STRONG BIRKHOFF ERGODIC THEOREM FOR SUBHARMONIC FUNCTIONS WITH IRRA TIONAL SHIFT AND ITS APPLICA TION TO ANALYTIC QUASI-PERIODIC COCYCLES

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Abstract. In this paper, we first prove the strong Birkhoff Ergodic Theorem for subharmonic functions with the irrational shift on the Torus. Then, it is applied to the analytic quasi-periodic Jacobi cocycles. We show that if the Lyapunov exponent of these cocycles is positive at one point, then it is positive on an interval centered at this point for suitable frequency and coupling numbers. We also prove that the Lyapunov exponent is Hölder continuous in $E$ on this interval and calculate the expression of its length. What’s more, if the coupling number of the potential is large, then the Lyapunov exponent is always positive for all irrational frequencies and Hölder continuous in $E$ for all finite Liouville frequencies. We also study the Lyapunov exponent of the Schrödinger cocycles, a special case of the Jacobi ones, and obtain its Hölder continuity in the frequency.

1. Introduction

By the Birkhoff Ergodic Theorem, if $T : X \to X$ is an ergodic transformation on a measurable space $(X, \Sigma, m)$ and $f$ is an $m$-integrable function, then the time average functions $f_n(x) = \frac{1}{n} \sum_{k=0}^{n-1} f(T^k x)$ converge to the space average $\langle f \rangle = \frac{1}{m(X)} \int_X f \, dm$ for almost every $x \in X$. But it doesn’t tell us how fast do they converge? So, we call a theorem the strong Birkhoff Ergodic Theorem, if it gives the convergence rate.

In this paper, we consider the strong Birkhoff Ergodic Theorem for subharmonic functions under the condition that the ergodic transformation is a shift on the Torus, i.e. $T : x \to x + \omega, \forall x \in T := [0,1]$. Specifically, assume that $u : \Omega \to \mathbb{R}$ is a subharmonic function on a domain $\Omega \subset \mathbb{C}$, $\partial \Omega$ consists of finitely many piece-wise $C^1$ curves and $T \subseteq \Omega$. Then, the Reisz Decomposition Theorem tells us that there exists a positive measure $\mu$ on $\Omega$ such that for any $\Omega_1 \supseteq \Omega_1$ (i.e., $\Omega_1$ is a compactly contained subregion of $\Omega$),

$$u(z) = \int_{\Omega_1} \log |z - \zeta| \, d\mu(\zeta) + h(z),$$

where $h$ is harmonic on $\Omega_1$ and $\mu$ is unique with this property.

In order to formulate our theorem, some notations about the shift $\omega$, which is always irrational in this paper, should be introduced. For any irrational $\omega$, there exist its continued fraction approximants $(\frac{p_s}{q_s})_{s=1}^{\infty}$, satisfying

$$\frac{1}{q_s(q_{s+1}+q_s)} < |\omega - \frac{p_s}{q_s}| < \frac{1}{q_s q_{s+1}}.$$

Define $\beta$ as the exponential growth exponent of $(\frac{p_s}{q_s})_{s=1}^{\infty}$ as follows:

$$\beta(\omega) := \limsup_s \frac{\log q_{s+1}}{q_s} \in [0, \infty].$$

Thus, if $\omega$ is a finite Liouville frequency, which means $\beta(\omega) < \infty$, then for any $\kappa > 0$, there exists $s_0 = s_0(\omega, \kappa) \geq 0$ such that for any $s \geq s_0$, $\log q_{s+1} \leq (\beta + \kappa)q_s$. Therefore, there exists a constant $\beta(\omega) < \infty$ such that for any $s \geq 0$,

$$\log q_{s+1} \leq \beta q_s.$$

Then, our strong Birkhoff Ergodic Theorem for irrational $\omega$ is as follows:

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Theorem 1. Let $u : \Omega \to \mathbb{R}$ be a subharmonic function on a domain $\Omega \subset \mathbb{C}$. Suppose that $\partial\Omega$ consists of finitely many piece-wise $C^1$ curves and $\mathbb{T}$ is contained in $\Omega \setminus \partial\Omega$. There exist a constant $C = C(\Omega_1)$ and an absolute constant $c$ such that for any $\delta > 0$ and irrational $\omega$, if $\beta(\omega) < \frac{\delta}{\mu(\Omega_1)}$, then for any positive $n$,

$$\text{mes}\left\{ x \in \mathbb{T} : \left| \frac{1}{n} \sum_{k=1}^{n} u(x + k\omega) - < u > \right| > \delta \right\} < \exp(-\frac{c}{\mu(\Omega_1)} \delta n),$$

where $\mu$ is the unique measure defined in (1.1).

Remark 1.1. It is obvious that $\beta(\omega) \geq \overline{\beta}(\omega)$. If we replace the assumption $\beta(\omega) < \frac{\delta}{\mu(\Omega_1)}$ by $\overline{\beta}(\omega) < \frac{\delta}{\mu(\Omega_1)} - \kappa$, then Theorem 1 still holds. But the absolute constant $c$ will depend on $\omega$. See details in Remark 2.5.

Remark 1.2. In [GS], Goldstein and Schlag proved that for any strong Diophantine $\omega$, which satisfies the strong Diophantine condition

$$\|n\omega\| \geq \frac{C_\omega}{n(\log n)^\nu} \quad \text{for all} \quad n \neq 0,$$

(1.3) holds when $\delta > \frac{(\log n)^{\nu+2}}{n}$. Obviously, this $\omega$ satisfies $\overline{\beta}(\omega) = 0$. Thus, we extend their conclusion to the finite Liouville frequency. What’s more, [ALSZ] shows that our result is also optimal, as (1.3) can not hold when $\overline{\beta}(\omega) = \infty$.

In this paper, we apply Theorem 1 to the following quasi-periodic analytic Jacobi operators $H_{x,\omega}$ on $\ell^2(\mathbb{Z})$:

$$(H_{x,\omega}\phi)(n) = -\lambda_xa(x + (n + 1)\omega)\phi(n + 1) - \lambda_xa(x + n\omega)\phi(n) + \lambda_xv(x + n\omega)\phi(n), \quad n \in \mathbb{Z},$$

where $v : \mathbb{T} \to \mathbb{R}$ is a real analytic function called potential, $a : \mathbb{T} \to \mathbb{C}$ is a complex analytic function and not identically zero, and $\lambda_x$ and $\lambda_v$ are real positive constants called coupling numbers. Then, their characteristic equations $H_{x,\omega}\phi = E\phi$ can be expressed as

$$\begin{pmatrix} \phi(n + 1) \\ \phi(n) \end{pmatrix} = \begin{pmatrix} 1 \\ \lambda_xa(x + (n + 1)\omega) \end{pmatrix} \begin{pmatrix} \lambda_xv(x + n\omega) - E & -\lambda_xa(x + n\omega) \\ \lambda_xa(x + (n + 1)\omega) & 0 \end{pmatrix} \begin{pmatrix} \phi(n) \\ \phi(n - 1) \end{pmatrix}.$$

Define

$$M(x, E, \omega) := \frac{1}{\lambda_xa(x + \omega)} \begin{pmatrix} \lambda_xv(x) - E & -\lambda_xa(x) \\ \lambda_xa(x + \omega) & 0 \end{pmatrix}$$

and call a map

$$(\omega, M) : (x, v) \mapsto (x + \omega, M(x)v)$$

a Jacobi cocycle. Due to the fact that an analytic function only has finite zeros, $M(x, E, \omega)$ and the $n$-step transfer matrix $M_n(x, E, \omega) := \prod_{k=0}^{n-1} M(x + k\omega, E)$ make sense almost everywhere. By the Kingman’s subadditive ergodic theorem, the Lyapunov exponent

$$L(E, \omega) = \lim_{n \to \infty} L_n(E, \omega) = \inf_{n \to \infty} L_n(E, \omega)$$

always exists, where

$$L_n(E, \omega) = \frac{1}{n} \int_{\mathbb{T}} \log \|M_n(x, E, \omega)\| dx.$$ 

To apply Theorem 1 and the Avalanche Principle (Proposition 5.1), we need to consider the following two matrices associated with $M_n$:

$$M'_n(x, E, \omega) = \prod_{j=1}^{n} \lambda_xa(x + j\omega) M_n(x, E, \omega) \quad \text{and} \quad M''_n(x, E, \omega) = \frac{M_n(x, E, \omega)}{|\det M_n(x, E, \omega)|^2}.$$
Note that an analytic function \( f(x) \) on \( \mathbb{T} \) has its complex analytic extension \( f(z) \) on the complex strip \( \mathbb{T}_\rho = \{ z : \text{Im} z < \rho \} \) and the complex analytic extension of \( \tilde{a}(x) \) should be defined on \( \mathbb{T}_\rho \) by

\[
\tilde{a}(z) := a\left(\frac{1}{z}\right).
\]

Then, the extensions of \( M(x, E, \omega), M_n(x, E, \omega), M_n^E(x, E, \omega) \) and \( M_n^E(z, \omega, E) \) are

\[
M(z, E, \omega) = \frac{1}{\lambda_0(d(z) + \omega)} \begin{pmatrix} \lambda_0 v(z) - E & -\lambda_0 \tilde{a}(z) \\ \lambda_0 a(z + \omega) & 0 \end{pmatrix}, \quad M_n(z, E, \omega) = \prod_{k=0}^{n-1} M(z + k\omega, E),
\]

(1.9)

\[
M_n^E(z, E, \omega) = \prod_{j=1}^{n} \lambda_0 a(z + j\omega) M_n(z, E, \omega) \quad \text{and} \quad M_n^E(z, \omega, E) = \frac{M_n(z, E, \omega)}{|\det M_n(z, E, \omega)|^2}.
\]

With fixed \( \omega \) and \( E \),

\[
u_n^E(z, E, \omega) = \frac{1}{n} \log ||M_n^E(z, E, \omega)||
\]

is a subharmonic function on \( \mathbb{T}_\rho \) such that Theorem 1 can be applied. We also consider the quantities \( u_n(z, E, \omega), u_n^E(z, E, \omega), L^n(E, \omega), L_n^E(E, \omega), L^n(E, \omega) \) and \( L_n^E(E, \omega) \) which are defined analogously. Based on the definitions, it is straightforward to check that

\[
\log ||M_n^E(z, E, \omega)|| = -\frac{1}{2} \sum_{j=0}^{n-1} d(z + j\omega, \omega) + \log ||M_n^E(z, E, \omega)||,
\]

where

\[
d(z, \omega) = \log |\lambda_0 a(z + \omega)\tilde{a}(z)|.
\]

It is also easily seen that \( L_n^E(\omega) = L_n(\omega, E) \geq 0 \), \( L^n(E, \omega) = L(E, \omega) \geq 0 \) and

\[
L(E, \omega) = L^n(E, \omega) - D,
\]

where

\[
D = \int_{\mathbb{T}} \log |\lambda_0 a(x)| dx = \int_{\mathbb{T}} \log |\lambda_0 \tilde{a}(x)| dx = \frac{1}{2} \int_{\mathbb{T}} d(x, \omega) dx.
\]

It is well known that \( L(E, \omega) \) is a \( C^\infty \) function of \( E \) on the resolvent set. So we only need to consider \( E \in \mathcal{E} \), where

\[
\mathcal{E} := [-2\lambda_0 ||a(x)||_{L^\infty(\mathbb{T})} - \lambda_0 ||v(x)||_{L^\infty(\mathbb{T})}, 2\lambda_0 ||a(x)||_{L^\infty(\mathbb{T})} + \lambda_0 ||v(x)||_{L^\infty(\mathbb{T})}].
\]

Simple computations yield that for any irrational \( \omega \) and \( 1 \leq n \in \mathbb{N} \),

\[
\sup_{E \in \mathcal{E}, x \in \mathbb{T}} \nu_n^E(x, E, \omega) \leq M_0,
\]

where

\[
M_0 := \log (3\lambda_0 ||a||_{L^\infty(\mathbb{T})} + 2\lambda_0 ||v||_{L^\infty(\mathbb{T})}) .
\]

If \( a(x) \equiv 1 \), then the Hamiltonian operators (1.5) become the following famous quasi-periodic analytic Schrödinger operators \( S_{x,\omega} \) on \( L^2(\mathbb{Z}) \):

\[
(S_{x,\omega} \phi)(n) = \phi(n + 1) + \phi(n - 1) + \lambda_0 v(x + n\omega) \phi(n), \quad n \in \mathbb{Z}.
\]

Then the n-step transfer matrix \( M_n^E(x, E, \omega) = \prod_{k=0}^{n-1} M^E(x + k\omega, E) \), where

\[
M^E(x, E, \omega) = \begin{pmatrix} \lambda_0 v(x) - E & -1 \\ 1 & 0 \end{pmatrix}.
\]
has the complex analytic extension $M^n_\omega(z, E, \omega)$ on $\mathbb{T}_p$, which is analytic and unimodular such that Theorem 1 and the Avalanche Principle can be applied directly. In 2001, Goldstein and Schlag([GS]) obtained the conclusion which we state in Remark 1.2 to prove the Lyapunov exponent of the Schrödinger operators

$$L'(E, \omega) := \lim_{n\to\infty} L'_n(E, \omega) = \lim_{n\to\infty} \frac{1}{n} \int \log \|M^n_\omega(x, E, \omega)\|dx$$

is Hölder continuous in $E$ for the strong Diophantine $\omega$. In that reference, they showed that the keys to prove the continuity of the Lyapunov exponent are two lemmas: the large deviation theorem (LDT for short) and the Avalanche Principle. We say an estimation is a LDT for continuity of the Lyapunov exponent are two lemmas: the large deviation theorem (LDT for short) and the Avalanche Principle. 

We say an estimation is a LDT for

$$u'_n(z, E, \omega) = \frac{1}{n} \log \|M^n_\omega(z, E, \omega)\|$$

if it satisfies

$$\text{mes}\{x : |u'_n(x, E, \omega) - L'_n(E, \omega)| > \delta\} < f(\delta, n).$$

They proved that

$$f(\delta, n) = \exp\left(-c\delta^2n\right)$$

for strong Diophantine $\omega$, where $c$ depends only on the potential $v(x)$. In 2002 Bourgain and Jitomirskaya showed in [BJ] that for any irrational $\omega$,

$$f(\delta, n) = \exp(-c\delta q),$$

where $0 < \delta < 1$ and $q$ is the denominator of $\omega$’s continued fraction approximants $\{\frac{p_n}{q_n}\}_{n=1}^{\infty}$ satisfying $q \leq n$. With the help of (1.18), they proved the joint continuity of $L'(E, \omega)$ in $(E, \omega)$ at every $(E, \omega_0)$ if $\omega_0$ is irrational. On the other hand, people always think that the Hölder continuity of $L'(E, \omega)$ in $E$ cannot hold when $\beta(\omega) \geq \log \lambda$, (Recently, Avila et al. declared the proof in the preparation reference [ALSZ]). So the question whether the Hölder continuity is true or not in the large coupling regime for the finite Liouville frequency $\omega$ is eagerly anticipated in our field. The first answer was [YZ] in 2014. They proved that there exist constants $\lambda_0$, $N_0$ which only depend on $\nu$ and small absolute constants $\epsilon'$ and $\epsilon''$ such that if $\beta(\omega) < \epsilon'$ and $\lambda_0 > \lambda_0$, then for any $n > N_0$

$$f\left(\frac{1}{100} \log \lambda_0, n\right) = \exp(-c''n)$$

and $L'(E, \omega)$ is Hölder continuous in $E$. Very recently, that question was finished by Han and Zhang in [HZ]. They proved that if the coupling number $\lambda_0$ is large and $\beta(\omega) \leq \log \lambda_0$, then the sharp LDT, which means

$$f(\delta, n) = \exp(-c\delta n),$$

and the Hölder continuity hold.

Compared to the Schrödinger cocycle, one of the distinguishing features of the Jacobi cocycle is that it is not $SL(2, \mathbb{C})$. It causes that Jitomirskaya and Marx proved the week Hölder continuity of the Lyapunov exponent of the analytic $GL(2, \mathbb{C})$ cocycles with Diophantine frequency in [JM]. But in [T], I showed that the continuity of the Lyapunov exponent of the Jacobi cocycles defined in (1.8) can be Hölder in $E$ with the strong Diophantine $\omega$. So far, from a technical perspective, the strong Birkhoff Ergodic Theorem is a necessary condition for applying the Avalanche Principle to the analytic quasi-periodic $GL(2, \mathbb{C})$ cocycles. The reason is that the Avalanche Principle can be applied only to the matrices whose determinants are not larger than 1.

