Quasi-periodic tiling with multiplicity: a lattice enumeration approach

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Abstract

The $k$-tiling problem is the problem of covering $\mathbb{R}^d$ with translates of a convex polytope $P$ using a discrete multiset $\Lambda$ of translation vectors. Thus, every point in $\mathbb{R}^d$ is covered exactly $k$ times, except possibly for the boundary of $P$ and its translates.

In this paper, we study the $k$-tiling problem when the tiling set $\Lambda$ is a finite union of translated lattices. This is motivated by the work of Kolountzakis [8] and Gravin, Kolountzakis, Robins, and Shiryaev [5], in which it was shown that with the exception of parallelograms in dimension 2 and two-flat zonotopes in dimension 3, the tiling set $\Lambda$ of a $k$-tiler $P$ in dimension 2 or 3 must be a finite union of translated lattices. Our approach in this paper is to study a certain lattice-point enumeration problem that is equivalent to the $k$-tiling problem, and we manage to find several situations in which a $k$-tiler $P$ can $m$-tile with a lattice, providing analogous results to McMullen’s [10] that a 1-tiler can 1-tile with a lattice.

1 Introduction

Multiple tiling problem can be described as follows, we want to cover every point in $\mathbb{R}^d$ exactly $k$ times by translating a convex polytope using a discrete multiset $\Lambda$ (also known as a tiling set) of translation vectors. However, in the process of trying to cover every point in $\mathbb{R}^d$, we may be forced to cover points in the boundary of $P$ of its translates for more than $k$ times. In order to avoid this technicality, we say that $P$ $k$-tiles $\mathbb{R}^d$ with $\Lambda$ if every point that does not belong to the boundary of any translate of $P$ is covered exactly $k$ times by translating $P$ using $\Lambda$. In the special case when $k$ is equal to 1, the multiple tiling problem becomes what is traditionally known as the transational tiling problem. For more details on the transational tiling problem, one can study the work of Alexandrov[1] and Gruber[7] to get a nice overview of the topic.

Several important structural results have been found for a polytope that 1-tiles in $\mathbb{R}^d$. In 1897, Minkowski [11] gave a necessary condition for a polytope to be a 1-tiler. He proved

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that if a convex polytope $P$ 1-tiles in $\mathbb{R}^d$, then $P$ must be centrally symmetric and all facets of $P$ are centrally symmetric. It was not until 50 years later that the converse was found by Venkov [14]. He proved that if a convex polytope $P$ is centrally symmetric, all facets of $P$ are centrally symmetric, and each belt of $P$ contains either 4 or 6 codimension 2 faces, then $P$ can 1-tile in $\mathbb{R}^d$. In 1980, McMullen [10] proved the same result independently from Venkov, and in addition to that he proved that if $P$ 1-tiles in $\mathbb{R}^d$, then $P$ can 1-tile $\mathbb{R}^d$ with a lattice as the tiling set.

In contrast to 1-tilers, there are still a lot of unsolved problems related to the structure of $k$-tilers in general. This is partly because $k$-tilers have a much richer structure compared to 1-tilers. For example, in two dimensions, there are only two types of convex polytopes that can 1-tile in $\mathbb{R}^2$, centrally symmetric parallelograms and centrally symmetric hexagons. In contrast, all centrally symmetric integer polygons can $k$-tile in $\mathbb{R}^2$ [6]. With that being said, there are several important results in multiple tiling that mirror the results in 1-tiling. In 1994, Bolle [4] provided a characterization of all polytopes that $k$-tile with a lattice in $\mathbb{R}^2$. In 2012, Gravin, Robins, and Shiryaev [6] proved that a $k$-tiler in $\mathbb{R}^3$ must be centrally symmetric and all its facets are centrally symmetric, providing an analogue to Minkowski’s result.

The study of the structure of the tiling set $\Lambda$ in multiple tiling setting was started by Kolountzakis [3] in 1999, in which he showed that if $P k$-tiles in $\mathbb{R}^2$, then with the exception of parallelograms, $P$ must $k$-tile with a finite union of translated lattices (not necessarily of the same lattice). In his Phd thesis, Shiryaev [12] extended Kolountzakis’ work to show that if $P$ can $k$-tile in $\mathbb{R}^2$, then $P$ can $m$-tile $\mathbb{R}^2$ for some $m$ with a lattice as the tiling set. A similar result to Kolountzakis’ was shown in three dimensions by Gravin, Kolountzakis, Robins, and Shiryaev [5], that with the exception of two flat-zonotopes, all convex polytopes that can $k$-tile in $\mathbb{R}^3$ must $k$-tile with a finite union of translated lattices. It is conjectured that all $k$-tilers in $\mathbb{R}^d$ other than an exceptional class of polytopes must $k$-tile with a finite union of translated lattices. This conjecture is still open, with the biggest hurdle being that we do not know what the exceptional class of polytopes should be for dimension 4 and above. Nevertheless, those results on the structure of the tiling set $\Lambda$ in dimension 2 and 3 motivates us to study the multiple tiling problem when $\Lambda$ is a finite union of translated lattices (also known as quasi-periodic sets) in general dimensions, and our final aim is to find an analogous result in multiple tiling setting to McMullen’s result [10] that 1-tilers in $\mathbb{R}^d$ can 1-tile with a lattice i.e. to show that all $k$-tilers in $\mathbb{R}^d$ can $m$-tile with a lattice for some $m$ (not necessarily equal to $k$).

In this paper, we study the quasi-periodic tiling problem by focusing on an equivalent lattice-point enumeration problem. This approach is motivated by a result in [6], in which it was shown that a polytope $P k$-tiles $\mathbb{R}^d$ with $\Lambda$ if the number of elements of $\Lambda$ contained in a translate of $P$ by an arbitrary vector is equal to a constant value $k$. This approach allows us to use several useful properties of a lattice-point enumerator that can be applied to quasi-periodic tiling problem. For more details regarding discrete-point enumeration of polytopes, the reader is invited to see the work of Beck and Sinai [3] or Barvinok [2].

Throughout this paper, we will use two different notions of general positions, one for vectors and one for lattices. Let $P$ be a a fixed convex polytope and let $\partial P$ denote the boundary of $P$, we say that a vector $v$ in $\mathbb{R}^d$ is in a general position to a discrete multiset $\Lambda$ if $v$ is not contained in $\partial P + \Lambda$, the union of translates of boundary of $P$ by $\Lambda$. The second
notion is defined for lattices in $\mathbb{R}^d$. Suppose that $Q$ is a union of $n$ translated lattices $L_1, L_2, \ldots, L_n$, we say that a lattice $L_i$ in $Q$ is in a **general position** to $Q$ if the set $\mathbb{R}^d \setminus H_i$ is path-connected, with $H_i$ is defined as:

$$H_i := \bigcup_{j=1, j\neq i}^{n} (\partial P + L_i) \cap (\partial P + L_j).$$  

We will refer to the hypothesis that $L_i$ is in a general position to $Q$ as Hypothesis 1 throughout this paper.

Our first result is to show that under Hypothesis 1, a result similar to McMullen’s can be concluded for the tiling set of a $k$-tiler.

**Theorem 1.1.** Suppose that a convex polytope $P$ $k$-tiles $\mathbb{R}^d$ with a discrete multiset $\Lambda$, and all elements of $\Lambda$ are contained in a quasi-periodic set $Q$. If a lattice $L$ in $Q$ is in a general position to $Q$, then $P$ $m$-tiles $\mathbb{R}^d$ with $L$ for some $m$.

Notice that in the assumption of Theorem 1.1, the tiling set $\Lambda$ is not required to be a quasi-periodic set. Indeed, the assumption imposed on the multiset $\Lambda$ in Theorem 1.1 is a weaker assumption, as we allow different elements of $\Lambda$ to have different multiplicity. So some elements of the quasi-periodic set $Q$ may not be contained in $\Lambda$ while some other elements of $Q$ may have multiplicity more than 1 in $\Lambda$, and thus the tiling set $\Lambda$ can have a very irregular structure. For dimension 2 and 3, it is known from [8] and [5] that other than one exceptional family, all $k$-tilers must $k$-tile with a quasi-periodic set, and hence the weaker assumption on $\Lambda$ does not provide us with more generality in dimension 2 and 3. However, in dimension 4 and above nothing is known about the structure of the tiling set $\Lambda$, and the weaker assumption provides us a much more general framework to work with. For example, when the quasi-periodic set in Theorem 1.1 is a lattice, Theorem 1.1 gives us an interesting corollary.

**Corollary 1.2.** Suppose that $P$ $k$-tiles $\mathbb{R}^d$ with a discrete multiset $\Lambda$, and all elements of $\Lambda$ are contained in a lattice $L$. Then $P$ can $m$-tile $\mathbb{R}^d$ with $L$ for some $m$.

We note that for Theorem 1.1 to hold, the hypothesis that the lattice in Theorem 1.1 is in a general position to the quasi-periodic $Q$ can not be omitted, as in Section 4 we present an example of a $k$-tiler $P$ that $k$-tiles $\mathbb{R}^d$ with a quasi-periodic set $Q$ such that every lattice in $Q$ is not in a general position to $Q$. In this example, every lattice contained in $Q$ is not a tiling set of $P$, thus showing that Hypothesis 1 is necessary for Theorem 1.1.

