

C_{2^n}\text{-EQUIVARIANT RATIONAL STABLE STEMS AND CHARACTERISTIC CLASSES}

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Abstract. In this short note, we compute the rational C_{2^n}\text{-equivariant stable stems and give minimal presentations for the } RO(C_{2^n})\text{-graded Bredon cohomology of the equivariant classifying spaces } BC_{2^n}S^1 \text{ and } BC_{2^n}Σ\text{ over the rational Burnside functor } AQ. \text{ We also examine for which compact Lie groups } L \text{ the maximal torus inclusion } T \to L \text{ induces an isomorphism from } H^\ast_C (BC_{2^n}L; AQ) \text{ onto the fixed points of } H^\ast_C (BC_{2^n}T; AQ) \text{ under the Weyl group action. We prove that this holds for } L = U(m) \text{ and any } n, m \geq 1 \text{ but does not hold for } L = SU(2) \text{ and } n > 1.

1. Introduction

This note is the followup to [Geo21c]. We start by computing the C_{2^n}\text{-equivariant rational stable stems; this is done in section 4. While the method employed here is the one used in [Geo21c] and goes back to [GM95], the result is quite a bit more complicated to state and requires the notation set up in sections 2 and 3.

We then attempt to generalize the results in [Geo21c] to groups C_{2^n}. In [Geo21c], we obtained minimal descriptions of the C_2\text{-equivariant Chern, Pontryagin and symplectic characteristic classes associated with genuine (Bredon) cohomology using coefficients in the rational Burnside Green functor } AQ. \text{ The idea was based on the maximal torus isomorphism: if } L \text{ is any one of } U(m), Sp(m), SO(m), SU(m), T \text{ is a maximal torus in } L \text{ and } W \text{ is the associated Weyl group then the inclusion } BC_2 T \to BC_2 L \text{ induces an isomorphism } H^\ast_C (BC_2 L; AQ) \to H^\ast_C (BC_2 T; AQ)^W. \text{ We then computed } H^\ast_C (BC_2 T; AQ) \text{ from } H^\ast_C (BC_2 S^1; AQ) \text{ and the Kunneth formula, which reduced us to the algebraic problem of computing a minimal presentation of the fixed points } H^\ast_C (BC_2 T; AQ)^W.

In section 5, we generalize the maximal torus isomorphism to groups } G = C_{2^n} \text{ when } L = U(m), \text{ but show that the maximal torus isomorphism is not true for...
\( G = C_{2n} \) and \( L = SU(2) \) when \( n > 1 \). We also compute the Green functor \( H^\bigstar_G(B_G S^1; \mathbb{A}_Q) \) which turns out to be algebraically quite a bit more complex compared to the \( C_2 \) case of [Geo21c]. For that reason, we do not attempt to follow the program in [Geo21c] and get minimal descriptions of \( H^\bigstar_G(B_G U(m); \mathbb{A}_Q) \) from the maximal torus isomorphism.

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2. **Rational Mackey functors**

The rational Burnside Green functor \( A_Q \) over a group \( G \) is defined on orbits as

\[ G/H \mapsto A(H) \otimes \mathbb{Q} \]

where \( A(H) \) is the Burnside ring of \( H \). A rational \( G \)-Mackey functor is by definition an \( A_Q \) module.

We shall use \( G \)-equivariant *unreduced* co/homology in \( A_Q \) coefficients. So if \( X \) is an unbased \( G \)-space, \( H^\bigstar_G(X) = H^\bigstar_G(X/G) \) is the rational \( G \)-Mackey functor defined on orbits as

\[ H^\bigstar_G(X)(G/H) = [S^\bigstar, X_+ \wedge HA_Q]^H \]

where \( HA_Q \) is the equivariant Eilenberg-MacLane spectrum associated to \( A_Q \) and the index \( \bigstar \) ranges over the real representation ring \( RO(G) \).

