Network Simulation solution of free convective flow from a vertical cone with combined effect of non-uniform surface heat flux and heat generation or absorption

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Abstract: A two dimensional mathematical model is formulated for the transitive laminar free convective, incompressible viscous fluid flow over vertical cone with variable surface heat flux combined with the effects of heat generation and absorption is considered. Using a powerful computational method based on thermoelectric analogy called Network Simulation Method (NSM), the solutions of governing nondimensional coupled, unsteady and nonlinear partial differential conservation equations of the flow that are obtained. The numerical technique is always stable and convergent which establish high efficiency and accuracy by employing network simulator computer code Pspice. The effects of velocity and temperature profiles have been analyzed for various factors, namely Prandtl number $Pr$, heat flux power law exponent $n$ and heat generation/absorption parameter $\Delta$ are analyzed graphically.

1. Introduction

Free convective flow due to the impact of gravitational force have been examined broadly because they happen often in nature and also in various fields of science and engineering. When a fluid is in contact with the heated surface is the result of temperature variation induces buoyancy force, which influences natural convection. Free convection flow occurs most frequently in cooling of nuclear reactors or in the investigation of the structure of stars and planets. Free convective flows caused by difference in temperature or concentration have been analysed by many authors. Numerous authors [1-4] from 1953 studied free convective flow from vertical cone for various boundary conditions (i.e. isothermal/non-isothermal temperature (heat flux)/concentration (mass flux)) in porous/non-porous medium with various methods (similarity/non-similarity, finite difference etc..) in steady/unsteady state.

To solve the present problem a well-tested, highly adaptable numerical procedure known as the Network Simulation Method (NSM) has been applied. Originally Nagel [5] formulated this method for applications in semiconductors at the University of California, Berkeley. In many engineering applications this method has been implemented subsequently. The fundamentals of NSM are based on thermoelectric analogy between thermal and electrical variables. NSM can be used for any kind of non-linearity due to boundary conditions, temperature differences of the thermal properties etc. This technique has been implemented recently to complex nonlinear thermo fluid dynamic problems. Zueco
and co-workers [8-15] formulated in its present form by utilizing a discretization procedure for the differential equations. NSM makes the partial differential equations that characterize the mathematical system of the physical process in terms of spatial discretization [6], yields the ordinary differential equations that are the fundamental to carry out the standard electrical network simulation model of each elemental control volume. In the discretized equations time is treated as a continuous variable. A network model is designed based on these equations. The whole medium and boundary conditions are expressed in terms of special electrical components connected in series to frame networks. The important advantage is that the network model consists of a couple of electrical components connected in series to which the boundary conditions are considered together to frame the entire model of the system. The design of network model is the conjunction of current control generators, resistors and capacitors with minimum programming rules. Once the complete network model is designed, a computer code Pspice [5] is applied to simulate it and to yield the numerical solution. [7] developed the relation between NSM and heat transfer. Beg et al. [8] presented a two dimensional mathematical model for the laminar free convective, steady, incompressible, boundary layer flow over a continuously moving plate embedded in a thermally stratified, non-Darcian porous medium and the governing conservation equations are solved by applying Network Simulation Method along with the Pspice algorithm. Beg et al. [9] obtained the numerical solutions of the sphere immersed in porous medium with composite impacts of heat generation, buoyancy, magnetic field and also Darcian and Forchheimer drag forces on the boundary layer using the NSM. Zueco et al. [10] investigate the axisymmetric convective heat and mass transfer boundary layer flow of vertical thin cylinder with constant heat and mass flux and solutions of boundary layer equations are obtained using Network Simulation Method. Zueco and Beg [11] analysed optically thick, magnetohydrodynamic, dissipative gas boundary layer flow past a non-isothermal permeable wedge immersed in a scattering, homogeneous, isotropic Darcy-Forchheimer porous regime with significance effects in thermal radiation together with heat sink/sources and surface transpiration. Beg et al. [12] developed a network model to simulate the unsteady, nonlinear buoyancy-driven double convective, grey, absorbing-emitting, incompressible fluid flow past an arbitrarily moving vertical plate. Zueco [13] studied unsteady natural free convective MHD flow of viscous dissipative fluid past a semi-infinite vertical plate with uniform heat flux using NSM. Beg et al. [14] developed the numerical solution for Magnetohydrodynamic, viscous, incompressible fluid flow over consisting two infinite parallel plates in rotating channel in Darcian porous medium with invariant pressure gradient using the Network Simulation Method. Zueco et al [15] presented network model for the laminar heat and mass transfer of an electrically conducting, heat generating/absorbing fluid past a permeable horizontal surface with the existence of viscous and joule heating and solved using the Network Simulation Method.

