INVARIANT THEORY OF FOLIATIONS
OF THE PROJECTIVE PLANE

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Abstract. We study the invariant theory of singular foliations of the projective plane. Our first main result is that a foliation of degree $m > 1$ is not stable only if it has singularities in dimension 1 or contains an isolated singular point with multiplicity at least $(m^2 - 1)/(2m + 1)$. Our second main result is the construction of an invariant map from the space of foliations of degree $m$ to that of curves of degree $m^2 + m - 2$. We describe this map explicitly in case $m = 2$.

1. Introduction

The study of (singular) foliations of the projective plane is an old one. It was central in works by Darboux [4] and Poincaré [11] in the XIX Century. More recently, the interest in the subject has been revived by Jouanolou [9]. It has been an active area of study ever since.

If we want to study foliations up to projective equivalence, we enter the realm of Invariant Theory. Though the motivation for this study is natural, and Invariant Theory is a classical subject, not much has been done so far in this direction. We can mention the work by Goméz-Mont and Kempf [8], who have shown that a foliation whose all singular points have Milnor number 1 is stable. (In fact, they showed the same result holds for singular foliations of higher dimension spaces as well.) Only recently, Alcántara [1], [2] has characterized the semi-stable foliations of degree 1 and 2, and studied their quotient spaces.

In these notes we propose to advance this study. Our first main result is Theorem 9 which says that a foliation of degree $m > 1$ is nonstable (resp. nonsemi-stable) only if it has singularities in dimension 1 or contains an isolated singular point of multiplicity at least (resp. greater than) $(m^2 - 1)/(2m + 1)$.

Our second main result is Theorem 10 which yields an invariant rational map $\Phi$ from the (projective) space of foliations of degree $m \geq 2$ to that of plane curves of degree $m^2 + m - 2$. Using this map, we can, in principle, produce invariants of foliations out of invariants of plane curves. However, though the invariants of plane curves can all be described by the symbolic method of the XIX Century, generators for the algebras of invariants are known only for very small degrees, not larger than 8. Since for $m \geq 3$, the curves have degree at least 10, the map $\Phi$ might be manageable only for $m = 2$, in which case we are dealing with quartics. In this case, we describe the map explicitly in Section 4.

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2. Singular foliations

1. Foliations. Given a smooth algebraic variety $X$ over an algebraically closed field $k$, a $d$-dimensional foliation of $X$ is a rank-$d$ subbundle of the tangent bundle of $X$. Typically though, these subbundles do not exist. For instance, take the projective plane $X := \mathbb{P}^2_k$. A subbundle of rank 1 of the tangent bundle would give rise to an exact sequence of locally free sheaves,

$$0 \to \mathcal{O}_{\mathbb{P}^2_k}(m) \to \Omega^1_{\mathbb{P}^2_k} \to \mathcal{O}_{\mathbb{P}^2_k}(n) \to 0$$

for certain integers $m$ and $n$, and this sequence would split because $H^1(\mathbb{P}^2_k, \mathcal{O}_{\mathbb{P}^2_k}(m-n)) = 0$.

Thus,

$$\Omega^1_{\mathbb{P}^2_k} \cong \mathcal{O}_{\mathbb{P}^2_k}(m) \oplus \mathcal{O}_{\mathbb{P}^2_k}(n).$$

Then it would follow that Euler sequence,

$$0 \to \Omega^1_{\mathbb{P}^2_k} \to \mathcal{O}_{\mathbb{P}^2_k}(-1)^{\oplus 3} \to \mathcal{O}_{\mathbb{P}^2_k} \to 0,$$

would split as well, as

$$H^1(\mathbb{P}^2_k, \mathcal{O}_{\mathbb{P}^2_k}(-m)) = H^1(\mathbb{P}^2_k, \mathcal{O}_{\mathbb{P}^2_k}(-n)) = 0,$$

giving rise to a nonzero global section of $\mathcal{O}_{\mathbb{P}^2_k}(-1)$, an absurd.

2. Singular foliations. One might ask however not for a subbundle, but for a subsheaf. This gives rise to a singular foliation. In other words, a singular foliation is a subsheaf of the tangent sheaf of $X$. Its dimension is the generic rank of the sheaf. For instance, take the projective plane $X := \mathbb{P}^2_k$. Given a singular foliation of dimension 1, we may replace the subsheaf by a possibly larger reflexive subsheaf. Since $X$ is smooth of dimension 2, this means that the subsheaf is locally free by [10], Lemma 1.1.10, p. 149. So, a singular foliation of $\mathbb{P}^2_k$ is a nonzero (thus injective) map

$$\eta: \mathcal{O}_{\mathbb{P}^2_k}(1-m) \to T_{\mathbb{P}^2_k},$$

where $T_{\mathbb{P}^2_k}$ is the tangent sheaf of $\mathbb{P}^2_k$. We will deal only with one-dimensional singular foliations of $\mathbb{P}^2_k$ from now on, and will thus drop the adjective “singular.”

Taking duals in the Euler sequence (1), we obtain the exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^2_k} \to \mathcal{O}_{\mathbb{P}^2_k}(1)^{\oplus 3} \to T_{\mathbb{P}^2_k} \to 0.$$

Since $\text{Ext}^1(\mathcal{O}_{\mathbb{P}^2_k}(1-m), \mathcal{O}_{\mathbb{P}^2_k}) = 0$, any map $\eta$ as in (2) lifts to a map

$$\bar{\eta}: \mathcal{O}_{\mathbb{P}^2_k}(1-m) \to \mathcal{O}_{\mathbb{P}^2_k}(1)^{\oplus 3},$$
which corresponds to a choice of three homogeneous polynomials $F$, $G$ and $H$ of degree $m$. In other words, $\eta$ induces a homogeneous vector field on the three-dimensional affine space $\mathbb{A}^3_k$:

\begin{equation}
D := F \frac{\partial}{\partial x} + G \frac{\partial}{\partial y} + H \frac{\partial}{\partial z}.
\end{equation}

Here $x$, $y$ and $z$ are the coordinates of $\mathbb{A}^3_k$. This vector field is not unique, as the lifting $\tilde{\eta}$ of $\eta$ is not, but any other vector field is obtaining from the above one by summing a multiple of the Euler field:

$$P\left(x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + z \frac{\partial}{\partial z}\right).$$

At any rate, we may harmlessly say that $D$, instead of $\eta$, is the foliation.

