INTEGRALS OF SUBHARMONIC FUNCTIONS AND THEIR DIFFERENCES WITH WEIGHT OVER SMALL SETS ON A RAY

1. Introduction.

1.1. Origins and research subject. One of the classical theorems of Rolf Nevanlinna can be considered as the original source of our results in this article [1, pp. 24–27]. Anatolii Asirovich Gol’dberg and Iosif Vladimiriovich Ostrovskii indicate the last reference in their classic monograph [2, Notes, Ch. 1]. The original source [1] remained inaccessible to us. But the mentioned theorem is stated with a complete proof in the monograph of A. A. Gol’dberg and I. V. Ostrovskii [2, Ch. 1, Theorem 7.2]. We give this result exactly in their formulation and notations.

Let \( f \) be a meromorphic function on the complex plane \( \mathbb{C} \) with the real axis \( \mathbb{R} \),

\[
M(r, f) := \max \left\{ |f(z)| : |z| = r \right\}, \quad r \in \mathbb{R}^+ := \{x \in \mathbb{R} : x \geq 0\},
\]

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and with the Nevanlinna characteristics

\[ T(r, f) := m(r, f) + N(r, f), \quad (1T) \]

\[ m(r, f) := \frac{1}{2\pi} \int_0^{2\pi} \ln^+ |f(re^{i\varphi})| \, d\varphi, \quad \ln^+ x := \max\{\ln x, 0\}, \quad (1m) \]

\[ N(r, f) := \int_0^r \frac{n(t, f) - n(0, f)}{t} \, dt + n(0, f) \ln r, \quad (1N) \]

where \( n(r, f) \) is the number of poles of \( f \) in the closed disc \( \overline{D}(r) := \{ z \in \mathbb{C} : |z| \leq r \} \), taking into account the multiplicity.

**Rolf Nevanlinna Theorem** ([2, Ch. 1, Theorem 7.2]). Let \( f(z) \) be a meromorphic function, \( k > 1 \) be a real number. Then

\[ \frac{1}{r} \int_0^r \ln^+ M(t, f) \, dt \leq C(k)T(kr, f), \quad (2) \]

where the constant \( C(k) > 1 \) depends on \( k \) only.

But [2, Ch. 1, proof of Theorem 7.2] uses [2, Chap. 1, Lemma 7.1] for \( R' := \sqrt{k}r \), which was proved only for \( R' > R > 1 \). It turns out to be essentially. So, for meromorphic function

\[ f(z) \equiv \frac{1}{z}, \quad M(r, f) \equiv \frac{1}{r}, \quad m(r, f) \equiv \ln^+ \frac{1}{r}, \quad N(r, f) \equiv \ln r, \quad T(r, f) \equiv \ln^+ r, \quad (3M) \]

for the left-hand side of (2), we have

\[ \frac{1}{r} \int_0^r \ln^+ M(t, f) \, dt \equiv \frac{1}{r} \int_0^r \ln^+ \frac{1}{t} \, dt \equiv 1 + \ln^+ \frac{1}{r} \geq 1. \quad (3I) \]

Thus, if (2) is satisfied, then \( C(k) \ln^+ kr \geq 1 \). But this is impossible if \( 0 \leq r \leq 1/k \). For the constant \( C(k) \) independent of \( r \), this is possible only if the requirement of the form \( r \geq r_0 > 0 \) is added to the inequality (2), and the constant \( C(k) \) also depends from the choice of a fixed number \( r_0 > 0 \). For the sake of fairness, we note that the known cases of application of the Rolf Nevanlinna Theorem are considered, as a rule, only for the cases \( r \to +\infty \) or \( r \geq 1 \).

