FORMAL CONNECTIONS FOR FAMILIES OF STAR PRODUCTS

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Abstract. We define the notion of a formal connection for a smooth family of star products with fixed underlying symplectic structure. Such a formal connection allows one to relate star products at different points in the family. This generalizes the formal Hitchin connection defined in [And12]. We establish a necessary and sufficient condition that guarantees the existence of a formal connection, and we describe the space of formal connections for a family as an affine space modelled by the derivations of the star products. Moreover we show that if the parameter space has trivial first cohomology group any two flat formal connections are related by an automorphism of the family of star products.

1. Introduction

1.1. Quantization and the Hitchin connection. In his seminal paper [Wit89] Witten investigated quantum Chern-Simons theory, a 3-dimensional topological quantum field theory (TQFT). There he also proposed the geometric quantization of the moduli space of flat connections on a Riemann surface $\Sigma$ as the 2-dimensional part of this theory. This moduli space has a natural symplectic structure $\omega$ and admits a prequantum line bundle, i.e. a Hermitian line bundle $L$ with a compatible connection, whose curvature is given by the symplectic form.

The Teichmüller space $T$ of the surface $\Sigma$ parametrizes complex structures on the moduli space, so for each point $\sigma \in T$ and each natural number $k$, called the level of quantization, we have the quantum state space of geometric quantization, which is the space

$$Q_k(\sigma) = H^0(M_\sigma; L^k)$$

of holomorphic sections of the $k$-th tensor power of the prequantum line bundle. These form the fibres of a vector bundle $Q$ over $T$, called the Verlinde bundle, and it was shown independently by Hitchin [Hit90] and Axelrod, Della Pietra and Witten [ADPW91] that this bundle admits a natural projectively flat connection, which we shall call the Hitchin connection (see also [And12] for a purely finite dimensional differential geometric approach to this connection). Consequently, the quantum spaces associated with different complex structures are identified, as projective spaces, through the parallel transport of this connection.

1.2. Formal connections. On a Poisson manifold $M$, a deformation quantization, or star product, is a $\mathbb{C}[[\hbar]]$-linear product on the space $C^\infty(M)[[\hbar]]$ that is associative, reduces to the pointwise product modulo $\hbar$, and such that the component of degree 1 in $\hbar$ of its commutator is the Poisson bracket.

The first named author of this paper has studied the asymptotic relationship between Toeplitz operators and the Hitchin connection in [And06], and extended his asymptotic analysis of the relationship between the Hitchin connection and the Toeplitz operators to higher orders [And12], which led him to define the following notion.

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Definition 1.1. If $M$ is a symplectic manifold equipped with a smooth family of star products parametrized by a manifold $\mathcal{T}$, a formal connection on $M$ is a connection on the bundle $\mathcal{T} \times C^\infty(M)[[h]] \to \mathcal{T}$ of the form:

\[ D_V f = V[f] + A(V)(f), \]

where $A$ is a smooth 1-form on $\mathcal{T}$ with values in differential operators on $M$ such that $A = 0 \mod h$, $f$ is a smooth section of the bundle, $V$ is any smooth vector field on $\mathcal{T}$, and $V[f]$ denotes the derivative of $f$ along $V$.

The formal connection is compatible with the family of star products $\{\star_\sigma\}_{\sigma \in \mathcal{T}}$, if it is a derivation of the products at any point in the base space $\mathcal{T}$, i.e.

\[ D_V (f \star_\sigma g) = D_V (f) \star_\sigma g + f \star_\sigma D_V (g), \]

for all $\sigma \in \mathcal{T}$ and all smooth functions $f$ and $g$ on $M$.

Recall that on a Kähler manifold $M$ we always have the so called Berezin-Toeplitz star products $\star_{BT}$, which can be constructed using the theory of Toeplitz operators, see [Sch00]. This leads us to consider the following class of examples of families of star products:

Definition 1.2. Let $M$ be a symplectic manifold with a family of compatible almost complex structures parametrized by a complex manifold $\mathcal{T}$, so that for any $\sigma \in \mathcal{T}$, the manifold $M_\sigma$ is a Kähler manifold, and let $\{\star_{BT_\sigma}\}_{\sigma \in \mathcal{T}}$ be the associated family of Berezin-Toeplitz star products. A formal Hitchin connection on $M$ is a formal connection that is compatible with this family of star products and that is flat.

In [And12] a particular formal Hitchin connection that is associated to the Hitchin connection from geometric quantization was studied, and it was shown that the projective flatness of the Hitchin connection implies the flatness of this formal Hitchin connection. An explicit expression for the 1-form $\tilde{A}(V)$ for the formal Hitchin connection associated to the Hitchin connection was given in [And12, AG11]. The formula reads

\[ \tilde{A}(V)(f) = -V[F] f + V[F] \star_{BT} f + h(E(V)(f) - H(V) \star_{BT} f), \]

where $E$ is a 1-form on $\mathcal{T}$ with values in differential operators on $M$, $H$ is the 1-form with values in $C^\infty(M)$ given by $H(V) = E(V)(1)$, and $F$ is the Ricci potential of the family. The construction will be recalled in more detail in Section 3.

It was furthermore noticed in [And12], that – provided certain cohomology groups of the mapping class group vanish – any formal Hitchin connection could be used to obtain a mapping class group equivariant deformation quantization on the moduli space.

With this motivation in mind, we return to the general case.

1.3. Classification of formal connections. Let $(M, \omega)$ be a symplectic manifold with a family of natural star products $\{\star_\sigma\}_{\sigma \in \mathcal{T}}$ that is parametrized by $\mathcal{T}$.

We first establish the following necessary and sufficient condition for the existence of a compatible formal connection:

Theorem 1.3. There exists a formal connection compatible with $\{\star_\sigma\}_{\sigma \in \mathcal{T}}$ if and only if the characteristic class $[\text{cl}(\star_\sigma)]$ of the star products is locally constant on $\mathcal{T}$.

In fact, we show that even more is true: we provide a natural way to associate a formal connection to every trivialization for the family of characteristic 2-forms associated to the family of star products and establish a formula for its curvature. The proof relies on Fedosov’s geometrical construction of star products [Fed94].
This result specializes to the case of a symplectic manifold equipped with a smooth family of compatible Kähler structures, where we take the family of Berezin-Toeplitz star products associated with them. In this situation Theorem 1.3 yields the existence of a compatible formal connection.

**Theorem 1.4.** Let \((M, \omega)\) be a compact, symplectic manifold, and let \(\mathcal{T}\) be a complex manifold parametrizing a family of compatible Kähler structures \(I_\sigma\) on \(M\), with \(\sigma \in \mathcal{T}\). The family of Berezin-Toeplitz star products associated with the family has constant characteristic class, and therefore admits a compatible formal connection.

Let us assume in the following that the family of star products \(\{\ast_\sigma\}_{\sigma \in \mathcal{T}}\) admits a compatible formal connection. We want to understand the space of all such formal connections. To this end, we denote the space of the derivations of the star product \(\ast_\sigma\) which are trivial modulo \(h\) by \(\text{Der}_0(M, \ast_\sigma)\). We shall see that the formal connections compatible with the family of star products form an affine space over the space of 1-forms on \(\mathcal{T}\) with values in \(\text{Der}_0(M, \ast_\sigma)\).

**Theorem 1.5.** Let \(M\) be a symplectic manifold equipped with a smooth family of star products \(\{\ast_\sigma\}_{\sigma \in \mathcal{T}}\) parametrized by \(\mathcal{T}\). The space \(\mathcal{F}(M, \ast_\sigma)\) of formal connections on \(M\) that are compatible with the family of star products is an affine space over the space of 1-forms on \(\mathcal{T}\) with values in the derivations of the star product that are trivial modulo \(h\), and it can then be written as:

\[
\mathcal{F}(M, \ast_\sigma) = D_0 + \Omega^1(\mathcal{T}, \text{Der}_0(M, \ast_\sigma)),
\]

for a fixed formal connection \(D_0\).

Furthermore, if we assume that \(H^1(M; \mathbb{R})\) vanishes, all derivations of a star product on \(M\) are inner, therefore they are parametrized by an element in \(\tilde{\mathcal{C}}^\infty(M)\), the space of formal functions on \(M\) modulo constants, which allows us to refine the previous result to the following.

**Theorem 1.6.** If \(M\) satisfies \(H^1(M; \mathbb{R}) = 0\), the space \(\mathcal{F}(M, \ast_\sigma)\) of formal connections on \(M\) that are compatible with the family of star products is an affine space over the 1-forms on \(\mathcal{T}\) with values in \(\tilde{\mathcal{C}}^\infty(M)\):

\[
\mathcal{F}(M, \ast_\sigma) \cong D_0 + \Omega^1(\mathcal{T}, \tilde{\mathcal{C}}^\infty(M))
\]

for a fixed formal connection \(D_0\).

### 1.4. Gauge transformations of formal connections

We study the action of gauge transformations on the space of formal connections \(\mathcal{F}(M, \ast_\sigma)\). The transformations which we consider are differential self-equivalences of the family of star products, since the gauge transformations have to preserve the compatibility with the family of star products. If we assume that the parameter space \(\mathcal{T}\) of the family of star products has trivial first cohomology group, i.e. \(H^1(\mathcal{T}, \mathbb{R}) = 0\), we obtain the following result.

**Theorem 1.7.** Let \(M\) be a symplectic manifold with a family of star products \(\{\ast_\sigma\}_{\sigma \in \mathcal{T}}\) parametrized by a smooth manifold \(\mathcal{T}\) with trivial first cohomology group. Let \(D, D' \in \mathcal{F}(M, \ast_\sigma)\) be formal connections for the family and let us assume that they are flat. Then they are gauge equivalent via a self-equivalence of the family of star products \(P \in C^\infty(\mathcal{T}, \mathcal{D}_h(M))\), meaning that

\[
D'_V = P^{-1}D_V P,
\]

for any vector field \(V\) on \(\mathcal{T}\).

In particular, it was shown in [And12] that the formal Hitchin connection associated to the Hitchin connection is flat whenever the Hitchin connection in geometric quantization is projectively flat. This implies the following corollary.
Corollary 1.8. Let $T$ be a smooth manifold with trivial first cohomology group, i.e. $H^1(T, \mathbb{R}) = 0$. If there exists a formal Hitchin connection $D$ in the bundle $T \times C^\infty(M)[[h]]$ over $T$, then it is unique up to gauge-equivalence.

1.5. The formal Hitchin connection at low orders. As mentioned above, a formal connection is called compatible with a family of star products if it is derivation with respect to the star products. Our main example of a formal connection is the formal Hitchin connection defined in [And12], which is known to be a derivation with respect to the family of Berezin-Toeplitz star products, [And12]. This fact relies on the existence of the Hitchin connection in geometric quantization, as well as the link between geometric and deformation quantization via Toeplitz operators.

This result assumes the existence of a Hitchin connection in geometric quantization, which puts several constraints on the objects involved in the construction, among them the condition that the family of Kähler structures be holomorphic and rigid. We shall see that these are quite strong requirements.

