Smoothed Podgórski Strength Criterion

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Abstract. A modification of the Podgórski limit state criterion is proposed in order to smoothen the plasticity surface allowing for application of the yield function in plastic evolution law. Convexity of the potential is discussed and calibration of five material parameters is carried out using common experimental tests, while two parameters remain arbitrary. Comparison of the model prediction with the available empirical data for concrete is presented and then recommendations are given in reference to the values of two arbitrary chosen parameters.

1. Introduction

Definition of the limit state criterion and the plastic potential for concrete and other frictional materials is still a widely researched problem. Assuming isotropic behaviour of the materials, the respective yield functions depend on three invariants of the stress tensor. Experimental data on failure of such materials in triaxial stress states show several challenging features which should be reproduced by theoretical models [1,3,7,8,9]. It is commonly accepted that continuous, differentiable and convex functions are the proper choice to describe both the onset of the plastic flow as well as its evolution [1,5,6,10].

In this paper, an attempt is being made to modify one of the best of the failure criteria available for frictional materials - the Podgórski criterion [10,11] being a generalization of another well-known criterion of Ottosen [9]. Both conditions serve as failure criteria with vertex singularity at equal triaxial tension state. The proposed modification converts those criteria to be represented by the surface smoothed at the singular point - gradient of the yield potential is a continuous function. However, the regularization leaves singularity of curvature at the surface apex. In the following sections we describe the proposed potential. Convexity considerations and free parameter calibration are presented, as well as a comparison of the criterion predictions with available empirical data for concrete is carried out.

2. Definition of the modified Podgórski strength criterion

The cylindrical invariants $\xi$, $r$ and $\Theta$ of the stress tensor $\sigma$ are defined as:

$$\xi = \frac{1}{\sqrt{3}} \text{tr}\sigma, \quad r = \|s\| = \sqrt{\text{tr}s^2}, \quad \Theta = \frac{1}{3} \arccos \left( \frac{\sqrt{6} \text{tr}s^3}{\sqrt{\text{tr}s^2}} \right)$$

and $s = \sigma - \xi k$, $k = I / \sqrt{3}$, where $I$ is the second order unit tensor.

It has been proposed the following modification of the Podgórski yield criterion:
\[ f(\xi, r, \Theta) = \lambda r^n + \alpha \left[ r g(\Theta) \right]^m + \xi - \beta = 0, \]

where \( g(\Theta) > 0 \) is the deviatoric section shape function defined in [10,11] and later considered in paper [1]:

\[ g(\Theta) = \cos \left[ \frac{1}{3} \arccos (\gamma \cos 3\Theta) - \frac{1}{3} \arccos (\delta) \right]. \]

Function \( g(\Theta) \) dependent on the Lode angle \( \Theta \) determines the shape of the deviatoric section of the yield locus. A curve located on the octahedral plane defined by (2) for given \( \xi \) has to be periodic of period \( 2\pi / 3 \) with the axes of symmetry described by \( \Theta = k \pi / 3 \) for \( k = 0,1,2 \). To maintain this property invariant \( \cos 3\Theta \) can be used in the definition of a shape function [1,10].

In (2) and (3) \( \alpha, \beta, \gamma, \delta, \lambda \) and \( m, n \) are material constants. The first five parameters can be calibrated based on five common experimental tests, while two remaining parameters \( m \) and \( n \) are free parameters with recommended values: \( m \approx 2 \) and regularization parameter \( n > 1 \) but close to one. The original Podgórski criterion is obtained for \( m = 2 \) and \( n = 1 \) [10,11]. Furthermore, for \( \delta = 1 \) formula (2) can be reduced to the Ottosen failure condition [9]. If \( \lambda = 0 \) and \( g = 1 \) the classical Mises-Schleicher criterion is obtained [4,12], and additionally for \( n = 1 \) the Drucker-Prager yield condition is recovered [2].

