On density of subgraphs of halved cubes

In memory of Michel Deza

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Abstract. Let $S$ be a family of subsets of a set $X$ of cardinality $m$ and VC-dim$(S)$ be the Vapnik-Chervonenkis dimension of $S$. Haussler, Littlestone, and Warmuth (Inf. Comput., 1994) proved that if $G_1(S) = (V,E)$ is the subgraph of the hypercube $Q_m$ induced by $S$ (called the 1-inclusion graph of $S$), then $|E| \leq \text{VC-dim}(S)$. Haussler (J. Combin. Th. A, 1995) presented an elegant proof of this inequality using the shifting operation.

In this note, we adapt the shifting technique to prove that if $S$ is an arbitrary set family and $G_{1,2}(S) = (V,E)$ is the 1,2-inclusion graph of $S$ (i.e., the subgraph of the square $Q^2_m$ of the hypercube $Q_m$ induced by $S$), then $|E| \leq \binom{d}{2}$, where $d := c\text{VC-dim}^*(S)$ is the clique-VC-dimension of $S$ (which we introduce in this paper). The 1,2-inclusion graphs are exactly the subgraphs of halved cubes and comprise subgraphs of Johnson graphs as a subclass.

1. INTRODUCTION

Let $S$ be a family of subsets of a set $X$ of cardinality $m$ and VC-dim$(S)$ be the Vapnik-Chervonenkis dimension of $S$. Haussler, Littlestone, and Warmuth [19, Lemma 2.4] proved that if $G_1(S) = (V,E)$ is the subgraph of the hypercube $Q_m$ induced by $S$ (called the 1-inclusion graph of $S$), then the following fundamental inequality holds: $|E| \leq \text{VC-dim}(S)$. They used this inequality to bound the worst-case expected risk of a prediction model of learning of concept classes $S$ based on the bounded degeneracy of their 1-inclusion graphs. Haussler [18] presented an elegant proof of this inequality using the shifting (push-down) operation. 1-Inclusion graphs have many other applications in computational learning theory, for example, in sample compression schemes [21]. They are exactly the induced subgraphs of hypercubes and in graph theory they have been studied under the name of cubical graphs [14]. Finding a densest $n$-vertex subgraph of the hypercube $Q_m$ (i.e., an $n$-vertex subgraph $G$ of $Q_m$ with the maximum number of edges) is equivalent to finding an $n$-vertex subgraph $G$ of $Q_m$ with the smallest edge-boundary (the number of edges of $Q_m$ running between $V$ and its complement in $Q_m$). This is the classical edge-isoperimetric problem for hypercubes [3][17]. Harper [16] nicely characterized the solutions of this problem: for any $n$, this is the subgraph of the hypercube induced by the initial segment of length $n$ of the lexicographic numbering of the vertices of the hypercube. One elegant way of proving this result is using compression [17].

Generalizing the density inequality $|E| \leq \text{VC-dim}(S)$ of [18][19] to more general classes of graphs is an interesting and important problem. In the current paper, we present a density result for 1,2-inclusion graphs $G_{1,2}(S)$ of arbitrary set families $S$. The 1,2-inclusion graphs are the subgraphs of the square $Q^2_m$ of the hypercube $Q_m$ and they are exactly the subgraphs of the halved cube $\frac{1}{2}Q_{m+1}$ (Johnson graphs and their subgraphs constitute an important subclass). Since 1,2-inclusion graphs may contain arbitrary large cliques for constant VC-dimension, we have to adapt the definition of classical VC-dimension to capture this phenomenon. For this purpose, we introduce the notion of clique-VC-dimension $c\text{VC-dim}^*(S)$ of any set family $S$. Here is the main result of the paper:
Let \( S \) be an arbitrary set family of \( 2^X \) with \(|X| = m\), let \( d = \text{cVC-dim}^*(S) \) be the clique-VC-dimension of \( S \) and \( G_{1,2}(S) = (V,E) \) be the 1,2-inclusion graph of \( S \). Then \( \frac{|E|}{|V|} \leq \binom{d}{2} \).

2. Related work

2.1. Haussler’s proof of the inequality \( \frac{|E|}{|V|} \leq \text{VC-dim}(S) \). We briefly review the notion of VC-dimension and the shifting method of \cite{18} of proving the inequality \( \frac{|E|}{|V|} \leq \text{VC-dim}(S) \) (the original proof of \cite{19} was by induction on the number of sets). In the same vein, see Harper’s proof \cite[Chapter 3]{17} of the isoperimetric inequality via compression. We will use the shifting method in the proof of Theorem 1.

Let \( S \) be a family of subsets of a set \( X = \{e_1, \ldots, e_m\} \); \( S \) can be viewed as a subset of vertices of the \( m \)-dimensional hypercube \( Q_m \). Denote by \( G_1(S) \) the subgraph of \( Q_m \) induced by the vertices of \( Q_m \) corresponding to the sets of \( S; G_1(S) \) is called the 1-inclusion graph of \( S \) \cite{18,19}. Vice-versa, for any subgraph \( G \) of \( Q_m \) there exists a family of subsets \( S \) of \( 2^X \) such that \( G \) is the 1-inclusion graph of \( S \). A subset \( Y \) of \( X \) is shattered by \( S \) if for all \( Y' \subseteq Y \) there exists \( S \in S \) such that \( S \cap Y = Y' \). The Vapnik-Chervonenkis’s dimension \cite{18} \( \text{VC-dim}(S) \) of \( S \) is the cardinality of the largest subset of \( X \) shattered by \( S \).

**Theorem 2** \cite{18,19}. If \( G := G_1(S) = (V,E) \) is the 1-inclusion graph of a set family \( S \subseteq 2^X \) with \( \text{VC-dim} \ VC-dim(S) = d \), then \( \frac{|E|}{|V|} \leq d \).

For a set family \( S \subseteq 2^X \), the shifting (push down or stabilization) operation \( \varphi_e \) with respect to an element \( e \in X \) replaces every set \( S \) of \( S \) such that \( S \setminus \{e\} \notin S \) by the set \( S \setminus \{e\} \). Denote by \( \varphi_e(S) \) the resulting set family and by \( G' = G_1(\varphi_e(S)) = (V',E') \) the 1-inclusion graph of \( \varphi_e(S) \). Haussler \cite{18} proved that the shifting map \( \varphi_e \) has the following properties:

1. \( \varphi_e \) is bijective on the vertex-sets: \(|V| = |V'|\),
2. \( \varphi_e \) is increasing the number of edges: \(|E| \leq |E'|\),
3. \( \varphi_e \) is decreasing the VC-dimension: \( \text{VC-dim}(S) \geq \text{VC-dim}(\varphi_e(S)) \).

Harper \cite[p.28]{17} called Steiner operations the set-maps \( \varphi : 2^X \rightarrow 2^X \) satisfying (1), (2), and the following condition:

4. \( S \subseteq T \) implies \( \varphi(S) \subseteq \varphi(T) \).

He proved that the compression operation defined in \cite[Subsection 3.3]{17} is a Steiner operation. Note that \( \varphi_e \) satisfies (4) (but is defined only on \( S \)).

After a finite sequence of shiftings, any set family \( S \) can be transformed into a set family \( S^* \), such that \( \varphi_e(S^*) = S^* \) holds for any \( e \in X \). The resulting set family \( S^* \), a complete shifting of \( S \), is downward closed (i.e., is a simplicial complex). Consequently, the 1-inclusion graph \( G_1(S^*) \) of \( S^* \) is a bouquet of cubes, i.e., a union of subcubes of \( Q_m \) with a common origin \( \varnothing \). Let \( G^* = G_1(S^*) = (V^*,E^*) \) and \( d^* = \text{VC-dim}(S^*) \). Since all shiftings satisfy the conditions (1)-(3), we conclude that \(|V^*| = |V|, |E^*| \geq |E|, \) and \( d^* \leq d \). Therefore, to prove the inequality \( \frac{|E|}{|V|} \leq d \) it suffices to show that \( \frac{|E^*|}{|V^*|} \leq d^* \). Haussler deduced it from Sauer’s lemma \cite{26}, however it is easy to prove this inequality directly, by bounding the degeneracy of \( G^* \). Indeed, let \( v_0 \) be the vertex of \( G^* \) corresponding to the origin \( \varnothing \) and let \( v \) be a furthest from \( v_0 \) vertex of \( G^* \). Then \( v_0 \) and \( v \) span a maximal cube of \( G^* \) (of dimension \( \leq d^* \)) and \( v \) belongs only to this maximal cube of \( G^* \). Therefore, if we remove \( v \) from \( G^* \), we will also remove at most \( d^* \) edges of \( G^* \) and the resulting graph will be again a bouquet of cubes \( G^- = (V^-,E^-) \) with one less vertex and dimension \( \leq d^* \). Therefore, we can apply the induction hypothesis to this bouquet \( G^- \) and deduce that \(|E^-| \leq |V^-|d^* \). Consequently, \(|E^*| \leq d^* + |E^-| \leq d^* + (|V^*| - 1)d^* = |V^*|d^* \).
To extend Haussler’s proof to subgraphs of halved cubes (and, equivalently, to subgraphs of squares of cubes), we need to appropriately define the shifting operation and the notion of VC-dimension, that satisfy the conditions (1)-(3). Additionally, the degeneracy of the 1,2-inclusion graph of the final shifted family must be bounded by a function of the VC-dimension. We will use the shifting operation with respect to pairs of elements (and not to single elements) and the notion of clique-VC-dimension instead of VC-dimension.

