Gelfand and Kolmogorov numbers of Sobolev embeddings of weighted function spaces II

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Abstract

We consider the Gelfand and Kolmogorov numbers of compact embeddings between weighted function spaces of Besov and Triebel-Lizorkin type with polynomial weights in the non-limiting case. Our main purpose here is to complement our previous results in \cite{32} in the context of the quasi-Banach setting, $0 < p, q \leq \infty$. In addition, sharp estimates for their approximation numbers in several cases left open in Skrzypczak (2005) \cite{25} are provided.

Key words: Gelfand numbers; Kolmogorov numbers; approximation numbers; weighted Besov and Triebel-Lizorkin spaces; Sobolev embeddings.

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1 Introduction

Since the 1950s, under the influence of the work Kolmogorov \cite{11}, a new perspective on approximation theory has developed. The study of widths, optimal recovery and computational complexity has received much attention, see \cite{1, 3, 5, 17, 18, 19, 20, 23, 27} for a survey. In particular, one of the major tasks is to determine the exact (or asymptotic exact) degree of various $n$-widths of some classical classes of functions in different computational settings, and find optimal algorithms.

The main aim of this paper is to complement our previous results obtained in \cite{32} on the Sobolev embeddings between weighted function spaces of Besov and Triebel-Lizorkin type. There we considered the case where the ratio of the weights $w(x)$ is of polynomial type, and established the asymptotic order of the Gelfand and Kolmogorov numbers of the corresponding embeddings where the spaces involved were Banach spaces, $1 \leq p, q \leq \infty$. The main reason for these

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restrictions was that there were still several gaps left open, especially on the estimates for Gelfand numbers of the Euclidean ball in the quasi-Banach case. Fortunately, we recently became aware of some surprising results from Foucart et al. [6] and Vybíral [31].

In the present paper, we extend completely the results of [25, 32] to the quasi-Banach space case $0 < p, q \leq \infty$ in the so-called non-limiting situation. In particular, we provide sharp estimates for the asymptotic behavior of the Gelfand and Kolmogorov numbers in several cases of the Banach space setting left open in [32]. Also, several gaps for the approximation numbers left open in [25] are closed.

The discretization technique adopted in [32] is still effective in the quasi-Banach setting. The characterization of weighted Besov spaces in terms of wavelets was proved by Haroske and Triebel [9] in the quasi-Banach case. Moreover, the operator ideal technique works also in this case. Historically, Pietsch [21, 22] developed the theory of operator ideals and $s$-numbers. The technique of estimating single $s$-numbers or entropy numbers via estimates of ideal (quasi-)norms derives from ideas of Carl [2]. In the 1980s this technique was frequently used in operator theory, in eigenvalue problems for Banach space operators, etc. However, the operator ideal technique remained unknown in the function spaces community for many years. As far as we know, it was applied for the first time in [12, 13], which both appeared in 2003.

In a sequel to [32], we concentrate on the Sobolev embeddings,

$$B_{p_1,q_1}^{s_1}(\mathbb{R}^d, w_\alpha) \hookrightarrow B_{p_2,q_2}^{s_2}(\mathbb{R}^d),$$

with polynomial weights

$$w_\alpha(x) := (1 + |x|^2)^{\alpha/2}$$

for some exponent $\alpha > 0$. The non-limiting case means that $\delta \neq \alpha$.

The organization of this paper is as follows. In Section 2, we recall some definitions and related properties, and present our main results. In Section 3, we collect several necessary estimates of Kolmogorov and Gelfand numbers of the Euclidean ball. Main proofs are shifted to Section 4. Finally, in Section 5 we complement the known results of Skrzypczak [25] for the approximation numbers, and compare these three quantities of the function space embeddings. Our main assertions are Theorem 2.6 and Theorem 2.9 which generalize the main results in [32].

Let us make an agreement throughout this paper,

$$-\infty < s_2 < s_1 < \infty, \ 0 < p_1, p_2, q_1, q_2 \leq \infty \text{ and } \delta = s_1 - s_2 - d\left(\frac{1}{p_1} - \frac{1}{p_2}\right) > 0$$

if no further restrictions are stated.

**Notation 1.1.** By the symbol ‘$\hookrightarrow$’ we denote continuous embeddings.

By $\mathbb{N}$ we denote the set of natural numbers, by $\mathbb{N}_0$ the set $\mathbb{N} \cup \{0\}$.

Identity operators will always be denoted by $\text{id}$. Sometimes we do not indicate the spaces where $\text{id}$ is considered, and likewise for other operators.

Let $X$ and $Y$ be complex quasi-Banach spaces and denote by $\mathcal{L}(X,Y)$ the class of all linear continuous operators $T : X \to Y$. If no ambiguity arises, we write $\|T\|$ instead of the more exact versions $\|T\|_{\mathcal{L}(X,Y)}$ or $\|T : X \to Y\|$. 


The symbol \( a_n \preceq b_n \) means that there exists a constant \( c > 0 \) such that \( a_n \leq cb_n \) for all \( n \in \mathbb{N} \). And \( a_n \succeq b_n \) stands for \( b_n \preceq a_n \), while \( a_n \sim b_n \) denotes \( a_n \preceq b_n \preceq a_n \).

All unimportant constants will be denoted by \( c \) or \( C \), sometimes with additional indices.

2 Main results

First let us recall the definitions of Kolmogorov and Gelfand numbers, cf. [23]. We use the symbol \( A \subset \subset B \) if \( A \) is a closed subspace of a topological vector space \( B \).

