Submodular and supermodular multi-labeling, and vertex happiness

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Abstract
In this paper, we investigate the submodular multi-labeling (Sub-ML) problem, a more general version of the submodular multiway partition (Sub-MP), which captures many cut problems as special cases, including the edge-/node-weighted multiway cut and the hypergraph multiway cut. We also study the complement of Sub-ML, the supermodular multi-labeling (Sup-ML) problem. We propose a convex (concave, respectively) relaxation for the Sub-ML (Sup-ML, respectively) problem based on the Lovász extension, which can be solved in polynomial time. By applying a randomized rounding, we prove that Sub-ML can be approximated within a factor of \(2 - \frac{2}{k}\) and Sup-ML can be approximated within a factor of \(\frac{2}{k}\), where \(k\) is the number of labels. In addition, we find that a recently studied vertex-coloring problem, the maximum happy vertices (MHV) problem, can be casted as a special case of Sup-ML. Hence, MHV can be approximated within \(\frac{2}{k}\), which improves the previous best \(\frac{1}{k}\)-approximation. Based on the associated LP relaxation, we further prove that the \(\frac{2}{k}\)-approximation is the best possible for MHV. For the complementary minimum unhappy vertices (MUHV) problem, casted as a special case of Sub-ML, it can be approximated within \(2 - \frac{2}{k}\) too; we prove that \(2 - \frac{2}{k}\) is the best possible based on the associated LP relaxation; lastly, we show that a \((2 - \frac{2}{k} - \epsilon)\)-approximation for MUHV is NP-hard under the Unique Games Conjecture, for any positive \(\epsilon\).

1 Introduction
Classification problems have been formulated as cuts, or partition, or labeling, or coloring, and have been widely studied for a very long time. In this paper, we study the submodular multi-labeling (Sub-ML) problem and its complement, the supermodular multi-labeling (Sup-ML) problem, which can be defined in the following. First, given a set \(V\), a non-negative submodular set function \(f: 2^V \to \mathbb{R}^+\) (\(\mathbb{R}^+\) is the set of positive real numbers), with \(f(\emptyset) = 0\), a set of labels \(L = \{1, 2, \ldots, k\}\), and a partial labeling function \(\ell: V \mapsto L\) which pre-assigns each label to at least one element of \(V\), the goal of the problem is to assign a label to each of the rest elements in \(V\) to minimize \(\sum_{i=1}^{k} f(S_i)\), where \(S_i\) is the subset of elements assigned with the label \(i\).

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The Sub-ML problem: Given a set \(V\), a non-negative submodular set function \(f: 2^V \to \mathbb{R}^+\), with \(f(\emptyset) = 0\), a set of labels \(L = \{1, 2, \ldots, k\}\), and a partial labeling function \(\ell: V \mapsto L\) which pre-assigns each label to at least one element of \(V\), the goal of the problem is to assign a label to each of the rest elements in \(V\) to minimize \(\sum_{i=1}^{k} f(S_i)\), where \(S_i\) is the subset of elements assigned with the label \(i\).
The **Sup-ML problem**: Given a set \( V \), a non-negative supermodular set function \( g : 2^V \to \mathbb{R}_+ \), with \( g(\emptyset) = 0 \), a set of labels \( L = \{1, 2, \ldots, k\} \), and a partial labeling function \( \ell : V \to L \) which pre-assigns each label to at least one element of \( V \), the goal of the problem is to assign a label to each of the rest elements in \( V \) to maximize \( \sum_{i=1}^{k} g(S_i) \), where \( S_i \) is the subset of elements assigned with the label \( i \).

The Sub-ML problem generalizes the **submodular multiway partition** (Sub-MP) problem, in the latter the given partial labeling function \( \ell \) pre-assigns each label to exactly one element. The Sub-MP problem was proposed by Zhao *et al.* [24], and it includes many well studied cut problems such as the (classic, edge-weighted) multiway cut [4], the node-weighted multiway cut [10] and the hypergraph multiway cut [18] as special cases. Analogously, the multiway uncut problem [14], the complement of the multiway cut, is a special case of the **supermodular multiway partition** (Sup-MP) problem. In the multiway uncut problem, the given \( k \) terminals can be considered as \( k \) elements each being pre-assigned with a different label from \( L = \{1, 2, \ldots, k\} \). A slight generalization of the multiway uncut problem, where the partial labeling function pre-assigns each label to at least one element, becomes the recently studied **maximum happy edges** (MHE) problem [23]. Clearly, MHE is a special case of the Sup-ML problem.

In the Sup-ML problem, if \( V \) denotes a set of items, \( L \) denotes a set of bidders, \( g \) is a monotone utility function such that \( g(X) \leq g(Y) \) for any two subsets \( X \subseteq Y \subseteq V \), and the partial function \( \ell \) is empty (meaning that no item has been pre-allocated to any bidder), then the problem becomes a special case of the optimal allocation (OA) problem [16] in combinatorial auctions, with its goal to maximize the sum of the utilities of all bidders. We note that in the general OA problem, \( g \) is not necessarily supermodular.

In a recently studied vertex-coloring problem by Zhang and Li [23], one is given an undirected graph \( G = (V, E) \) with a non-negative weight \( w(v) \) for each vertex \( v \in V \), a color set \( C = \{1, 2, \ldots, k\} \), and a partial vertex coloring function \( c : V \to C \), and the goal is to color all the uncolored vertices such that the total weight of happy vertices is maximized. A vertex is **happy** if it shares the same color with all its neighbors in the coloring scheme. The problem is referred to as the **maximum happy vertices** (MHV) [23], and its complement the **minimum unhappy vertices** (MUHV) problem can be defined analogously to minimize the total weight of unhappy vertices, where a vertex is **unhappy** if its color is different from at least one of its neighbors. We note that these two vertex-coloring problems are in fact labeling problems, and we use "color" and "label" interchangeably; they are different from the classic **graph coloring** problem [11], in which a feasible vertex coloring scheme must assign different colors to any adjacent vertices.

We next argue that MUHV is also a special case of the Sub-ML problem. Given a graph \( G = (V, E) \) with the vertex set \( V \) and the edge set \( E \), for any subset \( X \subseteq V \), define the **boundary** of \( X \), denoted as \( b(X) \), to be the subset of vertices of \( X \) each has at least one neighbor not in \( X \). When the vertices of \( X \) are all colored \( i \) but no vertex outside \( X \) is colored \( i \), then every vertex of \( b(X) \) is unhappy while all vertices of \( i(X) = X - b(X) \), called the **interior** of \( X \), are happy. Clearly, i) \( b(\emptyset) = \emptyset \); ii) \( b(X \cap Y) \subseteq b(X) \cup b(Y) \); iii) \( b(X \cup Y) \subseteq b(X) \cup b(Y) \); and iv) \( b(X \cap Y) \cap b(X \cup Y) \subseteq b(X) \cap b(Y) \), for any two subsets \( X, Y \subseteq V \).

Consider the Sub-ML problem, in which \( V \) takes the vertex set in a graph \( G = (V, E) \) and each vertex \( v \in V \) has a non-negative weight \( w(v) \). We extend the vertex weight function to subsets of vertices, that is, \( w(X) \triangleq \sum_{v \in X} w(v) \). We define the set function as
\[
f(X) = w(b(X)), \ \forall X \subseteq V.
\] (1)
One clearly sees that the function \( f \) is submodular, due to the above four properties associated
with the boundary \( b(\cdot) \). Then, the Sub-ML problem becomes the MUHV problem, to minimize the total weight of unhappy vertices in the graph \( G \).