In this paper, in order to extend the conclusion for the strong Diophantine frequency in [T] to the one for the finite Liouville frequency, we need to refer to a surprising discovery by Han and Zhang in [HZ] that the quantity $\mu(\Omega_0)$ in (1.1) for the subharmonic function $u'_n(z, E, \omega)$ defined in (1.16) does not depend on the large $\lambda_0$. With its help, we apply Theorem 1 to obtain the sharp LDT (1.19) and the Hölder continuity of $L(E, \omega)$ in $E$ for any finite Liouville $\omega$ when $\lambda_0$ and $|\log \lambda_0|$ are in the large regimes. We also improve the applications of the Avalanche Principle to get the lengths of the interval $I_{E_k}$ where $L(E, \omega)$ satisfies the Hölder condition for $E$. Moreover, if we consider the quasi-periodic analytic Schrödinger operators (1.14), then we can also obtain the Hölder continuity of the Lyapunov exponent in $\omega$. 

Now we begin to state the details of our conclusions. Choose
\begin{equation}
\Omega = \{ z : |\text{Re} z| < 1, |\text{Im} z| < \rho \}
\end{equation}
and
\begin{equation}
\Omega_1 = \{ z : |\text{Re} z| < \frac{2}{3}, |\text{Im} z| < \frac{\rho}{2} \}
\end{equation}
in Theorem 1. Then, the LDT for the Jacobi cocycles is as follows:

**Theorem 2.** There exist \( \lambda_0 = \lambda_0(v, \alpha, a) \) and \( c_{v,a} = c_{v,a}(v, a) \) such that for any \( \delta > 0 \), if \( \beta(v) < c_{v,a} \min(\delta, |D|) \) and \( \lambda_0 > \lambda_0 \), then
\begin{equation}
\text{mes} \{ x : |u_n(x, E, \omega) - L_{\omega}(E, \omega)| > \delta \} < \exp(-c_1 \delta^2 n) + \exp(-c_2 \delta n), \quad \forall n \geq n_1,
\end{equation}
where \( n_1 = n_1(\alpha, a, \lambda, v) \), \( c_1 = c_1(\alpha, a, \lambda, v) \) and \( c_2 = c_2(\alpha, a, \lambda, v) \).

**Remark 1.3.** We can give the expressions of \( \lambda_0 \) and \( c_{v,a} \) as follows:
\[
\lambda_0 := \max \left( \frac{\lambda_0}{|| v ||_{L^\infty(\Omega)}} \right) \quad \text{and} \quad c_{v,a} := \frac{1}{2CC(\Omega, \Omega_1)c_{v,a}},
\]
where \( e_0 \) defined in Lemma 3.1 depends only on \( v \), \( C \) is the constant from Theorem 1, \( C(\Omega, \Omega_1) \) is a constant which depends only on \( \Omega \) and \( \Omega_1 \), and \( c_{v,a} := \max \left( \log \frac{10||d||_{L^\infty(\Omega)}}{e_0}, \log \frac{10||d||_{L^\infty(\Omega)}}{e_0} \right) \).

**Remark 1.4.** Due to the fact that the Schrödinger operator is a special case of the Jacobi one, from now on to the end of this section, after every theorem for the Jacobi operators we will declare the corresponding conclusions for the Schrödinger ones in the next remark. The reason we do this is that, like the following LDT, we can get better results for the Schrödinger ones: There exist \( c_0 = c_0(v) := \frac{1}{\pi C_0} \) and \( \lambda_0' = \lambda_0'(v) := 2e^{-1} \) such that for any \( \delta > 0 \), if \( \beta(v) < c_0 \delta \) and \( \lambda_0 > \lambda_0' \), then for any positive \( n \),
\[
\text{mes} \{ x : |u_n(x, E, \omega) - L_n(E, \omega)| > \delta \} < \exp(-c_0 \delta^2 n),
\]
where \( c_0' = \tilde{c}_0(v, \lambda) := \frac{c_0'}{\text{sgn}(c_0)}, \quad C_0 := C(\Omega, \Omega_1) \log \frac{10||d||_{L^\infty(\Omega)}}{e_0}, \quad C \) and \( c \) are the absolute constants from Theorem 1, \( C(\Omega, \Omega_1) \) and \( e_0 \) are from Remark 1.3 and Lemma 3.1 respectively, and \( M_0' := \log (3 + 2\lambda_0 ||v||_{L^\infty(\Omega)}) \).

If the Lyapunov exponent \( L(E, \omega) \) is positive at one point \((E_0, \omega_0)\), then it is also positive on its neighborhood where we can have a better LDT, called the sharp one by Bourgain in [B]:

**Theorem 3.** Assume \( L(E_0, \omega_0) > 0 \). If \( \beta(\omega_0) < c_{v,a} \min (L(E_0, \omega_0), |D|) \) and \( \lambda_v > \lambda_0 \), where the constants \( c_{v,a} \) and \( \lambda_0 \) are defined in Theorem 2, then there exist \( r_E = r_E(\alpha, a, \lambda, v, E_0, \omega_0) \) and \( r_\omega = r_\omega(\alpha, a, \lambda, v, E_0, \omega_0) \) such that for any \( |E - E_0| \leq r_E \) and \( |\omega - \omega_0| \leq r_\omega \),
\[
\frac{3}{4} L(E_0, \omega_0) < L(E, \omega) < \frac{5}{4} L(E_0, \omega_0).
\]
Furthermore, if \( \beta(\omega) < \frac{1}{100}c_{v,a} L(E_0, \omega_0) \), then there exist \( \tilde{c}_{v,a} := \frac{c_{v,a}}{C(\Omega, \Omega_1)c_{v,a}} \) and \( \bar{\omega} = (\alpha, a, \lambda, v, E_0, \omega_0) \) such that
\begin{equation}
\text{mes} \{ x : |u_n(x, E, \omega) - L_n(E, \omega)| > \frac{1}{20} L(E, \omega) \} < \exp \left( -\frac{1}{12000} \tilde{c}_{v,a} L(E, \omega) n \right), \quad \forall n \geq \bar{\omega},
\end{equation}
where \( c \) is the absolute constant from Theorem 1, and \( C(\Omega, \Omega_1) \) and \( C_{v,a} \) are from Remark 1.3.

**Remark 1.5.** We apply the LDT (1.22) and the Avalanche Principle, not the continuity of Lyapunov exponent for the complex analytic cocycles with irrational \( \omega \) proved in [AJS], to prove that \( L(E, \omega_0) \) is positive on the interval \([E_0 - r_E, E_0 + r_E] \). The benefit is that we can calculate the expression of \( r_E \):
\begin{equation}
r_E = \frac{L(E_0, \omega_0)}{200 \bar{\omega}} \exp \left( (1 - \bar{\omega}) M_0 - 2\bar{\omega} |D| \right),
\end{equation}
where \( \bar{\omega} \), which appears in (1.23) and (1.24), is defined in (6.5). Due to the definition, it is easily seen that \( r_E \) is a continuous function in \( E_0 \). Thus, if \( L(E, \omega_0) \) is positive on \( \Sigma \times \{ \omega_0 \} \), then \( r_E := \inf_{E \in \Sigma} r_E \) exists and is positive.
Remark 1.6. We can not get the expression of $r_\omega$, as it comes from the compactness in $E$ and the joint continuity of $L(E, \omega)$, which was proved in [AJS].

Remark 1.7. The parameters $3/4$, $5/4$ and $1/20$ can be replaced in turn by $1 - k_1, 1 + k_2$ and $k_3$, where $0 < k_1, k_2, k_3 < 1$. Then, the new constants $c_3^\prime$ only differs from $c_3$ by a constant multiple of $400k^{-2}$, $r_\omega(k_1, k_2)$ and $r_\omega(k_1, k_2)$ depend on $k_1$ and $k_2$, and $\hat{h}_{k_1, k_2, k_3}$ depends on $k_1, k_2$ and $k_3$.

Remark 1.8. For the Schrödinger operators, we can calculate the expressions of $r_E^\prime$ and $r_\omega^\prime$: Assume $L^\prime(E_0, \omega_0) > 0$. If $\beta(\omega_0) < c_3 L^\prime(E_0, \omega_0)$ and $\lambda_0 > \lambda_0^\prime$, then exist $\tilde{h}$ defined in (6.12), $r_E^\prime = r_E^\prime(\lambda, v, E_0, \omega_0) := L^\prime(E_0, \omega_0) \exp(-LM_0^\prime \tilde{h})$, and $r_\omega^\prime = r_\omega^\prime(\lambda, v, E_0, \omega_0) := \frac{L^\prime(E_0, \omega_0)}{4 \max(c_\omega v(x)))} \exp(-5M_0^\prime \tilde{h})$, where $M_0^\prime$ comes from Remark 1.4, such that for any $|E - E_0| < r_E^\prime$ and $|\omega - \omega_0| \leq r_\omega^\prime$,

$$\frac{4}{5} L^\prime(E_0, \omega_0) < L^\prime(E, \omega) < \frac{6}{5} L^\prime(E_0, \omega_0).$$

Furthermore, if $\beta(\omega) < \frac{4}{100} c_3 L(E_0, \omega_0)$, then there exists $c_3^\prime := \frac{\tilde{c}_3}{4 \times 10^5}$ such that

$$\text{mes} \{ x : |\omega_0^\prime(x, E, \omega) - L_0^\prime(E, \omega)| > \frac{1}{20} L^\prime(E, \omega) < \exp(-c_3 L^\prime(E_0, \omega_0)) \} \forall n \geq \tilde{n}.$$ Here $c_3$ and $\tilde{c}_3$ are both from Remark 1.4.

Due to the positive Lyapunov exponent and the sharp LDT (1.23), the Avalanche Principle can be applied again to obtain the following Hölder continuity of Lyapunov exponent:

**Theorem 4.** Assume $L(E_0, \omega_0) > 0$, $\beta(\omega_0) < c_{v, \alpha} \min \left( \frac{L(E_0, \omega_0)}{15}, |D| \right)$ and $\lambda_0 > \lambda_0$, where the constants $c_{v, \alpha}$ and $\lambda_0$ are defined in Theorem 2. There exists $\tau = \tau(v, a) := \frac{\tilde{c}_3}{21 \times 10^5}$, where $\tilde{c}_3$ is from Theorem 3, such that for any $E_1, E_2 \in [E_0 - r_E, E_0 + r_E]$ and irrational $\omega \in [\omega_0 - r_\omega, \omega_0 + r_\omega]$ satisfying $\beta(\omega) < \frac{c_{v, \alpha} L(E_0, \omega_0)}{100}$, it has

$$|L(E_1, \omega) - L(E_2, \omega)| = |L^\prime(E_1, \omega) - L^\prime(E_2, \omega)| < (|E_1 - E_2|)^\tau.$$  

Remark 1.9. For the Schrödinger cocycles, we can prove the Hölder continuity in $\omega$ as follows: Assume $L^\prime(E_0, \omega_0) > 0$, $\beta(\omega_0) < \frac{c_3 L^\prime(E_0, \omega_0)}{15}$ and $\lambda_0 > \lambda_0^\prime$. There exists $\tau_\omega = \tau_\omega(v) := \frac{c_3}{21 \times 10^5}$ such that for any $E_1, E_2 \in [E_0 - r_E^\prime, E_0 + r_E^\prime]$ and $\omega_1, \omega_2 \in [\omega_0 - r_\omega^\prime, \omega_0 + r_\omega^\prime]$ satisfying $\max(\beta(\omega_1), \beta(\omega_2)) < \frac{c_3 L^\prime(E_0, \omega_0)}{100}$, it has

$$|L^\prime(E_1, \omega_1) - L^\prime(E_2, \omega_2)| \leq |E_1 - E_2|^{\tau_\omega} + |\omega_1 - \omega_2|^{\tau_\omega}.$$ Here the constants $c_\omega, \tilde{c}_\omega$ and $\lambda_0^\prime$ come from Remark 1.4.

It is well-known that Szers-Spencer [SS] gave a lower bound of the Lyapunov exponents of the Schrödinger operators in the large coupling regime. We prove that the similar result for the Jacobi ones also holds:

**Theorem 5.** For any $0 < \gamma < 1$, $E \in \mathcal{E}$ and irrational $\omega$, there exists $\lambda_\gamma = \lambda_\gamma(\lambda, a, v, \gamma)$ such that if $\lambda_\gamma > \lambda_\gamma$, then

$$L(E, \omega) > (1 - \gamma) \log \lambda_\gamma.$$ 

Remark 1.10. The setting of $\lambda_\gamma$ can be found in (3.7), which is a nondecreasing function of $\lambda_\gamma$. Moreover, if $\lambda_\gamma > \lambda_\gamma$, then

$$\left(1 - \frac{\gamma}{2}\right) \log \lambda_\gamma < L^\prime(E, \omega) < \left(1 + \frac{\gamma}{2}\right) \log \lambda_\gamma.$$  

Indeed, the proof of this theorem is obtained directly from the above two properties and (1.13).

Remark 1.11. Of course, we can also give the upper bound of $L(E, \omega)$, although it is useless in this paper. By (1.13), we need to show that $D > -\frac{\gamma}{2} \log \lambda_\gamma$. Define $D_\alpha := \exp \left( - \int_0^T \log |a(x)| dx \right)$. Then, if $\lambda_\gamma > \lambda_\gamma^\alpha D_\alpha^{-1}$, which is a decreasing function of $\lambda_\gamma$, then

$$L(E, \omega) < (1 + \gamma) \log \lambda_\gamma.$$
Thus, when \( \lambda_x > \lambda_p \), due to Theorem 4, we have that for any irrational \( \omega \), if \( \beta(\omega) < c_{r,a} \min \left( \frac{1}{\lambda_x} \log \lambda_x, |D| \right) \), then the Hölder continuity holds. Note that \( D < \frac{1}{\lambda} \log \lambda_x \) and \( L(E, \omega) > (1 - \gamma) \log \lambda_x \) at this time. Thus, if \( D \) is positive, then \( |D| < \frac{1}{\gamma} L(E, \omega) \) with \( \gamma = \frac{1}{\lambda} \). Conversely, if \( D \) is negative, then we need \( \beta(\omega) < -c_{r,a} D \) and \( \lambda_x > \max(\lambda_p, \lambda_{w}) \), where \( \lambda_{w} := \exp \left( \frac{15 \log(\omega)}{\log(1 - \gamma)} \right) \). Obviously, due to the definitions, \( \lambda_p \) is a decreasing function of \( \gamma \) with \( \lim_{\gamma \rightarrow 0} \lambda_p \rightarrow \infty \), and \( \lambda_{w} \) is an increasing function of \( \gamma \) with \( \lim_{\gamma \rightarrow 0} \lambda_{w} \rightarrow \infty \). Therefore, there exists \( 0 < \gamma_0 < 1 \) such that \( \lambda_p(\gamma_0) = \lambda_w(\gamma_0) \) and \( \max(\lambda_p(\gamma_0), \lambda_{w}(\gamma_0)) = \min(\lambda_p, \lambda_{w}(\gamma)) \) for any \( 0 < \gamma < 1 \).

In summary, we prove that for any finite Liouville \( \omega \), the Hölder continuity of Lyapunov exponent in \( E \) holds with suitable \( \lambda_x \) and \( \lambda_p \) as follows:

**Theorem 6.** For any finite Liouville \( \omega \), if \( \lambda_x > \frac{500}{\lambda} e^{\frac{50 \log(\omega)}{\log(1 - \gamma)}} \) and \( \lambda_p(\lambda_z, \alpha, v, \frac{1}{\lambda_x}) \), or \( \lambda_x > \frac{500}{\lambda} e^{\frac{50 \log(\omega)}{\log(1 - \gamma)}} \) and \( \lambda_p \), then for any \( |E_1 - E_2| < r_x \),

\[
|L(E_1, \omega) - L(E_2, \omega)| = |L^2(E_1, \omega) - L^2(E_2, \omega)| = (|E_1 - E_2|)^\gamma,
\]

where \( r_x \) comes from Remark 1.5, \( c_{r,a} \) is from Remark 1.3 and \( \tau \) is defined in Theorem 4.

**Remark 1.12.** For the Schrödinger operators in the large coupling regime, \( r_x^r \) and \( r_x^a \) only depend on \( \lambda_x \): For any finite Liouville \( \omega \), if \( \lambda_x > \max \left( \frac{200 \log(\omega)}{\log(1 - \gamma)} \right)^5, \exp \left( \frac{16 \beta(\omega)}{5 \log(1 - \gamma)} \right), 5 \max_{x \in \mathbb{Z}} |v'(x)| \right) \), where \( \eta_0 \) and \( \epsilon \) are from Lemma 3.1 and Remark 1.4 respectively, then for any \( E, E' \) and irrational \( \omega' \) satisfying \( |E - E'| < \lambda_x^{-800}, |\omega - \omega'| < \lambda_x^{-800} \) and \( \beta(\omega') \leq \beta(\omega) \), it has

\[
|L^2(E, \omega) - L^2(E', \omega')| < |E - E'|^{\gamma} + |\omega - \omega'|^{\gamma},
\]

where \( \tau_x \) is defined in Remark 1.9.

