LOCAL-ENTIRE CYCLIC COCYCLES FOR GRADED QUANTUM FIELD NETS

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Abstract. In a recent paper we studied general properties of super-KMS functionals on graded quantum dynamical systems coming from graded translation-covariant quantum field nets over $\mathbb{R}$, and we carried out a detailed analysis of these objects on certain models of superconformal nets. In the present article we show that these locally bounded functionals give rise to local-entire cyclic cocycles (generalized JLO cocycles) which are homotopy-invariant for a suitable class of perturbations of the dynamical system. Thus we can associate meaningful noncommutative geometric invariants to those graded quantum dynamical systems.

1. Introduction

KMS states on C*-algebras play a crucial role in quantum statistical mechanics and operator algebras, providing among other things a meaningful abstraction of thermodynamical equilibrium states on quantum systems, cf. [22, 2]. They have been considered in the framework of algebraic quantum field theory in several places like [5, 15] and more recently in [7, 8], and that is also the context we are interested in here.

Algebraic quantum field theory has been developed as an operator algebraic mathematically rigorous approach to quantum field theory using nets of operator algebras [15]. One of its many important aspects is supersymmetry, an internal symmetry between bosons and fermions, i.e., even and odd elements of the algebra, respectively. Although physical experimental confirmation is lacking so far, its mathematical structure is very rich and extends to the general context of C*-algebras, in which we are going to work here. A famous application of supersymmetry is found in Connes’s concept of spectral triples [14]. Given such a spectral triple with $\theta$-summability conditions, one constructs in a natural way a “super-Gibbs functional”, i.e., a supersymmetric or graded version of the usual Gibbs states (special cases of KMS states) from statistical mechanics. This super-Gibbs functional then gives rise to an entire cyclic cocycle, cf. also [13, 17]; it turns out to be a noncommutative geometric invariant for certain “regular perturbations” of the spectral triple and its corresponding dynamics. This construction is the starting point for a noncommutative geometric description of graded-local conformal nets of quantum fields over the circle $S^1$, as achieved in [10]. The cocycles there give rise to geometric invariants, which, in particular, recover parts of the representation theory of the graded-local conformal net, as partly already suggested in [21, 20, 12, 11].

It seems natural to ask whether this construction of entire cyclic cocycles can be generalized from spectral triples and super-Gibbs functionals to more general dynamics (in particular translations), supersymmetry and functionals, and whether it still gives rise to noncommutative geometric invariants and thermodynamical interpretations. In fact, Jaffe, Lesniewski and Wisniowski [15] and independently Kastler [19] took the first steps into that direction. They introduced a graded version of the KMS condition for states; we call it the super-KMS condition and the functionals satisfying this condition we call super-KMS functionals. They showed that bounded super-KMS functionals give rise to entire cyclic cocycles. Unfortunately, at that time several no-go theorems were still unknown, in particular that nontrivial

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super-KMS functionals for translation-covariant quantum field nets cannot be bounded \[6\], which turned their constructions out to be inapplicable here. It took several years to construct at least one first example of algebraic supersymmetry with unbounded but locally bounded super-KMS functionals and associated local-entire cyclic cocycle for the supersymmetric free field \[4\], which, however, still had to be put into the framework of graded nets of von Neumann algebras. In the preceding paper \[16\], we took precisely that step: having studied a few general aspects of super-KMS functionals, we carried out a detailed analysis of super-KMS functionals for the supersymmetric free field net and subsequently also for some other models. The meaning of those functionals relates to supersymmetric dynamics and phase transitions as briefly outlined there. Yet we have not treated relations to entire cyclic cohomology \[18\] so far, which was actually one of the initial motivations.

Our question is thus: Do super-KMS functionals for graded translation-covariant nets give rise to entire cyclic cocycles and geometric invariants of the net, generalizing \[10\]?

To this end, we start by presenting a general construction of local-entire cyclic cocycles out of local-exponentially bounded super-KMS functionals for graded translation-covariant nets over \(\mathbb{R}\), which is mainly due to \[4\]; in the main part we then show that these cocycles are nontrivial and form geometric invariants for “regular perturbations” of our dynamical system. As a problem this has been pointed out in \[4\, Sec.7\] for the specific model treated in Example 4.7 and more vaguely already in \[21\, Sec.6&7\]. A subsequent deeper investigation of the involved geometric invariants, probably related to index and K-theory and possibly recovering parts of the representation theory as recently achieved in \[10\] would be a natural task for future. We should stress that the setting in \[10\] is different and analytically substantially easier as we have a \(\theta\)-summable spectral triple and a (bounded) super-Gibbs functional (with respect to rotations) available.

After a short section on preliminaries as found in more detail in \[16\, Sec.2\], we provide the cocycle construction in Section 3, and discuss perturbations, homotopy-invariance, and a brief example in the final and main Section 4.

In the present paper, we deal with quantum field nets over \(\mathbb{R}\), whose physical meaning is that of a chiral component over a light-ray in two-dimensional Minkowski spacetime. One might, however, in some cases regard them also as restrictions to one light-ray of certain special nets over higher-dimensional manifolds. Moreover, such nets are deeply related to the concept of filtration in quantum probability, and a connection point and applications to that area are quite likely, yet beyond the scope of the present article. Thus we expect the ideas found here to be of interest in a much wider context.

### 2. Notation and preliminaries on super-KMS functionals

Let \(\mathcal{I}\) stand for the set of nontrivial bounded open intervals in \(\mathbb{R}\). For every interval \(I \in \mathcal{I}\) we write \(|I| := \sup\{|x| : x \in I\}|.\)

A **graded translation-covariant net** \(\mathcal{A}\) over \(\mathbb{R}\) is a map \(I \mapsto \mathcal{A}(I)\) from the set \(\mathcal{I}\) to the set of von Neumann algebras on a common infinite-dimensional separable Hilbert space \(\mathcal{H}\) satisfying the following properties:

- **Isotony.** \(\mathcal{A}(I_1) \subset \mathcal{A}(I_2)\) if \(I_1, I_2 \in \mathcal{I}\) and \(I_1 \subset I_2\).

- **Grading.** There is a fixed selfadjoint unitary \(\Gamma \neq 1\) (the grading unitary) on \(\mathcal{H}\) satisfying \(\Gamma \mathcal{A}(I) \Gamma = \mathcal{A}(I)\) for all \(I \in \mathcal{I}\). We write \(\gamma = \text{Ad} \Gamma\) and define the usual **graded commutator**

\[
[x, y] = xy + \frac{1}{4}(y - \gamma(y))(x - \gamma(x)) - \frac{1}{4}(y + \gamma(y))(x - \gamma(x))
- \frac{1}{4}(y - \gamma(y))(x + \gamma(x)) - \frac{1}{4}(y + \gamma(y))(x + \gamma(x)), \quad x, y \in \mathcal{A}(I).
\]

- **Translation-covariance.** There is a strongly continuous unitary representation on \(\mathcal{H}\) of the translation group \(\mathbb{R}\) with infinitesimal generator \(P\), commuting with \(\Gamma\), and
such that
\[ e^{it\mathcal{A}(I)} e^{-it\mathcal{A}} = \mathcal{A}(t + I), \quad t \in \mathbb{R}, I \in \mathcal{I}, \]
and the corresponding point-strongly continuous one-parameter automorphism group \((\alpha_t)_{t \in \mathbb{R}}\) (i.e., \(t \mapsto \alpha_t(x)\) is \(\sigma\)-weakly continuous, for every \(x \in \mathcal{A}(I)\)) restricts to *-isomorphisms from \(\mathcal{A}(I)\) to \(\mathcal{A}(t + I)\), for every \(t \in \mathbb{R}\) and \(I \in \mathcal{I}\), and is asymptotically graded-abelian:
\[
\lim_{t \to \infty} [x, \alpha_t(y)] = 0, \quad x, y \in \mathcal{A}(I), I \in \mathcal{I}.
\]
- **Positivity of the energy.** \(P\) is positive.

The universal or quasi-local \(C^*\)-algebra corresponding to a (graded) net \(\mathcal{A}\) over \(\mathbb{R}\) is defined as the \(C^*\)-direct limit
\[
\mathfrak{A} := \lim_{\rightarrow} \mathcal{A}(I)
\]
over \(I \in \mathcal{I}\), cf. also [2, 6, 15], noting that \(\mathcal{I}\) is directed, and its norm will be simply denoted by \(\| \cdot \|\). For all \(I \in \mathcal{I}\), \(\mathcal{A}(I)\) is naturally identified with a subalgebra of \(\mathfrak{A}\). Throughout this paper we use Gothic letters for the quasi-local \(C^*\)-algebra of the net with corresponding calligraphic letter. We write \(\alpha\) again for the induced group of automorphisms of \(\mathfrak{A}\).

Let \(\mathcal{A}\) be a graded translation-covariant net. A **superderivation** on \(\mathfrak{A}\) with respect to the grading \(\gamma\) and translation group \(\alpha\) is a linear map \(\delta : \text{dom}(\delta) \subset \mathfrak{A} \to \mathfrak{A}\) such that:

(i) \(\text{dom}(\delta) \subset \mathfrak{A}\) is an \(\alpha\-\gamma\)-invariant (i.e., globally invariant under the action of every \(\alpha_t\), \(t \in \mathbb{R}\), as well as \(\gamma\)) unital \(*\)-subalgebra, with
\[
\alpha_t \circ \delta(x) = \delta \circ \alpha_t(x), \quad \gamma \circ \delta(x) = -\delta \circ \gamma(x), \quad \delta(x^*) = \gamma(\delta(x)^*), \quad x \in \text{dom}(\delta), t \in \mathbb{R};
\]
(ii) \(\delta(xy) = \delta(x)y + \gamma(x)\delta(y),\) for all \(x, y \in \text{dom}(\delta)\),
(iii) \(\delta I := \delta |_{\text{dom}(\delta) \cap \mathcal{A}(I)}\) is a \((\sigma\)-weakly)-(\(\gamma\)-weakly) closed \(\sigma\)-weakly densely defined map with image in \(\mathcal{A}(I)\),
(iv) \(C^\infty(\delta I) := \bigcap_{n \in \mathbb{N}} \text{dom}(\delta^n) \subset \text{dom}(\delta_0) \cap \mathcal{A}(I) \subset \mathcal{A}(I)\) is \(\sigma\)-weakly dense,

By \(\text{dom}(\cdot)\), we always mean \(\text{dom}(\cdot) \cap \mathcal{A}(I)\) and thus \(\text{dom}(\delta I) = \text{dom}(\delta)_I\); \(\text{dom}(\cdot)_c\) stands for the union over \(I \in \mathcal{I}\) of \(\text{dom}(\cdot)_I\), which in some cases may actually be equal to \(\text{dom}(\cdot)\); in particular, \(C^\infty(\delta)_c = \bigcup_{I \in \mathcal{I}} \bigcap_{n \in \mathbb{N}} \text{dom}(\delta^n)_I\). We then call \((\mathfrak{A}, \gamma, (\alpha_t)_{t \in \mathbb{R}, \delta})\) a graded quantum dynamical system. We shall be interested in modifications of the usual KMS condition on \((\mathfrak{A}, \alpha)\), and we consider only the case of inverse temperature \(\beta = 1\); this can always be achieved by rescaling if \(\beta \neq 0, \infty\).

All \(*\)-algebras in this paper are understood to be unital with unit \(1\) and all Hilbert spaces separable.

Given \(t \in \mathbb{R}\), write
\[
\Delta^t_n := \{ s \in \mathbb{R}^n : 0 \leq \text{sgn}(t)s_1 \leq \ldots \leq \text{sgn}(t)s_n \leq |t| \};
\]
\[
\Delta_n := \Delta^1_n \text{ and the tube } \quad T^n := \{ s \in \mathbb{C}^n : \Im(s) \in \Delta_n \}.
\]
Notice that \(T^n\) is the standard closed strip in the complex plane.

