SMALL PRIME POWERS IN THE FIBONACCI SEQUENCE

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Abstract. It is shown that there are no non-trivial fifth-, seventh-, eleventh-, thirteenth- or seventeenth powers in the Fibonacci sequence.

For eleventh, thirteenth- and seventeenth powers an alternative (to the usual exhaustive check of products of powers of fundamental units) method is used to overcome the problem of having a large number of independent units and relatively high bounds on their exponents.

It is envisaged that the same method can be used to decide the question of the existence of higher small prime powers in the Fibonacci sequence and that the method can be applied to other binary recurrence sequences. The alternative method mentioned may have wider applications.

1. Introduction

The Fibonacci sequence \( \{F_n\}_{n=0}^{\infty} \) is defined by setting \( F_0 = 0 \), \( F_1 = 1 \) and, for \( n \geq 2 \), by setting \( F_n = F_{n-1} + F_{n-2} \).

Cohn [1] and Wylie [13] proved independently, by elementary means, that the only squares in the Fibonacci sequence are \( F_0 = 0, F_1 = F_2 = 1 \) and \( F_{12} = 144 \).

In [7], London and Finkelstein used previous results on solutions to two diophantine equations to show that the only cubes in the Fibonacci sequence are \( F_0 = 0, F_1 = F_2 = 1 \) and \( F_6 = 8 \). In [6], Lagarias and Weisser gave a complete determination of all Fibonacci numbers of the form \( 2^a 3^b \).

In [10], Pethő used linear forms in logarithms together with a computer search using congruence considerations to give an alternative proof of London and Finkelstein’s result.

Pethő ([9]) and Shorey and Stewart ([11]) proved independently that there are only finitely many perfect powers in any non-trivial binary recurrence sequence. In [8] Pethő states that if \( F_m = x^q \), for some positive integers \( x \) and \( q \), then \( q < 10^{98} \). In the same paper he also states that he used the same method that he used in [10] to show that the only fifth powers in the Fibonacci sequence are \( F_0 = 0 \) and \( F_1 = F_2 = 1 \).

In this paper the method outlined by Pethő in [10] is used as a starting point and then linear forms in logarithms together with the LLL algorithm...
are used to reprove the result for fifth powers and to prove that the only seventh-, eleventh-, thirteenth- or seventeenth powers in the Fibonacci sequence are \( F_0 = 0 \) and \( F_1 = F_2 = 1 \). An alternative method (to the usual exhaustive check of products of powers of fundamental units) is used to complete the search in the case of eleventh, thirteenth- and seventeenth powers.

It is envisaged that the same method can be used to decide the question of the existence of higher small prime powers in the Fibonacci sequence and that the method can be applied to search for prime powers in other binary recurrence sequences. The alternative method mentioned above may have wider applications.

Computations were performed using \textit{Magma}, \textit{Mathematica} and \textit{Pari-gp} and were carried out on Sun ultra 5- and Sun ultra 10 computers.

2. Elementary Considerations

It is easy to show that

\[
F_{m+1}^2 - F_{m+1}F_m - F_m^2 = (-1)^m.
\]

Pethő, in \cite{14}, gives the following lemma:

\textbf{Lemma 1.} Let \( q \geq 3 \) be a positive integer. If \( F_m = x^q \), for some positive integer \( x \), then \( m = 0, 1, 2, 6 \) or \( \exists \) a prime \( p \mid m \), such that \( F_p = x_1^q \), for some positive integer \( x_1 \).

Hence it can be assumed that \( m \) is an odd prime. Suppose \( F_m = x^n \) and \( F_{m+1} = y \), for some positive integers \( x, y \) and some prime \( n \geq 5 \). Then \( y^2 - yx^n - x^{2n} + 1 = 0 \), and regarding this equation as a quadratic in \( y \) and looking at its discriminant, it follows that \( 5x^{2n} - 4 = z^2 \), for some positive integer \( z \). Clearly \( z = 5v \pm 1 \), for some positive integer \( v \) and thus that

\[
(2.1) \quad x^{2n} = (2v)^2 + (v \pm 1)^2.
\]

It is clear that \((x, v) = 1\) and it is not difficult to show that \( x \) must be odd and \( v \) must be even. Looking at (2.1) in \( \mathbb{Z}[i] \), where \( i = \sqrt{-1} \), it can further be shown that \( v \pm 1 + 2iv \) has to be an \( n \)th power and that there exists integers \( A \) and \( B_1 \) such that

\[
(2.2) \quad v \pm 1 + 2iv = (A + B_1i)^n \Rightarrow x^2 = A^2 + B_1^2.
\]

\( A \) has to be odd and \( B_1 \) has to be even (= \( 2B \), say). Suppose \( n = 2m + 1 \). Comparing real and imaginary parts of the first equation in (2.2), it follows
that

\[ \pm 1 = B^n \sum_{j=0}^{m} (-1)^j 2^{2j} \left( \binom{n}{2j} \left( \frac{A}{B} \right)^{n-2j} - \binom{n}{2j+1} \left( \frac{A}{B} \right)^{n-2j-1} \right). \]

Let

\[ f_n(x) = \sum_{j=0}^{m} (-1)^j 2^{2j} \left( \binom{n}{2j} x^{n-2j} - \binom{n}{2j+1} x^{n-2j-1} \right). \]

Assume \( f_n(x) \) is irreducible (which can easily be proved for the cases examined: \( n = 5, 7, 11, 13 \) and 17) and let the roots be denoted \( \theta_1, \ldots, \theta_n \). Let \( \theta_i \) denote anyone of these roots. From (2.3) and (2.4) we have that

\[ \pm 1 = \prod_{k=1}^{n} (A - \theta_k B) = N_{K/Q}(A - \theta_1 B). \]

Hence \( A - \theta_1 B \) is a unit in \( \mathbb{Q}(\theta_i) \). For the cases examined it turns out that \( f_n(x) \) has all its roots real and that the rank of the group of units is \( n - 1 \).

Denote a set of fundamental units in \( \mathbb{Q}(\theta_i) \) by \( \epsilon_1^{(i)}, \ldots, \epsilon_{n-1}^{(i)} \) and let \( \beta_i := A - \theta_1 B \). Then there exists integers \( u_1, \ldots, u_{n-1} \) such that

\[ \beta_i = \pm \prod_{k=1}^{n-1} \epsilon_k^{(i)u_k}. \]

Let \( U = \max_{1 \leq k \leq n-1} |u_k| \). We next find an initial bound on \( U \).

3. Finding an initial bound on the exponents of the fundamental units

Suppose \( j \) is such that \( |\beta_j| = \min_{1 \leq i \leq n} |\beta_i| \) and let \( K = \mathbb{Q}(\theta_j) \). Define the following numbers:

\[ c_1 = \min_{i, i \neq j} |\theta_j - \theta_i|, \quad c_2 = \max_{r \neq s \neq t \neq r} \left| \frac{\theta_i - \theta_r}{\theta_t - \theta_s} \right|, \]

\[ c_{2a} = \max_{(s,t), j \neq s \neq t \neq j} \left| \frac{\theta_j - \theta_s}{\theta_j - \theta_t} \right|, \quad c_3 = \max_{i, i \neq j} |\theta_i - \theta_j|. \]

It is also assumed that

\[ |B| \geq \max \left\{ 4, \sqrt{2c_2^2 - \frac{2}{c_1}} \right\}. \]

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1It may be easy to show that \( f_n(x) \) is irreducible for all primes \( n \) and, using Sturm’s Theorem, that \( f_n(x) \) has all real roots for every prime \( n \), in which case the rank of the unit group is \( n - 1 \) for all prime \( n \). However, this is not examined here.
For \( i \neq j \), we have that

\[
|B||\theta_i - \theta_j| \leq |\beta_j| + |\beta_i| \leq 2|\beta_i| \implies \frac{|B|c_1}{2} \leq |\beta_i|
\]

This implies that

\[
|\beta_j| = \frac{1}{\prod_{i=1}^{n} \beta_i} \leq \left( \frac{2}{|B|c_1} \right)^{n-1}.
\]

Similarly, it follows from (3.2) that

\[
|\beta_i| \leq |B||\theta_i - \theta_j| + |\beta_j| \leq |B| \left( c_3 + \frac{c_1}{4c_2} \right) = c_4|B|,
\]

where \( c_4 = c_3 + c_1/(4c_2) \). Using (3.3), (3.2) and (3.5), it follows that

\[
|\log |\beta_i|| \leq c_5 \log |B|,
\]

where \( c_5 = \max \{ 1 + |\log(c_1/2)|/\log 4, 1 + |\log c_4|/\log 4 \} \).

