Finite-Sample Coverage Errors of the Cheap Bootstrap With Minimal Resampling Effort

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Abstract

The bootstrap is a popular data-driven method to quantify statistical uncertainty, but for modern high-dimensional problems, it could suffer from huge computational costs due to the need to repeatedly generate resamples and refit models. Recent work has shown that it is possible to reduce the resampling effort dramatically, even down to one Monte Carlo replication, for constructing asymptotically valid confidence intervals. We derive finite-sample coverage error bounds for these “cheap” bootstrap confidence intervals that shed light on their behaviors for large-scale problems where the curb of resampling effort is important. Our results show that the cheap bootstrap using a small number of resamples has comparable coverages as traditional bootstraps using infinite resamples, even when the dimension grows closely with the sample size. We validate our theoretical results and compare the performances of the cheap bootstrap with other benchmarks via a range of experiments.

1 Introduction

The bootstrap is a widely used method for statistical inference, notably confidence interval construction and uncertainty quantification (e.g. Efron and Tibshirani [1994], Davison and Hinkley [1997], Shao and Tu [2012], Hall and Martin [1988]). Its main idea is to resample data and use the distribution of resample estimates to approximate a sampling distribution. Typically, this approximation requires running many Monte Carlo replications to generate the resamples and refit models. This is affordable for classical small-scale problems, but for modern large-scale problems this repeated fitting could impose tremendous computation concerns. This issue motivates an array of recent works to curb the computation effort, mostly through a “subsampling” perspective that fits models on smaller data sets in the bootstrap process, e.g., Kleiner et al. [2012], Lu et al. [2020], Giordano et al. [2019], Schulam and Saria [2019], Alaa and Van Der Schaar [2020].

In contrast to subsampling, we consider in this paper the reduction in bootstrap computation cost by using fewer number of Monte Carlo replications or resamples. In particular, a recent initial
work Lam [2022] has shown that, to construct statistically valid confidence intervals, it is possible to reduce the resampling effort dramatically from standard bootstraps, even down to one Monte Carlo replication. The rough idea of this approach, which Lam [2022] called a “cheap” bootstrap, is to exploit the approximate independence among the resample estimates and original estimate instead of only the distributional closeness. This argument is asymptotic that unfortunately lacks finite-sample and dimensionality consideration. The latter is important for applications in modern machine learning and computational simulation problems, because it is precisely the scale of these problems that necessitates the saving of Monte Carlo costs.

Our main contribution in this paper is to develop finite-sample analyses for the cheap bootstrap procedure. Our results are three-fold. First, we present general bounds on the aggregation of \( B \) resample estimates, where \( B \) is a low number, that allow us to compare the cheap bootstrap coverage errors with more traditional quantile-based bootstrap methods including the basic and percentile bootstraps (e.g., Davison and Hinkley [1997] Section 5.2-5.3). These latter approaches require a large amount of resamples to approximate the sampling distribution. Our analyses show that the cheap bootstrap using as low as one resample achieves similar coverage error bounds as these approaches using infinite resamples, thus suggesting the capability to dramatically reduce computation cost.

Second, we specialize our bounds to the function-of-mean model that is customary in the high-dimensional Berry-Esseen and central limit theorem (CLT) literature (Pinelis and Molzon [2016], Zhilova [2020]). We derive bounds explicit on problem parameters, including the dimension \( p \), the sample size \( n \), and the number of resamples \( B \). These bounds reveal vanishing coverage errors for the cheap bootstrap when \( p = o(n) \), for any given \( B \geq 1 \). Note that the function-of-mean model does not capture all interesting problems, but it has been commonly used – and in fact, appears the only model used in deriving finite-sample CLT errors for technicality reasons. Our bounds shed light that, at least for this wide class of models, using a small number of resamples achieves a good coverage even in a dimension \( p \) growing closely with \( n \). Third, we also specialize our bounds to linear functions with weaker tail conditions, which have orders independent of \( p \) under certain conditions on the \( L_p \) norm or Orlicz norm of the linearly scaled random variable.

In addition to theoretical bounds, we test the cheap bootstrap and compare with other benchmarks on large-scale problems, including high-dimensional linear regression, high-dimensional logistic regression, computational simulation modeling, and a real-world data set RCV1-v2 (Lewis et al. [2004]). To give a sense of the comparisons, we briefly illustrate here the general conclusion observed in our experiments: Figure 1 (a) shows the coverage probabilities of 95%-level confidence intervals for three regression coefficients with corresponding true values 0, 2, −1 in a 9000-dimensional linear regression (in Section 4). We see the cheap bootstrap coverage probabilities are close to the nominal level 95% even with one resample, but the basic and percentile bootstraps only attain around 80%
coverage with ten resamples. In this example, one Monte Carlo replication to obtain each resample estimate takes around 4 minutes in the virtual machine e2-highmem-2 in Google Cloud Platform. Therefore, the cheap bootstrap only requires 4 minutes to obtain a statistically valid interval, but the standard bootstrap methods are still far from the nominal coverage even after more than a 40-minute run. Figure 1 (b) shows the average interval widths. This reveals the price of a wider interval for the cheap bootstrap when the Monte Carlo budget is very small, but considering the low coverages in the other two methods and the fast decay of the cheap bootstrap width for the first few number of resamples, such a price appears secondary.

Figure 1: Empirical coverage probabilities and confidence interval widths for different number of resamples in a linear regression

**Note**. For a random vector $X$, we write $X^k$ as the tensor power $X^\otimes k$. The vector norm is taken as the usual Euclidean norm. The matrix and tensor norm is taken as the operator norm. For a square matrix $M$, $tr(M)$ denotes the trace of $M$. $I_{p\times p}$ denotes the identity matrix in $\mathbb{R}^{p\times p}$ and $1_p$ denotes the vector in $\mathbb{R}^p$ whose components are all 1. $\Phi$ denotes the cumulative distribution function of the standard normal. $C^2(\mathbb{R}^p)$ denotes the twice continuously differentiable function on $\mathbb{R}^p$. Throughout the whole paper, we use $C > 0$ (without subscripts) to denote a universal constant which may vary each time it appears. We use $C_1, C_2, \ldots$ to denote constants that could depend on other parameters and we will clarify their dependence when using them.

### 2 Background on Cheap Bootstrap

We briefly review the cheap bootstrap method proposed in Lam [2022]. Suppose we are interested in estimating a target statistical quantity $\psi := \psi(P_X)$ where $\psi(\cdot) : \mathcal{P} \mapsto \mathbb{R}$ is a functional defined on the probability measure space $\mathcal{P}$. Given i.i.d. data $X_1, \ldots, X_n \in \mathbb{R}^p$ following the unknown
distribution $P_X$, we denote the empirical distribution as $\hat{P}_{X,n}(\cdot) := (1/n) \sum_{i=1}^{n} I(X_i \in \cdot)$. A natural point estimator is $\hat{\psi}_n := \psi(\hat{P}_{X,n})$.

The cheap bootstrap confidence interval for $\psi$ is constructed as follows. Conditional on $X_1, \ldots, X_n$, we independently resample, i.e., sample with replacement, the data for $B$ times to obtain resamples $\{X_1^{b}, \ldots, X_n^{b}\}$, $b = 1, \ldots, B$. Denoting $\hat{P}_{X,n}^{b}$ as the resample empirical distributions, we construct $B$ resample estimates $\hat{\psi}_n^{b} := \psi(\hat{P}_{X,n}^{b})$. A $(1 - \alpha)$-level confidence interval is then given by

$$
\left[ \hat{\psi}_n - t_{B,1-\alpha/2}S_{n,B}, \hat{\psi}_n + t_{B,1-\alpha/2}S_{n,B} \right],
$$

where $S_{n,B}^2 = (1/B) \sum_{b=1}^{B} (\hat{\psi}_n^{b} - \hat{\psi}_n)^2$, and $t_{B,1-\alpha/2}$ is the $(1-\alpha/2)$-th quantile of $t_B$, the $t$-distribution with degree of freedom $B$. The quantity $S_{n,B}^2$ resembles the sample variance of the resample estimates $\hat{\psi}_n^{b}$'s, in the sense that as $B \to \infty$, $S_{n,B}^2$ approaches the bootstrap variance $\text{Var}_*(\hat{\psi}_n^*)$ (where $\text{Var}_*(\cdot)$ denotes the variance of a resample conditional on the data). In this way, (1) reduces to the normality interval with a “plug-in” estimator of the standard error term when $B$ and $n$ are both large. However, the key here is that $B$ does not need to be large. In fact, Lam [2022] Theorem 1 shows that, under conditions on par with standard bootstrap methods, (1) is an asymptotically exact $(1 - \alpha)$-level confidence interval for any fixed $B \geq 1$, i.e.,

$$
P(\psi \in [\hat{\psi}_n - t_{B,1-\alpha/2}S_{n,B}, \hat{\psi}_n + t_{B,1-\alpha/2}S_{n,B}]) \to 1 - \alpha, \quad \text{as } n \to \infty \tag{2}
$$

where $P$ is the probability with respect to both the data and the randomness in the resampling process.

We briefly explain the asymptotic argument that gives rise to (2), and along the way we contrast with standard bootstrap methods. Under suitable conditions, the sampling distribution of an estimate $\hat{\psi}_n$ and the distribution of a resample estimate $\hat{\psi}_n^*$ are approximately equal. More formally, they are equal in the asymptotic sense of two CLTs $\sqrt{n}(\hat{\psi}_n - \psi) \overset{d}{\to} N(0, \sigma^2)$ for some $\sigma^2 > 0$, and $\sqrt{n}(\hat{\psi}_n^* - \hat{\psi}_n) \overset{d}{\to} N(0, \sigma^2)$ (conditional on $X_1, \ldots, X_n$ in probability) for the same $\sigma^2$. Note that the distribution of $\hat{\psi}_n - \psi$ allows us to construct a confidence interval for $\psi$. However, as this distribution is typically unknown, the standard bootstrap idea approximates it by the distribution of $\hat{\psi}_n^* - \hat{\psi}_n$ which can be readily simulated. Suppose we use the $\alpha/2$ and $(1 - \alpha/2)$-th quantiles of $\hat{\psi}_n^* - \hat{\psi}_n$, called $q_{\alpha/2}$ and $q_{1-\alpha/2}$, to construct $[\hat{\psi}_n - q_{1-\alpha/2}, \hat{\psi}_n - q_{\alpha/2}]$ as a $(1 - \alpha)$-level confidence interval. Then this is known as the basic bootstrap (Davison and Hinkley [1997] Section 5.2). Alternatively, suppose we obtain the $\alpha/2$ and $(1 - \alpha/2)$-th quantiles of $\hat{\psi}_n^*$, say $q_{\alpha/2}$ and $q_{1-\alpha/2}$ again, and use $[q_{\alpha/2}, q_{1-\alpha/2}]$. Then this becomes the percentile bootstrap (Davison and Hinkley [1997] Section 5.3), whose simplicity is justified by further exploiting the symmetry of the asymptotic normal distribution.

On the other hand, the cheap bootstrap interval (1) utilizes the following joint convergence

$$
\sqrt{n}(\hat{\psi}_n - \psi, \hat{\psi}_n^* - \hat{\psi}_n, \ldots, \hat{\psi}_n^{B} - \hat{\psi}_n) \overset{d}{\to} (\sigma Z_0, \sigma Z_1, \ldots, \sigma Z_B), \quad \text{as } n \to \infty \tag{3}
$$
where $Z_b, b = 0, \ldots, B$ are i.i.d. standard normal, a result can be derived by combining the two aforementioned CLTs. From (3), we can establish the convergence of a pivotal $t$-statistic $(\hat{\psi}_n - \psi)/S_{n,B} \xrightarrow{d} t_B$ which gives (2). The above shows that, with $B$ fixed as small as 1, (1) already offers a coverage close to the nominal level as $n \to \infty$. In this argument, the approximation accuracy of $(\hat{\psi}_n - \psi)/S_{n,B}$ by the $t_B$ random variable is crucial and will be the main focus in our finite-sample analysis.

3 Finite-Sample Coverage Error Bounds

We describe our finite-sample results at three levels, first under general assumptions (Section 3.1), then more explicit bounds under the function-of-mean model and sub-Gaussianity of $X$ (Section 3.2), finally bounds for a linear function under weaker tail assumptions on $X$ (Section 3.3).

3.1 General Finite-Sample Bounds

We have the following finite-sample bound for the cheap bootstrap:

**Theorem 1.** Suppose we have the finite-sample accuracy for the estimator $\hat{\psi}_n$

$$\sup_{x \in \mathbb{R}} \left| P(\sqrt{n}(\hat{\psi}_n - \psi) \leq x) - \Phi(x/\sigma) \right| \leq \mathcal{E}_1,$$

and with probability at least $1 - \beta$ we have the finite-sample accuracy for the bootstrap estimator $\hat{\psi}_n^*$

$$\sup_{x \in \mathbb{R}} \left| P^*(\sqrt{n}(\hat{\psi}_n^* - \hat{\psi}_n) \leq x) - \Phi(x/\sigma) \right| \leq \mathcal{E}_2,$$

where $\sigma > 0$, $\mathcal{E}_1$ and $\mathcal{E}_2$ are deterministic quantities, and $P^*$ denotes the probability on a resample conditional on the data. Then the cheap bootstrap coverage error satisfies, for any $B \geq 1$,

$$\left| P(\psi \in [\hat{\psi}_n - t_{B,1-\alpha/2}S_{n,B}, \hat{\psi}_n + t_{B,1-\alpha/2}S_{n,B}]) - (1 - \alpha) \right| \leq 2\mathcal{E}_1 + 2BE_2 + \beta. \quad (6)$$

Condition (4) is a Berry-Esseen bound (Bentkus [2003], Pinelis and Molzon [2016]) that gauges the normal approximation for the original estimate $\hat{\psi}_n$. Condition (5) manifests a similar normal approximation for the resample estimate $\hat{\psi}_n^*$, and has been a focus in the high-dimensional CLT literature (Zhilova [2020], Lopes [2020], Chernozhukov et al. [2020]). Both conditions (in their asymptotic form) are commonly used to establish the validity of standard bootstrap methods. Theorem 1 shows that, under these conditions, the coverage error of the cheap bootstrap with any $B \geq 1$ can be controlled. Note that Theorem 1 is very general in the sense that there is no direct assumption applied on the form of $\psi(\cdot)$ – All we assume is approximate normality in the sense of (4) and (5). Due to technical delicacies, in the bootstrap literature, finite-sample or higher-order coverage errors are typically obtainable only with specific models (Hall [2013], Zhilova [2020], Lopes [2020], Chernozhukov
et al. [2020]), the most popular being the function-of-mean model (see Section 3.2) or even simply the sample mean. In contrast, the bound in Theorem 1 that concludes the sufficiency in using a very small $B$ is a general statement that does not depend on the delicacies of $\psi(\cdot)$. The detailed proof of Theorem 1 is in Appendix C (and so are the proofs of all other theorems). Below we give a sketch of the main argument.

**Proof sketch of Theorem 1.** Step 1: We write the coverage probability as the expected value (with respect to data) of a multiple integral with respect to the distributions of $\sqrt{n}(\hat{\psi}_n - \hat{\psi}_n)$ (denoted by $Q^*$, conditional on data), i.e.,

$$
P(\psi \in [\hat{\psi}_n - t_{B,1-\alpha/2}S_n,B, \hat{\psi}_n + t_{B,1-\alpha/2}S_n,B])
= P \left( |\sqrt{n}(\hat{\psi}_n - \psi)| \leq t_{B,1-\alpha/2} \sqrt{\sum_{b=1}^{B} (\sqrt{n}(\hat{\psi}_n - \hat{\psi}_n))}^{2}/B \right)
= E \left[ \int \cdots \int |\sqrt{n}(\hat{\psi}_n - \psi)| \leq t_{B,1-\alpha/2} \sqrt{\sum_{b=1}^{B} z_b^2}/B dQ^*(z_B) \cdots dQ^*(z_1) \right]. \tag{7}
$$

Step 2: Suppose (5) happens and denote this event by $A$ which satisfies $P(A^c) \leq \beta$. For each $b = 1, \ldots, B$, given all other $z_{b'}, b' \neq b$, the integration region is of the form $z_b \in (-\infty, -q] \cup [q, \infty)$ for some $q \geq 0$. Then we can replace the distribution $Q^*$ by the distribution of $N(0, \sigma^2)$ (denoted by $P_0$) with controlled error given in (5) and obtain

$$
E \left[ \int \cdots \int |\sqrt{n}(\hat{\psi}_n - \psi)| \leq t_{B,1-\alpha/2} \sqrt{\sum_{b=1}^{B} z_b^2}/B dQ^*(z_B) \cdots dQ^*(z_1) \right]
= E \left[ \int \cdots \int |\sqrt{n}(\hat{\psi}_n - \psi)| \leq t_{B,1-\alpha/2} \sqrt{\sum_{b=1}^{B} z_b^2}/B dP_0(z_B) \cdots dP_0(z_1) \right] + R_1, \tag{8}
$$

where $|R_1| \leq 2BE_2 + \beta$ accounts for the error from (5) and the small probability event $A^c$.