Integrals over small subsets on arc or ray intervals are also widely used in the theory of entire and meromorphic functions. The starting point of these second-type estimates is the A. Edrei and W. H. J. Fuchs Lemma on Small Arcs [3, Sect. 2, Lemma III, Sect. 9], which has found important applications in the theory of meromorphic functions, reflected, in particular, in [2, Chap. 1, Theorems 7.3, 7.4]. A variation on the Edrei–Fuchs Lemma on Small Arcs is a Lemma on Small Intervals by A. F. Grishin and M. L. Sodin. The authors note that its proof repeats verbatim the proof of the Edrei–Fuchs Lemma on Small Arcs, and therefore
it is presented in [4, Lemma 3.1] without proof, but with interesting applications [4, Lemma 3.2, Theorem 3.1]. We formulate the Grishin–Sodin Lemma on Small Intervals also in the literal translation of the author’s version.

Denote by $\text{mes} E$ the linear Lebesgue measure of $E \subset \mathbb{R}$, and $E(R) := E \cap [1, R)$.

**Grishin–Sodin Lemma on Small Intervals** ([4, Lemma 3.1]). Let $f$ be a meromorphic function on $\mathbb{C}$, $E \subset [1, +\infty)$. Then

$$
\frac{1}{r} \int_{E(r)} \ln^+ M(t, f) \, dt \leq C \frac{k}{k - 1} \left( \frac{\text{mes} E(r)}{r} \ln \frac{2r}{\text{mes} E(r)} \right) T(kr, f),
$$

where $C$ is an absolute constant.

A version of the Grishin–Sodin Lemma on Small Intervals for subharmonic functions, but only of finite order, is proved in the joint work of A.F. Grishin and T.I. Malyutina [5]. The Grishin–Malyutina Lemma on Small Intervals found several important applications in proofs of key results of [5, Theorems 2, 4] and [6, 4.3, Lemma 4.2]. We do not give this version, because it is embedded in our joint with L.A. Gabdrakhmanova main result from [7, Theorem 1 (on small intervals)], where it is discussed in detail [7, Conclusion of the Grishin–Malyutina Theorem].

In this article, we consider integrals over subsets of intervals on a ray, but without estimates of integrals over small arcs on circles. In particular, we generalize the Rolf Nevanlinna Theorem, as well as the Grishin–Sodin Lemma on Small Intervals. Our Main Theorem on small intervals with integral $L^p$-norms for functions-multipliers is formulated in Subsection 1.2. Generalizations of the previous results are given in several directions. First, we consider differences of subharmonic functions, i.e., $\delta$-subharmonic functions of arbitrary growth on the plane. We give a simple correction to the summands dictated by counterexample (3). Second, in the integrand, a multiplier function of the class $L^p$ is allowed for $1 < p \leq \infty$, and the integrals are estimated using the $L^p$-norm with corresponding changes in the contribution of small subsets $E \subset \mathbb{R}^+$. Third, the estimates of integrals over intervals are in a certain sense uniform with respect to the class of all $(\delta)$-subharmonic functions with an integral normalization near zero. This will make it possible to translate them in the future into estimates of integrals of plurisubharmonic functions and their differences in a multidimensional complex space both for intervals on rays with common beginnings and for subsets on these intervals.

1.2. Basic definitions and a recent theorem on small intervals. This subsection can be referred to as needed. Our notations may differ from those used above. We write single-point sets without curly braces, if this does not cause confusion. So, $\mathbb{N} := \{1, 2, \ldots \}$ is the set of natural numbers and $\mathbb{N}_0 := \{0\} \cup \mathbb{N}$, $\mathbb{Z} := (-\mathbb{N}) \cup \mathbb{N}_0$ is the set of integers, and $\overline{\mathbb{R}}$ is the extended real axis with $-\infty := \inf \mathbb{R}$ and $+\infty := \sup \mathbb{R}$, $-\infty \leq x \leq +\infty$ for each $x \in \mathbb{R}$, $-(\pm \infty) = \mp \infty$, $\overline{\mathbb{R}}^+ := \mathbb{R}^+ \cup +\infty$, and

