Cauchy's almost forgotten Lagrangian formulation of the
Euler equation for 3D incompressible flow

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Two prized papers, one by Augustin Cauchy in 1815, presented to the French Academy and the other by Hermann Hankel in 1861, presented to Göttingen University, contain major discoveries on vorticity dynamics whose impact is now quickly increasing. Cauchy found a Lagrangian formulation of 3D ideal incompressible flow in terms of three invariants that generalize to three dimensions the now well-known law of conservation of vorticity along fluid particle trajectories for two-dimensional flow. This has very recently been used to prove analyticity in time of fluid particle trajectories for 3D incompressible Euler flow and can be extended to compressible flow, in particular to cosmological dark matter. Hankel showed that Cauchy’s formulation gives a very simple Lagrangian derivation of the Helmholtz vorticity-flux invariants and, in the middle of the proof, derived an intermediate result which is the conservation of the circulation of the velocity around a closed contour moving with the fluid. This circulation theorem was to be rediscovered independently by William Thomson (Kelvin) in 1869. Cauchy’s invariants were only occasionally cited in the 19th century — besides Hankel, foremost by George Stokes and Maurice Lévy — and even less so in the 20th until they were rediscovered via Emmy Noether’s theorem in the late 1960, but reattributed to Cauchy only at the end of the 20th century by Russian scientists.

I. INTRODUCTION

The motion of a fluid can be described either in a fixed frame of reference in terms of the spatio-temporal Eulerian coordinates \( x, y, z, t \) or by following individual fluid particles in terms of their initial or Lagrangian coordinates \( a, b, c, t \). The description of a viscous fluid is simpler in Eulerian coordinates, but for an ideal (inviscid) incompressible fluid there is a variational formulation in Lagrangian coordinates of the equations of motion, due precisely to Lagrange. Over most of the 19th and 20th century the Eulerian formulation became predominant.\(^1\)

In more recent years there has been a strong renewal of interest in Lagrangian approaches. In cosmology most of the matter is generally believed to be of the dark type, which is essentially collisionless and thus inviscid. Lagrangian perturbations methods have been developed since the nineties that shed light on the mechanisms of formation of cosmological large-scale structures. Closely related is the problem of reconstruction of the past Lagrangian history of the Universe from present-epoch observations of the distribution of galaxies. Novel fast photography techniques have been developed for the tracking of many particles seeded into laboratory flow that allow the reconstruction of a substantial fraction of the Lagrangian map that associates the positions of fluid particles at some initial time to their positions at later times. When the flow is electrically highly conducting and can support a magnetic field by the magnetohydrodynamic (MHD) dynamo effect, it has been shown that the long-time fate of such a magnetic field is connected to the issue of Lagrangian chaos, namely how fast neighbouring fluid particle trajectories separate in time.\(^2\)

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\(^1\) Lagrange, 1788. For more context on the early 18th century developments in fluid dynamics, see Truesdell, 1954b; Darrigol, 2005; Darrigol & Frisch, 2008. Regarding the history of Eulerian and Lagrangian coordinates, see Truesdell, 1954a.

\(^2\) For Lagrangian perturbations in cosmology, see, e.g., Moutarde et al., 1991; Buchert, 1992. For reconstruction of the Universe, see Brenier et al., 2003 and references therein. For Lagrangian experimental and numerical tracking techniques, see, e.g., Toschi & Bodenschatz, 2009. For MHD, see Arnold, Zeldovich, Ruzmaikin & Sokoloff, 1981; Vishik, 1989; Childress & Gilbert, 1995.
At a more fundamental level, over 250 years after Euler wrote the equations governing incompressible ideal three-dimensional (3D) fluid flow (generally known as the 3D Euler equations), we still do not know if the solutions remain well-behaved for all times or become singular and dissipate energy after some finite time; and this even when the initial data are very smooth, say, analytic. Many attempts have been made to tackle this problem by numerical simulations in a Eulerian coordinates framework, but the problem remains moot. Lagrange himself frequently preferred Eulerian coordinates, although the variational formulation he gave was formulated in Lagrangian coordinates: in modern mathematical language, the solutions to the equations of incompressible fluid flow are geodesics on the infinite-dimensional manifold SDiff of volume-preserving Lagrangian maps. In a Lagrangian framework one is “riding the eddy” and does not feel too much of the possible spatial roughness. Actually, fluid particle trajectories can be analytic in time even when the flow has only limited spatial smoothness.\(^3\)

Recently, borrowing some ideas of the cosmological Lagrangian perturbation theory, one of us and Vladislav Zheligovsky obtained a form of the incompressible 3D Euler equations in Lagrangian coordinates, from which simple recursion relations among the temporal Taylor coefficients of the Lagrangian map can be derived and analyticity in time of the Lagrangian trajectories can be proved in a rather elementary way. This Lagrangian approach can also be used to save significant computer time in high-resolution simulations of 3D ideal Euler flow. The Lagrangian equations used for this did not seem at first glance to be widely known, but after some time spent searching the past scientific literature, we found that the equations had been derived in 1815 by Augustin Cauchy in a long memoir that won a prize from the French Academy.\(^4\)

At first sight, Cauchy’s equations, to be presented in Section \[\text{II.C}\] — here called ‘Cauchy’s invariants equations’ to distinguish them from an important corollary, Cauchy’s vorticity formula, — seemed hardly cited at all. We then engaged in a much more systematic search. The surprising result can be summarized as follows: in the 19th century, Cauchy’s invariants equations are cited only in a small number of papers, the most important one being by Hermann Hankel; in the 20th century, Cauchy’s result seems almost completely uncited, except at the very end of the century.

The outline of the paper is as follows. Section \[\text{II}\] is devoted to Augustin Cauchy: in Section \[\text{II.A}\] we recall a few biographical elements, in particular those connected to the present study; Section \[\text{II.B}\] is devoted to the 1815 prize-winning \textit{Mémoire sur la propagation des ondes}; in Section \[\text{II.C}\] we analyze in detail the very beginning of its second part, which contains Cauchy’s Lagrangian formulation of the 3D ideal incompressible equations in terms of what would now be called invariants, a terminology we adopt here. Section \[\text{III}\] is about 19th century scientists who realized the importance of Cauchy’s Lagrangian formulation. Outstanding, here, is Hermann Hankel (Section \[\text{III.A}\]), a German mathematician whose keen interest in the history of mathematics allowed him not to miss Cauchy’s 1815 work and to understand in 1861 its potential in deriving Helmholtz’s results on vorticity in a Lagrangian framework, and discovering on the way what is known as Kelvin’s circulation theorem. Then, in Section \[\text{IV}\] we turn to the other 19th century scientist who discuss Cauchy’s invariants equations: Foremost George Stokes, then Maurice Lévy, Horace Lamb, Jules Andrade and Paul Appell and to a few others who mention the equations but may not be aware that they were obtained by Cauchy: Gustav Kirchhoff and Henri Poincaré.\(^5\)

Then, in Section \[\text{V}\] we turn to the 20th century and beyond. The first part (Section \[\text{V.A}\]) has Cauchy’s invariants equations apparently fallen into oblivion. In the second part (Section \[\text{V.B}\]) we shall see that, as a consequence of Emmy Noether’s theorem connecting continuous symmetries and invariants, a number of scientists were able to rederive Cauchy’s invariants equations for the 3D ideal incompressible flow, but without being aware of Cauchy’s

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\(^3\) Euler, 1757. On the issue of well-posedeness, see Eyink, Frisch, Moreau and Sobolevsky, 2008. On analyticity of fluid particle trajectories, see Serfati, 1995; Shnirelman, 2012; see also the next paragraph.

\(^4\) On analyticity of the Lagrangian map, see Frisch & Zheligovsky, 2013; Zheligovsky & Frisch, 2013. On a novel Lagrangian numerical method, see project NEImezzo, \textit{sub judice}, available on request. Cauchy, 1815/1827.