Step 3: Following the same logic in Step 2 and noticing that the integration region for $\sqrt{n}(\hat{\psi}_n - \psi)$ is $[-q, q]$ for some $q \geq 0$, we can also replace the distribution of $\sqrt{n}(\hat{\psi}_n - \psi)$ by the distribution $P_0$ with controlled error $|R_2| \leq 2E_1$ according to (4):

$$
E \left[ \int \cdots \int |\sqrt{n}(\hat{\psi}_n - \psi)| \leq t_{B,1-\alpha/2} \sqrt{\sum_{b=1}^{B} z_b^2}/B dP_0(z_B) \cdots dP_0(z_1) \right]
= \int \int \cdots \int |z_0| \leq t_{B,1-\alpha/2} \sqrt{\sum_{b=1}^{B} z_b^2}/B dP_0(z_B) \cdots dP_0(z_1) \cdots dP_0(z_0) + R_2 = 1 - \alpha + R_2. \tag{9}
$$

Step 4: Plugging (8) and (9) back into (7), we can express the coverage probability as a sum of the nominal level and the remainder term:

$$
P(\psi \in [\hat{\psi}_n - t_{B,1-\alpha/2}S_n,B, \hat{\psi}_n + t_{B,1-\alpha/2}S_n,B]) = 1 - \alpha + R_1 + R_2
$$

with error $|R_1 + R_2| \leq 2E_1 + 2BE_2 + \beta$. This gives our conclusion. \qed
Theorem 1 is designed to work well for small $B$ (our target scenario), but deteriorates when $B$ grows. However, in the latter case, we can strengthen the bound to cover the large-$B$ regime with additional conditions on the variance estimator (see Appendix A).

We compare with standard basic and percentile bootstraps using $B = \infty$. Below is a generalization of Zhilova [2020] which focuses only on the basic bootstrap under the function-of-mean model.

**Theorem 2.** Suppose the conditions in Theorem 1 hold. If $q_{\alpha/2}$, $q_{1-\alpha/2}$ are the $\alpha/2$-th and $(1-\alpha/2)$-th quantiles of $\hat{\psi}_n^* - \hat{\psi}_n$ respectively given $X_1, \ldots, X_n$, then a finite-sample bound on the basic bootstrap coverage error is given by

$$|P(\hat{\psi}_n - q_{1-\alpha/2} \leq \psi \leq \hat{\psi}_n - q_{\alpha/2}) - (1-\alpha)| \leq 2\mathcal{E}_1 + 2\mathcal{E}_2 + 2\beta.$$  \hspace{1cm} (10)

If $q_{\alpha/2}$, $q_{1-\alpha/2}$ are the $\alpha/2$-th and $(1-\alpha/2)$-th quantiles of $\hat{\psi}_n^*$ respectively given $X_1, \ldots, X_n$, then a finite-sample bound on the percentile bootstrap coverage error is given by

$$|P(q_{\alpha/2} \leq \psi \leq q_{1-\alpha/2}) - (1-\alpha)| \leq 2\mathcal{E}_1 + 2\mathcal{E}_2 + 2\beta.$$  \hspace{1cm} (11)

In view of Theorems 1 and 2, the cheap bootstrap with any fixed $B$ can achieve the same order of coverage error bound as the basic and percentile bootstraps with $B = \infty$, in the sense that

$$(1/2) \text{EB}_{\text{Quantile}} \leq \text{EB}_{\text{Cheap}} \leq B \text{EB}_{\text{Quantile}},$$ \hspace{1cm} (12)

where $\text{EB}_{\text{Cheap}}$ is the RHS error bound of (6) and $\text{EB}_{\text{Quantile}}$ is that of (10) or (11). This shows that, to attain a good coverage that is on par with standard basic/percentile bootstraps, it suffices to use the cheap bootstrap with a small $B$ which could save computation dramatically.

Besides coverage, another important quality of confidence interval is its width. To this end, note that for any fixed $B$, (5) ensures that $\sqrt{n}S_{n,B} \Rightarrow \sigma \sqrt{\chi^2_B / B}$ (unconditionally as $n \to \infty$ with proper model assumptions). Therefore, the half-width of (1) is approximately $t_{B,1-\alpha/2} \sigma \sqrt{\chi^2_B / (nB)}$ with expected value

$$E\left[t_{B,1-\alpha/2} \sigma \sqrt{\chi^2_B / nB}\right] = t_{B,1-\alpha/2} \sigma \sqrt{\frac{2}{Bn} \frac{\Gamma((B + 1)/2)}{\Gamma(B/2)}},$$ \hspace{1cm} (13)

where $\Gamma(\cdot)$ is the gamma function. Since the dimensional impact is hidden in $\sigma$ which is a common factor in the expected width as $B$ varies, we can see $p$ does not affect the relative width behavior as $B$ changes. In particular, from (13) we can readily see that the inflation of the expected width relative to the case $B = \infty$ is 417.3\% for $B = 1$, and dramatically reduces to 94.6\%, 24.8\% and 10.9\% for $B = 2, 5, 10$, thus giving an interval with both correct coverage and short width using a small computation budget $B$.

In the next sections, we will apply Theorem 1 to obtain explicit bounds for specific high-dimensional models. Here, in relation to (12), we briefly comment that the order of the coverage
error bounds for these models is of order $1/\sqrt{n}$, both for the cheap bootstrap (which we will derive) and state-of-the-art high-dimensional bootstrap. This is in contrast to the typical $1/n$ coverage error in two-sided bootstrap confidence intervals in low dimension (see Hall [2013] Section 3.5 for quantile-based bootstraps and Lam [2022] Section 3.2 for cheap bootstrap).

### 3.2 Function-Of-Mean Models

We now specialize to the function-of-mean model $\psi = g(\mu)$ for a mean vector $\mu = E[X] \in \mathbb{R}^p$ and function $g : \mathbb{R}^p \mapsto \mathbb{R}$, which allows us to construct more explicit bounds. The original estimate $\hat{\psi}_n$ and resample estimate $\hat{\psi}^*_n$ are now given by $g(\bar{X}_n)$ and $g(\bar{X}^*_n)$ respectively, where $\bar{X}_n$ denotes the sample mean of data and $\bar{X}^*_n$ denotes the resample mean of $X^*_1, \ldots, X^*_n$. We assume:

**Assumption 1.** The function $g(x) \in C^2(\mathbb{R}^p)$ has Hessian matrix $H_g(x)$ with uniformly bounded eigenvalues, that is, $\exists$ a constant $C_{H_g} > 0$ s.t. $\sup_{x \in \mathbb{R}^p} |a^\top H_g(x) a| \leq C_{H_g} |a|^2, \forall a \in \mathbb{R}^p$.

**Assumption 2.** $X$ is sub-Gaussian, i.e., there is a constant $\tau^2 > 0$ s.t. $E[\exp(a^\top (X - \mu))] \leq \exp(||a||^2 \tau^2/2), \forall a \in \mathbb{R}^p$. Furthermore, there exists an orthogonal matrix $T$ such that $TX$ has independent components and $X$ has a positive definite covariance matrix $\Sigma$ with the smallest eigenvalue $\lambda^{\Sigma} > 0$.

Based on Theorem 1, we derive the following explicit bound:

**Theorem 3.** Suppose the function $g$ satisfies Assumption 1 and random vector $X$ satisfies Assumption 2. Moreover, assume $||\nabla g(\mu)|| > C_{\nabla g} \sqrt{p}$ for some constant $C_{\nabla g} > 0$. Then we have

$$
\left| P(g(\mu) \in [g(\bar{X}_n) - t_{B,1-\alpha/2}S_{n,B}; g(\bar{X}_n) + t_{B,1-\alpha/2}S_{n,B}]) - (1 - \alpha) \right|
$$

$$
\leq \frac{6}{n} + BC \left( \frac{m_{31}}{\sqrt{n}\sigma^3} + \frac{C_{H_g} m_{31}^{1/3} \text{tr}(\Sigma)}{\sqrt{n}\sigma^2} + \frac{C_{H_g} m_{32}^{2/3}}{n^{5/6}\sigma} + \frac{C_{H_g} m_{31}^{1/3} m_{32}^{2/3}}{n\sigma^2} \right)
$$

$$
+ \frac{C_{H_g} \tau^2}{C_{\nabla g} \sqrt{\lambda^{\Sigma}}} \left( 1 + \log n \right) \frac{C_{H_g} \tau^2}{C_{\nabla g} \sqrt{\lambda^{\Sigma}}} \left( 1 + \log n \right)^{3/2} \frac{1}{\sqrt{n}}
$$

$$
+ \frac{\tau^4 \sqrt{p}}{\lambda^{\Sigma}} \left( 1 + \log n \right)^{1/2} + \frac{\tau^2 \sqrt{p}}{\lambda^{\Sigma}} \left( 1 + \log n \right)^{1/2} + \frac{\tau^3 \sqrt{p}}{\lambda^{\Sigma}} \left( 1 + \log n \right)^{1/2} \frac{\log n + \log p}{\sqrt{n}}
$$

$$
+ \frac{\tau^4 (\log n)^{3/2}}{\lambda^{\Sigma} \sqrt{n}} + \frac{\tau^2 (\log n)^{3/2}}{\lambda^{\Sigma} \sqrt{n}} + \frac{\tau^3 \sqrt{p}}{\lambda^{\Sigma} \sqrt{n}} \left( 1 + \log n \right)^{1/2} \frac{\log n + \log p}{\sqrt{n}},
$$

where $m_{31} = E[||\nabla g(\mu)^\top (X - \mu)||^3], m_{32} := E[||X - \mu||^3], \sigma^2 = \nabla g(\mu)^\top \Sigma \nabla g(\mu)$ and $C$ is a universal constant.

Theorem 3 is obtained by tracing the implicit quantities in Theorem 1 for the function-of-mean model, by extracting the dependence on problem parameters in the Berry-Esseen theorems for the
multivariate delta method (Pinelis and Molzon [2016]) and the standard bootstrap (Zhilova [2020]).
Assumption 2 potentially can be generalized to general dependence structure of absolutely continuous
X by using Theorem 4.2 in Zhilova [2020] to specialize the bootstrap accuracy (5). However, this
appears to require a generalization of the Hanson-Wright inequality to dependent-component random
vectors, which is still open to our best knowledge. The bound in Theorem 3 can be simplified with
reasonable assumptions on the involved quantities.

Corollary 1. Suppose the conditions in Theorem 3 hold. Moreover, suppose that the operator norm of the third order tensor
\[ E_3 \]
problem dimension grows slower than \(|\Sigma| = \Theta(1)\). Then as \( p, n \to \infty \),
\[
|P(g(\mu) \in [g(\bar{X}_n) - t_{B,1-\alpha/2}S_{n,B}, g(\bar{X}_n) + t_{B,1-\alpha/2}S_{n,B})] - (1 - \alpha)|
\]
\[
= B \times O \left( \left( 1 + \log n \right) \frac{p}{n} + \frac{1}{\sqrt{n}} \left( 1 + \frac{\log n}{p} \right)^{1/2} \right). 
\]

In Corollary 1, the cheap bootstrap coverage error shrinks to 0 as \( n \to \infty \) if \( p = o(n) \), i.e., the
problem dimension grows slower than \( n \) in any arbitrary fashion. Recall that \(|E[(X - \mu)^3]|\)
denotes the operator norm of the third order tensor \( E[(X - \mu)^3] \), and so the assumption \(|E[(X - \mu)^3]| = O(1)\) holds if the components of \( X \) are independent (or slightly weakly dependent). Other order
assumptions in Corollary 1 are natural. An example of the function-of-mean model is \( g(\mu) = |\mu|^2 \);
used also in Zhilova [2020], whose confidence interval becomes a simultaneous confidence region for
the mean vector \( \mu \).

3.3 Linear Functions

We consider a further specialization to linear \( g \) where, instead of sub-Gaussanity of \( X \), we are now
able to use weaker tail conditions. Assume \( g(x) = g_1^T x + g_2 \), where \( g_1 \in \mathbb{R}^p \) and \( g_2 \in \mathbb{R} \) are known.
Then \( g(\bar{X}_n) \) and \( g(\bar{X}_n^*) \) are essentially the sample mean and resample mean of i.i.d. random
variables \( g_1^T X_i + g_2, i = 1, \ldots, n \).

First, we consider the case where \( g_1^T (X - \mu) \) is sub-exponential, i.e., \( |g_1^T (X - \mu)|_{\psi_1} := \inf \{ \lambda > 0 : E[\psi_1(|g_1^T (X - \mu)|/\lambda)] \leq 1 \} < \infty \), where \(| \cdot |_{\psi_1} \) is the Orlicz norm induced by the function
\( \psi_1(x) = e^x - 1 \). Sub-exponential property is a weaker tail condition than sub-Gaussanity; see e.g.
Vershynin [2018] Sections 2.5 and 2.7. Under this condition, we have:

Theorem 4. Suppose \( g \) is a linear function in the form \( g(x) = g_1^T x + g_2 \). Assume that \( \sigma^2 = g_1^T \Sigma g_1 > 0 \) and \( |g_1^T (X - \mu)|_{\psi_1} < \infty \). Then for any \( n \geq 3 \) and for a universal constant \( C \), we have the following
finite-sample bound on the cheap bootstrap coverage error
\[
|P(g(\mu) \in [g(\bar{X}_n) - t_{B,1-\alpha/2}S_{n,B}, g(\bar{X}_n) + t_{B,1-\alpha/2}S_{n,B})] - (1 - \alpha)|
\]
\[
\leq \frac{C}{n} + BC \left( \frac{E[|g_1^T (X - \mu)|^3]}{\sigma^3 \sqrt{n}} + \frac{|g_1^T (X - \mu)|^4_{\psi_1} \log^{11}(n)}{\sigma^4 \sqrt{n}} \right) + CE[|g_1^T (X - \mu)|^3]. 
\]
Note that the bootstrap in Theorem 4 effectively applies on the univariate \( g_1^\top (X - \mu) \). Nonetheless, proving Theorem 4 requires tools from high-dimensional CLT (Lopes [2020], Chernozhukov et al. [2020]), as this turns out to be the only line of work that investigates finite-sample bootstrap errors (for mean estimation). The order of the bound in terms of \( p \) is controlled by \( g_1^\top (X - \mu) \), and so if the latter is well-scaled by its standard deviation \( \sigma \) in the sense that \( E[|g_1^\top (X - \mu)/\sigma|^q] < \infty \) (e.g., \( X \) follows a multivariate normal distribution), then the order is independent of \( p \), which means the error goes to 0 for any \( p \) as long as \( n \to \infty \). However, if the orders of \( E[|g_1^\top (X - \mu)/\sigma|^3] \) and \( |g_1^\top (X - \mu)/\sigma|_4 \) depend on \( p \), then the growing rate of \( p \) must be restricted by \( n \) to ensure the error goes to 0.

Next, we further weaken the tail condition on \( g_1^\top (X - \mu) \). We only assume \( E[|g_1^\top (X - \mu)|^q] < \infty \) for some \( q \geq 4 \). In this case, we have the following:

**Theorem 5.** Suppose \( g \) is a linear function in the form of \( g(x) = g_1^\top x + g_2 \). Assume that \( \sigma^2 = g_1^\top \Sigma g_1 > 0 \) and \( E[|g_1^\top (X - \mu)|^q] < \infty \) for some \( q \geq 4 \). Then for any \( n \geq 3 \), we have the following finite-sample bound on the cheap bootstrap coverage accuracy

\[
|P(g(\mu) \in [g(X_n) - t_{B,1-\alpha/2}S_{n,B}, g(X_n) + t_{B,1-\alpha/2}S_{n,B}]) - (1 - \alpha)|
\leq \frac{BC_1\sqrt{\log n}}{n^{1/2 - 3/(2q)}} \max \left\{ E[|g_1^\top (X - \mu)/\sigma|^q]^{1/4}, \sqrt{E[|g_1^\top (X - \mu)/\sigma|^4]} \right\} + \frac{CE[|g_1^\top (X - \mu)|^3]}{\sigma^3 \sqrt{n}},
\]

where \( C \) is a universal constant and \( C_1 \) is a constant depending only on \( q \).

The implication of Theorem 5 on the choice of \( p \) is similar to Theorem 4 so we omit it. In parallel to the above, explicit finite-sample bounds for standard quantile-based bootstrap methods can also be obtained by means of Theorem 2 under the assumptions in Theorems 3, 4 or 5 (see Appendix A).

### 4 Numerical Experiments

We consider six high-dimensional examples:

**Ellipsoidal estimation:** The estimation target is \( g(\mu) = ||\mu||^2 \), where \( \mu \) is the mean of \( X \in \mathbb{R}^p \) with ground-truth distribution \( N(0.021_p, 0.01I_p \times p) \). Sample size \( n = 10^5 \) and dimension \( p = 2.5 \times 10^4 \).

**Sinusoidal estimation:** The estimation target is \( g(\mu) = \sum_{i=1}^p \sin(\mu_i) \), where \( \mu = (\mu_i)_{i=1}^p \) is the mean of \( X \in \mathbb{R}^p \) with ground-truth distribution \( N(0, 0.01I_p \times p) \). Sample size \( n = 10^5 \) and dimension \( p = 2.5 \times 10^4 \).

