DIRAC’S CONTOUR REPRESENTATION FOR PARAPARTICLES

Sicong Jing\textsuperscript{1} and Charles A. Nelson\textsuperscript{2}

\textit{Department of Physics, State University of New York at Binghamton}

\textit{Binghamton, N.Y. 13902-6016}

\textbf{Abstract}

Dirac’s contour representation is extended to parabose and parafermi systems by use of deformed algebra techniques. In this analytic representation the action of the paraparticle annihilation operator is equivalent to a deformed differentiation which encodes the statistics of the paraparticle. In the parafermi case, the derivative’s ket-domain is degree $p$ polynomials.

\textsuperscript{1}On leave from: Department of Modern Physics, University of Science and Technology of China, Hefei, 230026, P.R. China.

\textsuperscript{2}Electronic address: cnelson@bingvmb.cc.binghamton.edu
1 Introduction

Recently for boson systems, there has been interest in the Dirac contour representation [1-3] because analytic representations are frequently important in the analysis of quantum field theoretic systems. The contour aspect of this representation is based on Heitler’s contour integral form of the $\delta$-function [4]. This representation provides a natural enlargement of the usual Hilbert space description because the expansion coefficients are distributions, instead of functions.

In this paper, we extend the representation to parabose and parafermi systems [5-6] by using deformed algebra techniques [7-9]. In formal appearance, the resulting contour representations are similar in the two cases. The non-trivial differences arise in the appearance of deformed brackets, $[n]$, and a deformed differentiation, $\frac{D}{Dz}$, in the parabose case, versus “curly braces” $\{n\}$ and $\frac{D}{Dz}$ in the parafermi case. When $p = 1$, the parafermi extension gives a contour representation for ordinary fermions.

In Sec. 2, the Dirac contour representation for parabose systems is constructed and its properties are studied. Sec. 3 is devoted to the extension to parafermi systems. Finally, in Sec. 4 there is a brief summary discussion.

2 Dirac’s Contour Representation for Parabosons

The trilinear commutation relations in the single-mode parabose case are

$$[a, \{a^\dagger, a\}] = 2a, \quad [a, (a^\dagger)^2] = 2a^\dagger.$$  \hfill (1)
where $a^\dagger$, $a$ are the parabose creation and annihilation operators. Actually, these relations can be replaced [8] by deformed bilinear ones

$$[a, a^\dagger] = 1 + (p - 1)(-)^{N_B}, \quad \{a, a^\dagger\} = p + 2N_B \tag{2}$$

where $N_B$ is the parabose number operator and $p$ is the order of the parastatistics. The number basis for the single-mode parabose system is

$$|n\rangle = \frac{(a^\dagger)^n}{\sqrt{n!}} \tag{3}$$

with

$$[n] = n + \frac{p - 1}{2} (1 - (-)^n), \quad [n]! = [n][n - 1] \cdots [1], \quad [0]! \equiv 1.$$  

Thus the eigenbras\(^3\) and eigenkets of $N_B$ are respectively represented by

$$\langle n| \rightarrow \frac{\sqrt{[n]!}}{z^{n+1}}, \quad |m\rangle \rightarrow \frac{z^m}{\sqrt{[m]!}} \tag{4}$$

in the Dirac contour representation associated with the parabose Hilbert space $\mathcal{H}_B$. In this representation, the “bra-ket” inner product is defined by

$$\langle n|m\rangle \rightarrow \oint_C \frac{dz}{2\pi i} \sqrt{\frac{[m]!}{[n]!}} \frac{z^n}{z^{m+1}} = \sqrt{\frac{[m]!}{[n]!}} \frac{1}{m!} \frac{d}{dz} z^n \big|_{z=0} = \delta_{mn} \tag{5}$$

where $C$ is a counterclockwise contour enclosing the origin of the complex $z$ plane. Arbitrary bra and ket states in $\mathcal{H}_B$, which can be expanded as $\langle f| = \sum_{n=0}^{\infty} f_n^* \langle n|$, $|f\rangle = \sum_{m=0}^{\infty} f_m |m\rangle$, are represented by

$$\langle f| \rightarrow \sum_{n=0}^{\infty} f_n^* \frac{\sqrt{[n]!}}{z^{n+1}} = f_k(z), \quad |f\rangle \rightarrow \sum_{m=0}^{\infty} f_m \frac{z^m}{\sqrt{[m]!}} = f_k(z) \tag{6}$$