**Definition 2.1.** Let \( T \in \mathcal{L}(X,Y) \).

(i) The \( n \)th Kolmogorov number of the operator \( T \) is defined by

\[
d_n(T,X,Y) = \inf\{\|Q^Y_N T\| : N \subset \subset Y, \dim(N) < n\},
\]

also written by \( d_n(T) \) if no confusion is possible. Here, \( Q^Y_N \) stands for the natural surjection of \( Y \) onto the quotient space \( Y/N \).

(ii) The \( n \)th Gelfand number of the operator \( T \) is defined by

\[
c_n(T,X,Y) = \inf\{\|TJ^X_M\| : M \subset \subset X, \text{codim}(M) < n\},
\]

also written by \( c_n(T) \) if no confusion is possible. Here, \( J^X_M \) stands for the natural injection of \( M \) into \( X \).

It is well-known that the operator \( T \) is compact if and only if \( \lim_n d_n(T) = 0 \) or equivalently \( \lim_n c_n(T) = 0 \), see [23].

The Kolmogorov and Gelfand numbers are dual to each other in the following sense, cf. [21, 23]: If \( X \) and \( Y \) are quasi-Banach spaces, then

\[
c_n(T^*) = d_n(T) \tag{2.1}
\]

for all compact operators \( T \in \mathcal{L}(X,Y) \) and

\[
d_n(T^*) = c_n(T) \tag{2.2}
\]

for all \( T \in \mathcal{L}(X,Y) \).

For later proofs, we shall recall some basic facts about approximation numbers. We define the \( n \)th approximation number of \( T \) by

\[
a_n(T) = \inf\{\|T - L\| : L \in \mathcal{L}(X,Y), \ \text{rank}(L) < n\}, \quad n \in \mathbb{N},
\]

where \( \text{rank}(L) \) denotes the dimension of \( L(X) \). We refer to [4, 21, 23] for detailed discussions of this concept and further references.

Both, Gelfand and Kolmogorov numbers, are subadditive and multiplicative \( s \)-numbers, as well as approximation numbers. One may consult Pietsch [22] (Sections 2.4, 2.5), for the proof in the Banach space case. Further, the generalization to \( p \)-Banach spaces follows obviously. Let \( s_n \) denote any of these three quantities \( a_n, d_n \) or \( c_n \), and let \( Y \) be a \( p \)-Banach space, \( 0 < p \leq 1 \). More precisely, we collect several common properties of them as follows,
and the set

Let

Definition 2.2. Let $0 < p, q \leq \infty$, and $s \in \mathbb{R}$. Then we put

\[ B^{s}_{p,q}(\mathbb{R}^{d}, w_{\alpha}) = \left\{ f \in S'(\mathbb{R}^{d}) : \| f \|_{B^{s}_{p,q}(\mathbb{R}^{d}, w_{\alpha})} = \| f w_{\alpha} \|_{B^{s}_{p,q}(\mathbb{R}^{d})} < \infty \right\}, \]

\[ F^{s}_{p,q}(\mathbb{R}^{d}, w_{\alpha}) = \left\{ f \in S'(\mathbb{R}^{d}) : \| f \|_{F^{s}_{p,q}(\mathbb{R}^{d}, w_{\alpha})} = \| f w_{\alpha} \|_{F^{s}_{p,q}(\mathbb{R}^{d})} < \infty \right\}, \]

Remark 2.3. If no ambiguity arises, then we can write $B^{s}_{p,q}(w_{\alpha})$ and $F^{s}_{p,q}(w_{\alpha})$ for brevity.
Theorem 2.9. Let \( 0 < p_1 < p_2 \leq \infty \). Further, suppose \( 0 < \alpha \). The embedding \( B^{s_1}_{p_1,q_1}(\mathbb{R}^d,w_\alpha) \to B^{s_2}_{p_2,q_2}(\mathbb{R}^d) \) is compact if and only if \( \min(\alpha, \delta) > d \max(\frac{1}{p_2} - \frac{1}{p_1}, 0) \).

The same assertion holds for \( F^s_{p,q} \)-spaces with the restriction \( p_1, p_2 < \infty \). We are now ready to formulate our main results.

Theorem 2.6. Let \( \alpha > 0 \), \( \delta \neq \alpha \), \( \theta = \frac{1/p_1 - 1/p_2}{1/2 - 1/p_2} \) and \( \frac{1}{p} = \frac{\mu}{d} + \frac{1}{p_1} \), where \( \mu = \min(\alpha, \delta) \).

Further, suppose \( 0 < p_1 \leq p_2 \leq \infty \) or \( \tilde{p} < p_2 < p_1 \leq \infty \).

