When Is an Ellipse Inscribed In a Quadrilateral Tangent at the Midpoint of Two or More Sides

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I. Introduction

Among all ellipses inscribed in a triangle, \( T \), the midpoint, or Steiner, ellipse is interesting and well-known \([2]\). It is the unique ellipse tangent to \( T \) at the midpoints of all three sides of \( T \) and is also the unique ellipse of maximal area inscribed in \( T \). What about ellipses inscribed in quadrilaterals, \( Q \)? Not surprisingly, perhaps, there is not always a midpoint ellipse—i.e., an ellipse inscribed in \( Q \) which is tangent at the midpoints of all four sides of \( Q \); In fact, in \([1]\) it was shown that if there is a midpoint ellipse, then \( Q \) must be a parallelogram.

That is, if \( Q \) is not a parallelogram, then there is no ellipse inscribed in \( Q \) which is tangent at the midpoint of all four sides of \( Q \); But can one do better than four sides of \( Q \)? In other words, if \( Q \) is not a parallelogram, is there an ellipse inscribed in \( Q \) which is tangent at the midpoint of three sides of \( Q \)? In Theorem 1 we prove that the answer is no. In fact, unless \( Q \) is a trapezoid (a quadrilateral with at least one pair of parallel sides), or what we call a midpoint diagonal quadrilateral (see the definition below), then there is not even an ellipse inscribed in \( Q \) which is tangent at the midpoint of two sides of \( Q \).

Definition: A convex quadrilateral, \( Q \), is called a midpoint diagonal quadrilateral (mdq) if the intersection point of the diagonals of \( Q \) coincides with the midpoint of at least one of the diagonals of \( Q \).

A parallelogram, \( P \), is a special case of an mdq since the diagonals of \( P \) bisect one another. In \([5]\) we discussed mdq’s as a generalization of parallelograms in a certain sense related to tangency chords and conjugate diameters of inscribed ellipses.

What about uniqueness? If \( Q \) is an mdq, then the ellipse inscribed in \( Q \) which is tangent at the midpoint of two sides of \( Q \) is not unique. Indeed we prove (Lemma 3) that in that case there are two such ellipses. However, if \( Q \) is a trapezoid, then the ellipse inscribed in \( Q \) which is tangent at the midpoint of two sides of \( Q \) is unique (Lemma 4).

Is there a connection with tangency at the midpoint of the sides of \( Q \) and the ellipse of maximal area inscribed in \( Q \) as with parallelograms? In \([3]\) we showed that the midpoint ellipse for a parallelogram also turns out to be the unique ellipse of maximal area inscribed in \( Q \). For trapezoids, we prove (Lemma 4) that the unique ellipse of maximal area inscribed in \( Q \) is the unique ellipse tangent to \( Q \) at the midpoint of two sides of \( Q \). However, for mdq’s, the unique ellipse of maximal area inscribed in \( Q \) need not be tangent at the midpoint of any side of \( Q \).

We use the notation \( Q(A_1, A_2, A_3, A_4) \) to denote the quadrilateral with vertices \( A_1, A_2, A_3, \) and \( A_4 \), starting with \( A_1 \) = lower left corner and going clockwise. Denote the sides of \( Q(A_1, A_2, A_3, A_4) \) by \( S_1, S_2, S_3, \) and \( S_4 \), going clockwise and starting with the leftmost side, \( S_1 \), and denote the diagonals of \( Q(A_1, A_2, A_3, A_4) \) by \( D_1 = A_1A_3 \) and \( D_2 = A_2A_4 \).
We note here that there are two types of mdq's: Type 1, where the diagonals intersect at the midpoint of \( D_2 \) and Type 2, where the diagonals intersect at the midpoint of \( D_1 \); Mdq's of types 1 and 2, respectively, are illustrated below.

Given a convex quadrilateral, \( Q = Q(A_1, A_2, A_3, A_4) \), which is not a parallelogram, it will simplify our work below to consider quadrilaterals with a special set of vertices. In particular, there is an affine transformation which sends \( A_1, A_2, \) and \( A_4 \) to the points \((0,0), (0,1), \) and \((1,0)\), respectively. It then follows that \( A_3 = (s,t) \) for some \( s, t > 0 \); Summarizing:

\[
Q_{s,t} = Q(A_1, A_2, A_3, A_4), \tag{1}
\]
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\[ A_1 = (0, 0), A_2 = (0, 1), A_3 = (s, t), A_4 = (1, 0). \]

