Satisfaction Problem of Consumers Demands measured by ordinary “Lebesgue measures” in $\mathbb{R}^\infty$

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1. Formulation of the main problem

In a wide class of general systems the so-called dynamical systems can be separated which, as usual, are used for description of the behaviour of various physical, economic and social processes during the time (see, for example [4]).

Let $(E, \rho)$ be a metric space. Recall, that a family of mappings $(\Phi_t)_{t \in \mathbb{R}}$ with $\Phi_t : E \to E$ for $t \in \mathbb{R}$ is called a dynamical system if it satisfies the following three conditions:

1) $\Phi_0(x) = x$ for each element $x \in E$ ;
2) A mapping $\Phi : E \times \mathbb{R} \to E$ defined by $\Phi(x, t) = \Phi_t(x)$ is continuous with respect to the variables $x$ and $t$;
3) if $x \in E$, $t_1 \in \mathbb{R}$ and $t_2 \in \mathbb{R}$, then $\Phi_{t_1}(\Phi_{t_2}(x)) = \Phi_{t_1+t_2}(x)$.

Satisfaction Rule of Consumers Demands under Dynamical System: Let $(\Phi_t)_{t \in \mathbb{R}}$ be some dynamical system defined in a metric space $(E, \rho)$ and let $\nu$ be a Borel measure on $E$. It is assumed that $t_1 < t_2 < \cdots < t_n$ and

*The present work was partially supported on the Shota Rustaveli National Science Foundation Grants: # GNFS 31/25, # GNFS / FR 1165-100 / 14.
m_k > 0 for 1 \leq k \leq m$. Supplier is a person who choose any Borel subset $Y \subseteq E$ of positive $\nu$ measure called an initial system and satisfies $n$ consumers demands as follows:

(i) At time $t_1$ supplier gives a set $\Phi_{t_1}(Y)$ and 1-th consumer at time $t_1$ choose a Borel subset $C_1 \subseteq \Phi_{t_1}(Y)$ of the measure $m_1$ (called 1-th demand) for which $\nu(C_1) = m_1$ as follows:

$$C_1 = \tau \{ X : X \in B(\Phi_{t_1}(Y)) \ & \ & \nu(X) = m_1 \},$$

where $B(\Phi_{t_1})$ denotes the class of all Borel subsets of $\Phi_{t_1}(Y)$ and $\tau$ denotes an operator of global choice.

(ii) At time $t_2$ supplier gives a set $\Phi_{t_2-t_1}(\Phi_{t_1}(Y) \setminus C_1)$ and 2-th consumer at time $t_2$ choose a Borel subset $C_2 \subseteq \Phi_{t_2-t_1}(\Phi_{t_1}(Y) \setminus C_1)$ of the measure $m_2$ (called 2-th demand) for which $\nu(C_2) = m_2$ as follows:

$$C_2 = \tau \{ X : X \in B(\Phi_{t_2-t_1}(\Phi_{t_1}(Y) \setminus C_1)) \ & \ & \nu(X) = m_2 \},$$

and so on.

**Satisfaction Problem of Consumers Demands (SPCD):** Assume that the supplier must satisfy $n$ consumers demands by the rule described above. What minimal measure of a measurable system which must take the supplier at the initial time $t = 0$ for satisfaction demands of all consumers?

We plan to consider SPCD in the case, when $(E, \rho)$ is an infinite-dimensional topological vector space $R^\infty$ equipped with Tychonov metric, $\nu$ is any ordinary “Lebesgue measure” in $R^\infty$ (cf. [7]) and $(\Phi_t)_{t \in R}$ in $R^\infty$ is a dynamical system defined by the one from the following differential equations:

- von Foerster-Lasota differential equation in $R^\infty$ (cf. [12]);
- The Black-Scholes equation (cf. [15]);
- Infinite generalised Maltusian growth equation in $R^\infty$ (cf. [6]);
- Fourier differential equation (cf. [15]).

The rest of the paper is the following.

In Section 2 we give constructions of ordinary “Lebesgue measures” in $R^\infty$. In the next sections we discuss the Satisfaction Problem of Consumers Demands for above mentioned mathematical models.

2. Auxiliary notions and propositions from measure theory and linear algebra

Let $(\beta_j)_{j \in N} \in [0, +\infty]^N$.

**Definition 2.1** We say that a number $\beta \in [0, +\infty]$ is an ordinary product of numbers $(\beta_j)_{j \in N}$ if

$$\beta = \lim_{n \to \infty} \prod_{i=1}^{n} \beta_i.$$
An ordinary product of numbers \((\beta_j)_{j \in \mathbb{N}}\) is denoted by \((O) \prod_{i \in \mathbb{N}} \beta_i\).