On the other hand, the explicit expression (3) that was obtained in [And12] makes sense in a more general situation, and therefore we can ask whether that expression in general gives a derivation of the Berezin-Toeplitz star product. This is a difficult question, because it involves the coefficients of the star product, which are in general hard to understand. But we can give an affirmative answer if we restrict our attention to the first order in the formal parameter. Moreover it is easy to check that the the expression (3) defines a flat formal connection up to first order, and we conclude that the expression obtained in [And12] gives a formal Hitchin connection up to order one.

Proposition 1.9. Let $M$ be a symplectic manifold with a family of compatible Kähler structures parametrized by a complex manifold $T$. Then the expression (3) defines a formal connection that, up to order one in the formal parameter, is a derivation of the family of Berezin-Toeplitz star products on $M$, and which is flat up to order one in the formal parameter. Therefore it defines up to order one a formal Hitchin connection in the sense of our definition above.

We plan to investigate the relationship between the formal Hitchin connection studied by Andersen and the formal connections obtained from Theorem 1.3 more closely in the future.

2. Deformation quantization

2.1. Star products. Let $M$ be a Poisson manifold. Recall that $\mathbb{C}[[h]]$ denotes the ring of formal power series with complex coefficients, and similarly $C^\infty_h(M) = C^\infty(M)[[h]]$ is the algebra of formal functions on $M$, which are formal power series with coefficients in $C^\infty(M)$. Then $C^\infty_h(M)$ is an algebra over $\mathbb{C}[[h]]$, and we can extend the Poisson bracket linearly to make $C^\infty_h(M)$ into a Poisson algebra. This allows us to formulate the following definition.

Definition 2.1. Let $(M, \{\cdot, \cdot\})$ be a Poisson manifold. A (formal) star product on (or deformation quantization of) $M$ is a $\mathbb{C}[[h]]$-bilinear map $\ast: C^\infty_h(M) \times C^\infty_h(M) \to C^\infty_h(M)$ written as

$$f \ast g = \sum_{k=0}^{\infty} c^k(f, g)h^k,$$

where, for $k \in \mathbb{N}$, the maps $c^k: C^\infty(M) \times C^\infty(M) \to C^\infty(M)$ are bilinear and called the coefficients of $\ast$. A star product is required to satisfy the following conditions:

1. associativity: $(f_1 \ast f_2) \ast f_3 = f_1 \ast (f_2 \ast f_3)$,
(2) $c^0(f_1, f_2) = f_1 f_2$,
(3) $c^1(f_1, f_2) - c^1(f_2, f_1) = i \{f_1, f_2\}$,
for all $f_1, f_2, f_3 \in C^\infty(M)$.

A star product is said to be differential if the coefficients $c^k$ are bidifferential operators, in the sense that, for a fixed $f \in C^\infty(M)$, both $c^k(f, \cdot)$ and $c^k(\cdot, f)$ are differential operators for all $k \in \mathbb{N}$. It is null on constants if $f \star 1 = 1 \star f = f$ for all $f \in C^\infty(M)$ (or, equivalently, if $c^k(1, f) = c^k(f, 1) = 0$ for all $k \geq 1$). We shall always assume that star products are null on constants, and this corresponds to the fact that the constant function 1 is the unit in the algebra of formal functions on $M$ equipped with the star product.

**Definition 2.2.** Two star products $\star, \star'$ on $M$ are said to be equivalent if there is a formal power series of linear maps

$$T = \sum_{k=0}^{\infty} T_k, \quad T_k : C^\infty_h(M) \to C^\infty_h(M), \quad k \in \mathbb{N},$$

such that $T_0 = \text{Id}$ and $T(f_1) \star' T(f_2) = T(f_1 \star f_2)$, for $f_1, f_2$ smooth functions on $M$.

In the following, we will restrict attention to natural star products. These are differential star products that satisfy the requirement that the coefficient $c^k$ is a bidifferential operator of order at most $k$. All star products which we will encounter are of this type.

### 2.2. Fedosov star products.

We review the main ingredients of Fedosov’s geometric construction of star products [Fed94]. In Section 5, we will make use of Fedosov’s framework to construct formal connections. Our exposition follows [Fed94] and Waldmann’s book [Wal07].

Let $(M, \omega)$ be a symplectic manifold of dimension $m = 2n$. Since any tangent space $T_x M$ is a symplectic vector space, we can consider the associated Weyl algebra.

**Definition 2.3.** The formal Weyl algebra $W_x$ associated to $T_x M$, for $x \in M$, is the associative $\mathbb{C}$-algebra with unit, whose elements are formal power series in $h$ with polynomials on $T_x M$ as coefficients. This means that an element in $W_x$ has the form:

$$a(y, h) = \sum_{k \in \mathbb{N}} h^k a_{k, \alpha} y^\alpha,$$

where $(y^1, \ldots, y^m)$ are local coordinates on $T_x M$ and $\alpha = (\alpha_1, \ldots, \alpha_m)$ is a multi-index.

The formal Weyl algebra is equipped with the following Moyal-Weyl product:

$$a \circ_{MW} b = \sum_{k=0}^{\infty} \left(\frac{ih}{2}\right)^k \frac{1}{k!} \pi^{i_1j_1} \cdots \pi^{i_kj_k} \frac{\partial^k a}{\partial y^{i_1} \cdots \partial y^{i_k}} \frac{\partial^k b}{\partial y^{j_1} \cdots \partial y^{j_k}},$$

where $\pi = \sum_{i<j} \pi^{ij} \frac{\partial}{\partial y^i} \wedge \frac{\partial}{\partial y^j}$ is the Poisson bivector dual to $\omega_x$. The Moyal-Weyl product is a deformation quantization of the linear symplectic space $(T_x M, \omega_x)$.

Let $W = \bigcup_{x \in M} W_x$. This defines a bundle of algebras over $M$, which is called the Weyl bundle. The space of smooth sections of this bundle, $\Gamma W$, gives an associative algebra with fibre-wise multiplication. This space of section can be thought of as a “quantized tangent bundle” of $M$. Note that the centre of $\Gamma W$ is formed by the elements that do not contain any $y^i$, and therefore is naturally identified with $C^\infty(M) = C^\infty(M)[[h]]$. 


We next define a useful degree on the Weyl algebra, called the total degree. It is given by assigning degree 1 to all the $y^i$, i.e. $\deg y^i = 1$ for any $i$, and $\deg h = 2$. The Moyal-Weyl product is additive with respect to the total degree.

A differential form on $M$ with values in $W$ is a section of $W \otimes \Lambda^q T^* M$, and can be expressed as:

$$a(x, y, h, dx) = \sum h^k a_{k, i_1, \ldots, i_p, j_1, \ldots, j_q} y^{i_1} \ldots y^{i_p} dx^{j_1} \wedge \cdots \wedge dx^{j_q}$$

in local coordinates, where the coefficients $a_{k, i_1, \ldots, i_p, j_1, \ldots, j_q}$ are symmetric in the $i$'s and anti-symmetric in the $j$'s. We denote the space of these differential forms by $\Omega(M, W)$.

Note that the Moyal-Weyl product on $\Gamma(W)$ extends to $\Omega(M, W)$. On the latter space we define an operator $\delta$ in the following way:

$$\delta(a) = \sum_i dx^i \wedge \frac{\partial a}{\partial y^i}, \quad \text{for all } a \in W \otimes \Lambda^q T^* M.$$ 

This operator can also be written as

$$\delta(a) = - \left[ \frac{i}{\hbar} \omega_{ij} dy^j dx^i, a \right] = - \frac{i}{\hbar} \text{ad}(\bar{\omega}),$$

where the commutator is with respect to the Moyal-Weyl product and $\bar{\omega}$ is $\omega$, seen as a section of $T^* M \otimes T^* M \subset W \otimes \Lambda^2 T^* M$.

An alternative interpretation of $\delta$ is as follows: we can define two commuting derivations $\delta$ and $\delta^*$ on $\Omega(M, W)$ by considering the identity morphism $TM \to TM$ as a section of $TM \otimes T^* M$. We can now insert the $TM$-part into either $\Lambda T^* M$ or $W$ and multiply the $T^* M$-part with the other factor, using the pointwise product on $W$. The operator $\delta$ corresponds to the latter case. We define $\delta^*$ to be the operator corresponding to the former.

The operators $\delta$ and $\delta^*$ are differentials, i.e. they square to zero. Moreover, one can use $\delta^*$ as a homotopy operator for $\delta$, which leads to the result that the cohomology of $\delta$ is concentrated in form-degree 0 and that $H^0(\Omega(M, W), \delta) = C^\infty_h(M)$.

**Definition 2.4.** A symplectic connection on a symplectic manifold $(M, \omega)$ is a linear connection $\nabla$ that is torsion-free and such that $\omega$ is parallel with respect to $\nabla$, i.e. $\nabla \omega = 0$.

Let us fix a symplectic connection $\nabla$ on $M$. Its curvature tensor is contracted with $\omega$ to yield an element $R \in \Omega^2(M, W)$. As usual, $\nabla$ extends to the covariant derivative $d_\nabla$ on $\Omega(M, TM)$, which we dualize and extend to $\Omega(M, W)$. It turns out that $d_\nabla$ is also a derivation for the fibre-wise Moyal-Weyl product. Moreover, $d_\nabla$ commutes with $\delta$ and squares to $-\frac{i}{\hbar} \text{ad}(R)$.

Fedosov’s idea is to correct the non-flatness of $-\delta + d_\nabla$ by finding an appropriate $r \in \Omega^1(M, W)$ such that the total operator

$$D_r := -\delta + d_\nabla + \frac{i}{\hbar} \text{ad}(r)$$

squares to zero. Regardless of flatness, an operator of the form of $D_r$ is a derivation of the Moyal-Weyl product defined above, i.e.

$$D_r(a \circ_{MW} b) = D_r(a) \circ_{MW} b + a \circ_{MW} D_r(b)$$

holds. Using Fedosov’s ansatz for $D_r$, one computes

$$D_r^2 = \frac{i}{\hbar} \text{ad} \left( -\omega - \delta r + R + d_\nabla r + \frac{i}{\hbar} r \circ_{MW} r \right).$$
Hence the flatness of $D_r$ is equivalent to the fact that
\begin{equation}
\alpha = \omega + \delta r - R - d\varphi - \frac{i}{h} r \circ_{MW} r
\end{equation}
lies in the center of $\Omega(M, W)$, which coincides with $\Omega(M)[[h]]$, where we see ordinary differential forms to be fibrewise constant polynomials on the tangent spaces.

We call a connection of the form (6) abelian if $\alpha$ satisfies this condition, i.e. is a scalar 2-form. If this is the case, then by the Bianchi identity we have that $d\alpha = D_r \alpha = 0$, and so $\alpha$ is closed (i.e. $\alpha \in Z^2(M)[[h]]$) and is called the Weyl curvature of $D_r$.

