A SOLUTION TO THE COMBINATORIAL PUZZLE OF
MAYERS VIRIAL EXPANSION

STEPHEN JAMES TATE

Abstract. Mayer’s second theorem in the context of a classical gas model
allows us to write the coefficients of the virial expansion of pressure in terms
of weighted two-connected graphs. Labelle, Leroux and Ducharme studied
the graph weights arising from the one-dimensional hard core gas model and
noticed that the sum of these weights over all two-connected graphs with n
vertices is \( -n(n-2)! \). The case of connected graphs has already been de-
scribed combinatorially by Bernardi and this article conveys, in a similar way
to Bernardi, how the two-connected version may also be obtained combina-
torially.

1. Introduction

This paper gives a proof, from a combinatorial perspective, for two identities
arising from Mayer’s theory of virial expansions for the one particle hard core gas
and the continuum hard core gas or Tonks gas.

The relationship for the one-particle hardcore gas is:

\[
\sum_{g \in \mathcal{B}[n]} (-1)^{e(g)} = -(n-2)!
\]

for the one particle hard core gas, where \( e(g) \) denotes the number of edges in the
graph \( g \).

Let the polytope corresponding to the graph \( g \) be defined as:

\[
\Pi_g := \{(x)_{[2,N]} \in \mathbb{R}^{N-1} | |x_i - x_j| < 1 \forall \{i,j\} \in E_g\}
\]

with \( x_1 = 0 \). The notation \((x)_{[2,N]} = (x_2, \cdots, x_N)\). Then we have, for the Tonks
gas:

\[
\sum_{g \in \mathcal{B}[n]} (-1)^{e(g)} \text{Vol}(\Pi_g) = -n(n-2)!
\]

Bernardi, in [4], gives the proof in the connected graph case.

The work of Mayer [17] introduced an important connection between the sub-
jects of combinatorics and statistical mechanics, in the form of representing co-
efficients of the important expansions of statistical mechanics - the partition

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purposes.
function, the cluster expansion and the virial expansion - in terms of weighted graphs of a particular type - simple graphs, connected graphs and two-connected graphs respectively. Most recently, the combinatorial species of structure framework introduced by Joyal [13], has been applied to statistical mechanics through the work of Ducharme, Labelle and Leroux [8, 15], Kaouche and Leroux [14] and Faris [9], in order to give an interpretation to these important connections. Useful developments of the subject of combinatorial species of structure can be found in the book by Bergeron Labelle and Leroux [7] and Flajolet and Sedgewick [11].

Such connections between the two subjects provides mutual exchanges. The main exchange that this paper focuses upon are the combinatorial identities indicated in the paper [8] afforded to us by using the combinatorial connection with two simplistic statistical mechanical models, namely the hardcore one-particle gas and the hardcore continuum gas. An explanation has already been given for the identities arising from the cluster expansion by Bernardi [4], which indicates that the two identities resolve to counting increasing trees and Cayley trees, respectively.

Interest in the virial expansion has recently been renewed with the papers by Pulvirenti and Tsagkarogiannis [21] and Morais and Procacci [19] in using the Canonical Ensemble as a method of achieving bounds, both on the virial coefficients and on the radius of convergence. Furthermore, the paper by Jansen [?]? suggests that at high temperatures the radius of convergence should be improved: actual improvements on the bounds of Lebowitz and Penrose [15] have been proposed recently [23].

The direction that this paper takes is to gain a better understanding of the combinatorics of two-connected graphs so that cancellations can be handled as efficiently as they are for connected graphs. A future target for this work is to be able to generalise the Matroid-structure from the Potts Model [22, 10] to a similar procedure for the more general posets. Proposing a natural Matroid model for two-connected graphs has some inherent difficulties in that minimal two-connected graphs do not have the same number of edges to be the required basis sets. However, the poset structure is clear and it is hoped with the additional ordering of edges introduced in this paper a similar concept of internally and externally active edges may be found that would enable the two-connected weights to be rewritten in terms of modified weights on the fixed two-connected graphs found here.

These future directions should mirror the development of tree-graph identities used for cluster expansions, which were developed by Brydges Battle and Federbush [5, 2, 3, 6]. A more general combinatorial interpretation of these identities are given in the paper by Abdesselam and Rivasseau [1].

The paper is organised so that, in Section 1, the two models of the one particle hard core gas and the Tonks gas are explained alongside what Mayer’s theorems give in these cases. This section may be passed over without effecting the understanding of the proof. The bc-tree used in the proof is also introduced in Section 2 and the explanation of polytopes and their decomposition into simplicies, attributed to Lass is indicated in Section 3 which is instrumental to the proof in the Tonks gas case. Readers already familiar with both concepts are able to skip these sections, or use them as reference. Section 5 presents the combinatorial
structures which describe the particular coefficients and gives a discussion on the meaning of the results. Sections 6 and 7 give the proofs of the one particle hard core and the Tonks gas case respectively. Section 8 ends with what is hoped to be future directions and context of these combinatorial interpretations and ideas of future work.

2. The Block Cutpoint Tree

This section briefly introduces the notion of the block cutpoint tree used in the proof.

- An articulation point in a connected graph is a vertex, which when it and its incident edges are removed, renders the graph disconnected. A synonym that is frequently used is a cutpoint.
- A two-connected graph is a connected graph without articulation points.
- A block is a maximal two-connected subgraph of a connected graph. Maximal in terms of edges and vertices it includes.

The block cutpoint tree (bc-tree) associated to a connected graph $g$ is a (bipartite) graph where the vertices represent the articulation points and the blocks in a connected graph. An edge, between an articulation point and a block, is present in this graph, when an articulation point is contained in a block. It is a tree, since if there were a cycle in this graph then the cycle itself would have been a block. An example of a block cutpoint tree is shown in Figure 1.

A bc-tree is bipartite with all leaves in one set (the blocks). It therefore has a unique centre. Two adjacent blocks are connected at a single articulation point and we count the articulation point in the block closest to the centre. There is precisely one articulation point in each block we omit in doing this and so in counting the total number of vertices, we add the number of vertices in each block minus one.
If an articulation point is the centre, we see that it is the one vertex omitted when counting like this. If the centre is an individual block then we would like to count all vertices in this block, but by ensuring we add the number of vertices in each block minus one, we have one less in this case as well. Hence, we have the identity:

\[ \sum_{i \in I} (k_i - 1) = n - 1 \]  

(2.1)

where \( I \) is the label set for the blocks \( k_i \) indicates the number of vertices in a block and \( n \) is the total number of vertices in the graph.

3. Polytopes and Simplices

In [3] there is a decomposition of the polytope \( \Pi_g \) into simplices, which is used in the proof given in [1]. Bernardi in [1] gives an explanation of this procedure. This section explains this procedure.

Consider \( (x)_2, n) \in \mathbb{R}^{n-1} \) and let \( h_i \) be the integer part of \( x_i \) and \( 0 \leq w_i < 1 \) be the fractional part such that \( h_i + w_i = x_i \). Let \( \sigma : [2, n] \to [2, n] \) be a bijection. We may define the simplex \( \pi(h, \sigma) \), by the set of \( x \) with integer part \( h \) and whose fractional parts satisfy: \( w_{\sigma(2)} < w_{\sigma(3)} < \cdots < w_{\sigma(n)} \). This simplex has volume \( \frac{1}{(n-1)!} \).