Let \(\alpha = (n_k)_{k \in \mathbb{N}} \in (\mathbb{N} \setminus \{0\})^N\). We set
\[
F_0 = [0, n_0] \cap \mathbb{N}, \quad F_1 = [n_0 + 1, n_0 + n_1] \cap \mathbb{N}, \ldots,
\]
\[
F_k = [n_0 + \cdots + n_{k-1} + 1, n_0 + \cdots + n_k] \cap \mathbb{N}, \ldots.
\]

**Definition 2.2** We say that a number \(\beta \in [0, +\infty]\) is an ordinary \(\alpha\)-product of numbers \((\beta_i)_{i \in \mathbb{N}}\) if \(\beta\) is an ordinary product of numbers \((\prod_{i \in F_k} \beta_i)_{k \in \mathbb{N}}\). An ordinary \(\alpha\)-product of numbers \((\beta_i)_{i \in \mathbb{N}}\) is denoted by \((O, \alpha) \prod_{i \in \mathbb{N}} \beta_i\).

**Definition 2.3** Let \(\alpha = (n_k)_{k \in \mathbb{N}} \in (\mathbb{N} \setminus \{0\})^N\). Let \((\alpha)\mathcal{OR}\) be the class of all infinite-dimensional measurable \(\alpha\)-rectangles \(R = \prod_{i \in \mathbb{N}} R_i (R_i \in \mathcal{B}(\mathbb{R}^{n_i}))\) for which an ordinary product of numbers \((m^{n_i}(R_i))_{i \in \mathbb{N}}\) exists and is finite, where \(m\) denotes a linear Lebesgue measure in \(R\).

**Definition 2.4** We say that a measure \(\lambda\) being the completion of a translation-invariant Borel measure is an ordinary \(\alpha\)-Lebesgue measure on \(\mathbb{R}^\infty\) (or, shortly, O\((\alpha)\)LM) if for every \(R \in (\alpha)\mathcal{OR}\) we have
\[
\lambda(R) = (O) \prod_{k \in \mathbb{N}} m^{n_k}(R_k).
\]

**Lemma 2.1** ([6], Theorem 1, p. 216) For every \(\alpha = (n_i)_{i \in \mathbb{N}} \in (\mathbb{N} \setminus \{0\})^N\), there exists a Borel measure \(\mu_\alpha\) on \(\mathbb{R}^\infty\) which is \(O(\alpha)\)LM.

**Lemma 2.2** ([8], Theorem 3, p. 9.) Let \(\alpha = (n_i)_{i \in \mathbb{N}} \in (\mathbb{N} \setminus \{0\})^N\), and let \(T^{n_i} : \mathbb{R}^{n_i} \to \mathbb{R}^{n_i}, i > 1\), be a family of linear transformation with Jacobians \(\Delta_i \neq 0\) and \(0 < \prod_{i=1}^\infty \Delta_i < \infty\). Let \(T^N : \mathbb{R}^N \to \mathbb{R}^N\) be the map defined by
\[
T^N(x) = (T^{n_1}(x_1, \cdots, x_{n_1}), T^{n_2}(x_{n_1+1}, \cdots, x_{n_1+n_2}), \cdots),
\]
where \(x = (x_i)_{i \in \mathbb{N}} \in \mathbb{R}^N\). Then for each \(E \in \mathcal{B}(\mathbb{R}^N)\), we have
\[
\mu_\alpha(T^N(E)) = \prod_{i=1}^\infty \Delta_i \mu_\alpha(E).
\]

In context with another interesting properties of partial analogs of the Lebesgue measures in \(\mathbb{R}^\infty\), the reader can see [1], [2], [5], [7], [9], [6].

In the sequel we identify the vector space \(\mathbb{R}^\infty\) of all real-valued sequences with the vector space of all real-valued infinite-dimensional vector-columns.
The need the following auxiliary proposition from linear algebra.

**Lemma 2.3** ([3], §6, Section 1) Let \( \alpha = (n_i)_{i \in N} \in (N \setminus \{0\})^N \) and, let \( A = (a_{ij})_{i,j \in N} \) be an infinite-dimensional real-valued \( \alpha \)-cellular matrix. Let us consider a linear autonomous differential equation of the first order

\[
\frac{d}{dt}((a_k)_{k \in N}) = A \times (a_k)_{k \in N}
\]

with an initial condition

\[
(a_k(0))_{k \in N} = (c_k)_{k \in N} \in \mathbb{R}^\infty.
\]

Then the solution of (2.1) – (2.2) is given by

\[
(a_k(t))_{k \in N} = \exp(tA) \times (c_k)_{k \in N}.
\]

**Proof.** Let us present the column \((a_k(t))_{k \in N}\) in the Maclaurin series as follows:

\[
(a_k(t))_{k \in N} = \sum_{m=0}^{\infty} \frac{(a_k^{(m)}(0))_{k \in N} t^m}{m!}.
\]

