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Volume 359, issue 10 (2021), p. 1217-1224

<https://doi.org/10.5802/crmath.256>

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On the maximum value of a confluent hypergeometric function

Bujar Xh. Fejzullahu

Abstract. We study the maximum value of the confluent hypergeometric function with oscillatory conditions of parameters. As a consequence, we obtain new inequalities for the Gauss hypergeometric function.

2000 Mathematics Subject Classification. 33C15, 33C20.

1. Introduction and main results

The confluent hypergeometric function $1\,F\,1(a; b; x)$, which is defined as

$$1\,F\,1(a; b; x) = \sum_{n=0}^{\infty} \frac{(a)_n}{(b)_n} \frac{x^n}{n!}$$

for $b \neq 0, -1, \ldots$, is a particular solution of the linear differential equation

$$xy'' + (b - x)y' - ay = 0. \quad (1)$$

When $a$ is a non-positive integer, the function $1\,F\,1(a; b; x)$ reduces to Laguerre polynomials, i.e.

$$L_{n}^{(b-1)}(x) = \frac{(b)_n}{n!} 1\,F\,1(-n; b; x), \quad n = 0, 1, 2, \ldots$$

The function $1\,F\,1$ is a special case of the generalized hypergeometric function

$$p\,F\,q(a_1, \ldots, a_p; b_1, \ldots, b_q; x) = \sum_{n=0}^{\infty} \frac{\prod_{i=1}^{p}(a_i)_n}{\prod_{j=1}^{q}(b_j)_n} \frac{x^n}{n!}, \quad (2)$$

$p$ and $q$ are non-negative integers, none of the numbers $b_j$ $(j = 1, \ldots, q)$ is equal to zero or to a negative integer. It is well known that the series $p\,F\,q(a_1, \ldots, a_p; b_1, \ldots, b_q; x)$ converges absolutely for all $x$ if $p \leq q$ and for $|x| < 1$ if $p = q + 1$, and it diverges for all $x \neq 0$ if $p > q + 1$. If one of the parameters $a_i$ equals zero or a negative integer, then the series (2) reduces to a polynomial.

The confluent hypergeometric function has been studied in great detail from its mathematical point of view (see, for instance, [12, 14, 18]). In particular, the estimate of the confluent hypergeometric function $1\,F\,1(a; b; x)$ has been widely and deeply studied when $x > 0$ and $b > a > 0$
Theorem 1. For $a < 0$ and $b > 1$

$$\max_{x \geq 0} e^{-x} |1_F_1(a; b; x)| = 1.$$  

Moreover, this maximum value is attained only when $x = 0$.

Corollary 2. When $a < 0$ and $b > 1$, let $\xi_k$, $k = 1, \ldots$, be the successive maxima of $y(x) = e^{-x} 1_F_1(a; b; x)$ arranged in increasing order, and let $j_{b,k}$ be the $k$-th positive zero of the Bessel function $J_b(x)$. Then,

$$y^2(\xi_i) - y^2(\xi_j) < \frac{b-a}{b^2} \left( \frac{2b-1}{2} \Delta \xi_{ij}^2 + 2 \frac{3}{4} \Delta \xi_{ij}^3 \right), \quad i < j,$$

where $\Delta \xi_{ij}^k = \xi_{i}^k - \xi_{j}^k$ and

$$\xi_k = \frac{j_{b,k}^2}{2b - 4a + 2} \left( 1 + \frac{2(b^2 - 1) + j_{b,k}^2}{3(2b - 4a + 2)^2} \right) + O\left( \frac{1}{a^5} \right), \quad \text{as} \quad a \to -\infty.$$
On the other hand, in the oscillation region of \( _1F_1(a; b; x) \), we have the following result analogous to (5).

\textbf{Theorem 3.} Let \( a < 0 \) and \( b > 1 \). For \( 0 \leq x < \frac{(2b-1)(b-2a)}{b} \)

\[ e^{-\frac{x}{b}} |_1F_1(a; b; x)| \leq 1. \quad (6) \]

Furthermore, for \( 0 \leq x < \frac{(2b-1)(b-2a)}{2b} \)

\[ e^{-\frac{x}{2b}} |_1F_1(a; b; x)| \leq \sqrt{M(x, b)}, \quad (7) \]

where

\[ M(x, b) = \frac{b-1}{\pi} \int_0^\pi \int_0^\sigma |\cos(x \sin \theta \cos \psi)| (\sin \psi)^{2b-3} (\sin \theta)^{2b-2} \, d\psi \, d\theta \]

and it has the property

\[ 0 < M(x, b) < M(0, x) = 1, \quad x > 0. \]

Consequently, by applying (4) and (5), we obtain the following inequalities for the Gauss hypergeometric function \( _2F_1 \).