2.1. Convexity

Introduction of the notation:

\[ \psi = \frac{1}{3} \arccos (\gamma \cos 3\Theta), \quad \chi = \frac{1}{3} \arccos (\delta) \quad \text{and} \quad \omega = \psi - \chi \]

implicates:

\[ \cos 3\chi = \delta \quad \text{with} \quad -1 \leq \delta \leq 1 \quad \text{and} \quad 0 \leq \chi \leq \pi / 3, \]

and:

\[ \cos 3\psi = \gamma \cos 3\Theta \quad \text{with} \quad -1 \leq \psi \leq 1, \quad 0 \leq \psi \leq \pi / 3 \quad \text{and} \quad -\pi / 3 \leq \omega \leq \pi / 3, \]

which results in \( \sin \psi \geq 0 \), \( \cos \psi \geq 0 \), \( \sin 3\psi \geq 0 \), \( 1/2 \leq \cos \omega \leq 1 \), and consequently in the following relationships:

\[ \frac{d\omega}{d\Theta} = \frac{d\psi}{d\Theta} = \frac{\gamma \sin 3\Theta}{\sqrt{1 - \gamma^2 \cos 3\Theta}} = \frac{\gamma \sin 3\Theta}{\sin 3\psi}, \]

Now, equation (3) can be rewritten as:

\[ g(\Theta) = \cos(\psi - \chi) = \cos \omega, \]

Using the theorem of convexity of a function being a sum of convex functions:

\[ f = f_1 + f_2, \quad \text{where} \quad f_1(\xi, r) = \lambda r^n + \xi - \beta, \quad f_2(r, \Theta) = \alpha \left[ r g(\Theta) \right]^m, \]

convexity of (2) can be proven. In case of the first function conditions are the following [1,5]:

\[ \psi, \chi \geq 0, \quad \gamma \geq 0, \quad 3\psi \geq 0, \quad 1/2 \leq \cos \omega \leq 1, \]

and:

\[ \psi, \chi \geq 0, \quad \gamma \geq 0, \quad 3\psi \geq 0, \quad 1/2 \leq \cos \omega \leq 1, \]

which results in \( \sin \psi \geq 0 \), \( \cos \psi \geq 0 \), \( \sin 3\psi \geq 0 \), \( 1/2 \leq \cos \omega \leq 1 \), and consequently in the following relationships:
\[ \lambda m (m-1)r^{m-2} \geq 0, \quad \lambda m r^{m-2} \geq 0, \quad \text{which is always met for } \lambda \geq 0 \text{ and } m \geq 1. \quad (10) \]

In the second case, the convexity requirements lead to the system of inequalities [1,5]:

\[
\alpha (n-1)r^{n-2}g^* \geq 0, \quad \alpha n r^{n-2}g^* - \left[ g^* + (n-1)(g')^2 + gg^* \right] \geq 0,
\alpha^2 n^2 (n-1)r^{2(n-1)}(g + g^*) \geq 0. \quad (11)
\]

Conditions (11) are fulfilled when:

\[
\alpha \geq 0, \quad n \geq 1, \quad g(\Theta) > 0 \quad \text{and} \quad g(\Theta) + g^*(\Theta) \geq 0,
\]

Using (6) and (7), the last condition in (12) can be written as:

\[
\left( 1 - \gamma^2 \right) (\cos \omega \sin 3\psi - 3 \sin \omega \cos 3\psi) \geq 0,
\]

Since \(-1 \leq \gamma \leq 1\), expression in the second bracket of (13) can be converted to the form:

\[
\chi \geq \psi - \arctg \left( 1 \text{tg} 3\psi \right) \quad \text{and} \quad \chi \leq \psi - \arctg \left( 1 \text{tg} 3\psi \right)
\]

for \(0 < \psi < \pi / 6\) and \(\pi / 6 < \psi < \pi / 3\), accordingly. Conditions (14) are satisfied when restrictions (5) and (6) are met. It can be shown that for \(\psi = 0\), \(\psi = \pi / 6\), \(\psi = \pi / 3\) convexity requirement is fulfilled. Note that for \(\gamma = \pm 1\) shape function (3) defines a surface with sharp corners for \(\Theta = 0\) and \(\Theta = \pi / 3\) [1,5,6,10]. That is the reason why those values are neglected in the above derivations.