Take into account the validity of the formula

\[
(a_k^{(m)}(0))_{k \in N} = \left(\frac{d^m a_k(t)}{dt^m}\right)_{k \in N} = A^m \times (a_k(0))_{k \in N},
\]

we get

\[
(a_k(t))_{k \in N} = \sum_{m=0}^{\infty} \frac{(a_k^{(m)}(0))_{k \in N} t^m}{m!} = \sum_{m=0}^{\infty} \frac{t A^m}{m!} \times (a_k(0))_{k \in N} = \exp(tA) \times (c_k)_{k \in N}
\]

(2.6)

In the sequel we will need some notions characterizing the behavior of some dynamical systems \((\Phi_t)_{t \geq 0}\) in \(\mathbb{R}^\infty\).

Let \( \nu \) be any “Lebesgue measure” in \(\mathbb{R}^\infty\) (see, for example, [5], [7]).

**Definition 2.5** We say that the dynamical system \((\Phi_t)_{t \geq 0}\) is stable in the sense of a “Lebesgue measure” \( \nu \) if the flow preserves the measure \( \nu \), i.e.

\[
(\forall t)(0 < t < \infty \& 0 < \nu(D) < \infty \rightarrow \nu(\Phi_t(D)) = \nu(D)).
\]

(2.7)

**Definition 2.6** We say that the dynamical system \((\Phi_t)_{t \geq 0}\) is expansible in the sense of a “Lebesgue measure” \( \nu \) if

\[
(\forall t_1)(\forall t_2)(0 < t_1 < t_2 < \infty \& 0 < \nu(D) < \infty \rightarrow \nu(\Phi_{t_1}(D)) < \nu(\Phi_{t_2}(D))).
\]

(2.8)
Definition 2.7 We say that the dynamical system \((\Phi_t)_{t \geq 0}\) is pressing in the sense of a “Lebesgue measure” \(\nu\) if the flow is dissipative, i.e.,

\[
(\forall t_1)(\forall t_2)(\forall D)(0 < t_1 < t_2 < \infty \& 0 < \nu(D) < \infty \rightarrow \nu(\Phi_{t_1}(D)) > \nu(\Phi_{t_2}(D))).
\]

(2.9)

Definition 2.8 We say that the dynamical system \((\Phi_t)_{t \geq 0}\) is totally expansible in the sense of a “Lebesgue measure” \(\nu\) if

\[
(\forall t)(\forall D)(0 < t < \infty \& 0 < \nu(D) < \infty \rightarrow \nu(\Phi_t(D)) = +\infty).
\]

(2.10)

Definition 2.9 We say that the dynamical system \((\Phi_t)_{t \geq 0}\) is totally pressing in the sense of a “Lebesgue measure” \(\nu\) if

\[
(\forall t)(\forall D)(0 < t < \infty \& 0 < \nu(D) < \infty \rightarrow \nu(\Phi_t(D)) = 0).
\]

(2.11)

3. Satisfaction Problem of Consumers Demands in von Foerster-Lasota model in \(R^\infty\)

In this section we consider a certain concept \([13]\) for a solution of some differential equations by “Maclaurin Differential Operators” in \(R^\infty\).

**Definition 3.1** “Maclaurin differential operator” \((M) \frac{\partial}{\partial x}\) in \(R^\infty\) is defined as follows:

\[
(M) \frac{\partial}{\partial x} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 & \ldots \\ 0 & 0 & 2 & 0 & \ldots \\ 0 & 0 & 0 & 3 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \times \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \end{pmatrix}. \tag{3.1}
\]

**Definition 3.2** “Maclaurin differential operator” \((M)x \frac{\partial}{\partial x}\) in \(R^\infty\) is defined as follows:

\[
(M)x \frac{\partial}{\partial x} ((a_k)_{k \in N}) = \begin{pmatrix} 0 & 0 & 0 & \ldots \\ 0 & 1 & 0 & 0 & \ldots \\ 0 & 0 & 2 & 0 & \ldots \\ 0 & 0 & 0 & 3 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \times \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \end{pmatrix}. \tag{3.2}
\]
Definition 3.3 “Maclaurin differential operator” \((\mathcal{M})x^2\frac{\partial^2}{\partial x^2}\) in \(\mathbb{R}^\infty\) is defined as follows:

\[
(\mathcal{M})x^2\frac{\partial^2}{\partial x^2}((a_k)_{k\in\mathbb{N}}) = \begin{pmatrix}
0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & \cdots \\
0 & 0 & 1 \times 2 & 0 & \cdots \\
0 & 0 & 0 & 2 \times 3 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix} \begin{pmatrix}
a_0 \\
a_1 \\
a_2 \\
\vdots
\end{pmatrix}.
\] (3.3)