Since \( Q_{st} \) is convex, \( s + t > 1 \). Also, if \( Q \) has a pair of parallel vertical sides, first rotate counterclockwise by 90°, yielding a quadrilateral with parallel horizontal sides. Since we are assuming that \( Q \) is not a parallelogram, we may then also assume that \( Q_{st} \) does not have parallel vertical sides and thus \( s \neq 1 \). Note that any trapezoid which is not a parallelogram may be mapped, by an affine transformation, to the quadrilateral \( Q_{s,1} \); Thus we may assume that \( (s,t) \in G \), where

\[ G = \{(s,t) : s,t > 0, s+t > 1, s \neq 1 \}. \]  

The following result gives the points of tangency of any ellipse inscribed in \( Q_{st} \) (see [4] where some details were provided). We leave the details of a proof to the reader.

For the rest of the paper we work with the quadrilateral \( Q_{st} \) defined above.

**Proposition 1:**

(i) \( E_0 \) is an ellipse inscribed in \( Q_{st} \) if and only if the general equation of \( E_0 \) is given by

\[ t^2x^2 + (4q^2(t-1)t + 2qt(s-t+2) - 2st)xy + \]

\[ (q(t-s) + s)^2y^2 - 2qt^2x - 2qt(q(t-s) + s)y + q^2t^2 = 0 \]  

for some \( q \in J = (0,1) \). Furthermore, (3) provides a one-to-one correspondence between ellipses inscribed in \( Q_{st} \) and points \( q \in J \).

(ii) If \( E_0 \) is an ellipse given by (3) for some \( q \in J \), then \( E_0 \) is tangent to the four sides of \( Q_{st} \) at the points

\[ P_i = \left(0, \frac{qt}{q(t-s) + s}\right) \in S_s, \quad P_i = \left(\frac{(1-q)s^2}{q(t-1)(s+t) + s}, \frac{t(s + q(t-1))}{q(t-1)(s+t) + s}\right) \in S_s, \]

\[ P_i = \left(\frac{s + q(1-t)}{q(s+t-2) + 1q(s+t-2) + 1}, \frac{(1-q)t}{1q(s+t-2) + 1}\right) \in S_s, \quad P_i = (q,0) \in S_s. \]

**Remark:** Using Proposition 1, it is easy to show that one can always find an ellipse inscribed in a quadrilateral, \( Q \), which is tangent to \( Q \) at the midpoint of at least one side of \( Q \), and this can be done for any given side of \( Q \).

The following lemma gives necessary and sufficient conditions for \( Q_{st} \) to be an mdq.

**Lemma 1:**

(i) \( Q_{st} \) is a type 1 midpoint diagonal quadrilateral if and only if \( s = t \).

(ii) \( Q_{st} \) is a type 2 midpoint diagonal quadrilateral if and only if \( s + t = 2 \).

**Proof:** The diagonals of \( Q_{st} \) are \( D_1 : y = \frac{t}{s}x \) and \( D_2 : y = 1 - x \), and they intersect at the point

\[ P = \left(\frac{s}{s+t}, \frac{t}{s+t}\right). \]  

Then midpoints of \( D_1 \) and \( D_2 \) are \( M_1 = \left(\frac{s}{2}, \frac{t}{2}\right) \) and \( M_2 = \left(\frac{1}{2}, \frac{1}{2}\right) \), respectively.
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\[ M_z = P \iff \frac{s}{s + t} = \frac{1}{2} \text{ and } \frac{t}{s + t} = \frac{1}{2}, \] both of which hold if and only if \( s = t; \) That proves (i);

\[ M_i = P \iff \frac{s}{s + t} = \frac{1}{s} \text{ and } \frac{t}{s + t} = \frac{1}{t}, \] both of which hold if and only if \( s + t = 2. \) That proves (ii).

The following lemma shows that affine transformations preserve the class of mdq's. We leave the details of the proof to the reader.

**Lemma 2:** Let \( T : \mathbb{R}^2 \to \mathbb{R}^2 \) be an affine transformation and let \( Q \) be a midpoint diagonal quadrilateral. Then \( Q' = T(Q) \) is also a midpoint diagonal quadrilateral.

**II. Main Results**

The following result shows that among non-trapezoids, the only quadrilaterals which have inscribed ellipses tangent at the midpoint of two sides are the mdq's.

**Lemma 3:** Let \( Q \) be a convex quadrilateral in the \( xy \) plane which is not a trapezoid.

(i) There is an ellipse inscribed in \( Q \) which is tangent at the midpoint of two or more sides of \( Q \) if and only if \( Q \) is a midpoint diagonal quadrilateral, in which case there are two such ellipses.