\textbf{Corollary 4.} Let \( a < 0 \), \( b > 1 \) and \( \text{Re}(\sigma) > 0 \).

(i) If \( a \) is not a negative integer and \( 0 \leq x < \text{Re}(z) \)

\[ |_2F_1\left(\sigma, a; b; \frac{x}{z}\right)| \leq \sqrt{\cosh(\pi \Im m(\sigma))} |z^\sigma| |\text{Re}(z) - x|^{-\text{Re}(\sigma)}. \quad (8) \]

In particular, for \( \sigma > 0 \) and \( 0 \leq x < 1 \)

\[ |_2F_1\left(\sigma, a; b; x\right)| \leq (1 - x)^{-\sigma}. \]

(ii) If \( a \) is a negative integer and \( 0 \leq x < 2\text{Re}(z) \)

\[ |_2F_1\left(\sigma, a; b; \frac{x}{z}\right)| \leq \sqrt{\cosh(\pi \Im m(\sigma))} |z^\sigma| \left| \text{Re}(z) - \frac{x}{2} \right|^{-\text{Re}(\sigma)}. \]

In particular, for \( \sigma > 0 \) and \( 0 \leq x < 2 \)

\[ |_2F_1\left(-n, \sigma; a; b; x\right)| \leq \left(1 - \frac{x}{2}\right)^{-\sigma}, \quad n \in \mathbb{N} \cup \{0\}. \]

Under condition \( a > 0 \), several lower and upper bound inequalities for \( _2F_1(\sigma, a; b; x) \) have been derived in the literature using different approaches (e.g. \cite{2–4,6,13,22} and references therein). For instance, in [13, Theorem 13], Luke gave the following two-sided bounds

\[ \left(1 - \frac{a}{b}\right)^{-\sigma} < _2F_1(\sigma, a; b; x) < 1 - \frac{a}{b} + \frac{a}{b} (1 - x)^{-\sigma}, \quad 0 < x < 1, 0 < \sigma, 0 < a < b, \]

whereas Karp and Sitnik \cite[Theorem 5]{6} showed that

\[ _2F_1(\sigma, a; b; x) < \left(1 - \frac{a}{b - 1}\right)^{-\sigma}, \quad 0 < x < 1, 0 < \sigma \leq 1, 1 < a + 1 < b. \]

On the other hand, in \cite{22} the authors derived some inequalities for the Gauss hypergeometric function \( _2F_1(\sigma, a; b; x) \) when \(-1 < a < 0, 1 < b < 2, 0 < \sigma < 1, \) and \( x \in (0, 1) \). We remark that when \( a \) is a negative integer or zero, the estimate of the polynomial \( _2F_1(a, \sigma; b; x) \) has been considered in several papers from different point of views (see for instance \cite{7,8} and references therein)

\section{2. Proof of the main results}

One of the main tool that we need for our purpose is the well-known Sonin–Pólya theorem (see \cite[footnote to Theorem 7.31.1]{21}) in the following form given by Szegö. Notice that this theorem was used by Szegö \cite{21} in a similar context to study the successive relative maxima of classical orthogonal polynomials.
The Sonin–Pólya theorem

Suppose that a function \( y = y(x) \) satisfies on an interval \( I \subset \mathbb{R} \) the differential equation

\[
(py')' + qy = 0,
\]

where \( p = p(x) > 0 \), \( q = q(x) > 0 \) and both \( p' \) and \( q' \) are continuous on that interval. Define Sonin’s function by

\[
S(x) := y^2(x) + \frac{p(x)}{q(x)} y^2(x),
\]

then we observe that

\[
S'(x) = -|p(x)q(x)|' \left| \frac{y'(x)}{q(x)} \right|^2,
\]

by which successive relative maxima of \( y^2 \) form an increasing or decreasing sequence according as \( pq \) decreases or increases on the corresponding interval.

Now, we can prove our main results.

