Multipartite quantum correlations in even and odd spin coherent states

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Abstract
The key ingredient of the approach presented in this paper is the factorization property of $SU(2)$ coherent states upon the splitting or decay of a quantum spin system. In this picture, the even and odd spin coherent states are viewed as comprising two, three or more spin sub-systems. From this perspective, we investigate the multipartite quantum correlations defined as the sum of the correlations of all possible bi-partitions. The pairwise quantum correlations are quantified by entanglement of formation and quantum discord. Special attention is devoted to tripartite splitting schemes. We explicitly derive the sum of the entanglement of formation for all possible bi-partitions. It coincides with the sum of all possible occurrences of pairwise quantum discord. The conservation relation between the distribution of entanglement of formation and quantum discord in the tripartite splitting scheme is discussed. We show that entanglement of formation and quantum discord possess the monogamy property for even spin coherent states, in contrast to odds ones which violate the monogamy relation when the overlap of the coherent states approaches unity.

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(Some figures may appear in colour only in the online journal)

1. Introduction

The characterization of nonclassical correlations and nonlocal correlations constitutes one of the main issues being intensively investigated in the field of quantum information science. The primary goal is to provide the best method for understanding the differences between quantum and classical physics. Quantum correlations constitute a relevant resource for managing information in several ways [1–3]. Different types of measures for quantifying the degree
of quantumness in a multipartite quantum system were introduced. In particular, entanglement of formation has been successfully employed in this sense. However, this measure does not account for all nonclassical aspects of correlations and unentangled mixed states can possess quantum correlations. In this respect, other measures beyond entanglement were proposed in the literature, such as for instance quantum discord, which was introduced in [4, 5]. It is defined as the difference between the total correlation and the classical correlation present in a bipartite system. The quantum discord coincides with entanglement for pure states. For mixed states, the explicit evaluation of quantum discord involves an optimization procedure which is in general a difficult task to achieve. There are a few two-qubit systems [6–12] for which analytical results were obtained. To overcome the difficulty in analytically evaluating quantum discord, a geometric method was introduced in [13]. Nowadays, entanglement of formation [14], quantum discord [4, 5] and its geometric variant [13] are typical examples of measures commonly used to determine the presence of quantum correlations between two different parts comprising a bipartite quantum system.

In recent years, efforts to identify and quantify quantum correlations were extended to correlated nonorthogonal states, such as for example Glauber coherent states and SU(2) and SU(1, 1) coherent states [15, 16] (for a review see [17]). Subsequently, many works have been devoted to investigating their role in quantum cryptography [18], quantum information processing [19] and quantum computing [20–22]. This is mainly motivated by the possibility of encoding quantum information in continuous variables [23]. For example, the even and odd Glauber coherent states, also termed Schrödinger cat states, can be considered as the basis states of a logical qubit [24, 25] and provide a practical way of implementing experimentally optical quantum systems which are useful for quantum information.

On the other hand, the structure of multipartite quantum systems is a complicated and challenging subject that has triggered much interest during the last decade (see [3] and references therein). In this paper, we shall strictly focus on the study of quantum correlations present in odd and even SU(2) coherent states. In fact, by considering the property according to which a spin-\( j \) coherent state \( | j, \eta \rangle \) can be factorized as a tensorial product of two SU(2) coherent states \( | j_1, \eta \rangle \) and \( | j_2, \eta \rangle \) with \( j = j_1 + j_2 \), it is possible to construct a picture where even and odd spin coherent states might be viewed as superpositions of two or more spin coherent systems. The idea of entanglement in a single particle, caused by quantum correlations between its intrinsic degrees of freedom, was discussed in [26–28]. Consequently, it seems natural to assume that an odd or even spin-\( j \) coherent state presents quantum correlations between its intrinsic parts resulting from the splitting of the spin-\( j \) into two or more subcomponents. In this scheme, one can analyze the properties of multipartite quantum correlations in many spin systems. The best way to approach this question is by the use of bipartite measures. This approach has the advantage of relying upon bipartite measures of entanglement of formation and quantum discord that are physically motivated and analytically computable. Also, another important question emerging in this context concerns the limitations of sharing quantum correlations. Indeed, the distribution of quantum correlations among the sub-systems of a multipartite quantum system is constrained by the so-called monogamy relation. It was firstly proposed by Coffman, Kundu and Wootters in 2001 [29] in analyzing the distribution of entanglement in a tripartite qubit system. Since then, the monogamy relation has been extended to other measures of quantum correlations. Unlike the squared concurrence [29], the entanglement of formation does not satisfy the monogamy relation [29] in a pure tripartite qubit system, but it was reported in [30, 31] that it can be satisfied in multimode Gaussian states. Furthermore, quantum correlations, measured by quantum discord, were shown to violate monogamy for some specific quantum states [32–36]. Now, there are many
attempts to establish the conditions under which a given quantum correlation measure is monogamous or not. One may quote, for instance, the results obtained in [37].

This paper is organized as follows. In section 2 we give the definitions of the bipartite measures: concurrence, entanglement of formation and quantum discord. We also introduce the measure of multipartite correlations in a given system as the sum of all possible bipartite correlations. Section 3 concerns even and odd spin coherent states. In particular we discuss the decomposition property of spin coherent states according to which they split in multipartite spin or qubit systems. In section 4, we derive the explicit expressions of pairwise quantum correlations present in even and odd spin coherent states decomposed in a pure bipartite system. An appropriate qubit mapping is introduced. The results of section 4 are extended in section 5 to the situation where the spin coherent state splits into three spin sub-systems. A qubit mapping is realized for all possible bi-partitions of the system. The total amount of entanglement of formation is derived in section 6. Similarly, in section 7, we explicitly evaluate the total amount of quantum discord present in even and odd spin coherent states viewed as a tripartite system. The sum of the amount of pairwise quantum discord is evaluated. It coincides with the total amount of bipartite entanglement of formation in agreement with the result obtained in [38]. This result originates from the conservation relation between the distribution of entanglement of formation and quantum discord proved in [39]. Limitations to sharing entanglement of formation as well as quantum discord are discussed. Some special cases are numerically examined to corroborate our analysis. The paper closed with concluding remarks.

2. Quantum correlations

The theoretical investigation of quantum correlations in a multipartite quantum system is motivated by the recent experimental progress in creating and manipulating highly correlated spin ensembles which provide experimentally accessible systems for quantum information processing. In general the analysis of the properties of quantum correlations in many spin systems is difficult. The simple way to approach this problem is by the use of bipartite measures that are explicitly computable, such as entanglement of formation and usual quantum discord. The definitions of each of these two measures is presented hereafter. For an arbitrary tripartite state, the quantum correlations present in the system can be computed by considering all possible bipartite splits. The whole system can be partitioned in two different ways. In the first bi-partition scheme, the system splits into two sub-systems, one containing one particle and the second comprising the two remaining particles. The second bi-partition is obtained by tracing out the degrees of freedom of the third subsystem. In this picture, the total amount of quantum correlations is given by the sum of all possible bipartite quantum correlations.

