NONPARAMETRIC TREATMENT EFFECT IDENTIFICATION IN SCHOOL CHOICE

JIAFENG CHEN

ABSTRACT. We study nonparametric identification and estimation of causal effects in centralized school assignment. We characterize the full set of identified treatment effects in common school choice settings, under unrestricted heterogeneity in individual potential outcomes. This exercise highlights two points of caution for practitioners: We find that lack of overlap poses a challenge to regression-based estimators; we also find that, asymptotically, regression-based estimators that aggregate across many treatment contrasts put zero weight on treatment effects identified from regression-discontinuity (RD) variation, when the mechanism allows for both RD and lottery-based variation. Due to the complex interplay between heterogeneous causal effects and school choice algorithms, we recommend empirical researchers clearly decompose aggregate causal effect estimates by sources of variation in these settings. Lastly, we provide estimators and accompanying asymptotic results for causal contrasts identified by RD variation in school choice.

Keywords: School choice, nonparametric identification, regression discontinuity, heterogeneous treatment effects

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1. Introduction

There is a rapidly growing empirical literature studying centralized school assignment (see, among others, Kapor et al., 2020; Abdulkadiroğlu et al., 2017a, 2022; Agarwal and Somaini, 2018; Fack et al., 2019; Calsamiglia et al., 2020; Abdulkadiroğlu et al., 2020, 2017b,c; Angrist et al., 2020; Beuermann et al., 2018). In particular, recent work by Abdulkadiroğlu et al. (2017a, 2022, 2020, 2017b) proposes using features of centralized school assignment for program evaluation of schools. Abdulkadiroğlu et al. (2017a) observe that certain centralized school assignment algorithms have inherent randomness, where ties between students are broken via lotteries. The randomness generates exogenous variation in school assignments that may be used to identify school effects. In subsequent work, Abdulkadiroğlu et al. (2022) observe that a wide class of school assignment mechanisms—based on deferred acceptance (Gale and Shapley, 1962)—have a particular cutoff structure where school assignments change discontinuously around a cutoff. They propose using such regression-discontinuity (RD) variation around the cutoffs to estimate RD-type treatment effects.

Centralized school choice has several complications that depart from the cross-sectional setting with multi-valued treatment selected based on observables (e.g., Goldsmith-Pinkham et al., 2021). First, in complex school choice mechanisms, treatment assignment is determined jointly via the algorithm, and thus two students’ treatments are generally not independent. The non-independence complicates identification and statistical properties of estimators. Second, conventional overlap conditions fail for a large portion of student-school pairs. For instance, a particular school might not appear on some students’ preference rankings, or students may qualify with certainty for schools they like better; in these cases, the potential outcomes of these students at the particular school are never observed. When combined with heterogeneity in potential outcomes and many treatment arms, overlap failure for specific student-school pairs makes interpreting aggregate causal effects complex.

To tackle these issues, we study causal effects and causal contrasts in school choice settings formally, and make three distinct contributions. First, we study school choice mechanisms that involve lottery tie-breakers from a design-based perspective. We show that existing tools from the design-based causal inference literature (Aronow and Middleton, 2013; Mukerjee et al., 2018) adapt well in this setting, without invoking large market approximations as Abdulkadiroğlu et al. (2017a) do. On the other hand, we show that regression-based estimators may suffer from lack of overlap. Since the regression system is underdetermined when overlap fails, automatic covariate-dropping may drive the implicit estimand.

Second, like Abdulkadiroğlu et al. (2022), we formally study more general school choice mechanisms that involve both lottery- and RD-type variation from a sampling-based perspective. In contrast to Abdulkadiroğlu et al. (2022), we do not directly assume that school assignments are independent across students, a property that only holds in the large-market limit. We also do

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1 Examples include variants of the student-proposing deferred acceptance algorithm and the immediate acceptance (Boston) mechanism.

2 Despite motivating all of our results in a school choice setting, our formal results extend easily to market design settings where agents are matched to objects or positions, especially if the assignment mechanism is deferred acceptance (Gale and Shapley, 1962), and the object of inference is the treatment effect of the agents’ assignments.
not assume constant treatment effects, and characterize the full set of conditional average treatment effect and treatment contrasts that are identified from the data. These treatment contrasts constitute the largest set of causal quantities that are point-identified under minimal assumptions.

Characterizing these causal quantities is a useful exercise for empirical research. On the one hand, it clarifies which causal quantities are identified and how these quantities relate to those that are relevant for policy- and decision-making. On the other hand, it also stress tests the robustness of common estimation approaches with respect to treatment effect heterogeneity. As a key example, we find that regression estimators based on Abdulkadiroğlu et al. (2022)’s local deferred acceptance propensity scores aggregate both RD-type and lottery-type variation. However, since RD-type variation has measure zero (identified via thin sets, in the terminology of Khan and Tamer, 2010), such aggregation leaves the lottery-type variation to asymptotically dominate the resulting causal estimand. We also show that schools that use test scores for ranking students are subjected to lottery-type variation only for students who disprefer these schools, compared to some school that uses lotteries. Since lottery-type variation dominates the regression estimand asymptotically, the regression estimand reflects causal effects of test-scores school only for students who dislike test-score schools, who may have less favorable potential outcomes to begin with. Based on these observations, we recommend empirical researchers to decompose the treatment effect estimates by sources of variation, a task that our characterization makes transparent.

Third, as a technical contribution, we provide estimators for RD-type treatment contrasts and demonstrate their consistency and asymptotic normality. Our asymptotic analysis does not abstract from the global dependence of treatment assignment, and thus departs from the standard analysis of regression discontinuity estimators (Hahn et al., 2001) as well as the propensity-score-based analysis of Abdulkadiroğlu et al. (2022). For school choice mechanisms, the dependence is induced by randomness in the RD cutoffs, as they are jointly determined. However, we find that this dependence—as it is asymptotically vanishing—contribute to a higher-order term in the stochastic expansion of the estimator, and as a result, the first-order behavior of these estimators remain standard, despite the nonstandard asymptotic sequence. In this sense, we provide formal justification for Abdulkadiroğlu et al. (2022)’s estimation approach.

Our work contributes to a recent methodological literature on causal inference in market design and school choice settings (Narita, 2021; Abdulkadiroğlu et al., 2022, 2017a; Diamond and Agarwal, 2017; Agarwal et al., 2020; Marinho et al., 2022). It is also related to natural experiment perspectives where treatments are assigned algorithmically (e.g. Narita and Yata, 2021), as well as causal inference in settings with equilibrium quantities (Munro et al., 2021). To the best of our knowledge, the asymptotics of RD estimators with random cutoffs appears novel in the RD literature as well (for an analysis assuming the cutoffs are ignorable, see Cattaneo et al., 2016).

Despite benefiting heavily from results developed by Abdulkadiroğlu et al. (2017a, 2022), our work is distinct and complementary in the following aspects. We provide a more formal analysis—for both identification and estimation—that does not ignore the dependence of treatment assignment, thereby formally justifying the approximations taken in previous work. We also do not assume

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3In parallel, there is a literature using tools from empirical industrial organization and econometrics of games in such settings; see Agarwal and Budish (2021); Agarwal and Somaini (2020) for recent reviews.
constant treatment effects. Lastly, we investigate properties of regression-based approaches undertaken in Abdulkadiroğlu et al. (2017a, 2022) and uncover certain properties that call for more care in interpreting aggregate estimands.

Finally, an important limitation to our results is that, in the absence of perfect compliance, these results only apply to intent-to-treat effects. We note that this limitation is inherent in our nonparametric perspective under a setting with multiple treatment arms (schools) and multiple instrument values, and is not specific to the school choice setting. General, nonparametric analyses of instrumental variable models with more than two treatment arms (Heckman et al., 2006; Kirkeboen et al., 2016; Behaghel et al., 2013) or with more than one instrument (Mogstad et al., 2020) are complex and areas of active research. As this paper focuses on issues specific to market design settings, we do not discuss compliance and focus on intent-to-treat effects.

This paper proceeds as follows. Section 2 introduces basic notation and school choice mechanisms. Section 3 school choice mechanisms based on lotteries from a design-based perspective, and discusses regression-based approaches in the literature. Our main results are in Section 4, which analyzes a broad class of school choice mechanisms from a sampling-based perspective. In particular, Sections 4.3 and 4.4 characterize the full set of causal quantities that are identified under minimal assumptions, and Section 4.5 illustrates the aggregation properties of the implicit estimand in Abdulkadiroğlu et al. (2022). Section 5 introduces asymptotic results for RD estimators in this setting. Section 6 concludes the paper.

2. Model, notation, and school choice mechanisms

Consider a finite set of students \( I = \{1, \ldots, N\} \) and a finite set of schools \( S = \{0, 1, \ldots, M\} \). The schools have capacities \( q_1, \ldots, q_M \in \mathbb{N} \). Assume the school 0 represents an outside option and has infinite capacity. Each student \( i \in I \) has observed (by the analyst) characteristics \( W_i = (X_i, Z_i) \). \( X_i \) is a vector of characteristics that are relevant for the assignment mechanism, and \( Z_i \) collects other observed characteristics that the analyst may condition on. In the ensuing analysis, we ignore \( Z_i \) by conditioning on a particular value \( Z_i = z \).

Each student \( i \in I \) is also associated with unobserved potential outcomes

\[
A_i = [Y_i(0), \ldots, Y_i(M)]'.
\]

Implicitly, the notation \( Y_i(s) \) defines the school \( s \) as a treatment. This rules out peer effects or other violations of the stable unit treatment value assumption (SUTVA), since the assignment statuses of other individuals \( j \neq i \) do not matter for the observed outcome for individual \( i \).\footnote{To account for potential peer effects, we note that for a known exposure mapping (Aronow and Samii, 2017), the design-based analyses extend in a natural manner.} The analyst additionally has access to the school assignments, and observes that each student \( i \) has been assigned to one of the \( M + 1 \) schools, which we may represent as a one-hot encoded binary vector \( D_i \) with a 1 at the entry corresponding to the school that \( i \) is assigned to, i.e. \( D_{is} = 1 \) if student

4In fact, Abdulkadiroğlu et al. (2022) “look forward to a more detailed investigation of the consequences of heterogeneous treatment effects for identification strategies of the sort considered here,” which is precisely the theme of this paper.
i is assigned to school s. We will slightly abuse notation and write \( Y_i = Y_i(D_i) \) as the observed outcome.

In full generality, an assignment mechanism \( \Pi_N \) is simply a map that takes the observable characteristics \( X_i \) as input and returns as output a distribution \( \Pi_N(X_1, \ldots, X_N) \) over the set of possible assignments that respect the capacity constraints at each school. The observed assignments are sampled from \( \Pi_N: (D_1, \ldots, D_N) \sim \Pi_N(X_1, \ldots, X_N) \). Moreover, we assume that \( \Pi_N(X_1, \ldots, X_N) \) is known, so that we may freely resample from \( \Pi_N \) if we wish. If \( \Pi_N(X_1, \ldots, X_N) \) is non-degenerate, we say that the mechanism is stochastic.

When the mechanism is stochastic, it is sometimes desirable to rely solely on the lottery-driven randomness for estimation of causal effects. This design-based approach treats information associated with the set of students \((W_i, A_i)_{i=1}^N\) as fixed, and treats uncertainty as arising solely from the randomness in the treatment assignment process \(\{D_i\}_{i=1}^N \sim \Pi_N(X_1, \ldots, X_N)\). We give a brief analysis in Section 3, highlighting some pitfalls of common regression-based estimators in such settings due to overlap failure in school choice settings.

However, when there is limited randomness in \( \Pi_N \), or when the analyst is interested in inference on causal effects beyond the observed set of students, the design-based approach may be undesirable. Many mechanisms have a cutoff structure that allows for RD-type variation, which provides causal identification in the absence of lotteries. In this case, estimation and inference—via a sampling-based perspective—are nevertheless possible but require specification of both a sampling process from a super-population and an asymptotic sequence.

To that end, when sampling-based results are discussed in Section 4, we assume

\[
(W_i, A_i)^\text{i.i.d.} \sim P_{W,A} \in \mathcal{P}.
\]

For a sample of \( N \), we also view each school capacity as a sequence \( q_{s,N} = \lfloor Nq_s \rfloor \), so that capacities, as proportions of \( N \), are held fixed. Importantly, for our asymptotic results in Section 5, we do not assume that the treatment is i.i.d., since for each sample of \( N \) students, the treatment variables are determined jointly through the assignment mechanism, and, as a result \( D_i \) and \( D_j \) are generally not independent. In contrast, existing approaches (Abdulkadiroğlu et al., 2022) often ignore such dependence, since the dependence is often not strong when \( N \) is large.\(^6\) The majority of our results are under this setting, which we turn to in Section 4.

### 3. Design-based analysis for stochastic mechanisms

In this section, we analyze, from a design-based perspective, mechanisms where \( \Pi_N \) has substantial randomness.\(^7\) The results in this section are self-contained, and readers may skip to the next section if they are only interested in sampling-based analyses.

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\(^6\)Ignoring the dependence is improper, but, fortunately, our formal results conclude that its statistical consequences are benign.

\(^7\)The literature on design-based inference is often limited to common forms of experimental designs—such as completely random or factorial designs. In contrast, a school assignment mechanism is considerably more general, and thus falls in the general setting of Mukerjee et al. (2018). This setting also bears some similarity to that of Borusyak and Hull (2020), though we consider the random assignment of a treatment rather than an instrument under monotonicity.
A highly general class of treatment effect parameters is the following set of linear treatment contrasts, indexed by the fixed and known coefficients $\lambda_{i,s}$:

$$
\tau = \sum_{i=1}^{N} \sum_{s=1}^{M} \lambda_{i,s} y_i(s),
$$

where we use lower-case $y_i(s)$ to denote potential outcomes, so as to emphasize that they are non-random. For instance, the sample average treatment effect between schools 1 and 2 is constructed by taking $\lambda_{i1} = 1/N$ and $\lambda_{i2} = -1/N$, whereas all other $\lambda_{i,s}$’s are zero. As another example, the sample average outcome of school 1 among those with observable information $Z_i = z$ is constructed by setting $\lambda_{i,1} = 1/N_z$, where $N_z = \# \{i : Z_i = z\}$. It is easy to see that linear contrasts nest various sample average treatment effects.

Perhaps unsurprisingly, linear contrasts of this form are identified if and only if they depend only on potential outcomes that are observed with positive probability.

**Proposition 3.1.** Let $\tau = \sum_{i=1}^{N} \sum_{s=1}^{M} \lambda_{i,s} y_i(s)$ be the parameter of interest. $\tau$ is identified if and only if, for all $i \in I, s \in S$, $\lambda_{i,s} = 0$ whenever student $i$ has no chance of being matched to school $s$:

$$
\pi_{is} \equiv E_{\Pi_N(X_1, \ldots, X_N)}[D_{is}] = 0.
$$

We relegate the proof to Appendix A.2. While straightforward, Proposition 3.1 does have implications for interpretation of estimated causal effects from lottery-based variation (Abdulkadiroğlu et al., 2017a). For instance, the average treatment effect between two schools is in general not identified, due to the presence of students for whom it is impossible that they are assigned to at least one of the schools. These students select into only one or neither of the two schools, and this selection may have policy implications. As an example, students with certain qualifiers—e.g. those who live near a particular school—are often prioritized by the school, making it more likely that they are assigned to the neighborhood school with probability one. Their counterfactual outcomes under changes to the neighborhood policy are not nonparametrically identified as a result.

Unbiased estimation of $\tau$ is straightforward via the Horvitz–Thompson estimator:

$$
\hat{\tau}_{HT} \equiv \sum_{i=1}^{N} \sum_{s=1}^{M} \lambda_{is} \frac{D_{is} y_i}{\pi_{is}} \quad \pi_{is} \equiv E_{\Pi_N(X_1, \ldots, X_N)}[D_{is}].
$$

In practice, the probabilities $\pi_{is}$ may be estimated to arbitrary precision via Monte Carlo or approximated via the asymptotic argument in Abdulkadiroğlu et al. (2017a). The Horvitz–Thompson estimator is exactly unbiased, and hold certain finite-sample optimality properties.\(^{10}\)

\(^{8}\)That is, $\tau$ is identified if no other value $\tau'$ is compatible with the distribution over the observed data $D_i, Y_i(D_i)$ induced by $\Pi_N$, holding $y_i(s)$ and $W_i$ fixed.

\(^{9}\)For instance, in 2013, Boston switched to a home-based system where students’ menus of school options depend on their residential address (Shi, 2015), where students only have access to the nearest schools. As a result, with the data from the home-based system alone, we cannot identify the effect of assigning students to schools that are farther from their addresses.

\(^{10}\)For instance, Horvitz and Thompson (1952) show that the estimator is the unique unbiased estimator among the class of homogeneous linear (in $D_{is} y_i$) estimators, whose weights do not depend on the realized assignment. Godambe and Joshi (1965) show that the estimation problem does not admit an UMVUE, but the Horvitz–Thompson estimator is admissible.
of $\hat{r}_{HT}$ depends on products of potential outcomes, and in general cannot be estimated. Conservative variance estimation is possible and the results of Aronow and Middleton (2013) apply.\textsuperscript{11}

3.1. Regression estimators under stochastic mechanisms. Instead of inverse weighting, in the context of stochastic mechanisms, Abdulkadiroğlu et al. (2017a) (expression (7)) consider using discretized assignment probabilities as covariates in a regression estimator. The estimator considered do not feature interaction between the treatment variable and the propensity score controls, which, in the typical binary-treatment, cross-sectional setting, estimates a weighted average of conditional average treatment effects (Angrist, 1998). However, Goldsmith-Pinkham et al. (2021) already show that with multiple treatments, the regression estimator without interactions estimates a non-convex weighting of the treatment effects.

Therefore, we analyze a regression estimator that does interact with the propensity score bins,\textsuperscript{12} and show that it is similar to a Horvitz–Thompson estimator for an estimand of the form (1), and derive the implicit unit-level weights $\lambda_{is}$. Such regressions estimate the maximal identifiable potential outcome average for treatment arm; however, since such averages involve different individuals for two different treatment arms, differences in the regression coefficients are not valid causal comparisons in general.

It is typical that certain sets of schools are grouped into a treatment arm. Suppose schools are partitioned into treatment levels $S_1, \ldots, S_K$. Correspondingly, for each treatment level $k$, let $e_{ik} = \sum_{s \in S_k} \pi_{is}$ be the assignment probability to the treatment level. Suppose we discretize the true assignment probabilities $e_{ik}$ into ordered bins, represented as indicator random variables $B_{k0}, \ldots, B_{kJ_k}$, where the bin $B_{k0}$ only contains $e_{ik} = 0$. This setup nests the case where the estimated propensity scores (Abdulkadiroğlu et al., 2017a) takes discrete values. Consider regressing the observed outcomes on treatment level indicators $L_{ik} = 1(s_i \in S_k)$ and their interaction with demeaned propensity score bin indicators $B_{i,k,j_k} - \bar{B}_{k,j_k}$:

$$\hat{\mu}_0, \ldots, \hat{\mu}_K = \arg \min_{\mu_0} \min_{\mu_j} \min_{j_k} \sum_{i=1}^N \left( Y_i - \sum_{k=0}^K \mu_k L_{ik} - \sum_{j_k=2}^J \sum_{k=0}^K \mu_{k,j_k} L_{ik} (B_{i,k,j_k} - \bar{B}_{k,j_k}) \right)^2.$$

We show in Lemma A.1 that $\hat{\mu}_k$ can be written as an approximate Horvitz–Thompson estimator:

$$\hat{\mu}_k = \sum_{i=1}^N \sum_{s=0}^M \frac{1(s \in S_k)\pi_{is}}{N_k \hat{e}_{ik}} \frac{D_{is} Y_i}{\pi_{is}}.$$

Here, $N_k$ is the number of students with nonzero probability for treatment arm $k$. Suppose student $i$ has $e_{ik}$ falling in the $j_k$th bin. Then $\hat{e}_{ik}$ uses the empirical proportion of those in the $j_k$th bin assigned to a school in treatment level $k$ to estimate $e_{ik}$. Estimating $e_{ik}$ may have finite-sample benefits over the Horvitz–Thompson estimator in terms of mean-squared error, at the cost of exact unbiasedness.

\textsuperscript{11}For distributional inference here, we would need a finite-population central limit theorem (Li and Ding, 2017). To the best of our knowledge, such a characterization is not available for complex market design settings, and we leave an exploration for future work.

\textsuperscript{12}Appendix A.3 shows that versions of the regression estimator without interactions are approximately unbiased under constant treatment effects.
The implicit estimand that corresponds to \( \hat{\mu}_k \) is thus the following linear treatment contrast, which weighs schools within treatment level \( k \) by their assignment probabilities at the individual level:

\[
\mu_k = \mathbb{E}_{\Pi_N(X_1,\ldots,X_N)} \left[ \sum_{i=1}^N \sum_{s=0}^M \lambda_{is}^{(k)} \frac{D_{is}Y_i}{\pi_{is}} \right] = \frac{1}{N_k} \sum_{i=1}^N \sum_{s \in S_k} \pi_{is} y_i(s),
\]

\( \mu_k \) is a maximal estimand in the sense that it only excludes individuals with \( e_{ik} = 0 \), whose causal effects for arm \( k \) are not identified. The weights \( \lambda_{is}^{(k)} \) are reasonable weights for causal means, in the sense that (i) \( \lambda_{is}^{(k)} \geq 0 \), (ii) \( \sum_s \lambda_{is}^{(k)} \in \{0,1/N_k\} \), and (iii) \( \sum_i \lambda_{is}^{(k)} = 1 \).

However, in general, for \( k \neq \ell \), the difference \( \lambda_{is}^{(k)} - \lambda_{is}^{(\ell)} \) are not reasonable weights for causal contrasts, since \( \sum_s (\lambda_{is}^{(k)} - \lambda_{is}^{(\ell)}) \neq 0 \), as the set of students with positive assignment probability to treatment type \( k \) may differ from those with the same property for type \( \ell \). Motivated by this, we might wish to design a regression estimator that estimates a causal contrast. Somewhat unexpectedly, canonical regression estimators for causal contrasts may not estimate valid causal contrasts when the overlap condition fails. Overlap failure is very common in the school choice setting. We illustrate this point with a general result, Lemma 3.2, about average treatment effect estimation under unconfoundedness with overlap failure.

**Lemma 3.2.** Consider real-valued random variables \( (Y(0), Y(1), D, X) \sim P \), where the distribution \( P \) has finite second moments. Suppose \( D \in \{0, 1\} \), \( X \in \{0, \ldots, J\} \). Let the observed outcome be \( Y = Y(D) \). Assume that treatment is unconfounded: \( Y(0), Y(1) \perp D \mid X \). Let \( X_j = \mathbb{1}(X = j) \), \( \nu_j = \mathbb{E}[X_j] \), \( \tilde{X}_j = X_j - \nu_j \), and \( \pi_j = \mathbb{P}(D = 1 \mid X_j = 1) \). Suppose the baseline level \( X = 0 \) has overlap: \( 0 < \pi_0 < 1 \). Then:

1. If, for all \( j \), \( 0 < \pi_j < 1 \), then the regression coefficient in the following population regression recovers the average treatment effect

\[
\beta_0 = \arg \min_{\beta} \min_{\alpha,\gamma,\delta} \mathbb{E} \left[ \left( Y - \alpha - D\beta - \sum_{j=1}^J \tilde{X}_j \gamma_j - \sum_{j=1}^J \delta_j D \tilde{X}_j \right)^2 \right] = \mathbb{E}[Y(1) - Y(0)].
\]

2. If, for any \( j \), \( \pi_j \in \{0, 1\} \), then the population projection coefficient \( \beta \) is not identified: i.e. \( D \) is a linear combination of \( 1 \) and \( \{\tilde{X}_j, D\tilde{X}_j\}_{j=1}^J \).

3. Let \( G = \{j : \pi_j \in \{0, 1\}\} \). The regression dropping interactions for indices in \( G \) estimates a particular aggregation of the conditional average treatment effects:

\[
\tilde{\beta}_0 = \arg \min_{\beta} \min_{\alpha,\gamma,\delta} \mathbb{E} \left[ \left( Y - \alpha - D\beta - \sum_{j=1}^J \tilde{X}_j \gamma_j - \sum_{j=1,j \notin G}^J \delta_j D \tilde{X}_j \right)^2 \right]
\]

\[
= \left( \nu_0 + \sum_{j \in G} \nu_j \right) \mathbb{E}[Y(1) - Y(0) \mid X = 0] + \sum_{j=1,j \notin G}^J \nu_j \mathbb{E}[Y(1) - Y(0) \mid X = j].
\]

In particular, \( \tilde{\beta}_0 \) is not invariant to the choice of the baseline covariate level \( X = 0 \).