Proof of Theorem 1. From (1), the corresponding differential equation for \( y(x) = e^{-x}F_1(a;b;x) \) is

\[
xy'' + (b + x)y' + (b - a)y = 0.
\]

By writing it in the self-adjoint form

\[
(x^b e^x y')' + (b - a)x^{b-1} e^x y = 0,
\]

we see that

\[
p(x) = x^b e^x, \quad q(x) = (b - a)x^{b-1} e^x
\]

and

\[
[p(x)q(x)]' = (b-a)(2b-1+2x)x^{2b-1} e^{2x}.
\]

Thus, if \( a < 0 \) and \( b > 1 \), the successive relative maxima of \( |e^{-x}F_1(a;b;x)| \) are decreasing on \([0,\infty)\) and

\[
|e^{-x}F_1(a;b;x)|^2 \leq S(x) \leq S(0) = y^2(0) = 1, \quad x \geq 0.
\]

This proves (4).

\[
\square
\]

Proof of Corollary 2. We observe that, using the differential equation

\[
\frac{d}{dx}[e^{-x}F_1(a;b;x)] = -\frac{b-a}{b} e^{-x}F_1(a;b+1;x),
\]

\[\xi_k = x_{a,b+1}, \text{ for all } k = 1, \ldots.\]

Thus, from (10) and (11) one has

\[
y^2(\xi_j) - y^2(\xi_i) = -\frac{1}{b-a} \int_{\xi_i}^{\xi_j} x(2b-1+2x)[y'(x)]^2 \, dx, \quad i < j.
\]

Now we can apply (4) and (12) to yield

\[
y^2(\xi_i) - y^2(\xi_j) < \frac{b-a}{b^2} \int_{\xi_i}^{\xi_j} x(2b-1+2x) \, dx
\]

\[
= \frac{b-a}{b^2} \left( \frac{2b-1}{2} \Delta \xi^2_{ij} + \frac{2}{3} \Delta \xi^3_{ij} \right),
\]

where \( \Delta \xi_{ij} = \xi_j - \xi_i \).

Finally, taking into account that the \( k \)-th positive zero \( x_{a,b}^k \) can be approximated by (see [18, Section 13.9])

\[
j_{b-1,k}^2 \left( 1 + \frac{2b(b-2) + j_{b-1,k}^2}{3(2b-4a)^2} \right) + O \left( \frac{1}{a^3} \right), \quad \text{as } a \to -\infty,
\]

we can achieve the proof of the corollary.

\[
\square
\]
Proof of Theorem 3. For the proof of (6), we proceed as in the proof of (4). According to (1), it is straightforward to check that the function $y(x) = e^{-\frac{x}{2}} F_1(a; b; x)$ satisfies
\begin{equation}
xy''(x) + by'(x) + \frac{2b - 4a - x}{4} y(x) = 0.
\end{equation}
In its self-adjoint form equation (13) becomes
\begin{equation}
(x^b y'(x))' + \frac{2b - 4a - x}{4} x^{b-1} y(x) = 0,
\end{equation}
which corresponds to equation (9) with
\begin{align*}
p(x) &= x^b, \\
q(x) &= \frac{2b - 4a - x}{4} x^{b-1},
\end{align*}
and
\begin{equation}
[p(x)q(x)]' = \frac{x^{2b-2}}{2} \left[(2b-1)(b-2a) - bx\right].
\end{equation}
Thus, for $a < 0$ and $b > 1$, the successive relative maxima of $|e^{-\frac{x}{2}} F_1(a; b; x)|$ are decreasing if $0 < x < (\frac{2b-1}{b-2a})$ and increasing if $(\frac{2b-1}{b-2a}) < x < 2b - 4a$. This completes the proof of (6).

We now continue with the proof of (7). Our starting point in the proof is the Glaeske [5] product formula for Laguerre functions, which in terms of the confluent hypergeometric functions can be written as
\begin{equation}
F_1(a; b; x) \cdot F_1(a; b; x) = \frac{\Gamma(b)}{\sqrt{\pi}} \int_0^\pi e^{-xy\cos\theta} (\sin\theta)^{2b-2} F_{b-\frac{1}{2}}(\sqrt{xy}\sin\theta) \\
\times F_1(a; b; x + y + 2\sqrt{xy}\cos\theta) \, d\theta, \quad (14)
\end{equation}
where $x, y \geq 0$, $\Re(b) > \frac{1}{2}$, and $F_{\nu}(z) := \left(\frac{z}{2}\right)^{-\nu} F_{\nu}(z)$. For $a = -n$, $n \in \mathbb{N}$, equation (14) was first obtained by Watson [23] and later on by several authors using quite different methods (see [5, 7, 16, 19, 20]), whereas in [15] Markett gave another, analytic proof of Glaeske’s result. In Appendix A, we give another simple proof of (14).