2.2. Other results. The inequality of Haussler et al. [19] as well as the notion of VC-dimension and Sauer lemma have been subsequently extended to subgraphs of Hamming graphs (i.e., from binary alphabets to arbitrary alphabets); see [20][23][25]. Cesa-Bianchi and Haussler [6] presented a graph-theoretical generalization of the Sauer lemma for the m-fold $F^m = F \times \cdots \times F$ Cartesian products of arbitrary undirected graphs $F$. In [9], we defined a notion of VC-dimension for subgraphs of Cartesian products of arbitrary connected graphs (hypercubes are Cartesian products of $K_2$) and we established a density result $\frac{|\mathcal{F}|}{|V|} \leq \text{VC-dim}(G) \cdot \alpha(H)$ for subgraphs $G$ of Cartesian products of graphs not containing a fixed graph $H$ as a minor ($\alpha(H)$ is a constant such that any graph not containing $H$ as a minor has density at most $\alpha(H)$; it is well known [12] that if $r := |V(H)|$, then $\alpha(H) \leq c r \sqrt{\log r}$ for a universal constant $c$).

For edge- and vertex-isoperimetric problems in Johnson graphs (which are still open problems), some authors [1][11] used a natural pushing to the left (or switching, or shifting) operation. Let $\mathcal{S}$ consists only of sets of size $r$. Given an arbitrary total order $e_1, \ldots, e_m$ of the elements of $X$ and two elements $e_i < e_j$, in the pushing to the left of $\mathcal{S}$ with respect to the pair $e_i, e_j$ each set $S$ of $\mathcal{S}$ containing $e_j$ and not containing $e_i$ is replaced by the set $S \setminus \{e_j\} \cup \{e_i\}$ if $S \setminus \{e_j\} \cup \{e_i\} \notin \mathcal{S}$. This operation preserves the size of $\mathcal{S}$, the cardinality $r$ of the sets and do not decrease the number of edges, but the degeneracy of the final graph is not easy to bound.

Bousquet and Thomassé [4] defined the notions of 2-shattering and 2VC-dimension and established the Erdős-Pósa property for the families of balls of fixed radius in graphs with bounded 2VC-dimension. These notions have some similarity with our concepts of c-shattering and clique-VC-dimension because they concern shattering not of all subsets but only of a certain pattern of subsets (of all pairs). Recall from [4] that a set family $\mathcal{S}$ 2-shatters a set $Y$ if for any 2-set $\{e_i, e_j\}$ of $Y$ there exists $S \in \mathcal{S}$ such that $Y \cap S = \{e_i, e_j\}$; the 2VC-dimension of $\mathcal{S}$ is the maximum size of a 2-shattered set.

Halved cubes and Johnson graphs host several important classes of graphs occurring from metric graph theory [2]: basis graphs of matroids are isometric subgraphs of Johnson graphs [22] and basis graphs of even $\Delta$-matroids are isometric subgraphs of halved cubes [7]. More general classes are the graphs isometrically embeddable into halved cubes and Johnson graphs. Similarly to Djoković’s characterization of isometric subgraphs of hypercubes [13], isometric subgraphs of Johnson graphs have been characterized in [8] (the problem of characterizing isometric subgraphs of halved cubes has been raised in [10] and is still open). Shpectorov [27] proved that the graphs admitting an isometric embedding into an $\ell_1$-space are exactly the graphs which admit a scale embedding into a hypercube and he proved that such graphs are exactly the graphs which are isometric subgraphs of Cartesian products of octahedra and of isometric subgraphs of halved cubes. For a presentation of most of these results, see the book by Deza and Laurent [10].

3. Preliminaries

3.1. Degeneracy. All graphs $G = (V, E)$ occurring in this note are finite, undirected, and simple. The degeneracy of $G$ is the minimal $k$ such that there exists a total order $v_1, \ldots, v_n$ of vertices of $G$ such that each vertex $v_i$ has degree at most $k$ in the subgraph of $G$ induced by
For any set family \( S \) of an even set family \( A \) and two sets \( A, B \) are adjacent in \( Q_m \) iff \( |A\Delta B| = 1 \). The \textit{halved cube} \( \frac{1}{2}Q_m \) has the subsets of \( S \) of even cardinality as vertices and two such vertices \( A, B \) are adjacent in \( \frac{1}{2}Q_m \) iff \( |A\Delta B| = 2 \) (one can also define halved cubes for subsets of odd size). Equivalently, the halved cube \( \frac{1}{2}Q_m \) is the square \( Q^2_{m-1} \) of the hypercube \( Q_{m-1} \), i.e., the graph formed by connecting pairs of vertices of \( Q_{m-1} \) whose distance is at most two in \( Q_{m-1} \). For an integer \( r > 0 \), the \textit{Johnson graph} \( J(r,m) \) has the subsets of \( S \) of size \( r \) as vertices and two such vertices \( A, B \) are adjacent in \( J(r,m) \) iff \( |A\Delta B| = 2 \). All Johnson graphs \( J(r,m) \) are (isometric) subgraphs of the corresponding halved cube \( \frac{1}{2}Q_m \). Notice also that the halved cube \( \frac{1}{2}Q_m \) and the Johnson graph \( J(r,m) \) are scale 2 embedded in the hypercube \( Q_m \).

Let \( S \) be a family of subsets of a set \( X = \{e_1, \ldots, e_m\} \). The \textit{1,2-inclusion graph} \( G_{1,2}(S) \) of \( S \) is the graph having \( S \) as the vertex-set and in which two vertices \( A, B \) are adjacent iff \( 1 \leq |A\Delta B| \leq 2 \), i.e., \( G_{1,2}(S) \) is the subgraph of the square \( Q^2_m \) of \( Q_m \) induced by \( S \). The graph \( G_{1,2}(S) \) comprises all edges of the 1-inclusion graph \( G_1(S) \) of \( S \) and of the subgraphs of the halved cubes induced by even and odd sets of \( S \). The latter edges of \( G_{1,2}(S) \) are of two types: \textit{vertical edges} \( SS' \) arise from sets \( S, S' \) such that \( |S| = |S'| + 2 \) or \( |S'| = |S| + 2 \) and \textit{horizontal edges} \( SS' \) arise from sets \( S, S' \) such that \( |S| = |S'| \).

If all sets of \( S \) have even cardinality, then we will call \( S \) an \textit{even set family}; in this case, the 1,2-inclusion graph \( G_{1,2}(S) \) coincides with the subgraph of the halved cube \( \frac{1}{2}Q_m \) induced by \( S \). Since \( Q^2_m \) is isomorphic to \( \frac{1}{2}Q_{m+1} \), any 1,2-inclusion graph is an induced subgraph of a halved cube. More precisely, any set family \( S \) of \( X \) can be lifted to an even set family \( S^+ \) of \( X \cup \{e_{m+1}\} \) in such a way that the 1,2-inclusion graphs of \( S \) and \( S^+ \) are isomorphic: \( S^+ \) consists of all sets of even size of \( S \) and of all sets of odd size of \( S \) to which the element \( e_{m+1} \) was added. The proof of the following lemma is straightforward:

**Lemma 1.** For any set family \( S \), the lifted family \( S^+ \) is an even set family and the 1,2-inclusion graphs \( G_{1,2}(S) \) and \( G_{1,2}(S^+) \) are isomorphic.

3.3. **Pointed set families and pointed cliques.** We will call a set family \( S \) a \textit{pointed set family} if \( \emptyset \in S \). Any set family \( S \) can be transformed into a pointed set family by the operation of twisting. For a set \( A \in S \), let \( S \Delta A := \{S \Delta A : S \in S\} \) and say that \( S \Delta A \) is obtained from \( S \) by applying a \textit{twisting} with respect to \( A \). Note that a twisting is a bijection between \( S \) and \( S \Delta A \) mapping the set \( A \) to \( \emptyset \) (and therefore \( S \Delta A \) is a pointed set family). Notice that any twisting of an even set family \( S \) is an even set family. As before, let \( G_1(S) \) denote the 1-inclusion graph of \( S \). The following properties of twisting are well-known and easy to prove:

**Lemma 2.** For any \( S \subseteq 2^X \) and any \( A \subseteq X \), \( G_1(S \Delta A) \simeq G_1(S) \) and \( \text{VC-dim}(S \Delta A) = \text{VC-dim}(S) \).