The fact that $P$ may not be able to $m$-tile with lattices contained in $Q$ naturally leads us to the following question: can a $k$-tiler $P$ $m$-tile for some $m$ with a lattice $L$ not necessarily contained in $Q$, even if every lattice in $Q$ is not in a general position to $Q$? We give a positive answer to this question in some specific cases, which is detailed below:

**Theorem 1.3.** Let $P$ be a convex polytope that $k$-tiles $\mathbb{R}^d$ with a discrete multiset $\Lambda$. Suppose that all elements of $\Lambda$ are contained in a union of two translated copies of one lattice. Then we can find a lattice $L$ in $\mathbb{R}^d$, such that $P$ can $m$-tile $\mathbb{R}^d$ with $L$ for some $m$. 
We conjecture that Theorem 1.3 can be generalized to the case in which elements of Λ are contained in an arbitrary quasi-periodic set.

The paper is organized as follows. Section 2 will be used to introduce definitions and notations used in this paper. We will use Section 3 to establish the connection between the \( k \)-tiling problem and a lattice-point enumeration problem. We will also introduce the notion of half-open polytopes in tiling setting in Section 3, which solves the technical problem with points in the boundary of the polytope mentioned in the beginning of the introduction. Section 4 will be devoted to prove Theorem 1.1, while Section 5 will be devoted to prove Theorem 1.3. Finally, in Section 6 we will discuss some important open problems that are related to the study of the structure of a tiling set.

2 Definitions and preliminaries

Given a convex polytope \( P \) in \( \mathbb{R}^d \), we will denote \( \text{Int}(P) \) as the interior of \( P \), and we will denote the boundary of \( P \) as \( \partial P \) (the closure of \( P \) minus the interior of \( P \)). Throughout this paper, \( \Lambda \) will denote a discrete multiset of vectors in \( \mathbb{R}^d \), \( L \) will denote a lattice in \( \mathbb{R}^d \), and \( Q \) will denote a quasi-periodic set, which is a finite union of translated lattices, not necessarily of the same lattice. We will use \( \#(A) \) to denote the cardinality of a finite multiset \( A \), and we will use \( A \cap S \) to denote the intersection of a multiset \( A \) and a set \( S \), which in our setting is the multiset containing elements of \( A \) that are contained in \( S \) with elements of \( A \cap S \) have the same multiplicity as their counterpart in \( A \). We define the multiset \( A \setminus S \) as the intersection of the multiset \( A \) and the complement set of \( S \).

A convex polytope \( P \subset \mathbb{R}^d \) is said to \( k \)-tile \( \mathbb{R}^d \) (\( k \) being a positive natural number) with a discrete multiset \( \Lambda \) of vectors in \( \mathbb{R}^d \) if

\[
\sum_{\lambda \in \Lambda} \mathbf{1}_{P+\lambda}(v) = k; \tag{2}
\]

for all \( v \notin \partial P + \Lambda \), where \( \mathbf{1}_P \) is the indicator function of the polytope \( P \). Notice that the multiplicity of elements in \( \Lambda \) can not exceed \( k \), as otherwise the left hand side of Equation 2 will exceed \( k \) for some \( v \) in \( \mathbb{R}^d \).

Throughout this paper, we assume that \( h \) is a fixed vector in \( \mathbb{R}^d \) such that every line with direction vector \( h \) meets \( \partial P \) at finitely many points. The \textbf{half-open} counterpart \( P^h \) of a convex polytope \( P \) is the subset of the closure of \( P \) that contains all points \( v \) such that for a sufficiently small \( \epsilon_v > 0 \) the ray \( r_v := \{v + ch \mid 0 < c < \epsilon_v\} \) is contained in \( \text{Int}(P) \). It is left as exercise to reader to check that \( \text{Int}(P) \) is contained in \( P^h \), and in addition to that, \( P^h \) contains a portion of \( \partial P \). In particular, when \( P \) is a cube, the half-open cube defined in [13] is a special case of the half-open polytope constructed from our definition by choosing a specific \( h \).

For a a discrete multiset \( \Lambda \) and a convex polytope \( P \) in \( \mathbb{R}^d \), the \textbf{A-point enumerator} of \( P \) is the integer \( \#(\Lambda \cap P) \), the number of points of \( \Lambda \) (counting multiplicity) contained in \( P \). When \( \Lambda \) is a lattice, we refer to \( \#(\Lambda \cap P) \) as a lattice-point enumerator instead. Due to a technical reason that will be evident in Section 3 we define two integer-valued functions \( L_\Lambda \) and \( L^h_\Lambda \) on every point \( v \) in \( \mathbb{R}^d \) as follows:

\[
L_\Lambda(v) := \#(\Lambda \cap \{-1 \cdot P + v\}), \quad L^h_\Lambda(v) := \#(\Lambda \cap \{-1 \cdot P^h + v\}),
\]
i.e. $L_{\Lambda}(v)$ is the number of points of $\Lambda$ contained in the translate of $-1 \cdot P$ by $v$ and $L_{\Lambda}^h(v)$ is the number of points of $\Lambda$ contained in the translate of $-1 \cdot P^h$ by $v$. If the intended multiset $\Lambda$ is evident from the context, we will use $L$ and $L^h$ as a shorthand for $L_{\Lambda}$ and $L_{\Lambda}^h$ respectively.

### 3 Lattice-point enumeration of polytopes

In this section we will present a lattice-point enumeration problem that is equivalent to $k$-tiling problem. The equivalence was first shown in [6], and in that paper the equivalence was used to show that all rational $k$-tilers can $m$-tile with a lattice for some $m$. However it would be inconvenient for us to use their equivalence directly, as somewhere in their equivalence a vector $v$ in $\mathbb{R}^d$ is required to be in a general position, which opens the door to all sorts of technical problems when applied to our approach. Hence we would like to remove that requirement to have a cleaner tool for our approach. This turns out to be possible by switching the polytope $P$ with its half-open counterpart $P^h$.

**Lemma 3.1.** A $d$-dimensional convex polytope $P$ $k$-tiles $\mathbb{R}^d$ with a discrete multiset $\Lambda$ if and only if its half-open counterpart $P^h$ $k$-tiles $\mathbb{R}^d$ with $\Lambda$. Moreover, if $P^h$ $k$-tiles $\mathbb{R}^d$ with $\Lambda$, then

$$\sum_{\lambda \in \Lambda} 1_{P^h + \lambda}(v) = k,$$

for all $v$ in $\mathbb{R}^d$.

**Proof.** The first part of the lemma is immediately implied by the fact that $P$ and $P^h$ have the same interior, which in turn implies that

$$\sum_{\lambda \in \Lambda} 1_{P^h + \lambda}(v) = \sum_{\lambda \in \Lambda} 1_{P + \lambda}(v),$$

for all $v \notin \partial P + \Lambda$. Therefore $P$ $k$-tiles with $\Lambda$ if and only if $P^h$ $k$-tiles with $\Lambda$.

To prove the second part of the claim, let $v$ be an arbitrary point in $\mathbb{R}^d$. The ray $r_{\epsilon_v} = \{v + c\mathbf{h} \mid 0 < c < \epsilon_v\}$ intersects only finitely many point of $\partial P + \Lambda$ (due to our assumption that every line with direction $\mathbf{h}$ will intersect $\partial P$ only at finitely many points), and hence for a sufficiently small $\epsilon_v > 0$, the ray $r_{\epsilon_v}$ does not intersect $\partial P + \Lambda$. This, coupled with the assumption that $P^h$(and hence $P$) $k$-tiles $\mathbb{R}^d$ with $\Lambda$, implies that there are exactly $k$ vectors $\lambda_1, \ldots, \lambda_k$ in $\Lambda$ such that $r_{\epsilon_v}$ is contained in the interior of $P + \lambda_i$ for all $i$. By the definition of half-open polytopes in Section 2, this means that $v$ is contained in $P^h + \lambda_i$ for all $i$, and hence we have:

$$\sum_{\lambda \in \Lambda} 1_{P^h + \lambda}(v) = k,$$

for all $v$ in $\mathbb{R}^d$. \[ \square \]

The lemma below has been shown in [6] for $v$ is in a general position in $\mathbb{R}^d$, and our proof is virtually identical with a minor adjustment for the case when $P$ is switched with $P^h$. 

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Lemma 3.2. (Gravin, Robins, Shiryaev [6]) A convex polytope $P_k$-tiles $\mathbb{R}^d$ with discrete multiset $\Lambda$ of vectors in $\mathbb{R}^d$ if and only $L_\Lambda(v)$ is equal to $k$ for every $v$ in a general position to $\Lambda$. If $P$ is being switched with $P^b$, then $P^b$ $k$-tiles $\mathbb{R}^d$ with $\Lambda$ if and only $L^b_\Lambda(v)$ is equal to $k$ for every $v$ in $\mathbb{R}^d$.

