GRAHAM THEOREM ON BOUNDED SYMMETRIC DOMAINS

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Abstract. Graham Theorem on the unit ball $B_n$ in $\mathbb{C}^n$ states that every invariant harmonic function $u \in C^n(B_n)$ must be pluriharmonic in $B_n$ [4]. This rigidity phenomenon of Graham have been studied by many authors (see, for examples, [6], [10], [11], etc.) In this paper, we prove that Graham theorem holds on classical bounded symmetric domains. Which include Type I domains, Type II domains, Type III domains $III(n)$ with even $n$ and some special Type IV domains.

1. Introduction

Let $(M^n, g)$ be a compact Riemannian with boundary $\partial M$ with Riemannian metric $g$. Let $\Delta_g$ be the Laplace-Beltrami operator associated to $g$. We consider the boundary value problem

(1.1) \[
\left\{ \begin{array}{ll}
\Delta_g u = 0, & \text{in } M, \\
u = \phi, & \text{on } \partial M.
\end{array} \right.
\]

When $\Delta_g$ is uniformly elliptic on $M$, the boundary value problem (1.1) is well understood (see, for examples, the books of Evans [2] and Gilbarg and Trudinger [3]). When $\Delta_g$ is not uniformly elliptic, the regularity of the solution $u$ of (1.1) becomes much more complicated. Typical examples we consider here are manifolds $(M^n, g)$ with bounded pseudoconvex domains $M$ in $\mathbb{C}^n$ and the Bergman metric $g$ of $M$. In particular, when $M$ is the unit ball $B_n$ in $\mathbb{C}^n$, it is well known from the books of Hua [7] that

(1.2) \[
\Delta_g = \left(1 - |z|^2\right)^n \sum_{\alpha, \beta=1}^n (\delta_{\alpha\beta} - z_{\alpha} \bar{z}_\beta) \frac{\partial^2}{\partial z_\alpha \partial \bar{z}_\beta}.
\]

For any $\phi \in C(\partial B_n)$, the Dirichlet boundary value problem (1.1) has a unique solution

(1.3) \[u(z) = \int_{\partial B_n} P(z, w) \phi(w) d\sigma(w) = \int_{\partial B_n} \frac{(1 - |z|^2)^n}{|1 - \langle z, w \rangle|^{2n}} \phi(w) d\sigma(w).
\]
When $\phi \in C^\infty (\partial B_n)$, it was proved by Graham [4] that the solution $u$ given by (1.3) can be expressed as

$$u (z) = G (z) + H (z) \left( 1 - |z|^2 \right)^n \log \left( 1 - |z|^2 \right), \quad z \in \mathbb{B}_n,$$

where $G, H \in C^\infty (\overline{B_n})$. When $n = 1$, $H \in C^\infty (B_n)$. However, when $n > 1$, in general, $H \not\equiv 0$ on $\partial B_n$. In particular, when $H = 0$ on $\partial B_n$, the following striking theorem was proved by Graham [4]:

**Theorem 1.1.** If $u \in C^n (\overline{B_n})$ is invariant harmonic ($\Delta_g u = 0$) in $B_n$, then $u$ is pluriharmonic in $B_n$.

Problem about whether Graham’s Phenomenon holds for more general domains $M$ and more general metric $g$ has been studied by several authors. For examples, Graham and Lee [6] studied the problem for $M$ being strictly pseudoconvex domains in $\mathbb{C}^n$ with smooth boundaries and Kähler metrics $g$ satisfying special symmetric property. In particular, they gave a characterization of CR-pluriharmonic functions on $\partial M$, which is a fundamental paper in the theory of the pseudo-Hermitian CR geometry. Li and Simon [10] proved a Graham type theorem for the polydisc in $\mathbb{C}^n$ with Bergman type metrics. In general, Graham’s phenomenon fails in $B_n$ with rotationally symmetric metrics when $n > 2$, counterexample was constructed by Graham and Lee [6]. Further information along this direction can be found in Li and Wei [11]. For more results on invariant harmonic functions and backgrounds we refer the reader to [1], [5], [8], [9], [12], [15]. However, the problem about whether Graham Theorem holds when $M$ is a classical bounded symmetric domain with Bergman metric $g$ is widely open. The main purpose of the paper is to investigate this problem.

For positive integers $m \leq n$, denoted by $M^{m,n} (\mathbb{C})$ the set of all $m \times n$ matrices with entries in $\mathbb{C}$. The classical bounded symmetric domains [7] are the following four types:

1. $\mathbf{I} (m,n) = \{ z \in M^{m,n} (\mathbb{C}) : I_m - zz^* > 0 \}$,
2. $\mathbf{II} (n) = \{ z \in \mathbf{I} (n,n) : z^t = z \}$,
3. $\mathbf{III} (n) = \{ z \in \mathbf{I} (n,n) : z^t = -z \}$

and

$$\mathbf{IV} (n) = \left\{ z \in \mathbb{C}^n : 2 |z|^2 - |zz^t|^2 - 1 < 0 \text{ and } |zz^t|^2 < 1 \right\}.$$

Denoted by $D_1 \cong D_2$ if $D_1$ and $D_2$ are biholomorphic equivalent. It is known from Lu [14] and Loos [13] that

1. $\mathbf{IV} (2) \cong \mathbf{IV} (1) \times \mathbf{IV} (1)$, $B_3 = \mathbf{I} (1,3) \cong \mathbf{III} (3)$,

and

1. $\mathbf{II} (2) \cong \mathbf{IV} (3)$, $\mathbf{I} (2,2) \cong \mathbf{IV} (4)$ and $\mathbf{III} (4) \cong \mathbf{IV} (6)$.

Let $D$ be a bounded domain with the Bergman metric $g^D$ of $D$. Let $\Delta_{g^D}$ denote the Laplace-Beltrami operator associated to $g^D$. Since the Bergman metric is biholomorphic invariant, we say that a function $u$ is invariant harmonic in $D$ if

$$\Delta_{g^D} u (z) = 0, \quad z \in D.$$
When $D$ is a bounded symmetric domain, $\Delta_g$ is called Hua operator. We use the following notations for Hua operators:

\begin{equation}
\Delta_1 = \Delta_g^{I(m,n)}, \Delta_2 = \Delta_g^{II(n)}, \Delta_3 = \Delta_g^{III(n)} \text{ and } \Delta_4 = \Delta_g^{IV(n)}.
\end{equation}

Denoted by $\mathcal{U}(D)$ the Šilov boundary or the characteristic boundary of $D$. For any $\phi \in C(\mathcal{U}(D))$, it was proved by Hua [7], the boundary value problem (1.1) has the unique solution

\begin{equation}
\label{eq13}
u(z) = \int_{\mathcal{U}(D)} P_D(z,w) \phi(w) d\sigma(w),
\end{equation}

where $P_D(z,w)$ is the Poisson-Szegö kernel given by

\begin{equation}
\label{eq14}
P_D(z,w) = (\det(I_m - zz^*))^{\kappa(D)} / |\det(I_m - zw^*)|^{2\kappa(D)}
\end{equation}

and

\begin{equation}
\kappa(D) = \begin{cases} 
n, & \text{if } D = I(m,n); 
n+1, & \text{if } D = II(n); 
n-1, & \text{if } D = III(n), n \text{ is even;} 
n, & \text{if } D = III(n), n \text{ is odd.}
\end{cases}
\end{equation}

The main results of the paper are the following theorem.

**Theorem 1.2.** Let $m, n \in \mathbb{N}$ and $m \leq n$.

(i) If $u \in C^n(I(m,n))$ is invariant harmonic then $u$ is pluriharmonic;

(ii) If $n$ is odd and if $u \in C^{n+1/2}(II(n))$ is invariant harmonic then $u$ is pluriharmonic;

(iii) If $n = 2k$ is even and if there is a $\alpha > 1/2$ such that $u \in C^{k,\alpha}(III(n))$ is invariant harmonic then $u$ is pluriharmonic;

(iv) If $n$ is even and if $u \in C^{n-1}(III(n))$ is invariant harmonic then $u$ is pluriharmonic.

**Remark** We note that the above smoothness assumptions are sharp. The condition $C^{k,\alpha}$ with $\alpha > 1/2$ in Part (iii) is the same as $\frac{n+1}{2} + \epsilon$ with $\epsilon > 0$. We need to add $\epsilon$ in Part (iii) rather than $\frac{n+1}{2}$ in Part (ii) because $\frac{n+1}{2}$ is not integer when $n$ is even.

In order to prove Parts (ii) and (iii) of Theorem 1.2 one of our key steps is to prove the following theorem in the unit ball.

**Theorem 1.3.** Let

\begin{equation}
\tilde{\Delta} := \sum_{j,k=1}^{n} \left( \delta_{jk} - |z|^2 z_j \bar{z}_k \right) \frac{\partial^2}{\partial z_j \partial \bar{z}_k}.
\end{equation}

Then the following two statements hold.

(i) If $n$ is odd and $u \in C^{n+1/2}(\overline{B_n})$ satisfying $\tilde{\Delta}u = 0$ in $B_n$, then $u$ is pluriharmonic in $B_n$;

(ii) If $n = 2k$ is even and if $u \in C^{k,\alpha}(\partial B_n)$ for some $\alpha > 1/2$, then $u$ is pluriharmonic in $B_n$.

