On the Optimal Control of Network LQR with Spatially-Exponential Decaying Structure

Runyu (Cathy) Zhang, Weiyu Li, Na Li

Abstract—This paper studies network LQR problems with system matrices being spatially-exponential decaying (SED) between nodes in the network. The major objective is to study whether the optimal controller also enjoys a SED structure, which is an appealing property for ensuring the optimality of decentralized control over the network. We start with studying the open-loop asymptotically stable system and show that the optimal LQR state feedback gain $K$ is ‘quasi’-SED in this setting, i.e. $||K_{ij}|| \sim O \left(e^{-\frac{\beta}{\text{poly} \ln(N)} \text{dist}(i,j)}\right)$. The decaying rate $\beta$ depends on the decaying rate and norms of system matrices and the open-loop exponential stability constants. Then the result is further generalized to unstable systems under a stabilizability assumption. Building upon the ‘quasi’-SED result on $K$, we give an upper-bound on the performance of $\kappa$-truncated local controllers, suggesting that distributed controllers can achieve near-optimal performance for SED systems. We develop these results via studying the structure of another type of controller, disturbance response control, which has been studied and used in recent online control literature; thus as a side result, we also prove the ‘quasi’-SED property of the optimal disturbance response control which serves as a contribution on its own merit.

I. INTRODUCTION

Multi-agent systems has been actively studied in recent years. In many real-life applications such as robotic swarms [1], power grids [2], smart buildings [3] etc., the system consists of a group of interactive agents whose dynamics are affected by each other, especially its local neighbors. In general, especially for large scale systems, because of the limited communication among agents, agents need to take control actions only based on local observations, motivating the study of distributed learning and control synthesis [4]–[10], where it seeks to find optimal distributed controllers that respect agents’ local information constraints.

However, although for certain types of systems (such as quadratic invariant system [7]) there exist efficient algorithms to find such optimal distributed controllers, not much is known about how the optimum within this information-constrained controller subset differs from the global optimal controller that has access to global information; that is, how much optimality we sacrifice by constraining the controller to be distributed. Answering this important question requires more detailed understanding of the structure of the global optimal controller. If the information pattern of the global optimal controller is distributed or close to distributed, then it can be expected that the optimal distributed controller could achieve a near-optimal performance as the global optimal controller.

The work by [11] first proposes to study this question under the setting of spatially invariant systems, where they leveraged the fact that spatially invariant systems can be decoupled under Fourier transform to show that the optimal linear quadratic regulator (LQR) controller is a convolution operator whose kernel has exponentially decaying structure, suggesting that the impact of far-way agents on the optimal control strategy decays exponentially with distance. However, the spatially invariant assumption is relatively restrictive and only holds for graphs with special patterns such as grid or lattice. Later [12] seeks to generalize the result to a broader class which they define as ‘spatially distributed’ class. The paper aims to show that the optimal LQR control gain matrix $K$ also lies in the same spatially distributed class given that the matrices $A, B, Q, R$ of the LQR problem are inside the spatially distributed class. However, there exist counter-examples (see e.g. [13] and Section II of this paper) such that $A, B, Q, R$ are spatially distributed, while the optimal controller is not, suggesting that additional conditions are needed in order for the optimal controller to preserve spatially decaying structure.

There is another line of literature that is related to characterizing the information pattern of optimal LQR controller. We can view the problem as a purely linear algebraic problem, where the objective is to show whether the LQR operator $\text{LQR}(A,B,Q,R)$ that calculates the optimal controller preserves the same information structure of the matrix inputs $A, B, Q, R$. There are many existing works in the field of matrix theory that study similar questions for different matrix operators $f$ [14]–[16], such as matrix exponential, matrix inverse and Lyapunov operators, etc. Typically these works try to understand the entrywise pattern of $f(A)$ given that $A$ is a banded or sparse matrix. Not much work is done under the setting where $A$ is spatially distributed other than banded or sparse. Further, as far as we know, existing literature mostly consider analytic $f$ which is a function on a single matrix $A$. It remains unclear how the results can be extended to LQR where the operator is a function on multiple system matrices $A, B, Q, R$.