Proof. For every $v$ in $\mathbb{R}^d$, we can write

$$\sum_{\lambda \in \Lambda} 1_{P+\lambda}(v) = \sum_{\lambda \in \Lambda} 1_{-1,P+v}(\lambda) = \#(\Lambda \cap \{-1 \cdot P + v\}) = L_\Lambda(v).$$

This, combined with Equation 2 in the definition of $k$-tilers, implies that $P_k$-tiles $\mathbb{R}^d$ if and only if $L_\Lambda(v)$ is equal to $k$ for every $v$ in a general position. Similarly, by Lemma 3.1, $P^b$-tiles $\mathbb{R}^d$ if and only if $L^b_\Lambda(v)$ is equal to $k$ for every $v$ in $\mathbb{R}^d$. \qed

The statement of Lemma 3.2 can be summarised as follows: The problem to check whether a polytope $P_k$-tiles $\mathbb{R}^d$ with a tiling set $\Lambda$ is equivalent to the problem to check whether the $\Lambda$-point enumerator of the half-open counterpart of a polytope $P$ is a constant function under translation. Throughout this paper, we will view $k$-tiling problem as the latter, as the latter approach works really well in the case quasi-periodic tiling. This is because a lattice-point enumerator has several useful properties that can be applied to approach the $k$-tiling problem. As a function that maps $\mathbb{R}^d$ to $\mathbb{Z}$, $L_\Lambda$ has several useful properties:

1. If $\mathcal{L}$ is a lattice in $\mathbb{R}^d$, then the function $L_\mathcal{L}$ is a periodic function of $\mathcal{L}$ i.e. we have $L_\mathcal{L}(v + \lambda) = L_\mathcal{L}(v)$ for all $v$ in $\mathbb{R}^d$ and for all $\lambda$ in $\mathcal{L}$. This is because lattice-point enumerator is invariant under translation by elements contained in the lattice.

2. The function $L_\Lambda$ is a constant function in a sufficiently small neighbourhood of $v$ in $\mathbb{R}^d$ if $v$ is in a general position with respect to $P$ and $\Lambda$. This is because if $v$ is not in $\partial P + \Lambda$, then $-1 \cdot P + v$ does not contain any points from $\Lambda$ in its boundary. Hence, moving $v$ in any sufficiently small direction will not change the value of $\Lambda$-point enumeration $\#(\Lambda \cap \{-1 \cdot P + v\})$, which is equal to $L_\Lambda(v)$.

It can be easily checked that the two properties above also hold for the function $L^b_\Lambda$.

4 Proof of Theorem 1.1

We will start this section by proving a functional equation involving the function $L_\Lambda(v)$ which will be used repeatedly throughout this paper. The proof borrows several ideas from asymptotic analysis of infinite sums in $\mathbb{R}^d$, as we are looking at the tiling problem from analysis viewpoint.

Lemma 4.1. Let $P$ be a convex polytope that $k$-tiles $\mathbb{R}^d$ with a discrete multiset $\Lambda$. Suppose that all elements of $\Lambda$ are contained in a finite union of translates of one lattice i.e. the set $\bigcup_{i=1}^n \mathcal{L}_i$, with $\mathcal{L}_i := a_i + \mathcal{L}$ is a translate of a lattice $\mathcal{L}$ in $\mathbb{R}^d$ and $a_i$ is a vector in $\mathbb{R}^d$. Then for an arbitrary vector $v \in \mathbb{R}^d$, we have the following equation:

$$\sum_{i=1}^n g_i \cdot L_{\mathcal{L}_i}^b(v - a_i) = k,$$

with $g_1, \ldots, g_n$ are non-negative real numbers.
Lemma 4.1 is a corollary of Lemma 3.2 in the particular case when $\Lambda$ is equal to the disjoint union of finitely many translates of one lattice, as we then have:

$$k = \#(\Lambda \cap \{-1 \cdot P^b + v\}) = \sum_{i=1}^{n} \#(\{a_i + \mathcal{L}\} \cap \{-1 \cdot P^b + v\})$$

$$= \sum_{i=1}^{n} \#(\mathcal{L} \cap \{-1 \cdot P^b + a_i\}) = \sum_{i=1}^{n} L^b_v (v - a_i),$$

for all $v$ in $\mathbb{R}^d$. However, this argument fails if we apply it to our original assumption on $\Lambda$ in Lemma 4.1. The main problem is because elements of the multiset $\Lambda$ are allowed to have different multiplicity, so some elements of $\mathcal{L}$ may not be contained in $\Lambda$ while some other elements may have multiplicity more than 1, causing the previous counting argument to fail. Hence, we require a different approach to prove Lemma 4.1.

**Proof.** Without loss of generality, we can assume that $\mathcal{L} = \mathbb{Z}^d$. We will not lose generality by doing so, as a lattice $\mathcal{L}$ in $\mathbb{R}^d$ is isomorphic to $\mathbb{Z}^d$.

For a real vector $w = (w_1, \ldots, w_d)$ in $\mathbb{R}^d$, we denote $\mathbb{R}^d_\geq w$ to be the subset of $\mathbb{R}^d$ containing $w' = (w'_1, \ldots, w'_d)$ satisfying $w'_i \geq w_i$ for all $i$, and let $\Lambda^\dagger := \Lambda \cap \mathbb{R}^d_{\geq 0}$. For any vector $v$ in $\mathbb{R}^d$, the fact that $\Lambda$ is a tiling set for $P^b$ means that there is a vector $\alpha(v) \in \mathbb{R}^d$ such that every point in $\mathbb{R}^d_{\geq \alpha(v)}$ is covered exactly $k$ times by $P^b + v + \Lambda^\dagger$. Also notice that since $\Lambda^\dagger$ is in the positive orthant, there is a vector $\beta(v)$ in $\mathbb{R}^d$ such that $P^b + v + \Lambda^\dagger$ does not contain elements from $\mathbb{R}^d_{\leq \beta(v)}$. Without loss of generality we can assume that both $\alpha(v)$ and $\beta(v)$ are integer vectors.

Let $\Gamma(v)$ be the multiset $(P^b + v + \Lambda^\dagger) \setminus \mathbb{R}^d_{\geq \alpha(v)}$. Since $P^b + v + \Lambda^\dagger$ covers every points in $\mathbb{R}^d_{\geq \alpha(v)}$ exactly $k$ times, we have the following equality:

$$\sum_{x \in P^b + v + \Lambda^\dagger} z^x = \sum_{x \in \mathbb{R}^d_{\geq \alpha(v)}} k z^x + \sum_{x \in \Gamma(v) \cap \mathbb{Z}^d} z^x, \quad (4)$$

with $z^x$ is the multivariable polynomial $z_1^{x_1} \cdots z_d^{x_d}$ for $x = (x_1, \ldots, x_d)$ an integer vector. We also set $|z_j| < 1$ so all the sums in Equation 4 converge to a well-defined value. We would like to derive Equation 3 by multiplying both sides of Equation 4 by $\prod_{j=1}^{d} (1 - z_j)$ and then taking the limit as $z$ goes to $(1, \ldots, 1)^{-}$, and with some modifications the limit will converge to Equation 3 as we will show below.

We define the multisets $\Lambda_i$, $i = 1, \ldots, n$ recursively be setting $\Lambda_i = (\Lambda^\dagger \cap \{a_i + \mathbb{Z}^d\}) \setminus \bigcup_{j=1}^{i-1} \Lambda_j$ for $1 \leq i \leq n$. Notice that we have $1_{\Lambda^\dagger} = \sum_{i=1}^{n} 1_{\Lambda_i}$ and elements of $\Lambda_i - a_i$ are contained in
$\mathbb{Z}^d$. With this, Equation 4 is now changed to
\[ \sum_{i=1}^{n} \sum_{x \in P^h + v + \Lambda_i} z^x = \sum_{x \in \mathbb{Z}^d_{\geq \alpha(v)}} k \cdot z^x + \sum_{x \in \Gamma(v) \cap \mathbb{Z}^d} z^x \]
\[ \sum_{i=1}^{n} \left[ \sum_{x \in P^h + v + a_i} z^x \cdot \sum_{x \in \Lambda_i - a_i} z^x \right] = \sum_{x \in \mathbb{Z}^d_{\geq \alpha(v)}} k \cdot z^x + \sum_{x \in \Gamma(v) \cap \mathbb{Z}^d} z^x, \]  
(5)
due to the fact that a point $x$ in $\mathbb{Z}^d$ that is contained in $P^h + v + \Lambda_i$ can be written as the sum $x_1 + x_2$, with $x_1$ is contained in $\{P^h + v + a_i\} \cap \mathbb{Z}^d$ and $x_2$ is contained in $\Lambda_i - a_i \subseteq \mathbb{Z}^d$.