We warn the reader of differing conventions that can be found in the literature: \( H^\bigstar(X) \) is sometimes used to denote the reduced homology Mackey functor (the group \( G \) being implicit), with \( H^G(X) \) denoting the value of this Mackey functor on the top level (i.e. the \( G/G \) orbit). In this paper, \( H^G(X) \) always denotes the Mackey functor and \( H^G(X)(G/G) \) always denotes the top level. This convention also applies when \( \bigstar = * \) ranges over the integers, in which case \( H_*(X) \) denotes the nonequivariant rational homology of \( X \).

All these conventions apply equally for cohomology \( H_G^\bigstar(X) \).

The \( RO(G) \)-graded Mackey functor \( H^G(X) \) is a module over the homology of a point \( H_G^\bigstar := H_G^\bigstar(\ast) \). This Green functor agrees with the equivariant rational stable stems:

\[ \pi^G_S \otimes \mathbb{Q} = H_G^\bigstar \]

Two facts about rational Mackey functors that we shall liberally use ([GM95]):

- All rational Mackey functors are projective and injective, so we have the Kunneth formula:

\[ H^G_G(X \times Y) = H^G_G(X) \boxtimes_{H^G_G} H^G_G(Y) \]

and duality formula:

\[ H^G_G(X) = \text{Hom}_{H^G_G}(H^G_G(X), H^G_G) \]

- For a \( G \)-Mackey functor \( M \) and a subgroup \( H \) of \( G \) consider the \( W_G H \) module \( M(G/H)/\text{Im}(\text{Tr}) \) where \( W_G H = N_G H/H \) is the Weyl group and \( \text{Im}(\text{Tr}) \) is the submodule spanned by the images of all transfer maps \( Tr_H^G \) for \( K \subseteq H \). If we let \( H \) vary over representatives of conjugacy classes of subgroups of \( G \) then we get a sequence of \( W_G H \) modules. This functor from rational \( G \)-Mackey functors
to sequences of $Q[H]$-modules is an equivalence of symmetric monoidal categories.

From now on, we specialize to the case $G = C_2^n$.

There are two 1-dimensional $Q[G]$ modules up to isomorphism: $Q$ with the trivial action and $Q$ with action $g \cdot 1 = -1$ where $g \in G$ is a generator. We shall denote the two modules by $Q$ and $Q_-$ respectively; every other module splits into a sum of these.

The representatives of conjugacy classes of $G = C_2^n$ are $H = C_2$, for every $0 \leq i \leq n$ thus the datum of a rational $G$-Mackey functor is equivalent to a sequence of rational $W_{C_i}H = C_2^n / C_2$ modules.

We let $M_i^+, 0 \leq i \leq n$, and $M_i^-, 0 \leq i < n$, be the Mackey functors corresponding to the sequences $C_2^n / C_2 \mapsto \delta_{ij} Q$ and $C_2^n / C_2 \mapsto \delta_{ij} Q_-$ respectively.

For example, $M_0^+, M_0^-$ are the constant Mackey functors corresponding to the modules $Q$ and $Q_-$ respectively.

Observe that:

- The $M_i^\pm$ are self-dual.
- $M_i^\pm \boxtimes M_j^\pm = 0$ if $i \neq j$.
- $M_i^\alpha \boxtimes M_j^\beta = M_i^{\alpha \beta}$ where $\alpha, \beta \in \{-1, 1\}$.

Henceforth we shall write $M_i$ for $M_i^+$.

The notation $M_i\{a\}$ shall mean a copy of $M_i$ with a choice of generator $a \in M_i(C_2^n / C_2) = Q$. The element $a$ generates $M_i\{a\}$ through its transfers:

$$M_i\{a\}(C_2^n / C_2) = \begin{cases} Q\{\text{Tr}_2^j(a)\} & \text{if } j \geq i \\ 0 & \text{if } j < i \end{cases}$$

We analogously define $M_i^{-}\{a\}$.