This paper is organized as follows. In Section 2, the strong Birkhoff Ergodic Theorem for subharmonic functions with irrational shift on the Torus is proved. Then we study the positive Lyapunov exponent of the Jacobi operators in the large potential coupling regimes, and obtain the independences of the quantities \( \mu(\Omega_1, 1) \) in (1.1) for two subharmonic functions \( \mu(\cdot, E, \omega) \) and \( d(\cdot, \omega) \), which are defined in (1.10) and (1.12) respectively, on the coupling numbers by Han and Zhang’s method to make Theorem 1 applied to the Jacobi operators with suitable frequencies and coupling numbers in Section 3. It help us in getting Theorem 2, the LDT for the Jacobi cocycles, in Section 4. Combining this LDT with the Avalanche Principle, we prove that the positive Lyapunov exponent can be extended from one point to an interval, and calculate its length in Section 5. Finally, the proofs of the rest theorems, Theorem 3, 4 and 6, are presented in the last section. In additional, the results of the Schrödinger operators stated in the remarks are proved under the proofs of the corresponding theorems of the Jacobi ones.

2. The Ergodic Theorem for Subharmonic Functions with Shift

Let \( \{x\} = x - [x] \). For any positive integer \( q \), complex number \( \xi = \xi + i\eta \) and \( 0 \leq \chi < 1 \), define

\[
f_{q,\xi}(x) = \sum_{0 \leq k < q} \log \|x + \frac{k}{q} - \xi\|, \quad F_{q,\xi}(x) = \sum_{0 \leq k < q} \log \|x + k\omega - \xi\| \quad \text{and} \quad I(\xi) = \int_0^1 \log |y - \xi|dy.
\]

Also define

\[
||x|| = \min_{n \in \mathbb{Z}} |x + n|,
\]

and the distance on \( T \):

\[
dist(x, y) = \|x - y\|.
\]

Given \( x \in [0, 1) \) and \( q = 1, 2, \cdots \). Set \( S = \{x + \frac{k}{q} : 0 \leq k < q\} \). We enumerate \( S \) as

1. \( 0 \leq \theta_0 < \theta_1 < \cdots < \theta_{q-1} \leq 1, \theta_j = \{x + \frac{k}{q}\}; \)
2. \( \theta_{j+1} = \theta_j + \frac{1}{q} \).
For any $\xi \in [0, 1)$, find the integers $j^+(x, q, \xi)$ and $j^-(x, q, \xi)$ such that if $\theta_0 \leq \xi < \theta_{q-1}$, then
\[
\theta_{j^+(x, q, \xi)} \leq \xi < \theta_{j^+(x, q, \xi)}^+ \quad \text{and} \quad j^+(x, q, \xi) - j^-(x, q, \xi) = 1,
\]
else $j^-(x, q, \xi) = q - 1$ and $j^+(x, q, \xi) = 0$. Let $k^-(x, q, \xi) = k_{j^+(x, q, \xi)}^-$ and $k^+(x, q, \xi) = k_{j^+(x, q, \xi)}^+$. Then, we have

**Lemma 2.1.** There exists an absolute constant $C$ such that 
\[
\left| \sum_{0 \leq k < q, k \neq k^+(x, q, \xi)} \log \|x + \frac{k}{q} - \zeta| - qI(\zeta) \right| \leq C \log q,
\]
where $C$ is an absolute constant.

**Proof.** Using the above notations one has
\[
\frac{1}{q} \sum_{0 \leq k < q, k \neq k^+(x, q, \xi)} \log |x + \frac{k}{q} - \zeta| = \frac{1}{q} \sum_{0 \leq j < q, j \neq j^+(x, q, \xi)} \log |\theta_j - \zeta - i\eta|.
\]
Also we have for any $0 \leq j < j^+(x, q, \xi)$,
\[
\left| (\theta_{j^+(x, q, \xi)} - \xi) - (\theta_j - \xi) \right| \leq \frac{j^+(x, q, \xi) - j}{q},
\]
and for any $j^+(x, q, \xi) < j \leq q - 1$,
\[
\left| (\theta_{j^+(x, q, \xi)} - \xi) - (\theta_j - \xi) \right| \leq \frac{j - j^+(x, q, \xi)}{q}.
\]
Note that if $1 \leq j < j^+(x, q, \xi)$, then
\[
\frac{1}{q} \log |\theta_j(x, q) - \zeta - i\eta| < \int_{\theta_{j^+(x, q, \xi)}}^{\theta_j(x, q)} \log |y - \zeta - i\eta|dy < \frac{1}{q} \log |\theta_{j^+(x, q, \xi)} - \xi - i\eta|,
\]
and if $j^+(x, q, \xi) < j \leq q - 2$, then
\[
\frac{1}{q} \log |\theta_j(x, q) - \zeta - i\eta| < \int_{\theta_{j(x, q)}}^{\theta_{j^+(x, q, \xi) + 1}} \log |y - \zeta - i\eta|dy < \frac{1}{q} \log |\theta_{j(x, q) + 1} - \xi - i\eta|.
\]
Thus
\[
\left| \frac{1}{q} \sum_{0 \leq j < q, j \neq j^+(x, q, \xi)} \log |\theta_j - \zeta - i\eta| - I(\zeta) \right| \leq \left| \sum_{0 \leq j < q, j \neq j^+(x, q, \xi)} \frac{1}{q} \log |\theta_j - \zeta - i\eta| - \frac{q - 1}{q} \int_{\theta_{j^+(x, q, \xi) + 1}}^{\theta_{j^+(x, q, \xi) + 1}} \log |y - \zeta - i\eta|dy \right|
\]
\[
\leq -4 \int_{y < 0} \log |y|dy < \frac{C}{q} \log q.
\]

**Remark 2.1.** Let $\|x + k'/q \| = \min_{z=1}^q \|x + k/z \|$, then $k'_0 = k^-(x, q, \xi)$ or $k'_0 = k^+(x, q, \xi)$. Note that $\|x + k^-(x, q, \xi)/q - \xi\|$ and $\|x + k^+(x, q, \xi)/q - \xi\|$ are larger than $\frac{1}{4q}$. Thus
\[
\left| \sum_{0 \leq k < q, k \neq k'_0} \log |x + \frac{k}{q} - \zeta| - qI(\zeta) \right| \leq C \log q,
\]
where $C$ is an absolute constant.

Let $\omega$ be irrational and $\left\{\frac{p_i}{q_i}\right\}_{i=1}^\infty$ be its continued fraction approximants. Then by (1.2), it has
\[
(2.2) \quad \frac{k}{q_i (q_i + 1)} < |k\omega - \frac{k p_i}{q_i}| < \frac{k}{q_i q_{i+1}} \leq \frac{1}{q_i + 1}, \quad 0 < k < q_i.
\]

**Lemma 2.2.** Let $\|x + k\omega \| = \min_{z=1}^{q(z)} \|x + k\omega \|$, then
\[
\left| F_{q, \zeta}(x, \zeta_I) - qI(\zeta) \right| \leq C \log q + \left| \log \|x + k\omega - \zeta\| \right|.
\]
Proof. We declare that if there exists 0 ≤ j < q_s such that \(|x + j\omega - \xi| \leq \frac{1}{2q_s} - \frac{1}{q_{s+1}}\), then \(j = k_0\). Actually, if \(|x + j\omega - \xi| \leq \frac{1}{2q_s} - \frac{1}{q_{s+1}}\) and \(j \neq k_0\), then \(|x + k_0\omega - \xi| \leq \frac{1}{2q_s} - \frac{1}{q_{s+1}}\), which implies
\[
|x + k_0\omega| - |x + j\omega| \leq \frac{1}{2q_s} - \frac{2}{q_{s+1}}.
\]
By (2.2), we have
\[
\left|\frac{x + j p_s}{q_s} - \frac{x + k_0 p_s}{q_s}\right| < \frac{1}{q_s}.
\]
It is a contraction. Thus there is at most one integer 0 ≤ k_0 < q_s such that \(|x + k_0\omega - \xi| < \frac{1}{2q_s} - \frac{1}{q_{s+1}}\) and
\[
|x + k\omega| - \xi| > \frac{1}{2q_s} - \frac{1}{q_{s+1}} > \frac{1}{4q_s}, \quad k \neq k_0.
\]
Due to (2.2), we have for any 0 ≤ k < q_s,
\[
\left|\frac{x + k p_s}{q_s} - \xi\right| < \frac{1}{q_{s+1}},
\]
or there exists only one integer 0 ≤ m < q_s such that
\[
\left|\frac{x + m \omega}{q_s} - \xi| + \left|\frac{x + m p_s}{q_s} - \xi\right| > 1 - \frac{1}{q_{s+1}},
\]
and the others satisfies (2.4). Notice that \(\{|x + \frac{k p_s}{q_s}| : 0 ≤ k < q_s\} = \{|x + \frac{k p_s}{q_s}| : 0 < k < q_s\}\). So, by Remark 2.1, we have
\[
|x + k p_s/q_s| - \xi| > \frac{1}{2qs}, \quad k \neq k_0'.
\]
Combining it with (2.4), we have
\[
\sum_{0 ≤ k < q_s, k \neq k_0, k_0'} \left|\log \frac{|x + k\omega - \xi| - \log |x + k p_s/q_s - \xi|\right| ≤ \sum_{0 ≤ k < q_s, k \neq k_0, k_0'} \left|\log \left(1 + \frac{|x + k\omega| - \xi|}{|x + k p_s/q_s| - \xi|}\right)\right|
\]
\[
\leq C \sum_{l=1}^{q_s} \frac{q_s^{-1} l q_s^{-1} q_s^{-1} \log q_s}{q_{s+1}} \leq C \log q_s.
\]
By Remark 2.1, it implies that
\[
\sum_{0 ≤ k < q_s, k \neq k_0, k_0'} \log \left|\frac{|x + k\omega - \xi|}{q_s I(\xi)}\right| ≤ C \log q.
\]
Then, combining it with (2.3), we complete the proof. \(\square\)

Lemma 2.3. Let \(n = l q_s < q_{s+1}\), then
\[
|F_{x,h}(\xi) - n I(\xi)| < C \log q_s + \log D(x - \xi, \omega, l q_s) + 2b n.
\]

Proof. Define \(x_h = x + h q_s \omega\) and \(|x_h + k_0 \omega - \xi| = \min_{0 ≤ k < q_s} |x_h + k \omega - \xi|\). Then due to Lemma 2.2, we have
\[
|F_{x_h}(\xi) - l q_s I(\xi)| ≤ \sum_{h=0}^{l-1} |F_{x_h}(\xi) - l q_s I(\xi)| \leq \sum_{h=0}^{l-1} \left|\log |x_h + k_0 \omega - \xi|\right| + C \log q_s.
\]
Note that
\[
\frac{1}{2q_{s+1}} < \frac{1}{q_s + q_{s+1}} < |q_s \omega - p_s| < \frac{1}{q_{s+1}}.
\]
Define \( Q = \left\lfloor \frac{2n}{q_s} \right\rfloor \) and let \( j \) be the number such that \( \|x_h + k_j \omega - \xi\| < \frac{1}{4q_{j+1}} \). Then by (2.7) and the declaration in the proof of Lemma 2.2, we have for any \( j - 2Q + 1 \leq h < j \) and \( j < h \leq j + 2Q - 1 \),

\[
\|x_h + k_j \omega - \xi\| > \frac{1}{4q_{j+1}}.
\]

Thus there are at most one point which is smaller than \( \frac{1}{4q_{j+1}} \). Recall that \( n = lq_s \). Then we have

\[
|F_{lq_s}(x) - lq_s I(\zeta)| \leq \sum_{h=0}^{l-1} \|\log \|x_h + k_j \omega - \xi\|\| + C l \log q_s \leq |\log D(x - \xi, -\omega, lq_s)| + C l \log q_s + |\log \frac{1}{4q_{j+1}}| \leq |\log D(x - \xi, -\omega, lq_s)| + C l \log q_s + 2l \log q_{j+1} \leq |\log D(x - \xi, -\omega, lq_s)| + C l \log q_s + 2\beta q_s \leq |\log D(x - \xi, -\omega, lq_s)| + C l \log q_s + 2\beta q_s.
\]

\( \square \)

**Lemma 2.4** (Lemma 3.2 in [GS]). Let \( \Omega \subset \mathbb{T} \) be an arbitrary finite set. Then

\[
\int_{\mathbb{T}} \exp(\sigma |\log \text{dist}(x, \Omega)|) dx \leq \frac{2^\sigma}{1 - \sigma} (|\Omega|)^\sigma
\]

for any \( 0 < \sigma < 1 \).

**Lemma 2.5.** Let \( n = lq_s \). Then for any \( 0 < \sigma < 1 \),

\[
\int_{\mathbb{T}} \exp(\sigma |F_{n,\zeta}(x) - nI(\zeta)|) dx < \exp(5\beta n).
\]

**Proof.** Set \( \Omega = \{ -m\omega : 0 \leq m < lq_s \} \). Then \( \mathbb{M} = lq_s \) and \( \text{dist}(x - \xi, \Omega) = D(x - \xi, -\omega, lq_s) \). Thus

\[
\int_{\mathbb{T}} \exp(\sigma |\log D(x - \xi, -\omega, lq_s)|) dx = \int_{\mathbb{T}} \exp(\sigma |\log \text{dist}(x, \Omega)|) dx \leq \frac{2^\sigma}{1 - \sigma} (|lq_s|)^\sigma.
\]

By Lemma 2.3, we have

\[
\int_{\mathbb{T}} \exp(\sigma |F_{n,\zeta}(x) - nI(\zeta)|) dx \leq \exp(2C \sigma \log(lq_s) + C \sigma l \log q_s + 2\sigma \beta n) < \exp(5\beta n).
\]

\( \square \)

**Remark 2.2.** It is easily seen that there exists a constant \( \tilde{C}(\zeta) \) such that for any \( n > 0 \) and \( 0 < \sigma < 1 \),

\[
\int_{\mathbb{T}} \exp(\sigma |F_{n,\zeta}(x) - nI(\zeta)|) dx < \exp(\tilde{C}(\zeta)n).
\]

So, the above lemmas show that for any \( n = lq_s \), \( q_s \geq q_{s+1} \), the large constant \( \tilde{C}(\zeta) \) can be changed by \( 5\beta \). It is also obvious that if \( n = lq_s \) is large, then \( \tilde{C}(\zeta) \) can be changed by \( 5\beta \).

Now for any \( n \), there exist \( q_s \) and \( q_{s+1} \) such that \( q_s \leq n < q_{s+1} \). Let \( n = l_s q_s + r_s \), where \( l_s = \left\lfloor \frac{n}{q_s} \right\rfloor \) and \( 0 \leq r_s = n - l_s q_s < q_s \). Then,

\[
\int_{0}^{1} \exp(\sigma |F_{n,\zeta}(x) - nI(\zeta)|) dx \leq \left[ \int_{0}^{1} \exp(2\sigma |F_{l_s q_s,\zeta}(x) - l_s q_s I(\zeta)|) dx \right]^\frac{1}{2} \times \left[ \int_{0}^{1} \exp(2\sigma |F_{r_s,\zeta}(x) - r_s I(\zeta)|) dx \right]^\frac{1}{2} \leq \exp(5\beta n) \left[ \int_{0}^{1} \exp(2\sigma |F_{r_s,\zeta}(x) - r_s I(\zeta)|) dx \right]^\frac{1}{2}.
\]
Let $r_s = l_{s-1}q_{s-1} + r_{s-1}$, where $l_{s-1} = \left[ \frac{r_{s-1}}{q_{s-1}} \right]$ and $0 \leq r_{s-1} = r_s - l_{s-1}q_{s-1} < q_{s-1}$. Then

$$\int_0^1 \exp(2\sigma|F_{r_s}(x) - r_sI(\zeta)|)dx \leq \left[ \int_0^1 \exp(2\sigma|F_{l_{s-1}q_{s-1}}(x) - l_{s-1}q_{s-1}I(\zeta))|dx \right]^{\frac{1}{2\sigma}}$$

$$\times \left[ \int_0^1 \exp(2\sigma|F_{r_{s-1}}(x) - r_{s-1}I(\zeta)|)dx \right]^{\frac{1}{2\sigma}}$$

$$\leq \exp(5\sigma\beta r_s) \left[ \int_0^1 \exp(2\sigma|F_{r_{s-1}}(x) - r_{s-1}I(\zeta)|)dx \right]^{\frac{1}{2\sigma}}.$$

We use induction here. Let $r_{s+1} = l_{s-1}q_{s-1} + r_{s-1}$, where $l_{s-1} = \left[ \frac{r_{s-1}}{q_{s-1}} \right]$ and $0 \leq r_{s-1} = r_{s+1} - l_{s-1}q_{s-1} < q_{s-1}$. Then

$$\int_0^1 \exp(2\sigma|F_{r_{s+1}}(x) - r_{s+1}I(\zeta)|)dx \leq \left[ \int_0^1 \exp(2\sigma|F_{l_{s-1}q_{s-1}}(x) - l_{s-1}q_{s-1}I(\zeta)|)dx \right]^{\frac{1}{2\sigma}}$$

$$\times \left[ \int_0^1 \exp(2\sigma|F_{r_{s-1}}(x) - r_{s-1}I(\zeta)|)dx \right]^{\frac{1}{2\sigma}}$$

$$\leq \exp(5\sigma\beta r_{s+1}) \left[ \int_0^1 \exp(2\sigma|F_{r_{s-1}}(x) - r_{s-1}I(\zeta)|)dx \right]^{\frac{1}{2\sigma}}.$$

