ON YAU RIGIDITY THEOREM FOR MINIMAL SUBMANIFOLDS IN SPHERES *

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Abstract

In this note, we investigate the well-known Yau rigidity theorem for minimal submanifolds in spheres. Using the parameter method of Yau and the DDVV inequality verified by Lu, Ge and Tang, we prove that if $M$ is an $n$-dimensional oriented compact minimal submanifold in the unit sphere $S^{n+p}(1)$, and if $K_M \geq \frac{\text{sgn}(p-1)p}{2(p+1)}$, then $M$ is either a totally geodesic sphere, the standard immersion of the product of two spheres, or the Veronese surface in $S^4(1)$. Here $\text{sgn}(\cdot)$ is the standard sign function. We also extend the rigidity theorem above to the case where $M$ is a compact submanifold with parallel mean curvature in a space form.

1 Introduction

It plays an important role in geometry of submanifolds to investigate rigidity of minimal submanifolds. After the pioneering rigidity theorem for closed minimal submanifolds in a sphere due to Simons [22], a series of striking rigidity results for minimal submanifolds were proved by several geometers [2, 13, 27]. Let $M^n$ be an $n$-dimensional compact Riemannian manifold isometrically immersed into an $(n+p)$-dimensional complete and simply connected Riemannian manifold $F^{n+p}(c)$ with constant curvature $c$. Denote by $K_M$ and $H$ the sectional curvature and mean curvature of $M$ respectively. In 1975, Yau [27] first proved the following celebrated rigidity theorem for minimal submanifolds in spheres under sectional curvature pinching condition.

**Theorem A.** Let $M^n$ be an $n$-dimensional oriented compact minimal submanifold in $S^{n+p}(1)$. If $K_M \geq \frac{p-1}{2p}$, then either $M$ is the totally geodesic sphere, the standard immersion of the product of two spheres, or the Veronese surface in $S^4(1)$.

The pinching constant above is the best possible in the case where $p = 1$, or $n = 2$ and $p = 2$. It’s better than the pinching constant of Simons [22] in the sense of the average of sectional curvatures. Later, Itoh [12] proved that if $M^n$ is an oriented compact minimal submanifold in $S^{n+p}(1)$ whose sectional curvature satisfies $K_M \geq \frac{p}{2(n+1)}$, then $M$ is the totally geodesic sphere or the Veronese submanifold. Further discussions in this direction

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have been carried out by many other authors (see [5, 10, 14, 21, 24, 25, 26]). An important problem is stated as follows.

**Open Problem B.** What is the best pinching constant for the rigidity theorem for oriented compact minimal submanifolds in a unit sphere under sectional (Ricci, scalar, resp.) curvature pinching condition?

Up to now, the problem above is still open. In particular, Lu's conjecture [18], a scalar curvature pinching problem for minimal submanifolds in a unit sphere, has not been verified yet. In this note, using Yau's parameter method [27] and the DDV conjecture proved by Lu, Ge and Tang [7, 16], we prove the following rigidity theorem for minimal submanifolds in spheres.

**Theorem 1.** Let $M^n$ be an $n$-dimensional oriented compact minimal submanifold in the unit sphere $S^{n+p}(1)$. If

$$K_M \geq \frac{\text{sgn}(p-1)p}{2(p+1)},$$

then $M$ is either a totally geodesic sphere, the standard immersion of the product of two spheres, or the Veronese surface in $S^4(1)$. Here $\text{sgn}(\cdot)$ is the standard sign function.

**Remark 1.** When $2 < p < n$, our pinching constant in Theorem 1 is better than ones given by Yau [27] and Itoh [12].

More generally, we obtain the following rigidity result for submanifolds with parallel mean curvature in spaces forms.

**Theorem 2.** Let $M^n$ be an $n$-dimensional oriented compact submanifold with parallel mean curvature ($H \neq 0$) in $F^{n+p}(c)$. If $c + H^2 > 0$ and

$$K_M \geq \frac{\text{sgn}(p-2)(p-1)}{2p}(c + H^2),$$

then $M$ is either a totally umbilical sphere $S^n(\frac{1}{\sqrt{c+H^2}})$ in $F^{n+p}(c)$, the standard immersion of the product of two spheres or the Veronese surface in $S^4(\frac{1}{\sqrt{c+H^2}})$.

