Special vortex in relativistic hydrodynamics

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Abstract. An exact solution of the Euler equations governing the flow of a compressible fluid in relativistic hydrodynamics is found and studied. It is a relativistic analogue of the Ovsyannikov vortex (special vortex) investigated earlier for classical gas dynamics. Solutions are partially invariant of Defect 1 and Rank 2 with respect to the rotation group. A theorem on the representation of the factor-system in the form of a union of a non-invariant subsystem for the function determining the deviation of the velocity vector from the meridian, and invariant subsystem for determination of thermodynamic parameters, the Lorentz factor and the radial velocity component is proved. Compatibility conditions for the overdetermined non-invariant subsystem are obtained.

A stationary solution of this type is studied in detail. It is proved that its invariant subsystem reduces to an implicit differential equation. For this equation, the manifold of branching of solutions is investigated, and a set of singular points is found.

1. Introduction
Equation of relativistic hydrodynamics governing the flow of a compressible fluid (gas dynamics) have the form [1]

\begin{align}
(\Gamma \rho)_{,t} + \nabla \cdot (\Gamma \rho \vec{u}) &= 0, \\
(\Gamma w \vec{u})_{,t} + \nabla \cdot (\Gamma w \vec{u} \times \vec{u}) + \nabla p &= 0, \\
(\Gamma w - p)_{,t} + \nabla \cdot (\Gamma w \vec{u}) &= 0.
\end{align}

In (1) vector $\vec{u}$ — gas velocity in laboratory coordinate system, $p$, $\rho$ — pressure and density, $w = 1 + \gamma \frac{\gamma - 1}{\gamma + 1} \frac{p}{\rho}$ — enthalpy, $\Gamma = (1 - |\vec{u}|^2)^{-1/2}$ — Lorentz factor are function of independent variables — the time $t$ and the spatial coordinates $\vec{x} = (x^1, x^2, x^3)$. A coordinate system is chosen such that the speed of light $c = 1$ and for the velocity modulus $|\vec{u}| = (\sum_{i=1}^{3} u_i^2)^{1/2}$ inequality $|\vec{u}| < 1$ is satisfied.

Equations (1) are written in the form of conservation laws. This is convenient for investigating the correctness of initial-boundary value problems and numerical modelling [2, 3, 4]. To construct and analyse the exact solution "special vortex", another form of equations (1) is more convenient.

Lemma 1 The Euler equations of compressible fluid for relativistic hydrodynamics can be written in the form

\begin{align}
b \mathbf{D} \vec{u} + p_{,t} \vec{u} + \nabla p &= 0, \\
Da + a \nabla \vec{u} &= 0, \\
a \mathbf{D}d &= p_{,t}.
\end{align}
In (2) \( a = \Gamma \rho, \ b = \Gamma^2 \rho w, \ d = b/a = \Gamma w, \ D = \partial_t + \vec{u} \cdot \nabla \) — total derivative.

A special vortex (Ovsyannikov vortex) is a solution of the differential equations, which is partially invariant with respect to the group of rotations \( SO(3) \) in the space \( \mathbb{R}^3(\vec{x}) \times \mathbb{R}^3(\vec{u}) \). In this representation, the group \( SO(3) \) has invariants \( r = |\vec{x}|^{1/2}, |\vec{u}| \) and \( \vec{x} \cdot \vec{u} \). Since there are only two invariants depending on the three velocity components \((u^1, u^2, u^3)\), we can construct a partially invariant solution. Invariant independent variables are \( t \) and \( r \). The choice of a non-invariant function is less obvious. L. V. Ovsyannikov in [5] proposed to take this function as an angle \( \omega = \omega(t, r, \theta, \varphi) \) measuring the deviation of the velocity vector \( \vec{u} \) from meridian of the sphere \( r = \text{const} \). We introduce spherical coordinates \((r, \theta, \phi)\) and \((U, V, W)\):

