Lattice Chiral Symmetry in Fermionic Interacting Theories and the Antifield Formalism

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Abstract

Recently we have discussed realization of an exact chiral symmetry in theories with self-interacting fermions on the lattice, based upon an auxiliary field method. In this paper we describe construction of the lattice chiral symmetry and discuss its structure in more detail. The antifield formalism is used to make symmetry consideration more transparent. We show that the quantum master equation in the antifield formalism generates all the relevant Ward-Takahashi identities including a Ginsparg-Wilson relation for interacting theories. Solutions of the quantum master equation are obtained in a closed form, but the resulting actions are found to be singular. Canonical transformations are used to obtain four types of regular actions. Two of them may define consistent quantum theories. Their Yukawa couplings are the same as those obtained by using the chiral decomposition in the free field algebra. Inclusion of the complete set of the auxiliary fields is briefly discussed.

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1 Introduction

Recently much attention has been paid to new realization of symmetries which are not compatible, in ordinary sense, with regularizations. The discovery of the exact chiral symmetry on the lattice [1]-[6] may be regarded as the prototype of such realization. (See, for example, ref.[7] for reviews of recent development.) The need for considering regularization-dependent symmetries is not restricted to lattice theories: A closely related issue in continuum theories is to realize gauge symmetries in the Wilsonian renormalization group (RG) [8], which introduces an infrared cutoff to define the Wilsonian effective action.

In the new realization of a symmetry, such as the lattice chiral symmetry or the gauge symmetry along the RG flow, symmetry transformation inevitably depends on cutoff parameter so as to make reconciliation with a regularization. For such a theory, neither an action nor path integral measure may remain invariant under the transformation. One has to determine the action and the symmetry transformation at the same time in such a way that total change arising from the action and the measure vanish. It is therefore difficult in general to give a nonperturbative formulation of the symmetry in the presence of interactions.

For the chiral symmetry on the lattice, the gauge interactions were incorporated into the Dirac operator in vector gauge theories. Recently, chiral gauge theories with anomaly-free fermion multiplets have been constructed in the abelian case [9]. In these theories, the Ginsparg-Wilson (GW) relation [1], the crucial algebraic identity in formulation of the symmetry, takes the same form as that in free field theories. The significance and consequences of this identity has been extensively discussed [2]-[7][9]-[19]. However, it seems to be not clear how the GW relation should be generalized for describing other interactions.

In a previous work [20], referred to as I hereafter, we discussed a lattice chiral symmetry in theories with fermionic self-interactions, introducing a scalar and a pseudoscalar fields as auxiliary fields. A GW relation for these theories was given. It depends on the Yukawa couplings to the auxiliary fields. The Yukawa interactions we obtained take different form from those considered in the literature [9][12][15][18][19]. The chiral transformation obtained from the GW relation becomes nonlinear. Under the transformation, none of the kinetic term of the Dirac fields, the Yukawa coupling term and the functional measure remains invariant. Although we gave chiral invariant partition function, these peculiar properties of the chiral symmetry remained to be studied.

The purpose of this paper is twofold: First, we consider construction of the lattice chiral symmetry discussed in I in more detail, using a framework of the antifield formalism originally developed by Batalin and Vilkovisky [21]. Second, besides the peculiar properties of the chiral symmetry described above, we argue that the the actions obtained in I are singular. In order to have regular actions, we introduce new dynamical variables, and it enables us to reconstruct the chiral symmetry.
The reason why we use the antifield formalism to describe the lattice chiral symmetry is as follows: The formalism has been recognized as the most general and powerful method of BRS quantization for theories with local as well as global symmetries. For regularization-dependent symmetries, it is a nontrivial problem to derive the Ward-Takahashi (WT) identities, and is certainly desirable to develop a general method for doing this task both in lattice and continuum theories. We believe that the antifield formalism is a strong candidate for the method. Actually, it has been applied to regularization-dependent symmetries realized along the RG flow \[22\][23]. In the antifield formalism, the presence of exact symmetries, irrespective of whether they are regularization-dependent or not, is expressed by the quantum master equation (QME) \[21\], \(\Sigma = 0\).

For the fermionic theories we consider, antifields are introduced at the stage of the block transformation \[1\] from a microscopic theory on a fine lattice to its macroscopic counterpart on a coarse lattice. In performing the transformation, we also use the auxiliary field method discussed in I. A scalar and a pseudo-scalar fields are introduced again to make an effective description of the fermionic self-interactions. Introduction of antifields for the Dirac and the auxiliary fields makes it possible to encode the chiral transformation in a natural way. This should be compared with the somehow ad hoc way of postulating the transformation rule in the conventional approach.

Since the fermionic interaction terms are replaced by the Yukawa term and a potential of the auxiliary fields, the fermionic sector in the macroscopic action is linearized. The price for the use of the auxiliary field method is that we only obtain a more weaker condition, vanishing of the expectation value \(\langle \Sigma \rangle = 0\), rather than the QME \(\Sigma = 0\). Although the condition \(\langle \Sigma \rangle = 0\) allows a wide class of solutions, we consider here only the solutions to the QME. We show that the QME generates all the relevant WT identities including the GW relation obtained in I. Under suitable assumptions, the QME can be solved in a closed form. The solutions are used to construct four types of chiral invariant partition functions.

The actions constructed with these sets of solutions turn out to have singularities in the Yukawa couplings as well as in the auxiliary field potential. The singularities in the Yukawa couplings arise at the momentum regions where would-be species doublers appear. In order to remove the singularities, we perform canonical transformations in the space of fields and antifields. Among four sets of the transformed regular actions, two of them contain massless doubler modes which decouple to the auxiliary fields. This decoupling occurs at tree level but could not be stable due to the quantum corrections. In other two sets of the actions, the doubler modes become massive, and decoupling to the auxiliary fields is ensured by the chiral symmetry. In these actions, the kinetic term of the Dirac fields, the Yukawa coupling term and the functional measure are all chiral invariant. The chiral transformation of the new variables takes the

\[\ast\]A similar kind of singularities has been discussed in different context in \[19\].
same form as the one for the free field theory. The Yukawa couplings coincide with those discussed in [7][15][18][19]. They can be obtained by using the chiral decomposition in the free field algebra [7][12][23]. The actions obey the classical master equation rather than the QME.

As for the auxiliary fields, we restrict ourselves mostly to a scalar and a pseudoscalar fields. We may include other auxiliary fields in a similar manner. We try here to give a formal argument of inclusion of the complete set of the fields with multi-flavor, though the constructed actions suffer from the singularities discussed above.

This paper is organized as follows. In section 2, we describe construction of macroscopic action for fermionic system introducing scalar and pseudoscalar auxiliary fields. Then, the block transformation [1] is reconsidered in the antifield formalism. In section 3, the QME is derived. Under suitable assumptions, we show that it yields the WT identities discussed in I. Two sets of particular solutions of the QME are given. Using these non-perturbative solutions, we give chiral invariant partition functions on the coarse lattice in section 4. In section 5, reconstruction of the chiral symmetry using canonical transformations is discussed. We perform inclusion of the complete set of the auxiliary fields in section 6. The section 7 is devoted to summary and discussion. Derivation of some formulae is given in Appendix.

2 Macroscopic action in the antifield formalism

In this section, we first construct a macroscopic action without introducing the antifields. Although this was discussed in I, a brief summary of the construction is given to make the present work self-contained. We then reconsider our microscopic as well as macroscopic theories in the antifield formalism, and give a phase-space extension of the block transformation.

2.1 Construction of macroscopic action

Let $A_c[\psi_c, \bar{\psi}_c]$ be a microscopic action of the Dirac fields $\psi_c(x), \bar{\psi}_c(x)$. The fields are defined on a $d$ (even) dimensional fine lattice whose positions are labeled by $x$. For simplicity, they are assumed to carry a single flavor. The microscopic action describes a certain class of fermionic self-interactions. Let $A[\Psi, \bar{\Psi}]$ be an effective action of the Dirac fields $\Psi_n, \bar{\Psi}_n$ defined on a coarse lattice. Indices $n, m$ are used for labeling sites of the lattice. The macroscopic action is obtained from the microscopic action via the block transformation

$$e^{-A[\Psi, \bar{\Psi}]} \equiv \int D\psi_c D\bar{\psi}_c e^{-A_c[\psi_c, \bar{\psi}_c] - \sum_n (\bar{\psi}_n - \bar{B}_n) \alpha (\Psi_n - B_n)}, \quad (2.1)$$

where $\alpha$ is a constant parameter proportional to inverse of the coarse lattice spacing $a$, $\alpha \propto a^{-1}$. The gaussian integral in (2.1) relates the macroscopic fields
\[\Psi_n, \bar{\Psi}_n\] to the block variables defined by

\[
\begin{align*}
B_n & \equiv \int d^d x \, f_n(x) \psi_c(x) \\
\bar{B}_n & \equiv \int d^d x \, \bar{\psi}_c(x) f_n^*(x)
\end{align*}
\]

(2.2)

where \(f_n(x)\) is an appropriate function for coarse graining. It is normalized as \(\int d^d x \, f_n^*(x) f_m(x) = \delta_{nm}\).

