Properties of Free Multiplicative Convolution

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Abstract
For given two Borel probability measures $\mu$ and $\nu$ on $\mathbb{R}_+ = [0, \infty)$, we derive properties of the free multiplicative convolution $\mu \boxtimes \nu$ via its Cauchy-Stieltjes transform. In particular we prove that $\mu \boxtimes \nu$ always has no singular continuous part and, under certain conditions, that the density of its absolutely continuous part is bounded by $x^{-1}$. We also consider a special case in which $\mu$ and $\nu$ are compactly supported Jacobi measures on $(0, \infty)$ having power law behavior with exponents in $(-1, 1)$. In this case, we prove that $\mu \boxtimes \nu$ is another such Jacobi measure whose density has square root decay at the edges of its support.

1 Introduction
The notion of free independence, introduced by Voiculescu in [29], has been the main object of numerous papers recently [27, 12, 18, 11, 16], especially after the discovery of its connection to random matrix theory in [31]. Specifically, free probability provides us a method to calculate the limiting spectral distribution of random matrix ensembles of the form $X + U^* Y U$ and $X^{1/2} U^* Y U X^{1/2}$ where $X$ and $Y$ have given limiting spectral distributions and $U$ is the Haar unitary matrix. The limiting distributions of such ensembles are given by the free additive and multiplicative convolutions (denoted respectively by $\boxplus$ and $\boxtimes$), which are the distributions of the sum and the product of two free independent random variables with given distributions, and therefore the convolutions themselves have been extensively studied. The purpose of this note is to derive certain properties of free multiplicative convolution of two probability measures on $[0, \infty)$.

As shown in [6, 7, 8], one of the most interesting features of the free additive and multiplicative convolutions is their regularity and, surprisingly, such regularity is not affected by the measures constituting convolutions unless one of them is a single point mass. The following typical example shows how this phenomenon distinguishes the free multiplicative convolution from the classical convolution: consider the probability measure $\mu = \delta_0/2 + \delta_2/2$ and two random variables $X$ and $Y$ both having distribution $\mu$. If $X$ and $Y$ are classically independent, the distribution of $XY$ is $3\delta_0/4 + \delta_4/4$, whereas the free multiplicative convolution $\mu \boxtimes \mu$, the distribution of $X^{1/2} Y X^{1/2}$ when $X$ and $Y$ are free, has absolutely continuous part and is explicitly given by

$$\mu \boxtimes \mu = \frac{1}{2} \delta_0(dx) + \frac{1}{2\pi} \sqrt{\frac{1}{x(x-4)}} 1_{(0,4)}(x) dx.$$ (1.1)
For generic measures $\mu$ and $\nu$, unlike the example above, it is hard to find any explicit formula of free multiplicative convolution. One of the reasons can be found in \cite{30}, namely, that all we can explicitly derive about $\mu \boxtimes \nu$ is its $S$-transform, whose definition involves the inverse mapping of the Cauchy-Stieltjes transform. Therefore calculating $\mu \boxtimes \nu$ amounts to solving the equation satisfied by its Stieltjes transform, and the equation itself is complex as it also includes the $S$-transforms of $\mu$ and $\nu$. Thus it is hard to derive microscopic properties of $\mu \boxtimes \nu$, such as regularity or asymptotic behavior of its density, which are often required in analysis of random matrices. In particular, when we consider deformed Wigner or sample covariance matrices so that $\mu$ is either Wigner’s semicircle law or Marchenko-Pastur distribution, limiting spectral measure having density with square root decay at the edges (see Theorem 2.4 for precise statements) is now known to be deeply related to edge universality, namely that the fluctuation of maximal eigenvalue follows Tracy-Widom distribution regardless of the distribution of matrix elements. For precise results and proofs of edge universality for deformed random matrix ensembles and how the square root behavior of density is related, we refer to \cite{28} and \cite{21} for deformed Gaussian unitary ensemble and deformed Wigner matrix, \cite{23} for sample covariance matrix with general population, and \cite{19} for sum of random projections. Also, when the square root behavior fails, the limiting distribution of largest eigenvalue of deformed Wigner matrix was covered in \cite{22}, which is different from the Tracy-Widom distribution.

As mentioned above, qualitative analysis of the free convolutions is even harder when both of the measures $\mu$ and $\nu$ are general. Nevertheless, a complex analytic method that can handle such difficulties was first introduced in \cite{32}, often referred as (analytic) subordination functions. In this paper, for compactly supported $\mu$ and $\nu$, Voiculescu defined the subordination functions to be the analytic self-maps $\omega_\mu$ and $\omega_\nu$ of the upper half plane $\mathbb{C}_+$ satisfying $m_{\mu\boxtimes\nu}(z) = m_\mu(\omega_\nu(z)) = m_\nu(\omega_\mu(z))$ where $m_\rho$ denotes the Stieltjes transform of probability measure $\rho$ on $\mathbb{R}$. Using the subordination functions, Voiculescu proved the monotonicity of $L^p$ norm of densities for free additive convolution, or more precisely, $\|\text{Im}\ m_{\mu\boxtimes\nu}(\bullet + i\epsilon)\|_p \leq \|\text{Im}\ m_\mu(\bullet + i\epsilon)\|_p$. Later, Biane \cite{13} showed the existence and uniqueness with full generality, also including multiplicative convolution, using both operator-algebraic and combinatorial approaches.

Other than the original definition of subordination functions, that is, the analytic continuation of $\omega_\mu = m_\nu^{-1} \circ m_{\mu\boxtimes\nu}$ in case of free additive convolution, Belinschi and Bercovici found a completely complex analytic approach in \cite{10}. They characterized the subordination functions of free convolutions as the attracting fixed point (or equivalently, Denjoy–Wolff point) of a complex analytic function on $\mathbb{C}_+$, and in fact their characterization has been used as an alternative definition of free convolutions itself by many authors. This approach, especially combined with the theory of boundary behavior of analytic self maps of $\mathbb{C}_+$, has been turned out to be useful in a sequence of papers \cite{4,7,8,9} by Belinschi. He mainly used the theory of cluster points, such as Seidel's theorem (see e.g. \cite{15}) and Lemma 3.6 in the present paper, to analyze the boundary behavior of Stieltjes transform of free convolutions, thereby proving various properties of the measures.

We also follow similar lines of proof for our first two main results, Theorems 2.3 and 2.4. Theorem 2.3 states that the singular continuous part of $\mu \boxtimes \nu$ is always zero provided neither of the factors is degenerate. Furthermore, Theorem 2.4 states that when $\mu \boxtimes \nu$ has no atoms and Stieltjes transforms of $\mu$ and $\nu$ behave continuously around 0 and $\infty$, the density is bounded by $1/x$ on $(0, \infty)$.

Having a closer look at the limiting distributions of various self-adjoint random matrix ensembles which involve matrix multiplication (see Table 1 for instance), we find that many of the distributions have density with square root decay at the upper edge. The last part of our paper proves, provided that both of the factors are compactly supported Jacobi measures on $(0, \infty)$ having power law behavior with exponents in $(-1, 1)$, that $\mu \boxtimes \nu$ is supported on a single interval and its density also has square root decay at the lower and upper edges. In particular by \cite{1}, an immediate corollary of our result is that as long as $t < 1$ and $n \in \mathbb{N}$, $(\mu^{\otimes t})^{\boxtimes n}$ always has an interval support and its density decays as square root at both edges, where $\pi$ is the free Poisson law. We also remark that an analogous result about free additive convolution was proved by Bao, Erdős, and Schnelli in \cite{4}, whose conclusion was required to prove optimal local law of multiplication
of random matrices at the spectral edges in [31] by the same authors.

Our proof mainly concerns the boundary behavior of $M$-functions of the measures $\mu, \nu$ and their free multiplicative convolution. To be specific, the function $M_\mu$ is defined simply as $\tau \circ \eta_\mu \circ \tau$ where $\tau(z) = z^{-1}$ and $\eta_\mu$ is the analytic self-map of $\mathbb{C} \backslash \mathbb{R}_+$ which has been conventionally used in the context of free probability (the $S$-transform is the inverse of $\eta$-transform); see [10]. Despite of being a simple conjugate of previously known transform, by introducing $M_\mu$, we find a particular similarity between the free additive and multiplicative convolutions along the proof of Theorem 2.6. Furthermore, we expect the $M$-function to be useful in later researches as it is more directly related to the Stieltjes transform than $\eta$-transform.

The paper consists of 6 sections in total. In Section 2, we state our main results, Theorems 2.3, 2.4, and 2.6. Also the same section contains new transform $M_\mu$ of probability measure on $\mathbb{R}_+$, which is then used to define new subordination functions correspondingly. Section 3 is dedicated to preliminary results about boundary behavior of generic analytic self maps of $\mathbb{C}_+$. and that of subordination functions proved in [7]. In Section 4 and 5, we use these results to prove Theorem 2.6 and 2.4 respectively. Finally, Theorem 2.6 is proved in Section 6.

Notational Remark 1.1. Throughout the paper, we denote the closed positive real axis $[0, \infty)$ by $\mathbb{R}_+$ and the set $\{x + iy \in \mathbb{C} : y > 0\}$ of complex numbers with positive imaginary part by $\mathbb{C}_+$. Unless otherwise indicated, for any subset $A$ of the Riemann sphere $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, $\overline{A}$ denotes its closure in $\hat{\mathbb{C}}$.

### 2 Definitions and Main Results

#### 2.1 Main results

**Definition 2.1.** For a Borel probability measure $\mu$, we denote its Lebesgue decomposition by

$$\mu = \mu^{pp} + \mu^{sc} + \mu^{ac},$$

where $\mu^{pp}$, $\mu^{sc}$, and $\mu^{ac}$ are the point mass, singular continuous, and absolutely continuous parts of $\mu$, respectively.

We first recall the definition of Stieltjes transform and define the $M$-function:

**Definition 2.2.** For a probability measure $\mu$ on $\mathbb{R}_+$, the Stieltjes transform $m_\mu$ of $\mu$ is the analytic self-map of $\mathbb{C} \backslash \mathbb{R}_+$ defined by

$$m_\mu(z) = \frac{1}{z} \frac{1}{x - z} d\mu(x), \quad z \in \mathbb{C} \backslash \mathbb{R}_+.\tag{2.2}$$

### Table 1:

| Distribution                  | Random matrix | Density                                      |
|------------------------------|---------------|----------------------------------------------|
| Marchenko-Pastur[21] (free Poisson[23]) | $X_1X_1^*$     | $\frac{1}{2\pi} \sqrt{\frac{4 - x}{x}} \mathbb{I}_{[0,4]}(x)$ |
| Kesten[17] (Free Meixner[14]) | $(U_1 + \cdots + U_k)(U_1 + \cdots + U_k)^*$ | $\frac{1}{2\pi} \sqrt{4k(k-1)x - k^2x^2}$ |
| Bures[24]                    | $(1 + U_1)XX^*(1 + U_1^*)$              | $\frac{\sqrt{3}}{\pi} \left[ (\frac{1}{x} + \sqrt{\frac{1}{x^2} + \frac{x}{y}}) - (\frac{1}{x} - \sqrt{\frac{1}{x^2} - \frac{1}{y^2}}) \right] \frac{\sqrt{2\sqrt{3}(\frac{\sqrt{27} + 3\sqrt{81 - 12k}}{\pi})^2 - 6\sqrt{27}}}{12\pi} x^\frac{3}{2} (27 + 3\sqrt{81 - 12k})^\frac{1}{2}$ |
| Fuss Catalan[26] $\pi^{(2)}$ | $X_1X_2X_1^*X_2^*$                        | |

Table 1: $N \times N$ random matrices $U_m$ and $X_m$ are drawn from Haar unitary and Ginibre ensembles, respectively. The rightmost side is the limiting distributions of each ensembles, properly normalized.
We also define analytic functions $M_\mu, \eta_\mu : \mathbb{C} \setminus \mathbb{R}_+ \to \mathbb{C}$ by
\[
M_\mu(z) := 1 - \frac{1}{zm_\mu(z) + 1} \quad \text{and} \quad \eta_\mu(z) := \frac{1}{M(1/z)}, \quad z \in \mathbb{C} \setminus \mathbb{R}_+.
\] (2.3)

Remark 2.1. Noting that
\[
zm_\mu(z) + 1 = \int \frac{x}{x - z} d\mu(x),
\]
we see that $M_\mu(z) - 1$ is the negative reciprocal of Stieltjes transform of the positive (not necessarily finite) measure $xd\mu(x)$, which maps $\mathbb{C}_+$ into itself.

Remark 2.2. Denoting
\[
\psi_\mu(z) := \int_{\mathbb{R}_+} \frac{x}{x - z} d\mu(x) = -1 - \frac{1}{z} \int_{\mathbb{R}_+} \frac{1}{x - 1/z} d\mu(x) = -1 - \frac{1}{z} m_\mu \left( \frac{1}{z} \right),
\]
we obtain
\[
\eta_\mu(z) = 1 + \frac{z}{m_\mu(1/z)} = 1 - \frac{1}{\psi_\mu(z) + 1},
\]
so that $\eta$ matches the definition given in [10].

The first result concerns the Lebesgue decomposition of $\mu \boxtimes \nu$, in analogy with the result of [8]:

**Theorem 2.3.** Let $\mu$ and $\nu$ be Borel probability measures on $\mathbb{R}_+$. Then

(i) $\mu \boxtimes \nu$ has an atom at $c \in (0, \infty)$ if and only if there exists $u, v \in (0, \infty)$ with $uv = c$ and $\mu(\{u\}) + \nu(\{v\}) > 1$. In this case, $(\mu \boxtimes \nu)(\{c\}) = \mu(\{u\}) + \nu(\{v\}) - 1$.

(ii) $(\mu \boxtimes \nu)(\{0\}) = \max(\mu(\{0\}), \nu(\{0\}))$.

(iii) The singular continuous part of $\mu \boxtimes \nu$ is zero.

(iv) The density $\frac{d(\mu \boxtimes \nu)^\text{ac}(x)}{dx}$ of absolutely continuous part of $\mu \boxtimes \nu$ is analytic whenever positive and finite, that is, there exists a Lebesgue measurable function $f : \mathbb{R} \to \mathbb{R}$ with $f(x)dx = d(\mu \boxtimes \nu)^\text{ac}(x)$ such that for any $x \in \mathbb{R}$ with $f(x) > 0$, $f$ is analytic in a neighborhood of $x$.

Remark 2.3. The first two assertions were proved in [7] and included for the sake of completeness. Also the proof of (iv) is contained in [8], but we propose a proof using $M$-functions.

Our second result is about the continuity and bound of density of the absolutely continuous part $(\mu \boxtimes \nu)^\text{ac}$.

**Theorem 2.4.** Let $\mu$ and $\nu$ be probability measures on $\mathbb{R}_+$ such that $M_\mu$ and $M_\nu$ extend continuously to 0 and $\infty$ with values
\[
M_\mu(\infty) = M_\nu(\infty) = \infty,
\]
\[
M_\mu(0) = -\frac{\mu(\{a\})}{1 - \mu(\{0\})}, \quad M_\nu(0) = -\frac{\nu(\{b\})}{1 - \nu(\{0\})}.
\] (2.7)
Furthermore, assume that $\mu(\{a\}) + \nu(\{b\}) < 1$ for all $a, b \in (0, \infty)$. Then the density $f$ of $(\mu \boxtimes \nu)^\text{ac}$ is continuous on $(0, \infty)$ and $xf(x)$ is uniformly bounded on $(0, \infty)$.

Remark 2.4.  
(i) Theorem 2.3 (i) readily proves that $\mu \boxtimes \nu$ can have point mass only at 0 under the assumptions of Theorem 2.4.

(ii) In [9] the author proved continuity and boundedness of density for the additive convolution $\mu \boxplus \nu$ under the absence of atoms. As it will be evident along the proof in Section 5, the crucial difference is that our bound is $C/x$, not a constant. In particular recalling the result of [11], the free Bessel law $\pi_{n1} = \pi_{2n}$, where $\pi$ is the free Poisson distribution, has density diverging at 0 with order $x^{-1+1/(n+1)}$. Therefore the bound $C/x$ is optimal at least around 0.
Now we give the assumption for Theorem 2.6.

