DEGENERACIES IN QUASI-CATEGORIES

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Abstract. In this note we show that a semisimplicial set with the weak Kan condition admits a simplicial structure, provided any object allows an idempotent self-equivalence. Moreover, any two choices of simplicial structures give rise to equivalent quasi-categories. The method is purely combinatorial and extends to semisimplicial objects in other categories; in particular to semisimplicial spaces satisfying the Segal condition (semi-Segal spaces).

1. Statement of the main results

A semisimplicial set (called ∆-set by [RS71]) is a functor (∆_{inj}^{op} → Set, where ∆_{inj} is the category of the totally ordered finite sets [n] = {0,...,n} and strictly monotone maps. Rourke–Sanderson [RS71] (see also [McC13]) showed that any semisimplicial set satisfying the Kan condition admits a simplicial structure. In this note we investigate under which condition there is a simplicial structure on a semisimplicial set which merely satisfies the “weak Kan condition” that all inner horns can be filled. For notational simplicity we will refer to such an object as quasi-semicategory. (Here “semi” stands for semisimplicial. We do not intend to suggest that such an object is, in general, a model for a non-unital infinity-category.)

Let X be a semisimplicial set. For f ∈ X_1, we write f: x → y where x = d_1 f and y = d_0 f. If f, g, h ∈ X_1, we write g ∘ f ≃ h if there is a 2-simplex σ such that d_1 σ = h, d_2 σ = f, and d_0 σ = g. The symbol Δ^n will denote the semisimplicial n-simplex (i.e., the presheaf represented by [n]), and Λ^n_i ⊂ Δ^n the (n, i)-horn.

Definition 1.1. (i) f: x → x in X is called idempotent if f ∘ f ≃ f holds.
(ii) A morphism f ∈ X_1 is called equivalence if f is both cartesian and co-cartesian – that is, if for any n ≥ 2 there is a filler for any horn Λ^n_i → X whose last edge is f and for any horn Λ^n_0 → X whose first edge is f.

Examples. (i) If X is a quasi-category, by Joyal [Joy02] this notion of equivalence agrees with the usual notion of equivalence (or quasi-isomorphism).
(ii) Let X = N(C) be the nerve of a non-unital category (so that X is a quasi-semicategory). It is not hard to see that f: x → y is an equivalence in our sense if and only if for any object z, the maps

− ∘ f: C(y, z) → C(x, z) and f ∘ −: C(z, x) → C(z, y)

are bijective.

If X is a quasi-category and x ∈ X_0, then the degeneracy s_0(x) of x has the property of being an idempotent equivalence of x. Our first result is a converse to this statement. We will say that a quasi-semicategory X has a simplicial structure if it is the underlying semisimplicial set of a simplicial set (which then is automatically a quasi-category).

Theorem 1.2. Let X be a quasi-semicategory and let s_0: X_0 → X_1 be any function such that for each x ∈ X_0, s_0(x) is an idempotent equivalence x → x. Then X has a simplicial structure whose degeneracy in degree 0 coincides with s_0.

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Corollary 1.3 (Rourke-Sanderson). Any semisimplicial set satisfying the Kan condition has a simplicial structure.

Theorem 1.2 comes with a relative version, see Theorem 2.1 below. From the relative version we will deduce:

Theorem 1.4. Let $C$, $C'$ be quasi-categories which have the same underlying semisimplicial set. Then $C$ and $C'$ are categorically equivalent.

In section 2 we will prove Theorems 1.2 and 1.4 and deduce Corollary 1.3. In section 3 we will generalize the results to semisimplicial objects in other categories.

The results of this paper will be used by the author in the proof of an analog of Waldhausen’s additivity theorem in the setup of cobordism categories [Ste]. The point is that cobordism categories are naturally categories without identities, just as cobordism spaces (considered by Quinn, Ranicki, Laures–McClure and others) are naturally semi-simplicial sets, while it is usually more convenient to work with simplicial objects.