Note that for any irrational $\omega$, there exists an absolute constant $\bar{C} > 1$ such that $q_{s+1} > C q_s$. Thus, there exists $m \geq 0$ such that $\bar{C}^{-m} \leq \beta$. Therefore, if $\zeta \in \Omega^\prime$, where $\Omega^\prime$ is a compact subregion of $C$, then

$$\int_0^1 \exp(\sigma|F_{r_s}(x) - nI(\zeta)|)dx \leq \exp\left(5\sigma\beta n + 5\sigma\beta(r_s + r_{s-1} + \cdots + r_{s-m+1})\right)$$

$$\times \left[ \int_0^1 \exp(2\sigma|F_{l_{s-1}q_{s-1}}(x) - r_{s-1}I(\zeta)|)dx \right]^{\frac{1}{2\sigma}}$$

$$\leq \exp\left(5\sigma\beta n + 5\sigma\beta(q_s + q_{s-1} + \cdots + q_{s-m+1}) + \bar{C}\sigma r_{s-m}\right)$$

$$\leq \exp\left(5\sigma\beta n + 5\sigma\beta(q_s + \bar{C}^{-1} q_s + \cdots + \bar{C}^{-m+1} q_s) + \bar{C}\sigma q_{s-m}\right)$$

$$\leq \exp\left(5\sigma\beta n + C\sigma q_s + \bar{C}\zeta \bar{C}^{-m}\sigma q_s\right)$$

$$\leq \exp\left(5\sigma\beta n + C\sigma q_s + \bar{C}\zeta \sigma q_s\right) < \exp(C\sigma\beta n),$$

where $\bar{C} = \sum_{k=0}^{\infty} \bar{C}^{-k} < \infty$ is an absolute constant and $C = C(\Omega^\prime)$. Thus

**Lemma 2.6.** There exists $c = c(\omega, \bar{C})$ such that for any positive $n$ and $0 < \sigma \leq c$, we have

$$\int_0^1 \exp(\sigma|F_{r_s}(x) - nI(\zeta)|)dx < \exp(C\sigma\beta n).$$

**Remark 2.3.** It is easy to see that the constant $c = 2^{\log \beta c}$ is an increasing function of $\beta$. So, $c_0 := 2^{\log \beta c_G} \leq c(\omega, \bar{C})$ for any irrational $\omega$, where $\beta_G = \beta(\omega_G)$ and $\omega_G = \frac{\sqrt{5} - 1}{2}$.

**Remark 2.4.** Note that $\log |x|$ is a subharmonic function. Thus, if $h$ is a 1-periodic harmonic function defined on a neighborhood of real axis, then for any positive $n$ and $0 < \sigma \leq c_0$, we have

$$(2.8) \quad \int_0^1 \exp(\sigma| \sum_{k=1}^n h(x + k\omega)) - n \int_0^1 h(y)dy)|dx < \exp(C\sigma\beta n).$$

Now let us recall the following Riesz's theorem proved in [GS1]:

Lemma 2.7. Let $u: \Omega \to \mathbb{R}$ be a subharmonic function on a domain $\Omega \subset \mathbb{C}$. Suppose that $\partial \Omega$ consists of finitely many piece-wise $C^1$ curves. There exists a positive measure $\mu$ on $\Omega$ such that for any $\Omega_1 \Subset \Omega$ (i.e., $\Omega_1$ is a compactly contained subregion of $\Omega$),

\begin{equation}
(2.9) \quad u(z) = \int_{\Omega_1} \log |z - \zeta| d\mu(\zeta) + h(z),
\end{equation}

where $h$ is harmonic on $\Omega_1$ and $\mu$ is unique with this property. Moreover, $\mu$ and $h$ satisfy the bounds

\begin{equation}
(2.10) \quad \mu(\Omega_1) \leq C(\Omega, \Omega_1) (\sup_{\Omega_1} u - \sup_{\Omega_1} u),
\end{equation}

\begin{equation}
(2.11) \quad \|h - \sup_{\Omega_1} u\|_{L^\infty(\Omega_1)} \leq C(\Omega, \Omega_1, \Omega_2) (\sup_{\Omega_1} u - \sup_{\Omega_1} u)
\end{equation}

for any $\Omega_2 \Subset \Omega_1$.

Notice that the ergodic measure for the shift on the Torus is the Lebesgue measure and $m(\mathbb{T}) = 1$. Then, $\langle u \rangle = \int_{\mathbb{T}} u(x)dx$, and

\[ \sum_{k=1}^{n} u(x + k\omega) - n < u > = \sum_{k=1}^{n} \int_{\Omega_1} \log |x + k\omega - \zeta| d\mu(\zeta) - n \int_{\Omega_1} I(\zeta) d\mu(\zeta) + \sum_{k=1}^{n} h(|x + k\omega|) - n \int_{0}^{1} h(y)dy. \]

Recall that

\[ \sum_{k=1}^{n} \int_{\Omega_1} \log |x + k\omega| - \zeta| d\mu(\zeta) = \int_{\Omega_1} F_{n,\epsilon}(x) d\mu(\zeta). \]

Then

\[ \int_{0}^{1} \exp\left( \sigma \sum_{k=1}^{n} u(x + k\omega) - n < u > \right) dx \leq \left[ \int_{0}^{1} \exp\left( \sigma \int_{\Omega_1} (F_{n,\epsilon}(x) - nI(\zeta)) d\mu(\zeta) \right) dx \right]^2, \]

\[ \times \left[ \int_{0}^{1} \exp\left( \sigma \sum_{k=1}^{n} h(|x + k\omega|) - n \int_{0}^{1} h(y)dy \right) dx \right]^2. \]

Since $\exp(\sigma \cdot)$ is a convex function, the Jensen’s inequality implies that

\[ \int_{0}^{1} \exp\left( \sigma \int_{\Omega_1} (F_{n,\epsilon}(x) - nI(\zeta)) d\mu(\zeta) \right) dx \leq \int_{0}^{1} \int_{\Omega_1} \exp\left( \sigma \mu(\Omega_1) |F_{n,\epsilon}(x) - nI(\zeta)| \frac{d\mu(\zeta)}{\mu(\Omega_1)} \right) dx d\mu(\Omega_1), \]

\[ = \int_{\Omega_1} \int_{0}^{1} \exp\left( \sigma \mu(\Omega_1) |F_{n,\epsilon}(x) - nI(\zeta)| \right) dx \frac{d\mu(\zeta)}{\mu(\Omega_1)} \]

\[ \leq \int_{\Omega_1} \exp(C\sigma \mu(\Omega_1) \beta n) \frac{d\mu(\zeta)}{\mu(\Omega_1)} \leq \exp(C\sigma \mu(\Omega_1) \beta n). \]

Thus, combining it with (2.8), we have for any $0 < \sigma \leq \frac{\delta_0}{\mu(\Omega_1)}$ and $\omega$,

\[ \int_{0}^{1} \exp\left( \sigma \sum_{k=1}^{n} u(x + k\omega) - n < u > \right) dx < \exp(C\sigma \mu(\Omega_1) \beta n). \]

Recall the Markov’s inequality: For any measurable extended real-valued function $f(x)$ and $\epsilon > 0$, we have

\[ \text{mes}\left( \{ x \in \mathbb{R} : |f(x)| \geq \epsilon \} \right) \leq \frac{1}{\epsilon} \int_{\mathbb{R}} |f| dx. \]

Let $f(x) = \exp\left( \sigma \sum_{k=1}^{n} u(x + k\omega) - n < u > \right)$ and $\epsilon = \exp(\sigma \delta n)$, then

\[ \text{mes}\left( \{ x \in \mathbb{R} : \sum_{k=1}^{n} u(x + k\omega) - n < u > > \delta \} \right) \leq \text{mes}\left( \{ x \in \mathbb{R} : \exp\left( \sigma \sum_{k=1}^{n} u(x + k\omega) - n < u > \right) \geq \exp(\sigma \delta n) \} \right). \]

\[ (2.12) \quad \leq \exp(-\sigma \delta n + C\sigma \mu(\Omega_1) \beta n). \]

Therefore, if $\beta < \frac{\delta}{2\mu(\Omega_1)}$, then we complete the proof of Theorem 1.
Remark 2.5. In this proof, we use the constant $\beta$ only to make the inequality $\log_q g_{r+1} < \beta q$, hold for any $s > 0$. Thus, if we change $\beta$ by $\tilde{\beta}$, then for large $n$, this inequality and Theorem 1 still hold. But in this condition, the integer $m$ and the constant $c$ will depend on $\tilde{\beta}$.

3. Positive Lyapunov Exponents and Strong Birkhoff Ergodic Theorem for Jacobi Operators

The fact that the determinant of $M_n^\omega(x, E, \omega)$, which is the transfer matrix of the Schrödinger operators and analytic in $x$, is always 1 makes $u_n^\omega(x, E, \omega) = \frac{1}{n} \log ||M_n^\omega(x, E, \omega)||$ have the upper and lower bounds. Correspondingly, $M_n(x, E, \omega)$ is the one of the Jacobi operators. But it is not analytic, and its determinant and the logarithm of its norm have no bounds. Therefore, in the introduction we define the matrix $M_n^\omega(x, E, \omega)$, which is analytic, so that Theorem 1 can be applied to it in this section. We also define the matrix $M_n^\omega(x, E, \omega)$, whose determinant is 1, so that the Avalanche Principle can be applied to it in Section 5 and 6. On the other hand, what we want to present in the theorems is the details especially the settings of $\lambda_i$ are very important to our paper.

Now we start to study the Lyapunov exponents of the analytic quasi-periodic Jacobi cocycles. Choose $\Omega$ in Lemma 2.7 as (1.20) and let $\lambda_\nu > \lambda_1 := \frac{\lambda_0 ||\partial \nu||_{L^\infty}}{\nu ||\partial \nu||_{L^\infty}}$. Then
\[
\sup_{E \in \mathcal{E}, z \in \Omega} u_2^\nu(z, E, \omega) \leq \log(5\lambda_\nu ||v||_{L^\infty(\Omega)}).
\]

And if $\lambda_\nu > \lambda_2 := (5||v||_{L^\infty(\Omega)})^{\frac{1}{2}}$, then
\[
L^2(E, \omega) \leq \sup_{E \in \mathcal{E}, z \in \Omega} u_2^\nu(z, E, \omega) \leq (1 + \gamma) \log \lambda_\nu.
\]

By the way, it is easy to see that if $\lambda_\nu > \max(\lambda_2, \lambda_D)$, where $\lambda_D := \frac{(\lambda_0 ||\partial \nu||_{L^\infty})^{\frac{1}{2}}}{\nu ||\partial \nu||_{L^\infty}}$, then
\[
D \leq \log \left(\lambda_D ||\partial \nu||_{L^2}\right) \leq \log \left(\lambda_D ||v||_{L^2}\right)^{\frac{1}{2}} \leq \frac{\gamma}{2} \log \lambda_\nu + \frac{\gamma^2}{2} \lambda_\nu < \gamma \log \lambda_\nu.
\]

To estimate the lower bound of $\sup_{z \in \Omega} u_2^\nu(z)$, we need the following lemma for the complex analytic function $v(z)$:

Lemma 3.1 (Lemma 14.5 in [BG]). For all $0 < \delta < \rho$, there is an $\epsilon_0 = \epsilon_0(\nu)$ such that
\[
\inf_{E, \nu} \sup_{|y| < \delta} \inf_{x \in [0,1]} |v(x + iy) - E_{\nu}| > \epsilon_0.
\]

Therefore, for any $E, \lambda_\nu$ and $0 < \delta < \rho$, there is $\frac{\delta}{2} < \gamma_0 < \delta$ such that
\[
\inf_{x \in [0,1]} |\lambda_\nu v(x + i\gamma_0) - E| > \lambda_\nu \epsilon_0.
\]

Let
\[
M_{n-1}^\omega(x + iy_0, E, \omega) \left(\begin{array}{c}1 \\ 0\end{array}\right) = \left(\begin{array}{c}g_{n-1} \\ h_{n-1}\end{array}\right).
\]

Then
\[
\left(\begin{array}{c}g_n \\ h_n\end{array}\right) = \left(\begin{array}{c}\lambda_\nu v(x + iy + (n-1)\omega) - E - \lambda_\nu \tilde{a}(x + iy + (n-1)\omega) \\ \lambda_\nu \tilde{a}(x + iy + n\omega)\end{array}\right) \left(\begin{array}{c}g_{n-1} \\ h_{n-1}\end{array}\right)
\]
\[
= \left(\begin{array}{c}(\lambda_\nu v(x + iy + (n-1)\omega) - E) g_{n-1} - \lambda_\nu \tilde{a}(x + iy + (n-1)\omega) h_{n-1} \\ \lambda_\nu \tilde{a}(x + iy + n\omega) h_{n-1}\end{array}\right),
\]
Set \( \lambda_3 = \lambda_3(v, \lambda_a, \alpha) = 2\lambda_a\|a\|_{L^\infty(\Omega)}^{-1} \). If \( \lambda_\nu > \lambda_3 \), then for any \( E \in \mathcal{E} \), it implies

\[
\inf_{x \in [0,1]} |\lambda_\nu(v(x + iy)) - E| > \lambda_\nu \varepsilon_\nu > 2\lambda_a\|a\|_{L^\infty(\Omega)}.
\]

Now we use the induction to show that

\[
|g_n| \geq |h_n|, \quad \text{and} \quad |g_n| \geq (\lambda_\nu \varepsilon_\nu - \lambda_a\|a\|_{L^\infty(\Omega)})|g_{n-1}| \geq (\lambda_\nu \varepsilon_\nu - \lambda_a\|a\|_{L^\infty(\Omega)})^n, \quad n \geq 1.
\]

Due to (3.4) and (3.5), it has that \( g_0 = 1 \), \( h_0 = 0 \) and

\[
|g_1| = |\lambda_\nu (x + iy) - E| > \lambda_\nu \varepsilon_\nu, \quad |h_1| = |\lambda_a a(x + n\omega)| \leq \lambda_a\|a\|_{L^\infty(\Omega)}.
\]

Let \( |g_n| \geq |h_n| \) and \( |g_n| > (\lambda_\nu \varepsilon_\nu - \lambda_a\|a\|_{L^\infty(\Omega)})|g_{n-1}| > (\lambda_\nu \varepsilon_\nu - \lambda_a\|a\|_{L^\infty(\Omega)})^n \). Then, we finish this induction by

\[
|g_{n+1}| \geq (\lambda_\nu (x + iy + t\omega) - E)|g_n| > (\lambda_\nu \varepsilon_\nu - \lambda_a\|a\|_{L^\infty(\Omega)})^{n+1},
\]

and

\[
|h_{n+1}| \leq |\lambda_a a(x + n\omega)g_n| < \lambda_a\|a\|_{L^\infty(\Omega)}|g_n| \leq |g_{n+1}|.
\]

Therefore, we have

\[
\|M^s_n(x + iy_0, E, \omega)\| \geq \left( \left( \begin{array}{cc} M^s_n(x + iy_0, E, \omega) & (1) \\ 1 & 0 \end{array} \right) \right)^n = |g_n| > (\lambda_\nu \varepsilon_\nu - \lambda_a\|a\|_{L^\infty(\Omega)})^n \geq \left( \frac{1}{2} \lambda_a \varepsilon_\nu \right)^n.
\]

It implies that

\[
u_\nu^*(x + iy_0, E, \omega) = \frac{1}{n} \log \|M^s_n(x + iy_0, E, \omega)\| \geq \log \left( \frac{1}{2} \lambda_a \varepsilon_\nu \right).
\]

Write \( \mathbb{H} = \{ z : \text{Im} z > 0 \} \) for the upper half-plane and \( \mathbb{H}_t \) for the strip \( \{ z = x + iy : 0 < y < \frac{t}{2} \} \). Then denote by \( \mu(z, \mathbb{E}, \mathbb{H}) \) the harmonic measure of \( \mathbb{E} \) at \( z \in \mathbb{H} \) and \( \mu_t(y_0, \mathbb{E}_t, \mathbb{H}_t) \) the harmonic measure of \( \mathbb{E}_t \) at \( y_0 \) in \( \mathbb{H}_t \). Note that \( \psi(z) = \exp \left( \frac{2\pi}{\rho} z \right) \) is a conformal map from \( \mathbb{H}_t \) onto \( \mathbb{H} \). Due to [GM], we have

\[
\mu_t(y_0, \mathbb{E}_t, \mathbb{H}_t) = \mu_t(\psi(y_0), \psi(\mathbb{E}_t), \mathbb{H}),
\]

and

\[
\mu(z = x + iy, \mathbb{E}, \mathbb{H}) = \int_{\mathbb{E}} \frac{y}{(t - x)^2 + y^2} \frac{dt}{\pi}.
\]