The condition \( |x_i - x_j| < 1 \) is equivalent to \( h_i - h_j \in \{0, \text{sign}(w_j - w_i)\} \). We therefore have that \( \pi(h, \sigma) \subset \Pi_G \) if and only if for all \( (i, j) \in G \), we have that \( h_i - h_j \in \{0, \text{sign}(\sigma^{-1}(j) - \sigma^{-1}(i))\} \) with \( h_1 = 0 \) and \( \sigma(1) = 1 \).

Lemma 3.1. For any graph \( G \) on \( [n] \), the value \( (n-1)! \text{Vol}(\Pi_G) \) counts the pairs \( h \in \mathbb{Z}^{n-1} \) and \( \sigma \in S_{n-1} \) such that \( \pi(h, \sigma) \) is a subpolytope of \( \Pi_G \).

We may rearrange the sums over connected or two-connected graphs of the graph weights by first casting the sum as a sum over the pairs \( (h, \sigma) \) and symmetrising the weight over isomorphic graphs. The symmetrisation procedure can be understood by considering a permutation \( \sigma \) of \( [2, n] \) and defining for any vector \( h = (h_2, \ldots, h_n) \in \mathbb{Z}^{n-1} \), \( \sigma(h) = (h_{\sigma(2)}, \ldots, h_{\sigma(n)}) \) and for any graph \( G \) with labels in \( [n] \), the graph \( \sigma(G) \), which replaces each label \( i \) with \( \sigma(i) \). Observe that \( \pi(h, \sigma) \subset \Pi_G \) if and only if \( \pi(\sigma^{-1}(h), \text{Id}) \subset \Pi_{\sigma(G)} \) for any permutation \( \sigma \) of \( [2, n] \), hence, we can rewrite:

\[
\sum_{h \in \mathbb{Z}^{n-1}, G \in \mathcal{H}[n]} (-1)^{e(G)} = \sum_{h \in \mathbb{Z}^{n-1}, G \in \mathcal{H}[n]} \sum_{\pi(h, \sigma) \subset \Pi_G} (-1)^{e(G)}
\]

\[
= \sum_{h \in \mathbb{Z}^{n-1}, G \in \mathcal{H}[n]} \sum_{\pi(h, \sigma^{-1}(h), \text{Id}) \subset \Pi_{\sigma(G)}} (-1)^{e(\sigma^{-1}(G))}
\]

\[
= \sum_{h \in \mathbb{Z}^{n-1}, G \in \mathcal{H}[n]} \sum_{\pi(h, \text{Id}) \subset \Pi_G} (-1)^{e(G)}
\]

(3.1)
We let $H$ denote either $C$ or $B$, then:

$$
\sum_{G \in H[n]} w(G) = \sum_{G \in H[n]} (-1)^{\epsilon(G)} \frac{\text{Vol}(\Pi_G)}{(n-1)!} \sum_{h \in \mathbb{Z}^{n-1}} (-1)^{\epsilon(G)} \\
= \sum_{h \in \mathbb{Z}^{n-1} \cap H[n]} (-1)^{\epsilon(G)}
$$

(3.2)

We define the centroid of the vector $h$, by $\bar{h} = (\bar{h}_1, \cdots, \bar{h}_n)$, where $\bar{h}_i = h_i + \frac{i-1}{n}$. We define $G_h$ as the graph on $[n]$ where the edges are all pairs $(i, j)$ such that $|\bar{h}_i - \bar{h}_j| < 1$. We define $H_h[n] := \{ G \in H[n] | E_G \cap E_{G_h} = E_G \}$ where $H$ can be replaced by $C$ or $B$.

The final sum indicates that we need to count pairs $h$ and $G$ such that $\pi(h, Id) \subset \Pi_G$. This means that the centroid $\bar{h} \in \Pi_G$, since $\bar{h}$ is in the interior of $\pi(h, Id)$.

This means that for $h \in \Pi_G$, we require that:

$$
\forall (i, j) \in E_G \ | \bar{h}_i - \bar{h}_j| < 1
$$

(3.3)

This means that we can rewrite our sum as:

$$
\sum_{h \in \mathbb{Z}^{n-1}} \sum_{G \in H_h} (-1)^{\epsilon(G)}
$$

(3.4)

we can thus consider the total sum as first a sum over the subset of graphs $H_h$ for each $h$ and add the results. This leads to considering separate $\Psi_h : B_h \rightarrow B_h$ which are involutions and finding their fixed points.

### 4. The Two Models from Statistical Mechanics

In a classical gas system of $n$ indistinguishable interacting particles in a vessel $\Lambda \subset \mathbb{R}^d$ with only two-body interactions and no external potential, we may write the Hamiltonian as:

$$
H(\mathbf{p}, \mathbf{q}) = \sum_{i=1}^{n} \frac{p_i^2}{2m} + \sum_{1 \leq i < j \leq n} \phi(q_i, q_j)
$$

(4.1)

where $\mathbf{q}$ represents the generalised coordinates and $\mathbf{p}$ the conjugate momenta. The canonical partition function of the gas model is:

$$
Z(\Lambda, \beta, n) = \frac{1}{n!} \prod_{i=1}^{n} \left( \int_{\Lambda} d^d q_i \int_{\mathbb{R}^d} d^d p_i \right) \exp(-\beta H)
$$

(4.2)

Integrating out the Gaussian integrals for the momenta, we obtain a factor $\frac{1}{\lambda^n}$, where $\lambda$ is the thermal wavelength. The partition function is therefore:

$$
Z(\Lambda, \beta, n) = \frac{1}{n! \lambda^n} \prod_{i=1}^{n} \left( \int_{\Lambda} d^d q_i \right) \prod_{1 \leq i < j \leq n} \exp(-\beta \phi(q_i, q_j))
$$

(4.3)

The Mayer trick [17], allows us to rewrite the canonical partition function in terms of weighted graphs. The first stage is to define the Mayer $f$-function:

$$
f(q_i, q_j) := \exp(-\beta \phi(q_i, q_j)) - 1
$$

(4.4)
We realise that the product of exponentials in (4.3) may be rewritten as:
\[
\prod_{1 \leq i < j \leq n} \exp(-\beta \varphi(q_i, q_j)) = \prod_{1 \leq i < j \leq n} (1 + f(q_i, q_j)) = \sum_{g \in \mathcal{G}[n]} \prod_{(i,j) \in E_g} f(q_i, q_j) (4.5)
\]
where \( \mathcal{G}[n] \) is the set of simple graphs (no multiple edges or loops) on \( n \) points.