Super-KMS functionals are some of the central objects of this paper and we choose the following definition, which was motivated by the corresponding ones in [4, 6] but is actually stronger and more suitable for the theory and examples developed in this paper and in [10].

**Definition 2.1.** A super-KMS functional (in short \(s\text{KMS functional}\)) \(\phi\) on a graded quantum dynamical system \((\mathfrak{A}, \gamma, (\alpha_t)_{t \in \mathbb{R}, \delta})\) for a given translation-covariant net \(\mathcal{A}\) is a linear functional defined on a \(*\)-subalgebra \(\text{dom}(\phi) \subset \mathfrak{A}\) such that:

(S0) Domain properties: \(\phi(x^*) = \bar{\phi(x)}\), for all \(x \in \text{dom}(\phi)\); \(\text{dom}(\phi)_I \subset \mathcal{A}(I)\) is \(\sigma\)-weakly dense, for all \(I \in \mathcal{I}\), and \(\text{dom}(\phi)\) is globally \(\alpha\-\gamma\)-invariant.
(S1) Local normality: \(\phi_I := \phi |_{\text{dom}(\phi)_I}\) is bounded and extends to a normal (i.e., \(\sigma\)-weakly continuous) linear functional on \(\mathcal{A}(I)\), denoted again \(\phi_I\), for all \(I \in \mathcal{I}\).
Definition 2.1 for a generic graded quantum dynamical system $s$. Henceforth, let permits to associate a (local-)entire cyclic cocycle again, although involving certain analytical.

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Some $s$KMS functionals exhibit the following additional property:

Let $\phi$, $\rho$, $\delta$.

Theorem 2.2. Let $(A, \gamma, \alpha)$ be a graded translation-covariant net and $(\phi, \text{dom}(\phi))$ a functional on $(A, \gamma, \alpha)$ satisfying $(S_0)$-$(S_3)$. Then the following holds:

1. $\phi$ is translation and grading invariant, i.e.,

$$\phi \circ \alpha_t = \phi = \phi \circ \gamma, \quad t \in \mathbb{R}.$$ 

2. $\phi$ is neither positive nor bounded.

3. The functionals $|\phi|_t$ and $\phi_t^\pm := \frac{1}{2}(|\phi| \pm \phi_t)$ obtained through restriction are individually well-defined, bounded and positive, but they do not form a directed system with respect to restriction, so they do not give rise to positive (unbounded) functionals on $A$.

The original motivation for sKMS functionals comes from the studies of supersymmetric dynamics and “phase transitions” between bosons and fermions. This is discussed within the context of the free field model in [16, Sec.3], together with a conditional uniqueness and existence proof. Our main application of sKMS functionals here lies in local-entire cyclic cohomology as carried out in the next section.

3. JLO cocycles for super-KMS functionals

Given an sKMS functional for a graded quantum dynamical system, Jaffe, Lesniewski and Wisniowski have found a natural way to associate an entire cyclic cocycle [18], generalizing their famous construction of JLO cocycles for super-Gibbs functionals with supercharges [17]. Unfortunately, [18] works only for bounded sKMS functionals (e.g. as is the case with the rotation group in [16, Sec.4] for nets over $S^1$), which for the translation group under usual assumptions do not exist according to Theorem [2.2]. Buchholz and Grundling showed, however, that in a special model a local-exponential bound similar to $(S_6)$ is satisfied, which permits to associate a (local-)entire cyclic cocycle again, although involving certain analytical difficulties [4, Sec.6]. Their result and proof actually generalize to sKMS functionals as in Definition 2.1 for a generic graded quantum dynamical systems. Henceforth, let $A$ be a generic graded translation-covariant net $A$ on $\mathbb{R}$ and $((A, \| \cdot \|), \gamma, (\alpha_t)_{t \in \mathbb{R}}, \delta)$ a corresponding generic graded quantum dynamical system.

Given a normed algebra $(A, \| \cdot \|')$, we recall that the induced norm of an $n$-linear functional $\rho_n$ on $A$ is

$$\|\rho_n\|' = \sup_{x_i \in A} \frac{|\rho_n(x_0, \ldots, x_n)|}{\|x_0\|' \cdots \|x_n\|'}.$$
Definition 3.1. A local-entire cochain on a *-subalgebra $A \subseteq \mathfrak{A}$ is given by a sequence $(\rho_n)_{n \in \mathbb{N}_0}$ of $(n+1)$-linear maps $\rho_n$ on $A$ with $\rho_n(x_0, \ldots, x_n) = 0$ if $x_i \in \mathbb{C}1$ for some $i = 1, \ldots, n$, such that

$$\lim_{n \to \infty} n^{1/2} \|\rho_n [A \cap A(I)] \|^{1/n} = 0, \quad I \in \mathcal{I}.$$ 

The even cochains are those with $\rho_{2n+1} = 0$ and the odd cochains those with $\rho_{2n} = 0$, for all $n \in \mathbb{N}_0$. A local-entire cochain is a local-entire cyclic cocycle if $\partial \rho = 0$, where $\partial := B + b$ maps even into odd local-entire chains and v.v., and

$$(bp)_n(x_0, \ldots, x_n) := \sum_{j=0}^{n-1} (-1)^j \rho_{n-1}(x_0, \ldots, x_j, x_{j+1}, \ldots, x_n) + (-1)^n \rho_{n-1}(x_nx_0, x_1, \ldots, x_{n-1})$$

$$(B\rho)_n(x_0, \ldots, x_n) := \sum_{j=0}^{n} (-1)^{nj} \rho_{n+1}(1, x_j, \ldots, x_{j-1}), \quad x_i \in A.$$ 

We consider here for $A$ in the above definition the *-algebra $C^\infty(\delta)^\gamma_0$ (the fixed points of $C^\infty(\delta)$, under $\gamma$) equipped with the graph norm $\|\|_\ast := \|\cdot\| + ||\delta(\cdot)||$.

Theorem 3.2. Given a local-exponentially bounded sKMS functional $\phi$ for the graded quantum dynamical system $(\mathfrak{A}, \gamma, (\alpha_t)_{t \in \mathbb{R}}, \delta)$, the expression

$$\tau_n(x_0, \ldots, x_n) := \text{anal. cont}_{\gamma \to 1} \int_{\Delta_n} \phi(x_0a_{s_1}(\delta(x_1))) \cdots a_{s_n}(\delta(x_n))) \, d^n s, \quad x_i \in C^\infty(\delta)^\gamma_0,$$

for even $n \in \mathbb{N}_0$, and $\tau_n = 0$, for odd $n \in \mathbb{N}_0$, is welldefined, and gives rise to an even local-entire cyclic cocycle $(\tau_n)_{n \in \mathbb{N}_0}$ on $C^\infty(\delta)^\gamma_0$, called the JLO cocycle.

Proof. The proof is basically given in [3, Th.6.3&6.4], but for the reader’s convenience and since some aspects of our setting are slightly different and required again in the following section, we include it here with the corresponding adjustments. The first part deals with local-entireness, the second with the algebraic cocycle condition.

(Part 1.) Given $n \in \mathbb{N}_0$ and $x_i \in \text{dom}(\phi)_c$, $i = 0, \ldots, n$, as in the assumptions, there is $I \in \mathcal{I}$ such that all $x_i$ lie in $\text{dom}(\phi)_I$. We fix such an $I$ and write $A := \text{dom}(\phi)_I$ in the first part of the proof. Then, for every $t_i \in \mathbb{R}$, we have $\alpha_{t_i}(x_i) \in \mathcal{A}(I + t_i)$ and $||\alpha_{t_i}(x_i)\|| = ||x_i\||$ by the local *-property of $\alpha_t$; property (S0) implies then

$$\left|\phi(\alpha_{t_0}(x_0) \cdots \alpha_{t_n}(x_n))\right| \leq C_1 e^{C_2(\|I\| + \sup_i |t_i|)^2} \|x_0\| \cdots \|x_n\|.$$ 

We would like to find an upper bound for this in terms of analytic functions. One can easily check that $(|I| + \sup_i |t_i|)^2 \leq |I|^2 + 2|I| + (1 + 2|I|) \sum_{i=1}^{n} t_i^2$, so

$$\left|\phi(\alpha_{t_0}(x_0) \cdots \alpha_{t_n}(x_n))\right| \leq B_1 \exp\left(B_2 \sum_{i=1}^{n} t_i^2\right) \|x_0\| \cdots \|x_n\|$$

with constants $B_1 = C_1 e^{C_2(|I|^2 + 2|I|)}$ and $B_2 = C_2(1 + 2|I|)$. We are interested in an analytic continuation of this as a function of $t$. For all $s_i \in \mathbb{R}$, define:

$$F_{x_0, \ldots, x_n}(s_1, \ldots, s_n) := \exp\left(-B_2 \sum_{k=1}^{n} (s_1 + \cdots + s_k)^2\right) \cdot \phi\left(x_0a_{s_1}(x_1) \cdots a_{s_1+\cdots+s_n}(x_n)\right).$$

Consequently, $|F_{x_0, \ldots, x_n}(s_1, \ldots, s_n)| \leq B_1 \|x_0\| \cdots \|x_n\|$, for all $s \in \mathbb{R}^n$, and by the sKMS property (S2) of $\phi$, the function $F_{x_0, \ldots, x_n}$ can be analytically continued in each variable $s_j$ to the strip $T^1$. This way, we obtain functions $F_{x_0, \ldots, x_n}^{(j)}$ which are analytic in the flat tubes $T^{(j)} := \mathbb{R}^2 \times T^1 \times \mathbb{R}^{n-j}$, for all $j = 1, \ldots, n$; using the flat tube theorem [3, Lem.A.2] inductively, we thus obtain a unique analytic continuation of $F_{x_0, \ldots, x_n}$ into the tube $T := \{z \in \mathbb{C}^n : 0 \leq 3z_j \leq 1, \ j = 1, \ldots, n; \sum_{j=1}^{n} 3z_j \leq 1\}$ (the convex hull of $\bigcup_{j=1,\ldots,n} T^{(j)}$) coinciding with each $F_{x_0, \ldots, x_n}^{(j)}$ on $T^{(j)}$. For several purposes like the entireness condition
we need a bound for this analytic function $F_{x_0,...,x_n}$ on $T$, and we start by finding bounds for $F_{x_0,...,x_n}$. Let $G_{x_0,...,x_n}(s_1,\ldots,s_n) := \phi\left(x_0\alpha_{s_1}(x_1)\cdots\alpha_{s_1+\cdots+s_n}(x_n)\right)$ which has just been shown to have a unique analytic continuation to each $T^{(j)}$, and by the growth condition in (S2) we know that $|G_{s_1,...,s_n}(s_1,\ldots,s_j + i r_j,\ldots,s_n)| \leq C_0 (1 + |r_j|)^{p_0}$ with certain scalars $C_0 \in \mathbb{R}_+$ and $p_0 \in \mathbb{N}$ independent of $s_j$ and $r_j \in [0,1]$. Then by the above definition

\[ F_{x_0,...,x_n}(s_1,\ldots,s_j + i r_j,\ldots,s_n) = G_{x_0,...,x_n}(s_1,\ldots,s_j + i r_j,\ldots,s_n) \times \exp\left(B_2 r_j^2(n + 1 - j) - B_2 \sum_{k=1}^{n} s_k^2 + i \theta_{s,r}\right), \]

for some $\theta_{s,r} \in \mathbb{R}$. From the above polynomial growth property of $G_{x_0,...,x_n} in s_j$ we conclude that $F_{x_0,...,x_n}$ is bounded. Thus according to the Phragmen-Lindelöf theorem, cf. [2 Prop.5.3.5], the bound of the analytic function $F_{x_0,...,x_n}$ is attained on the boundary of $T^{(j)}$. On the real part of the boundary of $T^{(j)}$ we have computed above: $|F_{x_0,...,x_n}(s_1,\ldots,s_n)| \leq B_1 \|x_0\| \cdots \|x_n\|$. By the sKMS property (S2) and translation invariance of $\phi$ we have on the other part $\mathbb{R}^{j-1} \times (i+\mathbb{R}) \times \mathbb{R}^{n-j}$:

\[ |G_{x_0,...,x_n}(s_1,\ldots,s_j + i,\ldots,s_n)| = |\phi\left(\alpha_{s_1+\cdots,s_j}(x)\cdots\alpha_{s_1+\cdots+s_n}(x)\phi_0\alpha_{s_1}(x_1)\cdots\alpha_{s_1+\cdots+s_n-1}(x_{j-1})\right)| \leq B_1 \exp\left(B_2 \sum_{k=1}^{n} (s_1 + \cdots + s_k)^2 \cdot \|x_0\| \cdots \|x_n\|\right), \]

hence by (3.1):

\[ |F_{x_0,...,x_n}(s_1,\ldots,z_j,\ldots,s_n)| \leq B_1 e^{B_2 n} \|x_0\| \cdots \|x_n\|, \quad z_j \in T. \]