## 2 Notation and lemmas

Throughout this paper, let $M^n$ be an $n$-dimensional compact Riemannian manifold isometrically immersed into an $(n + p)$-dimensional complete and simply connected space form $F^{n+p}(c)$ of constant curvature $c$. We shall make use of the following convention on the range of indices:

$$1 \leq A, B, C, \ldots \leq n + p; \ 1 \leq i, j, k, \ldots \leq n; \ n + 1 \leq \alpha, \beta, \gamma, \ldots \leq n + p.$$
Choose a local field of orthonormal frames \( \{ e_A \} \) in \( F^{n+p}(c) \) such that, restricted to \( M \), the \( e_i \)'s are tangent to \( M \). Let \( \{ \omega_A \} \) and \( \{ \omega_{AB} \} \) be the dual frame field and the connection 1-forms of \( F^{n+p}(c) \) respectively. Restricting these forms to \( M \), we have

\[
\omega_{\alpha i} = \sum_j h^\alpha_{ij} \omega_j, \quad h^\alpha_{ij} = h^\alpha_{ji},
\]

\[
h = \sum_{\alpha, i, j} h^\alpha_{ij} \omega_i \otimes \omega_j \otimes e_\alpha, \quad \xi = \frac{1}{n} \sum_{\alpha, i} h^\alpha_{ij} e_\alpha,
\]

\[
R_{ijkl} = c(\delta_{ik} \delta_{jl} - \delta_{il} \delta_{jk}) + \sum_\alpha (h^\alpha_{ik} h^\alpha_{jl} - h^\alpha_{il} h^\alpha_{jk}),
\]

\[
R_{\alpha \beta kl} = \sum_i (h^\alpha_{ik} h^\beta_{il} - h^\alpha_{il} h^\beta_{ik}),
\]

where \( h, \xi, R_{ijkl}, R_{\alpha \beta kl} \), and \( R_{ABCD} \) are the second fundamental form, the mean curvature vector, the curvature tensor, the normal curvature tensor of \( M \), and the curvature tensor of \( N \), respectively. We define

\[
S = |h|^2, \quad H = |\xi|, \quad H_\alpha = (h^\alpha_{ij})_{n \times n}.
\]

The scalar curvature \( R \) of \( M \) is given by

\[
R = n(n - 1)c + n^2 H^2 - S.
\]

Denote \( K_M(p, \pi) \) the sectional curvature of \( M \) for tangent 2-plane \( \pi \subset T_p M \) at point \( p \in M \). Set \( K_{\text{min}}(p) = \min_{\pi \subset T_p M} K_M(p, \pi) \). From [27], we have the following lemma.

**Lemma 1.** If \( M^n \) is a submanifold with parallel mean curvature and positive sectional curvature in \( F^{n+p}(c) \), then \( M \) is a pseudo-umbilical submanifold.

Let \( M \) be a submanifold with parallel mean curvature vector \( \xi \). Choose \( e_{n+1} \) such that it is parallel to \( \xi \), and

\[
tr H_{n+1} = nH, \quad tr H_\alpha = 0, \quad \alpha \neq n + 1.
\]

Set

\[
S_H = tr H^2_{n+1}, \quad S_I = \sum_{\alpha \neq n+1} tr H^2_\alpha.
\]

When \( M \) is a pseudo-umbilical submanifold, we have

\[
S_H = tr H^2_{n+1} = nH^2.
\]

Denoting the first and second covariant derivatives of \( h^\alpha_{ij} \) by \( h^\alpha_{ijk} \) and \( h^\alpha_{ijkl} \) respectively, we have

\[
\sum_k h^\alpha_{ijk} \omega_k = dh^\alpha_{ij} - \sum_k h^\alpha_{ik} \omega_k - \sum_k h^\alpha_{kj} \omega_k - \sum_\beta h^\beta_{ij} \omega_\beta, \quad \sum_l h^\alpha_{i j k l} = dh^\alpha_{ijk} - \sum_l h^\alpha_{ij} \omega_k - \sum_l h^\alpha_{ij} \omega_l - \sum_l h^\alpha_{ijk} \omega_l - \sum_\beta h^\beta_{ijk} \omega_\beta.
\]
Then we have

