On long-time behavior of solutions of the Zakharov–Rubenchik/Benney–Roskes system

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Abstract
We study decay properties for solutions to the initial value problem associated with the one-dimensional Zakharov–Rubenchik/Benney–Roskes (ZR/BR) system. We prove time-integrability in growing compact intervals of size \( r' \), \( r < 2/3 \), centered on some characteristic curves coming from the underlying transport equations associated with the ZR/BR system. Additionally, we prove decay to zero of the local energy-norm in so-called far-field regions. Our results are independent of the size of the initial data and do not require any parity condition.

Keywords: Zakharov–Rubenchik/Benney–Roskes, long-time behavior, scattering, decay
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1. Introduction and main results

1.1. The model

In this work we seek to show decay properties for solutions of the initial value problem associated with the Zakharov–Rubenchik/Benney–Roskes (ZR/BR) system in one space dimension...
\[
\begin{cases}
  i\partial_t \psi + \omega \partial_x^2 \psi = \gamma \left( \eta - \frac{1}{2} \alpha \rho + q|\psi|^2 \right) \psi, \\
  \theta \partial_t \rho + \partial_x \left( \eta - \alpha \rho \right) = -\gamma \partial_x (|\psi|^2), \\
  \theta \partial_t \eta + \partial_x (\beta \rho - \alpha \eta) = \frac{1}{2} \alpha \gamma \partial_x (|\psi|^2), \\
  \psi(0, x) = \psi(0, x), \quad \rho(0, x) = \rho_0(x), \quad \eta(0, x) = \eta_0(x).
\end{cases}
\] (1.1)

Here \(\psi(t, x)\) denotes a complex-valued function, while \(\rho(t, x)\) and \(\eta(t, x)\) are both real-valued functions, and \(t, x \in \mathbb{R}\). All Greek letters \((\omega, \alpha, \beta, \gamma, \theta)\) denote real parameters, and in the sequel we shall always assume that

\[
\omega > 0, \quad \beta > 0, \quad \gamma > 0, \quad \beta - \alpha^2 > 0, \quad 0 < \theta < 1, \quad \text{and} \quad q := \gamma + \frac{\alpha(\alpha \gamma - 1)}{2(\beta - \alpha^2)}.
\]

Model (1.1) corresponds to the one-dimensional case of the most general system derived by Zakharov and Rubenchik [19] to describe the interaction of spectrally narrow high-frequency wave packets of small amplitude with low-frequency acoustic type oscillations. This system was also independently found by Benney and Roskes [1] in the context of gravity waves, and in the three-dimensional case has the following form

\[
\begin{cases}
  i\partial_t \psi + iv_x \partial_x \psi = -\frac{\omega''}{2} \partial_x^2 \psi - \frac{v_y}{2k} \Delta_{\perp} \psi + (q|\psi|^2 + \beta \rho + \alpha \partial_x \eta)|\psi|^2, \\
  \partial_t \rho + \rho_0 \Delta \eta + \alpha \partial_x |\psi|^2 = 0, \\
  \partial_t \eta + \frac{c^2}{\rho_0} \rho + \beta |\psi|^2 = 0,
\end{cases}
\] (1.2)

where \(\Delta_{\perp} = \partial_x^2 + \partial_y^2\). In this context, \(\psi(t, x)\) stands for the amplitude of the carrying (high frequency) waves with wave number \(k\), frequency \(\omega = \omega(k)\) and \(v_x = v^x(k)\) stands for its group velocity. On the other hand, \(\rho(t, x)\) and \(\eta(t, x)\) correspond to the density fluctuation and the hydrodynamic potential respectively.

System (1.2) has also been derived in several other physical situations, such as for example, in the study of Alfvén waves (transverse oscillations of the magnetic fields) in the magnetohydrodynamics equations (see for instance [2, 17]). Moreover, system (1.2) contains various important models as limiting cases, such as the classical (scalar) Zakharov system and the Davey–Stewartson systems. We refer to [3] for a rigorous justification of the Zakharov limit (supersonic limit) of the ZR/BR system. However, the rigorous proof of the Davey–Stewartson limit from system (1.2) remains still open.

In the one-dimensional case the situation is a little better understood. In fact, in this case we can also consider the adiabatic limit, that is, to take \(\theta \to 0\) in (1.1), from where we can formally see that \(\rho(t, x)\) and \(\eta(t, x)\) satisfy now the following relations

\[
\rho = -\frac{\gamma \alpha}{2(\beta - \alpha^2)} |\psi|^2, \quad \eta = -\frac{\gamma}{\beta - \alpha^2} |\psi|^2.
\]

Then, we infer that the complex amplitude \(\psi\) solves the cubic nonlinear Schrödinger equation

\[
i\partial_t \psi + \omega \partial_x^2 \psi = -\frac{\gamma \alpha}{3(\beta - \alpha^2)} |\psi|^2 \psi.
\]
A rigorous justification of such limit (for well-prepared initial data) was proved by Oliveira in [16]. Therefore, we can certainly see that the ZR/BR system is thus richer than those models.

On the other hand, the ZR/BR system (1.1) and (1.2) possess a Hamiltonian structure [19], and hence, it follows (at least formally) that the energy of system (1.1) is conserved along the trajectory, which in the one-dimensional case can be written as

\[
E(\psi(t), \rho(t), \eta(t)) := \int_{\mathbb{R}} \left( \omega|\psi_x|^2 + \frac{\gamma}{2} |\psi|^4 + \frac{\beta}{2} \rho^2 + \frac{1}{2} \eta^2 + \frac{\gamma}{2} (2\eta - \alpha \rho) |\psi|^2 - \alpha \rho \eta \right) dx
= E(\psi_0, \rho_0, \eta_0).
\]

Moreover, the ZR/BR system (1.1) also conserves (formally) the mass and the momentum of the solution, which are given by the following relations (respectively)

\[
M(\psi(t), \rho(t), \eta(t)) := \int_{\mathbb{R}} |\psi(t, x)|^2 dx = M(\psi_0, \rho_0, \eta_0), \quad \text{and,}
\]

\[
P(\psi(t), \rho(t), \eta(t)) := \text{Im} \int_{\mathbb{R}} \overline{\psi_x} - \theta x \int_{\mathbb{R}} \rho(t, x) \eta(t, x) dx = P(\psi_0, \rho_0, \eta_0).
\]

Additionally, related to these conservation laws, the ZR/BR system (1.1) is invariant under space-time translations, as well as invariant under phase rotations.

Regarding the existence of solitary waves, in the case \( \beta - \alpha^2 > 0, \gamma > 0 \) and \( \theta < 1 \), Oliveira has proved in [15] the existence and the orbital stability of solitary waves of the form

\[
(\psi, \rho, \eta)(t, x) := \left( e^{i\lambda t} e^{i\omega x/2\omega} R(x - ct), a(c)|R(x - ct)|^2, b(c)|R(x - ct)|^2 \right), \tag{1.3}
\]

where \( \lambda \in \mathbb{R}, c \geq 0 \) and \( R(\cdot) \) is an positive, even and exponentially decaying complex-valued function, while \( a(c) \) and \( b(c) \) are given by the following formulas

\[
a(c) := \frac{-\gamma(c\theta + \frac{\alpha}{2})}{\beta - (c\theta + \alpha)^2}, \quad b(c) := \frac{-\gamma(c\theta + \frac{\alpha}{2})}{\beta - (c\theta + \alpha)^2}.
\]

In particular, the analysis carried out by Oliveira shows that a necessary condition for these solitary waves to exists is that the following two inequalities must be satisfied

\[
a(c) - \frac{\alpha}{2} b(c) + q < 0 \quad \text{and} \quad \frac{c^2}{4\omega} - \lambda < 0. \tag{1.4}
\]

On the other hand, recently in [7] Luong et al studied the existence of the so-called bright and dark solitons for system (1.1). They proved their existence under some conditions on the coefficients of the equations (similar to the one in (1.4)). Then, they used these solitons to construct line-solitons for the higher-dimensional case. However, none of these solitons belongs to the energy space since they do not decay at \( \pm \infty \) (see [7] for further details).

Finally, concerning the well-posedness for system (1.1), Oliveira [15] proved local and global well-posedness for the one-dimensional case in \( H^2(\mathbb{R}) \times H^1(\mathbb{R}) \times H^1(\mathbb{R}) \). Later, Linares and Matheus [5] extended the result given by Oliveira showing local (and then global) well-posedness for initial data in the energy space \( H^1(\mathbb{R}) \times L^2(\mathbb{R}) \times L^2(\mathbb{R}) \). Additionally, a polynomial bound for the growth of the \( H^s \)-norm of \( \psi \) was stated in [5]. More specifically, they proved that, for smooth initial data, solutions to system (1.1) satisfies the following property:

\[
\|\psi\|_{H^s(\mathbb{R})} \lesssim 1 + |t|^{(s-1)+}.
\]
In fact, Linares and Matheus used this property to show that system (1.1) is globally well-posed in $H^s(\mathbb{R}) \times H^{s+\frac{1}{2}}(\mathbb{R}) \times H^{s+\frac{1}{2}}(\mathbb{R})$ for all $k \geq 0$. Moreover, regarding the higher-dimensional cases, Ponce et al [18] have proved that (1.2) is locally well posed in $H^s(\mathbb{R}^d) \times H^{s+2}(\mathbb{R}^d) \times H^{s+2}(\mathbb{R}^d)$, for $s > d/2$, where the space-dimension $d = 2, 3$. Lastly, we mention that Luong et al have recently proved the well-posedness (under some extra conditions) of system (1.2) in the background of a line-soliton [7].

1.2. Main results

In the remainder of this work we focus in decay properties for general solutions of (1.1) in the energy space. Our first main result states that there exists two specific characteristic curves such that, along them, there is an additional time-integrability property on growing compact sets.

**Theorem 1.1.** Let $v_\pm := \pm \theta^{-1}(\sqrt{\beta} \pm \alpha)$ fixed. Consider $(\psi, \rho, \eta) \in C(\mathbb{R}, H^1 \times L^2 \times L^2)$ to be any solution to system (1.1) emanating from an initial data $(\psi_0, \rho_0, \eta_0) \in H^1 \times L^2 \times L^2$. Then, for any $c \in \mathbb{R}_+$, the following inequality holds

$$\int_0^{+\infty} \frac{1}{\mu_{\pm}(t)} \int_{\Omega_{\pm}(t)} |\psi(t, x)|^2 \, dx \, dt < +\infty,$$

where $\Omega_{\pm}(t) := \{ x \in \mathbb{R} : -c \lambda(t) \leq x - v_\pm t \leq c \lambda(t) \}$, $\kappa := 10^{100}$ and

$$\lambda(t) := t^{1/3} \log \log^{-2/3}(\kappa + t) \quad \text{and} \quad \mu_{\pm}(t) := t \log(\kappa + t) \log(\kappa + t).$$

Furthermore, we have the following scenarios:

(a) If $\pm \alpha < 0$, then, the following inequality holds

$$\int_0^{+\infty} \frac{1}{\mu_{\pm}(t)} \int_{\Omega_{\pm}(t)} \left( |\psi_{\pm}(t, x)|^2 + |\psi(t, x)|^2 + \rho^2(t, x) + \eta^2(t, x) \right) \, dx \, dt < +\infty.$$

In particular, we have that

$$\liminf_{t \to +\infty} \int_{\Omega_{\pm}(t)} \left( |\psi_{\pm}(t, x)|^2 + |\psi(t, x)|^2 + \rho^2(t, x) + \eta^2(t, x) \right) \, dx = 0.$$

(b) If $\alpha = 0$, then, the following inequality holds

$$\int_0^{+\infty} \frac{1}{\mu_{\pm}(t)} \int_{\Omega_{\pm}(t)} \left( |\psi_{\pm}(t, x)|^2 + |\psi(t, x)|^4 + \eta^2(t, x) + \rho^2(t, x) \right) \, dx \, dt < +\infty,$$

where $\lambda$ and $\mu_{\pm}$ defined as above and $\Omega_0(t) := \{ x \in \mathbb{R} : c \lambda(t) \leq |x| \leq C \lambda(t) \}$. In particular, the following is satisfied

$$\liminf_{t \to +\infty} \int_{\Omega_{\pm}(t)} \left( |\psi_{\pm}(t, x)|^2 + |\psi(t, x)|^4 + \eta^2(t, x) + \rho^2(t, x) \right) \, dx = 0.$$

**Remark 1.1.** In the previous statement, the condition $\pm \alpha < 0$ must be understood according to the sets $\Omega_{\pm}$. In other words, if $+\alpha < 0$, then both results for $\Omega_+$ hold, while if $-\alpha < 0$, then both results for $\Omega_-$ hold. Notice that if $\alpha < 0$, the result for $\Omega_-$ is not necessarily true.

**Remark 1.2.** It is important to notice that, as soon as $\alpha \neq 0$ we cannot deduce any time-integrability nor decay property on compacts sets centered at the origin. Of course, this is
consequence of (and consistent with) the existence of the standing-wave solution presented in (1.3). On the other hand, when \( \alpha = 0 \), condition (1.4) does not allow standing-wave solutions to exists, more specifically, the first inequality in (1.4) is not satisfied when \( c = \alpha = 0 \), and hence item 2 is not contradictory with the existence of such family of solutions.

Our second main result states that, in the so-called far-field region, solutions (in the energy space) must decay to zero.

