On a nonlinear elliptic system from Maxwell-Chern-Simons vortex theory

Tonia Ricciardi

Dipartimento di Matematica e Applicazioni
Università di Napoli Federico II
Via Cintia
80126 Naples, Italy
fax: +39 081 675665
e-mail: tonia.ricciardi@unina.it

July 30, 2002

Abstract

We define an abstract nonlinear elliptic system, admitting a variational structure, and including the vortex equations for some Maxwell-Chern-Simons gauge theories as special cases. We analyze the asymptotic behavior of its solutions, and we provide a general simplified framework for the asymptotics previously derived in those special cases. As a byproduct of our abstract formulation, we also find some new qualitative properties of solutions.

KEY WORDS: nonlinear elliptic system, Chern-Simons vortex theory

MCS 2000 SUBJECT CLASSIFICATION: 35J60

0 Introduction

Motivated by the analysis of vortex configurations for the self-dual $U(1)$ Maxwell-Chern-Simons model introduced in [7] (see also Yang [11], Dunne [5], Jaffe and Taubes [6]), we considered in [9] solutions $(u, v)$ for the system:

\begin{align}
-\Delta u &= q(v - e^u) - 4\pi \sum_{j=1}^{n} \delta_{p_j} \quad \text{on } \Sigma \\
-\Delta v &= q \{ e^u(1 - v) - q(v - e^u) \} \quad \text{on } \Sigma,
\end{align}

where $\Sigma$ is a compact Riemannian 2-manifold without boundary, $n \geq 0$ is an integer, $p_j \in \Sigma$ for $j = 1, \ldots, n$, $\Delta$ denotes the Laplace-Beltrami operator and $q > 0$ is a constant. It is of both mathematical and physical interest to understand the asymptotic behavior of solutions to (1)–(2) as $q \to +\infty$. In [9]

*Partially supported by PRIN 2000 “Variational Methods and Nonlinear Differential Equations”
we provided a rigorous proof of the formal asymptotics derived in [7], in any relevant norm. Namely, we showed that if \((u, v)\) are (distributional) solutions for (1)–(2) with \(q \to +\infty\), then there exists a solution \(u_\infty\) for the equation

\[
-\Delta u_\infty = e^{u_\infty}(1 - e^{u_\infty}) - 4\pi \sum_{j=1}^{n} \delta_{p_j} \quad \text{on } \Sigma
\]

such that \((e^u, v) \to (e^{u_\infty}, e^{u_\infty})\) in \(C^h(\Sigma) \times C^h(\Sigma)\), for any \(h > 0\). (Note that \(e^u, e^{u_\infty}\) are smooth). Such a result completed our previous convergence result obtained with Tarantello [10], where the asymptotics for \(v\) was established in the \(L^2\)-sense only. See also Chae and Kim [2].

More recently, Chae and Nam [3] analyzed an elliptic system, whose solutions describe vortex configurations for the self-dual \(CP(1)\) Maxwell-Chern-Simons model introduced in [4]. Their system (in a special case) is given by:

\[
\begin{align*}
\Delta U &= 2Q(-V + S - \frac{1 - e^U}{1 + e^U}) + 4\pi \sum_{j=1}^{n} \delta_{p_j} \quad \text{on } \Sigma \\
\Delta V &= -4Q^2(-V + S - \frac{1 - e^U}{1 + e^U}) + Q \frac{4e^U}{(1 + e^U)^2} V \quad \text{on } \Sigma
\end{align*}
\]

where \(\Sigma\) and \(p_1, \ldots, p_n\) are as in (1)–(2), \(U, V\) are the unknown functions and \(S \in \mathbb{R}, Q > 0\) are given constants. Among other results, they derive an asymptotic behavior as \(Q \to +\infty\) analogous to that of system (1)–(2).

Thus it is natural to seek a common underlying structure for (1)–(2) and (4)–(5), which would allow such asymptotic behaviors.

Our aim in this note is to identify a general nonlinear system including (1)–(2) and (4)–(5) as special cases, and to show that the asymptotic behaviors described above are in fact a general property of its solutions. We believe that our proof of the asymptotics for our abstract system simplifies the previous approaches. Our formulation will also allow us to find some new qualitative properties of solutions.

More precisely, we consider (distributional) solutions \((\tilde{u}, v)\) for the system:

\[
\begin{align*}
-\Delta \tilde{u} &= q(v - f(e^{\tilde{u}})) - 4\pi \sum_{j=1}^{n} \delta_{p_j} \quad \text{on } \Sigma \\
-\Delta v &= q \left[ f'(e^{\tilde{u}})e^{\tilde{u}}(s - v) - q(v - f(e^{\tilde{u}})) \right] \quad \text{on } \Sigma.
\end{align*}
\]

Here \(\Sigma\) and \(p_1, \ldots, p_n\) are as in (1)–(2), \(f = f(t), t \geq 0\) is smooth and strictly increasing, \(s \in \mathbb{R}\) satisfies \(f(0) < s < \sup_{t>0} f(t)\). We shall later show that when \(n = 0\), system (6)–(7) only admits the trivial solution \((f(e^{\tilde{u}}), v) = (s, s)\). Without loss of generality, we assume \(\text{vol} \Sigma = 1\).