The aim of the present study is to analyse the problem of unsteady free convective flow from a non-isothermal vertical cone with variable surface heat flux together with the combined effect of heat generation/absorption was not attempted by any researcher in literature survey. Hence, the present work investigated the natural convective flow over a non-isothermal vertical cone with the effects as mentioned above. The solutions of the governing conservation equations are obtained by applying Network Simulation Method. The effects of velocity and temperature for various quantities of parameters $A$, $Pr$ and $n$ are studied.
2. Mathematical Formulation

Figure 1: Physical Model and co-ordinate system

An axi-symmetric transient natural convective flow of a viscous incompressible fluid flow over vertical cone with variable surface heat flux together with the effects of heat absorption or generation. Further it is considered that the impact due to pressure gradient and viscous dissipation along the boundary layer is insignificant. The temperature at surface of the cone and the surrounding fluid which is at rest are with equal temperature \( T_\infty \). When time \( t' > 0 \), the temperature of the surface of the cone is raised to \( T' = T'_{\infty} + \frac{Q_0}{\rho c_p} \) retained at the same level. The coordinate system is taken (as presented in Figure 1) such that \( r \) represents the distance from the apex \( (x = 0) \) along the cone surface and \( y \) represents the distance along outward normal. The properties of fluid are considered as constant except the variations in density which causes a buoyancy force in momentum equation. Using Boussinesq approximation the governing conservation equations of continuity, momentum and energy are given below:

Continuity equation:

\[
\frac{\partial}{\partial x}(ru) + \frac{\partial}{\partial y}(rv) = 0
\]  
(1)

Momentum equation:

\[
\frac{\partial u}{\partial t'} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \nu \frac{\partial^2 u}{\partial y^2} + g \beta (T' - T'_\infty) \cos \phi
\]

(2)

Energy equation:

\[
\frac{\partial T'}{\partial t'} + u \frac{\partial T'}{\partial x} + v \frac{\partial T'}{\partial y} = \frac{Q_0}{\rho c_p} (T' - T'_{\infty}) + \alpha \frac{\partial^2 T'}{\partial y^2}
\]

(3)

Where \( u \) and \( v \) are the velocity factors corresponding to \( x \) and \( y \) directions, \( T' \) surface temperature, \( T'_{\infty} \) temperature outside surface of the cone, \( t' \) is time, \( g \) acceleration due to gravity, \( \beta \) is the volumetric coefficient of thermal expansion, \( \nu \) Kinematic viscosity, \( Q_0 \) Dimensional heat generation/absorption coefficient, \( \alpha \) Thermal diffusivity, \( \rho \) is the density, \( c_p \) Specific heat at constant pressure, \( k \) Thermal conductivity.

The initial and boundary conditions are as follows

\( t' \leq 0: u = 0, \quad v = 0, \quad T' = T'_{\infty} \) for all \( x \) and \( y \).
Local Skin-friction and local Nusselt number are given by
\[ \tau_x = \mu \left( \frac{\partial u}{\partial y} \right)_{y=0}, \quad \text{and} \quad Nu_x = -\frac{x}{T_w - T_\infty} \left( \frac{\partial T'}{\partial y} \right)_{y=0} \] (5)

Applying the following dimensionless quantities:
\[ R = \frac{rL}{x}, \quad X = \frac{x}{L}, \quad Y = \frac{y}{L} \left( Gr_L \right)^{-\frac{1}{3}}, \quad \text{where} \quad r = x \sin \phi, \]
\[ V = \frac{\nu L}{U} \left( Gr_L \right)^{-\frac{1}{3}}, \quad U = \frac{\nu L}{U} \left( Gr_L \right)^{-\frac{1}{3}}, \quad t = \frac{\nu L^2}{T_w} \left( Gr_L \right)^{-\frac{1}{3}}, \quad Pr = \frac{\nu}{\alpha}, \]
\[ T = \frac{(T_w - T_\infty)}{Ld_{(l,l)}} \left( Gr_L \right)^{-\frac{1}{3}}, \quad Gr_L = \frac{g \beta (q_w (L) / k) L^4 \cos \phi}{\nu^2}, \quad \Delta = \frac{Q_w L^2}{C_p \mu (Gr_L)^{-\frac{1}{3}}} \] (6)

Where \( L \) Reference length, \( Gr_L \) Thermal Grashof number, \( Pr \) Prandtl number, \( \Delta \) Dimensionless heat generation and absorption parameter, \( \phi \) Semi-vertical angle of the cone.

