A GENERALIZED WOLFF’S IDEAL THEOREM ON
certain subalgebras of $H^\infty(D)$

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Abstract. We prove the generalized Wolff’s Ideal Theorem on
certain uniformly closed subalgebras of $H^\infty(D)$ on which the Corona
Theorem is already known to hold.

1. Introduction

Carleson’s celebrated proof of the Corona Theorem [1], which gives
necessary and sufficient conditions for unit membership in the ideal
of $H^\infty(D)$ generated by a given set of functions, opened the door for
several new questions. The first we will consider, which we call a “ge n-
eralized ideal problem,” asks whether we can find weaker conditions
under which a given function $h$ is included in the ideal. If not, can we
at least find some $p > 1$ so that $h^p$ belongs to the ideal? Also, are there
other algebras for which a result similar to Carleson’s holds?

The first two questions were proposed and answered (at least in part)
by Wolff [21] in a result we refer to as “Wolff’s Theorem.” The third
has been a topic of research over the years, with varying results. (For
examples of algebras on which a corona theorem holds, see Tolokon-
nikov [15] and Nikolski [7], as well as Costea-Sawyer-Wick [3]; for some
negative examples, see Scheinberg [13] and Trent [20].)

Carleson’s Corona Theorem states that the ideal $I$ generated by a
finite set of functions $\{f_i\}_{i=1}^n \subset H^\infty(D)$ is the entire space $H^\infty(D)$
provided that there exists $\delta > 0$ such that

$$\left( \sum_{i=1}^n |f_i(z)|^2 \right)^{1/2} \geq \delta \text{ for all } z \in D. \quad (1)$$

This result can be extended to hold for infinitely many functions $\{f_i\}_{i=1}^\infty$
(see [10], [13]).

Under what conditions, then, could we expect a given function $h \in
H^\infty(D)$ to be found in $I$? One might suppose, based on Carleson’s
result, that a sufficient condition would be

\[
\left( \sum_{i=1}^{n} |f_i(z)|^2 \right)^{\frac{1}{2}} \geq |h(z)| \quad \text{for all } z \in \mathbb{D}.
\]  

(2)

Although necessary, Rao proved that (2) is not sufficient (see Garnett [5]). Wolff, however, proved that, given (2), \( h^3 \in \mathcal{I} [5] \).

**Theorem A (Wolff).** If

\[
\{f_j\}_{j=1}^{n} \subset H^\infty(\mathbb{D}), h \in H^\infty(\mathbb{D}) \quad \text{and}
\]

\[
\left( \sum_{j=1}^{n} |f_j(z)|^2 \right)^{\frac{1}{2}} \geq |h(z)| \quad \text{for all } z \in \mathbb{D},
\]

then

\[ h^3 \in \mathcal{I}(\{f_j\}_{j=1}^{n}), \]

the ideal generated by \( \{f_j\}_{j=1}^{n} \) in \( H^\infty(\mathbb{D}) \).

Treil [16] has since shown that Wolff’s Theorem fails when the exponent “3” is replaced with “2”.

Note that if we consider the radical of the ideal \( \mathcal{I}(\{f_j\}_{j=1}^{n}) \),

\[
\text{Rad}(\{f_j\}_{j=1}^{n}) = \{h \in H^\infty(\mathbb{D}) : \exists q \in \mathbb{N} \text{ with } h^q \in \mathcal{I}(\{f_j\}_{j=1}^{n}) \},
\]

then (2) gives a characterization of radical ideal membership. That is, \( h \in \text{Rad}(\{f_j\}_{j=1}^{n}) \) if and only if there exists \( M < \infty \) and \( q \in \mathbb{N} \) such that \( |h^q(z)| \leq M \left( \sum_{i=1}^{n} |f_i(z)|^2 \right)^{\frac{1}{2}} \) for all \( z \in \mathbb{D} \).

For \( \{f_j\}_{j=1}^{\infty} \subset H^\infty(\mathbb{D}) \), let \( F(z) = (f_1(z), f_2(z), ...) \). If we assume

\[
1 \geq (F(z)F(z)^*)^{\frac{1}{2}} \geq |h(z)| \quad \text{for all } z \in \mathbb{D},
\]

then we get \( h \in H^\infty(\mathbb{D}) \). Can we improve this estimate and still obtain ideal membership for \( h \)? In other words, for what increasing function \( \psi \) does the condition

\[
|h(z)| \leq F(z)F(z)^* \psi(F(z)F(z)^*), \quad \text{for all } z \in \mathbb{D}
\]

imply that \( h \in \mathcal{I} \)? Many authors, independently, have considered this question, including Cegrell [2], Pau [9], Trent [19], and Treil [17]. It is Treil who has given the best known sufficient condition for ideal membership.

We give Treil’s Theorem as follows:
Theorem B (Treil). Let $F(z) = (f_1(z), f_2(z), \ldots), \{f_j\}_{j=1}^{\infty} \subset H^\infty(\mathbb{D})$, $F(z)F(z)^* \leq 1$ for all $z \in \mathbb{D}$, and $h \in H^\infty(\mathbb{D})$ such that

$$1 \geq F(z)F(z)^* \psi(F(z)F(z)^*) \geq |h(z)| \text{ for all } z \in \mathbb{D},$$

where $\psi : [0, 1] \to [0, 1]$ is an increasing function such that $\int_0^1 \frac{\psi(t)}{t} \, dt < \infty$. Then there exists $G(z) \in \bigoplus_{1}^{\infty} H^\infty(\mathbb{D})$ such that

$$F(z)G(z)^T = h(z), \text{ for all } z \in \mathbb{D}.$$

For our paper, we consider three types of subalgebras of $H^\infty(\mathbb{D})$. We use the fact that both the Corona Theorem and Wolff’s Theorem hold on $H^\infty(\mathbb{D})$ to find solutions contained within the given subalgebras. However, it should be noted that Scheinberg’s results [13] show that the Corona Theorem, and thus Wolff’s Ideal Theorem, fails for some unital closed subalgebras of $H^\infty(\mathbb{D})$.