Clearly, when \(f(t) = t\) and \(s = 1\), system (6)–(7) reduces to (1)–(2). On the other hand, setting \(v := V - S, s := -S, q := 2Q\), system (6)–(7) reduces to system (4)–(5) with \(f\) defined by \(f(t) = (t - 1)/(t + 1)\).

As already mentioned, we are interested in the asymptotic behavior of solutions as \(q \to +\infty\). By a formal analysis of (6)–(7) we expect that, up to subsequences, \((\tilde{u}, v)\) converges to some solution \((u_\infty, f(e^{u_\infty}))\), for the equation

\[
-\Delta u_\infty = f'(e^{u_\infty})e^{u_\infty}(s - f(e^{u_\infty})) - 4\pi \sum_{j=1}^{n} \delta_{p_j} \quad \text{on } \Sigma.
\]
Our main result states that this is indeed the case, with respect to any relevant norm:

**Theorem 0.1.** Let \((\tilde{u}, v)\) be (distributional) solutions to (6)–(7), with \(q \to +\infty\). There exists a (distributional) solution \(\tilde{u}_\infty\) to (8) such that a subsequence, still denoted \((\tilde{u}, v)\), satisfies:

\[
(e^{\tilde{u}}, v) \to (e^{\tilde{u}_\infty}, f(e^{\tilde{u}_\infty})) \quad \text{in } C^h(\Sigma) \times C^h(\Sigma), \forall h \geq 0.
\]

In order to work in suitable Sobolev spaces, it is standard (see [11]) to define a “Green’s function” \(u_0\), solution for the problem

\[
-\Delta u_0 = 4\pi \left( n - \sum_{j=1}^{n} \delta_{p_j} \right) \quad \text{on } \Sigma \\
\int_{\Sigma} u_0 = 0
\]

(see [1] for the unique existence of \(u_0\)). Setting \(\tilde{u} = u_0 + u\), we obtain the equivalent system for \((u, v) \in H^1(\Sigma) \times H^1(\Sigma):

\[
\begin{align*}
(9) \quad -\Delta u &= q(v - f(e^{u_0+u})) - 4\pi n & \text{on } \Sigma \\
(10) \quad -\Delta v &= q \left[ f'(e^{u_0+u})e^{u_0+u}(s - v) - q(v - f(e^{u_0+u})) \right] & \text{on } \Sigma,
\end{align*}
\]

where \(e^{u_0}\) is smooth. We also note that system (9)–(10) admits a variational formulation. Indeed, solutions \((u, v)\) to (9)–(10) correspond to critical points \(v \in H^2(\Sigma)\) for the functional:

\[
I(u) = \frac{1}{2q^2} \int (\Delta u)^2 + \frac{1}{2} \int |\nabla u|^2 + \frac{1}{q} \int f'(e^{u_0+u})e^{u_0+u}|\nabla(u_0 + u)|^2 + \frac{1}{2} \int (f(e^{u_0+u}) - s)^2 + 4\pi n \int u.
\]

Since

\[
e^{u_0}|\nabla u_0|^2 = \Delta e^{u_0} + 4\pi n,
\]

the function \(e^{u_0}|\nabla u_0|^2\) is smooth, and \(I\) is well-defined on \(H^2(\Sigma)\) by Sobolev embeddings. To see how critical points for \(I\) correspond to solutions for (9)–(10), we solve (9) for \(v\):

\[
v = q^{-1}(-\Delta u + 4\pi n) + f(e^{u_0+u}).
\]

Substituting into (10), we obtain the fourth-order equation:

\[
\frac{1}{q^2} \Delta^2 u - \Delta u - \frac{1}{q} \left[ \Delta f(e^{u_0+u}) + f'(e^{u_0+u})e^{u_0+u} \Delta(u_0 + u) \right]
+ f'(e^{u_0+u})e^{u_0+u}(f(e^{u_0+u}) - s) + 4\pi n = 0 \quad \text{on } \Sigma.
\]
Integration by parts shows that
\[
\frac{d}{dt} |_{t=0} \left[ f'(e^{u_0+u+t\phi})e^{u_0+u+t\phi} \nabla(u_0 + u + t\phi) \right]^2 = - \int \{ \Delta f(e^{u_0+u}) + f'(e^{u_0+u})e^{u_0+u} \Delta (u_0 + u) \},
\]
and thus critical points for $I$ correspond to solutions for (11). We shall exploit this variational structure in order to study the multiplicity of solutions to (6)–(7) in a forthcoming note.

The remaining part of this note is devoted to the proof of Theorem 0.1. The main point of the proof is to obtain a priori estimates for $\tilde{u}$ in $H^1$ and for $v$ independent of $q \to +\infty$ in the Sobolev spaces $H^k$, for every $k \geq 0$. To this end, in Section 1 we first establish some preliminary estimates in $L^\infty$ and $H^1$. In Section 2, exploiting the specific structure of system (9)–(10), we set up an iteration in the framework of the Banach algebras $H^k \cap L^\infty$, for $k \geq 0$, which yields the desired estimates.