**Assumption 2.5.** Let \( \mu \) and \( \nu \) be compactly supported probability measure on \( (0, \infty) \) with means 1. Furthermore, we assume that \( \mu \) and \( \nu \) are absolutely continuous with respect to the Lebesgue measure with densities \( f_\mu \) and \( f_\nu \), respectively, and that the density functions satisfy the following:

(i) Both of \( f_\mu \) and \( f_\nu \) have single non-empty intervals as supports, denoted by \( [E_\mu^-, E_\mu^+] \) and \( [E_\nu^-, E_\nu^+] \), respectively.

(ii) The density functions have a power law behavior: there are constants \( t_\mu^+, t_\nu^+ \in (-1, 1) \) such that

\[
C^{-1} \leq \frac{f_\mu(x)}{(x - E_\mu^-)^{t_\mu^+} (E_\mu^+ - x)^{t_\mu^-}} \leq C, \quad \text{for a.e. } x \in [E_\mu^-, E_\mu^+],
\]

\[
C^{-1} \leq \frac{f_\nu(x)}{(x - E_\nu^-)^{t_\nu^+} (E_\nu^+ - x)^{t_\nu^-}} \leq C, \quad \text{for a.e. } x \in [E_\nu^-, E_\nu^+]
\]

holds for some constant \( C > 1 \).

**Remark 2.5.** By Theorem 2.3, under Assumption 2.5, the free multiplicative convolution \( \mu \boxtimes \nu \) is absolutely continuous with respect to the Lebesgue measure and its density is analytic whenever positive and finite.

**Remark 2.6.** We remark that the assumptions on power laws of \( \mu \) and \( \nu \) can be weakened to more technical divergence conditions around the edges of supports (See Remark 6.3).

**Theorem 2.6.** Let \( \mu \) and \( \nu \) satisfy Assumption 2.5 and let \( f \) be the density of \( \mu \boxtimes \nu \). Then \( \mu \boxtimes \nu \) is also supported on a single compact interval in \( (0, \infty) \), denoted by \( [E_-, E_+] \). Moreover, there exists a constant \( C > 1 \) such that

\[
C^{-1} \leq \frac{f(x)}{\sqrt{x - E_- \sqrt{E_+ - x}}} \leq C, \quad \forall x \in [E_-, E_+].
\]

### 2.2 Subordination in free multiplicative convolution

**Theorem 2.7** (Theorem 3.3 of [10]). For two probability measure \( \mu \) and \( \nu \) on \( \mathbb{R}_+ \) both not \( \delta_0 \), there exist unique analytic functions \( \omega_\mu, \omega_\nu : \mathbb{C} \setminus \mathbb{R}_+ \to \mathbb{C} \setminus \mathbb{R}_+ \) such that

(i) \( \omega_\nu(0-) = \omega_\mu(0-) = 0 \),

(ii) \( \omega_\mu \) maps \( \mathbb{C}_+ \) into \( \mathbb{C}_+ \), and for every \( z \in \mathbb{C}_+ \) we have \( \omega_\mu(z) = \overline{\omega_\nu(\overline{z})} \) and \( \arg \omega_\mu(z) \geq \arg z \). The same statements hold also for \( \omega_\nu \).

(iii) For any \( z \in \mathbb{C} \setminus \mathbb{R}_+ \), \( \eta_\mu \bar{\omega}_\nu(z) = \eta_\nu \omega_\mu(z) = \eta_\mu(\omega_\nu(z)) \).

(iv) \( \omega_\mu(z) \omega_\nu(z) = z \eta_\mu \bar{\eta}_\nu(z) \).

As explained in Section 1, we conjugate the subordination functions and the \( \eta \)-transforms by the inversion \( z \mapsto z^{-1} \) to give another, yet equivalent, definition of subordination functions.

**Proposition 2.8.** Let \( \mu \) and \( \nu \) be probability measures on \( \mathbb{R}_+ \), both not \( \delta_0 \). Define two analytic functions \( \Omega_\mu \) and \( \Omega_\nu \) on \( \mathbb{C} \setminus \mathbb{R}_+ \) as follows:

\[
\Omega_\mu(z) := \frac{1}{\omega_\mu(1/z)} \quad \text{and} \quad \Omega_\nu(z) := \frac{1}{\omega_\nu(1/z)},
\]

where \( \omega_\mu, \omega_\nu \) are as given in Theorem 2.7. Then \( \Omega_\mu \) and \( \Omega_\nu \) map \( \mathbb{C}_+ \) into itself and \( \Omega_\mu, \Omega_\nu \) satisfy the following:
Proof. By the remark above, we get
\[
\Im \frac{1}{\eta_\mu(1/z)} = \Im \frac{zm_\mu(z)}{zm_\mu(z) + 1} = -\Im \frac{1}{zm_\mu(z) + 1} = |zm_\mu(z) + 1|^{-2} \Im zm_\mu(z)
\]
\[
= |zm_\mu(z) + 1|^{-2} \Im \int_{\mathbb{R}^+} \frac{z}{x-z} \, \, d\mu(x) = \frac{\Im z}{|zm_\mu(z) + 1|^2} \int_{\mathbb{R}^+} \frac{x}{|x-z|^2} \, \, d\mu(x) \geq 0. \tag{2.11}
\]

Now the equation for \( \Omega \) follows from
\[
M_\mu(\Omega_\mu(z)) = \eta_\mu \left( \frac{1}{\Omega_\mu(z)} \right)^{-1} = \eta_\mu (\omega_\mu(1/z))^{-1} = \eta_\mu \omega_\nu(1/z)^{-1} = M_\mu \omega_\nu(z), \tag{2.12}
\]
and
\[
\Omega_\mu(z)\Omega_\nu(z) = \frac{1}{\omega_\mu(1/z)\omega_\nu(1/z)} = \left( \frac{1}{z \eta_\mu \omega_\nu(1/z)} \right)^{-1} = z M_\nu \omega_\nu(z). \tag{2.13}
\]
Finally, from the fact that \( \omega_\mu(\bar{z}) = \bar{\omega}_\mu(z) \), we have
\[
\arg \omega_\mu(z) = -\arg \omega_\mu(1/z) = \arg \omega_\mu(1/\bar{z}) \geq \arg(1/\bar{z}) = z \tag{2.14}
\]
whenever \( z \in \mathbb{C}_+ \). \( \Box \)

Remark 2.7. A direct consequence of Proposition 2.8 and the definition of \( M_\mu(z) \) is the following identity:
\[
\int \frac{x}{x-z} \, \, d(\mu \boxtimes \nu)(x) = zm_\mu \omega_\nu(z) + 1 = \Omega_\nu(z)m_\mu(\Omega_\nu(z)) + 1 = \int \frac{x}{x-\Omega_\nu(z)} \, \, d\mu(x), \tag{2.15}
\]
which yields
\[
\Im(zm_\mu \omega_\nu(z) + 1) = \Im \Omega_\nu(z) \int \frac{x}{|x-\Omega_\nu(z)|^2} \, \, d\mu(x) \tag{2.16}
\]
Now we can simply translate the following two results of [8, 9] concerning \( \omega_\mu \) and \( \omega_\nu \) in terms of \( \Omega_\mu \) and \( \Omega_\nu \):

Lemma 2.9 (Remark 3.3 of [7]). The subordination function \( \Omega_\mu \) and \( \Omega_\nu \) extends continuously to \((\mathbb{R} \cup \{\infty\}) \setminus \{0\}\), with values in \( \mathbb{C}_+ \).

Lemma 2.10 (Theorem 1.28 of [8]). Let \( x \in \mathbb{R} \setminus \{0\} \). If \( \Omega_\mu(x) \) or \( \Omega_\nu(x) \) is in \( \mathbb{C}_+ \), both of \( \Omega_\mu \) and \( \Omega_\nu \) extends analytically through \( x \).

3 Preliminary Results

Notational Remark 3.1. For notational simplicity, we denote \( \rho := \mu \boxtimes \nu \) in the rest of the paper.

In order to follow the scheme of [8, 9] and convert the properties of Stieltjes transform \( m_\mu(z) \) into that of \( \rho \), we require the definition of nontangential limits:
Definition 3.1. Let \( f : \mathbb{C}_+ \to \mathbb{C} \) be a function, \( c \in \mathbb{R} \), and \( \ell \in \mathcal{C} \). Define
\[
\Gamma_{\alpha,c} := \{ z \in \mathbb{C}_+ : |\text{Re} z - c| < \alpha |\text{Im} z \}
\]
(3.1)
if \( c \neq \infty \), and \( \Gamma_{\alpha,\infty} = \Gamma_{\alpha,0} \). If we have
\[
\lim_{z \to c, z \in \Gamma_{\alpha,c}} f(z) = \ell
\]
for all \( \alpha > 0 \), we say that \( f \) has nontangential limit \( \ell \) at \( c \) and write
\[
< \lim_{z \to c} f(z) = \lim_{z \to c, z \in \Gamma_{\alpha,c}} f(z) = \ell.
\]
(3.3)

If \( f \) is defined on \( \mathbb{C} \setminus \mathbb{R}_+ \), we also say that \( f \) has nontangential limit \( \ell \) at \( 0 \) in \( \mathbb{C} \setminus \mathbb{R}_+ \) if \( < \lim_{z \to 0, z \in \mathbb{C} \setminus \mathbb{R}_+} f(z) \) is equal to \( \ell \), where the square root maps \( \mathbb{C} \setminus \mathbb{R}_+ \) to \( \mathbb{C}_+ \). In this case we write
\[
\lim_{z \to 0, z \in \mathbb{C} \setminus \mathbb{R}_+} f(z) = \ell.
\]
(3.4)

The nontangential limit at \( \infty \) in \( \mathbb{C} \setminus \mathbb{R}_+ \) is defined analogously.

Our first preliminary result concerns the relationship between a probability measure \( \mu \) and the nontangential limits of \( m_\mu \) on \( \mathbb{R} \):

Lemma 3.2 (Lemma 2.17 of [8]). Let \( \mu \) be a Borel probability measure on \( \mathbb{R} \).

(i) For \( \mu^c \)-almost all \( x \in \mathbb{R} \), the nontangential limit \( < \lim_{x \to z} \text{Im} m_\mu(z) \) is infinite.

(ii) We have \( \mu(\{x\}) = -< \lim_{z \to x} (z - x) m_\mu(z) \).

(iii) If \( f(t) = \frac{d\mu^c(t)}{dt} \), for almost all \( x \in \mathbb{R} \), we have \( \pi f(x) = < \lim_{x \to z} \text{Im} m_\mu(x) \).

We will also frequently use the classical results below, concerning nontangential limits of analytic functions on the upper half plane. Most of them were about analytic functions on the unit disc in their first versions, and we can translate the results to functions on \( \mathbb{C}_+ \) using M"obius transform (or Cayley transform). For a proof, we refer to [8].

Lemma 3.3 (Theorem 2.5–2.7 of [8]).

(i) (Fatou) Let \( f : \mathbb{C}_+ \to \overline{\mathbb{C}}_+ \) be an analytic function. Then the set of points \( x \in \mathbb{R} \) at which the nontangential limit of \( f \) fails to exists in \( \mathbb{C} \) is of Lebesgue measure zero.

(ii) (Privalov) Let \( f : \mathbb{C}_+ \to \mathbb{C} \) be an analytic function. If there exists \( A \subset \mathbb{R} \) of nonzero Lebesgue measure such that the nontangential limit of \( f \) exists and equals zero at each point of \( A \). Then \( f \equiv 0 \) on \( \mathbb{C}_+ \).

(iii) (Lindel"of) Let \( f : \mathbb{C}_+ \to \mathcal{C} \) be a meromorphic function with \( |\mathcal{C} \setminus f(\mathbb{C}_+)\| > 2 \), and let \( x \in \mathbb{R} \). If there exists a path \( \gamma : [0, 1] \to \mathbb{C}_+ \) such that \( \lim_{t \to 1} \gamma(t) = x \) and \( \ell = \lim_{t \to 1} f(\gamma(t)) \) exists in \( \mathcal{C} \), then the nontangential limit of \( f \) at \( x \) exists and equals \( \ell \).

Lemma 3.4 (Lemma 2.13 of [8]). Let \( F : \mathbb{C}_+ \to \mathbb{C}_+ \) be analytic and let \( a \in \mathbb{R} \). If
\[
\lim_{z \to a} F(z) = c \in \mathbb{R},
\]
(3.5)
then
\[
\lim_{z \to a} \frac{F(z) - c}{z - a} = \lim_{z \to a} \frac{\text{Im} F(z)}{\text{Im} z},
\]
(3.6)
where the equality is considered in \( \hat{C} \). Conversely, if

\[
\liminf_{z \to a} \frac{\Im F(z)}{\Im z} < \infty, \tag{3.7}
\]

then \( \angle \lim_{z \to a} F(z) \) exists and belongs to \( \mathbb{R} \cup \{ \infty \} \). Moreover, if \( F \) is not a constant, then we have

\[
\liminf_{z \to a} \Im F(z)/\Im z > 0.
\]

Our proofs of Theorem 2.3 and 2.4 proceed by way of contradiction, and large portion of it relies on a remarkable lemma, proved and developed by Belinschi in a sequence of papers [7, 8, 9]. In order to give its precise statement, we require the definition of cluster sets:

**Definition 3.5.** For a function \( f : C_+ \to C_+ \) and \( x \in \overline{C} \), we define the cluster set of \( f \) at \( x \) as

\[
C_{C_+}(f, x) \equiv C(f, x) := \{ w \in \hat{C} : \exists \{ z_n \} \in C_+ \text{ s.t. } z_n \to x, f(z_n) \to w \}. \tag{3.8}
\]

Given the definition of cluster sets, the formal statement of the lemma is given below:

**Lemma 3.6** (Lemma 5 of [9]). Let \( f \) be a nonconstant analytic self map of \( C_+ \). Assume that \( x \in \overline{C} \) is so that \( C(f, x) \cap \mathbb{R} \) is infinite. Then there exist an interval \( [a, b] \subset \mathbb{R} \cap C(f, x) \) and a sequence of mutually disjoint segments \( \{[z_n, w_n]\}_{n \in \mathbb{N}} \subset C_+ \) so that the following holds:

(i) \( \lim_{n \to \infty} z_n = \lim_{n \to \infty} w_n = x \),

(ii) \( \lim_{n \to \infty} f(z_n) = a < b = \lim_{n \to \infty} f(w_n) \),

(iii) The sets \( \{f([z_n, w_n])\}_{n \in \mathbb{N}} \) are mutually disjoint in \( C_+ \),

(iv) For any \( c \in (a, b) \), there is an \( n_c \in \mathbb{N} \) such that

\[
f([z_n, w_n]) \cap (c + i\mathbb{R}_+) \neq \emptyset, \quad \forall n \geq n_c,
\]

(v) For any \( [c, d] \subset (a, b) \),

\[
\lim_{n \to \infty} \sup \{\max\{\Im v : v \in f([z_n, w_n]) \cap (t + i\mathbb{R}_+)\} : t \in [c, d]\} = 0, \tag{3.9}
\]

(vi) For any \( c \in \mathbb{R} \setminus [a, b] \), there is an \( n_c \in \mathbb{N} \) such that

\[
f([z_n, w_n]) \cap (c + i\mathbb{R}_+) = \emptyset, \quad \forall n \geq n_c. \tag{3.10}
\]

**Remark 3.1.** We directly see that an immediate corollary of the lemma is that whenever \( C(f, x) \cap \mathbb{R} \) is infinite, there exists an interval \( H \subset C(f, x) \cap \mathbb{R} \) of nonzero Lebesgue measure so that for any \( c \in H \), we can find a sequence \( \{z_n^{(c)}\} \) with \( z_n \to x \) and \( f(z_n) \to c \) in \( c + i\mathbb{R}_+ \) (in other words, vertically).

