The Replica Symmetric Solution for Potts Models on \(d\)-Regular Graphs

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Received: 24 July 2012 / Accepted: 16 December 2013
Published online: 13 March 2014 – © Springer-Verlag Berlin Heidelberg 2014

Abstract: We establish an explicit formula for the limiting free energy density (log-partition function divided by the number of vertices) for ferromagnetic Potts models on uniformly sparse graph sequences converging locally to the \(d\)-regular tree for \(d\) even, covering all temperature regimes. This formula coincides with the Bethe free energy functional evaluated at a suitable fixed point of the belief propagation recursion on the \(d\)-regular tree, the so-called replica symmetric solution. For uniformly random \(d\)-regular graphs we further show that the replica symmetric Bethe formula is an upper bound for the asymptotic free energy for any model with permissive interactions.

1. Introduction

Let \(G = (V, E)\) be a finite undirected graph with vertices \(V\) and edges \(E\), and \(\mathcal{X}\) a finite alphabet of spins. A factor model on \(G\) is a probability measure on the space of (spin) configurations \(\sigma \in \mathcal{X}^V\) of the form

\[
\nu_G^\psi(\sigma) \equiv \frac{1}{Z_G(\psi)} \prod_{(ij) \in E} \psi(\sigma_i, \sigma_j) \prod_{i \in V} \bar{\psi}(\sigma_i),
\]

where \(\psi\) is a non-negative symmetric function on \(\mathcal{X}^2\), \(\bar{\psi}\) is a positive function on \(\mathcal{X}\), and \(Z_G(\psi) \equiv Z_G\) is the normalizing constant, called the partition function (with its logarithm called the free energy). The pair \(\psi \equiv (\psi, \bar{\psi})\) is called a specification for the factor model (1), and we assume it to be permissive, meaning there exists \(\sigma^P \in \mathcal{X}\) with \(\min_\sigma \psi(\sigma, \sigma^P) > 0\).

Research partially supported by NSF grants A. Dembo, A. Montanari, N. Sun: DMS-1106627 and A. Montanari: CCF-0743978, A. Sly: Alfred P. Sloan Research Fellowship, and N. Sun: Department of Defense NDSEG Fellowship.
A primary example we consider in this paper is the \textit{q-state Potts model} on $G$ with inverse temperature $\beta$ and magnetic field $B$, given by specification
\begin{equation}
\psi(\sigma, \sigma') = e^{\beta \mathbb{1}[\sigma = \sigma']}, \quad \bar{\psi}(\sigma) = e^{B \mathbb{1}[\sigma = 1]}, \quad \mathcal{X} = [q] \equiv \{1, \ldots, q\}.
\end{equation}
We write $\nu_{G}^{\beta, B}$ for the corresponding measure on $[q]^V$. The model is said to be \textit{ferromagnetic} if $\beta \geq 0$, and \textit{anti-ferromagnetic} otherwise. The case $q = 2$ corresponds to the \textit{Ising} model.

In this paper we study the asymptotics of the free energy for factor models (1) on graph sequences $G_n = (V_n, E_n)$ converging locally to the $d$-regular tree $T_d$ ($d \geq 3$) in the sense of Benjamini–Schramm [BS01] (see Definition 1.1). This class includes in particular any sequence of $d$-regular graphs with girth (minimal cycle length) diverging to infinity.

The study of statistical mechanics on regular trees has a long history going back to Bethe [Bet35]. While tree graphs do not capture the finite-dimensional structure of actual physical systems, models on trees are often amenable to exact analysis. Further, it is often argued that they are a good approximation to models on the lattice $\mathbb{Z}^d$ for large $d$ or for long interaction range [Wei48,ATA73,CLR79,Tho86]. According to this expectation, models on trees provide a flexible and well-defined approach for investigating \textit{mean-field theory} (i.e., the behavior of statistical mechanics models in high dimensions).

While this expectation proves to be correct in a number of examples, it has recently become clear that, in many cases, models on trees fail to capture the “correct” mean-field behavior. Spin glasses provide an important example of this phenomenon: a fairly natural class of spin glasses on trees was introduced by Thouless [Tho86] and further characterized by Chayes et al. [CCST86]. However, the thermodynamic behavior observed there is very different from the widely accepted mean-field theory of spin glasses, as obtained from analysis of the Sherrington–Kirkpatrick (SK) model [MPV87,Tal11]. In particular, the low-temperature phase of the tree models defined in [Tho86] does not exhibit replica symmetry breaking (in contrast with SK). A similar discrepancy was observed in the case of Anderson localization by Aizenman–Warzel [AW06].

In the case of spin glasses, Mézard–Parisi [MP01] argued that this difference arises because of a particular feature of tree graphs: in the subgraph induced by the first $\ell$ levels of the regular tree, the leaves constitute a non-vanishing fraction of the vertices as $\ell \to \infty$. They suggested that mean-field theory ought instead to be defined by considering graphs that are not themselves trees, but “look like regular trees” in the neighborhood of a typical vertex (which fails for the depth-$\ell$ subtree of the regular tree)—the canonical example being the (uniformly) random $d$-regular graph ensemble. This approach allows one to reconcile discrepancies in several known cases. In particular, spin glasses on random regular graphs are expected to exhibit replica symmetry breaking with features analogous to the SK model (see [MP01] and [MM09, Ch. 17]).

Let us also mention that the study of statistical mechanics models on locally tree-like graphs has attracted renewed interest because of the connection with random combinatorial problems, such as $k$-\textit{sat} and graph coloring. Statistical physicists were indeed able to compute threshold locations for these models by analyzing suitable Gibbs measures on locally tree-like structures [MPZ02,KMR+07,MM09]. Rigorous verification of these predictions is an outstanding mathematical challenge.

In this paper we consider the existence and value of the \textit{free energy density} (asymptotic free energy per spin)
\begin{equation}
\phi \equiv \lim_{n \to \infty} \phi_n \equiv \lim_{n \to \infty} n^{-1} \mathbb{E}_n[\log Z_n], \quad Z_n \equiv Z_{G_n}(\psi),
\end{equation}