Decoherent Histories for Spacetime Domains

J.J. Halliwell
Blackett Laboratory
Imperial College
London, SW7 2BZ
UK
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Abstract

The decoherent histories approach is a natural medium in which to address problems in quantum theory which involve time in a non-trivial way. This article reviews the various attempts and difficulties involved in using the decoherent histories approach to calculate the probability for crossing the surface $x = 0$ during a finite interval of time. The commonly encountered difficulties in assigning crossing times arise here as difficulties in satisfying the consistency (no-interference) condition. This can be overcome by introducing an environment to produce decoherence, and probabilities exhibiting the expected classical limit are obtained. The probabilities are, however, dependent to some degree on the decohering environment. The results are compared with a recently proposed irreversible detector model. A third method is introduced, involving continuous quantum measurement theory. Some closely related work on the interpretation of the wave function in quantum cosmology is described.

1 Introduction

Although opinions differ as to the value and achievements of attempts to quantize the gravity, it is undeniable that this endeavour has inspired a considerable amount of work in a variety of related fields. In particular, the quantization of gravity puts considerable pressure on both the mathematical and conceptual foundations of quantum theory, so it is perhaps not surprising that many researchers in quantum gravity have been drawn into working on the foundations of quantum mechanics.

One of the key issues that arises in the quantization of gravity is the “Problem of Time”. In the quantization of cosmological models, the wave function of the universe satisfies not a Schrödinger equation, but the Wheeler–DeWitt
The wave function $\Psi$ depends on the three-metric $h_{ij}$ and the matter field configurations $\phi$ on a closed spacelike three-surface [35, 37, 25]. There is no time label. Its absence is deeply entwined with the four-dimensional diffeomorphism invariance of general relativity. It is often conjectured that “time” is somehow already present amongst the dynamical variables $h_{ij}, \phi$, although to date it has proved impossible to extract a unique, globally defined time variable.

Although a comprehensive scheme for interpreting the wave function is yet to be put forward, a prevalent view is that the interpretation will involve treating all the dynamical variables $h_{ij}, \phi$ on an equal footing, rather than trying to single out one particular combination of them to act as time. For this reason, it is of interest to see if one can carry out a similar exercise in non-relativistic quantum mechanics. That is, to see what the predictions quantum mechanics makes about spacetime regions, rather than regions of space at fixed moments of time.

Such predictions are not the ones that quantum mechanics usually makes. In standard non-relativistic quantum mechanics, the probability of finding a particle between points $x$ and $x + dx$ at a fixed time $t$ is given by

$$p(x, t)dx = |\Psi(x, t)|^2 dx$$

(2)

where $\Psi(x, t)$ is the wave function of the particle. More generally, the variety of questions one might ask about a particle at a fixed moment of time may be represented by a projection operator $P_\alpha$, which is exhaustive

$$\sum_\alpha P_\alpha = 1$$

(3)

and mutually exclusive

$$P_\alpha P_\beta = \delta_{\alpha\beta} P_\alpha$$

(4)

The projection operator appropriate to asking questions about position is $P = |x\rangle\langle x|$. The probability of a particular alternative is given by

$$p(\alpha) = \text{Tr} (P_\alpha \rho)$$

(5)

where $\rho$ is the density operator of the system at the time in question.

The key feature of the above standard formulae is that they do not treat space and time on an equal footing. Suppose one asks, for example, the same sort of question with space and time interchanged. That is, what is the probability of finding the particle at point $x$ in the time interval $t$ to $t + dt$? The point is that the answer is not given by $|\Psi(x, t)|^2 dt$. The reason for this is that, unlike the value of $x$ at fixed $t$, the value of $t$ at fixed $x$ does not refer to an exclusive set of alternatives. The position of a particle at fixed time is a well-defined
quantity in quantum mechanics, but the time at which a particle is found at a fixed position is much more difficult to define because of the possibility of multiple crossings.

This question is clearly a physically relevant one since time is measured by physical devices which are generally limited in their precision. It is therefore never possible to say that a physical event occurs at a precise value of time, only that it occurs in some range of times. Furthermore, there has been considerable recent experimental and theoretical interest in the question of tunneling times. This is the question, given that a particle has tunneled through a barrier region, how much time did it spend inside the barrier?

Spacetime questions tend to be rather non-trivial. As stressed by Hartle, who has carried out a number of investigations in this area, time plays a “peculiar and central role” in non-relativistic quantum mechanics. It is not represented by a self-adjoint operator and there appears to be no obstruction to assuming that it may be measured with arbitrary precision. It enters the Schrödinger equation as an external parameter. As such, it is perhaps best thought of as a label referring to a classical, external measuring device, rather than as a fundamental quantum observable. Yet time is measured by physical systems, and all physical systems are believed to be subject to the laws of quantum theory.

Given these features, means more elaborate that those usually employed are required to define quantum-mechanical probabilities that do not refer to a specific moment of time, and the issue has a long history. One may find in the literature a variety of attempts to define questions of time in a quantum-mechanical way. These include attempts to define time operators, the use of internal physical clocks and path integral approaches. The literature on tunneling times is a particularly rich source of ideas on this topic. Many of these attempts also tie in with the time-energy uncertainty relations. For a nice review of many of these issues, see Ref. 3.

This article is concerned with the attempts to solve problems of a spacetime nature using the decoherent histories approach to quantum theory. It is perhaps of interest to note that, in addition to inspiring work on the question involving time, considerations of quantum gravity were also partly responsible for the development of the decoherent histories approach. For our purposes, the particular attraction of this approach is that it addresses directly the notion of a “history” or “trajectory” and in particular shows how to assign probabilities to them. It is therefore very suited to the question of spacetime probabilities considered here. This is because the question of whether a particle did or did not enter a given region at any time in a given time interval clearly cannot be reduced to a question about the state of the particle at a fixed moment of time, but depends on the entire history of the system during that time interval.

The decoherent histories approach, for spacetime questions, turns out to be
most clearly formulated in terms of path integrals over paths in configuration space. The desired spacetime amplitudes are obtained by summing \( \exp(i \bar{\hbar} S[x(t)]) \), where \( S[x(t)] \) is the action, over paths \( x(t) \) passing through the spacetime region in question, and consistent with the initial state. The probabilities are obtained by squaring the amplitudes in the usual way. (The decoherent histories approach is not inextricably tied to path integrals, however. Operator approaches to the same questions are also available, but are often more cumbersome.)

When computed according to the path integral scheme outlined above, the probability of entering a spacetime region added to the probability of not entering that region is not equal to 1, in general. This is because of interference. The question of whether a particle enters a spacetime region, when carefully broken down, is actually a quite complicated combination of questions about the positions of the particle at a sequence of times. It is therefore, in essence, a complicated combination of double slit situations. Not surprisingly, there is therefore interference and probabilities cannot be assigned.

From the point of view of the decoherent histories approach to quantum theory, therefore, the probability of entering a spacetime region is quite simply not defined in general for a simple point particle system, due to the presence of interference. It is here that the decoherent histories approach, like all the other approaches to defining time in quantum theory, runs up against its own particular brand of difficulties.

It is, however, a common feature of the decoherent histories approach that most of the histories of interest cannot be defined due to interference – histories defined by position at more than one time for example. It is well known that the interference may be removed by coupling to environment, typically a bath of harmonic oscillators in a thermal state. We will therefore consider the above spacetime problem in the presence of an environment.

The decoherent histories approach is reviewed in Section 2 and its application to simple spacetime questions is discussed in Section 3. The inclusion of the environment to induce decoherence is described in Section 4.

The probabilities produced by the decoherent histories approach are in some sense somewhat abstract since they do not refer to a particular measuring device. In Section 5 we therefore introduce a model measuring device for the purposes of comparison. The decoherence model of Section 4 consists of quite a crude environment, which has, however, been very successful in producing decoherence and emergent classicality. The measurements it effectively carries out are of a rather robust and crucially, irreversible, nature. Hence the most important sort of comparison is with an irreversible detector model. Interestingly, most of the arrival time models discussed in the literature are not of this type. It is therefore of interest to develop a model detector, not dissimilar to the decoherence model, but sufficiently modified to carry out a more precise measurement. The comparison between the decoherent histories approach and the detector model is then carried out in Section 6. This also leads to the introduction of a third
candidate for the crossing time probability, derived from continuous quantum measurement theory.

