GLOBAL ASPECTS OF SYMMETRIES IN SIGMA MODELS WITH TORSION

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ABSTRACT †

It is shown that non-trivial topological sectors can prevent the quantum mechanical implementation of the symmetries of the classical field equations of sigma models with torsion. The associated anomaly is computed, and it is shown that it depends on the homotopy class of the topological sector of the theory and the group action on the sigma model manifold that generates the symmetries of the classical field equations.

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1. Introduction

Two-dimensional sigma models have been extensively studied the last ten years because of their applications to string theory and the theory of integrable systems. More recently, attention has been focused on sigma models with symmetries. Such sigma models arise naturally in the investigation of sigma model duality (see for example refs. [1]) and the study of supersymmetric sigma models with potentials [2,3,4].

The fields of a sigma model are maps $\phi$ from the two-dimensional space-time $\Sigma$ into a Riemannian manifold $(M, h)$ which is called sigma model manifold or target space, and the couplings are tensors on $M$. In the following, we will focus on sigma models with couplings a metric $h$, a scalar function $V$, and an antisymmetric two-form $b$. The form $b$ may be locally defined on $M$ and the part of the sigma model action $I$ that contains this coupling is called Wess-Zumino (WZ) term [5] or WZ action. The action $I$ is either locally or globally defined depending on whether the WZ term $b$ is a locally or globally defined form correspondingly. The classical field equations, however, are always globally defined on $M$ because the coupling $b$ enters in them through its exterior derivative $H = \frac{3}{2} db$ and $H$ is a globally defined closed three-form on $M$ (see section 3). From the classical field theory point of view, the latter is enough to guarantee that the above theory is covariant under reparameterisations of the sigma model manifold and so well defined. In the path integral quantisation of this theory, however, $\exp(2\pi i I)$ is required to be globally defined as well, i.e. the sigma model action $I$ must be globally defined up to an integer. If the action $I$ is not globally defined, this additional requirement leads to a certain quantisation condition for the WZ coupling $b$ [6,7].

The symmetries of sigma models that we will examine are those induced by vector fields on the sigma model manifold. A sufficient condition for the transformations generated by vector fields on the sigma model manifold to be symmetries of the field equations is to leave the tensors $h, V, H$ invariant. In the quantum theory though, we need to know the conditions for the invariance of the sigma model action under the above transformations. This is straightforward for the terms in the action that contain couplings $h$ and $V$; the conditions are the same as those for the invariance of the associated terms in the field equations. New conditions are required though for the invariance of the WZ action in addition to those necessary for the invariance of the corresponding term in the field equations; I will call these new conditions anomalies. This is due to the fact that the WZ action may be locally defined on $M$ and the symmetries of the field equations may leave this action invariant up to surface terms that cannot be integrated away. (The latter can happen even in the special case where $b$ is globally defined on $M$).

The main point of this talk is to present the conditions under which the action of a sigma model with a WZ term is invariant (up to an integer) under the transformations generated by a group $G$ acting on the sigma model manifold $M$. I will then introduce the (1,1)-supersymmetric two-dimensional massive sigma model and show that the above anomaly cancels for the symmetries that arise naturally in this model.

The conditions for the invariance of the WZ action have been presented before by the author in ref. [8]. However the original publication does not contain the proof of key statements regarding these conditions and there is no mention of the conditions that are necessary for the WZ action to be invariant under infinitesimal transformations. These
will be included here. The action of massive (1,1)-supersymmetric sigma models was given in ref. [3] in collaboration with C.M. Hull and P.K. Townsend.

In section two, I will briefly review the conditions for the symmetries of the equations of motion of a charged particle coupled to a magnetic field to be implemented in the quantum theory of the system. In section three, I will give the action of a bosonic sigma model with a WZ term and a scalar potential, and the conditions for its field equations to be invariant under transformations generated by a group acting on the sigma model manifold. In section four, I will present a global definition of the WZ action and give the conditions for this action to be invariant under the symmetries of its field equations, and in section five, I will discuss the symmetries of supersymmetric massive sigma models.