\[
x = r \sin \theta \cos \varphi, \quad y = r \sin \theta \sin \varphi, \quad z = r \cos \theta,
\]

such that \( r \geq 0, \ 0 \leq \theta \leq \pi, \ 0 \leq \varphi \leq 2\pi; \ U, V, W \) — radial, altitude and longitude components of the velocity. We introduce new coordinates for the velocity components, namely polar coordinates for tangent component \( \vec{u}_r = (V, W) \) of the velocity vector:

\[
V = H \cos \omega, \quad W = H \sin \omega,
\]

such that \( H = \sqrt{V^2 + W^2}, \ \omega = \arctan W/V \). Special vortex is a solution of equations (2) in which a special dependence of the functions on the independent variables is realized, namely:

\[
U = U(r, t), \quad \Gamma = \Gamma(r, t), \quad H = H(r, t), \\
p = p(r, t), \quad \rho = \rho(r, t), \quad w = w(r, t), \\
\omega = \omega(t, r, \theta, \varphi).
\]

From the general theory of group analysis of differential equations [6], after substitution of the representation (4) in (2), we obtain composition of a factor-system in the form of a union of an overdetermined system of differential equations for function \( \omega \) and a system of equations for invariant functions \( U, \Gamma, H, p, \rho, w \).

Relativistic hydrodynamics equations are written in various forms, adapted to solve specific problems [4]. In [2] they are reduced to a symmetrized form. Some particular solutions of these equations are investigated, for example, in [7, 8, 9].

2. Special vortex equations for relativistic gas dynamics

After substitution of (4) in (2), written in spherical coordinates \((r, \theta, \phi)\) and \((U, H, \omega)\), we obtain following statement

**Lemma 2** Special vortex equations for relativistic gas dynamics are represented as a union of invariant subsystem

\[
ad(D_0 U - \frac{1}{r}H^2) + p_t U + p_r = 0,
\]

\[
D_0 H + \frac{U}{r} H + p_t H = 0,
\]

\[
ad_0 d = p_t,
\]

where \( D_0 = \partial_t + U \partial_r \) — invariant part of the total derivative, and overdetermined subsystem for function \( \omega \):

\[
k \sin \theta \partial_0 \omega + \sin \theta \cos \omega \partial_\theta \omega + \sin \omega \partial_\varphi \omega + \cos \theta \sin \omega = 0,
\]

\[
\sin \theta \sin \omega \partial_\theta \omega - \cos \omega \partial_\varphi \omega = h \sin \theta + \cos \theta \cos \omega,
\]

where

\[
k = r/H, \quad h = k(a^{-1}D_0 a + r^{-2}(r^2U)_r).
\]
It is remarkable that (6) and (7) coincide exactly with the corresponding equations for classical gas dynamics [5, 10]. But function \( h \) is different, it is related to other physical quantities. This fact emphasizes the extremely successful choice of variables made by Ovsyannikov in [5]. A consequence of this fact is

**Lemma 3 (Ovsyannikov compatibility condition)** Compatibility condition of overdetermined system (6) has the form

\[ kD_0h = h^2 + 1. \]  

Equations (8) complement invariant subsystem (5). Thus, all mathematical results proved for the special vortex in classical gas dynamics [5, 10] are carried over to special vortex in relativistic gas dynamics.

### 3. Stationary special vortex for relativistic gas dynamics

Consider a stationary special vortex, i.e. solution which is partially invariant with respect to the group \( \langle \partial_t, SO(3) \rangle \). In representation (4) and in equations of Lemma 2, it is necessary to remove the dependence on time, then \( D_0 = U \frac{d}{dR} \). Consider, in the following, monatomic gas with \( \gamma = 5/3 \), which is interesting for applications. Equations (5) can be partially integrated and the following Lemma takes place

