Thermodynamics and weak cosmic censorship conjecture in the Kerr-AdS black hole

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Abstract:
We investigate the laws of thermodynamics and weak cosmic censorship conjecture in the normal and extended phase space of a Kerr-AdS black hole by analyzing the energy-momentum relation of the absorbed fermion dropping into the horizon. In the normal phase space, the first law, second law as well as the weak cosmic censorship conjecture are found to be valid in all the initial states of the black hole. However, in the extended phase space, although the first law and weak cosmic censorship conjecture are still valid, the second law becomes more subtle. We find that the validity or violation of the second law depends on the spin parameter, the radius of the AdS spacetime, and their variations. In addition, in the extended phase space, the configurations of the extremal and near-extremal Kerr-AdS black holes are unchanged as the fermions are absorbed since the final and initial states are the same.
1 Introduction

Investigation on the properties of black holes is always an intriguing topic in general relativity. In particular, the discovery of gravitational waves [1] reignited this research in recent years. Now we already know that a black hole is not only a celestial body, but also a thermodynamic system due to the pioneering work of Hawking [2] who proved that black holes can emit radiation. Just like the ordinary thermodynamic systems, there are also four laws for the black hole thermodynamic systems [3], which are valid only at the horizon for the thermodynamic quantities, such as temperature and entropy. The importance of thermodynamics of black holes is that it provides fundamental relations between theories such as gravity, thermodynamics and quantum field theory.

Thermodynamics in the AdS space is more prevalent for the existence of a cosmological parameter. Initially, the cosmological parameter in the thermodynamic system is treated as a constant. However, more and more evidences have shown that this parameter may be considered as a dynamical variable [4], moreover Ref. [5] put forward to regarding it as a state function. Especially, Refs. [6–8] proposed that the cosmological constant can be regarded as the pressure of the thermodynamic system, while its thermodynamic conjugate variable can be defined naturally as volume. The thermodynamic phase space...
thus is extended. In the extended phase space, the Smarr relation derived by the scaling method is also valid besides the first law of thermodynamics. The mass, of course, is not the internal energy but rather the enthalpy of the thermodynamic system. The emergent property in the extended phase space is that the phase structures of a black holes are more abundant, and one can observe the van der Waals-like phase transition [9–14], reentrant phase transitions [15–18].

Another reason for the importance of the thermodynamics in AdS space arises from the AdS/CFT correspondence [19–22], which states that the gravity theory in $d$-dimensional AdS spacetime is dual to the conformal field theory in $(d - 1)$-dimension. According to this duality, the temperature of the black holes is dual to the temperature of the conformal field theory. Various holographic applications have been proposed in the last two decades, for example, the holographic superconductors [23–26], holographic thermalizations [27–30], holographic Fermi/non-Fermi liquids [31–33], and so on.

For the general relativity, the existence of singularities are inevitable. If singularities are naked, causality in the spacetimes may break down, and physics will lose its predictive power. Penrose conjectured that the singularity will not be naked for it is always covered by an event horizon [34], which is the so-called weak cosmic censorship conjecture. Since there are no general ways to prove this conjecture, we thus should investigate each type of black holes. Wald proposed firstly a gedanken experiment to check this conjecture by throwing a particle into the Kerr-Newman black hole, and he found that the weak cosmic censorship conjecture was valid for the extremal Kerr-Newman black holes [35]. However, for the near-extremal Reissner-Nordström [36] black hole and near-extremal Kerr black hole, this conjecture were found to be violated. Later it was pointed out that the invalidity of the weak cosmic censorship conjecture in [36, 37] was that the backreaction and self-force effects are neglected[38]. So far, the weak cosmic censorship conjecture has been checked in various black holes with and without the self-force effect by adding a particle [39–55].

As mentioned above, there are many works to investigate the thermodynamics of black holes in the extended phase space, however all of them focus only on the first law of thermodynamics. There is little work to discuss the second law of thermodynamics and the weak cosmic censorship conjecture in the extended phase space. Recently, Ref. [56] suggested that the first law, second law as well as the weak cosmic censorship conjecture can be investigated in the extended phase space under charged particles absorption. Their results showed that though the first law and the weak cosmic censorship conjecture were valid, the second law was violated for the near-extremal Reissner-Nordström-AdS black holes. Subsequently, the idea in [56] was extended to Born-Infeld-anti-de Sitter black hole [57], torus-like black hole [58] and charged fermions absorption [59–61]. In this paper, we are going to investigate the laws of thermodynamics and weak cosmic censorship conjecture in the Kerr-AdS black hole. Different form previous investigations, the topology of this black hole is axisymmetric. In fact, Refs. [62, 63] have investigated the laws of thermodynamics of this black hole in the normal phase space. In order to get the first law as well as the second law, a reference energy of the particle in the asymptotic region was imposed. In this paper, we will explore whether the first law and second law may be obtained by the energy-momentum relation of the particles without any imposition. In addition, until now
there is no work investigating the laws of thermodynamics and weak cosmic censorship conjecture of the rotating black holes in the extended phase space, therefore, we are going to explore whether our method can apply to this type of black holes.

This paper is arranged as follows. In section 2, we introduce firstly the thermodynamics of the Kerr-AdS black hole, and then investigate the dynamics of the spinning fermions in this black hole. In section 3, we investigate the first law, the second law as well as the weak cosmic censorship conjecture in the normal phase space. The law of thermodynamics and weak cosmic censorship conjecture in the extended phase space are studied in section 4. Section 5 is devoted to our conclusions. Throughout this paper, we will set the gravitational constant $G$ and the light velocity $c$ to be one.

