AN OVERTWISTED DISK IN A VIRTUAL CONTACT STRUCTURE
AND THE WEINSTEIN CONJECTURE

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Abstract. Hofer proved in [Hof] the Weinstein conjecture for a closed contact 3-manifold with an overtwisted disk. In this article we extend it to the virtual contact structure and provide a new explicit example of the virtual contact structure with an overtwisted disk via a Lutz twist.

1. Introduction

A virtual contact structure which naturally appear in the study of magnetic flows is a generalization of a contact structure. Dynamical properties of a virtual contact structure including stability, displaceability, periodic orbits, and leaf-wise intersections are studied in the literatures [Pat, Mel, CFP, Me2, BF, Bae], when the virtual contact structure arises as an energy hypersurface of a twisted cotangent bundle. In this article we focus on the topological aspect of a virtual contact structure.

Definition 1.1. A Hamiltonian structure on an oriented $(2n+1)$-dimensional manifold $M$ is a closed 2-form $\omega$ of maximal rank, i.e. $\omega^n$ vanishes nowhere. So $\ker \omega_x := \{ v \in T_x M \mid \iota_v \omega_x = 0 \}$ gives an 1-dimensional foliation on $M$. By using the orientation on $M$, we orient $\ker \omega$ and choose a non-vanishing vector field $X_\omega$ on $M$ such that $RX_\omega = \ker \omega$.

Definition 1.2. A Hamiltonian structure $(M, \omega)$ is called virtual contact if there exists a covering $p : \hat{M} \to M$ and a primitive $\lambda \in \Omega^1(\hat{M})$ of $p^*\omega$ satisfying the following conditions:

$$\| \lambda \|_{C^0} \leq C < \infty, \quad \inf_{x \in \hat{M}} \lambda(x)(\hat{X}(x)) \geq \epsilon > 0 \quad (1.1)$$

for some $C, \epsilon \in \mathbb{R}$. Here $\hat{X}$ is the lift of $X$ and $\| \cdot \|_{C^0}$-norm is given by the lifted metric $\hat{m} := p^*m$ on $\hat{M}$ where $m$ is a Riemannian metric on $M$.

A virtual contact structure $(p : \hat{M} \to M, \omega, \lambda)$ on a smooth manifold $M$ is called smooth if all higher covariant derivatives $\nabla_Y \lambda$ are exist and uniformly bounded. Here $Y$ is $l$-pairs of $G$-invariant smooth vector fields and $\nabla$ is a $G$-invariant connection on $\hat{M}$, where $G$ is the deck-transformation group of the covering $p : \hat{M} \to M$.

One of the main question in contact geometry is (the intrinsic version of) the Weinstein conjecture [Wei] which says that every closed contact manifold $(Q, \zeta)$ has a closed orbit for any Reeb vector field. Indeed, we need a contact 1-form $\alpha$ for $\zeta$, i.e. $\ker \alpha = \zeta$, to define the Reeb vector field $X^{\alpha}$ which satisfies

$$d\alpha(X^{\alpha}, \cdot) = 0, \quad \alpha(X^{\alpha}) = 1.$$ 

We also have a Hamiltonian structure $(Q, d\alpha)$ and the vector field $X^{d\alpha}$ in Definition 1.1 turns out to be a rescaling of $X^{\alpha}$.

The Weinstein conjecture was first proved by Hofer in [Hof] for a closed contact 3-manifold $M$ with an overtwisted disk or with $\pi_2(M) \neq 0$ or $M = S^3$. Later, Taubes [Tau] proved the conjecture for any closed contact 3-manifold, see also [Hut]. There are several extensions of the conjecture including the strong Weinstein conjecture in [ACH, GZ] and the Weinstein conjecture.

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for the stable Hamiltonian structure in [HT]. The following question is raised by G. Paternain as another generalization of the Weinstein conjecture in the virtual contact structure.

**Question 1.3.** Let \((p : \hat{M} \to M, \omega, \lambda)\) be a virtual contact structure on a closed manifold \(M\). Does \(X^\omega\) admit a periodic orbit?

There is a fundamental dichotomy of contact topology on 3-manifolds, tight and overtwisted. In order to state the result we need to extend these concepts to 3-dimensional virtual contact structures.

**Definition 1.4.** An embedded disk \(F\) in a contact 3-manifold \((Q, \zeta)\) is an overtwisted disk if \(\partial F \subset \zeta|_{\partial F}\) and \(TF \cap \zeta|_F\) defines a smooth 1-dimensional characteristic foliation on \(F\) except a unique elliptic singular point \(e \in \text{int}F\) with \(T_eF = \zeta_e\). A virtual contact structure \((p : \hat{M} \to M, \omega, \lambda)\) is called overtwisted if \((\hat{M}, \ker \lambda)\) contains an overtwisted disk.

Let us briefly mention Hofer’s argument of deriving a contractible periodic orbit from an overtwisted disk. By the filling method he constructed a family of pseudoholomorphic disks, a Bishop family, in the symplectization of a closed contact 3-manifold. The overtwisted disk guarantees a gradient exploding sequence in the Bishop family and by the rescaling argument we obtain a finite energy plane. The failure of the finite energy plane to be a sphere produce a periodic orbit as we desired. By extending the above argument we obtain the following result:

**Main Theorem.** Let \((p : \hat{M} \to M, \omega, \lambda)\) be a virtual contact structure on a closed 3-manifold \(M\). If \((p : \hat{M} \to M, \omega, \lambda)\) is smooth and overtwisted then \(X^\omega\) has a contractible periodic orbit.

Hofer’s argument was initiated from the Eliashberg’s filling technique in [Eli] which induces an obstruction to symplectic fillings. A similar question about symplectic filling is possible in the virtual contact structure. We can also expect a higher dimensional generalization of the main theorem as achieved in [AlH] [NR].

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2. An almost complex structures on a virtual contact structure

Let \((p : \hat{M} \to M, \omega, \lambda)\) be a virtual contact structure on a \((2n + 1)\)-dimensional Riemannian manifold \((M, m)\), then \(\lambda \in \Omega^1(\hat{M})\) is a contact form on \(\hat{M}\), i.e. \(\lambda \wedge d\lambda^n > 0\). Let us denote \(X^\lambda\) by the Reeb vector field for the contact 1-form \(\lambda\)

\[
d\lambda(X^\lambda, -) = 0, \quad \lambda(X^\lambda) = 1.
\]

By the virtual contact condition there are positive constants \(\epsilon', C'\) such that \(\epsilon' \leq |X^\lambda|_{\hat{m}} \leq C'\). Note that \(X^\lambda\) is a rescaling of the vector field \(\tilde{X}\) in Definition 1.2. Associated to the virtual contact structure, we have the following canonical decomposition:

\[
T\hat{M} = (\ker, X^\lambda) \oplus (\xi := \ker \lambda, \omega).
\]  \hspace{1cm} (2.1)

Here \(\ker \omega\) is a line bundle with the section \(X^\lambda\) and \((\ker \lambda, \omega)\) is a symplectic bundle on \(\hat{M}\).

Now we choose an almost complex structure \(J : \xi \to \xi\) such that

\[
\tilde{m}_x(\pi_\lambda k_1, \pi_\lambda k_2) = d\lambda_x(\pi_\lambda k_1, J(x)\pi_\lambda k_2),
\]  \hspace{1cm} (2.2)

where \(k_1, k_2 \in T_x\hat{M}, \pi_\lambda : T\hat{M} \to \xi\) be the fibrewise projection map along the Reeb direction \(X^\lambda\). Let \(\tilde{J}\) be the associated almost complex structure on \(\mathbb{R} \times \hat{M}\) defined by

\[
\tilde{J}(a, u)(h, k) := (-\lambda(u)k, J(u)\pi_\lambda k + h \cdot X^\lambda(u)).
\]  \hspace{1cm} (2.3)
Now define a corresponding Riemannian metric on $\mathbb{R} \times \hat{M}$ by

$$m_\lambda((h_1,k_1),(h_2,k_2)) := h_1h_2 + \lambda(k_1)\lambda(k_2) + \hat{m}(\pi_\lambda k_1,\pi_\lambda k_2).$$  \hfill (2.4)

**Remark 2.1.** Suppose that the deck transformation group $G$ is finite, then by averaging $\lambda$ we obtain a 1-form

$$\Lambda = \frac{1}{|G|} \sum_{g \in G} g^*\lambda$$

on $\hat{M}$. Since $\Lambda$ is $G$-invariant, it descends to a 1-form $\lambda \in \Omega^1(M)$ such that $\Lambda = p^*\lambda$ and $d\lambda = \omega$. So $\Lambda$ becomes a contact 1-form on $M$ and hence the dynamics of a contact manifold $(\hat{M},\Lambda)$ determines the dynamics of the virtual contact structure $(p: \hat{M} \to M,\omega,\lambda)$.

From now on we mainly consider the case that $|G|$ is infinite i.e. $\hat{M}$ is noncompact, and $\lambda$ is not $G$-invariant. In such a case, $G$ does not preserve $X^\lambda$, $\xi$ and hence $J$, $\hat{J}$, $m_\lambda$ are not preserved by the $G$-action, while $\hat{\omega}$ and $\mathbb{R}X^\lambda = \ker \hat{\omega}$ are $G$-invariant.

By virtue of the relation (2.2), $J$ and $\hat{J}$ behave well under the $G$-action even though they are not $G$-invariant, see (4.12) in the proof of Lemma 4.5. If we require only the almost complex structure $J$ on $\xi$ to be $d\lambda$-compatible instead of (2.2), then we cannot control the limit behavior of $J$ and $\hat{J}$ on the unbounded region of $\hat{M}$.

### 3. A Bishop family from an overtwisted disk

In this section we introduce a Bishop family in the virtual contact structure and recall its known properties from [Hof AH HWZ]. In this section we provide a gradient exploding sequence of pseudoholomorphic disks from an overtwisted disk. All constructions, theorems, and lemmas in this section are direct consequences of the ones in the above references, so we omit the proof here.

Let $(p: \hat{M} \to M,\omega,\lambda)$ be a virtual contact structure on a smooth 3-manifold $M$ and $F \subset \hat{M}$ be an overtwisted disk with the elliptic singular point $e \in \int F$. We consider a family of pseudoholomorphic disks $\tilde{u}_\tau = (a_\tau,u_\tau) : \mathbb{D} = \{ z \in \mathbb{C} \mid |z| \leq 1 \} \to \mathbb{R} \times \hat{M}$, $\tau \in (0,\tau_0)$

$$\partial_s \tilde{u}_\tau + \hat{J}(\tilde{u}_\tau)\partial_t \tilde{u}_\tau = 0$$ \hfill (3.1)

satisfying the following conditions:

1. $\tilde{u}_0 \equiv (0,e)$;
2. $\tilde{u}_\tau(\partial \mathbb{D}) \subset \{0\} \times (\int F \setminus \{e\})$ for all $\tau \in (0,\tau_0)$;
3. $\bigcup_{0<\tau<\tau_0} u_\tau(\partial \mathbb{D})$ is an open neighborhood of $e$ in $F$;
4. $\tilde{u}_\tau(\mathbb{D}) \cap \tilde{u}_\rho(\mathbb{D}) = \emptyset$ if $\tau \neq \rho$;
5. $\tilde{u}_\tau$ is an embedding for $\tau \in (0,\tau_0)$ and $u_\tau(\partial \mathbb{D})$ winds once around $e$.