Thus

\[
\mu_t(y = \frac{\rho}{2}, 5) = \frac{5\pi y_0}{\pi \rho} < \frac{5\delta}{\rho},
\]

By the subharmonicity and (3.1), it yields that if \( \lambda_\nu > \max(\lambda_1, \lambda_3) \), then

\[
\log \left( \frac{1}{2} \lambda_a \varepsilon_\nu \right) < \nu_\nu^*(iy_0, E, \omega) \leq \int_{y = 0}^{y = \frac{\rho}{2}} u_\nu^*(z, E, \omega) \mu_t(dz)
\]

\[
= \int_{y = 0}^{\rho} u_\nu^*(x, E, \omega) \mu_t(dx) + \int_{y = \frac{\rho}{2}}^{\frac{\rho}{2}} u_\nu^*(x + iy, E, \omega) \mu_t(dx)
\]

\[
\leq \int_{\mathbb{R}} u_\nu^*(x, E, \omega) \mu_t(dx) + \frac{5\delta}{\rho} \sup_{y = \frac{\rho}{2}} u_\nu^*(x + iy, E, \omega)
\]

\[
\leq \int_{\mathbb{R}} u_\nu^*(x, E, \omega) \mu_t(dx) + \frac{5\delta}{\rho} \log \left( 5\lambda_a\|v\|_{L^\infty(\Omega)} \right).
\]

So, if \( \delta < \frac{\rho}{10} \) and \( \lambda_\nu > \lambda_4 := 5\|v\|_{L^\infty(\Omega)} \left( \frac{2}{\rho} \right)^2 \), then

\[
(3.6) \quad \int_{\mathbb{R}} u_\nu^*(x, E, \omega) \mu_t(dx) \geq \log \left( \frac{1}{2} \lambda_a \varepsilon_\nu \right) - \frac{5\delta}{\rho} \log \left( 5\lambda_a\|v\|_{L^\infty(\Omega)} \right) > (1 - \gamma) \log \lambda_\nu.
\]

Set

\[
u_\nu^*(x) = u_\nu^*(x + h), \quad h \in T.
\]
Then, due to Lemma 3.1, it is obvious that (3.6) also holds for \( u_n^\alpha(x) \). So, for any \( h \in \mathbb{T} \), it has
\[
\int_{\mathbb{R}} u_n^\alpha(x + h) \mu_s(dx) > (1 - \gamma) \log \lambda_v.
\]
Integrating in \( h \in \mathbb{T} \) implies that
\[
L_n^*(E, \omega) = \int_{0}^{1} u_n^\alpha(x + h, E, \omega)dh \geq \left( \int_{\mathbb{R}} \mu_s(dx) \right) \left( \int_{0}^{1} u_n^\alpha(x + h, E, \omega)dh \right)
= \int_{0}^{1} \int_{\mathbb{R}} u_n(x + h, E, \omega) \mu_s(dx)dxdh
> (1 - \gamma) \log \lambda_v, \quad \forall n \geq 0.
\]
Thus, combining it with (3.2) and (3.3), we finish the proof of Theorem 5 with \( n \to +\infty \) and
\[
(3.7) \quad \lambda_v > \lambda_p := \max \left\{ 5|v|^2, 2, \frac{\lambda_0}{\|v\|_{L^2(\Omega)}}, \frac{(\lambda_0 - \|v\|_{L^2(\Omega)})^2}{\|v\|_{L^2(\Omega)}}, \left( \frac{2}{\epsilon_0} \right)^{\frac{1}{2}} \right\} \geq \max(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_D).
\]

Remark 3.1. For the Schrödinger operators, due to Lemma 3.1, \( |\|v\|_{L^2(\Omega)}| > \epsilon_0 \). Therefore, there exists \( \lambda_p' := \left( \frac{20|\|v\|_{L^2(\Omega)}|}{\epsilon_0} \right)^{\frac{1}{2}} \) such that if \( \lambda_v > \lambda_p' \), then \( L_n^*(E, \omega) \) is positive for any irrational \( \omega \) and \( E \in \mathcal{E}_\alpha \) as follows:
\[
(1 - \gamma) \log \lambda_v < L_n^*(E, \omega) < (1 + \gamma) \log \lambda_v, \quad \forall n \geq 1,
\]
where \( \mathcal{E}_\alpha := [-2 - \|v\|_{L^2(\Omega)}, 2 + \|v\|_{L^2(\Omega)}] \). What’s more, for any irrational \( \omega, E \in \mathcal{E}_\alpha \) and \( x \in \mathbb{T} \),
\[
u_n^\alpha(x, E, \omega) \leq M_\alpha \leq (1 + \gamma) \log \alpha.
\]

On the other hand, if we choose \( \Omega_1 \) in Lemma 2.7 as (1.21) and \( \delta = \frac{\xi}{\epsilon_0} \) in Lemma 3.1, then
\[
\sup_{E \in \mathcal{E}_\delta, x \in \Omega_1} u_n^\alpha(x, E, \omega) \geq \log \left( \frac{1}{2}, \epsilon_0 \right), \quad \lambda_v > \lambda_3.
\]
Combining it with (2.10) and (3.1), we have the following lemma:

**Lemma 3.2.** There exist \( \lambda_0 = \lambda_0(v, \lambda_0, \epsilon) := \max(\lambda_1, \lambda_3) \) and \( C_v = C_v(\epsilon) = C(\Omega, \Omega_1) \log \frac{10\|v\|_{L^2(\Omega)}}{\epsilon_0} \) such that for \( \lambda_v > \lambda_0 \),
\[
\mu_\alpha(\Omega_1) \leq C_v,
\]
where \( \mu_\alpha \) is the unique measure for \( u_n^\alpha(z, E, \omega) \) in Lemma 2.7.

Thus, Theorem 1 can be applied to \( u_n^\alpha(x, E, \omega) \) as follows:

**Lemma 3.3.** There exists \( c_v = c_v(\epsilon) := \frac{1}{\epsilon \epsilon_0} \) such that if \( \beta < c_v \delta \) and \( \lambda_v > \lambda_0 \), then for any positive \( k \) and \( n \),
\[
\mes \left( \left\{ x \in \mathbb{T} : \left| \frac{1}{k} \sum_{j=1}^{k} u_n^\alpha(x + j\omega) - L_n^*(E, \omega) \right| > \delta \right\} \right) < \exp(-\tilde{c}_v \delta k),
\]
where \( \tilde{c}_v = \frac{\xi}{\epsilon_0} \).

**Remark 3.2.** For the Schrödinger operators\( (\lambda, a \equiv 1) \), we have that if \( \beta < c_v \delta \) and \( \lambda_v > \lambda_0^\beta(v) := 2\epsilon_0^{-1} \), then for any positive \( k \) and \( n \),
\[
\mes \left( \left\{ x \in \mathbb{T} : \left| \frac{1}{k} \sum_{j=1}^{k} u_n^\alpha(x + j\omega) - L_n^*(E, \omega) \right| > \delta \right\} \right) < \exp(-\tilde{c}_v \delta k).
\]
Similar computations show that for any $\lambda_d \neq 0$,

$$\mu_d(\Omega_1) \leq C(\Omega, \Omega_1) \log \frac{\|d\|_{L^\infty(\Omega)}}{\|d\|_{L^\infty(\Omega_1)}} := C_a,$$

where $\mu_d$ is the unique measure in Lemma 2.7 for $d(\zeta, \omega)$ defined in (1.12). Correspondingly, the following two strong Birkhoff Ergodic Theorems both hold:

**Lemma 3.4.** There exists $c_a = c_a(a) = \frac{1}{2c_a}$ such that if $\beta < c_a\delta$, then for any positive $k$,

$$\operatorname{mes}\left\{ x \in T : \left| \frac{1}{k} \sum_{j=1}^{k} \log |a(x + j\omega)| - D \right| > \delta \right\} < \exp(-c_a\delta k),$$

and

$$\operatorname{mes}\left\{ x \in T : \left| \frac{1}{k} \sum_{j=1}^{k} d(x + j\omega, \omega) - 2D \right| > \delta \right\} < \exp(-c_a\delta k),$$

where $c_a = \frac{c}{c_a}$.

**Remark 3.3.** By (1.11), Lemma 3.3 and Lemma 3.4, it implies that if $\beta < \min(c_v, c_a)$ and $\lambda_v > \lambda_0$, then

$$\operatorname{mes}\left\{ x \in T : \left| \frac{1}{k} \sum_{j=1}^{k} u_j^\beta(x + j\omega) - L_m(E, \omega) \right| > \delta \right\} < \exp(-c_v\delta k) + \exp(-c_a\delta k).$$

Set $C_{v,a} = \max(\log \frac{10\|d\|_{L^\infty(\Omega)}}{\|d\|_{L^\infty(\Omega)}}), \log \frac{10\|d\|_{L^\infty(\Omega)}}{\|d\|_{L^\infty(\Omega)}})$. Then, there exist $c_{v,a} = \frac{1}{2c_{v,a}}$ and $\tilde{c}_{v,a} = \frac{c}{c_{v,a}}$ such that for any $\delta > 0$, if $\beta < c_{v,a}\delta$ and $\lambda_v > \lambda_0$, then

$$\operatorname{mes}\left\{ x \in T : \left| \frac{1}{k} \sum_{j=1}^{k} u_j^\beta(x + j\omega) - L_m(E, \omega) \right| > \delta \right\} < \exp(-\tilde{c}_{v,a}\delta k).$$

And to reduce the plugging of too many symbols, we can use $c_{v,a}$ and $\tilde{c}_{v,a}$ instead of $c_v$ and $\tilde{c}_v$ in Lemma 3.3, and of $c_a$ and $\tilde{c}_a$ in Lemma 3.4, as $c_{v,a} = \min(c_v, c_a)$ and $\tilde{c}_{v,a} = \min(c_v, \tilde{c}_a)$.

4. THE PROOF OF THEOREM 2

Define

$$\mathcal{X}_m = \left\{ x \in T : \left| \frac{1}{m} \sum_{j=0}^{m-1} d(x + j\omega) - 2D \right| > \frac{k}{m} \right\}.$$ 

By Lemma 3.4, we have that if $\beta < c_{v,a}\delta$, then $\operatorname{mes}\left( \mathcal{X}_m \right) = \exp(-\tilde{c}_{v,a}\delta k)$ for any $1 \leq m \leq k$. It implies that

$$\operatorname{mes}\left\{ x \in T : \left| \frac{1}{k} \sum_{j=0}^{k-1} d(x + j\omega) - (k+1)D \right| > k\delta \right\} < k \exp(-\tilde{c}_{v,a}\delta k).$$

**Corollary 2.3** in [T] proved that

$$-\frac{2Mk}{n} + \sum_{j=0}^{k-1} \frac{k-j}{nk} d(x + j\omega) \leq u_j^\beta(x, E, \omega) - \frac{1}{k} \sum_{j=1}^{k} u_j^\beta(x + k\omega, E, \omega) \leq \frac{2Mk}{n} - \sum_{j=0}^{k-1} \frac{k-j}{nk} d(x + (n + j-1)\omega).$$

So, we define

$$\mathcal{Y}_- = \left\{ x \in T : -\frac{2Mk}{n} + \sum_{j=0}^{k-1} \frac{k-j}{nk} d(x + j\omega) < -\delta \right\}$$

and let $k = C_1\delta n$ and $4C_1M \leq 1$. Then

$$\mathcal{Y}_- \subset \left\{ x \in T : \sum_{j=0}^{k-1} \frac{k-j}{k} d(x + j\omega) < -\frac{\delta n}{2} = -\frac{k}{2C_1} \right\}.$$
Assume $6C_1 |D| \leq 1$ to make $\frac{1}{2C_1} + D = C_2 > 2|D| \geq 0$ and
\[ \frac{k}{2} + (k + 1)D = C_2k + D \geq |D|k. \]
It implies that if $\beta < c_{v,a}|D|$, then
\[ \text{mes}(\mathcal{Y})_+ = \text{mes} \left\{ x \in \mathbb{T} : \frac{2Mk}{n} - \sum_{j=0}^{k-1} \frac{k-j}{nk} d(x + n\omega) > \delta \right\} < k \exp(-\tilde{c}_{v,a}|D|k) \]
Because $\exp(-\zeta y) \leq \zeta^{-1}$ for any $y, \zeta > 0$, so
\[ C_1 \delta n \exp(-\tilde{c}_{v,a}|D|\delta n) = C_1 \delta n \exp(-\tilde{c}_{v,a}|D|/2 \exp(-\tilde{c}_{v,a}|D|/2) \leq \frac{2}{\tilde{c}_{v,a}|D|} \exp(-\tilde{c}_{v,a}|D|/2 C_1 \delta n) \]
where $\tilde{c}_1 = \tilde{c}_1(v, a, |D|, M_0) = \frac{\tilde{c}_{v,a}(D)}{4} < \tilde{c}_{v,a}$ and $n_1 = n_1(a, v, |D|, M_0, \delta)$ satisfying
\[ n_1 = \frac{-\log \tilde{c}_{v,a}|D|}{\tilde{c}_{v,a}|D| C_1 \delta}. \]

Similar calculations show that
\[ \text{mes}(\mathcal{Y})_+ = \text{mes} \left\{ x \in \mathbb{T} : \frac{2Mk}{n} - \sum_{j=0}^{k-1} \frac{k-j}{nk} d(x + n\omega) > \delta \right\} < \exp(-\tilde{c}_1 \delta n). \]
Therefore, we have the deviation estimation as follows:

**Lemma 4.1.** For any $\delta > 0$, if $\beta < c_{v,a} \min(\delta, |D|)$, then
\[ \text{mes} \left\{ x \in \mathbb{T} : |u_n^e(x, E, \omega) - \frac{1}{k} \sum_{j=1}^{k} u_n^e(x + j\omega, E, \omega) | > \delta \right\} < 2 \exp(-\tilde{c}_1 \delta n), \forall n \geq n_1. \]

Combining it with Lemma 3.3, we have the following LDT for $u_n^e(x, E, \omega)$:

**Lemma 4.2.** For any $\delta > 0$, if $\beta < c_{v,a} \min(\delta, |D|)$ and $\lambda_\infty > \lambda_0$, then
\[ \text{mes} \left\{ x \in \mathbb{T} : |u_n^e(x, E, \omega) - L_n^e(E, \omega) | > \frac{3\delta}{4} \right\} < \exp(-\tilde{c}_a, C_1 \left( \frac{\delta}{4} \right)^2 n) + 2 \exp(-\tilde{c}_1 \delta n), \forall n \geq n_1. \]

**Remark 4.1.** For the Schrödinger operators, $d(x) \equiv 1$ and
\[ |u_n^e(x, E, \omega) - u_n^e(x + k\omega, E, \omega) | \leq \frac{2M_0^2 k}{n}. \]
Then, due to Remark 3.2 and the setting $k = \frac{\delta n}{4M_0^2}$, we have if $\beta < c_\lambda \delta$ and $\lambda_\infty > \lambda_0$, then there exists $\tilde{c}_\lambda = \tilde{c}_\lambda(\lambda, \nu) := \frac{1}{8M_0^2} \tilde{c}_\nu$ such that for any positive $n$,
\[ \text{mes} \left\{ x \in \mathbb{T} : |u_n^e(x, E, \omega) - L_n^e(E, \omega) | > \delta \right\} < \exp(-\tilde{c}_\lambda \delta n) = \exp(-\tilde{c}_\lambda \delta n). \]

**Proof of Theorem 2.** The theorem is obtained directly by the setting of $\tilde{c}_1$, (1.9), Lemma 3.4 and Lemma 4.2. \qed

With the similar process by changing (1.9) to (1.11), we have the following LDT for $u_n^e(x, E, \omega)$, which will be applied to satisfy the assumption (5.1) in the Avalanche Principle:
Lemma 4.3. For any $\delta > 0$ and $E \in \mathcal{E}$, if $\beta < c_{\alpha,a} \min(\delta, |D|)$ and $\lambda_\nu > \lambda_0$, then

$$\mes \left( \left\{ x \in \mathbb{T} : |u_n^a(x, E, \omega) - L_n(E, \omega)| > \delta \right\} \right) < \exp(-\tilde{c}_a C \left( \frac{\delta}{\epsilon} \right)^2 n) + 3 \exp(-\tilde{c}_1 \delta n), \forall n \geq n_1.$$  

Remark 4.2. If $\delta < \delta_0 = \left( \frac{\log \lambda}{\log \lambda_v} \right)$, then

$$\mes \left( \left\{ x \in \mathbb{T} : |u_n^a(x, E, \omega) - L_n(E, \omega)| > \delta \right\} \right) < \exp(-\tilde{c}_a \delta^2 n), \forall n \geq n_1,$$

where $\tilde{c}_a = \frac{c_{\alpha,a}}{20}.$

5. Applications of Avalanche Principle and the Positive Lyapunov Exponents on an Interval

Avalanche Principle is the following:

Proposition 5.1 (Avalanche Principle). Let $A_1, \ldots, A_n$ be a sequence of $2 \times 2$–matrices whose determinants satisfy

$$\max_{1 \leq j \leq n} |\det A_j| \leq 1.$$  

Suppose that

$$\min_{1 \leq j \leq n} \|A_j\| \geq \gamma > n$$  

and

$$\max_{1 \leq j \leq n} \left[ \log \|A_{j+1}\| + \log \|A_j\| - \log \|A_{j+1}A_j\| \right] < \frac{1}{2} \log \gamma.$$  

Then

$$\left| \log \|A_n \cdot \ldots \cdot A_1\| + \sum_{j=2}^{n-1} \log \|A_j\| - \sum_{j=1}^{n-1} \log \|A_{j+1}A_j\| \right| < C \frac{a^n}{\gamma}$$

with some absolute constant $C$.