2. A Quantum Mechanical Model

It is instructive to present the conditions under which the symmetries of the equations of motion of a charged particle coupled to a magnetic field can be implemented quantum mechanically [9]. This is because there is a close relation between these conditions and some of the conditions that I will derive in section 4 for the case of the two-dimensional sigma model with a WZ term. The action of a charged particle coupled to a magnetic field $b$ is

$$I = \int dt \left( \frac{1}{2} h_{ij} \partial_t \phi^i \partial_t \phi^j + b_i \partial_t \phi^i \right), \quad (2.1)$$

where $t$ is a parameter of the world-line, $\phi$ are the co-ordinates of the particle and $h$ is the metric of the manifold $\mathcal{M}$ in which the particle propagates. The coupling $b$ is a locally defined one-form on $\mathcal{M}$ with patching conditions $b_1 = b_2 + da_{12}$ on the intersection $U_1 \cap U_2$ of any two open sets $U_1, U_2$ of $\mathcal{M}$, and $a_{12}$ is a function on $U_1 \cap U_2$. Because of these patching conditions the last term of the above action is locally defined on $\mathcal{M}$. The equations of motion are

$$\nabla_t \partial_t \phi^i - h^{ij} \omega_{jk} \partial_t \phi^k = 0 , \quad (2.2)$$

where

$$\omega = db , \quad (2.3)$$

$\nabla_i$ is the Levi-Civita covariant derivative of the metric $h$ and $\nabla_t \equiv \partial_t \phi^i \nabla_i$. The equations of motion are globally defined on $\mathcal{M}$ because $\omega \equiv db$ is a globally defined closed two-form, i.e. $\omega_1 = \omega_2$ on the intersection of any two open sets $U_1$ and $U_2$ of $\mathcal{M}$.

Let $G$ be a group and $f : G \times \mathcal{M} \rightarrow \mathcal{M}$ be a group action of $G$ on $\mathcal{M}$; $f_{gg'} = fgf_{g'}$ and $f_e = Id_{\mathcal{M}}$ where $g, g', e \in G$ ($e$ is the identity element of $G$). Sufficient conditions for the invariance of the equations of motion (2.2) under the group action $f$ are

$$(f^*_gh)_{ij} = h_{ij} , \quad (f^*_g\omega)_{ij} = \omega_{ij} , \quad g \in G , \quad (2.4)$$

i.e. the transformations $f_g$ are isometries and leave the closed two-form $\omega$ invariant.

In the quantum theory, it is required that $\omega$ be the curvature of a line bundle over the manifold $\mathcal{M}$ and the wave functions be sections of this line bundle; this property of $\omega$ is a quantisation condition for the coupling $b$ of the action (2.1) and it is called Dirac’s quantisation condition. If in addition this theory has symmetries generated by a group
action as above, the conditions (2.4) are not enough to guarantee that these symmetries can be implemented with unitary transformations on the Hilbert space of the theory. For this, \textit{additional} conditions are necessary. To describe the additional conditions, let $[\omega]$ be the cohomology class of the curvature $\omega$ in $H^2(M,\mathbb{Z})$ and $G$ be compact and connected. We first pull-back $[\omega]$ using the group action $f$ on the manifold $G \times M$ and then decompose the pulled-back cohomology class $f^* [\omega]$ as

$$f^* [\omega] = [\omega] + [\sigma_1] + [\sigma_2],$$

where $[\sigma_1] \in H^1(G, H^1(M,\mathbb{Z}))$ and $[\sigma_2] \in H^2(G,\mathbb{Z})$. (We have used the K"unneth formula to perform this decomposition). Apart from (2.4), the \textit{additional} conditions to implement the classical symmetries with unitary transformations on the Hilbert space of the above theory are

$$[\sigma_1] = 0, \quad [\sigma_2] = 0.$$  \hfill (2.6)

After some computation, we can show that a consequence of the first condition of (2.6) is that the $\exp(2i\pi I)$ of the action $I$ (eqn. (2.1)) and the charges of this theory associated with the above symmetries are globally defined on the manifold $M$, and a consequence of the second condition of (2.6) is that the Poisson bracket algebra of these charges is isomorphic to the Lie algebra of $G$. For the proof of all the above statements as well as the study of the case where $G$ is disconnected using the theory of universal classifying spaces see refs. [9].

In section 4, I will examine the analogue of the first condition of (2.6) for the case of two-dimensional sigma models with WZ term. It is worth pointing out though that a condition similar to the second of (2.6) appears in the case of two-dimensional sigma models as well but this will be presented elsewhere [10].