Henceforth we denote by $C > 0$ a general constant independent of $q$, which may vary from line to line. Unless otherwise specified, all equations are defined on $\Sigma$ and all integrals are taken over $\Sigma$ with respect to the Lebesgue measure.

1 A priori estimates

Our aim in this section is to establish estimates in $H^1$ and in $L^\infty$ for $e^{\tilde{u}}$ and $v$, as stated in the following

**Proposition 1.1.** There exists a constant $C > 0$ independent of $q \to +\infty$, such that:

\[
\|e^{\tilde{u}}\|_{H^1 \cap L^\infty} + \|v\|_{H^1 \cap L^\infty} + \|q(v - f(e^{\tilde{u}}))\|_{L^2} \leq C. \tag{12}
\]

We shall first obtain some pointwise estimates, which depend on the increasing monotonicity of $f$ in an essential way:

**Lemma 1.1.** The following estimates hold, pointwise on $\Sigma$:

(i) $f(0) \leq f(e^{\tilde{u}}) \leq s$

(ii) $f(0) \leq v \leq s$.

**Corollary 1.1.** If $n = 0$, then $(e^{\tilde{u}}, v) = (f^{-1}(s), s)$.

**Proof.** Suppose $n = 0$. Integrating (6) and (7) we find that

\[
q \int (v - f(e^u)) = 0 = \int f'(e^u)e^u(s - v).
\]

By Lemma 1.1–(ii) we have $v \leq s$. Since $f'(e^u)e^u > 0$, the above identity implies $v \equiv s$. Then $\Delta v \equiv 0$ and thus (7) implies $q(s - f(e^u)) \equiv 0$, that is, $f(e^u) \equiv s$, as asserted.

As a consequence of Lemma 1.1, the nonlinearity $f$ may be truncated. Therefore in what follows, without loss of generality, we assume that:

\[
\sup_{t > 0} \{ |f(t)| + |f'(t)| + |f''(t)| \} \leq C. \tag{13}
\]
Proof of Lemma 1.1. Let \( \bar{x} \in \Sigma \) be such that \( \bar{u}(\bar{x}) = \max_{\Sigma} \bar{u} \). Then \( \varpi \neq p_j \) for all \( j = 1, \ldots, n \) and (6) implies that
\[
f(e^{\bar{u}(\bar{x})}) \leq v(\bar{x}).
\]
Now we equivalently rewrite equation (7) in the form:
\[
-\Delta v + q^2 \left( 1 + \frac{1}{q} f'(e^{\bar{u}}) e^{\bar{u}} \right) v = q^2 \left( f(e^{\bar{u}}) + \frac{s}{q} f'(e^{\bar{u}}) e^{\bar{u}} \right).
\]
Let \( \varpi, y \in \Sigma \) such that \( v(\varpi) = \max_{\Sigma} v, \, v(y) = \min_{\Sigma} v \). Then, the maximum principle applied to (14) implies that
\[
\frac{f(e^{\tilde{u}(\varpi)}) + \frac{s}{q} f'(e^{\tilde{u}(\varpi)}) e^{\tilde{u}(\varpi)}}{1 + \frac{1}{q} f'(e^{\tilde{u}(\varpi)}) e^{\tilde{u}(\varpi)}} \leq v \leq \frac{f(e^{\tilde{u}(\varpi)}) + \frac{s}{q} f'(e^{\tilde{u}(\varpi)}) e^{\tilde{u}(\varpi)}}{1 + \frac{1}{q} f'(e^{\tilde{u}(\varpi)}) e^{\tilde{u}(\varpi)}},
\]
pointwise on \( \Sigma \). If \( \varpi = p_j \) for some \( j = 1, \ldots, n \), then \( e^{\tilde{u}(\varpi)} = 0 \) and therefore the second inequality in (15) implies: \( v(\varpi) \leq f(0) \). Since \( f(0) < s \) by assumption, the second part of (ii) is established in this case. (In fact, we can show that \( \varpi \neq p_j \), for all \( j = 1, \ldots, n \), see Remark 1.1 below). If \( \varpi \neq p_j \) for all \( j = 1, \ldots, n \), then we observe that by increasing monotonicity of \( f \) we have:
\[
f(e^{\tilde{u}(\varpi)}) \leq f(e^{\bar{u}(\bar{x})}) \leq v(\bar{x}) \leq v(\varpi).
\]
Inserting into the second inequality in (15), we derive:
\[
\left( 1 + \frac{1}{q} f'(e^{\tilde{u}(\varpi)}) e^{\tilde{u}(\varpi)} \right) v(\varpi) \leq v(\varpi) + \frac{s}{q} f'(e^{\tilde{u}(\varpi)}) e^{\tilde{u}(\varpi)},
\]
which, recalling that \( f' > 0 \), in turn yields:
\[
\tilde{u}(\varpi) v(\varpi) \leq s e^{\tilde{u}(\varpi)},
\]
with \( e^{\tilde{u}(\varpi)} > 0 \). Hence (i) and the second part of (ii) follow. It remains to show that \( v \geq f(0) \). By (i), we know that \( s - f(e^{\tilde{u}(\varpi)}) \geq 0 \). Therefore, by the increasing monotonicity of \( f \):
\[
\frac{s - f(e^{\tilde{u}(\varpi)})}{1 + \frac{1}{q} f'(e^{\tilde{u}(\varpi)}) e^{\tilde{u}(\varpi)}} \leq s - f(e^{\tilde{u}(\varpi)}) \leq s - f(0).
\]
Consequently, combining the first inequality in (15) and (17), we obtain:
\[
v(\varpi) \geq \frac{f(e^{\tilde{u}(\varpi)}) + \frac{s}{q} f'(e^{\tilde{u}(\varpi)}) e^{\tilde{u}(\varpi)}}{1 + \frac{1}{q} f'(e^{\tilde{u}(\varpi)}) e^{\tilde{u}(\varpi)}} = s - \frac{s - f(e^{\tilde{u}(\varpi)})}{1 + \frac{1}{q} f'(e^{\tilde{u}(\varpi)}) e^{\tilde{u}(\varpi)}} \geq f(0),
\]
and the proof of (ii) is complete.