\(^3\)Note that the dual space $\langle n|$ is not obtained by Hermitian conjugation, see [1-3].
where the subscripts \( b(k) \) refer to bra (ket). When \( \sum |f_n|^2 = 1 \), the normalized function \( f_k(z) \) is an analytic function in the complex plane \( z \in \mathcal{C} \). If \( \mathcal{C} \) is enlarged to include the point at infinity (\( \bar{\mathcal{C}} = \mathcal{C} + \infty \)), the function \( g_b(z) \) is also an analytic function in the neighborhood of the infinity point in \( \bar{\mathcal{C}} \). From (5) the inner product of two such states is

\[
< f | g > \rightarrow \oint_{\mathcal{C}} \frac{dz}{2\pi i} f_b(z) g_k(z) = \sum_{n=0}^{\infty} f^*_n g_n
\]

(7)

An arbitrary operator \( \Theta \) in \( \mathcal{H}_B \) with \( \Theta = \sum_{m,n=0}^{\infty} \Theta_{mn} |m><n|, \quad \Theta_{mn} = <m|\Theta|n> \), has the Dirac contour representation

\[
\Theta \rightarrow \Theta(z, w) = \sum_{m,n=0}^{\infty} \Theta_{mn} \sqrt{\frac{[n]!}{[m]!}} \frac{z^m}{w^{n+1}}.
\]

(8)

So,

\[
\Theta |g > \rightarrow \oint_{\mathcal{B}} \frac{dw}{2\pi i} \Theta(z, w) g_k(w), \quad < f | \Theta \rightarrow \oint_{\mathcal{B}} \frac{dw}{2\pi i} f_b(w) \Theta(w, z)
\]

(9)

represent new states where the counterclockwise contour \( \mathcal{B} \) encloses the origin of the complex \( w \) plane. The product of two operators \( \Theta_1, \Theta_2 \) in \( \mathcal{H}_B \) takes the form of a generalized convolution

\[
\Theta_1 \Theta_2 \rightarrow \oint_{\mathcal{B}} \frac{dv}{2\pi i} \Theta_1(z, v) \Theta_2(v, w)
\]

(10)

In the parabose number representation, \( a^\dagger |n > = \sqrt{[n+1]} |n + 1 > \), which gives \( (a^\dagger)_{mn} = \sqrt{[n+1]} \delta_{m,n+1} \), e.g. see [6]. So from (8)

\[
a^\dagger \rightarrow \sum_{m,n=0}^{\infty} \sqrt{[n+1]} \delta_{m,n+1} \sqrt{\frac{[n]!}{[m]!}} \frac{z^m}{w^{n+1}} = \sum_{n=0}^{\infty} z^{n+1}
\]

which converges to \( z(w - z)^{-1} \) when \( |w| > |z| \). For \( |z| > |w| \), the sum diverges. However, then the point \( z \) in first equation in (9) lies outside the contour \( \mathcal{B} \) and so (9) gives zero since \( g_k(w) \) is
an arbitrary analytic function. So in this contour representation, we write

\[ a^\dagger \rightarrow \frac{z}{w-z} \theta(|w| - |z|), \tag{11} \]

Also note, by (9),

\[ a^\dagger |n > \rightarrow \oint_B \frac{dw}{2\pi i} \frac{z}{w-z} \frac{w^n}{\sqrt{|n|!}} = \frac{z^{n+1}}{\sqrt{|n|!}} = \frac{\sqrt{n+1}}{\sqrt{|n+1|!}} \]

which agrees with \( a^\dagger |n > = \sqrt{|n+1|} |n+1 > \), and

\[ < 0 | a^\dagger \rightarrow \oint_C \frac{dz}{2\pi i} \frac{1}{z} \frac{z}{w-z} = 0, \quad (|w| > |z|) \tag{12} \]

so \( a^\dagger \) does annihilate the bra state \(< 0 | \rightarrow \frac{1}{z} >\).