Lemma 5.1. Assume $L_n(E, \omega) > \delta = \min \left( 1, \delta_0, \frac{1}{15} L_n(E, \omega) \right)$, $L_{2n}(E, \omega) > \frac{\delta}{10} L_n(E, \omega)$, $\beta < c_{\alpha,a} \min(\frac{L_n(E, \omega)}{15}, |D|)$ and $\lambda_\nu > \lambda_0$. Let $N = mn$, $m \in \mathbb{N}$ and $\exp(\frac{\tilde{c}_a}{3} \delta^2 n) \leq m \leq \exp(\frac{1}{3} \delta^2 n) + 1$. There exists $n_2 = n_2(\lambda_\nu, a, \lambda_\nu, \nu, \delta)$ such that for any $n \geq n_2$,

$$\left| L_N(E, \omega) + L_n(E, \omega) - 2L_{2n}(E, \omega) \right| \leq \exp(-\frac{\tilde{c}_a \delta^2 n}{20}).$$

Proof. By (4.4), we have, for $0 \leq j \leq m - 1$ and $\forall x \in \mathcal{G},$

$$|u_n^a(x + j \nu, E, \omega) - L_n(E, \omega)| < \frac{L_n(E, \omega)}{15},$$

$$|u_n^a(x + j \nu, E, \omega) - L_{2n}(E, \omega)| < \frac{L_n(E, \omega)}{15},$$

with

$$\mes (\mathbb{T} \setminus \mathcal{G}) \leq 2m \times \exp \left( -\frac{\tilde{c}_a \delta^2 n}{3} \right) < 2 \exp \left( -\frac{2\tilde{c}_n}{3} \delta^2 n \right).$$

Thus, when $x \in \mathcal{G},$

$$\|M_n^a(x + j \nu, E, \omega)\| > \exp\left( \frac{14}{15} nL_n(E, \omega) \right),$$

and

$$\left| \log \|M_n^a(x + j \nu, E, \omega)\| + \log \|M_n^a(x + (j + 1) \nu, E, \omega)\| - \log \|M_n^a(x + j \nu, E, \omega)M_n^a(x + (j + 1) \nu, E, \omega)\| \right|$$

$$< 4n \frac{L_n(E, \omega)}{100} + 2n|L_n(E, \omega) - L_{2n}(E, \omega)| + \frac{14}{15} nL_n(E, \omega).$$
Therefore, Avalanche Principle applies for \( \gamma = \exp(\frac{14}{15}nL_n(E)) \). Integrating over \( G \), we obtain

\[
\begin{align*}
(5.6) \quad & \left| \int_G u_n(x, E, \omega)dx + \frac{1}{m} \int_G \sum_{j=2}^{n-1} u_n(x + (j - 1)n\omega, E, \omega)dx - \frac{2}{m} \int_G \sum_{j=1}^{n-1} u_a(x + (j - 1)n\omega, E, \omega)dx \right| \\
& \leq C \frac{m}{N} \exp(-\frac{14}{15}nL_n(E, \omega)).
\end{align*}
\]

We want to replace integration over \( G \) by integration over \( T \). By the Cauchy-Schwartz inequality, it has for any \( E, \omega \) and \( n \),

\[
\left| \int_{T \setminus G} u_n^m(x, E, \omega)dx \right| \leq \|u_n^m(\cdot, E, \omega)\|_{L^2(T)} \left( \text{mes} \left( T \setminus G \right) \right)^{\frac{1}{2}} < \|u_n^m(\cdot, E, \omega)\|_{L^2(T)} \exp(-\frac{c_u \delta^2}{3}n).
\]

Thus

\[
\begin{align*}
& \left| \int_{T \setminus G} u_n^m(x, E, \omega)dx + \frac{1}{m} \int_{T \setminus G} \sum_{j=2}^{n-1} u_n^m(x + (j - 1)n\omega, E, \omega)dx - \frac{2}{m} \int_{T \setminus G} \sum_{j=1}^{n-1} u_a(x + (j - 1)n\omega, E, \omega)dx \right| \\
& \leq 4\|u_n^m(\cdot, E, \omega)\|_{L^2(T)} \exp(-\frac{c_u \delta^2}{3}n).
\end{align*}
\]

Combining it with (5.6), we have

\[
\begin{align*}
& |L_n(E, \omega) + \frac{m-2}{m}L_n(E, \omega) - \frac{2}{m}L_{2n}(E, \omega)| \leq 4\|u_n^m(\cdot, E, \omega)\|_{L^2(T)} \exp(-\frac{c_u \delta^2}{3}n) + C \frac{m}{N} \exp(-\frac{14}{15}nL_n(E, \omega)).
\end{align*}
\]

Thus, if \( \exp(\frac{c_u \delta^2}{3}n) \leq m \leq \exp(\frac{c_u \delta^2}{3}n) + 1 \), then

\[
\begin{align*}
& |L_n(E, \omega) + L_n(E, \omega) - 2L_{2n}(E, \omega)| \\
& \leq 4\|u_n^m(\cdot, E, \omega)\|_{L^2(T)} \exp(-\frac{c_u \delta^2}{3}n) + C \frac{m}{N} \exp(-\frac{14}{15}nL_n(E, \omega)) + \frac{2}{m} |L_n(E, \omega) - L_{2n}(E, \omega)| \\
& < 4 \sup_{EE} \|M^x(\cdot, E, \omega)\|_{L^2(T)} \exp(-\frac{c_u \delta^2}{4}n) + C \frac{m}{N} \exp(-\frac{14}{15}nL_n(E, \omega)) + \frac{1}{5m} L_n(E, \omega) \\
& \leq \exp(-\frac{c_u \delta^2}{4}n), \quad \forall n \geq n_2,
\end{align*}
\]

where

\[
(5.7) \quad n_2 = n_2(\alpha, \alpha, \lambda_0, \nu, \delta) = \frac{12 \log \left( 5 \sup_{EE} \|M^x(\cdot, E, \omega)\|_{L^2(T)} \right)}{c_u \delta^2}.
\]

Lemma 5.2. Assume \( L_n(E, \omega) > 0, \delta = \min \{ 1, \delta_0, \frac{1}{10}L_n(E, \omega) \}, L_{2n}(E, \omega) > \frac{2}{10}L_n(E, \omega), \beta < c_{v, n} \min \left( \frac{L_n(E, \omega)}{5}, |D| \right) \) and \( \lambda_n > \lambda_0 \). Then, for any \( n \geq n_2 \),

\[
(5.8) \quad |L(E, \omega) + L_n(E, \omega) - 2L_{2n}(E, \omega)| < \exp\left( -\frac{c_u \delta^2 n}{4} \right).
\]

Proof. By lemma 5.1 for \( N_0 = n, N_1 = mN_0 \) and \( \exp(\frac{c_u \delta^2 N_0}{4}) \leq m < \exp(\frac{c_u \delta^2 N_0}{4}) + 1 \), we have

\[
(5.9) \quad |L_{N_1}(E, \omega) + L_{N_0}(E, \omega) - 2L_{2N_0}(E, \omega)| < \exp(-\frac{c_u \delta^2 N_0}{4}).
\]

In particular

\[
|L_{N_1}(E, \omega) - L_{2N_0}(E, \omega)| < 2 \exp(-\frac{c_u \delta^2 N_0}{4}).
\]

Since \( 0 \leq L_{N_0}(E, \omega) - L_{2N_0}(E, \omega) < \frac{1}{10}L_{N_0}(E, \omega) \) and (5.9), we obtain that

\[
L_{N_1}(E, \omega) > L_{N_0}(E, \omega) - 2(L_{N_0}(E, \omega) - L_{2N_0}(E, \omega)) - \exp(-\frac{c_u \delta^2 N_0}{4}) > \frac{4}{5}L_{N_0}(E, \omega) - \exp(-\frac{c_u \delta^2 N_0}{4}) > 79\delta,
\]
and
\[ |L_{N_1}(E, \omega) - L_2N_1(E, \omega)| \leq 2 \exp\left(-\frac{\bar{c}_u}{20} \delta^2 N_0\right) < 2 \delta < \frac{2}{79} L_{N_1}(E, \omega) < \frac{1}{10} L_{N_1}(E, \omega). \]

Set \( \delta' = \frac{1}{60} \delta \). Then \( L_{N_1}(E, \omega) > 15 \delta' \), and Lemma 5.1 applies for \( N_2 = m_1 N_1 \) and \( \exp\left(\frac{\bar{c}_u}{60} \delta^2 N_1\right) \leq m_1 < \exp\left(\frac{\bar{c}_u}{60} \delta^2 N_1\right) + 1 \). Therefore,
\[ |L_{N_1}(E, \omega) + L_{N_1}(E, \omega) - 2L_{2N_1}(E, \omega)| \leq \exp\left(-\frac{\bar{c}_u}{4} \delta^2 N_1\right). \]

\[ L_{N_1}(E, \omega) > L_{N_1}(E, \omega) - 2|L_{N_1}(E, \omega) - L_2N_1(E, \omega)| - \exp\left(-\frac{\bar{c}_u}{4} \delta^2 N_1\right) > \frac{4}{5} L_{N_1}(E, \omega) - 6 \exp\left(-\frac{\bar{c}_u}{4} \delta^2 N_0\right) > 79 \delta > 100 \delta', \]

\[ |L_{2N_1}(E, \omega) + L_{N_1}(E, \omega) - 2L_{2N_1}(E, \omega)| \leq \exp\left(-\frac{\bar{c}_u}{4} \delta^2 N_1\right), \]

and
\[ |L_{N_1}(E, \omega) - L_{2N_1}(E, \omega)| < 2 \exp\left(-\frac{\bar{c}_u}{4} \delta^2 N_1\right). \]

Since \( N_1 > 8N_0 \), we have
\[ \exp\left(-\frac{\bar{c}_u}{4} \delta^2 N_1\right) = \exp\left(-\frac{\bar{c}_u}{4} \delta^2 N_0\right) < \left(\exp\left(-\frac{\bar{c}_u}{4} \delta^2 N_0\right)\right)^2 < \left(\frac{\delta}{12}\right)^2. \]

This implies in particular that
\[ |L_{N_1}(E, \omega) - L_{2N_1}(E, \omega)| < 2 \exp\left(-\frac{\bar{c}_u}{4} \delta^2 N_1\right) < 2 \delta < \frac{1}{10} L_{N_1}(E, \omega). \]

Then Lemma 5.1 applies for \( N_3 = m_2 N_2 \) and \( \exp\left(\frac{\bar{c}_u}{60} \delta^2 N_2\right) \leq m_2 < \exp\left(\frac{\bar{c}_u}{60} \delta^2 N_2\right) + 1 \). E.T.C.. We obtain \( N_{i+1} = m_i N_i \) and \( \exp\left(\frac{\bar{c}_u}{60} \delta^2 N_i\right) \leq m_i < \exp\left(\frac{\bar{c}_u}{60} \delta^2 N_i\right) + 1 \). Then
\[ |L_{N_{i+1}}(E, \omega) + L_{N_i}(E, \omega) - 2L_{2N_i}(E, \omega)| \leq \exp\left(-\frac{\bar{c}_u}{4} \delta^2 N_i\right), \]

\[ L_{N_{i+1}}(E, \omega) > L_{N_i}(E, \omega) - 2|L_{N_i}(E, \omega) - L_{2N_i}(E, \omega)| - \exp\left(-\frac{\bar{c}_u}{4} \delta^2 N_i\right) > \frac{4}{5} L_{N_i}(E, \omega) - \sum_{j=1}^{1} \frac{1}{2} \delta^j \geq 79 \delta > 50 \delta = 100 \delta', \]

\[ |L_{2N_{i+1}}(E, \omega) + L_{N_i}(E, \omega) - 2L_{2N_i}(E, \omega)| \leq \exp\left(-\frac{\bar{c}_u}{4} \delta^2 N_i\right), \]

\[ |L_{N_{i+1}}(E, \omega) - L_{2N_{i+1}}(E, \omega)| 
< 2 \delta < \frac{1}{10} L_{N_{i+1}}(E, \omega). \]

Moreover,
\[ |L_{N_{i+1}}(E, \omega) - L_{N_i}(E, \omega)| \leq |L_{N_{i+1}}(E, \omega) + L_{N_i}(E, \omega) - 2L_{2N_i}(E, \omega)| + 2|L_{N_i}(E, \omega) - L_{2N_i}(E, \omega)| < \exp\left(-\frac{\bar{c}_u}{4} \delta^2 N_i\right) + 4 \exp\left(-\frac{\bar{c}_u}{4} \delta^2 N_{i-1}\right) < 5 \exp\left(-\frac{\bar{c}_u}{4} \delta^2 N_{i-1}\right), \ i \geq 2, \]

and
\[ |L_{N_i}(E, \omega) - L_{N_i}(E, \omega)| < 5 \exp\left(-\frac{\bar{c}_u}{4} \delta^2 N_0\right). \]
Since \( L_{N_i} \to L(E, \omega) \) with \( i \to \infty \), we have
\[
|L(E, \omega) + L_{N_0}(E, \omega) - 2L_{2N_0}(E, \omega)|
\leq \sum_{i \geq 1} |L_{N_{i+1}}(E, \omega) - L_{N_i}(E, \omega)| + |L_{N_i}(E, \omega) + L_{N_0}(E, \omega) - 2L_{2N_0}(E, \omega)|
\leq \sum_{i \geq 2} |L_{N_{i+1}}(E, \omega) - L_{N_i}(E, \omega)| + |L_{N_i}(E, \omega) - L_{N_1}(E, \omega)| + |L_{N_1}(E, \omega) + L_{N_0}(E, \omega) - 2L_{2N_0}(E, \omega)|
\leq \sum_{i \geq 2} 5 \exp\left(-\frac{\tilde{c}_u}{4} \delta^2 N_0\right) + \sum_{i \geq 2} \exp\left(-\frac{\tilde{c}_u}{4} \delta^2 N_0\right) + \exp\left(-\frac{\tilde{c}_u}{4} \delta^2 N_0\right)
\leq \exp\left(-\frac{\tilde{c}_u}{4} \delta^2 N_0\right).
\]

\[\square\]

**Lemma 5.3.** Assume \( L(E_0, \omega_0) > 0 \). There exists \( n_3 = n_3(\alpha, \lambda, \mu, \nu, L(E_0, \omega_0)) \) such that for any \( n \geq n_3 \), if \( |E - E_0| < \frac{L(E_0, \omega_0)}{2n_3} \), then
\[
|L_n(E_0, \omega_0) - L_n(E, \omega_0)| \leq \frac{L(E_0, \omega_0)}{100}.
\]

**Proof.** Note that
\[
\|M_n^q(x, E_0, \omega) - M_n^q(x, E, \omega)\| \leq \|M_n^q(x, E_0, \omega) - M_n^q(x, E, \omega)\|
\leq \sum_{j=0}^{n-1} \|M^q(x + (n - 1)\omega, E_0, \omega) \times \cdots \times M^q(x + j\omega, E_0, \omega) \times \|M^q(x + (n - 1)\omega, E, \omega) \times \cdots \times M^q(x, E, \omega)\|
\leq ne^{(n-1)M_0}|E_0 - E|.
\]

By (1.9), we have
\[
\|M_n^q(x, E_0, \omega)\| - \|M_n^q(x, E, \omega)\| \leq \exp\left(-\frac{1}{2} \sum_{j=0}^{n-1} d(x + j\omega, \omega)\right) \|M_n^q(x, E_0, \omega)\| - \|M_n^q(x, E, \omega)\|
\leq \exp\left(-\frac{1}{2} \sum_{j=0}^{n-1} d(x + j\omega, \omega)\right) |E_0 - E|.
\]