We write a graph \( g = (E_g, V_g) \), where \( E_g \subset [n]^{(2)} \) is the edge set and \( V_g = [n] \) is the vertex set. The notation is \( [n] = \{1, \ldots, n\} \) and \([n]^{(2)} = \{\{i, j\}|i \neq j \in [n]\}\). This motivates the graph weight:
\[
W(g) = \prod_{i=1}^{n} \left( \int_{\Lambda} d^d q_i \right) \prod_{(k,l) \in E_g} f(q_k, q_l) (4.6)
\]
We can therefore write the partition function as:
\[
Z(\Lambda, \beta, n) = \frac{1}{n! \lambda^n} \sum_{g \in \mathcal{G}[n]} W(g) (4.7)
\]
In order to obtain the grand canonical partition function we sum:
\[
\Xi(\Lambda, \beta, z) = \sum_{n=0}^{\infty} z^n \lambda^n Z(\lambda, \beta, n) (4.8)
\]
where \( z = e^{\beta \mu} \) the activity and \( \mu \) is the chemical potential. In terms of graphs, we write this as:
\[
\Xi(\Lambda, \beta, z) = \sum_{n=0}^{\infty} \frac{z^n}{n!} \sum_{g \in \mathcal{G}[n]} W(g) =: \mathcal{G}_W(z) (4.9)
\]
The pressure is defined to be:
\[
\beta P = \lim_{|\Lambda| \to \infty} \frac{1}{|\Lambda|} \log \Xi(\Lambda, \beta, z) (4.10)
\]
If we define the new weight \( w(g) = \lim_{|\Lambda| \to \infty} \frac{1}{|\Lambda|} W(g) \), then the pressure function can be written in terms of connected graphs:
\[
\beta P = C_w(z) = \sum_{n=1}^{\infty} \frac{z^n}{n!} \sum_{g \in \mathcal{C}[n]} w(g) (4.11)
\]
This is the content of Mayer’s First Theorem [17] and is explained in the paper [8]. The density \( \rho \) is:
\[
\rho = z \frac{\partial}{\partial z} \beta P = C^*(z) (4.12)
\]
where \( C^* \) denotes a rooted connected graph. From Mayer’s Second Theorem [17] or by the Dissymmetry Theorem [8], we are able to obtain a series expansion for pressure in terms of density, in which the coefficients are, up to a prefactor, the \( w \)-weighted two-connected graphs.
\[
\beta P = \rho - \sum_{n=2}^{\infty} \frac{(n-1) \rho^n}{n!} \sum_{g \in \mathcal{B}[n]} w(g) (4.13)
\]
One may also consult the book by McCoy [18] for an explanation of the derivation of these two theorems.
4.1. One Particle Hard Core Gas. The potential for a one-particle hard core gas is:
\[ \varphi(q_i, q_j) = \infty \]  
(4.14)
so that \( \exp(-\beta \varphi(q_i, q_j)) = 0 \) and \( f(q_i, q_j) = -1 \). The grand canonical partition function is
\[ \Xi(z) = 1 + z \]  
(4.15)
The statistical mechanical relationships give pressure and density as:
\[ \beta P = \log(1 + z) \]  
(4.16)
\[ \rho = \frac{z}{1 + z} \]  
(4.17)
We may invert (4.17), to obtain:
\[ z = \frac{\rho}{1 - \rho} \]  
(4.18)
and substitute for \( z \) in (4.16), to obtain:
\[ \beta P = -\log(1 - \rho) \]  
(4.19)
The two series expansions derived from statistical mechanics are:
\[ \beta P = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}z^n}{n} \]  
(4.20)
\[ \beta P = \sum_{n=1}^{\infty} \frac{\rho^n}{n} \]  
(4.21)
If we compare these two power series with (4.11) and (4.13) respectively, using the graph weight \( w(g) = (-1)^{c(g)} \), where \( c(g) \) is the number of edges in graph \( g \), we obtain:
\[ \sum_{g \in \mathcal{C}[n]} (-1)^{c(g)} = (-1)^{n-1}(n - 1)! \]  
(4.22)
\[ \sum_{g \in \mathcal{B}[n]} (-1)^{c(g)} = -(n - 2)! \]  
(4.23)
4.2. Continuum Hard Core Gas - Tonks Gas. For a continuum hard core gas in one dimension with diameter 1, the potential is:
\[ \varphi(q_i, q_j) = \begin{cases} \infty & \text{if} \quad |q_i - q_j| < 1 \\ 0 & \text{otherwise} \end{cases} \]  
(4.24)
The exponential and Mayer \( f \)-functions are:
\[ \exp(-\beta \varphi(q_i, q_j)) = \begin{cases} 0 & \text{if} \quad |q_i - q_j| < 1 \\ 1 & \text{otherwise} \end{cases} \]  
(4.25)
\[ f(q_i, q_j) = \begin{cases} -1 & \text{if} \quad |q_i - q_j| < 1 \\ 0 & \text{otherwise} \end{cases} \]  
(4.26)
We therefore have the graph weight:

$$w(g) = (-1)^{e(g)} \int_{\mathbb{R}^{n-1}} \prod_{(i,j) \in E_g} \chi(|x_i - x_j| < 1) \, dx_2 \cdots dx_n$$  \hspace{1cm} (4.27)

where $x_1 = 0$ and $\chi$ is the indicator function.

In [8], this is interpreted as the volume of a convex polytope $\Pi_g$ in $\mathbb{R}^{n-1}$. The polytope is defined by:

$$\Pi_g = \{(x)_{[2,n]} \in \mathbb{R}^{n-1} ||x_i - x_j| < 1 \forall (i,j) \in E_g \mid x_1 = 0\}$$

Where the notation $[2,n] = \{2, 3, \ldots, n\}$ and $(x)_{[2,n]} = (x_2, \ldots, x_n)$.

Hence the graph weight may be written as:

$$w(g) = (-1)^{e(g)} \text{Vol}(\Pi_g)$$  \hspace{1cm} (4.28)

The derivation of the cluster and virial expansions, using statistical mechanics, are more difficult in this case, but they are done in [8] and we achieve:

$$\beta P = W(z) = \sum_{n=1}^{\infty} \frac{(-n)^{n-1} z^n}{n!}$$  \hspace{1cm} (4.29)

$$\beta P = \frac{\rho}{1 - \rho} = \sum_{n=1}^{\infty} \rho^n$$  \hspace{1cm} (4.30)

where $W(z)$ is the Lambert $W$-function.

If we compare these to the results of Mayer’s First and Second Theorems (4.11) and (4.13), we obtain the combinatorial relationships:

$$\sum_{g \in C[n]} (-1)^{e(g)} \text{Vol}(\Pi_g) = (-1)^{n-1} n^{n-1}$$  \hspace{1cm} (4.31)

$$\sum_{g \in B[n]} (-1)^{e(g)} \text{Vol}(\Pi_g) = -n(n-2)!$$  \hspace{1cm} (4.32)

5. Results

The main results of this article are the interpretations combinatorially of how the cancellations for the virial expansion coefficients work in two particular cases.

**Theorem 5.1** (Combinatorial Identity from the one-particle hard-core model). We have the combinatorial identity:

$$\sum_{g \in B[n]} (-1)^{e(g)} = -(n-2)!$$  \hspace{1cm} (5.1)

This is proved through an involution $\Psi$, which effectively pairs graphs differing by only one edge, leaving some small collection of graphs fixed, which give the $(n-2)!$ factor. The fixed graphs are described inductively:

- take the base graph as the single edge between two points with labels 1 and 2.
- Given a fixed graph on $n$ vertices, we shift its labels by $1 (i \mapsto i + 1)$ and add a vertex with label 1. We add from this new vertex an edge to the vertex now labelled 2 and another edge to any of the $n - 1$ remaining vertices.
These fixed graphs all have $2n - 3$ edges, giving the $-1$ factor and if $a_n$ denotes the number of them, we have the recursion: $a_n = (n - 2)a_{n-1}$, which gives the $(n - 2)!$, taking $a_2 = 1$.

The fixed graphs for $n = 3, 4$ are shown in Figure 2.