Let \( \{ i_k : k = 1, \cdots, n-1 \} \) denote the set \( \{ 1, 2, \cdots, n \} \setminus \{ j \} \). From (2.5),

\[
\begin{pmatrix}
\log |\beta_{i_1}| \\
\vdots \\
\log |\beta_{i_{n-1}}|
\end{pmatrix} =
\begin{pmatrix}
\log |\epsilon_{i_1}^{(i_1)}| & \cdots & \log |\epsilon_{i_{n-1}}^{(i_{n-1})}|
\end{pmatrix}
\begin{pmatrix}
u_1 \\
v_2 \\
\vdots \\
v_{n-1}
\end{pmatrix}
= M
\begin{pmatrix}
u_1 \\
v_2 \\
\vdots \\
v_{n-1}
\end{pmatrix},
\]

\[
\implies
\begin{pmatrix}
u_1 \\
v_2 \\
\vdots \\
v_{n-1}
\end{pmatrix} = M^{-1}
\begin{pmatrix}
\log |\beta_{i_1}| \\
\vdots \\
\log |\beta_{i_{n-1}}|
\end{pmatrix}.
\]

Suppose \( M^{-1} := (m_{r,s}) \), \( 1 \leq r \leq n-1, 1 \leq s \leq n-1 \). Then

\[
U \leq \max_{1 \leq r \leq n-1} \{|m_{r,1}| + \cdots + |m_{r,n-1}|\} \times \max_{1 \leq t \leq n-1} \{|\log |\beta_{i_t}||| \leq c_6 \log |B|,
\]
where \( c_6 = \max_{1 \leq r \leq n-1} \{ |m_{r,1}| + \cdots + |m_{r,n-1}| \} \times c_5 \). Thus

\[
(3.7) \quad \exp \left( \frac{U}{c_6} \right) \leq |B|.
\]

Let

\[
(3.8) \quad \Lambda = \log \left| \frac{\theta_j - \theta_k}{\theta_j - \theta_l} \right| + \sum_{r=1}^{n-1} u_r \log \left| \frac{e_r^{(l)}}{e_r^{(k)}} \right| = \log \left| \frac{\theta_j - \theta_k}{\theta_j - \theta_l} \frac{A - \theta_l B}{A - \theta_k B} \right|.
\]

By Siegel’s identity,

\[
(3.9) \quad \left| \frac{\theta_l - \theta_k}{\theta_l - \theta_j} \frac{A - \theta_j B}{A - \theta_k B} \right| = \left| \frac{\theta_j - \theta_k}{\theta_j - \theta_l} \frac{A - \theta_l B}{A - \theta_k B} - 1 \right|.
\]

From the definition of the \( \beta_i \)'s, (3.1), (3.3) and (3.4), it follows that

\[
(3.10) \quad \left| \frac{\theta_l - \theta_k}{\theta_l - \theta_j} \frac{A - \theta_j B}{A - \theta_k B} \right| \leq c_2 \left( \frac{2}{|B|c_1} \right)^n < \frac{1}{2}.
\]

From (3.9) and (3.10), it follows that

\[
(3.11) \quad \frac{\theta_j - \theta_k}{\theta_j - \theta_l} \frac{A - \theta_l B}{A - \theta_k B} > \frac{1}{2} > 0
\]

\[
(3.12) \quad \Rightarrow \Lambda = \log \left| \frac{\theta_j - \theta_k}{\theta_j - \theta_l} \frac{A - \theta_l B}{A - \theta_k B} \right|
\]

\[
(3.13) \quad \Rightarrow |\Lambda| < 2c_2 \left( \frac{2}{|B|c_1} \right)^n.
\]

The last inequality follows, in the case \( \Lambda > 0 \), from (3.10), (3.9) and the fact that \( e^x - 1 > x \), for \( x > 0 \). In the case \( \Lambda < 0 \), we also use (3.11). Note that \( \Lambda \neq 0 \), or else the right side of (3.9) is zero, implying, on the left side of (3.9), that either \( \theta_j \) is rational, or \( \theta_l = \theta_k \). However, both of these are impossible, since \( \theta_j \) is algebraic of degree \( n \), and \( f_n(x) \) has distinct roots.

Combining this last inequality for \( |\Lambda| \) with (3.7) it follows that

\[
(3.14) \quad \frac{c_1^2}{2^{n+1}c_2} \exp \left( \frac{nU}{c_6} \right) < \frac{1}{|\Lambda|}.
\]

Next, the following theorem of Baker and Wüstholz (2) gives an upper bound on \( 1/|\Lambda| \):
Theorem 1. Denote by $\alpha_1, \ldots, \alpha_n$ algebraic numbers, not 0 or 1, by $\log \alpha_1, \ldots, \log \alpha_n$ determinations of their logarithms, by $d$ the degree over $\mathbb{Q}$ of the number field $Q(\alpha_1, \ldots, \alpha_n)$ and by $b_1, \ldots, b_n$ rational integers, not all 0 and let $B = \max \{ |b_1|, \ldots, |b_n|, e^{1/d} \}$.

Define $\log A_i = \max \{ h(\alpha_i), (1/d) \log |\alpha_i|, 1/d \}$ $(1 \leq i \leq n)$, where $h(\alpha)$ denotes the absolute logarithmic Weil height of $\alpha$. Assuming the number $\Lambda = b_1 \log \alpha_1 + \cdots + b_n \log \alpha_n$ does not vanish, then

$$|\Lambda| \geq \exp \{ -C(n, d) \log A_1 \cdots \log A_n \log B \},$$

where $C(n, d) = 18(n+1)n^{n+1}(32d)^{n+2}\log(2nd)$.

Here

$$h(\alpha) = \frac{1}{[Q(\alpha) : \mathbb{Q}]} \log \left| a_0 \prod_{r=1}^{s} \max \{ 1, |\alpha^i| \} \right|,$$

where the minimal polynomial of $\alpha$ has leading coefficient $a_0$ and $\alpha = \alpha^1, \ldots, \alpha^s$ are the conjugates of $\alpha$.

In our application, it can be seen from (3.8) that $n$ has the same meaning as previously, that $b_1 = 1$, that $\alpha_1 = |(\theta_j - \theta_k)/(\theta_j - \theta_l)|$, and, for $j = 2, \ldots, n$, that $b_j = u_{j-1}$ and $\alpha_j = |\epsilon_j^{(l)} / \epsilon_j^{(k)}|$.

