Asymptotic Profile of Solutions for Strongly Damped Klein-Gordon Equations

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Abstract

We consider the Cauchy problem in \( \mathbb{R}^n \) for strongly damped Klein-Gordon equations. We derive asymptotic profiles of solutions with weighted \( L^1(\mathbb{R}^n) \) initial data by a simple method introduced by R. Ikehata. The obtained results show that the wave effect will be weak because of the mass term, especially in the low dimensional case \( (n = 1, 2) \) as compared with the strongly damped wave equations without mass term \( (m = 0) \), so the most interesting topic in this paper is the \( n = 1, 2 \) cases.

1 Introduction.

We are concerned with the Cauchy problem for the so called Klein-Gordon type of equation in \( \mathbb{R}^n \) \( (n \geq 1) \) with the structural damping and mass terms

\[
\begin{align*}
  u_{tt}(t, x) - \Delta u(t, x) + m^2 u(t, x) - \Delta u_t(t, x) &= 0, & (t, x) &\in (0, \infty) \times \mathbb{R}^n, \\
  u(0, x) &= u_0(x), & u_t(0, x) &= u_1(x), & x &\in \mathbb{R}^n,
\end{align*}
\]

where \( m > 0 \). The initial data \( u_0 \) and \( u_1 \) are also chosen from the usual energy space

\[
[u_0, u_1] \in H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n).
\]

Then, it is well-known (cf. see [13]) that the problem (1.1)-(1.2) has a unique weak solution

\[
u \in C([0, \infty); H^1(\mathbb{R}^n)) \cap C^1([0, \infty); L^2(\mathbb{R}^n)).\]

We first mention several known facts concerning the strongly damped wave equations without mass term \( (i.e., m = 0) \) such that

\[
u_{tt}(t, x) - \Delta u(t, x) - \Delta u_t(t, x) = 0, & (t, x) &\in (0, \infty) \times \mathbb{R}^n.
\]

At the first stage, Ponce [19] and Shibata [20] studied the decay structure of the \( L^p \)-norm of solutions (including their derivatives) to the problem (1.3) and (1.2). Recently, by the results from Ikehata-Todorova-Yordanov [13], Ikehata [9] and Ikehata-Onodera [11] one can know that

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the asymptotic profile of the solution to the equation (1.3) is the so called diffusion waves, that is,
\[ \hat{u}(t, \xi) \sim P_1 e^{-t|\xi|^2/2} \frac{\sin(t|\xi|)}{|\xi|} \quad (t \to \infty), \]
where
\[ P_j := \int_{\mathbb{R}^n} u_j(x) dx, \quad (j = 0, 1), \]
and
\[ \hat{u}(t, \xi) := \mathcal{F}(u(t, \cdot))(\xi) := \frac{1}{\sqrt{(2\pi)^n}} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} u(t, x) dx \]
represents the partial Fourier transform of the solution \( u(t, x) \) with respect to the \( x \)-variable. In this connection, concerning the higher order asymptotic expansion of the solution to problem (1.3) and (1.2), one can cite quite recent results due to Michihisa [15] (see also Ikehata-Takeda [12] for an asymptotic behavior of solutions for a more generalized equations of (1.3) from a different point of view). Furthermore, as an application to the nonlinear problem of the equation (1.3) one can cite D’Abbicco-Reissig [5] and the references therein. Anyway, from the results obtained by [9] and [11] one can know that as \( t \to \infty \), in the case when \( n \geq 3 \) one has
\[ C|P_1|t^{-\frac{n-2}{4}} \leq \|u(t, \cdot)\| \leq C^{-1}(\|u_1\|_{1,1} + \|u_0\|_{1,1} + \|u_1\| + \|u_0\|_{H^1})t^{-\frac{n-2}{4}}, \]
while in the case when \( n = 1, 2 \) it is true that
\[ C|P_1|\sqrt{\log t} \leq \|u(t, \cdot)\| \leq C^{-1}(\|u_1\|_{1,1} + \|u_0\|_{1,1} + \|u_1\| + \|u_0\|_{H^1})\sqrt{\log t}, \quad (n = 2) \]
\[ C|P_1|\sqrt{t} \leq \|u(t, \cdot)\| \leq C^{-1}(\|u_1\|_{1,1} + \|u_0\|_{1,1} + \|u_1\| + \|u_0\|_{H^1})\sqrt{t}, \quad (n = 1). \]
Interestingly, through the solution to the strongly damped wave equation (1.3), (at least) in the one and two dimensional whole space cases we can find that the so called Poincaré inequality never hold, because one can also get the decay estimates of the total energy (see Ikehata-Natsume [10] Theorem 1.1], and also Charão-da Luz-Ikehata [11]) such that for all \( n \geq 1 \)
\[ \|u(t, \cdot)\| + \|\nabla u(t, \cdot)\| \leq C(1 + t)^{-\frac{\alpha}{2}} \|u_1\|_{1} + C(1 + t)^{-\frac{n+2}{4}} \|u_0\|_{1} + Ce^{-\alpha t}(\|u_1\| + \|\nabla u_0\|). \]
This phenomena is coming from an observation on the asymptotic profile itself such that in the low dimensional case \( n = 1, 2 \) the effect of the wave part of the profile coming from the factor \( \sin(t|\xi|)/|\xi| \) is extremely strong, so that the diffusive structure of the solution vanishes any more as time goes to infinity. This interesting phenomena shows a quite different aspect as compared with the usual weakly damped wave equation case, which is nowadays very popular due to many papers published by D’Abbicco-Ebert [3], Hosono-Ogawa [7], Karch [14], Narazaki [15], Nishihara [17], Takeda [21], Wakasugi [23] and the references therein:
\[ u_{tt}(t, x) - \Delta u(t, x) + u_t(t, x) = 0, \quad (t, x) \in (0, \infty) \times \mathbb{R}^n. \] (1.4)
In the weakly damped wave equation case (1.4) the \( L^2 \)-norm of solutions necessarily decays when the total energy decays with some rate. This is because of the diffusive aspect of the equation (1.4). In this sense, the strongly damped wave equation (1.3) does not have a diffusive structure anymore at least in the low dimensional case \( n = 1, 2 \).

In this connection, from these observations one can find that when we want to get the "decay" property of the solution to the Cauchy problem of the equation (1.3), we have to impose rather a stronger assumption on the initial velocity \( u_1(x) \) such that \( P_1 = 0 \). People sometimes impose this zero average condition on the initial data when they study the decay property of solutions to
the equation (1.4) in the case when \( n = 2 \) (and/or \( n = 1 \)). However, this zero average condition \( P_j = 0 \) \((j = 0, 1)\) seems to be just a technical one (at least) in the case of (1.4), while in the case of (1.3) such a zero average condition \( P_1 = 0 \) seems to be essential in the low dimensional case \( n = 1, 2 \) in order to get the \( L^2 \)-decay of solutions. It should be emphasized that the paper due to D’Abbicco-Ebert-Picon \([4]\) have also pointed out its importance of the zero average condition \( P_j = 0 \) \((j = 0, 1)\) in order to get the decay estimates of solutions to the equations

\[
 u_{tt}(t, x) + (-\Delta)^\sigma u(t, x) + (-\Delta)^\delta u_t(t, x) = 0,
\]

in terms of Hardy spaces.

If one considers the following weakly damped Klein-Gordon equation in place of (1.1):

\[
 u_{tt}(t, x) - \Delta u(t, x) + m^2 u(t, x) + u_t(t, x) = 0, \quad (t, x) \in (0, \infty) \times \mathbb{R}^n,
\]

one has many previous papers, and one can cite D’Abbicco \([2]\), da-Luz-Ikehata-Charão \([6]\), Zuazua \([24]\), and the references therein (For non-damped Klein-Gordon equations, one can cite several papers due to Nunes-Bastos \([18]\) and the references therein concerning the local energy decay property and its related research).

By adding the mass term, the purpose of this paper is to study what kind of decay structure the equation (1.1) has. Our first result can be stated as follows.

**Theorem 1.1** Let \( n \geq 1 \). If \([u_0, u_1] \in (H^1(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)) \times (L^2(\mathbb{R}^n) \cap L^1(\mathbb{R}^n))\), then it is true that

\[
\|u_t(t, \cdot)\| + \|u(t, \cdot)\| + \|\nabla u(t, \cdot)\| \leq C(1 + t)^{-\frac{\sigma}{2}}(\|u_1\|_1 + \|u_0\|_1) + Ce^{-\alpha t}(\|u_1\| + \|u_0\|_{H^1}),
\]

for any \( t \geq 0 \), where \( C > 0 \) is a constant depending on \( m > 0 \), and \( \alpha > 0 \) is a independent constant of \( m \).