**Linear regression:** Consider the true model \( Y = X^\top \beta + \epsilon \), where \( X \in \mathbb{R}^p \) follows \( N(0, 0.01I_p \times p) \) and \( \epsilon \sim N(0, 1) \) independent of \( X \). The first, second and last 1/3 components of \( \beta = (\beta_i)_{i=1}^p \) are \( 0, 2, -1 \) respectively. We estimate \( \beta \) given i.i.d. data \( (X_i, Y_i)_{i=1}^n \) with \( n = 10^5 \) and \( p = 9000 \).

**Logistic regression:** Consider the true model \( Y \in \{0, 1\}, X \in \mathbb{R}^p, P(Y = 1|X) = \exp(X^\top \beta)/(1+ \exp(X^\top \beta)) \), where \( X \sim N(0, 0.01I_p \times p) \). The first 300 components of \( \beta_i \)'s are 1, the second 300 components \(-1\) and the rest 0. As suggested in Sur and Candès [2019], we choose such values of \( \beta_i \)'s to
make sure $\text{Var}(X^T \beta) = 6$ does not increase with $p$ so that $P(Y = 1|X)$ is not trivially equal to 0 or 1 in most cases. We estimate $\beta$ given i.i.d. data $(X_i, Y_i)_{i=1}^n$ with $n = 10^5$ and $p = 9000$.

**Stochastic simulation model:** Consider a stochastic computer communication model used to calculate the steady-state average message delay (Cheng and Holland [1997], Lin et al. [2015], Lam and Qian [2021]; see Appendix C for details). This problem can be cast as computing $\psi(P_1, \ldots, P_p)$ where $\psi$ represents this expensive simulation model (due to the need to run very long time in order to reach steady state) and $P_j$’s denote the input distributions, $p = 13$ in total. The data sizes for observing these 13 input models range from $3 \times 10^4$ to $6 \times 10^4$.

**A real data example:** We run logistic regression on the RCV1-v2 data in Lewis et al. [2004]. This dataset contains $n = 804414$ manually categorized newswire stories with a total of $p = 47236$ features. “Economics” (“ECAT”) is chosen as the +1 label. We target coefficient estimation.

**Setups and comparison benchmarks.** In each example above, our targets are 95%-level confidence intervals for the target parameters. We test four bootstrap confidence intervals: 1) cheap bootstrap (1); 2) basic bootstrap described in Section 2; 3) percentile bootstrap described in Section 2; 4) standard error bootstrap that uses standard normal quantile and standard deviation of $\hat{\psi}_n^{(b)}$’s in lieu of $t_{B,1-\alpha/2}$ and $S_{n,B}$ respectively in (1). For each setup except the real-data example, we run 1000 experimental repetitions, each time generating a new data set from the ground truth distribution and construct the intervals. We report the empirical coverage and average interval width over these repetitions. For examples with more than one target estimation quantity, we further average the coverages and widths over all these targets. We vary the number of resamples $B$ from 1 to 10 in all examples.

**Results and discussions.** Table 1 describes our results, where we report $B = 1, 2, 5, 10$.

**Coverage probability:** The cheap bootstrap performs the best in terms of the coverage probabilities in almost all cases (except the real-data example where we cannot validate and only report the interval widths). In all but two entries, the cheap bootstrap gives the closest coverages to the nominal 95% among all considered bootstrap methods, and in all but three entries the cheap bootstrap coverages are above 95%. In contrast, other approaches are substantially below the nominal level except for very few cases with $B = 10$. For example, in the ellipsoidal estimation, cheap bootstrap coverage probabilities are above 95% for all considered $B$’s, while the highest coverage among other bootstrap methods is 82.1% even for $B = 10$. These observations corroborate with theory since unlike standard bootstrap methods, the cheap bootstrap gives small coverage errors even with very small $B$. Note that when $B = 1$, the entries of other bootstrap methods are all “N.A.” since quantile-based approaches cannot even output two distinct finite numbers using one resample, and standard error bootstrap uses $B−1$ in the denominator of the sample variance.

**Interval width:** Cheap bootstrap intervals are wider than other bootstrap intervals. However, these widths appear to decay very fast for the first few $B$’s. In all examples, they decrease by around
2/3 from $B = 1$ to $B = 2$ and by around 4/5 from $B = 1$ to $B = 10$, hinting a quick enhancement of statistical efficiency as the computation budget increases. Note that while the other bootstrap intervals are shorter, they fall short in attaining the nominal coverage. It is thus reasonable to see the larger widths of the cheap bootstrap intervals which appear to push up the coverages by the right amounts.

5 Discussions and Other Connections

We close this paper by discussing our results in the context of several lines of literature. First, our work is related to bootstrap coverage analysis. The commonest approach is to use the Edgeworth expansion that reveals the asymptotic higher-order terms in the coverage errors; see the comprehensive monograph Hall [2013]. It is only until recently where finite-sample bounds appear, mostly in the high-dimensional CLT literature where the target is sample mean (Chernozhukov et al. [2017], Lopes [2020], Chernozhukov et al. [2020] and references therein). They aim to prove a uniform finite-sample bound of normal approximation of the sample mean over all hyperrectangles. An alternative approach is to use Stein’s method (Fang and Koike [2021]).

Second, within the bootstrap framework, various approaches have been proposed to reduce the Monte Carlo sampling effort by, e.g., variance reduction such as importance sampling (Booth and Do [1993]), or analytic approximation especially when applying iterated bootstraps (Booth and Hall [1994], Lee and Young [1995]). These methods, however, require additional knowledge such as an explicit way to calculate variance, or focus on tail estimation issue. The closest work to the cheap bootstrap idea we utilize in this paper is Hall [1986] who investigates the number of resamples for one-sided basic bootstrap intervals. Nonetheless, Hall [1986] suggests a minimum of 19 for $B$ in a 95%-level interval, obtained via an order-statistics calculation.

Finally, there are approaches outside the bootstrap framework that are computationally light. The so-called batch means method (Glynn and Iglehart [1990], Schmeiser [1982], Schruben [1983]) divides data into batches to construct batched estimates which are then aggregated to generate confidence intervals. This approach has been used commonly in simulation analysis to tackle situations with serial dependence. However, it typically faces a tradeoff between the number of batches and the sample size per batch, which ideally should both be large to obtain good performances. Compared to batching, the cheap bootstrap that we consider in this paper is free of this tradeoff as the resample size can be chosen independent of the number of resamples.

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Table 1: Coverage probabilities (Pro.) and confidence interval widths (Wid.) of the numerical examples. The closest coverage probability to the nominal 95% level among all methods are bolded.

| Example       | $B$ | Cheap Bootstrap | Basic Bootstrap | Percentile Bootstrap | Standard Error Bootstrap |
|---------------|-----|------------------|------------------|-----------------------|--------------------------|
|               |     | Pro. | Wid. | Pro. | Wid. | Pro. | Wid. | Pro. | Wid. | Pro. | Wid. |
| Ellipsoidal   | 1   | 96.0% | 0.069 | N.A. | N.A. | N.A. | N.A. | N.A. | N.A. |
| estimation    | 2   | 97.3% | 0.026 | 32.2% | 0.002 | 5.5% | 0.002 | 55.1% | 0.006 |
|               | 5   | 97.4% | 0.016 | 66.0% | 0.005 | 13.6% | 0.005 | 70.1% | 0.007 |
|               | 10  | 97.5% | 0.014 | 82.1% | 0.006 | 20.8% | 0.006 | 73.6% | 0.008 |
| Sinusoidal    | 1   | 94.4% | 0.958 | N.A. | N.A. | N.A. | N.A. | N.A. | N.A. |
| estimation    | 2   | 95.2% | 0.384 | 29.6% | 0.051 | 35.2% | 0.051 | 63.2% | 0.142 |
|               | 5   | 93.6% | 0.248 | 71.2% | 0.117 | 66.4% | 0.117 | 86.4% | 0.187 |
|               | 10  | 94.4% | 0.222 | 84.0% | 0.156 | 83.2% | 0.156 | 89.6% | 0.196 |
| Linear        | 1   | 95.1% | 0.68  | N.A. | N.A. | N.A. | N.A. | N.A. | N.A. |
| regression    | 2   | 95.1% | 0.256 | 33.5% | 0.038 | 33.5% | 0.038 | 70.2% | 0.105 |
|               | 5   | 95.2% | 0.164 | 67.0% | 0.078 | 67.0% | 0.078 | 88.1% | 0.123 |
|               | 10  | 95.2% | 0.146 | 82.2% | 0.103 | 82.2% | 0.103 | 92.1% | 0.128 |
| Logistic      | 1   | 96.1% | 2.866 | N.A. | N.A. | N.A. | N.A. | N.A. | N.A. |
| regression    | 2   | 96.9% | 1.074 | 39.7% | 0.147 | 31.7% | 0.147 | 73.4% | 0.407 |
|               | 5   | 97.9% | 0.685 | 77.9% | 0.302 | 63.3% | 0.302 | 91.0% | 0.479 |
|               | 10  | 98.4% | 0.609 | 91.9% | 0.400 | 77.7% | 0.400 | 94.6% | 0.496 |
| Stochastic    | 1   | 96.9% | 1.757×10⁻³ | N.A. | N.A. | N.A. | N.A. | N.A. | N.A. |
| simulation    | 2   | 98.8% | 6.417×10⁻⁴ | 21.9% | 6.962×10⁻⁵ | 47.0% | 6.962×10⁻⁵ | 68.7% | 1.930×10⁻⁴ |
|               | 5   | 99.7% | 4.044×10⁻⁴ | 43.2% | 1.428×10⁻⁴ | 90.4% | 1.428×10⁻⁴ | 87.1% | 2.269×10⁻⁴ |
|               | 10  | 100%  | 3.591×10⁻⁴ | 55.6% | 1.915×10⁻⁴ | 99.8% | 1.915×10⁻⁴ | 92.6% | 2.375×10⁻⁴ |
| Real          | 1   | N.A. | 3.594 | N.A. | N.A. | N.A. | N.A. | N.A. | N.A. |
| data          | 2   | N.A. | 1.361 | N.A. | 0.201 | N.A. | 0.201 | N.A. | 0.556 |
|               | 5   | N.A. | 0.877 | N.A. | 0.414 | N.A. | 0.414 | N.A. | 0.658 |
|               | 10  | N.A. | 0.779 | N.A. | 0.547 | N.A. | 0.547 | N.A. | 0.682 |
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### A Additional Theoretical Results

This section provides additional theoretical results. Appendix A.1 establishes an alternative finite-sample bound for the cheap bootstrap that generalizes Theorem 1 to cover the large-$B$ regime. Section A.2 provides finite-sample bounds for standard quantile-based bootstrap methods under the conditions in Sections 3.2 and 3.3.

#### A.1 Further Finite Sample Bound for the Cheap Bootstrap

The following result generalizes Theorem 1 to include both the small and large-$B$ regimes:

**Theorem 6.** Suppose we have the finite sample accuracy for the estimator $\hat{\psi}_n$

$$
\sup_{x \in \mathbb{R}} \left| P(\sqrt{n}(\hat{\psi}_n - \psi) \leq x) - \Phi \left( \frac{x}{\sigma} \right) \right| \leq \mathcal{E}_1,
$$

and with probability at least $1 - \beta$ we have the finite sample accuracy for the bootstrap estimator $\hat{\psi}_n^*$

$$
\sup_{x \in \mathbb{R}} \left| P^*(\sqrt{n}(\hat{\psi}_n^* - \hat{\psi}_n) \leq x) - \Phi \left( \frac{x}{\sigma} \right) \right| \leq \mathcal{E}_2,
$$

where $\sigma > 0$ and $P^*$ denotes the probability on a resample conditional on the data. Further, suppose that the following concentration inequality holds

$$
P \left( \left| \sqrt{\frac{1}{B}} \sum_{b=1}^{B} (\sqrt{n}(\hat{\psi}_n^b - \hat{\psi}_n))^2 - \sigma \right| \geq \mathcal{E}_3 \right) \leq \mathcal{E}_4,
$$
where $\mathcal{E}_3$ is deterministic and $\sigma - \mathcal{E}_3 > 0$. Then we have the following finite sample bound on the cheap bootstrap coverage error

$$|P(\psi \in [\hat{\psi}_n - t_{B,1-\alpha/2}S_{n,B}, \hat{\psi}_n + t_{B,1-\alpha/2}S_{n,B}] - (1-\alpha))|$$

$$\leq \min \left\{ 2\mathcal{E}_1 + 2B\mathcal{E}_2 + \beta, 2\mathcal{E}_1 + 2\mathcal{E}_4 + \sqrt{\frac{2}{\pi}}|t_{B,1-\alpha/2} - z_{1-\alpha/2}| + \sqrt{\frac{2}{\pi} \frac{\mathcal{E}_3}{\sigma}}t_{B,1-\alpha/2} \right\},$$

where $z_{1-\alpha/2}$ is the $(1-\alpha/2)$-quantile of the standard normal.

The finite sample accuracy in Theorem 6 consists of two parts. The first one $2\mathcal{E}_1 + 2B\mathcal{E}_2 + \beta$ works well when $B$ is small as shown in Sections 3.2 and 3.3 but it deteriorates when $B$ grows. In contrast, the second part

$$2\mathcal{E}_1 + 2\mathcal{E}_4 + \sqrt{\frac{2}{\pi}}|t_{B,1-\alpha/2} - z_{1-\alpha/2}| + \sqrt{\frac{2}{\pi} \frac{\mathcal{E}_3}{\sigma}}t_{B,1-\alpha/2} \quad (16)$$

vanishes as $B,n \to \infty$ but does not if $B$ is bounded even if $n \to \infty$. Its behavior for bounded $B$ is easy to see: The third term $\sqrt{\frac{2}{\pi}}|t_{B,1-\alpha/2} - z_{1-\alpha/2}|$ in (16) is bounded away from zero if $B$ is bounded and thus (16) never converges to zero even if $n \to \infty$. To explain why (16) vanishes as $B,n \to \infty$, first note that the first term $2\mathcal{E}_1$ is independent of $B$ and satisfies $\mathcal{E}_1 \to 0$ as $n \to \infty$ by the Berry-Esseen Theorem for a reasonable model $\psi(\cdot)$ such as the function-of-mean model in Section 3.2. Second, notice that $\sqrt{(1/B)\sum_{b=1}^{B}(\sqrt{n}(\hat{\psi}_b - \hat{\psi}_n))^2}$ is the bootstrap estimator of the asymptotic standard deviation $\sigma$. Therefore, (15) is the concentration inequality for the bootstrap principle applied to the estimation of $\sigma$ and would hold with a choice of $\mathcal{E}_3$ and $\mathcal{E}_4$ satisfying $\mathcal{E}_3 \to 0, \mathcal{E}_4 \to 0$ as $B,n \to \infty$. Lastly, since $t_{B} \xrightarrow{d} N(0,1)$ as $B \to \infty$, by Lemma 21.2 in Van der Vaart [2000], we have $t_{B,1-\alpha/2} \to z_{1-\alpha/2}$ as $B \to \infty$. Therefore, we can see the second bound (16) converges to zero as $B,n \to \infty$ at any rate.

Under concrete assumptions on $X$ as in Sections 3.2 and 3.3, explicit forms of $\mathcal{E}_3$ and $\mathcal{E}_4$ depending on $B$, $n$ and the distribution of $X$ can be derived, based on similar arguments as the explicit bounds in Theorems 3-5. Then by studying the order of these explicit bounds with respect to $B,p$ and $n$, we can deduce a proper growth rate of dimension $p = p(B,n)$ which ensures a vanishing error as $B,n \to \infty$. The concentration inequality (15) seems unexploited in the literature and we leave it as future work.

### A.2 Explicit Finite-Sample Bounds for the Quantile-Based Bootstrap Methods

In this section, in parallel to Theorems 3-5 for the cheap bootstrap, we provide a few explicit bounds for standard quantile-based bootstrap methods under the same conditions.