We have to show that this bound holds on all $T$. To this end let $C := B_1 e^{B_2 n} \|x_0\| \cdots \|x_n\|$ and define, for every $\alpha \in [0,2\pi]$:

\[ f_\alpha(z_1,\ldots,z_n) := (F_{x_0,...,x_n}(z_1,\ldots,z_n) - e^{i \alpha} C)^{-1}, \]

for all $z \in T$. Since $z_j \mapsto f_\alpha(s_1,\ldots,z_j,\ldots,s_n)$ is analytic on the strip $T$, for every $j = 1,\ldots,n$, the flat tube theorem [2 Lem. A.2] again implies that $f_\alpha$ has a unique analytic continuation to $T$, and hence cannot have any singularities in $T$, i.e., $F_{x_0,...,x_n}(s_1,\ldots,z_n) \neq e^{i \alpha} C$ for all $\alpha$. Since $F_{x_0,...,x_n}$ is continuous, its image set $F_{x_0,...,x_n}(T)$ must be connected, and since it has some points inside the circle of radius $C$, the entire image set is inside that circle, so

\[ |F_{x_0,...,x_n}(z_1,\ldots,z_n)| \leq B_1 e^{B_2 n} \|x_0\| \cdots \|x_n\|, \quad z \in T, \quad x \in A. \]

We finally perform a change of variables and summarize the above as follows:

For all $I \in \mathcal{I}$ and $x_i \in \text{dom}(\phi)_I$, the function

\[ s \in \mathbb{R}^n \mapsto \phi(x_0\alpha_{s_1}(x_1)\cdots\alpha_{s_n}(x_n)) \]

has a unique analytic continuation to the tube $T^n$, for which we write shortly

\[ z \in T^n \mapsto \phi(x_0\alpha_{z_1}(x_1)\cdots\alpha_{z_n}(x_n)), \]

although $\alpha_{z_i}(x_i)$ itself makes no sense for complex $z_i$. The continuation is bounded by

\[ |\phi(x_0\alpha_{z_1}(x_1)\cdots\alpha_{z_n}(x_n))| \leq C_1 e^{2C_2(|I|+1)^2(n+1)} e^{2C_2(|I|+1)^2 \sum_{k=1}^{n} |z_k|^2} \|x_0\| \cdots \|x_n\|, \]

for all $z \in T^n$.

With the special choice $(z_1,\ldots,z_n) = i(r_1,\ldots,r_n) \in i \Delta_n$, we arrive at

\[ |\tau_n(x_0,\ldots,x_n)| \leq \frac{C_1}{n^2} e^{4C_2(|I|+1)^2/2(n+1)} \|x_0\| \cdots \|x_n\| \]

since the volume of $\Delta_n$ is $1/n!$. Thus

\[ n^{1/2} \|\tau_n\|^{1/n} \leq n^{1/2} (C_1/n!)^{1/n} e^{4C_2(|I|+1)^2/2(n+1)/n} \sim n^{-1/2} C_1 e^{4C_2(|I|+1)^2} \to 0, \quad n \to \infty, \]
which concludes the proof of local-entireness.

(Part 2.) In order to prove the algebraic cocycle condition, we first claim that

$$\phi\left(\delta(x_0)\alpha_{i_1}(\delta(x_1)) \cdots \alpha_{i_n}(\delta(x_n))\right) = \sum_{j=1}^{n} \frac{\partial}{\partial x_j} \phi\left(\left(\gamma(x_0)\alpha_{i_1}(\gamma\delta(x_1)) \cdots \alpha_{i_j}(\gamma\delta(x_j)) \cdots \alpha_{i_{s_j-1}}(\gamma\delta(x_{j-1}))\alpha_{i_j}(x_j)\alpha_{i_{s_j+1}}(\delta(x_{j+1})) \cdots \alpha_{i_n}(\delta(x_n))\right)\right)$$

for $x_i \in C^\infty(\delta)_I$ and $s \in \Delta_n$.

Proof of the claim. Recall that the superderivation property (ii) implies

$$\delta(x_0) \cdots \delta(x_n) = \delta(x_0\delta(x_1) \cdots \delta(x_n)) - \sum_{j=1}^{n} \gamma(x_0\delta(x_1) \cdots \delta(x_{j-1}))\delta^2(x_j)\delta(x_{j+1}) \cdots \delta(x_n),$$

and that $\alpha_{t_j}$ commutes with $\delta$. Combining this first with property $(S_4)$ and then with $(S_5)$, we find

$$\phi\left(\delta(x_0)\alpha_{t_1}(\delta(x_1)) \cdots \alpha_{t_n}(\delta(x_n))\right)$$

$$= - \sum_{j=1}^{n} \phi\left(\gamma(x_0)\alpha_{t_1}(\gamma\delta(x_1)) \cdots \alpha_{t_{j-1}}(\gamma\delta(x_{j-1}))\alpha_{t_j}(\delta^2(x_j))\alpha_{t_{j+1}}(\delta(x_{j+1})) \cdots \alpha_{t_n}(\delta(x_n))\right)$$

$$= \sum_{j=1}^{n} i \frac{\partial}{\partial x_j} \phi\left(\left(\gamma(x_0)\alpha_{t_1}(\gamma\delta(x_1)) \cdots \alpha_{t_{j-1}}(\gamma\delta(x_{j-1}))\alpha_{t_j}(x_j)\alpha_{t_{j+1}}(\delta(x_{j+1})) \cdots \alpha_{t_n}(\delta(x_n))\right)\right).$$

As in (3.2), the two functions

$$t \in \mathbb{R}^d \mapsto \phi\left(\delta(x_0)\alpha_{t_1}(\delta(x_1)) \cdots \alpha_{t_n}(\delta(x_n))\right),$$

$$t \in \mathbb{R}^d \mapsto \sum_{j=1}^{n} i \frac{\partial}{\partial x_j} \phi\left(\left(\gamma(x_0)\alpha_{t_1}(\gamma\delta(x_1)) \cdots \alpha_{t_{j-1}}(\gamma\delta(x_{j-1}))\alpha_{t_j}(x_j)\alpha_{t_{j+1}}(\delta(x_{j+1})) \cdots \alpha_{t_n}(\delta(x_n))\right)\right)$$

extend uniquely to analytic functions on the tubes $\mathcal{T}^n$ coinciding on $\mathbb{R}^n$. Thus they extend to the same function on $\mathcal{T}^n$, with the same argument as in (3.2). With the special choice $z = s \in i\Delta_n$, we obtain (3.4), thus the claim is proved.

The sKMS condition $(S_2)$ and analyticity property (3.2) together yield the following relations (for a detailed proof cf. [4, Lem.8.5] replacing the algebra denoted there by $\mathcal{A}_0$ with $A = C^\infty(\delta) \cap \mathcal{A}(I)$ in the present setting):

Lemma 3.3. In the above setting and with $x_i \in A$, the following equalities hold:

(i) $\int_{\Delta_n} \phi\left(x_0\alpha_{i_1}(x_1) \cdots \alpha_{i_n}(x_n)\right) d^n s = \int_{\Delta_n} \phi\left(\gamma(x_0)\alpha_{i_1}(x_0) \cdots \alpha_{i_n}(x_{n-1})\right) d^n s.$

(ii) For $j = 2, \ldots, n$ we have:

$$\int_{\Delta_n+1} \frac{\partial}{\partial s_j} \phi\left(x_0\alpha_{i_1}(x_1) \cdots \alpha_{i_{n+1}}(x_{n+1})\right) d^{n+1} s$$

$$= \left[ \frac{\partial}{\partial s_j} \phi\left(x_0\alpha_{i_1}(x_1) \cdots \alpha_{i_{j-1}}(x_{j-1}) \cdots \alpha_{i_n}(x_{n+1})\right) - \phi\left(x_0\alpha_{i_1}(x_1) \cdots \alpha_{i_{j-1}}(x_{j-1}) \cdots \alpha_{i_n}(x_{n+1})\right) \right] d^n s,$$

$$\int_{\Delta_{n+1}} \frac{\partial}{\partial s_1} \phi\left(x_0\alpha_{i_1}(x_1) \cdots \alpha_{i_{n+1}}(x_{n+1})\right) d^{n+1} s$$

$$= \left[ \frac{\partial}{\partial s_1} \phi\left(x_0\alpha_{i_1}(x_1) \cdots \alpha_{i_{n+1}}(x_{n+1})\right) - \phi\left(x_0\alpha_{i_1}(x_1) \cdots \alpha_{i_{n+1}}(x_{n+1})\right) \right] d^n s.$$
Let us continue now with the proof of the cocycle condition, starting with $B\tau_{n+1}$. Recalling that $\tau$ is an even chain, we may restrict to odd $n$. From the definition of $B$ we then have, for $x_i \in A^n$ and using $\delta(1) = 0$:

\[
(B\tau_{n+1})(x_0, \ldots, x_n) = \int_{\Delta_n+1} \sum_{j=0}^{n} (-1)^{nj} \phi \left( 1 \alpha_{i_1 s_1} (\delta(x_j)) \cdots \alpha_{i_n s_n-1} (\delta(x_j)) \alpha_{i_n s_n+1} (\delta(x_j)) \cdots \alpha_{i_n s_n+1} (\delta(x_{j-1})) \right) d^{n+1} s.
\]

Applying Lemma 3.3(i) repetitively and using the fact that $\gamma(\delta(x_j)) = -\delta(x_j)$ for all $j$ and that $(-1)^{nj} = (-1)^j$ for odd $n$, we obtain

\[
(B\tau_{n+1})(x_0, \ldots, x_n) = \int_{\Delta_n} \phi \left( \delta(x_0) \alpha_{i_1 s_1} (\delta(x_1)) \cdots \alpha_{i_s s_{j-1}} (\delta(x_{j-1})) \alpha_{i_s s_j} (\delta(x_j)) \cdots \alpha_{i_s s_n} (\delta(x_n)) \right) d^n s
\]

Now we consider $b\tau_{n-1}$, again with odd $n$ of course. By definition of $b$, we have

\[
(b\tau_{n-1})(x_0, \ldots, x_n) = \int_{\Delta_{n-1}} \left( \sum_{j=0}^{n-1} (-1)^j \phi \left( x_0 \alpha_{i_1 s_1} (\delta(x_1)) \cdots \alpha_{i_s s_{j}} (\delta(x_{j+1})) \cdots \alpha_{i_s s_{n-1}} (\delta(x_n)) \right) - \phi \left( x_n x_0 \alpha_{i_1 s_1} (\delta(x_1)) \cdots \alpha_{i_s s_{n-1}} (\delta(x_{n-1})) \right) \right) d^{n-1} s.
\]