Let $\gamma_i = \epsilon_i^{(l)} / \epsilon_i^{(k)}$, with conjugates $\gamma_i = \gamma_1^1, \ldots, \gamma_1^n$. Since $\gamma_i$ is a unit its minimal polynomial has its leading coefficient $a_0 = 1$ and since $f_n(x)$ has all real roots, $|\gamma_i^r| = \pm \gamma_i^r$.

$$|\gamma_i^r| \leq \eta_i := \frac{\max \{ |\epsilon_i^{(r)}| : 1 \leq r \leq n \} }{\min \{ |\epsilon_i^{(r)}| : 1 \leq r \leq n \} } \implies h(\gamma_i) \leq \log \eta_i.$$

Let $\delta_{jkl} = (\theta_j - \theta_k)/(\theta_j - \theta_l)$ and suppose the minimum polynomial of $\delta_{jkl}$ is $g(x)$. The conjugates of $\delta_{jkl}$ are bounded by $c_{2n}$. For small primes $n$ the Galois group associated to the polynomial $f_n(x)$ can be determined using a computer Algebra system like Magma. Let $p(x) = \prod (x - \delta_{rst})(\theta_r - \theta_t)$ where the product is taken over all conjugates $\delta_{rst}$ of $\delta_{jkl}$. $p(x) \in \mathbb{Z}[x]$ and $g(x) \mid p(x)$. For low values of $n$, $p(x)$ can be calculated numerically and $g(x)$ and thus $d_g = \deg(g(x))$ and $a_0$ can be determined explicitly. In fact, for the cases examined, $d_g = n(n-1)$ and $a_0 = 2^{2(n-1)}$. Then

$$h(\delta_{jkl}) \leq \log 2^{2(n-1)} \cdot n(n-1) + \log c_{2n}.$$

$$\left[ \mathbb{Q} \left( \delta_{jkl}, \epsilon_1^{(l)} / \epsilon_1^{(k)}, \ldots, \epsilon_i^{(l)} / \epsilon_i^{(k)} \right) : \mathbb{Q} \right] \leq \left[ \mathbb{Q}(\theta_1, \ldots, \theta_n) : \mathbb{Q} \right] =: D.$$

For the values of $n$ examined, $D = n(n-1)$. 
Let $C(n, D)$ be as defined in the theorem. Then
\[
|\Lambda| > \exp \left( -C(n, D) \prod_i \log \eta_i \left( \frac{\log 2(2^{(n-1)})}{n(n-1)} + \log c_{2a} \right) \log U \right)
= \exp(-c_7 \log U) = U^{-c_7},
\]
where $c_7 = C(n, D) \prod_i \log \eta_i (\frac{\log 2(2^{(n-1)})}{n(n-1)} + \log c_{2a})$. Combining this inequality with (3.14) it follows that
\[
\frac{c_1^n}{2n+1c_2} \exp(nU/c_6) < U^{c_7}.
\]
If it is assumed that $|U| \geq 4$ then
\[
\frac{U}{\log U} \leq \frac{c_0c_7}{n} + \frac{c_6 \left| \log \frac{c_1^n}{2n+1c_2} \right|}{n \log 4}.
\]
Thus an upper bound can be found for $U$. Denote this upper bound by $K_3$. This bound is generally too large to enable the remaining cases to be tested so it is next reduced, using the LLL algorithm.

4. Reducing the bound

To reduce the bound a version of the LLL algorithm is applied, as outlined in the paper of Tzanakis and De Weger [12]. Using the notation of their paper:

Let
\[
\Lambda = \delta + a_1\mu_1 + \cdots + a_q\mu_q,
\]
where the $a_i$’s are integers and the $\mu$’s and $\delta$ are real, with $\delta \neq 0$. Let
\[
A = \max_{1 \leq i \leq q} |a_i|. \ K_1, K_2 \text{ and } K_3 \text{ are positive numbers satisfying}
\]
\[
|\Lambda| < K_1 \exp(-K_2 A), \ A < K_3.
\]
Choose $c_0 = \sigma_1 K_3^q$ where $\sigma_1 > 1$. Consider the lattice $\Gamma$ associated with the matrix
\[
A = \begin{pmatrix}
1 & 0 & \cdots & 0 & 0 \\
0 & 1 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 1 & 0 \\
[c_0\mu_1] & [c_0\mu_2] & \cdots & [c_0\mu_{q-1}] & [c_0\mu_q]
\end{pmatrix}
\]
Find a reduced basis $b_1, \ldots, b_q$ for this basis and let $\mathcal{B}$ be the matrix associated with this basis.

Let $x = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \in \mathbb{Z}^q$ and let $s = \begin{pmatrix} s_1 \\ \vdots \\ s_{q-1} \\ s_q \end{pmatrix} = \mathcal{B}^{-1}x$. 

Let $|y|$ denote the distance from $y$ to the nearest integer.

**Proposition 1.** Let $i^* = \max\{i : 1 \leq i \leq q \text{ and } s_i \notin \mathbb{Z}\}$. If

$$2^{-(q-1)/2}||s_{i^*}||b_1| \geq \sqrt{\left(4q^2 + 3q - \frac{3}{4}\right)K_3},$$

then every solution of (4.1) satisfying (4.2) satisfies

$$A < \frac{1}{K_2} \log \left(\frac{c_0K_1}{qK_3}\right).$$

From (3.14), this proposition can be applied with $a_r = u_r$, for $1 \leq r \leq n - 1$, and

$q = n - 1, \quad A = U, \quad K_1 = \frac{2^{n+1}c_2}{c_1^n},$

$$K_2 = \frac{n}{c_6}, \quad \delta = \log \left|\frac{\theta_j - \theta_k}{\theta_j - \theta_l}\right|, \quad \mu_r = \log \left|\frac{\epsilon_r^{(l)}}{\epsilon_r^{(k)}}\right|.$$

Once a new lower bound is found the proposition is then applied again by now setting $K_3 = 1/K_2 \log (c_0K_1/(qK_3))$ and this is repeated until the bound is reduced as far as possible.

### 5. Completing the Search

Once $K_3$ has been reduced as much as the LLL algorithm will allow, a computer search is done of products of powers of the fundamental units, with these powers bounded by this final value of $K_3$, as on the right hand side of (2.5), to see if any of these products have the form of the left side of (2.5). However, see later for the cases $n = 11, 13$ and 17.

Remark: These calculations have to be carried out for each value of $j$ – in other words all of the $c_i$’s, apart from $c_2$ are dependent on the choice of $j$.

These theoretical results are now applied for $n = 5, 7, 11, 13$ and 17.

### 6. Small prime powers in the Fibonacci sequence

Remark: As noted above, most of the $c_i$ depend on the choice of $j$ and hence are given as vectors (the first component being the value got by letting $j = 1$ and so on). For a fixed $j$, $k$ is chosen to be $j + 1 \mod n$ and $l$ is chosen to be $j + 2 \mod n$. In what follows, $\theta$ denotes a root of $f_n(x)$.

1) The case $n = 5$: $D = 20$ and

$$f_5(x) = -16 + 80x + 40x^2 - 40x^3 - 5x^4 + x^5.$$

The zeroes of $f_5(x)$ are

$$\{-4.64105, -1.1869, 0.185992, 1.75785, 8.88411\}.$$

A set of fundamental units in $\mathbb{Q}(\theta)$ is

$$\left\{1/16\theta^3 + 3/16\theta^2 - 1/4\theta - 1/4, \right\}$$
The zeroes of fifth powers in the Fibonacci sequence other than the trivial ones.

Initially, \( K_2 = 14 \), and \( c_1 = \{3.4541, 1.3728, 1.3728, 1.5718, 7.1262\} \),
\( c_2 = 7.3356 \),
\( c_{2a} = \{3.9156, 7.3356, 6.3356, 4.5336, 1.8979\} \),
\( c_3 = \{13.5251, 10.0710, 8.6981, 7.1262, 13.5251\} \),
\( c_5 = \{2.8850, 2.6694, 2.5642, 2.4219, 2.8916\} \),
\( c_6 = \{1.8086, 1.6734, 1.4252, 1.3461, 1.4617\} \),
\( c_7 = \{1.7353 \times 10^{32}, 2.3987 \times 10^{32}, 2.2438 \times 10^{32}, 1.89018 \times 10^{32}, 9.70057 \times 10^{31}\} \).