**Remark 1.1** If one applies the previous results introduced by \([6]\) Corollary 2.1 and Corollary 2.2 based on Theorem 2.2 with (i) of Hypothesis 1, one can easily derive

\[
\|u_t(t, \cdot)\| + \|u(t, \cdot)\| + \|\nabla u(t, \cdot)\| \leq Ct^{-\frac{\delta}{2} + \delta}(\|u_1\|_1 + \|u_0\|_1) + Ce^{-\alpha t}(\|u_1\| + \|u_0\|_{H^1}),
\]

for any \( \delta > 0 \) and \( t \gg 1 \). A new point of view of Theorem 1.1 is in the preciseness of the statement with \( \delta = 0 \) and \( t \geq 0 \).

Concerning the profile in asymptotic sense is read as follows.

**Theorem 1.2** Let \( n \geq 1 \). If \([u_0, u_1] \in (H^1(\mathbb{R}^n) \cap L^{1,1}(\mathbb{R}^n)) \times (L^2(\mathbb{R}^n) \cap L^{1,1}(\mathbb{R}^n))\), then the solution \( u(t, x) \) to problem (1.1)-(1.2) satisfies

\[
\int_{\mathbb{R}^n} \left| F(u(t, \cdot))(\xi) - \left( P_1 e^{-t|\xi|^2/2} \sin(t\sqrt{|\xi|^2 + m^2}) \sqrt{|\xi|^2 + m^2} + P_0 e^{-t|\xi|^2/2} \cos(t\sqrt{|\xi|^2 + m^2}) \right) \right|^2 d\xi
\]

\[
\leq C(\|u_1\|_{1,1}^2 + \|u_0\|_{1,1}^2)(1 + t)^{-\frac{n+2}{2}} + C(\|u_1\|^2 + \|u_0\|^2_{H^1})e^{-\omega t} + C(\|u_0\|_1^2 + \|u_1\|_1^2)e^{-\kappa t} \quad (t \geq 0),
\]

where \( C > 0, \omega > 0, \) and \( \kappa > 0 \) are constants, which depend only on \( m > 0 \).
Furthermore, we denote the Fourier transform \( \hat{L} \) decay rate of the method introduced by [9]. As an application, we will discuss the optimality concerning the method in the Fourier space due to [22], and in section 3 we prove Theorem 1.2 by the use of \( \| \cdot \| \) norms, in particular, we use an exponent together with dissipative structure to the equation

\[
\frac{\sin(t\sqrt{|\xi|^2+m^2})}{\sqrt{|\xi|^2+m^2}} \text{ of the profile, that is, this factor is no effective in the low frequency region } |\xi| \ll 1.
\]

Based on Theorem 1.2 one can check the optimality of the decay rate just obtained in Theorem 1.1.

**Theorem 1.3** Let \( n \geq 1 \). If \( [u_0, u_1] \in (H^1(\mathbb{R}^n) \cap L^{1,1}(\mathbb{R}^n)) \times (L^2(\mathbb{R}^n) \cap L^{1,1}(\mathbb{R}^n)), \) then the corresponding solution \( u(t, x) \) to problem (1.1)-(1.2) satisfies

\[
C^{-1}(|P_0| + |P_1|) t^{-\frac{n}{2}} \leq \| u(t, \cdot) \| \leq C t^{-\frac{n}{4}} (\| u_0 \|_1 + \| u_1 \|_1 + \| u_0 \|_{H^1} + \| u_1 \|)
\]

for large \( t \gg 1 \), where the constant \( C > 0 \) depends on \( m > 0 \) and \( n \).

**Remark 1.3** When one compares the result of Theorem 1.3 for (1.1) with the one of (1.3), in particular, one has a significant difference in the low dimensional case \( n = 1, 2 \), that is, in the Klein-Gordon equation case, the wave effect is extremely weak as compared with that of diffusive one, so that one can get the decay property of the \( L^2 \)-norm of solutions even in the low dimensional case. This is because of the existence of the mass term \( m > 0 \).

In our forthcoming paper, we will study semi-linear problem to find the so called critical exponent together with dissipative structure to the equation

\[
u_{tt}(t, x) - \Delta u(t, x) + m^2 u(t, x) - \Delta u(t, x) = f(u(t, x)).
\]

(1.6)

Our plan in this paper is as follows. In section 2, we shall prove Theorems 1.1 by the energy method in the Fourier space due to [22], and in section 3 we prove Theorem 1.2 by the use of the method introduced by [9]. As an application, we will discuss the optimality concerning the decay rate of the \( L^2 \)-norm of solutions in Section 4 to prove Theorem 1.3.

**Notation.** Throughout this paper, \( \| \cdot \|_q \) stands for the usual \( L^q(\mathbb{R}^n) \)-norm. For simplicity of notations, in particular, we use \( \| \cdot \| \) instead of \( \| \cdot \|_2 \).

\[
f \in L^{1,\gamma}(\mathbb{R}^n) \iff f \in L^1(\mathbb{R}^n), \| f \|_1,\gamma := \int_{\mathbb{R}^n} (1 + |x|)^\gamma |f(x)| dx < +\infty, \quad \gamma \geq 0.
\]

Furthermore, we denote the Fourier transform \( \hat{\phi}(\xi) \) of the function \( \phi(x) \) by

\[
\mathcal{F}(\phi)(\xi) := \hat{\phi}(\xi) := \frac{1}{(2\pi)^n/2} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} \phi(x) dx,
\]

(1.7)

where \( i := \sqrt{-1} \), and \( x \cdot \xi = \sum_{i=1}^{n} x_i \xi_i \) for \( x = (x_1, \cdots, x_n) \) and \( \xi = (\xi_1, \cdots, \xi_n) \), and the inverse Fourier transform of \( \mathcal{F} \) is denoted by \( \mathcal{F}^{-1} \). When we estimate several functions by applying the Fourier transform sometimes we can also use the following definition in place of (1.7)

\[
\mathcal{F}(\phi)(\xi) := \int_{\mathbb{R}^n} e^{-ix\cdot\xi} \phi(x) dx
\]

without loss of generality. We also use the notation

\[
v_t = \frac{\partial v}{\partial t}, \quad \nu_{tt} = \frac{\partial^2 v}{\partial t^2}, \quad \Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}, \quad x = (x_1, \cdots, x_n).
\]
2 Proofs of Theorems 1.1.

To begin with, let us start with proving Theorem 1.1 by relying on the method due to [22]. In order to use the energy method in the Fourier space, we shall prepare the following notation.

\[ E_0(t, \xi) := \frac{1}{2} |\hat{u}_t|^2 + \frac{1}{2} (|\xi|^2 + m^2)|\hat{u}|^2, \]

\[ E(t, \xi) := \frac{1}{2} |\hat{u}_t|^2 + \frac{1}{2} (|\xi|^2 + m^2)|\hat{u}|^2 + \beta \rho(\xi) \Re(\hat{u}_t \overline{\hat{u}}) + \frac{1}{2} \beta \rho(\xi)|\xi|^2 |\hat{u}|^2 \]

\[ = E_0(t, \xi) + \beta \rho(\xi) \Re(\hat{u}_t \overline{\hat{u}}) + \frac{1}{2} \beta \rho(\xi)|\xi|^2 |\hat{u}|^2, \]

\[ F(t, \xi) := |\xi|^2 |\hat{u}_t|^2 + \beta \rho(\xi) (|\xi|^2 + m^2)|\hat{u}|^2, \]

\[ R(t, \xi) := \beta \rho(\xi)|\hat{u}_t|^2. \]

We define a key function \( \rho : \mathbb{R}^n_\xi \rightarrow \mathbb{R} \) by

\[ \rho(\xi) = \frac{|\xi|^2}{1 + |\xi|^2}. \]

The discovery of this key function is a crucial point in this section.