Conversely, given $D$ as in (4), one can describe the foliation $\eta$ in very concrete terms: the direction given by $\eta$ at a point $(x : y : z) \in \mathbb{P}^2_k$ is that of the line passing through $(x : y : z)$ and $(F(x, y, z) : G(x, y, z) : H(x, y, z))$, whenever these two points are distinct.

3. **The space of foliations.** There are thus many (singular) foliations. In fact, identifying foliations that differ one from the other by multiplication by a nonzero constant, we obtain a projective space, 

$$F_m := \mathbb{P}(H^0(\mathbb{P}^2_k, T_{\mathbb{P}^2} \otimes \mathcal{O}_{\mathbb{P}^2_k}(m - 1))).$$

It follows from the long exact sequence in cohomology associated to (3) that 

$$\dim F_m = 3h^0(\mathbb{P}^2_k, \mathcal{O}_{\mathbb{P}^2_k}(m)) - h^0(\mathbb{P}^2_k, \mathcal{O}_{\mathbb{P}^2_k}(m - 1)) - 1$$

$$= 3\binom{m + 2}{2} - \binom{m + 1}{2} - 1$$

$$= m^2 + 4m + 2.$$ 

4. **Singular points.** The map $\eta$ in (2), though injective, does not give rise to a subbundle. In other words, the degeneracy scheme of the map is nonempty. The degeneracy scheme is called the **singular locus** of the foliation, and its points the **singular points or singularities** of the foliation. Since $\eta \neq 0$, the dimension of this locus is at most 1. If the dimension is 1, then $\eta$ decomposes in a unique way as 

$$\mathcal{O}_{\mathbb{P}^2_k}(1 - m) \rightarrow \mathcal{O}_{\mathbb{P}^2_k}(1 - n) \rightarrow T_{\mathbb{P}^2},$$

where the first map is multiplication by a homogeneous polynomial of degree $m - n$, for a certain $n < m$, and the second is a foliation with finite singular locus. In this case, we say that $\eta$ has singularities in dimension 1.

If the dimension is zero each singularity appears with a certain length in the singular locus, called its **Milnor number**. Then we can use Porteous Formula (see
\[ \delta = \int_{\mathbb{P}^2_k} c_2(T_{\mathbb{P}^2_k} \otimes \mathcal{O}_{\mathbb{P}^2_k}(m - 1)) \cap [\mathbb{P}^2_k] \]
\[ = \int_{\mathbb{P}^2_k} \left[ \frac{c(\mathcal{O}_{\mathbb{P}^2_k}(m))^3}{c(\mathcal{O}_{\mathbb{P}^2_k}(m - 1))} \right]^2 \cap [\mathbb{P}^2_k] \quad \text{(Sequence \ref{eq:sequence} and Whitney Formula)} \]
\[ = \int_{\mathbb{P}^2_k} \left[ \frac{(1 + mh)^3}{1 + (m - 1)h} \right]^2 \cap [\mathbb{P}^2_k] \quad \text{(where } h := c_1(\mathcal{O}_{\mathbb{P}^2_k}(1))) \]
\[ = \int_{\mathbb{P}^2_k} [(1 + 3mh + 3m^2h^2)(1 - (m - 1)h + (m - 1)^2h^2)]_2 \cap [\mathbb{P}^2_k] \]
\[ = (m - 1)^2 - 3m(m - 1) + 3m^2 \]
\[ = m^2 + m + 1. \]

Another important invariant of a singular point of the foliation is its \textit{multiplicity}, the maximum power of the maximal ideal of the local ring of \( \mathbb{P}^2 \) at the point containing the ideal of the singular locus of the foliation.

5. \textit{The degree}. Given a singular foliation \( \eta \) as in \( \ref{eq:eta} \), the integer \( m \), clearly non-negative, has a geometric interpretation. Indeed, \( m \) is the number of tangencies of \( \eta \) to a general line. More precisely, given a line \( L \) on \( \mathbb{P}^2_k \), we may look at the set of points where \( \eta \) is either singular or assigns a line equal to \( L \). Given a general line, this is a finite set. (Just pick a nonsingular point \( P \) of \( \eta \), and choose \( L \) transversal to the line at \( P \) given by \( \eta \).) The number of points \( s \) of this set, counted with the appropriate weights, is given by Porteous Formula, as the length of the degeneracy scheme of the map of vector bundles
\[ \mathcal{O}_{\mathbb{P}^2_k}(1 - m)|_L \oplus T_L \xrightarrow{(\eta|_L,\beta)} T_{\mathbb{P}^2_k}|_L, \]
where \( \beta \) is the natural inclusion between tangent bundles. Thus
\[ s = \int_L (c_1(T_{\mathbb{P}^2_k}|_L) - c_1(\mathcal{O}_{\mathbb{P}^2_k}(1 - m)|_L) - c_1(T_L)) \cap [L] \]
\[ = \int_L (3h - (1 - m)h - 2h) \quad \text{(where } h \text{ is the class of a point)} \]
\[ = m. \]