As already mentioned in the proof of Lemma 1.1, we can actually show that \( v \) does not attain its maximum at \( p_j, j = 1, \ldots, n \):

**Remark 1.1.** If \( v \) is constant, then \( n = 0 \) and \( (e^{\bar{u}}, v) = (f(s), s) \). In particular, if \( n > 0 \), then \( v \) cannot be a constant. Furthermore, if \( n > 0 \), then \( v \) attains its maximum on \( \Sigma \setminus \{p_1, \ldots, p_n\} \).
Proof. Suppose \( v \equiv k = \text{constant} \). Then integrating (6) and (7) we obtain
\[
\int f'(e^\tilde{u}) e^\tilde{u} (s - k) = q \int (k - f(e^\tilde{u})) = 4\pi n
\]
and therefore
\[
k = s - \frac{4\pi n}{\int f'(e^\tilde{u}) e^\tilde{u}} \leq s.
\]
If \( k = s \), then \( n = 0 \) and by Corollary 1.1 we have \( k = s = f^{-1}(e^\tilde{u}) \). Thus, the statement of the lemma is established in this case.

Therefore we assume \( k < s \). In particular \( n > 0 \), and thus \( \tilde{u} \) is not constant. Furthermore, \( t = e^\tilde{u} \) attains values in \([0, \delta]\) for some \( \delta > 0 \). Setting \( \phi(\tilde{u}) = f(e^\tilde{u}) \), we have from (7) that \( \phi \) satisfies the differential equation
\[
(s - k)\phi' = q(k - \phi) \quad \tilde{u} \leq -M
\]
for some \( M > 0 \) and thus
\[
\phi(\tilde{u}) = Ce^{-q\tilde{u}/(s-k)} + k.
\]
Recalling the definition of \( \phi \), it follows that \( f \) has the form
\[
f(t) = Ct^{-q/(s-k)} + k,
\]
which is singular at \( t = 0 \), contradiction. Now, if \( v \) attains its maximum at some \( p_j \), then by (15) necessarily \( v \equiv f(0) \). Hence, \( n = 0 \).

The next estimate will be used to derive \( H^1 \)-bounds for \( e^\tilde{u} \) and \( v \):

Lemma 1.2. We have:
\[
\int e^\tilde{u} |\nabla \tilde{u}|^2 \leq C.
\]

Proof. Multiplying equation (6) by \( e^\tilde{u} \) and integrating by parts, we obtain
\[
q \int e^\tilde{u} (v - f(e^\tilde{u})) = \int e^\tilde{u} |\nabla \tilde{u}|^2 \geq 0.
\]
By the pointwise estimates in Lemma 1.1, it follows that:
\[
\frac{1}{q} \int e^\tilde{u} |\nabla \tilde{u}|^2 \leq C.
\]
Multiplying (7) by \( e^\tilde{u} \) and integrating, we find
\[
q \int e^\tilde{u} (v - f(e^\tilde{u})) = \int e^{2\tilde{u}} f'(e^\tilde{u})(s - v) + \frac{1}{q} \int e^\tilde{u} \Delta v.
\]
Integration by parts yields:
\[
\frac{1}{q} \int e^\tilde{u} \Delta v = -\int ve^\tilde{u} (v - f(e^\tilde{u})) + \frac{1}{q} \int ve^\tilde{u} |\nabla \tilde{u}|^2.
\]
Hence, by the pointwise estimates as in Lemma 1.1, and taking into account (18), we conclude that

\[
\frac{1}{q} \left| \int e^{\tilde{u}} \Delta v \right| \leq C.
\]

Inserting into (19), recalling Lemma 1.1, we derive that

\[
q \int e^{\tilde{u}} (v - f(e^{\tilde{u}})) \leq C,
\]

and thus it follows that

\[
\int e^{\tilde{u}} |\nabla \tilde{u}|^2 = q \int e^{\tilde{u}} (v - f(e^{\tilde{u}})) \leq C.
\]

The next identity is the main step in deriving the \(H^1\)-estimate for \(v\) and the \(L^2\)-estimate for \(q(v - f(e^{\tilde{u}}))\):