Conversely, consider the Sup-ML problem, in which \( V \) takes the vertex set in a graph \( G = (V, E) \) and each vertex \( v \in V \) has a non-negative weight \( w(v) \). We define the set function as

\[
g(X) = w(i(X)), \quad \forall X \subseteq V.
\]

Then \( g \) is supermodular, due to \( f(X) + g(X) = w(X) \) for any subset \( X \subseteq V \). Thus, the Sup-ML problem becomes the MHV problem, to maximize the total weight of happy vertices in the graph \( G \).

### 1.1 Related works

For the Sub-MP problem, Zhao et al. [24] presented a \((k - 1)\)-approximation algorithm. Years later, Chekuri and Ene [3] proposed a convex relaxation for Sub-MP by using the Lovász extension, and they presented a 2-approximation based on this relaxation. This was further improved to a \((2 - \frac{1}{k})\)-approximation shortly after by Ene et al. [5]. On the inapproximability, Ene et al. [6] proved that a \((2 - \frac{1}{k} - \epsilon)\)-approximation for Sub-MP requires exponentially many value queries for any \( \epsilon > 0 \), or otherwise it implies \( NP = RP \). Additionally, Ene et al. [6] proved that the hypergraph multiway cut and the node-weighted multiway cut, both special cases of Sub-MP, are Unique Games-hard to achieve a \((2 - \frac{1}{k} - \epsilon)\)-approximation for any \( \epsilon > 0 \), where \( k \) is the number of terminals.

As a special case of Sub-MP, the multiway cut problem is \( NP \)-hard for \( k \geq 3 \) even if all edges have unit weight [4], with many approximation algorithms designed and analyzed [3][2][9][12][1][19]. Most of these approximation results are based on the linear program (LP) relaxation presented by Călinescu et al. [2], and the current best approximation ratio is 1.2965 [19]. However, for the multiway uncut problem, it seems only studied by Langberg et al. [14], who presented a 0.8535-approximation based on an LP relaxation. For the MHE problem, Zhang and Li [23] presented a \( \frac{1}{2} \)-approximation based on a simple division strategy; by using an extended LP relaxation of the one presented for the multiway uncut, Zhang et al. [22] later proved that MHE can be approximated within \( \frac{1}{2} + \frac{\sqrt{2}}{4} \log k \geq 0.8535 \), where \( h(k) \geq 1 \) is a function in \( k \).

For the OA problem, the case where the utility function \( g \) is submodular has been well studied [13][8][9][7][13], while the case where the utility function \( g \) is supermodular, a special case of Sup-ML, hasn’t drawn much attention in the literature, except that Shioura and Suzuki [20] studied the special case where \( g \) is quadratic, i.e., \( g(X) = \sum_{u,v \in X} a_1(u,v) + \sum_{v \in X} a_2(v) \), for any subset \( X \subseteq V \). Shioura and Suzuki [20] showed that when the number of bidders \( k = 2 \), the OA problem with a quadratic supermodular utility function can be solved in \( O(n^3/\log n) \) time, while it is \( NP \)-hard when \( k \geq 3 \); they also presented a \( \frac{1}{2} \)-approximation algorithm for \( k \geq 3 \) based on a randomized LP rounding technique, and for \( k = 3 \) specifically, they obtained a \( \frac{1}{2} \)-approximation.

For the MHV problem, Zhang and Li [23] proved that it is polynomial time solvable for \( k = 2 \) and it becomes \( NP \)-hard for \( k \geq 3 \); for \( k \geq 3 \), they presented two approximation algorithms: a greedy algorithm with an approximation ratio of \( \frac{1}{2} \), and an \( \Omega(\frac{1}{\Delta}) \)-approximation based on a subset-growth technique, where \( \Delta \) is the maximum vertex degree of the input graph. Recently, Zhang et al. [22] presented an improved algorithm with an approximation ratio of \( \frac{1}{2 \Delta + 1} \) based on a combination of randomized LP rounding techniques. In summary, the current best approximation ratio for the MHV problem is \( \max\{\frac{1}{2}, \frac{1}{\Delta + 1}\} \). For the MUHV problem, to the best of our knowledge, it hasn’t been particularly studied in the literature.
1.2 Our contributions

We study the Sub-ML problem in Section 2. Given a set $V = \{v_1, v_2, \ldots, v_n\}$, $y_j \triangleq y(v_j)$ is a real variable that maps the element $v_j$ to the closed unit interval $[0, 1]$. For a set function $f : 2^V \rightarrow \mathbb{R}_+$, its Lovász extension is a function $\hat{f} : [0, 1]^V \rightarrow \mathbb{R}_+$ such that

$$\hat{f}(y) = \sum_{j=1}^{n-1} (y_{\pi_j} - y_{\pi_{j+1}}) f(\{v_{\pi_1}, v_{\pi_2}, \ldots, v_{\pi_j}\}),$$

where $y = (y_1, y_2, \ldots, y_n) \in [0, 1]^V$ and $\pi$ is a permutation on $\{1, 2, \ldots, n\}$ such that $1 = y_{\pi_1} \geq y_{\pi_2} \geq \ldots \geq y_{\pi_n} = 0$ [17, 21].

Motivated by the use of the Lovász relaxation for Sub-MP presented in [3], we present a convex relaxation for Sub-ML based on the same Lovász extension in Section 2; we then show in Section 2.1 that by adopting the randomized rounding technique in [6], Sub-ML can be approximated within a factor of $(2 - \frac{2}{k})$. In Section 3, we analogously study the Sup-ML problem and present a concave relaxation based on the Lovász extension. Applying the same randomized rounding technique, we show in Section 3.1 that Sup-ML can be approximated within a factor of $\frac{2}{k}$.

In Section 2.3, we propose an LP relaxation, called LP-MUHV, for the MUHV problem; by the submodular set function $f$ defined in Eq. (1), we show that the convex relaxation for Sub-ML based on the Lovász extension is equivalent to LP-MUHV. In Section 2.4, we prove a lower bound of $2 - \frac{2}{k}$ on the integrality gap of LP-MUHV; thus we conclude that the $(2 - \frac{2}{k})$-approximation for Sub-ML is the best possible for MUHV, based on LP-MUHV. In Section 2.5, furthermore, we prove that it is Unique Games-hard to achieve a $(2 - \frac{2}{k} - \epsilon)$-approximation for MUHV, for any $\epsilon > 0$.

In Section 3.3, by the supermodular set function $g$ defined in Eq. (2), we show that the LP relaxation for MHV presented in [22], called LP-MHV, is equivalent to the concave relaxation for Sup-ML based on the Lovász extension; in Section 3.4, we present an upper bound of $\frac{2}{k}$ on the integrality gap of LP-MHV; we conclude that the $\frac{2}{k}$-approximation for Sup-ML is the best possible for MHV, based on LP-MHV.

2 The submodular multi-labeling (Sub-ML) problem

Recall that the Sub-ML problem is a restricted Sub-ML, in that the given labeling function $\ell$ assigns each label of $L = \{1, 2, \ldots, k\}$ to exactly one element of $V$. We extend the convex relaxation for Sub-MP based on the Lovász extension by Chekuri and Ene [3], in the following, to a relaxation for Sub-ML.

