Abstract

Given a fusion 2-category and a suitable module 2-category, the dual tensor 2-category is the associated 2-category of module 2-endofunctors. In order to study the properties of this 2-category, we begin by proving that the relative tensor product of modules over a separable algebra in a fusion 2-category exists. We use this result to construct the Morita 3-category of separable algebras in a fusion 2-category. Then, we explain how module 2-categories form a 3-category, and we prove that, over a fusion 2-category, the 2-adjoint of a left module 2-functor carries a canonical left module structure. We define separable module 2-categories over a fusion 2-category, and prove that the Morita 3-category of separable algebras is equivalent to the 3-category of separable module 2-categories. Finally, we show that the dual tensor 2-category with respect to a separable module 2-category is a multifusion 2-category.

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Introduction

The definition of the dual tensor 1-category to a fusion 1-category with respect to a suitable module 1-category originated in [Ost03], where it was used to define and study Morita equivalences between fusion 1-categories. This notion of Morita equivalence has many classical applications in the theory of fusion 1-categories (for instance, see [ENO05] and [ENO09]), but also in the theory of subfactors (see [Müg03]), and in conformal field theory (see [FRS02]). In particular, it is important to understand when the dual tensor 1-category to a fusion 1-category is again a fusion 1-category. This is guaranteed by requiring that the module 1-categories under consideration be separable in the sense of [DSPS21], restriction which features implicitly in the earlier work of [Müg03] and [FRS02].

In the present article, we examine the categorified version of this last question. That is we study separable module 2-categories, and dual tensor 2-categories to a fusion 2-category with respect to such module 2-categories. The need to examine these objects has multiple origins. Firstly, as recalled above, this constitutes a necessary first steps towards defining and studying Morita equivalences between fusion 2-categories, point to which we will return in [Déc]. Secondly, it was conjectured in [DR18] that fusion 2-categories are the object of a symmetric monoidal 4-category with duals (in the sense of [Lur10]). Proving this conjecture undoubtedly requires a thorough understanding of this 4-category. But, by analogy with the decategorified setting studied in [DSPS21], the 1-morphisms of the aforementioned 4-category are separable bimodule 2-categories. Thirdly, at the moment, the only construction available to produce new fusion 2-categories out of the ones that are already known is the 2-Deligne tensor product introduced in [Déc21a]. Taking the dual tensor 2-category to a fusion 2-category with respect to a separable module 2-category provides a new method to build interesting fusion 2-categories. In a somewhat different direction, we will see that separable module 2-categories are intimately linked to the Morita theory of separable algebras in fusion 2-categories. Further, it was shown in [Déc22b]...
that many familiar objects in the theory of fusion 1-categories such as $G$-graded fusion 1-categories and $G$-crossed fusion 1-categories over a finite group $G$ are separable algebras in certain fusion 2-categories. Thus, as a byproduct of our investigations, we recover the equivariant Morita theory of $G$-graded fusion 1-categories introduced in [GJS21], but also obtain appropriate versions of Morita theory for $G$-crossed fusion 1-categories and various other flavours of fusion 1-categories.

Let us now recall the notion of the dual tensor 1-category to a fusion 1-category in detail. Let $C$ be a multifusion 1-category (over an algebraically closed field of characteristic zero), and let $M$ be a finite semisimple left $C$-module 1-category. Following [Ost03], the dual tensor 1-category to $C$ with respect to $M$ is the rigid monoidal 1-category $\text{End}_C(M)$ of left $C$-module endofunctors of $M$, also denoted by $C^*_M$. This 1-category admits another description. Namely, it was shown in [Ost03] that there exists an algebra $A$ in $C$ such that $M \simeq \text{Mod}_C(A)$. Thus, there is a monoidal equivalence between $\text{End}_C(M)$ and $\text{Bimod}_C(A)^{op}$, the monoidal 1-category of $A$-$A$-bimodules in $C$. Following [DSPS21], let us call the module 1-category $M$ separable if the the algebra $A$ is separable. Provided $M$ is separable, we have that $\text{End}_C(M)$ is a multifusion 1-category, so that the dual of a multifusion 1-category with respect to a separable module 1-category is again a multifusion 1-category. In fact, this property characterizes separable module 1-categories precisely. Let us also remark that the above discussion remains sensible over an field. Further, every finite semisimple module 1-category is separable over a field of characteristic zero, but that this property does not hold over fields of positive characteristic (see [DSPS21]).

Our goal is to categorify the results recalled in the last paragraph. More precisely, for now, let us work over an algebraically closed field of characteristic zero, and recall from [DR18] that a multifusion 2-category is a finite semisimple rigid monoidal 2-category, and that a fusion 2-category is a multifusion 2-category whose monoidal unit is a simple object. We fix a multifusion 2-category $\mathcal{C}$ together with an algebra $A$ in $\mathcal{C}$ that is separable, which implies that the 2-category $\text{Bimod}_\mathcal{C}(A)$ of $A$-$A$-bimodules in $\mathcal{C}$ is finite semisimple (see [Dec22b]).

We now wish to endow $\text{Bimod}_\mathcal{C}(A)$ with a monoidal structure. As expected, the desired monoidal structure is given by the relative tensor product of modules over the separable algebra $A$, which generalizes the relative tensor product of finite semisimple module 1-categories over a fusion 1-category introduced in [ENO9]. We establish more generally the existence of the relative tensor product of modules over a separable algebra in any monoidal 2-category $\mathcal{D}$ that is Karoubi complete in the sense of [GJT19].

Theorem 3.1.6. Let $B$ be a separable algebra in a Karoubi complete monoidal 2-category $\mathcal{D}$. Then, the relative tensor product of any right $B$-module $M$, and left $B$-module $N$ in $\mathcal{D}$ exists.

In fact, elaborating on the above result, we construct the Morita 3-category $\text{Mor}^{sep}(\mathcal{D})$ of separable algebras, bimodules, and their morphisms in $\mathcal{D}$. Related 3-categories have previously been considered in [Hau17] and [GJT19].
We then turn our attention towards the 2-category \( \text{End}_\mathcal{C}(\mathcal{M}) \) of left \( \mathcal{C} \)-module 2-endofunctors on the left \( \mathcal{C} \)-module 2-category \( \mathcal{M} \). We show that this 2-category has a canonical monoidal structure given by composition. More generally, for any fixed monoidal 2-category \( \mathcal{D} \), we will construct a 3-category \( \text{LMod}(\mathcal{D}) \) of left \( \mathcal{D} \)-module 2-categories, left \( \mathcal{D} \)-module 2-functors, left \( \mathcal{D} \)-module 2-natural transformations, and left \( \mathcal{D} \)-module modification, by promoting the 3-category of 2-categories considered in [Gur13]. Further, if \( \mathcal{D} \) is rigid, we will show that if a left \( \mathcal{D} \)-module 2-functor has a 2-adjoint as a plain 2-functor, it has a 2-adjoint as a \( \mathcal{D} \)-module 2-functor. In particular, for any left \( \mathcal{D} \)-module 2-category \( \mathcal{N} \), the monoidal 2-category \( \text{End}_\mathcal{C}(\mathcal{N}) \) is a rigid if every (plain) 2-endofunctor on \( \mathcal{N} \) has a 2-adjoint.

Now, it was shown in [Dec21c] that the 2-category \( \text{Mod}_\mathcal{C}(A) \) of right \( A \)-module in \( \mathcal{C} \) admits a canonical left \( \mathcal{C} \)-module structure. By analogy with the decategorified setting, we wish to compare the monoidal 2-categories \( \text{Bimod}_\mathcal{C}(A) \) and \( \text{End}_\mathcal{C}(\text{Mod}_\mathcal{C}(A)) \). We will do so in more generality by working over an arbitrary field, and letting \( \mathcal{C} \) be a compact semisimple tensor 2-category in the sense of [Dec21b]. Over an algebraically closed field of characteristic zero, this recovers precisely the notion of a multifusion 2-category. Under these hypotheses, we say that a left \( \mathcal{C} \)-module 2-category is separable if it is equivalent to the 2-category of modules over a separable algebra, and write \( \text{LMod}^{sep}(\mathcal{C}) \) for the full sub-3-category of \( \text{LMod}(\mathcal{C}) \) on the separable module 2-categories. We then prove the following twice categorified version of the classical Eilenberg-Watts theorem.

**Theorem 5.1.2.** Let \( \mathcal{C} \) be a compact semisimple 2-category. There is a contravariant linear 3-functor

\[
\text{Mod}_\mathcal{C} : \text{Mor}^{sep}(\mathcal{C}) \to \text{LMod}^{sep}(\mathcal{C})
\]

that sends a separable algebra in \( \mathcal{C} \) to the associated separable left \( \mathcal{C} \)-module 2-category of right modules. Moreover, this 3-functor is an equivalence.

Let us mention that various particular cases of the above theorem have already appeared in the literature (see theorem 3.2.2 of [Dec22a], the finite semisimple case of theorem 4.16 of [GJS21], and corollary 3.1.5 of [Dec21b]). Finally, under a mild assumption on the compact semisimple 2-category tensor \( \mathcal{C} \), we can bring together the various results of this article in order to obtain the following theorem.

**Theorem 5.3.2.** Let \( k \) be a perfect field, and \( A \) a separable algebra in a locally separable compact semisimple tensor 2-category \( \mathcal{C} \). Then,

\[
\text{End}_\mathcal{C}(\text{Mod}_\mathcal{C}(A)) \simeq \text{Bimod}_\mathcal{C}(A)^{\mathcal{C}, op}
\]

is a compact semisimple tensor 2-category.

In particular, given a separable module 2-category \( \mathcal{M} \), we call \( \text{End}_\mathcal{C}(\mathcal{M}) \) the dual tensor 2-category to \( \mathcal{C} \) with respect to \( \mathcal{M} \), which we denote by \( \mathcal{C}^{\mathcal{M}} \). Specializing
the above theorem to the case of algebraically closed fields of characteristic zero, we obtain the following sought-after result, which will be used in the subsequent article \[D\varepsilon\] to set up the Morita theory of fusion 2-categories.

**Corollary 5.3.5.** Let \( k \) be an algebraically closed field of characteristic zero, \( \mathcal{C} \) a multifusion 2-category, and \( \mathcal{M} \) a separable left \( \mathcal{C} \)-module 2-category. Then, \( \mathcal{C}^\ast_\mathcal{M} \), the dual tensor 2-category to \( \mathcal{C} \) with respect to \( \mathcal{M} \), is a multifusion 2-category.

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1 Preliminaries

1.1 Graphical Conventions

The main objects of study of the present article are (weak) 2-categories with additional structures. In this context, it is convenient to use the graphical calculus originally developed in [GS16], and subsequently modified in [Dec21c]. More precisely, we use string diagrams, in which regions correspond to objects, strings to 1-morphisms, and coupons to 2-morphisms. Our diagrams are to be read from top to bottom, which gives the composition of 1-morphisms, and from left to right, which gives the composition of 2-morphisms. We use the symbol 1 to denote the identity 1-morphism on an object, but will omit it from the notations if it is not necessary. To illustrate our conventions, let \( \mathcal{C} \) be a 2-category, and let \( f : A \to B \), and \( g, h : B \to C \) be 1-morphisms. Given a 2-morphism \( \gamma : g \Rightarrow h \), the composite 2-morphism \( \gamma \circ f \) is represented in our graphical calculus by the following diagram:

\[
\begin{array}{c}
\gamma \\
\downarrow \\
\end{array}
\]

Throughout, we will work with a monoidal 2-category \( \mathcal{C} \) in the sense of [SP11]. In particular, we write \( \boxtimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C} \) for the monoidal product of \( \mathcal{C} \) and \( I \) for its monoidal unit. In fact, we will most often assume that \( \mathcal{C} \) is strict cubical, which is not a loss of generality thanks to [Gur13]. More precisely, a strict cubical monoidal 2-category is a strict 2-category \( \mathcal{C} \), such that the monoidal product \( \boxtimes \) is strictly associative and the unit \( I \) is strict. We will therefore systematically omit \( I \) from the notations in this case. In addition, the 2-functor \( \boxtimes \) is strict in either variable separately. In general, the 2-functor \( \boxtimes \) is not strict though. In detail, given pairs of composable 1-morphisms \( f_1, f_2 \) and \( g_1, g_2 \) in \( \mathcal{C} \), the 2-isomorphism

\[
\phi_{(f_2 \boxtimes g_2) \cdot (f_1 \boxtimes g_1)} : (f_2 \boxtimes g_2) \circ (f_1 \boxtimes g_1) \cong (f_2 \circ f_1) \boxtimes (g_2 \circ g_1)
\]
witnessing that □ preserves the composition of 1-morphisms, called the inter-
changer, is not trivial. Nevertheless, the strict cubical hypothesis guarantees
that $\phi^{□}_{(f_2, g_2), (f_1, g_1)}$ is trivial when either $f_2 = 1$ or $g_1 = 1$. Given $f$ and $g$ two
1-morphisms in $C$, the 2-isomorphism

$$\phi^{□}_{(f, 1), (1, g)} : (f □ 1) \circ (1 □ g) \cong (1 □ g) \circ (f □ 1)$$

will be depicted using the diagram below on the left, and its inverse using the
diagram on the right:

![Diagram](image)

In particular, note that we have omitted the symbol □. We will systemically do
so in order to improve the readability of our diagrams.

In section 4, we will also consider 2-functors and 2-natural transformations,
and we now recall from [Dec21c] how to extend the above graphical calculus
to these objects. In detail, given $F : \mathfrak{A} \to \mathfrak{B}$ a (weak) 2-functor, we write
$\phi^F_A : Id_{F(A)} \cong F(Id_A)$ for the 2-isomorphism witnessing that $F$
preserves the identity 1-morphism on the object $A$ in $\mathfrak{A}$, and $\phi^F_{g, f} : F(g) \circ F(f) \cong F(g \circ f)$
for the 2-isomorphism witnessing that $F$ preserves the composition of the two
composable 1-morphisms $g$ and $f$ in $\mathfrak{A}$. These 2-isomorphisms satisfy well-known
compatibility conditions. Now, given any 2-morphism $\upsilon : g \circ f \Rightarrow k \circ h$ in $\mathfrak{A}$, we set:

$$F(\upsilon) = (\phi^F_{h, k})^{-1} \cdot F(\upsilon) \cdot \phi^F_{g, f}.$$

We extend this convention in the obvious way to the image under a 2-functor of
a general 2-morphism, and note that it is well-defined thanks to the coherence
axioms for a 2-functor.

Now, let $F, G : \mathfrak{A} \to \mathfrak{B}$ be two 2-functors, and let $\tau : F \Rightarrow G$ be 2-natural
transformation. This means that, for every object $A$ in $\mathfrak{A}$, we have a 1-morphism
$\tau_A : F(A) \to G(A)$, and for every 1-morphism $f : A \to B$ in $\mathfrak{A}$, we have a
2-isomorphism

$$F(A) \xrightarrow{\tau_A} AB \xleftarrow{\tau_f} G(f) \xrightarrow{\tau_B} G(B),$$

The collection of these 2-isomorphisms has to satisfy the obvious coherence
relations. In our graphical language, we will depict the 2-isomorphism $\tau_f$ using
We review the definition of a 2-condensation monad introduced in [GJF19] as a categorification of the notions of an idempotent (see also [DR18]). More precisely, we recall the unpacked version of this definition given in section 1.1 of [Déc22a].

**Definition 1.2.1.** A 2-condensation monad in a 2-category $\mathcal{C}$ is an object $A$ of $\mathcal{C}$ equipped with a 1-morphism $e : A \to A$ and two 2-morphisms $\mu : e \circ e \Rightarrow e$ and $\delta : e \circ e \Rightarrow e$ such that $\mu$ is associative, $\delta$ is coassociative, the Frobenius relations hold (i.e. $\delta$ is a 2-morphism of $e$-$e$-bimodules) and $\mu \cdot \delta = \text{Id}_e$.

Categorifying the notion of split surjection, [GJF19] gave the definition of a 2-condensation, which we recall below. Further, we also review the definition of the splitting of a 2-condensation monad by a 2-condensation, which is spelled out in [Déc22a].

**Definition 1.2.2.** A 2-condensation in a 2-category $\mathcal{C}$ is a pair of objects $A, B$ in $\mathcal{C}$ together with two 1-morphisms $f : A \to B$ and $g : B \to A$ and two 2-morphisms $\phi : f \circ g \Rightarrow \text{Id}_B$ and $\gamma : \text{Id}_B \Rightarrow f \circ g$ such that $\phi \cdot \gamma = \text{Id}_B$.

**Definition 1.2.3.** Let $\mathcal{C}$ be a 2-category, and $(A, e, \mu^e, \delta^e)$ a 2-condensation monad in $\mathcal{C}$. A splitting of $\mathcal{C}$ is a 2-condensation $(A, B, f, g, \phi, \gamma)$ together with a 2-isomorphism $\theta : g \circ f \cong e$ such that

\[
\mu^e = \theta \cdot (g \circ \phi \circ f) \cdot (\theta^{-1} \circ \theta^{-1}) \quad \text{and} \quad \delta^e = (\theta \circ \theta) \cdot (g \circ \gamma \circ f) \cdot \theta^{-1}.
\]

**Remark 1.2.4.** Let $\mathcal{C}$ be a 2-category whose $\text{Hom}$-categories are idempotent complete. It was shown in theorem 2.3.2 of [GJF19] that the 2-category of splittings of a fixed 2-condensation monads in $\mathcal{C}$ is either empty or a contractible 2-groupoid.

Following [GJF19], we will call a 2-category locally idempotent complete if its $\text{Hom}$-categories are idempotent complete, that is idempotents splits. Further, when working over a fixed field $k$, we will call a $k$-linear 2-category locally Cauchy complete if its $\text{Hom}$-categories are Cauchy complete, that is they have direct sums and idempotents splits.