In Section 7 we briefly discuss another type of non-trivial time question, namely, given that a system is in an energy eigenstate, what is the probability that it will pass through a given region in configuration space at any time? The reason this is of interest is that it is, in essence, the question one needs to answer in order to interpret solutions to the Wheeler-DeWitt equation (4).

We summarize and conclude in Section 8.

2 Decoherent Histories Approach to Quantum Theory

In this Section we give a brief summary of the decoherent histories approach to quantum theory. It has been described in considerable depth in many other places [17, 18, 20, 22, 23, 37, 40, 56, 13].

In quantum mechanics, propositions about the attributes of a system at a fixed moment of time are represented by sets of projections operators. The projection operators $P_{\alpha}$ effect a partition of the possible alternatives $\alpha$ a system may exhibit at each moment of time. They are exhaustive and exclusive, as noted in Eqs. (3), (4). A projector is said to be fine-grained if it is of the form $|\alpha\rangle\langle\alpha|$, where $\{|\alpha\rangle\}$ are a complete set of states. Otherwise it is coarse-grained. A quantum-mechanical history (strictly, a homogeneous history [40]) is characterized by a string of time-dependent projections, $P_{\alpha_1}^{1}(t_1), \cdots P_{\alpha_n}^{n}(t_n)$, together with an initial state $\rho$. The time-dependent projections are related to the time-independent ones by

$$P_{\alpha_k}^{k}(t_k) = e^{iH(t_k-t_0)}P_{\alpha_k}^{k}e^{-iH(t_k-t_0)}$$

where $H$ is the Hamiltonian. The candidate probability for these homogeneous histories is

$$p(\alpha_1, \alpha_2, \cdots \alpha_n) = \text{Tr} \left( P_{\alpha_n}^{n}(t_n) \cdots P_{\alpha_1}^{1}(t_1) \rho P_{\alpha_1}^{1}(t_1) \cdots P_{\alpha_n}^{n}(t_n) \right)$$

It is straightforward to show that (7) is both non-negative and normalized to unity when summed over $\alpha_1, \cdots \alpha_n$. However, (7) does not satisfy all the axioms of probability theory, and for that reason it is referred to as a candidate probability. It does not satisfy the requirement of additivity on disjoint regions of sample space. More precisely, for each set of histories, one may construct coarser-grained histories by grouping the histories together. This may be achieved, for example, by summing over the projections at each moment of time,

$$\hat{P}_{\alpha} = \sum_{\alpha \in \alpha} P_{\alpha}$$

5
(although this is not the most general type of coarse graining – see below). The additivity requirement is then that the probabilities for each coarser-grained history should be the sum of the probabilities of the finer-grained histories of which it is comprised. Quantum-mechanical interference generally prevents this requirement from being satisfied. Histories of closed quantum systems cannot in general be assigned probabilities.

There are, however, certain types of histories for which interference is negligible, and the candidate probabilities for histories do satisfy the sum rules. These histories may be found using the decoherence functional:

$$D(\alpha, \alpha') = \text{Tr} \left( P^m_{\alpha_n}(t_n) \cdots P^1_{\alpha_1}(t_1) \rho P^1_{\alpha'_1}(t_1) \cdots P^m_{\alpha'_n}(t_n) \right)$$  \hspace{1cm} (9)$$

Here $\alpha$ denotes the string $\alpha_1, \alpha_2, \cdots, \alpha_n$. Intuitively, the decoherence functional measures the amount of interference between pairs of histories. It may be shown that the additivity requirement is satisfied for all coarse-grainings if and only if

$$\text{Re} D(\alpha, \alpha') = 0$$  \hspace{1cm} (10)$$

for all distinct pairs of histories $\alpha, \alpha'$. Such sets of histories are said to be consistent, or weakly decoherent. The consistency condition (10) is typically satisfied only for coarse–grained histories, and this then often leads to satisfaction of the stronger condition of decoherence

$$D(\alpha, \alpha') = 0$$  \hspace{1cm} (11)$$

for $\alpha \neq \alpha'$. The condition of decoherence is associated with the existence of so-called generalized records. This means that it is possible to add a projector $R_\beta$ at the end of the chain such that decoherence is preserved and such that the label $\beta$ is perfectly correlated with the history alternatives $\alpha_1, \cdots, \alpha_n$. There is therefore in principle some physical measurement that could be carried out at the end of the history from which complete information about the entire history can be recovered [18, 19, 26].

For histories characterized by projections onto ranges of position at different times, the decoherence functional may be represented by a path integral:

$$D(\alpha, \alpha') = \int_{\alpha} Dx \int_{\alpha'} Dy \exp \left( \frac{i}{\hbar} S[x] - \frac{i}{\hbar} S[y] \right) \rho(x_0, y_0)$$  \hspace{1cm} (12)$$

The integral is over paths $x(t)$, $y(t)$ starting at $x_0$, $y_0$, and both ending at the same final point $x_f$, where $x_f$, $x_0$ and $y_0$ are all integrated over, and weighted by the initial state $\rho(x_0, y_0)$. The paths are also constrained to pass through spatial gates at a sequence of times corresponding to the projection operators.

However, the path integral representation of the decoherence functional also points the way towards asking types of questions that are not represented by homogeneous histories [33]. In this article we are particularly interested in the
following question. Suppose a particle starts at $t = 0$ in some quantum state. What is the probability that the particle will either cross or never cross $x = 0$ during the time interval $[0, \tau]$? In the path integral of the form (12) it is clear how to proceed. One sums over paths that, respectively, either always cross or never cross $x = 0$ during the time interval.

How does this look in operator language? The operator form of the decoherence functional is

$$D(\alpha, \alpha') = \text{Tr} \left( C_\alpha \rho C_{\alpha'}^\dagger \right)$$

(13)

where

$$C_\alpha = P_{\alpha_n}(t_n) \cdots P_{\alpha_1}(t_1)$$

(14)

The histories that never cross $x = 0$ are represented by taking the projectors in $C_\alpha$ to be onto the positive $x$-axis, and then taking the limit $n \to \infty$ and $t_k - t_{k-1} \to 0$. The histories that always cross $x = 0$ are then represented by the object

$$\bar{C}_\alpha = 1 - C_\alpha$$

(15)

This is called an inhomogenous history, because it cannot be represented as a single string of projectors. It can however, be represented as a sum of strings of projectors \[33, 40\].

The proper framework in which these operations, in particular (15), are understood, is the so-called generalized quantum theory of Hartle \[33\] and Isham et al. \[40\]. It is called “generalized” because it admits inhomogeneous histories as viable objects, whilst standard quantum theory concerns itself entirely with homogeneous histories. We will make essential use of inhomogeneous histories in what follows.

In practice, for point particle systems, decoherence is readily achieved by coupling to an environment. Here, we will use the much studied case of the quantum Brownian motion model, in which the particle is linearly coupled through position to a bath of harmonic oscillators in a thermal state at temperature $T$ and characterized by a dissipation coefficient $\gamma$. The details of this model may be found elsewhere \[10, 14, 22, 23\].

We consider histories characterized only by the position of the particle and the environmental coordinates are traced out. The path integral representation of the decoherence functional then has the form

$$D(\alpha, \alpha') = \int_\alpha \mathcal{D}x \int_{\alpha'} \mathcal{D}y \, \exp \left( \frac{i}{\hbar} S[x] - \frac{i}{\hbar} S[y] + \frac{i}{\hbar} W[x, y] \right) \rho(x_0, y_0)$$

(16)

where $W[x, y]$ is the Feynman–Vernon influence functional phase, and is given by

$$W[x, y] = -m \gamma \int \, dt \, (x - y)(\dot{x} + \dot{y}) + i \frac{2m \gamma kT}{\hbar} \int \, dt \, (x - y)^2$$

(17)

The first term induces dissipation in the effective classical equations of motion. The second term is responsible for thermal fluctuations. It is also responsible
for suppressing contributions from paths \( x(t) \) and \( y(t) \) that differ widely, and produces decoherence of configuration space histories.

The corresponding classical theory is no longer the mechanics of a single point particle, but a point particle coupled to a heat bath. The classical correspondence is now to a stochastic process which may be described by either a Langevin equation, or by a Fokker-Planck equation for a phase space probability distribution \( w(p, x, t) \):

\[
\frac{\partial w}{\partial t} = -\frac{p}{m} \frac{\partial w}{\partial x} + 2\gamma \frac{\partial (pw)}{\partial p} + D \frac{\partial^2 w}{\partial p^2}
\]

(18)

where \( w \geq 0 \) and

\[
\int dp \int dx \ w(p, x, t) = 1
\]

(19)

When the mass is sufficiently large, this equation describes near–deterministic evolution with small thermal fluctuations about it.