Let $V = \{v_1, v_2, \ldots, v_n\}$, $f$ be the non-negative submodular set function, and $T_i \subset V$ be the subset of elements pre-labeled $i$, for each $i \in L$. (We note that $|T_i| \geq 1$.) Define a binary variable $y^i_j \triangleq y^i(v_j)$ for the couple of the element $v_j$ and the label $i$, such that $y^i_j = 1$ if and only if the element $v_j$ is labeled $i$; $y^i_j$ is then relaxed to be a real variable in the closed unit interval $[0, 1]$. Let $y_i = (y^i_1, y^i_2, \ldots, y^i_n)$, which is a vector of $[0, 1]^V$. Let $\hat{f} : [0, 1]^V \rightarrow \mathbb{R}_+$ be the Lovász extension of $f$, defined in Eq. (3). A relaxation based on the Lovász extension for
Sub-ML can be written as follows.

\[
\begin{align*}
\text{minimize} & \quad \sum_{i=1}^{k} \hat{f}(y_i) \\
\text{subject to} & \quad \sum_{i=1}^{k} y_j^i = 1, \quad \forall v_j \in V \\
& \quad y_j^i = 1, \quad \forall v_j \in T_i, \quad i \in L \\
& \quad y_j^i \geq 0, \quad \forall v_j \in V, \quad i \in L
\end{align*}
\]  

(Constraint (4) embeds the vertex \(v_j\) into the \(k\)-dimensional simplex.) It has been proved by Lovász [17] that a set function is submodular (supermodular, respectively) if and only if its Lovász extension is convex (concave, respectively). Therefore, (CP-Sub-ML) is a convex program (CP) and thus it can be solved in polynomial time.

2.1 A \((2 - \frac{2}{k})\)-approximation algorithm

We next present an approximation algorithm \(\text{RR}\) for Sub-ML, that first solves the convex program (CP-Sub-ML), followed by a randomized rounding. Such a randomized rounding approach has been used in [6] for approximating Sub-MP. In more details, we use a uniformly random variable \(\theta\) in the closed unit interval \([0, 1]\), and define the following \(k + 3\) sets:

\[
\begin{align*}
S_i(\theta) &= \{v_j \mid y_j^i > \theta\}, \text{ for each } i \in L, \\
S(\theta) &= \bigcup_{i=1}^{k} S_i(\theta), \\
R(\theta) &= V - S(\theta), \\
Q(\theta) &= R(1 - \theta).
\end{align*}
\]

From the definition of Lovász extension in Eq. (3) and the definition of \(S_i(\theta)\) above, one clearly sees that

\[
\hat{f}(y_i) = \sum_{j=1}^{n-1} (y_{\pi_j}^i - y_{\pi_{j+1}}^i) f(S_i(y_{\pi_{j+1}}^i)) = \int_{0}^{1} f(S_i(\theta)) d\theta.
\]

It follows that the optimal solution to (CP-Sub-ML) has a value

\[
\text{OPT}(\text{CP-Sub-ML}) = \sum_{i=1}^{k} \hat{f}(y_i) = \sum_{i=1}^{k} \int_{0}^{1} f(S_i(\theta)) d\theta.
\]

**Algorithm RR**

1. Solve (CP-Sub-ML) to obtain an optimal fractional solution \(\{y_j^i \mid v_j \in V, i \in L\}\).
2. Pick a parameter \(\theta \in (\frac{1}{2}, 1]\) uniformly at random.
3. Assign all elements of \(S_i(\theta)\) the label \(i\), for each \(i \in L\).
4. Pick a label \(i'\) from \(L\) uniformly at random, assign all elements of \(R(\theta)\) the label \(i'\).

From the description of Algorithm RR, we see that the parameter \(\theta\) is chosen from \((\frac{1}{2}, 1]\); due to Constraint (4) of (CP-Sub-ML), all the \(k\) sets \(S_i(\theta), i = 1, 2, \ldots, k\), are pairwise disjoint. Clearly, the probability that \(S_i(\theta)\) contains all the elements labeled \(i\) is \(1 - \frac{1}{k}\), since
the elements of $R(\theta)$ are labeled $i$ with a probably $\frac{1}{k}$. Therefore, the value of the solution returned by Algorithm RR for the Sub-ML problem is

$$\text{SOL}(\text{Sub-ML}, \theta) = \left(1 - \frac{1}{k}\right) \sum_{i=1}^{k} f(S_i(\theta)) + \frac{1}{k} \sum_{i=1}^{k} f(S_i(\theta) \cup R(\theta)).$$

For any $\theta \in (\frac{1}{2}, 1]$, $S_i(\theta) \subseteq S_i(1 - \theta)$; on the other hand, if an element $v_j \in S_i(1 - \theta)$, then $v_j \notin S_i(\theta)$ for any $i' \neq i$, which implies that either $v_j \in S_i(\theta)$ or $v_j \in R(\theta)$. Thus, we have

$$S_i(\theta) \cup R(\theta) = S_i(1 - \theta) \cup R(\theta) = S_i(1 - \theta) \cup Q(1 - \theta). \quad (9)$$

Using Eq. (9), the expected value of the solution achieved by Algorithm RR for Sub-ML is

$$E[\text{SOL}(\text{Sub-ML}, \theta)]$$

$$= 2 \int_{\frac{1}{2}}^{1} \text{SOL}(\text{Sub-ML}, \theta) d\theta$$

$$= \left(2 - \frac{2}{k}\right) \sum_{i=1}^{k} \int_{\frac{1}{2}}^{1} f(S_i(\theta)) d\theta + \frac{2}{k} \sum_{i=1}^{k} \int_{\frac{1}{2}}^{1} f(S_i(\theta) \cup R(\theta)) d\theta$$

$$= \left(2 - \frac{2}{k}\right) \sum_{i=1}^{k} \int_{\frac{1}{2}}^{1} f(S_i(\theta)) d\theta + \frac{2}{k} \sum_{i=1}^{k} \int_{\frac{1}{2}}^{1} f(S_i(1 - \theta) \cup Q(1 - \theta)) d\theta$$

$$= \left(2 - \frac{2}{k}\right) \sum_{i=1}^{k} \int_{\frac{1}{2}}^{1} f(S_i(\theta)) d\theta + \frac{2}{k} \sum_{i=1}^{k} \int_{0}^{\frac{1}{2}} f(S_i(\theta) \cup Q(\theta)) d\theta. \quad (10)$$

The following two inequalities have been proved in [3] (as Lemmas 2.5 and 2.6, respectively). Their proofs use the submodularity of the function $f$, the convexity of the Lovász extension $\hat{f}$, and Theorem 1.5 in [3].

$$\sum_{i=1}^{k} \int_{0}^{\frac{1}{2}} f(S_i(\theta)) d\theta \geq \int_{0}^{\frac{1}{2}} f(Q(\theta)) d\theta. \quad (11)$$

$$\sum_{i=1}^{k} \int_{0}^{\frac{1}{2}} f(S_i(\theta)) d\theta \geq \sum_{i=1}^{k} \int_{0}^{\frac{1}{2}} f(S_i(\theta) \cup Q(\theta)) d\theta - (k - 2) \int_{0}^{\frac{1}{2}} f(Q(\theta)) d\theta. \quad (12)$$

**Theorem 1.** Algorithm RR is a $(2 - \frac{2}{k})$-approximation for Sub-ML.

**Proof.** Combining Eqs. (11) and (12), we obtain

$$\sum_{i=1}^{k} \int_{0}^{\frac{1}{2}} f(S_i(\theta) \cup Q(\theta)) d\theta \leq \sum_{i=1}^{k} \int_{0}^{\frac{1}{2}} f(S_i(\theta)) d\theta + (k - 2) \int_{0}^{\frac{1}{2}} f(Q(\theta)) d\theta$$

$$\leq \sum_{i=1}^{k} \int_{0}^{\frac{1}{2}} f(S_i(\theta)) d\theta + (k - 2) \sum_{i=1}^{k} \int_{0}^{\frac{1}{2}} f(S_i(\theta)) d\theta$$

$$= (k - 1) \sum_{i=1}^{k} \int_{0}^{\frac{1}{2}} f(S_i(\theta)) d\theta.$$
Then, using Eqs. (10) [8], the expected value of the solution achieved by Algorithm RR for Sub-ML becomes

\[
E[\text{SOL(Sub-ML, } \theta)] \\
\leq \left(2 - \frac{2}{k}\right) \sum_{i=1}^{k} \int_{\frac{1}{2}}^{1} f(S_i(\theta))\,d\theta + \frac{2}{k} (k - 1) \int_{0}^{\frac{1}{2}} f(S_i(\theta))\,d\theta \\
= \left(2 - \frac{2}{k}\right) \sum_{i=1}^{k} \int_{0}^{1} f(S_i(\theta))\,d\theta \\
= \left(2 - \frac{2}{k}\right) \cdot \text{OPT(CP-Sub-ML)}.
\]