$$
 x + (+\infty) = +\infty \text{ for } x \in \overline{\mathbb{R}} \setminus -\infty, x + (-\infty) = -\infty \text{ for } x \in \overline{\mathbb{R}} \setminus +\infty, \\
 x \cdot (\pm \infty) := \pm \infty =: (-x) \cdot (\mp \infty) \text{ for } x \in \overline{\mathbb{R}}^+ \setminus 0, \\
\frac{\pm x}{0} := \pm \infty \text{ for } x \in \overline{\mathbb{R}}^+ \setminus 0, \quad \frac{x}{\pm \infty} := 0 \text{ for } x \in \mathbb{R}, \quad \text{but } 0 \cdot \pm \infty := 0,
$$

(5)
unless otherwise stated; \( x^+ := \max\{0, x\} = (-x) \) for \( x \in \mathbb{R} \); \( \inf \emptyset := +\infty \) and \( \sup \emptyset := -\infty \) for the empty set \( \emptyset \). An interval \( I \subset \mathbb{R} \) is a connected set with ends \( \inf I \in \mathbb{R} \) and \( \sup I \in \mathbb{R} \); \( (a, b) := \{ x \in \mathbb{R} : a < x < b \} \), \( [a, b] := \{ x \in \mathbb{R} : a \leq x \leq b \} \), \( (a, b] := [a, b) \backslash a \). \( D(z, r) := \{ z' \in \mathbb{C} : |z' - z| < r \} \) is an open disc, \( \overline{D}(z, r) := \{ z' \in \mathbb{C} : |z' - z| \leq r \} \) is a closed disc, \( \partial D(z, r) := \{ z' \in \mathbb{C} : |z' - z| = r \} \) is the circle with center \( z \in \mathbb{C} \) of radius \( r \in \mathbb{R}^+ \); \( D(z, 0) = \emptyset \), \( \overline{D}(z, 0) = \partial D(z, 0) = z \), \( D(z, +\infty) = \mathbb{C} \). Besides, \( D(r) := D(0, r) \), \( \overline{D}(r) := \overline{D}(0, r) \).

Given a function \( f : X \to \mathbb{R} \), \( f^+ := \sup \{ 0, f \} \) and \( f^- := (-f)^+ \) are positive and negative parts of function \( f \), respectively; \( |f| := f^+ + f^- \).

Given \( S \subset \mathbb{C} \), \( \text{sbh}(S) \) is the class of all subharmonic on an open neighbourhood of \( S \). The class \( \text{sbh}_\neq(S) \) contains the trivial \((-\infty)\)-function \(-\infty\). We set \( \text{sbh}_\neq(S) := \text{sbh}(S) \backslash -\infty \) for connected subset \( S \subset \mathbb{C} \).

By \( \lambda \) we denote the linear Lebesgue measure on \( \mathbb{R} \) and its restrictions on arbitrary \( \lambda \)-measurable subsets \( S \subset \mathbb{R} \), by setting \( \lambda(\pm \infty) := 0 \). We also use the notation \( \lambda(S) \) from Subsection 1.1. The concepts almost everywhere, measurability and integrability mean \( \lambda \)-almost everywhere, \( \lambda \)-measurability and \( \lambda \)-integrability, respectively.

We denote by
\[
\text{ess sup}_{S} f := \inf \{ a \in \mathbb{R} : \lambda(\{ x \in S : f(x) > a \}) = 0 \}
\]
(6)
the essential upper bound of measurable function \( f \) defined almost everywhere on \( S \), and
\[
L^\infty(S) := \{ f : \| f \|_{L^\infty(S)} := \text{ess sup}_{S} |f| < +\infty \},
\]
(7\(\infty\))
and for numbers \( p > 1 \),
\[
L^p(S) := \{ f : \| f \|_{L^p(S)} := \left( \int_{S} |f|^p \, d\lambda \right)^{1/p} < +\infty \}, \quad p \in (1, +\infty),
\]
(7\(p\))
together with the numbers \( q \) associated with \( p \) by equality
\[
\frac{1}{p} + \frac{1}{q} = 1, \quad 1 < q = \frac{p}{p - 1} < +\infty, \quad \text{but } q := 1 \text{ if } p = \infty.
\]
(7\(q\))