Initially, \( K_3 = \{10^{34}, 10^{34}, 10^{34}, 10^{34}, 10^{34}\} \), and eventually \( K_3 = \{11, 12, 10, 10, 8\} \). Finally, a check on products of powers of fundamental units, with the power being bounded in absolute value by 12 shows that there are no fifth powers in the Fibonacci sequence other than the trivial ones.

2) The case \( n = 7 \): \( D = 42 \) and
\( f_7(x) = 64 - 448x - 336x^2 + 560x^3 + 140x^4 - 84x^5 - 7x^6 + x^7. \)
The zeroes of \( f_7(x) \) are
\( \{-6.68663, -2.19286, -0.804777, 0.132665, 1.13197, 2.88015, 12.5395\} \).

A set of fundamental units for \( \mathbb{Q}(\theta) \) is
\( \{5/512\theta^6 - 33/512\theta^5 - 211/256\theta^4 + 59/64\theta^3 + 31/8\theta^2 - 73/32\theta + 5/16, 1/512\theta^6 - 7/512\theta^5 - 11/64\theta^4 + 25/64\theta^3 + 27/32\theta^2 - 39/32\theta + 1/4, 3/512\theta^6 - 37/512\theta^5 - 13/64\theta^4 + 177/64\theta^3 - 171/32\theta^2 + 87/32\theta + 1/4, 5/256\theta^6 - 7/64\theta^5 - 433/256\theta^4 - 9/32\theta^3 + 69/32\theta^2 + 9/8\theta - 1/16, 7/256\theta^6 - 33/256\theta^5 - 81/32\theta^4 - 75/32\theta^3 + 87/16\theta + 11/16, 1/256\theta^6 - 7/256\theta^5 - 5/16\theta^4 + 11/16\theta^3 + 39/16\theta^2 - 21/16\theta - 2\} \).

\( c_1 = \{4.49377, 1.38808, 0.937441, 0.937441, 0.999309, 1.74818, 9.65932\} \),
\( c_2 = 14.2348 \),
\( c_{2a} = \{4.27839, 10.6135, 14.2348, 13.2348, 11.4154, 5.52537, 1.99042\} \),
Fibonacci sequence other than the trivial ones. The following code produces a set of fundamental units:

\[
\begin{align*}
K_1 &= \{0.0984719, 367.017, 5727.71, 5727.71, 3661.77, 73.0283, 0.000464476\}, \\
K_2 &= \{3.41253, 3.27862, 3.36051, 3.42691, 3.78542, 4.33539, 3.33104\}
\end{align*}
\]

\(K_3\) is initially \(\{10^{10}, 10^{19}, 10^{19}, 10^{10}, 10^{19}, 10^{19}\}\) and eventually \(\{16, 17, 17, 18, 17, 15, 16\}\). A check on products of powers of fundamental units, with the power being bounded in absolute value by 18 produces no such products which are linear in \(\theta\) and thus shows that there are no seventh powers in the Fibonacci sequence other than the trivial ones.

Remark: A set of fundamental units for the cases \(n = 11, 13\) and 17 are not included here because they are so large. However, they can be found in Appendix I. They can also be easily generated in GP/PARI. For \(n = 11\), the following code produces a set of fundamental units:

\[
\begin{align*}
\{ f = Pol(1024 - 11264x - 14080x^2 + 42240x^3 + 21120x^4 \\
- 29568x^5 - 7392x^6 + 5280x^7 + 660x^8 - 220x^9 - 11x^{10} + x^{11}, x) ; \\
bnfinit(f, 1)[8][5]\}
\end{align*}
\]

\[\text{Guillaume Hanrot pointed out to me that my use of the GP/PARI bnfinit command in a preprint version of this paper (where I used it without the flag "1") did not necessarily produce a system of fundamental units. Thus it is necessary to a little more to show, for each } n \text{ in question, that the system of units I used, } \{\epsilon_i\}_{i=1}^{n-1} \text{, is indeed a fundamental system. Let the system of fundamental units generated by } bnfinit(f_n, 1) \text{ in GP/PARI, version 2.10, be denoted by } \{\alpha_i\}_{i=1}^{n-1}. \text{ For the cases } n = 5 \text{ and } n = 7, \text{ this set equals } \{\epsilon_i\}_{i=1}^{n-1}. \text{ For } n = 11,\]

\[
\{\epsilon_i\}_{i=1}^{10} = \left\{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \frac{\alpha_2\alpha_6\alpha_7}{\alpha_3\alpha_4}, \alpha_{10}\right\}.
\]

For \(n = 13,\)

\[
\{\epsilon_i\}_{i=1}^{12} = \left\{\alpha_8, \alpha_9, \alpha_2, \alpha_1, \alpha_3, \alpha_4, -\frac{\alpha_3\alpha_4}{\alpha_6}, \alpha_7, \frac{1}{\alpha_{10}}, -\alpha_4\alpha_9, \frac{\alpha_1\alpha_3\alpha_5\alpha_11}{\alpha_3\alpha_4\alpha_7}, \alpha_{12}\right\}.
\]

For \(n = 17,\)

\[
\{\epsilon_i\}_{i=1}^{16} = \left\{\frac{\alpha_7^2\alpha_2\alpha_5\alpha_6^2\alpha_{11}\alpha_{16}}{\alpha_4^4\alpha_5^5\alpha_9^8\alpha_{12}}, \alpha_2, \alpha_4, \alpha_3, \alpha_5, \alpha_6, \alpha_7, \alpha_8, \alpha_9, \alpha_{12}, \right\}.
\]

\[
\left\{\alpha_1\alpha_2\alpha_3\alpha_11, \frac{\alpha_1\alpha_2\alpha_3\alpha_{11}}{\alpha_4}, \alpha_{13}, \alpha_{10}, \alpha_{14}, -\frac{\alpha_2}{\alpha_{10}\alpha_{15}}, \frac{1}{\alpha_1}\right\}.
\]

Each of these sets is clearly also a set of fundamental units.
The case $n = 11$: $D = 110$ and

$$f_{11}(x) = 1024 - 11264x - 14080x^2 + 42240x^3 + 21120x^4 - 29568x^5$$
$$- 7392x^6 + 5280x^7 + 660x^8 - 220x^9 - 11x^{10} + x^{11}.$$  

The zeroes of $f_{11}(x)$ are

$$\{-10.6902, -3.93193, -2.12056, -1.16928, -0.496752, 0.0843495, 0.680024, 1.40783, 2.51488, 4.91791, 19.8037\}.$$ 

$c_1 = \{6.75826, 1.81137, 0.951281, 0.672528, 0.581101, 0.581101, 0.595674, 0.727806, 1.10705, 2.40303, 14.8858\}.$

$c_2 = 34.9345.$

$c_{2a} = \{4.5121, 13.1037, 23.0471, 31.1853, 34.9345, 33.9345, 32.1043, 25.2758, 15.6171, 6.49517, 2.04852\}.$

$c_3 = \{30.4939, 23.7356, 21.9243, 20.973, 20.3005, 19.7194, 19.1237, 18.3959, 17.2888, 15.6081, 30.4939\}.$

$c_5 = \{3.46637, 3.28489, 3.22745, 3.1954, 3.17187, 3.15092, 3.12881, 3.10086, 3.05622, 2.98291, 3.46774\}.$

$c_6 = \{3.03767, 2.38601, 3.34699, 3.39081, 2.34158, 3.44318, 3.15266, 3.38847, 2.68879, 2.40831, 3.06079\}.$