Our Contributions. In this paper we consider the standard infinite-horizon discrete time network LQR problem, where there are $N$ agents in the network and each agent has its own local state and local control actions. We focus on the case where $A, B, Q, R$ are spatially-exponential decaying (SED, Definition 1) and $A$ is exponentially stable, and show that the optimal LQR state feedback gain $K$ is ‘quasi’-SED in this setting, i.e. $||K_{ij}|| \sim O \left(e^{-\frac{\beta}{\text{poly} \ln(N)} \text{dist}(i,j)}\right)$ (Theorem 1). The rate $\beta$ is written out explicitly in terms of the SED.
rate and norms of system parameters $A, B, Q, R$ and the exponential stability of $A$. Our result can also be extended to the case where matrix $A$ is unstable under the SED stabilizable assumption (due to space limit, this part of the result is deferred to the ArXiv version of this paper [17]). As far as we know, we are the first that give concrete decaying rate analysis on the optimal controller for LQR problem with spatially decaying structure. Our result not only gives an answer to the theoretical question of whether spatially distributed LQR also obtains a spatially distributed optimal controller, but also sheds light on how the decaying rates depends on different factors, which provides insights in real applications on how to design the pattern of the information constraints so that the distributed controller can approximate the global optimal as much as possible.

Building upon the ‘quasi’-SED result on $K$, we give an upper-bound on the performance of $k$-truncated local controllers, suggesting that for SED systems, agents can achieve $c$-optimal performance by only knowing the state information of their $O(poly \ln(N) \ln(1/c))-neighbors$.

Additionally, our proof approaches the problem via disturbance response parameterization, thus as a side result, we also prove the ‘quasi’-SED property of the optimal disturbance response controller. Given that disturbance response controller has its own advantage compared with state feedback controller and is gaining more and more attention in the online learning and control community [18]–[22], we also believe that it is a contribution on its own merit.

The work that is the most relevant to our paper is one recent ArXiv preprint [23], where they proved the spatially-exponential decaying property of the optimal controller $K$ for network LQR problem where the matrix $A$ is sparse and $B, Q, R$ are block-diagonal. Our work differs from theirs in the following aspects: (i) the setting considered in this paper is broader compared with [23] (ii) The proof techniques are very different. Results in [23] are derived by leveraging KKT condition, while our proofs are mainly based on disturbance response parameterization. As a result, we also characterize the spatial decaying structure for disturbance response controllers, which is not considered in [23].

**Notations:** Throughout the paper, we use $\| \cdot \|$ to denote the $\ell_2$ norm of a vector as well as the induced $\ell_2$ norm of a matrix. $\lambda_{\text{max}}(X), \lambda_{\text{min}}(X)$ denotes the maximum and minimum eigenvalue of a square matrix $X$ respectively.

Due to space limit, we put the supplementary materials to the ArXiv version of this paper [17].

**II. Problem Settings and Preliminaries**

We consider the infinite-horizon discrete time network linear quadratic regulator (LQR) problem with $N$ agents $[N] = \{1, 2, \ldots, N\}$ that are embedded in a network. We assume that there’s a distance function defined on the $N$-agent network, i.e., $\text{dist}(\cdot, \cdot) : [N] \times [N] \rightarrow \mathbb{R}^{\geq 0}$, such that $\text{dist}(i, j) = \text{dist}(j, i)$, and that triangle inequality holds $\text{dist}(i, j) \leq \text{dist}(i, k) + \text{dist}(k, j), \forall i, j, k \in [N]$. For example, if the $N$ agents are embedded in an undirected graph, then $\text{dist}(i, j)$ can be taken as the graph distance, which is the length of the shortest path from agent $i$ to $j$. Without further explanation, $\text{dist}(\cdot, \cdot)$ refers specifically to the graph distance throughout the paper, but our results also hold for other types of distances such as Euclidean distance.