If \( S := I \) is an interval with ends \( a \leq b \), then for Lebesgue integral of \( f \) over the interval \( I \) we will use two forms of notation
\[
\int_{I} f \, d\lambda := \int_{a}^{b} f(t) \, dt.
\]
(8)

For \( r \in \mathbb{R}^+ \) and an arbitrary function \( v : \partial D(0, r) \to \mathbb{R} \), we define
\[
M_v(r) := \sup_{|z|=r} v(z), \quad C_v(r) := \frac{1}{2\pi} \int_{0}^{2\pi} v(re^{i\alpha}) \, d\alpha
\]
(9)
The latter is the average over the circle \( \partial D(0, r) \) for \( v \), if the function \( s \mapsto re^{i\alpha} \) is integrable on \([0, 2\pi]\); \( C_v(0) := M_v(0) = C_v(0) = v(0) \). For the properties of the characteristics \( M_v \) and
\[ M(r, f) = M_{ln|f|}(r), \quad m(r, f) = C_{ln^+|f|}(r). \]

**Theorem on small intervals with weight** ([7, Theorem 1, Remark 1.1]). There is an absolute constant \( a \geq 1 \) such that

\[ \int_E M_{|u|} g \, d\lambda \leq left( \frac{a}{b} \ln \frac{a}{b} right) \left( M_u((1 + b)R) + 2C_u^-(r_0) \right) \|g\|_{L^\infty(E)} \times \]

\[ \times \left( \text{mes } E + \min\{\text{mes } E, 3bR\} \ln \frac{3ebR}{\min\{\text{mes } E, 3bR\}} \right) \]

where \( m_{\infty}(E; R, b) \leq 2 \text{mes } E \) when \( \text{mes } E > 3bR \), and

\[ m_{\infty}(E; R, b) \leq 2 \text{mes } E \ln \frac{3ebR}{\text{mes } E}, \quad \text{if } \text{mes } E \leq 3bR. \]

**2. Main Theorem.** For a Borel subset \( S \subset \mathbb{C} \), the set of all Borel, or Radon, positive measures \( \mu \geq 0 \) on \( S \) is denoted by \( \text{Meas}^+(S) \), and \( \text{Meas}(S) := \text{Meas}^+(S) - \text{Meas}^+(S) \) is the set of all charges, or signed measures, on \( S \). For a measure \( \mu \in \text{Meas}^+(\bar{D}(R)) \), we set

\[ \mu^{rad}(r) := \mu(\bar{D}(r)) \in \mathbb{R}^+, \quad 0 \leq r \leq R, \]

\[ N_{\mu}(r, R) := \int_r^R \frac{\mu^{rad}(t)}{t} \, dt \in \mathbb{R}^+, \quad 0 \leq r \leq R, \]

For \( 0 \leq r \leq R \in \mathbb{R}^+ \) and an arbitrary function \( v: \partial\bar{D}(r) \cup \partial\bar{D}(R) \to \mathbb{R} \), we define

\[ C_v(r, R) := C_v^+(R) - C_v^-(r) = \frac{1}{2\pi} \int_0^{2\pi} \left( v(Re^{i\alpha}) - v(re^{i\alpha}) \right) \, ds \]

provided that \( C_v^+(R) \) and \( C_v^-(r) \) are well defined.