Denote by \( d_n \) the \( n \)th Kolmogorov number of the embedding (1.1). Then \( d_n \sim n^{-\kappa} \), where

\[
(i) \quad \kappa = \frac{d}{2} \quad \text{if} \quad 0 < p_1 \leq p_2 \leq 2 \quad \text{or} \quad 2 < p_1 = p_2 \leq \infty,
\]

\[
(ii) \quad \kappa = \frac{d}{2} + \frac{1}{p_1} - \frac{1}{p_2} \quad \text{if} \quad \tilde{p} < p_2 < p_1 \leq \infty,
\]

\[
(iii) \quad \kappa = \frac{d}{2} + \frac{1}{2} - \frac{1}{p_2} \quad \text{if} \quad 0 < p_1 < 2 < p_2 \leq \infty \quad \text{and} \quad \mu > \frac{d}{p_2},
\]

\[
(iv) \quad \kappa = \frac{\mu}{d} \cdot \tilde{p}_2 \quad \text{if} \quad 0 < p_1 < 2 < p_2 < \infty \quad \text{and} \quad \mu < \frac{d}{p_2},
\]

\[
(v) \quad \kappa = \frac{d}{2} + \frac{1}{p_1} - \frac{1}{p_2} \quad \text{if} \quad 2 \leq p_1 < p_2 \leq \infty \quad \text{and} \quad \mu > \frac{d}{2} \theta,
\]

\[
(vi) \quad \kappa = \frac{d}{2} \cdot \tilde{p}_2 \quad \text{if} \quad 2 \leq p_1 < p_2 < \infty \quad \text{and} \quad \mu < \frac{d}{p_2} \theta.
\]

Remark 2.7. We shift the proof of the above theorem to Subsection 4.1. And we wish to mention that both points, (iv) and (vi), vanish if \( p_2 = \infty \).

Remark 2.8. As is pointed in [32] (Remark 2.6), Similar conclusions on the estimation of Kolmogorov numbers of Sobolev embeddings on bounded domains could be made for Corollary 19 in [28]. The counterexample to our new part (iv) could be also made for the limiting case \( \delta = \frac{d}{p_2} \) according to [27].

For \( 0 < p \leq \infty \), we set

\[
p' = \begin{cases} 
\frac{p}{p-1} & \text{if} \quad 1 < p < \infty, \\
1 & \text{if} \quad p = \infty, \\
\infty & \text{if} \quad 0 < p \leq 1.
\end{cases}
\]

Theorem 2.9. Let \( \alpha > 0 \), \( \delta \neq \alpha \), \( \theta_1 = \frac{1/p_2 - 1/p_1}{1/2 - 1/p_1} \) and \( \frac{1}{p} = \frac{\mu}{d} + \frac{1}{p_1} \), where \( \mu = \min(\alpha, \delta) \).

Further, suppose \( 0 < p_1 \leq p_2 \leq \infty \) or \( \tilde{p} < p_2 < p_1 \leq \infty \).

Denote by \( c_n \) the \( n \)th Gelfand number of the embedding (1.1). Then \( c_n \sim n^{-\kappa} \), where

\[
(i) \quad \kappa = \frac{d}{2} \quad \text{if} \quad 2 \leq p_1 \leq p_2 \leq \infty \quad \text{or} \quad 0 < p_1 = p_2 < 2,
\]

\[
(ii) \quad \kappa = \frac{d}{2} + \frac{1}{p_1} - \frac{1}{p_2} \quad \text{if} \quad \tilde{p} < p_2 < p_1 \leq \infty,
\]
\( \kappa = \frac{\mu}{d} + \frac{1}{p_1} - \frac{1}{2} \) if \( 0 < p_1 < 2 < p_2 \leq \infty \) and \( \mu > \frac{d}{p_1} \),

\( \kappa = \frac{\mu}{d} \cdot \frac{d}{p_1} \) if \( 1 < p_1 < 2 < p_2 \leq \infty \) and \( \mu < \frac{d}{p_1} \),

\( \kappa = \frac{\mu}{d} + \frac{1}{p_1} - \frac{1}{2} \) if \( 0 < p_1 < p_2 \leq 2 \) and \( \mu > \frac{d}{p_1} \theta_1 \),

\( \kappa = \frac{\mu}{d} \cdot \frac{d}{p_1} \) if \( 1 < p_1 < p_2 \leq 2 \) and \( \mu < \frac{d}{p_1} \theta_1 \).

**Remark 2.10.** We shift the proof of this assertion to Subsection 4.2. Note that both points, (iv) and (vi), vanish if \( 0 < p_1 \leq 1 \).

**Remark 2.11.** As well as in the Banach space case, we observe that

\[
d_n(\text{id}, B_{p_1,q_1}(\mathbb{R}^d,v_1), B_{p_2,q_2}(\mathbb{R}^d,v_2)) \sim d_n(\text{id}, B_{p_1,q_1}(\mathbb{R}^d,v_1/v_2), B_{p_2,q_2}(\mathbb{R}^d)),
\]

in the quasi-Banach case, where \( v_1, v_2 \) are admissible weight functions. Moreover, the same formula holds for the Gelfand numbers. Therefore, without loss of generality we can assume that the target space is an unweighted space.

**Corollary 2.12.** Theorem 2.6 and Theorem 2.9 remain valid if instead of the embedding (1.1) we have any one of the following embeddings:

\[
F_{s_1,q_1}(\mathbb{R}^d,w_\alpha) \hookrightarrow B_{p_2,q_2}(\mathbb{R}^d) \quad \text{if } p_1 < \infty,
\]

\[
B_{s_1,q_1}(\mathbb{R}^d,w_\alpha) \hookrightarrow F_{p_2,q_2}(\mathbb{R}^d) \quad \text{if } p_2 < \infty,
\]

\[
F_{s_1,q_1}(\mathbb{R}^d,w_\alpha) \hookrightarrow F_{p_2,q_2}(\mathbb{R}^d) \quad \text{if } p_1,p_2 < \infty.
\]

**Remark 2.13.** We shift the short proof of this corollary to Subsection 4.3.

**Remark 2.14.** As is noted in [32] (Remark 2.10), for the limiting case \( \delta = \alpha \), the exact order of related \( n \)-widths may possibly depend on \( q_1 \) and \( q_2 \). Some ideas from [9, 14] may be helpful to further research in this situation.