The rational Burnside $G$-Green functor is

$$A_Q(C_2^n / C_2) = \frac{Q[x_{i,j}]}{x_{i,j} \cdot x_{i,k} = 2^{i - \max(j,k)} x_{i,\min(j,k)}}$$

where $x_{i,j} = [C_2^n / C_2] \in A(C_2^n)$ for $0 \leq j < i$. To complete the Mackey functor description, we note that:

$$\text{Tr}_2^{j+1}(x_{i,j}) = x_{i+1,j} \quad \text{and} \quad \text{Tr}_2^{j+1}(1) = x_{i+1,i}$$

Let

$$y_i = \begin{cases} 1 - \frac{x_{i,i-1}}{2} & \text{if } i \geq 1 \\ 1 & \text{if } i = 0 \end{cases}$$

living in $A_Q(C_2^n / C_2)$). We can see that $y_i$ spans a copy of $M_i$ in $A_Q$ and:

$$A_Q = \oplus_{i=0}^n M_i\{y_i\}$$

This is an isomorphism of Green functors, where the RHS becomes a Green functor by setting the product of elements from different summands to be 0 and furthermore setting the $y_i$ to be idempotent ($y_i^2 = y_i$).
3. **Euler and orientation classes**

The real representation ring $RO(C_{2\pi})$ is spanned by the irreducible representations $1, \sigma, \lambda_k$ where $\sigma$ is the 1-dimensional sign representation and $\lambda_k$ is the 2-dimensional representation given by rotation by $2\pi s(m/2^n)$ degrees for $1 \leq m$ dividing $2^{n-2}$ and odd $1 \leq s < 2^n/m$. Note that 2-locally, $S^\lambda_{s,m} \simeq S^{\lambda_{s,m}}$ as $C_{2\pi}$-equivariant spaces, by the $s$-power map. Therefore, to compute $H_{G_{2\pi}}^{2}(X)$ it suffices to only consider $\star$ in the span of $1, \sigma, \lambda_k := \lambda_{1,2^k}$ for $0 \leq k \leq n - 2$ ($\lambda_0 = 2\sigma$ and $\lambda_n = 2\sigma$).

We shall now define generating classes for $H_G^\star$.

We first have Euler classes $a_{\sigma} : S^0 \hookrightarrow S^\sigma$ and $a_{\lambda_k} : S^0 \hookrightarrow S^{\lambda_k}$ given by the inclusion of the north and south poles; under the Hurewicz map these classes are $a_{\sigma} \in H_G^G(G/G)$ and $a_{\lambda_k} \in H_G^{G_{\lambda_k}}(G/G)$.

There are also orientation classes $u_{\sigma} \in H_{1-\sigma}^G(C_{2n}/C_{2n-1})$, $u_{2\sigma} \in H_{2-2\sigma}^G(G/G)$ and $u_{\lambda_k} \in H_{2-\lambda_k}^G(G/G)$ but we shall need a small computation in order to define them.

Using the cofiber sequence $C_{2n}/C_{2n-1} \rightarrow S_0 \stackrel{a_{\sigma}}{\rightarrow} S^\sigma$ we get:

$$\tilde{H}^G_*(S^\sigma) = M_n \{ a_{\sigma} \}$$

$$\tilde{H}^G_*(S^\sigma) = \bigoplus_{i=0}^{n-1} M_i \{ y_i \text{Res}_{2i}^{2^{n-1}}(u_{\sigma}) \}$$

where $\tilde{H}_*(X)$ denotes the reduced homology of a based $G$-space $X$. We can further see that $\tilde{H}^G_*(S^\sigma)$ is generated as a Green functor module by a class $u_{\sigma} \in H_{1-\sigma}^G(C_{2n}/C_{2n-1})$. So we get

$$\tilde{H}^G_*(S^{2\sigma}) = M_n \{ a_{\sigma} \} \bigoplus \bigoplus_{i=0}^{n-1} M_i \{ y_i \text{Res}_{2i}^{2^{n-1}}(u_{\sigma}) \}$$