**Definition 1.2.5.** A locally idempotent complete 2-category is Karoubi complete if every 2-condensation monad splits. A locally Cauchy complete $k$-linear 2-category is Cauchy complete if it is Karoubi complete and has direct sums for objects.

**Remark 1.2.6.** It is always possible to Karoubi complete an arbitrary locally idempotent complete 2-category (see [DR18] and [GJF19]). Further, this process satisfies a precise 3-universal property as explained in [Déc22a].
1.3 Compact Semisimple 2-Categories

Let \( k \) be a field. We now review the definition of a semisimple 2-category, given in [DR18] over algebraically closed field of characteristic zero. We will then recall the notion of a compact semisimple 2-category introduced in [Déc21b].

**Definition 1.3.1.** A \( k \)-linear 2-category is semisimple if it is locally semisimple, has right and left adjoints for 1-morphisms, and is Cauchy complete.

An object \( C \) of a semisimple 2-category \( \mathcal{C} \) is called simple if the identity 1-morphism \( \text{Id}_C \) is a simple object of the 1-category \( \text{End}_\mathcal{C}(C) \). We say that two simple object \( C, D \) of \( \mathcal{C} \) are in the same connected component if there exists a non-zero 1-morphism between them. As explained in section 1 of [Déc21b], this defines an equivalence relation on the set of simple object, whose equivalence classes are called the connected components of \( \mathcal{C} \).

**Definition 1.3.2.** A semisimple \( k \)-linear 2-category is compact if it is locally finite semisimple and has finitely many connected component.

As was shown in [Déc21b], the notion of compact semisimple 2-category is the appropriate categorification of the definition of a finite semisimple 1-category. Namely, following [DR18], a finite semisimple 2-category is a semisimple 2-categories which is locally finite semisimple and has finitely many equivalence classes of simple objects. However, it was proven in [Déc21b] that, over a general field, there does not exist any finite semisimple 2-category, but there always exists compact semisimple 2-categories. Let us note that, over algebraically closed fields or real closed fields, they do show that every compact semisimple 2-category is in fact finite.

Finally, we recall the definitions of a tensor 2-category and of a fusion 2-category, as introduced in [DR18] over algebraically closed fields of characteristic zero. We proceed to give some examples.

**Definition 1.3.3.** A tensor 2-category is a rigid monoidal \( k \)-linear 2-category.

A fusion 2-category is finite semisimple tensor 2-category, whose monoidal unit is simple.

**Example 1.3.4.** A perfect (\( k \)-linear) 1-category is a finite semisimple (\( k \)-linear) 1-category, for which the algebra of endomorphisms of any object is separable. Note that if \( k \) is algebraically closed or has characteristic zero, then every finite semisimple 1-category is perfect. We write \( 2\text{Vect} \) for the 2-category of perfect finite semisimple 1-categories, also called perfect 2-vector spaces. The Deligne tensor product endows \( 2\text{Vect} \) with the structure of a fusion 2-category.

**Example 1.3.5.** Let \( G \) be a finite group. We use \( 2\text{Vect}_G \) to denote the compact semisimple 2-category of \( G \)-graded perfect 2-vector spaces. The convolution product turns \( 2\text{Vect}_G \) into a compact semisimple tensor 2-category. Furthermore, given a 4-cocycle \( \pi \) for \( G \) with coefficients in \( k^\times \), we can form the fusion 2-category \( 2\text{Vect}^\pi_G \) by twisting the structure 2-isomorphisms of \( 2\text{Vect}_G \) using \( \pi \) (see construction 2.1.16 of [DR18] or [Déc21]).
Example 1.3.6. Let us fix $C$ a finite semisimple tensor 1-category (over $k$). Following [DSPS21], we say that a finite semisimple right $C$-module 1-category is separable if it is equivalent to the 1-category of left modules over a separable algebra in $C$. If $k$ has characteristic zero, every finite semisimple $C$-module 1-category is separable. We write $\text{Mod}(C)$ for the compact semisimple 2-category of separable right $C$-module 1-categories. If $B$ is a braided finite semisimple tensor 1-category, then the relative Deligne tensor product over $B$ endows the 2-category $\text{Mod}(B)$ with a rigid monoidal structure, so that $\text{Mod}(B)$ is a compact semisimple tensor 2-category (see [Dec21b]).

Example 1.3.7. Let $G$ be a finite group whose order is coprime to $\text{char}(k)$. We write $B_G$ for the 2-category with one object $\ast$, and $\text{End}_{B_G}(\ast) = G$. We may consider the compact semisimple 2-category $\text{Fun}(B_G, \mathbf{2Vect})$ of (finite perfect) 2-representations of $G$, denoted by $\mathbf{2Rep}(G)$. Said differently, the objects of $\mathbf{2Rep}(G)$ are perfect 2-vector spaces equipped with a $G$-action. The symmetric monoidal structure of $\mathbf{2Vect}$ endows $\mathbf{2Rep}(G)$ with the structure of a symmetric compact semisimple 2-category. More precisely, given $V$ and $W$ two 2-vector spaces with a $G$-action, their monoidal product is given by the Deligne tensor product $V \boxtimes W$ endowed with the diagonal $G$-action. The compact semisimple 2-category $\mathbf{2Rep}(G)$ is fact rigid as can be seen either directly or from lemma 1.3.8 below.

The next lemma gives an alternative description of the symmetric monoidal 2-category $\mathbf{2Rep}(G)$ of perfect 2-representations of a finite group $G$. To this end, let us write $\text{Rep}(G)$ for the symmetric fusion 1-category of finite dimensional representations of $G$.

Lemma 1.3.8. Let $G$ be a finite group whose order is coprime to $\text{char}(k)$. The symmetric monoidal compact semisimple 2-categories $\text{Mod}(\text{Rep}(G))$ and $\mathbf{2Rep}(G)$ are equivalent. In particular, $\mathbf{2Rep}(G)$ is rigid.

Proof. This follows from a slight elaboration on theorem 8.5 of [Gre10]. For completeness, we give a proof using the theory of compact semisimple tensor 2-categories. By construction, the monoidal unit $I$ of $\mathbf{2Rep}(G)$ is $\mathbf{Vect}$, the 1-category of finite $k$-vector spaces, equipped with the trivial $G$-action. Now, note that the underlying 2-category of $\mathbf{2Rep}(G)$ is exactly $\text{Mod}(\mathbf{Vect}_{G^{op}})$. Namely, a $G$-action on a perfect 2-vector is precisely the data of a right $\mathbf{Vect}_{G^{op}}$-module structure. In particular, $\mathbf{2Rep}(G)$ is a connected compact semisimple 2-category. It then follows from theorem 3.14 of [Dec21b] and the fact that $\mathbf{Vect}$ induces a Morita equivalence between the separable finite semisimple tensor 1-categories $\mathbf{Vect}_{G^{op}}$ and $\text{Rep}(G)$ that the 2-functor
\[
\text{Hom}_{\mathbf{2Rep}(G)}(\mathbf{Vect}, -) : \mathbf{2Rep}(G) \rightarrow \text{Mod}(\text{Rep}(G))
\]
is an equivalence of 2-categories. But, we have $\text{End}_{\mathbf{2Rep}(G)}(\mathbf{Vect}) \simeq \text{Rep}(G)$ as symmetric finite semisimple tensor 1-categories. Therefore, it follows from proposition 3.3.4 of [Dec21b] (see also proposition 2.4.7 of [Dec22a]) that the 2-functor $\text{Hom}_{\mathbf{2Rep}(G)}(\mathbf{Vect}, -)$ induces an equivalence of symmetric monoidal 2-categories. This finishes the proof. □
2 Algebras & Modules

We review some key definitions using our graphical calculus. More precisely, we begin recalling the definition of an algebra in a (strict cubical) monoidal 2-category. We go on to review the definitions of right and left modules as well as that of bimodules. We end this section by recollecting the definitions of rigid and separable algebras, and giving plenty of examples in fusion 2-categories.

2.1 Algebras

Throughout, we work with a fixed strict cubical monoidal 2-category $\mathcal{C}$. We begin by recalling the definition of an algebra (also called pseudo-monoid in [DS97]) in $\mathcal{C}$ in the form of definition 1.2.1 of [Déc22b]. For the definition of an algebra in an arbitrary monoidal 2-category expressed using our graphical language, we refer the reader to definition 3.1.1 of [Déc21c].

**Definition 2.1.1.** An algebra in $\mathcal{C}$ consists of:

1. An object $A$ of $\mathcal{C}$;
2. Two 1-morphisms $m : A \square A \to A$ and $i : I \to A$;
3. Three 2-isomorphisms

\[
\begin{align*}
\lambda : & A A A \to A, \\
\mu : & A A \to A, \\
\rho : & A \to A
\end{align*}
\]

satisfying:

a. We have

\[
\begin{align*}
1m & = \lambda (m 1), \\
1m & = \mu (1 m)
\end{align*}
\]

satisfying:

b. We have:

\[
\begin{align*}
1m & = \lambda (m 1), \\
1m & = \mu (1 m)
\end{align*}
\]
We will make use of the following coherence results for algebras derived in section 6.3 of [Hou07]. We will also use the analogue of equation (3) for $\rho$, which follows from lemma 2.2.2 below.

**Lemma 2.1.2.** Given any algebra $A$, the following two equalities hold:

\[
\begin{align*}
\lambda m & = m \lambda, \\
\rho m & = m \rho.
\end{align*}
\]

(3)

(4)

### 2.2 Modules

Let us fix an algebra $A$ in the strict cubical monoidal 2-category $\mathcal{C}$. We now recall the notion of a right $A$-module in $\mathcal{C}$ given in definition 1.2.3 of [Déc22b]. We invite the reader to consult definition 3.2.1 of [Déc21c] for a version of this definition in a general monoidal 2-category.

**Definition 2.2.1.** A right $A$-module in $\mathcal{C}$ consists of:

1. An object $M$ of $\mathcal{C}$;
2. A 1-morphism $n^M : M \square A \rightarrow M$;
3. Two 2-isomorphisms

\[
\begin{align*}
MA A & \xrightarrow{n^{M\lambda}_M} MA \\
\lambda M & \xrightarrow{1_M} \mu^M
\end{align*}
\]

satisfying:

a. We have

\[
\begin{align*}
\lambda m & = m \lambda, \\
\rho m & = m \rho.
\end{align*}
\]

(5)
b. We have:

\[
\begin{array}{c}
\begin{tikzpicture}
\node (m) {$n^M$};
\node (n) [right of=m] {$n^M$};
\node (m1) [above of=m] {$n^M$};
\node (n1) [above of=n] {$n^M$};
\node (l) [left of=m, below of=m] {$lm$};
\node (l1) [left of=n, below of=n] {$lm$};
\node (l2) [right of=m1, below of=m1] {$l1$};
\node (l3) [right of=n1, below of=n1] {$l1$};
\draw[->] (l) to (m);
\draw[->] (l1) to (n);
\draw[->] (l2) to (m1);
\draw[->] (l3) to (n1);
\end{tikzpicture}
\end{array}
\]

For later use, let us recall the following coherence result established in lemma 1.2.8 of [Dec22b].

**Lemma 2.2.2.** Given any right $A$-module $M$, we have the following equality:

\[
\begin{array}{c}
\begin{tikzpicture}
\node (m) {$n^M$};
\node (n) [right of=m] {$n^M$};
\node (m1) [above of=m] {$n^M$};
\node (n1) [above of=n] {$n^M$};
\node (l) [left of=m, below of=m] {$lm$};
\node (l1) [left of=n, below of=n] {$lm$};
\node (l2) [right of=m1, below of=m1] {$l1$};
\node (l3) [right of=n1, below of=n1] {$l1$};
\draw[->] (l) to (m);
\draw[->] (l1) to (n);
\draw[->] (l2) to (m1);
\draw[->] (l3) to (n1);
\end{tikzpicture}
\end{array}
\]

Finally, let us recall definitions 3.2.6 and 3.2.7 of [Dec21c].

**Definition 2.2.3.** Let $M$ and $N$ be two right $A$-modules. A right $A$-module 1-morphism consists of a 1-morphism $f : M \to N$ in $\mathcal{C}$ together with an invertible 2-morphism

\[
\begin{array}{c}
\begin{tikzpicture}
\node (m) {$MA$};
\node (n) [right of=m] {$M$};
\node (l) [left of=m, below of=m] {$f_1$};
\node (l1) [left of=n, below of=n] {$f_1$};
\node (l2) [right of=m, below of=m] {$\psi_f\zeta$};
\node (l3) [right of=n, below of=n] {$\psi_f\zeta$};
\draw[->] (l) to (m);
\draw[->] (l1) to (n);
\draw[->] (l2) to (m);
\draw[->] (l3) to (n);
\end{tikzpicture}
\end{array}
\]

subject to the coherence relations:

a. We have:

\[
\begin{array}{c}
\begin{tikzpicture}
\node (m) {$n^M$};
\node (n) [right of=m] {$n^M$};
\node (l) [left of=m, below of=m] {$n^M$};
\node (l1) [left of=n, below of=n] {$n^M$};
\node (l2) [right of=m, below of=m] {$\psi_f\zeta$};
\node (l3) [right of=n, below of=n] {$\psi_f\zeta$};
\draw[->] (l) to (m);
\draw[->] (l1) to (n);
\draw[->] (l2) to (m);
\draw[->] (l3) to (n);
\end{tikzpicture}
\end{array}
\]

b. We have:

\[
\begin{array}{c}
\begin{tikzpicture}
\node (m) {$n^N$};
\node (n) [right of=m] {$n^N$};
\node (l) [left of=m, below of=m] {$n^N$};
\node (l1) [left of=n, below of=n] {$n^N$};
\node (l2) [right of=m, below of=m] {$\psi_f\zeta$};
\node (l3) [right of=n, below of=n] {$\psi_f\zeta$};
\draw[->] (l) to (m);
\draw[->] (l1) to (n);
\draw[->] (l2) to (m);
\draw[->] (l3) to (n);
\end{tikzpicture}
\end{array}
\]
Definition 2.2.4. Let $M$ and $N$ be two right $A$-modules, and $f, g : M \to M$ two right $A$-module 1-morphisms. A right $A$-module 2-morphism $f \Rightarrow g$ is a 2-morphism $\gamma : f \Rightarrow g$ in $\mathcal{C}$ that satisfies the following equality:

$$f \Rightarrow g = f \Rightarrow g.$$ 

The above structures can be assembled into a 2-category as was proven in lemma 3.2.10 of [Déc21c]. In fact, as we have assumed that $\mathcal{C}$ is strict cubical, this 2-category is strict.

Lemma 2.2.5. Right $A$-modules, right $A$-module 1-morphisms, and right $A$-module 2-morphisms in $\mathcal{C}$ form a strict 2-category, which we denote by $\text{Mod}_\mathcal{C}(A)$.

Let us now recall the definition of left $A$-module in $\mathcal{C}$ from definition A.1.1 of [Déc22b].

Definition 2.2.6. A left $A$-module in $\mathcal{C}$ consists of:

1. An object $M$ of $\mathcal{C}$;
2. A 1-morphism $l^M : A \Box M \to M$;
3. Two 2-isomorphisms

satisfying:

a. We have:

$$\mu^M = \mu^M.$$ (10)

b. We have:

$$\mu^M = \mu^M.$$ (11)
**Definition 2.2.7.** Let $M$ and $N$ be two left $A$-modules. A left $A$-module 1-morphism consists of a 1-morphism $f : M \to N$ in $\mathcal{C}$ together with an invertible 2-morphism

\[
\begin{array}{ccc}
AM & \xrightarrow{\iota^M} & M \\
\downarrow{\chi} & \cong & \downarrow{f} \\
AN & \xrightarrow{\iota^N} & N,
\end{array}
\]

subject to the coherence relations:

a. We have:

\[
\begin{tikzcd}
\text{11f} & m_1 \\
\mu^N & \mu^M \\
\mu^N & \mu^M
\end{tikzcd}
\]

b. We have:

\[
\begin{tikzcd}
\text{id} & 1f \\
\mu^N & \mu^M \\
\mu^N & \mu^M
\end{tikzcd}
\]

**Definition 2.2.8.** Let $M$ and $N$ be two left $A$-modules, and $f,g : M \to M$ two left $A$-module 1-morphisms. A left $A$-module 2-morphism $f \Rightarrow g$ is a 2-morphism $\gamma : f \Rightarrow g$ in $\mathcal{C}$ that satisfies the following equality:

\[
\begin{tikzcd}
\text{id} & 1f \\
\tau & \tau \\
\tau & \tau
\end{tikzcd}
\]

A slight variant of the proof of lemma 3.2.10 of [Déc21c] shows that left $A$-modules and their morphisms can be assembled into a 2-category. As $\mathcal{C}$ is strict cubical, this 2-category is strict.

**Lemma 2.2.9.** Left $A$-modules, left $A$-module 1-morphisms, and left $A$-module 2-morphisms in $\mathcal{C}$ form a strict 2-category, which we denote by $\text{LMod}_\mathcal{C}(A)$. 
2.3 Bimodules

Let \((A, m^A, i^A, \lambda^A, \mu^A, \rho^A)\) and \((B, m^B, i^B, \lambda^B, \mu^B, \rho^B)\) be algebras in the strict cubical monoidal 2-category \(\mathcal{C}\). We now review the notion of an \(A\text{-}B\)-bimodule in \(\mathcal{C}\).