3. Two-dimensional Sigma Models with Torsion

The action of a two-dimensional sigma model with WZ term $b$ and target space a Riemannian manifold $(M, h)$ is

$$I = \int d^2 x (h + b)_{ij} \partial_i \phi^j \partial_j \phi^i - V(\phi),$$  \hfill (3.1)

where the sigma model fields $\phi$ are maps from the two-dimensional space-time $\Sigma$ with light-cone co-ordinates $\{x^+, x^-= t + x, t = t - x\}$ into $M$ and $V$ is a real function on $M$. The two-form $b$ is \textit{locally} defined on $M$ with patching conditions $b_1 = b_2 + dm_{12}$ at the intersection $U_1 \cap U_2$ of any two open sets $U_1, U_2$ of $\mathcal{M}$ where $m_{12}$ is an one-form defined on $U_1 \cap U_2$. The WZ term in the action is then defined on $M$. One can define a closed three-form $H$ on $M$ as

$$H = \frac{3}{2} db.$$  \hfill (3.2)

Observe that $H$ is globally defined on $M$ and it is called torsion for reasons that will become apparent below. The field equations are

$$\nabla_{\Sigma}^{(+)} \partial_+ \phi^i + \frac{1}{2} h^{ij} \partial_j V = 0,$$  \hfill (3.3)
where the connections of the covariant derivatives $\nabla^{(\pm)}$ are
\[
\Gamma^{(\pm)i}_{jk} = \{i\}_{jk} \pm H^i_{jk} .
\] (3.4)

The tensor $H$ is the torsion of the connection $\Gamma^{(+)}$ and it is globally defined on $\mathcal{M}$.

Let $G$ be a group and $f : G \times \mathcal{M} \to \mathcal{M}$ be a group action of $G$ on the target manifold $\mathcal{M}$ as in the previous section; $f_{gg'} = f_g f_{g'}$ and $f_e = Id_\mathcal{M}$ where $g, g', e \in G$. Sufficient conditions for the invariance of the field equations (3.3) under the group action $f$ are
\[
(f^*h)_{ij} = h_{ij}, \quad (f^*H)_{ijk} = H_{ijk}, \quad f^*V = V, \quad g \in G ,
\] (3.5)
i.e. the group action leaves invariant the metric $h$ (so the group action $f$ generates isometries of the Riemannian manifold $(\mathcal{M}, h)$), the closed three-form $H$ and the scalar potential $V$. Let in addition $G$ be a Lie group with Lie algebra $\mathcal{L}(G)$. The infinitesimal form of the above conditions is then
\[
(L_a h)_{ij} = 0, \quad (L_a H)_{ijk} = 0, \quad L_a V = 0 ,
\] (3.6)
where $\{L_a; a = 1, \ldots, \dim \mathcal{L}(G)\}$ is the Lie derivative with respect to the vector field $\{X_a; a = 1, \ldots, \dim \mathcal{L}(G)\}$ generated by the group action $f$ on $\mathcal{M}$. The vector fields $X_a$ are Killing ($\nabla (i X_j) a = 0$).

In the path-integral quantisation of this theory, one needs to know the conditions for the action (3.1) to be invariant (up to an integer) under the transformations generated by a group action. The invariance of the terms in the action (3.1) involving the metric $h$ and the scalar potential $V$ follows directly from the conditions on $h, V$ given in eqn. (3.5) for the invariance of the field equations. However for the invariance of the Wess-Zumino action in (3.1), we need additional conditions besides those of eqn. (3.5) due partly to the fact that this term is not globally defined on $\mathcal{M}$. There are two main approaches to define globally the Wess-Zumino term. The first is the homotopy approach due to Wess and Zumino [5], and Rohm and Witten [6], and, the second is the Čech Cohomology approach due to O. Alvarez [7]. In the next section, I will use the former to examine the symmetries of the Wess-Zumino term. A study of the symmetries of the WZ action in the Čech Cohomology approach will be presented elsewhere [10].