### 3 Preliminaries

#### 3.1 Discretization of function spaces

We use the discrete wavelet transform in order to transfer these problems from function spaces to the corresponding sequence spaces, and then solve the task for the sequence spaces. Afterwards, the results are transferred back to function spaces. The crucial point in this discretization technique is that the asymptotic order of the estimates is preserved.

**Proposition 3.1.** Let \( s \in \mathbb{R} \) and \( 0 < p, q \leq \infty \). Assume

\[
r > \max(s, \frac{2d}{p} + \frac{d}{2} - s).
\]
Then for every weight \( w_\alpha \) there exists an orthonormal basis of compactly supported wavelet functions \( \{ \varphi_{j,k} \}_{j,k} \cup \{ \psi_{i,j,k} \}_{i,j,k} \), \( j \in \mathbb{N}_0 \), \( k \in \mathbb{Z}^d \) and \( i = 1, \ldots, 2^d - 1 \), such that a distribution \( f \in \mathcal{S}'(\mathbb{R}^d) \) belongs to \( B^s_{p,q}(w_\alpha) \) if and only if

\[
\|f|B^s_{p,q}(w_\alpha)\| \overset{\bullet}{=} \left( \sum_{k \in \mathbb{Z}^d} |\langle f, \varphi_{0,k} \rangle w_\alpha(k)|^p \right)^{1/p} \\
+ \sum_{i=1}^{2^d-1} \left\{ \sum_{j=0}^{\infty} 2^j (s^d + \frac{d}{2} - \frac{1}{p})q \left( \sum_{k \in \mathbb{Z}^d} |\langle f, \psi_{i,j,k} \rangle w_\alpha(2^{-j}k)|^p \right)^{q/p} \right\}^{1/q} < \infty.
\] (3.1)

Furthermore, \( \|f|B^s_{p,q}(w_\alpha)\| \overset{\bullet}{=} \) may be used as an equivalent quasi-norm in \( B^s_{p,q}(w_\alpha) \).

**Remark 3.2.** The proof of this proposition in its full generality may be found in Haroske and Triebel [9]. One can also consult [14] for historical remarks on various techniques of decompositions.

Let \( 0 < p, q \leq \infty \). Based on Proposition 3.1 we will work with the following weighted sequence spaces

\[
\ell_q(2^{js} \lambda) := \left\{ \lambda = (\lambda_{j,k})_{j,k} : \lambda_{j,k} \in \mathbb{C}, \right. \\
\left. \|\lambda|\ell_q(2^{js} \lambda)\| = \left( \sum_{j=0}^{\infty} 2^{jsq} \left( \sum_{k \in \mathbb{Z}^d} |\lambda_{j,k} w_{j,k}|^p \right)^{q/p} \right)^{1/q} < \infty \right\},
\] (3.2)

(usual modification if \( p = \infty \) and/or \( q = \infty \)), where \( w_{j,k} = w_\alpha(2^{-j}k) \). If \( s = 0 \) we will write \( \ell_q(\lambda) \).

### 3.2 Kolmogorov numbers of the Euclidean ball

To begin with, we shall make preparations for the estimates of Kolmogorov numbers of related function space embeddings in the quasi-Banach setting with \( 0 < p_1 < 1 \) or \( 0 < p_2 < 1 \), and for several cases left over in the Banach setting with \( p_2 = \infty \). The following result, Lemma 3.3, is due to Garnaev and Gluskin [7], Kashin [10] and Vybijal [31].

**Lemma 3.3.** Let \( N \in \mathbb{N} \) and \( n \leq N \).

(i) If \( 1 \leq p < 2 \) and \( n \leq \frac{N}{4} \) then

\[
n^{-1/2} \leq d_n (\text{id}, \ell^N_p, \ell^N_\infty) \leq n^{-1/2} \left( \log \left( \frac{eN}{n} \right) \right)^{3/2}.
\]

(ii) If \( 2 \leq p < \infty \) then

\[
\frac{1}{4} \min \left\{ 1, \left( c_1 \frac{\log(1 + \frac{N}{n-1})}{n-1} \right)^{1/p} \right\} \leq d_n (\text{id}, \ell^N_p, \ell^N_\infty) \leq \min \left\{ 1, \left( c_2 \frac{\log(1 + \frac{N}{n-1})}{n-1} \right)^{1/p} \right\}
\]

are valid for certain absolute constants \( c_1 > 0 \) and \( c_2 > 0 \).
(iii) If $0 < p_1 < 1$ and $p_1 < p_2 \leq \infty$ then
\[ d_n(\text{id}, \ell_{p_1}^N, \ell_{p_2}^N) = d_n(\text{id}, \ell_{\min(p_1, p_2)}^N, \ell_{p_2}^N). \]

The following lemma is a simple corollary of Lemma 3.3. And the proof mimics that of Lemma 10 in [25].

**Lemma 3.4.** Let $1 \leq p_1 < 2$ and $N = 1, 2, 3, \ldots$. Then there is a positive constant $C$ independent of $N$ and $n$ such that
\[
    d_n(id, \ell_{p_1}^N, \ell_{p_2}^\infty) = d_n(id, \ell_{\min(1, p_2)}^N, \ell_{p_2}^\infty) \leq C \min\left\{ \frac{1}{p_1}, \frac{1}{n^{-1/p_2 - 1/p_1}} \right\}, \quad n \in \mathbb{N},
\]
where $[cn]$ denotes the upper integer part of $cn$.