Note that points in $\Gamma(v)$ (i.e. the points in $P^h + v + \Lambda^i$ not contained in $\mathbb{R}^d_{\geq \alpha(v)}$) is contained in $\mathbb{R}^d \setminus (\mathbb{R}^d_{\geq \alpha(v)} \cup \mathbb{R}^d_{\leq \beta(v)})$. It is easy to see that $\mathbb{R}^d \setminus (\mathbb{R}^d_{\geq \alpha(v)} \cup \mathbb{R}^d_{\leq \beta(v)})$ is contained in the set $\bigcup_{i=1}^{d} R_i(v)$, with $R_i(v)$ is defined as

$$R_i(v) := \{(a_1, \ldots, a_d) \mid \beta(v)_i \leq a_i < \alpha(v)_i \text{ and } \alpha(v)_j \leq a_j \text{ for all } j \neq i, \ 1 \leq j \leq d \}.$$  

Since we assume $|z_i| < 1$ for all $i$, an easy computation shows that we have the closed-form expression for the following two sums:

$$\sum_{x \in \mathbb{Z}^d_{\geq \alpha(v)}} z^x = \prod_{j=1}^{d} \frac{z^{\alpha(v)_j}}{1 - z_j}, \quad \sum_{x \in R_i(v) \cap \mathbb{Z}^d} z^x = (z^{\beta(v)_i - \alpha(v)_i} - 1) \cdot \prod_{j=1}^{d} \frac{z^{\alpha(v)_j}}{1 - z_j}. \]  
(6)

Since $\Gamma(v)$ is contained in $\bigcup_{i=1}^{d} R_i(v)$, we can conclude the following limit:

$$\lim_{z \rightarrow (1, \ldots, 1)^{-}} \sum_{x \in \Gamma(v) \cap \mathbb{Z}^d} z^x \cdot \prod_{j=1}^{d} (1 - z_j) \leq \lim_{z \rightarrow (1, \ldots, 1)^{-}} \sum_{i=1}^{d} \sum_{x \in R_i(v) \cap \mathbb{Z}^d} k \cdot z^x \cdot \prod_{j=1}^{d} (1 - z_j)$$

$$= \lim_{z \rightarrow (1, \ldots, 1)^{-}} \sum_{i=1}^{d} (z^{\beta(v)_i - \alpha(v)_i} - 1) z^{\alpha(v)_i} = 0. \]  
(7)

Multiplying the right hand side of Equation 5 with $\prod_{j=1}^{d} (1 - z_j)$ and then taking the limit as $z$ goes to $(1, \ldots, 1)^{-}$, we get

$$\lim_{z \rightarrow (1, \ldots, 1)^{-}} \sum_{x \in \mathbb{Z}^d_{\geq \alpha(v)}} k \cdot z^x \cdot \prod_{j=1}^{d} (1 - z_j) + \sum_{x \in \Gamma(v) \cap \mathbb{Z}^d} z^x \cdot \prod_{j=1}^{d} (1 - z_j) = k, \]  
(8)

by plugging in Equation 6 and Equation 7. We will do the same thing to the the left hand side of Equation 5. Multiplying the left hand side of Equation 5 by $\prod_{j=1}^{d} (1 - z_j)$ and then
taking the limit as $z$ goes to $(1, \ldots, 1)^-$, we get the following expression

$$
\sum_{i=1}^{n} \left[ \lim_{z \to (1, \ldots, 1)^-} \sum_{x \in P^h + a_i + v} z^x \cdot \sum_{x \in \Lambda_i - a_i} z^x \cdot \prod_{j=1}^{d} (1 - z_j) \right]
$$

$$
= \sum_{i=1}^{n} \#(\mathcal{L} \cap \{P^h + a_i + v\}) \cdot \lim_{z \to (1, \ldots, 1)^-} \sum_{x \in \Lambda_i - a_i} z^x \cdot \prod_{j=1}^{d} (1 - z_j) \tag{9}
$$

Let $s_i(z)$ denote $\sum_{x \in \Lambda_i - a_i} z^x \cdot \prod_{j=1}^{d} (1 - z_j)$. We would like that the limit $s_i(z)$ exists for all $i$ as $z$ goes to $(1, \ldots, 1)^-$, so that we can derive the left hand side of Equation 3 from Equation 9. However, the limit of $s_i(z)$ may not exist in general if $\Lambda_i - a_i$ is an arbitrary subset of $\mathbb{Z}_d^d$ (an example of $s_i(z)$ that does not have a limit is included in Appendix A), and we need to do some modification to our argument to derive Equation 3. We will do so by choosing a special sequence of vectors $(z^u)_{u=1,2,\ldots}$ in $\mathbb{R}^d$ that converges to $(1, \ldots, 1)^-$, and then evaluating $\lim_{u \to \infty} s_i(z^u)$ (instead of evaluating the limit of $s_i(z)$ directly) to derive Equation 3.

Notice that $s_i(z)$ is always positive when $0 < z_i < 1$ for all $i$, as with that condition both $\sum_{x \in \Lambda_i - a_i} z^x$ and $\prod_{j=1}^{d} (1 - z_j)$ are positive. Also notice that the multiplicity of all elements in $\Lambda_i$ can not exceed $k$, as the multiplicity of all elements in $\Lambda$ can not exceed $k$. This, together with the fact that all elements of $\Lambda_i - a_i$ are contained in $\mathbb{Z}_d^d$ (because $\Lambda^1$, and hence $\Lambda_i$, have all elements contained in $\mathbb{Z}_d^d$), gives us the following inequality:

$$
s_i(z) = \sum_{x \in \Lambda_i - a_i} z^x \cdot \prod_{j=1}^{d} (1 - z_j) \leq \sum_{x \in \mathbb{Z}_d^{d-a_i}} k z^x \cdot \prod_{j=1}^{d} (1 - z_j) \tag{10}
$$

$$
= k z^{-a_i} \cdot \prod_{j=1}^{d} \frac{1}{1 - z_j} \cdot \prod_{j=1}^{d} (1 - z_j) = k z^{-a_i} < k,
$$

when $z$ is sufficiently close to $(1, \ldots, 1)^-$, and $\lfloor v \rfloor$ is the integer part of the real vector $v$ in $\mathbb{R}^d$. Hence, as $z$ goes to $(1, \ldots, 1)^-$, the value of $s_i(z)$ is bounded between 0 and $k$. So there exists a sequence $(z^u)_{u=1,2,\ldots}$ that converges to $(1, \ldots, 1)^-$, such that $\lim_{u \to \infty} s_i(z^u)$ exists for all $i$. We now define $g_i := \lim_{u \to \infty} s_i(z^u)$ for all $i$. Note that $g_i$ is non-negative as $s_i(z)$ are always positive for $z$ sufficiently close to $(1, \ldots, 1)^-$. Also from the definition of $s_i(z)$ (which does not involve $v$), it is easy to see that $g_i$ is a constant that is independent from $v$.

Now, multiplying both sides of Equation 3 with $\prod_{j=1}^{d} (1 - z_j)$ and then plugging in $z = z^u, u = 1, 2, \ldots$ and then taking the limit as $u$ goes to infinity, we will get the following equation:

$$
\sum_{i=1}^{n} \#(\mathcal{L} \cap \{P^h + a_i + v\}) \cdot g_i = k. \tag{10}
$$

To get the equation in the statement of the lemma, notice that $\#(\mathcal{L} \cap \{P^h + a_i + v\})$ is equal to $\#(-1 \cdot \mathcal{L} \cap \{-1 \cdot (P^h + a_i + v)\})$, which is equal to $\#(\mathcal{L} \cap \{-1 \cdot P^h - a_i - v\} = L_{\Lambda}(-a_i - v)$ (by the fact that $\mathcal{L} = -1 \cdot \mathcal{L}$). Substituting $v$ with $-v$ to get $L_{\Lambda}(v - a_i)$, and then plugging
\#(\mathcal{L} \cap \{P^h + a_i + v\}) = L^h_\Lambda(v - a_i)\) into Equation 10 we get Equation 3 in the lemma and the proof is now complete.

Remark. The proof of Lemma 4.1 will fail to work if the condition on \(\Lambda\) is weakened to elements of \(\Lambda\) are contained in a finite union of translated lattices, as the assumption that elements of \(\Lambda\) are contained in a finite union of translates of one lattice is essential to derive Equation 3 from Equation 4.

Lemma 4.1 is what allows us to weaken our assumption from \(\Lambda\) being a quasi-periodic set to elements of \(\Lambda\) are contained in a quasi-periodic set. We will describe the idea rigorously in the proof of Theorem 4.2 below.

Theorem 4.2. Let \(P\) be a convex polytope that \(k\)-tiles \(\mathbb{R}^d\) with a discrete multiset \(\Lambda\). If all elements in \(\Lambda\) are contained in a finite union of translates of one lattice \(\mathcal{L}_1, \ldots, \mathcal{L}_n\), then \(P\) can \(m\)-tile in \(\mathbb{R}^d\) for some \(m\) with a finite union of copies of \(\mathcal{L}_1, \mathcal{L}_2, \ldots, \) and \(\mathcal{L}_n\).

Proof. Let elements of \(\Lambda\) be contained in the union of \(n\) translates \(a_i + \mathcal{L}\), with \(a_i\) are vectors in \(\mathbb{R}^d\) and \(\mathcal{L}\) is a lattice in \(\mathbb{R}^d\). Notice that \(\Lambda\) satisfies the hypothesis in Lemma 4.1. Hence, we have the following formula:

\[
\sum_{i=1}^{n} g_i \cdot L^h_\mathcal{L}(v - a_i) = k, \tag{11}
\]

for some non-negative numbers \(g_1, \ldots, g_n\) and all vectors \(v\) in \(\mathbb{R}^d\). We will show that we can in fact choose \(g_1, \ldots, g_n\) to be non-negative integers.