**Remark 1.4.** We point out here that the operator $\tilde{\Delta}$ is not included in $\Delta_g$ with $g$ is rotation symmetric metrics in Graham and Lee [6] or Li and Wei [11].
The paper is organized as follows. In section 2, we study the fundamental properties of the Poisson-Szegö kernels, we will prove that they satisfy a system of differential equations. We will prove Theorem 1.3 in Section 3. As applications of results in Section 2, Graham’s theorem and Theorem 1.3, we will prove Theorem 1.2 in Section 4. Finally, in Section 5, we will prove Graham’s phenomenon fails on IV (2) and will give some remarks on the problem over III (3) and IV (4).

2. System of Differential Equations

Let $\Delta_1, \Delta_2, \Delta_3$ and $\Delta_4$ denote Hua operators, the Laplace-Beltrami operators associated to the Bergman metrics on the classical bounded symmetric domains $I(m, n)$, $\Pi(n)$, $\PiI(n)$ and $\PiIV(n)$, respectively. According to the books of Hua [7] and Lu [14], the following proposition holds.

**Proposition 2.1.** For $z \in M^{m,n}(\mathbb{C})$ and $V(z) = I_m - zz^*$, we let

\[
V_{jk} = [V(z)]_{jk} = \delta_{jk} - \sum_{\ell=1}^n z_j \bar{z}_k \ell.
\]

Then the Hua operators are given by

\[
\Delta_1 = \sum_{j,k=1}^m V_{jk} \Delta_1^{jk}, \quad \Delta_1^{jk} := \sum_{\alpha,\beta=1}^n \delta_{\alpha\beta} - \sum_{\ell=1}^m z_{\ell\alpha} \bar{z}_{\ell\beta} \frac{\partial^2}{\partial z_\alpha \partial \bar{z}_\beta};
\]

\[
\Delta_2 = \frac{1}{4} \sum_{j,k=1}^n V_{jk} \Delta_2^{jk}, \quad \Delta_2^{jk} := \sum_{\alpha,\beta=1}^n \frac{V_{\alpha\beta}}{(1 - \delta_{\alpha j}/2) (1 - \delta_{k\beta}/2)} \frac{\partial^2}{\partial z_\alpha \partial \bar{z}_\beta};
\]

\[
\Delta_3 = \frac{1}{4} \sum_{j,k=1}^n V_{jk} \Delta_3^{jk}, \quad \Delta_3^{jk} := \sum_{\alpha,\beta=1}^n V_{\alpha\beta} (1 - \delta_{\alpha j}) (1 - \delta_{\beta k}) \frac{\partial^2}{\partial z_\alpha \partial \bar{z}_\beta};
\]

and

\[
\Delta_4 = \sum_{j,k=1}^n [r(z) (\delta_{jk} - 2z_j \bar{z}_k) + 2 (\bar{z}_j - s(z) z_j) (z_k - s(z) \bar{z}_k)] \frac{\partial^2}{\partial z_j \partial \bar{z}_k},
\]

where $z \in \mathbb{C}^n$ and

\[
s(z) = \sum_{j=1}^n z_j^2 \quad \text{and} \quad r(z) = 1 - 2|z|^2 + 2s(z).
\]

The main purpose of this section is to prove the following theorem.

**Theorem 2.2.** Let $D$ be a bounded symmetric domain and let $u \in C^2(D) \cap C(\overline{D})$. Then

(i) $\Delta_1 u = 0$ on $I(m, n)$ if and only if

$\Delta_1^{jk} u(z) = 0, \; \text{for all} \; 1 \leq j, k \leq m.$

(ii) $\Delta_2 u = 0$ on $\Pi(n)$ if and only if

$\Delta_2^{jk} u(z) = 0, \; 1 \leq j, k \leq n.$

(iii) When $n$ is even, $\Delta_3 u(z) = 0$ on $\PiI(n)$ if and only if

$\Delta_3^{jk} u(z) = 0, \; 1 \leq j, k \leq n.$
Part (i) of Theorem 2.2 was proved by Hua [7] using the technique of Lie group. We will divide the proof of the rest of the above theorem into several lemmas.

Define

\[ W(z, w) = I_m - zw^* \] and \[ V(z) = W(z, z) \].

By (1.14), the Poisson-Szegö kernel on \( D \) can be written as

\[ P^D(z, w) = \frac{(\det V(z)^\kappa(D))}{|\det W(z, w)|^{2\kappa(D)}}. \]

**Proposition 2.3.** With the notations above, \( P = P^D(z, w) \) and \( \kappa = \kappa(D) \), one has

\[ \frac{1}{\kappa^2 P} \frac{\partial^2 P}{\partial z_{j\alpha} \partial \bar{z}_{k\beta}} = \frac{1}{\kappa} \frac{\partial^2 \log \det V(z)}{\partial z_{j\alpha} \partial \bar{z}_{k\beta}} + \left( b_{j\alpha} \bar{c}_{k\beta} + c_{j\alpha} c_{k\beta} - b_{j\alpha} \bar{c}_{k\beta} - c_{j\alpha} \bar{c}_{k\beta} \right), \]

where

\[ b_{j\alpha} := \frac{\partial \log \det V(z)}{\partial z_{j\alpha}} \] and \( c_{j\alpha} := \frac{\partial \log \det W(z, w)}{\partial z_{j\alpha}}. \)

**Proof.** Notice that \( \det W(z, w) = \det W(w, z) \) and (2.8), one has

\[ \log P^D(z, w) = \kappa (\log \det V(z) - \log \det W(z, w) - \log \det W(w, z)) \]

and then

\[ \kappa \frac{\partial^2 \log V(z)}{\partial z_{j\alpha} \partial \bar{z}_{k\beta}} = \frac{\partial^2 \log P^D(z, w)}{\partial z_{j\alpha} \partial \bar{z}_{k\beta}} = \frac{1}{P} \frac{\partial^2 P}{\partial z_{j\alpha} \partial \bar{z}_{k\beta}} - \frac{\partial \log P}{\partial z_{j\alpha}} \frac{\partial \log P}{\partial \bar{z}_{k\beta}}. \]

Therefore

\[ \frac{1}{P} \frac{\partial^2 P}{\partial z_{j\alpha} \partial \bar{z}_{k\beta}} = \kappa \frac{\partial^2 \log V(z)}{\partial z_{j\alpha} \partial \bar{z}_{k\beta}} + \frac{\partial \log P}{\partial z_{j\alpha}} \frac{\partial \log P}{\partial \bar{z}_{k\beta}}. \]

A simple computation gives the proof of the proposition. \( \square \)

Let \( M(\cdot) = [M_{jk}(\cdot)] \) be an \( n \times n \) matrix and \( M^{-1}(\cdot) = [M_{jk}^{-1}(\cdot)] \). Here \( j \) represents row index while \( k \) represents column index. Then

\[ \frac{\partial M(z)}{\partial z_{j\alpha}} = \left[ \frac{\partial M_{jk}(z)}{\partial z_{j\alpha}} \right]. \]

Denoted by \( E_{jk} \) the \( n \times n \) matrix with \((j, k)\)-entry 1, other entries 0.

**Lemma 2.4.** (i) For \( z \in \Pi(n) \) and \( w \in \Pi(n) \), one has

\[ c_{j\alpha}(z, w) = \frac{\partial \log \det W(z, w)}{\partial z_{j\alpha}} = -(2 - \delta_{j\alpha}) \left[ w^* W^{-1}(z, w) \right]_{j\alpha} \]

and

\[ \bar{c}_{k\beta}(z, w) = -(2 - \delta_{k\beta}) \left[ W^{-1}(w, z) w \right]_{k\beta}. \]

(ii) For \( z \in \Pi(n) \) and \( w \in \Pi(n) \), one has

\[ c_{j\alpha}(z, w) = \frac{\partial \log \det W(z, w)}{\partial z_{j\alpha}} = 2 \left[ w^* W^{-1}(z, w) \right]_{j\alpha} \]

and

\[ \bar{c}_{k\beta}(z, w) = -2 \left[ W^{-1}(w, z) w \right]_{k\beta}. \]
Proof. On $\mathbf{II}(n)$, $z^t = z$ and $w^t = w$, one can easily check that

$$\label{2.17}(w^*W^{-1}(z,w))^t = w^*W^{-1}(z,w).$$

Thus

$$c_{ja}(z,w) = \text{tr} \left( W^{-1}(z,w) \frac{\partial W(z,w)}{\partial z_{ja}} \right)$$

$$= - \text{tr} \left( W^{-1}(z,w) \left( 1 - \frac{\delta_{ja}}{2} \right) (E_{ja} + E_{aj}) W^* \right)$$

$$= - \left( 1 - \frac{\delta_{ja}}{2} \right) \text{tr} \left( E_{ja} w^* W^{-1}(z,w) + w^* W^{-1}(z,w) E_{aj} \right)$$

$$= - \left( 1 - \frac{\delta_{ja}}{2} \right) \left( \text{tr} \left( E_{ja} w^* W^{-1}(z,w) \right) + \text{tr} \left( E_{aj} w^* W^{-1}(z,w) \right) \right)$$

$$= - \left( 2 - \delta_{ja} \right) \left( \text{tr} \left( E_{ja} w^* W^{-1}(z,w) \right) \right)$$

$$= - \left( 2 - \delta_{ja} \right) \left[ w^* W^{-1}(z,w) \right]_{ja}.$$

By \eqref{2.17}, one has \eqref{2.13} holds and Part (i) is proved.