Using that $S^{2\sigma} = S^\sigma \wedge S^\sigma$ and the Kunneth formula, we get a class $u_{2\sigma}$ restricting to $u_{\sigma}$ and:

$$\tilde{H}^G_*(S^{2\sigma}) = M_n \{ a_{\sigma} \} \bigoplus \bigoplus_{i=0}^{n-1} M_i \{ y_i \text{Res}_{2i}^{2^{n}}(u_{2\sigma}) \}$$

For $0 \leq k \leq n - 2$ we have a $G$-CW decomposition $S^0 \subseteq X \subseteq S^{\lambda_k}$ where $X$ consists of the points $(x_1, x_2, x_3) \in S^{\lambda_k} \subseteq \mathbb{R}^3$ with $x_1 = 0$ or $x_2 = 0$. From this decomposition we can see that:

$$\tilde{H}^G_*(S^{\lambda_k}) = \bigoplus_{i=0}^{n-1} M_i \{ y_i \text{Res}_{2i}^{2^{n}}(u_{\lambda_k}) \}$$

for a class $u_{\lambda_k} \in H_{2-\lambda_k}^G(G/G)$. This also works for $k = n - 1$ and $\lambda_{n-1} = 2\sigma$ giving a different way of obtaining $a_{2\sigma} = a_2^2$ and $u_{2\sigma}$.

The classes $u_{\sigma}, u_{\lambda_k}, 0 \leq k \leq n - 1$, have not been canonically defined so far. Once we fix orientations for the spheres $S^{\lambda_k}$, the $u_{\lambda_k}$ are uniquely determined by the following two facts:

- A $G$-self-equivalence of $S^{\lambda_k}$ induces the identity map on the Mackey functor $H^G_*(S^{\lambda_k})$ if it does so on its bottom level $H^G_1(S^{\lambda_k})(G/e)$.
- An orientation of $S^{\lambda_k}$ determines a generator for $\mathbb{Z} = H_2(S^2; \mathbb{Z})$ and consequently a generator for $Q = H_2(S^2; Q) = H^G_2(S^{\lambda_k})(G/e)$. 

4
The first fact is proven using that the Mackey functor $H^G_S(S^\lambda_k)$ is generated by the transfers of $y_i\, \text{Res}^{2n}_{2i}(u_{\lambda_k})$ where $i \leq k$, so we only need to check that the induced map is the identity on $H^G_S(S^\lambda_k)(G/C_{2i}) = H^C_{2i}(S^\lambda_k)(C_{2i}/C_{2i})$ which follows from the fact that $C_{2i}$ acts trivially on $S^\lambda_k$ when $i \leq k$.

We can similarly uniquely determine $u_r$ upon fixing an orientation of $S^r$ that is compatible with the orientation for $S^{\lambda_k-1} = S^{2^r}$, meaning that $\text{Res}^{2n}_{2^{r-1}}(u_{2r}) = u^G_r$.

The discussion regarding orientation classes can also be performed integrally, defining $A\mathbb{Z}$-orientation classes $u_r, u_{2r}, u_{\lambda_k}$ upon fixing orientations for $S^r, S^{2^r}, S^{\lambda_k}$ as above. The $A\mathbb{Z}$-orientation classes map to the corresponding $\mathbb{Z}$-orientation classes of [HHR16] under the map $HA\mathbb{Z} \to H\mathbb{Z}$ where $\mathbb{Z}$ is the constant Green functor corresponding to the trivial $G$-module $\mathbb{Z}$.

4. Rational stable stems

In this section we shall give a presentation of the Green functor $H^G_\star$ with generators and relations.

The generators are elements $r_k \in H^G_{V_k}(C_{2^n}/C_{2^{n-1}})$ spanning $M_{i_k}$, where $\epsilon_k = +$ or $-$, such that every element of $\prod_{H \subseteq G, \star \in RO(G)} H^G_{\star}(G/H)$ can be obtained from the $r_k$ using the operations of addition, multiplication, restriction, transfer and scalar multiplication (where the scalars are elements of $\prod_{H \subseteq G} A_\mathbb{Q}(G/H)$).