2. The relative existence theorem

We start by recalling some terminology. A semisimplicial map $p: X \to Y$ is called inner fibration if any commutative diagram of semisimplicial sets

\[
\begin{array}{ccc}
\Lambda^n_i & \xrightarrow{h} & X \\
\downarrow & & \downarrow^p \\
\Delta^n & \xleftarrow{k} & Y
\end{array}
\]

has a diagonal lift as dotted in the diagram, provided $0 < i < n$. An element $a \in X_1$ is called $p$-cartesian if any commutative diagram (1) has a diagonal lift, provided $i = n$ and the last edge of $h$ is $a$; it is called $p$-cocartesian if it is $p^\text{op}$-cocartesian as an element of $X_1^\text{op}$. These definitions are in accordance with the usual simplicial notions.

If $Y$ has a simplicial structure, and $f: x \to x$ is a 1-simplex in $X$, then we call $f$ $p$-idempotent if there is a 2-simplex $\sigma \in X_2$ all whose boundaries are $f$, and which projects to the degenerate simplex $s^n_0(p(x)) \in Y$.

Theorem 2.1. Let $p: X \to Y$ be an inner fibration of semisimplicial sets and $f: A \to X$ the inclusion of a semisimplicial subset; assume that $Y$ and $A$ have simplicial structures such that $p \circ f$ is a simplicial map. Let $s_0: X_0 \to X_1$ be a map, compatible with the degeneracies $s_0$ on $A$ and $Y$, and such that for all $x \in X_0$, $s_0(x)$ is $p$-idempotent, $p$-cartesian, and $p$-cocartesian.

Then $s_0: X_0 \to X_1$ extends to a simplicial structure on $X$ such that $f$ and $p$ are simplicial.

Addendum 2.2. If $p$ is a Kan fibration, then a map $s_0: X_0 \to X_1$ as required in the Theorem exists always, so that a compatible simplicial structure on $X$ exists without further hypotheses.

Theorem 1.2 is a special case of Theorem 2.1 where $Y$ is the terminal object and $A = \emptyset$; Corollary 1.3 follows from the Addendum. The relative existence theorem also implies Theorem 1.4. Let $J$ be the groupoid with two objects 0 and 1 and two non-identity morphisms. We apply Theorem 2.1 with $X = \mathcal{C} \times J$, $A = \mathcal{C} \times \{0, 1\}$, $Y = J$, and $p$ the projection map, where $A$ carries the simplicial structure of $\mathcal{C}$ over 0 and of $C'$ over 1. We conclude that $\mathcal{C} \times J$ has a simplicial structure compatible with $\mathcal{C}$ over 0 and with $C'$ over 1, such that $p$ is simplicial. Now note that $p$ is a cartesian fibration over $J$ so the pull-backs over 0 and 1 are categorically equivalent [Lur09 3.3.1.3].
We come to the proof of Theorem 2.1 which is a modification of the strategy from [McC13]. Throughout this section $X$ and $A$ will be as in the assumption of Theorem 2.1. For notational brevity we will give the proof only in the case where $Y$ is the terminal object $\{^{*}\}$ so that the datum of $p$ and $Y$ may be ignored. The proof in the general case is identical, if “filling a horn” is replaced by “choosing a diagonal lift”.

Recall the simplicial identities:

\begin{align}
  d_i d_k &= d_{k-1} d_i \ (i < k); \\
  d_i s_k &= \begin{cases} 
    s_{k-1} d_i & (i < k), \\
    \text{id} & (i = k, k + 1), \\
    s_k d_{i-1} & (i > k + 1);
  \end{cases} \\
  s_i s_k &= s_{k+1} s_i \ (i \leq k).
\end{align}

The construction of degeneracy maps is by induction. Let us call an \textit{N-good system} a system of maps $(s_k: X_n \to X_{n+1})$ $(n \geq 0, 0 \leq k \leq \min(n, N))$ that satisfies the simplicial identities whenever they apply, and that extends the given maps on $A$ and $X_0$. Clearly a $(1)$-good system exists; we wish to prove that any $(N - 1)$-good system $(s_0, \ldots, s_{N-1})$ can be extended to an $N$-good one.

We proceed in two steps. Let us call an \textit{almost $N$-good system} a system of maps $(s_k: X_n \to X_{n+1})$ $(n \geq 0, 0 \leq k \leq \min(n, N))$ satisfying the condition for being $N$-good, except that we do not require the identity $d_{N+1} s_N = \text{id}$ to hold.