2.1. Bipartite measures of entanglement of formation and quantum discord

We shall first briefly review the concept of quantum discord [4, 5]. The total correlation is usually quantified by the mutual information, usually expressed in terms of von Neumann entropy, as

\[ I(\rho_{AB}) = S(\rho_A) + S(\rho_B) - S(\rho_{AB}), \]

where \( \rho_{AB} \) is the state of a bipartite quantum system composed of the sub-systems A and B, the operator \( \rho_{A(B)} = \text{Tr}_{B(A)}(\rho_{AB}) \) is the reduced state of A(B) and \( S(\rho) \) is the von Neumann
entropy of a quantum state $\rho$. The mutual information $I(\rho_{AB})$ contains both quantum and classical correlations. It decomposes as

$$I(\rho_{AB}) = D(\rho_{AB}) + C(\rho_{AB}),$$

and the quantum discord $D(\rho_{AB})$ is defined as the difference between the total correlation $I(\rho_{AB})$ and the classical correlation $C(\rho_{AB})$, present in the bipartite system $AB$. The classical part $C(\rho_{AB})$ can be determined by optimizing the local measurement procedure as follows. Let us consider a von Neumann type measurement, on the subsystem $A$. For the classical correlation of a bipartite state $\rho_{AB}$, the von Neumann measurement yields the statistical ensemble $\{p_{B,k}, \rho_{B,k}\}$ such that

$$\rho_{AB} \rightarrow \frac{(M_k \otimes \mathbb{I})\rho_{AB}(M_k \otimes \mathbb{I})}{p_{B,k}}$$

where the measurement operation is written as [40]

$$M_k = U_k \Pi_k U_k^\dagger$$

where $\Pi_k = |k\rangle\langle k|$ ($k = 0, 1$) is the one-dimensional projector for subsystem $A$ along the computational base $|k\rangle$, $U \in SU(2)$ is a unitary operator and

$$p_{B,k} = \text{tr}[(M_k \otimes \mathbb{I})\rho_{AB}(M_k \otimes \mathbb{I})].$$

The amount of information acquired about particle $B$ is then given by

$$S(\rho_B) - \sum_k p_{B,k} S(\rho_{B,k}).$$

which depends on measurement $M$. To remove the measurement dependence, a maximization over all possible measurements is performed and the classical correlation is written as

$$C(\rho_{AB}) = \max_M \left[ S(\rho_B) - \sum_k p_{B,k} S(\rho_{B,k}) \right] = S(\rho_B) - \tilde{S}_{\text{min}}$$

(3)

where $\tilde{S}_{\text{min}}$ denotes the minimal value of the conditional entropy

$$\tilde{S} = \sum_k p_{B,k} S(\rho_{B,k}).$$

(4)

When optimization is taken over all perfect measurement, the quantum discord is

$$D(\rho_{AB}) = I(\rho_{AB}) - C(\rho_{AB}) = S(\rho_A) + \tilde{S}_{\text{min}} - S(\rho_{AB}).$$

(5)

The explicit evaluation of quantum discord (5) requires the analytical computation of $\tilde{S}_{\text{min}}$. This quantity was explicitly derived for only a few exceptional two-qubit quantum states. One may quote for instance the results obtained in [7, 41] (see also [11, 12, 42]). In this paper, we shall be mainly concerned with two-rank quantum states for which the minimization of the conditional entropy (4) can be exactly performed by purifying the density matrix $\rho_{AB}$ and making use of the Koashi–Winter relation [43] (see also [44]). This relation establishes the connection between the classical correlation of a bipartite state $\rho_{AB}$ and the entanglement of formation of its complement $\rho_{BC}$. Hereafter, we shall briefly discuss this method. We assume that the density matrix $\rho_{AB}$ has two non-vanishing eigenvalues (two-rank matrix). It decomposes as

$$\rho_{AB} = \lambda_+ |\phi_+\rangle_{AB} \langle \phi_+| + \lambda_- |\phi_-\rangle_{AB} \langle \phi_-|$$

(6)

where $\lambda_+$ and $\lambda_-$ are the eigenvalues of $\rho_{AB}$ and the corresponding eigenstates are denoted by $|\phi_+\rangle_{AB}$ and $|\phi_-\rangle_{AB}$, respectively. The purification of the mixed state $\rho_{AB}$ is realized by attaching a qubit $C$ to the two-qubit system $A$ and $B$. This yields

$$|\phi\rangle_{ABC} = \sqrt{\lambda_+} |\phi_+\rangle_{AB} \otimes |0\rangle_C + \sqrt{\lambda_-} |\phi_-\rangle_{AB} \otimes |1\rangle_C$$

(7)
such that the whole system ABC is described by the pure density matrix \( \rho_{ABC} = |\phi\rangle_{ABC}\langle\phi| \) from which one has the bipartite densities \( \rho_{AB} = \text{Tr}_C\rho_{ABC} \) and \( \rho_{BC} = \text{Tr}_A\rho_{ABC} \). According to the Koashi–Winter relation [43], the minimal value of the conditional entropy coincides with the entanglement of formation of \( \rho_{BC} \):

\[
\tilde{S}_{\min} = E(\rho_{BC})
\]

which is given by

\[
\tilde{S}_{\min} = E(\rho_{BC}) = H\left(\frac{1}{2} + \frac{1}{2}\sqrt{1 - |C(\rho_{BC})|^2}\right)
\]

where \( H(x) = -x \log_2 x - (1 - x) \log_2 (1 - x) \) is the binary entropy function and \( C(\rho_{BC}) \) is the concurrence of the density matrix \( \rho_{BC} \). We recall that for \( \rho_{12} \), the density matrix for a pair of qubits 1 and 2, which may be pure or mixed, the concurrence is [45]

\[
C_{12} = \max \{\lambda_1 - \lambda_2 - \lambda_3 - \lambda_4, 0\}
\]

for \( \lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \lambda_4 \) the square roots of the eigenvalues of the ‘spin-flipped’ density matrix

\[
\rho_{12} = \rho_{12}(\sigma_y \otimes \sigma_y)\rho_{12}^\ast(\sigma_y \otimes \sigma_y),
\]

where the star stands for complex conjugation in the basis \( \{|00\rangle, |01\rangle, |10\rangle, |11\rangle\} \) where the Pauli matrix is \( \sigma_y = i |1\rangle \langle 0| - i |0\rangle \langle 1| \). Nonzero concurrence traduces the entanglement between the qubits 1 and 2; otherwise they are separable. Using equations (5) and (9), the quantum discord is written as

\[
D_{AB} \equiv D_{\text{AB}}^\rightarrow = S_A - S_{AB} + E_{BC}.
\]

In the same manner, when the measurement is performed on the subsystem B, it is simply verified that the quantum discord takes the form

\[
D_{BA} \equiv D_{\text{BA}}^\rightarrow = S_B - S_{AB} + E_{AC}.
\]

Notice that for a pure density matrix \( \rho_{AB} \), the quantum discord reduces to entanglement of formation \( E(\rho_{AB}) \).

### 2.2. Multipartite quantum correlations

The measure of multipartite quantum correlations constitutes an important issue in the context of quantum information. Some attempts to provide a precise way to quantify and characterize the genuine multipartite correlations were discussed in the literature, yielding different approaches [38, 46–48]. In particular, Rulli and Sarandy [48] defined the multipartite measure of quantum correlation as the maximum of the quantum correlations existing between all possible bi-partition of the multipartite quantum system. In a similar way, Ma and coworkers [38] suggested a slightly different definition to quantify the global multipartite quantum correlation. It is defined as the sum of the correlations in all possible bi-partitions. In this paper, in parallel with the treatment discussed in [38], we shall quantify the global quantum correlations present in even and odd spin coherent states as follows. For a tripartite spin coherent state system \((j_1, j_2, j_3)\) arising from the decomposition of a spin- \( j \) coherent state with \( j = j_1 + j_2 + j_3 \), the total amount of quantum correlation is defined by

\[
Q(j_1, j_2, j_3) = \frac{1}{12} (Q_{j_1|j} + Q_{j_2|j} + Q_{j_1|j} + Q_{j_1|j} + Q_{j_2|j} + Q_{j_2|j} + Q_{j_1|j} + Q_{j_1|j} + Q_{j_2|j} + Q_{j_2|j} + Q_{j_1|j} + Q_{j_1|j})
\]

where the bipartite measure \( Q \) stands for entanglement of formation or quantum discord. More details concerning the remarkable splitting property of spin coherent states will be presented in the next section. On the other hand, as we shall be dealing with tripartite quantum states,
it is natural to investigate the intriguing monogamy relation of quantum correlation present in spin coherent states. The concept of monogamy can be introduced as follows. Let $Q_{AB}$ denote the shared correlation $Q$ between $A$ and $B$. Similarly, let us denote by $Q_{AC}$ the measure of the correlation between $A$ and $C$ and $Q_{ABC}$, the correlation shared between $A$ and the composite subsystem $BC$ comprising $B$ and $C$. The bipartite measure of correlations $Q$ is monogamous if $Q_{ABC}$ is greater that the sum of $Q_{AB}$ and $Q_{AC}$:

$$Q_{ABC} \geq Q_{AB} + Q_{AC}.$$  

This inequality imposes severe limitations to sharing quantum correlations. The monogamy of entanglement of formation and quantum discord in tripartite spin coherent states are examined in sections 6 and 7. It must be emphasized that the conditions under which any measure of quantum correlations that comprise and go beyond entanglement of formation was discussed by Fanchini et al in [49] for an arbitrary pure tripartite state. In particular, the authors developed an elegant operational approach based on the discrepancy between classical and quantum correlations to set up the constraints that any pure tripartite state must satisfy, such that the entanglement of formation follows the monogamy property. This approach also allows us to understand the result obtained by Giorgi [32], according to which the entanglement of formation and quantum discord obey the same monogamous relation.