Definition 3.4 Formally, we set that the factorial of each negative integer number is equal to \(+\infty\). Then “Maclaurin differential operator” \((\mathcal{M})x^n\frac{\partial^n}{\partial x^n}\) in \(\mathbb{R}^\infty\) is defined as follows:

\[
(\mathcal{M})x^n\frac{\partial^n}{\partial x^n}((a_k)_{k\in\mathbb{N}}) = \begin{pmatrix}
0! & 0 & 0 & 0 & \cdots \\
0 & 1! \frac{1}{(1-n)!} & 0 & \cdots \\
0 & 0 & 2! \frac{2}{(2-n)!} & 0 & \cdots \\
0 & 0 & 0 & 3! \frac{3}{(3-n)!} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix} \begin{pmatrix}
a_0 \\
a_1 \\
a_2 \\
\vdots
\end{pmatrix}.
\] (3.4)

Theorem 3.3 ([14], Theorem 11.2, p.139) Let \((A_n)_{0\leq n\leq m}(m \in \mathbb{N})\) be a sequence of real numbers. Let consider a non-homogeneous “Maclaurin differential operators” equation of the first order

\[
\frac{d}{dt}((a_k)_{k\in\mathbb{N}}) = \sum_{n=0}^{m} A_n (\mathcal{M})x^n\frac{\partial^n}{\partial x^n} \times (a_k)_{k\in\mathbb{N}} + (f_k(t))_{k\in\mathbb{N}}
\] (3.5)

with initial condition

\[
(a_k(0))_{k\in\mathbb{N}} = (C_k)_{k\in\mathbb{N}},
\] (3.6)

where

(i) \((C_k)_{k\in\mathbb{N}} \in \mathbb{R}^\infty\);

(ii) \((f_k(t))_{k\in\mathbb{N}}\) is the sequence of continuous functions on \(\mathbb{R}\).

Then

\[
(a_k(t))_{k\in\mathbb{N}} = (e^t \sum_{n=0}^{m} A_n \frac{1}{n!} C_k + \int_0^t e^{(t-\tau)} \sum_{n=0}^{m} A_n \frac{1}{n!} f_k(\tau) d\tau)_{k\in\mathbb{N}}.
\] (3.7)

We have the following consequence of Theorem 3.3.

Corollary 3.1 Let consider the von Foerster-Lasota operator equation in \(\mathbb{R}^\infty\) defined by

\[
\frac{d}{dt}((a_k)_{k\in\mathbb{N}}) = -(\mathcal{M})\left(x\frac{\partial}{\partial x}\right)(a_k)_{k\in\mathbb{N}} + \gamma(a_k)_{k\in\mathbb{N}}
\] (3.8)
with initial condition

\[ (a_k(0))_{k \in N} = (C_k)_{k \in N} \in \mathbb{R}^\infty. \]  

(3.9)

Then

\[ (a_k(t))_{k \in N} = (e^{t(\gamma-k)}C_k)_{k \in N}. \]  

(3.10)

Satisfaction Problem of Consumers Demands in von Foerster-Lasota model.

Let consider \((1,1,\cdots)\)-ordinary "Lebesgue measure" \(\mu_{(1,1,\cdots)}\). By Lemma 2.2 we know that \(\mu_{(1,1,\cdots)}(\Phi_t(X)) = e^{\sum_{k=0}^{\infty} t(\gamma-k)}C_k_{(1,1,\cdots)}(X)\), where the von Foerster-Lasota motion \(\Phi_t : \mathbb{R}^\infty \to \mathbb{R}^\infty\) is defined by

\[ \Phi_t((C_k)_{k \in N}) = (e^{t(\gamma-k)}C_k)_{k \in N} \]  

(3.11)

for \((C_k)_{k \in N} \in \mathbb{R}^\infty\).

Since \(e^{\sum_{k=0}^{\infty} t(\gamma-k)} = 0\), we claim for an arbitrary initial system \(S_0 \in B(\mathbb{R}^\infty)\) and \(t > 0\), the set \(\mu_{(1,1,\cdots)}(\Phi_t(S_0)) = 0\). Hence the first consumer can not choose a Borel subset \(C_1 \subseteq S_t = \Phi_t(S_0)\) for which \(\mu_{(1,1,\cdots)}(C_1) = m_1 > 0\). The latter relation means that Satisfaction Problem of Consumers Demands in von Foerster-Lasota model has no any solution.