Now, let us apply the Fourier transform to the both sides of (1.1) together with the initial data (1.2). Then in the Fourier space \( \mathbb{R}^n_\xi \) one has the reduced problem

\[ \hat{u}_{tt}(t, \xi) + (|\xi|^2 + m^2)\hat{u}(t, \xi) + |\xi|^2 \hat{u}_t(t, \xi) = 0, \quad (t, \xi) \in (0, \infty) \times \mathbb{R}^n_\xi, \]

\[ \hat{u}(0, \xi) = \hat{u}_0(\xi), \quad \hat{u}_t(0, \xi) = \hat{u}_1(\xi), \quad \xi \in \mathbb{R}^n_\xi. \]

Multiply both sides of (2.1) by \( \overline{\hat{u}_t} \), and further \( \beta \rho(\xi) \overline{\hat{u}} \). Then, by taking the real part of the resulting identities one has

\[ \frac{d}{dt} E_0(t, \xi) + |\xi|^2 |\hat{u}_t|^2 = 0, \]

\[ \frac{d}{dt} \{ \beta \rho(\xi) \Re(\hat{u}_t \overline{\hat{u}}) + \frac{1}{2} \beta |\xi|^2 \rho(\xi)|\hat{u}|^2 \} + \beta (|\xi|^2 + m^2) \rho(\xi)|\hat{u}|^2 = \beta \rho(\xi)|\hat{u}_t|^2. \]

By adding (2.3) and (2.4), one has

\[ \frac{d}{dt} E(t, \xi) + F(t, \xi) = R(t, \xi). \]

We prove

**Lemma 2.1** For \( \beta > 0 \), it is true that

\[ R(t, \xi) \leq \beta F(t, \xi), \quad \xi \in \mathbb{R}^n_\xi. \]

**Proof.** Noting the facts that

\[ \rho(\xi) \leq 1, \quad \rho(\xi) \leq |\xi| \]

for all \( \xi \in \mathbb{R}^n_\xi \), the statement is true. \( \Box \)

It follows from (2.5) and Lemma 2.1 that

\[ \frac{d}{dt} E(t, \xi) + (1 - \beta) F(t, \xi) \leq 0, \]

provided that the parameters \( \beta > 0 \) are small enough such that \( 1 - \beta > 0 \).
Lemma 2.2 There is a constant $M > 0$ depending only on small $\beta > 0$ such that for all $\xi \in \mathbb{R}^n$, it follows that

$$\rho(\xi) E(t, \xi) \leq MF(t, \xi).$$

Proof. We first note the inequality

$$\Re(\hat{u}_t \bar{u}) \leq \frac{1}{2} (|\hat{u}_t|^2 + |\hat{u}|^2). \quad (2.8)$$

Then, it follows (2.6) and (2.8) that

$$\rho(\xi) E(t, \xi) \leq \frac{1}{2} |\hat{u}_t|^2 + \frac{1}{2\beta} \beta \rho(\xi) (|\xi|^2 + m^2) |\hat{u}|^2$$

$$+ \frac{\beta}{2} \rho(\xi) |\xi|^2 |\hat{u}|^2 + \beta \rho(\xi) \Re(\hat{u}_t \bar{u})$$

$$\leq \frac{1}{2} |\xi|^2 |\hat{u}_t|^2 + \frac{1}{2\beta} F(t, \xi) + \frac{1}{2} F(t, \xi) + \frac{\beta \rho(\xi)^2}{2} (|\hat{u}_t|^2 + |\hat{u}|^2)$$

$$\leq \frac{1}{2} F(t, \xi) + \frac{1}{2\beta} F(t, \xi) + \frac{1}{2} F(t, \xi) + \frac{\beta}{2} |\xi|^2 |\hat{u}_t|^2 + \frac{\beta}{2} |\xi|^2 \rho(\xi) |\hat{u}|^2$$

$$\leq (1 + \frac{1}{2\beta} + \frac{\beta}{2}) F(t, \xi) + \frac{1}{2} \beta \rho(\xi) (|\xi|^2 + m^2) |\hat{u}|^2$$

$$= (1 + \frac{1}{2\beta} + \frac{\beta}{2} + \frac{1}{2}) F(t, \xi), \quad (\forall \xi \in \mathbb{R}^n),$$

which implies the desired estimate by setting

$$M := \left( \frac{1}{2\beta} + \frac{\beta}{2} + \frac{3}{2} \right).$$

Lemma 2.2 and (2.7) imply

$$\frac{d}{dt} E(t, \xi) + (1 - \beta) \rho(\xi) M^{-1} E(t, \xi) \leq 0 \quad (2.9)$$

for any $\xi \in \mathbb{R}^n$. From (2.9) we find

$$E(t, \xi) \leq e^{-\alpha \rho(\xi)t} E(0, \xi), \quad \xi \in \mathbb{R}^n$$

(2.10)

where $\alpha := (1 - \beta) M^{-1} > 0$ with small $\beta \in (0, 1)$.

On the other hand, in the case of $\xi \neq 0$, since we have

$$\pm \beta \rho(\xi) \Re(\hat{u}_t \bar{u}) \leq \frac{\beta}{2} |\xi|^2 \rho(\xi) |\hat{u}|^2 + \frac{\beta \rho(\xi)}{2} |\hat{u}_t|^2,$$

(2.11)

it follows from the definition of $E(t, \xi)$ and (2.11) with the minus sign that

$$E(t, \xi) \geq \frac{1}{2} \left( 1 - \frac{\beta \rho(\xi)}{|\xi|^2} \right) |\hat{u}_t|^2 + \frac{1}{2} (|\xi|^2 + m^2) |\hat{u}|^2 \quad (\xi \neq 0).$$

(2.12)

And also, from (2.6) again we see

$$\frac{\beta \rho(\xi)}{|\xi|^2} \leq \frac{\beta}{|\xi|^2} |\xi|^2 = \beta,$$
so that one has
\[ 1 - \beta \rho(\xi) \geq 1 - \beta > 0, \] (2.13)
for \( \xi \neq 0 \) if we choose small \( \beta \in (0, 1) \). So, we obtain
\[ E(t, \xi) \geq (1 - \beta) E_0(t, \xi), \quad \xi \neq 0. \] (2.14)
Since \( E(t, 0) = E_0(t, 0) \), (2.14) holds true for all \( \xi \in \mathbb{R}_T^n \). Thus, from (2.10) one has
\[ E_0(t, \xi) \leq (1 - \beta)^{-1} e^{-\alpha t} E(0, \xi), \quad \xi \in \mathbb{R}_T^n. \] (2.15)
While, because of (2.11) with plus sign, for \( \xi \neq 0 \) one has
\[
E(t, \xi) \leq \frac{1}{2} |\tilde{u}_t|^2 + \frac{1}{2} (|\xi|^2 + m^2)|\tilde{u}|^2 + \frac{\beta}{2} \rho(\xi)|\tilde{u}_t|^2 + \frac{\beta}{2} |\xi|^2 |\tilde{u}|^2
\leq \frac{1}{2} (1 + \beta)|\tilde{u}_t|^2 + \frac{1}{2} (|\xi|^2 + m^2)|\tilde{u}|^2 + \frac{\beta}{2} (|\xi|^2 + m^2)|\tilde{u}|^2
\leq (1 + 2\beta) E_0(t, \xi),
\]
which implies
\[ E(t, \xi) \leq C E_0(t, \xi) \] (2.16)
for all \( \xi \in \mathbb{R}_T^n \) with some constant \( C := (1 + 2\beta) > 0 \). By (2.15) and (2.16) with \( t = 0 \) one has arrived at the significant estimate.

**Lemma 2.3** Let \( \beta > 0 \) be a small number. Then, there is a constant \( C = C(\beta) > 0 \) and \( \alpha = \alpha(\beta) > 0 \) such that for all \( \xi \in \mathbb{R}_T^n \) it is true that
\[ E_0(t, \xi) \leq C e^{-\alpha t} E_0(0, \xi). \]

**Proof of Theorem 1.1.** By lemma 2.3 and the Plancherel theorem one has
\[
\int_{\mathbb{R}_T^n} (|u_t|^2 + |\nabla u|^2 + m^2 |u|^2) dx \leq C \int_{\mathbb{R}_T^n} (|\tilde{u}_t|^2 + (|\xi|^2 + m^2)|\tilde{u}|^2) d\xi \leq C \int_{\mathbb{R}_T^n} E_0(t, \xi) d\xi
\leq C \left( \int_{|\xi| \leq 1} + \int_{|\xi| \geq 1} \right) e^{-\alpha t} (|\tilde{u}_1(\xi)|^2 + (|\xi|^2 + m^2)|\tilde{u}_0(\xi)|^2) d\xi =: C(I_{low} + I_{high}). \] (2.17)

We first prepare the following standard formula.
\[ \int_{|\xi| \leq 1} e^{-\gamma |\xi|^2} |\xi|^k d\xi \leq C (1 + t)^{-\frac{k+n}{2}} \quad (t \geq 0), \] (2.18)
for each \( k \in \mathbb{N} \cup \{0\} \), and \( \gamma > 0 \).