For an arbitrary \(v\) and \(w\) in \(\mathbb{R}^d\), we define the integer

\[
l_i(v, w) := L^h_\mathcal{L}(v - a_i) - L^h_\mathcal{L}(w - a_i),
\]

and \(V\) a vector space in \(\mathbb{R}^n\) defined by:

\[
V := \langle (l_1(v, w), l_2(v, w), \ldots, l_n(v, w)) \mid v, w \in \mathbb{R}^d \rangle_{\mathbb{R}}.
\]

Note that the orthogonal complement \(V^\perp\) of \(V\) has non-zero elements, as by Equation 11 we have the vector \((g_1, \ldots, g_n)\) in the orthogonal complement. Note that by definition of \(V\), generators of \(V\) are rational vectors, and hence the basis of \(V\) and \(V^\perp\) can be chosen to be integer vectors. This, added with fact that a non-negative non-zero vector \((g_1, \ldots, g_n)\) is contained in \(V^\perp\), implies that there exists a non-negative non-zero integer vector \((g'_1, \ldots, g'_n)\) that is contained in \(V^\perp\). By the construction of the vector space \(V\), a vector \((g'_1, \ldots, g'_n)\) orthogonal to \(V\) will mean that:

\[
m = \sum_{i=1}^{n} g'_i \cdot \#(\mathcal{L} \cap \{-1 \cdot P^h + v - a_1\}) = \sum_{i=1}^{n} g'_i \cdot \#(a_i + \mathcal{L} \cap \{-1 \cdot P^h + v\}). \tag{12}
\]

for some positive integers \(m\) and for all \(v\) in \(\mathbb{R}^d\). By Lemma 3.2 this implies that \(P\) \(m\)-tiles \(\mathbb{R}^d\) with the union of the translated lattices \(\mathcal{L}_i, i = 1, \ldots, n\) with each lattice \(\mathcal{L}_i\) having multiplicity \(g_i\). Now the proof is complete. \(\square\)
In dimension 2 and 3, it was shown in [8] and [6] that all $k$-tilers other than one exceptional family must $k$-tile with a finite union of translated lattices, so Theorem 4.2 does not tell us anything new for dimension 2 and 3. However, nothing is known about the structure of the tiling set $\Lambda$ for dimension 4 and above, so Theorem 4.2 provides us with some new insight on the structure of the tiling set $\Lambda$. In one particular case when all elements of $\Lambda$ are contained in a lattice, Theorem 4.2 will give us the following corollary:

**Corollary 4.2.** Suppose that $P$ $k$-tiles $\mathbb{R}^d$ with a discrete multiset $\Lambda$, and all elements of $\Lambda$ are contained in a lattice $L$. Then $P$ can $m$-tile $\mathbb{R}^d$ with $L$ for some $m$. $\blacksquare$

Corollary 4.2 is a very handy tool to prove if a $k$-tiler $P$ can $m$-tile with a lattice for some $m$, as showing that $P$ can $m$-tile with translation vectors taken from a lattice is an easier task compared to showing that $P$ can $m$-tile with a lattice itself. For example, Corollary 4.2 will be used in the proof of Theorem 1.1 to show that $P$ can $m$-tile with a lattice.

We are now getting very close to prove Theorem 1.1. The idea of the proof of Theorem 1.1 is to reduce the problem from the case in which elements of $\Lambda$ contained in a finite union of translated lattices to the case in which elements of $\Lambda$ are contained in a lattice, and we can then use Corollary 4.2 to show that $P$ can $k$-tile $\mathbb{R}^d$ with a lattice. In order to do so, we will need one more lemma, which is stated below.

**Lemma 4.3.** Suppose that $P$ $k$-tiles $\mathbb{R}^d$ with a discrete multiset $\Lambda$, and suppose that $\Lambda$ is the disjoint union of two discrete multisets $\Lambda_1$ and $\Lambda_2$. If the set $\mathbb{R}^d \setminus H$ is path connected, with $H$ is defined as the set $(\partial P + \Lambda_1) \cap (\partial P + \Lambda_2)$, then $P$ can $m$-tile $\mathbb{R}^d$ with $\Lambda_1$ for some $m$.

**Proof.** We will start by mentioning a functional equation that relates the lattice-point enumerator of $\Lambda_1$ and $\Lambda_2$. The equation is derived from Lemma 3.2 as a direct result of $P$ $k$-tiles $\mathbb{R}^d$ with $\Lambda_1 \sqcup \Lambda_2$:

$$L^h_{\Lambda_1}(v) + L^h_{\Lambda_2}(v) = L^h_{\Lambda}(v), \quad (13)$$

for all $v$ in $\mathbb{R}^d$. Intuitively, the idea of the proof is that we want to construct a path $\mathcal{P}$ of points in $\mathbb{R}^d$, and then we show that the function $L^h_{\Lambda_1}$ is a constant function on the path $\mathcal{P}$. The reason why such a path exist is because the set $\mathbb{R}^d \setminus H$ is path connected, so we can constructed a path between two points from $\mathbb{R}^d \setminus H$ such that the path avoids $H$, which is the set in which the function $L^h_{\Lambda_1}$ can change value.

Let $v_1$ and $v_2$ be two points in $\mathbb{R}^d \setminus H$. Since $\mathbb{R}^d \setminus H$ is path-connected, there is a path $\mathcal{P} : [0, 1] \rightarrow \mathbb{R}^d$ that starts at $v_1$ and ends at $v_2$, and $\mathcal{P}$ does not contain points from $H$. We will show that the function $L^h_{\Lambda_1}(\mathcal{P}(x))$ is a constant function of $x$ as $x$ goes from 0 to 1. Suppose that $L^h_{\Lambda_1}(\mathcal{P}(x))$ is not a constant function, then that means there is a constant $\alpha$ in $[0, 1]$, such that the function $L^h_{\Lambda_1}(\mathcal{P}(x))$ is not a constant function in all open neighbourhood of $\alpha$. Since $\mathcal{P}(\alpha)$ is not contained in $(\partial P + \Lambda_1) \cap (\partial P + \Lambda_2)$, then one of the following scenarios must happen:

- $\mathcal{P}(\alpha)$ is not contained in $\partial P + \Lambda_1$. This means that $\mathcal{P}(\alpha)$ is in a general position to $\Lambda_1$. By the Property 2 of the function $L^h_{\Lambda_1}$ mentioned in Section 3, $L^h_{\Lambda_1}$ is a constant function in a sufficiently small neighbourhood of $\mathcal{P}(\alpha)$, a contradiction to the assumption on $\alpha$.

- $\mathcal{P}(\alpha)$ is not contained in $\partial P + \Lambda_2$. By similar argument as above, the function $L^h_{\Lambda_1}$ is a constant function in a sufficiently small neighbourhood of $\alpha$. By Equation 13 we have
$L_{\Lambda}^b = k - L_{\Lambda}^b$. This means that $L_{\Lambda}^b$ is also a constant function in a sufficiently small neighbourhood of $\mathcal{P}(\alpha)$, again a contradiction to the assumption on $\alpha$.

Hence we conclude that the function $L_{\Lambda}^b$ has a constant value $m$ for all $v$ in $\mathbb{R}^d \setminus H$, and we are one step away from proving the lemma.

First, note that the function $L_{\Lambda}^b(v)$ is a constant function in a sufficiently small neighbourhood of $v$ if $v$ is in a general position to $\Lambda_1$ by Property 2 in Section 3. Also note that the set $H = (\partial P + \Lambda_1) \cap (\partial P + \Lambda_2)$ is a closed set with no interior, and hence there is always a point $w$ in $\mathbb{R}^d \setminus H$ that is also contained in the open neighbourhood of $v$. This means that $L_{\Lambda}^b(v) = L_{\Lambda}^b(w) = m$ for all $v$ in a general position to $\Lambda_1$. By Lemma 3.2 we conclude that $P^b$ (and hence $P$) $m$-tiles $\mathbb{R}^d$ with $\Lambda_1$.

We will now proceed to the proof of Theorem 1.1. For the convenience of the reader, we will restate the definition of Hypothesis 1 from Section 1. Let $\mathcal{Q}$ be a union of $n$ translated lattices $(\mathcal{L}_i), i = 1, \ldots, n$ in $\mathbb{R}^d$, and we denote by $H_i$ the set

$$H_i := \bigcup_{j=1, j \neq i}^n (\partial P + \mathcal{L}_i) \cap (\partial P + \mathcal{L}_j).$$

We say that the lattice $\mathcal{L}_i$ is in a general position to $\mathcal{Q}$ if $\mathbb{R}^d \setminus H_i$ is path-connected, and we refer to the hypothesis that $\mathcal{L}_i$ is in a general position as Hypothesis 1. An example of a lattice in a general position in $\mathbb{R}^2$ is shown in Example 1 below.