In the number representation, \( a|n > = \sqrt{|n|} |n-1 > \), so \( a_{mn} = \sqrt{|n|} \delta_{m,n-1} \) and

\[ a \rightarrow \sum_{m,n=0}^\infty \sqrt{|n|} \delta_{m,n-1} \sqrt{|m|!} \frac{z^m}{|m|!} w^{m+1} = \frac{1}{w^2} \sum_{n=0}^\infty [n] \left( \frac{z}{w} \right)^{n-1}, \]

\[ = \left( \frac{1}{(w-z)^2} + \frac{p-1}{2z} \frac{1}{w-z} - \frac{p-1}{2z} \frac{1}{w+z} \right) \theta(|w| - |z|). \tag{13} \]

Obviously, (13) reduces to Dirac’s boson result [1] when \( p = 1 \). With the definition [9] of deformed differentiation (c.f. Eq.(3) above)

\[ \frac{D}{Dz} f(z) \equiv \frac{d}{dz} f(z) + \frac{p-1}{2z} (f(z) - f(-z)), \tag{14} \]

we can write (13) as

\[ a \rightarrow \left( \frac{D}{Dz} \frac{1}{w-z} \right) \theta(|w| - |z|), \tag{15} \]

Again, from (9) we obtain

\[ a|n > \rightarrow \oint_B \frac{dw}{2\pi i} \left( \frac{1}{(w-z)^2} + \frac{p-1}{2z} \frac{1}{w-z} - \frac{p-1}{2z} \frac{1}{w+z} \right) \frac{w^n}{\sqrt{|n|!}} \]

\(^4\)Here \( \theta(x) = 1 \) for \( x > 0 \) and \( \theta(x) = 0 \) for \( x < 0 \).
\( = \left( n + \frac{p-1}{2} - \frac{p-1}{2}(-)^n \right) \frac{z^{n-1}}{\sqrt{|n|!}} = \sqrt{|n|} \frac{z^{n-1}}{\sqrt{|n-1|!}} \) \tag{16}

which agrees with \( a|n> = \sqrt{|n|} |n-1> \).

Then from the convolution formula (10), we obtain

\[
a^\dagger a \to \oint_C \frac{dv}{2\pi i} \frac{z}{v-z} \left( \frac{1}{(w-v)^2} + \frac{p-1}{2w(w-v)} - \frac{p-1}{2v(w+v)} \right)
\]

\[
= \left( \frac{z}{(w-z)^2} + \frac{p-1}{2(w-z)} - \frac{p-1}{2(w+z)} \right) \theta(|w| - |z|) \tag{17}
\]

and

\[
aa^\dagger \to \oint_C \frac{dv}{2\pi i} \left( \frac{1}{(v-z)^2} + \frac{p-1}{2z(v-z)} - \frac{p-1}{2z(v+z)} \right) \frac{v}{w-v}
\]

\[
= \left( \frac{w}{(w-z)^2} + \frac{p-1}{2(w-z)} + \frac{p-1}{2(w+z)} \right) \theta(|w| - |z|). \tag{18}
\]

In the contour integration in (18), the points \( z \) and \( -z \) both lie inside the contour \( C \). From (17-18),

\[
(N_B + \frac{p}{2}) |n> = \frac{1}{2} (a^\dagger a + aa^\dagger) |n>
\]

\[
\to \oint_B \frac{dw}{2\pi i} \frac{1}{2} \left( \frac{z+w}{(w-z)^2} + \frac{p-1}{w-z} \right) \frac{w^n}{\sqrt{|n|!}} = (n + \frac{p}{2}) \frac{z^n}{\sqrt{|n|!}},
\]

\[
< n| (N_B + \frac{p}{2}) = (n) \frac{1}{2} (a^\dagger a + aa^\dagger)
\]

\[
\to \oint_B \frac{dw}{2\pi i} \frac{\sqrt{|n|!}}{2w^{n+1}} \left( \frac{z+w}{(z-w)^2} + \frac{p-1}{z-w} \right) = \frac{\sqrt{|n|!}}{2n!} \frac{d^n}{dw^n} \left( \frac{z+w}{(z-w)^2} + \frac{p-1}{z-w} \right) |_{w=0}
\]

\[
= (n + \frac{p}{2}) \frac{\sqrt{|n|!}}{2z^{n+1}},
\]

(20)

which explicitly verifies that \( |n> \) and \( < n| \) are eigenstates of the parabose number operator \( N_B \).