The last preliminary result required is the following classical representation theorem of Nevanlinna-Pick functions, which characterizes all analytic self-maps of \( C_+ \) as a Stieltjes transform of some positive Borel measure on \( \mathbb{R} \):

**Lemma 3.7** (Nevanlinna-Pick representation). Let \( F : C_+ \to C_+ \) be analytic. Then there exists unique triple \( (a, b, \rho) \) of \( a \in \mathbb{R}, b > 0 \) and a positive Borel measure \( \rho \) on \( \mathbb{R} \) such that

\[
F(z) = a + bz + \int_{\mathbb{R}} \frac{1}{x - z} - \frac{x}{1 + ax^2} d\rho(x), \quad \int_{\mathbb{R}} \frac{1}{1 + ax^2} d\rho(x) < \infty. \tag{3.11}
\]
Conversely, for any such triple \((a, b, \rho)\), the formula \((3.11)\) defines a unique analytic function \(F : \mathbb{C}_+ \to \mathbb{C}_+\). Furthermore, if the function \(F\) satisfies
\[
\sup_{\eta \geq 1} |\eta F(\eta)| < \infty,
\]
then the measure \(\rho\) above is finite and \(F\) satisfies
\[
F(z) = \int_{\mathbb{R}} \frac{1}{x - z} \, d\rho(x).
\]

4 Lebesgue Decomposition of \(\mu \otimes \nu\)

In order to prove (iii), we assume that \(\rho^{sc}\) is nonzero. Then by Lemma 3.2 there must be a Borel measurable subset \(H\) of \(\mathbb{R} \setminus \{0\}\) satisfying the following:

- \(\rho(H) = \rho^{sc}(\mathbb{R}) = \rho^{sc}(H)\).
- \(H\) is uncountable.
- The nontangential limit \(< \lim_{z \to x} \text{Im} zm_\rho(z)\) is infinite for any \(x \in H\).

Note that the first follows from the definition, the second from \(\rho^{sc} \perp \rho^{pp}\), and the last follows from (i) of Lemma 3.2 applied to the measure \(x \, d\rho^{sc}(x)\). At each point \(c \in H\), we claim the following two assertions:

(a) \(< \lim_{z \to c} \Omega_\mu(z) = \nu \) and \(< \lim_{z \to c} \Omega_\nu(z) = \nu \), where \(\nu, \nu \in \mathbb{R}\).

(b) \(\nu \nu = c\) and \(\mu(\nu) + \nu(\nu) = 1\).

We first see how the assertions above lead to a contradiction. Part (b) of the claim implies that at least one of \(\{\nu(c) : c \in H\}\) and \(\{\nu(c) : c \in H\}\) must be uncountable, and hence either \(\{\nu(c) : c \in H, \nu(\nu(c)) > 0\}\) or \(\{\nu(c) : c \in H, \mu(\nu(c)) > 0\}\) is also uncountable. Since a probability measure cannot have uncountably many atoms, we obtain contradiction.

From Lemma 2.9 we see that \(\nu\) and \(\nu\) exist in the extended complex plane. If \(\nu \in \mathbb{C}_+\), Lemma 2.10 implies that both of the subordination functions extends analytically through \(c\). Then \(M_\rho = M_\nu \circ \Omega_\mu\) also analytically extends through \(c\), and \(M_\rho(c) = M_\nu(\mu(c)) \in \mathbb{C}_+\), contradicting \(< \lim_{z \to c} \text{Im} m_\rho(z) = \infty\). Thus \(\nu, \nu \in \mathbb{R}\) and it suffices to show \(\nu, \nu \in \mathbb{R}\) to prove (a). To this end, suppose \(\nu = \infty\). Then one has
\[
1 = \lim_{y \to 0} M_\rho(c + iy) = \lim_{y \to 0} M_\nu(\Omega_\mu(c + iy)) = \lim_{z \to \infty} M_\nu(z) = \infty,
\]
where we used Lemma 3.3 (iii) in the third equality. Hence we obtain \(\nu, \nu < \infty\).

Now we turn to the proof of (b). We first observe that
\[
\lim_{z \to \infty} \left| \frac{1}{zm_\rho(z) + 1} \right| = \lim_{z \to \infty} \left| \frac{1}{\text{Im}(zm_\rho(z) + 1)} \right| = 0,
\]
from which we obtain
\[
c = \lim_{z \to \infty} zM_\rho(z) = \nu \nu.
\]

Also, we observe
\[
1 = \lim_{z \to \infty} M_\rho(z) = \lim_{z \to \infty} M_\nu(\Omega_\mu(z)) = \lim_{z \to \nu} M_\nu(z),
\]
where we again used Lemma 3.3 (iii) in the last equality. Then from Lemma 3.2 (ii) and Lemma 3.4 for any fixed \( x \in (0, \infty) \) satisfying \(< \lim_{z \to x} M_\mu(z) = 1,\)

\[
0 < \frac{1}{\mu(\{x\})} = -x \lim_{z \to x} \frac{(z - x)(zm(\mu(z) + 1))^{-1}}{z - x} = x \lim_{z \to x} \frac{M_\mu(z) - 1}{M_\mu(z) z - x} = x \lim_{z \to x} \frac{\frac{M_\mu(z)}{M_\mu(z)} - 1}{z - x} = \lim_{z \to x} \frac{\text{Im} M_\mu(z)/|M_\mu(z)|}{\text{Im} z/|z|} = \lim_{z \to x} \frac{\sin \arg M_\mu(z)}{\sin \arg z} \leq \lim_{z \to x} \frac{\arg M_\mu(z)}{\arg z}. \tag{4.5}
\]

We prove that the last inequality is actually an equality. When \( \mu(\{x\}) = 0 \) the proof is immediate as the left-hand side is infinite. On the other hand if \( \mu(\{x\}) > 0 \), there must be a sequence \( z_n \in \mathbb{C}_+ \) with \( z_n \to x \) and

\[
\liminf_{z \to x} \frac{\sin \arg M_\mu(z_n)}{\arg z_n} = \frac{1}{\mu(\{x\})}, \tag{4.6}
\]

so that \( \sin \arg M_\mu(z_n) \searrow 0 \) as \( n \to \infty \), which in turn implies \( \sin \arg M_\mu(z_n)/\arg M_\mu(z_n) \to 1 \) as \( n \to \infty \). Thus we have

\[
\liminf_{z \to x} \frac{\arg M_\mu(z)}{\arg z} \leq \liminf_{z \to x} \frac{\sin \arg M_\mu(z)}{\sin \arg z}. \tag{4.7}
\]

Now choosing \( x = v_\nu \), we get

\[
\frac{1}{\mu(\{v_\nu\})} - 1 = \liminf_{z \to v_\nu} \frac{\arg M_\mu(z) - \arg z}{\arg z} \leq \liminf_{y \to 0} \frac{\arg M_\mu(\Omega_\nu(c + iy)) - \arg \Omega_\nu(c + iy)}{\arg \Omega_\nu(c + iy)} = \liminf_{y \to 0} \frac{\arg \Omega_\nu(c + iy) - \arg(c + iy)}{\arg \Omega_\nu(c + iy)} \leq \liminf_{y \to 0} \frac{\arg \Omega_\nu(c + iy)}{\arg \Omega_\nu(c + iy)}. \tag{4.8}
\]

where we used Proposition 2.8 (iv) in the second equality. By symmetry, we also have corresponding inequality for \( 1/\nu(\{v_\mu\}) \), and multiplying two inequalities we obtain

\[
\left(\frac{1}{\mu(\{v_\nu\})} - 1\right) \left(\frac{1}{\nu(\{v_\mu\})} - 1\right) \leq 1, \tag{4.9}
\]

which implies \( \mu(\{v_\nu\}) + \nu(\{v_\mu\}) \geq 1 \). As \( \rho \) does not have point mass at \( c = v_\nu \), the first part of Theorem 2.8 gives us \( \mu(\{v_\nu\}) + \nu(\{v_\mu\}) = 1 \).

For the last part, by Lemma 3.3 (i), there exists a subset \( E \subset \mathbb{R} \) of zero Lebesgue measure such that \( 0 \in E \) and for all \( x \in \mathbb{R} \setminus E \) the nontangential limits \( \lim_{y \to 0} m_\rho(x + iy) \), \( \lim_{y \to 0} M_\rho(x + iy) \), \( \lim_{y \to 0} \Omega_\rho(x + iy) \), and \( \lim_{y \to 0} \Omega_\nu(x + iy) \) exist and are finite. Furthermore, by Lemma 3.2 (iii), we may specifically take the following function to be the density of \( \rho^\alpha \):

\[
f(x) := \begin{cases} \frac{1}{\pi} \lim_{y \to 0} \text{Im } m_\rho(x + iy) & \text{if the limit exists and } x \in \mathbb{R} \setminus E, \\ 0 & \text{otherwise.} \end{cases} \tag{4.10}
\]

As \( E \) was of zero Lebesgue measure, we have the desired result.
5 Boundedness of Density

This section is devoted to the proof Theorem 2.4 which can be derived from the following proposition:

**Proposition 5.1.** Let $\mu$ and $\nu$ be Borel probability measures on $\mathbb{R}_+$ such that $M_\mu$ and $M_\nu$ are continuous at infinity and 0, that is,

\[
C_{\mathbb{C}_+}(M_\mu, \infty) = \{\infty\} = C_{\mathbb{C}_+}(M_\nu, \infty),
\]

\[
C_{\mathbb{C}_+}(M_\mu, 0) = -\frac{\mu(\{0\})}{1-\mu(\{0\})}, \quad C_{\mathbb{C}_+}(M_\nu, 0) = -\frac{\nu(\{0\})}{1-\nu(\{0\})}.
\]

If $\{zm_\mu(z) : z \in \mathbb{C}_+\} = \{(1-M_\mu(z))^{-1} : z \in \mathbb{C}_+\}$ is unbounded, then there exists $u, v \in \mathbb{R}$ such that $\mu(\{u\}) + \nu(\{v\}) \geq 1$.

Based on the proposition above, we will prove the theorem in Section 5.3.

5.1 Behavior of subordination functions at 0

In order to prove the proposition, we need the behavior of subordination functions at two distinguished singularities, namely 0 and $\infty$. Recalling Lemma 5.3 we already know that $\Omega_\mu$ and $\Omega_\nu$ extend continuously to $\infty$ with value $\infty$. Henceforth we focus on the behavior at 0, in particular, we will prove that the subordination functions extend continuously to 0.

**Definition 5.2.** For any probability measure $\tau$ on $\mathbb{R}_+$, we denote

\[
H_\tau(z) := \frac{M_\tau(z)}{z}.
\]

**Lemma 5.3.** Within the notation of Proposition 5.1, $C(\Omega_\mu, 0) \cap \mathbb{C}_+ = C(\Omega_\nu, 0) \cap \mathbb{C}_+ = \emptyset$.

**Proof.** Suppose that $l \in C(\Omega_\mu, 0) \cap \mathbb{C}_+$, so that there exists a sequence $\{z_n\}_{n \in \mathbb{N}} \subset \mathbb{C}_+$ which satisfies $z_n \to 0$ and $\Omega_\mu(z_n) \to l$ as $n$ tends to $\infty$. First, we observe that $H_\nu(\Omega_\mu(z_n))$ has strictly positive imaginary part as $n \to \infty$ and the limit $H_\nu(l)$ is also in $\mathbb{C}_+$. Moreover we have

\[
0 < H_\nu(\Omega_\mu(z)) < \arg(zH_\nu(\Omega_\mu(z))) = \arg z + \arg H_\nu(\Omega_\mu(z)) < \pi - \arg \Omega_\mu(z) + \arg z < \pi.
\]

In particular, this implies that $z_nH_\nu(\Omega_\mu(z_n))$ converges to 0 nontangentially in $\mathbb{C}_- \setminus \mathbb{R}_+$ (we are not excluding the case in which $\{z_n\}$ is asymptotically tangent to $\mathbb{R}_+$). As $\mu$ is supported on $\mathbb{R}_+$, we obtain

\[
\lim_{n \to \infty} M_\mu(z_nH_\nu(\Omega_\mu(z_n))) = -\frac{\delta_\mu}{1-\delta_\mu},
\]

and hence

\[
l = \lim_{n \to \infty} \Omega_\mu(z_n) = \lim_{n \to \infty} \frac{M_\mu(zH_\nu(\Omega_\mu(z)))}{H_\nu(\Omega_\mu(z))} = -\frac{\delta_\mu}{H_\nu(l)(1-\delta_\mu)}.
\]

Now multiplying both sides by $H_\nu(l)$, we find that

\[
-\frac{\delta_\mu}{1-\delta_\mu} = M_\nu(l) \in \mathbb{C}_+
\]

which is contradiction as $l$ was assumed to be in $\mathbb{C}_+$.

\[\Box\]
Lemma 5.4. The subordination functions $\Omega_\mu$ and $\Omega_\nu$ have nontangential limits at 0, given by

$$
\lim_{z \to 0, z \in \mathbb{C} \setminus \mathbb{R}_+} \Omega_\mu(z) = \begin{cases} 
0 & \text{if } \nu(\{0\}) \geq \mu(\{0\}), \\
M^{-1}_\nu \left( \frac{-\mu(\{0\})}{1 - \mu(\{0\})} \right) & \text{if } \mu(\{0\}) > \nu(\{0\}), 
\end{cases}
$$

(5.7)

Also they have nontangential limits $\infty$ at $\infty$.

Proof. Considering the function $z \mapsto \Omega_\mu(z^2)^{1/2}$ which is an analytic self map of $\mathbb{C}_+$ where the square root maps $\mathbb{C} \setminus \mathbb{R}_+$ to $\mathbb{C}_+$, Lemma 3.3 (iii) implies that if the limits

$$
\lim_{z \to 0} \Omega_\mu(z) \quad \text{and} \quad \lim_{z \to -\infty} \Omega_\mu(z)
$$

exist, they must be the nontangential limits. Thus we can restrict our attention to the limits along negative real axis.

Now we see that for any probability measure $\tau$ on $\mathbb{R}_+$, the function $M_\tau$ is strictly increasing analytic function on $(-\infty, 0)$ as

$$
M'_\tau(z) = \left( \int \frac{x}{x - z} \, d\tau(x) \right)^{-2} \left( \int \frac{x}{(x - z)^2} \, d\tau(x) \right) \in (0, 1), \quad \forall z \in (-\infty, 0).
$$

(5.9)

Also, the image $M_\tau(-\infty, 0)$ is precisely $(-\infty, -\frac{\tau(\{0\})}{\tau(\{0\}) - 1})$. In particular, $M_\tau(z)$ restricted to $(-\infty, 0)$ has an analytic, strictly increasing inverse on $(-\infty, -\frac{\tau(\{0\})}{\tau(\{0\}) - 1})$.

Now for $\lim_{z \to 0}\Omega_\mu(z)$, Theorem 2.3 (ii) implies

$$
M_{\mu \Omega_\nu}(-\infty, 0) = (-\infty, -\frac{\mu(\{0\})}{1 - \mu(\{0\})}) = (-\infty, -\frac{\mu(\{0\})}{1 - \mu(\{0\})}) \cap (-\infty, -\frac{\nu(\{0\})}{1 - \nu(\{0\})}),
$$

(5.10)

so that using Proposition 2.5 (iii) we get

$$
\Omega_\mu(z) = M^{-1}_{\nu} \circ M_{\mu \Omega_\nu}(z)
$$

(5.11)

on the whole open line $(-\infty, 0)$. Thus, we obtain

$$
\lim_{z \to 0} \Omega_\mu(z) = \lim_{z \to -\frac{\mu(\{0\})}{1 - \mu(\{0\})}} M^{-1}_\nu(z) = \begin{cases} 
0 & \text{if } \rho(\{0\}) = \nu(\{0\}), \\
M^{-1}_\nu \left( \frac{-\rho(\{0\})}{1 - \rho(\{0\})} \right) & \text{if } \rho(\{0\}) > \nu(\{0\}), 
\end{cases}
$$

(5.12)

and similarly

$$
\lim_{z \to -\infty} \Omega_\mu(z) = \lim_{z \to -\infty} M^{-1}_\nu(z) = -\infty.
$$

(5.13)

Proposition 5.5. The subordination functions $\Omega_\mu$ and $\Omega_\nu$ are continuous at 0.