4. Satisfaction Problem of Consumers Demands in Black-Sholes Model in \(\mathbb{R}^\infty\)

The Black-Scholes differential equation in \(\mathbb{R}^\infty\) has the following form:

\[ \frac{d}{dt}(a_k)_{k \in N} = -\frac{1}{2}\sigma^2(M)(x^2(\frac{\partial}{\partial x^2}))(a_k)_{k \in N} - r(M)(x\frac{\partial}{\partial x})(a_k)_{k \in N} + r(a_k)_{k \in N} \]  

(4.1)

Notice that (4.1) is a particular case of (3.5) for which \(m = 2\), \(A_0 = r\), \(A_1 = -r\), \(A_2 = -\frac{\sigma^2}{2}\). Following Theorem 3.3, the solution of (4.1) has the form

\[ (a_k(t))_{k \in N} = (e^{t(\frac{\sigma^2}{2} \frac{k^2}{2} - \frac{k^2}{2} \frac{r^2}{2}}) C_k)_{k \in N} \]  

(4.2)

Satisfaction Problem of Consumers Demands in Black-Sholes model.

Let consider \((1,1,\cdots)\)-ordinary "Lebesgue measure" \(\mu_{(1,1,\cdots)}\). By Lemma 2.2 we know that \(\mu_{(1,1,\cdots)}(\Phi_t(X)) = e^{\sum_{k=0}^{\infty} t(\gamma-k)}C_k_{(1,1,\cdots)}(X)\), where the von Foerster-Lasota motion \(\Phi_t : \mathbb{R}^\infty \to \mathbb{R}^\infty\) is defined by

\[ \Phi_t((C_k)_{k \in N}) = (e^{t(\gamma-k)}C_k)_{k \in N} \]  

(4.3)

for \((C_k)_{k \in N} \in \mathbb{R}^\infty\).
Since $\sum_{k \in \mathbb{N}} r \frac{k!}{(k-0)!} - r \frac{k!}{(k-1)!} - \frac{1}{2} \sigma^2 \frac{k!}{(k-2)!} = -\infty$, we claim for an arbitrary initial system $S_0 \in \mathcal{B}(R^\infty)$ and $t > 0$, the set $\mu_{(1,1,\ldots)}(\Phi_t(S_0)) = 0$. Hence first consumer can not choose a Borel subset $C_1 \subseteq S_{t_1} = \Phi_{t_1}(S_0)$ for which $\mu_{(1,1,\ldots)}(C_1) > m_1 > 0$. The latter relation means that Satisfaction Problem of Consumers Demands in Black-Scholes model has no any solution.

5. Satisfaction Problem of Consumers Demands in infinite continuous generalised Malthusian growth model in $R^\infty$

Let us consider an infinite non-antagonistic family of populations and let $\Psi_k(t)$ be the population function of the $k$-th Population. Then the generalised continuous Malthusian growth model for an infinite family of non-antagonistic populations is described by the following linear differential equation

$$\frac{d((a_k(t))_{k \in \mathbb{N}})}{dt} = A \times (a_k(t))_{k \in \mathbb{N}}$$ (5.1)

with an initial condition

$$(a_k(0))_{k \in \mathbb{N}} = (a_k)_{k \in \mathbb{N}} \in R^\infty,$$ (5.2)

where $A$ is an infinite-dimensional real-valued diagonal matrix with diagonal elements $(\lambda_k)_{k \in \mathbb{N}}$.

By Lemma 2.3 we know that the solution of (5.1) is given by

$$(a_k(t))_{k \in \mathbb{N}} = (e^{t\lambda_k}a_k)_{k \in \mathbb{N}}.$$ (5.3)

Satisfaction Problem of Consumers Demands in infinite continuous generalised Malthusian growth model in $R^\infty$.

Let consider $(1,1,\cdots)$-ordinary "Lebesgue measure" $\mu_{(1,1,\ldots)}$. By Lemma 2.2 we know that

$$\mu_{(1,1,\ldots)}(\Phi_t(X)) = e^{t \sum_{k=0}^{\infty} \lambda_k} \mu_{(1,1,\ldots)}(X),$$ (5.4)

where the infinite continuous generalised Malthusian growth motion $\Phi_t : R^\infty \to R^\infty$ is defined by

$$\Phi_t((a_k)_{k \in \mathbb{N}}) = (e^{t\lambda_k}a_k)_{k \in \mathbb{N}}$$ (5.5)

for $(a_k)_{k \in \mathbb{N}} \in R^\infty$.

Case 1. $\sum_{k=0}^{\infty} \lambda_k$ is divergent. In that case we do not know whether SPCD in infinite continuous generalised Malthusian growth model has any solution.

Case 2. $\sum_{k=0}^{\infty} \lambda_k = -\infty$. In that case SPCD in infinite continuous generalised Malthusian growth model has no any solution.

Case 3. $\sum_{k=0}^{\infty} \lambda_k = +\infty$. In that case SPCD in infinite continuous generalised Malthusian growth model also has no any solution because if the supplier take an arbitrary measurable system of the positive $(1,1,\cdots)$-ordinary
"Lebesgue measure" \( \mu_{(1,1,\ldots)} \), then demands of all consumers will be always satisfied.