4. Let \( \tilde{X}_j = X_j - \frac{\nu_j}{1 - \sum_{k \in G} \nu_k} \). The regression in (3) with \( \tilde{X}_j \) instead of \( X_j \) recovers the maximal identifiable average treatment effect \( \mathbb{E}[Y(1) - Y(0) \mid X \notin G] \).
The first claim of Lemma 3.2 restates the well-known result (e.g. Imbens and Wooldridge, 2009) that the interacted regression with properly demeaned covariates recovers the average treatment effect. The second and third claim show that, when the overlap condition fails, the interacted regression is non-identified due to collinearity; in this case, automatic covariate dropping in typical statistical software routines recovers convex-weighted average treatment effects that are not invariant to the choice of the baseline covariate level. The fourth claim states that modifying the covariate-demeaning—which amounts to dropping problematic observations—recovers the maximal identifiable causal contrast.

To practitioners, Lemma 3.2 offers a warning that regression estimators may have unreasonable estimands when overlap fails, unless care is taken to manually purge observations that do not select into the relevant treatment arms with positive probability. It is perhaps more transparent and simpler using the appropriate Horvitz–Thompson for these causal contrasts instead.13

4. SAMPLING-BASED ANALYSIS

We now turn to causal analysis from a sampling-based perspective. That is, we think of $(W_i, A_i)^{i.i.d.} \sim P_{W,A} \in \mathcal{P}$ as drawn from a super-population; for each sample of $N$ students, the treatments are drawn from $\Pi_N(X_1,\ldots,X_N)$. The dependence of treatment assignments, mediated through the mechanism, deserves some care. As a result, we begin with defining causal identification in Section 4.1 for a class of mechanisms with large-market structure. In Section 4.2, we introduce the class of mechanisms that we specifically consider. After setting up these primitives, Section 4.3 introduces a running example and characterizes identified treatment effects in the stylized example, as a prelude to formal analyses in Section 4.4. Finally, in Section 4.5, we illustrate some potentially undesirable properties of regression estimators that aggregate over different treatment effects.

4.1. Large-market approximation and identification. We recall that treatment assignments are necessarily dependent due to centralized matching. Dealing with this dependence of treatment assignments can be challenging. As often noted in previous work (Abdulkadiroğlu et al., 2017a, 2022), in many real-world mechanisms, the dependence is limited as it vanishes as $N \to \infty$ (Azevedo and Leshno, 2016). We formalize this notion of a large-market approximation in the definition below, and make a remark on its implication for the identification of causal effects. Although these ideas are not new, stating them precisely is, to our knowledge, novel. Assignment mechanisms, for which the results of Azevedo and Leshno (2016) apply, satisfy Definition 4.1.

**Definition 4.1.** We say a mechanism $\Pi_N$ has a large-market approximation under the sampling model $\mathcal{P}$ if for every $P_{W,A} \in \mathcal{P}$,

1. The assignment mechanism is such that $\Pi_N(X_1,\ldots,X_N)$ depends on a real-valued vector $C_N$, called cutoffs, computed from $X_1,\ldots,X_N$:

   $$\Pi_N(X_1,\ldots,X_N) = \Pi(X_1,\ldots,X_N; C_N).$$

2. $C_N \overset{p}{\to} c(P_{W,A})$ under the asymptotic sequence $(W_i, A_i)^{i.i.d.} \sim P_{W,A}$.

13 Or other weighting estimators with potentially superior variance properties (Khan and Ugander, 2021).
(3) For a fixed $c$, let 
\[
(D_i^*(c))_{i=1}^N \mid (X_1, \ldots, X_N) \sim \Pi_N(X_1, \ldots, X_N; c),
\]
then the induced joint distribution on $(W_i, A_i, D_i^*(c))_{i=1}^N$ is i.i.d. over $i$.

Since the proper asymptotic sequence here has a triangular array structure—$(W_i, A_i)$ are i.i.d., but the treatment assignments are jointly determined for every sample of $N$—some refinements to the conventional definition of identification\(^{14}\) are called for. Certain parameters may be point-identified conventionally but not consistently estimable, since the variation required to identify such a parameter may vanish as $N \to \infty$. We give an example in Appendix A.1 to illustrate the conceptual difficulties.

To avoid these difficulties, we use the following refined notion of identification for the rest of the paper.

**Definition 4.2.** We say a parameter $\tau(P_{W,A})$ is identified at $P_{W,A} \in \mathcal{P}$ if

1. The mechanism $\Pi_N$ has a large market approximation.
2. Let $Q(P) \overset{d}{=} (W_i, D_i^*(c(P_{W,A})), Y_i(D_i^*(c(P_{W,A}))))$ be the distribution of the observed large-market data—that is, data if the cutoffs were fixed at $c(P_{W,A})$, when $(W, A) \sim P$. Then there does not exist some $P'_{W,A} \in \mathcal{P}$ where $\tau(P_{W,A}) \neq \tau(P'_{W,A})$ but $Q(P_{W,A}) \sim Q(P'_{W,A})$.

### 4.2. Mechanisms with deferred acceptance priority scores

In this subsection, we describe a large class of assignment mechanisms that satisfy Definition 4.1, following examples in Abdulkadiroğlu et al. (2017a, 2022). These mechanisms can be viewed as first computing a set of (possibly random) priority scores from student characteristics $X_1, \ldots, X_N$, and then use the priority scores to generate school rankings of students via the deferred acceptance algorithm (Gale and Shapley, 1962; Roth and Sotomayor, 1992).

For these mechanisms, we assume the assignment-relevant observable information $X_i$ contains the student’s preferences $\succ_i$ and *eligibility information* $R_i$ related to the student’s priority order at schools: $X_i = (\succ_i, R_i)$. The mechanism $\Pi_N$ then assigns each student a priority score at each school $s$\(^{15}\)

\[
V_{is} = g_s(\succ_i, R_i, U_{is}) \in [0, 1] \quad U_{is} \sim \text{Unif}[0, 1]
\]

where $U_{is}$ is a lottery tie-breaker drawn by the mechanism. Assume that for two different individuals $i, j$, $V_{is} \neq V_{js}$ almost surely. Then the priority scores induce an ordering $\succ_s$ at each school, where $i \succ_s j$ if and only if $V_{is} > V_{js}$. The orderings $\{\succ_i : i \in I\}$ and $\{\succ_s : s \in S\}$ are then inputs to the deferred acceptance algorithm (Gale and Shapley, 1962):

1. Initially, all students are unmatched, and they have not been rejected from any school.
2. At the beginning of stage $t$, every unmatched student proposes to her favorite school, according to $\succ_i$, from which she has not been rejected.

\(^{14}\)That is, a parameter is point-identified if no two distinct values of the parameter are compatible with a the joint distribution of observed data $(D_i, W_i, Y_i(D_i))_{i=1}^N$ induced by $P_{W,A} \in \mathcal{P}$ and $\Pi_N$.

\(^{15}\)The restriction of $V_{is}$ to $[0, 1]$ is without loss of generality, and the functions $g_s$ are known to the analyst.
(3) Each school \(s\) considers the set of students tentatively matched to \(s\) after stage \(t - 1\), as well as those who propose to \(s\) at stage \(t\), and tentatively accepts the most preferable students, up to capacity \(q_s\), ranked according to \(\succ_s\), while rejecting the rest.

(4) Stage \(t\) concludes. If there is an unmatched student who have not been rejected from every school on her list, then stage \(t + 1\) begins and we return to step (2); otherwise, the algorithm terminates, and outputs the tentative matches at the conclusion of stage \(t\).

Mechanisms in this class have outputs that may be rationalized by a vector of cutoffs \(C_N\). To wit, consider \(C_N = (C_{1,N}, \ldots, C_{M,N})\) where each cutoff \(C_{s,N}\) is the priority of the student matched to the school who has the least priority:

\[
C_{s,N} = \begin{cases} 
\min \{V_{is} : i \text{ is matched to } s\} & \text{if } s \text{ is oversubscribed, i.e. it is matched to } q_s \text{ students} \\
0 & \text{otherwise.}
\end{cases}
\]

We may rationalize the school assignment made by the algorithm as though each student \(i\) is assigned to her favorite school among those schools \(s\) such that \(V_{is} \geq C_{s,N}\). In other words, schools whose cutoffs are lower than student \(i\)'s priority scores constitute \(i\)'s choice set, and \(i\) is simply matched to her favorite school in her choice set.

Proposition 3 of Azevedo and Leshno (2016) states that if \(\{(\succ_{is}, V_{is}) : s \in S\}\) are independently and identically distributed across students \(i\), then under mild conditions,\(^{16}\) the cutoffs \(C_N\) concentrate to some population counterpart \(c\) at the parametric rate:

\[
\|C_N - c\|_\infty = O_P(N^{-1/2}).
\]

These features of deferred acceptance show that mechanisms with deferred acceptance priority scores satisfy Definition 4.1, which we summarize below.

**Definition 4.3.** We say a mechanism \(\Pi_N\) has deferred acceptance priority scores if its output agrees with deferred acceptance allocations from student preferences \(\succ_i\) and school priorities \(\succ_s\), where \(\succ_s\) ranks \(V_{is}\) from (4).

Next, we introduce a special case of mechanisms with deferred acceptance priority scores. These mechanisms specialize by choosing \(g_s\) of a particular form. We call these mechanisms *mixed mechanisms*, since they allow for certain schools to use lottery tie-breakers and certain schools to use test score tie-breakers. Our subsequent analysis is about Example 1, where some results about the more general Definition 4.1 are relegated to Appendix A.5.

**Example 1 (Mixed mechanism).** Assume \(R_i = ([Q_{i1}, \ldots, Q_{iM}], [R_{i1}, \ldots, R_{iT}])\), where the \(R_{it}\)'s are test scores and the \(Q_{is}\)'s are discrete qualifiers. Discrete qualifiers usually encode information such as whether a student \(i\) has a sibling at school \(s\). In many school choice mechanisms (e.g. Boston Public Schools), these qualifiers enable students to receive priority. The test scores \(R_i\) take value in \([0,1]^T\) and the discrete qualifiers \(Q_{is}\) take value in \(\{0, \ldots, \tau_s\}\).

\(^{16}\)Precisely speaking, Azevedo and Leshno (2016) define a notion of deferred acceptance matching acting on the *continuum economy*—which is the distribution of \(\{(\succ_{is}, V_{is}) : s \in S\}\). The additional regularity condition is that this continuum version of deferred acceptance matching admits a unique stable matching. In other words, it is a very mild regularity condition on the distribution of \(\{(\succ_{is}, V_{is}) : s \in S\}\).
Suppose there are lottery schools and test-score schools. A lottery school $s$ uses a lottery indexed by $\ell_s \in \{1, \ldots, L\}$. A test-score school $s$ uses a test score indexed by $t_s \in \{1, \ldots, T\}$. The priority scores are determined via

$$
g_s(\succ_i, R_i, U_{is}) = \begin{cases} 
g_s(Q_{is}, R_{it_s}) & s \text{ is a test-score school} \\
g_s(Q_{is}, U_{it_s}) & s \text{ is a lottery school, } U_{it_s} \in [0,1] 
\end{cases}
$$

where the maps

$$g_s(q, v) = \frac{q + v}{q_s + 1}$$

In general, $g_s$ represents a lexicographic ordering in $(q,v)$. The restriction of $g_s(q,v)$ to an affine function is without loss of generality. ■

As Example 1 is a realistic description of many school choice mechanisms, the analysis in Section 4 focuses on analyzing treatment effects in this case. Whenever we refer to mixed mechanisms, we also maintain the following assumption, Assumption 4.4, on the sampling process, which is a support and continuity condition for the test scores $R$, as well as a normalization on $g_s$. Assumption 4.4 may be weakened, but is maintained for simplicity.

**Assumption 4.4** (Mixed mechanism). (1) The distribution of the test scores $(R_1, \ldots, R_T)$, conditional on discrete qualifiers $Q$, preferences $\succ$, potential outcomes $A$, and other observable characteristics $Z$, is absolutely continuous with respect to the Lebesgue measure on $[0,1]^T$ and has positive density.

(2) For lottery schools $s$, the distribution of the lottery numbers $U_{it}, \ell \in \{1, \ldots, L\}$ is also absolutely continuous with respect to the Lebesgue measure on $[0,1]^L$, and independent of the student characteristics $(Q,A,\succ,Z)$, though the lottery numbers may be non-independent across $\ell$ for the same student.

(3) For all $s$, the $[0,1] \to [0,1]$ function $r \mapsto g_s(q,r)$ is affine and increasing.

4.3. Causal identification: an example. In the next subsection, we are interested in characterizing the set of causal effects and causal contrasts that are identified (in the sense of Definition 4.2) in mixed mechanisms. By a causal effect, we mean some conditional expectation of the potential outcomes $Y(s)$. By a causal contrast, we mean conditional expectation of the differences in potential outcomes $Y(s_1) - Y(s_2)$ for two schools $s_1, s_2$. By the law of iterated expectations, it suffices (\begin{footnotesize}
\footnote{17This is a construction in Abdulkadiroğlu et al. (2017a)}
\footnote{18The mechanisms satisfying Definition 4.3—and mixed mechanisms in particular—are exactly studied in Abdulkadiroğlu et al. (2022) and are thus sufficiently general for many school choice mechanisms in practice, including immediate acceptance, China’s parallel mechanism, England’s first-preference-first mechanisms, and the Taiwan mechanism (see footnote 5 in Abdulkadiroğlu et al., 2022). Importantly, since Definition 4.1 only requires the mechanism allocations to be computed by student-proposing deferred acceptance, perhaps after some transformations of the student submitted preferences, it accommodates mechanisms that are not strategyproof nor stable, such as immediate acceptance (Abdulkadiroğlu and Sönmez, 2003). The additional requirement that Proposition 3 in Azevedo and Leshno (2016) is satisfied for such a mechanism is mild, and likewise maintained by the previous literature (Abdulkadiroğlu et al., 2017a, 2022).}
\footnote{19Assumption 1(ii) in Abdulkadiroğlu et al. (2022) is similar. The assumption can be weakened—for instance, we do not require any restrictions on behavior of $R$ far in the tails for identification and estimation. The restriction that $R$ is positive density and rectangular support does not rule out dependence. Moreover, if two test scores are perfectly correlated, we can handle that case by reducing the set of test scores $1, \ldots, T$.}
to focus on the finest conditional means, namely
\[
E[Y(s) \mid X, Z] = E[Y(s) \mid \succ, R, Z] \text{ and } E[Y(s_1) - Y(s_2) \mid \succ, R, Z]
\]
for mixed mechanisms. Since the observable covariates \(Z\) can always be conditioned upon and do not affect allocations, we suppress their dependence and think of the subsequent results as conditioning on some value \(Z = z\).

The problem of characterizing the set of identified causal effects and contrasts can be rephrased as: For a given preference \(\succ\), what is the region for \(R\) on which the effect or contrast is identified?

Before answering this question formally in generality, we turn to a stylized example that illustrates the key intuitions.

Suppose there are four schools \(s_0, s_1, s_2, s_3\) and three types of students denoted by \((A, B, C)\). The students have the following preferences and the same discrete priorities \(Q\):

| Preferences | Discrete qualifiers \(Q\) |
|-------------|--------------------------|
| \(A\)       | \(s_2 \succ s_3 \succ s_1 \succ s_0\) | 0 |
| \(B\)       | \(s_2 \succ s_1 \succ s_3 \succ s_0\) | 0 |
| \(C\)       | \(s_3 \succ s_2 \succ s_1 \succ s_0\) | 0 |

Additionally,

1. The schools \(s_1, s_2\) are test-score schools using the same test score \(R \in [0, 1]\).
2. \(s_3\) is a lottery school, and \(s_0\) is an undersubscribed (lottery) school with sufficient capacity.
3. Since every student has the same discrete qualifier, it is without loss to assume the priority score is simply the test score \(g_s(\succ, Q, R) = R\).
4. In the spirit of the large-market definition of identification (Definition 4.2), assume the number of students is sufficiently large so that the cutoffs \(c_1, c_2\) are fixed.
5. Since everyone prefers \(s_2 \succ s_1\), school 2 has a more stringent cutoff: \(c_2 > c_1\).
6. Since the distribution of \(R\) is unspecified, it is without loss to assume \(c_2 = \frac{2}{3}, c_1 = \frac{1}{3}\).
7. Finally, assume that \(s_3\) is oversubscribed—the probability of qualifying for \(s_3\) for any student is not zero or one.\(^{20}\)

Now, let us consider the choice sets and eventual matches for a student of type \(A\), as a function of their test score \(R\). In general, we would consider \(R = (Q, R)\), but since the discrete qualifiers \(Q\) do not vary across students, it is without loss to consider \(R\).

- When \(R \in [0, \frac{1}{3})\), they do not qualify for either \(s_1\) or \(s_2\), and they are assigned to \(s_3\) if they win the lottery at \(s_3\). Otherwise, they are assigned to \(s_0\). Since they have positive probability of being assigned to either \(s_0\) or \(s_3\), unsurprisingly, \(E[Y(s_0) \mid \succ_A, R]\) and \(E[Y(s_3) \mid \succ_A, R]\) are identified for \(R\) in this region.
- When \(R \in [\frac{1}{3}, \frac{2}{3})\), they do not qualify for \(s_2\), but they do qualify for \(s_1\). Since they prefer \(s_3\) to \(s_1\), they are assigned to \(s_3\) if they win the lottery at \(s_3\). Otherwise, they are assigned to \(s_1\). Since

\[^{20}\text{Note that if certain students do have discrete priority over others, then it may be the case that they qualify for } s_3 \text{ with probability one, even if } s_3 \text{ is oversubscribed.}\]
they have positive probability of being assigned to either \( s_1 \) or \( s_3 \), naturally, \( \mathbb{E}[Y(s_0) \mid \succ_A, R] \) and \( \mathbb{E}[Y(s_3) \mid \succ_A, R] \) are identified for \( R \) in this region.

- When \( R \in [\frac{2}{3}, 1] \), they qualify for \( s_2 \). Since \( s_2 \) is their favorite school, they are assigned to \( s_2 \). Since they have positive probability of being assigned only to \( s_2 \), only \( \mathbb{E}[Y(s_2) \mid \succ_A, R] \) is identified in this region.

For each school \( s_j \), we can consider the region of \( R \) on which \( \mathbb{E}[Y(s_j) \mid \succ_A, R] \) is identified. We shall call such regions \( s_j \)-eligibility sets, denoted \( E_{s_j}(A, c_1, c_2) \). It is easy to see that \( s_j \)-eligibility sets are simply regions on which the probability of being assigned to \( s_j \) is positive:

\[
E_{s_j}(A, c_1, c_2) = \left\{ R : \Pr(Q(D^*_j(c_1, c_2) = 1 \mid \succ_A, R) > 0 \right\}
\]

Here, the probability is over the distribution of the observed data in the large experiment limit, \( Q = Q(P_{W,A}) \), defined in Definition 4.2. For students of type \( A \), revisiting our computation above, we find that \( E_{s_0}(A, c_1, c_2) = [0, 1/3) \), \( E_{s_1}(A, c_1, c_2) = [1/3, 2/3) \), \( E_{s_2}(A, c_1, c_2) = [2/3, 1] \), and \( E_{s_3}(A, c_1, c_2) = [0, 2/3) \).

Computing these \( s_j \)-eligibility sets for every type of student, we have the following:

| Preference type | \( E_{s_0} \) | \( E_{s_1} \) | \( E_{s_2} \) | \( E_{s_3} \) |
|----------------|------------|------------|------------|------------|
| \( A \)        | \([0, \frac{1}{3})\) | \([\frac{1}{3}, \frac{2}{3})\) | \([\frac{2}{3}, 1]\) | \([0, \frac{2}{3})\) |
| \( B \)        | \([0, \frac{1}{3})\) | \([\frac{1}{3}, \frac{2}{3})\) | \([\frac{2}{3}, 1]\) | \([0, \frac{1}{3})\) |
| \( C \)        | \([0, \frac{1}{3})\) | \([\frac{1}{3}, \frac{2}{3})\) | \([\frac{2}{3}, 1]\) | \([0, 1]\) |

Finally, if we are willing to assume that \( \mathbb{E}[Y(s_j) \mid \succ_T, R] \) is continuous in \( R \) for each type \( T \), then we can extend identification to the closure of \( E_{s_j}(T, c) \). This extrapolation via continuity is exactly the RD-type variation that Abdulkadiroğlu et al. (2022, 2017b) use.

We now turn to causal contrasts for pairs of schools. A causal contrast \( \mathbb{E}[Y(s_j) - Y(s_k) \mid \succ_T, R] \) is identified if and only if both conditional expectations are identified. Hence, it is identified if and only if \( R \in \overline{E}_{s_j}(T, c) \cap \overline{E}_{s_k}(T, c) \). From the table of \( E_{s_j} \) above, it is straightforward to compute \( \overline{E}_{s_i} \cap \overline{E}_{s_j} \) for each pair of schools \( (s_i, s_j) \):

| Preference type | \((s_0, s_1)\) | \((s_0, s_2)\) | \((s_0, s_3)\) | \((s_1, s_2)\) | \((s_1, s_3)\) | \((s_2, s_3)\) |
|----------------|-------------|-------------|-------------|-------------|-------------|-------------|
| \( A \)        | \(\{\frac{1}{3}\}\) | \(\emptyset\) | \([0, \frac{1}{3})\) | \(\{\frac{2}{3}\}\) | \([0, \frac{2}{3})\) | \(\{\frac{2}{3}\}\) |
| \( B \)        | \(\{\frac{1}{3}\}\) | \(\emptyset\) | \([0, \frac{1}{3})\) | \(\{\frac{2}{3}\}\) | \(\{\frac{1}{3}\}\) | \(\emptyset\) |
| \( C \)        | \(\{\frac{1}{3}\}\) | \(\emptyset\) | \([0, \frac{1}{3})\) | \(\{\frac{2}{3}\}\) | \([0, \frac{2}{3})\) | \([\frac{2}{3}, 1]\) |

We see that there is important heterogeneity, in at least two senses. Ignoring the rich heterogeneity may lead to inaccurate interpretations of causal estimates. First, different pairs of schools \((s_i, s_j)\) are comparable on different regions of the test score \( R \). Imagine that we are interested in the causal contrasts between the test score schools \( s_1, s_2 \) and the undersubscribed school \( s_0 \). Because \( s_0, s_1 \) are comparable at \( R = \frac{1}{3} \), but \( s_2, s_0 \) are nowhere comparable, such causal contrasts actually reflect conditional average treatment contrasts for a narrow slice of students with test scores at \( \frac{1}{3} \),
and these contrasts only represent substitution between $s_0$ and $s_1$. Interpreting these contrasts as causal effects of $s_1$ or $s_2$ against the undersubscribed school $s_0$ for the typical student could lead to suboptimal policy decisions related to $s_2$.

Second, regions of $R$ that admit comparisons for a given pair of schools differ substantially across student types—this is true in this example for comparisons with between $(s_2, s_3)$:

- Students of type $A$—who prefer $s_2$ to $s_3$—admits valid comparisons between $s_2, s_3$ at a single point $\{\frac{2}{3}\}$, which has zero measure.
- There is no variation for students of type $B$ between $s_2, s_3$.
- Students of type $C$—who prefer $s_3$ to $s_2$—have variation between $(s_2, s_3)$ for the set $[\frac{2}{3}, 1]$, which has positive measure.

Again, ignoring heterogeneity on this front may lead to suboptimal decision-making. Suppose we directly pool over these causal contrasts between $s_2$ and $s_3$, then the resulting average only reflects the causal effects for students of type $C$ with test scores in $[\frac{2}{3}, 1]$, as the margin of substitution for students of type $A$ is zero-measure. However, because students of type $C$ prefer $s_3$ to $s_2$, if these preferences are partially due to knowledge of their potential outcomes, we might expect these causal contrasts to be more favorable towards $s_3$ than the causal contrasts for students of type $A$. In fact, whenever we identify a causal contrast between a test-score school and a lottery school, if such a causal contrast is over a region with positive measure, then it must be for students who prefer the lottery school to the test-score school.

We should expect the heterogeneity in both senses to be even more complex in general, as, for this example, we only include 3 out of the 24 possible preferences and only a single type of test score. In light of the heterogeneity, estimates of treatment effects that pool over multiple schools, multiple student preference types, and multiple test score values may not be transparent with respect to the implicit weighting assigned to the pairwise comparisons, conditional on student preferences and test score values. To understand these estimates, we characterize the eligibility sets $E$s as well as pairwise treatment contrasts in the next section, as well as making formal the two observations from this example:

- Causal effects $\mathbb{E}[Y(s) \mid \succ, R = (Q, r)]$ are identified if $r$ belongs to the closure of the $s_j$ eligibility set $E_s(\succ, Q, c)$, defined as the set of $r$ for which treatment assignment to $s$ has positive probability.
- Causal contrasts $\mathbb{E}[Y(s_1) - Y(s_2) \mid \succ, R = (Q, r)]$ are identified if $r$ belongs to the intersection $\overline{E}_{s_1}(\succ, Q, c) \cap \overline{E}_{s_2}(\succ, Q, c)$.