\(^5\) Hankel, 1861. Helmholtz, 1858. Thomson (Lord Kelvin), 1869.
work. In Section V.C we shall find that Russian scientists, followed by others, reminded us that all this had been started a long time ago with Cauchy. In Section VI we make some concluding remarks.\(^6\)

Since one of our key goals is to understand how important work such as Cauchy’s 1815 formulation of the hydrodynamical equations in Lagrangian coordinates managed to get nearly lost, we are obliged to pay attention to who cites whom. This is a delicate matter, given that present-day ethical rules of citing definitely did not apply in past centuries. But without fast communications, a peer-review system to point out missing references and a much larger scientific population, the rules had to be different. Here, we shall do our best to mention each instance of citation of previous work by the author being discussed, when such work is relevant to our paper.

II. CAUCHY

A. Biographical elements

These elements are just intended to give a background on the circumstances which led to Cauchy’s 1815 work. Our main sources have been the biography of Augustin-Louis Cauchy by Bruno Belhoste, the biography by Claude Alphonse Valson, written just a few years after the death of Cauchy and the minutes (Procès-Verbaux) of the meetings of the French Academy of Sciences, referred to as PV, followed by the date of the corresponding meeting.\(^7\)

\(^6\) Noether, 1918. For the rediscovery of Cauchy, see Abrashkin, Zen’kovich and Yakubovich, 1996; Zakharov & Kuznetso, 1997. 

\(^7\) Belhoste, 1991; cf also his PhD thesis, Belhoste, 1982; Valson, 1868; Procès-Verbaux des Séances de l’Académie, 1915.
Cauchy was born in 1789, turbulent times, but this did not affect his ability to get the best theoretical and practical training available in the early 19th century, attending successively École Polytechnique and École des Ponts et Chaussées. His first employment was as a junior engineer in a major harbour project in Cherbourg in 1810 and then as an engineer at the Ourcq Canal project in Paris in 1813. During his engineering years he already displayed keen interest in deep mathematical questions, several of which he solved in a way that sufficiently impressed the mathematicians at the Academy of Sciences, (called “First Class of the Institute of France” until King Louis XVIII restored the old name of “Academy of sciences”; here, we refer to it just as “Academy”). In 1813, the Geometry section of the Academy ranked Cauchy second for an election to the Academy but the vote of the rest of the members of the Academy went heavily against him. Anyway, during the years 1812–1815 Cauchy had a strong coupling to the Academy and his name appears about one hundred times in the minutes of the Academy meetings. His awarding of the 1815 mathematics prize by the Academy (see Section [ILB]) is is one more evidence that he was a rising star.8

Eventually, the King took advantage of the reorganization of the Academy to remove Lazare Carnot and Gaspard Monge from the Academy and appoint Cauchy as a member in March 1816. This unusual way of entering the Academy produced some friction with regular members.9

Cauchy, who was soon to become a world-dominant figure in mathematics and mathematical physics, was no easy-going personality, he was however willing to suffer considerably for his ideas, particularly those grounded in his Christian beliefs. For example, he went into exile in 1830 for eight long years in order not to have to swear an oath of allegiance to King Louis Philippe, considered by Cauchy as not legitimate.10

B. The 1815 mathematics prize

The differential equations given by the author are rigorously applicable only to the case where the depth of the fluid is infinite; but he succeeded in obtaining their general integrals in a form allowing to discuss the results and comparing them to experiments.11

Thus reads the beginning of the statement made by the Academy during the public ceremony of January 8, 1816 at which Cauchy received the 1815 Grand Prix (mathematics prize). The events leading to this prize started on December 27, 1813 when the Academy committee in charge of proposing a subject for a mathematics prize decided for “le problème des ondes à la surface d’un liquide de profondeur indéfinie” (the problem of waves on the surface of a liquid of arbitrary depth). The committee put Laplace in charge of defining the scientific programme.12

On October 2, 1815 the Academy received two anonymous manuscripts – as usual in those circumstances, distinguished by an epigraph for later identification. Cauchy’s manuscript had the epigraph Nosse quot ionii veniant ad littora fluctus (Virgil, Geor. II, 108; translation: To know how many waves come rolling shoreward from the Ionian sea).

On December 26, 1815 the committee proposed giving the 1816 prize to Cauchy’s manuscript.13

The manuscript of the prize was not published until 1827, when appeared the first volume of Mémoires des Savans étrangers (Memoirs of non-member scientists) printed since the 1816 reorganization of the Academy. It comprised a hefty 310 pages, including 189 pages with 20 technical notes (the last 7 where added at various dates after 1815, but all the

8 PV: 1812–1815. For the 1813 election, see Belhoste, 1991: 37.
9 Belhoste, 1991: 46–47.
10 Belhoste, 1991: Chaps. 9 and 10.
11 Les équations différentielles données par l’auteur ne s’appliquent rigoureusement qu’au cas où la profondeur du fluide est infinie; mais il est parvenu à obtenir leurs intégrales générales sous une forme qui permet d’en discuter les résultats et de les comparer à l’expérience. Institut de France, 1816.
12 PV: 1815, 27 décembre.
13 PV: 1815, 2 octobre, 26 décembre.
material not contained in the 1815 manuscript is clearly identified by the author).  

It is interesting that Cauchy’s prized manuscript begins with a statement of the problem to be solved:

*A massive fluid, initially at rest, and of an arbitrary depth, has been put in motion by the effect of a given cause. One asks, after a determined time, the shape of the external surface of the fluid and the speed of each molecule located at the same surface.*

Although we have not found this sentence in any of the minutes of the Academy or at its archives, it is likely that it constitutes the Academy’s detailed formulation of the problem, which had been entrusted to Laplace.

Actually, Cauchy gave a more general treatment than had been requested by the Academy, since he obtained results not only for the *surface* of the fluid, but also for its *bulk*. The memoir itself has three parts: the solution is in the third part, whereas the first two actually contain, in the intention of the Author, a sort of preparatory material describing the initial state of the whole fluid and its later evolution.

Here, it is the first section of the second part that interests us; it is entitled *On the equations that subsist at any instant of the motion for all the points within the mass of the fluid.* Our focus will be entirely on this section and even more so on its very beginning, where Cauchy obtains his Lagrangian formulation of the 3D incompressible Euler equations in terms of three invariants and uses it immediately to derive what is called Cauchy’s vorticity formula relating the current and initial vorticity fields through the Jacobian matrix of the Lagrangian map. In the next section we turn to these matters.

### C. Cauchy’s invariants equations and the Cauchy vorticity formula

There is of course no real substitute for reading Section I of the second part of Cauchy’s paper. It can be mostly understood without prior reading of the first part. The notation is not too different from the modern one, except that, of course, no vectors were used. For illustration, Figure 2 gives Cauchy’s key equation (from the point of view of the present paper) as it was published in 1827. (Our attempts to retrieve the orginal hand-written manuscript of 1815 have failed.) In our description of the work we shall use modernized notation.

Cauchy considers a 3D ideal incompressible fluid subject to an external force. Analyzing the various forces acting on a “molecule” (i.e. a fluid particle), he derives his eq. (4), which in our notation reads

$$\partial_t v + (v \cdot \nabla)v = -\nabla p + F,$$

where, $v$ is the flow velocity, $F$ the external force, $p$ the pressure (divided by the constant fluid density, taken here unity for convenience) and $t$ the time. He then points out that these equations coincide with those obtained by Lagrange by another method. They coincide also in their form and method of derivation with those obtained by Euler and are presently called the Euler equations.

Cauchy, then changes to Lagrangian variables, denoted by him $(a, b, c, t)$ (here, $(a, t)$). The Eulerian position $x$ becomes then a function $x(a, t)$. Nowadays, the representation of

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14 The publication reference is found in our reference list as Cauchy, 1815/1827, but what we discuss here is, according to Cauchy, unchanged from the 1815 prized manuscript. On the circumstances of publication, see Belhoste, 1991; for the delay before publication, see also Smithies, 1997: §2.1.

15 Original in French: Une masse fluide pesante, primitivement en repos, et d’une profondeur indéfinie, a été mise en mouvement par le fait d’une cause donnée. On demande, au bout d’un temps déterminé, la forme de la surface extérieure du fluide, et la vitesse de chacune des molécules situées à cette même surface.

16 For a short description of Cauchy’s memoir, see Darrigol, 2005: 43–45. For a detailed presentation of Cauchy’s paper in the context of 19th century work on wave propagation, see Risser, 1925; Dalmedico, 1989.