2 Relation between the energy and momentum of the spinning fermions

2.1 Review of the Kerr-AdS black holes

In the Boyer-Lindquist coordinates, the line element of the Kerr-AdS black hole can be written as

$$ds^2 = -\frac{\Delta}{\rho^2}\left(dt - \frac{a\sin^2 \theta}{\Xi} d\varphi\right)^2 + \frac{\rho^2}{\Delta} dr^2 + \frac{\rho^2}{\Delta} d\theta^2 + \frac{\Delta \theta}{\rho^2}\left(adt - \frac{r^2 + a^2}{\Xi} d\varphi\right)^2,$$

with the metric functions given by

$$\Delta = (r^2 + a^2)(1 + r^2/l^2) - 2mr,$$

$$\rho^2 = r^2 + a^2 \cos^2 \theta,$$

$$\Xi = 1 - \frac{a^2}{l^2},$$

$$\Delta_\theta = 1 - a^2/l^2 \cos^2 \theta,$$

in which, $m$, $a$ and $l$ are the black hole mass parameter, spin parameter and the AdS radius respectively. The cosmological constant is related to the AdS radius as $\Lambda = -3/l^2$. The metric (2.1) describes the Kerr black hole (Schwarzschild-AdS black hole) in the limit of $l = \infty$ ($a = 0$), respectively.

For the Kerr-AdS black holes, due to the presence of rotation, the event horizon does not coincide with the infinite red-shift surface, leading to an ergosphere between the event horizon and infinite red-shift surface. The matter near the horizon will be dragged inevitably by the gravitational field with an azimuthal angular velocity, defined by

$$\bar{\Omega} = \frac{a\Xi(\Delta_\theta(r^2 + a^2) - \Delta)}{\Delta_\theta(r^2 + a^2)^2 - \Delta a^2 \sin^2 \theta}.$$  

The angular velocities at the event horizon and infinity can be readily obtained from Eq.(2.6) as

$$\Omega_h = \frac{a\Xi}{r_h^2 + a^2},$$

(2.7)
\[ \Xi_{\infty} = -\frac{a}{l^2}, \quad (2.8) \]

where \( r_h \) is the event horizon of the black hole by imposing \( \Delta(r_h) = 0 \). It is obvious that the angular velocity at infinity does not vanish, which is different from that in the asymptotically flat case, where \( \bar{\Omega}_{\infty} = 0 \). The net angular velocity that contributes to the thermodynamics is given by [5]

\[ \Omega_h = \bar{\Omega}_h - \bar{\Omega}_{\infty} = \frac{a\Xi}{r_h^2 + a^2} + \frac{a}{l^2}. \quad (2.9) \]

The Hawking temperature and Bekenstein-Hawking entropy of the Kerr-AdS black holes are

\[ T_h = \frac{r_h}{4\pi(r_h^2 + a^2)} \left( 1 + \frac{a^2}{l^2} + 3r_h^2 - \frac{a^2}{r_h^2} \right), \quad (2.10) \]
\[ S = \pi(r_h^2 + a^2) \frac{\Xi}{2}. \quad (2.11) \]

The black hole mass \( M \) and angular momentum \( J \) can be expressed by the mass parameter \( m \) and spin parameter \( a \) as

\[ M = \frac{m}{\Xi^2}, \quad J = \frac{am}{\Xi^2}. \quad (2.12) \]

Thus, from Eqs. (2.10), (2.11), and (2.12), we can get the first law of thermodynamics

\[ dM = T_h dS + \Omega_h dJ. \quad (2.13) \]

In Eq. (2.13), the cosmological parameter is treated as a constant, and the phase space of the thermodynamics in this case is called as normal phase space. Recent investigations have shown that the normal phase space can be extended by treating the cosmological parameter as an extensive variable, thus we can construct an extended phase space [6–8]. For the Kerr-AdS black holes, the first law of thermodynamics in the extended phase space can be expressed as [65, 66]

\[ dM = T_h dS + \Omega_h dJ + V dP, \quad (2.14) \]

in which \( P \) and \( V \) are respectively the pressure and volume of the thermodynamic system, which are defined by

\[ P = \frac{3}{8\pi l^2}, \quad V = \frac{2\pi (a^2 + r_h^2) (a^2 l^2 - a^2 r_h^2 + 2l^2 r_h^2)}{3l^2 \Xi^2 r_h}. \quad (2.15) \]

Based on the first law in the extended phase space, the phase structures of the Kerr-AdS black holes have been investigated extensively. It was shown that in the \( P - V \) plane, there were phase transitions between the small black holes and large black holes [65, 66], which is similar to the van der Waals phase transition.
2.2 Motion of the spinning fermions in the Kerr-AdS black holes

When particles are absorbed by spherical symmetric black holes, it has been confirmed that they satisfy the first law of thermodynamics as the energy-momentum relations of particles are employed. We will adopt the similar strategy to investigate the laws of thermodynamics of Kerr-AdS black holes, when the absorbed particles are spinning fermions. The Dirac equation now is

\[ i \gamma^\mu (\partial_\mu + \Omega_\mu) \psi - \frac{u}{\hbar} \psi = 0, \]  

(2.16)

in which \( u \) is the rest mass of the fermion, while \( \Omega_\mu = \frac{i}{2} \Gamma^{\alpha\beta}_\mu \Pi_{\alpha\beta} \), \( \Pi_{\alpha\beta} = \frac{i}{4} [\gamma^\alpha, \gamma^\beta] \), and Dirac matrices \( \gamma^\mu \) satisfy \( \{ \gamma^\mu, \gamma^\nu \} = 2g^\mu\nu \).