On $\mathbf{III}(n)$, it is easily to see that $(w^*W^{-1}(z,w))^t$ and $(W^{-1}(w,z)w)^t$ are anti-symmetric and

$$c_{ja}(z,w) = \text{tr} \left( W^{-1}(z,w) \frac{\partial W(z,w)}{\partial z_{ja}} \right)$$

$$= - \text{tr} \left( W^{-1}(z,w) (E_{ja} - E_{aj}) w^* \right)$$

$$= -2 \text{tr} \left( E_{ja} w^* W^{-1}(z,w) \right)$$

$$= -2 \left[ w^* W^{-1}(z,w) \right]_{aj}$$

$$= 2 \left[ w^* W^{-1}(z,w) \right]_{ja}$$

and $c_{k\beta}(z,w) = -2 \left[ W^{-1}(w,z) w k_{\beta} \right]$. Therefore, Part (ii) is proved and so is the lemma. \hfill \square

**Lemma 2.5.**

(i) For $z \in \mathbf{II}(n)$, one has

$$A^{jk}_2(z) := \frac{2}{n+1} \Delta^{jk}_2 \log \det V = -4 V^{kj}$$

and

$$B^{jk}_2(z) := \sum_{\alpha,\beta} \frac{V_{\alpha\beta}}{(1 - \delta_{ja}/2)(1 - \delta_{k\beta}/2)} b_{ja} b_{k\beta} = 4 \left[ V^{-1}(z^*) - I_n \right]_{jk}.$$  \hfill (2.19)

(ii) For $z \in \mathbf{III}(n)$, one has

$$A^{jk}_3(z) := \frac{1}{\kappa} \Delta^{jk}_3 \log \det V(z) = -2 (n-1) V^{kj},$$

where $\kappa = \kappa \left( \mathbf{III}(n) \right)$ and

$$B^{jk}_3(z) := \sum_{\alpha,\beta} V_{\alpha\beta} (1 - \delta_{ja}) (1 - \delta_{k\beta}) b_{ja} b_{k\beta} = 4 \left[ V^{-1}(z^*) - I_n \right]_{jk}.$$  \hfill (2.21)

**Proof.** On $\mathbf{II}(n)$, it is known from Lu \cite{14}

$$\frac{\partial^2 \log \det V(z)}{\partial z_{ja} \partial z_{k\beta}} = -2 \left( 1 - \frac{\delta_{ja}}{2} \right) \left( 1 - \frac{\delta_{jk}}{2} \right) (V^{ji} V^{k\alpha} + V^{\beta \alpha} V^{kj}).$$
This implies that
\[
A_{kj}^2 (z) = \frac{2}{n+1} \Delta_{kj}^2 \log \det V (z)
= \frac{2}{n+1} \sum_{\alpha, \beta=1}^n \frac{V_{\alpha \beta}}{(1 - \delta_{j\alpha}/2) (1 - \delta_{k\beta}/2)} \frac{\partial^2 \log \det V (z)}{\partial z_j \partial \bar{z}_k}
= \frac{4}{n+1} \sum_{\alpha, \beta=1}^n V_{\alpha \beta} \left( V^{\beta j} V^{\alpha k} + V^{\beta k} V^{\alpha j} \right)
= \frac{4}{n+1} (V^{kj} + nV^{kj})
= -4V^{kj}.
\]

With the notations \( b_{j\alpha}(z) = c_{j\alpha}(z, z) \), \( V(z) = W(z, z) \) and the fact that \( V^{-1}(z) z \) is symmetric, Part (i) of Lemma 2.4 implies
\[
B_{kj}^1 (z) = \sum_{\alpha, \beta} \frac{V_{\alpha \beta}}{(1 - \delta_{j\alpha}/2) (1 - \delta_{k\beta}/2)} c_{j\alpha}(z, z) c_{k\beta}(z, z)
= 4 \sum_{\alpha, \beta} \frac{V_{\alpha \beta}}{(1 - \delta_{j\alpha}/2) (1 - \delta_{k\beta}/2)} \left( 1 - \frac{\delta_{j\alpha}}{2} \right) \left( 1 - \frac{\delta_{k\beta}}{2} \right) \left[ z^* V^{-1}(z) \right]_{j\alpha} \left( 1 - \frac{\delta_{k\beta}}{2} \right) \left[ V^{-1}(z) z \right]_{k\beta}
= 4 \sum_{\alpha, \beta} V_{\alpha \beta} \left[ z^* V^{-1}(z) \right]_{j\alpha} \left[ V^{-1}(z) z \right]_{k\beta}
= 4 \left[ z^* V^{-1}(z) V(z) V^{-1}(z) z \right]_{jk}
= 4 \left[ z^* V^{-1}(z) z \right]_{jk}
= 4 \left[ V^{-1}(z^*) - I_n \right]_{jk}.
\]

On III \((n)\), according to Lu [14], one has
\[
\frac{\partial^2 \log \det V (z)}{\partial z_j \partial \bar{z}_k} = 2 \left( V^{\beta j} V^{\alpha k} - V^{\beta k} V^{\alpha j} \right).
\]

This implies
\[
A_{jk}^3 (z) = \frac{1}{\kappa} \Delta_{jk}^3 \log \det V (z)
= \frac{1}{\kappa} \sum_{\alpha, \beta=1}^n (1 - \delta_{j\alpha}) (1 - \delta_{k\beta}) V_{\alpha \beta} \left( V^{\beta j} V^{\alpha k} - V^{\beta k} V^{\alpha j} \right)
= 2 \sum_{\alpha=1}^n (1 - \delta_{j\alpha}) \left[ V^{\alpha k} \delta_{j\alpha} - V^{kj} - V^{kj} V^{\alpha k} + V^{kj} V^{\alpha k} V^{kj} \right]
= -2 \frac{(n-1)}{\kappa} V^{kj}.
\]

Notice that \( z^* V^{-1}(z) \) is anti-symmetric, one has
\[
(1 - \delta_{j\alpha}) \left[ z^* V^{-1}(z) \right]_{j\alpha} = \left[ z^* V^{-1}(z) \right]_{j\alpha}.
\]
Part (ii) of Lemma 2.4 implies

\[ B^{jk}_{3} (z) = \sum_{\alpha, \beta} V_{\alpha\beta} (1 - \delta_{j\alpha}) (1 - \delta_{k\beta}) c_{\alpha\alpha} (z, z) \overline{c_{k\beta}} (z, z) \]

\[ = -4 \sum_{\alpha, \beta} V_{\alpha\beta} (1 - \delta_{j\alpha}) (1 - \delta_{k\beta}) [z^{*} V^{-1} (z)]_{j\alpha} [V^{-1} (z) z]_{k\beta} \]

\[ = 4 [z^{*} V^{-1} (z) V (z) V^{-1} (z) z]_{jk} \]

\[ = 4 [z^{*} V^{-1} (z) z]_{jk} \]

\[ = 4 [V^{-1} (z^{*}) - I_{n}]_{jk}. \]

Therefore, the proof of the lemma is complete. \qed

Lemma 2.6. Let

\[ C^{jk}_{2} (z, w) := \sum_{\alpha, \beta} V_{\alpha\beta} c_{\alpha\alpha} (z, w) \overline{c_{k\beta}} (z, w) \]

(2.22)

and

\[ C^{jk}_{3} (z, w) := \sum_{\alpha, \beta} V_{\alpha\beta} (1 - \delta_{j\alpha}) (1 - \delta_{k\beta}) c_{\alpha\alpha} (z, w) \overline{c_{k\beta}} (z, w). \]

Then

(i) For \( w \in U (\Pi (n)) \) and \( z \in \Pi (n) \), one has

\[ C^{jk}_{2} (z, w) = 4 [W^{-1} (z^{*}, w^{*}) + W^{-1} (w^{*}, z^{*}) - I_{n}]_{jk}. \]

(2.24)

(ii) For \( w \in U (\Pi I (n)) \) and \( z \in \Pi I (n) \), one has

\[ C^{jk}_{3} (z, w) = 4 [W^{-1} (z^{*}, w^{*}) + W^{-1} (w^{*}, z^{*}) - I_{n} - F (z, w)]_{jk}, \]

where

\[ F (z, w) := W^{-1} (w^{*}, z^{*}) (I_{n} - w^{*} w) W^{-1} (z^{*}, w^{*}) . \]

(2.26)

In particular, when \( n \) is even and \( w \in U (\Pi I (n)) \), one has \( I_{n} - w^{*} w = 0 \) and \( F (z, w) = 0 \).

Proof. On \( \Pi (n) \), notice that \( w w^{*} = I_{n} \).

\[ w^{*} W^{-1} (z, w) = W^{-1} (z, w)^{t} w^{*} \quad \text{and} \quad W^{-1} (w, z) w = w W^{-1} (w, z)^{t}, \]

one has

\[ w^{*} W^{-1} (z, w) V (z) W^{-1} (w, z) w \]

\[ = W^{-1} (z, w)^{t} w^{*} (I_{n} - z z^{*}) w W^{-1} (w, z)^{t} \]

\[ = W^{-1} (w^{*}, z^{*}) (I_{n} - w^{*} z z^{*}) w W^{-1} (z^{*}, w^{*}) \]

\[ = W^{-1} (w^{*}, z^{*}) (W (w^{*}, z^{*}) + (I_{n} - W (w^{*}, z^{*})) W (z^{*}, w^{*})) W^{-1} (z^{*}, w^{*}) \]

\[ = (I_{n} + (W^{-1} (w^{*}, z^{*}) - I_{n}) W (z^{*}, w^{*})) W^{-1} (z^{*}, w^{*}) \]

\[ = W^{-1} (z^{*}, w^{*}) + W^{-1} (w^{*}, z^{*}) - I_{n}. \]
By Part (i) of Lemma 2.4 and the identity above, one has
\[ C_{2}^{jk} (z, w) = \sum_{\alpha, \beta = 1}^{n} \frac{V_{\alpha \beta}}{(1 - \delta_{j\alpha}/2)(1 - \delta_{k\beta}/2)} c_{j\alpha} (z, w) c_{k\beta} (z, w) \]
\[ = 4 \sum_{\alpha, \beta} V_{\alpha \beta} [w^* W^{-1} (z, w)]_{j\alpha} [W^{-1} (w, z) \cdot w]_{k\beta} \]
\[ = 4 [w^* W^{-1} (z, w) V (z) W^{-1} (w, z) \cdot w]_{jk} \]
\[ = 4 [W^{-1} (z^*, w^*) + W^{-1} (w^*, z^*) - I_n]_{jk}. \]

Therefore, Part (i) is proved.