**Case 4.** \( \sum_{k=0}^{\infty} \lambda_k \) is convergent and \( -\infty < \sum_{k=0}^{\infty} \lambda_k < +\infty \).

Let show that the solution of SPCD in infinite continuous generalised Malthusian growth model is defined by

\[
m = \sum_{k=1}^{n} m_k e^{-t_k \sum_{i=1}^{\infty} \lambda_k} \quad (5.6)
\]

Indeed, let choose an arbitrary Borel subset \( S_0 \subset \mathbb{R}^\infty \) with \( \mu_{(1,1,\ldots)}(S_0) = m \). At the moment \( t = t_1 \) the set \( S_0 \) is transformed into set \( \Phi_{t_1}(S_0) \) whose \( \mu_{(1,1,\ldots)} \) measure is equal to

\[
e^{t_1 \sum_{i=1}^{\infty} \lambda_k} m = e^{t_1 \sum_{i=1}^{\infty} \lambda_k} \left( \sum_{k=1}^{n} m_k e^{-t_k \sum_{i=1}^{\infty} \lambda_k} \right) = \sum_{k=1}^{n} m_k e^{(t_1-t_k) \sum_{i=1}^{\infty} \lambda_k} = m_1 + \sum_{k=2}^{n} m_k e^{(t_1-t_k) \sum_{i=1}^{\infty} \lambda_k} \quad (5.7)
\]

When the demand \( C_1 \) of the first consumer will be satisfied, we obtain the set \( \Phi_{t_1}(S_0) \setminus C_1 \) for which

\[
\mu_{(1,1,\ldots)}(\Phi_{t_1}(S_0) \setminus C_1) = \sum_{k=2}^{n} m_k e^{(t_1-t_k) \sum_{i=1}^{\infty} \lambda_k} \quad (5.8)
\]

At moment \( t = t_2 \) the set \( \Phi_{t_1}(S_0) \setminus C_1 \) is transformed into set \( \Phi_{t_2-t_1}(\Phi_{t_1}(S_0) \setminus C_1) \) whose \( \mu_{(1,1,\ldots)} \) measure is equal to

\[
e^{(t_2-t_1) \sum_{i=1}^{\infty} \lambda_k} \mu_{(1,1,\ldots)}(\Phi_{t_2}(S_0) \setminus C_1) = e^{(t_2-t_1) \sum_{i=1}^{\infty} \lambda_k} \sum_{k=2}^{n} m_k e^{(t_2-t_k) \sum_{i=1}^{\infty} \lambda_k} = e^{(t_2-t_1) + (t_1-t_k) \sum_{i=1}^{\infty} \lambda_k} \sum_{k=2}^{n} m_k e^{(t_2-t_k) \sum_{i=1}^{\infty} \lambda_k} = \sum_{k=2}^{n} m_k e^{(t_2-t_k) \sum_{i=1}^{\infty} \lambda_k} = m_2 + \sum_{k=3}^{n} m_k e^{(t_2-t_k) \sum_{i=1}^{\infty} \lambda_k} \quad (5.9)
\]

When the demand \( C_2 \) of the second consumer will be satisfied, we obtain the set \( \Phi_{t_2-t_1}(\Phi_{t_1}(S_0) \setminus C_1) \setminus C_2 \) for which

\[
\mu_{(1,1,\ldots)}(\Phi_{t_2-t_1}(\Phi_{t_1}(S_0) \setminus C_1) \setminus C_2) = \sum_{k=3}^{n} m_k e^{(t_2-t_k) \sum_{i=1}^{\infty} \lambda_k} \quad (5.10)
\]

and so on.

Now it obvious that at the moment \( t = t_n \) we obtain a set whose \( \mu_{(1,1,\ldots)} \) measure exactly coincides with the positive number \( m_n \) and hence the demand of the \( n \)-th consumer will be satisfied.
Observation 5.1 We have showed that Satisfaction Problem of Consumers Demands in infinite continuous generalised Malthusian growth model in $\mathbb{R}^\infty$ has the solution if dynamical system defined by (5.3) is pressing, expansible or stable in the sense of the measure $\mu_{(1,1,\cdots)}$. When $(\Phi_t)_{t \in \mathbb{R}}$ is totally pressing or totally expansible the the same problem has no any solution.