In order to investigate the motion of the fermions in the rotating black holes, one should carry out the dragging coordinate transformation with the definition \( \phi = \varphi - \bar{\Omega} t \). In this case, the metric in Eq.(2.1) becomes

\[ ds^2 = -F(r) dt^2 + \frac{1}{G(r)} dr^2 + H(r) d\theta^2 + K(r) d\phi^2, \]  

(2.17)

in which

\[ F(r) = \frac{\Delta \Delta_{\theta} \rho^2}{\Delta \theta (r^2 + a^2)^2 - \Delta a^2 \sin^2 \theta}, \quad G(r) = \frac{\Delta}{\rho^2}, \]  

(2.18)

\[ H(r) = \frac{\rho^2}{\Delta \theta}, \quad K(r) = \frac{\sin^2 \theta}{\rho^2 \Delta \theta} (r^2 + a^2)^2 \Delta_{\theta} - \Delta a^2 \sin^2 \theta). \]  

(2.19)

From Eq.(2.17), we know that the event horizon and infinite red-shift surface coincide. This coincidence is important to investigate the tunneling radiation of the particles, for in this case Landau’s condition of the coordinate clock synchronization is satisfied.

To solve the Dirac equation in Eq.(2.17), we choose \( \gamma^\mu \) matrices as [67]

\[ \gamma^t = \frac{1}{\sqrt{F}} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \gamma^r = \sqrt{G} \begin{pmatrix} 0 & \sigma^3 \\ \sigma^3 & 0 \end{pmatrix}, \]  

(2.20)

\[ \gamma^\theta = \frac{1}{\sqrt{H}} \begin{pmatrix} 0 & \sigma^1 \\ \sigma^1 & 1 \end{pmatrix}, \quad \gamma^\phi = \frac{1}{\sqrt{K}} \begin{pmatrix} 0 & \sigma^2 \\ \sigma^2 & 0 \end{pmatrix}. \]

(2.21)

in which \( \sigma^i \) are the Pauli sigma matrices with the following definition

\[ \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \]

For a fermion with spin 1/2, the wave function have spin up and down state. In this paper, we are interested in the spin up state since the spin down case is similar. The wave function for the spin up state is supposed to be

\[ \psi = \begin{pmatrix} C(t, r, \theta, \phi) \\ 0 \\ D(t, r, \theta, \phi) \end{pmatrix} \exp \left( \frac{i}{\hbar} I(t, r, \theta, \phi) \right). \]  

(2.22)
Inserting Eq.(2.22) as well as Eq.(2.20) into Eq.(2.16), and applying the WKB approximation, we are led to the following equations

\[-\left(\frac{iC}{\sqrt{F}} \partial_t I + D\sqrt{G} \partial_r I\right) + uC = 0,\]  
\[\left(\frac{iD}{\sqrt{F}} \partial_t I - C\sqrt{G} \partial_r I\right) + uD = 0,\]  
\[-D\left(\frac{1}{\sqrt{H}} \partial_b I + \frac{i}{\sqrt{K}} \partial_b I\right) = 0,\]  
\[-C\left(\frac{1}{\sqrt{H}} \partial_b I + \frac{i}{\sqrt{K}} \partial_b I\right) = 0,\]

in which we only take the leading order of \(\hbar\). In order to solve these equations, one should separate the quantity \(I\). According to the symmetries of the spacetime, the quantity \(I\) in the dragging coordinate system can be re-expressed as [67]

\[I = -(\omega - j\Omega_h) t + I (r) + L\phi + \Theta(\theta),\]  

where \(\omega\) and \(j\) are respectively fermion’s energy and angular momentum measured by the observer at the infinity. Substituting Eq.(2.27) into Eqs.(2.23) and (2.24), we obtain

\[C\left(u + \frac{i}{\sqrt{F}} (\omega - j\Omega_h)\right) - D\sqrt{G} \partial_r I = 0,\]  
\[-C\sqrt{G} \partial_r I + D\left(u - \frac{i}{\sqrt{F}} (\omega - j\Omega_h)\right) = 0,\]

Eqs.(2.28) and (2.29) have non-trivial solutions for \(C\) and \(D\) only if the determinant of the coefficient matrix vanishes. From this condition, we can get \(\partial_r I\), and further the radial momentum \(p^r = g^{rr} \partial_r I,\)

\[p^r = \pm \sqrt{\frac{G}{F}} \sqrt{(\omega - j\Omega_h)^2 + \mu^2 F}.\]  

In order to study the laws of thermodynamics, we need to focus on the near horizon region. Thus, in this case Eq.(2.30) can be simplified as

\[\omega = \frac{p^r_h}{a^2 + r^2_h} |p^r_h| + j\Omega_h,\]  

in which \(p^2_h = a^2 \cos^2 \theta + r^2_h\) and \(p^r_h \equiv p^r |_{r_h}\). From Eq.(2.31), we know that when \(\omega < j\Omega_h\), the superradiation occurs and the energy of the black hole flows out of the horizon, since the sign in front of \(|p^r_h|\) is negative. In this paper, we assume that the superradiation does not occur, which implies \(\omega > j\Omega_h\) and the sign in front of \(|p^r_h|\) is positive.