(ii) There is no ellipse inscribed in \( Q \) which is tangent at the midpoint of three sides of \( Q \).

**Proof:** By Lemma 2 and standard properties of affine transformations, we may assume that \( Q = Q_{s,t} \), the quadrilateral given in (1) with \((s,t) \in G; \) The midpoints of the sides of \( Q_{s,t} \) are given by \( MP_1 = \left(0, \frac{1}{2}\right) \in S_1, MP_2 = \left(\frac{s}{2}, \frac{1 + t}{2}\right) \in S_2, \) and \( MP_3 = \left(\frac{1}{2}, 0\right) \in S_4 \); Now let \( E_0 \) denote an ellipse inscribed in \( Q_{s,t} \), and let \( P_j \in S_j, j = 1, 2, 3, 4 \) denote the points of tangency of \( E_0 \) with the sides of \( Q_{s,t} \); By Proposition 1(ii),

\[ P_1 = MP_1 \iff \frac{qt}{q(t - s) + s} = \frac{1}{2}, \] (4)

\[ P_2 = MP_2 \iff \frac{(1 - q)s}{q(t - 1)(s + t) + s} = \frac{1}{2}, \] (5)

and

\[ \frac{t(s + q(t - 1))}{(q(t - 1)(s + t) + s)} = \frac{1 + t}{2}, \] (6)

\[ P_3 = MP_3 \iff \frac{s + q(t - 1)}{q(s + t - 2) + 1} = \frac{1 + s}{2}, \] (7)

and

\[ \frac{(1 - q)t}{q(s + t - 2) + 1} = \frac{t}{2}, \] (8)

\[ P_4 = MP_4 \iff q = \frac{1}{2}. \] (9)

Equations (4) and (9) each have the unique solutions \( q_1 = \frac{s}{s + t} \) and \( q_4 = \frac{1}{2} \), respectively. The system of equations in (5) and (6) has unique solution \( q_2 = \frac{s}{t^2 + st + s - t} \), and the system of equations in (7) and (8) has unique solution \( q_3 = \frac{1}{s + t} \); It is trivial that \( q_1, q_3, q_4 \in J = (0, 1) \); Since \((s, t) \in G, t(s + t - 1) > 0, \) which implies that \( q_2 \in J \). We now check which pairs of midpoints of sides of \( Q_{s,t} \) can be points of tangency of \( E_0 \); Note that different values of \( q \) yield distinct inscribed ellipses by the one-to-one correspondence between ellipses inscribed in \( Q_{s,t} \) and points \( q \in J \).

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(a) \( S_1 \) and \( S_2 \): \( q_1 = q_2 \Leftrightarrow \frac{s}{t^2 + st + s - t} - \frac{s}{s + t} = 0 \Leftrightarrow \frac{st(s + t - 2)}{(ts + t^2 + s - t)(s + t)} = 0 \Leftrightarrow s + t = 2. \)

(b) \( S_1 \) and \( S_3 \): \( q_1 = q_3 \Leftrightarrow \frac{1}{s + t} - \frac{s}{s + t} = 0 \Leftrightarrow \frac{s - 1}{s + t} = 0 \), which has no solution since \( s \neq 1. \)

(c) \( S_2 \) and \( S_3 \): \( q_2 = q_3 \Leftrightarrow \frac{1}{s + t} - \frac{s}{t^2 + st + s - t} = 0 \Leftrightarrow \frac{(t - s)(s + t - 1)}{(ts + t^2 + s - t)(s + t)} = 0 \Leftrightarrow s = t. \)

(d) \( S_1 \) and \( S_4 \): \( q_1 = q_4 \Leftrightarrow \frac{1}{2} - \frac{s}{s + t} = 0 \Leftrightarrow \frac{1}{2} - \frac{1}{2} = 0 \Leftrightarrow s = t. \)

(e) \( S_2 \) and \( S_4 \): \( q_2 = q_4 \Leftrightarrow \frac{1}{2} - \frac{s}{t^2 + st + s - t} = 0 \Leftrightarrow \frac{1}{2} - \frac{1}{2} = 0 \Leftrightarrow s + t = 2. \)

(f) \( S_3 \) and \( S_4 \): \( q_3 = q_4 \Leftrightarrow \frac{1}{2} - \frac{s}{s + t} = 0 \Leftrightarrow \frac{1}{2} - \frac{1}{2} = 0 \Leftrightarrow s + t = 2. \)