6. \textit{The dual point of view}. Let
\[ \omega := \bigwedge^2 \Omega^1_{\mathbb{P}^2_k} \cong \mathcal{O}_{\mathbb{P}^2_k}(-3). \]
The natural product map
\[ \Omega^1_{\mathbb{P}^2_k} \otimes \Omega^1_{\mathbb{P}^2_k} \to \omega \]
gives rise to an isomorphism
\[ \Omega^1_{\mathbb{P}^2_k} \to T_{\mathbb{P}^2_k} \otimes \omega. \]
Under this isomorphism, a map \( \eta \) as in (2), which corresponds to a section of \( T_{\mathbb{P}^2_k} \otimes \mathcal{O}_{\mathbb{P}^2_k}(m-1) \), corresponds to a section
\[
\tau \in H^0(\mathbb{P}^2_k, \mathcal{O}_{\mathbb{P}^2_k}(m+2)).
\]
Because of (1), this section corresponds to three homogeneous polynomials \( A, B \) and \( C \) of degree \( m+1 \) satisfying the relation
\[
xA + yB + zC = 0.
\]
We may view the polynomials as giving a homogeneous form on \( \mathbb{A}^3_k \):
\[
w := Adx + Bdy + Cdz.
\]
If \( \eta \) is given by \( D \) as in (4), then \( w \) is obtained from the determinant:
\[
\begin{vmatrix}
x & y & z \\
F & G & H \\
dx & dy & dz
\end{vmatrix}
\]
In other words, \( A = yH - zG, B = zF - xH \) and \( C = xG - yF \). Of course, the assignment \( \eta \mapsto \tau \) gives rise to a (linear) isomorphism:
\[
\mathbf{P}(H^0(\mathbb{P}^2_k, T_{\mathbb{P}^2_k}(m-1))) \rightarrow \mathbf{P}(H^0(\mathbb{P}^2_k, \mathcal{O}_{\mathbb{P}^2_k}(m+2))).
\]
We may view \( \mathbf{F}_m \) as the space on the left-hand side or that on the right-hand side, at our convenience. And we may harmlessly say that \( \tau \) or \( w \) is the foliation.

Geometrically, for each point \((a : b : c)\) of \( \mathbb{P}^2_k \) the direction at the point given by \( \eta \) is that of the line with equation:
\[
A(a, b, c)x + B(a, b, c)y + C(a, b, c)z = 0.
\]
And the singular locus of the foliation is given by \( A = B = C = 0 \).

Notice that, because of (6), the singular locus is locally given by two equations. So the following inequality holds relating the Milnor number \( \mu_P \) and the multiplicity \( e_P \) of a singularity \( P \) of the foliation:
\[
\mu_P \geq \frac{(e_P + 1)e_P}{2} + e_P - 1 = \frac{e_P^2 + 3e_P - 2}{2}.
\]

3. THE ACTION

7. The action. The group of automorphisms of \( \mathbb{P}^2_k \), namely \( \text{PGL}(3) \), acts in a natural way on the space of foliations. The action can be described very simply in geometric terms: Let \( \phi \) be an automorphism of \( \mathbb{P}^2_k \); given a foliation \( \eta \), the new foliation \( \phi \cdot \eta \) assigns to every point \( P \in \mathbb{P}^2_k \) the line \( \phi(L) \), where \( L \) is the line given by \( \eta \) at \( \phi^{-1}(P) \). Algebraically, let \( g \) be a 3-by-3 matrix corresponding to \( \phi \), and let \( w \) as in (7) correspond to \( \eta \). Then \( \phi \cdot \eta \) corresponds to \( g \cdot w \), where
\[
g \cdot w = \begin{bmatrix} A^g & B^g & C^g \end{bmatrix} g^{-1} \begin{bmatrix} dx \\ dy \\ dz \end{bmatrix}.
\]
(Given any polynomial $P \in k[x, y, z]$, we denote by $P^g$ the polynomial which, viewed as a function on $A^3_k$, interpreted as the space of column vectors of dimension 3, satisfies

$$P^g(v) = P(g^{-1}v) \text{ for each } v \in A^3_k.$$)

8. Stable points. The action of $\text{PGL}(3)$ produces the same orbits as the action by $\text{SL}(3)$, the special linear group, that of 3-by-3 matrices with determinant 1, induced by the natural surjection $\text{SL}(3) \to \text{PGL}(3)$. So we will consider this induced action.

Geometric Invariant Theory tells us that there is a categorical quotient of a certain open subset of $F_m$, that of semi-stable points. The semi-stable points are those for which there is an invariant homogeneous polynomial on the coordinates of $F_m$ not vanishing at the point. And the quotient is simply the projective scheme associated to the (graded) algebra of invariants. Furthermore, a smaller open subset of $F_m$, consisting of stable points, whose orbits in the semi-stable locus are closed, admits even a geometric quotient, which is thus an orbit space; see [6].

To understand the quotient, it is crucial to describe the semi-stable points. However, it is not easy to determine them from the definition. A lot more manageable than the definition is the Hilbert–Mumford Numerical Criterion, by means of one-parameter subgroups.

It was using this criterion that Goméz-Mont and Kempf [8] have shown that a foliation whose all singular points have Milnor number 1 is stable, that is, corresponds to a stable point of $F_m$. And Alcántara [11], [2] has characterized the semi-stable foliations of degrees 1 and 2.

In our case, a one-parameter subgroup is a nontrivial homomorphism of algebraic groups $\lambda: G_m \to \text{SL}(3)$, where $G_m$ is the multiplicative group of the field $k$. Every such homomorphism is diagonalizable: there is $g \in \text{SL}(3)$ such that

$$g^{-1} \lambda(t)g = \lambda_{r_1, r_2, r_3}(t), \quad \text{where } \lambda_{r_1, r_2, r_3}(t) = \begin{bmatrix} t^{r_1} & 0 & 0 \\ 0 & t^{r_2} & 0 \\ 0 & 0 & t^{r_3} \end{bmatrix}$$

for each $t \in G_m$. Since $\det \lambda(t) = 1$ for every $t$, the $r_i$ are integers such that

$$r_0 + r_1 + r_2 = 0.$$

We may also assume that $r_1 \geq r_2 \geq r_3$. Since $\lambda$ is nontrivial, $r_1 > 0 > r_3$.