Similarly, for integer powers of \( a \) and \( a^\dagger \)

\[
(a^\dagger)^m \to \frac{z^m}{w-z} \theta(|w| - |z|),
\]

\[
(a)^n \to \left( \frac{D^n}{Dz^n} \frac{1}{w-z} \right) \theta(|w| - |z|).
\]

(21)
So from convolution formula, the representations for their normal-ordered and antinormal-ordered products are

\[(a^\dagger)^m a^n \rightarrow \oint_C \frac{dw}{2\pi i} (\frac{z^m}{w-z}) \frac{1}{w-v} = z^m (\frac{D^n}{Dz^n} \frac{1}{w-z}) \theta(|w| - |z|),\]

\[(a)^n (a^\dagger)^m \rightarrow \oint_C \frac{dw}{2\pi i} (\frac{D^n}{Dz^n} \frac{1}{w-v}) \frac{v^m}{w-v} = \frac{D^n}{Dz^n} \oint_C \frac{dw}{2\pi i} (\frac{1}{v-z}) \frac{w^m}{w-v}\]

(22)

Furthermore, for an arbitrary ket \(|g >\) in \(\mathcal{H}_B\),

\[(a^\dagger)^m a^n |g > \rightarrow \oint_B \frac{dw}{2\pi i} z^m \frac{D^n}{Dz^n} g_k(w)\]

\[= z^m \frac{D^n}{Dz^n} \oint_B \frac{dw}{2\pi i} g_k(w) = z^m \frac{D^n}{Dz^n} g_k(z),\]

(23)

\[(a)^n (a^\dagger)^m |g > \rightarrow \oint_B \frac{dw}{2\pi i} (\frac{D^n}{Dz^n} \frac{z^m}{w-z}) g_k(w)\]

\[= \frac{D^n}{Dz^n} (z^m \oint_B \frac{dw}{2\pi i} g_k(w)) = \frac{D^n}{Dz^n} (z^m g_k(z)),\]

which shows that in the Dirac contour representation (as well as in the Bargmann representation [10]) the action of the parabose operators \(a^\dagger\) and \(a\) is respectively equivalent to multiplication by \(z\) and deformed differentiation, \(\frac{D}{Dz}\), of the analytic function \(g_k(z)\).

As the last topic in this section, we give the Dirac contour representation for the unnormalized parabose coherent state \(|\alpha >\):

\[|\alpha > = \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n > \rightarrow \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} \frac{z^n}{\sqrt{n!}} = E(\alpha z)\]

(24)

where \(E(z) \equiv \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}}\) is an entire function. By (9),

\[a |\alpha > \rightarrow \oint_B \frac{dw}{2\pi i} (\frac{D}{Dz} \frac{1}{w-z}) E(\alpha w) = \alpha E(\alpha z),\]

(25)

\[5\text{An instructive exercise is to verify the trilinear commutation relations; note in general, } \frac{D}{Dz} (fg) \neq \left(\frac{D}{Dz} f\right) g + f(\frac{D}{Dz} g).\]
which verifies $a |\alpha > = \alpha |\alpha >$. By (4), the corresponding bra state $< \alpha |$ in the Dirac contour representation is

\[ < \alpha | = \sum_{n=0}^{\infty} \frac{1}{\sqrt{[n]!}} \sum_{n=0}^{\infty} \frac{(\alpha^*)^n}{z^{n+1}} \frac{\sqrt{[n]!}}{\sqrt{[n]!}} = \frac{1}{z - \alpha^*}, \quad (|z| > |\alpha|). \quad (26) \]

So by (9),

\[ < \alpha |a^\dagger \rightarrow \oint_B \frac{dw}{2\pi i} \left( \frac{1}{w - \alpha^*} \right) \frac{w}{z - w} = \frac{\alpha^*}{z - \alpha^*} \quad (|z| > |\alpha|) \]

which shows that $< \alpha |a^\dagger = < \alpha | \alpha^*$. This means that in the Dirac contour representation, the dual vectors $< \alpha |$ of the parabose coherent states are the eigenbra vectors of the parabose creation operator $a^\dagger$ with eigenvalue $\alpha$.

The contour form of the resolution of unity [1,2] associated with $< \alpha |$ and the $a^\dagger$ eigenvectors $|z >'$ is eq.(13) in [12].