Using Poisson’s integral (see [23, 3.3])
\begin{equation}
\mathcal{F}_\nu(z) = \frac{1}{\sqrt{\pi\Gamma(\nu + \frac{1}{2})}} \int_0^\pi e^{iz\cos\psi} (\sin\psi)^{2\nu} \, d\psi, \quad \Re(\nu) > -\frac{1}{2},
\end{equation}
the product formula (14) becomes
\begin{equation}
F_1(a; b; x) \cdot F_1(a; b; x) = \frac{b-1}{\pi} \int_0^\pi \int_0^\pi e^{-xy\cos\theta + iy\sin\theta\cos\psi} (\sin\psi)^{2b-3} (\sin\theta)^{2b-2} \\
\times F_1(a; b; x + y + 2\sqrt{xy}\cos\theta) \, d\psi \, d\theta
\end{equation}
\begin{align*}
&= \frac{b-1}{\pi} \int_0^\pi \int_0^\pi e^{-xy\cos\theta} \cos(\sqrt{xy}\sin\psi\cos\theta)(\sin\psi)^{2b-3} \\
&\quad \times (\sin\theta)^{2b-2} F_1(a; b; x + y + 2\sqrt{xy}\cos\theta) \, d\psi \, d\theta.
\end{align*}
We put $x = y$ in (15) and multiply the obtained relation by $e^{-x}$. As a result, we obtain
\begin{equation}
\left[ e^{-\frac{x}{2}} F_1(a; b; x) \right]^2 = \frac{b-1}{\pi} \int_0^\pi \int_0^\pi \cos(x\sin\theta\cos\psi)(\sin\psi)^{2b-3} (\sin\theta)^{2b-2} \\
\times e^{-x(1 + \cos\theta)} F_1(a; b, 2x(1 + \cos\theta)) \, d\psi \, d\theta.
\end{equation}
Then, taking into account (6), for $0 \leq x < \frac{(2b-1)(b-2a)}{2b}$
\begin{equation}
\left[ e^{-\frac{x}{2}} F_1(a; b; x) \right]^2 \leq \frac{b-1}{\pi} \int_0^\pi \int_0^\pi \cos(x\sin\theta\cos\psi)(\sin\psi)^{2b-3} (\sin\theta)^{2b-2} \, d\psi \, d\theta = \mathcal{M}(x, b).
\end{equation}
Finally, based on equation (20), we have
\begin{equation}
0 < \mathcal{M}(x, b) < \mathcal{M}(0, b), \quad x > 0.
\end{equation}
The proof of Theorem 3 is completed. \hfill \Box
Proof of Corollary 4. (i). Applying inequality (4) to the confluent hypergeometric function appearing in the Laplace transform of the Gauss hypergeometric function (see [12, p. 59])

\[ {}_2F_1(\sigma, a; b; \frac{x}{z}) = \frac{z^\sigma}{\Gamma(\sigma)} \int_0^\infty e^{-zt} t^{\sigma-1} {}_1F_1(a; b; xt) \, dt, \]

where \( 0 \leq x < \Re(z) \) and \( \Re(\sigma) > 0 \), we have

\[ \left| {}_2F_1(\sigma, a; b; \frac{x}{z}) \right| \leq \frac{|z^\sigma|}{|\Gamma(\sigma)|} \int_0^\infty e^{-t|\Re(z)-x|} t^{\Re(\sigma)-1} \, dt = \frac{\Gamma(\Re(\sigma))}{|\Gamma(\sigma)|} |z^\sigma| |\Re(z)-x|^{-\Re(\sigma)}. \]

Finally, using inequality (see [18, Section 5.6])

\[ |\Gamma(p + i q)| \geq \frac{\Gamma(p)}{\sqrt{\cosh(q\pi)}} \]

we get (8).

(ii). By making use of (5), the proof of case (ii) can be completed by following the proof of case (i). \( \square \)