Proof. Suppose $\mathcal{C}(\Omega_\mu, 0) \subset \mathbb{R}$ is infinite. Then for any $c \in \mathcal{C}(\Omega_\mu, 0) \setminus \{0, \infty\}$ for which the nontangential limit $\lim_{z \to c} M_{\mu}(c)$ exists and is finite, we use Lemma 3.0 to take a sequence $\{z_n^{(c)}\}_{n \in \mathbb{N}}$ such that $z_n^{(c)} \to 0$, $\Omega_{\mu}(z_n^{(c)}) \to c$, $\Omega_{\mu}(z_n^{(c)}) \in c + i \mathbb{R}_+$. Then

$$
\frac{M_{\mu}(\Omega_{\mu}(z_n^{(c)}))}{\Omega_{\mu}(z_n^{(c)})} \to 0,
$$

(5.14)
and hence we obtain
\[
\lim_{z \to c} M_\nu(z) = \lim_{n \to \infty} M_\nu(\Omega_\mu (z_n)) = \lim_{n \to \infty} M_\mu(\Omega_\nu (z_n))
\]
\[
= \lim_{n \to \infty} M_\mu \left( z_n (\frac{M_\nu(\Omega_\mu (z_n))}{\Omega_\mu (z_n)}) \right) = \lim_{z \to 0} M_\mu(z) = -\frac{\mu(\{0\})}{1 - \mu(\{0\})},
\]
which is a contradiction by Lemma 3.3 (ii). The continuity of \( \Omega_\nu \) follows from the same proof.

\[\square\]

5.2 Proof of Proposition 5.1

Suppose on the contrary that there exists a sequence \( \{z_n\}_{n \in \mathbb{N}} \) in \( \mathbb{C}_+ \) such that \( \lim_{n \to \infty} M_\rho(z_n) = 1 \). Taking a subsequence, we assume that the sequence \( \{z_n\} \) converges to \( c \in \mathbb{C}_+ \). Also, since \( M_\rho \) is analytic in \( \mathbb{C} \setminus \mathbb{R}_+ \) with
\[
1 - M_\rho(z) = \left( \int \frac{x}{x - z} d\rho(x) \right)^{-1} > 1, \quad \forall z \in (-\infty, 0),
\]
\( M_\rho(\mathbb{C}_+) \subset \mathbb{C}_+ \), and \( M_\rho(z) = M_\rho(z) \), we may assume that \( c \in [0, \infty) \). We show that \( c = 0 \) and \( \infty \) lead to contradiction, and \( c \in (0, \infty) \) implies that \( \mu(\{v\}) + \nu(\{u\}) \geq 1 \).

5.2.1 \( c = 0 \) or \( \infty \)

We first assume \( c = 0 \). Letting \( \mu(\{0\}) \geq \nu(\{0\}) \) without loss of generality, by Proposition 5.5, we see that \( \lim_{n \to \infty} \Omega_\nu(z_n) = 0 \). Using the continuity of \( M_\mu \), this in turn implies
\[
\lim_{n \to \infty} M_\rho(z_n) = \lim_{n \to \infty} M_\mu(\Omega_\nu(z_n)) = \lim_{z \to 0} M_\mu(z) = -\frac{\mu(\{0\})}{1 - \mu(\{0\})} \neq 1.
\]
Now we assume \( c = \infty \). Again by Lemma 2.9 and 2.4, we have \( \lim_{n \to \infty} \Omega_\nu(z_n) = \infty \) so that the continuity of \( M_\mu \) at \( \infty \) implies
\[
\lim_{n \to \infty} M_\rho(z_n) = \lim_{n \to \infty} M_\mu(\Omega_\nu(z_n)) = \lim_{z \to \infty} M_\mu(z) = \infty.
\]

5.2.2 \( c \in (0, \infty) \)

Now we assume \( c > 0 \). From Lemma 2.9, we already know that
\[
\Omega_\mu(c) := \lim_{z \to c} \Omega_\mu(z) \quad \text{and} \quad \Omega_\nu(c) := \lim_{z \to c} \Omega_\nu(z)
\]
both exists. In particular, as
\[
\arg \Omega_\mu(z_n) \leq \arg M_\rho(z_n) \to 0,
\]
we have \( \Omega_\mu(c) \in [0, \infty] \) and the same argument shows \( \Omega_\nu(c) \in [0, \infty] \).

If \( \Omega_\mu(c) \) were infinity or \( 0 \), again the continuity of \( M_\nu \) would imply
\[
\infty = \lim_{z \to \infty} M_\nu(z) = \lim_{n \to \infty} M_\nu(\Omega_\mu(z_n)) = \lim_{n \to \infty} M_\rho(z_n(c)) = 1,
\]
\[
\frac{\nu(\{0\})}{1 - \nu(\{0\})} = \lim_{z \to 0} M_\nu(z) = \lim_{n \to \infty} M_\nu(\Omega_\mu(z_n)) = \lim_{n \to \infty} M_\rho(z_n) = 1,
\]
leading to contradiction. Similarly \( \Omega_\nu(c) \in (0, \infty) \). Then Proposition 2.8 (iv) together with Lemma 2.9 implies \( M_\rho \) also continuously extends to \( c \) with value 1. Thus
\[
\lim_{n \to 0} M_\rho(\Omega_\mu(c + i\eta)) = \lim_{n \to 0} M_\rho(c + i\eta) = 1,
\]

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and Lemma 3.3 (iii) implies existence of the nontangential limits

\[ \lim_{z \to \Omega_\mu(z)} M_\mu(z) = \lim_{z \to \Omega_\nu(c)} M_\mu(z) = 1. \] (5.23)

Given the existence of nontangential limits, Lemma 3.3 implies

\[ \lim_{z \to \Omega_\nu(c)} \frac{M_\mu(z) - 1}{z - \Omega_\mu(c)} = \liminf_{z \to \Omega_\nu(c)} \frac{\arg M_\mu(z)}{\arg z} \] (5.24)

and similar equality for \( \Omega_\nu \) and \( M_\mu \). Following the lines below (4.5), we again obtain

\[ \mu(\{\Omega_\nu(c)\}) + \nu(\{\Omega_\mu(c)\}) \geq 1. \] (5.25)

5.3 Proof of Theorem 2.4

By Proposition 5.1, \( \{zm_\mu(z)\} : z \in \mathbb{C}_+ \) is bounded, so that \( \frac{1}{z} M_\mu(z) \) is bounded by Stieltjes inversion.

For the continuity, we first focus on the point 0. By Lemma 5.4 and Proposition 5.5, both of \( \Omega_\mu \) and \( \Omega_\nu \) are continuous at 0 and either of them must have value 0 at 0. Thus Proposition 2.8 (iii) implies that \( C(M_\mu(z), 0) = 1 \).

Now for \( c > 0 \), Lemma 2.9 and Proposition 2.8 (iv) implies that \( \left| C(M_\mu(z), c) \right| = 1 \). Note that we are allowing the value \( \infty \) for \( M_\mu \), but nevertheless \( \mathbb{R} \ni t \mapsto tm_\mu(t) \) is a well defined continuous function being bounded.

6 Support and Edge Behavior of the Free Multiplicative Convolution

In this section we focus on the proof of Theorem 2.6. The most important ingredient is Proposition 6.10, which states that \( \Omega_\mu \) and \( \Omega_\nu \) stay away from \( \text{supp} \nu \) and \( \text{supp} \mu \), respectively. Such phenomenon, in particular the lower bound of the distance between \( \Omega_\mu(z) \) and \( \text{supp} \nu \), is often called the stability bound, as it is directly related to the stability of subordination equations in Proposition 2.8: see [20, 2] for an instance in free additive convolution. In the remaining two sections, we utilize the stability to prove our theorem.

Notational Remark 6.1. As in the previous sections, we also denote \( \rho := \mu \boxtimes \nu \) for simplicity. Also by Remark 2.6, the measure \( \rho \) is absolutely continuous and we abuse the notation to denote its density by \( \rho \) as well. Specifically, we take the following function as the density:

\[ \rho(x) := \begin{cases} \frac{1}{\pi} \lim_{y \to 0} \text{Im} \ m_\rho(x + iy) & \text{if the limit exists}, \\ 0 & \text{otherwise.} \end{cases} \] (6.1)

In fact, we shall see that the limit exists everywhere.

Along the proof, several difficulties arise within calculation which stem from the fact that \( M_\mu(z) \) contains the reciprocal of Stieltjes transform of \( \mu \), not the transform itself. Following [4], a typical application of Nevanlinna-Pick representation to \( M_\mu(z) \) enables us to conceive \( M_\mu(z) \) as a Stieltjes transform of another, yet closely related, measure. Recalling the fixed point approach in [10], we see that \( \arg \Omega_\mu(z), \arg \Omega_\nu(z) \geq \arg z \) fundamentally accounts to the inequality \( \arg M_\mu(z), \arg M_\nu(z) \geq \arg z \). Thus the most natural function to which we should apply Nevanlinna representation, in order to have a closer look at the density via Lemma 3.4, must be \( M_\mu(z)/z \).

Lemma 6.1. For any probability measure \( \tau \) on \( \mathbb{R}_+ \) which is not a point mass, \( \frac{1}{z} M_\tau(z) \) maps \( \mathbb{C}_+ \) into itself.
Proof. We directly see that
\[
\text{Im} \left[ \frac{1}{z} M_\nu(z) \right] = \text{Im} \frac{m_\nu(z)}{z m_\nu(z) + 1} = \frac{1}{|zm_\nu(z) + 1|^2} \text{Im} \left[ \overline{z} |m_\nu(z)|^2 + m_\nu(z) \right] = \frac{\text{Im} z}{|zm_\nu(z) + 1|^2} \left[ \int \frac{1}{|x - z|^2} d\tau(x) - \int \frac{1}{x - z} d\tau(x) \right] \geq 0, \quad (6.2)
\]
and the equality holds for some \( z \in \mathbb{C}_+ \) if and only if \( 1/(x - z) \) is a constant \( \tau \)-a.e. or, equivalently, \( \tau \) is a point mass.

Now we present below the representation of \( M_\mu(z) \):

**Lemma 6.2** (Nevanlinna-Pick representation of \( M_\mu(z) \)). Let \( \mu \) and \( \nu \) be the probability measures on \( \mathbb{R}_+ \) satisfying Assumption \( \mathbb{R}_+ \). Then there exist unique finite Borel (not necessarily probability) measures \( \widehat{\mu} \) and \( \widehat{\nu} \) on \( \mathbb{R}_+ \) such that
\[
\frac{M_\mu(z)}{z} = 1 + \int_{\mathbb{R}_+} \frac{1}{x - z} d\widehat{\mu}(x), \quad \widehat{\mu}(\mathbb{R}) = \text{Var} [\mu] = \int x^2 d\mu(x) - 1, \quad \text{and} \quad \text{supp} \mu = \text{supp} \widehat{\mu},
\]
and the same set of equality holds for \( \widehat{\nu} \).

Proof. We denote \( \mu \) by \( \mu \) for simplicity, and the same argument proves the assertion for \( \nu \). By the last assertion of Lemma \( \mathbb{R}_+ \) as \( \frac{1}{z} M_\mu(z) \) maps \( \mathbb{C}_+ \) into itself, it suffices to check that
\[
\sup_{\eta \geq 1} |M_\mu(\eta i) - \eta| < \infty \quad (6.4)
\]
to prove the first assertion.

We calculate the limit as follows:
\[
\lim_{\eta \to \infty} \eta \left( \frac{M(\eta i)}{\eta i} - 1 \right) = \lim_{\eta \to \infty} \frac{\text{Im} m_\mu(\eta i) - \eta + \eta^2 m_\mu(\eta i)}{\text{Im} m_\mu(\eta i) + 1} = - \lim_{\eta \to \infty} \text{Im} m_\mu(\eta i) + 1 + \lim_{\eta \to \infty} \text{Im} m_\mu(\eta i) + 1 + \eta^2 m_\mu(\eta i) = - \text{Var} [\mu],
\]
where we used the fact that
\[
\lim_{\eta \to \infty} \text{Im} m_\mu(\eta i) = -\mu(\mathbb{R}) = -1, \quad \lim_{\eta \to \infty} \text{Im} m_\mu(\eta i) + 1 = - \int x d\mu(x) = -1. \quad (6.6)
\]

Given the representation, the second equality directly follows:
\[
\widehat{\mu}(\mathbb{R}) = \lim_{\eta \to \infty} -\eta \int \frac{1}{x - \eta i} d\widehat{\mu}(x) = - \lim_{\eta \to \infty} (M_\mu(\eta i) - \eta i) = \int x^2 d\mu(x) - 1 = \text{Var} [\mu] \quad (6.7)
\]

Now we prove that \( \text{supp} \mu = \text{supp} \widehat{\mu} \). Note that \( \mu \) being supported on a single interval implies that \( zm_\mu(z) + 1 \neq 0 \) for any \( z \in \mathbb{C} \setminus \text{supp} \mu \). Therefore \( zm_\mu(z) + 1 \) is nonzero away from the support of \( \mu \) and hence \( M_\mu(z) - z \) is analytic on \( \mathbb{C} \setminus \text{supp} \mu \) with \( \text{Im}(M_\mu(z) - z) = 0 \) for \( z \in \mathbb{R} \setminus \text{supp} \mu \). Thus, whenever \( x \in \mathbb{R} \setminus \text{supp} \mu \), we have
\[
\lim_{\eta \to 0} \text{Im} m_\mu(x + \eta i) = 0, \quad (6.8)
\]
which proves \( \text{supp} \widehat{\mu} \subset \text{supp} \mu \).
Finally, we have to prove the reverse inclusion $\text{supp } \mu \subset \text{supp } \hat{\mu}$. Suppose not, and take a non-empty open interval $I \subset \text{supp } \mu \setminus \text{supp } \hat{\mu}$. By Stieltjes inversion, for almost every $E \in I$, we have

$$\lim_{\eta \to 0^+} \text{Im } m_\mu(E + i\eta) = \pi f_\mu(E) > 0$$

(6.9)

On the other hand, $I \not\subset \text{supp } \hat{\mu}$ implies that $\frac{M_\mu(z)}{z}$ extends to $I$ with real-values on $I$, so that $\frac{M_\mu(z)}{z} - 1$ extends to a analytic function defined on $D := \{ E + i\eta : E \in I \}$ by Schwarz reflection. Also $M_\mu(z) - 1$ is not identically zero in $D$. for $M_\mu(z) - 1 = 0$ implies $(zm_\mu(z) + 1)^{-1} = 0$, contradicting the assumption $\mu \neq \delta_0$. Thus there are only finitely many solutions of the equation $M_\mu(z) = 1$ and the formula

$$zm_\mu(z) + 1 = \frac{1}{1 - M_\mu(z)}$$

(6.10)

defines a meromorphic extension of $zm_\mu(z) + 1$ on $D$, which is real-valued on $I$. Therefore for almost every $E \in I \setminus \{0\}$ for which $1 - M_\mu(E) \neq 0$ we get

$$\lim_{\eta \searrow 0} m_\mu(E + i\eta) = \frac{M_\mu(z)}{E(1 - M_\mu(E))} \in \mathbb{R},$$

(6.11)

contradicting the fact that $\lim_{\eta \searrow 0} \text{Im } m_\mu(E + i\eta) > 0$.

\[ \square \]

**Definition 6.3.** Let $\hat{\mu}$ and $\hat{\nu}$ be the finite Borel measures on $\mathbb{R}$, respectively corresponding to $\mu$ and $\nu$ by means of Lemma 6.2. Also we define

$$I_\mu(z) := \int \frac{x}{|x-z|^2} d\mu(x), \quad I_\nu(z) := \int \frac{x}{|x-z|^2} d\nu(x),$$

$$\hat{I}_\mu(z) := \int \frac{1}{|x-z|^2} d\hat{\mu}(x), \quad \hat{I}_\nu(z) := \int \frac{1}{|x-z|^2} d\hat{\nu}(x),$$

(6.12)

whenever $z$ is not in the support of the measure in each integral.

