Two-dimensional characteristic polynomials in the direct calculation of optical phase sum and difference

M. Miranda, B.V. Dorrío, J. Blanco, J. Díz-Bugarín and F. Ribas

Applied Physics Dpt, University of Vigo, Campus Universitario, 36310, Vigo, Spain.
E-mail: marta_miranda@uvigo.es

Abstract. Two-stage phase shifting algorithms make possible to directly recover the sum or the difference of the encoded optical phase of two different fringe patterns. These algorithms can be constructed, for example, by combining known phase shifting algorithms in a non-linear way. In this work two-stage phase shifting algorithms are linked to a two-dimensional characteristic polynomial to qualitatively analyse their behaviour against the main systematic error sources in an analysis protocol like that used for phase shifting algorithms. This tool enables us to understand the propagation of properties from precursor phase shifting algorithms to new evaluation algorithms that can be built from them.

1. Introduction

In countless applications of scientific and technological interest [1-7] the information of the measurand is linked to the sum or the difference of the optical phase encoded between an original fringe pattern $s(r, \phi)$ and another modified one $t(r, \phi+\Delta \phi)$. In principle, the phase sum $2\phi(r)+\Delta \phi(r)$ or the phase difference $\Delta \phi(r)$ between both patterns can be retrieved by duplicating the evaluation process (where the unwrapping [8] of the phases can be necessary) and adding or subtracting, respectively, the results obtained. In this way, the phase $\phi(r)$ of the first pattern can be recovered [9] as a linear combination of $M$ irradiance values $s_{m}(r, \phi, \alpha_{m})$, with $\alpha_{m}=2\pi(m-1)/M$ [10,11], (i.e., equally shifted in phase), and subsequently, making a linear combination of them that is introduced in the argument of an arctan function

$$\phi(r) = \arctan \left( \frac{\sum_{m=1}^{M} n_{m} s_{m}(r, \phi, \alpha_{m})}{\sum_{m=1}^{M} d_{m} s_{m}(r, \phi, \alpha_{m})} \right) = \arctan \left( \frac{\sum_{m=1}^{M} h_{n_{m}}(u_{1m}) s_{n_{m}}(r, \phi, \alpha_{m})}{\sum_{m=1}^{M} h_{d_{m}}(u_{1m}) s_{d_{m}}(r, \phi, \alpha_{m})} \right)$$

(1)

where $(n_{m},d_{m})$ are the coefficients of the linear combination that define the phase shifting algorithm (PSA) and that can be regarded as sampling amplitudes. It can also be expressed as a discretization of the irradiance signal with sampling functions $h_{N,D}(u_{1})$ [12,13] – where $u_{1}$ is the spatial or temporal variable for the original signal – at each sampling point $u_{1}=u_{1m}$ where $\alpha_{m}=2\pi f_{s}u_{1m}$ with $f_{s}$ its reference frequency. Likewise, the phase $\phi(r)+\Delta \phi(r)$ of the modified pattern $t(r, \phi+\Delta \phi)$ is obtained by the combination of $P$ irradiance values $t_{p}(r, \phi+\Delta \phi, \beta_{p})$
with \((n_p,d_p)\) the sampling amplitudes at the point analysed \(u_2\) as \(\beta_p=2\pi f_u u_2\) where \(f_u\) is its reference frequency. The sensitivity of each PSA, corresponding to different error sources, can be given by the behaviour in reciprocal space of their characteristic spectra [14-19]

\[
H_{N,D}(f_{1,2}) = \text{FT}\left[h_{N,D}(u_{1,2})\right]
\]

where FT stands for Fourier Transform. An alternative but equivalent characterization is obtained relating the calculation of the optical phase of a PSA to an associated complex number [20-25]. So equation (1) can be rewritten as

\[
V(\phi) = \sum_{m=1}^{M} (d_m + jn_m) s_m(r,\phi,u_m) = \\
\sum_{k=-\infty}^{\infty} a_k e^{j k \phi} \sum_{m=1}^{M} (d_m + jn_m) e^{j k \alpha (m-1)} = \sum_{k=-\infty}^{\infty} a_k e^{j k \phi} P(e^{j k \alpha})
\]