4.4. Identification in mixed mechanisms. We now characterize the $s$-eligibility sets in generality for mixed mechanisms (Example 1), maintaining Assumption 4.4. As a reminder, in such a mechanism, the students’ eligibility information takes the form $R_i = ([Q_{i1}, \ldots, Q_{iM}, R_{it1}, \ldots, R_{itT}])$. The $R_{it}$’s are test scores and the $Q_{is}$’s are discrete priorities. The schools are divided into lottery schools and test-score schools, such that school priorities are determined by the priority score

$$V_{is} \equiv g_s(\succ, R_i, U_{is}) = \begin{cases} g_s(Q_{is}, R_{its}) & s \text{ is a test-score school that uses test } t_s \\ g_s(Q_{is}, U_{its}) & s \text{ is a lottery school} \end{cases}$$
and the maps $g_s(q, v)$ represent a lexicographic ordering in $(q, v)$, which can be taken to be a
known affine function (Assumption 4.4). Formally, we fix $P_{W,A} \in \mathcal{P}$, and maintain the following
assumptions about $P_{W,A}$.

**Assumption 4.5.** Assume that the conditional causal effects

$$r \mapsto \mathbb{E}[Y(s) \mid (\succ, Q, r)] \equiv \mu_s(\succ, Q, r)$$

are continuous for every $s$ and $P_{W,A}$-almost every $(\succ, Q)$.

Assumption 4.5 is a standard assumption for RD identification, which we maintain throughout.\footnote{In non-strategyproof mechanisms, Marinho et al. (2022) show that strategic reporting of preferences may result in bunching at the cutoff that render Assumption 4.5 implausible. However, such bunching has testable implications; moreover, for such bunching to occur, students must be good at predicting the jointly determined school cutoffs, which can be demanding.}

We also assume that the distribution $P_{W,A}$ satisfies the conditions for Azevedo and Leshno (2016)’s Proposition 3.

**Assumption 4.6.** $P_{W,A}$ is such that the cutoffs on priority scores $V_is$, $\{C_{s,N}\}$, satisfy

$$\max_{s \in S} |C_{s,N} - c_s| = O_p(N^{-1/2})$$

for some fixed $c = (c_s : s \in S) \equiv c(P_{W,A})$.

We now formalize the first informal result of Section 4.3.

**Proposition 4.7.** Consider a mixed mechanism. Fix some $(\succ, Q)$ with positive probability under $P_{W,A}$. Let $c = c(P_{W,A})$ be the limiting cutoffs, and let $Q = Q(P_{W,A})$ be the distribution of the observed data with cutoff $c$. Define the $s$-eligibility sets as

$$E_s(\succ, Q, c) \equiv \{r : \mathbb{Q}[D^*_s(c) = 1 \mid (\succ, Q, r)] > 0 \} \subset [0, 1]^T$$

Then, under Assumptions 4.5 and 4.6, $\mu_s(\succ, Q, r)$ is identified if and only if $r \in E_s(\succ, Q, c)$. The if-direction follows from a standard inverse weighting argument. We note that if $r \in E_s(\succ, Q, c)$, we can write

$$\mu_s(\succ, Q, r) = \mathbb{E}_Q \left[ \frac{D^*_s(c)Y(D^*_s(c))}{\mathbb{Q}(D^*_s(c) = 1 \mid (\succ, Q, r))} \mid (\succ, Q, r) \right], \quad (5)$$

where the right-hand side is a function of the joint distribution of observed outcomes in the large market limit $Q$. We leave the only-if direction to Appendix A.5.

With the identification result Proposition 4.7 in hand, we are now ready to characterize identification of causal contrasts. Proposition 4.7 shows that these causal effects take the form of aggregations of conditional average treatment effects:

$$\tau_{s_1,s_0}^{\text{fine}}(\succ, Q, r) \equiv \mathbb{E}[Y(s_1) - Y(s_0) \mid (\succ, Q), R = r] \text{ where } r \in E_{s_1}(\succ, Q, c) \cap E_{s_0}(\succ, Q, c) \quad (6)$$

Thus it is convenient to explicitly compute $E_s(\succ, Q, c)$ as well as intersections $E_{s_1}(\succ, Q, c) \cap E_{s_0}(\succ, Q, c)$. We do so in Theorem 4.8 below.

Theorem 4.8 formalizes and generalizes the ad hoc computation in Section 4.3; we note that the statement is notationally cumbersome due to its generality, but it does not require substantial conceptual leaps relative to the calculations in Section 4.3. For the rest of this subsection, we first
Example 2

Assumption 4.6 takes one of three forms: (a) some value transports the cutoff. Then we have the following properties regarding variation qualifies the student to a school she prefers to.

Abdulkadiroğlu et al. (2022) Theorem 4.8. Lastly, we Abdulkadiroğlu et al. Assume 4.5 Theorem 4.8. Consider a mixed mechanism under Assumption 4.5 and Assumption 4.6. Let \( (\succ, Q, R) \) be in the support of the distribution of student observable characteristics, and let \( c \) be the limiting cutoff under Assumption 4.6. Then we have the following properties regarding variation induced by lottery:

(1) Define the support supp\((\succ, Q, R) = \{s : R \in E_s(\succ, Q, c)\} \) as the set of schools that have positive probability of being assigned at characteristics \((\succ, Q, R)\). Then supp\((\succ, Q, R)\) contains at most one test-score school, and if it does, then the test score school is the least preferred element in supp\((\succ, Q, R)\) according to \( \succ \).

(2) If \( s_1, s_2 \) are two test-score schools, then \( E_{s_1}(\succ, Q, c) \cap E_{s_2}(\succ, Q, c) = \emptyset \).

Moreover, for a school \( s \) and a test \( t \), define the following objects:

- Suppose \( t_s = t \). Let the most disfavored test score that clears the cutoff \( c_s \) at school \( s \) be\(^{22}\)
  \[
  r_t(s, Q_s, c_s) = g_s^{-1}(Q_s, c_s) \equiv \inf \{r \in [0, 1] : g_s(Q_s, r) \geq c_s\}.
  \]
- Let the most lenient test score \( t \) among those schools preferred to \( s \) be\(^{23}\)
  \[
  R_t(s, \succ, Q; c) = \min_{t_s=t} r_t(s_1, Q_s, c_s)
  \]
- Let \( L(Q, c) \) be the set of lottery schools at which a student with discrete priority \( Q \) qualifies with probability one.

Then, we have the following explicit form of eligibility sets for test-score schools and the intersections of their closures, suppressing the dependence on \( \succ, Q, c \):

(3) For a test-score school \( s_0 \), the \( s_0 \)-eligibility set takes the following form:

\[
E_{s_0}(\succ, Q, c) = \begin{cases} 
[r_{t_0}(s_0), R_{t_0}(s_0)] \times \left( \times_{t \neq t_0} [0, R_t(s_0)] \right), & \text{if } s_0 \succ L(Q, c) \\
\emptyset, & \text{otherwise},
\end{cases}
\]
and its closure takes the following form

\[
\overline{E}_{s_0}(\succ, Q, c) = \begin{cases} 
[r_{t_0}(s_0), R_{t_0}(s_0)] \times \left( \times_{t \neq t_0} [0, R_t(s_0)] \right), & \text{if } E_{s_0} \neq \emptyset \\
\emptyset, & \text{otherwise}.
\end{cases}
\]

These are either hyperrectangles of dimension \( T = \dim R \) or empty sets.

\(^{22}\)Effectively, \( r_t \) transports the cutoff \( c_s \) from the space of priority scores \( V_s \) to the space of test scores \( R_t \). Depending on the value of \( Q_s \), \( r_t \) takes one of three forms: (a) some value \( r_{t,s}(c) \in (0, 1) \), which does not depend on \( Q_s \), (b) 0, indicating that \( Q_s \) is sufficiently compelling to qualify the student at \( s \) regardless of her test scores, or (c) 1, indicating that \( Q_s \) is sufficiently low that the student cannot be seated at \( s \) regardless of her test scores.

\(^{23}\)\( R_t(s) \) is, loosely speaking, what Abdulkadiroğlu et al. (2022) refer to as the most informative disqualification. Echoing Abdulkadiroğlu et al. (2022), the test score distribution of students matched to \( s \) are truncated by \( R_t \) on the right, since any test score \( R_t \) higher than \( R_{t_0} \) qualifies the student to a school she prefers to \( s \). The definition differs from most informative disqualification in Abdulkadiroğlu et al. (2022) slightly and immaterially, as we operate in the space of the test scores \( R_t \), while Abdulkadiroğlu et al. (2022) operate in the space of the priority scores \( V_s \).
(4) For two test-score schools $s_0, s_1$, where $s_1 \succ s_0$, the intersection $E_{s_0} \cap E_{s_1}$ takes the following form
\[
E_{s_0} \cap E_{s_1} = \begin{cases} 
E_{s_0} \cap \{r_{s_1}(s_1)\} \times [0,1]^{T-1} & \text{if } R_{s_1}(s_0) = r_{s_1}(s_1) \text{ and } E_{s_0}, E_{s_1} \neq \emptyset \\
\emptyset & \text{otherwise},
\end{cases}
\]
which is either a hyperrectangle of dimension $T - 1$ or an empty set.

**Example 2** (Specializing Theorem 4.8 to Section 4.3). In the context of the running example Section 4.3, we enumerate some instances of the first two claims of Theorem 4.8:

1. For students of type $A$ ($s_2 \succ s_3 \succ s_1 \succ s_0$), the test score $R = \frac{1}{2}$ falls inside the eligibility set for both $s_1$ and $s_3$, indicating that $\text{supp}(\succ_A, Q_A, \frac{1}{2}) = \{s_1, s_3\}$. It indeed contains only one test score school ($s_1$), which is less preferred by $A$ to $s_3$.

2. There are only two test-score schools, $s_1$ and $s_2$. For every student, it is easy to check that $E_{s_1} \cap E_{s_2} = \emptyset$.

To illustrate the next two claims, let us consider students of type $B$ ($s_2 \succ s_1 \succ s_3 \succ s_0$) with schools $s_1, s_2$:

3. The third claim of Theorem 4.8 states that
\[
E_{s_1}(\succ_B, Q_B, c) = \overline{[r(s_1), R(s_1)]},
\]
if $s_1 \succ_B L(Q_B, c)$, where we omit the $t$ subscript as there is a single test score. The set $L(Q_B, c)$ is empty in this case, as there are no discrete priority differences, and hence students of type $B$ do not qualify for any lottery schools for sure. Moreover, no lottery school is preferred to $s_1$ by students of type $B$. The quantity $\overline{r(s_1)}$ is simply $s_1$’s cutoff in test-score space, which is $\frac{1}{3}$. The quantity $\overline{R(s_1)}$ is the most lenient test score cutoff for test schools preferred to $s_1$—in this case $\{s_2\}$. Hence $\overline{R(s_1)} = \overline{r(s_2)} = \frac{2}{3}$. We conclude that $E_{s_1}$ for students of type $B$ is equal to $[\frac{1}{3}, \frac{2}{3}]$, as claimed in Section 4.3.

4. The fourth claim of Theorem 4.8 states that
\[
E_{s_2}(\succ_B, Q_B, c) \cap E_{s_2}(\succ_B, Q_B, c) = \overline{[r(s_1), R(s_1)]} \cap \{r(s_2)\} = \overline{[\frac{1}{3}, \frac{2}{3}]} \cap \overline{\{\frac{2}{3}\}} = \overline{\{\frac{2}{3}\}},
\]
which agrees with the calculation in Section 4.3. ■

The first two claims of Theorem 4.8 concern the identification induced by the randomized lottery number. The first claim observes that the support $\text{supp}(\succ, Q, R)$—the set of schools that $(\succ, Q, R)$ may be randomly assigned to—contains at most one test-score school. This is because the random lottery number $U$ cannot induce randomized assignment between two schools that do not use lottery numbers for assignment.

Moreover, if the support does contain a test-score school $s$, then $s$ must be the $\succ$-least preferred school in $\text{supp}(\succ, Q, R)$. To see this, note that the student with characteristics $R = (Q, R)$ can attend $s$ if she so desires; thus having positive probability of attending $s'$ implies that $s' \succ s$. Therefore, the first claim implies that most lottery-driven comparisons between schools are comparisons between lottery schools. Lottery-driven comparisons between a lottery school and a test-score school are also possible, but always imply that the lottery school is preferred to the test-score
school. In the presence of both lottery schools and test-score schools, more students are subjected to lottery-driven variation. Thus, an estimand that pools over many pairwise treatment contrasts tends to reflect the potential outcomes of students who prefer lottery schools.

The second claim of Theorem 4.8 observes that the eligibility sets of two test-score schools are disjoint, because lottery comparisons between two test-score schools are impossible. As a result, causal comparisons between two test-score schools necessarily rely on the possibly non-disjoint closures of the eligibility sets. Continuity of the potential outcome conditional means (Assumption 4.5) is therefore critical for identification. Since the intersection of the closures is typically a measure-zero set, the resulting causal estimands are irregularly identified, in the sense of Khan and Tamer (2010), and hence are estimated at a slower rate than $N^{-1/2}$.

The third and fourth claims characterize regions of observable characteristics on which causal comparisons between two test-score schools are possible. They illustrate that these regions are easily computed. The third claim characterizes $s_0$-eligibility sets for a school $s_0$. The form (7) of the $s_0$-eligibility set for a test-score school comes from three simple facts. First, if any school in $L(Q, c)$ is preferable to $s_0$, then $s_0$ cannot possibly be assigned with positive probability under any test score configurations, and hence the $s_0$-eligibility set is empty. Second, for the student to be eligible at $s_0$, she must have test scores that clear the cutoff $r_{t_0}$, explaining the lower bound in the $t_0^{th}$ coordinate. Third, for every test $t$, if the score $R_t$ exceeds $R_{t_0}$, then there is some other school $s \succ s_0$ that the student qualifies for, leaving it impossible for the student to be matched to $s_0$; hence, $R_t < R_{t_0}$ for $R \in E_{s_0}$, explaining the upper bound in the $t^{th}$ coordinate. The result for a lottery school $s_0$ follows analogously.

The fourth claim characterizes the (measure-zero) region of observables on which the causal contrast between $s_1, s_0$—two test-score schools where $s_1 \succ s_0$—is identified. Combined with Proposition 4.7, the fourth claim makes explicit the identified causal effects between $s_0$ and $s_1$, which are aggregations of (6). When (9) is nonempty, it is simply the slice of $E_{s_0}$ where the $t_1^{th}$ coordinate equals the threshold $r_{t_1}$.

Taken together, Theorem 4.8 highlights the complex heterogeneity of identified treatment effects. The individuals whose treatment effects and treatment contrasts are identified have very predictable and distinct traits. For instance,

- Claim (1) implies that the lottery variation applies to test-score schools only for those who do not prefer the test-score schools.
- Claim (3) implies that test-score school causal effects are only identified for those whose test scores do not qualify for schools that are preferred.
- Claim (4) implies that the causal contrast between two test score schools $s_0, s_1$ are only identified for those whose scores are near the cutoff on the more-preferred school and whose scores do not qualify them for any schools they prefer to both $s_0$ and $s_1$.

In practice, we may suspect these subpopulations may have different causal effects than populations a particular policy targets. At the very least, it seems prudent to assess the heterogeneity when reporting estimates that aggregate over the kaleidoscope of conditional average treatment effects and contrasts. To that end, Theorem 4.8 precisely delineates these causal objects, aiding in their
interpretation. Likewise, our asymptotic results in Section 5 provides guidance for their estimation and inference.

We conclude this subsection with a discussion of the policy relevance of the identified treatment effects. Aggregations of (6) cover both variations induced by lottery- and RD-type variation.\footnote{In this sense, they are more general than the pure lottery-driven causal effects in Proposition 3.1. Of course, Proposition 3.1 concerns sample average treatment effects where (6) are population average treatment effects, and so they are not strictly comparable.} For simplicity, let us consider the identified treatment effect between two schools \(s_0, s_1\). Proposition 4.7 implies that the only (conditional average) treatment contrasts that are identified are aggregations of (6). One natural policy interpretation of effects like the above is the following. Expanding school \(s_1\) by a small amount generates direct treatment effects that are exactly aggregations of \(\tau_{\text{fine}}^{s_1,s_0}\) over choices of \(s_0\) and \(\succ\) such that \(s_1 \succ s_0\), representing the effect of students who prefer \(s_1\) sorting into \(s_1\). Of course, such an expansion of capacities would also create indirect effects due to sorting into schools which are newly vacant due to expanding capacity at \(s_1\). Despite the complexity, these indirect effects are also aggregations of effects of the form (6).

However, treatment effects (6) do not inform counterfactual outcomes that involve global interventions, without further smoothness or homogeneity assumptions. This represents inherent limitations of school choice data. For instance, treatment effects (6) are not in general sufficient of large increases or reductions in the school capacity, closure of schools, or substantial changes to the school’s admission policy, as such interventions can easily involve comparisons on students with observables \((\succ, Q, R)\) between schools where \(R \notin E_{s_1} \cap E_{s_0}\). Nevertheless, in the presence of stronger identifying assumptions, homogeneity of treatment effects, or parametric structural models, the nonparametrically identified treatment effects (6) may serve as moments for estimation or for validation.

4.5. **RD variation is asymptotically vanishing in regression estimation with local deferred acceptance propensity scores.** In mixed mechanisms, Abdulkadiroğlu et al. (2022) propose local deferred acceptance (DA) propensity scores and estimators that condition on estimated values of these propensity scores. Here, in the context of Section 4.3, we calculate the local DA propensity scores. The details of this computation, as well as a formal connection between Theorem 4.8 and local DA scores are in Appendix A.7. We also characterize the implicit estimand of a condition-on-propensity-score estimator of the causal effect between two groups of schools.

For the estimand to represent a causal contrast, the bandwidth parameter \(\delta\) must be approximately zero. However, importantly, in the presence of variation driven by lotteries, all causal effects whose identification relies on RD-type variation contribute a vanishing amount to the estimand as the bandwidth \(\delta \to 0\). Thus, asymptotically, the estimand only aggregates over variation driven by lotteries rather than by RD. Then, Theorem 4.8(1) implies that such an estimand only reflects treatment contrasts for students who disprefer test-score schools.

Let us continue with the simplified setting of Section 4.3, imposing a few further assumptions. Recall that we have three types of students, \(\{A, B, C\}\), differing only in preferences, with the same discrete priority \(Q \equiv 0\). There are four schools \(s_0, \ldots, s_3\), such that \(s_1, s_2\) are test-score schools
using a single test score $R$, and $s_0, s_3$ are lottery schools. School $s_0$ is undersubscribed. School $s_1$ has cutoff $c_1 = \frac{1}{3}$ and school $s_2$ has cutoff $c_2 = \frac{2}{3}$.

Here, we additionally assume that the probability of qualifying for $s_3$ for any student is approximately equal to $\frac{1}{2}$, in order to compute the local DA propensity scores. Moreover, for ease of analysis, we binarize the treatment and assume constant potential outcomes within a treatment arm. Let $Y(s_0) = Y(s_1) = Y_C$ and $Y(s_2) = Y(s_3) = Y_T$, where the potential outcomes $Y_C, Y_T$ may vary freely across students. Correspondingly, let $D = D_2 + D_3$ be the indicator for whether the student is assigned to a treatment school $(s_2, s_3)$.

We partition the space of test scores into five regions, with a bandwidth parameter $\delta > 0$. Regions II and IV are small bands around the cutoffs $c_1, c_2$, and regions I, III, V are large regions in between the cutoffs:

|    | I     | II    | III   | IV    | V     |
|----|-------|-------|-------|-------|-------|
|    | $(0, \frac{1}{3} - \delta)$ | $(\frac{1}{3} - \delta, \frac{1}{3} + \delta)$ | $(\frac{1}{3} + \delta, \frac{2}{3} - \delta)$ | $(\frac{2}{3} - \delta, \frac{2}{3} + \delta)$ | $(\frac{2}{3} + \delta, 1)$ |

The local deferred acceptance propensity scores, as a function of the region that the test score $R$ falls into, are as follows:

|    | I     | II    | III   | IV    | V     |
|----|-------|-------|-------|-------|-------|
| $A$ | 0.5   | 0.5   | 0.5   | 0.75  | 1     |
| $B$ | 0.5   | 0.25  | 0     | 0.5   | 1     |
| $C$ | 0.5   | 0.5   | 0.5   | 0.75  | 1     |

The following remark below illustrates how the scores are computed.

**Remark 1** (Computation details of local DA scores and connection to $s$-eligibility sets). We walk through the computation (Abdulkadiroğlu et al., 2022) for, say, students of type $B$ ($s_2 \succ s_1 \succ s_3 \succ s_0$) with test scores in region II. Loosely speaking, the probability that a student is assigned to a school $s$ is the product of (i) the probability that the student fails to clear the cutoff of any test score schools that she likes better, (ii) the probability that the student fails to qualify for any lottery school that she likes better, and (iii) the (conditional) probability that she qualifies for $s$. The key here is that when the test score falls in regions II and IV, we act as if there is a probability of $1/2$ that the test score falls on the left-side of the cutoff. Then, for a student of type $B$ with test scores in II:

- She fails to qualify for $s_2$ when her test score is in II with probability one.
- She qualifies for $s_1$ with probability 0.5 since her test score is in II.
- $s_1$ is the only control school preferred to $s_3$
- She qualifies for $s_3$ (via lottery) with probability 0.5.
- Thus her probability of being treated is $\psi = 0 + (1 - 0.5) \cdot 0.5 = 0.25$. ■
We now turn to analyzing the natural estimand following the propensity score approach. The treatment effect estimand via conditioning on the computed propensity scores \( \psi \) is

\[
\tau(\delta) \equiv \sum_{x \in \{0.25, 0.5, 0.75\}} \left( \mathbb{E}[Y \mid D = 1, \psi = x] - \mathbb{E}[Y \mid D = 0, \psi = x] \right) \Pr(\psi = x) \tag{10}
\]

\[
= \sum_{T \in \{A, B, C\}} \sum_{S \in \{I, \ldots, V\}} \Pr(T, S) \left\{ \frac{\mathbb{E}[Y_T \mid D = 1, T, S] \Pr(D = 1 \mid T, S)}{\Pr(D = 1 \mid \psi = \psi(T, S))} - \frac{\mathbb{E}[Y_C \mid D = 0, T, S] \Pr(D = 0 \mid T, S)}{\Pr(D = 0 \mid \psi = \psi(T, S))} \right\} \tag{11}
\]

where (11) decomposes \( \tau(\delta) \) as a weighted average over student types and the regions test scores fall in. The details of this argument are in Appendix A.7.

In order to interpret (10) as a causal contrast, we must have \( \delta \to 0 \). Note that for students who fall in regions II and IV, their treatment statuses are not independent of their unobservables, even conditional on the local DA scores \( \psi \). Since they are treated if and only if the running variable \( R \) falls on one side of the cutoffs (\( \frac{1}{3} \) or \( \frac{2}{3} \)), their treatment statuses would be correlated with their potential outcomes, for instance, if both are correlated with \( R \). This selection effect disappears as \( \delta \to 0 \). Thus, for \( \tau(\delta) \) to be approximately a causal contrast, we must have \( \delta \approx 0 \).

However, if \( \delta \approx 0 \), then (11) puts very little weight on the regions II and IV—the areas where the variation is driven by RD—since these regions have very little measure. Translated to finite samples, this observation implies that the asymptotic unbiasedness of propensity-score estimators requires \( \delta = \delta_N \to 0 \) at appropriate rates, yet the part of the estimator driven by variation from RD becomes asymptotically negligible as \( \delta \to 0 \), so long as there is variation driven by lotteries. In light of our characterization of the causal comparisons (Theorem 4.8), this finding is not surprising, as all comparisons between test-score schools rely on sets of zero measure, and are thus necessarily estimated at rates that are slower than the parametric \( \sqrt{N} \) rate (Khan and Tamer, 2010). Indeed, the observation is confirmed by our estimation results in Section 5, where the locally linear regression estimator for RD-type variation converges at the rates no faster than \( N^{-2/5} \gg N^{-1/2} \).

As we have so far alluded to, there are reasons to suspect that the lottery-driven variation—which dominates asymptotically in (10)—may not be representative for policy questions, in the presence of heterogeneity. We now give a particular example in which the aggregation may be misleading.

Claim (1) in Theorem 4.8 shows that the only lottery variation available to a test-score school is driven by the student losing the lottery at a school they prefer. As a result, the estimand (10) only includes test-score schools through this variation, and thus exclusively reflects the treatment contrasts of test-score schools for those students who disprefer the test-score school. If high-achieving schools also tend to be test-score schools, and if student preferences correlate positively with potential outcomes, then (10) reflects causal contrasts for high-achieving schools only among the students that may have good reasons to believe that they would not succeed academically there.