Appendix A. Proof of the Watson–Glaeske formula

Substituting the integral representation for \( {}_1F_1 \) (see [14, Section 6.5])

\[ {}_1F_1(a; b; z) = \frac{2^{1-b} \Gamma(b) e^{z/2}}{\Gamma(a) \Gamma(b-a)} \int_1^t e^{-zt} (1-t)^{b-2} \left(1 + \frac{t}{1-t}\right)^{a-1} \, dt, \]

where \( \Re(b) > \Re(a) > 0 \), into Bailey’s product formula for confluent hypergeometric functions (see [1])

\[ {}_1F_1(a; b; x) \cdot {}_1F_1(a; b; y) = \sum_{k=0}^{\infty} \frac{(-1)^k (a) (b-a)_k}{k! (b)_k (b+2k)_k} (xy)^k {}_1F_1(a+k; b+2k; x+y) \]

we get, after interchanging the order of summation and integration

\[ {}_1F_1(a; b; x) \cdot {}_1F_1(a; b; y) = 2^{1-b} \Gamma(b) e^{xy} \frac{xy}{\Gamma(a) \Gamma(b-a)} \int_1^t e^{-zt} (1-t)^{b-2} \left(1 + \frac{t}{1-t}\right)^{a-1} \left[ \sum_{k=0}^{\infty} \frac{(-1)^k (\sqrt{xy}(1-t^2))^{2k}}{k! \Gamma(b+k)} \right] \, dt. \]

Using the series expansion

\[ \mathcal{J}_v(z) = \sum_{k=0}^{\infty} \frac{(-1)^k \left( \frac{z}{2}\right)^{2k}}{k! \Gamma(v + k + 1)}, \]

we obtain

\[ {}_1F_1(a; b; x) \cdot {}_1F_1(a; b; y) = 2^{1-b} \Gamma(b) e^{xy} \frac{xy}{\Gamma(a) \Gamma(b-a)} \int_1^t e^{-zt} (1-t)^{b-2} \left(1 + \frac{t}{1-t}\right)^{a-1} \times \mathcal{J}_{b-1}(\sqrt{xy(1-t^2)}) \, dt. \]

On the other hand, from Gegenbauer’s double integral representation for \( \mathcal{J}_v \) (see [23, Section 3.33])

\[ \mathcal{J}_v(\omega) = \frac{1}{\pi \Gamma(v)} \int_0^{\pi} \int_0^{\pi} e^{itz \cos \theta - i|z| \cos \phi} (\cos \phi \sin \theta \cos \psi + \sin \phi \sin \theta \cos \psi) (\sin \psi)^{2v-1} (\sin \theta)^{2v} \, d\psi \, d\theta, \]

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where \( \omega^2 = Z^2 + z^2 - zZ \cos \Phi \) and \( v > 0 \), for \( Z = -it \sqrt{xy} \), \( z = -\sqrt{xy} \), \( \Phi = \frac{\pi}{2} \) and \( v = b - 1 \) we have
\[
\mathcal{J}_{b-1}(\sqrt{xy}(1-t^2)) = \frac{1}{\pi \Gamma(b-1)} \int_0^\pi \int_0^\pi e^{i \sqrt{xy} \cos \theta + i \sqrt{xy} \sin \theta \cos \psi} \times (\sin \psi)^{2b-3}(\sin \theta)^{2b-2} \, d\psi \, d\theta. \tag{18}
\]
Now substituting (18) into (17) and taking into account (16) yields
\[
1 \, F_1(a; b; x) \cdot 1 \, F_1(a; b; y) = \frac{2^{1-b}(b-1) \Gamma(b) e^{\frac{xy}{b}}}{\pi \Gamma(a) \Gamma(b-a)} \int_0^\pi \int_0^\pi e^{i \sqrt{xy} \cos \theta + i \sqrt{xy} \sin \theta \cos \psi} (\sin \psi)^{2b-3}(\sin \theta)^{2b-2}
\]
\[
\times \left[ \int_0^1 e^{\frac{x+yt}{\sqrt{xy}} \cos \theta} (1-t)^{b-2} \left( \frac{1+t}{1-t} \right)^{a-1} \, dt \right] \, d\psi \, d\theta \tag{19}
\]
\[
= \frac{b-1}{\pi} \int_0^\pi \int_0^\pi e^{-\sqrt{xy} \cos \theta + i \sqrt{xy} \sin \theta \cos \psi} (\sin \psi)^{2b-3}(\sin \theta)^{2b-2}
\]
\[
\times 1 \, F_1(a; b, x + y + 2 \sqrt{xy} \cos \theta) \, d\psi \, d\theta.
\]
By using analytic continuation, equation (19) can be extended to \( a \in \mathbb{C} \) and \( \Re(b) > 1 \). This proves equation (15) and completes the proof of Glaeske’s result.

In particular, putting \( x = y = 0 \) in (19) yields
\[
1 = \frac{b-1}{\pi} \int_0^\pi \int_0^\pi (\sin \psi)^{2b-3}(\sin \theta)^{2b-2} \, d\psi \, d\theta. \tag{20}
\]

Acknowledgements

The author thanks the referee for a careful reading of the manuscript and for his suggestions.

