Abstract

We propose differentiable quantization (DQ) for efficient deep neural network (DNN) inference where gradient descent is used to learn the quantizer’s step size, dynamic range and bitwidth. Training with differentiable quantizers brings two main benefits: first, DQ does not introduce hyperparameters; second, we can learn for each layer a different step size, dynamic range and bitwidth. Our experiments show that DNNs with heterogeneous and learned bitwidth yield better performance than DNNs with a homogeneous one. Further, we show that there is one natural DQ parametrization especially well suited for training. We confirm our findings with experiments on CIFAR-10 and ImageNet and we obtain quantized DNNs with learned quantization parameters achieving state-of-the-art performance.

1 Introduction

Quantized DNNs apply quantizers \( Q : \mathbb{R} \rightarrow \{ q_1, ..., q_I \} \) to discretize the weights and/or activations of a DNN \([1,7]\). They require considerably less memory and have a lower computational complexity, since discretized values \( \{ q_1, ..., q_I \} \) can be stored, multiplied and accumulated efficiently. This is particularly relevant for inference on mobile or embedded devices with limited computational power.

However, gradient based training of quantized DNNs is difficult, as the gradient of a quantization function vanishes almost everywhere, i.e., backpropagation through a quantized DNN almost always returns a zero gradient. Different solutions to this problem have been proposed in the literature: A first possibility is to use DNNs with stochastic weights from a categorical distribution and to optimize the evidence lower bound (ELBO) to obtain an estimate of the posterior distribution of the weights. As proposed in \([8,10]\), the categorical distribution can be relaxed to a concrete distribution – a smoothed approximation of the categorical distribution – such that the ELBO becomes differentiable under reparametrization.

A second possibility is to use the straight through estimator (STE) \([11]\). STE allows the gradients to be backpropagated through the quantizers and, thus, the network weights can be adapted with standard gradient descent \([12]\). Compared to STE based methods, stochastic methods suffer from
large gradient variance, which makes training of large quantized DNNs difficult. Therefore, STE based methods are more popular in practice.

More recent research [13-16] focuses on methods which can also learn the optimal quantization parameters, e.g., the stepsize, dynamic range and bitwidth, in parallel to the network weights. This is a promising approach, as DNNs with learned quantization parameters almost always outperform DNNs with naive chosen or handcrafted parameters.

Recently, and in parallel to our work, [13] explored the use of STE to define the gradient with respect to the quantizers’s dynamic range. The authors applied a per-tensor quantization and used the quantization range as an additional trainable parameter also learned with gradient descent. Similarly, [14] learned the stepsize using gradient descent. However, neither of them learned the optimal bitwidth of the quantizers. We will show that directly learning the bitwidth using gradient methods is not optimal. Instead, we propose to reparametrize the problem and to learn the stepsize and dynamic range. The bitwidth can then be inferred from them. Alternatively, [15,16] tried to learn the bitwidth with reinforcement learning, i.e., they learn an optimal bitwidth assignment policy. Their experiments show that a DNN with a learned and heterogeneous bitwidth assignment outperforms quantized DNNs with a fixed bitwidth assignment. However, such methods have a high computational complexity as the bitwidth policy must be learned, which involves training many quantized DNNs.

In this paper, we discuss how to learn the stepsize, dynamic range and bitwidth with gradient descent. Therefore, we consider quantizers which can be differentiated with respect to their parameters. Compared to [15,16], our method has the advantage that training quantized DNNs has nearly the same computational complexity as standard float32 training. The contributions of this paper are three-fold:

1. We discuss how to use differentiable quantization (DQ) to train quantized DNNs which are able to learn all quantization parameters using per-tensor quantization and global memory constraints. We formulate training as a constrained optimization problem and show how to solve it such that we learn the optimal stepsize, dynamic range and bitwidth for each tensor. To our knowledge, this has never been done before.

2. We show that there are different parametrizations for uniform and power-of-two DQ and that in both cases there exists one natural DQ parametrization with gradients particularly well suited for training. The other parametrizations have the problem of yielding gradients with coupled components. Some even have unbounded values causing training instability. These problems make it particularly hard to learn the bitwidth $b$.

3. We confirm our findings with experiments on CIFAR-10 and ImageNet. For example, we train a heterogeneously quantized MobileNetV2 on ImageNet requiring a total of only 1.65MB to store the weights and only 0.57MB to store its largest feature map. This is equivalent to a homogeneous 4bit quantization of both weights and activations. However, our network learns to allocate the bitwidth heterogeneously in an optimal way. Our MobileNetV2 achieves an error of 30.26% compared to 29.82% for the floating point baseline. This is state-of-the-art for such a heavily quantized MobileNetV2.

We use the following notation throughout this paper: $x, x, X$ and $\mathcal{X}$ denote a scalar, a (column) vector, a matrix and a tensor with three or four dimensions, respectively; $\lfloor \cdot \rfloor$ and $\lceil \cdot \rceil$ are the floor and ceiling operators. Finally, $\delta(\cdot)$ denotes the Dirac delta function.

## 2 Differentiable quantization (DQ)

We now introduce differentiable quantization (DQ), a method to train quantized DNNs. The application to training with memory constraints will be discussed in Sec.3.

Let $Q(x; \theta)$ be a quantizer with the parameters $\theta$ mapping $x \in \mathbb{R}$ to discrete values $\{q_1, ..., q_I\}$. We consider different parametrizations of $Q(x; \theta)$ for uniform quantization and power-of-two quantization and analyze how well the corresponding straight-through estimates for $\partial_x Q(x; \theta)$ and $\nabla_\theta Q(x; \theta)$ are suited to train quantized DNNs.
We will show in Sec. 2.3 that this implies a diagonal Hessian, which results in a better convergence.

θ | which is non-zero in the interesting region

A symmetric uniform quantizer \( Q_U(x; \theta) \) which maps a real value \( x \in \mathbb{R} \) to one of \( I = 2k + 1 \) quantized values \( q \in \{-kd, ..., 0, ..., kd\} \) computes

\[
q = Q_U(x; \theta) = \text{sign}(x) \begin{cases} 
    d \left\lfloor \frac{|x|}{d} + \frac{1}{2} \right\rfloor & |x| \leq q_{\text{max}} \\
    q_{\text{max}} & |x| > q_{\text{max}}
\end{cases},
\]

(1)

where the parameter vector \( \theta = [d, q_{\text{max}}, b]^T \) consists of the step size \( d \in \mathbb{R} \), the maximum value \( q_{\text{max}} \in \mathbb{R} \) and the number of bits \( b \in \mathbb{N} \) used to encode the quantized values \( q \). The elements of \( \theta \) depend on each other, i.e., \( q_{\text{max}} = (2^{b-1} - 1)d \).

The exact derivative \( \partial_x Q_U(x; \theta) = \sum_{k=-2^{b-1}-1}^{2^{b-1}-2} \delta (x - d \left( k + \frac{1}{2} \right)) \) is not useful to train quantized DNNs because it vanishes almost everywhere. A common solution is to define the derivative using STE [11], ignoring the floor operation in [1]. This leads to

\[
\partial_x Q_U(x) = \begin{cases} 
    1 & |x| \leq q_{\text{max}} \\
    0 & |x| > q_{\text{max}}
\end{cases},
\]

(2)

which is non-zero in the interesting region \( |x| \leq q_{\text{max}} \) and which turned out to be very useful to train quantized DNNs in practice [17].