The first type is the collection of subalgebras of the form

$$\mathbb{C} + BH^\infty(\mathbb{D}) = \{\alpha + Bg : \alpha \in \mathbb{C}, g \in H^\infty(\mathbb{D})\},$$

where $B$ is a fixed Blaschke product. We can regard $\bigoplus_{1}^{\infty} (\mathbb{C} + BH^\infty(\mathbb{D}))$, denoted by $(\mathbb{C} + BH^\infty(\mathbb{D}))_{1^2}$, as a closed subalgebra of $H^\infty_{1^2}(\mathbb{D})$.

It should be noted that this algebra was introduced and function problems were considered in J. Solasso [14], M. Ragupathi [8], and Davidson, Paulsen, Ragupathi, and Singh [4]. In [6], Mortini, Sasane, and Wick proved the Corona Theorem for a finite number of generators, whereas the infinite version is due to Ryle and Trent [11] [12]. One can easily check that the same Rao’s example serves as a counter example for this subalgebra also. So, condition (2) is not sufficient to characterize the ideal membership in the algebra $\mathbb{C} + BH^\infty(\mathbb{D})$.

One of the goals of this paper is to solve the generalized ideal problem and extend Wolff’s ideal theorem to $\mathbb{C} + BH^\infty(\mathbb{D})$.

Theorem 1.1. Let $F(z) = (f_1(z), f_2(z), \ldots) \in (\mathbb{C} + BH^\infty(\mathbb{D}))_{1^2}$, $F(z)F(z)^* \leq 1$ for all $z \in \mathbb{D}$ and $h(z) \in \mathbb{C} + BH^\infty(\mathbb{D})$, with

$$1 \geq F(z)F(z)^* \psi(F(z)F(z)^*) \geq |h(z)| \text{ for all } z \in \mathbb{D},$$

where $\psi$ is a function given as in Theorem B. Then there exists $V(z) = (v_1(z), v_2(z), \ldots) \in (\mathbb{C} + BH^\infty(\mathbb{D}))_{1^2}$ such that

$$F(z)V(z)^T = h(z) \text{ for all } z \in \mathbb{D}.$$
Corollary 1.1. Let $F(z) = (f_1(z), f_2(z), \ldots) \in (\mathbb{C} + BH^\infty(\mathbb{D}))_{12}$ and $h(z) \in \mathbb{C} + BH^\infty(\mathbb{D})$, with
\[
1 \geq |F(z)F(z)^*|^{\frac{1}{2}} \geq |h(z)| \quad \text{for all} \quad z \in \mathbb{D}.
\]
Then there exists $V(z) = (v_1(z), v_2(z), \ldots) \in (\mathbb{C} + BH^\infty(\mathbb{D}))_{12}$ such that
\begin{align*}
(a) & \quad F(z)V(z)^T = h^3(z) \quad \text{for all} \quad z \in \mathbb{D}, \text{ and} \\
(b) & \quad \|V\|_\infty \leq (1 + \frac{1}{\|F(0)\|_1})C_0,
\end{align*}
where $C_0 = (1 + 4\sqrt{e} + 8\sqrt{2e} + 72e^3)$ and $\alpha$ is a zero of $B(z)$.

For the second type of subalgebra, let $K \subset \mathbb{Z}_+$ and define
\[
H^\infty_K(\mathbb{D}) = \{f \in H^\infty(\mathbb{D}) : f^{(j)}(0) = 0 \quad \text{for all} \quad j \in K\}.
\]
We consider those sets $K$ for which $H^\infty_K(\mathbb{D})$ is an algebra under the usual product of functions. Obviously, not every set $K$ defines an algebra; for example, let $K = \{2\}$. Though there is not a complete characterization of the set $K$ for which $H^\infty_K(\mathbb{D})$ is an algebra, Ryle and Trent have given certain criteria that the set $K$ must meet. In Lemma 2.1, we will state some of these criteria. For our purposes we assume $K$ is finite. (We justify this assumption in the next section.)

We define algebras comprised of vectors with entries in $H^\infty_K(\mathbb{D})$ as follows:
\[
\mathcal{H}^\infty_{K,n}(\mathbb{D}) = \{f_j\}_{j=1}^n : f_j(z) \in H^\infty_K(\mathbb{D}) \text{ for } j = 1, 2, \ldots, n
\]
and
\[
\sup_{z \in \mathbb{D}} \sum_{j=1}^n \|f_j(z)\|^2 < \infty.
\]
Multiplication here is entrywise, and $n$ can be either a positive integer or $\infty$. We write the elements of $\mathcal{H}^\infty_{K,n}(\mathbb{D})$ as row vectors, so that $F(z) \in \mathcal{H}^\infty_{K,n}(\mathbb{D})$ means that $F(z) \in \mathcal{M}(\mathbb{C}^n, \mathbb{C})$ for fixed $z$. If $n = \infty$, then $F(z) \in \mathcal{M}(\ell^2, \mathbb{C})$.