### 3 Dirac’s Contour Representation for Parafermions

The parafermi trilinear commutation relation in the single-mode case is

\[ [f, [f^\dagger, f]] = 2f \quad (28) \]

where $f^\dagger, f$ are the parfermi creation and annihilation operators. This trilinear relation can also be rewritten in bilinear form in terms of deformed oscillators[7]

\[ \{f, f^\dagger\} = p + 2pN_f - 2N_f^2, \quad [f, f^\dagger] = p - 2N_f \]

where $N_f$ is the parafermi number operator. We also define a deformed parafermi number operator

\[ \{N_f\} = N_f(p + 1 - N_f) \quad (30) \]
To obtain the ordinary fermions for $p = 1$, we set $N_f = \{N_f\}$ which implies $N_f^2 = N_f$ and then the bilinear relations (29) imply $N_f = f^\dagger f$ and the ordinary fermi algebra $\{f, f^\dagger\} = 1$. The eigenstates of $N_f$ are

$$|n> = \frac{(f^\dagger)^n}{\sqrt{\{n\}!}}|0>, \quad N_f|n> = n|n>,$$  \hspace{1cm} (31)

with

$$\{n\} = n(p + 1 - n), \quad \{n\}! = \{n\}\{n-1\}\cdots\{1\}, \quad \{0\}! = 1$$  \hspace{1cm} (32)

From (32), $\{0\} = \{p + 1\} = 0$ and $\{n\} < 0$ for $n > p + 1$. So for parafermi systems, we consider $n$ as non-negative integers in the range $0 \leq n \leq p$. In this number basis, we also have

$$f|n> = \sqrt{\{n\}}|n - 1>, \quad f^\dagger|n> = \sqrt{\{n+1\}}|n + 1> \quad (0 \leq n \leq p)$$  \hspace{1cm} (33)

In order to construct the contour representation for parafermi systems, we introduce a polynomial basis[7]. Let $\mathcal{H}$ be the set of analytic functions

$$f(z) = \sum_{n=0}^{\infty} c_n z^n,$$

in the complex plane $C$ and $\mathcal{J}_p$ be the projection operator which projects the function $f(z)$ to the polynomial $\mathcal{J}_p f(z)$ of degree $p$

$$\mathcal{J}_p f(z) = \sum_{n=0}^{p} c_n z^n \in \mathcal{J}_p \mathcal{H}.$$  \hspace{1cm} (34)

Thereby, the space spanned by the parafermi number basis $|n>$ is identified with the polynomial space $\mathcal{J}_p \mathcal{H}$ spanned by the basis $\frac{z^n}{\sqrt{\{n\}!}}$,

$$|n> \rightarrow \frac{z^n}{\sqrt{\{n\}!}}, \quad (n = 0, 1, \cdots p).$$  \hspace{1cm} (35)

We denote the dual space of $\mathcal{H}$ by $\mathcal{H}^*$, which is the set of functions

$$g(z) = \sum_{n=0}^{\infty} \frac{d_n}{z^{n+1}}.$$
When $\sum |d_n|^2 = 1$, these functions are also analytic in the neighborhood of the infinity point in $\bar{C}$. In $\mathcal{H}^*$ we similarly introduce a projection operator $\mathcal{J}_p^*$ which projects the function $g(z)$ to a truncated one $\mathcal{J}_p^*g(z)$ of minus degree $p+1$

$$\mathcal{J}_p^*g(z) = \sum_{n=0}^{p} \frac{d_n}{z^{n+1}} \epsilon \mathcal{J}_p^*\mathcal{H}^*.$$  \hspace{1cm} (36)

Thus, the space spanned by the dual vectors $<n|$ of the parafermi number basis $|n>$ is identified with the space $\mathcal{J}_p^*\mathcal{H}^*$ spanned by

$$<n| \to \sqrt{\frac{|n|!}{z^{n+1}}}, \quad (n = 0, 1, \cdots p).$$  \hspace{1cm} (37)

In this representation, we define the inner product of $<n|$ and $|m>$ by

$$<n|m> \to \oint_C \frac{dz}{2\pi i} \sqrt{\frac{|n|!}{|m|!}} \frac{z^m}{z^{n+1}} = \delta_{mn}.$$  \hspace{1cm} (38)

where the contour $C$ is defined as in (5).

Again guided by (30) with $\{N\} \to z \frac{D}{Dz}$ and $N \to z \frac{d}{dz}$, in the parafermi case we define the deformed derivative

$$\frac{D}{Dz} f(z) \equiv \frac{d}{dz} (p + 1 - z \frac{d}{dz})$$  \hspace{1cm} (39)

which gives for instance

$$\frac{D}{Dz} z^n = \{n\} z^{n-1}, \quad (n = 0, 1, \cdots p).$$  \hspace{1cm} (40)

