TAME PARAHORIC NONABELIAN HODGE CORRESPONDENCE IN POSITIVE CHARACTERISTIC OVER ALGEBRAIC CURVES

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Abstract. Let $G$ be a reductive group, and let $X$ be an algebraic curve over an algebraically closed field $k$ with positive characteristic. We prove a version of nonabelian Hodge correspondence for tame $G$-local systems over $X$ and logarithmic $G$-Higgs bundles over the Frobenius twist $X'$. To obtain a full description of the correspondence for the noncompact case, we introduce the language of parahoric group schemes to establish the correspondence.

1. Introduction

1.1. Background. Let $X$ be a smooth projective variety. When the characteristic is zero, Simpson gave a correspondence between Higgs bundles and local systems on $X$ [21]. This correspondence is analytic in nature but it does not preserve the algebraic structure. In positive characteristic, the work of Ogus–Vologodsky and Chen–Zhu show that the connection between Higgs bundles and local systems is much closer [7, 17]. Especially, when $X$ is an algebraic curve, Chen–Zhu shows that the stack of $G$-local systems on $X$ is a twisted version of the stack of $G$-Higgs bundles on $X'$, which is the Frobenius twist of $X$. This result is a crucial ingredient in the proof of the geometric Langlands conjecture in positive characteristic on curves [8].

For the noncompact case, Simpson established such a correspondence under “tameness” condition in characteristic zero, and he introduced filtered (parabolic) Higgs bundles and filtered local systems to give a precise description of this correspondence [20]. Inspired by Simpson’s work, people first introduced parabolic $G$-Higgs bundles and parabolic $G$-local systems to establish this correspondence for principal bundles [4]. However, the parabolic objects are not enough to establishing the correspondence completely. With a careful discussion of the local data, Boalch introduced the language of parahoric subgroups and give a precise description of the correspondence locally [5, §6]. With the help of the local study, it is believed that the language of parahoric group is the correct one to give a full description of the nonabelian Hodge correspondence on noncompact curves for $G$-local systems and $G$-Higgs bundles [4, 12, 13, 14].

In positive characteristic, one may ask a similar question: how to construct the nonabelian Hodge correspondence on noncompact curves under “tameness” condition? In this paper, we establish a version of this correspondence: tame parahoric nonabelian Hodge correspondence in positive characteristic over curves. The word “tameness” has two stories here:

1. First, tameness is a condition on Higgs fields introduced in [20], which can be regarded as regular singularities on connections or logarithmic case of Higgs bundles when extended to the compactification. The tameness condition is a crucial property to establish the correspondence.

2. Second, since we work in positive characteristic, it is related to the concept of tamely ramified coverings. In characteristic zero, Balaji–Seshadri found that parahoric torsors over $X$ correspond to $\Gamma$-equivariant bundles over $Y$, where $Y \to X$ is a Galois covering with Galois group $\Gamma$ [3]. When generalizing this approach to positive characteristic $p$, we suppose that the order of $\Gamma$ and the characteristic $p$ are coprime, which means that the covering $Y \to X$ is tamely ramified and the stack $[Y/\Gamma]$ is tame [1].

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This problem has been investigated in [19] for local systems with nilpotent residues. Compared with this work, we do not impose any condition on the residue of logarithmic connections. Moreover, some of the investigations are motivated by the results in [5], especially in the local case.

1.2. Main Results.

1.2.1. Local Case. Let $G$ be a connected reductive linear algebraic group with maximal torus $T$ over an algebraically closed field $k$. Let $O := k[[z]]$ and $K := k((z))$. A parahoric (sub)group is usually understood as a subgroup of $G(K)$ (see [9, 18]). In characteristic zero, given a weight (rational cocharacter) $\theta$ of $T$, Boalch constructed a special parahoric group $P_\theta(O) \subseteq G(K)$ to study the tame Riemann-Hilbert correspondence locally [5]. As a special case, when $\theta = 0$, then $P_\theta(O)$ is exactly $G(O)$, which goes back to the case of $G$-bundles.

We first study this problem locally on a formal disc $\mathcal{D} := \text{Spec}(O)$ with $\theta = 0$. In this case, a logahoric $P_\theta(O)$-connection is exactly a logarithmic $G$-connection (or a tame $G$-local system) over $\mathcal{D}$, i.e.

$$d - A\frac{dz}{z}, \quad A = \sum_{i \geq 0} a_i z^i \in \mathfrak{g}(O),$$

and a logahoric $P_\theta(O)$-Higgs field is a logarithmic $G$-Higgs field over $\mathcal{D}$. The word “logahoric” is a blend of logarithmic and parahoric introduced in [5]. Fixing a splitting of the Lie algebra $\pi : t_{\mathfrak{g}_p} \hookrightarrow t$, we can decompose $a_0$ as

$$a_0 = s + n = \tau + \sigma + n,$$

where $s$ (resp. $n$) is the semisimple part (resp. nilpotent part) of $a_0$, and $\tau \in t_{\mathfrak{g}_p}$ is called the rational part of $s$, while $\sigma$ is called the irrational part of $s$. By generalizing a classical calculation to positive characteristic, a logarithmic $G$-connection is equivalent to one in the form $d - B\frac{dz}{z}$ under the gauge action, where $B = \sum_{i \geq 0} b_i z^i$ such that $b_i$ lies in the generalized eigenspace of the operator $[b_0, -]$ with eigenvalue $i$ (see Lemma 4.2). Therefore, we can assume that given a logarithmic $G$-connection $d - A\frac{dz}{z}$, the term $a_i$ lies in the generalized eigenspace of the operator $\text{tor} \{\tau, -\}$ with eigenvalue $i$.

We first consider the case that the semisimple part of $a_0$ is irrational, i.e. $\tau = 0$.

- Locsys$^{\text{tame}}_{G, \text{irr}}(\mathcal{D})$ is the category of tame $G$-local systems $(E, \nabla)$ on the formal disc $\mathcal{D}$ such that the semisimple part of the residue of $\nabla$ is irrational (under gauge action).
- Higgs$^{\text{tame}}_{G, \text{irr}}(\mathcal{D}')$ is the category of logarithmic $G$-Higgs bundles $(E, \phi)$ on the Frobenius twist of $\mathcal{D}$ such that the semisimple part of the residue of $\phi$ is irrational (under adjoint action).

We prove that these two categories are equivalent (Proposition 4.7)

$$\text{Locsys}^{\text{tame}}_{G, \text{irr}}(\mathcal{D}) \cong \text{Higgs}^{\text{tame}}_{G, \text{irr}}(\mathcal{D}').$$

If the rational part $\sigma$ of $a_0$ is nontrivial, we define the category Locsys$^{\text{tame}}_{G, \tau}(\mathcal{D})$ similarly. We prove the equivalence

$$\text{Locsys}^{\text{tame}}_{G, \tau}(\mathcal{D}) \cong \text{Higgs}^{\text{tame}}_{G, \tau}(\mathcal{D}'),$$

where Higgs$^{\text{tame}}_{G, \tau}(\mathcal{D}')$ is the category of logahoric $G_\tau'(O)$-Higgs bundles over $\mathcal{D}'$, where $G_\tau'(O)$ is regarded as a parahoric group over $\mathcal{D}'$ (Lemma 4.8), such that semisimple part of the residue is irrational (Proposition 4.10). This finishes the discussion of the local case and the correspondence we obtain is called local tame nonabelian Hodge correspondence.

Next, we come to the parahoric case over $\mathcal{D}$. We fix a tame weight $\theta$, which is a weight such that the denominator $d$ is coprime to $p$. Similar to the above discussion, we define two categories:

- Locsys$^{\text{tame}}_{P, \theta}(\mathcal{D})$ is the category of tame $P_\theta(O)$-local systems on $\mathcal{D}$ such that the semisimple part of the residue is $\tau$ (under gauge action);
- Higgs$^{\text{tame}}_{G, \theta, \text{irr}}(\mathcal{D}')$ is the category of logahoric $G_\theta'(O)$-Higgs bundles on $\mathcal{D}'$ such that the semisimple part of the residue is irrational (under adjoint action).
The following diagram

\[
\begin{array}{ccc}
\text{Locsys}^\text{tame}_{G,\theta+\tau}(\mathbb{D}_x) & \xrightarrow{\text{Proposition 3.9}} & \text{Locsys}^\text{tame}_{G,\theta+\tau+\tau}(\mathbb{D}_y/\Gamma) \\
\downarrow & & \downarrow \\
\text{Higgs}^\text{tame}_{G,\theta+\tau,\text{irr}}(\mathbb{D}_x) & \xrightarrow{\text{Theorem 3.7}} & \text{Higgs}^\text{tame}_{G,\theta+\tau,\text{irr}}(\mathbb{D}_y/\Gamma)
\end{array}
\]

implies the equivalence of these two categories. Actually, each arrow in the diagram can be understood as an equivalence and the horizontal arrows are discussed in [3] based on the correspondence studied by Balaji–Seshadri [3].

**Theorem 1.1 (Theorem 4.15).** We have an equivalence of categories

\[
\text{Locsys}^\text{tame}_{G,\theta+\tau}(\mathbb{D}) \cong \text{Higgs}^\text{tame}_{G,\theta+\tau}(\mathbb{D}).
\]

Moreover, the p-curvature of the parahoric \( \mathcal{P}_\theta(\mathcal{O}) \)-connection is zero if and only if the corresponding parahoric \( G_{\theta+\tau} \)-Higgs bundle has zero Higgs field.

This correspondence is called the local tame parahoric nonabelian Hodge correspondence.

1.2.2. Global Case. After the local description of tame parahoric nonabelian Hodge correspondence, we move to the global case and consider the case \( \theta = 0 \) first. Let \( X \) be a smooth algebraic curve with a given reduced effective divisor \( D \). Let \( \mathcal{L} := \Omega_X(D) \), and denote by \( \mathcal{L}' \) the corresponding line bundle over \( X' \), which is the Frobenius twist of \( X \), and we have \( Fr^*\mathcal{L}' \cong \mathcal{L}^p \), where \( Fr : X \to X' \) is the Frobenius morphism. Let \( B_{\mathcal{L}'} \) be the \( \mathcal{L}' \)-twisted Hitchin base. Then we have a natural morphism

\[
h_p : \text{Locsys}^\text{tame}_{G} \to B_{\mathcal{L}'},
\]

which is called \( p \)-Hitchin morphism (see [1] Proposition 3.1) or [3]. The \( p \)-Hitchin morphism helps us to construct a group schemes \( J^p \) over \( X \times B_{\mathcal{L}'} \), equipped with a natural action

\[
\text{Locsys}^\text{tame}_{J^p} \times_{B_{\mathcal{L}'}} \text{Locsys}^\text{tame}_{G} \to \text{Locsys}^\text{tame}_{G}.
\]

Now we define a substack \( \mathcal{A} \subseteq \text{Locsys}^\text{tame}_{J^p} \), of which the connections are with vanishing \( p \)-curvature, and let \( \mathcal{X} \) be the stack parametrizing \((E, \nabla, \Psi)\) such that

- \((E, \nabla)\) is a tame \( G \)-local system with zero \( p \)-curvature,
- \( \Psi \) is a horizontal section of \( \text{Ad}(E) \otimes Fr^*\mathcal{L}' \)

Clearly, \( \mathcal{X} \) is an algebraic stack over \( B_{\mathcal{L}'} \). In the meantime, we can construct a vector bundle \( \mathcal{B}_{\mathcal{L}'} \) over \( B_{\mathcal{L}'} \), of which the fiber is \( H^0(X', \text{Lie}({J}^p'_0) \otimes \mathcal{L}') \) for each \( b' \in B_{\mathcal{L}'} \). We prove that \( \text{Locsys}^\text{tame}_{J^p} \) is smooth over \( \mathcal{B}_{\mathcal{L}'} \) (Lemma 5.10). Then, we define the stack \( \mathcal{H} \)

\[
\mathcal{H} \quad \text{......} \quad \text{Locsys}^\text{tame}_{J^p} \quad \text{......} \quad \mathcal{B}_{\mathcal{L}'} \]

as the pullback of the diagram. With the help of the local study in [3] we follow Chen–Zhu’s approach to prove the main results in this paper.

**Theorem 1.2 (Theorem 5.19).** There exists a canonical isomorphism of stacks over \( B_{\mathcal{L}'} \):

\[
\mathcal{H} \times^\mathcal{A} \mathcal{X} \to \text{Locsys}^\text{tame}_{G}.
\]

Moreover, the structure of \( \mathcal{X} \) can be described by the rational semisimple residues \( \tau = \{ \tau_x | x \in D \} \).

**Proposition 1.3 (Proposition 5.24).** The stack \( \mathcal{X} \) is the disjoint union of \( \mathcal{X}_\tau \), where the union ranges over all rational semisimple conjugacy classes of \( \tau \), and we have

\[
\mathcal{X}_\tau \cong \text{Higgs}^\text{tame}_{G_\tau}.
\]
For the parahoric case, we fix a collection of tame weights \( \theta = \{ \theta_x, x \in D \} \). With respect to the data \((X, D, \theta)\), we can construct two objects:

- a parahoric group scheme \( P_\theta \) over \( X \);
- a tamely ramified covering \( Y \to X \).

Balaji–Seshadri showed that parahoric torsors are equivalent to \( \Gamma \)-equivariant \( G \)-bundles \([3]\) in characteristic zero. Their approaches can be generalized to Higgs bundles and local systems in mixed characteristic (see \([3, 13]\)). We show the following isomorphisms of stacks (Theorem 3.7 and Proposition 3.9):

\[
\text{Higgs}_{G, \rho}^\text{tame}(\{Y/\Gamma\}) \cong \text{Higgs}^\text{tame}_{P_\theta}(X),
\]

\[
\text{Locsys}_{G, \rho}^\text{tame}(\{Y/\Gamma\}) \cong \text{Locsys}^\text{tame}_{P_\theta}(X),
\]

where \( \rho \) is a topological data corresponding to weights \( \theta \) and \( \{Y/\Gamma\} \) is the quotient stack, of which the coarse moduli space is \( X \). This equivalence shows that proving the tame parahoric nonabelian Hodge correspondence on \( X \) is equivalent to considering the correspondence over \( \{Y/\Gamma\} \) (or an \( \Gamma \)-equivariant version over \( Y \)), which is a direct result of Theorem 5.19.

Theorem 1.4 (Theorem 5.25 and Proposition 5.27). There exists a canonical isomorphism of stacks over \( B_{L'} \):

\[
\mathcal{H} \times^\mathbb{A} \mathcal{X}_{P_\theta} \to \text{Locsys}_{P_\theta}^\text{tame}.
\]

The stack \( \mathcal{X}_{P_\theta} \) is the disjoint union of \( \mathcal{X}_{P_\theta, \tau} \) (over \( \tau \)) such that

\[
\mathcal{X}_{P_\theta, \tau} \cong \text{Higgs}^\text{tame}_{G_{P_\theta}}(X')
\]

This correspondence is called the **tame parahoric nonabelian Hodge correspondence**.

1.3. Structure of the Paper. In \([3]\) we briefly review some necessary backgrounds about parahoric groups. In \([4]\) we generalize the correspondence studied by Balaji–Seshadri \([24]\) to positive characteristics. More precisely, let \( Y \to X \) be a covering with Galois group \( \Gamma \), and we prove that there is a correspondence between parahoric Higgs bundles (resp. local systems) over \( X \) and \( \Gamma \)-equivariant Higgs bundles (resp. local systems) over \( Y \). In \([5]\) we construct the tame parahoric nonabelian Hodge correspondence on a formal disc \( D \) (Theorem 4.15). Based on the local study, we prove the **tame parahoric nonabelian Hodge correspondence** (Theorem 5.19 and 5.25).