In Theorem 1.2 and the Corollary 1.2, we will extend the analogue of Theorem 1.1 and the Corollary 1.1, respectively, in this algebra. However, we need the additional assumption that $F(0) \neq 0$.

Theorem 1.2. Let $F(z) = (f_1(z), f_2(z), \ldots) \in \mathcal{H}^\infty_{K,n}(\mathbb{D})$ and $h(z) \in H^\infty_K(\mathbb{D})$, with $1 \geq |F(z)F(z)^*\psi(F(z)F(z)^*)| \geq |h(z)| \forall z \in \mathbb{D}$. Suppose also that $F(0) \neq 0$. Then there exists $V(z) = (v_1(z), v_2(z), \ldots) \in \mathcal{H}^\infty_{K,n}(\mathbb{D})$ such that
\[
F(z)V(z)^T = h(z) \forall z \in \mathbb{D}.
\]
Corollary 1.2. Let $F(z) = (f_1(z), f_2(z), \ldots) \in \mathcal{H}^\infty_{K,n}(\mathbb{D})$ and $h(z) \in H^\infty_K(\mathbb{D})$, with $[F(z)F(z)^*]^\frac{1}{2} \geq |h(z)| \quad \forall \ z \in \mathbb{D}$. Suppose also that $F(0) \neq 0$. Then there exists $V(z) = (v_1(z), v_2(z), \ldots) \in \mathcal{H}^\infty_{K,n}(\mathbb{D})$ such that

(a) $F(z)V(z)^T = h^3(z) \quad \forall \ z \in \mathbb{D}$, and

(b) $\|V\|_\infty \leq 2C_0$

where $C_0 = (1 + 4\sqrt{e} + 8\sqrt[8]{2}e + 72e^{\frac{1}{2}})$.

For the third type of algebra, let $K = \{k_1, \ldots, k_p\}$ be a nontrivial finite subset of $\mathbb{Z}_+$ such that $H^\infty_K(\mathbb{D})$ is an algebra, with $k_1 < \cdots < k_p$. For a fixed Blaschke product, $B$, we define

$$H^\infty_K(B)(\mathbb{D}) = \left\{ \sum_{j \notin K} a_j B^j + B^{k_p+1}g : g \in H^\infty(\mathbb{D}) \text{ and } a_j \in \mathbb{C} \right\}.$$

For $F \in \mathcal{H}^\infty_{K(B),n}(\mathbb{D})$, denote

$$F(z) = \sum_{j \notin K} B^j(z)F_j + B^{k_p+1}(z)F_{k_p+1}(z).$$

We define $\mathcal{H}^\infty_{K(B),n}(\mathbb{D})$ similarly to $\mathcal{H}^\infty_{K,n}(\mathbb{D})$.

Theorem 1.3. Let $F(z) = (f_1(z), f_2(z), \ldots) \in \mathcal{H}^\infty_{K(B),n}(\mathbb{D})$ and $h(z) \in H^\infty_K(B)(\mathbb{D})$, with $1 \geq F(z)F(z)^* \psi(F(z)F(z)^*) \geq |h(z)| \quad \forall \ z \in \mathbb{D}$. Suppose also that $F_0 \neq 0$. Then there exists $V(z) = (v_1(z), v_2(z), \ldots) \in \mathcal{H}^\infty_{K(B),n}(\mathbb{D})$ such that

$$F(z)V(z)^T = h(z) \quad \text{for all } z \in \mathbb{D}.$$

Corollary 1.3. Let $F(z) = (f_1(z), f_2(z), \ldots) \in \mathcal{H}^\infty_{K(B),n}(\mathbb{D})$ and $h(z) \in H^\infty_K(B)(\mathbb{D})$, with $[F(z)F(z)^*]^\frac{1}{2} \geq |h(z)| \quad \forall \ z \in \mathbb{D}$. Suppose also that $F_0 \neq 0$. Then there exists $V(z) = (v_1(z), v_2(z), \ldots) \in \mathcal{H}^\infty_{K(B),n}(\mathbb{D})$ such that

(a) $F(z)V(z)^T = h^3(z) \quad \forall \ z \in \mathbb{D}$, and

(b) $\|V\|_\infty \leq (1 + \frac{1}{\|F(0)\|_\infty})C_0$,

where $C_0 = (1 + 4\sqrt{e} + 8\sqrt[8]{2}e + 72e^{\frac{1}{2}})$ and $\alpha$ is a zero of $B(z)$.

Suppose we take the hypotheses of Theorem 1.2, but we allow $F(0) = 0$. Since $F(z)$ is a vector of holomorphic functions, we have $F(z) = z^m F_m(z)$ for some $m \in \mathbb{N}$ where the entries of $F_m(z)$ are holomorphic on $\mathbb{D}$ and $F_m(0) \neq 0$. One might attempt to continue in the vein of the proof to Theorem 1.2 with $F_m$ in place of $F$. Unfortunately, we need not expect $F_m$ to lie in $\mathcal{H}^\infty_{K,n}(\mathbb{D})$ or any subalgebra thereof. We
encounter a similar problem if we allow $F_0 = 0$ in \[1.3\] and factor $B(z)$ off of $F(z)$.

However, there are conditions under which generalized ideal membership (and thus Wolff’s Theorem) still hold even if we allow the vectors above to be zero. For a set $K$ such that $H_K^\infty(\mathbb{D})$ is an algebra and $m \in \mathbb{N}$, $m \notin K$, define $K - m = \{j - m : j \in K \text{ and } j > m\}$.