On III(n), \( w^* W^{-1} (z, w) \) and \( W^{-1} (w, z) \cdot w \) are anti-symmetric, by Part (ii) of Lemma 2.4 one has
\[ C_{3}^{jk} (z, w) = \sum_{\alpha, \beta = 1}^{n} (1 - \delta_{j\alpha}) (1 - \delta_{k\beta}) V_{\alpha \beta} c_{j\alpha} (z, w) c_{k\beta} (z, w) \]
\[ = -4 \sum_{\alpha, \beta} (1 - \delta_{j\alpha}) (1 - \delta_{k\beta}) V_{\alpha \beta} [w^* W^{-1} (z, w)]_{j\alpha} [W^{-1} (w, z) \cdot w]_{k\beta} \]
\[ = 4 [w^* W^{-1} (z, w) V (z) W^{-1} (w, z) \cdot w]_{jk} \]
\[ = 4 [W^{-1} (z, w)^t w^* V (z) w W^{-1} (w, z)^t]_{jk} \]
and
\[ W^{-1} (z, w)^t w^* V (z) w W^{-1} (w, z)^t \]
\[ = W^{-1} (w^*, z^*) (w^* w - w^* z z - W^{-1} (z^*, w^*) \]
\[ = W^{-1} (w^*, z^*) (w^* w - I_n + I_n - w^* w + w^* z - w^* z z - w^* z z) W^{-1} (z^*, w^*) \]
\[ = W^{-1} (z^*, w^*) + W^{-1} (w^*, z^*) w^* z - W^{-1} (w^*, z^*) (I_n - w^* w) W^{-1} (z^*, w^*) \]
\[ = W^{-1} (z^*, w^*) + W^{-1} (w^*, z^*) - I_n - W^{-1} (w^*, z^*) (I_n - w^* w) W^{-1} (z^*, w^*). \]

This gives (2.25) and (2.26). Therefore, the proof of the lemma is complete. \( \square \)

**Lemma 2.7.** The following statements hold.

(i) For \( z \in \Pi(n) \) and \( w \in \Pi(n) \), one has
\[ D_{2}^{jk} := \sum_{\alpha, \beta} \frac{V_{\alpha \beta} b_{j\alpha} c_{k\beta}}{(1 - \delta_{j\alpha}/2)(1 - \delta_{k\beta}/2)} = 4 [W^{-1} (z^*, w^*) - I_n]_{jk} \]

and
\[ E_{2}^{jk} := \sum_{\alpha, \beta} \frac{V_{\alpha \beta} c_{j\alpha} b_{k\beta}}{(1 - \delta_{j\alpha}/2)(1 - \delta_{k\beta}/2)} = 4 [W^{-1} (w^*, z^*) - I_n]_{jk}. \]

(ii) For \( z \in III(n) \) and \( w \in III(n) \), one has
\[ D_{3}^{jk} := \sum_{\alpha, \beta} V_{\alpha \beta} (1 - \delta_{j\alpha})(1 - \delta_{k\beta}) b_{j\alpha} c_{k\beta} = 4 [W^{-1} (z^*, w^*) - I_n]_{jk} \]

and
\[ E_{3}^{jk} := \sum_{\alpha, \beta} V_{\alpha \beta} (1 - \delta_{j\alpha})(1 - \delta_{k\beta}) c_{j\alpha} b_{k\beta} = 4 [W^{-1} (w^*, z^*) - I_n]_{jk}. \]
Proof. On $\Pi (n)$, by Part (i) of Lemma 2.4, one has

\[
D_{jk} = 4 \sum_{\alpha, \beta = 1}^{n} V_{\alpha \beta} \left[ z^* V^{-1} (z) \right]_{j \alpha} \left[ W^{-1} (w, z) w \right]_{k \beta}
\]

\[
= 4 \left[ z^* (W^{-1} (w, z) w)^T \right]_{jk}
\]

\[
= 4 \left[ z^* w W^{-1} (z^*, w^*) \right]_{jk}
\]

\[
= 4 \left[ W^{-1} (z^*, w^*) - I_n \right]_{jk}
\]

and

\[
E_{jk} = 4 \sum_{\alpha, \beta = 1}^{n} V_{\alpha \beta} \left[ w^* W^{-1} (z, w) \right]_{j \alpha} \left[ V^{-1} (z) z \right]_{k \beta}
\]

\[
= 4 \left[ w^* W^{-1} (z, w) V (z) V^{-1} (z) z \right]_{jk}
\]

\[
= 4 \left[ w^* W^{-1} (z, w) z \right]_{jk}
\]

\[
= 4 \left[ W^{-1} (w^* , z^*) - I_n \right]_{jk},
\]

Part (i) is proved.

On $\Pi (n)$, $z^* V^{-1} (z)$ and $W^{-1} (w, z) w$ are anti-symmetric, one has

\[
D_{jk} = -4 \sum_{\alpha, \beta = 1}^{n} (1 - \delta_{j \alpha}) (1 - \delta_{k \beta}) V_{\alpha \beta} \left[ z^* V^{-1} (z) \right]_{j \alpha} \left[ W^{-1} (w, z) w \right]_{k \beta}
\]

\[
= 4 \left[ z^* V^{-1} (z) V (z) w W^{-1} (z^*, w^*) \right]_{jk}
\]

\[
= 4 \left[ z^* w W^{-1} (z^*, w^*) \right]_{jk}
\]

\[
= 4 \left[ W^{-1} (z^*, w^*) - I_n \right]_{jk}
\]

and

\[
E_{jk} = -4 \sum_{\alpha, \beta = 1}^{n} (1 - \delta_{j \alpha}) (1 - \delta_{k \beta}) V_{\alpha \beta} \left[ w^* W^{-1} (z, w) \right]_{j \alpha} \left[ V^{-1} (z) z \right]_{k \beta}
\]

\[
= 4 \left[ w^* W^{-1} (z, w) V (z) V^{-1} (z) z \right]_{jk}
\]

\[
= 4 \left[ w^* W^{-1} (z, w) z \right]_{jk}
\]

\[
= 4 \left[ w^* z W^{-1} (w^*, z^*) \right]_{jk}
\]

\[
= 4 \left[ W^{-1} (w^*, z^*) - I_n \right]_{jk}.
\]

Therefore, the proof of the lemma is complete. \(\square\)

The proof of Theorem 2.2

Proof. Part (i) of Theorem 2.2 was proved by Hua [7]. We start to prove Part (ii).
On $\Pi(n)$, by Propositions 2.1 and 2.3, Lemmas 2.5, 2.6 and 2.7 for $z \in \Pi(n)$ and $w \in U(\Pi(n))$, one has
\[
\Delta_{jk}^\Pi(n)(z, w) = A_{jk}^\Pi(n) + B_{jk}^\Pi(n) + C_{jk}^\Pi(n) + D_{jk}^\Pi(n) + E_{jk}^\Pi(n)
\]
\[
= -4V_{kj} + 4\left[V^{-1}(z^*) - I_n\right]_{jk} + 4\left[W^{-1}(z^*, w^*) + W^{-1}(w^*, z^*) - I_n\right]_{jk} = 0.
\]

On $\Pi(n)$ and $n$ is even, by Proposition 2.1 and 2.3, Lemmas 2.5, 2.6 and 2.7 for $z \in \Pi(n)$ and $w \in U(\Pi(n))$, one has
\[
\Delta_{jk}^\Pi(n)(z, w) = A_{jk}^\Pi(n) + B_{jk}^\Pi(n) + C_{jk}^\Pi(n) + D_{jk}^\Pi(n) + E_{jk}^\Pi(n)
\]
\[
= -4V_{kj} + 4\left[V^{-1}(z^*) - I_n\right]_{jk} + 4\left[W^{-1}(z^*, w^*) + W^{-1}(w^*, z^*) - I_n\right]_{jk} = 0.
\]

Therefore, the proof of the theorem is complete by the Poisson integral formula (1.13) for $u$. \hfill \Box

3. Proof of Theorem 1.3

In this section, we will prove Theorem 1.3 by using the idea based on the argument in Graham [4].

Denoted by $\tilde{\Delta}$ the modified Laplace-Beltrami operator in the unit ball $B^n \subset \mathbb{C}^n$:
\[
\tilde{\Delta} := \sum_{j,k=1}^n \left( \delta_{jk} - |z|^2 \bar{z}_j \bar{z}_k \right) \frac{\partial^2}{\partial z_j \partial \bar{z}_k}.
\]

This is a new operator which is not included in the cases of the Laplace-Beltrami operators studied in Graham and Lee [6].