Now, let us start with estimating both \( I_{low} \) and \( I_{high} \) based on the shape of \( \rho(\xi) \). In fact, in the case when \( |\xi| \leq 1 \), since one has
\[ \rho(\xi) \geq \frac{|\xi|^2}{2}, \]
it follows that
\[
I_{low} \leq \int_{|\xi| \leq 1} e^{-\eta |\xi|^2} |\tilde{u}_1(\xi)|^2 d\xi + \int_{|\xi| \leq 1} e^{-\eta |\xi|^2} (|\xi|^2 + m^2)|\tilde{u}_0|^2 d\xi
\leq C \|u_1\|_1^2 (1 + t)^{-\frac{n}{2}} + C \|u_0\|_1^2 (1 + t)^{-\frac{n}{2}},
\]

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Lemma 3.1 Let us solve (3.1)-(3.2) directly under the condition that $0 < \eta < \frac{\sigma}{2}$. Here, we have just used the formula (2.18).

Next, we shall estimate the high frequency part. First, one also has

$$|\eta| \geq 1 \Rightarrow \frac{|\eta|^2}{|\eta|^2 + 1} \geq \frac{1}{2}. $$

Hence, it follows that

$$I_{\text{high}} = \int_{|\eta| \geq 1} e^{-\alpha \eta \eta^t(t)|\hat{u}_1|^2 + (|\eta|^2 + m^2)|\hat{u}_0|^2}d\eta \leq \int_{|\eta| \geq 1} e^{-\eta^t\eta(\eta)|\hat{u}_1|^2 + (|\eta|^2 + m^2)|\hat{u}_0|^2}d\eta$$

$$\leq C e^{-\eta^t\eta(\eta)|u_1|^2 + \|u_0\|_{H^1}^2},$$

with some constant $C > 0$. This completes the proof of Theorem 1.1.

3 Proof of Theorems 1.2.

In this section, let us prove Theorem 1.2 by employing the method due to [8] [9]. We first establish the asymptotic profile of the solution in the low frequency region $|\eta| \ll 1$, which is an essential ingredient.

Lemma 3.1 Let $n \geq 1$. Then, it is true that there exists a constant $C > 0$ depending on $m > 0$ such that for $t \geq 0$

$$\int_{|\eta| \leq \delta_0} |F(u(t, \cdot))(\eta) - \left( P_1 e^{-t|\eta|^2/2} \sin \left( t \sqrt{|\eta|^2 + m^2} \right) + P_0 e^{-t|\eta|^2/2} \cos \left( t \sqrt{|\eta|^2 + m^2} \right) \right) |^2 d\eta$$

$$\leq C(|u_1|_{1,1}^2 + \|u_0\|_{H^1}^2)(1 + t)^{-\frac{n+2}{2}}$$

with a small positive constant $\delta_0 \ll 1$.

In order to prove Lemma 3.1 we apply the Fourier transform with respect to the space variable $x$ of the both sides of (1.1)-(1.2). Then in the Fourier space $R^d_\delta$ one has the reduced problem:

$$\hat{u}_{tt}(t, \eta) + (|\eta|^2 + m^2)\hat{u}(t, \eta) + |\eta|^2\hat{u}_t(t, \eta) = 0, \quad (t, \eta) \in (0, \infty) \times R^d_\delta,$$

$$\hat{u}(0, \eta) = \hat{u}_0(\eta), \quad \hat{u}_t(0, \eta) = \hat{u}_1(\eta), \quad \eta \in R^d_\delta. \quad (3.2)$$

Let us solve (3.1)-(3.2) directly under the condition that $0 < |\eta| \leq \delta_0 \ll 1$. In this case we get

$$\hat{u}(t, \eta) = \hat{u}_0(\eta) e^{-\sigma_2 t} + \hat{u}_1(\eta) e^{-\sigma_1 t} \quad (3.3)$$

$$\hat{u}(t, \eta) = \frac{e^{\sigma_1 t} - e^{\sigma_2 t}}{\sigma_1 - \sigma_2} \hat{u}_1(\eta) + \frac{\sigma_1 e^{\sigma_2 t} - \sigma_2 e^{\sigma_1 t}}{\sigma_1 - \sigma_2} \hat{u}_0(\eta), \quad (3.4)$$

where $\sigma_j \in C (j = 1, 2)$ have forms:

$$\sigma_1 = -\frac{|\eta|^2 + i \sqrt{4(|\eta|^2 + m^2) - |\eta|^4}}{2}, \quad \sigma_2 = -\frac{|\eta|^2 - i \sqrt{4(|\eta|^2 + m^2) - |\eta|^4}}{2}.$$

In this connection, the smallness $0 < |\eta| \leq \delta_0 \ll 1$ of $|\eta|$ is assumed to guarantee

$$D := 4(|\eta|^2 + m^2) - |\eta|^4 > 0.$$
Now, we use the decomposition of the initial data based on the idea due to [9]:

\[ \hat{u}_j(\xi) = A_j(\xi) - iB_j(\xi) + P_j \quad (j = 0, 1), \]  

(3.5)

where

\[ A_j(\xi) := \int_{\mathbb{R}^n} (\cos(x \cdot \xi) - 1) u_j(x) \, dx, \quad B_j(x) := \int_{\mathbb{R}^n} \sin(x \cdot \xi) u_j(x) \, dx, \quad (j = 0, 1). \]

From (3.4) and (3.5) we see that

\[ \hat{u}(t, \xi) = P_1 \left( \frac{e^{\sigma_1 t} - e^{\sigma_2 t}}{\sigma_1 - \sigma_2} \right) + P_0 \left( \frac{\sigma_1 e^{\sigma_1 t} - \sigma_2 e^{\sigma_2 t}}{\sigma_1 - \sigma_2} \right) + (A_1(\xi) - iB_1(\xi)) \left( \frac{e^{\sigma_1 t} - e^{\sigma_2 t}}{\sigma_1 - \sigma_2} \right) + (A_0(\xi) - iB_0(\xi)) \left( \frac{\sigma_1 e^{\sigma_1 t} - \sigma_2 e^{\sigma_2 t}}{\sigma_1 - \sigma_2} \right), \]

(3.6)

for all \( \xi \) satisfying \( 0 < |\xi| \leq \delta_0 \). It is easy to check that

\[ \frac{e^{\sigma_1 t} - e^{\sigma_2 t}}{\sigma_1 - \sigma_2} = \frac{2e^{-t|\xi|^2/2} \sin\left(\frac{\sqrt{D}t}{2}\right)}{\sqrt{D}}, \]  

(3.7)

\[ \frac{\sigma_1 e^{\sigma_1 t} - \sigma_2 e^{\sigma_2 t}}{\sigma_1 - \sigma_2} = \frac{|\xi|^2 e^{-t|\xi|^2/2} \sin\left(\frac{\sqrt{D}t}{2}\right)}{\sqrt{D}} + e^{-t|\xi|^2/2} \cos\left(\frac{t\sqrt{D}}{2}\right). \]

(3.8)

If we set

\[ K_1(t, \xi) := P_0 \frac{|\xi|^2 e^{-t|\xi|^2/2} \sin\left(\frac{\sqrt{D}t}{2}\right)}{\sqrt{D}}, \]

\[ K_2(t, \xi) := (A_1(\xi) - iB_1(\xi)) \left( \frac{e^{\sigma_1 t} - e^{\sigma_2 t}}{\sigma_1 - \sigma_2} \right), \]

\[ K_3(t, \xi) := (A_0(\xi) - iB_0(\xi)) \left( \frac{\sigma_1 e^{\sigma_1 t} - \sigma_2 e^{\sigma_2 t}}{\sigma_1 - \sigma_2} \right), \]

then it follows from (3.6), (3.7) and (3.8) that

\[ \hat{u}(t, \xi) = 2P_1 \frac{e^{-t|\xi|^2/2} \sin\left(\frac{\sqrt{D}t}{2}\right)}{\sqrt{D}} + P_0 e^{-t|\xi|^2/2} \cos\left(\frac{t\sqrt{D}}{2}\right) + K_1(t, \xi) + K_2(t, \xi) + K_3(t, \xi), \quad 0 < |\xi| \leq \delta_0. \]

(3.9)