The fact that the $r_k$ span $M_{i_k}$ gives all the additive (Mackey functor) relations, but also implies certain multiplicative relations by means of the Kunneth formula: if $i_k < i_l$ then $r_k \cdot \text{Res}^{2i_l}_{2i_k}(r_l) = 0$ and if $i_k = i_l$ then $r_k r_l$ spans $M_{i_k}^{\epsilon_k \epsilon_l}$.

Finally, if $r \in H^G_{V_k}(C_{2^n}/C_{2^{n-1}})$ and there exists a unique $r' \in H^G_{V_k}(C_{2^n}/C_{2^{n-1}})$ with $rr' = y_i$, then we shall use the notation $y_i/r$ to denote $r'$. If $r, y_i/r$ are generators then we have the implicit relation $r \cdot (y_i/r) = y_i$.

**Proposition 4.1.** The Green functor $H^G_\star$ has a presentation whose generating set is the union of the following four families:

- $y_i\, \text{Res}^{2i-1}_{2i}(u_{\epsilon})$ and $y_i/\text{Res}^{2i-1}_{2i}(u_{\epsilon})$ spanning $M_i^-$, where $0 \leq i < n$.
- $y_i\, \text{Res}^{2n}_{2i}(u_{\lambda_k})$ and $y_i/\text{Res}^{2n}_{2i}(u_{\lambda_k})$ spanning $M_i$, where $0 \leq i \leq k$ and $0 \leq k \leq n-2$.
- $y_i\, \text{Res}^{2n}_{2i}(a_{\lambda_k})$ and $y_i/\text{Res}^{2n}_{2i}(a_{\lambda_k})$ spanning $M_i$, where $k < i \leq n$ and $0 \leq k \leq n-2$.
- $a_{\epsilon} (= y_n a_{\epsilon})$ and $y_n/a_{\epsilon}$ spanning $M_n$.

We have implicit relations of the form $(y_i/y) \cdot (y_i/y) = y_i$ in each of the four families. The remaining multiplicative relations can be obtained using the Kunneth formula.

Two observations:

- For $0 \leq i < n$, the square of $y_i\, \text{Res}^{2i-1}_{2i}(u_{\epsilon})$ is $y_i\, \text{Res}^{2i}_{2i}(u_{2\epsilon})$ and spans $M_i$.
- The ring $H^G_\star(G/G)$ has multiplicative relations: $a_{\epsilon} u_{2\epsilon} = 0$, $a_{\epsilon} u_{\lambda_k} = 0$ and $a_{\lambda_k} u_{\lambda_k} = 0$ for $s \leq k$.

The Green functor presentation also gives us an additive decomposition of $H^G_\star$ into $M_i, M_i^-$ but to state it explicitly, we’ll need some notation: For each integer
tuple \( t = (j_0, \ldots, j_{n-1}, j'_0, \ldots, j'_{n-1}) \) let

\[
k(t) = \begin{cases} 
  n & \text{if } j_k = 0 \text{ for all } k \\
  \min\{k : j_k \neq 0\} & \text{otherwise}
\end{cases}
\]

and

\[
k'(t) = \begin{cases} 
  -1 & \text{if } j'_{k'} = 0 \text{ for all } k' \\
  \max\{k' : j'_{k'} \neq 0\} & \text{otherwise}
\end{cases}
\]

and consider the representation

\[
V_i^\pm = \sum_{k=0}^{n-2} (j_k(2 - \lambda_k) - j'_k\lambda_k) + j_{n-1}(1 - \sigma) - j'_{n-1}\sigma
\]

where the sign \( \pm \) in \( V_i^\pm \) is \( + \) if \( j_{n-1} \) is even and \( - \) if \( j_{n-1} \) is odd.