Analogously to the proof of the first assertion of Lemma 2, one can easily show that:

**Lemma 3.** For any set family \( S \subseteq 2^X \) and any \( A \subseteq X \), \( G_{1,2}(S \Delta A) \simeq G_{1,2}(S) \).

We will say that a clique \( C \) of \( \frac{1}{2}Q_m \) is a \textit{pointed clique} if \( C \) is a pointed set family.

**Lemma 4.** By a twisting, any clique \( C \) of \( \frac{1}{2}Q_m \) can be transformed into a pointed clique.

\textit{Proof.} Let \( C \) be a clique of \( \frac{1}{2}Q_m \). Let \( A \) be a set of maximal size which is a vertex of \( C \). Then the twisting of \( C \) with respect to \( A \) maps \( C \) into a pointed clique \( C \Delta A \) of \( \frac{1}{2}Q_m \): indeed, if \( C', C'' \)
Lemma 5. Any pointed maximal clique $C$ of a halved cube $\frac{1}{2}Q_m$ is (a) a sporadic 4-clique of the form $\{\emptyset, \{e_i, e_j\}, \{e_i, e_k\}, \{e_j, e_k\}\}$ for arbitrary elements $e_i, e_j, e_k \in X$, or (b) a clique of size $m$ of the form $\{\emptyset\} \cup \{\{e_i, e_j\} : e_j \in X \setminus \{e_i\}\}$ for an arbitrary but fixed element $e_i \in X$. \hfill \square

Proof. Since $C$ is a pointed clique, $\emptyset$ is a vertex of $C$, denote it $C_0$. All other neighbors of $C_0$ in $\frac{1}{2}Q_m$ are sets of the form $\{e_i, e_j\}$ with $e_i, e_j \in X$, i.e., the neighborhood of $C_0$ in the halved cube $\frac{1}{2}Q_m$ is the line-graph of the complete graph $K_m$ having $X$ as the vertex-set. In particular, the clique $C_0 := C \setminus \{C_0\}$ corresponds to a set of pairwise incident edges of $K_m$. It can be easily seen that this set of edges defines either a triangle or a star of $K_m$. Indeed, pick an edge $e_ie_j$ of $K_m$ corresponding to a pair $\{e_i, e_j\} \in C_0$. If the respective set of edges is not a star, then necessarily $C_0$ contains two pairs of the form $\{e_i, e_k\}$ and $\{e_j, e_l\}$, both different from $\{e_i, e_j\}$. But then $k = l$, otherwise the edges $e_ie_k$ and $e_je_l$ would not be incident. Thus $C_0$ contains the three pairs $\{e_i, e_j\}, \{e_i, e_k\}$, and $\{e_j, e_k\}$. If $C_0$ contains yet another pair, then this pair will be necessarily disjoint from one of the three previous pairs, a contradiction. Thus in this case, $C = \{\emptyset, \{e_i, e_j\}, \{e_i, e_k\}, \{e_j, e_k\}\}$. Otherwise, if the respective set of edges is a star with center $e_i$, then $C_0$ is a clique of size $m - 1$ of the form $\{\{e_i, e_j\} : e_j \in X \setminus \{e_i\}\}$. \hfill \square

We describe now the structure of pointed cliques in halved cubes.

4. THE CLIQUE-VC-DIMENSION

As we noticed above, the classical VC-dimension of set families cannot be used to bound the density of their 1,2-inclusion graphs. Indeed, the 1,2-inclusion graph of the set family $S_0 := \{\{e_j\} : e_j \in X\}$ is a complete graph, while the VC-dimension of $S_0$ is 1 (notice also that the 2VC-dimension of $S_0$ is 0).

We will define a notion that is more appropriate for this purpose, which we will call clique-VC-dimension. The idea is to use the form of pointed cliques of $\frac{1}{2}Q_m$ established above and to shatter them. In view of Lemma 1, it suffices to define the clique-VC-dimension for even set families. First we present a generalized definition of classical shattering.

Let $X = \{e_1, \ldots, e_m\}$ and $S \subseteq 2^X$. Let $Y$ be a subset of $X$. Denote by $Q[Y]$ the subcube of $Q_m$ consisting of all subsets of $Y$. Analogously, for two sets $Y'$ and $Y$ such that $Y' \subset Y$, denote by $Q[Y', Y]$ the smallest subcube of $Q_m$ containing the sets $Y'$ and $Y$: $Q[Y', Y] = \{Z \subset X$:
an extension containing a surjective function i.e., pairs of the form \{e, f\} ∈ \mathcal{S}

and let \(X\) be a family of 2-sets of \(\mathcal{S}\). We will say that a pair \((e, Y)\) is c-shattered by \(\mathcal{S}\) if each 2-set \(Y' \subseteq Z \subseteq Y\) is c-shattered by \(\mathcal{S}\) and \(Y' \subseteq Z\) contains a set \(e\) and all the 2-sets of \(P(e, Y)\). For simplicity, we will denote this cube by \(Q(e, Y)\).

We will say that a pair \((e_i, Y)\) with \(Y \subseteq X\) and \(e_i \notin Y\) is c-shattered by \(\mathcal{S}\) if there exists a surjective function \(f : \pi_{Q(e_i, Y)}(\mathcal{S}) \to P(e_i, Y)\) such that for any \(S \in \pi_{Q(e_i, Y)}(\mathcal{S})\) the inclusion \(f(S) \subseteq S\) holds. In other words, \((e_i, Y)\) is c-shattered by \(\mathcal{S}\) if each 2-set \(\{e_i, e_j\} \in P(e_i, Y)\) admits an extension \(S_j \in \pi_{Q(e_i, Y)}(\mathcal{S})\) such that \(\{e_i, e_j\} \subseteq S_j\) and for any two 2-sets \(\{e_i, e_j\}, \{e_i, e_{j'}\} \in P(e_i, Y)\) the sets \(S_j\) and \(S_{j'}\) are distinct. Since \(\emptyset \in \mathcal{S}\), the empty set \(\emptyset\) is always shattered by \(\mathcal{S}\).

For a pointed even set family \(\mathcal{S}\), the clique-VC-dimension is

\[c\text{-VC-dim}(\mathcal{S}) := \max\{|Y| + 1 : Y \subseteq X \text{ and } \exists e_i \in X \setminus Y \text{ such that } (e_i, Y) \text{ is c-shattered by } \mathcal{S}\}\]

We continue with some simple examples of clique-VC-dimension:

\[\text{Example 1: } \mathcal{S} = \{\emptyset, X\}\]

\[\text{Example 2: } \mathcal{S} = \{\emptyset, \{e\}, X\}\]
Example 1. For set family \( S_0 = \{ \{ e_j \} : e_j \in X \} \) introduced above, let \( S_0^+ = \{ \{ e_j, e_{m+1} \} : e_j \in X \} \) be the lifting of \( S_0 \) to an even set family. For an arbitrary (but fixed) element \( e_i \), let \( S_1 := \{ \emptyset \} \cup \{ e_i, e_j : e_j \neq e_i \} \). Then \( S_1 \) coincides with \( S_0 \cup \{ e_i \} \) and with \( S_0^+ \cup \{ e_i, e_{m+1} \} \). \( S_1 \) is an even set family; its 1,2-inclusion graph is a pointed clique, and \( \text{cVC-dim}(S_1) = 2 + 1 = 3. \)

Example 2. Let \( S_2 = \{ \emptyset, \{ e_1, e_2 \}, \{ e_1, e_3 \}, \{ e_2, e_3 \} \} \) be the sporadic 4-clique from Lemma \( \ref{lem:sporadic} \).

In this case, one can c-shatter any two of the pairs \( \{ e_1, e_2 \}, \{ e_1, e_3 \}, \{ e_2, e_3 \} \) but not all three. This shows that \( \text{cVC-dim}(S_2) = 2 + 1 = 3. \)

Example 3. For arbitrary even integers \( m \) and \( k \), let \( X \) be a ground set of size \( m + km \) which is the disjoint union of \( m + 1 \) sets \( X_0, X_1, \ldots, X_m \), where \( X_0 = \{ e_1, \ldots, e_m \} \) and \( X_i = \{ e_{i1}, \ldots, e_{ik} \} \) for each \( i = 1, \ldots, m \). Let \( S_3 \) be the pointed even set family consisting of the empty set \( \emptyset \), the set \( X_i \), and for each \( i = 1, \ldots, m \) of all the 2-sets of \( P(e_i, X_i) = \{ \{ e_i, e_{i1} \}, \ldots, \{ e_i, e_{ik} \} \} \) of size \( k + 1 \) and these cliques pairwise intersect in a single vertex \( \emptyset \). We assert that \( \text{cVC-dim}(S_3) = k + 2 \).