References

[1] W. N. Bailey, “On the product of two Legendre polynomials with different arguments”, Proc. Lond. Math. Soc. 41 (1936), p. 215-220.
[2] R. W. Barnard, K. C. Richards, H. C. Tiedeman, “A survey of some bounds for Gaus’ hypergeometric function and related bivariate means”, J. Math. Inequal. 4 (2010), no. 1, p. 45-52.
[3] B. C. Carlson, “Some inequalities for hypergeometric functions”, Proc. Am. Math. Soc. 17 (1966), p. 32-39.
[4] T. Ebner, “Inequalities for hypergeometric functions”, Arch. Ration. Mech. Anal. 4 (1960), p. 341-351.
[5] H-J. Glaeske, “Die Laguerre-Pinney-Transformation”, Aequationes Math. 22 (1981), p. 73-85.
[6] D. Karp, S. M. Sitnik, “Inequalities and monotonicity of ratios for generalized hypergeometric function”, J. Approx. Theory 161 (2009), no. 1, p. 337-352.
[7] T. Koornwinder, A. Kostenko, G. Teschl, “Jacobi polynomials, Bernstein-type inequalities and dispersion estimates for the discrete Laguerre operator”, Adv. Math. 333 (2018), p. 796-821.
[8] I. Krasikov, “On an upper bound on Jacobi polynomials”, J. Approx. Theory 149 (2007), no. 2, p. 116-130.
[9] I. Krasikov, A. Zarkh, “Equioscillatory property of the Laguerre polynomials”, J. Approx. Theory 162 (2010), no. 11, p. 2021-2047.
[10] E. Levin, D. Lubinsky, “Orthogonal polynomials for exponential weights \( x^{2p}e^{-2Q(x)} \) on \([0, d]\)”, J. Approx. Theory 139 (2006), no. 1-2, p. 107-143.
[11] E. R. Love, “Inequalities for Laguerre functions”, J. Inequal. Appl. 1 (1997), no. 3, p. 293-299.
[12] Y. L. Luke, The special functions and their approximations, Vol. I, Mathematics in Science and Engineering, vol. 53, Academic Press Inc., 1969.
[13] ———, “Inequalities for generalized hypergeometric functions”, J. Approx. Theory 5 (1972), p. 41-65.
[14] W. Magnus, F. Oberhettinger, R. P. Soni, Formulas and theorems for the special functions of Mathematical Physics, Grundhren der Mathematischen Wissenschaften, vol. 52, Springer, 1966.
[15] C. Markett, “Product Formulas for Bessel, Whittaker, and Jacobi Functions via the Solution of an Associated Cauchy Problem”, in Anniversary Volume on Approximation Theory and Functional Analysis, ISNM. International Series of Numerical Mathematics, vol. 65, Birkhäuser, 1984, p. 449-462.
[16] ———, “A new proof of Watson’s product formula for Laguerre polynomials via a Cauchy problem associated with a singular differential operator”, SIAM J. Math. Anal. 17 (1986), p. 1010-1032.
[17] H. N. Mhaskar, E. B. Saff, “Where does the sup norm of a weighted polynomial live? (A generalization of incomplete polynomials)”, *Constr. Approx.* 1 (1985), p. 71-91.

[18] F. W. J. Olver, A. B. Olde Daalhuis, D. W. Lozier, B. I. Schneider, R. F. Boisvert, C. W. Clark, B. R. Miller, B. V. Saunders, H. S. Cohl, M. A. McClain (eds.), *NIST Digital Library of Mathematical Functions*, http://dlmf.nist.gov/.

[19] J. Peetre, “The Weyl transform and Laguerre polynomials”, *Matematiche* 27 (1973), p. 301-323.

[20] K. Stempak, “A new proof of a Watson’s formula”, *Can. Math. Bull.* 31 (1988), no. 4, p. 414-418.

[21] G. Szegö, *Orthogonal polynomials*, Colloquium Publications, vol. 23, American Mathematical Society, 1975.

[22] F. Wang, F. Qi, “Monotonicity and sharp inequalities related to complete $(p, q)$-elliptic integrals of the first kind”, *C. R. Math. Acad. Sci. Paris* 358 (2020), no. 8, p. 961-970.

[23] G. N. Watson, “Another note on Laguerre polynomials”, *J. Lond. Math. Soc.* 14 (1939), p. 19-22.