The path-integral over the microscopic fields in (2.1) will generate fermionic self-interaction terms in the macroscopic action \(A[\Psi, \bar{\Psi}]\). Instead of dealing with such terms directly, we introduce some auxiliary fields on the coarse lattice to describe the fermionic interactions. The maximal number of the auxiliary fields to be introduced is equal to the dimension of the Clifford algebra, i.e., \(2^d\) for the \(d\) (even)-dimensional Dirac fields. In section 6, we discuss the inclusion of the complete set of the auxiliary fields. For simplicity, we restrict ourselves here to a scalar \(\sigma_n\) and a pseudoscalar field \(\pi_n\), because the scalar and pseudoscalar interactions are recognized as the most important couplings to describe chiral symmetry and its spontaneous breaking in the effective theory. The macroscopic action we consider then takes of the form

\[
A[\Psi, \bar{\Psi}] = \sum_{nm} \left\{ \bar{\Psi}_n(D_0)_{nm} \Psi_m + V[\bar{\Psi}_n(\delta_{nm} + h(\nabla))_{nm}]_{\Psi_m} \Psi_n \gamma_5(\delta_{nm} + h(\nabla))_{nm} \Psi_m \right\}
\]

(2.3)

where \(D_0\) is the Dirac operator for the kinetic term, and \(V\) denotes fermionic interactions which consist of contact terms as well as non-contact ones with the difference operators \(h(\nabla)_{nm}\). We may obtain the action (2.3) by performing integration over the auxiliary fields in a new macroscopic action:

\[
e^{-A[\Psi, \bar{\Psi}]} = \int \mathcal{D} \pi \mathcal{D} \sigma \ e^{-\sum_{nm} \bar{\Psi}_n(D_0 + (\delta + h(\nabla)))_{nm}(i\gamma_5 \pi + \sigma)_{m} \Psi_m - A_X[\pi, \sigma].}
\]

(2.4)

It is noted that the Dirac fields appear only bilinearly in the new action. All the fermionic interactions are cast into the Yukawa couplings with the auxiliary fields and the potential term \(A_X[\pi, \sigma]\). In summary, the block transformation is given by

\[
\int \mathcal{D} \pi \mathcal{D} \sigma \ e^{-\sum_{nm} \bar{\Psi}_n \tilde{D}(\pi, \sigma)_{nm} \Psi_m - A_X[\pi, \sigma]} = \int \mathcal{D} \psi_c \mathcal{D} \bar{\psi}_c \ e^{-A_c[\psi_c, \bar{\psi}_c] - \sum_{n} (\bar{\Psi}_n - \bar{B}_n) \alpha(\Psi_n - B_n)},
\]

(2.5)

with the total Dirac operator

\[
\tilde{D}(\pi, \sigma)_{nm} = (D_0)_{nm} + (\delta + h(\nabla))_{nm} (i\gamma_5 \pi + \sigma)_{m}
\]

\[\equiv D(\pi, \sigma)_{nm} + (i\gamma_5 \pi + \sigma)_{n}.
\]

(2.6)
Here $\bar{D}(\pi, \sigma)$ is assumed to be at most linear in $\pi$ and $\sigma$. We now reconsider the block transformation (2.3) in the antifield formalism.

2.2 The antifield formalism and the block transformation

The antifield formalism [21] describes any local or global symmetry as a “BRS” symmetry. It defines a kind of “canonical structure” for a given action of fields by adding their “momentum variables” called antifields. In our previous papers [22][23], the formalism has been used for realization of symmetries along the Wilsonian RG flow. The purpose of this subsection is to give a lattice version of the formalism in the context of chiral symmetry.

In the antifield formalism, the chiral transformation in the microscopic theory takes the form of BRS transformation:

$$
\delta_B \psi_c(x) = i C \gamma_5 \psi_c(x),
$$
$$
\delta_B \bar{\psi}_c(x) = i C \bar{\psi}_c(x) \gamma_5,
$$

(2.7)

where $C$ is a constant ghost. It is Grassmann odd, therefore $C^2 = 0$. For the Dirac fields $\phi^a \equiv \{\psi_c, \bar{\psi}_c\}$, one introduces anti-Dirac fields $\phi^*_a \equiv \{\psi^*_c, \bar{\psi}^*_c\}$. Although the antifields are unphysical, they play an important role for encoding chiral symmetry, and should be eliminated only at the final stage of our calculation. In order to include the antifields, one considers an extended microscopic action,

$$
S_c[\phi, \phi^*] \equiv A_c[\psi_c, \bar{\psi}_c] + \int d^d x [\psi_c^*(x) \delta_B \psi_c(x) + \delta_B \bar{\psi}_c(x) \bar{\psi}_c^*(x)].
$$

(2.8)

It is noted here that the BRS transformation operator $\delta_B$ is Grassmann odd and carries one unit of ghost number. Therefore, the antifields $\phi^*_a$ regarded as source terms for $\delta_B \phi^a$ are Grassmann even and carry ghost number $-1$. The canonical structure in the theory with antifields is specified by the antibracket. For any functions $F[\phi, \phi^*]$ and $G[\phi, \phi^*]$, it is defined by

$$
(F, G)_\phi = \int d^d x \left[ \frac{\partial F}{\partial \psi_c(x)} \frac{\partial G}{\partial \psi^*_c(x)} - \frac{\partial F}{\partial \bar{\psi}_c(x)} \frac{\partial G}{\partial \bar{\psi}^*_c(x)} + \frac{\partial F}{\partial \psi_c(x)} \frac{\partial G}{\partial \psi^*_c(x)} - \frac{\partial F}{\partial \bar{\psi}_c(x)} \frac{\partial G}{\partial \bar{\psi}^*_c(x)} \right].
$$

(2.9)

The chiral transformation of $F$ is described as

$$
\delta_B F = (F, S_c)_\phi.
$$

(2.10)

Note that this is an operation from the right. If the original action $A_c[\psi_c, \bar{\psi}_c]$ is chiral invariant $\delta_B A_c = 0$, the extended action $S_c[\phi, \phi^*]$ is so, too. It is expressed by the classical master equation,

$$
(S_c, S_c)_\phi = 0.
$$

(2.11)
For the Dirac fields on the coarse lattice, we introduce their anti-fields \( \Psi^*_n \) and \( \bar{\Psi}^*_n \). We also include anti-auxiliary fields: \( \pi^*_n \) and \( \sigma^*_n \). Let \( \Phi^A \equiv \{ \Psi_n, \bar{\Psi}_n, \pi_n, \sigma_n \} \) be all the fields on the coarse lattice, and \( \Phi^*_A \equiv \{ \Psi^*_n, \bar{\Psi}^*_n, \pi^*_n, \sigma^*_n \} \) be their antifields. Then, a phase-space extension of (2.5) is given by

\[
\int D\pi D\sigma D\pi^* D\sigma^* \prod_n \delta(\pi^*_n)\delta(\sigma^*_n) e^{-S[\Phi, \Phi^*]}
\]

\[= \int D\phi D\phi^* \prod_x \delta\left( \sum_n \Psi^*_n f_n(x) - \psi^*_c(x) \right) \delta\left( \sum_n \bar{\Psi}^*_n f_n^*(x) - \bar{\psi}^*_c(x) \right) \times e^{-S^\text{total}_c[\phi, \phi^*]}, \tag{2.12}\]

where the block transformation for the antifield sector is described by using the \( \delta \) functions. The total microscopic action in (2.12),

\[S^\text{total}_c[\phi, \phi^*] \equiv S_c[\phi, \phi^*] + \sum_n (\bar{\Psi}^*_n - \bar{B}_n)\alpha(\Psi_n - B_n), \tag{2.13}\]

has terms linear in the microscopic Dirac fields:

\[
\int d^d x \sum_n (f_n^*(x)\bar{\psi}^*_c(x)\alpha(\Psi + i C\gamma_5\alpha^{-1}\bar{\Psi}^*_n) + f_n(x)(\bar{\Psi} - \Psi^* i C\gamma_5\alpha^{-1})\alpha\psi^*_c(x)).
\]

It implies that the effective source terms for \( \bar{\psi}^*_c \) and \( \psi^*_c \) are proportional to \((\Psi + i C\gamma_5\alpha^{-1}\bar{\Psi}^*_n)_n\) and \((\bar{\Psi} - \Psi^* i C\gamma_5\alpha^{-1})_n\), respectively. Thus, we find that the macroscopic action takes the form

\[
S[\Phi, \Phi^*] = \sum_{nm} (\Psi - \Psi^* i C\gamma_5\alpha^{-1})_n (\bar{\Psi}_m + \Psi m + A_X[\pi, \sigma] + \sum_n (\pi^*_n \delta_B \pi_n + \sigma^*_n \delta_B \sigma_n)), \tag{2.14}\]

where the anti-auxiliary fields are included. They are multiplied by the BRS transformed auxiliary fields, \( \delta_B \pi_n \) and \( \delta_B \sigma_n \), which are to be determined later. We have used the total Dirac operator \( \tilde{D}(\pi, \sigma) \) given in (2.4).

It is noted that the chiral transformation for the Dirac fields \( \Psi, \bar{\Psi} \) is automatically encoded due to the presence of the anti-Dirac fields \( \Psi^*, \bar{\Psi}^* \) in (2.14):

\[
\delta_B \Psi_n = (\Psi_n, S[\Phi, \Phi^*])_{\Phi} = i C\gamma_3 \left(1 - \alpha^{-1}\tilde{D}\right)_{nm} \Psi_m, \tag{2.15}\]

\[
\delta_B \bar{\Psi}_n = (\bar{\Psi}_n, S[\Phi, \Phi^*])_{\Phi} = i C\bar{\Psi}_m \left(1 - \alpha^{-1}\tilde{D}\right)_{mn} \gamma_5, \tag{2.15}\]

where \( (\ , \ )_{\Phi} \) denotes the antibracket for the macroscopic sector. It is given by

\[(F, G)_\Phi = (F, G)_D + (F, G)_X.\]
\[
(F, G)_D = \sum_n \left[ \frac{\partial^r F}{\partial \Psi_n} \frac{\partial^l G}{\partial \Psi_n} - \frac{\partial^r F}{\partial \bar{\Psi}_n} \frac{\partial^l G}{\partial \bar{\Psi}_n} \right] + \left[ \frac{\partial^r F}{\partial \Psi_n} \frac{\partial^l G}{\partial \bar{\Psi}_n} - \frac{\partial^r F}{\partial \bar{\Psi}_n} \frac{\partial^l G}{\partial \Psi_n} \right],
\]

\[
(F, G)_X = \sum_n \left[ \frac{\partial^r F}{\partial \pi_n} \frac{\partial^l G}{\partial \pi_n} - \frac{\partial^r F}{\partial \bar{\pi}_n} \frac{\partial^l G}{\partial \bar{\pi}_n} \right] + \left[ \frac{\partial^r F}{\partial \sigma_n} \frac{\partial^l G}{\partial \sigma_n} - \frac{\partial^r F}{\partial \bar{\sigma}_n} \frac{\partial^l G}{\partial \bar{\sigma}_n} \right].
\]

(2.16)

In the antifield formalism, one can use canonical transformations for the phase-space variables. They are defined as transformations from \(\{\Phi^A, \Phi^*_A\}\) to \(\{\Phi'^A, \Phi'^*_A\}\) that render the antibrackets invariant: \( (F, G)_\Phi = (F, G)_{\Phi'} \). One thing which should be remarked is that the path-integral measure \( D\Phi D\bar{\Phi} \) is not left invariant in general under the canonical transformations. Since there is no Liouville measure in the phase-space, one has to take account of the associated Jacobian factor in quantum theory.