**Theorem 5.2** (Combinatorial Identity from the continuum hardcore gas). We have the combinatorial identity:

$$\sum_{g \in B[n]} (-1)^{\epsilon(g)} \text{Vol}(\Pi_g) = -n(n - 2)! \quad (5.2)$$

This is proved through a collection of involutions $(\Psi_h)_{h \in \mathbb{Z}^{n-1}}$. The index $h$ is related to the partition of the polytopes into areas of equal volume attributed to Lass in [4, 8]. The meaning of $h$ is explained in subsection 3. The fixed points of these involutions occur only when $h$ is of the form $(0, \cdots, 0, -1, \cdots, -1)$, meaning that any edge is possible. There are precisely $n$ possibilities of these sequences, which corresponds to the $n$ positions of the last zero. The particular $h$ provides a bijection $\sigma : [n] \to [n]$ on which the fixed graphs correspond to an increasing tree (given by the order $\sigma(i) < \sigma(j)$ if and only if $i < j$) on the labels $\{\sigma(1), \cdots, \sigma(n - 1)\}$. This is paired with every edge from $\sigma(n)$ to the vertices $\{\sigma(1), \cdots, \sigma(n - 1)\}$.

The number of these increasing trees on $n - 1$ vertices is $(n - 2)!$ and hence we obtain the factor $n(n - 2)!$. We notice that these graphs are on $n - 2 + n - 1 = 2n - 3$ edges, which provides the minus sign. An example of a fixed graph is given in Figure 3.

**Figure 2.** The Fixed Points for $\Psi$ for $n = 3, 4$
Remark 1 (Connection to Bernardi’s Interpretation). It is worth noting that the increasing tree idea present in [4] also appears as an important idea in the fixed points of this involution.

Remark 2 (Alternative Version of Theorem 5.1). The understanding of the fixed points of the involution $\Psi_0$ provides us with an alternative method of proving Theorem 5.1. We can see that the fixed points may also correspond to the collection of increasing trees on $[n-1]$ with vertex $n$ having an edge to all other vertices.

Remark 3 (Complications for two-connected graphs). The two-connected case is necessarily more complicated than the connected case. First of all, minimal two-connected graphs do not all have the same number of edges for a fixed number of vertices as trees (minimal connected graphs) do. Simply removing edges appropriately down to a minimal graph cannot provide a combinatorial understanding as there will still be sign differences to take care of. Furthermore, the sign of the factor is constant - the number of edges must always be odd for whatever value of $n$ we take.

Remark 4 (Why to expect $2n - 3$ as an important number of edges). It can be shown that any two-connected graph on $n$ vertices with greater than or equal to $2n - 3$ edges necessarily has a chord. The chord can be removed, keeping the graph two-connected, which indicates that a graph with this number of edges cannot be minimally two-connected. It is also possible to construct a graph with $n$ vertices and $2n - 4$ edges that is minimally two-connected, as shown in Figure 4. The number of edges being $2n - 3$ marks some transition in the possibility of being minimal.

Proof. Any two-connected graph on $n$ vertices with greater than $2n - 3$ edges necessarily has a chord.

This is done by induction on the number of vertices $n$.

The cases $n = 2, 3$ are vacuous and one can see from the examples in Figure 5 that this holds when $n = 4$. 

Figure 3. Example of Fixed Graph for $\Psi_h$ on 7 vertices with the bijection $\sigma$
Let $d_1$ be the degree of the vertex labelled 1. If we remove the vertex 1 and incident edges, then we have a connected graph, which may be decomposed into its bc-tree. Each block with $l$ vertices in the tree has to have $\leq 2l - 4$ edges or else we have a smaller graph which has a chord by induction. We note here that blocks of size 2 or 3 need to be treated separately. We let $l_i$ denote the size of the $i$th block not of size 2 or 3 and $b_2$ and $b_3$ denote the number of blocks of size 2 and 3 respectively. We know the relationship:

$$\sum_i (l_i - 1) + b_2 + 2b_3 = n - 2 \quad (5.3)$$

The total number of edges in the graph must then not exceed:

$$\sum_i 2(l_i - 1) - 2b_{\geq 4} + b_2 + 3b_3 \leq 2n - 4 - b_2 - b_3 - 2b_{\geq 4} \quad (5.4)$$

where $b_{\geq 4}$ indicates the number of blocks with more than four vertices. We know that the number of edges must be greater than $2n - 3 - d_1$ and so we obtain the inequality:

$$d_1 \geq 1 + b_2 + b_3 + 2b_{\geq 4} \geq 1 + \text{total number of blocks} \quad (5.5)$$

By the pigeonhole principle, 1 must meet a block at two vertices, call these $\alpha$ and $\beta$. If we have two blocks then we have a third neighbour of 1, call this $\gamma$.

If all three are in the same block then we can find a path $\alpha \to \beta \to \gamma$ and thus the edge $(1, \beta)$ is a chord. Otherwise $\gamma$ is in another block. Let $A$ be the articulation point of the block containing $\alpha$ and $\beta$ closest to $\gamma$. We have a path from $A$ to $\gamma$ outside of this block since it is a connected graph. We are also able to construct a path $\alpha \to \beta \to A$ since they are all in one block. Concatenating these paths gives again a path $\alpha \to \beta \to \gamma$ from which we determine $(1, \beta)$ is a chord.

If we have only one block and two neighbours of one, we may use induction, since the graph without 1 is therefore two-connected and satisfies the same inequality, since we have taken away two edges.

$\square$
The key idea, which is present in the paper by Bernardi [4], is that a graph involution involving removing/adding edges means that we may rewrite the weighted sum over two-connected graphs as:

\[
\sum_{g \in \mathcal{B}[n]} (-1)^{c(g)} = \sum_{g \in \mathcal{B}[n]} (-1)^{c(\Psi(g))}
\]  

(6.1)
since $\Psi$ is a bijection. The sum of these is therefore:

$$2 \sum_{g \in \mathcal{B}[n]} (-1)^{c(g)} = \sum_{g \in \mathcal{B}[n]} ((-1)^{c(g)} + (-1)^{\Psi(g)})$$

$$= 2 \sum_{g \in \mathcal{B}[n]|\Psi(g) = g} (-1)^{c(g)} \quad (6.2)$$

Hence we are left with just enumerating the fixed points of the involution.

We first define the lexicographic order on edges $e \in [n]^2$ by:

$$(i, j) < (k, l) \text{ if } \begin{cases} \min\{i, j\} < \min\{k, l\} \\ \text{or } \min\{i, j\} = \min\{k, l\} \text{ and } \max\{i, j\} < \max\{k, l\} \end{cases}$$

The involution $\Psi : \mathcal{B} \to \mathcal{B}$ is described as follows:

[i] For a graph on $n$ vertices and strictly more than $2n - 3$ edges

Consider each edge $e$ (not necessarily in the graph) in lexicographic order. Check whether the endpoints of the edge $e$ are contained in a cycle $C$ within the graph where each edge $f \in C$ satisfies $f > e$. Take the maximal edge $e$ for which this is true and:

- add it if it is not in the graph
- remove it if it is in the graph

[ii] For a graph on $n$ vertices and strictly less than $2n - 3$ edges

We consider first whether there are any edges satisfying the condition required for > $2n - 3$ edges. If there are, we add/remove the maximal such chord. If we do not have such a chord, then we consider each edge $e$ (not necessarily in the graph) in lexicographic order

[a] If $e \in G$

Check whether there is a cycle in the graph containing the endpoints of $e$ but not the edge $e$. If no such cycle exists, do nothing, otherwise check whether, upon removal of this edge, it is possible to add a smaller edge $f < e$, which does not create a cycle containing the endpoints of a smaller edge. If it is possible, leave the edge, otherwise remove it.