Let \( T \) be the set of all tuples \( t \) with \( k'(t) < k(t) \); as \( t \) ranges over \( T \), the \( V_i^\pm \) are pairwise non-isomorphic virtual representations. We can now state the additive description:

**Proposition 4.2.** The \( C_{2n} \) equivariant rational stable stems are:

\[
H^G_\star = \begin{cases} 
  \bigoplus_{k'(<k(t))} M_i & \text{if } \star = V_i^+ \text{ for } t \in T \\
  \bigoplus_{k'(<k(t))} M_i^- & \text{if } \star = V_i^- \text{ for } t \in T \\
  0 & \text{otherwise}
\end{cases}
\]

**Proof.** (Of Proposition 4.1). Any representation sphere \( S^r \) is the smash product of \( S^r, S^{\lambda_k} \) and their duals \( S^{-r}, S^{-\lambda_k} \). By duality,

\[
\tilde{H}^G_\star(S^{-r}) = \tilde{H}^G_\star(S^r) = M_n \oplus \oplus_{i=0}^{n-1} M_i^- \{y_i, \text{Res}_{2^i}(u_{\sigma}^{-1})\}
\]

Let \( t \) be a generator for this copy of \( M_n \); then

\[
\tilde{H}^G_\star(S^0) = \tilde{H}^G_0(S^r) \otimes \tilde{H}^G_0(S^{-r}) \oplus \tilde{H}^G_1(S^r) \otimes \tilde{H}^G_1(S^{-r})
\]

On the left hand side we have a factor \( M_n \{y_n\} \) and on the right hand side we have \( M_n \{a_\sigma\} \otimes M_n \{t\} = M_n \{a_\sigma t\} \) so \( y_n = \lambda a_\sigma t \) for \( \lambda \in \mathbb{Q}^\times \). Thus we can pick \( t = y_n/a_\sigma \). The result then follows from the Kunneth formula. \( \square \)

We note that taking geometric fixed points inverts all Euler classes, annihilating all orientation classes and setting \( y_i = 1 \). Therefore:

\[
\Phi^{C_{2n}Q}(HAQ)_\star = \mathbb{Q}[a_\sigma^\pm, a_\lambda^\pm]_{0 \leq k \leq n-2}
\]

hence \( \Phi^{C_{2n}Q}HAQ = HQ \) as nonequivariant spectra. The homotopy fixed points, homotopy orbits and Tate fixed points are computed using that \( HAQ \to HQ \) is a nonequivariant equivalence, where \( Q = M_0 \) is the constant Green functor. Thus:

\[
(HAQ)_{hC_{2n}}^\star = (HAQ)^{hC_{2n}}_\star = \mathbb{Q}[u_2^\pm, u_\lambda^\pm]_{0 \leq k \leq n-2}
\]

and \( (HAQ)^{C_{2n}}_\star = * \).
5. \( C_{2^n} \) rational characteristic classes

**Proposition 5.1.** As a Green functor algebra over the homology of a point:

\[
H_G^\bullet(B_G S^1) = \frac{H_G^\bullet[u, \alpha_{m,j}]_{1 \leq m \leq n, 1 \leq j < 2^n}}{\alpha_{m,j} \alpha_{m',j'} = \delta_{m,m'} \delta_{j,j'} \alpha_{m,j}, \ Res_{2m-1}^m(\alpha_{m,j}) = 0}
\]

for \(|u| = 2\) and \(|\alpha_{m,j}| = 0\).

**Proof.** Note that

\[
H_G^\bullet(X) = H_G^\bullet(X) \otimes_{A_Q} H_G^\bullet
\]

so it suffices to describe the integer graded cohomology.

For an explicit model of \( B_G S^1 \) we take \( \mathbb{C} P^\infty \) with a \( G \) action that can be described as follows: Let \( V_1, ..., V_{2^n} \) be an ordering on the irreducible complex \( G \)-representations and set \( V_{k+2^n} = V_k \) for any \( m \in \mathbb{Z}, 1 \leq k \leq 2^n \). The action of \( g \in G \) on homogeneous coordinates is \( g(z_1 : z_2 : \cdots) = (gz_1 : gz_2 : \cdots) \) where \( g \) acts on \( z_i \) as it does on \( V_i \).