### 3 Thermodynamics and weak cosmic censorship conjecture in the normal phase space

We will employ the energy-momentum relation derived in Eq.(2.31) to study the first law of thermodynamics in the normal phase space of Kerr-AdS black hole in this section. In addition, we will also discuss the second law and the weak cosmic censorship conjecture in the Kerr-AdS background.
3.1 The first law of thermodynamics in the normal phase space

We assume that the absorbed fermion has energy $\omega$ and angular momentum $j$. As it drops into the black hole, the internal energy $M$ and angular momentum $J$ of the black hole would change according to

$$\omega = dM, \quad j = dJ,$$

(3.1)

Due to the first law of thermodynamics, Eq.(2.31) can be rewritten as

$$dM = j\Omega_h + \frac{\rho_h^2}{a^2 + r_h^2} p_h^r.$$

(3.2)

In the normal phase space, the initial state of the black hole is represented by $(m, a, r_h)$, while the final state is represented by $(m + dm, a + da, r_h + dr_h)$. That is, the absorbed fermions will change the configurations of the black holes. However, the horizon is always determined by the function $\Delta(r)$. In other words, for the horizon the relation $\Delta(r_h + dr_h) = 0$ always holds. Thus, the change of the horizon should satisfy

$$d\Delta_h = \frac{\partial \Delta_h}{\partial m} dm + \frac{\partial \Delta_h}{\partial a} da + \frac{\partial \Delta_h}{\partial r_h} dr_h = 0.$$

(3.3)

In addition, by considering Eq.(2.12), Eq.(3.2) can be rewritten as

$$\frac{\partial m}{\partial a} \Xi^2 da + \frac{\partial m}{\partial m} \Xi^2 dm - \left( \frac{a \Xi}{a^2 + r_h^2} + \frac{a}{l^2} \right) \left( \frac{\partial am}{\partial \Xi^2} da + \frac{\partial am}{\partial m} \Xi^2 dm \right) - \frac{\rho_h^2}{a^2 + r_h^2} p_h^r = 0.$$  

(3.4)

After solving Eq.(3.4), we reach

$$dm = \frac{al^2 \Xi (l^2 - 3r_h^2) \Xi da}{4l^2 r_h^4 (l^2 - a^2)} + \frac{a^2 p_h^r (a - 2l^2) + a^2 p_h^r (l^2 - 2r_h^2) + a^4 p_h^r}{(a^2 + r_h^2) (l^2 - a^2)} \left[ \frac{l^2 r_h^4 (l^2 - a^2)}{a^2 + r_h^2} \right].$$

(3.5)

Inserting Eq.(3.5) into Eq.(3.3), we arrive at

$$dr_h = \frac{al^2 da}{a^2 r_h - l^2 r_h} + \frac{ar_h da}{a^2 - l^2} + \frac{2p_h^r (a - l) (a^2 + r_h^2) \rho_h^2}{(a^2 + r_h^2) [a^2 l^2 - r_h^2 (a^2 + l^2) - 3r_h^4]}.$$  

(3.6)

Based on the above formula, one can readily obtain the variation of entropy by making use of Eq.(2.11), which is

$$dS = \frac{\partial S}{\partial a} da + \frac{\partial S}{\partial r_h} dr_h = \frac{4\pi l^2 r_h (a^2 + r_h^2) \rho_h^2 p_h^r}{(a^2 + r_h^2) [a^2 l^2 - r_h^2 (a^2 + l^2) - 3r_h^4]}.$$  

(3.7)

With the help of Eqs.(2.10) and (3.7), we can get the following relation

$$T_h dS = \frac{\rho_h^2}{a^2 + r_h^2} p_h^r.$$  

(3.8)

Therefore, the internal energy in Eq.(3.2) can be rewritten as

$$dM = T_h dS + \Omega_h dJ,$$

(3.9)

which is consistent with that in Eq.(2.13). Obviously, we can see that as a spinning fermion drops into the Kerr-AdS black hole, the first law of thermodynamics is valid in the normal phase space.
3.2 The second law of thermodynamics in the normal phase space

From the Eq.(3.7), we can also check that the second law of thermodynamics in Kerr-AdS black hole, which states that the entropy of the black holes never decrease. The process that a fermion drops into the black hole is irreversible, thus the entropy of the black hole should increase.

For the extremal black holes, the inner horizon and outer horizon coincide and the temperature vanishes at the horizon. From Eq.(2.10), we can get the radius of the extremal black hole, that is

\[ r_{\text{extreme}} = \frac{\sqrt{-a^2 + \sqrt{a^4 + 14a^2l^2 + l^4}} - l}{\sqrt{6}}. \]

(3.10)

The second law of the extremal black holes is meaningless for the temperature of the thermodynamics system vanishes. In fact, from Eq.(3.8), we can know that the variation of the entropy is divergent in this case. For the non-extremal black hole, the temperature is larger than zero for \( r > r_{\text{extreme}} \), thus \( dS \) in Eq.(3.7) is positive, implying that the second law of thermodynamics is also valid in the normal phase space for the non-extremal Kerr-AdS black holes.

3.3 Weak cosmic censorship conjecture in the normal phase space

In the normal phase space, it is also necessary to check the validity of the weak cosmic censorship conjecture since the Kerr-AdS background will change as the fermions are absorbed. The validity of the weak cosmic censorship can be tested by computing the minimum value of the function \( \Delta(r) \) after the absorption of fermions. A non-positive minimum value of the function \( \Delta(r) \) implies the existence of horizons indicating that the cosmic censorship conjecture is valid. On the contrary, a positive minimum value of \( \Delta(r) \) implies the violation of cosmic censorship conjecture since the spacetime singularity is naked. The absorbed particles will change the configurations of the black holes, thus the minimum value of \( \Delta(r) \) may move arbitrarily. Therefore, in the following we will explore how the minimum of \( \Delta(r) \) moves as a fermion is absorbed by the Kerr-AdS black hole in order to make sure whether there exists an event horizon.