4. Symmetries and the Wess-Zumino Action

Let $\Sigma$, the two-dimensional space-time, be a closed manifold, i.e. compact and without boundary, and $[\Sigma, \mathcal{M}]$ be the homotopy classes of maps from $\Sigma$ into the sigma model target manifold $\mathcal{M}$. To define the Wess-Zumino term in the homotopy approach, we choose a ‘background’ map $\phi_0$ from $\Sigma$ into $\mathcal{M}$ such that $\phi_0$ is homotopic to $\phi$, i.e. there is a map $F$ ($F : [0, 1] \times \Sigma \to \mathcal{M}$) and
\[
F(s, x) = \begin{cases} 
\phi_0(x) & s=0 , \\
\phi(x) & s=1 .
\end{cases}
\] (4.1)
The homotopy $F$ interpolates between $\phi_0$ and $\phi$. In the following, we will use the notation $\phi_0 \simeq_F \phi$ to denote that the map $\phi_0$ is homotopic to $\phi$ with respect to $F$. The action of the Wess-Zumino term is defined [6] as

$$S_{WZ}[\phi, \phi_0; F] = \int_{[0,1] \times \Sigma} F^* H .$$  \hspace{1cm} (4.2)

As indicated, the action of the Wess-Zumino term depends on the choice of the homotopy $F$ that interpolates between $\phi$ and $\phi_0$. To determine the dependence of $S_{WZ}$ on $F$, we take two different homotopies $F_1$ and $F_2$ that interpolate between $\phi$ and $\phi_0$ and compute the difference

$$\Delta S_{WZ} = S_{WZ}[\phi, \phi_0; F_1] - S_{WZ}[\phi, \phi_0; F_2] .$$  \hspace{1cm} (4.3)

This difference can be rewritten as

$$\Delta S_{WZ} = S_{WZ}[\phi_0, \phi_0; F_3] = \int_{S^1 \times \Sigma} F_3^* H ,$$  \hspace{1cm} (4.4)

where

$$F_3(s_3, x) = \begin{cases} F_1(2s_3, x) & 0 \leq s_3 \leq \frac{1}{2} \\ F_2(-2s_3 + 2, x) & \frac{1}{2} \leq s_3 \leq 1 \end{cases} .$$  \hspace{1cm} (4.5)

Note that $F_3(0, x) = F_3(1, x) = \phi_0(x)$. The difference $\Delta S_{WZ}$ of (4.4) is the integral of a closed three-form over a compact three-manifold without boundary and in general its value is a real number. However if $[H] \in H^3(M, \mathbb{Z})$, the difference is an integer and thus the functional

$$A[\phi, \phi_0] = e^{2i\pi S_{WZ}[\phi, \phi_0; F]}$$  \hspace{1cm} (4.6)

becomes independent of the choice of homotopy $F$. This property of $A$ is sufficient for the consistency of the path-integral quantisation of this theory. In the following, we will take $[H] \in H^3(M, \mathbb{Z})$ and so the WZ action will be independent from the choice of homotopy $F \mod 1$.

The invariance of the field equations of a sigma model with a Wess-Zumino term under the group action $f$ of a (connected) group $G$ on $M$ does not necessarily imply the invariance of the action $S_{WZ}[\phi, \phi_0; F]$. The transformation $\phi^g \equiv f_g(\phi)$ of the field $\phi$ induces the transformation $S_{WZ}[\phi^g, \phi_0; F_4]$ on the Wess-Zumino action which can be rewritten as

$$S_{WZ}[\phi^g, \phi_0; F_4] = S_{WZ}[\phi^g, \phi_0; F_3] + \text{integer} = S_{WZ}[\phi^g, \phi_0^g; F_2] + S_{WZ}[\phi_0^g, \phi_0; F_1] + \text{integer} ,$$  \hspace{1cm} (4.7)

where

$$F_3(s, x) = \begin{cases} F_1(2s, x) & 0 \leq s \leq \frac{1}{2} \\ F_2(2s - 1, x) & \frac{1}{2} \leq s \leq 1 \end{cases} .$$  \hspace{1cm} (4.8)

The first equality in (4.7) follows from the observation that $\phi_0 \simeq_{F_3} \phi^g$ and the property of the WZ action to be independent of the choice of homotopy $F$ up to an integer, and the second from the definition of the WZ term. Using again the property of the WZ action to be independent of the choice of homotopy $F \mod 1$, we can choose $F_2 = F^g$ and then use the
property of $H$ to be invariant under the group action $f$ (this comes from the invariance of the field equations eqn. (3.5)) to reexpress (4.7) as

$$S_{WZ}[\phi^g, \phi_0; F_4] = S_{WZ}[\phi, \phi_0; F] + S_{WZ}[\phi^g_0, \phi_0; F_1] + \text{integer}' .$$

From (4.9) it is clear that in addition to the conditions (3.5) for the invariance of the field equations, the vanishing of

$$c[\phi_0, g] = S_{WZ}[\phi^g_0, \phi_0; F_1] \mod 1 .$$

is a necessary condition [8] in order for the group action $S_{WZ}[\phi, \phi_0; F]$. Note that $c[\phi_0, g]$ is independent of the choice of homotopy $F_1$. I will refer to $c[\phi_0, g]$ as anomaly.