That proves that there is an ellipse inscribed in \( Q_{ij} \), which is tangent at the midpoints of \( S_1 \) and \( S_2 \)

or at the midpoints of \( S_3 \) and \( S_4 \) if and only if \( s + t = 2 \), and there is an ellipse inscribed in \( Q_{ij} \), which is tangent at the midpoints of \( S_2 \) and \( S_3 \) or at the midpoints of \( S_1 \) and \( S_4 \) if and only if \( s = t \). Furthermore, if \( s \neq t \) or if \( s + t \neq 2 \), then there is no ellipse inscribed in \( Q_{ij} \), which is tangent at the midpoint of two sides of \( Q_{ij} \). That proves (i) by Lemma 1. To prove (ii), to have an ellipse inscribed in \( Q_{ij} \), which is tangent at the midpoint of three sides of \( Q_{ij} \), those three sides are either \( S_1, S_2, \) and \( S_3 \); \( S_1, S_2, \) and \( S_4 \); \( S_1, S_3, \) and \( S_4 \); or \( S_2, S_3, \) and \( S_4 \). By (a)-(f) above, that is not possible.

For trapezoids inscribed in \( Q \) we have the following result.

**Lemma 4:** Assume that \( Q \) is a trapezoid which is not a parallelogram. Then

(i) There is a unique ellipse inscribed in \( Q \) which is tangent at the midpoint of two sides of \( Q \), and that ellipse is the unique ellipse of maximal area inscribed in \( Q \).

(ii) There is no ellipse inscribed in \( Q \) which is tangent at the midpoint of three sides of \( Q \).

**Proof:** Again, by affine invariance, we may assume that \( Q = Q_{i,j} \), the quadrilateral given in (1) with \( t = 1 \);

Note that \( 0 < s \neq 1 \). Now let \( P_0 \) denote an ellipse inscribed in \( Q_{i,j} \). Letting \( M_{P_j} \in S_j, j = 1, 2, 3, 4 \) denote the corresponding midpoints of the sides and using Proposition 1(ii) again, with \( t = 1 \), we have

\[
P_1 = M_{P_1} \Leftrightarrow \frac{q}{(1 - s)q + s} = \frac{1}{2}, \quad (10)
\]

\[
P_2 = M_{P_2} \Leftrightarrow (1 - q)s = \frac{s}{2}, \quad (11)
\]

\[
P_3 = M_{P_3} \Leftrightarrow \frac{s}{(s - 1)q + 1} = \frac{1 + s}{2}, \quad \text{and} \quad (12)
\]

\[
1 - q = \frac{1}{(s - 1)q + 1}, \quad (13)
\]

\[
P_4 = M_{P_4} \Leftrightarrow q = \frac{1}{2}, \quad (14)
\]

The unique solution of the equations in (11) and in (14) is \( q = \frac{1}{2} \in J \); the unique solution of the equation in (10) is \( q = \frac{s}{1 + s} \in J \), and the unique solution of the system of equations in (12) and (13) is \( q = \frac{1}{1 + s} \in J \).
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We now check which pairs of midpoints of sides of $Q_{r,l}$ can be points of tangency with $E_0$:

(a) $q = \frac{1}{2}$ gives tangency at the midpoints of $S_2$ and $S_4$.

(b) $S_1$ and $S_2$ or $S_1$ and $S_3$:
$$\frac{s}{1+s} = \frac{1}{2} \Leftrightarrow s = 1.$$

(c) $S_3$ and $S_2$ or $S_3$ and $S_4$:
$$\frac{1}{1+s} = \frac{1}{2} \Leftrightarrow s = 1.$$

(d) $S_1$ and $S_3$:
$$\frac{s}{1+s} = \frac{1}{2} \Leftrightarrow s = 1.$$

Since we have assumed that $s \neq 1$, the only way to have an ellipse inscribed in $Q_{r,l}$ which is tangent at the midpoint of two sides of $Q_{r,l}$ is if those sides are $S_2$ and $S_4$ or $S_3$ and $S_4$.

$$1.11s = \frac{1}{2} \Rightarrow s = 1.$$
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(2) Let $Q$ be the trapezoid with vertices $(0,0), (0,1), (2,1),$ and $(1,1)$; The ellipse with equation
\[
\left(x - \frac{5}{4}\right)^2 - 3\left(x - \frac{5}{4}\right)\left(y - \frac{1}{2}\right) + \frac{25}{4} \left(y - \frac{1}{2}\right)^2 = 1
\]
is tangent to $Q$ at $(2,1)$ and at $\left(\frac{1}{2}, 0\right)$, the midpoints of $S_2$ and $S_4$, respectively. See Figure 2 below.

Figure 2

References

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