Such a family of pseudo-holomorphic disks $\mathcal{B}$ exists and we call $\mathcal{B}$ a Bishop family. We state the implicit function theorem near an embedded pseudoholomorphic disk as follows:

**Theorem 3.1 (Hof).** Let $(p: \hat{M} \to M,\omega,\lambda)$ be a 3-dimensional virtual contact structure with an overtwisted disk $F \subset \hat{M}$. Moreover, let $\tilde{u}_0$ be a pseudoholomorphic disk in $(\mathbb{R} \times \hat{M},\hat{J})$ satisfying the condition (2), (5) of the Bishop family. Then there exists a smooth embedding $U : (-\epsilon,\epsilon) \times \mathbb{D} \to \mathbb{R} \times \hat{M}$ such that with $\tilde{u}(\tau)(z) := U(\tau,z)$ we have

$$\tilde{u}(\tau)(z) \in F \text{ for all } z \in \partial \mathbb{D};$$

$$\partial J\tilde{u}(\tau) = 0 \text{ for all } \tau \in (-\epsilon,\epsilon);$$

$$\tilde{u}(0) = \tilde{u}_0.$$ 

Moreover, the associated disk family $\tau \mapsto \tilde{u}(\tau)(\mathbb{D})$ is unique up to parametrization of $\mathbb{D}$.

1 More precisely, we consider a non-amenable group which means that there is no averaging operation on bounded functions, see [CFP] for the details.

2 In the original statement, $(\mathbb{R} \times \hat{M},\{0\} \times F)$ can be generalized to an almost complex manifold with a totally real submanifold. Here the condition (2) of the Bishop family guarantees the totally real boundary condition. Then the condition (5) should be replaced by the Maslov index condition, $\mu(\tilde{u}_0) = 2.$
If $\mathcal{B} = (\tilde{u}_\tau)_{\tau \in [0, \tau_0)}$ is a Bishop family and $(\phi_\tau : \mathbb{D} \to \mathbb{D})_{\tau \in [0, \tau_0)}$ is a $\tau$-parametrized family of conformal maps then $(\tilde{u}_\tau \circ \phi_\tau)_{\tau \in [0, \tau_0)}$ is also a Bishop family. In order to fix a parametrization of $\tilde{u}_\tau(\mathbb{D})$, let us first parametrize the leaves of the characteristic foliation, in Definition 1.4.

Moreover, this normalization condition (3.2) prohibits the existence of a gradient explosion sequence on the boundary as follows:

**Theorem 3.2 (HWZ).** Let $(p : \hat{M} \to M, \omega, \lambda)$ be a $3$-dimensional virtual contact structure with an overtwisted disk $F \subset \hat{M}$. Let $\mathcal{B} = (u_\tau)_{\tau \in [0, \tau_0)}$ be a Bishop family with the normalization condition (3.2) as in the above. Then there exists $\epsilon > 0$ such that on the annulus $A_\epsilon = \{z \in \mathbb{D} \mid 1 - \epsilon \leq |z| \leq 1\}$ we have

$$\sup_{0 \leq \tau < \tau_0} \sup_{z \in A_\epsilon} |\nabla u_\tau(z)| < \infty.$$  

**Remark 3.3.** Even though Theorem 3.2 is proved only for closed contact $3$-manifolds, it is still valid for the non-compact case. Suppose that there is a gradient exploding sequence $(\tilde{u}_k, z_k)$ such that $z_k$ converges to $\partial \mathbb{D}$. Then $u_k(z_k)$ should converge to a point in $F$ and so it cannot escape to the unbounded region of $\hat{M}$. The non-compactness of $\hat{M}$ causes no additional difficulties in the proof of Theorem 3.2.

The following observation is crucial when we produce a gradient explosion sequence from a given normalized Bishop family:

**Lemma 3.4 (Ho).** Let $(p : \hat{M} \to M, \omega, \lambda)$ be a $3$-dimensional virtual contact structure with an overtwisted disk $F \subset \hat{M}$. Let $\mathcal{B}$ be a Bishop family and take an embedded disk $\tilde{u}_\tau = (a_\tau, u_\tau) \in \mathcal{B}$. Then $u_\tau|_{\partial \mathbb{D}} : \partial \mathbb{D} \to F$ is transversal to the foliation $TF \cap \xi|_F$ on $F$.

By the definition of the overtwisted disk $F$, Definition 1.4, the boundary $T\partial F$ is contained in the foliation $TF \cap \xi|_F$ on $F$. So Lemma 3.4 informs us that $u_\tau(\partial \mathbb{D})$ cannot meet $\partial F$. Since our Bishop family $\mathcal{B} = (\tilde{u}_\tau)_{\tau \in [0, \tau_0)}$ emanated from the singular point $e \in \text{int} F$, $u_\tau(\partial \mathbb{D})$ never touch $\partial F$ nor escape it. In other words, $e \in \text{int} F$ enables us to create $\mathcal{B}$, while $\partial F$ gives us an obstruction to extend $\mathcal{B}$.

We may assume that $\mathcal{B} = (\tilde{u}_\tau)_{\tau \in [0, \tau_0)}$ is a maximal Bishop family with the normalization condition without loss of generality. Suppose that $\|\nabla \tilde{u}_\tau\|_{C^0(\mathbb{D})}$ is $\tau$-uniformly bounded, then by the elliptic estimate of a pseudoholomorphic disk we have a $\tau$-uniform $C^\infty$-bound. The uniform gradient bound guarantees that the image $\tilde{u}_\tau(\mathbb{D})$ is also uniformly bounded in $\mathbb{R} \times \hat{M}$. We then conclude by the Arzelà-Ascoli theorem for every sequence $\tau_k \to \tau_0$ there is a $C^\infty$-convergent subsequence of $\tilde{u}_{\tau_k}$. By the implicit function theorem, Theorem 3.1, we extend our Bishop family further. But this contradicts the maximality of our initial Bishop family $\mathcal{B}$. This proves the following result:

**Theorem 3.5 (AH).** Let $(p : \hat{M} \to M, \omega, \lambda)$ be a $3$-dimensional virtual contact structure with an overtwisted disk $F \subset \hat{M}$. Let $\mathcal{B} = (\tilde{u}_\tau)_{0 \leq \tau < \tau_0}$ be a normalized maximal Bishop family on $\mathbb{R} \times \hat{M}$ which emerges from $F$. Then we have

$$\sup_{\tau \in [0, \tau_0)} \|\nabla \tilde{u}_\tau\|_{C^0(\mathbb{D})} = \infty.$$  

4. Existence of a finite energy plane

The gradient explosion in the previous section will guarantee the existence of a finite energy plane by the rescaling argument. The non-compactness of $\mathbb{R}$ and $\hat{M}$ causes analytical difficulties in
the rescaling process. It is already studied that the escape phenomenon of the gradient exploding disks in $\mathbb{R}$-direction. In this section we mainly discuss the analytical issue from $\hat{M}$-direction.

For a $\hat{J}$-holomorphic map $\tilde{u} = (a, u) : \mathbb{D} \to \mathbb{R} \times \hat{M}$

$$\partial_t \tilde{u} + \tilde{J}(\tilde{u}) \partial_t \tilde{u} = 0,$$  \hspace{1cm}  (4.1)

we define an energy $E(\tilde{u})$ by

$$E(\tilde{u}) := \sup_{\varphi \in \Sigma} \int_{\mathbb{D}} \tilde{u}^* d(\varphi \lambda),$$ \hspace{1cm}  (4.2)

where $\Sigma = \{ \varphi \in C^\infty(\mathbb{R}, [0, 1]) \mid \varphi \geq 0 \}$. Recall that (4.1) is equivalent to

$$\pi_\lambda \partial_t u + J(u) \pi_\lambda \partial_t u = 0$$

$$u^* \lambda \circ i = da$$

and we remark that the integrand in (4.2) is nonnegative. By a simple computation we check that

$$\tilde{u}^* d(\varphi \lambda) = \frac{1}{2} \varphi(a) [a^2 + a_1^2 + (\lambda(u)u_1)^2 + (\lambda(u)u_2)^2] \, ds \wedge dt$$

$$= \sup$$

$$+ \frac{1}{2} \varphi(a) [\|\pi_\lambda u_1\|^2 + \|\pi_\lambda u_2\|^2] \, ds \wedge dt \geq 0. \hspace{1cm} (4.3)$$

Note that the Reeb direction of $du$ contributes to $\circ$-term, while $\ast$-term comes from the contact plane part. By the boundary condition of the Bishop family we have the following uniform energy bound.

**Lemma 4.1** ([Ho]). Let $F$ be an overtwisted disk as above and let $\tilde{u}$ be a solution of (4.1) satisfying the boundary condition in the definition of the Bishop family. Then there exists a constant $C = C(\lambda, F) > 0$ so that

$$E(\tilde{u}) \leq C.$$\hspace{1cm} Especially $C$ does not depend on $\tilde{u}$.

Before stating the existence of a finite energy plane we introduce the following helpful lemma, so called Hofer’s lemma, which will be used to find a suitable sequence in the rescaling argument.

**Lemma 4.2** ([Ho]). Let $(W, m)$ be a complete metric space and $R : W \to [0, \infty)$ a continuous function. Assume $x_0 \in W$ and $\epsilon_0 > 0$ are given. Then there exist $x \in B_{2\epsilon_0}(x_0)$ and $\epsilon \in (0, \epsilon_0]$ satisfying

$$R(x_0)\epsilon_0 \leq R(x)\epsilon;$$

$$R(y) \leq 2R(x) \text{ for } y \in B_\epsilon(x).$$

**Theorem 4.3.** Let $M$ be a closed 3-manifold equipped with a smooth virtual contact structure $(p : \hat{M} \to M, \omega, \lambda)$ and an overtwisted disk $F \subset \hat{M}$. Let $\mathcal{B} = (\tilde{u}_\tau)_{0 \leq \tau < \tau_0}$ be a normalized maximal Bishop family on $(\mathbb{R} \times \hat{M}, \hat{J})$ which emerges from $F$. Moreover we have $\sup_{\tau \in [0, \tau_0]} \|\nabla \tilde{u}_\tau\| = \infty$. Then there exist an almost complex structure $\tilde{J}^\infty$ on $\mathbb{R} \times \hat{M}$ and a non-constant $\tilde{J}^\infty$-holomorphic map $\tilde{v}^\infty = (b^\infty, v^\infty) : \mathbb{C} \to \mathbb{R} \times \hat{M}$ with finite energy.