At time step $t$, each agent $i$ has its local state $x_i^t \in \mathbb{R}^{n_i}$ and local control action $u_i^t \in \mathbb{R}^{n_i}$. We use $x_i \in \mathbb{R}^{n_i}$, $u_i \in \mathbb{R}^{n_i}$ to denote the joint state and control action of the $N$ agents, i.e., $x_t = \sum_{i=1}^{N} n_i x_i^t, u_t = \sum_{i=1}^{N} n_i u_i^t$ and $x_t = [x_1^t; x_2^t; \ldots; x_N^t], \ u_t = [u_1^t; u_2^t; \ldots; u_N^t].$

The dynamics of agent $i$’s state $x_i^t$ is governed by the linear equation

$$x_i^{t+1} = \sum_{j=[N]} [A]_{ij} x_j^t + [B]_{ij} u_j^t + w_i^t. \quad (1)$$

Here the submatrix notation $[X]_{ij}$ for a matrix $X$ denotes the submatrix of $X$ where its row indexes correspond to the role indexes of agent $i$ and its column indexes correspond to the indexes of agent $j$. $w_i^t \sim N(0, I)$’s are standard i.i.d. Gaussian noises.

Each agent $i$ also has its local stage cost $c_i^t := \sum_{j=[N]} [Q]_{ij} x_j^t + [R]_{ij} u_j^t + 2 u_j^t [S]_{ij} x_i^t$, which is a quadratic function on $x_i$ and $u_i$. The total stage cost is the summation of all local costs, i.e.,

$$x^t Q x_t + u^t R u_t + 2 u^t S x_t = \sum_{i=[N]} c_i^t.$$  

Using the system matrices $A, B$ and the cost matrices $Q, R, S$, the network LQR problem is same as the classical LQR formulation shown as below,

$$\min_{(u_t)_{t=0}^{\infty}} \lim_{T \to \infty} \frac{1}{T} \mathbb{E} \sum_{t=0}^{T-1} x^T Q x_t + u^T R u_t + 2 u^T S x_t \quad s.t. \quad x_{t+1} = A x_t + B u_t + w_t, \ w_t \sim N(0, I). \quad (2)$$

Throughout the paper, we make the following assumption on the LQR problem.

**Assumption 1.** $Q \succ 0, \ R - SQ^{-1} S^T \succ 0.$

When matrices $(A, B)$ are stabilizable, the solution (2) is given by a linear state feedback controller where

$$K = (R + B^T P B)^{-1}(B^T P A + S). \quad (3)$$

Here, the cost-to-go matrix $P$ is the solution to the algebraic Ricatti equation

$$P = A^T P A - (A^T P B + B^T S)(R + B^T P B)^{-1}(B^T P A + S) + Q. \quad (4)$$

Without any special structure of the matrices $A, B, Q, R, S$, the controller $K$ is often a “dense” matrix in the sense that each $[K]_{ij}$ plays an important role, meaning that for each controller $u_i$, it would need the global state information $x_1^t, \ldots, x_i^t$ of the system. In this paper, we would like to study the information pattern of the global optimal $K$ for special classes of system matrices $A, B, Q, R, S$ and hope the corresponding optimal $K$ would have some structure (close to) being distributed/local which, hence, will ensure near-optimal

1By indexes of agent $i$, we mean that, if the total index length is $n_e$, then the indexes of agent $i$ is of range $[\sum_{i=1}^{n_e} n_i + 1, \sum_{i=1}^{n_e} n_i]$. The same definition also applies for total index length of $n_e$. 

performance of optimal distributed/local controllers.

A. The spatially-exponential decaying structure

In this paper, the special class of network LQR problem we consider are the problems with matrices \( A, B, Q, R, S \) satisfying the following special decaying property.

**Definition 1** (spatially-exponential decaying (SED)). A matrix \( X \in \mathbb{R}^{n \times n'} \) \((n', n = n_x \text{ or } n_u)\) is \((c, \gamma)\)-spatially-exponential decaying (SED) if

\[
||[X]_{ij}|| \leq c \cdot e^{-\gamma \dim(i,j)}, \quad \forall \ i, j \in [N].
\]

We make the following assumption.

**Assumption 2** (SED system). There exists \( \gamma_{sys} > 0 \) and constant \( a, b, q, r, s > 0 \) such that \( A, B, Q, R, S \) are \((a, \gamma_{sys}), (b, \gamma_{sys}), (q, \gamma_{sys}), (r, \gamma_{sys}), (s, \gamma_{sys})\)-SED respectively. Here for convenience, we assume without loss of generality that \( a, b, q, r \geq 1 \).

Assumption 2 implies that both the system dynamics and the quadratic cost are decoupled across agents in the sense that if dist\((i, j)\) is large, then \( x_j, u_j \) have exponentially small effect on the dynamics of \( x_i \) and on the cost of \( x_i \).