**Remark 6.1.** Using the definition above, we can write

$$\text{Im}(zm_\mu(z) + 1) = \text{Im } \int \frac{x}{x-z} d\rho(x) = \text{Im } \int \frac{x}{x-z} d\rho(x)$$

$$= \text{Im } \int \frac{x}{x-\Omega_\mu(z)} d\nu(x) = \text{Im } \Omega_\mu(z) \int \frac{x}{x-\Omega_\mu(z)}^2 d\nu(x) = \text{Im } \Omega_\mu(z) I_\nu(\Omega_\mu(z))$$

(6.13)

**Remark 6.2.** We have

$$M_\mu(z) = M_\mu(\Omega_\nu(z)) = \Omega_\nu(z)(m_\hat{\mu}(\Omega_\nu(z)) + 1) = \frac{\Omega_\mu(z)\Omega_\nu(z)}{z}$$

(6.14)

thus

$$\frac{\Omega_\mu(z)}{z} = m_\hat{\mu}(\Omega_\nu(z)) + 1 \quad \text{and} \quad \frac{\Omega_\nu(z)}{z} = m_\hat{\nu}(\Omega_\mu(z)) + 1.$$  

(6.15)

Therefore taking imaginary parts of both sides we get

$$\text{Im } \frac{\Omega_\mu(z)}{z} = \hat{I}_\mu(\Omega_\nu(z)) \text{Im } \Omega_\nu(z)$$

(6.16)

and the corresponding equality for $\text{Im}(\Omega_\nu(z)/z)$. Multiplying the equations, we obtain

$$\hat{I}_\mu(\Omega_\nu(z)) \hat{I}_\nu(\Omega_\mu(z)) = \frac{\text{Im } \frac{\Omega_\mu(z)}{z} \text{Im } \frac{\Omega_\nu(z)}{z}}{\text{Im } \Omega_\mu(z) \text{Im } \Omega_\nu(z)}.$$  

(6.17)
6.1 Stability bounds

The main result of this section is the following proposition:

**Proposition 6.4.** Let $\mu$ and $\nu$ satisfy Assumption 2.2 and let $D \subset \mathbb{C} \cup \mathbb{R}$ be compact. Then there exists a constant $C \geq 1$ such that for all $z \in D$,

$$C^{-1} \text{Im}(zm_\nu(z)) \leq \text{Im} \Omega_\mu(z) \leq C \text{Im}(zm_\nu(z)), \quad (6.18)$$

$$C^{-1} \text{Im}(zm_\nu(z)) \leq \text{Im} \Omega_\nu(z) \leq C \text{Im}(zm_\nu(z)).$$

In order to prove the result, we observe from Proposition 2.8 that

$$\text{Im}(zm_\nu(z) + 1) = \text{Im}(\Omega_\nu(z)M_\rho(\Omega_\nu(z)) + 1) = I_\mu(\Omega_\nu(z)) \text{Im} \Omega_\nu(z) \quad (6.19)$$

and similar equality for $\Omega_\mu(z)$, so that finding the constant $C$ in (6.18) is equivalent to finding the upper and lower bounds of $I_\mu(\Omega_\nu(z))$ and $I_\nu(\Omega_\mu(z))$.

The lower bound follows from the lemma below:

**Lemma 6.5.** Let $\mu$ and $\nu$ satisfy Assumption 2.2 and let $D \subset \overline{\mathbb{C}} \setminus \{0, \infty\}$ be compact. Then there exists a constant $C > 1$ such that for all $z \in D$,

$$C^{-1} \leq |\Omega_\mu(z)| \leq C, \quad C^{-1} \leq |\Omega_\nu(z)| \leq C. \quad (6.20)$$

**Proof.** Recalling Lemma 2.9 it suffices to prove that for any $c \in \mathbb{R} \setminus \{0\}$, $\Omega_\mu(c)$ and $\Omega_\nu(c)$ cannot be zero or infinity.

Suppose on the contrary that $\Omega_\mu(c) = 0$ for some $c \in \mathbb{R} \setminus \{0\}$. Then

$$\lim_{\eta \to 0} \frac{\Omega_\nu(c + i\eta)}{c + i\eta} = \lim_{\eta \to 0} \frac{M_\nu(\Omega_\mu(c + i\eta))}{\Omega_\mu(c + i\eta)} = \lim_{z \to 0} \frac{M_\nu(z)}{z} = \int_1^\infty \frac{d\nu(x)}{x} \in (0, \infty), \quad (6.21)$$

which in turn implies $\Omega_\nu(c) \neq 0, \infty$. On the other hand, as $\mu$ has a single interval support with density which is strictly positive in its interior, the Stieltjes transform $m_\mu(z)$ is bounded below in $D$. Then we have

$$M_\mu(\Omega_\nu(c + i\eta)) = M_\rho(c + i\eta) = \frac{\Omega_\mu(c + i\eta)\Omega_\nu(c + i\eta)}{c + i\eta} \to 0, \quad (6.22)$$

as $\eta \to 0$. This in turn implies, from the definition of $M_\rho$, that

$$\Omega_\nu(c + i\eta)m_\mu(\Omega_\nu(c + i\eta)) \to 0, \quad (6.23)$$

and hence

$$m_\mu(\Omega_\nu(c + i\eta)) \to 0, \quad (6.24)$$

leading to contradiction.

Now suppose that $\Omega_\mu(c) = \infty$. As $\nu$ has mean 1 with compact support, we have

$$\lim_{\eta \to 0} \frac{\Omega_\nu(c + i\eta)}{c + i\eta} = \lim_{\eta \to 0} \frac{M_\nu(\Omega_\mu(c + i\eta))}{\Omega_\mu(c + i\eta)} = \lim_{z \to \infty} \frac{M_\nu(z)}{z} = 1, \quad (6.25)$$

by the same reasoning as above. Also, since $zd\mu(x)$ has strictly positive density in the interior of its support, $zm_\mu(z) + 1$ is bounded below on $D$. But using Proposition 2.8 we have

$$M_\mu(\Omega_\nu(c + i\eta)) = M_\rho(c + i\eta) = \frac{\Omega_\nu(c + i\eta)\Omega_\nu(c + i\eta)}{c + i\eta} \to \infty, \quad (6.26)$$

so that

$$\Omega_\nu(c + i\eta)m_\mu(\Omega_\nu(c + i\eta)) + 1 = (1 - M_\mu(\Omega_\nu(c + i\eta)))^{-1} \to 0. \quad (6.27)$$

As $zm_\mu(z) + 1$ is bounded below, we obtain contradiction. \(\square\)
Now we tend to proof of the upper bound, which directly follows given the lower bound of $\text{dist}(\Omega_\mu, \text{supp} \nu)$ and $\text{dist}(\Omega_\nu, \text{supp} \mu)$. In order to prove it, we first bound $\tilde{I}_\mu(\Omega_\nu)$ and $\tilde{I}_\nu(\Omega_\mu)$ from above and below. In particular, by the following absolute inequality, the lower bound implies the upper bound.

**Lemma 6.6.** For any $z \in \mathbb{C}_+$, we have

$$|z|^2 \tilde{I}_\mu(\Omega_\nu(z)) \tilde{I}_\nu(\Omega_\mu(z)) \leq 1.$$  

**(Proof.** We consider the following quantity:

$$|z|^2 \tilde{I}_\mu(\Omega_\nu(z)) \tilde{I}_\nu(\Omega_\mu(z)) - 1 = |z|^2 \frac{\text{Im} \frac{\Omega_\nu(z)}{z} \text{Im} \frac{\Omega_\nu(z)}{z} - \text{Im} \Omega_\mu(z) \text{Im} \Omega_\nu(z)}{\text{Im} \Omega_\mu(z) \text{Im} \Omega_\nu(z)}.  \quad (6.29)$$

Denoting $\text{arg } z = \theta, \text{arg } \Omega_\mu(z) = \theta_\mu$, and $\text{arg } \Omega_\nu(z) = \theta_\nu$, the numerator is equal to the following:

$$\sin(\theta_\nu - \theta) \sin(\theta_\mu - \theta) - \sin \theta_\mu \sin \theta_\nu = \frac{\cos(\theta_\mu + \theta_\nu) - \cos(\theta_\mu + \theta_\nu - 2\theta)}{2} = \frac{\cos((\theta_\mu + \theta_\nu - \theta) + \theta) - \cos((\theta_\mu + \theta_\nu - \theta) - \theta)}{2} = \frac{\cos(\theta_\mu + \theta_\nu - \theta) \cos \theta - \sin(\theta_\mu + \theta_\nu - \theta) \sin \theta - \cos(\theta_\mu + \theta_\nu - \theta) \cos \theta - \sin(\theta_\mu + \theta_\nu - \theta) \sin \theta}{2} = - \sin \theta \sin(\theta_\mu + \theta_\nu - \theta).  \quad (6.30)$$

Noting that $\theta_\mu + \theta_\nu - \theta = \text{arg}(\Omega_\mu(z) \Omega_\nu(z)/z) = \text{arg}(M_\nu(z))$. As $z \in \mathbb{C}_+$ implies $M_\nu(z) \in \mathbb{C}_+$, we conclude that the last quantity is negative as desired. □

Combining the lemmas above, we can directly prove the following assertion:

**Lemma 6.7.** Let $\mu$ and $\nu$ satisfy Assumption 2.5 and let $D \subset \overline{\mathbb{C}_+} \setminus \{0, \infty\}$ be compact. Then there exists constants $c_1, c_2, c_3 > 0$ such that the following holds:

$$\inf_{z \in D} I_\mu(\Omega_\nu(z)) \geq c_1, \quad \inf_{z \in D} \tilde{I}_\mu(\Omega_\nu(z)) \geq c_2, \quad \sup_{z \in D} \tilde{I}_\mu(\Omega_\nu(z)) \leq c_3, \quad \text{and} \quad \inf_{z \in D} I_\nu(\Omega_\mu(z)) \geq c_1, \quad \inf_{z \in D} \tilde{I}_\nu(\Omega_\mu(z)) \geq c_2, \quad \sup_{z \in D} \tilde{I}_\nu(\Omega_\mu(z)) \leq c_3.  \quad (6.31)$$

**(Proof.** The first two assertions follows directly from (6.3) and Lemma 6.6. Now given the first two inequalities, (6.33) follows by merely noting that $|z|$ is bounded below and above in $D$. □

The last estimate needed to prove the lower bound of $\text{dist}(\Omega_\mu, \text{supp} \nu)$ and $\text{dist}(\Omega_\nu, \text{supp} \mu)$ is given in the following computational lemma:

**Lemma 6.8** (Lemma 3.4 of [3]). Let $z = E + i\eta$ with $\eta \geq 0$ and $|z| \leq \theta$ for some small $\theta > 0$. For $-1 < t < 1$, the following holds:

$$\int_0^\theta \frac{x^t}{(x-E)^2 + \eta^2} \, dt \sim \begin{cases} \frac{E^t}{\eta} |t|^{-1} \sim |E|^{t-1} & \text{if } E > \eta, \\ |z|^t \sim |E|^{t-1} & \text{if } E < -\eta, \\ \eta^{-1} & \text{if } \eta > |E|, \end{cases}  \quad (6.34)$$

where we write $C(z) \sim D(z)$ whenever there exists a constant $c > 1$ such that $D(z)/c < C(z) < cD(z)$ uniformly on $\{z : |z| \leq \theta\}$.  \hfill \hfill \hfill \hfill 18
Definition 6.9. For \( z \in \mathbb{C} \), we define
\[
d_\mu(z) := \text{dist}(z, \text{supp } \mu), \quad d_\nu(z) := \text{dist}(z, \text{supp } \nu).
\] (6.35)

Finally we prove the main result of this subsection.

**Proposition 6.10.** Let \( \mu \) and \( \nu \) be probability measures on \((0, \infty)\) satisfying Assumption \[2.5\]. Then there exists a constant \( c > 0 \) such that
\[
\inf_{z \in \mathbb{C}^+} d_\mu(\Omega_\mu(z)) \geq c, \quad \inf_{z \in \mathbb{C}^+} d_\nu(\Omega_\nu(z)) \geq c.
\] (6.36)

**Proof.** If \( z \) is sufficiently close to 0 or \( \infty \), as \( \mu \) and \( \nu \) satisfy the assumptions of Theorem \[2.4\], Lemma \[5.4\] and Proposition \[6.5\] readily prove the result. Thus, we may restrict our attention to a compact subset \( \overline{\mathbb{C}^+ \setminus \{0, \infty\}} \).

We will prove that \( \text{dist}(\Omega_\mu(\mathcal{D})), \text{supp } \mu) = 0 \) implies that \( \{\widehat{I}_\mu(\Omega_\nu(z)) : z \in \mathcal{D}\} \) is unbounded, which is contradiction by Lemma \[6.4\]. In order to do so, we first assume that \( \Omega_\mu(z) \) converges to a point in \([E^\mu_-, E^\mu_- + \delta]\) for small enough \( \delta > 0 \) to be chosen.

Recall that
\[
\widehat{I}_\mu(z) = \frac{\text{Im} M_\mu(z)}{\text{Im } z} + \frac{1}{\text{Im } z} \text{Im} \left[ \frac{1}{z} - \frac{1}{z (zm_\mu(z) + 1)} \right] = -\frac{1}{|z|^2} + \frac{\text{Im}(zm_\mu(z) + 1)}{(\text{Im } z)|z|^2 |zm_\mu(z) + 1|^2}.
\] (6.37)

Observe that if \( z \in \mathbb{C}_+ \cup \mathbb{R} \setminus \text{supp } \mu \) satisfies \( |z - E^\mu_0| \leq \delta \) for sufficiently small \( \delta \), we have
\[
|z m_\mu(z) + 1| \leq C + C' \int_0^{2\delta} \frac{x^\mu}{x + E^\mu_- - z} \, dx \leq C + C'(d_\mu(z))^t^\mu.
\] (6.38)

Similarly, for \( |z - E^\mu_0| \leq \delta \), Lemma \[6.8\] gives us
\[
\int \frac{xd_\mu(x)}{|x - z|^2} \geq c \begin{cases} (E - E^\mu_\eta)^t^\mu & \text{if } E - E^\mu_\eta > \eta, \\ (E^\mu_\eta - E)^t^\mu - 1 & \text{if } E - E^\mu_\eta < -\eta, \\ \eta^t^\mu - 1 & \text{if } \eta > |E - E^\mu_\eta|, \end{cases}
\] (6.39)

where we denote \( z = E + i\eta \).