with additional phase \(\alpha=\alpha_{m+1}-\alpha_m=\alpha_m/(m-1)\) and similarly for the modified pattern (equation (2))

\[
V(\phi + \Delta\phi) = \sum_{p=1}^{P} (d_p + jn_p) t_p(r,\phi + \Delta\phi,u_p) = \\
\sum_{g=-\infty}^{\infty} b_g e^{j g (\phi + \Delta\phi)} \sum_{p=1}^{P} (d_p + jn_p) e^{j g \beta (p-1)} = \sum_{g=-\infty}^{\infty} b_g e^{j g (\phi + \Delta\phi)} P(e^{j g \beta})
\]

where \(\beta=\beta_{m+1}-\beta_m=\beta_m/(p-1)\). Both equations are also expressed according to the characteristic polynomials (CP) of the original pattern \(P[\exp(jk\alpha)]\) (4) and the modified one \(P[\exp(jg\beta)]\) (5).

Thus the multiplicity and localization of the roots of each CP, just like their characteristic spectra, indicate their sensitivity to the main error sources. In this way, the complete characterisation of a PSA’s insensitivity can be done in terms of the characteristic spectra theory which is completely convertible in terms of the CP theory and vice versa, obtaining coinciding results in all cases given that:

\[
H_D(f_1) + jH_N(f_1) = \sum_{m=1}^{M} d_m e^{-j \alpha_m f_1} + j \sum_{m=1}^{M} n_m e^{-j \alpha_m f_1} = \sum_{m=1}^{M} (d_m + jn_m) e^{m-1(-j \alpha_m)} = P(e^{-j \alpha m})(6a)
\]

\[
H_D(f_2) + jH_N(f_2) = \sum_{p=1}^{P} d_p e^{-j \beta_p f_2} + j \sum_{p=1}^{P} n_p e^{-j \beta_p f_2} = \sum_{p=1}^{P} (d_p + jn_p) e^{p-1(-j \beta_p)} = P(e^{-j \beta p})(6b)
\]

However, instead of duplicating the evaluation and characterization mechanism, the phase sum or the phase difference can be recovered using a direct calculation with two-stage phase shifting algorithms (TSPSAs) that provide directly continuous values of the phase difference \(\Delta\phi(r)\) when a period is not completed in the whole pattern area and thus avoid the unwrapping process. TSPSAs can
be obtained, for instance, by combining the numerators and denominators from the original (1) and modified (2) PSAs in a non-linear way in the argument of an arctan function [26-31]:

\[
[\phi(r) + \Delta\phi(r)] \pm \phi(r) = \arctan \left( \sum_{m=1}^{M} \sum_{p=1}^{P} n_m s_m(r, \phi, \alpha_m) n_p t_p(r, \phi + \Delta\phi, \beta_p) + \sum_{m=1}^{M} \sum_{p=1}^{P} d_m s_m(r, \phi, \alpha_m) d_p t_p(r, \phi + \Delta\phi, \beta_p) \right) \tag{7}
\]

In this way, TSPSAs design also obeys a least squares fit, which leads to investigation of inherited behaviours and shared analysis tools with PSAs. Moreover, its generic equation can also be written in the same way as for the its PSA precursors as [30]

\[
[\phi(r) + \Delta\phi(r)] \pm \phi(r) = \arctan \left( \sum_{m=1}^{M} \sum_{p=1}^{P} n_{m,p} s_{m,p}(r, \phi, \alpha_m) t_{p}(r, \phi + \Delta\phi, \beta_p) + \sum_{m=1}^{M} \sum_{p=1}^{P} d_{m,p} s_{m,p}(r, \phi, \alpha_m) t_{p}(r, \phi + \Delta\phi, \beta_p) \right) = \arctan \left( \sum_{m=1}^{M} \sum_{p=1}^{P} h_N(u_{1,m,p}, u_{2,p}) s_{m,p}(r, \phi, \alpha_m) t_{p}(r, \phi + \Delta\phi, \beta_p) \right) \tag{8}
\]