For the contributions with $j = 0, 1$ we find

\[
\int_{\Delta_{n-1}} \phi \left( x_0 x_1 \alpha_{i_1 s_1} (\delta(x_2)) \cdots \alpha_{i_s s_{n-1}} (\delta(x_n)) \right) d^{n-1} s
\]

\[
- \int_{\Delta_{n-1}} \phi \left( x_0 \alpha_{i_1 s_1} (\delta(x_1) x_2) \alpha_{i_2 s_2} (\delta(x_3)) \cdots \alpha_{i_s s_{n-1}} (\delta(x_n)) \right) d^{n-1} s
\]

\[
= - \int_{\Delta_{n-1}} \phi \left( x_0 \alpha_{i_1 s_1} (\delta(x_1) x_2 + x_1 \delta(x_2)) \alpha_{i_2 s_2} (\delta(x_3)) \cdots \alpha_{i_s s_{n-1}} (\delta(x_n)) \right) d^{n-1} s
\]

\[
- \phi \left( x_0 x_1 \alpha_{i_1 s_1} (\delta(x_2)) \alpha_{i_2 s_2} (\delta(x_3)) \cdots \alpha_{i_s s_{n-1}} (\delta(x_n)) \right) d^{n-1} s
\]

\[
= - \int_{\Delta_{n-1}} \phi \left( x_0 \alpha_{i_1 s_1} (\delta(x_1) x_2) \alpha_{i_2 s_2} (\delta(x_3)) \cdots \alpha_{i_s s_{n-1}} (\delta(x_n)) \right) d^{n-1} s
\]

\[
- \int_{\Delta_{n-1}} \frac{\partial}{\partial s_1} \phi \left( x_0 \alpha_{i_1 s_1} (x_1) \alpha_{i_2 s_2} (\delta(x_2)) \alpha_{i_s s_3} (\delta(x_3)) \cdots \alpha_{i_s s_n} (\delta(x_n)) \right) d^n s
\]
where we made use of (3.6) in the last step. For $2 \leq j \leq n - 1$ we find, using (3.5):

$$(-1)^j \int_{\Delta_{n-1}} \phi(x_0 \alpha_{i_1 s_1} (\delta(x_1)) \cdots \alpha_{i_j s_j} (\delta(x_j)) \cdot \alpha_{i_{n-1} s_{n-1}} (\delta(x_{n-1}))) \, d^{n-1} s$$

$$= (-1)^j \int_{\Delta_{n-1}} \phi(x_0 \alpha_{i_1 s_1} (\delta(x_1)) \cdots \alpha_{i_j s_j} (\delta(x_j)x_{j+1}) + x_j \delta(x_{j+1})) \cdots$$

$$\cdots \alpha_{i_{n-1} s_{n-1}} (\delta(x_{n-1}))) \, d^{n-1} s$$

$$= (-1)^j \int_{\Delta_{n-1}} (\phi(x_0 \alpha_{i_1 s_1} (\delta(x_1)) \cdots \alpha_{i_j s_j} (\delta(x_j)x_{j+1}) \cdots \alpha_{i_{n-1} s_{n-1}} (\delta(x_{n-1})))$$

$$+ \phi(x_0 \alpha_{i_1 s_1} (\delta(x_1)) \cdots \alpha_{i_{j-1} s_{j-1}} (\delta(x_{j-1})x_j) \cdots \alpha_{i_{n-1} s_{n-1}} (\delta(x_{n-1})))) \, d^{n-1} s$$

$$+ (-1)^j \int_{\Delta_{n}} \frac{\partial}{\partial s_j} \phi(x_0 \alpha_{i_1 s_1} (\delta(x_1)) \cdots \alpha_{i_j s_j} (\delta(x_j)) \cdots \alpha_{i_{n-1} s_{n-1}} (\delta(x_{n-1}))) \, d^n s.$$

In these equations, the first and second term on the right-hand side cancel between subsequent summands of the sum (3.3) over $j$. Putting all together, we obtain

$$(b \tau_{n-1})(x_0, \ldots, x_n)$$

$$= (-1)^{n-1} \int_{\Delta_{n-1}} \phi(x_0 \alpha_{i_1 s_1} (\delta(x_1)) \cdots \alpha_{i_{n-1} s_{n-1}} (\delta(x_{n-1}))) \, d^{n-1} s$$

$$+ \sum_{j=1}^{n-1} (-1)^j \int_{\Delta_{n}} \frac{\partial}{\partial s_j} \phi(x_0 \alpha_{i_1 s_1} (\delta(x_1)) \cdots \alpha_{i_j s_j} (\delta(x_j)) \cdots \alpha_{i_{n-1} s_{n-1}} (\delta(x_{n-1}))) \, d^n s$$

$$+ (-1)^n \int_{\Delta_{n-1}} \phi(x_n x_0 \alpha_{i_1 s_1} (\delta(x_1)) \cdots \alpha_{i_{n-1} s_{n-1}} (\delta(x_{n-1}))) \, d^{n-1} s$$

$$= \sum_{j=1}^{n} (-1)^j \int_{\Delta_{n}} \frac{\partial}{\partial s_j} \phi(x_0 \alpha_{i_1 s_1} (\delta(x_1)) \cdots \alpha_{i_j s_j} (\delta(x_j)) \cdots \alpha_{i_{n-1} s_{n-1}} (\delta(x_{n-1}))) \, d^n s$$

$$= - \int_{\Delta_{n}} \sum_{j=1}^{n} \frac{\partial}{\partial s_j} \phi(x_0 \alpha_{i_1 s_1} (\gamma \delta(x_1)) \cdots \alpha_{i_{j-1} s_{j-1}} (\gamma \delta(x_{j-1}))) \alpha_{i_j s_j} (\delta(x_j)) \cdots \alpha_{i_{n-1} s_{n-1}} (\delta(x_{n-1}))) \, d^n s.$$

Finally combining this with (3.4) and (3.7) proves

$$(b \tau_{n-1})(x_0, \ldots, x_n) = - \int_{\Delta_{n}} \phi(\delta(x_0) \alpha_{i_1 s_1} (\delta(x_1)) \cdots \alpha_{i_{n-1} s_{n-1}} (\delta(x_{n-1}))) \, d^n s$$

$$= - (B \tau_{n+1})(x_0, \ldots, x_n), \quad x_i \in A^\gamma,$$

i.e., $(B + b) \tau = 0$. Since this holds for every choice of $I \in \mathcal{I}$, we see that $\tau$ is a local-entire cyclic cocycle on $C^\infty(\delta)^\gamma_c$.

4. Perturbations of super-KMS functionals and homotopy-invariance of their JLO cocycles

Let us study the general situation of a perturbation of a given graded quantum dynamical system $(\mathfrak{A}, \gamma, (\alpha_t)_{t \in \mathbb{R}}, \delta)$ by an odd selfadjoint $Q \in C^\infty(\delta)^c$. Some of the ideas followed here are found in [13], which however works in an analytically different context. Let us start by making our concepts of perturbation more precise:
Proposition 4.1. Let $\phi$ be a local-exponentially bounded sKMS functional for a graded quantum dynamical system $(\mathcal{A}, \gamma, (\alpha_t)_{t \in \mathbb{R}})$. For every $I \in \mathcal{I}$ and odd selfadjoint $Q \in C^\infty(\delta)_I$ and $r \in [0, 1]$, let $\delta_r := \delta + r[Q, \cdot]$ and $\alpha_r := r\delta(Q) + r^2Q^2 \in C^\infty(\delta)_I$, so $\delta_r^2 = \delta^2 + \text{ad} \alpha_r$. Define formally

$$\alpha_t^r(x) := \sum_{n \in \mathbb{N}_0} (it)^n \int_{\Delta_n} \text{ad}(\alpha_{s_1}(a_r)) \cdots \text{ad}(\alpha_{s_n}(a_r))(\alpha_t(x)) \, d^n s,$$

and

$$\gamma_t^r(x) := \sum_{n \in \mathbb{N}_0} (it)^n \int_{\Delta_n} \alpha_{s_1}(a_r) \cdots \alpha_{s_n}(a_r) \alpha_t(x) \, d^n s, \quad x \in \text{dom}(\phi), t \in \mathbb{R}.$$

Then the sums converge and define one-parameter groups, which commute with $\gamma$ and are continuous in the following sense:

$$-i \frac{d}{dt} \phi(x \alpha_t^r(y)z) \big|_{t=0} = \phi(x(\delta^2 + \text{ad} \alpha_r)(y)z), \quad -i \frac{d}{dt} \phi(x \gamma_t^r(y)z) \big|_{t=0} = \phi(x(\delta^2 + \alpha_r)(y)z),$$

for every $x, z \in \text{dom}(\phi)$ and $y \in C^\infty(\delta)$. Moreover, $\alpha_t^r$ are $^*$-automorphisms, and we have the following behavior under perturbation: there are constants $C_1, C_2 > 0$ such that

$$\|\alpha_t^r(x) - \alpha_t^r(y)\|, \|\gamma_t^r(x) - \gamma_t^r(y)\| \leq 2|q - r| \|\delta(Q)\| + \|Q^2\| |t| e^{2\|\delta(Q)\| + \|Q^2\|} \|x\|,$$

for every $t \in \mathbb{R}$, $x \in \text{dom}(\phi)$, and $q, r \in [0, 1]$.

Proof. We consider only $\alpha_t^r$ since $\gamma_t^r$ can be treated analogously. Since $\|\alpha_s(a_r)\| = \|\alpha_r\|$ is uniformly bounded in $r \in [0, 1]$ and $s \in \mathbb{R}$, we see that each summand is bounded by $\frac{1}{n!} |t|^n \|2\alpha_r\| \|x\|$, so the sum converges. In case of $\alpha_t^r$, it follows moreover from the $^*$-property of $\alpha_t$ and selfadjointness of $\alpha_r$ that $\alpha_t^r$ has the $^*$-property.

We want to check the group property of $\alpha^r$:

$$\alpha_{t_1}^r \alpha_{t_2}^r(x) = \sum_{n_1 \in \mathbb{N}_0} \sum_{n_2 \in \mathbb{N}_0} (it_1)^{n_1} (it_2)^{n_2} \int_{\Delta_{n_1}} \int_{\Delta_{n_2}} \text{ad}(\alpha_{t_1s_1}(a_r)) \cdots \text{ad}(\alpha_{t_1s_{n_1}}(a_r)) \alpha_{t_1t_2}(x) \, d^{n_2} s \, d^{n_1} s$$

$$= \sum_{n \in \mathbb{N}_0} \sum_{n_1=0}^n (it_1)^n (it_2)^{n_1} \int_{\Delta_{n_1}} \int_{\Delta_{n-n_1}} \text{ad}(\alpha_{t_1s_1}(a_r)) \cdots \text{ad}(\alpha_{t_1s_{n_1}}(a_r)) \alpha_{t_1t_2}(x) \, d^{n-n_1} s \, d^{n_1} s$$

$$= \sum_{n \in \mathbb{N}_0} (it_1 + t_2)^n \int_{\Delta_n} \text{ad}(\alpha_{t_1+t_2s_1}(a_r)) \cdots \text{ad}(\alpha_{t_1+t_2s_n}(a_r)) \alpha_{t_1+t_2}(x) \, d^n s \quad (\ast)$$

$$= \alpha_{t_1+t_2}^r(x).$$

If $\text{sgn} \ t_1 = \text{sgn} \ t_2$, the one but last line $\ast$ is obvious. To cover the general case, it suffices then to show $\ast$ for $t_2 = -t_1 < 0$ (or analogously $t_2 = -t_1 > 0$) since all other cases can be reduced to a composition of the latter one and the case $\text{sgn} \ t_1 = \text{sgn} \ t_2$. To this end, notice that $(t_1/t_2)^{n_1} = (-1)^{n_1}$ is alternating, leading to cancellation of mutually consecutive terms with fixed $n$; this leaves us only with the term $n = 0$. Thus in particular, $\alpha_t^r \alpha_t^r(x) = \alpha_0(x) = \alpha_0^r(x)$, for every $t \in \mathbb{R}$, proving the group property for $(\alpha_t^r)_{t \in \mathbb{R}}$.