Then in each case of \( t^\mu \) being nonnegative or negative, we analyze the quotient
\[
\left( \int \frac{x}{|x - z|^2} \, dx \right) / \left( \int \frac{x}{x - z} \, dx \right).
\] (6.40)

If \( t^\mu \geq 0 \), the RHS of \( (6.38) \) is bounded, so that the quotient is bounded below by
\[
c \begin{cases} (E - E^\mu_\eta)^t^\mu & \text{if } E - E^\mu_\eta > \eta, \\ (E^\mu_\eta - E)^t^\mu - 1 & \text{if } E - E^\mu_\eta < -\eta, \\ \eta^t^\mu - 1 & \text{if } \eta > |E - E^\mu_\eta|, \end{cases}
\] (6.41)

On the other hand if \( t^\mu < 0 \), the last quantity in \( (6.38) \) diverges with order \( (d_\mu(z))^t^\mu \) as \( z \) approaches to \( \text{supp } \mu \). In particular, from
\[
d_\mu(z) \leq 2 \begin{cases} E - E^\mu_\eta & \text{if } E - E^\mu_\eta > \eta, \\ |E - E^\mu_\eta| & \text{if } E - E^\mu_\eta < -\eta, \\ \eta & \text{if } \eta > |E - E^\mu_\eta|, \end{cases}
\] (6.42)
we conclude that the quotient has following lower bound:

\[
    c \begin{cases} 
        \frac{(E-E^\mu)^{-\nu}}{\eta} & \text{if } E - E^\mu > \eta, \\
        \frac{(E^\mu - E)^{-\nu}}{\eta} & \text{if } E - E^\mu < -\eta, \\
        \frac{\eta^{-\nu}}{1} & \text{if } \eta > |E - E^\mu|.
    \end{cases} \tag{6.43}
\]

We see that in both cases, the quotient in \[6.40\] diverges as \(\Omega_\mu(z)\) approaches to \([E^\mu_-, E^\mu_+ + \delta]\). Similar reasoning proves the analogous assertion for the upper edge.

Now we consider the case in which \(\Omega_\mu(z)\) converges to a point in the bulk of \(\text{supp } \mu\), namely, \([E^\mu_+ + \delta, E^\mu_+ - \delta]\). Take \(\delta' \in (0, \delta)\). Since the density \(f_\mu\) of \(\mu\) is strictly positive on \([E^\mu_+ + \delta', E^\mu_+ - \delta']\), there exists a constant \(C > 0\) depending on \(\delta'\) such that

\[
    \text{Im}(\omega m_\mu(\omega) + 1) = \text{Im } \omega \int \frac{x \mu(x)}{|x - \omega|^2} \geq C, \quad \forall \omega \in \mathbb{C}_+ \cup \mathbb{R} \setminus \text{supp } \mu \cap \{\omega : \text{dist}(\omega, [E^\mu_+ + \delta, E^\mu_+ - \delta]) < \delta'\}. \tag{6.44}
\]

Also, for density being bounded in the bulk, there also exist constants \(C_1, C_2 > 0\) depending on \(\delta'\) such that

\[
    |\omega m_\mu(\omega) + 1| \leq C_1 + \int_{E^\mu_+ - \delta}^{E^\mu_+ - \delta} \frac{dx}{|x - \omega|} \leq C_1 + C_2 \int_{E^\mu_+ - \delta}^{E^\mu_+ - \delta} \frac{dx}{|x - E| + \eta} \leq C_1 + C_2 |\log \eta| \tag{6.45}
\]

for all \(\omega \in \mathbb{C}_+ \cup \mathbb{R} \setminus \text{supp } \mu\) with \(\text{dist}(\omega, [E^\mu_+ + \delta, E^\mu_+ - \delta]) < \delta'\). By the same reasoning as above, we conclude that \(\bar{I}_\mu(\omega)\) must diverge to infinity as \(\omega\) tends to \([E^\mu_+ + \delta, E^\mu_+ - \delta]\).

But by Lemma \[6.7\], \(\bar{I}_\mu(\Omega_\nu(z))\) must remain bounded as long as \(z\) stays within any compact subset of \(\mathbb{C}_+ \setminus \{0, \infty\}\), leading to contradiction. \(\Box\)

**Remark 6.3.** As easily seen, the power law behavior of \(\mu\) and \(\nu\) are used only in the proof of Proposition \[6.10\]. Nonetheless, we remark that the same proof can be applied even if the power laws are replaced by the divergence of

\[
    \left| \int \frac{x}{x - z} \mu(x) \right|^{-2} \left( \int \frac{x}{|x - z|^2} \mu(x) \right) = \infty \tag{6.46}
\]

for \(z\) tending to the support of \(\mu\).

### 6.2 Characterization of endpoints

**Lemma 6.11.** Let \(\mu\) and \(\nu\) satisfy Assumption \[2.5\] and let \(E \in \mathbb{R}\). Then

\[
    \lim_{z \to E, z \in \mathbb{C}_+} \frac{\text{Im } \Omega_\mu(z)}{\text{Im } \Omega_\nu(z)} = \frac{I_\mu(\Omega_\nu(E))}{I_\nu(\Omega_\mu(E)).} \tag{6.47}
\]

Furthermore, the limit is a strictly positive and continuous function in \(E\).

**Proof.** For any \(z \in \mathbb{C}_+\), we have

\[
    \frac{\text{Im } \Omega_\mu(z)}{\text{Im } \Omega_\nu(z)} = \frac{\text{Im}(zm_\mu(z) + 1) I_\mu(\Omega_\nu(z))}{\text{Im}(zm_\nu(z) + 1) I_\nu(\Omega_\mu(z))} = \frac{I_\mu(\Omega_\nu(z))}{I_\nu(\Omega_\mu(z))}, \tag{6.48}
\]

and continuity of subordination functions and Proposition \[6.10\] enable us to obtain the equality by taking the limit \(z \to E\) in \(\mathbb{C}_+\), via dominated convergence. The limit is strictly positive and continuous again by Proposition \[6.10\]. \(\Box\)

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**Definition 6.12.** In order to characterize the edge of \( \text{supp } \mu \boxplus \nu \), we investigate a larger set \( V \), defined as follows:

\[
V := \partial \{ x \in \mathbb{R} : \rho(x) > 0 \}. 
\]  

**Proposition 6.13.** Let \( \mu \) and \( \nu \) satisfy Assumption 2.2. Then for all \( z \in \mathbb{C}_+ \cup \mathbb{R} \), the following inequality holds:

\[
\left| \left( \frac{\Omega_\nu(z)}{M_\mu(\Omega_\nu(z))} M'_{\mu}(\Omega_\nu(z)) - 1 \right) \left( \frac{\Omega_\mu(z)}{M_\nu(\Omega_\mu(z))} M'_{\nu}(\Omega_\mu(z)) - 1 \right) \right| \leq 1. \tag{6.50}
\]

Furthermore, for \( z = E + i\eta \in \mathbb{C}_+ \cup \mathbb{R} \), the equality in (6.50) holds if and only if \( E \in V \) and \( \eta = 0 \). In this case, we also have

\[
\left( \frac{\Omega_\nu(z)}{M_\mu(\Omega_\nu(z))} M'_{\mu}(\Omega_\nu(z)) - 1 \right) \left( \frac{\Omega_\mu(z)}{M_\nu(\Omega_\mu(z))} M'_{\nu}(\Omega_\mu(z)) - 1 \right) = 1. \tag{6.51}
\]

**Proof.** We first observe that from Lemma 6.2 that

\[
M'_{\mu}(z) = \frac{d}{dz}(zm_\mu(z) + z) = \int \frac{x}{(x-z)^2} d\hat{\mu} + 1,
\]  

so that

\[
\frac{\Omega_\nu(z)}{M_\mu(\Omega_\nu(z))} M'_{\mu}(\Omega_\nu(z)) - 1 = \frac{1}{M_\mu(\Omega_\nu(z))} \left( \int \frac{x\Omega_\nu(z)}{(x-\Omega_\nu(z))^2} d\hat{\mu}(x) + \Omega_\nu(z) - M_\mu(\Omega_\nu(z)) \right)
\]

\[
= \frac{1}{M_\mu(\Omega_\nu(z))} \int \left[ \frac{x\Omega_\nu(z)}{(x-\Omega_\nu(z))^2} - \frac{\Omega_\nu(z)}{x-\Omega_\nu(z)} \right] d\hat{\mu}(x)
\]

\[
= \frac{\Omega_\nu(z)^2}{M_\mu(\Omega_\nu(z))} \int \frac{1}{(x-\Omega_\nu(z))^2} d\hat{\mu}(x). \tag{6.53}
\]

Thus, we have

\[
\left| \left( \frac{\Omega_\nu(z)}{M_\mu(\Omega_\nu(z))} M'_{\mu}(\Omega_\nu(z)) - 1 \right) \left( \frac{\Omega_\mu(z)}{M_\nu(\Omega_\mu(z))} M'_{\nu}(\Omega_\mu(z)) - 1 \right) \right| \leq \left| \frac{\hat{\mu}(\Omega_\nu(z))}{M_\mu(\Omega_\nu(z))} \right| \left| \hat{\mu}(\Omega_\mu(z)) \right| \leq |z|^2 \left| \hat{\mu}(\Omega_\nu(z)) \right| \left| \hat{\mu}(\Omega_\mu(z)) \right| \leq 1, \tag{6.54}
\]

by (6.53).

Now we suppose \( z = E + i\eta \in \mathbb{C}_+ \) is such that the equality holds in (6.50) and hence inequalities (i) and (ii) in (6.51) are both equalities. Then we first observe

\[
\left| \int \frac{1}{(x-\Omega_\nu(z))^2} d\hat{\mu}(x) \right| = \int \frac{1}{|x-\Omega_\nu(z)|^2} d\hat{\mu}(x), \tag{6.55}
\]

which implies \((x-\Omega_\nu(z))^2 = |x-\Omega_\nu(z)|^2\) for \( \hat{\mu} \)-a.e. \( x \in \mathbb{R} \). In particular, for \( \Omega_\nu(z) \) being bounded, we have \( \text{Im } \Omega_\nu(z) = 0 \). By a similar reasoning we also have \( \text{Im } \Omega_\mu(z) = 0 \).

Then we have

\[
\frac{\Omega_\mu(z)}{z} = \frac{M_\mu(\Omega_\nu(z))}{\Omega_\nu(z)} = \frac{\int_{x-\Omega_\nu(z)}^1 \mu(x) \, dx}{\int_{x-\Omega_\nu(z)}^1 \mu(x) \, dx} > 0, \tag{6.56}
\]

and similarly

\[
\frac{\Omega_\nu(z)}{z} = \frac{M_\nu(\Omega_\mu(z))}{\Omega_\mu(z)} > 0, \tag{6.57}
\]

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using Proposition 6.10 and \( \text{Im} \Omega_{\mu}(z) = \text{Im} \Omega_\nu(z) = 0 \). Thus \( z, \Omega_\nu(z), \Omega_\mu(z) \) and \( M_\rho(z) \) are real numbers with the same sign or all zero.

If \( z \leq 0 \) so that \( \Omega_\mu(z), \Omega_\nu(z) \) and \( M_\rho(z) \) are all non-positive, from the definition of \( M_\mu \) we find that

\[
\frac{\Omega_\nu(z)}{M_\mu(\Omega_\nu(z))} \frac{M'_\nu(\Omega_\nu(z))}{M'_\nu(\Omega_\nu(z))} = \left( \int \frac{x}{x - \Omega_\nu(z)} \text{d} \mu(x) \right)^{-1} \left( \int \frac{1}{x - \Omega_\nu(z)} \text{d} \mu(x) \right)^{-1} \left( \int \frac{x}{(x - \Omega_\nu(z))^2} \text{d} \mu(x) \right) \in (0, 1],
\]

(6.58)

where the upper bound follows from Cauchy-Schwarz inequality applied to two functions \( (x/(x - z))^{1/2} \) and \( (1/(x - z))^{1/2} \). Thus both of the factors in (6.51) are strictly positive, from the left-hand side and the first factor are. Also for \( \Omega_\mu(z) \) and \( \Omega_\nu(z) \) converging to positive numbers \( \Omega_\mu(E) \) and \( \Omega_\nu(E) \), \( \text{Im} \theta_\alpha \) and \( \text{Im} \theta_\beta \) must converge to 0 as \( \theta \) tends to 0. Therefore Proposition 2.8 implies that the last limit should be bounded by 1. Multiplying the equalities yields

\[
0 < E \hat{T}_\mu(\Omega_\nu(E)) = \lim_{\theta \to 0} \frac{\text{Im} \Omega_\nu(E e^{i \theta})}{\text{Im} \Omega_\nu(E)} = \lim_{\theta \to 0} \frac{\text{Im} \Omega_\nu(E e^{i \theta})}{\text{Im} \Omega_\nu(E e^{i \theta})} \frac{\text{Im} \Omega_\nu(E e^{i \theta})}{\text{Im} \Omega_\nu(E e^{i \theta})} = \text{Im} \Omega_\nu(E) \lim_{\theta \to 0} \frac{\sin(\theta_\alpha - \theta)}{\sin \theta_\alpha}
\]

(6.59)

and similarly

\[
E \hat{T}_\nu(\Omega_\mu(E)) = \frac{\text{Im} \Omega_\mu(E)}{\text{Im} \Omega_\mu(E)} \lim_{\theta \to 0} \frac{\sin(\theta_\beta - \theta)}{\sin \theta_\beta},
\]

(6.60)

where we denote \( \text{arg} \Omega_\mu(E e^{i \theta}) = \theta_\alpha \) and \( \text{arg} \Omega_\nu(E e^{i \theta}) = \theta_\beta \). It should be noted that the second factor is strictly positive, as the left-hand side and the first factor are. Also for \( \Omega_\mu(z) \) and \( \Omega_\nu(z) \) converging to positive numbers \( \Omega_\mu(E) \) and \( \Omega_\nu(E) \), \( \sin \theta_\alpha \) and \( \sin \theta_\beta \) must converge to 0 as \( \theta \) tends to 0.

To summarize, so far we have proved that for any \( z = E + i \eta \in \mathbb{C}_+ \cup \mathbb{R} \) at which the equality holds in (6.50), we have \( z = E \in (0, \infty), \text{Im} \Omega_\mu(z) = \text{Im} \Omega_\nu(z) = 0 \), and

\[
\lim_{\theta \to 0} \frac{\sin(\theta_\alpha - \theta)}{\sin \theta_\alpha} = \lim_{\theta \to 0} \frac{\sin(\theta_\beta - \theta)}{\sin \theta_\beta} = \lim_{\theta \to 0} \frac{\theta_\beta - \theta}{\theta_\beta} = 1.
\]

(6.62)

First, we see that \( \text{Im} \Omega_\mu(E) = 0 \) implies \( \text{Im}(Em_\rho(E) + 1) = 0 \), and thus \( E \rho(E) = 0 \). Now if we suppose that \( E \in (V \cup \{ x \in \mathbb{R} : \rho(x) > 0 \}) \) or, equivalently, dist\((E, V \cup \{ x \in \mathbb{R} : \rho(x) > 0 \}) > 0 \), then for a small enough \( \theta_0 > 0 \) there exists a constant \( c > 0 \) such that

\[
\frac{\text{Im}(E e^{i \theta} m_\rho(E e^{i \theta}) + 1)}{E \sin \theta} = \frac{\text{Im}(E e^{i \theta} m_\rho(E e^{i \theta}) + 1)}{\text{Im} E e^{i \theta}} = \frac{1}{c} \frac{x}{|x - E e^{i \theta}|^2} \text{d} \rho(x) > c
\]

(6.64)

for all \( \theta \in [0, \theta_0] \). This in turn implies

\[
\frac{\sin \theta}{\sin \theta_\alpha} = \frac{\frac{\text{Im} \Omega_\mu(E e^{i \theta})}{E}}{\text{Im}(E e^{i \theta} m_\rho(E e^{i \theta}) + 1)} \frac{E \sin \theta}{\text{Im}(E e^{i \theta} m_\rho(E e^{i \theta}) + 1)} \frac{\text{Im} \Omega_\mu(E e^{i \theta})}{\text{Im} \Omega_\mu(E e^{i \theta})} \leq c \frac{\text{Im} \Omega_\mu(E e^{i \theta})}{E} \text{Im}(\Omega_\mu(E)) \leq C,
\]

(6.65)
which contradicts \( \theta/\theta_\alpha \to 0 \).

In order to prove the converse, suppose \( z = E \in \mathcal{V} \). As the density \( \rho \) is continuous and bounded by Theorem 2.4, we readily see that \( \rho(E) = 0 \) implies \( \text{Im } m_\rho(E) = 0 \) and thus \( \text{Im}(Em_\rho(E)+1) = 0 \). Furthermore \( \rho \) being compactly supported in \((0, \infty)\), \( \mathcal{V} \) must also be contained in \((0, \infty)\). Recalling

\[
\text{Im } \Omega_\mu(E)I_\nu(\Omega_\mu(E)) = \text{Im}(Em_\rho(E)+1),
\]

we also get \( \text{Im } \Omega_\mu(E) = \text{Im } \Omega_\nu(E) = 0 \), proving (i). Using the same argument as above we can also prove that \( \Omega_\mu(z) \), \( \Omega_\nu(z) \) are positive.