$$h_{ijk}^\alpha = h_{kj}^\alpha, \quad h_{ijkl}^\alpha - h_{ikjl}^\alpha = \sum_m h_{im}^\alpha R_{mjkl} + \sum_m h_{mj}^\alpha R_{mikl} - \sum_\beta h_{j\beta}^\beta R_{\alpha\beta kl}. \quad (6)$$

$$\Delta h_{ij}^\alpha = \sum_k h_{ikj}^\alpha = \sum_k h_{kij}^\alpha + \sum_k \left( \sum_m h_{km}^\alpha R_{mijk} + \sum_m h_{mi}^\alpha R_{mkjk} - \sum_\beta h_{ki}^\beta R_{\alpha\beta jk} \right). \quad (6)$$

The following lemma will be used in the proof of our main results.

**Lemma 2** ([27]). If $M^n$ is a submanifold with parallel mean curvature in $F^{n+p}(c)$, then either $H \equiv 0$ or $H$ is non-zero constant and $H_{n+1}^\alpha H_\alpha = H_\alpha H_{n+1}$ for all $\alpha$.

The DDVV inequality proved by Lu, Ge and Tang ([7] [16] is stated as follows.

**DDVV Inequality.** Let $B_1, ..., B_m$ be symmetric $(n \times n)$-matrices, then

$$\sum_{r,s=1}^m \|[B_r, B_s]\|^2 \leq \left( \sum_{r=1}^m \|B_r\|^2 \right)^2, \quad (7)$$

where the equality holds if and only if under some rotation all $B_r$’s are zero except two matrices which can be written as

$$\tilde{B}_r = P \begin{pmatrix} 0 & \mu & 0 & \cdots & 0 \\ \mu & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} P^t, \quad \tilde{B}_s = P \begin{pmatrix} \mu & 0 & 0 & \cdots & 0 \\ 0 & -\mu & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} P^t,$$

where $P$ is an orthogonal $(n \times n)$-matrix. Here $\| \cdot \|^2$ denotes the sum of squares of entries of the matrix and $[A, B] = AB - BA$ is the commutator of the matrices $A$, $B$.

For further discussions about the DDVV inequality, we refer to see [4] [7] [15] [16] [17] [18].

### 3 Proof of the theorems

When $M^n$ be a minimal submanifold in $S^{n+p}(1)$, we have $tr H_\alpha = 0$ for all $\alpha$ and $\sum_i h_{iikl}^\alpha = 0$. It follows from (6) that

$$\Delta h_{ij}^\alpha = \sum_{k,m} h_{km}^\alpha R_{mijk} + \sum_{k,m} h_{mi}^\alpha R_{mkjk} - \sum_{k,\beta} h_{ki}^\beta R_{\alpha\beta jk}. \quad (8)$$

Thus

$$\sum_{i,j,\alpha} h_{ij}^\alpha \Delta h_{ij}^\alpha = \sum_{i,j,k,m,\alpha} (h_{km}^\alpha h_{ij}^\alpha R_{mijk} + h_{mi}^\alpha h_{ij}^\alpha R_{mkjk}) - \sum_{i,j,k,\alpha,\beta} h_{ij}^\alpha h_{ki}^\beta R_{\alpha\beta jk}. \quad (9)$$
Proof of Theorem 1. By using (1) and (2), we get

\[ \sum_{i,j,k,m,\alpha} h_{ij}^\alpha h_{km}^\alpha R_{mijk} + \sum_{i,j,k,m,\alpha} h_{ij}^\alpha h_{mi}^\alpha R_{mkjk} = nS + \sum_{\alpha,\beta} tr H_\alpha^2 \cdot tr (H_\alpha^2 H_\beta) - \sum_{\alpha,\beta} [tr (H_\alpha H_\beta)]^2 - \sum_{\alpha,\beta} [tr (H_\alpha^2 H_\beta^2) - tr (H_\alpha H_\beta)^2], \]

and

\[ \sum_{i,j,k,\alpha,\beta} h_{ij}^\alpha h_{ki}^\beta R_{\alpha\beta kj} = \sum_{\alpha,\beta} [tr (H_\alpha^2 H_\beta^2) - tr (H_\alpha H_\beta)^2]. \]

