On the joint residence time of $N$ independent two-dimensional Brownian motions.

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Abstract

We study the behavior of several joint residence times of $N$ independent Brownian particles in a disc of radius $R$ in two dimensions. We consider: (i) the time $T_N(t)$ spent by all $N$ particles simultaneously in the disc within the time interval $[0, t]$; (ii) the time $T^{(m)}_N(t)$ which at least $m$ out of $N$ particles spend together in the disc within the time interval $[0, t]$; and (iii) the time $\tilde{T}^{(m)}_N(t)$ which exactly $m$ out of $N$ particles spend together in the disc within the time interval $[0, t]$. We obtain very simple exact expressions for the expectations of these three residence times in the limit $t \to \infty$.

1 Introduction

One of important and physically meaningful characteristics of individual Brownian particle trajectories is the time $T(t)$ spent in a finite domain $S$ within some observation time $t$. This time is referred to in the literature as the "occupation" [11] or the "residence time" [3 4 5 6 7]. Note that contrary to the first exit or the first passage times out of $S$, the residence time accounts for multiple exits from, and entries into $S$. As a matter of fact, the first passage times can be obtained, once the residence times are known [7]. The long time properties of $T(t)$ are essentially dependent on the dimension of the embedding space. In one and two dimensions all moments of $T(t)$ diverge with time, which mirrors the fact that here the Brownian motion is "compactly" exploring the space [8]. In two dimensions, in particular, the long time behavior of $T(t)$ obeys the Kallianpur-Robbins’ law [9], which
states that the scaling variable $T'(t)$, defined as $T'(t) = \frac{4\pi DT(t)}{S \ln t}$, where $D$ stands for the particle’s diffusion coefficient and $S$ is the area of the domain, is asymptotically distributed according to the exponential law. On the contrary, in higher dimensions, these moments tend to finite limiting values when $t \to \infty$. Unfortunately, explicit expressions in these transient cases are not always available. A notable exception is the special case of a three-dimensional spherical domain, a case in which explicit values of the moments of the limiting values of the occupation time have been obtained [6].

Recently, motivated in part by the development of new experimental techniques, such as, e.g., fluorescence correlation spectroscopy (FCS), which enables to register single particle events in many particle systems (see for example Ref. [15]), there has been a considerable interest in the behavior of related properties of a set of independently diffusing particles (see Ref. [10] and references therein). For instance, the number of distinct sites visited in $n$ steps by $N$ random walks on a lattice [11], the time spent together by the first $j$ out of a total number $N$ particles, before they escape from a given region [12, 13, 14] or the order statistics for $N$ diffusing particles in the trapping problem [10] have been extensively analysed. It has been recognized that, despite the absence of any physical interaction between the diffusing particles, in many instances a highly cooperative behavior emerges.

In this paper we study collective statistical properties of the mean residence time of $N$ independent Brownian particles in a finite domain $S$. We focus here on the behavior in a two-dimensional continuum and suppose that $S$ is a disc of radius $R$ centered at the origin. We consider three kinds of occupation times:

- The time $T_N(t)$ spent by all $N$ particles simultaneously in the domain $S$ within the time interval $[0, t]$;
- The time $T_N^{(m)}(t)$ which at least $m$ out of $N$ particles spend together in the domain $S$ within the time interval $[0, t]$;
- The time $\tilde{T}_N^{(m)}(t)$ which exactly $m$ out of $N$ particles spend together in the domain $S$ within the time interval $[0, t]$.

We obtain here very simple expressions for the mean values of these times in the limit $t \to \infty$.

We note that this type of functionals of random walks has not received much attention up to now, except for two particular cases. The first one concerns the occupation time of harmonically bounded Brownian particles in two dimensions [16]. In the second case the problem of the occupancy of a single lattice site by a concentration of random walkers has been analysed [17]. Note that in both cases one finds that the moments of the residence time diverge as $t \to \infty$. 