As \( E \in \mathcal{V} \), there must be a sequence \( \{\epsilon_n\}_{n \in \mathbb{N}} \) of real numbers increasing or decreasing to 0 such that \( \rho(E + \epsilon_n) > 0 \) for all \( n \in \mathbb{N} \). For each fixed \( n \in \mathbb{N} \), Using the Stieltjes inversion and the continuity of \( zm_\rho(z) + 1 \), we first observe that

\[
0 < (E + \epsilon_n)\rho(E + \epsilon_n) = \lim_{\eta \to 0} \text{Im } [(E + \epsilon_n + i\eta)m_\rho(E + \epsilon_n + i\eta) + 1] \]

\[
= \lim_{\theta \to 0} \text{Im } [(E + \epsilon_n)e^{i\theta}m_\rho((E + \epsilon_n)e^{i\theta}) + 1] = \lim_{\theta \to 0} (E + \epsilon_n)\sin \int \frac{x}{|x - (E + \epsilon_n)e^{i\theta}|^2} d\rho(x). \tag{6.67}
\]

Then Proposition 6.10 implies

\[
\lim_{\theta \to 0} \frac{\text{Im } \Omega_\mu((E + \epsilon_n)e^{i\theta})}{(E + \epsilon_n)\sin \theta} = \lim_{\theta \to 0} \frac{1}{I_\nu(\Omega_\mu((E + \epsilon_n)e^{i\theta}))} \int \frac{x}{|x - (E + \epsilon_n)e^{i\theta}|^2} d\rho(x) = \infty, \tag{6.68}
\]

so that

\[
\lim_{\theta \to 0} \frac{\theta}{\theta_\alpha} \leq \lim_{\theta \to 0} \frac{\theta}{\sin \theta \sin \arg \Omega_\mu((E + \epsilon_n)e^{i\theta})} = \lim_{\theta \to 0} \frac{\sin \theta}{\sin \arg \Omega_\mu((E + \epsilon_n)e^{i\theta})} = 0, \tag{6.69}
\]

where we used \( |\Omega_\mu(E + \epsilon_n)| < \infty \) in the last equality.

Finally we conclude

\[
\lim_{\theta \to 0} \frac{\sin(\theta - \theta)}{\sin \theta_\alpha} = 1 - \lim_{\theta \to 0} \frac{\theta}{\theta_\alpha} = 1 \tag{6.70}
\]

and similar equality for \( \theta_\beta \), which directly give (ii).

For the last part, we observe that \( \Omega_\mu(z) \) and \( \Omega_\nu(z) \) are real numbers when the equivalent conditions hold, so that (i) is equality even if we do not take absolute value of the left side.

Now that we are given Proposition 6.13, we can now characterize points in \( \mathcal{V} \) as the solutions of equation (6.51). Yet we still do not know whether the set \( \mathcal{V} \) consists of exactly two points, \( E_- \) and \( E_+ \). In order to exhaust the possibility of a non-edge point in \( \mathcal{V} \), i.e., the existence of isolated zero (see 4), we need representations of the subordination functions corresponding to Lemma 6.2.

**Lemma 6.14.** Let \( \Omega_\mu \) and \( \Omega_\nu \) be the subordination functions corresponding to \( \mu \) and \( \nu \) by means of Proposition 2.8. Then there exists finite Borel measures \( \tilde{\mu} \) and \( \tilde{\nu} \) on \( \mathbb{R}_+ \) such that the following holds:

(i) \( \frac{\Omega_\mu(z)}{z} = 1 + m_\mu(z) \) and \( \frac{\Omega_\nu(z)}{z} = 1 + m_\nu(z) \) whenever \( z \notin \text{supp } \tilde{\mu} \) and \( z \notin \text{supp } \tilde{\nu} \), respectively,

(ii) \( \tilde{\mu} (\mathbb{R}) = \text{Var } [\mu] \) and \( \tilde{\nu} (\mathbb{R}) = \text{Var } [\nu] \),

(iii) \( \text{supp } \tilde{\mu} = \text{supp } \tilde{\nu} = \text{supp } \rho \).

**Proof.** We start from the following identity:

\[
\frac{\Omega_\mu(z)}{z} = \frac{M_\mu(\Omega_\nu(z))}{\Omega_\nu(z)}, \tag{6.71}
\]
Since \( M_\mu(z)/z \) has strictly positive imaginary part for \( z \in \mathbb{C}_+ \) as \( \mu \) is not a point mass, we see that \( \Omega_\mu(z)/z \) is an analytic self-map of \( \mathbb{C}_+ \). Furthermore, we see that Lemma 2.9 and 5.4 implies that \( \Omega_\nu(\eta) \) and \( \Omega_\mu(\eta) \) tends to infinity in \( \mathbb{C}_+ \) as \( \eta \to \infty \), so that

\[
\lim_{\eta \to \infty} \frac{\Omega_\mu(\eta)}{\eta} = \lim_{\eta \to \infty} \frac{M_\mu(\Omega_\nu(\eta))}{\Omega_\nu(\eta)} = \lim_{z \to \infty, z \in \mathbb{C}_+} \frac{M_\mu(z)}{z} = 1 \quad (6.72)
\]

and similarly \( \Omega_\nu(\eta)/(\eta) \to 1 \).

Thus we can again use Lemma 3.4 and hence it suffices to prove that

\[
\sup_{\eta \geq 1} |\Omega_\mu(z) - z| < \infty. \quad (6.73)
\]

Indeed, by Lemma 6.2 we see that

\[
\lim_{\eta \to \infty} \Omega_\mu(\eta) - \eta = \lim_{\eta \to \infty} \frac{\eta}{\Omega_\nu(\eta)} (M_\mu(\Omega_\nu(\eta)) - \Omega_\nu(\eta)) = -\text{Var} [\mu], \quad (6.74)
\]

which proves the first two assertions.

In order to prove the last part, we first suppose that \( E \notin \text{supp} \tilde{\mu} \). Then \( \Omega_\mu \) is analytic in a neighborhood of \( E \), so that \( M_\mu = M_\nu \circ \Omega_\mu \) is also analytic in the neighborhood as \( \Omega_\mu(E) \notin \text{supp} \nu \) by Proposition 6.10. Now it directly follows that \( \text{Im } Em_\rho(E) = 0 \) as \( \text{Im } \Omega_\mu(E) = 0 \). Thus \( E \notin \text{supp} \tilde{\mu} \) gives us a neighborhood \( U \) of \( E \) in \( \mathbb{R} \) on which \( \text{Im } zm_\rho(z) + 1 = 0 \), hence \( U \subset \text{supp} \rho^c \). Therefore we have the inclusion \( \text{supp} \rho \subset \text{supp} \tilde{\mu} \).

For the converse, suppose \( E \in \mathbb{R} \setminus \text{supp} \rho \). Then \( M_\mu(E) \in \mathbb{R} \), which together with Proposition 6.10 implies \( \text{Im } \Omega_\mu(E) = \text{Im } \Omega_\nu(E) = 0 \). Furthermore, if \( E = 0 \),

\[
\lim_{z \to 0} \frac{\Omega_\mu(E)}{E} = \lim_{z \to 0} \frac{M_\mu(\Omega_\nu(z))}{\Omega_\nu(z)} = \lim_{w \to 0} \frac{M_\mu(w)}{w} = \int \frac{1}{x} d\mu(x). \quad (6.75)
\]

Thus for a neighborhood \( I \) of \( E \), \( \text{Im } \frac{\Omega_\mu(x)}{x} = 0 \) whenever \( x \in I \). Therefore \( E \notin \text{supp} \tilde{\mu} \), concluding the proof.

Now we are ready to prove that \( \mathcal{V} \) in fact is exactly two endpoints of \( \text{supp } \rho \).

**Proposition 6.15.** Let \( \mu \) and \( \nu \) satisfy Assumption 2.5. Then there exists two positive real numbers \( E_- < E_+ \) such that \( \mathcal{V} = \{ E_-, E_+ \} \) and

\[
\{ x \in \mathbb{R} : \rho(x) > 0 \} = (E_-, E_+). \quad (6.76)
\]

**Proof.** We have seen that \( \text{Im } \Omega_\mu(E) = \text{Im } \Omega_\nu(E) = 0 \) for any \( E \in \mathcal{V} \) in the proof of Lemma 6.13. Thus using Proposition 6.10, we divide the possible locations of \( \Omega_\mu(E) \) and \( \Omega_\nu(E) \) for \( E \in \mathcal{V} \) as follows:

(i) \( \Omega_\mu(E) < E_\mu^\nu, \Omega_\nu(E) < E_\nu^\nu, \)

(ii) \( \Omega_\mu(E) > E_\mu^\nu, \Omega_\nu(E) > E_\nu^\nu, \)

(iii) \( \Omega_\mu(E) < E_\mu^\nu, \Omega_\nu(E) > E_\nu^\nu \) or \( \Omega_\mu(E) > E_\mu^\nu, \Omega_\nu(E) < E_\nu^\nu. \)

Among these cases, we shall prove that in each of (i) and (ii) the equation (6.51) have exactly one solution, while in the last case it does not have any.

For simplicity, recalling (6.51), we define

\[
f(E) := E^2 \tilde{f}_\mu(\Omega_\nu(E)) \tilde{f}_\nu(\Omega_\mu(E)), \quad \forall E > 0. \quad (6.77)
\]
Using Lemma 6.14 we find that
\[ \Omega_\mu'(z) = \frac{d}{dz}(z + zm_\mu(z)) = 1 + \int \frac{x}{(x-z)^2} \, d\tilde{\mu}(x), \tag{6.78} \]
so that \( \Omega_\mu \) is increasing on \((\text{supp } \tilde{\mu})^c = (\text{supp } \rho)^c\). Similarly \( \Omega_\nu \) is increasing on \((\text{supp } \rho)^c\). We also need another representation of \( f(E) \) below:
\[ f(E) = \frac{1}{M_\mu(\Omega_\nu(E))M_\nu(\Omega_\mu(E))} \left( \int \frac{\Omega_\nu(E)^2}{(x-\Omega_\nu(E))^2} \, d\tilde{\mu}(x) \right) \left( \int \frac{\Omega_\mu(E)^2}{(x-\Omega_\mu(E))^2} \, d\tilde{\nu}(x) \right). \tag{6.79} \]

We first prove the existence and uniqueness in the cases (i) and (ii). For existence, let \( E_\pm \) and \( E_+ \) to be the leftmost and rightmost components of \( \Omega_\mu(\mathbb{R}) \cap \mathbb{R} \) and \( \Omega_\nu(\mathbb{R}) \cap \mathbb{R} \) so that
\[ \Omega_\mu(E_-) < E_-^\mu, \quad \Omega_\nu(E_-) < E_-^\nu, \quad \text{and} \quad \Omega_\mu(E_+) > E_+^\mu, \quad \Omega_\nu(E_+) > E_+^\nu. \tag{6.80} \]

To prove the uniqueness, suppose that \( E_0 \) satisfies (i). If \( E_0 < E_- \), as \( \hat{\mu}(\Omega_{\nu}(E_0)) \) and \( \hat{\nu}(\Omega_{\mu}(E_0)) \), we see that
\[ f(E_0) = E_0^2 \hat{\mu}(\Omega_{\nu}(E_0))\hat{\nu}(\Omega_{\mu}(E_0)) < E_0^2 \hat{\mu}(\Omega_{\nu}(E_-))\hat{\nu}(\Omega_{\mu}(E_-)) = f(E_-) = 1, \tag{6.81} \]
which is contradiction. On the other hand if \( E_0 \) is larger than \( E_- \) yet satisfies (i) and (6.81), we must have
\[ \hat{\mu}(\Omega_{\nu}(E_0))\hat{\nu}(\Omega_{\mu}(E_0)) < \hat{\mu}(\Omega_{\nu}(E_-))\hat{\nu}(\Omega_{\mu}(E_-)), \tag{6.82} \]
so that either \( \Omega_\mu(E_0) < \Omega_\mu(E_-) \) or \( \Omega_\nu(E_0) < \Omega_\nu(E_-) \) must hold as \( \hat{\mu} \) and \( \hat{\nu} \) are increasing. Supposing the former without loss of generality, it follows that \( \Omega_\mu(E_0) \) coincides with \( \Omega_\mu(E_1) \) for some \( E_1 \in (0, E_-) \), as \( \Omega_\mu \) is a continuous, strictly increasing function which maps \((-\infty, E_-)\) onto \((-\infty, \Omega_\mu(E_-))\). Then we see that \( \Omega_\nu(E_0) \) must also be the same as \( \Omega_\nu(E_1) \), as both of them are the unique solution of \( M_\mu(\Omega) = M_\nu(\Omega_{\nu}(E_0)) = M_\nu(\Omega_{\nu}(E_1)) \) in \((-\infty, E_\mu^\nu)\). Now we observe from (6.79) that \( f(E_0) = f(E_1) < 1 \), which is a contradiction.

Similarly, let \( E_0 \) satisfy (ii) and (ii). We first recall that \( M_\mu \) and \( M_\nu \) are positive and increasing on \((E_+^\mu, \infty)\) and \((E_-^\nu, \infty)\), and that
\[ \frac{d}{dw} \frac{w^2}{(x-w)^2} = 2 - \frac{w}{x-w} \frac{x}{(x-w)^2} < 0, \quad \forall w > x > 0. \tag{6.83} \]
Therefore from (6.79) we see that it is decreasing in \( E \) for \( E > E_+ \). Thus as above, \( E_0 > E_+ \) implies \( f(E_0) < f(E_+) = 1 \). On the other hand if we suppose that \( E_0 \) solves (6.81) and is less than \( E_+ \), obtain
\[ \hat{\mu}(\Omega_{\nu}(E_0))\hat{\nu}(\Omega_{\mu}(E_0)) > \hat{\mu}(\Omega_{\nu}(E_+))\hat{\nu}(\Omega_{\mu}(E_+)). \tag{6.84} \]
Combining this inequality with (ii), either \( \Omega_\mu(E_0) < \Omega_\mu(E_+) \) or \( \Omega_\nu(E_0) < \Omega_\nu(E_+) \) must hold, and by Proposition 2.8 (iii) one implies the other. Therefore we have
\[ E_0^\mu < \Omega_\nu(E_0) < \Omega_\nu(E_+), \quad E_0^\nu < \Omega_\nu(E_0) < \Omega_\nu(E_+). \tag{6.85} \]
Now using (6.79), this would imply \( f(E_0) > f(E_+) = 1 \), which is a contradiction,
It remains to prove that there is no solution to \((6.31)\) satisfying (iii). To this end, we suppose \(E_0\) is such solution, in particular satisfying \[
\Omega_{\mu}(E_0) < E_-^\mu, \quad \Omega_{\nu}(E_0) > E_+^\mu. \tag{6.86}
\]
We also note that above readily implies \(E_0 \in (E_-, E_+), \) for if not we would end up either (i) or (ii). Then we have \[
M_{\mu}(E_0) = 1 - \left( \int \frac{x}{x - \Omega_{\mu}(E_0)} \, d\nu(x) \right)^{-1} < 1, \tag{6.87}
\]
yet at the same time \[
M_{\mu}(E_0) = 1 - \left( \int \frac{x}{x - \Omega_{\nu}(E_0)} \, d\mu(x) \right)^{-1} > 1. \tag{6.88}
\]
The same argument for the other case leads to a contradiction, concluding the proof. \(\square\)

### 6.3 Square root behavior at the edges

In this section, we prove that the subordination functions \(\Omega_{\mu}\) and \(\Omega_{\nu}\) have square root behavior at the edges \(E_-\) and \(E_+\), so that \(M_{\mu}(z) = M_{\nu}(\Omega_{\mu}(z)) = M_{\mu}(\Omega_{\nu}(z))\) also does.