To learn the optimal quantization parameters \( \theta \), we define the gradient \( \nabla_{\theta} Q_U(x; \theta) \) using STE whenever we need to differentiate a non-differentiable floor operation. As the elements of \( \theta \) depend on each other, we can choose from three equivalent parametrizations \( Q_U(x; \theta) \), which we call “U1” to “U3”. Interestingly, they differ in their gradients:

**Case U1:** Parametrization with respect to \( \theta = [b, d]^T \), using \( q_{\text{max}} = q_{\text{max}}(b, d) \) gives

\[
\nabla_{\theta} Q_U(x; \theta) = \begin{bmatrix} 
\partial_{b} Q_U(x; \theta) \\
\partial_{d} Q_U(x; \theta)
\end{bmatrix} = \begin{cases} 
0 & |x| \leq (2^{b-1} - 1)d \\
\frac{1}{2^{b-1} \log(2) d} \left( Q_U(x; \theta) - x \right) \text{sign}(x) & |x| > (2^{b-1} - 1)d
\end{cases}
\]

(3a)

**Case U2:** Parametrization with respect to \( \theta = [b, q_{\text{max}}]^T \), using \( d = d(b, q_{\text{max}}) \) gives

\[
\nabla_{\theta} Q_U(x; \theta) = \begin{bmatrix} 
\partial_{b} Q_U(x; \theta) \\
\partial_{q_{\text{max}}} Q_U(x; \theta)
\end{bmatrix} = \begin{cases} 
-\frac{2^{b-1} \log(2)}{2^{b-1} - 1} & |x| \leq q_{\text{max}} \\
0 & |x| > q_{\text{max}}
\end{cases}
\]

(3b)

**Case U3:** Parametrization with respect to \( \theta = [d, q_{\text{max}}]^T \), using \( b = b(d, q_{\text{max}}) \) gives

\[
\nabla_{\theta} Q_U(x; \theta) = \begin{bmatrix} 
\partial_{d} Q_U(x; \theta) \\
\partial_{q_{\text{max}}} Q_U(x; \theta)
\end{bmatrix} = \begin{cases} 
\frac{1}{q_{\text{max}}} & |x| \leq q_{\text{max}} \\
0 & |x| > q_{\text{max}}
\end{cases}
\]

(3c)

Fig. 1 shows the gradients for the three parametrizations. Ideally, they should have the following two properties: First, the gradient magnitude should be bounded and not vary a lot, as exploding magnitudes force us to use small learning rates to avoid divergence. Second, the gradient vectors should be unit vectors, i.e., only one partial derivative inside \( \nabla_{\theta} Q_U(x; \theta) \) is non-zero for any \( (x, \theta) \). We will show in Sec. 2.3 that this implies a diagonal Hessian, which results in a better convergence.

![Figure 1: Derivatives for the three different parametrizations of \( Q_U(x; \theta) \).](image-url)
behavior of gradient descent.

Parametrization U1 has none of these properties, since \( \| \nabla_\theta Q_U(x; \theta) \|_2 \) grows exponentially with increasing \( b \) and the elements of \( \nabla_\theta Q_U(x; \theta) \) depend on each other. Also U2 has none of them, since \( \partial_b Q_U(x; \theta) \in [-d \log 2, d \log 2] \). Additionally, \( \partial_b Q_U(x; \theta) \) and \( \partial_{q_{\text{max}}} Q_U(x; \theta) \) depend on each other. Case U3 is most promising, as it has bounded gradient magnitudes with \( \partial_b Q_U(x; \theta) \in [-\frac{1}{2}, \frac{1}{2}] \) and \( \partial_{q_{\text{max}}} Q_U(x; \theta) \in \{-1, 1\} \). Furthermore, the gradient vectors are unit vectors, which leads to a diagonal Hessian matrix.

Previous works did not encounter the problem that different parametrizations of DQ can lead to optimization problems with pathological curvature as in case of U1 and U2. They only defined \( \partial_{q_{\text{max}}} Q(x; \theta) \) \(^1^3\) or \( \partial_d Q(x; \theta) \) \(^1^4\). We show that \( b \) can be learned for the optimal choice U3.

Similar considerations can be made for the power-of-two quantization \( Q_P(x; \theta) \), which maps a real-valued number \( x \in \mathbb{R} \) to a quantized value \( q \in \{ \pm 2^k : k \in \mathbb{Z} \} \) by

\[
q = Q_P(x; \theta) = \text{sign}(x) \begin{cases} 
q_{\text{min}} & |x| \leq q_{\text{min}} \\
2^{\lfloor 0.5 + \log_2 |x| \rfloor} & q_{\text{min}} < |x| \leq q_{\text{max}} \\
q_{\text{max}} & |x| > q_{\text{max}} 
\end{cases}
\]

(4)

where \( q_{\text{min}} \) and \( q_{\text{max}} \) are the minimum and maximum absolute values of the quantizer for a bitwidth of \( b \) bit. Power-of-two quantization is an especially interesting scheme for DNN quantization, since a multiplication of quantized values can be implemented as an addition of the exponents.

Using the STE for the floor operation, the derivative \( \partial_x Q_P(x; \theta) \) is given by

\[
\partial_x Q_P(x) = \begin{cases} 
0 & |x| \leq q_{\text{min}} \\
2^{\lfloor 0.5 + \log_2 |x| \rfloor} & q_{\text{min}} < |x| \leq q_{\text{max}} \\
0 & |x| > q_{\text{max}} 
\end{cases}
\]

(5)

The power-of-two quantization has the following three parameters \([b, q_{\text{min}}, q_{\text{max}}] =: \theta\) which depend on each other: \( q_{\text{max}} = 2^{2^{b-1}-1}q_{\text{min}} \). Therefore, we have again three different parametrizations with \( \theta = [b, q_{\text{min}}] \), \( \theta = [b, q_{\text{max}}] \) or \( \theta = [q_{\text{min}}, q_{\text{max}}] \), respectively. Similarly to the uniform case, one parametrization \( \theta = [q_{\text{min}}, q_{\text{max}}] \) leads to a gradient of a very simple form

\[
\nabla_\theta Q_P(x; \theta) = \begin{bmatrix} \partial_{q_{\text{min}}} Q_U(x; \theta) \\ \partial_{q_{\text{max}}} Q_U(x; \theta) \end{bmatrix} = \begin{bmatrix} [1, 0]^T & |x| \leq q_{\text{min}} \\
[0, 0]^T & q_{\text{min}} < |x| \leq q_{\text{max}} \\
[0, 1]^T & |x| > q_{\text{max}} \end{bmatrix}
\]

(6)

which has again a bounded gradient magnitude and independent components and is, hence, best suited for first order gradient based optimization.

### 2.2 Hardware constraints on \( \theta \)

In practice, for an efficient hardware implementation, we need to ensure that the quantization parameters only take specific discrete values: for uniform quantization, only integer values are allowed for the bitwidth \( b \), and the stepsize \( d \) must be a power-of-two, see e.g. \([3]\); for power-of-two quantization, the bitwidth must be an integer, and the minimum and maximum absolute values \( q_{\text{min}} \) and \( q_{\text{max}} \) must be powers-of-two.

We fulfill these constraints by rounding the parameters in the forward pass to the closest integer or power-of-two value. In the backward pass we update the original float values, i.e., we use again the STE to propagate the gradients.

### 2.3 Experimental comparison of DQ parametrizations

In the following we compare the parametrizations using two experiments.