**Lemma 1.3.** The following identity holds:

\[
(20) \quad \int |\nabla v|^2 + q^2 \int (v - f(e^{\tilde{u}}))^2 = \int (s - v) \left( f''(e^{\tilde{u}}) e^{\tilde{u}} + f'(e^{\tilde{u}}) \right) e^{\tilde{u}} |\nabla \tilde{u}|^2.
\]

**Proof.** We compute:

\[
\Delta f(e^{\tilde{u}}) = \left( f''(e^{\tilde{u}}) e^{\tilde{u}} + f'(e^{\tilde{u}}) \right) e^{\tilde{u}} |\nabla \tilde{u}|^2 + f'(e^{\tilde{u}}) e^{\tilde{u}} \Delta \tilde{u}.
\]

Therefore \(f(e^{\tilde{u}})\) satisfies the equation:

\[
(21) \quad -\Delta f(e^{\tilde{u}}) + q f'(e^{\tilde{u}}) e^{\tilde{u}} f(e^{\tilde{u}}) = q f'(e^{\tilde{u}}) e^{\tilde{u}} v - \left( f''(e^{\tilde{u}}) e^{\tilde{u}} + f'(e^{\tilde{u}}) \right) e^{\tilde{u}} |\nabla \tilde{u}|^2.
\]

Integrating (21), we obtain

\[
(22) \quad q \int f'(e^{\tilde{u}}) e^{\tilde{u}} (v - f(e^{\tilde{u}})) = \int \left( f''(e^{\tilde{u}}) e^{\tilde{u}} + f'(e^{\tilde{u}}) \right) e^{\tilde{u}} |\nabla \tilde{u}|^2
\]

Now we multiply (7) by \(v - f(e^{\tilde{u}})\) and integrate to obtain:

\[
\int -\Delta v (v - f(e^{\tilde{u}})) = q \int f'(e^{\tilde{u}}) e^{\tilde{u}} (s - v) (v - f(e^{\tilde{u}})) - q^2 \int (v - f(e^{\tilde{u}}))^2.
\]

Integrating by parts and using (21) we find:

\[
\int -\Delta v (v - f(e^{\tilde{u}})) = \int |\nabla v|^2 + \int v \Delta f(e^{\tilde{u}})
\]

\[
= \int |\nabla v|^2 - q \int v f'(e^{\tilde{u}}) e^{\tilde{u}} (v - f(e^{\tilde{u}})) + \int v \left( f''(e^{\tilde{u}}) e^{\tilde{u}} + f'(e^{\tilde{u}}) \right) e^{\tilde{u}} |\nabla \tilde{u}|^2.
\]

Equating left hand sides in the last two identities, we obtain

\[
\int |\nabla v|^2 + q^2 \int (v - f(e^{\tilde{u}}))^2 + \int v \left( f''(e^{\tilde{u}}) e^{\tilde{u}} + f'(e^{\tilde{u}}) \right) e^{\tilde{u}} |\nabla \tilde{u}|^2
\]

\[
= sq \int f'(e^{\tilde{u}}) e^{\tilde{u}} (v - f(e^{\tilde{u}})),
\]

and thus identity (20) is established. \(\square\)
Now we can finally provide the proof of Proposition 1.1. Lemma 1.1 readily implies \( \|e^{\tilde{u}}\|_{L^\infty} \leq C \) and \( \|v\|_{L^\infty} \leq C \). In order to obtain the \( H^1 \)-estimate for \( e^{\tilde{u}} \), it suffices to observe that by Lemma 1.1–(i) and by Lemma 1.2 we have:

\[
\int |\nabla e^{\tilde{u}}|^2 = \int e^{2\tilde{u}}|\nabla \tilde{u}|^2 \leq C \int e^{\tilde{u}}|\nabla \tilde{u}|^2 \leq C.
\]

Therefore, we are left to estimate \( \|v\|_{L^2} \) and \( \|q(v - f(e^{\tilde{u}}))\|_{L^2} \). Using identity (20), we have:

\[
\int |\nabla v|^2 + q^2 \int (v - f(e^{\tilde{u}}))^2 \leq \|s - v\|_{L^\infty} \|f''(e^{\tilde{u}})\|_{L^\infty} \int e^{\tilde{u}}|\nabla \tilde{u}|^2 
\]

\[
\leq C \int e^{\tilde{u}}|\nabla \tilde{u}|^2 \leq C,
\]

where we again used Lemma 1.1 and Lemma 1.2 in order to derive the last step.

\[\square\]

2 Iteration

The aim of this section is to obtain bounds for solutions in \( H^k \), for every \( k \geq 0 \), as given in the following

Proposition 2.1. For all \( k \geq 0 \) there exists a constant \( C > 0 \) (possibly depending on \( k \)) such that:

\[
\|\tilde{u} - u_0\|_{H^k} + \|v\|_{H^k} \leq C.
\]