The following theorem shows how to construct the appropriate $r$ in order to obtain an abelian $D_r$.

**Theorem 2.5. (Fedosov)** Let $\nabla$ be a symplectic connection on $M$ and 
\[ \alpha = \omega + h\alpha_1 + h^2\alpha_2 + \cdots \in Z^2(M)[[h]] \]
be a closed formal 2-form that is a perturbation of the symplectic form. Then there exists a unique $r \in \Omega^2(M, W)$, such that the $D_r$ given by (6) is an abelian connection with Weyl curvature $\alpha$ and $\delta^* r = 0$.

We are now in position to define the Fedosov star product corresponding to a given Fedosov connection. One first shows that every element $f \in C^\infty(M)[[h]]$ extends uniquely to an element
\[ \tau(f) \in \Omega^0(M, W)[[h]] \]
which is parallel with respect to the Fedosov connection $D_r$. The proof amounts to breaking up the equation $D_r \tau(f) = 0$ into its homogeneous pieces with respect to the total degree. Cohomological considerations similar to the ones in the proof of Theorem 2.5 guarantee that the extension $\tau(f)$ exists and is unique.

The Fedosov star product $\star_\nabla, \alpha$ associated to the closed formal 2-form $\alpha$ and to the symplectic connection $\nabla$ on $M$ is given by the following formula:
\begin{equation}
f \star_\nabla, \alpha g : = p(\tau(f) \circ_{MW} \tau(g)),
\end{equation}
where $p : \Gamma(W) \to \mathcal{C}^\infty_h(M)$ is the projection of a section of the Weyl bundle to its fibrewise constant part. Since $D_r$ is a derivation with respect to $\circ_{MW}$, this defines an associative product and one checks inductively that one actually obtains a natural star product, i.e. $\star_\nabla, \alpha$ is given by bidifferential operators whose component or order $h^k$ is of differential-order at most $k$ in each argument.

**Definition 2.6.** The characteristic 2-form of a Fedosov star product $\star_\nabla, \alpha$ is
\[ \text{cl}(\star_\nabla, \alpha) = \alpha \in Z^2(M; \mathbb{R})[[h]], \]
where $\alpha$ is the Weyl curvature of the corresponding Fedosov connection $D_r$.

The class of $\text{cl}(\star_\nabla, \alpha)$ is called the characteristic class of $\star_\nabla, \alpha$.

The definition of the characteristic class of Fedosov star products generalizes to arbitrary differential star products since every differential star product is equivalent to one of Fedosov type and two Fedosov star products are equivalent if and only if their characteristic classes coincide. Hence one just defines the characteristic class of an arbitrary differential star product to be the characteristic class of an equivalent star product of Fedosov type. For a more extended treatment of classification results, we refer the reader to [Del95], [GR99], [Fed94] and [Wal07].

We note that if we restrict attention to natural star products, a stronger statement holds, see [GR99, Theorem 4.1]: every such star product is equivalent to a preferred Fedosov star product through a preferred equivalence. This allows one to assign not only a characteristic class, but a characteristic 2-form to a natural
star product – just take it to be the characteristic 2-form of the preferred equivalent Fedosov star product.

2.3. The Berezin-Toeplitz star product. We describe the link between geometric quantization and deformation quantization due to [Sch00].

Definition 2.7. A prequantum line bundle over a symplectic manifold \((M, \omega)\) is the data of a complex line bundle \(L\), equipped with a Hermitian metric \(h\) and a compatible connection \(\nabla\) whose curvature satisfies:

\[ F_\nabla = -i\omega. \]

We say that a symplectic manifold is prequantizable if it admits a prequantum line bundle. If we assume that \(M\) is a compact prequantizable Kähler manifold we can make the following definition.

Definition 2.8. Let \(f \in C^\infty(M)\). The Toeplitz operator \(T_{f}^{(k)} : C^\infty(M; L^k) \to H^0(M; L^k)\) is the map defined by

\[ T_{f}^{(k)}(s) = \pi^{(k)}(fs), \]

mapping a smooth section \(s\) on \(L^k\) to its projection to the subspace of holomorphic sections.

Schlichenmaier showed in [Sch00] that any compact Kähler manifold admits a natural star product on it, namely the Berezin-Toeplitz star product.

Theorem 2.9 (Schlichenmaier). There exists a unique star product \(\ast_{BT}\) for \(M\), called the Berezin-Toeplitz star product, and expressed by:

\[ f_1 \ast_{BT} f_2 = \sum_{k=0}^{\infty} c^{(k)}(f_1, f_2)h^k, \]

with \(c^{(k)}(f_1, f_2) \in C^\infty(M)\) determined by the requirement that for all \(f_1, f_2 \in C^\infty(M)\) and for any positive integer \(L\) the following estimate holds:

\[ \left\| T_{f_1,\sigma}^{(k)} T_{f_2,\sigma}^{(k)} - \sum_{l=0}^{L} T_{\omega_l^{(k)}(f_1, f_2),\sigma}^{(k)} k^{-l} \right\| = O(k^{-(L+1)}). \]

Karabegov and Schlichenmaier proved [KS01] that the Berezin-Toeplitz star product is differential, null on constants and of Wick type. The last property means that for any locally defined functions \(f\) and \(g\) with \(f\) anti-holomorphic and \(g\) holomorphic, and any function \(h\) one has

\[ f \ast h = fh \quad \text{and} \quad h \ast g = hg. \]

Remark 2.10. A star product is said to be with separation of variables if it has the same property characterizing star products of Wick type, but with the role of holomorphic and anti-holomorphic switched. For any star product \(\ast\), we can define the opposite star product \(\ast_o\) by: \( f \ast_o g = g \ast f \) for any formal functions \(f\) and \(g\). Therefore, if \(\ast\) is a of Wick type, then \(\ast_o\) is with separation of variables.

Gammelgaard [Gam14] showed that the Berezin-Toeplitz star product can be locally expressed in a graph theoretical way, with weights determined by the geometry of the graph.

The formal Hitchin connection that we define in the next section is related to the Berezin-Toeplitz star product.
3. The Hitchin connection

Here we review briefly the construction of the Hitchin connection, in the differential geometric version of [And12]. The Hitchin connection was introduced by Hitchin in [Hit90], as a connection over the Teichmüller space in the bundle one obtains by applying geometric quantization to the moduli spaces of flat $SU(n)$ connections. Furthermore Hitchin proved that this connection is projectively flat. Hitchin’s construction was motivated by Witten’s study [Wit89] of quantum Chern-Simons theory in 2 + 1 dimensions. In [And12] a differential geometric construction of the Hitchin connection which works for a more general class of manifolds was provided and its existence under certain assumptions on the manifolds was given.

3.1. Smooth families of Kähler structures. Let $(M,\omega)$ be a symplectic manifold and let $T$ be a smooth manifold that parametrizes smoothly a family of Kähler structures on $M$. This means that we have a smooth map

$$I: T \rightarrow C^\infty(M, \text{End}(TM))$$

that associates to each $\sigma \in T$ an integrable and compatible almost complex structure on $M_\sigma$, since $\omega$ is $I_\sigma$ invariant. We can use this isomorphism to define the bivector field:

$$\tilde{\omega} = -(i_\omega^{-1} \otimes i_\omega^{-1})(\omega),$$

which satisfies the identity $\omega \cdot \tilde{\omega} = \tilde{\omega} \cdot \omega = \text{Id}$, where the dot indicated contraction of tensors in their entries closest to the dot, which is relevant when working with non-symmetric tensors. For example when we write $\omega \cdot \tilde{\omega}$ we mean that the right-most entry of $\omega$ is contracted with the left-most one of $\tilde{\omega}$.

Similarly we get a type-interchanging isomorphism $i_{g_\sigma}: TM_C \rightarrow TM_C^\ast$, induced by the Kähler metric on $M_\sigma$. The two isomorphisms are related by the equation:

$$i_{g_\sigma} = I_\sigma i_\omega.$$  

From the fact that $g$ and $\omega$ have type $(1,1)$ it follows that these two isomorphisms exchange types. As done for $\omega$, we can define the inverse metric tensor by

$$\tilde{g} = (i_g^{-1} \otimes i_g^{-1})(g),$$

which gives a symmetric bivector field satisfying the relation $g \cdot \tilde{g} = \tilde{g} \cdot g$. This bivector field is related to the bivector field associated to $\omega$ by the relation $\tilde{\omega} = I \cdot \tilde{g}$.

On an (almost) complex manifold $M$ we have a natural decomposition of the complexified tangent bundle:

$$TM_C = T'^M_I \oplus T''M_I,$$

where the two summands are the eigenspaces of the endomorphism $I$ respectively for the eigenvalues $i$ and $-i$:

$$T'^M_I = \ker(I - i \text{Id}), \quad T''M_I = \ker(I + i \text{Id}).$$

Section of the first subspace are said to be a vector fields of type $(1,0)$, and sections of the second subspace are vector fields of type $(0,1)$.

The decomposition is explicitly given by the projections to the two subspaces:

$$\pi_{1,0}^I = \frac{1}{2}(\text{Id} - iI), \quad \pi_{0,1}^I = \frac{1}{2}(\text{Id} + iI),$$

and for a vector field $X$ we denote its decomposition by $X = X_I^I + X_I''$. 
3.2. A symmetric bivector field. Let us now assume that $(M, \omega)$ is a symplectic manifold equipped with a smooth family of compatible almost complex structures $I$ parametrized by $\mathcal{T}$. We can define a bivector field $\tilde{G}(V) \in C^\infty(M, TM_\sigma \otimes TM_\sigma)$ by requiring that the relation

\begin{equation}
V[I] = (\Id \otimes i_\omega)(\tilde{G}(V))
\end{equation}

holds for all vector fields $V$. If we differentiate the identity $\tilde{g} = -I \cdot \tilde{\omega}$ along a vector field $V$ on $\mathcal{T}$, we get that

$$V[\tilde{g}] = -V[I] \cdot \tilde{\omega} = -\tilde{G}(V),$$

and since $\tilde{g}$ is a symmetric bivector field, so is $\tilde{G}(V)$. Moreover, because of the types of $V[I]$ and $\tilde{\omega}$, we get a decomposition

$$\tilde{G}(V) = G(V) + \tilde{G}(V),$$

where $G(V)_\sigma \in C^\infty(M, S^2(T^*M_\sigma))$ and $\tilde{G}(V)_\sigma \in C^\infty(M, S^2(T''M_\sigma)).$

Recalling the identity $g = \omega \cdot I$, we obtain a formula for the variation of the Kähler metric by differentiation:

$$V[g] = \omega \cdot V[I] = \omega \cdot \tilde{G}(V) \cdot \omega,$$

and the $(1,1)$-part of $V[g]$ vanishes because of the types of $\omega$ and $\tilde{G}(V)$.