1) Quantization of Gaussian data — In our first experiment we use DQ to learn the optimal quantization parameters \( \theta^* \) which minimize the mean squared error (MSE) \( E \left[ (Q(x; \theta) - x)^2 \right] \) with gradient descent and compare the convergence speed for all possible parametrizations of a uniform and power-of-two quantizer. We choose this example as the gradient \( \nabla_\theta Q(x; \theta) = \)
As the quantization parameters with SGD using momentum and a learning rate schedule that
and where we used the outer-product approximation \[18\] in order to simplify our consider-
and the error surface has fewer steep ridges where gradient descent starts oscillating. Fig.
and velocity \[\theta\] starts oscillating. Fig. 2 shows how the MSE evolves during optimization. The experiment clearly
is best suited to optimize the quantization parameters of the uniform quantizer.

It is interesting to study the Hessian \[H = \nabla_\theta \nabla_\theta^T E \left[ (Q(x; \theta) - x)^2 \right] \in \mathbb{R}^{2 \times 2}\] of the MSE, which is
and activations on CIFAR-10 \[20\] using the same settings as proposed by \[19\]. Fig. 3 shows the evolution of the training and validation error during training for the case of uniform quantization. The plots for power-of-two quantization can be found in the supplementary (Fig. S.4). We train this network starting from either a random or a pre-trained float network initialization and optimize the weights as well as the quantization parameters with SGD using momentum and a learning rate schedule that reduces the learning rate by a factor of 10 after 80 and 120 epochs.

2) CIFAR-10 In our second experiment we train a ResNet-20 \[19\] with quantized parameters and

Figure 2: Mean squared error curves for uniform and power-of-two quantization of data \[x \sim N(0, 1)\].

Figure 3: Mean squared error surfaces for uniform quantization. Only U3 reaches the optimum \[\theta^*\].

\[E \left[ (Q(x; \theta) - x)^2 \right] \text{ is just a scaled version of } \nabla_\theta Q(x; \theta), \text{ i.e., the gradient direction depends directly on the parametrization of } Q(x; \theta) \text{ and thus the effects of changing the parametrization can be observed. We generate data by drawing } 10^4 \text{ samples from } N(0, 1). \text{ Please note that the same example was studied in } [13]. \]

Fig. 3 shows the corresponding error surfaces for the three different parametrizations for the case of uniform quantization. The red curve shows the path through the parameter space taken by gradient descent in order to optimize the MSE, starting with the initial values \[b = 2, \ d = q_{\text{max}} = 1\]. The optimum \[\theta^*\] is located at \[b = 16, \ d \leq 2^{-13}, \ q_{\text{max}} = 4\], since we allow a maximal bitwidth of 16bit and the largest sample magnitude in our dataset is \[\max\{x_1, ..., x_N\} \leq 4\]. In each of the cases U1-U3, the error surface is composed of steep ridges and large flat regions. The steep ridges force us to use small learning rates to avoid divergence. However, since the optimum \[\theta^*\] lies within a large flat region, reachability depends on the initialization. For cases U1 and U2, we need to traverse this flat region in order to attain \[\theta^*\]. However, for U3, \[\theta^*\] lies at the border of a flat region and can be easily reached. Furthermore, case U3 shows a much faster and more stable convergence without oscillation, since the gradient magnitudes are bounded and the error surface has fewer steep ridges where gradient descent starts oscillating. Fig. 2 shows how the MSE evolves during optimization. The experiment clearly shows that the parametrization \[\theta = [d, q_{\text{max}}]^T\] is best suited to optimize the quantization parameters of the uniform quantizer.

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\[H = E \left[ \nabla_\theta Q(x; \theta) \nabla_\theta Q(x; \theta)^T + (Q(x; \theta) - x) \nabla_\theta \nabla_\theta^T Q(x; \theta) \right] = E \left[ \nabla_\theta Q(x; \theta) \nabla_\theta Q(x; \theta)^T \right]\]

and where we used the outer-product approximation \[18\] in order to simplify our consider-
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We now discuss how to train quantized DNNs with memory constraints. Such constraints appear in the following ways during training, i.e., the network can learn to use a large number of parameters \( \theta \) parametrization of DQ, i.e., using either \( \theta = [b, d]^T \) or \( \theta = [d, q_{\text{max}}]^T \). The remaining quantization parameters are chosen such that we start from an initial bitwidth of \( b = 4 \) bit. We define no memory constraints during training, i.e., the network can learn to use a large number of bits to quantize weights and activations of each layer. From Fig. 4, we again observe that the parametrization \( \theta = [d, q_{\text{max}}]^T \) for the case of uniform quantization is best suited to train a uniformly quantized DNN as it converges to a better local optimum. Furthermore, we observe the smallest oscillation of the validation error for this parametrization.

Table 1 compares the best validation error for all parametrizations of the uniform and power-of-two quantizations. We trained networks with only quantized weights and full precision activations as well as with both being quantized. In case of activation quantization with power-of-two, we use one bit to explicitly represent the value \( x = 0 \). This is advantageous as the ReLU nonlinearity will map many activations to this value. We can observe that training the quantized DNN with the optimal parametrization of DQ, i.e., using either \( \theta = [d, q_{\text{max}}]^T \) or \( \theta = [q_{\text{max}}, q_{\text{max}}]^T \) results in a network with the lowest validation error. This result again supports our theoretical considerations from Sec. 2.1.

### Table 1: Comparison of different DQ parametrizations for ResNet-20 on CIFAR-10.

| Quantization | Float32 | Uniform quantization | Power-of-two quantization |
|--------------|---------|----------------------|---------------------------|
| Weights      | 8.50%/7.29% | 17.8% / 17.7% | 8.80% / 7.44% |
| Weights+Activations | 28.9% / 9.03% | 9.43% / 7.74% | 9.23% / 7.40% |
| Weights+Activations | 9.86% | 10.61%/7.56% | 15.10%/9.86% |

In case of randomly initialized weights, we use an initial stepsize \( d_l = 2^{-3} \) for the quantization of weights and activations. Otherwise, we initialize the weights using a pre-trained floating point network and the initial stepsize for a layer is chosen to be \( d_l = 2^{\lfloor \log_2(\text{max} |W_l|/(2^{b-1}) - 1) \rfloor} \). The oscillation of the validation error for this parametrization.

3 Training quantized DNNs with memory constraints

We now discuss how to train quantized DNNs with memory constraints. Such constraints appear in many applications when the network inference is performed on an embedded device with limited computational power and memory resources.

A quantized DNN consists of layers which compute

\[
X_l = f_l(Q(W_l; \theta_l^w) \ast Q(X_{l-1}; \theta_{l-1}^w) + Q(c_l; \theta_l^c)) \quad \text{with} \quad l = 1, ..., L, \tag{7}
\]

where \( f_l(\cdot) \) denotes the nonlinear activation function of layer \( l \) and \( Q(\cdot; \theta) \) is a per-tensor quantization with parameters \( \theta \) applied separately to the input and output tensors \( X_{l-1} \in I_l \) and \( X_l \in I_l \), and also to both the weight tensors \( W_l \in P_l \) and the bias vector \( c_l \in \mathbb{R}^{M_l} \). For a fully connected layer, \( I_{l-1} = \mathbb{R}^{M_{l-1}}, I_l = \mathbb{R}^{M_l} \) are vectors, \( P_l = \mathbb{R}^{M_l \times M_{l-1}} \) are matrices and \( A \ast B \) is a matrix-vector product. In case of a convolutional layer, \( I_{l-1} = \mathbb{R}^{M_{l-1} \times N_{l-1} \times N_{l-1}}, I_l = \mathbb{R}^{M_l \times N_l \times N_l}, P_l = \mathbb{R}^{M_l \times M_{l-1} \times K_l \times K_l} \) are tensors and \( A \ast B \) is a set of \( M_{l-1} M_l \) 2D convolutions, where the convolution is performed on square-sized feature maps of size \( N_{l-1} \times N_{l-1} \) using square-sized kernels of size \( K_l \times K_l \).