If \( D \subset \mathbb{C} \) is a domain and \( u \in \text{sbh}_u(D) \), then there is its Riesz measure

\[ \Delta_u := \frac{1}{2\pi} \Delta u \in \text{Meas}^+(D), \]

where \( \Delta \) is the Laplace operator acting in the sense of the theory of distribution or generalized functions. This definition of the Riesz measures carries over naturally to \( u \in \text{sbh}_u(S) \).
for connected subsets $S \subset \mathbb{C}$. If $v \in \text{sbh}_u(D(R))$, then, by the Poisson–Jensen–Privalov formula, we have

$$C_v(r, R) = N_{\Delta_v}(r, R) \quad \text{for all } 0 < r < R < +\infty. \quad (14)$$

Let $U = u - v$ be a difference of subharmonic functions $u, v \in \text{sbh}_a(D(0, R))$, i.e., a $\delta$-subharmonic non-trivial ($\neq \pm \infty$) function on $\overline{D}(R)$ with the Riesz charge $\Delta_U = \Delta_u - \Delta_v$, [10], [11], [12], [13, 3.1]. A representation $U = u - v$ with $u, v \in \text{sbh}_a(D(0, R))$ is canonical if the Riesz measure $\Delta_{u, v}$ of $u$ is the upper variation $\Delta_U^+$ of $\Delta_U$ and the Riesz measure $\Delta_{v, u}$ of $v$ is the lower variation $\Delta_U^-$ of $\Delta_U$. The canonical representation for $U$ is defined up to the harmonic function added simultaneously to each of the representing subharmonic functions $u$ and $v$. We define a characteristic function of this $\delta$-subharmonic function $U$ as a function of two variables

$$T_U(r, R) := C_{\sup_{\Delta_u, \Delta_v}}(r, R) = C_{U^+}(r, R) + C_{U^-}(r, R) = C_{U^+}(r, R) + N_{\Delta_U^+}(r, R), \quad 0 < r \leq R \in \mathbb{R}^+. \quad (15)$$

This characteristic function $T_U$ is already uniquely defined for all $0 < r \leq R < +\infty$ by positive values in $\mathbb{R}^+$, and is also increasing and convex with respect to $\ln$ in the second variable $R$, but is decreasing in the first variable $r \leq R$.

**Main Theorem.** Let $0 < r_0 < r < +\infty$, $1 < k \in \mathbb{R}^+$, $E \subset [0, r]$ be measurable, $g \in L^p(E)$, where $1 < p \leq \infty$ and $q \in [1, +\infty)$ is from (7q), $U \neq \pm \infty$ be a $\delta$-subharmonic non-trivial functions on $\mathbb{C}$, and $u \neq -\infty$ be a subharmonic function on $\mathbb{C}$. Then

$$\frac{1}{r} \int_E M_U^+(t)g(t) \, dt \leq 4q\frac{k}{k - 1} \left( T_U(r_0, kr) + C_{U^+}(r_0) \right) \|g\|_{L^p(E)} \frac{\sqrt{\text{mes} E}}{r} \ln \frac{4kr}{\text{mes} E}, \quad (16T)$$

$$\frac{1}{r} \int_E M_{\mu}(t)g(t) \, dt \leq 5q\frac{k}{k - 1} \left( M_{\mu^+}(kr) + C_{\mu^-}(r_0) \right) \|g\|_{L^p(E)} \frac{\sqrt{\text{mes} E}}{r} \ln \frac{4kr}{\text{mes} E}. \quad (16M)$$

### 3. Lemmata and Proof of Main Theorem.

**Lemma 1.** Let $\mu \in \text{Meas}^+(\overline{D}(R))$. Then

$$\mu^\text{rad}(r) \leq \frac{R}{R - r} N_{\mu}(r, R) \quad \text{for each } 0 \leq r \leq R. \quad (17)$$

**Proof.** By definitions (11), we have

$$\mu^\text{rad}(r) = \int_r^R \frac{\mu^\text{rad}(r)}{t} \, dt \leq \int_r^R \frac{1}{t} \, dt \leq N_{\mu}(r, R) \leq \int_r^R \frac{1}{R} \, dt = \frac{R}{R - r} N_{\mu}(r, R),$$

and we obtain (17). \( \square \)