Assume, for instance, that \( \|M_n^q(x, E_0, \omega)\| \geq \|M_n^q(x, E, \omega)\| \). Then
\[
\log \|M_n^q(x, E_0, \omega)\| - \log \|M_n^q(x, E, \omega)\| = \log(1 + \frac{\|M_n^q(x, E_0, \omega)\| - \|M_n^q(x, E, \omega)\|}{\|M_n^q(x, E, \omega)\|})
\leq \frac{1}{\|M_n^q(x, E_0, \omega)\| - \|M_n^q(x, E, \omega)\|} (\|M_n^q(x, E_0, \omega)\| - \|M_n^q(x, E, \omega)\|)
\leq \frac{1}{\|M_n^q(x, E_0, \omega)\| - \|M_n^q(x, E, \omega)\|} \|M_n^q(x, E_0, \omega)\| - \|M_n^q(x, E, \omega)\|
\leq ne^{(n-1)M_0} \exp\left(-\frac{1}{2} \sum_{j=0}^{n-1} d(x + j\omega, \omega)\right) |E_0 - E|.
\]

(5.11)
Thus, due to Lemma 3.4, we have that if $\beta \leq c_{v,a}|D|$, then there exists $\mathbb{B}_D$ satisfying $\exp(\tilde{c}_{v,a}|D|n)$ such that when $x \notin \mathbb{B}_D$,

$$
\sum_{j=0}^{n-1} \frac{1}{2}d(x + j\omega) > nD - |D| \geq -2n|D|.
$$

The same estimate holds if $\|M_n^a(x, E_0, \omega)\| \leq \|M_n^b(x, E, \omega)\|$. So

$$(5.12) \quad \quad \log \|M_n^a(x, E_0, \omega)\| - \log \|M_n^a(x, E, \omega)\| \leq n \exp((n-1)M_0 \exp(2|D|n))D_0 - E, $$

when $x \notin \mathbb{B}_D$. Set $r_E(n) = \frac{L(E, \omega_0)}{200}$ $n \exp((n-1)M_0 \exp(2|D|n))^{-1}$. Therefore, if $|E - E_0| \leq r_E(n)$, then

$$
\left| \int_{\mathbb{B}_D} u_n^a(x, E_0, \omega_0) - \int_{\mathbb{B}_D} u_n^a(x, E, \omega_0) \right| < \frac{L(E_0, \omega_0)}{200}.
$$

By the Cauchy-Schwarz inequality, it has for any $E, \omega$ and $n$,

$$
\left| \int_{\mathbb{B}_D} u_n^a(x, E, \omega)dx \right| \leq \|u_n^a(\cdot, E, \omega)\|_{L^2(\mathbb{T})} \sup_{E \in \mathbb{E}} \|M_n^a(\cdot, E, \omega_0)\|^{-1} < \sup_{E \in \mathbb{E}} \|M_n^a(\cdot, E, \omega_0)\|^{-1} \exp(-\frac{\tilde{c}_{v,a}}{2}|D|n).
$$

Thus, there exists $n_3 = n_3(\lambda_0, \lambda, \nu, L_0, \omega_0)$ satisfying

$$(5.13) \quad \quad n_3 = \frac{2}{\tilde{c}_{v,a}|D|} \log \left[ \frac{L(E_0, \omega_0)}{200} \sup_{E \in \mathbb{E}} \|M_n^a(\cdot, E, \omega_0)\|^{-1} \right]
$$

such that for any $n \geq n_3$, if $|E - E_0| < r_E(n)$, then

$$
|L_n(E_0, \omega_0) - L_n(E, \omega_0)| \leq \frac{L(E_0, \omega_0)}{100}.
$$

\[ \square \]

Now, we can get an interval centered at $E_0$, where the Lyapunov exponent is always positive.

Lemma 5.4. Assume $L(E_0, \omega_0) > 0$, $\beta(\omega_0) < c_{v,a} \min \left( \frac{L(E_0, \omega_0)}{15}, |D| \right)$ and $\lambda_0 > \lambda_0$. There exists $r_E > 0$ such that for any $|E - E_0| < r_E$.

$$
\frac{6}{5}L(E_0, \omega_0) > L(E, \omega_0) > \frac{4}{5}L(E_0, \omega_0).
$$

Proof. By the subadditive property, there exists $n_4 = n_4(\lambda_0, \lambda, \nu, \omega_0, E_0)$ such that for any $n \geq n_4$,

$$
L_n(E_0, \omega_0) - L(E_0, \omega_0) < \frac{L(E_0, \omega_0)}{100} \quad \text{and} \quad L_n(E_0, \omega_0) - L_{2n}(E_0, \omega_0) < \frac{L(E_0, \omega_0)}{100}.
$$

Thus, for any $|E - E_0| \leq r_E(2n_4)$, it has

$$
L_n(E, \omega_0) \geq L(E, \omega_0) - |L_n(E, \omega_0) - L_n(E_0, \omega_0)| - |L_n(E_0, \omega_0) - L(E, \omega_0)|
$$

$$
> L(E, \omega_0) - \frac{L(E_0, \omega_0)}{100} - \frac{L(E_0, \omega_0)}{100} = \frac{49}{50}L(E_0, \omega_0) > 0,
$$

and

$$
|L_n(E, \omega_0) - L_{2n}(E, \omega_0)| \leq |L_n(E, \omega_0) - L_n(E_0, \omega_0)| + |L_n(E_0, \omega_0) - L_{2n}(E_0, \omega_0)| + |L_{2n}(E_0, \omega_0) - L_{2n}(E, \omega_0)|
$$

$$
< \frac{L(E_0, \omega_0)}{100} + \frac{L(E_0, \omega_0)}{100} + \frac{L(E_0, \omega_0)}{100} = \frac{3}{10}L(E_0, \omega_0) < \frac{1}{10}L_n(E, \omega_0).
$$

Thus, Lemma 5.2 applies for $L_n(E, \omega_0)$. It implies that if $n > n_4$, then

$$
L(E, \omega_0) > L_n(E, \omega_0) - |L_n(E, \omega_0) - L_{2n}(E, \omega_0)| - \exp(-\tilde{c}_{v,a}n) > \frac{441}{500}L(E_0) - \exp(-\tilde{c}_{v,a}n) > \frac{4}{5}L(E_0, \omega_0).
$$

Let $r_E = r_E(2n_4)$ and $L(E, \omega_0) < \frac{5}{4}L(E_0, \omega_0)$ by similar computations. \[ \square \]

Remark 5.1. Due to the compactness in $E$ and the joint continuity of $L(E, \omega)$, there exists $r_\omega$ such that for any $|\omega - \omega_0| \leq r_\omega$ and $|E - E_0| \leq r_E$,

$$
\frac{5}{4}L(E_0, \omega_0) > L(E, \omega) > \frac{3}{4}L(E_0, \omega_0).
$$
When we consider the Schrödinger operators, we can calculate the expression of $r_\omega$:

**Lemma 5.5.** Assume $L'(E_0, \omega_0) > 0$, $\beta(\omega_0) < \frac{1}{100}L'(E_0, \omega_0)$ and $\lambda > \lambda'_0$. There exist $r'_E = r'_E(\lambda_0, E_0, \omega_0)$ such that for any $[\omega - \omega_0] < r'_E$, $\beta(\omega) < \frac{1}{100}L'(E_0, \omega_0)$.

$$\frac{6}{5} L'(E_0, \omega_0) > L'(E, \omega) > \frac{4}{5} L'(E_0, \omega_0).$$

**Proof.**

(5.14) $$\left| \left| M'_n(x, E, \omega_0) \right| - \left| M'_n(x, E, \omega_0) \right| \right| \leq \left| M'_n(x, E, \omega_0) - M'_n(x, E, \omega) \right|$$

$$\leq \sum_{j=0}^{n-1} \left( \left| M'(x + (n-j)\omega_0, E, \omega_0) \times \cdots \times M'(x + j\omega_0, E, \omega_0) \right| \right) \times \left| M'(x + j\omega_0, E, \omega_0) \right| \times \left| M'(x + (j+1)\omega_0, E, \omega) \right| \times \left| M'(x + (j+1)\omega_0, E, \omega) \right|$$

$$\leq n^2 \lambda(V \exp(n - 1)M'_0) |\omega_0 - \omega|,$$

where $V = \max_{T} (\nu'(x))$. Like the proof of Lemma 5.3, similar computations show that for any $n \geq n'_4$, it has

$$\left| L'_n(E_0, \omega) - L'_n(E_0, \omega_0) \right| \leq \frac{L'(E_0, \omega_0)}{100},$$

when $|\omega - \omega_0| \leq r'_E(n)$, where $r'_E(n) = \frac{L'(E_0, \omega_0)}{100} \exp(-nM'_0)$. Similarly,

$$\left| L'_n(E_0, \omega) - L'_n(E, \omega_0) \right| \leq \frac{L'(E_0, \omega_0)}{100},$$

when $|E - E_0| \leq r'_E(n)$, where $r'_E = \frac{L'(E_0, \omega_0)}{100} \exp(-nM'_0)$. Combining them with (5.10), we have

$$\left| L'_n(E, \omega) - L'_n(E_0, \omega_0) \right| \leq \frac{L'(E_0, \omega_0)}{50},$$

when $|\omega - \omega_0| \leq r_0(n)$ and $|E - E_0| \leq r_0(n)$. Thus, Lemma 5.2 holds for $L'_n(E, \omega)$ and this lemma is proved similarly as Lemma 5.4 with the settings $r'_0 = r'_0(\max(n'_2, 2n'_4))$ and $r'_E = r'_E(\max(n'_2, 2n'_4))$, where

$$n'_2 := \min(1, \frac{60M'_0}{c_0})$$

and $n'_4$ is the integer which makes $L'_n(E_0, \omega_0) - L'(E_0, \omega_0) < \frac{L'(E_0, \omega_0)}{100}$ for any $n \geq n'_4$. \qed

6. PROOFS OF THE REST THEOREMS

Before showing the proofs, we first need the following Lemma (Theorem 1.5 in [AJS]) to get the uniform convergence of $u_n'(x, E, \omega)$:

**Lemma 6.1.** The functions $\mathbb{R} \times C^0(\mathbb{T}, \mathcal{L}(\mathbb{C}^d, \mathbb{C}^d)) \ni (\omega, A) \mapsto L(A, \omega) \in [-\infty, \infty]$ are continuous at any $(\omega', A')$ with $\omega' \in \mathbb{R} \setminus \mathbb{Q}$. Here $C^0(\mathbb{T}, \mathcal{L}(\mathbb{C}^d, \mathbb{C}^d))$ means the set of the functions which are complex analytic from $\mathbb{T}$ to $\mathcal{L}(\mathbb{C}^d, \mathbb{C}^d)$.

**Lemma 6.2.** Assume $L(E_0, \omega_0) > 0$, $\beta(\omega_0) < c_{0, \omega} \min \left( \frac{L(E_0, \omega_0)}{15}, |D| \right)$ and $\lambda > \lambda_0$. Then there exists $n_5 = n_5(\omega_0, E_0, r_0, \omega, \lambda_0, a, \lambda_0, v)$ such that for any $n > n_5$, $x \in \mathbb{T}$, $E \in [E_0 - r_0, E_0 + r_0]$ and irrational $\omega \in [\omega_0 - r_0, \omega_0 + r_0]$,

$$u_n'(x, E, \omega) \leq \frac{6}{5} L(E, \omega).$$

**Proof.** Furman [F] proved the uniformity in $x$ for any continuous cocycle on a uniquely ergodic system. Then, due to the continuity of $L^d$ in $E$ by Lemma 6.1 and the compactness, we have

$$\lim_{n \to \infty} u_n'(x, E, \omega) \leq L^d(E, \omega).$$
uniformly in \( x \in \mathbb{T} \) and \( E \in [E_0 - r_E, E_0 + r_E] \). Similarly,
\[
\limsup_{n \to \infty} \frac{1}{2n} \sum_{j=0}^{n-1} d(z + j\omega, \omega) \leq D
\]
uniformly in \( x \in \mathbb{T} \). Thus, this lemma follows directly from (1.11). \(\square\)

**Remark 6.1.** By Theorem 5, if \( \lambda_c > \lambda_p \), then \( L(E, \omega) \) is always positive. Thus, (6.1) holds for any \( E \in \mathcal{E} \) and irrational \( \omega \).

**Remark 6.2.** For the Schrödinger operators, due to the joint continuity of Lyapunov exponent in \((E, \omega)\), we have that if \( L'(E_0, \omega_0) > 0, \beta(\omega_0) < \frac{1}{24} L'(E_0, \omega_0) \) and \( \lambda_s > \lambda_0' \), then for any \( n > n_5' \), \( x \in \mathbb{T}, |E - E_0| \leq r_E' \) and \( |\omega - \omega_0| \leq r_\omega' \), it has
\[
(6.2) \quad u_n'(x, E, \omega) \leq \frac{6}{5} L'(E, \omega).
\]
What’s more, by Remark 3.1, if \( \lambda > \lambda_0'(\frac{1}{3}) \), then for any \( n \geq 1, x \in \mathbb{T}, E \in \mathcal{E}_n \) and irrational \( \omega \),
\[
(6.3) \quad u_n'(x, E, \omega) < \frac{51}{50} \log \lambda \quad \text{and} \quad L_{2n}^t(E, \omega) > \frac{9}{10} L_n^t(E, \omega).
\]

**Proof of Theorem 3.** Remark 5.1 proves the property that \( L(E, \omega) \) is positive in the neighborhood of \((E_0, \omega_0)\). Now we start the proofs of the sharp LDTs for \( u_n'(x, E, \omega) \) and \( u_n(x, E, \omega) \).