Fix a subgroup \( H = C_{2^n} \) of \( G \). The fixed points under the \( H \)-action are:

\[
(B_G S^1)^H = \bigoplus_{j=1}^{2^n} \mathbb{C} P^\infty
\]

To understand the indexing, let \( W_1, ..., W_{2^n} \) be an ordering on the irreducible complex \( C_{2^n} \)-representations; the \( j \)-th \( \mathbb{C} P^\infty \) in \( (B_G S^1)^H \) corresponds to the set of points with homogeneous coordinates \( (z_1 : z_2 : \cdots) \) such that \( z_k = 0 \) if \( \text{Res}^{2^n}(V_k) \neq W_j \).

By [GM95] we have:

\[
H_G^\bullet(B_G S^1) = \bigoplus_{n=0}^{\infty} H^\bullet((B_G S^1) C_{2^n}) \mathbb{C} / C_{2^n}
\]

where \( H^\bullet(X) \) is nonequivariant cohomology in \( Q \) coefficients. The action of \( C_{2^n} / C_{2^n} \) on nonequivariant cohomology is trivial since it’s determined in degree \( * = 2 \) and thus on the 2-skeleton, which itself is the disjoint union of copies of \( S^2 = \mathbb{C} P^1 \) and for each \( S^2 \) the action is a rotation hence has degree 1. Thus

\[
H_G^\bullet(B_G S^1) = \bigoplus_{m=0}^{\infty} \bigoplus_{j=1}^{2^m} H^\bullet(\mathbb{C} P^\infty) = \bigoplus_{m=0}^{\infty} \bigoplus_{j=1}^{2^m} Q[e_{m,j}]
\]

where each \( e_{m,j} \) spans \( M_m \). Set \( \alpha_{m,j} = e_{m,j}^0 \) and \( u = \sum_m e_{m,j} \); then

\[
\sum_{j=1}^{2^m} \alpha_{m,j} = \frac{\text{Tr}_{2^m}(y_m)}{2^m}
\]

so the \( \alpha_{m,2^m} \) are superfluous. Thus we can take \( 1 \leq m \leq n \) and \( 1 \leq j < 2^n \) in the indexing for \( \alpha_{m,j} \).

We can similarly prove that:

**Proposition 5.2.** We have an isomorphism of Green functor algebras over \( H_G^\bullet \):

\[
H_G^\bullet(B_G \Sigma_2) = \frac{H_G^\bullet(B_G S^1)}{u}
\]

where the quotient map \( H_G^\bullet(B_G S^1) \to H_G^\bullet(B_G \Sigma_2) \) is induced by complexification: \( B_G \Sigma_2 = B_G O(1) \to B_G U(1) = B_G S^1 \).
The set of generators \( \{ u, \alpha_{m,i} \} \) for \( H^*_G(B_GS^1) \) is not minimal. Indeed, whenever we have generators \( e_1, \ldots, e_s \) with \( e_ie_j = \delta_{ij}e_i \), we can replace them by a single generator defined by \( e = e_1 + 2e_2 + \cdots + se_s \):

\[
\frac{Q[e_1, \ldots, e_s]}{e_ie_j = \delta_{ij}e_i} = \frac{Q[e]}{e(e-1)\cdots(e-s)}
\]

This isomorphism follows from the fact that any polynomial \( f \) on \( e_1, \ldots, e_n \) satisfies:

\[
f(e) = f(0) + (f(1) - f(0))e_1 + \cdots + (f(s) - f(0))e_s
\]

and thus

\[
e_i = \frac{f_i(e)}{f_i(i)} \text{ where } f_i(x) = \frac{x(x-1)\cdots(x-s)}{x-i}
\]

In this way, \( H^*_G(B_GS^1) \) is generated as an \( A_Q \) algebra by two elements \( u, \alpha \) but now with \( \alpha \) satisfying some rather complicated relations. If \( n = 1 \) i.e. \( G = C_2 \), we only have one \( \alpha_{m,i} \) element, namely \( \alpha = \alpha_{1,1} \) satisfying \( \alpha^2 = \alpha \).