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2. Preliminaries

Let \( k \) be an algebraically closed field of positive characteristic. Let \( G \) be a connected reductive linear algebraic group over \( k \). We fix a maximal torus \( T \subseteq G \). Denote by \( \mathfrak{g} \) and \( \mathfrak{t} \) the Lie algebras of \( G \) and \( T \) respectively. Let \( X^*(T) := \text{Hom}(T, \mathbb{G}_m) \) be the group of characters, and let \( X_*(T) := \text{Hom}(\mathbb{G}_m, T) \) be the group of cocharacters. The adjoint action of \( T \) on \( \mathfrak{g} \) gives a decomposition of the Lie algebra

\[
\mathfrak{g} = \bigoplus_{\alpha \in X^*(T)} \mathfrak{g}_\alpha.
\]

A root is a nonzero character \( \alpha \), of which \( \mathfrak{g}_\alpha \neq 0 \), and \( \mathfrak{g}_\alpha \) is called a **root space**. Denote by \( R \) the set of roots. We define a natural pairing

\[
\langle \cdot, \cdot \rangle : X_*(T) \times X^*(T) \to \mathbb{Z}
\]

This pairing can be extended to \( \mathbb{Q} \) naturally. Thus, \( \langle \theta, \alpha \rangle \) is a well-defined rational number, where \( \theta \) is a weight and \( \alpha \) is a root. We also use the notation \( \alpha(\theta) \) for this number.
Tame Weights.
A rational cocharacter \( \theta \in X_*(T) \otimes \mathbb{Z} \mathbb{Q} \) is called a weight. In characteristic zero, a weight \( \theta \) can be regarded as an element in \( t_\mathbb{Q} \) under differentiation. In positive characteristic, we need some extra conditions.

**Definition 2.1.** A weight \( \theta \) is tame if its denominator is coprime to \( p \).

Given a tame weight \( \theta = \vartheta \otimes \frac{g}{p} \), we have \( (b, p) = 1 \). Then \( \theta \) corresponds to a well-defined element in \( t_{F_p} \). Abusing the notation, a tame weight can either be a rational cocharacter or an element in \( t_{F_p} \).

Now given \( a \in g \), let \( a = s + n \) be its Jordan decomposition, where \( s \) is the semisimple part and \( n \) is the nilpotent part. Denote by \( t \subseteq g \) the corresponding Lie algebras. Now we fix a splitting of Lie algebras \( \pi : t_{F_p} \hookrightarrow t \) and decompose \( s = \tau + \sigma \) such that under an appropriate conjugation, we have \( \tau \in t_{F_p} \) and the projection of \( \sigma \) onto \( t_{F_p} \) is zero. We say that \( \tau \) is the rational part of \( s \) and \( \sigma \) is the irrational part of \( s \).

For convenience, we sometimes assume that \( \tau \in t_{F_{a_p}} \), and then we have \( \alpha(\tau) \in \mathbb{F}_p \), while \( \alpha(\sigma) \notin \mathbb{F}_p \) for every root \( \alpha \).

**Grading of Lie algebras.**
Given a weight \( \theta \), it also induces a (graded) decomposition of the Lie algebra \( g \)

\[
g = \bigoplus_{\lambda \in \mathbb{Q}} g_\lambda,
\]

where \( g_\lambda = \bigoplus_{\alpha(\theta) = \lambda} g_\alpha \). Note that this decomposition is not given by the eigenspace of the differentiation of the adjoint action \( \text{Ad}(\theta) \), of which the eigenvalues are in \( \mathbb{F}_p \). For any integer \( k \), we define

\[
g_{\geq k} := \bigoplus_{\lambda \geq -k} g_\lambda \subseteq g.
\]

As a special case, \( g_{\geq 0} = p_\theta \) is the Lie algebra of the parabolic subgroup \( P_\theta \subseteq G \) associated to \( \theta \), and \( g_0 = t_0 \) is the Lie algebra of the Levi component \( L_\theta \) of \( P_\theta \).

**Parahoric groups.**
Given a root \( \alpha \in R \), there is a natural isomorphism

\[
\text{Lie}(G_\alpha) \to g_\alpha.
\]

This isomorphism induces a natural homomorphism

\[
\alpha_\alpha : G_\alpha \to G,
\]

such that \( t_\alpha(a)u_\alpha = u_\alpha(t_\alpha(a)) \) for \( t \in T \) and \( a \in G_\alpha \). Denote by \( U_\alpha \) the image of \( u_\alpha \), which is a closed subgroup. Then, we define \( O := k[[z]] \) and \( K := k((z)) \). Denote by \( LG := G(K) \) the loop group and \( Lg := g(K) \) the loop Lie algebra (see \([18]\) for more details).

**Definition 2.2.** Given a weight \( \theta \), the parahoric (sub)group \( \mathcal{P}_\theta(O) \subseteq LG \) is defined as

\[
\mathcal{P}_\theta(O) := \langle T(O), U_\alpha(z^{m_\alpha(\theta)}O), \alpha \in R \rangle,
\]

where \( m_\alpha(\theta) := \lceil -\alpha(\theta) \rceil \) and \( \lceil \cdot \rceil \) is the ceiling function. The Levi subgroup \( \mathcal{L}_\theta(O) \) of \( \mathcal{P}_\theta(O) \) is defined as

\[
\mathcal{L}_\theta(O) := \langle T(O), U_\alpha(O), \alpha \in R \text{ and } \alpha(\theta) = 0 \rangle.
\]

As a special case, when \( \theta = 0 \), then \( \mathcal{P}_\theta(O) = G(O) \). Fixing a weight \( \theta \), we define a grading of \( Lg \)

\[
Lg_{\geq k} := \{ \sum_{i \in \mathbb{Z}} a_i z^i \in Lg \mid a_i \in g_{\geq -k} \},
\]

where \( k \in \mathbb{Q} \). If \( k \geq 0 \), denote by \( G_{\geq k} \) the associated group in \( LG \). Here are some examples. When \( k = 0 \), \( \mathcal{P}_\theta(O) := Lg_{\geq 0} \) is the Lie algebra of the parahoric group \( \mathcal{P}_\theta(O) \). If \( \theta = 0 \), the filtration \( Lg_{\geq k} \) is the natural filtration of \( Lg \) based on the degree.
**Parahoric group schemes.**

The definition of parahoric groups is a local picture of parahoric group schemes. Let $X$ be a smooth algebraic curve over $k$, and we also fix a reduced effective divisor $D$ on $X$, which is a set of $s$ distinct points. For each point $x \in D$, we equip it with a weight $\theta_x \in Y(T) \otimes \mathbb{Q}$. Let $\theta := \{\theta_x, x \in D\}$ be the collection of weights over points in $D$.

**Definition 2.3.** We define a group scheme $P_\theta$ over $X$ by gluing the following local data

$$P_\theta|_{X \setminus D} \cong G \times (X \setminus D), \quad P_\theta|_{\mathbb{D}_x} \cong P_{\theta_x}(O), \quad x \in D,$$

where $\mathbb{D}_x$ is a formal disc around $x$. This group scheme $P_\theta$ is called a *parahoric group scheme*.

By [9, Lemma 3.18], the group scheme $P_\theta$ defined above is a smooth affine group scheme of finite type, flat over $X$. By definition, we know $P_\theta|_{X \setminus D} \simeq G \times (X \setminus D)$, and then there is a natural $D_X$-scheme structure on $G \times X$. Let $\mathcal{L} := \Omega_X(D)$. These facts induce a natural connection $\nabla_{P_\theta}$ with first order pole on $\mathcal{O}_{P_\theta}$:

$$\mathcal{O}_{P_\theta} \xrightarrow{\nabla_{P_\theta}} \mathcal{O}_{P_\theta} \otimes_{\mathcal{O}_X} \mathcal{L}$$

such that $\nabla_{P_\theta}$ satisfies the condition $\nabla_{P_\theta}(fg) = g \otimes df + f \nabla_{P_\theta}(g)$ for $f$ is a local section of $\mathcal{O}_X$ and $g$ is a local section of $\mathcal{O}_{P_\theta}$.

**Tame Parahoric Local systems and Logahoric Higgs bundles.**

Let $E$ be a $P_\theta$-torsor over $X$, which can be understood by gluing the local data: $G$-torsor $E|_{X \setminus D}$ and $P_{\theta_x}$-torsor $E|_{\mathbb{D}_x}$. A *logahoric $P_\theta$-connection* is a connection with first order pole on $E$. More precisely, it is a derivation

$$\nabla : E \rightarrow \mathcal{O}_E \otimes_{\mathcal{O}_X} \mathcal{L}$$

such that the following diagram commutes:

$$\begin{array}{ccc}
\mathcal{O}_E & \xrightarrow{\nabla_E} & \mathcal{O}_{P_\theta} \otimes_{\mathcal{O}_X} \mathcal{O}_E \\
\mathcal{O}_E \otimes_{\mathcal{O}_X} \mathcal{L} & \xrightarrow{\nabla_{P_\theta} \otimes \text{id} + \text{id} \otimes \nabla_{P_\theta}} & \mathcal{O}_{P_\theta} \otimes_{\mathcal{O}_X} \mathcal{O}_E \otimes_{\mathcal{O}_X} \mathcal{L}.
\end{array}$$

**Definition 2.4.** A *tame $P_\theta$-local system* over $X$ is a pair $(E, \nabla)$, where $E$ is a $P_\theta$-torsor and $\nabla$ is a logahoric $P_\theta$-connection on $E$. Denote by $\text{Locsys}_{P_\theta}^{tame}(X)$ the stack of tame $P_\theta$-local systems over $X$. If there is no ambiguity, we omit $X$ in the notation $\text{Locsys}_{P_\theta}^{tame}(X)$.

Now let $\text{Ad}(E)$ be the adjoint bundle of $E$. A section $\phi \in H^0(X, \text{Ad}(E) \otimes_{\mathcal{O}_X} \mathcal{L})$ is called a *logarithmic $P_\theta$-Higgs field*.

**Definition 2.5.** A *logarithmic $P_\theta$-Higgs bundle* is a pair $(E, \phi)$, where $E$ is a $P_\theta$-torsor and $\phi \in H^0(X, \text{Ad}(E) \otimes \mathcal{L})$ is a logarithmic $P_\theta$-Higgs field. Denote by $\text{Higgs}_{P_\theta}^{tame}(X)$ the stack of logarithmic $P_\theta$-Higgs bundles over $X$. Sometimes, we add the subscript $\mathcal{L}$ and use the notation $\text{Higgs}_{P_\theta}^{tame}(X)$ to emphasize that it is for $\mathcal{L}$-twisted $P_\theta$-Higgs bundles.

### 3. Correspondence: Parahoric vs. Equivariant

Balaji and Seshadri gives the correspondence between parahoric torsors and equivariant bundles [9]. A similar correspondence also works for Higgs bundles and local systems [2] [10]. Although the correspondence is only given in characteristic zero, it can be naturally generalized to prime characteristic under some necessary conditions (tame weights). In this section, we first give the correspondence of parahoric torsors and equivariant bundles in positive characteristic, which is a direct generalization of Balaji and Seshadri’s work, and then we give the correspondences for Higgs bundles and local systems.

We introduce some notations first. Let $X$ and $Y$ be two smooth algebraic curves over $k$. In fact, we assume that $Y \rightarrow X$ is a covering with Galois group $\Gamma$. Let $x$ (resp. $y$) be a point in $X$ (resp.
3.1. **Torsors.** Let $\Gamma$ be a cyclic group of order $d$ with a generator $\gamma$. In this paper, we always assume that the order $d$ and the characteristic $p$ are coprime. Choose a $d$-th root of unity $\zeta$, and we have a natural $\Gamma$-action on $D_y$ such that $\gamma w = \zeta w$.

**Definition 3.1.** A $\Gamma$-equivariant $G$-torsor is a $G$-torsor $F$ over $D_y$ together with a lift of the action of $\Gamma$ on $F$ preserving the $G$-action. For simplicity, a $\Gamma$-equivariant $G$-torsor is called a $(\Gamma, G)$-torsor.

Since we work on $D_y$ now, we may assume that $F$ is the trivial $G$-torsor, and

$$F(D_y) \cong G(O_w).$$

Therefore, given a $(\Gamma, G)$-torsor $F$, we get a morphism $\rho : \Gamma \to G$ well-defined up to conjugation. If a $(\Gamma, G)$-torsor $F$ corresponds to $\rho$, then we say that $F$ is of type $\rho$. By [24, Lemma 2.5], the $\Gamma$-equivariant structure is uniquely determined by the representation $\rho$ and one may assume that the $\Gamma$-action on $D_y \times G$ is of the form

$$\gamma \cdot (w, h) = (\gamma w, \rho(\gamma)h^{-1}(\gamma)).$$

Since $\Gamma$ is a cyclic group, the representation $\rho$ factors through $T$ under a suitable conjugation. Note that

$$\text{Hom}(\Gamma, T) \cong \text{Hom}(X_*(T), X_*(\Gamma_y)) = \text{Hom}(X_*(T), \mathbb{Z}/d\mathbb{Z}) = X_*(T)/d \cdot X_*(T).$$

Then, a representation $\rho : \Gamma \to T$ corresponds to an element in $X_*(T) \otimes \mathbb{Q}$ with denominator $d$, which is a tame weight. Although this correspondence is not unique in general, a representation $\rho$ corresponds to a unique tame small weight $\theta$, which means that the rational number is smaller than one. On the other hand, given a tame weight $\theta$, it corresponds to a unique representation $\rho : \Gamma \to T$ given by $\rho(\gamma) = \zeta^{d \theta}$. Furthermore, we can find an element $\Delta(w) := w^{d \theta} \in T(K_w)$, which satisfies

$$\Delta(\gamma w) = \rho(\gamma)\Delta(w).$$

Let $F$ be a $(\Gamma, G)$-torsor of type $\rho$, and let $\theta$ be a tame weight corresponding to $\rho$. Denote by $\text{Aut}_{(\Gamma, G)}(F)$ the automorphism group. An automorphism $\sigma \in \text{Aut}_{(\Gamma, G)}(F)$ is equivalent to an element in $G(\mathbb{C}[w])^\Gamma$. Let $\Delta(w)$ be the element satisfying $\Delta(\gamma w) = \rho(w)\Delta(w)$. We define $\zeta := \Delta^{-1}\sigma\Delta$. Clearly, we have

$$\zeta(\gamma w) = \zeta(w),$$

which means that $\zeta$ is $\Gamma$-invariant. Therefore, it can be descended to an element $G(K_z)$ by substituting $z = w^d$. Moreover, for each root $\alpha \in R$, we have

$$\zeta(w)_{\alpha} = \sigma(w)_{\alpha}w^{-d\alpha(\theta)},$$

where the subscript $\alpha$ means that the element is in $U_{\alpha}(O_w)$. Substituting $z = w^d$, we have

$$\zeta(z)_{\alpha} = \sigma(z)_{\alpha}z^{-\alpha(\theta)}.$$

Since $\zeta(z)_{\alpha}$ is $\Gamma$-invariant, we have $\zeta(z)_{\alpha} \in U_{\alpha}(z^{m(\theta)}O_z)$ for each $\alpha \in R$. In conclusion, the element $\zeta(z)$ is in $P_{\theta}(O_z)$, and the above discussion implies the following lemma:

**Lemma 3.2** (Theorem 2.3.1 in [3]). Let $F$ be a $(\Gamma, G)$-torsor of type $\rho$, and let $\theta$ be a weight corresponding to $\rho$. Then, we have

$$\text{Aut}_{(\Gamma, G)}(F) \cong P_{\theta}(O_z).$$

More precisely, the automorphism group $\Delta^{-1}\text{Aut}_{(\Gamma, G)}(F)\Delta$ can be descended to $D_x$, which is isomorphic to $P_{\theta}(O_z)$.

This isomorphism gives us the following equivalence of categories.
Lemma 3.3. The category of $(\Gamma, G)$-torsors of type $\rho$ over $D_y$ is equivalent to the category of $P_\theta(O_z)$-torsors over $D_x$.