Therefore, the expected performance ratio of Algorithm RR is \(2 - \frac{2}{k}\). The running time of Algorithm RR is clearly polynomial. This finishes the proof of the theorem. ▲

2.2 The minimum unhappy vertices (MUHV) problem

Recall that in the MUHV problem, one is given an undirected graph \(G = (V, E)\) with a non-negative vertex weight function \(w(\cdot)\), a color set \(C = \{1, 2, \ldots, k\}\), and a partial vertex coloring function \(c : V \rightarrow C\), and the goal is to color all the uncolored vertices such that the total weight of unhappy vertices is minimized. A vertex is unhappy if its color is different from at least one of its neighbors. For any subset \(X \subseteq V\), its boundary \(b(X)\) is defined as the subset of vertices of \(X\) each has at least one neighbor not in \(X\). Define a non-negative set function \(f : 2^V \rightarrow \mathbb{R}_+\) as in Eq. (10), that is, \(f(X) = w(b(X))\). The goal of MUHV becomes to minimize \(\sum_{i=1}^{k} f(S_i)\), where \(S_i\) is the subset of vertices colored \(i\) (that is, \(b(S_i)\) contains all unhappy vertices that are colored \(i\)). Since \(f\) is submodular, MUHV is a special case of Sub-ML.

2.3 An LP relaxation for MUHV

Let \(V = \{v_1, v_2, \ldots, v_n\}\), a binary variable \(y^i_j \triangleq y^i(v_j)\) denote whether or not the vertex \(v_j\) is colored \(i\), and \(y_i = (y^1_i, y^2_i, \ldots, y^n_i)\). We use \(w_j = w(v_j)\) to denote the weight of the vertex \(v_j\). We formulate a novel LP relaxation for the MUHV problem as follows.

\[
\text{minimize } \sum_{j=1}^{n} w_j x_j \tag{LP-MUHV}
\]

subject to

\[
\sum_{i=1}^{k} y^i_j = 1, \quad \forall v_j \in V 
\]

\[
y^i_j = 1, \quad \forall v_j \in V, \forall i \in C \text{ s.t. } c(v_j) = i \tag{14}
\]

\[
x^i_j \geq y^i_j - y^i_h, \quad \forall v_j \in V, \forall v_h \in N(v_j), \forall i \in C \tag{15}
\]

\[
x_j = \sum_{i=1}^{k} x^i_j, \quad \forall v_j \in V \tag{16}
\]

\[
y^i_j, x^i_j, x_j \geq 0, \quad \forall v_j \in V, \forall i \in C \tag{17}
\]

where \(x_j\) indicates whether the vertex \(v_j\) is unhappy, and \(N(v_j)\) is the set of neighbors of \(v_j\).

For each color \(i\), since there is at least one vertex pre-colored \(i\) and at least one vertex pre-colored another color, we let \(\pi\) be the permutation on \(\{1, 2, \ldots, n\}\) for \(y_i\) such that
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We construct an instance \( I = (G, EW, C, w) \) based on the definition of the Lovász extension in Eq. (3), the objective function of the relaxation \( \text{(CP-Sub-ML)} \) becomes

\[
\sum_{i=1}^{k} \hat{f}(y_i) = \sum_{i=1}^{k} \left( \sum_{j=1}^{n-1} (y_{\pi_j}^i - y_{\pi_{j+1}}^i) f(\{v_{\pi_1}, v_{\pi_2}, \ldots, v_{\pi_i}\}) \right) = \sum_{i=1}^{k} \sum_{j=1}^{n-1} (y_{\pi_j}^i - y_{\pi_{j+1}}^i) \sum_{v_h \in B(\{v_{\pi_1}, v_{\pi_2}, \ldots, v_{\pi_i}\})} w_h.
\]

(18)

For each vertex \( v_p \in V \), let \( v_q \) denote its neighbor that appears the last in the permutation \( (v_{\pi_1}, v_{\pi_2}, \ldots, v_{\pi_n}) \). Assume \( p = \pi_{j_1} \) and \( q = \pi_{j_2} \). Clearly, \( v_p \in B(\{v_{\pi_1}, v_{\pi_2}, \ldots, v_{\pi_j}\}) \) if and only if i) \( p \in \{\pi_1, \pi_2, \ldots, \pi_j\} \) and ii) \( q \notin \{\pi_1, \pi_2, \ldots, \pi_j\} \), that is, we must have \( j_1 \leq j < j_2 \).

It follows that for the vertex \( v_p \in V \), the coefficient of \( w_p \) in Eq. (18) is

\[
\sum_{i=1}^{k} \sum_{j=j_1}^{j_2-1} (y_{\pi_j}^i - y_{\pi_{j+1}}^i) = \sum_{i=1}^{k} (y_{\pi_p}^i - y_{\pi_q}^i) = \sum_{i=1}^{k} x_p^i = x_p,
\]

where the last two equalities hold due to Constraints (15, 16) of \( \text{(LP-MUHV)} \). This indicates that by setting the submodular set function \( f \) as defined in Eq. (1), \( \text{(CP-Sub-ML)} \) is the same as \( \text{(LP-MUHV)} \). Therefore, we have the following theorem.

\[\blacktriangleright \textbf{Theorem 2. Algorithm RR is a } (2 - \frac{2}{k})\text{-approximation for MUHV.}\]

2.4 A lower bound on the integrality gap for \( \text{LP-MUHV} \)

We construct an instance \( I = (G = (V, E), w(\cdot), C = \{1, 2, \ldots, k\}, c) \) of the MUHV problem to obtain a lower bound on the integrality gap of \( \text{(LP-MUHV)} \), the LP relaxation for the MUHV problem.

- Let \( T = \{t_1, t_2, \ldots, t_k\} \) be a set of \( k \) pre-colored vertices, called terminals; all terminals have the same weight \( w_i \geq 0 \), and the terminal \( t_i \) is pre-colored \( i \), i.e. \( c(t_i) = i \).
- Associated with each pair of distinct terminals \( t_i \) and \( t_j, i < j \), there is a vertex \( b_{(ij)} \).
  - Let \( V_0 = \{b_{(ij)} \mid i < j\} \), then \( |V_0| = \binom{k}{2} \); all vertices of \( V_0 \) have the same weight \( w_0 \geq 0 \), and none of them is pre-colored.
- The vertex set \( V = T \cup V_0 \); the edge set \( E = \{\{t_i, b_{(ij)}\}, \{t_j, b_{(ij)}\} \mid i < j\} \). Clearly, \( |V| = k + \binom{k}{2} \) and \( |E| = 2 \binom{k}{2} \).