For the Kerr-AdS black holes, we label the radial coordinate of the minimum value for \( \Delta(r) \) as \( r_l \). At \( r_l \), the following relations always satisfy

\[ \Delta(r)|_{r=r_l} = \Delta_l = \lambda \leq 0, \]

(3.11)

\[ \partial_r \Delta(r)|_{r=r_l} = \Delta_l' = 0. \]

(3.12)

For the extremal black holes, \( \lambda = 0 \), the horizon and the location of the minimum value are coincident. For the near-extremal black holes, \( \lambda \) is a small quantity, the location of the minimum value is located between the inner horizon and outer horizon. As a fermion with mass \( \omega \) and angular momentum \( j \) drops into the black hole, the mass and angular momentum of the black hole will increase as \( M + \omega, J + j \) respectively. Correspondingly,
the location of the minimum value and event horizon change into \( r_l + dr_l, r_h + dr_h \). There is also a shift of \( \Delta(r) \), which is labeled as \( d\Delta_l \). At \( r_l + dr_l \), we have

\[
\Delta(r_l + dr_l) = \Delta_l + d\Delta_l = \lambda + \left( \frac{\partial \Delta_l}{\partial m} dm + \frac{\partial \Delta_l}{\partial a} da \right),
\]

(3.13)

where we have used Eq.(3.11). For the case of \( \Delta(r_l + dr_l) > 0 \), there is no horizon while for the case of \( \Delta(r_l + dr_l) \leq 0 \), there are always horizons. In the following we will focus on finding the final form of Eq.(3.13). First, we consider the extremal black holes, for which the horizon is located at \( r_l \) and \( \Delta_l = 0 \). In this case, Eq.(3.5) can be adopted, thus we reach

\[
d\Delta_l = \frac{-2ada(l^2 r_l (a^2 - r_l(3m + r_l)) + a^2 r_l^3 + l^4 (m - r_l))}{l^2 r_l (l^2 - a^2)}
- \frac{2 (a^2 - l^2)^2 (a^2 + r_l^2) (a^2 \cos^2 \theta + r_l^2)}{l^2 r_l (l^2 - a^2) (a^2 + r_l^2)}. \]

(3.14)

In addition, from Eq.(2.2) the mass of the black hole can be written as

\[
m = \frac{(a^2 + r_l^2) (l^2 + r_l^2)}{2l^2 r_l}. \]

(3.15)

Substituting Eqs.(3.15) and (3.10) into Eq.(3.14), we finally get

\[
d\Delta_l = \frac{\sqrt{\frac{2}{3}(a - l)(a + l) \left( a^2 \left( 6 \cos^2 \theta - 1 \right) + \sqrt{a^4 + 14a^2 l^2 + l^4 - l^2} \right)}}{l^2 \sqrt{-a^2 + \sqrt{a^4 + 14a^2 l^2 + l^4 - l^2}}}.
\]

(3.16)

The values of \( d\Delta_l \) are shown in Figure (1) for various \( l \)'s. It is obvious that for a given value of \( l \), the value of \( d\Delta_l \) can be positive or negative. Take the example of \( l = 0.7 \), we can see that for the case of \( a < l \), \( d\Delta_l \) is negative; while for the case of \( a > l \), \( d\Delta_l \) is positive. For other values of \( l \)'s, we can observe similar behaviors of \( d\Delta_l \). Therefore, we can conclude that for the case of \( a < l \), the weak cosmic censorship conjecture is valid while for the case of \( a > l \), the weak cosmic censorship conjecture is violated. Our results are consistent with those in Ref. [5], in which the authors found that the Kerr-AdS solution was valid only for \( a < l \) since the singularity will be naked for \( a \geq l \).
4 Thermodynamics and weak cosmic censorship conjecture in the extended phase space

In this section, we are going to check the thermodynamics and weak cosmic censorship conjecture in the extended phase space of Kerr-AdS black hole under a fermion absorption. We will explore how the pressure and volume affect the thermodynamics and weak cosmic censorship conjecture.

4.1 The first law of thermodynamics in the extended phase space

In the extended phase space, the mass is no longer related to the internal energy but rather the enthalpy of the thermodynamic system. The relation between the enthalpy $M$, internal energy $U$, pressure $P$ and volume $V$ are

$$ M = U + PV. \quad (4.1) $$

As the fermions are absorbed by the black holes, the energy and angular momentum are supposed to be conserved. Thus, the energy and angular momentum of the spinning fermion equal to the varied energy and angular momentum of the black hole, which leads to

$$ \omega = dU = d(M - PV), \quad j = dJ, \quad (4.2) $$

where we have used Eq.(3.1). In this case, Eq.(2.31) changes correspondingly as

$$ d(M - PV) = j\Omega_h + \frac{\rho_h^2}{a^2 + r_h^2}p_h. \quad (4.3) $$

Considering Eq.(2.12), Eq.(4.3) can be rewritten as

$$ \frac{\partial}{\partial a} \frac{m}{\Xi^2} \frac{da}{\Xi^2} + \frac{\partial}{\partial l} \frac{m}{\Xi^2} dl + \frac{\partial}{\partial m} \frac{m}{\Xi^2} dm - V \frac{\partial P}{\partial l} dl - PY - p_h X $$

$$ - \left( \frac{a\Xi}{a^2 + r_h^2} + \frac{a}{l^2} \right) \left( \frac{\partial}{\partial a} \frac{am}{\Xi^2} da + \frac{\partial}{\partial l} \frac{am}{\Xi^2} dl + \frac{\partial}{\partial m} \frac{am}{\Xi^2} dm \right) = 0, \quad (4.4) $$
respectively, which implies the initial state and final state, the horizons are determined by the initial state should be characterized by the extended phase space, the AdS radius is also a parameter of the black hole. Thus, the substituted Eq.(4.7) into Eq.(4.8), we can solve the final configuration of the black hole becomes

\[ Y \equiv dV = \frac{\partial V}{\partial a} da + \frac{\partial V}{\partial l} dl + \frac{\partial V}{\partial r_h} dr_h, \quad (4.5) \]

\[ X = \frac{a^2 \cos^2 \theta + r_h^2}{a^2 + r_h^2}. \quad (4.6) \]