**Proposition 6.16.** Let \(\mu\) and \(\nu\) be probability measures on \((0, \infty)\) satisfying Assumption \(\mathcal{B}_0\). Let \(\text{supp} \tilde{\mu} \otimes \nu = [E_-, E_+]\), following from Proposition \(\mathcal{B}_1\). Then there exist strictly positive constants \(\gamma_-, \gamma_+, \gamma_-\), and \(\gamma_+\) such that \[
\Omega_{\mu}(z) = \Omega_{\mu}(E_-) + \gamma_- \sqrt{E_- - z} + O(|z - E_-|^{3/2}), \tag{6.89}
\]
for \(z\) in a neighborhood of \(E_-\) with the principal branch of square root (with \(\sqrt{-1} = i\)). Similarly for \(E_+\), we have \[
\Omega_{\mu}(z) = \Omega_{\mu}(E_+) + \gamma_+ \sqrt{z - E_+} + O(|z - E_+|^{3/2}), \tag{6.90}
\]
for \(z\) in a neighborhood of \(E_+\) with the same branch of square root. The same holds with \(\alpha\) replaced by \(\beta\).

**Proof.** For simplicity, we focus on the behavior of \(\Omega_{\mu}\). The corresponding results for \(\Omega_{\nu}\) can be proved analogously.

We first note that \[
M_{\mu}'(\Omega) = \frac{1}{(\Omega m_{\mu}(\Omega) + 1)^2} \int \frac{x}{(x - \Omega)^2} \, d\mu \neq 0 \tag{6.91}
\]
whenever \(\Omega \notin \text{supp} \mu\). Since \(\Omega_{\nu}(E_-) \in (0, E_-^\mu)\), the inverse \(M_{\mu}^{-1}\) is well-defined and analytic in a neighborhood of \(M_{\mu} \circ \Omega_{\nu}(E_-) = M_{\nu} \circ \Omega_{\mu}(E_-)\), hence the function \[
\bar{z}_-(\Omega) := \frac{M_{\mu}^{-1} \circ M_{\nu}(\Omega)}{M_{\nu}(\Omega)} \tag{6.92}
\]
is analytic in a neighborhood of \(\Omega_{\mu}(E_-)\). Furthermore we find that if \(z\) is sufficiently close to \(E_-\), \[
M_{\mu}^{-1} \circ M_{\nu} \circ \Omega_{\mu}(z) = M_{\mu}^{-1} \circ M_{\mu} \circ \Omega_{\nu}(z) = \Omega_{\nu}(z), \tag{6.93}
\]
so that \[
\bar{z}_-(\Omega_{\mu}(z)) = \Omega_{\mu}(z) \frac{M_{\mu}^{-1} \circ M_{\nu} \circ \Omega_{\mu}(z)}{M_{\mu}(z)} = \frac{\Omega_{\mu}(z) \Omega_{\nu}(z)}{M_{\mu}(z)} = z. \tag{6.94}
\]
Now given the fact that \(\bar{z}_-\) is analytic in a neighborhood of \(\Omega_{\mu}(E_-)\), we consider its Taylor expansion \[
\bar{z}_-(\Omega) = E_- + \bar{z}_-'(\Omega_{\mu}(E_-))(\Omega - \Omega_{\mu}(E_-)) + \frac{1}{2} \bar{z}_''(\Omega_{\mu}(E_-))(\Omega - \Omega_{\mu}(E_-))^2 + R_-(\Omega) \tag{6.95}
\]
around \( \Omega_\mu(E_-) \), where \( R_-(\Omega) = O(|\Omega - \Omega_\mu(E_-)|^3) \). We shall prove that

\[
\bar{z}_+''(\Omega_\mu(E_-)) = 0, \quad \text{and} \quad \bar{z}_-''(\Omega_\mu(E_-)) \neq 0. \tag{6.96}
\]

Along the proof, values at \( \Omega_\mu(E_\pm) \) of the functions \( M_\nu \) and its derivative are used repeatedly, which can be derived from either the definition of \( M_\mu \) or Lemma 6.2. For reader’s convenience, they are listed below:

\[
0 < \Omega_\mu(E_-) < E_\nu' < \Omega_\mu(E_+) \quad \text{and} \quad 0 < \Omega_\mu(E_-) < E_\nu'' < \Omega_\mu(E_+);
\]

\[
M_\nu(\Omega_\mu(E_-)) = 1 - \left( \int_{x - \Omega_\mu(E_-)}^{x} d\nu(x) \right)^{-1} \in (0, 1), \quad \text{and} \quad M_\nu(\Omega_\mu(E_+)) \in (1, \infty);
\]

\[
M_\nu'(\Omega_\mu(E_\pm)) = 1 + \frac{x}{(x - \Omega_\mu(E_\pm))^2} d\nu(x) > 1;
\]

\[
M_\nu''(\Omega_\mu(E_-)) = \frac{x}{(x - \Omega_\mu(E_-))^4} d\nu(x) > 0 \quad \text{and} \quad M_\nu''(\Omega_\mu(E_+)) < 0.
\]

For the first derivative, by the definition of \( \bar{z}_- \) and the fact that \( E_- \in \mathcal{V} \) satisfies (6.51), we see that

\[
\bar{z}_-''(\Omega_\mu(E_-)) \]

\[
= \left[ \frac{M_\mu^{-1} \circ M_\nu(\Omega)}{M_\nu(\Omega)} + \Omega \frac{M_\nu'(\Omega)}{M_\nu(\Omega)} + \Omega \frac{M_\nu''(\Omega)}{M_\nu(\Omega)^2} \right]_{\Omega = \Omega_\mu(E_-)}
\]

\[
= \frac{1}{M_\nu'(\Omega_\mu(E_-))} \left[ \frac{\Omega_\nu(E_-)}{M_\nu'(\Omega_\mu(E_-))} M_\nu'(\Omega_\nu(E_-)) + \frac{\Omega_\nu(E_-)}{M_\nu'(\Omega_\nu(E_-))} M_\nu'(\Omega_\mu(E_-)) \right.
\]

\[
- \frac{\Omega_\nu(E_-) M_\nu'(\Omega_\nu(E_-)) M_\nu'(\Omega_\mu(E_-))}{M_\rho'(\Omega_\nu(E_-))^2}
\]

\[
= 0, \tag{6.98}
\]

since \( M_\nu'(\Omega_\nu(E_-)) \neq 0 \).

Differentiating once again we obtain

\[
\bar{z}_-'''(\Omega) \]

\[
= 2 \left[ \frac{M_\nu'(\Omega)}{M_\nu(\Omega)} M_\nu'(\Omega_\nu(E_-)) + \frac{M_\nu'(\Omega)}{M_\nu(\Omega)^2} M_\nu''(\Omega) + \frac{M_\nu''(\Omega)}{M_\nu'(\Omega)} \right]
\]

\[
+ \Omega \left[ \frac{M_\nu''(\Omega)}{M_\nu'(\Omega)} + \frac{M_\nu'(\Omega)}{M_\nu(\Omega)^2} M_\nu''(\Omega) \right] - \frac{M_\nu'(\Omega)}{M_\nu(\Omega)^2}
\]

\[
= 0. \tag{6.99}
\]

Plugging in \( \Omega = \Omega_\mu(z) \), above simplifies to

\[
2 \left[ \frac{M_\nu'(\Omega_\mu(z))}{M_\rho'(\Omega_\nu(z))} - \frac{\Omega_\nu(z) M_\nu'(\Omega_\mu(z))}{M_\rho'(\Omega_\nu(z))^2} M_\rho'(\Omega_\nu(z))^2 \right]
\]

\[
+ \frac{\Omega_\nu(z)}{M_\rho'(z)} \left[ \frac{M_\nu'(\Omega_\mu(z))}{M_\rho'(\Omega_\nu(z))} - \frac{M_\nu'(\Omega_\nu(z)) M_\nu''(\Omega_\mu(z))}{M_\rho'(\Omega_\nu(z))^3} \right] - \Omega_\nu(z) M_\rho'(z) \left[ \frac{M_\nu''(\Omega_\mu(z))}{M_\rho'(z)^2} - 2 \frac{M_\nu'(\Omega_\mu(z))^2}{M_\rho'(z)^3} \right]. \tag{6.100}
\]

Now if \( z = E_- \), as \( E_- \) satisfies (6.51), combining the first three terms and the last term we have

\[
\frac{M_\nu'(\Omega_\mu(E_-))}{M_\rho'(\Omega_\nu(E_-))} - \frac{\Omega_\nu(E_-) M_\nu'(\Omega_\mu(E_-))}{M_\rho'(E_-)^2} = \frac{M_\nu'(\Omega_\mu(E_-))}{M_\rho'(E_-) M_\rho'(\Omega_\nu(E_-))}. \tag{6.101}
\]
Similarly, considering the fourth and sixth terms,
\[
\frac{\Omega_{\mu}(E_-)M'_{\nu}(\Omega_{\mu}(E_-))}{M_{\nu}(E_-)M'_{\nu}(\Omega_{\mu}(E_-))} - \frac{\Omega_{\mu}(E_-)M''_{\nu}(\Omega_{\mu}(E_-))}{M_{\nu}(E_-)^2} = -\frac{\Omega_{\nu}(E_-)M''_{\mu}(\Omega_{\mu}(E_-))}{M_{\nu}(E_-)M'_{\nu}(\Omega_{\mu}(E_-))}.
\]
(6.102)

Therefore, we have
\[
\tilde{z}''(\Omega_{\mu}(E_-)) = 2 \frac{M'_{\mu}(\Omega_{\mu}(E_-))}{M_{\mu}(E_-)} - \frac{\Omega_{\nu}(E_-)M''_{\mu}(\Omega_{\mu}(E_-))}{M_{\nu}(E_-)M'_{\nu}(\Omega_{\mu}(E_-))} - \frac{\Omega_{\mu}(E_-)M'_{\nu}(\Omega_{\mu}(E_-))^2M''_{\nu}(\Omega_{\nu}(E_-))}{M_{\mu}(E_-)M'_{\nu}(\Omega_{\nu}(E_-))^3}.
\]
(6.103)

Considering (6.51) again, we have
\[
M'_{\mu}(\Omega_{\nu}(E_-)) \frac{\Omega_{\nu}(E_-)}{M_{\nu}(E_-)} \left( \frac{\Omega_{\mu}(E_-)M'_{\nu}(\Omega_{\mu}(E_-))}{M_{\mu}(E_-)} - 1 \right) = \frac{\Omega_{\mu}(E_-)M'_{\mu}(\Omega_{\mu}(E_-))}{M_{\mu}(E_-)},
\]
so that
\[
2 \frac{M'_{\mu}(\Omega_{\mu}(E_-))}{M_{\mu}(E_-)M'_{\mu}(\Omega_{\nu}(E_-))} - \frac{\Omega_{\nu}(E_-)M''_{\mu}(\Omega_{\mu}(E_-))}{M_{\nu}(E_-)M'_{\nu}(\Omega_{\mu}(E_-))} = \frac{\Omega_{\mu}(E_-)}{M_{\mu}(E_-)} \left( 2 \frac{M'_{\mu}(\Omega_{\mu}(E_-))}{M_{\mu}(E_-)} - \frac{2}{\Omega_{\mu}(E_-)} - \frac{M''_{\mu}(\Omega_{\mu}(E_-))}{M'_{\mu}(\Omega_{\mu}(E_-))} \right).
\]
(6.105)

Now recalling Lemma 6.2, we see that
\[
M'_{\mu}(z) = \frac{d}{dz} \left( z + \int \frac{z}{x-z} d\tilde{\mu}(x) \right) = \frac{d}{dz} \left( z - \tilde{\mu}[\mathbb{R}] + \int \frac{x}{x-z} d\tilde{\mu}(x) \right) = 1 + \int \left( \frac{1}{x-z} + z \frac{1}{(x-z)^2} \right) \tilde{\mu}(x).
\]
(6.106)

Similarly
\[
M''_{\mu}(z) = 2 \int \frac{x}{(x-z)^3} d\tilde{\mu}(x) = 2 \int \left( \frac{1}{(x-z)^2} + z \frac{1}{(x-z)^3} \right) d\tilde{\mu}(x).
\]
(6.107)

Denoting \( \int (x-z)^{-k}d\tilde{\mu}(x) = m_k(z) \equiv m_k \), we see that
\[
2z M'_{\mu}(z)^2 - 2M_{\mu}(z) M'_\mu(z) - 2z M_{\mu}(z) M''_{\mu}(z)
\]
\[= 2z(1 + m_1 + zm_2)^2 - 2(z + zm_1)(1 + m_1 + zm_2) - 2z(z + zm_1)(m_2 + zm_3) = 2z^3(-m_3 - m_1m_3 + m_2^2).
\]
(6.108)

Since \( 0 < \Omega_{\mu}(E_-) < E'_- \), Cauchy-Schwarz inequality implies \( m_1m_3 > m_2^2 \) for \( z = \Omega_{\mu}(E_-) \). Therefore by (6.54), we see that
\[
\tilde{z}''(\Omega_{\mu}(E_-)) < 0.
\]
(6.109)

Thus defining \( \gamma_{\alpha} = -2/(\tilde{z}''(\Omega_{\mu}(E_-))) > 0 \), from the Taylor series expansion of \( \tilde{z}_-(\Omega) \) we see that
\[
z - E_- = -\frac{1}{\gamma_{\alpha}}(\Omega_{\mu}(z) - \Omega_{\mu}(E_-))^2 + R_-(\Omega_{\mu}(z))
\]
(6.110)

for \( z \) in a neighborhood of \( E_- \), by continuity of \( \Omega_{\mu} \). Inverting the expansion concludes the proof for lower edge.

For the upper edge, we similarly define \( \tilde{z}_+ \) to be the inverse function of \( \Omega_{\mu}(z) \) in a neighborhood of \( E_+ \), so that its derivatives have the exactly same form as those of \( \tilde{z}_- \). Thus \( \tilde{z}'_+(\Omega_{\mu}(E_+)) = 0 \) can be proved.
in a completely analogous manner. On the other hand, observing that \( M''_{\nu}(\Omega_{\mu}(E_+)) \) and \( M''_{\mu}(\Omega_{\nu}(E_+)) \) are strictly negative, we can immediately see that \( \tilde{z}_+''(E_+) > 0 \) from (6.103). Now the result follows again from Taylor expansion of \( \tilde{z}_+'' \) around \( \Omega_{\mu}(E_+) \). Note the difference of signs of \( \tilde{z}_+'' \) induces that of branches of square roots in (6.89) and (6.90).

Proof of Theorem 2.6. We first recall that,

\[
\text{Im}(zm_{\nu}(z) + 1) = \text{Im}(\Omega_{\mu}(z)m_{\nu}(z) + 1) = I_{\nu}(\Omega_{\mu}(z)) \text{Im} \Omega_{\mu}(z), \quad \forall z \in \mathbb{C}_+.
\]

(6.111)

From Proposition 6.10, we also find that \( I_{\nu}(\Omega_{\mu}(z)) \) is bounded below and above uniformly in any compact set \( D \subset \mathbb{C}_+ \cup \mathbb{R} \setminus \{0\} \). Thus we see that

\[
\frac{x\rho(x)}{\sqrt{x - E_-}} = \frac{\text{Im}(xm_{\nu}(x) + 1)}{\sqrt{x - E_-}}
\]

(6.112)
is bounded above and below by

\[
\frac{\text{Im} \Omega_{\mu}(x)}{\sqrt{x - E_-}} = \begin{cases} 
\text{Im} \sqrt{E_- - x} + O(|z - E_-|) = 1 + O(|z - E_-|) & \text{if } x > E_-,
0 & \text{otherwise},
\end{cases}
\]

(6.113)

around \( E_- \). Similar reasoning for \( E_+ \) proves the assertion.

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