### 3. Spin coherent states as multi-qubit systems

#### 3.1. The multi-qubit structure of Bloch coherent spin states

An arbitrary spin system is described by the $su(2)$ algebra generated by the operators $J_+, J_-$ and $J_3$ satisfying the following structure relations

$$[J_3, J_\pm] = \pm J_\pm, \quad [J_-, J_+] = -2J_3.$$  

The different irreducible representation classes of the group $SU(2)$ are completely determined by the quantum angular momentum $j$ which may take integer or half integer values ($j = \frac{1}{2}, 1, \frac{3}{2}, \ldots$). The $(2j + 1)$-dimensional Hilbert space is spanned by the irreducible tensorial set $|j, m\rangle$, $m = -j, -j + 1, \ldots, j - 1, j$ characterizing the spin-$j$ representations of the group $SU(2)$. The standard $SU(2)$ coherent states are obtained by the action of an element of the coset space $SU(2)/U(1)$

$$D_j(\xi) = \exp(\xi J_+ - \xi^* J_-),$$  

on the extremal state $|j, -j\rangle$. This action gives the states

$$|j, \eta\rangle = D_j(\xi)|j, -j\rangle = \exp(\xi J_+ - \xi^* J_-)|j, -j\rangle = (1 + |\eta|^2)^{-j}\exp(\eta J_+)|j, -j\rangle,$$

where $\eta = (\xi/|\xi|) \tan |\xi|$. In the basis $|j, m\rangle$, they are written

$$|j, \eta\rangle = (1 + |\eta|^2)^{-j}\sum_{m=-j}^{j} \left[ \frac{(2j)!}{(j+m)!(j-m)!} \right]^{1/2} \eta^{j+m}|j, m\rangle.$$

They satisfy the resolution to the identity property

$$\int d\mu(j, \eta) |j, \eta\rangle \langle j, \eta| = I,$$

$$d\mu(j, \eta) = \frac{2j + 1}{\pi} (1 + |\eta|^2)^{2j}.$$  

The spin coherent states are not orthogonal to each other:

$$\langle j, \eta_1 | j, \eta_2 \rangle = (1 + |\eta_1|^2)^{-j}(1 + |\eta_2|^2)^{-j}(1 + \eta_1^* \eta_2)^{2j}.$$  

$$\langle j, \eta_1 | j, \eta_2 \rangle = (1 + |\eta_1|^2)^{-j}(1 + |\eta_2|^2)^{-j}(1 + \eta_1^* \eta_2)^{2j}.$$  

$$\langle j, \eta_1 | j, \eta_2 \rangle = (1 + |\eta_1|^2)^{-j}(1 + |\eta_2|^2)^{-j}(1 + \eta_1^* \eta_2)^{2j}.$$
The resolution to identity makes it possible to expand an arbitrary state in terms of the coherent states $|j, \eta\rangle$. In the special case $j = \frac{1}{2}$, the spin coherent states (19) reduce to

$$|\eta\rangle = \frac{1}{\sqrt{1 + \eta^2}} |\downarrow\rangle + \frac{\eta}{\sqrt{1 + \eta^2}} |\uparrow\rangle. \quad (22)$$

Here and in the following $|\eta\rangle$ is short for the spin-$\frac{1}{2}$ coherent state $|\frac{1}{2}, \eta\rangle$ with $|\uparrow\rangle \equiv |\frac{1}{2}, \frac{1}{2}\rangle$ and $|\downarrow\rangle \equiv |\frac{1}{2}, -\frac{1}{2}\rangle$. It is important to note that the tensorial product of two $SU(2)$ coherent states $|j_1, \eta_1\rangle$ and $|j_2, \eta_2\rangle$ produces a spin-$(j_1 + j_2)$ coherent state labeled by the same variable:

$$|j_1, \eta_1\rangle \otimes |j_2, \eta_2\rangle = (D_{j_1} \otimes D_{j_2}) ((|j_1, j_1\rangle \otimes |j_2, j_2\rangle) = D_{j_1 + j_2} |j_1 + j_2, j_1 + j_2\rangle = |j_1 + j_2, \eta\rangle. \quad (23)$$

Only coherent states possess this remarkable property. It allows us to write any spin-$j$ coherent states as a $2j$ tensorial product of spin-$\frac{1}{2}$ coherent states:

$$|j, \eta\rangle = (|\eta\rangle)^{\otimes 2j} = \left( \frac{1}{\sqrt{1 + \eta^2}} |\downarrow\rangle + \frac{\eta}{\sqrt{1 + \eta^2}} |\uparrow\rangle \right)^{\otimes 2j} \quad = (1 + \bar{\eta} \eta)^{-j} \sum_{m=-j}^{+j} \left( \frac{2j}{j + m} \right)^{\frac{1}{2}} \eta^{j + m} |j, m\rangle,$$

reflecting that a spin-$j$ coherent state may be viewed as a multipartite state containing $2j$ qubits.

### 3.2. Even and odd coherent states

The even and odd spin coherent states are defined by

$$|j, \eta, m\rangle = \mathcal{N}_m (|j, \eta\rangle + e^{i\pi m} |j, -\eta\rangle) \quad (24)$$

where the integer $m \in \mathbb{Z}$ takes the values $m = 0 \text{ (mod } 2\text{)}$ and $m = 1 \text{ (mod } 2\text{)}$. The normalization factor $\mathcal{N}_m$ is

$$\mathcal{N}_m = \left[ 2 + 2p^{2j} \cos m\pi \right]^{-1/2}$$

where $p$ denotes the overlap between the states $|\eta\rangle$ and $|-\eta\rangle$. It is given as

$$p = (\eta - \bar{\eta} \eta) = \frac{1 - \bar{\eta} \eta}{1 + \bar{\eta} \eta}. \quad (25)$$

For $j = \frac{1}{2}$, the even and odd coherent states coincide with $|\uparrow\rangle$ and $|\downarrow\rangle$. They can be identified with basis states for a logical qubit as $|0\rangle \rightarrow |\uparrow\rangle$ and $|1\rangle \rightarrow |\downarrow\rangle$. This line of reasoning can be extended to higher spin values and provides a scheme for encoding information in superpositions of arbitrary spin coherent states, especially even and odd ones. Indeed, the states $|j, \eta, 0\rangle$ and $|j, \eta, 1\rangle$ define a two-dimensional orthogonal basis and give a first possible encoding scheme. Thus, one can identify the even state $|j, \eta, 0\rangle$ and the odd state $|j, \eta, 1\rangle$ as the basis of a logical qubit as

$$|j, \eta, 0\rangle \rightarrow |0\rangle_j \quad |j, \eta, 1\rangle \rightarrow |1\rangle_j.$$ 