By solving Eq.(4.4), we obtain

\[ dm = \frac{U_1 + U_2 + U_3 + U_4}{8\pi l^2 r_h^3 (l^2 - a^2)}, \quad (4.7) \]

where

\[ U_1 = 8\pi a l^4 m r_h (l^2 - 3r_h^2) da + l^3 r_h^3 (-8\pi r_h^3 dl + 8\pi l^3 p_r^2 X + 3lY), \]

\[ U_2 = a^6 (4\pi l (r_h^2 - l^2) dl + r_h (8\pi l^2 p_r^2 X + 3Y)), \]

\[ U_3 = a^2 l r_h (4\pi r_h^2 (l^2 (8m - 5r) + r_h^3) dl + l (l^2 - 2r_h^2)), \]

\[ U_4 = a^4 r_h (r_h^2 - 2l^2) (8\pi l r_h dl + 8\pi l^2 p_r^2 X + 3Y). \]

In the extended phase space, the AdS radius is also a parameter of the black hole. Thus, the initial state should be characterized by \((m, a, r_h, l)\). As a fermion drops into the black hole, the final configuration of the black hole becomes \((m + dm, a + da, r_h + dr_h, l + dl)\). At the initial state and final state, the horizons are determined by \(\Delta(r_h) = 0\) and \(\Delta(r_h + dr_h) = 0\) respectively, which implies

\[ d\Delta_h = \frac{\partial \Delta_h}{\partial m} dm + \frac{\partial \Delta_h}{\partial a} da + \frac{\partial \Delta_h}{\partial r_h} dr_h + \frac{\partial \Delta_h}{\partial l} dl = 0. \quad (4.8) \]

Substituting Eq.(4.7) into Eq.(4.8), we can solve \(dr_h\) directly as

\[ dr_h = \frac{V_1 + V_2 + V_3 + V_4}{4\pi l^2 r_h (l^2 - a^2) (a^2 (l^2 - r_h^2) - r_h^2 (l^2 + 3r_h^2))}, \quad (4.9) \]

where

\[ V_1 = 4\pi a l^2 r_h^2 (l^4 + 4l^2 r_h^2 + 3r_h^4) da - l^4 r_h^3 (8\pi l^2 p_r^2 X + 3Y), \]

\[ V_2 = a^6 (4\pi l (l^2 - r_h^2) dl - r_h (8\pi l^2 p_r^2 X + 3Y)), \]

\[ V_3 = 4\pi a^3 l^2 (r_h^4 - l^4) da - a^2 l r_h (4\pi r_h^3 (l^2 + 3r_h^2) dl + l (l^2 - 2r_h^2) (8\pi l^2 p_r^2 X + 3Y)), \]

\[ V_4 = a^4 (2l^2 r_h - r_h^3) (8\pi l^2 p_r^2 X + 3Y) - 16\pi l r_h^3 dl). \quad (4.10) \]

From Eq.(2.11), the variation of entropy can be expressed as

\[ dS = \frac{\partial S}{\partial a} da + \frac{\partial S}{\partial l} dl + \frac{\partial S}{\partial r_h} dr_h. \quad (4.11) \]

With the help of Eqs.(2.10), (2.15), (4.6), and (4.16), we find that

\[ T_h dS - P dV = \frac{p_r^2}{a^2 + r_h^2}. \quad (4.12) \]
In this case, the internal energy in Eq.(4.3) can be rewritten as

\[ d(M - PV) = T_h dS + \Omega_h dJ - P dV. \]  

(4.13)

From Eq.(4.1), we can obtain the differential relation between the enthalpy and internal energy as

\[ dM = dU + PdV + VdP. \]  

(4.14)

Substituting Eq.(4.14) into Eq.(4.13), we get

\[ dM = T dS + \Omega_h dJ + VdP, \]  

(4.15)

which is consistent with that in Eq.(2.14). Therefore, we conclude that as a fermion drops into the Kerr-AdS black hole, the first law of thermodynamics is also valid in the extended phase space.

### 4.2 The second law of thermodynamics in the extended phase space

In the normal phase space, we have known that both the first law and second law are valid as a fermion drops into the Kerr-AdS black holes. However, this is not always true for all the cases. The satisfaction of the first law of thermodynamics does not mean that the second law is also satisfied, especially in the extended phase space [56]. In this section, we will employ Eq.(4.16) to check the second law of Kerr-AdS black holes in the extended phase space. In order to compute Eq.(4.16), we should solve \( dr_h \) in Eq.(4.9). Substituting Eq.(4.6) into Eq.(4.9), we can get

\[ dr_h. \]

And then substituting it further into Eq.(4.16), the variation of entropy thus can be rewritten as

\[ dS = \frac{E}{(a^2 - l^2)^2 (a^2 (r_h^2 + l^2) + 2a^2 r_h^4 - a^2 (2l^4 r_h^2 + 9l^2 r_h^4 + 3r_h^6) + 2l^4 r_h^4)}, \]  