**Proof.** Since $\sup_{\tau \in [0, \tau_0]} \|\nabla \tilde{u}_\tau\| = \infty$, we choose a sequence $(\tilde{u}_k)_{k \in \mathbb{N}}$ from our Bishop family $\mathcal{B}$ satisfying

$$\lim_{k \to \infty} \|\nabla \tilde{u}_k\| = \infty.$$\hspace{1cm} We pick a sequence $(z_k)_{k \in \mathbb{N}}$ in $\mathbb{D}$ so that $R_k := |\nabla \tilde{u}_k(z_k)| \to \infty$ as $k \to \infty$.

If the image of the sequence $(u_k(z_k))_{k \in \mathbb{N}}$ is bounded in $\hat{M}$ then we are able to directly apply the Hofer’s argument to guarantee the existence of a finite energy plane. However, there is no a priori reason that $(u_k(z_k))_{k \in \mathbb{N}}$ is contained in a bounded region. To remedy this situation we will use the compactness of $M$ via the projection $p : \hat{M} \to M$. If we consider the sequence $u_\lambda(z_k) := p \circ u_k(z_k) \in M$ then it has a convergent subsequence on $M$, we still denote $u_k, z_k$. Let

$^5$Here the energy is given by the contact form $\lambda_\infty$ in (4.15).
us fix a fundamental domain \( M \subset \hat{M} \) with respect to the deck-transformation on \( \hat{M} \). Now we choose a sequence of deck-transformations \( g_k \in G \) such that each \( g_k^{-1} \circ u_k(z_k) \) is contained in the fixed fundamental domain \( \hat{M} \).

In order to rescale the gradient explosion, take a sequence \( \epsilon_k \to 0 \) so that \( R_k \epsilon_k \to \infty \). Using Lemma 4.2 by slightly changing \( z_k \) and \( \epsilon_k \), we may assume in addition that \( |\nabla \tilde{u}_k(z)| \leq 2R_k \) for all \( z \in D \) with \( |z-z_k| \leq \epsilon_k \). We define a sequence of maps \( \tilde{v}_k : B_{R_k}(-R_kz_k) \to \mathbb{R} \times \hat{M} \) by

\[
\tilde{v}_k(z) := (b_k, \tilde{v}_k) := \left( a_k(z_k + \frac{z}{R_k}) - a_k(z_k), g_k^{-1} \circ u_k(z_k + \frac{z}{R_k}) \right),
\]

so that

\[
b_k(0) = 0, \quad \tilde{v}_k(0) \in M
\]

and

\[
|\nabla \tilde{v}_k(0)| = 1.
\]

Note that a sequence of domains

\[
B^k := B_{R_k}(-R_kz_k) \cap B_{\epsilon_k} \in \mathbb{C}
\]

satisfies \( \bigcup_{k \in \mathbb{N}} B^k = \mathbb{C} \) and we have a uniform gradient bound

\[
|\nabla \tilde{v}_k(z)| \leq 2 \quad \text{for} \quad z \in B^k.
\]

Now we consider a sequence of restrictions \( (\tilde{v}_k|_{B^1})_{k \in \mathbb{N}} \). The conditions 4.5, 4.7 imply that the image \( \tilde{v}_k(B^1) \) is uniformly bounded in \( \mathbb{R} \times \hat{M} \). By applying the Arzel`a-Ascoli theorem we have a subsequence, again denote \( \tilde{v}_k|_{B^1} \), and a continuous map \( \tilde{v}^1 : B^1 \to \mathbb{R} \times \hat{M} \) such that \( \tilde{v}_k|_{B^1} \) converges uniformly to \( \tilde{v}^1 \). Recall from Remark 2.1 that the almost complex structure \( \tilde{J} \) on \( \mathbb{R} \times \hat{M} \) is not invariant under the \( G \)-action, and hence \( g_k^{-1}\tilde{u}_k := (a_k, g_k^{-1} \circ u_k) \), \( \tilde{v}_k \), and \( \tilde{v}^1 \) cannot be \( \tilde{J} \)-holomorphic. So we need to find new almost complex structures which make \( \tilde{v}_k, \tilde{v}^1 \) to be pseudo-holomorphic.

First choose a compact subset \( E^1 \subset \hat{M} \) containing \( \bigcup_k v_k(B^1(0)) \). Let us define a sequence of almost complex structures \( (\tilde{J}_k)_{k \in \mathbb{N}} \) on \( \mathbb{R} \times \hat{M} \) by

\[
\tilde{J}_k(a, u)(h, l) := dg_k^{-1}[\tilde{J}(a, g_ku)(h, dg_kl)],
\]

then \( g_k^{-1}\tilde{u}_k \) becomes \( \tilde{J}_k \)-holomorphic and so does \( \tilde{v}_k \). In order to understand the sequence of almost complex structures \( (\tilde{J}_k)_{k \in \mathbb{N}} \) and their limits we consider the following sequences

\[
\lambda_k := g_k^*\lambda, \quad \xi_k := \ker \lambda_k, \quad X_k := (g_k^{-1})_*X^\lambda, \quad J_k := \tilde{J}_k|_{\xi_k}.
\]

because of the construction of \( \tilde{J} \). \((2.2)\) and \((2.3)\). More precisely, we have

\[
\lambda_k(x)v = \lambda(g_kx)dg_kv; \quad \xi_k(x) = \{dg_k^{-1}w \mid w \in \xi(g_kx)\}; \quad X_k(x) = dg_k^{-1}X^\lambda(g_kx); \quad J_k(x)v = dg_k^{-1}[J(g_kx)dg_kv],
\]

where \( x \in \hat{M} \) and \( v \in T_x\hat{M} \). Note that \( X_k \) is a Reeb vector field of \( \lambda_k \) and \( J_k \) is an almost complex structure on \( \xi_k \).

For convenience, we choose a smooth coframe field \( \{\rho_1, \rho_2, \rho_3\} \) of the closed 3-manifold \( M \).\(^7\)

Then its lift \( \{\tilde{\rho}_1, \tilde{\rho}_2, \tilde{\rho}_3\} \) gives an induced coframe on \( \hat{M} \). There are coefficient functions \( c_1, c_2, c_3 \in C^\infty(\hat{M}) \) for \( \lambda \in \Omega^1(\hat{M}) \) satisfying

\[
\lambda = c_1\tilde{\rho}_1 + c_2\tilde{\rho}_2 + c_3\tilde{\rho}_3.
\]

\(^6\)Here we use Theorem 4.2. \(^7\)Here we use that the tangent bundle \( TQ \) of any closed 3-manifold \( Q \) is trivial.
By the smooth condition for the virtual contact structure in Definition 1.2 and the compactness of \( M \), all higher directional derivatives \( \nabla_Y c_i \), \( i = 1, 2, 3 \) are exist and uniformly bounded. Here \( Y \) is again \( l \)-pairs of \( G \)-invariant smooth vector fields. Note that the following simple observation

\[
\nabla_Y (c_i \circ g_k)(x) = \nabla_{d g_k(Y)} c_i(g_k(x)) = \nabla_Y c_i(g_k(x))
\]

holds for \( x \in \widehat{M} \), \( g_k \in G \), and \( i = 1, 2, 3 \). So all higher directional derivatives of \( c_i \circ g_k \) are also exist and uniformly bounded. Now we apply the Arzelà-Ascoli theorem to the sequence of \( 1 \)-forms

\[
\lambda_k|_{E^1} = \sum_{i=1}^{3} (c_i \circ g_k) \hat{\nu}_i|_{E^1},
\]

in order to obtain a subsequence, still denote \( \lambda_k \), which converges to \( \lambda^1 \in \Omega^1(E^1) \) in \( C^\infty \).

**Lemma 4.4.** The limit 1-form \( \lambda^1 \) is a contact 1-form on \( \text{int} E^1 \).

**Proof.** Since \( \lambda \) is a contact 1-form on \( \widehat{M} \), \( \lambda_k = g_k^* \lambda \) satisfies

\[
\lambda_k \land d \lambda_k > 0 \quad \text{on int} \: E^1
\]

for each \( k \in \mathbb{N} \). As a limit of \( \lambda_k \), however, \( \lambda^1 \) may degenerate and could satisfy \( \lambda^1 \land d \lambda^1 \geq 0 \) so it is needed to exclude the case \( \lambda^1 \land d \lambda^1 = 0 \). Suppose that there is \( x \in \text{int} E^1 \) such that \( \lambda^1(x) \land d \lambda^1(x) = 0 \). By the construction of \( \lambda^1 \), its exterior derivative \( d \lambda_k|_{E^1} \) also converges to \( d \lambda^1 \). Thus we have

\[
d \lambda_k|_{E^1} = d g_k^* \lambda|_{E^1} = g_k^* d \lambda|_{E^1} = g_k^* \hat{\omega}|_{E^1} = \hat{\omega}|_{E^1} \longrightarrow d \lambda^1,
\]

which implies \( \lambda^1(x) \land \hat{\omega}(x) = 0 \). Let us recall the decomposition \( T_x \widehat{M} \cong \mathbb{R} \widehat{X}(x) \oplus \ker \lambda(x) \) from (2.1) where \( \widehat{X}(x) \) is defined in Definition 1.2 which generates \( \ker \hat{\omega}(x) \) and \( G \)-invariant. So we deduce

\[
0 = \iota_{\widehat{X}(x)} (\lambda^1(x) \land \hat{\omega}(x))
\]

\[
= \lambda^1(x) (\widehat{X}(x)) \cdot \hat{\omega}(x) + \lambda^1(x) \land \iota_{\widehat{X}(x)} \hat{\omega}(x)
\]

\[
= \lambda^1(x) (\widehat{X}(x)) \cdot \hat{\omega}(x).
\]

Since \( \hat{\omega}(x) \neq 0 \) on \( \ker \lambda(x) \), we deduce \( \lambda^1(x)(\widehat{X}(x)) = 0 \). From the definition of \( \lambda^1, \lambda_i \) we obtain

\[
0 = \lambda^1(x)(\widehat{X}(x))
\]

\[
= \lim_{k \to \infty} \lambda_k(x)(\widehat{X}(x))
\]

\[
= \lim_{k \to \infty} (g_k^* \lambda)(x)(\widehat{X}(x))
\]

\[
= \lim_{k \to \infty} \lambda(g_k x)(d g_k(\widehat{X}(x)))
\]

\[
= \lim_{k \to \infty} \lambda(g_k x)(\hat{X}(g_k x)),
\]

where the last equality comes from that \( \hat{X} \) is \( G \)-invariant. But this cannot be possible because of the virtual contact condition in Definition 1.2

\[
\inf_{x \in \widehat{M}} \lambda(x)(\widehat{X}(x)) \geq \epsilon > 0.
\]

\[\square\]

By this contact 1-form \( \lambda^1 \) we can construct the corresponding contact structure \( \ell^1 := \ker \lambda^1 \), the Reeb vector field \( X^1 \) for \( \lambda^1 \) on \( E^1 \) with the decomposition \( TE^1 = \mathbb{R} X^1 \oplus \xi^1 \) and the projection \( \pi^1 : TE^1 \to \xi^1 \). By the same construction we define the almost complex structures \( J^1, \hat{J}^1 \) on \( \xi^1 \), \( T(\mathbb{R} \times E^1) \) by satisfying

\[
\hat{m}_x (\pi^1 k_1, \pi^1 k_2) = \hat{\omega}_x (\pi^1 k_1, J^1(x) \pi^1 k_2),
\]

\[
\hat{J}^1(a, u)(h, k) = (-\lambda^1(u) k, J^1(u) \pi^1 k + h \cdot X^1(u)).
\]

Note here that the vector field \( X^1 \) is different from \( X^\lambda|_{E^1} \), but both vector fields generate \( \ker \hat{\omega}|_{E^1} \).
Lemma 4.5. As in the above setting, $\lambda_k|_{E^1} \overset{C^\infty}{\longrightarrow} \lambda^1$ implies $\tilde{J}_k|_{E^1} \overset{C^\infty}{\longrightarrow} \tilde{J}^1$.

