Langevin dynamics of vortex lines in the counterflowing He II. Talk given at the Low Temperature Conference, Kazan, 2015

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The problem of the statistics of a set of chaotic vortex lines in a counterflowing superfluid helium is studied. We introduced a Langevin-type force into the equation of motion of the vortex line in presence of relative velocity \( v_{ns} \). This random force is supposed to be Gaussian satisfying the fluctuation-dissipation theorem. The corresponding Fokker-Planck equation for probability functional in the vortex loop configuration space is shown to have a solution in the form of Gibbs distribution with the substitution \( E\{s\} \to E\{(s) \} - P(v_n - v_s) \), where \( E\{s\} \) is the energy of the vortex configuration \( \{s\} \), and \( P \) is the Lamb impulse. Some physical consequences of this fact are discussed.

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I. INTRODUCTION AND SCIENTIFIC BACKGROUND

The Langevin approach and, relating to it, the Fokker-Planck equation are the very powerful methods for both analytical and numerical studies of stochastic processes in various fields of theoretical physics [2]. These methods are also applied for study of nonequilibrium states, in particular, for classical turbulence (see for details the book [3]). They allow e.g. to develop renormalization procedure and, modifying the shape of correlator of the stirring force, to test various physical situations. For instance, if the stirring force has a correlator proportional to the squared wave vector, then the pumping locally equilibrates viscous damping and the system is driven into a thermal equilibrium state. However, when stirring force are concentrated on large scales (small wave numbers) the according procedure results in the solution with Kolmogorov cascade (although these results are not very rigorous and there are many open questions left).

In the dynamics of quantum vortices the Langevin and the Fokker-Planck approaches were applied first to describe the growth of small ideally circular vortex rings under thermal activation (see for details and bibliography the book [4]). For an ensemble of vortices this theory had been used for dynamic processes in films of \(^4\)He, variant of nonstationary Kosterlitz-Thouless phase transition [5]. A combination of the above two examples was the use of the Fokker-Planck equations for gas of circular vortex rings to describe lambda transition in superfluid helium (for review and bibliography see e.g. [6]). In papers [7],[8] there was an attempt to study both numerically and analytically the stochastic behavior of single, non-circular vortex loop (disregarding reconnections) in the local induction approximation and driven by low-frequency noise.

The problem of the equilibrium statistics of a set of chaotic vortex lines in a counterflowing superfluid had been studied [8],[10],[11] some time ago. Considering vortex filament undergoing random forcing, the Langevin-type equation of motion of the line had been obtained. The respective Fokker-Planck equation for probability functional \( \mathcal{P}(\{s(\xi)\}) \) in vortex loop configuration space is shown to have a solution of the form \( \mathcal{P}(\{s(\xi)\}) = \mathcal{N} \exp(-H\{s\}/T) \), where \( \mathcal{N} \) is a normalizing factor and \( H\{s\} \) is energy of vortex line configurations.

In the present paper we study dynamics of the vortex filament under action of the Langevin-type random force in presence of relative velocity \( v_{ns} \). Motivation of this topic is due to fact that chaotic set of vortex filaments, the so called quantum turbulence develops in the counterflowing superfluid helium without random stirring, so it is important to compare both mechanisms.

II. FOKKER-PLANCK EQUATION

We consider the Langevin dynamics of vortex loops in three-dimensional space with no boundaries. The equation of motion of the vortex line elements reads

\[
\dot{s} = B(\xi) + v_s + \alpha s' \times (v_{ns} - B(\xi)) + \alpha' s' \times s' \times (v_{ns} - B(\xi)) + f(\xi_0, t).
\] (1)

Here \( B(\xi) \) is the self induced, Biot-Savart, propagation velocity of the vortex filament at a point \( s \), determined by Biot-Savart law; \( s(\xi, t) \) is the radius-vector of the vortex line points; \( \xi \) is a label parameter, in this case it coincides with the arc length; \( v_{ns} \) is the relative velocity between the normal and superfluid component helium; \( s' \) is the
derivative with respect to the arc length, $\alpha$, $\alpha'$ are the friction coefficients, describing interaction of vortex filament with normal component, and $f(\xi,t)$ is the Langevin force. The Langevin force is supposed to be a white noise with the following correlator

$$F = \langle f_i(\xi_1,t_1)f_j(\xi_2,t_2) \rangle = \frac{2k_B T\alpha}{\rho_\kappa} \delta_{ij}(t_1-t_2)\delta(\xi_1-\xi_2).$$  \hspace{1cm} (2)

Here $i,j$ are the spatial components; $t_1,t_2$ are the arbitrary time moments; $\xi_1,\xi_2$ denote any points on the vortex line. The “temperature”, related to this force via the fluctuation dissipation theorem is an artificial quantity, having nothing to do with real temperature. In this sense we study a model problem, which has application to a number of problems on chaotic vortex lines, including superfluid turbulence. We can claim, that it would be considered as 3D variant of the famous Onsager model \[12\]. The only case, where this approach can be applied for real situation, (at least for quantum fluids) is the vicinity of phase transition (due to the fact, that the superfluid density is very small there).