(4.16)

in which

\[ E = 2\pi l \left( -a^{10} (r_h^2 + l^2) dl - 4a^2 p_h^r (a^2 - l^2)^3 \cos^2(\theta) r_h^3 + a^3 l r_h^4 (r_h^2 + l^2)^2 da + 4l^7 p_h^r r_h^5 \right) 
+ 2\pi l \left( -3a^8 r_h^2 (r_h^2 + l^2) dl + a^7 l (r_h^2 + l^2)^2 da + 2a^5 l^2 r_h^2 (r_h^2 + l^2)^2 da - 12a^2 l^5 p_h^r r_h^5 \right) 
+ 2\pi l \left( a^6 (-r_h^4) (3 (r_h^2 + l^2) dl + 4l^5 p_h^r r_h) - a^4 r_h^5 (l^2 r_h dl + r_h^3 dl - 12l^3 p_h^r) \right). \]  

(4.17)

From Eq.(4.16), we see that the variation of the entropy depends on \( a, l, da, dl \) while fixing \( r_h \). Because we are interested in the second laws of black holes, thus \( a < l \) should be satisfied, otherwise the singularity will be naked and the thermodynamic system will break down.

Since the formula of \( dS \) in (4.16) consists of a bunch of parameters, for simplicity and without loss of generality, we will impose \( p_h^r = 1, \theta = 0, dl = 0.01, l = 1 \) in this subsection in order to see the effects of \( da, a \) and \( r_h \) to the variation of the entropy. Of course one can also set other parameters, but the general conclusions will be similar. In Fig.2 we show the 3-dimensional plots for the \( dS \) vs. \( da \) and \( r_h \) while fixing various \( a \)’s. We see that \( dS \) will
Figure 2. Three dimensional visualizations of $dS$ with respect to $da$ and $r_h$ for various fixed $a$’s. The horizontal gray plane is at $dS \equiv 0$. We can readily see that $dS > 0$ and $dS < 0$ can all exist in different parameter regimes.

be positive or negative for different parameter regimes. For instance of $a \equiv 0.7$ and $r_h = 20$ (the right panel of top row of Fig. 2), there exists a critical value of $da$ such that $dS > 0$ if $da < 0.0067$ and $dS > 0$ if $da > 0.0067$. In fact, the critical values of $da$ and $r_h$ that will
render $dS = 0$ can be easily obtained from Eq.(4.16) as,

$$
da = \frac{a^8 (r_h^2 + 1) + 2a^6 r_h^2 (r_h^2 + 200r_h + 1) + a^4 r_h^3 (r_h^2 + r_h - 1200) + 1200a^2 r_h^3 - 400r_h^2}{100a^3 (r_h^2 + 1)^2 (a^2 + r_h^2)}.
$$

(4.18)

As $a$ is small, for example $a \equiv 0.1$ (left panel of bottom row of Fig.2), $dS > 0$ if $r_h < 8$ whatever $da$ is. This means the second law of thermodynamics will always be satisfied if the black hole is near-extremal (with smaller $r_h$) independent of $da$. In the opposite limit, i.e., $r_h > 8$, the second law of thermodynamics will be violated.

Therefore, in the extended phase space, the second law of thermodynamics may be valid or violated, depending on the values of parameters $a, da$ and $r_h$ (given that we have fixed the values of $p_h^r, \theta, dl$ and $l$), which is different from that in the normal phase space.

### 4.3 Weak cosmic censorship conjecture in the extended phase space

In the normal phase space, we have investigated the weak cosmic censorship conjecture under fermions absorption and found that it was valid since the final state was also a Kerr-AdS black hole. In this section, we will study the weak cosmic censorship conjecture in the extended phase space. We need to explore what the final state is as the fermions are absorbed by the Kerr-AdS black holes. Like the case in the normal phase space, we are going to discuss the minimal value of the function $\Delta(r)$ since a non-positive minimum value implies the existence of horizons while a positive one does not.

Different from that in the normal phase space, $l$ now is a variable and the initial state is represented by $(m, a, l)$. As a fermion drops into the black hole, the final state will change into $(m + dm, a + da, l + dl)$. Correspondingly, there are also shifts for the locations of the minimum value and event horizon, $r_l \rightarrow r_l + dr_l, r_h \rightarrow r_h + dr_h$. Thus, the shift for $\Delta(r)$ is

$$
d\Delta_l = \Delta_l (r_l + dr_l) - \Delta_l = \frac{\partial \Delta_l}{\partial m} dm + \frac{\partial \Delta_l}{\partial a} da + \frac{\partial \Delta_l}{\partial l} dl = -\frac{2r_m^2 (a^2 + r_m^2) dl}{l^3} + 2a \left( \frac{r_m^2}{l^2} + 1 \right) da - 2r_m dm,
$$

(4.19)

where we have used $\Delta'_l = 0$ in Eq.(3.11). Our next step is to find the explicit result of Eq.(4.19).

For the extremal black holes, we have $r_h = r_l$ so that $\Delta_l = 0$. In this case, Eq.(4.4) can be used. Substituting $m$ in Eq.(3.15) and $p_h^r$ in Eq.(4.12) into Eq.(4.19), we obtain

$$
dm = a^2 \left( -2r_m^3 dl - l^3 dr_m + lr_m^2 \right) dr_m + a \left( \frac{l^2 + r_m^2}{l^2 r_m} \right) da + \frac{l^2 r_m^2}{2l^3 r_m^2} \left( l^2 + 3r_m^2 \right) dr_m - 2r_m^5 dl.
$$

(4.20)

Inserting Eq.(4.20) into Eq.(4.19), we get

$$
d\Delta_l = a^2 \left( \frac{1}{r_m} - \frac{r_m}{l^2} \right) dr_m + r_m \left( -\frac{3r_m^2}{l^2} - 1 \right) dr_m.
$$

(4.21)
For the extremal black holes, Eq.(3.10) is applicable. Thus, Eq.(4.21) can be simplified further to be
\[ d\Delta_l = 0. \] (4.22)
From Eq.(4.22), we see that there is no shift in \( \Delta_l \) for the extremal black holes. The configuration of extremal black holes does not change, thus the extremal black holes are still extremal black holes as fermions are absorbed.