17 In modern scientific language: The equations that describe the fluid motion within its bulk at any time.

18 Cauchy, 1815/1827: 35–49.

19 Euler, 1755; Lagrange, 1788: 453.
the flow in terms of the coordinates \((x, t)\) is called \textit{Eulerian} and, when the coordinates \((a, t)\) are used, it is called \textit{Lagrangian}; the (time-dependent) map \(a \mapsto x\) is called the \textit{Lagrangian map}. The velocity and the acceleration of a fluid particle are then \(\dot{x}(a, t)\) and \(\ddot{x}(a, t)\), respectively, where the dot denotes the Lagrangian time derivative. The Euler equations \([1]\) state that the acceleration minus the external force is balanced by minus the Eulerian gradient of the pressure. Making use of the set of nine partial derivatives of the \(x_k\)s with respect to the \(a_k\)s (now called the Jacobian matrix) Cauchy transforms the Eulerian pressure gradient into a Lagrangian pressure gradient, here denoted \(\nabla^L p\), and obtains his eq. \((7)\)

\[
\sum_{k=1}^{3} (\ddot{x}_k - F_k) \nabla^L x_k = -\nabla^L p, \tag{2}
\]

where \(x_k\) and \(F_k\) denote the \(k\)th components of \(x\) and \(F\), respectively. This is precisely Lagrange’s eq. \((D)\). Note that Lagrange first wrote the equations in Lagrangian coordinates and then switched to Eulerian coordinates; Cauchy did it the other way round.\(^{20}\)

Then, Cauchy considers the condition of incompressibility, which he first writes in Lagrangian coordinates. In modern terms, the Jacobian of the Lagrangian map should be equal to unity for all \((a, t)\) (his eq. \((9)\))

\[
\det (\nabla^L x) = 1. \tag{3}
\]

He also writes it in Eulerian coordinates (his eq. \((10)\))

\[
\nabla \cdot v = 0, \tag{4}
\]

an equation already found in Euler but which has been derived earlier in the axisymmetric case by d’Alembert.\(^{21}\)

Then, Cauchy observes that the two equations \((2)-(3)\) are not integrable, but if one restricts oneself to external forces deriving from a potential \(\lambda\), then \((2)\) can be integrated once, as he will show. Cauchy thus writes (his eq. \((11)\))

\[
F = \nabla \lambda. \tag{5}
\]

He then rewrites \((2)\) as (his eq. \((13)\))

\[
\sum_{k=1}^{3} \ddot{x}_k \nabla^L x_k = -\nabla^L (p - \lambda). \tag{6}
\]

Observe that the r.h.s is a Lagrangian gradient. Cauchy then applies what we now call a (Lagrangian) curl to cancel out the r.h.s. He thus obtains his eq. \((14)\)

\[
\nabla^L \times \sum_{k=1}^{3} \ddot{x}_k \nabla^L x_k = 0. \tag{7}
\]

He then notices that the three components of the l.h.s. of \((7)\) are exact time derivatives of three quantities which thus must be time-independent. He easily identifies their constant values to what we now call the initial vorticity \(\omega_0 = \nabla^L \times v_0\). This way, Cauchy obtains his eq. \((15)\):

\[
\sum_{k=1}^{3} \nabla^L \ddot{x}_k \times \nabla^L x_k = \sum_{k=1}^{3} \nabla^L v_k \times \nabla^L x_k = \omega_0, \tag{8}
\]

and states: \textit{Telles sont les intégrales que nous avions annoncées} (Such are the integrals that we had announced). Indeed, in Lagrangian coordinates, the r.h.s. is time-independent.

\(^{20}\) Lagrange, 1788: 446.

\(^{21}\) For a discussion of Euler and d’Alembert’s contribution to the incompressibility condition, see Darrigol & Frisch, 2008: §§3, 4.
The constant quantities in the l.h.s. are now usually called “the Cauchy invariants,” a terminology we shall adopt. As we shall see in Section III.A they are closely connected to the circulation invariants of Helmholtz and Kelvin and are the three-dimensional generalization of the two-dimensional vorticity invariant. As to the Cauchy equations (3) and (8) for the Lagrangian map, which play a central role in the present paper, we shall refer to them as “Cauchy’s invariants equation”, to avoid any possible confusion with “Cauchy’s vorticity formula,” discussed below. Occasionally, we shall also refer to the quantities that remain constant along fluid particle trajectories as “Cauchy invariants.”

Cauchy was obviously aware that he had succeeded in partially integrating the equations of motion. However, modern concepts such as invariants and their relation to symmetry/invariance groups would emerge only about one century later (see Section V.B). Nonetheless, for his invariants equations Cauchy immediately found an application, which would become quite famous (much more, so far, than Cauchy’s invariants).

Starting from (3), written in terms of the velocity, Cauchy reexpresses its Lagrangian space derivatives in terms of the Eulerian ones and the Jacobian matrix. He obtains for the l.h.s. of (8) expressions which are linear in the components of the vorticity $\omega$ (evaluated at the current time $t$) and quadratic in the Jacobian matrix. He then solves these linear equations, using the fact the Jacobian is unity. He thus obtains his eq. (17):

$$\omega = \omega_0 \cdot \nabla^L x,$$

(9)

or, with indices

$$\omega_i = \sum_j \omega_{0j} \nabla^L_j x_i.$$  

(10)

In modern terms this “Cauchy vorticity formula” states that the current vorticity is obtained by multiplying the initial vorticity by the Jacobian matrix. Cauchy gives a rather low-key application of his formula, which is consistent with the context of the prize: in the first part of his memoir, he had envisaged a mechanism of setting the fluid in motion impulsively that would produce a flow initially potential and thus with no vorticity. His formula then implies that the flow would have no vorticity at any instant of time. In the language of the time this was expressed by stating that $v \cdot dx$ is a “complete differential.”

Nonetheless, Cauchy was certainly aware of Lagrange’s theorem, which states that an ideal flow initially potential, stays potential at later times. Lagrange’s proof used Eulerian coordinates and assumed that the velocity could be Taylor-expanded in time to arbitrary orders. Lagrange then showed that if the vorticity vanishes initially, so will its time-Taylor coefficients of arbitrary orders.\textsuperscript{22}

\textsuperscript{22} Lagrange, 1788: 458–463.
Cauchy’s proof requires only a limited smoothness of the flow (he does not state how much) and it must have appeared to the readers at the time that, as long as the Jacobian matrix exists, then the persistence of potential flow will hold. Stokes observed that the vanishing of all the Taylor coefficients does not imply the vanishing of a function (giving well-known examples such as $e^{-1/t^2}$ near $t = 0$); he thus considered Cauchy’s proof more general than that of Lagrange. Today, we know that a flow with an initial velocity field that is “moderately smooth in space” (just a little more regular than once differentiable in space), will stay so for at least a finite time, during which its temporal smoothness in time in Eulerian coordinates is not better than its spatial smoothness, rendering Lagrange’s argument inapplicable, whereas Cauchy’s proof only requires spatial differentiability of the Lagrangian map, which is now known to hold with a moderately smooth initial velocity field.\(^{23}\)

One reason why Cauchy’s vorticity formula (9) is very well known today is that it applies not only to the vorticity in an ideal fluid but also to a magnetic field in ideal conducting fluid flow governed by the magnetohydrodynamic (MHD) equations. In modern mathematical language, both the vorticity and the magnetic field are transported 2-forms. In an ordinary fluid, one cannot prescribe the velocity and the vorticity independently, but in a conducting fluid one can prescribe the velocity and the initial magnetic field independently in the limit of weak magnetic fields, when studying the kinematic dynamo problem.\(^{24}\)

All this explains that there has been a strong interest, particularly in recent years, in Cauchy’s vorticity formula (9). Our main focus in this paper are the Cauchy invariants equations (8). One cannot describe the history of the Cauchy’s vorticity formula without mentioning his invariants equations, and thus we cannot completely disentangle the histories of their citations. However, nowadays, most derivations of Cauchy’s vorticity formula use a much shorter route, based on the Eulerian vorticity equation of Helmholtz

$$\partial_t \omega + v \cdot \nabla \omega = \omega \cdot \nabla v,$$

and thus bypass Cauchy’s invariants equations.\(^{25}\)

The rest of Cauchy’s Section I of Part II (pp. 44-49) is devoted to the case of potential flow and does not concern us here.