Before defining the Hitchin connection, we need to define a certain differential operator associated to a bivector field. From a symmetric holomorphic bivector field $Z \in C^\infty(M, S^2(T^*M_\sigma))$ we can obtain a holomorphic bundle map $Z: T^*M_\sigma \to T^*M_\sigma$ by contraction. We define the operator $\Delta_Z$ to be the composition:

$$C^\infty(M, \mathcal{L}^k) \xrightarrow{\nabla_{(1,0)}^k} C^\infty(M, T^*M_\sigma \otimes \mathcal{L}) \xrightarrow{Z \otimes \Id} C^\infty(M, T^*M_\sigma \otimes \mathcal{L}) \xrightarrow{\nabla_{(1,0)}^{k,0} \otimes \Id + \Id \otimes \nabla_{(1,0)}} C^\infty(M, T^*M_\sigma \otimes T^*M_\sigma \otimes \mathcal{L}^k) \to C^\infty(M, \mathcal{L}^k),$$

where $\nabla_{(1,0)}^k$ is the holomorphic part of the Levi-Civita connection, and the last arrow is the trace.

This operator can be expressed in a more concise way as follows. Define the operator

$$\nabla_{X,Y}^Z = \nabla_X \nabla_Y - \nabla_{X,Y},$$

which is tensorial and symmetric in the vector fields $X$ and $Y$. Hence it can be evaluated on a symmetric bivector field and we have:

$$\Delta_Z = \nabla_Z^2 + \nabla_{\delta(Z)},$$

where $\delta(Z)$ denotes the divergence of the bivector field $Z$.

The previous construction can be done for any line bundle $\mathcal{L}$ over $M$. In particular, if we consider the trivial line bundle over $M$ with the trivial connection, then the sections are just functions on $M$, and the operator $\delta_Z$ (where $\delta_Z$ denotes the bivector field obtained by raising both indices of the metric tensor) coincides with the Laplace–Beltrami operator $\Delta$.

3.3. Holomorphic and rigid families. The explicit construction of a Hitchin connection in [And12] is for a compact symplectic manifold equipped with a smooth family of Kähler structures that satisfy two additional properties, which we shall explain below.

Assuming that the manifold $\mathcal{T}$ has a complex structure, it makes sense to require the family $I$ is a holomorphic map from $\mathcal{T}$ to the space of complex structures. We make this requirement precise as follows:
Definition 3.1. Let $\mathcal{T}$ be a complex manifold, and $I$ a family of Kähler structures on $M$ that is parametrized $\mathcal{T}$. We say that $I$ is holomorphic if:

$$V'[I] = V[I]' \quad \text{and} \quad V''[I] = V[I]'',$$

for any vector field $V$ on $\mathcal{T}$.

The second condition is the rigidity of the family of Kähler structures.

Definition 3.2. We say that the family $I$ of Kähler structures on $M$ is rigid if:

$$\nabla_{X''} G(V) = 0,$$

for all vector fields $V$ on $\mathcal{T}$ and $X$ on $M$.

In other words, the family $I$ is rigid if $G(V)$ is a holomorphic section of $S^2(T'M)$, for any vector field $V$ on $\mathcal{T}$.

Remark 3.3. The expression rigid family is used in this context for the following reason (the notion was first introduced in [And12]): it might be possible to extend a rigid family to a bigger family of rigid structures whose dimension at $\sigma$ in the family is given by $\dim H^0(M, S^2(T'M))$, but beyond this the family does not deform any further. Thus it is rigid in this sense.

3.4. The Hitchin connection. The prequantum space $P_k = \mathcal{C}^\infty(M, L^k)$ forms the fibre of a trivial vector bundle over $\mathcal{T}$ of infinite rank,

$$(\hat{P}_k) = \mathcal{T} \times P_k.$$

Let $\nabla^t$ denote the trivial connection on this bundle.

Definition 3.4. A Hitchin connection in the bundle $\hat{P}_k$ is a connection of the form

$$\hat{\nabla} = \nabla^t + a,$$

where $a \in \Omega^1(\mathcal{T}, \mathcal{D}(M, L^k))$ is a one-form on $\mathcal{T}$ with values in the space of differential operators on sections of $L^k$, such that $\hat{\nabla}$ preserves the quantum subspaces

$$Q_k(\sigma) = H^0(M_\sigma, L^k)$$

of holomorphic sections of the $k$-th power of the prequantum line bundle, inside each fibre of $\hat{P}_k$.

The existence of a Hitchin connection in the bundle $\hat{P}_k$ implies that the subspaces $Q_k(\sigma)$ form a subbundle $\hat{Q}_k$, because it can be trivialized locally through parallel transport by $\nabla$.

We are now ready to state the main result of [And12], showing the existence of the Hitchin connection.

Theorem 3.5 (Andersen). Let $(M, \omega)$ be a compact, prequantizable, symplectic manifold which satisfies that there exists an $n \in \mathbb{Z}$ such that the first Chern class of $(M, \omega)$ is $n \left[\frac{\omega^2}{2\pi}\right] \in H^2(M; \mathbb{Z})$ and $H^1(M; \mathbb{R}) = 0$. Moreover suppose that $I$ is a rigid holomorphic family of Kähler structures on $M$, parametrized by a complex manifold $\mathcal{T}$. Then there exists a Hitchin connection in the bundle $\hat{Q}_k$ over $\mathcal{T}$, given by the following expression:

$$\hat{\nabla}_V = \nabla_V + \frac{1}{4k + 2n}(\Delta_G(V) + 2\nabla_G(V) dF + 4kV'[F]),$$

where $\nabla_V$ is the trivial connection in $\hat{P}_k$, and $V$ is any smooth vector field on $\mathcal{T}$. 
4. Formal connections

The idea of formal connections arises as a generalization of the formal Hitchin connection that was defined in [And12] as the analogue of the Hitchin connection from geometric quantization. In that paper it was shown that the formal Hitchin connection is flat under certain conditions, and its trivialization up to first order has been given by the first named author of this paper together with Gammelgaard in [AG11].

Let $M$ be a symplectic manifold and $\mathcal{T}$ a smooth manifold parametrizing a family of star products on $M$. We let $C^h$ be the trivial fibre bundle over $\mathcal{T}$ with fibre $C^\infty(M)[[h]]$, i.e.

$$C^h = \mathcal{T} \times C^\infty(M)[[h]].$$

Let $\mathcal{D}(M)$ denote the space of differential operators on $M$, and let $\mathcal{D}_h(M) = \mathcal{D}(M)[[h]]$ denote formal differential operators on $M$, which are formal power series with coefficients in $\mathcal{D}(M)$.

**Definition 4.1.** A formal connection $D$ is a connection in the bundle $C^h$ over $\mathcal{T}$ that can be written as

$$D_V f = V[f] + A(V)(f),$$

where $A$ is a smooth 1-form on $\mathcal{T}$ with values in $\mathcal{D}_h(M)$ such that $A = 0 \mod h$, $f$ is a smooth section of $C^h$, $V$ is any smooth vector field on $\mathcal{T}$, and $V[f]$ denotes the derivative of $f$ along $V$.

The operator $A(V)(f)$ can be expressed as a series of differential operators

$$A(V)(f) = \sum_{k=1}^{\infty} A^k(V)(f)h^k,$$

where each $A^k$ is a smooth 1-form on $\mathcal{T}$ with values in $\mathcal{D}(M)$.

Normally we are interested in looking at formal connections in the presence of a family of star products on the manifold, and then we require the following compatibility:

**Definition 4.2.** Let $\{\ast_\sigma\}_{\sigma \in \mathcal{T}}$ be a family of star products. We say that a formal connection $D$ is compatible with the family of star products $\ast_\sigma$ if $D_V$ is a derivation of $\ast_\sigma$ for every vector field $V$ and every $\sigma \in \mathcal{T}$, that is, the following equality holds:

$$D_V(f \ast_\sigma g) = D_V(f) \ast_\sigma g + f \ast_\sigma D_V(g).$$

If the family of star products is natural, we also require the 1-form $A$ to consist of natural differential operators, i.e. the degree as a differential operator of the component $A^k$ is bounded by $k$.

4.1. The formal Hitchin connection associated to geometric quantization.

**Definition 4.3.** Let $M$ be a symplectic manifold with a family of compatible almost complex structures parametrized by a complex manifold $\mathcal{T}$, so that for any $\sigma \in \mathcal{T}$, the manifold $M_\sigma$ is Kähler. Let $\{\ast_{BT}^{\sigma}\}_{\sigma \in \mathcal{T}}$ be the associated family of Berezin-Toeplitz star products, see Subsection 2.3. A formal Hitchin connection for $\mathcal{T}$ is a formal connection which is compatible with $\{\ast_{BT}^{\sigma}\}_{\sigma \in \mathcal{T}}$ and which is flat.

The Hitchin connection $\hat{\nabla}$ in $\hat{Q}_k$ which we discussed in the previous section induces a connection $\hat{\nabla}^e$ in the endomorphism bundle $\text{End}(\hat{Q}_k)$. The following result establishes the existence of a formal Hitchin connection under the same assumptions as in Theorem 3.5.
Theorem 4.4 (Andersen). There is a unique formal connection $D$, written as $D_V = V + \hat{A}(V)$, which satisfies
\begin{equation}
\hat{\nabla}_V T^{(k)}_f \sim T^{(k)}_{(D_V f)(1/(2k+n))}
\end{equation}
for all smooth section $f$ of $C_h$ and all smooth vector fields $V$ on $T$. Here the symbol $\sim$ has the following meaning: for any positive integer $L$ we have that
\[
\left\| \hat{\nabla}_V T^{(k)}_f - \left( T^{(k)}_{V[f]} + \frac{L}{\Delta_{\hat{G}(V)} F} \frac{1}{(2k+n)} \right) \right\| = O(k^{-L+1})
\]
uniformly over compact subsets of $T$ for all smooth maps $f : T \to C^\infty(M)$.

In [And12] the following explicit formula for $\hat{A}$ is given:
\begin{equation}
\hat{A}(V)(f) = -V[F]f + \frac{1}{4} \Delta_{\hat{G}(V)} f - 2\nabla_{\hat{G}(V)} df f - 2\Delta_{\hat{G}(V)} F f - 2nV[F]f.
\end{equation}

From this equation we immediately get an expression for $H$:
\[
H(V) = E(V)(1) = \frac{1}{2} (\Delta_{\hat{G}(V)} F + nV[F]).
\]

We can summarize the previous results by writing the following formula for the formal Hitchin connection studied by Andersen:
\begin{equation}
D_V f = V[f] - \frac{1}{4} h\Delta_{\hat{G}(V)} f + \frac{1}{2} h\nabla_{\hat{G}(V)} df f + V[F] * BT f - V[f] f
\end{equation}
\begin{equation}
- \frac{1}{2} h(\Delta_{\hat{G}(V)} F) * BT f - nV[F] * BT f - \Delta_{\hat{G}(V)} F f - nV[F] f.
\end{equation}

The following two propositions, proved in [And12], show respectively that the formal connection constructed in Theorem 3.5 is a derivation with respect to the Berezin-Toeplitz star product (thus it is compatible with the family of Berezin-Toeplitz star products) and that it is flat whenever the Hitchin connection is projectively flat.