Other encoding schemes involving more qubits are also possible. They can be realized using the factorization or the splitting property of spin coherent states (23). In fact, the states (24) can also be expressed as

$$|j, \eta, m\rangle = \mathcal{N}_m (|j_1, \eta_1\rangle \otimes |j_2, \eta_2\rangle + e^{i\pi m} |j_1, -\eta_1\rangle \otimes |j_2, -\eta_2\rangle) \quad (26)$$
with \( j = j_1 + j_2 \). They can be rewritten as a two-qubit state in the basis
\[
|j, \eta, 0\rangle \rightarrow |0\rangle_j, \quad |j, \eta, 1\rangle \rightarrow |1\rangle_j, \quad i = 1, 2.
\]
defined by means of odd and even spin coherent states associated with the angular momenta \( j_1 \) and \( j_2 \). This construction is easily generalizable to three and more qubits. In this manner, the states \(|j, \eta, m\rangle\) can be viewed as multipartite fermionic coherent states:
\[
|j, \eta, m\rangle = N_m((|\eta\rangle)^{\otimes 2j} + e^{i\text{int}}(|-\eta\rangle)^{\otimes 2j}). \tag{27}
\]
Furthermore, the logical qubits of \(|j, \eta, 0\rangle\) (even) and \(|j, \eta, 1\rangle\) (odd) spin coherent states behave like a multipartite state of Greenberger–Horne–Zeilinger (GHZ)-type \([50]\) in the asymptotic limit \(p \rightarrow 0\). In this special limiting case, the states \(|\eta\rangle\) and \(|-\eta\rangle\) approach orthogonality and an orthogonal basis can be defined such that \(|0\rangle \equiv |\eta\rangle\) and \(|1\rangle \equiv |-\eta\rangle\). Thus, the state \(|j, \eta, m\rangle\) becomes of GHZ-type
\[
|j, \eta, m\rangle \sim |\text{GHZ}_{2j}\rangle = \frac{1}{\sqrt{2}} (|0\rangle \otimes |0\rangle \otimes \cdots \otimes |0\rangle + e^{i\text{int}}|1\rangle \otimes |1\rangle \otimes \cdots \otimes |1\rangle). \tag{28}
\]
The second limiting case corresponds to the situation when \(p \rightarrow 1\) (or \(\eta \rightarrow 0\)). In this case it is simple to check that the state \(|j, \eta, m = 0(\text{mod} \ 2)\rangle\) (27) reduces to the ground state of a collection of \(2j\) fermions
\[
|j, 0, 0(\text{mod} \ 2)\rangle \sim |\downarrow\rangle \otimes |\downarrow\rangle \otimes \cdots \otimes |\downarrow\rangle, \tag{29}
\]
and the state \(|j, \eta, 1(\text{mod} \ 2)\rangle\) becomes a multipartite state of \(W\) type \([51]\)
\[
|j, 0, 1(\text{mod} \ 2)\rangle \sim |W_{2j}\rangle = \frac{1}{\sqrt{2j}}(|\uparrow\rangle \otimes |\downarrow\rangle \otimes \cdots \otimes |\downarrow\rangle + |\downarrow\rangle \otimes |\uparrow\rangle \otimes \cdots \otimes |\uparrow\rangle). \tag{30}
\]
The even spin coherent states \(|j, \eta, m = 0(\text{mod} \ 2)\rangle\) interpolate continuously between GHZ\(_{2j}\) states \((p \rightarrow 0)\) and the completely separable state \(|\downarrow\rangle \otimes |\downarrow\rangle \otimes \cdots \otimes |\downarrow\rangle(p \rightarrow 1)\). In the odd case, corresponding to \(|j, \eta, m = 1(\text{mod} \ 2)\rangle\), we obtain states interpolating between states of GHZ\(_{2j}\) type \((p \rightarrow 0)\) and states of \(W_{2j}\) type \((p \rightarrow 1)\).

The decomposition property (23) provides us with a picture where even and odd spin coherent states can be considered as comprising multipartite spin sub-systems. This is our main motivation to investigate the quantum correlations present in a single spin coherent state. This issue is discussed as follows.

### 4. Bipartite splitting and bipartite correlations

In this section, we first discuss the bipartite splitting described by equation (26). In this scheme, the entire system contains two sub-systems characterized by the angular momenta \(j_1\) and \(j_2\) such that \(j = j_1 + j_2\). Accordingly, \((2j - 1)\) possible bipartite splittings are possible:
\[
j_1 = j - \frac{s}{2}, \quad j_2 = \frac{s}{2}, \quad s = 1, 2, \ldots, 2j - 2, 2j - 1,
\]
and subsequently it is interesting to compare the pairwise quantum correlations in each possible bipartite splitting.

#### 4.1. Bipartite entanglement of formation

As discussed in the previous section, for each bi-partition \(s = 1, 2, \ldots, 2j - 1\), the coherent state \(|j, \eta, m\rangle\) can be expressed as a state of two logical qubits. In this sense, for each subsystem, an orthogonal basis \(|0\rangle_i, |1\rangle_i\rangle\) with \(l = j_1\) or \(j_2\), can be defined as
\[
|0\rangle_l = \frac{|l, \eta\rangle + |l, -\eta\rangle}{\sqrt{2(1 + p^2)}}, \quad |1\rangle_l = \frac{|l, \eta\rangle - |l, -\eta\rangle}{\sqrt{2(1 - p^2)}}. \tag{31}
\]
The bipartite density matrix $\rho = |j, \eta, m\rangle\langle j, \eta, m|$ is pure. In this situation, the quantum discord for the pure state $\rho_{AB} = \rho$ coincides with the entanglement of formation. It is given by the von Neumann entropy of the subsystem characterized by the spin $j_1$:

$$D(\rho) = E(\rho) = S(\rho_{j_1})$$

where $\rho_{j_1} = \text{Tr}_{j_2}(\rho)$ is the reduced density matrix of the first subsystem obtained by tracing out the spin $j_2$. Thus, the quantum discord is written as

$$D(\rho) = -\lambda_+ \log_2 \lambda_+ - \lambda_- \log_2 \lambda_-$$

in terms of the eigenvalues of the reduced density matrix $\rho_{j_1}$ given by

$$\lambda_{\pm} = \frac{1}{2} (1 \pm \sqrt{1 - C^2}).$$

In equation (34), $C$ is the concurrence between the two sub-systems given by

$$C = \frac{\sqrt{1 - p^{2j_1}} \sqrt{1 - p^{2j_2}}}{1 + p^{2j_1} \cos m \pi}$$

which is simply obtained by using the qubit mapping (31). It follows that the entanglement of formation is written

$$E_{j_1, j_2} \equiv E(\rho) = H \left( \frac{1}{2} + \frac{1}{2} \frac{p^{2j_1} + p^{2j_2} \cos m \pi}{1 + p^{2j_1} \cos m \pi} \right).$$

where $H$ stands for the binary entropy defined above. Notice that the entanglement of formation satisfies the symmetry relation

$$E_{j_1, j_2} = E_{j_2, j_1}$$

as expected. For $p \to 0$, the state (26) reduces to a bipartite state of GHZ-type which is maximally entangled ($C = 1$) and the entanglement of formation is $E(\rho) = 1$. The limiting case $p \to 1$ is slightly different. In fact, we have $E(\rho) = 0$ for $m$ even (i.e. symmetric pure states). The spin odd coherent states (i.e. $m$ odd) become of $W$ type when $p \to 1$ and the bipartite concurrence is written

$$C = 2 \frac{\sqrt{j_1 j_2}}{j_1 + j_2}.$$

It follows that the corresponding pairwise quantum entanglement takes the form

$$E(\rho) = D(\rho) = H \left( \frac{1}{2} + \frac{1}{2} \frac{j_1 - j_2}{j_1 + j_2} \right).$$

The entanglement of formation in $W$ states is maximal when $j_1 = j_2$ ($E(\rho) = 1$). On the other hand, in a splitting scheme such as $j_2 \ll j_1$ or $j_1 \ll j_2$, the states of $W$ type are unentangled ($E(\rho) = 0$).

### 4.2. Illustration

To exemplify the above results, we consider the even and odd coherent states associated with the spin $j = 2$. The three possible bipartite splitting schemes are

$$\left( j_1 = \frac{1}{2}, j_2 = \frac{1}{2} \right), \quad \left( j_1 = 1, j_2 = 1 \right), \quad \left( j_1 = \frac{1}{2}, j_2 = \frac{3}{2} \right).$$

Using equation (36) and the relation (37), one gets

$$E_{\frac{1}{2}, \frac{1}{2}} = E_{\frac{1}{2}, \frac{3}{2}} = H \left( \frac{1}{2} + \frac{p}{2} \frac{p^2 + \cos m \pi}{1 + p^2 \cos m \pi} \right).$$
Figure 1. The pairwise entanglement of formation $E = E_{j_1, j_2}$ versus the overlap $p$ for $(j_1 = \frac{3}{2}, j_2 = \frac{1}{2})$ and $(j_1 = 1, j_2 = 1)$ with $m = 0$.