The anomaly $c[\phi_0, g]$ has some novel properties. In particular, we can show that $c[\phi_0, g]$ depends on the homotopy class $[\phi_0]$ of $\phi_0$ rather than $\phi_0$ itself. To prove this, let $\phi_1 \simeq_{F_1} \phi_0$. We write

$$S_{WZ}[\phi^g_0, \phi_0; F] = S_{WZ}[\phi^g_0, \phi_0; F_5] + \text{integer}$$

where

$$F_5(s_5, x) = \begin{cases} F_4(3s_5, x) & 0 \leq s_5 \leq \frac{1}{3} , \\ F_3(3s_5 - 1, x) & \frac{1}{3} \leq s_5 \leq \frac{2}{3} , \\ F_2(3s_5 - 2, x) & \frac{2}{3} \leq s_5 \leq 1 . \end{cases}$$

Observe from the expression for $F_5$ that $\phi_0 \simeq_{F_5} \phi^g_0$. Using the property of the Wess-Zumino action to be independent of the choice of homotopy mod 1 and the invariance of $H$, we write $S_{WZ}[\phi^g_0, \phi^g; F_2] = S_{WZ}[\phi_0, \phi; F_1] + \text{integer}$. The equation (4.11) then becomes

$$S_{WZ}[\phi^g_0, \phi_0; F] = S_{WZ}[\phi_0, \phi_1; F_1] + S_{WZ}[\phi^g_1, \phi_1; F_3] + S_{WZ}[\phi_1, \phi_0; F_4] + \text{integer}'$$

$$+ S_{WZ}[\phi_1, \phi_0; F_4] + \text{integer}'' .$$

So if $\phi_0$ is homotopic to $\phi_1$, we have shown that $c[\phi_1, g] = c[\phi_0, g]$ and thus the anomaly $c$ is a map from $[\Sigma, \mathcal{M}] \times G$ into $\mathbb{R}/\mathbb{Z}$.

We can also show that

$$c[[\phi], g_1, g_2] = (c[[\phi], g_1] + c[[\phi], g_2]) \mod 1 , \quad g_1, g_2 \in G .$$

This immediately follows from

$$S_{WZ}[\phi^{g_1, g_2}, \phi_0; F] = S_{WZ}[\phi^{g_1, g_2}, \phi_0; F_3] + \text{integer}$$

where

$$=S_{WZ}[(\phi^{g_1}_0)^{g_2}, \phi^{g_2}_0; F_2] + S_{WZ}[\phi^{g_2}_0, \phi_0; F_1] + \text{integer}'$$

$$=S_{WZ}[\phi^{g_2}_0, \phi_0; F_4] + S_{WZ}[\phi^{g_2}_0, \phi_0; F_1] + \text{integer}'' .$$
where $F_3$ is related to $F_1$ and $F_2$ as in (4.8) and $F_2 = F^g_2$. In addition, $c[[\phi], e] = 0$, so the anomaly is represented by a group homomorphism from $G$ into $\mathbb{R}/\mathbb{Z}$ for every homotopy class of maps from $\Sigma$ into $\mathcal{M}$.

An important consequence of the above properties of the anomaly $c$ is that $c$ vanishes whenever it is evaluated on the trivial topological sector of the theory, i.e. on the trivial class of $[\Sigma, \mathcal{M}]$. To prove this, we observe that the trivial topological sector can be represented by the constant maps from $\Sigma$ into $\mathcal{M}$, where $\pi_2$ is characterized by the trivial homotopy class, hence any constant map from $\Sigma$ into $\mathcal{M}$, can be taken to be any path in $\mathcal{M}$ that interpolates between them. In which case, $F$ is a map of one variable and therefore the pull-back form $F^*H$ of the three-form $H$ with respect to the homotopy $F$ that interpolates between $\phi^0_0$ and $\phi_0$. Since $\phi^0_0$ and $\phi_0$ are constants maps from $\Sigma$ into $\mathcal{M}$, they can be thought of as points in $\mathcal{M}$ and so we can take $F$ to be any path in $\mathcal{M}$ that interpolates between them. In fact using the same arguments, we can show that the anomaly $c$ vanishes for all the topological sectors $[\phi]$, i.e. for the theory that admit a homotopy $F$ which interpolates between $\phi$ and $\phi^g$ with differential map $dF : T([0, 1] \times \Sigma) \to T(\mathcal{M})$ that has rank less than three. Thus only the non-trivial topological sectors can break the symmetries of the field equations of a sigma model with a WZ term.