The equations (1), (2) and (3) can be transformed to the following dimensionless form:
\[ \frac{\partial (UR)}{\partial X} + \frac{\partial (VR)}{\partial Y} = 0 \] (7)
\[ \frac{\partial U}{\partial t} + U \frac{\partial U}{\partial X} + V \frac{\partial U}{\partial Y} = \frac{\partial^2 U}{\partial Y^2} + T \] (8)
\[ \frac{\partial T}{\partial t} + U \frac{\partial T}{\partial X} + V \frac{\partial T}{\partial Y} = \frac{1}{Pr} \frac{\partial^2 T}{\partial Y^2} + \Delta T \] (9)

The initial and boundary conditions corresponds to non-dimensional form are
\[ t \leq 0: \quad U = 0, \quad V = 0, \quad T = 0 \quad \text{for all} \quad X \quad \text{and} \quad Y \]
\[ t > 0: \quad U = 0, \quad V = 0, \quad \frac{\partial T}{\partial Y} = -X'' \quad \text{at} \quad Y = 0 \]
\[ U = 0, \quad T = 0 \quad \text{at} \quad X = 0 \]
\[ U \rightarrow 0, \quad T \rightarrow 0 \quad \text{as} \quad Y \rightarrow \infty \] (10)

The dimensionless quantities of local Skin-friction and local Nusselt number are
\[ \tau_x = Gr_L^{\frac{2}{3}} \left( \frac{\partial U}{\partial Y} \right)_{Y=0}, \quad Nu_x = X \left( \frac{-\partial T}{\partial Y} \right)_{Y=0} Gr_L^{\frac{1}{3}} \] (11)

Taking semi-infinite vertical cone slant height as \( L = 1 \), it is considered a rectangular domain in which \( X \) varies from 0 to 1 and \( Y \) varies from 0 to \( Y_{\text{max}} = 20 \), where \( X = L \) represents the slant height of the vertical cone and \( Y_{\text{max}} \) is considered as \( \infty \), where \( Y_{\text{max}} \) lies apart from the momentum, thermal boundary layers. The region of integration is regarded as a rectangle having mesh sizes \( \Delta X = 0.25 \) and \( \Delta Y = 0.25 \).
3. Solution Procedure

The numerical solutions of the transient nonlinear coupled partial differential equations (7)-(9) together with the initial and boundary conditions (10) are obtained using Network Simulation Method. The discretization of the conservation equations is formulated on the finite difference procedure. It necessary requires discretization of the spatial co-ordinates considering time remaining as a real continuous parameter.

Based on these assumptions a suitable design of electrical network circuit is formulated. Electric analogy is applied in which the variable voltage (V) is corresponds to velocities (U, V), temperature (T) and the variable electric current (J) corresponds to the velocity fluxes and the temperature fluxes.

For each dimensionless boundary layer equation, two circuits are developed. The solution of whole network is obtained by implementing a suitable computer program code with electric circuit simulator Pspice [5].

3.1 Design of the Network Model

The design of the network model is given as follows; Refer Alhamaand González-Fernandez [7], Zueco [13] for more information. Executing finite difference formulation the resulting dimensionless boundary layer equations becomes

\[
(U_{i+1,j} - U_{i,j})/(\Delta X) + (V_{i,j} - V_{i+1,j})/(\Delta Y) + \frac{U_{i,j}}{i \Delta X} = 0
\]  

\[
\Delta Y \frac{dU_{i,j}}{dt} + \Delta Y \frac{dV_{i,j}}{dt} + \alpha U_{i,j} = \alpha \frac{U_{i,j}}{i \Delta X}
\]  

\[
\Delta Y \frac{dT_{i,j}}{dt} + \Delta Y \frac{dU_{i,j}}{dt} + \alpha U_{i,j} = \alpha \frac{T_{i,j}}{i \Delta X}
\]

Defining the following currents:

(i) Equation of Momentum balance

\[
j_{0i,j} = \left( U_{i,j} - U_{i+1,j} \right)/(\Delta Y)
\]

\[
j_{1i,j} = \left( U_{i,j} - U_{i+1,j} \right)/(\Delta Y)
\]

\[
j_{2i,j} = \frac{\Delta Y \left( U_{i+1,j} - U_{i,j} \right)}{\Delta X}
\]

\[
j_{3i,j} = \frac{\Delta Y \left( U_{i+1,j} - U_{i,j} \right)}{\Delta X}
\]

\[
j_{4i,j} = \frac{\Delta Y \left( U_{i+1,j} - U_{i,j} \right)}{\Delta X}
\]

(j) Equation of Energy balance

\[
j_{0i,j} = \left( T_{i,j} - T_{i+1,j} \right)/(\Delta X)
\]

Where \( j_{0i,j}, j_{1i,j}, j_{2i,j}, j_{3i,j}, j_{4i,j} \) are the currents that enter and leave the cell for the friction term of \( U \), \( j_{0i,j} \) the current corresponding to the buoyancy term, \( j_{0i,j} \) and \( j_{0i,j} \) are the currents corresponding to the inertia terms of \( U \) and \( V \), respectively, while \( j_{0i,j} \) is the transitory term. The currents \( j_{0i,j} \) and \( j_{0i,j} \) are executed with two resistors \( R_{0i,j} \) of values “\( \Delta Y/2 \)” ; while the currents \( j_{0i,j}, j_{0i,j}, j_{0i,j} \) are executed with voltage control current generators. Therefore, the current \( j_{0i,j} \) is executed by means of generator \( G_{0i,j} \) with the control action “\( \Delta Y \) (\( T_{i,j} \))”, while the remaining currents are implemented with the generators, \( G_{0i,j}, G_{0i,j}, G_{0i,j} \) correspondingly, by means of the voltages “\( V_{i,j} \left( U_{i+1,j} - U_{i,j} \right) \)” and “\( \Delta Y \left( U_{i+1,j} - U_{i,j} \right)/(\Delta X) \)”. In momentum equation \( U_{i,j}, U_{i,j}, U_{i,j}, U_{i,j} \) are the voltages (velocities) of the nodes “\( i - \Delta X, j \), “\( i + \Delta X, j \), “\( i, j - \Delta X \), “\( i, j + \Delta X \)” in the cell and \( U_{i,j} \) represent the velocity that corresponds to the centre of this cell (i, j) and \( j_{0i,j} \) is the current which is executed by capacitor of value \( C_{i,j} \) = \( \Delta Y \), linked to the centre.
\[ J_{r,i,j,xy} = \left( T_{i,j,xy} - T_{i,j} \right) / (\Delta Y / 2) \]
\[ J_{t,i,j,xy} = (T_{i,j} - T_{i,j,xy}) / (\Delta Y / 2) \]
\[ J_{x,i,j} = \rho \Delta Y \frac{U_{i,j}}{T_{i,j} - T_{i,j,xy}} / (\Delta X) \]
\[ J_{y,i,j} = \rho \Delta Y \frac{V_{i,j}}{T_{i,j} - T_{i,j,xy}} / (\Delta X) \]
\[ J_{1,i,j} = \rho \Delta Y \Delta T_{i,j} / \Delta t \]
\[ J_{2,i,j} = \rho \Delta Y \Delta T_{i,j} / \Delta t \]

where \( J_{r,i,j,xy} \) and \( J_{t,i,j,xy} \) are the currents which enter and leaves the cell corresponds the conduction intransversal direction, viscous dissipation is implemented by the current \( J_{1,i,j} \). The currents correspond the convective terms of \( U \) and \( V \) and \( J_{2,i,j} \) respectively, while \( J_{1,i,j} \) is the transilatory term. The currents \( J_{r,i,j,xy} \) and \( J_{t,i,j,xy} \) are executed by means two resistances \( R_{r,i,j,xy} \) of values \( \Delta Y / 2 \); while the currents \( J_{x,i,j} \) and \( J_{y,i,j} \) corresponds to the voltages \( \rho \Delta Y U_{i,j} / (T_{i,j} - T_{i,j,xy}) \) and \( \rho \Delta Y V_{i,j} / (T_{i,j} - T_{i,j,xy}) \) respectively; The temperatures at the nodes \( i - \Delta X, j \), \( i + \Delta X, j \), \( i, j - \Delta Y \), \( i, j + \Delta Y \) are implemented with the voltage \( T_{i,j,xy}, T_{i+\Delta X,j,xy}, T_{i,j-\Delta Y,xy}, T_{i,j+\Delta Y,xy} \) in each cell of the energy balance equation, where the temperature in the centre of this cell \( (i,j) \) represented by \( T_{i,j} \) and \( J_{1,i,j} \) is the current executed with a capacitor whose value is \( C_{r,i,j} = \rho \Delta Y \), linked to the centre of each cell.