There is one specific feature. The term with $\alpha'$ doesn’t enter into the fluctuation dissipation theorem. Its role is reduced to that, that it just changes the self-induced velocity (Biot-Savart term), but doesn’t affect the effective temperature. It could be said that for dissipation of energy only the $\alpha$ term is responsible, while the reactive mutual-friction $\alpha'$ just slightly renormalizes the inertial term of conventional hydrodynamics. This situation (for different reason) had been discussed earlier in paper \[13\]. Further we omit term with $\alpha'$.

Let us consider the probability distribution functional \[10\] defined as

$$P(\{s(\xi)\},t) = \langle \delta(s(\xi) - s(\xi),t) \rangle. \hspace{1cm} (3)$$

The Fokker-Planck equation for the time evolution of quantity $P(\{s(\xi)\}, t)$ can be derived from equation of motion \[11\] in standard way (see e.g. \[2,10\])

$$\frac{\partial P}{\partial t} + \int d\xi \frac{\delta}{\delta s(\xi)} \left[ \{B(\xi) + v_\xi + \alpha s'(\xi) \times (v_n - v_s - B(\xi))\} + F \frac{\delta}{\delta s(\xi)} \right] P = 0 \hspace{1cm} (4)$$

**III. EQUILIBRIUM SOLUTION TO FOKKER-PLANCK EQUATION**

We are looking for solution of the following form the Gibbs distribution

$$P(\{s(\xi)\},t) = N \exp(-\frac{H\{s\}}{k_B T}), \hspace{1cm} (5)$$

Where the Hamiltonian $H\{s\}$ is

$$H\{s\} = E\{s\} - P(v_n - v_s). \hspace{1cm} (6)$$

The energy $E(s)$ of vortex configuration $\{s\}$ is

$$E(s) = \frac{\rho_\kappa^2}{8\pi} \int_\Gamma \int_{\Gamma'} \frac{s'(\xi)s'(\xi')}{|s(\xi)-s(\xi')|} d\xi d\xi'. \hspace{1cm} (7)$$

On the role of momentum we take the so called Lamb Impulse defined as

$$P = \frac{\rho_\kappa}{2} \int s(\xi) \times s'(\xi) \ d\xi. \hspace{1cm} (8)$$

In usual statistical mechanics when $P$ is true momentum of particle (or quasiparticle) Eq. (5) is obvious and follows from the Galilean transformation. Since the Lamb impulse is not “real” momentum, then Eq. (5) is not obvious and needs in the verification. Let’s give the proof the realization of the Gibbs distribution in form Eq. (5).

First of all we give two identities. The variational derivative of energy of vortices is related to the self induced Biot-Savart propagation velocity $B(\xi)$ in following manner

$$B(\xi) = \frac{1}{\rho_\kappa \kappa} \frac{s'(\xi) \times \delta E(s)}{\delta s(\xi),t} \Rightarrow \rho_\kappa \kappa s'(\xi) \times B(\xi) = \frac{\delta E(s)}{\delta s(\xi),t}. \hspace{1cm} (9)$$
The latter equality requires \(s'(\xi)B(\xi) = 0\), which can be reached by suitable reparametrization. We take parametrization \(s(\xi, t)\), where \(s'(\xi)\delta E(s)/\delta s(\xi, t) = 0\) and \(s'(\xi)s'(\xi) = 1\).

Another identity concerns of variational derivative of quantity \(P(v_n - v_s)\). For constant vector \(v_{ns}\) the following relation takes place:

\[
\rho_s s'(\xi) \times v_{ns} = \delta(Pv_{ns})/\delta s(\xi, t).
\]

(10)

Now we have to substitute the Gibbs distribution (5) in the Fokker-Planck equation (1). We take that \(v_n\) is the time independent and we work in a frame where \(v_n\) is zero. Let us check the non dissipative terms, the first two terms in square brackets of the Fokker-Planck equation (4). It is readily verified that

\[
\int d\xi \left[ \delta \left( s'(\xi) \times v_{ns} \right) \right] \left( s'(\xi) \times \delta E(s)/\delta s(\xi, t) + v_s \right) \exp \left( -\frac{H(s)}{k_B T} \right)
\]

\[
-\frac{1}{k_B T} \int d\xi \left( s'(\xi) \times \delta E(s)/\delta s(\xi, t) + v_s \right) \left( -\frac{\delta E(s)}{\delta s(\xi, t)} + s'(\xi) \times v_s \right) \exp \left( -\frac{H(s)}{k_B T} \right)
\]