Figure 2. The pairwise entanglement of formation $E = E_{j_1, j_2}$ versus the overlap $p$ for $(j_1 = \frac{3}{2}, j_2 = \frac{1}{2})$ and $(j_1 = 1, j_2 = 1)$ with $m = 1$.

and

$$E_{1,1} = H\left(\frac{1}{2} + \frac{p^2}{2} \frac{1 + \cos m\pi}{1 + p^2 \cos m\pi}\right). \quad (39)$$

The behavior of the entanglement of formation $E_{\frac{3}{2}, \frac{1}{2}}$ and $E_{1,1}$ versus the overlap $p$ is plotted in figures 1 and 2, corresponding to even ($m = 0$) and odd ($m = 1$) spin coherent states, respectively. As seen from the figures, in both cases the entanglement of formation in the splitting scheme $2 \rightarrow (1, 1)$ is greater than the one existing between the spin sub-systems arising from the decomposition $2 \rightarrow (\frac{3}{2}, \frac{1}{2})$ for any value of $p$. In general, for a given spin $j$, the maximal value of entanglement of formation $E_{j_1, j_2}$ is reached in the bi-partition where $j_1 = j_2 = \frac{1}{2}$. In figure 2, for odd spin coherent states, we have $E_{1,1} = 1$ as can be verified from expression (39).
5. Three-mode splitting and qubit mapping

Analogously to the bipartite case, we consider in this section the tripartite splitting of even and odd spin coherent states \((24)\). The entire system decays into three sub-systems, one subsystem describing a particle of spin \(j_1\), the second referring to a particle of spin \(j_2\) and the remaining particle being of spin \(j_3 = j - j_1 - j_2\). In this scheme, the state \(|j, \eta, m\rangle\) is written as

\[
|j, \eta, m\rangle = N_m(|j_1, \eta\rangle \otimes |j_2, \eta\rangle \otimes |j_3, \eta\rangle + e^{im\pi} |j_1, -\eta\rangle \otimes |j_2, -\eta\rangle \otimes |j_3, -\eta\rangle).
\]

To evaluate the bipartite quantum correlations in present coherent states, decomposed as in \((40)\), two different bi-partitions are considered. The first one yields pure bipartite states and the second one involves mixed two-qubit states.

5.1. Bipartite pure states

The pure bi-partitions of the state \((40)\) can be introduced in three different ways. In the first one, the state \(|j, \eta, m\rangle\) is written as

\[
|j, \eta, m\rangle_{j,j-j_1} = N_m(|j_1, \eta\rangle \otimes |j - j_1, -\eta\rangle + e^{im\pi} |j_1, -\eta\rangle \otimes |j - j_1, -\eta\rangle).
\]

Similarly, the state \((40)\) can also be partitioned as

\[
|j, \eta, m\rangle_{j,j-j_2} = N_m(|j_2, \eta\rangle \otimes |j - j_2, -\eta\rangle + e^{im\pi} |j_2, -\eta\rangle \otimes |j - j_2, -\eta\rangle).
\]

The third bi-partition is given by

\[
|j, \eta, m\rangle_{j,j-j_3} = N_m(|j_3, \eta\rangle \otimes |j - j_3, -\eta\rangle + e^{im\pi} |j_3, -\eta\rangle \otimes |j - j_3, -\eta\rangle).
\]

For each bi-partition, the state \(|j, \eta, m\rangle\) can be converted into a state of two logical qubits. This is achieved by introducing, for the first subsystem, the orthogonal basis \(|0\rangle\), \(|1\rangle\), with \(l = j_1, j_2\) or \(j_3\), defined as

\[
|0\rangle_l = \frac{|l, \eta\rangle + |l, -\eta\rangle}{\sqrt{2(1 + p^{2l})}}, \quad |1\rangle_l = \frac{|l, \eta\rangle - |l, -\eta\rangle}{\sqrt{2(1 - p^{2l})}},
\]

and, for the second subsystem, the orthogonal basis \(|0\rangle_{j,\eta-j_1}\), \(|1\rangle_{j,\eta-j_1}\) given by

\[
|0\rangle_{j,j-1} = \frac{|j - l, \eta\rangle + |j - l, -\eta\rangle}{\sqrt{2(1 + p^{2(j-l)})}}, \quad |1\rangle_{j,j-1} = \frac{|j - l, \eta\rangle - |j - l, -\eta\rangle}{\sqrt{2(1 - p^{2(j-l)})}}.
\]

Reporting the equations \((44)\) and \((45)\) in \((41)\), \((42)\) and \((43)\), one has the expression of the pure state \(|j, \eta, m\rangle_{l,j-j_1}\) in the basis \(|0\rangle_l \otimes |0\rangle_{j,j-1}, |0\rangle_l \otimes |1\rangle_{j,j-1}, |1\rangle_l \otimes |0\rangle_{j,j-1}, |1\rangle_l \otimes |1\rangle_{j,j-1}\). It is given by

\[
|j, \eta, m\rangle_{l,j-j_1} = \sum_{\alpha=0,1} \sum_{\beta=0,1} C_{\alpha,\beta} |\alpha\rangle_l \otimes |\beta\rangle_{j,j-1}
\]

where the coefficients \(C_{\alpha,\beta}\) are

\[
C_{0,0} = N_m(1 + e^{im\pi})a_0b_{j-j_1}, \quad C_{0,1} = N_m(1 - e^{im\pi})a_0b_{j-j_1} \quad C_{1,0} = N_m(1 - e^{im\pi})a_1b_{j-j_1}, \quad C_{1,1} = N_m(1 + e^{im\pi})b_1b_{j-j_1}
\]

in terms of the quantities

\[
a_k = \sqrt{\frac{1 + p^{2k}}{2}}, \quad b_k = \sqrt{\frac{1 - p^{2k}}{2}} \quad \text{for} \quad k = l, j - l
\]

involving the overlap \(p\) \((25)\) which is related to the non-orthogonality of two spin coherent states of equal amplitude and opposite phase.
5.2. Bipartite mixed states

The second class of bipartite density matrices can be realized from the state (40) by considering the reduced density matrices $\rho_{l_1l_2}$ that are obtained by tracing out the degrees of freedom of the third subsystem. There are three different density matrices $\rho_{j_1j_2}$, $\rho_{j_2j_2}$, and $\rho_{j_1j_1}$. Explicitly, they are given by

$$\rho_{j_1j_2} = \text{Tr}_2(|j, \eta, m\rangle\langle j, \eta, m|)$$

$$= \mathcal{N}_m^2(|\eta, \eta\rangle\langle \eta, \eta| + | - \eta, -\eta\rangle\langle - \eta, -\eta| + e^{i\pi q}| - \eta, -\eta\rangle)$$

with $q \equiv p^{2(j-l_1-l_2)} = p^{2S}$ and

$$| \pm \eta, \pm \eta\rangle = |l_1, \pm \eta\rangle \otimes |l_2, \pm \eta\rangle.$$

It is interesting to note that the density matrix $\rho_{l_1l_2}$ is a two-rank operator. Indeed, it can be rewritten as

$$\rho_{l_1l_2} = \frac{1}{2}(1 + q)^\frac{N^2_m}{N^2_e} |\phi_+\rangle\langle \phi_+| + \frac{1}{2}(1 - q)^\frac{N^2_m}{N^2_e} |\phi_-\rangle\langle \phi_-|$$

where

$$|\phi_\pm\rangle = \mathcal{N}_e^2(|l_1, \eta\rangle \otimes |l_2, \eta\rangle \pm e^{i\pi m} |l_1, -\eta\rangle \otimes |l_2, -\eta\rangle)$$

and

$$\mathcal{N}^2_e = 2 \pm 2p^{2(l_1+l_2)} \cos m\pi.$$

In this case, the density matrix $\rho_{l_1l_2}$ can also be converted into a two-qubit system by an appropriate qubit mapping. For this, we introduce an orthogonal pair $|0\rangle_l, |1\rangle_l$ as

$$|0\rangle_l = \frac{|l, \eta\rangle + |l, -\eta\rangle}{\sqrt{2(1 + p^{2l})}} \quad |1\rangle_l = \frac{|l, \eta\rangle - |l, -\eta\rangle}{\sqrt{2(1 - p^{2l})}}$$

where $l = l_1$ for the first subsystem and $l = l_2$ for the second. Substituting equation (49) into (47), we obtain the density matrix

$$\rho_{l_1l_2} = \mathcal{N}^2 \begin{pmatrix}
2a^2b^2(1 + q \cos m\pi) & 0 & 2a_1b_1a_2b_2(1 + q \cos m\pi) \\
0 & 2a^2b^2(1 - q \cos m\pi) & 2a_1b_1a_2b_2(1 - q \cos m\pi) \\
2a_1b_1a_2b_2(1 + q \cos m\pi) & 0 & 2a^2b^2(1 - q \cos m\pi)
\end{pmatrix}$$

in the basis $\{|0\rangle_l, |1\rangle_l, |0\rangle_{l_2}, |1\rangle_{l_2}, |0\rangle_{l_1}, |1\rangle_{l_1}\}$ where the quantities $a_1, b_1, a_2, b_2$ are defined by

$$a_i = \sqrt{\frac{1 + p^{2l_i}}{2}}, \quad b_i = \sqrt{\frac{1 - p^{2l_i}}{2}} \quad \text{for } i = 1, 2.$$