### III. 19TH CENTURY

#### A. Hermann Hankel

Hermann Hankel (1839–1873) was a German mathematician who studied with August Ferdinand Moebius, Bernhard Riemann, Karl Weierstrass and Leopold Kronecker. Our sources on his life are the obituary by Wilhelm von Zahn, a 19th century biography by Moritz Cantor, a 20th century short biography by Michael Crowe and an assessment of his mathematical contribution from a modern perspective by Antonie Frans Monna.\(^{26}\)

Hankel is quite well-known for work on Hankel matrices, on Hankel transforms and on Hankel functions. He was in addition also involved very seriously in the history of mathematics and has left a book Zur Geschichte der Mathematik in Alterthum und Mittelalter (On the History of Mathematics in the Antiquity and the Middle Ages) which includes, among other things, one of the first studies bringing out the major contributions of Indian mathematics.\(^ {27}\) What interests us here is Hankel’s work on fluid dynamics contained in a manuscript prized by Göttinngen University Zur allgemeinen Theorie der Bewegung der Flüssigkeiten (On the general theory of motion of fluids).

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\(^{23}\) Stokes, 1845: 305. On spatial differentiability see, e.g. Zheligovsky & Frisch, 2013 and references therein.

\(^{24}\) On MHD and dynamo theory, see Moffatt, 1978.

\(^{25}\) Helmholtz, 1858.

\(^{26}\) von Zahn, 1869. Cantor, 1879. Crowe, 2008. Monna, 1973.

\(^{27}\) Hankel, 1874.
To understand the context of this prize, let us recall that in 1858 Hermann Helmholtz (1821–1894) wrote a major paper about vortex motion in three-dimensional incompressible ideal flow that generated considerable interest.\textsuperscript{28}

A key result of his work, stated in modern language, is that the flux of the vorticity through an infinitesimal piece of surface is a Lagrangian invariant. This result, known as Helmholtz’s second theorem, immediately implies, by adding up infinitesimal surface elements, that the same holds for a finite surface; moreover, by using Stokes’s theorem one obtains Kelvin’s circulation theorem. Helmholtz’s derivation of his result is to a large extent resting on an Eulerian approach and begins with the establishment of the aforementioned Eulerian vorticity equation. Furthermore, Helmholtz’s derivation, mostly written for physicists, was a bit heuristic.

On June 4, 1860 Göttingen University (Philosophische Facultät der Georgia Augusta) set up a prize, intended to stimulate interest in Lagrangian approaches and in particular to give such a derivation of Helmholtz’s invariants:

\textit{The general equations for determining fluids motions may be given in two ways, one of which is Eulerian, the other one is Lagrangian. The illustrious Dirichlet pointed out in the posthumous unpublished paper “On a problem of hydrodynamics” the until now almost completely overlooked advantages of the Lagrangian}

\textsuperscript{28}Helmholtz, 1858; for his work on vortex dynamics, see also the biography by Königsberger, 1902–1903, English version 1906: 167–171. For an extensive bibliography of vortex flow since Helmholtz, see Meleshko & Aref, 2007.
The prominent reference to Dirichlet can be understood as follows. Johann Peter Gustav Lejeune Dirichlet (1805–1859) was an important German mathematician who, from 1855 to his death, succeeded Carl Friedrich Gauss in Göttingen. He had also a strong interest in hydrodynamics. In 1856–1867 he wrote an unfinished paper “Untersuchungen über ein Problem der Hydrodynamik” (Investigation of a problem in hydrodynamics). Dirichlet asked the German mathematician Richard Dedekind (1831–1916), at that time professor in Göttingen, to help him with the work, but was not able to finish completely before he died. Major pieces of the work were found by Dedekind who published them in 1859 with the above title, followed by “From his legacy, edited by R. Dedekind.” In the Introduction Dirichlet pointed out that Lagrange himself was not too keen to advocate the use of what was later called Lagrangian coordinates, found by Lagrange to be a bit complicated. Eulerian coordinates quickly grew in favour. Dirichlet however observed that Eulerian coordinates have their own drawbacks, particularly when the volume occupied by the fluid changes in time.

We now turn to Hankel’s prize-winning manuscript. It carried the epigraph Tanto utiliores sunt notae, quanto magis exprimunt rerum relationes (The more signs express relations among things, the more useful they are). The manuscript was written in Latin, but Hankel got the permission to print a slightly edited German translation, which was also published as a book. In 1863 a four-page review was published in Fortschrifte der Physik (Progress in Physics); it was signed, anonymously as “HL.” and will be cited below as Fortschrifte.

This work is of particular interest, not only because it is the first time that it was shown that the Helmholtz invariants are directly connected with the Cauchy invariants, but also because it gives the first derivation of what is generally called the Kelvin circulation theorem. Hankel discusses both compressible and incompressible fluids, but here it suffices to consider the latter.

To derive the Helmholtz invariants, Hankel first establishes Cauchy’s invariants equations (§5), for which he refers to Cauchy’s 1815/1827 prized paper. Hankel then rewrites this, in modernised notation,

\[
\nabla^L \times \sum_{k=1}^{3} \dot{x}_k \nabla^L x_k = \omega_0,
\]

where \( \dot{v}_k = \dot{x}_k \) are the components of the velocity \( \dot{v} \). This is eq. (3) of Hankel’s §6. The equation is not to be found in Cauchy’s 1815/1827 paper but, given that Cauchy obtained his invariants equations (here, §5) by taking the Lagrangian curl of (2) and then integrating over the time variable, it is not really surprising that the left-hand-side of (8) is a Lagrangian curl.

Hankel’s next step is to consider in Lagrangian space a connected (zusammenhängende) surface, here denoted by \( S_0 \), whose boundary is a curve, here denoted by \( C_0 \), and to apply to (2) what was later to be called the Stokes theorem, relating the flux of the curl of a vector field across a surface to the circulation of the vector field along the boundary of this surface. Hankel could not easily be aware of what Thomson and Stokes had done before on the subject and of the much earlier work of Ostrogradski and thus he devotes his §7 to proving the Stokes theorem.

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29 Aequationes generales motui fluidorum determinando inservientes duobus modis exhiberi possunt, quorum alter Eulero, alter Lagrangio debetur. Lagrangiani modi utilitates adhuc fere penitus neglecti clarissimus Dirichlet indicavit in commentatione postuma “de problemate quodam hydrodynamico” inscripta; sed abs explicatione earum ubiore morbo supremo impeditus esse videtur. Itaque postulat ordo theoriam motus fluidorum aequationibus Lagrangianis superstructam eamque eo saltem perductam, ut leges motus rotatorii a clarissimo Helmholtz allo modo eratae inde reddunt.

30 Dirichlet, 1859. On the role of Euler in introducing both the “Eulerian” and the “Lagrangian” coordinates, see Hankel, 1861: 3 and Truesdell, 1954: 30 (footnote 2).

31 Hankel, 1861. Anonymous, 1863.

32 Concerning the history of the Stokes theorem, see Katz, 1979.
In § 8, Hankel then infers that the flux through $S_0$ of the r.h.s. of (12), namely the initial vorticity, is given by the circulation along $C_0$ of the initial velocity $v_0$:

$$\int_{C_0} v_0 \cdot da = \int_{S_0} \omega_0 \cdot n_0 d\sigma_0,$$

(13)

where $n_0$ denotes the local unit normal to $S_0$ and $d\sigma_0$ the surface element. This is the unnumbered equation near the top of his p. 38. Then, in § 9, he similarly handles the l.h.s. of (12) and first notices that

$$\sum_{k=1}^{3} v_k \nabla^L x_k \cdot da = v \cdot dx.$$

(14)

This is the third equation before eq. (2) of his § 9. He thus obtains the Eulerian circulation, an integral over the curve $C$ where are located at the present time the fluid particles initially on $C_0$:

$$\int_C v \cdot dx = \int_{S_0} \omega_0 \cdot n_0 d\sigma_0,$$

(15)

which is the unnumbered equation just before eq. (2) of his § 9. Eq. (15), together with (13) is clearly the standard circulation theorem, generally associated to the name of Kelvin.