**Lemma 2.3.** Any $(N - 1)$-good system extends to an almost $N$-good system.

**Proof.** The construction of $s_N: X_n \to X_{n+1}$ is by induction on $n$, starting at $n = N$. In the case $N = 0$, the induction beginning is provided by the map $s_0: X_0 \to X_1$ which exists by assumption. The induction step and, in the case $N \neq 0$, also the induction beginning, are proven by the same construction which we now explain.

Assume that we have an $(N - 1)$-good system $(s_0, \ldots, s_{N-1})$ and maps $s_N: X_\ell \to X_{\ell+1}$ for $N \leq \ell \leq n - 1$, satisfying the condition for being almost $N$-good whenever they apply. We wish to define $s_N: X_n \to X_{n+1}$ so that the conditions for being almost $N$-good hold whenever they apply; that is, (3) and (4) should hold for $k = N$ except we do not require $d_{N+1} s_N = \text{id}$.

For $x \in X_n$, the equations in (3) for $k = N$, $i \neq N + 1$, are $(n+1)$-many equations that together prescribe the restriction of $s_N(x)$ to the horn $\Lambda^{n+1}_{N+1} \subset \Delta^{n+1}$. Therefore we will define $s_N(x)$ as a filler for the horn $\Lambda^{n+1}_{N+1} \to X$ which is defined by the right-hand sides of the relevant equations in (3). In more detail, we let

\[ x_i = \begin{cases} 
  s_{N-1} d_i(x), & (i < N), \\
  x, & (i = N), \\
  s_N d_{i-1}(x), & (i > N + 1)
\end{cases} \]

where the operator $s_N$ in the last case acts on $X_{n-1}$ and is given by hypothesis. We claim that

\[ d_j(x_i) = d_{i-1}(x_j), \quad (j < i, j, i \neq N + 1)\]

so that the sequence $x_i$ for $i \neq N + 1$ defines a horn $\Lambda^{n+1}_{N+1}$ in $X$. The equations (5) can be easily verified by hand using the relevant equations of (3), making a case by case distinction.

If $n > N$ (the induction step case), the horn defined in this way is an inner horn so a filler exists because $X$ is a quasi-semicategory. If $n = N$ (the induction beginning case), this is a right horn, but applying (3) iteratively, we see that

\[ d_0^N s_N(x) = s_0 d_0^N (x)\]
so the last edge of the horn is in the image of \( s_0 : X_0 \to X_1 \) and therefore a cartesian morphism by our assumptions. So the horn has a filler in this case again, by definition of being cartesian.

We would like to define \( s_N(x) \) as a choice of filler for this horn; however this definition is a little too crude in that we didn’t ensure that the restriction of \( s_N \) to \( A_n \) is as required, nor that the simplicial identities hold. This can be rectified as follows: First, if \( x = f(x') \) for some \( x' \in A_n \), we have to and do choose \( f s_N(x') \) as a filler for the horn in order to make \( f \) simplicial. Second, if \( x \in X_n \) is of the form \( x = s_i(y) \) for some \( i < N \), then the equation \( s_N s_i = s_i s_{N-1} \) from \( \text{(4)} \) forces us to choose \( s_N(x) := s_i s_{N-1}(y) \) as a filler for the horn.

To complete the proof, we need to show this rule is well-defined; that is, if \( x = s_i(y) = s_j(y') \) or if \( x = f(x') = s_i(y) \), any of the choices leads to the the same value of \( s_N(x) \). To justify this, we use the following Lemma, which we prove at the end of the section.

**Lemma 2.4.** In an \((N-1)\)-good system, and for \( i < j < N \), \( k < N \), the commutative squares

\[
\begin{array}{ccc}
X_{n-2} & s_{j-1} & X_{n-1} \\
\downarrow s_i & & \downarrow s_i \\
X_{n-1} & s_j & X_n
\end{array}
\quad \begin{array}{ccc}
A_{n-1} & f & X_{n-1} \\
\downarrow s_k & & \downarrow s_k \\
A_n & f & X_n
\end{array}
\]

are pull-back squares.