**Definition 2.3.1.** An \(A\text{-}B\)-bimodule in \(\mathcal{C}\) consists of:

1. An object \(P\) of \(\mathcal{C}\);
2. The data \((P, l_P^P, \lambda_P^P, \kappa_P^P)\) of a left \(A\)-module structure on \(P\);
3. The data \((P, n_P^P, \nu_P^P, \rho_P^P)\) of a right \(B\)-module structure on \(P\);
4. A 2-isomorphism

\[
\begin{array}{ccc}
APB & \xrightarrow{l_P^P} & PB \\
\downarrow \beta_P^P & & \downarrow \gamma_P^P \\
AP & \xrightarrow{i_P^P} & M,
\end{array}
\]

satisfying:

a. We have:

\[
\begin{array}{c}
1_{n_P^P} l_P^P \beta_P^P = 1_{n_P^P} m_A^A l_P^P = \gamma_P^P 1_{n_P^P} l_P^P, \quad (14)
\end{array}
\]

b. We have:

\[
\begin{array}{c}
1_{n_B^B} l_P^P \beta_P^P = 1_{n_B^B} m_B^B l_P^P = \gamma_P^P 1_{n_B^B} l_P^P, \quad (15)
\end{array}
\]

**Definition 2.3.2.** Let \(P\) and \(Q\) be two \(A\text{-}B\)-bimodules in \(\mathcal{C}\). An \(A\text{-}B\)-bimodule 1-morphism consists of a 1-morphism \(f : P \to Q\) in \(\mathcal{C}\) together with the data \((f, \xi_f)\) of a left \(A\)-module structure and \((f, \psi_f)\) of a right \(B\)-module structure satisfying:

\[
\begin{array}{c}
1_{n_f^P} l_f^P \beta_f^P = 1_{n_f^P} m_A^A l_f^P = \gamma_f^P 1_{n_f^P} l_f^P, \quad (16)
\end{array}
\]
**Definition 2.3.3.** Let \( P \) and \( Q \) be two \( A\)-\( B \)-bimodules, and \( f, g : P \to Q \) two \( A\)-\( B \)-bimodules 1-morphisms in \( \mathcal{C} \). An \( A\)-\( B \)-bimodule 2-morphism \( f \Rightarrow g \) is a 2-morphism \( \gamma : f \Rightarrow g \) in \( \mathcal{C} \), which is both a left \( A \)-module 2-morphism and a right \( B \)-module 2-morphism.

A slight elaboration on the proof of lemma 3.2.10 of [Dec21c] proves that \( A\)-\( B \)-bimodules in \( \mathcal{C} \) and their morphisms can be assembled into a 2-category. Further, as \( \mathcal{C} \) is strict cubical, this 2-category is in fact strict.

**Lemma 2.3.4.** Given two algebras \( A \) and \( B \) in \( \mathcal{C} \), \( A\)-\( B \)-bimodule 1-morphisms, and \( A\)-\( B \)-bimodule 2-morphisms form a strict 2-category, which we denote by \( \text{Bimod}_\mathcal{C}(A,B) \).

### 2.4 Rigid and Separable Algebras

Let \( \mathcal{C} \) be a strict cubical monoidal 2-category. A rigid algebra in \( \mathcal{C} \) is an algebra \( A \) whose multiplication 1-morphism \( m : A \Box A \to A \) has a right adjoint as an \( A\)-\( A \)-bimodule 1-morphism. In particular, we wish to emphasize that this is a property of an algebra, and not additional structure. Let us also remark that this notion was first introduced in [Gai12], and was first considered in the study of fusion 2-categories in [JFR21]. Before giving examples of this notion in the next section, we review the unpacked version of this definition given in section 2.1 of [Dec22b].

**Definition 2.4.1.** A rigid algebra in \( \mathcal{C} \) consists of:

1. An algebra \( A \) in \( \mathcal{C} \) as in definition 2.1.1;
2. A right adjoint \( m^* : A \to A \Box A \) in \( \mathcal{C} \) to the multiplication map \( m \) with unit \( \eta^m \) and counit \( \epsilon^m \) (depicted below as a cup and a cap);
3. Two 2-isomorphisms

\[
\begin{align*}
AA & \xrightarrow{m^*} A \\
A & \xrightarrow{m^*} A
\end{align*}
\]

\[
\begin{align*}
AAA & \xrightarrow{1m^*} AAA \\
AAA & \xrightarrow{m^*} AA
\end{align*}
\]

satisfying:

a. The 2-morphism \( \psi^l \) endow \( m^* \) with the structure of a left \( A \)-module 1-morphism:

\[
\begin{align*}
\begin{tikzpicture}
  \node (1) at (0,0) {$m^*$};
  \node (2) at (1,0) {$\mu$};
  \node (3) at (2,0) {$m$};
  \node (4) at (3,0) {$m^*$};
  \node (5) at (4,0) {$m_1$};
  \node (6) at (5,0) {$m_1$};

  \draw[->] (1) to (2);
  \draw[->] (2) to (3);
  \draw[->] (3) to (4);
  \draw[->] (4) to (5);
  \draw[->] (5) to (6);
  \draw[->] (1) to (3);
  \draw[->] (2) to (4);
  \draw[->] (3) to (5);
  \draw[->] (4) to (6);
\end{tikzpicture}
\end{align*}
\]
b. The 2-morphism $\psi$ endow $m^*$ with the structure of a right $A$-module 1-morphism:

\begin{align}
(18) \quad m^* \lambda & \quad (m^* \lambda) \\
& = (m^* \lambda) \\
& = (m^* \lambda) \\
& = (m^* \lambda)
\end{align}

\begin{align}
(19) \quad m^* \mu \quad m^* \mu \\
& = (m^* \mu) \\
& = (m^* \mu) \\
& = (m^* \mu)
\end{align}

\begin{align}
(20) \quad \mu \quad m^* \mu \\
& = (\mu) \\
& = (\mu) \\
& = (\mu)
\end{align}

c. The structures of left and right $A$-module 1-morphisms on $m^*$ constructed above are compatible, i.e. they turn $m^*$ into an $A$-$A$-bimodule 1-morphism:

\begin{align}
(21) \quad m \lambda \quad m \lambda \\
& = (m \lambda) \\
& = (m \lambda) \\
& = (m \lambda)
\end{align}

d. The 2-morphism $\epsilon$, depicted below as a cap, is an $A$-$A$-bimodule 2-morphism:

\begin{align}
(22) \quad m \quad m \\
& = (m) \\
& = (m) \\
& = (m)
\end{align}

\begin{align}
(23) \quad m \quad m \\
& = (m) \\
& = (m) \\
& = (m)
\end{align}
e. The 2-morphism $\eta^m$, depicted below as a cup, is an $A$-$A$-bimodule 2-morphism:

\[
\begin{align*}
\begin{array}{c}
\begin{tikzpicture}
\node (m1) at (0,0) {$m$};
\node (m) at (1,0) {$m$};
\node (m*) at (1,-1) {$m^*$};
\node (m1*) at (0,-1) {$m$};
\node (m*1) at (0,1) {$m$};
\path (m1) edge[bend right] node[auto] {$\mu$} (m);
\path (m) edge[bend left] node[auto] {$\nu$} (m1);
\path (m) edge node[auto] {$1m$} (m1);
\path (m*) edge node[auto] {$m^*$} (m1*);
\end{tikzpicture}
\end{array}
\end{align*}
\]

Following [JFR21], a rigid algebra $A$ in $\mathcal{C}$ is called separable if the $A$-$A$-bimodule 2-morphism $\epsilon^m : m \circ m^* \Rightarrow Id_A$ as in the above definition has a section as an $A$-$A$-bimodule 2-morphism. Let us again highlight that being separable is a property of an algebra. We now recall the detailed definition of a separable algebra given in definition 2.1.2 of [Dec22b].

**Definition 2.4.2.** A separable algebra in $\mathcal{C}$ is a rigid algebra $A$ in $\mathcal{C}$ equipped with a 2-morphism $\gamma^m : Id_A \Rightarrow m \circ m^*$ such that:

a. The 2-morphism $\gamma^m$ is a section of $\epsilon^m$, i.e. $\epsilon^m \cdot \gamma^m = Id_A$,

b. The 2-morphism $\gamma^m$ is an $A$-$A$-bimodule 2-morphism:

\[
\begin{align*}
\begin{array}{c}
\begin{tikzpicture}
\node (m1) at (0,0) {$m$};
\node (m) at (1,0) {$m$};
\node (m*) at (1,-1) {$m^*$};
\node (m1*) at (0,-1) {$m$};
\node (m*1) at (0,1) {$m$};
\path (m1) edge[bend right] node[auto] {$\mu$} (m);
\path (m) edge[bend left] node[auto] {$\nu$} (m1);
\path (m) edge node[auto] {$1m$} (m1);
\path (m*) edge node[auto] {$m^*$} (m1*);
\end{tikzpicture}
\end{array}
\end{align*}
\]

Let $k$ be a field, and let assume that $\mathcal{C}$ is a monoidal compact semisimple $k$-linear 2-category. The properties of rigid and separable algebras in $\mathcal{C}$ have been investigated in details in [Dec22b]. In particular, theorem 3.1.6 of [Dec22b] shows that if $A$ is a rigid algebra in $\mathcal{C}$, then $A$ is separable if and only if $\text{Bimod}_\mathcal{C}(A)$ is compact semisimple. Further, if either of these conditions is satisfied, both $\text{Mod}_\mathcal{C}(A)$, and $\text{LMod}_\mathcal{C}(A)$ are compact semisimple 2-categories.
2.5 Examples

Let $k$ be a field. Following [Déc22b], we examine rigid and separable algebras in some of the examples of compact semisimple tensor 2-categories given in section 1.3. We emphasize that these compact semisimple tensor 2-categories are not strict cubical monoidal 2-category, so that we really have to use the fully weak definition of an algebra in a monoidal 2-category.

**Example 2.5.1.** Algebras in $\mathbf{2Vect}$ are precisely perfect monoidal ($k$-linear) 1-categories, and rigid algebras are precisely perfect tensor 1-categories, i.e. perfect monoidal 1-categories whose objects have right and left duals. Corollary 3.1.7 of [Déc22b] shows that a perfect tensor 1-category $C$ yields a separable algebra in $\mathbf{2Vect}$ if and only if its Drinfel’d center $Z(C)$ is a finite semisimple 1-category. If $k$ has characteristic zero, it follows from corollary 2.6.8 of [DSPS21] that finite semisimple tensor 1-categories give all separable algebras in $\mathbf{2Vect}$.

**Example 2.5.2.** Let $G$ be a finite group. Algebras in $\mathbf{2Vect}_G$ are precisely perfect $G$-graded monoidal 1-categories, and rigid algebras are exactly perfect $G$-graded tensor 1-categories. If $k$ has characteristic zero, it is straightforward to check that finite semisimple $G$-graded tensor categories yield all separable algebras in $\mathbf{2Vect}_G$. More generally, given a 4-cocycle for $G$ with coefficients in $k^\times$, algebras in $\mathbf{2Vect}_G^\pi$ should be thought of as perfect $\pi$-twisted $G$-graded monoidal 1-categories. If $H \subseteq G$ is a subgroup, and $\gamma$ is a 3-cocochain for $H$ such that $d\gamma = \pi|_H$, we can consider the algebra $\mathbf{Vect}_H^\pi$ in $\mathbf{2Vect}_G^\pi$. It follows from corollary 3.3.7 of [Déc22b] that $\mathbf{Vect}_H^\pi$ yields a rigid algebra in $\mathbf{2Vect}_G^\pi$, which is separable if and only if the characteristic of $k$ does not divide the order of $H$.

**Example 2.5.3.** Let $B$ be a braided finite semisimple tensor 1-category. In the terminology of [BJS21], a $B$-central monoidal 1-category is a monoidal 1-category $C$ equipped with a braided monoidal functor $F : B \to Z(C)$ to the Drinfel’d center of $C$. Note that this induces in particular a right $B$-module structure on $C$. This notion has also appeared under different names in [DGNO10], [HPT16] and [MPP18]. It follows from proposition 3.2 of [BJS21] that algebras in $\mathbf{Mod}(B)$ correspond exactly to finite semisimple $B$-central monoidal 1-categories, which are separable as right $B$-module 1-categories. By lemma 2.1.4 of [Déc22b], every $B$-central finite semisimple tensor 1-category, which is separable as right $B$-module 1-category, is a rigid algebra in $\mathbf{Mod}(B)$. If $k$ has characteristic zero, one may check that separable algebras in $\mathbf{Mod}(B)$ are precisely $B$-central finite semisimple tensor 1-categories.

**Example 2.5.4.** Let $G$ be a finite group of order coprime to $\text{char}(k)$. Algebras in $\mathbf{2Rep}(G)$ are given exactly by perfect monoidal 1-categories with a $G$-action. Further, rigid algebras $\mathbf{2Rep}(G)$ are precisely perfect tensor 1-categories with a $G$-action, and it follows from lemma 3.3.5 of [Déc22b] that such a rigid algebra is separable if and only if the underlying perfect tensor 1-category is separable.

**Remark 2.5.5.** Lemma 1.3.8 has one particularly noteworthy consequence, which we now explain. As $\mathbf{Mod}(\mathbf{Rep}(G))$ and $\mathbf{2Rep}(G)$ are equivalent as symmetric
monoidal 2-categories, the associated (symmetric monoidal) 2-categories of
gerber, 1-morphisms of gerber and 2-morphisms of gerber are equivalent. In
particular, this induces an equivalence between the full sub-2-categories on the
rigid gerber. If we assume that \( k \) is an algebraically closed field of characteristic
zero, we therefore get an equivalence between the 2-category of multifusion 1-
categories with a \( G \)-action and \( \text{Rep}(G) \)-central multifusion 1-categories. In
the theory of fusion 1-categories, this is a well-known result (see theorem 4.18 of
\cite{DGNO10}). In addition, we also get an equivalence between the (symmetric
monoidal) 2-categories of braided rigid gerber. That is there is an equivalence
between the 2-category of braided multifusion 1-categories with a braided \( G \-
action and braided multifusion 1-categories equipped with a braided functor from \( \text{Rep}(G) \). This is also a classical result (see proposition 4.22 of \cite{DGNO10}).

3 The Relative Tensor Product over Separable
Algebras

Throughout this section, we work with a fixed monoidal 2-category \( \mathcal{C} \), which
we assume to be strict cubical without loss of generality. Our first goal is to
explain the 2-universal property of the relative tensor product of a right and a
left module over an arbitrary algebra \( A \). We then prove that if \( \mathcal{C} \) is Karoubi
complete and \( A \) is separable, then the relative tensor product over \( A \) always
exists. Using this fact, we construct the Morita 3-category of separable algebras,
bimodules, and their morphisms in \( \mathcal{C} \).

3.1 Definition & Existence

Let \( A \) be an algebra in \( \mathcal{C} \). We fix \( M \) a right \( A \)-module in \( \mathcal{C} \), \( N \) a left \( A \)-module
in \( \mathcal{C} \). We begin by defining \( A \)-balanced 1-morphisms and 2-morphisms out of
the pair \((M, N)\).

**Definition 3.1.1.** Let \( C \) be an object of \( \mathcal{C} \). An \( A \)-balanced 1-morphism
\((M, N) \to C\) consists of:

1. A 1-morphism \( f : M \boxtimes N \to C \) in \( \mathcal{C} \);
2. A 2-isomorphisms

\[
\begin{array}{ccc}
MAN & \xrightarrow{n} & MN \\
\downarrow & \cong & \downarrow \beta \\
M \times N & \xrightarrow{f} & C,
\end{array}
\]

satisfying:

a. We have:
b. We have:

\[
\begin{align*}
\begin{tikzpicture}
  \node (lM) at (0,0) {$\psi^M$};
  \node (lN) at (1,0) {$\psi^N$};
  \node (N) at (1,1) {$\phi^N$};
  \node (M) at (0,1) {$\phi^M$};
  \node (f) at (2,0) {$f$};
  \draw (lM) edge (N)
  \draw (lN) edge (M)
  \draw (M) edge (f)
  \draw (N) edge (f)
  \draw (lM) edge[bend left] (f)
  \draw (lN) edge[bend left] (f);
\end{tikzpicture}
\end{align*}
\]

\[=\]

\[
\begin{align*}
\begin{tikzpicture}
  \node (lM) at (0,0) {$\psi^M$};
  \node (lN) at (1,0) {$\psi^N$};
  \node (N) at (1,1) {$\phi^N$};
  \node (M) at (0,1) {$\phi^M$};
  \node (f) at (2,0) {$f$};
  \draw (lM) edge (N)
  \draw (lN) edge (M)
  \draw (M) edge (f)
  \draw (N) edge (f)
  \draw (lM) edge[bend left] (f)
  \draw (lN) edge[bend left] (f);
\end{tikzpicture}
\end{align*}
\]

\[=\]

\[
\begin{align*}
\begin{tikzpicture}
  \node (lM) at (0,0) {$\psi^M$};
  \node (lN) at (1,0) {$\psi^N$};
  \node (N) at (1,1) {$\phi^N$};
  \node (M) at (0,1) {$\phi^M$};
  \node (f) at (2,0) {$f$};
  \draw (lM) edge (N)
  \draw (lN) edge (M)
  \draw (M) edge (f)
  \draw (N) edge (f)
  \draw (lM) edge[bend left] (f)
  \draw (lN) edge[bend left] (f);
\end{tikzpicture}
\end{align*}
\]

\[=\]

\[
\begin{align*}
\begin{tikzpicture}
  \node (lM) at (0,0) {$\psi^M$};
  \node (lN) at (1,0) {$\psi^N$};
  \node (N) at (1,1) {$\phi^N$};
  \node (M) at (0,1) {$\phi^M$};
  \node (f) at (2,0) {$f$};
  \draw (lM) edge (N)
  \draw (lN) edge (M)
  \draw (M) edge (f)
  \draw (N) edge (f)
  \draw (lM) edge[bend left] (f)
  \draw (lN) edge[bend left] (f);
\end{tikzpicture}
\end{align*}
\]

Definition 3.1.2. Let $C$ be an object of $\mathcal{C}$, and $f, g : (M, N) \to C$ be two $A$-balanced 1-morphisms. An $A$-balanced 2-morphism $f \Rightarrow g$ is a 2-morphism $\gamma : f \Rightarrow g$ in $C$ such that

\[
\begin{align*}
\begin{tikzpicture}
  \node (lM) at (0,0) {$\psi^M$};
  \node (lN) at (1,0) {$\psi^N$};
  \node (N) at (1,1) {$\phi^N$};
  \node (M) at (0,1) {$\phi^M$};
  \node (f) at (2,0) {$f$};
  \draw (lM) edge (N)
  \draw (lN) edge (M)
  \draw (M) edge (f)
  \draw (N) edge (f)
  \draw (lM) edge[bend left] (f)
  \draw (lN) edge[bend left] (f);
\end{tikzpicture}
\end{align*}
\]

Definition 3.1.3. The relative tensor product of $M$ and $N$ over $A$, if it exists, is an object $M □_A N$ of $C$ together with an $A$-balanced 1-morphism $t_A : (M, N) \to M □_A N$ satisfying the following 2-universal property:

1. For every $A$-balanced 1-morphism $f : (M, N) \to C$, there exists a 1-morphism $\tilde{f} : M □_A N \to C$ in $C$ and a 2-isomorphism $\xi : \tilde{f} \circ t_A \cong f$.