Hankel does not seem to wish highlighting this result, hence the unnumbered equations. Also, the result is not mentioned in the Fortschritte review. Nevertheless, the fact that Hankel proved the circulation theorem eight years before Kelvin did not escape the attention of Truesdell, who even proposed calling it the “Hankel–Kelvin circulation theorem.” Truesdell did not however explain how Hankel proceeded and furthermore never cited Cauchy’s invariants equations, but just the Cauchy’s vorticity formula. This could be the reason why Truesdell’s rather justified suggestion did not seem to have many followers, one exception being a book on ship propellers by Breslin and Andersen who were aware of Truesdell’s suggestion. One further application by Hankel of the Stokes theorem, gives him the constancy in time of the flux of the vorticity through any finite surface moving with the fluid. Finally, he lets this surface shrink to an infinitesimal element and obtains Helmholtz’s theorem.

To conclude this section on Hankel, we ask: how much was his work on hydrodynamics remembered? An interesting case is that of Heinrich Martin Weber (1842–1913), who was quite close to Riemann. In 1868, Weber wrote a paper titled Über eine Transformation der hydrodynamischen Gleichungen (On a transformation of the equations of hydrodynamics) which, from the point of view of its scientific content, is very closely related to Cauchy’s 1815 invariants equations and even more so to Hankel’s 1861 reformulation (12). Specifically, by “decurling” (12) in Lagrangian coordinates, one obtains

$$\sum_{k=1}^{3} \dot{x}_k \nabla^L x_k = v_0 - \nabla^L W,$$

(16)

where $v_0$ is the initial velocity and $W$ a scalar function, here called “the Weber function”. Actually, Weber showed that $W$ is the time integral from 0 to $t$, in Lagrangian coordinates, of $p - (1/2)|v|^2$, where $p$ is the pressure and $v$ the velocity. Weber derived his equation by a clever transformation of Lagrange’s equation (17), now called the “Weber transform.” Weber did cite Hankel but without an actual reference and just as a person who had pointed out that the so-called Eulerian and Lagrangian coordinates both were first introduced by Euler (a statement made by Hankel but attributed by him to his advisor, Riemann).34

Felix Auerbach (1856–1933),35 a German scientist with wide-ranging interests in all areas of physics, in hydrodynamics, in architecture and painting, wrote in his early days Die theoretische Hydrodynamik nach dem Gange ihrer Entwicklung in der neuesten Zeit, in Kürze

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33 On Thomson’s (Lord Kelvin) own derivation of the circulation theorem, see Thomson (Lord Kelvin), 1869.

34 Weber, 1868.

35 He and his wife took their lives on February 26, 1933.
dargestellt (A brief presentation of theoretical hydrodynamics, following its evolution in the most recent times). This manuscript won the Querini Stampalia Foundation prize of the Royal Venetian Institute of Sciences, Letters and Arts (Atti del Reale Istituto Veneto di Scienze, Lettere ed Arti) on the assigned theme of “Essential progress of theoretical hydrodynamics.” Several pages are devoted to Hankel’s hydrodynamics work and the manuscript contains also a brief reference to Cauchy on p. 34.\textsuperscript{36}

IV. STOKES, LÉVY, LAMB, ANDRADE AND APPELL

An Irish physicist and mathematician, George Gabriel Stokes (1819–1903) spent all his career at the University of Cambridge in England and was considered a leading British scientist, particularly so for many contributions to the dynamics of both ideal and viscous fluids.\textsuperscript{37}

Stokes followed rather closely the work of French mathematicians and physicists and made genuine efforts to cite other scientists’s work. To the best of our knowledge, he was the first to realize the importance of the discovery of the Cauchy invariants (called by Stokes “integrals”) and of the ensuing Cauchy formula \textsuperscript{[4]} for the vorticity. In three papers in the late 1840s, Stokes discussed various proofs of Lagrange’s theorem on the persistence in time of potentiality for 3D incompressible flow.\textsuperscript{38}

In particular, in the 1848 paper “Notes on Hydrodynamics IV” Stokes describes in detail

\textsuperscript{36} Auerbach, 1881.
\textsuperscript{37} For the biography of Stokes, see Parkinson, 2008, Wood, 2003, Wilson’s 2011 two volumes on the correspondence between Stokes and Kelvin with many biographical elements (Stokes & Thomson, 1846–1909, 1870–1903), and the obituary by Rayleigh (Strutt (Lord Rayleigh), 1904).
\textsuperscript{38} Stokes, 1845, 1846, 1848.
Cauchy’s proof of Lagrange’s theorem and also gives an alternative proof of his own. Concerning Cauchy’s proof, Stokes writes:

*The theorem considered follows as a particular consequence from M. Cauchy’s integrals. As however the equations employed in obtaining these integrals are rather long, and the integrals themselves do not seem to lead to any result of much interest except the theorem enunciated at the beginning of this article. Stokes also gave an alternative proof of his own, not using the Cauchy integrals.*

However, as observed by Meleshko and Aref, in 1883 when Stokes edited his “Mathematical and Physical Papers”, he introduced a footnote, referring to an added note at the end of the paper. This note begins as follows:

*It may be noticed that two of Helmholtz’s fundamental propositions respecting vortex motion follow immediately from Cauchy’s integrals; or rather, two propositions the same as those of Helmholtz merely generalized so as to include elastic fluids follow from Cauchy’s equations similarly generalized.*

The two propositions are (i) that “the same loci of particles which at one moment are vortex lines remain vortex lines throughout the motion” and (ii) in modernised language, that the product of the modulus of the vorticity and of the area of a perpendicular section of an infinitesimal vortex tube does not change in time while following the Lagrangian motion. Actually Stokes’s statement should not be misread: he mentions “Cauchy’s integrals” but, in 1883, Stokes understands by this only the Cauchy vorticity formula, which of course was derived from Cauchy’s invariants equations.

Maurice Lévy (1838–1910) was a French engineer and specialist of continuum mechanics. In 1890 he gave a lecture on “Modern hydrodynamics and the hypothesis of action at a distance” at the Collège de France where he was a professor. On the first page of the published version, Lévy writes:

*The admirable properties of vortices were discovered only in 1858 by Helmholtz, although they merely*
express the intermediate integrals of Lagrange’s hydrodynamical equations, discovered by Cauchy…

Lévy observed that Cauchy wrote three (scalar) conservation laws; together with the condition of incompressibility this makes four equations for the three components of the Lagrangian map. Lévy stated that the equations are actually compatible (this follows from the fact that the Lagrangian divergence of Cauchy’s three invariants vanishes). It is of particular interest that Lévy very much highlighted what we would today call the nonlocal character of the equations of incompressible fluid dynamics. This did not seem to him in violation of any known mechanical principle. Of course, such observations were made fifteen years before the birth of relativity theory. Today, we know that the nonlocal character stems from taking the limit of vanishing Mach number for a slightly compressible fluid, a limit that amounts to letting the speed of sound tend to infinity.

Horace Lamb (1849–1934), a British applied mathematician, wrote one of the most authoritative treatises on hydrodynamics, with editions ranging from 1879 to 1932 (the title “Hydrodynamics” was used only from 1895). From 1895, Lamb had Cauchy’s invariants equations (8) but only as an intermediate step to obtain Cauchy’s vorticity formula (9) and Cauchy’s derivation of Lagrange’s theorem. There is no mention of “integrals” or “invariants”, although mere inspection of the equations makes it clear that we are here dealing with integrals of motion, as Cauchy himself pointed out in 1815.42

Paul Émile Appell (1855–1930), a French mathematician with a talent for simple and illuminating writing, published in 1897 a paper where he gave an elementary and immediate interpretation of Cauchy’s equations, leading to the fundamental theorems of the theory of vortices. He first rederived Cauchy’s invariants equations (8). He then considered the following first-order differential form

$$v \cdot dx - v_0 \cdot da,$$  \hspace{1cm} (17)

where $v_0$ is the initial velocity, and showed that as a consequence of (8) it is an exact differential $dV$ of some function $V$. (Actually $V = -W$ where $W$ is the Weber function defined near the end of Section III.A.) Since the integral of such an exact form on a closed contour vanishes with suitable connectedness and regularity assumptions, Appell immediately obtained the Hankel–Kelvin circulation theorem. Appell pointed out that here he was just following Poincaré’s Théorie des tourbillons (Lectures on vortices). Actually, Poincaré’s derivation is a bit more mathematical and quite close to Hankel’s derivation of the circulation theorem (see Section III.A). The derivation is again based on the Cauchy invariants equations (8) for which Poincaré cites Kirchhoff’s “Lectures on mathematical physics (mechanics)”7. The latter writes indeed Cauchy’s invariants equations (8) [Lecture 15, §3, eq. (14) on p. 165] but does not give any reference.43