### 3 Spacetime Coarse Grainings

We are generally interested in spacetime coarse grainings which consist of asking for the probability that a particle does or does not enter a certain region of space during a certain time interval. However, the essentials of this question boil down to the following simpler question: what is the probability that the particle will either cross or not cross \( x = 0 \) at any time in the time interval \([0, \tau] \)? We will concentrate on this question.

We briefly review the results of Yamada and Takagi [68], Hartle [33, 36, 37] and Micanek and Hartle [51]. We will compute the decoherence functional using the path integral expression (12), which may be written

\[
D(\alpha, \alpha') = \int dx_f \Psi_\tau^\alpha(x_f) \left( \Psi_\tau^\alpha(x_f) \right)^* = \int dx_f \Psi_\tau^\alpha(x_f) \left( \Psi_\tau^\alpha(x_f) \right)^*
\]

(20)

where \( \Psi_\tau^\alpha(x_f) \) denotes the amplitude obtained by summing over paths ending at \( x_f \) at time \( \tau \), consistent with the restriction \( \alpha \) and consistent with the given initial state, so we have

\[
\Psi_\tau^\alpha(x_f) = \int_\alpha dx(t) \exp \left( \frac{i}{\hbar} S[x] \right) \Psi_0(x_0)
\]

(21)

Suppose the system starts out in the initial state \( \Psi_0(x) \) at \( t = 0 \). The amplitude for the particle to start in this initial state, and end up at \( x \) at time \( \tau \), but without ever crossing \( x = 0 \), is

\[
\Psi_\tau^r(x) = \int_{-\infty}^{\infty} dx_0 \ g_r(x, \tau|x_0, 0) \Psi_0(x_0)
\]

(22)
where \( g_r \) is the restricted Green function, \( i.e. \), the sum over paths that never cross \( x = 0 \). For the free particle considered here (and also for any system with a potential symmetric about \( x = 0 \)), \( g_r \) may be constructed by the method of images:

\[
g_r(x, \tau \mid x_0, 0) = \left[ \theta(x) \theta(x_0) + \theta(-x) \theta(-x_0) \right] \times (g(x, \tau \mid x_0, 0) - g(x, \tau \mid -x_0, 0))
\] (23)

where \( g(x, \tau \mid x_0, 0) \) is the unrestricted propagator.

The amplitude to cross \( x = 0 \) is

\[
\Psi^c_\tau(x) = \int_{-\infty}^{\infty} dx_0 \, g_c(x, \tau \mid x_0, 0) \, \Psi_0(x_0)
\] (24)

where \( g_c(x, \tau \mid x_0, 0) \) is the crossing propagator, \( i.e. \), the sum over paths which always cross \( x = 0 \). This breaks up into two parts. If \( x \) and \( x_0 \) are on opposite sides of \( x = 0 \), it is clearly just the usual propagator \( g(x, \tau \mid x_0, 0) \). If \( x \) and \( x_0 \) are on the same side of \( x = 0 \), it is given by \( g(-x, \tau \mid x_0, 0) \). This may be seen by reflecting the segment of the path after last crossing about \( x = 0 \). (Alternatively, this is just the usual propagator minus the restricted one).

Hence,

\[
g_c(x, \tau \mid x_0, 0) = \left[ \theta(x) \theta(-x_0) + \theta(-x) \theta(x_0) \right] \, g(x, \tau \mid x_0, 0) + \left[ \theta(x) \theta(x_0) + \theta(-x) \theta(-x_0) \right] \, g(-x, \tau \mid x_0, 0)
\] (25)

The crossing propagator may also be expressed in terms of the so-called path decomposition expansion, a form which is sometimes useful [3, 5, 24, 29].

Inserting these expressions in the decoherence function, Yamada and Takagi found that the consistency condition may be satisfied exactly by states which are antisymmetric about \( x = 0 \). The probability of crossing \( x = 0 \) is then 0 and the probability of not crossing is 1. What is happening in this case is that the probability flux across \( x = 0 \), which clearly has non-zero components going both to the left and the right, averages to zero.

Less trivial probabilities are obtained in the case where one asks for the probability that the particle remains always in \( x > 0 \) or not, with an initial state with support along the entire \( x \)-axis [37]. The probabilities become trivial again, however, in the interesting case of an initial state with support only in \( x > 0 \).

Yamada and Takagi have also considered the case of the probability of finding the particle in a spacetime region \([68]\). That is, the probability that the particle enters, or does not enter, the spatial interval \( \Delta \), at any time during the time interval \([0, \tau]\). Again the consistency condition is satisfied only for very special initial states and the probabilities are then rather trivial.

In an attempt to assign probabilities for arbitrary initial states, Micanek and Hartle considered the above results in the limit that the time interval \([0, \tau]\)
becomes very small \[51\]. Such an assignment must clearly be possible in the limit \(\tau \to 0\). They found that both the off-diagonal terms of the decoherence functional \(D\) and the crossing probability \(p\) are of order \(\epsilon = (\hbar t/m)^{\frac{1}{2}}\) for small \(t\), and the probability \(\bar{p}\) for not crossing is of order 1. Hence \(p + \bar{p} \approx 1\). They therefore argued that probabilities can be assigned if \(t\) is sufficiently small. On the other hand, we have the exact relation,

\[ p + \bar{p} + 2\text{Re}D = 1 \]  

(26)

\(\text{Re}D\) represents the degree of fuzziness in the definition of the probabilities. Since it is of the same order as \(\bar{p}\), one may wonder whether it is then valid to claim approximate consistency. Another condition that may be relevant is the condition

\[ |D|^2 \ll p\bar{p} \]  

(27)

which was suggested in Ref. \[13\] as a measure of approximate decoherence, and is clearly satisfied in this case.

We conclude from these various studies that for a system consisting of a single point particle, crossing probabilities can be assigned to histories only in a limited class of circumstances.

There is one particularly important case in which this lack of probability assignment is perhaps unsettling. Consider a wave packet that starts at \(x_0 > 0\) moving towards the origin. The amplitude for not crossing is given by the restricted amplitude \[22\] and the restricted propagator \[23\]. However, in the case where the centre of the wave packet reaches the origin during the time interval, it is easily seen from the propagator \[23\] that after hitting the origin there is a piece of the wave packet which is reflected back into \(x > 0\) (this is the image wave packet that has come from \(x < 0\)). This means that we have the counterintuitive result that the probability for remaining in \(x > 0\) is not in fact close to zero \[33, 67\] as one would expect. It is unsettling because one sometimes thinks of wave packets as being the closest thing quantum theory has to a classical path, yet the behaviour of the wave packet in this case is utterly different to the corresponding expected classical behaviour.

Although counterintuitive, it is not that disturbing, since with this initial state, the histories for crossing and not crossing do not satisfy the consistency condition, so we should not expect them to agree with our physical intuition. Still, it would be reassuring to see that the formalism set up so far yields the intuitively expected classical limit under appropriate circumstances. To obtain that, we need a decoherence mechanism, and this we now consider.
4 Decoherence of Spacetime Coarse-Grained Histories in the Quantum Brownian Motion Model

We have seen that crossing probabilities can only be assigned in the decoherent histories approach for very special initial states, and furthermore, we do not get an intuitively sensible classical limit for wave packet initial states. It is, however, well-known that most sets of histories of interest do not in fact exhibit decoherence without the presence of some physical mechanism to produce it. In this Section, we therefore discuss a modified situation consisting of a point particle coupled to a bath of harmonic oscillators in a thermal state. This model, the quantum Brownian motion model [1], produces decoherence of histories of positions in a variety of situations.

This explicit modification of the single particle system means that the corresponding classical problem (to which the quantum results should reduce under certain circumstances) is in fact a stochastic process described by either a Langevin equation or by a Fokker-Planck equation. It is therefore appropriate to first study the crossing problem in the corresponding classical stochastic process (see for example, Refs.[64, 8, 9, 50, 69], and references therein).