2. For any 1-morphisms $g, h : M □_A N \to C$ in $C$, and any $A$-balanced 2-morphism $\gamma : g \circ t_A \Rightarrow h \circ t_A$, there exists a unique 2-morphism $\zeta : g \Rightarrow h$ such that $\zeta \circ t_A = \gamma$.

Remark 3.1.4. Observe that, for any object $C$ in $\mathcal{C}$, $A$-balanced 1-morphisms and 2-morphisms out of $(M, N)$ form a 1-category, which we denote by $Bal_A(M, N ; C)$. Furthermore, this assignment is functorial in $M$, $N$, and $C$. Definition 3.1.3 may be rephrased as asserting that precomposition with $t_A$ induces an equivalence of 1-categories

\[Home(\mathcal{C})(M □_A N, C) \simeq Bal_A(M, N ; C),\]

which is natural in the object $C$ in $\mathcal{C}$. Let us also note that it follows readily from the definition that the 2-category of relative tensor products $M □_A N$ is either empty or a contractible 2-groupoid.
Remark 3.1.5. Over an algebraically closed field, with $\mathcal{C} = \mathbf{2Vect}$, and $\mathcal{C}$ a multifusion 1-category, then definition 3.1.3 recovers the relative tensor product over $\mathcal{C}$ as in definition 3.3 of [ENO09]. As $\mathcal{C}$ is automatically separable in this case, theorem 3.1.6 below recovers the well-known statement that the relative tensor product of finite semisimple module 1-categories over $\mathcal{C}$ exists and is a finite semisimple 1-category. Other particular cases of definition 3.1.3 have already appeared as definition 3.2 [DSPS19] and definition 3.3 of [BZBJ18].

Theorem 3.1.6. Let $A$ be a separable algebra in a Karoubi complete 2-category monoidal 2-category $\mathcal{C}$. Then, the relative tensor product of any right $A$-module $M$ and any left $A$-module $N$ in $\mathcal{C}$ exists.

Proof. Let us consider the 2-condensation monad $(M \boxtimes N, e, \mu, \delta)$ in $\mathcal{C}$ given by

$$e := (M \boxtimes l^N) \circ (n^M \boxtimes A \boxtimes N) \circ (M \boxtimes (m^* \circ i) \boxtimes N),$$

and

$$\mu := \mu,$$

$$\delta := \delta.$$
moving the two indicated coupons labeled $1\kappa^N$ and $\nu^{M^{-1}}1$ to the left along the corresponding arrows, which brings us to figure 2. We then use equation (10) on the blue coupons and equation (5) on the green coupons to arrive at figure 3. We go on by moving the coupon labeled $11\kappa^N$ up, as well as the coupons labeled $\nu^{M^{-1}}1111$, $\nu^{M^{-1}}111$, and $1\mu^{-1}111$ to the left along the green arrow. Having arrived at figure 4, we move the coupon labeled $\nu^{M^{-1}}111$ up, and that labelled $\nu^{M^{-1}}111$ down. Further, we also move the left most cap along the red arrow, and in doing so, use equations (23) and (22), which brings us to figure 5. Now, we use equations (21) on the blue coupons and cancel the green coupons to arrive at figure 6. We then move the coupon labeled $1\mu^{-1}1111$ to the right, as well as the coupon labeled $1\mu^{-1}1$ up in order to apply equation (19) to the green coupons, and use equation (17) on the red coupons, bringing us to figure 7. We can then make use of equation (3) on the blue coupons, and cancel the green coupons to arrive at figure 8. Finally, reorganising the diagram along the depicted arrows leads us to figure 9, which represents $\mu \cdot (\mu \circ e)$. Thence, we have established the associativity of $\mu$ as desired. The coassociativity of $\delta$ can be proven similarly.

Let us now move on to proving that $(\mu \circ e) \cdot (e \circ \delta) = \delta \cdot \mu$ using diagrams depicted in section A.1. Figure 10 depicts the left hand-side of this equality. By moving the coupons labeled $1\kappa^N$ and $\nu^M11$ to the right, we arrive at figure 11. Then, applying equation (10) to the blue coupons, and equation (5) to the green ones, we get to contemplate figure 12. We proceed to move some coupons along the depicted arrows, and use equation (19) on the blue coupons, and equation (17) on the green coupons, which brings us to figure 13. Using equation (3) on the blue coupons, and moving the coupons labeled $1\psi^{-1}1$ and $1\kappa^{-1}1$ to the right yields the diagram given in figure 14. Then, we first apply equation (21) to the blue coupons, and then equation (1) on the green coupon together with the coupon labeled $1\mu1$, which was just created. This brings us to figure 15. Finally, using in succession equation (23) on the blue coupons, equation (22) on the green coupons, and equation (3) on the red coupons, leads us to figure 16, which depicts $\delta \cdot \mu$. This proves the desired equality. The equality $(e \circ \mu) \cdot (\delta \circ e) = \delta \cdot \mu$ can be proven using a similar argument.

In order to prove that the relative tensor product of $M$ and $N$ over $A$ exists, we will use the reformulation given in remark 3.1.4. To this end, recall that 2-condensation monads are preserved by all 2-functors, so that applying $\text{Hom}_C(-, C)$ to $(M \Box N, e, \mu, \delta)$ yields a 2-condensation monad on the 1-category $\text{Hom}_C(M \Box N, C)$. In fact, this yields a 2-condensation monad on the 2-functor $\text{Hom}_C(M \Box N, -)$. We claim that $\text{Bal}_A(M, N; C)$ is a splitting this 2-condensation monad. Namely, let $U : \text{Bal}_A(M, N; C) \to \text{Hom}_C(M \Box N, C)$ be the forgetful functor, and $E : \text{Hom}_C(M \Box N, C) \to \text{Bal}_A(M, N; C)$ be the functor given by $f \mapsto f \circ e$, with $A$-balanced structure on the composite $f \circ e$ supplied by the 2-isomorphism $\beta^{\text{Bal}}$ given by
The fact that this defines an $A$-balanced structure can be seen as follows. Let us start with the right hand-side of equation (28) for $\beta foe$. We begin by applying equation (21) after having moved some coupons, then we use equation (2). We continue by appealing to equations (5) and (10), followed by (19) and (17). At last, we can use equations (3) and (7) for $A$ as well as reorganise the string diagram to get to the left hand-side of (28). Equation (29) for $\beta foe$ follows similarly. Now, observe that both $U$ and $A$ are 2-natural in $C$. Further, let us define natural transformations $p : E \circ U \Rightarrow Id$ and $s : Id \Rightarrow E \circ U$ by

$$p_f := \begin{array}{c}
\mu \\
\lambda \\
n^{M1} \\
\xi \\
\mu \\
\lambda \\
\xi \\
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\delta \\
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\text{for every } A\text{-balanced 1-morphism } f : M \square N \to C. \text{ Again, note that } s \text{ and } p \text{ are 2-natural in } C. \text{ Further, we have } p \cdot s = Id, \text{ so that }

\[(Hom_k(M \square N, -), Bal_A(M, N; -), E, U, p, s)\]

is a 2-condensation. It remains to check that this splits the 2-condensation monad on $Hom_k(M \square N, -)$ induced by $(M \square N, e, \mu, \delta)$. To see this, it is enough to prove that for every 1-morphism $f : M \square N \to C$, we have $p_{foe} = f \cdot \mu$ and $s_{foe} = f \cdot \delta$. The first equality follows by first applying equations (5), (17), (19), and (17) again. The second equality follows by first applying equations (3) and (7) for $A$ as well as reorganising the string diagram to get to the left hand-side of (28).
followed by equation (7) for $A$, and then by using successively equations (6), (18), and (1). The second equality is obtained in a similar fashion.

Finally, as $\mathcal{C}$ is Karoubi complete, the 2-condensation monad $(M □N, e, \mu, \delta)$ admits a splitting in $\mathcal{C}$, which we denote by $M □_A N$. Now, the splitting of a 2-condensation monad is preserved by any 2-functor, so that $\text{Hom}_\mathcal{C}(M □_A N, −)$ is also a splitting of the 2-condensation monad on $\text{Hom}_\mathcal{C}(M □N, −)$ induced by $(M □N, e, \mu, \delta)$. But, the 2-category of splittings of a 2-condensation monad is a contractible 2-groupoid, so that we get the desired equivalence.

**Remark 3.1.7.** In the language of [CMV02], the 1-category $\text{Bal}_A(M, N; C)$ is the pseudo-coequalizer for the descent object

$$\text{Hom}_\mathcal{C}(MN, C) \xrightarrow{\text{Hom}_\mathcal{C}(MAN, C)} \text{Hom}_\mathcal{C}(MAAN, C)$$

obtained by applying $\text{Hom}_\mathcal{C}(−, C)$ to the canonical codescent object

$$M □A □A □N \xrightarrow{\text{id}} M □AN \xleftarrow{\text{id}} M □N.$$

Theorem 3.1.6 shows that the 2-functor $\text{Bal}_A(M, N; −)$ is corepresented by $M □_A N$, so that $M □_A N$ is the pseudo-coequalizer of the above codescent object.

Thanks to the definition of the relative tensor product using a 2-universal property, the following result is an immediate consequence of the above theorem.

**Corollary 3.1.8.** If $\mathcal{C}$ is a Karoubi complete 2-category, and $A$ is a separable algebra, the relative tensor product over $A$ defines a 2-functor

$$□ : \text{Mod}_\mathcal{C}(A) \times \text{LMod}_\mathcal{C}(A) \to \mathcal{C}.$$  

**Remark 3.1.9.** For completeness, let us note that if $\mathcal{C}$ is a linear monoidal 2-category, then it follows from the 2-universal property of $□_A$ and the fact that $□$ is a bilinear 2-functor that $□_A$ is a bilinear 2-functor.

### 3.2 The Morita 3-Category

Our goal is now to explain how to construct the Morita 3-category of separable algebras in a Karoubi complete monoidal 2-category. In order to do so, we need to generalize the setup of the previous section to bimodules.

**Definition 3.2.1.** Let $A, B, C$ be algebras in $\mathcal{C}$, and let $M$ be an $A$-$B$-bimodule, $N$ be a $B$-$C$-bimodule, and $P$ be an $A$-$C$-bimodule. A $B$-balanced $A$-$C$-bimodule 1-morphism $(M, N) \to P$ is an $A$-$C$-bimodule 1-morphism $f : M □N \to P$ together with an $A$-$C$-bimodule 2-isomorphism $\beta^f : f \circ (M □N) \cong f \circ (n^M □N)$ providing $f$ with an $A$-balanced structure. A $B$-balanced $A$-$C$-bimodule 2-morphism is an $A$-$C$-bimodule 2-morphism that is also $B$-balanced.

**Proposition 3.2.2.** Let $A, B, C$ be algebras in $\mathcal{C}$, with $B$ separable. Let $M$ be an $A$-$B$-bimodule, and $N$ be a $B$-$C$-bimodule, the relative tensor product $t_B : M □N \to M □B N$ can be endowed with an $A$-$C$-bimodule structure such that it is 2-universal with respect to $B$-balanced $A$-$C$-bimodule morphisms.
Proof. Note that if $M$ and $N$ are bimodules in the proof of theorem 3.1.6 then the 2-condensation monad $(M\Box N, e, \mu, \delta)$ in $\mathcal{C}$ can be upgraded to a 2-condensation monad in $\text{Bimod}_\mathcal{C}(A, C)$. The remainder of the proof can be straightforwardly adapted to accommodate for the bimodule case. The only noteworthy change is that one needs to use the fact that $\text{Bimod}_\mathcal{C}(A, C)$ is Karoubi complete, which follows from the proof of proposition 3.3.4 of [Déc21c] as $\mathcal{C}$ is Karoubi complete. In particular, this constructs a 2-universal $B$-balanced $A$-$C$-bimodule 1-morphism $\tilde{t}_B : M\Box N \to M\Box B N$. But, as splittings of 2-condensation monads are preserved by all 2-functors, the underlying $B$-balanced 1-morphism $\tilde{t}_B : M\Box N \to M\Box B N$ in $\mathcal{C}$ satisfies the 2-universal property of $\Box_B$. This finishes the proof of the proposition.

Remark 3.2.3. Let us sketch an alternative proof of proposition 3.2.2. It follows from the construction of theorem 3.1.6 and the fact that 2-condensation are preserved by all 2-functors that $A\Box t_B : M\Box N \to A\Box(M\Box B N)$ is 2-universal with respect to $B$-balanced 1-morphisms. The 2-universal property of the relative tensor product over $B$ can then be used repeatedly to endow $t_B : M\Box N \to M\Box B N$ with a left $A$-module structure. Similarly, we can construct a right $C$-module structure on $t_B$, which is compatible with the left $A$-module structure. Finally, one can directly check that the $B$-balanced $A$-$C$-bimodules 1-morphism $t_B$ is 2-universal with respect to $B$-balanced $A$-$C$-bimodule morphisms.

Corollary 3.2.4. Let $A, B, C$ be arbitrary algebras in $\mathcal{C}$ with $B$ separable. The relative tensor product over $B$ induces a 2-functor

$$\Box_B : \text{Bimod}_\mathcal{C}(A, B) \times \text{Bimod}_\mathcal{C}(B, C) \to \text{Bimod}_\mathcal{C}(A, C).$$

We now prove a unitality property of the relative tensor product that will play a crucial role later on.

Lemma 3.2.5. Let $A$ and $B$ be arbitrary algebras in $\mathcal{C}$. There is a 2-natural adjoint equivalence

$$l_P^M : A\Box A P \simeq P$$

for any $A$-$B$-bimodule $P$ in $\mathcal{C}$.

Proof. Let $P, Q$ be two $A$-$B$-bimodule in $\mathcal{C}$. Observe that $l_P^M : A\Box P \to P$ is an $A$-balanced $A$-$B$-bimodule 1-morphism via $\beta^{l_P^M} := \kappa^P$. We claim that this 1-morphism satisfies the 2-universal property defining the relative tensor product. Namely, given $f : A\Box P \to Q$ an $A$-balanced $A$-$B$-bimodule 1-morphism, we define $g$ as the composite right $B$-module 1-morphism

$$g : P \xrightarrow{\Box_B} A\Box P \xrightarrow{f} Q.$$
endows $g$ with a compatible left $A$-module structure. Further, it follows from the definitions that the 2-isomorphism $\xi : g \circ l^P \cong f$ given by

$$
\xi := \begin{array}{c}
\lambda^{-1} \\
\beta \end{array}
$$

is an $A$-balanced $A-B$-bimodule 2-morphism as desired. Now, let $g, h : P \to Q$ be two $A-B$-bimodule 1-morphisms, and $\gamma : g \circ l^P \Rightarrow h \circ l^P$ be an $A$-balanced $A-B$-bimodule 2-morphisms, then it is not hard to check that $\zeta := \gamma \circ (i \Box P)$ is an $A-B$-bimodule 2-morphism satisfying $\zeta \circ l^P = \gamma$. This finishes the proof of the claim. Finally, using the 2-universal property of the relative tensor product, one can readily construct the desired adjoint 2-natural equivalence $lM$.

For our purposes, it is also necessary to examine the relative tensor product of multiple bimodules.

**Definition 3.2.6.** Let $A, B, C, D$ be algebras in $\mathcal{C}$, and let $M$ be an $A-B$-bimodule, $N$ be a $B-C$-bimodule, $P$ be a $C-D$-bimodule, and $Q$ an $A-D$-bimodule in $\mathcal{C}$. A $(B,C)$-balanced $A-D$-bimodule 1-morphism $(M,N,P) \to Q$ consists of:

1. An $A-D$-bimodule 1-morphism $f : M \Box N \Box P \to Q$,

2. Two $A-D$-bimodule 2-isomorphisms $\beta_B^f$ and $\beta_C^f$ given by

$$
\begin{array}{c}
MBNP \xrightarrow{n^{M11}_N} MNP \\
\beta_B^f \\
MNP \xrightarrow{f} Q,
\end{array}
\quad
\begin{array}{c}
MBNP \xrightarrow{1n^{N1}_P} MNP \\
\beta_C^f \\
MNP \xrightarrow{f} Q,
\end{array}
$$

satisfying:

a. The 2-isomorphism $\beta_B^f$ endows $f : (M, N \Box P) \to Q$ with a $B$-balanced structure,

b. The 2-isomorphism $\beta_C^f$ endows $f : (M \Box N, P) \to Q$ with a $C$-balanced structure,

c. The 2-isomorphisms $\beta_B^f$ and $\beta_C^f$ commute in the sense that

$$
\begin{array}{c}
\begin{array}{c}
MBNP \xrightarrow{n^{M11}_N} MNP \\
\beta_B^f \\
MNP \xrightarrow{f} Q,
\end{array}
\quad
\begin{array}{c}
MBNP \xrightarrow{1n^{N1}_P} MNP \\
\beta_C^f \\
MNP \xrightarrow{f} Q.
\end{array}
\end{array}
$$

A $(B,C)$-balanced $A-D$-bimodule 2-morphism is an $A-D$-bimodule 2-morphism that is both $B$-balanced and $C$-balanced.