It will be convenient to define the Banach spaces \( X^0 := L^\infty \), \( X^k := H^k \cap L^\infty \) for \( k \geq 1 \), endowed with the norms \( \|\cdot\|_{X^k} := \|\cdot\|_{H^k} + \|\cdot\|_{L^\infty} \). We recall the well-known Sobolev-Gagliardo-Nirenberg inequality, see e.g. [8]:

\[
\|D^j u\|_{L^p(\mathbb{R}^n)} \leq C \|D^k u\|_{L^r(\mathbb{R}^n)} \|u\|_{L^s(\mathbb{R}^n)}^{1-a} \|u\|_{L^\infty}^a \forall u \in C_c^\infty(\mathbb{R}^n),
\]

where

\[
\frac{1}{p} = \frac{j}{k} + a \left( \frac{1}{r} - \frac{m}{n} \right) + (1-a) \frac{1}{q},
\]

\[
\frac{j}{k} \leq a \leq 1.
\]

Taking \( k = 2 \), \( a = j/k \), \( q = \infty \), and using partitions of unity on \( \Sigma \), we obtain:

(23) \[
\|D^j u\|_{L^{2j/k}} \leq C \|D^k u\|_{L^r} \|u\|_{L^\infty}^{1-j/k} \forall u \in C^\infty(\Sigma).
\]

By (23) and the Hölder inequality that if \( u_1, \ldots, u_t \in X^k \) and \( \beta_1, \ldots, \beta_t \) are multi-indices such that \( |\beta_1| + \cdots + |\beta_t| = k \), then the product \( D^{\beta_1} u_1 \cdots D^{\beta_t} u_t \in L^2 \) and

\[
\|D^{\beta_1} u_1 \cdots D^{\beta_t} u_t\|_{L^2} \leq C \|u_1\|_{X^k} \cdots \|u_t\|_{X^k}.
\]

In particular, \( X^k \) is a Banach algebra for every \( k \geq 0 \), i.e.,

\[
\|u_1 u_2\|_{X^k} \leq C \|u_1\|_{X^k} \|u_2\|_{X^k}.
\]

We shall need the following
Lemma 2.1. Let $F \in C^\infty(\Sigma \times \mathbb{R})$, $G \in C^\infty(\Sigma \times \mathbb{R} \times \mathbb{R}^2)$. Then for all $k \geq 0$ there exists constants $C_k = C_k(||u||_{L^\infty})$, $C'_k = C'_k(||u||_{L^\infty}, ||v||_{L^\infty}, ||\nabla u||_{L^\infty})$, such that:

$$
\|F(x,u)\|_{X^k} \leq C_k(1 + \|u\|_{X^k}^k)
$$

$$
\|G(x,u,v,\nabla u)\|_{X^{k-1}} \leq C'_k(1 + \|u\|_{X^k}^{k-1} + \|v\|_{X^{k-1}}).
$$

Proof. Denote by $\alpha$ a multi-index such that $|\alpha| = k$. It suffices to observe that

$$
D^\alpha F(x,u) = \sum_{|\alpha_1| + \cdots + |\alpha_k| = |\alpha|} F^{(b)}(x,u) D^{\alpha_1} u \cdots D^{\alpha_k} u
$$

and therefore, recalling (23), we have

$$
\|D^\alpha F(x,u)\|_{L^2} \leq C(||u||_{L^\infty}) \sum_{|\alpha_1| + \cdots + |\alpha_k| = |\alpha|} \|D^{\alpha_1} u \cdots D^{\alpha_k} u\|_{L^2}
$$

$$
\leq C(||u||_{L^\infty}) \sum_{|\alpha_1| + \cdots + |\alpha_k| = |\alpha|} \|D^{\alpha_1} u\|_{L^{2(|\alpha_1|/|\alpha_1|)} \cdots \|D^{\alpha_k} u\|_{L^{2(|\alpha_k|/|\alpha_k|)}}
$$

$$
\leq C(||u||_{L^\infty})(1 + \|u\|_{X^k}).
$$

Since obviously $\|F(x,u)\|_{L^\infty} \leq C(||u||_{L^\infty})$, the first estimate is established. The estimate for $G(x,u,v,\nabla u)$ is obtained analogously. \(\square\)

Now we observe that (10) is of the form:

$$
-\Delta u + q^2(1 + \frac{1}{q})u = q^2 f.
$$

We shall need some a priori estimates for solutions to (24). The next two results state that, under suitable assumptions, a solution $u$ for (24) satisfies the same regularity properties as the right hand side $f$, independently of $q \to +\infty$.

Lemma 2.2. Suppose $u$ is a solution for (24) with $c \in L^\infty$ and $f \in L^p$ for some $1 \leq p \leq +\infty$. Then there exist $q_k > 0$ and $C > 0$ independent of $u$ such that

$$
\|u\|_{L^p} \leq C||f||_{L^p},
$$

for all $q \geq q_k$.