where \((n_{m,p}, d_{m,p})\) are their sampling amplitudes, which are precisely the values that make good recovery of either the phase sum, when \(n_{m,p} = \sin(\alpha_m + \beta_p)\) and \(d_{m,p} = \cos(\alpha_m + \beta_p)\), or the phase difference, \(n_{m,p} = \sin(\alpha_m - \beta_p)\) and \(d_{m,p} = \cos(\alpha_m - \beta_p)\). The goodness of a TSPSA is determined by its sensitivities to the main error sources that is given in the reciprocal space by their two-dimensional characteristic spectra:

\[
H_{N,D}(f_1, f_2) = FT\left[h_{N,D}(u_1, u_2)\right] \tag{9}
\]

In this work we describe the phase characterization process of a TSPSAs using as a tool a two-dimensional characteristic polynomial (TDCP) that provides valuable information on the transmission of sensitivities from precursor PSAs to TSPSAs related with the most significant systematic errors.

2. Two-Dimensional Characteristic Polynomial

The qualitative characterisation of the phase sum \(2\phi(r) + \Delta\phi(r)\) and the phase difference \(\Delta\phi(r)\) can be carried out by associating a TDCP to the TSPSAs by relating a complex number to the calculation of the phase by means of equation (8):

\[
V(\phi + \Delta\phi \pm \phi) = \sum_{m=1}^{M} \sum_{p=1}^{P} (d_{m,p} + jn_{m,p}) s_m(r, \phi, \alpha_m) t_p(r, \phi + \Delta\phi, \beta_p) \tag{10}
\]

We consider the system affected by the presence of undesired harmonics in the recovered signal, the \(k\)-coefficients \(a_k\) and the \(g\)-coefficients \(b_g\) with \(k, g > 1\), and by phase shift errors

\[
E^e \alpha_m = E^e \alpha_m + \alpha_m + \sum_{q=1}^{\infty} E^q \alpha^q_m \frac{\alpha^q_m}{q\pi^{q-1}} \tag{11a}
\]
\[ E \beta_p = \beta_p + E \beta_p = \beta_p + \sum_{r=1}^{\infty} \frac{\beta'_r}{r\pi^{r-1}} \]

(11b)

with errors \( E \alpha_m \) and \( E \beta_r \) quantified by the \( q \)th and \( r \)th terms \( \varepsilon_q \) and \( \chi_r \). So, the original \( s_m(\mathbf{r}, \phi, \alpha_m) \) and modified \( t_p(\mathbf{r}, \phi + \Delta \phi, \beta_p) \) patterns are altered with both sensitivities

\[ s_m(\mathbf{r}, \phi, \alpha_m) = \sum_{k=0}^{\infty} a_k(\mathbf{r}) \cos[k(\phi + E \alpha_m)] = \sum_{k=-\infty}^{\infty} \frac{a_k(\mathbf{r})}{2} e^{jk(E \alpha_m + \phi)} \]

(12a)

and introduced into equation (10), the complex representation of the TSPSA, obtaining

\[ V(\phi + \Delta \phi \pm \phi) = \sum_{k=-\infty}^{\infty} \sum_{g=-\infty}^{\infty} \frac{a_k b_g}{4} e^{i\phi(\phi + E \alpha_m)} P(e^{ik\alpha}, e^{ig\beta}) + \]

\[ + \sum_{q=1}^{\infty} \sum_{\alpha=1}^{\infty} \frac{jk\alpha e^{jk\alpha}}{q\pi^{q-1}} \frac{\partial^q P(e^{jk\alpha}, e^{ig\beta})}{\partial(e^{jk\alpha})^q} + \sum_{r=1}^{\infty} \sum_{\beta=1}^{\infty} \frac{jr\beta e^{jr\beta}}{r\pi^{r-1}} \frac{\partial^r P(e^{jk\alpha}, e^{ig\beta})}{\partial(e^{jk\alpha})^r} \]

(13)

that is expressed according to the \((M-1)(P-1)\) order TDCP:

\[ P(e^{ik\alpha}, e^{ig\beta}) = \sum_{m=1}^{M} \sum_{p=1}^{P} (d_{m,p} + jn_{m,p}) e^{i[k(m-1)\alpha + g(p-1)\beta]} \]