Indeed, let \( Y \) be the set consisting of \( X_i \) for a given \( i \in \{ 1, \ldots, m \} \) plus the singleton \( \{ e_{(i+1)1} \} \). Then the pair \( (e_i, Y) \) is c-shattered by \( S_3 \). The c-shattering map \( f : \pi_{Q(e_i, Y)}(S_3) \to P(e_i, Y) \) is defined as follows: every 2-set of \( P(e_i, X_i) \subset Q(e_i, Y) \) is in \( S_3 \) and is thus mapped to itself, \( X \cap (Y \cup \{ e_i \}) = Y \cup \{ e_i \} \) is an extension of the remaining 2-set \( \{ e_i, e_{(i+1)1} \} \in Q(e_i, Y) \) and thus \( f(Y \cup \{ e_i \}) := \{ e_i, e_{(i+1)1} \} \). Since \( |Y| = k + 1 \), we showed that \( \text{cVC-dim}(S_3) \geq k + 2 \). On the other hand, \( \text{cVC-dim}(S_3) \leq k + 2 \) because every element \( e \) from \( X \) is in at most \( k + 1 \) sets of \( S_3 \). Therefore, \( \text{cVC-dim}(S_3) = k + 2 \).

4.2. The clique-VC-dimension of even and arbitrary set families. The **clique-VC-dimension** \( \text{cVC-dim}^*(S) \) of an even set family \( S \) is the minimum of the clique-VC-dimensions of the pointed even set families \( S \Delta A \) for \( A \in S \):

\[
\text{cVC-dim}^*(S) := \min\{ \text{cVC-dim}(S \Delta A) : A \in S \}.
\]

The **clique-VC-dimension** \( \text{cVC-dim}^*(S) \) of an arbitrary set family \( S \) is the clique-VC-dimension of its lifting \( S^+ \).

**Remark 1.** A simple analysis shows that for the even set families from Examples \( \ref{ex:sporadic} \), \( \ref{ex:example2} \), \( \ref{ex:example3} \), we have \( \text{cVC-dim}(S_1) = m \), \( \text{cVC-dim}(S_2) = 3 \), and \( \text{cVC-dim}(S_3) = k + 2 \).

**Remark 2.** In fact, the set family \( S_3 \) shows that the maximum degree of a 1,2-inclusion graph \( G_{1,2}(S) \) of an even set family \( S \) can be arbitrarily larger than \( \text{cVC-dim}^*(S) \). Indeed, \( \emptyset \) is the vertex of maximum degree of \( G_{1,2}(S_3) \) and its degree is \( km \).
The family $S_3$ also explains why in the definition of the clique-VC-dimension of $S$ we take the minimum over all $S \Delta A, A \in S$. Consider the twisting of $S_3$ with respect to the set $X \in S_3$. Then one can see that $\text{cVC-dim}(S_3 \Delta X) \geq (m-1)k + 1$. Indeed, $S_3 \Delta X = \{ \emptyset, X \} \cup (\bigcup_{(i,j) \in \{1, \ldots, m\} \times \{1, \ldots, k\}} \{X \setminus \{e_i, e_j\}\})$. Let $Y := \{e_{ij} : i \in \{1, \ldots, m\} - 1\}$ and $j \in \{1, \ldots, k\}$. We assert that $(e_1, Y)$ is c-shattered by $S_3 \Delta X$. We set $S'_3 := \pi_{Q(e_1, Y)}(S_3 \Delta X), S_{ij} := X \setminus \{e_i, e_j\}$, and $S'_{ij} := \pi_{Q(e_1, Y)}(S_{ij})$. Let $f : S'_3 \rightarrow P(e_1, Y)$ be such that for all $i \in \{2, \ldots, m\}$ and $j \in \{1, \ldots, k\}$, we have $f(S'_{ij}) = \{e_1, e_{(i-1)j}\}$. Clearly, every $\{e_1, e_{(i-1)j}\}$ has an extension $S'_{ij}$ with a non-empty fiber $(S'_{ij} \in \mathcal{F}(S'_{ij}))$, and for all $S_{il} \neq S_{ij}$, we have $S'_{il} \neq S'_{ij}$, hence $f$ is a surjection. Therefore, $(e_1, Y)$ is c-shattered. Since $|Y| = (m-1)k$, whence $\text{cVC-dim}(S_3 \Delta X) \geq (m-1)k + 1$.

5. PROOF OF THEOREM

After the preparatory work done in previous three subsections, here we present the proof of our main result. We start the proof by defining the double shifting ($d$-shifting) as a natural adaptation of the shifting to pointed even families. We show that, similarly to classical shifting operation, $d$-shifting satisfies the conditions (1)-(3) and that the result of a complete sequence of $d$-shiftings is a bouquet of halved cubes (which is a particular pointed even set family). We show that the degeneracy of the 1,2-inclusion graph of such a bouquet $\mathcal{B}$ is bounded by $\binom{d}{2}$, where $d := \text{cVC-dim}(\mathcal{B})$. We conclude the proof of the theorem by considering arbitrary even set families $S$ and applying the previous arguments to the pointed family $S \Delta A$, where $A$ is a set of $S$ such that $\text{cVC-dim}(S \Delta A) = \text{cVC-dim}^{*}(S)$.

5.1. Double shifting of pointed even families. For a pointed even set family $S \subseteq 2^X$, the double shifting (d-shifting for short) with respect to a 2-set $\{e_i, e_j\} \subseteq X$ is a map $\varphi_{ij} : S \rightarrow 2^X$ which replaces every set $S$ of $S$ such that $\{e_i, e_j\} \subseteq S$ and $S \setminus \{e_i, e_j\} \notin S$ by the set $S \setminus \{e_i, e_j\}$:

$$\varphi_{ij} : S \rightarrow 2^X$$

$$S \rightarrow \begin{cases} S \setminus \{e_i, e_j\}, & \text{if } \{e_i, e_j\} \subseteq S \text{ and } S \setminus \{e_i, e_j\} \notin S \\ S, & \text{otherwise.} \end{cases}$$

Proposition 1. Let $S \subseteq 2^X$ be a pointed even set family, let $\{e_i, e_j\} \subseteq X$ be a 2-set, and let $G_{1,2}(S) = G = (V, E)$ and $G_{1,2}(\varphi_{ij}(S)) = G' = (V', E')$ be the subgraphs of the halved cube induced by $S$ and $\varphi_{ij}(S)$, respectively. Then $|V| = |V'|$ and $|E| \leq |E'|$ hold.

Proof. The fact that a $d$-shifting $\varphi_{ij}$ preserves the number of vertices of an induced subgraph of halved cube immediately follows from the definition. Therefore we only need to show that $\varphi_{ij}$ cannot decrease the number of edges, i.e., that there exists an injective map $\psi_{ij} : E \rightarrow E'$. We will call an edge $SS'$ of $G$ stable if $\varphi_{ij}(S) = S$ and $\varphi_{ij}(S') = S'$ hold and shiftable otherwise. For each stable edge $SS'$ we will set $\psi_{ij}(SS') := SS'$.

Now, pick any shiftable edge $SS'$ of $G$. Notice that in this case $\{e_i, e_j\} \subseteq S$ or $\{e_i, e_j\} \subseteq S'$. To define $\psi_{ij}(SS')$, we distinguish two cases depending on whether $\{e_i, e_j\}$ is a subset of only one of the sets $S, S'$ or of both of them.

Case 1'. $\{e_i, e_j\} \subseteq S$ and $\{e_i, e_j\} \not\subseteq S'$ (the case $\{e_i, e_j\} \subseteq S'$ and $\{e_i, e_j\} \not\subseteq S$ is similar).

Since $\{e_i, e_j\} \not\subseteq S'$, necessarily $\varphi_{ij}(S') = S'$. Since $SS'$ is shiftable, $\varphi_{ij}(S) \not= S$, i.e., $\varphi_{ij}(S) = S \setminus \{e_i, e_j\} = Z$. We consider two cases depending on whether one of the elements $e_i$ or $e_j$ belongs to $S'$ or not.