**3.3 Gelfand numbers of the Euclidean ball**

In this subsection we collect some known results on $c_n(id, \ell_{p_1}^N, \ell_{p_2}^N)$ for later use, cf. [6, 10, 16, 21, 31].

The following result is due to Foucart et al. [6]. Note that the $n$-th Gelfand number is identical to the $(n-1)$-th Gelfand width of $T$ defined in [6], see also Pinkus [23]. Here we recall it in our pattern.

**Lemma 3.5.** If $0 < p_2 \leq p_1 \leq \infty$, then there is a constant $c$, $0 < c \leq 1$, such that
\[
    d_{[cn]+1}(id, \ell_{p_1}^{2n}, \ell_{p_2}^{2n}) \geq n^{1/p_2 - 1/p_1}, \quad n \in \mathbb{N},
\]
where $[cn]$ denotes the upper integer part of $cn$. 

We wish to mention that, in contrast to Lemma 3.9, the estimate
\[
    d_n(id, \ell_{p_1}^N, \ell_{p_2}^\infty) = (N - n + 1)^{-1/p_2 - 1/p_1}, \quad 1 \leq n \leq N \leq \infty,
\]
is not valid for Kolmogorov numbers if $0 < p_2 \leq p_1 \leq \infty$ and $p_2 < 1$. The following estimate from below was proved by Vybíral [31].

**Lemma 3.6.** Let $1 \leq n \leq N < \infty$.

(i) If $0 < p_1 \leq 1$ and $2 < p_2 \leq \infty$ then there exist constants $C_1, C_2 > 0$ depending only on $p_1$ and $p_2$ such that
\[
    C_1 \min\left\{ \frac{1}{n-1}, \ln\left(\frac{N}{n-1}\right) + 1 \right\}^{1/p_1 - 1/p_2} \leq c_n(id, \ell_{p_1}^N, \ell_{p_2}^N) \leq C_2 \min\left\{ \frac{1}{n-1}, \ln\left(\frac{N}{n-1}\right) + 1 \right\}^{1/p_1 - 1/p_2}. \]

(ii) If $0 < p_1 \leq 1$ and $p_1 < p_2 \leq 2$ then there exist constants $C_1, C_2 > 0$ depending only on $p_1$ and $p_2$ such that
\[
    C_1 \min\left\{ \frac{1}{n-1}, \ln\left(\frac{N}{n-1}\right) + 1 \right\}^{1/p_1 - 1/p_2} \leq c_n(id, \ell_{p_1}^N, \ell_{p_2}^N) \leq C_2 \min\left\{ \frac{1}{n-1}, \ln\left(\frac{N}{n-1}\right) + 1 \right\}^{1/p_1 - 1/p_2}. \]

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Remark 3.7. For the upper bounds considered above, there is another result given by Vybíral [31], cf. Lemma 4.11, with a slight difference between them on the log-factors. But they are equivalent to each other for our upper estimates in Theorem 2.9.

Lemma 3.8. Let $n \in \mathbb{N}$.

(i) If $0 < p_1 \leq 1$ and $2 \leq p_2 \leq \infty$ then
$$c_n \left( \text{id}, \ell^{2n}_{p_1}, \ell^{2n}_{p_2} \right) \geq n^{1/2 - 1/p_1}. \tag{3.4}$$

(ii) If $0 < p_1 \leq 1$ and $p_1 < p_2 \leq 2$ then
$$c_n \left( \text{id}, \ell^{2n}_{p_1}, \ell^{2n}_{p_2} \right) \geq n^{1/p_2 - 1/p_1}. \tag{3.5}$$

The proof of the above lemma follows literally [31, p. 567], by the multiplicativity of Gelfand numbers. In fact, The point (ii) in Lemma 3.6 may also imply point (ii) of Lemma 3.8.

Lemma 3.9. If $1 \leq n \leq N < \infty$ and $0 < p_2 \leq p_1 \leq \infty$, then
$$a_n \left( \text{id}, \ell^N_{p_1}, \ell^N_{p_2} \right) = c_n \left( \text{id}, \ell^N_{p_1}, \ell^N_{p_2} \right) = (N - n + 1)^{1/p_2 - 1/p_1}.$$  

The proof of this lemma follows literally [21, Section 11.11.4, see also [23]. Indeed the original proof is used only to deal with the Banach setting. However, the same proof works also in the quasi-Banach setting $0 < p_2 \leq p_1 \leq \infty$.

4 Proofs

4.1 Proof of Theorem 2.6

In [32] we were able to prove this theorem in the Banach space case $1 \leq p, q \leq \infty$, with the assumption that $p_2 < \infty$ holds if $p_1 < p_2$. It is remarkable that the results there do not depend on the fine indices $q_1$ and $q_2$. And the proof may be directly generalized to the quasi-Banach setting $0 < q_1, q_2 \leq \infty$, with $1 \leq p_1, p_2 \leq \infty$. Afterwards, the restrictions $q_1, q_2 \geq 1$ could be lifted.

Therefore, we may concentrate on the proof of

♣ (i) if $0 < p_1 < 1$ and $p_1 \leq p_2 \leq 2$,

(⪀) (ii) if $0 < \tilde{p} < p_2 < p_1 \leq \infty$ and $0 < p_2 < 1$,

♣ (iii) and (v) if $1 \leq p_1 < p_2 = \infty$,

(♦) (iii) and (iv) if $0 < p_1 < 1$ and $2 < p_2 \leq \infty$.

