Light-front approach to relativistic electrodynamics

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Abstract

We illustrate how our recent light-front approach simplifies relativistic electrodynamics with an electromagnetic (EM) field $F^{\mu\nu}$ that is the sum of a (even very intense) plane travelling wave $F^{\mu\nu}(ct-z)$ and a static part $F^{\mu\nu}_s(x,y,z)$; it adopts the light-like coordinate $\xi = ct-z$ instead of time $t$ as an independent variable. This can be applied to several cases of extreme acceleration, both in vacuum and in a cold diluted plasma hit by a very short and intense laser pulse (slingshot effect, plasma wave-breaking and laser wake-field acceleration, etc.)

1 Introduction

The equation of motion of a particle of rest mass $m$, electric charge $q$ in an external EM field

\begin{align}
\dot{p}(t) &= qE[ct, x(t)] + \frac{p(t)}{\sqrt{m^2c^2 + p^2(t)}} \wedge qB[ct, x(t)], \\
\dot{x}(t) &= \frac{qE(t)}{\sqrt{m^2c^2 + p^2(t)}}
\end{align}

in its general form is non-autonomous and highly nonlinear in the unknowns $x(t), p(t)$. Here $x, p$ are the position and relativistic momentum of the particle, while $E, B$ are the electric and magnetic field; we use Gauss CGS units. In the situations characterizing many standard applications [1] can be solved or simplified making one or more of the following assumptions:

1. $E, B$ are constant or vary “slowly” in space/time;
2. the motion of the particle keeps non-relativistic;
3. $E, B$ are so small that nonlinear effects in $E, B$ are negligible;
4. $E, B$ are monochromatic waves, or very slow modulations thereof.
If the role of the charges (isolated or in the form of a continuum) as sources of the EM field cannot be neglected, these equations are to be coupled with the Maxwell equations; the resulting system is much more complicated. In many interesting situations none of the above assumptions is satisfied. For instance, many violent astrophysical electrodynamic processes occur in the presence of extremely intense and rapidly varying electromagnetic fields (see e.g. [1] and references therein). On the other hand, the on-going, amazing developments of laser technologies leading to compact sources of extremely intense and short coherent EM waves\(^1\) allow extreme accelerations through the Laser Wake Field Acceleration (LWFA) mechanism \(^2\) in plasmas\(^2\). To deal with such complex problems it is almost unavoidable to resort to numerical resolution methods, e.g. particle-in-cell (PIC) techniques for plasmas. PIC or other codes in general involve huge and costly computations for each choice of the free parameters; exploring the parameter space blindly to pinpoint the interesting regions is prohibitive, if not accompanied by some analytical insight that can simplify the work, at least in special cases or in a limited space-time region. It is therefore important to look for new approaches simplifying the study of the dynamics induced by (1) when conditions (i-iv) are not fulfilled.

Here we summarize an approach \cite{9} that systematically applies the light-front formalism \cite{10}; it is especially fruitful if in the spacetime region of interest \(E, B\) are the sum of static parts and a plane transverse travelling wave propagating in a fixed (\(z\), say) direction:

\[
\begin{align*}
E(t, x) &= \epsilon^\perp (ct - z) + E_s(x), \\
B(t, x) &= k \wedge \epsilon^\perp (ct - z) + B_s(x).
\end{align*}
\]

We decompose 3-vectors such as \(x\) in the form \(x = x_i + y_j + z_k\) \((x, y, z\) are the cartesian coordinates) and denote as \(x \equiv (ct, x)\) the set of coordinates of Minkowski spacetime points

\(^1\)The technique of Chirped Pulse Amplification \cite{2, 3}, awarded with the Nobel Prize in physics in 2018, yields pulses of intensity up to \(10^{23}\) Watt per square centimeter and duration down to femtoseconds. Huge investments (e.g. for the Extreme Light Infrastructure program within the EU ESFRI roadmap) in new technologies (thin film or relativistic mirror compression, etc. \cite{4, 1}) will yield even more intense/short pulses, at a lower cost.

\(^2\)In fact, huge investments aiming at the construction of table-top particle accelerators based on Wake Field Acceleration are being made all over the world. For instance, in Europe the large network of research centers “European Plasma Research Accelerator with eXcellence In Applications” (EUPRAXIA) has been created to develop the associated technologies and has just issued its expected Conceptual Design Report of reliable accelerators of this kind \cite{6, 7, 8}. Among the present and future applications of accelerators we mention:

- **Medicine**: diagnostics (PET,...), cancer therapy by accelerated particles (electrons, protons, ions) or locally induced radioisotope production,...;
- **Research**: elementary particle physics, materials science, structural biology, inertial nuclear fusion, electron beams for X-ray free electron laser,...;
- **Industry**: atomic scale lithography, surface treatment of materials, sterilization, additive-layer (i.e. 3D print) manufacturing, detection systems across shields,...;
- **Environmental remediation**: flue gas cleanup, petroleum cracking, transmutation of nuclear wastes,...
with respect to the laboratory frame. We assume only that $\epsilon^\perp(\xi)$ is piecewise continuous and

$$a\) \quad \epsilon^\perp \text{ has a compact support } [0, l],$$

or a') $\epsilon^\perp \in L^1(\mathbb{R})$, (3)

and treat on the same footing all $\epsilon^\perp$ fulfilling (3) regardless of their Fourier analysis, in particular:

1. A modulated monochromatic wave (here $a_1^2 + a_2^2 = 1$):

$$\epsilon^\perp(\xi) = \epsilon(\xi) \left[ i a_1 \cos(k\xi + \varphi) + j a_2 \sin(k\xi) \right].$$

(modulation carrier wave $\epsilon^\perp(\xi)$) (4)

2. A superposition of a finite number of waves of type 1.

3. An ‘impulse’ (few, one, or even a fraction of oscillation) [11, 4].

The starting point of our approach is: since no particle can reach the speed of light $c$, then $\xi(t) = ct - z(t)$ is strictly growing, and we can recast the eq. of motion (1) in an equivalent form (1) making the change of independent variable $t \mapsto \xi = ct - z$; the term $\epsilon^\perp[ct - z(t)]$, where the unknown $z(t)$ is in the argument of the highly nonlinear and rapidly varying $\epsilon^\perp$, becomes the known forcing term $\epsilon^\perp(\xi)$, what makes (9-12) more manageable than (1). More radically, we can apply Hamilton’s principle to the action functional $S(\lambda)$ of the particle parametrizing the worldline $\lambda$ (fig. 1) by $\xi$ instead of $t$; the associated Euler-Lagrange and Hamilton equations are equivalent and have the same features as (9-12). The Hamilton equations involve also a more convenient set of canonically conjugated momenta as unknown (dependent) variables.

We apply the approach first to an isolated particle (section 2), then to a cold diluted plasma initially at rest and hit by a plane EM wave (section 3).

2 Electrodynamics of a single particle

2.1 Set-up and general results

Given points spacetime $x_0, x_1$ with $x_1$ in the causal cone of $x_0$, let $\Lambda$ be the set of time-like curves from $x_0$ to $x_1$. Given a $\lambda \in \Lambda$, we can equivalently use either $t$ or $\xi$ as a parameter on $\lambda$ in the corresponding action functional of the particle:

$$S(\lambda) = - \int_\lambda mc^2 dt + qA(x) = - \int_{t_0}^{t_1} d\tau \gamma \frac{mc^2 + qu^\mu A_\mu}{L[x, \tilde{x}, t]} = - \int_{\xi_0}^{\xi_1} d\xi \frac{mc^2 + q\tilde{u}^\mu \tilde{A}_\mu}{\mathcal{L}[\tilde{x}, \tilde{x}', \xi]}. \quad (5)$$
Here \( A(x) = A_\mu(x)dx^\mu = A^0(x)cdt - A(x) \cdot dx \) is the EM potential 1-form (the dot is the scalar product in Euclidean \( \mathbb{R}^3 \)), which is related to the EM fields by \( E = -\partial_0 A/c - \nabla A^0 \) and \( B = \nabla \wedge A \) (\( E^i = F^{i0}, \ B^1 = F^{32}, \) etc in terms of the EM tensor \( F^{\mu\nu} \)). For any given function \( f(ct, x) \) we denote \( \hat{f}(\xi, \hat{x}) \equiv f(\xi + \hat{z}, \hat{x}) \), abbreviate \( \dot{f} \equiv df/dt, \ \dot{f}' \equiv df/d\xi \) (total derivatives). In particular, \( \hat{x}(\xi) \) is the particle position seen as a function of \( \xi \); it is determined by \( \hat{x}(\xi) = x(t) \). We raise and lower greek letter indices by the Minkowski metric \( \eta_{\mu\nu} = \eta^{\mu\nu} \), with \( \eta_{00} = 1, \ \eta_{11} = -1, \) etc.; \( (cd\tau)^2 = (cdt)^2 - d\mathbf{x}^2 \) is the square of the infinitesimal Minkowski distance (\( \tau \) is the proper time of the particle), \( dt/d\tau = \gamma = 1/\sqrt{1-\beta^2} \) (with \( \beta \equiv \dot{x}/c \)) is the Lorentz relativistic factor, \( u = (u^0, \mathbf{u}) \equiv (\gamma, \gamma \beta) = \left( \frac{p^0}{mc^2}, \frac{\mathbf{p}}{mc} \right) \) is the 4-velocity, i.e. the dimensionless version of the 4-momentum, and

\[
\frac{d\xi}{d(ct)} = u^0 - \dot{z} = u^- = \gamma(1 - \beta^2) > 0
\]

is the light-like component \( u^- \) of \( u \), as well as the Doppler factor experienced by the particle, and is positive-definite (the first = in (5) follows from \( \gamma = dt/d\tau, \ \dot{p}^2 = md\dot{z}/d\tau \)). We name \( s \) the light-like relativistic factor, or shortly the \( s \)-factor. In terms of the “hatted” coordinates and their derivatives\(^3\)

\[
\hat{u} = \hat{s} \hat{x}', \quad \frac{1}{\hat{s}} = \sqrt{1+2\hat{z}' - \hat{x}'^2}.
\]

\footnote{In fact, since \( cd(\xi) = \xi + \hat{z} \) implies \( cd'(\xi) = 1 + \hat{z}'(\xi) > 0 \), we find

\[
\frac{1}{\hat{s}} = \frac{d\tau}{dt} \frac{d(ct)}{d\xi} = \frac{1}{\gamma} \frac{d(ct)}{d\xi} = \sqrt{1-\left(\frac{dx}{cdt}\right)^2 \frac{d(ct)}{d\xi}^2} = \sqrt{\left(\frac{cdt}{d\xi}\right)^2 - \left(\frac{dx}{d\xi}\right)^2} = \sqrt{(1+\hat{z}')^2 - \hat{x}'^2} = \sqrt{1+2\hat{z}' - \hat{x}'^2}.
\]}

Figure 1: Every worldline \( \lambda \) and hyperplane \( \xi = \text{const} \) in Minkowski space intersect once. The wave-particle interaction occurs only along the intersection of \( \lambda \) with the support of the EM wave (painted pink), which assuming (3) is delimited by the \( \xi = 0 \) and \( \xi = l \) hyperplanes.
\(\dot{\gamma}, \dot{\gamma}^z, \dot{\beta}, \dot{x}'\) can be expressed as rational functions of \(\dot{u}^\perp, \dot{s}\):

\[
\dot{\gamma} = \frac{1 + \dot{u}^\perp + s^2}{2s}, \quad \dot{\gamma}^z = \tilde{\gamma} - \dot{s}, \quad \dot{\beta} = \frac{\dot{u}^\perp}{\tilde{\gamma}},
\]

\[
\dot{x}^\perp' = \frac{\dot{u}^\perp}{\tilde{s}}, \quad \dot{z}' = \frac{1 + \dot{u}^\perp + \frac{1}{2}}{2s^2} - \frac{1}{2}
\]  

(8)

By Hamilton’s principle, any extremum \(\lambda\) of \(S\) is the worldline of a possible motion of the particle with initial position \(x_0\) at time \(t_0\) and final position \(x_1\) at time \(t_1\). Hence it fulfills Euler-Lagrange equations in both forms \(\frac{d}{dt} \frac{\partial L}{\partial \dot{x}} = \frac{\partial L}{\partial x}\) and \(\frac{d}{dt} \frac{\partial \tilde{L}}{\partial \dot{x}} = \frac{\partial \tilde{L}}{\partial x}\), equivalent to (1). The Legendre transforms yield the Hamiltonians \(H \equiv \dot{x} \cdot \frac{\partial L}{\partial \dot{x}} - L = \gamma mc^2 + qA^0\) and \(\tilde{H} \equiv x' \cdot \frac{\partial \tilde{L}}{\partial \dot{x}} - \tilde{L} = \gamma mc^2 + q\tilde{A}^0\) respectively. In fact, the change of ‘time’ \(t \rightarrow \xi\) induces a generalized canonical (i.e. contact) transformation mapping \(H \mapsto \tilde{H}\). The latter is a rational function of \(\dot{x}, \tilde{H} \equiv \frac{\partial \tilde{L}}{\partial \dot{x}}\), or, equivalently, of \(\tilde{s}, \dot{u}^\perp\):

\[
\tilde{H}(\dot{x}, \tilde{\Pi}; \xi) = mc^2 \frac{1 + \dot{u}^\perp + s^2}{2s} + q\tilde{A}^0 (\xi, \dot{x}),
\]

where

\[
\begin{align*}
mc^2 \dot{u}^\perp &= \tilde{\Pi} - q\tilde{A}^0 (\xi, \dot{x}), \\
mc^2 \dot{s} &= -\tilde{\Pi}^z - q(\tilde{A}^0 - \tilde{A}^z)(\xi, \dot{x}),
\end{align*}
\]

(11)

while \(H(x, P, t) = \sqrt{\tilde{m}^2 c^4 + (cP - qA)^2} + qA^0\) (where \(P \equiv \frac{\partial L}{\partial \dot{x}} = p + \frac{q}{c}A\) is not). Eq. (1) are also equivalent to the Hamilton equations \(\dot{x}' = \frac{\partial H}{\partial \dot{p}}\), \(\dot{p}' = -\frac{\partial H}{\partial \dot{x}}\). These amount to (9) and

\[
\begin{align*}
\dot{u}^\perp' &= \frac{q}{mc^2} \left[ s \tilde{E} + \dot{u}^\perp \wedge \dot{B} \right] \perp, \\
\dot{s}' &= \frac{q}{mc^2} \left[ \frac{\dot{u}^\perp}{s} - \tilde{E} \cdot \tilde{\dot{E}}^\perp - \frac{\left( \dot{u}^\perp \wedge \dot{B} \right)^\perp}{\tilde{s}} \right]
\end{align*}
\]

(12)

with \(\tilde{\gamma}\) as given in (8). All the new equations, in particular these ones, can be also obtained more directly from the old ones by putting a caret on all dynamical variables and replacing \(d/dt\) by \((c\dot{s}/\gamma) d/d\xi\). Along the solutions \(\tilde{H}\) gives the particle energy as a function of \(\xi\). Once solved (9,12), analytically or numerically, we just need to invert \(\tilde{t}(\xi) = \xi + \tilde{z}(\xi)\) and set \(x(t) = \dot{x}[\xi(t)]\) to obtain the solution as a function of \(t\).