$c_7 = \{2.8731 \times 10^{78}, 4.7491 \times 10^{78}, 5.7427 \times 10^{78}, 6.2748 \times 10^{78}, 6.4746 \times 10^{78}, 6.4235 \times 10^{78}, 6.326 \times 10^{78}, 5.9051 \times 10^{78}, 5.0579 \times 10^{78}, 3.5141 \times 10^{78}, 1.4836 \times 10^{78}\}$

$K_1 = \{0.0001, 207.753, 247867., 1.1241 \times 10^7, 5.6085 \times 10^7, 5.6085 \times 10^7, 4.271 \times 10^7, 4.7146 \times 10^6, 46750.2, 9.2739, 1.7994 \times 10^{-8}\}.$

$K_2 = \{3.6212, 4.61021, 3.28653, 3.24406, 4.69768, 3.19472, 3.48911, 3.2463, 4.09106, 4.56752, 3.59385\}.$

Initially,

$$K_3 = \{10^{81}, 10^{81}, 10^{81}, 10^{81}, 10^{81}, 10^{81}, 10^{81}, 10^{81}, 10^{81}, 10^{81}\},$$

and finally $K_3 = \{32, 27, 41, 43, 29, 47, 40, 43, 32, 26, 28\}.$

Checking all products of powers of ten independent units and with these powers being bounded absolutely by 47 would take some time but there is
a much quicker way, which is now described. The same method is applied for \( n = 13 \) and 17.

Let \( p = \sum_{i=0}^{10} a_i \theta^i, q = \sum_{i=0}^{10} b_i \theta^i \) be two numbers in \( \mathbb{Q}(\theta) \). Then \( p \times q = \sum_{i=0}^{10} c_i \theta^i \), for some \( c_i \in \mathbb{Q} \) and if the \( a_i \) are bounded in absolute value by \( K_a \) and the \( b_i \) are bounded in absolute value by \( K_b \) then a bound for the \( c_i \) can be found in terms of \( K_a \) and \( K_b \). In fact the \( c_i \) are bounded by 16564181057933828\( K_a K_b \). Another way to see this is to regard \( p \) and \( q \) as polynomials in \( \theta \) and reduce their product modulo \( f_{11}(\theta) \). Let \( M = 16564181057933828 \).

Next, pick out the coefficient \( v \) that is largest in absolute value in the following list of powers of the fundamental units: \( \{\epsilon_i^r : 1 \leq r \leq 10, -47 \leq i \leq 47\} \), regarding the members of this set as polynomials in \( \theta \) (This involves checking 950 units rather than 95\(^{10}\)). We have

\[
v = 1/512 \times (20107468130152762104958655475357868066478593
\quad 50616256350654502987105724151326017006680926209502
\quad 49932605099826766485654586806568806547) .
\]

Thus the coefficient of any power of \( \theta \) in any expression of the form \( \prod_{i=1}^{10} \epsilon_i^{r_i} \) where \(-47 \leq i_r \leq 47\) is less than \( M^9 v^{10} \). In particular, \( \beta_i = A + B \theta^i \) must have \( A \) and \( B \) less than \( M^9 v^{10} \) and thus, from (2.2),

\[
x \leq \sqrt{5} M^9 v^{10} \quad \text{and thus} \quad F_m = x^{11} \leq (\sqrt{5} M^9 v^{10})^{11} \leq 7.7943 \times 10^{15864} .
\]

Since \( F_m \leq (1 + \sqrt{5})/2)^{10} \sqrt{5} \Rightarrow m \leq 75913 \). The odd terms in the Fibonacci sequence up to \( F_{75913} \) can be checked directly to see if any are eleventh powers and no non-trivial eleventh powers are found.

More efficiently, one can use the standard trick that if an eleventh power exists, it must be an eleventh power residue modulo every prime. Choose, say, ten primes \( p_1, \ldots, p_{10} \equiv 1 \pmod{11} \) and calculate the eleventh power residues in each case. Using the fact that \( F_{i+3} = 3F_{i+1} - F_{i-1} \) (it being necessary to consider \( F_j \) for \( j \) odd) and working modulo each of the ten primes in parallel, it is a matter of seconds to check up \( j = 75913 \) for eleventh powers (by checking if \( F_j \mod p_i \) is an eleventh power residue for \( p_i \), for each \( i \)).

4) The case \( n = 13 \): \( D = 156 \) and

\[
f_{13}(x) = -4096 + 53248 x + 79872 x^2 - 292864 x^3
- 183040 x^4 + 329472 x^5 + 109824 x^6 - 109824 x^7
- 20592 x^8 + 11440 x^9 + 1144 x^{10} - 312 x^{11} - 13 x^{12} + x^{13} .
\]

The roots of \( f_{13}(x) \) are

\[
\{-12.6754, -4.75486, -2.68723, -1.64838, -0.960337, -0.41792, 0.0713607, 0.569323, 1.14244, 1.90337, 3.13069, 5.90001, 23.4269\} .
\]
\[c_1 = \{7.92054, 2.06763, 1.03885, 0.688043, 0.542417, 0.48928, 0.48928, 0.497963, 0.573112, 0.760936, 1.22732, 2.76932, 17.52696209 \}.\]

\[c_2 = 48.7346.\]

\[c_{2a} = \{4.55807, 13.63, 25.1375, 36.4444, 44.9604, 48.7346, 47.7346, 45.9023, 38.8833, 28.2856, 16.537, 6.70758, 2.059821752.\}\]

\[c_3 = \{36.1023, 28.1818, 26.1142, 25.0753, 24.3873, 23.8449, 23.3556, 22.8576, 22.2845, 21.5236, 20.2963, 18.5754, 36.102352.\}\]

\[c_5 = \{3.58782, 3.40862, 3.35353, 3.3242, 3.30411, 3.28788, 3.27293, 3.25738, 3.23908, 3.21405, 3.17179, 3.10821, 3.5888^{23.52}.\}\]

\[c_6 = \{5.2067, 3.88525, 4.86669, 3.30557, 3.62425, 3.62461, 3.28336, 3.98813, 2.97576, 3.41762, 3.86076, 3.09418, 4.097533.52.\}\]

\[c_7 = \{1.30674 \times 10^{95}, 2.18839 \times 10^{95}, 2.68104 \times 10^{95}, 2.97999 \times 10^{95}, 3.14901 \times 10^{95}, 3.21389 \times 10^{95}, 3.1972 \times 10^{95}, 3.1657 \times 10^{95}, 3.03213 \times 10^{95}, 2.77601 \times 10^{95}, 2.34399 \times 10^{95}, 1.6177 \times 10^{95}, 6.67449 \times 10^{94}.\}\]

\[K_1 = \{1.65365 \times 10^{-6}, 63.2569, 486473., 1.03099 \times 10^{8}, 2.26948 \times 10^{9}, 8.66973 \times 10^{9}, 8.66973 \times 10^{9}, 6.89763 \times 10^{9}, 1.10954 \times 10^{9}, 2.78439 \times 10^{7}, 55693.3, 1.41715, 5.421 \times 10^{-11}.\}\]

\[K_2 = \{2.49678, 3.34599, 2.67122, 3.93275, 3.58695, 3.5866, 3.95935, 3.25967, 4.36864, 3.80382, 3.36721, 4.20144, 3.17265\}.\]

Initially,

\[K_3 = \{10^{98}, 10^{98}, 10^{98}, 10^{98}, 10^{98}, 10^{98}, 10^{98}, 10^{98}, 10^{98}, 10^{98}, 10^{98}, 10^{98}, 10^{98}, 10^{98}, 10^{98}, 10^{98}, 10^{98}, 10^{97}, 10^{97}\}\]

and eventually \(K_3 = \{53, 45, 58, 49, 55, 55, 43, 52, 44, 50, 55, 41, 47\}.