The first result is in parallel to Theorem 3 under the function-of-mean model:
Theorem 7. Suppose the conditions in Theorem 3 hold. If \( q_{\alpha/2}, q_{1-\alpha/2} \) are the \( \alpha/2 \)-th and \( (1-\alpha/2) \)-th quantiles of \( g(\bar{X}_n^*) - g(\bar{X}_n) \) respectively given \( X_1, \ldots, X_n \), then the finite-sample bound on the basic bootstrap coverage error is given by

\[
|P(g(\bar{X}_n) - q_{1-\alpha/2} \leq g(\mu) \leq g(\bar{X}_n) - q_{\alpha/2}) - (1 - \alpha)|
\leq \frac{12}{n} + C \left( \frac{m_{31}}{\sqrt{n}\sigma^3} + \frac{C_H \lambda_{31}^{1/3} \text{tr}(\Sigma)}{\sqrt{n}\sigma^2} + \frac{C_H \lambda_{32}^{2/3}}{n^{5/6}\sigma} + \frac{C_H \lambda_{31}^{1/3} \lambda_{32}^{2/3}}{n}\sigma^2 \right)
+ \frac{C_H \tau^2}{\sqrt{\lambda\Sigma}} \left( 1 + \log\left( \frac{n}{p} \right) \right)^{3/2} + \frac{\tau^2 \sqrt{p}}{\lambda\Sigma n} \left( 1 + \log\left( \frac{n}{p} \right) \right)^{3/2} + \frac{\tau^3}{\lambda_{32}^{3/2}} \left( 1 + \log\left( \frac{n}{p} \right) \right)^{3/2} \left( \log n + \log p \right),
\]

where \( C \) is a universal constant and \( C_1 \) is a constant only depending on \( C_X \). If \( q_{\alpha/2}, q_{1-\alpha/2} \) are the \( \alpha/2 \)-th and \( (1-\alpha/2) \)-th quantiles of \( g(\bar{X}_n^*) \) respectively given \( X_1, \ldots, X_n \), then the finite-sample bound on the percentile bootstrap coverage error is given by

\[
|P(q_{\alpha/2} \leq g(\mu) \leq q_{1-\alpha/2}) - (1 - \alpha)|
\leq \frac{12}{n} + C \left( \frac{m_{31}}{\sqrt{n}\sigma^3} + \frac{C_H \lambda_{31}^{1/3} \text{tr}(\Sigma)}{\sqrt{n}\sigma^2} + \frac{C_H \lambda_{32}^{2/3}}{n^{5/6}\sigma} + \frac{C_H \lambda_{31}^{1/3} \lambda_{32}^{2/3}}{n}\sigma^2 \right)
+ \frac{C_H \tau^2}{\sqrt{\lambda\Sigma}} \left( 1 + \log\left( \frac{n}{p} \right) \right)^{3/2} + \frac{\tau^2 \sqrt{p}}{\lambda\Sigma n} \left( 1 + \log\left( \frac{n}{p} \right) \right)^{3/2} + \frac{\tau^3}{\lambda_{32}^{3/2}} \left( 1 + \log\left( \frac{n}{p} \right) \right)^{3/2} \left( \log n + \log p \right),
\]

where \( C \) is a universal constant and \( C_1 \) is a constant only depending on \( C_X \).

Our discussion below Theorem 2 shows that the cheap bootstrap error bound with any given \( B \) and quantile-based bootstrap error bounds with \( B = \infty \) only differ up to a constant. Therefore, the order analysis for the cheap bootstrap in Corollary 1 also applies here, that is, under the conditions in Corollary 1, the quantile-based bootstrap coverage errors shrink to 0 as \( n \to \infty \) if \( p = o(n) \).

The second result is in parallel to Theorem 4 under the sub-exponential assumption and linearity of \( g \):

Theorem 8. Suppose the conditions in Theorem 4 hold. If \( q_{\alpha/2}, q_{1-\alpha/2} \) are the \( \alpha/2 \)-th and \( (1-\alpha/2) \)-th quantiles of \( g(\bar{X}_n^*) - g(\bar{X}_n) \) respectively given \( X_1, \ldots, X_n \), then the finite-sample bound on the basic
bootstrap coverage error is given by

\[
|P(g(\bar{X}_n) - q_{1-\alpha/2} \leq g(\mu) \leq g(\bar{X}_n) - q_{\alpha/2}) - (1 - \alpha)|
\leq C \left( \frac{1}{n} + \frac{E[|g_1^T(X - \mu)|^3]}{\sigma^3 \sqrt{n}} + \frac{||g_1^T(X - \mu)||^4_{\psi_1} \log^{11}(n)}{\sigma^4 \sqrt{n}} \right) + \frac{CE[|g_1^T(X - \mu)|^3]}{\sigma^3 \sqrt{n}},
\]

where \( C \) is a universal constant. If \( q_{\alpha/2}, q_{1-\alpha/2} \) are the \( \alpha/2 \)-th and \( (1 - \alpha/2) \)-th quantiles of \( g(\bar{X}_n^*) \) respectively given \( X_1, \ldots, X_n \), then the finite-sample bound on the percentile bootstrap coverage error is given by

\[
|P(q_{\alpha/2} \leq g(\mu) \leq q_{1-\alpha/2}) - (1 - \alpha)|
\leq C \left( \frac{1}{n} + \frac{E[|g_1^T(X - \mu)|^3]}{\sigma^3 \sqrt{n}} + \frac{||g_1^T(X - \mu)||^4_{\psi_1} \log^{11}(n)}{\sigma^4 \sqrt{n}} \right) + \frac{CE[|g_1^T(X - \mu)|^3]}{\sigma^3 \sqrt{n}},
\]

where \( C \) is a universal constant.

The last result is in parallel to Theorem 5 under moment conditions and linearity of \( g \):

**Theorem 9.** Suppose the conditions in Theorem 5 hold. If \( q_{\alpha/2}, q_{1-\alpha/2} \) are the \( \alpha/2 \)- and \( (1 - \alpha/2) \)-quantiles of \( g(\bar{X}_n^*) - g(\bar{X}_n) \) respectively given \( X_1, \ldots, X_n \), then the finite-sample bound on the basic bootstrap coverage error is given by

\[
|P(g(\bar{X}_n) - q_{1-\alpha/2} \leq g(\mu) \leq g(\bar{X}_n) - q_{\alpha/2}) - (1 - \alpha)|
\leq \frac{2}{\sqrt{n}} + C_1 \max \left\{ \frac{E[|g_1^T(X - \mu)|/\sigma|^{1/q}] \sqrt{E[|g_1^T(X - \mu)/\sigma|^{4/q}]}}{\sqrt{n}}, \left( \frac{(\log n)^{3/2}}{\sqrt{n}} + \frac{\sqrt{\log n}}{n^{1/2-3/(2q)}} \right) \right\} \left( \frac{\log n}{n^{1/2-3/(2q)}} \right)
\]

where \( C \) is a universal constant and \( C_1 \) is a constant depending only on \( q \). If \( q_{\alpha/2}, q_{1-\alpha/2} \) are the \( \alpha/2 \)- and \( (1 - \alpha/2) \)-quantiles of \( g(\bar{X}_n^*) \) respectively given \( X_1, \ldots, X_n \), then the finite-sample bound on the percentile bootstrap coverage error is given by

\[
|P(q_{\alpha/2} \leq g(\mu) \leq q_{1-\alpha/2}) - (1 - \alpha)|
\leq \frac{2}{\sqrt{n}} + C_1 \max \left\{ \frac{E[|g_1^T(X - \mu)|/\sigma|^{1/q}] \sqrt{E[|g_1^T(X - \mu)/\sigma|^{4/q}]}}{\sqrt{n}}, \left( \frac{(\log n)^{3/2}}{\sqrt{n}} + \frac{\sqrt{\log n}}{n^{1/2-3/(2q)}} \right) \right\} \left( \frac{\log n}{n^{1/2-3/(2q)}} \right)
\]

where \( C \) is a universal constant and \( C_1 \) is a constant depending only on \( q \).

The order analysis for the cheap bootstrap also applies to Theorems 8 and 9. If \( g_1^T(X - \mu) \) is well-scaled by its standard deviation \( \sigma \) in the sense that the \( L_p \) norm and Orlicz norm \( || \cdot ||_{\psi_1} \) is independent of \( p \), then the errors shrink to 0 for any \( p \) as \( n \to \infty \). Otherwise, the growth rate of \( p \) should depend on \( n \) to obtain a vanishing error.
B Details of Numerical Experiments and Additional Numerical Results

In this section, we present additional results and details of the experiments in Section 4. We also report some additional experiments. The following subsections, Sections B.1, B.2, B.3 and B.4 refer to each example presented in Section 4 other than the first two function-of-mean models. Section B.5 further validates our performances by a simulation study with a lower nominal level 70%. Finally, Section B.6 studies the coverage error behavior as $B$ and $n$ vary for the sinusoidal estimation.

B.1 Linear Regression

Figure 2 shows the box plot of the coverage probabilities and confidence interval widths of all $\beta_i$’s with $B = 1, 2, 5, 10$, where “CB”, “BB”, “PB” and “SEB” stand for the cheap bootstrap, basic bootstrap, percentile bootstrap and standard error bootstrap respectively to generate 95%-level confidence intervals. In terms of coverage probability, the cheap bootstrap has coverages close to the nominal level 95% for almost all $\beta_i$’s for any $B$. On the other hand, standard error bootstrap coverages are above 90% only when $B = 10$ while the quantile-based bootstrap coverages are still below 85% for most of the $\beta_i$’s even for $B = 10$. We also observe that the cheap bootstrap has wider confidence interval widths than other methods, which helps lift the coverage probability to the nominal level. Besides, cheap bootstrap widths decrease at a fast rate for the first few $B$’s, showing a desirable swift gain in statistical efficiency as $B$ moves away from 1. For almost all $\beta_i$’s, the widths decrease by nearly $2/3$ from $B = 1$ to $B = 2$ and by nearly $4/5$ from $B = 1$ to $B = 10$.

Figure 2: Box plot of empirical coverage probabilities and confidence interval widths of all $\beta_i$’s for different number of resamples in a linear regression
B.2 Logistic Regression

Figure 3 presents the coverage probabilities and confidence interval widths of 95%-level confidence intervals for three typical choices of parameters: $\beta_1 = 1$, $\beta_{301} = -1$ and $\beta_{601} = 0$. We observe that all cheap bootstrap coverage probabilities are close to or larger than the nominal level 95% while other bootstrap method coverages are below 90% except for the standard error bootstrap for $\beta_{601} = 0$ and $B \geq 5$. Besides, cheap bootstrap interval widths are larger than others but decay very fast for the first few $B$’s, in line with our observation in the previous linear regression example. In fact, it is already quite close to other bootstrap widths for $\beta_{601} = 0$ and $B = 10$. Figure 4 reports the box plot of the coverage probabilities and confidence interval widths of all $\beta_i$’s with $B = 1, 2, 5, 10$. We distinguish between $\beta_i \neq 0$ and $\beta_i = 0$ since the former has wider widths than the latter. For $\beta_i \neq 0$, the cheap bootstrap widths shrink more slowly so that almost all cheap bootstrap coverage probabilities are 100% but other bootstrap method coverages are still below 90% in almost all cases. For $\beta_i = 0$, the cheap bootstrap with any $B$, standard error bootstrap with $B = 5, 10$ and basic bootstrap with $B = 10$ have coverage probabilities close to the nominal level 95%. In other cases, most of the coverage probabilities are below 85%. A similar decay rate for the cheap bootstrap interval width is also observed here: it decreases by around $2/3$ from $B = 1$ to $B = 2$ and by around $4/5$ from $B = 1$ to $B = 10$.

To study the impact of the dimension $p$ on the behavior of the cheap bootstrap, we consider another setting of the logistic regression where $p$ is increased from 9000 to $2.5 \times 10^4$ and others remain the same as in Section 4 (e.g., $n = 10^5$). Table 2 reports the average empirical coverage and width over all parameters. From the over-coverage probabilities (of the cheap bootstrap and standard error bootstrap), under-coverage probabilities (of the basic bootstrap and percentile bootstrap), and the huge confidence interval widths (of all bootstrap methods), we see that all the bootstrap methods fail to converge. The different behaviors for $p = 9000$ versus $p = 2.5 \times 10^4$ support, to some extent, the implication of Corollary 1 that $p$ should be chosen such that $p = o(n)$ to ensure a vanishing coverage error.

B.3 Computer Network

We detail the specifications of the computer communication network simulation model; similar models have been used in Cheng and Holland [1997], Lin et al. [2015], Lam and Qian [2021]. This network can be represented by an undirected graph in Figure 5. The four nodes denote message processing units and the four edges are transport channels. For every pair of nodes $i, j$ ($i \neq j$), there are external messages which enter into node $i$ from the external and are to be transmitted to node $j$ through a prescribed path. Their arrival time follows a Poisson process with parameter $\lambda_{i,j}$ showed in Table 3. All the message lengths (unit: bits) are i.i.d. following a common exponential distribution with mean 300 bits. Suppose each unit spends 0.001 second to process a message passing it. We assume
Figure 3: Empirical coverage probabilities and confidence interval widths for different number of resamples in a logistic regression.
Figure 4: Box plot of empirical coverage probabilities and confidence interval widths of all $\beta_i$'s for different number of resamples in a logistic regression

Table 2: Coverage probabilities (Pro.) and confidence interval widths (Wid.) of the logistic regression with $p = 2.5 \times 10^4$ and $n = 10^5$.

| $B$ | Cheap Bootstrap | Basic Bootstrap | Percentile Bootstrap | Standard Error Bootstrap |
|-----|-----------------|-----------------|----------------------|-------------------------|
|     | Pro. | Wid. | Pro. | Wid. | Pro. | Wid. | Pro. | Wid. | Pro. | Wid. |
| 1   | 99.9% | 201.611 | N.A. | N.A. | N.A. | N.A. | N.A. | N.A. | N.A. | N.A. |
| 2   | 100%  | 74.595  | 36.6% | 7.597 | 35.3% | 7.597 | 98.8% | 21.057 |       |       |
| 5   | 100%  | 47.205  | 72.5% | 15.737 | 70.1% | 15.737 | 100%  | 24.922 |       |       |
| 10  | 100%  | 41.583  | 86.8% | 20.819 | 84.7% | 20.819 | 100%  | 25.750 |       |       |
the node storage is unlimited but the channel storage is restricted to 275000 bits. Message speed in transport channels is 150000 miles per second and channel $i$ has length $100i$ miles. Therefore, it takes $l/275000 + 100i/150000$ seconds for a message with length $l$ bits to pass channel $i$. Suppose the network is empty at the beginning. The performance measure of interest is the steady-state average delay for the messages where delay means the time from the entering node to the destination node. It has approximate true value $7.05 \times 10^{-3}$. This example has 13 unknown input distributions, i.e., 12 inter-arrival time distributions $\text{Exp}(\lambda_{i,j})$ and one message length distribution $\text{Exp}(1/300)$, for which we have data sizes from $3 \times 10^4$ to $6 \times 10^4$. Given input distributions $P_1, \ldots, P_{13}$, the performance measure of this system can be computed accurately by

$$\psi(P_1, \ldots, P_{13}) = E_{P_1, \ldots, P_{13}} \left[ \frac{1}{9500} \sum_{k=501}^{10000} D_k \right],$$

where $D_k$ is the delay for the $k$-th message. The point estimator of $\psi(P_1, \ldots, P_{13})$ is taken as $\hat{\psi} = \psi(\hat{P}_{1,n_1}, \ldots, \hat{P}_{13,n_{13}})$ where each $\hat{P}_{i,n_i}$ is the empirical distribution of $n_i$ i.i.d. samples $\{X_{i,j}, j = 1, \ldots, n_i\}$ from the $i$-th input distribution $P_i$.

Next we construct the bootstrap estimator $\hat{\psi}^{*b}$. For each $b = 1, \ldots, B$ and $i = 1, \ldots, 13$, we sample with replacement the data $\{X_{i,j}, j = 1, \ldots, n_i\}$ to obtain the bootstrap resamples $\{X_{i,j}^{*b}, j = 1, \ldots, n_i\}$ and denote the resample empirical distribution by $\hat{P}_{i,n_i}^{*b}$. The sampling procedure is conducted independently for different $b$ and $i$. The bootstrap estimator is taken as $\hat{\psi}^{*b} = \psi(\hat{P}_{1,n_1}^{*b}, \ldots, \hat{P}_{13,n_{13}}^{*b})$.

The cheap bootstrap confidence interval is still constructed as in (1).

![Figure 5: A computer network with four nodes and four channels](image)

The results for the above configuration can be found in the row “Stochastic simulation” of Table 1 and the corresponding discussions can be found in Section 4.

To investigate the robustness of the cheap bootstrap or other methods, we consider an alternative configuration where computer network is the same but the input models are different. More concretely, all 13 input models (12 inter-arrival time distributions and one message length distribution) are changed to Gamma distributions $\text{Gamma}(\alpha, \beta)$ which have densities of the form

$$f(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}, x > 0.$$
Table 3: Arrival rates $\lambda_{i,j}$ of messages to be transmitted from node $i$ to node $j$

| Node $i$ | 1  | 2  | 3  | 4  |
|---------|----|----|----|----|
| 1       | N.A. | 40 | 30 | 35 |
| 2       | 50  | N.A. | 45 | 15 |
| 3       | 60  | 15  | N.A. | 20 |
| 4       | 25  | 30  | 40  | N.A. |

The message length distribution follows Gamma($2.5, 1/200$) and the parameters for the inter-arrival time distributions Gamma($\alpha_{i,j}, \beta_{i,j}$) are given in Table 4. Under the new input distributions, the true steady-state mean delay is approximately 0.0109. Figure 6 reports the results. The cheap bootstrap coverage probabilities are close to the nominal level 95% for any $B$ while those of the basic bootstrap and standard error bootstrap are below 60% and 90% respectively even for $B = 10$. Percentile bootstrap also performs well when $B \geq 7$ perhaps because of the skewness of the estimates. But this is not always the case in view of the previous numerical results.