Abdulkadiroğlu et al. (2022) find that Grade A (high-performing) schools in New York City have low estimated treatment effects. They also find that test-score Grade A schools have similar treatment effects with lottery-based Grade A schools. These observations are consistent with
the hypothesis that test-score Grade A schools are higher-achieving on average, but for students subjected to lottery variation at these schools—who disprefer such schools—the treatment effects are low and comparable to lottery-based Grade A schools. Of course, whether the evidence is dispositive of such a hypothesis remains to be empirically verified.

Ultimately, whether the variation driven by RD matters independently depends on what policy questions the causal contrasts are meant to inform. If they matter independently, then empirical researchers should isolate variation from these regions (II and IV) in this example, either by ex post reweighting or by presenting separate estimates. We note, however, the statistical uncertainty in estimates of the discontinuity-based causal effects is likely to be orders of magnitudes larger than their lottery-driven counterparts.

5. ASYMPTOTIC RESULTS FOR SAMPLING-BASED ANALYSIS

Having characterized the identified causal effects, we now turn to formally deriving asymptotic properties of a locally linear regression estimator (Hahn et al., 2001) for a family of estimands whose identification depends on RD-type variation. Our asymptotic arguments explicitly account for the fact that $C_N \rightarrow c$ in probability, and show that when $C_N = c + O_p(1/\sqrt{N})$, the randomness in $C_N$ does not affect the estimator to first order. As a result, the estimator is asymptotically normal and unbiased, under the usual undersmoothing bandwidths. This analysis is novel, and formally justifies inference procedures in the literature.

Consider a mixed mechanism (Example 1 and maintaining Assumption 4.4). Let $s_1, s_0$ be two test-score schools with tests $t_0, t_1$, respectively. We are interested in the $s_1$-minus-$s_0$ treatment effect contrast of students near school $s_1$’s cutoff, for those students with $s_1 \succ s_0$ and whose effects are identified. Indeed, students for which such a treatment effect is identified have observable characteristics such that, all else equal, they are matched to school $s_1$ if $R_{t_1}$ is slightly higher, and they are matched to school $s_0$ if $R_{t_1}$ is slightly lower.

Formally speaking, we consider the estimand
\[
\tau_{s_0, s_1} = \mathbb{E} \left[ Y(s_1) - Y(s_0) \mid s_1 \succ s_0, R \in \mathcal{E}_{s_0}(\succ, Q; c) \cap \mathcal{E}_{s_1}(\succ, Q; c) \right],
\]
\[
= \int \tau_{s_1, s_0}^{\text{fine}}(\succ, Q, r) dP_{W,A}(\succ, Q, R \mid s_1 \succ s_0, R \in \mathcal{E}_{s_0}(\succ, Q; c) \cap \mathcal{E}_{s_1}(\succ, Q; c))
\]
\[
\equiv \mathbb{E}[Y(s_1) - Y(s_0) | J_i(c) = 1, R_{t_1} = \rho(c)],
\]
where we may write the conditioning as sample selection indicator $J_i(c) = 1$ and the conditioning that $R_{t_1}$ is at some cutoff $R_{t_1} = \rho(c)$ that we define shortly.\textsuperscript{25} The second line indicates that the estimand is a conditional-probability weighted average of the identified treatment effects $\tau_{s_1, s_0}^{\text{fine}}$ in (6), pooling over those who prefer $s_1$ to $s_0$ and whose $(s_1, s_0)$-causal contrast is identified. The condition $R \in \mathcal{E}_{s_0}(\succ, Q; c) \cap \mathcal{E}_{s_1}(\succ, Q; c)$ exactly restricts that these students are on the cusp of being matched to $s_1$, while their best alternative is $s_0$.

$\tau_{s_0, s_1}$ is the coarsest conditional treatment effect between schools $s_0$ and $s_1$ driven by variation near $s_1$’s cutoff, for those whose best alternative is $s_0$.\textsuperscript{26} In finite samples, the coarsest effect

\textsuperscript{25}The precise definition of $J_i(c)$ is notationally cumbersome, which we lay out explicitly in (B.20) in Appendix B.

\textsuperscript{26}In fact, $\tau_{s_0, s_1}$ and $-\tau_{s_1, s_0}$ form the maximal aggregation of the identified conditional average treatment effects (6).
has the most data with which to estimate, and thus the asymptotic analysis of estimation for such effects is likely to be practically relevant. Moreover, estimation of treatment effects identified from randomization, as opposed to RD, may be analyzed with simple inverse propensity weighting arguments or even in a design-based framework as in Section 3. Thus, estimation of the coarsest RD effects is both practically relevant and non-obvious, and so we focus on it here. Nonetheless, finer conditional average treatment effects are identified, and analogous estimators may be proposed by modifying the indicator \( J_i(c) \). Their asymptotic analyses are analogous to our analysis of the coarsest level of effect.

In particular, we consider a locally linear regression estimator for \( \tau_{s_0,s_1} \) with a uniform kernel. The estimator takes the form of a difference

\[
\hat{\tau}(h_N) = \hat{\beta}_{+,0}(h_N) - \hat{\beta}_{-,0}(h_N)
\]

indexed by some bandwidth \( h_N \). The estimator for the right-limit, \( \hat{\beta}_{+,0}(h_N) \), is the intercept in a linear regression of some reweighted outcome measure \( Y^{(1)}_i(C_N) \) on the centered running variable \( (R_{it_1} - \rho(C_N)) \), among those observations with \( R_{it_1} \in [\rho(C_N), \rho(C_N) + h_N] \) and \( J_i(C_N) = 1 \):

\[
\hat{\beta}_{+}(h_N) = \arg \min_{b_0,b_1} \sum_{i=1}^{N} J_i(C_N) \mathbb{1}(R_{it_1} \in [\rho(C_N), \rho(C_N) + h_N]) \left( Y^{(1)}_i(C_N) - b_0 - (R_{it_1} - \rho(C_N))b_1 \right)^2.
\]

We define \( \hat{\beta}_{-}(h_N) \) analogously. We emphasize, despite the cumbersome notation, that the estimator is nothing more than local linear regression on a easily-computed subsample of the data, with respect to an outcome variable constructed from the data.

To complete the description of the estimator, it remains to define the cutoff \( \rho(c) \) and the reweighted outcome \( Y^{(1)}_i(c) \). For a school \( s \) that uses the \( u \)th test or lottery number, we define \( r_{w,s}(c) \) as the corresponding cutoff in the space of test scores or lottery numbers under cutoffs \( c \), and we define \( \rho(c) = r_{t_1,s_1}(c) \):

\[
r_{w,s}(c) = \max \left\{ g_s^{-1}(q,c_s) : g_s^{-1}(q,c_s) < 1, q = 1, \ldots, q_{s_1} \right\} \in [0,1]; \quad \rho(c) = r_{t_1,s_1}(c).
\]

The object \( r_{w,s}(c) \) is more intuitive than its definition seems. Excluding knife-edge cases (ruled out shortly in Assumption 5.1), for cutoffs \( c \) in the space of priority scores \( V_{i,s} \), there is a single level of the discrete qualifer \( q \) where some students with \( Q_{is} = q \) qualify for school \( s \) and others do not. Those who qualify have test score or lottery numbers above some cutoff, and those who do not qualify have those running variables below the cutoff. That cutoff is precisely \( r_{w,s}(c) \).

On the other hand, the reweighted outcome measure \( Y^{(1)}_i(c) \) is defined as the inverse-propensity weighting \( Y^{(1)}_i(c) = \frac{D_i^{(1)}(c)Y_i}{\pi_i^{(1)}(c)} \), where: (i) 0/0 is interpreted as zero; (ii) \( D_i^{(1)}(c) \) is an indicator for the student failing to qualify for all lottery schools that she prefers to school \( s_1 \) under cutoffs \( c \):

\[
D_i^{(1)}(c) = \prod_{s:t_s \neq 0, s > s_1} \mathbb{1}(U_{its} < g_s^{-1}(Q_i,c_s)) ;^{27} \text{and} \quad \pi_i^{(1)}(c) = \mathbb{E}[D_i^{(1)}(c) \mid (>_{i}, Q_i)].
\]

Since the only other random variable in the expectation, \( U_{i} \), is independent of \( (>_{i}, Q_i) \), this amounts to integrating over the known lottery distribution.

\[ \text{27The object } D_i \text{ is essentially equal to } D_{i,Q_i}^{(1)} \text{ (defined in Definition } 4.1 \text{; and } \pi_i \text{ is essentially equal to } \mathbb{E}[D_{i,Q_i}^{(1)} \mid (>_{i}, Q_i)], \text{ which is formalized in Lemma B.1.} \]
Having introduced the estimator, we turn to assumptions. The first assumption is a substantive restriction on school capacities and the population distribution of student characteristics, such that the large-market cutoffs are not in certain knife-edge configurations.

**Assumption 5.1 (Population cutoffs are interior).** The distribution of student observables satisfies Assumption 4.6, where the population cutoffs \( c \) satisfy:

1. School \( s_1 \) is not undersubscribed: \( \rho(c) > 0 \).
2. For each school \( s \), either (i) there is a unique \( q_s^*(c) \) under which the tie-breaker cutoff is in the open interval \((0, 1)\), \( 0 < r_{t,s}(c) \equiv g_s^{-1}(q_s^*(c), c_s) < 1 \), or (ii) \( c_s = 0 \) and for all \( q \), \( r_{t,s}(c) = g_s^{-1}(q, c_s) = 0 \); in this case, \( q_s^*(c) = -\infty \).
3. If \( c_s = 0 \), then \( s \) is eventually undersubscribed: \( \Pr(C_{s,N} = 0) \to 1 \).
4. If two schools \( s_3, s_4 \) uses the same test \( t \), then their test score cutoffs are different, unless both are undersubscribed: \( r_{t,s_3}(c) = r_{t,s_4}(c) \implies r_{t,s_3}(c) = r_{t,s_4}(c) = 0 \).

Assumption 5.1 rules out populations where the large-market cutoffs from Azevedo and Leshno (2016) are on the boundary of certain sets. The first assumption simply says that the school \( s_1 \) is not undersubscribed so that the treatment effect \( \tau_{s_0,s_1} \) is identified. The second assumption says that the cutoff for each school \( s \) does not lie exactly on the boundary between some \( Q = q \) and \( Q = q+1 \).

It rules out a scenario where everyone with \( Q = q \) do not qualify for \( s \) regardless of their tie-breakers and everyone with \( Q = q + 1 \) qualify for \( s \) regardless of their tie-breakers. The third assumption says that undersubscribed schools are eventually undersubscribed, and so the population capacity of a school is not exactly at the threshold making the school undersubscribed.\(^{28}\) Lastly, the fourth assumption assumes that the population cutoffs are not exactly the same for two schools that uses the same test.\(^{29}\)

Additionally, we maintain a few technical assumptions, Assumptions B.2 to B.5, stated in Appendix B, on the distribution of test scores and lotteries, as well as the conditional means and higher moments of the potential outcomes. Under these technical assumptions, the estimator \( \hat{\tau}_{s_0,s_1} \) is first-order equivalent to an oracle estimator \( \hat{\tau}_{s_0,s_1} = \hat{\beta}_+ - \hat{\beta}_- \), where

\[
\hat{\beta}_+(h_N) \equiv \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix} = \arg\min_{b_0,b_1} \sum_{i=1}^N J_i(c) \mathbb{1}(R_{t_i} \in [\rho(c), \rho(c) + h_N]) \left( Y_i^{(1)}(c) - b_0 - (R_{t_i} - \rho(c))b_1 \right)^2
\]

and similarly for \( \hat{\beta}_- \). The oracle estimator \( \hat{\tau}_{s_0,s_1} \) is a standard RD estimator on observations with \( J_i(c) = 1 \), whose asymptotic properties are well-known (among others, Imbens and Lemieux, 2008; Hahn et al., 2001). Thus, \( \hat{\tau}_{s_0,s_1} \) is asymptotically normal with estimable asymptotic variance.

---

\(^{28}\)This assumption is stronger than the \( O_p(N^{-1/2}) \) convergence of the cutoffs that we assume. However, the assumption holds generically for sufficiently large population school capacities. Precisely speaking, suppose some school \( s \) with population capacity \( q_s \) is undersubscribed in the population, but the probability that it is undersubscribed in samples of size \( N \) does not tend to one (and so violates the third assumption). Then we may add a little slack to the school capacity—for any \( \epsilon > 0 \), making the capacity \( q_s + \epsilon \) instead—to guarantee that the third assumption holds. Intuitively, adding \( \epsilon \) to the capacity adds \( O(\epsilon N) \) seats to the school in finite samples, but the random fluctuation of the market generates variation in student assignments of size \( O(\sqrt{N}) \).

\(^{29}\)This is Assumption 2 in Abdulkadiroğlu et al. (2022).
Theorem 5.2. Under Assumptions 5.1, 4.6, and B.2 to B.5, assuming $N^{-1/2} = o(h_N)$ and $h_N = o(1)$,

$$\sqrt{Nh_N}(\hat{\tau}_{s_0,s_1} - \tau_{s_0,s_1}) = o_p(1).$$

In particular, if $h_N = O(N^{-d})$ with $1/4 < d < 1/5$, the discrepancy is smaller than the bias of the oracle estimator: $\sqrt{Nh_N}(\hat{\tau}_{s_0,s_1} - \tau_{s_0,s_1}) = o_p(\sqrt{Nh_N}h_N^2) = o_p(1)$. Moreover, under undersmoothing, i.e. $h_N = o(N^{-1/5})$, $\hat{\tau}_{s_0,s_1}$ is asymptotically normal:

$$\hat{\tau}_{s_0,s_1}^{-1}(\tau_{s_0,s_1} - \tau_{s_0,s_1}) \overset{d}{\to} N(0,1).$$

The variance estimate $\hat{\sigma}^2_{s_0,s_1}$ is the sum $\hat{\sigma}^2_{s_0,s_1} = \hat{\sigma}^2_{+,N} + \hat{\sigma}^2_{-,N}$, where (i) $\hat{\sigma}^2_{+,N}$ is defined as

$$\hat{\sigma}^2_{+,N} = \frac{4Nh_N}{N_+} \left( \frac{1}{N_+} \sum_{i=1}^N W_i(C_N, h_N) Y_i^{(1)}(C_N)^2 - \hat{\beta}^2_{+,0} \right);$$

(ii) the sample size is

$$N_+ \equiv \sum_{i=1}^N W_i(C_N, h_N) \quad W_i(C_N, h_N) \equiv J_i(C_N) \mathbb{1}(R_{ti} \in [\rho(C_N), \rho(C_N) + h_N])$$

and (iii) $\hat{\sigma}^2_{-,N}$ is defined analogously.

The proof of Theorem 5.2 is relegated to Appendix B. 30 We give a brief sketch here. Within the bandwidth $h_N$, there are roughly $O(Nh_N)$ students. On running variables other than $t_1$, the discrepancy $\|C_N - c\|_{\infty} = O_p(N^{-1/2})$ leads to discrepancies $J_i(C_N) \neq J_i(c)$ in about $O(Nh_N \cdot N^{-1/2}) = O(\sqrt{Nh_N})$ students. Thus the asymptotic bias incurred due to discrepancies in $J_i$ is of order $O(\sqrt{Nh_N}/\sqrt{Nh_N}) = O(\sqrt{h_N})$. On the other hand, for the running variable $R_{ti}$, the discrepancy in $C_{s,N} - c_s$ incurs a discrepancy in $O(\sqrt{N})$ students, all of whom are contained in the bandwidth. Thankfully, the accrued error in these students is of order $O((\sqrt{N})^{1/2}) = O(N^{1/4})$, via an application of Kolmogorov’s maximal inequality, a key step in our argument. The total asymptotic discrepancy, due to $c \neq C_N$, is then $O((N^{1/4} + N^{1/2}h_N)/\sqrt{Nh_N})$, leading to the rates in the result.

Theorem 5.2 provides a basis for Wald inference with locally linear regression estimators, through standard undersmoothing arguments. This supplies theoretical justification for confidence intervals reported in empirical work (Abdulkadiroğlu et al., 2022), as well as various normal-limit hypothesis tests. Moreover, the higher-order computation in Theorem 5.2 opens the path to bandwidth choices (Calonico et al., 2014; Imbens and Wager, 2019), though we leave a more detailed analysis to future work.

6. Conclusion

Detailed administrative data from school choice settings provide an exciting frontier for causal inference in observational data. Remarkably, school choice markets are engineered (Roth, 2002) to have desirable properties for market participants, and yet it may yield natural-experiment variation that inform program evaluation and policy objectives. Credibility of empirical studies using such

30See Theorem B.6 for the first-order equivalence and Theorem B.9 for the variance estimation.
variation demands an understanding of what counterfactual queries the data can and cannot answer absent further assumptions. Our analyses here provide a step towards that understanding.

As a review, we provide a detailed analysis of identified treatment effects in school choice settings. This exercise highlights the complex features that drive identification in school choice settings. We find that common estimators may be subject to certain pitfalls. Regression estimators in stochastic mechanisms may estimate different estimands as a function of which covariates are dropped, due to very limited overlap. Regression estimators in mixed mechanisms may pool over different treatment contrasts, and asymptotically assigns zero weight to those identified via RD variation. Since lottery- and RD-type variation applies to individuals with predictably different characteristics, such pooling may be unrepresentative when heterogeneity is present. Lastly, we provide asymptotic results for estimation and inference, under a proper asymptotic sequence that does not ignore treatment assignment dependence.

There are many questions for future research, both theoretical and empirical. We enumerate a few here. Motivated by the complex heterogeneity in the treatment effects in mixed mechanisms, how do we efficiently aggregate them into a policy relevant treatment effect (Heckman and Vytlacil, 2001)? Each treatment effect is likely poorly estimated with finite data—how do we pool similar estimates to improve efficiency without having the lottery effects dominate? Under heterogeneous treatment effects, these effects are only partially informative for school value-added; nevertheless, these effects may validate school value-added estimates. It also remains an empirical question how large the heterogeneity is and what it implies about choice behavior. Finally, we leave a similar characterization of treatment effects in top-trading cycles—an algorithm that does not fall in our framework—to future work, since the latter also admits a cutoff structure (Leshno and Lo, 2021).
Appendix to “Nonparametric Treatment Effect Identification in School Choice”

Jiafeng Chen
Harvard University
jiafengchen@g.harvard.edu

APPENDIX A. MISCELLANEOUS DISCUSSIONS

A.1. An example illustrating that identification notions are not straightforward. As an example, consider $X_1, \ldots, X_N \sim \mathcal{N}(0, 1)$ and a treatment where everyone above median is treated:

$$D_i = 1(X_i \geq \text{Med}(X_1, \ldots, X_N)).$$

Note that for any finite $N$, the conditional ATE

$$\tau(x) = \mathbb{E}[Y(1) - Y(0) \mid X = x]$$

is identified, since the observed conditional expectations are equal to conditional expectations of the potential outcomes

$$\mathbb{E}[Y_i \mid D_i = d, X_i = x] = \mathbb{E}\mathbb{E}[Y_i(d) \mid X_i, X_i = x, D_i = d] \mid D_i = d, X_i = x]$$

and since $\Pr(D_i = d \mid X_i = x) > 0$ for all $x \in \mathbb{R}$. However, there is a sense in which the only morally identified parameter is $\tau(0)$ since $\text{Med}(X_1, \ldots, X_N) \xrightarrow{a.s.} 0$. For any $c < 0$, it becomes vanishingly unlikely that a unit with $X_i = c$ is treated, and similarly for $c > 0$. However, we would need $N \to \infty$ for the conditional expectations to be estimated, and hence $\tau(x)$ cannot be estimated if $x \neq 0$.

A.2. Identification in stochastic mechanisms (details). We prove Proposition 3.1, reproduced here.

**Proposition 3.1.** Let $\tau = \sum_{i=1}^{N} \sum_{s=1}^{M} \lambda_{i,s} y_i(s)$ be the parameter of interest. $\tau$ is identified if and only if, for all $i \in I, s \in S$, $\lambda_{i,s} = 0$ whenever student $i$ has no chance of being matched to school $s$:

$$\pi_{i,s} = \mathbb{E}_{\Pi_N(X_1, \ldots, X_N)}[D_{i,s}] = 0.$$

**Proof.** (“If” part) Note that the random variable $\frac{D_i \alpha Y_i}{\pi_{i,s}}$ has expectation (over $D \sim \Pi_N$) $y_i(s)$, so long as $\pi_{i,s} \neq 0$. Therefore, the set of numbers $y_i(s)$ where $\pi_{i,s} \neq 0$ is identified. Hence, if $\tau$ does not depend on those $y_i(s)$ for which $\pi_{i,s} = 0$, $\tau$ would be identified.

(“Only if” part) We prove the contrapositive. Suppose $\lambda_{i,s} \neq 0$ where $\pi_{i,s} = 0$. Note that $y_i(s)$ is never observed. Hence, changes in the value of $y_i(s)$ are not reflected in the distribution of the observed data. However, such changes are reflected in $\tau$, resulting in observationally equivalent $\tau$ values. Hence $\tau$ is not identified. \qed

A.3. More on regression estimators in stochastic mechanisms. In this section, we expand on Lemma A.1 by analyzing a non-interacted regression estimator. In contrast to the specification (2), a much more common specification does not interact the controls with the treatment:

$$Y_i = \alpha + \sum_{k=1}^{K} \beta_k L_{ik} + \gamma' B_i + v_i$$

(A.12)
where \( B_i \) is a vector of covariates, which may be the full collection of assignment probability bins \([B_{ikj}: k = 0, \ldots, K, j_k = 1, \ldots, J_K]\). From a superpopulation perspective, when there is a single unconfounded treatment—i.e. \( K = 1 \)—the estimand \( \beta_k = \beta_1 \) is a weighted average treatment effect. However, as Goldsmith-Pinkham, Hull and Kolesár (2021) point out, with multiple treatments \( \beta_k \) is no longer a convex-weighted average treatment effect. This issue of the specification (A.12) exists outside of our setting, and thus we do not explore it and instead assume constant treatment effects for our analysis. Of course, if the treatment categories \( L_k \) contains multiple schools, constant treatment effect is an even stronger assumption than usual.

Nevertheless, suppose the unit-level potential outcomes are of the form

\[ y_i(s) = y_i(0) + \tau_k \text{ for } s \in S_k \]

where \( \tau_k \) does not depend on \( i \). As a result, all the heterogeneity is contained in variation of the baseline potential outcomes. We may compute via the Frisch–Waugh–Lovell theorem that

\[
\hat{\beta} = \left[ \frac{\tau_1}{\tau_K} \right] + \left[ \sum_{i=1}^{N} \hat{L}_i \hat{L}'_i \right]^{-1} \sum_{i=1}^{N} \hat{L}_i y_i(0) \equiv \tau + \epsilon_N
\]

where \( \hat{L}_i \) is the projection residual from a regression of \( L_i \equiv [L_{i1}, \ldots, L_{iK}]' \) on a constant and the covariates \( B_i \). The OLS estimator \( \hat{\beta} \) can be decomposed into the target parameter \( \tau \) plus a noise term \( \epsilon_N \), where if \( N \) is sufficiently large, the noise term has mean:

\[
\mathbb{E} [\epsilon_N] \approx \left( \frac{1}{N} \sum_{i=1}^{N} \hat{L}_i \hat{L}'_i \right)^{-1} \frac{1}{N} \sum_{i=1}^{N} \mathbb{E} [L_i - B_i \hat{\Pi}] y_i(0),
\]

where \( B_i \hat{\Pi} \) is the fitted value from the \( L \)-on-\( X \) regression. \( \mathbb{E} [\epsilon_N] \) is close to zero if the best prediction from \( X_i \) is close to the assignment probability: \( B_i \hat{\Pi} \approx [e_{i1}, \ldots, e_{iK}]' \), suggesting that \( B_i \) may take the form of the propensity score bins, or even directly the propensity scores \( B_i = [e_{i1}, \ldots, e_{iK}]' \). \(^{31}\)

### A.4. Regression estimation proofs

Before introducing Lemma A.1, we define some notation. It is typical that certain sets of schools are grouped into a treatment arm. Therefore, we coarsen treatments into the levels \( L_{i0}, \ldots, L_{iK} \), where the indicator for assignment to category \( k \) is defined as \( L_{ik} = \sum_{s \in S_k} D_{is} \), summing over some partition of the set of schools: \( \bigcup_{k=1}^{K} S_k = S \). Correspondingly, let the assignment probability to each category of schools be \( e_{ik} = \sum_{s \in S_k} \pi_{is} \), where \( \pi_{is} = \Pr (X_i | X_{i1}, \ldots, X_{iN}) (D_{is} = 1) \). We consider discretizing \( e_{ik} \) into bins \( \{B_{k0}, \ldots, B_{kJ_k} : k = 1, \ldots, K \} \), where \( B_{ikj_k} = 1 \) denotes that student \( i \)'s assignment probability to the \( k^{th} \) category of schools, \( e_{ik} \), falls in the \( j_k^{th} \) bin, where (i) the bins contain similar values of the assignment probabilities; (ii) the bins are ordered such that assignment probabilities in the \( j_k^{th} \) bin are lower than those in the \( (j_k + 1)^{th} \) bin; (iii), the \( 0^{th} \) bin contains only \( e_{ik} = 0 \); and, (iv) any value of the assignment probability falls into some bin. Note that the partition over the assignment probabilities may differ by school. \(^{32}\)

**Lemma A.1.** The regression coefficients (2) are explicitly written as

\[
\hat{\mu}_k = \frac{1}{N_k} \sum_{i=1}^{N} (1 - B_{i,k,0}) \frac{L_{ik} Y_i}{e_{ik}} = \sum_{i=1}^{N} \sum_{s=0}^{M} I(s \in S_k) \pi_{is} \frac{D_{is} Y_i}{N_k e_{ik}} \pi_{is}
\]

(A.13)

\(^{31}\)These two estimators are not numerically equivalent in sample, but they should be close when \( N \) is large.