We shall give the proof of the estimates from above and below in following four steps.

Step 1: Proof of (⪀).
To shorten notation define $1/p = 1/p_2 - 1/p_1$. 

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We use the relation, 
\[ d_n(\text{id}, \ell_1^{N_1}, \ell_2^{N_2}) \leq a_n(\text{id}, \ell_1^{N_1}, \ell_2^{N_2}), \text{ cf. } (2.4) \].

Note that Step 1 of the proof of Proposition 15 in \[25\] may be directly generalized to the quasi-Banach setting, where 
\[ 0 < p_2 < p_1 \leq \infty \] and \[ 0 < q_1, q_2 \leq \infty \]. So similar arguments give the estimates from above as required.

For the estimates from below, we obtain by Lemma 3.5 that
\[ d_n(\text{id}, \ell_1^{N_1}, \ell_2^{N_2}) \succeq n^{1/p}, \] (4.1)
for \( n = \left[ \frac{c_2}{2} \cdot N \right] \), where \( c \) is the constant from Lemma 3.5. We can deal with the estimates in a similar manner as in Step 4 of the proof of Proposition 11 in \[25\], by using (4.1). Note that in order to guarantee the compactness of the embeddings here we only need to consider two cases, 
\[ \frac{d}{p} < \alpha < \delta \text{ or } \frac{d}{p} < \delta < \alpha, \] instead of four cases.

**Step 2:** Proof of (♣).
If \( p_2 \geq 1 \), we obtain by Lemma 3.3 that
\[ d_n(\text{id}, \ell_1^{N_1}, \ell_2^{N_2}) = d_n(\text{id}, \ell_1^{N_1}, \ell_2^{N_2}). \] (4.2)

Then the estimates may be shifted immediately to the Banach case, \( 1 = p_1 \leq p_2 \leq 2 \). Similar arguments give the sharp two-sided estimates.

If \( p_2 < 1 \) and \( n \leq N \), then
\[ d_n(\text{id}, \ell_1^{N_1}, \ell_2^{N_2}) = d_n(\text{id}, \ell_1^{N_1}, \ell_2^{N_2}). \]

Thereby, the proof of the upper and lower estimates follows in the same way as in the first step. Two cases, \( 0 < \alpha < \delta \) or \( 0 < \delta < \alpha \), are considered instead.

**Step 3:** Proof of (♠).
For the estimates from above, we still adopt the operator ideal. \[3.3\] and Lemma 3.3 (ii) imply that
\[ L_{s,\infty}^{(d)}(\text{id}, \ell_1^{N_1}, \ell_\infty^{N_2}) \leq C \begin{cases} 
N^{1/s-1/2} & \text{if } 1 \leq p_1 < 2 \text{ and } \frac{1}{s} > \frac{1}{2}, \\
N^{1/s-1/p_1} & \text{if } 2 \leq p_1 < \infty \text{ and } \frac{1}{s} > \frac{1}{p_1}.
\end{cases} \] (4.3)

Note that related computations above is similar to that of ideal quasi-norms of entropy numbers, cf. \[ 4, 15 \]. Next we proceed as in the proof of Proposition 11 in \[25\]. As to the definition of \( P_{i,j}, i,j \in \mathbb{N}_0 \), we refer to the counterpart there again. For any given \( M \in \mathbb{N}_0 \), we also put
\[ P := \sum_{m=0}^{M} \sum_{j+i=m} P_{j,i} \quad \text{and} \quad Q := \sum_{m=M+1}^{\infty} \sum_{j+i=m} P_{j,i}. \] (4.4)
Set \( \beta = \max(2, p_1) \). For \( L_{s,\infty}^{(d)}(P) \), we choose \( s \) such that \( \frac{1}{s} > \frac{4}{3} + \frac{1}{p_1} \). And for \( L_{s,\infty}^{(d)}(Q) \), we select \( s \) satisfying \( \frac{1}{s} < \frac{4}{3} < \frac{3}{2} + \frac{1}{p_1} \). Once more the upper estimates are finished.

For the lower estimates, we obtain by Lemma 3.5 that, for \( n = \left[ \frac{N}{4} \right] \),
\[ d_n(\text{id}, \ell_1^{N_1}, \ell_\infty^{N_2}) \geq n^{-1/\beta}, \text{ where } \beta = \max(2, p_1). \] (4.5)
Then we only consider two cases, $0 < \alpha < \delta$ or $0 < \delta < \alpha$ instead. Again, we follow Step 4 of the proof of Proposition 11 in [25], now using (4.3) instead.

**Step 4: Proof of (○).**

If $p_2 < \infty$, the estimates may be shifted immediately by point (iii) of Lemma 3.3 to the Banach case, $1 = p_1 < 2 \leq p_2 < \infty$, considered in [32]. Similar arguments give the sharp two-sided estimates.

If $p_2 = \infty$, then the point (iv) vanishes, and for (○) in point (iii) we follow trivially the third step by Lemma 3.3 (iii).

4.2 Proof of Theorem 2.9

Arguments similar to the last proof lead us to lift the restrictions $q_1, q_2 \geq 1$ in those results for Gelfand numbers obtained in [32]. So we can concentrate on the proof of

(♠) (i) if $0 < p_1 = p_2 < 1$,

(♡) (ii) if $0 < \tilde{p} < p_2 < p_1 \leq \infty$ and $0 < p_2 < 1$,

(♣) (iii) if $0 < p_1 \leq 1$ and $2 < p_2 \leq \infty$,

(♢) (v) if $0 < p_1 \leq 1$ and $p_1 < p_2 \leq 2$.