Concerning continuity in $t$, we recall that both $\alpha_t$ and $y$ lie in $C^\infty(\delta)_I$, by assumption. Thus term-by-term differentiation of the series

$$\phi(x \alpha_t^r(y)z) = \sum_{n \in \mathbb{N}_0} i^n \int_{\Delta_n} \phi(x \text{ad}(\alpha_{s_1}(a_r)) \cdots \text{ad}(\alpha_{s_n}(a_r))(\alpha_t(y))z) \, d^n s$$
in \( t = 0 \) yields a convergent series again with nonzero contribution only for the zeroth and first summand, namely

\[
- i \frac{d}{dt} \phi(x \alpha_t^2(y) z) \big|_{t=0} = - i \frac{d}{dt} \phi(x \alpha_t(y) z) \big|_{t=0} - i \frac{d}{dt} \int_0^t \phi(x \text{ad}(\alpha_t(a_r)) (\alpha_t(y)) z) \, ds \big|_{t=0}
\]

\[
= \phi(x \delta_t^2(y) z) + \phi(x (\text{ad}(\alpha_t)(y)) z) = \phi(x \delta_t^2(y) z),
\]

making use of the weak supersymmetry property \((S_3)\) of \(((\alpha_t))_{t \in \mathbb{R}, \delta}\). We call \( \delta_t^2 = \delta^2 + \text{ad}(\alpha_t) \) the \( \delta\)-weak generator of \( \alpha^r \).

Now let us turn to the difference \( \alpha_t^r(x) - \alpha_t^q(x) \).

Remembering \( a_r \in C^\infty(\delta)_c \) and \((3.3)\), we have for the \( n \)-th term in the sum the following upper bound:

\[
|t|^n \left| \int_{\Delta_n} \text{ad}(\alpha_{ts_1}(a_r)) \cdots \text{ad}(\alpha_{ts_n}(a_r))(x) - \text{ad}(\alpha_{ts_1}(a_q)) \cdots \text{ad}(\alpha_{ts_n}(a_q)) \alpha_t(x) \, d^n s \right|
\]

\[
\leq |t|^n \int_{\Delta_n} \left| \sum_{k=1}^n x \text{ad}(\alpha_{ts_1}(a_r)) \cdots \text{ad}(\alpha_{ts_k}(a_r)) \text{ad}(\alpha_{ts_{k+1}}(a_r)) \cdots \text{ad}(\alpha_{ts_n}(a_q)) \alpha_t(x) \right| \, d^n s
\]

\[
- \text{ad}(\alpha_{ts_1}(a_r)) \cdots \text{ad}(\alpha_{ts_{k-1}}(a_r)) \text{ad}(\alpha_{ts_k}(a_q)) \cdots \text{ad}(\alpha_{ts_n}(a_q)) \alpha_t(x) \right| \, d^n s
\]

\[
\leq \frac{|t|^n}{n!} \sum_{k=1}^n \|a_r\|^{k-1} \|a_r - a_q\| \|a_q\|^{n-k} \|x\|
\]

\[
\leq \frac{|t|^n}{(n-1)!} [(\|a_r\|^{n-1} + \|a_q\|^{n-1}) \|a_r - a_q\| \|x\|].
\]

Summing over \( n \) and using the power series expansion of the exponential function and \( \|a_r - a_q\| \leq 2|r - q| (\|Q\| + \|Q^2\|) \), we obtain the stated upper bound.

Finally, we have to check that every \( \alpha_t^r \) (but not \( \gamma_t^r \)) is multiplicative. This will follow immediately from multiplicity of \( \alpha_t \) and the subsequent Lemma \((4.3)(2)\&(3)\), which does not make use of multiplicity, namely

\[
\alpha_t^r(xy) = \gamma_t^r(1) \alpha_t(x) \alpha_t(y) \gamma_t^r(1)^* = \gamma_t^r(1) \alpha_t(x) \gamma_t^r(1)^* \gamma_t^r(1) \alpha_t(y) \gamma_t^r(1)^* = \alpha_t^r(x) \alpha_t^r(y),
\]

for all \( x, y \in \text{dom}(\phi)_c \). We conclude that \( \alpha_t^r \) is a \(*\)-automorphism.

\[
\square
\]

**Definition 4.2.** Given a graded quantum dynamical system \((\mathfrak{A}, \gamma, (\alpha_t))_{t \in \mathbb{R}, \delta}\) and an odd selfadjoint element \( Q \in C^\infty(\delta)_c \), let

\[
\delta_r := \delta + r[Q, _-], \quad r \in [0, 1].
\]

Then \((\mathfrak{A}, \gamma, (\alpha_t^r))_{t \in \mathbb{R}, \delta_r}\), for every \( r \in [0, 1] \), is called a perturbed graded quantum dynamical system for \((\mathfrak{A}, \gamma, (\alpha_t))_{t \in \mathbb{R}, \delta}\). If \( \phi \) is an sKMS functional for the original system, then the corresponding perturbed functional is given by

\[
\phi^r(x) := \phi(x \gamma_t^r(1)), \quad x \in \text{dom}(\phi^r) := \text{dom}(\phi)_c.
\]

Note first that \( t \in \mathbb{R} \mapsto \gamma_t^r(1) \) need not be analytically continuable, but the above expression is just a sloppy notation for the analytic continuation of the composed function \( t \mapsto \phi(x \gamma^r_t(1)) \), which will be proved in Proposition \((4.3)\) to be well-defined. Second, for \( r \neq 0 \), \( \alpha^r \) loses its geometric interpretation: in general,

\[
\alpha_t^r(\mathcal{A}(I)) \not\subset \mathcal{A}(t+I), \quad t \in \mathbb{R}, I \in \mathcal{I}.
\]

Only for \( I \) sufficiently large and \( t \) small such that \( Q \in \mathcal{A}(I_0) \) and \( I_0 \subset I \cap (t+I) \), this inclusion still holds.

**Lemma 4.3.** In the above setting, we have the following equalities:

1. \( \gamma_t^r(x) = \gamma_t^r(1) \alpha_t \gamma_{t-x}^r(x) \), for all \( s, t \in \mathbb{R} \) and \( x \in \text{dom}(\phi)_c \).
2. \( \gamma_t^r(1)^* = \alpha_t(\gamma_{t-t}^r(1)) \) and \( \gamma_t^r(1)^* \gamma_t^r(1)^* = \gamma_t^r(1)^* \gamma_t^r(1) = 1 \), for all \( t \in \mathbb{R} \).
\( (3) \ \alpha_t^x(x) = \gamma_t^x(\textbf{1})\alpha_t(x)\gamma_t^x(\textbf{1})^*, \) for all \( t \in \mathbb{R} \) and \( x \in \text{dom}(\phi). \)

\( (4) \ \alpha_t^x(y) = \gamma_t^x(xy), \) for all \( t \in \mathbb{R} \) and \( x, y \in \text{dom}(\phi). \)

**Proof.** (1) follows from the one-parameter group property and the definition of \( \gamma_t^x: \)

\[ \gamma_t^x(x) = \gamma_t^x(\gamma_t^{-s}(1)) = \gamma_t^x(1)\alpha_s(\gamma_t^{-s}(x)). \]

(2) The first statement is obvious from the definition of \( \gamma_t^x \) and the self-adjointness of \( \alpha_t, \) checked summand-wise. Combining it with statement (1), we get

\[ \gamma_t^x(1)\gamma_t^x(1)^* = \gamma_t^x(1)\alpha_t(\gamma_t^{-t}(1)) = \alpha_t(\gamma_t^x(1)) = 1. \]

(3) Considering the defining sum of \( \gamma_t^x(1)\alpha_t(x)\gamma_t^x(1)^* \), we have to compute

\[ \sum_{n=0}^{\infty} \sum_{m=0}^{n} (it)^n(-1)^m \int_{\Delta} \int_{\Delta} \alpha_{sx}(a_r) \cdots \alpha_{sx-n}(a_r) \alpha_t(x) \alpha_{y}(a_r) \cdots \alpha_{y}(a_r) d^n s \ d^m q. \]

Now we need a bit of combinatoric thinking. Notice that, for fixed \( m \) and every \( s \in \Delta_{n-m}, q \in \Delta_m, j \in \{1, \ldots, m\} \) there is \( k_j \in \{0, \ldots, n-m\} \) such that \( s_{k_j} \leq q_j \leq s_{k_{j-1}} \), and inserting the components of \( q \) between those of \( s \) according to this ordering, we obtain an element \( u \in \Delta_n \). Varying \( s \) and \( q \) with fixed such \( k = (k_1, \ldots, k_m) \) produces all \( u \in \Delta_n \). Varying \( k \) produces another copy of \( \Delta_n \) and there are exactly \( \binom{n}{m} \) ways (indexed by the tuples \( k \)). The integral in the above sum over the summand with given \( n \) and \( m \) and varying \( k \) corresponds precisely to all the \( \binom{n}{m} \) summands obtained by writing out \( \text{ad}(\alpha_t)(z) = yz - zy \) in

\[ (it)^n \int_{\Delta} \text{ad}(\alpha_t(a_r)) \cdots \text{ad}(\alpha_t(a_r))(\alpha_t(x)) d^n s, \]

with \( m \) terms \( \alpha_{a_r}(a_r) \) on the right and \( n-m \) on the left of \( \alpha_t(x) \). Summing then over \( m \) and \( n \) concludes the proof.

(4) is now a direct consequence of (1), (2) and (3):

\[ \alpha_t^x(x)\gamma_t^x(y) = \gamma_t^x(1)\alpha_t(x)\gamma_t^x(1)^* \gamma_t^x(1)\alpha_t(y) = \gamma_t^x(1)\alpha_t(xy) = \gamma_t^x(xy). \]

\[ \square \]

**Lemma 4.4.** In the above setting, let \( \phi \) be an sKMS functional for \( (\mathfrak{A}, \gamma, (\alpha_t)_{t \in \mathbb{R}}, \delta). \)

(1) The function \( t \in \mathbb{R}^{n+1} \mapsto \phi(\alpha_t(x_1) \cdots \alpha_{t_n}(x_n)\alpha_{t_{n+1}}(x_0)) \) has a unique analytic continuation to \( T^{n+1} \) and, for every \( x_i \in \text{dom}(\phi) \) and \( z \in T^n \), we have

\[ \phi(\alpha_{z_1}(x_1) \cdots \alpha_{z_n}(x_n)\alpha_t(0)) = \phi(x_0\alpha_{z_1}(\gamma(x_1)) \cdots \alpha_{z_n}(\gamma(x_n))). \]

(2) For all \( x_i \in \text{dom}(\phi), \) the function

\[ z \in T^n \mapsto \phi(\alpha_{z_1}(x_1) \cdots \alpha_{z_n}(x_n)) \]

is analytic and we have

\[ \phi(\alpha_{z_n}(x_1) \cdots \alpha_{z_1}(x_1)) = \phi(\alpha_{z_1}(x_1) \cdots \alpha_{z_n}(x_n)). \]

**Proof.** (1) The unique analytic continuation has been obtained in [17]. Keeping the first \( n \) variables real, the sKMS property implies

\[ \phi(\alpha_{t_1}(x_1) \cdots \alpha_{t_n}(x_n)\alpha_t(0)) = \phi(x_0\gamma(\alpha_{t_1}(x_1) \cdots \alpha_{t_n}(x_n))) = \phi(x_0\alpha_{t_1}(\gamma(x_1)) \cdots \alpha_{t_n}(\gamma(x_n))), \]

for all \( t \in \mathbb{R}^n \). From the uniqueness of the analytic continuation to \( T^n \) we obtain the statement.

(2) Let

\[ G_{x_1, \ldots, x_n}(t_1, \ldots, t_n) := \phi(\alpha_{t_1}(x_1)\alpha_{t_2}(x_2) \cdots \alpha_{t_n}(x_n)), \]

where \( x_1, \ldots, x_n \) are real numbers and \( t_1, \ldots, t_n \) are inside the period of \( T^n \).
for \( t \in \mathbb{R}^n \). It has a unique analytic continuation to \( T^n \) according to (3.2). Moreover, \( \alpha \) invariance of \( \phi \) shows that \( G_{x_1, \ldots, x_n}(t_1, \ldots, t_n) = G_{x_1, \ldots, x_n}(t_1 + t, t_2 + t, \ldots, t_n + t) \), for all \( t \in \mathbb{R} \), hence it actually extends uniquely to an analytic function on \( T_1^{n-1} = \{ z \in \mathbb{C}^n : \exists z_j \leq \exists z_j+1, j = 1, \ldots, n-1, \exists z_n - \exists z_1 \leq 1 \} \). Notice that \((\bar{z}_n, \ldots, \bar{z}_1) \in T_1^{n-1} \) if \( z \in T_1^{n-1} \).