6. Quantum entanglement in the three-splitting scheme

6.1. Entanglement of formation

In the pure bipartite splitting scheme, the concurrence is given by

$$\mathcal{C}(\rho_{k_1k_2}) = \frac{\sqrt{1 - p^{4l_1}}\sqrt{1 - p^{4(l_1-k_1)}}}{1 + p^{2l} \cos m\pi}$$
where the triplet \((k_1, k_2, k_3)\) stands for \((j_1, j_2, j_3)\), \((j_2, j_1, j_3)\) and \((j_3, j_1, j_2)\), corresponding to the states (41), (42) and (43), respectively. Subsequently, the entanglement of formation is written

\[
E(\rho_{ijkl_3}) = H \left( \frac{1}{2} + \frac{1}{2} \frac{p^{2(l_3 + k_1)} + p^{2(j - k_1)l_3} \cos m\pi}{1 + p^{2j} \cos m\pi} \right) .
\]  

(52)

For mixed bipartite states belonging to the second bi-partitioning class (47), the concurrence is given by

\[
C(\rho_{ij}) = p^{2(j - l_1 - l_2)} \sqrt{\frac{(1 - p^{4l_1})(1 - p^{4l_2})}{1 + p^{2j} \cos m\pi}}
\]

(53)

where the reduced density matrix \(\rho_{ij}\) stands for \(\rho_{j_1j_2}\), \(\rho_{j_2j_1}\), and \(\rho_{j_1j_3}\). The entanglement of formation is written as

\[
E(\rho_{ij}) = H \left( \frac{1}{2} + \frac{1}{2} \sqrt{\frac{p^{4l_1}(1 - p^{4l_2})(1 - p^{4l_3})}{(1 + p^{2j} \cos m\pi)^2}} \right) .
\]

(54)

6.2. Multipartite entanglement of formation

When the bipartite quantum correlations are quantified by entanglement of formation, the definition (14) gives

\[
E(j_1, j_2, j_3) = \frac{1}{6} \left( E(\rho_{j_1j_2}) + E(\rho_{j_1j_3}) + E(\rho_{j_2j_1}) + E(\rho_{j_2j_3}) + E(\rho_{j_3j_1}) + E(\rho_{j_3j_2}) \right) .
\]

(55)

Using the results (52) and (54), the total amount of quantum entanglement is explicitly given by

\[
E(j_1, j_2, j_3) = \frac{1}{6} \left[ H \left( \frac{1}{2} + \frac{1}{2} \frac{p^{2j_1} + p^{2(j_2 + j_3)} \cos m\pi}{1 + p^{2j} \cos m\pi} \right) + H \left( \frac{1}{2} + \frac{1}{2} \sqrt{\frac{1 - p^{4j_1}(1 - p^{4j_2})(1 - p^{4j_3})}{(1 + p^{2j} \cos m\pi)^2}} \right) \right]
\]

\[
+ H \left( \frac{1}{2} + \frac{1}{2} \frac{p^{2j_2} + p^{2(j_1 + j_3)} \cos m\pi}{1 + p^{2j} \cos m\pi} \right) + H \left( \frac{1}{2} + \frac{1}{2} \sqrt{\frac{1 - p^{4j_2}(1 - p^{4j_1})(1 - p^{4j_3})}{(1 + p^{2j} \cos m\pi)^2}} \right) \right]
\]

\[
+ H \left( \frac{1}{2} + \frac{1}{2} \frac{p^{2j_3} + p^{2(j_1 + j_2)} \cos m\pi}{1 + p^{2j} \cos m\pi} \right) + H \left( \frac{1}{2} + \frac{1}{2} \sqrt{\frac{1 - p^{4j_3}(1 - p^{4j_1})(1 - p^{4j_2})}{(1 + p^{2j} \cos m\pi)^2}} \right) \right] 
\]

(56)

which is completely symmetric in \(j_1, j_2, j_3\). This quantity will be compared with the sum of pairwise quantum discord of all possible bi-partitions of the state (40) and its behavior in terms of the overlap \(p\) in some particular cases is examined in section 7.

6.3. Monogamy of entanglement of formation

The entanglement shared by more than two parties constitutes a subtle issue in investigating multipartite correlations. Thus, considering the limitations of sharing entanglement in the orthogonal case, we study the monogamy of entanglement of formation in tripartite spin coherent states. In this respect, we analyze the situations where the following inequality

\[
E(\rho_{j_1j_2}) + E(\rho_{j_1j_3}) \leq E(\rho_{j_2j_3})
\]

is satisfied or violated. The notations are as above. Clearly, to decide if the entanglement of formation is monogamous or not in spin coherent states, we shall treat some particular cases. We first consider the splitting \(j_1 = j_2 = j_3 = \frac{1}{2}\) which arises from the decomposition of even
and odd coherent states associated with the spin $j = \frac{3}{2}$. The behaviors of the entanglement of the formation difference

$$\Delta E = E(\rho_{j_1 j_2 j_3}) - E(\rho_{j_1 j_2}) - E(\rho_{j_1 j_3}),$$

for even and odd spin coherent states, are reported in figure 3. They show that the entanglement of formation always satisfies the monogamy relation in the even case ($m = 0$) but ceases to be monogamous in the odd case ($m = 1$) when the overlap $p$ is greater than 0.8. This also indicates that the monogamy relation is violated in three-qubit states of $W$ type obtained in the limiting case $p \to 1$. Similarly, we also considered the two tripartite splitting $(j_1 = \frac{1}{2}, j_2 = \frac{1}{2}, j_3 = 1)$ and $(j_1 = 1, j_2 = \frac{1}{2}, j_3 = \frac{1}{2})$ which can originate from the splitting of the spin $j = 2$. Figure 4 reveals that the monogamy relation is satisfied for even spin coherent states ($m = 0$). However, for odd spin coherent states ($m = 1$), the entanglement of formation does not follow the monogamy as $p$ approaches unity (see figure 5). This agrees
with the result of figure 3 and confirms that in a \(W\) state comprising three-qubits, the monogamy of entanglement of formation is violated.

7. Quantum discord in the three-splitting scheme

7.1. Quantum discord

In the pure bipartite splitting scheme defined by (41), (42) and (43), the quantum discord and entanglement of formation as a measure of bipartite quantum correlations are identical and we have

\[
D(\rho_{j_1|j_2j_3}) = E(\rho_{j_1|j_2j_3}) \quad D(\rho_{j_2|j_1j_3}) = E(\rho_{j_2|j_1j_3}) \quad D(\rho_{j_3|j_1j_2}) = E(\rho_{j_3|j_1j_2})
\]

where the entanglement of formation is given by (52) modulo some obvious substitutions.