Under the EM field (2) eqs (12) amount to

\[
\begin{align*}
\dot{u}^\perp' &= \frac{q}{mc^2} \left[ (1 + \tilde{z}') \tilde{E}_{\perp}^\perp + (\dot{x}' \wedge \dot{B}_s) \wedge + \epsilon^\perp (\xi) \right], \\
\dot{s}' &= -\frac{q}{mc^2} \left[ \tilde{E}_{s}^\perp - \dot{x}^\perp \cdot \tilde{E}_{s}^\perp + (\dot{x}' \wedge \dot{B}_s) \wedge \right],
\end{align*}
\]

(13)

while the energy gain (normalized to \(mc^2\)) in the interval \([\xi_0, \xi_1]\) is

\[
\mathcal{E} \equiv \frac{\tilde{H}(\xi_1) - \tilde{H}(\xi_0)}{mc^2} = \int_{\xi_0}^{\xi_1} d\xi \frac{q \epsilon^\perp}{mc^2} \cdot \frac{\dot{u}^\perp}{\tilde{s}}.
\]

(14)
We can obtain the fields \( [2] \) from an EM potential of the same form, \( A^\mu(x) = \alpha^\mu(ct-z) + A^\mu_0(x) \); in the Landau gauges \((\partial_\mu A^\mu = 0)\) \( A_\mu \) must fulfill the Coulomb gauges \((\nabla \cdot A_\mu = 0)\), and it must be \( \alpha^{\mu\nu} = \alpha^{0\nu}, \quad \epsilon^{\mu} = -\alpha^{\perp\mu}, \quad E_\mu = -\nabla A_\mu^0, \quad B_\mu = \nabla \times A_\mu \). We shall set \( \alpha^\perp = \alpha^0 = 0 \), as \( \alpha^\perp, \alpha^0 \) appear neither in the observables \( E, B \) nor in the equations of motion. We fix \( \alpha^\perp(\xi) \) uniquely by requiring that it vanish as \( \xi \to -\infty \); this leads to

\[
\alpha^\perp(\xi) \equiv -\int_{-\infty}^{\xi} dy \, \epsilon^\perp(y). \tag{15}
\]

Under rather general assumptions on \( \epsilon \) \([1]\) implies

\[
\alpha^\perp(\xi) = -\frac{\epsilon(\xi)}{k} \epsilon^\perp(y) + O\left(\frac{1}{k^2}\right) \simeq -\frac{\epsilon(\xi)}{k} \epsilon^\perp(y) \tag{16}
\]

where \( \epsilon^\perp(\xi) := \epsilon^\perp(\xi + \pi/2k) \); in the appendix of \([2]\) we recall upper bounds for the remainder \( O(1/k^2) \). For very slow modulations (i.e. \( |\epsilon| \ll |k\epsilon| \) namely the modulating amplitude \( \epsilon \) does not vary significantly over the wavelength \( \lambda \equiv 2\pi/k \) - like the ones characterizing most conventional applications (radio broadcasting, ordinary laser beams, etc.) - the right estimate is very good. Consequently, if \( \epsilon(\xi) \) goes to zero also as \( \xi \to \infty \), then \( \alpha^\perp(\xi) \) approximately does as well. In particular, in case \( a \) eq. (15) implies \( \alpha^\perp(\xi) = 0 \) if \( \xi \leq 0 \), \( \alpha^\perp(\xi) = \alpha^\perp(l) \simeq 0 \) if \( \xi \geq l \).

While the usual Hamilton equations \( \dot{P} = -\partial H/\partial x, \quad \dot{x} = \partial H/\partial P \), which amount to

\[
\dot{u}(t) = \frac{q}{mc} \left\{ E_s + \frac{\dot{x}}{c} \times B_s + \left( \frac{\dot{x}}{c} \cdot \epsilon^{\perp}[ct-z(t)] \right) k - \frac{1}{c} \frac{d}{dt} \alpha^\perp[ct-z(t)] \right\}, \tag{17}
\]

and \( \dot{x} = u/\sqrt{1+u^2} \), have no rational form and contain the unknown \( ct-z(t) \) in the argument of the rapidly varying function \( \epsilon^{\perp}, \alpha^\perp \), \([9][13]\) on the contrary are rational in the unknowns and contain \( \epsilon^\perp(\xi) \) as a known forcing term. This simplifies their study and the determination of \( \mathcal{E} \).

### 2.2 Dynamics under an EM potential of the form \( A^\mu = A^\mu(t,z) \)

Eq. (12) are further simplified if \( A^\mu \) does not depend on transverse coordinates. This applies in particular if \( E_s = E_s^z(z) k, \quad B_s = B_s^z(z) \), choosing \( A^0 = -\int dz E_s^z(\zeta), \quad A^z = \alpha^z - k \wedge \int dz B^z(\zeta), \quad A^\perp \equiv 0 \). As \( \partial H/\partial \dot{z}^\perp = 0 \), we find \( \dot{\Pi}^\perp = q \mathbf{K} \equiv \text{const} \), i.e. the known result \( \frac{mc}{q} \ddot{z}^\perp = \mathbf{K} - \dot{A}^\perp(\xi,z) \). Setting \( v := \dot{u}^\perp \) and replacing in (13) we obtain the first order system in the unknowns \( \dot{\zeta}, \dot{s} \)

\[
\zeta' = \frac{1}{2s^2} \frac{1}{2} - \frac{q}{mc} E_s^z(\zeta) - \frac{1}{2s} \frac{\partial v}{\partial \zeta}, \tag{18}
\]

Once solved system (18) for \( \dot{\zeta}(\xi), \dot{s}(\xi) \), the other unknowns are obtained integrating (9):

\[
\dot{x}(\xi) = x_0 + \int_{\xi_0}^{\xi} dy \frac{\dot{u}(y)}{s(y)} \tag{19}
\]
[the z-component of (19) amounts to (18a) with initial condition $\hat{z}(\xi_0) = z_0$]. If in addition $B_s \equiv 0$, then $A_s \equiv 0$ (in the Coulomb gauge), implying that $\hat{u}^+(\xi) = \frac{q}{mc^2} [K^z - \alpha^+(\xi)]$ and $\hat{v} = \hat{u}^2$ in (18) are already known. Thus the system (18) to be solved simplifies to

$$\begin{align*}
\hat{z}' &= \frac{1+\hat{v}}{2s^2} - \frac{1}{2}, \\
\hat{s}' &= \frac{-q}{mc^2} E_s^z(\hat{z}).
\end{align*}$$

(20)

Remarks. Some noteworthy properties of the corresponding solutions are [9]:

1. Where $\epsilon^\perp(\xi) = 0$ then $\hat{v}(\xi) = v_c = \text{const}$, $\hat{H}$ is conserved, (20) is solved by quadrature.

2. In case (3a) the final transverse momentum is $mc\hat{u}^+(l)$. If $\epsilon$ of (4) varies very slowly and $\hat{u}^+(0) = 0$, then by (16) $\hat{u}^+(l) \simeq 0$.

3. Fast oscillations of $\epsilon^\perp$ make $\hat{z}(\xi)$ oscillate much less than $\hat{x}(\xi)$, and $\hat{s}(\xi)$ even less: as $\hat{s} > 0$, $\hat{v} = \hat{u}^2 \geq 0$, integrating (20a) averages the fast oscillations of $\hat{u}$ to yield much smaller relative oscillations of $\hat{z}$, while integrating (20b) averages the residual small oscillations of $E_s^z(\hat{z})$ to yield an essentially smooth $\hat{s}(\xi)$. On the contrary, $\gamma(t), \beta(t), \hat{u}(t), ..., \hat{\gamma}(t), \hat{\beta}(t), \hat{u}(t), ..., \hat{\gamma}(t), \hat{\beta}(t), \hat{u}(t), ...$ are recovered via (8) and (16), which are recovered via (8), oscillate fast, and so do also $\gamma(t), \beta(t), \hat{u}(t), ...$ See e.g. fig. 2, 4, 6.

4. If $\hat{u}^+(0) = 0$ and the EM wave is a very slowly modulated (4)-(3a), integrating (14) by parts across $[0, l]$ and using (16) we find that the final energy gain is given by

$$E_f = \int_0^l d\xi \frac{\hat{v}'(\xi)}{2s(\xi)} \simeq \int_0^l d\xi \frac{\hat{v}(\xi)\hat{s}'(\xi)}{2s^2(\xi)}; \quad (21)$$

this will be automatically positive (resp. negative) if $\hat{s}(\xi)$ is growing (resp. decreasing) in all of $[0, l]$. Correspondingly, the interaction with the EM wave can be used to accelerate (resp. decelerate) the particle.