**Theorem 1.2.** Let \((\psi, \rho, \eta) \in C(\mathbb{R}, H^1 \times L^2 \times L^2)\) be any solution to system (1.1) emanating from an initial data \((\psi_0, \rho_0, \eta_0) \in H^1 \times L^2 \times L^2\). Then, for any pair of constants \(c_1, c_2 > 0\) the following properties holds:

(a) Consider any non-negative function \(\zeta \in C^4(\mathbb{R})\) satisfying that, there exists \(\delta > 0\) such that, for all \(t > 0\) it holds

\[
\zeta(t) \gtrsim t \log(\kappa + t)^{1+\delta} \quad \text{and} \quad \zeta'(t) \gtrsim \log(\kappa + t)^{\delta+1}.
\]

Then, setting \(\Omega_t := \{x \in \mathbb{R} : c_1 \zeta(t) \leq |x| \leq c_2 \zeta(t)\}\), the following limit holds

\[
\lim_{t \to \infty} \|\psi(t)\|_{L^2(\Omega_t)} = 0. \tag{1.5}
\]

(b) Assume additionally that \((\psi, \rho, \eta) \in C(\mathbb{R}, H^2 \times H^1 \times H^1)\) is a solution emanating from an initial data \((\psi_0, \rho_0, \eta_0) \in H^2 \times H^1 \times H^1\). Then, for any non-negative \(\zeta \in C^4(\mathbb{R})\) satisfying that, there exists \(\delta > 0\) such that, for all \(t > 0\),

\[
\zeta(t) \gtrsim t^{\delta+\frac{1}{2}} \quad \text{and} \quad \zeta'(t) \gtrsim t^{\delta+1},
\]

the following decay for the local energy norm holds

\[
\lim_{t \to \infty} \left( \|\psi(t)\|_{H^1(\Omega_t)} + \|\rho(t)\|_{L^2(\Omega_t)} + \|\eta(t)\|_{L^2(\Omega_t)} \right) = 0.
\]

**Remark 1.3.** Note that none of the above theorems require any smallness assumption in terms of the initial data \(\|\psi_0\|_{H^1} \ll 1\). Moreover, they do not require any parity assumption either (as their counterparts founded in [11, 12]), nor any extra decay hypotheses in terms of weighted Sobolev norms, such as \(\|x\psi\|_{L^2} \ll 1\) for example.

**Remark 1.4.** One important difference between the results above and those in [11, 12] is that, in both of those works, the equations under study preserve the oddness of the initial data (for the Schrödinger component \(\psi\)), while system (1.1) does not. Hence, the analysis presented there assuming parity conditions on the initial data cannot be applied to system (1.1).

**Remark 1.5.** To the best of our knowledge, these are the first results dealing with global properties of system (1.1) in the one-dimensional case.

Finally, it is worth mentioning that the techniques involved in the proof of theorems 1.1 and 1.2 have already been used before in some other contexts. We refer to [11] for the use of some of these ideas in context of the one-dimensional Schrödinger equation, and to [12] for scalar Zakharov system (as well as the Klein–Gordon Zakharov system). On the other hand, for other type of systems that have served us for motivations we refer to [4, 6, 14]. However, as previously described, system (1.1) has some important differences with respect to the above cases (see remark 1.4), what does not allow us to apply the same ideas. In particular, the presence of some transport equations in (1.1) breaks the symmetry properties used in previous works to study Schrödinger-type equations/systems with these specific techniques. Finally, it is important to mention that most of these ideas come from classical works, such as [8–10, 13].
2. Preliminary lemmas

2.1. Virial identities

In this section we seek to establish the key virial identities required in our analysis. In order to do that we consider the following weight function

\[ \Phi(x) := \tanh(x), \quad \text{and hence} \quad \Phi' = \text{sech}^2(x). \]

Additionally, we consider time-dependent scaling functions \( \lambda_1(t) \), \( \lambda_2(t) \) and \( \mu(t) \) given by

\[
\begin{align*}
\lambda_1(t) &:= (\kappa + t)^{2/3} \log \log^{-2/3}(\kappa + t), \\
\lambda_2(t) &:= (\kappa + t)^{2/3} \log \log^{4/3}(\kappa + t), \\
\mu(t) &:= (\kappa + t)^{1/3} \log(\kappa + t) \log \log^{5/3}(\kappa + t),
\end{align*}
\]

(2.1)

where \( \kappa := 10^{100} \). The role of \( \mu(t) \) and \( \lambda_i(t) \) is to provide some extra time-decay so that we can somehow neglect bad terms (with no sign) that prevent us to conclude the properties claimed in the above theorems. Additionally, one can think of \( \lambda_1(t) \) as the rate of growth of the set \( \Omega_\pm(t) \) and \( \Omega_0(t) \) defined in theorem 1.1. The key idea for considering exactly these definitions for \( \mu(t) \) and \( \lambda_i(t) \) is that

\[
\frac{1}{\mu(t)\lambda_2(t)} \frac{\lambda_1'(t)}{\mu(t)\lambda_1(t)} \frac{\mu'(t)}{\mu^2(t)} \in L^1(\mathbb{R}_+), \quad \text{while} \quad \frac{1}{\mu(t)\lambda_1(t)} \notin L^1(\mathbb{R}_+).
\]

In the sequel we shall exploit these two properties. Moreover, for the sake of simplicity we introduce the following useful notation

\[
u_- := -\theta^{-1}\left(\sqrt{\beta} - \alpha\right) \quad \text{and} \quad \nu_+ := \theta^{-1}\left(\sqrt{\beta} + \alpha\right).
\]

(2.2)

Then, with all of the above notations, we define the modified mean functionals \( \mathcal{J}_1(t) \) and \( \mathcal{J}_2(t) \), adapted to the curves \( x - \nu_+ t \), which are given by

\[
\begin{align*}
\mathcal{J}_1(t) &:= \frac{\theta}{\mu(t)} \int_{\mathbb{R}} \left(\sqrt{\beta} \rho(t, x - \nu_- t) + \eta(t, x - \nu_- t)\right) \Phi\left(\frac{x}{\lambda_1(t)}\right) \Phi'\left(\frac{x}{\lambda_2(t)}\right) \, dx, \\
\mathcal{J}_2(t) &:= \frac{\theta}{\mu(t)} \int_{\mathbb{R}} \left(\sqrt{\beta} \rho(t, x - \nu_+ t) - \eta(t, x - \nu_+ t)\right) \Phi\left(\frac{x}{\lambda_1(t)}\right) \Phi'\left(\frac{x}{\lambda_2(t)}\right) \, dx.
\end{align*}
\]

The reason why we evaluate solutions on these translated points \( x - \nu_\pm t \) is to be able to take advantage of the characteristics of the underlying transport equations associated to (1.1). Moreover, another key quantity that shall play a fundamental role in our proof is the modified momentum functional \( \mathcal{I}(t) \), which is given by

\[
\mathcal{I}(t) := \frac{1}{\mu(t)} \text{Im} \int_{\mathbb{R}} \psi(t, x) \overline{\psi(t, x)} \Phi\left(\frac{x}{\lambda_1(t)}\right) \, dx - \frac{\theta}{\mu(t)} \int_{\mathbb{R}} \rho(t, x) \eta(t, x) \Phi\left(\frac{x}{\lambda_1(t)}\right) \, dx.
\]
For the sake of simplicity and the clarity of computations, we split the previous functional into two parts, namely,

\[ I_1(t) := \frac{1}{\mu(t)} \Im \int_{\mathbb{R}} \psi(t, x) \overline{\psi_x(t, x)} \Phi \left( \frac{x}{\lambda_1(t)} \right) \, dx, \]

\[ I_2(t) := \frac{\theta}{\mu(t)} \int_{\mathbb{R}} \rho(t, x) \eta(t, x) \Phi \left( \frac{x}{\lambda_1(t)} \right) \, dx. \]

We anticipate that, thanks to the explicit form of \( \Phi \) and the conservation of the momentum and the energy, if \((\psi, \rho, \eta)\) is a solution to system (1.1) belonging to the class \( C(\mathbb{R}, H^1 \times L^2 \times L^2) \), then, all the modified functionals above \( J_1(t), J_2(t) \) and \( I(t) \) are well defined for all times \( t \in \mathbb{R} \) (see lemma 2.3 for further details).

The following three lemmas give us the first basic virial identities satisfied by the modified functionals \( J_1(t), J_2(t) \) and \( I(t) \) above.

**Lemma 2.1.** Let \((\psi, \rho, \eta) \in C(\mathbb{R}, H^1 \times L^2 \times L^2) \) be any solution to system (1.1). Then, for all \( t \in \mathbb{R} \), the following identity holds

\[-\frac{d}{dt} I(t) = \frac{2 \omega}{\mu \lambda_1} \int |\psi_x|^2 \Phi' - \omega \frac{\mu}{2 \mu \lambda_1} \int |\psi|^2 \Phi'' + \frac{1}{2 \mu \lambda_1} \int \eta^2 \Phi' + \frac{\beta}{2 \mu \lambda_1} \int \rho^2 \Phi' + \frac{\gamma q}{2 \mu \lambda_1} \int |\psi|^4 \Phi' - \frac{\alpha}{\mu \lambda_1} \int \rho \eta \Phi' + \frac{\gamma}{\mu \lambda_1} \int \left( \eta - \frac{\alpha}{2} \right) |\psi|^2 \Phi' - \frac{\mu \eta}{\mu \lambda_1} \int \rho \eta \Phi' + \frac{\mu}{\mu \lambda_1} \Im \int \overline{\psi} \psi_x \Phi + \frac{\lambda_1}{\mu \lambda_1} \Im \int \left( \frac{x}{\lambda_1} \right) \overline{\psi} \psi_x \Phi' - \frac{\theta \lambda_1}{\mu \lambda_1} \int \rho \eta \Phi'. \]

**Proof.** The proof is somehow straightforward and follows from direct computations; we shall only proceed formally. Notice that the following reasoning can be made rigorously by standards approximation and density arguments.

Directly differentiating the definition of the functional \( I_1 \), using system (1.1) and performing several integration by parts we obtain

\[ \frac{d}{dt} I_1(t) = \frac{1}{\mu} \Im \int \psi x \overline{\psi} \Phi + \frac{1}{\mu} \Im \int \psi \overline{\psi_x} \Phi - \frac{\lambda_1}{\mu \lambda_1} \Im \int \left( \frac{x}{\lambda_1} \right) \psi \overline{\psi_x} \Phi' - \frac{\mu}{\mu \lambda_1} \Im \int \psi \overline{\psi_x} \Phi \]

\[ = \frac{2}{\mu} \Im \int \psi \overline{\psi_x} \Phi + \frac{1}{\mu \lambda_1} \Im \int \psi \overline{\psi_x} \Phi' - \frac{\lambda_1}{\mu \lambda_1} \Im \int \left( \frac{x}{\lambda_1} \right) \psi \overline{\psi_x} \Phi' + \frac{\mu}{\mu \lambda_1} \Im \int \psi \overline{\psi_x} \Phi \]

\[ = \frac{2}{\mu} \Re \int \left( \omega \psi_x + \gamma \left( \eta - \frac{\alpha}{2} \right) + q |\psi|^2 \right) \overline{\psi} \psi_x + \frac{1}{\mu \lambda_1} \Re \int \left( \omega \psi_x + \gamma \left( \eta - \frac{\alpha}{2} \right) + q |\psi|^2 \right) \overline{\psi} \psi_x \Phi' + \frac{\lambda_1}{\mu \lambda_1} \Im \int \left( \frac{x}{\lambda_1} \right) \psi \overline{\psi_x} \Phi' \]

\[ = -\frac{2 \omega}{\mu \lambda_1} \int |\psi|^2 \Phi' + \frac{\gamma}{\mu \lambda_1} \int \left( \eta - \frac{\alpha}{2} \right) |\psi|^2 \Phi' - \frac{\mu}{\mu \lambda_1} \Im \int |\psi|^2 \Phi'' - \frac{\mu}{\mu \lambda_1} \Im \int |\psi|^2 \Phi + \frac{\omega}{2 \mu \lambda_1} \int |\psi|^2 \Phi'' \]

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Now we compute the time-derivative of the second functional $I_2$. In fact, by direct differentiation again, using system (1.1) and performing several integration by parts we get

$$\frac{d}{dt} I_2 = \frac{\theta}{\mu} \int \rho \eta \Phi + \frac{\theta}{\mu} \int \rho \eta \Phi - \frac{\theta \lambda_1}{\mu \lambda_1} \int \left( \frac{x}{\lambda_1} \right) \rho \eta \Phi' - \frac{\theta \mu}{\mu^2} \int \rho \eta \Phi'$$

$$= \frac{1}{2 \mu \lambda_1} \int \eta^2 \Phi' + \frac{\alpha}{\mu} \int \rho \eta \Phi + \frac{\gamma}{\mu} \int \eta |\psi|^2 \Phi$$

$$+ \frac{\beta}{2 \mu \lambda_1} \int \rho^2 \Phi' + \frac{\alpha}{\mu} \int \rho \eta \Phi' - \frac{\alpha \gamma}{2 \mu} \int \rho \eta |\psi|^2 \Phi$$

$$- \frac{\alpha \gamma}{2 \mu \lambda_1} \int \rho |\psi|^2 \Phi' - \frac{\theta \mu'}{\mu^2} \int \rho \eta \Phi'$$

$$= \frac{1}{2 \mu \lambda_1} \int \eta^2 \Phi' - \frac{\alpha}{\mu \lambda_1} \int \rho \eta \Phi' + \frac{\gamma}{\mu} \int \eta |\psi|^2 \Phi$$

$$+ \frac{\gamma}{\mu \lambda_1} \int \eta |\psi|^2 \Phi' - \frac{\theta \mu'}{\mu^2} \int \rho \eta \Phi'$$

$$+ \frac{\beta}{2 \mu \lambda_1} \int \rho^2 \Phi' - \frac{\alpha \gamma}{2 \mu} \int \rho \eta |\psi|^2 \Phi - \frac{\alpha \gamma}{2 \mu \lambda_1} \int \rho |\psi|^2 \Phi'$$

$$- \frac{\theta \lambda_1}{\mu \lambda_1} \int \left( \frac{x}{\lambda_1} \right) \rho \eta \Phi'$$.