6. Satisfaction Problem of Consumers Demands for dynamical system defined by the Fourier differential equation in $\mathbb{R}^\infty$

**Definition 6.1.** "Fourier differential operator" $(\mathcal{F}) \frac{d}{dx} : \mathbb{R}^\infty \to \mathbb{R}^\infty$ is defined as follows:

\[
(\mathcal{F}) \frac{d}{dx} \begin{pmatrix}
\sigma_1 & \sigma_2 & \sigma_3 & \cdots \\
\beta_1 & \beta_2 & \beta_3 & \cdots \\
\gamma_1 & \gamma_2 & \gamma_3 & \cdots \\
\delta_1 & \delta_2 & \delta_3 & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{pmatrix} = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & \frac{k\pi}{l} & 0 & 0 & 0 & \cdots \\
0 & -\frac{k\pi}{l} & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & -\frac{2k\pi}{l} & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & -\frac{3k\pi}{l} & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix} \times \begin{pmatrix}
\sigma_1 & \sigma_2 & \sigma_3 & \cdots \\
\beta_1 & \beta_2 & \beta_3 & \cdots \\
\gamma_1 & \gamma_2 & \gamma_3 & \cdots \\
\delta_1 & \delta_2 & \delta_3 & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{pmatrix}.
\]

(6.1)

Suppose that $(A_n)_{n \in \mathbb{N}} \in \mathbb{R}^N$ be a sequence of real numbers such that

\[
\sigma_k = \sum_{n=0}^{\infty} (-1)^n A_{2n} \left(\frac{k\pi}{l}\right)^{2n}
\]

(6.2)

and

\[
\omega_k = \sum_{n=0}^{\infty} (-1)^n A_{2n+1} \left(\frac{k\pi}{l}\right)^{2n+1}
\]

(6.3)

are convergent for each $k \geq 1$.

**Corollary 6.1.** (cf. [11]) Let us consider a partial differential equation of the first order

\[
\frac{\partial}{\partial t} ((a_k)_{k \in \mathbb{N}}) = \left( \sum_{n=0}^{\infty} A_n \left( (\mathcal{F}) \frac{\partial}{\partial x} \right)^n \right) \times (a_k)_{k \in \mathbb{N}}
\]

(6.4)

with initial condition

\[
(a_k(0))_{k \in \mathbb{N}} = (C_k)_{k \in \mathbb{N}},
\]

(6.5)

where $(C_k)_{k \in \mathbb{N}} \in \mathbb{R}^\infty$.

Suppose that the sequence of real numbers $(\sigma_k)_{k \in \mathbb{N}}$ and $(\omega_k)_{k \in \mathbb{N}}$ defined by (6.2) – (6.3) are convergent.
Then the solution \((\Phi_t)_{t \in R}\) of (6.4) – (6.5) is defined by

\[
\Phi_t((C_k)_{k \in N}) = e^{t(\sum_{n=0}^{\infty} A_n \left( (\mathcal{F} \partial_x)^n \right) \times (C_k)_{k \in N}}
\]

(6.6)

where \(e^{t(\sum_{n=0}^{\infty} A_n \left( (\mathcal{F} \partial_x)^n \right)}\) denotes an exponent of the matrix \(t(\sum_{n=0}^{\infty} A_n \left( (\mathcal{F} \partial_x)^n \right)}\) and it exactly coincides with an infinite-dimensional \((1, 2, 2, \ldots)\)-cellular matrix \(D(t)\) with cells \((D_k(t))_{k \in N}\) for which \(D_0(t) = (e^{tA_0})\) and

\[
D_k(t) = e^{\sigma_k t} \begin{pmatrix} \cos(\omega_k t) & \sin(\omega_k t) \\ -\sin(\omega_k t) & \cos(\omega_k t) \end{pmatrix}.
\]

(6.7)

By Lemma 2.2 and Corollary 6.1, one can easily establish the validity of the following assertions.

**Observation 6.1** Suppose that \((\Phi_t)_{t \in R}\) is the dynamical system in \(R^\infty\) which comes from Corollary 6.1. Then \((\Phi_t)_{t \in R}\) is:

a) stable in the sense of an ordinary \((1, 2, 2, \ldots)\)-Lebesgue measure \(\mu_{(1, 2, 2, \ldots)}\) in \(R^\infty\) if and only if \(A_0 + 2 \sum_{k=1}^{\infty} \sigma_k = 0\).

b) extensible in the sense of an ordinary \((1, 2, 2, \ldots)\)-Lebesgue measure \(\mu_{(1, 2, 2, \ldots)}\) in \(R^\infty\) if and only if \(-\infty < A_0 + 2 \sum_{k=1}^{\infty} \sigma_k < 0\).

c) pressing in the sense of an ordinary \((1, 2, 2, \ldots)\)-Lebesgue measure \(\mu_{(1, 2, 2, \ldots)}\) in \(R^\infty\) if and only if \(-\infty < A_0 + 2 \sum_{k=1}^{\infty} \sigma_k < 0\) and the series \(A_0 + 2 \sum_{k=1}^{\infty} \sigma_k\) is absolutely convergent.