Proposition 4.5. The formal operator $D_V$ is a derivation with respect to the star product $*_{BT}$ for each $\sigma \in T$, meaning that it satisfies the relation:
\begin{equation}
D_V (f_1 *_{BT} f_2) = D_V (f_1) *_{BT} f_2 + f_1 *_{BT} D_V (f_2)
\end{equation}
for all $f_1, f_2 \in C^\infty(M)$.

Proposition 4.6. If the Hitchin connection $\hat{\nabla}$ in $\hat{Q}_k$ is projectively flat, then the formal Hitchin connection $D_V = V + \hat{A}(V)$ associated to it is flat.

The fact that this formal Hitchin connection is a derivation of the Berezin-Toeplitz star product, stated in 4.5, relies on the theory of geometric quantization and Toeplitz operators, and their link to deformation quantization. Consequently, the validity of this result relies on the existence of a Hitchin connection in geometric quantization, which puts many requirements on the objects involved – in particular, one needs the compatible Kähler structures to be rigid and holomorphic.

Remark 4.7. If we adopt the assumptions of Theorem 3.5 and if the Hitchin connection $\hat{\nabla}$ is projectively flat, the previous two propositions imply that formal Hitchin connection from [And12] is a formal Hitchin connection according to Definition 4.3.
4.2. **Low orders of the formal Hitchin connection.** The explicit expression of the formal Hitchin connection (18) makes sense in a more general setting. Therefore the question arises to which extent this expression actually defines a formal connection compatible with the Berezin-Toeplitz star products. In other words, we wonder whether it defines a derivation of the Berezin-Toeplitz star products. Here we shall answer this question up to order one by a direct computation, for which we need not assume that a Hitchin connection in the framework of geometric quantization exists. We include two preliminary lemmata about properties of the coefficients of the Berezin-Toeplitz star product.

Recall that we use the special notation $c^{(k)}$ for the coefficients of $\star_\sigma$, when we consider Berezin-Toeplitz star products.

**Lemma 4.8.** Let $M$ be a symplectic manifold with a family of compatible Kähler structures parametrized by a manifold $T$ on $M$, and let $c^{(k)}$ denote the coefficients of the Berezin-Toeplitz star product associated to the complex structure for a certain $\sigma \in T$. Then we have:

\[
V[c^{(1)}](f, g) = \frac{1}{4} \left( \Delta_{\tilde{G}(V)}(fg) - \Delta_{\tilde{G}(V)}(f)g - \Delta_{\tilde{G}(V)}(g)f \right).
\]

**Proof.** By a result of Karabegov [Kar96] we know that the degree 1 coefficient of the Berezin-Toeplitz star product can be written as:

\[
c^{(1)}(f, g) = g(\delta f, \bar{\delta} g) = i\nabla_{X_f}(f),
\]

for any functions $f, g \in C^\infty(M)$, where $X_f$ is the Hamiltonian vector field associated to $f$.

By differentiating equation (21), we get the following relation:

\[
V[c^{(1)}](f, g) = \frac{1}{2} df \tilde{G}(V)dg = \frac{1}{2} \nabla_{\tilde{G}(V)}dg(f).
\]

The operator $\Delta_{\tilde{G}(V)}$ is written as $\Delta_{\tilde{G}(V)} = \nabla_{\tilde{G}(V)}^2 + \nabla_{\delta \tilde{G}(V)}$, thus we get that

\[
\Delta_{\tilde{G}(V)}(fg) - \Delta_{\tilde{G}(V)}(f)g - \Delta_{\tilde{G}(V)}(g)f = \nabla_{\tilde{G}(V)}^2(fg) - \nabla_{\tilde{G}(V)}^2(f)g - \nabla_{\tilde{G}(V)}^2(g)f,
\]

since the sum of the order one terms vanishes. We can express the symmetric bivector field $\tilde{G}(V)$ as $\sum_j (X_j \otimes Y_j)$ for vector fields $X_j$ and $Y_j$, and rewrite the right hand side of (20) as:

\[
\frac{1}{4} \left( \sum_j \nabla_{X_j} \nabla_{Y_j}(fg) - g \sum_j \nabla_{X_j} (\nabla_{Y_j} f) - f \sum_j \nabla_{X_j} (\nabla_{Y_j} g) \right)
\]

\[
= \frac{1}{4} \sum_j \left( \nabla_{Y_j} (f) \nabla_{X_j} (g) + \nabla_{X_j} (f) \nabla_{Y_j} (g) \right)
\]

\[
= \frac{1}{2} df \tilde{G}(V)dg,
\]

where we use the symmetry of the bivector field. This concludes the proof. \hfill \Box

**Remark 4.9.** Let us note that the expression we obtained for $V[c^{(1)}]$ also shows that it is symmetric in the two variables.

We remark that because the Berezin-Toeplitz star product is natural, the first order coefficient $c^{(1)}$ is a differential operator of order 1 and hence is a derivation with respect to both arguments.

We now show that the expression (18) gives a derivation of the Berezin-Toeplitz star product up to order 1 in $h$. The derivation relation (19) can be written as:

\[
f \, V[\star_{BT}] \, g = \tilde{A}(V)(f) \star_{BT} g + f \star_{BT} \tilde{A}(V)(g) - \tilde{A}(V)(f \star_{BT} g),
\]
where $V[\star^{BT}]$ denotes the product with coefficients $V[e^{i\hbar}]$. Modulo $\hbar^2$ and using $\tilde{A}^0 = 0$, we arrive at the following condition that has to hold for all vector fields $V$ on $T$ and smooth functions $f$ and $g$ on $M$:

$$V[e^{(1)}](f,g) = -\tilde{A}^{1}(V)(fg) + \tilde{A}^{1}(V)(f)g + f \tilde{A}^{1}(V)(g).$$

To check this, we extract the expression for $\tilde{A}^{1}$ from (18), which gives us the following:

$$\tilde{A}^{1}(V)(f) = -\frac{1}{4} \Delta \tilde{G}(V)(f) + e^{(1)}(V[F],f) + V[e^{(1)}](F,f).$$

We can now substitute this into (24) to get:

$$V[e^{(1)}](f,g) = \frac{1}{4} \left( \Delta \tilde{G}(V)(fg) - g \Delta \tilde{G}(V)(f) - f \Delta \tilde{G}(V)(g) \right),$$

since the summands of order one in $f$ vanish. But this is precisely the expression we obtained in Lemma 4.8, and so we have checked that the derivation property holds.

Moreover we can check that the expression (18) defines a formal connection that is flat up to order one in $\hbar$. This amounts to showing that its curvature vanishes modulo $\hbar^2$. Note that the only terms we see when we compute the curvature modulo $\hbar^2$ are those coming from $d \tau \tilde{A}^{1}$, since the commutator terms in the curvature are of order at least $\hbar^2$. Therefore $\tilde{A}^{1}$ defines a flat connection up to order one if and only if it is closed with respect to $d \tau$. The closeness is most easily established by defining the 0-form $P = e^{(1)}(F,f) - \frac{1}{4} \Delta$ and checking that $V[P] = \tilde{A}^{1}(V)$ for any vector field on $T$.

The discussion above can be summed up in the following proposition.

**Proposition 4.10.** Let $M$ be a symplectic manifold with a family of compatible Kähler structures parametrized by a complex manifold $T$. Then expression (18) defines a formal connection that, up to order one in the formal parameter, is a derivation of the family of Berezin-Toeplitz star products on $M$, and that is flat up to order one in the formal parameter. Therefore it defines a formal Hitchin connection up to order one in the sense of Definition 4.3.

### 4.3. Derivations of star products.

We shall now put aside the formal Hitchin connection and look at formal connections in general. We aim at describing the space of formal connections on a symplectic manifold.

We begin by studying the space of derivations of a star product. Recall that a map $B: \hbar C^\infty(M) \to \hbar C^\infty(M)$ is a derivation with respect to a fixed star products if it satisfies the relation:

$$B(f \star g) = B(f) \star g + f \star B(g),$$

for any $f, g$ smooth (formal) functions on $M$. Let us now introduce some notation that will be useful in the following.

**Definition 4.11.** If $\star$ is a star product on $(M, \omega)$, we define the star commutator by:

$$[f, g]_\star = f \star g - g \star f,$$

for $f, g$ smooth (formal) functions on $M$.

Let us note that this definition makes $\hbar C^\infty(M)$ into a Lie algebra with adjoint representation

$$\text{ad}_\star f = [f, \cdot]_\star.$$
Remark 4.12. Since the term of order 0 in the star product is the commutative pointwise product, \( \text{ad}_s f \) is always divisible by the formal parameter \( h \), and the first term in the expansion is the Poisson bracket:

\[
\text{ad}_s f(g) = i h \{ f, g \} + \ldots
\]

The following lemma is proved by Gutt and Rawnsley in [GR99]:

Lemma 4.13 (Gutt, Rawnsley). If \( \ast \) is a differential star product on a symplectic manifold \((M, \omega)\), then the space of derivations modulo inner derivations can be identified with \( H^1(M, \mathbb{R})[[h]] \).

As a corollary, we see that if the first de Rham cohomology of \( M \) vanishes, every derivation is inner, i.e. can be written as \( \frac{1}{h} \text{ad}_s b \) for a formal function \( b \).

In the following \( \text{Der}(M, \ast) \) denotes the space of derivations of the star product \( \ast \) on \( M \), and \( \text{Der}_0(M, \ast) \) denote the subset of the derivations that are zero modulo \( h \).

Proposition 4.14. Let \((M, \omega)\) be a symplectic manifold satisfying \( H^1(M; \mathbb{R}) = 0 \), and let \( \ast \) be a differential star product on \( M \). The space of formal functions on \( M \) modulo the constants is isomorphic to the space of derivations of \( \ast \) that are zero modulo \( h \) via the map:

\[
\tilde{C}_h^\infty(M) \to \text{Der}_0(M, \ast)
\]

\[
b \mapsto \text{ad}_s(b).
\]

Proof. It is immediate to check that, for any formal smooth function \( b \in C_h^\infty(M) \), the star commutator \( \text{ad}_s(b) \) is a derivation of \( \ast \), and that it is trivial modulo \( h \). If we add to \( b \) any constant \( b_0 \in \mathbb{C}[[h]] \), then clearly \( \text{ad}_s(b + b_0) = \text{ad}_s(b) \), since the star product is null on constants, so the map is well defined on the quotient \( \tilde{C}_h^\infty(M) \) and it is linear. To check injectivity we assume that \( \text{ad}_s(b) \) is trivial for a formal smooth function \( b \). Let us assume that

\[
b = h^l b_l + O(h^{l+1})
\]

where \( b_l \in C^\infty(M) \). It then follows that \( \{ b_l, f \} = 0 \) for all \( f \in C_h^\infty(M) \), and this implies that \( b_l \) is a constant. We subtract this constant from \( b \) and proceed to order \( l + 1 \), where we can repeat the exact same argument. Hence we end up with the conclusion that \( b \) is a constant. Surjectivity follows from Lemma 4.13: if \( A \) is any derivation in \( \text{Der}_0(M, \ast) \), then it is divisible by \( h \), and \( A \) is again a derivation. So by the previous proposition, we can find \( b \in C_h^\infty(M) \) such that \( h^{-1}A = h^{-1} \text{ad}_s(b) \), and therefore \( A = \text{ad}_s(b) \).