Hence, gathering both previous identities we conclude the desired result.$\square$

**Lemma 2.2.** Let $(\psi, \rho, \eta) \in C(\mathbb{R}, H^1 \times L^2 \times L^2)$ be any solution to system (1.1). Then, for all $t \in \mathbb{R}$, the following identities hold:

$$(1) \quad \frac{d}{dt} J_1(t) = \gamma \frac{(2 \sqrt{3} - \alpha)}{2 \mu(t) \lambda_1(t)} \int |\psi(t, x - \ldots t)|^2 \Phi' \left( \frac{x}{\lambda_1(t)} \right) \Phi' \left( \frac{x}{\lambda_2(t)} \right)$$

$$+ \gamma \frac{(2 \sqrt{3} - \alpha)}{2 \mu(t) \lambda_2(t)} \int |\psi(t, x - \ldots t)|^2 \Phi \left( \frac{x}{\lambda_1(t)} \right) \Phi' \left( \frac{x}{\lambda_2(t)} \right)$$

$$- \frac{\theta \mu'(t)}{\mu^2(t)} \int \left( \sqrt{3} \rho + \eta \right) (t, x - \ldots t) \Phi' \left( \frac{x}{\lambda_1(t)} \right) \Phi' \left( \frac{x}{\lambda_2(t)} \right)$$

$$- \frac{\theta \lambda_1(t)}{\mu(t) \lambda_1(t)} \int \left( \frac{x}{\lambda_1(t)} \right) \left( \sqrt{3} \rho + \eta \right) (t, x - \ldots t)$$
\[
\begin{align*}
&\times \Phi'(x) \Phi'(x) \Phi'(x) - \frac{\theta \lambda_1'(t)}{\mu(t)\lambda_2(t)} \int \frac{x}{\lambda_2(t)} \\
&\times \left( \sqrt{\beta \rho + \eta} \right)(t,x-u_t) \Phi \left( \frac{x}{\lambda_1(t)} \right) \Phi'' \left( \frac{x}{\lambda_2(t)} \right).
\end{align*}
\]

(2) \[
\frac{d}{dt} J_2(t) = \frac{\gamma(2\sqrt{\beta} + \alpha)}{2\mu(t)\lambda_1(t)} \int \psi \left( t,x-u_t \right) \left[ \Phi' \left( \frac{x}{\lambda_1(t)} \right) \Phi' \left( \frac{x}{\lambda_2(t)} \right) \right] \\
+ \frac{\gamma(2\sqrt{\beta} + \alpha)}{2\mu(t)\lambda_2(t)} \int \psi \left( t,x-u_t \right) \left[ \Phi' \left( \frac{x}{\lambda_1(t)} \right) \Phi' \left( \frac{x}{\lambda_2(t)} \right) \right] \\
- \frac{\theta \mu'(t)}{\mu^2(t)} \int \left( \sqrt{\beta \rho - \eta} \right)(t,x-u_t) \Phi \left( \frac{x}{\lambda_1(t)} \right) \Phi' \left( \frac{x}{\lambda_2(t)} \right) \\
- \frac{\theta \lambda_1'(t)}{\mu(t)\lambda_1(t)} \int \left( \frac{x}{\lambda_1(t)} \right) \left( \sqrt{\beta \rho - \eta} \right)(t,x-u_t) \\
\times \Phi' \left( \frac{x}{\lambda_1(t)} \right) \Phi' \left( \frac{x}{\lambda_2(t)} \right) - \frac{\theta \lambda_1'(t)}{\mu(t)\lambda_2(t)} \int \left( \frac{x}{\lambda_2(t)} \right) \\
\times \left( \sqrt{\beta \rho - \eta} \right)(t,x-u_t) \Phi \left( \frac{x}{\lambda_1(t)} \right) \Phi'' \left( \frac{x}{\lambda_2(t)} \right).
\]

**Proof.** Similarly to the previous lemma, we proceed by a direct computation. For the sake of simplicity, throughout this proof we will denote by \( \Phi_i, \Phi'_i \) and \( \Phi''_i \) the functions given by \( \Phi_i(x):= \Phi \left( \frac{x}{\gamma_{10}} \right), \Phi'_i(x):= \Phi' \left( \frac{x}{\gamma_{10}} \right) \) and \( \Phi''_i(x):= \Phi'' \left( \frac{x}{\gamma_{10}} \right), \) with \( i = 1, 2 \). Also, we shall only write \( \rho, \eta, \Phi_1 \) and \( \Phi_2 \), omitting their arguments.

Indeed, taking the time derivative of the functional, using system (1.1) and performing some integration by parts we obtain

\[
\begin{align*}
\frac{d}{dt} J_i(t) &= \frac{\sqrt{\beta} \theta}{\mu} \int \rho_i \Phi_i \Phi_i' + \frac{\beta - \alpha \sqrt{\beta}}{\mu} \int \rho_i \Phi_i \Phi_i'' \\
&- \frac{\theta \sqrt{\beta} \mu'}{\mu^2} \int \rho \Phi_i \Phi_i' + \frac{\theta}{\mu} \int \eta_i \Phi_i \Phi_i' \\
&+ \frac{\sqrt{\beta} - \alpha}{\mu} \int \eta_i \Phi_i \Phi_i' - \frac{\theta \mu'}{\mu^2} \int \eta_i \Phi_i \Phi_i' - \frac{\sqrt{\beta} \theta \lambda_1'}{\mu \lambda_1} \int \left( \frac{x}{\lambda_1} \right) \rho \Phi_i \Phi_i' \\
&- \frac{\theta \lambda_1'}{\mu \lambda_1} \int \left( \frac{x}{\lambda_1} \right) \eta_i \Phi_i \Phi_i'' + \frac{\sqrt{\beta} \theta \lambda_2'}{\mu \lambda_2} \int \left( \frac{x}{\lambda_2} \right) \rho \Phi_i \Phi_i'' \\
&= \frac{\alpha \sqrt{\beta}}{\mu} \int \rho_i \Phi_i \Phi_i' - \frac{\sqrt{\beta}}{\mu} \int \eta_i \Phi_i \Phi_i' - \frac{\gamma \sqrt{\beta}}{\mu} \int \left( |\psi|^2 \right) \Phi_i \Phi_i' \\
&+ \frac{\alpha}{\mu} \int \eta_i \Phi_i \Phi_i' + \frac{\beta - \sqrt{\beta} \alpha}{\mu} \int \rho_i \Phi_i \Phi_i' - \frac{\sqrt{\beta} \mu'}{\mu^2} \int \rho_i \Phi_i \Phi_i' \\
&- \frac{\beta}{\mu} \int \rho_i \Phi_i \Phi_i' - \frac{\theta \mu'}{\mu^2} \int \eta_i \Phi_i \Phi_i' + \frac{\alpha \gamma}{2 \mu} \int \left( |\psi|^2 \right) \Phi_i \Phi_i'.
\end{align*}
\]
three of them. Specifically, taking advantage of the conservation of both mass and energy, we

\[
\frac{\sqrt{\beta} - \alpha}{\mu} \int \eta \Phi \Phi' - \frac{\sqrt{\beta} \theta \lambda_1'}{\mu \lambda_1} \int \left( \frac{x}{\lambda_1} \right) \rho \Phi' \Phi' \\
- \frac{\theta \lambda_1'}{\mu \lambda_1} \int \left( \frac{x}{\lambda_1} \right) \eta \Phi' \Phi' - \frac{\sqrt{\beta} \theta \lambda_2'}{\mu \lambda_2} \int \left( \frac{x}{\lambda_2} \right) \rho \Phi' \Phi' \\
- \frac{\theta \lambda_2'}{\mu \lambda_2} \int \left( \frac{x}{\lambda_2} \right) \eta \Phi' \Phi' \\
= \frac{\gamma \sqrt{\beta}}{\mu \lambda_1} \int |\psi|^2 \Phi' \Phi' + \frac{\gamma \sqrt{\beta}}{\mu \lambda_2} \int |\psi|^2 \Phi \Phi' - \frac{\sqrt{\beta} \mu \lambda_1'}{\mu \lambda_1} \int \rho \Phi' \Phi' \\
- \frac{\theta \mu \lambda_2'}{\mu \lambda_2} \int \eta \Phi' \Phi' - \alpha \frac{\mu \lambda_1'}{\mu \lambda_1} \int |\psi|^2 \Phi' \Phi' \\
- \alpha \frac{\mu \lambda_2'}{\mu \lambda_2} \int |\psi|^2 \Phi' \Phi' - \frac{\sqrt{\beta} \theta \lambda_1'}{\mu \lambda_1} \int \left( \frac{x}{\lambda_1} \right) \rho \Phi' \Phi' \\
- \frac{\theta \lambda_1'}{\mu \lambda_1} \int \left( \frac{x}{\lambda_1} \right) \eta \Phi' \Phi' - \frac{\sqrt{\beta} \theta \lambda_2'}{\mu \lambda_2} \int \left( \frac{x}{\lambda_2} \right) \rho \Phi' \Phi' \\
- \frac{\theta \lambda_2'}{\mu \lambda_2} \int \left( \frac{x}{\lambda_2} \right) \eta \Phi' \Phi'.
\]

Thus, by gathering terms we conclude the proof of the lemma. The proof of the second formula follows the same arguments. Hence, we omit it. \(\square\)

Remark 2.1. We emphasise that none of these virial lemmas require the explicit definition of \(\Phi\) nor the one for the scaling functions \(\lambda_i \) and \(\mu\) that we gave at the beginning of this section. In fact, in section 4 we shall exploit (2.3) for completely different definitions of \(\Phi, \lambda \) and \(\mu\) (as soon as all quantities are well-defined). However, unless stated otherwise, throughout the proof of theorem 1.1 we shall always assume that we are referring to the functions defined at the beginning of this section.

2.2. Uniform boundedness of the energy norm

The following lemma is a direct consequence of the conservation laws and give us the time-uniform boundedness of the \(H^1 \times L^2 \times L^2\)-norm.

Lemma 2.3. Let \((\psi, \rho, \eta) \in C(\mathbb{R}, H^1 \times L^2 \times L^2)\) be a global solution emanating from an initial data \((\psi_0, \rho_0, \eta_0) \in H^1 \times L^2 \times L^2\). Then, there exists a constant \(C \in \mathbb{R}_+\), depending only on the norm of the initial data, such that the following global bound holds

\[
\|\psi(t)\|_{H^1}^2 + \|\rho(t)\|_{L^2}^2 + \|\eta(t)\|_{L^2}^2 \leq C, \quad \forall \ t \in \mathbb{R}.
\]

Proof. The idea of the proof is to use the conservation of the energy, re-constructing such conserved quantity from the energy norm. In fact, first of all let us recall that from the conservation of the energy we have

\[
\int_{\mathbb{R}} \left( \sqrt{\omega} |\psi| \right)^2 + \frac{\beta}{2} \rho^2 + \frac{1}{2} \eta^2 + \frac{\gamma q}{2} |\psi|^2 + \frac{\gamma}{2} \left( 2 \eta - \alpha \rho \right) |\psi|^2 - \alpha \rho \eta \right) \, dx = E(0). \quad (2.4)
\]

Hence, essentially we have to show that we can control the last three addends with the first three of them. Specifically, taking advantage of the conservation of both mass and energy, we
would like to find appropriate constants \( c > 0, \alpha, \beta \in \mathbb{R} \) such that
\[
\| \psi(t) \|_{H^1}^2 + \| \rho(t) \|_{L^2}^2 + \| \eta(t) \|_{L^2}^2 \leq c \left( E(0)^a + M(0)^b \right).
\]

First, we notice that to control the crossed term \( \rho \eta \), it is enough to use Young inequality for products, from which we get
\[
\alpha \int_\mathbb{R} \rho(t,x) \eta(t,x) \, dx \leq \frac{\beta + \alpha^2}{4} \int \rho^2(t,x) \, dx + \frac{\alpha^2}{2(\beta + \alpha^2)} \int \eta^2(t,x) \, dx.
\]  
(2.5)

Then, gathering the corresponding quadratic terms with respect to \((\rho, \eta)\) appearing in the energy, we have
\[
\int \left( \frac{\beta}{2} \rho^2(t,x) + \frac{1}{2} \eta^2(t,x) - \alpha \rho(t,x) \eta(t,x) \right) \, dx \geq \frac{\beta - \alpha^2}{4} \| \rho(t) \|_{L^2}^2 + \frac{\beta}{2(\beta + \alpha^2)} \| \eta(t) \|_{L^2}^2.
\]

We continue by bounding the contribution of the \( L^4 \)-norm of \( \psi(t) \). Indeed, by using Gagliardo–Nirenberg interpolation inequality, as well as Young inequality in the resulting right-hand side, we obtain
\[
\int \| \psi(t,x) \|^4 \, dx \leq \| \psi(t) \|_{H^1} \| \psi(t) \|_{L^2}^3 \leq \varepsilon \| \psi(t) \|_{L^2}^4 + \frac{1}{\varepsilon} M(0).
\]

Once again, due to the conservation of mass, it is enough to choose \( \varepsilon \in (0,1) \) sufficiently small so that we can absorb \( \varepsilon \| \psi(t) \|_{L^2}^4 \) by using the first term in (2.4). Finally, it only remains to bound
\[
\frac{\gamma}{2} \int \left( 2 \eta(t,x) - \alpha \rho(t,x) \right) | \psi(t,x) |^2 \, dx.
\]

However, notice that this term can be controlled by the previous ones. In fact, we have
\[
\frac{\gamma}{2} \int_\mathbb{R} \left( 2 \eta(t,x) - \alpha \rho(t,x) \right) | \psi(t,x) |^2 \, dx \leq \frac{\beta}{16(\beta + \alpha^2)} \| \eta(t) \|_{L^2}^4 + \frac{\beta - \alpha^2}{16} \| \rho(t) \|_{L^2}^2
\]
\[
+ \left( \frac{\gamma^2(\beta + \alpha^2)}{8 \beta} + \frac{2 \alpha^2}{\beta - \alpha^2} \right) \| \phi(t) \|_{L^4}^4.
\]

Therefore, gathering all the above estimates we conclude the proof of the lemma. \( \square \)

As a consequence of the previous lemma, we conclude the uniform boundedness of all the modified functionals.