[b] If $e \notin G$

Check whether there is a cycle in the graph containing the endpoints of $e$ but not the edge $e$. If no such cycle exists, do nothing, otherwise if adding this edge does not create a cycle containing the endpoints of a smaller edge $f < e$, but not the edge $f$ itself (where $f \in G$), then add it, otherwise do nothing.

Note that we stop this process as soon as we reach an edge with which we can do something.

[iii] For a graph with $n$ vertices and $2n - 3$ edges:

- For each present edge, treat it like a graph on strictly less than $2n - 3$ edges
- For each absent edge, treat it like a graph on strictly greater than $2n - 3$ edges
- Take the minimal choice of which edge to add/remove.

In order to prove that this mapping $\Psi$ does what we require, we break things down into separate lemmas:
• Firstly we show that $\Psi$ is a mapping $\mathcal{B} \rightarrow \mathcal{B}$
• We show $\Psi$ is indeed an involution and thus a bijection
• We indicate no graph on $n$ vertices with strictly greater than $2n - 3$ edges can be fixed
• We indicate no graph on $n$ vertices with strictly less than $2n - 3$ edges can be fixed
• We find the subset of graphs on $n$ vertices with $2n - 3$ edges which are fixed

Lemma 6.1. The image of the mapping $\Psi : \mathcal{B} \rightarrow \mathcal{G}$ is contained within $\mathcal{B}$.

Proof. The mapping $\Psi$ can add an edge, remove an edge or leave the graph fixed.
If the graph is left fixed, then trivially we get a two-connected graph.
If an edge is added to a two-connected graph, then it remains two-connected.
If an edge $e = (i, j)$ is removed from the graph $G$, the definition of the mapping $\Psi$ ensures that there is a cycle in $G \setminus (i, j)$ containing $i$ and $j$.
$G \setminus (i, j)$ has a cycle including $i$ and $j$ and so they must be in the same block.
If we have another block in the graph, then adding the edge $(i, j)$ back in cannot reduce the number of blocks, since it is internal to a block and this would imply $G$ is not two-connected. Therefore, $G \setminus (i, j)$ has to be two-connected.

Lemma 6.2. The mapping $\Psi$ is an involution and thus bijective.

Proof. If we remove an edge, we retain the cycle for the edge and do not create any more cycles. In particular, $\Psi(G)$ will have a subset of the important cycles for either case, but will include the crucial cycle to determine the point to add/remove.
If we add an edge and:
• $|E| \geq 2n - 3$
  We cannot create a cycle corresponding to a larger edge, since we are adding an edge which cannot be involved in a larger cycle.
• $|E| < 2n - 3$
  We have the restriction that we do not allow edges that create smaller cycles.
The considered edge is therefore paired - if we add it to $G$, then we remove it from $\Psi(G)$ and similarly if we remove it from $G$, we add it to $\Psi(G)$.

Lemma 6.3. No graph on $n$ vertices and strictly greater than $2n - 3$ edges is fixed.

Proof. It is sufficient to prove that any two-connected graph with $\geq 2n - 2$ edges contains a cycle where the two minimal vertices are not neighbours, since then there is a possible edge to add or remove and we can find the maximum of the non-empty set of these.
This is proved by induction on the number of vertices $n$, for $n \geq 4$.
For $n = 4$, the only such graph we have is the complete graph, which certainly has such a cycle as illustrated in Figure 6.
For general $n$, by induction, assume it is true for all $l$ such that $4 \leq l < n$.
Removing the vertex labelled 1 from such a graph, we obtain a connected graph with a corresponding bc-tree. Each of the blocks in this subgraph has (strictly)
less than \( n \) vertices. If this graph were to not have such a cycle then either: it is the single edge graph, if the block has two vertices; the triangle graph, if the block has three vertices; otherwise, applying the induction hypothesis, we require \( \leq 2k_i - 3 \) edges for a block of size \( k_i \geq 4 \), since otherwise we would have such a cycle in a subgraph. Let \( b_2 \) = number of blocks of size two (with one edge) and \( b_3 \) = number of blocks of size three (with three edges).

Counting the vertices in this subgraph, we have:

\[
\sum_{\text{blocks of size } \geq 4} (k_i - 1) + 1 = n - 1 - b_2 - 2b_3 \tag{6.3}
\]

as described in Section 2.

If we now count the edges in this subgraph, we have:

\[
e(G) \leq \sum_{\text{blocks of size } \geq 4} (2k_i - 3) + b_2 + 3b_3 \\
= 2 \sum_{\text{blocks of size } \geq 4} (k_i - 1) - b_{\geq 4} + b_2 + b_3 \\
= 2(n - 2 - b_2 - 2b_3) - b_{\geq 4} + b_2 + 3b_3 \\
= 2(n - 2) - \text{number of blocks} \tag{6.4}
\]

We may also count the total number of edges in this subgraph as \( \geq 2n - 2 - d_1 \), where \( d_1 \) is the degree of vertex 1. We thus achieve the inequality:

\[
d_1 \geq \text{number of blocks} + 2 \tag{6.5}
\]

We know that the number of blocks \( \geq 1 \) and so \( d_1 \geq 3 \). By pigeonhole principle, one block has at least two edges from 1. Furthermore, either two blocks have two edges or one block has three.

If a block has three edges from 1, then let \( \alpha \) be the label of the minimal such vertex. We prove now that we have a cycle containing 1 and \( \alpha \) but not the edge \((1, \alpha)\). We note that it could also be possible to have another vertex in the path constructed below, which is smaller than \( \alpha \) in this case it would be smaller than the other two neighbours of 1 used in this block and hence, we could add the edge from 1 to this point.

Let \( \alpha < \beta < \gamma \) be the labels of these three vertices. We indicate the existence of a path \( \beta \rightarrow \alpha \rightarrow \gamma \) inside this block.

Since we are in a two-connected graph, we have two disjoint paths between any pair of vertices. We may follow a path \( \beta \rightarrow \alpha \) until it first coincides with any of the two disjoint paths \( \alpha \rightarrow \gamma \) we then follow this first path to \( \alpha \) and then the disjoint path from \( \alpha \rightarrow \gamma \). This gives a path (with no repeated edges). We may add the two edges \((1, \beta)\) and \((1, \gamma)\) to this to obtain the required cycle.

If we have two neighbours of 1 in block \( B \) and another two in a disjoint block \( C \), we let \( \alpha \) be the smallest label of these four and \( \beta \) its partner and \( \gamma \) and \( \delta \) the labels of the other pair. In the bc-tree there is a single path from \( B \) to \( C \). Let \( A_B \) and \( A_C \) be the articulation points adjacent to \( B \) and \( C \) respectively in the path. By above, we can construct paths: \( \beta \rightarrow \alpha \rightarrow A_B \) inside \( B \); \( A_B \rightarrow A_C \) outside of the blocks; and \( A_C \rightarrow \gamma \) in block \( C \). We can therefore add the edges \((1, \beta)\) and \((1, \gamma)\) and we have the required cycle.
Lemma 6.4. No graph with \( n \) vertices and strictly less than \( 2n - 3 \) edges is fixed.