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Lemma 3.2.7. Let $A$, $B$, $C$, $D$ be algebras in $\mathcal{C}$, and let $M$ be an $A$-$B$-bimodule, $N$ be a $B$-$C$-bimodule, and $P$ be a $C$-$D$-bimodule. If $\mathcal{C}$ is Karoubi complete, and $B$, $C$ are separable algebras, then both $M \Box_B (N \Box_C P)$ and $(M \Box_B P) \Box_C$ are 2-universal with respect to $(B, C)$-balanced $A$-$D$-bimodule morphisms. In particular, there exists an adjoints 2-natural equivalence

$$\alpha_{M,N,P}^M : (M \Box_B N) \Box_C P \simeq M \Box_B (N \Box_C P).$$

Proof. Let us show that the $(B, C)$-balanced $A$-$D$-bimodule 1-morphism

$$M \Box N \Box P \xrightarrow{M \Box t_C} M \Box (N \Box_C P) \xrightarrow{f_B} M \Box_B (N \Box_C P)$$

is 2-universal with respect to $(B, C)$-balanced $A$-$D$-bimodule morphisms. In order to prove this, note that the $C$-balanced $A$-$D$-bimodule 1-morphism $M \Box t_C : M \Box N \Box P \to M \Box (N \Box_C P)$ is 2-universal with respect to $C$-balanced $A$-$D$-bimodule morphism.

Now, let $Q$ be an $A$-$D$-bimodule, and let $f : (M, N, P) \to Q$ be a $(B, C)$-balanced $A$-$D$-bimodule 1-morphism. This gives us the solid arrow part of the diagram below

As $f$ is in particular a $C$-balanced $A$-$D$-bimodule 1-morphisms, there exists an $A$-$D$-bimodule 1-morphism $f' : M \Box (N \Box_C P) \to Q$, and a $C$-balanced $A$-$D$-bimodule 2-isomorphism $\xi_C : f' \circ (M \Box t_C) \cong f$. But, thanks to equation (30) and the 2-universal property of $M \Box t_C$, the $B$-balanced structure of $f$ induces a $B$-balanced structure on $f'$. Thus, there exists an $A$-$D$-bimodule 1-morphisms $\tilde{f} : M \Box_B (N \Box_C P) \to Q$, and a $B$-balanced $A$-$D$-bimodule 2-isomorphism $\xi_B : \tilde{f} \circ t_B \cong f'$. It follows from the definitions that the composite $A$-$D$-bimodule 2-isomorphism $\xi_B \cdot (\xi_B \circ (M \Box t_C))$ is $(B, C)$-balanced, so that $\tilde{f}$ is the sought after factorization of $f$.

Finally, let $g, h : M \Box_B (N \Box_C P) \to Q$ be two $A$-$D$-bimodule 1-morphisms, and $\gamma : g \circ t_B \circ (M \Box t_C) \Rightarrow h \circ t_B \circ (M \Box t_C)$ a $(B, C)$-balanced $A$-$D$-bimodule 2-morphism. It follows immediately from the 2-universal property of $M \Box t_C$ that there exists an $A$-$D$-bimodule 2-morphism $\zeta' : g \circ t_B \Rightarrow h \circ t_B$ such that $\zeta \circ (M \Box t_C) = \gamma$. But, using the 2-universal property of $M \Box t_C$ again together with the fact that $\gamma$ is $B$-balanced, we find that $\zeta'$ is necessarily $B$-balanced. Thence, by the 2-universal property of $t_B$, there exists an $A$-$D$-bimodule 2-morphism $\zeta : g \Rightarrow h$ such that $\zeta \circ t_B = \zeta'$. Putting everything together, we find that $\zeta \circ t_B \circ (M \Box t_C) = \gamma$ as desired. This proves that $M \Box_B (N \Box_C P)$ is 2-universal with respect to $(B, C)$-balanced $A$-$D$-bimodule morphisms.
One proceeds analogously to show that the \((B,C)\)-balanced \(A\)-\(D\)-bimodule 1-morphism

\[
M □ N □ P \xrightarrow{\imath_B □ P} (M □ B N) □ P \xrightarrow{\iota_C} (M □ B N) □ C P
\]

is 2-universal with respect to \((B,C)\)-balanced \(A\)-\(D\)-bimodule morphisms. The second part of the statement then follows readily by appealing to the 2-universal property.

We are now ready to explain the main construction of this section.

**Theorem 3.2.8.** Let \(\mathcal{C}\) be a Karoubi complete monoidal 2-category. Separable algebras in \(\mathcal{C}\), bimodules, bimodule 1-morphisms, and bimodule 2-morphisms form a 3-category, which we denote by \(\text{Mor}^{\text{sep}}(\mathcal{C})\).

**Proof.** Let \(A, B, C\), be separable algebras in \(\mathcal{C}\). We set

\[
\text{Hom}_{\text{Mor}^{\text{sep}}(\mathcal{C})}(B, A) := \text{Bimod}(A, B).
\]

Then, the bilinear 2-functor

\[
□_B : \text{Bimod}_\mathcal{C}(A, B) \times \text{Bimod}_\mathcal{C}(B, C) \to \text{Bimod}_\mathcal{C}(A, C)
\]

of corollary 3.2.4 provides us with the necessary composition 2-functor. Further, the identity 1-morphism on the algebra \(A\) is given by the canonical \(A\)-\(A\)-bimodule \(A\). It remains to prove that these operations can be made suitably coherent in the sense of definition 4.1 of [Gur13]. Firstly, note that lemma 3.2.5 provides us with an adjoint 2-natural equivalence \(\iota_M\). Using a similar argument, one can construct a 2-natural equivalence \(\iota_M\) given on the \(A\)-\(B\)-bimodule \(P\) by \(\iota_M : P □ B \simeq P\). Moreover, lemma 3.2.7 provides us with an adjoints 2-natural equivalence \(\alpha_M\) witnessing associativity of the composition of 1-morphisms.

Secondly, we have to supply invertible modifications \(\lambda_M, \mu_M, \rho_M, \pi_M\) between specific composites of \(\iota_M, \iota_M, \alpha_M\). Let us explain how to construct \(\lambda_M\). Let \(M\) be an \(A\)-\(B\)-bimodule and \(N\) a \(B\)-\(C\)-bimodule in \(\mathcal{C}\), and consider the diagram

\[
\begin{array}{ccc}
A □ M □ N & \xrightarrow{\iota_M □ N} & M □ B N \\
(A □ A M) □ B N & \xrightarrow{\iota_M □ B N} & M □ B N \\
A □ A (M □ B N), & \xrightarrow{\iota_M □ B N} & M □ B N
\end{array}
\]

where the three unlabeled arrows are the canonical \((A, B)\)-balanced \(A\)-\(B\)-bimodule 1-morphisms, and the three top triangles are filled by canonical \((A, B)\)-balanced \(A\)-\(B\)-bimodule 2-isomorphisms. Thanks to the 2-universal property of \(A □ M □ N \to (A □ A M) □ B N\), there exists an \(A\)-\(B\)-bimodule 2-isomorphism

\[
\lambda_{M, N} : \iota^M □ B N \simeq \iota^M □ B N \circ \alpha^M_{A □ A M, N}.
\]
Using the 2-universal property again, it is easy to check that these 2-isomorphisms define an invertible modification. The invertible modifications $\mu^M$ and $\rho^M$ are constructed similarly.

It remains to construct the invertible modification $\pi^M$. Given separable algebras $A$, $B$, $C$, $D$, $E$, one defines $(B,C,D)$-balanced $A$-$E$-bimodule morphisms by adapting definition 3.2.6 in the obvious way. Following the proof of lemma 3.2.7, one then shows that for any $A$-$B$-bimodule $M$, $B$-$C$-bimodule $N$, $C$-$D$-bimodule $P$, and $D$-$E$-bimodule $Q$, the canonical $(B,C,D)$-balanced $A$-$E$-bimodule 1-morphisms to the different ways of parenthesising $M \square_B N \square_C P \square_D Q$ are all 2-universal with respect $(B,C,D)$-balanced $A$-$E$-bimodule morphisms.

Analogously to the above arguments, $\pi^M$ is constructed using this 2-universal property.

Finally, one has to check that the equation between these invertible modifications given in definition 4.1 of [Gur13] are satisfied. All of them follow readily from the 2-universal property of the relative tensor product over either three or four algebras.

\[\square\]

Remark 3.2.9. Over a perfect field, the 3-category $\text{Mor}^{sep}(2\text{Vect})$ constructed above is the underlying 3-category of the symmetric monoidal 3-category $\text{TC}^{sep}$ of separable multifusion 1-categories considered in [DSPS21]. Over an algebraically closed field of characteristic zero, and given $B$ a braided fusion 1-category, the 3-category $\text{Mor}^{sep}(\text{Mod}(B))$ corresponds to the $\text{Hom}$-3-category from $B$ to $\text{Vect}$ in the symmetric monoidal 4-category $\text{BrFus}$ of braided fusion 1-categories considered in [BJS21].

Remark 3.2.10. Let $\mathcal{C}$ be a Karoubi complete monoidal 2-category. In [GJF19], the authors outlined the construction of a 3-category $\text{Kar}(\mathcal{C})$ of 3-condensation monads, condensation bimodules, condensation bimodule 1-morphisms, and condensation bimodule 2-morphisms. Using variants of the results proven in section 3 of [GJF19], we expect that one can prove that the 3-category $\text{Mor}^{sep}(\mathcal{C})$ considered above is equivalent to $\text{Kar}(\mathcal{C})$.

Remark 3.2.11. Our proof of theorem 3.2.8 also applies to other setups. Namely, given any monoidal 2-category $\mathcal{C}$ and any set $\mathcal{A}$ of algebras in $\mathcal{C}$ such that for any algebras $A$, $B$, and $C$ in $\mathcal{A}$ the relative tensor product over $B$ of any $A$-$B$-bimodule and $B$-$C$-bimodule exists. The above proof constructs a 3-category $\text{Mor}^{\mathcal{A}}(\mathcal{C})$ of algebras in $\mathcal{A}$, bimodules between them, and their bimodule morphisms. In particular, if every codescent diagram admits a pseudo-coequalizer in $\mathcal{C}$, and that $\square$ commutes with them, then it follows from remark 3.1.7 that the relative tensor product over any algebra in $\mathcal{C}$ exists. In this case, we can therefore consider the 3-category $\text{Mor}(\mathcal{C})$ of all algebras in $\mathcal{C}$, bimodules and their bimodule morphisms. We note that this last example has already been thoroughly examined in [Hau17] in an $\infty$-categorical context.
4 Module 2-Categories

We recall the definitions of a module 2-category, module 2-functor, module 2-natural transformation, and module modification and show that, over a fixed monoidal 2-category, these objects assemble into a 3-category. We then review the definition of a 2-adjunction between two 2-functors, and explain how this concepts interacts with that of a module 2-functor over a rigid monoidal 2-category. Theses results are quite technical in nature, but will play a determining role in the last part of the present article.

4.1 The 3-Category of Module 2-Categories

Let $C$ be a cubical monoidal 2-category. Our goal is to construct a 3-category whose objects are left $C$-module 2-categories in the sense of definition 2.1.3 of [Dec21c]. Now, it follows from proposition 2.2.8 of [Dec21c] that every pair $(C, M)$ consisting of a monoidal 2-category $C$ and a left $C$-module 2-category $M$ is equivalent to a pair in which both $C$ and $M$ are strict cubical (see definition 4.1.1 below). Thus, there is no loss of generality in assuming that $C$ and $M$ are strict cubical. In fact, by remark 2.2.9 of [Dec21c], this strictification procedure holds for any set of module 2-categories.

Definition 4.1.1. Let $M$ be a strict 2-category. A strict cubical left $C$-module 2-category structure on $M$ is a strict cubical 2-functor $\square : C \times M \to M$ such that:

1. The induced 2-functor $I\square(-) : M \to M$ is exactly the identity 2-functor,
2. The two 2-functors $((-)\square(-))\square(-) : C \times C \times M \to M$, and $(-)\square((-)\square(-)) : C \times C \times M \to M$

are equal on the nose.

Notation 4.1.2. It is straightforward to extend the graphical conventions introduced in 1.1 for strict cubical monoidal 2-categories to strict cubical left $C$-module 2-categories. Throughout this section, we use this extended graphical language.

Remark 4.1.3. If $k$ is a field, and $C$ is a monoidal $k$-linear 2-category, then, by definition, $\square : C\square C \to C$ is a bilinear 2-functor. Likewise, if $M$ is a $k$-linear 2-category left $C$-module 2-category, we require that $\square : C \times M \to M$ is a bilinear 2-functor.

Definition 4.1.4. Let $M$ and $N$ be two strict cubical left $C$-module 2-categories. A left $C$-module 2-functor is a (not necessarily strict) 2-functor $F : M \to N$ together with:

1. An adjoint 2-natural equivalence $k^F$ given on $A$ in $C$, and $M$ in $M$ by

\[ k^F_{A,M} : A\square F(M) \to F(A\square M) ; \]
2. Two invertible modifications $\omega^F$, and $\gamma^F$ given on $A, B$ in $\mathcal{C}$ and $M$ in $\mathcal{M}$ by

$$
\begin{align*}
&\begin{tikzcd}
A \square F(B \square M) \\
A \square B \square F(M) \\
F(A \square B \square M),
\end{tikzcd}
\end{align*}
$$

$$
\gamma^F_M : k^F_{f,M} \Rightarrow Id_{F(M)};
$$

Subject to the following relations:

a. For every $A, B, C$ in $\mathcal{C}$, and $M$ in $\mathcal{M}$, the equality

$$
\begin{align*}
&\begin{tikzcd}
A \square B \square C \square F(M) \\
A \square B \square M \\
F(A \square B \square C \square M),
\end{tikzcd}
\end{align*}
$$

holds in $\text{Hom}_\mathcal{M}(A \square B \square C \square F(M), F(A \square B \square C \square M))$,

b. For every $A$ in $\mathcal{C}$, and $M$ in $\mathcal{M}$, the equality

$$
\begin{align*}
&\begin{tikzcd}
A \square C \square F(M) \\
A \square M \\
F(A \square C \square M),
\end{tikzcd}
\end{align*}
$$

holds in $\text{Hom}_\mathcal{M}(A \square C \square F(M), F(A \square C \square M))$;

c. For every $B$ in $\mathcal{C}$, and $M$ in $\mathcal{M}$, the equality

$$
\begin{align*}
&\begin{tikzcd}
I \square B \square F(M) \\
I \square M \\
F(I \square B \square M),
\end{tikzcd}
\end{align*}
$$

holds in $\text{Hom}_\mathcal{M}(I \square B \square F(M), F(B \square M))$.

**Definition 4.1.5.** Let $F, G : \mathcal{M} \to \mathcal{N}$ be two left $\mathcal{C}$-module 2-functors as in definition 4.1.4. A left $\mathcal{C}$-module 2-natural transformation is a 2-natural transformation $\theta : F \Rightarrow G$ equipped with an invertible modification $\Pi^\theta$ given on $A$ in $\mathcal{C}$, and $M$ in $\mathcal{M}$ by

$$
\begin{align*}
AG(M) & \xrightarrow{\theta^A} AF(M) \\
G(AM) & \xleftarrow{\theta^A} F(AM);
\end{align*}
$$

Subject to the following relations:
a. For every $A, B$ in $\mathcal{C}$, and $M$ in $\mathcal{M}$, the equality
\[ = \]
holds in $\text{Hom}_\mathcal{N}(A \Box B \Box F(M), G(A \Box B \Box M))$;

b. For every $M$ in $\mathcal{M}$, the equality
\[ = \]
holds in $\text{Hom}_\mathcal{N}(I \Box F(M), G(M))$.

**Definition 4.1.6.** Let $\theta, \tau : F \Rightarrow G$ be two left $\mathcal{C}$-module 2-natural transformations. A left $\mathcal{C}$-module modification is a modification $\Xi : \theta \Rightarrow \tau$ such that for every $A$ in $\mathcal{C}$, and $M$ in $\mathcal{M}$ the equality
\[ = \]
holds in $\text{Hom}_\mathcal{N}(A \Box F(M), G(A \Box M))$.

Fixing two strict cubical left $\mathcal{C}$-module 2-categories $\mathcal{M}$ and $\mathcal{N}$, it was shown in proposition 2.2.1 of [Dec21c] that left $\mathcal{C}$-module 2-functors, left $\mathcal{C}$-module 2-natural transformations, and left $\mathcal{C}$-module modifications form a 2-category, which we denote by $\text{Fun}_\mathcal{C}(\mathcal{M}, \mathcal{N})$. In particular, given $\theta, \overline{\theta}, \overline{\tau} : F \Rightarrow G$ left $\mathcal{C}$-module 2-natural transformations, and two left $\mathcal{C}$-module modifications $\Xi : \theta \Rightarrow \overline{\theta}$, $Z : \overline{\theta} \Rightarrow \overline{\tau}$, the vertical composite $Z \bullet \Xi$ is a left $\mathcal{C}$-module modification. Further, given two left $\mathcal{C}$-module 2-natural transformations $\theta : F \Rightarrow G$ and $\tau : G \Rightarrow H$, their composite is the 2-natural transformation $\tau \cdot \theta$ equipped with the invertible modification
\[ \Pi^\tau \cdot \theta := (\Pi^\tau \cdot \theta) \cdot (\tau \cdot \Pi^\theta). \]

Thanks to our strictness hypotheses, the above composition of left $\mathcal{C}$-module 2-natural transformations is in fact strictly associative and unital. Thence, $\text{Fun}_\mathcal{C}(\mathcal{M}, \mathcal{N})$ is in fact a strict 2-category. For later use, we need to assemble all of these 2-categories together.