**Corollary 2.4.** Let \((\psi, \rho, \eta) \in C(\mathbb{R}, H^1 \times L^2 \times L^2)\) be any solution to system (1.1) emanating from an initial data \((\psi_0, \rho_0, \eta_0) \in H^1 \times L^2 \times L^2\). Consider \( \lambda_1(t), \lambda_2(t) \) and \( \mu(t) \) defined as in (2.1). Then, the following bound holds
\[
\sup_{t \in [0, +\infty)} \left( |J_1(t)| + |J_2(t)| + |J_3(t)| + |J_4(t)| \right) < +\infty.
\]

**Proof.** First of all, notice that the time-uniform boundedness of \( J_1 \) and \( J_2 \) follows directly from Hölder inequality as well as the previous lemma. In the same fashion, to bound \( J_3 \) we proceed by Hölder inequality. However, since \( J_3 \) is of order 1 in \((\rho, \eta)\), in this case we obtain
\[
|J_3(t)| \lesssim \frac{1}{\mu(t)} \| \rho(t) + \eta(t) \|_{L^2} \| \Phi \|_{L^\infty} \| \Phi'(\frac{\cdot}{\lambda_2(t)}) \|_{L^2} \lesssim \frac{\lambda_1(t)^{1/2}}{\mu(t)} < C, \quad (2.6)
\]
where \( C > 0 \) only depends on the initial data \((\psi_0, \rho_0, \eta_0)\).

To conclude, we notice that the same procedure also provides a time-uniform bound for \( \mathcal{J}_2(t) \). The proof is complete. \( \square \)

**Remark 2.2.** Inequality (2.6) is precisely the condition that does not allow us to choose \( \lambda_1(t) \) growing any faster. In particular, this is the reason why we cannot choose \( \lambda_1(t) = t^{1+} \), for example.

### 3. Proof of theorem 1.1

#### 3.1. Time integrability of \(|\psi|^2\)

In this section we seek to use the previously found virial identities to prove the time integrability of the solution. In order to do that, we split the analysis in several steps. First, we shall show the time integrability (in the region given in theorem 1.1) only for \(|\psi(t,x)|^2\), which is proved in the following proposition.

**Proposition 3.1.** Let \((\psi, \rho, \eta) \in C(\mathbb{R}, H^1 \times L^2 \times L^2)\) be any solution to system (1.1) emanating from an initial data \((\psi_0, \rho_0, \eta_0) \in H^1 \times L^2 \times L^2\). Then, for \( \lambda_1(t), \lambda_2(t), \mu(t) \) and \( v_+ \) defined as in (2.1) and (2.2), the following inequality holds

\[
\int_0^{+\infty} \frac{1}{\mu(t)\lambda_1(t)} \int \left| \psi(t, x - v_\pm t) \right|^2 \operatorname{sech}^2 \left( \frac{x}{\lambda_1(t)} \right) \, dx \, dt < +\infty. \tag{3.1}
\]

**Proof.** Let us first consider the case of \( v_- \). The case for \( v_+ \) follows from the same bounds up to trivial modifications. Indeed, we define

\[
\mathcal{F}_-(t) := \frac{d}{dt} \mathcal{J}_1(t) - \frac{\gamma(2\sqrt{\beta} - \alpha)}{2\mu(t)\lambda_2(t)} \int \left| \psi(t, x - v_- t) \right|^2 \Phi \left( \frac{x}{\lambda_1(t)} \right) \Phi' \left( \frac{x}{\lambda_2(t)} \right)
\]

\[
+ \theta \frac{\mu'(t)}{\mu^2(t)} \int \sqrt{\beta \rho + \eta} (t, x - v_- t) \Phi \left( \frac{x}{\lambda_1(t)} \right) \Phi' \left( \frac{x}{\lambda_2(t)} \right)
\]

\[
+ \theta \frac{\lambda_1'(t)}{\mu(t)\lambda_1(t)} \int \left( \sqrt{\beta \rho + \eta} \right) (t, x - v_- t) \Phi \left( \frac{x}{\lambda_1(t)} \right) \Phi' \left( \frac{x}{\lambda_2(t)} \right)
\]

\[
\times \Phi' \left( \frac{x}{\lambda_1(t)} \right) \Phi' \left( \frac{x}{\lambda_2(t)} \right) + \theta \frac{\lambda_2'(t)}{\mu(t)\lambda_2(t)} \int \left( \sqrt{\beta \rho + \eta} \right) (t, x - v_- t) \Phi \left( \frac{x}{\lambda_1(t)} \right) \Phi' \left( \frac{x}{\lambda_2(t)} \right)
\]

\[
= I + II + III + IV + V.
\]

Then, from lemma 2.2 we infer that

\[
\int \frac{\gamma(2\sqrt{\beta} - \alpha)}{2\mu(t)\lambda_1(t)} \int \left| \psi(t, x - v_- t) \right|^2 \Phi \left( \frac{x}{\lambda_1(t)} \right) \Phi' \left( \frac{x}{\lambda_2(t)} \right) \, dx = \mathcal{F}_-(t).
\]

Hence, the problem is reduced to prove that we can integrate \( \mathcal{F}_-(t) \) on \((0, +\infty)\). In fact, first of all notice that, from corollary 2.4 we infer that

\[
\left| \int_0^{+\infty} I(t) \, dt \right| \lesssim \limsup_{t \to +\infty} |\mathcal{J}_1(t) - \mathcal{J}_1(0)| < +\infty.
\]
Moreover, from lemma 2.3 as well as the explicit definitions of \( \mu(t) \) and \( \lambda_2(t) \), it immediately follows that \( \Pi \in L^1(\mathbb{R}^+) \). On the other hand, from lemma 2.3 along with Hölder inequality, we can bound \( \text{III}(t) \) by

\[
|\text{III}(t)| \lesssim \frac{\mu(t)\|\Phi(t, \cdot)\|_{L^1}}{\mu^2(t)} \lesssim \frac{\nu^{1/2}(t)}{\mu^2(t)} \lesssim \frac{1}{(\kappa + t) \log(\kappa + t) \log \log^{3/2} (\kappa + t)} \in L^1(\mathbb{R}^+).
\]

In the same fashion, applying lemma 2.3 and Hölder inequality, we can bound \( |\text{IV}(t)| \) and \( |\text{V}(t)| \) pointwisely by integrable function as

\[
|\text{IV}(t)| \lesssim \frac{\nu^{1/2}(t) \lambda_1(t)}{\mu(t) \lambda_1(t)} \lesssim \frac{1}{(\kappa + t) \log(\kappa + t) \log \log^2 (\kappa + t)} \in L^1(\mathbb{R}^+),
\]

\[
|\text{V}(t)| \lesssim \frac{\nu^{1/2}(t) \lambda_2(t)}{\mu(t) \lambda_2(t)} \lesssim \frac{1}{(\kappa + t) \log(\kappa + t) \log \log^{3/2} (\kappa + t)} \in L^1(\mathbb{R}^+).
\]

Therefore, gathering all the above inequalities, we conclude the proof of (3.1) in the case of \( v_- \). Notice that the same proof (up to trivial modifications) also works for \( v_+ \). The proof is complete.

**Remark 3.1.** The proof of the previous proposition does not depend on the value of \( \alpha \in \mathbb{R} \), and hence, this concludes the first inequality in theorem 1.1.

### 3.2. Time integrability of the full solution

In this section we seek to extend the analysis to the full solution, that is, to include the corresponding integral terms associated to \((\psi, \rho, \eta)\). From now on we split the analysis in two cases concerning the values of \( \alpha \in \mathbb{R} \).

#### 3.2.1. Case \( \alpha \neq 0 \)

In order to take advantage of the previous analysis, we consider a different version of the modified momentum functional adapted to this region. More specifically, we define modified momentum functional adapted to the characteristics \( x - v_\pm t \), that is,

\[
\tilde{L}_\pm(t) := \frac{1}{\mu(t)} \Im \int_{\mathbb{R}} \psi(t, x - v_\pm t) \overline{\psi}(t, x - v_\pm t) \Phi \left( \frac{x}{\nu_1(t)} \right) \, dx
\]

\[
- \frac{\theta}{\mu(t)} \int_{\mathbb{R}} \rho(t, x - v_\pm t) \eta(t, x - v_\pm t) \Phi \left( \frac{x}{\nu_1(t)} \right) \, dx.
\]

Notice that, as a direct consequence of lemma 2.1, we have the following identity

\[
- \frac{d}{dt} \tilde{L}_\pm(t) = \frac{2\omega}{\mu \lambda_1} \int |\psi|^2 \Phi'' - \frac{\omega}{2\mu \lambda_1} \int |\psi|^2 \Phi'''
\]

\[
+ \frac{1}{2\mu \lambda_1} \int \eta^2 \Phi' + \frac{\beta}{2\mu \lambda_1} \int \rho^2 \Phi' + \frac{\gamma q}{2\mu \lambda_1} \int |\psi|^4 \Phi'
\]

\[
- \frac{\alpha}{\mu \lambda_1} \int \rho \eta \Phi' + \frac{\gamma}{\mu \lambda_1} \int \left( \eta - \frac{\alpha}{2} \rho \right) |\psi|^2 \Phi' - \frac{\theta \mu'}{\mu^2} \int \rho \eta \Phi
\]

\[
+ \frac{\mu^2}{\mu} \Im \int \psi \overline{\psi} \Phi + \frac{\lambda_1}{\mu \lambda_1} \Im \int \left( \frac{x}{\nu_1} \right) \psi \overline{\psi} \Phi'.
\]
\[-\frac{\theta \lambda'_1}{\mu \lambda_1} \int \left( \frac{x}{\lambda_1} \right) \rho \partial_x \Phi' \pm \frac{\sqrt{\beta} \pm \alpha}{\theta \mu \lambda_1} \IM \int \overline{\psi} \partial_x \Phi' \pm \frac{\sqrt{\beta} \pm \alpha}{\mu \lambda_1} \int \rho \partial_x \Phi', \tag{3.2}\]

where we have used the fact that, since \( \Phi \) is real-valued, we have

\[-\IM \int \overline{\psi} \partial_x \Phi \left( \frac{x}{\lambda_1} \right) = \frac{1}{\lambda_1} \IM \int \overline{\psi} \partial_t \Phi \left( \frac{x}{\lambda_1} \right).\]

With all of this at hand, we are ready to prove the time integrability of the full solution in weighted spaces along these characteristics. The following proposition concludes the proof the theorem 1.1 in the case \( \alpha \neq 0 \).

**Proposition 3.2.** Let \((\psi, \rho, \eta) \in C(\mathbb{R}, H^1 \times L^2 \times L^2)\) be any solution to system (1.1) emanating from an initial data \((\psi_0, \rho_0, \eta_0) \in H^1 \times L^2 \times L^2\). For \( \lambda_1(t), \mu(t) \) and \( \nu_\pm \) defined as in (2.1) and (2.2), we have the following two cases:

(a) If \( \alpha > 0 \), then the following holds

\[ \int_0^{+\infty} \frac{1}{\mu(t) \lambda_1(t)} \int_\mathbb{R} \left( |\psi|^2 + |\psi|^2 + \rho^2 + \eta^2 \right) (t, x - v_\pm t) \sech^2 \left( \frac{x}{\lambda_1(t)} \right) \, dx \, dt < +\infty. \]

(b) If \( \alpha < 0 \), then the following holds

\[ \int_0^{+\infty} \frac{1}{\mu(t) \lambda_1(t)} \int_\mathbb{R} \left( |\psi|^2 + |\psi|^2 + \rho^2 + \eta^2 \right) (t, x - v_\pm t) \sech^2 \left( \frac{x}{\lambda_1(t)} \right) \, dx \, dt < +\infty. \]

**Proof.** We shall proceed in a similar fashion as in proposition 3.1. However, notice that in this case we have several quadratic terms with no definite sign. The idea is to use proposition 3.1 to deal with the terms involving \(|\psi|^2\), and to absorb the crossed terms of the form \(\rho \eta\) with the ones with \(\rho^2\) and \(\eta^2\). In fact, first of all notice that, thanks to lemma 2.3 and the explicit definitions of \(\mu\) and \(\lambda_1\), it is not difficult to see that

\[ \frac{1}{\mu \lambda_1} \int |\psi|^2 \Phi'' \in L^1(\mathbb{R}_+), \quad \frac{\mu'}{\mu^2} \int \rho \Phi \in L^1(\mathbb{R}_+), \quad \frac{\mu'}{\mu^2} \IM \int \overline{\psi} \Phi \in L^1(\mathbb{R}_+). \]