Proof. We prove this by induction on \( n \). We initiate at \( n = 4 \) and we have only cycles. All these cycles have an edge to add as indicated by the dashed line in Figure 7. Consider a given two-connected graph on \( n \) vertices and \( \leq 2n - 4 \) edges. If we remove 1 and its incident edges, then we are left with a connected graph on the rest of the labels.

We consider again the block cutpoint decomposition of this graph. By Lemma 6.3, none of the blocks with \( k_i \) vertices have \( \geq 2k_i - 3 \) edges, or else we would have a cycle with a chord that can be added or removed. Furthermore, each block of size \( k_i \) must have \( 2k_i - 3 \) edges or else, by induction, we will have a subgraph where the condition is satisfied, indicating that it will be satisfied by the whole graph.

The key point why having a block that can have an edge added or removed in this case is that if we wish to add an edge to an individual block, it can only create a new chord within the block and by the rules applied to the block it wouldn’t cause an issue. If we wanted to remove an edge in a block, by the rules applied to the block, we have no smaller edge inside the block, so it would be outside the block and be able to be removed anyway.

If we count edges, first by considering that we remove 1:

\[
e(G) \leq 2n - 4 - d_1
\]

Then consider counting the edges by the blocks:

\[
e(G) = \sum_{i \in I} (2k_i - 3) = 2(n - 2) - \text{number of blocks}
\]

We then obtain \( d_1 \leq \text{number of blocks} \).

As a simple inequality, we know that the number of blocks in a graph is always bounded above by \( n - 1 \), where \( n \) is the number of vertices in the graph. Since \( n - 2 \geq \text{number of blocks} \geq d_1 \), we therefore have a vertex to which 1 is not connected to.

If we have a vertex not in the neighbourhood of 1, which is in a block which is not a leaf block or it is an articulation point between two leaf blocks in the bc-tree, then we can add the smallest such possibility. This will not create any extra cycles.

Figure 6. The Cycle Found in the Graph \( K_4 \)
involving 1 and another vertex connected to 1, since we could just add the path from the leaf block further away from the vertex in question to this vertex instead of the direct link from 1 to this vertex, thus meaning it was already removable. Hence in this case there is an edge to add or remove.

If all the vertices contained in every block that is not a leaf are neighbours of 1, there is a vertex in a leaf block that is not a neighbour of 1. We may remove the smallest edge \((1, j)\) in one of the internal blocks unless we have a smaller vertex that is not neighbour of 1. This must appear in a leaf block. In this leaf block we have a neighbour of 1, or else the graph with 1 would not be two-connected.

If we have a vertex with smaller label than this neighbour of 1, \(\nu\), in the leaf vertex, call it \(\zeta\), then the edge \((1, \zeta)\) may be added, since we may find a cycle \(1 \rightarrow \nu \rightarrow \zeta \rightarrow \eta \rightarrow 1\), where \(\eta\) is the neighbour of 1 in the closest block to \(\zeta\) and is necessarily larger than \(\zeta\) or else we would not have such an example.

If all vertices in leaf blocks are larger than the corresponding neighbour of 1 in the block, then none of these edges could be added and removing the smallest edge is fine.

\[\square\]

**Figure 7.** The Four Examples on \(k = 4\) Vertices with Additional Edge

**Lemma 6.5.** The fixed points of \(\Psi\) on \(n\) vertices with \(2n - 3\) edges are as described in Theorem 5.1.

**Proof.** [i] The bc-decomposition - \(d_1 = 1 + \text{number of blocks}\)

First of all we consider the graph without the vertex 1 and its bc-tree decomposition.

We know from Lemma 6.3 that it is not possible to have a block with \(k_i\) vertices and \(\geq 2k_i - 2\) edges, or else we have a cycle with a chord that can be added or removed by either rule.

This therefore implies that the number of edges counted by blocks is:

\[e(G) \leq \sum_{i \in I} (2k_i - 3) = 2(n - 2) - \text{number of blocks} \quad (6.6)\]
The number of edges is: $e(G) = 2n - 3 - d_1$ and so $d_1 \geq 1 + \text{number of blocks.}$

If we have $d_1 \geq 2 + \text{number of blocks}$, then we either have a block with three neighbours of 1 or two blocks with two neighbours of 1 and we can use the same construction as in Lemma 6.3 and can therefore add or remove an edge according to the rules in this case, since we give priority in both cases to some edge which is a chord in a cycle made of larger edges.

This leaves us with $d_1 = 1 + \text{number of blocks}$ and since if we have two blocks with two neighbours of one or one block with three neighbours of 1, we precisely have the case described in Lemma 6.3 and can remove an edge.

Therefore, for a graph fixed under $\Psi$, we have that precisely one block has two neighbours of 1 and the rest have precisely one. We know that 2 must also be a neighbour of 1 or else it may be added. We also have to exclude the possibility that 1 and 2 can appear in a cycle without the edge $(1, 2)$. This requires that 2 appears in a leaf block in the bc-tree and it is the only neighbour of 1 in this block.

[ii] To show we must only have one block

We suppose for contradiction that $d_1 \geq 3$, which is equivalent to assuming we have at least two blocks. We have one block containing two neighbours of 1 and a neighbouring block containing one. We call these vertices $\alpha \beta$ and $\gamma$, and the corresponding articulation point $A$. From the proof of Lemma 6.3 we can construct in the first block the path $\alpha \rightarrow \beta \rightarrow A$ and then in the second block $A \rightarrow \gamma$, thus giving that $\beta$ is a chord. We therefore have at least one present edge $(1, l) \in G$ with endpoints in a cycle, not including the edge. We call the smallest such $l$, $j$.

Either, this edge may be removed, in which case it is not a fixed graph, or it cannot. Supposing that it cannot, then there is some $i < j$ such that $(1, i)$ can be added to $G \setminus \{(1, j)\}$.

Firstly, if $(1, i)$ is in a cycle excluding smaller edges, then it may be added and this graph is not fixed.

We know that the only edges involving 1 which therefore may not be removed as a chord are those found on their own in leaf blocks. We note that $i$ is necessarily smaller than any other neighbour of 1 and that if two leaf blocks are adjacent then that is the whole graph and we would have two neighbours of 1 in one of these blocks, so we may exclude such a possibility, therefore $(1, i)$ has to have all its cycles including at least one of these leaf-block vertices and one such vertex $\mu$ has to appear in all of them. This means that there is an articulation point, in the graph with 1 removed, between $i$ and the neighbours of 1, excluding $\mu$. Hence, $i$ must be in the same block as $\mu$.

If $\mu < i$, then, as before, we can create a path $i \rightarrow \mu \rightarrow A \rightarrow$ the closest vertex attached to 1. This would mean that we cannot add $1 - i$ in the first case and so removing $1 - j$ is fine.

If $\mu > i$, then, since we are assuming no two leaf blocks are adjacent, we have an adjacent block, via articulation point $A$, with a vertex $l > i$, which is a neighbour of 1. The cycle $1 \rightarrow \mu \rightarrow i \rightarrow A \rightarrow l \rightarrow 1$ is thus made up of larger edges than $(1, i)$ and it could be added in the first place.

Hence a fixed graph cannot have more than one block in the bc-tree without 1 and 1 must have degree two.
[iii] The final induction step
Now that we know 1 has degree two, we may build up the collection of fixed graphs inductively. If we remove 1, then we have a two-connected graph on $2n - 3 = 2(n - 1) - 3$ edges and $n - 1$ vertices and we have the fixed points by induction. We note that 1 has to connect to 2 or else we could always add the edge $(1, 2)$ and so we have a choice from $(n - 2)$ points to also attach to 1. This gives the formula $(n - 1)!$, since for $n = 3$ the fixed point is just the 3-cycle.