**Lemma 2.** Let $0 \leq r < R < +\infty$, $E \subset [0, r]$ be measurable, $U = u - v$ be a difference of subharmonic functions $u, v \in \text{sbh}_a(D(R))$, $\Delta_v$ be the Riesz measure of $v$, and $g \in L^p(E)$. Then

$$\int_E M_U^+(t)g(t) \, dt \leq \left( \frac{R + r}{R - r} C_{U^+}(R) \sqrt{\text{mes} E} + \Delta_{U^+}(R) \sup_{0 \leq x \leq R} \left\| \ln \frac{2R}{|x - x|} \right\|_{L^q(E)} \right) \|g\|_{L^p(E)}. \quad (18)$$
Proof. For \( w \in E \subset \overline{D}(r) \), by the Poisson–Jensen formula ([8, 4.5]), we have
\[
U(te^{ia}) = \frac{1}{2\pi} \int_0^{2\pi} U(Re^{is}) \frac{Re^{is} + te^{ia}}{Re^{is} - te^{ia}} \, ds - \int_{D(R)} \ln \left| \frac{R^2 - zte^{-ia}}{R(Re^{ia} - z)} \right| \, d\Delta(z) + \\
\quad + \int_{D(R)} \ln \left| \frac{R^2 - zte^{-ia}}{R(Re^{ia} - z)} \right| \, d\Delta(z) \leq \frac{R + r}{R - r} C_{U^+}(R) + \int_{D(R)} \ln \left| \frac{2R}{t - |z|} \right| \, d\Delta(z)
\]
where the right-hand side of the inequality is positive and independent of \( a \in [0, 2\pi) \). Hence, by integrating, we get
\[
\int_E M^+(t)g(t) \, dt \leq \frac{R + r}{R - r} C_{U^+}(R) \int_E |g|(t) \, dt + \int_{D(R)} \int_E \ln \left| \frac{2R}{t - |z|} \right| |g|(t) \, dt \, d\Delta(z).
\]
Therefore, by Hölder’s inequality, we obtain
\[
\int_E M^+(t)g(t) \, dt \leq \frac{R + r}{R - r} C_{U^+}(R) \|g\|_{L^p(E)} (\text{mes } E)^{1/q} + \\
\quad + \Delta(D(R)) \|g\|_{L^p(E)} \sup_{z \in D(R)} \left\| \ln \left| \frac{2R}{t - |z|} \right| \right\|_{L^q(E)}.
\]
The latter gives (18).

\[\Box\]

Lemma 3. Let \( q \in \mathbb{R}^+, 0 < A \in \mathbb{R}^+, \text{ and } a \in (0, A/e] \). Then
\[
\int_0^a \ln^q \frac{A}{x} \, dx \leq (1 + q^{q+1})a \ln^q \frac{A}{a}.
\]

Proof. We denote by \( |q| := \max\{n \in \mathbb{Z} : n \leq q\} \) the integer part of \( q \). Evidently,
\[
\ln \frac{A}{x} \geq 1 \quad \text{if } x \in (0, A/e],
\]

We integrate the integral from (19) by parts \( |q| + 1 \) times:
\[
\int_0^a \ln^q \frac{A}{x} \, dx = a \ln^q \frac{A}{a} + qa \ln^{q-1} \frac{A}{a} + q(q-1)a \ln^{q-2} \frac{A}{a} + \ldots
\]
\[
\quad \cdot + q(q-1) \ldots (q - |q| + 1) \ln^{q-|q|} \frac{A}{a} + q(q-1) \ldots (q - |q|) \int_0^a \ln^{q-|q|-1} \frac{A}{x} \, dx \leq
\]
\[
\leq \left( a \ln^q \frac{A}{a} \right) \left( 1 + \frac{q}{\ln^2 \frac{A}{a}} + \frac{q(q-1)}{\ln^3 \frac{A}{a}} + \ldots \right)
\]
\[
\quad \cdot + \frac{q(q-1) \ldots (q - |q| + 1)}{\ln^{q-|q|+1} \frac{A}{a}} + \frac{q(q-1) \ldots (q - |q|)}{a \ln^{q-|q|} \frac{A}{a}} \int_0^a \ln^{q-|q|-1} \frac{A}{x} \, dx \right) \leq
\]
\[
\leq \left( 1 + q + q(q-1) + \ldots + q(q-1) \ldots (q - |q| + 1) + q(q-1) \ldots (q - |q|) \right) a \ln^q \frac{A}{a}.
\]