Moreover, from lemma 2.3 we also infer that we can integrate \(\tilde{T}(t)\) on \((0, +\infty)\). On the other hand, by Hölder inequality, the explicit definitions of \(\mu\) and \(\lambda_1\), as well as lemma 2.3, we obtain

\[ \frac{\lambda_1}{\mu \lambda_1} \IM \int \left( \frac{x}{\lambda_1} \right) \overline{\psi} \partial_x \Phi \left( \frac{x}{\lambda_1} \right) \leq \frac{\lambda_1}{\mu \lambda_1} \int |\psi|^2 \Phi'' \leq \frac{\mu'}{\mu^2} \int \rho \Phi \leq L^1(\mathbb{R}_+), \quad \text{and,} \]

\[ \frac{\lambda_1}{\mu \lambda_1} \int \left( \frac{x}{\lambda_1} \right) \rho \Phi \left( \frac{x}{\lambda_1} \right) \leq \frac{\lambda_1}{\mu \lambda_1} \int |\rho||\Phi''| \leq L^1(\mathbb{R}_+). \]
Now, for the term involving $|\psi|^4$ in (3.2), we use proposition 3.1 and lemma 2.3 as well as Sobolev embedding, from where we get\footnote{Notice that here we are actually using proposition 3.1 with a different definition of $\Phi(x)$, so that $\Phi'(x) = \text{sech}^4(x)$. By taking the integration constant equal to zero we get $\Phi \in L^\infty$.}

\[
\frac{1}{\mu \lambda_1} \int |\psi|^4 \Phi' \lesssim \frac{1}{\mu \lambda_1} \|\psi\|^2_{L_{x}^{2\infty} H_{\mu}} \int |\psi|^2 \Phi' \in L^1(\mathbb{R}_+) \tag{3.3}
\]

Besides, from Young inequality for products, it is not difficult to see that

\[
\frac{2\omega}{\mu \lambda_1} \int |\psi_{\alpha}|^2 \Phi' \pm (\frac{\sqrt{\beta} \pm \alpha}{\mu \lambda_1}) \text{Im} \int \bar{\psi} \psi \Phi' \geq \frac{\omega}{\mu \lambda_1} \int |\psi_{\alpha}|^2 \Phi' - (\frac{\sqrt{\beta} - \alpha^2}{4\omega \mu \lambda_1}) \int |\psi|^2 \Phi'.
\]

Thanks to proposition 3.1, the last term in the right-hand side above, belongs to $L^1(\mathbb{R}_+)$. Also, from Young inequality for products again, we additionally infer that

\[
\frac{\gamma}{\mu \lambda_1} \int \left(\eta - \frac{\alpha}{2}\right) |\psi|^2 \Phi' \geq -\frac{\epsilon_1}{\mu \lambda_1} \int \eta^2 \Phi' - \frac{\epsilon_2}{\mu \lambda_1} \int \rho^2 \Phi' - \frac{K}{\mu \lambda_1} \int |\psi|^4 \Phi' \tag{3.4}
\]

where $\epsilon_1, \epsilon_2 > 0$ denote sufficiently small numbers that shall be fixed later. Here $K = K(\epsilon_1, \epsilon_2) > 0$ is a large number, however, proceeding in the same fashion as in (3.3) we infer that this term belongs to $L^1(\mathbb{R}_+)$ no matter what the value of $K$ is.

Finally, it only remains to control the quadratic terms in $\rho$ and $\eta$. Unfortunately, due to the factor $\mp(\sqrt{\beta} \pm 2\alpha) \rho \eta$ appearing in (3.2), we cannot obtain the required positivity in both directions $v_\pm$ for the terms involving $\rho^2$, $\eta^2$ and $\rho \eta$. Thus, we split the analysis in two cases regarding the sign of $\alpha$.

**Case** $\alpha > 0$: we aim to prove the following claim that provide us the required positivity:

There exists two constants $\epsilon_1, \epsilon_2 > 0$, such that\footnote{Notice that here we are actually using proposition 3.1 with a different definition of $\Phi(x)$, so that $\Phi'(x) = \text{sech}^4(x)$. By taking the integration constant equal to zero we get $\Phi \in L^\infty$.}

\[
\frac{1}{2\mu \lambda} \int \eta^2 \Phi' + \frac{\beta}{2\mu \lambda} \int \rho^2 \Phi' + \frac{\sqrt{\beta} - 2\alpha}{\mu \lambda} \int \rho \eta \Phi' \geq \frac{1}{2\mu \lambda} \int \eta^2 \Phi' \tag{3.5}
\]

In this case we shall take advantage of $\tilde{L}_-(t)$. In fact, first of all, notice that, if $\sqrt{\beta} - 2\alpha = 0$, then there is nothing to prove. Then, in the sequel we assume $\sqrt{\beta} - 2\alpha \neq 0$. Indeed, consider the parameter $\epsilon_1 \in \mathbb{R}$ given by

\[
\epsilon_1 := \frac{1}{2} \left(1 + \frac{\beta}{(\sqrt{\beta} - 2\alpha)^2}\right) > 0.
\]

Then, by using Young inequality for products, with parameter given by $\epsilon_1$, we infer

\[
\frac{1}{2\mu \lambda} \int \eta^2 \Phi' + \frac{\beta}{2\mu \lambda} \int \rho^2 \Phi' + \frac{\sqrt{\beta} - 2\alpha}{\mu \lambda} \int \rho \eta \Phi' \geq \frac{1}{2\mu \lambda} \int \eta^2 \Phi' \tag{3.5}
\]

Moreover, notice that, since $\beta - \alpha^2 > 0$ and $\alpha > 0$, we infer that $\epsilon_1 > 1$, and hence $1 - \epsilon_1^{-1} > 0$. On the other hand, by direct computations we see that

\[
\beta - (\sqrt{\beta} - 2\alpha)^2 \epsilon_1 > \beta - \frac{(\sqrt{\beta} - 2\alpha)^2 \beta}{(\sqrt{\beta} - 2\alpha)^2} = 0.
\]
Then, plugging the previous computations into (3.5), we conclude the proof of the claim.

**Case** $\alpha < 0$: in this case we aim to prove the following claim that provide us the required positivity: there exists two constants $c_1, c_2 > 0$, such that

$$\frac{1}{2\mu\lambda} \int \eta^2 \Phi' + \frac{\beta}{2\mu\lambda} \int \rho^2 \Phi' - \frac{\sqrt{\beta} + 2\alpha}{\mu\lambda} \int \rho \eta \Phi' \geq c_1 \int \eta^2 \Phi' + c_2 \int \rho^2 \Phi'. \quad (3.6)$$

In contrast with the previous case, in this case we shall take advantage of $\tilde{L}_+(t)$. In fact, first of all, notice that, if $\sqrt{\beta} + 2\alpha = 0$, then there is nothing to prove. Thus, in the sequel we assume $\sqrt{\beta} + 2\alpha \neq 0$. Indeed, define $\varepsilon_2 \in \mathbb{R}$ as

$$\varepsilon_2 := \frac{1}{2} \left(1 + \frac{\beta}{(\sqrt{\beta} + 2\alpha)^2}\right) > 0.$$

Then, by using Young inequality for products, with parameter given by $\varepsilon_2$, we infer

$$\frac{1}{2\mu\lambda} \int \eta^2 \Phi' + \frac{\beta}{2\mu\lambda} \int \rho^2 \Phi' - \frac{\sqrt{\beta} + 2\alpha}{\mu\lambda} \int \rho \eta \Phi' \geq \frac{1}{2\mu\lambda} (1 - \varepsilon_2^1) \int \eta^2 \Phi' + \frac{1}{2\mu\lambda} \left(\beta - (\sqrt{\beta} + 2\alpha)^2\varepsilon_2\right) \int \rho^2 \Phi'.$$

Moreover, by using both $\beta - \alpha^2 > 0$ and $\alpha < 0$, proceeding in exactly the same fashion as in the previous claim, we conclude that both factors in front of each of the integral on the right-hand side of the latter inequality are strictly positive. Thus, we conclude the proof of the claim.

Finally, it only remains to set the definition of $\varepsilon_{1+}^*$ and $\varepsilon_{2+}^*$ in (3.4). Notice that we have to give a different definition depending on the case $\alpha \geq 0$. In fact, from the above analysis it follows that it is enough to consider

$$\varepsilon_{1+}^* := 10^{-10} \left(\beta - (\sqrt{\beta} - 2\alpha)^2\varepsilon_1\right), \quad \varepsilon_{2+}^* := 10^{-10} (1 - \varepsilon_1^{-1}),$$

$$\varepsilon_{1-}^* := 10^{-10} \left(\beta - (\sqrt{\beta} + 2\alpha)^2\varepsilon_2\right), \quad \varepsilon_{2-}^* := 10^{-10} (1 - \varepsilon_2^{-1}),$$

where $\varepsilon_{1+}^*$ stands for the case where $\alpha$ is positive or negative respectively. Hence, we conclude the proof of the proposition.

\[ \square \]

3.2.2. **Case** $\alpha = 0$. In the case when $\alpha = 0$, we can give a much simpler and shorter proof. In fact, in this case, from lemma 2.1 we can easily deduce the following result.

**Proposition 3.3.** Let $(\psi, \rho, \eta) \in C(\mathbb{R}, H^1 \times L^2 \times L^2)$ be any solution to system (1.1) emanating from an initial data $(\psi_0, \rho_0, \eta_0) \in H^1 \times L^2 \times L^2$. For $\lambda_1(t)$, $\mu(t)$ and $\nu_\pm$ defined as in (2.1) and (2.2), the following inequality holds

$$\int_0^{+\infty} \frac{1}{\mu(t)\lambda_1(t)} \int \left(|\psi(t, x)|^2 + |\psi_\nu(t, x)|^2 + \eta^2(t, x) + \rho^2(t, x) \operatorname{sech}^2 \left(\frac{x}{\lambda_1(t)}\right)\right) dx \, dt < +\infty.$$

**Proof.** In fact, by using the standard modified momentum function $\mathcal{I}(t)$ (instead of $\tilde{\mathcal{I}}(t)$ as before), we proceed in the same fashion as in the previous proposition, using lemma 2.1 and...
noticing that, under our current assumptions, \( q = \gamma > 0 \), from where we infer that

\[
\frac{3\gamma q}{4\mu \lambda_1} \int |\psi|^4 \Phi' + \frac{1}{2\mu \lambda_1} \int \eta^2 \Phi' + \frac{\gamma}{\mu \lambda_1} \int \eta |\psi|^2 \Phi' > \frac{\gamma^2}{100\mu \lambda_1} \int |\psi|^4 \Phi' + \frac{1}{100\mu \lambda_1} \int \eta^2 \Phi'.
\]

Then, the proof follows by gathering the latter inequality with (2.3) and recalling that \( \lambda_1^{-3} \in L^1(\mathbb{R}_+) \). In order to avoid over-repeated computations we omit the details. \( \square \)

4. Decay in far field regions

In this section we seek to prove pointwise decay in far field regions by taking advantage of some suitable virial identities, as before. The analysis is similar (in spirit) to that shown in the previous section. However, in this case, the idea will be somewhat the opposite, in the sense that now the important terms shall come from the derivative of the weight \( \Phi \), instead of the derivative of the solution, as in the previous section. To do so, we consider both the modified mass functional as well as the modified energy functional, which are given by (respectively)

\[
\mathcal{M}_\pm(t) := \int_\mathbb{R} \Phi \left( \frac{x_\pm + \zeta(t)}{\lambda(t)} \right) |\psi(t, x)|^2 \, dx,
\]

\[
\mathcal{E}_\pm(t) := \int_\mathbb{R} \Phi \left( \frac{x_\pm + \zeta(t)}{\lambda(t)} \right) \left( \omega |\psi_x|^2 + \frac{\gamma q}{2} |\psi|^4 + \frac{\beta}{2} \rho^2 + \frac{1}{2} \eta^2 + \frac{\gamma}{2}(2\eta - \alpha \rho) |\psi|^2 - \alpha \rho \eta \right) \, dx.
\]

Here, \( \Phi \) stands for a smooth and bounded weight (not necessarily decaying at \( \pm \infty \)), which shall be completely different to the one chosen in the previous section (see (4.9) and (4.10) for the exact definition). Notice also that, in contrast with the modified mean functional, now we only require \( \Phi \) belonging to \( L^\infty(\mathbb{R}) \) in order for \( \mathcal{M}_\pm \) and \( \mathcal{E}_\pm \) to be well-defined and uniformly bounded (for solutions in the energy space). Additionally, in this case we define the scaling \( \lambda(t) \) and the shift \( \zeta(t) \) as

\[
\lambda(t) := (1 + t)^{\frac{1}{2} + \delta}, \quad \zeta(t) \geq c_1 \lambda(t), \quad \zeta'(t) \geq c_2 \lambda'(t), \quad c_1, c_2 > 0, \quad \delta > 0.
\]

In a similar spirit as in the previous section, the main motivation to consider these specific definitions of \( \lambda \) and \( \zeta \) is to obtain

\[
\frac{1}{\lambda} \frac{1}{\zeta} \in L^1(\mathbb{R}_+), \quad \text{however} \quad \frac{\lambda'}{\lambda} \frac{\zeta'}{\zeta} \notin L^1(\mathbb{R}_+).
\]

which shall allow us to neglect some bad terms (in some sense). We emphasise that, in this case, we are considering a scaling factor \( \lambda(t) \) growing faster than linear (in contrast with the previous sections). This changes the behavior of some important terms (with respect to the above analysis) that we intend to take advantage of.
On the other hand, to simplify computations, we split the modified energy functional $\mathcal{E}$ into the following functionals

$$
\mathcal{E}_{\pm,1}(t) := \int_{\mathbb{R}} \Phi' \left( \frac{\pm x + \zeta(t)}{\lambda(t)} \right) \left( \omega |\psi_1|^2 + \frac{\gamma q}{2} |\psi|^4 \right) \, dx,
$$

$$
\mathcal{E}_{\pm,2}(t) := \int_{\mathbb{R}} \Phi' \left( \frac{\pm x + \zeta(t)}{\lambda(t)} \right) \left( \frac{\beta}{2} \rho^2 + \frac{1}{2} \eta^2 + \frac{\gamma}{2} (2\eta - \alpha \rho) |\psi|^2 - \alpha \rho \eta \right) \, dx.
$$

Before going further, let us compute the virial identities associated with our current functionals, that shall give us the fundamental information for the following analysis.