**Theorem 3.1.** Let $n > 1$, $p, q \in \mathbb{N} \cup \{0\}$. Let $f_{p,q}(z) = \sum_{|\alpha|=p, |\beta|=q} a_{\alpha \beta} z^\alpha \bar{z}^\beta$ be harmonic in $B_n$ and $u \in C^2(B_n)$ such that
\[
\begin{cases}
\tilde{\Delta} u = 0, & \text{in } B_n; \\
u = f_{p,q}, & \text{on } \partial B_n.
\end{cases}
\]

Then

(i) If $n$ is odd and $u \in C^{n+1} \left(\overline{B_n}\right)$ then $pq = 0$;

(ii) If $n$ is even and $u \in C^{2+\alpha} \left(\overline{B_n}\right)$ for some $\alpha > 1/2$ then $pq = 0$.

**Proof.** Following the argument of Graham [4], we consider $h(t)$ on $[0,1]$ such that $h(1) = 1$ and
\[
\tilde{\Delta} \left(h \left(|z|^4\right) f_{p,q}(z)\right) = 0, \quad z \in B_n.
\]

Notice that
\[
\frac{\partial^2 \left( f_{p,q}(z) h \left( |z|^4 \right) \right)}{\partial z_j \partial \bar{z}_k} = f_{p,q}(z) \left[ 2h' \left( |z|^4 \right) \left( z_k \bar{z}_j + |z|^2 \delta_{jk} \right) + 4 |z|^4 \bar{z}_j z_k h'' \left( |z|^4 \right) \right] \\
+ h \left( |z|^4 \right) \frac{\partial^2 f_{p,q}}{\partial z_j \partial \bar{z}_k} + 2 |z|^2 h' \left( |z|^4 \right) \left( z_k \frac{\partial f_{p,q}}{\partial z_j} + \bar{z}_j \frac{\partial f_{p,q}}{\partial \bar{z}_k} \right),
\]

\[
\sum_{j,k=1}^n \left( \delta_{jk} - |z|^2 z_j \bar{z}_k \right) \left[ 2h' \left( |z|^4 \right) \left( z_k \bar{z}_j + |z|^2 \delta_{jk} \right) + 4 |z|^4 \bar{z}_j z_k h'' \left( |z|^4 \right) \right] \\
= 4h'' \left( |z|^4 \right) |z|^6 \left( 1 - |z|^4 \right) + 2h' \left( |z|^4 \right) \left( |z|^2 - |z|^6 \right) + |z|^2 \left( n - |z|^4 \right)
\]

and

\[
\sum_{j,k=1}^n \left( \delta_{jk} - |z|^2 z_j \bar{z}_k \right) \left[ 2 |z|^2 h' \left( |z|^4 \right) \left( z_k \frac{\partial f_{p,q}}{\partial z_j} + \bar{z}_j \frac{\partial f_{p,q}}{\partial \bar{z}_k} \right) + h \left( |z|^4 \right) \frac{\partial^2 f_{p,q}}{\partial z_j \partial \bar{z}_k} \right] \\
= 2 |z|^2 h' \left( |z|^4 \right) (p + q) \left( 1 - |z|^4 \right) f_{p,q}(z) - |z|^2 h \left( |z|^4 \right) pq f_{p,q}(z).
\]

Therefore,

\[
0 = \Delta \left( h \left( |z|^4 \right) \right) f_{p,q}(z) \\
= 4h'' \left( |z|^4 \right) |z|^6 \left( 1 - |z|^4 \right) f_{p,q}(z) + 2h' \left( |z|^4 \right) \left( |z|^2 - |z|^6 \right) + |z|^2 \left( n - |z|^4 \right) f_{p,q}(z) \\
+ 2 |z|^2 h' \left( |z|^4 \right) (p + q) \left( 1 - |z|^4 \right) f_{p,q} - |z|^2 h \left( |z|^4 \right) pq f_{p,q} \\
= 4h'' \left( |z|^4 \right) |z|^6 \left( 1 - |z|^4 \right) f_{p,q}(z) \\
+ 2h' \left( |z|^4 \right) \left( |z|^2 (n + 1 + (p + q)) - |z|^6 (p + q + 2) \right) f_{p,q}(z) \\
- |z|^2 h \left( |z|^4 \right) pq f_{p,q}(z).
\]

With \( t = |z|^4, h(t) \) satisfies the equation:

\[
t (1 - t) h'' (t) + h' (t) \left[ \frac{p}{2} + \frac{q}{2} + \frac{n + 1}{2} - \left( \frac{p}{2} + \frac{q}{2} + 1 \right) t \right] - \frac{pq}{2} h(t) = 0.
\]

By the standard hypergeometric function theory \[4\] and \[13\], the smooth solution at \( t = 0 \) must be

\[
h(t) = \frac{F \left( \frac{p}{2}, \frac{q}{2}, \frac{p+q+n+1}{2}; \frac{t}{2} \right)}{F \left( \frac{p}{2}, \frac{q}{2}, \frac{p+q+n+1}{2}; 1 \right)}
\]

where

\[
F(a, b, c; t) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} t^n
\]

and

\[
(a)_n = \alpha (\alpha + 1) \cdots (\alpha + n - 1).
\]

Assuming that \( p, q > 0 \), we will study the behavior of \( h(t) \) near \( t = 1 \) according to the value of \( n \).
By the definition of \( F(a, b, c; t) \) given by (3.3), it is easy to verify that
\[
\frac{d}{dt} F(a, b, c; t) = \frac{ab}{c} F(a+1, b+1, c+1; t) \quad \text{if} \quad abc \neq 0.
\]
and the following lemma about hypergeometric function can be found in [16].

**Lemma 3.2.** For \( a, b, s > 0 \) with \( a > s \) and \( b > s \), one has
\[
\lim_{t \to 1^-} \frac{F(a, b, a+b; t)}{\log(1-t)} = \frac{\Gamma(a+b)}{\Gamma(a) \Gamma(b)}.
\]

**Euler’s identity:**
\[
F(a, b, a+b-s; t) = (1-t)^{-s} F(b-s, a-s, a+b-s; t)
\]
and
\[
\lim_{t \to 1^-} (1-t)^s F(a, b, a+b-s; t) = \frac{\Gamma(a+b-s) \Gamma(s)}{\Gamma(a) \Gamma(b)}.
\]

To complete the proof of Theorem 3.1, we need the following lemma.

**Lemma 3.3.** For \( k \in \mathbb{N} \) and \( p, q > 0 \), one has
(i) There exist \( H \in C^k([0,1]) \) and \( G \in C^k([0,1]) \) with \( G(1) \neq 0 \) such that
\[
F\left(\frac{p}{2}, \frac{q}{2}, \frac{2k+p+q}{2}; t\right) = H(t) + G(t) (1-t)^k \log(1-t).
\]
(ii) There exist \( H \in C^{k+1}([0,1]) \), a constant \( c \neq 0 \) such that
\[
F\left(\frac{p}{2}, \frac{q}{2}, \frac{2k+p+q+1}{2}; t\right) = H(t) + c (1-t)^{k+\frac{1}{2}} G_1(t),
\]
where
\[
G_1(t) := F\left(k + \frac{q+1}{2}, k + \frac{p+1}{2}, \frac{4k+p+q+3}{2}; t\right).
\]
Moreover,
\[
(1-t)^{k+\frac{1}{2}} G_1(t) \notin C^{k+1}([0,1]).
\]

**Proof.** Part (i) can be found in Graham [4]. The proof of Part (ii) may be found through reading materials in [16]. For convenience for readers, we sketch a proof here. By (3.5), one has
\[
F_\ell(t) := \frac{d^\ell}{dt^\ell} F\left(\frac{p}{2}, \frac{q}{2}, \frac{2k+p+q+1}{2}; t\right)
= c_\ell F\left(\ell + \frac{p}{2}, \ell + \frac{q}{2}, \ell + \frac{2k+p+q+1}{2}; t\right),
\]
where
\[
c_0 = 1 \quad \text{and} \quad c_\ell = \frac{\left(\frac{\ell}{2}\right)_\ell \left(\frac{\ell}{2}\right)_\ell}{\left(\frac{2k+p+q+1}{2}\right)_\ell}.
\]
Notice that \( G_1(t) \in C([0,1]) \). By (3.5) and (3.7), there exists \( \hat{c} \neq 0 \) such that
\[
(1-t)^{\frac{1}{2}} \frac{d}{dt} G_1(t) = \hat{c} F\left(k + 1 + \frac{p}{2}, k + 1 + \frac{q}{2}, 2k + \frac{p+q+5}{2}\right) \in C([0,1]).
\]
Let
\[ H_1 (t) = c_k + \frac{c_{k+1}}{2} \int_0^t (1-s)^{\frac{1}{2}} \frac{dG_1 (s)}{ds} ds. \]
Then \( H_1 (t) \in C^1 ([0,1]) \). By (3.7), again,
\[ F_{k+1} (t) = c_{k+1} (1-t)^{-\frac{1}{2}} G_1 (t). \]
The definition of \( F_t \) implies
\[
F_k (t) = c_k + \int_0^t F_{k+1} (s) ds = c_k + c_{k+1} \int_0^t (1-s)^{\frac{1}{2}} G_1 (s) ds \\
= H_1 (t) - \frac{c_{k+1}}{2} (1-t)^{\frac{1}{2}} G_1 (t).
\]
Let
\[ H_2 (t) = c_{k-1} + \int_0^t \left[ H_1 (s) - \frac{1}{3} (1-s)^{\frac{1}{2}} \frac{d}{ds} G_1 (s) \right] ds.
\]
Then \( H_2 (t) \in C^2 ([0,1]) \) and
\[ F_{k-1} = H_2 (t) + \frac{c_{k+1}}{3} (1-t)^{\frac{1}{2}} G_1 (t). \]
By induction, there exist \( H \in C^{k+1} ([0,1]) \) and a constant \( c \neq 0 \) such that
\[
F \left( \frac{p}{2}, \frac{q}{2}, \frac{2k+p+q+1}{2}; t \right) = H (t) + c (1-t)^{k+\frac{1}{2}} G_1 (t).
\]
And
\[
\frac{d}{dt} \left( (1-t)^{\frac{1}{2}} G_1 (t) \right) = \frac{1}{2} (1-t)^{-\frac{1}{2}} G_1 (t) + (1-t)^{\frac{1}{2}} \frac{dG_1 (t)}{dt} \notin C ([0,1]).
\]
This implies that \( (1-t)^{k+\frac{1}{2}} G_1 (t) \notin C^{k+1} ([0,1]) \) and the lemma is proved. \( \square \)