4.1 The Crossing Time Problem in Classical Brownian Motion

Classical Brownian motion may be described by the Fokker-Planck equation (18) for the phase space probability distribution \( w(p, x, t) \). For simplicity we will work in the limit of negligible dissipation, hence the equation is,

\[
\frac{\partial w}{\partial t} = -\frac{p}{m} \frac{\partial w}{\partial x} + D \frac{\partial^2 w}{\partial p^2}
\]  

(28)

where \( D = 2m\gamma kT \). The Fokker-Planck equation is to be solved subject to the initial condition

\[
w(p, x, 0) = w_0(p, x)
\]  

(29)

Consider now the crossing time problem in classical Brownian motion. The question is this. Suppose the initial state is localized in the region \( x > 0 \). What is the probability that, under evolution according to the Fokker-Planck equation (28), the particle either crosses or does not cross \( x = 0 \) during the time interval \([0, \tau]\)?

A useful way to formulate spacetime questions of this type is in terms of the Fokker-Planck propagator, \( K(p, x, \tau|p_0, x_0, 0) \). The solution to (28) with the initial condition (29) may be written in terms of \( K \) as,

\[
w(p, x, \tau) = \int_{-\infty}^{\infty} dp \int_{-\infty}^{\infty} dx \ K(p, x, \tau|p_0, x_0, 0) \ w_0(p, x)
\]  

(30)
The Fokker-Planck propagator satisfies the Fokker-Planck equation (28) with respect to its final arguments, and satisfies delta function initial conditions,

\[ K(p, x, 0|p_0, x_0, 0) = \delta(p - p_0) \delta(x - x_0) \]  

(31)

For the free particle without dissipation, it is given explicitly by

\[ K(p, x, \tau|p_0, x_0, 0) = N \exp \left( -\alpha (p - p_0)^2 - \beta (x - x_0 - \frac{p_0 \tau}{m})^2 \right) + \epsilon (p - p_0) (x - x_0 - \frac{p_0 \tau}{m}) \]  

(32)

where \( N, \alpha, \beta \) and \( \epsilon \) are given by

\[ \alpha = \frac{1}{D \tau}, \quad \beta = \frac{3m^2}{D \tau^2}, \quad \epsilon = \frac{3m}{D \tau^2}, \quad N = \left( \frac{3m^2}{4\pi D^2 \tau^4} \right)^{\frac{1}{2}} \]  

(33)

(with \( D = 2m \gamma kT \)). An important property it satisfies is the composition law

\[ K(p, x, \tau|p_0, x_0, 0) = \int_{-\infty}^{\infty} dp_1 \int_{-\infty}^{\infty} dx_1 K(p, x, \tau|p_1, x_1, t_1) K(p_1, x_1, t_1|p_0, x_0, 0) \]  

(34)

where \( \tau > t_1 > 0 \).

For our purposes, the utility of the Fokker-Planck propagator is that it may be used to assign probabilities to individual paths in phase space. Divide the time interval \([0, \tau]\) into subintervals, \( t_0 = 0, t_1, t_2, \ldots, t_{n-1}, t_n = \tau \). Then in the limit that the subintervals go to zero, and \( n \to \infty \) but with \( \tau \) held constant, the quantity

\[ \prod_{k=1}^{n} K(p_k, x_k, t_k|p_{k-1}, x_{k-1}, t_{k-1}) \]  

(35)

is proportional to the probability for a path in phase space. The probability for various types of coarse grained paths (including spacetime coarse grainings) can therefore be calculated by summing over this basic object.

We are interested in the probability \( w_r(p_n, x_n, \tau) \) that the particle follows a path which remains always in the region \( x > 0 \) during the time interval \([0, \tau]\) and ends at the point \( x_n > 0 \) with momentum \( p_n \). The desired total probabilities for crossing or not crossing can then be constructed from this object. \( w_r \) is clearly given by the continuum limit of the expression

\[ w_r(p_n, x_n, \tau) = \int_0^{\infty} dx_{n-1} \cdots \int_0^{\infty} dx_1 \int_{-\infty}^{\infty} dx_0 \int_{-\infty}^{\infty} dp_{n-1} \cdots \int_{-\infty}^{\infty} dp_1 \int_{-\infty}^{\infty} dp_0 \times \prod_{k=1}^{n} K(p_k, x_k, t_k|p_{k-1}, x_{k-1}, t_{k-1}) w_0(p_0, x_0) \]  

(36)

Now it is actually more useful to derive a differential equation and boundary conditions for \( w_r(p, x, \tau) \), rather than attempt to evaluate the above multiple
integral. First of all, it is clear from the properties of the propagator that $w_r(p, x, \tau)$ satisfies the Fokker-Planck equation (28) and the initial condition (29). However, we also expect some sort of condition at $x = 0$. From the explicit expression for the propagator (32), (33), we see that in the continuum limit, the propagator between $p_{n-1}, x_{n-1}$ and the final point $p_n, x_n$ becomes proportional to the delta function

$$\delta (x_n - x_{n-1} - p_n \tau / m)$$

(37)

Since $x_{n-1} \geq 0$, when $x_n = 0$ this delta function will give zero when $p_n > 0$, but could be non-zero when $p_n < 0$. Hence we deduce that the boundary condition on $w_r(p, x, t)$ is

$$w_r(p, 0, t) = 0, \text{ if } p > 0$$

(38)

This is the absorbing boundary condition usually given for the crossing time problem [50, 66] (although this argument for it does not seem to have appeared elsewhere).

It is now convenient to introduce a restricted propagator $K_r(p, x, \tau|p_0, x_0, 0)$, which propagates $w_r(p, x, \tau)$. That is, $K_r$ satisfies the delta function initial conditions (38) and the same boundary conditions as $w_r$, Eq.(38). Since the original Fokker-Planck equation is not invariant under $x \to -x$, we cannot expect that a simple method of images (of the type used in Section 3), will readily yield the restricted propagator $K_r$. $K_r$ has recently been found [9], using a modified method of images technique due to Carslaw [11], and we briefly summarize those results.

Consider first the usual Fokker-Planck propagator (32). Introducing the coordinates

$$X = \frac{p}{m} \frac{3x}{2\tau}, \quad Y = \frac{\sqrt{3}x}{2\tau}$$

(39)

$$X_0 = \frac{-p_0}{2m} \frac{3x_0}{2\tau}, \quad Y_0 = \frac{\sqrt{3}}{2} \left( \frac{p_0}{m} + \frac{x_0}{\tau} \right)$$

(40)

the propagator (32) becomes,

$$K = \frac{\sqrt{3}}{2\pi \tilde{t}^2} \exp \left( \frac{-\frac{1}{\tilde{t}} (X - X_0)^2 - \frac{Y - Y_0)^2}{\tilde{t}} \right)$$

(41)

Here, $\tilde{t} = D\tau/m^2$. Now go to polar coordinates,

$$X = r \cos \theta, \quad Y = r \sin \theta$$

(42)

$$X_0 = r' \cos \theta', \quad Y_0 = r' \sin \theta'$$

(43)

Then from (43), it is possible to construct a so-called multiform Green function [11],

$$g(r, \theta; r', \theta') = \frac{\sqrt{3}}{2\pi^{3/2} \tilde{t}^2} \exp \left( -\frac{r^2 + r'^2 - 2rr'\cos(\theta - \theta')}{\tilde{t}} \right) \int_{-\infty}^{\infty} d\lambda \, e^{-\lambda^2}$$

(44)
where

\[ a = 2 \left( \frac{rr'}{l} \right)^\frac{1}{2} \cos \left( \frac{\theta - \theta'}{2} \right) \]  

Like the original Fokker-Planck propagator, this object is a solution to the Fokker-Planck equation with delta function initial conditions, but differs in that it has the property that it is defined on a two-sheeted Riemann surface and has period $4\pi$. The desired restricted propagator $K_r$ is then given by

\[ K_r(p, x, \tau | p_0, x_0, 0) = g(r, \theta, r', \theta') - g(r, \theta, r', -\theta') \]  

The point $x = 0$ for $p > 0$ is $\theta = 0$ in the new coordinates, and the above object indeed vanishes at $\theta = 0$. Furthermore, the second term in the above goes to zero at $\tau = 0$, whilst the first one goes to a delta function as required.

The probability of not crossing the surface during the time interval $[0, t]$ is then given by

\[ p_r = \int_{-\infty}^{\infty} dp \int_{-\infty}^{\infty} dx \int_{0}^{\infty} dp_0 \int_{0}^{\infty} dx_0 \ K_r(p, x, \tau | p_0, x_0, 0) \ w_0(p_0, x_0) \]  

The probability of crossing must then be $p_c = 1 - p_r$, which can also be written,

\[ p_c = \int_{-\infty}^{0} dp \int_{-\infty}^{\infty} dp_0 \int_{0}^{\infty} dx_0 \ \frac{p}{m} \ K_r(p, x = 0, \tau | p_0, x_0, 0) \ w_0(p_0, x_0) \]  

This completes the discussion of the classical stochastic problem.