**Lemma 4.1.** Let $(\psi, \rho, \eta) \in C(\mathbb{R}, H^1 \times L^2 \times L^2)$ be any solution to system (1.1). Then, for all $t \in \mathbb{R}$, the following identity holds

$$
\frac{d}{dt} \mathcal{M}_\pm(t) = -\frac{\lambda'}{\lambda} \int_{\mathbb{R}} \Phi' \left( \frac{\pm x + \zeta(t)}{\lambda(t)} \right) \left( \frac{\pm x + \zeta(t)}{\lambda(t)} \right) \left( \omega |\psi_1|^2 + \frac{\gamma q}{2} |\psi|^4 \right) \, dx
$$

$$
+ \frac{\lambda'}{\lambda} \int_{\mathbb{R}} \Phi' \left( \frac{\pm x + \zeta(t)}{\lambda(t)} \right) \left( \frac{\beta}{2} \rho^2 + \frac{1}{2} \eta^2 \right) \, dx
$$

$$
+ \frac{\gamma}{2} (2\eta - \alpha \rho) |\psi|^2 - \alpha \rho \eta \right) \, dx - \frac{\lambda'}{\lambda} \int_{\mathbb{R}} \Phi' \left( \frac{\pm x + \zeta(t)}{\lambda(t)} \right) \left( \omega |\psi_1|^2 + \frac{\gamma q}{2} |\psi|^4 \right) \, dx
$$

$$
+ \frac{\gamma}{2} (2\eta - \alpha \rho) |\psi|^2 - \alpha \rho \eta \right) \, dx
$$

$$
\pm \frac{2\omega}{\lambda} \text{Im} \int_{\mathbb{R}} \Phi' \left( \frac{\pm x + \zeta(t)}{\lambda(t)} \right) \bar{\psi}_x \psi_x \, dx
$$

$$
\pm \frac{2 q \omega}{\lambda} \text{Im} \int_{\mathbb{R}} \Phi' \left( \frac{\pm x + \zeta(t)}{\lambda(t)} \right) |\psi|^2 |\psi_1|^2 \, dx
$$

$$
\pm \frac{\alpha}{\theta \lambda} \int_{\mathbb{R}} \Phi' \left( \frac{\pm x + \zeta(t)}{\lambda(t)} \right) (\beta |\rho|^2 + |\eta|^2) \, dx
$$

$$
\pm \frac{\beta + \alpha^2}{\theta \lambda} \int_{\mathbb{R}} \Phi' \left( \frac{\pm x + \zeta(t)}{\lambda(t)} \right) \rho \eta \, dx.
$$

**Proof.** The proof follows from direct computations. We omit this proof. \qed

**Lemma 4.2.** Let $(\psi, \rho, \eta) \in C(\mathbb{R}, H^2 \times H^1 \times H^1)$ be any solution to system (1.1). Then, for all $t \in \mathbb{R}$, the following identity holds,
\[
\pm \frac{\gamma}{\theta \lambda} \int_{\mathbb{R}} \phi' \left( \frac{\pm x + \zeta(t)}{\lambda(t)} \right) \rho |\psi|^2 \, dx \\
\pm \frac{3 \gamma \alpha}{2 \theta \lambda} \int_{\mathbb{R}} \phi' \left( \frac{\pm x + \zeta(t)}{\lambda(t)} \right) \eta |\psi|^2 \, dx \\
\pm \frac{\gamma^2 \alpha}{\theta \lambda} \int_{\mathbb{R}} \phi' \left( \frac{\pm x + \zeta(t)}{\lambda(t)} \right) |\psi|^4 \, dx \\
\pm \frac{\gamma \omega}{\lambda} \text{Im} \int_{\mathbb{R}} \phi' \left( \frac{\pm x + \zeta(t)}{\lambda(t)} \right) (2\eta - \alpha \rho) \overline{\psi} \psi_x \, dx. \\
(4.5)
\]

**Proof.** First of all, in order to simplify the computations, we split the derivative of $E_{\pm,i}$ into the sum of the derivatives of $E_{\pm,i}, i = 1, 2$, treating separately each of these functionals and then summing-up the corresponding results. In fact, directly differentiating $E_{\pm,1}$, using that $(\psi, \rho, \eta)$ solves system (1.1) and then performing several integration by parts, we obtain

\[
\frac{d}{dt} E_{\pm,1}(t) = \frac{\zeta}{\lambda} \int_{\mathbb{R}} \phi' \left( \frac{\pm x + \zeta(t)}{\lambda(t)} \right) \left( \omega |\psi_x|^2 + \frac{3q}{2} |\psi|^4 \right) \, dx \\
- \frac{\lambda'}{\lambda} \int_{\mathbb{R}} \phi' \left( \frac{\pm x + \zeta(t)}{\lambda(t)} \right) \left( \frac{\pm x + \zeta(t)}{\lambda(t)} \right) \left( \omega |\psi_x|^2 + \frac{3q}{2} |\psi|^4 \right) \, dx \\
\pm \frac{2\omega^2}{\lambda} \text{Im} \int_{\mathbb{R}} \phi' \left( \frac{\pm x + \zeta(t)}{\lambda(t)} \right) \overline{\psi} \psi_x \, dx \\
\pm \frac{2\gamma q \omega}{\lambda} \text{Im} \int_{\mathbb{R}} \phi' \left( \frac{\pm x + \zeta(t)}{\lambda(t)} \right) |\psi|^2 \overline{\psi} \psi \, dx \\
+ 2\omega \gamma \text{Im} \int_{\mathbb{R}} \phi \left( \frac{\pm x + \zeta(t)}{\lambda(t)} \right) \overline{\psi} \psi_x \left( \eta - \frac{1}{2} \alpha \rho + q |\psi|^2 \right) \, dx \\
+ 2\gamma q \omega \text{Im} \int_{\mathbb{R}} \phi \left( \frac{\pm x + \zeta(t)}{\lambda(t)} \right) (|\psi|^2)_x \overline{\psi} \psi \, dx \\
=: R_{\pm,1} + R_{\pm,2} + R_{\pm,3} + R_{\pm,4} + R_{\pm,5} + R_{\pm,6}. \\
(4.6)
\]

We now proceed with $E_{\pm,2}$. In fact, in a similar fashion as before, directly differentiating $E_{\pm,2}$, using that $(\psi, \rho, \eta)$ solves (1.1), and then performing several integration by parts, we get

\[
\frac{d}{dt} E_{\pm,2}(t) = \frac{\zeta}{\lambda} \int_{\mathbb{R}} \phi' \left( \frac{\pm x + \zeta(t)}{\lambda(t)} \right) \left( \beta |\rho|^2 + \frac{1}{2} \rho^2 + \frac{\gamma}{2} (2\eta - \alpha \rho) |\psi|^2 - \alpha \rho \eta \right) \, dx \\
- \frac{\lambda'}{\lambda} \int_{\mathbb{R}} \phi' \left( \frac{\pm x + \zeta(t)}{\lambda(t)} \right) \left( \frac{\pm x + \zeta(t)}{\lambda(t)} \right) \left( \beta |\rho|^2 + \frac{1}{2} \rho^2 \right) \, dx \\
+ \frac{\gamma}{2} (2\eta - \alpha \rho) |\psi|^2 - \alpha \rho \eta \, dx \\
\pm \frac{\alpha}{\theta \lambda} \int_{\mathbb{R}} \phi' \left( \frac{\pm x + \zeta(t)}{\lambda(t)} \right) |\psi|^2 \, dx \\
\times (|\beta| |\rho|^2 + |\eta|^2) \, dx \pm \frac{\beta + \alpha^2}{\theta \lambda} \int_{\mathbb{R}} \phi' \left( \frac{\pm x + \zeta(t)}{\lambda(t)} \right) \rho |\eta|^2 \, dx \\
\pm \frac{\gamma (\beta + \alpha^2/2)}{\theta \lambda} \int_{\mathbb{R}} \phi' \left( \frac{\pm x + \zeta(t)}{\lambda(t)} \right) \rho |\psi|^2 \, dx.
\]
that (4.2) holds, we have

\[ \frac{3}{2} \gamma \alpha \int_{\mathbb{R}} \Phi' \left( \frac{\pm x + \zeta(t)}{\lambda(t)} \right) \eta |\psi|^2 \, dx \]

\[ \frac{\gamma^2 \alpha}{2} \int_{\mathbb{R}} \Phi' \left( \frac{\pm x + \zeta(t)}{\lambda(t)} \right) |\psi|^4 \, dx \]

\[ \pm \frac{\gamma \omega}{\lambda} \text{Im} \int_{\mathbb{R}} \Phi' \left( \frac{\pm x + \zeta(t)}{\lambda(t)} \right) (2\eta - \alpha \rho) \overline{\psi}_x \, dx \]

\[ + \frac{\gamma \omega}{\lambda} \int_{\mathbb{R}} \Phi \left( \frac{\pm x + \zeta(t)}{\lambda(t)} \right) (2\eta - \alpha \rho)x \overline{\psi}_x \, dx. \]  

(4.7)

Finally, for the sake of simplicity let us define \( R_{\pm,7} \) as the following quantity

\[ R_{\pm,7} := \gamma \omega \text{Im} \int_{\mathbb{R}} \Phi \left( \frac{\pm x + \zeta(t)}{\lambda(t)} \right) (2\eta - \alpha \rho)x \overline{\psi}_x \, dx. \]

Then, it is not difficult to see that with these definitions we have the relation \( R_{\pm,5} = R_{\pm,6} + R_{\pm,7} \).

Therefore, summing up all the previous computations, and then using the above relation, we conclude the proof of (4.5).

\[ \square \]

### 4.1. Time integrability of the weighted \( L^2 \)-norm

In this subsection we restrict ourselves to the simpler case of the time integrability of the weighted \( L^2 \)-norm for the Schrödinger part of the solution \( \psi(t, x) \). The integrability (and decay) of this weighted-norm is a fundamental part of the analysis since (as we shall see) it triggers the decay of the whole weighted energy norm. In fact, let us start by recalling the relation

\[ - \frac{d}{dt} M_{\pm}(t) \pm \frac{2 \omega}{\lambda} \text{Im} \int_{\mathbb{R}} \Phi' \left( \frac{\pm x + \zeta(t)}{\lambda(t)} \right) \overline{\psi}_x \, dx \]

\[ = \frac{\lambda'}{\lambda} \int_{\mathbb{R}} \Phi' \left( \frac{\pm x + \zeta(t)}{\lambda(t)} \right) |\psi|^2 \, dx - \frac{\zeta'}{\lambda} \int_{\mathbb{R}} \Phi' \left( \frac{\pm x + \zeta(t)}{\lambda(t)} \right) |\psi|^2 \, dx. \]  

(4.8)

Then, notice that the left-hand side of the above relation is time-integrable on \( \mathbb{R}_+ \), provided that \( \Phi' \in L^\infty(\mathbb{R}) \). In fact, if \( \Phi' \) is bounded, then from Hölder inequality, lemma 2.3 and the fact that (4.2) holds, we have

\[ \frac{2}{\lambda} \text{Im} \int_{\mathbb{R}} \Phi' \left( \frac{\pm x + \zeta(t)}{\lambda(t)} \right) \overline{\psi}_x \, dx \leq \frac{1}{\lambda} \in L^1(\mathbb{R}_+). \]

Motivated by the time-integrability above, as well as identity (4.8), we shall give a suitable definition for \( \Phi \in C^\infty(\mathbb{R}) \) so that we are able to obtain a convenient sign-property in the right-hand side of (4.8).