Subcase 1'.1. $e_i \in S'$ and $e_j \not\in S'$ (the case $e_j \in S'$ and $e_i \not\in S'$ is similar). In this case, there is an element $e_k \in X$ such that $S \Delta S' = \{e_k, e_k\}$. Observe that $S \not\subseteq S'$ since $e_j \not\in S'$ and $e_j \in S$. Hence either $S' \subseteq S$ or there exists $A \subset X$ such that $S' = A \cup \{e_k\}$ and $S = A \cup \{e_j\}$. In the
former case, we have $S = S' \cup \{e_j, e_k\}$, $Z = S' \cup \{e_k\} \setminus \{e_i\}$, and $Z \Delta S' = \{e_i, e_k\}$. In the later case, we have $Z = A \setminus \{e_i\}$ and $Z \Delta S' = \{e_i, e_k\}$. In both cases, $|Z \Delta S'| = 2$ and $Z S' \in E'$. We set $\psi_{ij}(SS') := Z S'$.

**Subcase 1'.** $e_i \notin S'$ and $e_j \notin S'$. Then $S \Delta S' = \{e_i, e_j\}$ and $S \setminus \{e_i, e_j\} = Z = S'$. We obtain a contradiction that $SS'$ is shiftable (i.e., $Z = \varphi_{ij}(S)$ cannot be in $S$).

**Case 2'.** $\{e_i, e_j\} \subseteq S$ and $\{e_i, e_j\} \subseteq S'$.

Set $Z := S \setminus \{e_i, e_j\}$ and $Z' := S' \setminus \{e_i, e_j\}$. Then both sets $Z, Z'$ belong to $\varphi_{ij}(S)$ and $Z Z'$ defines an edge of $G'$. Since $SS'$ is shiftable, at least one of the sets $Z, Z'$ does not belong to $S$.

**Subcase 2'.1.** $Z, Z' \notin S$. Then $\varphi_{ij}(S) = Z$ and $\varphi_{ij}(S') = Z'$ and $Z Z'$ is an edge of $G'$. In this case, we set $\psi_{ij}(SS') := Z Z'$.

**Subcase 2'.2.** $Z \in S$ and $Z' \notin S$ (the case $Z \notin S$ and $Z' \in S$ is similar). Then $\varphi_{ij}(S) = S, \varphi_{ij}(S') = Z'$, and $Z Z'$ is an edge of $G'$ but not of $G$. In this case, we set $\psi_{ij}(SS') := Z Z'$.

It remains to show that the map $\psi_{ij} : E \rightarrow E'$ is injective. Suppose by way of contradiction that $G'$ contains an edge $ZZ'$ for which there exist two distinct edges $SS'$ and $CC'$ of $G$ such that $\psi_{ij}(SS') = \psi_{ij}(CC') = ZZ'$. Since at least one of the edges $SS'$ and $CC'$ is different from $ZZ'$, from the definition of d-shifting we conclude that $ZZ'$ is not an edge of $G$, say $Z' \notin S$. This also implies that $SS'$ and $CC'$ are shiftable edges of $G$.

**Case 1''.** $Z \notin S$.

From the definition of the map $\psi_{ij}$ and since $Z, Z' \notin S$, both edges $SS'$ and $CC'$ are in Subcase 2'. This shows that $Z = S \setminus \{e_i, e_j\}$, $Z' = S' \setminus \{e_i, e_j\}$, and $Z = C \setminus \{e_i, e_j\}$, $Z' = C' \setminus \{e_i, e_j\}$, yielding $S = C$ and $S' = C'$, a contradiction.

**Case 2''.** $Z \in S$.

After an appropriate renaming of the sets $S, S'$ and $C, C'$, we can suppose that $\varphi_{ij}(S) = \varphi_{ij}(C) = Z$ and $\varphi_{ij}(S') = \varphi_{ij}(C') = Z'$. Since $Z' \notin S$, from the definition of the map $\psi_{ij}$, we deduce that $S' = Z' \cup \{e_i, e_j\} = C'$. On the other hand, since $Z \in S$, we have either $S = C = Z$ which contradicts the choice of $SS' \neq CC'$, or $S \setminus \{e_i, e_j\} = C = Z$ (or the symmetric possibility $C \setminus \{e_i, e_j\} = S = Z$) which contradicts the fact that $SS'$ (or $CC'$) is shiftable.

This shows that the map $\psi_{ij} : E \rightarrow E'$ is injective, thus $|E| \leq |E'|$. □

**Lemma 6.** If $\varphi_{ij}$ is a d-shifting of a pointed even family $S \subset 2^X$, then $cVC\dim(\varphi_{ij}(S)) \leq cVC\dim(S)$.

**Proof.** Let $(e, Y)$ be c-shattered by $S_{ij} := \varphi_{ij}(S)$ (recall that $Y \subset X$ and $e \notin Y$). Let $S' := \pi_{Q(e,Y)}(S)$ and $S' := \pi_{Q(e,Y)}(\varphi_{ij}(S))$. By definition of c-shattering, there exists a surjective function $f$ associating every element of $S'_{ij}$ to a 2-set $\{e, e'\} \in P(e, Y)$. We will define a surjective function $g$ from $S'$ to $S'_{ij}$, and derive from $f$ a c-shattering function $f' := f \circ g$ from $S'$ to $P(e, Y)$. Let $S_{e'} \subseteq S_{ij}$ be a set such that $f(S_{e'}) = \{e, e'\}$. If $S_{e'} \in S'$, then the 2-set $\{e, e'\}$ also has an extension in $S'$ and we can set $g(S_{e'}) := S_{e'}$. If $S_{e'} \notin S'$, it means that there exists a set $S \in S$ such that $S \neq \varphi_{ij}(S)$ and $\varphi_{ij}(S)$ is in the fiber $F(S_{e'})$ of $S_{e'}$ with respect to $\pi_{Q(e,Y)}$ in $S_{ij}$. The set $S$ is in the fiber $F(S')$ of some set $S' \in S'$ with respect to $\pi_{Q(e,Y)}$. Since $\varphi_{ij}(S) \subseteq S$, we have $S_{e'} \subseteq S'$ and $S' \subseteq S'$ is an extension of the 2-set $\{e, e'\}$. We set $g(S') := S_{e'}$. Moreover, for every set $S' \subseteq S' \setminus S_{ij}$, there is a set $S \in F(S')$ such that $\varphi_{ij}(S) \neq S$. In this case, there is a set $S_{e'} \subseteq S_{ij}$ such that $\varphi_{ij}(S) \subseteq F(S_{e'})$. We set $g(S') := S_{e'}$. We have $S_{e'} \subseteq S'$ since $\varphi_{ij}(S) \subseteq S$.

The function $g$ is surjective by definition and maps every set of $S'$ either on itself or on a subset of it. Since $f$ is a c-shattering function, so is $f' := f \circ g$ and $(e, Y)$ is c-shattered by $S$. 9
Figure 4. To the proof of Lemma 6.

Consequently, we have $cVC\text{-dim}(\varphi_{ij}(S)) \leq cVC\text{-dim}(S)$ since every $(e, Y)$ c-shattered by $S_{ij}$ is also c-shattered by $S$.

5.2. Bouquets of halved cubes. A bouquet of cubes (called usually a downward closed family or a simplicial complex) is a set family $B \subseteq 2^X$ such that $S \in B$ and $S' \subseteq S$ implies $S' \in B$. Obviously $B$ is a pointed family. Note that any bouquet of cubes $B$ is the union of all cubes of the form $Q[\varnothing, S]$, where $S$ is an inclusion-wise maximal subset of $B$.

A bouquet of halved cubes is an even set family $B \subseteq 2^X$ such that for any $S \in B$, any subset $S'$ of $S$ of even size is included in $B$. In other words, a bouquet of halved cubes $B$ is the union of all halved cubes spanned by $\varnothing$ and inclusion-wise maximal subsets $S$ of $B$.

Lemma 7. After a finite number of d-shiftings, any pointed even set family $S$ of $2^X$ can be transformed into a bouquet of halved cubes.

Proof. Let $S_0, S_1, S_2, \ldots$ be a sequence of even set families such that $S_0 = S$ and, for any $i \geq 1$, $S_i$ was obtained from $S_{i-1}$ by a d-shifting and $S_i \neq S_{i-1}$. This sequence is necessarily finite because each d-shifting strictly decreases the sum of sizes of the sets in the family. Let $S_r$ denote the last family in the sequence. This means that the d-shifting of $S_r$ with respect to any pair of elements of $X$ leads to the same set family $S_r$. Therefore, for any set $S \in S_r$ and for any pair $\{e_j, e_k\} \subseteq S$, the set $S \setminus \{e_j, e_k\}$ belongs to $S_r$, i.e., $S_r$ is a bouquet of halved cubes.

We continue with simple properties of bouquets of halved cubes.