To get the explicit expressions of quantum discord in bipartite mixed states \(\rho_{l_1l_2}\) of the form (50), we evaluate the mutual information entropy and the minimum of conditional entropy according to the general algorithm discussed in section 2. We first calculate the mutual information. The non-vanishing eigenvalues of the density matrix \(\rho_{l_1l_2}\) are

\[
\lambda_{\pm} = \frac{1}{2} \left( 1 \pm p^{2(j_1-l_1)}(1 \pm p^{2(j_2-l_2)} \cos(m\pi)) \right),
\]

and the joint entropy is

\[
S(\rho_{l_1l_2}) = h(\lambda_+) + h(\lambda_-) = H(\lambda_+).
\]

The eigenvalues of the marginal \(\rho_{l_1} = \text{Tr}_{l_2}\rho_{l_1l_2}\) are

\[
\lambda_{1,\pm} = \frac{1}{2} \left( 1 \pm p^{2(j_1-l_1)}(1 \pm p^{2l_1} \cos(m\pi)) \right),
\]

and the marginal entropy reads

\[
S(\rho_{l_1}) = h(\lambda_{1,+}) + h(\lambda_{1,-}) = H(\lambda_{1,+}).
\]

The eigenvalues of the marginal \(\rho_{l_2} = \text{Tr}_{l_1}\rho_{l_1l_2}\) are

\[
\lambda_{2,\pm} = \frac{1}{2} \left( 1 \pm p^{2(j_2-l_2)}(1 \pm p^{2l_2} \cos(m\pi)) \right),
\]
and the corresponding entropy is given by
\[ S(\rho_{l_2}) = h(\lambda_{2+}) + h(\lambda_{2-}) = H(\lambda_{2+}). \] (61)
It follows that the mutual information defined by (1) takes the form
\[ I(\rho_{l_2l_3}) = H(\lambda_{1+}) + H(\lambda_{2+}) - H(\lambda_+). \] (62)
The second important step in deriving pairwise quantum discord requires the explicit calculation of the minimal amount of the conditional entropy (4). According to the general discussion presented in the second section, it is necessary to purify the density matrix \( \rho_{l_2l_3} \) and determine the entanglement of formation of its complement. This algorithm can be achieved as follows. The matrix \( \rho_{l_2l_3} \) is a two-qubit state and subsequently decomposes as
\[ \rho_{l_2l_3} = \lambda_+ |\phi_+\rangle |\phi_+\rangle + \lambda_- |\phi_-\rangle |\phi_-\rangle \] (63)
where the eigenvalues \( \lambda_+ \) and \( \lambda_- \) are given by (58) and the corresponding eigenstates \( |\phi_+\rangle \) and \( |\phi_-\rangle \) are written as
\[ |\phi_+\rangle = \frac{\sqrt{(1 + p^2l_1)(1 + p^2l_2)}}{\sqrt{2(1 + p^{2l_1+2l_2})}} |0_{l_1}, 0_{l_2}\rangle + \frac{\sqrt{(1 - p^2l_1)(1 - p^2l_2)}}{\sqrt{2(1 + p^{2l_1+2l_2})}} |1_{l_1}, 1_{l_2}\rangle \] (64)
\[ |\phi_-\rangle = \frac{\sqrt{(1 + p^2l_1)(1 - p^2l_2)}}{\sqrt{2(1 - p^{2l_1+2l_2})}} |0_{l_1}, 1_{l_2}\rangle + \frac{\sqrt{(1 - p^2l_1)(1 + p^2l_2)}}{\sqrt{2(1 - p^{2l_1+2l_2})}} |1_{l_1}, 0_{l_2}\rangle \] (65)
in the basis (49). Attaching a qubit 3 to the two-qubit system (12) \( \equiv (l_1l_2) \), we write the purification of \( \rho_{l_2l_3} \) as
\[ |\phi\rangle = \sqrt{\lambda_+} |\phi_+\rangle \otimes |0\rangle + \sqrt{\lambda_-} |\phi_-\rangle \otimes |1\rangle \] (66)
such that the whole system (123) is described by the pure density matrix \( \rho_{l_1l_2l_3} \equiv |\phi\rangle \langle \phi| \).
Using the Koashi–Winter relation (9), we have
\[ S_{\text{min}}(\rho_{23}) = E(\rho_{23}) = H\left(\frac{l_1}{2} + \frac{l_2}{2}\sqrt{1 - |C(\rho_{23})|^2}\right) \] (67)
where the concurrence of the density matrix \( \rho_{23} \equiv \rho_{l_1l_3} \) is
\[ |C(\rho_{23})|^2 = \frac{p^{2l_1}(1 - p^{2l_2})(1 - p^{2l_1-l_2-l_3})}{(1 + p^{2l_1} \cos m\pi)^2}. \] (68)
It follows that the quantum discord is then given by
\[ D(\rho_{l_2l_3}) = S(\rho_{l_1l_2}) - S(\rho_{l_1l_3}) + E(\rho_{l_1l_3}). \] (69)
Using the equations (59), (60) and (67), it can be rewritten explicitly as
\[ D^\pm(\rho_{l_2l_3}) = H\left(\frac{1}{2} \left(\frac{1}{1 + p^{2l_1}} \cos(m\pi)\right)^{l_1} \right) \]
\[ - H\left(\frac{1}{2} \left(\frac{1}{1 + p^{2l_2}} \cos(m\pi)\right)^{l_2}\right) \]
\[ + H\left(\frac{1}{2} + \frac{1}{2} \sqrt{1 - \frac{p^{2l_1}(1 - p^{2l_2})(1 - p^{2l_1-l_2-l_3})}{(1 + p^{2l_1} \cos m\pi)^2}}\right) \] (70)
where the pair \((l_1, l_2)\) stands for \((j_1, j_2), (j_1, j_3)\) and \((j_2, j_3)\). Similarly, the measure of quantum discord obtained by measuring the second qubit \( B \equiv l_2 \) is
\[ D^\pm(\rho_{l_2l_3}) = H\left(\frac{1}{2} \left(\frac{1}{1 + p^{2l_1}} \cos(m\pi)\right)^{l_1} \right) \]
\[ - H\left(\frac{1}{2} \left(\frac{1}{1 + p^{2l_2}} \cos(m\pi)\right)^{l_2}\right) \]
\[ + H\left(\frac{1}{2} + \frac{1}{2} \sqrt{1 - \frac{p^{2l_1}(1 - p^{2l_2})(1 - p^{2l_1-l_2-l_3})}{(1 + p^{2l_2} \cos m\pi)^2}}\right). \] (71)
It is interesting to note that
\[
D^-(\rho_{l|lc}) = D^+(\rho_{lc|l}).
\]
(71)
It is clear that for \( l_1 = l_2 \), the quantum discord is symmetric, i.e. \( D^-(\rho_{ll}) = D^+(\rho_{ll}) \). Using equation (69), one obtains the following conservation relations
\[
\begin{align*}
D^-(\rho_{ljl_2}) + D^-(\rho_{jl_2l}) &= E_{ji,jl} + E_{ji,jl}, \\
D^-(\rho_{ljl_2}) + D^+(\rho_{jl_2l}) &= E_{ji,jl} + E_{ji,jl}, \\
D^-(\rho_{ljl_2}) + D^-(\rho_{jl_2l}) &= E_{ji,jl} + E_{ji,jl}. \\
\end{align*}
\]
(72)
Similar conservation relations hold for the measures of quantum discord given by (70). They can be easily derived from the relation (71). Using the conservation relations (72), we have
\[
D^-(\rho_{ljl_2}) + D^-(\rho_{jl_2l}) + D^+(\rho_{jl_2l}) = E_{ji,jl} + E_{ji,jl} + E_{ji,jl_2}.
\]
This reflects that the sum of pairwise quantum discord for all bipartite mixed states coincides with the sum of entanglement of formation. It must be noted that the conservation relations of type (72) involving entanglement of formation and quantum discord were first derived in [38].