Now we consider the correspondence globally. Let $Y \to X$ be a covering of smooth algebraic curves with Galois group $\Gamma$. Suppose that the order of $\Gamma$ is not divided by $p$. Here is the definition of $\Gamma$-equivariant $G$-torsor over $Y$.

Definition 3.4. A $(\Gamma, G)$-torsor over $Y$ is a $G$-torsor $F$ together with a lift $\Gamma$-action on $F$ preserving the action of $G$.

Equivalently, a $(\Gamma, G)$-torsor over $Y$ is a $G$-torsor over the quotient stack $[Y/\Gamma]$. Denote by $\text{Bun}_G([Y/\Gamma])$ the stack of $(\Gamma, G)$-torsors over $Y$.

Given $y \in Y$, let $\Gamma_y$ be the stabilizer group of the point $y$, and we suppose that $\Gamma_y$ is a cyclic group. Denote by $R$ the set of points in $Y$, of which the stabilizer groups are nontrivial. This set $R$ is regarded as the set of ramifications, and then denote by $D \subseteq X$ the branch divisor. Let $F$ be a $(\Gamma, G)$-torsor over $Y$. As was discussed above, the $\Gamma$-action around $y \in R$ is given by a representation $\rho_y : \Gamma_y \to T$. Denote by $\rho := \{\rho_y, y \in R\}$ the collection of representations. Let $\theta_y$ be a corresponding weight of $\rho_y$, and then denote by $\theta$ the collection of weights, which gives a parahoric group scheme $P_\theta$ over $X$.

A $(\Gamma, G)$-torsor $F$ over $Y$ can be understood by gluing the following local data. We define $F_y := D_y \times G$, such that the $\Gamma_y$-action is defined as

$$\gamma \cdot (u, g) \to (\gamma u, \rho_y(\gamma)g), \quad u \in D_y, \gamma \in \Gamma_y,$$

and define $F_0 := (Y \setminus R) \times G$ with the $\Gamma_y$-structure

$$\gamma \cdot (u, g) \to (\gamma u, g), \quad u \in Y \setminus R, \gamma \in \Gamma_y.$$

Therefore, a $(\Gamma, G)$-torsor $F$ being of type $\rho$, is equivalent to giving $(\Gamma, G)$-isomorphisms

$$\Theta_y : F_y|_{D_y^y} \to F_0|_{D_y^y}, \quad y \in R.$$

With respect to the local picture we discussed, a $(\Gamma, G)$-bundle $F$ over $Y$ corresponds to a $G_\theta$-torsor over $X$.

Theorem 3.5 (Theorem 5.3.1 in [3]). The category of $(\Gamma, G)$-torsors of type $\rho$ over $Y$ is equivalent to the category of $P_\theta$-torsors over $X$. Furthermore, the equivalence of categories gives the equivalence of stacks, i.e.

$$\text{Bun}_G^\rho([Y/\Gamma]) \cong \text{Bun}_{P_\theta}(X).$$

3.2. Logarithmic Higgs Bundles. Let $F$ be a $(\Gamma, G)$-bundle over $D_y$, and denote by $E$ the corresponding $P_\theta(O_z)$-torsor over $D_x$. Denote by $\text{Ad}(F)$ the adjoint bundle. Without loss of generality, suppose that $\text{Ad}(F) = g(O_u)$. Let $\phi$ be an element in $g(O_u)\frac{dw}{w}$, which is regarded as a section $D_y \to \text{Ad}(F) \otimes \Omega_{D_y}(y)$ and is called a logarithmic Higgs field. Assume that $\phi$ is $\Gamma$-equivariant, i.e.

$$\phi(\gamma w) = \rho(\gamma)\phi(w)\rho^{-1}(\gamma).$$

With the same notations as in §3.1, we define

$$\varphi = \Delta^{-1}\phi\Delta.$$

Clearly, $\varphi$ is $\Gamma$-invariant, i.e.

$$\varphi(\gamma w) = \varphi(w).$$

Therefore, $\varphi(w)$ can be descended to a section $D_x \to \text{Ad}(E) \otimes \Omega_{D_x}(x)$ by substituting $z = w^d$, which implies that $\varphi(z) \in P_\theta(O_x)\frac{dz}{z}$. Abusing the notation, we still use $\varphi : D_x \to \text{Ad}(E) \otimes \Omega_{D_x}(x)$ for the corresponding section. It is easy to check that a $\Gamma$-equivariant logarithmic Higgs field $\phi$ of $F$ corresponds to a unique logarithmic $P_\theta(O_z)$-Higgs field $\varphi$ of $E$. With respect to the above discussion, we have a one-to-one correspondence between logarithmic $(\Gamma, G)$-Higgs bundles over $D_y$ and logarithmic $P_\theta$-Higgs bundles over $D_x$. This local discussion can be generalized globally. With the same setup as in §3.1, we introduce the following definition.
Definition 3.6. A logarithmic \((\Gamma, G)\)-Higgs bundle over \(Y\) is a pair \((F, \phi)\), where \(F\) is a \((\Gamma, G)\)-torsor over \(Y\) and \(\phi \in H^0(Y, \text{Ad}(F) \otimes \Omega_Y(R))\) is a \(\Gamma\)-equivariant section.

Theorem 3.7 (Theorem 3.6 in \[14\]). The stack of logarithmic \((\Gamma, G)\)-Higgs bundles of type \(\rho\) over \(Y\) is isomorphic to the stack of logarithmic \(\mathcal{P}_\theta\)-Higgs bundles over \(X\), i.e.
\[
\text{Higgs}^\text{tame,\rho}(\mathcal{Y}/\Gamma) \cong \text{Higgs}^\text{tame}_\mathcal{P}_\theta(X).
\]

We refer the reader to \[22\] \[23\] \[25\] for more details about the algebraic stacks Higgs \(\mathcal{Y}\)-torsor, \(\mathcal{P}_\theta\)-torsor.

3.3. Tame Local Systems. We follow the same notations as above.

Definition 3.8. A tame \((\Gamma, G)\)-local system of type \(\rho\) over \(Y\) is a \((\Gamma, G)\)-torsor \(F\) together with a logarithmic \(\Gamma\)-equivariant \(G\)-connection \(\nabla\). A logarithmic \(\Gamma\)-equivariant \(G\)-connection is also called a logarithmic \((\Gamma, G)\)-connection.

Let \(y\) be a ramification point on \(Y\), \(x\) be its image on \(X\). Over a neighborhood of \(y\), the connection \(\nabla\) can be written as \(d - A \frac{dw}{w}\), where \(A \in \mathfrak{g}(\mathcal{O}_w)\). Then the condition that \(\nabla\) is \(\Gamma\)-invariant (under the gauge action) means that
\[
d\rho(\gamma)\rho(\gamma)^{-1} + \text{Ad}(\rho(\gamma))A(w)\frac{dw}{w} = A(\gamma w)\frac{dw}{w},
\]
where \(\rho : \Gamma \to T\) is the representation we studied in \(3.1\). Since \(\rho(\gamma)\) is a constant matrix, therefore we have
\[
\text{Ad}(\rho(\gamma))A(w)\frac{dw}{w} = A(\gamma w)\frac{dw}{w},
\]
which is equivalent to say that \(A\) lies in \(\mathfrak{g}(\mathcal{O}_w)^\Gamma = \mathfrak{p}_\theta(\mathcal{O}_z)\). With the same idea as Theorem \(3.6\) and \(3.7\) we get the following.

Proposition 3.9. The stack of tame \((\Gamma, G)\)-local systems of type \(\rho\) over \(Y\) is equivalent to the stack of tame \(\mathcal{P}_\theta\)-local systems over \(X\), i.e.
\[
\text{Locsys}^\text{tame,\rho}(\mathcal{Y}/\Gamma) \cong \text{Locsys}^\text{tame}_\mathcal{P}_\theta(X).
\]

Proof. We only give the correspondence locally. As we discussed in \(3.1\) a \((\Gamma, G)\)-torsor \(F\) of type \(\rho\) over \(\mathbb{D}_y\) corresponds to a unique \(\mathcal{P}_\theta(\mathcal{O}_z)\)-torsor over \(\mathbb{D}_x\). A tame \((\Gamma, G)\)-local system on \(\mathbb{D}_y\) can be regarded as a logarithmic connection \(A \frac{dw}{w}\), where \(A \in \mathfrak{g}(\mathcal{O}_w)\). Under the gauge action by \(\Delta^{-1}\), the logarithmic \((\Gamma, G)\)-connection \(A \frac{dw}{w}\) corresponds to following
\[
\Delta^{-1} \cdot A \frac{dw}{w} = d \cdot (-\theta + \text{Ad}(\Delta^{-1})A)\frac{dw}{w}.
\]
Note that \(\text{Ad}(\Delta^{-1})A\) is \(\Gamma\)-invariant, which can be descended to an element in \(\mathfrak{p}_\theta(\mathcal{O}_z)\) by substituting \(z = w^d\). As we explained above, the element \(-\theta + \text{Ad}(\Delta^{-1})A\) is in \(\mathfrak{p}_\theta(\mathcal{O}_z)\). Therefore, we get a logarithmic \(\mathcal{P}_\theta(\mathcal{O}_z)\)-connection on \(\mathbb{D}_x\). The other direction can be proved similarly. \(\square\)

4. LOCAL TAME PARAHORIC NONABELIAN HODGE CORRESPONDENCE

In this section, we study tame \(G\)-local systems and logarithmic \(G\)-Higgs bundles over a formal disc \(\mathbb{D}\). We prove the tame nonabelian Hodge correspondence over \(\mathbb{D}\) and generalize this correspondence to a parahoric version.

Fixing a local coordinate \(z\) of \(\mathbb{D} = \text{Spec}(\mathcal{O})\), a tame \(G\)-local systems over \(\mathbb{D}\) is a logarithmic connection \(d - A \frac{dz}{z}\) (indeed over the trivial \(G\)-torsor) , where \(A \in \mathfrak{g}(\mathcal{O})\). We say that \(d - A \frac{dz}{z}\) is equivalent to \(d - B \frac{dz}{z}\) (under the gauge action) if there exists \(g \in G(\mathcal{O})\) such that
\[
dgg^{-1} + \text{Ad}(g)A \frac{dz}{z} = B \frac{dz}{z}.
\]
This action is known as the gauge action, and denote it by \( g \ast A \frac{dz}{z} = B \frac{dz}{z} \). Therefore, the category of logarithmic G-connections over \( \mathbb{D} \) is regarded as the equivalence classes of logarithmic G-connections \( d - A \frac{dz}{z} \) under the gauge action.

Now we come to the side of logarithmic G-Higgs bundles. A logarithmic G-Higgs field over \( \mathbb{D} \) is \( A \frac{dz}{z} \), where \( A \in g(\mathcal{O}) \). The action of \( G(\mathcal{O}) \) on Higgs fields is defined as the adjoint action, and \( A \frac{dz}{z} \) is equivalent to \( B \frac{dz}{z} \) if there exists \( g \in G(\mathcal{O}) \) such that \( A = \text{Ad}(g)B \). Thus, the category of logarithmic G-Higgs bundles over \( \mathbb{D} \) is considered as the equivalence classes of Higgs fields \( A \frac{dz}{z} \) under the adjoint action.

4.1. Standard Form of Tame Connections. In this subsection we would like to prove an analogue of the results in [5] in positive characteristic. For the case of characteristic zero, we also refer the reader to [11] for more details. First let us consider logarithmic connections for the reductive group \( G \).

**Lemma 4.1.** Let \( \theta \) be a weight. Let \( X_k \in Lg_{\geq k} \) where \( k > 0 \). Then there exists an element \( g_k \in G_{\geq k} \) such that

\[
\text{Ad}(g_k) = \text{id} + [X_k, -] \mod Lg_{> k}
\]

as an operator on \( Lg_{\geq 0} \).

**Proof.** Let \( X_k = \sum_0^\alpha X_k^\alpha \), where \( X_k^\alpha \in g_\alpha(\mathcal{K}) \). Since \( k > 0 \), it is easy to verify that if we have constructed \( g_k^\alpha \) for each \( X_k^\alpha \), then the product \( g_k = \prod g_k^\alpha \) satisfies the requirement. So we can assume \( X_k = X_k^\alpha \notin Lg_{> k} \). Furthermore, it is enough to consider the case when \( X_k^\alpha \) is of the form \( z^k (\theta, \alpha) g_\alpha \). If \( \alpha \neq 0 \), then one can consider a morphism from \( SL_2 \) to \( G(\mathcal{K}) \) corresponds to the root \( \alpha + k - (\theta, \alpha) \), then the statement essentially reduces to \( SL_2 \) representations, so we can choose \( g_k^\alpha \in U_\alpha(z^{-(\theta, \alpha) + k}\mathcal{O}) \) that satisfies our requirement if \( \alpha \neq 0 \). If \( \alpha = 0 \), we can choose \( g_k^\alpha \in T(\mathcal{O}) \) that satisfies our requirement. \( \square \)

**Lemma 4.2 (Standard Form).** Let \( A = \sum_{i \geq 0} a_i z^i \in g(\mathcal{O}) \). There exists \( g \in G(\mathcal{O}) \) such that

\[
d_{gg^{-1}} + \text{Ad}(g)A \frac{dz}{z} = B \frac{dz}{z}
\]

where \( B = \sum_{i \geq 0} b_i z^i \) and \( b_i \) lies in the generalized eigenspace of the operator \([b_0, -]\) with eigenvalue \( i \).

**Proof.** We shall construct a sequence of elements \( g_k \), where \( g_k \) lies in the \( k \)-th congruence subgroup of \( G(\mathcal{O}) \) such that if we take \( g = g_k g_{k-1} \cdots g_0 \), then after applying gauge action of \( g \), we have

\[
g \ast A \frac{dz}{z} = \left( \sum_{i \geq 0} b_i z^i \right) \frac{dz}{z},
\]

such that \( b_i \) lies in the generalized eigenspace of \( b_0 \) with eigenvalue \( i \) for all \( i \leq k \). For \( k = 0 \), we can take \( g_0 = 1 \). Suppose now we have chosen the elements \( g_0, g_1, \ldots, g_k \in G(\mathcal{O}) \) such that we get \( \left( \sum_{i \geq 0} b_i z^i \right) \frac{dz}{z} \), where \( b_i \) lies in the generalized eigenspace of \( b_0 \) with eigenvalue \( i \) for all \( i \leq k \). Abusing the notation, let

\[
A = b_0 + b_1 z + \cdots + b_k z^k + a_{k+1} z^{k+1} + \cdots.
\]

Taking an arbitrary element \( y \in g \), let \( x \in G(\mathcal{O}) \) be the element satisfying the condition of Lemma 4.1 for \( y z^{k+1} \). Note that

\[
d_{xx^{-1}} = (k + 1) y z^{k+1} \frac{dz}{z} \mod z^{k+2}g(\mathcal{O}) \frac{dz}{z}, \quad \text{Ad}(x)A = (A + [y z^{k+1}, A]) \frac{dz}{z} \mod z^{k+2}g(\mathcal{O}) \frac{dz}{z}.
\]

After gauge transform by \( x \), we get

\[
x \ast A = d_{xx^{-1}} + \text{Ad}(x)A \frac{dz}{z} = \left( -(k + 1) y z^{k+1} + (A + [y z^{k+1}, A]) \right) \frac{dz}{z} \mod z^{k+2}g(\mathcal{O}) \frac{dz}{z}.
\]

\[
= \left( \sum_{i=0}^k b_i z^i \right) \frac{dz}{z} \mod z^{k+2}g(\mathcal{O}) \frac{dz}{z}.
\]
We can choose an element \( y \in \mathfrak{g} \) such that \((k + 1)y + a_{k+1} + [y, b_0]\) lies in the generalized eigenspace of \( b_0 \) with eigenvalue \((k + 1)\). This finishes the proof of this lemma.