We can rewrite the fermionic part of the action in (2.14) by performing a canonical transformation,

\[
\Psi'_n = \Psi_n + i C\gamma_5 \alpha^{-1} \bar{\Psi}'_n,
\]

\[
\bar{\Psi}'_n = \bar{\Psi}_n + \Psi^*_n i C\gamma_5 \alpha^{-1},
\]

\[
\Psi'^*_n = \Psi^*_n,
\]

\[
\bar{\Psi}'^*_n = \bar{\Psi}^*_n.
\]

(2.17)

It is easy to see that the Jacobian factor associated with this canonical transformation is trivial. Hereafter, we use the new set of variables in construction of the lattice chiral symmetry, and represent it by \(\{\Psi, \bar{\Psi}, \Psi^*, \bar{\Psi}^*\}\) removing primes. Then, using these variables, the macroscopic extended action (2.14) is expressed as

\[
S[\Phi, \Phi^*] = S_D + S_X,
\]

\[
S_D = \sum_{nm} \bar{\Psi}_n \tilde{D}(\pi, \sigma)_{nm} \Psi_m
\]

\[
+ \sum_{nm} \left[ \Psi^*_n i C\gamma_5 (1 - 2\alpha^{-1} \tilde{D})_{nm} \Psi_m - \bar{\Psi}_n i C\gamma_5\delta_{nm} \bar{\Psi}^*_m \right],
\]

\[
S_X = A_X[\pi, \sigma] + \sum_n \left( \pi^*_n \delta_B \pi_n + \sigma^*_n \delta_B \sigma_n \right).
\]

(2.18)

It leads to the asymmetric form of the chiral transformation:

\[
\delta_B \Psi_n = i C\gamma_5 \left(1 - 2\alpha^{-1} \tilde{D}\right)_{nm} \Psi_m,
\]

\[
\delta_B \bar{\Psi}_n = i C\bar{\Psi}_n \gamma_5.
\]

(2.19)
Here $\bar{\Psi}$ obeys the standard chiral transformation, while $\Psi$ does not. Instead, we may consider the chiral transformation

$$
\delta_B \Psi_n = i C \gamma_5 \Psi_n,
$$

$$
\delta_B \bar{\Psi}_n = i C \bar{\Psi}_m \left(1 - 2\alpha^{-1} \bar{D}\right)_{mn} \gamma_5,
$$

(2.20)

where $\Psi$ obeys the standard chiral transformation, while $\bar{\Psi}$ does not. The Dirac action which leads to (2.20) is given by

$$
S_D = \sum_{nm} \bar{\Psi}_n \bar{D} (\pi, \sigma)_{nm} \Psi_m
$$

$$
+ \sum_{nm} \left[ \Psi_n^* i C \gamma_5 \delta_{nm} \Psi_m - \bar{\Psi}_n^* i C \left(1 - 2\alpha^{-1} \bar{D}\right)_{nm} \gamma_5 \bar{\Psi}_m \right],
$$

(2.21)

which can be obtained from (2.14) via another canonical transformation.

In this section, we have given the block transformation (2.12) in the antifield formalism. It relates symmetry properties in the microscopic theory to those in the macroscopic theory. After a canonical transformation, the extended action (2.18) or that with the action for the Dirac fields (2.21) has been obtained. We consider below the WT identities for this action.

### 3 The QME and its solutions

In the antifield formalism, the basic object which detects the presence of symmetry in a given quantum system is the WT operator. For a path-integral $\int D\phi e^{-W[\phi, \phi^*]}$ with an action $W[\phi, \phi^*]$, the WT operator is defined as $\Sigma[\phi, \phi^*] = (W, W)_{\phi}/2 - \Delta_{\phi} W = e^W \Delta_{\phi} e^{-W}$, where $\Delta$ denotes the “divergence” operator whose explicit expression is given below. The WT operator $\Sigma[\phi, \phi^*]$ can be interpreted as follows: Consider a change of variables $\phi \rightarrow \phi + (\phi, W)_{\phi}$. It induces changes in the action by $(W, W)_{\phi}/2$ and those arising from the functional measure by $\Delta_{\phi} W$. Invariance of the path-integral requires cancellation of these two contributions: $\Sigma[\phi, \phi^*] = 0$. This is the QME, which ensures the presence of BRS symmetry in the quantum system. In this section, we derive the QME in our macroscopic theory, and then solve it.

#### 3.1 The QME in the macroscopic theory

For the microscopic action, the WT operator reads

$$
\Sigma[\phi, \phi^*] \equiv \frac{1}{2} (S_c, S_c)_{\phi} - \Delta_{\phi} S_c,
$$

(3.1)

where the $\Delta$-derivative is given by

$$
\Delta_{\phi} = \int d^4x \left[ \frac{\partial^r}{\partial \psi_c (x)} \frac{\partial^r}{\partial \psi^*_c (x)} - \frac{\partial^r}{\partial \psi_c (x)} \frac{\partial^r}{\partial \psi^*_c (x)} \right].
$$

(3.2)
We now discuss the relation between the WT operator for the microscopic action and that for the macroscopic action. To this end, we consider the functional average of the WT operator in the microscopic theory

\[ \langle \Sigma[\phi, \phi^*] \rangle_{\phi} \]

\[ = \left[ \int D\phi D\phi^* \prod_n \delta\left( \sum_n \Psi_n^* f_n(x) - \psi_n^*(x) \right) \delta\left( \sum_n \bar{\psi}_n^* f_n(x) - \bar{\psi}_n^*(x) \right) \right. \]

\[ \times e^{-(S_{\text{total}} - S_c)} \Delta_\phi e^{-S_c} \left[ \int D\phi D\phi^* \prod_n \delta\left( \sum_n \Psi_n^* f_n(x) - \psi_n^*(x) \right) \right. \]

\[ \times \left. \delta\left( \sum_n \bar{\psi}_n^* f_n(x) - \bar{\psi}_n^*(x) \right) e^{-S_{\text{total}}} \right] \left. -1 \right], \tag{3.3} \]

where the actions \( S_c \) and \( S_{\text{total}} \) are those defined in \( (2.8) \) and \( (2.13) \). Performing integration by parts in \( (3.3) \) and using \( \Delta \), one obtains

\[ \langle \Sigma[\phi, \phi^*] \rangle_{\phi} = \frac{\Delta_D \int D\pi D\pi^* D\sigma^* \prod_n \delta(\pi_n^*) \delta(\sigma_n^*) \int D\pi D\pi^* D\sigma^* \prod_n \delta(\pi_n^*) \delta(\sigma_n^*) e^{-S[\Phi, \Phi^*]}}{\int D\pi D\pi^* D\sigma^* \prod_n \delta(\pi_n^*) \delta(\sigma_n^*) e^{-S[\Phi, \Phi^*]}} \], \tag{3.4} \]

where the \( \Delta_D \) is the \( \Delta \)-derivative for the Dirac-field sector:

\[ \Delta_D = \sum_n \left( \frac{\partial^r}{\partial \Psi_n^*} \frac{\partial^r}{\partial \Psi_n^*} + \frac{\partial^r}{\partial \sigma_n^*} \frac{\partial^r}{\partial \sigma_n^*} \right). \tag{3.5} \]

In order to include the contributions from the auxiliary fields in the WT operator, we note that there is a trivial identity

\[ \int D\pi D\pi^* D\sigma^* \prod_n \delta(\pi_n^*) \delta(\sigma_n^*) \Delta_X e^{-S[\Phi, \Phi^*]} = 0, \tag{3.6} \]

where the \( \Delta_X \) is the \( \Delta \)-derivative for the auxiliary-field sector:

\[ \Delta_X = - \sum_n \left( \frac{\partial^r}{\partial \pi_n^*} \frac{\partial^r}{\partial \pi_n^*} + \frac{\partial^r}{\partial \sigma_n^*} \frac{\partial^r}{\partial \sigma_n^*} \right). \tag{3.7} \]

Let us define the WT operator for the macroscopic action \( S[\Phi, \Phi^*] \) in \( (2.18) \):

\[ \Sigma[\Phi, \Phi^*] = \frac{1}{2}(S, S) - (\Delta_D + \Delta_X)S = \frac{1}{2}(S, S) - \Delta_\phi S. \tag{3.8} \]

Adding \( (3.6) \) to \( (3.4) \) and using \( \Delta_\phi \equiv \Delta_D + \Delta_X \), one finds that

\[ \langle \Sigma[\phi, \phi^*] \rangle_{\phi} = \frac{\int D\pi D\pi^* D\sigma^* \prod_n \delta(\pi_n^*) \delta(\sigma_n^*) \Delta_\phi e^{-S[\Phi, \Phi^*]}}{\int D\pi D\pi^* D\sigma^* \prod_n \delta(\pi_n^*) \delta(\sigma_n^*) e^{-S[\Phi, \Phi^*]}} \]

\[ = \frac{\int D\pi D\pi^* D\sigma^* \prod_n \delta(\pi_n^*) \delta(\sigma_n^*) e^{-S[\Phi, \Phi^*]} \Sigma[\Phi, \Phi^*]}{\int D\pi D\pi^* D\sigma^* \prod_n \delta(\pi_n^*) \delta(\sigma_n^*) e^{-S[\Phi, \Phi^*]}} \]

\[ = \langle \Sigma[\Phi, \Phi^*] \rangle_X. \tag{3.9} \]
This is our fundamental relation between the WT operators in both theories.