4.2 The Crossing Time Problem in Quantum Brownian Motion

We now consider the analogous problem in the quantum case. We therefore attempt to repeat the analysis of Section 3, but using instead of (18), the decoherence functional appropriate to the quantum Brownian motion model. It may be written,

\[ D(\alpha, \alpha') = \text{Tr} (\rho_{\alpha\alpha'}) \]  

where

\[ \rho_{\alpha\alpha'}(x_f, y_f) = \int_{\alpha} \mathcal{D}x \int_{\alpha'} \mathcal{D}y \ \exp \left( \frac{i}{\hbar} S[x] - \frac{i}{\hbar} S[y] + \frac{i}{\hbar} W[x, y] \right) \ \rho_0(x_0, y_0) \]  

Here, $W[x, y]$ is the influence functional phase (8), but with the dissipation term neglected. The sum is over all paths $x, y$ which are consistent with the coarse graining $\alpha, \alpha'$, and end at the final points $x_f, y_f$.

We will concentrate on the case in which the initial density operator has support only on the positive axis, and we ask for the probability that the particle
either crosses or never crosses $x = 0$ during the time interval $[0, \tau]$. The history label $\alpha$ takes two values, which we denote $\alpha = c$ and $\alpha = r$ for, respectively, crossing and not crossing.

The objects $\rho_{\alpha \alpha'}$ defined in Eq.(50) actually obeys a master equation,

$$i\hbar \frac{\partial \rho}{\partial t} = -\frac{\hbar^2}{2m} \left( \frac{\partial^2 \rho}{\partial x^2} - \frac{\partial^2 \rho}{\partial y^2} \right) - i\hbar D(x - y)^2 \rho$$  \hspace{1cm} (51)

This is the usual master equation for the evolution of the density operator of quantum Brownian motion \[10\]. The objects $\rho_{\alpha \alpha'}$ are then found by solving this equation subject to matching the initial state $\rho_0$, and also to the following boundary conditions (which follow from the path integral representation):

$$\rho_{rr}(x, y) = 0, \text{ for } x \leq 0 \text{ and } y \leq 0 \hspace{1cm} (52)$$

$$\rho_{rc}(x, y) = 0, \text{ for } x \leq 0 \hspace{1cm} (53)$$

$$\rho_{cr}(x, y) = 0, \text{ for } y \leq 0 \hspace{1cm} (54)$$

Given $\rho_{rr}, \rho_{rc}, \rho_{cr}$, the quantity $\rho_{cc}$ may be calculated from the relation,

$$\rho_{rr} + \rho_{rc} + \rho_{cr} + \rho_{cc} = \rho \hspace{1cm} (55)$$

In the unitary case, this problem was solved very easily using the method of images. The problem in the non-unitary case treated here, however, is that the master equation is not invariant under $x \to -x$ (or under $y \to -y$), hence $\rho(-x, y)$ and $\rho(x, -y)$ are not solutions to the master equation. The method of images is therefore not applicable in this case (contrary to the claim in Ref.[33]). As far as an analytic approach goes, this represent a very serious technical problem. Restricted propagation problems are very hard to solve analytically in the absence of the method of images. However, the presence of the decohering environment allows for an approximate solution of the problem. This is described in detail in Ref.[32]. The results are intuitively clear and we summarize them here.

First of all, decoherence of position histories in this model is extremely good, so $\rho_{rc} \approx 0, \rho_{cr} \approx 0$. We may therefore assign probabilities for not crossing and for crossing, and these are equal respectively to $\text{Tr} \rho_{rr}$ and $\text{Tr} \rho_{cc}$. To see what these probabilities are, we make use of the Wigner representation of the density operator \[3\]:

$$W(p, x) = \frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} d\xi \ e^{-\frac{i}{\hbar} p\xi} \rho(x + \frac{\xi}{2}, x - \frac{\xi}{2}) \hspace{1cm} (56)$$

The Wigner representation is very useful in studies of the master equation, since it is similar to a classical phase space distribution function. Indeed, for quantum Brownian motion model with a free particle, the Wigner function obeys the same Fokker-Planck equation \[28\] as the analogous classical phase space distribution function. What makes it fail to be a classical phase space distribution
is that it can take negative values. However, it can be shown that the Wigner function becomes positive after a short time (typically the decoherence time), and numerous authors have discussed its use as an approximate classical phase space distribution, under these conditions [31].

Given approximate decoherence, it was shown at some length in Ref. [32] using the path integral (50) that the Wigner transform of $\rho_{rr}$ is given by

$$W_{rr}(m\dot{X}_f, X_f) = \int_r D\dot{X} \exp \left( -\frac{m}{8\gamma kT} \int dt \ddot{X}^2 \right) W_0(m\dot{X}_0, X_0)$$  (57)

where the functional integral over $X(t)$ is over paths which lie in $X > 0$, and match $X_f$ and $\dot{X}_f$ at the final time. If the paths $X(t)$ were not restricted, Eq. (57) would in fact be a path integral representation of the Fokker-Planck propagator (32) [43]. With the restriction $X > 0$, it may be shown that it is a representation of the restricted Fokker-Planck propagator (46) or (36).

It then follows that the probabilities for not crossing and for crossing $x = 0$ are given, to a good approximation, by the classical stochastic results (47), (48), with the classical phase space distribution function $w_0$ replaced by the initial Wigner function $W_0$ in the quantum case. This result is the expected and intuitively obvious one, although as outlined in Ref. [32], it is a non-trivial matter to show that the boundary conditions on $\rho_{\alpha\alpha'}$ in the quantum case reduce to the boundary conditions on $W$ appropriate to the classical stochastic problem.

### 4.3 Properties of the Solution

Some simple properties of our results may be seen by examining the path integral form of the solution (57). The important case to consider is the motion of a wavepacket, since this is the situation that gave problematic results in Section 3. We take an initial state consisting of a wavepacket concentrated at some $x > 0$, and moving towards the origin. We are interested in the probability of whether it will cross $x = 0$ or not during some time interval, under the evolution by the path integral (57).

The integrand in (57) is peaked about the unique path for which $\ddot{X} = 0$ with the prescribed values of $X_0$ and $\dot{X}_0$. This is of course the classical path with the prescribed initial data. From (37), the spatial width $(\Delta X)^2$ of the peak is of order $\gamma kT/(m\tau^3)$. If the classical path does not cross $x = 0$ and approaches $x = 0$ no closer than a distance $\Delta X$ during the time interval, then it will lie well within the integration range $X > 0$, and the propagation is essentially the same as unrestricted propagation, since the dominant contribution to the integral comes from the region $X > 0$. It is then easy to see, from the normalization of the Wigner function, that the probability of not crossing is approximately 1, the intuitively expected result.

If the classical path crosses $x = 0$ during the time interval, it will lie outside the integration range of $X$ for time slices after the time at which it crossed.
If it crosses sufficiently early that an entire wave packet of width $\Delta X$ may enter $x < 0$ before time $\tau$, then the functional integration will sample only the exponentially small tail of the integrand, so $W_{rr}$ will be very small. The probability of not crossing will therefore be close to zero, again the intuitively expected result.

The inclusion of the environment therefore restores the intuitively sensible classical limit to the quantum case of Section 3.

In the above simple examples, the crossing probabilities are independent of the details of the environment, to a leading order approximation. It is clear that in a more precise expression, the crossing probabilities will in fact depend on the features of the environment (e.g., its temperature). One might find this slightly unsettling, at least in comparison to quantum-mechanical probabilities at a fixed moment of time, which depend only on the state at that time and not on the details of how the property in question might be measured. This is in keeping with an opinion sometimes expressed on questions of time in quantum mechanics – that to specify times one has to specify the physical mechanism by which it is measured [47].

5 A Detector Model

Although the results of the previous sections produced mathematically viable candidates for the probabilities of crossing and not crossing $x = 0$, it is by no means clear how they correspond to a particular type of measurement. As noted in Section 2, general theorems exist showing that decoherence of histories implies the existence at the final end of the histories of a record storing the information about the decohered histories [18, 19]. This means that there is some quantity at a fixed moment of time which is correlated with the property of crossing or not crossing $x = 0$ during the time interval $[0, \tau]$, and which could in principle be measured. Records associated with decoherence have, however, been explicitly found only in a few simple cases (see Ref. [26], for example). For these reasons, it is of interest to compare the approaches involving the decoherent histories approach with a completely different approach involving a specific model of a detector.