The anomaly $c$ vanishes for all sigma models with torsion for which $[\Sigma, \mathcal{M}]$ has only one element, the trivial class. For example, this is the case for the sigma models with $\Sigma = S^2$ and $\mathcal{M} = SU(N)$ because $\pi_2(SU(N)) = 0$.

It is well known that the homotopy classes of maps $[\Sigma, \mathcal{M}]$ under certain conditions can be given a group structure. So it is natural to ask whether or not the anomaly $c$ is a group homomorphism from $[\Sigma, \mathcal{M}] \times G$ into $\mathbb{R}/\mathbb{Z}$. We have already shown above that $c[[\phi], g] = 0$ if $[\phi]$ is the trivial class of $[\Sigma, \mathcal{M}]$. So it remains to show whether $c[[\phi_1, \phi_2], g] = (c[[\phi_1], g] + c[[\phi_2], g])$ mod 1. Examination of the particle model in section 2 indicates that indeed the anomaly $c$ is a group homomorphism but this will not be pursued further here.

Next we will calculate the form of the anomaly for infinitesimal transformations. For this, let us consider an one-parameter subgroup of $G$ generated by the vector $v \in \mathcal{L}(G)$.

The elements of this subgroup can be written as $g(r) = \exp(rv)$ where $r$ is a real number and $g$ can be thought as a map from the real line $\mathbb{R}$ into $G$. Using this subgroup, we define another map that we will call again $g$ from $[0, 1] \times (-\epsilon, \epsilon)$ into $G$ by setting $r = sz$, i.e. $g(s, z) = \exp(szv)$, where $\epsilon$ is a positive real number. We observe that $g(s, z)$ has the following properties:

$$g(s, z) = \begin{cases} e & \text{if either } s=0 \text{ or } z=0 \\ g(z) = \exp(zv) & s=1 \end{cases} \quad (4.16)$$

The ‘infinitesimal’ anomaly is given by

$$\Delta[\phi, v] := \frac{d}{dz} S_{WZ} [\phi^g(z), \phi; F]_{z=0} \quad ,$$  

where

$$F(s, x) := \phi^g(s, z)(x) \quad .$$  

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Because of (4.16), the homotopy $F$ interpolates between $\phi$ and $\phi^{g(z)}$. Using the definition (4.17) of the infinitesimal anomaly and the Wess-Zumino action, we can show after some further computation that

$$\Delta[\phi, v] = \int_{\Sigma} v^a \phi^* (\eta_a) ,$$

where

$$\eta_a \equiv i_a H ,$$

and $i_a$ is the inner derivation with respect to the Killing vector field $X_a$. Using $dH = 0$ from (3.2), $L_a H = 0$ from (3.6) and $L_a = di_a + i_a d$, we can show that $\eta_a \equiv i_a H$ is a closed two-form

$$d\eta_a = 0 .$$

Therefore the infinitesimal anomaly $\Delta[[\phi], v]$ is the integral of the pull-back with the map $\phi$ of a closed two-form of $\mathcal{M}$ with domain the space-time $\Sigma$ and so it depends on the homotopy class $[\phi]$ of $\phi$, i.e. $\Delta = \Delta[[\phi], v]$.

The infinitesimal anomaly $\Delta[[\phi], v]$ vanishes whenever $[\phi] \in [\Sigma, \mathcal{M}]$ is the trivial class (in agreement with the behaviour of $c[[\phi], g]$ discussed above). The anomaly $\Delta[[\phi], v]$ also vanishes provided that the two-forms $\{\eta_a\}$ are exact. Note though that $\Delta[[\phi], v]$ does not vanish if $b$ is merely globally defined on $\mathcal{M}$.