For the microscopic theory, we have assumed that the original fermionic
action \( A_c \) is chiral invariant. This leads to the classical master equation (2.11).
Using (2.8) and (3.2), one can directly verify that \( \Delta \phi S_c = 0 \). Therefore, the
microscopic action satisfies the QME,
\[
\Sigma[\phi, \phi^*] = 0. \tag{3.10}
\]
Using (3.9), we obtain \( \langle \Sigma[\Phi, \Phi^*] \rangle_X = 0 \) for the macroscopic theory. This implies
the integral of the WT operator \( \Sigma[\Phi, \Phi^*] \) over the auxiliary fields gives zero. It
is allowed a wide class of solutions for which the WT operator becomes \( \pi \) or \( \sigma \)
derivative of something. We consider here more restrict class of solutions for
which the macroscopic action obeys the QME
\[
\Sigma[\Phi, \Phi^*] = \frac{1}{2} (S_D + S_X, S_D + S_X)_\Phi \\
- [\Delta D + \Delta X] [S_D + S_X] = 0. \tag{3.11}
\]
In order to further reduce (3.11), we assume that \( \delta_B \pi_n \) and \( \delta_B \sigma_n \) are given
by \( C \) times functions only of \( \pi_n \) and \( \sigma_n \). Since there appear no fermionic con-
tributions in \( \Delta \phi S_c \), the quantum master equation can be decomposed into two
conditions:
\[
\frac{1}{2} (S_D, S_D)_D + (S_D, S_X)_X = 0, \tag{3.12}
\]
\[
\frac{1}{2} (S_X, S_X)_X - [\Delta D S_D + \Delta X S_X] = 0. \tag{3.13}
\]
For any macroscopic fields \( \Phi^A \), the BRS transform \( \delta_B \Phi^A \) is proportional to the
ghost \( C \). It is convenient here to introduce Grassmann even counterpart \( \delta \) of
the odd operator \( \delta_B \). We may define it by
\[
\delta_B \Phi^A = -\delta \Phi^A \ C, \quad \Delta X S_X = -\delta J_X \ C, \tag{3.14}
\]
where \( \delta J_X \) is the change in the functional measure \( D\pi D\sigma \) induced by the chiral
transformation \( \delta \pi \) and \( \delta \sigma \). On the other hand, the change in the fermionic
functional measure is calculated to be
\[
\Delta D S_D = 2i \sum_n \text{Tr} \left( \gamma_5 - \gamma_5 \alpha^{-1} \tilde{D} \right)_{nn} \ C. \tag{3.15}
\]
The above relations (3.14) and (3.15) can be used to show that the QME
leads to the WT identities given in I. Actually, one finds that (3.12) and (3.13)
yield
\[
\frac{1}{2} (S_D, S_D)_D + (S_D, S_X)_X
\]
\[
= -i \Psi_n \left[ \{ \gamma_5, \hat{D} \} - 2\alpha^{-1} \hat{D} \gamma_5 \hat{D} - i \delta \hat{D} \right]_{nm} \Psi_m \ C = 0, \quad (3.16)
\]
\[
\frac{1}{2} \left( S_X, S_X \right)_X - [\Delta_D S_D + \Delta_X S_X]
= - \left[ \delta A_X + 2i \sum_n \text{Tr} \left( \gamma_5 - \gamma_5 \alpha^{-1} \hat{D} \right)_{nn} - \delta J_X \right] C = 0. \quad (3.17)
\]

Having obtained the WT identities from the QME, we are now in a position to solve them. For notational simplicity, we take below \( \alpha = 1 \), unless otherwise stated.

### 3.2 Solutions to the QME

Let us consider first (3.16) which reduces to
\[
\left\{ \gamma_5, \hat{D} \right\} - 2 \hat{D} \gamma_5 \hat{D} - i \delta \hat{D} \right]_{nm} = 0, \quad (3.18)
\]
where
\[
\hat{D}_{nm} = D_{nm} + \delta_{nm} X_n
\]
\[
X_n = (i \gamma_5 \pi + \sigma)_n. \quad (3.19)
\]
This is the GW relation for our system with auxiliary fields. As discussed in I, it is straightforward to determine \( \delta X_n \) owing to the locality assumption:
\[
\left\{ \begin{array}{l}
\delta X_n = -2i \gamma_5 (X_n - X_n X_n) \\
\delta \pi_n = -2 \sigma_n + 2 (\sigma_n^2 - \pi_n^2) \\
\delta \sigma_n = 2 \pi_n - 4 \sigma_n \pi_n
\end{array} \right. \quad (3.20)
\]
Note that \( X \) commutes with \( \gamma_5 \), and obeys is a nonlinear transformation. Using this result, the GW relation reduces to
\[
\left\{ \gamma_5, D \right\} - 2 D \gamma_5 D - 2 D \gamma_5 X - 2 X \gamma_5 D - i \delta D \right]_{nm} = 0. \quad (3.21)
\]
In order to solve (3.21), we make an ansatz for the Dirac operator:
\[
\hat{D} \equiv D_0 + (1 + \mathcal{L}(D_0)) X (1 + \mathcal{R}(D_0)) = D + X,
\]
\[
D = D_0 + \mathcal{L}(D_0)X + X \mathcal{R}(D_0) + \mathcal{L}(D_0)X \mathcal{R}(D_0), \quad (3.22)
\]
where the \( D_0 \) is the Dirac operator in the free theory. It satisfies the original GW relation
\[
\left\{ \gamma_5, D_0 \right\} = 2 D_0 \gamma_5 D_0. \quad (3.23)
\]
\[\dagger\]The matrix \( \gamma_5 \) satisfies \( \gamma_5^2 = 1 \).
Let us suppose that a solution for $D_0$ such as the Neuberger’s type $[2]$ is given, and determine the functions $L(D_0)$ and $R(D_0)$. One substitutes (3.22) into (3.21) using (3.20) for $\delta X$. Then, the resulting expression for l.h.s. of (3.21) can be expanded in powers of $X$: There appear linear and quadratic terms in $X$. As shown in Appendix, both of these terms vanish if the following conditions are satisfied:

$$ (1 - 2D_0) \gamma_5 (1 + L) = (1 + L) \gamma_5, \quad (3.24) $$

$$ (1 + R) \gamma_5 (1 - 2D_0) = \gamma_5 (1 + R), \quad (3.25) $$

$$ (1 + R) \gamma_5 (1 + L) \gamma_5 = \gamma_5 (1 + L) \gamma_5 (1 + R) = 1. \quad (3.26) $$

We find two sets of solutions to these equations given by

$$ L(D_0) = -D_0 + \frac{1}{1 - \gamma_5 D_0 \gamma_5} \gamma_5 D_0 \gamma_5 D_0 $$

$$ R(D_0) = -D_0, \quad (3.27) $$

or

$$ L(D_0) = -D_0, $$

$$ R(D_0) = -D_0 + D_0 \gamma_5 D_0 \gamma_5 \frac{1}{1 - \gamma_5 D_0 \gamma_5} $$

$$ = -D_0 \frac{1 - 2\gamma_5 D_0 \gamma_5}{1 - \gamma_5 D_0 \gamma_5}. \quad (3.28) $$

In Appendix, we show that $L$ and $R$ in (3.27) solve (3.24) $\sim$ (3.26). In summary, the Dirac operator which solves the GW relation (3.18) is given by

$$ \tilde{D} = D_0 + \gamma_5 \left( \frac{1}{1 - D_0} \right) \gamma_5 X (1 - D_0), \quad (3.29) $$

or

$$ \tilde{D} = D_0 + (1 - D_0) X \gamma_5 \left( \frac{1}{1 - D_0} \right) \gamma_5. \quad (3.30) $$

Let us consider (3.29) and (3.30). Since a matrix notation is used, the singularities arising from $D_0 = 1$ cannot be eliminated with the factor $(1 - D_0)$. The momenta satisfying the condition $D_0 = 1$ are those at which the species doublers appear. Therefore, the Yukawa terms suffer from the singularities. This kind of singularities have been discussed in ref. [19] in a different context.