Example 1. Let $P$ be a rectangle in $\mathbb{R}^2$, formed by taking $(0,0), (0,1), (1, 0)$ and $(1, 1)$ as the vertices of $P$. Define $\mathcal{Q}$ as the union of two lattices $\mathcal{L}_1$ and $\mathcal{L}_2$, with $\mathcal{L}_1 := \mathbb{Z}^2$ and $\mathcal{L}_2 := (\frac{1}{2}, 1) + \mathbb{Z}^2$. It can be easily checked that $P$ 2-tiles $\mathbb{R}^2$ with $\mathcal{Q}$. Note that $\partial P + \mathcal{L}_1$ is equal to $\{(x,y) \in \mathbb{R}^2 \mid x \in \mathbb{Z} \text{ or } y \in \mathbb{Z}\}$, and $\partial P + \mathcal{L}_2$ is equal to $\{(x,y) \in \mathbb{R}^2 \mid x \in \frac{1}{2} + \mathbb{Z} \text{ or } y \in \frac{1}{2} + \mathbb{Z}\}$. Both of those are discrete union of lines in $\mathbb{R}^2$, and their intersection is the set $H_1 = ((\mathbb{Z} \times (\frac{1}{2} + \mathbb{Z})) \cup ((\frac{1}{2} + \mathbb{Z}) \times \mathbb{Z})$, which is a discrete union of points in $\mathbb{R}^2$. One can now easily see that $\mathbb{R}^2 \setminus H_1$ is path-connected, and thus $\mathcal{L}_1$ is in a general position to $\mathcal{Q}$.

With all the tools present at hand now, we will now proceed to prove Theorem 1.1.

Theorem 1.1. Suppose that a convex polytope $P$ $k$-tiles $\mathbb{R}^d$ with a multiset $\Lambda$, such that all elements of $\Lambda$ are contained in the union of $n$ translated lattices $\mathcal{L}_i, i = 1, \ldots, n$, which we denote by $\mathcal{Q}$. If the lattice $\mathcal{L}_i$ is in a general position to $\mathcal{Q}$, then $P$ $m$-tiles $\mathbb{R}^d$ with $\mathcal{L}_i$ for some $m$.

The condition that $\mathbb{R}^d \setminus H_i$ is path-connected turns out to be a necessary part of the theorem, as if that condition is removed, then it can happen that $P$ can not $k$-tile $\mathbb{R}^d$ with every lattice $\mathcal{L}_i$ contained in the quasi-periodic set $\mathcal{Q}$, as demonstrated by the example below.

Example 2. Let $P$ be a rectangle in $\mathbb{R}^2$ formed by taking $(0,0), (0,\frac{1}{2}), (1, 0), (1, \frac{1}{2})$ as the vertices of $P$. It can be easily checked that $P$ 1-tiles $\mathbb{R}^2$ with $\mathbb{Z}^2 \cup (\frac{1}{2}, \frac{\sqrt{2}}{2}) + \mathbb{Z}^2$. However, $P$ can not $m$-tile $\mathbb{R}^2$ for any $m$ with $\mathbb{Z}^2$ as the point $(\frac{3}{4}, \frac{3}{4})$ and a small neighbourhood around that point are not contained in $P + \mathbb{Z}^2$ (similarly, $P + (\frac{1}{2}, \frac{\sqrt{2}}{2}) + \mathbb{Z}^2$ can not $m$-tile for any $m$ because $(\frac{3}{4} + \frac{1}{2}, \frac{3}{4} + \frac{\sqrt{2}}{2})$ and a small neighbourhood around that point are not contained in...
Notice that \( H_1 = H_2 \) is equal to \( \{(x, y) \in \mathbb{R}^2 \mid x \in \mathbb{Z}\} \), a discrete union of parallel lines in \( \mathbb{R}^2 \) and thus \( \mathbb{R}^2 \setminus H_1 \) and \( \mathbb{R}^2 \setminus H_2 \) are not path-connected, which is why Theorem 1.1 cannot be applied to this example.

Proof of Theorem 1.1. Without loss of generality, we can assume that \( \mathcal{L}_i \) is a lattice instead of a translate of a lattice. This is because \( P \) \( k \)-tiles \( \mathbb{R}^d \) with \( \Lambda \) if and only if \( P \) \( k \)-tiles \( \mathbb{R}^d \) with \( v + \Lambda \) for an arbitrary \( v \in \mathbb{R}^d \). Hence, we do not lose any generality by assuming that \( \mathcal{L}_i \) is a lattice. Let \( \Lambda_1 = \Lambda \cap \mathcal{L}_i \) and \( \Lambda_2 = \Lambda \setminus \Lambda_1 \). Then \( \Lambda \) is a disjoint union of \( \Lambda_1 \) and \( \Lambda_2 \), and notice that the set \( H = (\partial P + \Lambda_1) \cap \partial(P + \Lambda_2) \) is contained in \( H_i \) (by the distributive law). This means that \( \mathbb{R}^d \setminus H \) is path-connected, and hence by Lemma 4.3, \( P \) can \( m \)-tile \( \mathbb{R}^d \) with \( \Lambda_1 \) for some \( m \), with elements of \( \Lambda_1 \) are contained in \( \mathcal{L}_i \). By Corollary 1.2, \( P \) can \( m' \)-tile with \( \mathcal{L}_i \) for some \( m' \).

Remark. In [8], it was shown that in two dimensions, a \( k \)-tiler that is not a parallelogram \( k \)-tiles with a finite union of translated lattices. A similar result for three dimensions was shown in [5], in which a \( k \)-tiler that is not a two-flat zonotope \( k \)-tiles with a finite union of translated lattices. If we can show that the finite union of translated lattices used by \( k \)-tilers that are not parallellograms or two-flat zonotopes satisfies Hypothesis 1 in Theorem 1.1, then we can conclude that all \( k \)-tilers in two and three dimensions can \( m \)-tile with a lattice for some \( m \). Unfortunately, it is not known whether the quasi-periodic set used by \( k \)-tilers in dimension 2 and 3 that are not parallellograms or two-flat zonotopes satisfies the technical condition in Theorem 1.1 (the polytope \( P \) in Example 2 is a parallelogram), as the proofs in [8] and [5] are existential proofs and not constructive.

The fact that a \( k \)-tiler \( P \) may not be able to \( m \)-tile with lattices contained in \( \mathcal{Q} \) as shown in Example 2 naturally leads us to the following question: can a \( k \)-tiler \( P \) \( m \)-tile for some \( m \) with a lattice \( \mathcal{L} \) not necessarily contained in \( \mathcal{Q} \), even if every lattice in \( \mathcal{Q} \) does not satisfy Hypothesis 1? We will give a partial answer to this question in the next section.

5 Quasi-periodic tiling without Hypothesis 1

In this section, we would like to discuss quasi-periodic tiling in general when we drop Hypothesis 1 from Theorem 1.1. As has been discussed previously in Example 2, if a lattice \( \mathcal{L} \) contained in the tiling set \( \Lambda \) is not in a general position, then \( P \) may not be able to \( m \)-tile \( \mathbb{R}^d \) with \( \mathcal{L} \). However, if we are simply interested in whether a \( k \)-tiler \( P \) can \( m \)-tile \( \mathbb{R}^d \) with some lattice (not necessarily contained in \( \Lambda \)) to show an analogue to McMullen’s result [10] for 1-tiling, then we can hope that \( P \) can still \( m \)-tile for some \( m \) with a lattice (not necessarily contained in \( \Lambda \)) even if every lattice contained in \( \Lambda \) is not in a general position. For example, the rectangle in Example 2 can 2-tile \( \mathbb{R}^2 \) with \( (\frac{1}{2} \mathbb{Z})^2 \), which is not contained in \( \Lambda = \mathbb{Z}^2 \cup (\frac{1}{2}, \frac{\sqrt{2}}{2}) + \mathbb{Z}^2 \).

In the next theorem we will give a partial answer to this question.

Theorem 1.3. Let \( P \) be a convex polytope that \( k \)-tiles \( \mathbb{R}^d \) with a discrete multiset \( \Lambda \). Suppose that all elements in \( \Lambda \) are contained in \( \mathcal{L}_1 \cup \mathcal{L}_2 \), with \( \mathcal{L}_1 \) and \( \mathcal{L}_2 \) are translates of one lattice in \( \mathbb{R}^d \), then there is a lattice \( \mathcal{L} \) such that \( P \) can \( m \)-tile with \( \mathcal{L} \) for some \( m \).
In particular, Theorem 1.3 can be applied to show that the rectangle in Example 2 can be m-tile \( \mathbb{R}^d \) with a lattice.

**Proof.** Without loss of generality, we can assume that \( L_1 = \mathbb{Z}^d \) and \( L_2 = a + \mathbb{Z}^d \) for some \( a \) in \( \mathbb{R}^d \). We will not lose generality by doing this, as we can transform the original lattice to this particular case by translating the lattice and applying a linear transformation to \( \mathbb{R}^d \). We can also assume that \( a \) is not a rational vector, as if that is the case, we can let \( N \) be the least common multiple of the denominator of entries of \( a \). We will then have \( \mathbb{Z}^d \) and \( a + \mathbb{Z}^d \) contained in \((\frac{1}{N}\mathbb{Z})^d\) and by Corollary 1.2 we have that \( L \) for some \( m \).