The multiplication of the function $\mathcal{J}_p f(z)$ by $z$ can be regarded as a map from the subspace $\mathcal{J}_p\mathcal{H}$ into the subspace $\mathcal{J}_p\mathcal{H} - \mathcal{J}_0\mathcal{H}$ since

$$z \sum_{n=0}^{p} c_n z^n = \sum_{n=1}^{p} c_{n-1} z^n \epsilon (\mathcal{J}_p\mathcal{H} - \mathcal{J}_0\mathcal{H}).$$  \hspace{1cm} (41)

Also the derivative $\frac{d}{dz}$ is a map from $\mathcal{J}_p\mathcal{H}$ into $\mathcal{J}_{p-1}\mathcal{H}$ since

$$\frac{d}{dz} \sum_{n=0}^{p} c_n z^n = \sum_{n=0}^{p-1} (n + 1) c_{n+1} z^n \epsilon \mathcal{J}_{p-1}\mathcal{H}.$$  \hspace{1cm} (42)
Similarly, multiplication of $J^*_p g(z)$ by $z$ is a map from $J^*_p \mathcal{H}$ into $J^*_p \mathcal{H}^* + J_0 \mathcal{H}$,

$$z \sum_{n=0}^{p} \frac{d_n}{z^{n+1}} = d_0 + \sum_{n=0}^{p-1} \frac{d_{n+1}}{z^{n+1}} \epsilon (J^*_p \mathcal{H}^* + J_0 \mathcal{H}), \quad (43)$$

and the derivative $\frac{d}{dz}$ is a map from $J^*_p \mathcal{H}$ into $J^*_p \mathcal{H}^* - J^*_0 \mathcal{H}^*$,

$$\frac{d}{dz} \sum_{n=0}^{p} \frac{d_n}{z^{n+1}} = - \sum_{n=1}^{p} \frac{n d_{n-1}}{z^{n+1}} \epsilon (J^*_p \mathcal{H}^* - J^*_0 \mathcal{H}^*). \quad (44)$$

Note that when $p = 1$, parafermi deformed differentiation reduces to ordinary differentiation in the subspace $J_0 \mathcal{H}$.

For simplicity, we shall omit the explicit projection symbols $J_p$ and $J^*_p$ in the following so in the parafermi case it is to be remembered that the domains are the subspaces $J_p \mathcal{H}$ and $J^*_p \mathcal{H}$. With this understanding, the formalism is analogous to that for parabose systems: Arbitrary bra and ket states in the parafermi Hilbert space $\mathcal{H}_f$ which can be expanded as $<g| = \sum_{n=0}^{p} g^n < n|$, $|g> = \sum_{m=0}^{p} g_m |m>$, are represented by

$$<g| \rightarrow \sum_{n=0}^{p} g^n \frac{\sqrt{\{n\}}!}{z^{n+1}} = g^{(f)}(z), \quad |g> \rightarrow \sum_{m=0}^{p} g_m \frac{z^m}{\sqrt{\{m\}}!} = g^{(k)}(z) \quad (45)$$

and the inner product of two such vectors is

$$<f|g> \rightarrow \oint_C \frac{dz}{2\pi i} f^{(f)}(z) g^{(f)}(z) = \sum_{n=0}^{p} f^n_n g_n. \quad (46)$$

For an operator $A$ defined on $\mathcal{H}_f$ by $A = \sum_{m=0}^{p} A_{mn} |m><n|$, there is the contour representation

$$A \rightarrow A(z, w) = \sum_{m,n=0}^{p} A_{mn} \frac{\{n\}!}{\{m\}!} \frac{z^m}{z^{n+1}}. \quad (47)$$

so Eqs.(9-10) again follow. As in the parabose case, we find that

$$f^\dagger \rightarrow \frac{z}{w - z} \theta(|w| - |z|),$$
\[ \begin{align*} \phi & \to \left( \frac{D}{Dz} \frac{1}{w-z} \right) \theta(|w| - |z|), \\
\phi \phi^\dagger & \to z \left( \frac{D}{Dz} \frac{1}{w-z} \right) \theta(|w| - |z|), \\
\phi \phi^\dagger & \to \left( \frac{D}{Dz} \frac{z}{w-z} \right) \theta(|w| - |z|). \end{align*} \] (48)