Let us shave the needless factors of the right hand side of (3.9) to get a precise shape of the asymptotic profile by using the mean value theorem. In fact, from the mean value theorem it follows that

\[ \sin\left(\frac{t\sqrt{D}}{2}\right) = \sin(t\sqrt{E}) + \frac{t}{2} \left( \sqrt{D} - 2\sqrt{E} \right) \cos\left( \epsilon(t, \xi) \right), \]

\[ \cos\left(\frac{t\sqrt{D}}{2}\right) = \cos\left(\sqrt{E} \right) - \frac{t}{2} \left( \sqrt{D} - 2\sqrt{E} \right) \sin\left( \eta(t, \xi) \right), \]

\[ \frac{2}{\sqrt{D}} = \frac{1}{\sqrt{E}} + \frac{|\xi|^4}{\sqrt{(4E - |\xi|^4\theta_3)^3}}, \]
so that from (3.9) one has arrived at the identity

\[ E := |\xi|^2 + m^2, \]

\[ e(t, \xi) := \frac{t\sqrt{D}}{2} \theta_1 + t \sqrt{E}(1 - \theta_1), \quad \exists \theta_1 \in (0, 1), \]

\[ \eta(t, \xi) := \frac{t\sqrt{D}}{2} \theta_2 + t \sqrt{E}(1 - \theta_2) \quad \exists \theta_2 \in (0, 1), \]

so that from (3.9) one has arrived at the identity

\[ \dot{u}(t, \xi) = P_1 e^{-t|\xi|^2/2} \frac{1}{\sqrt{E}} \sin(t\sqrt{E}) + P_0 e^{-t|\xi|^2/2} \cos(t\sqrt{E}) \]

\[ + 2P_1|\xi|^4 e^{-t|\xi|^2/2} \frac{1}{\sqrt{(4E - |\xi|^4 \theta_3)^3}} \sin(t\sqrt{E}) + P_1 t e^{-t|\xi|^2/2} \frac{\sqrt{D} - 2\sqrt{E}}{\sqrt{E}} \cos(e(t, \xi)) \]

\[ + P_1 t|\xi|^4 e^{-t|\xi|^2/2} \frac{\sqrt{D} - 2\sqrt{E}}{\sqrt{(4E - |\xi|^4 \theta_3)^3}} \sin(e(t, \xi)) - \frac{P_0 t}{2} e^{-t|\xi|^2/2} \frac{(\sqrt{D} - 2\sqrt{E}) \sin(\eta(t, \xi)) + \sum_{j=1}^{3} K_j(t, \xi)}{\sqrt{(4E - |\xi|^4 \theta_3)^3}}. \]

Set

\[ K_4(t, \xi) := 2P_1|\xi|^4 e^{-t|\xi|^2/2} \frac{1}{\sqrt{(4E - |\xi|^4 \theta_3)^3}} \sin(t\sqrt{E}) \]

\[ K_5(t, \xi) := P_1 t e^{-t|\xi|^2/2} \frac{\sqrt{D} - 2\sqrt{E}}{\sqrt{E}} \cos(e(t, \xi)), \]

\[ K_6(t, \xi) := P_1 t|\xi|^4 e^{-t|\xi|^2/2} \frac{\sqrt{D} - 2\sqrt{E}}{\sqrt{(4E - |\xi|^4 \theta_3)^3}} \sin(e(t, \xi)), \]

and

\[ K_7(t, \xi) := -\frac{P_0 t}{2} e^{-t|\xi|^2/2} (\sqrt{D} - 2\sqrt{E}) \sin(\eta(t, \xi)). \]

Then, one has arrived at the significant identity, which holds in the low frequency region $|\xi| \leq \delta_0$:

\[ \dot{u}(t, \xi) = P_1 e^{-t|\xi|^2/2} \frac{\sin(t\sqrt{|\xi|^2 + m^2})}{\sqrt{|\xi|^2 + m^2}} + P_0 e^{-t|\xi|^2/2} \cos(t\sqrt{|\xi|^2 + m^2}) + \sum_{j=1}^{7} K_j(t, \xi). \quad (3.10) \]

In the following part, let us check decay orders of the remainder terms $K_j(t, \xi)$ $(j = 1, 2, 3, 4, 5, 6, 7)$ of (3.10) based on the formula (2.18). Note that $\delta_0 > 0$ is a sufficiently small number such that $4m^2 > \delta_0^4$.

(I) **Estimate for $K_1(t, \xi)$**.

\[ \int_{|\xi| \leq \delta_0} d\xi \leq |P_0|^2 \int_{|\xi| \leq \delta_0} \frac{|\xi|^4 e^{-t|\xi|^2/2} \sin(t\sqrt{D}/2)^2}{4(|\xi|^2 + m^2) - |\xi|^4} d\xi \]

\[ \leq \frac{|P_0|^2}{4m^2 - \delta_0^4} \int_{|\xi| \leq \delta_0} |\xi|^4 e^{-t|\xi|^2/2} d\xi \leq \frac{|P_0|^2}{4m^2 - \delta_0^4} (1 + t)^{-\frac{3\delta_0^4}{4m^2}}. \quad (3.11) \]

(II) **Estimate for $K_4(t, \xi)$**.

\[ \int_{|\xi| \leq \delta_0} d\xi \leq 4P_1^2 \int_{|\xi| \leq \delta_0} \frac{|\xi|^8 e^{-t|\xi|^2/2} \sin(t\sqrt{E})^2}{\sqrt{(4(|\xi|^2 + m^2) - |\xi|^4 \theta_3)^6}} d\xi \]

\[ \leq 4P_1^2 \frac{1}{(4m^2 - \delta_0^4)^3} \int_{|\xi| \leq \delta_0} |\xi|^8 e^{-t|\xi|^2/2} d\xi \leq 4P_1^2 \frac{1}{(4m^2 - \delta_0^4)^3} (1 + t)^{-\frac{3\delta_0^4}{4m^2}}. \quad (3.12) \]
Here, we shall prepare the elementary inequality such that
\[
|\sqrt{D} - 2\sqrt{E}| = \frac{|\xi|^4}{\sqrt{D} + 2\sqrt{E}} \leq \frac{|\xi|^4}{|\xi|^2 + m^2} \leq \frac{|\xi|^4}{m} \quad (|\xi| \leq \delta_0).
\]
(III) Estimate for $K_5(t, \xi)$.
It follows from (3.13) again that
\[
\int_{|\xi| \leq \delta_0} |K_5(t, \xi)|^2 d\xi \leq P_1^2 \int_{|\xi| \leq \delta_0} e^{-t|\xi|^2} \left| \frac{\sqrt{D} - 2\sqrt{E}}{4E - |\xi|^4} \cos (\varepsilon(t, \xi)) \right|^2 d\xi 
\leq m^{-4} P_1^2 \int_{|\xi| \leq \delta_0} e^{-t|\xi|^2} |\xi|^8 d\xi \leq m^{-4} P_1^2 (1 + t)^{-\frac{a+4}{2}}. \quad (3.14)
\]

(IV) Estimate for $K_6(t, \xi)$.
It follows from (3.13) again that
\[
\int_{|\xi| \leq \delta_0} |K_6(t, \xi)|^2 d\xi \leq |P_1|^2 \int_{|\xi| \leq \delta_0} |\xi|^2 e^{-t|\xi|^2} \left| \frac{\sqrt{D} - 2\sqrt{E}}{4E - |\xi|^4} \cos (\varepsilon(t, \xi)) \right|^2 d\xi 
\leq \frac{|P_1|^2 (1 + t)^2}{m^2} |4m^2 - \delta_0^4|^{-3} \int_{|\xi| \leq \delta_0} e^{-t|\xi|^2} |\xi|^16 d\xi 
\leq \frac{|P_1|^2 (1 + t)^2}{m^2} |4m^2 - \delta_0^4|^{-3} (1 + t)^{-\frac{a+12}{2}}. \quad (3.15)
\]

(V) Estimate for $K_7(t, \xi)$.
Because of (3.13) again one has
\[
\int_{|\xi| \leq \delta_0} |K_7(t, \xi)|^2 d\xi \leq |P_1|^2 (1 + t)^2 m^{-2} \int_{|\xi| \leq \delta_0} |\xi|^2 e^{-t|\xi|^2} \left| \frac{\sin (\eta(t, \xi))}{\sin (\theta(t, \xi))} \right|^2 d\xi 
\leq |P_1|^2 (1 + t)^2 m^{-2} \int_{|\xi| \leq \delta_0} |\xi|^2 e^{-t|\xi|^2} |\xi|^2 d\xi \leq m^{-2} |P_1|^2 (1 + t)^{-\frac{a+4}{2}}. \quad (3.16)
\]

In order to estimate further $K_j$ ($j = 2, 3$), we prepare the following lemma, introduced in [5].