2.3 Dynamics under travelling waves and static fields $E_s, B_s$

If $E_s, B_s = \text{const}$ then eq. (13) are immediately integrated to yield

$$\begin{align*}
\hat{u}^+ &= \frac{q}{mc^2} [K^z - \alpha^+(\xi)] + (\xi + \hat{z})E_s^z + (\hat{x} \wedge B_s)^z], \\
\hat{s} &= \frac{-q}{mc^2} [K^z + \xi E_s^z - \hat{x} \cdot E_s^z + (\hat{x} \wedge B_s)^z]
\end{align*}$$

(22)

(the integration constants $K^j$ are fixed by the initial conditions), or more explicitly

$$\begin{align*}
\hat{u}^x &= w^x(\xi) + (e^x - b^y)\hat{z} + b\hat{y}, \\
\hat{u}^y &= w^y(\xi) + (e^y + b^x)\hat{z} - b\hat{x}, \\
\hat{s} &= w^z(\xi) + (e^x - b^y)\hat{x} + (e^y + b^x)\hat{y};
\end{align*}$$

(23)

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here we have introduced the dimensionless functions $w^+(\xi) := q [K^+ - \alpha^+ (\xi) + \xi E_z^+ ] / mc^2$, $w^-(\xi) := - q (K^2 + \xi E_z^2) / mc^2$ and the constants $e^+ := q E_z^+ / mc^2$, $b^+ + bk := q B_s / mc^2$. Hence, if we adopt for simplicity the initial conditions $x, y, z$, then, setting $\kappa := q E_x^+ / mc^2$, we have introduced the dimensionless functions $x, y, z$, then, setting $\kappa := q E_x^+ / mc^2$, $x_0 := \hat{x}(\xi_0)$, $y_0 := \hat{y}(\xi_0)$, $z_0 := \hat{z}(\xi_0)$, $s_0 := s(\xi_0)$, we can put the solutions of (9) in the compact form

\[
(\hat{x} + i \hat{y})(\xi) = (s_0 - \kappa \xi)^{ib/\kappa} \left[ (x_0 + iy_0) + \int_{\xi_0}^{\xi} \frac{d\xi}{s_0 - \kappa \xi} \left( \frac{x^2 + i y^2}{s_0 - \kappa \xi} \right) \right],
\]

\[
\hat{s}(\xi) = s_0 - \kappa \xi, \quad \hat{u}^z(\xi) = \frac{1}{2} \left( \frac{x^2 + i y^2}{s_0 - \kappa \xi} \right) \hat{w}^+ (\xi) - 1,
\]

\[
\hat{u}^+ (\xi) = (s_0 - \kappa \xi) \hat{x}^+ (\xi), \quad \hat{\gamma}(\xi) = s_0 - \kappa \xi + \hat{u}^z (\xi),
\]

\[
\hat{z}(\xi) = z_0 + \int_{\xi_0}^{\xi} \frac{d\xi}{2(s_0 - \kappa \xi)^2} \left( \frac{1}{s_0 - \kappa \xi} \hat{w}^+ (\xi) + \hat{x}^+ (\xi) \right) - 1.
\]  

Proposition 9 in [9]. If $E_s, B_s$ are constant fulfilling the only condition $B_s^+ = k \land E_s^+$ then, setting $\kappa := q E_s^+ / mc^2$, $x_0 := \hat{x}(\xi_0)$, $y_0 := \hat{y}(\xi_0)$, $z_0 := \hat{z}(\xi_0)$, $s_0 := s(\xi_0)$, we can put the solutions of [9] in the compact form

\[
(\hat{x} + i \hat{y})(\xi) = (s_0 - \kappa \xi)^{ib/\kappa} \left[ (x_0 + iy_0) + \int_{\xi_0}^{\xi} \frac{d\xi}{s_0 - \kappa \xi} \left( \frac{x^2 + i y^2}{s_0 - \kappa \xi} \right) \right],
\]

\[
\hat{s}(\xi) = s_0 - \kappa \xi, \quad \hat{u}^z(\xi) = \frac{1}{2} \left( \frac{x^2 + i y^2}{s_0 - \kappa \xi} \right) \hat{w}^+ (\xi) - 1,
\]

\[
\hat{u}^+ (\xi) = (s_0 - \kappa \xi) \hat{x}^+ (\xi), \quad \hat{\gamma}(\xi) = s_0 - \kappa \xi + \hat{u}^z (\xi),
\]

\[
\hat{z}(\xi) = z_0 + \int_{\xi_0}^{\xi} \frac{d\xi}{2(s_0 - \kappa \xi)^2} \left( \frac{1}{s_0 - \kappa \xi} \hat{w}^+ (\xi) + \hat{x}^+ (\xi) \right) - 1.
\]

Up to our knowledge such general solutions have not appeared in the literature before Ref. [2], but reduce to known ones under additional assumptions on $E_s, B_s$. Now we briefly review some of them, adopting for simplicity the initial conditions $x_0 = 0 = u_0$, whence $s_0 = 1$.

### 2.3.1 Zero static fields case: $E_s = B_s = 0$.

Then (24) becomes [12][13]:

\[
\hat{s}(\xi) = 1, \quad \hat{u}^+ = \frac{-q a^+}{mc^2}, \quad \hat{u}^z = \frac{\hat{u}^z}{2}, \quad \hat{\gamma} = 1 + \hat{u}^z
\]

\[
\hat{z}(\xi) = \int_{\xi_0}^{\xi} \frac{dy \hat{u}^z (y)}{2}, \quad \hat{x}^z (\xi) = \int_{\xi_0}^{\xi} dy \hat{u}^z (y).
\]

The solutions (25) induced by two $x$-polarized pulses and the corresponding $e^-$ trajectories in the $zx$ plane are shown in fig. 2. Note that:

- The maxima of $u^z$, $\alpha^+$ (and approximately also of $\epsilon(\xi)$, if $\epsilon(\xi)$ is slowly varying) coincide.

- Since $u^z \geq 0$, the $z$-drift is positive-definite. Rescaling $\epsilon^+ \mapsto a \epsilon^+$, we find that $\hat{x}^+ \mapsto \hat{x}^+$, $\hat{u}^z \mapsto \hat{u}^z$ scale like $a$, while $\hat{z}, \hat{u}^z$ scale like $a^2$; hence the trajectory goes to a straight line in the limit $a \to \infty$. This is due to magnetic force $q f B \wedge \hat{B}$ in (4).

- **Corollary** The final 4-velocity $u$ and energy gain read

\[
u^+_f = \hat{u}^+ (\infty), \quad u^+_f = E_f = \frac{1}{2} \hat{u}^+_f = \gamma_f - 1;
\]

Both are very small if the pulse modulation $\epsilon$ is slow [extremely small if $\epsilon \in S(\mathbb{R})$ or $\epsilon \in C_c(\mathbb{R})$].
Figure 2: Solutions (25) and $e^-$ trajectories in the $zx$ plane induced by two $x$-polarized pulses with carrier wavelength $\lambda \equiv 2\pi/k = 0.8\mu m$, gaussian modulation $\epsilon(\xi) = E_{M}^{\perp} \exp[-\xi^2/2\sigma]$, $\sigma = 20\mu m^2$, $a_0 \equiv \epsilon E_{M}^{\perp}/kmc^2 = 0.8$ (left) or $a_0 = 3.3$ (right).

We compare this result with the so-called Lawson-Woodward or General Acceleration Theorem [14, 15, 16, 17]. This states that, in spite of large energy variations during the interaction, the final energy gain $E_f$ of a charged particle $P$ interacting with an EM field is zero if:

1. the interaction occurs in $\mathbb{R}^3$ vacuum (no boundaries);
2. $E_s = B_s = 0$ and $\epsilon^{\perp}$ is very slowly modulated;
3. $v^z \simeq c$ along the whole acceleration path;
4. nonlinear (in $\epsilon^{\perp}$) effects $q\beta \wedge B$ are negligible;
5. the power radiated by $P$ is negligible.

Our Corollary, as Ref. [18], states that the final energy gain is zero also if we relax iii), iv), but the EM field is a plane travelling wave. To obtain a non-zero $E_f$ one has to violate some other conditions of the theorem, as e.g. we see in next cases.
2.3.2 Case $E_s = 0$, $B_s = B_s^z k$, and cyclotron autoresonance.