Now we continue the proof of Theorem 3.1. By (3.2),
\[
\left. \begin{array}{l}
\left. \begin{array}{l}
\alpha > \frac{1}{2}.
\end{array} \right\} \end{array} \right\} \end{array}
\]
\[ u (z) = h \left( |z|^4 \right) f_{p,q} (z) = \frac{F \left( \frac{p}{2}, \frac{q}{2}, \frac{p+q+n+1}{2}; |z|^4 \right)}{F \left( \frac{p}{2}, \frac{q}{2}, \frac{p+q+n+1}{2}; 1 \right)} f_{p,q} (z). \]
We have the following two cases:
(i) When \( n \) is odd and \( pq \neq 0 \), by Part (i) of Lemma 3.2 with \( k = \frac{n+1}{2} \),
\[ h (t) = H (t) + G (t) (1-t)^{-\frac{n+1}{2}} \log (1-t) \]
with \( H, G \in C^\infty ([0,1]) \) and \( G(1) \neq 0 \). Since \( u \in C^{\frac{n+2}{2}} \left( B_n \right) \) implies \( h (t) \in C^{\frac{n+1}{2}} ([0,1]) \). This is a contradiction, which implies that \( pq = 0 \). This proves the Part (i) of Theorem 3.1.
(ii) When \( n = 2k \) is even and \( pq \neq 0 \), by Part (ii) of Lemma 3.2 for any \( \alpha > 1/2 \), we know that \( h \in C^\infty ([0,1]) \cap C^k ([0,1]) \) and
\[ (1-t)^{1-\alpha} \frac{d^{k+1}}{dt^{k+1}} h (t) \]
is unbounded on \([0,1] \). The assumption of Part (ii) of Theorem 3.1 implies \( h \in C^{k,\alpha} ([0,1]) \cap C^\infty ([0,1]) \) for some \( \alpha > \frac{1}{2} \). This is a contradiction, which implies that \( pq = 0 \). This proves Part (ii) of Theorem 3.1.

Therefore, the proof of Theorem 3.1 is complete. \( \square \)
Proof of Theorem 1.3

Proof. By the spherical harmonic expansions for $u$ on $\partial B_n$,

$$ u(z) = \sum_{p,q=0}^{+\infty} f_{p,q}(z), \quad z \in \partial B_n $$

where $f_{p,q}$ is a spherical harmonic function in $B_n$ of homogenous degrees $(p, q)$. Then

$$ u(z) = \sum_{p,q=0}^{+\infty} h_{p,q}(\lvert z \rvert^4) f_{p,q}(z). $$

By Theorem 3.1 and the assumption of Theorem 1.3, one has that $f_{p,q} = 0$ if $pq \neq 0$. This implies $u$ is pluriharmonic in $B_n$, and the proof of Theorem 1.3 is complete. □

4. Proof of Theorem 1.2

For a bounded domain $D \subset \mathbb{C}^N$, we use $\text{Aut}(D)$ to denote the automorphism group on $D$. We say that $D$ is transitive or homogeneous if any two points $z, w \in D$ there is a $\phi \in \text{Aut}(D)$ such that $\phi(z) = w$. $D$ is symmetric if for any $z \in D$, there is $S_z \in \text{Aut}(D)$ such that $z$ is an isolated fixed point for $(S_z)^2$.

Proposition 4.1. Let $D$ be a transitive domain in $\mathbb{C}^N$. Let $A(D)$ be a subset of $C^2(D)$ such that for any $\phi \in \text{Aut}(D)$ and $u \in A(D)$ one has $u \circ \phi \in A(D)$. If there is a point $z_0 \in D$ such that

$$ \frac{\partial^2 u(z_0)}{\partial z_j \partial \overline{z}_k} = 0, \quad 1 \leq j, k \leq N, \quad u \in A(D), $$

then $A(D)$ is a subset of pluriharmonic functions on $D$.

Proof. Let $u \in A(D)$ be an arbitrary element. Then for any $w \in D$, since $D$ is transitive, there is a $\phi \in \text{Aut}(D)$ such that $\phi(z_0) = w$.

Since $A(D)$ is invariant under automorphism, one has that $u \circ \phi \in A(D)$ and

$$ \frac{\partial^2 u \circ \phi}{\partial z_j \partial \overline{z}_k}(z_0) = 0, \quad 1 \leq j, k \leq N. $$

Let $H_u$ be the complex Hessian matrix of $u$ and let $\phi'(z) = \left[ \frac{\partial \phi_k}{\partial z_j} \right]$ be the Jacobian matrix with index $j$ represents the row and $k$ represents the column. Then

$$ H_{\phi \circ \phi}(z_0) = \phi'(z_0) H_u(\phi(z_0)) \phi'(z_0)^*. $$

Therefore

$$ H_u(w) = \phi'(z_0)^{-1} H_{\phi \circ \phi}(z_0)(\phi'(z_0)^*)^{-1} = 0. $$

This proves that $u$ is pluriharmonic in $D$. □

Lemma 4.2. If $A$ is an $n \times n$ matrix over $\mathbb{C}$ such that

$$ \langle \xi, \xi A \rangle = 0, \quad \text{for all } \xi \in \partial B_n, $$

then $A = 0$. 
Proof. Applying (4.2) to $\xi = e_k$, one has $[A]_{kk} = 0$ for all $1 \leq k \leq n$. Then applying the identity (4.2) to $\xi = \frac{1}{\sqrt{2}}(e_k + e_j)$ and to $\xi = \frac{1}{\sqrt{2}}(e_k + \sqrt{-1}e_j)$, respectively, one has

$$[A]_{jk} + [A]_{kj} = 0 \quad \text{and} \quad [A]_{jk} - [A]_{kj} = 0 \quad \text{for} \quad k \neq j.$$

This implies $A = 0$. $\square$

**Theorem 4.3.** Let $m \leq n$ and $u \in C^n(\mathbf{I}(m,n))$ be invariant harmonic in $\mathbf{I}(m,n)$. Then

$$\frac{\partial^2 u}{\partial z_j \partial z_k}(0) = 0, \quad 1 \leq j, k \leq m, \quad 1 \leq \alpha, \beta \leq n.$$  

Proof. For any $\lambda = (\lambda_1, \cdots, \lambda_n) \in B_n$ and $\xi = (\xi_1, \cdots, \xi_m) \in \partial B_m$ is fixed, we let

$$z = z(\lambda) := \lambda_i \overline{\lambda}_j.$$  

Let $g(\lambda) = u(z(\lambda))$. Then $g \in C^n(B_n)$ and

$$\frac{\partial^2 g(\lambda)}{\partial \lambda_i \partial \lambda_j} = \sum_{k,\ell=1}^{m} \sum_{\alpha,\beta=1}^{n} \frac{\partial^2 u(z)}{\partial z_k \partial z_\ell} \frac{\partial (\xi_k \lambda_\alpha)}{\partial \lambda_i} \frac{\partial (\xi_\ell \lambda_\beta)}{\partial \lambda_j} = \sum_{k,\ell=1}^{m} \frac{\partial^2 u(z)}{\partial z_k \partial z_\ell} \xi_k \xi_\ell.$$  

Therefore,

$$\sum_{i,j=1}^{n} (\delta_{ij} - \lambda_i \overline{\lambda}_j) \frac{\partial^2 g(\lambda)}{\partial \lambda_i \partial \lambda_j} = \sum_{k,\ell=1}^{m} \sum_{i,j=1}^{n} (\delta_{ij} - \lambda_i \overline{\lambda}_j) \frac{\partial^2 u}{\partial z_k \partial z_\ell} = \sum_{k,\ell=1}^{m} \sum_{i,j=1}^{n} (\delta_{ij} - \sum_{p=1}^{m} z(\lambda)_p \overline{z(\lambda)}_p) \frac{\partial^2 u}{\partial z_k \partial z_\ell} = \sum_{k,\ell=1}^{m} \xi_k \xi_\ell (\Delta_{1}^2 u) \circ (z(\lambda)) = 0.$$  

Graham Theorem for invariant harmonic function on $B_n$ implies that $g$ is pluriharmonic in $B_n$. In particular, by (4.6), one has

$$0 = \frac{\partial^2 g}{\partial \lambda_i \partial \lambda_j}(0) = \sum_{k,\ell=1}^{m} \frac{\partial^2 u(0)}{\partial z_k \partial z_\ell} \xi_k \xi_\ell, \quad 1 \leq k, \ell \leq m, \quad 1 \leq i, j \leq n$$  

for all $\xi \in B_n$. Combining this and Lemma 4.2, one has

$$\frac{\partial^2 u}{\partial z_k \partial z_\ell}(0) = 0, \quad 1 \leq k, \ell \leq m, 1 \leq i, j \leq n.$$  