We therefore introduce, following Ref. [27], a model detector which is coupled to the particle in the region $x < 0$, and such that it undergoes a transition when the coupling is switched on. Such detectors have certainly been considered before (see, e.g., Ref. [4]). The particle could, for example, be coupled to a simple two-level system that flips from one level to the other when the particle is detected. One of the difficulties of many detector models, however, is that if they are modeled by unitary quantum mechanics, the possibility of the reverse transition exists. Because quantum mechanics is fundamentally reversible, the detector could return to the undetected state under its self-dynamics, even when the particle has interacted with it.
To get around this difficulty, we appeal to the fact that realistic detectors have a very large number of degrees of freedom, and are therefore effectively irreversible. They are designed so that there is an overwhelming large probability for them to make a transition in one direction rather than its reverse. We consider a simple model detector that has this property. This is achieved by coupling a two-level system detector to a large environment, which makes its evolution effectively irreversible.

The detector is a two-level system, with levels $|1\rangle$ and $|0\rangle$, representing the states of no detection and detection, respectively. Introduce the raising and lowering operators

$$\sigma_+ = |1\rangle \langle 0|, \quad \sigma_- = |0\rangle \langle 1|$$

and let the Hamiltonian of the detector be $H_d = \frac{1}{2}\hbar \omega \sigma_z$, where

$$\sigma_z = |1\rangle \langle 1| - |0\rangle \langle 0|$$

so $|0\rangle$ and $|1\rangle$ are eigenstates of $H_d$ with eigenvalues $-\frac{1}{2}\hbar \omega$ and $\frac{1}{2}\hbar \omega$ respectively. We would like to couple the detector to a free particle in such a way that the detector makes an essentially irreversible transition from $|1\rangle$ to $|0\rangle$ if the particle enters $x < 0$, and remains in $|1\rangle$ otherwise. This can be arranged by coupling the detector to a large environment of oscillators in their ground state, with a coupling proportional to $\theta(-x)$. This means that if the particle enters the region $x < 0$, the detector becomes coupled to the large environment causing it to undergo a transition. Since the environment is in its ground state, if the detector initial state is the higher energy state $|1\rangle$ it will, with overwhelming probability, make a transition from $|1\rangle$ to the lower energy state $|0\rangle$. A possible Hamiltonian describing this process for the three-component system is

$$H = H_s + H_d + H_E + V(x)H_{dE}$$

where the first three terms are the Hamiltonians of the particle, detector and environment respectively, and $H_{dE}$ is the interaction Hamiltonian of the detector and its environment. The simplest choice of environment is a collection of harmonic oscillators,

$$H_E = \sum_n \hbar \omega_n a_n^\dagger a_n$$

and we take the coupling to the detector to be via the interaction

$$H_{dE} = \sum_n \hbar (\kappa_n^* \sigma_- a_n^\dagger + \kappa_n \sigma_+ a_n)$$

An environment consisting of an electromagnetic field, for example, would give terms of this general form. $V(x)$ is a potential concentrated in $x < 0$ (and we will eventually make the simplest choice, $V(x) = \theta(-x)$, but for the moment we keep it more general). The important feature is that the interaction between
the detector and its environment, causing the detector to undergo a transition, is switched on only when the particle is in $x < 0$.

A similar although more elaborate model particle detector has been previously studied by Schulman [61] (see also Refs. [62]). The advantage of the present model is that it is easier to solve explicitly.

We are interested in the reduced dynamics of the particle and detector with the environment traced out. Hence we seek a master equation for the reduced density operator $\rho$ of the particle and detector. With the above choices for $H_E$ and $H_{dE}$, the derivation of the master equation is standard [12, 16] and will not be repeated here. There is the small complication of the factor of $V(x)$ in the interaction term, but this is readily accommodated. We assume a factored initial state, and we assume that the environment starts out in the ground state. In a Markovian approximation (essentially the assumption that the environment dynamics is much faster than detector or particle dynamics), and in the approximation of weak detector-environment coupling, the master equation is

$$\dot{\rho} = -\frac{i}{\hbar} [H_s + H_d, \rho] - \frac{\gamma}{2} (V^2(x)\sigma_+\sigma_-\rho + \rho\sigma_+\sigma_-V^2(x) - 2V(x)\sigma_-\rho\sigma_+V(x))$$

(63)

Here, $\gamma$ is a phenomenological constant determined by the distribution of oscillators in the environment and underlying coupling constants. The frequency $\omega$ in $H_d$ is also renormalized to a new value $\omega'$.

Eq. (63) is the sought-after description of a particle coupled to an effectively irreversible detector in the region $x < 0$. In the dynamics of the detector plus environment only (i.e., with $V = 1$ and $H_s = 0$), it is readily shown that every initial state tends to the state $|0\rangle\langle 0|$ on a timescale $\gamma^{-1}$. With the particle coupled in, if the initial state of the detector is chosen to be $|1\rangle\langle 1|$, it undergoes an irreversible transition to the state $|0\rangle\langle 0|$ if the particle enters $x < 0$, and remains in its initial state otherwise.

Eq. (63) is in fact of the Lindblad form (the most general Markovian master equation preserving density operator properties [48]). A similar detection scheme based on a postulated master equation similar to (63) was previously considered in Ref. [41].

The master equation (63) is easily solved by writing

$$\rho = \rho_{11} \otimes |1\rangle\langle 1| + \rho_{01} \otimes |0\rangle\langle 1| + \rho_{10} \otimes |1\rangle\langle 0| + \rho_{00} \otimes |0\rangle\langle 0|$$

(64)

We suppose that the particle starts out in an initial state $|\Psi_0\rangle$, hence the master equation is to be solved subject to the initial condition,

$$\rho(0) = |\Psi_0\rangle\langle \Psi_0| \otimes |1\rangle\langle 1|$$

(65)

The probability that the detector does not register during $[0, \tau]$ is

$$p_{nd} = \text{Tr} \rho_{11} = \int_{-\infty}^{\infty} dx \int_{0}^{\tau} \rho_{11}(x, x, \tau)$$

(66)
and the probability that it registers is

$$p_d = \text{Tr}\rho_{00} = \int_{-\infty}^{\infty} dx \rho_{00}(x, x, \tau) \quad (67)$$

(where the trace is over the particle Hilbert space). Clearly \(p_{nd} + p_d = 1\), since \(\text{Tr}\rho = 1\).

The explicit solution to the master equation is straightforward and was carried out in Ref. [27]. There, it was shown that, when \(V(x) = \theta(-x)\), the solution for \(\rho_{11}\) may be written

$$\rho_{11}(t) = \exp \left( -\frac{i}{\hbar}H_{st} - \frac{\gamma}{2}V t \right) \rho_{11}(0) \exp \left( \frac{i}{\hbar}H_{st} - \frac{\gamma}{2}V t \right) \quad (68)$$

What is particularly interesting about this expression is that it can be factored into a pure state. Let \(\rho_{11} = |\Psi\rangle\langle\Psi|\). Then, noting that \(\rho_{11}(0) = |\Psi_0\rangle\langle\Psi_0|\), Eq. (68) is equivalent to

$$|\Psi(t)\rangle = \exp \left( -\frac{i}{\hbar}H_{st} - \frac{\gamma}{2}V t \right) |\Psi_0\rangle \quad (69)$$

The probability for no detection is then

$$p_{nd} = \int_{-\infty}^{\infty} dx \ |\Psi(x, \tau)|^2 \quad (70)$$

The pure state (69) evolves according to a Schrödinger equation with an imaginary contribution to the potential, \(-\frac{1}{2}i\hbar\gamma V\). Complex potentials of precisely this type have been used previously in studies of arrival times, as phenomenological devices, to imitate absorbing boundary conditions (see, for example Refs. [2, 54, 57]). Here, the appearance of a complex potential is derived from the master equation of a particle coupled to an irreversible detector, which in turn may be derived from the unitary dynamics of the combined particle–detector–environment system.

In summary, this detector model nicely reproduces earlier phenomenological results on arrival times. In Ref. [55] it is also shown that the expression (69), (70), is very closely related to the “ideal” arrival time distribution of Kijowski [44]. An improved more physically realistic irreversible detector model (although more difficult to solve analytically) was recently put forward by Muga et al. [53].