Table 4: Parameters ($\alpha_{i,j}, \beta_{i,j}$) for the inter-arrival time distribution of messages to be transmitted from node $i$ to node $j$

| Node $i$ | 1     | 2     | 3     | 4     |
|---------|-------|-------|-------|-------|
| 1       | N.A.  | (1.5, 60) | (0.7, 40) | (1.3, 50) |
| 2       | (2, 80) | N.A.  | (1.5, 65) | (0.6, 20) |
| 3       | (3, 100) | (0.5, 25) | N.A.  | (1.2, 30) |
| 4       | (0.8, 40) | (1.1, 50) | (0.9, 35) | N.A.  |

B.4 Real World Problem

The data we use is the RCV1-v2 data in Lewis et al. [2004]. This dataset contains $n = 804414$ manually categorized newswire stories with a total of $p = 47236$ features. We compare the confidence interval widths of the logistic regression parameters for the four bootstrap methods, by using all the observations to run the logistic regression and estimate the parameters. That is, the observation matrix is of the size $804414 \times 47236 \approx 4 \times 10^{10}$. There are up to 103 different categories for all these newswire stories. As in Singh et al. [2009] and Balakrishnan and Madigan [2008], we only use the
“economics” (“ECAT”) as the +1 label, i.e., the label $Y$ is 1 if the newswire story is in “economics” and 0 if not, which leads to 119920 positive labels. Besides, we add $l_2$ regularization to this logistic regression as in Singh et al. [2009] and Balakrishnan and Madigan [2008].

This logistic regression takes about 30-40 minutes to run one bootstrap resample when we use sklearn.linear_model.LogisticRegression (a machine learning package in Python) in the virtual machine c2-standard-8 in Google Cloud Platform. Therefore, the common bootstrap methods which require $B = 50$ or 100 would be computationnally expensive.

In Section 4, we report and discuss the average interval widths over all $\beta_i$’s for the four bootstrap methods. Here we display the results for three individual parameters, namely the first three $\beta_i$’s, in Figure 7. Since we are only able to run one experimental repetition in this real world example, the confidence interval widths contain some noises and thus we cannot observe the monotonicity of the widths when $B$ increases. But we still see the cheap bootstrap confidence interval widths are wider than others, with general trends that resemble the average interval widths in our synthetic examples. This suggests that the cheap bootstrap confidence intervals would have higher and closer-to-nominal coverages than the other methods.

B.5 Numerical Experiments with a Lower Nominal Level

In this section, we conduct a simulation study with the nominal level 70% to further support the validity of the cheap bootstrap. We choose the ellipsoidal estimation and sinusoidal estimation presented in Section 4 as our model. All the settings are the same except that we use a different sample size $n = 4 \times 10^4$ and a different dimension $p = 9000$. Table 5 presents the empirical coverage and average interval width over 1000 experimental repetitions. We can observe that the cheap bootstrap
Figure 7: Confidence interval widths of the first three $\beta$’s for different number of resamples in a real world logistic regression

Table 5: Coverage probabilities (Pro.) and confidence interval widths (Wid.) of the numerical examples. The closest coverage probability to the nominal 70% level among all methods are bolded.

| Example       | $B$ | Cheap Bootstrap | Basic Bootstrap | Percentile Bootstrap | Standard Error Bootstrap |
|---------------|-----|------------------|-----------------|----------------------|--------------------------|
|               | Pro. | Wid.             | Pro.            | Wid.                 | Pro.                     | Wid.                     |
| Ellipsoidal   | 1    | 73.0%            | 9.828 × 10^{-3} | N.A.                 | N.A.                     | N.A.                     | N.A.                     |
| estimation    | 2    | 70.0%            | 7.568 × 10^{-3} | 30.9%                | 2.197 × 10^{-3}          | 7.7%                     | 2.197 × 10^{-3}          | 34.3%                     | 3.220 × 10^{-3}          |
| Sinusoidal    | 5    | 68.5%            | 6.639 × 10^{-3} | 64.7%                | 4.429 × 10^{-3}          | 16.5%                    | 4.429 × 10^{-3}          | 42.3%                     | 3.717 × 10^{-3}          |
| estimation    | 10   | 66.9%            | 6.323 × 10^{-3} | 60.1%                | 3.756 × 10^{-3}          | 10.8%                    | 3.756 × 10^{-3}          | 41.8%                     | 3.804 × 10^{-3}          |
| Sinusoidal    | 1    | 70.3%            | 0.148           | N.A.                 | N.A.                     | N.A.                     | N.A.                     | N.A.                     |
| estimation    | 2    | 72.0%            | 0.116           | 34.9%                | 0.052                    | 33.3%                    | 0.052                    | 52.4%                     | 0.077                    |
| Sinusoidal    | 5    | 72.2%            | 0.104           | 67.6%                | 0.108                    | 68.6%                    | 0.108                    | 65.6%                     | 0.091                    |
| Sinusoidal    | 10   | 72.3%            | 0.100           | 65.8%                | 0.093                    | 63.7%                    | 0.093                    | 68.8%                     | 0.094                    |

Coverage probabilities are still close to the nominal level, and are the closest among all methods in all cases except two (sinusoidal $B = 5$ and $B = 10$) where percentile and standard error bootstraps each outperforms slightly. Regarding these exceptional cases, we note that the outperformance of the percentile bootstrap is likely a coincidence, because as a quantile-based method it cannot construct a symmetric 70% level confidence interval from only 5 resamples. We have used the minimum and maximum of the 5 resamples to construct this percentile bootstrap confidence interval, whose actual nominal level should be close to 100%. So it is likely by accident that percentile bootstrap coverage is closest to 70% and in fact this coverage is far away from its actual nominal level around 100%.
B.6 Coverage Error Behavior with Respect to $B$ and $n$

In this section, we numerically study the cheap bootstrap coverage error behavior with respect to $B$ and $n$ and illustrate how it aligns with our theoretical bounds. We choose the model as the sinusoidal estimation in Section 4. We fix the dimension $p = 9000$ and vary $B$ and $n$. Figure 8 displays the colormap of the absolute value of empirical coverage errors (nominal level 95%), where the $x$-axis represents $B$ and $y$-axis represents $n$. We cut the results of the basic and percentile bootstraps for the first few $B$’s because their errors are too large. From Figure 8 (a), it appears that the cheap bootstrap coverage error does not change much in this regime of $n$. This matches to some extent Theorem 3 and Corollary 1 that guarantee the coverage error would decrease as $n$ increases with a slow rate $1/\sqrt{n}$. Further, when we fix $n$, the cheap bootstrap coverage error seems to lack clear trend and otherwise be quite stable as $B$ changes. On the other hand, the basic and percentile bootstrap coverage errors show a clear decreasing trend as $B$ increases. Their different behaviors are attributed to the different ideas behind them. The basic bootstrap and percentile bootstrap are quantile-based methods. As $B$ increases, the bootstrap quantile estimate is closer and closer to the true quantile, which leads to the improvement on the coverage error. However, the cheap bootstrap method relies on a totally different idea, i.e., it relies on approximate independence of the resamples from the original estimator and thus a $t$-distribution-based (with degree of freedom $B$) confidence interval can be constructed. Different $B$ just means a different pivotal $t$-distribution. There is no evident reason that the pivotal $t$-distribution with a larger degree of freedom $B$ will lead to a smaller coverage error.

C Proofs

Proof of Theorem 1: We define $Q^*$ as the distribution of $\sqrt{n}(\hat{\psi}_n^* - \hat{\psi}_n)$ conditional on $X_1,\ldots,X_n$. With the repeated bootstrap resampling, we have that $\sqrt{n}(\hat{\psi}_n^{*b} - \hat{\psi}_n)$, $b = 1,\ldots,B$ are independent conditional on $X_1,\ldots,X_n$. Then we can write the coverage probability as

$$P(\psi \in [\hat{\psi}_n - t_{B,1-\alpha/2}S_{n,B}, \hat{\psi}_n + t_{B,1-\alpha/2}S_{n,B}])$$

$$= P \left( \left| \frac{\sqrt{n}(\hat{\psi}_n - \psi)}{\frac{1}{B} \sum_{b=1}^{B} (\sqrt{n}(\hat{\psi}_n^{*b} - \hat{\psi}_n))^2} \right| \leq t_{B,1-\alpha/2} \right)$$

$$= E \left[ \prod_{b=1}^{B} dQ^*(z_{B}) \cdots dQ^*(z_{1}) \right]$$

where the expectation $E$ is taken with respect to $X_1,\ldots,X_n$. If we write $A$ as the event that

$$\sup_{x \in \mathbb{R}} \left| Q^*((-\infty, x]) - \Phi \left( \frac{x}{\sigma} \right) \right| \equiv \sup_{x \in \mathbb{R}} \left| P^*(\sqrt{n}(\hat{\psi}_n^* - \hat{\psi}_n) \leq x) - \Phi \left( \frac{x}{\sigma} \right) \right| \leq \mathcal{E}_2,$$  \hspace{1cm} (17)
Figure 8: Colormap of the absolute value of empirical coverage errors (nominal level 95%) for the sinusoidal estimation.
then we know that $P(A^c) \leq \beta$. We consider the coverage probability intersected with $\mathcal{A}$, i.e.,

$$E \left[ \int \cdots \int \left[ \frac{\sum_{b=1}^{B} z_b^2}{B} \right] dQ^*(z_B) \cdots dQ^*(z_1); \mathcal{A} \right].$$ (18)

Note that conditional on $X_1, \ldots, X_n$ and given $z_1, \ldots, z_{B-1}$, the integral region for $z_B$ can be written as

$$\left\{ z_B : |\sqrt{n}(\hat{\psi}_n - \psi)| \leq t_{B,1-\alpha/2}\sqrt{\frac{1}{B} \sum_{b=1}^{B} z_b^2} \right\} = (-\infty, -q) \cup [q, \infty)$$

for some $q \geq 0$. Therefore, applying (17), we have

$$\left| \int_{|\sqrt{n}(\hat{\psi}_n - \psi)| \leq t_{B,1-\alpha/2}\sqrt{\frac{1}{B} \sum_{b=1}^{B} z_b^2}} dQ^*(z_B) - \int_{|\sqrt{n}(\hat{\psi}_n - \psi)| \leq t_{B,1-\alpha/2}\sqrt{\frac{1}{B} \sum_{b=1}^{B} z_b^2}} dP_0(z_B) \right| \leq 2\mathcal{E}_2,$$

where $P_0$ is the distribution of $N(0, \sigma^2)$. Plugging it into (18), we have

$$E \left[ \int \cdots \int [\sqrt{n}(g(\hat{X}_n) - g(\mu))] \leq t_{B,1-\alpha/2}\sqrt{\frac{1}{B} \sum_{b=1}^{B} z_b^2} \right] dQ^*(z_B) \cdots dQ^*(z_1); \mathcal{A} \right]$$

$$= E \left[ \int \cdots \int [\sqrt{n}(\hat{\psi}_n - \psi)] \leq t_{B,1-\alpha/2}\sqrt{\frac{1}{B} \sum_{b=1}^{B} z_b^2} \right] dP_0(z_B) dQ^*(z_{B-1}) \cdots dQ^*(z_1); \mathcal{A} \right] + R_B,$$

where the error $R_B$ satisfies

$$|R_B| \leq E \left[ \int \cdots \int 2\mathcal{E}_2 dQ^*(z_{B-1}) \cdots dQ^*(z_1); \mathcal{A} \right] \leq 2\mathcal{E}_2.$$

By the same argument, we can further replace the remaining $Q^*(z_i)$’s by $P_0(z_i)$’s and obtain

$$E \left[ \int \cdots \int [\sqrt{n}(\hat{\psi}_n - \psi)] \leq t_{B,1-\alpha/2}\sqrt{\frac{1}{B} \sum_{b=1}^{B} z_b^2} \right] dQ^*(z_B) \cdots dQ^*(z_1); \mathcal{A} \right]$$

$$= E \left[ \int \cdots \int [\sqrt{n}(\hat{\psi}_n - \psi)] \leq t_{B,1-\alpha/2}\sqrt{\frac{1}{B} \sum_{b=1}^{B} z_b^2} \right] dP_0(z_B) \cdots dP_0(z_1); \mathcal{A} \right] + \sum_{b=1}^{B} R_b,$$

where each error $R_b$ satisfies

$$|R_b| \leq 2\mathcal{E}_2.$$

Therefore, the coverage probability satisfies

$$E \left[ \int \cdots \int [\sqrt{n}(\hat{\psi}_n - \psi)] \leq t_{B,1-\alpha/2}\sqrt{\frac{1}{B} \sum_{b=1}^{B} z_b^2} \right] dQ^*(z_B) \cdots dQ^*(z_1)$$

$$= E \left[ \int \cdots \int [\sqrt{n}(\hat{\psi}_n - \psi)] \leq t_{B,1-\alpha/2}\sqrt{\frac{1}{B} \sum_{b=1}^{B} z_b^2} \right] dQ^*(z_B) \cdots dQ^*(z_1); \mathcal{A} \right]$$

$$+ E \left[ \int \cdots \int [\sqrt{n}(\hat{\psi}_n - \psi)] \leq t_{B,1-\alpha/2}\sqrt{\frac{1}{B} \sum_{b=1}^{B} z_b^2} \right] dQ^*(z_B) \cdots dQ^*(z_1); \mathcal{A}^c \right].$$
\[
\begin{aligned}
&= E \left[ \int \cdots \int_{|\sqrt{\psi} - \psi| \leq t_{B,1-\alpha/2} \sqrt{\frac{1}{\pi} \sum_{b=1}^{B} z_b^2}} dP_0(z_B) \cdots dP_0(z_1); A \right] + \sum_{b=1}^{B} R_b \\
&+ E \left[ \int \cdots \int_{|\sqrt{\psi} - \psi| \leq t_{B,1-\alpha/2} \sqrt{\frac{1}{\pi} \sum_{b=1}^{B} z_b^2}} dQ^*(z_B) \cdots dQ^*(z_1); A^c \right] \\
&= E \left[ \int \cdots \int_{|\sqrt{\psi} - \psi| \leq t_{B,1-\alpha/2} \sqrt{\frac{1}{\pi} \sum_{b=1}^{B} z_b^2}} dP_0(z_B) \cdots dP_0(z_1) \right] + R_{A^c} + \sum_{b=1}^{B} R_b, \quad (19)
\end{aligned}
\]

where the additional error \( R_{A^c} \) is given by

\[
R_{A^c} = E \left[ \int \cdots \int_{|\sqrt{\psi} - \psi| \leq t_{B,1-\alpha/2} \sqrt{\frac{1}{\pi} \sum_{b=1}^{B} z_b^2}} dQ^*(z_B) \cdots dQ^*(z_1); A^c \right] \\
- E \left[ \int \cdots \int_{|\sqrt{\psi} - \psi| \leq t_{B,1-\alpha/2} \sqrt{\frac{1}{\pi} \sum_{b=1}^{B} z_b^2}} dP_0(z_B) \cdots dP_0(z_1); A^c \right]
\]

and it satisfies \( |R_{A^c}| \leq P(A^c) \leq \beta \). Now we will handle the distribution of \( \sqrt{n}(\hat{\psi}_n - \psi) \) which is denoted by \( Q_0 \). Note that by Fubini’s theorem we have

\[
E \left[ \int \cdots \int_{|\sqrt{\psi} - \psi| \leq t_{B,1-\alpha/2} \sqrt{\frac{1}{\pi} \sum_{b=1}^{B} z_b^2}} dP_0(z_B) \cdots dP_0(z_1) \right] \\
= \int \cdots \int_{|z_0| \leq t_{B,1-\alpha/2} \sqrt{\frac{1}{\pi} \sum_{b=1}^{B} z_b^2}} dQ_0(z_0) dP_0(z_B) \cdots dP_0(z_1).
\]

Given \( z_1, \ldots, z_B \), consider the innermost integral with respect to \( Q_0 \). By the finite-sample accuracy for \( Q_0 \), i.e.,

\[
\sup_{x \in \mathbb{R}} |P(\sqrt{n}(\hat{\psi}_n - \psi) \leq x) - \Phi \left( \frac{x}{\sigma} \right) | \equiv \sup_{x \in \mathbb{R}} |Q_0((-\infty, x]) - P_0((-\infty, x])| \leq \mathcal{E}_1,
\]

we have

\[
\left| \int_{|z_0| \leq t_{B,1-\alpha/2} \sqrt{\frac{1}{\pi} \sum_{b=1}^{B} z_b^2}} dQ_0(z_0) - \int_{|z_0| \leq t_{B,1-\alpha/2} \sqrt{\frac{1}{\pi} \sum_{b=1}^{B} z_b^2}} dP_0(z_0) \right| \leq 2\mathcal{E}_1.
\]

Therefore,

\[
E \left[ \int \cdots \int_{|\sqrt{\psi} - \psi| \leq t_{B,1-\alpha/2} \sqrt{\frac{1}{\pi} \sum_{b=1}^{B} z_b^2}} dP_0(z_B) \cdots dP_0(z_1) \right] \\
= \int \cdots \int_{|z_0| \leq t_{B,1-\alpha/2} \sqrt{\frac{1}{\pi} \sum_{b=1}^{B} z_b^2}} dP_0(z_0) dP_0(z_B) \cdots dP_0(z_1) + R_0 \\
= 1 - \alpha + R_0, \quad (20)
\]

where the second equality follows from

\[
\frac{Z_0}{\sqrt{\frac{1}{\pi} \sum_{b=1}^{B} Z_b^2}} \overset{d}{=} t_B
\]