By permuting the coordinates, we can assume that \( a := (\alpha_1, \alpha_2, \ldots, \alpha_k, \beta_{k+1}, \ldots, \beta_d) \), with the requirement that \( \alpha_1, \ldots, \alpha_k \) are irrational numbers linearly independent over \( \mathbb{Q} \) and \( \beta_{k+1}, \ldots, \beta_d \) are contained in \( \mathbb{Q}[\alpha_1, \ldots, \alpha_k] \). Note that since \( a \) is not a rational vector, we have \( k \geq 1 \). For \( k + 1 \leq i \leq d \), let \( \beta_i = c_{i,1}\alpha_1 + \ldots + c_{i,k}\alpha_k + c_{i,k+1} \), with \( c_{i,j} \in \mathbb{Q} \) (this can be done because \( \beta_i \) is contained in \( \mathbb{Q}[\alpha_1, \ldots, \alpha_k] \)). Let \( n \) be the least common multiple of the denominators of \( c_{i,j} \) with \( i \) ranging from \( k + 1 \) to \( d \) and \( j \) ranging from \( 1 \) to \( k + 1 \). Note that by definition, we have that \( \beta_i \) is contained in \( \frac{1}{n}\mathbb{Z}[\alpha_1, \ldots, \alpha_k] \) for \( k + 1 \leq i \leq d \).

In the rest of the proof, we will show that \( L^h(\mathbb{Z})^d(v) \) has a constant value \( m \) for all \( v \) in \( \mathbb{R}^d \) in a general position to \((\frac{1}{n}\mathbb{Z})^d\). By Lemma 3.2 this implies that \( P^h \) (and hence \( P \)) \( m \)-tiles \( \mathbb{R}^d \) with \((\frac{1}{n}\mathbb{Z})^d\) for some \( m \), which finishes the proof. In the rest of this proof, we will use \( L^h(v) \) as a shorthand for \( L^h_{(\frac{1}{n}\mathbb{Z})^d}(v) \) to avoid messy typesetting.

By Lemma 4.1 the following equation holds for all \( v \) in \( \mathbb{R}^d \):

\[
g_1 \cdot L^h(v) + g_2 \cdot L^h(v - a) = k, \tag{14}
\]

for some non-negative real numbers \( g_1 \) and \( g_2 \). If we have \( g_1 = 0 \), then \( L^h(v - a) = \frac{k}{g_2} \) for all \( v \) in \( \mathbb{R}^d \), and setting \( m = \frac{k}{g_2} \) completes the proof. The proof for when \( g_2 = 0 \) is identical. Hence we can assume that both \( g_1 \) and \( g_2 \) are non-zero.

Let \( L_j \) be defined as \( L_j := L^h(v - a \cdot j) - \frac{k}{g_1+g_2} \) for all integers \( j \). Substituting \( v \) with \( v - a \cdot j \) to Equation (14) we get the following relation for all integers \( j \):

\[
L_j g_1 + L_{j+1} g_2 = 0, \tag{15}
\]

Without loss of generality, we can assume that \( g_2 \leq g_1 \).

First, suppose that \( g_2 < g_1 \), note that we have \( L_0 = (-\frac{g_2}{g_1})^j L_j \) for all integers \( j \). As \( j \) tends to infinity, \( (-\frac{g_2}{g_1})^j L_j \) tends to 0 because \( L_j = L^h(v - a) - \frac{k}{g_1+g_2} \) is a bounded function of \( v \) (because the function \( L^h \) is a periodic function, as explained by Property 1 in Section 3). This implies that \( L_0 = 0 \), which means that \( L^h(v) = \frac{k}{g_1+g_2} \) for all \( v \) in \( \mathbb{R}^d \), and the proof is complete for the case when \( g_2 < g_1 \).

Suppose that \( g_1 = g_2 \). This implies that \( L_{2j+1} = L_1 \) and \( L_{2j} = L_0 \) for all integers \( j \). In the rest of the proof, we will show that \( L_{2j+1} = L_0 \) for some integers \( j \), which implies that \( L_0 = L_1 \). This, together with the fact that \( L_0 + L_1 = 0 \) (from Equation (15)), implies that \( L_0 = L^h(v) - \frac{k}{g_1+g_2} = 0 \), which completes the proof.

For any two points \( v \) and \( w \) in \( \mathbb{R}^d \) and a constant \( \epsilon > 0 \), we say that \( v \) is \( \epsilon \)-close to \( w \) modulo \( \mathbb{Z}^d \) if \( |v - w\lambda| < \epsilon \) for some \( \lambda \) in \( \mathbb{Z}^d \). Remember that in the beginning of the proof we have \( a := (\alpha_1, \ldots, \alpha_k, \beta_{k+1}, \ldots, \beta_d) \) with \( \alpha_1, \ldots, \alpha_k \) are irrational numbers linearly independent
over $\mathbb{Q}$, and $\beta_{k+1}, \ldots, \beta_d$ are contained in $\frac{1}{N}\mathbb{Z}[\alpha_1, \ldots, \alpha_k, 1]$. Let $a' = (\alpha_1, \ldots, \alpha_k)$ be a vector in $\mathbb{R}^k$, we will show that given $\epsilon > 0$, there exists an integer $j$ such that $(2j + 1) \cdot a'$ is $\epsilon$-close to 0 modulo $\mathbb{Z}^k$. To do so, we will borrow a very powerful tool from Number Theory, namely the Weyl Criterion for multidimensional case. We say that a sequence $(x_n)$, $n = 1, 2, \ldots$ of vectors in $\mathbb{R}^k$ is dense modulo $\mathbb{Z}^k$ in $\mathbb{R}^k$ if for any point $v$ in $\mathbb{R}^k$ and a constant $\epsilon > 0$, there exists a natural number $j$ such that $v$ is $\epsilon$-close to $x_j$ modulo $\mathbb{Z}^k$.

**Theorem 5.1.** (a weaker form of Weyl Criterion, Theorem 6.2 [9]) A sequence $(x_n)$, $n = 1, 2, \ldots$ with $x_i \in \mathbb{R}^k$ is dense modulo $\mathbb{Z}^k$ in $\mathbb{R}^k$ if for every lattice point $h \in \mathbb{Z}^k$, $h \neq 0$,

$$
\lim_{M \to \infty} \frac{1}{M} \sum_{n=1}^{M} e^{2\pi i (h, x_n)} = 0.
$$

The sequence of vectors $(2j \cdot a'), j = 0, 1, 2, \ldots$ satisfies the condition in Weyl Criterion, because $\alpha_1, \ldots, \alpha_k$ are irrational numbers linearly independent over $\mathbb{Q}$, which means that $\langle h, a' \rangle$ is never equal to 0 for all values of $h$ in $\mathbb{Z}^k$. Thus, the limit of the sum in Weyl Criterion is equal to

$$
\lim_{M \to \infty} \frac{1}{M} \sum_{n=1}^{M} e^{2\pi i (h, x_n)} = \lim_{M \to \infty} \frac{1}{M} \sum_{j=0}^{M-1} e^{2j \cdot 2\pi i (h, a')} = \lim_{M \to \infty} \frac{1}{M} \cdot \frac{1}{1 - e^{4\pi i (h, a')}} = 0.
$$

Hence we conclude that $(2j \cdot a'), j = 0, 1, 2, \ldots$ is dense modulo $\mathbb{Z}^k$ in $\mathbb{R}^k$ and hence for a constant $\epsilon > 0$, there exists an integer $j$ such that $-a'$ is $\epsilon$-close to $2j \cdot a'$ modulo $\mathbb{Z}^k$, which means that $(2j + 1) \cdot a'$ is $\epsilon$-close to 0 modulo $\mathbb{Z}^k$.

The fact that there exists an odd number $2j + 1$ such that $(2j + 1) \cdot a'$ is $\epsilon$-close to 0 modulo $\mathbb{Z}^k$ implies that $(2j + 1) \alpha_1, \ldots, (2j + 1) \alpha_k$ are all $\epsilon$-close to an integer. Since $\beta_i$ is contained in $\frac{1}{N}\mathbb{Z}[\alpha_1, \ldots, \alpha_k, 1]$, this also implies that $(2j + 1) \beta_i$ is $O(\epsilon)$-close to $\frac{1}{N} \mathbb{Z}$ for $k + 1 \leq i \leq d$. Hence we conclude that we can find an odd number $2j + 1$ such that $(2j + 1) \cdot a$ is $O(\epsilon)$-close to $\frac{1}{N} \mathbb{Z}^d$.

We will now show that $L_{2j+1} = L_0$ as promised. Remember that we assume $v$ is in a general position to $\frac{1}{N} \mathbb{Z}^d$, and hence by Property 2 there is a sufficiently small neighbourhood of $v$ such that $L^b$ is a constant function in the neighbourhood. By our previous argument, for any $\epsilon > 0$, there is an odd number $2j + 1$ such that $(2j + 1) \cdot a$ is $O(\epsilon)$-close to $\frac{1}{N} \mathbb{Z}^d$. This means that with a sufficiently small $\epsilon$, $v - (2j + 1) \cdot a$ is contained in the neighbourhood of $v$ in which $L^b$ is a constant function. This implies that $L_{2j+1} = L^b(v - (2j + 1) \cdot a) = L^b(v) = L_0$, and the proof is complete.