Now, the space of forms $w$ as in (7), satisfying (6), has a basis of the form:

$$w_\alpha^1 := x^{\alpha_1}y^{\alpha_2}z^{\alpha_3}(-ydx + xdy),$$

$$w_\beta^2 := x^{\beta_1}y^{\beta_2}z^{\beta_3}(-zdx + xdz),$$

$$w_\gamma^3 := y^{\gamma_2}z^{\gamma_3}(-zdy + ydz),$$

where $\alpha := (\alpha_1, \alpha_2, \alpha_3)$ and $\beta := (\beta_1, \beta_2, \beta_3)$ (resp. $\gamma := (\gamma_2, \gamma_3)$) run through all triples (resp. pairs) of nonnegative integers summing up to $m$. This basis
diagonalizes the action of \( \lambda_{r_1,r_2,r_3} \). More precisely,
\[
\begin{align*}
\lambda_{r_1,r_2,r_3}(t) \cdot w_{\alpha}^1 &= t^{-r_1(\alpha_1+1)-r_2(\alpha_2+1)-r_3\alpha_3} w_{\alpha}^1, \\
\lambda_{r_1,r_2,r_3}(t) \cdot w_{\beta}^2 &= t^{-r_1(\beta_1+1)-r_2\beta_2-r_3(\beta_3+1)} w_{\beta}^2, \\
\lambda_{r_1,r_2,r_3}(t) \cdot w_{\gamma}^3 &= t^{-r_2(\gamma_2+1)-r_3(\gamma_3+1)} w_{\gamma}^3,
\end{align*}
\]

Finally, consider a point of \( F_m \), corresponding to \( w \) as in (7). Then, for each \( g \in \text{SL}(3) \),
\[
(9) \quad g \cdot w = \sum_{\alpha} a_\alpha(g) w_{\alpha}^1 + \sum_{\beta} b_\beta(g) w_{\beta}^2 + \sum_{\gamma} c_\gamma(g) w_{\gamma}^3,
\]
for unique \( a_\alpha(g) \), \( b_\beta(g) \) and \( c_\gamma(g) \) in \( k \). Then the Hilbert–Mumford Numerical Criterion says that \( w \) is not stable, that is, the corresponding point on \( F_m \) is not stable, if and only if there are \( g \in \text{SL}(3) \) and integers \( r_1, r_2, r_3 \) satisfying \( r_1+r_2+r_3 = 0 \) and \( 0 < r_1 \geq r_2 \geq r_3 < 0 \) such that all of the following conditions hold:
\[
(10) \quad \begin{align*}
& r_1(\alpha_1+1) + r_2(\alpha_2+1) + r_3\alpha_3 \leq 0 \quad \text{if} \ a_\alpha(g) \neq 0, \\
& r_1(\beta_1+1) + r_2\beta_2 + r_3(\beta_3+1) \leq 0 \quad \text{if} \ b_\beta(g) \neq 0, \\
& r_2(\gamma_2+1) + r_3(\gamma_3+1) \leq 0 \quad \text{if} \ c_\gamma(g) \neq 0.
\end{align*}
\]
Furthermore, \( w \) is nonsemi-stable if in addition all the inequalities above are strict.

**Theorem 9.** A foliation of degree \( m > 1 \) is nonstable (resp. nonsemi-stable) only if it has singularities in dimension 1 or contains an isolated singular point with multiplicity at least (resp. greater than) \( (m^2 - 1)/(2m + 1) \).

**Proof.** Let \( w \) as in (7) correspond to the foliation. Assume first that \( w \) is nonstable. Then there are \( g \in \text{SL}(3) \) and integers \( r_1, r_2, r_3 \) satisfying
\[
(11) \quad r_1 + r_2 + r_3 = 0 \quad \text{and} \quad 0 < r_1 \geq r_2 \geq r_3 < 0
\]
such that (10) holds. Since \( w \) is stable if and only \( g \cdot w \) is, and the foliation \( w \) has singularities in dimension 1 or contains an isolated singular point with a certain multiplicity if and only if the same holds for \( g \cdot w \), we may assume that \( g = 1 \), and simplify the notation:
\[
\begin{align*}
& a_\alpha := a_\alpha(1), \quad b_\beta := b_\beta(1), \quad c_\gamma := c_\gamma(1).
\end{align*}
\]
We claim that either the foliation has singularities in dimension 1 or
\[
(12) \quad r_2 \leq \frac{-r_3}{m+1}.
\]
Indeed, suppose (12) does not hold. Let \( \alpha = (\alpha_1, \alpha_2, \alpha_3) \) be a triple of nonnegative integers with \( \alpha_3 = 0 \) and \( \alpha_1+\alpha_2 = m \). Then, since \( \alpha_1, r_1-r_2 \geq 0 \) and \( -r_3, r_2 > 0 \),
\[
\begin{align*}
0 < (r_1-r_2)\alpha_1 + r_2m - r_3 &= (r_1-r_2)\alpha_1 + r_2(\alpha_1+\alpha_2) - r_3 \\
&= r_1(\alpha_1+1) + r_2(\alpha_2+1) + r_3\alpha_3.
\end{align*}
\]
Thus (10) yields $a_\alpha = 0$.

Also, let $\beta = (\beta_1, \beta_2, \beta_3)$ be a triple of nonnegative integers with $\beta_3 = 0$ and $\beta_1 + \beta_2 = m$. Then, since $\beta_1, r_1 - r_2 \geq 0$ and $r_2, m - 1 > 0$,

$$0 < (r_1 - r_2)\beta_1 + r_2(m - 1) = (r_1 - r_2)\beta_1 + r_2(\beta_1 + \beta_2) - r_2 = r_1(\beta_1 + 1) + r_2\beta_2 + r_3(\beta_3 + 1).$$

Thus (10) yields $b_\beta = 0$.

Similarly:

If $\alpha_1 > m^2 - 1$ then $a_\alpha = 0$.

Indeed, if $\alpha_1 > (m^2 - 1)/(2m + 1)$ then

$$r_1(\alpha_1 + 1) + r_2(\alpha_2 + 1) + r_3\alpha_3 = (r_1 - r_3)\alpha_1 + (r_2 - r_3)\alpha_2 + r_3(m - 1)$$

$$\geq -r_3\frac{2m + 1}{m + 1}\alpha_1 + r_3(m - 1) > 0,$$

where for the first inequality above we used that $\alpha_3 = m - \alpha_1 - \alpha_2$ and $r_3 = -r_1 - r_2$, and for the first inequality we used (13), $\alpha_2 \geq 0$ and $r_2 \geq r_3$. Thus $a_\alpha = 0$ from (10).

Similarly:

If $\beta_1 > \frac{m^2 + m + 1}{2m + 1}$ then $b_\beta = 0$.