We shall give the proof of the estimates from above and below in following four steps.

**Step 1: Proof of (♠).**

If $0 < p_1 = p_2 < 1$ and $n \leq N$, then it holds obviously that

$$c_n (\text{id}, \ell_{p_1}^N, \ell_{p_2}^N) = c_n (\text{id}, \ell_{p_2}^N, \ell_{p_2}^N) = 1.$$  

The proof of (♠) follows literally that of Proposition 13 in [25].

**Step 2: Proof of (♡).**

By Lemma 3.9, the proof of (♡) follows exactly as in the proof for the Banach case given in [32], see also [25] (Proposition 15) for further details.

**Step 3: Proof of (♣).**

We can deal with the proof of (♣) in a way similar to that of (♣) in Theorem 2.6.

For the upper estimates, we first obtain by Lemma 3.6 that

$$L^{(c)}_{s,\infty} (\text{id}, \ell_{p_1}^N, \ell_{p_2}^N) \leq CN^{1/s-(1/p_1-1/2)} \quad \text{if} \quad \frac{1}{s} > \frac{1}{p_1} - \frac{1}{2}. \quad (4.6)$$

Again, we adopt the notations $P$ and $Q$ from [25]. Choose $s$ such that $\frac{1}{s} > \frac{\mu}{d} + \frac{1}{p_1} - \frac{1}{2}$ for $L^{(c)}_{s,\infty} (P)$, and $\frac{1}{p_1} - \frac{1}{2} < \frac{1}{s} < \frac{\mu}{d} + \frac{1}{p_1} - \frac{1}{2}$ for $L^{(c)}_{s,\infty} (Q)$, respectively. Once more the upper estimates are complete.

For the lower estimates, we use (3.4) with $n = \lceil \frac{N}{2} \rceil$. Once more we follow the pattern of Step 4 in the proof of Proposition 11 in [25], now dealing with two cases, $0 < \alpha < \delta$ or $0 < \delta < \alpha$.

**Step 4: Proof of (♢).**

The proof of (♢) follows literally as in the third step, with Lemma 3.6 (i) replaced by Lemma 3.6 (ii) for the upper bounds, and (3.4) replaced by (3.5) for the lower bounds. \qed
4.3 Proof of Corollary 2.12

We denote by $A^s_{p,q}(\mathbb{R}^d, w_\alpha) (A^s_{p,q}(\mathbb{R}^d))$ either $B^s_{p,q}(\mathbb{R}^d, w_\alpha)(B^s_{p,q}(\mathbb{R}^d))$ or $F^s_{p,q}(\mathbb{R}^d, w_\alpha)(F^s_{p,q}(\mathbb{R}^d))$, with the restraint that for the $F$-spaces $p < \infty$ holds. The proof follows by the relations as below, see [4, Section 2.2.3, p. 44] and [28] for further details,

$$B^s_{p,u}(\mathbb{R}^d) \hookrightarrow F^s_{p,q}(\mathbb{R}^d) \hookrightarrow B^s_{p,v}(\mathbb{R}^d),$$

where $0 < u \leq \min(p, q)$ and $\max(p, q) \leq v \leq \infty$.

For the embeddings of these three types, we prove the upper bounds, by virtue of the multiplicativity property of Kolmogorov and Gelfand numbers, and the following embeddings

$$A^{s_1}_{p_1,q_1}(\mathbb{R}^d, w_\alpha) \hookrightarrow B^{s_1}_{p_1,\infty}(\mathbb{R}^d, w_\alpha) \hookrightarrow B^{s_2}_{p_2,u_1}(\mathbb{R}^d) \hookrightarrow A^{s_2}_{p_2,q_2}(\mathbb{R}^d),$$

where $u_1 = \min(p_2, q_2)$.

For the estimate from below we can consider the following embeddings

$$B^{s_1}_{p_1,u_2}(\mathbb{R}^d, w_\alpha) \hookrightarrow A^{s_1}_{p_1,q_1}(\mathbb{R}^d, w_\alpha) \hookrightarrow A^{s_2}_{p_2,q_2}(\mathbb{R}^d) \hookrightarrow B^{s_2}_{p_2,\infty}(\mathbb{R}^d),$$

where $u_2 = \min(p_1, q_1)$.

5 Comparisons with approximation numbers

In this closing section we wish to compare the approximation, Gelfand and Kolmogorov numbers of Sobolev embeddings between weighted function spaces of Besov and Triebel-Lizorkin type in the non-limiting situation.

Let us first complement the known results for the approximation numbers. Specifically, in [25, 26] the exact estimates of approximation number were established in almost all cases. However, the problem was still open in case when $0 < p_1 \leq 1$ and $p_2 = \infty$. Here we are able to close the gaps in the non-limiting situation.

**Lemma 5.1.** Let $0 < p \leq 1$ and $N \in \mathbb{N}$.

(i) Let $0 < \lambda < 1$. Then there exists a constant $C_\lambda > 0$ depending only on $\lambda$ such that

$$a_n(id, \ell_p^N, \ell_\infty^N) \leq \begin{cases} 1 & \text{if } n \leq N^\lambda, \\ C_\lambda n^{-1/2} & \text{if } N^\lambda < n \leq N, \\ 0 & \text{if } n > N. \end{cases}$$

(ii) There exists a constant $C > 0$ independent of $n$ such that for any $n \in \mathbb{N}$

$$a_n(id, \ell_p^{2n}, \ell_\infty^{2n}) \geq Cn^{-1/2}.$$
**Theorem 5.2.** Let $\alpha > 0$, $\delta \neq \alpha$, $t = \min(p_1', p_2)$ and $\frac{1}{p} = \frac{\mu}{d} + \frac{1}{p_1}$, where $\mu = \min(\alpha, \delta)$. Further, suppose $0 < p_1 \leq p_2 \leq \infty$ or $\tilde{p} < p_2 < p_1 \leq \infty$.