**Theorem 4.1.7.** Let $\mathcal{C}$ be a monoidal 2-category. Left $\mathcal{C}$-module 2-categories, left $\mathcal{C}$-module 2-functors, left $\mathcal{C}$-module 2-natural transformations, and left $\mathcal{C}$-module modifications form a 3-category, which we denote by $\text{LMod}(\mathcal{C})$. 

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Proof. In section 5.1 of [Gur13], the author constructs a 3-category of 2-categories. Our proof is precisely a left $\mathcal{C}$-module version of this argument. In order to do this, it is enough to upgrade the structures defined in section 5.1 of [Gur13] with suitable left $\mathcal{C}$-module actions. Furthermore, thanks to proposition 2.2.8 and remark 2.2.9 of [Déc21c], we may assume without loss of generality that $\mathcal{C}$ and every left $\mathcal{C}$-module 2-category is strict cubical.

Let $\mathcal{M}, \mathcal{N}$, and $\mathcal{P}$ be strict cubical left $\mathcal{C}$-module 2-categories. We begin by constructing the 2-functor

$$\circ : \text{Fun}_\mathcal{C}(\mathcal{N}, \mathcal{P}) \times \text{Fun}_\mathcal{C}(\mathcal{M}, \mathcal{N}) \to \text{Fun}_\mathcal{C}(\mathcal{M}, \mathcal{P})$$

providing us with the composition of left $\mathcal{C}$-module 2-functors. Given two left $\mathcal{C}$-module 2-functors $F : \mathcal{M} \to \mathcal{N}$ and $G : \mathcal{N} \to \mathcal{P}$, we endow their composite $G \circ F$ with a left $\mathcal{C}$-module structure using the following assignments. We define the adjoint 2-natural equivalence $k_{G \circ F}$ by

$$k_{G \circ F}^{A, M} := G(k^F_{A, M}) \circ k^G_{A, F(M)},$$

for every $A$ in $\mathcal{C}$ and $M$ in $\mathcal{M}$, and the two invertible modifications $\omega_{G \circ F}$, and $\gamma_{G \circ F}$ by

$$\omega_{G \circ F} := \begin{array}{c}
1k^G \\
1G(k^F) \\
G(k^F)
\end{array} \quad \begin{array}{c}
k^G \\
G(k^F) \\
G(k^F)
\end{array} \quad \begin{array}{c}
G(\omega) \\
G(\omega')
\end{array} \quad \begin{array}{c}
G(\omega') \\
G(\omega')
\end{array},$$

$$\gamma_{G \circ F} := \begin{array}{c}
k^G \\
G(k^F)
\end{array} \quad \begin{array}{c}
G(\gamma) \\
G(\gamma')
\end{array}.$$

It is not hard to show that the above data satisfies the axioms of definition 4.1.4.

Then, given a left $\mathcal{C}$-module 2-natural transformation $\theta : F_1 \Rightarrow F_2$ between two left $\mathcal{C}$-module 2-functors $F_1, F_2 : \mathcal{M} \to \mathcal{N}$, we endow the 2-natural transformation $G \circ \theta$ with a left $\mathcal{C}$-module structure by setting

$$\Pi^{G \circ \theta} := \begin{array}{c}
1G(\theta) \\
1G(k^F) \\
G(k^F)
\end{array} \quad \begin{array}{c}
k^G \\
G(k^F) \\
G(\theta)
\end{array} \quad \begin{array}{c}
G(\Pi') \\
G(\Pi')
\end{array} \quad \begin{array}{c}
G(\Pi') \\
G(\Pi')
\end{array}.$$

Likewise, given a left $\mathcal{C}$-module 2-natural transformation $\tau : G_1 \Rightarrow G_2$ between two left $\mathcal{C}$-module 2-functors $G_1, G_2 : \mathcal{N} \to \mathcal{P}$, we may similarly define a left $\mathcal{C}$-module structure on the 2-natural transformation $\tau \circ F$ by
Now, recall from the proof of proposition 5.1 of [Gur13] that \( \tau \circ \theta = (G_2 \circ \theta) \cdot (\tau \circ F_1) \), so that the 2-natural transformation \( \tau \circ \theta \) inherits a \( \mathbf{C} \)-module structure. These assignments can be straightforwardly extended to left \( \mathbf{C} \)-module modifications, so that we obtain a functor

\[
Nat_{\mathbf{C}}(G_1, G_2) \times Nat_{\mathbf{C}}(F_1, F_2) \to Nat_{\mathbf{C}}(G_1 \circ F_1, G_2 \circ F_2)
\]

between the 1-categories of left \( \mathbf{C} \)-module 2-natural transformations. The additional structure constraints needed to define a 2-functor are the invertible modifications given in proposition 5.1 of [Gur13], and one checks easily that they respect the relevant left \( \mathbf{C} \)-module structures defined above. Thus, the unit on \( \mathbf{M} \) for the composition of left \( \mathbf{C} \)-module 2-functors is given by the identity 2-functor \( \text{Id} : \mathbf{M} \to \mathbf{M} \) with its canonical left \( \mathbf{C} \)-module structure.

Proposition 5.3 of [Gur13] defines an adjoint 2-natural equivalence \( \alpha \) witnessing the associativity of the composition of (plain) 2-functors. Now, let \( \mathbf{M}, \mathbf{N}, \mathbf{P}, \text{ and } \mathbf{Q} \) be strict cubical left \( \mathbf{C} \)-module 2-categories, and let \( F : \mathbf{M} \to \mathbf{N}, G : \mathbf{N} \to \mathbf{P}, \text{ and } H : \mathbf{P} \to \mathbf{Q} \) be left \( \mathbf{C} \)-module 2-functors. It follows from proposition 5.3 of [Gur13] that \( \alpha_{H,G,F} : (H \circ G) \circ F \simeq H \circ (G \circ F) \) is the identity 2-natural transformation. Further, the left \( \mathbf{C} \)-module structures of \( (H \circ G) \circ F \) and \( H \circ (G \circ F) \) are equal, so that we can upgrade \( \alpha_{H,G,F} \) to a left \( \mathbf{C} \)-module adjoint 2-natural equivalence using the identity modification. The collection of these assignments promote \( \alpha \) to an adjoint 2-natural equivalence witnessing the associativity of the composition of left \( \mathbf{C} \)-module 2-functors.

Analogously, the adjoint 2-natural equivalences \( l \) and \( r \) constructed in proposition 5.5 of [Gur13], witnessing that composition of 2-functors is unital, can be promoted to left \( \mathbf{C} \)-module adjoint 2-natural equivalences. Namely, as we are working with strict 2-categories, these adjoint 2-natural equivalences are in fact both given by the identity 2-natural adjoint equivalence. Thus, given a \( \mathbf{C} \)-module 2-functor \( F : \mathbf{M} \to \mathbf{N} \) between strict cubical left \( \mathbf{C} \)-module 2-categories, the 2-natural transformations \( l_F \) and \( r_F \) can canonically be upgraded to left \( \mathbf{C} \)-module adjoint 2-natural equivalences. With these additional pieces of data, \( l \) and \( r \) define adjoint 2-natural equivalence witnessing the unitality of the composition of left \( \mathbf{C} \)-module 2-functors. The proof is then completed by checking that the invertible modification \( \pi, \mu, \lambda, \text{ and } \rho \) given in proposition 5.6 of [Gur13] are compatible with the left \( \mathbf{C} \)-module structures we have defined. This is immediate as it follows from our strictness assumptions that \( \pi, \mu, \lambda, \text{ and } \rho \) are all identity modifications.

\[
\text{Remark 4.1.8.} \text{ It follows immediately from our proof of theorem } 4.1.7 \text{ that there is a forgetful 3-functor } \text{LMod}(\mathbf{C}) \to \text{2Cat} \text{ to the 3-category of 2-categories.}
\]
Corollary 4.1.9. Let $\mathcal{M}$ be a left $\mathcal{C}$-module 2-category. Then, $\text{End}_{\mathcal{C}}(\mathcal{M})$ is a monoidal 2-category. Further, given $\mathcal{M}$ be any left $\mathcal{C}$-module 2-category, the 2-category $\text{Fun}_{\mathcal{C}}(\mathcal{M}, \mathcal{N})$ is an $\text{End}_{\mathcal{C}}(\mathcal{M})\text{-}\text{End}_{\mathcal{C}}(\mathcal{N})$-bimodule 2-category.

We end this first section on module 2-categories with the following proposition, which will constitute a key ingredient in our study of the Morita theory of fusion 2-categories.

Proposition 4.1.10. Let $\mathcal{M}$ be a left $\mathcal{C}$-module 2-category. The action of $\text{End}_{\mathcal{C}}(\mathcal{M})$ on $\mathcal{M}$ given by evaluation defines a left $\text{End}_{\mathcal{C}}(\mathcal{M})$-module structure on $\mathcal{M}$. Further, this structure is compatible with the left $\mathcal{C}$-module one, so that $\mathcal{M}$ has a left $\text{End}_{\mathcal{C}}(\mathcal{M}) \times \mathcal{C}$-module 2-category.

Proof. One may directly check that evaluation of 2-functors $\text{End}_{\mathcal{C}}(\mathcal{M}) \times \mathcal{M} \to \mathcal{M}$ provides $\mathcal{M}$ with a left $\text{End}_{\mathcal{C}}(\mathcal{M})$-module structure. By definition, this left $\text{End}_{\mathcal{C}}(\mathcal{M})$-module structure on $\mathcal{M}$ is compatible with the left $\mathcal{C}$-module structure, so that $\mathcal{M}$ has a left $\text{End}_{\mathcal{C}}(\mathcal{M}) \times \mathcal{C}$-module structure.

4.2 Module 2-Functors and 2-Adjunctions

The goal of this section is to study the interaction between the notion of a module 2-functor recalled above, and that of a 2-adjunction between 2-functors, which we now recall. Let $\mathcal{M}$ and $\mathcal{N}$ be two 2-categories, and $F : \mathcal{M} \to \mathcal{N}$ and $G : \mathcal{N} \to \mathcal{M}$ be two 2-functors.

Definition 4.2.1. A 2-adjunction between $F$ and $G$ consists of two 2-natural transformations $u^F$, called the unit, and $c^F$, called the counit, given on $M$ in $\mathcal{M}$ and $N$ in $\mathcal{N}$ by

$$u^F_M : M \to G(F(M)), \quad c^F_N : F(G(N)) \to N,$$

together with invertible modifications $\Phi^F$ and $\Psi^F$, called triangulators, given on $M$ in $\mathcal{M}$ and $N$ in $\mathcal{N}$ by

$$\Phi^F_M : c^F_{F(M)} \circ F(u^F_M) \cong \text{Id}_{F(M)}, \quad \Psi^F_N : G(c^F_N) \circ u^F_G \cong \text{Id}_{G(N)}.$$

We also say that $F$ is a left 2-adjoint to $G$, or that $G$ is a right 2-adjoint to $F$.

Let $\mathcal{C}$ be a rigid monoidal 2-category, and assume that both $\mathcal{M}$ and $\mathcal{N}$ are left $\mathcal{C}$-module 2-categories. The categorified version of corollary 2.13 of [DSPS19] holds, as we show in the next two propositions. In fact, our proof also establishes the categorifications of their lemmas 2.10 and 2.11.

Proposition 4.2.2. Let $\mathcal{C}$ be a rigid monoidal 2-category, and let $F : \mathcal{M} \to \mathcal{N}$ be a left $\mathcal{C}$-module 2-functor between left $\mathcal{C}$-module 2-categories. If $F$ has a right 2-adjoint $G$, then $G$ is canonically a left $\mathcal{C}$-module 2-functor.
Proof. Thank to proposition 2.2.8 and remark 2.2.9 of [Déc21c], we may assume that \( C \) is strict cubical, and that both \( \mathfrak{M} \) and \( \mathfrak{N} \) are strict cubical left \( C \)-module 2-categories. We begin by proving that \( G \) can be endowed with a lax left \( C \)-module structure. Given \( A \) in \( C \), and \( M \) in \( \mathfrak{M} \), we let the 2-natural transformation \( k^G \) be given by

\[
k^G_{A,M} : A \Box G(M) \xrightarrow{u^F} G(F(A \Box G(M))) \xrightarrow{G((k^F)^*)} G(A \Box F(G(M))) \xrightarrow{G(1c^F)} G(A \Box M),
\]

where \((k^F)^*\) is the pseudo-inverse of \( k^F \) provided in the data of a left \( C \)-module 2-functor. The invertible modifications \( \omega^G \) and \( \gamma^G \) are given by

\[
\omega^G := \frac{u^F}{G(1c^F)} \quad \quad \quad \gamma^G := \frac{G((k^F)^*)}{G(1c^F)}.
\]

Using the axioms of definition 4.1.4 for \( F \), it is easy to check that \( \omega^G \) and \( \gamma^G \) satisfy the axioms of 4.1.4.

It remains to show that \( k^G \) can be upgraded to an adjoint 2-natural equivalence. As every 2-natural equivalence can be upgraded to an adjoint 2-natural equivalence (see section 1 of [Gur12]), it is enough to exhibit for every \( A \) in \( C \) and \( M \) in \( \mathfrak{M} \), a pseudo-inverse \((k^G)_{A,M}^\bullet \) for the 1-morphism \( k^G_{A,M} \). Let \( \sharp A \) be a left dual for \( A \) in \( C \) with unit 1-morphism \( i_A : I \to A \Box \sharp A \), counit 1-morphism \( e_A : \sharp A \Box A \to I \) and 2-isomorphisms \( C_A : (e_A \Box \sharp A) \circ (\sharp A \Box i_A) \Rightarrow Id_{\sharp A} \), and \( D_A : Id_{\sharp A} \Rightarrow (A \Box e_A) \circ (i_A \Box A) \). We define

\[
(k^G)_{A,M}^\bullet : G(A \Box M) \xrightarrow{i_A^1} A \Box \sharp A \Box G(A \Box M) \xrightarrow{1u^F} A \Box G(\sharp A \Box F(A \Box M)) \xrightarrow{1G((k^F)^*)} A \Box G(\sharp A \Box FG(A \Box M)) \xrightarrow{1G(1c^F)} A \Box G(\sharp A \Box A \Box M) \xrightarrow{1G(e_A^1)} A \Box G(M),
\]

where \((k^F)^*\) denotes the canonical pseudo-inverse of \( k^F \) supplied by the definition of a module 2-functor. Let us denote by \((\omega^{F^{-1}})^*\) and \((\gamma^{F^{-1}})^*\) the 2-isomorphisms given by
where the cups and the caps denote the unit and counit 2-isomorphisms witnessing that \( k^F \) and \((k^F)^\bullet\) form an adjoint 2-natural equivalence. Finally, the two 2-isomorphisms

witness that \((k^G)^\bullet_{A,M}\) is a pseudo-inverse for \(k^G_{A,M}\) as desired. □

**Corollary 4.2.3.** Let \( F : \mathcal{M} \to \mathcal{M} \) be a left \( \mathcal{C} \)-module 2-functor. If \( F \) has a right 2-adjoint \( G \), then the left \( \mathcal{C} \)-module 2-functor \( G \) is a right dual for \( G \) in \( \text{End}_C(\mathcal{M}) \).

**Proof.** It is enough to upgrade \( u^F \) and \( c^F \) to left \( \mathcal{C} \)-module 2-natural transformations, and show that \( \Phi^F \) and \( \Psi^F \) define invertible left \( \mathcal{C} \)-module modifications.
For the first part, we endow $u^F$ and $c^F$ with left $\mathcal{C}$-module structures using the modifications $\Pi_{u^F}$ and $\Pi_{c^F}$ specified by

\[
\Pi_{u^F} := \begin{array}{c}
\Pi_{u^F} \\
G((k^F)^*) \\
G(\iota^F) \\
G(k^F) \\
\end{array}
\]

\[
\Pi_{c^F} := \begin{array}{c}
\Pi_{c^F} \\
F(u^F) \\
FG((k^F)^*) \\
FG(\iota^F) \\
\end{array}
\]

It is easy to check that $\Pi_{u^F}$ and $\Pi_{c^F}$ satisfy the axioms of definition 4.1.5. Finally, using naturality together with axioms a and b of definition 4.1.4 for $G$, one can readily check that $\Phi^F$ and $\Psi^F$ are compatible with the left $\mathcal{C}$-module structure on $u^F$ and $c^F$ defined above, which finishes the proof of the result.

Proposition 4.2.2 above shows that the right 2-adjoint of a left $\mathcal{C}$-module 2-functor has a canonical left $\mathcal{C}$-module structure. This result has a clear analogue for left adjoints, which we spell out below.

**Proposition 4.2.4.** Let $G : \mathcal{M} \to \mathcal{M}$ be a left $\mathcal{C}$-module 2-functor. If $G$ has a left 2-adjoint $F$, then $F$ is canonically a left $\mathcal{C}$-module 2-functor.

**Proof.** We write $\mathcal{C}^{\text{op}}$ for the monoidal 2-category obtained from $\mathcal{C}$ by reversing the direction of the 1-morphisms. Note that if $\mathcal{C}$ is strict cubical, then $\mathcal{C}^{\text{op}}$ is strict opcubical. It is clear that $\mathcal{M}^{\text{op}}$ and $\mathcal{N}^{\text{op}}$ are left $\mathcal{C}^{\text{op}}$-module 2-categories. Further, $G^{\text{op}} : \mathcal{M}^{\text{op}} \to \mathcal{N}^{\text{op}}$ is a left $\mathcal{C}^{\text{op}}$-module 2-functor, and our hypothesis guarantees that $G^{\text{op}}$ has a right adjoint given by $F^{\text{op}}$. Then, by proposition 4.2.2, $F^{\text{op}}$ is a left $\mathcal{C}^{\text{op}}$-module 2-functor, so that $F$ is a left $\mathcal{C}$-module 2-functor as desired.