6 A Comparison of the Decoherent Histories Result with the Detector Result

We may now compare the two candidate expressions for the crossing time probabilities, one from decoherent histories with an environment, the other from an
irreversible detector model. We will quickly see that the two results are not in fact very close, but it is perhaps of interest to see exactly why, and how they may be improved.

We first massage the decoherent histories result into a more suitable form. Consider the probability for remaining in \( x > 0 \). From (50) it is given by
\[
P_r = \int_r Dx(t) \int_r Dy(t) \exp \left( \frac{i}{\hbar} S[x(t)] - \frac{i}{\hbar} S[y(t)] \right) \times \exp \left( -a \int dt \ (x - y)^2 \right) \rho_0(x_0, y_0) \tag{71}
\]
where \( a = D/\hbar^2 \). Following Ref. [32], we make the observation that the last exponential may be deconvolved:
\[
\exp \left( -a \int dt \ (x - y)^2 \right) = \int D\bar{x} \exp \left( -2a \int dt \ (x - \bar{x})^2 - 2a \int dt \ (y - \bar{x})^2 \right) \tag{72}
\]
Hence, assuming a pure initial state, the probability (71) may be written,
\[
P_r = \int D\bar{x}(t) \int_r Dx(t) \exp \left( \frac{i}{\hbar} S[x(t)] - 2a \int dt \ (x - \bar{x})^2 \right) \Psi_0(x_0) \times \int D\bar{y}(t) \exp \left( -\frac{i}{\hbar} S[y(t)] - 2a \int dt \ (y - \bar{x})^2 \right) \Psi_0^*(y_0) \tag{73}
\]
In these integrals, \( \bar{x}(t) \) is integrated over an infinite range, but \( x(t) \) and \( y(t) \) are integrated only over the positive real line. This restriction is quite difficult to implement in practice [32]. However, because of the exponential factors, negative values of \( \bar{x}(t) \) are strongly suppressed, so we may take its range to be over positive values only, with exponentially small error. Furthermore, having done this we may then (for technical simplicity) allow the range of \( x(t) \) and \( y(t) \) to be over the entire real line, again with exponentially small error. Therefore, we have that
\[
P_r \approx \int_r D\bar{x}(t) \int Dx(t) \exp \left( \frac{i}{\hbar} S[x(t)] - 2a \int dt \ (x - \bar{x})^2 \right) \Psi_0(x_0) \times \int D\bar{y}(t) \exp \left( -\frac{i}{\hbar} S[y(t)] - 2a \int dt \ (y - \bar{x})^2 \right) \Psi_0^*(y_0) \tag{74}
\]
This may finally be written,
\[
P_r \approx \int_r D\bar{x}(t) \langle \Psi_{\bar{x}} | \Psi_{\bar{x}} \rangle \tag{75}
\]
where
\[
\Psi_{\bar{x}}(x_f, \tau) = \int Dx(t) \exp \left( \frac{i}{\hbar} S[x(t)] - 2a \int dt \ (x - \bar{x})^2 \right) \Psi_0(x_0) \tag{76}
\]
Written in this way the probability has a natural interpretation in terms of continuous quantum measurement. Eq.(76) is the wave function for a system undergoing continuous measurement of its position along a trajectory $\vec{x}(t)$ to within a precision proportional to $a^{-\frac{1}{2}}$. The probability for any such trajectory is $\langle \Psi_\vec{x} | \Psi_\vec{x} \rangle$, hence the probability to remain in the region $x > 0$ is obtained by integrating over $\vec{x}(t) > 0$. The probability (71), derived from the decoherent histories approach, is therefore, to an excellent approximation, the same as the result naturally obtained from continuous quantum measurement theory.

Now we compare with the detector model. The probability for no detection is computed from the wave function (69). In a path integral representation, this may be written,

$$\Psi_{nd}(x, \tau) = \int \mathcal{D}x(t) \exp \left( \frac{i}{\hbar} S[x(t)] - \frac{\gamma}{2} \int_0^\tau dt V(x(t)) \right) \Psi_0(x_0)$$

(77)

The sum is over all paths $x(t)$ connecting $x_0$ at $t = 0$ to $x_f$ at $t = \tau$. The probability for no detection is then quite simply

$$p_{nd} = \langle \Psi_{nd} | \Psi_{nd} \rangle$$

(78)

Whilst the two different expressions, (75), (76), versus (77), (78) are similar in some ways, they are not obviously close and suffer from a rather key difference. Eq.(73) is obtained by summing the probability for any path $\vec{x}(t)$ over positive values of $\vec{x}$. In Eqs.(77), (78), by contrast, the restriction to paths in $x > 0$ is already imposed in the amplitude. The difference between the probabilities provided by the detector and those provided by the decoherent histories approach is, therefore, the difference between summing amplitudes and squaring, versus squaring and then summing.

In the decoherent histories approach, the coupling to the environment produces an effective measurement of the system that is much finer than is required for the crossing time problem. It effectively measures the entire trajectory, which is clearly much more information than is required to determine whether or not the particle enters $x < 0$. In this sense this particular decoherent histories model is much cruder than the detector model, since it destroys far more interference than it really needs to in order to define the crossing time. This is due to the form of the particle-environment coupling which is linear in the particle’s position. It would be of interest to explore a decoherent histories model with a more refined type of coupling which is more specifically geared to the crossing time problem.

It is of interest to note that continuous quantum measurement theory in fact suggests another candidate expression for the probability of not crossing which is closer to the detector model. Suppose that before squaring, we sum the amplitude (76) over positive $\vec{x}(t)$:

$$\Psi_+(x_f, \tau) = \int \mathcal{D}x(t) \exp \left( \frac{i}{\hbar} S[x(t)] \right)$$
The probability is then

\[ p_+ = \langle \Psi_+ | \Psi_+ \rangle \]  

(80)

This expression for the probability of not entering \( x < 0 \) is completely natural from the point of view of continuous quantum measurement theory. It does not follow from either the detector model or from the decoherent histories approach presented here, but one can regard it as yet another proposal with which to define the arrival time probability. The amplitude (79) is now more closely analogous to the detector result (77). To see this, introduce the effective potential \( V_{\text{eff}}(x) \) defined by

\[
\exp \left( -\int dt \, V_{\text{eff}}(x(t)) \right) = \int \mathcal{D} \bar{x}(t) \left( -2a \int dt \, (x - \bar{x})^2 \right) \]  

(81)

The integral can be evaluated exactly, but it is clear that \( V_{\text{eff}}(x) \sim 0 \) for \( x >> 0 \), and \( V_{\text{eff}}(x) \sim 2ax^2 \) for \( x << 0 \). Eq.(79) therefore has the same general form as (77). The potential is not exactly the same, but has the same physical effect, which is to suppress paths in \( x < 0 \).

7 Timeless Questions in Quantum Theory

We now briefly consider a related question in quantum theory that involves time in a non-trivial way, which is in fact more closely related to the Wheeler-DeWitt equation of quantum cosmology, (1). This equation may be thought of as the statement that the wave function of the system is in an energy eigenstate. As stated in the Introduction, the equation contains no notion of time, and indeed “time” and the notion of trajectories are thought to somehow emerge from the wave function. To test this idea, and hence to provide some sort of interpretation for the wave function, we need to find an answer to the question, “What is the probability associated with a given region \( \Delta \) of configuration space when the system is in an energy eigenstate, without any reference to time?”.

Classically, the question is well-defined. A system with fixed energy consists of a set of classical trajectories, perhaps with some probability distribution on them. The classical trajectories are just curves in configuration space, and the question is then quite simply one of determining whether or not these curves intersect the given region \( \Delta \). But, like the arrival time problem in non-relativistic quantum mechanics, the problem is considerably harder to phrase in quantum theory.

To see the beginnings of the difficulties, we briefly consider the following simple question for a two-dimensional system with coordinates \( x_1, x_2 \): given that the system is in an energy eigenstate, what is the value of \( x_1 \) given the value of \( x_2 \)? Slightly rephrased, what is the probability that the system intersects
the surface $x_2 = \text{constant}$ between $x_1$ and $x_1 + dx_1$, at any time? An operator approach to the problem, for example, takes the following form. For a free particle, the classical trajectories are

$$x_1(t) = x_1 + \frac{p_1 t}{m}, \quad x_2(t) = x_2 + \frac{p_2 t}{m}$$

and we may eliminate $t$ between them to write,

$$x_1(t) = x_1 + \frac{p_1}{p_2} (x_2(t) - x_2)$$

This is the classical answer to the question, what is the value of $x_1$ at a given value of $x_2$? One may attempt to raise this to the status of an operator in the quantum theory. It commutes with the free particle Hamiltonian,

$$H = \frac{1}{2}(p_1^2 + p_2^2)$$

so is in this sense an observable of the theory – measuring it will not displace the system from an energy eigenstate of $H$. This approach encounters problems, however, in defining (83). It cannot be made into a self-adjoint operator, due to the presence of the $1/p_2$ factor. In this way it is very similar to the problem of defining a time operator.