Below we give one system example that satisfy the SED structure. More motivation of studying SED structure is also provided in Appendix H in [17].

**Example 1** (Heat equation on the cyclic graph \( \mathbb{Z}_N \)). Consider the case where the \( N \) agents are located on the cyclic graph \( \mathbb{Z}_N \), where agent \( i \) is only connected with \( i - 1 \) and \( i + 1 \) (modular \( N \)). The heat equation dynamic is defined as

\[
x_{i+1}^t = x_i^t + \eta \left(-2x_i^t + x_{i+1}^{t+1} + x_{i-1}^{t+1} + b_1u_i^t + w_i^t\right),
\]

thus \( A = I - \eta L \), where \( L \) is the graph Laplacian of \( \mathbb{Z}_N \), and \( B = \eta \text{diag}\{b_1^N\} \) is a diagonal matrix whose diagonal entries are \( \eta b_1 \)'s. The control objective is to let the state vector goes to zero while keeping the control energy low, which can be modeled as

\[
\min \lim_{T \to \infty} \frac{1}{T} \sum_{t=1}^{T} \sum_{i=1}^{N} q_i x_i^2 + r_i u_i^2,
\]

thus \( Q, R \) matrices in this setting are \( Q = \text{diag}\{q_i^N\} \), \( R = \text{diag}\{r_i^N\} \). With small \( \eta < 1 \), we can set \( \gamma_{sys} = \ln(\frac{1}{\eta}) \), \( a = 1 \), \( b = \max\{\eta \max\{b_1, 1\}\} \), \( q = \max\{\max\{q_i, 1\}\} \) and \( r = \max\{\max\{r_i, 1\}\} \) in Assumption 2.

**Remark 1.** In the example, we directly start with a decoupled discrete-time dynamics (5) whose sparse patterns of \( A, B \) are aligned with the graph structure. Note that (5) is a first-order Euler approximation of the continuous time heat equation (c.f. [24]). Yet if we use higher-order approximation or directly integrated from the original continuous-time dynamics, agents that are not directly connected on the graph will also have effect on each other even within a small discretization period \( \Delta t \). Fortunately, it can be shown that the effect decays exponentially with the distance of agents (see more in Appendix H in [17]). This is serves as one of the motivations for us to study SED systems rather than simply locally-interacted systems.

Unfortunately, a SED system doesn’t necessarily imply that its optimal control gain \( K \) is SED without additional conditions, as shown in the counter-example below.

**B. A counter-example**

We can construct the following example, where \( A, B, Q, R, S \in \mathbb{R}^{N \times N} \) which are all SED, whereas the optimal control gain \( K \) is not.

\[
A=1.1I, \quad Q=R=I, \quad S=0, \quad B=
\begin{bmatrix}
1 & 1 & 1 \\
\vdots & \ddots & \ddots \\
1 & 1 & 1
\end{bmatrix}
\] (6)

Since \( A, Q, R \) are scalar matrices, and \( B \) is a banded matrix, clearly they are SED matrices with respect to the graph distance defined by \( \mathbb{Z}_N \), hence it satisfies Assumption 2.

We pick \( N = 100 \) and directly call the d\texttt{lr} function in \texttt{MATLAB} to solve for the optimal controller. To visualize the structure of the optimal controller \( K \), Figure 1 plots the heatmap of the absolute value of entries in \( K \); Figure 2 plots the absolute value of the entries of a row of \( K \). The two figures demonstrate that the upper-triangular entries have non-negligible absolute values, implying that \( K \) is not SED. Thus this serves as a counter-example showing that the optimal controller \( K \) is not SED even if \( A, B, Q, R, S \) are SED.

### III. Main Results

The previous counter-example suggests that additional conditions are needed in order for \( K \) to also be SED. In this paper, we first look into the open-loop exponential stable system (Assumption 3). (See the ArXiv version of this paper [17] for an extension to the unstable system.) Then, we illustrate how our results ensure a small optimality gap between the ‘\( \kappa \)-truncated local controllers’ and the optimal global controller \( K \).