7.2. Multipartite quantum correlations

Based on the asymmetric definition of quantum discord, two interesting quantities were defined by Fanchini et al [52]. In our context, they are written as
\[
\Delta_{l_1|l_2}^+ = \frac{1}{2}(D^+(\rho_{l_1l_2}) - D^+(\rho_{l_2l_1})),
\]
(73)
and
\[
\Delta_{l_1|l_2}^- = \frac{1}{2}(D^-(\rho_{l_1l_2}) - D^-(\rho_{l_2l_1})).
\]
(74)
The sum \( \Delta_{l_1|l_2}^+ \) is the average of locally inaccessible information when the measurements are performed on the sub-systems \( l_1 \) and \( l_2 \). It quantifies the disturbance caused by any local measurement. The difference \( \Delta_{l_1|l_2}^- \) is the balance of locally inaccessible information and quantifies the asymmetry between the sub-systems in responding to the measurement disturbance. Using equation (69), it is easy to verify that the average and the balance of quantum discord satisfy the following identities
\[
\Delta_{j_1|j_2}^+ + \Delta_{j_1|j_3}^+ + \Delta_{j_2|j_3}^+ = E_{j_1,j_3} + E_{j_1,j_2} + E_{j_2,j_3},
\]
(75)
and
\[
\Delta_{j_1|j_2}^- + \Delta_{j_1|j_3}^- + \Delta_{j_2|j_3}^- = 0.
\]
(76)
Using the main definition (14), it is interesting to note that the total amount of quantum discord present in the state (40) can be simply written in terms of the average of locally inaccessible information (73). Indeed, we have
\[
D(j_1, j_2, j_3) = \frac{1}{2}\left(\Delta_{j_1|j_2}^+ + \Delta_{j_1|j_3}^+ + \Delta_{j_2|j_3}^+ + \Delta_{j_1,(j_2,j_3)}^+ + \Delta_{j_2,(j_1,j_3)}^+ + \Delta_{j_3,(j_1,j_2)}^+\right)
\]
(77)
where the quantity \( \Delta_{j_1,(k,k)}^+ \) coincides with the entanglement of formation \( E(\rho_{j_1|k,k}) \) given by (52). Furthermore, using the conservation relation (75), one gets
\[
D(j_1, j_2, j_3) = E(j_1, j_2, j_3)
\]
(78)
where \( E(j_1, j_2, j_3) \) is given by (56). This result coincides with that obtained in [38]. It reflects that the sum of quantum discord present in all possible bi-partitions is exactly the total amount of bipartite entanglement of formation in the entire system. Since for a spin-\( j \) coherent state there are different tripartite splitting possibilities denoted here by \( (j_1, j_2, j_3) \) such that \( j_1 + j_2 + j_3 = j \), it seems natural to compare the total amount
of multipartite correlations in each splitting scheme. To illustrate this, we consider a situation where \( j = 3 \). The tripartite quantum discord \( D(j_1, j_2, j_3) \) (78) is totally symmetric in \( j_1, j_2 \) and \( j_3 \). Thus, for \( j = 3 \), three inequivalent splitting schemes are of special interest. They correspond to \((j_1 = 1, j_2 = 1, j_3 = 1), (j_1 = \frac{1}{2}, j_2 = \frac{1}{2}, j_3 = 2)\) and \((j_1 = \frac{1}{2}, j_2 = 1, j_3 = \frac{3}{2})\). In figures 6 and 7, we plot the quantity \( D(j_1, j_2, j_3) \) as a function of the overlap \( p \) for each case.

From figures 6 and 7, one can see that the tripartite quantum discord \( D(j_1 = 1, j_2 = 1, j_3 = 1), D(j_1 = \frac{1}{2}, j_2 = \frac{1}{2}, j_3 = 2) \) and \( D(j_1 = \frac{1}{2}, j_2 = 1, j_3 = \frac{3}{2}) \) are all equal for \( p \simeq 0.5 \). Note also that for \( p \leq 0.5 \), the sum of all occurrences pairwise quantum discord obtained in the splitting scheme \((j = 3) \rightarrow (j_1 = 1, j_2 = 1, j_3 = 1)\) is minimal in comparison with the two others. This behavior changes when \( p \geq 0.5 \) and the quantity \( D(j_1 = 1, j_2 = 1, j_3 = 1) \) becomes maximal. For even spin coherent states \((m = 0)\), the measure of tripartite quantum correlations vanishes when \( p \rightarrow 1 \), as expected (see equations (56) and (78)).
7.3. Monogamy of quantum discord

In the pure tripartite state (40), the quantum discord satisfies the monogamy relation when the following condition

$$D^{-}(\rho_{j_{1}j_{2}}) + D^{-}(\rho_{j_{1}j_{3}}) \leq D^{-}(\rho_{j_{1}|j_{2}j_{3}})$$

is satisfied. As for entanglement of formation, we shall focus on some special cases to determine the positivity of the function

$$\Delta D = D^{-}(\rho_{j_{1}|j_{2}j_{3}}) - D^{-}(\rho_{j_{1}j_{2}}) - D^{-}(\rho_{j_{1}j_{3}})$$

when the overlap varies from 0 to 1. We first consider the situation where \((j_{1} = \frac{1}{2}, j_{2} = \frac{1}{2}, j_{3} = \frac{1}{2})\). The function \(\Delta D\) is plotted in figure 8. In this case the quantum discord is monogamous for the even spin coherent state. However, for the odd spin coherent state, the monogamy relation is satisfied only when \(p \leq 0.8\). We also consider the situations where \((j_{1} = 1, j_{2} = \frac{1}{2}, j_{3} = \frac{1}{2})\), \((j_{1} = \frac{1}{2}, j_{2} = 1, j_{3} = \frac{1}{2})\), and \((j_{1} = \frac{1}{2}, j_{2} = \frac{1}{2}, j_{3} = 1)\), are associated with the spin \(j = 2\). The behavior of the function \(\Delta D\) for even coherent states \((m = 0)\) is reported in figure 9. Clearly, the monogamy relation is satisfied.
Figure 10. The function $\Delta D$ versus the overlap $p$ when $(j_1 = 1, j_2 = \frac{1}{2}, j_3 = \frac{1}{2})$ and $(j_1 = \frac{1}{2}, j_2 = \frac{1}{2}, j_3 = 1)$ for $m = 1$.

representing the function $\Delta D$ for the odd case ($m = 1$), reveals that the quantum discord ceases to be monogamous for $p$ approaching unity. Remark that in figures 9 and 10, we have $\Delta D(\frac{1}{2}, 1, \frac{1}{2}) = \Delta D(\frac{1}{2}, \frac{1}{2}, 1)$ as expected. It is interesting to note that the behavior of $\Delta D$ versus $p$ is identical to the ones obtained for $\Delta E$ in the previous section (figures 3, 4 and 5). This is essentially due to the conservation relations between quantum discord and entanglement of formation (72) [38]. Finally, it is interesting to note that the odd tripartite coherent states ($m = 1$) interpolate continuously between the three-qubit GHZ$_3$ states when $p \to 0$ and W$_3$ states for $p \to 1$. It follows from figure 10 that the GHZ$_3$ states follow monogamy and the W$_3$ states do not.

8. Concluding remarks

The main motivation for investigating the multipartite quantum correlations in even and odd coherent states is the decomposition (or factorization) property given by (23). In this way, a single $j$-spin coherent state is viewed as comprising two, three or, in general, $2j$ qubits. Moreover, this decomposition property allows us to investigate the pairwise quantum correlations in a single spin coherent state. In this paper, we mainly focused on bipartite and tripartite decomposition. For each case, the spin coherent states were mapped to a two- or three-qubit system. We have considered the multipartite quantum correlation in even and odd spin coherent states measured by entanglement of formation and quantum discord. We defined the total amount of quantum correlation in spin coherent states, viewed as a multi-components system, as the sum of all pairwise quantum correlations. We explicitly derived the expressions of multipartite entanglement of formation and quantum discord for even and odd spin coherent states. The sum of all possible occurrences of pairwise entanglement of formation in an even or odd spin coherent state, viewed as a pure tripartite state, is explicitly derived and coincides with the sum of occurrences of pairwise quantum discord of all possible bi-partitions, as shown in [38]. This peculiar result originates from the conservation relation between the entanglement of formation and quantum discord given by (72). We also examined the monogamy relation of the entanglement of formation and quantum discord. Remarkably, in the simplest cases that we considered, these two measures are monogamous for the even spin coherent state, in contrast to the odd case, where the monogamy relation is violated for states involving an
overlap $\rho$ approaching unity. In particular, we have shown that the entanglement of formation and quantum discord follow the monogamy relation in the three-qubit Greenberger–Horne–Zeilinger states in contrast to the three-qubit states of $W$ type. As concerns continuation of the present work, it will be an important issue to extend the present approach to other coherent and squeezed states. Further thought in this direction might be worthwhile in investigating genuine multipartite quantum correlations. Finally, it is interesting to examine the relation between the spin coherent state factorization (23) and the tensor product decomposition of two fermions developed in [53].

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