Given any logarithmic connection \( d - A \frac{dz}{z} \), where \( A = \sum_{i \geq 0} a_i z^i \), we can assume that under gauge transformation, \( a_i \) lies in the generalized eigenspace of the operator \([a_0, -]\) by Lemma 4.2 and a logarithmic connection in this form will be called in standard form. We will also need a more general version of Lemma 4.2 which works over an Artinian local algebra over \( k \):

**Lemma 4.3.** Let \((R, \mathfrak{m})\) be an Artinian local algebra over \( k \). Let \( A = \sum_{i \geq 0} a_i z^i \in \mathfrak{g}(O_R) \) be an element, where \( O_R := R[[z]] \). Suppose that \( a_0 \in \zeta + \mathfrak{m} \otimes \mathfrak{g} \), where \( \zeta \in \mathfrak{g} \). Let \( g_\lambda \) be the generalized eigenspace of adjoint action of \( \zeta \) with eigenvalue \( \lambda \). Then there exists \( g \in G(O_R) \) such that

\[
dgg^{-1} + \text{Ad}(g) A \frac{dz}{z} = B \frac{dz}{z},
\]

where \( B = \sum_{i \geq 0} b_i z^i \) and each \( b_i \) lies in the \( \mathfrak{g}_i \otimes R \).

**Proof.** There exists a filtration of \( R \) by ideals \( I_i \) such that \( I_i/I_{i+1} \) is annihilated by \( \mathfrak{m} \). Using this filtration, one can show that for any element \( y \in \mathfrak{g}_i \otimes R \) and any integer \( n \neq \lambda \), there exists \( P \in \mathfrak{g} \otimes R \) such that \( y = nP - [\zeta, P] \). Now one can use the same argument as in Lemma 4.2 to finish the proof. \( \square \)

**4.2. Irrational Case.** Let \( d - A \frac{dz}{z} \) be a logarithmic connection, where \( A = \sum_{i \geq 0} a_i z^i \). Let \( a_0 = \tau + \sigma + n \), where \( \tau \) is the rational part and \( \sigma \) is the irrational part (see \( \preceq \)). We first consider a special case that the eigenvalues of \([a_0, -]\) are not rational, i.e. \( \tau = 0 \).

**Corollary 4.4.** Let \( A = \sum_{i \geq 0} a_i z^i \), where \( a_0 = \sigma + n \) with trivial rational part \((\tau = 0)\). Then there exists \( g \in G(\mathcal{O}) \) such that \( dgg^{-1} + \text{Ad}(g) A \frac{dz}{z} \in \mathfrak{g}(\mathcal{O}^p) \frac{dz}{z} \).

**Proof.** This is a direct result of Lemma 4.2. \( \square \)

This corollary shows that a logarithmic connection in this case is equivalent to a logarithmic connection in \( \mathfrak{g}(\mathcal{O}^p) \frac{dz}{z} \), which is in standard form. On the other hand, if \( a_0 \) has non-zero rational eigenvalues, then it is unclear whether \( A \) can be put into \( \mathfrak{g}(\mathcal{O}^p) \) via \( G(\mathcal{O}) \). But nonetheless one can show that \( A \) can be put into \( \mathfrak{g}(\mathcal{K}^p) \frac{dz}{z} \) via \( G(\mathcal{K}) \):

**Lemma 4.5.** Let \( A = \sum_{i \geq 0} a_i z^i \in \mathfrak{g}(\mathcal{O}) \). Then there exists \( g \in G(\mathcal{K}) \) such that

\[
dgg^{-1} + \text{Ad}(g) A \frac{dz}{z} = C \frac{dz}{z},
\]

where \( C \in \mathfrak{g}(\mathcal{K}^p) \). Moreover, if we choose a splitting of the embedding \( t_{\mathfrak{g}} \hookrightarrow t \) as well as a set of representatives for the semisimple orbits for the adjoint action of \( G \) on \( \mathfrak{g} \) denoted by \( \mathcal{D} \), then one can assume that \((C_0)_{st} \in \mathcal{D} \).

**Proof.** By Lemma 4.2 we can assume that each \( a_i \) lies in the generalized eigenspace of \( a_0 \) with eigenvalue \( i \). Let \( a_0 = \tau + \sigma + n \), where \( \tau \) is the rational part and \( \sigma \) is the irrational part. Then we have \([\tau, a_i] = ia_i\). One may choose a weight \( \theta_\tau \in X_*(T) \otimes \mathbb{Q} \) such that

\[
\langle \theta_\tau, \alpha \rangle \in \mathbb{Z} \quad \text{and} \quad \langle \theta_\tau, \alpha \rangle = \alpha(\tau) \mod p,
\]

for any root \( \alpha \). Now taking the gauge action by \( z^{\theta_\tau} \), we get the desired form. \( \square \)

**Lemma 4.6.** If \( g \in G(\mathcal{K}) \) such that \( g * C \frac{dz}{z} = D \frac{dz}{z} \), where \( C \) and \( D \) are of the form in Lemma 4.3. Then \( g \in G(\mathcal{K}^p) \).

**Proof.** We may assume that the semisimple part of \( c_0 \) and \( d_0 \) are in \( t \), and we also choose an embedding of \( G \) into \( \text{GL}_n \). Thus, we shall view \( g, C, D \) as elements in \( \text{GL}_n(\mathcal{O}) \) and \( \mathfrak{g}_n(\mathcal{O}) \). Write \( g = \sum_{i \geq 0} g_i z^i \).

Let \( j \) be the smallest index such that \( j \) does not divide \( p \) and \( g_j \neq 0 \). Then using equation \( dg + gA = Bg \),
we conclude that $jg_j + g_jc_0 = d_0g_j$. Consider the operator on $g_t$, given by
\[ T \to d_0T - Tc_0. \]
Since the semisimple parts of $a_0$ and $b_0$ are in $t$ and that $a_0$ and $b_0$ has no nonzero rational eigenvalues, we conclude that the only rational eigenvalue of this operator is zero. Therefore, $j = 0$ and then $j$ is divisible by $p$. \hfill \square

Let $\text{Locsys}_{G, \text{irr}}^{\text{tame}}(\mathcal{D})$ be the category of tame $G$-local systems on the formal disc $\mathcal{D}$ such that the semisimple part of the residue is irrational. Similarly, let $\text{Higgs}_{G, \text{irr}}^{\text{tame}}(\mathcal{D}')$ be the category of logarithmic $G$-Higgs bundles on $\mathcal{D}'$, the Frobenius twist of $\mathcal{D}$, such that the semisimple part of the residue is irrational.

**Proposition 4.7.** The category $\text{Locsys}_{G, \text{irr}}^{\text{tame}}(\mathcal{D})$ is equivalent to $\text{Higgs}_{G, \text{irr}}^{\text{tame}}(\mathcal{D}')$.

**Proof.** Take an element $A\frac{dz}{z} \in \text{Locsys}_{G, \text{irr}}^{\text{tame}}(\mathcal{D})$, where $A = \sum_{i \geq 0} a_i z^i$. Since the semisimple part of $a_0$ is irrational, this element $A\frac{dz}{z}$ is equivalent to $A'\frac{dz}{z} \in g(O')\frac{dz}{z}$ under the gauge action by Corollary 4.4. By the Frobenius twist, the logarithmic connection $A'\frac{dz}{z}$ can be considered as a logarithmic Higgs field over $\mathcal{D}'$, and we use the same notation $A'\frac{dw}{w}$. Furthermore, by Lemma 4.6 we get a well-defined functor
\[ \text{Locsys}_{G, \text{irr}}^{\text{tame}}(\mathcal{D}) \to \text{Higgs}_{G, \text{irr}}^{\text{tame}}(\mathcal{D}') \]
\[ A \to A'. \]
It is easy to check that this functor induces an equivalence of categories. \hfill \square

**4.3. Rational Case.** For any semisimple rational element $\tau \in g$, we equip the trivial $G$-torsor over $\mathcal{D}$ with a logarithmic $G$-connection given by $\tau\frac{dz}{z}$. Let $G'_{r}(\mathcal{O}) \subset G(\mathcal{O})$ be the group of automorphisms of $d - \tau\frac{dz}{z}$. Moreover, the elements in the Lie algebra $g'_{r}(\mathcal{O})$ of $G'_{r}(\mathcal{O})$ can be written as $A = \sum_{i \geq 0} a_i z^i$, where each $a_i$ lies in the generalized eigenspace of $\tau$ with eigenvalue $i$. Since for any $g \in G'_{r}(\mathcal{O})$, we have
\[ dgg^{-1} + g\tau g^{-1}\frac{dz}{z} = \tau\frac{dz}{z}, \]
then gauge action of $G'_{r}(\mathcal{O})$ on its Lie algebra can be written as
\[ g * A\frac{dz}{z} = dgg^{-1} + gAg^{-1}\frac{dz}{z} = \tau\frac{dz}{z} + g(A - \tau)g^{-1}\frac{dz}{z}. \]

**Lemma 4.8.** The automorphism group $G'_{r}(\mathcal{O})$ can be identified with a parahoric subgroup over $\mathcal{D}'$.

**Proof.** With the same approach as Lemma 4.3, let $\theta$ be a weight such that
\[ \langle \theta, \alpha \rangle \in \mathbb{Z} \quad \text{and} \quad \langle \theta, \alpha \rangle = a(\tau) \mod p. \]
Then from the definition of $G'_{r}(\mathcal{O})$, it is easy to see that the Lie algebra of $\text{Ad}(z^{-\theta})(G'_{r}(\mathcal{O}))$ is the Lie algebra of the parahoric subgroup of $G(k((z^p)))$ over $\mathcal{D}'$ defined by the weight $\theta \frac{dz}{z}$. It remains to show $G'_{r}(\mathcal{O})$ is connected. It is easy to see that the image of the evaluation morphism $G'_{r}(\mathcal{O}) \to G(\mathcal{O}) \to G$ is equal to $Z_G(\tau)$, which is a Levi subgroup of $G$, hence connected. This proves the claim. \hfill \square

**Lemma 4.9.** Take two elements
\[ A = \sum_{i \geq 0} a_i z^i \in g(\mathcal{O}), \quad B = \sum_{i \geq 0} b_i z^i \in g(\mathcal{O}) \]
in standard form. Assume $a_0$ and $b_0$ has the same rational semisimple part $\tau$. Suppose that $g \in G(\mathcal{O})$ such that $dgg^{-1} + gAg^{-1}\frac{dz}{z} = B\frac{dz}{z}$, then $g \in G'_{r}(\mathcal{O})$. 

Proof. First, we represent \( g \) as \( g = g_1 g_0 \) where \( g_0 \in G \) and \( g_1 \) lies in the kernel of the evaluation map \( G(\mathcal{O}) \to G \). By the condition \( dg + A g^{-1} \frac{dz}{z} = B g \frac{dz}{z} \), one concludes that \( g_0 \in Z_G(\tau) \). Since \( Z_G(\tau) \subseteq G'_s(\mathcal{O}) \), one may assume that \( a_0 = b_0 \).

Now we fix a representation of \( G \) and view \( A \) as a matrix. The condition \( dg + A g^{-1} \frac{dz}{z} = B g \frac{dz}{z} \) gives us

\[
dg + g A \frac{dz}{z} = B g \frac{dz}{z}.
\]

By the property of \( G'_s(\mathcal{O}) \), the element \( g \in G'_s(\mathcal{O}) \) if and only if \( [\tau, g_i] = i g_i \) when \( g_i \neq 0 \), where \( g = \sum_{i \geq 0} g_i z^i \). Let \( k \) be the smallest index such that \( g_k \neq 0 \) and \( g_k \) is not in the eigenspace of \( \tau \) with eigenvalue \( k \). Then we have

\[
k y_k + \sum_{i+j=k} g_i a_j = \sum_{i+j=k} b_j g_i.
\]

If \( i < k \) and \( a_j g_i \neq 0 \) or \( g_i b_j \neq 0 \), then by our choice of \( k \), \( a_j g_i \) satisfies

\[
[\tau, a_j g_i] = (i+j)a_j g_i = k a_j g_i.
\]

A similar formula also holds for \( g_i b_j \). Thus, the element \( k y_k + g_k a_0 - a_0 g_k \) lies in the generalized eigenspace of \( a_0 \) with eigenvalue \( k \). By the assumption that \( g_0 = 0 \) in this subspace, we arrive at a contradiction. \( \Box \)

We fix a rational semisimple element \( \tau \) in \( g \). Let \( \text{Locsys}^{\text{tame}}_{G, \tau}(\mathbb{D}) \) be the category of tame \( G \)-local systems on the formal disc such that the rational semisimple part of the residue is conjugate to \( \tau \). More precisely, if \( d - A \frac{dz}{z} \in \text{Locsys}^{\text{tame}}_{G, \tau}(\mathbb{D}) \) where \( A = \sum_{i \geq 0} a_i z^i \), then the rational semisimple part of \( a_0 \) is conjugate to \( \tau \). Let \( \text{Higgs}^{\text{tame}, \text{irr}}_{G'_s, \tau}(\mathbb{D}') \) be the category of logarithmic \( G'_s(\mathcal{O}) \)-Higgs bundles on the Frobenius twist of \( \mathbb{D} \) such that the semisimple part of the residue is irrational (up to conjugation). Here we regard \( G'_s(\mathcal{O}) \) as the parahoric group \( \mathcal{P}_{\tau, \mathbb{D}}(\mathcal{O}) \) determined in Lemma 4.8.

**Proposition 4.10.** The category \( \text{Locsys}^{\text{tame}}_{G, \tau}(\mathbb{D}) \) is equivalent to \( \text{Higgs}^{\text{tame}}_{G'_s, \tau}(\mathbb{D}') \). Moreover, let \( \Gamma \) be a cyclic group with order \( d \) coprime to \( p \), and then the category \( \text{Locsys}^{\text{tame}}_{G, \tau}(\mathbb{D}/\mathbb{Q}) \) is equivalent to \( \text{Higgs}^{\text{tame}}_{G'_s, \tau}(\mathbb{D}'/\Gamma) \).

**Proof.** As we discussed at the beginning of this subsection, we find that

\[
g * A \frac{dz}{z} = dg^{-1} + A g^{-1} \frac{dz}{z} = \tau \frac{dz}{z} + g(A - \tau) g^{-1} \frac{dz}{z},
\]

where \( g \in G'_s(\mathcal{O}) \). We use the same approach as in Proposition 4.7 to construct a map

\[
\text{Locsys}^{\text{tame}}_{G, \tau}(\mathbb{D}) \to \text{Higgs}^{\text{tame}}_{G'_s, \tau}(\mathbb{D}'), \quad A \to \text{Ad}(z^{-\theta} \tau)(A - \tau).
\]

By Lemma 4.9 as well as the action of \( G'_s(\mathcal{O}) \) on its Lie algebra, it is easy to check that this map is well-defined and bijective.