Hence, if we can write \( x = s_i(y) = s_j(y') \) for \( i < j < N \), there exists a \( z \in X_{n-2} \) such that \( y = s_{j-1}(z) \) and \( y' = s_i(z) \). Then we have

\[
s_i s_{N-1}(y) = s_i s_{N-1} s_{j-1}(z) = s_j s_{N-1} s_i(z) = s_j s_{N-1}(y')
\]

provided \( i < j < N \) and the system is \((N-1)\)-good, so that both possible definitions of \( s_N(x) \) agree. Similarly, if \( x = s_i(y) = f(x') \), then there exists \( y' \in A_{n-1} \) with \( y = f(y') \) and \( s_i(y') = x' \) so

\[
s_i s_{N-1}(y) = s_i s_{N-1} f(y') = f s_i s_{N-1}(y') = f s_N s_i(y') = f s_N(x')
\]

and again the two possible definitions agree. \( \square \)

Next we come to the second step of our construction.

**Lemma 2.5.** If \((s_0, \ldots, s_N)\) is an almost \(N\)-good system, then there is a collection of maps \( \sigma_N : X_n \to X_{n+1} \), \( n \geq N \), such that \((s_0, \ldots, s_{N-1}, \sigma_N)\) is \(N\)-good.

**Proof.** We construct maps \( T_N : X_n \to X_{n+2} \) for \( n \geq N \) such that

\[
\begin{align*}
d_i T_N &= \begin{cases} s_{N-1}^2 d_i, & (i < N), \\
s_N, & (i = N + 1, N + 2), \\
T_N d_{i-2}, & (i > N + 2); \end{cases} \\
T_N s_i &= s_N^2 s_i, & (i < N).
\end{align*}
\]

One should think of the map \( T_N \) as a candidate for the double degeneracy \( \sigma_N^2 \). Indeed, if \( s_N \) is already \(N\)-good, then the operators \( T_N := s_N^2 \) satisfy the above properties (plus the equation \( s_N = d_N T_N \)). On the other hand, if we are given an almost \(N\)-good system \((s_0, \ldots, s_N)\), and maps \( T_N \) satisfying \( \text{(6)} \) and \( \text{(7)} \), then by setting \( \sigma_N := d_N T_N \), we obtain an \(N\)-good system.

The construction of the collection \((T_N)\) is very analogous to the construction in the previous step and is by induction on \( n \geq N \). In the case \( N = 0 \), the
induction beginning is given by any map $T_0: X_0 \to X_2$ that sends $x \in X_0$ to a 2-simplex expressing the fact that $s_0(z)$ is $p$-idempotent, where we also assume that on $A_0 \subset X_0$, the map is actually given by $s_0$. The induction beginning in the other cases and the induction step are by the same construction as follows:

Assume that we have an almost $N$-good system $(s_0, \ldots, s_N)$ and maps $T_N: X_\ell \to X_{\ell+2}$ for $\ell \leq n - 1$, satisfying the conditions (6) and (7). We wish to define $T_N: X_n \to X_{n+2}$ so that (6) and (7) are again satisfied.

For $x \in X_n$, the $(n+2)$-simplex $\Lambda_n^{n+2} \to X$. In more detail, if we let

$$x_i = \begin{cases} s_{n-1}^2 d_i(x), & (i < N), \\
 s_N(x), & (i = N + 1, N + 2), \\
 T_N d_{i-2}(x), & (i > N + 2) \end{cases}$$

then again a case-by-case calculation shows that the horn equations

(8) \hspace{1cm} d_j(x_i) = d_{i-1}(x_j), \hspace{1cm} (j < i, j, i \neq N)

hold.

If $N > 0$, the horn $\Lambda_N^{n+2}$ is an inner horn which can be filled by an $(n+2)$-simplex $X_{n+2}$ we call $T_N(x)$. If $N = 0$, then (6) shows that the first edge is $s_0$ of the first vertex, which is cocartesian by assumption. So we can fill in the horn as well to get an element $T_0(z) \in X_{n+2}$.

Again we need to modify this construction in two ways: First, if $x = f(x')$, we choose as filler the element $f s_N^2(x')$ provided by the simplicial set structure of $A$. Second, if $x \in X_n$ degenerate, then the choice of filler $T_n(x)$ is forced to us by (7).