Let us postpone discussion on the above singularities, and turn to the other condition (3.17). It reduces to three equations:

$$ \delta A_X^{(0)}[\pi, \sigma] = 0, \quad (3.31) $$

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\[ \delta A^{(1)}_X[\pi, \sigma] = \delta J_X[\pi, \sigma] = -8 \sum_n \pi_n \]
\[ = i2^{2-d/2} \sum_n \text{Tr} \gamma_5 (X_n - X_n^\dagger), \quad (3.32) \]
\[ \delta A^{(2)}_X[\pi, \sigma] = -2i \sum_n \text{Tr} (\gamma_5 - \gamma_5 \tilde{D})_{nn} \]
\[ = 2i \sum_n \text{Tr} (\gamma_5 (1 + L) X (1 + R))_{nn} \]
\[ = 2i \sum_n \text{Tr} (\gamma_5 X)_{nn}, \quad (3.33) \]

where we have used (3.26) to obtain the last expression of (3.33). The factor \(2^{-d/2}\) in (3.32) is needed to normalize the trace, denoted by Tr, over the spinor indices. In (3.31), \(A^{(0)}_X\) corresponds to an invariant potential. The terms \(A^{(1)}_X\) and \(A^{(2)}_X\) are counter terms needed to cancel \(\Delta_X S_X\) and \(\Delta_D S_D\). Solutions of the above conditions are given by

\[ A^{(0)}_X[\pi, \sigma] = \sum_n h \left( \frac{\sigma_n^2 + \sigma_{\pi n}^2}{1 - 2\sigma_n + \pi_n^2 + \sigma_n^2} \right), \quad (3.34) \]
\[ A^{(1)}_X[\pi, \sigma] = \sum_n 2 \ln \left( 1 - 2\sigma_n + \pi_n^2 + \sigma_n^2 \right) \]
\[ = 2^{1-d/2} \sum_n \text{Tr} \ln \left( (1 - X_n) (1 - X_n^\dagger) \right), \quad (3.35) \]
\[ A^{(2)}_X[\pi, \sigma] = \sum_n \text{Tr} \left( \ln (1 - X) \right)_{nn}, \quad (3.36) \]

where a function \(h\) is introduced to describe the invariant potential. We notice that all these three terms become singular at \(X = 1\).

In this section, we have solved the QME (3.11) to determine the effective action on the coarse lattice. The solutions we have obtained are used to construct a lattice chiral symmetry. Let us discuss its structure in the next section.

4 Lattice chiral symmetry in the macroscopic theory

Let us first summarize our results. In contrast to I, we discuss quantization of the system in the antifield formalism. The partition function for the macroscopic theory is given by

\[ Z_{\text{MACRO}} = \int D\Phi D\Phi^* \prod_A \delta(\Phi_A^* \Phi_A) \exp -S[\Phi, \Phi^*], \quad (4.1) \]
with the total macroscopic action

\[
S[\Psi, \Psi^*] = S_D + S_X,
\]

\[
S_X = A_X^{(0)} + A_X^{(1)} + A_X^{(2)} + 2 \sum_n \left[ \pi_n^* \{ \sigma_n - (\sigma_n^2 - \pi_n^2) \} C + \sigma_n^* \{ -\pi_n + 2\sigma_n \pi_n \} C \right] .
\]  

\[ (4.2) \]

There arise four types of the action \( S_D \) for the Dirac fields:

\( (I) \)

\[
S_D = \sum_{nm} \left[ \bar{\Psi}_n \tilde{D}_{nm} \Psi_m + \Psi_m^* i C \gamma_5 \left( 1 - 2\tilde{D} \right)_{nm} \Psi_m - \Psi_m^* C \gamma_5 \delta_{nm} \bar{\Psi}_n \right],
\]

\[
\tilde{D} = D_0 + \gamma_5 \frac{1}{1 - D_0} \gamma_5 \ X \ (1 - D_0) ,
\]

\[ (4.3) \]

\( (II) \)

\[
S_D = \sum_{nm} \left[ \bar{\Psi}_n \tilde{D}_{nm} \Psi_m + \Psi_m^* i C \gamma_5 \delta_{nm} \Psi_m - \Psi_m^* C \gamma_5 \Psi_m \right],
\]

\[
\tilde{D} = D_0 + \gamma_5 \frac{1}{1 - D_0} \gamma_5 \ X \ (1 - D_0),
\]

\[ (4.4) \]

\( (III) \)

\[
S_D = \sum_{nm} \left[ \bar{\Psi}_n \tilde{D}_{nm} \Psi_m + \Psi_m^* i C \gamma_5 \delta_{nm} \Psi_m - \Psi_m^* C \gamma_5 \Psi_m \right],
\]

\[
\tilde{D} = D_0 + (1 - D_0) X \gamma_5 \frac{1}{1 - D_0} \gamma_5,
\]

\[ (4.5) \]

\( (IV) \)

\[
S_D = \sum_{nm} \left[ \bar{\Psi}_n \tilde{D}_{nm} \Psi_m + \Psi_m^* i C \gamma_5 \delta_{nm} \Psi_m - \Psi_m^* C \gamma_5 \Psi_m \right],
\]

\[
\tilde{D} = D_0 + (1 - D_0) X \gamma_5 \frac{1}{1 - D_0} \gamma_5.
\]

\[ (4.6) \]

The potential terms of the auxiliary fields, \( A_X^{(0)}, A_X^{(1)} \) and \( A_X^{(2)} \) are given in \((3.34) \sim (3.36)\). Under the BRS transformation

\[
\delta_B \Psi_n = i C \gamma_5 \left( 1 - 2\tilde{D} \right)_{nm} \Psi_m,
\]

\[
\delta_B \bar{\Psi}_n = i C \bar{\Psi}_n \gamma_5,
\]

\[
\delta_B X_n = 2i \gamma_5 (X_n - X_n X_n) \ C,
\]

\[ (4.7) \]

or

\[
\delta_B \Psi_n = i C \gamma_5 \bar{\Psi}_n,
\]

\[
\delta_B \bar{\Psi}_n = i C \Psi_m \left( 1 - 2\tilde{D} \right)_{mn} \gamma_5,
\]

\[
\delta_B X_n = 2i \gamma_5 (X_n - X_n X_n) \ C,
\]

\[ (4.8) \]
the functional measure in (4.1) multiplies by the contributions from the counter
action,
\[ D\Phi D\Phi^* \prod_A \delta(\Phi^*_A) e^{-A^{(1)}(\pi,\sigma) - A^{(2)}(\pi,\sigma)} \]
\[ = \prod_n d\Psi_n d\bar{\Psi}_n d\pi_n d\sigma_n \prod_m d\Psi^*_m d\bar{\Psi}^*_m d\pi^*_m d\sigma^*_m \delta(\Psi^*_m) \delta(\bar{\Psi}^*_m) \delta(\pi^*_m) \delta(\sigma^*_m) \]
\[ \times e^{-A^{(1)}(\pi,\sigma) - A^{(2)}(\pi,\sigma)}, \quad (4.9) \]
remains invariant. The remaining part of the action
\[ S_D + S_X - A^{(1)}_X - A^{(2)}_X, \]
is also left invariant under (4.7) or (4.8). In other words, the macroscopic action
(4.2) solves the QME, \( \Sigma[\Phi, \Phi^*] = (S, S)/2 - \Delta S = 0 \), which demonstrates
the presence of an exact chiral symmetry in the quantum system on the coarse
lattice. In (4.4), the chiral transformation of the macroscopic fields is encoded
as the BRS transformation due to the presence of the antifields. Once the
transformation rule is known, one may eliminate the antifields by the integration
to obtain the partition function given in I.

We have shown that the QME is solved in a closed form. It should be noted,
however, that the resulting actions \( (4.3) \sim (4.6) \) are singular as discussed above.
Moreover, the chiral symmetry is realized in a peculiar way. The macroscopic
fields (except for \( \bar{\Psi} \) or \( \Psi \)) transform nonlinearly: \( \delta_B \Psi \) or \( \delta_B \bar{\Psi} \) contains \( X \), and
\( \delta_B X \) has a quadratic term of \( X \). As a result, in the fermionic action \( S_D \) given in
(4.2), neither the kinetic term \( \bar{\Psi} D_0 \Psi \) nor the Yukawa coupling \( \bar{\Psi}(1 + \mathcal{L})X(1 + \mathcal{R})\Psi \) is chiral invariant, while their sum \( (\bar{\Psi} D_0 \Psi) + (\bar{\Psi}(1 + \mathcal{L})X(1 + \mathcal{R})\Psi) \) becomes
invariant. Turn to the functional measure \( D\Phi \) of the macroscopic fields, it is not
chiral invariant so that the counter terms should be included. Furthermore, the
basic invariant made out of the auxiliary fields is nonpolynomial as in (3.34).

Because of these problems, we would like to reconstruct the chiral symmetry
in such a way that (1) the actions are free from the singularities, (2) the kinetic
term of the Dirac fields, the Yukawa coupling term and the functional measure
are all chiral invariant, and (3) the auxiliary field potential becomes polynomial.
The reconstruction of the symmetry satisfying the above conditions can be done
by employing a new set of canonical variables for each type of actions \( (4.3) \sim (4.6) \). We argue that two of the transformed actions define consistent quantum
theories but the remaining two cases may not.

5 Reconstruction of chiral symmetry in terms
of new canonical variables

Let \( \hat{\Phi}^A = \{ \Theta, \bar{\Theta}, \hat{X} = i\gamma_5 \pi + \hat{\sigma} \} \) be new fields. In the antifield formalism, we
obtain the new fields by considering a canonical transformation from \( \{ \Phi^A, \Phi^*_A \} \)
to \{\hat{\Phi}^A, \hat{\Phi}_A^*\}, where \(\hat{\Phi}_A^* = \{\Theta^*, \bar{\Theta}^*, \hat{X}^*\}\). The generator is given by
\[
G[\Phi, \hat{\Phi}^*] = \sum_{nm} \Theta^*_n Y(X)_{nm} \Psi_m + \sum_n \bar{\Psi}_n U(X)_{nm} \hat{\Theta}^*_m + \sum_n \text{Tr} \left[ \hat{X}^*_n W(X)_n \right],
\]
where \(\hat{X}^* = 2^{-d/2}[-i\gamma_5 \hat{\pi}^* + \hat{\sigma}^*]\). The matrices \(Y, U\) and \(W\) are functions of \(X\), and symmetric in the spinor indices. The new fields are obtained by \(\hat{\Phi}^A = \partial G/\partial \hat{\Phi}_A^*\), while the old set of antifields are given by \(\Phi^*_A = \partial G/\partial \Phi^A\):
\[
\Theta_n = [Y(X)\Psi]_n, \\
\bar{\Theta}_n = [\bar{\Psi}U(X)]_n, \\
\hat{X}_n = W(X)_n, \\
\Psi^*_n = [\Theta^* Y(X)]_n, \\
\bar{\Psi}^*_n = [U(X)\hat{\Theta}^*]_n, \\
X^*_n = \text{Tr} \left[ \hat{X}^*_n \partial W(X)_n \right] + \sum_{ml} \Theta^*_m \partial Y(X)_{ml} \Psi_l \\
+ \sum_{ml} \bar{\Psi}_m \partial Y(X)_{ml} \bar{\Theta}^*_l. \tag{5.2}
\]

There is a variety of choices for the matrices \(Y, U\) and \(W\) with which the transformed actions are free from the singularities. Among them, we discuss the following four cases corresponding to the actions (4.3) \sim (4.6).