With the same notation that was used for the case \(n = 11\),

\[v = 1/4096 \times (93158647867090656840416856127516852294230637148702, 851086532124807140957259454209260273172314431029910278429059765, 0839320632215240547355005877176619694735203879318746044181).\]
$M = 316357820342343521286$, and so $F_m \leq (\sqrt{5}M^{11} v^{12})^{13} \leq 5.4892 \times 10^{20199}$ and so $m \leq 139720$. A check shows that $F_j$ is not a thirteenth power for $3 \leq j \leq 139720$.

5) The case $n = 17$: $D = 272$ and

$$f_{17}(x) = -65536 + 1114112 x + 2228224 x^2 - 11141120 x^3 - 9748480 x^4 + 25346048 x^5 + 12673024 x^6 - 19914752 x^7 - 6223360 x^8 + 6223360 x^9 + 1244672 x^{10} - 792064 x^{11} - 99008 x^{12} + 38080 x^{13} + 2720 x^{14} - 544 x^{15} - 17 x^{16} + x^{17}.$$  

The zeroes of $f_{17}(x)$ are

$$\{-16.6323, -6.36449, -3.75619, -2.50343, -1.72576, -1.16412,$$
$$-0.712712, -0.317684, 0.0545603, 0.430621, 0.838223, 1.31512,$$
$$1.92569, 2.80429, 4.30707, 7.83505, 30.6661\}.$$  

$c_1 = \{10.2678, 2.60829, 1.25276, 0.777669, 0.561638, 0.451412,$
$$0.395027, 0.372245, 0.372245, 0.376061, 0.407602, 0.476899, 0.61057,$$
$$0.878601, 1.50278, 3.52799, 22.831\},$

$c_2 = 83.2349.$

$c_2a = \{4.60646, 14.1973, 27.4771, 42.6525, 57.6738, 70.5125, 79.4345,$
$$83.2349, 82.2349, 80.4005, 73.1789, 61.5455, 47.0714, 31.7115,$$
$$17.5402, 6.93523, 2.07167\}.$

$c_3 = \{47.2984, 37.0306, 34.4223, 33.1695, 32.3918, 31.8302, 31.3788,$
$$30.9838, 30.6115, 30.2355, 29.8279, 29.351, 28.7404, 27.8618,$$
$$26.359, 24.4674, 47.298467\}.$

$c_5 = \{3.78233, 3.60548, 3.55271, 3.52594, 3.50882, 3.49619, 3.48589,$
$$3.47675, 3.46803, 3.45911, 3.44932, 3.4377, 3.42255, 3.40018,$$
$$3.36024, 3.30671, 3.782967\}.$

$c_6 = \{6.96297, 6.95734, 5.89133, 6.24564, 4.71335, 4.94999, 5.6139,$
$$7.93478, 7.87795, 4.98754, 6.01398, 7.84567, 6.87894, 5.26067,$$
$$5.08113, 5.13209, 8.364577\}.$

$c_7 = \{2.15293 \times 10^{126}, 3.65902 \times 10^{126}, 4.54254 \times 10^{126}, 5.13093 \times 10^{126},$
$$5.53464 \times 10^{126}, 5.80357 \times 10^{126}, 5.96299 \times 10^{126}, 6.02552 \times 10^{126},$$
$$5.4892 \times 10^{20199}\}.$
6.00935 \times 10^{126}, 5.97916 \times 10^{126}, 5.85324 \times 10^{126}, 5.62158 \times 10^{126},
5.26283 \times 10^{126}, 4.73432 \times 10^{126}, 3.94195 \times 10^{126}, 2.7004 \times 10^{126},
1.08369 \times 10^{126}).

K_1 = \{1.3922 \times 10^{-10}, 1.82303, 473234., 1.56794 \times 10^9, 3.9635 \times 10^{11},
1.62592 \times 10^{13}, 1.57104 \times 10^{14}, 4.3128 \times 10^{14}, 4.3128 \times 10^{14}, 3.62626 \times 10^{14},
9.22206 \times 10^{13}, 6.39148 \times 10^{12}, 9.57958 \times 10^{10}, 1.96963 \times 10^{8}, 21460.7,
0.01073, 1.75352 \times 10^{-16}\}.

K_2 = \{2.44149, 2.44346, 2.88559, 2.7219, 3.60678, 3.43435, 3.0282,
2.14247, 2.15792, 3.40849, 2.82675, 2.1668, 2.47131, 3.23152, 3.34572,
3.31249, 2.03238\}.

Initially,
K_3 = \{10^{134}, 10^{134}, 10^{134}, 10^{134}, 10^{134}, 10^{134}, 10^{134}, 10^{134}, 10^{134}, 10^{134}, 10^{134},
10^{134}, 10^{134}, 10^{134}, 10^{134}, 10^{134}, 10^{134}, 10^{134}\}

and eventually
K_3 = \{93, 103, 91, 100, 76, 81, 93, 135, 134, 82, 102,
132, 113, 83, 77, 73, 106\}.

With the same notation as above M = 416654165624561667592653373446

and
v = 1/\sqrt{5} M_{16} v_{16}^{17} \leq 3.2504 \times 10^{128042} and thus m \leq 616986. A check shows that F_j is not a seventeenth power in the range 3 \leq j \leq 616986.

7. Conclusion

The same method could be used to extend the results about prime powers in other binary recurrence sequences, in particular the Lucas sequence. It is possible the alternative method used to overcome the problems of have
a large number of independent units and a large bound on their exponents may be applied in other situations.

Acknowledgements: I wish to thank Guillaume Hanrot for several helpful comments. He drew my attention to his paper \cite{Hanrot}, in which he describes a method for solving Thue equations without the full unit group, a method can be applied to the problem of finding small prime powers in the Fibonacci sequence. He also points out that, once the LLL reduction of the bound on the powers of the units has been completed, that there several methods for shortening the final check of remaining possible cases (See \cite{Hanrot}).

Finally, he pointed out that the “thueinit” and “thue” commands in PARI/GP can be used to solve the associated Thue equation, when the degree is small.

8. Appendix I: Fundamental Units in $\mathbb{Q}(\theta)$ for the cases $n = 11, 13$ and 17.

For the case $n = 11$ a set of fundamental units in the associated field is the following:

\[
\begin{align*}
\{ & 209 x^8 + 90 x^7 + 7449 x^3 - 21179 x^4 + 4375 x^5 + 4257 x^6 + 2711 x^7 - \\
& 5987 x^8 - 3233 x^9 + 7 x^{10} + 421 x^6 + 1643 x^7 + 2313 x^8 - 3605 x^9 \\
& 5311 x^4 - 4537 x^5 + 2571 x^6 - 2048 x^7 + 13 x^{10} \\
& - \frac{3}{128} + \frac{207 x}{256} - \frac{271 x^2}{256} - \frac{39 x^3}{256} + \frac{801 x^4}{1024} - \frac{715 x^5}{2048} - \frac{27 x^6}{4096} + \frac{113 x^7}{32768} \\
& 4209 x^5 - 1089 x^6 + 369 x^7 + 297 x^8 + 27 x^9 + 147 x^{10} \\
& 2048 x^7 + 4096 x^8 + 65536 x^9 \\
& 2577 x^3 + 803 x^4 + 5 x^5 + 1353 x^6 + 463 x^7 + 187 x^8 + 17 x^9 \\
& - \frac{31}{256} + \frac{2423 x}{256} + \frac{2271 x^2}{256} + \frac{31 x^3}{256} + \frac{21377 x^4}{1024} - \frac{4845 x^5}{2048} + \frac{16773 x^6}{4096} + 1491 x^7 \\
& 16384 x^8 + 503 x^9 + 49 x^{10} + 17 x^{11} + 909 x^{12} + 23 x^{13} + 2757 x^{14} + 737 x^{15} \\
& - \frac{1705 x^5}{1024} + 55 x^6 + 659 x^7 - 7 x^8 - 31 x^9 + x^{10} + \frac{501}{256} + 2155 x^{11} \\
& 13771 x^2 + 2787 x^3 + 69915 x^4 + 23013 x^5 + 9541 x^6 + 1145 x^7 + \frac{12909 x^8}{512} \\
& 1273 x^9 + 117 x^{10} + 113 x^{11} + 4569 x^{12} + 2559 x^{13} + 927 x^{14} + 61735 x^{15} + 1721 x^{16} \\
& 131072 + 131072, 256 + 256 + 64 - 512 - 2048 - 512 + \frac{\theta}{512} + \frac{\theta^2}{256} + \frac{\theta^3}{128} + \frac{\theta^4}{64} + \frac{\theta^5}{32} + \frac{\theta^6}{16} + \frac{\theta^7}{8} + \frac{\theta^8}{4} + \frac{\theta^9}{2} + \frac{\theta^{10}}{1}.
\end{align*}
\]
\[
\left\{ \frac{5429 x^6}{1024} + \frac{3943 x^7}{8192} + \frac{14059 x^8}{65536} + \frac{641 x^9}{65536} + \frac{31 x^{10}}{32768} \right\} + \frac{7401 x}{256} + \frac{1651 x^2}{256} + \frac{3747 x^3}{128} - \frac{19629 x^4}{2048} - \frac{14289 x^5}{2048} + \frac{8857 x^6}{4096} + \frac{439 x^7}{1024} - \frac{6255 x^8}{65536} - \frac{365 x^9}{65536} + \frac{15 x^{10}}{32768} \right\}.
\]