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The analysis of such functionals of Brownian trajectories might be of importance for several physical processes. Consider for example $N$ molecules diffusing on a surface, which contains a receptor sensible to the presence of a given number of molecules, say $m$, with $m \leq N$, i.e. we suppose that there is a kind of sensitivity threshold. Suppose next that the activity of the receptor is proportional to the time during which it is active, that is to the time when there are at least $m$ molecules in the vicinity of the receptor. We finally have a situation where the response of the receptor at time $t$ is proportional to the time $T_N^{(m)}$ defined previously. Another application can be found in the FCS which is, as mentioned, a single molecule microscopy method. In these experiments, a given region of space is illuminated by a laser beam, and one observes the fluorescence signal of the molecules going through that region. In order to observe a single molecule, it might be important to limit events corresponding to the presence of more than one molecule inside the illuminated region, by decreasing the extension of the beam [15]. The evaluation of the time spent when there are at least 2 particles out of the total number of molecules inside the illuminated region, that is precisely the quantity $T_N^{(2)}$, could give a quantitative measurement of these undesirable events, and thus could furnish an indication of the required extension of the beam.

2 The model.

Consider $N$ independent Brownian particles diffusing on an infinite two-dimensional plane. Let $r_j(t)$ denote the positions of these particles at time $t$, while $D_j$ stand for the corresponding diffusion coefficients, which are not necessarily equal to each other. We also assume that all particles are initially at the origin, i.e. $r_j^{(0)} = 0$.

Next, let us introduce three auxiliary indicator functions $1_S(r)$, $I_S^{(m)}(\{r_j\})$ and $\tilde{I}_S^{(m)}(\{r_j\})$ which have the following properties

$$1_S(r) = \begin{cases} 1, & \text{if } |r| \leq R, \\ 0, & \text{otherwise}, \end{cases}$$

(1)

$$I_S^{(m)}(\{r_j\}) = \begin{cases} 1, & \text{if at least } m \text{ of } N \text{ particles are in } S, \\ 0, & \text{otherwise}, \end{cases}$$

(2)

and

$$\tilde{I}_S^{(m)}(\{r_j\}) = \begin{cases} 1, & \text{if exactly } m \text{ of } N \text{ particles are in } S, \\ 0, & \text{otherwise}. \end{cases}$$

(3)
The occupation times $T_N(t)$, $T_N^{(m)}(t)$ and $\tilde{T}_N^{(m)}(t)$, defined in the Introduction, can be then formally written as

$$T_N(t) = \int_0^t dt' \left( \prod_{j=1}^N \mathbb{1}_{S}(r_j(t')) \right), \quad (4)$$

$$T_N^{(m)}(t) = \int_0^t dt' T_s^{(m)}(\{r_j(t')\}), \quad (5)$$

and

$$\tilde{T}_N^{(m)}(t) = \int_0^t dt' \tilde{T}_s^{(m)}(\{r_j(t')\}). \quad (6)$$

Using the Poincare-type formulae [19], the two last equations can be cast into a more convenient form:

$$T_N^{(m)}(t) = \int_0^t dt' \sum_{k=m}^N (-1)^{(k-m)} \binom{k-1}{m-1} \sum_{1 \leq i_1 < ... < i_k \leq N} \left( \prod_{j=1}^k \mathbb{1}_{S}(r_j(t')) \right), \quad (7)$$

$$\tilde{T}_N^{(m)}(t) = \int_0^t dt' \sum_{k=m}^N (-1)^{(k-m)} \binom{k}{m} \sum_{1 \leq i_1 < ... < i_k \leq N} \left( \prod_{j=1}^k \mathbb{1}_{S}(r_j(t')) \right), \quad (8)$$

where the sums extend over all ordered $k$-uptlets in $\{1, \ldots, N\}$, and $N \geq m \geq 2$. Eqs. (7) and (8) show that $T_N^{(m)}(t)$ and $\tilde{T}_N^{(m)}(t)$ are both functionals of $T_N(t)$. Consequently, we focus our analysis on the behavior of $\langle T_N(t) \rangle$. Results for $\langle T_N^{(m)}(t) \rangle$ and $\langle \tilde{T}_N^{(m)}(t) \rangle$ can be straightforwardly obtained once $\langle T_N(t) \rangle$ is known and will be presented here without derivation.