Again, Lemma 2.4 ensures that this is well-defined. \]

Lemmas 2.3 and 2.5 together prove the induction step and therefore Theorem 2.1. We now give the postponed Lemma 2.4. It builds on the following Lemma, which is valid in an arbitrary category and whose proof is an easy exercise.

**Lemma 2.6.** Suppose that in the commutative square

$$
\begin{array}{ccc}
A & \xrightarrow{i} & B \\
\downarrow{j} & & \downarrow{g} \\
X & \xrightarrow{j} & Y
\end{array}
$$

the morphism $i$ is a retract of $j$, and that $j$ is injective. Then the diagram is a pull-back diagram.

(Here, being a retract means that there exist morphisms $F: X \to A$ and $G: Y \to B$ such that $Ff = \text{id}_A$, $Gg = \text{id}_B$, and $iF = Gj$.) Lemma 2.4 follows from this result by choosing $d_i$ as vertical retractions.

**Proof of the Addendum.** By the Kan condition for the horn $\Lambda_1^1 \subset \Delta^1$, any $x \in X_0$ is $d_0$ of some 1-simplex $e$. Then filling in the $(2, 2)$-horn

$$
\begin{array}{ccc}
\ & \ & x \\
\ & \nearrow \ & f \\
x & \searrow \ & e \\
 & \ & y
\end{array}
$$
yields an edge \( f: x \to x \); filling in the \((3,0)\)-horn

\[
\begin{array}{c}
\text{\(x\)} \\
\downarrow^f \\
\downarrow^e \\
\text{\(e\)}
\end{array}
\begin{array}{c}
\text{\(f\)} \\
\downarrow \\
\downarrow \\
\text{\(x\)}
\end{array}
\begin{array}{c}
\text{\(x\)} \\
\uparrow_f \\
\uparrow_e \\
\text{\(e\)}
\end{array}
\begin{array}{c}
\text{\(x\)} \\
\downarrow^f \\
\downarrow^e \\
\text{\(e\)}
\end{array}
\begin{array}{c}
\text{\(x\)} \\
\uparrow f \\
\uparrow e \\
\text{\(e\)}
\end{array}
\]

shows that \( f \) is idempotent (and an equivalence, as any edge in a Kan semi-simplicial set). Thus the correspondence \( x \mapsto f \) provides a function \( s_0 \) as required. \( \square \)

3. A generalization

The proof above works for semi-simplicial objects in other categories than the category of sets. Indeed, let \( \mathcal{C} \) be any category, closed under limits, and provided with a subclass of morphisms called “cofibrations”, satisfying axioms A and B below.

A: A split-injection is a cofibration.

For a collection of morphisms \( X_i \to X \) \( (i \in \{1, \ldots, N\}) \), their “union” \( \bigcup_{i=1}^{N} X_i \) is defined to be the colimit of the objects \( X_i \) over their “intersections” \( X_{ij} := X_i \times_X X_j \), that is, the colimit of the diagram formed by the objects \( X_i \) and the objects \( X_{ij} \) (for \( i < j \)), together with the projection maps \( X_{ij} \to X_i \) and \( X_{ij} \to X_j \).

With this notation, the second axiom reads:

B: If \((c_i: X_i \to X)_{i \in \{1, \ldots, N\}}\) is a finite family of cofibrations, then their “union” exists and the induced map \( \bigcup_{i=1}^{N} X_i \to X \) is a cofibration.

In the last section we studied the case where \( \mathcal{C} \) is the category of sets and the cofibrations are the injective maps. In this situation, one easily verifies that the “union” \( \bigcup_{n} X_n \) maps injectively into \( X \), with image the actual union of the subsets \( c_n(X_n) \subset X \), which justifies our notation.

As usual, we call a morphism in \( \mathcal{C} \) an “acyclic fibration” if it has the right lifting property against all cofibrations. Clearly the collection of acyclic fibrations is closed under compositions and pull-backs. In our previous example, the category of sets, a map is an acyclic fibration if and only if it is surjective.