**Theorem 1.4.** Let $F(z) = (f_1(z), f_2(z), \ldots) \in \mathcal{H}_K^n(\mathbb{D})$ and $h(z) \in H_K^\infty(\mathbb{D})$, with $1 \geq F(z)F(z)^* \psi(F(z)F(z)^*) \geq |h(z)|$ for all $z \in \mathbb{D}$. Suppose also that $F(z) = z^m F_m(z)$ with $F_m(0) \neq 0$. If either

(i) $K - m$ defines an algebra $H_{K-m}^\infty(\mathbb{D})$, or

(ii) $m > k_p$,

then there exists $V(z) = (v_1(z), v_2(z), \ldots) \in \mathcal{H}_{K,n}^\infty(\mathbb{D})$ such that

$$F(z)V(z)^T = h(z) \quad \text{for all } z \in \mathbb{D}.$$ 

**Theorem 1.5.** Let $F(z) = (f_1(z), f_2(z), \ldots) \in \mathcal{H}_{K(B)}^\infty(\mathbb{D})$ and $h(z) \in H_{K(B)}^\infty(\mathbb{D})$, with $1 \geq F(z)F(z)^* \psi(F(z)F(z)^*) \geq |h(z)|$ for all $z \in \mathbb{D}$. Suppose also that $F_0 = 0$, and let $j_1 > 0$ be the greatest power of $B$ common to all terms of $F$. If either

(i) $K - j_1$ defines an algebra $H_{K-j_1}^\infty(\mathbb{D})$, or

(ii) $j_1 > k_p$,

then there exists $V(z) = (v_1(z), v_2(z), \ldots) \in \mathcal{H}_{K(B),n}^\infty(\mathbb{D})$ such that

$$F(z)V(z)^T = h(z) \quad \text{for all } z \in \mathbb{D}.$$ 

2. Preliminaries

Integral to the proofs of our theorems are “Q-operators” which are derived from the Koosul complex [11]. As these operators have already been discussed in several papers, we will only give the pertinent results here. Proofs of these results may be found in [11].

We let $H \wedge K$ denote the exterior product between two Hilbert spaces $H$ and $K$, and $l^2_{(n)} = \wedge_{i=1}^n l^2$. In keeping with this notation, $l^2_{(0)} = \mathbb{C}$.

Let $\{e_i\}_{i=1}^\infty$ denote the standard basis in $l^2$. If $I_n$ denotes increasing $n$-tuples of positive integers and if $(i_1, i_2, \ldots, i_n) \in I_n$, we let $\pi_n = (i_1, i_2, \ldots, i_n)$ and, abusing notation, we write $\pi_n \in I_n$. If we define $e_{\pi_n} = e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_n}$, then $\{e_{\pi_n}\}_{\pi_n \in I_n}$ is defined to be the standard basis for $l^2_{(n)}$.

Let $H(E)$ be a reproducing kernel Hilbert space on a set $E$, and let $\mathcal{A} = M(H(E))$, the multiplier algebra on $H(E)$. Let $F(z) = (f_1(z), f_2(z), \ldots)$, where $\{f_n\}_{n=1}^\infty \subset \mathcal{A}$, such that $F(z)F(z)^* \leq 1 \forall z \in$
Finally, we have the relation
\[
Q^{(n)_\ast}_{F(z)}(z) : l^2_0 \rightarrow l^2_{(n+1)}
\]
by
\[
Q^{(n)_\ast}_{F(z)}(z)(w_n) = F(z) \land w_n,
\]
where \(w_n \in l^2_0\). Note that \(Q^{(0)_\ast}_{F(z)}(z) = F(z)\).

There are a few facts about these operators we wish to employ. Obviously, \(\text{ran } Q^{(n)_\ast}_{F(z)}(z) \subset \ker Q^{(n+1)_\ast}_{F(z)}\), so
\[
\text{ran } Q^{(n+1)_\ast}_{F(z)} \subset \ker Q^{(n)_\ast}_{F(z)}. \tag{3}
\]

Also, the entries of \(Q^{(n)}_{F(z)}\) belong to the set \(\{0, \pm f_1(z), \pm f_2(z), \ldots\}\). Finally, we have the relation
\[
(F(z)G(z)^T)I = G(z)^T F(z) + Q^{(n)}_{F(z)} Q^{T}_{F(z)} \tag{4}
\]
where \(Q_A = Q^{(1)}_A\) and \(I\) is an identity matrix.

We also draw on the following results from [11]:

**Lemma 2.1.** Let \(K \subseteq \mathbb{N}\) such that \(H^\infty_K(\mathbb{D})\) is an algebra. Then

(i) \(k_0 \in K\) if and only if \(\varphi(z) = z^{k_0} \in H^\infty_K(\mathbb{D})\).

(ii) If \(j, k \notin K\), then \(j + k \notin K\).

(iii) Suppose \(k_0 \in K\). If \(1 < j < k_0\) satisfies \(j \notin K\), then \(k_0 - j \in K\).