Indeed, if $\beta_1 > (m^2 + m + 1)/(2m + 1)$ then

$$r_1(\beta_1 + 1) + r_2\beta_2 + r_3(\beta_3 + 1) = (r_1 - r_3)\beta_1 + (r_2 - r_3)\beta_2 + r_3m - r_2$$

$$\geq -r_3\frac{2m + 1}{m + 1}\beta_1 + r_3\left(m + \frac{1}{m + 1}\right) > 0,$$

where for the first inequality above we used (12). Thus $b_\beta = 0$ from (10).

Now, using (5) to expand (9), we get $w = Adx + Bdy + Cdz$, where

$$B = \sum_{\alpha} a_\alpha x^{\alpha_1 + 1} y^{\alpha_2} z^{\alpha_3} - \sum_{\gamma} c_\gamma y^{\gamma_2} z^{\gamma_3 + 1},$$

$$C = \sum_{\beta} b_\beta x^{\beta_1 + 1} y^{\beta_2} z^{\beta_3} + \sum_{\gamma} c_\gamma y^{\gamma_2 + 1} z^{\gamma_3}.$$
Let \( P := (1 : 0 : 0) \). Since \( xA + yB + zC = 0 \), the ideal of the singular locus of the foliation at \( P \) is generated by \( B(1, y/x, z/x) \) and \( C(1, y/x, z/x) \). Since \( \gamma_2 + \gamma_3 = m \), it follows that the multiplicity of the foliation at \( P \) is \( \min(m + 1, \xi) \) where

\[
\xi := \min\left( \min(\alpha_2 + \alpha_3 | a_\alpha \neq 0), \min(\beta_2 + \beta_3 | b_\beta \neq 0) \right)
\]

\[
= \min\left( \min(m - \alpha_1 | a_\alpha \neq 0), \min(m - \beta_1 | b_\beta \neq 0) \right)
\]

\[
= m - \max\left( \max(\alpha_1 | a_\alpha \neq 0), \max(\beta_1 | b_\beta \neq 0) \right).
\]

(The minimum (resp. maximum) of the empty set is \( +\infty \) (resp. \( -\infty \)) by convention.) Thus, it follows from (14) and (15) that

\[
(16) \quad \xi \geq m - \max\left( \frac{m^2 - 1}{2m + 1}, \frac{m^2 + m + 1}{2m + 1} \right) = m - \frac{m^2 + m + 1}{2m + 1} = \frac{m^2 - 1}{2m + 1}.
\]

If \( w \) is nonsemi-stable then the same proof works with the following modifications: the inequality in (12) is strict while those in (13), (14), (15) and (16) are not.

\[ \square \]

4. The Dual Discriminant Curve

**Theorem 10.** Given a foliation of \( \mathbb{P}^2_k \) of degree \( m \geq 2 \) whose singular locus does not contain a double curve, the lines tangent to the foliation with multiplicity at least 2 are parameterized by a curve on the dual plane \( \mathbb{P}^2_k \) of degree \( m^2 + m - 2 \).

**Proof.** Let \( \bar{x}, \bar{y} \) and \( \bar{z} \) be coordinates of \( \mathbb{P}^2_k \) dual to \( x, y \) and \( z \). The incidence variety \( I \subset \mathbb{P}^2_k \times \mathbb{P}^2_k \) is thus given by

\[
\bar{x}x + \bar{y}y + \bar{z}z = 0.
\]

Let \( D \) as in (1) correspond to the foliation. Requiring the above line to be tangent to the foliation at \((x : y : z)\) is to impose that

\[
\bar{x}F + \bar{y}G + \bar{z}H = 0.
\]

Let \( V \subseteq I \) be the subscheme of \( I \) given by the above equation. It parameterizes the pairs \((P, L)\) where \( L \) is a line on \( \mathbb{P}^2_k \) and \( P \) is a point on \( L \) where the foliation is singular or tangent to \( L \). Let \( \pi: V \to \mathbb{P}^2_k \) denote the projection, and let \( D \subseteq V \) be the degeneracy locus of the natural map \( \pi^*\Omega^1_{\mathbb{P}^2_k} \to \Omega^1_V \). Since the singular locus of the foliation contains no double curve, \( \pi(D) \neq \mathbb{P}^2_k \). Let \( C \subseteq \mathbb{P}^2_k \) be the curve such that \( \pi_*[D] = [C] \) as cycles. This is the curve parameterizing lines tangent to the foliation with multiplicity at least 2.

We claim that \( \deg C = m^2 + m - 2 \). Indeed, let \( h_1 \) (resp. \( h_2 \)) be the pullback to \( \mathbb{P}^2_k \times \mathbb{P}^2_k \) of the hyperplane class \( h \) on \( \mathbb{P}^2_k \) (resp. \( \bar{h} \) on \( \mathbb{P}^2_k \)). Then

\[
[I] = h_1 + h_2 \quad \text{and} \quad [V] = (h_1 + h_2)(mh_1 + h_2)
\]

in the Chow ring of \( \mathbb{P}^2_k \times \mathbb{P}^2_k \). Now,

\[
[D] = c_1(\Omega^1_V) \cap [V] - c_1(\pi^*\Omega^1_{\mathbb{P}^2_k}) \cap [V].
\]
It follows from the Whitney Sum Formula ([7], Thm. 3.2(e), p. 50) and the Euler exact sequence that
\[ \pi^*c_1(\Omega^1_{\mathbb{P}^2_k}) \cap [\mathbb{P}^2_k \times \mathbb{P}^2_k] = -3h_2. \]

In addition, \( \Omega^1_V \) sits in the natural exact sequence,
\[ 0 \rightarrow \mathcal{O}_V(-1,-1) \oplus \mathcal{O}_V(-m,-1) \rightarrow \Omega^1_{\mathbb{P}^2_k \times \mathbb{P}^2_k}|_V \rightarrow \Omega^1_V \rightarrow 0. \]

Thus, applying the Whitney Sum Formula again,
\[ c_1(\Omega^1_V) \cap [V] = (-3(h_1 + h_2) + (h_1 + h_2) + (mh_1 + h_2))[V] = ((m - 2)h_1 - h_2)[V]. \]

So,
\[ [D] = ((m - 2)h_1 + 2h_2)[V] = ((m - 2)h_1 + 2h_2)(h_1 + h_2)(mh_1 + h_2). \]