(14)

The localization and multiplicity of the roots of the TDCP indicate the sensitivities of the TSPSA in a similar way to that employed in the one-dimensional case for PSAs. Thus the only harmonic that should be detected in each case is the reference one, \((k,g)=(1,1)\) for the sum and \((k,g)=(-1,1)\) for the difference, the rest must be null in order to avoid the presence of undesired harmonics. If \((k,g)=(0,0)\) is detected, this indicates sensitivity to local average irradiance. Accepting that the patterns should not have detuning errors it demands mathematically that \( e^{i\phi(\phi + E \alpha_m)} \) and \( e^{i\phi(\phi + E \alpha_m + \Delta \phi \pm \phi)} \) are double roots. Similarly the harmonic component \((k,g)\) may be detected with detuning insensitivity if \( e^{i\phi(\phi + E \alpha_m)} \) and \( e^{i\phi(\phi + E \alpha_m + \Delta \phi \pm \phi)} \) are double roots. Likewise, in order to avoid error in the \( q \) and \( r \) order phase shifts, their derivatives of this order must be null.

Calculating the zeros of a TDCP can be considerably simplified by relating the two-dimensional sampling amplitudes with those for their PSA precursors according to equations (4) and (5). Thus the additive TDCP can be expressed as the direct multiplication of the CP of the PSA precursors

\[ P(e^{ik\alpha}, e^{ig\beta}) = \sum_{m=1}^{M} \sum_{p=1}^{P} (d_{m,p} + jn_{m,p}) e^{i[k(m-1)\alpha + g(p-1)\beta]} \]

(15)

and the same for the differential TDCP but recovering \(-\phi\) in the original pattern:
In this way it is also possible to directly relate the TDCP with the two-dimensional combined characteristic spectrum

\[ P(e^{j\alpha}, e^{j\beta}) = \sum_{m=1}^{M} \sum_{p=1}^{P} (d_m - jn_m)(d_p + jn_p) e^{j(k(m-1)\alpha)} e^{j(p-1)\beta} = P^*(e^{-j\alpha})P(e^{j\beta}) \]  

(16)

where the negative sign refers to the difference \( \Delta \phi(r) \) and the positive to the sum \( 2\phi(r) + \Delta \phi(r) \). In this way the propagation of the PSA precursor sensitivities to the TSPSAs is obtained in an obvious way according to equations (15) and (16). Furthermore, if \( \Delta \phi(r) \) is null, equations (8) and (16) tell us that the TSPSA is actually a new multiple averaged PSA [32] with probable improved properties with regards to their PSA precursors and that these properties appear directly by multiplying their corresponding CP, thus clearly accounting for the propagation.

3. Examples of Two-Dimensional Characteristic Polynomial

There are infinite possible PSA combinations, equation (7), that construct TSPSAs and each one of them with sensitivities that the TDCP can determine by analysing the corresponding TDCP roots. By way of an example we characterise the Schwider-Hariharan [33,34] TSPSA, which uses the same PSA in the original and modified pattern, in order to study well-known inherited behaviours:

\[ (\phi + \Delta \phi) \mp \phi = \arctan \left( \frac{2s_1 - s_3 - s_s (2t_2 - 2t_4)}{2s_2 - 2s_4 (2t_2 - 2t_4) \mp (2s_3 - s_1 - s_s (2t_2 - t_1 - t_3)} \right) \]  

(18)

The additive TDCP can be found according to equation (15) simply as the multiplication of the CPs [17] of the Schwider-Hariharan PSA

\[ P(e^{j\alpha}, e^{j\beta}) = \left( e^{j\alpha} - 1 \right) \left( e^{j\alpha} + 1 \right) \left( e^{j\beta} + j \right)^2 \left( e^{j\beta} - 1 \right) \left( e^{j\beta} + j \right)^2 \]  

(19)

and the differential TDCP, equation (16), by inverting the sign of the original pattern

\[ P(e^{j\alpha}, e^{j\beta}) = \left( e^{j\alpha} - 1 \right) \left( e^{j\alpha} + 1 \right) \left( e^{j\beta} - j \right)^2 \left( e^{j\beta} + 1 \right) \left( e^{j\beta} + j \right)^2 \]  