**Lemma 2.2.** If \(H^\infty_K(\mathbb{D})\) is an algebra, then there exists \(d \in \mathbb{N}\), a finite set \(\{n_1\}_{i=1}^p \subseteq \mathbb{N}\) with \(n_1 < \cdots < n_p\) and \(\gcd(n_1, \ldots, n_p) = 1\), and a positive integer \(N_0 > n_p\) so that
\[
\mathbb{N} - K = \{n_1d, n_2d, \ldots, n_pd, N_0d, (N_0 + j)d : j \in \mathbb{N}\}.
\]

Lemma 2.2 tells us that the nontrivial sets \(K \subset \mathbb{N}\) for which \(H^\infty_K(\mathbb{D})\) is an algebra are the sets \(K\) for which there exist \(l_1 < \cdots < l_r \) in \(\mathbb{N}\) with \(\gcd(l_1, \ldots, l_r) = d\) so that \(\mathbb{N} - K\) is the semigroup of \(\mathbb{N}\) generated by \(\{l_1, \ldots, l_r\}\) under addition.

Thus the elements of \(H^\infty_K(\mathbb{D})\) have the form
\[
F(z) = f_0 + f_1z^{n_1d} + \cdots + f_jz^{n jd} + f_{j+1}z^{(n_j+1)d} + f_{j+2}z^{(n_j+2)d} + \cdots
\]
where \(f_i \in \mathbb{C}\). Letting \(w = z^d\) yields
\[
F_1(w) = f_0 + f_1w^{n_1} + \cdots + f_{j-1}w^{n_{j-1}} + \sum_{k=0}^{\infty} f_{j+k}w^{n_j+k}.
\]
Thus $F_1(w)$ is contained in the algebra $H^\infty_{K_1}(\mathbb{D})$, where
\[ K_1 = \{1, \ldots, n_1 - 1, n_1 + 1, \ldots, n_2 - 1, n_2 + 1, \ldots, n_j - 1\} \]
is a finite set.

The above argument suggests us that the problem of finding a solution to the ideal problem in $H^\infty_{K_1}(\mathbb{D})$, where $K$ is infinite, can be reduced to two simpler steps. First, solve the corresponding problem in $H^\infty_{K_1}(\mathbb{D})$, where $K_1$ is finite as above. Then, take those solutions in $H^\infty_{K_1}(\mathbb{D})$ and compose them with $z^d$ in order to get the solution in $H^\infty_{K_1}(\mathbb{D})$.

3. The Proofs

Our approach for each proof is similar. Since we are dealing with subspaces of $H^\infty(\mathbb{D})$, we use Treil’s (Wolff’s) Theorem to find a solution $G(z) \in H^\infty_{\mathcal{F}}(\mathbb{D})$ to $F(z)G(z)^T = h(z)$ ($F(z)G(z)^T = h^3(z)$) on $\mathbb{D}$. The trick is to find a solution that is contained in the appropriate subalgebra. By [3], $V(z)^T = G(z)^T + Q_{F(z)}X(z)^T$ is also a solution, so we seek a vector $X(z)$ such that $V(z)$ is in the appropriate subalgebra.

Proof of Theorem [1.1] Let $F(z) \in (\mathbb{C} + BH^\infty(\mathbb{D}))H^2$, $h(z) \in \mathbb{C} + BH^\infty(\mathbb{D})$, and suppose
\[ 1 \geq F(z)F(z)^* \psi(F(z)F(z)^*) \geq |h(z)| \quad \text{for all } z \in \mathbb{D}. \]
By Treil’s theorem, there exists $G(z) \in H^\infty_{\mathcal{F}}(\mathbb{D})$ such that
\[ F(z)G(z)^T = h(z) \quad \text{for all } z \in \mathbb{D}. \]

Write $F(z) = F_c + B(z)F_B(z)$, where $F_c \in H^2$ and $F_B(z) \in H^\infty_{\mathcal{F}}(\mathbb{D})$. Also, write $h(z) = h_c + B(z)h_B(z)$, with $h_c \in \mathbb{C}$ and $h_B \in H^\infty(\mathbb{D})$. We consider two cases.

Suppose first that $F_c \neq 0$.

By (4), we have
\[ h(z)I = (F(z)G(z)^T)I = G(z)^T F(z) + Q_{F(z)}Q_{G(z)}^T \]
\[ \implies (h_c + B(z)h_B(z))I = G(z)^T (F_c + B(z)F_B(z)) + Q_{F(z)}Q_{G(z)}^T. \]

Thus
\[ (h_c + B(z)h_B(z))F_c^* = G(z)^T (F_c + B(z)F_B(z))F_c^* + Q_{F(z)}Q_{G(z)}^T F_c^* \]
\[ \implies h_cF_c^* + B(z)(h_B(z) - G(z)^TF_B(z))F_c^* = G(z)^T F_cF_c^* + Q_{F(z)}Q_{G(z)}^T F_c^* \]
\[ \implies \frac{h_c}{\|F_c\|^2} F_c^* + B(z)(h_B(z) - G(z)^TF_B(z)) \frac{F_c^*}{\|F_c\|^2} = G(z)^T + Q_{F(z)}Q_{G(z)}^T \frac{F_c^*}{\|F_c\|^2}. \]

The right hand side of the last equation is clearly a solution $V(z)^T$ to $F(z)V(z)^T = h(z)$, while the left hand side shows this solution is in $(\mathbb{C} + BH^\infty(\mathbb{D}))H^2$. Thus we take $X(z)^T = Q_{G(z)}^T \frac{F_c^*}{\|F_c\|^2}$. 