**Lemma 4** For stationary special vortex for relativistic gas dynamics invariant subsystem is reduced to ordinary implicit differential equation

\[ F(R, h, p; m_0, s_0) \equiv q^{3/2} - R^2 p \left( 3m_0 + s_0 \frac{p^2}{1 + h^2} \right) q + 3m_0^2 R^4 p^2 q^{1/2} - m_0^3 R^6 p^3 = 0, \]  

where \( p = \frac{d}{dR} \); \( s_0 > 0, 0 < m_0 < 1 \) — constants characterizing physics of the problem, \( R > 1 \) — normalized distance, \( q = q(R, h, p) \) has the form

\[ q(R, h, p) = R^2(R^2 - 1)p^2 - (1 + h^2)^2. \]  

Invariant functions have following form in terms of function \( h \) and its derivative \( h_R \):

\[ U = \frac{1 + h^2}{R^2 h_R}, \quad a = \frac{a_0}{a_0} \frac{h_R}{\sqrt{1 + h^2}}, \]

\[ \Gamma = \left[ 1 - \frac{1}{R^2} \left( 1 + \frac{1 + h^2}{R h_R} \right)^2 \right]^{-1/2}, \]

\[ d = d_0, \quad H = \frac{1}{R}, \quad w = d_0 / \Gamma, \]  

where \( a_0, a_0, d_0 \) are constant.

Thus, the determination of the special vortex for relativistic gas dynamics reduces to solving equation (9) and overdetermined system (6). This system is integrated in finite form in [5], its solution \( \omega \) is given by an implicit function. Geometric interpretation and domain investigation are given in [11].

Equation (9) belongs to a class of equations that are unresolved with respect to derivatives. Now such equations are usually called implicit differential equations. The present state of the theory is presented in [12], detailed historical review is given in [13, 14]. The specificity of equations of this type is the existence of manifold of branching solutions, the presence of trajectories bundle starting from singular points of different degrees of degeneracy [12]. Applications of this theory to gas dynamics can be found, for example, in [5, 15].
4. Investigation of singular points of equation (9)
Implicit equation (9) can be resolved with respect to derivative $p$ at all points of $\mathbb{R}^3(R, h, p)$, except for the points of manifold

$$F(R, h, p) = 0, \quad F_p(R, h, p) = 0. \quad (12)$$

Curve (12) is called a criminant of equation (9), it is manifold of branching of integral curves. It consists of singular points of equation (9), called regular singular points. In general, equation (9) can have singular points of two types — folded and collected. Folded singular points can be found from the system of equations

$$F(R, h, p) = 0, \quad F_p(R, h, p) = 0, \quad F_R(R, h, p) + pF_h(R, h, p) = 0. \quad (13)$$

Collected singular points are determined from the system of equations

$$F(R, h, p) = 0, \quad F_p(R, h, p) = 0, \quad F_{pp}(R, h, p) = 0. \quad (14)$$

Projection of the criminant curve onto plane $\mathbb{R}^2(h, R)$ is called a discriminant curve.

The essence of the geometric approach to study of implicit differential equations, proposed by Poincaré, is to raise the equation to a vector field

$$R_\tau = F_p, \quad h_\tau = pF_p, \quad p_\tau = -(F_R + pF_h), \quad (15)$$

where $\tau$ — new parameter along integral curve. With this interpretation, the integral curves of equation (9) are located on different sheets of surface $F = 0$. These integral curves can be projected onto the plane $\mathbb{R}^2(h, R)$ with overlapping.

The investigation of the singular points of equations (9) is connected with large computational difficulties. The solution of the systems of equations (13) and (14), determined by the resultants of the corresponding polynomials, can be obtained by means of a system of symbolic and numerical computations. Even with the use of such computing systems, the calculation can be quite long.

5. Numerical techniques
We will deal with $F^2$ because it is more convenient from the computational point of view (everywhere below by $F$ we mean $F^2$).