Since \((tr(H_\alpha H_\beta))\) is a symmetric \((p \times p)\)-matrix, we can choose the normal frame fields \(\{e_\alpha\}\) such that

\[ tr (H_\alpha H_\beta) = tr H_\alpha^2 \cdot \delta_{\alpha\beta}. \]

This implies

\[ \sum_{\alpha,\beta} [tr (H_\alpha H_\beta)]^2 = \sum_{\alpha} (tr H_\alpha^2)^2. \] (10)

From above equalities, we obtain

\[ \sum_{i,j,\alpha} h_{ij}^\alpha \Delta h_{ij}^\alpha = -anS + (1 + a) \sum_{i,j,k,m,\alpha} (h_{ij}^\alpha h_{km}^\alpha R_{mijk} + h_{ij}^\alpha h_{mi}^\alpha R_{mkjk}) + (a - 1) \sum_{\alpha,\beta} [tr (H_\alpha^2 H_\beta^2) - tr (H_\alpha H_\beta)^2] + a \sum_{\alpha,\beta} (tr H_\alpha^2)^2, \] (11)

for all real number \(a\). For fixed \(\alpha\), we choose the orthonormal frame fields \(\{e_i\}\) such that \(h_{ij}^\alpha = \lambda_i^\alpha \delta_{ij}\). Hence, we get

\[ \sum_{i,j,k,m,\alpha} h_{ij}^\alpha h_{km}^\alpha R_{mijk} + \sum_{i,j,k,m,\alpha} h_{ij}^\alpha h_{mi}^\alpha R_{mkjk} = \sum_{i,k} \lambda_i^\alpha \lambda_k^\alpha R_{kiik} + \sum_{i,k} \lambda_i^\alpha \lambda_i^\alpha R_{ikik} \]

\[ = \frac{1}{2} \sum_{i,j} (\lambda_i^\alpha - \lambda_j^\alpha)^2 R_{ijij} \]

\[ \geq \frac{1}{2} K_{\min} \sum_{i,j} (\lambda_i^\alpha - \lambda_j^\alpha)^2 \]

\[ = nK_{\min} (tr H_\alpha^2), \] (12)

which implies that

\[ \sum_{i,j,k,m,\alpha} h_{ij}^\alpha h_{km}^\alpha R_{mijk} + \sum_{i,j,k,m,\alpha} h_{ij}^\alpha h_{mi}^\alpha R_{mkjk} \geq nK_{\min} S. \] (13)

On the other hand, by a direct computation and the DDVV inequality, we obtain

\[ \sum_{\alpha,\beta} tr (H_\alpha^2 H_\beta^2) - tr (H_\alpha H_\beta)^2 = \frac{1}{2} \sum_{\alpha,\beta} tr (H_\alpha H_\beta - H_\beta H_\alpha)^2 \]

\[ \leq \frac{1}{2} sgn(p - 1) (\sum_{\alpha} tr H_\alpha^2)^2 \]

\[ = \frac{1}{2} sgn(p - 1) S^2, \] (14)
where \( sgn(\cdot) \) is the standard sign function. It follows from (11), (13) and (14) that

\[
\frac{1}{2} \Delta S = \sum_{i,j,\alpha} (h_{ij}^\alpha)^2 + \sum_{i,j,\alpha} h_{ij}^\alpha \Delta h_{ij}^\alpha
\]

\[
\geq \sum_{i,j,k,\alpha} (h_{ijk}^\alpha)^2 - anS + (1 + a)nK_{\min}S + \left[ \frac{a}{p} + \frac{sgn(p-1)}{2}(a-1) \right] S^2, \tag{15}
\]

for \( 0 \leq a < 1 \). Taking \( a = sgn(p-1) \frac{p}{p+2} \), we get

\[
\frac{1}{2} \Delta S \geq nS\left[ (1 + sgn(p-1) \frac{p}{p+2})K_{\min} - sgn(p-1) \frac{p}{p+2} \right].
\]

It follows from the assumption and the maximum principal that \( S \) is a constant, and

\[
S\left[ (1 + sgn(p-1) \frac{p}{p+2})K_{\min} - sgn(p-1) \frac{p}{p+2} \right] = 0.
\]

If there is a point \( q \in M \) such that \( K_{\min}(q) > sgn(p-1) \frac{p}{2(p+1)} \), then \( S = 0 \), i.e., \( M \) is totally geodesic. If \( K_{\min} = sgn(p-1) \frac{p}{2(p+1)} \), then inequalities in (13), (14) and (15) become equalities. From the DDVV inequality we obtain \( p \leq 2 \). This together with Theorem A implies \( M \) is the product of two spheres or the Veronese surface in \( S^4(1) \). This completes the proof of Theorem 1.