Lemma 3.2 Let $n \geq 1$. Then it holds that for all $\xi \in \mathbb{R}^n$
\[
|A_j(\xi)| \leq L |\xi||u_j|_{1,1} \quad (j = 0, 1),
|B_j(\xi)| \leq M |\xi||u_j|_{1,1} \quad (j = 0, 1),
\]
where
\[
L := \sup_{\theta \neq 0} \frac{|1 - \cos \theta|}{|\theta|} < +\infty, \quad M := \sup_{\theta \neq 0} \frac{|\sin \theta|}{|\theta|} < +\infty,
\]
and both $A_j(\xi)$ and $B_j(\xi)$ are defined in (3.5).

Basing on Lemma 3.2 we check decay rates of $K_j(t, \xi)$ with $j = 2, 3$. This part is essential in this paper.

(VI) Estimate for $K_2(t, \xi)$.
It follows from (3.7) and Lemma 3.2 that
\[
\int_{|\xi| \leq \delta_0} |K_2(t, \xi)|^2 d\xi \leq C (L^2 + M^2) |u_1|_{1,1}^2 \int_{|\xi| \leq \delta_0} |\xi|^2 e^{-t|\xi|^2} \left( \frac{\sin^2 \left( \frac{\sqrt{D}}{2} \right)}{2(|\xi|^2 + m^2) - |\xi|^4} \right) d\xi,
\leq C |u_1|_{1,1}^2 (4m^2 - \delta_0^4)^{-1} \int_{|\xi| \leq \delta_0} |\xi|^2 e^{-t|\xi|^2} d\xi \leq C |u_1|_{1,1}^2 (1 + t)^{-\frac{a+4}{2}}. \quad (3.17)
\]
(VII) Estimate for $K_3(t, \xi)$.

Similarly,

$$\int_{|\xi| \leq \delta_0} |K_3(t, \xi)|^2 d\xi \leq C(L^2 + M^2)\|u_0\|_{1,1}^2 \int_{|\xi| \leq \delta_0} |\xi|^6 e^{-t|\xi|^2} \sin^2 \left( \frac{t \sqrt{D}}{2} \right) d\xi$$

$$+ C(L^2 + M^2)\|u_0\|_{1,1}^2 \int_{|\xi| \leq \delta_0} |\xi|^2 e^{-t|\xi|^2} \cos^2 \left( \frac{t \sqrt{D}}{2} \right) d\xi$$

$$\leq C\|u_0\|_{1,1}^2 \int_{|\xi| \leq \delta_0} |\xi|^6 e^{-t|\xi|^2} d\xi + C\|u_0\|_{1,1}^2 \int_{|\xi| \leq \delta_0} |\xi|^2 e^{-t|\xi|^2} d\xi$$

$$\leq C\|u_0\|_{1,1}^2 (1 + t) - \frac{n+6}{2} + C\|u_0\|_{1,1}^2 (1 + t) - \frac{n+2}{2}, \quad (t \geq 0), \quad (3.18)$$

Under these preparations let us prove Lemma 3.1.

Proof of Lemma 3.1. It follows from (3.10), (3.13), (3.15), (3.16), (3.17), (3.18) and (3.18) that

$$\int_{|\xi| \leq \delta_0} |\mathcal{F}(u(t, \cdot))(\xi) - \left( P_1 e^{-t|\xi|^2/2} \sin \left( t \frac{\sqrt{\xi^2 + m^2}}{\sqrt{\xi^2 + m^2}} \right) + P_0 e^{-t|\xi|^2/2} \cos \left( t \frac{\sqrt{\xi^2 + m^2}}{\sqrt{\xi^2 + m^2}} \right) \right) |^2 d\xi$$

$$\leq C\|u_1\|_{1,1}^2 (1 + t) - \frac{n+2}{2} + C\|u_0\|_{1,1}^2 (1 + t) - \frac{n+2}{2}, \quad (t \geq 0),$$

which implies the desired estimate.

Proof of Theorem 1.2. First we decompose the integrand to be estimated into two parts such that one is the low frequency part, and the other is high frequency one as follows.

$$\int_{\mathbb{R}^n} |\mathcal{F}(u(t, \cdot))(\xi) - \left( P_1 e^{-t|\xi|^2/2} \sin \left( t \frac{\sqrt{\xi^2 + m^2}}{\sqrt{\xi^2 + m^2}} \right) + P_0 e^{-t|\xi|^2/2} \cos \left( t \frac{\sqrt{\xi^2 + m^2}}{\sqrt{\xi^2 + m^2}} \right) \right) |^2 d\xi$$

$$\left( \int_{|\xi| \leq \delta_0} + \int_{|\xi| \geq \delta_0} \right) |\hat{u}(t, \xi) - \left( P_1 e^{-t|\xi|^2/2} \sin \left( t \frac{\sqrt{\xi^2 + m^2}}{\sqrt{\xi^2 + m^2}} \right) + P_0 e^{-t|\xi|^2/2} \cos \left( t \frac{\sqrt{\xi^2 + m^2}}{\sqrt{\xi^2 + m^2}} \right) \right) |^2 d\xi$$

$$=: I_1(t) + I_h(t).$$

To begin with, as a direct consequence of Lemma 3.1 we can see

$$I_1(t) \leq C\|u_1\|_{1,1}^2 + \|u_0\|_{1,1}^2 (1 + t) - \frac{n+2}{2}. \quad (3.19)$$

On the other hand, it follows from Lemma 2.3 that

$$|\tilde{u}_t(t, \xi)|^2 + (|\xi|^2 + m^2)|\tilde{u}(t, \xi)|^2 \leq C e^{-\alpha \rho(\xi)t} (|\tilde{u}_1(\xi)|^2 + (|\xi|^2 + m^2)|\tilde{u}_0(\xi)|^2), \quad \xi \in \mathbb{R}^n. \quad (3.20)$$

In this case, we see that if $\delta_0 \leq |\xi| \leq 1$, then

$$\rho(\xi) = \frac{|\xi|^2}{1 + |\xi|^2} \geq \frac{|\xi|^2}{2} \geq \frac{\delta_0^2}{2},$$

and if $|\xi| \geq 1$, then

$$\rho(\xi) = \frac{|\xi|^2}{1 + |\xi|^2} \geq \frac{|\xi|^2}{2|\xi|^2} = \frac{1}{2} \geq \frac{\delta_0^2}{2}.$$
Therefore, from (3.20) it follows that
\[
m^2 \int_{|\xi| \geq \delta_0} |\hat{u}(t, \xi)|^2 d\xi \leq C \int_{|\xi| \geq \delta_0} e^{-\alpha \rho(\xi)t} \left( |\hat{u}_1(\xi)|^2 + (|\xi|^2 + m^2)|\hat{u}_0(\xi)|^2 \right) d\xi
\]
\[
\leq \int_{|\xi| \geq \delta_0} e^{-\frac{\alpha \rho^2}{2}t} \left( |\hat{u}_1(\xi)|^2 + (|\xi|^2 + m^2)|\hat{u}_0(\xi)|^2 \right) d\xi
\]
\[
\leq Ce^{-\omega t} \left( \|u_1\|^2 + \|u_0\|^2_{H^1} \right) \quad (t \geq 0),
\]
where \( \omega := \alpha \delta_0^2/2 \), and the constant \( C > 0 \) depends on \( \delta_0 \), so that \( m > 0 \).

On the other hand,
\[
P_1^2 \int_{|\xi| \geq \delta_0} e^{-t|\xi|^2} \left| \frac{\sin \left( t \sqrt{|\xi|^2 + m^2} \right)}{\sqrt{|\xi|^2 + m^2}} \right|^2 d\xi + P_0^2 \int_{|\xi| \geq \delta_0} e^{-t|\xi|^2} \cos \left( t \sqrt{|\xi|^2 + m^2} \right)^2 d\xi
\]
\[
\leq C \left( \frac{1}{m^2} + 1 \right) (\|u_1\|^2 + \|u_0\|^2_{H^1}) \int_{|\xi| \geq \delta_0} e^{-t|\xi|^2} d\xi
\]
\[
= C \left( \frac{1}{m^2} + 1 \right) (\|u_1\|^2 + \|u_0\|^2_{H^1}) \int_{|\xi| \geq \delta_0} e^{-\frac{1}{m^2} e^{-\frac{t}{2}} e^{-\frac{t}{2}} d\xi}
\]
\[
\leq C (\|u_1\|^2 + \|u_0\|^2_{H^1}) e^{-\frac{t}{2}} (1 + t)^{-\frac{\alpha}{2}} \leq C (\|u_1\|^2 + \|u_0\|^2_{H^1}) e^{-\kappa t} \quad (t \gg 1),
\]
with some constants \( C > 0 \) and \( \kappa > 0 \). Therefore, by evaluating \( I_h(t) \) based on (3.21) and (3.22), and combining it with (3.19) one can arrive at
\[
\int_{\mathbb{R}^n} |\mathcal{F}(u(t, \cdot))(\xi)| - \left\{ \int_{\mathbb{R}^n} e^{-t|\xi|^2/2} \sin \left( t \sqrt{|\xi|^2 + m^2} \right) + P_0 e^{-t|\xi|^2} \cos \left( t \sqrt{|\xi|^2 + m^2} \right) \right\}^2 d\xi
\]
\[
\leq C (\|u_1\|^2_{L^1} + \|u_0\|^2_{L^1}) (1 + t)^{-\frac{\alpha}{2}} + C e^{-\omega t} (\|u_1\|^2 + \|u_0\|^2_{H^1})
\]
\[
+ C e^{-\kappa t} (\|u_0\|^2_{L^1} + \|u_1\|^2_{L^1}) \quad (t \geq 0).
\]
This implies the desired estimate. \( \square \)