Figure 3: The electron motion (27) (up) and the $zx$-projection of the corresponding trajectory (down) induced in a longitudinal magnetic field $B^z = 10^5$ G by a circularly polarized modulated EM wave (4) with wavelength $\lambda = 2\pi/k = 1mm$, $b = k = 58.6\text{cm}^{-1}$, gaussian enveloping amplitude $\epsilon(\xi) = E^\perp_M \exp[-\xi^2/2\sigma]$ with $\sigma = 3\text{cm}^2$ and $a_0 \equiv eE^\perp_M/kmc^2 = 0.15$, trivial initial conditions ($x_0 = u_0 = 0$), giving $\dot{E}_f \approx 28.5$.

Eq. (24) becomes $\dot{s} \equiv 1$ and, since $(1-\kappa\xi)^{ib/\kappa} = \exp [(ib/k) \log (1-\kappa\xi)] = \exp [-ib/\kappa \kappa \xi + O(k)]$ reduces to $e^{-ib\xi}$ as $\kappa \to 0$,

\[
\begin{align*}
(\dot{x} + i\dot{y})(\xi) &= \int_0^\xi d\zeta \ e^{ib(\zeta-\xi)} (w^x+iw^y)(\zeta), \\
\hat{u}^z &= \hat{x}' = \hat{u}^{z,2}, \\
\hat{u}^y = \hat{z}' = \frac{\hat{u}^{y,2}}{2} &\equiv \mathcal{E} = \gamma - 1, \\
\hat{z}(\xi) &= \int_0^\xi d\zeta \ \frac{\hat{u}^{z,2}(\zeta)}{2}.
\end{align*}
\]  

(27)

The solution (27) reduces to that of [19, 20] if $\epsilon^\perp$ is monochromatic. This leads to cyclotron autoresonance if $-b - k = 2\pi/\chi \gg \frac{k}{\xi}$. In fact, this conditions ensures that the cyclotron frequency associated to $B^z_s$ equals the EM wave frequency, implying an accelerated transverse motion as in a cyclotron (with a spiral trajectory), since also the transverse magnetic field oscillates with the same frequency, the associated (longitudinal) magnetic force keeps always the same sign and thus accelerates the particle in the $z$-direction during the whole EM pulse. If for simplicity $\epsilon^\perp$ is slowly modulated, circularly polarized then $w^x(\xi) + iw^y(\xi) \simeq e^{ik\xi}w(\xi)$, where $w(\xi) \equiv qe(\xi)/kmc^2$, and

\[
(\dot{x} + i\dot{y})(\xi) \simeq iW(\xi)e^{ik\xi}, \quad (\dot{u}^x + i\dot{u}^y)(\xi) \simeq -ke^{ik\xi}W(\xi),
\]

where $W(\xi) \equiv \int_0^\xi d\zeta w(\zeta) > 0$ grows with $\xi$. In particular if $\epsilon^\perp(\xi) = 0$ for $\xi \geq l$, then for
such $\xi$

$$\dot{z}'(\xi) \simeq \frac{k^2}{2} W^2(l) \simeq 2E_f; \quad \frac{|\dot{x}^+(\xi)|}{\dot{z}'(\xi)} \simeq \frac{2}{kW(l)} \ll 1;$$

the final energy gain and collimation are noteworthy resp. by the first, second relation.

2.3.3 Case $E_s = E_z^s k$, $B_s = 0$

Then the solution (24) reduces to

$$\dot{s}(\xi) = 1 - \kappa \xi,$$

if $e^\pm$ is slowly modulated then by (21) the final energy gain $E_f \simeq \int_0^\xi d\xi \frac{-\dot{\kappa}^2(\xi)}{2(1 - \kappa \xi)^2}$ is negative if $\kappa > 0$, positive if $\kappa \leq 0$ and has a unique maximum point $\kappa_M < 0$ if $e(\xi)$ has a finite

4Clearly the domain of definition is $\xi \in [0, \xi_f]$, where $\xi_f \leq 1/\kappa$; however $\dot{t}(\xi_f) = \infty$, so that as a function of $t$ the solution is defined in all of $[0, \infty[$, as required [9].
support with a unique maximum. An acceleration device based on this solution could be as follows: at \( t = 0 \) the particle lies at rest with \( z_0 \lesssim 0 \), just at the left of a metallic grating \( G \) contained in the \( z = 0 \) plane and set at zero electric potential; another metallic plate \( P \) contained in a plane \( z = z_p > 0 \) is set at electric potential \( V = V_p \). A short laser pulse \( \epsilon^\perp \) hitting the particle boosts it beyond \( G \) through the ponderomotive force; choosing \( qV_p > 0 \) implies \( \kappa = -qV_p / z_p mc^2 < 0 \), and a backward longitudinal electric force \( qE^\perp z \). If \( qV_p \) is large enough, then \( z(t) \) reaches a maximum smaller than \( z_p \), then is accelerated backwards and exits the grating with energy \( \mathcal{E}_f \) and negligible transverse momentum. However, a large \( \mathcal{E}_f \) requires extremely large \( |V_p| \), far beyond the material breakdown threshold, what prevents its realization as a static field (namely, sparks between \( G, P \) would arise and rapidly reduce \( |V_p| \)). Alternatively, one can make the pulse itself generate such large \( |E^\perp z| \) within a plasma at the right time, so as to induce the slingshot effect \([21, 22, 23]\); this is briefly explained at the end of next section.

3 Impact of a short laser pulse onto a cold diluted plasma

![Figure 5](image)

Figure 5: Left: a plane EM wave of finite length approaching normally a plasma in equilibrium. Right: normalized EM wave \( \epsilon^\pm \) (blue) with carrier wavelength \( \lambda = 0.8 \mu m \), linear polarization, gaussian modulation \( \epsilon(\xi) = E_M^\perp \exp(-\xi^2/2\sigma) \) with \( \sigma = 20 \mu m^2 \); \( a_0 \equiv eE_M^\perp / kmc^2 = 3.3 \) (whence the average pulse intensity is \( 10^{19} \text{W/cm}^2 \)); the associated \( u^\pm_\perp \) is painted purple. \( l \) is the length of the \( z \)-interval where the amplitude \( \epsilon \) overcomes all ionization thresholds of the atoms of the gas yielding the plasma; here we have chosen helium, whence \( l \approx 40 \lambda = 32 \mu m \), and the thresholds for 1\(^{st}\) and 2\(^{nd}\) ionization are overcome almost simultaneously.

3.1 Our simplified hydrodynamic model and its range of validity

We distinguish the types of particles (electrons and ions, possibly of various kinds) composing the plasma by an index \( h \), and denote by \( q_h, m_h \) their charges, masses. Assume that the plasma is initially in hydrodynamic conditions with all initial data not depending on \( x^\perp \);
these data consist of EM fields of the form (2) as well as the set of initial values of the Eulerian velocity \( \mathbf{v}_h \) and density \( n_h \) of each fluid consisting of the same type of particles. Consequently also the solutions of the Lorentz-Maxwell and continuity equations for \( \mathbf{B}, \mathbf{E}, \mathbf{u}_h, n_h \) do not depend on \( \mathbf{x}^i \), nor do the displacements \( \Delta \mathbf{x}_h \equiv \mathbf{x}_h(t, \mathbf{X}) - \mathbf{X} \) on \( \mathbf{X}^+ \). Here \( \mathbf{x}_h(t, \mathbf{X}) \) is the position at time \( t \) of the material element of the \( h \)-th fluid with initial position \( \mathbf{X} \equiv (X, Y, Z) \); \( \mathbf{X}_h(t, \mathbf{x}) \) is the inverse of \( \mathbf{x}_h(t, \mathbf{X}) \) (at fixed \( t \)); \( \beta_h = \mathbf{v}_h/c \), etc. For brevity we refer to the particles (of the \( h \)-th type) contained: in this material element, as the ‘\( h \) particles’ (e.g. ‘\( X \) electrons’); in a region \( \Omega \), as the ‘\( \Omega \) particles’; in the layer between \( Z, Z + dZ \), as the ‘\( Z \) particles’. More specifically, we consider (fig. 5) a very short and intense EM plane wave (laser pulse) (3a) hitting normally a cold plasma initially in equilibrium, possibly immersed in a static and uniform magnetic field \( \mathbf{B}_s \) (actually, the plasma may be locally obtained from a gas ionized by the very high electric field of the pulse itself). The initial conditions are:

\[
\begin{align*}
n_h(0, \mathbf{x}) = 0 & \quad \text{if } z \leq 0, \quad \mathbf{u}_h(0, \mathbf{x}) = 0, \quad j^0(0, \mathbf{x}) = \sum q_h n_h(0, \mathbf{x}) = 0, \\
\mathbf{E}(0, \mathbf{x}) &= \mathbf{\epsilon}^+(-z), \quad \mathbf{B}(0, \mathbf{x}) = k \wedge \mathbf{\epsilon}^+(-z) + \mathbf{B}_s,
\end{align*}
\]

whence the 4-current density \( j = (j^0, \mathbf{j}) = \sum q_h n_h(1, \beta_h) \) is zero at \( t = 0 \). Then the Maxwell equations \( \nabla \cdot \mathbf{E} = 4\pi j^0, \partial_t \mathbf{E}/c + 4\pi j^z = (\nabla \wedge \mathbf{B})^z = 0 \) imply \( 13 \)

\[
E^z(t, z) = 4\pi \sum q_h \tilde{N}_h[Z_h(t, z)], \quad \tilde{N}_h(Z) := \int_0^Z d\zeta n_h(0, \zeta);
\]

using (30) to express \( E^z \) in terms of the (still unknown) longitudinal motion \( [Z_h(t, \cdot) \) is the inverse of \( z_h(t, \cdot) \) we reduce the number of unknowns by one.

Define \( \mathbf{a}^\parallel \) as in (15). In the Landau gauges (29) compatible with the following initial conditions for the gauge potential:

\[
\begin{align*}
\mathbf{A}(0, \mathbf{x}) &= \mathbf{a}^\parallel(-z) + \mathbf{B}_s \wedge \mathbf{x}/2, \quad \partial_t \mathbf{A}(0, \mathbf{x}) = -c \mathbf{\epsilon}^+(-z), \\
A^0(0, \mathbf{x}) &= \partial_t A^0(0, \mathbf{x}) = 0. \quad \text{Eq (29,31) and causality imply that } \mathbf{x}_h(t, \mathbf{X}) = \mathbf{X}, \mathbf{A}^\parallel(t, \mathbf{x}) = \mathbf{B}_s \wedge \mathbf{x}/2 \text{ if } ct \leq z, \quad \mathbf{j} \equiv 0 \text{ if } ct \leq |z|. \quad \mathbf{A}^\parallel \text{ is coupled to the current through } \Box \mathbf{A}^\parallel = 4\pi \mathbf{j}^\parallel.
\end{align*}
\]

Equipped with (31) the latter amounts to the integral equation

\[
\mathbf{A}^\parallel - \mathbf{a}^\parallel - \frac{1}{2} (\mathbf{B}_s \wedge \mathbf{x})^\parallel = 2\pi \int dsd\zeta \theta(ct - s - |z - \zeta|) \theta(s) j^\parallel \left( \frac{s}{c}, \zeta \right);
\]

here we have used the Green function of the d’Alembertian \( \partial_t^2/c^2 - \partial_z^2 \) in dimension 2 (\( \theta \) stands for the Heaviside step function). The right-hand side (rhs) is zero for \( t \leq 0 \) (\( t = 0 \) is the beginning of the laser-plasma interaction). Within short time intervals \([0, t']\) (to be determined \textit{a posteriori}) we can thus: approximate \( \mathbf{A}^\parallel(t, z) \approx \mathbf{a}^\parallel(ct - z) + (\mathbf{B}_s \wedge \mathbf{x})^\parallel \); also neglect the motion of ions with respect to the motion of the (much lighter) electrons. Hence we set \( z_p(t, Z) \equiv Z \), and the proton density \( n_p \) (due to ions of all kinds) equals the initial one and therefore the initial electron density \( \tilde{n}_q(z) := n_e(0, z) \), by the initial electric neutrality of the plasma.
We now adopt $\xi$ instead of $t$ as the independent variable. Setting $\tilde{N}(Z) := \int_0^Z d\zeta \tilde{n}_0(\zeta)$, the equations \( \ref{eq:32} \) & initial conditions for the electron fluid amount to

\[
\begin{align*}
mc^2 \ddot{e}_e(\xi, Z) & = 4\pi e^2 \left[ \tilde{N}(\tilde{e}_e) - \tilde{N}(Z) \right] + e(\Delta \ddot{x}_e + \dot{B}_e) \quad (\text{33}), \\
mc^2 \ddot{\mathbf{u}}_e'(\xi, Z) & = e\alpha'/t - e(\Delta \mathbf{x}_e + \dot{B}_e)^{\perp}, \\
\Delta \mathbf{x}_e(0, \mathbf{X}) & = 0, \quad \mathbf{u}_e(0, \mathbf{X}) = 0 \quad \Rightarrow \quad \ddot{e}(0, \mathbf{X}) = 1. \quad (\text{34})
\end{align*}
\]

This makes eq. \( \text{(33)} \) a family parametrized by $Z$ of decoupled ODEs (of the type considered in section \( \ref{sec:2} \)) in the unknowns $\Delta \mathbf{x}_e, \ddot{e}_e, \mathbf{u}_e^{\perp}$, rather than a set of PDEs. After solving these equations and inverting $\ddot{e}_e(\mathbf{x}, Z)$, all Eulerian fields $f(ct, z)$ will be obtained from $f(\xi, Z)$ by the replacement $(\xi, Z) \mapsto (ct - z, \tilde{Z}(ct - z, z))$.

If $\mathbf{B}_e = 0$, then as in section \( \ref{sec:2.2} \) \( \text{(33b)} \) is solved by $\mathbf{u}_e^{\perp}(\xi, Z) = e\alpha'(\xi)/mc^2$, while abbreviating $\Delta \equiv \Delta \ddot{e}_e$ and $v \equiv \mathbf{u}_e^{\perp}$, \( \text{(33a)} \) and the $z$-component of \( \text{(33c)} \) take \( \ref{eq:22} \) \( \ref{eq:23} \) the form of \( \text{(20)} \),

\[
\dot{\Delta}' = \frac{1 + v}{2s^2} - \frac{1}{2}, \quad \mathbf{s}'_e = \frac{4\pi e^2}{mc^2} \left\{ \tilde{N}[\tilde{e}_e] - \tilde{N}(Z) \right\}. \quad \text{(35)}
\]

For every $Z$ \( \text{(35)} \) have the form of Hamilton equations $q' = \partial \hat{H}/\partial p$, $p' = -\partial \hat{H}/\partial q$ of a 1-dim system: $\xi, \Delta, -\ddot{e}_e$ play the role of $t, q, p$, while the Hamiltonian is rational in $\ddot{e}_e$ and reads

\[
\begin{align*}
\hat{H}(\Delta, \ddot{e}_e, \xi; Z) & \equiv \frac{\ddot{e}_e^2 + 1 + v(\xi)}{2\ddot{e}_e} + U(\Delta; Z), \\
U(\Delta; Z) & \equiv \frac{4\pi e^2}{mc^2} \left[ \tilde{N}(Z + \Delta) - \tilde{N}(Z) - \tilde{N}(Z)\Delta \right], \\
\tilde{N}(Z) & \equiv \int_0^Z d\zeta \tilde{N}(\zeta) = \int_0^Z d\zeta \tilde{n}_0(\zeta) (Z - \zeta);
\end{align*}
\]

we have defined the potential energy $U$ fixing the free additive constant so that $U(0, Z) \equiv 0$. Eq.s \( \text{(35,34)} \) can be solved numerically, or by quadrature where $e^\perp(\xi) = 0$ (i.e. after the pulse). Finally, the equation $\dot{\mathbf{s}}_e = \mathbf{u}_e/\ddot{e}_e$ with initial conditions \( \text{(34)} \) is solved by

\[
\dot{\mathbf{x}}_e^\perp(\xi, \mathbf{X}) - \mathbf{X}^\perp = \int_0^\xi \mathbf{u}_e^\perp(\eta)/\ddot{e}_e(\eta, \mathbf{Z}), \quad \dot{e}_e(\xi, Z) - Z = \Delta(\xi, Z). \quad (\text{37})
\]