Jules Andrade (1857–1933), a French specialist of mechanics and chronometry, published in 1898 “Leçons de mécanique physique” (Lectures on physical mechanics). Its Chapter VI was devoted to fluid dynamics. On p. 242, Andrade derived the Cauchy invariants equations (8), which he called “Cauchy’s intermediate integrals”. Andrade also derived Cauchy’s vorticity formula (9) and Lagrange’s theorem, closely following Cauchy. Andrade then stated Les théorèmes d’Helmholtz sont aussi refermés dans ces équation, mais Cauchy ne les a pas aperçus. (Helmholtz’s theorems are also contained in these equations but Cauchy did not perceive them). He then showed how to derive Helmholtz’s result along more or less the lines used by Stokes in his 1883 added note (see above).44

41 Les admirables propriétés des tourbillons n’ont été découvertes qu’en 1858 par Helmholtz, bien qu’elles n’expriment pas autre chose que les intégrales intermédiaires des équations de l’Hydrodynamique de Lagrange, découvertes par Cauchy…
42 Lamb, 1932: §146, p. 204 first unnumbered equation; also §143, p. 225 first unnumbered equation of the 1895 edition.
43 Appell, 1897. Poincaré, 1893. Kirchhoff, 1876.
44 Andrade, 1898.
V. 20TH CENTURY

The material is here separated into three subsections: Section V.A has not only Cauchy’s work on the invariants equations forgotten, but the invariants themselves never mentioned. In Section V.B we find the independent rediscovery of the Cauchy invariants by application of Noether’s theorem. Eventually, in Section V.C everything will reconnect in the late 20th century.

A. The (Cauchy) invariants fallen into oblivion

The 20th century was to see a tremendous rise of research in fluid dynamics, driven to a significant part by the needs of the blossoming aeronautical industry. For this, the study of flow constrained by external or internal boundaries – with viscous boundary layers of the kind introduced by Prandtl in 1905 – was essential.

Inclusion of viscous effects requires the use of the Navier–Stokes equations, which are somewhat easier to study in Eulerian coordinates. Mathematical issues, relating to ideal and viscous fluid flow, such as the well-posedness of the fluid dynamical equations, started being addressed with the new tools of functional analysis. For example, Lichtenstein gave the first proof of the well-posedness for at least a finite time of the three-dimensional incompressible Euler equations with sufficiently smooth initial data. Then, Hölder and Wolibner independently showed that, under suitable conditions, the two-dimensional incompressible Euler equations constitute a well-posed problem for all times. Leray obtained similar results in the viscous case and introduced the important concept of “weak solutions” which need not be differentiable.

We do not give here other details. We stress that these results were generally obtained using Eulerian coordinates, with an occasional excursion into Lagrangian coordinates by Lichtenstein. It seems that during the 20th century Cauchy’s invariants equations were hardly used for mathematical studies. On the one hand, this could be because of the general belief at that time, going back to Lagrange, that (what we now call) Lagrangian coordinates are unnecessarily complicated; as we already stated, the questionable character of this belief was underlined by Dirichlet.

On the other hand, it may be that Cauchy’s Lagrangian formulation through (8) just drifted into oblivion.

In order to understand better what had happened, we examined, in addition to the mathematical papers already cited, a considerable number of major fluid mechanics textbooks published in the 20th century and looked for citations of Cauchy’s 1815/1827 work. A fully relevant citation would not only have Cauchy’s invariants equations, but also stress, as Cauchy did, that they define Lagrangian invariants. Here a word of caution is required: since Stokes in 1883, several authors have referred to the Cauchy vorticity formula as “Cauchy’s integrals.” Cauchy used the word “integrals” in connection with (8) and not (9). We failed to find any truly relevant citations before the very end of the 20th century, although Cauchy’s invariants equations (or a 2D instance) were rediscovered independently. Hereafter, we indicate some of the partially relevant findings.

Lamb’s treatment of Cauchy remained exactly what it was in 1895 (see Section IV) with little emphasis on (8).

Lichtenstein, in addition to his pioneering papers on the mathematical theory of ideal flow, published several books. In 1929 he produced volume XXX Grundlagen der Hydrodynamik (Foundations of hydrodynamics) of an encyclopedia of mathematics with emphasis on applications, edited by Richard Courant. Here, in Chapter 10, Cauchy’s invariants equations appear briefly [his eq. (54) for the case \( \rho = 1 \)] but are not directly attributed to Cauchy and no use is made of the invariance other than deriving the Cauchy vorticity formula.
Sommerfeld’s 1945 “Mechanics of deformable bodies” and Landau and Lifshitz’s 1944 first edition of “Fluid Mechanics” seem to contain neither the Cauchy invariants nor the Cauchy vorticity formula. In his Encyclopedia of Physics article on the foundations of fluid dynamics, Oswatitsch follows rather closely Cauchy’s original derivation of the vorticity formula but skips the Cauchy invariants (a similar treatment with some allowance for turbulent fluctuations is made by Goldstein). We have already mentioned in Section II.A that Truesdell cited Hankel quite extensively but cited Cauchy only for his vorticity formula. Ramsey has (8) but, again, only as an intermediate step in proving Cauchy’s vorticity formula. Batchelor in his “Introduction to Fluid Dynamics” derived the vorticity formula and attributed it to Cauchy. Finally, Stuart and Tabor in their introductory paper to the Theme Issue of Philosophical Transactions, devoted to the Lagrangian description of fluid motions, have Cauchy’s invariants equations: these are their equations (2.13)–(2.15), which were here derived from the Cauchy vorticity formula; this amounts to retracing Cauchy’s steps in reverse. It is not mentioned that the resulting equations were already in Cauchy 1815/1827.50

B. The (Cauchy) invariants rediscovered and Noether’s theorem

As we shall see, the rebirth of Cauchy’s invariants at the very end of the 20th century and the beginning of the 21st (Section V.C) was preceded by rediscoveries (without Cauchy being named). The most widely known of these rediscoveries used a novel tool developed in the early 20th century. Emmy Noether (1882–1935) is recognized as one of the most important mathematicians of all times for her work in algebra.51

In other fields of mathematics and in mathematical physics, she is also known for major contributions. Here, we are concerned with a theorem (now called “Noether’s theorem”), which she proved in 1915 and published in 1918, that relates continuous symmetry groups and invariants for mechanical systems possessing a Lagrangian variational formulation. With the development of quantum mechanics and field theory in the 20th century, this theorem was to acquire a central role and is covered in most textbooks on analytical mechanics or

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50 Sommerfeld, 1945. Landau & Lifshitz, 1944. Oswatitsch, 1959. Goldstein, 1938. Chap. 5, § 84. Truesdell, 1954. Ramsey, 1913: Chap. 2, 22. Batchelor, 1966: 376. Stuart & Tabor, 1990.
51 See, e.g., Dick, 1981 and the Wikipedia article at http://en.wikipedia.org/wiki/Emmy_noether
on field theory.\footnote{Noether, 1918.}