The proof of Theorem 4.3 is complete. $\square$
Theorem 4.4. If $n > 1$ is either odd and $u \in C^{\frac{1}{2}}(\Pi(n))$ or $n = 2k$ is even and $u \in C^{k,\alpha}(\Pi(n))$ for some $\alpha > 1/2$ and if $u$ is invariant harmonic in $\Pi(n)$, then

$$\frac{\partial^2 u}{\partial z_{j \alpha} \partial \bar{z}_{k \beta}}(0) = 0, \quad 1 \leq j, \alpha, k, \beta \leq n.$$ (4.7)

Proof. Let $\lambda = (\lambda_1, \cdots, \lambda_n) \in B_n$ and $U = [U_{jk}]$ is a unitary matrix. Let

$$z(\lambda) = (\lambda U)^t(\lambda U) \in \Pi(n)$$ (4.8)

and

$$v(\lambda) = u(z(\lambda)).$$ (4.9)

Since $z(\lambda)$ is holomorphic in $\lambda$ and $z(\lambda)$ is symmetric, we have

$$\frac{\partial^2 v(\lambda)}{\partial \lambda_\alpha \partial \lambda_\beta} = \sum_{i,j=1}^{n} \frac{1}{2 - \delta_{k \ell}} \frac{\partial}{\partial \lambda_\alpha} \left( \frac{\partial u}{\partial \bar{z}_{k \ell}}(z(\lambda)) \left( \sum_{q} \lambda_q U_{q \ell} \lambda_q U_{k \ell} + U_{q k} \lambda_q U_{q \ell} \right) \right)$$

$$= \sum_{i,j=1}^{n} \sum_{k,\ell=1}^{n} \frac{1}{(2 - \delta_{k \ell})(2 - \delta_{ij})} \frac{\partial^2 u}{\partial z_{ij} \partial \bar{z}_{k \ell}}(z(\lambda)) \left( \sum_{p} \lambda_p U_{p j} \lambda_p U_{i \alpha} + U_{p i} U_{i j} \right) \left( \sum_{q} \lambda_q U_{q \ell} \lambda_q U_{k \ell} + U_{q k} \lambda_q U_{q \ell} \right)$$

$$= \sum_{p,q} \lambda_p \lambda_q \sum_{i,k=1}^{n} U_{pi} U_{qk} \sum_{j,\ell=1}^{n} \frac{1}{(2 - \delta_{k \ell})(2 - \delta_{ij})} \frac{\partial^2 u}{\partial z_{ij} \partial \bar{z}_{k \ell}}(z(\lambda)) U_{i j} \lambda_{\alpha j} U_{\beta \ell}$$

and

$$\sum_{\alpha,\beta=1}^{n} (\delta_{\alpha \beta} - |\lambda|^2 \lambda_\alpha \bar{\lambda}_\beta) U_{\alpha j} U_{\beta \ell} = \delta_{j \ell} - |\lambda|^2 \sum_{\alpha=1}^{n} \lambda_\alpha U_{\alpha j} \sum_{\beta=1}^{n} \lambda_\beta U_{\beta \ell}$$

$$= \delta_{j \ell} - |\lambda|^2 |\lambda U_{1 j}|^2 |\lambda U_{1 \ell}|$$

$$= \delta_{j \ell} - \left[ \lambda U_{1 j} \right]_{1 j} \lambda U_{1 \ell} \cdot (\lambda U_{1 \ell})^*$$

$$= \delta_{j \ell} - \sum_{\alpha=1}^{n} z(\lambda)_{p j} z(\lambda)_{p \ell}$$

$$= V_{j \ell}(z(\lambda)).$$
Therefore,

\[
\sum_{\alpha, \beta = 1}^{n} \left( \delta_{\alpha \beta} - |\lambda|^2 \bar{\lambda}_\alpha \lambda_\beta \right) \frac{\partial^2 v(\lambda)}{\partial \lambda_\alpha \partial \lambda_\beta} = \sum_{p, q}^{n} \lambda_p \bar{\lambda}_q \sum_{i, k}^{n} U_{pi} \bar{U}_{p q} \sum_{j, \ell}^{n} \left( \frac{1}{(1 - \frac{1}{2} \delta_{k\ell})(1 - \frac{1}{2} \delta_{ij})} \frac{\partial^2 u}{\partial z_{ij} \partial \bar{z}_{k\ell}}(z(\lambda)) \right)
\]

\[
\sum_{p, q}^{n} \lambda_p \bar{\lambda}_q \sum_{i, k}^{n} U_{pi} \bar{U}_{p q} \cdot \Delta u(z(\lambda)) = 0.
\]

Applying Theorem 1.3 to \( v \) on \( B_n \), one has \( v \) is pluriharmonic in \( B_n \). Thus,

\[
0 = \frac{\partial v(\lambda)}{\partial \lambda_\alpha} \frac{\partial v(\lambda)}{\partial \bar{\lambda}_\beta} = \sum_{p, q}^{n} \lambda_p \bar{\lambda}_q \sum_{i, k}^{n} U_{pi} \bar{U}_{p q} \cdot \Delta u(z(\lambda)) U_{\alpha j} U_{\beta \ell}.
\]

For any \( \xi \in \partial B_n \) and \( \omega \in (0, 1) \), by letting \( \lambda = \omega \xi U^* \) one has

\[
\omega^2 \sum_{i, k}^{n} \xi_i \bar{\xi}_k \sum_{j, \ell}^{n} \left( \frac{1}{(1 - \frac{1}{2} \delta_{k\ell})(1 - \frac{1}{2} \delta_{ij})} \frac{\partial^2 u}{\partial z_{ij} \partial \bar{z}_{k\ell}}(\omega^2 \xi^t \xi) U_{\alpha j} U_{\beta \ell} = 0.
\]

Let \( \omega \to 0 \), one obtain

\[
\sum_{i, k}^{n} \xi_i \bar{\xi}_k \sum_{j, \ell}^{n} \left( \frac{1}{(1 - \frac{1}{2} \delta_{k\ell})(1 - \frac{1}{2} \delta_{ij})} \frac{\partial^2 u}{\partial z_{ij} \partial \bar{z}_{k\ell}}(0) U_{\alpha j} U_{\beta \ell} = 0.
\]

Let

\[
A = [A_{ki}] := \left[ \sum_{j, \ell}^{n} \left( \frac{1}{(1 - \frac{1}{2} \delta_{k\ell})(1 - \frac{1}{2} \delta_{ij})} \frac{\partial^2 u}{\partial z_{ij} \partial \bar{z}_{k\ell}}(0) U_{\alpha j} U_{\beta \ell} \right) \right].
\]

Then

\[
\langle \xi A, \xi \rangle = 0.
\]

By Lemma 4.2, this implies \( A = 0 \). Therefore,

\[
\sum_{j, \ell}^{n} \left( \frac{1}{(1 - \frac{1}{2} \delta_{k\ell})(1 - \frac{1}{2} \delta_{ij})} \frac{\partial^2 u}{\partial z_{ij} \partial \bar{z}_{k\ell}}(0) U_{\alpha j} U_{\beta \ell} \right) = 0.
\]

Take \( \alpha = \beta \), Lemma 4.2 implies that

\[
\frac{1}{(1 - \frac{1}{2} \delta_{k\ell})(1 - \frac{1}{2} \delta_{ij})} \frac{\partial^2 u}{\partial z_{ij} \partial \bar{z}_{k\ell}}(0) = 0, \quad 1 \leq i, j, k, \ell \leq n.
\]

Therefore,

\[
\frac{\partial^2 u}{\partial z_{ij} \partial \bar{z}_{k\ell}}(0) = 0, \quad 1 \leq i, j, k, \ell \leq n,
\]

and the proof of the theorem is complete. \( \square \)
Theorem 4.5. Let $n > 1$ be even and $u \in C^{n-1} \left( \mathbf{III}(n) \right)$ is invariant harmonic in $\mathbf{III}(n)$. Then

\begin{equation}
\frac{\partial^2 u(0)}{\partial z_j \partial \overline{z}_k} = 0, \quad 1 \leq j, k, \alpha, \beta \leq n.
\end{equation}

Proof. Let $z(\lambda) : B_{n-1} \to \mathbf{III}(n)$ be defined by

\begin{equation}
z(\lambda) = \begin{bmatrix}
0 & \lambda \\
-\lambda^t & O_{n-1}
\end{bmatrix},
\end{equation}

where $O_{n-1}$ is an $(n-1) \times (n-1)$ zero matrix. Let $g(\lambda) = u(z(\lambda)), \quad \lambda = (\lambda_2, \cdots, \lambda_n) \in B_{n-1}$.