The case (i): For (4.3), we take
\[
Y(X)_{nm} = \left[ \frac{1}{1-D_0} (1-X)(1-D_0) \right]_{nm}, \\
U(X)_{mn} = \delta_{nm}, \\
W(X)_n = \frac{X_n}{1-X_n}. \tag{5.3}
\]

One can confirm that the Jacobian factor associated with the change of variables from \{\pi, \sigma\}_n to \{\hat{\pi}_n, \hat{\sigma}_n\} exactly cancels the contribution from the counter term \(A_X^{(1)}:\)
\[
D\pi D\sigma e^{-A_X^{(1)}} = \prod_n d\pi_n d\sigma_n e^{-A_X^{(1)}} = D\hat{\pi} D\hat{\sigma}. \tag{5.4}
\]

Likewise, since
\[
\sum_n \text{Tr} \ln Y_{nn} - A_X^{(2)} = 0, \tag{5.5}
\]

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one can also see that the fermionic measure with the counter term \( A^{(2)}_X \) becomes

\[
\mathcal{D}\Psi \mathcal{D}\bar{\Psi} \exp[A^{(2)}_X] = \mathcal{D}\Theta \mathcal{D}\bar{\Theta} \exp[\sum_n \text{Tr}(\ln Y)_{nn} - A^{(2)}_X] = \mathcal{D}\Theta \mathcal{D}\bar{\Theta}.
\]

(5.6)

Therefore, the transformed theory is described by the partition function

\[
\begin{align*}
Z_{\text{old}} &= \int \mathcal{D}\Phi \mathcal{D}\bar{\Phi} \prod_{A} \delta(\Phi_A^*) \exp(-S[\Phi, \Phi^*]) \\
&= Z_{\text{new}} = \int \mathcal{D}\hat{\Phi} \mathcal{D}\bar{\hat{\Phi}} \prod_{A} \delta(\hat{\Phi}_A^*) \exp(-S[\hat{\Phi}, \hat{\Phi}^*]),
\end{align*}
\]

(5.7)

where the new action is given by

\[
S[\hat{\Phi}, \hat{\Phi}^*] = S_D + S_X,
\]

\[
S_D = \sum_{nm} \bar{\Theta}_n \left[D_0 + \hat{X}(1 - D_0)\right]_{nm} \Theta_m
+n \Theta_n^* C i \gamma_5 (1 - 2D_0)_{nm} \Theta_m - \sum_{n} \bar{\Theta}_n C i \gamma_5 \Theta_n^*,
\]

\[
S_X = \sum_{n} h(\hat{\pi}_n^2 + \hat{\sigma}_n^2) + 2 \sum_{n} (\hat{\pi}_n^* \hat{\sigma}_n - \hat{\sigma}_n^* \hat{\pi}_n) C.
\]

(5.8)

We have used here the relations

\[
\begin{align*}
\bar{\Psi}_n D_{nm} \Psi_m &= \bar{\Theta}_n (\bar{D}Y^{-1})_{nm} \Theta_m \\
&= \bar{\Theta}_n \left[D_0 + \hat{X}(1 - D_0)\right]_{nm} \Theta_m
\end{align*}
\]

\[
\frac{\pi_n^2 + \sigma_n^2}{(1 - \sigma_n)^2 + \pi_n^2} = \frac{\hat{\pi}_n^2 + \hat{\sigma}_n^2}{(1 - \hat{\sigma}_n)^2 + \hat{\pi}_n^2}.
\]

(5.9)

The partition function (5.7) is invariant under

\[
\begin{align*}
\delta_B \Theta_n &= i C\gamma_5 (1 - 2D_0)_{nm} \Theta_m, \\
\delta_B \bar{\Theta}_n &= i C\Theta_n \gamma_5, \\
\delta_B \hat{X}_n &= 2i \gamma_5 \hat{X}_n C.
\end{align*}
\]

(5.10)

It is noticed that the Yukawa couplings in the action (5.8) are the same as those discussed by many authors [12][13][15][18][19]:

\[
\mathcal{O}_{\text{Yukawa}} = \sum_{nm} \Theta_n [(i\gamma_5 \hat{\pi} + \hat{\sigma})(1 - D_0)]_{nm} \Theta_m
\]

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\[ \psi_n \left[ (i \gamma_5 \pi + \sigma) (1 - D_0) \right]_{nm} \psi_m \]

\[ = \sum_{nm} \Theta_n \bar{\Theta}_n (1 - D_0)_{nm} \Theta_m \hat{\sigma}_n + \sum_{nm} \Theta_n (1 - D_0)_{nm} \Theta_m \hat{\sigma}_n. \tag{5.11} \]

is form invariant, and chiral invariant:

\[ \delta_B \mathcal{O}_{Yukawa} = 0. \tag{5.12} \]

The case (ii): For (4.4), we may choose

\[ Y(X)_{nm} = \delta_{nm} \]

\[ U(X)_{nm} = \left[ \gamma_5 \frac{1}{1 - D_0} (1 - X) \gamma_5 \right]_{nm} \]

\[ W(X)_{n} = \frac{X_n}{1 - X_n}. \tag{5.13} \]

In this case, the Jacobian factor for the Dirac fields generates an additional factor \( \text{Det}(1 - D_0)^{-1} \), we may define the partition function as

\[ Z_{\text{old}} = \int \mathcal{D} \Phi \mathcal{D} \Phi^* \prod_A \delta(\Phi^*_A) \exp \left( -S[\Phi, \Phi^*] \right) \]

\[ = \frac{1}{\text{Det}(1 - D_0)} Z_{\text{new}} \]

\[ Z_{\text{new}} = \int \mathcal{D} \hat{\Phi} \mathcal{D} \hat{\Phi}^* \prod_A \delta(\hat{\Phi}^*_A) \exp \left( -S[\hat{\Phi}, \hat{\Phi}^*] \right), \tag{5.14} \]

where

\[ S[\Phi, \Phi^*] = S_D + S_X, \]

\[ S_D = \sum_{nm} \Theta_n \left[ D_0 (1 - \gamma_5 D_0 \gamma_5) \right]_{nm} \Theta_m + \sum_{nm} \Theta_n \left[ \hat{X} (1 - \gamma_5 D_0 \gamma_5) \right]_{nm} \Theta_m \]

\[ + \sum_n \Theta_n^* C \ i \gamma_5 \Theta_n - \sum_n \bar{\Theta}_n C \ i \gamma_5 \bar{\Theta}_n^*, \]

\[ S_X = \sum_n \hbar (\hat{\sigma}_n^2 + \hat{\sigma}_n^2) + 2 \sum_n (\hat{\pi}_n^* \hat{\sigma}_n - \hat{\pi}_n \hat{\sigma}_n^*) C. \tag{5.15} \]

The partition function \( Z_{\text{new}} \) is invariant under

\[ \delta_B \Theta_n = i C \gamma_5 \Theta_n, \]

\[ \delta_B \bar{\Theta}_n = i C \bar{\Theta}_n \gamma_5, \]

\[ \delta_B \hat{X}_n = 2i \gamma_5 \hat{X}_n C. \tag{5.16} \]
which is the same as the standard form of the chiral transformation in continuum (or microscopic) theories.

The case (iii): For the action (5.13), the same results as (5.14), (5.15) and (5.16) for the case (ii) can be obtained with

\[
Y(X)_{nm} = \left[ \gamma_5 \frac{1}{1-D_0} (1 - X) \gamma_5 \right]_{nm},
\]
\[
U(X)_{nm} = \delta_{nm},
\]
\[
W(X)_n = \frac{X_n}{1 - X_n}.
\]  

(5.17)

The case (iv): For the action (4.4), we consider the matrices similar to the case (i) as

\[
Y(X)_{nm} = \delta_{nm},
\]
\[
U(X)_{mn} = \left[ (1 - D_0)(1 - X) \frac{1}{1-D_0} \right]_{nm},
\]
\[
W(X)_n = \frac{X_n}{1 - X_n}.
\]  

(5.18)

Then, one obtains the partition function (5.7) with the Dirac action

\[
S_D = \sum_{nm} \bar{\Theta}_n \left[ D_0 + (1 - D_0) \hat{X} \right]_{nm} \Theta_m
\]
\[+ \sum_n \Theta_n C i \gamma_5 \Theta_n - \sum_{nm} \bar{\Theta}_n C i (1 - 2D_0)_{nm} \gamma_5 \bar{\Theta}^*_m. \]  

(5.19)

The chiral transformation takes the form

\[
\delta_B \Theta_n = i C \gamma_5 \Theta_n,
\]
\[
\delta_B \bar{\Theta}_n = i C \bar{\Theta}_n (1 - 2D_0) \gamma_5,
\]
\[
\delta_B \hat{X}_n = 2i \gamma_5 \hat{X}_n C.
\]  

(5.20)

Let us discuss some physical consequences for the cases (i) \sim (iv) listed above. The four are classified into two two groups: (i) and (iv), (ii) and (iii). Actually, (ii) and (iii) share the same action (5.13) and the chiral transformation (5.16). The kinetic term of the Dirac fields as well as the Yukawa term in this action contains the factor \((1 - \gamma_5 D_0 \gamma_5)\) in front of the Dirac fields. Since \(D_0 = 1\) at the momenta where the doubler modes appear, this factor vanishes. Thus, the doubler modes remain massless, and decouple with the auxiliary fields. However, this is the case only at tree level, and decoupling could not persist at quantum level: There are other chiral invariant Yukawa terms such as

\[
\bar{\Theta}_n \hat{X}_n \Theta_n, \quad \bar{\Theta}_n \hat{X}_n (\gamma_5 D_0 \gamma_5 D_0)_{nm} \Theta_m,
\]
in which the doublers couple with the auxiliary fields. There are no reasons to exclude these terms in the quantum corrections. Therefore, (ii) and (iii) cannot give a consistent theory.