By derivation we obtain several useful relations, in particular

\[
\frac{\partial Z_e}{\partial z}(t, z) = \frac{\dot{\ddot{e}}_e}{\ddot{e}_e} \frac{\partial Z_e}{\partial Z}(t, z) \bigg|_{(\xi, Z) = (ct - z, \tilde{Z}(ct - z, z))}. \quad (\text{38})
\]

Hence the maps $\mathbf{x}_e(\xi, \cdot) : \mathbf{X} \mapsto \mathbf{x}_e, \quad \mathbf{x}_e(t, \cdot) : \mathbf{X} \mapsto \mathbf{x}_e$ are invertible as long as $\partial Z \ddot{e}_e \equiv \partial \ddot{e}_e/\partial Z$ is positive. The approximation on $A^+(t, z)$ is acceptable as long as the so determined motion makes $|\text{rhs}(\text{32})| \ll |\alpha^+ + B_e/2 \wedge \mathbf{x}|$; otherwise rhs \( \text{(32)} \) determines the first correction to $A^+$; and so on.

Summarizing, the present hydrodynamic model is justified in a sufficiently short time interval $[0, t']$ where $\partial Z \ddot{e}_e > 0$, $|\text{rhs}(\text{32})| \ll |\alpha^+ + B_e/2 \wedge \mathbf{x}|$, and the motion of ions is negligible.
3.2 Some general features of the motions ruled by (35-34)

\( \bar{N}(Z) \) grows with \( Z \), and so does the rhs (35b) with \( \dot{\hat{\Delta}} \). As soon as \( v(\xi) \) becomes positive for \( \xi > 0 \), then so do also \( \Delta \) and \( \dot{s}_e - 1 \): by (37), all electrons reached by the pulse start to oscillate transversely and drift forward (pushed by the ponderomotive force); the \( Z \approx 0 \) electrons leave behind themselves a layer of ions \( L_e \) of finite thickness \( \zeta(t) = \Delta(t, 0) = \hat{\Delta} [\hat{\xi}(t, 0), 0] \) completely deprived of electrons, see fig. 7. \( \dot{s}_e \) keeps growing as long as \( \hat{\Delta} \geq 0 \), making the rhs (35a) vanish at the first \( \hat{\xi}(Z) > 0 \) where \( \hat{\xi}^2(\xi, Z) = 1 + v(\xi) \) and become negative for \( \xi > \hat{\xi} \). Hence \( \hat{\Delta}(\xi, Z) \) reaches a positive maximum at \( \xi = \hat{\xi}(Z) \) and then starts decreasing towards negative values (electrons are attracted back by ions in \( L_e \)). For \( \xi > l \) energy is conserved, the paths in \( (\Delta, s_e) \) phase space are the level curves (parametrized by \( Z \)) fulfilling \( \dot{H}(\Delta, s_e; Z) = h(Z) = 1 + \int_0^1 d\xi v'(\xi)/s(\xi, Z) \) [by (21)]; the dependence of \( \dot{P} \equiv (\hat{\Delta}, \dot{s}_e) \) on \( \xi \) is obtained by quadrature. Since \( \dot{U}(\Delta; 0) = 0 \) for \( \Delta \leq 0 \), then \( \hat{\Delta}(\xi, 0) \to -\infty \) as \( \xi \to \infty \): the \( Z = 0 \) electrons escape to \( z_e = -\infty \) infinity. Whereas if \( Z > 0 \) then \( \dot{U}(\Delta; Z) \to \infty \) as \( |\Delta| \to \infty \), the path is a cycle, and \( P(\xi, Z) \) is periodic in \( \xi \), with period

\[
c T(Z) \equiv \Xi(Z) = \int_{\Delta_m}^{\Delta_M} \frac{2 d\Delta}{\sqrt{1 - \mu^2/[h(Z) - \dot{U}(\Delta; Z)]^2}}, \quad \mu^2 \equiv 1 + v(l) : (39)
\]

all \( \hat{\Delta}(\cdot, Z) \) oscillate around zero, i.e. all \( Z > 0 \) electrons do \( T(Z) \)-periodic longitudinal oscillations about their initial positions \( z = Z \).

Assume now that the electron density has an upper bound and becomes a constant well inside the bulk: \( 0 < \tilde{n}_0(z) \leq n_b \) if \( z > 0 \) and \( \tilde{n}_0(z) = n_0 \) if \( z \geq z_s \), for some constant \( n_b \geq n_0, z_s \geq 0 \). Then there exist \( Z_b > 0 \) and \( Z_d > Z_b, z_s \) such that:

1. the \( Z \in [0, Z_b[ \) electrons exit and re-enter the bulk;
2. the \( Z \geq Z_b \) electrons remain inside the bulk, i.e. \( \dot{s}_e \geq 0 \) for all \( \xi \);
3. the \( Z \geq Z_d \) electrons fulfill \( \dot{s}_e \geq z_s \) for all \( \xi \).

For the latter \( \tilde{n}_0(\xi) \equiv n_0, \dot{U}(\Delta, Z) \equiv M \Delta^2/2 \), so that (35-34) no longer depends on \( Z \) and reduces to the same Cauchy problem for all \( Z \geq Z_d \):

\[
\Delta' = \frac{1 + v}{2s^2} - \frac{1}{2}, \quad s' = M \Delta, \quad M := \frac{4\pi e^2 n_0}{mc^2} \equiv \omega_p^2/c^2, \quad \Delta(0) = 0, \quad s(0) = 1.
\]

Eq. (40) is the equation of motion of a relativistic harmonic oscillator (with a forcing term \( v \) for \( 0 < \xi < l \)); consequently, also the final energy \( h \) is the same for all \( Z \geq Z_d \). We denote as \( \Xi_{\mu}(n_0) = cT_{\mu}(n_0) \) the corresponding plasma period (39) (recall that \( T_{\mu} \geq T_{\mu}^{nr} \equiv 2\pi/\omega_p = \sqrt{\pi m/n_0 e^2} \), which is the non-relativistic limit of \( T_{\mu} \)). Thus for all \( Z > Z_d \) electrons it is

\[[5] \text{When } \dot{s} = 0 \text{ then (40) implies } \Delta'' = -M \Delta/s^3. \text{ In the nonrelativistic regime } \dot{s} \simeq 1, \dot{\xi}(t) \simeq ct, \text{ and this becomes the nonrelativistic harmonic equation } \hat{\Delta} = -\omega_p^2 \Delta. \]
Figure 6: a) Normalized gaussian EM pulse of full width at half maximum $l_{fwhm} = 10.5\lambda$, linear polarization, peak amplitude $a_0 \equiv 2\pi eE_M^{\perp}/mc^2k = 2$ (leading to a peak intensity $I = 1.7 \times 10^{19}$W/cm$^2$ if $\lambda = 0.8\mu$m). b) Corresponding solution of (40-41) for $Z \geq Z_d$, with $\tilde{n}_0(Z) = n_0\theta(Z)$, $n_0 \equiv 10^{18}$cm$^{-3}$ (whence $h = 1.36$); as anticipated, $\hat{s}$ is indeed insensitive to fast oscillations of $\epsilon^{\perp}_\perp$. c) Corresponding normalized electron density inside the bulk as a function of $z$ at $ct = 200\lambda$. d) Phase portraits for the same $\tilde{n}_0(Z)$, $h = 1.36$, $\mu = 1$.