Proof. For $p = +\infty$, the statement follows by the maximum principle:

$$
\|u\|_{L^\infty} \leq \left\| \frac{f}{1 + \frac{1}{q}} \right\|_{L^\infty} \leq \frac{||f||_{L^\infty}}{1 - \frac{1}{q} ||c||_{L^\infty}},
$$

hence for large $q$ we find:

$$
\|u\|_{L^\infty} \leq C||f||_{L^\infty}.
$$

Now we assume $2 \leq p < +\infty$. Multiplying (24) by $|u|^{p-2}u$ and integrating by parts, we find

$$
(p - 1) \int |u|^{p-2} |\nabla u|^2 + \int (1 + \frac{1}{q} c)|u|^p = \int f|u|^{p-2} u
$$
It follows that
\[ \|u\|_{L^p}^p \leq \frac{\int |f||u|^{p-1}}{1 - \frac{1}{q} \|c\|_{L^\infty}} \]
and therefore, by the Hölder inequality,
\[ \|u\|_{L^p} \leq \frac{\|f\|_{L^p}}{1 - \frac{1}{q} \|c\|_{L^\infty}}, \]
and hence the asserted estimate is established in the case \(2 \leq p \leq +\infty\). In the remaining case \(1 \leq p < 2\), we proceed by duality. Let \(\varphi\) be defined by
\[ -\Delta \varphi + (1 + \frac{1}{q} c) \varphi = |u|^{p-2} u. \]
Then (25) with \(p \geq 2\) yields \(\|\varphi\|_{L^{p'}} \leq C \|u\|_{L^p}^{p-1}\). Multiplying (24) by \(\varphi\) and integrating, we find:
\[ \int |u|^p = \int -\Delta \varphi + \int (1 + \frac{1}{q} c) u \varphi = \int f \varphi. \]
Consequently,
\[ \int |u|^p \leq \|f\|_{L^p} \|\varphi\|_{L^{p'}} \leq C \|f\|_{L^p} \|u\|_{L^p}^{p-1}, \]
and the asserted estimate is established also for \(1 \leq p < 2\).

**Lemma 2.3.** Let \(c, f \in X^k\) and suppose that \(u\) satisfies: (24). For every \(k \geq 0\) there exist \(q_k > 0\), \(C_k > 0\) such that
\[ \|u\|_{X^k} \leq C_k \|f\|_{X^k}, \]
for all \(q \geq q_k\).

**Proof.** Denote by \(\alpha\) a multi-index, \(|\alpha| = k\). Multiplying (24) by \(D^{2\alpha} u\) and integrating by parts, we obtain:
\[ \int |\nabla D^\alpha u|^2 + q^2 \int D^\alpha[(1 + \frac{1}{q} c) u] D^\alpha u = q^2 \int D^\alpha u D^\alpha f, \]
Therefore,
\[ \int D^\alpha[(1 + \frac{1}{q} c) u] D^\alpha u \leq \int D^\alpha f D^\alpha u, \]
and thus we estimate:
\[ \int (D^\alpha u)^2 \leq \int D^\alpha f D^\alpha u - \frac{1}{q} \int D^\alpha(c u) D^\alpha u \]
\[ \leq \|D^\alpha f\|_{L^2} \|D^\alpha u\|_{L^2} + \frac{1}{q} \|D^\alpha(c u)\|_{L^2} \|D^\alpha u\|_{L^2} \]
\[ \leq \|u\|_{X^k} \|f\|_{X^k} + \frac{1}{q} \|c u\|_{X^k} \|u\|_{X^k} \]
\[ \leq \|u\|_{X^k} \|f\|_{X^k} + \frac{1}{q} \|c\|_{X^k} \|u\|_{X^k}^2, \]
where we have used (23) to derive the last line. Since $\alpha$ is an arbitrary multi-index satisfying $|\alpha| = k$, we conclude from the above and (26) that

$$\|u\|_{X^k} \leq C(\|f\|_{X^k} + \frac{1}{q}\|c\|_{X^k}\|u\|_{X^k}).$$

Now the asserted estimate follows easily. $\square$

At this point, it is useful to note that $q(v - f(e^{u_0 + u}))$ also satisfies an equation of the form (24). In fact, it is convenient to set

$$w := q(v - f(e^{u_0 + u}))$$

and to consider $w$ as a third unknown function. Then the triple $(u, v, w)$ satisfies a system of the following simple form:

\begin{align*}
(27) & \quad -\Delta u = w - 4\pi n \\
(28) & \quad -\Delta v + q^2[1 + \frac{1}{q}c(x, u)]v = q^2 F_q(x, u) \\
(29) & \quad -\Delta w + q^2[1 + \frac{1}{q}c(x, u)]w = q^2 G_q(x, u, v, \nabla u)
\end{align*}

where

\begin{align*}
c(x, u) &= f'(e^{u_0 + u})e^{u_0 + u} \\
F_q(x, u) &= f(e^{u_0 + u}) + \frac{s}{q}f'(e^{u_0 + u})e^{u_0 + u} \\
G_q(x, u, v, \nabla u) &= f'(e^{u_0 + u})e^{u_0 + u}(s - v) \\
&\quad + \frac{1}{q}(f''(e^{u_0 + u})e^{u_0 + u} + f'(e^{u_0 + u}))e^{u_0 + u}|\nabla(u_0 + u)|^2.
\end{align*}

Proposition 2.1 will follow by a bootstrap argument applied to (27)–(28)–(29).