Since \( h_1^3 = h_2^3 = 0 \), we get
\[ [D] = (2m + m(m - 2) + (m - 2))h_1^2h_2 + ((m - 2) + 2 + 2m)h_1h_2^2 = (m^2 + m - 2)h_1^2h_2 + 3mh_1h_2^2, \]
and thus \( \pi_*[D] = (m^2 + m - 2)h. \)

\[ \square \]

11. Degree 2. Let
\[ C_d := \mathbb{P}(R^0(\mathbb{P}^2_k, \mathcal{O}_{\mathbb{P}^2_k}(d))), \]
the projective space parameterizing plane curves of degree \( d \). It has dimension \((d^2 + 3d)/2\). By Theorem 10, there is a rational map
\[ \Phi: F_m \dashrightarrow C_{m^2 + m - 2}. \]

In case \( m = 2 \), both the target and the source of \( \Phi \) have the same dimension, as
\[ m^2 + 4m + 2 = \frac{(m^2 + m - 2)^2 + 3(m^2 + m - 2)}{2} = 14. \]

In this case, the dimensions are small enough that \( \Phi \) can be explicitly described, using CoCoA [3] (assuming the ground field \( k \) has characteristic 0).

Consider a point of \( F_2 \) given by \( w \) as in (7). As in Section 3 we may write
\[ w = \sum_\alpha a_\alpha w^1_\alpha + \sum_\beta b_\beta w^2_\beta + \sum_\gamma c_\gamma w^3_\gamma, \]
for unique \( a_\alpha, b_\beta \) and \( c_\gamma \) in \( k \), where \( \alpha := (\alpha_1, \alpha_2, \alpha_3) \) and \( \beta := (\beta_1, \beta_2, \beta_3) \) (resp. \( \gamma := (\gamma_2, \gamma_3) \)) run through all triples (resp. pairs) of nonnegative integers summing up to 2, and the \( w^1_\alpha, w^2_\beta \) and \( w^3_\gamma \) are given in [3].
The coefficients $a_\alpha$, $b_\beta$, and $c_\gamma$ can be seen as coordinates of $\mathbb{F}_2 \cong \mathbb{P}^2_k$. Then the associated quartic to $w$ is given by:

$$
\left( c_{(2,0)}c_{(0,2)} - \frac{c_{(1,1)}^2}{4} \right) \bar{x}^4 + \left( \frac{c_{(1,1)}b_{(0,1,1)}}{2} - c_{(0,2)}b_{(0,0,2)} - c_{(2,0)}b_{(0,0,2)} \right) \bar{x}^3 \bar{y} \\
+ \left( b_{(0,0,2)}b_{(0,0,2)} + c_{(0,2)}b_{(1,1,0)} - \frac{b_{(0,1,1)}^2}{4} - \frac{c_{(1,1)}b_{(1,0,1)}}{2} \right) \bar{x}^2 \bar{y}^2 \\
+ \left( \frac{b_{(1,0,1)}b_{(1,0,1)}}{2} - c_{(0,2)}b_{(2,0,0)} - b_{(0,0,2)}b_{(1,1,0)} \right) \bar{x} \bar{y}^3 + \left( b_{(2,0,0)}b_{(0,0,2)} - \frac{b_{(1,0,1)}^2}{4} \right) \bar{y}^4 \\
+ \left( c_{(0,2)}a_{(0,0,2)} + c_{(2,0)}a_{(0,0,2)} - \frac{c_{(1,1)}a_{(0,1,1)}}{2} \right) \bar{x}^3 \bar{z} \\
+ \left( \frac{c_{(1,1)}a_{(1,0,1)}}{2} + \frac{b_{(0,1,1)}a_{(0,1,1)}}{2} + c_{(2,0)}b_{(1,0,1)} - \frac{c_{(1,1)}b_{(1,1,0)}}{2} - b_{(0,0,2)}a_{(0,0,2)} \\
- b_{(2,0,0)}a_{(0,0,2)} - c_{(2,0)}a_{(1,1,0)} \right) \bar{x}^2 \bar{y} \bar{z} \\
+ \left( \frac{b_{(1,1,0)}b_{(1,1,0)}}{2} + c_{(0,2)}a_{(2,0,0)} + b_{(1,1,0)}a_{(0,0,2)} + b_{(0,0,2)}a_{(1,1,0)} + c_{(1,1)}b_{(2,0,0)} \\
- \frac{b_{(0,1,1)}a_{(1,0,1)}}{2} - \frac{b_{(1,0,1)}a_{(1,0,1)}}{2} - b_{(0,2,0)}b_{(1,0,1)} \right) \bar{x} \bar{y} \bar{z}^2 \\
+ \left( \frac{b_{(1,0,1)}b_{(1,0,1)}}{2} + \frac{b_{(1,0,1)}a_{(1,0,1)}}{2} \right) \bar{y}^3 \bar{z} \\
+ \left( a_{(2,0,0)}a_{(0,0,2)} + \frac{c_{(1,1)}a_{(1,1,0)}}{2} - \frac{a_{(0,1,1)}^2}{4} - c_{(2,0)}a_{(0,1,0)} \right) \bar{x}^2 \bar{z}^2 \\
+ \left( \frac{a_{(0,1,1)}a_{(0,1,1)}}{2} + b_{(1,0,1)}a_{(2,0,0)} + b_{(0,2,0)}a_{(1,1,0)} - \frac{b_{(0,1,1)}a_{(1,1,0)}}{2} - \frac{b_{(1,1,0)}a_{(0,1,1)}}{2} - \frac{b_{(0,2,0)}a_{(2,0,0)} - c_{(1,1)}a_{(2,0,0)} - a_{(0,0,2)}a_{(1,1,0)} \right) \bar{x} \bar{y} \bar{z}^2 \\
+ \left( b_{(2,0,0)}b_{(0,2,0)} + b_{(1,1,0)}a_{(2,0,0)} + a_{(2,0,0)}a_{(0,0,2)} + b_{(2,0,0)}a_{(0,0,2)} - \frac{b_{(1,1,0)}^2}{4} - \frac{a_{(1,1,0)}^2}{4} \\
- \frac{b_{(1,0,1)}a_{(1,1,0)}}{2} - \frac{b_{(1,1,0)}a_{(1,1,0)}}{2} \right) \bar{y}^2 \bar{z}^2 \\
+ \left( \frac{a_{(1,1,0)}a_{(0,1,1)}}{2} + c_{(2,0)}a_{(2,0,0)} - a_{(0,0,2)}a_{(1,1,0)} \right) \bar{x} \bar{z}^3 \\
+ \left( \frac{b_{(1,1,0)}a_{(1,1,0)}}{2} + \frac{a_{(1,1,0)}a_{(1,1,0)}}{2} \right) \bar{y} \bar{z}^3 \\
- \frac{b_{(0,0,2)}a_{(2,0,0)} - b_{(2,0,0)}a_{(0,0,2)} - a_{(2,0,0)}a_{(0,1,1)} \right) \bar{y} \bar{z}^3 \\
+ \left( a_{(2,0,0)}a_{(0,0,2)} - \frac{a_{(1,1,0)}^2}{4} \right) \bar{z}^4 = 0,
$$