When \( M^n \) is a submanifold with parallel mean curvature in \( F^{n+p}(c) \), we have \( \xi = He_{n+1} \), and \( \sum_i h_{ikl}^\alpha = 0 \) for \( \alpha \neq n+1 \). It follows from (6) and Lemma 2 that

\[
\Delta h_{ij}^\alpha = \sum_{k,m} h_{km}^\alpha R_{mijk} + \sum_{k,m} h_{mi}^\alpha R_{mkjk} - \sum_{k,\beta \neq n+1} h_{k\beta}^\beta R_{\alpha\beta\beta}, \quad \alpha \neq n+1. \tag{16}
\]

Thus

\[
\sum_{i,j,\alpha \neq n+1} h_{ij}^\alpha \Delta h_{ij}^\alpha = \sum_{i,j,k,m,\alpha \neq n+1} \left( h_{ij}^\alpha h_{km}^\alpha R_{mijk} + h_{ij}^\alpha h_{mi}^\alpha R_{mkjk} \right)
\]

\[
- \sum_{i,j,k,\alpha,\beta \neq n+1} h_{ij}^\alpha h_{k\beta}^\beta R_{\alpha\beta\beta}. \tag{17}
\]

**Proof of Theorem 2.** Applying (1) and (2), we get

\[
\sum_{i,j,k,m,\alpha \neq n+1} h_{ij}^\alpha h_{km}^\alpha R_{mijk} + \sum_{i,j,k,m,\alpha \neq n+1} h_{ij}^\alpha h_{mi}^\alpha R_{mkjk}
\]

\[
= ncS_1 + \sum_{\alpha \neq n+1,\beta} trH_{\beta} \cdot tr(H_{\alpha}^2 H_{\beta}) - \sum_{\alpha \neq n+1,\beta} [tr(H_{\alpha} H_{\beta})]^2
\]

\[
- \sum_{\alpha,\beta \neq n+1} [tr(H_{\alpha}^2 H_{\beta}^2) - tr(H_{\alpha} H_{\beta})^2],
\]

and

\[
\sum_{i,j,k,\alpha,\beta \neq n+1} h_{ij}^\alpha h_{k\beta}^\beta R_{\alpha\beta\beta} = \sum_{\alpha,\beta \neq n+1} [tr(H_{\alpha}^2 H_{\beta}^2) - tr(H_{\alpha} H_{\beta})^2].
\]
Since $\alpha, \beta \neq n + 1$, $(tr(H_{\alpha}H_{\beta}))$ is a symmetric $(p - 1) \times (p - 1)$-matrix. We choose the normal vector fields $\{e_{\alpha}\}_{\alpha \neq n + 1}$ such that

$$tr(H_{\alpha}H_{\beta}) = trH_{\alpha}^{2} \cdot \delta_{\alpha\beta},$$

which implies

$$\sum_{\alpha,\beta \neq n + 1} |tr(H_{\alpha}H_{\beta})|^{2} = \sum_{\alpha \neq n + 1} tr(H_{\alpha}^{2}).$$

(18)

For any real number $a$, we have

$$\sum_{i,j,\alpha \neq n + 1} h_{ij}^{\alpha} \Delta h_{ij}^{\alpha} = (1 + a) \sum_{i,j,k,m,\alpha \neq n + 1} (h_{ij}^{\alpha} h_{km}^{\alpha} R_{mijk} + h_{ij}^{\alpha} h_{mi}^{\alpha} R_{mijk}) - ancS_{I}$$

$$+ (a - 1) \sum_{\alpha \neq n + 1} [tr(H_{\alpha}^{2} H_{\alpha}^{2}) - tr(H_{\alpha}H_{\beta})^{2}] + a \sum_{\alpha \neq n + 1} (trH_{\alpha}^{2})^{2}$$

$$+ a \left\{ - \sum_{\alpha \neq n + 1} tr(H_{\alpha}^{2} H_{n+1}) \cdot trH_{n+1} + \sum_{\alpha \neq n + 1} [tr(H_{\alpha}H_{n+1})]^{2} \right\}.$$ (19)

When $p = 1$, $M$ is a compact hypersurface with nonzero constant mean curvature and nonnegative sectional curvature in $F^{n+1}(c)$. The assertion was proved by Nomizu and Synth for $c \geq 0$ and by Walter for $c < 0$, respectively.