\(^{32}\)Of course, this notation nests the case where the estimated propensity scores take discrete values, as in Abdulkadirgül et al. (2017a).
where

1. \(N_k = \sum_{i=1}^{N} (1 - B_{i,k,0})\) is the number of students with nonzero probability for arm \(k\),
2. \(n_{jk} = \sum_{i=1}^{N} B_{i,k,jk}\) is the number of students whose assignment probabilities for arm \(k\) falls into bin \(jk\),
3. \(n_{k,jk} = \sum_{\ell=1}^{N} L_{\ell k} B_{\ell,k,jk}\) is the number of students who are assigned to arm \(k\) and whose assignment probabilities for arm \(k\) fall into bin \(jk\), and finally
4. \(\hat{e}_{ik} = \sum_{jk=1}^{K} B_{i,k,jk} \frac{n_{k,jk}}{n_{k,k}}\) is student \(i\)'s estimated assignment probability for arm \(k\), defined as the proportion of individuals assigned to arm \(k\), among those whose assignment probabilities for arm \(k\) fall into the same bin as \(i\)’s.

Proof. We may reparametrize the regression (2) into

\[
\min_{\hat{\mu}_k} \sum_{i} \left( Y_i - \sum_{k=0}^{K} \sum_{j=1}^{J_k} \nu_{k,jk} L_{ik} B_{ik} \right)^2 
\]

to find that

\[
\hat{\mu}_k = \sum_{jk=1}^{K} \frac{n_{k,jk}}{N_k} \hat{\nu}_{k,jk},
\]

where \(\hat{\nu}_{k,jk}\) is the within-cell mean, among those who have \(L_{ik} = 1\) and \(B_{i,k,jk} = 1\):

\[
\hat{\nu}_{k,jk} = \sum_{i=1}^{N} \frac{B_{ik} L_{ik} Y_i}{n_{k,jk}}.
\]

The claim then follows from explicitly computing \(\hat{\mu}_k\). \(\square\)

We prove Lemma 3.2, restated below, here

**Lemma 3.2.** Consider real-valued random variables \((Y(0), Y(1), D, X) \sim P\), where the distribution \(P\) has finite second moments. Suppose \(D \in \{0, 1\}, X \in \{0, \ldots, J\}\). Let the observed outcome be \(Y = Y(D)\). Assume that treatment is unconfounded: \(Y(0), Y(1) \perp D \mid X\). Let \(X_j = \mathbb{1}(X = j), \nu_j = \mathbb{E}[X_j], \bar{X}_j = X_j - \nu_j\), and \(\pi_j = \Pr(D = 1 \mid X_j = 1)\). Suppose the baseline level \(X = 0\) has overlap: \(0 < \pi_0 < 1\). Then:

1. If, for all \(j\), \(0 < \pi_j < 1\), then the regression coefficient in the following population regression recovers the average treatment effect

\[
\beta_0 = \arg \min_{\beta} \min_{\alpha, \gamma, \delta} \mathbb{E} \left[ \left( Y - \alpha - D\beta - \sum_{j=1}^{J} \bar{X}_j \gamma_j - \sum_{j=1}^{J} \delta_j D \bar{X}_j \right)^2 \right] = \mathbb{E}[Y(1) - Y(0)].
\]

2. If, for any \(j\), \(\pi_j \in \{0, 1\}\), then the population projection coefficient \(\beta\) is not identified: i.e. \(D\) is a linear combination of 1 and \((\bar{X}_j, D \bar{X}_j)^{J}_{j=1}\).

3. Let \(G = \{ j : \pi_j \in \{0, 1\} \}\). The regression dropping interactions for indices in \(G\) estimates a particular aggregation of the conditional average treatment effects:

\[
\hat{\beta}_0 = \arg \min_{\beta} \min_{\alpha, \gamma, \delta} \mathbb{E} \left[ \left( Y - \alpha - D\beta - \sum_{j=1}^{J} \bar{X}_j \gamma_j - \sum_{j=1, j \notin G}^{J} \delta_j D \bar{X}_j \right)^2 \right]
\]

\[
= \left( \nu_0 + \sum_{j \in G} \nu_j \right) \mathbb{E}[Y(1) - Y(0) \mid X = 0] + \sum_{j=1, j \notin G}^{J} \nu_j \mathbb{E}[Y(1) - Y(0) \mid X = j].
\]

In particular, \(\hat{\beta}_0\) is not invariant to the choice of the baseline covariate level \(X = 0\).
(4) Let $\bar{X}_j = X_j - \frac{\mu_j}{\sum_{k \in G} \nu_k}$. The regression in (3) with $\bar{X}_j$ instead of $\tilde{X}_j$ recovers the maximal identifiable average treatment effect $E[Y(1) - Y(0) \mid X \not\in G]$.

**Proof.** (1) The claim is well-known. See, e.g., expression (21.35) in Wooldridge (2010).
(2) Suppose $\pi_j = 1$. Then $D\bar{X}_j = D(X_j - \nu_j) = X_j - \nu_j D$. Since
\[
D = -\nu_j^{-1} \left(D\bar{X}_j - \tilde{X}_j - \nu_j\right)
\]
is a linear combination of 1, $D\bar{X}_j$, $\bar{X}_j$, the coefficient on $D$ is not identified.
Suppose $\nu_j = 0$, then $D\bar{X}_j = -\nu_j D$, and is therefore collinear with $D$.
(3) Let $\mu_{dk} = E[Y \mid D = d, X_k = 1] = E[Y(d) \mid X_k = 1]$. (Certain $\mu_{dk}$’s are not defined since the conditioning is probability zero.) For $k \not\in G$, the regression predicts
\[
m_{dk} = \alpha - \sum_{k=1}^J \gamma_k \nu_k + \left(\beta + \delta_k - \sum_{k=1, k \in G} \delta_k \nu_k\right) d + \gamma_k \equiv \tilde{\alpha} + (\tilde{\beta} + \gamma_k) d + \gamma_k
\]
If $k \in G$ and $\pi_k = 0$, then
\[
m_{0k} = \tilde{\alpha} + \gamma_k
\]
If $k \in G$ and $\pi_k = 1$, then
\[
m_{1k} = \tilde{\alpha} + \tilde{\beta} + \gamma_k.
\]
Finally, for $k = 0$, we have
\[
\mu_{d0} = \tilde{\alpha} + \tilde{\beta} d
\]
Now, it is well-known that the regression problem is equivalent to projecting $\mu_{DX}$ onto $m_{DX}$, where the latter is a function of the parameters:
\[
E[(\mu_{DX} - m_{DX})^2] = \sum_{k=0}^J \nu_k \pi_k (\mu_{1k} - m_{1k})^2 + \nu_k (1 - \pi_k)(\mu_{1k} - m_{1k})^2.
\]
The minimum value of zero in the above is achieved by setting
\[
\tilde{\alpha} = \mu_{00}, \quad \tilde{\beta} = \mu_{10} - \mu_{00}, \quad \gamma_k = \begin{cases} \mu_{0k} - \mu_{00} & \pi_k < 1 \\ \mu_{1k} - \mu_{10} & \pi_k = 1 \end{cases}, \quad \text{and} \quad \delta_k = (\mu_{1k} - \mu_{0k}) - (\mu_{10} - \mu_{00}),
\]
and thus they correspond to the population OLS coefficients. Note that there are $2(J + 1) - |G|$ $\mu_{dk}$’s and $2 + J + J - |G| = 2(J + 1) - |G|$ coefficients. Hence the solution of the coefficients is unique.
The coefficient $\beta$ is
\[
\beta = \tilde{\beta} + \sum_{k=1, k \in G} \delta_k \nu_k (\mu_{10} - \mu_{00}) + \sum_{k=1, k \not\in G} \nu_k (\mu_{1k} - \mu_{0k}) - \sum_{k=1, k \not\in G} \nu_k (\mu_{10} - \mu_{00})
\]
\[
= \left(1 + \sum_{k \in G} \nu_k\right) (\mu_{10} - \mu_{00}) + \sum_{k=1, k \not\in G} \nu_k (\mu_{1k} - \mu_{0k}),
\]
as claimed.
(4) The result follows readily from the argument in (3) by replacing $\nu_k$ with $\nu_k/(1 - \sum_{j \in G} \nu_j)$.

A.5. **Identification of conditional means of potential outcomes.** In this subsection, we characterize the set of causal effects that are identified at $P_{W,A}$, in the sense of Definition 4.2. By a causal effect, we mean some conditional average of the potential outcomes $Y(s)$. By the law of iterated expectations, it suffices to consider the finest conditioning for the conditional means. Furthermore, since extending our results to
conditioning on observable but non-mechanism-relevant covariates \( Z \) is straightforward,\(^{33}\) we suppress dependence on such covariates \( Z \). We assume that the following conditional expectations of potential outcomes is continuous in \( \mathbf{R} \).\(^{34}\)

**Assumption A.2.** The map
\[
\mathbf{R} \mapsto \mathbb{E}[Y(s) \mid (\succ, \mathbf{R})] = \mu_s(\succ, \mathbf{R}) \tag{A.14}
\]
is continuous for every \( s \in S \) and every \( \succ \).

By the restricted notion of identification defined in Section 4.1 we may consider the school cutoffs as fixed \( \{c_s : s \in S\} \), maintaining the following assumption that ensures the conclusion of Azevedo and Leshno (2016) holds.

**Assumption A.3.** The population of students is such that the cutoffs on priority scores \( V_is, \{C_{s,N} \} \), satisfy max\(s\in S \mid C_{s,N} - c_s = O_p(N^{-1/2}) \) for some fixed \( c = (c_s : s \in S) \).

\( \mu_s(\succ, \mathbf{R}) \) is the finest level of causal effects, and we characterize the set of \( (\succ, \mathbf{R}) \) combinations for which these effects are identified. Under Assumptions 4.6 and A.2, we show that \( \mu_s(\succ, \mathbf{R}) \) is identified if and only if \( \mathbf{R} \) falls in a set \( \overline{E}_s(\succ, c) \), which is the closure of what we term the \( s \)-eligibility set for \( \succ \).

**Proposition A.4.** Consider a mechanism with deferred acceptance priority scores, described in Section 4.2. Let the cutoffs of test schools be \( c = c(P_{W,A}) \) as in Definition 4.1. Fix a preference order \( \succ \) that occurs with positive probability under \( P_{W,A} \). Under Assumptions A.2 and A.3, the conditional expectation of potential outcomes \( \mu_s(\succ, \mathbf{r}) \) is identified at \( P_{W,A} \) if and only if \( \mathbf{r} \in \overline{E}_s(\succ, c) \), where:

1. Let \( Q = Q(P_{W,A}) \) as in Definition 4.2 be the limiting distribution of the observed data. Define \( E_s(\succ, c) = \{r : Q[D_s^*(c) = 1 \mid \succ, \mathbf{R} = r] > 0\} \) as the \( s \)-eligibility set for preference \( \succ \).
2. \( \overline{E}_s \) is the closure of \( E_s \) with respect to the metric on the values \( \mathbf{R} \).

**Proof.** (If) The potential randomness in the mechanism (driven by lottery numbers \( U_{is} \)) induces a distribution \( Q(D^* \mid \succ, \mathbf{R} = \mathbf{r}) \) over treatment values \( D^*(c) = [D_{1}^*, \ldots, D_{M}^*]' \). These treatments are ignorable by virtue of the lottery:
\[
D^*(c) \equiv \{(Y(s) : s \in S), Z) \mid (\succ, \mathbf{R})\}.
\]

Since the treatment values are with respect to a fixed set of cutoffs \( c = \{c_s\} \), these treatment values are independently drawn for each student. The \( s \)-eligibility set \( E_s(\succ, c) \) is then defined as the set of \( \mathbf{R} \) values where a student with preference \( \succ \) has positive probability of being assigned school \( s \), at the cutoff values \( c \). Naturally, since \( D^*(c) \) is ignorable, the mean potential outcome for \( s \), \( \mu_s(\succ, \mathbf{R}) \), is identified whenever the overlap condition holds for \( s \), i.e. when \( \mathbf{R} \in E_s(\succ, c) \). Explicitly, for instance, we may consider the inverse propensity weighting:
\[
\mu_s(\succ, \mathbf{R}) = \mathbb{E} \left[ \frac{D_s^* Y}{\Pr[D_s^*(c) \mid \succ, \mathbf{R}] \mid \succ, \mathbf{R}} \right]. \tag{A.15}
\]

Since \( \mu_s \) is assumed to be continuous in \( \mathbf{R} \) (Assumption A.2), we may extend the identification to the closure \( \overline{E}_s(\succ, c) \).

(Only if) Fix cutoffs at \( c \). Consider \( \succ \) with positive measure and suppose \( \mathbf{r} \notin \overline{E}_s(\succ, c) \). Since the complement of a closed set is open, there is a neighborhood \( U \) of \( \mathbf{r} \) such that \( U \cap \overline{E}_s(\succ, c) = \emptyset \). Consider two continuous

---

\(^{33}\)We may simply understand the analysis as conditioning on some fixed value \( Z = z \).

\(^{34}\)The notion of continuity is with respect to some metric on the domain. When we specialize to \( \mathbf{R} = (Q, R) \) in mixed mechanisms, the metric can be taken as
\[
d((Q_1, R_1), (Q_2, R_2)) = 1(Q_1 = Q_2) + \|R_1 - R_2\|.
\]
Proposition 4.7

Abdulkadiroğlu et al. and Theorem 4.8 transports the cutoff follows immediately from the more general takes one of three forms: (a) some value , by plugging in qualifies the student to a school she prefers to (Abdulkadiroğlu et al. Abdulkadiroğlu et al. Proposition A.4 Abdulkadiroğlu et al.

\[ \mu_s(\succ, r, Z) \neq \lambda_s(\succ, r, Z). \]

They exist since \( U \) is open. Note that under the cutoff \( c \) and corresponding treatment \( D^\ast(c) \) has that \( \Pr(D^\ast(c) \mid \succ, r', Z) = 0 \) for all \( r' \in U \), as a result, the observed outcome is never \( Y_i(s) \) when \( r' \in U \). Naturally, \( \mu_s, \lambda_s \) are observationally equivalent. Hence \( \mu_s(\succ, r, Z) \) is not identified. □

"As a reminder, we use the notation \( D^\ast \) to distinguish from the fact that these treatment values are contingent on a fixed cutoff. In particular, \( D^\ast(c) s = 1 \) if the student is matched to school \( s \) when the cutoff is \( c \), which is a deterministic function of the student’s observable characteristics as well as her lottery numbers.

The proof of Proposition 4.7 follows immediately from the more general Proposition A.4, by plugging in the metric on the space of \( R = (Q, R) \).

A.6. Proof of Theorem 4.8.

Theorem 4.8. Consider a mixed mechanism under Assumption 4.5 and Assumption 4.6. Let \( (\succ, Q, R) \) be in the support of the distribution of student observable characteristics, and let \( c \) be the limiting cutoff under Assumption 4.6. Then we have the following properties regarding variation induced by lottery:

1. Define the support \( \text{supp}(\succ, Q, R) = \{ s : R \in E_s(\succ, Q, c) \} \) as the set of schools that have positive probability of being assigned at characteristics \( (\succ, Q, R) \). Then \( \text{supp}(\succ, Q, R) \) contains at most one test-score school, and if it does, then the test score school is the least preferred element in \( \text{supp}(\succ, Q, R) \) according to \( \succ \).

2. If \( s_1, s_2 \) are two test-score schools, then \( E_{s_1}(\succ, Q, c) \cap E_{s_2}(\succ, Q, c) = \emptyset \).

Moreover, for a school \( s \) and a test \( t \), define the following objects:

- Suppose \( t_s = t \). Let the most disfavored test score that clears the cutoff \( c_s \) at school \( s \) be\(^{35}\)

\[ r_t(s, Q_s, c_s) = g_s^{-1}(Q_s, c_s) \equiv \inf \{ r \in [0, 1] : g_s(Q_s, r) \geq c_s \}. \]

- Let the most lenient test score \( t \) among those schools preferred to \( s \) be\(^{36}\)

\[ R_t(s, \succ, Q; c) = \min_{s_1, t_1 \succ s \atop t_1 = t} r_t(s_1, Q_s, c_s) \]

- Let \( L(Q, c) \) be the set of lottery schools at which a student with discrete priority \( Q \) qualifies with probability one.

Then, we have the following explicit form of eligibility sets for test-score schools and the intersections of their closures, suppressing the dependence on \( \succ, Q, c \):

3. For a test-score school \( s_0 \), the \( s_0 \)-eligibility set takes the following form:

\[
E_{s_0}(\succ, Q, c) = \begin{cases} 
[r_{t_0}(s_0), R_{t_0}(s_0)] \times \left[ X_{t_0} \neq 0, R_{t_0}(s_0) \right], & \text{if } s_0 \succ L(Q, c) \\
\emptyset, & \text{otherwise,} 
\end{cases}
\]

\(^{35}\)Effectively, \( r_t \) transports the cutoff \( c_s \) from the space of priority scores \( V_s \) to the space of test scores \( R_t \). Depending on the value of \( Q_s \), \( r_t \) takes one of three forms: (a) some value \( r_{t_0}(c) \in (0, 1) \), which does not depend on \( Q_s \), (b) 0, indicating that \( Q_s \) is sufficiently compelling to qualify the student at \( s \) regardless of her test scores, or (c) 1, indicating that \( Q_s \) is sufficiently low that the student cannot be seated at \( s \) regardless of her test scores.

\(^{36}\)\( R_t(s) \) is, loosely speaking, what Abdulkadiroğlu et al. (2022) refer to as the most informative disqualification. Echoing Abdulkadiroğlu et al. (2022), the test score distribution of students matched to \( s \) are truncated by \( R_t \) on the right, since any test score \( R_t \) higher than \( R_t \) qualifies the student to a school she prefers to \( s \). The definition differs from most informative disqualification in Abdulkadiroğlu et al. (2022) slightly and immaterially, as we operate in the space of the test scores \( R_t \), while Abdulkadiroğlu et al. (2022) operate in the space of the priority scores \( V_s \).
and its closure takes the following form

\[ \overline{E}_{s_0}(\succ, Q, c) = \begin{cases} \left[ r_{t_0}(s_0), R_{t_0}(s_0) \right] \times \left( \times_{t \neq t_0} [0, R_t(s_0)] \right), & \text{if } E_{s_0} \neq \emptyset \\ \emptyset, & \text{otherwise}. \end{cases} \]  

(8)

These are either hyperrectangles of dimension \( T = \dim R \) or empty sets.

(4) For two test-score schools \( s_0, s_1 \), where \( s_1 \succ s_0 \), the intersection \( \overline{E}_{s_0} \cap \overline{E}_{s_1} \) takes the following form

\[ \overline{E}_{s_0} \cap \overline{E}_{s_1} = \begin{cases} \overline{E}_{s_0} \cap \left( \left\{ t_1(1) \right\} \times [0, 1]^{T-1} \right) \quad \text{if } R_{t_1}(s_0) = r_{t_1}(s_1) \text{ and } E_{s_0}, E_{s_1} \neq \emptyset \\ \emptyset, & \text{otherwise}, \end{cases} \]  

(9)

which is either a hyperrectangle of dimension \( T - 1 \) or an empty set.

**Proof.** (1) At any \((\succ, Q, R)\) value, the treatment \( D^* \) must only put weight on at most a single test-score school, since there cannot be lottery variation between two test-score schools. Hence the support contains at most one test-score school. Suppose \( D^* \) does put weight on a test-score school \( s \). Then at \((\succ, Q, R)\), the student qualifies for \( s \). Suppose \( s \succ s' \) for some lottery school \( s' \). Then \( D^* \) cannot put weight on \( s' \), since the student qualifies for \( s \) and must be assigned at a school no worse than \( s \).

(2) This is a consequence of the fact that only 1 test-score school is in the support.

(3) Fix \((\succ, Q)\) in the support and test-score school \( s_0 \). If \( s_0 \) does not dominate every school in \( L(Q, c) \), then the student can pick any school in \( L(Q, c) \) and must not be assigned to \( s_0 \) regardless of \( R \) values. Hence \( s_0 \succ L(Q, c) \) is a necessary condition for \( E_{s_0} \) being non-empty.

The student qualifies at \( s_0 \) iff \( r_{t_0}(s_0) \leq R_{t_0} \). The student fails to qualify for any test-score school that she prefers iff \( R_t < R_{t_0}(s_0) \) for all \( t \). Hence, when \( E_{s_0} \) is as stated (7), \( R \in E_{s_0} \) implies that \( s_0 \) is the student’s favorite test score school among those in her choice set, and she does not qualify for sure at any lottery school preferred to \( s_0 \). This is exactly the condition needed for the probability of being assigned to \( s_0 \) under \( D^* \) to be positive.

(4) Assume \( E_{s_0}, E_{s_1} \neq \emptyset \). Note that \( R_{t_1}(s_0) \leq R_{t_1}(s_1) \) since \( s_1 \succ s_0 \). Thus the intersection only depends on \( E_{s_1} \) on the \( t_1 \)th coordinate. Note that \( r_{t_1}(s_1) \geq R_{t_1}(s_0) \). Hence the intersection is only nonempty if the \( \geq \) is an equality. In that case, the intersection is simply \( \overline{E}_{s_0} \), but with \( \{ r_{t_1}(s_1) \} \) on the \( t_1 \)th coordinate.

\( \square \)

**A.7. Connection to local deferred acceptance propensity scores.** We collect our computations in the following remark for Section 4.5. In summary, for \( \tau(\delta) \) to be approximately a causal contrast, we must have \( \delta \approx 0 \). However, this means that the variation driven by RD (in II, IV) becomes negligible.

**Remark 2** (Details of argument in Section 4.5). (1) Let \( C \in \{A, B, C\} \times \{1, \ldots, V\} \) denote a student preference-by-test score region pair. We may express, by law of total probability, \( \tau(\delta) \) as a weighted average of weighted treatment contrasts over \( C \):

\[
\tau(\delta) = \sum_C \Pr(C) \cdot \left\{ \frac{\mathbb{E}[Y_T \mid D = 1, C] \Pr(D = 1 \mid C)}{\Pr(D = 1 \mid \psi = \psi(C))} - \frac{\mathbb{E}[Y_C \mid D = 0, C] \Pr(D = 1 \mid C)}{\Pr(D = 1 \mid \psi = \psi(C))} \right\}.
\]

(9)

(9)

\[ \equiv \sum_C \Pr(C) \left\{ w_1(C, \delta) \mathbb{E}[Y_T \mid D = 1, C] - w_0(C, \delta) \mathbb{E}[Y_T \mid D = 0, C] \right\} \]

(2) One source of bias is that \( w_0(C, \delta) \neq w_1(C, \delta) \). The weights associated with \( D = d \),

\[
w_d(C, \delta) = \frac{\Pr(D = d \mid C)}{\sum_{C'} \Pr(D = d \mid C') \Pr(C' \mid \psi = \psi(C))},
\]

are equal to one if the true assignment probability is exactly equal to \( \psi(C) \) for all \( C \). However, it is unlikely that the threshold-crossing probabilities are exactly 1/2 in regions II, IV for \( \delta > 0 \), which causes the estimand
The student’s propensity score is zero if
\[ Pr(B, II) \tau(\delta, B, II) \equiv Pr(B, II) \cdot (E[Y_T | D = 1, B, II] - E[Y_C | D = 0, B, II]). \]
An analysis of the conditions for treating students of type (B, II) finds that
\[
\tau(\delta, B, II) = E \left[ Y_T - Y_C \mid B, \frac{1}{3} - \delta < R < \frac{1}{3} \right] + (1 - w_L(\delta)) \left( E \left[ Y_C \mid B, \frac{1}{3} - \delta < R < \frac{1}{3} \right] - E \left[ Y_C \mid B, \frac{1}{3} \leq R < \frac{1}{3} + \delta \right] \right).
\]
where
\[ w_L(\delta) = \frac{1}{2} Pr(\frac{1}{3} - \delta < R < \frac{1}{3} | B, II) \]
is the conditional probability of failing to qualify for both schools s_1 and s_3 (thereby assigned to school s_0), conditional on the student being untreated. This form comes from the fact that (i) students of this type are only treated when \( \frac{1}{3} - \delta < R < \frac{1}{3} \) and (ii) they are untreated either when \( R > \frac{1}{3} \), or when \( \frac{1}{3} - \delta < R < \frac{1}{3} \) but they fail to qualify via lottery at s_3. The selection bias term may be small when \( \delta \) is small, under suitable continuity conditions on \( r \mapsto E[Y_C \mid R = r, B] \).