3 Results.

The first moment of $T_N(t)$, i.e. the mean time spent in $S$ by $N$ particles simultaneously within the time interval $[0, t]$, is given by

$$\mu_N(t) = \langle T_N(t) \rangle = \int_0^t \left( \prod_{j=1}^N \langle \mathbb{1}_{S}(r_j(t')) \rangle \right) dt'. \quad (9)$$

Turning to the limit $t \to \infty$, and performing averaging over the realizations of Brownian motions, we obtain

$$\mu_N \equiv \lim_{t \to \infty} \mu_N(t) = \int_0^\infty dt' \left\{ \prod_{j=1}^N \left[ 1 - \exp \left( -\frac{R^2}{4t'D_j} \right) \right] \right\} \quad (10)$$

We take next the diffusion coefficient $D_1$ of the first particle as the reference. Introducing the corresponding time scale $\tau = R^2/4D_1$, as well as the dimensionless parameters $\lambda_i = D_i/D_1$, $i = 1, \ldots, N$, we find that $\mu_N$ obeys

$$\mu_N = \tau \int_0^1 \frac{\prod_{j=1}^N \left( 1 - v^{\lambda_j} \right)}{v(\ln v)^2} \, dv. \quad (11)$$
Next, expanding the product $\prod_{j=1}^{N} (1 - v^{\lambda_j})$ in powers of $v$, we arrive, by performing integrations by parts, at the following explicit expression:

$$
\mu_N = \tau \sum_{k=1}^{N} (-1)^k \sum_{1 \leq i_1 < i_2 < \ldots < i_k \leq N} (\lambda_{i_1} + \ldots + \lambda_{i_k}) \ln (\lambda_{i_1} + \ldots + \lambda_{i_k}).
$$

which does not seem to be an a priori trivial result.

In the particular case when only two Brownian particles, $N = 2$, are present in the system, Eq. (12) yields the following symmetric compact expression:

$$
\mu_2 = \frac{R^2}{4} \left[ \frac{1}{D_1} \ln \left( 1 + \frac{D_1}{D_2} \right) + \frac{1}{D_2} \ln \left( 1 + \frac{D_2}{D_1} \right) \right].
$$

When one of the particles moves much faster than the second one, that is, for instance, when $D_1 \gg D_2$, Eq. (13) reduces to

$$
\mu_2 \sim \frac{R^2}{4D_1} \ln \left( \frac{D_1}{D_2} \right).
$$

Note that such a result appears to be quite reasonable from the physical point of view. To a first approximation, one estimates that the joint residence time is given by the joint residence time before the first exit time of the slow particle out of the disc. This time is of order $t_2 = R^2/D_2$. Meanwhile, the fast particle leaves the disc and returns back several times. The time spent at each return of the fast particle inside the disc is of order $\tau = R^2/D_1$, and the order of magnitude of the number of returns at time $t_2$ of the particle one inside the cercle is given by $\ln \left( t_2/\tau \right)$ (cf. the Kallianpur-Robbins’ law mentioned in the Introduction). As a consequence, we expect that for $D_1 \gg D_2$ one has

$$
\mu_2 \sim \tau \ln \left( \frac{t_2}{\tau} \right) \sim \frac{R^2}{D_1} \ln \left( \frac{D_1}{D_2} \right),
$$

which reproduces the dependence of $\mu_2$ on the diffusion coefficients $D_1$ and $D_2$ given by Eq. (14).

Consider now the special case when all of the particles have the same diffusion coefficient $D_1$. Here, Eq. (12) becomes

$$
\mu_N = \tau \sum_{k=1}^{N} (-1)^k k \binom{N}{k} \ln(k),
$$

which in the asymptotical limit $N \to \infty$ behaves as

$$
\mu_N \sim \frac{\tau}{\ln(N)},
$$

displaying a very slow decay as a function of $N$.