(iv) All the graphs of this form are fixed
We must now indicate that all graphs of the prescribed form are fixed. We know that such a graph has no chord $e$ that is within a cycle containing edges $f > e$. Any cycle involving the vertices $a$ and $b$ with $a < b$ must contain either an edge $(\alpha, \beta)$ with $\min\{\alpha, \beta\} < a$ or $(a, a+1)$, since the edges from $a$ either go to a smaller vertex, to $a + 1$ or to precisely one other larger vertex. Therefore we always have a smaller edge, since if $b > a + 1$ then either of the two cases will do. If $b = a + 1$, then we couldn’t use the edge $(a, a + 1)$ if it were to be a chord and we would have a smaller edge.

We have a chord to remove, but for such a chord there is always a smaller chord to add if we remove it. Say $(\alpha, \beta)$ is a chord we are wanting to remove with $\alpha < \beta$. $\alpha$ is only a neighbour of two vertices with a larger label. The cycle involving $\alpha$ and $\beta$ but not the edge $(\alpha, \beta)$, must therefore contain a vertex smaller than $\alpha$ as a neighbour of $\alpha$. Call this vertex $\gamma$. This vertex can have $\beta$ as its other neighbour, in which case, we have a path $\beta \rightarrow \alpha$ in this cycle involving a distinct vertex $\delta$, which is not a neighbour of $\gamma$ and so $(\gamma, \delta) < (\alpha, \beta)$ and is a chord in a cycle and may be added. If $\gamma$ and $\beta$ are not neighbours, then they are in a cycle with each other and the chord $(\gamma, \beta) < (\alpha, \beta)$ and so could have been added. This means that we are unable to remove the chord $(\alpha, \beta)$ as we would have the opportunity to add a smaller chord and hence this graph is fixed.

□

7. The Tonks Gas - Proof of Theorem 5.2

We define an involution $\Psi_h$ for each $h \in \mathbb{Z}^{n-1}$ on the set of two connected graphs, which are compatible with the vector $h$, $B_h$. In order to do this we must define some order on the edges, which emphasises the compatibility with $h$. The order depends on $h$ in the sense that we first order edges by the value of $|\bar{h}_i - \bar{h}_j|$, which we call the edge length, and then order within these subsets by the lexicographic order on pairs $(\bar{h}_i, \bar{h}_j)$. The important thing is that for a graph on $n$ vertices, we can understand that this relates to a bijection $\sigma : [n] \rightarrow [n]$ in which $\sigma(i)$ indicates the label of the $i$th smallest entry in $h$. We note also that edges with $|\bar{h}_i - \bar{h}_j| > 1$ are forbidden in this set up. The terms larger and smaller used in this section refer to this order on the edges.

Now that the order has been defined, we must indicate what the involution does. This is analogous to the previous case and the fact it is an involution on two-connected graphs follows in the same way. Given a two-connected graph on $n$ vertices, we perform the involution according to the three cases below:

[i] For strictly greater than $2n - 3$ edges
We add/remove an edge $e$ if and only if its endpoints are contained in a cycle involving edges $f > e$. From all possibilities of edges to add/remove, we add/remove the maximum.

**[ii] For strictly less than $2n - 3$ edges**

We give first priority to adding/removing the maximal chord $e$, which is contained in a cycle of edges $f > e$. If this is not possible, then:

Considering each edge in turn in the order given above:

[a] If $e \in G$

If $e$ is a chord in the graph, then we may remove it, providing it doesn’t allow a smaller edge to be added.

[b] If $e \notin G$

We may add this edge, providing that it does allow for a smaller edge to be removed.

**[iii] For $2n - 3$ edges**

For present edges, we follow the removal procedure as for strictly less than $2n - 3$ and for absent edges, we follow the adding procedure of strictly greater than $2n - 3$ edges.

In order to prove this result we are required to prove:

- The image of $B_h$ under $\Psi_h$ is contained in $B_h$.
- $\Psi_h$ is an involution
- No graphs on $n$ vertices with strictly greater than $2n - 3$ edges are fixed
- No graphs on $n$ vertices with strictly less than $2n - 3$ edges are fixed
- Those graphs on $n$ vertices and $2n - 3$ edges that are fixed are of the prescribed type

The first two requirements follow straightforwardly as in the previous case. The first is just a restriction, noting that no forbidden edges are ever added when using $\Psi_h$. The rules are the same as in the previous case, but with different orderings and so it is an involution.

**Lemma 7.1.** No graph on $n$ vertices with strictly greater than $2n - 3$ edges is fixed.

*Proof.* We proceed by induction on $n$. We indicate that the collection of possible edges to add/remove is non-empty, thereby indicating that we will have a maximum and thus add/remove some edge.

Consider the minimal edge in the above ordering. Either this edge is absent, in which case it may be added, since the endpoints will be in a cycle, and we are done, or else it is present. If it is present and its endpoints are in a cycle without the edge (i.e., it is a chord), then it may be removed and we are done. Therefore, we assume that the minimal edge is present and not a chord. If we remove this edge, we have a connected graph with at least two blocks, since if the two endpoints were in the same block, then we would have a cycle including them both. Each block of size $l$ must have $\leq 2l - 3$ edges, or else we can use induction and find a required edge. The total number of edges in this graph is $\geq 2n - 3$ and if we count
these block by block, we achieve:

\[ 2n - 3 \leq \sum_{i \in I} (2k_i - 3) \]

\[ = 2 \sum_{i \in I} (k_i - 1) - \text{number of blocks} \]

\[ = 2n - 2 - \text{number of blocks} \quad (7.1) \]

Therefore, \( 1 \geq \text{number of blocks} \), which gives us a contradiction. Hence we have an edge to add/remove.

\[ \square \]

Lemma 7.2. The only possible fixed graphs under the above involution are those on \( 2n - 3 \) edges where \( \sigma(n) \) is attached to all other vertices and on the subset of vertices \( \{ \sigma(1), \cdots, \sigma(n - 1) \} \) we have an increasing tree, where the vertex order is defined by \( \sigma(i) < \sigma(j) \) if and only if \( i < j \). We call the class of graphs \( B_T \)-graphs. The proof also covers the unproved case that no graph on \( n \) vertices with \( < 2n - 3 \) edges can be fixed by \( \Psi \).

Proof. We proceed by induction. For the induction assumption, we have that the only graphs fixed under \( \Psi \) are those of the form \( B_T \). For any \( 4 \leq k < n \), the initiation at 4 can be seen in the appendix, Section A.

[i] The case when \( e(G) = 2n - 3 \)

If we consider the minimal edge of a graph \( G \) on \( 2n - 3 \) vertices and call it \((\sigma(i), \sigma(j))\). We know that if the edge is not present in the graph then we may add it. If this edge is a chord in a cycle, then it can be removed. We thus assume that the edge is not a chord in any cycle and so that means that the graph without this edge, \( H = G \setminus \{ (\sigma(i), \sigma(j)) \} \), cannot be two-connected. It must be connected as neither \( i \) nor \( j \) can be articulation points. Hence we are left with a connected graph with at least two blocks.

Each of the blocks, formed by removing this edge, must have \( 2l - 3 \) edges, where \( l \) is the number of vertices in the block. If it did not, then we could add/remove edges from the subgraphs by following the process for blocks with number of edges not equal to \( 2l - 3 \). This is by the induction assumption.