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for i.i.d. \( Z_i \sim N(0, \sigma^2) \), \( i = 0, \ldots, B \) and the error \( R_0 \) satisfies

\[
|R_0| \leq 2\varepsilon_1.
\]

Plugging (20) into (19), we have

\[
E \left[ \int \cdots \int _{|\sqrt{n}(\hat{\psi}_n - \psi)| \leq t_{B,1 - \alpha/2} \sqrt{\frac{1}{n} \sum _{b=1} ^B z_b^2}} dQ^*(z_B) \cdots dQ^*(z_1) \right] = 1 - \alpha + R_{A^c} + \sum _{b=0} ^B R_b := 1 - \alpha + R,
\]

where the overall error satisfies

\[
|R| \leq 2\varepsilon_1 + 2B\varepsilon_2 + \beta.
\]

Proof of Theorem 2: Recall that for a cumulative distribution function \( F \) of a random variable, the \( q \)-th quantile is defined as \( F^{-1}(q) = \inf \{ x : F(x) \geq q \} \). We first prove a useful result: if the cumulative distribution functions of two random variables \( X \) and \( Y \) satisfy

\[
\sup _{t \in \mathbb{R}} |F_X(t) - F_Y(t)| \leq \varepsilon,
\]

then for any \( \alpha \in [0,1] \),

\[
F_Y^{-1}(\alpha - \varepsilon) \leq F_X^{-1}(\alpha) \leq F_Y^{-1}(\alpha + \varepsilon). \tag{22}
\]

To prove it, we note that if \( \alpha - \varepsilon < 0 \) then \( -\infty = F_Y^{-1}(\alpha - \varepsilon) \leq F_X^{-1}(\alpha) \) trivially holds and if \( \alpha + \varepsilon > 1 \) then \( F_X^{-1}(\alpha) \leq F_Y^{-1}(\alpha + \varepsilon) = \infty \) trivially holds. So we assume \( 0 \leq \alpha - \varepsilon \leq \alpha + \varepsilon \leq 1 \). Now let’s prove the first inequality \( F_Y^{-1}(\alpha - \varepsilon) \leq F_X^{-1}(\alpha) \). By the definition of \( F_X^{-1} \) and right-continuity of \( F_X \), we know that \( F_X(F_X^{-1}(\alpha)) \geq \alpha \). Therefore, by (21), we know that \( F_Y(F_X^{-1}(\alpha)) \geq \alpha - \varepsilon \), which implies \( F_Y^{-1}(\alpha - \varepsilon) \leq F_X^{-1}(\alpha) \) by the definition of \( F_Y^{-1}(\alpha - \varepsilon) \). This proves the first inequality in (22). Interchanging the role of \( X \) and \( Y \), we have \( F_X^{-1}(\alpha - \varepsilon) \leq F_Y^{-1}(\alpha) \). Replacing \( \alpha \) by \( \alpha + \varepsilon \), we obtain the second inequality \( F_X^{-1}(\alpha) \leq F_Y^{-1}(\alpha + \varepsilon) \).

Now we consider the basic bootstrap. We write \( A \) as the event

\[
\sup _{x \in \mathbb{R}} \left| P^\ast(\sqrt{n}(\hat{\psi}_n - \bar{\psi}_n) \leq x) - \Phi \left( \frac{x}{\sigma} \right) \right| \leq \varepsilon_2.
\]

By our assumption, we have \( P(A^c) \leq \beta \). Note that if \( q_{\alpha/2} \) and \( q_{1 - \alpha/2} \) are the \( \alpha/2 \)-th and \( 1 - \alpha/2 \)-th quantiles of \( \hat{\psi}_n - \bar{\psi}_n \) given \( X_1, \ldots, X_n \), then \( \sqrt{n}q_{\alpha/2} \) and \( \sqrt{n}q_{1 - \alpha/2} \) are the \( \alpha/2 \)-th and \( 1 - \alpha/2 \)-th quantiles of \( \sqrt{n}(\hat{\psi}_n - \bar{\psi}_n) \) respectively given \( X_1, \ldots, X_n \). By the inequality (22), when \( A \) happens, we have

\[
\sigma z_{\alpha/2 - \varepsilon_2} \leq \sqrt{n}q_{\alpha/2} \leq \sigma z_{\alpha/2 + \varepsilon_2}.
\]
where \( z_q \) is the \( q \)-th quantile of the standard normal. This inequality implies

\[
P(\sqrt{n}(\hat{\psi}_n - \psi) < \sigma z_{\alpha/2 - \varepsilon_2}; A) \leq P(\sqrt{n}(\hat{\psi}_n - \psi) < \sqrt{n}q_{\alpha/2}; A) \tag{23}
\]

and

\[
P(\sqrt{n}(\hat{\psi}_n - \psi) < \sqrt{n}q_{\alpha/2}; A) \leq P(\sqrt{n}(\hat{\psi}_n - \psi) < \sigma z_{\alpha/2 + \varepsilon_2}; A). \tag{24}
\]

Next, we notice that

\[
\sup_{x \in \mathbb{R}} \left| P(\sqrt{n}(\hat{\psi}_n - \psi) \leq x) - \Phi \left( \frac{x}{\sigma} \right) \right| \leq \varepsilon_1
\]

\[\iff\]

\[
\sup_{x \in \mathbb{R}} P(\sqrt{n}(\hat{\psi}_n - \psi) < x) - \Phi \left( \frac{x}{\sigma} \right) \leq \varepsilon_1.
\]

Thus, (23) implies that

\[
P(\sqrt{n}(\hat{\psi}_n - \psi) < \sqrt{n}q_{\alpha/2}; A)
\]

\[
\geq P(\sqrt{n}(\hat{\psi}_n - \psi) < \sigma z_{\alpha/2 - \varepsilon_2}; A)
\]

\[
\geq P(\sqrt{n}(\hat{\psi}_n - \psi) < \sigma z_{\alpha/2 - \varepsilon_2}) - P(A^c)
\]

\[
\geq \Phi \left( \frac{\sigma z_{\alpha/2 - \varepsilon_2}}{\sigma} \right) - \varepsilon_1 - \beta
\]

\[
= \frac{\alpha}{2} - \varepsilon_1 - \varepsilon_2 - \beta.
\]

Similarly, (24) implies that

\[
P(\sqrt{n}(\hat{\psi}_n - \psi) < \sqrt{n}q_{\alpha/2}; A) \leq P(\sqrt{n}(\hat{\psi}_n - \psi) < \sigma z_{\alpha/2 + \varepsilon_2}) \leq \frac{\alpha}{2} + \varepsilon_1 + \varepsilon_2.
\]

Therefore, we have the following two-sided bound

\[
\frac{\alpha}{2} - \varepsilon_1 - \varepsilon_2 - \beta \leq P(\sqrt{n}(\hat{\psi}_n - \psi) < \sqrt{n}q_{\alpha/2}; A) \leq \frac{\alpha}{2} + \varepsilon_1 + \varepsilon_2.
\]

For the \((1 - \alpha/2)\)-th quantile, we can also derive a similar bound

\[
1 - \frac{\alpha}{2} - \varepsilon_1 - \varepsilon_2 - \beta \leq P(\sqrt{n}(\hat{\psi}_n - \psi) \leq \sqrt{n}q_{1-\alpha/2}; A) \leq 1 - \frac{\alpha}{2} + \varepsilon_1 + \varepsilon_2.
\]

So we have

\[
|P(\sqrt{n}q_{\alpha/2} \leq \sqrt{n}(\hat{\psi}_n - \psi) \leq \sqrt{n}q_{1-\alpha/2}; A) - (1 - \alpha)| \leq 2\varepsilon_1 + 2\varepsilon_2 + \beta,
\]

which gives rise to

\[
|P(\sqrt{n}q_{\alpha/2} \leq \sqrt{n}(\hat{\psi}_n - \psi) \leq \sqrt{n}q_{1-\alpha/2}) - (1 - \alpha)|
\]

\[
\leq |P(\sqrt{n}q_{\alpha/2} \leq \sqrt{n}(\hat{\psi}_n - \psi) \leq \sqrt{n}q_{1-\alpha/2}; A) - (1 - \alpha)|
\]

\[
+ P(\sqrt{n}q_{\alpha/2} \leq \sqrt{n}(\hat{\psi}_n - \psi) \leq \sqrt{n}q_{1-\alpha/2}; A^c)
\]

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This completes our proof.

or equivalently

\[ |P(\hat{\psi}_n - q_{1-\alpha/2} \leq \psi \leq \hat{\psi}_n - q_{\alpha/2}) - (1 - \alpha)| \leq 2\mathcal{E}_1 + 2\mathcal{E}_2 + 2\beta. \]

The result for the percentile bootstrap follows similarly but we need to use the symmetry of \( N(0, \sigma^2) \). Note that

\[
\sup_{x \in \mathbb{R}} \left| P(\sqrt{n}(\hat{\psi}_n - \psi) \leq x) - \Phi \left( \frac{x}{\sigma} \right) \right| \leq \mathcal{E}_1
\]

\[
\Leftrightarrow \sup_{x \in \mathbb{R}} \left| P(\sqrt{n}(\hat{\psi}_n - \psi) < x) - \Phi \left( \frac{x}{\sigma} \right) \right| \leq \mathcal{E}_1
\]

and the latter can be rewritten as

\[
\sup_{x \in \mathbb{R}} \left| P(\sqrt{n}(\hat{\psi}_n - \psi) < x) - \Phi \left( \frac{x}{\sigma} \right) \right| = \sup_{x \in \mathbb{R}} \left| P(\sqrt{n}(\hat{\psi}_n - \psi) < -x) - \Phi \left( \frac{-x}{\sigma} \right) \right| = \sup_{x \in \mathbb{R}} \left| P(\sqrt{n}(\psi - \hat{\psi}_n) > x) - \Phi \left( \frac{-x}{\sigma} \right) \right|
\]

\[
= \sup_{x \in \mathbb{R}} \left| (1 - P(\sqrt{n}(\psi - \hat{\psi}_n) > x)) - (1 - \Phi \left( \frac{-x}{\sigma} \right)) \right|
\]

\[
= \sup_{x \in \mathbb{R}} \left| P(\sqrt{n}(\psi - \hat{\psi}_n) \leq x) - \Phi \left( \frac{x}{\sigma} \right) \right| \leq \mathcal{E}_1,
\]

where the last equality uses the symmetry of \( N(0, \sigma^2) \), i.e., \( 1 - \Phi(-x/\sigma) = \Phi(x/\sigma) \). Moreover, if \( q_{\alpha/2} \) and \( q_{1-\alpha/2} \) are the \( \alpha/2 \)-th and \( (1 - \alpha/2) \)-th quantiles of \( \hat{\psi}_n^* \) given \( X_1, \ldots, X_n \), then \( \sqrt{n}(q_{\alpha/2} - \hat{\psi}_n) \) and \( \sqrt{n}(q_{1-\alpha/2} - \hat{\psi}_n) \) are the \( \alpha/2 \)-th and \( (1 - \alpha/2) \)-th quantiles of \( \sqrt{n}(\hat{\psi}_n^* - \hat{\psi}_n) \) given \( X_1, \ldots, X_n \). Therefore, the proof for the basic bootstrap also applies if we replace \( \sqrt{n}q_{\alpha/2}, \sqrt{n}q_{1-\alpha/2} \) and \( \sqrt{n}(\hat{\psi}_n - \psi) \) in that proof by \( \sqrt{n}(q_{\alpha/2} - \hat{\psi}_n), \sqrt{n}(q_{1-\alpha/2} - \hat{\psi}_n) \) and \( \sqrt{n}(\psi - \hat{\psi}_n) \) respectively. In particular, the final result now reads as follows

\[
|P(\sqrt{n}(q_{\alpha/2} - \hat{\psi}_n) \leq \sqrt{n}(\psi - \hat{\psi}_n) \leq \sqrt{n}(q_{1-\alpha/2} - \hat{\psi}_n)) - (1 - \alpha)| \leq 2\mathcal{E}_1 + 2\mathcal{E}_2 + 2\beta
\]

or equivalently

\[
|P(q_{\alpha/2} \leq \psi \leq q_{1-\alpha/2}) - (1 - \alpha)| \leq 2\mathcal{E}_1 + 2\mathcal{E}_2 + 2\beta.
\]

This completes our proof. \( \square \)

To prove Theorem 3, we first need to prove two lemmas regarding (4) and (5) respectively.

The following lemma is from Theorem 2.11 in Pinelis and Molzon [2016] which establishes the Berry-Esseen theorem in the multivariate delta method in the form of (4).
Lemma 1. Suppose that $X_1, \ldots, X_n$ are i.i.d. random vectors in $\mathbb{R}^p$ satisfying $E[X] = \mu$, $\text{Var}(X) = \Sigma$, $m_{31} := E[\|\nabla g(\mu)^\top (X - \mu)\|^3] < \infty$ and $m_{32} := E[\|X - \mu\|^3] < \infty$. Suppose $g(x)$ satisfies Assumption 1 and $\sigma^2 := \nabla g(\mu)^\top \Sigma \nabla g(\mu)^\top > 0$. Then there is a universal constant $C > 0$ s.t.

$$
\sup_{x \in \mathbb{R}} \left| P\left( \frac{\sqrt{n}(g(\bar{X}_n) - g(\mu))}{\sigma} \leq x \right) - \Phi \left( \frac{\bar{x}}{\sigma} \right) \right| 
\leq C \left( \frac{m_{31}}{\sqrt{n} \sigma^3} + \frac{C_{H_g} m_{31}^{1/3} \text{tr}(\Sigma)}{\sqrt{n} \sigma^2} + \frac{C_{H_g} m_{32}^{2/3}}{n^{5/6} \sigma} + \frac{C_{H_g} m_3 m_{32}^{2/3}}{n \sigma^2} \right).
$$

Proof of Lemma 1: Define $f(x) = g(x + \mu) - g(\mu)$ and its linearization $L(x) = \nabla g(\mu)^\top x$. Then by the second order Taylor expansion of $f(x)$ and boundedness property of $H_g$ in Assumption 1, we can see (2.1) in Pinelis and Molzon [2016] holds for $M_r = C_{H_g}$ and any $\epsilon > 0$. By Theorem 2.11 in Pinelis and Molzon [2016] with $V = X - \mu$, $c_* = 1/2$ and $\epsilon \to \infty$, we have

$$
\sup_{x \in \mathbb{R}} \left| P\left( \frac{\sqrt{n}(g(\bar{X}_n) - g(\mu))}{\sigma} \leq x \right) - \Phi \left( \frac{\bar{x}}{\sigma} \right) \right| 
\leq \frac{\hat{R}_0 + \hat{R}_1 m_{31}/(\sigma^3)}{\sqrt{n}} + \frac{\hat{R}_20 + \hat{R}_21 m_{31}^{1/3}/\sigma \text{tr}(\Sigma)}{\sqrt{n}} + \frac{\hat{R}_30 + \hat{R}_31 m_{31}^{1/3}/\sigma m_{32}^{2/3}}{\sqrt{n}},
$$

(25)

where the additional term $\hat{R}_1$ in Theorem 2.11 vanishes as $\epsilon \to \infty$. By the definition of these $\hat{R}$'s with $c_* = 1/2$ in (2.30) in Pinelis and Molzon [2016], we can see there is a universal constant $C > 0$ s.t.

$$
\hat{R}_0 \leq C, \hat{R}_1 \leq C, \hat{R}_20 \leq C \frac{C_{H_g}}{\sigma}, \hat{R}_21 \leq C \frac{C_{H_g}}{\sigma n^{1/3}}, \hat{R}_30 \leq C \frac{C_{H_g}}{\sigma n^{1/2}}, \hat{R}_31 \leq C \frac{C_{H_g}}{\sigma n^{1/2}}.
$$

Moreover, by Holder's inequality, $m_{31} = E[\|\nabla g(\mu)^\top (X - \mu)\|^3] \geq E[\|\nabla g(\mu)^\top (X - \mu)\|^2]^{3/2} = \sigma^3$, which implies that $\hat{R}_0, \hat{R}_20$ can be absorbed into $\hat{R}_1 m_{31}/(\sigma^3), \hat{R}_21 m_{31}^{1/3}/\sigma$ respectively by choosing a larger $C$. Therefore, (25) can be written as

$$
\sup_{x \in \mathbb{R}} \left| P\left( \frac{\sqrt{n}(g(\bar{X}_n) - g(\mu))}{\sigma} \leq x \right) - \Phi \left( \frac{\bar{x}}{\sigma} \right) \right| 
\leq C \left( \frac{m_{31}}{\sqrt{n} \sigma^3} + \frac{C_{H_g} m_{31}^{1/3} \text{tr}(\Sigma)}{\sqrt{n} \sigma^2} + \frac{C_{H_g} m_{32}^{2/3}}{n^{5/6} \sigma} + \frac{C_{H_g} m_3 m_{32}^{2/3}}{n \sigma^2} \right).
$$

This concludes our proof.

Next we prove the finite-sample accuracy (5) for the bootstrap estimator by extracting the dependence on problem parameters in Theorem 4.2 in Zhilova [2020] and combining it with Lemma 1.