When $p = 2$, $K_{M} \geq 0$ and $H = constant \neq 0$. We know from Theorem 9 in [27] that $M$ is a minimal hypersurface in the totally umbilical sphere $S^{n+1}(\frac{1}{\sqrt{c + H^{2}}})$. This together with Theorem A implies that $M$ is either a totally umbilical sphere or the standard immersion of the product of two spheres.

When $p \geq 3$, it follows from Lemma 1 and the assumption that $M$ is pseudo-umbilical, i.e., $h_{ij}^{n+1} = H \delta_{ij}$. Hence, we have

$$\sum_{\alpha \neq n + 1} tr(H_{\alpha}^{2} H_{n+1}) \cdot trH_{n+1} - \sum_{\alpha \neq n + 1} [tr(H_{\alpha}H_{n+1})]^{2}$$

$$= \sum_{i,j,k,m,\alpha \neq n + 1} h_{ij}^{\alpha} h_{km}^{\alpha} h_{mj}^{n+1} h_{kk}^{n+1} - \sum_{i,j,k,m,\alpha \neq n + 1} h_{ij}^{\alpha} h_{km}^{\alpha} h_{nk}^{n+1} h_{ij}^{n+1}$$

$$= nH^{2} \sum_{i,j,\alpha \neq n + 1} (h_{ij}^{\alpha})^{2} - H^{2} \sum_{\alpha \neq n + 1} (trH_{\alpha})^{2}$$

$$= nH^{2} S_{I}. \quad (20)$$

On the other hand, we get from (12)

$$\sum_{i,j,k,m,\alpha \neq n + 1} h_{ij}^{\alpha} h_{km}^{\alpha} R_{mijk} + \sum_{i,j,k,m,\alpha \neq n + 1} h_{ij}^{\alpha} h_{mi}^{\alpha} R_{mijk} \geq nK_{min} S_{I}. \quad (21)$$

By a direct computation and the DDVV inequality, we obtain

$$\sum_{\alpha,\beta \neq n + 1} tr(H_{\alpha}^{2} H_{\beta}^{2}) - tr(H_{\alpha}H_{\beta})^{2} = \frac{1}{2} \sum_{\alpha,\beta \neq n + 1} tr(H_{\alpha}H_{\beta} - H_{\beta}H_{\alpha})^{2}$$

$$\leq \frac{1}{2} \left( \sum_{\alpha \neq n + 1} trH_{\alpha}^{2} \right)^{2}$$

$$= \frac{1}{2} S_{I}^{2}. \quad (22)$$
It follows from (19), (20), (21) and (22) that
\[
\frac{1}{2} \Delta S_{I} = \sum_{i,j,\alpha \neq n+1} (h_{ijk}^{\alpha})^{2} + \sum_{i,j,\alpha \neq n+1} h_{ij}^{\alpha} \Delta h_{ij}^{\alpha} \\
\geq (1 + a)nK_{\min}S_{I} + a \sum_{\alpha \neq n+1} (trH_{\alpha}^{2})^{2} + \frac{1}{2} (a - 1)S_{I}^{2} - an(c + H^{2})S_{I} \\
\geq (1 + a)nK_{\min}S_{I} + \left( \frac{a}{p - 1} + \frac{a - 1}{2} \right)S_{I}^{2} - an(c + H^{2})S_{I} \\
= S_{I} \left[ (1 + a)nK_{\min} + \left( \frac{a}{p - 1} + \frac{a - 1}{2} \right)S_{I} - an(c + H^{2}) \right],
\]
(23)
for \(0 \leq a < 1\). Taking \(a = \frac{p-1}{p+1}\), we get
\[
\frac{1}{2} \Delta S_{I} \geq nS_{I} \left[ (1 + a)K_{\min} - a(c + H^{2}) \right] \\
= nS_{I} \left[ \left( 1 + \frac{p - 1}{p + 1} \right)K_{\min} - \frac{p - 1}{p + 1}(c + H^{2}) \right].
\]
It follows from the assumption and the maximum principal that \(S_{I}\) is a constant, and
\[
S_{I} \left[ \left( 1 + \frac{p - 1}{p + 1} \right)K_{\min} - \frac{p - 1}{p + 1}(c + H^{2}) \right] = 0.
\]
If there is a point \(q \in M\) such that \(K_{\min}(q) > \frac{(p-1)(c+H^{2})}{2p}\), then \(S_{I} = 0\), i.e., \(M\) is a compact hypersurface with nonzero constant mean curvature and positive sectional curvature in a totally geodesic submanifold \(F^{n+1}(c)\). Therefore, \(M\) is a totally umbilical sphere \(S^{n}(\frac{1}{\sqrt{c+H^{2}}}c)\).