Let \( s\mathcal{C} \) denote the category of semi-simplicial objects in \( \mathcal{C} \), and let \( X \in s\mathcal{C} \). Since \( \mathcal{C} \) is closed under limits, the contravariant functor \( X \) extends along the Yoneda embedding \( \Delta^{inj} \to s\mathcal{Set} \), via the formula

\[
X(A) := \lim_{\Delta^{inj} \to A} X_n \quad (A \in \mathcal{Set}^{(\Delta^{inj})^{op}})
\]

where the limit is indexed over the category of simplices of \( A \). With this definition, the canonical map \( X_n \to X(\Delta^n) \) is an isomorphism.

**Definition 3.1.** Let \( X, Y \in s\mathcal{C} \) and \( T \in \mathcal{C} \).

(i) A semisimplicial map \( p: X \to Y \) is an inner fibration (resp., a Kan fibration) if the canonical maps

\[
X_n \to X(\Lambda^n_i) \times_Y (\Lambda^n_i) Y_n
\]

in \( \mathcal{C} \) are acyclic fibrations for \( 0 < i < n \) (resp., for \( 0 \leq i \leq n \)).

(ii) Suppose further that \( Y \) has a simplicial structure. A map \( f: T \to X_1 \) is p-idempotent if there exists a map \( T \to X_2 \) which agrees with \( f \) on all three boundaries, and whose image \( T \to Y_2 \) in \( Y \) factors through the degeneracy \( Y_0 \to Y_2 \).
(iii) A map \( f : T \to X_1 \) is \( p \)-cartesian if for any \( n > 0 \), the canonical map
\[
T \times_{X_1} X_n \to T \times_{X_1} X(\Lambda^n_0) \times_{Y(\Lambda^n_0)} Y_0
\]
in \( C \) is an acyclic fibration, where \( X(\Lambda^n_0) \) maps to \( X_1 \) by the last edge map.
The notion of \( p \)-cocartesianness is defined dually.

With these notions, we have the following generalizations of Theorem 2.1 and Addendum 2.2.

**Theorem 3.2.** Let \( p : X \to Y \) and \( f : A \to X \) be morphisms in \( sC \) where \( p \) is an inner fibration and \( f \) an injective cofibration in each semi-simplicial degree, and where \( Y \) and \( A \) have simplicial structures such that \( p \circ f \) is simplicial. Let \( s_0 : X_0 \to X_1 \) be a \( p \)-idempotent, \( p \)-cartesian and \( p \)-cocartesion morphism in \( C \) which is compatible with the degeneracies \( s_0 \) on \( A \) and \( Y \).

Then \( s_0 : X_0 \to X_1 \) extends to a simplicial structure on \( X \) such that \( f \) and \( p \) are simplicial.

**Addendum 3.3.** If \( p \) is a Kan fibration, then a map \( s_0 : X_0 \to X_1 \) as required in the Theorem exists always, so that a compatible simplicial structure on \( X \) exists without further hypotheses.

The proof of Theorem 3.2 is identical to the proof of Theorem 2.1. In terms of this section, the proof of Lemma 2.3 constructs a commutative solid square

\[
\begin{array}{ccc}
B & \to & X_{n+1} \\
\downarrow & & \downarrow \\
X_n & \to & X(\Lambda^n_{N+1}) \times_{Y(\Lambda^n_{N+1})} Y_{n+1}
\end{array}
\]

where \( B \) is the “union” of the cofibrations \( s_i : X_{n-1} \to X_n, \ i < N \), and the map \( f : A_n \to X_n \); note that Lemma 2.3 precisely identifies the “intersections” of \( s_i \) with \( s_j \) and with \( f \); and the calculation following the Lemma shows that the map \( B \to X_{n+1} \) is well-defined. Therefore \( s_N : X_n \to X_{n+1} \) exists by definition of inner fibration (in the case \( n > N \)) and by definition of \( p \)-cartesian (in the case \( n = N \)).

By induction this gives rise to an almost \( N \)-good structure just as in the proof of Theorem 3.2. The same re-writing can be made for Lemma 2.5 and the second step of the proof. The proof of the Addendum is completely analogous. Here are two examples.

3.1. **Semi-Segal spaces.** We take \( C \) the category of simplicial sets, with the usual notion of cofibration (level-wise injective maps). Then a map is an acyclic fibration in our sense if and only if it is a Kan fibration and a weak equivalence (after realization). We call an object in \( sC \) a semisimplicial space for short.