The Noether charges associated with the symmetries of the action (3.1) generated by the group action $f$ are

$$Q_a = \int dx \ (X_{ai} \partial_t \phi^i + u_{ai} \partial_x \phi^i) ,$$

where the one-forms $\{u_a\}$ are locally defined and they are related to the closed two-forms $\{\eta_a\}$ as follows:

$$\eta_a = du_a .$$

If the two-forms $\{\eta_a\}$ are exact, then $\{u_a\}$ are globally defined and consequently the Noether charges $\{Q_a\}$ are globally defined as well. However even if $\{u_a\}$ are globally defined, it is not expected that the Poisson bracket algebra of these charges to be necessarily isomorphic to $\mathcal{L}(G)$. Recall that the Poisson bracket algebra of the charges of the particle model in section 2 has similar behaviour.

The methods developed above to examine the symmetries of the WZ action for two-dimensional sigma models can be extended to the case of n-dimensional ones [8].

5. Concluding Remarks

Symmetries generated by vector fields on the sigma model manifold arise naturally in the study of (p,q)-supersymmetric two-dimensional massive sigma models because the charges of such symmetries appear as central charges in the Poisson bracket algebra of supersymmetry charges of these models. The above vector fields also enter in the expression for the scalar potential.

The simplest massive supersymmetric sigma model with a central charge is the one with (1,1)-supersymmetry. Let $X$ be a Killing vector field that leaves invariant both the
torsion $H$ of eqn. (3.2) and $u$ of eqn. (4.23). It can be shown that the most general action of a massive sigma model with (1,1) supersymmetry [3] is

$$I = \int d^2 x \left\{ \partial_+ \phi^i \partial_\phi \phi^j (h_{ij} + b_{ij}) + ih_{ij} \lambda_+^i \nabla_+ (\phi) \lambda_+^j - ih_{ij} \psi_-^i \nabla_\psi \psi_-^j \right. $$

$$- \frac{1}{2} \psi_-^k \psi_-^j \lambda_+^i \lambda_+^j R_{ijkl}^{-} + m \nabla_i^{-} (u - X)_j \lambda_+^i \psi_-^j - V(\phi) \} , \quad (5.1)$$

where $\lambda_+$ and $\psi_-$ are real chiral fermions and

$$V(\phi) = \frac{m^2}{4} h_{ij} (u - X)_i (u - X)_j , \quad (5.2)$$

is the scalar potential. The rest of the notation follows from sections 3 and 4. Observe that the requirement for the action (5.1) to be (1,1)-supersymmetric imposes strong restrictions on the form of the scalar potential $V(\phi)$.

To define globally the field equations derived from this action on the sigma model target space $\mathcal{M}$, we have to assume that $u$ is globally defined one-form on $\mathcal{M}$. As we have established in the previous section, this is a sufficient condition for the WZ action to be invariant under the infinitesimal transformations generated by $X$.

Computation of the Poisson bracket algebra of charges of the above model reveals that

$$\{ S_+, S_+ \} = E + P, \quad \{ S_, S_- \} = E - P, \quad \{ S_+, S_- \} = Q_X . \quad (5.3)$$

where $S_+$, $S_-$ are the supersymmetry charges, $E$ is the energy, $P$ is the momentum and $Q_X$ is the Noether charge associated with the symmetries generated by $X$ (eqn. (4.22)). To derive (5.3), we have assumed $X^i u_i = 0$. Observe that $Q_X$ appears in the Poisson bracket of left- with right- supersymmetry charges. The form of the charges as well as details of this computation can be found in refs. [4].

In conclusion, non-trivial topological sectors can break quantum mechanically the symmetries of the field equations of a sigma model with torsion. This is partly due to the WZ action which may not be globally defined and/or may be invariant up to surface terms that cannot be integrated away. An associated anomaly is computed and it is found to be a WZ-like action that depends on the homotopy class of the topological sector of the theory and the group action on the sigma model manifold that generates the symmetries of the field equations. Sufficient conditions for the vanishing of this anomaly are also given. In particular, it is shown that the anomaly vanishes whenever the theory has only one topological sector, i.e. the trivial one. Finally it is worth pointing out that, as in the case for the one-dimensional sigma model, other conditions are necessary in addition to the vanishing of the above anomaly in order for the symmetries of the field equations of $n$-dimensional ($n \geq 2$) sigma models with torsion to be implemented quantum mechanically.

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