We also connect our results in Theorem 4.8 to Abdulkadiroğlu et al. (2022)’s local deferred acceptance propensity score (defined in Theorem 1 in Abdulkadiroğlu et al. (2022)). The propensity score for assignment at test-score school s_0 is computed as follows. Consider a student with discrete qualifiers Q, preferences \( \succ \), and test scores \( R = [R_1, \ldots, R_T] \). Consider a slightly augmented version of the eligibility set (8)
\[
\tilde{E}_{s_0}(\succ, Q, c; \delta) = \left\{ \begin{array}{ll} \left[ r_{L_0}(s_0) - \delta, R_{L_0}(s_0) + \delta \right] \times \left( X_{t \neq t_0} \{ 0, R_t(s_0) + \delta \} \right), & \text{if } s_0 \succ L(Q, c) \\ \emptyset, & \text{otherwise,} \end{array} \right. \]
where all the boundaries are expanded by some bandwidth \( \delta \).\(^{37}\) The student’s propensity score is zero if \( R \not\in \tilde{E}_{s_0} \).

Suppose \( R \in \tilde{E}_{s_0} \). Consider the following probability
\[
\hat{\sigma}_{s_0}(\succ, Q, R; \delta) \equiv \hat{Pr}(\text{Failing to qualify for any test-score school } s \succ s_0)
\]
\[ = \left( \frac{1}{2} \right)^{m_{s_0}(\succ, Q, R; \delta)}, \quad (A.17) \]
where \( m_{s_0}(\succ, Q, R) \equiv \sum_{s \succeq s_0; t \neq t_0} 1[ R_t > r_{L_0}(s) - \delta ] \) is the number of test-score schools preferred to s_0, for which the student’s corresponding test score falls in the bandwidth. From a randomization perspective, \( m_{s_0} \) is roughly the number of test-score schools for which the student’s scores are close enough to the cutoffs to treat assignment as random. Indeed, the quantity \( \hat{\sigma} \) approximates qualification at \( s \succ s_0 \) as outcomes
\(^{37}\)Assume \( \delta \) is sufficiently small so that adding or subtracting \( \delta \) doesn’t collide with the boundary of \([0, 1]\).
of independent coin flips, if \( R_t \) is within \( \delta \) of the cutoff.\(^{38} \) Such an approximation is valid in the limit, as Abdulkadiroğlu et al. (2022) assume continuous densities of the running variable. The probability

\[
\hat{\lambda}_{s_0}(\succ, Q, R; \delta) \equiv \text{Pr}(\text{Failing to qualify for lottery schools } s \succ s_0)
\]
is readily computed via redrawing the lottery numbers in simulation, or via a large-market approximation with fixed \( c_s \), as Abdulkadiroğlu et al. (2022) do. Putting it together, the local propensity score for being assigned to the test-score school \( s_0 \) is thus

\[
\hat{\psi}_s(\succ, Q, R; \delta) \equiv \begin{cases} 
\frac{1}{2} \hat{\sigma}_{s_0}(\succ, Q, R; \delta) \cdot \hat{\lambda}_{s_0}(\succ, Q, R; \delta) & R_{t_0} < r_{t_0}(s_0) + \delta \text{ and } \tilde{E}_{s_0} \neq \emptyset \\
\hat{\sigma}_{s_0}(\succ, Q, R; \delta) \cdot \hat{\lambda}_{s_0}(\succ, Q, R; \delta) & R_{t_0} \geq r_{t_0}(s_0) + \delta \text{ and } \tilde{E}_{s_0} \neq \emptyset \\
0 & \text{otherwise.}
\end{cases}
\]

\(^{(A.18)} \)

\( \hat{\psi}_s \) is a product of \( \hat{\sigma} \) and \( \hat{\lambda} \), since due to the exogeneity of lottery numbers, failing to qualify for lottery schools and failing to qualify for test-score schools are independent events.

Similarly, for a lottery school \( s_0 \), if \( \tilde{E}_{s_0} \neq \emptyset \), the propensity score may be approximated via

\[
\hat{\sigma}_{s_0}(\succ, Q, R; \delta) \cdot \hat{\lambda}_{s_0}(\succ, Q, R; \delta) \cdot \hat{\pi}_{s_0}(\succ, Q, R; \delta),
\]

\(^{(A.19)} \)

where the third term \( \hat{\pi}_{s_0}(\succ, Q, R; \delta) \) approximates the conditional probability

\[
\text{Pr}(\text{Qualifies for } s_0 \mid \text{Failing to qualify for lottery schools } s \succ s_0),
\]

via, again, simulation or analytic approximation.

\(^{38} \)The upper bound condition \( R_{t_0} < r_{t_0}(s) + \delta \) is satisfied because \( R \in \tilde{E}_{s_0} \)
This section contains details for estimation and inference for mixed mechanisms (Section 5). Various lemmas that bound various terms are relegated to the subsections, and the section contains the main flow of the argument. Recall that we consider test-scores schools \( s_0, s_1 \) with tests \( t_0, t_1 \). We have the following treatment effect
\[
\tau = \tau_{s_0, s_1} = \mathbb{E} [Y(s_1) - Y(s_0) \mid s_1 \succ s_0, R \in \mathcal{E}_{s_0}(\succ, Q; c) \cap \mathcal{E}_{s_1}(\succ, Q; c)].
\]
Recall that \( r_t(s; c) \) is the interior \((0, 1)\) value of \( r_t(s; Q) \) that does not depend on \( Q \). For convenience on the test-score cutoff of school \( s_1 \), let \( \rho(c) = r_{t_1, s_1}(c) \).

Let us first decompose the conditioning event into restrictions on \( t_1 \) and restrictions not on \( t_1 \). Recall that the intersection
\[
\mathcal{E}_{s_0}(\succ, Q; c) \cap \mathcal{E}_{s_1}(\succ, Q; c)
\]
takes the form of (9), which is a Cartesian product of intervals corresponding to the following conditions on the vector of test scores \( R = [R_1, \ldots, R_T] \):
\[
\begin{align*}
R_{t_1} &= r_{t_1}(s_1; Q, \succ, c) \\
R_{t_1} &\leq R_{t_1}(s_0; Q, \succ, c) \\
R_{t_0} &\in [r_{t_0}(s_0; Q, \succ, c), R_{t_0}(s_0; Q, \succ, c)] \\
R_t &\leq R_t(s_0; Q, \succ, c), t \neq t_1, t_0
\end{align*}
\]
as well as the condition that \( s_0 \succ L(Q, c) \). If \( t_1 = t_0 \), then the \( R_{t_0} \) condition should be replaced with the following condition
\[
R_{t_1}(s_0; Q, \succ, c) = r_{t_1}(s_0; Q, \succ, c).
\]

Define the following indicator random variables (functions of \( R, \succ, Q \)) that correspond to the above restrictions, with the restrictions on \( t_1 \) relaxed with the bandwidth parameter \( h \):

- \( I_1^+(c, h) = \mathbb{1} (R_{t_1} \in [\rho(c), \rho(c) + h]) \)
- \( I_1(c, h) = \mathbb{1} (\rho(c) + h < R_{t_1}(s_1; Q, \succ, c)) \)
- \( I_{10}(c) = \mathbb{1} (\rho(c) \leq R_{t_1}(s_0; Q, \succ, c)) \)
- If \( t_1 \neq t_0 \), then \( I_0(c) = \mathbb{1} (R_{t_0} \in [r_{t_0}(s_0; Q, \succ, c), R_{t_0}(s_0; Q, \succ, c)]) \).

Otherwise \( I_0(c) = 1 \).

- For \( t \neq t_0, t_1 \), define \( I_t(c) = \mathbb{1} (0 < R_t \leq R_t(s_0; Q, \succ, c)) \)
- \( I(c) = \mathbb{1} (s_0 \succ L(Q, c), s_1 \succ s_0) \)
- Let the sample selection indicator be defined as
\[
J(c, h) = I(c) I_{10}(c) \cdot I_1(c) I_0(c) \cdot \prod_{t \neq t_0, t_1} I_t(c),
\]
such that, for fixed \((R, Q, \succ)\),
\[
s_1 \succ s_0 \text{ and } R \in \mathcal{E}_{s_0}(\succ, Q; c) \cap \mathcal{E}_{s_1}(\succ, Q; c) \iff \lim_{h \to 0} J(c, h) I_1^+(c, h) = 1.
\]

We only consider the asymptotic behavior of the estimator for the right-limit. The result for that of the left-limit is exactly analogous. A natural estimator of the right-limit,
\[
\lim_{h \to 0} \mathbb{E} [Y \mid J(c, h) = 1, I_1^+(c, h) = 1] = \mathbb{E} [Y(s_1) \mid s_1 \succ s_0, R \in \mathcal{E}_{s_0}(\succ, Q; c) \cap \mathcal{E}_{s_1}(\succ, Q; c)],
\]

37
is a locally linear estimator with uniform kernel and bandwidth $h_N$:

$$\hat{\beta}(h_N) = [\hat{\beta}_0, \hat{\beta}_1]' = \arg\min_{\beta_0, \beta_1} \sum_{i=1}^{N} W_i(C_N, h_N) [Y_i(C_N) - \beta_0 - \beta_1 (R_{it} - \rho(C_N))],$$

where, recall, that the outcome and weighting are defined as:

- $Y_i(C_N) = \frac{D_i^1(C_N)Y_i(s_{i1})}{\pi_i(C_N)}$, where, for a fixed $c$, $D_i^1(c)$ is the indicator for failing to qualify for lottery schools that $s_i$ prefers:

$$D_i^1(c) = \prod_{s,t: c \neq s, s \succ c} 1 \left( U_{it} < g_{s}^{-1}(Q_i,c) \right)$$

and $\pi_i(c)$ is the corresponding probability, integrating solely over the lottery numbers $U_i$:

$$\pi_i(c) = \Pr \left[ \forall s : t_s \neq \emptyset, s \succ c, U_{it} < g_{s}^{-1}(Q_i,c) \mid Q_i \succ c \right].$$

Note that $\pi_i(c) = 0$ implies that $D_i^1(c) = 0$, and we define $0/0 = 0$ in this case.

As a quick digression, we immediately have the following fact about $Y_i(c)$, indicating that it can be replaced with the inverse propensity weighting $\frac{D_i^1(c)Y_i(s_{i1})}{\pi_i(C_N)}$.

**Lemma B.1.** For a fixed $c$, suppose $W_i(c, h_N) = 1$. (This implies $\rho(c) < R_{it}(s_{i1}; \succ c, Q_i,c)$.) Moreover, suppose $h_N$ is sufficiently small such that $\rho(c) + h_N < R_{it}(s_{i1}; \succ c, Q_i,c)$. Then $D_i^1(c) = D_i^1(c)$ and

$$\pi_i(c) = \mathbb{E} \left[ D_i^1(c) \mid \succ c, Q_i, R_i \right]$$

for all $R_i \succ c, Q_i$ such that $W_i(c, h_N) = 1$.

**Proof.** Under the assumptions and fixed cutoff $c$, the treatment is $s_{i1}$ if and only if the student fails to obtain admission to any lottery school that she prefers, since $W_i = 1$ indicates that she qualifies for $s_{i1}$ and fails to qualify for any test-score school that she prefers to $s_{i1}$. Hence $D_i^1(c) = D_i^1(c)$ and the corresponding probability agree. $\square$

- $W_i(C_N, h_N) = J_i(C_N, h_N)I_{W_i}(C_N, h_N)$.

Let $x_i(C_N) = [1, R_{it} - \rho(C_N)]'$ collect the right-hand side variable in the weighted least-squares regression. Then the locally linear regression estimator is

$$\hat{\beta}_0(h_N) = e_1' \left( \sum_{i=1}^{N} W_i(C_N, h_N)x_i(C_N)Y_i(C_N) \right) \left( \sum_{i=1}^{N} W_i(C_N, h_N)x_i(C_N)x_i(C_N)' \right)^{-1} \left( \sum_{i=1}^{N} W_i(C_N, h_N)x_i(C_N)Y_i(C_N) \right)$$

There is a natural oracle estimator

$$\hat{\beta}_0(h_N) = e_1' \left( \sum_{i=1}^{N} W_i(c, h_N)x_i(c)x_i(c)' \right) \left( \sum_{i=1}^{N} W_i(c, h_N)x_i(c)x_i(c)' \right)^{-1} \left( \sum_{i=1}^{N} W_i(c, h_N)x_i(c)Y_i(c) \right)$$

whose asymptotic properties are well-understood. Our goal is to show that the difference between the two estimators is small:

$$\sqrt{nh_N} \left( \hat{\beta}_0(h_N) - \hat{\beta}_0(h_N) \right) = o_p(1).$$

First, let us recall the following assumption to avoid certain knife-edge populations.

**Assumption 5.1 (Population cutoffs are interior).** The distribution of student observables satisfies Assumption 4.6, where the population cutoffs $c$ satisfy:

1. School $s_{i1}$ is not undersubscribed: $\rho(c) > 0$.  

39 $e_1 = [1, 0]'$. 

38
(2) For each school $s$, either (i) there is a unique $q^*_s(c)$ under which the tie-breaker cutoff is in the open interval $(0, 1)$, $0 < r_{t,s}(c) \equiv g_s^{-1}(q^*_s(c), c_s) < 1$, or (ii) $c_s = 0$ and for all $q$, $r_{t,s}(c) = g_s^{-1}(q, c_s) = 0$; in this case, $q^*_s(c) = -\infty$.

(3) If $c_s = 0$, then $s$ is eventually undersubscribed: $\Pr(C_{s,N} = 0) \rightarrow 1$.

(4) If two schools $s_3, s_4$ uses the same test $t$, then their test score cutoffs are different, unless both are undersubscribed: $r_{t,s_3}(c) = r_{t,s_4}(c) \implies r_{t,s_3}(c) = r_{t,s_4}(c) = 0$.

Let us also state the following technical conditions

**Assumption B.2 (Bounded densities).** For some constant $0 < B < \infty$,

1. The density of $(R_i \mid \succ_i, Q_i, A_i)$ with respect to the Lebesgue measure is positive and bounded by $B$, uniformly over the conditioning variables.
2. The density of $U_i = [U_{it,s}, \ell_s \neq \emptyset]$ with respect to the Lebesgue measure is positive and bounded by $B$.

**Assumption B.3 (Moment bounds).**

1. Let $\epsilon^{(1)}_i = Y_i(s_1) - \mu_(r)$. For some $\varepsilon > 0$, the $(2 + \varepsilon)^{th}$ moment exists and is bounded uniformly:

\[ \mathbb{E}[(\epsilon^{(1)}_i)^{2+\varepsilon} \mid J_i(c) = 1, R_{dit} = r] < B_V(\varepsilon) < \infty. \]

Similarly, the same moment bounds hold for $Y_i(s_0)$. Note that this implies that the second moment is bounded uniformly by some $B_V = B_V(0)$.

2. The conditional variance $\text{Var}(\epsilon_i \mid J_i(c) = 1, R_{dit} = r)$ is right-continuous at $\rho(c)$ with right-limit $\sigma^2_\rho > 0$. Similarly, the conditional variance for $Y_i(s_0)$ is also continuous with left-limit $\sigma^2_{\rho} > 0$.

3. The conditional first moment is bounded uniformly: $\mathbb{E}[|Y_i(s_k)| \mid R_i, \succ_i, Q_i] < B_M < \infty$ for $k = 0, 1$.

**Assumption B.4 (Smoothness of mean).** The maps $\mu_+(r), \mu_-(r)$ are thrice continuously differentiable with bounded third derivative $\|\mu'''_+(r)\|_\infty, \|\mu'''_-(r)\|_\infty < B_D < \infty$.

**Assumption B.5 (Continuously differentiable density).** The density $f(r) = p(R_{dit} = r \mid J_i(c) = 1)$ is continuously differentiable at $\rho(c)$ and strictly positive.

We introduce the main theorem.

**Theorem B.6.** Under Assumptions 5.1, 4.6, and B.2 to B.5, assuming $N^{-1/2} = o(h_N)$ and $h_N = o(1)$, then the feasible estimator and the oracle estimator are equivalent in the first order

\[ \sqrt{N}h_N(\hat{\beta} - \bar{\beta}) = O_p \left( h^{1/2}_N + N^{-1/4} h^{-1/2}_N + N^{-1/2} h^{-1}_N \right) = o_p(1). \]

**Corollary B.7.** Under Theorem B.6, we immediately have that the discrepancy is $o_p(\sqrt{Nh_N^2})$ if $h_N = O(N^{-d})$ with $d \in [0.2, 0.25]$.

Let $\hat{\beta} = \hat{A}_{1N}^{-1} \hat{A}_{2N}$ and let $\bar{\beta} = \tilde{A}_{1N}^{-1} \tilde{A}_{2N}$ for matrices $\hat{A}_{kN}, \tilde{A}_{kN}$. The theorem follows from the following proposition.

**Proposition B.8.** Under Assumptions 5.1, 4.6, and B.2 to B.5, assuming $N^{-1/2} = o(h_N)$ and $h_N = o(1)$, then

1. The matrix

\[ \hat{A}_{1N} = \begin{bmatrix} O_p(1) & O_p(h_N) \\ O_p(h_N) & O_p(h^2_N) \end{bmatrix} \]
and, as a result,

\[ (\hat{A}_{1N} + b_N)^{-1} = \hat{A}_{1N}^{-1} + \begin{bmatrix} O_p(b_N) & O_p(b_N/h_N^2) \\ O_p(b_N/h_N^2) & O_p(b_N/h_N^2) \end{bmatrix}. \]

Similarly,

\[ \hat{A}_{2N} = \begin{bmatrix} O_p(1) \\ O_p(h_N) \end{bmatrix}. \]

(2) Let

\[ \tilde{\beta} = \left( \frac{1}{Nh_N} \sum_{i=1}^{N} J_i(c) I_{i}^+(C_N, h_N) x_i(C_N) x_i(C_N)^t \right)^{-1} \left( \frac{1}{Nh_N} \sum_{i=1}^{N} J_i(c) I_{i}^+(C_N, h_N) x_i(C_N) Y_i(c) \right) \]

\[ = \hat{A}_{1N}^{-1} \hat{A}_{2N}. \]

Then

\[ \hat{A}_{1N} = \tilde{A}_{1N} = \begin{bmatrix} O_p(N^{-1/2}/h_N) & O_p(N^{-1/2}) \\ O_p(N^{-1/2}) & O_p(N^{-1/2}h_N) \end{bmatrix} \]

and

\[ \hat{A}_{2N} = \tilde{A}_{2N} + \begin{bmatrix} N^{-1/2}/h_N \\ N^{-1/2} \end{bmatrix} = \begin{bmatrix} O_p(1) \\ O_p(h_N) \end{bmatrix}. \]

(3) Moreover, \( \sqrt{Nh_N}(\beta_0 - \tilde{\beta}_0) = O_p(\sqrt{h_N}) \).

(4) We may write the discrepancy as

\[ \hat{A}_{1N}^{-1} \sqrt{Nh_N} \tilde{A}_{2N} - \hat{A}_{1N}^{-1} \sqrt{Nh_N} \hat{A}_{2N} = \hat{A}_{1N}^{-1} \hat{B}_{2N} - \hat{A}_{1N}^{-1} \tilde{B}_{2N} \]

for some \( \hat{B}_{2N}, \tilde{B}_{2N} \) where (a) \( \hat{B}_{2N} = [O_p(1), O_p(h_N)]' \) and (b)

\[ \tilde{B}_{2N} = \tilde{B}_{2N} + \begin{bmatrix} O_p(h_N^{3/2} + N^{-1/4}h_N^{-1/2}) \\ O_p(h_N^{5/2} + N^{-1/4}h_N^{-1/2}) \end{bmatrix}. \]

Remark 3. We write \( J_i(c) \) instead of \( J_i(c, h_N) \) since it does not depend on \( h_N \) for sufficiently small \( h_N \)—see Corollary B.12. ■

Proof. [Proof of Theorem B.6 assuming Proposition B.8]

We multiply out, by parts (2) and (4):

\[ \hat{A}_{1N} \hat{B}_{2N} = \left( \hat{A}_{1N} + \begin{bmatrix} O_p(N^{-1/2}/h_N) & O_p(N^{-1/2}) \\ O_p(N^{-1/2}) & O_p(N^{-1/2}h_N) \end{bmatrix} \right)^{-1} \left( \hat{B}_{2N} + \begin{bmatrix} O_p(h_N^{3/2} + N^{-1/4}h_N^{-1/2}) \\ O_p(h_N^{5/2} + N^{-1/4}h_N^{-1/2}) \end{bmatrix} \right) \]

The first term is

\[ \hat{A}_{1N}^{-1} + \begin{bmatrix} O_p(N^{-1/2}/h_N) & O_p(N^{-1/2}/h_N^2) \\ O_p(N^{-1/2}/h_N^2) & O_p(N^{-1/2}/h_N^2) \end{bmatrix} \]

Multiplying out, we have that the RHS is

\[ \hat{A}_{1N}^{\frac{3}{2}} \hat{B}_{2N} + \begin{bmatrix} O_p \left( N^{-1/2}/h_N + h_N^{3/2} + N^{-1/4}h_N^{-1/2} \right) \\ O_p \left( h_N^{-1} \cdot \left( N^{-1/2}/h_N + h_N^{3/2} + N^{-1/4}h_N^{-1/2} \right) \right) \]

The total discrepancy between \( \hat{\beta} \) and \( \tilde{\beta} \), in the first entry, by (3), is then

\[ O_p \left( N^{-1/2}/h_N + h_N^{3/2} + N^{-1/4}h_N^{-1/2} + \sqrt{h_N} \right) = O_p \left( N^{-1/2}/h_N + h_N^{1/2} + N^{-1/4}h_N^{-1/2} \right). \]
**Proof of Proposition B.8.** We prove Proposition B.8 in the remainder of this section. The first part is a direct application of Lemma A.2 in Imbens and Kalyanaraman (2012), which is a routine approximation of the sum $\hat{A}_{1N}$ with its integral counterpart.

Proof. [Proof of Proposition B.8(1)] The claim follows directly from Lemma B.20, which is a restatement of Lemma A.2 in Imbens and Kalyanaraman (2012). The inversion part follows from $1/(a+b) = 1/a + O(b/a^2)$. □

Next, the proof of part (2) follows from bounds of the discrepancy between $\hat{A}_{1N}$ and $\tilde{A}_{1N}$, detailed in Lemma B.16.

Proof. [Proof of Proposition B.8(2)] Lemma B.16 directly shows that

$$\hat{A}_{1N} = \tilde{A}_{1N} + O_p(N^{-1/2}) \begin{bmatrix} 1/h_N & 1 \\ 1/h_N & 1 \\ \end{bmatrix}$$

when we expand

$$A_{1N} = \begin{bmatrix} S_{0N} & S_{1N} \\ S_{1N} & S_{2N} \end{bmatrix}$$

in the notation of Lemma B.16. The part about $\tilde{A}_{2N}$ follows similarly from Corollary B.17. □

Next, the proof of part (3) follows from bounds of the discrepancy between $\hat{A}_{kN}$ and $\tilde{A}_{kN}$, detailed in Lemmas B.14 and B.15.