d) stable in the sense of a standard \((1, 2, 2, \ldots)\)-Lebesgue measure \(\nu_{(1, 2, 2, \ldots)}\) in \(R^\infty\) if and only if \(A_0 + 2 \sum_{k=1}^{\infty} \sigma_k = 0\) and the series \(A_0 + 2 \sum_{k=1}^{\infty} \sigma_k\) is absolutely convergent.

e) extensible in the sense of a standard \((1, 2, 2, \ldots)\)-Lebesgue measure \(\nu_{(1, 2, 2, \ldots)}\) in \(R^\infty\) if and only if \(0 < A_0 + 2 \sum_{k=1}^{\infty} \sigma_k < +\infty\) and the series \(A_0 + 2 \sum_{k=1}^{\infty} \sigma_k\) is absolutely convergent.

f) pressing in the sense of a standard \((1, 2, 2, \ldots)\)-Lebesgue measure \(\nu_{(1, 2, 2, \ldots)}\) in \(R^\infty\) if and only if \(-\infty < A_0 + 2 \sum_{k=1}^{\infty} \sigma_k < 0\) and the series \(A_0 + 2 \sum_{k=1}^{\infty} \sigma_k\) is absolutely convergent.

\[
g)\] totally extensible in the sense of a standard \((1, 2, 2, \ldots)\)-Lebesgue measure \(\nu_{(1, 2, 2, \ldots)}\) in \(R^\infty\) if and only if the series \(A_0 + 2 \sum_{k=1}^{\infty} \sigma_k\) is not absolutely convergent and \(\sum_{k \in S_+} \sigma_k > -\infty\), where \(S_+\) denotes a set of all natural numbers for which \(\sigma_k < 0\).

h) totally pressing in the sense of a standard \((1, 2, 2, \ldots)\)-Lebesgue measure \(\nu_{(1, 2, 2, \ldots)}\) in \(R^\infty\) if and only if the series \(A_0 + 2 \sum_{k=1}^{\infty} \sigma_k\) is not absolutely convergent and \(\sum_{k \in S_-} \sigma_k = -\infty\), where \(S_-\) denotes a set of all natural numbers for which \(\sigma_k < 0\).

**Satisfaction Problem of Consumers Demands for dynamical system \((\Phi_t)_{t \in R}\) in \(R^\infty\) defined by (6.6).**

Let consider \((1, 2, 2, \ldots)\)-ordinary ”Lebesgue measure” \(\mu_{(1, 2, 2, \ldots)}\). By Lemma 2.2 we know that

\[
\mu_{(1, 2, 2, \ldots)}(\Phi_t(X)) = e^{t(A_0 + 2 \sum_{k=0}^{\infty} \sigma_k)} \mu_{(1, 2, 2, \ldots)}(X).
\]

(6.8)
Case 1. $A_0 + 2 \sum_{k=0}^{\infty} \sigma_k$ is divergent. In that case we do not know whether SPCD for dynamical system $(\Phi_t)_{t \in \mathbb{R}}$ in $\mathbb{R}^\infty$ defined by (6.6) has any solution.

Case 2. $A_0 + 2 \sum_{k=0}^{\infty} \sigma_k = -\infty$. In that case SPCD for dynamical system $(\Phi_t)_{t \in \mathbb{R}}$ in $\mathbb{R}^\infty$ defined by (6.6) has no any solution.

Case 3. $A_0 + 2 \sum_{k=0}^{\infty} \sigma_k = +\infty$. In that case SPCD for dynamical system $(\Phi_t)_{t \in \mathbb{R}}$ in $\mathbb{R}^\infty$ defined by (6.6) has no any solution because if the supplier take an arbitrary measurable system of the positive $\mu_{1,1,\ldots}$ measure, then demands of all consumers will be always satisfied.

Case 4. $A_0 + 2 \sum_{k=0}^{\infty} \sigma_k$ is convergent and $-\infty < A_0 + 2 \sum_{k=0}^{\infty} \sigma_k < +\infty$.

By the scheme used in Case 4 of the Section 5, one can easily show that the solution of SPCD for dynamical system $(\Phi_t)_{t \in \mathbb{R}}$ in $\mathbb{R}^\infty$ defined by (6.6) is given by

$$m = \sum_{k=1}^{n} m_k e^{-t_s (A_0 + 2 \sum_{k=0}^{\infty} \sigma_k)}$$

(6.9).

Observation 6.2 We have showed that Satisfaction Problem of Consumers Demands for dynamical system $(\Phi_t)_{t \in \mathbb{R}}$ in $\mathbb{R}^\infty$ defined by (6.6) has the solution if $(\Phi_t)_{t \in \mathbb{R}}$ is pressing, expansible or stable in the sense of the measure $\mu_{1,2,\ldots}$.

When $(\Phi_t)_{t \in \mathbb{R}}$ is totally pressing or totally expansible the the same problem has no any solution.

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