Similarly, one derives the following corollary.

**Corollary 4.2.5.** Let $G : \mathcal{M} \to \mathcal{M}$ be a left $\mathcal{C}$-module 2-functor. If $G$ has a left 2-adjoint $F$, then the left $\mathcal{C}$-module 2-functor $F$ is a left dual for $G$ in $\text{End}_\mathcal{C}(\mathcal{M})$.  

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5 Dual Tensor 2-Categories

In general, it is difficult to work with arbitrary compact semisimple module 2-categories over a fixed compact semisimple tensor 2-category $\mathcal{C}$. Motivated by the decategorified situation studied in [DSPS21], we therefore restrict our attention to a particularly nice class of compact semisimple module 2-categories called separable module 2-categories. We prove that the 3-category of separable algebras in $\mathcal{C}$ is equivalent to the 3-category of separable left $\mathcal{C}$-module 2-categories. Under a mild assumption on $\mathcal{C}$, we then show that the monoidal 2-category of bimodules over a separable algebra in $\mathcal{C}$ is a compact semisimple tensor 2-category. This allows us to define the dual tensor 2-category of $\mathcal{C}$ with respect to a separable module 2-category. Throughout, we work over a fixed field $k$, meaning that all categories and functors under consideration are $k$-linear.

5.1 Separable Module 2-Categories

Let us fix $\mathcal{C}$ a compact semisimple tensor 2-category, and $\mathcal{M}$ a compact semisimple left $\mathcal{C}$-module 2-category.

**Definition 5.1.1.** The compact semisimple left $\mathcal{C}$-module 2-category $\mathcal{M}$ is called separable if there exists a separable algebra $A$ in $\mathcal{C}$ such that

$$\mathcal{M} \cong \text{Mod}_\mathcal{C}(A)$$



as left $\mathcal{C}$-module 2-categories.

In theorem 4.1.7 we have proven that left $\mathcal{C}$-module 2-categories form a 3-category, which we denote by $\text{LMod}(\mathcal{C})$. We will write $\text{LMod}^{\text{sep}}(\mathcal{C})$ for the full sub-3-category whose objects are the separable module 2-categories. We are now ready to state our main theorem, which was conjectured in remark 5.4.9 of [Déc21c]. Let us also mention that when $k$ is algebraically closed of characteristic zero and $\mathcal{C} = \mathbf{2Vect}$, then we recover the main result of [Déc22a] as every multifusion 1-category is separable. With $k$ algebraically closed of characteristic zero and $\mathcal{C} = \mathbf{2Vect}_G$ for some finite group $G$, we also obtain the $G$-graded multifusion 1-category case of theorem 4.16 of [GJS21]. Further, if $k$ is perfect and $\mathcal{C} = \mathbf{2Vect}$, the theorem below also recovers corollary 3.1.5 of [Déc21b] thanks to proposition 2.5.10 of [DSPS21].

**Theorem 5.1.2.** Let $\mathcal{C}$ be a compact semisimple 2-category. There is a contravariant linear 3-functor

$$\text{Mod}_\mathcal{C} : \text{Mor}^{\text{sep}}(\mathcal{C}) \to \text{LMod}^{\text{sep}}(\mathcal{C})$$

that sends a separable algebra in $\mathcal{C}$ to the associated separable left $\mathcal{C}$-module 2-category of right modules. Moreover, this 3-functor is an equivalence.

**Proof.** Without loss of generality, we may assume that $\mathcal{C}$ is strict cubical. The monoidal unit $I$ of $\mathcal{C}$ is canonically a separable algebra in $\mathcal{C}$. Thanks to theorem 3.2.8 this yields a contravariant linear 3-functor

$$\text{Hom}_{\text{Mor}^{\text{sep}}(\mathcal{C})}(-, I) : \text{Mor}^{\text{sep}}(\mathcal{C}) \to \text{2Cat}_k$$

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to the 3-category of \( k \)-linear 2-categories. But, \( \textbf{End}_{\text{Mor}^{sep}(\mathcal{C})}(I, I) = \text{Bimod}_\mathcal{C}(I) = \mathcal{C} \) as monoidal 2-categories thanks to our strictness hypothesis. In particular, for every separable algebra \( A \) in \( \mathcal{C} \), the 2-category \( \text{Hom}_{\text{Mor}^{sep}(\mathcal{C})}(A, I) \) has a canonical left \( \mathcal{C} \)-module structure, which is compatible with bimodule morphisms in the variable \( A \). Further, \( \text{Hom}_{\text{Mor}^{sep}(\mathcal{C})}(A, I) = \text{Bimod}_\mathcal{C}(I, A) = \text{Mod}_\mathcal{C}(A) \) is a separable left \( \mathcal{C} \)-module 2-category. Thus, the 3-functor \( \text{Hom}_{\text{Mor}^{sep}(\mathcal{C})}(-, I) \) can canonically be lifted to a 3-functor

\[
\text{Mod}_\mathcal{C} : \text{Mor}^{sep}(\mathcal{C}) \to \text{LMod}(\mathcal{C}).
\]

It remains to prove that \( \text{Mod}_\mathcal{C} \) is an equivalence of 3-categories, i.e. that it is essentially surjective and induces equivalences on \( \text{Hom} \)-2-categories. Essential surjectivity follows immediately from the definition of a separable left \( \mathcal{C} \)-module 2-category. Therefore, it only remains to prove that for every separable algebras \( A, B \) in \( \mathcal{C} \), the 2-functor

\[
F : \text{Bimod}_\mathcal{C}(A, B) \to \text{Fun}_\mathcal{C}(\text{Mod}_\mathcal{C}(A), \text{Mod}_\mathcal{C}(B))
\]

\[
P \mapsto (-) \square_A P
\]

induced by \( \text{Mod}_\mathcal{C} \) is an equivalence of 2-categories. In order to exhibit a pseudo-inverse to \( F \), consider the following 2-functor

\[
B : \text{Fun}_\mathcal{C}(\text{Mod}_\mathcal{C}(A), \text{Mod}_\mathcal{C}(B)) \to \text{Bimod}_\mathcal{C}(A, B),
\]

\[
F \mapsto F(A)
\]

where the left \( A \)-module structure on \( F(A) \) arises from the canonical \( A-A \)-bimodule structure on \( A \). This assignment can straightforwardly be extended to left \( \mathcal{C} \)-module 2-natural transformations and left \( \mathcal{C} \)-module modifications. Further, for any \( A-B \)-bimodule \( P \) in \( \mathcal{C} \), we have that \( B \circ F(P) = A \square_A P \) as an \( A-B \)-bimodule in \( \mathcal{C} \). Thence, by lemma [3.2.5] we find that \( B \circ F \simeq \text{Id} \) as desired.

Now, let \( F : \text{Mod}_\mathcal{C}(A) \to \text{Mod}_\mathcal{C}(B) \) be a left \( \mathcal{C} \)-module 2-functor. By definition, for any right \( A \)-module \( M \), we have that \( (F \circ B)(M) = M \square_A F(A) \). We claim that \( M \square_A F(A) \simeq F(M) \). Namely, the 1-morphism

\[
f^F := F(n^M) \circ k^F_{M,A} : M \square F(A) \to F(M)
\]

equipped with the 2-isomorphism

\[
\beta f^F := \begin{tikzcd}
1^F \arrow[r, tail] \arrow[dr, tail] & F(n^M) \arrow[r, tail] \arrow[dr, tail] & F(n^M) \\
1F(m) \arrow[r, tail] & k^F \arrow[r, tail] & k^F
\end{tikzcd}
\]

is an \( A \)-balanced right \( B \)-module 1-morphism. Moreover, it is easy to check that it is 2-universal with respect to \( A \)-balanced right \( B \)-module morphisms.
Using this 2-universal property, we readily find that $F \circ B(F) \simeq F$. Finally, the definition of the $A$-balanced right $B$-module 1-morphism $f^F$ is 2-natural in $F$, so that we get $F \circ B \simeq Id$ from the 2-universal property. This finishes proving that $F$ and $B$ are pseudo-inverses.

We now prove an alternative characterization of separability for a compact semisimple left $\mathcal{C}$-module 2-category $\mathcal{M}$ under mild assumptions on the underlying 2-category of $\mathcal{C}$. More precisely, following [Dec21b], if $k$ is perfect, we say that $\mathcal{C}$ is locally separable if for every simple object $C$ of $\mathcal{C}$, the finite semisimple tensor 1-category $\text{End}_\mathcal{C}(C)$ is separable in the sense of [DSPS21].

**Proposition 5.1.3.** Let $k$ be a perfect field, and $\mathcal{C}$ a locally separable compact semisimple tensor 2-category. The compact semisimple left $\mathcal{C}$-module 2-category $\mathcal{M}$ is separable if and only if $\text{End}_\mathcal{C}(\mathcal{M})$ is a compact semisimple 2-category.

**Proof.** The forward direction follows by combining theorem 5.1.2 above with proposition 3.1.3 of [Dec22b]. Conversely, let us assume that $\text{End}_\mathcal{C}(\mathcal{M})$ is compact semisimple. Thanks to theorem 5.1.5 and remark 5.1.11 of [Dec21c], there exists an algebra $A$ in $\mathcal{C}$ such that $\text{Mod}_\mathcal{C}(A) \simeq \mathcal{M}$ as left $\mathcal{C}$-module 2-categories. We will use the 2-functor $B : \text{End}_\mathcal{C}(\text{Mod}_\mathcal{C}(A)) \rightarrow \text{Bimod}_\mathcal{C}(A)$, sending a left $\mathcal{C}$-module 2-functor to its value on the canonical $A$-$A$-bimodule $A$. Firstly, observe that the image under $B$ of the identity 2-functor $\text{Id}$ on $\text{Mod}_\mathcal{C}(A)$ is given by $A$. Further, if we write $F : \text{Mod}_\mathcal{C}(A) \rightarrow \text{Mod}_\mathcal{C}(A)$ for the canonical left $\mathcal{C}$-module 2-functor given by $M \mapsto M \Box A$, then we have $B(F) = F(A) = A \Box A$. Secondly, observe that for any right $A$-module $M$, $n^M : M \Box A \rightarrow M$ defines a left $\mathcal{C}$-module 2-natural transformation $n : F \Rightarrow \text{Id}$ such that $B(n) = m : A \Box A \rightarrow A$ with its canonical $A$-$A$-bimodule structure. But $\text{End}_\mathcal{C}(\mathcal{M})$ has right adjoints for 1-morphisms by hypothesis, so that $n$ has a right adjoint $n^\ast$ with counit $e^n$. As right adjoint are preserved by 2-functors, $B(n^\ast)$ is a right adjoint for $m$ as an $A$-$A$-bimodule 1-morphism with counit $B(e^n)$. This implies that $A$ is rigid. Thirdly, note that the 2-morphism $e^n : n \cdot n^\ast \Rightarrow \text{Id}$ is surjective. Namely, for every simple object $M$ of $\mathcal{M} \simeq \text{Mod}_\mathcal{C}(A)$, the 2-morphism $e^n_M$ is surjective as $n^M : M \Box A \rightarrow M$ is a non-zero 1-morphism. But $\text{End}_\mathcal{C}(\mathcal{M})$ is compact semisimple, so that $e^n$ has a section $\gamma^n$ as a left $\mathcal{C}$-module modification. As sections of 2-morphisms are preserved by arbitrary 2-functors, we find that $B(\gamma^n)$ is a section of $B(e^n)$ as an $A$-$A$-bimodule 2-morphism. Consequently, $A$ is separable, and the proof is complete.

The proof of the above proposition also establishes the following result (over any field $k$ and compact semisimple tensor 2-category $\mathcal{C}$).

**Corollary 5.1.4.** Assume that $\mathcal{M}$ is a separable left $\mathcal{C}$-module 2-category, and let $B$ be any algebra such that $\mathcal{M} \simeq \text{Mod}_\mathcal{C}(B)$ as left $\mathcal{C}$-module 2-categories, then $B$ is separable.
Remark 5.1.5. Let us call two algebras $A$, and $B$ in $\mathcal{C}$ Morita equivalent if $\text{Mod}_\mathcal{C}(A) \simeq \text{Mod}_\mathcal{C}(B)$ as left $\mathcal{C}$-module 2-categories. Note that this is clearly an equivalence relation. The above corollary may then be succinctly reformulated as the statement that separability is a Morita invariant property. Further, rigidity is also a Morita invariant property. On one hand, if $A$ is a rigid algebra, then $\text{Mod}_\mathcal{C}(A)$ has right adjoints by theorem 2.2.8 of [Dec21b], so that $\text{End}_\mathcal{C}(\text{Mod}_\mathcal{C}(A))$ has right adjoints. On the other hand, the proof of proposition 5.1.3 above, proves that if $\text{End}_\mathcal{C}(\text{Mod}_\mathcal{C}(A))$ has right adjoints, then $A$ is rigid.

We end this section by examining an example.

Example 5.1.6. Let $G$ be a finite group whose order is coprime to $\text{char}(k)$. Then, the monoidal forgetful 2-functor $\text{2Vect}_G \to \text{2Vect}$ provides $\text{2Vect}$ with a canonical structure of a left $\text{2Vect}_G$-module 2-category. We claim that $\text{End}_{\text{2Vect}_G}(\text{2Vect}) \simeq \text{2Rep}(G)$ as 2-categories. Namely, let $F : \text{2Vect} \to \text{2Vect}$ be a left $\text{2Vect}_G$-module 2-functor. As $\text{2Vect}$ is generated by $\text{Vect}$ under direct sums and splittings of 2-condensation monads, the underlying linear 2-functor $F_0$ is determined by $V := F(\text{Vect})$, a perfect 1-category. Further, unfolding the definition, we find that the left $\text{2Vect}_G$-module structure on $F$ yields a $G$-action on $V$. But $\text{2Vect}_G$ is the Cauchy completion of the monoidal 2-category $G \times \text{2Vect}$, so that this $G$-action on $V$ characterizes $F_0$ completely up to equivalence. A similar argument deals with $\text{2Vect}_G$-module 2-natural transformations and $\text{2Vect}_G$-module modifications, establishing the desired equivalence $\text{End}_{\text{2Vect}_G}(\text{2Vect}) \simeq \text{2Rep}(G)$ of 2-categories. An immediate consequence of the above equivalence is that $\text{Vect}$ is a separable $\text{2Vect}_G$-module 2-category.

Over an algebraically closed field of characteristic zero, this equivalence was first observed in section 3.2 of [Del21].

We now wish to upgrade this to an equivalence of monoidal 2-categories. Observe that the identity $\text{2Vect}_G$-module 2-endofunctor of $\text{2Vect}$ corresponds to the object $I = \text{Vect}$ of $\text{2Rep}(G)$ under the above equivalence. It follows from the proof of lemma 1.3.8 that $\text{2Rep}(G)$ is a connected compact semisimple 2-category. By proposition 3.3.4 of [Dec21b], the monoidal structure on $\text{2Rep}(G)$ is therefore completely determined by the braiding $\beta$ on the finite semisimple tensor 1-category $\text{End}_{\text{2Rep}(G)}(I) \simeq \text{Rep}(G)$. This means that, if we write $\text{2Rep}^\beta(G) := \text{Mod}(\text{Rep}^\beta(G))$, then we have $\text{End}_{\text{2Vect}_G}(\text{2Vect}) \simeq \text{2Rep}^\beta(G)$ as monoidal 2-categories. But, by proposition 4.1.10, $\text{2Vect}$ is a left $\text{2Rep}^\beta(G)$-module 2-category. Thus, by definition, there exists a braided monoidal functor $\text{Rep}^\beta(G) \to \text{2(Vect)} = \text{Vect}$. As this functor is necessarily faithful, this forces the braiding $\beta$ to be the trivial one, so that $\text{End}_{\text{2Vect}_G}(\text{2Vect}) \simeq \text{2Rep}(G)$ as monoidal 2-categories.

5.2 Indecomposable Module 2-Categories

It is useful to know when the compact semisimple monoidal 2-category $\text{End}_\mathcal{C}(\mathcal{M})$ has simple monoidal unit. We now explain when this is the case.

Definition 5.2.1. A compact semisimple left $\mathcal{C}$-module 2-category $\mathcal{M}$ is indecomposable if there exists a simple object $M$ of $\mathcal{M}$ such that for any simple object
There exists an object \( C \) in \( \mathcal{C} \) and a non-zero 1-morphisms \( C \boxtimes M \rightarrow N \).

**Example 5.2.2.** A left \( 2\text{Vect} \)-module 2-category is indecomposable if and only if the underlying compact semisimple 2-category is connected. More generally, if \( \mathcal{C} \) is a connected compact semisimple tensor 2-category, then a left \( \mathcal{C} \)-module 2-category is indecomposable if and only if the underlying compact semisimple 2-category is connected.

**Lemma 5.2.3.** Let \( \mathcal{M} \) be a compact semisimple left \( \mathcal{C} \)-module 2-category. There exists a decomposition

\[
\mathcal{M} \cong \bigoplus_{i=1}^{n} \mathcal{M}_i
\]

into a direct sum of indecomposable left \( \mathcal{C} \)-module 2-categories.