We will not pursue this approach any further here. Instead we briefly report on two other approaches, which, exactly like the approaches described in this article, use decoherent histories, or a detector model.

The decoherent histories approach to the question involves summing over paths in configuration space which either enter or do not enter a given region $\Delta$ at any moment of time. In practice this is achieved by summing over paths which either enter or do not enter during a fixed time interval $[0, \tau]$, and then summing $\tau$ over an infinite range. The detailed construction of this is described in Ref.[30]. As in the crossing time problem described in Section 4, a decohering environment is required to make the probabilities well-defined, and we then expect the final result to be a reasonably simple formula involving the Wigner function, closely analogous to the classical case. The full details of this have yet to be worked out, but is perhaps useful to give here the classical result (which, although well-defined, is not totally trivial).

We consider a $2n$-dimensional phase space with coordinates $p, x$. Denote the classical trajectories by $x^{cl}(t)$, and suppose that they match the initial data $p_0, x_0$ at some fiducial initial point $t = t_0$ (which is arbitrary). For a free particle,

$$x^{cl}(t) = x_0 + \frac{p_0}{m} (t - t_0)$$

Let $f_\Delta(x)$ be a characteristic function for the region $\Delta$ so is 1 inside $\Delta$ and zero outside. We suppose that the classical system is described by a phase space
distribution function $w(p, x)$. To be a true analogue of an energy eigenstate in the quantum case, $w$ has to be stationary, so

$$w(p(t), x(t)) = w(p(t + t_1), x(t + t_1))$$

for any $t_1$.

We may now write down the probability for a classical trajectory entering the region $\Delta$. It is,

$$p_{\Delta} = \int d^n p_0 d^n x_0 \, w(p, x) \, \theta \left( \int_{-\infty}^{\infty} dt \, f_{\Delta}(x^{cl}(t)) - \epsilon \right)$$

(87)

Here, $\epsilon$ is a small parameter which is taken to zero through positive values, and is present to avoid ambiguities in the $\theta$ function at zero argument. The integral inside the $\theta$ function is the total time spent by the trajectory $x^{cl}(t)$ inside the region $\Delta$, but we are only interested in whether this time is positive or zero. The initial data $p_0, x_0$ are therefore effectively integrated only over values for which the trajectory spends a time in excess of $\epsilon$ in the region $\Delta$. It is easy to see that the whole construction is invariant under shifting the fiducial point $t_0$. This is the analogue of reparametrization invariance (or more generally, diffeomorphism invariance) in the Wheeler-DeWitt equation Eq.(1).

It is expected that a decoherent histories analysis will yield a result of the approximate form (87) (with $w$ replaced by the Wigner function).

The other approach to the question posed at the beginning of this section is to use a detector model (this is described in detail in Ref.[28]). The detector model arises from Barbour’s observation [7] that a substantial insight into the Wheeler-DeWitt equation may be found in Mott’s 1929 analysis of alpha-particle tracks in a Wilson cloud chamber [52]. Mott’s paper concerned the question of how the alpha-particle’s outgoing spherical wave state, $e^{ikR}/R$, could lead to straight line tracks in a cloud chamber. His explanation was to model the cloud chamber as a collection of atoms that may be ionized by the passage of the alpha-particle. They therefore act as detectors that measure the alpha-particle’s trajectory. The probability that certain atoms are ionized is indeed found to be strongly peaked when the atoms lie along a straight line through the point of origin of the alpha-particle.

Mott had in mind a time-evolving process, but he actually solved the time-independent equation

$$(H_0 + H_d + \lambda H_{int}) |\Psi\rangle = E |\Psi\rangle$$

(88)

Here $H_0$ is the alpha-particle Hamiltonian, $H_d$ is the Hamiltonian for the ionizing atoms, and $H_{int}$ describes the Coulomb interaction between the alpha-particle and the ionizing atoms (where $\lambda$ is a small coupling constant). Now the interesting point, as Barbour notes, is that Mott derived all the physics from this equation with little reference to time. Mott’s calculation is therefore an
excellent model for many aspects of the Wheeler-DeWitt equation. In Ref. [28] a model of this type is considered with a series of detectors, and it is shown how to produce a plausible formula for the probability that the system enters a series of regions in configuration space without reference to time. A comparison of this approach with the anticipated decoherent histories result (87) is yet to be carried out.

8 Discussion

We have reviewed a number of approaches to the crossing time problem in non-relativistic quantum theory, primarily using the decoherent histories approach. We have also briefly reviewed some attempts to extend these ideas to models more closely related to the Wheeler-DeWitt equation. On the face of it, the decoherent histories approach appears to be particularly well adapted to this problem, since it naturally incorporates the notion of trajectory, and hence readily accommodates questions of a non-trivial temporal nature. Having said that, however, good expressions for the crossing time probability are not acquired very easily.

As described in Section 3, the decoherence or consistency conditions are satisfied only for very special classes of initial states. For a system consisting of a single point particle, therefore, the decoherent histories approach does not supply an answer to the crossing time problem for arbitrary initial states. This is rectified by the inclusion of a thermal environment, as described in Section 4, and probabilities for the crossing time can then be obtained for arbitrary initial system states. They do, however, depend to some extent on the environment producing the decoherence, and moreover, they are essentially the same as the classical stochastic results. One might therefore criticize this result on the grounds that it is "not very quantum". This is largely true, but the essential achievement of Section 4 is to show that the decoherent histories approach can be made to give the anticipated classical result. This was not true of the earlier approaches reviewed in Section 3.

In Section 5, a detector model was introduced to give an alternative expression for the crossing time probability, for the purposes of comparison with the decoherent histories result. The detector model gave a better result, in that it agreed and substantiated an earlier result of Allcock [2], which in turn is closely related to the ideal distribution of Kijowski [44].

On comparison with the decoherent histories result, in Section 6, it was easy to see that the environment in Section 4 produced far more decoherence than is necessary to define the arrival time, and in that sense, that particular environment is a very crude model for the measurement of time. The comparison did, however, inspire the proposal of a third candidate expression from which the arrival time probability could be calculated, namely Eq. (79), which is based on continuous quantum measurement theory. This expression does not seem to
have been considered previously and will be explored in more detail elsewhere. One might be led from these results to a somewhat negative assessment of the decoherent histories approach’s ability to provide the crossing time probability. The somewhat crude nature of the results of Section 4, are however, due to the choice of a rather indiscriminate system-environment coupling, which effectively measures the entire trajectory. It seems likely that a much-improved result could be obtained through choice of a more refined coupling better suited to this particular problem.

Furthermore, there is another aspect to the decoherent histories approach in this context which has not yet been explored. Many approaches to the arrival time problem are based on model measuring devices, that is, physical systems in which one of the dynamical variables is correlated with time in some way. The detector model of Section 5 was of this type: one could think of the two-state system as being some kind of clock or detector attached to the particle, which switches on when the particle enters the region \( x < 0 \). By physically measuring the two-state system at the end of the time interval \([0, \tau]\) of interest, one expects to be able to deduce that the particle was in \( x < 0 \), or not, during the time interval. The outstanding question, however, is this: how do we really know that the detector state is correlated with whether or not the particle entered \( x < 0 \)?

This is where the decoherent histories approach comes in. We consider a system consisting of the particle and a detector (and possibly also an environment, if necessary). We then look at histories in which both the final state of the detector and the particle alternatives (whether or not it entered \( x < 0 \) during \([0, \tau]\)) are specified. If these histories are decoherent, we then obtain a joint probability distribution for the histories of the particle and the final state of the detector, and we can ask to what degree these two things are correlated. If they are perfectly correlated, then the detector probability is exactly the same as the probability of the detector and the particle alternatives.

In brief, therefore, the decoherent histories approach will be a useful tool in assessing the extent to which a proposed detector really does its job. Many model detectors are proposed essentially on the basis of classical arguments, but the decoherent histories approach allows their effectiveness to be checked in a genuinely quantum way. This possibility does not appear to have been explored in the context of arrival times, but will be considered elsewhere.

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