Lemma 8. Let $B \subseteq 2^X$ be a bouquet of halved cubes of clique-VC-dimension $d := cVC\text{-dim}(B)$. Then the following properties hold:

(i) for any element $e_i \in X$, $|\{e_i, e_j\} \in B : e_j \in X \setminus \{e_i\}| \leq d - 1$;
(ii) if $S$ is a set of $B$, then $|S| \leq d$;
(iii) if $S$ is a set of $B$ maximal by inclusion, then $B \setminus \{S\}$ is still a bouquet of halved cubes.

Proof. The inequality $|\{e_i, e_j\} \in S : e_j \in X \setminus \{e_i\}| \leq d - 1$ directly follows from the definition of $cVC\text{-dim}(B)$. The property (iii) immediately follows from the definition of a bouquet of halved cubes. To prove (ii), suppose by way of contradiction that $|S| > d$. Since $B$ is a bouquet of halved cubes, every subset of $S$ of even cardinality belongs to $B$. Therefore, if we pick any $e \in S$ and if we set $Y := S \setminus \{e\}$, then all the 2-sets of the form $\{e, e'\}$ with $e' \in Y$ are subsets of $S$,
and thus are sets of $\mathcal{B}$. Consequently $(e, Y)$ is c-shattered by $\mathcal{B}$. Since $|Y| = |S| - 1 > d - 1$, this contradicts the assumption that $d = c\text{VC-dim}(\mathcal{B})$. 

\section{Degeneracy of bouquets of halved cubes}

In this subsection we prove the following upper bound for degeneracy of 1,2-inclusion graphs of bouquets of halved cubes:

**Proposition 2.** Let $\mathcal{B} \subset 2^X$ be a bouquet of halved cubes of clique-VC-dimension $d := c\text{VC-dim}(\mathcal{B})$, and let $G := G_{1,2}(\mathcal{B})$. Then the degeneracy of $G$ is at most $\binom{d}{2}$.

**Proof.** Let $S$ be a set of maximal size of $\mathcal{B}$. By Lemma\ref{lemma:3}(iii), $\mathcal{B} \setminus \{S\}$ is a bouquet of halved cubes. Thus, it suffices to show that the degree of $S$ in $G$ is upper bounded by $\binom{d}{2}$. From Lemma\ref{lemma:2}(ii), we know that $|S| \leq d$. This implies that $S$ is incident in $G$ to at most $\binom{d}{2}$ vertical edges. Therefore, it remains to bound the number of horizontal edges sharing $S$. The following lemma will be useful for this purpose:

**Lemma 9.** If $|S| = d - k \leq d$, then $S$ is incident in $G$ to at most $(d - k)k$ horizontal edges.

**Proof.** Pick any $s \in S$ and set $Y := S \setminus \{s\}$. For an element $e \in X \setminus S$, let $S^e_s := Y \cup \{e\}$. Notice that such $S^e_s$ are exactly the neighbors of $S$ in $\frac{1}{2}Q_m$ connected by a horizontal edge. Let $X^s := \{e \in X \setminus S : S^e_s \in B\}$.

Pick any element $y \in Y$. Then $y \in S^e_s$ for any $e \in X'$. Since $\mathcal{B}$ is a bouquet of halved cubes, each of the $d - k - 1$ pairs $(e, e')$ with $e', e \in S \setminus \{y\}$ belongs to $\mathcal{B}$ (yielding $P(y, S \setminus \{y\}) \subseteq B$). To each set $S^e_s, e \in X'$, corresponds the unique pair $(y, e)$ and $(y, e) \in B$ because $y, e \in S^e_s$. Therefore $P(y, X') \subseteq B$. Since $|P(y, S \setminus \{y\})| + |P(y, X')| \leq d - 1$ and $|P(y, S \setminus \{y\})| = d - k - 1, |X'| = |P(y, X')|$, we conclude that $|X'| \leq k$. Therefore, for a fixed element $s \in S$, $S$ has at most $k$ neighbors of the form $S^e_s$ with $e \in X'$. Since there are $|S| = d - k$ possible choices of the element $s$, $S$ has at most $(d - k)k$ neighbors of cardinality $|S|$. 

We now continue the proof of Proposition 2. Let $|S| = d - k \leq d$. Then $S$ has $\binom{d-k}{2}$ neighbors of the form $S \setminus \{e, e'\}$ with $e \neq e' \in S$, i.e., $S$ has $\binom{d-k}{2}$ incident vertical edges. It remains to bound the number of neighbors of $S$ of the form $S \setminus \{e\} \cup \{e'\}$ with $e \in S$ and $e' \in X \setminus S$. By Lemma\ref{lemma:9} $S$ has at most $(d - k)k$ such neighbors. Summarizing, $S$ possesses $(d - k)k + \binom{d-k}{2} = \frac{1}{2}(d^2 - d - k^2 + k)$ neighbors in $G$, and this number is maximal for $k = 0$ because

$$\frac{1}{2}(d^2 - d - k^2 + k) = \frac{1}{2}(d^2 - d) - \frac{1}{2}(k^2 - k) = \binom{d}{2} - \binom{k}{2} \leq \binom{d}{2}.$$

Hence, the degree of $S$ in $G$ is at most $\binom{d}{2}$, as asserted.

\section{Proof of Theorem 1.}

First, let $\mathcal{S}$ be an even set family over $X$ with $|X| = m$, $d = c\text{VC-dim}^*(\mathcal{S})$ be the clique-VC-dimension of $\mathcal{S}$, and $G_{1,2}(\mathcal{S}) = (V, E)$ be the 1,2-inclusion graph of $\mathcal{S}$. We have to prove that $\frac{|E|}{|V|} \leq \binom{d}{2} = D$.

Let $A$ be a set of $\mathcal{S}$ such that $c\text{VC-dim}(\mathcal{S}A) = c\text{VC-dim}^*(\mathcal{S}) = d$. By Lemma\ref{lemma:12} $G_{1,2}(\mathcal{S}A) \cong G_{1,2}(\mathcal{S})$. Thus it suffices to prove the inequality $\frac{|E|_{G_{1,2}(\mathcal{S}A)}}{|V|_{G_{1,2}(\mathcal{S}A)}} \leq D$. Consider a complete sequence of d-shiftings of $\mathcal{S}A$ and denote by $(\mathcal{S}A)^*$ the resulting set family. Since $\mathcal{S}A$ is a pointed even set family, applying Lemma\ref{lemma:10} to each d-shifting, we deduce that $c\text{VC-dim}((\mathcal{S}A)^*) \leq c\text{VC-dim}(\mathcal{S}A) = d$. By Lemma\ref{lemma:11}, $(\mathcal{S}A)^*$ is a bouquet of halved cubes, thus, by Proposition 2, the degeneracy of its 1,2-inclusion graph $G^* := G_{1,2}(\mathcal{S}A)^*$ is at most $D$. Therefore, if $G^* = (V^*, E^*)$, then $\frac{|E^*|}{|V^*|} \leq D$ (here we used the fact that the degeneracy of a graph $G = (V, E)$ is an upper bound for the ratio $\frac{|E|}{|V|}$). Applying Proposition 1 to each of the d-shiftings and taking into account that $G_{1,2}(\mathcal{S}A) \cong G_{1,2}(\mathcal{S})$, we conclude that $\frac{|E|}{|V|} \leq \frac{|E^*|}{|V^*|}$.
yielding the required density inequality $\frac{|E|}{|V|} \leq D$ and finishing the proof of Theorem 1 in case of even set families. If $S$ is an arbitrary set family, then $c\text{VC-dim}^*(S) = c\text{VC-dim}^*(S^+)$, where $S^+$ is the lifting of $S$ to an even set family. Since by Lemma 1 $S$ and $S^+$ have isomorphic 1,2-inclusion graphs, the density result for $S$ follows from the density result for $S^+$. This concludes the proof of Theorem 1.