For the version of stacks, it is equivalent to consider logarithmic \( (\Gamma, G) \)-connection on \( \mathbb{D} \) such that the rational semisimple part of the residue is \( \tau \). As we discussed in §3.5 a logarithmic \( G \)-connection \( d - A \frac{dz}{z} \) is \( \Gamma \)-equivariant means that \( A \in \mathfrak{g}(\mathcal{O}) \) is \( \Gamma \)-equivariant. Clearly, \( (A - \tau) \frac{dz}{z} \) is also \( \Gamma \)-equivariant as a logarithmic Higgs field. Therefore, the correspondence also holds for stacks. \( \Box \)

For future use, let us note the following:

**Lemma 4.11.** Let \( A = \sum_{i \geq 0} a_i z^i \) be an element in \( g'_s(\mathcal{O}) \frac{dz}{z} \), where \( \tau \) is the rational part of the \( a_0 \). Then the logarithmic \( G \)-connection \( d - A \frac{dz}{z} \) has zero p-curvature if and only if \( A = \tau \). More generally, let \( A \in \mathfrak{g}(\mathcal{O}_R) \), where \( (R, \mathfrak{m}) \) is an Artinian local algebra over \( k \). Suppose that the rational part of \( a_0 \) is \( \tau \) when modulo \( \mathfrak{m} \). Then the connection \( d - A \frac{dz}{z} \) has zero p-curvature if and only if there exists \( g \in G(\mathcal{O}_R) \) such that \( dg^{-1} + A g^{-1} \frac{dz}{z} = \tau \frac{dz}{z} \).
Proof. It is well-known that in the logarithmic case, we have \((zd\theta)^{(p)} = z\theta\) \cite{10 \S1.2.2}. Suppose that the logarithmic connection is \(d - A\frac{dz}{z}\). It has zero \(p\)-curvature if and only if \(A^p - A = 0\). Based on this fact, it is clear that when \(A = \tau\), the logarithmic connection \(d - A\frac{dz}{z}\) has zero \(p\)-curvature. Then we consider the other direction. Suppose that \(d - A\frac{dz}{z}\) has zero \(p\)-curvature, where \(A \in g'_r(\mathcal{O})\) with rational part \(\tau\). With the same approach as Lemma \[8.5\text{ and } 8.8\], we choose a weight \(\theta \in X_*(T) \otimes \mathbb{Z} \mathbb{Q}\). Let \(B = z^{-\theta} \ast A\frac{dz}{z} = Ad(z^{-\theta})(A - \tau)\frac{dz}{z}\). Clearly, the logarithmic connection \(d - B\frac{dz}{z}\) also has zero \(p\)-curvature. Note that \(B\) is of the form \(\sum b_i z^i\) by Proposition \[4.7\]. Thus, \((zd\theta)(B) = 0\). Together with the fact \(B^p - B = 0\), one concludes that \(B = 0\). Therefore, we have \(A = \tau\).

In the general situation, one can first use Lemma \[4.3\] to convert \(A\) into the form

\[
A = \sigma + \text{higher degree terms } \in g'_r(\mathcal{O}_R),
\]

where the semisimple part of \(\sigma\) modulo \(m\) is irrational. Then the condition that \(p\)-curvature of \(A\) equals to zero implies that \(A = \tau\) by the same argument as in the case over a field.

\[\blacksquare\]

4.4. Parahoric Case. In this subsection, we would like to generalize the results above to the case of parahoric subgroups. Namely, let \(\mathcal{P}_\theta(\mathcal{O})\) be a parahoric subgroup of \(G(K)\) with a tame weight \(\theta \in X_*(T) \otimes \mathbb{Z} \mathbb{Q}\) and the weight \(\theta\) can be regarded as an element in \(I_f\). In a similar way, we consider the gauge action of \(\mathcal{P}_\theta(\mathcal{O})\) on \(p_\theta(\mathcal{O})\frac{dz}{z}\). Then one may adapt the analysis on \(g(\mathcal{O})\frac{dz}{z}\) to \(p_\theta(\mathcal{O})\frac{dz}{z}\), but we shall use a slightly different approach which is based on \[3\].

Recall that a weight \(\theta\) induces a natural decomposition of \(g = \bigoplus_{\lambda \in \mathbb{Q}} g_\lambda\) indexed by \(\mathbb{Q}\), where \(g_\lambda\) is the \(\lambda\)-th graded piece. This decomposition also induces a filtration of the loop Lie algebra \(Lg\) (see \[2\]). The following is a reformulation of the proof of \[5\] Theorem 6\] in our setting:

**Lemma 4.12.** Let \(A \in p_\theta(\mathcal{O})\frac{dz}{z}\) be an element. Suppose that the weight zero component \(A(0)\) is equal to \((\tau + \sigma + \sum a_i z^i)\frac{dz}{z}\), where \(\tau\) is semisimple rational element and \(\sigma\) is a semisimple irrational element such that

\[
[\tau, a_i] = ia_i, \quad [\sigma, a_i] = 0, \quad \sum a_i \text{ is nilpotent}.
\]

Then there exists \(g \in \mathcal{P}_\theta(\mathcal{O})\) such that

\[
dg g^{-1} + Ad(g)A = (\tau + \sigma + \sum A_i z^i)\frac{dz}{z}
\]

where \(A_i \in g_{\geq i}, [\tau, A_i] = ia_i, [\sigma, A_i] = 0\).

Now we take an element in \(A = \sum_{i \geq 0} a_i z^i \in p_\theta(\mathcal{O}_2)\), where we use \(z\) to emphasize the local coordinate. By Proposition \[3.3\], the gauge action of \(\mathcal{P}_\theta(\mathcal{O}_2)\) on the Levi quotient \(\mathfrak{l}_\theta(\mathcal{O}_2)\) gets transformed into adjoint action of \(G(\mathcal{O}_w)\) on \(g\). Fixing a choice of \(d\)-th root of unity \(\zeta\), we get the following:

**Lemma 4.13.** There is a one-to-one correspondence between orbits of \(\mathcal{O}_2(\mathcal{O}_2)\frac{dz}{z}\) under the gauge action of \(\mathcal{P}_\theta(\mathcal{O}_2)\) and the orbits of \(\mathfrak{h}_\theta \frac{dw}{w}\) under the adjoint action of \(Z_G(\zeta^{d\theta})\).

**Proof.** The proof is similar to \[5\] Lemma 4\]. \[\blacksquare\]

Given an element \(A \in p_\theta(\mathcal{O}_2)\), let \(A = \sum_{i \geq 0} a_i z^i\), where \(a_0 = \tau + \sigma\), and denote by \(d + A\frac{dz}{z}\) the corresponding connection on \(\mathbb{D}_x\). Then, by Lemma \[4.12\] and \[4.13\], it corresponds to a \(\Gamma\)-equivariant connection \(d(\theta + \tau + \sigma + \sum a_i)\frac{dw}{w}\) on \(\mathbb{D}_y\). Now we choose a tame weight \(\theta\) with the property that

\[
\langle \theta_\tau, \alpha \rangle \equiv \alpha(\tau) \mod p
\]

for all roots \(\alpha\) and that \(\langle \theta_\tau, \alpha \rangle = 0\) whenever \(\alpha(\tau) = 0\). We also assume that \(d\theta\) is integral where \(d \in \mathbb{N}\) such that \((d, p) = 1\). Consider the following diagram:

\[
\begin{align*}
\mathcal{P}_\theta(\mathcal{O}_2) & \longrightarrow \mathcal{G}_d(\mathcal{O}_w) \\
\downarrow & \\
G(\mathcal{O}_w) & \longrightarrow G(\mathcal{O}_w)
\end{align*}
\]
Taking an element \( g \in \mathcal{P}_0(O_z) \), we consider \( g \) as an element in \( G(K_w) \) by substituting \( z = w^d \). Then, we can identify \( \mathcal{P}_0(O_z) \) with a subgroup of \( G(O_w) \) (actually, it is not a subgroup, they are the same under the conjugation) via

\[
\mathcal{P}_0(O_z) \subseteq G(K_w) \to G(O_w), \quad g \to \text{Ad}(w^d)g.
\]

With this idea in mind, we are ready to prove the following lemma:

**Lemma 4.14.** The group \( G'_{d+\theta+\tau}(O_w) \) can be identified with the parahoric group \( \mathcal{P}_{\theta+\phi+}\) over \( \mathbb{D}' \).

**Proof.** The proof of this lemma is similar to that of Lemm 4.8. First, let us look at the situation at the level of Lie algebras. \( \mathfrak{p}_0(O_z) \) consists of elements of the form \( \sum a_i z^i \) where \( a_i \in \mathfrak{g}_\alpha \) such that \( \langle \theta, \alpha \rangle + i \geq 0 \). With respect to the discussion above, if the element \( \sum a_i z^i \) lies in the intersection of Lie algebras of \( \mathcal{P}_0(O_z) \) and \( G'_{d+\theta+\tau}(O_w) \), we get the following condition on \( \sum a_i z^i \):

\[
(d\theta, \alpha) + di = (d\theta + d\theta_\tau, \alpha) + kp
\]

where \( k \in \mathbb{Z} \). Since \( \theta_\tau \) is integral and \( (d, p) = 1 \), this implies that \( d \mid k \). Let us write \( k = md \). The condition \( (d, \alpha) + i \geq 0 \) translates into \( \left( \frac{d\theta + d\theta_\tau}{p}, \alpha \right) + m \geq 0 \). This proves the claim at the level of Lie algebras. Moreover, the condition \( (d, \alpha) + i = 0 \) translates into \( \left( \frac{d\theta + d\theta_\tau}{p}, \alpha \right) + m = 0 \), this implies that the Levi quotient of the Lie algebra of \( \mathcal{P}_{\theta+\phi+}\) can be identified with the Lie algebra of the centralizer of \( d\theta + d\theta_\tau \) in the connected reductive group \( Z_G(\zeta^d) \). Since the intersection of the Levi quotient of \( \mathcal{P}_0(O_z) \) and \( G'_{d+\theta+\tau}(O_w) \) is the centralizer of the semisimple element \( d\theta + d\theta_\tau \) in the connected reductive group \( Z_G(\zeta^d) \), which is connected, this implies the claim. \( \square \)

Let \( \text{Locsys}_{\mathcal{P}_0,\tau}^{tame}(\mathbb{D}) \) be the category of logahoric \( \mathcal{P}_0(\mathcal{O}) \)-connections on \( \mathbb{D} \) such that the rational semisimple part of the residue is \( \tau \) (up to conjugation), and let \( \text{Higgs}^{tame}_{G'_{\theta+\tau,\text{irr}}}(\mathbb{D}') \) be the category of logahoric \( G'_{\theta+\tau}(\mathcal{O}) \)-Higgs bundles on \( \mathbb{D}' \) such that the semisimple part of the residue is irrational (up to conjugation). Now we are ready to prove the local tame parahoric nonabelian Hodge correspondence.

**Theorem 4.15.** The category \( \text{Locsys}_{\mathcal{P}_0,\tau}^{tame}(\mathbb{D}) \) is equivalent to \( \text{Higgs}^{tame}_{G'_{\theta+\tau,\text{irr}}}(\mathbb{D}') \). Moreover, the p-curvature of the logahoric \( \mathcal{P}_0(\mathcal{O}) \)-connection is zero if and only if the corresponding logahoric \( G'_{\theta+\tau}(\mathcal{O}) \)-Higgs bundle has zero Higgs field.

**Proof.** In the proof, we will use the correspondence between equivariant bundles and parahoric torsors, and we follow the notations \( \mathbb{D}_x = \text{Spec}(O_z) \) and \( \mathbb{D}_y = \text{Spec}(O_w) \) in \( \boxtimes \). Let \( d \) be a positive integer such that \( z = w^d \), \( \theta_\tau \) is integral and \( (d, p) = 1 \). Denote by \( \Gamma \) the cyclic group of order \( d \). The following diagram gives the idea of the proof.

\[
\begin{array}{ccc}
\text{Locsys}_{\mathcal{P}_0,\tau}^{tame}(\mathbb{D}_x) & \xrightarrow{\text{Proposition 4.10}} & \text{Locsys}_{G'_{d+\theta+\tau}}^{tame}(\mathbb{D}_y/\Gamma) \\
\downarrow & & \downarrow \\
\text{Higgs}^{tame}_{G'_{\theta+\tau,\text{irr}}}(\mathbb{D}_x') & \xleftarrow{\text{Theorem 4.15}} & \text{Higgs}^{tame}_{G'_{d+\theta+\tau,\text{irr}}}(\mathbb{D}_y'/\Gamma)
\end{array}
\]

Take a logarithmic connection in \( \text{Locsys}_{\mathcal{P}_0,\tau}^{tame}(\mathbb{D}_x) \), and assume that the connection is represented by \( \Delta - A \frac{dz}{z} \), where \( A \in \mathcal{P}_0(\mathcal{O}) \). By Lemma 4.17, one may assume that \( A \) is of the form \( (\tau + \sigma + \sum a_i z^i) \frac{dz}{z} \), where \( \tau \) is rational semisimple, \( \sigma \) is irrational semisimple and \([\tau, a_i] = ia_i, [\sigma, a_i] = 0\) for all \( i \).

By Proposition 4.10, the category \( \text{Locsys}_{\mathcal{P}_0,\tau}^{tame}(\mathbb{D}_x) \) is equivalent to \( \text{Locsys}_{G'_{d+\theta+\tau}}^{tame}(\mathbb{D}_y'/\Gamma) \). We just want to remind the reader that the residue \( \tau \) changes to \( d(\theta + \tau) \) via the transformation \( \Delta(W) = w^{d\theta} \) and the substitution \( z = w^d \), which is easily observed from the proof of Proposition 3.9.

Next, using the identification in Lemma 3.2 and Lemma 4.8, we see that \( G'_{d+\theta+\tau}(O_w) \equiv \mathcal{P}_{d(\theta + \phi + \tau)}^{\phi}(O'_w) \).
over $O'_w$. By Proposition 4.10 the category $\text{Locsys}^{tame}_{G,d(\theta+\tau)}([D'_y/\Gamma])$ is equivalent to the category $\text{Higgs}^{tame}_{G,d(\theta+\tau),\text{irr}}([D'_y/\Gamma])$.

For the last step, we have

$$G_{d(\theta+\tau)}(O_w) \cong P_{\theta+\tau}(O'_w) \cong G_{\theta+\tau}(O_z)$$

by Proposition 4.14. Therefore, we apply Theorem 3.7 and obtain the equivalence of categories between $\text{Higgs}^{tame}_{G_{d(\theta+\tau),\text{irr}}}(O'_w)$ and $\text{Higgs}^{tame}_{G_{\theta+\tau},\text{irr}}([D'_y/\Gamma])$. This finishes the proof of this theorem. □

5. TAME PARAHORIC NONABELIAN HODGE CORRESPONDENCE

In this section, we establish the global version of tame parahoric nonabelian Hodge correspondence on curves in positive characteristic. With the help of the local study in §4, we mostly follow Chen–Zhu’s approach to give the correspondence (Theorem 5.19 and 5.23). Furthermore, we apply the local results in §4 and give a more precise description of the stack for Higgs bundles in the correspondence (Propositions 5.24 and 5.27).

5.1. Artin-Schreier Map. Let $k[\mathfrak{g}]$ and $k[\mathfrak{t}]$ be the algebras of regular functions on $\mathfrak{g}$ and $\mathfrak{t}$. Chevalley restriction theorem shows that we have an isomorphism $k[\mathfrak{g}]^G \cong k[\mathfrak{t}]^W$, where $W$ is the Weyl group. Let $\mathfrak{c} = \text{Spec}(k[\mathfrak{t}]^W)$, and denote by $p : \mathfrak{t} \to \mathfrak{c}$ the projection. Let $\mathfrak{t}^{irr}$ be the set of irrational elements in $\mathfrak{t}$. A semisimple conjugacy class in $\mathfrak{g}$ is irrational if it is the conjugacy class of an element $x \in \mathfrak{t}^{irr}$. Let $\mathfrak{c}^{irr}$ be the set of irrational semisimple conjugacy class. It is easy to see that $\mathfrak{c}^{irr}$ is an open subset of $\mathfrak{c}$.

Lemma 5.1. Let $\mathfrak{t}'$ (resp. $\mathfrak{c}'$) be the Frobenius twist of $\mathfrak{t}$ (resp. $\mathfrak{c}$). Then the following diagram is Cartesian:

$$\begin{array}{ccc}
\mathfrak{t}^{irr} & \overset{\text{AS}}{\longrightarrow} & \mathfrak{t}' \\
p \downarrow & & \downarrow p' \\
\mathfrak{c}^{irr} & \longrightarrow & \mathfrak{c}'
\end{array}$$

where AS stands for the Artin-Schreier map. Moreover, the morphism $\mathfrak{c}^{irr} \to \mathfrak{c}'$ is surjective.