As a matter of fact, this expression coincides exactly with the result of Refs. [20, 21] obtained for the mean first exit time out of the disc by one of $N$ particles. This is, of course,
not counterintuitive, since, after the departure of one particle, the probability that all \(N\) particles arrive altogether inside the disc goes to zero when \(N\) tends to infinity.

Note also that the result in Eq.(17) is completely different from its lattice counterpart, e.g., from the result obtained for the mean joint occupation time of a single lattice site by \(N\) independent random walks. Here, for example, for continuous time random walks (with jump frequency \(\omega\)) on a \(d\)-dimensional hypercubic lattice, one finds (see Ref.[17]):

\[
\mu_N = \int_0^\infty \prod_{j=1}^N p_j(0, t'|0, 0)dt' = \int_0^\infty e^{-N\omega t'} \left[ I_0 \left( \frac{\omega t'}{d} \right) \right]^{N_d} dt',
\]

(18)

where \(p_j(0, t'|0, 0)\) denotes the probability that the \(j\) walker, which starts its random walk at the origin, returns to the origin at time \(t'\), and \(I_0(z)\) is the modified Bessel function. The large-\(N\) asymptotic behavior of \(\mu_N\) thus follows [18]

\[
\mu_N = \frac{1}{N\omega} \left( 1 + \frac{1}{N d} + \frac{3}{4(N d)^2} + \frac{3}{2(N d)^3} + ... \right)
\]

(19)

In this case, \(\mu_N\) decreases with the increase in the number of walkers \(N\) at a much faster rate than in continuum, Eq.(17). A similar result has been already reported in the context of the behavior of first passage times in finite systems [12].

We finally consider an important special case in which only one of the particles has the diffusion coefficient \(D_1\), while the remaining \(N - 1\) have the diffusion coefficient \(D_2\) - a case of an "impure" particle among a set of \(N - 1\) "pure" particles. In this special case Eq.(12) becomes, explicitly,

\[
\mu_N = \frac{R^2}{4} \left\{ \frac{N - 1}{D_2} \sum_{k=1}^{N-1} (-1)^k \ln \left( \frac{k}{k + D_2/D_1} \right) \left( \frac{N - 2}{k - 1} \right) \right. \\
+ \left. \frac{1}{D_1} \sum_{k=0}^{N-1} (-1)^{k-1} \ln \left( k + D_2/D_1 \right) \left( \frac{N - 1}{k} \right) \right\}.
\]

(20)

We turn now to the behavior of \(\mu^{(m)}_N = \lim_{t \to \infty} \langle T_{N}^{(m)}(t) \rangle\). Assuming that all particles have the same diffusion coefficient \(D_1\), we find, using Eq.(7), as well as the result obtained previously for \(\mu_N\), Eq.(16), that

\[
\mu^{(m)}_N = \tau \sum_{k=m}^{N} (-1)^{(k-m)} \frac{(k-1)}{(m-1)} \frac{(N)}{k} \sum_{j=1}^{k} (-1)^j j \frac{k}{j} \ln(j).
\]

(21)

Changing the order of summations, we finally obtain the following result:

\[
\mu^{(m)}_N = \tau m \frac{(N)}{m} \sum_{k=0}^{m-1} (-1)^{m-k} \frac{(m-1)}{k} \ln(N - k).
\]

(22)
In particular, for $m = 2$ (which corresponds to the FCS example mentioned in the Introduction), the last equation yields

$$
\mu_N^{(2)} = \tau N(N - 1) \ln \left( \frac{N}{N - 1} \right),
$$

where $\tau = R^2/D_1$. The asymptotic limit $N \to \infty$ with $m$ finite in Eq.(22) leads to

$$
\mu_N^{(m)} \sim \frac{N}{m - 1} \tau,
$$

which is compatible, for $m = 2$, with the result in Eq.(23) when $N \to \infty$.