Proof. [Proof of Proposition B.8(3)] Note that (1) and (2) implies that

$$\hat{A}_{1N} = \begin{bmatrix} O_p(1) & O_p(h_N) \\ O_p(h_N) & O_p(h_N^2) \end{bmatrix}.$$  

Lemma B.14 shows that

$$\hat{A}_{1N} = \tilde{A}_{1N} + O_p(N^{-1/2}) \begin{bmatrix} 1 \\ h_N \\ \end{bmatrix}.$$  

and Lemma B.15 shows that

$$\hat{A}_{2N} = \tilde{A}_{2N} + O_p(N^{-1/2}) \begin{bmatrix} 1 \\ h_N \\ \end{bmatrix}.$$  

The inverse is then

$$\hat{A}_{1N}^{-1} = \tilde{A}_{1N}^{-1} + O_p(N^{-1/2}) \begin{bmatrix} 1/h_N \\ 1/h_N^2 \\ \end{bmatrix}.$$  

Multiplying the terms out, we have that

$$\hat{A}_{1N}^{-1} \hat{A}_{2N} = \tilde{A}_{1N}^{-1} \tilde{A}_{2N} + \begin{bmatrix} N^{-1/2} \\ N^{-1/2}/h_N \end{bmatrix}.$$  

Scaling by $\sqrt{Nh_N}$ yields the bound $\sqrt{Nh_N}$ in (3). □

Lastly, we consider the fourth claim. To that end, we recall that

$$\mathbb{E}[Y_i(c) \mid J_i(c) = 1, R_{it_s} = r] = \mathbb{E}[Y_i(s_1) \mid J_i(c) = 1, R_{it_s} = r] \equiv \mu_+(r).$$

Let $\epsilon_i = Y_i(c) - \mu_+{R_{it_s}}$. Now, observe that $\mathbb{E}[\epsilon_i \mid R_{it_s}, J_i(c) = 1] = 0$. We first do a Taylor expansion of $\mu_+$. Assumption B.4 implies that

$$\mu_+(r) = \mu_+(\rho(c)) + \mu'_+(\rho(c))(r - \rho) + \frac{1}{2} \mu''_+(\rho(c))(r - \rho)^2 = \nu(r; c, \rho),$$

where $|\nu(r; c, \rho)| < B_D(r - \rho)^3 + B_\mu(c)(|r - \rho(c)| + |\rho(c) - \rho|)|\rho(c) - \rho|$ for some constant $B_\mu(c)$.  

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Proof. [Proof of Proposition B.8(4)] In the notation of Lemmas B.16 and B.18, we can write \( \tilde{A}_{2N} = \frac{1}{Nh_N} \sum_{i=1}^{N} J_i(c) I_{i1}^{\ast}(C_N, h_N) x_i(C_N) Y_i(c) \) as
\[
\begin{bmatrix}
\mu_+(c) S_{0N} + \mu_+(c) S_{1N} + \frac{\mu''(c)}{2} S_{2N} \\
\mu_+(c) S_{1N} + \mu_+(c) S_{2N} + \frac{\mu''(c)}{2} S_{3N}
\end{bmatrix} + \tilde{\nu}_N + \begin{bmatrix} T_{0N} \\ T_{1N} \end{bmatrix},
\]
where the argument \( \rho = \rho(C_N) \) for \( S_{kN} \). Let
\[
\tilde{B}_{2N} = \sqrt{Nh_N} \begin{bmatrix}
\frac{\mu''(c)}{2} S_{2N} \\
\frac{\mu''(c)}{2} S_{3N}
\end{bmatrix} + \tilde{\nu}_N + \begin{bmatrix} T_{0N} \\ T_{1N} \end{bmatrix}.
\]
Let \( \tilde{B}_{2N} \) be similarly defined. Note that
\[
\tilde{A}_{1N}^{\ast} \tilde{A}_{2N} = \begin{bmatrix}
\mu_+(c) \\
\mu_+(c)
\end{bmatrix} + \frac{1}{\sqrt{Nh_N}} \tilde{A}_{1N}^{\ast} \tilde{B}_{2N}
\]
and similarly
\[
\tilde{A}_{1N} \tilde{A}_{2N} = \begin{bmatrix}
\mu_+(c) \\
\mu_+(c)
\end{bmatrix} + \frac{1}{\sqrt{Nh_N}} \tilde{A}_{1N} \tilde{B}_{2N}
\]
Thus it remains to show that
\[
\tilde{B}_{2N} = \tilde{B}_{2N} + \begin{bmatrix} O_p(h_N^{3/2} + N^{-1/4} h_N^{-1/2}) \\ O_p(h_N^{3/2} + N^{-1/4} h_N^{-1/2}) \end{bmatrix}.
\]
The above claim follows immediately from the bounds in Lemmas B.16, B.18, and B.19. \( \square \)

Central limit theorem and variance estimation. Under the Taylor expansion (Assumption B.4) of \( \mu_+(r) \), we have that, so long as \( h_N = o(N^{-1/5}) \),
\[
\sqrt{Nh_N}(\hat{\beta}_0 - \mu_+(c)) = \frac{1}{\sqrt{Nh_N}} \sum_{i=1}^{N} \frac{\nu_2 - \nu_1 R_{i1} - \nu_1 \rho(c)}{h_N} f(\rho(c)) \Pr(J_i(c) = 1) \cdot W_i(c, h_N) \cdot (Y_i(c) - \mu_+(R_{i1})) + o_p(1)
\]
via a standard argument. See, for instance, Imbens and Lemieux (2008); Hahn et al. (2001); Imbens and Kalyanaraman (2012).

Theorem B.9. When \( h_N = o(N^{-1/5}) \), under Assumptions B.3 to B.5, we have the following central limit theorem:
\[
\hat{\sigma}_N^{-1}(\hat{\beta}_0 - \mu_+(c)) \overset{d}{\rightarrow} \mathcal{N}(0, 1)
\]
where the variance estimate is
\[
\hat{\sigma}_N^2 = \frac{4Nh_N}{N_+} \left( \frac{1}{N_+} \sum_{i=1}^{N} W_i(C_N, h_N) Y_i(C_N)^2 - \hat{\beta}_0^2 \right)
\]
\( N_+ = \sum_{i=1}^{N} W_i(C_N, h_N) \).

Proof. The central limit theorem follows from Lemma B.22, which shows normality of \( Z_N \) under Lyapunov conditions, and Lemma B.24, which shows consistency of \( \hat{\sigma}_N^2 \). \( \square \)

Guide to the lemmas. We conclude the main text of this appendix section with a guide to the lemmas that are appended in the rest of the section (Appendices B.1 to B.5). The key to the bounds is placing ourselves in an event that is well-behaved, in the sense that the ordering of the sample cutoffs \( C_N \) agrees with its population counterpart. This is dealt with in Appendix B.1. Under such an event, all but \( \sqrt{N} \) of students’ qualification statuses in sample disagree with those in population, yielding bounds related to \( J_i \) (Appendix B.2) and \( J_i Y_i(C_N) \) (Appendix B.3). Having dealt with \( J_i(C_N, h_N) \neq J_i(c) \), we can bound the
discrepancy due to $\rho(C_N) \neq \rho(c)$, and those are in Corollary B.17 and Lemma B.18 in Appendix B.4. Lastly, Appendix B.5 contains lemmas that are useful for the CLT and variance estimation parts of the argument.

B.1. Placing ourselves on well-behaved events.

Lemma B.10. Let $0 \leq M_N \to \infty$ diverge. Let $A_N = A_N(M_N, h_N)$ be the following event:

1. (c and $C_N$ agree on all $q^*_s$) For any school $s$ and any $q \in \{0, \ldots, \mathcal{Q}_s\}$, $g^{-1}_s(q, C_N) \in (0, 1)$ if and only if $g^{-1}_s(q, c_s) \in (0, 1)$. If $g^{-1}_s(q, C_N) \notin (0, 1)$, then $g^{-1}_s(q, C_N) = g^{-1}_s(q, c_s)$.

2. The cutoffs converge for every $s, q$:

$$\max_{s} \max_{q} |g^{-1}_s(q, c_s) - g^{-1}_s(q, C_N)| \leq M_N N^{-1/2}$$

3. (c and $C_N$ agree on the ordering of $r_{t,s}$) For any schools $s_1, s_2$ which uses the same test $t$, $r_{t,s_1}(C_N)$ and $r_{t,s_2}(C_N)$ are exactly ordered as $r_{t,s_1}(c)$ and $r_{t,s_2}(c)$.

4. For all $\triangleright, Q$, $\rho(C_N) + h_N < R_{t_1}(s_1; \triangleright_1, Q, C_N)$ if and only if $\rho(c) < R_{t_1}(s_1; \triangleright_1, Q, c)$.

Under Assumptions 5.1 and 4.6 and $h_N \to 0$, $A_N$ occurs almost surely eventually:

$$\lim_{N \to \infty} \Pr(A_N) = 1.$$

Proof. Since intersections of eventually almost sure events are eventually almost sure, it suffices to show that the following types of events individually occur with probability tending to one:

1. For any fixed $s$ and any $q \in \{0, \ldots, \mathcal{Q}_s\}$, $g^{-1}_s(q, C_{s,N}) \in (0, 1)$ if and only if $g^{-1}_s(q, c_s) \in (0, 1)$. If $g^{-1}_s(q, C_{s,N}) \notin (0, 1)$, then $g^{-1}_s(q, C_{s,N}) = g^{-1}_s(q, c_s)$.

   • Suppose $q^*_s(c)$ in (1) in Assumption 5.1 is not $-\infty$. Then for $q = q^*_s(c)$, $g^{-1}_s(q, c_s) \in (0, 1)$. By the linearity (and hence continuity) of $r \mapsto g_s(q, r)$, there exists some $\epsilon > 0$ such that $g^{-1}_s(q, c') \in (0, 1)$ for $c' \in [c_s - \epsilon, c_s + \epsilon]$.

   Note that the event $C_{N,s} \in [c_s - \epsilon, c_s + \epsilon]$ implies that (a) $g^{-1}_s(q, C_{s,N}) \in (0, 1)$, (b) $g^{-1}_s(q', C_{s,N}) = 0$ for $q' > q^*$, and (c) $g^{-1}_s(q', C_{s,N}) = 1$ for $q' < q^*$. These agree with $g^{-1}_s(q, c_s)$. The event happens eventually almost surely since

$$\Pr(C_{s,N} \in [c_s - \epsilon, c_s + \epsilon]) \to 1$$

by Assumption 4.6.

   • On the other hand, suppose $q^*_s(c) = -\infty$ and $s$ is undersubscribed. Assumption 5.1 (2) implies that $C_{s,N} = 0$ eventually almost surely, meaning that $g^{-1}_s(q, C_{s,N}) = g^{-1}_s(q, c_s)$ for every $q$ with probability tending to 1.

2. For fixed $s, q$, $|g^{-1}_s(q, c_s) - g^{-1}_s(q, C_{s,N})| \leq M_N N^{-1/2}$

   • If $q \neq q^*_s$, with probability tending to 1 $|g^{-1}_s(q, c_s) - g^{-1}_s(q, C_{s,N})| = 0$.

   • If $q = q^*_s$, then since $\max_{s} |c_s - C_{s,N}| = O_p(N^{-1/2})$ and $g_s(q, \cdot)$ is affine, the preimage is also $O_p(N^{-1/2})$ (uniformly over $s$).

3. For fixed schools $s_1, s_2$ which uses the same test $t$, $r_{t,s_1}(C_N)$ and $r_{t,s_2}(C_N)$ are exactly ordered as $r_{t,s_1}(c)$ and $r_{t,s_2}(c)$.

   • Suppose $r_{t,s_1}(c) > r_{t,s_2}(c)$. Note that for any $\epsilon > 0$, $\Pr[r_{t,s_1}(C_N) > r_{t,s_1}(c) - \epsilon] \to 1$ by Assumption 4.6. Similarly, $\Pr[r_{t,s_2}(C_N) < r_{t,s_2}(c) + \epsilon] \to 1$. Therefore, we may take $\epsilon = \lim_{N \to \infty} r_{t,s_1}(c) - r_{t,s_2}(c)$.

   • Suppose $r_{t,s_1}(c) = r_{t,s_2}(c)$. Then by (3) in Assumption 5.1, both schools are undersubscribed. In that case, (2) in Assumption 5.1 implies that $r_{t,s_1}(C_N) = r_{t,s_2}(C_N)$ eventually almost surely.

4. For a fixed $\triangleright, Q$ and $h_N \to 0$, $\rho(C_N) + h_N < R_{t_1}(s_1; \triangleright, Q, C_N)$ if and only if $\rho(c) < R_{t_1}(s_1; \triangleright, Q, c)$. 43
• By Assumption 5.1, \( \rho(c) \in (0, 1) \), and hence \( \rho(c) \neq R_{\leq}(s_1; >, Q, c) \) for any \( >, Q \). The event in (2) implies
\[
|\rho(C_N) - \rho(c)| < M_N N^{-1/2}
\]
and \( |R_{\leq}(s_1; >, Q, C_N) - R_{\leq}(s_1; >, Q, c)| < M_N N^{-1/2} \). Under this event, since \( h_N \to 0 \), we have \( \rho(C_N) + h_N < R_{\leq}(s_1; >, Q, C_N) \) for all sufficiently large \( N \). Hence if (2) occurs almost surely eventually, then (4) must also.

\[\square\]

**Remark 4.** We work with nonstochastic sequences of the bandwidth parameter \( h_N \). If the bandwidth parameter is a stochastic \( H_N \), then we can modify by appending to \( A_N \) the event \( H_N < M_N h_N \) for some nonstochastic sequence \( h_N \). If \( H_N = O_p(h_N) \), then \( \Pr(H_N < M_N h_N) \to 1 \); as a result, our subsequent conclusions are not affected.

**B.2. Bounding discrepancy in \( J_i \).** By studying the implications of the event \( A_N \)—all score cutoffs are induced by \( C_N \) agrees with that induced by \( c \) and all almost-sure-qualification statuses also agree—we immediately have the following result, which, roughly speaking, implies that the event \( J_i(C_N) \neq J_i(c) \) is a subset of an event where \( R_i \) belongs to a set of Lebesgue measure at most \( M_N N^{-1/2} \).

**Lemma B.11.** On the event \( A_N \),

1. For all \( i, I_i(c) = I_i(C_N) \).
2. For all \( i, I_{1i}(c, h) = I_{1i}(C_N, h_N) \).
3. For all \( i, I_{0i}(c) = I_{0i}(C_N) \).
4. For \( t \neq t_0, t_1, I_{1i}(c) \neq I_{1i}(C_N) \) implies that (a) \( R_{i}(s_0; Q_i, >i, C_N) \in (0, 1), (b) \)
\[
|\rho(s_0; Q_i, >i, C_N) - \rho(s_0; Q_i, >i, c)| \leq M_N N^{-1/2},
\]
and (c) \( R_{i1} \) is between \( R_{i1}(s_0; Q_i, >i, C_N) \) and \( R_{i1}(s_0; Q_i, >i, c) \).
5. If \( t_1 = t_0 \), then \( I_{0i}(C_N) = I_{0i}(c) \).
6. For \( t_1 \neq t_0, I_{0i}(c) \neq I_{0i}(C_N) \) implies that (a) \( R_{i0}(s_0; Q_i, >i, C_N) \in (0, 1), (b) \)
\[
|\rho(s_0; Q_i, >i, C_N) - \rho(s_0; Q_i, >i, c)| \leq M_N N^{-1/2},
\]
(c) \( |r_{i0}(s_0; Q_i, >i, C_N) - r_{i0}(s_0; Q_i, >i, c)| \leq M_N N^{-1/2}, \) and (d) either \( R_{i0} \) is between \( R_{i0}(s_0; Q_i, >i, C_N) \) and \( R_{i0}(s_0; Q_i, >i, c) \), or \( R_{i0} \) is between \( r_{i0}(s_0; Q_i, >i, c) \) and \( r_{i0}(s_0; Q_i, >i, c) \).
7. Under Assumption 5.1, for all \( i, \pi_i(c) = 0 \) if and only if \( \pi_i(C_N) = 0 \). Moreover, if \( \pi_i(c) > 0 \), then
\[
|\pi_i(C_N) - \pi_i(c)| \leq LBM_N N^{-1/2}.
\]

Proof. Every claim is immediate given the definition of \( A_N \) in Lemma B.10.

\[\square\]

**Corollary B.12.** On the event \( A_N \), the disagreement \( J_i(C_N, h_N) \neq J_i(c, h_N) \) implies that \( (R_{i1} : t \neq t_1) \in K(>i, Q_i; c), \) where \( \mu_{2^r - 1}(K(>i, Q_i; c)) \leq TM_N N^{-1/2} \). Moreover, under Assumption 5.1, for all sufficiently small \( h_N \), \( J_i(c, h_N) = J_i(c) \) does not depend on \( h_N \).

Proof. \( J_i(C_N, h_N) \neq J_i(c, h_N) \) implies that at least one of \( I_i, I_{1i}, I_{0i} \) have a disagreement over \( c \) and \( C_N \). On \( A_N \), disagreements of \( I_{0i} \) imply that \( R_{i0} \) is contained in a region of measure at most \( M_N N^{-1/2} \). If \( t_1 = t_0 \), then the union of disagreements over \( I_{0i} \) is a region of size at most \( (T - 1)M_N N^{-1/2} \), and \( I_i, I_{0i} \) has no disagreement. If \( t_1 \neq t_0 \), disagreements of \( I_{0i} \) imply that \( R_{i0} \) is contained in a region of measure at most...
2M_NN^{-1/2}, and hence the union of disagreements is a region of size at most \( TM_NN^{-1/2} \). Neither case implies anything about \( R_{t1}\).

The only part of \( J_i \) that depends on \( h_N \) is \( I_i(C_N, h_N) \), which does not depend on \( h_N \) when \( h_N \) is sufficiently small due to Assumption 5.1.

**Corollary B.13.** Suppose \( M_N = o(N^{1/2}) \). On the event \( A_N \), there exists some \( \eta > 0 \), independently of \( M_N \), such that for all sufficiently large \( N \), if \( \pi_i(C_N) > 0 \) then \( \pi_i(C_N) \geq \eta \). Equivalently, \( 1/\pi_i(C_N) < 1/\eta \) whenever defined.

**Proof.** Let \( \eta = \min \{ v = \pi_i(c) : v > 0 \} / 2 > 0 \). Under \( A_N \), \( \pi_i(c) = 0 \) if and only if \( \pi_i(C_N) = 0 \) and \( \pi_i(C_N) \) is uniformly \( o(1) \) away from \( \pi_i(c) \). Hence for sufficiently large \( N \), the discrepancy between \( \pi_i(c) \) and \( \pi_i(C_N) \) is bounded above by \( \eta \), thereby \( \pi_i(C_N) > \eta \) as long as \( \pi_i(C_N) > 0 \).

**Lemma B.14.** Let \( \Gamma_i \geq 0 \) be some random variable at the student level where \( \mathbb{E}[\Gamma_i \mid R_i, r_i, Q_i, Z_i] < B_M < \infty \) almost surely. Under Assumptions 5.1, 4.6, and B.2, assuming \( N^{-1/2} = o(h_N) \), the discrepancy of the sample selection is of the following stochastic order:

\[
F_N \equiv \frac{1}{\sqrt{Nh_N}} \sum_{i=1}^{N} |J_i(C_N, h_N) - J_i(c, h_N)|I_i^+(C_N, h_N)\Gamma_i = O_p(1).
\]

**Proof.** On the event \( A_N \), \( |\rho(c) - \rho(C_N)| \leq M_NN^{-1/2} \). Then, on \( A_N \),

\[
\sqrt{Nh_N}F_N = \sum_{i=1}^{N} |J_i(C_N) - J_i(c)|I_i^+(C_N, h_N)\Gamma_i \\
\leq \sum_{i=1}^{N} \mathbb{I} (|R_{i1} : t \neq t_1| \in K(\gamma_i, c)) \mathbb{I} (R_{i1} \in [\rho(c) - M_NN^{-1/2}, \rho(c) + M_NN^{-1/2} + h_N])\Gamma_i \\
\equiv \sqrt{Nh_N}G_N(M_N).
\]

Hence, under Lemma B.10, for any sequence \( M_N \to \infty \), since the corresponding \( \mathbb{P}(A_N) \to 1 \),

\[
F_N = F_NA_N + o_p(1) \leq G_N(M_N)A_N + o_p(1) \leq G_N(M_N) + o_p(1).
\]

Since \( G_N \geq 0 \) almost surely, by Markov’s inequality, Assumption B.2, and \( N^{-1/2} = o(h_N) \),

\[
G_N = O_p(\mathbb{E}[G_N]) = \frac{1}{\sqrt{Nh_N}} \cdot O_p \left( N \cdot (TM_NN^{-1/2}) \cdot (2M_NN^{-1/2} + h_N) \cdot B \right) \leq M_N^2O_p(1).
\]

Note that

\[
\mathbb{E}[G_N] = \frac{1}{\sqrt{Nh_N}} \cdot \mathbb{E} \left[ R_{i1} \in \tilde{K}(\gamma_i, c) \right] \mathbb{E}[\Gamma_i \mid R_{i1} \in \tilde{K}(\gamma_i, c)] \\
\leq \frac{\sqrt{N}}{h_N} \cdot (TM_NN^{-1/2}) \cdot (2M_NN^{-1/2} + h_N) \cdot B_M \\
= O(M_N^2)
\]

Therefore, for any \( M_N \to \infty \), no matter how slowly, \( F_N = O_p(M_N^2) \). This implies that \( F_N = O_p(1) \).

**B.3. Bounding discrepancy in terms involving \( Y_i(C_N) \).**

**Lemma B.15.** Fix \( M_N \to \infty \). Suppose that \( \mathbb{E}[|Y_i(s_1) \mid R_i, r_i, Q_i, Z_i] < B_M < \infty \) almost surely. Then the difference

\[
\left| \sum_{i=1}^{N} J_i(C_N, h_N)I_i^+(C_N, h_N)Y_i(C_N) - \sum_{i=1}^{N} J_i(C_N, h_N)I_i^+(C_N, h_N)Y_i(C_N) \right| \leq \Delta_1N + \Delta_2N + \Delta_3N
\]

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where
\[
\begin{align*}
\Delta_{1N} &= \sum_i I_i^+(C_N, h_N)J_i(c, h_N)\left\{ \frac{D_i^1(C_N)}{\pi_i(C_N)} - \frac{D_i^1(c)}{\pi_i(c)} \right\} Y_i(s_1) \\
\Delta_{2N} &= \sum_i I_i^+(C_N, h_N)\frac{D_i^1(c)}{\pi_i(c)}|J_i(C_N, h_N) - J_i(c, h_N)|Y_i(s_1) \\
\Delta_{3N} &= \sum_i I_i^+(C_N, h_N)|J_i(C_N, h_N) - J_i(c, h_N)|\left\{ \frac{D_i^1(C_N)}{\pi_i(C_N)} - \frac{D_i^1(c)}{\pi_i(c)} \right\} Y_i(s_1).
\end{align*}
\]

Moreover, under Assumptions 5.1, 4.6, and B.2, for all sufficiently large \(N\), \(\frac{\Delta_{ijN}}{\sqrt{Nh_N}} = O_p(1)\). As a result,
\[
\left| \sum_{i=1}^N J_i(C_N, h_N)I_i^+(C_N, h_N)Y_i(C_N) - \sum_{i=1}^N J_i(C_N, h_N)I_i^+(C_N, h_N)Y_i(C_N) \right| = O_p\left(N^{1/2}h_N\right).
\]

**Proof.** The part before “moreover” follows from adding and subtracting and triangle inequality.

To prove the claim after “moreover,” first, note that by Corollary B.13, for all sufficiently large \(N\), the inverse propensity weight \(1/\pi_i < 1/n\). Immediately, then, \(\Delta_{2N}, \Delta_{3N}\) are bounded above by
\[
\sum_i I_i^+(C_N, h_N)|J_i(C_N, h_N) - J_i(c, h_N)| \cdot Y_i(s_1) = O_p(\sqrt{Nh_N})
\]

via Lemma B.14.