4.4. The affine space of formal connections. Let \( D \) and \( D' \) be two formal connections on \( M \) for the same family of star products parametrized by \( T \). It is immediate to see that

\[
D'_V - D_V = A'(V) - A(V) = (A' - A)(V).
\]

Hence their difference is a 1-form on \( T \) with values in the derivations of the star products of the family, and it is zero modulo \( h \).

If \( M \) is a symplectic manifold equipped with a family of star products \( \{ \ast_{\sigma} \}_{\sigma \in T} \) parametrized by \( T \), let us denote by \( \mathcal{F}(M, \ast_{\sigma}) \) the space of the formal connections that are compatible with the family. We see that \( \mathcal{F}(M, \ast_{\sigma}) \) is an affine space over the space of 1-forms on \( T \) with values in the derivations of the star product that are zero modulo \( h \), and thus it can be written as:

\[
\mathcal{F}(M, \ast_{\sigma}) = D_0 + \Omega^1(T, \text{Der}_0(M, \ast_{\sigma})),
\]

for a fixed formal connection \( D_0 \).
If we assume that \( H^1(M; \mathbb{R}) \) vanishes, then Proposition 4.14 tells us that all derivations of \( * \) are inner, and therefore they are parametrized by an element in \( C_0^\infty(M) \), the space of formal functions on \( M \) modulo the constants. Therefore the compatible formal connections form an affine space modelled on the 1-forms on \( T \) with values in \( C_0^\infty(M) \).

\[
\mathcal{F}(M, \star_\sigma) \cong D_0 + \Omega^1(T, \hat{C}_R^\infty(M)),
\]

for a fixed compatible formal connection \( D_0 \).

4.5. **Gauge transformations of formal connections.** We shall study gauge transformations on the space of formal connections \( \mathcal{F}(M, \star_\sigma) \). The transformations we consider are self-equivalences of the family of star products, since the connections should still act as derivations after we transform them. This means that we look at \( P \in C^\infty(T, D_\hbar(M)) \) such that

\[
P_\sigma(f \star_\sigma g) = P_\sigma(f) \star_\sigma P_\sigma(g),
\]

for any \( \sigma \in \mathcal{T} \) and any smooth function \( f \) and \( g \) and such that \( P_\sigma \) is invertible for any \( \sigma \in \mathcal{T} \).

We are now ready to prove the following theorem.

**Theorem 4.15.** Let \( M \) be a symplectic manifold with a family of star products \( \{\star_\sigma\}_{\sigma \in \mathcal{T}} \) parametrized by a smooth manifold \( \mathcal{T} \) with \( H^1(\mathcal{T}, \mathbb{R}) = 0 \). Let \( D, D' \in \mathcal{F}(M, \star_\sigma) \) be formal connections for the family and let us assume that they are flat. Then they are gauge equivalent via a self-equivalence of the family of star products \( P \in C^\infty(T, D_\hbar(M)) \), meaning that

\[
D'_V = P^{-1}D_V P,
\]

for any vector field \( V \) on \( \mathcal{T} \).

**Proof.** By the definition of a formal connection, we have that

\[
D_V = V + A(V)
\]

\[
D'_V = V + A'(V),
\]

for two 1-forms \( A, A' \in \Omega(\mathcal{T}, D_\hbar(M)) \) with values in formal differential operators on \( M \), and any vector field \( V \) on \( \mathcal{T} \). Then we can rewrite (27) plugging in a section \( f \) of the bundle as:

\[
V[f] + A'(V)(f) = P^{-1}(V + A(V))P(f)
\]

\[
= P^{-1}(V[P](f) + P(V[f]) + A(V)(P(f))).
\]

Therefore, if we apply \( P \) on both sides we get the following equation:

\[
V[P] = PA'(V) - A(V)P.
\]

If we can find a \( P \) that solves the equation, then we get the wanted gauge transformation.

Let us define a 1-form \( \alpha \) on \( \mathcal{T} \) with values in formal differential operators on \( M \) by \( \alpha(V) = PA'(V) - A(V)P \), for any vector field \( V \) on \( \mathcal{T} \). The existence of \( P \) is equivalent to the exactness of \( \alpha \), which is the same as \( \alpha \) being closed, since the first cohomology of \( \mathcal{T} \) vanishes. Therefore it is enough to show that \( d_\mathcal{T}\alpha \) vanishes in order to obtain the wanted conclusion. At any point of \( \mathcal{T} \) we can choose two vector fields \( V \) and \( W \) that commute. Then we have that:

\[
d_\mathcal{T}\alpha(V, W) = V[\alpha(W)] - W[\alpha(V)]
\]

\[
= V[PA'(W) - A(W)P] - W[PA'(V) - A(V)P]
\]

\[
= P(A'(V)A'(W) - A'(W)A'(V) + V[A'(W)] - W[A'(V)])
\]

\[
- (A(V)A(W) - A(W)A(V) + V[A(W)] - W[A(V)]) P.
\]
Note that to obtain the last equality we substitute again the expression for $V[P]$ and $W[P]$ from (29).

Let us now compute the expressions for the curvature of $D$, which we are assuming is zero:

$$0 = F_D(V, W) = D_V D_W - D_W D_V - D_{[V, W]}.$$ 

The last summand vanishes because we chose commuting vector fields, hence we get, for any section:

$$0 = A(V) A(W)(f) - A(W) A(V)(f) - W[A(V)(f)]$$

$$+ A(V) W[f] + V[A(W)(f)] - A(W) V[f]$$

$$= A(V) A(W)(f) - A(W) A(V)(f) - W[A(V)](f) + V[A(W)](f),$$

which is the same as:

$$0 = F_D(V, W) = A(V) A(W) - A(W) A(V) + V[A(W)] - W[A(V)].$$

By computing the curvature in the same way for $D'$ we obtain:

$$0 = F_{D'}(V, W) = A'(V) A'(W) - A'(W) A'(V) + V[A'(W)] - W[A'(V)].$$

By comparing with (30), we see that flatness implies precisely that $\alpha$ is closed, and this concludes the proof. \hfill \Box

**Remark 4.16.** At first glance Theorem 4.15 seems too strong, in the sense that one might naively expect the vanishing of the fundamental group, instead of the first cohomology, to be the relevant condition. However, two factors improve the situation: First of all, the bundle under consideration is assumed to be trivial. Second, the Lie algebra in which our connection takes values in is the Lie algebra of formal differential operators which vanish modulo $h$. This Lie algebra is filtered by those formal differential operators which start at order $h^k$. The associated graded Lie algebra is easily seen to be abelian. This two facts show that we are effectively dealing with the case of connection with values in an abelian Lie algebra. Hence flatness reduces to closedness and gauge-equivalence reduces to a shift by an exact one-form. This explains the purely cohomological nature of our result.

We obtain the following corollary:

**Corollary 4.17.** Let $T$ be a manifold with trivial first cohomology group, i.e. $H^1(T, \mathbb{R}) = 0$. If there exists a formal Hitchin connection $D$ in the bundle $C_h = T \times C^\infty_h(M)$ on $T$, then it is unique up to gauge equivalence.

5. **Existence of formal connections**

In this section we study the question of existence of formal connections on a symplectic manifold $(M, \omega)$, equipped with a smooth family of natural star products $\{*_{\sigma}\}_{\sigma \in \mathcal{T}}$. As we will see, this problem can be reduced to a cohomological condition in terms of the corresponding family $\text{cl}(*_{\sigma})$ of characteristic 2-forms, see Subsection 2.2.

We briefly review the Hochschild complex $HC^\bullet(A)$ of an algebra $A$ with values in itself, relying on [Ger63, Wal07]. It is given by

$$HC^k(A) := \text{Hom}(A^\otimes k, A),$$

the space of multilinear maps from $A$ to itself. The graded vector space $HC^\bullet(A)$ comes equipped with the Gerstenhaber bracket $[, ]_G$. For $\psi \in HC^{r+1}(A)$ and
\( \varphi \in HC^{s+1}(A) \), it is given by

\[
[\psi, \varphi](a_0, \ldots, a_{r+s}) = \sum_{i=0}^{r} (-1)^{i} \psi(a_0, \ldots, a_{i-1}, \varphi(a_i, \ldots, a_{i+s}), a_{i+s+1}, \ldots, a_{r+s})
\]

\[-(-1)^{r} \sum_{j=0}^{s} (-1)^{j-r} \varphi(a_0, \ldots, a_{j-1}, \psi(a_j, \ldots, a_{j+r}), a_{j+r+1}, \ldots, a_{r+s}).\]

The bracket \([\cdot, \cdot]_G\) makes \(HC(A)\) a graded Lie algebra if we assign to \(\psi \in HC^{r+1}(A)\) the degree \(r\). Moreover, the associativity of the product \(m : A \otimes A \to A\) of \(A\) can be rewritten as

\[
[m, m]_G = 0.
\]

As a consequence, the map \(d_H(\varphi) := [m, \varphi]_G\) defines a coboundary operator on \(HC^\bullet(A)\), which is referred to as the Hochschild differential. The cohomology of \(HC^\bullet(A)\) with respect to \(d_H\) is called the Hochschild cohomology of \(A\) and is known to control the infinitesimal deformations of the algebra \(A\).

We will be interested in the case of \(A\) being \(C^\infty(M)\), seen as a module over \(\mathbb{C}[[h]]\). To get a reasonable cohomology theory, one has to take the Fréchet-topology into account (thus modifying the tensor product and the space of homomorphisms). Alternatively, one can restrict to the space of multi-differential operators \(D^m(M)\), where \(m\) is the number of arguments. The Gerstenhaber bracket \([\cdot, \cdot]_G\) and the Hochschild differential \(d_H\) restrict to \(D^\bullet(M) = \oplus_{m \geq 0} D^m(M)\). We will also consider the space of formal multi-differential operators \(D^\bullet_M(M)\) and the \(\mathbb{C}[[h]]\)-linear extensions of \([\cdot, \cdot]_G\) and \(d_H\), respectively.