**Example 4.** As in the case of classical VC-dimension and Theorem 2 the inequality from Theorem 1 between the density of 1,2-inclusion graph $G_{1,2}(S)$ and the clique-VC-dimension of $S$ is sharp in the following sense: there exist even set families $S$ such that the degeneracy of $G_{1,2}(S)$ equals to $\binom{d}{2}$. For example, the sporadic clique $S_2$ has clique VC-dimension 3 (see Examples 2 and remark 1), degeneracy 3, and density $\frac{3}{2}$. Notice that $G_{1,2}(S_2)$ is the halved cube $\frac{1}{2}Q_3$. More generally, let $S_1$ be the even set family consisting of all even subsets of an $m$-set $X$. Clearly $d := c\text{VC-dim}^*(S_4) = |X| = m$ and $S_4$ induces the halved cube $\frac{1}{2}Q_m$. We assert that $\frac{1}{2}Q_m$ has degeneracy $\binom{d}{2}$. Indeed, every $S \in S_4$ is incident to $\binom{|X| - |S|}{2}$ supersets of cardinality $|S| + 2$, to $\binom{|S|}{2}$ subsets of cardinality $|S| - 2$, and to $|S|(|X| - |S|)$ sets of cardinality $|S|$. Setting $s := |S|$, we conclude that each set $S$ has degree

$$\frac{(m-s)(m-s-1) + s(s-1)}{2} + s(m-s) = \frac{1}{2}(m^2 - m) = \frac{1}{2}(d^2 - d) = \binom{d}{2}.$$

**Remark 3.** In the following table, for pointed even set families $S_0, S_1, \ldots, S_4$ defined in Examples 1,2 and 4, we present their VC-dimension, the two clique VC-dimensions, the 2VC-dimension, the degeneracy, and the density.

| $S$ | VC-dim | cVC-dim | cVC-dim* | degeneracy | density | 2VC-dim |
|-----|--------|---------|----------|------------|---------|---------|
| $S_0$ | 1 | - | $m$ | $m-1$ | $\frac{m-1}{2}$ | 0 |
| $S_1$ | 1 | $m$ | $m$ | $m-1$ | $\frac{m-1}{2}$ | 2 |
| $S_2$ | 2 | 3 | 3 | 3 | $\frac{3}{2}$ | 3 |
| $S_3$ | 2 | $k+2$ | $k+2$ | $k$ | $\frac{k}{2} + o(1)$ | 2 |
| $S_3 \triangle X$ | 2 | $(m-1)k+1$ | $k+2$ | $k$ | $\frac{k}{2} + o(1)$ | 2 |
| $S_4$ | $m-1$ | $m$ | $m$ | $\frac{m}{2}$ | $\frac{1}{2}\binom{m}{2}$ | $m$ |

6. **Final discussion**

In this note, we adapted the shifting techniques to prove that if $S$ is an arbitrary set family and $G_{1,2}(S) = (V,E)$ is the 1,2-inclusion graph of $S$, then $\frac{|E|}{|V|} \leq \binom{d}{2}$, where $d := c\text{VC-dim}^*(S)$ is the clique-VC-dimension of $S$. The essential ingredients of our proof are Proposition 1 (showing that d-shiftings preserve the number of vertices and do not decrease the number of edges), Lemma 6 (showing that d-shiftings do not increase the clique-VC-dimension), and Proposition 2 (bounding the density of bouquets of halved cubes, resulting from complete d-shiftings), all established for even set families. While Propositions 1 and 2 are not very sensitive to the chosen definition of the clique-VC-dimension (but they require using the definition of 1,2-inclusion graphs as the subgraphs of the halved cube $\frac{1}{2}Q_m$), Lemma 6 strongly depends on how the clique-VC-dimension is defined. For example, this lemma does not hold for the notion of 2VC-dimension of $S$ discussed in Section 2. Notice also that, differently from the classical VC-dimension and similarly to our notion of clique-VC-dimension, 2VC-dimension is not invariant under twistings.
In analogy to 2-shattering and 2VC-dimension, we can define the concepts of star-shattering and star-VC-dimension, which might be useful for finding sharper upper bounds (than those obtained in this paper) for density of 1,2-inclusion graphs. Let \( Y \subset X \) and \( e \notin Y \). We say that a set family \( S \) star-shatters (or s-shatters) the pair \((e,Y)\) if for any \( y \in Y \) there exists a set \( S \in S \) such that \( S \cap (Y \cup \{e\}) = \{e, y\} \). The star-VC-dimension of a pointed set family \( S \) is

\[
\text{sVC-dim}(S) := \max\{|Y| + 1 : Y \subset X \text{ and } \exists e_i \in X \setminus Y \text{ such that } (e_i, Y) \text{ is s-shattered by } S\}.
\]

The difference with c-shattering is that, in the definition of s-shattering, a pair \((e,Y)\) is s-shattered if all 2-sets of \( P(e,Y) \) have non-empty fibers, i.e., if \( P(e,Y) \subseteq \pi_{Q(e,Y)}(S) \). Consequently, any s-shattered pair \((e,Y)\) is c-shattered, thus \text{sVC-dim}(S) \leq \text{cVC-dim}(S)\). Since \text{sVC-dim}(S_{2\Delta X}) = 3 \) and \( G_{1,2}(S_{2\Delta X}) \) contains a clique of size \( k+1 \), \text{sVC-dim}(S) cannot be used directly to bound the density of 1,2-inclusion graphs. We can adapt this notion by taking the maximum over all twistings with respect to sets of \( S \): the star-VC-dimension \text{sVC-dim}^*(S) \) of an arbitrary set family \( S \) is \( \max\{\text{sVC-dim}(S\Delta A) : A \in S\} \). Even if \text{sVC-dim}(S) \leq \text{cVC-dim}(S) \) holds for pointed families, as the following examples show, there are no relationships between \text{cVC-dim}^*(S) \) and \text{sVC-dim}^*(S) \) for even families.

**Example 5.** Let \( X = \{1,2,\ldots,2m-1,2m\} \), where \( m \) is an arbitrary even integer, and let \( S_5 := \{\emptyset\} \cup \{\{i,2i-1,2i\} : i = 1,\ldots,m\} \). The nonempty sets of \( S_5 \) can be viewed as intervals of even length of \( N \) with a common origin. The 1,2-inclusion graph of \( S_5 \) is a path of length \( m \). For any set \( \{1,2,\ldots,2j\} \), the twisted family \( S_5^j := S_5 \Delta \{1,2,\ldots,2j\} \) is the union of the set families \( S' := \{\emptyset, \{2i+1,2i+2\}, \ldots, \{2i+1,2i+2,\ldots,2m\}\} \) and \( S'' := \{\{i,2i-1,2i\}, \ldots, \{2i-1,2i\}\} \). We assert that for any \( i = 1,\ldots,m \), we have \text{sVC-dim}(S_5^i) \leq 3 \) and \text{cVC-dim}(S_5^i) = \max\{i, m-i \} + 1 \). Indeed, for any element \( j \in X \), \( S_5^i \) cannot simultaneously s-shatter two pairs \( \{j,l_1\}, \{j,l_2\} \) with \( j < l_1 < l_2 \) because every set of \( S_5 \) containing \( l_2 \) also contains \( l_1 \). Analogously, \( S_5^i \) cannot s-shatter two pairs \( \{j,l_1\} \) and \( \{j,l_2\} \) with \( l_2 < l_1 < j \). Consequently, if the pair \( (j,Y) \) is s-shattered by \( S_5 \), then \( |Y| \leq 2 \). This shows that \text{sVC-dim}^*(S_5) \leq 3 \).

To see that \text{cVC-dim}(S_5^i) = \max\{i, m-i \} + 1 \), notice that \( S' \) c-shatters the pair \( (2i+1,Y') \) with \( Y' := \{2i+2,2i+4,\ldots,2m\} \) and \( S'' \) c-shatters the pair \( (2i,Y'') \) with \( Y'' := \{1,3,\ldots,2i-1\} \). Since the minimum over all \( i = 1,\ldots,m \) of \( \max\{i, m-i \} + 1 \) is attained for \( i = \frac{m}{2} \), we conclude that \text{cVC-dim}^*(S_5) = \frac{m}{2} + 1 \). Therefore \text{sVC-dim}^*(S) \) can be arbitrarily smaller than \text{cVC-dim}^*(S) \).

**Example 6.** Let \( X = X_1 \cup X_2 \) with \( X_1 = \{e_1,\ldots,e_m\} \) and \( X_2 = \{x_1,\ldots,x_m\} \), and let \( S_6 := \{\emptyset, \{e_1,x_1\}\} \cup \{\{e_1,e_i,x_1,x_i\} : 2 \leq i \leq m\} \). The 1,2-inclusion graph of \( S_6 \) is a star. One can easily see that \text{sVC-dim}(S_6) = m \). On the other hand, for the twisted family \( S_6' := S_6 \Delta \{e_1,x_1\} = \{\emptyset\} \cup \{\{e_i,x_i\} : 1 \leq i \leq m\} \), one can check that \text{cVC-dim}(S_6') = 2 \), showing that \text{cVC-dim}^*(S_6) = 2 \) and \text{sVC-dim}^*(S_6) = m \). Therefore \text{sVC-dim}^*(S) \) can be arbitrarily larger than \text{cVC-dim}^*(S) \).

Therefore, it is natural to ask whether in Theorem 1 one can replace \text{cVC-dim}^*(S) \) by \text{sVC-dim}^*(S) \). However, we were not able to decide the status of the following question:

**Question 1.** Is it true that for any (even) set family \( S \) with the 1,2-inclusion graph \( G_{1,2}(S) = (V,E) \) and star-VC-dimension \( d = \text{sVC-dim}^*(S) \), we have \( \frac{|E|}{|V|} = O(d^2) \)?