For the norm estimate, we have
\[ \|V\|_\infty \leq \left(1 + \frac{1}{\|F_c\|^2}\right) \|G\|_\infty. \]

Now suppose \( F_c = 0 \). We thus have
\[ |B(z)|^2 F_B(z) F_B(z)^* \psi \left(|B(z)|^2 F_B(z) F_B(z)^* \right) \geq |h_c + B(z) h_B(z)| \quad \text{for all } z \in \mathbb{D}. \]

Letting \( z = \alpha \), where \( \alpha \) is a zero of \( B(z) \), we see that \( h_c = 0 \). Thus
\[ |B(z)|^2 F_B(z) F_B(z)^* \psi \left(|B(z)|^2 F_B(z) F_B(z)^* \right) \geq |B(z)| |h_B(z)| \]
This implies that we can factor at least one more \( B \) out from \( h_B \). Since \( \psi \) is increasing on \([0, 1]\) and \( |B(z)| \leq 1 \) on \( \mathbb{D} \), we get
\[ |F_B(z) F_B(z)^* \psi \left(F_B(z) F_B(z)^* \right) \geq |h_{B_1}(z)|, \]
where \( h_{B_1} = B h_{B_1} \). We should note that \( h_{B_1} \) may contain more \( B \)'s.

By Treil’s Theorem, there exists \( G_1(z) \in H_{p_2}^\infty(\mathbb{D}) \) such that
\[ F_B(z) G_1(z)^T = h_{B_1}(z) \]
\[ \implies F(z) B(z) G_1(z)^T = B^2(z) h_{B_1}(z) = h(z) \quad \text{for all } z \in \mathbb{D}. \]

Thus, \( B(z) G_1(z)^T \) is the solution we seek.

We also see that \( \|B(z) G_1(z)^T\|_\infty \leq \|G_1(z)\|_\infty \). That means the size of our \((C + B H^\infty(\mathbb{D}))_{l_2}\) solution is always less than that of \((H^\infty(\mathbb{D}))_{l_2}\) solution.

\[ \square \]

**Proof of Corollary 1.1.** The proof of this Corollary is similar to the proof of Theorem 1.1. We replace \( h \) with \( h^3 \) and use Wolff’s Theorem in \( H^\infty(\mathbb{D}) \). As in Theorem 1.1, we get a solution \( V(z) \) of \( F(z) V(z)^T = h^3(z) \) in \((C + B H^\infty(\mathbb{D}))_{l_2}\) satisfying
\[ \|V\|_\infty \leq \left(1 + \frac{1}{\|F_c\|^2}\right) \|G_1\|_\infty, \]
where \( G_1(z) \) is the \((H^\infty(\mathbb{D}))_{l_2}\) solution of \( F(z) G(z)^T = h^3(z) \). For the norm estimate, we draw upon the estimate in [19] with \( \psi(t) = t^{\frac{3}{2}} \). \[ \square \]

The proofs of Theorems 1.2 through 1.5 are by induction. Since in each case a solution in \( H^\infty(\mathbb{D}) \) exists, by Treil, our use of induction is justified. Similarly, Corollaries can be obtained using Wolff’s Theorem instead of Treil’s, as in Corollary 1.1.

We denote \( K_{p-1} = K \setminus \{k_p\} \), where \( k_p \) is the largest member of \( K \). If \( H^\infty_K(\mathbb{D}) \) is an algebra, then so is \( H^\infty_{K_{p-1}}(\mathbb{D}) \).
Proof of Theorem 1.2. Let \( F(z) \in \mathcal{H}_{K,n}^\infty(\mathbb{D}) \), \( h(z) \in H_K^\infty(\mathbb{D}) \) such that \( F(0) \neq 0 \) and
\[
1 \geq F(z)F(z)^*\psi(F(z)F(z)^*) \geq |h(z)| \text{ for all } z \in \mathbb{D}.
\]
By induction, there exists \( G(z) \in \mathcal{H}_{K-1,n}^\infty(\mathbb{D}) \) with
\[
F(z)G(z)^T = h(z) \text{ for all } z \in \mathbb{D}.
\]
We denote “\( k_p \)” by “\( k \)”, and we let
\[
X(z)^T = \frac{1}{k!} \frac{Q_{F(0)} G^{(k)}(0)^T}{F(0)F(0)^*} z^k \in H_{K-1}^\infty(\mathbb{D}).
\]
We consider
\[
V(z)^T = G(z)^T - Q_{F(z)} X(z)^T.
\]
We see that
\[
F(z)V(z)^T = h(z) \text{ for all } z \in \mathbb{D} \text{ and } V(z) \in \mathcal{H}_{K-1,n}^\infty(\mathbb{D}).
\]
We must show that \( V^{(k)}(0) = 0 \). But by (1),
\[
V^{(k)}(0)^T = G^{(k)}(0)^T - \sum_{j=0}^{k} \binom{k}{j} Q_{F(0)}^{(k-j)} X^{(j)}(0)^T
\]
\[
= G^{(k)}(0)^T - Q_{F(0)} X^{(k)}(0)^T
\]
\[
= G^{(k)}(0)^T - Q_{F(0)} Q_{F(0)}^* G^{(k)}(0)^T
\]
\[
= \frac{F(0)^* F(0)}{F(0) F(0)^*} G^{(k)}(0)^T.
\]
Our proof thus depends on establishing that \( F(0) G^{(k)}(0)^T = 0 \). But \( F(z) G(z)^T = h(z) \) on \( \mathbb{D} \), and \( h(z) \in H_K^\infty(\mathbb{D}) \). Differentiating \( k \) times and evaluating at \( 0 \), we obtain
\[
\sum_{j=1}^{k} \binom{k}{j} F^{(k-j)}(0) G^{(j)}(0)^T = 0.
\]  
(5)
Since \( G(z) \in \mathcal{H}_{K-1,n}^\infty(\mathbb{D}) \), we have \( G^{(j)}(0) = 0 \) for all \( j \in K_{p-1} \). If \( j \notin K \) and \( j < k \), we have \( k - j \in K \), so \( F^{(k-j)}(0) = 0 \). Thus (5) becomes
\[
F(0) G^{(k)}(0)^T = 0
\]
which is the desired result. \( \square \)
Proof of Theorem 1.4. Observe first that \( m \notin K \), or else we would have \( 0 = \frac{d^{m}}{dz^{m}}[z^{m}F_{m}(z)]|_{z=0} = m!F_{m}(0) \), and \( F_{m}(0) \neq 0 \), by assumption. Similarly to Theorem 1.1 in the case where \( F \), we have, by Treil, \( F_{m}(z)G_{m}(z)^{T} = h_{m}(z) \forall z \in \mathbb{D} \), where \( G_{m} \in H^{\infty}(\mathbb{D}) \) and \( h(z) = z^{2m}h_{m}(z) \). Thus