Lemma 2. Suppose the conditions in Theorem 3 hold. Then with probability at least $1 - 6/n$ we have

$$
\sup_{x \in \mathbb{R}} \left| P^*\left( \frac{\sqrt{n}(g(\bar{X}_n^*) - g(\bar{X}_n))}{\sigma} \leq x \right) - \Phi \left( \frac{\bar{x}}{\sigma} \right) \right| 
$$

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\[
\leq C \left( \frac{m_{31}}{\sqrt{n} \sigma^3} + \frac{C_{H_3} m_{31}^{1/3} \text{tr}(\Sigma)}{\sqrt{n} \sigma^2} + \frac{C_{H_3} m_{32}^{2/3}}{n^{5/6} \sigma} + \frac{C_{H_3} m_{31} m_{32}^{1/3}}{n \sigma^2} \right) + \frac{C_{H_3} \tau^2}{C_{\nabla g} \sqrt{\lambda_{\Sigma}}} \left( 1 + \log \frac{n}{p} \right) \sqrt{\frac{p}{n}} + \frac{\tau^2 \sqrt{p}}{\lambda_{\Sigma} n} \left( 1 + \log \frac{n}{p} \right) \right) \\
+ \frac{\tau^4 \sqrt{p}}{\lambda_{\Sigma}^2} \left( 1 + \log \frac{n}{p} \right)^{1/2} + \tau^2 \sqrt{p} \left( 1 + \log \frac{n}{p} \right)^{1/2} + \frac{\tau^3 \sqrt{p}}{\lambda_{\Sigma}^3} \left( 1 + \log \frac{n}{p} \right) \right) \\
\leq \tau^4 \left( \frac{3}{\lambda_{\Sigma}^2} + \frac{2}{\sqrt{n}} \right) \frac{1}{\sqrt{\lambda_{\Sigma}^2} \tau \sqrt{n}} + \frac{1}{\lambda_{\Sigma}^2} \tau \frac{1}{\sqrt{n}} + R_{1,n,3}
\]

where \(m_{31}, m_{32}\) and \(\sigma^2\) are defined in Lemma 1 and \(C\) is a universal constant.

**Proof of Lemma 2:** We use Theorem 4.2 in Zhilova [2020] to prove this lemma. First, we verify the conditions of Theorem 4.2 with \(K = 3\). For \(K = 3\), by Remark 2.1, we can take \(U_i \equiv 0, Y_i - \mu = Z_i - \mu \sim N(0, \Sigma_z)\) independent of \(X_1, \ldots, X_n\) with \(\Sigma_z = \Sigma\) which satisfies (2.1) and (2.2) for approximating \(X_i - \mu\) (since \(X_i\) is not centered in Theorem 4.2, \(Y_i\) should also be non-centered). In this case, we can see \(C_z := ||\Sigma_z^{-1/2}|| = ||\Sigma^{-1/2}|| = 1/\sqrt{\lambda_\Sigma}\). Similarly, \(C_X := ||\Sigma^{-1/2}|| = 1/\sqrt{\lambda_\Sigma}\). Other conditions about \(f\) in Theorem 4.2 have already been assumed in the statement of this lemma.

Thus, by Theorem 4.2, it holds with probability at least \(1 - 6e^{-x^2}\) for \(x > 0\):

\[
\sup_{t \in \mathbb{R}} |P(\sqrt{n}(g(\bar{X}_n) - g(\mu)) \leq t) - P^*(\sqrt{n}(g(\bar{X}_n^*) - g(\mu)) \leq t)|
\leq 2C_{H_3} \sqrt{\frac{1}{\lambda_\Sigma} \tau^2} \left( 1 + 2 \frac{\sqrt{x}}{p} \right) \frac{1}{C_{\nabla g} \sqrt{n}} + C_{B,i.i.d.} \frac{1}{\lambda_{\Sigma}^{3/2}} C_{M,3} \frac{1}{\sqrt{n}}
\]

\[
+ 2C_{B,i.i.d.} \left( 1 + 2 \frac{\sqrt{x}}{p} \right)^{3/2} \frac{1}{\lambda_{\Sigma}^{3/2}} \tau \frac{1}{\sqrt{n}} + R_{1,n,3}
\]

(26)

where \(C_{B,i.i.d.} > 0\) is a constant only depending on \(K\) and thus is a universal constant for \(K = 3\), \(C_{M,3}\) is defined as

\[
C_{M,3} = ||E[(X - \mu)^3]|| + ||E[(Y_1 - \mu)^3]||
\]

and \(R_{1,n,3}\) is defined in (B.14) as

\[
R_{1,n,3} = \frac{\tau^4 C_{x,2}}{\lambda_{\Sigma}^2 \sqrt{2n}} + \frac{4\tau^2 C_{x,2}}{\lambda_{\Sigma} \sqrt{2n}} + \frac{\tilde{C}_{\phi,2} \tau^3 C_{x,3}}{2 \lambda_{\Sigma}^{3/2} \sqrt{n}}.
\]

(27)

Since \(Y_1 - \mu \sim N(0, \Sigma_z)\), the tensor power \((Y_1 - \mu)^3\) has expectation zero and thus \(C_{M,3}\) can be simplified as \(C_{M,3} = ||E[(X - \mu)^3]||\). \(\tilde{C}_{\phi,2}\) in (27) is defined in (A.13) and according to Remark A.1, it depends on the choice of \(\phi(t)\) which is a \(K = 3\) times continuously differentiable smooth approximation of the indicator function \(1\{t \leq 0\}\). Once we fix such \(\phi(t)\), \(\tilde{C}_{\phi,2}\) is a universal constant. \(C_{x,2}\) and \(C_{x,3}\) in (27) are defined in (B.27) of Theorem B.1. But notice that in Assumption 2, instead of assuming \(X\) has a bounded density as in Theorem B.1, we assume that \(TX\) has independent
components for some orthogonal matrix $T$. In this case, the concentration inequality (B.32) reduces to the Hanson-Wright inequality (see Theorem 1.1 in ?), which holds for our $X$ since $||TAT^\top||_{HS} = ||A||_{HS}, ||TAT^\top|| = ||A||$ for any matrix $A$. Therefore, we have

$$C_{x,2} = C((x + \log n) \vee \sqrt{x + \log n})\sqrt{2x} + 2\sqrt{\frac{p}{n}} \left(1 + 2\sqrt{\frac{x + \log n}{p}} + \frac{2(x + \log n)}{p}\right)^{1/2},$$

$$C_{x,3} = C \left(\left(1 + 2\sqrt{\frac{x + \log n + \log p}{p}} + \frac{2(x + \log n + \log p)}{p}\right)\right)^{1/2} \times ((x + \log n + \log p) \vee \sqrt{x + \log n + \log p})\sqrt{2x} + 3\sqrt{\frac{p}{n}} \left(1 + 2\sqrt{\frac{x + \log n}{p}} + \frac{2(x + \log n)}{p}\right),$$

where $C$ is a universal constant based on the replacement of (B.32) with Hanson-Wright inequality in the proof of Theorem B.1. Besides, since we assume $n \geq 3$, we know that $x + \log n \geq \sqrt{x + \log n}$ and $x + \log n + \log p \geq \sqrt{x + \log n + \log p}$ holds for any $x > 0$. Therefore, $C_{x,2}$ and $C_{x,3}$ can be simplified as

$$C_{x,2} = C(x + \log n)\sqrt{2x} + 2\sqrt{\frac{p}{n}} \left(1 + 2\sqrt{\frac{x + \log n}{p}} + \frac{2(x + \log n)}{p}\right)^{1/2},$$

$$C_{x,3} = C \left(\left(1 + 2\sqrt{\frac{x + \log n + \log p}{p}} + \frac{2(x + \log n + \log p)}{p}\right)\right)^{1/2} (x + \log n + \log p)\sqrt{2x} + 3\sqrt{\frac{p}{n}} \left(1 + 2\sqrt{\frac{x + \log n}{p}} + \frac{2(x + \log n)}{p}\right).$$

Moreover, for any $y > 0$, we always have $1 + y \leq 1 + 2\sqrt{y} + 2y \leq 4(1 + y)$. Therefore, the remainder term (27) can be bounded in a more compact way up to some constants as follows:

$$R_{1,n,3} \leq C \left(\frac{\tau^4}{\lambda_\Sigma^2 \sqrt{n}} (x + \log n)\sqrt{x} + \frac{\tau^2}{\lambda_\Sigma \sqrt{n}} (x + \log n)\sqrt{x}\right)$$

$$+ \frac{\tau^3}{\lambda_\Sigma^{3/2} \sqrt{n}} \left(1 + \frac{x + \log n + \log p}{p}\right)^{1/2} (x + \log n + \log p)\sqrt{x}$$

$$+ \frac{\tau^4 \sqrt{p}}{\lambda_\Sigma^2 n} \left(1 + \frac{x + \log n}{p}\right)^{1/2} + \frac{\tau^3 \sqrt{p}}{\lambda_\Sigma^2 n} \left(1 + \frac{x + \log n}{p}\right)^{1/2} + \frac{\tau^3 \sqrt{p}}{\lambda_\Sigma^{3/2} n} \left(1 + \frac{x + \log n}{p}\right),$$

where $C$ is a universal constant. Plugging it back to (26), we can similarly write (26) in a more compact way:

$$\sup_{t \in \mathbb{R}} |P(\sqrt{n}(g(\bar{X}_n) - g(\mu)) \leq t) - P^*(\sqrt{n}(g(\bar{X}_n^*) - g(\bar{X}_n)) \leq t)|$$

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\[
\begin{align*}
\leq C \left( \frac{C_H \tau^2}{C_{\nabla g} \sqrt{\lambda_\Sigma}} \left( 1 + \frac{x}{p} \right) \sqrt{\frac{p}{n}} + \frac{\|E[(X - \mu)^3]\|}{\lambda_{3/2}^2} \frac{1}{\sqrt{n}} + \frac{\tau^3}{\lambda_{3/2}^2} \left( 1 + \frac{x}{p} \right)^{3/2} \frac{1}{\sqrt{n}} \right) \\
+ \frac{\tau^4 \sqrt{p}}{\lambda_{3/2}^2} \left( 1 + \frac{x + \log n}{p} \right)^{1/2} + \frac{\tau^2 \sqrt{p}}{\lambda_\Sigma} \left( 1 + \frac{x + \log n}{p} \right)^{1/2} + \frac{\tau^3 \sqrt{p}}{\lambda_{3/2}^2 n} \left( 1 + \frac{x + \log n}{p} \right) \\
+ \frac{\tau^4}{\lambda_{3/2}^2 \sqrt{n}} (x + \log n) \sqrt{x} + \frac{\tau^2}{\lambda_\Sigma \sqrt{n}} (x + \log n) \sqrt{x} \\
+ \frac{\tau^3}{\lambda_{3/2}^2 \sqrt{n}} \left( 1 + \frac{x + \log n + \log p}{p} \right)^{1/2} (x + \log n + \log p) \sqrt{\log n}\right),
\end{align*}
\]

where \(C\) is a universal constant. Now we choose \(x = \log n\). Then with probability at least \(1 - 6/n\), we have

\[
\begin{align*}
\sup_{t \in \mathbb{R}} |P(\sqrt{n}(g(\bar{X}_n) - g(\mu)) \leq t) - P^*(\sqrt{n}(g(\bar{X}_n) - g(\bar{X}_n)) \leq t)|
\leq C \left( \frac{C_H \tau^2}{C_{\nabla g} \sqrt{\lambda_\Sigma}} \left( 1 + \frac{x}{p} \right) \sqrt{\frac{p}{n}} + \frac{\|E[(X - \mu)^3]\|}{\lambda_{3/2}^2} \frac{1}{\sqrt{n}} + \frac{\tau^3}{\lambda_{3/2}^2} \left( 1 + \frac{x}{p} \right)^{3/2} \frac{1}{\sqrt{n}} \right) \\
+ \frac{\tau^4 \sqrt{p}}{\lambda_{3/2}^2} \left( 1 + \frac{x + \log n}{p} \right)^{1/2} + \frac{\tau^2 \sqrt{p}}{\lambda_\Sigma} \left( 1 + \frac{x + \log n}{p} \right)^{1/2} + \frac{\tau^3 \sqrt{p}}{\lambda_{3/2}^2 n} \left( 1 + \frac{x + \log n}{p} \right) \\
+ \frac{\tau^4 (\log n)^{3/2}}{\lambda_{3/2}^2 \sqrt{n}} + \frac{\tau^2 (\log n)^{3/2}}{\lambda_\Sigma \sqrt{n}} + \frac{\tau^3}{\lambda_{3/2}^2 \sqrt{n}} \left( 1 + \frac{x + \log n + \log p}{p} \right)^{1/2} (\log n + \log p) \sqrt{\log n}\right),
\end{align*}
\]

where \(C\) is a universal constant and we have absorbed \(\log p/p\) into the constant term \(1\) due to \(\log p/p \leq 1\).

Now we combine the above bound with Lemma 1 in this paper. Since \(X\) is sub-Gaussian, the moment conditions in Lemma 1 hold. Moreover, since \(\Sigma\) is positive definite and \(\|\nabla g(\mu)\| > 0\), \(\sigma^2 = \nabla g(\mu) \Sigma \nabla g(\mu) > 0\) is also satisfied. Therefore, by Lemma 1 and triangular inequality, we obtain the desired bound with probability at least \(1 - 6/n\)

\[
\begin{align*}
\sup_{x \in \mathbb{R}} |P^*(\sqrt{n}(g(\bar{X}_n) - g(\bar{X}_n)) \leq x) - \Phi \left( \frac{x}{\sigma} \right)|
\leq C \left( \frac{m_{31}}{\sqrt{n} \sigma^3} + \frac{C_{H_0} m_{31}^{1/3} (r(\Sigma))}{\sqrt{n} \sigma^2} + \frac{C_{H_0} m_{32}^{1/3} m_{33}^{1/3}}{n^{5/6} \sigma} + \frac{C_{H_0} m_{31}^{1/3} m_{33}^{1/3}}{n \sigma^2} \right) \\
+ \frac{C_{H_0} \tau^2}{C_{\nabla g} \sqrt{\lambda_\Sigma}} \left( 1 + \frac{x}{p} \right) \sqrt{\frac{p}{n}} + \frac{\|E[(X - \mu)^3]\|}{\lambda_{3/2}^2} \frac{1}{\sqrt{n}} + \frac{\tau^3}{\lambda_{3/2}^2} \left( 1 + \frac{x}{p} \right)^{3/2} \frac{1}{\sqrt{n}} \right) \\
+ \frac{\tau^4 \sqrt{p}}{\lambda_{3/2}^2} \left( 1 + \frac{x + \log n}{p} \right)^{1/2} + \frac{\tau^2 \sqrt{p}}{\lambda_\Sigma} \left( 1 + \frac{x + \log n}{p} \right)^{1/2} + \frac{\tau^3 \sqrt{p}}{\lambda_{3/2}^2 n} \left( 1 + \frac{x + \log n}{p} \right) \\
+ \frac{\tau^4 (\log n)^{3/2}}{\lambda_{3/2}^2 \sqrt{n}} + \frac{\tau^2 (\log n)^{3/2}}{\lambda_\Sigma \sqrt{n}} + \frac{\tau^3}{\lambda_{3/2}^2 \sqrt{n}} \left( 1 + \frac{x + \log n + \log p}{p} \right)^{1/2} (\log n + \log p) \sqrt{\log n}\right),
\end{align*}
\]

where \(C\) is a universal constant.
Proof of Theorem 3: Plugging the bounds in Lemma 1 and Lemma 2 into Theorem 1, we obtain the desired finite sample bound for the cheap bootstrap coverage accuracy. Besides, the error $E_1$ can be absorbed in $E_2$. \hfill \Box

Proof of Corollary 1: It suffices to show that if $\tau = O(1)$ and $||\nabla g(\mu)||^2 = O(p)$, then $tr(\Sigma) = O(p)$, $m_{31} = O(p^{3/2})$ and $m_{32} = O(p^{3/2})$. In fact, if these orders hold, with other orders assumed in Corollary 1, we can easily get the desired order after absorbing the small order terms into large order terms.