Since \( \phi(x^n) = \phi(x) \), we have
\[
G_{x_1, \ldots, x_n}(t_1, \ldots, t_n) = G_{x_{n-1}, \ldots, x_1}(t_n, \ldots, t_1) = G_{x_{n-1}, \ldots, x_1}(t_n, \ldots, t_1),
\]
for all \( t \in \mathbb{R}^n \). Since the left-hand side has a unique analytic continuation to \( T_1^{n-1} \), so must the right-hand side, namely
\[
G_{\bar{x}_1, \ldots, \bar{x}_n}(\bar{z}_1, \ldots, \bar{z}_n) = G_{\bar{x}_{n-1}, \ldots, \bar{x}_1}(\bar{z}_n, \ldots, \bar{z}_1) = G_{\bar{x}_{n-1}, \ldots, \bar{x}_1}(\bar{z}_n, \ldots, \bar{z}_1).
\]

With these tools at hand, let us study perturbations of sKMS functionals.

**Proposition 4.5.** Suppose \( \phi \) is a local-exponentially bounded sKMS functional for a graded quantum dynamical system \((\mathcal{A}, \gamma, (\alpha_t)_{t \in \mathbb{R}}, \delta)\) and let \( Q \in \mathcal{C}^\infty(\delta)_c \) be an odd selfadjoint perturbation. For every \( r \in [0, 1] \), the corresponding perturbed functional \((\phi^r, \text{dom}(\phi^r))\) is a well-defined sKMS functional with respect to the perturbed system \((\mathcal{A}, \gamma, (\alpha_t^r)_{t \in \mathbb{R}}, \delta_r)\) in Definition 4.3, but in general not satisfying the bounds in (S2) and (S6), nor local normality.

**Proof.** Notice first that, for every \( x, y, z \in \text{dom}(\phi)_c \), the function
\[
(t, u) \in \mathbb{R}^2 \mapsto \phi(x \gamma_t^r(y) \alpha_u(z)) = \left( \int_{\Delta_n} (t^n) \phi(xa_{s_1}(a_r) \cdots a_{s_n}(a_r) \alpha_t(y) \alpha_u(z)) d^n s \right)
\]
has a unique analytic continuation to the tube \( T^2 = \{(t, u) \in \mathbb{C}^2 : 0 \leq \Im(t) \leq \Im(u) \leq 1 \} \) which is analytic on the interior of \( T^2 \). This can be seen as follows: Each of the summands on the right-hand side has a unique analytic continuation to \( T^2 \), which is essentially a consequence of (3.2); moreover those continuations are bounded as in (3.3). Thus the sum of those continuations converges compactly and defines an analytic continuation of \( t \mapsto \phi(x \gamma_t^r(y) \alpha_u(z)) \) to \( T^2 \) according to Weierstrass’ convergence criterion. In the same way but more generally, we see that, for every \( n \in \mathbb{N} \) and \( x_i \in \text{dom}(\phi)_c \),
\[
(t_1, \ldots, t_n) \in \mathbb{R}^n \mapsto \phi(x_0 \gamma_{t_1}^r(x_1) \alpha_{t_1}(\gamma_{t_2-t_1}^r(x_1)) \cdots \alpha_{t_{n-1}}(\gamma_{t_n-t_{n-1}}^r(x_n)))
\]
has a unique analytic continuation to \( T^n \).

Let us keep on record an explicit local bound for the analytic continuation. Let \( I \) be large enough so that \( Q \in \text{dom}(\phi)_I \). Then, for every \( r \in [0, 1] \) and \( x \in \text{dom}(\phi)_I \), we find
\[
|\phi(x \gamma_t^r(1))| \leq \sum_{n=0}^{\infty} \left| \int_{\Delta_n} \phi(xa_{s_1}(a_r) \cdots a_{s_n}(a_r)) d^n s \right| \leq C_I e^{4C_2(1+|I|)^2(n+1)} \|a_r\| \|x\| \leq C_I \|x\|. \tag{4.2}
\]
with \( C_I := C_I \exp(4C_2(1+|I|)^2 + \|a_r\| e^{4C_2(1+|I|)^2}) > 0 \) using (3.3). Thus we have local boundedness of \( \phi^r \) for sufficiently large interval \( I \), hence for all intervals (by isotony). We expect, however, neither local-exponential boundedness nor local normality for \( \phi^r \), so only a weaker but for our purposes sufficient version of (S1) and (S6).

We have to check the sKMS property (S2), and we may do this summand-wise again owing to the above reasoning. Given \( x, z \in \text{dom}(\phi)_c \) and applying Lemma 4.3(4), we see that
\[
\phi(x \alpha_t^r(z) \gamma_1^r(1)) = \phi(x \gamma_t^r(z) \alpha_t^r(\gamma_1^r(1)))
\]
has a unique analytic continuation in \((t, u)\) to \( T^2 \) according to (4.1). Then
\[
F_{x, z}(t) := \phi(x \alpha_t^r(z) \gamma_1^r(1)), \quad t \in \mathbb{R},
\]
has a unique analytic continuation to \( T^1 \), and
\[
F_{x,t}(t + i) = \phi(x\alpha_{t+1}^r(z)\gamma_t^r(1)) = \phi(x\alpha_t^r(\alpha_t^r(z))\gamma_t^r(1))
\]
\[
= \phi(x\gamma_t^r(1)\alpha_t(\alpha_t^r(z))
\]
\[
= \sum_{n=0}^{\infty} (-1)^n \int_{\Delta_n} \phi(x\alpha_{t,s_1}(a_r)\cdots\alpha_{t,s_n}(a_r)\alpha_t(\alpha_t^r(z))) \, d^n s
\]
(4.3)
\[
= \sum_{n=0}^{\infty} (-1)^n \int_{\Delta_n} \phi(\alpha_t^r(z)\gamma(x)\alpha_{t,s_1}(\gamma(a_r))\cdots\alpha_{t,s_n}(\gamma(a_r))) \, d^n s
\]
\[
= \phi(\alpha_t^r(z)\gamma(x)\gamma_t^r(1)).
\]
Here the second line follows from the analytic continuation of the function
\[
s \mapsto \phi(x\alpha_{t+1}^r(z)\gamma_t^r(1)) = \phi(x\alpha_t^r(\alpha_t^r(z))\gamma_t^r(1)) = \phi(x\alpha_t^r(1)\alpha_t(\alpha_t^r(z)))
\]
applying Lemma 4.4(4); the fourth one follows from Lemma 4.4(1), while the last line is clear
since \( \gamma(a_r) = a_r \). Thus the sKMS property \((S_2)\) (except for the polynomial growth condition)
holds for \( \phi^r \).

Let us check the remaining conditions for sKMS functionals. We may choose \( \text{dom}(\phi^r) := \text{dom}(\phi) \) since \( \gamma_t^r(1) \) is smooth and localized. Then, for all \( x \in \text{dom}(\phi) \),
\[
\phi^r(x^*) = \sum_{n=0}^{\infty} (-1)^n \int_{\Delta_n} \phi(x^*\alpha_{t,s_1}(a_r)\cdots\alpha_{t,s_n}(a_r)) \, d^n s
\]
\[
= \sum_{n=0}^{\infty} (-1)^n \int_{\Delta_n} \phi(x^*\alpha_{t,s_1}(\gamma(a_r))\cdots\alpha_{t,s_n}(\gamma(a_r))) \, d^n s
\]
\[
= \sum_{n=0}^{\infty} (-1)^n \int_{\Delta_n} \phi(\alpha_{t,s_1}(a_r)\cdots\alpha_{t,s_n}(a_r)\alpha_t(x^*)) \, d^n s
\]
\[
= \sum_{n=0}^{\infty} (-1)^n \int_{\Delta_n} \phi(\alpha_{t,s_1}(a_r)\cdots\alpha_{t,s_n}(a_r)) \, d^n s'
\]
\[
= \phi(x_{\gamma_t^r}(1)) = \phi^r(x).
\]
Here the third line follows from Lemma 4.4(1), the fourth one from Lemma 4.4(2), and the
fifth one from (constant) analytic continuation to \( \mathbb{C} \) of the constant function \( t \in \mathbb{R} \mapsto \phi(\alpha_t(y)) \),
i.e., from \( \phi \circ \alpha_{t,1}(y) = \phi(y) \). We conclude with a change of variable \( s'_k = 1 - s_{n+1-k} \). Thus
Hermitianity and all other properties in \((S_0)\) are clear.

The normalization property \( \phi^r(1) = 1 \) will be shown in (4.13) as a corollary of the proof
of Theorem 4.3 which does not make use of \((S_3)\) but instead only of the finiteness of \( \phi^r(1) \).

Concerning \((S_4)\), we have to show \( \phi^r \circ \delta_z(z) = 0 \), for \( z \in C^\infty(\delta_z \circ c) \). We claim that
\[
\phi(ze(t)) = 0, \quad e(t) := \delta(\gamma_t^r(1)) + rQ\gamma_t^r(1) - r\gamma_t^r(1)\alpha_t(Q), \quad (4.4)
\]
for all \( t \in \mathbb{R} \), which implies in particular that \( t \in \mathbb{R} \mapsto \phi(ze(t)) \) has a unique and trivial
analytic continuation to \( \mathbb{C} \). Using then first \( \phi \circ \delta = 0 \) and analytic continuation for the first
term on the right-hand side below and \( \phi \circ \gamma = \gamma \) and Lemma 4.4(1) with a similar reasoning.
as in (4.3) for the second one, we obtain

\[ \phi^r \circ \delta_r(z) = \phi(\delta(z)\gamma^r_1(1)) + r\phi(Qz\gamma^r_1(1)) - r\phi(\gamma(z)Q\gamma^r_1(1)) \]

\[ = \phi(-\gamma(z)\delta(\gamma^r_1(1))) + r\phi(\gamma(z)\gamma^r_1(1)\alpha_t(Q)) - r\phi(\gamma(z)Q\gamma^r_1(1)) \]

\[ = -\phi(\gamma(z)e(i)) = 0. \]

**Proof of the claim.** According to (4.1), \( E : t \in \mathbb{R} \mapsto \phi(ze(t)) \) extends to an analytic function on \( T^1 \); moreover, it is differentiable on \( \mathbb{R} \) as follows from the definition of \( \gamma^r_1 \) in Proposition 4.4. Differentiation by \( t \) together with properties (S1) and (S3) for \( \phi \) yields

\[ -i \frac{d}{dt} E(t) = \phi(z(\delta^2 + a_r)(\gamma^r_1(1)) - z(\delta^2 + a_r)(\gamma^r_1(1))\alpha_t(rQ)) \]

\[ = \phi(\delta^2(ze(t))) - \phi(\delta^2(z)e(t)) + \phi(za_re(t)) \]

\[ = \phi((\delta^2 + a_r)(z^*e(t)), \quad t \in \mathbb{R}. \]

Recursively one finds \( z_{r,n} := (\delta^2 + a_r)^n(z^*) \in C^\infty(\delta)_c \) such that

\[ (-i)^n \frac{d^n}{dt^n} E(t) = \phi(z_{r,n}e(t)), \quad n \in \mathbb{N}_0, t \in \mathbb{R}. \]

Since all \( z_{r,n}e(t) \) are localized (uniformly for \( t \) in bounded intervals) and \( \phi \) is locally bounded, all derivatives of \( E \) are continuous; in other words, \( E \) is smooth on \( \mathbb{R} \) and furthermore, according to our preceding discussion, analytic on the interior of \( T^1 \). Since \( \epsilon(0) = 0 \), we get for all derivatives: \( E^{(n)}(0) = 0, n \in \mathbb{N}_0 \). Applying the \( C^\infty \)-version of Schwarz’ reflection principle [1], Th.1 shows that \( E \equiv 0 \) on the whole strip \( T^1 \), thus (4.1), which proves the claim.