The main difficulty here is that a d-shifting may increase the star-VC-dimension, i.e., Lemma 4 does no longer hold. The difference between the s-shattering and c-shattering is that a 2-set \( \{e,y\} \) with \( y \in Y \) can be s-shattered only by a set \( S \in S \) which belongs to the fiber \( F(\{e,y\}) \) (the requirement \( Y \cap S = \{e,y\}\)), while \( \{e,y\} \) can be c-shattered by a set \( S \) if \( S \) just includes

\[^1\text{As noticed by one referee and O. Bousquet, in this form, the star-VC-dimension minus one coincides with the notion of star number that has been studied in the context of active learning [15 Definition 2].}\]
this set (the requirement \(\{e, y\} \subseteq S\)). When performing a d-shifting \(\varphi_{ij}\) with respect to a pair \(\{e_i, e_j\}\) such that \(\{e_i, e_j\} \cap \{e, y\} = \emptyset\), a set \(S \in \mathcal{S}\) can be mapped to a set \(\varphi_{ij}(S)\) belonging to the fiber \(F(\{e, y\})\). If \(\varphi_{ij}(S)\) is used to c-shatter the 2-set \(\{e, y\}\) by \(\varphi_{ij}(S)\), then \(S\) can be used to shatter \(\{e, y\}\) by \(S\) (the proof of Lemma 6). However, this is no longer true for s-shattering, because initially \(S\) may not necessarily belong to \(F(\{e, y\})\).

Also we have not found a counterexample to the following question (where the square of the clique-VC-dimension or of the star-VC-dimension is replaced by the product of the classical VC-dimension of \(\mathcal{S}\) and the clique number of \(G_{1,2}(\mathcal{S})\)):

**Question 2.** Is it true that for any set family \(\mathcal{S}\) with 1,2-inclusion graph \(G_{1,2}(\mathcal{S}) = (V, E)\), \(d = \text{VC-dim}(\mathcal{S})\), and clique number \(\omega = \omega(G_{1,2}(\mathcal{S}))\), we have \(\frac{|E|}{|V|} = O(d \cdot \omega)\)?

Hypercubes are subgraphs of Johnson graphs, therefore they are 1,2-inclusion graphs. This shows the necessity of both parameters (VC-dimension and clique number) in the formulation of Question 2. As above, the bottleneck in solving Question 2 via shifting is that this operation may increase the clique number of 1,2-inclusion graphs.

An alternative approach to Questions 1 and 2 is to adapt the original proof of Theorem 2 given in [19]. In brief, for a set family \(\mathcal{S}\) of VC-dimension \(d\) and an element \(e\), let \(\mathcal{S}_e = \{S' \subseteq X \setminus \{e\} : S' = S \cap X\text{ for some }S \in \mathcal{S}\}\) and \(\mathcal{S}^e = \{S' \subseteq X \setminus \{e\} : S' \text{ and } S' \cup \{e\} \text{ belong to } S\}\). Then \(|\mathcal{S}| = |\mathcal{S}_e| + |\mathcal{S}^e|\), \(\text{VC-dim}(\mathcal{S}_e) \leq d\), and \(\text{VC-dim}(\mathcal{S}^e) \leq d - 1\) hold. Denote by \(G_e\) and \(G^e\) the 1-inclusion graphs of \(\mathcal{S}_e\) and \(\mathcal{S}^e\). Then \(|E(G_e)| \leq |V(G_e)| = |\mathcal{S}_e|\) and \(|E(G^e)| \leq (d - 1)|V(G^e)| = (d - 1)|\mathcal{S}^e|\) by induction hypothesis. The proof of the required density inequality follows by induction from the equality \(|V(G)| = |\mathcal{S}| = |\mathcal{S}_e| + |\mathcal{S}^e| = |V(G_e)| + |V(G^e)|\) and the inequality \(|E(G)| \leq |E(G_e)| + |E(G^e)| + |V(G^e)|\). Unfortunately, as was the case for shifting, the clique number of \(G_{1,2}(\mathcal{S}_e)\) may be strictly larger than the clique number of \(G_{1,2}(\mathcal{S})\). Also the inequality \(|E(G)| \leq |E(G_e)| + |E(G^e)| + |V(G^e)|\) is no longer true in this form if instead of 1-inclusion graphs one consider 1,2-inclusion graphs.

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**References**

[1] R. Ahlswede and N. Cai, A counterexample to Kleitman’s conjecture concerning an edge-isoperimetric problem, Combinatorics, Probability and Computing 8 (1999), 301–305.

[2] H.-J. Bandelt and V. Chepoi, Metric graph theory and geometry: a survey, in: J. E. Goodman, J. Pach, R. Pollack (Eds.), Surveys on Discrete and Computational Geometry. Twenty Years later, Contemp. Math., vol. 453, AMS, Providence, RI, 2008, pp. 49–86.

[3] S. L Bezrukov, Edge isoperimetric problems on graphs, Proc. Bolyai Math. Studies 449 (1998).

[4] N. Bousquet and S. Thomassé, VC-dimension and Erdős-Pósa property, Discr. Math. 338 (2015), 2302–2317.

[5] A.E. Brouwer, A.M. Cohen, and A. Neumaier, Distance Regular Graphs, Springer-Verlag Berlin, New York, 1989.

[6] N. Cesa-Bianchi and D. Haussler, A graph-theoretic generalization of Sauer-Shelah lemma, Discr. Appl. Math. 86 (1998), 27–35.

[7] V. Chepoi, Basis graphs of even Delta-matroids, J. Combin. Th. Ser B 97 (2007), 175–192.

[8] V. Chepoi, Distance-preserving subgraphs of Johnson graphs, Combinatorica (to appear).

[9] V. Chepoi, A. Labourel, and S. Ratel, On density of subgraphs of Cartesian products, (in preparation).

[10] M. Deza and M. Laurent, Geometry of Cuts and Metrics, Springer-Verlag, Berlin, 1997.

[11] V. Diego, O. Serra, and L. Vena, On a problem by Shapozenko on Johnson graphs. arXiv:1604.05084, 2016.

[12] R. Diestel, Graph Theory, Graduate texts in mathematics, Springer New York, Berlin, Paris, 1997.

[13] D. Z. Djoković, Distance-preserving subgraphs of hypercubes, J. Combin. Th. Ser. B 14 (1973), 263–267.
[14] M.R. Garey and R.L. Graham, On cubical graphs, J. Combin. Th. B 18 (1975), 84–95.
[15] S. Hanneke and L. Yang, Minimax analysis of active learning, J. Mach. Learn. Res. 16 (2015), 3487–3602.
[16] L.H. Harper, Optimal assignments of numbers to vertices, SIAM J. Appl. Math., 12 (1964), 131–135.
[17] L.H. Harper, Global Methods for Combinatorial Isoperimetric Problems, Cambridge Studies in Advanced Mathematics (No. 90), Cambridge University Press 2004.
[18] D. Haussler, Sphere packing numbers for subsets of the Boolean $n$-cube with bounded Vapnik-Chervonenkis dimension, J. Comb. Th. Ser. A 69 (1995), 217–232.
[19] D. Haussler, N. Littlestone, and M. K. Warmuth, Predicting \{0,1\}-functions on randomly drawn points, Inf. Comput. 115 (1994), 248–292.
[20] D. Haussler and P.M. Long, A generalization of Sauer’s lemma. J. Combin. Th. Ser. A 71 (1995), 219–240.
[21] D. Kuzmin and M.K. Warmuth, Unlabelled compression schemes for maximum classes, J. Mach. Learn. Res. 8 (2007), 2047–2081.
[22] S.B. Maurer, Matroid basis graphs I, J. Combin. Th. Ser. B 14 (1973), 216–240.
[23] B. K. Natarajan, On learning sets and functions, Machine Learning 4 (1989), 67–97.
[24] D. Pollard, Convergence of Stochastic Processes, Springer Science & Business Media, 2012.
[25] B.I. Rubinstein, P.L. Bartlett, and J.H. Rubinstein, Shifting: one-inclusion mistake bounds and sample compression, J. Comput. Syst. Sci. 75 (2009), 37–59.
[26] N. Sauer, On the density of families of sets, J. Combin. Th., Ser. A 13 (1972), 145–147.
[27] S.V. Shpectorov, On scale embeddings of graphs into hypercubes, Europ. J. Combin. 14 (1993), 117–130.
[28] V.N. Vapnik and A.Y. Chervonenkis, On the uniform convergence of relative frequencies of events to their probabilities, Theory Probab. Appl. 16 (1971), 264–280.