DNNs with quantized weights and activations have a smaller memory footprint and are also computationally cheaper to evaluate since \( Q(\alpha; \theta) : Q(\beta; \theta) \) for \( \alpha, \beta \in \mathbb{R} \) requires only an integer multiplication for the case of uniform quantization or an integer addition of the exponents for power-of-two quantization. Furthermore, \( Q(\alpha; \theta) + Q(\beta; \theta) \) for \( \alpha, \beta \in \mathbb{R} \) only requires an integer addition.

Table 2 compares the computational complexity and the memory footprint of layers which apply uniform or power-of-two quantization to weights and activations.

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This paper, we use “weights” to refer to \( W \) and \( c \).
We consider the following memory characteristics of the DNN, constraining them during training:

As proposed in Sec. 3, we use the best parametrizations for uniform and power-of-two DQ, i.e.,

$$\lambda$$

where

$$J$$

To train the quantized DNN with memory constraints, we need to solve the optimization problem

$$C$$

Table 2 gives $$S^w_l$$ for the case of fully connected and convolutional layers. Each layer’s memory requirement $$S^w_l(\theta^w_l)$$ is smaller than a certain maximum weight memory size $$S^w_0$$. Table 2 gives $$S^z_l(\theta^z_l)$$ for the case of fully connected and convolutional layers. Such a constraint is important if we use pipelining for accelerated inference, i.e., if we evaluate multiple layers with several consecutive inputs in parallel. This can, e.g., be the case for FPGA implementations.

3. Maximum activation memory $$\hat{S}^z_l(\theta^z_1, ..., \theta^z_L) = \max_{i=1, ..., L} S^z_l(\theta^z_l)$$ to store the largest feature map: We use the constraint

$$g_3(\theta^z_1, ..., \theta^z_L) = S^z(\theta^z_1, ..., \theta^z_L) - \hat{S}^z_0 \leq 0$$

(8c)

to ensure that the maximum activation size $$\hat{S}^z$$ does not exceed a given limit $$\hat{S}^z_0$$. This constraint is relevant for DNN implementations where layers are processed sequentially.

To train the quantized DNN with memory constraints, we need to solve the optimization problem

$$\min_{\mathbf{W}, c_i, \theta^w_l, \theta^z_l} \mathbb{E}_p(\mathbf{x}, \mathbf{y}) \left[ J(\mathbf{X}_L, \mathbf{Y}) \right]$$

s.t. $$g_j(\theta^w_1, ..., \theta^w_l, \theta^z_1, ..., \theta^z_L) \leq 0$$ for all $$j = 1, ..., 3$$

(9)

where $$J(\mathbf{X}_L, \mathbf{Y})$$ is the loss function for yielding the DNN output $$\mathbf{X}_L$$ although the ground truth is $$\mathbf{Y}$$. Eq. (8) learns the weights $$\mathbf{W}_l$$, $$c$$ as well as the quantization parameters $$\theta^z_1$$, $$\theta^z_L$$. In order to use simple stochastic gradient descent solvers, we use the penalty method [22] to convert (9) into the unconstrained optimization problem

$$\min_{\mathbf{W}_l, c_i, \theta^w_l, \theta^z_l} \mathbb{E}_p(\mathbf{x}, \mathbf{y}) \left[ J(\mathbf{X}_L, \mathbf{Y}) \right] + \sum_{j=1}^J \lambda_j \max(0, g_j(\theta^w_1, ..., \theta^w_l, \theta^z_1, ..., \theta^z_L))^2$$

(10)

where $$\lambda_j \in \mathbb{R}^+$$ are individual weightings for the penalty terms.

4 Experiments

We conduct experiments to demonstrate that DQ can learn the optimal bitwidth of popular DNNs. We show that DQ quantized models are superior compared to same sized models with homogenous bitwidth.

As proposed in Sec. 3, we use the best parametrizations for uniform and power-of-two DQ, i.e.,

$$\theta_U = [d, q_{max}]^T$$ and $$\theta_P = [q_{min}, q_{max}]^T$$, respectively. Both parametrizations do not directly depend on the bitwidth $$b$$. Therefore, we compute it by using $$b(\theta_U) = \left\lceil \log_2 \left( \frac{q_{max}}{q_{min}} + 1 \right) \right\rceil$$ and $$b(\theta_P) = \left\lceil \log_2 \left( \frac{q_{min}}{q_{max}} + 1 \right) \right\rceil$$. All quantized networks use a pre-trained float32 network for initialization and all quantizers are initialized as described in Sec. 2.3. Please note that we quantize all layers opposed to other papers which use a higher precision for the first and/or last layer.
Table 3: Homogeneous vs. heterogeneous quantization of ResNet-20 on CIFAR-10.

| Bitwidth | q_{max} | Size | Uniform quant. | Power-of-two quant. |
|----------|---------|------|----------------|---------------------|
| Weight/Activ. | Weight/Activ. | Weight/Activ. (max)//Activ. (sum) | Validation error | Validation error |
| Baseline | 32bit/32bit | – | 1048KB/64KB/736KB | 7.29% |
| Fixed | 2bit/32bit | fixed/– | 65KB/64KB/736KB | 10.81% | 8.99% |
| Homogeneous as in [13] | 2bit/32bit | learned/– | 65KB/64KB/736KB | 9.47% | 8.79% |
| Heterogeneous (w/ constr. (8a)) | 2bit/32bit | learned/learned | 70KB/64KB/736KB | 8.59% | 8.53% |
| Heterogeneous (w/ constr. (8a) and (8c)) | 2bit/4bit | learned/learned learned/learned | 70KB/8KB/92KB | 8.59% | 8.53% |
| Heterogeneous (w/o constr.) | 2bit/4bit | learned/learned learned/learned | 70KB/8KB/– | 8.58% | 11.23% |

Table 4: Homogeneous vs. heterogeneous quantization of MobileNetV2 and ResNet-18 on ImageNet.

| Bitwidth | q_{max} | MobileNetV2 | ResNet-18 |
|----------|---------|-------------|-----------|
| Weight/Activ. | Weight/Activ. | Size | Validation Error | Size | Validation Error |
| Baseline | 32bit/32bit | – | 13.23MB/4.59MB | 29.82% | 44.56MB/3.04MB | 29.72% |
| Fixed | 4bit/4bit | fixed/– | 1.65MB/0.57MB | 36.27% | 5.57MB/0.38MB | 34.15% |
| Homogeneous as in [13] | 4bit/4bit | learned/learned | 1.65MB/0.57MB | 32.21% | 5.57MB/0.38MB | 30.49% |
| Heterogeneous (w/ constr. (8c)) | 4bit/4bit | learned/learned learned/learned | 1.55MB/0.57MB | 30.26% | 5.40MB/0.38MB | 29.92% |
| Heterogeneous (w/o constr.) | learned/learned | learned/learned | 3.14MB/1.58MB | 29.41% | 10.50MB/1.05MB | 29.34% |

First, in Table 3, we train a ResNet-20 on CIFAR-10 with quantized weights and float32 activations. We start with the most restrictive quantization scheme with fixed q_{max} and b (“Fixed”). Then, we allow the model to learn q_{max} while b remains fixed as was done in [13] (“Homogeneous”). Finally, we learn both q_{max} and b with the constraint that the weight size is at most 70KB (“Heterogeneous”). This allows the model to allocate more than two bits to some layers as can be seen from Fig. 5. From Table 3, we observe that the error is smallest when we learn all quantization parameters.