Let \( c^* \) denote a coloring function that completes the given partial coloring function \( c \), that is, \( c^* \) assigns a color for each vertex and it assigns the color \( i \) to the terminal \( t_i \), for each \( i \in C \). Then,

- all vertices of \( V_0 \) must be unhappy, since the vertex \( b_{(ij)} \) is adjacent to two terminals \( t_i \) and \( t_j \) colored with distinct colors;
- the terminal \( t_i \) is adjacent to \( k - 1 \) vertices \( \{b_{(ij)} \mid j \neq i\} \), while the vertex \( b_{(ij)} \) is adjacent to the terminals \( t_i \) and \( t_j \); it follows that if \( t_i \) is happy, then all vertices of \( \{b_{(ij)} \mid j \neq i\} \) are colored \( i \), subsequently none of the other terminals can be happy; in other words, at most one of the \( k \) terminals can be happy, regardless of what the coloring function \( c^* \) is.
Let OPT(MUHV) denote the optimum of the above constructed instance \( I \). We have

\[
\text{OPT}(\text{MUHV}) \geq (k - 1)w_i + \left(\frac{k}{2}\right)w_b. \tag{19}
\]

Let us consider the following fractional feasible solution to the instance \( I \) in the LP relaxation \( \text{LP-MUHV} \):

- for each terminal \( t_i \in T \), \( y^i(t_i) = 1 \) and \( y^j(t_i) = 0 \) for all \( j \neq i \);
- for each vertex \( b_{(ij)} \in V_b \), \( y^i(b_{(ij)}) = y^j(b_{(ij)}) = \frac{1}{2} \) and \( y^\ell(b_{(ij)}) = 0 \) for all \( \ell \neq i, j \);
- for each terminal \( t_i \in T \), we set \( x^i(t_i) = y^i(t_i) - y^j(b_{(ij)}) = \frac{1}{2} \), \( x^j(t_i) = 0 \) for all \( j \neq i \), and \( x(t_i) = \sum_{\ell=1}^{k} x^\ell(t_i) = \frac{1}{2} \);
- for each vertex \( b_{(ij)} \in V_b \), we set \( x^i(b_{(ij)}) = y^i(b_{(ij)}) - y^j(t_j) = \frac{1}{2} \), \( x^j(b_{(ij)}) = y^i(b_{(ij)}) - y^j(t_j) = \frac{1}{2} \), \( x^\ell(b_{(ij)}) = 0 \) for all \( \ell \neq i, j \), and \( x(b_{(ij)}) = \sum_{\ell=1}^{k} x^\ell(b_{(ij)}) = 1 \).

Let OPT(\text{LP-MUHV}) denote the optimum of the instance \( I \) in the LP relaxation \( \text{LP-MUHV} \). It is no greater than the value of the above fractional feasible solution, that is,

\[
\text{OPT(\text{LP-MUHV})} \leq \frac{1}{2}kw_i + \left(\frac{k}{2}\right)w_b. \tag{20}
\]

Combining Eqs. \( \text{19} \) and \( \text{20} \) and setting \( w_b = 0 \), it gives a lower bound on the integrality gap of \( \text{LP-MUHV} \):

\[
\frac{\text{OPT(MUHV)}}{\text{OPT(\text{LP-MUHV})}} \geq \frac{k - 1}{\frac{1}{2}k} = 2 - \frac{2}{k}.
\]

We thus have proved the following theorem.

\textbf{Theorem 3.} The integrality gap of \( \text{LP-MUHV} \) has a lower bound of \( 2 - \frac{2}{k} \).

Theorems 2 and 3 together imply that the \( (2 - \frac{2}{k}) \)-approximation algorithm RR for the MUHV problem is the best possible based on the LP relaxation \( \text{LP-MUHV} \).

### 2.5 Inapproximability of MUHV

In the hypergraph multiway cut (Hyp-MC) problem, we are given a hypergraph \( H = (V_H, E_H) \) with a non-negative weight \( w(e) \) for each hyperedge \( e \in E_H \) and a set of \( k \) terminals \( T = \{t_1, t_2, \ldots, t_k\} \subseteq V \), the goal is to remove a minimum-weight set of hyperedges so that every two terminals are disconnected. Ene et al. \cite{ene2011inapproximability} proved that a \( (2 - \frac{2}{k}) \)-approximation for Hyp-MC is NP-hard, for any \( \epsilon > 0 \), assuming the Unique Games Conjecture. Our main result in this section is that it is also Unique Games-hard to achieve a \( (2 - \frac{2}{k} - \epsilon) \)-approximation for the MUHV problem.

\textbf{Theorem 4.} No \( (2 - \frac{2}{k} - \epsilon) \)-approximation algorithm for the MUHV problem exists, for any \( \epsilon > 0 \), assuming the Unique Games Conjecture.

\textbf{Proof.} We prove the theorem by constructing an approximation preserving reduction from the Hyp-MC problem to the MUHV problem.

Given an instance \( (H = (V_H, E_H), w(.), T = \{t_1, t_2, \ldots, t_k\}) \) of the Hyp-MC problem, we construct an instance \( (G = (V, E), w'(.), C = \{1, 2, \ldots, k\}, c) \) of MUHV as follows:

- for each hyperedge \( e \in E_H \), we create a vertex \( v_e \); let the vertex set be \( V = V_H \cup V_E \), where \( V_E = \{v_e \mid e \in E_H\} \); call \( T = \{t_1, t_2, \ldots, t_k\} \subseteq V \) the terminal set;
for each vertex \( v \in V_H \), its weight is \( w'(v) = 0 \); for each vertex \( v_e \in V_E \), its weight is \( w'(v_e) = w(e) \);

- for each vertex \( v_e \in V_E \), it is adjacent to every vertex of \( e \); let the edge set be \( E = \{ \{v_e, v\} \mid e \in E_H, v \in e \} \);

- let the color set be \( C = \{1, 2, \ldots, k\} \) and let the partial coloring function \( c : V \mapsto C \)

We note that the graph \( G \) is actually bipartite, and the two parts of vertices are \( V_H \) and \( V_E \).

Consider a simple path \( P \) connecting two terminals \( t_i \) and \( t_j \) in the hypergraph \( H = (V_H, E_H) \). Every two consecutive vertices on \( P \) must belong to a common hyperedge; therefore, the path \( P \) one-to-one corresponds to a simple path in the constructed graph \( G = (V, E) \) connecting the two vertices \( t_i \) and \( t_j \), which is also denoted as \( P \) without any ambiguity. For any coloring function \( c^* \) that completes the given partial coloring function \( c \), we have \( c^*(t_i) = i \) for each \( i = \{1, 2, \ldots, k\} \). It follows that any simple path \( P \) connecting \( t_i \) and \( t_j \) must contain at least one vertex \( v_e \in V_E \) such that its preceding vertex and its succeeding vertex, both in \( V_H \), are colored differently. The vertex \( v_e \) is thus unhappy under the coloring scheme \( c^* \). In the hypergraph \( H \), removing the corresponding hyperedge \( e \) breaks the path \( P \), thus disconnecting \( t_i \) and \( t_j \) via the path \( P \). Therefore, removing all the hyperedges whose corresponding vertices in the graph \( G \) are unhappy disconnects all pairs of terminals. In other words, any solution to the constructed instance of the MUHV problem can be transferred into a feasible solution to the given instance of the Hyp-MC problem; the transfer is done in linear time and the two solutions have exactly the same value.

Conversely, given a subset \( E_H^* \) of hyperedges in the hypergraph \( H = (V_H, E_H) \) whose removal disconnects all pairs of terminals, let \( V_H^* \) and \( E_H^* \) denote the subsets of vertices and hyperedges in the connected component of the remainder hypergraph \( (V_H, E_H - E_H^*) \) that contains the terminal \( t_i \), for each \( i = 1, 2, \ldots, k \). Denote the vertex subsets in the constructed graph \( G = (V, E) \) corresponding to \( V_H^* \) and \( E_H^* \) as \( V_H^i \) and \( V_E^i \), respectively, for \( i = 1, 2, \ldots, k \). We complete the partial coloring function \( c \) by coloring all vertices of \( V_H^i \cup V_E^i \) with the color \( i \), for \( i = 1, 2, \ldots, k \), and coloring all the other remaining vertices of \( V \) with the color 1. Clearly, all vertices of \( \{v_e \mid e \in E_H - E_H^* \} \) are happy; due to every vertex of \( V_H \) has weight 0 (such that we may ignore its happiness), we conclude that the total weight of unhappy vertices in this coloring scheme is no more than \( w(E_H^*) = \sum_{e \in E_H^*} w(e) \).