In the counter-example in Equation (6), if we replace \( A = 1.1I \) with \( A = 0.9I \), the optimal control gain \( K \) immediately becomes a SED matrix (Figure 3). This observation suggests that stability indeed plays an important role in the decaying structure of \( K \), and inspires us to first consider the open-loop stable systems. We focus on systems that are asymptotically stable, which is equivalent to exponentially stable for linear systems [25]. To characterize
the exact decaying rate of the optimal controller \( K \), we provide the following definition of exponential stability.

**Definition 2** \(((\tau, \epsilon^{-\rho})\)-stability). For \( \tau \geq 1, \rho > 0 \), we define a matrix \( A \) as \((\tau, \epsilon^{-\rho})\)-stable if \( \|A^k\| \leq \tau \epsilon^{-\rho k} \).

**Assumption 3** (Open-loop Stability). The linear system considered in (2) is open-loop asymptotically stable.

Additionally, for LQR problems with Assumption 1 and 3, the optimal \( K \) always asymptotically stabilizes the system (c.f. [25]). Thus without loss of generality we assume that there exist some \( \tau \geq 1, \rho > 0 \) such that both \( A \) and \( A - BK \) are \((\tau, \epsilon^{-\rho})\)-stable, i.e.

\[
\|A^k\|, \|(A - BK)^k\| \leq \tau \epsilon^{-\rho k}.
\]

(7)

With Assumption 3, we are able to prove the following theorem. For compactness, here we only state the main result and briefly discuss the implication. The intuition and proof sketches are provided in the later sections.

**Theorem 1** (Proof and formal statement see Appendix E). Under Assumption 1, 2, and 3, the optimal control gain \( K \) for problem (2) is \((c_K, \gamma_K)\)-SED, where

\[
c_K \lesssim O\left(\text{poly}\left(N, \tau, \frac{1}{1 - e^{-2\rho}}, \lambda_{\min}(R - SQ^{-1}S^T)\right)\right)
\]

\[
\|B\|, \|Q\|, \|S\|, \|K\|, b, q, s
\]

\[
\gamma_K \gtrsim \gamma_{\text{sys}} \frac{\text{poly}\left(\tau, \frac{1}{1 - e^{-2\rho}}, \lambda_{\min}(R - SQ^{-1}S^T)\right)}{\lambda_{\max}(R), \|B\|, \|Q\|, \|S\|, \text{ln} \{N, a, b, q, r, s, \gamma_{\text{sys}}\}}
\]

where \( \tau, \rho \) are defined as in (7).

**Remark 2** (Extension to unstable systems). We would also like to emphasize that Theorem 1 can be further extended to the unstable system with a stabilizing assumption. Due to space limit, the result is deferred to the ArXiv version [17].

**Remark 3** (Discussion on the SED parameter \((c_K, \gamma_K)\)). Since this paper considers the finite dimension problem, any matrix is \((c, \gamma)\)-SED as long as \( c \) is picked to be arbitrarily large and \( \gamma \) arbitrarily close to zero. For example, for a matrix \( K \) defined on a cyclic graph \( Z_n \), we can always choose \( \gamma = 1/N \), \( c = \max_{ij} \|K_{ij}\| \), \( e \), then \( \|K_{ij}\| \leq ce^{-\gamma_{\text{dist}(ij)}} \); thus merely saying that a matrix \( K \) is spatially-exponential decaying without checking \((c_K, \gamma_K)\) is not informative. Therefore in order for the SED rate to be meaningful, \( \gamma_K \) should not `scale with \( N \)` (or at least scale better than \( N \)) when we increase the dimension \( N \) of the problem. Hence we would like to emphasize that, Theorem 1 needs to be applied with caution. It is necessary to check that how all the factors in Theorem 1, i.e., \( \rho, \tau, \gamma_{\text{sys}}, \lambda_{\min}(R), \lambda_{\max}(R), \|B\|, \|Q\| \), . . . , grow with \( N \). To make this more concrete, we calculate these factors explicitly for the Heat Equation system in [17].