Proof. Let us define $\mu_k := \lambda_k|_{E^1} - \lambda^1 \in \Omega^1(E^1)$ which converges to 0 in $C^\infty$ and recall that $X_k$ the Reeb vector field of $\lambda_k$. Since $d\lambda_k|_{E^1} = \omega|_{E^1} = d\lambda^1$, we infer $\mathbb{R}X_k = \ker \omega = \mathbb{R}X^1$ and hence $X_k|_{E^1}$ is a rescaling vector field of $X^1$. More precisely,

$$X_k|_{E^1} = \frac{1}{1 + \mu_k(X^1)} X^1.$$  

Here $X^1$ is a bounded smooth vector field on $E^1$ and hence $X_k|_{E^1}$ also $C^\infty$-converges to $X^1$. Note that $\pi_k : TE^1 \to \xi_k|_{E^1}$ is equal to $\mathbb{I}_{TE^1} - \lambda_k(-)X_k$ and $\pi^1 = \mathbb{I}_{TE^1} - \lambda^1(-)X^1$. Since

$$\lambda_k|_{E^1} \overset{C^\infty}{\longrightarrow} \lambda^1, \quad X_k|_{E^1} \overset{C^\infty}{\longrightarrow} X^1, \quad (4.10)$$

$\pi_k|_{E^1}$ converges to $\pi^1$ in $C^\infty$. Now consider a sequence of metrics $(\hat{m}(\pi_k{-}, \pi_k{-}))_{k \in \mathbb{N}}$ on $\xi_k$ then we subsequently have

$$\hat{m}(\pi_k{-}, \pi_k{-}) \overset{C^\infty}{\longrightarrow} \hat{m}(\pi^1{-}, \pi^1{-}) \quad (4.11)$$
as symmetric bilinear forms on $E^1$.

We now define a metric $m_k$ on $\xi_k$ by

$$(m_k)_x(\pi_k h, \pi_k l) := \hat{\omega}_x(\pi_k h, J_k(x) \pi_k l),$$

where $J_k$ is the almost complex structure in (4.8). For $h, l \in E^1$ we then have

$$(m_k)_x(\pi_k h, \pi_k l) = \hat{\omega}_x(\pi_k h, J_k(\pi_k l)$$

$$= \hat{\omega}_x(g_k(\pi_k h, d g_k^{-1} J(g_k(x)) d g_k \pi_k l)$$

$$= (g_k^{-1} \hat{\omega})_x(g_k \pi_k h, d g_k^{-1} J(g_k(x)) d g_k \pi_k l)$$

$$= \hat{\omega}_g(x)(d g_k \pi_k h, J(g_k(x)) d g_k \pi_k l)$$

$$= m_g(x)(d g_k \pi_k h, d g_k \pi_k l)$$

$$= (g_k)^* m(x)(\pi_k h, \pi_k l)$$

$$= \hat{m}_x(\pi_k h, \pi_k l).$$

Here the 3rd and last equality come from the $G$-invariance of $\hat{\omega}$, $\hat{m}$ respectively. By combining $4.9, 4.11, 4.12$ we deduce

$$\hat{\omega}(\pi_k{-}, J_k \pi_k{-}) \overset{C^\infty}{\longrightarrow} \hat{\omega}(\pi^1{-}, J^1 \pi^1{-})$$
as 2-forms on $E^1$ and hence

$$J_k \circ \pi_k|_{E^1} \overset{C^\infty}{\longrightarrow} J^1 \circ \pi^1 \quad (4.13)$$
as (1,1)-forms on $E^1$.

Now we are ready to compare the almost complex structures $\tilde{J}_k|_{E^1}$ and $\tilde{J}^1$

$$\tilde{J}_k(a, u)(h, l) = (-\lambda_k(u) l, J_k(u) \pi_k l + h \cdot X_k(u)),$$

$$\tilde{J}_k(a, u)(h, l) = (-\lambda_k(u) l, J^1(u) \pi^1 l + h \cdot X^1(u)), \quad (4.14)$$

where $(a, u) \in \mathbb{R} \times E^1$, $(h, l) \in T_{(a, u)}(\mathbb{R} \times E^1)$. By $4.10, 4.13$ and $4.14$ we finally conclude that $\tilde{J}_k|_{E^1}$ converges to $\tilde{J}^1$ in $C^\infty$-topology. □

Up to now, we have a sequence of triples $(v_k|_{B^1}, \lambda_k|_{E^1}, \tilde{J}_k|_{\mathbb{R} \times E^1})_{k \in \mathbb{N}}$ and $(v^1, \lambda^1, \tilde{J}^1)$ with the following convergence:

$$v_k|_{B^1} \overset{C^0}{\longrightarrow} v^1, \quad \lambda_k|_{E^1} \overset{C^\infty}{\longrightarrow} \lambda^1, \quad \tilde{J}_k|_{\mathbb{R} \times E^1} \overset{C^\infty}{\longrightarrow} \tilde{J}^1.$$

Let us recall the sequence of bounded domains $B^n = B_{R_n}(-R_n z_n) \cap B_{\epsilon_n R_n}(0)$ from 4.6 satisfying $\bigcup_{n \in \mathbb{N}} B^n = \mathbb{C}$ and consider a sequence of compact subsets $E^n \subset \tilde{M}$ satisfying

Suppose that our metric $\hat{m}$ is not $G$-invariant then $\lim_{k \to \infty} g_k^* \hat{m}$ may not be a metric anymore. Moreover, in such a case, we can not define $J^1$ as in 4.9.
\( E^n \subset E^{n+1} \) for all \( n \in \mathbb{N} \);
\( \bigcup_{k \in \mathbb{N}} v_k(B^n) \subset E^n \) for all \( n \in \mathbb{N} \);
\( \bigcup_{n \in \mathbb{N}} E^n = \tilde{M} \).

Now we apply the Arzelà-Ascoli theorem to the triple \((v_k, \lambda_k, \tilde{J}_k)_{k \in \mathbb{N}}\) inductively. For \( n \geq 2 \), we pick a subsequence \( k_n \) of \( k_{n-1} \) such that there exists a continuous map \( v^n : B^n \to \mathbb{R} \times E^n \) with
\[
v_k^n|_{B^n} \xrightarrow{C^0} v^n \quad \text{as} \quad j \to \infty.
\]
By the same argument as in Lemma 4.4, we obtain a contact form \( \lambda^n \in \Omega^1(E^n) \) and an almost complex structure \( \tilde{J}^n \) on \( \mathbb{R} \times E^n \) satisfying
\[
\lambda_{k_n(j)}|_{E^n} \xrightarrow{C^\infty} \lambda^n, \quad \tilde{J}_{k_n(j)}|_{\mathbb{R} \times E^n} \xrightarrow{C^\infty} \tilde{J}^n \quad \text{as} \quad j \to \infty.
\]
As a consequence of the above construction, we have
\[
\bar{v}^{n+1}|_{B^n} = \bar{v}^n, \quad \lambda^{n+1}|_{E^n} = \lambda^n, \quad \tilde{J}^{n+1}|_{\mathbb{R} \times E^n} = \tilde{J}^n \quad \text{for} \quad n \in \mathbb{N}.
\]
Especially note that \( \lambda^n \) determines \( \lambda^\infty \) as in (4.9) and \( v^n \) is \( \tilde{J}^n \)-holomorphic for all \( n \in \mathbb{N} \).

We now consider the diagonal sequence of triples \((\bar{v}_{k_j(j)}|_{B^j}, \lambda_{k_j(j)}|_{E^j}, \tilde{J}_{k_j(j)}|_{\mathbb{R} \times E^j})_{j \in \mathbb{N}}\), then we obtain a continuous map \( \bar{v}^\infty : \mathbb{C} \to \mathbb{R} \times \tilde{M} \), a contact form \( \lambda^\infty \in \Omega^1(\tilde{M}) \) and an almost complex structure \( \tilde{J}^\infty \) on \( \mathbb{R} \times \tilde{M} \) satisfying
\[
\bar{v}_{k_j(j)}|_{B^j} \xrightarrow{C^0} \bar{v}^\infty, \quad \lambda_{k_j(j)}|_{E^j} \xrightarrow{C^\infty} \lambda^\infty, \quad \tilde{J}_{k_j(j)}|_{\mathbb{R} \times E^j} \xrightarrow{C^\infty} \tilde{J}^\infty \quad \text{as} \quad j \to \infty. \tag{4.15}
\]
and
\[
\bar{v}^\infty|_{B^n} = \bar{v}^n, \quad \lambda^\infty|_{E^n} = \lambda^n, \quad \tilde{J}^\infty|_{\mathbb{R} \times E^n} = \tilde{J}^n \quad \text{for} \quad n \in \mathbb{N}.
\]
Moreover, the limit 1-form \( \lambda^\infty \) defines a new virtual contact structure as follows:

**Lemma 4.6.** As in the above setting. Let \((p : \tilde{M} \to M, \omega, \lambda)\) be a smooth virtual contact structure, then so is \((p : \tilde{M} \to M, \omega, \lambda^\infty)\).