If \(K_{\min} = \frac{(p-1)(c+H^{2})}{2p}\), then inequalities in (21), (22) and (23) become equalities. This together with the DDVV inequality implies that \(p = 3\) and \(K_{\min} = \frac{c + H^{2}}{3}\). Taking \(a = 0\) in (23), we get \(S_{I} = 2n(\frac{1}{\sqrt{c+H^{2}}})\). By the same argument as in [2], we conclude that \(n = 2\). Hence, \(K_{M} = \frac{c + H^{2}}{3}\) and \(M\) is the Veronese surface in \(S^{4}(\frac{1}{\sqrt{c+H^{2}}})\). This completes the proof of Theorem 2.

Combing Theorems 1, 2 and rigidity results in [12], [21], [26], we present a general version of the Yau rigidity theorem.

**Generalized Yau Rigidity Theorem.** Let \(M^{n}\) be an \(n\)-dimensional oriented compact submanifold with parallel mean curvature in \(F^{n+p}(c)\), where \(c + H^{2} > 0\). Set \(k(m, n) = \min\{sgn(m - 1)m, n\}\). Then we have

(i) if \(H = 0\) and
\[
K_{M} \geq \frac{k(p, n)c}{2[k(p, n) + 1]},
\]
then \(M\) is either a totally geodesic sphere, the standard immersion of the product of two spheres, or the Veronese submanifold;
(ii) if $H \neq 0$ and
\[
K_M \geq \frac{k(p-1,n)(c+H^2)}{2[k(p-1,n)+1]},
\]
then $M$ is either a totally umbilical sphere $S^n(\frac{1}{\sqrt{c+H^2}})$ in $F^{n+p}(c)$, the standard immersion of the product of two spheres, or the Veronese submanifold.

Recently Andrews and Baker [1] generalized a weaker version of Huisken’s convergence theorem [8] for mean curvature flow of convex hypersurfaces in $\mathbb{R}^{n+1}$ to higher codimensional cases. Motivated by Generalized Yau Rigidity Theorem, we would like to propose the following conjecture on mean curvature flow in higher codimensions, which can be considered as a generalization of the Huisken convergence theorem [8].

**Conjecture.** Let $M_0 = F_0(M)$ be an $n$-dimensional compact submanifold in an $(n+p)$-dimensional space form $F^{n+p}(c)$ with $c+H^2 > 0$. If the sectional curvature of $M_0$ satisfies
\[
K_M > \frac{k(p,n)(c+H^2)}{2[k(p,n)+1]},
\]
then the mean curvature flow
\[
\begin{cases}
\frac{\partial}{\partial t} F(x,t) = n\xi(x,t), & x \in M, \ t \geq 0, \\
F(\cdot, 0) = F_0(\cdot).
\end{cases}
\tag{24}
\]
has a unique smooth solution $F : M \times [0,T) \to F^{n+p}(c)$ on a finite maximal time interval, and $F_t(\cdot)$ converges uniformly to a round point $q \in F^{n+p}(c)$ as $t \to T$.

When $p = 1$ and $c = 0$, the conjecture was verified by Huisken [8]. When $p = 1$ and $c = 1$, a weaker version of the conjecture was proved by Huisken [9].

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