To take advantage of the structure of the virial identities, from now on we consider \( \Phi \) to be any non-increasing smooth function such that it satisfies the following conditions

\[ \{ \Phi(s) = 1, \, s \leq -1 \}, \quad \{ \Phi(s) = 0, \, s \geq 0 \} \quad \text{and} \quad \{ \Phi' \equiv -1 \, \text{on} \, \left[ -\frac{9}{10}, \frac{1}{10} \right] \}. \]

(4.9)
Notice that, as a particular consequence of its definition, we have the following inequalities
\[ \forall s \in \mathbb{R}, \quad \Phi'(x) \leq 0 \quad \text{and} \quad s\Phi'(s) \geq 0. \tag{4.10} \]

As already mentioned, we now focus in studying the right-hand side of (4.8). Notice that, thanks to (4.9) and (4.10) we infer that, for all \( t \geq 0 \), the following sign-properties are satisfied
\[ \frac{\lambda'}{\lambda} \int_{\mathbb{R}} \Phi' \left( \frac{\pm x + \zeta}{\lambda} \right) \left( \frac{\pm x + \zeta}{\lambda} \right) |\psi|^2 \, dx \geq 0 \quad \text{and} \quad -\frac{\lambda'}{\lambda} \int_{\mathbb{R}} \Phi' \left( \frac{\pm x + \zeta}{\lambda} \right) |\psi|^2 \, dx \geq 0. \]

Consequently, due to the fact that the left-hand side of (4.8) is integrable in time, we can compute the time-integral over \( \mathbb{R}_+ \) and get
\[ \int_0^\infty \left( \frac{\lambda'}{\lambda} \int_{\mathbb{R}} \Phi' \left( \frac{\pm x + \zeta}{\lambda} \right) \left( \frac{\pm x + \zeta}{\lambda} \right) |\psi|^2 \, dx \right) \, dt < \infty. \tag{4.11} \]

Then, gathering this latter inequality with the sign-property above, we deduce in particular
\[ \int_0^\infty \frac{\zeta'(t)}{\lambda(t)} \int_{\mathbb{R}} \Phi' \left( \frac{\pm x + \zeta(t)}{\lambda(t)} \right) |\psi(t, x)|^2 \, dx \, dt < \infty. \]

**Remark 4.1.** Notice that, as a particular consequence of the latter inequality, recalling also that \( \lambda^{-1}(t) \zeta' \notin L^1(\mathbb{R}_+) \), we infer the existence of a sequence of times \( \{t_n\}_{n \in \mathbb{R}}, \) satisfying \( t_n \to \infty \), such that
\[ \lim_{n \to +\infty} \int_{\Omega(t_n)} |\psi(t_n, x)|^2 \, dx = 0, \tag{4.12} \]
where the set \( \Omega(t) \) can be defined, for example, as
\[ \Omega(t) := \{ x \in \mathbb{R} : \frac{1}{10} \lambda(t) + \zeta(t) \leq |x| \leq \frac{9}{10} \lambda(t) + \zeta(t) \}. \tag{4.13} \]

Moreover, notice that from the above
\[ \int_0^\infty \frac{\lambda'(t)}{\lambda(t)} \int_{\Omega(t)} |\psi(t, x)|^2 \, dx < +\infty \quad \text{and} \quad \int_0^\infty \frac{\zeta'(t)}{\lambda(t)} \int_{\Omega(t)} |\psi(t, x)|^2 \, dx < +\infty. \tag{4.14} \]

Hence, from now on, we can use properties (4.12) and (4.14) without depending on weights \( \Phi \) satisfying (4.9). In particular, in the sequel we shall use (4.12) for compactly supported weight functions encoding the same (or strictly contained) regions as in (4.13).

### 4.2. Decay of the \( L^2 \)-norm

In this section we seek to prove the pointwise decay of \( L^2 \)-norm of the Schrödinger component of the solution \( \psi(t, x) \) restricted to the far-field regions. In fact, let us start by considering \( \Psi \in C_0^\infty(\mathbb{R}) \) to be any non-negative function such that
\[ \text{supp}(\Psi) \subset \left[ -\frac{3}{4}, \frac{1}{4} \right] \quad \text{with} \quad \Psi \equiv 1 \text{ on } \left[ -\frac{3}{5}, \frac{2}{5} \right]. \tag{4.15} \]
Notice that, in particular, this implies that $\text{supp}(\Psi) \subset \text{supp}(\Phi')$. Additionally, we assume that $\Psi$ satisfies the following pointwise properties
\[
\forall s \in \mathbb{R}, \quad \Psi(s) \lesssim |\Phi'(s)| \quad \text{and} \quad |\Psi'(s)| \lesssim |\Phi'(s)|.
\] (4.16)

Now, we re-write the previous virial identity (4.4) in terms of our new weight function $\Psi$ (instead of using $\Phi$). Then, using lemma 2.3 it is not difficult to see that
\[
\left| \frac{d}{dt} \int_{\mathbb{R}} \Psi \left( \frac{x + \zeta}{\lambda} \right) |\psi|^2 \, dx \right| \lesssim \frac{1}{\lambda} + \frac{\lambda'}{\lambda} \int_{\mathbb{R}} \left| \Phi' \left( \pm \frac{x + \zeta}{\lambda} \right) \right| |\psi|^2 \, dx
\]
\[
+ \frac{\lambda'}{\lambda} \int_{\mathbb{R}} \left| \Phi' \left( \pm \frac{x + \zeta}{\lambda} \right) \right| |\psi|^2 \, dx.
\] (4.17)

Now, recall that, as stated in remark 4.1, there exists a sequence of time $\{t_n\}_{n \in \mathbb{N}}$, satisfying $t_n \to \infty$, such that (4.12) holds. As a consequence, we also have that
\[
\lim_{n \to \infty} \left( \int_{\mathbb{R}} \Psi \left( \frac{x + \zeta}{\lambda} \right) |\psi|^2 \, dx \right) (t_n) = 0.
\]

Therefore, we can integrate both sides of (4.17) in time over the interval $[t, t_n]$, and then take the limit $t_n \to \infty$, what lead us to
\[
\int_{\mathbb{R}} \Psi \left( \frac{x + \zeta}{\lambda} \right) |\psi|^2 \, dx \lesssim \int_{t}^{\infty} \frac{ds}{\lambda(s)} + \int_{t}^{\infty} \frac{\lambda'}{\lambda} \int_{\mathbb{R}} \left| \Phi' \left( \pm \frac{x + \zeta}{\lambda} \right) \right| |\psi|^2 \, dx \, ds
\]
\[
+ \int_{t}^{\infty} \frac{\lambda'}{\lambda} \int_{\mathbb{R}} \left| \Phi' \left( \pm \frac{x + \zeta}{\lambda} \right) \right| \left( \pm \frac{x + \zeta}{\lambda} \right) |\psi|^2 \, dx \, ds.
\]

Finally, by using both time-integrabilities in (4.14), we can take now the limit $t \to \infty$, from where we conclude that
\[
\lim_{t \to \infty} \int_{\mathbb{R}} \Psi \left( \frac{x + \zeta}{\lambda} \right) |\psi|^2 \, dx = 0.
\]

**Remark 4.2.** Notice that the proof of the integrability (and subsequent decay) of the $L^2$-norm of $\psi$ also works for other definitions of $\lambda$ as well as other definitions of $\Phi$ and $\Psi$. For example, in the above analysis we have only used the fact that the scaling $\lambda(t)$ satisfies
\[
\frac{1}{\lambda} \in L^1(\mathbb{R}_+) \quad \text{and} \quad \frac{\lambda'}{\lambda} \notin L^1(\mathbb{R}_+).
\]

In consequence, the proof still holds for any $\lambda$ with such property. As a result, we can take for example $\lambda(t) = ct^p$, for any $p > 2$ and any $c \in \mathbb{R}_+$, and then following the above computations we obtain the desired result.

### 4.3. Time-integrability of the full solution

In this section we seek to show the time integrability of the local energy norm for the remaining terms. Now, in order to make computations simpler, let us break down expression (4.5), so that we can write the cleaner formula
\[
-\frac{d}{dt} E_{\pm}(t) + R_{\pm}(t) = -E_{\pm}(t).
\] (4.18)
More specifically, we define the functionals $\mathcal{E}_\pm$ and $\mathcal{R}_\pm$ given by
\[
\mathcal{E}_\pm := \frac{\gamma'}{\lambda} \int R^\prime \left( \frac{\pm x + \zeta(t)}{\lambda(t)} \right) \left( \omega \psi_x^2 + \frac{\gamma q}{2} |\psi|^2 \right) dx
\]
\[
- \frac{\gamma'}{\lambda} \int R^\prime \left( \frac{\pm x + \zeta(t)}{\lambda(t)} \right) \left( \frac{\pm x + \zeta(t)}{\lambda(t)} \right) \left( \omega \psi_x^2 + \frac{\gamma q}{2} |\psi|^2 \right) dx
\]
\[
+ \frac{\gamma'}{\lambda} \int R^\prime \left( \frac{\pm x + \zeta(t)}{\lambda(t)} \right) \left( \frac{\pm x + \zeta(t)}{\lambda(t)} \right) \left( \omega \psi_x^2 + \frac{\gamma q}{2} |\psi|^2 \right) dx
\]
\[
\times \left( \frac{\beta}{2} \rho^2 + \frac{1}{2} |\eta|^2 + \frac{\gamma}{2} (2\eta - \alpha \rho) |\psi|^2 - \alpha \rho \eta \right) dx,
\]
\[
\mathcal{R}_\pm(t) := \pm 2 \frac{\omega^2}{\lambda} \text{Im} \int R^\prime \left( \frac{\pm x + \zeta(t)}{\lambda(t)} \right) \bar{\psi} \psi_t dx
\]
\[
= \pm 2 \frac{\gamma q \omega}{\lambda} \text{Im} \int R^\prime \left( \frac{\pm x + \zeta(t)}{\lambda(t)} \right) \bar{\psi} \psi_t \psi dx
\]
\[
= \pm \frac{\alpha}{\theta \lambda} \int R^\prime \left( \frac{\pm x + \zeta(t)}{\lambda(t)} \right) (\beta |\rho|^2 + |\eta|^2) dx
\]
\[
= \pm \frac{\beta + \alpha^2}{\theta \lambda} \int R^\prime \left( \frac{\pm x + \zeta(t)}{\lambda(t)} \right) \rho |\psi|^2 dx
\]
\[
= \pm \frac{\gamma (\beta + \alpha^2/2)}{\theta \lambda} \int R^\prime \left( \frac{\pm x + \zeta(t)}{\lambda(t)} \right) \rho |\psi|^2 dx
\]
\[
= \pm \frac{3 \beta + 2 \alpha^2}{\theta \lambda} \int R^\prime \left( \frac{\pm x + \zeta(t)}{\lambda(t)} \right) \rho |\psi|^2 dx
\]
\[
\times \left( \frac{\beta}{2} \rho^2 + \frac{1}{2} |\eta|^2 + \frac{\gamma}{2} (2\eta - \alpha \rho) |\psi|^2 - \alpha \rho \eta \right) dx.
\]

On the other hand, notice that, since $(\psi, \rho, \eta)$ is a solution to system (1.1) belonging to the class $C(\mathbb{R}, H^2 \times H^1 \times H^1)$, then the energy associated to this solution $E(\psi(t), \rho(t), \eta(t))$ is finite. This means that, because the weight $\Phi$ considered in the modified functional $\mathcal{E}$ is bounded, one has
\[
\int_{\mathbb{R}^+} \frac{d}{dt} \mathcal{E}_\pm dt < \infty.
\]

Now, we treat the remaining term.

Let us consider $\Phi \in C^\infty(\mathbb{R})$ to be any non-increasing function such that (4.9) holds, and hence, satisfying also (4.10). Now, we intend to bound term by term the right-hand side of (4.5). In fact, first, since $\Phi'$ is bounded, using Young inequality and Sobolev embedding along
with lemma 2.3, we see that

\[ |R_{\pm,2}| \leq \frac{2\gamma \omega |q|}{\lambda(t)} \left( \|\psi\|_{L^2}^2 + \|\psi_t\|_{L^2}^2 \right) \lesssim \frac{1}{\lambda(t)} \in L^1(\mathbb{R}^+) \]

Also, immediately from proposition (2.3), we obtain

\[ |R_{\pm,3}| \leq \frac{|\alpha|}{\theta \lambda(t)} \int_\mathbb{R} \left| \Phi^\prime \left( \frac{x + \zeta(t)}{\lambda(t)} \right) \right| (\beta |\rho|^2 + |\eta|^2) \, dx \lesssim \frac{1}{\lambda(t)} \in L^1(\mathbb{R}^+) \]

On the other hand, for \( R_{+,4} \), by using Young inequality for products and lemma 2.3 we get

\[ |R_{+,4}| \leq \frac{\beta + \alpha^2}{2\theta} \frac{1}{\lambda(t)} \left( \|\rho(t)\|_{L^2}^2 + \|\eta(t)\|_{L^2}^2 \right) \lesssim \frac{1}{\lambda(t)} \in L^1(\mathbb{R}^+) \]

We point out that all the remaining terms \( R_{\pm,i}, i = 5, \ldots, 8 \), can be treated in the very same fashion as for the previous terms. In fact, from Hölder inequality, Sobolev embedding and then using lemma 2.3 in the resulting right-hand side, we deduce that

\[ \sum_{i=5}^{8} |R_{\pm,i}| \lesssim \frac{1}{\lambda(t)} \left( \|\eta(t)\|_{L^2}^2 + \|\rho(t)\|_{L^2}^2 + \|\psi(t)\|_{H^2}^2 \right) \lesssim \frac{1}{\lambda(t)} \in L^1(\mathbb{R}^+) \]

Finally, to complete the analysis regarding the time-integrability of \( R_{\pm} \), it only remains to consider \( R_{\pm,1} \). In order to do that, we first need to recall one of the main results proven in [5] which give us the polynomial growth of the \( H^2 \)-norm of \( \psi(t) \). In fact, from [5, proposition 1.1] we have that

\[ \|\psi(t)\|_{H^2(\mathbb{R})} \lesssim 1 + |t|^{1+}. \quad (4.19) \]

As a consequence, recalling the explicit form of \( \lambda(t) \) in (4.1), we conclude that

\[ \frac{1}{\lambda(t)}\|\psi(t)\|_{H^2(\mathbb{R})} \in L^1(\mathbb{R}^+) \]

Therefore, by using Hölder inequality as well as lemma 2.3 and (4.19), we infer that

\[ |R_{\pm,1}| \lesssim \frac{1}{\lambda(t)} \|\psi_t(t)\|_{L^2(\mathbb{R})} \|\psi_{xx}(t)\|_{L^2(\mathbb{R})} \in L^1(\mathbb{R}^+) \]

In conclusion, we can integrate over \( \mathbb{R}^+ \) in time both sides of (4.18), from where we obtain

\[ \int_{\mathbb{R}^+} - \mathcal{E}_\pm(t) \, dt < \infty. \quad (4.20) \]