Proof. First, we have a $W$-equivariant commutative diagram:

$$\begin{array}{ccc}
\mathfrak{t} & \overset{\text{AS}}{\longrightarrow} & \mathfrak{t}' \\
p \downarrow & & \downarrow p' \\
\mathfrak{c} & \longrightarrow & \mathfrak{c}'
\end{array}$$

It remains to show that when restricted to $\mathfrak{c}^{irr}$, the fibers of $p$ and $p'$ can be canonically identified. Indeed, if $\alpha(x) = \lambda$, then $\alpha(\mathfrak{AS}(x)) = \lambda^p - \lambda$. If $x \in \mathfrak{t}^{irr}$, then $\lambda^p - \lambda = 0$ if and only if $\lambda = 0$. Hence we have $w(x) = x$ if and only if $w(\mathfrak{AS}(x)) = \mathfrak{AS}(x)$ for all $w \in W$. This proves the claim. □

By Chevalley restriction theorem, we have a natural map $\chi : \mathfrak{g} \to \mathfrak{c}$ induced by $k[\mathfrak{c}] \to k[\mathfrak{g}]$. Denote by $\text{kos} : \mathfrak{c} \to \mathfrak{g}$ the Kostant section. We define the group scheme $I$ over $\mathfrak{g}$ as

$$I = \{(g, x) \in G \times \mathfrak{g} \mid \text{Ad}_g(x) = x\}.$$

Then, define $J = \text{kos}^*I$. There is a canonical isomorphism $\chi^*J|_{\mathfrak{g}^{reg}} \cong I|_{\mathfrak{g}^{reg}}$, where $\mathfrak{g}^{reg}$ is the open subset of regular elements, and the morphisms $I \to \mathfrak{g}$ and $J \to \mathfrak{c}$ are $G_m$-equivariant. Furthermore, we have a tautological section $\tau : \mathfrak{c} \to J$, which is also $G_m$-equivariant (see [7, §2.3]).

Corollary 5.2. Let $J'$ be the regular centralize group scheme over $\mathfrak{c}'$ under the Frobenius twist. The pullback of $\text{Lie}(J')$ is canonically isomorphic to $\text{Lie}(J)$ over the open set $\mathfrak{c}^{irr}$, i.e.

$$\text{Fr}^*(\text{Lie}(J'))|_{\mathfrak{c}^{irr}} \cong \text{Lie}(J)|_{\mathfrak{c}^{irr}}.$$

Proof. This follows from Lemma 5.1 and [10, Proposition 12.5]. □
5.2. \textit{p-Hitchin Map for Tame Local Systems.} Recall that we use the notation \( \mathcal{L} := \Omega_X(D) \) for the logarithmic cotangent sheaf. Regarding the line bundle \( \mathcal{L} \) as a \( \mathbb{G}_m \)-torsor, we can twist every Lie algebras and group schemes by \( \mathcal{L} \). For example, denote by \( \mathfrak{g}_\mathcal{L} := \mathfrak{g} \times_{\mathbb{G}_m} \mathcal{L}^\times \). With respect to the data above, the stack of logarithmic \( G \)-Higgs bundles (\( \mathcal{L} \)-twisted Higgs bundles) over \( X \) can be regarded as the stack of sections

\[ \text{Higgs}_{G,\mathcal{L}}^{\text{tame}} = \text{Sect}(X, [\mathfrak{g}_\mathcal{L}/G]), \]

where we add the subscript \( \mathcal{L} \) to emphasize that it is \( \mathcal{L} \)-twisted. Furthermore, the \textit{Hitchin base} \( B_{\mathcal{L}} := \text{Sect}(X, \mathcal{c}_\mathcal{L}) \) is regarded as the stack of sections, which is also a scheme. The morphism \( \chi : \mathfrak{g} \to \mathfrak{c} \) induces a natural map \( [\chi_\mathcal{L}/G] : [\mathfrak{g}_\mathcal{L}/G] \to \mathfrak{c} \), and then we have

\[ h : \text{Higgs}_{G,\mathcal{L}}^{\text{tame}} = \text{Sect}(X, [\mathfrak{g}_\mathcal{L}/G]) \to \text{Sect}(X, \mathcal{c}_\mathcal{L}) = B_{\mathcal{L}}, \]

which is the \textit{Hitchin map}.

Let \((E, \phi) \in \text{Higgs}_{G,\mathcal{L}}^{\text{tame}} \) be a logarithmic \( G \)-Higgs bundle, and denote by \( h_{E,\phi} : X \to [\mathfrak{g}_\mathcal{L}/G] \) the corresponding section with image \( b : X \to \mathcal{c}_\mathcal{L} \) in the Hitchin base \( B_{\mathcal{L}} \). Taking pullback of the following diagram

\[
\begin{array}{ccc}
\mathfrak{g}_\mathcal{L} & \to & \mathfrak{g} \\
\downarrow & & \downarrow \\
\mathfrak{c}_\mathcal{L} & \to & \mathfrak{c}
\end{array}
\]

we get a smooth group scheme \( J_b := b^* \mathcal{L} \) over \( X \). On the other hand, the morphism \( \chi^* J \to I \) induces the morphism \( [\chi_\mathcal{L}/G]^* \mathcal{L} \to [\mathfrak{g}_\mathcal{L}/G] \) of group schemes over \( [\mathfrak{g}_\mathcal{L}/G] \). Pulling back the morphism to \( X \) via \( h_{E,\phi} \), we get

\[ a_{E,\phi} : J_b \to h_{E,\phi}^*[\mathfrak{g}_\mathcal{L}/G] = \text{Aut}(E, \phi) \subseteq \text{Aut}(E). \]

Thus, we can twist \((E, \phi) \in h^{-1}(b) \) by a \( J_b \)-torsor.

Recall that the stack of tame \( G \)-local systems \( \text{Locsys}_{G}^{\text{tame}} \) parametrizes pairs \((E, \nabla)\), where \( E \) is a \( G \)-torsor and \( \nabla \) is a logarithmic \( G \)-connection. In the logarithmic case, one still has an analogue notion of the \( p \)-curvature as in [7 A.6]. Let \( X' \) be the Frobenius twist of \( X \) with natural morphism \( Fr : X \to X' \). Define \( D' = Fr(D) \) and \( \mathcal{L}' := \Omega_{X'}(D') \). Clearly, \( Fr^* \mathcal{L}' \cong \mathcal{L}^p \), and the \( p \)-curvature \( \Psi(\nabla) \) of a logarithmic \( G \)-connection \( \nabla \) is a horizontal section of \( \text{Ad}(E) \otimes \mathcal{L}^p \). Then, there is a unique morphism

\[ h_p : \text{Locsys}_{G}^{\text{tame}} \to B_{\mathcal{L}'}, \]

which is called the \textit{p-Hitchin map}, such that the following diagram commutes

\[
\begin{array}{ccc}
\text{Locsys}_{G}^{\text{tame}} & \xrightarrow{h_p} & B_{\mathcal{L}'} \\
\downarrow & & \downarrow \\
\text{Higgs}_{G,\mathcal{L}'}^{\text{tame}} & \to & B_{\mathcal{L}'},
\end{array}
\]

where \( \text{Higgs}_{G,\mathcal{L}'}^{\text{tame}} := \text{Sect}(X, [\mathfrak{g}_\mathcal{L}'/G]) \) and \( B_{\mathcal{L}'} := \text{Sect}(X, \mathcal{c}_\mathcal{L}') \) [7 Proposition 3.1, Lemma 3.2]. Now let \((E, \nabla)\) be a tame \( G \)-local system. Denote by \( \Psi := \Psi(\nabla) \) the \( p \)-curvature and set \( b' = h_p(E, \nabla) \in B_{\mathcal{L}'} \) with \( b' \) the image in \( B_{\mathcal{L}'} \). Then, we obtain a natural morphism

\[ a_{E,\Psi} : J_{b'} \to \text{Aut}(E). \]

Moreover, we have \( Fr^*(J_{b'}) = J_{b''} \), and there is a canonical connection \( \nabla \) on \( J_{b''} \) such that \( (J_{b''})^\nabla = J_{b'} \).

Based on the above discussion, We have the following lemma.

\begin{lemma}
The homomorphism \( a_{E,\Psi} \) is horizontal.
\end{lemma}

\begin{proof}
It is enough to show that the restriction of \( a_{E,\Psi} \) to \( X \setminus D \) is horizontal, which is proven in [7 Lemma 3.3]. \hfill \square
\end{proof}
The group schemes $J_{\theta}$ can be realized a family of group schemes $J_{P}$ over $X \times B_{\mathcal{L}'}$, which is equipped with a natural connection along $X$. From Lemma 5.3, we conclude that for any tame $J_{P}$-local system $(P, \nabla_{P})$ and a tame $G$-local system $(E, \nabla)$, one may apply [4, A.5] to get a tame $G$-local system:

\[ ((P, \nabla_{P}), (E, \nabla)) \mapsto P \otimes E := (a_{E, \theta})_{*} P \otimes E. \]

This actually defines an action:

\[
\text{Locsys}_{J_{P}}^{\text{tame}} \times \text{Locsys}_{G}^{\text{tame}} \to \text{Locsys}_{G}^{\text{tame}}.
\]

**Lemma 5.5.** Given a tame $G$-local system $(E, \nabla)$, let $b' = h_{P}(E, \nabla)$. If $(P, \nabla_{P})$ is a tame $J_{P}$-local system such that its $p$-curvature is zero, then $h_{P}(P \otimes E, \nabla_{P \otimes E}) = b'$.

**Proof.** Similar to the proof of Lemma 5.3, one only need to check this over $X \setminus D$, in which case one may apply [7, Lemma 3.4]. \[\square\]

Now we consider an analogue of all the constructions above for the parahoric case. Let $\theta$ be a collection of tame weights and let $\mathcal{P}_{\theta}$ be the parahoric group scheme over $X$ corresponding to $\theta$. Let $\text{Higgs}_{\mathcal{P}_{\theta}, \mathcal{L}}^{\text{tame}}$ be the stack of logahoric $\mathcal{P}_{\theta}$-Higgs bundles on $X$. First we have the following:

**Lemma 5.6.** There exists a parahoric Hitchin fibration $h: \text{Higgs}_{\mathcal{P}_{\theta}, \mathcal{L}}^{\text{tame}} \to \text{Sect}(X, \mathcal{E}_{\mathcal{L}})$.

**Proof.** The parahoric version of the Hitchin morphism has been considered in [25, §4], and we give a slightly different proof in the view point of equivariant Higgs bundles. Let $(E, \phi)$ be a parahoric $\mathcal{P}_{\theta}$-Higgs bundle. Away from the support of $D$, the structure group of $E$ is identified with $G$. Then, we have a morphism $\text{Ad}(E) \otimes \mathcal{L} \to \mathcal{E}_{\mathcal{L}}$ away from the support of $D$. It remains to show that this morphism can be extended to $T$. Then, we can work locally on $X$ and assume that $\mathcal{L}$ is trivial and there exists a cover $Y$ over $X$ of degree $d$ such that $(E, \phi)$ can be identified with a logarithmic $(\Gamma, G)$-Higgs bundle on $Y$. If we identify $\phi$ with an element in $\text{Sect}(Y, \mathcal{G}_{\mathcal{L}}, \mathcal{L}_{Y})$, where $\mathcal{L}_{Y}$ is the pullback of $\mathcal{L}$ to $Y$, then we get an element in $\text{Sect}^{\mathcal{L}}(Y, \mathcal{E}_{\mathcal{L}}, \mathcal{G}_{\mathcal{L}})$, which is the same as $\text{Sect}(X, \mathcal{E}_{\mathcal{L}})$. This finishes the proof of this lemma. \[\square\]

Given a logahoric $\mathcal{P}_{\theta}$-Higgs bundle $(E, \phi)$ on $X$, let $b$ be its image in $\text{Sect}(X, \mathcal{E}_{\mathcal{L}})$. Let $J_{b}$ be the regular centralizer group scheme over $X$.

**Lemma 5.7.** Let $(E, \phi)$ be a logahoric $\mathcal{P}_{\theta}$-Higgs bundle and let $b = h(E, \phi)$. Let $J_{b}$ be the regular centralizer group scheme on $X$. Then we have a group homomorphism $J_{b} \to \text{Aut}(E)$ over $X$.

**Proof.** Away from the support of $D$, $(E, \phi)$ is identified with a $G$-Higgs bundle, and thus we get a morphism $J_{b} \to \text{Aut}(E)$ on $X \setminus D$. We claim that this morphism extends to $X$. The approach is exactly the same as Lemma 5.6 and we only need to look at formal neighborhood of each point in the support of $D$. Let us assume $X = \text{Spec}(k[[z]])$ and $Y = \text{Spec}(k[[w]])$ where $w^{d} = z$. The element $b$ can be viewed as either a point in $\text{Sect}(X, \mathcal{E}_{\mathcal{L}})$ or $\text{Sect}(Y, \mathcal{E}_{\mathcal{L}})$, so we get regular centralizer group schemes $J_{X, b}$ and $J_{Y, b}$ over $X$ and $Y$ respectively. We have $J_{Y, b} \simeq J_{X, b} \times_{X} Y$. The Higgs bundle $(E, \phi)$ can be identified with a $(\Gamma, G)$-equivariant Higgs bundle on $Y$ as discussed in §3.2, so we get a morphism $J_{Y, b} \to \text{Aut}(E)$. Passing to $\Gamma$-invariant sections, we get $J_{X, b} \to \text{Aut}(E)$, where $J_{b} = J_{X, b}$. \[\square\]

Next we look at tame parahoric local systems. Recall that $\text{Locsys}_{\mathcal{P}_{\theta}}^{\text{tame}}$ is the stack of tame $\mathcal{P}_{\theta}$-local systems on $X$.

**Lemma 5.8.** Let $(E, \nabla)$ be a tame $\mathcal{P}_{\theta}$-local system on $X$. The $p$-curvature of $(E, \nabla)$ is actually in $\text{Sect}(X', \mathcal{E}_{\mathcal{L}}')$, and then we obtain a natural morphism

\[ h_{p}: \text{Locsys}_{\mathcal{P}_{\theta}}^{\text{tame}} \to \text{Sect}(X', \mathcal{E}_{\mathcal{L}}'), \]

which is called the $p$-Hitchin morphism. More generally, if $\Psi$ is a horizontal section of $\text{Ad}(E) \otimes F^{*}(\mathcal{L}')$, then the image of $\Psi$ in $\text{Sect}(X, \mathcal{E}_{F^{*}(\mathcal{L})'})$ is also horizontal.
Proof. Away from the support of $D$, $\mathcal{P}_b$ can be identified with $G$, so the arguments in [7 Proposition 3.1 and Lemma 3.2] shows that $h(E, \Psi)$ is a horizontal section of $\text{Ad}(E) \otimes F^*(\mathcal{L}')$ with respect to the canonical connection on $F^*\mathcal{L}'$. With the same proof as in Lemma 5.6, we can extend it to $X$ and obtain a horizontal section in $\text{Sect}(X, \zeta_{\text{Fr}^*(\mathcal{L}'))}$. \hfill $\square$

Let $(E, \nabla)$ be a tame $\mathcal{P}_b$-local system, denote by $b'$ its image in $\text{Sect}(X', \zeta_{\mathcal{E}'}_0)$. Similar to the case of principal bundles, the element $b'$ defines a regular centralizer group scheme $J'_{b'}$ over $X'$, and let $J_{b'}$ be its pullback to $X$. Then one gets a group homomorphism $a_{E, \Psi} : J_{b'} \to \text{Aut}(E)$, where we use the same notation. Note that $J_{b'}$ is equipped with a canonical connection as the pullback of $J'_{b'}$, while $\text{Aut}(E)$ is equipped with a logarithmic connection. Then, we obtain the following lemma as an analogue of Lemma 5.3.

Lemma 5.9. The homomorphism $a_{E, \Psi}$ is horizontal.