For the near-extremal black holes, Eq.(3.10) is not applicable at \( r_m \) since it is valid only at \( r_h \). However, we can expand it near \( r_m \). Replacing \( r_h \) in Eq.(4.20) with \( r_m + \epsilon \) and expanding it to the first order of \( \epsilon \), we reach
\[ dM = A + B \epsilon + O(\epsilon^2), \] (4.23)
in which
\[ A = \frac{a^2}{2l^2} - \frac{a^2}{2r_m^2} + a \frac{da}{r_m} + \frac{dr_m}{2} - \frac{a^2 r_m dl}{l^3} + \frac{a r_m dl}{l^3} - \frac{r_m^3 dl}{l^3} + \frac{3r_m^2 dr_m}{2l^2}, \]
and
\[ B = -\frac{a^2}{l^3} + \frac{a^2}{r_m^3} + \frac{ada}{r_m^2} - \frac{3r_m^2 dl}{l^3} + 3r_m \frac{dr_m}{l^2}. \]
Substituting Eq.(4.23) into Eq.(4.19), we obtain
\[ d\Delta_l = a^2 \left( \frac{1}{r_m} - \frac{r_m}{l^2} \right) dr_m + r_m \left( -\frac{3r_m^2}{l^2} - 1 \right) dr_m + 2 \left( a^2 (r_m^2 dl - l^2) dr_m + a r_m (l^2 - r_m^2) da + 3r_m^4 (r_m dl - ldr) \right) \epsilon + O(\epsilon)^2. \] (4.24)
In the following, we will focus on simplifying Eq.(4.24). Since at the horizon \( \Delta(r_h) = 0 \), thus one can replace \( r_h \) with \( r_m + \epsilon \) and expand \( \Delta(r_h) \) to the first order of \( \epsilon \),
\[ \Delta(r_h) = a^2 \left( 1 - \frac{r_m^2}{l^2} \right) - 3r_m^4 - r_m^2 = 0, \] (4.25)
where we have employed Eq.(3.15). Solving Eq.(4.25), we obtain
\[ l = \frac{r_m \sqrt{a^2 + 3r_m^2}}{\sqrt{a^2 - r_m^2}}. \] (4.26)
Differentiating both sides of Eq.(4.26), we can further get
\[ dl = \frac{a^4 dr_m + 6a^2 r_m^2 dr_m - 4a r_m^3 da - 3r_m^4 dr_m}{(a^2 - r_m^2)^{3/2} \sqrt{a^2 + 3r_m^2}}. \] (4.27)
Substituting Eqs.(4.26) and (4.27) into Eq.(4.24), we finally arrive at
\[ d\Delta_l = O(\epsilon)^2. \] (4.28)
Since \( O(\epsilon)^2 \) is the higher order of the small quantity \( \epsilon \), thus it can be omitted. In this case, we see that the final states of the near-extremal black holes are still the near-extremal black holes, which is similar to that of the extremal black holes. Therefore, we can conclude that the weak cosmic censorship conjecture still holds for both the extremal and near-extremal black holes in the extended phase space.
5 Conclusions

We investigated the laws of thermodynamics and weak cosmic censorship conjecture by considering a spinning fermion absorbed by the Kerr-AdS black hole. The dynamics of the fermion was investigated by the Dirac equation in the dragging coordinate system. As a result, we obtained the energy-momentum relation of the spinning fermion near the event horizon, which was the basic of our investigation.

In the normal phase space, we derived firstly the first law of thermodynamics by considering the energy-momentum relation as well as the energy and angular momentum conservation. Then we investigated the second law of thermodynamics by discussing the variation of the entropy. As predicted, it was found that the variation of the entropy is always positive. The validity of the weak cosmic censorship conjecture was also checked in the normal phase space by studying the variation of the function $\Delta(r)$ at the minimal point. For the case that the initial state was a black hole, the final state was found to be a black hole as well, thus the weak cosmic censorship conjecture was valid as well in the Kerr-AdS black hole.

Later, we investigated the laws of thermodynamics and weak cosmic censorship conjecture in the extended phase space, which has not been reported previously as far as we know. In the extended phase space, the cosmological parameter is not a constant any more, however, the pressure of the thermodynamic system is considered as a constant. The first law thus contains the contribution from pressure and volume. From the energy-momentum relation as well as the energy and angular momentum conservation, we also derived the first law successfully. The second law, however, is more subtle since we found that it depends on the values of the spin parameter, AdS radius as well as the their variations. As the value of the AdS radius is fixed, we found that the variation of entropy are different for different spin parameters. And for a fixed spin parameter, the variation of the spin parameter also affects the variation of entropy, which can be positive or negative. Therefore, the second law in the extended phase space may be valid or violated. We also checked the validity of the weak cosmic censorship conjecture in the extended phase space by studying the variation of the function $\Delta(r)$ at the minimal point. For the initial states were extremal and near-extremal black holes, the final states were found to be extremal and near-extremal black holes as well. The absorbed fermion thus would not change the location of the minimum, thus the weak cosmic censorship conjecture is still valid in the extended phase space.

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