Apart from these system factors, there’s one term in the bound of \( \gamma_K \) that always scales with \( N \), which is the \( O(\text{poly}\ln(N)) \) in the denominator, making the decaying rate of the entries \( [K]_{ij} \) `quasi-`SED, i.e., \( O\left(\exp\left(-\frac{\beta_{\text{dist}(ij)}}{\text{poly}\ln(N)}\right)\right) \), which is slightly worse than exponential decay. It is still an open question to us whether the \( O(\text{poly}\ln(N)) \) factor is fundamental or a proof artifact, which is left as future work. Nevertheless, we remark that the `quasi-`exponential bound is still good because \( \ln(N) \) grows very slowly with \( N \). As long as \( \text{dist}(i, j) \gtrsim O(N^\epsilon) \) given any fixed constant \( \epsilon > 0 \), we can still get that \( \|K_{ij}\| \) is close to zero since \( \lim_{N \to +\infty} \frac{\ln(N)}{\text{poly}\ln(N)} = +\infty \). Thus the bound works for a wide range of systems, such as the cyclic graph \( Z_N \) in Example 1, or lattice graph etc. More generally speaking, the bound is valid as long as the graph is not well-connected. □

A. Performance of the truncated local controller

The above results shed light on how well distributed controllers can approximate the global optimal performance for SED systems. Consider the following \( \kappa \)-truncated local controller:

\[
[K^{\kappa}_{\text{trunc}}]_{ij} := \begin{cases} [K]_{ij} & \text{if dist}(i, j) \leq \kappa - 1 \\ 0 & \text{otherwise} \end{cases}
\]

The \( \kappa \)-truncated local controller is desirable for distributed control because each node only requires the state information from its \( \kappa \)-hop neighborhood to calculate its control action. We measure the performance in terms of the LQR cost, i.e., for any exponentially stable controller \( K' \), its LQR cost \( C(K') \) is given as:

\[
C(K') := \min_{\{u_t\}_{t=0}^{\tau} \in \mathbb{R}^n} \lim_{T \to +\infty} \frac{1}{T} \mathbb{E} \sum_{t=0}^{T-1} x_t^T Q x_t + u_t^T R u_t + 2 u_t^T S x_t
\]

\[
s.t. \quad x_{t+1} = A x_t + B u_t + w_t, \quad w_t \sim \mathcal{N}(0, I), \quad u_t = K' x_t
\]

**Theorem 2** (Proof in Appendix F in [17]). Assume that the optimal control gain \( K \) is \((c_K, \gamma_K)\)-SED, then for \( \kappa \geq \ln\left(\frac{2\tau_{K^{\kappa}_{\text{trunc}}}}{1 - e^{-\rho}}\right) \), we have that

\[
C(K^{\kappa}_{\text{trunc}}) - C(K)
\]

\[
\leq \frac{2\tau_{K^{\kappa}_{\text{trunc}}}}{1 - e^{-\rho}} \|R + B^T P B\| \sqrt{N} \min\{n_x, n_u\} e^{-\gamma K} 
\]

where \( P \) is defined in (4).

Theorem 2 together with Theorem 1 implies that for SED systems that satisfy Assumption 3, in order to achieve \( \epsilon \)-optimal performance in terms of the LQR cost, we only need to set \( \kappa \sim \text{poly} \ln(N) \ln(1/\epsilon) \). That is, each agent only need to know the information of its \( \kappa \)-hop neighbors, which is only a negligible proportion of the total number of agents (for graphs that are not well-connected), to achieve near-optimal performance.

IV. PROOF ENABLER: DISTURBANCE RESPONSE CONTROLLER

The rest of the paper mainly focuses on providing insights and proof sketches of the main results, especially Theorem
1. One of the major technical difficulties is that the algebraic Riccati equation (4) is a nonlinear matrix equation, rendering it hard to verify how the spatially decaying structure is preserved given that the matrix coefficients $A, B, Q, R, S$ in (4) are SED. Hence, we study the problem from a different perspective — doing convex reparameterization of (2) using disturbance response. In this section, we consider the following finite-truncated disturbance response controller optimization problem of (2)

$$
\min_{L(H)} \quad C \left( L(H) := \left[ L_1(H); \ldots; L_K(H) \right] \right) \\
= \lim_{T \to \infty} \mathbb{E} \sum_{t=0}^{T-1} x_t^T Q x_t + u_t^T R u_t + 2u_t^T S x_t 
$$

s.t. \quad x_{t+1} = Ax_t + Bu_t + w_t, \quad w_t \sim \mathcal{N}(0, I) \\
u_t = L(H) w_{t-1} + \cdots + L(H) w_{t-H} \quad (w_s = 0 \text{ for } s < 0).