Now, let us break down this expression so that we can analyze the conflicting terms of \( \mathcal{E}_\pm(t) \) without sign. More specifically, we would like to absorbs or discard the part of the expression that does not constitute the weighted energy norm. In fact, in similar fashion as in the proof of proposition 2.3, we have that

\[ \frac{\beta}{2} \rho^2 + \frac{1}{2} \eta^2 - \alpha \rho \eta \geq \frac{\beta - \alpha^2}{4} |\rho|^2 + \frac{\beta}{2(\beta + \alpha^2)} |\eta|^2 > 0, \]
where we used the fact that \( \beta - \alpha^2 > 0 \) and \( \beta > 0 \). Also, we have that

\[
\frac{\gamma}{2} (2\eta - \alpha\rho) |\psi|^2 \leq \frac{\beta - \alpha^2}{2\beta} |\rho|^2 + \frac{\beta}{16(\beta + \alpha^2)} |\eta|^2 + \left( \frac{\gamma^2(\beta + \alpha^2)}{8\beta} \right) |\psi|^4.
\]

Gathering both equations above, we get

\[
\frac{\beta}{2} \rho^2 + \frac{1}{2} \eta^2 - \alpha\rho\eta + \frac{\gamma}{2} (2\eta - \alpha\rho) |\psi|^2 \geq \frac{3(\beta - \alpha^2)}{16} |\rho|^2 + \frac{7\beta}{16(\beta + \alpha^2)} |\eta|^2 - \left( \frac{\gamma^2(\beta + \alpha^2)}{8\beta} \right) |\psi|^4.
\]

Finally, to deal with the remaining uncontrolled terms (involving the \( L^4 \)-norm of \( \psi \)), we make use of (4.11). Indeed, notice that by Sobolev embedding,

\[
\int_{\mathbb{R}} \left[ \frac{\lambda'}{\lambda} \Phi' \left( \frac{\pm x + \zeta(t)}{\lambda(t)} \right) \left( \frac{\pm x + \zeta(t)}{\lambda(t)} \right) - \frac{\zeta'}{\lambda} \Phi' \left( \frac{\pm x + \zeta(t)}{\lambda(t)} \right) \right] |\psi|^4 \, dx
\]

\[
\leq \| \psi \|^2_{L^1(\mathbb{R})} \int_{\mathbb{R}} \left[ \frac{\lambda'}{\lambda} \Phi' \left( \frac{\pm x + \zeta(t)}{\lambda(t)} \right) \left( \frac{\pm x + \zeta(t)}{\lambda(t)} \right) - \frac{\zeta'}{\lambda} \Phi' \left( \frac{\pm x + \zeta(t)}{\lambda(t)} \right) \right] |\psi|^2 \, dx.
\]

Then, thanks to (4.11), we can conclude that

\[
\int_{\mathbb{R}} \left[ \frac{\lambda'}{\lambda} \Phi' \left( \frac{\pm x + \zeta(t)}{\lambda(t)} \right) \left( \frac{\pm x + \zeta(t)}{\lambda(t)} \right) - \frac{\zeta'}{\lambda} \Phi' \left( \frac{\pm x + \zeta(t)}{\lambda(t)} \right) \right] |\psi|^4 \, dx
\]

is integrable in time over \( \mathbb{R}_+ \). Therefore, one can write the following

\[
\int_{\mathbb{R}} \left[ \frac{\lambda'}{\lambda} \Phi' \left( \frac{\pm x + \zeta(t)}{\lambda(t)} \right) \left( \frac{\pm x + \zeta(t)}{\lambda(t)} \right) - \frac{\zeta'}{\lambda} \Phi' \left( \frac{\pm x + \zeta(t)}{\lambda(t)} \right) \right] \omega |\psi|^2 \, dx
\]

\[
+ \int_{\mathbb{R}} \left[ \frac{\lambda'}{\lambda} \Phi' \left( \frac{\pm x + \zeta(t)}{\lambda(t)} \right) \left( \frac{\pm x + \zeta(t)}{\lambda(t)} \right) - \frac{\zeta'}{\lambda} \Phi' \left( \frac{\pm x + \zeta(t)}{\lambda(t)} \right) \right] 3(\beta - \alpha^2) |\rho|^2 \, dx
\]

\[
+ \int_{\mathbb{R}} \left[ \frac{\lambda'}{\lambda} \Phi' \left( \frac{\pm x + \zeta(t)}{\lambda(t)} \right) \left( \frac{\pm x + \zeta(t)}{\lambda(t)} \right) - \frac{\zeta'}{\lambda} \Phi' \left( \frac{\pm x + \zeta(t)}{\lambda(t)} \right) \right] \frac{7\beta}{16(\beta + \alpha^2)} |\eta|^2 \, dx
\]

\[
\leq -\varepsilon \pm + K \int_{\mathbb{R}} \left[ \frac{\lambda'}{\lambda} \Phi' \left( \frac{\pm x + \zeta(t)}{\lambda(t)} \right) \left( \frac{\pm x + \zeta(t)}{\lambda(t)} \right) - \frac{\zeta'}{\lambda} \Phi' \left( \frac{\pm x + \zeta(t)}{\lambda(t)} \right) \right] |\psi|^4 \, dx,
\]

(4.22)
where \( K \) is the absolute value of a constant depending on \( \beta, \alpha, \gamma, q \). This way, we conclude that the right-hand side of the inequality above can be integrated in time over \( \R_+ \). Moreover, since
\[
\Phi'(s)s \geq 0 \quad \text{and} \quad \Phi'(s) \leq 0 \quad \forall \ s \in \R,
\]
we also infer that
\[
\int_{\R_+} \frac{C}{\lambda} \int_{\R} \left| \frac{\pm x + \zeta(t)}{\lambda(t)} \right| \left( (|\psi_1|^2 + |\rho|^2 + |\eta|^2) \right) dx \, dt < \infty.
\]

**Remark 4.3.** We notice that, thanks to the fact that \( \frac{C}{\lambda} \notin L^1(\R_+) \), one infers that there exists a sequence \( \{t_n\} \), with \( \{t_n\} \to \infty \), such that
\[
\left( \int_{\R} \left| \frac{\pm x + \zeta(t_n)}{\lambda(t_n)} \right| \left( (|\psi_1|^2 + |\rho|^2 + |\eta|^2) \right) dx \right)(t_n) \to 0, \quad \text{as} \quad t_n \to \infty. \tag{4.23}
\]

### 4.4. Decay of the full solution

Finally, in this subsection we devote ourselves to prove decay of solutions in the energy space along the curves \( \pm \zeta \). The idea is the same as for the decay of the \( L^2 \)-norm in subsection 4.2. We proceed by taking a convenient weight \( \Psi \) such that \( (4.15) \) holds. Then, we have that \( \text{supp}(\Psi) \subseteq \text{supp}(\Phi') \) and \( (4.16) \) is satisfied. Next, we consider the virial identity \( (4.2) \) with the weight \( \Psi \) instead of \( \Phi \). Thus, taking into account the previous estimations stated in subsection 4.3 along with the pointwise properties \( (4.15) \) and \( (4.16) \), we have that
\[
\frac{d}{dt} \int_{\R} \left| \frac{\pm x + \zeta(t)}{\lambda(t)} \right| \left( (|\psi_1|^2 + |\rho|^2 + |\eta|^2) \right) dx
\]

\[
+ \frac{\gamma q}{2} |\psi|^4 + \frac{\beta}{2} \rho^2 + \frac{1}{2} \eta^2
\]

\[
\leq \frac{C}{\lambda} \int_{\R} \left| \frac{\pm x + \zeta(t)}{\lambda(t)} \right| \left( (|\psi_1|^2 + |\rho|^2 + |\eta|^2) \right) dx
\]

\[
- \frac{\lambda'}{\lambda} \int_{\R} \left| \frac{\pm x + \zeta(t)}{\lambda(t)} \right| \left( \frac{\pm x + \zeta(t)}{\lambda(t)} \right) \left( (|\psi_1|^2 + |\rho|^2 + |\eta|^2) \right) dx \, dt
\]

\[
+ \frac{\zeta'}{\lambda} \int_{\R} \left| \frac{\pm x + \zeta(t)}{\lambda(t)} \right| \left( \frac{\pm x + \zeta(t)}{\lambda(t)} \right) \left( \frac{\pm x + \zeta(t)}{\lambda(t)} \right) \left( (|\psi_1|^2 + |\rho|^2 + |\eta|^2) \right) dx
\]

\[
\times \left( \beta \rho^2 + \frac{1}{2} \eta^2 + \frac{\gamma}{2} (2\eta - \alpha \rho) |\psi|^2 - \alpha \rho \eta \right)
\]

\[
+ \frac{C}{\lambda(t)} \left( 1 + \| \psi(t) \|_{H^1(\R)}^2 \right),
\]

where we recall that \( \lambda^{-1} \) and \( \lambda^{-1} \| \psi \|_{H^1}^2 \) are both time-integrable in \( \R_+ \). Moreover, the whole right-hand side of the last inequality is integrable. Indeed, because \( \Psi \) satisfies \( (4.15) \) and \( (4.16) \)
then (4.22) along with Young inequality implies that
\[
\frac{\lambda'}{\lambda} \psi' \left( \frac{x + \zeta(t)}{\lambda(t)} \right) \left( \omega |\psi_x|^2 + \frac{\gamma q}{2} |\psi|^4 \right) dx
\]
\[- \frac{\lambda'}{\lambda} \psi' \left( \frac{x + \zeta(t)}{\lambda(t)} \right) \left( \frac{x + \zeta(t)}{\lambda(t)} \right) \left( \omega |\psi_x|^2 + \frac{\gamma q}{2} |\psi|^4 \right) dx dr
\]
\[+ \frac{\lambda'}{\lambda} \psi' \left( \frac{x + \zeta(t)}{\lambda(t)} \right) \left( \frac{\beta}{2} \rho^2 + \frac{1}{2} \eta^2 + \frac{\gamma}{2} (2\eta - \alpha \rho)|\psi|^2 - \alpha \rho \eta \right) dx
\]
\[- \frac{\lambda'}{\lambda} \psi' \left( \frac{x + \zeta(t)}{\lambda(t)} \right) \left( \frac{x + \zeta(t)}{\lambda(t)} \right)
\times \left( \frac{\beta}{2} \rho^2 + \frac{1}{2} \eta^2 + \frac{\gamma}{2} (2\eta - \alpha \rho)|\psi|^2 - \alpha \rho \eta \right) dx
\]
\[\leq C \left( -\varphi_{\pm} + K \int \left\{ \frac{\lambda'}{\lambda} \psi' \left( \frac{x + \zeta(t)}{\lambda(t)} \right) \left( \frac{x + \zeta(t)}{\lambda(t)} \right) \right\} \right. |\psi|^4 dx
\]
\[\left. - \frac{\lambda'}{\lambda} \psi' \left( \frac{x + \zeta(t)}{\lambda(t)} \right) \right\} \right. |\psi|^4 dx
\]
Now, recall that we have already shown in the previous section that the right-hand side of the above inequality is time-integrable in \(\mathbb{R}_+\). Therefore, we conclude that there exists a time-integrable function \(g : \mathbb{R} \to \mathbb{R}\) such that we can write
\[
\left| \frac{d}{dr} \int_{\mathbb{R}} \left( \frac{x + \zeta(t)}{\lambda(t)} \right) \left( \omega |\psi_x|^2 + \frac{\gamma q}{2} |\psi|^4 + \frac{\beta}{2} \rho^2 + \frac{1}{2} \eta^2
\right.
\]
\[+ \frac{\gamma}{2} (2\eta - \alpha \rho)|\psi|^2 - \alpha \rho \eta \right) dx \right| \leq g(t).
\]
Consequently, we are entitled to integrate over the time interval \([t, t_n]\) and, because of (4.23), taking \(t_n \to \infty\), we get
\[
\int_{\mathbb{R}} \left( \frac{x + \zeta(t)}{\lambda(t)} \right) \left( \omega |\psi_x|^2 + \frac{\gamma q}{2} |\psi|^4 + \frac{\beta}{2} \rho^2 + \frac{1}{2} \eta^2 + \frac{\gamma}{2} (2\eta - \alpha \rho)|\psi|^2 - \alpha \rho \eta \right) dx
\]
\[\leq \int_{t}^{\infty} g(\tau) d\tau.
\]
Now, notice that, by using inequality (4.21) we can re-write the expression above in terms of the energy norm, as
\[
\int_{\mathbb{R}} \left( \frac{x + \zeta(t)}{\lambda(t)} \right) \left( \omega |\psi_x|^2 + \frac{\gamma q}{2} |\psi|^4 + \frac{\beta}{2} \rho^2 + \frac{7\beta}{16(\beta + \alpha^2)} \eta^2 \right) \left( t, x \right) dx
\]
\[\leq \int_{t}^{\infty} g(\tau) d\tau + \int_{\mathbb{R}} \psi \left( \frac{x + \zeta(t)}{\lambda(t)} \right) |\psi(t)|^4 dx.
\]
Finally, notice that the latter integral involving \(|\psi(t, x)|^4\) converges to zero as \(t \to \infty\). In fact, this is a consequence of the decay of the \(L^2\)-norm (1.5) and lemma (2.3), along with the Gagliardo–Nirenberg inequality, that allows us to bound the \(L^4\)-norm with the \(H^1\)-norm and
$L^2$-norm. Then, to conclude, we take $t \to \infty$ in the latter inequality above, from where we obtain the decay

$$\lim_{t \to \infty} \int_{\mathbb{R}} \Psi \left( \frac{\pm x + \zeta(t)}{\lambda(t)} \right) \left( |\psi_x(t, x)|^2 + \rho^2(t, x) + \eta^2(t, x) \right) (t, x) dx = 0.$$  

The proof is complete.

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