It has been known since Lagrange’s “Mécanique Analitique” of 1788 that the motion of an incompressible three-dimensional fluid possesses a variational formulation. In modern language, if the fluid occupies the whole space \( \mathbb{R}^3 \) and the Lagrangian map is specified at time \( t = 0 \) (the identity) and at some time \( T > 0 \), the Lagrangian map \( \mathbf{x}(a, t) \) at intermediate times is an extremum of the action integral

\[
S = \int_0^T dt \int_{\mathbb{R}^3} d^3a \, L(\dot{x}), \quad \text{where} \quad L(\dot{x}) \equiv \frac{1}{2} |\dot{x}|^2,
\]

with the constraint of incompressibility that

\[
\det(\nabla^L \mathbf{x}(a, t)) = 1 \quad \text{for all} \quad 0 \leq t \leq T,
\]

where we recall that \( \nabla^L \) denotes the Lagrangian gradient. Indeed if, following Lagrange, we introduce infinitesimal variations \( \delta \mathbf{x}(a, t) \), vanishing at \( t = 0 \) and at \( t = T \), we find from \footnote{Newcomb, 1967.} that the vanishing of the variation of the action requires (after an integration by parts over time) that

\[
\delta S = -\int_0^T dt \int_{\mathbb{R}^3} d^3a \, \ddot{\mathbf{x}}(a, t) \cdot \delta \mathbf{x}(a, t) = 0,
\]

for all variations \( \delta \mathbf{x} \) consistent with incompressibility. This constraint is more easily written in Eulerian coordinates by defining

\[
\delta^E \mathbf{x}(x, t) \equiv \delta \mathbf{x}(a(x, t), t),
\]

where \( a(x, t) \) is the inverse of the Lagrangian map. The incompressibility constraint for infinitesimal variations is then simply \( \nabla \cdot \delta^E \mathbf{x}(x, t) = 0 \). (In particular, by taking \( \delta \mathbf{x} = \dot{x} dt \), one has \( \nabla \cdot \mathbf{v} = 0 \), where \( \mathbf{v} = \dot{x} \).) It thus follows that the variation of the action \footnote{Newcomb, 1967.} must vanish for all \( \delta^E \mathbf{x} \) of zero Eulerian divergence. Hence the acceleration \( \ddot{x} \) must be the Eulerian gradient of a suitable function (actually the negative of the pressure):

\[
\ddot{x} = -\nabla p.
\]

It has been known for a long time that the obvious invariances of the Lagrangian \( (1/2)|\dot{x}|^2 \), such as invariance under time and space translations and under rotations, are connected, by Noether’s theorem, to standard mechanical invariants, viz. the conservation of kinetic energy, of momentum and of angular momentum. In 1967 William A. Newcomb, a theoretical physicist well-known for the “Newcomb paradox”, noticed a new continuous invariance group he called “exchange invariance” (which is now mostly called “relabeling symmetry”). From this he inferred, by Noether’s theorem, new invariants which have been identified with the Cauchy invariants many years later (see Section \ref{sec:Cauchy}).

Newcomb observed that the action is preserved if we change the original Lagrangian coordinates \( a \) to new Lagrangian coordinates \( a' \), provided the map from the \( a \)s to the \( a' \)s conserves volumes. An infinitesimal version, needed to apply Noether’s theorem, is

\[
a \rightarrow a + \delta a(a), \quad \nabla^L \cdot \delta a(a) = 0.
\]

The resulting change in the Lagrangian map at time \( t \) is then (in components with summation over repeated indices) \( \delta x_i = \nabla^L_j x_i \delta a_j \) and the change in the action is

\[
\delta S = \int_0^T dt \int_{\mathbb{R}^3} d^3a \, \dot{x}_i \partial^L_i \left( \nabla^L_j x_i \right) \delta a_j,
\]
where $\partial_t^L$ denotes the Lagrangian time derivative, when a dot would be too cumbersome. Setting this variation equal to zero for all perturbations $\delta a$ of vanishing divergence, we find that $\dot{x}_i \partial_t^L (\nabla_j^L x_i)$ should be a Lagrangian gradient. Thus its Lagrangian curl should vanish, that is

$$\nabla^L \times [\partial_t^L (\dot{x}_i \nabla_j^L x_i) - \ddot{x}_i \nabla_j^L x_i] = 0.$$  \hspace{1cm} (25)

By (22), the second term in the square bracket is the Lagrangian gradient of the pressure, whose Lagrangian curl vanishes. Thus (25) is equivalent to the Cauchy invariants equations, when they are written in Hankel’s form, with a curl in front, as in (12).

Actually, Newcomb was preceded in 1960 by Eckart, a specialist of quantum mechanics who applied variational methods to fluid dynamics. Eckart rederived the circulation theorem and obtained the Cauchy invariants equations (his equations (3.9), (4.4) and (4.5)) without any explicit use of Noether’s theorem, but in another 1963 paper Eckart points out that “The general theorems just mentioned are also consequences of this invariance, a fact that does not seem to have been noted before.” (By “this invariance” he understands the unimodular group of volume-conserving transformations of the Lagrangian coordinates.) In 1963 Calkin did apply Noether’s theorem to hydodynamics and magnetohydrodynamics and recovered various known invariants, but apparently not the Cauchy invariants. The subject of the relabeling symmetry was reviewed in Salmon’s Annual Review of Fluid Mechanics paper, which also contains further references, none of which mentions the Cauchy invariants.

Finally, we should mention that a special case of Cauchy’s reformulation of the equations in Lagrangian coordinates was rediscovered in the fifties without any use of Noether’s theorem. In a classical book on water waves, Stoker pointed out that certain hydrodynamical problems involving a free surface are better handled using Lagrangian than Eulerian coordinates. He then described the PhD work of his student Pohle in the early fifties at the Courant Institute of Mathematical Sciences in New York. Pohle established Cauchy’s invariant equation (5) (without citing Cauchy) in the special case of two dimensions when there is a single invariant, while pointing out that “similar results hold for the three dimensional case.” He then assumed analyticity in time of the Lagrangian map and obtained recurrence relations among the corresponding Taylor coefficients. Here, two-dimensionality allowed him to use a complex-variable method to obtain special solutions of relevance to the breakup of a dam. Stoker stated that the assumed time-analyticity could probably be established “at least for a finite time” and pointed out that “the convergence of developments of this kind in some simpler problems in hydodynamics has been proved by Lichtenstein.” This happened indeed, but only recently. Cauchy’s 1815/1827 paper was cited many times in Stoker’s book, in connection with waves, and not in the section discussing the Lagrangian formulation. In the mid-eighties Stoker’s 1957 work on the two-dimensional Lagrangian equations was cited by Abrashkin and Yakubovich, among the persons who in the late nineties would be involved in correctly identifying Cauchy’s role in the discovery of the invariants (see, Section C.55).

C. The renaissance of Cauchy’s invariants

It was only when the 20th century was nearing its end that Cauchy’s name was again associated to his invariants, thanks to Russian scientists. Of course Russia has had for a long time a very strong tradition in fluid mechanics. In 1966 Abrashkin, Zen’kovich and Yakubovich, from the Institute of Applied Physics in Nizhny Novgorod, wrote a paper about a new matrix reformulation of the 3D Euler equations in Lagrangian coordinates. Their eqs. (4) are Cauchy’s invariants equations written as three scalar equations, just as in Cauchy’s 1815/1827 paper. The time-independent right hand sides are qualified by them as “integrals of motion” and attributed to Cauchy (only by referring to Lamb). On the next page, they refer to them as “Cauchy invariants.” In a 1997 review paper on nonlinear waves with significant emphasis on plasma physics, Zakharov and Kuznetsov from the Landau

54 Eckart, 1960; 1963: 1038. Calkin, 1963. Salmon, 1988: § 4.
55 Stoker, 1957: § 12.1. Pohle, 1951. Abrashkin & Yakubovich, 1985.
Institute in Moscow discussed the relabeling symmetry and the corresponding conservation law. Their eq. (7.11) gives the Cauchy invariants equations in vector notation. They do point out, including in their Abstract, that these are “the Cauchy invariants.” Again, they cite Cauchy’s work with an indirect reference via Lamb.\textsuperscript{56}

The name “Cauchy invariant” is of course fully appropriate, given the modern meaning of “invariant”, a local or global quantity conserved in the course of time. This name was already used in Russia several years prior to the 1996–1997 publications. Evsey Yakubovich (2013 private communication, transmitted through Evgenii Kuznetsov) confirmed that it was used in internal discussions at the Institute of Applied Physics in the early nineties. Two further publications containing the name “Cauchy invariants”\textsuperscript{57} were published by the Nizhnii Novgorod group.