4 Optimality of decay rates.

In this section, we shall study the optimality of the decay rates of the solution obtained in Theorem 1.1. This is studied based on the results obtained in Theorems 1.1 and 1.2. We first compute two simple estimates from above on the quantities below. Set
\[
I_1(t) := \int_{\mathbb{R}^n_{\rho}} e^{-t|\xi|^2} \left| \frac{\sin \left( t \sqrt{|\xi|^2 + m^2} \right)}{\sqrt{|\xi|^2 + m^2}} \right|^2 d\xi,
\]
\[
I_0(t) := \int_{\mathbb{R}^n_{\rho}} e^{-t|\xi|^2} \left| \cos \left( t \sqrt{|\xi|^2 + m^2} \right) \right|^2 d\xi.
\]
Now, it follows that
\[
I_1(t) \leq \frac{1}{m^2} \int_{\mathbb{R}^n_{\rho}} e^{-t|\xi|^2} d\xi \leq C m^{-2} (1 + t)^{-\frac{\alpha}{2}},
\]
(4.1)
and
\[ I_0(t) \leq \int_{\mathbb{R}^n_t} e^{-t|k|^2} d\xi \leq C(1 + t)^{-\frac{n}{2}}. \] (4.2)

Next, in order to get the estimates from below about \( I_j(t) (j = 0, 1) \), we use the polar co-ordinate transformation as follows:

\[ I_1(t) = |\omega_n| t^{-\frac{n}{2}} \int_0^\infty \sigma^{n-1} e^{-\sigma^2} \frac{t \sin^2 \left( t \sqrt{\frac{\sigma^2}{t} + m^2} \right)}{tm^2 + \sigma^2} d\sigma \]
\[ = \frac{|\omega_n|}{2} t^{-\frac{n}{2}} \int_0^\infty \sigma^{n-1} e^{-\sigma^2} \frac{t}{\sigma^2 + tm^2} d\sigma - \frac{|\omega_n|}{2} t^{-\frac{n}{2}} \int_0^\infty \sigma^{n-1} e^{-\sigma^2} \frac{t \cos(2t \sqrt{\frac{\sigma^2}{t} + m^2})}{\sigma^2 + tm^2} d\sigma, \] (4.3)

where the surface measure of the \( n \)-dimensional unit ball is defined by
\[ |\omega_n| := \int_{|\omega|=1} d\omega. \]

Let us estimate (4.3) as follows. Since
\[ 0 \leq \sigma^{n-1} e^{-\sigma^2} \frac{t}{\sigma^2 + tm^2} \leq \frac{1}{m^2} \sigma^{n-1} e^{-\sigma^2} \in L^1(0, \infty), \]
and
\[ \lim_{t \to \infty} \sigma^{n-1} e^{-\sigma^2} \frac{t}{\sigma^2 + tm^2} = \frac{1}{m^2} \sigma^{n-1} e^{-\sigma^2}, \]
it follows from the Lebesgue convergence theorem that
\[ \lim_{t \to \infty} \int_0^\infty \sigma^{n-1} e^{-\sigma^2} \frac{t}{\sigma^2 + tm^2} d\sigma = \frac{1}{m^2} \int_0^\infty \sigma^{n-1} e^{-\sigma^2} d\sigma =: K_n > 0, \]
so that there exists a constant \( t_0 \gg 1 \) such that for all \( t \geq t_0 \) one has
\[ \int_0^\infty \sigma^{n-1} e^{-\sigma^2} \frac{t}{\sigma^2 + tm^2} d\sigma \geq \frac{1}{2} K_n. \]
This implies
\[ I_1(t) \geq \frac{K_n |\omega_n|}{4} t^{-\frac{n}{2}} - \frac{|\omega_n|}{2} t^{-\frac{n}{2}} \int_0^\infty \sigma^{n-1} e^{-\sigma^2} \frac{t \cos(2t \sqrt{\frac{\sigma^2}{t} + m^2})}{\sigma^2 + tm^2} d\sigma \] (4.4)
for all \( t \geq t_0 \). Set
\[ C_1(t) := \int_0^\infty \sigma^{n-1} e^{-\sigma^2} \frac{t}{\sigma^2 + tm^2} \cos(2t \sqrt{\frac{\sigma^2}{t} + m^2}) d\sigma. \]
Then, it holds that
\[ C_1(t) \]
\[ = \frac{1}{m^2} \int_0^\infty \sigma^{n-1} e^{-\sigma^2} \cos(2t \sqrt{\frac{\sigma^2}{t} + m^2}) d\sigma - \frac{1}{m^2} \int_0^\infty \sigma^{n-1} e^{-\sigma^2} \frac{\sigma^2}{\sigma^2 + tm^2} \cos(2t \sqrt{\frac{\sigma^2}{t} + m^2}) d\sigma. \] (4.5)
Here, by the Lebesgue convergence theorem it follows again that
\[ 0 \leq \int_0^\infty \sigma^{n-1} e^{-\sigma^2} \frac{\sigma^2}{\sigma^2 + tm^2} \cos(2t \sqrt{\frac{\sigma^2}{t} + m^2}) d\sigma \leq \int_0^\infty \sigma^{n-1} e^{-\sigma^2} \frac{\sigma^2}{\sigma^2 + tm^2} d\sigma \to 0 \] (4.6)
as $t \to \infty$. Furthermore, it follows from the Lebesgue theorem that (see Appendix below)

$$\lim_{t \to \infty} \int_0^\infty \sigma^{n-1} e^{-\sigma^2} \cos(2t \sqrt{\frac{\sigma^2}{t} + m^2}) d\sigma = 0.$$  \hspace{2cm} (4.7)

Therefore, it follows from (4.5)-(4.7) that

$$\lim_{t \to \infty} C_1(t) = 0.$$

(4.8)

Because of (4.4) and (4.8) one has

$$I_1(t) \geq \frac{K_n |\omega|}{4} t^{-\frac{n}{2}} - \frac{|\omega|}{2} t^{-\frac{n}{2}} o(1) \quad (t \to \infty).$$

Thus, there exists a constant $t_1(\geq t_0)$ such that for all $t \geq t_1$

$$I_1(t) \geq \frac{K_n |\omega|}{8} t^{-\frac{n}{2}}.$$  \hspace{2cm} (4.9)

Next, concerning $I_0(t)$, similarly to the computation for $I_1(t)$ one can get

$$I_0(t) = \frac{|\omega|}{2} t^{-\frac{n}{2}} \int_0^\infty \sigma^{n-1} e^{-\sigma^2} d\sigma + \frac{|\omega|}{2} t^{-\frac{n}{2}} \int_0^\infty \sigma^{n-1} e^{-\sigma^2} \cos(2t \sqrt{\frac{\sigma^2}{t} + m^2}) d\sigma,$$

so that by the Riemann-Lebesgue theorem it follows that

$$I_0(t) \geq L_n |\omega| t^{-\frac{n}{2}}, \quad (t \geq t_1)$$

(4.10)

where

$$L_n := \int_0^\infty \sigma^{n-1} e^{-\sigma^2} d\sigma.$$

On the other hand, set

$$I_2(t) := \int_{\mathbb{R}^d} \frac{e^{-t|\xi|^2}}{\sqrt{|\xi|^2 + m^2}} \sin\left(2t \sqrt{\frac{|\xi|^2}{t} + m^2}\right) d\xi.$$

Then, by the polar-coordinate transform one has

$$I_2(t) = |\omega| t^{-\frac{n}{2}} \int_0^\infty \frac{e^{-\sigma^2} \sigma^{n-1}}{\sqrt{\sigma^2/t + m^2}} \sin(2t \sqrt{\sigma^2/t + m^2}) d\sigma.$$  \hspace{2cm} (4.11)