By the same argument where we bound \(1/\pi_i\),
\[
\Delta_{1N} = \sum_{i=1}^N I_i^+(C_N, h_N)J_i(c, h_N)\left\{ \frac{1}{\pi_i(C_N)} - \frac{1}{\pi_i(c)} \right\} D_i^1(c)Y_i(s_1) + O_p(\sqrt{Nh_N}).
\]

By Lemma B.11,
\[
\left\{ \frac{1}{\pi_i(C_N)} - \frac{1}{\pi_i(c)} \right\} < M_N N^{-1/2}.
\]

On \(A_N\), since
\[
\sum_i I_i^+(C_N, h_N)J_i(c, h_N)D_i^1(c)Y_i(s_1) \leq \sum_i \mathbb{1}(R_i \in \tilde{K}(r_i, Q_i; c))Y_i(s_1)
\]

where \(\sup_{r_i, Q_i} \mu(\tilde{K}(r_i, Q_i; c)) = O(h_N)\). We have again by Markov’s inequality and the bound on the conditional first moment of \(Y_i(s_1)\),
\[
\sum_i I_i^+(C_N, h_N)J_i(c, h_N)D_i^1(c)Y_i(s_1) = O_p(Nh_N).
\]

Hence \(\Delta_{1N} = O_p(\sqrt{Nh_N})\). \(\square\)

**B.4. Bounding terms involving \(x_i\) and \(I_{1i}^+\).**

**Lemma B.16.** Suppose \(N^{-1/2} = o(h_N)\). Consider
\[
S_{k,N}(\rho) = \frac{1}{Nh_N} \sum_{i=1}^N J_i(c)(R_{it} - \rho)^k \mathbb{1}(R_{it} \in [\rho, \rho + h_N]).
\]

Then, under Assumptions 5.1, 4.6, and B.2,
\[
|S_{k,N}(\rho(C_N)) - S_{k,N}(\rho(c))| = O_p\left(h_N^{k-1}N^{-1/2}\right) = o_p(h_N^k).
\]

**Proof.** Suppose \(N\) is sufficiently large such that \(M_N N^{-1/2} < h_N\). On the event \(A_N\),
\[
|S_{k,N}(\rho(C_N)) - S_{k,N}(\rho(c))| \leq \sup \left\{ |S_{k,N}(\rho) - S_{k,N}(\rho(c))| : \rho \in [\rho(c) - M_N N^{-1/2}, \rho(c) + M_N N^{-1/2}] \right\}.
\]

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For a fixed $\rho \in [\rho(c) - MN^{-1/2}, \rho(c) + MN^{-1/2}]$, the difference
\[
|S_{k,N}(\rho(CN)) - S_{k,N}(\rho(c))| \leq \frac{1}{Nh_N} \sum_{i=1}^{N} J_i(c) \mathbb{1}(R_{it_1} \in \rho \omega(c) + h_N)\Delta_{ik} \\
+ \frac{1}{Nh_N} \sum_{i=1}^{N} J_i(c) \mathbb{1}(R_{it_1} \in \Delta_2) (R_{it_1} - \rho)^k \\
+ \frac{1}{Nh_N} \sum_{i=1}^{N} J_i(c) \mathbb{1}(R_{it_1} \in \Delta_2) \Delta_{ik}
\]
where $\Delta_{ik} = |(R_{it_1} - \rho)^k - (R_{it_1} - \rho(c))^k|$ and $\Delta_2 = [\rho, \rho(c)] \cup [\rho + h_N, \rho(c) + h_N]$ if $\rho < \rho(c)$ and $[\rho(c), \rho] \cup [\rho(c) + h_N, \rho + h_N]$ otherwise.

Note that $\Delta_{ik} = 0$ if $k = 0$. If $k > 0$ then
\[
\Delta_{ik} < |\rho - \rho(c)| k (2MN^{-1/2} + h_N)^{k-1} < B_k M_N N^{-1/2} h_N^{k-1}
\]
for some constants $B_k$, by the difference of two $k^{th}$ powers formula. Let $B_0 = 0$, then the first term is bounded by
\[
B_k M_N N^{-1/2} h_N^{k-1} \cdot \frac{1}{Nh_N} \sum_{i=1}^{N} J_i(c) \mathbb{1}(R_{it_1} \in \rho \omega(c) + h_N).
\]
The second term is bounded by
\[
B_k h_N^k \frac{1}{Nh_N} \sum_{i=1}^{N} J_i(c) \mathbb{1}(R_{it_1} \in \Delta_2)
\]
for some constants $B_k$ where $B_0 = 1$. The third term is bounded by
\[
B_k M_N N^{-1/2} h_N^{k-1} \frac{1}{Nh_N} \sum_{i=1}^{N} J_i(c) \mathbb{1}(R_{it_1} \in \Delta_2)
\]
These bounds hold regardless of $\rho$, and hence taking the supremum over $\rho$ yields that, for any $M_N \to \infty$,
\[
|S_{k,N}(\rho(CN)) - S_{k,N}(\rho(c))| = O_p \left( B_k M_N N^{-1/2} h_N^{k-1} + h_N^{k-2} M_N N^{-1/2} + B_k M_N N^{-1} h_N^{k-2} \right)
\]
\[
= O_p \left( M_N h_N^{k-1} N^{-1/2} \right).
\]
Hence $|S_{k,N}(\rho(CN)) - S_{k,N}(\rho(c))| = O_p \left( h_N^{k-1} N^{-1/2} \right)$.

\[\square\]

**Corollary B.17.** The conclusion of Lemma B.16 continues to hold if each term of $S_{k,N}(\rho)$ is multiplied with some independent $\Gamma_i$ where $\mathbb{E}[|\Gamma_i| \mid R_{it_1}, \epsilon_i, Q_t, Z_t] < B_M < \infty$ almost surely.

**Proof.** The bounds continue to hold where the right-hand side involves terms like
\[
\frac{1}{Nh_N} \sum_{i=1}^{N} J_i(c) \mathbb{1}(R_{it_1} \in \Delta_2) |\Gamma_i|.
\]
The last step of the proof to Lemma B.16 uses Markov’s inequality, which incurs a constant of $B_M$ since terms like
\[
\mathbb{E}[|\Gamma_i| \mid J_i(c) \mathbb{1}(R_{it_1} \in \Delta_2) = 1] \leq B_M.
\]
\[\square\]

**Lemma B.18.** Suppose $N^{-1/2} = o(h_N)$. Suppose $\epsilon_i$ are independent over $i$ with $\mathbb{E}[\epsilon_i \mid J_i(c), R_{it_1}] = 0$ and $\text{Var}[\epsilon_i \mid J_i(c), R_{it_1}] < B_V < \infty$ almost surely. Consider
\[
T_{k,N}(\rho) \equiv \frac{1}{Nh_N} \sum_{i=1}^{N} J_i(c, h_N)(R_{it_1} - \rho)^k \mathbb{1}(R_{it_1} \in [\rho, \rho + h_N]) \epsilon_i.
\]

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Then, under Assumptions 5.1, 4.6, and B.2, for \( k = 0, 1 \),
\[
|T_{k,N}(\rho(C_N)) - T_{k,N}(\rho(c))| = O_p \left( N^{-1/4} \cdot N^{-1/2} \cdot h_N^{k-1} \right).
\]

**Proof.** Suppose \( N \) is sufficiently large such that \( M_NN^{-1/2} < h_N \). On the event \( A_N \),
\[
|T_{k,N}(\rho(C_N)) - T_{k,N}(\rho(c))| \leq \sup \left\{ |T_{k,N}(\rho) - T_{k,N}(\rho(c))| : \rho \in [\rho(c) - M_NN^{-1/2}, \rho(c) + M_NN^{-1/2}] \right\}.
\]
For a fixed \( \rho \in [\rho(c) - M_NN^{-1/2}, \rho(c) + M_NN^{-1/2}] \), the difference
\[
|T_{k,N}(\rho(C_N)) - T_{k,N}(\rho(c))| \leq \frac{1}{Nh_N} \left| \sum_{i=1}^{N} J_i(c) \mathbb{1}(R_{it_1} \in [\rho(c), \rho(c) + h_N]) \right| \Delta_{ik} \epsilon_i
\]
\[
+ \frac{1}{Nh_N} \left| \sum_{i=1}^{N} J_i(c) \mathbb{1}(R_{it_1} \in \Delta_2)(R_{it_1} - \rho)^k \epsilon_i \right|
\]
\[
+ \frac{1}{Nh_N} \left| \sum_{i=1}^{N} J_i(c) \mathbb{1}(R_{it_1} \in \Delta_2) \right| \Delta_{ik} \epsilon_i
\]
where \( \Delta_{ik} = |(R_{it_1} - \rho)^k - (R_{it_1} - \rho(c))^k| \) and \( \Delta_2 = [\rho, \rho(c)] \cup [\rho + h_N, \rho(c) + h_N] \) if \( \rho < \rho(c) \) and \( \rho(c), \rho] \cup [\rho(c) + h_N, \rho + h_N] \) otherwise. Note that \( \Delta_{ik} = 0 \) if \( k = 0 \) and \( \Delta_{ik} = |\rho - \rho(c)| \leq M_NN^{-1/2} \) if \( k = 1 \).

We first show that
\[
\left| \sum_{i=1}^{N} \mathbb{1}(R_{it_1} \in \Delta_2) \eta_i \right| = O_p(N^{1/4}) \quad \eta_i \equiv J_i(c, h_N) \epsilon_i
\]
Note that the event
\[
\sum_{i=1}^{N} \mathbb{1}(R_{it_1} \text{ is between } \rho(c) \text{ and } \rho \text{, for some } \rho \in [\rho(c) - M_NN^{-1/2}, \rho(c) + M_NN^{-1/2}])
\]
occurs with probability tending to 1, and so does the event
\[
\sum_{i=1}^{N} \mathbb{1}(R_{it_1} \text{ is between } \rho(c) + h_N \text{ and } \rho + h_N \text{, for some } \rho \in [\rho(c) - M_NN^{-1/2}, \rho(c) + M_NN^{-1/2}]) < K_N.
\]
On both events, the sum
\[
\left| \sum_{i=1}^{N} \mathbb{1}(R_{it_1} \in \Delta_2) \eta_i \right| \leq \sup_{U_1 < K_N} \left| \sum_{1 \leq u_1 \leq U_1} \eta_1(u_1) \right| + \sup_{U_2 < K_N} \left| \sum_{1 \leq u_2 \leq U_2} \eta_2(u_2) \right|
\]
where we label the observation such that \( \eta_1(u) \) is the \( u^{th} \) \( \eta_i \) with \( R_{it_1} \) closest to \( \rho(c) \) and \( \eta_2(u) \) is the \( u^{th} \) \( \eta_i \) with \( R_{it_1} \) closest to \( \rho(c) + h_N \). Observe that \( Z_{1U} \equiv \sum_{1 \leq u \leq U} \eta_1(u) \) is a martingale adapted to the filtration \( \mathcal{F}_U = \sigma \{ (R_{it_1})_{i=1}^N, \eta_1(u) : u \leq U \} \). By Kolmogorov’s maximal inequality,
\[
\Pr \left( \sup_{U \leq K_N} |Z_{1U}| \geq t \right) \leq \frac{\mathbb{E}[Z_{1K_N}^2]}{t^2} \leq \frac{K_N B_{U}}{t^2}.
\]
Similarly, we obtain the same bound for the terms involving \( \epsilon_2(u) \). Hence
\[
\sup_{U_1 < K_N} \left| \sum_{1 \leq u_1 \leq U_1} \eta_1(u_1) \right| + \sup_{U_2 < K_N} \left| \sum_{1 \leq u_2 \leq U_2} \eta_2(u_2) \right| = O_p(\sqrt{K_N}) = M_N O_p(N^{1/4}).
\]
Therefore, since for any arbitrarily slowly diverging \( M_N \), the three events that we place ourselves on occurs with probability tending to 1, \( \sum_{i=1}^{N} \mathbb{1}(R_{it_1} \in \Delta_2) \eta_i = O_p(N^{1/4}) \).
Now, we bound the three terms on the RHS. The second term is bounded above by
\[ \frac{h_n^{k-1}}{N} \sum_{i=1}^{N} J_i(c, h_N) I(R_{it_1} \in \Delta_2) \eta_i = O_p(h_n^{k-1} N^{-3/4}). \]

The third term is also \( O_p(h_n^{k-1} N^{-3/4}) \) since \( \Delta_{1ik} < M_N N^{-1/2} = O(1) \) uniformly over \( i \). The first term is zero if \( k = 0 \). If \( k = 1 \), the first term is bounded above by
\[ M_N N^{-1/2} h_n^{k-1} \sum_{i} \mathbb{1}(R_{it_1} \in [\rho(c), \rho(c) + h_N]) \eta_i. \]

Chebyshev’s inequality suggests that
\[ \sum_{i} \mathbb{1}(R_{it_1} \in [\rho(c), \rho(c) + h_N]) \eta_i = O_p \left( \sqrt{N} \sqrt{\text{Var}(\mathbb{1}(R_{it_1} \in [\rho(c), \rho(c) + h_N]) \eta_i)} \right) = O_p(\sqrt{Nh_N}), \]
thus bounding the first term with \( O_p \left( N^{-1} h_n^{-1/2} \right) = o_p(h_n^{k-1} N^{-3/4}) \). Hence, since the above bounds are uniform over \( \rho \in [\rho(c) - M_N N^{-1/2}, \rho(c) + M_N N^{-1/2}] \), the bound is \( O_p(N^{-3/4} h_n^{k-1}) \) on the difference \( |T_k,C(\rho(C)) - T_{k,N}(\rho(c))| \). \qed

**Lemma B.19.** Suppose \( \nu(r; c, \rho) \) is such that
\[ |\nu(r; c, \rho)| < B_D(r - \rho(c))^3 + B_M(|r - \rho(c)| + |\rho(c) - \rho|)|\rho(c) - \rho|. \]

Then the difference
\[ \mathcal{L}_N(C_N) - \mathcal{L}_N(c) \]
\[ = \frac{1}{Nh_N} \sum_{i=1}^{N} J_i(c') I_{1,i}(C_N, h_N) x_i(C_N) \nu(R_{it_1}; c, \rho(C)) - \frac{1}{Nh_N} \sum_{i=1}^{N} J_i(c') I_{1,i}(c, h_N) x_i(c) \nu(R_{it_1}; c, \rho(c)) \]
\[ = o_p(h_n N^{-1/2}), \]
assuming \( N^{-1/2} = o(h_N) \).

**Proof.** On \( A_N \), when \( I_{1,i} = 1 \), the \( \nu \) terms are uniformly bounded by
\[ B_D(h_n + 2M_N N^{-1/2})^3 + 10B_M(c)(h_n + M_N N^{-1/2})M_N N^{-1/2} = O(h_n N^{-1/2} + h_n^3). \]
Thus, by Lemma B.16, the difference is bounded by
\[ O_p(h_n N^{-1/2} + h_n^3) \left[ \frac{N^{-1/2}/h_n}{N^{-1/2}} \right] = o_p(h_n N^{-1/2}). \]

**Lemma B.20** (A modified version of Lemma A.2 in Imbens and Kalyanaraman (2012)). Consider \( S_{k,N} = S_{k,N}(c) \) in Lemma B.16. Then, under Assumption B.5,
\[ S_{k,N} = \text{Pr}(J_i(c) = 1) \cdot f(\rho(c))h_n^k \int_0^{1/2} t^{j} dt + o_p(h_n^k), \]
and, as a result,
\[ \begin{bmatrix} S_{0,N} & S_{1,N} \\ S_{1,N} & S_{2,N} \end{bmatrix}^{-1} = \begin{bmatrix} a_2 & -a_1/h_N \\ -a_1/h_N & a_2/h_N^2 \end{bmatrix} + \begin{bmatrix} o_p(1) & o_p(1/h_N) \\ o_p(1/h_N) & o_p(1/h_N^2) \end{bmatrix} \]

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where the constants are\(^{40}\)
\[
    a_k = \frac{\nu_k}{\Pr(J_i(c) = 1) f(\rho(c))(\nu_0 \nu_2 - \nu_1^2)} = \frac{12/(k + 1)}{\Pr(J_i(c) = 1) f(\rho(c))} \quad \nu_k = \int_0^1 t^k \, dt = \frac{1}{k + 1}
\]
and \(f(\rho(c)) = p(R_{it1} = \rho(c) | J_i(c) = 1)\) is the conditional density of the running variable at the cutoff point.

**Proof.** The presence of \(J_i(c)\) adds \(\Pr(J_i(c) = 1)\) to the final result, via conditioning on \(J_i(c) = 1\). The rest of the result follows directly from Lemma A.2 in Imbens and Kalyanaraman (2012) when working with the joint distribution conditioned on \(J_i(c) = 1\).\(\square\)

### B.5. Central limit theorem and variance estimation.

**Lemma B.21.** Let
\[
    Z_N = \frac{1}{\sqrt{Nh_N}} \sum_{i=1}^N W_i(c, h_N) \frac{4 - 6 R_{it1} - \rho(c)}{h_N} \frac{f(\rho(c))}{\Pr(J_i(c) = 1) f(\rho(c))} \epsilon_i
\]
Then, under Assumptions B.3 and B.5,
\[
    \text{Var}(Z_N) \to \frac{4}{\Pr(J_i(c) = 1) f(\rho(c))} \sigma_+^2\text{ as } N \to \infty.
\]

**Proof.** It suffices to compute the limit
\[
    \mathbb{E} \left[ \left( \frac{4 - 6 R_{it1} - \rho(c)}{h_N} \left( 2 - 3 \frac{R_{it1} - \rho(c)}{h_N} \right) \epsilon_i^2 \right) | J_i(c) = 1 \right] = \mathbb{E} \left[ \left( \frac{4 - 6 R_{it1} - \rho(c)}{h_N} \left( 2 - 3 \frac{R_{it1} - \rho(c)}{h_N} \right) \epsilon_i^2 \right) | J_i(c) = 1, R_{it1} \right] = \mathbb{E} \left[ \left( \frac{4 - 6 R_{it1} - \rho(c)}{h_N} \left( 2 - 3 \frac{R_{it1} - \rho(c)}{h_N} \right) \epsilon_i^2 \right) | J_i(c) = 1 \right]
\]
\[
    = \frac{1}{h_N} \int_{\rho(c)}^{\rho(c) + h_N} \left( 2 - 3 \frac{r - \rho(c)}{h_N} \right) \sigma_+^2(r) f(r) \, dr
\]
\[
    \rightarrow \sigma_+^2 f(\rho(c)) \cdot \int_0^1 (2 - 3v)^2 \, dv
\]
\[
    = \sigma_+^2 f(\rho(c))\text{ (Dominated convergence and continuity)}
\]
Thus, the limiting variance is
\[
    \frac{1}{Nh_N} \cdot N \cdot \Pr(J_i(c) = 1) \cdot \frac{4h_N}{\Pr(J_i(c) = 1)^2 f(\rho(c))^2} (\sigma_+^2 f(\rho(c)) + o(1)) \rightarrow \frac{4}{\Pr(J_i(c) = 1) f(\rho(c))}.
\]
\(\square\)

**Lemma B.22** (Lyapunov). Let
\[
    Z_N = \frac{1}{\sqrt{Nh_N}} \sum_{i=1}^N W_i(c, h_N) \frac{4 - 6 R_{it1} - \rho(c)}{h_N} \frac{f(\rho(c))}{\Pr(J_i(c) = 1) f(\rho(c))} \epsilon_i \equiv \sum_{i=1}^N Z_{N,i}.
\]
Then, under Assumptions B.3 and B.5 \(\text{NE}[Z_{N,i}]^{2+\varepsilon} \to 0\) where \(\varepsilon\) is given in Assumption B.3. Hence
\[
    Z_N \overset{d}{\to} \mathcal{N} \left( 0, \frac{4}{\Pr(J_i(c) = 1) f(\rho(c))} \sigma_+^2 \right).
\]

\(^{40}\)The constants \(\nu_k\) depends on the kernel choice, which we fix to be the uniform kernel \(K(x) = \mathbb{1}(x < 1/2)\).
Proof. The part after “hence” follows directly from the Lyapunov CLT for triangular arrays. Now,
\[
E|Z_{N,i}|^{2+\varepsilon} = \frac{\text{Pr}(J_i(c) = 1)}{N h_N (N h_N)^{\varepsilon/2}} \cdot E \left[ \mathbb{1}(R_{it_1} \in [\rho(c), \rho(c) + h_N]) \cdot \left( 2 - \frac{3 R_{it_1} - \rho(c)}{h_N} \right)^{2+\varepsilon} \varepsilon^{2+\varepsilon} \right] \left( J_i(c) = 1 \right)
\]
Since the \(2+\varepsilon\) moment of \(\varepsilon_i\) is uniformly bounded, and \(R_{it_1} - \rho(c) < h_N\) whenever \(\mathbb{1}(R_{it_1} \in [\rho(c), \rho(c) + h_N]) = 1\), the above is bounded above by
\[
B_{\text{CLT}} \leq \frac{1}{N(N h_N)^{\varepsilon/2}} = o(N)
\]
for some constant \(B_{\text{CLT}}\).

\[\square\]

**Lemma B.23** (WLLN for triangular arrays, Durrett (2019) Theorem 2.2.11). For each \(n\) let \(X_{n,k}\) be independent for \(1 \leq k \leq n\). Let \(b_n > 0\) with \(b_n \to \infty\). Let \(\overline{X}_{n,k} = X_{n,k} \mathbb{1}(|X_{n,k} \leq b_n|)\). Suppose that as \(n \to \infty\),

1. \(\sum \text{Pr} \{ |X_{n,k}| > b_n \} \to 0\)
2. \(b_n^{-2} \sum_{k=1}^{n} E[X_{n,k}^2] \to 0\).

Let \(S_n = \sum_k X_{n,k}\) and let \(\mu_n = E[X_{n,k}]\), then
\[
\frac{1}{b_n}(S_n - \mu_n) \xrightarrow{p} 0.
\]

**Lemma B.24** (Variance estimation). Let \(N_+\) be the number of observations with \(J_i(C_N) = 1\) and \(R_{it_1} \in [\rho(C_N), \rho(C_N) + h_N]\). Then, under Assumptions B.3 and B.5, and that \(\hat{\beta}_0 = \mu_+(\rho(c)) + o_p(1)\),
\[
\frac{Nh_N}{N_+} \left( \frac{1}{N_+} \sum_{i=1}^{N} W_i(C_N, h_N) Y_i(C_N)^2 - \hat{\beta}_0^2 \right) \xrightarrow{p} \frac{\sigma^2}{\text{Pr}(J_i(c) = 1) f(\rho(c))}.
\]

Proof. Note that
\[
\frac{1}{Nh_N} N_+ = \frac{1}{Nh_N} \sum_i W_i(C_N, h_N) = \frac{1}{Nh_N} \sum_i W_i(c, h_N) + o_p(1) = \text{Pr}(J_i(c) = 1) f(\rho(c)) + o_p(1)
\]
(Lemmas B.14 and B.16)

By Lemma B.15 and Corollary B.17, we have that
\[
\frac{1}{Nh_N} \sum_{i=1}^{N} W_i(C_N, h_N) Y_i^2(C_N) = \frac{1}{Nh_N} \sum_{i=1}^{N} W_i(c, h_N) Y_i^2(c) + o_p(1)
\]
\[
= \text{Pr}(J_i(c) = 1) \mathbb{E} \left[ \frac{\mathbb{1}(R_{it_1} \in [\rho(c), \rho(c) + h_N])}{h_N} Y_i(c)^2 \mid J_i(c) = 1 \right] + o_p(1)
\]
\[
\to \text{Pr}(J_i(c) = 1) \mathbb{E}[Y_i(c)^2 \mid J_i(c) = 1, R_{it_1} = \rho(c)] f(\rho(c)).
\]
The second equality follows from Lemma B.23, which requires some justification. Barring that, the claim follows via Slutsky’s theorem, noting that \(\hat{\beta}_0 = \mu_+(\rho(c)) + o_p(1)\).

To show the second equality above, let \(X_{k,N} = W_k(c, h_N) Y_i^2(c)\) and let \(b_N = Nh_N\). Note that by Markov’s inequality and Assumption B.3,
\[
\text{Pr}(X_{k,N} > b_N) = \text{Pr} \left( W_k(c, h_N) Y_i(c)^2 > b_N^{1+\varepsilon/2} \right) \lesssim \frac{\mathbb{E}[W_k(c, h_N)]}{b_N^{1+\varepsilon/2}} \lesssim \frac{b_N}{b_N^{1+\varepsilon/2}}.
\]

Thus the first condition of Lemma B.23 is satisfied:
\[
\sum_k \text{Pr}[X_{k,N} > b_N] \lesssim b_N / b_N^{1+\varepsilon/2} \to 0.
\]
Note that $E[X_{k,N}] \lesssim h_N$ since $E[X_{k,N} \mid X_{k,N} \neq 0] < \infty$. Note that (Lemma 2.2.13 Durrett, 2019)

$$E[X_{k,N}^2] = \int_0^{b_N} 2y \Pr(X_{k,N} > y) \, dy \lesssim \int_0^{b_N} 2y \frac{h_N}{y^{1+\varepsilon/2}} \, dy$$

via the same Markov’s inequality argument. Calculating the integral shows that

$$b_N^{-2} \sum_k E[X_{k,N}] \to 0$$

and thus the second condition follows. The implication of Lemma B.23 is that

$$\frac{1}{Nh_N} \sum_{i=1}^N W_i(c, h_N)Y_i^2(c) = \mathbb{E} \left[ \frac{1}{Nh_N} \sum_{i=1}^N W_i(c, h_N)Y_i^2(c)1(Y_i^2(c) < Nh_N) \right] + o_p(1)$$

Since $\mathbb{E}[Y_i^2(c) \mid J_i(c) = 1, R_{c_i} = r] < B < \infty$,

$$\mathbb{E} \left[ \frac{1}{Nh_N} \sum_{i=1}^N W_i(c, h_N)Y_i^2(c)1(Y_i^2(c) < Nh_N) \right] = \mathbb{E} \left[ \frac{1}{Nh_N} \sum_{i=1}^N W_i(c, h_N)Y_i^2(c) \right] + o(1),$$

concluding the proof. \qed
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