This problem is a special case of Youla parameterization [26] for the open-loop stable LQR case. The reason that we consider problem (8) is twofold: (i) There is a simple relationship of the solution of (8) with the optimal control gain $K$ [27]; (ii) Compared with solving the Riccati equation for $K$, (8) is a quadratic unconstrained optimization problem w.r.t. $L(H)$, so it can be expected that the problem obtains an explicit solution by solving a system of linear equations, whose SED structure is easier to analyze.

### A. Relationship with the optimal control gain $K$

**Lemma 1** ([27]), Let $K$ be the optimal control gain of the LQR problem (2), then under Assumption 3, $L(H)$ solved from (10) satisfies

$$
\|K + L_1(H)\| \leq \frac{2\tau^3 \|B\|^2 \|K\| \|Q\| + \|B\| \|K\| \|S\|}{\lambda_{\min}(R - SQ^{-1}S^T)} (1 - e^{-2\rho}/2) e^{-H\rho},
$$

(9)

Lemma 1 suggests that, rather surprisingly, $L_1(H)$ is a good approximation of $K$, and the approximation error decays exponentially with the truncation horizon $H$. As a consequence, in order to understand the SED structure for $K$, we could instead study the SED structure for $L_1(H)$, which is essentially easier.

### B. SED structure of $L(H)$

We first state the main theorem of this section:

**Theorem 3** (SED structure of $L(H)$), Under Assumption 2 and 3, the optimal $L(H)$ that solves (8) satisfies that $L_k(H)$ is $(c_L, \gamma_L)$-SED for any $1 \leq k \leq K$, where

$$
eq \text{poly}\left( N, \tau, \frac{1}{1 - e^{-2\rho}}, \frac{1}{\lambda_{\min}(R - SQ^{-1}S^T)}, \|B\|, \|Q\|, \|S\|, b, q, s \right),$$

$$
\gamma_L(H) \geq \gamma_{\text{sys}} \sqrt{O} \left( \lambda_{\min}(R), \|B\|, \|Q\|, \|S\|, \ln \{N, H, a, b, q, r, s \} \right).
$$

The full proof as well as formal statement of Theorem 3 is in Appendix D in [17]. We also view Theorem 3 itself as an interesting finding. There are many recent works that use disturbance response controllers for online adaptive control [18]–[22]. It would be an exciting direction to see how the SED structure of the optimal disturbance response controller could lead to more efficient learning/control algorithms in these problems.

### V. Proof Sketches

The previous section has introduced the relationship of disturbance response controller and the state feedback controller (Lemma 1) and shown the SED structure of the disturbance response controller (Theorem 3). Combining these two insights finishes the proof of our main result Theorem 1. Thus this section focuses on a detailed proof sketch of Theorem 3, which is the key step in proving the main results.

**A. Solution of problem (8)**

Note that (8) is a quadratic unconstrained optimization problem w.r.t. $L(H)$, so it can be expected that the problem obtains an explicit solution. As an initial step, we first identify the explicit solution, which is the main focus of this section. We first state the main lemma:

**Lemma 2** (Proof in Appendix B in [17]), The optimal $L(H)$ of (8) solves

$$
M(H)L(H) + J(H) = 0, \quad (10)
$$

where $M(H) \in \mathbb{R}^{K_n \times K_n}$ and $J(H) \in \mathbb{R}^{K_n \times n_s}$ are defined as

$$
M(H) := \begin{bmatrix} M_{11} & M_{12} & \cdots & M_{1H} \\
M_{21} & M_{22} & \cdots & M_{2H} \\
\vdots & \vdots & \ddots & \vdots \\
M_{H1} & M_{H2} & \cdots & M_{HH} \end{bmatrix}, \quad J(H) := \begin{bmatrix} J_1 \\
\vdots \\
J_H \end{bmatrix}, \quad (11)
$$

with submatrices $M_{km} \in \mathbb{R}^{n_k \times n_k}$, $J_k \in \mathbb{R}^{n_k \times n_s}$ defined as:

$$
M_{km} := \begin{bmatrix} B^T G B + R, & k = m \\
B^T G A^k - B S A^{k-1} B, & k > m \end{bmatrix}, \quad (12)
$$

$$
J_k := B^T G A^k + S A^{k-1}. \quad (13)
$$

Here $G \in \mathbb{R}^{n_s \times n_s}$ is defined as:

$$
G := \sum_{t=0}^{\infty} (A^t)^T Q A^t. \quad (14)
$$

Although the definition of variables may seem heavy at first glance, the key takeaways of Lemma 2 are actually simple and straightforward: (i) The optimal $L(H)$ solves a system of linear equations (Eq (10)); (ii) The submatrices $M_{km}, J_k$ of $M(H), J(H)$ are matrix polynomials with respect to the LQR parameters $A, B, Q, R, S$ as well as $G$.