**Proof.** Since \( \lambda_k \) converges to \( \lambda^\infty \) in \( C^\infty_{\text{loc}} \), its exterior derivative \( d\lambda_k \) also converges to \( d\lambda^\infty \). This implies that \( \lambda^\infty \) is again a primitive of \( \tilde{\omega} \). By the construction of \( \lambda^\infty \) there is a sequence \((g_i)_{i \in \mathbb{N}}\) of deck transformations which satisfies the following estimate for any \( x \in \tilde{M} \), \( x \in \tilde{M} \), and \( x \in \tilde{M} \), and \( i \)-pairs of \( G \)-invariant smooth vector fields \( Y = (Y_1, Y_2, \ldots, Y_l) \):
\[
|\nabla_Y \lambda^\infty(x)|_{\tilde{m}} \leq \sup_{x \in \tilde{M}} |\nabla_Y \lambda(x)|_{\tilde{m}} \leq C,
\]
where \( g_i Y = (dg_i Y_1, dg_i Y_2, \ldots, dg_i Y_l) \). Here the 2nd, 4th, and 6th (in)equality come from the \( G \)-invariance of \( \nabla \), \( m \), and \( Y \) respectively and the last inequality is induced by the smooth condition

\footnote{Here taking a subsequence is equivalent to choose an increasing and unbounded function \( s_n : \mathbb{N} \to \mathbb{N} \) satisfying \( k_n = k_{n-1} \circ s_n : \mathbb{N} \to \mathbb{N} \). Note that \( k_1 : \mathbb{N} \to \mathbb{N} \) is the identity map.}
in Definition 1.2. By the similar argument as in Lemma 4.4 we have

\[\lambda^\infty(x)(\tilde{X}(x)) = \lim_{i \to \infty} g_i^* \lambda(x)(\tilde{X}(x)) = \lim_{i \to \infty} \lambda(g_i x)(dg_i(\tilde{X} x)) = \lim_{i \to \infty} \lambda(g_i x)(\tilde{X}(g_i x)) \geq \inf_{x \in \tilde{M}} \lambda(x)(\tilde{X}(x)) \geq \epsilon.\]

The above two estimates (4.16), (4.17) show that the virtual contact structure \((\tilde{M}, \omega, \lambda^\infty)\) is smooth.

Therefore we have a continuous \(J^\infty\)-holomorphic map \(\tilde{v}^\infty : \mathbb{C} \to \mathbb{R} \times \tilde{M}\). The \(C^0_{\text{loc}}\)-convergence of \(\tilde{v}_k_{\mid B^j}\) to \(\tilde{v}^\infty\) can be improved to \(C^0_{\text{loc}}\)-convergence by applying the elliptic bootstrapping argument, see [MS, Theorem B.4.2]. Finally we have a smooth \(J^\infty\)-holomorphic plane

\[\tilde{v}^\infty = (b^\infty, v^\infty) : \mathbb{C} \to \mathbb{R} \times \tilde{M}\]

which is non-constant in view of \(|\nabla \tilde{v}^\infty(0)| = 1\).

Now it remains to show the finiteness of the energy

\[E(\tilde{v}^\infty) = \sup_{\varphi \in \Sigma} \int_{\mathbb{C}} \tilde{v}^\infty^\ast d(\varphi \lambda^\infty),\]

where \(\Sigma = \{ \varphi \in C^\infty(\mathbb{R}, [0, 1]) \mid \theta' \geq 0 \}\). For any compact set \(K \subset \mathbb{C}\) there is \(j \in \mathbb{N}\) sufficiently large so that \(K \subset B^j\). Then we have

\[
\sup_{\varphi \in \Sigma} \int_K \tilde{v}^\ast_j d(\varphi \lambda_j) \leq \sup_{\varphi \in \Sigma} \int_{B^j} \tilde{v}^\ast_j d(\varphi \lambda_j) \leq \sup_{\varphi \in \Sigma} \int_{\mathbb{C}} \tilde{u}^\ast_j d(\varphi \lambda) = E(\tilde{u}_j).
\]

Let \(j \to \infty\) so that \(\tilde{v}_j, \lambda_j\) converge to \(\tilde{v}^\infty, \lambda^\infty\) in \(C^\infty_{\text{loc}}\) and take the supremum over all compact set \(K \subset \mathbb{C}\) then we obtain

\[E(\tilde{v}^\infty) \leq \lim_{j \to \infty} E(\tilde{u}_j) \leq C,\]

where the constant \(C\) comes from Lemma 4.1.

5. FROM A FINITE ENERGY PLANE TO A PERIODIC ORBIT

We will use the following notations for simplicity:

\[\alpha := \lambda^\infty, \quad \zeta := \xi^\infty, \quad \pi_\alpha := \pi^\infty, \quad \hat{I} := \hat{J}^\infty, \quad \hat{I} := \hat{J}^\infty, \quad b := b^\infty, \quad v := v^\infty, \quad \tilde{v} := \tilde{v}^\infty.\]

Lemma 4.6 says that \((\tilde{M}, \tilde{M}, \omega, \alpha)\) is a smooth virtual contact structure with the decomposition \(TM = \mathbb{R} \tilde{X} \oplus \zeta\), the projection \(\pi_\alpha : \tilde{M} \to \zeta\) along the Reeb direction \(\tilde{X}\), and the almost complex structures \(\hat{I}, \hat{I} \) on \(\zeta, \mathbb{R} \times \tilde{M}\). Moreover, Theorem 4.3 implies that \(\tilde{v} : \mathbb{C} \to \mathbb{R} \times \tilde{M}\) is a non-constant \(\hat{I}\)-holomorphic plane with finite energy. In other words, \(\tilde{v} = (b, v)\) is a solution of

\[\pi_\alpha \partial_s v + I(v) \pi_\alpha \partial_t v = 0, \quad (v^\ast \alpha) \circ i = db, \quad (5.1)\]

and

\[0 < E(\tilde{v}) = \sup_{\varphi \in \Sigma} \int_{\mathbb{C}} \tilde{v}^\ast d(\varphi \alpha) < \infty,\]

where \(\Sigma = \{ \varphi \in C^\infty(\mathbb{R}, [0, 1]) \mid \theta' \geq 0 \}\). Note here that \(M\) is compact and hence the projection \(\underline{v} := p \circ v : \mathbb{C} \to M\) has a compact image, while \(v(\mathbb{C})\) maybe non-compact.
The main aim of this section is to find a periodic orbit from the above finite energy plane.

**Theorem 5.1.** Let \((p : \hat{M} \to M, \omega, \alpha)\) be a smooth virtual contact structure and \(\tilde{v} = (b, v) : \mathbb{C} \to \mathbb{R} \times M\) be a solution of (5.1) satisfying
\[
0 < E(\tilde{v}) < \infty, \quad \int_{\mathbb{C}} v^*\omega > 0
\]
then for every sequence \(R_k \to \infty\) there exists a subsequence \((R_{k'})_{k' \in \mathbb{N}}\) such that the \(C^\infty\)-limit
\[
x(t) := \lim_{k' \to \infty} v(R_{k'} e^{2\pi it})
\]
exists and its projection \(x(t) = p \circ x(t)\) defines a non-constant closed periodic solution of
\[
\dot{x}(t) = X(x(t))
\]
where \(X\) is a non-vanishing vector field generating \(\ker \omega\).

**Proposition 5.2** ([Hof]). Let \(\tilde{v} = (b, v) : \mathbb{C} \to \mathbb{R} \times \hat{M}\) solves (5.1) with finite energy. If
\[
\int_{\mathbb{C}} v^*\omega = 0, \quad \tag{5.2}
\]
then \(\tilde{v}\) is constant.

**Remark 5.3.** The assumption (5.2) implies that the energy from the contact plane, like \(*\)-term in (4.3), vanishes and then \(b : \mathbb{C} \to \mathbb{R}\) becomes a harmonic function because of the \(I\)-holomorphic equation (5.1). Indeed, \(b\) can be regarded as a real part of the holomorphic function \(\Psi := b + i\beta : \mathbb{C} \to \mathbb{C}\) where \(\beta : \mathbb{C} \to \mathbb{R}\) is a primitive of \(v^*\alpha \in \Omega^1(\mathbb{C})\). Now suppose that \(\tilde{v}\) is non-constant then so is \(b\). Essentially, Liouville’s theorem for \(\Psi\) implies that the energy \(E(\tilde{v})\) is infinite. This is a contradiction. The above argument is still valid when \(\hat{M}\) is non-compact and we omit the detailed proof.

Let \(\phi : \mathbb{R} \times S^1 \to \mathbb{C} \setminus \{0\}\) be a holomorphic map defined by
\[
\phi(s, t) = e^{2\pi(s + it)},
\]
where \(S^1 = \mathbb{R} / \mathbb{Z}\). For the later purpose we consider a \(\bar{I}\)-holomorphic cylinder instead of a plane. So we define
\[
\tilde{v}_\phi := (b_\phi, v_\phi) := (b, v) \circ \phi : \mathbb{R} \times S^1 \to \mathbb{R} \times \hat{M}, \tag{5.3}
\]
then we have
\[
\partial_s \tilde{v}_\phi + \bar{I}(\tilde{v}_\phi) \partial_t \tilde{v}_\phi = 0;
\]
\[
\int_{\mathbb{R} \times S^1} v_\phi^*\omega > 0;
\]
\[
0 < E(\tilde{v}_\phi) = \sup_{\varphi \in \Sigma} \int_{\mathbb{R} \times S^1} \tilde{v}_\phi^*d(\varphi\alpha) < \infty, \tag{5.4}
\]
where \(\Sigma\) is as before.

**Proposition 5.4.** Let \(\tilde{v}_\phi\) be a solution of (5.4), then there exists some constant \(l > 0\) such that
\[
|\nabla \tilde{v}_\phi(s, t)| \leq l
\]
for all \((s, t) \in \mathbb{R} \times S^1\).