Now as we did for tame $G$-local systems, we still have an action of tame $J_{b'}$-local systems on tame parahoric local systems:

\begin{equation}
\text{Locsys}_{m}^{\text{tame}} \times \text{Locsys}_{p}^{\text{tame}} \to \text{Locsys}_{p}^{\text{tame}}.
\end{equation}

Lemma 5.9 still holds in the parahoric case. Moreover, we define two substacks $\mathcal{A}_{0} \subseteq \mathcal{A} \subseteq \text{Locsys}_{\mathcal{P}_{b}}^{\text{tame}}$.

- $\mathcal{A}$ is the stack of tame $J'_{b}$-local systems with zero $p$-curvature.
- $\mathcal{A}_{0} \subseteq \mathcal{A}$ is the substack of $(P, \nabla_{P}) \in \mathcal{A}$, of which $\nabla_{P}$ has no poles.

Clearly, $\mathcal{A}$ and $\mathcal{A}_{0}$ are group stacks over $B_{\mathcal{E}'}$. Lemma 5.3 implies the following:

Corollary 5.11. There exists a natural action of $\mathcal{A}$ on $\text{Locsys}_{G}^{\text{tame}}$ which preserves the $p$-Hitchin map for $\text{Locsys}_{G}^{\text{tame}}$.

5.3. Vector Bundle $\mathcal{B}_{\mathcal{E}'}$. Now we construct a vector bundle $\mathcal{B}_{\mathcal{E}'}$ over $B_{\mathcal{E}'}$, of which the fiber is $H^{0}(X', \text{Lie}(J'_{b'}) \otimes \mathcal{L}')$ for each $b' \in B_{\mathcal{E}'}$. The fiber is actually the space of $p$-curvatures for logarithmic $J'_{b'}$-connections. Then the stack $\text{Locsys}_{\mathcal{P}_{b}}^{\text{tame}}$ is equipped with a natural morphism to $\mathcal{B}_{\mathcal{E}'}$. In this subsection, we will prove that this morphism is smooth.

Let $\mathcal{G}'$ be a smooth affine commutative group scheme over $X'$ and denote by $\mathcal{G}$ its Frobenius pullback to $X$. The following sequence is an analogue of [7, Proposition A.7] in our setting:

Lemma 5.12. On $X'_{\text{et}}$, we have a sequence of sheaves

\begin{equation}
0 \to \mathcal{G}' \to \mathcal{F}_{r} \mathcal{G} \xrightarrow{Fr_{r} \text{log}} \text{Lie}^{\text{tame}}_{\mathcal{G}'} \otimes \mathcal{F}_{r} \mathcal{L}' \xrightarrow{h_{p}} \text{Lie}^{\text{tame}}_{\mathcal{G}'} \otimes \mathcal{L} \to 0.
\end{equation}

It is exact except at the position $\text{Lie}^{\text{tame}}_{\mathcal{G}'} \otimes \mathcal{F}_{r} \mathcal{L}$.

Proof. By Proposition A.7 of [7], we have the following sequence:

\begin{equation}
0 \to \mathcal{G}' \to \mathcal{F}_{r} \mathcal{G} \xrightarrow{Fr_{r} \text{log}} \text{Lie}^{\text{tame}}_{\mathcal{G}'} \otimes \mathcal{F}_{r} \Omega_{X} \xrightarrow{h_{p}} \text{Lie}^{\text{tame}}_{\mathcal{G}'} \otimes \Omega_{X} \to 0,
\end{equation}

which is exact. Recall that $\mathcal{L} = \Omega_{X}(D)$. So it remains to show $h_{p}$ is surjective as a morphism from $\text{Lie}^{\text{tame}}_{\mathcal{G}'} \otimes \mathcal{F}_{r} \Omega_{X}(x)$ to $\text{Lie}^{\text{tame}}_{\mathcal{G}'} \otimes \Omega_{X}(x)$ for each $x \in D$. Using the surjectivity of $h_{p}$ in [6.13], we only need to show the restriction of $h_{p}$ to $x$ is surjective. By the description of $h_{p}$ in [7, Lemma A.8], we have $h_{p}(\eta_{x}) = \eta_{x}^{[p]} - \eta_{x}$, where $\eta_{x} \in \text{Lie}^{\text{tame}}_{\mathcal{G}'}|_{x}$ and $(-)^{[p]}$ is the $p$-power operator. Since $\mathcal{G}$ is a smooth affine commutative group scheme, we may assume $\mathcal{G}'|_{x}$ is either $G_{m}$ or $G_{a}$ and the assertion follows directly. \hfill $\square$

Now we assume that $\mathcal{G}'$ is a regular centralizer group scheme (see [10]).

Lemma 5.14. Let $\eta$ be an element in $\text{Lie}^{\text{tame}}_{\mathcal{G}'}|_{x}$ such that $\eta^{[p]} - \eta = 0$. Then, in the étale topology, there exists $\omega \in \text{Lie}^{\text{tame}}_{\mathcal{G}'} \otimes \mathcal{F}_{r} \Omega_{X}(x)$ such that $h_{p}(\omega) = 0$ and the residue of $\omega$ at $x$ is equal to $\eta$. 
Proof. One may assume $X'$ is affine and that $\mathcal{G}'$ is the regular centralizer determined by an element $\phi \in \mathfrak{g}(X')$. Let $L$ be the Levi subgroup of $G := \mathcal{G}'|_{x'}$ determined by the semisimple part of the fiber $\phi_{x'}$. Then $\eta$ can be identified with an integral element in $Z(L)$. By Lemma B.0.3, one may pass to an étale cover of $X'$ so that $\mathcal{G}'$ is the regular centralizer for the Levi subgroup $L$. In this case, $Z(L)$ can be identified with a subgroup of $\mathcal{G}'$, hence locally there exists an element $f \in \mathcal{G}'(X' \setminus x')$ such that the residue of $\omega = Fr_\ast \text{dlog}(f)$ is $\eta$ and $h_p(\omega) = 0$.

**Corollary 5.15.** Let $q \in \operatorname{Lie}(\mathcal{G}') \otimes \Omega_{X'}(x')$. We take an element $\eta \in \operatorname{Lie}(\mathcal{G}')|_{x'}$ such that $h_p(\eta)$ is equal to the residue of $q$ at $x'$. Then the category of tame $\mathcal{G}$-local systems on $X$ such that the $p$-curvature equal to $q$ and the residue equal to $\eta$ is a gerbe over $\mathcal{B}_{\mathcal{G}'}$.

**Proof.** Combine Lemma 5.12 and Lemma 5.14, one concludes that locally over $X'$, there exists $\omega \in \operatorname{Lie}(\mathcal{G}') \otimes Fr_\ast \Omega_X(x)$ such that $h_p(\omega) = q$ and that the residue of $\omega$ is equal to $\eta$. Moreover, two different choices of such $\omega$ satisfies the following conditions: $\omega_1 - \omega_2 \in \operatorname{Lie}(\mathcal{G}') \otimes Fr_\ast \Omega_X$ and $h_p(\omega_1 - \omega_2) = 0$. Now the claim follows from the exactness of 5.13. $\square$

Now we are ready to prove the following:

**Lemma 5.16.** Locsys$^{\text{tame}}_{J_P}$ is smooth over $\mathcal{B}_{\mathcal{L}'}$.

**Proof.** Fixing a point $b' \in B_{\mathcal{L}'}$, the fibers of Locsys$^{\text{tame}}_{J_P}$ and $\mathcal{B}_{\mathcal{L}'}$ over $b$ are smooth. We will show that

$$\text{Locsys}_{J_P}^{\text{tame}} \rightarrow H^0(X', \operatorname{Lie}(J_{b'}) \otimes \mathcal{L}')$$

is smooth. It suffices to prove that the induced morphism on tangent space is surjective. Let $K^\bullet$ be the complex given by:

$$K^\bullet : \operatorname{Lie}(J_{b'}) \rightarrow \operatorname{Lie}(J_{b'}) \otimes \mathcal{L}' .$$

The tangent space of Locsys$^{\text{tame}}_{J_P}$ is given by $H^1(X', K^\bullet)$. Let $H^i(K^\bullet)$ be the cohomology group of $K^\bullet$ at degree $i$. Then by Lemma 5.12, the cohomology $H^1(K^\bullet)$ admits a surjective morphism to $\operatorname{Lie}(J_{b'}) \otimes \mathcal{L}'$ such that the kernel is supported at $x'$. Thus, one may view $H^1(K^\bullet)$ as a coherent sheaf on $X'$ and that $\operatorname{Lie}(J_{b'}) \otimes \mathcal{L}'$ is a direct summand of $H^1(K^\bullet)$. Then the morphism

$$H^0(X', H^1(K^\bullet)) \rightarrow H^0(X', \operatorname{Lie}(J_{b'}) \otimes \mathcal{L}')$$

is surjective. Since $H^2(X', H^0(K^\bullet)) = 0$, we have $H^1(X', K^\bullet) \rightarrow H^0(X', H^1(K^\bullet))$ is surjective. Therefore, we have

$$H^1(X', K^\bullet) \rightarrow H^0(X', \operatorname{Lie}(J_{b'}) \otimes \mathcal{L}')$$

is surjective. This finishes the proof. $\square$

5.4. **Tame Nonabelian Hodge Correspondence.** By Lemma 5.10 we have a smooth morphism Locsys$^{\text{tame}}_{J_P} \rightarrow \mathcal{B}_{\mathcal{L}'}$. The tautological section $\eta \rightarrow \operatorname{Lie}(J)$ induces a canonical section $\tau' : B_{\mathcal{L}'} \rightarrow \mathcal{B}_{\mathcal{L}'}$. Then we define $\mathcal{H}$ as the pullback of the following diagram

$$\begin{array}{ccc}
\mathcal{H} & \rightarrow & \text{Locsys}_{J_P}^{\text{tame}} \\
\downarrow & & \downarrow \\
B_{\mathcal{L}'} & \rightarrow & \mathcal{B}_{\mathcal{L}'} ,
\end{array}$$

For each point $b' \in B_{\mathcal{L}'}$, the fiber of $\mathcal{B}_{\mathcal{L}'} \rightarrow B_{\mathcal{L}'}$ is $H^0(X', \operatorname{Lie}(J_{b'}) \otimes \mathcal{L}')$. One may consider the residue morphism:

$$H^0(X', \operatorname{Lie}(J_{b'}) \otimes \mathcal{L}') \xrightarrow{\text{Res}} \operatorname{Lie}(J_{b'}) \otimes \mathcal{L}'|_{x'} .$$

By Lemma 5.10 we get a morphism by taking over all $b' \in B_{\mathcal{L}'}$

Locsys$^{\text{tame}}_{J_P} \rightarrow \operatorname{Lie}(J) \otimes \mathcal{L}'|_{x'}$. 

Similarly, by taking the residue of logarithmic connections, we get a morphism

$$\text{Locsys}_{J^p}^{\text{tame}} \to \text{Lie}(J') \otimes \mathcal{L}|_x.$$  

Then under the Artin-Schreier map, we have the following commutative diagram:

$$\begin{array}{ccc}
\text{Locsys}_{J^p}^{\text{tame}} & \longrightarrow & \text{Lie}(J') \otimes \mathcal{L}|_x \\
\downarrow \text{AS} & & \downarrow \text{AS} \\
\text{Lie}(J') \otimes \mathcal{L}'|_{x'} & \longrightarrow & \text{Lie}(J') \otimes \mathcal{L}'|_{x'}
\end{array}$$

where AS is induced from the map

$$\text{Lie}(J') \to \text{Lie}(J'), \quad g \to g^{[p]} - g,$$

where $(-)^{[p]}$ is the $p$-power operator. With respect to the above discussion, we get the following picture:

$$\begin{array}{ccc}
\mathcal{H} & \longrightarrow & \text{Locsys}_{J^p}^{\text{tame}} \\
\downarrow \tau' & & \downarrow \text{AS} \\
B_{\mathcal{L}'} & \longrightarrow & \text{Lie}(J') \otimes \mathcal{L}'|_{x'}
\end{array}$$

Here we view $\text{Lie}(J') \otimes \mathcal{L}|_x$ and $\text{Lie}(J') \otimes \mathcal{L}'|_{x'}$ as vector bundles over $c\mathcal{L}|_x$. The diagram (5.17) implies that we have the induced morphism

$$\mathcal{H} \to B_{\mathcal{L}'} \times_{\text{Lie}(J') \otimes \mathcal{L}'|_{x'}} \text{Lie}(J') \otimes \mathcal{L}|_x.$$ 

To simplify the notations, let us denote

$$B_{\mathcal{L}'}^{\text{ext}} := B_{\mathcal{L}'} \times_{\text{Lie}(J') \otimes \mathcal{L}'|_{x'}} \text{Lie}(J') \otimes \mathcal{L}|_x.$$ 

Since AS in (5.17) is étale and surjective, the morphism $B_{\mathcal{L}'}^{\text{ext}} \to B_{\mathcal{L}'}$ is also étale and surjective.

**Lemma 5.18.** The stack $\mathcal{H}$ is an $\mathcal{A}$-torsor over $B_{\mathcal{L}'}$, and it is also an $(A_0 \times_{B_{\mathcal{L}'}} B_{\mathcal{L}'}^{\text{ext}})$-torsor over $B_{\mathcal{L}'}^{\text{ext}}$.

**Proof.** First, the fibers of $\mathcal{H} \to B_{\mathcal{L}'}^{\text{ext}}$ (resp. $\mathcal{H} \to B_{\mathcal{L}'}$) are torsors over fibers of $A_0 \times_{B_{\mathcal{L}'}} B_{\mathcal{L}'}^{\text{ext}}$ (resp. $A \to B_{\mathcal{L}'}$). Then, since $B_{\mathcal{L}'}^{\text{ext}}$ is étale over $B_{\mathcal{L}'}$, $\mathcal{H}$ is smooth over both $B_{\mathcal{L}'}^{\text{ext}}$ and $B_{\mathcal{L}'}$ by Lemma 5.16. Also, $B_{\mathcal{L}'}^{\text{ext}}$ maps surjectively to $B_{\mathcal{L}'}$. Thus, it is enough to show $\mathcal{H} \to B_{\mathcal{L}'}^{\text{ext}}$ is surjective, which is a consequence of Corollary 5.15 as well as [7, Lemma 3.15 and Lemma 3.16].

Let $\mathcal{X}$ be the stack that parameterizes triples $(E, \nabla, \Psi)$ such that

$\bullet$ $(E, \nabla)$ is a tame $G$-local systems with zero $p$-curvature,

$\bullet$ $\Psi$ is a horizontal section of $\text{Ad}(E) \otimes Fr^* \mathcal{L}'$

Clearly, $\mathcal{X}$ is an algebraic stack over $B_{\mathcal{L}'}$.

**Theorem 5.19.** There exists a canonical isomorphism of stacks over $B_{\mathcal{L}'}$:

$$\mathcal{H} \times^\mathcal{A} \mathcal{X} \to \text{Locsys}_{G}^{\text{tame}}.$$ 

Similarly, there exists a canonical isomorphism of stacks over $B_{\mathcal{L}'}^{\text{ext}}$:

$$\mathcal{H} \times^\mathcal{A_0}^{\text{ext}} \mathcal{X}^{\text{ext}} \to \text{Locsys}_{G}^{\text{tame}} \times_{B_{\mathcal{L}'}} B_{\mathcal{L}'}^{\text{ext}},$$

where $A_0^{\text{ext}}$ and $\mathcal{X}^{\text{ext}}$ stands for the base change of $A_0$ and $\mathcal{X}$ to $B_{\mathcal{L}'}^{\text{ext}}.$
Proof. The morphism $\mathcal{H} \times^A \mathcal{X} \to \text{Locsys}_{\text{tame}}^G$ is induced by the action of $\text{Locsys}_{\text{tame}}^G$ on $\text{Locsys}_{\text{tame}}^G$ as in (5.10). The inverse morphism is given by:

$$\mathcal{H}(-\tau') \times^A \text{Locsys}_{\text{tame}}^G \to \mathcal{X},$$

where $\mathcal{H}(-\tau')$ is the pullback of $\text{Locsys}_{\text{tame}}^G$ to $B_{\mathcal{X}'}$ via the section $B_{\mathcal{X}'} \to \mathcal{Y}'$. Given that we have established Lemma 5.18 and Lemma 5.5 the proof that these two morphisms are inverse of each other is almost the same as the proof of [7, Proposition 3.9 and Theorem 3.12] based on the local study in 5.4. The situation for $\mathcal{X}_{\text{ext}}$ is similar.