Finally, (S5) is shown using the above analytic continuation properties together with the expression for the \( \phi \)-weak generator of \( \alpha^r \) in Proposition 4.1

\[ -i \frac{d}{dt} \phi^r(x\alpha^r_t(y)z |_{t=0} = -i \frac{d}{dt} \phi(x\alpha^r_t(y)z\gamma^r_1(1)) |_{t=0} = \phi(x\delta^2(y)z\gamma^r_1(1)) |_{t=0} = \phi^r(x\delta^2(y)z). \]

We are now ready for the main result:

**Theorem 4.6.** Given a local-exponentially bounded sKMS functional \( \phi \) for a graded quantum dynamical system \((\mathfrak{A}, \gamma, (\alpha_t)_{t \in \mathbb{R}}, \delta)\), the even JLO cochain \( \tau \) over \( C^\infty(\delta)_c \) is a local-entire cyclic cocycle. Moreover, it is homotopy-invariant: given an odd selfadjoint perturbation \( Q \in C^\infty(\delta)_c \), the corresponding perturbed functionals \( \phi^r \) for the perturbed system \((\mathfrak{A}, \gamma, (\alpha^r_t)_{t \in \mathbb{R}}, \delta_r)\) in Definition 4.2 give rise to even JLO local-entire cyclic cocycles \( \tau^r \) again, which are mutually cohomologous, for all \( r \in [0, 1] \).

**Proof.** The first statement is just Theorem 3.2.

Let us consider the perturbed functionals, and let \( I \in \mathfrak{I} \) be an arbitrary fixed interval such that \( Q \in C^\infty(\delta)_I \). Then by definition of \( \gamma^r_1 \) and \( \alpha^r_t = \gamma^r_1(1)\alpha_t(\gamma^r_1(1))^* \), we have, for every \( s \in \mathbb{R}^{n+1} \) and \( x_i \in C^\infty(\delta)_c \):

\[ \phi(x_0\alpha^r_{s_1}(x_1)\cdots\alpha^r_{s_n}(x_n)\gamma^r_{s_{n+1}}(1)) \]

\[ = \sum_{k \in \mathbb{N}_0^{n+1}} |k| \int_{\Delta^k_{s_1}} \cdots \int_{\Delta^k_{s_{n+1}}} \phi(x_0 \text{ad}(\alpha_{t_1,1}(a_r)) \cdots \text{ad}(\alpha_{t_{1,k_1}(a_r)}) \text{ad}(\alpha_{s_1}(x_1)) \cdots \]

\[ \cdots \text{ad}(\alpha_{s_n+t_{n+1,k_{n+1}}(a_r)} \cdots \text{ad}(\alpha_{s_n+t_{n+1,k_{n+1}}(a_r)})\alpha_{s_{n+1}}(1)) \text{d}^k_{t_{n+1} \cdots t_1}. \]

Following the argument of (3.2) or (4.1), each of the integrands has a unique analytic continuation to \( T^{|k|} \) (with the usual multi-index notation), and the integrals thus have a continuation
Moreover, according to (3.3) we obtain the corresponding properties of the original system as just done. This way, Lemma 3.3, in the case of the perturbed system, we obtain continuation and upper bound from a precise growth factor in (3.4). For the corresponding JLO cochain \( \tau \), the latter are bounded by

\[
|\phi(x_0 \alpha^r_{s_1}(x_1) \cdots \alpha^r_{s_m}(x_n)) \gamma^r_{n+1}(1)| \leq \sum_{k \in \mathbb{N}^{n+1}_0} C_1 e^{2C_2(1+|I|)^2(\|k\|+1)} e^{C_2(1+|I|)\sum_{i=1}^{n+1} k_i |z_i|^2} \frac{1}{k!} |z|^k \|2a_r\| |\bar{k} : \|x_0\| \cdots \|x_n\|.
\]

Thus (4.5) is the sum of analytically continuable functions and the sum of the continuations is compactly convergent and hence analytic. In fact, we have for \( z \in \mathcal{T}^{n+1} \):

\[
\sum_{k \in \mathbb{N}^{n+1}_0} C_1 e^{2C_2(1+|I|)^2(\|k\|+1)} e^{C_2(1+|I|)\sum_{i=1}^{n+1} k_i |z_i|^2} \frac{1}{k!} |z|^k \|2a_r\| |\bar{k} : \|x_0\| \cdots \|x_n\|.
\]

Integration over \( i \Delta_n \) then gives rise to the well-defined

\[
F^r_n(x_0, ..., x_n) := \int_{\Delta_n} \phi^r(x_0 \alpha^r_{s_1}(x_1) \cdots \alpha^r_{s_m}(x_n)) d^n s, \quad x_i \in C^\infty(\delta)_c,
\]

and for the corresponding JLO cocycle \( \tau^r \) we therefore find, for every \( I \in \mathcal{I} \):

\[
\sqrt{n} ||\tau^r||_{C^\infty(\delta)_c} ||s||^{1/n} \leq \sqrt{n} \left( \frac{1}{n} C_1 e^{2C_2(1+|I|)^2} \exp \left( (n+1) e^{4C_2(1+|I|)^2} \|2a_r\| \right) \right)^{1/n}
\]

\[
\sim \frac{1}{\sqrt{n}} \exp \left( e^{4C_2(1+|I|)^2} \|2a_r\| \right), \quad n \to \infty,
\]

which converges to 0 for \( n \to \infty \), so \( \tau^r \) is in fact local-entire. The cyclic cocycle condition is purely algebraic and literally goes like (Part 2) of the proof of Theorem 3.2 based on the fact that \( \phi^r \) satisfies the sKMS condition for the perturbed system \((\mathfrak{A}^r, \gamma^r, (\alpha^r_s)_{s \in \mathcal{B}}, \delta_r)\). The precise growth factor in (S2), which is different for the perturbed functional, does not play a role here; it is needed in order to obtain the analytic continuations (3.2) and the bound (3.3); in the case of the perturbed system, we obtain continuation and upper bound from the corresponding properties of the original system as just done. This way, Lemma 3.3 reformulated for arbitrary \( r \in [0,1] \), implies the following equalities for \( x_i \in C^\infty(\delta)_c \) and \( k = 1, ..., n-1 \):

\[
F^r_n(x_0, ..., x_n) = F^r_n(\gamma(x_n), x_0, ..., x_{n-1})
\]

\[
F^r_n(x_0, x_1, ..., \delta^r_r(x_k), ..., x_n) = F^r_{n-1}(x_0, ..., x_{k-1}x_{k+1}, ..., x_n) - F^r_{n-1}(x_0, ..., x_kx_{k+1}, ..., x_n)
\]

\[
F^r_n(x_0, x_1, ..., x_{n-1}, \delta^r_r(x_n)) = F^r_{n-1}(x_0, ..., x_{n-1}x_n) - F^r_{n-1}(\gamma(x_n)x_0, x_1, ..., x_{n-1})
\]

\[
\sum_{j=0}^{n} F^r_{n+1}(1, x_j, ..., x_n; \gamma(x_0), ..., \gamma(x_{j-1})) = F^r_n(x_0, ..., x_n).
\]

Moreover, together with (3.4) we obtain

\[
\sum_{j=0}^{n} F^r_n(\gamma(x_0), ..., \gamma(x_{j-1}), \delta_r(x_j), x_{j+1}, ..., x_n) = 0.
\]

Concerning perturbation invariance of the cyclic cocycle \( \tau \), we would like to show that

\[
G^r_{n-1}(x_0, ..., x_{n-1}) := \sum_{k=0}^{n-1} (-1)^k F^r_n(x_0, \delta_r(x_1), ..., \delta_r(x_k), Q, ..., \delta_r(x_{n-1})), \quad x_i \in C^\infty(\delta)_c,
\]
for even and \( G^r_{n-1} = 0 \) for odd \( n \in \mathbb{N} \), defines a local-entire cochain on \( C^\infty(\delta^c)^r_{c} = C^\infty(\delta^r)^c \) such that
\[
\frac{d}{d\tau} \tau^r = \partial G^r. \tag{4.12}
\]

This would imply that, for every \( q, r \in [0, 1] \), the cochains \( \tau^q \) and \( \tau^r \) differ by a coboundary, i.e., are cohomologous.

We first notice that the cochain \( (G^r_n)_{n \in \mathbb{N}_0} \) above is clearly well-defined. The local-entireness condition is verified in the same way as for the JLO cochain \( (\tau^r_n)_{n \in \mathbb{N}_0} \) above, which becomes clear when writing \( \tau^r \) in terms of \( F^r_n \), with \( n \in \mathbb{N}_0 \).

In order to prove (4.12), we have to calculate \( \partial G^r \). Applying (4.9) to the definition of the operator \( B \) in Definition 3.1, we obtain, for \( x_i \in C^\infty(\delta^c)^r \), thus \( \gamma(x_i) = x_i \), \( \gamma(\delta_r(x_i)) = -\delta_r(x_i) \) and \( \gamma(Q) = -Q \) and \( n \in 2\mathbb{N}_0 \):

\[
BG^r_{n+1}(x_0, \ldots, x_n) = \sum_{j=0}^n G^r_{n+1}(1, x_j, \ldots, x_{j-1})
\]
\[
= \sum_{j=0}^n \left( \sum_{k=0}^{j-1} (-1)^{k+2-j} F^r_{n+1}(1, \delta_r(x_j), \ldots, \delta_r(x_k), Q, \ldots \delta_r(x_{j-1})) \right)
\]
\[
+ \sum_{k=j}^n (-1)^{k+1-j} F^r_{n+1}(1, \delta_r(x_j), \ldots, \delta_r(x_k), Q, \ldots \delta_r(x_{j-1})) \right)
\]
\[
= \sum_{k=0}^n (-1)^{k+1} \left( \sum_{j=0}^k (-1)^j F^r_{n+1}(1, \delta_r(x_j), \ldots, \delta_r(x_k), Q, \ldots \delta_r(x_{j-1})) \right)
\]
\[
+ \sum_{j=k+1}^n (-1)^{j+1} F^r_{n+1}(1, \delta_r(x_j), \ldots, \delta_r(x_k), Q, \ldots \delta_r(x_{j-1})) \right)
\]
\[
= - \sum_{k=0}^n (-1)^k F^r_n(\delta_r(x_0), \ldots, \delta_r(x_k), Q, \ldots \delta_r(x_n)),
\]

using (4.9) in the last line together with the fact that all \( \delta_r(x_i) \) and \( Q \) are homogeneously odd. In the case of \( b \) we find:

\[
bG^r_{n-1}(x_0, \ldots, x_n) = \sum_{j=0}^{n-1} (-1)^j G^r_{n-1}(x_0, \ldots, x_j x_{j+1}, \ldots, x_n) + (-1)^n G^r_{n-1}(x_n x_0, x_1, \ldots, x_{n-1})
\]
\[
= \sum_{j=0}^{n-1} \left( \sum_{k=0}^{j-1} (-1)^{j+k} F^r_n(x_0, \delta_r(x_1), \ldots, \delta_r(x_k), Q, \ldots \delta_r(x_j x_{j+1}), \ldots, \delta_r(x_n)) \right)
\]
\[
+ \sum_{k=j+1}^n (-1)^{j+k-1} F^r_n(x_0, \delta_r(x_1), \ldots, \delta_r(x_j x_{j+1}), \ldots, \delta_r(x_k), Q, \ldots \delta_r(x_n)) \right)
\]
\[
+ \sum_{k=0}^{n-1} (-1)^{n+k} F^r_n(x_n x_0, \delta_r(x_1), \ldots, \delta_r(x_k), Q, \ldots \delta_r(x_{n-1}))
\]
\[
\begin{align*}
&= \sum_{j=0}^{n-1} \left( \sum_{k=0}^{j-1} (-1)^{j+k} F^r_n(x_0, \delta_r(x_1), \ldots, \delta_r(x_k), Q, \ldots, \delta_r(x_j) x_{j+1} + x_j \delta_r(x_{j+1}), \ldots, \delta_r(x_n)) \\
&\quad + \sum_{k=j+1}^{n} (-1)^{j+k-1} F^r_n(x_0, \delta_r(x_1), \ldots, \delta_r(x_j) x_{j+1} + x_j \delta_r(x_{j+1}), \ldots, \delta_r(x_k), Q, \ldots, \delta_r(x_n)) \right) \\
&\quad + \sum_{k=0}^{n-1} (-1)^{n+k} F^r_n(x_n x_0, \delta_r(x_1), \ldots, \delta_r(x_k), Q, \ldots, \delta_r(x_{n-1})) \\
&= \sum_{k=0}^{n} \left( - \sum_{j=0}^{k} (-1)^{j+k-1} F^r_{n+1}(x_0, \delta_r(x_1), \ldots, \delta^2_r(x_j), \ldots, \delta_r(x_k), Q, \ldots, \delta_r(x_n)) \right) \\
&\quad - \sum_{j=k+1}^{n} (-1)^{j+k} F^r_{n+1}(x_0, \delta_r(x_1), \ldots, \delta_r(x_k), Q, \ldots, \delta^2_r(x_j), \ldots, \delta_r(x_n)) \right) \\
&\quad + \sum_{k=1}^{n} F^r_n(x_0, \delta_r(x_1), \ldots, \delta_r(x_k-1), [Q, x_k], \ldots, \delta_r(x_n)) \\
&= \sum_{k=0}^{n} (-1)^{k} F^r_{n+1}(\delta_r(x_0), \delta_r(x_1), \ldots, \delta_r(x_k), Q, \ldots, \delta_r(x_n)) \\
&\quad + \sum_{k=0}^{n} (-1)^{k+k} F^r_{n+1}(0, \delta_r(x_1), \ldots, \delta_r(x_k), \delta_r(Q), \ldots, \delta_r(x_n)) \\
&\quad + \sum_{k=1}^{n} F^r_n(x_0, \delta_r(x_1), \ldots, [Q, x_k], \ldots, \delta_r(x_n)),
\end{align*}
\]

using (4.10) in the last and (4.7), (4.8) in the second but last equality.

Let us turn to the left-hand side of (4.12). From the definition of \( \alpha^r_t \) and \( \gamma^r_t \), we see that the functions \( r \mapsto \phi(x \alpha^r_t(y) z) \) and \( r \mapsto \phi(x \gamma^r_t(1)) \) are differentiable and that

\[
\frac{d}{dr} \phi(x \alpha^r_t(y) z) = \sum_{n \in \mathbb{N}_0} (it)^n \frac{d}{dr} \int_{\Delta_n} \phi(x \text{ad}(\alpha_{\text{tp}_1}(ar)) \cdots \text{ad}(\alpha_{\text{tp}_n}(ar)) \alpha(y) z) \ d^n \rho
\]

\[
= \sum_{n \in \mathbb{N}_0} \sum_{k=0}^{n} (is)^k (it - is)^{n-k} \int_{\Delta_k} \int_{\Delta_{n-k}} \phi(x \text{ad}(\alpha_{sp_1}(ar)) \cdots \text{ad}(\alpha_{sp_k}(ar)) \\
\times \alpha_s(\text{ad} (\delta_r) \text{ad}(\alpha_{(t-s)p_{k+1}}(ar)) \cdots \text{ad}(\alpha_{(t-s)p_n}(ar)) \alpha_{t-s}(y))) z) \ d^{n-k} \ p \ d^k \ d s
\]

\[
= \int_0^t \phi(x \alpha^r_s([\delta^r_r, \alpha^r_{t-s}(y)]) z) \ d s = \int_0^t \phi(x [\alpha^r_s(\delta_r(Q)), \alpha^r_t(y)] z) \ d s,
\]

for all \( t \in [0,1] \).
using \( \hat{\alpha}_r = \delta_r(Q) \). Analogously,

\[
\frac{d}{dr} \phi(x_{\gamma_r}(1)) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} (i \pi)^k (i t - i \pi)^{n-k} \int_0^t \int_{\Delta_{n-k}} \phi(x_{\alpha_{sp_1}(\hat{\alpha}_r) \cdots \alpha_{sp_k}(\hat{\alpha}_r)} \times \alpha_s(\hat{\alpha}_r \alpha_{(t-s)p_{k+1}}(\hat{\alpha}_r) \cdots \alpha_{(t-s)p_n}(\hat{\alpha}_r) \alpha_{(t-s)}(x)) \, d^{n-k} p \, d^{k} p \, d s
\]

\[
= \int_0^t \phi(x_{\gamma_r}(1) \alpha_s(\hat{\alpha}_r) \alpha_s(\gamma_{r-s}(1))) \, d s
\]

\[
= \int_0^t \phi(x_{\gamma_r}(1) \alpha_s(\hat{\alpha}_r) \gamma_s^*(1) \gamma_r^*(1)) \, d s = \int_0^t \phi(x_{\alpha_s(\delta_r(Q))} \gamma_r(1)) \, d s,
\]

using the Lemma 4.3 in the last two steps.

Consider now, for given \( x_0, \ldots, x_n \in C^\infty(\delta_r^n) \), the functions

\[
r \in [0, 1] \mapsto K_{x_0, \ldots, x_n}(r; t_1, \ldots, t_n, u) = \frac{d}{dr} \phi(x_0 \alpha_{t_1}^r(\delta_r(x_1)) \cdots \alpha_{t_n}^r(\delta_r(x_n)) \gamma_u^r(1)), \quad t_i, u \in \mathbb{R}.
\]

Then we have, for \( t \in \Delta_n^u \):

\[
K_{x_0, \ldots, x_n}(r; t_1, \ldots, t_n, u) = \sum_{j=1}^{n} \phi(x_0 \alpha_{t_1}^r(\delta_r(x_1)) \cdots \alpha_{t_j}^r(\delta_r(x_j))) \alpha_{t_{j+1}}^r(\delta_r(x_{j+1})) \gamma_u^r(1)) + \int_{0}^{t_j} \int_{t_{j+1}}^{t} \phi(x_0 \alpha_{t_1}^r(Q, x_1) \cdots \alpha_{t_{j+1}}^r(\delta_r(x_{j+1}) \alpha_{t_{j+2}}^r(\delta_r(Q)) \cdots \alpha_{t_{n+1}}^r(\delta_r(x_{n+1})) \gamma_u^r(1)) \, d s
\]

In the same way as (4.5), this has a unique analytic continuation to the tube \( T^{n+1} \), and we obtain, for \( t \in \Delta_n^u \):

\[
K_{x_0, \ldots, x_n}(r; t_1, \ldots, t_n, i) = \sum_{j=1}^{n} \phi(x_0 \alpha_{t_1}^r(\delta_r(x_1)) \cdots \alpha_{t_{j+1}}^r(\delta_r(x_{j+1}) \gamma_u^r(1)) + \int_{0}^{t_j} \int_{t_{j+1}}^{t} \phi(x_0 \alpha_{t_1}^r(Q, x_1) \cdots \alpha_{t_{j+1}}^r(\delta_r(x_{j+1}) \alpha_{t_{j+2}}^r(\delta_r(Q)) \cdots \alpha_{t_{n+1}}^r(\delta_r(x_{n+1})) \gamma_u^r(1)) \, d s,
\]

where \( t_0 = 0 \) and \( t_{n+1} = 1 \). Together with the definition of \( \tau_r \) we thus have

\[
\frac{d}{dr} \tau_r^r(x_0, \ldots, x_n) = \int_{\Delta_n} K_{x_0, \ldots, x_n}(r; t_1, \ldots, t_n, 1) \, d^n t
\]

\[
= \sum_{k=1}^{n} F^r_{n, \delta_r(x_1), \ldots, [Q, x_k], \ldots, \delta_r(x_n)} + \sum_{k=0}^{n-1} F^r_{n+1, \delta_r(x_1), \ldots, \delta_r(x_k), \delta_r(Q), \ldots, \delta_r(x_n)} = (BG^r_{n+1} + bG^r_{n-1})(x_0, \ldots, x_n) = (\partial G^r)_n(x_0, \ldots, x_n).
\]

As a corollary of the proof we find

\[
\phi^r(1) = \tau^r_0(1) = \tau_0(1) + \int_0^r (\partial G^q)_0(1) \, dq = \tau_0(1) + 0 = \phi(1) = 1, \tag{4.13}
\]

which completes the proof of property \( (S_0) \) in Proposition 4.3.
For the reader familiar with [16] or at least standard examples of superconformal nets and the corresponding notation, we provide two short illustrations of the above results:

**Example 4.7. Supersymmetric free field net.** The supersymmetric free field net $\mathcal{A}$ and an associated sKMS functional $\langle \phi, \text{dom}(\phi) \rangle$ have been extensively studied in [16 Sec.3], and we refer to the notation introduced there, in particular the construction of the superderivation $\delta$ and the sKMS functional; $J$ and $F$ stand for the corresponding bosonic and fermionic free field currents.

The corresponding even JLO cocycle $\tau$ on $\mathcal{C}^\infty(\delta)_c$ is nontrivial because

$$\tau(1) = \tau_0(1) = \phi(1) = 1,$$

owing to the normalization condition $(S_3)$ on $\phi$.

An example of an admissible perturbation $\delta_r = \delta + r \mathrm{ad} Q$ which leaves the class of $\tau$ invariant is given by

$$Q = \int_{\mathbb{R}} \alpha_t \left( (J(f) - i)^{-1} F(f) (J(f) + i)^{-1} \right) h(t) \, dt,$$

with arbitrary but fixed $f, h \in \mathcal{C}^\infty_c(\mathbb{R})$, i.e., compactly supported $\mathbb{R}$-valued smooth functions on $\mathbb{R}$. This perturbation is selfadjoint, odd, localized and smooth, i.e., $Q \in \mathcal{C}^\infty(\delta)_c$.

It seems interesting to study explicit (co-)homology classes and pairings with $K_0$-theory (and corresponding projections) and understand their physical meaning. However, the computations are very tedious and left for future study. The conceptually interesting projections investigated in [16 Sec.5] or [9, Sec.4] are unfortunately global, hence not in $\mathfrak{A}$ and not applicable here.

Obviously, we may replace the free field net by its rational extension as discussed in [16 Th.3.8], and all the above results concerning the JLO cocycle and perturbations should extend to that setting.

**Example 4.8. Super-Virasoro net.** Since the super-Virasoro net with central charge $c \geq 3/2$ is a subnet of the free field net, this example becomes a consequence of the preceding one by restricting the cocycle to $\mathfrak{A}_{SVir} \cap \mathcal{C}^\infty(\delta)_c$, denoted by $\tau_{SVir}$. Note that also the perturbed cocycle defined by the perturbation $Q$ in the preceding example (probably not in $\mathfrak{A}_{SVir} \cap \mathcal{C}^\infty(\delta)_c$) restricts to a local-entire cyclic cocycle on $\mathfrak{A}_{SVir} \cap \mathcal{C}^\infty(\delta)_c$, which is cohomologous to $\tau_{SVir}$. Whether $\tau_{SVir}$ is really meaningful and whether there are possible perturbations in $\mathfrak{A}_{SVir} \cap \mathcal{C}^\infty(\delta)_c$ depends on whether $\mathfrak{A}_{SVir}(I) \cap \mathcal{C}^\infty(\delta)_c \subset \mathfrak{A}_{SVir}(I)$ is actually larger than $\mathcal{C}I$. We expect this to be true, but a proof is missing so far, cf. [16 Th.3.10].

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