**Proof.** Let \(\rho : \mathbb{C} \to \mathbb{R} \times S^1 : (s + it) \mapsto (s, e^{2\pi it})\) be a 1-periodic map in the \(S^1\)-coordinate and let us define
\[
\tilde{v}_\rho := (b_\rho, v_\rho) := (b_\phi, v_\phi) \circ \rho : \mathbb{C} \to \mathbb{R} \times \hat{M}.
\]
So it is equivalent to show that \(|\nabla \tilde{v}_\rho(s, t)|\) is bounded. Suppose that there is a sequence \(z_k \in \mathbb{C}\) such that
\[
R_k := |\nabla \tilde{v}_\rho(z_k)| \to \infty.
\]
Note that $\text{Re}(z_k) \to \infty$ since the gradient is bounded on $\text{Re}(z_k) \leq 0$. By applying Lemma 4.2 to positive real sequences $(R_k)_{k \in \mathbb{N}}, (\epsilon_k)_{k \in \mathbb{N}}$ with
\[ \epsilon_k \to 0, \quad \epsilon_k R_k \to \infty, \]
we additionally assume that
\[ |\nabla \tilde{v}_\rho(z)| \leq 2R_k \quad \text{for} \ z \in B_{\epsilon_k}(z_k) \subset \mathbb{C}. \tag{5.5} \]
The sequence $(v_\rho(z_k))_{k \in \mathbb{N}}$ may escape to the unbounded region of $\hat{M}$. So we pick a sequence of deck-transformations $(f_k)_{k \in \mathbb{N}}$ in $G$ so that
\[ f_k^{-1} \circ v_\rho(z_k) \in \hat{M} \]
where $\hat{M}$ is a fixed fundamental domain in $\hat{M}$ with respect to $G$. Now we define $\tilde{\mu}_k : B_{R_k}(-R_k z_k) \to \mathbb{R} \times \hat{M}$ by
\[ \tilde{\mu}_k(z) := (\beta_k(z), \mu_k(z)) \]
\[ := \left( b_\rho(\frac{z}{R_k} + z_k) - b_\rho(z_k), f_k^{-1} \circ v_\rho(\frac{z}{R_k} + z_k) \right). \tag{5.6} \]
We also define a sequence of contact forms $\alpha_k := f_k^* \alpha$ and a sequence of almost complex structures
\[ \tilde{I}_k(a, u)(r, l) := df_k^{-1} \left[ \tilde{I}(a, f_k u)(r, df_k l) \right]. \]
By the similar argument as in Theorem 4.3 there are a suitable subsequence $(k')_{k' \in \mathbb{N}}$, a smooth map $\tilde{\mu}' : \mathbb{C} \to \mathbb{R} \times \hat{M}$, a contact form $\alpha' \in \Omega^1(\hat{M})$ and an almost complex structure $\tilde{I}'$ on $\mathbb{R} \times \hat{M}$ satisfying
\[ \tilde{\mu}_{k'} \xrightarrow{C^\infty} \tilde{\mu}', \quad \alpha_{k'} \xrightarrow{C^{\infty}} \alpha', \quad \tilde{I}_{k'} \xrightarrow{C^{\infty}} \tilde{I}'. \]
Moreover, the limit contact form $\alpha'$ determines the limit almost complex structure $\tilde{I}'$ as in Lemma 4.5 and $\tilde{\mu}'$ is $\tilde{I}'$-holomorphic, i.e.
\[ \partial_s \tilde{\mu}' + \tilde{I}'(\tilde{\mu}') \partial_s \tilde{\mu}' = 0. \]
From (5.6), (5.5) we deduce
\[ |\nabla \tilde{\mu}'(0)| = 1, \quad |\nabla \tilde{\mu}'(z)| \leq 2 \quad \text{for} \ z \in \mathbb{C}. \]
With a sequence of functions $\varphi_k(s) := \varphi(s - b(z_k))$ in $\Sigma$, we estimate
\[ \int_{B_{\epsilon_k}(z_k)} \tilde{\mu}_{k*} d(\varphi \alpha_k) = \int_{B_{\epsilon_k}(z_k)} \tilde{v}_{\rho*} d(\varphi \alpha_k) \leq \int_{\mathbb{R} \times [0,1]} \tilde{v}_{\rho*} d(\varphi \alpha) \leq E(\tilde{v}_\rho) < \infty. \]
Replace $k$ with $k'$ and let $k' \to \infty$ then we deduce
\[ E(\tilde{\mu}') \leq E(\tilde{v}_\rho) < \infty. \]
We know
\[ T := \int_{\mathbb{R} \times S^1} v_{\rho*} \alpha = \int_{\mathbb{R} \times S^1} \tilde{v}_{\rho*} d(\varphi_0 \alpha) \leq E(\tilde{v}_\rho) < \infty \tag{5.7} \]
where $\varphi_0 \equiv 1$. If $q := \int \mu'^* d\alpha'$ is positive, then there is a subsequence $l$ such that
\[ \epsilon_l \leq \frac{1}{2}, \quad \text{and} \quad B_{\epsilon_l}(z_l) \text{ are disjoint}. \]
So we obtain a following contradiction
\[ \infty = \sum_l \frac{q}{2} \leq \sum_l \int_{B_{\epsilon_l}(z_l)} v_{\rho*} \alpha \leq \int_{\mathbb{R} \times S^1} v_{\rho*} \alpha < \infty. \]
Consequently we have a non-constant $\tilde{I}'$-holomorphic map $\tilde{\mu}' : \mathbb{C} \to \mathbb{R} \times \hat{M}$ with a finite energy and $\int \mu'^* d\alpha' = 0$. By Proposition 5.2 such a map $\tilde{\mu}'$ cannot be possible. Therefore $|\nabla \tilde{v}_\rho(s, t)|$ is uniformly bounded. \[ \square \]
Proof of Theorem 5.1. Let $\tilde{v}_\phi = (b_\phi, v_\phi): \mathbb{R} \times S^1 \to \mathbb{R} \times \hat{M}$ be the map from the previous proposition satisfying (5.4). We pick a sequence of real numbers $(s_k)_{k \in \mathbb{N}}$ with $\lim_{k \to \infty} s_k = \infty$ and note that $v_\phi(s_k, 0) \in \hat{M}$ may escape to the unbounded region as $k \to \infty$. For the fixed fundamental region $\hat{M} \subset \hat{M}$, we choose a sequence of deck-transformations $(h_k)_{k \in \mathbb{N}}$ so that

$$h_k^{-1} \circ v_\phi(s_k, 0) \in \hat{M}.$$  

Now we define a sequence of cylinder maps $\hat{w}_k := (c_k, w_k): \mathbb{R} \times S^1 \to \mathbb{R} \times \hat{M}$ by

$$\hat{w}_k(s, t) := \left( b_\phi(s + s_k, t) - b_\phi(s_k, 0), h_k^{-1} \circ v_\phi(s + s_k, t) \right).$$ 

Let us also define a corresponding sequence of contact forms $\alpha_k := h_k^* \alpha$ and a sequence of almost complex structures

$$\hat{I}_k(a, u)(r, l) := dh_k^{-1}[\hat{I}(a, h_k u)(r, dh_k l)].$$

By the similar procedure as in Theorem 4.3 and Proposition 5.4 we choose a suitable subsequence $(k')_{k' \in \mathbb{N}}$ so that there exist a smooth map

$$\hat{w} = (c, w): \mathbb{R} \times S^1 \to \mathbb{R} \times \hat{M},$$

a contact form $\alpha_{\infty} \in \Omega^1(\hat{M})$, and an almost complex structure $\hat{I}_{\infty}$ on $\mathbb{R} \times \hat{M}$ satisfying

$$(\hat{w}_k', \hat{w}_k) \overset{C_{loc}}{\to} (\hat{w}, \alpha_{\infty}), \quad (\hat{I}_k', \hat{I}_k) \overset{C_{loc}}{\to} \hat{I}_{\infty}.$$  

In addition, the contact form $\alpha_{\infty}$ governs the almost complex structure $\hat{I}_{\infty}$ in the view of Lemma 4.5 and $\hat{w}$ is an $\hat{I}_{\infty}$-holomorphic cylinder, i.e.

$$\partial_s \hat{w} + \hat{I}_{\infty}(\hat{w}) \partial_t \hat{w} = 0 \quad \text{on } \mathbb{R} \times S^1.$$  

Moreover, Proposition 5.4 implies

$$|\nabla \hat{w}(s, t)| \leq t \quad \text{for } (s, t) \in \mathbb{R} \times S^1.$$ 

From (5.7) we know $T = \int_{\mathbb{R} \times S^1} v_\phi^* d\alpha < \infty$ and hence for every $R > 0$ we have

$$\int_{[-R, R] \times S^1} w_k^* d\alpha_k = \int_{[-R + s_k, R + s_k] \times S^1} v_\phi^* d\alpha \to 0$$

as $k \to \infty$. Thus we obtain

$$\int_{\mathbb{R} \times S^1} w^* d\alpha_{\infty} = 0.$$  

We know by the construction of $\tilde{v}_\phi = (b_\phi, v_\phi)$ in (5.3) that $v_\phi(s, t)$ converges to some point in $\hat{M}$ as $s \to -\infty$. Then for any $s_0 \in \mathbb{R}$

$$\int_{\{s_0\} \times S^1} w_k^* d\alpha_k = \int_{(-\infty, s_0] \times S^1} w_k^* d\alpha_k = \int_{(-\infty, s_0 + s_k] \times S^1} v_\phi^* d\alpha$$

converges to $T = \int_{\mathbb{R} \times S^1} v^* d\alpha > 0$ as $k \to \infty$. Replace $k$ with $k'$ and passing to the limit $k' \to \infty$ we obtain

$$\int_{\{s_0\} \times S^1} w^* d\alpha_{\infty} = \int_{\mathbb{R} \times S^1} v_\rho^* d\alpha = T > 0.$$  

Now we consider an $\hat{I}_{\infty}$-holomorphic map $\hat{w}_\rho := (c_\rho, w_\rho) := (c, w) \circ \rho: \mathbb{C} \to \mathbb{R} \times \hat{M}$ satisfying

$$\pi_{\alpha_{\infty}} \partial_s w_\rho + I_{\infty}(w_\rho) \partial_t w_\rho = 0$$

$$w_\rho^* \alpha_{\infty} \circ i = dc_\rho.$$  

Here $\pi_{\alpha_{\infty}}: \hat{M} \to \ker \alpha_{\infty}$ be the projection along the Reeb direction and $I_{\infty} = \hat{I}_{\infty}|_{\ker \alpha_{\infty}}$. By the same argument as in Remark 5.3 we deduce from (5.10) that $w^* d\alpha_{\infty} = 0$ and hence $w_\rho^* \alpha_{\infty} \in \Omega^1(\mathbb{C})$ is exact. We introduce a primitive $q: \mathbb{C} \to \mathbb{R}$ of $w_\rho^* \alpha_{\infty}$ so that

$$\Phi := c_\rho + iq: \mathbb{C} \to \mathbb{C}$$

is holomorphic. Moreover, $w^* d\alpha_{\infty} = 0$ implies that

$$\pi_{\alpha_{\infty}} \circ dq_\rho(z): \mathbb{C} \to \ker \alpha_{\infty}(w_\rho(z))$$
is the zero map for all $z \in \mathbb{C}$.