For \( n = 13 \) a set of fundamental units in the associated field is:

\[
\left\{ \frac{23}{1024} - \frac{10109 \theta}{2048} - \frac{44003 \theta^2}{4096} + \frac{124839 \theta^3}{8192} + \frac{136209 \theta^4}{4096} + \frac{3917 \theta^5}{1384} + \frac{243 \theta^6}{32768} - \frac{61481 \theta^7}{65536} + \frac{369007 \theta^8}{262144} + \frac{49647 \theta^9}{524288} - \frac{39319 \theta^{10}}{1048576} - \frac{2917 \theta^{11}}{2097152} + \frac{243 \theta^{12}}{2097152}, \right.
\]
\[
- \frac{3727}{1024} - \frac{7181 \theta}{2048} - \frac{21621 \theta^2}{4096} - \frac{121847 \theta^3}{8192} + \frac{655867 \theta^4}{4096} + \frac{119945 \theta^5}{1384} - \frac{991747 \theta^6}{32768}, \right.
\]
\[
\left. \frac{1024}{1024} + \frac{128}{4096} - \frac{4096}{1024} + \frac{128}{8192} - \frac{37475 \theta^7}{4096} + \frac{888013 \theta^8}{262144} + \frac{7157 \theta^9}{524288} + \frac{101367 \theta^{10}}{1048576} + \frac{1149 \theta^{11}}{2097152} + \frac{335 \theta^{12}}{32768}, \right.
\]
\[
\left. \frac{1407}{1024} - \frac{2625 \theta}{2048} - \frac{160297 \theta^2}{4096} + \frac{2013 \theta^3}{8192} + \frac{395753 \theta^4}{1024} + \frac{6531 \theta^5}{16384} + \frac{524081 \theta^6}{32768}, \right.
\]
\[
\left. \frac{195}{1024} - \frac{15321 \theta}{2048} + \frac{58735 \theta^2}{4096} + \frac{5855 \theta^3}{8192} + \frac{289521 \theta^4}{16384} + \frac{76579 \theta^5}{32768} + \frac{191731 \theta^6}{65536}, \right.
\]
\[
\left. \frac{256}{1024} - \frac{2048}{2048} - \frac{2048}{8192} + \frac{8192}{8192} + \frac{8192}{16384} + \frac{8192}{16384}, \right.
\]
\[
\left. \frac{92625 \theta^7}{65536} + \frac{19531 \theta^8}{16384} + \frac{50851 \theta^9}{524288} + \frac{16651 \theta^{10}}{524288} - \frac{2097152}{2097152} + \frac{2097152}{1048576}, \right.
\]
\[
\left. \frac{1}{16} \frac{639 \theta}{512} - \frac{175 \theta^2}{64} + \frac{3547 \theta^3}{2048} + \frac{821 \theta^4}{2048} + \frac{2439 \theta^5}{4096} - \frac{243 \theta^6}{128}, \right.
\]
\[
\left. \frac{1129 \theta^7}{16384} + \frac{3441 \theta^8}{131072} + \frac{1581 \theta^9}{4096} + \frac{23 \theta^{10}}{524288} + \frac{23 \theta^{11}}{524288} + \frac{9 \theta^{12}}{524288}, \right.
\]
\[
\left. \frac{243}{1024} - \frac{2135 \theta}{2048} - \frac{22321 \theta^2}{4096} + \frac{64117 \theta^3}{8192} + \frac{53391 \theta^4}{8192} + \frac{60195 \theta^5}{48321 \theta^6} - \frac{48321 \theta^6}{16384}, \right.
\]
\[
\left. \frac{1024}{1024} - \frac{2048}{4096} + \frac{4096}{8192} + \frac{8192}{8192} + \frac{8192}{16384} - \frac{32768}{16384}, \right.
\]
\[
\left. \frac{27909 \theta^7}{32768} + \frac{22815 \theta^8}{262144} + \frac{6291 \theta^9}{262144} - \frac{1053 \theta^{10}}{1048576} + \frac{10071 \theta^{11}}{1048576}, \right.
\]
\[
\left. \frac{243}{1024} - \frac{1659 \theta}{26361 \theta^2} + \frac{5907 \theta^3}{4096} + \frac{149841 \theta^4}{8192} + \frac{1048576}{1048576}, \right.
\]
\[
\left. \frac{1024}{1024} - \frac{512 \theta}{4096} + \frac{512}{8192} + \frac{512}{8192} - \frac{32768}{8192} - \frac{32768}{8192}, \right.
\]
\[
\left. \frac{525 \theta^7}{525} + \frac{190759 \theta^8}{262144} + \frac{5441 \theta^9}{131072} + \frac{20001 \theta^{10}}{1048576} + \frac{89 \theta^{11}}{1048576} + \frac{61 \theta^{12}}{1048576}, \right.
\]
\[
\left. \frac{593}{1024} + \frac{19249 \theta}{2048} + \frac{51101 \theta^2}{4096} + \frac{123517 \theta^3}{2048} + \frac{66827 \theta^4}{8192} + \frac{124229 \theta^5}{16384} + \frac{185417 \theta^6}{16384}, \right.
\]
\[
\left. \frac{512}{32768} - \frac{85397 \theta^7}{131072} + \frac{159199 \theta^8}{262144} + \frac{35493 \theta^9}{524288} + \frac{17713 \theta^{10}}{1048576} + \frac{1529 \theta^{11}}{1048576} + \frac{115 \theta^{12}}{1048576}, \right.\]
Finally, for the case \( n = 17 \) a set of fundamental units in the associated field is:

\[
\begin{align*}
\left\{ & 1669 + 1013441 \theta^4 - 831099 \theta^2 - 15553837 \theta^3 + 9705537 \theta^4 + 7095341 \theta^5 + 524288 \\
& 3305551 \theta^6 - 124377673 \theta^7 + 21555899 \theta^8 + 90616595 \theta^9 + 22540321 \theta^{10} + 4194304 \\
& 262144 - 2097152 + 524288 + 8388608 + 256021 \theta^{14} + 66067 \theta^{15} \\
& 33554432 + 16777216 + 134217728 + 67108864 + 536879012 + 3807 \theta^{16} + 536870912, \\
& 3381 + 1443 \theta - 1104995 \theta^2 - 596807 \theta^3 + 12341635 \theta^4 + 1742777 \theta^5 + 16777216 \\
& 16384 + 512 - 65536 - 32768 + 262144 + 65536 \\
& 39959399 \theta^6 + 6905549 \theta^7 + 50345477 \theta^8 + 1358191 \theta^9 + 25774521 \theta^{10} \\
& 1048576 + 524288 + 4194304 + 91457 \theta^{13} + 285757 \theta^{14} + 4507 \theta^{15} \\
& 8388608 + 67108864 + 16777216 + 268435456 + 134217728 + 527 \theta^{16} \\
& 13 - 11119 \theta - 5847 \theta^2 - 359545 \theta^3 + 899457 \theta^4 + 810279 \theta^5 + 268435456 \\
& 64 - 4096 - 512 + 16384 + 16384 + 65536 \\
& 1690541 \theta^6 + 148823 \theta^7 + 434743 \theta^8 + 1161099 \theta^9 + 544193 \theta^{10} \\
& 32768 + 262144 + 262144 + 1048576 + 262144 \\
& 616077 \theta^{11} + 406671 \theta^{12} + 87947 \theta^{13} + 11269 \theta^{14} + 2581 \theta^{15} \\
& 4194304 + 4194304 + 16777216 - 8388608 - 67108864 + \ldots
\end{align*}
\]
\[
\begin{align*}
3177 & - 37351 \theta + 260767 \theta^2 + 854595 \theta^3 + 5165539 \theta^4 - 160099 \theta^5 - \\
4096 & - 8192 - 16384 + 32768 + 65536 - 131072 - \\
17391923 \theta^6 - 5968849 \theta^7 + 1880835 \theta^8 + 6236715 \theta^9 - 8469197 \theta^{10} - \\
262144 & - 524288 + 1048576 + 2097152 + 35177 \theta^{11} - 414304 - \\
2025815 \theta^{11} & + 1527105 \theta^{12} + 33554432 - 5339 \theta^{15} + \\
838608 & + 16777216 + 6710864 - 134217728 + \\
\frac{161 \theta^{16}}{67108864},
\end{align*}
\]

\[
\begin{align*}
35177 & + 87353 \theta - 1685455 \theta^2 - 1286573 \theta^3 + 18808867 \theta^4 + 10349213 \theta^5 + \\
4096 & - 8192 - 16384 + 32768 + 65536 - 131072 - \\
62922435 \theta^6 - 26769697 \theta^7 + 79984865 \theta^8 + 25086155 \theta^9 - 40730397 \theta^{10} + \\
262144 & - 524288 + 1048576 + 2097152 + 35177 \theta^{11} - 414304 - \\
8793575 \theta^{11} & + 7786305 \theta^{12} + 33554432 - 5339 \theta^{15} + \\
838608 & + 16777216 + 6710864 - 134217728 + \\
\frac{134217728}{1609 \theta^{16}},
\end{align*}
\]

\[
\begin{align*}
8353 & + 82469 \theta - 334519 \theta^2 - 2389817 \theta^3 + 4045291 \theta^4 + 1646549 \theta^5 - \\
16384 & - 32768 + 65536 - 131072 - 262144 - 524288 - \\
21174147 \theta^6 - 41242981 \theta^7 + 41290081 \theta^8 + 33247615 \theta^9 - 838608 - \\
1048576 & - 2097152 + 99323 \theta^{13} + 441361 \theta^{14} - 26891 \theta^{15} + \\
9888859 \theta^{11} & - 556713 \theta^{12} + 335425 \theta^{14} - 536870912 + \\
33554432 & + 6710864 + 134217728 + 1269 \theta^{16} + \\
\frac{536870912}{17529 \theta^{16}},
\end{align*}
\]

\[
\begin{align*}
17529 & + 199385 \theta - 2570675 \theta^2 - 1099275 \theta^3 + 22040999 \theta^4 + 14557163 \theta^5 - \\
8192 & + 16384 + 32768 + 65536 - 131072 - 262144 - \\
67044783 \theta^6 - 35241615 \theta^7 + 82863837 \theta^8 + 30445541 \theta^9 + 41990713 \theta^{10} - \\
524288 & - 1048576 + 2097152 + 177216 \theta^{16} + \\
10034992 \theta^{11} & - 8051613 \theta^{12} + 6710864 + 134217728 + 1269 \theta^{16} + \\
16777216 & + 33554432 + 6710864 + 134217728 + \\
\frac{536870912}{1683 \theta^{16}},
\end{align*}
\]

\[
\begin{align*}
-15439 & + 7457 \theta - 772229 \theta^2 - 225567 \theta^3 + 19970281 \theta^4 + 5384407 \theta^5 - \\
-16384 & + 4096 - 65536 + 4096 + 262144 + 5384407 \theta^5 + \\
83962553 \theta^6 - 5203139 \theta^7 + 118428135 \theta^8 + 7939383 \theta^9 - 63912463 \theta^{10} - \\
1048576 & - 131072 + 4194304 - 1048576 - 16777216 - 2983 \theta^{15} + \\
149561 \theta^{11} & - 1291267 \theta^{12} + 4194304 - 1048576 - 16777216 - 2983 \theta^{15} + \\
262144 & + 6710864 + 16777216 + 1375 \theta^{16} + \\
\frac{268435456}{268435456},
\end{align*}
\]
\[
\begin{align*}
37669 & \quad 782207 \theta - 393153 \theta^2 + 2915471 \theta^3 + 3884888 \theta^4 - 12052419 \theta^5 - \\
16384 & \quad 32768 - 65536 - 130172 - 262144 - 524288 + \\
105613751 \theta^6 & \quad + 15873183 \theta^7 + 4194304 \theta^8 - 77583405 \theta^9 + 4455329 \theta^{10} + \\
1048576 & \quad + 2097152 + 438056 \theta^9 - 8388608 + 313501 \theta^{10} - 16777216 + \\
12880225 \theta^{11} & \quad + 6875085 \theta^{12} + 134217728 - 313501 \theta^{14} + 7711 \theta^{15} + \\
33554432 & \quad + 67108864 + 268435456 - 536870912 + \\
& \quad 536870912.
\end{align*}
\]
| \( \theta^6 \) | \( \theta^7 \) | \( \theta^8 \) | \( \theta^9 \) | \( \theta^{10} \) |
|----------------|----------------|----------------|----------------|----------------|
| 24349275       | 524288         | 1560157        | 8388608        | 1048576        |
| 524288         | 2097152        | 43097 \( \theta^{12} \) | 4545933 \( \theta^{14} \) | 10569 \( \theta^{15} \) |
| 1560157        | 2097152        | 43097 \( \theta^{12} \) | 10569 \( \theta^{15} \) | 383 \( \theta^{16} \) |
| 8388608        | 1560157        | 8388608        | 134217728      | 67108864       |
| 17285          | 8388608        | 1048576        | 16777216       | 134217728      |
| 16384          | 524288         | 8388608        | 134217728      | 33554432       |
| 152102419 \( \theta^6 \) | 52147849 \( \theta^7 \) | 187354341 \( \theta^8 \) | 51864899 \( \theta^9 \) | 93440845 \( \theta^{10} \) |
| 1048576        | 2097152        | 4194304        | 8388608        | 16777216       |
| 1875327 \( \theta^{11} \) | 17634053 \( \theta^{12} \) | 2246239 \( \theta^{13} \) | 992033 \( \theta^{14} \) | 59731 \( \theta^{15} \) |
| 33554432       | 67108864       | 134217728      | 268435456      | 536870912      |
| 536870912      | 3601 \( \theta^{16} \) | \( \theta^{16} \) | \( \theta^{16} \) | \( \theta^{16} \) |

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