Let \(\{*_\sigma\}_{\sigma \in \mathcal{T}}\) be a family of star products for \((M, \omega)\), smoothly parametrized by \(\mathcal{T}\). Given a vector field \(V\) on \(\mathcal{T}\), we write \(V[*_\sigma]\) for the product whose \(i\)-th coefficient is \(V[e_i]\), i.e. the variation of \(\{*_\sigma\}_{\sigma \in \mathcal{T}}\) in the direction of \(V\). The operator

\[
B_V(f, g) = f V[*_\sigma] g
\]

can be seen as a family of elements in \(D^\bullet_M(M)\). Since associativity of \(\{*_\sigma\}_{\sigma \in \mathcal{T}}\) can be written as

\[
[*_\sigma, *_\sigma]_G = 0,
\]

we obtain

\[
d_H B_V = [*_\sigma, V[*_\sigma]]_G = \frac{1}{2} V([*_\sigma, *_\sigma]_G) = 0,
\]

i.e. \(B_V\) is closed with respect to the Hochschild differential associated to the family of star products \(\{*_\sigma\}_{\sigma \in \mathcal{T}}\).

We now reconsider the compatibility requirement between a formal connection \(D\) and \(\{*_\sigma\}_{\sigma \in \mathcal{T}}\) from Definition 4.2. Using the decomposition \(D_V = V + A(V)\), the compatibility can be further rewritten as

\[
A(V)(f_*) g + f_* A(V)(g) - A(V)(f_* g) = f V[*_\sigma] g.
\]

We see that the above equation can be rewritten as

\[
d_H A(V) = V[*_\sigma]
\]

and we arrive at the following interpretation of (32):

**Proposition 5.1.** Let \((M, \omega)\) be a symplectic manifold, equipped with a family of natural star products \(\{*_\sigma\}_{\sigma \in \mathcal{T}}\) on \(M\), smoothly parametrized by \(\mathcal{T}\).

There is a one-to-one correspondence between:

1. Formal connections \(D_V = V + A\) on \(\mathcal{T}\) which are compatible with \(\{*_\sigma\}_{\sigma \in \mathcal{T}}\).
Families of formal differential operators $A(V) \in \mathcal{D}_1^h(M)$, with $A(V) \equiv 0 \mod h$ that satisfy
\[ d_H A(V) = V[∗_σ], \]
where $d_H$ denotes the Hochschild coboundary operator with respect to the family of star products $\{∗_σ\}_{σ \in \mathcal{T}}$.

In particular, a family of star products $\{∗_σ\}_{σ \in \mathcal{T}}$ admits a compatible formal connection if and only if the family of cocycles given by $B(σ, V) = V[∗_σ]$ is exact.

By work of Weinstein-Xu [WX98], the Hochschild cohomology of $C^\infty_h(M)$ with respect to a star product is isomorphic to the de Rham cohomology $hH^•(M, \mathbb{R})[[h]]$. Consequently, one expects that the cohomological condition on $V[∗_σ]$ from Proposition 5.1 translates into a cohomological condition on the characteristic 2-forms $cl(∗_σ)$. We will show that this is indeed the case. Instead of applying results of [WX98], we directly work in Fedosov’s framework, which we modify for our purposes. This will lead to the proof of the following theorem, but let us first introduce
\[ P = \{ β \in Ω^1(\mathcal{T}, Ω^1(M)[[h]]) \mid d_M i_V β = V[cl(∗_σ)] \} \]
and
\[ C = \{ A \in hΩ^1(\mathcal{T}, \mathcal{D}_1^h(M)) \mid d_H i_V A = V[∗_σ] \}. \]

**Theorem 5.2.** Let $(M, ω)$ be a symplectic manifold which is equipped with a family of natural star products $\{∗_σ\}$, smoothly parametrized by $\mathcal{T}$.

There is a natural map
\[ C : P \to C \]
As a consequence of Proposition 5.1 and Theorem 5.2, we obtain

**Theorem 5.3.** Let $(M, ω)$ be a symplectic manifold which is equipped with a family of natural star products $\{∗_σ\}$, smoothly parametrized by $\mathcal{T}$.

Then the following statements are equivalent:

1. The cohomology class of $cl(∗_σ)$ is locally constant in $\mathcal{T}$.
2. There is a 1-form $A \in Ω^1(\mathcal{T}, h\mathcal{D}_1^h(M))$ with values in formal differential operators on $M$ such that for any vector field $V$ on $\mathcal{T}$, and any smooth functions $f$ and $g$ on $M$ the identity
\[ fV[∗_σ]g = A(V)(f) ∗ g + f ∗ A(V)(g) − A(V)(f ∗ g) \]
holds.
3. The family of star products admits a formal connection.

**Proof.** We start with the implication (1) $\Rightarrow$ (2). Assuming Theorem 5.2, the only part of the statement that needs additional arguing is that, given family of closed 2-forms $α_σ := cl(∗_σ)$ with locally constant cohomology class, one can find a smooth 1-form $β$ on $\mathcal{T}$ with values in $Ω^1(M)$ such that for all vector fields $V$ on $\mathcal{T}$ the identity
\[ d_M i_V β = V[α_σ] \]
holds. To this end, we assume without loss of generality that $\mathcal{T}$ is connected. We fix a base-point $t_0 \in \mathcal{T}$ and consider the family
\[ α = α_{t_0}, \]
which is exact. Hence there is a family of 1-forms $γ$ on $M$ such that
\[ d_M γ = α − α_{t_0} \]
holds.\textsuperscript{1} If we apply the Lie derivative with respect to a vector field $V$ on $\mathcal{T}$, we obtain:

$$-d_{M}i_{V}(d\tau) = V[\alpha].$$

Hence $\beta := -d\tau$ is the desired 1-form on $\mathcal{T}$ with values in $\Omega^{1}(M)$.

The implication (2) $\Rightarrow$ (3) is precisely the content of Proposition 5.1.

The implication (3) $\Rightarrow$ (1) relies on the parallel transport of the connection $D_{V} = V + A(V)$. Given two points $\sigma$ and $\sigma'$ in the same connected component of $\mathcal{T}$, we can choose a smooth path $\gamma$ which starts at $\sigma$ and ends at $\sigma'$. We pull back the family $\ast_{\gamma}$ and the connection $D_{V}$ along $\gamma$. We claim that the parallel transport $\Phi(t)$ of $D_{V}$ along $\gamma$ exists and is unique. If we consider a fixed order of $h$, this reduces to the existence and uniqueness of solutions to an ODE of the form

$$\frac{d}{dt}F(t) = G(t),$$

with $G(t)$ a given smooth one-parameter family of differential operators on $M$. Modulo constants (with respect to $t$) there is clearly a unique solution, given by integration with respect to $t$.

We can now consider the family of star products

$$\Phi(t) \circ \ast_{\sigma} \circ \Phi(t)^{-1} \circ \Phi(t)^{-1}.$$

At $t = 0$, it coincides with $\ast_{\gamma(t)}$ and satisfies the same ordinary differential equation, which is

$$\frac{d}{dt}\ast_{\gamma(t)} = \ast_{\gamma(t)} \circ (A(\frac{d\gamma}{dt}) \otimes \text{id} + \text{id} \otimes A(\frac{d\gamma}{dt})) - A(\frac{d\gamma}{dt}) \circ \ast_{\gamma(t)}.$$

By uniqueness of the solution to this ODE, we conclude

$$\ast_{\gamma(t)} = \Phi(t) \circ \ast_{\sigma} \circ \Phi(t)^{-1} \circ \Phi(t)^{-1}.$$

In particular, $\ast_{\sigma}$ and $\ast_{\sigma'}$ are equivalent star products. Consequently their characteristic classes coincide. \hfill $\square$

The discussion above specializes to the case of a compact symplectic manifold $M$ with a family of compatible Kähler structures parametrized by $\mathcal{T}$. As in Subsection 2.3, one can consider the family of Berezin-Toeplitz star products $\{\ast_{\sigma}\}_{\sigma \in \mathcal{T}}$. The characteristic class of $\ast_{\sigma}$ is proportional to the first Chern class of $M$, see [Haw00]. Since the first Chern class is independent of the compatible complex structure $\sigma$, we may apply Theorem 5.3 to the family $\{\ast_{\sigma}\}_{\sigma \in \mathcal{T}}$ and obtain the existence of a formal connection which is compatible with $\{\ast_{\sigma}\}_{\sigma \in \mathcal{T}}$. Even better, we might directly apply Theorem 5.2: The only input data is a family $\beta_{\sigma}$ of 1-forms on $\mathcal{T}$ with values in $\Omega(M)^{1}[[h]]$ such that the condition from Theorem 5.2 holds. Using Hodge-theory with respect to the family of Kähler metrics yields a preferred such family. We hence obtain the following result:

**Theorem 5.4.** Let $(M, \omega)$ be a compact, symplectic manifold equipped with a family of compatible Kähler structures parametrized by a manifold $\mathcal{T}$. Let us consider the corresponding family of Berezin-Toeplitz star products $\{\ast_{\sigma}\}_{\sigma \in \mathcal{T}}$. Then the family admits a preferred formal connection.

\textsuperscript{1} The existence of such a smooth family follows from the existence of an operator

$$d^{*} : d(\Omega^{k-1}(M)) \rightarrow \Omega^{k-1}(M),$$

called the anti-differential, such that $dd^{*} \beta = \beta$ holds, and which maps smooth families to smooth families. For $M$ compact, $d^{*}$ might be constructed using Hodge-theory. For arbitrary $M$ one can use the Čech-de Rham double complex to construct such an operator, as is done in [AG14].
We hope to compare the formal connection from Theorem 5.4 to the expression for the formal Hitchin connection given in Equation (18), Subsection 4.1, more closely in future work.

5.1. Proof of Theorem 5.2. This subsection contains the proof of Theorem 5.2. We will use Fedosov’s framework for the deformation quantization of symplectic manifolds, which we reviewed in Subsection 2.2. The first step of the proof is in fact the reduction to families of Fedosov star products via [GR99, Theorem 4.1]: Gutt and Rawnsley show there that every natural star product is equivalent to a preferred star product of Fedosov-type through a preferred equivalence. Hence we assume from now on without loss of generality that each member $\star_\sigma$ of the family of star products which we consider equals $\star_{\nabla_{\sigma},\alpha_\sigma}$, where

- $\nabla_\sigma$ is a symplectic connection for $(M,\omega)$,
- $\alpha_\sigma$ is an element of $\omega + hZ^2(M,\mathbb{R})[[h]]$.

Moreover, these data fit together into smooth families parametrized by $\mathcal{T}$.

Our aim is to understand the dependence of Fedosov’s construction from Subsection 2.2 on the data $\{\nabla_\sigma,\alpha_\sigma\}_{\sigma \in \mathcal{T}}$. As in the proof of Theorem 5.2 we can fix a 1-form $\beta$ on $\mathcal{T}$ with values in $\Omega^1(M)[[h]]$ such that for all vector fields $V$ on $\mathcal{T}$ the equation

$$d_Mi_V\beta = V[\alpha_\sigma]$$

holds.