In Table 4, weights and activations are quantized. For activation quantization, we consider two cases as discussed in Sec. 3. The first one constrains the total activation memory S^x while the second constrains the maximum activation memory S^x such that both have the same size as a homogeneously quantized model with 4bit activations. Again, we observe that the error is smallest when we learn all quantization parameters.

We also use DQ to train quantized ResNet-18 [19] and MobileNetV2 [23] on ImageNet [24] with 4bit uniform weights and activations or equivalent-sized networks with learned quantization parameters. This is quite aggressive and, thus, a fixed quantization scheme loses more than 6% accuracy while our heterogeneous quantization loses less than 0.5% compared to a float32 precision network.

Note that we constrain the activation memory by (8c), i.e., the maximum activation memory S^x such that both have the same size as a homogeneously quantized model with 4bit activations. Again, we observe that the error is smallest when we learn all quantization parameters.

Our DQ results compare favorably to other recent quantization approaches. To our knowledge, the best result for a 4bit ResNet-18 was reported by [14] (29.91% error). This is very close to our performance (29.92% error). However, [14] did not quantize the first and last layers. In Fig. 5, it is interesting to observe that our method actually learns such an assignment. Besides, our network is smaller than the model in [14] since our size constraint includes the first and last layers. Moreover, [14] learns stepsizes which are not restricted to powers-of-two. Uniform quantization with power-of-two stepsize leads to more efficient inference, effectively allowing to efficiently compute any multiplication with an integer multiplication and bit-shift. To our knowledge only [15] reported results of MobileNetV2 quantized to 4 bit. They keep the baseline performance constraining the network to the same size as the 4bit network. However, they do not quantize the activations in this case. Overall, the experiments show that our approach is competitive to other recent quantization methods while it does not require to retrain the network multiple times in contrast to reinforcement learning approaches [15, 16]. This makes DQ training efficient and one epoch on ImageNet takes 37min for MobileNetV2 and 18min for ResNet-18 on four Nvidia Tesla V100.
5 Conclusions

In this paper we discussed differentiable quantization and its application to the training of compact DNNs with memory constraints. In order to fulfill memory constraints, we introduced penalty functions during training and used stochastic gradient descent to find the optimal weights as well as the optimal quantization values in a joint fashion. We showed that there are several possible parametrizations of the quantization function. In particular, learning the bitwidth directly is not optimal; therefore, we proposed to parametrize the quantizer with the stepsize and dynamic range instead. The bitwidth can then be inferred from them. With this approach, we obtained state-of-the-art results on CIFAR-10 and ImageNet for quantized DNNs.

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S.1 Derivation of the gradients for differentiable quantization (DQ)

In the following sections, we will give the derivatives \( \frac{\partial}{\partial x} Q(x; \theta) \) and gradients \( \nabla_{\theta} Q(x; \theta) \) for the uniform and the power-of-two quantizers. The results are summarized in Sec. 3 of the main paper.

We use the straight-through gradient estimate whenever we need to differentiate a non-differentiable floor function, i.e., we assume

\[
\frac{\partial}{\partial x} \lfloor x \rfloor = 1. \tag{S.1}
\]

S.1.1 Derivatives of the uniform quantizer

Fig. S.1(a) shows a symmetric uniform quantizer \( Q_U(x; \theta) \) which maps a real value \( x \in \mathbb{R} \) to one of \( I = 2^k + 1 \) quantized values \( q \in \{-kd, ..., 0, ..., kd\} \) by computing

\[
q = Q_U(x; \theta) = \text{sign}(x) \begin{cases} 
0 & |x| \leq q_{\text{max}} \\
\lfloor \frac{|x|}{d} + \frac{1}{2} \rfloor & |x| > q_{\text{max}} 
\end{cases}
\]  \hspace{1cm} \tag{S.2}

using the parameters \( \theta = [d, q_{\text{max}}, b]^T \) where \( d \in \mathbb{R} \) is the stepsize, \( q_{\text{max}} \in \mathbb{R} \) is the maximum value and \( b \in \mathbb{N} \) is the number of bits that we use to encode the quantized values \( q \). The elements of \( \theta \) are dependent as there is the relationship \( q_{\text{max}} = (2^b - 1 - 1) \cdot d \).

S.1.1.1 Case U1: Parametrization with respect to \( b \) and \( d \)

For the parametrization with respect to the bitwidth \( b \) and steps size \( d \), \( \text{S.2} \) is given by

\[
q = Q_U(x; b, d) = \text{sign}(x) d \begin{cases} 
0 & |x| \leq (2^b - 1 - 1) \cdot d \\
\lfloor \frac{|x|}{2d} + \frac{1}{2} \rfloor & |x| > (2^b - 1 - 1) \cdot d 
\end{cases}
\]  \hspace{1cm} \tag{S.3}

and the derivatives are given by

\[
\frac{\partial Q_U(x; b, d)}{\partial b} = \text{sign}(x) \frac{2^b - 1 - 1}{2^{b-1} - 1} \begin{cases} 
0 & |x| \leq (2^b - 1 - 1) \cdot d \\
\frac{(2^b - 1 - 1)}{d} & |x| > (2^b - 1 - 1) \cdot d 
\end{cases}
\]  \hspace{1cm} \tag{S.4a}

\[
\frac{\partial Q_U(x; b, d)}{\partial d} = \text{sign}(x) \frac{1}{d} \begin{cases} 
\frac{|x|}{2} & |x| \leq (2^b - 1 - 1) \cdot d \\
\frac{d}{2^{b-1} - 1} - |x| & |x| > (2^b - 1 - 1) \cdot d 
\end{cases}
\]  \hspace{1cm} \tag{S.4b}

S.1.1.2 Case U2: Parametrization with respect to \( b \) and \( q_{\text{max}} \)

For the parametrization with respect to the bitwidth \( b \) and maximum value \( q_{\text{max}} \), \( \text{S.2} \) is given by

\[
q = Q_U(x; b, q_{\text{max}}) = \text{sign}(x) q_{\text{max}} \begin{cases} 
\frac{q_{\text{max}}}{2^{b-1} - 1} & |x| \leq q_{\text{max}} \\
1 & |x| > q_{\text{max}} 
\end{cases}
\]  \hspace{1cm} \tag{S.5}

Preprint. Under review.
The power-of-two quantization has the three parameters \( q \) on each other, i.e.,

\[
\frac{\partial Q}{\partial q} = \begin{cases} 
\frac{q_{\max}}{2^{b-1}} & |x| \leq q_{\max}, \\
0 & |x| > q_{\max}, 
\end{cases}
\]

Using the STE for the floor operation, the derivative

\[
\frac{\partial Q}{\partial b} = \begin{cases} 
\frac{q_{\max}}{2^{b-1}} |x|^{b-1} + \frac{1}{2} & |x| \leq q_{\max}, \\
0 & |x| > q_{\max}, 
\end{cases}
\]

where \( \beta_1 = \frac{q_{\max}}{2^{b-1}} \frac{\partial |x|^{b-1} + \frac{1}{2}}{q_{\max}} = |x| \), if we use the straight-through gradient estimate for the floor function.