**Proof.** Let \( \mathcal{O}(\mathcal{M}) \) denote the finite set of equivalence classes of simple objects of \( \mathcal{M} \). Let \( M, N \) be two (equivalence classes of) simple objects in \( \mathcal{M} \), we write \( M \sim N \) if there exists an object \( C \) of \( \mathcal{C} \) and a non-zero 1-morphism \( C \boxtimes M \rightarrow N \). This relation is evidently reflexive, symmetry follows from lemma 2.2.10 of [Dec21a], and transitivity from lemma 2.2.11 of [Dec21a]. Let us write \( \mathcal{O}(\mathcal{M})/\sim = \{ X_1, \ldots, X_n \} \), and let \( \mathcal{M}_i \) be the full compact semisimple sub-2-category of \( \mathcal{M} \) generated under direct sums and splittings of 2-condensation monads by the simple objects in \( X_i \). As the relation \( \sim \) is coarser than that of being connected, the sub-2-categories \( \mathcal{M}_i \) and \( \mathcal{M}_j \) do not contain any common simple object. Furthermore, it is immediate from the definition of \( \sim \) that \( \mathcal{M}_i \) inherits a left \( \mathcal{C} \)-module structure, under which it is indecomposable. Thence, we find \( \mathcal{M} \cong \bigoplus_{i=1}^{n} \mathcal{M}_i \) as left \( \mathcal{C} \)-module 2-categories.

**Lemma 5.2.4.** Let \( \mathcal{M} \) be a compact semisimple left \( \mathcal{C} \)-module 2-category. Then, the identity left \( \mathcal{C} \)-module 2-functor on \( \mathcal{M} \) splits as the direct sum of the projectors onto the \( \mathcal{M}_i \). Further, if \( \mathcal{M} \) is separable, every such projector is a simple object of \( \text{End}_{\mathcal{C}}(\mathcal{M})^* \).

**Proof.** The first assertion is immediate. Let us assume that \( \mathcal{M} \) is separable, and \( \mathcal{M} \cong \bigoplus_{i=1}^{n} \mathcal{M}_i \) be a decomposition of \( \mathcal{M} \) as a direct sum of indecomposable compact semisimple left \( \mathcal{C} \)-module 2-categories. We wish to prove that the projection \( P_i : \mathcal{M} \rightarrow \mathcal{M}_i \rightarrow \mathcal{M} \) is a simple object of \( \text{End}_{\mathcal{C}}(\mathcal{M}) \). To this end, let \( Q, R : \mathcal{M} \rightarrow \mathcal{M} \) be two \( \mathcal{C} \)-module 2-functor such that \( P_i = Q \boxtimes R \). Let us additionally assume that \( Q(M) \) is no-zero for some \( M \) (necessarily in \( \mathcal{M}_i \)). Then, it follows from the proof of lemma 5.2.3 that given any simple object \( N \) of \( \mathcal{M}_i \), there exists an object \( C \) of \( \mathcal{C} \) and a non-zero 1-morphism \( C \boxtimes N \rightarrow M \). In particular, \( M \) is the splitting of a 2-condensation monad supported on \( C \boxtimes N \). But splittings of 2-condensation monads are preserved by all 2-functors, so that \( M \) is the splitting of a 2-condensation monad on \( Q(C \boxtimes N) \). As \( M \) is non-zero, so must be \( Q(C \boxtimes N) \). Now, \( Q \) is a left \( \mathcal{C} \)-module 2-functor, so that \( Q(C \boxtimes N) \cong C \boxtimes Q(N) \), which implies that \( Q(N) \) is non-zero. Finally, we have \( N = P_i(N) = Q(N) \boxtimes R(N) \), and \( N \) is simple, so that \( R(N) = 0 \) by proposition 1.1.7 of [Dec21b]. As \( N \) was arbitrary, we find that \( R = 0 \), which finishes the proof of the lemma.
Corollary 5.2.5. Let \( \mathfrak{M} \) be a separable left \( \mathcal{C} \)-module 2-category. Then \( \mathfrak{M} \) is indecomposable if and only if \( \text{End}_{\mathcal{C}}(\mathfrak{M}) \) has simple monoidal unit.

Given the equivalence of 3-categories established in theorem 5.1.2, it is only natural to examine what property of a separable algebra corresponds to indecomposability of the associated module 2-category.

Definition 5.2.6. Let \( A \) be a separable algebra. We say that \( A \) is indecomposable if \( A \) is simple as an \( A-A \)-bimodule.

Corollary 5.2.7. A separable algebra \( A \) is indecomposable if and only if \( \text{Mod}_{\mathcal{C}}(A) \) is indecomposable.

Remark 5.2.8. In particular, this shows that being indecomposable is a Morita invariant property of separable algebras in \( \mathcal{C} \). Further, it follows from lemma 5.2.6 that any separable algebra \( A \) may be split into a direct sum of indecomposable separable algebras. A direct proof of this fact is given in the proof of theorem 3.1.6 of [Déc22b].

5.3 Dual Tensor 2-Categories

In this section, we assume throughout that \( k \) is perfect. The following technical result is needed to prove our main theorem over fields of positive characteristic. Before stating it, we need to recall some terminology from [Déc21b]. We say that a compact semisimple 2-category \( \mathcal{C} \) is locally separable if for every simple object \( C \) of \( \mathcal{C} \), the finite semisimple tensor 1-category \( \text{End}_{\mathcal{C}}(C) \) is separable in the sense of [DSPS21]. We remark that, over fields of characteristic zero, they show that every finite semisimple tensor 1-category is separable, so that every compact semisimple 2-category is locally separable in this case.

Proposition 5.3.1. Let \( \mathcal{C} \) be a locally separable compact semisimple monoidal 2-category, and \( A \) a separable algebra in \( \mathcal{C} \). Then, \( \text{Mod}_{\mathcal{C}}(A) \) is locally separable.

Proof. By theorem 1.4.7 of [Déc21b], there exists a separable finite semisimple tensor 1-category \( \mathcal{C} \) such that \( \text{Mod}(\mathcal{C}) \simeq \mathcal{C} \) as 2-categories. It follows from theorem 5.1.2 that

\[
\text{End}(\mathcal{C}) \simeq \text{Bimod}(\mathcal{C})^{\mathcal{C}^{\text{op}}}.
\]

Further, observe that algebras in \( \text{Bimod}(\mathcal{C}) \) are precisely given by finite semisimple monoidal 1-categories \( \mathcal{D} \) equipped with a monoidal functor \( \mathcal{C} \to \mathcal{D} \). Moreover, a variant of the proof of lemma 2.1.4 of [Déc22b] shows that such an algebra \( \mathcal{D} \) is rigid if and only \( \mathcal{D} \) is a tensor 1-category. Also note that the 2-category \( \mathcal{D}-\mathcal{D} \)-bimodules in \( \text{Bimod}(\mathcal{C}) \) is canonically equivalent to \( \text{Bimod}(\mathcal{D}) \), so that \( \mathcal{D} \) is separable as an algebra in \( \text{Bimod}(\mathcal{C}) \) if and only if it is separable as a finite semisimple tensor 1-category. Further, observe that the separable algebra \( A \) in \( \mathcal{C} \) yields a separable algebra \( A \) in \( \text{End}(\mathcal{C}) \simeq \text{Bimod}(\mathcal{C})^{\mathcal{C}^{\text{op}}} \) via \( C \mapsto C \square A \). It follows from the definition that

\[
\text{Mod}_{\mathcal{C}}(A) \simeq \text{Mod}_{\mathcal{C}}(A),
\]
where, on the right hand-side, we use the canonical right \( \text{End}(\mathcal{C})^{\text{op}} \)-module structure on \( \mathcal{C} \).

Finally, note that \( \text{End}(\mathcal{C}) \), the finite semisimple tensor 1-category of linear endofunctors of \( \mathcal{C} \), is a rigid algebra in \( \text{Bimod}(\mathcal{C}) \) via the left action of \( \mathcal{C} \) on itself. Further, there are equivalences of right \( \text{Bimod}(\mathcal{C}) \)-module 2-categories

\[
\text{LMod}_{\text{Bimod}(\mathcal{C})}(\text{End}(\mathcal{C})) \simeq \text{Bimod}(\text{End}(\mathcal{C}), \mathcal{C}) \simeq \text{Mod}(\mathcal{C}) \simeq \mathcal{C},
\]

as \( \text{End}(\mathcal{C}) \) and \( \text{Vect} \) are Morita equivalent finite semisimple tensor 1-categories. Putting everything together, we find that there are equivalences of 2-categories

\[
\text{Mod}_\mathcal{C}(\text{Mod}_{\mathcal{C}}(A)) \simeq \text{Bimod}_{\mathcal{C}}(\text{End}(\mathcal{C}), A) \simeq \text{Bimod}(\text{End}(\mathcal{C}), A) \simeq \text{Mod}(\text{End}(\mathcal{C})^{\text{op}} \boxtimes A).
\]

But it follows from corollary 2.5.11 of [DSPS21] that \( \text{End}(\mathcal{C})^{\text{op}} \boxtimes A \) is a separable finite semisimple tensor 1-category, so that \( \text{Mod}(\text{End}(\mathcal{C})^{\text{op}} \boxtimes A) \) is locally separable by theorem 1.4.6 of [Déc21b].

We are now in the position to prove our main theorem 5.1.2.

**Theorem 5.3.2.** Let \( k \) be a perfect field, and \( A \) a separable algebra in a locally separable compact semisimple tensor 2-category \( \mathcal{C} \). Then,

\[
\text{End}_\mathcal{C}(\text{Mod}_A(\mathcal{C})) \simeq \text{Bimod}_{\mathcal{C}}(\text{End}(\mathcal{C}), A)^{\square \text{A}^{\text{op}}}
\]

is a compact semisimple tensor 2-category.

**Proof.** The equivalence of monoidal 2-categories is an immediate consequence of theorem 5.1.2. Furthermore, it follows from theorem 3.1.6 of [Déc22b] that the underlying 2-category \( \text{Bimod}_\mathcal{C}(A) \) is compact semisimple. Thus, it only remains to establish the existence of duals. We will show that \( \text{End}_\mathcal{C}(\text{Mod}_\mathcal{C}(A)) \) has this property. Namely, it follows from proposition 5.3.1 that \( \text{Mod}_\mathcal{C}(A) \) is locally separable. Then, corollary 3.2.3 of [Déc21b] shows that every linear 2-functor \( \text{Mod}_\mathcal{C}(A) \to \text{Mod}_\mathcal{C}(A) \) has a right 2-adjoint 2-functor. In particular, corollary 1.2.3 applies to every object of \( \text{End}_\mathcal{C}(\text{Mod}_\mathcal{C}(A)) \), which proves that \( \text{End}_\mathcal{C}(\text{Mod}_\mathcal{C}(A)) \) has right duals. By corollary 1.3.4 of [Déc22b], \( \text{End}_\mathcal{C}(\text{Mod}_\mathcal{C}(A)) \) also has left duals, which concludes the proof of the result.

**Remark 5.3.3.** The assumption that \( \mathcal{C} \) be locally separable in theorem 5.3.2 might not be necessary. Namely, we believe that it is possible to show direct that for any separable algebra \( A \) in a compact semisimple tensor 2-category, the monoidal 2-category \( \text{Bimod}_\mathcal{C}(A) \) has duals. We leave it to the interested readers to pursue this line of investigation further.

Thanks to the above theorem, the following definition is sensible.

**Definition 5.3.4.** Let \( \mathcal{C} \) be a locally separable compact semisimple tensor 2-category, and let \( \mathcal{M} \) be a separable left \( \mathcal{C} \)-module 2-category. We write \( \mathcal{C}_\mathcal{M} \) for the compact semisimple tensor 2-category \( \text{End}_\mathcal{C}(\mathcal{M}) \), and call it the dual tensor 2-category to \( \mathcal{C} \) with respect to \( \mathcal{M} \).
Combining the theorem 5.3.2 above with corollary 2.2.4 of [Déc21b], we obtain the following result, which is the starting point of the study of Morita equivalences of fusion 2-category that we will develop in [Déc].

**Corollary 5.3.5.** Let \( k \) be an algebraically closed field of characteristic zero, \( C \) a multifusion 2-category, and \( M \) a separable left \( C \)-module 2-category. Then, \( C^*_M \), the dual tensor 2-category to \( C \) with respect to \( M \), is a multifusion 2-category.

The following corollary now follows from the discussion given in example 2.5.2.

**Corollary 5.3.6.** Let \( G \) be a finite group, and \( \pi \) a 4-cocycle for \( G \) with coefficients in \( k \). Further, let \( H \subseteq G \) be a subgroup of order coprime to \( \text{char}(k) \) and \( \gamma \) a 3-cochain for \( H \) with coefficients in \( k^\times \). Then, \( \text{Bimod}_{\text{Vect}_C}(\text{Vect}_{H}) \) is a compact semisimple tensor 2-category.

We end this section by examining some examples of dual tensor 2-categories.

**Example 5.3.7.** Let \( k \) be an algebraically closed field, and let \( C \) be a separable fusion 1-category. The compact semisimple 2-category \( \text{Mod}(C) \) admits a canonical left \( \text{2Vect} \)-module structure. We claim that \( \text{2Vect}_{\text{Mod}(C)} \simeq \text{Mod}(\mathcal{Z}(C)^{\text{op}}) \) as monoidal 2-categories. For the reader’s convenience, we supply a different proof. Note that theorem 5.1.2 provides us with an equivalence of monoidal 2-categories \( \text{2Vect}_{\text{Mod}(C)} \simeq \text{Bimod}(C)^{\text{op}} \). But, theorem 1.3 of [Gre10] implies that \( \text{Bimod}(C) \simeq \text{Mod}(\mathcal{Z}(C)) \) as monoidal 2-categories, which concludes the proof.

Alternatively, as \( C \) is a fusion 1-category, \( C \otimes^{\text{op}} C \) is also a fusion 1-category, so that the compact semisimple 2-category \( \text{Bimod}(C) \simeq \text{Mod}(C \otimes^{\text{op}} C) \) is connected. Thus, by proposition 3.3.4 of [Déc21b], in order to determine the monoidal structure on \( \text{Bimod}(C) \), it is enough to understand the braiding on the finite semisimple tensor 1-category \( \text{End}_{\text{C}}(C) = \mathcal{Z}(C) \) of endomorphism the monoidal unit. Given that the monoidal structure of \( \text{Bimod}(C) \) is given by the relative Deligne tensor product \( \otimes \), the braiding on \( \mathcal{Z}(C) \) is the canonical one.

**Example 5.3.8.** We now discuss a generalization of example 5.3.7. For simplicity, we will assume that \( k \) is an algebraically closed field of characteristic zero. Let \( B \) be a non-degenerate braided fusion 1-category, and \( C \) a \( B \)-central fusion 1-category. We claim that there is an equivalence of monoidal 2-categories \( \text{Bimod}(C) \simeq \text{Mod}(\mathcal{Z}(C)^{\text{op}}) \), where \( A \) is the centralizer of \( B \) in \( \mathcal{Z}(C) \), which is non-degenerate by theorem 3.13 of [DGNO10]. By corollary 5.9 of [DMNO13], this implies further that \( B \) and \( A \) are Witt equivalent non-degenerate braided fusion 1-categories.

In order to prove the claim, observe that, as \( C \) is a separable algebra in \( \text{Mod}(B) \), corollary 5.3.5 establishes that \( \text{Mod}(B)^{\star}_{\text{Mod}(C)} \simeq \text{Bimod}_{\text{Mod}(B)}(C)^{\text{op}} \) is a fusion 2-category. Moreover, after having unfolded the definitions, we find \( \text{Bimod}_{\text{Mod}(B)}(C) \simeq \text{Mod}(C)^{\text{op}} \otimes_{B} C \) as finite semisimple 2-categories. As \( B \) is non-degenerate, it follows from theorems 2.26 and 3.20 of [BJSS21] that there is an equivalence

\[
C^{\text{op}} \otimes_{B} \mathcal{C} \simeq \text{End}_{A}(\mathcal{C})
\]
of multifusion 1-categories. In particular, $\text{Bimod}_{\text{Mod}(B)}(C) \simeq \text{Mod}(C^{\otimes \text{op}} \boxtimes_B C)$ is a connected finite semisimple 2-category. The above equivalence of multifusion 1-categories implies that $\mathcal{A} \simeq \text{End}_{C^{\otimes \text{op}} \boxtimes_B C}(C)$ as fusion 1-categories, so that the endomorphism fusion 1-category of $C$ in $\text{Bimod}_{\text{Mod}(B)}(C)$ is given by $\mathcal{A}$. But $C$ is the monoidal unit of $\text{Bimod}_{\text{Mod}(B)}(C)$. Thence, appealing to proposition 2.4.7 of [Dec22c], it is enough to understand the braiding on the fusion 1-category $\mathcal{A}$ of endomorphisms of $C$ in $\text{Bimod}_{\text{Mod}(B)}(C)$.

To this end, note that the forgetful 2-functor $\text{Bimod}_{\text{Mod}(B)}(C) \to \text{Bimod}(C)$ induces the canonical inclusion of fusion 1-categories $\mathcal{A} \hookrightarrow \mathcal{Z}(C)$. But the forgetful monoidal 2-functor

$$\text{End}_{\text{Mod}(B)}(\text{Mod}(C)) \to \text{End}(\text{Mod}(C))$$

is identified via theorem 5.1.2 to $\text{Bimod}_{\text{Mod}(B)}(C) \to \text{Bimod}(C)$. This shows that the monoidal inclusion $\mathcal{A} \hookrightarrow \mathcal{Z}(C)$ is in fact braided, thereby completing the proof of the claim.

Remark 5.3.9. As one can readily observe from example 5.1.6, the non-degeneracy condition in example 5.3.8 above can not be omitted in general. We will return to this point in detail in [Dec].
A Appendix

A.1 Diagrams for the proof of theorem 3.1.6

Figure 1: Associativity (Part 1)
Figure 2: Associativity (Part 2)
Figure 3: Associativity (Part 3)
Figure 4: Associativity (Part 4)
Figure 6: Associativity (Part 6)
Figure 7. Associativity (Part 7)
Figure 8: Associativity (Part 8)
Figure 9: Associativity (Part 9)
Figure 10: Left Frobenius (Part 1)
Figure 11: Left Frobenius (Part 2)
Figure 12: Left Frobenius (Part 3)
Figure 13: Left Frobenius (Part 4)
Figure 15: Left Frobenius (Part 6)
Figure 16: Left Frobenius (Part 7)
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