\[\text{(20)}\]
We have a recurrent formula $P_q = 1 + qP_{q-1}$ for $1 \leq q \in \mathbb{R}^+$. Therefore,

$$P_q = \sum_{j=0}^{[q]} q^j + q^{[q]} (q - [q]) \leq 1 + q^{[q]+1} \leq 1 + q^{q+1} \quad \text{for each } p \in \mathbb{R}^+. $$

Thus, we obtain (19). \hfill \Box

**Lemma 4.** For $E \subset [0, r] \subset [0, R]$, $q \geq 1$, and $0 \leq x \leq R$, we have

$$ \left\| \ln \frac{2R}{t-x} \right\|_{L^q(E)} \leq 2q \sqrt{\text{mes } E} \ln \frac{4R}{\text{mes } E}, \quad (21) $$

**Proof.** We use

**Lemma A** ([2, Lemma 7.2]). Let $f : (-a, a) \to \mathbb{R}$ be an even integrable function on $(-a, a)$ decreasing on $(0, a)$, $E \subset (-a, a)$ be a measurable subset. Then

$$ \int_E f \, d\lambda \leq 2 \int_0^{\lambda(E)/2} f(t) \, dt. \quad (22) $$

By Lemma A we obtain

$$ \int_E \ln^q \frac{2R}{|t-x|} \, dt = \int_{E-x} \ln^q \frac{2R}{|t|} \, dt \leq 2 \int_0^{\lambda(E-x)/2} \ln^q \frac{2R}{t} \, dt = 2 \int_0^{\lambda(E)/2} \ln^q \frac{2R}{t} \, dt, $$

where $a := \lambda(E)/2 \leq r/2 \leq R/2 \leq 2R/e =: A/e$. Hence, by Lemma 3, we have

$$ \int_E \ln^q \frac{2R}{|t-x|} \, dt \leq 2(1 + q^{q+1}) \frac{\lambda(E)}{2} \ln^q \frac{2R}{\lambda(E)/2} = (1 + q^{q+1})(\text{mes } E) \ln^q \frac{4R}{\text{mes } E} $$

and $(1 + q^{q+1})^{1/q} \leq (2q^{q+1})^{1/q} = q((2q)^{1/2q})^2 \leq 2q$ for $q \geq 1$, since the function $x \mapsto x^x$ is decreasing in $[1/e, +\infty)$. Thus,

$$ \left\| \int_E \ln^q \frac{2R}{|t-x|} \right\|_{L^q(E)} \leq \sqrt{1 + q^{q+1}} \sqrt{\text{mes } E} \ln \frac{4R}{\text{mes } E} \leq 2q \sqrt{\text{mes } E} \ln \left( \frac{4R}{\text{mes } E} \right) $$

which gives (21). \hfill \Box

**Main Lemma.** Let $0 < r < +\infty$, $0 < b \in \mathbb{R}^+$, $E \subset [0, r]$ be measurable, $U = u - v$ be a difference of subharmonic functions $u, v \in \text{sbh}_a \left( \mathcal{D}((1+b)^2r) \right)$, and $g \in L^p(E)$, where $1 < p \leq +\infty$ and $q \in [1, +\infty)$ is from (7q). Then

$$ \int_E M^+_E(t)g(t) \, dt \leq 2q \frac{1+b}{b} \left( C_U ((1+b)r) + N_{\Delta^+} ((1+b)r, (1+b)^2r) \right) \times $$