A map \( p \) in \( sC \) is a *Reedy fibration* if for any inclusion \( A \subset B \) of semi-simplicial sets, the induced map
\[
X(B) \to Y(B) \times_{Y(A)} X(A)
\]
is a Kan fibration. The space \( X \) is called *Reedy fibrant* if the projection \( X \to \{ \ast \} \) is a Reedy fibration.

Let \( I_n \subset \Delta^n \) be the semi-simplicial subset spanned by the edges \( (i, i + 1) \), where \( i = 0, \ldots, n - 1 \). By definition, \( X \) is a *semi-Segal space* if it is Reedy fibrant and if for each \( n > 1 \), the map \( X_n \to X(I_n) \) induced by the inclusion \( I_n \to \Delta^n \) is a weak equivalence of spaces.

The following is a variation of [JT07, 3.4].

**Lemma 3.4.** A Reedy fibration \( p : X \to Y \) between semi-Segal spaces is an inner fibration in our sense.
Proof. The map in question is a fibration by the fact that $p$ is a Reedy fibration. To show that it is a weak equivalence, we show that for each inner horn $\Lambda^n_k$, $0 < k < n$, in the square

$$
\begin{array}{ccc}
X_n & \xrightarrow{p} & Y_n \\
\downarrow & & \downarrow \\
X(\Lambda^n_k) & \xrightarrow{p} & Y(\Lambda^n_k)
\end{array}
$$

the vertical maps are weak equivalences.

Recall that the forgetful functor from simplicial sets to semisimplicial sets has a left adjoint $A \mapsto A^+$ which is an embedding of categories. Let $A \subset C$ be the class of injective semi-simplicial maps (that is, injective simplicial maps that are of the form $f^+: A^+ \to B^+$) such that $f^+: Y(B) \to Y(A)$ is a weak equivalence (hence an acyclic fibration). As $Y$ is Reedy fibrant by assumption, the class $A$ contains the inclusion $L_n \to [n]$; by [JT07, Lemma 3.5] it contains therefore every inner horn inclusion $\Lambda^n_k \to [n]$, too. Thus $Y_n \to Y(\Lambda^n_k)$ is an acyclic fibration. The same argument applies to $X$.

\[\square\]

As a consequence, we deduce from Theorem 3.2 the following result. (Recall that here “space” means “simplicial set”.)

**Theorem 3.5.** Let $p: X \to Y$ be an inner fibration of semi-Segal spaces and $f: A \to X$ the inclusion of a semisimplicial subspace; assume that $Y$ and $A$ have simplicial structures such that $p \circ f$ is a simplicial map. Let $s_0: X_0 \to X_1$ be a map, compatible with the degeneracies $s_0$ on $A$ and $Y$, and such that $s_0$ is $p$-idempotent, $p$-cartesian, and $p$-cocartesian.

Then $s_0: X_0 \to X_1$ extends to a simplicial structure on $X$ such that $f$ and $p$ are simplicial.

We close this section by giving a criterion for $p$-(co-)cartesianness. For $\sigma \in X(A)$ and $A \subset B$, we denote by $X(B)/\sigma \subset X(B)$ the subspaces of all elements mapping to $\sigma \in X(A)$ under the map induced by the inclusion $A \subset B$.

**Lemma 3.6.** For a Reedy fibration $p: X \to Y$ of semi-Segal spaces, and $f: T \to X_1$, the following are equivalent:

(i) $f$ is $p$-cartesian.

(ii) For any $t \in T_0$, the composite $\{\ast\} \xrightarrow{\tau} T \xrightarrow{f} X_1$ is $p$-cartesian.

(iii) For any $t \in T_0$, with $e := f(t): x' \to x$, the following commutative square

$$
\begin{array}{ccc}
X_2/e & \xrightarrow{d_1} & X_1/x \\
\downarrow{p} & & \downarrow{p} \\
Y_2/p(e) & \xrightarrow{d_1} & Y_1/p(x)
\end{array}
$$

is a homotopy pull-back:

**Remark.** By the Segal condition, the map $d_2: X_2/e \to X_1/x'$ is a weak equivalence so the horizontal maps in the diagram may be thought of as “postcomposition by $e$ and $p(e)$”, respectively.