Because
\[
M_n^u(x + k\omega, E, \omega)M_n^u(x, E, \omega) = M_n^u(x + n\omega, E, \omega)M_n^u(x, E, \omega)
\]
and \( \|A^{-1}\| = \|A\| \geq 1 \) if \( \det A = 1 \), so
\[
\left| \log \|M_n^u(x + k\omega, E, \omega)\| - \log \|M_n^u(x, E, \omega)\| \right| \leq \log \|M_n^u(x, E, \omega)\| + \log \|M_n^u(x + n\omega, E, \omega)\|.
\]
Due to (6.1), if \( k^2 > \frac{2n_5'(M_0-D)}{L(E_0, \omega_0)} \geq n_5^2 \), \( |E - E_0| \leq r_E \) and \( |\omega - \omega_0| \leq r_\omega \), then
\[
\left| u_n'(x + k\omega, E, \omega) - u_n'(x, E, \omega) \right| \leq \frac{12k}{5n} L(E, \omega).
\]
It implies that
\[
\left| u_n'(x, E, \omega) - \frac{1}{k} \sum_{j=1}^{k} u_n'(x + j\omega, E, \omega) \right| \leq \frac{1}{k} \left( \sum_{j=1}^{n_5} + \sum_{j=n_5+1}^{k} \right) \left[ u_n'(x, E, \omega) - u_n'(x + j\omega, E, \omega) \right]
\]
\[
\leq \frac{1}{nk} \left[ \sum_{j=1}^{n_5} 2jM_0 + \sum_{j=n_5+1}^{k} \frac{12}{5} jL(E, \omega) \right] - \frac{1}{2nk} \left[ \sum_{j=1}^{n_5} \sum_{m=0}^{j-1} d(x + m\omega) - 2jD \right] + \sum_{j=1}^{n_5} \left( \sum_{m=0}^{j-1} d(x + (m + n)\omega) - 2jD \right).
\]
If
\[
\sum_{j=1}^{n_5} \left( \sum_{m=0}^{j-1} d(x + m\omega) - 2jD \right) < -\frac{n_5}{4} kL(E, \omega),
\]
there exists \( 1 \leq j \leq n_5 \) such that
\[
\sum_{m=0}^{j-1} d(x + m\omega) - 2jD < -\frac{n}{4} kL(E, \omega).
\]
By Lemma 3.4, we have that if \( \beta(\omega) < \frac{c_{v,a}}{4} \kappa L(E, \omega) \), then
\[
\mathrm{mes}\left\{ x \in \mathbb{T} : \sum_{m=0}^{j-1} d(x + m\omega) - 2jD < -\frac{n}{4} \kappa L(E, \omega) \right\} \leq \exp\left(-\frac{c_{v,a}}{4} \kappa L(E, \omega) j\right) = \exp\left(-\frac{c_{v,a}}{4} \kappa L(E, \omega) n\right).
\]
Thus, there exist
\[
 (6.4) \quad n_6 := \frac{16 \log \frac{32}{\bar{c}_{v,a} \kappa L(E_0, \omega_0)}}{\bar{c}_{v,a} \kappa L(E_0, \omega_0)}
\]
and \( \mathcal{B} \) satisfying
\[
\mathrm{mes}\left(\mathcal{B}\right) \leq 2n_5 \exp\left(-\frac{\bar{c}_{v,a}}{4} \kappa L(E, \omega) n\right) \leq \exp\left(-\frac{\bar{c}_{v,a}}{8} \kappa L(E, \omega) n\right),
\]
such that if \( x \notin \mathcal{B} \) and \( n = \frac{2k}{\eta} > n_6 \), then
\[
-\frac{1}{2nk} \left[ \sum_{j=1}^{n_6} \sum_{m=0}^{j-1} d(x + m\omega) - 2jD \right] + \frac{1}{nk} \left[ \sum_{j=1}^{n_6} \sum_{m=0}^{j-1} d(x + (m+n)\omega) - 2jD \right] \leq \frac{1}{4} \kappa L(E, \omega)
\]
and
\[
\left| u_n^\ast(x, E, \omega) - \frac{1}{k} \sum_{j=1}^{k} u_n^\ast(x + j\omega, E, \omega) \right| \leq \frac{1}{nk} \left[ \sum_{j=1}^{n_6} 2j(M_0 - D) + \sum_{j=n_6+1}^{k} \frac{12}{5} jL(E, \omega) \right] + \frac{1}{4} \kappa L(E, \omega)
\]
\[
\leq \frac{n_6^2 (M_0 - D)}{nk} + \frac{6k}{5n} L(E, \omega) + \frac{1}{4} \kappa L(E, \omega)
\]
\[
\leq \frac{\kappa}{8} L(E_0, \omega_0) + \frac{11}{20} \kappa L(E, \omega) < \frac{3}{4} \kappa L(E, \omega).
\]
Let \( \delta = \frac{\kappa}{4} L(E, \omega) \) in Remark 3.3 and
\[
\hat{h} = \max(n_1, n_2, n_3, 2n_4, n_5, n_6),
\]
where \( n_1, n_2, n_3, n_4, n_5 \) and \( n_6 \) are defined in (4.3), (5.7), (5.13), the proof of Lemma 5.4, the proof of Lemma 6.2 and (6.4), respectively. Then, redefine \( r_E \) as
\[
r_E = r_E(\hat{h}) = \frac{L(E_0, \omega_0)}{200\hat{h}} \exp\left( ((1 - \hat{h})M_0 - 2|D|\hat{h}) \right).
\]
Therefore, if \( \beta < c_{v,a} \min\left(\frac{\delta L(E, \omega)}{12}, \frac{L(E, \omega)}{12}, |D|\right) \), \( \lambda_0 \) and \( n > \hat{h} \), then for any \( |E - E_0| \leq r_E \) and \( |\omega - \omega_0| \leq r_\omega \),
\[
(6.5) \quad \mathrm{mes}\left\{ x : \left| u_n^\ast(x, E, \omega) - L_n(E, \omega) \right| < \kappa L(E, \omega) \right\} < \exp\left(\frac{\bar{c}_{v,a}}{8} \kappa L(E, \omega) n\right) + \exp\left(-\frac{\bar{c}_{v,a}}{25} \kappa^2 L(E, \omega) n\right)
\]
\[
< \exp\left(-\frac{\bar{c}_{v,a}}{30} \kappa^2 L(E, \omega) n\right).
\]
Similarly, we can get the sharp LDT for \( u_n(x, E, \omega) \) by the following relationship between \( u_n(x, E, \omega) \) and \( u_n^\ast(x, E, \omega) \):
\[
u_n(x, E, \omega) = u_n^\ast(x, E, \omega) + \frac{1}{n} \sum_{j=0}^{n-1} \left( \log |\lambda_\alpha(x + j\omega)| - \log |\lambda_\alpha(x + (j + 1)\omega)| \right)
\]
\[
\square
\]
Remark 6.3. It is easily seen that if \( \lambda_0 > \lambda_p \), then the sharp LDT (1.23) holds for any \( E \in \mathcal{E} \) and irrational \( \omega \).

Remark 6.4. For the Schrödinger operators, we have similarly that if \( k = \frac{\kappa}{5} n_5 > n_6 \left(\frac{2M_1}{\bar{c}_{v,a} L(E_0, \omega_0)}\right)^{\frac{1}{2}} \), \( |E - E_0| \leq r_E^0 \) and \( |\omega - \omega_0| \leq r_\omega^0 \), then
\[
\left| u_n^\ast(x + k\omega, E, \omega) - u_n^\ast(x, E, \omega) \right| \leq \frac{12k}{5n} L^\ast(E, \omega)
\]
and

\[ (6.7) \quad \left| u_n^x(x, E, \omega) - \frac{1}{k} \sum_{j=1}^{k} u_n^j(x + j\omega, E, \omega) \right| \leq \frac{1}{k} \left( \sum_{j=1}^{n_k} + \sum_{j=n_k+1}^{k} \right) \left| u_n^x(x, E, \omega) - u_n^j(x + j\omega, E, \omega) \right| \leq \frac{1}{n_k} \left( \sum_{j=1}^{n_k} 2jM_0^j + \sum_{j=n_k+1}^{k} \frac{12}{5} jL^j(E, \omega) \right) \leq \frac{n_k^2 M_0^k}{nk} + \frac{6k}{5n} L^k(E, \omega) \leq \frac{3}{5} \kappa L^k(E, \omega). \]

Combining it with Remark 3.2, we have that if \( \beta(\omega) < \frac{1}{4} \epsilon, \kappa L^k(E, \omega) \) and \( \lambda > 2\epsilon_0^{-1} \), then

\[ (6.8) \quad \min \left\{ x : \left| u_n^x(x, E, \omega) - L_n^x(E, \omega) \right| < \kappa L^k(E, \omega) \right\} < \exp(-\frac{\epsilon}{10^2} L^k(E, \omega)). \]

**Proof of Theorem 4.** Similar to the proofs of Lemma 5.1 and Lemma 5.2, by the sharp LDT (6.6) with \( \kappa = \frac{1}{10} \), the Avalanche Principle can be applied again, and we have

\[ (6.9) \quad \left| L(E, \omega) + L_0(E, \omega) - 2L_2(E, \omega) \right| < \exp(-\frac{1}{48000} \bar{\epsilon}_{v,a} L(E, \omega)n) \leq \exp(-10^{-5} \bar{\epsilon}_{v,a} L(E_0, \omega_0)n). \]

On the other hand, by (5.10), (5.11) and Lemma 6.1, it implies that

\[ \left| \log \| M_n^x(x, E_1, \omega) \| - \log \| M_n^x(x, E_2, \omega) \| \right| \leq \frac{\left( \| M_n^x(x, E_1, \omega) \| - \| M_n^x(x, E_2, \omega) \| \right)}{\prod_{j=1}^{n_k} d(x + j\omega)^+} \leq \left( \sum_{j=0}^{n_k} \left( \prod_{m=1}^{n_k-j} M^a(x + (n - m)\omega, E_1, \omega) \right) \right) \times \exp(-nD) \times \exp \left( -\sum_{j=0}^{n_k-1} \frac{1}{2} b(x + j\omega) + nD \right) \times |E_1 - E_2| \]

\[ = \left( \sum_{j=0}^{n_k} \left( \prod_{m=1}^{n_k-j} M^a(x + (n - m)\omega, E_1, \omega) \right) \right) \times \exp(-nD) \times \exp \left( -\sum_{j=0}^{n_k-1} \frac{1}{2} b(x + j\omega) + nD \right) \times |E_1 - E_0| \]

\[ \leq \left\{ 2 \sum_{j=1}^{n_k} \exp \left( (M_0 - D)n_5 + \frac{6}{5} \max_{j=1,2} (L(E_j, \omega)n) \right) \right\} \times \exp \left( -\sum_{j=0}^{n_k-1} \frac{1}{2} b(x + j\omega) + nD \right) \times |E_1 - E_2| \]

Due to Lemma 3.4, we have that there exists \( \mathcal{B}' \) satisfying \( \min \left\{ \mathcal{B}' : \exp(-\bar{\epsilon}_{v,a} L(E_0, \omega_0)n) \right\} \) such that if \( x \notin \mathcal{B}' \), then

\[ \left| \log \| M_n^x(x, E_1, \omega) \| - \log \| M_n^x(x, E_2, \omega) \| \right| \leq n \exp(2L(E_0, \omega_0)n) |E_1 - E_2|. \]
Thus
\[ |L_n(E_1, \omega) - L_n(E_2, \omega)| = \int_{\mathbb{T}^d} |u^n_0(x, E_1, \omega) - u^n_0(x, E_2, \omega)| dx + \int_{\mathbb{T}^d} |u^n_0(x, E_1, \omega) - u^n_0(x, E_2, \omega)| dx \]
\[ < \exp(2L(E_0, \omega_0)n)|E_1 - E_2| + 2 \sup_{E \mathcal{E}} \|M'(|E, \omega_0)|_{\mathcal{L}^2_\omega} \exp(-\frac{\tilde{c}_{\text{tra}}}{2}L(E_0, \omega_0)n). \]

Combining it with (6.9), we have
\[ |L(E_1, \omega) - L(E_2, \omega)| \leq \left| L(E_1, \omega) + L_n(E_1, \omega) - 2L_{2n}(E_1, \omega) \right| + \left| L(E_2, \omega) + L_n(E_2, \omega) - 2L_{2n}(E_2, \omega) \right| \]
\[ + 2 \exp(-10^{-5}\tilde{c}_{\text{v,a}}L(E_0, \omega_0)n) + 2 \exp(4L(E_0, \omega_0)n)|E_1 - E_2| \]
\[ + 4 \sup_{E \mathcal{E}} \|M'(|E, \omega_0)|_{\mathcal{L}^2_\omega} \exp(-\frac{\tilde{c}_{\text{v,a}}}{2}L(E_0, \omega_0)n) \]
\[ < 3 \exp(-10^{-5}\tilde{c}_{\text{v,a}}L(E_0, \omega_0)) + 2 \exp(4L(E_0, \omega_0)n)|E_1 - E_2|. \]

Note that \( M_0 - D \geq L(E_0, \omega_0) \). Thus, if \( |E_1 - E_2| < 2r_\delta \), then there exists an integer \( n > \bar{n} \) such that
\[ \exp\left((10^{-5}\tilde{c}_{\text{v,a}} + 4)L(E_0, \omega_0)(n + 1)\right) < |E_1 - E_2| \leq \exp\left((-10^{-5}\tilde{c}_{\text{v,a}} + 4)L(E_0, \omega_0)n\right). \]

Then
\[ |L(E, \omega_0) - L(E_0, \omega_0)| < 5 \exp(-10^{-5}\tilde{c}_{\text{v,a}}L(E_0, \omega_0)n) < \exp(-2 \times 10^5)\tilde{c}_{\text{v,a}}L(E, \omega_0)(n + 1) \leq |E - E_0|, \]
where \( \tau = \frac{\tilde{c}_{\text{v,a}}}{\Omega_{\omega_0} \times 8 \times 10^5} \).

Remark 6.5. Let us outline the proof of Remark 1.9. If \( n > \frac{10M_0}{L(E_0, \omega_0)}n_0^* \), then
\[ \|\log \|M'_n(x, E, \omega_0)\| - \log \|M'_n(x, E, \omega_2)\| \leq \|M'_n(x, E, \omega_1) - M'_n(x, E, \omega_2)\| \]
\[ \leq \sum_{j=1}^{n_1} \sum_{j=n_1+1}^{n-n_1} \sum_{j-n_1+1}^{n} \left( \left\| \prod_{m=1}^{n-j} M'(x + (n - m)\omega_1, E, \omega_1) \right\| \times \prod_{m=1}^{0} M'(x + m\omega_2, E, \omega_2) \right) \]
\[ \leq 2 \sum_{j=1}^{n_1} \exp\left( n_3 M_0 + 6 \max_{i=1,2} (L'(E, \omega_i))(n - n_3) \right) + \sum_{j=n_1+1}^{n} \exp\left( 6 \max_{i=1,2} (L'(E, \omega_i))(n) \right) \times \lambda_s nV|\omega_1 - \omega_2| \]
\[ \leq \sum_{j=1}^{n} \exp\left( \frac{7}{5} L'(E_0, \omega_0)n \right) \times \lambda_s nV|\omega_1 - \omega_2|. \]

Thus, there exists
\[ n_1^* := -\frac{5 \log \frac{L(E_0, \omega_0)}{L'(E_0, \omega_0)}}{\lambda_s nV} \]
such that for any \( n \geq n_1^* , \)
\[ |L_n(E, \omega_1) - L_n(E, \omega_2)| \leq \lambda nV \exp\left( \frac{7}{5} L'(E_0, \omega_0)n \right) \times |\omega_1 - \omega_2| < \exp(2L'(E_0, \omega_0)n) \times |\omega_1 - \omega_2|. \]

Similarly, for any \( n > \frac{10M_0}{L(E_0, \omega_0)}n_0^* , \)
\[ |L_n(E_1, \omega) - L_n(E_2, \omega)| \leq \exp\left( \frac{7}{5} L'(E_0, \omega_0)n \right) \times |E_1 - E_2|. \]

On the other hand, applying (6.8) with \( \kappa = \frac{1}{2} \) to the Avalanche Principle, we have
\[ |L'(E, \omega) + L'_n(E, \omega) - 2L_{2n}(E, \omega)| \leq \exp(-10^{-5}\tilde{c}_{\text{v,a}}L'(E_0, \omega_0)n). \]
Let
\begin{equation}
\tilde{n}_s := \max\left(n_2^s, 2n_3^s, 80\left(\frac{2M_0^s}{L^r(E_0, \omega_0)}\right)^{\frac{1}{2}} n_5^s, \frac{10M_0^s}{L^r(E_0, \omega_0)} n_5^s, n_7^s\right),
\end{equation}
where \(n_5^s\) is defined in (5.15), \(n_4^s\) in Lemma 5.5, \(n_3^s\) in Remark 6.2 and \(n_2^s\) in (6.11). Then, we redefine \(r^s_E\) and \(r^s_\omega\) as follows:
\begin{align*}
r^s_E &= \frac{L^r(E_0, \omega_0)}{200\tilde{n}_s} \exp(-5M_0^s\tilde{n}_s), \\
r^s_\omega &= \frac{L^r(E_0, \omega_0)}{400\max_{\tau(x)}(v'(x))\tilde{n}_s} \exp(-5M_0^s\tilde{n}_s).
\end{align*}
Note that \(5M_0^s > (10^{-5}c_v + 4)L^r(E_0, \omega_0)\). Thus, for any \(E_1, E_2 \in [E_0 - r^s_E, E_0 + r^s_E]\), \(\omega_1, \omega_2 \in [\omega_0 - r^s_\omega, \omega_0 + r^s_\omega]\) satisfying \(\max(\beta(\omega_1), \beta(\omega_2)) < \frac{1}{10}L^r(E_0, \omega_0)\), there exist \(n_6^s\) and \(n_7^s\) such that
\[
|E_1 - E_2| \sim \exp\left(-(10^{-5}c_v + 4)L^r(E_0, \omega_0)n_6^s\right), |\omega_1 - \omega_2| \sim \exp\left(-(10^{-5}c_v + 4)L^r(E_0, \omega_0)n_7^s\right).
\]

Therefore,
\[
|L^r(E_1, \omega_1) - L^r(E_2, \omega_2)| \leq |L^r(E_1, \omega_1) + L^r(E_2, \omega_2)| + |L^r(E_1, \omega_1) - L^r(E_2, \omega_2)| + |L^r(E_1, \omega_1) - L^r(E_2, \omega_2)| + |L^r(E_1, \omega_1) - L^r(E_2, \omega_2)|
\]
\[
< 2 \exp(-10^{-5}c_vL(E_0, \omega_0)n_6^s) + 2 \exp(4L(E_0, \omega_0)n_7^s)|\omega_1 - \omega_2|
\]
\[
< 4 \exp(-10^{-5}c_v,\omega L(E_0, \omega_0)n_7^s) < |\omega_1 - \omega_2|^7,
\]
and similarly,
\[
|L^r(E_1, \omega_1) - L^r(E_2, \omega_2)| < |E_1 - E_2|^7.
\]

Remark 6.6. If \(\lambda_s > \max\left(L^r_p(\frac{1}{50}), SV\right)\), then \(L^r(E, \omega)\) is always positive for any \(E \in E_s\) and irrational \(\omega\). Thus, we do not need to apply the LDT and the Avalanche Principle to obtain the interval where the Lyapunov exponent is positive. Furthermore, due to Remark 6.2, we have
\[
n_5^s = 1, 80\left(\frac{2M_0^s}{L^r(E_0, \omega_0)}\right)^{\frac{1}{2}} n_5^s < 160, 10M_0^s \frac{10M_0^s}{L^r(E_0, \omega_0)} n_5^s < 20, n_7^s < 20.
\]
Overall, the integers \(n_5^s\) and \(n_7^s\) only need to be larger than 160. Therefore, (1.26) holds for any \(|E_1 - E_2| < \lambda_s^{-800}\) and \(|\omega_1 - \omega_2| < \lambda_s^{-800}\) satisfying \(\max(\beta(\omega_1), \beta(\omega_2)) < \frac{1}{10}L^r(E_0, \omega_0)\).

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