Then

\begin{align*}
\frac{\partial g(\lambda)}{\partial \lambda_p} &= \sum_{j<\alpha} \frac{\partial u(z(\lambda))}{\partial z_{ja}} \frac{\partial (z_{ja}(\lambda) - z_{\alpha j}(\lambda))}{\partial \lambda_p} \\
&= \sum_{j, \alpha = 1}^n \frac{\partial u(z(\lambda))}{\partial z_{ja}} \frac{\partial z_{ja}(\lambda)}{\partial \lambda_p} \\
&= 2 \frac{\partial u(z(\lambda))}{\partial z_{1p}}
\end{align*}

and

\begin{equation}
\frac{\partial^2 g(\lambda)}{\partial \lambda_p \partial \overline{\lambda_q}} = 4 \frac{\partial^2 u(z(\lambda))}{\partial z_{1p} \partial \overline{z}_q}.
\end{equation}

Since $V(z(\lambda)) = I_n - z(\lambda)z(\lambda)^*$, one has

\begin{equation}
V_{p1} = (1 - |\lambda|^2)\delta_{p1}, \quad V_{1p} = (1 - |\lambda|^2)\delta_{1p}
\end{equation}

and

\begin{equation}
V_{\alpha\beta} = \delta_{\alpha\beta} - \lambda_{\alpha} \overline{\lambda}_{\beta} \quad \text{for} \quad \alpha, \beta \geq 2.
\end{equation}

Therefore,

\begin{align*}
\sum_{p,q=2}^n (\delta_{pq} - \lambda_p \overline{\lambda_q}) \frac{\partial^2 g(\lambda)}{\partial \lambda_p \partial \overline{\lambda_q}}
&= 4 \sum_{p,q=2}^n (\delta_{pq} - \lambda_p \overline{\lambda_q}) \frac{\partial^2 u(\lambda)}{\partial z_{1p} \partial \overline{z}_q} \\
&= 4 \Delta_3^{11} u(\lambda(\lambda)) = 0.
\end{align*}

By the Graham’s theorem on $B_{n-1}$, one have $g$ is pluriharmonic in $\lambda \in B_{n-1}$. For any unitary matrix $U$ since $u_U(z) = u(U^t z U)$

is also invariant harmonic in $\mathbf{III}(n)$. By the argument, if we let

$g(\lambda) = u(U^t z(\lambda) U)$

then

\begin{equation}
\frac{\partial^2 g(0)}{\partial \lambda_p \partial \lambda_q} = 0.
\end{equation}
Notice that
\[
\frac{\partial g(\lambda)}{\partial \lambda_p} = \sum_{j<\alpha}^{n} \frac{\partial u(z(\lambda))}{\partial z_{j\alpha}} \frac{\partial (z_{j\alpha}(\lambda) - z_{\alpha j}(\lambda))}{\partial \lambda_p}
= \sum_{j,\alpha=1}^{n} \frac{\partial u(z(\lambda))}{\partial z_{j\alpha}} \frac{\partial z_{j\alpha}(\lambda)}{\partial \lambda_p}
= \sum_{j,\alpha=1}^{n} \frac{\partial u(z(\lambda))}{\partial z_{j\alpha}} (U_{1j}U_{\alpha p} - U_{pj}U_{1\alpha}),
\]
and since \(z\) is anti-symmetric, one has
\[
0 = \sum_{p,q=1}^{n} U_{pq} U_{pmn} \frac{\partial^2 g(0)}{\partial \lambda_p \partial \lambda_q}
= \sum_{j,\alpha=1}^{n} \sum_{k,\beta=1}^{n} \frac{\partial^2 u(0)}{\partial z_{j\alpha} \partial z_{k\beta}} (U_{1j}U_{\alpha p} - U_{pj}U_{1\alpha})(U_{1k}U_{q\beta} - U_{qk}U_{1\beta})
= \sum_{j,k=1}^{n} \frac{\partial^2 u(0)}{\partial z_{jm} \partial z_{k\ell}} U_{1j}U_{1k} - \sum_{j,\beta=1}^{n} \frac{\partial^2 u(0)}{\partial z_{jm} \partial z_{k\beta}} U_{1j}U_{1\beta}
- \sum_{\alpha,\beta=1}^{n} \frac{\partial^2 u(0)}{\partial z_{m\alpha} \partial z_{\ell\beta}} U_{1\alpha}U_{1\beta} + \sum_{\alpha,\beta=1}^{n} \frac{\partial^2 u(0)}{\partial z_{m\alpha} \partial z_{\ell\beta}} U_{1\alpha}U_{1\beta}
= 4 \sum_{j,k=1}^{n} \frac{\partial^2 u(0)}{\partial z_{jm} \partial z_{k\ell}} U_{1j}U_{1k}.\]

Lemma 4.2 implies
\[
\frac{\partial^2 u(0)}{\partial z_{jm} \partial z_{k\ell}} = 0.
\]

The proof of the theorem is complete. \(\square\)

Proof of Theorem 1.2

**Proof.**
1) If \(u \in C^n(I(m,n))\) is invariant harmonic in \(I(m,n)\). By Theorem 4.3 and Proposition 1.1, one has
\[
\frac{\partial^2 u}{\partial z_{j\alpha} \partial z_{k\beta}}(z) = 0, \quad z \in I(m,n), \quad 1 \leq j, k \leq m, 1 \leq \alpha, \beta \leq n.
\]
This means that \(u\) is pluriharmonic in \(I(m,n)\).
2) If \( n > 1 \) is odd and if \( u \in C^{\frac{n+1}{2}}(\Pi(n)) \) or \( n = 2k > 1 \) is even and if \( u \in C^{k,\alpha}(\Pi(n)) \) for some \( \alpha > 1/2 \) and if \( u \) is invariant harmonic in \( \Pi(n) \).

By Theorem 4.4 and Proposition 4.1, one has
\[
\frac{\partial^2 u}{\partial z_j \partial \overline{z}_k}(z) = 0, \quad z \in \Pi(n), \quad 1 \leq j, k \leq n.
\]
This means that \( u \) is pluriharmonic in \( \Pi(n) \).

3) If \( n \) is even and if \( u \in C^{n-1}(\Pi(n)) \) is invariant harmonic in \( \Pi(n) \). By Theorem 4.5 and Proposition 4.1, one has
\[
\frac{\partial^2 u}{\partial z_j \partial \overline{z}_k}(z) = 0, \quad z \in \Pi(n), \quad 1 \leq j, k \leq n.
\]
This means that \( u \) is pluriharmonic in \( \Pi(n) \).

Therefore, the proof of Theorem 1.2 is complete.

5. Remarks on \( \Pi(3), \Pi(4) \) and \( \Pi(2) \)

First, let us make a remark on \( \Pi(2k+1) \) when \( k = 1 \). It is known from Lu [14] that \( \Pi(3) \) is biholomorphic to \( B_3 \). In fact, if let
\[
\varphi(z) = \begin{bmatrix} 0 & z_1 & z_2 \\ -z_1 & 0 & z_3 \\ -z_2 & -z_3 & 0 \end{bmatrix}
\]
then it is easy to verify that \( \varphi : B_3 \to \Pi(3) \) is a biholomorphic map.

By Theorem 1.2 or the Graham’s theorem on \( B_3 \), one has

**Corollary 5.1.** If \( u \in C^2(\Pi(3)) \) is invariant harmonic then \( u \) is pluriharmonic in \( \Pi(3) \).

Second, it is known from Lu [14] that \( \Pi(4) \) is biholomorphic to \( \Pi(2,2) \). By Theorem 1.2, one has the following corollary.

**Corollary 5.2.** If \( u \in C^2(\Pi(4)) \) is invariant harmonic in \( \Pi(4) \) then \( u \) is pluriharmonic in \( \Pi(4) \).

Finally, it is known from Lu [14] that \( \Pi(2) \) is biholomorphic to the polydisc \( D(0,1)^2 \) in \( \mathbb{C}^2 \). Moreover, one can verify that the following map
\[
(w_1, w_2) = \varphi(z) = (z_1 + iz_2, z_1 - iz_2) : D(0,1)^2 \to \Pi(2)
\]
is a biholomorphic map. Applying the result in Li-Simon [10], one has the following result.

**Corollary 5.3.** If \( u \in C(\Pi(2)) \) is invariant harmonic in \( \Pi(2) \) then
\begin{enumerate}
  
(i) \( u \) is harmonic (in the regular sense) in \( \Pi(2) \);
  
(ii) \( 2 \Re \frac{\partial^2 u}{\partial w_1 \partial w_2} = 0 \) in \( \Pi(2) \);
  
(iii) \( u \) is not pluriharmonic in general.
\end{enumerate}

**Proof.** Since
\[
(5.2) \quad z_1 = \frac{w_1 + w_2}{2}, \quad z_2 = \frac{w_1 - w_2}{2i}
\]
and let
\[
(5.3) \quad v(z) = u(w).
\]
Then $v$ is invariant harmonic in $D(0, 1)^2$ and continuous up to the boundary. By the result in [10], we have

$$\frac{\partial^2 v}{\partial z_1 \partial \bar{z}_1} = \frac{\partial^2 v}{\partial z_2 \partial \bar{z}_2} = 0, \quad z \in D(0, 1)^2. \tag{5.4}$$

Notice that

$$4 \frac{\partial^2 v}{\partial z_1 \partial \bar{z}_1} = \frac{\partial^2 u}{\partial w_1 \partial \bar{w}_1} + \frac{\partial^2 u}{\partial w_1 \partial \bar{w}_2} + \frac{\partial^2 u}{\partial w_2 \partial \bar{w}_1} + \frac{\partial^2 u}{\partial w_2 \partial \bar{w}_2}, \tag{5.5}$$

and

$$-4 \frac{\partial^2 v}{\partial z_2 \partial \bar{z}_2} = \frac{\partial^2 u}{\partial w_1 \partial \bar{w}_1} - \frac{\partial^2 u}{\partial w_1 \partial \bar{w}_2} - \frac{\partial^2 u}{\partial w_2 \partial \bar{w}_1} + \frac{\partial^2 u}{\partial w_2 \partial \bar{w}_2}. \tag{5.6}$$

By (5.3) – (5.6), one can easily see Parts (i) and (ii) of the above corollary holds.

In order to prove Part (iii), we let

$$u(z_1, z_2) = |z_1|^2 - |z_2|^2. \tag{5.7}$$

One can verify that $u$ is invariant harmonic in $\mathbf{IV}(2)$, but it is clearly that $u$ is not pluriharmonic in $\mathbf{IV}(2)$. Therefore, the proof of the corollary is complete. □

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