We now go through Fedosov’s construction, seen as fibred over the parameter space $\mathcal{T}$. In the first step, we realize each star product $\star_{\nabla_{\sigma},\alpha_\sigma}$ with the help of a Fedosov connection $D_{r(\sigma)}$. In the following, we interpret $r$ as a family of elements of $\Omega^2(M,W)$. Recall that $W$ denotes the Weyl bundle over $M$. By definition, the star product $\star_{\nabla_{\sigma},\alpha_\sigma}$ is given by

$$f \star_{\nabla_{\sigma},\alpha_\sigma} g = p(\tau_{r(\sigma)}(f) \circ MW \tau_{r(\sigma)}(g)),$$

where $\tau_{r(\sigma)}(f)$ is the unique extension of $f$ to $\Gamma(W)$ which is constant with respect to the connection $D_{r(\sigma)}$ and $p$ is the canonical projection $\Gamma(W) \to C^\infty(M)$.

We claim that there is a 1-form $s$ on $\mathcal{T}$ with values in $\Omega^2(M,W)$ such that the operator

$$\tilde{D}_s := d_r + \frac{i}{h} \text{ad}(s)$$

commutes with $D_r$ in the graded sense, i.e.

$$[\tilde{D}_s, D_r] := \tilde{D}_s \circ D_r + D_r \circ \tilde{D}_s = 0.$$

Using the form of $D_r$ and our ansatz for $\tilde{D}_s$, we obtain the following expression for the graded commutator

$$i_V[\tilde{D}_s, D_r] = \frac{i}{h} \text{ad}(D_r(i_V s) + V[r]) + V[d\nabla_{\sigma}]$$

where $V$ is an arbitrary vector field on $\mathcal{T}$. Recall that $\nabla_{\sigma}$ is a family of symplectic connections. These form an affine space over the subspace $S$ of $\Omega^1(M,\text{End}(TM))$ which corresponds to totally symmetric contravariant 3-tensors after contraction with $\omega$. Hence the variation of $\nabla_{\sigma}$ is encoded by a 1-form $S_{\sigma}$ on $\mathcal{T}$ with values in $S$. In the following we use $\omega$ to turn $S_{\sigma}$ into a 1-form on $\mathcal{T}$ with values in $\Omega^1(M,W)$.

Lemma 5.5. The variation $V[d\nabla_{\sigma}]$ of the covariant derivative $d\nabla_{\sigma}$ on $\Omega(M,W)$ can be expressed as

$$V[d\nabla_{\sigma}] = \frac{i}{2h} \text{ad}(i_V S_{\sigma}).$$
Proof. If we consider $a \in W$ which does not depend on $\hbar$, only odd powers of $\hbar$ will appear in the adjoint action $\text{ad}(a)$ under the Moyal-Weyl product. Hence we only need to compute $\text{ad}(i_{V}S_{\sigma})$ to first order, which is essentially given by contracting $i_{V}S_{\sigma}$ by the Poisson bivector field dual to $\omega$, just undoing the contraction by $\omega$ that occurred before. □

Thanks to the lemma, we arrive at the following formula for the commutator:

$$i_{V}[\hat{D}_{s}, D_{r}] = \frac{i}{\hbar} \text{ad} \left( -D_{r}(i_{V}s) + V[r] + \frac{1}{2}i_{V}S_{\sigma} \right).$$

Our strategy is now the same as in Fedosov’s construction: we have to choose $s$ such that the expression in the bracket on the right hand side of the above equation lies in the center of $\Omega(M, W)$, which is $\Omega(M)[[\hbar]]$, i.e. we impose

$$- D_{r}(i_{V}s) + V[r] + \frac{1}{2}i_{V}S_{\sigma} = i_{V}\beta$$

for $\beta$ a 1-form on $T$ with values in $\Omega^{1}(M)$. One finds a necessary condition for Equation (33) which reads

$$V[\alpha_{s}] = d_{M}i_{V}\beta,$$

and coincides with the condition we imposed on $\beta$ at the beginning.

We have the following analogon to Theorem 2.5:

**Proposition 5.6.** There is a unique $s \in \Omega^{1}(T, \Omega^{1}(M, W))$ such that

$$- D_{r}(i_{V}s) + V[r] + \frac{1}{2}i_{V}S_{\sigma} = i_{V}\beta$$

and $\delta^{*}(i_{V}s) = 0$ hold for all vector fields $V$ on $T$.

Proof. We rewrite the equation as

$$D_{r}(i_{V}s) = V[r] + \frac{1}{2}i_{V}S_{\sigma} - i_{V}\beta.$$

By general cohomological considerations, it suffices to now prove that the right hand side of the above equation is closed with respect to $D_{r}$.

By $0 = [D_{r}, [d_{T}, D_{s}]] = \pm \frac{i}{\hbar} \text{ad}(D_{s}(V[r] + \frac{1}{2}i_{V}S_{\sigma} - i_{V}\beta))$, we know that $D_{s}(V[r] + \frac{1}{2}i_{V}S_{\sigma} - i_{V}\beta)$ is central. If we compute the component of this element in the center, we obtain $V[\alpha_{s}] - d_{M}i_{V}\beta$, which vanishes by assumption.

We now know that there is an appropriate solution $i_{V}s$. The condition $\delta^{*}(i_{V}s) = 0$ singles out a unique one. □

**Lemma 5.7.** Let $V$ be a vector field on $T$. The variation of $\tau_{r}(f)$ in the direction of $V$ is given by

$$V[\tau_{r}(f)] = \frac{i}{\hbar} (\tau \circ p - \text{id})([i_{V}s, \tau_{r}(f)]).$$

Proof. Let us define $\mu(f)$ to be $\hat{D}_{s}\tau_{r}(f)$. Since $D_{r}$ and $\hat{D}_{s}$ commute, $\mu(f)$ is closed with respect to $D_{r}$. Moreover, the image under $p$ computes to

$$p(\mu(f)) = p((d\tau + \frac{i}{\hbar} \text{ad}(s))\tau_{r}(f)) = \frac{i}{\hbar} p([s, \tau_{r}(f)]).$$

Since $D_{r}$-closed elements are determined by their image under $p$, we obtain

$$\mu(f) = \frac{i}{\hbar} (\tau \circ p)([s, \tau_{r}(f)]).$$

Inserting $\mu(f) = (d\tau + \frac{i}{\hbar} \text{ad}(s))\tau_{r}(f)$ into the equality yields

$$d\tau \tau_{r}(f) = \frac{i}{\hbar} (\tau \circ p - \text{id})([s, \tau_{r}(f)]).$$
Proposition 5.8. Let \( s \in \Omega^1(T, \Omega^0(M, W)) \) be as in Proposition 5.6.

Then
\[
A(V) := i \frac{1}{\hbar} \mathcal{P}([i V s, \tau_r(\cdot)])
\]
defines a 1-form on \( T \) with values in \( \mathcal{D}^*_h(M) \) that satisfies the following:

1. If \( X_f \) denotes the Hamiltonian vector field of \( f \) and \( \beta_1 \) is the component of \( \beta \) of order \( \hbar \), we have
\[
A(V)(f) = -h(i V X_f \beta_1) \quad (\text{mod } h^2).
\]

2. The identity
\[
d_H i V A(V) = V[\*_{\nabla_s, \alpha_s}]
\]
holds.

Proof. That \( A(V) \) is a differential operator follow from the fact that the value of \( \tau_r(f) \) at \( x \) depends only on the jet of \( f \) at \( x \) and that the Moyal-Weyl product only acts fibrewise.

To verify the claim about the lowest order terms of \( A(V) \), we have to consider the lowest orders of \( s \) and \( \tau_r(f) \) with respect to the total degree, which is given by the polynomial degree in the Weyl-algebra plus twice the power in \( \hbar \). The expansion up to order 1 of \( \tau_r(f) \) is \( f + df \), where \( df \) is seen as a function on \( T \) with values in \( W \). Since \( f \in C^\infty(M) \) lies in the center of the Weyl-algebra, only the term \( df \) is relevant for our considerations.

Concerning \( s \), we notice that we only need to consider its component in \( T^*M \subset W \) because we are only interested in \( p([s, df]) \), and all other components lead to terms that project to zero under \( p \). Inspecting this component in lowest order, we find an element \( a \) which is uniquely determined by
\[
-\delta(i V a) = -h i V \beta_1, \quad \text{and } \delta^* a = 0,
\]
where \( \beta_1 \) is the term of order \( \hbar \) in \( \beta \). The solution to this equation is given by
\[
a = h \delta^* i V \beta_1.
\]
In total, we obtain that the lowest order term of \( A(V) \) is given by
\[
i \frac{1}{\hbar} \text{ad}(a)(df) = i[\delta^* i V \beta_1, df] = -h(i V X_f \beta_1),
\]
where \( X_f \) denotes the Hamiltonian vector field of \( f \).

It remains to verify that the image of \( A(V) \) under the Hochschild differential \( d_H \) is \( V[\*_{\cdot}] \). In fact we compute:
\[
V[f \*_{\nabla_s, \alpha_s} g] = p(V[\tau_r(f)] \circ_{MW} \tau_r(g)) + p(\tau_r(f) \circ_{MW} V[\tau_r(g)]).
\]
We now use the expression for the variation of \( \tau_r(f) \) and \( \tau_r(g) \) we obtained in Lemma 5.7 and arrive at
\[
V[f \*_{\nabla_s, \alpha_s} g] = A(V)(f) \*_{\nabla_s, \alpha_s} g - f \*_{\nabla_s, \alpha_s} A(V)(g) - A(V)(f \*_{\nabla_s, \alpha_s} g),
\]
which is exactly \( V[\*_{\nabla_s, \alpha_s}] = d_H A(V) \).

Combining the formulae in Proposition 5.6 and 5.8, we get the map \( C \) and thereby a complete proof of Theorem 5.2.

The following result expresses the curvature of the formal connection \( D_V = V + A(V) \) in terms of the element \( s \):

Proposition 5.9. Let \( A(V) := \frac{1}{\hbar} \mathcal{P}([i V s, \tau_r(\cdot)]) \) be the connection 1-form associated to the element \( s \) from Proposition 5.6.
The curvature of the formal connection $D_V = V + A(V)$ equals the 2-form $\Omega_s$ on $T$ with values in formal differential operators on $M$ given by

$$\Omega_s(f) := \frac{i}{\hbar} p([d_T s + \frac{i}{\hbar} s \circ MW s, \tau_r(f)]).$$

**Proof.** We first compute $\, (d_T A)(f) = d_T \frac{i}{\hbar} p([s, \tau_r(f)]).$ By Lemma 5.7, we obtain

$$\frac{i}{\hbar} p([d_T s, \tau_r(f)] - [s, \frac{i}{\hbar} (\tau \circ p - \text{id})[s, \tau_r(f)]].$$

On the other hand, applying $A \wedge A$ to $f$ yields

$$\left(\frac{i}{\hbar}\right)^2 p([s, \tau_r([s, \tau_r(f)])),$$

which cancels with one of the terms from $\, (d_T A)(f).$ The remaining terms are

$$\frac{i}{\hbar} p([d_T s, \tau_r(f)] + \frac{i}{\hbar} [s, [s, \tau_r(f)]]).$$

In order to arrive at the claimed expression, we apply the identity $[s, [s, X]] = \frac{1}{2} [[s, s], X] = [s \circ MW s, X].$ \hfill \qed

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