Unlike these cases, the doubler modes in (i) and (iv) are massive and decouple with the auxiliary fields because of the factor \(1 - D_0\) in the Yukawa couplings. The chiral invariant Yukawa terms always contain this factor, and therefore the chiral invariance protects the couplings of the doublers to the auxiliary fields.

This can be seen by use of the chiral decomposition \[7\][12][23]:

\[
\hat{\Theta}_R = \frac{1 + \hat{\gamma}_5}{2} \Theta,
\hat{\Theta}_L = \frac{1 - \hat{\gamma}_5}{2} \Theta,
\bar{\Theta}_R = \bar{\Theta} \frac{1 + \gamma_5}{2},
\bar{\Theta}_L = \bar{\Theta} \frac{1 - \gamma_5}{2}.
\]

where

\[
\hat{\gamma}_5 = \gamma_5 (1 - 2D_0).
\]

Using

\[
\delta \hat{\Theta}_R = i C \hat{\Theta}_R, \quad \delta \hat{\Theta}_L = -i C \hat{\Theta}_L,
\delta \bar{\Theta}_R = i C \bar{\Theta}_R, \quad \delta \bar{\Theta}_L = -i C \bar{\Theta}_L,
\]

we may construct the the Yukawa term by using the chiral projection. One finds that resulting Yukawa term is exactly the same as \(O_{\text{Yukawa}}\) in (5.11):

\[
\mathcal{F}_{\text{Yukawa}} \equiv \bar{\Theta}_R \hat{\Theta}_R (\hat{\sigma} + i \hat{\pi}) + \bar{\Theta}_L \hat{\Theta}_L (\hat{\sigma} - i \hat{\pi})
= \bar{\Theta} \hat{X} (1 - D_0) \Theta = O_{\text{Yukawa}}
\]

In the transformed theory, the integration over the new auxiliary fields can be performed explicitly. The last expression of the Yukawa term in (5.11) can be used to do it. One then obtains

\[
Z_{\text{MACRO}} = \int \mathcal{D}\Theta \mathcal{D}\bar{\Theta} \exp \left( -\sum_{nm} \bar{\Theta}_n (D_0)_{nm} \Theta_m - \hbar \left( O_{\text{4-fermi}}(\Theta, \bar{\Theta}) \right) \right),
\]

where the antifields are integrated, too. The four-fermi interaction operator \(O_{\text{4-fermi}}\) is given by

\[
O_{\text{4-fermi}}(\Theta, \bar{\Theta}) = \left( \sum_{nm} \bar{\Theta}_n i \gamma_5 (1 - D_0)_{nm} \Theta_m \right)^2
\]
For the simplest case \( h(x) = x \), we obtain the Nambu-Jona-Lasinio model.

In this section, we have reconstructed the lattice chiral symmetry using the canonical transformations. Since these transformations are singular, the transformed theories are only equivalent to the original ones up to the singularities.

6 Inclusion of the complete set of the auxiliary fields

In the above formulation of the lattice chiral symmetry, the auxiliary fields we have considered are restricted to a scalar and a pseudoscalar fields. We discuss in this section inclusion of the complete set of the auxiliary fields. Let \( \Gamma^i \) \((i = 1 \sim 2^d)\) be the complete set of the Clifford algebra in \( d \) (=even) dimensional space. The Dirac fields carry \( N_F \) flavors, and form the fundamental representation of \( u(N_F) \) algebra with a basis \( T^a \) \((a = 1 \sim N_F^2)\) satisfying \( T^a \dagger = T^a \). Let \( \lambda^A \) \((A = 1 \sim N \equiv 2^d N_F^2)\) be the direct product of the above two sets of the matrices,

\[
\lambda^A = \lambda_i^a = \Gamma^i \otimes T^a,
\]

(normalized by

\[
\text{tr}(\lambda_i^a \lambda^b) = \text{tr}(\Gamma^i) \text{tr}(T^a T^b) = 2^d N_F \delta^{ij} \delta^{ab} = 2^d N_F \delta^{ab}. (6.2)
\]

Here \( \text{tr}(\Gamma) \) and \( \text{tr}(T) \) denote the traces in the spinor and the flavor spaces, respectively.

We introduce the complete set of the auxiliary fields \( x^A \), and define

\[
x^A = x^A \lambda^A,
\]

\[
x^A = 2^{-\frac{d}{2}} N_F^{-1} \text{Tr} \left( \lambda_i^a X \right),
\]

which is the extension of (3.31).

Let us consider the block transformation suppressing the antifields for simplicity.

\[
\int \mathcal{D}X \exp \left( - \sum_{nm} \bar{\Psi}_n \hat{D}(X)_{nm} \Psi_m - A_X[X] \right)
\]

\[= \int \mathcal{D}\psi \mathcal{D}\bar{\psi} \exp \left( -A_c[\psi, \bar{\psi}] - \sum_n (\bar{\Psi}_n - \bar{B}_n) \alpha (\Psi_n - B_n) \right), \quad (6.4)
\]
where $DX = \prod_{A=1}^{N} \prod_{n} dx_n^A$. The $\tilde{D}(X)$ is again assumed to be linear in $X$. Hereafter, we take again $\alpha = 1$. We may define the chiral transformation of the macroscopic fields as

$$
\delta \Psi_n = i\gamma_5 \left( 1 - 2\tilde{D} \right)_{nm} \Psi_m,
\delta \bar{\Psi}_n = \bar{\Psi}_n i\gamma_5,
\delta X_n = -i \{ \gamma_5, X_n \} + 2iX_n\gamma_5X_n.
$$

(6.5)

The WT identities associated with (6.5) are given by

$$
\left( i\{ \gamma_5, \tilde{D} \} - 2i\tilde{D}\gamma_5\tilde{D} + \delta \tilde{D} \right)_{nm} = 0,
$$

(6.6)

$$
\delta A_X^{(0)}[X] = 0,
\delta A_X^{(1)}[X] = \delta J_X[X] = 4N^2 \frac{2}{NF} \sum_n \text{Tr} \left( i\gamma_5X_n \right),
$$

(6.7)

$$
\delta A_X^{(2)}[X] = \delta J_{\Psi,\bar{\Psi}} = -2i \sum_n \text{Tr} \left( \gamma_5 - \gamma_5\tilde{D}(X) \right)_{nm},
$$

(6.8)

where the potential term $A_X$ is decomposed as $A_X = A_X^{(0)} + A_X^{(1)} + A_X^{(2)}$. It should be noted that the chiral transformation and the WT identities essentially take the same form as those for the truncated case, except that $X$ does not commute with $\gamma_5$ here. For the Dirac operator

$$
\tilde{D}(X) = D_0 + (1 + L(D_0))(1 + R(D_0)),
$$

(6.9)

the GW relation (6.6) can be solved with the $D_0$, $L$ and $R$ given in (3.26) and (3.27) or (3.28). Likewise, one obtains the counter terms

$$
A_X^{(1)}[X] = 2N^2 \frac{2}{NF} \sum_n \text{Tr} \ln (1 - X_n),
$$

$$
A_X^{(2)}[X] = \sum_n \text{Tr} \ln (1 - X_n)
$$

(6.10)

In order to construct the invariant potential $A_X^{(0)}$, we may define new set of the auxiliary fields:

$$
\hat{X}_n = \frac{X_n}{1 - X_n},
$$

(6.11)

which obeys a linear transformation as

$$
\delta \hat{X}_n = -i \{ \gamma_5, \hat{X}_n \}.
$$

(6.12)
Since $\hat{X} = \hat{x}^A X^A$ transforms linearly, it is easy to obtain a quadratic invariant:

$$f_{\text{inv}}^I(\hat{x}^A_n) = G_I^{AB} \hat{x}^A_n \hat{x}^B_n, \quad (I = 1, 2, \cdots), \quad (6.13)$$

where $G_I^{AB}$ are suitable coefficients. Using these invariants, we have

$$A_X^{(0)}[X] = \sum_n h(f_{\text{inv}}^I(\hat{x}^A_n)). \quad (6.14)$$

In summary, a chiral invariant partition function $Z$ is given by

$$Z = \int [D\hat{X} D\psi D\bar{\psi}] e^{-A_X^{(1)} - A_X^{(2)}} e^{-\sum_{nm} \bar{\psi}_n \hat{D}_{nm} \psi_m - A_X^{(0)}}, \quad (6.15)$$

where

$$\hat{D}(X) = D_0 + (1 + L)X(1 + R),$$

$$A_X^{(0)}[X] = \sum_n h(f_{\text{inv}}(\hat{x}^A_n)),$$

$$A_X^{(1)}[X] + A_X^{(2)}[X] = \left(2N^2 - \frac{4}{N_F} + 1\right) \sum_n \text{Tr} \ln \left(1 - \frac{X_n}{\alpha}\right). \quad (6.16)$$