Denote by $a_n$ the $n$th approximation number of the Sobolev embedding

$$A^{\delta_1}_{p_1,q_1}(\mathbb{R}^d, w_\alpha) \hookrightarrow A^{\delta_2}_{p_2,q_2}(\mathbb{R}^d).$$

Then $a_n \sim n^{-\kappa}$, where

(i) $\kappa = \frac{d}{2}$ if $0 < p_1 \leq p_2 \leq 2$ or $2 \leq p_1 \leq p_2 \leq \infty$,

(ii) $\kappa = \frac{d}{2} + \frac{1}{p_1} - \frac{1}{p_2}$ if $\tilde{p} < p_2 < p_1 \leq \infty$,

(iii) $\kappa = \frac{d}{2} + \frac{1}{r} - \frac{1}{s}$ if $0 < p_1 < 2 < p_2 \leq \infty$ and $\mu > \frac{d}{2}$,

(iv) $\kappa = \frac{d}{2} + \frac{1}{r}$ if $0 < p_1 < 2 < p_2 \leq \infty$ and $\mu < \frac{d}{2}$.

**Proof.** We only sketch the proof in the case when $0 < p_1 \leq 1$ and $p_2 = \infty$. For the proofs in the other cases, one can consult [25], cf. also [4] for the asymptotic estimates of the Euclidean ball in the quasi-Banach case. The crucial point in the following step is the proper choice of $\lambda$.

**Step 1** (Upper estimates). $0 < p_1 \leq 1$ and $p_2 = \infty$ imply $t = \infty$. We select $0 < \lambda < 1$ such that $\frac{\lambda}{2(1-\lambda)} < \frac{\min(\alpha, \delta)}{d}$. The inequality $\lambda \cdot \frac{1}{s} \leq \frac{1}{s} - \frac{1}{2}$ holds if and only if $\frac{1}{s} \geq \frac{1}{2} \cdot \frac{1}{2(1-\lambda)}$, where $0 < \lambda < 1$. Then, we find by (5.1) that for any $N \in \mathbb{N}$

$$L^{(a)}_{s,\infty}(\text{id}, \ell^N_{p_1}, \ell^N_{p_2}) \leq C \left( \frac{\lambda}{N^{2(1-\lambda)}} \right), \quad \text{if} \quad \frac{1}{s} = \frac{1}{2(1-\lambda)},$$

(5.4)

Next, our proof may mimic that of Proposition 11 in [25]. As to the precise definitions of $P$ and $Q$, we refer to the counterpart there again. For the estimation of $a_n(P)$, we choose $s$ such that $\frac{1}{s} > \frac{1}{2(1-\lambda)}$ and $d\left(\frac{1}{s} - \frac{1}{2}\right) > \min(\alpha, \delta)$, and proceed by using (5.5). For the estimation of $a_n(Q)$, we choose $s = h = 2(1-\lambda)$, and use (5.4) instead. Note that $\lambda \cdot \frac{1}{s} = \frac{1}{h} - \frac{1}{2} < \frac{\min(\alpha, \delta)}{d}$ for the above choice of $\lambda$.

**Step 2** (Lower estimates). We only need to consider two cases, $0 < \delta \leq \alpha$ or $0 < \alpha \leq \delta$. And we choose $\ell = \left[ \frac{N}{2} \right]$ where $N$ is taken in the same way as in point (i) or (ii) of Step 4 of Proposition 11 in [25], respectively, and use (5.2).

**Remark 5.3.** Note that in the above assertion point (iv) vanishes if $0 < p_1 \leq 1$ and $p_2 = \infty$. Moreover, the two function spaces in the embedding (5.3) may be of different types, i.e., one is the Besov space, and the other is the Triebel-Lizorkin space.

The comparison of these above theorems, shows that, for Sobolev embeddings of weighted function spaces of Besov and Triebel-Lizorkin type, with $p < \infty$ for the $F$-spaces,

(i) $a_n \sim c_n$ if either

(a) $2 \leq p_1 < p_2 \leq \infty$ or,

(b) $\tilde{p} < p_2 \leq p_1 \leq \infty$ or,

(c) $1 \leq p_1 < p_1' \leq p_2 \leq \infty$ and $\min(\alpha, \delta) \neq \frac{d}{p_1}$.
(ii) \( a_n \sim d_n \) if either
   \( (a) 0 < p_1 < p_2 \leq 2 \) or,
   \( (b) \tilde{p} < p_2 \leq p_1 \leq \infty \),
   \( (c) 0 < p_1 < 2 < p_2 \leq p_1' \leq \infty \) and \( \min(\alpha, \delta) \neq \frac{d}{p_2} \).

(iii) \( c_n \sim d_n \) if either
   \( (a) \tilde{p} < p_2 \leq p_1 \leq \infty \),
   \( (b) 1 \leq p_1 < p_1' = p_2 \leq \infty \) and \( \min(\alpha, \delta) \neq \frac{d}{p_2} \).

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