In summary, the Hyp-MC problem is polynomial-time reducible to the MUHV problem, and our reduction preserves the value of any feasible solution and consequently preserves the approximation ratio.

### 3 The supermodular multi-labeling (Sup-ML) problem

Similar to [CP-Sub-ML], let \( V = \{v_1, v_2, \ldots, v_n\} \), \( g \) be the non-negative supermodular set function, and \( T_i \subset V \) be the subset of elements pre-labeled \( i \), for \( i \in L = \{1, 2, \ldots, k\} \). (We note that \(|T_i| \geq 1\). Define a binary variable \( y^i_j \triangleq y^i(v_j) \) for the couple of the element \( v_j \) and the label \( i \), such that \( y^i_j = 1 \) if and only if the element \( v_j \) is labeled \( i \); \( y^i_j \) is then relaxed to be a real variable in the closed unit interval \([0, 1]\). Let \( \hat{g} : [0, 1]^V \rightarrow \mathbb{R}_+ \) be the Lovász extension of \( g \), defined in Eq. (3). A relaxation
based on the Lovász extension for Sup-ML can be written as follows.

\[
\text{maximize } \sum_{i=1}^{k} \hat{g}(y_i) \quad \text{(CP-Sup-ML)}
\]

subject to \( \sum_{i=1}^{k} y_j^i = 1, \quad \forall v_j \in V \) \hspace{1cm} \text{(21)}

\( y_j^i = 1, \quad \forall v_j \in T_i, \ i \in L \)

\( y_j^i \geq 0, \quad \forall v_j \in V, \ i \in L \) \hspace{1cm} \text{(22)}

\( y_j^i \geq 0, \quad \forall v_j \in V, \ i \in L \)

\( \text{(CP-Sup-ML)} \) is a concave program and thus it can be solved in polynomial time.

3.1 A \( \frac{2}{k} \)-approximation algorithm

The only difference between \( \text{(CP-Sup-ML)} \) and \( \text{(CP-Sub-ML)} \) is the supermodular set function \( g \) versus the submodular set function \( f \). Thus, to approximate Sup-ML, we can use the same randomized rounding Algorithm \( RR \) designed for Sub-ML, except that in Step 1 we solve \( \text{(CP-Sup-ML)} \) to obtain an optimal fractional solution.

For the uniformly random variable \( \theta \) in the closed unit interval \([0, 1]\), we define the same \( k+3 \) sets \( S_i(\theta), S(\theta), R(\theta) \) and \( Q(\theta) \) as in Eq. (7) in Section 2.1. Then, analogously we have

\[
\hat{g}(y_i) = \int_{0}^{1} g(S_i(\theta)) d\theta.
\]

It follows that the value of the fractional optimal solution to \( \text{(CP-Sup-ML)} \) is

\[
\text{OPT}(\text{CP-Sup-ML}) = \sum_{i=1}^{k} \hat{g}(y_i) = \sum_{i=1}^{k} \int_{0}^{1} g(S_i(\theta)) d\theta.
\]

Also, similarly as Eq. (10), the expected value of the solution achieved by Algorithm \( RR \) for the Sup-ML is

\[
E[\text{SOL(Sup-ML, } \theta)] = \left( 2 - \frac{2}{k} \right) \sum_{i=1}^{k} \int_{0}^{1} g(S_i(\theta)) d\theta + \frac{2}{k} \sum_{i=1}^{k} \int_{0}^{\frac{1}{2}} g(S_i(\theta) \cup Q(\theta)) d\theta.
\]

Lemma 5. For any \( \delta \in [\frac{1}{2}, 1] \),

\[
\sum_{i=1}^{k-1} \int_{0}^{1} g \left( \bigcup_{j=1}^{i} S_j(\theta) \right) \cap S_{i+1}(\theta) d\theta \leq \int_{0}^{1} g(R(\theta)) d\theta.
\]

Proof. We drop the dependence \( (\theta) \) for simplicity. Due to the non-negativity of the set function \( g \), we prove the lemma in the following only for \( \delta = 1 \).

For any two vectors \( u, v \in [0, 1]^V \), we define two vector operations:

\[
u \vee v = (\min\{u_1, v_1\}, \min\{u_2, v_2\}, \ldots, \min\{u_n, v_n\}),
\]

\[
u \wedge v = (\max\{u_1, v_1\}, \max\{u_2, v_2\}, \ldots, \max\{u_n, v_n\}).
\]

Then clearly, \( (u \vee v) + (u \wedge v) = u + v \). From \( \hat{g}(y_i) = \int_{0}^{1} g(S_i) d\theta \), we have

\[
\hat{g} \left( \bigvee_{j=1}^{i} y_j \wedge y_{i+1} \right) = \int_{0}^{1} g \left( \bigcup_{j=1}^{i} S_j \right) \cap S_{i+1} d\theta.
\]
We observe the fact on the $k$ vectors $y_i$, $i = 1, 2, \ldots, k$, that

\[
\sum_{i=1}^{k-1} \left( \bigvee_{j=1}^{i} y_j \right) \wedge y_{i+1} + \bigvee_{j=1}^{k} y_j = \sum_{i=1}^{k-2} \left( \bigvee_{j=1}^{i} y_j \right) \wedge y_{i+1} + \bigvee_{j=1}^{k-1} y_j + \ldots = \sum_{i=1}^{k} y_i = 1.
\]

Since $\hat{g}$ is concave, it follows from Eq. (20) that

\[
\int_{0}^{1} g(R) d\theta = \hat{g} \left( 1 - \bigvee_{j=1}^{k} y_j \right)
\]

\[
= \hat{g} \left( \sum_{i=1}^{k-1} \left( \bigvee_{j=1}^{i} y_j \right) \wedge y_{i+1} \right)
\]

\[
\geq \sum_{i=1}^{k-1} \hat{g} \left( \bigvee_{j=1}^{i} y_j \wedge y_{i+1} \right)
\]

\[
= \sum_{i=1}^{k-1} \int_{0}^{1} g \left( \bigcup_{j=1}^{i} S_j \right) \cap S_{i+1} d\theta.
\]

This proves the lemma. ▶

Analogous to Eq. (12) for Sub-ML, which has been proved in [6], the following lemma holds for Sup-ML.

\begin{lemma}
\end{lemma}

\[
\sum_{i=1}^{k} \int_{0}^{\frac{1}{2}} g(S_i(\theta)) d\theta \leq \sum_{i=1}^{k} \int_{0}^{\frac{1}{2}} g(S_i(\theta) \cup Q(\theta)) d\theta - (k - 2) \int_{0}^{\frac{1}{2}} g(Q(\theta)) d\theta.
\]

\textbf{Proof.} We drop the dependence $(\theta)$ for simplicity.