(11)

Let us check the first line, which includes differentiation of terms inside of the square brackets

\[
\int d\xi \left\{ \frac{\delta}{\delta s(\xi)} \left[ s'(\xi) \times \delta E(s)/\delta s(\xi, t) + v_s \right] \right\} \exp \left( -\frac{H(s)}{k_B T} \right)
\]

(12)

Performing functional differentiation and using tensor notation, the first term in square brackets is proportional to

\[\epsilon^{\alpha\beta\gamma} \delta \frac{\delta E(s)}{\delta s(\xi, t)} \left( s'_{\beta}(\xi) \frac{\delta H(s)}{\delta s_{\gamma}(\xi, t)} \right) .\]

(13)

The functional derivative \(\delta s'_\alpha(\xi)/\delta s_{\alpha}(\xi) \propto \delta_{\beta\alpha}\), therefore this term vanishes due to symmetry. Further, the functional derivative from constant velocity obviously also vanishes \(v_s\), \(\delta v_s/\delta s(\xi) = 0\). The rest terms are

\[
-\frac{1}{k_B T} \int d\xi \left( \delta E(s)/\delta s(\xi, t) \left( s'(\xi) \times \delta E(s)/\delta s(\xi, t) - \rho_s K(s'(\xi) \times v_s)(s'(\xi) \times \delta E(s)/\delta s(\xi, t)) \right) \right.
\]

\[
\left. + \frac{\delta E(s)}{\delta s(\xi, t)} v_s - \rho_s K(s'(\xi) \times v_s) v_s \right) \exp \left( -\frac{H(s)}{k_B T} \right)
\]

(14)

Again the first terms vanishes due to symmetry and the forth term is zero, since production \(s'(\xi) \times v_s\) is normal to \(v_s\). The rest ones lead to the following expression (we omit factor \(-1/k_B T\), and choose parametrization where and \(s'(\xi)s'(\xi) = 1\))

\[
\int d\xi \frac{\delta E(s)}{\delta s(\xi, t)} v_s - \rho_s K(s'(\xi) \times v_s)(s'(\xi) \times \delta E(s)/\delta s(\xi, t)) \exp \left( -\frac{H(s)}{k_B T} \right) =
\]

\[
\int d\xi \left[ -v_s \frac{\delta E(s)}{\delta s(\xi, t)} + (s'(\xi)v_s)(s'(\xi)\delta E(s)/\delta s(\xi, t)) \right] \exp \left( -\frac{H(s)}{k_B T} \right)
\]

(15)

In parametrization where \(s'(\xi)\delta E(s)/\delta s(\xi, t) = 0\) all terms vanish.

Thus, the reversible (not associated with dissipation) term in the original equation of motion (11) does not contribute to the functional dynamics of the probability distribution. We say that it is the divergents free term. It is understood, that the foregoing relates only to the case of thermal equilibrium, i.e. valid only for solutions (5).

The last term in the integrand (11) can be converted by using the fluctuation-dissipation theorem (2) as follows:

\[
\int d\xi \int d\xi' k_B T \alpha n \frac{\delta E(s)}{\delta s(\xi, t)} \delta t_1 \delta t_2 \delta n \frac{\delta E(s)}{\delta s(\xi', t)} \exp \left( -\frac{H(s)}{k_B T} \right)
\]

(16)

Applying identities relation (19) and (110) we can easily verify that the resulting expression exactly compensates the dissipative term in the integrand in (11)(we recall that we work in a frame where \(v_n\) is zero). This implies that the Gibbs distribution with the Hamiltonian \(H\{s\}\) (16) is indeed the solution of the Fokker–Planck equation, as it must be in accordance with the general physical principles.
IV. CONCLUSION

Relations (5)-(8) are assigned to evaluate partition function and to calculate various statistical properties of the vortex tangle. A considerable simplification in the evaluation of the partition function can be reached with the use of the Edwards trick (see, for details [14], [15]). Namely, the quantity \( \exp \left( -\frac{E\{s\} - P(v_n - v_s)}{k_B T} \right) \) can be written as a Gaussian path integral over an auxiliary vector field \( \mathbf{A}(r) \), which is readily evaluated. With the use of this technique we intend to calculate the structure factors of quantum turbulence, e.g. average polarization of the vortex loops composing the vortex tangle in the counterflowing helium II, as well as anisotropy and mean curvature. These quantities were earlier obtained only in the numerical work by Schwarz [16]. The results on properties of vortex tangle, which we plan to obtained on the basis of the formalism, derived in the present work are supposed to compared with numerous data on the quantum turbulence.

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