**Proof of Lemma 3.6.** (i) implies (ii) because acyclic fibrations are stable under pull-back. For the converse direction, we note that in the map under consideration,

$$
T \times X \xrightarrow{p} T \times Y \xrightarrow{Y(\Lambda^n_0) \times Y(\Lambda^n_0)} Y
$$

both domain and target are Kan fibrations over $T$. Therefore, to show that the map is a weak equivalence, it suffices to test on all fibers of $T_0$. But this is condition (ii).
For the equivalence between (ii) and (iii), we consider the following diagram (for \( n > 0 \)):

\[
\begin{array}{c}
X_{n+1}/e \xrightarrow{d_n} X(\Lambda_{n+1}^+)/e \xrightarrow{p} X_0/y \\
\downarrow p \quad \downarrow p \\
Y_{n+1}/p(e) \xrightarrow{d_n} Y(\Lambda_{n+1}^+)/p(e) \xrightarrow{p} Y_0/p(y)
\end{array}
\]

and notice that condition (ii) is equivalent to the left square being a homotopy pull-back, for any \( e = f(t) \in X_1 \). Now we note that for \( n = 1 \), the right horizontal maps are isomorphisms. Hence, if (ii) holds, then the total square is a homotopy pull-back for \( n = 1 \), that is (iii) holds.

Conversely assume that (iii) holds. We consider the commutative diagram

\[
\begin{array}{c}
X_{n+1}/e \xrightarrow{d_n} X_0/y \\
\downarrow \approx \quad \downarrow \approx \\
X_{n-1} \times X_0 X_2/e \xrightarrow{id \times d_1} X_{n-1} \times X_0 X_1/y
\end{array}
\]

where the vertical arrows are equivalences by the Segal condition and the lower horizontal map is one by (iii). It follows that the total square in (9) is a homotopy pull-back for all \( n > 0 \).

We show by induction on \( n \) that the left square is a homotopy pull-back, too. If \( n = 1 \), then we remarked above that the right horizontal maps are isomorphisms which immediately implies the claim. For the induction step, we note that the inclusion \( \Delta^n \subset \Lambda_0^+ \) of the \( n \)-th face is obtained by filling in horns \( \Lambda_k^0 \) for \( k \leq n \), with last edge \( (n, n+1) \). (By induction on \( k \), fill in all pairs of type \( (i_1, \ldots, i_k, n) \) and \( (i_1, \ldots, i_k, n+1) \); for each such pair this corresponds to filling in a horn as required.) By the inductive assumption, it follows that the right square is a homotopy pull-back, hence so is the left.

3.2. Multi-semisimplicial sets. We take \( C = s^k\text{Set} \) the category of \( k \)-fold semisimplicial sets, where a morphism defined to be a cofibration if it is injective in each multi-semisimplicial level. We say that an object \( X \) of \( C \) satisfies the Kan condition if, after writing \( s^k\text{Set} = s(s^{k-1}\text{Set}) \) by singling out any of the \( k \) simplicial directions, any map

\[
X_n \rightarrow X(\Lambda^n)
\]

induced by a horn inclusion \( \Lambda^n \subset \Delta^n \) is an acyclic fibration in the sense of this section. (This is equivalent to [McC13 Definition 5.2].)

Theorem 3.7 ([McC13]). Any \( k \)-fold semisimplicial set which satisfies the Kan condition, has a \( k \)-fold simplicial structure.

Proof. We show more generally that any \( k \)-fold semisimplicial \( l \)-fold simplicial set \( X \), satisfying the Kan condition, has a \( (k+l) \)-fold simplicial structure. The proof is by induction on \( k \), where the induction beginning \( k = 0 \) holds obviously. For the induction step, we view \( X \) as a semisimplicial object in the category of \( (k-1) \)-fold semisimplicial \( l \)-fold simplicial sets. By Theorem [McC13], this can be promoted to a simplicial object, corresponding to a \( (k-1) \)-fold \((l+1)\)-fold simplicial set. But this admits a simplicial structure by induction hypothesis.

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