Suppose that $\Phi$ is constant or $c_\rho$ is constant. Then we have

$$w_\rho^* \alpha^\infty \circ i = dc_\rho = 0$$

which means that $\partial_s w_\rho, \partial_t w_\rho$ have no Reeb direction component. Since $\pi^\infty_\alpha \circ dw_\rho$ is also zero, $w_\rho$ must be a constant map. But this contradicts

$$\int_{\{s_0\} \times S^1} w_\rho^* \alpha^\infty = T > 0$$

and we conclude $\Phi$ is non-constant. The gradient of $\Phi$ is bounded because of

$$\sup_\mathbb{C} |\nabla \Phi| = 2 \sup_\mathbb{C} |\nabla c_\rho| \leq 2 \sup_\mathbb{C} |\nabla \tilde{w}_\rho| = 2 \sup_\mathbb{C} |\nabla \tilde{w}| < \infty.$$

As a consequence, $\Phi$ should be an affine non-constant holomorphic map, i.e.

$$\Phi(z) = l z + m$$

where $l, m \in \mathbb{C}$, $l = l_1 + i l_2 \neq 0$, $m = m_1 + i m_2$. Then

$$c_\rho(z) = c_\rho(s, t) = l_1 s - l_2 t + m_1 = l_1 s + m_1$$

since $c_\rho$ is 1-periodic in $t$. Let $X^\alpha_\infty$ be a Reeb vector field for the contact form $\alpha^\infty$ then we have

$$\partial_s w_\rho = \pi^\infty_\alpha \partial_s w_\rho + \alpha^\infty(\partial_s w_\rho) X^\alpha_\infty (w_\rho)$$

$$= \alpha^\infty(\partial_s w_\rho) X^\alpha_\infty (w_\rho)$$

$$= -\partial_t c_\rho X^\alpha_\infty (w_\rho)$$

$$= 0$$

and

$$\partial_t w_\rho = \pi^\infty_\alpha \partial_t w_\rho + \alpha^\infty(\partial_t w_\rho) X^\alpha_\infty (w_\rho)$$

$$= \alpha^\infty(\partial_t w_\rho) X^\alpha_\infty (w_\rho)$$

$$= \partial_s c_\rho X^\alpha_\infty (w_\rho)$$

$$= l_1 X^\alpha_\infty (w_\rho).$$

Here we use $w_\rho^* \alpha^\infty \circ i = dc_\rho$ in (5.12) for the 3rd equations. Since $w_\rho$ is 1-periodic in the $t$-coordinate, we finally obtain a non-constant closed orbit $x(t) := w(s_0, l_1^{-1} t)$ of the Reeb vector field $X^\alpha_\infty$ which generate ker $\hat{\omega}$. Consequently a reparameterization of $\tilde{x}(t) = p \circ x(t)$ gives us a non-constant contractible periodic orbit of the vector field $X^\omega$ on $M$ which generates ker $\omega$. 

\begin{proof}[Proof of Main Theorem]
By the overtwisted disk in $\vec{M}$ and Theorem 3.5 we have a normalized maximal Bishop family $(\vec{u}_\tau)_{0 \leq \tau < \tau_0}$ on $\mathbb{R} \times \vec{M}$ with

$$\sup_{\tau \in [0, \tau_0]} \|\nabla \vec{u}_\tau\|_{C^0(D)} = \infty.$$

From the rescaling argument with deck-transformation in the proof of Theorem 4.3 we construct an almost complex structure $\vec{J}^\infty$ as a limit and a finite energy $\vec{J}^\infty$-holomorphic plane

$$\vec{v} = (b, v) : \mathbb{C} \to \mathbb{R} \times \vec{M}.$$

Finally Theorem 5.1 guarantees a contractible periodic orbit for the vector field $X^\omega$ in Definition 1.1 by considering the projection of the limit

$$x(t) = \lim_{R \to \infty} v(R_k e^{2\pi i t}).$$

\end{proof}
6. EXAMPLES OF A VIRTUAL CONTACT STRUCTURE

The well-known example of a virtual contact structure is an energy hypersurface of a twisted cotangent bundle above the Mañé critical value, as mentioned in the introduction.

Example 6.1. On the cotangent bundle \( \tau : T^* N \to N \) of a closed manifold \( N \) we define autonomous Hamiltonian systems given by a convex Hamiltonian

\[
H(q, p) = \frac{1}{2} |p|^2 + U(q)
\]

with a twisted symplectic form

\[
\omega_\sigma = dp \wedge dq + \tau^* \sigma.
\]

Here \( dp \wedge dq \) is the canonical symplectic form on \( T^* N \), \( |p| \) denote the dual norm of a Riemannian metric \( g \) on \( N \), \( U : N \to \mathbb{R} \) is a smooth potential, and \( \sigma \) is a closed 2-form on \( N \). When the pullback \( \pi^* \sigma \) to the cover \( \pi : \tilde{N} \to N \) is exact, the Mañé critical value is defined as

\[
c = c(g, \sigma, U, \pi) := \inf_{\theta \in \Omega^1(\tilde{N})} \sup_{q \in \tilde{N}} \tilde{H}(q, \theta_0),
\]

where the infimum is taken over all \( \theta \in \Omega^1(\tilde{N}) \) satisfying \( d\theta = \pi^* \sigma \) and \( \tilde{H} \) is the lift of \( H \) to \( \tilde{N} \). Then \( \Sigma_k := H^{-1}(k), k > c \) admits a virtual contact structure

\[
(\pi : \pi^{-1}(\Sigma_k) \to \Sigma_k, \omega_\sigma, \pi^*(p \wedge dq) + \theta_0)
\]

for some primitive \( \theta_0 \) of \( \pi^* \sigma \), see [CFP] Lemma 5.1] for the detail.

Other example is given by the mapping torus construction.

Example 6.2. Let \( (L, \omega_L) \) be a closed symplectic manifold and assume that \( \omega_L \) admits a bounded primitive on the universal cover \( p : \tilde{L} \to L \). Now take a Hamiltonian diffeomorphism \( \varphi \) on \( (L, \omega_L) \) and consider the induced mapping torus

\[
L_\varphi := \frac{L \times [0, 1]}{(l, 0) \sim (\varphi(l), 1)}.
\]

Then we can endow \( L_\varphi \) with a virtual contact structure, see [Ba2] Remark 3.4] for the construction.
In the remaining section we construct a virtual contact structure with an overtwisted disk.

**Example 6.3.** Let $\Sigma$ be a closed surface of genus $g \geq 2$ then $\Sigma$ can be represented by $\mathbb{D}/G$ with the universal covering map $p_{\Sigma} : \mathbb{D} \to \Sigma$. Here $\mathbb{D} = \{x + iy \in \mathbb{C} \mid x^2 + y^2 < 1\}$ is the Poincaré disk with the Poincaré metric

$$ds^2 = \frac{dx^2 + dy^2}{(1 - x^2 - y^2)^2}$$

and $G$ is a discrete group of isometry of $\mathbb{D}$. Let $\omega_{\Sigma}$ be a volume form on $\Sigma$ we then may assume

$$p_{\Sigma}^* \omega_{\Sigma} = \frac{2dx \wedge dy}{(1 - x^2 - y^2)^2} = \frac{2rdr \wedge d\varphi}{(1 - r^2)^2},$$

where $(r, \varphi)$ are the polar coordinates for $\mathbb{C}$. Now consider $M = S^1 \times \Sigma$ with the covering $p : S^1 \times \mathbb{D} \to M$ and the projection $\pi_{\Sigma} : M \to \Sigma$ which fit into the following diagram:

$$\begin{array}{ccc}
S^1 \times \mathbb{D} & \xrightarrow{p} & M \\
\downarrow \pi & & \downarrow \pi_{\Sigma} \\
\mathbb{D} & \xrightarrow{p_{\Sigma}} & \Sigma
\end{array}$$

The Hamiltonian structure $(M, \omega_M := \pi_{\Sigma}^* \omega_{\Sigma})$ admits a virtual contact structure by choosing a covering map $p : S^1 \times \mathbb{D} \to M$ and a primitive

$$\alpha = dt + \frac{x dy - y dx}{1 - x^2 - y^2} = dt + \frac{r^2 d\varphi}{1 - r^2}$$

of $\omega := p^* \omega_M$ where $t$ is the coordinate for $S^1$ Note here that $\omega \in \Omega^2(S^1 \times \mathbb{D})$ is $G$-invariant but its primitive $\alpha \in \Omega^1(S^1 \times \mathbb{D})$ is not.

In order to do a Lutz twist on the virtual contact structure $(M, \omega_M, \alpha)$, we start with a small constant $\delta \in (0, \frac{1}{3}]$ satisfying $g(B_{3}(0)) \cap B_{\delta}(0) = \emptyset$ for all $g \in G$. Next we choose $\epsilon \in (0, \frac{\delta}{10}]$, $C = \max\{||\alpha||_{\infty}, 1\}$ and then consider $C^1$-functions $f_1, f_2$ meeting the following conditions:

- $f_1(r) = -1$, $f_2(r) = \frac{r^2}{1 - r^2}$ for $r \in [0, \frac{1}{2})$;
- $f_1(r) = 1$, $f_2(r) = \frac{r^2}{1 - r^2}$ for $r \in (\delta - \frac{1}{2}, \delta]$;
- $f_3(\frac{1}{2}) = -4C$, $f_3'(\frac{1}{2}) = 0$.

The followings are not essential but we require additional conditions for simplicity:

- $f_1', f_2'$ are piece-wise linear on $r \in \left[\frac{1}{2}, \delta - \frac{1}{2}\right]$;
- $f_1'$ is constant on $r \in (\epsilon, \delta - \epsilon]$;
- $f_2'$ is constant on $r \in (\epsilon, \delta - \epsilon)$;
- $f_1'$ is differentiable except at $\left(\frac{1}{2}, \epsilon, \delta - \epsilon, \delta - \frac{1}{2}\right)$;
- $f_2'$ is differentiable except at $\left(\frac{1}{2}, \epsilon, \frac{1}{2} - \epsilon, \frac{1}{2} + \epsilon, \delta - \frac{1}{2}\right)$. 

Now we consider the 1-form

$$\alpha^\delta := f_1(r) dt + f_2(r) d\varphi$$

(6.1)

on $\pi^{-1}(B_{\delta}(0))$ and $\eta_{\delta} \in \Omega^1(S^1 \times \mathbb{D})$ which equals $\alpha^\delta - \alpha$ on $\pi^{-1}(B_{\delta}(0))$ and vanishes elsewhere. From the locally supported 1-form $\eta_{\delta}$, we define a $G$-invariant 1-form $\eta \in \Omega^1(S^1 \times \mathbb{D})$ by

$$\eta := \sum_{g \in G} g^* \eta_{\delta}.$$ 

By the above construction there is a 1-form $\eta$ on $M$ such that $p^* \eta = \eta$. Since $\eta$ is bounded, it immediately follows that $\eta$ is also bounded with respect to the product metric on $S^1 \times \mathbb{D}$.

**Proposition 6.4.** As in the above setting, a triple $(p : S^1 \times \mathbb{D} \to M, \omega^L := \omega_M + d\eta, \alpha^L := \alpha + \eta)$ is a virtual contact structure equipped with an overtwisted disk.