**B. SED structure of $M_{km}, J_k$’s**

**Lemma 3** (SED structure of $M_{km}, J_k$’s, Proof in Appendix C in [17]), Under Assumption 1, 2, and 3, for any $k, m \geq 1$, $M_{km}$ is $(c_M, \gamma_M)$-SED and $J_k$ is $(c_J, \gamma_M)$-SED ($M_{km}, J_k$ defined in (13)), where the absolute constants are defined as

$$
c_M = b^2 N^2 \left( \frac{\tau^2 \|Q\|}{1 - e^{-2\rho}} + 2q \right) + b N (s + \tau \|S\|) + r,
$$

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\[ c_J = bN \left( \frac{\tau^2 \|Q\|}{1-e^{-2\rho}} + 2q \right) + s + \tau \|S\|, \gamma_M = \frac{\gamma_{sys} \rho}{(\rho + \ln(aN))} \] .

We provide a brief intuition of the lemma. The key fact is that \( G \) in (14), which shows up in every \( M_{km} \) and \( J_k \), is (‘quasi’-)SED. Although as far as we know we are not aware whether exact same results exist in literature, the proof technique resembles previous works that study the setting where \( A, Q \) are sparse or banded matrices [16]. Then, using similar arguments, we can show that \( M_{km}, J_k \)‘s are also SED.

C. SED structure of \( L(H) \)

Now we are ready to present a proof sketch of Theorem 3. The proof depends on the Taylor expansion of \( (M(H)^{\ell}) \). It is not hard to show that \( M(H) \) is a positive definite matrix with bounded eigenvalues, namely, \( 0 < \lambda_{\text{min}} \leq M(H) \leq \lambda_{\text{max}} I \) for some \( \lambda_{\text{min}}, \lambda_{\text{max}} \) (the value of \( \lambda_{\text{min}} \) and \( \lambda_{\text{max}} \) is formally calculated in Lemma 8 in Appendix D in [17]). Here for convenience we assume that \( \lambda_{\text{max}} \geq 1 \). Normalize \( M(H) \) as \( M(H)^{\ell} := I - \frac{M(H)}{\lambda_{\text{max}}} \), then we have that \( 0 \leq M(H)^{\ell} \leq (1 - \frac{\lambda_{\text{min}}}{\lambda_{\text{max}}}) I \). Applying Taylor expansion to Lemma 2, we can write \( L(H) \) as:

\[ L(H) = -\left( \frac{1}{\lambda_{\text{max}}} \right) \sum_{s=0}^{\infty} \lambda_{\text{max}}^{s} (I - M(H)^{\ell})^{-s} J(H) \]

Then the rest of the proof is proving that the Taylor series is quasi-SED. The proof technique resembles the proof for the SED structure for matrix \( G \) in the previous section. A more detailed proof sketch is provided in [17].

VI. CONCLUSIONS AND FUTURE DIRECTIONS

This paper explores the spatially decaying structure for infinite-horizon discrete time LQR problem which is spatially-exponential decaying (SED). We show that the optimal LQR state feedback gain \( K \) is ‘quasi’-SED in this setting under certain stability conditions. Based on this result, we also analyze the near-optimal performance of \( \kappa \)-truncated local controllers. Additionally, as a side result of our proof, we also show that the optimal disturbance response controller is also ‘quasi’-SED. There are many interesting open questions; to name a few, whether we could improve the ‘quasi’-SED rate so that the poly \( \ln(N) \) term can be removed, how to handle infinite number of agents (i.e. \( N \to \infty \)) and continuous time settings, and how to efficiently learn a \( \kappa \)-truncated local controller in the sample-based settings etc. In the long run, we hope to incorporate the results in this papers to more clever design of distributed learning and control of multi-agent systems.

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