Let us have blocks of sizes \( (k_i)_{i \in I} \), where \( I \) is an index set for the blocks, then we know that \( \sum_{i \in I} (k_i - 1) = n - 1 \) and we may add the edges in this graph \( H \) in two ways: firstly, by the edges in the blocks:

\[ \sum_{i \in I} (2k_i - 3) = 2 \sum_{i \in I} (k_i - 1) - |I| = 2n - 2 - \text{number of blocks} \quad (7.2) \]

and secondly, from knowing we have just removed an edge from a graph with \( 2n - 3 \) edges, we must have \( 2n - 4 \) edges. This gives that the number of blocks is 2.

These two smaller blocks by induction must be of the form given by \( B_T \), otherwise we can add or remove edges inside the blocks. We want to consider the single articulation point in this connected graph. We wish to first prove that this articulation point is necessarily \( \sigma(n) \).

If we suppose for contradiction that the articulation point in the graph is some \( a \neq \sigma(n) \). We let \( \max_1 \) and \( \max_2 \) be the maximum vertices in the two blocks. We then know by induction that each of these must be connected to all vertices
in their respective blocks. At least in one of the two blocks \( a \neq \max_1 \). Consider
the minimal edge adjacent to \( a \), which is necessarily in one of the two blocks. If
the minimal edge \((a, \mu_1)\) is in the first block and \( a \neq \max_1 \), then we may consider
the cycle: \( \max_2 \to a \to \max_1 \to \mu_1 \to \sigma(i) \to \sigma(j) \to \max_2 \). \( \to \) represents a
direct edge and \( \to \) indicates following a path in the increasing tree in the block.
If \( a = \max_1 \), then we do not need the edge \( a \to \max_1 \) and can replace it with just
a single vertex.

Thus we have an edge that may be removed or indeed an even smaller edge
that can be removed. If we remove this edge, we need to indicate that no smaller
edge can be added. We know we cannot add an edge between the two blocks as
this would make \((\sigma(i), \sigma(j))\) removable. A smaller edge then has to involve two
vertices in a single block.

Hence, we have a removable edge and so the graph is not fixed. The articula-
tion point must therefore be \( \sigma(n) \). We therefore require that \( \sigma(n) \) is attached to
everything. This implies that we cannot have any impossible edges in our graph.
We note that requiring the graph to not have any impossible edges is equivalent
to saying that \( h \) has to be of the form \((0, \ldots, 0, -1, \ldots, -1)\) and so the smallest
edge is necessarily \((\sigma(1), \sigma(2))\), the first two points to be \(-1\), or if only the final
entry is \(-1\) it would be \(n\) and \(1\) respectively. If all entries are \(0\), then \( \sigma \) is the
identity. We have that \( \sigma(n) \) has all other vertices in its neighbourhood and that
there are increasing trees from \( \sigma(1) \) and \( \sigma(2) \). If we just add an edge between \( \sigma(1) \)
and \( \sigma(2) \) then we still have an increasing tree on the whole graph and hence the
fixed graph has to be of this form.

[ii] The case \( e(g) < 2n - 3 \)
We consider removing the minimal edge \((\sigma(i), \sigma(j))\) of the graph \( G \) and under-
stand the block-cutpoint decomposition of this graph, which must have at least
two blocks. The blocks all have less vertices than the whole graph and so must
satisfy the induction assumption - that is be of form \( B_T \) and in particular for a
block with \( l \) vertices have \( 2l - 3 \) edges. If we count the number of edges through
the blocks and through the inequality, we obtain:

\[
e(g) = \sum_{i \in I} (2k_i - 3) = 2(n - 1) - \text{number of blocks} \\
\leq 2n - 5
\]

In particular, we have that there are at least three blocks. We therefore know
that there are a pair of blocks who do not have the articulation point as \( \sigma(n) \).
In this case, we may repeat the proof as above, using that we have a path from
the articulation points that are not in common between the two blocks to \( \sigma(i) \)
and \( \sigma(j) \) instead of the immediate connection \((\sigma(i), \sigma(j))\). The only difference is
whether we may add a smaller edge between two blocks. We cannot add such an
edge as the only cycle it can be a chord in involves this minimal edge that we
removed. Therefore, because priority is given to edges \( e \) appearing as chords in
cycles with edges \( f > e \), then the edge described would be able to be added or at
least the maximal such possibility in the block.

[iii] Graphs of the form \( B_T \) are fixed
If a graph is of the form described by $B_T$, then we need to indicate that it is fixed. The only possible edges to add are between non-adjacent vertices in the increasing tree. If we try to add such an edge, then the edges in the increasing tree, which are smaller and are required to form the cycle, mean that it cannot be a chord in a cycle with larger edges. The only edges that may be removed are between $\sigma(n)$ and another vertex. We note that $|\tilde{h}_{\sigma(i)} - \tilde{h}_{\sigma(j)}| = \frac{|i - j|}{n}$ and so the edges are ordered by the differences of the vertex labels. We note that $\sigma(n - 1)$ must be a leaf in this tree and so the edge $(\sigma(n), \sigma(n - 1))$ cannot be removed in any case. Therefore, the length of the removable edge is at least $\frac{2}{n}$. We need to indicate that there is a missing edge with a smaller length. If two consecutive labels are not neighbours in the tree then we are done, since they have edge length of $\frac{1}{n}$, otherwise the increasing tree is linear and we can add $(\sigma(1), \sigma(3))$ if we remove any edge from $\sigma(n)$, since it appears earlier in the lexicographic order.

\[ \square \]

8. Outlook and Conclusions

The cancellations in two simple models of the virial expansion can be understood combinatorially by understanding the involutions given in this paper. It would be helpful to understand how the particular graphs, which are fixed points, can help to give an understanding of better bounds for the virial expansion. The parallel that is useful to draw here is that for the cluster expansion, we have the increasing and Cayley trees as the combinatorial objects representing the two cases above. It has been shown by Groeneveld [12] that these examples provide the extreme cases for positive potentials and an adaptation is available for stable potentials.

Much work has been done on the expansions or graph-tree identities in this case. The use of the notion of externally active edges in the paper by Bernardi [4] is connected to the use of matroids in the q-state Potts Model [22]. Edges in graphs can be viewed as externally active and these are the edges that are able to be identified with particular factors. They are able to be bounded in both the stable and positive potential cases and thus become only some extra factor to consider. The combinatorial part of the argument then relies on counting trees. This has already been understood as the Penrose partition [20]. Is it then possible that we can use internally and externally active edges to find a similar conclusion for positive potentials with these increasing trees and the special vertex adjacent to everything? This should certainly be the next stage of investigation and the the importance of a combinatorial understanding of these coefficients. Generalising this sort of structure to the poset of graphs with a total order for the edges should be able to provide the connection to use these identities.

Appendix A. The Examples for $n = 4$

Below are given the tables explaining which graphs are fixed for $n = 4$ for different compatible $h$-values, which are listed beside the graphs. The tables indicate what edges are added/removed according to the rules in Section [7] and the letters indicate the pairings of the $(G, h)$ pairs through the involution $\Psi_h$. 
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**DEPARTMENT OF MATHEMATICS, UNIVERSITY OF WARWICK, COVENTRY, CV4 7AL, UNITED KINGDOM**

_E-mail address: s.j.tate@warwick.ac.uk_