S.1.1.3 Case U3: Parametrization with respect to \( d \) and \( q_{\max} \)

Eq. (S.2) gives the quantization with respect to the step size \( d \) and maximum value \( q_{\max} \). The derivatives are

\[
\frac{\partial Q}{\partial d} = \begin{cases} 
\frac{d}{d} & |x| \leq q_{\max}, \\
0 & |x| > q_{\max}, 
\end{cases}
\]

\[
\frac{\partial Q}{\partial q_{\max}} = \begin{cases} 
\frac{1}{q_{\max}} & |x| \leq q_{\max}, \\
0 & |x| > q_{\max}. 
\end{cases}
\]

S.1.2 Derivatives of the power-of-two quantizer

Power-of-two quantization \( Q_P(x; \theta) \) maps a real-valued number \( x \in \mathbb{R} \) to a quantized value \( q \in \{ \pm 2^k : k \in \mathbb{Z} \} \) by

\[
q = Q_P(x; \theta) = \begin{cases} 
\min_{x \in \{ \pm 2^k : k \in \mathbb{Z} \}} & |x| \leq q_{\min}, \\
q_{\max} & |x| > q_{\max}, 
\end{cases}
\]

where \( q_{\min} \) and \( q_{\max} \) are the minimum and maximum (absolute) values of the quantizer for a bitwidth of \( b \) bits. Fig. [S.1b] shows the quantization curve for this quantization scheme.

Using the STE for the floor operation, the derivative \( \partial_x Q_P(x; \theta) \) is given by

\[
\frac{\partial_x Q_P(x)}{x} = \begin{cases} 
0 & |x| \leq q_{\min}, \\
|2^{b-1} \log_2(x^2)| & q_{\min} < |x| \leq q_{\max}, \\
0 & |x| > q_{\max}. 
\end{cases}
\]

The power-of-two quantization has the three parameters \( \theta = [b, q_{\min}, q_{\max}] \), which are dependent on each other, i.e., \( q_{\max} = 2^{b-1} q_{\min} \). Therefore, we have again three different parametrizations with \( \theta = [b, q_{\min}] \), \( \theta = [b, q_{\max}] \) or \( \theta = [q_{\min}, q_{\max}] \), respectively. The resulting partial derivatives for...
each parametrization are shown in Fig. S.2 and summarized in the following sections. Similar to the uniform case, one parametrization \( \theta = [q_{\max}, q_{\min}] \) leads to a gradient with the nice form

\[
\nabla_{\theta} Q_P(x; \theta) = \begin{bmatrix}
\partial_{q_{\min}} Q_U(x; \theta) \\
\partial_{q_{\max}} Q_U(x; \theta)
\end{bmatrix} = \begin{cases}
[1, 0]^T & |x| \leq q_{\min} \\
[0, 0]^T & q_{\min} < |x| \leq q_{\max} \\
[0, 1]^T & |x| > q_{\max}
\end{cases},
\]

which has a bounded gradient magnitude and independent components and is, hence, well suited for first order gradient based optimization.

**S.1.2.1 Case P1:** Parametrization with respect to \( b \) and \( q_{\max} \)

For the parametrization with \( \theta = [b, q_{\max}] \), (S.8) is given by

\[
Q_P(x; b, q_{\max}) = \begin{cases}
2^{-2b^{-1}+1} q_{\max} & |x| \leq 2^{-2b^{-1}+1} q_{\max} \\
2[0.5 + \log_2|x|] - 2^{-2b^{-1}+1} q_{\max} & 2^{-2b^{-1}+1} q_{\max} < |x| \leq q_{\max} \\
q_{\max} & |x| > q_{\max}
\end{cases},
\]

and the partial derivatives are

\[
\frac{\partial Q_P(x; b, q_{\max})}{\partial b} = \begin{cases}
-2^{-2b^{-1}+b}(\log 2)^2 q_{\max} & |x| \leq -2^{-2b^{-1}+1} q_{\max} \\
0 & -2^{-2b^{-1}+1} q_{\max} < |x| \leq q_{\max} \\
0 & |x| > q_{\max}
\end{cases},
\]

\[
\frac{\partial Q_P(x; b, q_{\max})}{\partial q_{\max}} = \begin{cases}
2^{-2b^{-1}+1} & |x| \leq -2^{-2b^{-1}+1} q_{\max} \\
0 & -2^{-2b^{-1}+1} q_{\max} < |x| \leq q_{\max} \\
1 & |x| > q_{\max}
\end{cases}.
\]

**S.1.2.2 Case P2:** Parametrization with respect to \( b \) and \( q_{\min} \)

For the parametrization with \( \theta = [b, q_{\min}] \), (S.8) is given by

\[
Q_P(x; b, q_{\min}) = \begin{cases}
q_{\min} & |x| \leq q_{\min} \\
2[0.5 + \log_2|x|] - 2^{b^{-1}-1} q_{\min} & 2^{b^{-1}-1} q_{\min} < |x| \leq q_{\min} \\
2^{b^{-1}-1} q_{\min} & |x| > q_{\min}
\end{cases},
\]

and the partial derivatives are

\[
\frac{\partial Q_P(x; b, q_{\min})}{\partial b} = \begin{cases}
0 & q_{\min} \leq |x| \leq 2^{b^{-1}-1} q_{\min} \\
2^{b^{-1}+b-2}(\log 2)^2 q_{\min} & |x| > 2^{b^{-1}-1} q_{\min}
\end{cases},
\]

\[
\frac{\partial Q_P(x; b, q_{\min})}{\partial q_{\min}} = \begin{cases}
1 & q_{\min} \leq |x| \leq 2^{b^{-1}-1} q_{\min} \\
0 & 2^{b^{-1}-1} q_{\min} < |x| \leq 2^{b^{-1}-1} q_{\min}
\end{cases}.
\]
S.1.2.3 Case P3: Parametrization with respect to $q_{\min}$ and $q_{\max}$

Eq. (S.8) gives the parametrization of $Q(x; \theta)$ with respect to the minimum value $q_{\min}$ and maximum value $q_{\max}$. The derivatives are

\[
\frac{\partial Q_P(x; q_{\min}, q_{\max})}{\partial q_{\min}} = \text{sign}(x) \begin{cases} 
1 & |x| \leq q_{\min} \\
0 & q_{\min} < |x| \leq q_{\max} \\
0 & |x| > q_{\max} \end{cases},
\]

(S.15a)

\[
\frac{\partial Q_P(x; q_{\min}, q_{\max})}{\partial q_{\max}} = \text{sign}(x) \begin{cases} 
0 & |x| \leq q_{\min} \\
0 & q_{\min} < |x| \leq q_{\max} \\
1 & |x| > q_{\max} \end{cases}.
\]

(S.15b)

S.1.3 Experimental validation for power-of-two quantization

In Sec. 3.3 of the paper, we compared the three different parametrizations of the uniform quantizer. Due to space limitations, we could not discuss the power-of-two quantization in the main paper and therefore give the results here.

The experimental setup is the same as in Sec. 3.3, i.e., we use DQ to learn the optimal quantization parameters of a power-of-two quantizer, which minimize the expected quantization error $\min_\theta \mathbb{E}[|x - Q(x; \theta)|^2]$. We use three different parametrizations for the power-of-two quantizer, adapt the quantizer’s parameters with gradient descent and compare the convergence speed as well as the final quantization error. As an input, we generate $10^4$ samples from $N(0, 1)$.

Fig. S.3 shows the corresponding error surfaces. Again, the optimum $\theta^*$ is not attained for two parametrizations, namely P1 and P2, as $\theta^*$ is surrounded by a large, mostly flat region. For these two cases, gradient descent tends to oscillate at steep ridges and tends to be unstable. However, gradient descent converges to a point close to $\theta^*$ for parametrization P3, where $\theta = [q_{\min}, q_{\max}]$.