First, by supermodularity, we have

\[
\sum_{i=1}^{k} (g(S_i) + g(Q)) \leq \sum_{i=1}^{k} (g(S_i \cup Q) + g(S_i \cap Q)) = \sum_{i=1}^{k} g(S_i \cup Q) + \sum_{i=1}^{k} g(S_i \cap Q),
\]

and

\[
\sum_{i=1}^{k} g(S_i \cap Q) \leq \sum_{i=1}^{k-1} g \left( \bigcup_{j=1}^{i} S_j \right) \cap S_{i+1} \cap Q \right) + g \left( \bigcup_{i=1}^{k} (S_i \cap Q) \right).
\]

From the definitions of the sets $S_i(\theta)$, $S(\theta)$, $R(\theta)$ and $Q(\theta)$ in Eq. (17), we know that when a vertex $v_j \in S_i(\theta)$, then $y_j' > \theta$ and thus $y_j' < 1 - \theta$ for any $i' \neq i$. Therefore, if $v_j \in S_i(\theta) \cap S_i'(\theta)$ with $i \neq i'$, then $v_j \in R(1 - \theta) = Q(\theta)$. This shows that $(\bigcup_{j=1}^{i} S_j) \cap S_{i+1} \subseteq Q$, and consequently Eq. (28) becomes

\[
\sum_{i=1}^{k} g(S_i \cap Q) \leq \sum_{i=1}^{k-1} g \left( \bigcup_{j=1}^{i} S_j \right) \cap S_{i+1} \right) + g \left( S_i \cap Q \right).
\]
Integrating Eq. (29) from 0 to $\frac{1}{2}$, using Lemma 5 at $\delta = \frac{1}{2}$ and the supermodularity, we have

$$\sum_{i=1}^{k} \int_{0}^{\frac{1}{2}} g(S_i \cap Q) d\theta \leq \sum_{i=1}^{k-1} \int_{0}^{\frac{1}{2}} g \left( \bigcup_{j=1}^{i} S_j \cap S_{i+1} \right) d\theta + \int_{0}^{\frac{1}{2}} g(S \cap Q) d\theta$$

$$\leq \int_{0}^{1} g(R) d\theta + \int_{0}^{\frac{1}{2}} g(S \cap Q) d\theta$$

$$= \int_{0}^{\frac{1}{2}} g(Q) d\theta + \int_{0}^{\frac{1}{2}} g(R) d\theta + \int_{0}^{\frac{1}{2}} g(S \cap Q) d\theta$$

$$= \int_{0}^{\frac{1}{2}} g(Q) d\theta + \int_{0}^{\frac{1}{2}} g(R) d\theta + \int_{0}^{\frac{1}{2}} g(S \cap Q) d\theta$$

$$\leq \int_{0}^{\frac{1}{2}} g(Q) d\theta + \int_{0}^{\frac{1}{2}} \left( g(R \cup (S \cap Q)) + g(R \cap S \cap Q) \right) d\theta$$

$$= \int_{0}^{\frac{1}{2}} g(Q) d\theta + \int_{0}^{\frac{1}{2}} g(R \cup (S \cap Q)) d\theta$$

$$= \int_{0}^{\frac{1}{2}} g(Q) d\theta + \int_{0}^{\frac{1}{2}} g(R \cup Q) d\theta$$

$$= 2 \int_{0}^{\frac{1}{2}} g(Q) d\theta.$$

Therefore, integrating Eq. (27) from 0 to $\frac{1}{2}$ gives

$$\sum_{i=1}^{k} \int_{0}^{\frac{1}{2}} g(S_i) d\theta \leq \sum_{i=1}^{k} \int_{0}^{\frac{1}{2}} \left( g(S_i \cup Q) + g(S_i \cap Q) - g(Q) \right) d\theta$$

$$\leq \sum_{i=1}^{k} \int_{0}^{\frac{1}{2}} g(S_i \cup Q) d\theta - (k - 2) \int_{0}^{\frac{1}{2}} g(Q) d\theta.$$

This proves the lemma.

Theorem 7. Algorithm RR is a $\frac{2}{k}$-approximation for Sup-ML.
Submodular and supermodular multi-labeling

Proof. Combining Eqs. (25) and (24), and using Lemma 6, we have

\[
E[\text{SOL}(\text{Sup-ML}, \theta)] \\
= \left(2 - \frac{2}{k}\right) \sum_{i=1}^{k} \int_{0}^{\frac{1}{2}} g(S_{i}(\theta)) d\theta + 2 \left(\frac{1}{k}\right) \sum_{i=1}^{k} \int_{0}^{\frac{1}{2}} g(S_{i}(\theta) \cup Q(\theta)) d\theta
\]

\[
\geq \left(2 - \frac{2}{k}\right) \sum_{i=1}^{k} \int_{0}^{\frac{1}{2}} g(S_{i}(\theta)) d\theta + \frac{1}{k} \left(\sum_{i=1}^{k} \int_{0}^{\frac{1}{2}} g(S_{i}(\theta)) d\theta + k \int_{0}^{\frac{1}{2}} g(Q(\theta)) d\theta\right)
\]

\[
= \frac{2}{k} \sum_{i=1}^{k} \int_{0}^{\frac{1}{2}} g(S_{i}(\theta)) d\theta + \left(2 - \frac{4}{k}\right) \left(\sum_{i=1}^{k} \int_{0}^{\frac{1}{2}} g(S_{i}(\theta)) d\theta + \int_{0}^{\frac{1}{2}} g(Q(\theta)) d\theta\right)
\]

\[
= \frac{2}{k} \sum_{i=1}^{k} \int_{0}^{\frac{1}{2}} g(S_{i}(\theta)) d\theta + \frac{1}{k} \left(\sum_{i=1}^{k} \int_{0}^{\frac{1}{2}} g(S_{i}(\theta)) d\theta + \int_{0}^{\frac{1}{2}} g(R(\theta)) d\theta\right)
\]

\[
\geq \frac{2}{k} \sum_{i=1}^{k} \int_{0}^{\frac{1}{2}} g(S_{i}(\theta)) d\theta
\]

(30)

\[
= \frac{2}{k} \cdot \text{OPT(} \text{CP-Sup-ML)}
\]

where the last Inequality (30) holds due to the non-negativity of the set function \( g \). Therefore, Algorithm RR is a \( \frac{2}{k} \)-approximation for the Sup-ML problem. \hfill \( \blacksquare \)

3.2 The maximum happy vertices (MHV) problem

Recall that in the MHV problem, we are given a graph \( G = (V, E) \) with a non-negative weight \( w(v) \) for each vertex \( v \in V \), a color set \( C = \{1, 2, \ldots, k\} \), and a partial vertex coloring function \( c : V \mapsto C \), and our goal is to color all the uncolored vertices such that the total weight of happy vertices is maximized. A vertex is happy if it shares the same color with all its neighbors. By taking the set \( V \) as the vertex set of the graph \( G \) and defining a set function \( g : 2^{V} \mapsto \mathbb{R}_{+} \) as in Eq. (2), \( g \) is supermodular and the Sup-ML problem becomes the MHV problem, where we use “color” and “label” interchangeably. Under the hat of Sup-ML, the goal is to color all the uncolored vertices such that \( \sum_{i=1}^{k} g(S_{i}) \) is maximized, where \( S_{i} \) is the set of all vertices colored with \( i \).