Lastly, for $\tilde{\mu}_N^{(m)} = \lim_{t \to \infty} \langle \tilde{T}_N^{(m)}(t) \rangle$, i.e. the mean limiting time spent simultaneously by exactly $m$ out of $N$ particles within $S$, we find, using Eqs.(8) and (16) (or the relation $\tilde{\mu}_N^{(m)} = \mu_N^{(m)} - \mu_N^{(m+1)}$),

$$
\tilde{\mu}_N^{(m)} = \tau \left( \frac{N}{m} \right) \sum_{k=0}^{m} (-1)^{m-k} (N - k) \binom{m}{k} \ln(N - k),
$$

(25)

We note parenthetically that, curiously enough, $\tilde{\mu}_N^{(m)}$ appears to be a non-monotonic function of $m$, as suggested by numerical analysis of Eq.(25). In the asymptotic limit $N \to \infty$ with $m$ finite, Eq.(25) yields

$$
\tilde{\mu}_N^{(m)} \sim \frac{N}{m(m - 1)} \tau,
$$

(26)

which is not an apriori trivial result.

4 Conclusions.

To conclude, we have studied several types of joint residence times of a disc of radius $R$ by $N$ independent Brownian particles: The time $T_N(t)$ spent by all $N$ particles simultaneously in the domain $S$ within the time interval $[0, t]$; the time $T_N^{(m)}(t)$ which at least $m$ out of $N$ particles spend together in the domain $S$ within the time interval $[0, t]$; and finally, the time $\tilde{T}_N^{(m)}(t)$ which exactly $m$ out of $N$ particles spend together in the domain $S$ within the time interval $[0, t]$. We have shown that in case when all the particles are initially located at the center of the disc, it is possible to obtain the mean values of such joint residence times exactly for arbitrary values of the particle number $N$.

References

[1] D. A. Darling and M. Kac, Trans. Am. Math. Soc. 84, 444 (1957)
[2] G. H. Weiss, *Aspects and Applications of the Random Walk*, (North-Holland, Amsterdam, 1994)

[3] A. Blumen and G. Zumofen, J. Chem Phys. **75**, 892 (1981)

[4] G. Zumofen and A. Blumen, J. Chem Phys. **76**, 3713 (1982)

[5] N. Agmon, J. Chem. Phys. **81**, 3644 (1984)

[6] A. M. Berezhkovskii, V. Zaloj and N. Agmon, Phys. Rev. E **57**, 3937 (1998)

[7] A. Bar-Haim and J. Klafter, J. Chem. Phys. **109**, 5187 (1998)

[8] P.G.de Gennes, J. Chem. Phys. **76**, 3316 (1982)

[9] G. Kallianpur and H. Robbins, Proc. Nat. Acad. Sci. **39**, 525 (1953)

[10] B. Yuste and L. Acedo, Phys. Rev. E **64**, 061107 (2001)

[11] H. Larralde, P. Trunfio, S. Havlin, H. E. Stanley and G. H. Weiss, Phys. Rev. A **45**, 7128 (1992)

[12] G. H. Weiss, K. E. Shuler and K. Lindenberg, J. Stat. Phys. **31**, 255 (1983)

[13] S. B. Yuste and K. Lindenberg, J. Stat. Phys. **85**, 501 (1996)

[14] J. Dragé and J. Klafter, Phys. Rev. E **60**, 6503 (1999)

[15] M. Eigen and R. Rigler, Proc. Natl. Acad. Sci. USA **91**, 570 (1994)

[16] O. Bénichou and J. Desbois, J. Phys. A **33**, 6655 (2000)

[17] M. Boguñá, A. M. Berezhkovskii and G. H. Weiss, Phys. Rev. E **62**, 3250 (2000)

[18] B. H. Hughes, *Random Walks and Random Environments*, Random Walks Vol. 1 (Oxford, Clarendon Press, 1995)

[19] W. Feller, *An Introduction to Probability Theory and its Applications*, Vol.I, (Wiley, N.Y., 1966)

[20] B. Yuste and L. Acedo, J. Phys. A **33**, 507 (2000)

[21] B. Yuste, L. Acedo and K. Lindenberg, Phys. Rev. E **64**, 052102 (2001)