Since $e^{-\sigma^2} \in L^1(0, \infty)$ for $k \in \mathbb{N} \cup \{0\}$, one can also get

$$\int_0^\infty \frac{e^{-\sigma^2} \sigma^{n-1}}{\sqrt{\sigma^2/t + m^2}} \sin(2t \sqrt{\sigma^2/t + m^2}) d\sigma = (o(1) + O(t^{-1})) \quad (t \to \infty),$$

so that from (4.11) one can get

$$I_2(t) = |\omega| t^{-\frac{n}{2}} (o(1) + O(t^{-1})) \quad (t \to \infty).$$  \hspace{2cm} (4.12)

Indeed, because of the mean value theorem it follows that

$$\frac{1}{\sqrt{\frac{\sigma^2}{t} + m^2}} = \frac{1}{m} - \frac{\sigma^2}{2t (\theta \sigma^2/t + m^2)^{3/2}}$$
for some $\theta \in (0,1)$, so that one has
\[
\int_0^\infty \frac{e^{-\sigma^2 \sigma^{-1}}}{\sqrt{(\sigma^2/t) + m^2}} \sin(2t \sqrt{(\sigma^2/t) + m^2}) d\sigma
\]
\[
= \frac{1}{m} \int_0^\infty e^{-\sigma^2 \sigma^{-1}} \sin(2t \sqrt{(\sigma^2/t) + m^2}) d\sigma
\]
\[
- \frac{1}{2t} \int_0^\infty e^{-\sigma^2 \sigma^{n+1}} \frac{1}{((\theta \sigma^2/t) + m^2)^{3/2}} \sin(2t \sqrt{(\sigma^2/t) + m^2}) d\sigma.
\]
Here, one can estimate as follows:
\[
\frac{1}{2t} \int_0^\infty e^{-\sigma^2 \sigma^{n+1}} \frac{1}{((\theta \sigma^2/t) + m^2)^{3/2}} \sin(2t \sqrt{(\sigma^2/t) + m^2}) d\sigma
\]
\[
\leq \frac{1}{2m^3t} \int_0^\infty e^{-\sigma^2 \sigma^{n+1}} d\sigma = O(t^{-1}) \quad (t \to \infty),
\]
so that
\[
I_2(t) = |\omega_n| t^{-\frac{n}{2}} \left( \frac{1}{m} \int_0^\infty e^{-\sigma^2 \sigma^{-1}} \sin(2t \sqrt{(\sigma^2/t) + m^2}) d\sigma + O(t^{-1}) \right),
\]
which implies the validity of (4.12), where one has just used the similar argument to deriving (4.7) (see also Appendix below).

Now, noting the identity
\[
\|P_1 e^{-\frac{\xi^2}{2}} \sin(\sqrt{\xi^2 + m^2}) + P_0 e^{-\frac{\xi^2}{2}} \cos(\sqrt{\xi^2 + m^2})\|^2
\]
\[
= |P_1|^2 I_1(t) + |P_0|^2 I_0(t) + P_1 P_0 I_2(t),
\]
it follows from (4.9), (4.10) and (4.12) that
\[
\|P_1 e^{-\frac{\xi^2}{2}} \sin(\sqrt{\xi^2 + m^2}) + P_0 e^{-\frac{\xi^2}{2}} \cos(\sqrt{\xi^2 + m^2})\|^2
\]
\[
\geq \frac{K_n}{8} |\omega_n| t^{-\frac{1}{2}} + \frac{L_n}{4} |\omega_n| t^{-\frac{1}{2}} + P_1 P_0 |\omega_n| t^{-\frac{1}{2}} (o(1) + O(t^{-1}))
\]
for large $t \gg 1$. This shows
\[
\|P_1 e^{-\frac{\xi^2}{2}} \sin(\sqrt{\xi^2 + m^2}) + P_0 e^{-\frac{\xi^2}{2}} \cos(\sqrt{\xi^2 + m^2})\|^2
\]
\[
\geq |P_1|^2 \frac{K_n |\omega_n|}{16} t^{-\frac{1}{2}} + |P_0|^2 \frac{L_n |\omega_n|}{8} t^{-\frac{1}{2}}
\]
for large $t \gg 1$.

Based on the estimates (4.13) and Theorems 1.1 and 1.2 combined with the Plancherel theorem, one can prove Theorem 1.3 as follows. This is rather standard nowadays. Indeed, one has
\[
\|u(t, \cdot)\| = \|\dot{u}(t, \cdot)\| \geq \|P_1 e^{-\frac{\xi^2}{2}} \sin(\sqrt{\xi^2 + m^2}) + P_0 e^{-\frac{\xi^2}{2}} \cos(\sqrt{\xi^2 + m^2})\|
\]
\[- \| \hat{u}(t, \cdot) - \left( P_1 e^{-tu^2 \frac{\sin(t\sqrt{|\xi|^2 + m^2})}{\sqrt{|\xi|^2 + m^2}}} + P_0 e^{-tu^2 \frac{\cos(t\sqrt{|\xi|^2 + m^2})}{\sqrt{|\xi|^2 + m^2}}} \right) \| \]
\[\geq \left( |P_1|^2 \frac{K_n |\omega_n|}{16} + |P_0|^2 \frac{L_n |\omega_n|}{8} \right)^{1/2} t^{-\frac{n}{4}} + O(t^{-\frac{n+2}{4}})\]

for large $t \gg 1$. Anyway, under the assumption of Theorem 1.3, one can get the estimate of $\| u(t, \cdot) \|$ from below. Concerning the estimates from above, one can use Theorem 1.1 directly. This completes the proof of Theorem 1.3. \[\Box\]

5 Appendix.

In this section, we will check the validity of (4.7) by using the Lebesgue convergence theorem. In this connection, in the case when $m = 0$ (massless type) (4.7) is a direct consequence of the Riemann-Lebesgue theorem because of $\sigma^{n-1} e^{-\sigma^2} \in L^1(0, \infty)$. In the case of $m > 0$, one needs a little troublesome computations by using a specialty of the functions $\sigma^{n-1} e^{-\sigma^2}$ as follows.

We set

\[I(t) := \int_0^\infty \sigma^{n-1} e^{-\sigma^2} \cos \left( 2t \sqrt{\sigma^2 + m^2} \right) d\sigma.\]

Then, one has

\[I(t) = t^n \int_0^\infty \sigma^{n-1} e^{-\sigma^2} \cos \left( 2t \sqrt{\sigma^2 + m^2} \right) d\sigma,\]

so that one can get

\[0 \leq I(t) \leq t^n \int_0^\infty \sigma^{n-1} e^{-\sigma^2} d\sigma = \left( \int_0^1 + \int_1^\infty \right) t^n \sigma^{n-1} e^{-\sigma^2} d\sigma =: I_{low}(t) + I_{high}(t).\]

Let us estimate $I_{low}(t)$ and $I_{high}(t)$, respectively.

To begin with, one can get

\[\lim_{t \to \infty} t^n \sigma^{n-1} e^{-\sigma^2} = 0 \quad (a.e. \sigma \in [0, \infty)) \tag{5.1}\]

So, in order to apply the Lebesgue dominated convergence theorem, one has to get a priori bounds for the $t$-dependent functions $t^n \sigma^{n-1} e^{-\sigma^2}$ in each interval $[0, 1]$ and $[1, \infty)$.

**Case I:** In the case of $\sigma \in (0, 1]$, there exists a constant $M > 0$ such that

\[0 \leq t^n \sigma^{n-1} e^{-\sigma^2} \leq M,\]

where $M$ is independent of any $t \gg 1$ and $\sigma \in (0, 1]$. Together with (5.1), one can get

\[\lim_{t \to \infty} I_{low}(t) = 0.\]

**Case II:** In the case when $\sigma \in [1, \infty)$, if we choose $\ell \in \mathbb{N}$ large enough such that $2\ell > n$, then one can get

\[0 \leq t^n \sigma^{n-1} e^{-\sigma^2} \leq \frac{\sigma^{n-1} t^n}{\ell!} \leq t^{-\frac{n+2}{2}\ell!} \ell! \sigma^{-2\ell+n-1} \leq \ell! \sigma^{-2\ell+n-1}.\]
with \( \sigma^{-2t+n-1} \in L^1(1, \infty) \) for \( t \gg 1 \). Together with (5.1), one can also get
\[
\lim_{t \to \infty} I_{\text{high}}(t) = 0.
\]

Anyway, from the argument above, one can check the validity of (4.7).

\[\square\]

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