$$ \times \|g\|_{L^p(E)} \sqrt{\text{mes } E} \ln \frac{4(1+b)r}{\text{mes } E}. \quad (23) $$
Proof. We set $R := (1 + b)r$. By Lemma 2, Lemma 1 with $(1 + b)^2 r$ instead of $R$ and $R$ instead of $r$, and Lemma 4, we have

\[
\int_E M^+_u(t)g(t) \, dt \leq \left( \frac{2 + b}{b} C_{u^+}((1 + b)r) \right) \sqrt{\text{mes } E} + \frac{1 + b}{b} N_{\Delta_e}((1 + b)r, (1 + b)^2 r) 2q \sqrt{\text{mes } E} \ln \left( \frac{4(1 + b)r}{\text{mes } E} \right) \|g\|_{L^p(E)} \leq 2 \frac{1 + b}{b} \left( C_{u^+}((1 + b)r) + N_{\Delta_e}((1 + b)r, (1 + b)^2 r) \right) \|g\|_{L^p(E)} 2q \sqrt{\text{mes } E} \ln \left( \frac{4(1 + b)r}{\text{mes } E} \right),
\]

which gives (23).

\[\square\]

Proof of the Main Theorem. We can assume that $U = u - v$ is the canonical representation of $U$. Consider a number $b > 0$ such that $(1 + b)^2 = k$. By the Main Lemma, we have

\[
\int_E M^+_u(t)g(t) \, dt \leq 2q \frac{\sqrt{k}}{k - 1} \left( C_{u^+}(\sqrt{kr}) + N_{\Delta_e}(\sqrt{kr}, k r) \right) \|g\|_{L^p(E)} \sqrt{\text{mes } E} \ln \left( \frac{4\sqrt{kr}}{\text{mes } E} \right) \leq 2q \frac{2k}{k - 1} \left( C_{u^+}(r, kr) + N_{\Delta_e}(r, kr) \right) \|g\|_{L^p(E)} \sqrt{\text{mes } E} \ln \left( \frac{4kr}{\text{mes } E} \right),
\]

and, by definition (15), obtain (16T).

Evidently, for any function $u$ with values in $\mathbb{R}$, we have

\[M^+_u = M^+_u, \quad M_u = M^+_u + M^-_u + M^- (-u),\]

(24)

If $u \in \text{sph}_u(\mathbb{C})$, then, under conditions of the Main Theorem, the function $M^+_u$ is increasing, and, by H"older’s inequality,

\[
\int_E M^+_u(t)g(t) \, dt \leq M^+_u(r) \|g\|_{L^p(E)} \sqrt{\text{mes } E}.
\]

(25)

For $U_u := 0 - u$, the difference $0 - u$ is the canonical representation of $\delta$-subharmonic non-trivial function $U_u$ and we have

\[
T_{U_u}(r, R) \overset{(15)}{=} C_{\sup\{0, u\}}(r, R) = C_{u^+}(r, R), \quad C_{u^+}(r, R) \leq C_{u^+}(R) \leq M^+_u(R).
\]

(26)

Hence, by the Main Theorem in part (16T) for $U_u$ in the role of $U$, we obtain

\[
\frac{1}{r} \int_E M^{(-u)+}(t)g(t) \, dt \overset{(24)}{=} \frac{1}{r} \int_E M^{(-u)+}_u(t)g(t) \, dt \leq \frac{4kq}{k - 1} \left( T_{U_u}(r, kr) + C_{U_u^+}(r, kr) \right) \|g\|_{L^p(E)} \sqrt{\text{mes } E} \ln \left( \frac{4kr}{\text{mes } E} \right) \leq \frac{4kq}{k - 1} \left( M^+_u(kr) + C_{u^+}(r, kr) \right) \|g\|_{L^p(E)} \sqrt{\text{mes } E} \ln \left( \frac{4kr}{\text{mes } E} \right)
\]

(26)

The latter together with (25) gives (16M).
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