(20)

thus a factorised vision is obtained of the TSPSA roots. To see the sensitivities of each TSPSA the module of their respective TDCPs can be found (Figure 1, 2).
The additive TDCP (figure 1) shows that the reference frequency $2\phi+\Delta\phi$ is effectively recovered as $(k,g)=(1,1)$ is not a root. All the even harmonics of the sum where $k=g$ are not detected (since their derivatives are null over abscise axis) and neither are the harmonics of $\Delta\phi(r)$. There do exist many other harmonics components that interfere with the signal. Furthermore, given that at $(k,g)=(-1,-1)$ the additive TDCP is zero with third derivative null, it is insensitive to detuning. Moreover, the fifth and ninth harmonics are also detected with detuning insensitivity since at $(k,g)=(-5,-5)$ and $(k,g)=(-9,-9)$ it is zero with third derivative null.

The differential TDCP (figure 2) has a similar representation to the additive TDCP except that it is $\pi$ displaced. It detects the signal at $(k,g)=(-1,1)$ and at $k=1$ and $g=-1$ has double roots which means it is also insensitive to detuning. The same reasoning can be applied to fifth and ninth harmonics. Also, among others, the even harmonics of $\Delta\phi(r)$ with $k=g$ are not detected, and neither are the harmonics of the sum.

**Figure 1.** Module of additive TDCP for the Schwider-Hariharan TSPSA.

**Figure 2.** Module of differential TDCP for the Schwider-Hariharan TSPSA.
The Schwider-Hariharan PSA has the same sensitivities as both TSPSAs from which it can be understood that there is a transfer of sensitivities from the precursor PSAs [31].

Conclusions
Current metrological inspection techniques demand tools that quickly and accurately provide information about the measurement. TSPSAs provide the optical phase sum or difference in a direct calculation. The new TDCP provides an immediate qualitative characterisation of the TSPSA that inversely should allow the design of a TSPSA with the sensitivities required in each system. Direct relations between TDCP theory and Fourier description of TSPSAs are also provided. By knowing the CPs of the PSAs we can construct the TDCP simply by multiplying. Moreover, the sensitivities of the PSAs that construct the TSPSA are intimately related to those of each PSA and the additive TDCP. This tool makes it possible to efficiently construct new PSAs with tailored properties simply by combining PSAs with known properties.

Acknowledgements
The authors are grateful for funding received from Xunta de Galicia (07DPI002CT) and Ministerio de Ciencia e Innovación (DPI2008-06818-C2-01/DPI).