Finally, we also did a comparison of the different power-of-two quantizations on CIFAR-10. Fig S.4 shows the evolution of the training and validation error if we start from a random or a pre-trained float network initialization. We can observe that $\theta = [q_{\min}, q_{\max}]$ has the best convergence behavior and thus also results in the smallest validation error (cf. Table 1 in the main paper). The unstable behavior of P2 is expected as the derivative $\frac{\partial Q_P}{\partial q_{\min}}$ can take very large (absolute) values.
S.2 Implementation details for differentiable quantization

The following code gives our differentiable quantizer implementation in NNabla \cite{1}. The source code for reproducing our results will be published after the review process has been finished.
S.2.1 Uniform quantization

S.2.1.1 Case U1: Parametrization with respect to \( b \) and \( d \)

```python
def parametric_fixed_point_quantize_d_b(x, sign,
    n_init, n_min, n_max,
    d_init, d_min, d_max,
    fix_parameters=False):
    """Parametric version of 'fixed_point_quantize' where the bitwidth 'b' and stepsize 'd' are learnable parameters.
    ""

    Returns:
    ~ nnabla.Variable: N-D array.
    """

def clip_scalar(v, min_value, max_value):
    return F.minimum_scalar(F.maximum_scalar(v, min_value), max_value)

def broadcast_scalar(v, shape):
    return F.broadcast(F.reshape(v, (1,) * len(shape), inplace=False), shape=shape)

def quantize_pow2(v):
    return 2 ** F.round(F.log(v) / np.log(2.))

def get_parameter_or_create(name, shape,
    initializer, need_grad=True, as_need_grad=False):
    n = F.round(clip_scalar(n, n_min, n_max))
    if sign:
        n = n - 1

    d = quantize_pow2(clip_scalar(d, d_min, d_max))

    # ensure that dynamic range is in specified range
    xmax = d * (2 ** n - 1)

    # compute min/max value that we can represent
    if sign:
        xmin = -xmax
    else:
        xmin = nn.Variable((1,), need_grad=False)
        xmin.d = 0.

    # broadcast variables to correct size
    d = broadcast_scalar(d, shape=x.shape)
    xmin = broadcast_scalar(xmin, shape=x.shape)
    xmax = broadcast_scalar(xmax, shape=x.shape)

    # apply fixed-point quantization
    return d * F.round(F.clip_by_value(x, xmin, xmax) / d)
```

6
S.2.1.2 Case U2: Parametrization with respect to $b$ and $q_{\text{max}}$

```python
def parametric_fixed_point_quantize_b_xmax(x, sign,
    n_init, n_min, n_max,
    xmax_init, xmax_min, xmax_max,
    fix_parameters=False):
    
    """Parametric version of 'fixed_point_quantize' where the bitwidth 'b' and dynamic range 'xmax' are learnable parameters."
    
    Returns:
    ~ nnabla.Variable: N-D array.
    """
    def clip_scalar(v, min_value, max_value):
        return F.minimum_scalar(F.maximum_scalar(v, min_value), max_value)
    
    def broadcast_scalar(v, shape):
        return F.broadcast(F.reshape(v, (1,) * len(shape), inplace=False), shape=shape)
    
    def quantize_pow2(v):
        return 2 ** F.round(F.log(v) / np.log(2.))
    
    n = get_parameter_or_create("n", () ,
        ConstantInitializer(n_init),
        need_grad=True,
        as_need_grad=not fix_parameters)
    xmax = get_parameter_or_create("xmax", () ,
        ConstantInitializer(xmax_init),
        need_grad=True,
        as_need_grad=not fix_parameters)
    
    # ensure that bitwidth is in specified range and an integer
    n = F.round(clip_scalar(n, n_min, n_max))
    if sign:
        n = n - 1
    
    # ensure that dynamic range is in specified range
    xmax = clip_scalar(xmax, xmax_min, xmax_max)
    
    # compute step size from dynamic range and make sure that it is a pow2
    d = quantize_pow2(xmax / (2 ** n - 1))
    
    # compute min/max value that we can represent
    if sign:
        xmin = -xmax
    else:
        xmin = nn.Variable((1,), need_grad=False)
        xmin.d = 0.
    
    # broadcast variables to correct size
    d = broadcast_scalar(d, shape=x.shape)
    xmin = broadcast_scalar(xmin, shape=x.shape)
    xmax = broadcast_scalar(xmax, shape=x.shape)
    
    # apply fixed-point quantization
    return d * F.round(F.clip_by_value(x, xmin, xmax) / d)
```

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S.2.1.3 Case U3: Parametrization with respect to $d$ and $q_{\text{max}}$

```python
def parametric_fixed_point_quantize_d_xmax(x, sign,  
d_init, d_min, d_max,  
xmax_init, xmax_min, xmax_max,  
fix_parameters=False):
    """ Parametric version of 'fixed_point_quantize' where the  
    stepsize 'd' and dynamic range 'xmax' are learnable parameters.  
    """  
    Returns:  
    - nnabla.Variable: N-D array.  
    
def clip_scalar(v, min_value, max_value):
        return F.minimum_scalar(F.maximum_scalar(v, min_value), max_value)

def broadcast_scalar(v, shape):
    return F.broadcast(F.reshape(v, (1,) * len(shape), inplace=False), shape=shape)

def quantize_pow2(v):
    return 2 ** F.round(F.log(v) / np.log(2.))

d = get_parameter_or_create("d", (),  
    ConstantInitializer(d_init),  
    need_grad=True,  
    as_need_grad=not fix_parameters)
    xmax = get_parameter_or_create("xmax", (),  
    ConstantInitializer(xmax_init),  
    need_grad=True,  
    as_need_grad=not fix_parameters)

    # ensure that stepsize is in specified range and a power of two
    d = quantize_pow2(clip_scalar(d, d_min, d_max))

    # ensure that dynamic range is in specified range
    xmax = clip_scalar(xmax, xmax_min, xmax_max)

    # compute min/max value that we can represent
    if sign:
        xmin = -xmax
    else:
        xmin = nn.Variable((1,), need_grad=False)
        xmin.d = 0.

    # broadcast variables to correct size
    d = broadcast_scalar(d, shape=x.shape)
    xmin = broadcast_scalar(xmin, shape=x.shape)
    xmax = broadcast_scalar(xmax, shape=x.shape)

    # apply fixed-point quantization
    return d * F.round(F.clip_by_value(x, xmin, xmax) / d)
```

