all \( \ell \)-stresses on \( \Delta \) for \( \ell > i \).

**Corollary 3.4.** Let \( \Delta \) be a cs CM complex of dimension \( d \) with \( h_i(\Delta) = \binom{d}{i} \). Then for any \( 1 \leq j \leq d - i \) and any non-zero \((i+j)\)-stress \( w \), \( w \) can be written as a weighted sum of \((i+j)\)-stresses of the form \( \prod_{k=1}^{i+j} (x_{v_k} + x_{-v_k}) \), where all \( \pm v_1, \ldots, \pm v_{i+j} \in V \) are distinct. In particular, \( w \) is symmetric.

**Proof:** The proof is by induction on \( j \). Let \( u_0 \) be any vertex participating in the non-zero \((i+j)\)-stress \( w \) and let \( w_0 := \partial_{x_{u_0}} w \). Since \( w \) is an \((i+j)\)-stress, it follows that \( w_0 \) is an \((i+j-1)\)-stress that lives on \( \text{st}(u_0) \). Thus, if \( j = 1 \), then by Corollary 3.3, \( w_0 \) is a weighted sum of \((i+j-1)\)-stresses of the form \( \prod_{k=1}^{i+j-1} (x_{v_k} + x_{-v_k}) \) where all \( \pm v_1, \ldots, \pm v_{i+j-1} \) are distinct and different from \( \pm u_0 \), while if \( j > 1 \), then the same statement holds by the inductive hypothesis and Lemma 3.1.

Let \( u_1 \) be one of the vertices participating in \( w_0 \), and let \( w_1 \) be a substress of \( w_0 \) consisting of all the terms that involve \( x_{u_1} \) or \( x_{-u_1} \), that is, \( w_1 \) is of the form

\[
w_1 = (x_{u_1} + x_{-u_1}) \sum_{v_2 \ldots v_{i+j-1}} c_{v_2 \ldots v_{i+j-1}} \prod_{k=2}^{i+j-1} (x_{v_k} + x_{-v_k}).
\]

Then \( w \) can be written as

\[
w = x_{u_0} w_1 + x_{u_0} w_2 + w_3,
\]

where the polynomial \( w_2 \) involves neither \( x_{u_1} \) nor \( x_{-u_1} \) and the polynomial \( w_3 \) does not involve \( x_{u_0} \). Applying \( \partial_{x_{u_1}} \) to \( w \), we conclude that

\[
\partial_{x_{u_1}} w = x_{u_0} \left( \sum_{v_2 \ldots v_{i+j-1}} c_{v_2 \ldots v_{i+j-1}} \prod_{k=2}^{i+j-1} (x_{v_k} + x_{-v_k}) \right) + \partial_{x_{u_1}} w_3.
\]

As \( \partial_{x_{u_1}} w \) is a non-zero \((i+j-1)\)-stress, it must be symmetric. (Indeed, for \( j > 1 \) this holds by the inductive hypothesis, while for \( j = 1 \) this holds because \( h_i(\Delta) = \binom{d}{i} \).) It then follows that

\[
(x_{u_0} + x_{-u_0}) \sum_{v_2 \ldots v_{i+j-1}} c_{v_2 \ldots v_{i+j-1}} \prod_{k=2}^{i+j-1} (x_{v_k} + x_{-v_k})
\]

is a substress of \( \partial_{x_{u_1}} w \), and an analogous computation shows that it is also a substress of \( \partial_{x_{-u_1}} w \). Thus

\[
(x_{u_0} + x_{-u_0})(x_{u_1} + x_{-u_1}) \sum_{v_2 \ldots v_{i+j-1}} c_{v_2 \ldots v_{i+j-1}} \prod_{k=2}^{i+j-1} (x_{v_k} + x_{-v_k})
\]

is a substress of \( w \). It is symmetric; it is also a weighted sum of stresses of the form \( \prod_{k=1}^{i+j} (x_{a_k} + x_{-a_k}) \). The induction on the number of terms of \( w \) then finishes the proof of the inductive step. \( \square \)

We are now in a position to finish the proof of Conjecture 1.2.
Theorem 3.5. Fix integers \( d \geq 2 \) and \( 0 < i < d \). Let \( \Delta \) be a cs CM complex of dimension \( d - 1 \) with \( h_i(\Delta) = \binom{d}{i} \). Then \( h_j(\Delta) = \binom{d}{j} \) for all \( j \geq i \).

**Proof:** By Corollary 3.4, for all \( j > i \), all \( j \)-stresses on \( \Delta \) are symmetric. Corollary 2.2 then finishes the proof. \( \square \)

We close with the following consequence of Corollary 3.4 that might be of independent interest:

**Corollary 3.6.** If \( \Delta \) is a \( (d - 1) \)-dimensional cs CM simplicial complex with \( h_i(\Delta) = \binom{d}{i} \), then \( \Delta \) contains a subcomplex \( \Gamma \) isomorphic to \( \partial C^*_d \); furthermore \( S(\Delta)_j = S(\Gamma)_j \) for all \( j \geq i \).

**Proof:** Since by Theorem 2.1, \( h_d(\Delta) > 0 \), it follows that there is a non-zero \( d \)-stress \( w \) on \( \Delta \). But by Corollary 3.3, the support of such \( w \) is the union of the boundary complexes of \( d \)-cross-polytopes. Hence \( \Delta \) must contain \( \Gamma \cong \partial C^*_d \) as a subcomplex. Then \( S(\Delta)_j \supseteq S(\Gamma)_j \) for all \( j \), and comparing the dimensions we see that, in fact, \( S(\Delta)_j = S(\Gamma)_j \) for all \( j \geq i \). \( \square \)

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