\[
F(z)z^{m}G_{m}(z)^{T} = z^{2m}h_{m}(z) = h(z) \forall z \in \mathbb{D}.
\]

We wish to show \( z^{m}G_{m}(z) \in \mathcal{H}_{K,m}^{\infty}(\mathbb{D}) \). If \( m > k_{p} \), the result is immediate.

If \( m < k_{p} \), then suppose that \( K - m \) defines an algebra \( H_{K-m}^{\infty}(\mathbb{D}) \). Since \( m \notin K \), \( K - m \subset K_{p-1} \) and thus \( H_{K_{p-1}}^{\infty}(\mathbb{D}) \subset H_{K-m}^{\infty}(\mathbb{D}) \). Using induction, we can thus take \( G \in H_{K-m}^{\infty}(\mathbb{D}) \). Now, \( z^{m}G_{m}(z) \in \mathcal{H}_{K,m}^{\infty}(\mathbb{D}) \), for if \( j \in K \), then

\[
\left. \frac{d^{j}}{dz^{j}}(z^{m}G_{m}(z)) \right|_{z=0} = \binom{j}{m} m!G_{m}^{(j-m)}(0) = 0.
\]

(We assume here that \( j > m \). If \( j < m \), the result is trivial.)

Finally, \( \|V\|_{\infty} = \|G_{m}\|_{\infty} \).

\( \square \)

For \( F \in H_{K(B)}^{\infty}(\mathbb{D}) \), denote \( F(z) = B^{j_{1}}(z)F_{j_{1}} + \cdots + B^{j_{n}}(z)F_{j_{n}} + B^{k_{p}+1}(z)F_{k_{p}+1}(z) \), where \( F_{j_{i}} \in \mathbb{C} \) for \( i = 1, \ldots , n \) and \( F_{k_{p}+1}(z) \in (H^{\infty}(\mathbb{D}))^{n} \). We inductively assume there exists a solution \( G(z) = B^{j_{1}}(z)G_{j_{1}} + \cdots + B^{j_{n}}(z)G_{j_{n}} + B^{j_{n}+1}(z)G_{j_{n}+1}(z) \). One can check to see that \( G \) belongs to a subalgebra containing \( H_{K(B)}^{\infty}(\mathbb{D}) \).

Proof of Theorem 1.3. Since \( F_{0} \neq 0 \), denote \( j_{1} = 0 \). Proceeding as in the proof of Theorem 1.1 in the case \( F_{c} \neq 0 \), we obtain

\[
[h_{0} + B^{j_{2}}(z)h_{j_{2}} + \cdots + B^{j_{n}}(z)h_{j_{n}} + B^{k_{p}+1}(z)h_{k_{p}+1}(z)]F_{0}^{*} = 0,
\]

\[
G(z)^{T}[B^{j_{2}}(z)F_{j_{2}} + \cdots + B^{j_{n}}(z)F_{j_{n}} + B^{k_{p}+1}(z)F_{k_{p}+1}(z)]F_{0}^{*} = 0.
\]

The remainder of the proof consists of showing that the left-hand side of this equation lies in \( \mathcal{H}_{K(B),n}^{\infty}(\mathbb{D}) \). We observe that

\[
G(z)^{T}[B^{j_{2}}(z)F_{j_{2}} + \cdots + B^{j_{n}}(z)F_{j_{n}} + B^{k_{p}+1}(z)F_{k_{p}+1}(z)]
\]

\[
= [G_{0} + B^{j_{2}}(z)G_{j_{2}} + \cdots + B^{j_{n}-1}(z)G_{j_{n}-1}]^{T}
\times [B^{j_{2}}(z)F_{j_{2}} + \cdots + B^{j_{n}}(z)F_{j_{n}} + B^{k_{p}+1}(z)F_{k_{p}+1}(z)]
\]

\[
+ B^{j_{n}}(z)G_{j_{n}}(z)^{T}[B^{j_{2}}(z)F_{j_{2}} + \cdots + B^{j_{n}}(z)F_{j_{n}} + B^{k_{p}+1}(z)F_{k_{p}+1}(z)]
\]

\[
= [G_{0} + B^{j_{2}}(z)G_{j_{2}} + \cdots + B^{j_{n}-1}(z)G_{j_{n}-1}]^{T}
\times [B^{j_{2}}(z)F_{j_{2}} + \cdots + B^{j_{n}}(z)F_{j_{n}} + B^{k_{p}+1}(z)F_{k_{p}+1}(z)]
\]