where $\bar{x}$, $\bar{y}$ and $\bar{z}$ are the coordinates of $\mathbb{P}^2_k$ dual to $x$, $y$ and $z$. 
12. Invariants and instability. Invariants for degree-2 foliations can thus be obtained from invariants for plane quartics by composition. However, the latter invariants are not completely known. In [5], Thm. 3.2, p. 286, assuming \( k \) is the field of complex numbers, Dixmier produced a homogeneous system of parameters for the algebra of invariants of the quartics: seven homogeneous invariants of degrees 3, 6, 9, 12, 15, 18 and 27. More invariants should be necessary. According to [5], p. 280, the algebra of invariants can be generated by 56 invariants, though Shioda [12], p. 1046, conjectured that 13 should be enough.

At any rate, if the foliation is not semi-stable, neither is the corresponding quartic. This can be seen directly from our explicit description of the associated quartic, as follows. If \( w \) is not semi-stable, there are \( g \in \text{SL}(3) \) and integers \( r_1, r_2, r_3 \) satisfying

\[
r_1 + r_2 + r_3 = 0 \quad \text{and} \quad 0 < r_1 \geq r_2 \geq r_3 < 0
\]

such that (10) holds and the inequalities are strict. As in the proof of Theorem 9, assume \( g = 1 \). Then, reasoning as in the proof of that theorem, we can show that

\[
a_{(2,0,0)} = a_{(1,1,0)} = a_{(1,0,1)} = b_{(2,0,0)} = b_{(1,1,0)} = 0.
\]

Furthermore, either \( b_{(1,0,1)} = 0 \) or

\[
a_{(0,2,0)} = a_{(0,1,1)} = b_{(0,2,0)} = 0.
\]

Thus, using (17) to simplify the equation of the quartic, we get:

\[
\begin{align*}
\left( c_{(2,0)}c_{(0,2)} - \frac{c_{(1,1)}^2}{4} \right) & \ddot{x}^4 + \left( \frac{c_{(1,1)}b_{(0,1,1)}}{2} - c_{(0,2)}b_{(0,2,0)} - c_{(2,0)}b_{(0,0,2)} - c_{(2,0)}b_{(0,2,0)} - c_{(2,0)}b_{(0,0,2)} \right) \ddot{y}^3 \ddot{z} \\
+ \left( b_{(0,2,0)}b_{(0,0,2)} - \frac{b_{(0,1,1)}^2}{4} - \frac{c_{(1,1)}b_{(1,0,1)}}{2} \right) & \ddot{x}^2 \ddot{y}^2 + \left( \frac{b_{(1,0,1)}b_{(0,1,1)}}{2} \right) \ddot{x} \ddot{y}^3 \\
- \left( \frac{b_{(1,0,1)}^2}{4} \right) & \ddot{y}^4 + \left( c_{(0,2)}a_{(0,2,0)} + c_{(2,0)}a_{(0,0,2)} - \frac{c_{(1,1)}a_{(0,1,1)}}{2} \right) \ddot{x}^3 \ddot{z} \\
+ \left( \frac{b_{(0,1,1)}a_{(0,1,1)}}{2} \right) & + c_{(2,0)}b_{(1,0,1)} - b_{(0,0,2)}a_{(0,0,2)} - \frac{b_{(2,0,0)}a_{(0,0,2)}}{2} \ddot{x}^2 \ddot{y} \ddot{z} \\
- \left( b_{(0,2,0)}b_{(1,0,1)} + \frac{b_{(1,0,1)}a_{(0,1,1)}}{2} \right) & \ddot{x} \ddot{y}^2 \ddot{z} + \left( a_{(0,2,0)}a_{(0,0,2)} - \frac{a_{(1,1)}^2}{4} \right) \ddot{x}^2 \ddot{z}^2 \\
+ \left( b_{(1,0,1)}a_{(0,2,0)} \right) & \ddot{x} \ddot{y} \ddot{z}^2 = 0.
\end{align*}
\]

Then \((0:0:1)\) is a singular point of the quartic. Furthermore, if \( b_{(1,0,1)} \neq 0 \), then (18) holds, and the equation becomes:

\[
\begin{align*}
\left( c_{(2,0)}c_{(0,2)} - \frac{c_{(1,1)}^2}{4} \right) & \ddot{x}^4 + \left( \frac{c_{(1,1)}b_{(0,1,1)}}{2} - c_{(2,0)}b_{(0,0,2)} \right) \ddot{x}^3 \ddot{y} \\
- \left( \frac{b_{(2,0,0)}^2}{4} + \frac{c_{(1,1)}b_{(1,0,1)}}{2} \right) & \ddot{x}^2 \ddot{y}^2 + \left( \frac{b_{(1,0,1)}b_{(0,1,1)}}{2} \right) \ddot{x} \ddot{y}^3 - \left( \frac{b_{(0,1,1)}^2}{4} \right) \ddot{y}^4 \\
+ \left( c_{(2,0)}a_{(0,0,2)} \right) & \ddot{x}^3 \ddot{z} + \left( c_{(2,0)}b_{(1,0,1)} \right) \ddot{x}^2 \ddot{y} \ddot{z} = 0.
\end{align*}
\]
In this case, the quartic has a triple point at \((0 : 0 : 1)\) with two equal tangent lines, or a quadruple point, whence is not semi-stable, according to loc. cit., p. 80. On the other hand, if \(b_{1(0,1)} = 0\), then the equation of the quartic becomes