Let us give a special case of $d = 2$ and $N_F = 1$: $X = i\gamma_5\pi + \sigma + i\gamma_\mu V^\mu$. The chiral transformation is given by

$$\delta \pi_n = -2\sigma_n - 2\pi_n^2 + 2\sigma_n^2 + 2V^\mu_n^2,$$

$$\delta \sigma_n = 2\pi_n - 4\pi_n\sigma_n,$$

$$\delta V^\mu_n = -4V^\mu_n\pi_n. \quad (6.17)$$

The new fields $\hat{x}^A$ defined in (6.11) and their transformation are given by

$$\tilde{\pi}_n = \frac{\pi_n}{(1 - \sigma_n)^2 + (\pi_n)^2 + (V^\mu_n)^2}, \quad \delta \tilde{\pi}_n = -2\tilde{\sigma}_n,$$

$$\tilde{\sigma}_n = \frac{\sigma_n - \pi_n^2 - \pi_n^2 - V^\mu_n^2}{(1 - \sigma_n)^2 + (\pi_n)^2 + (V^\mu_n)^2}, \quad \delta \tilde{\sigma}_n = 2\tilde{\pi}_n,$$

$$\tilde{V}^\mu_n = \frac{V^\mu_n}{(1 - \sigma_n)^2 + (\pi_n)^2 + (V^\mu_n)^2}, \quad \delta \tilde{V}^\mu_n = 0. \quad (6.18)$$

We have two quadratic invariants:

$$f_{\text{inv}}^1 = \tilde{\pi}_n^2 + \tilde{\sigma}_n^2 = \frac{\pi_n^2 + (\sigma_n - \pi_n^2 - \pi_n^2 - V^\mu_n^2)^2}{(1 - \sigma_n)^2 + (\pi_n)^2 + (V^\mu_n)^2},$$

$$f_{\text{inv}}^2 = (\tilde{V}^\mu_n)^2 = \frac{V^\mu_n^2}{(1 - \sigma_n)^2 + (\pi_n)^2 + (V^\mu_n)^2}. \quad (6.19)$$
In this section, we have discussed an extension of the auxiliary method. We have obtained a formal expression of chiral invariant partition function (6.16). Its action, however, is not free from the singularities discussed in section 3, and it seems to be difficult to construct the canonical transformation which removes these singularities.

7 Summary and discussion

There have been known nontrivial examples where exact chiral symmetries are realized in interacting theories on the lattice. One was given by Lüscher who discussed chiral gauge theories. The lattice chiral symmetry in fermionic interacting system discussed in this paper may provide another example. For our fermionic system with auxiliary fields, there arise two sets of the WT identities: One is the GW relation which tells us how to define chiral transformation for the Dirac as well as the auxiliary fields on the coarse lattice. Under the suitable locality assumption for the auxiliary fields, we have determined the chiral transformation for the macroscopic fields. The transformation rule is used to construct chiral invariant actions. The other WT identity can be interpreted as an anomaly matching relation between the microscopic and the macroscopic theories. This identity contains contributions arising from the transformation of the functional measure, and is used to construct counter terms needed to make the functional measure on the coarse lattice chiral invariant. In the antifield formalism, these WT identities are obtained from the QME Σ = 0.

Owing to the auxiliary field method, the fermionic sector of our system is linearized. The price for it is that the integration over the auxiliary fields remains in the condition < Σ > = 0. We have considered in this paper the QME Σ = 0, and found four types of actions which solve the QME. However, they are found to have singularities in the Yukawa couplings and the potential of the auxiliary fields. Those in the Yukawa couplings are related to the presence of doubler modes. Beside these singularities, none of the kinetic term of the Dirac fields, the Yukawa couplings and the functional measure becomes chiral invariant in the realization of the symmetry with the block variables. In order to avoid these problems, we have used more suitable sets of variables obtained by (singular) canonical transformations. We have discussed four types of the transformed actions. In all cases, the new fields transform linearly, and have chiral invariant functional measure. Among these actions, only two of them define consistent quantum theories. They are exactly equivalent to those obtained by using the representation method for chiral algebra arising from free field theory: One defines the chiral transformation in the free theory, and then constructs chiral invariant Yukawa couplings from the chiral decomposition. The chiral invariance in such systems is expressed by the classical master equation rather than the QME. Since the new auxiliary fields belong to the conventional SO(2) multiplet, they are readily integrated to give purely fermionic system. The Nambu-Jona-
Lasinio model emerges as the simplest one. Our results may give justification of the representation method for formulating lattice chiral symmetry in theories with generic interactions.

Let us discuss some implications of our results for realization of regularization-dependent symmetry in lattice or continuum theory. The block transformation on the lattice or its continuum analog plays an important rôle in inheriting symmetry properties of the macroscopic theory from those of the microscopic theory. In general, the functional measure for the original block variables may not be invariant under the cutoff-dependent symmetry transformations. The induced change in the functional measure corresponds to the $\Delta$ derivative of the extended action, whose explicit expression depends on the UV regularization scheme. However, unless nontrivial anomaly is present in the given theory, one can always find the counter action needed to cancel the $\Delta$ derivative contribution. The counter action is regularization dependent, but is expected to be antifield independent. Then, a canonical transformation will be performed in such a way that new fields have invariant measure. This corresponds to the reduction of the QME to the classical master equation.

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A Derivation of some formulae

Let us first derive (3.24), (3.25) and (3.26). The reduced GW relation (3.21) is divided into the local terms and the nonlocal terms which depend on $\mathcal{L}$ and/or $\mathcal{R}$. The nonlocal terms becomes

\[
\begin{align*}
\{\gamma_5, \mathcal{L}X + X\mathcal{R} + \mathcal{L}X\mathcal{R}\} \\
-2D_0\gamma_5 (X + \mathcal{L}X + X\mathcal{R} + \mathcal{L}X\mathcal{R}) \\
-2(X + \mathcal{L}X + X\mathcal{R} + \mathcal{L}X\mathcal{R})\gamma_5 D_0 \\
-2X\gamma_5(\mathcal{L}X + X\mathcal{R} + \mathcal{L}X\mathcal{R}) \\
-2(\mathcal{L}X + X\mathcal{R} + \mathcal{L}X\mathcal{R})\gamma_5 X \\
-2(\mathcal{L}X + X\mathcal{R} + \mathcal{L}X\mathcal{R})\gamma_5(\mathcal{L}X + X\mathcal{R} + \mathcal{L}X\mathcal{R}) \\
-2 (\mathcal{L}\gamma_5 (X - X^2) + \gamma_5(X - X^2)\mathcal{R} + \mathcal{L}\gamma_5 (X - X^2)\mathcal{R} ) \\
= KKX + X\mathcal{H} + KX\mathcal{R} + \mathcal{L}X\mathcal{H} - 2X\mathcal{G}X - 2X\mathcal{G}X\mathcal{R} - 2\mathcal{L}X\mathcal{G}X - 2\mathcal{L}X\mathcal{G}X\mathcal{R} = 0.
\end{align*}
\]
where

\[
\begin{align*}
\mathcal{K} &= \gamma_5 \mathcal{L} - \mathcal{L} \gamma_5 - 2D_0 \gamma_5 - 2D_0 \gamma_5 \mathcal{L}, \\
\mathcal{H} &= -\gamma_5 \mathcal{R} + \mathcal{R} \gamma_5 - 2\gamma_5 D_0 - 2\mathcal{R} \gamma_5 D_0, \\
\mathcal{G} &= \gamma_5 \mathcal{L} + \mathcal{R} \gamma_5 + \mathcal{R} \gamma_5 \mathcal{L}.
\end{align*}
\]

(A.2)

The conditions \( \mathcal{K} = \mathcal{H} = \mathcal{G} = 0 \) lead to (3.24), (3.25) and (3.26). Let us show that \( L \) and \( R \) in (3.27) are solutions of (3.24) \(-\sim\) (3.26). \( K \) with \( L \) in (3.27) is the GW relation in free theory (3.23). \( H \) with \( R \) in (3.27) becomes

\[
\begin{align*}
\mathcal{H} &= -\gamma_5 D_0 - D_0 \gamma_5 + 2D_0 \gamma_5 D_0 \\
&= \gamma_5 D_0 \gamma_5 D_0 \gamma_5 \frac{1}{1 - \gamma_5 D_0 \gamma_5} + D_0 \gamma_5 D_0 \gamma_5 \frac{1}{1 - \gamma_5 D_0 \gamma_5} \gamma_5 (1 - 2D_0) \\
&= -D_0 \gamma_5 D_0 \frac{1}{1 - \gamma_5 D_0 \gamma_5} + D_0 \gamma_5 D_0 \frac{1}{1 - D_0} (1 - 2D_0) \\
&= D_0 \gamma_5 D_0 \frac{1}{1 - \gamma_5 D_0 \gamma_5} (- (1 - D_0) + (1 - \gamma_5 D_0 \gamma_5) (1 - 2D_0)) \frac{1}{1 - D_0} \\
&= 0,
\end{align*}
\]

where we have used \( \gamma_5 D_0 \gamma_5 D_0 \gamma_5 = D_0 \gamma_5 D_0 \). Finally, \( G \) turns to be

\[
\begin{align*}
\mathcal{G} &= -\gamma_5 D_0 + \left( -D_0 + D_0 \gamma_5 D_0 \gamma_5 \frac{1}{1 - \gamma_5 D_0 \gamma_5} \right) \gamma_5 \\
&= (-\gamma_5 D_0 - D_0 \gamma_5 + 2D_0 \gamma_5 D_0) \\
&= D_0 \gamma_5 D_0 \left( -1 + \frac{1}{1 - \gamma_5 D_0 \gamma_5} \right) \gamma_5 - D_0 \gamma_5 D_0 \gamma_5 \frac{1}{1 - \gamma_5 D_0 \gamma_5} \gamma_5 D_0 \\
&= D_0 \gamma_5 D_0 \gamma_5 \frac{1}{1 - \gamma_5 D_0 \gamma_5} \gamma_5 D_0 - D_0 \gamma_5 D_0 \gamma_5 \frac{1}{1 - \gamma_5 D_0 \gamma_5} \gamma_5 D_0 \\
&= 0.
\end{align*}
\]

(A.3)

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