\[
+ B^{j_{n}}(z)G_{j_{n}}(z)^{T}[B^{j_{2}}(z)F_{j_{2}} + \cdots + B^{j_{n}}(z)F_{j_{n}} + B^{k_{p}+1}(z)F_{k_{p}+1}(z)]
\]

\[
\]
The first term is clearly in $B(\mathbb{C}^n, \mathcal{H}^\infty_{K(B),n}(\mathbb{D}))$. Since, for $i = 2, \ldots, n$, $j_n + j_i \notin K$, we must have $j_n + j_1 > k_\alpha$. This shows that the second term is also in $B(\mathbb{C}^n, \mathcal{H}^\infty_{K_n}(\mathbb{D}))$. Thus $G(z)^T + Q_{F(z)}Q_T^{\ast}G(z)\frac{F^{\ast}}{\|F_0\|_r} \in \mathcal{H}^\infty_{K(B),n}(\mathbb{D})$.

**Proof of Theorem 1.5.** Since $F_0 = 0$, we have $F(z) = B^{j_1}(z)F_\alpha(z)$, where $F_\alpha(z) \in (\mathcal{H}^\infty(\mathbb{D}))^n$ and $F_\alpha(z)$ has a nonzero constant term. As in the proof of Theorem 1.1 in the case where $F_\alpha = 0$, there exists $G_\alpha(z) \in \mathcal{H}^\infty(\mathbb{D})$ such that $F_\alpha(z)G_\alpha(z)^T = h_\alpha(z) \forall z \in \mathbb{D}$, where $B^{j_1}(z)h_\alpha = h(z)$. Thus

$$F(z)B^{j_1}(z)G_\alpha(z)^T = B^{2j_1}(z)h_\alpha(z) = h(z) \forall z \in \mathbb{D}.$$ 

If $j_1 > k_\alpha$, then $B^{j_1}(z)G_\alpha(z) = B^{k_\alpha+1}(z)[B^{j_1-k_\alpha-1}(z)G_\alpha(z)] \in \mathcal{H}^\infty_{K(B)}(\mathbb{D})$, and we are done.

If $j_1 < k_\alpha$, then suppose $K - j_1$ defines an algebra $\mathcal{H}_{K-j_1}^\infty(\mathbb{D})$. Then $F_\alpha \in \mathcal{H}^\infty_{(K-j_1)(B)}(\mathbb{D})$ and since the constant term of $F_\alpha$ is nonzero, then by Theorem 1.3 we may assume $G_\alpha \in \mathcal{H}^\infty_{(K-j_1)(B)}(\mathbb{D})$. Then $B^{ji}(z)G_\alpha(z) \in \mathcal{H}_{K_j}^\infty(\mathbb{D})$.

Finally, $\|V\|_\infty = \|G_\alpha\|_\infty$. 

\[\square\]

4. Further Results and Questions

4.1. Radical Ideals. As noted in the introduction, equation (2) provides a characterization for membership in the radical of the ideal generated by the functions $f_i$, denoted $\text{Rad}((f_i)_{i=1}^n)$. We obtain similar characterizations for algebras of form $\mathbb{C} + B\mathcal{H}^\infty(\mathbb{D})$ and $\mathcal{H}^\infty_K(\mathbb{D})$. For the first type of algebra this result is immediate, but the second type of algebra requires a bit of discussion.

Let $F(z) \in \mathcal{H}_{K_n}^\infty(\mathbb{D})$ and $h(z) \in H^\infty_K(\mathbb{D})$. Clearly, if $h \in \text{Rad}(\mathcal{I})$, where $\mathcal{I}$ is the ideal generated by the entries in $F(z)$, then

$$M[F(z)F^*(z)]^{\frac{1}{2}} \geq |h^q(z)| \forall z \in \mathbb{D}$$

for some $M > 0$, $q \in \mathbb{N}$. For the converse, we have two cases. If $F(0) \neq 0$, the result follows from Theorem 1.2. If $F(0) = 0$, then $F(z) = z^mF_m(z)$ and $h(z) = z^mh_m(z)$ as in the proof of Theorem 1.4. We use Wolff’s Theorem to obtain a $G(z) \in (\mathcal{H}^\infty(\mathbb{D}))^n$ and $q \in \mathbb{N}$ such that $F(z)G(z)^T = h^q(z) \forall z \in \mathbb{D}$. Take $L \in \mathbb{N}$ such that $mL > k_\alpha$. Then

$$h^{p+L}(z) = F(z)[h^L(z)G(z)^T] = F(z)[z^{mL}h_m(z)G(z)^T].$$

We thus take $U(z)^T = z^{mL}h_m(z)G(z)^T$. Since $mL > k_\alpha$, $U(z) \in H^\infty_K(\mathbb{D})$. This shows that $h \in \text{Rad}(\mathcal{I})$. 

\[\square\]
4.2. A Full Extension of Wolff’s Theorem. The added assumption in Theorem 1.4 that $H^\infty_{K^m}(\mathbb{D})$ is an algebra (as well as the similar assumption in Theorem 1.5) was necessary for our proof. For example, if $K = \{1, 2, 5\}$, then $H^\infty_K(\mathbb{D})$ is an algebra, but $K - 3 = \{2\}$ does not define an algebra. It remains an open question whether the generalized ideal result and Wolff’s Theorem can be fully extended to the subalgebras $H^\infty_K(\mathbb{D})$ and $H^\infty_K(\mathbb{B})(\mathbb{D})$.

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