2.2. Calibration

Material parameters \(\alpha, \beta, \gamma, \delta, \lambda\) are calibrated based on five commonly used experimental tests listed in table 1 [7-9]. Two of them are located on the tensile meridian (\(\Theta = 0\)), the next two - on the compressive meridian (\(\Theta = \pi / 3\)) and the last one on the shear meridian (\(\Theta = \pi / 6\)).

| Principal stresses, invariants | Uniaxial tension | Uniaxial compression | Biaxial compression | Triaxial compression | Biplane shear |
|-------------------------------|----------------|---------------------|--------------------|---------------------|-------------|
| \(\sigma_1\)                 | \(\sigma_T\)   | 0                   | \(-\sigma_{BC}\)   | \(-\sigma_{TC}\cdot \eta > 1\) | 0           |
| \(\sigma_2\)                 | 0               | 0                   | \(-\sigma_{BC}\)   | \(-\sigma_{TC}\)   | \(-\sigma_{BS} / 2\) |
| \(\sigma_3\)                 | 0               | \(-\sigma_c\)       | \(-\sigma_{BC}\)   | \(-\eta \sigma_{TC}\) | \(-\sigma_{BS}\) |
| \(\xi\)                      | \(\frac{\sigma_T}{\sqrt{3}}\) | \(\xi_C = \frac{\sigma_c}{\sqrt{3}}\) | \(\xi_{BC} = \frac{2\sigma_{BC}}{\sqrt{3}}\) | \(\xi_{TC} = \frac{\sigma_{TC} + \eta}{\sqrt{3}}\) | \(\xi_{BS} = \frac{\sigma_{BS}}{\sqrt{3}}\) |
| \(r\)                        | \(\frac{\sigma_T}{\sqrt{3}}\) | \(\frac{\sigma_c}{\sqrt{3}}\) | \(\frac{\sigma_{BC}}{\sqrt{3}}\) | \(\frac{\sigma_{TC} + \eta}{\sqrt{3}}\) | \(\frac{\sigma_{BS}}{\sqrt{3}}\) |
| \(\Theta\)                   | 0               | \(\Theta = \pi / 3\) | 0                 | \(\Theta = \pi / 3\) | \(\Theta = \pi / 6\) |

Denoting by \(g_T\), \(g_C\) and \(g_S\) values of the shape function (3) for the characteristic meridians, one can obtain:
\[ g_r = g(0) = \cos(\varphi - \chi), \quad g_c = g\left(\frac{\pi}{3}\right) = \cos\left(\frac{\pi}{3} - \varphi - \chi\right), \quad g_s = g\left(\frac{\pi}{6}\right) = \cos\left(\frac{\pi}{6} - \chi\right) , \]  

(15)

where additional notation is introduced, and the following useful identity is exploited:

\[ \varphi(\gamma) = \frac{1}{3} \arccos \gamma, \quad \varphi(-\gamma) = \frac{\pi}{3} - \varphi(\gamma) , \]  

(16)

Using the values of the stress invariants from table 1, notation (15) and (16) in the strength criterion (2) one can derive formula for the material parameter:

\[ \lambda = \frac{(\xi_T - \xi_C) r_{bc}^m r_{TC}^m + (\xi_c - \xi_{BC}) r_{TC}^m r_{TC}^m + (\xi_B - \xi_{IC}) r_{TC}^m r_{TC}^m - (\xi_T - \xi_{IC}) r_{bc}^m r_{bc}^m}{(r_{TC}^m - r_{bc}^m) r_{bc}^m r_{TC}^m + (r_{BC}^m - r_{TC}^m) r_{TC}^m r_{TC}^m + (r_{IC}^m - r_{TC}^m) r_{TC}^m r_{TC}^m - (r_{bc}^m - r_{TC}^m) r_{bc}^m r_{bc}^m} , \]

(17)

and then for the ratios:

\[ \frac{g_r^n}{g_c^n} = \frac{(\xi_T - \xi_{BC}) r_{bc}^m - \lambda (r_{bc}^m - r_{bc}^m) r_{bc}^m}{(\xi_c - \xi_{BC}) r_{BC}^m - \lambda r_{BC}^m r_{bc}^m + (r_{bc}^m - r_{BC}^m) r_{bc}^m} , \]  

(18)

\[ \frac{g_s^n}{g_c^n} = \frac{(\xi_T - \xi_{BC}) r_{bc}^m - \lambda (r_{bc}^m - r_{bc}^m) r_{bc}^m}{(\xi_c - \xi_{BC}) r_{BC}^m - \lambda r_{BC}^m r_{bc}^m + (r_{bc}^m - r_{BC}^m) r_{bc}^m} , \]  

(19)

Next, using (15), (16) and the ratios of the characteristic values of the shape function defined as:

\[ p = \frac{g_r}{g_c} , \quad k = \frac{g_r}{g_s} , \]

(20)