In order to start the procedure, we need:

**Lemma 2.4.** The following estimates hold:

\begin{enumerate}
\item \(\|u\|_{X^1} + \|v\|_{X^1} + \|w\|_{X^0} \leq C\)  
\item \(\|u\|_{L^\infty} \leq C\)  
\item \(\|\nabla u\|_{L^\infty} \leq C\)
\end{enumerate}

**Proof.** Proof of (i). Multiplying (9) by \(u - \int u\) and integrating, we have:

$$\int |\nabla u|^2 = q \int (v - f(e^{\tilde{u}}))(u - \int u)$$

$$\leq \|q(v - f(e^{\tilde{u}}))\|_2 \|u - \int u\|_2 \leq C\|\nabla u\|_2,$$

where the last inequality follows by Lemma 1.1 and by the Poincaré inequality. Hence \(\|\nabla u\|_2 \leq C\). By Lemma 1.1–(ii), we have that \(e^{\tilde{u}} \leq C\), and thus we only have to show that \(\int u \geq -C\). To this end, we first observe that integrating (9) and (10) we obtain:

$$\int f'(e^{u_0 + u})e^{u_0 + u}(s - v) = q \int (v - f(e^{u_0 + u})) = 4\pi n.$$
On the other hand, we have in a straightforward manner:

\[
\int f'(e^{u_0+u})e^{u_0+u}(s-v) \leq C \int e^{u_0+u} \leq Ce^f u \|e^{u_0}\|_\infty \int e^{u-f} u \leq C \int e^{u-f} u.
\]

Hence, recalling the Moser-Trudinger inequality (see [1]) and the estimate for \(\|\nabla u\|_2\), we conclude that

\[
4\pi n \leq Ce^f u \int e^{u-f} u \leq Ce^f u e^{\gamma/|\nabla u|^2} \leq Ce^f u,
\]

which establishes (i). Proof of (ii). Since \(\|w\|_{L^2} \leq C\), by (i) and elliptic regularity we obtain \(\|u\|_{H^2} \leq C\). Then Sobolev embeddings yield \(\|\nabla u\|_{L^p} \leq C\), for any \(1 \leq p < +\infty\) and \(\|u\|_{L^\infty} \leq C\), which establishes (ii). Proof of (iii). By (29), \(\|\nabla u\|_{L^p} \leq C\) and Lemma 2.2 imply that \(\|u\|_{L^p} \leq C\), for any \(1 \leq p < +\infty\). Then (27) and Sobolev embeddings yield \(\|u\|_{W^{2,p}} \leq C\), for any \(1 \leq p < +\infty\). For \(p > 2\), the Sobolev embeddings yield (iii).

Now we can provide the

Proof of Proposition 2.1. We argue by induction on \(k \in \mathbb{N}_0\).

**CLAIM A:** There holds:

\[
\|u\|_{X^1} + \|v\|_{X^1} + \|w\|_{X^0} \leq C.
\]

The above follows by Proposition 1.1 and by Lemma 2.4.

**CLAIM B:** Suppose:

\[
\|u\|_{X^k} + \|v\|_{X^k} + \|w\|_{X^{k-1}} \leq C_k.
\]

Then:

\[
\|u\|_{X^{k+1}} + \|v\|_{X^{k+1}} + \|w\|_{X^k} \leq C_{k+1}.
\]

Indeed,

\[
\|w\|_{X^{k-1}} \leq C \Rightarrow \|u\|_{X^{k+1}} \leq C \quad \text{by (27) and standard elliptic regularity}
\]

\[
\Rightarrow \|v\|_{X^{k+1}} \leq C \quad \text{by (28), Lemma 2.1 and Lemma 2.3}
\]

\[
\Rightarrow \|w\|_{X^k} \leq C \quad \text{by (29), Lemma 2.1 and Lemma 2.3}.
\]

Now Claim A, Claim B and a standard induction argument conclude the proof.

Proof of Theorem 0.1. Let \((u, v)\) be solutions to system (9)–(10), with \(q \to +\infty\). By the a priori estimates as stated in Proposition 2.1 and by standard compactness arguments, there exist \(u_\infty, v_\infty\) such that up to subsequences \(u \to u_\infty\) and \(v \to v_\infty\) in \(C^h\), for all \(h \geq 0\). We write (9) in the form:

\[
v = f(e^{u_0+u}) + \frac{1}{q}(-\Delta u + 4\pi n).
\]
Taking limits, we find \( v_\infty = f(e^{u_0 + u_\infty}) \). Furthermore, taking limits in (10), we obtain
\[
q(v - f(e^{u_0 + u_\infty})) \rightarrow f'(e^{u_0 + u_\infty})e^{u_0 + u_\infty}(s - f(e^{u_0 + u_\infty})),
\]
where the convergence holds in \( C^h \), for any \( h \geq 0 \). Consequently, taking limits in (9), we find that \( u_\infty \) satisfies:
\[
-\Delta u_\infty = f'(e^{u_0 + u_\infty})e^{u_0 + u_\infty}(s - f(e^{u_0 + u_\infty})) - 4\pi \sum_{j=1}^{n} \delta_{p_j}.
\]
Setting \( \tilde{u}_\infty = u_0 + u_\infty \), we conclude the proof of Theorem 0.1.

Acknowledgements

I am grateful to Professors Danielle Hilhorst, Masayasu Mimura and Gabriella Tarantello for interesting and stimulating discussions.

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