**Remark.** The reader may be tempted to ask whether the proof of Theorem 1.3 can be strengthened to show that $P$ can m-tile with $\mathbb{Z}^d$ instead of $\frac{1}{N} \mathbb{Z}^d$. This is not possible, as shown by Example 2 in which $P$ can not m-tile with $\mathbb{Z}^d$, and the argument in the proof of Theorem 1.3 will fail to work if we try to prove that $P$ can m-tile with $\mathbb{Z}^d$ (instead of $\frac{1}{N} \mathbb{Z}^d$). This is because in the second-to-last paragraph of the proof, we can not conclude that there exists an odd number $2j + 1$ such that $(2j + 1) \cdot a$ is $O(\epsilon)$-close to $\mathbb{Z}^d$. The best we can do is to conclude that $(2j + 1)N \cdot a$ is $O(\epsilon)$-close to $\mathbb{Z}^d$, but if $N$ is even then $(2j + 1)N$ is even and the argument stops working because we need $(2j + 1)N$ to be odd to conclude that $L_n = L_0$ for some odd number $n$, which is crucial to the proof.
6 Some open questions

We conclude this paper with some interesting open questions in which the results of this paper can be applied to continue the research of the structure of the tiling set of a $k$-tiler.

Conjecture 6.1. Prove or disprove that if a convex polytope $P$ $k$-tiles $\mathbb{R}^d$, then $P$ can $m$-tile $\mathbb{R}^d$ with a lattice for some $m$.

This conjecture had been asked in [6]. This conjecture is an analogue to McMullen’s result [10], that all 1-tilers in $\mathbb{R}^d$ can 1-tile with a lattice. [12] gave a positive answer in dimension 2.

Conjecture 6.2. Suppose that a polytope $P$ $k$-tiles $\mathbb{R}^d$ with a discrete multiset $\Lambda$, and all elements of $\Lambda$ are contained in a finite union of translated lattices $Q$. Prove or disprove that $P$ $m$-tiles $\mathbb{R}^d$ with a lattice $L$ for some $m$.

Conjecture 6.2 is a generalization of Theorem 1.1 by removing the hypothesis that $L$ is a lattice in $Q$ that is in a general position to $Q$. For the particular case when all elements of $\Lambda$ are contained in the union of two translates of one lattice, Theorem 1.3 gives a positive answer. A weaker form of this conjecture would be to prove the claim for the case when all elements of $\Lambda$ are contained in a finite union of translates of one lattice, in which we have Lemma 4.1 as a good starting point.

For dimension 2 and 3, Conjecture 6.1 is a corollary of Conjecture 6.2 if the latter is true, as [8] and [5] showed that all $k$-tilers in dimension 2 and 3 can $k$-tile with a finite union of translated lattices.

Conjecture 6.3. Suppose that $P$ $k$-tiles $\mathbb{R}^d$ with a discrete multiset $\Lambda$, and all elements of $\Lambda$ are contained in a finite union of translated lattices $Q$. Prove or disprove that other than one exceptional class of polytopes, the set $Q$ satisfies Hypothesis 1 in Theorem 1.1.

For dimension 2 we conjecture that the exceptional class is parallelograms, and for dimension 3 case we conjecture that the exceptional class is two-flat zonotopes. This conjecture is motivated by results of [8] and [5] that polytopes in dimension 2 and 3 not from the exceptional class must $k$-tile with a finite union of translated lattices. So far we have not been able to construct a counter-example that is not a parallelogram and two-flat zonotopes in dimension 2 and 3 respectively (the polytope in Example 2 is a parallelogram). We also do no have any idea what the exceptional class should be for dimension 4 and above.

Conjecture 6.3, if proven to be true, will give at least a partial answer to Conjecture 6.2 as by Theorem 1.1 it means that other than one exceptional class of polytopes, all polytopes that can $k$-tile with a finite union of translated lattices will also be able to $m$-tile with a lattice.

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Appendix A  Nonconvergence of $s_i(z)$

Let $\Lambda_i - a_i$ denote an arbitrary subset of $\mathbb{Z}^d_{\geq 0}$, we define the function $s_i(z)$ as

$$s_i(z) := \sum_{x \in \Lambda_i - a_i} z^x \cdot \prod_{j=1}^{d} (1 - z_i),$$
with $z^x$ is the multivariable polynomial $z_1^{x_1} \ldots z_d^{x_d}$ for $x = (x_1, \ldots, x_d)$ an integer vector. We also set $|z_j| < 1$ in order for the sum to converge to a well-defined value.

Here we show an example of $\Lambda_i - a_i$ that causes $s_i(z)$ not to have a limit when $z$ goes to $(1, 1, \ldots, 1)^{-}$. Let $d = 1$ and $\Lambda_i - a_i$ to be equal to the set $\mathcal{Z}$, which is defined as

$$\mathcal{Z} = \{ x \in \mathbb{Z} \mid \exp^2(2u) \leq x < \exp^2(2u + 1), \ u \in \mathbb{Z}_{\geq 0} \},$$

with $\exp^2(n)$ is an iterated exponent $2^n$. We will show that $s_i(z)$ does not have a well-defined limit when $z$ goes to $1^-$. With the set $\Lambda_i - a_i$ as defined above, the function $s_i(z)$ is equal to:

$$s_i(z) = \sum_{u=0}^{\infty} z^{\exp^2(2u)} - z^{\exp^2(2u+1)}.$$ \hfill (16)

We denote the integer $z_n := (0.9)^{\exp^2(-\exp^2(n))}$, and we will show that $\lim_{n \to \infty} s_i(z_{2n})$ converges to 0.9 while $\lim_{n \to \infty} s_i(z_{2n+1})$ converges to 0.1, which shows that $\lim_{n \to \infty} s_i(z_{n})$ does not exist. Let $[x]$ denotes the integer part of a real number $x$, note that the sum of first $[\frac{n}{2}]$ terms of the sum $s_i(z_{n})$ is equal to:

$$\sum_{u=0}^{[\frac{n}{2}]-1} z_n^u = \sum_{u=0}^{[\frac{n}{2}]-1} (0.9)^{\exp^2(2u)-\exp^2(n)} - (0.9)^{\exp^2(2u+1)-\exp^2(n)}$$

$$< \sum_{u=0}^{[\frac{n}{2}]-1} 1 - (0.9)^{\exp^2(2u+1)-\exp^2(n)}$$

$$= \left[ \frac{n}{2} \right] \cdot (1 - (0.9)^{\exp^2(n-1)-\exp^2(1)})$$

$$= \left[ \frac{n}{2} \right] \cdot (1 - (0.9)^{\exp^2(n-1)})$$

$$< \left[ \frac{n}{2} \right] \cdot 0.1 \cdot \left[ \sum_{u=0}^{\exp^2(n-1)-1} 0.9^{u} \exp^2(-\exp^2(n-1)) \right]^{-1}$$

$$< \frac{\left[ \frac{n}{2} \right] \cdot 0.1}{0.9 \cdot \exp^2(n-1)},$$ \hfill (17)

while the sum of the rest of the terms of $s_i(z_{2n})$ excluding the $[\frac{n}{2}]$-th term is equal to:

$$\sum_{u=[\frac{n}{2}]+1}^{\infty} z_n^u = \sum_{u=[\frac{n}{2}]+1}^{\infty} (0.9)^{\exp^2(2u)-\exp^2(n)} - (0.9)^{\exp^2(2u+1)-\exp^2(n)}$$

$$< (0.9)^{\exp^2(2[n/2]+2)-\exp^2(n)} \cdot \sum_{u=0}^{\infty} (0.9)^u$$

$$< (0.9)^{\exp^2(n+1)-\exp^2(n)} \cdot 0.1 \cdot (0.9)^{\exp^2(n)} \cdot 0.1$$ \hfill (18)

Note that as $n$ goes to infinity, the right hand side of Equation (17) and Equation (18) goes to 0. This, combined with the fact that all terms in the sum $s_i(z)$ are always positive, means that the limit of the sum in (17) and Equation (18) converges to 0 as $n$ goes to infinity. Hence
the limit of $s_i(z_n)$ as $n$ goes to infinity is equal to the limit of the $\left\lfloor \frac{n}{2} \right\rfloor$-th term of the sum $s_i(z)$, i.e. :

$$\lim_{n \to \infty} s_i(z_n) = \lim_{n \to \infty} (0.9)^{\exp_2(\exp_2(2\left\lfloor \frac{n}{2} \right\rfloor) - \exp_2(n))} - (0.9)^{\exp_2(\exp_2(2\left\lfloor \frac{n}{2} \right\rfloor + 1) - \exp_2(n))}.$$  \hfill (19)

When $n$ is an even number, the right hand side of Equation 19 becomes $0.9 - (0.9)^{\exp_2(n)}$, which converges to 0.9 as $n$ goes to infinity. On the other hand, when $n$ is an odd number, the right hand side of Equation 19 becomes $0.9^{\exp_2(-\exp_2(n-1))} - 0.9$, which converges to 0.1 as $n$ goes to infinity. This shows that $\lim_{n \to \infty} s_i(z_n)$ does not exist, which completes our argument.

For an example of $\Lambda_i - a_i$ in general dimensions $d$ that causes $s_i(z)$ not to have a limit as $z$ goes to $(1, \ldots, 1)^{-}$, one can take $\Lambda_i - a_i$ to be equal to $Z^d$, and using the same argument above one can show that the limit of $s_i(z)$ oscillates between $(0.1)^d$ and $(0.9)^d$. 