The finite difference formulation corresponds to continuity equation (10) is
\[ V_{i,j} = U_{i,j} \Delta Y / (i\Delta X) + \left( U_{i+\Delta X,j} - U_{i,j,xy} \right) \Delta Y / (2 \Delta X) - V_{i,j,xy} \]  
\[ \text{Equation of momentum} \]
\[ \text{Equation of energy} \]

Applying Kirchhoff’s law for electric network model corresponds to the equations [(13) – (14)] can be represented by
\[ \dot{J}_{r,i,j,xy} + J_{r,i,j,xy} - J_{t,i,j,xy} + J_{t,i,j} + J_{x,i,j} + J_{y,i,j} = 0 \]
\[ \dot{J}_{t,i,j,xy} + J_{t,i,j,xy} - J_{r,i,j,xy} + J_{r,i,j} + J_{x,i,j} + J_{y,i,j} = 0 \]  
\[ \text{Equation of continuity} \]
\[ \text{Equation of continuity} \]

In order to execute the boundary conditions (at \( X = 0 \) and as \( Y \to \infty \)) of temperature and velocity, ground elements are implemented. Finally to simulate the non-uniform heat flux \( \frac{\partial T}{\partial Y} = -X^* \) constant voltage and constant current and at \( Y = 0 \) are employed. For the initial condition, corresponds to the voltages \( U = T = 0 \) for \( t \leq 0 \) are implemented to the two capacitors \( C_{T,j,i} \) and \( C_{U,j,i} \) respectively.

Figure 2. Network model for control volume
Figure 3. Network model for control volume

Equation of momentum–Equation of energy
4. Result and Discussion

The suitable network model has been formulated and analyzed natural convective flow over non-isothermal vertical cone with variable surface heat flux together with the effect of heat generation/absorption. The results are plotted for velocity and temperature profiles for several parameters Prandtl number ($Pr$), surface heat flux power law exponent ($n$) and the heat absorption/generation parameter ($\Delta$).

Figure 4 shows the velocity profile for various values of $Pr$. The momentum boundary layer thickness increases with the increase of $Pr$. The influence of the heat absorption or generation parameter $\Delta$ on the velocity field is shown in Figure 5. Figure 6 presents the velocity profile for different values of heat flux power law exponent $n$, the velocity increases for the smaller values of the controlling parameters $n$. Figure 7 illustrates the effects of $Pr$ on temperature profile, thermal boundary layer becomes thin with increasing $Pr$. Figure 8 depicts the variation in temperature profile due to the influence of heat generation or absorption parameter $\Delta$ on. The temperature increases for higher values of $\Delta$ and the thermal boundary layer thickness increases for lower values of $\Delta$. The influence of heat flux power law exponent $n$ on temperature is presented in Figure 9; the gradient of heat flux along the vertex of the cone decreases with the increasing values of $n$.

![Figure 4](image1.png)

![Figure 5](image2.png)

![Figure 6](image3.png)

![Figure 7](image4.png)

![Figure 8](image5.png)

![Figure 9](image6.png)

**Figure 4.** Velocity profile at $X=1.0$ for various quantities of $Pr$.

**Figure 5.** Velocity profile at $X=1.0$ for various quantities of $\Delta$.


Figure 6. Velocity profiles at X=1.0 for various quantities of \( \nu \) and \( Pr \).

Figure 7. Temperature profiles at X=1.0 for various quantities of \( \Delta \) and \( \nu \).

Figure 8: Temperature profiles at X=1.0 for various quantities of \( \Delta \).

Figure 9: Temperature profiles at X=1.0 for various quantities of \( n \).

5. Conclusion
In the present investigation, formulated a mathematical model for the free convective flow from a vertical cone subject to variable surface heat flux with the effect of heat generation/absorption. A parametric study is carried out to illustrate the effects of thermo physical parameters on the profiles of velocity, temperature.

1. The velocity of the fluid increases for larger quantities of $\alpha$ and smaller quantities of $Pr$ and $n$.
2. The temperature decreases for smaller quantities of $\alpha$ and higher quantities of $Pr$ and $n$.

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