\( g_r, g_c \) and \( g_s \) are expressed via \( p \) and \( k \) parameters:

\[ g_s = \sqrt{\frac{4p^2 - k^2 (1+p)^2}{4p(p-k^2)}} , \quad g_r = kg_s, \quad g_c = \frac{k}{p} g_s , \]  

(21)

Using trigonometric identities, deviatoric section shape parameters \( \gamma \) and \( \delta \) can be found as a function of \( p \) and \( k \), i.e.:

\[ \gamma = \frac{k^2 (1+p)^2 - p^2 4p^2 - k^2 (1+p)^2}{2p^3} , \quad \delta = \frac{k(p-1)[3p^2 - k^2 (p^2 + p + 1)]}{2\sqrt{(p-k^2)^3}} , \]  

(22)

Obtained results (21) and (22) are supposed to meet the following restrictions:

\[ 1 \leq p < 2 , \quad \frac{\sqrt{3p}}{1+p} \leq k \leq \frac{p}{\sqrt{p^2 + p + 1}} \quad \text{then} \quad -1 < \gamma < 1 , \quad -1 \leq \delta \leq 1 , \]

(23)

Limits on \( p \) parameter follow its range reported in experiments for concrete and frictional materials [3,7,8,9]. Finally, the two remaining parameters can be evaluated:
\[ \alpha = \frac{\varepsilon_T - \varepsilon_{BC} - \lambda (r^{m}_T - r^{n}_T)}{r^{n}_T}, \quad \beta = \frac{\varepsilon_T r^{m}_T - \varepsilon_{BC} r^{n}_T + \lambda (r^{m}_T r^{m}_T - r^{n}_T r^{m}_T)}{r^{m}_T - r^{n}_T}, \]  
(24)

2.3. Plane stress conditions

In case of the plane stress conditions the following invariants of the plane stress tensor \( \sigma \) and its deviator \( s \) are used:

\[ \bar{\varepsilon} = \frac{1}{\sqrt{2}} \text{tr} \bar{\sigma} = \frac{1}{\sqrt{2}} (\sigma_1 + \sigma_2), \quad \bar{r}^2 = \text{tr} \bar{s}^2 = \frac{1}{2} (\sigma_1 - \sigma_2)^2. \]  
(25)

Then invariants (1) can be expressed as:

\[ \xi = \sqrt{\frac{2}{3}} \bar{\varepsilon}, \quad r^2 = \bar{r}^2 + \frac{1}{3} \bar{\varepsilon}^2, \quad \cos 3\Theta = \frac{\bar{\varepsilon} (9\bar{r}^2 - \bar{\varepsilon}^2)}{\sqrt{(3\bar{r}^2 + \bar{\varepsilon}^2)}}. \]  
(26)

Using relationships (26) in (2) and (3) the yield function and the shape function can be expressed via new arguments, i.e.: \( \bar{f}(\xi, \bar{r}) = 0 \) and \( \bar{g}(\xi, \bar{r}) \).

3. Results and discussions

In this study, five parameters \( \alpha, \beta, \gamma, \delta, \lambda \) are determined based on the chosen tests, while two remaining parameters \( m \) and \( n \) are arbitrary. Typical strength ratios for concrete are taken from literature: \( \sigma_T = 0.1 \sigma_C, \sigma_{BC} = 1.6 \sigma_C, \sigma_{BS} = 1.28 \sigma_C, \sigma_{TC} = 1.25 \sigma_C \) with \( \eta = 4.91 \) [3-7-9]. Yield limit in the uniaxial compression test \( \sigma_C \) is treated as a scaling factor in the following graphical presentation of the obtained results. Results of the calibration for several values of \( m \) and \( n \) parameters are presented in table 2.