---

10In Example 6.2 let $(L, \omega_L)$ be a closed surface $(\Sigma, \omega_{\Sigma})$ of genus $g \geq 2$ and take a Hamiltonian diffeomorphism $\varphi = \text{id}$, then we have the above setting.
Figure 2. graphs of $f_1(r)$, $f_2(r)$ and their differentials

PROOF. We already know that $\alpha^L$ is bounded with respect to $G$-invariant metric. Before checking the non-vanishing condition, we first need to specify the non-vanishing vector field $R^L$ which generates $\ker d\alpha^L$. For convenience, let $h : [\delta - \frac{\epsilon}{2}, \delta] \to \mathbb{R}$ be a positive function satisfying $h(\delta - \frac{\epsilon}{2}) = f'_2(\delta - \frac{\epsilon}{2})$, $h(\delta) = 1$. Then we may choose our Reeb-like vector field $R^L$ as follows:

$$R^L = f'_2(\frac{\epsilon}{2})\partial_t \quad \text{for } r \in [0, \frac{\epsilon}{2}];$$

$$R^L = f'_2(r)\partial_t - f'_1(r)\partial_\phi \quad \text{for } r \in [\frac{\epsilon}{2}, \delta - \frac{\epsilon}{2}];$$

$$R^L = h(r)\partial_t \quad \text{for } r \in [\delta - \frac{\epsilon}{2}, \delta];$$

$$R^L = \partial_\phi \quad \text{for } x \in S^1 \times \left( \mathbb{D} \setminus \bigcup_{g \in G} g(B_\delta(0)) \right).$$

Recall that $R^L$ is $G$-invariant and hence $R^L$ on $S^1 \times B_\delta(0)$ determines $R^L$ on $S^1 \times g(B_\delta(0))$ for any $g \in G$.

Let us begin with the case $x \in S^1 \times \left( \mathbb{D} \setminus \bigcup_{g \in G} g(B_\delta(0)) \right)$. In this region there is no change under the Lutz twist and $g \in G$ acts trivially on $\partial_t$, thus we have

$$\alpha^L(x)[R^L(x)] = 1.$$
Next we consider the case $r \in [0, \frac{\delta}{2}]$ where $\alpha^L = -dt - \frac{r^2}{4r^2}d\varphi$ and $R^L = f_2'(\frac{r}{2})\partial_t$. For $x \in S^1 \times (\bigcup_{g \in G} g(B_{\frac{\delta}{2}}(0)))$ we obtain
\[
\alpha^L(x)[R^L(x)] = -f_2'\left(\frac{r}{2}\right) > 0.
\]
Now we move to the case $r \in [\delta - \frac{\delta}{2}, \delta]$ where $\alpha^L = dt + \frac{r^2}{4r^2}d\varphi$ and $R^L = h(r)\partial_t$. Let $A = \{(r, \varphi) \in \mathbb{D} | \delta - \frac{\delta}{2} \leq r \leq \delta\}$ and $x \in S^1 \times (\bigcup_{g \in G} g(A))$ then
\[
\alpha^L(x)[R^L(x)] \geq \min_{r \in [\delta - \frac{\delta}{2}, \delta]} h(r) > 0.
\]

It remains to verify the case $r \in [\frac{\delta}{2}, \delta - \frac{\delta}{2}]$ and we need the following preparation. Let $g \in G$ be a Möbius transformation which acts on $\mathbb{D}$ and denote $g(0) = z_0$. Then it is necessary to know the lower bound of $\alpha^L(R^L)$ on $\pi^{-1} \circ g(B_{\delta}(0))$. The following simple observation
\[
\alpha^L(R^L)|_{\pi^{-1}(z_0)} = \alpha^L(g,R^L)|_{g(\pi^{-1}(0))} = g^*\alpha^L(R^L)|_{\pi^{-1}(0)}
\]
informs us that it suffices to investigate $g^*\alpha^L(R^L)$ on $\pi^{-1}(B_{\delta}(0))$. Now we compare $g^*\alpha^L$ with $\alpha^L$ as follows:
\[
g^*\alpha^L(R^L) = g^*(\alpha + \eta)(R^L) = (g^*\alpha + \eta)(R^L) = (g^*\alpha - \alpha + \alpha^L)(R^L).
\]
Thus it is important to estimate $(g^*\alpha - \alpha)(R^L)$. Since $G$ acts trivially on the $t$-coordinate, $g^*\alpha - \alpha$ has no $dt$-part and by the construction of $C$ we know $\|\alpha\| \leq C$. For $r \in [\frac{\delta}{2}, \delta - \frac{\delta}{2}]$ we then have
\[
|\alpha^L(R^L)| \leq 2C|f_1'(r)| \leq 2Cf_1'(r).
\]
Here the last inequality comes from the following estimate:
\[
|\partial_\varphi| = \| - r \sin \varphi \partial_x + r \cos \varphi \partial_y \|
\leq \|r \partial_x\| + \|r \partial_y\|
\leq \frac{2r}{1 - r^2} \bigg|_{r = \delta - \frac{\delta}{2}} \leq 1.
\]
The point of the estimate [6.3] is that the last term does not depend on $f_2'(r)$ and $g \in G$. By combining [6.1], [6.2] and [6.3], we estimate
\[
g^*\alpha^L(R^L) \geq \alpha^L(R^L) - |(g^*\alpha - \alpha)(R^L)|
\geq \alpha^L(R^L) - |f_1'(r)f_2'(r) - f_2'(r)f_1'(r) - 2Cf_1'(r)|
\geq \alpha^L(R^L) - |f_2'(r)| - 2Cf_1'(r).
\]
for $r \in [\frac{\delta}{2}, \delta - \frac{\delta}{2}]$.

**Case i :** $r \in [\frac{\delta}{2}, \epsilon]$.  
Recall that $C \geq 1$, $\delta \in (0, \frac{3}{4})$, $\epsilon \in (0, \frac{\delta}{16})$. By the choice of $\delta, \epsilon$ we estimate
\[
f_1(r) = \frac{4}{2\delta - 3\epsilon^2}(r - \frac{\epsilon}{2})^2 - 1 \leq -\frac{7}{8};
\]
\[
f_1'(r) = \frac{8}{2\delta - 3\epsilon^2}(r - \frac{\epsilon}{2}) \leq \frac{5}{2\delta}(r - \frac{\epsilon}{2});
\]
\[
f_2'(r) \leq \frac{-16C + 2\epsilon}{\epsilon\delta}(r - \frac{\epsilon}{2}) - \epsilon.
\]
and
\[
L(r) > f_1(r)f'_2(r) - 2Cf'_1(r)
\]
\[
\geq -\frac{7}{8} \left( -16C + 2\epsilon \delta \right) \left( r - \frac{\epsilon}{2} - \epsilon \right) - 2C \frac{5}{2\epsilon \delta} (r - \frac{\epsilon}{2})
\]
\[
= \frac{36C - 7\epsilon \delta}{4\epsilon \delta} (r - \frac{\epsilon}{2}) + \frac{7}{8\epsilon}
\]
\[
\geq \frac{7}{8} \epsilon.
\]

**Case ii:** \( r \in [\epsilon, \frac{\delta}{2} - \epsilon] \).

We have the following estimate
\[
f_1(r) = \frac{4}{2\delta - 3\epsilon} \left( r - \frac{\delta}{2} \right);
\]
\[
f'_1(r) = \frac{4}{2\delta - 3\epsilon};
\]
\[
f_2(r) \leq -\frac{8C}{\delta} (r - \epsilon);
\]
\[
f'_2(r) \leq -\frac{8C}{\delta}
\]

and
\[
L(r) \geq \frac{4}{2\delta - 3\epsilon} \left( r - \frac{\delta}{2} \right) \frac{8C}{\delta} (r - \epsilon) \frac{4}{2\delta - 3\epsilon} \frac{4}{2\delta - 3\epsilon}
\]
\[
= \frac{8C}{2\delta - 3\epsilon} \left( 1 - \frac{4\epsilon}{\delta} \right)
\]
\[
\geq \frac{8C}{2\delta - 3\epsilon} \cdot \frac{3}{5}
\]
\[
\geq \frac{12C}{5\delta}.
\]

**Case iii:** \( r \in [\frac{\delta}{2} - \epsilon, \frac{\delta}{2} + \epsilon] \).

We obtain
\[
f_1(r) = \frac{4}{2\delta - 3\epsilon} \left( r - \frac{\delta}{2} \right);
\]
\[
f'_1(r) = \frac{4}{2\delta - 3\epsilon};
\]
\[
f_2(r) \leq -3C;
\]
\[
|f'_2(r)| \leq \left| \frac{8C}{\epsilon \delta} \left( r - \frac{\delta}{2} \right) \right|
\]

and
\[
L(r) \geq \frac{4}{2\delta - 3\epsilon} \left( r - \frac{\delta}{2} \right) \frac{8C}{\epsilon \delta} \left( r - \frac{\delta}{2} \right) + 3C \frac{4}{2\delta - 3\epsilon} - 2C \frac{4}{2\delta - 3\epsilon}
\]
\[
= \frac{4C}{2\delta - 3\epsilon} \left( \frac{8C}{\epsilon \delta} \left( r - \frac{\delta}{2} \right)^2 + 1 \right)
\]
\[
\geq \frac{4C}{2\delta - 3\epsilon}
\]
\[
\geq \frac{2C}{\delta}.
\]

The cases \( r \in [\frac{\delta}{2} + \epsilon, \delta - \epsilon], \ r \in [\delta - \epsilon, \delta - \frac{\epsilon}{2}] \) are similar to **Case ii, Case i** respectively.

Therefore we finally have \( \inf_{x \in S^1 \times D} \psi_l(R^l) > 0 \) which conclude that the triple \((p : S^1 \times D \to M, \omega^L, \alpha^L)\) is a virtual contact structure. Moreover, \( \pi^{-1}(B_\delta(0)) \) contains an overtwisted disk. This proves the lemma. \(\square\)
In the above construction the 1-form $\alpha^L$ and the vector field $R^L$ are not smooth. In fact $f_1', f_2'$ are not smooth on finite points and $R^L$ is also not smooth on the corresponding region. Now we consider a sequence of $C^\infty$ functions $(h_{1,n},h_{2,n})_{n\in\mathbb{N}}$ which converges to $(f_1, f_2)$ in $C^1$. Then by the same construction as above we have a sequence of 1-forms $(\lambda_n^L)_{n\in\mathbb{N}}$ and a sequence of smooth vector fields $(X_n^L)_{n\in\mathbb{N}}$ satisfying the property that $\lambda_n^L(X_n^L)$ converges uniformly to $\alpha^L(R^L)$. For sufficiently large $n \in \mathbb{N}$ we choose a smooth 1-form $\lambda^L$ such that $\inf_{x \in S^1 \times \mathbb{D}} \lambda^L(X^L) > 0$. Again by the construction, each derivative of $\lambda_n^L$ is determined by $h_{1,n}, h_{2,n}$. We may require that all derivatives of $h_{1,n}, h_{2,n}$ are uniformly bounded with respect to the Poincaré metric, without loss of generality. These conclude that $(p : S^1 \times \mathbb{D} \to M, \omega^L, \lambda^L)$ is a smooth virtual contact structure.

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