5.5. Structure of $\mathcal{X}$. We will give a description of $\mathcal{X}$ in terms of $X'$. It turns out that the structure of $\mathcal{X}$ depends on the residue of the logarithmic connection.

Lemma 5.20. Let $(E, \nabla, \Psi)$ be an element in $\mathcal{X}$. For each point $x \in D$, the residue of $\nabla$ at $x$ is a rational semisimple element.

Proof. Since $\nabla$ has zero $p$-curvature, Lemma 4.11 implies the result.

Let $\tau \in t$ be a rational semisimple element, which can be considered as a representative of the corresponding conjugacy class. Any such element $\tau$ defines a natural weight $\theta_\tau$ parahoric subgroup $G'_\tau$ of $G$ over $X'$ as we discussed in 4.1. Let $(\theta_\tau)_*O_X(x)$ be the corresponding $T$-torsor. Since $O_X(x)$ is equipped with a natural connection with a pole at $x$, the $T$-torsor $(\theta_\tau)_*O_X(x)$ is also equipped with a logarithmic connection $\nabla_\tau$ such that the residue is $\tau$. Let $G'_\tau$ be the group of automorphisms of the tame $G$-local system $((\theta_\tau)_*O_X(x), \nabla_\tau)$. Lemma 4.11 implies that $G'_\tau$ can be identified with a parahoric group scheme over $X'$ such that the Levi factor is equal to the centralizer of $\tau$ in $G$.

Now we take a collection of rational semisimple elements $\tau := \{\tau_x, x \in D\}$. Let $\text{Locsys}_{\text{tame}}^G \subseteq \text{Locsys}_{\text{tame}}^{G, \tau}$ be the stack classifying tame $G$-local systems $(E, \nabla)$ on $X$ such that

- $\nabla$ is of zero $p$-curvature;
- the residue of $\nabla$ at $x$ is conjugate to $\tau_x$ for each point $x \in D$.

Denote by $G'_\tau$ the corresponding parahoric group scheme over $X'$.

Lemma 5.21. The stack $\text{Locsys}_{\text{tame}}^{G, \tau}$ is equivalent to the stack $\text{Bun}_{G'_\tau}$ of $G'_\tau$-torsors over $X'$.

Proof. It is enough to prove this lemma for a single point $D = \{x\}$, and let $\tau$ be the given rational semisimple element at $x$. As we stated above, the $G$-torsor $(\theta_\tau)_*O_X(x)$ has a natural logarithmic connection $\nabla_\tau$ such that the residue is equal to $\tau$ (under conjugation) and the $p$-curvature is zero. We will prove that for any $(E, \nabla) \in \text{Locsys}_{\text{tame}}^{G, \tau}$, we have that $(E, \nabla)$ is locally isomorphic to $((\theta_\tau)_*O_X(x), \nabla_\tau)$ as tame local systems on $X$, which will imply this lemma.

Let $\text{Iso}(((\theta_\tau)_*O_X(x), \nabla_\tau), (E, \nabla))$ be the group of isomorphisms between $((\theta_\tau)_*O_X(x), \nabla_\tau)$ and $(E, \nabla)$. The isomorphism group $\text{Iso}$ is an affine scheme over $X'$. In fact, $\text{Iso}$ is smooth and maps surjectively over $X'$. Away from $x$, this follows from Cartier descent. Lemma 4.11 implies that $x'$ is in the image of the morphism $\text{Iso} \to X'$. The smoothness of $\text{Iso}$ over $x'$ is a consequence of Lemma 4.3 Lemma 4.11 and the following Lemma 5.22.

Lemma 5.22. Let $U$ be a finite type affine scheme over a curve $X$. Given $x \in X$, let $z$ be a local coordinate at $x$. Then $U$ is smooth over $x$ if the functor

$$(5.23) \quad H(R) = \text{Hom}_X(\text{Spec}(R[[z]]), Y)$$

on Artinian local algebras over $k$ is formally smooth.

Proof. Let $R, R'$ be two Artinian local algebras over $k$ such that we have a closed embedding $\text{Spec}(R) \to \text{Spec}(R')$. Given a diagram

$$
\begin{array}{ccc}
\text{Spec}(R) & \to & \text{Spec}(R') \\
& f & \downarrow \\
& & \text{Spec}(X)
\end{array}
$$
we need to show that $f$ lifts to $f'$. Composing $f$ with $\text{Spec}(R[[z]]) \to \text{Spec}(R)$, one can view $f$ as an element in $H(R)$. If $H$ is formally smooth, then $f$ lifts to $f' \in H(R')$. Now the morphism $\text{Spec}(R') \to X$ endows $\text{Spec}(R')$ with the structure as a subscheme of $\text{Spec}(R'[z])$. Therefore, one finds a lift of $f$ to $f'$.

Let $\mathcal{X}_\tau$ be the substack of $\mathcal{X}$, which parametrizes triples $(E, \nabla, \Psi)$ such that the residue of $\nabla$ at $x$ is conjugate to $\tau_x$.

**Proposition 5.24.** We have

$$\mathcal{X}_\tau \cong \text{Higgs}_{G_\tau}^{tame},$$

and the stack $\mathcal{X}_\tau$ is an open substack of $\mathcal{X}$. Hence $\mathcal{X}$ is the disjoint union of $\mathcal{X}_\tau$, where the union ranges over all rational semisimple conjugacy classes.

**Proof.** Let $(E, \nabla, \Psi) \in \mathcal{X}_\tau$ be an element. By Lemma 5.21 the pair $(E, \nabla)$ determines a unique $G'_\tau$-torsor $E'$. Thus, we only need to show that if $\Psi \in H^0(X, \text{Ad}(E) \otimes Fr^*\mathcal{L})$ is a horizontal section, then $\Psi \in \text{Ad}(E') \otimes \mathcal{L}''$. Using Cartier descent again, one only needs to prove this statement over the formal disc at $x$, so one may assume that the connection is of the form $d + \tau_x \frac{dz}{z}$. An element $A \frac{dz}{z} \in g(O_x) \frac{dz}{z}$ is horizontal if and only if $dA + [A, \tau_x] \frac{dz}{z} = 0$. By the definition of $G'_\tau$, this implies that $A \in g'_\tau(O)$. This finishes the proof for the isomorphism. The fact that $\mathcal{X}_\tau$ is an open substack follows from the fact that the condition that the residue is fixed is stable under deformations by Lemma 4.11.

**5.6. Tame Parahoric Nonabelian Hodge Correspondence.** Now we consider the more general case of tame parahoric local systems. Let $\mathcal{P}_\theta$ be the parahoric group scheme on $X$ corresponding to a given collection of tame weights $\theta$. The first is the analogue of Theorem 5.19. Let $\mathcal{X}_{\mathcal{P}_\theta}$ be the stack parameterizes triples $(E, \nabla, \Psi)$ such that

- $(E, \nabla)$ is a tame $\mathcal{P}_\theta$-local system with zero $p$-curvature,
- $\Psi$ is a horizontal section in $\text{Ad}(E) \otimes Fr^*\mathcal{L}$.

**Theorem 5.25.** There exists a canonical isomorphism of stacks over $B_{\mathcal{L}}$: $$\mathcal{H}^A \times^A \mathcal{X}_{\mathcal{P}_\theta} \to \text{Locsys}_{\mathcal{P}_\theta}^{tame}.$$ 

**Proof.** The action

$$\text{Locsys}_{\mathcal{P}_\theta}^{tame} \times \text{Locsys}_{\mathcal{P}_\theta}^{tame} \to \text{Locsys}_{\mathcal{P}_\theta}^{tame}.$$ 

given in (5.10) induces a morphism

$$\mathcal{H}^A \times^A \mathcal{X}_{\mathcal{P}_\theta} \to \text{Locsys}_{\mathcal{P}_\theta}^{tame}.$$ 

By the canonical section $\tau': B_{\mathcal{L}} \to R_{\mathcal{L}}$ defined in (5.4) the “inverse” morphism is also given by

$$\mathcal{H}(-\tau') \times^A \text{Locsys}_{\mathcal{P}_\theta}^{tame} \to \mathcal{X}.$$ 

Then, based on the local study in (5.10) the same argument as in [7] Proposition 3.9 and Theorem 3.12 finishes the proof of this theorem.

**Lemma 5.26.** $\text{Locsys}_{\mathcal{P}_\theta, \tau}(X)$ is isomorphic to $\text{Bun}_{G'_d(\theta + \tau)}(X')$.

**Proof.** We go with the following diagram to prove this lemma.
\[
\text{Locsys}_{\text{P}, \tau}^{\text{tame}}(X) \xrightarrow{\text{Proposition 5.19}} \text{Locsys}_{G,d(\theta+\tau)}^{\text{tame}}([Y/\Gamma])
\]

\[
\downarrow \quad \text{Lemma 5.21}
\]

\[
\text{Bun}_{G_{\theta+\tau}}(X') \xrightarrow{\text{Theorem 5.26}} \text{Bun}_{G_{\tau'(\theta+\tau)}}([Y'/G])
\]

In the proof of Theorem 4.15 we show that

\[
\text{Locsys}_{\text{P}, \tau}^{\text{tame}}(D_x) \cong \text{Locsys}_{G,d(\theta+\tau)}^{\text{tame}}([D_y/\Gamma]).
\]

By Proposition 5.21 this local picture can be naturally generalized as follows:

\[
\text{Locsys}_{\text{P}, \tau}^{\text{tame}}(X) \cong \text{Locsys}_{G,d(\theta+\tau)}^{\text{tame}}([Y/\Gamma]).
\]

Lemma 5.21 gives the isomorphism \(\text{Locsys}_{G,d(\theta+\tau)}^{\text{tame}}([Y/\Gamma]) \cong \text{Bun}_{G_{\tau'(\theta+\tau)}}([Y'/G]).\) Note that although the lemma is proved for \(G\)-bundles, the result can be naturally generalized to a \(\Gamma\)-equivariant version. Finally, by Theorem 5.26 we have the desired isomorphism.

Let \(\mathcal{X}_{\text{P}, \tau}\) be the substack of \(\mathcal{X}_{\text{P}}\) such that \((E, \nabla, \Psi) \in \mathcal{X}_{\text{P}, \tau}\) if the residue of the logarithmic connection \(\nabla\) at \(x \in D\) is conjugate to \(\tau_x\). Sometimes we use the notation \(\mathcal{X}_{\text{P}, \tau}(X)\) to emphasize that it is defined over \(X\). Following the diagram below,

\[
\mathcal{X}_{\text{P}, \tau}(X) \xrightarrow{\text{Proposition 5.19}} \text{Higgs}_{G_{\theta+\tau}}^{\text{tame}}(X') \xrightarrow{\text{Theorem 5.26}} \text{Higgs}_{G_{\tau'(\theta+\tau)}}^{\text{tame}}([Y'/G])
\]

we apply similar arguments as in Proposition 5.24 and give a description of the structure of \(\mathcal{X}_{\text{P}, \tau}\).

**Proposition 5.27.** We have

\[
\mathcal{X}_{\text{P}, \tau} \cong \text{Higgs}_{G_{\theta+\tau}}^{\text{tame}}(X'),
\]

and the stack \(\mathcal{X}_{\text{P}}\) is the disjoint union of all \(\mathcal{X}_{\text{P}, \tau}\), where \(\tau\) ranges over a set of representatives of rational semisimple conjugacy class of \(g\) under the adjoint action of \(Z_G(\zeta^{d\theta})\).

**References**

[1] D. Abramovich, M. Olsson, A. Vistoli, *Tame stacks in positive characteristic*. Ann. Inst. Fourier (Grenoble) 58, no. 4, 1057-1091 (2008).

[2] V. Balaji, I. Biswas, Y. Pandey, *Connections on parahoric torsors over curves*. Publ. Res. Inst. Math. Sci. 53(4), 551-585 (2017).

[3] V. Balaji, C. S. Seshadri, *Moduli of parahoric G-torsors on a compact Riemann surface*. Journal of Algebraic Geometry 24, 1-49 (2015).

[4] O. Biquard, O. García-Prada, I. Mundet i Riera *Parabolic Higgs bundles and representations of the fundamental group of a punctured surface into a real group*. Adv. Math. 372, 107305, 70pp. (2020).

[5] P. Boalch, *Riemann-Hilbert for tame complex parahoric connections*. Transformation Groups 16, 27-50 (2011).

[6] F. Bruhat, J. Tits, *Groupes réductifs sur un corps local. II. Schémas en groupes*. Existance d’une donnée radicielle valuée. Inst. Hautes Études Sci. Publ. Math. 60, 197-376 (1984).

[7] T. H. Chen, X. Zhu, *Non-abelian Hodge theory for algebraic curves in characteristic p*. Geom. Funct. Anal. 25, 1706-1733 (2015).

[8] T. H. Chen, X. Zhu, *Geometric Langlands in prime characteristic*. Compos. Math. 153(2), 395 - 452 (2017).

[9] V. Chernousov, P. Gille, A. Pianzola, *Torsors over the punctured affine line*. Amer. J. Math. 134(6), 1541-1583 (2012).

[10] R. Donagi, D. Gaitsgory, *The gerbe of Higgs bundles*. Transform Groups, 7, 109-153 (2002).

[11] A. F. Herrero, *Reduction theory for connections over the formal punctured disc*. arXiv:2003.00008 (2020).

[12] P. Huang, G. Kydonakis, H. Sun, L. Zhao, *Tame Parahoric Nonabelian Hodge Correspondence on Curves*. arXiv: 2205.15475 (2022).

[13] P. Huang, H. Sun, *Meromorphic parahoric Higgs torsors and filtered Stokes G-local systems on curves*. Adv. Math. 429, Paper No. 109183, 38 pp (2023).
[14] G. Kydonakis, H. Sun, L. Zhao, Logahoric Higgs torsors for a complex reductive group. Math. Ann. 388, no.3, 3183–3228 (2024).
[15] M. Li, The Poincare line bundle and autoduality of Hitchin fibers. arXiv:2005.01396 (2020).
[16] A. Ogus, F-crystals, Griffiths transversality, and the Hodge decomposition. Astérisque, no.221, ii+183 pp (1994).
[17] A. Ogus, V. Vologodsky, Nonabelian Hodge theory in characteristic p Publ. Math. Inst. Hautes Etud. Sci. 106, 1–138 (2007).
[18] G. Pappas, M. Rapoport, Twisted loop groups and their affine flag varieties. Adv. Math. 219(1), 118-198 (2008).
[19] S. Shen, Tamely ramified geometric Langlands correspondence in positive characteristic. arXiv:1810.12491.
[20] C. T. Simpson, Harmonic bundles on noncompact curves. J. Amer. Math. Soc. 3(3), 713-770 (1990).
[21] C. T. Simpson Higgs bundles and local systems. Inst. Hautes Etudes Sci. Pub. Math. 75, 5–95 (1992).
[22] H. Sun, Moduli Problem of Hitchin Pairs over Deligne-Mumford Stacks. Proc. Amer. Math. Soc. 150, no.1, 131–143 (2022).
[23] H. Sun, Moduli Space of A-modules on Projective Deligne-Mumford Stacks. arXiv:2003.11674 (2020).
[24] C. Teleman, C. Woodward, Parabolic bundles, products of conjugacy classes, and quantum cohomology. Annales de L’Institut Fourier 3, 713-748 (2003).
[25] Z. Yun, Global Springer theory. Adv. Math. 228, 266-328 (2011).

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