| Parameter | \( n=1 \) | \( n=1.1 \) | \( n=1.1 \) |
|-----------|--------|--------|--------|
| \( m=1.8 \) | \( m=2 \) | \( m=2.2 \) | \( m=1.8 \) | \( m=2 \) | \( m=2.2 \) |
| \( \lambda \sigma_{C}^{3-m} \) | 0.1798 | 0.1152 | 0.0811 | 0.1181 | 0.0783 | 0.0543 |
| \( p \) | 1.845 | 1.793 | 1.765 | 1.720 | 1.689 | 1.672 |
| \( k \) | 1.138 | 1.133 | 1.130 | 1.121 | 1.117 | 1.115 |
| \( g_S \) | 0.884 | 0.825 | 0.8848 | 0.8923 | 0.8950 | 0.8964 |
| \( g_C \) | 0.5419 | 0.5577 | 0.5666 | 0.5814 | 0.5921 | 0.5980 |
| \( g_T \) | 1.000 | 1.000 | 1.000 | 0.9999 | 0.9999 | 0.9999 |
| \( \gamma \) | 0.9974 | 0.9948 | 0.9928 | 0.9922 | 0.9892 | 0.9874 |
| \( \delta \) | 0.9971 | 0.9948 | 0.9932 | 0.9863 | 0.9832 | 0.9814 |
| \( \alpha \sigma_{C}^{1-m} \) | 1.439 | 1.496 | 1.532 | 1.470 | 1.511 | 1.536 |
| \( \beta / \sigma_C \) | 0.1771 | 0.1806 | 0.1831 | 0.1525 | 0.1543 | 0.1556 |

During the calibration, limits on parameters must be carefully checked due to the convexity requirement, since those conditions can be violated for the given experimental data or assumed values of \( m \) and \( n \). For example, in figure 1 a graph of function \( \lambda(n) \) for \( m=1 \) is shown, which passes through zero when
\( n = 1.2498 \). Therefore convexity condition for the given data is violated for \( n > 1.2498 \). Similar issue occurs when dependence of \( p(n,m) \) is investigated. In the next graph of figure 1 upper limit \( p = 2 \) is crossed when \( m = 1.5629 \) for \( n = 1 \), or when \( m = 1.4169 \) with \( n = 1.1 \). Shape parameter \( k(n,m) \) is shown in the third graph with upper and lower limits set in (23), with similar values of \( m \) causing loss of convexity.

**Figure 1.** Dependence of \( \lambda, p \) and \( k \) on the assumed values of \( m, n \)

In figure 2 a graph of vertex zone of the plasticity surface in meridional section is shown for three values of \( n \) parameter. In the same figure a graph of radial component of the yield function gradient \( f_r \) is plotted, showing no singularity at the vertex for \( n > 1 \).

**Figure 2.** Meridional cross-section and gradient component \( f_r \) in the vicinity of the vertex for \( m = 2 \)

In figure 3 partial graphs of the deviatoric section of the plasticity surface are shown for three values of \( n \) parameter. For fixed \( m = 2 \) cross-sections are plotted for hydrostatic pressure levels \( \xi = \xi_r \) and \( \xi = \xi_c \). It can be observed that with increasing value of \( n \) the deviatoric section becomes less triangular (i.e. with more rounded corners) than for \( n = 1 \).

**Figure 3.** Change of shape of the deviatoric cross-section for \( \xi = \xi_r \) and \( \xi = \xi_c \) with \( m = 2 \)
In figure 4 plane stress cross-sections of the yield surface are presented for three values of $n$ or $m$ parameter. It is apparent that with increasing $n$, the curvature is modified, particularly in the regions of biaxial compression and of biaxial tension. On the other hand, a change in $m$ influences only biaxial compression zone of the cross-section. Those improvements are useful when refining the existing models, especially considering the fact that plane stress conditions are often encountered in various types of constructions.

![Figure 4. Dependence of shape of the yield curves for the plane stress on the values of $n$ and $m$](image)

In figure 5 and figure 6 prediction of the failure criterion is compared with experimental data available in literature [3,7,8]. Calibration points are symbolically marked in the figures. Based on the comparison, values of $n = 1.05$ and $m = 1.85$ are recommended to use in applications. As shown in figure 5, for the chosen values the yield surface covers a broad range of the stress states, up to $\xi / \sigma_c \sim 20$. However, for the plane stress condition, the value of $m$ could be adjusted to the class of concrete, as presented in figure 6. For the low strength concrete $m > 2$ is recommended, while for the normal concrete $m = 1.85$ is more suitable.

![Figure 5. Comparison of predictions of the criterion with experimental data [3,7,8] for the tensile and compressive meridians for the recommended values of $n$ and $m$](image)
4. Conclusions
In conclusion, introduction of two free parameters $n$ and $m$ into the original model facilitates the use of the smoothed Podgórski criterion in plasticity theory and then in structural analysis. Moreover, it was shown that using this criterion one can improve agreement of the prediction with available experimental results over a wide range of data.

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