Also, the unnormalized parafermi coherent state \(| \beta > \phi \) is represented by

\[ | \beta > \phi = \sum_{n=0}^{p} \frac{\beta^m}{\sqrt{n}!} | n > \rightarrow \sum_{n=0}^{p} \frac{\beta^m}{\{n\}!} \equiv E_f(\beta z) \] (49)

where \( E_f(z) \) is a polynomial. By (9), \( f | \beta > \phi = \beta | \beta > \phi \). Furthermore, by (45) the corresponding bra state in this contour representation is

\[ \phi < \beta | \phi^\dagger = \sum_{n=0}^{p} \frac{(\beta^*)^m}{\sqrt{n}!} < n | \rightarrow \sum_{n=0}^{p} \frac{(\beta^*)^m}{\{n\}!} = \frac{1}{z - \beta^*}, \quad (|z| > |\beta|), \] (50)

which satisfies \( \phi < \beta | \phi^\dagger = \phi < \beta | \beta^* \). The contour form of the resolution of unity associated with \( \phi < \beta \) is formally the same as that for parabosons, see eq.(13) in [12], but it is restricted to the parafermi subspace.

### 4 Discussion

In summary, in this paper we extend the Dirac contour representation to the single-mode parabose and parafermi systems. In these extensions we use deformed bilinear commutation relations to replace the intrinsic trilinear relations of paraparticles. Thus the formalism is similar to that for ordinary bose systems, the non-trivial differences include (i) the appearance of \( \sqrt{n}! \) or \( \sqrt{\{n\}!} \) in parasystems, in place of \( \sqrt{n}! \), in expressions which represent states and operators versus the number basis, and (ii) the representation of the action of the paraparticle annihilation operator by a deformed differentiation \( \frac{D}{Dz} \) or \( \frac{D}{Dz} \), instead of by ordinary differentiation. In a similar manner,
the Dirac contour representation can also be extended to many other deformed oscillator systems such as the q-deformed harmonic oscillator, the Arik-Coon oscillator, and the q-deformed parabose and parafermi oscillators[11]. The action of the oscillator’s annihilation operator is equivalent to a deformed differentiation which encodes the statistical properties of the oscillator. As in the case of parafermions, see (39-44), the definition of the deformed derivative always must include specification of its domain. Deformed integration, i.e. the inverse operation, can therefore also be defined.

This work was partially supported by the National Natural Science Foundation of China and by U.S. Dept. of Energy Contract No. DE-FG 02-96ER40291.

References

[1] P.A.M. Dirac, Commun. Dublin Inst. Adv. Studies Ser. A1, 1(1943); J. Schwinger, Quantum Kinematics and Dynamics, Sec. 4.8, (W.A. Benjamin, N.Y., 1970).

[2] H.Y. Fan and J.R. Klauder, Mod. Phys. Lett. A9, 1291(1994); H.Y. Fan, Z.W. Liu and T.N. Ruan, Commun. Theor. Phys. (Beijing), 3 175(1984).

[3] A. Vourdas and R.F. Bishop, Phys. Rev. A53, R1205(1996).

[4] W. Heitler The Quantum Theory of Radiation, Sec. II.8, (Claredon Press; 3rd ed., 1954).

[5] H.S. Green, Phys. Rev. 90, 270(1953); D.V. Volkov Sov. Phys. JETP 11, 375(1960); O.W. Greenberg and A.M.L. Messiah, Phys. Rev. 138, B1155(1965).
[6] Y. Ohnuki and S. Kamefuchi, *Quantum Field Theory and Parastatistics* (Berlin: Springer-Verlag, 1982).

[7] D. Bonatsos and C. Daskaloyannis, J. Phys. A: Math. Gen. **26**, 1589(1993).

[8] A.J. Macfarlane, J. Phys. A: Math. Gen. **35**, 1054(1994).

[9] S. Jing, “A new kind of deformed calculus and parabosonic coordinate representation”, to appear in J. Phys. A: Math. Gen. **31**, (1998).

[10] V. Bargmann, Commun. Pure Appl. Math. **14**, 180, 187(1961); *ibid.* **20**, 1(1967).

[11] L.C. Biedenharn, J. Phys. A: Math. Gen. **22**, L873(1989); A.J. Macfarlane, *ibid.* **22**, 4581(1989); M. Arik and D. Coon, J. Math. Phys. **17**, 524(1976); R. Floreanini and L. Vinet, J. Phys. A: Math. Gen. **23**, L1019(1990); and K. Odaka, T. Kishi and S. Kamefuchi, *ibid.* **24**, L591(1991).

[12] S. Jing and C.A. Nelson, SUNY-BING 5/21/98, hep-th/9805214; SUNY BING 7/2/98, hep-th/9807048.