$\hat{z}_e(\xi, Z) = Z + \Delta(\xi)$ and hence $z_e(t, Z) = Z + \Delta(ct-z)$ for all $\xi$. The restriction of $z_e(t, \cdot)$ to $[Z_d, \infty]$ is invertible [with inverse $Z_e(t, z) = z - \Delta(ct-z)$]: no $Z$-electron layer with $Z \geq Z_d$ can collide with another one. This justifies there the hydrodynamical picture used so far. All Eulerian fields are found to depend on $t, z$ only through $ct-z$, e.g. $u(t, z) = \hat{u}(ct-z),...$: a plasma travelling-wave with spacial period $\Xi_H(n_0)$ and phase velocity $c$ trails the pulse for $z \geq Z_d$ [24]. On the other hand, plasma wave breakings [24], i.e. collisions among (sufficiently close) $Z$-electron layers with $Z < Z_d$, occur only at sufficiently large times, due to the nontrivial $Z$-dependence of (39) in the regions where $\tilde{n}_0(Z)$ is non-homogenous (in particular near the vacuum-plasma interface $Z \sim 0$). There our hydrodynamic description is globally self-consistent for $t < t_c$ ($t_c$ stands for the time of the first wave-breaking) and allows to determine [24] $t_c$ and where wave-breakings occur; the use of kinetic theory (a statistical description of the plasma in phase space that takes collisions into account) is necessary after the first wave-breaking. As known, a moderate wave-breaking is actually welcome as a possible injection mechanism in the plasma wave of a bunch of electrons fast enough and in phase to be accelerated ‘surfing’ the wave, according to the LWFA mechanism [5]; a phase of the right sign arises where $\tilde{n}_0(Z)$ decreases.

Let $T_H(n_b)$ is the plasma period (39) associated to the electron density $\tilde{n}_0(Z) \equiv n_b$. If

$$2l \lesssim \Xi_H(n_b) = cT_H(n_b)$$

(42)
Figure 7: Conditions as in fig. 6 ($\tilde{\eta}_0(Z) = n_0\theta(Z)$). Left: Normalized total charge density $1 - n_e/n_0$ at $ct = 20\lambda$; the yellow part denotes the positive ion layer $L_I$, the thin blue stripes denote the negative parts (due to an excess of electrons). Right: The corresponding worldlines of $Z$-electrons; for $Z \geq Z_d \approx 10\lambda$ they do not intersect, whereas the $Z < Z_d$ ones that first intersect do about after $5/4$ oscillations induced by the pulse, here at $t = t_c \approx 84\lambda/c = 224\text{fs}$. In pink the support $0 \leq ct - z \leq l$ of the pulse, in red the 'effective support' $l_{\text{fwhm}}/2 \leq ct - z \leq l_{\text{fwhm}}/2$ of the pulse, in yellow the evolution of the pure-ion layer $L_I$. 


(i.e. the pulse is sufficiently short) then \( \Delta(l,Z) \geq 0 \) for all \( Z \): the pulse overcomes all electrons before they overshoot their initial \( z \)-coordinate; in particular, it has completely entered the bulk before any small-\( Z \) electron gets out of it, namely before \( L_t \) is refilled. Condition (42) also secures that the spacial period of the plasma wave is larger that the pulse length.

In fig. 6 we plot the solution corresponding to the pulse of fig. 5-right (with \( l \approx 40 \mu m, \ l_fwhm \approx 10.5 \mu m \)) and to \( n_0 = 10^{18} \text{cm}^{-3} \); \( s(\xi) \) is indeed insensitive to the fast oscillations of \( \epsilon_\perp \) (see remark 2.2.3); \( \Delta(\xi) \) grows positive for small \( \xi \). The other unknowns are obtained through (19). After the pulse is passed the solution becomes periodic with period \( \Xi_H \approx 47 \mu m \).

These \( l_fwhm, n_0 \) fulfill (42). Replacing these solutions in the rhs(32) we find that \( A_\perp \approx \alpha_\perp \) is indeed verified at least for \( t < t_c \approx 5 \xi_H/c \). In fig. 7 we plot the charge density just after the pulse has hit the plasma and the worldlines of the \( Z \)-electrons for \( 0 < Z < 30 \lambda \) and \( 0 \leq t \leq 100 \lambda/c \approx 267 \text{fs} \).

### 3.3 Comparison with the ‘comoving frame’ approach

The Lorentz-Maxwell and continuity equations in a (homogeneous) plasma admit [26] travelling wave solutions where the EM field (plane wave laser pulse) has the form \( E(t,x) = \epsilon_\perp(v_l t - z), B(t,x) = k \wedge \epsilon_\perp(v_l t - z) \); i.e. the pulse travels with a (constant) velocity \( v_l = v_l k, \ v_l \leq c \). If \( v_l < c \), in studying the electrodynamics (plasma waves, LWFA, ...) induced by such a pulse it is common and useful to adopt the ‘comoving frame’ approach. This consists in describing the physics not w.r.t. the inertial reference frame \( \mathcal{R} \) of the laboratory, but w.r.t. a one \( \mathcal{R}' \) comoving with the pulse, i.e. moving with velocity \( v_l \) with respect to (w.r.t.) \( \mathcal{R} \). Setting \( \beta_l = v_l/c, \gamma_l^{-1} = \sqrt{1 - \beta_l^2}, \) the spacetime coordinates w.r.t. \( \mathcal{R}, \mathcal{R}' \) are related by the Lorentz transformation

\[
ct' = \gamma_l(ct - \beta_l z), \quad x'^\perp = x^\perp, \quad z' = \gamma_l(z - \beta_l ct), \quad (43)
\]

and more generally the components of all 4-tensors [in particular the canonically conjugated 4-momentum \((H, \mathbf{P})\) and the EM tensor \((F^{\mu\nu})\)] w.r.t. \( \mathcal{R}, \mathcal{R}' \) are related by this Lorentz transformation and can be expressed in terms of \( x' \); the pulse part \( \epsilon^{\perp'} \) of the EM field is a function of \( t' \) only. Therefore adopting \( t' \) as the independent variable has the same advantage described before, i.e. \( \epsilon^{\perp'}(t') \) appears in the equations of motion of all charged particles (including the plasma constituents) as a known forcing term; correspondingly, the known expression \( A^{\perp'}(x') = \alpha^{\perp'}(t') \) appears in the Hamiltonian of all charged particles. However, this approach is no more applicable when \( v = c \), for which the transformation (43) is ill-defined.

On the contrary, although \( \xi = ct - \beta_l z \propto t' \) can no more be interpreted as the time coordinate w.r.t. an inertial reference frame, in our approach adopting \( \xi \) instead of \( t' \) as the independent variable keeps this advantage and allows to take the limit \( v \to c \).
3.4 Finite laser spot radius corrections, slingshot effect, and discussion

The above predictions are based on idealizing the laser pulse as a plane EM wave. In a more realistic picture the laser pulse is cylindrically symmetric around the $\vec{z}$-axis and has a finite spot radius $R$. The first rough correction to the above predictions is that only the $Z \simeq 0$ electrons inside a cylinder of radius $R$ are pushed forward and leave behind themselves a cylinder of ions of the same radius and finite height $\zeta(t) = \Delta(t, 0) = \Delta[\xi(t, 0), 0]$ completely deprived of electrons. Using causality and heuristic arguments we can compute [22] further (rough) corrections, in particular estimate the motion of the lateral electrons (LE), just outside $c_t$, toward the axis of $c_t$, attracted by the ions. If $R$ is not too small the $Z$-electrons with smallest $Z$ and closest to the axis succeed in exiting the bulk before the LE reach the axis and close their way out, and proceed indefinitely in the negative $z$ direction. As a result, for carefully tuned $R, \tilde{n}_0(Z)$ [fulfilling [42], in particular] the impact of a very short and intense laser pulse on the surface of a cold low-density plasma (or gas, ionized into a plasma by the pulse itself), as considered e.g. in fig. 5-right, may induce, beside a plasma traveling-wave trailing the pulse, also the slingshot effect [22, 23, 21], i.e. the backward acceleration and expulsion from the plasma of some surface electrons (those with smallest $Z$ and closest to the $\vec{z}$-axis) with remarkable energy. For reviews see also [27, 28]. On the other hand, if $R$ is smaller the LE may close the rear part of $c_t$ and make it into an ion bubble (completely deprived of electrons), before any electron gets out of the bulk. Depending on the conditions, the bubble may then disappear or trail the pulse; in the latter case the LWFA takes place in the particularly favourable bubble regime [29, 30, 31, 32].

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