Now let us prove $\tau = O(p)$, $m_{31} = O(p^{3/2})$ and $m_{32} = O(p^{3/2})$ provided $\tau = O(1)$ and $||\nabla g(\mu)||^2 = O(p)$. Recall that $X$ is assumed to be sub-Gaussian, i.e.,

$$
E[\exp(a^T(X - \mu))] \leq \exp(||a||^2/2), \forall a \in \mathbb{R}^p.
$$

for some $\tau^2 > 0$. Therefore, $X_i - \mu_i, i = 1, \ldots, p$ are sub-Gaussian random variables with sub-Gaussian norm $\tau$ up to a universal constant (see Vershynin [2018] Section 2.5). For simplicity, we write $a \lesssim b$ if $a \leq Cb$ for a universal constant $C > 0$. By Proposition 2.5.2 (ii) in Vershynin [2018], $E[|X_i - \mu_i|^2] = \Sigma_{ii} \lesssim \tau^2$ and $E[|X_i - \mu_i|^4] \lesssim \tau^4$. Therefore, we can see $tr(\Sigma) \lesssim \tau^2p = O(p)$. By H"older’s inequality,

$$
m_{32} = E[||X - \mu||^3] \leq E[||X - \mu||^4]^{3/4} = \left( \sum_{i,j=1}^p E[(X_i - \mu_i)^2(X_j - \mu_j)^2] \right)^{3/4} \leq \left( \sum_{i,j=1}^p \sqrt{E[(X_i - \mu_i)^4]E[(X_j - \mu_j)^4]} \right)^{3/4} \lesssim \left( \sum_{i,j=1}^p \tau^4 \right)^{3/4} = \tau^3p^{3/2} = O(p^{3/2}).
$$

Moreover, (28) also implies that $\nabla g(\mu)^T(X - \mu)$ is sub-Gaussian with sub-Gaussian norm $||\nabla g(\mu)||\tau$ up to a universal constant. By Proposition 2.5.2 (ii) in Vershynin [2018], we have $m_{31} = E[||\nabla g(\mu)^T(X - \mu)||^3] \lesssim ||\nabla g(\mu)||\tau^3 = O(p^{3/2})$. This concludes our proof. \hfill \Box

Proof of Theorem 4: We will apply Theorem 1 to prove this theorem. The finite-sample bound (4) can be obtained by the Berry-Esseen theorem:

$$
\sup_{x \in \mathbb{R}} \left| P(\sqrt{n}(g_1^\top X_n - g_1^\top \mu) \leq x) - \Phi \left( \frac{x}{\sigma} \right) \right| \leq \frac{CE[|g_1^\top(X - \mu)|^3]}{\sigma^3 \sqrt{n}}.
$$

Next, we need to find a bound for

$$
\sup_{x \in \mathbb{R}} \left| P^*(\sqrt{n}(g_1^* X_n^* - g_1^* \mu) \leq x) - \Phi \left( \frac{x}{\sigma} \right) \right|.
$$

We consider the i.i.d. centered random variables $g_1^\top(X_i - \mu), i = 1, \ldots, n$ which have non-degenerate variance $\sigma^2 = g_1^\top \Sigma g_1 > 0$. We apply Theorem 2.5 in Lopes [2020] to $g_1^\top(X_i - \mu)$’s by choosing
\[ Y = N(0, g_1^\top \Sigma g_1) \] such that \( q = 1, \Delta = 0, \omega_1 = \|g_1^\top (X - \mu)/\sigma\|_{\psi_1} = \|g_1^\top (X - \mu)\|_{\psi_1}/\sigma \) and obtain that with probability at least \( 1 - C/n \),

\[
\sup_{x \in \mathbb{R}} \left| P^*(\sqrt{n}(g_1^\top X_n^* - g_1^\top \bar{X}_n) \leq x) - P(\sqrt{n}g_1^\top (X_n - \mu) \leq x) \right| \leq \frac{C\|g_1^\top (X - \mu)\|_{\psi_1}^4 \log^{11}(n)}{\sigma^4 \sqrt{n}},
\]

where \( C > 0 \) is a universal constant. By the triangle inequality and Berry-Esseen bound (29), the following holds with probability at least \( 1 - C/n \)

\[
\sup_{x \in \mathbb{R}} \left| P^*(\sqrt{n}(g_1^\top X_n^* - g_1^\top X_n) \leq x) - \Phi\left( \frac{x}{\sigma} \right) \right| \leq \frac{CE[\|g_1^\top (X - \mu)\|^3]}{\sigma^3 \sqrt{n}} + \frac{C\|g_1^\top (X - \mu)\|_{\psi_1}^4 \log^{11}(n)}{\sigma^4 \sqrt{n}},
\]

where \( C > 0 \) is a universal constant.

**Proof of Theorem 5:** We use Theorem 1 to prove this theorem. As in the proof of Theorem 4, (4) is given by the Berry-Esseen theorem in (29) and we only need to bound

\[
\sup_{x \in \mathbb{R}} \left| P^*(\sqrt{n}(g_1^\top \bar{X}_n^* - g_1^\top \bar{X}_n) \leq x) - \Phi\left( \frac{x}{\sigma} \right) \right|.
\]

In this regard, we will use the results in Chernozhukov et al. [2020]. Note that their results only apply for at least three-dimensional random vectors so we consider the following setting. Suppose we have \( 3n \) i.i.d. random variables \( g_1^\top (X_{ij} - \mu), i = 1, \ldots, n, j = 1, 2, 3 \) where each \( X_{ij} \) has the same distribution as \( X_1 \). Then we can construct \( n \) three-dimensional i.i.d. random vectors as \( \tilde{X}_i := (g_1^\top (X_{i1} - \mu), g_1^\top (X_{i2} - \mu), g_1^\top (X_{i3} - \mu))^\top, i = 1, \ldots, n \) whose components have common variance \( \sigma^2 = g_1^\top \Sigma g_1 \). Then we can see that conditions (E.3) and (M) are satisfied for

\[
B_n = \max \left\{ 3E[\|g_1^\top (X - \mu)/\sigma\|_{q}/\sqrt{n}, \sqrt{E[\|g_1^\top (X - \mu)/\sigma\|_{4}]} \right\}.
\]

Therefore, by Corollary 3.2 in Chernozhukov et al. [2020] \( (\sigma_{s,W} = 1 \text{ since } \bar{X}_i \text{ has independent components}) \), we have with probability at least \( 1 - 1/\sqrt{n} \) that

\[
\sup_{A \in \mathbb{R}} |P^*(\sqrt{n}(\tilde{X}_n^* - \bar{X}_n) \in A) - P(N(0, \sigma^2 I_{3 \times 3}) \in A)|
\]

\[ \leq C_1 B_n \left( \frac{\log 3 \log n \sqrt{\log(3\sqrt{n})}}{\sqrt{n}} + \frac{\log 3 \sqrt{\log(3n)}}{n^{1/2 - 3/(2q)}} \right) \]

\[ \leq C_1 \max \left\{ 3E[\|g_1^\top (X - \mu)/\sigma\|_{q}/\sqrt{n}, \sqrt{E[\|g_1^\top (X - \mu)/\sigma\|_{4}]} \right\} \left( \frac{\log n)^{3/2}}{\sqrt{n}} + \frac{\sqrt{\log n}}{n^{1/2 - 3/(2q)}} \right), \]

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where \( C_1 > 0 \) denotes a constant depending only on \( q \) which are different for its two appearances and \( \mathcal{R} \) contains all the hyperrectangles in \( \mathbb{R}^3 \). In particular, if we only focus on the first component of \( \tilde{X}_i \), that is, we choose \( A = (-\infty, x] \times \mathbb{R} \times \mathbb{R} \), we have with probability at least 1 – 1/\( \sqrt{n} \) that

\[
\sup_{x \in \mathbb{R}} \left| P^*(\sqrt{n}(g_1^\top \tilde{X}_n^* - g_1^\top \tilde{X}_n) \leq x) - \Phi \left( \frac{x}{\sigma} \right) \right| \\
\leq C_1 \max \left\{ 3E[|g_1^\top (X - \mu)/\sigma|^{1/q}, \sqrt{E[|g_1^\top (X - \mu)/\sigma|^q]} \right\} \left( \frac{(\log n)^{3/2}}{\sqrt{n}} + \frac{\sqrt{\log n}}{n^{1/2-3/(2q)}} \right). \quad (31)
\]

By Theorem 1 and the bounds (29) and (31), we then obtain

\[
|P(g(\mu) \in [g(\tilde{X}_n) - t_{B,1-\alpha/2}S_{n,B}, g(\tilde{X}_n) + t_{B,1-\alpha/2}S_{n,B})] - (1 - \alpha)| \\
\leq \frac{2}{\sqrt{n}} + BC_1 \max \left\{ E[|g_1^\top (X - \mu)/\sigma|^{1/q}, \sqrt{E[|g_1^\top (X - \mu)/\sigma|^q]} \right\} \times \left( \frac{(\log n)^{3/2}}{\sqrt{n}} + \frac{\sqrt{\log n}}{n^{1/2-3/(2q)}} \right) + CE\!\! \left[|g_1^\top (X - \mu)|^3 \right] \sigma^3 \sqrt{n},
\]

where \( C \) is a universal constant and \( C_1 \) is a constant depending only on \( q \). Finally notice that

\[
\frac{(\log n)^{3/2}}{\sqrt{n}} = o \left( \frac{\sqrt{\log n}}{n^{1/2-3/(2q)}} \right),
\]

and \( E[|g_1^\top (X - \mu)|^3]/\sigma^3 \geq 1 \). We can absorb \( (\log n)^{3/2}/\sqrt{n} \) into \( \sqrt{\log n}/n^{1/2-3/(2q)} \) and absorb \( 2/\sqrt{n} \) into \( CE\!\! \left[|g_1^\top (X - \mu)|^3 \right] \sigma^3 \sqrt{n} \) (with larger constants \( C_1 \) and \( C \)), which leads to

\[
|P(g(\mu) \in [g(\tilde{X}_n) - t_{B,1-\alpha/2}S_{n,B}, g(\tilde{X}_n) + t_{B,1-\alpha/2}S_{n,B})] - (1 - \alpha)| \\
\leq BC_1 \max \left\{ E[|g_1^\top (X - \mu)/\sigma|^{1/q}, \sqrt{E[|g_1^\top (X - \mu)/\sigma|^q]} \right\} \frac{\sqrt{\log n}}{n^{1/2-3/(2q)}} + CE\!\! \left[|g_1^\top (X - \mu)|^3 \right] \sigma^3 \sqrt{n}.
\]

\[ \square \]

**Proof of Theorem 6:** In view of Theorem 1, it suffices to show

\[
|P(\psi \in [\hat{\psi}_n - t_{B,1-\alpha/2}S_{n,B}, \hat{\psi}_n + t_{B,1-\alpha/2}S_{n,B}]) - (1 - \alpha)| \\
\leq 2\mathcal{E}_1 + 2\mathcal{E}_4 + \sqrt{\frac{2}{\pi}} |t_{B,1-\alpha/2} - z_{1-\alpha/2}| + \sqrt{\frac{2}{\pi}} \frac{\mathcal{E}_3}{\sigma} t_{B,1-\alpha/2}.
\]

Then taking the minimum of the two bounds, we can get the desired result. We write \( \mathcal{A} \) as the event that

\[
\left| \frac{1}{B} \sum_{b=1}^{B} (\sqrt{n}(\hat{\psi}_n - \hat{\psi}_n))^2 - \sigma \right| \leq \mathcal{E}_3 \\
\Leftrightarrow \sigma - \mathcal{E}_3 \leq \sqrt{\frac{1}{B} \sum_{b=1}^{B} (\sqrt{n}(\hat{\psi}_n - \hat{\psi}_n))^2} \leq \sigma + \mathcal{E}_3.
\]

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Then we have $P(A^c) \leq \mathcal{E}_4$. Note that the confidence interval can be written as

$$
\psi \in \left[ \hat{\psi}_n - t_{B,1-\alpha/2} S_{n,B}, \hat{\psi}_n + t_{B,1-\alpha/2} S_{n,B} \right]
$$

which implies

$$
\Rightarrow \left\{ \psi : \left| \frac{\sqrt{n}(\hat{\psi}_n - \psi)}{\sqrt{B \sum_{b=1}^B (\sqrt{n}(\hat{\psi}_n^b - \hat{\psi}_n))^2}} \right| \leq t_{B,1-\alpha/2} \right\}
$$

Therefore, we have

$$
P\left( \left| \frac{\sqrt{n}(\hat{\psi}_n - \psi)}{\sigma - \mathcal{E}_3} \right| \leq t_{B,1-\alpha/2}; A \right)
$$

$$
\leq P(\psi \in \left[ \hat{\psi}_n - t_{B,1-\alpha/2} S_{n,B}, \hat{\psi}_n + t_{B,1-\alpha/2} S_{n,B} \right]; A)
$$

$$
\leq P\left( \left| \frac{\sqrt{n}(\hat{\psi}_n - \psi)}{\sigma + \mathcal{E}_3} \right| \leq t_{B,1-\alpha/2}; A \right).
$$

For the upper bound, we can further bound it as follows

$$
P\left( \left| \frac{\sqrt{n}(\hat{\psi}_n - \psi)}{\sigma + \mathcal{E}_3} \right| \leq t_{B,1-\alpha/2}; A \right)
$$

$$
\leq P\left( \frac{\sqrt{n}(\hat{\psi}_n - \psi)}{\sigma + \mathcal{E}_3} \leq t_{B,1-\alpha/2} \right)
$$

$$
= P(- (\sigma + \mathcal{E}_3) t_{B,1-\alpha/2} \leq \sqrt{n}(\hat{\psi}_n - \psi) \leq (\sigma + \mathcal{E}_3) t_{B,1-\alpha/2}).
$$

By means of the finite-sample accuracy in (14), we have

$$
P(- (\sigma + \mathcal{E}_3) t_{B,1-\alpha/2} \leq \sqrt{n}(\hat{\psi}_n - \psi) \leq (\sigma + \mathcal{E}_3) t_{B,1-\alpha/2})
$$

$$
\leq \Phi \left( \frac{\sigma + \mathcal{E}_3}{\sigma} t_{B,1-\alpha/2} \right) - \Phi \left( \frac{- \sigma + \mathcal{E}_3}{\sigma} t_{B,1-\alpha/2} \right) + 2\mathcal{E}_1
$$

$$
\leq \Phi(z_{1-\alpha/2}) - \Phi(-z_{1-\alpha/2}) + 2\mathcal{E}_1 + \sqrt{\frac{2}{\pi}} \left| \frac{\sigma + \mathcal{E}_3}{\sigma} t_{B,1-\alpha/2} - z_{1-\alpha/2} \right|
$$

$$
\leq 1 - \alpha + 2\mathcal{E}_1 + \sqrt{\frac{2}{\pi}} |t_{B,1-\alpha/2} - z_{1-\alpha/2}| + \sqrt{\frac{2}{\pi}} \frac{\mathcal{E}_3}{\sigma} t_{B,1-\alpha/2},
$$

where $z_{1-\alpha/2}$ is the $(1-\alpha/2)$-th quantile of the standard normal and the second inequality is due to the $1/\sqrt{2\pi}$-Lipschitz property of $\Phi(\cdot)$. For the lower bound, by a similar argument we can obtain

$$
P\left( \left| \frac{\sqrt{n}(\hat{\psi}_n - \psi)}{\sigma - \mathcal{E}_3} \right| \leq t_{B,1-\alpha/2}; A \right)
$$

$$
\geq P\left( \left| \frac{\sqrt{n}(\hat{\psi}_n - \psi)}{\sigma - \mathcal{E}_3} \right| \leq t_{B,1-\alpha/2} \right) - P(A^c)
$$

$$
\geq 1 - \alpha - 2\mathcal{E}_1 - \sqrt{\frac{2}{\pi}} |t_{B,1-\alpha/2} - z_{1-\alpha/2}| - \sqrt{\frac{2}{\pi}} \frac{\mathcal{E}_3}{\sigma} t_{B,1-\alpha/2} - P(A^c)\)
\[ \geq 1 - \alpha - 2\mathcal{E}_1 - \sqrt{\frac{2}{\pi}}|t_{B,1-\alpha/2} - z_{1-\alpha/2}| - \sqrt{\frac{2}{\pi}} \mathcal{E}_3 t_{B,1-\alpha/2} - \mathcal{E}_4. \]

Thus, by combining the upper and lower bounds, we have the following bound for the coverage error when \( \mathcal{A} \) happens

\[ |P(\psi \in [\hat{\psi}_n - t_{B,1-\alpha/2}S_{n,B}, \hat{\psi}_n + t_{B,1-\alpha/2}S_{n,B}]; \mathcal{A}) - (1 - \alpha)| \]
\[ \leq 2\mathcal{E}_1 + \mathcal{E}_4 + \sqrt{\frac{2}{\pi}}|t_{B,1-\alpha/2} - z_{1-\alpha/2}| + \sqrt{\frac{2}{\pi}} \sigma t_{B,1-\alpha/2}. \]

Finally, the overall coverage error can be bounded by

\[ |P(\psi \in [\hat{\psi}_n - t_{B,1-\alpha/2}S_{n,B}, \hat{\psi}_n + t_{B,1-\alpha/2}S_{n,B}]) - (1 - \alpha)| \]
\[ \leq |P(\psi \in [\hat{\psi}_n - t_{B,1-\alpha/2}S_{n,B}, \hat{\psi}_n + t_{B,1-\alpha/2}S_{n,B}]; \mathcal{A}) - (1 - \alpha)| + P(\mathcal{A}^c) \]
\[ \leq 2\mathcal{E}_1 + 2\mathcal{E}_4 + \sqrt{\frac{2}{\pi}}|t_{B,1-\alpha/2} - z_{1-\alpha/2}| + \sqrt{\frac{2}{\pi}} \sigma t_{B,1-\alpha/2}, \]

which, combined with Theorem 1, gives us the desired bound.

\[ \square \]

**Proof of Theorem 7:** By means of Lemmas 1 and 2, this directly follows from Theorem 2. Besides, the error \( \mathcal{E}_1 \) can be absorbed into \( \mathcal{E}_2 \).

\[ \square \]

**Proof of Theorem 8:** Plugging the bounds (29) and (30) into Theorem 2, we get the desired result.

\[ \square \]

**Proof of Theorem 9:** Plugging the bounds (29) and (31) into Theorem 2, we get the desired result.