References
[1] Kemao Q, Seah H S and Asundi A K 2003 Algorithm for directly retrieving the phase difference: a generalization Opt. Eng. 42 1721-1724
[2] Novak M, Millerd J, Brock N, North-Morris M, Hayes J and Wyant J 2005 Analysis of a micropolarizer array-based simultaneous phase-shifting interferometer Appl. Opt. 44 6861-6868
[3] Viotti M R, Dolinko A E, Galizzi G E and Kaufmann G H 2006 A portable digital speckle pattern interferometry device to measure residual stresses using the hole drilling technique Opt. Las. Eng. 44 1052-1066
[4] Kiire T, Nakadate S and Shibuya M 2008 Simultaneous formation of four fringes by using a polarization quadrature phase-shifting interferometer with wave plates and a diffraction grating Appl. Opt. 47 4787-4792
[5] Bhaduri B, Kothiyal M P and Mohan N K, 2008 A comparative study of phase-shifting algorithms in digital speckle pattern interferometry Optik 119 147-152
[6] Toto-Arellano N I, Rodriguez-Zúrita G, Meneses-Fabian C and Vazquez-Castillo J F 2008 Phase shifts in the Fourier spectra of phase gratings and phase grids: an application for oneshot phaseshifting interferometry Opt. Express 16 19330-19341
[7] Cordero R R, François M, Lira I and Vial-Edwards C 2005 Whole-field analysis of uniaxial tensile tests by Moiré interferometry Opt. Lasers Eng. 43 919-936
[8] Ghiglia D C and Pritt M D 1998 Two dimensional phase unwrapping Wiley
[9] Dorri B V and Fernández J L 1999 Phase-evaluation methods in whole-field optical measurement techniques Meas. Sci. Technol. 10 33-35
[10] Morgan C J 1982 Least-squares estimation in phase-measurement interferometry Opt. Lett. 7 368-370
[11] Greivenkamp J E 1984 Generalized data reduction for heterodyne interferometry Opt. Eng. 23 350-352
[12] Freischlad K and Koliopoulos C L 1990 Fourier description of digital phase-measuring interferometry J. Opt. Soc. Am. A 7 542-551
[13] Malacara D, Servín M and Malacara Z 2005 Interferogram Analysis for Optical Testing, Taylor&Francis Group: Florida
[14] Servín M, Estrada J C and Quiroga J A 2009 Spectral analysis of phase shifting algorithms Optics Express 17 16423-16427
[15] Servín M, Estrada J C and Quiroga J A 2009 The general theory of phase shifting algorithms Optics Express 17 21867-21881
[16] Servín M, Estrada J C, Quiroga J A, Mosiño J F and M. Cywiak 2009 Noise in phase shifting interferometry Optics Express 11 8789-8794
[17] Mosiño J F, Servín M and Quiroga J A 2009 Phasorial analysis of detuning error in temporal phase shifting algorithms Optics Express 17 5618-5623
[18] Mosiño J F, Malacara Doblado D and Malacara Hernández D 2009 Calculus of exact detuning phase shift error in temporal phase shifting algorithms Optics Express 17 15766-15771
[19] Mosiño J F, Malacara Doblado D and Malacara-Hernández D 2009 A method to design tunable quadrature filters in phase shifting interferometry Optics Express 17 15772-15777
[20] Surrel Y 1996 Design of algorithms for phase measurements by the use of phase stepping Appl. Opt. 35 51-60
[21] Surrel Y 1997 Additive noise effect in digital phase detection Appl. Opt. 36 271-276
[22] Surrel Y 1997 Design of phase-detection algorithms insensitive to bias modulation Appl. Opt. 36 805-807
[23] Surrel Y 1998 Phase-shifting algorithms for nonlinear and spatially nonuniform phase shifts: comment J. Opt. Soc. Am. A 15 1227-1233
[24] Surrel Y 1998 Extended averaging and data windowing techniques in phase-stepping measurements: An approach using the characteristic polynomial theory Opt. Eng. 37 2314-2319
[25] Surrel Y 2000 Fringe Analysis Top. Appl. Phys. 77 55-102
[26] Stetson K A and Brohinsky W R 1985 Electrooptic holography and its application to hologram interferometry Appl. Opt. 24 3631-3637
[27] Saldner H O, Molin N-E and Stetson K A 1996 Fourier transform evaluation of phase data in spatially phase-biased TV holograms Appl. Opt. 35 332-336
[28] Burke J and Helmers H 1998 Complex division as a common basis for calculating phase differences in electronic speckle pattern interferometry in one step Appl. Opt. 37 2589-2590
[29] Miranda M and Dorrio B V 2008 Error behaviour in Differential Phase-Shifting Algorithms SPIE: Optical Fabrication, Testing, and Metrology III 7102 71021B-1-71021B-9
[30] Miranda M and Dorrio B V 2010Fourier analysis of two-stage phase-shifting algorithms J. Opt. Soc. Am. A 27 276-285
[31] Miranda M and Dorrio B V 2010 Error propagation in differential phase evaluation Optics Express 18 3199-3209
[32] Schmit J and Creath K 1995 Extended averaging technique for derivation of error-compensating algorithms in phase-shifting interferometry Appl. Opt. 34 3610-3619
[33] Schwider J, Burow R, Elssner K E, Grzanna J, Spolaczyk R and Merkel K 1983 Digital wavefront measuring interferometry: some systematic error sources Appl. Opt. 22 3421-3432
[34] Hariharan P, Oreb B F and Eiju T 1987 Digital phase-shifting interferometry: a simple error-compensating phase calculation algorithm Appl. Opt. 26 2504-2506