5.1. Discriminant curve
The discriminant curve is given by the following system of equations

$$F(R, h, p) = 0, \quad F_p(R, h, p) = 0. \quad (16)$$

To solve this and other similar systems, the following technique is used. Since the equations of the system (16) are polynomials in $p$, we can eliminate $p$ by constructing the resultant of two these polynomials. Denote by $\Phi_1 = F$, $\Phi_2 = F_p$. Then

$$R_{12}(R, h) = \text{Resultant}(\Phi_1, \Phi_2). \quad (17)$$

Further, in the $(R, h)$-plane, we can numerically solve the algebraic equation $R_{12}(R, h) = 0$ for $h$. Thus, for given $m_0$, $s_0$ and $R$, we can calculate the discriminant curve. As a result of the numerical experiment, the following statement was obtained.

**Theorem 1 (about discriminant curve)** The discriminant curve consists of one or two components.

The discriminant curve always has its “main” component, but for small $s_0$ ($s_0 \approx 10^{-2}$ and less) an additional (“secondary”) component of the discriminant curve appears (Figure 1).
5.2. Folded singular points

Folded singular points are found from the system of equations (13). To solve this system, we apply the same technique as for system of equations (16). Denote by $\Phi_1 = F, \Phi_2 = F_p, \Phi_3 = F_R + pF_h$. In order to solve this system, we construct the following resultants

\[
R_{12}(R, h) = \text{Resultant}(\Phi_1, \Phi_2), \\
R_{13}(R, h) = \text{Resultant}(\Phi_1, \Phi_3).
\] (18)

Intersection of curves $R_{12} = 0$ and $R_{13} = 0$ gives us a set of folded singular points. To find $R_{12}(R, h)$, it is necessary to calculate determinant of order 19 ($11! = 39,916,800$ terms, without zeros). After simplification, the degree of the polynomial obtained is 60 and 108 for $R$ and $h$, respectively. To find $R_{13}(R, h)$ we calculate determinant of order 20 ($15! = 1,307,674,368,000$ terms, without zeros). The degree of $R_{13}(R, h)$ is 64 for $R$ and 170 for $h$.

As a result of the numerical experiment, the following statement was obtained.

**Theorem 2 (existence and uniqueness of folded singular points)** For any values of parameters $s_0$ and $m_0$ there is a unique folded singular point.

Figure 2 shows the standard arrangement of $R_{12}$ and $R_{13}$.
5.3. Collected singular points

Collected singular points are found from the system of equations (14). Note that we can use previous result — \( R_{12} = \text{Resultant}(\Phi_1, \Phi_2) \), see (18). Denoting \( \Phi_4 = F_{pp} \), we obtain one more resultant for finding collected singular points

\[
R_{14}(R, h) = \text{Resultant}(\Phi_1, \Phi_4).
\]  (19)

Intersection of curves \( R_{12} = 0 \) and \( R_{14} = 0 \) gives us a set of collected singular points. To find \( R_{14}(R, h) \) we calculate determinant of order 18 (11! terms). After simplification, the degree of \( R_{14}(R, h) \) is 88 for \( R \) and 144 for \( h \).

As a result of numerical simulation we obtained the following statement.

**Theorem 3 (classification of collected singular points)** For all parameters \( s_0 > 0 \) and \( 0 < m_0 < 1 \) only one of the following cases is realized:

(i) there are no collected points,

(ii) there is one collected point (on the “main” component),

(iii) there are two collected points (one on the “main” component and one on the “secondary” component).

It is found that when \( s_0 \) moves from 0 to 1, the curve \( R_{14} = 0 \) moves to the direction of increasing \( R \). The curve \( R_{12} = 0 \) has exactly the same behaviour when \( m_0 \) is moving from 0 to 1. For identical \( m_0 \) and \( s_0 \) they coincide. The third case is realized for small \( s_0 \) (\( s_0 \approx 10^{-5} \) and less).

![Figure 3](image-url) Zero, one and two collected singular points. The intersection points of curves \( R_{12} \) and \( R_{14} \) are collected singular points.

All calculations are performed in Wolfram Mathematica.

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