**Proposition 5.3.** The inclusion \( B_GU(1)^m \to B_GU(m) \) induces an isomorphism of Green functor algebras over \( H^*_G \):

\[
H^*_G(B_GU(m)) = (\otimes^m H^*_G(B_GU(1)))^{\Sigma_m}
\]

**Proof.** Let \( V_i \) be the complex \( G \)-representation corresponding to the root of unity \( e^{2\pi i/n} \). The Grassmannian model for \( B_GU(m) \) uses complex \( m \)-dimensional subspaces of \( \mathbb{C}^{\infty \times \infty} \); a \( G \)-fixed point \( W \) of \( B_GU(m) \) is then a \( G \)-representation and thus as splits as \( W = \bigoplus_{i=1}^{2^m} k_i V_i \) for \( k_i = 0, 1, \ldots \) with \( \sum k_i = m \). An automorphism of \( W \) is made out automorphisms for each \( k_i V_i \) hence

\[
B_GU(m)^G = \coprod_{\Sigma_{k_i = m}} \bigotimes_{i=1}^{2^m} BU(k_i)
\]

Following [Geo21c] and inducting on the \( n \) in \( G = C_{2^n} \), it suffices to show that

\[
H^*(B_GU(m)) \to H^*(\bigotimes_{i=1}^{2^m} BU(k_i))
\]

is an isomorphism after taking \( \Sigma_m \) fixed points on the RHS. Spelling this out, we have:

\[
\prod_{\Sigma_{k_i = m}} \otimes_{i=1}^{2^n} H^*(BU(k_i)) \to \prod_{i=1}^{2^m} \otimes^m H^*(BS^1)
\]

where the product on the right is indexed on configurations \( (V_{r_1}, \ldots, V_{r_m}) \). If we fix \( k_i \) with \( \sum_i k_i = m \) then we get

\[
\otimes_{i=1}^{2^n} H^*(BU(k_i)) \to \prod_{i=1}^{2^m} \otimes^m H^*(BS^1)
\]

where the product on the right is indexed on configurations \( (V_{r_1}, \ldots, V_{r_m}) \) where \( k_i \) many of the \( r_i \)'s are equal to \( i \). Taking \( \Sigma_m \) fixed points is equivalent to fixing a configuration and then taking \( \Sigma_{k_1} \times \cdots \times \Sigma_{k_2^n} \) fixed points, where each \( \Sigma_{k_i} \) permutes the \( k_i \) many coordinates that are \( V_i \) in the configuration. Thus we are reduced to the nonequivariant isomorphism:

\[
H^*(BU(k_i)) = (\otimes^k H^*(BS^1))^{\Sigma_{k_i}}
\]

\( \square \)
For $n = 1$, $G = C_2$ and $H^*_G(B_GS^1)$ has a simple enough description to allow the computation of an explicit minimal presentation of $H^*_G(B_GU(m)) = H^*_G(B_GU(U(1)^m))$. Due to the greater algebraic complexity of $H^*_G(B_GS^1)$ for $n \geq 2$ ($G = C_{2^n}$), we do not attempt to generalize this and the rest of [Geo21c] to groups $G = C_{2^n}$ for $n \geq 2$.

We note that the maximal torus isomorphism does not work $C_{2^n}$ equivariantly for the Lie group $L = SU(2) = Sp(1)$ and $n \geq 2$. The reason is that a $C_{2^n}$ representation in $SU(2)$ is $2, 2\sigma$ or $V_i \oplus V_{-i}, 1 \leq i < 2^{n-1}$, using the notation of the proof above. Thus:

$$B_GSU(2)^G = BSU(2) \coprod_{i=1}^{2^{n-1}-1} BS^1$$

so $H^0(B_GSU(2)^G)$ has dimension $2^n + 1$. On the other hand, the maximal torus is $U(1) \subseteq SU(2)$ with Weyl group $C_2$ and

$$B_GU(1)^G = \coprod_{i=1}^{2^n} BS^1$$

The $C_2$ action does not affect $H^0(B_GU(1)^G)$ which has dimension $2^n$. Finally, $2^n + 1 = 2^n$ only when $n = 1$.

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9