S.2.2 Power-of-two quantization

S.2.2.1 Case P1: Parametrization with respect to \( b \) and \( q_{\text{max}} \)

```python
def parametric_pow2_quantize_b_xmax(x, sign, with_zero,
    n_init, n_min, n_max,
    xmax_init, xmax_min, xmax_max,
    fix_parameters=False):
    """Parametric version of 'pow2_quantize' where the
    bitwidth 'n' and range 'xmax' are learnable parameters.
    ""
    Returns:
    ~ nnabla.Variable: N-D array.
    
    def clip_scalar(v, min_value, max_value):
        return F.minimum_scalar(F.maximum_scalar(v, min_value), max_value)
    
    def broadcast_scalar(v, shape):
        return F.broadcast(F.reshape(v, (1,) * len(shape), inplace=False), shape=shape)
    
    def quantize_pow2(v):
        return 2 ** F.round(F.log(F.abs(v)) / np.log(2.))
    
    n = get_parameter_or_create("n", (),
        ConstantInitializer(n_init),
        need_grad=True,
        as_need_grad=not fix_parameters)
    xmax = get_parameter_or_create("xmax", (),
        ConstantInitializer(xmax_init),
        need_grad=True,
        as_need_grad=not fix_parameters)
    
    # ensure that bitwidth is in specified range and an integer
    n = F.round(clip_scalar(n, n_min, n_max))
    if sign:
        n = n - 1
    if with_zero:
        n = n - 1
    
    # ensure that dynamic range is in specified range and an integer
    xmax = quantize_pow2(clip_scalar(xmax, xmax_min, xmax_max))
    
    # compute min value that we can represent
    xmin = (2 ** (-2 ** n + 1)) * xmax
    
    # broadcast variables to correct size
    xmin = broadcast_scalar(xmin, shape=x.shape)
    xmax = broadcast_scalar(xmax, shape=x.shape)
    
    # if unsigned, then quantize all negative values to zero
    if not sign:
        x = F.relu(x)
    
    # compute absolute value/sign of input
    ax = F.abs(x)
    sx = F.sign(x)
    
    if with_zero:
        # prune smallest elements (in magnitude) to zero if they are smaller
        # than 'x_min / sqrt(2)'
        x_threshold = xmin / np.sqrt(2)
        
        idx1 = F.greater_equal(ax, x_threshold) * F.less(ax, xmin)
        idx2 = F.greater_equal(ax, xmin) * F.less(ax, xmax)
        idx3 = F.greater_equal(ax, xmax)
        
        else:
            idx1 = F.less(ax, xmin)
            idx2 = F.greater_equal(ax, xmin) * F.less(ax, xmax)
            idx3 = F.greater_equal(ax, xmax)
    
    # do not backpropagate gradient through indices
    idx1.need_grad = False
    idx2.need_grad = False
    idx3.need_grad = False
    
    # do not backpropagate gradient through sign
    sx.need_grad = False
    
    # take care of values outside of dynamic range
    return sx * (xmin * idx1 + quantize_pow2(ax) * idx2 + xmax * idx3)
```

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S.2.2.2 Case P2: Parametrization with respect to $b$ and $q_{\text{min}}$

```python
def parametric_pow2_quantize_b_xmin(x, sign, with_zero,
    n_init, n_min, n_max,
    xmin_init, xmin_min, xmin_max,
    fix_parameters=False):
    """ Parametric version of 'pow2_quantize' where the
    bitwidth 'n' and the smallest value 'xmin' are learnable parameters.
    ""
    Returns:
    """nnabla.Variable: N-D array.
    """
    def clip_scalar(v, min_value, max_value):
        return F.minimum_scalar(F.maximum_scalar(v, min_value), max_value)
    def broadcast_scalar(v, shape):
        return F.broadcast(F.reshape(v, (1,) * len(shape), inplace=False), shape=shape)
    def quantize_pow2(v):
        return 2 ** F.round(F.log(F.abs(v)) / np.log(2.))
    n = get_parameter_or_create("n", (),
        ConstantInitializer(n_init),
        need_grad=True,
        as_need_grad=not fix_parameters)
    xmin = get_parameter_or_create("xmin", (),
        ConstantInitializer(xmin_init),
        need_grad=True,
        as_need_grad=not fix_parameters)

    # ensure that bitwidth is in specified range and an integer
    n = F.round(clip_scalar(n, n_min, n_max))
    if sign:
        n = n - 1
    if with_zero:
        n = n - 1

    # ensure that minimum dynamic range is in specified range and a power-of-two
    xmin = quantize_pow2(clip_scalar(xmin, xmin_min, xmin_max))

    # compute min/max value that we can represent
    xmax = xmin * (2 ** ((2 ** n) - 1))

    # broadcast variables to correct size
    xmin = broadcast_scalar(xmin, shape=x.shape)
    xmax = broadcast_scalar(xmax, shape=x.shape)

    # if unsigned, then quantize all negative values to zero
    if not sign:
        x = F.relu(x)

    # compute absolute value/sign of input
    ax = F.abs(x)
    sx = F.sign(x)

    if with_zero:
        # prune smallest elements (in magnitude) to zero if they are smaller
        # than '$x_{\text{min}} / \sqrt{2}'$
        x_threshold = xmin / np.sqrt(2)
        idx1 = F.greater_equal(ax, x_threshold) * F.less(ax, xmin)
        idx2 = F.greater_equal(ax, xmin) * F.less(ax, xmax)
        idx3 = F.greater_equal(ax, xmax)
        else:
            idx1 = F.less(ax, xmin)
            idx2 = F.greater_equal(ax, xmin) * F.less(ax, xmax)
            idx3 = F.greater_equal(ax, xmax)

    # do not backpropagate gradient through indices
    idx1.need_grad = False
    idx2.need_grad = False
    idx3.need_grad = False

    # do not backpropagate gradient through sign
    ax.need_grad = False

    # take care of values outside of dynamic range
    return ax * (xmin * idx1 + quantize_pow2(ax) * idx2 + xmax * idx3)
```
S.2.2.3 Case P3: Parametrization with respect to $q_{\text{min}}$ and $q_{\text{max}}$

```python
def parametric_pow2_quantize_xmin_xmax(x, sign, with_zero, 
    xmin_init, xmin_min, xmin_max, 
    xmax_init, xmax_min, xmax_max, 
    fix_parameters=False):
    # Parametric version of 'pow2_quantize' where the
    # min value 'xmin' and max value 'xmax' are learnable parameters.
    Returns:
    nnabla.Variable: N-D array.

def clip_scalar(v, min_value, max_value):
    return F.minimum_scalar(F.maximum_scalar(v, min_value), max_value)

def broadcast_scalar(v, shape):
    return F.broadcast(F.reshape(v, (1,) * len(shape), inplace=False), shape=shape)

def quantize_pow2(v):
    return 2. ** F.round(F.log(F.abs(v)) / np.log(2.))

xmin = get_parameter_or_create("xmin", (), 
    ConstantInitializer(xmin_init), 
    need_grad=True, 
    as_need_grad=not fix_parameters)
xmax = get_parameter_or_create("xmax", (), 
    ConstantInitializer(xmax_init), 
    need_grad=True, 
    as_need_grad=not fix_parameters)

# ensure that minimum dynamic range is in specified range and a power-of-two
xmin = quantize_pow2(clip_scalar(xmin, xmin_min, xmin_max))

# ensure that minimum dynamic range is in specified range and a power-of-two
xmax = quantize_pow2(clip_scalar(xmax, xmax_min, xmax_max))

# broadcast variables to correct size
xmin = broadcast_scalar(xmin, shape=x.shape)
xmax = broadcast_scalar(xmax, shape=x.shape)

# if unsigned, then quantize all negative values to zero
if not sign:
    x = F.relu(x)

# compute absolute value/sign of input
ax = F.abs(x)
sx = F.sign(x)

if with_zero:
    # prune smallest elements (in magnitude) to zero if they are smaller
    # than 'x_min / \sqrt(2)'.
    x_threshold = xmin / np.sqrt(2)
    idx1 = F.greater_equal(ax, x_threshold) * F.less(ax, xmin)
    idx2 = F.greater_equal(ax, xmin) * F.less(ax, xmax)
    else:
        idx1 = F.less(ax, xmin)
        idx2 = F.greater_equal(ax, xmin) * F.less(ax, xmax)
        idx3 = F.greater_equal(ax, xmax)

# do not backpropagate gradient through indices
idx1.need_grad = False
idx2.need_grad = False
idx3.need_grad = False

# do not backpropagate gradient through sign
sx.need_grad = False

# take care of values outside of dynamic range
return sx * (xmin * idx1 + quantize_pow2(ax) * idx2 + xmax * idx3)
```

References

[1] Sony: Neural Network Libraries (NNabla). (https://github.com/sony/nnabla)