\[
\left( c_{(2,0)} c_{(0,2)} - \frac{c^2_{(1,1)}}{4} \right) \bar{x}^4 + \left( \frac{c_{(1,1)} b_{(0,1,1)}}{2} - c_{(0,2)} b_{(0,2,0)} - c_{(2,0)} b_{(0,0,2)} \right) \bar{x}^3 \bar{y} + \left( b_{(0,2,0)} b_{(0,0,2)} - \frac{b^2_{(0,1,1)}}{4} \right) \bar{x}^2 \bar{y}^2 + \left( c_{(2,0)} a_{(0,0,2)} + c_{(2,0)} a_{(0,0,2)} - \frac{c_{(1,1)} a_{(0,1,1)}}{2} \right) \bar{x}^3 \bar{z} + \left( \frac{b_{(0,1,1)} a_{(0,1,1)}}{2} - b_{(0,0,2)} a_{(0,0,2)} - b_{(0,2,0)} a_{(0,0,2)} \right) \bar{x}^2 \bar{y} \bar{z} + \left( a_{(0,2,0)} a_{(0,0,2)} - \frac{a^2_{(0,1,1)}}{4} \right) \bar{x}^2 \bar{z}^2 = 0,
\]

whence the union of a double line, \(\bar{x} = 0\), and a conic, thus again not semi-stable, according to loc. cit.. However, there are nonsemi-stable quartics with milder singularities that do not correspond to nonsemi-stable foliations. For instance, if we set \((17)\), we end up with Equation \((19)\) for the quartic. If we further set \(a_{(0,2,0)} = a_{(0,1,1)} = 0\), we get

\[
\left( c_{(2,0)} c_{(0,2)} - \frac{c^2_{(1,1)}}{4} \right) \bar{x}^4 + \left( c_{(1,1)} b_{(0,1,1)} - c_{(0,2)} b_{(0,2,0)} - c_{(2,0)} b_{(0,0,2)} \right) \bar{x}^3 \bar{y} + \left( b_{(0,2,0)} b_{(0,0,2)} - \frac{b^2_{(0,1,1)}}{4} \right) \bar{x}^2 \bar{y}^2 + \left( c_{(2,0)} a_{(0,0,2)} + c_{(2,0)} a_{(0,0,2)} - \frac{c_{(1,1)} a_{(0,1,1)}}{2} \right) \bar{x}^3 \bar{z} + \left( \frac{b_{(0,1,1)} a_{(0,1,1)}}{2} - b_{(0,0,2)} a_{(0,0,2)} - b_{(0,2,0)} a_{(0,0,2)} \right) \bar{x}^2 \bar{y} \bar{z} + \left( a_{(0,2,0)} a_{(0,0,2)} - \frac{a^2_{(0,1,1)}}{4} \right) \bar{x}^2 \bar{z}^2 = 0,
\]

This quartic has a triple or quadruple point, and is thus nonsemi-stable. If we choose the remaining coordinates of \(\mathbb{F}_2\) such that

\[
b_{(1,0,1)} b_{(0,2,0)} \neq 0 \quad \text{and} \quad b_{(1,0,1)} c_{(2,0)} + a_{(0,0,2)} b_{(0,2,0)} \neq 0,
\]

then the triple point has distinct tangent lines. If we now let none of these lines be contained in the quartic, which is an open condition on the parameters that can be satisfied, as it can be easily verified with CoCoA [3], then \((0 : 0 : 1)\) is the unique singular point of the quartic. Thus, the quartic arises from a semi-stable foliation. The above simple example shows that there are invariants of degree-2 foliations that do not arise from invariants of quartics.

**References**

[1] C. Alcántara, *The good quotient of the semi-stable foliations of \(\mathbb{CP}^2\) of degree 1*, Results Math. **53** (2009), 1–7.

[2] C. Alcántara, *Geometric invariant theory for holomorphic foliations on \(\mathbb{CP}^2\) of degree 2*, Glasgow Math. J. **53** (2011), 153–168.

[3] CoCoATeam, *CoCoA: a system for doing Computations in Commutative Algebra*. Available at [http://cocoa.dima.unige.it](http://cocoa.dima.unige.it)
[4] G. Darboux, Mémoire sur les équations différentielles algébriques du premier ordre et du premier degré (Mélanges), Bull. Sci. Mathématiques 2ème série 2 (1878), 60–96; 123–144; 151–200.
[5] J. Dixmier, On the projective invariants of quartic plane curves, Adv. in Math. 64 (1987), 279–304.
[6] J. Fogarty, F. Kirwan and D. Mumford, Geometric Invariant Theory, Ergebnisse der Mathematik und ihrer Grenzgebiete (2), vol. 34, Springer-Verlag, Berlin, 1994.
[7] W. Fulton, Intersection Theory, Ergebnisse der Mathematik und ihrer Grenzgebiete (3), vol. 2, Springer-Verlag, Berlin, 1984.
[8] X. Goméz-Mont and G. Kempf, Stability of meromorphic vector fields in projective spaces, Comment. Math. Helv. 64 (1989), 462–473.
[9] J.-P. Jouanolou, Équations de Pfaff algébriques, Lecture Notes in Mathematics, vol. 708, Springer-Verlag, Berlin, 1979.
[10] C. Okonek, M. Schneider and H. Spindler, Vector bundles on complex projective spaces, Progress in Mathematics, vol. 3, Birkhäuser, Boston, 1980.
[11] H. Poincaré, Sur l’intégration algébrique des équations différentielles du premier ordre et du premier degré, I and II, R. Circ. Mat. Palermo 5 (1891), 161–191; 11 (1897), 193–239.
[12] T. Shioda, On the graded ring of invariants of binary octavics, Amer. J. Math. 89 (1967), 1022–1046.

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