3.3 An LP relaxation for MHV

The following LP relaxation for the MHV problem on a given graph \( G = (V, E) \) is formulated in Zhang et al. [22], where \( V = \{v_1, v_2, \ldots, v_n\} \), a binary variable \( y_{ij} \triangleq y_{ij}^i(v_j) \) denotes whether or not the vertex \( v_j \) is colored \( i \), and \( y_i = (y_{i1}, y_{i2}, \ldots, y_{in}) \). We use \( w_j = w(v_j) \) to denote the weight of the vertex \( v_j \).
\[ \text{maximize } \sum_{j=1}^{n} w_j z_j \quad \text{(LP-MHV)} \]

subject to

\[ \sum_{i=1}^{k} y_{ij}^i = 1, \quad \forall v_j \in V \quad (31) \]
\[ y_{ij}^i = 1, \quad \forall v_j \in V, \; \forall i \in C \; \text{s.t. } c(v_j) = i \quad (32) \]
\[ z_j^i = \min_{v_h \in N[v_j]} \{y_{ih}^i\}, \quad \forall v_j \in V, \; \forall i \in C \quad (33) \]
\[ z_j = \sum_{i=1}^{k} z_j^i, \quad \forall v_j \in V \quad (34) \]
\[ z_j^i, z_j, y_{ij}^i \geq 0, \quad \forall v_j \in V, \; \forall i \in C \quad (35) \]

where \( z_j^i \) indicates whether the vertex \( v_j \) is happy by color \( i \), \( z_j \) indicates whether the vertex \( v_j \) is happy, and \( N[v_j] \) is the closed neighborhood of the vertex \( v_j \).

For each color \( i \), since there is at least one vertex pre-colored \( i \) and at least one vertex pre-colored another color, we let \( \pi \) be the permutation for \( y_{ij}^i \) such that \( 1 = y_{i\pi_1}^i \geq y_{i\pi_2}^i \geq \ldots \geq y_{i\pi_n}^i = 0 \). In the concave relaxation \((\text{CP-Sup-ML})\) based on the Lovász extension for \( \text{Sup-ML} \), when we set the supermodular set function \( g \) as in Eq. (2), the objective function of \((\text{CP-Sup-ML})\) becomes

\[ \sum_{i=1}^{k} \sum_{j=1}^{n-1} \left( y_{i\pi_j}^i - y_{i\pi_{j+1}}^i \right) g(\{v_{\pi_1}, v_{\pi_2}, \ldots, v_{\pi_j}\}) = \sum_{i=1}^{k} \sum_{j=1}^{n-1} \left( y_{i\pi_j}^i - y_{i\pi_{j+1}}^i \right) \sum_{v_h \in i(\{v_{\pi_1}, v_{\pi_2}, \ldots, v_{\pi_j}\})} w_h. \quad (36) \]

For each vertex \( v_p \in V \), let \( v_q \) denote its neighbor that appears the last in the permutation \((v_{\pi_1}, v_{\pi_2}, \ldots, v_{\pi_n})\). Assume \( p = \pi_j \) and \( q = \pi_{j+1} \). Clearly, \( v_p \in i(\{v_{\pi_1}, v_{\pi_2}, \ldots, v_{\pi_j}\}) \) if and only if \( p, q \in \{\pi_1, \pi_2, \ldots, \pi_j\} \), that is, we must have \( j_1, j_2 \leq j \). It follows that for the vertex \( v_p \in V \), the coefficient of \( w_p \) in Eq. (36) is

\[ \sum_{i=1}^{k} \sum_{j=\max(j_1, j_2)}^{n} \left( y_{i\pi_j}^i - y_{i\pi_{j+1}}^i \right) = \sum_{i=1}^{k} z_p^i = z_p, \]

where the last two equalities hold due to Constraints \((33, 34)\) of \( \text{LP-MHV} \). This shows that by setting the supermodular set function \( g \) as defined in Eq. (2), \((\text{CP-Sup-ML})\) is the same as \((\text{LP-MHV})\). Therefore, we have the following theorem.

\[ \blacktriangledown \text{Theorem 8. Algorithm RR is a } \frac{2}{k} \text{-approximation for MHV.} \]

### 3.4 An upper bound on the integrality gap of LP-MHV

We use the instance constructed for MUHV in Subsection 2.4 to derive an upper bound. Note that MHV and MUHV have the complementary optimization goals on the same input instance. Let \( \text{OPT(MHV)} \) denote the value of an optimal solution to the constructed instance \( I \); from Eq. (19) we obtain

\[ \text{OPT}(\text{MHV}) \leq w_t, \quad (37) \]
that is, any coloring scheme may make at most one terminal happy.

Consider the following fractional feasible solution to the instance $I$ in the LP relaxation \( \text{LP-MHV} \), which is very similar to the one for \( \text{LP-MUHV} \) in Section 2.4:

- for each terminal $t_i \in T$, $y_i(t_i) = 1$ and $y_j(t_i) = 0$ for all $j \neq i$;
- for each vertex $b_{ij} \in V_b$, $y_i(b_{ij}) = y_j(b_{ij}) = \frac{1}{2}$ and $y_\ell(b_{ij}) = 0$ for all $\ell \neq i, j$;
- for each terminal $t_i \in T$, we set $z_i(t_i) = y_i(b_{ij}) = \frac{1}{2}$, $z_j(t_i) = 0$ for all $j \neq i$, and $z(t_i) = \sum_{\ell=1}^k z_\ell(t_i) = \frac{1}{2}$;
- for each vertex $b_{ij} \in V_b$, we set $z_\ell(b_{ij}) = 0$ for all $\ell \in C$, and $z(b_{ij}) = 0$.

Let \( \text{OPT}(\text{LP-MHV}) \) denote the optimum of the instance $I$ in the LP relaxation \( \text{LP-MHV} \). It is greater than or equal to the value of the above fractional feasible solution, that is,

\[
\text{OPT}(\text{LP-MHV}) \geq \frac{1}{2} k w_t. \tag{38}
\]

Combining Eqs. (37) and (38), it gives an upper bound on the integrality gap of \( \text{LP-MHV} \):

\[
\frac{\text{OPT}(\text{MHV})}{\text{OPT}(\text{LP-MHV})} \leq \frac{1}{k} \leq \frac{2}{k}.
\]

We thus have proved the following theorem.

\textbf{Theorem 9.} The integrality gap of \( \text{LP-MHV} \) has an upper bound of \( \frac{2}{k} \).

Theorems 8 and 9 together imply that the $\frac{2}{k}$-approximation algorithm RR for the MHV problem is the best possible based on the LP relaxation \( \text{LP-MHV} \).

\section{Conclusions}

We studied the Sub-ML problem and its complement, the Sup-ML problem. Motivated by the Lovász relaxation presented by Chekuri and Ene [3], we presented a convex relaxation and a concave relaxation for Sub-ML and Sup-ML, respectively. Adopting the randomized rounding approach used in [6], we showed that Sub-ML and Sup-ML can be approximated within a factor of $2 - \frac{2}{k}$ and $\frac{2}{k}$, respectively.

We showed that the MUHV and MHV problems are special cases of the Sub-ML and Sup-ML problems, respectively. For MUHV, we formulated a novel LP relaxation and proved that it is equivalent to the convex relaxation based on the Lovász extension for Sub-ML, and showed a lower bound of $2 - \frac{2}{k}$ on the integrality gap of the LP relaxation; these suggest that the $(2 - \frac{2}{k})$-approximation for Sub-ML is also a $(2 - \frac{2}{k})$-approximation for MUHV, and it is the best possible based on the LP relaxation. We then proved that this $(2 - \frac{2}{k})$-approximation is optimal for MUHV, assuming the Unique Games Conjecture.

For MHV, we showed that the LP relaxation presented by Zhang et al. [22] is equivalent to the concave relaxation based on the Lovász extension for Sup-ML, and proved an upper bound of $\frac{2}{k}$ on the integrality gap of the LP relaxation; analogously, these suggest that the $\frac{2}{k}$-approximation for Sup-ML is also a $\frac{2}{k}$-approximation for MHV, and it is the best possible based on the LP relaxation. This $\frac{2}{k}$-approximation for MHV improves the previous best approximation algorithm with ratio $\max\{\frac{1}{k}, \frac{\Delta + 1}{\Delta + 2}\}$ [23, 22], where $\Delta$ is the maximum vertex degree of the input graph.
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