Cauchy’s role in introducing the invariants having thus finally been recognized, several works discussing the Cauchy invariants have appeared since the year 2000.\textsuperscript{58}

VI. CONCLUDING REMARKS

What have we learned from exploring this more than two-century-long (1788–2014) history of the Lagrangian formulation for the incompressible Euler equations? Actually, we have a situation (here in hydrodynamics), for which a discovery made two centuries back and hardly ever used since, has emerged as particularly relevant for modern research developments. We have in mind Pohle’s early-fifties work on the breakup of dams, the invariants obtained from Noether’s theorem in the sixties and the recent proof of time-analyticity of Lagrangian fluid particle trajectories. In all these instances, Cauchy’s invariants equations do play a key role, but were actually rediscovered in different ways, for example by a transposition to incompressible hydrodynamics of a Lagrangian perturbation method that cosmologists have developed since the nineties.\textsuperscript{59}

Probably, Cauchy made his 1815 discovery too early to be adequately appreciated, because the study of invariants and conservation laws would not emerge as an important paradigm for another century. Only a corollary of \textsuperscript{5}, namely Cauchy’s vorticity formula \textsuperscript{9} was attracting attention in the early 19th century, because of Lagrange’s theorem. The first serious opportunity to understand some of the importance of Cauchy’s invariants equations came in Germany around 1860 after Dirichlet stated that Lagrangian coordinates are important (a statement not heard much again until late in the 20th century), when Göttingen University, possibly prompted by Riemann, pushed for studies based on Lagrangian coordinates to get more insight into Helmholtz’s vorticity theorems and, last but not least, when Hankel found how to make full use of Cauchy’s invariants equations and in the process came across the circulation theorem some years before Kelvin.

Later, in the first decades of the 20th century, Cauchy’s invariants equations faded into oblivion or were viewed a mere intermediate step in proving Cauchy’s vorticity formula. Eventually, developments in theoretical mechanics, closely connected to the rise of quantum mechanics and later of quantum field theory, led to a rediscovery of the invariants in the late 1960 through application of Noether’s theorem. Another 30 years elapsed, during which developments in nonlinear physics were increasingly making use of symmetries and invariants, until the crucial importance of Cauchy’s work could be appreciated.

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\textsuperscript{56} Abrashkin, Zen’kovich and Yakubovich, 1996. Lamb, 1932. Zakharov & Kuznetsov, 1997.
\textsuperscript{57} Yakubovich & Zenkovich, 2001. Abrashkin & Yakubovich, 2006.
\textsuperscript{58} Friedlander & Lipton-Lifschitz, 2003. Bennett, 2006. Kuznetsov, 2006. Eyink, 2013. Ohkitani, 2014.
\textsuperscript{59} Frisch & Zheligovsky, 2013. Zheligovsky & Frisch, 2013.
\textsuperscript{On the relation of Cauchy’s invariants equations and cosmology, see Zheligovsky & Frisch, 2013: §3.
References

Abrashkin, A.A., Yakubovich, E.I. 1981 ‘Nonstationary vortex flows of an ideal incompressible fluid.’ J. Appl. Mech. Tech. Phys., 26, 2, 202–208. Translated from Zh. Prikl. Mekh. Tekh. Fiz., 2, 57–64, 1985, in Russian.

Abrashkin, A.A. & Yakubovich, E.I. 2006 Vortex Dynamics in the Lagrangian Description, Fizmatlit, Moscow.

Abrashkin, A.A., Zenz’kovich, D.A., Yakubovich, E.I. 1996 ‘Matrix formulation of hydrodynamics and extension of ptolemaic flows to three-dimensional motions.’ Radiophys. Quantum El., 39, 6, 518–526. Translated from Izv. Vuz. Radiof., 39, 6, 783–796, 1996, in Russian.

Andrade, Jules 1898 Leçons de Mécanique Physique

Anonymous, (signed as Hl.), 1863 ‘Aufsatz über die allgemeine Theorie der Bewegung der Flüssigkeiten. Eine von der philosophischen Facultät der Georg-August am 4. Juni 1861 gekrönte Preisschrift, Göttingen’ in Die Fortschritte der Physik im Jahre 1861 produced by Physikalische Gesellschaft zu Berlin, 57–61. https://play.google.com/books/reader?id=zt0EAAAmAAJ&printsec=frontcover&output=reader&authuser=0&hl=en&pg=GBS.PA57

Appell, Paul 1897 ‘Sur les équations de l’Hydrodynamique et la théorie des tourbillons.’ Journal de mathématiques pures et appliquées, 5e série, 3, 5–16. http://portail.mathdoc.fr/JMPA/PDF/JMPA_1897_5_3_A1_0.pdf

Arnold, V.I., Zeldovich, Y.B., Ruzmaikin, A.A., Sokoloff, D.D. 1981 ‘A magnetic field in a stationary flow with stretching in a Riemannian space.’ Sov. Phys. JETP., 54, 1083–1086. Translated from Zh. Eksp. Teor. Fiz., 81, 2052–2058, in Russian.

Auerbach, Felix 1881 Die theoretische Hydrodynamik nach dem Gange ihrer Entwicklung in der neuesten Zeit in Kürze dargestellt: von dem K. Venetianischen Institute der Wissenschaften gekrönte Preisschrift, F. Vieweg und Sohn, Braunschweig. https://archive.org/stream/dietheoretische01auergoog#page/n7/mode/2up

Batchelor, G.K. 1967 An Introduction to Fluid Mechanics, Cambridge University Press, Massachusetts.

Belhoste, B. 1982 Augustin-Louis Cauchy et la pratique des sciences exactes en France au XIXème siècle, Thèse de 3ème cycle, Université Paris I.

Belhoste, B. 1991 Augustin-Louis Cauchy: a Biography, New York etc., Springer-Verlag.

Bennett, A. 2006 Lagrangian Fluid Dynamics, Cambridge University Press.

Brenier, Y., Frisch, U., Hénon, M., Loeper, G., Matarrese, S., Mohayaee, R., Sobolevski, A. 2003 ‘Reconstruction of the early Universe as a convex optimization problem.’ Mon. Not. R. Astron. Soc., 346, 501–524.

Brezin, J.P. & Andersen, P. 1996 Hydrodynamics of Ship Propellers, Cambridge University Press.

Buchert, T. 1992 ‘Lagrangian theory of gravitational instability of Friedman-Lemaître cosmologies and the Zeldovich approximation.’ Mon. Not. R. Astron. Soc., 254, 729–737.

Calkin, M.G. 1963 ‘An action principle for magnetohydrodynamics.’ Can. J. Phys., 41, 12, 2241–2251.

Cantor, Moritz 1879 ‘Hankel Hermann H.’ in Allgemeine deutsche Biographie, 10, 516–519. http://daten.digitale-sammlungen.de/0000/bsb00008368/images/index.html?fip=193.174.98.30&id=00008368&seite=518

Cauchy, Augustin-Louis 1815/1827 ‘Théorie de la propagation des ondes à la surface d’un fluide pendant d’une profondeur indéfinie – Prix d’analyse mathématique remporté par M. Augustin-Louis Cauchy, ingénieur des Ponts et Chaussées. (Concours de 1815).’ Mémoires présentés par divers savans à l’Académie royale des sciences de l’Institut de France et imprimés par son ordre. Sciences mathématiques et physiques. Tome I, imprimé par autorisation du Roi l’Imprimerie royale, 5–318. http://gallica.bnf.fr/ark:/12148/bpt6k000941x/f14.image.r=Oeuvres%20complètes%20d%27Augustin%20Cauchy.langFR

Childress, S. & Gilbert, A.D. 1995 Stretch, Twist, Fold: The Fast Dynamo, Vol. 37, Springer Verlag.

Crowe, M.J. 2008 ‘Hankel, Hermann’ in The History of Modern Mathematics, Vol.2, Proceedings of the Symposium on the History of Modern Mathematics, New-York, June, 20-24, 1988, edited by D.E. Rowe, J.Mc Cleary, 129–168.

Dalmedico, A.D. 1989 ‘La propagation des ondes en eau profonde et ses développements mathématiques (Poisson, Cauchy 1815–1825).’ In The History of Modern Mathematics, Vol.2, Proceedings of the Symposium on the History of Modern Mathematics, New-York, June, 20-24, 1988, edited by D.E. Rowe, J.Mc Cleary, 129–168.

Darrigol, O. 2005 Worlds of flow: A history of hydrodynamics from the Bernoulli to Prandtl, Oxford University Press.

Darrigol, O. & Frisch, U. 2008 ‘From Newtons mechanics to Eulers equations.’ Physica D, 237, 14, 1855–1869.

Dick, Auguste 1970 Emmy Noether: 1882-1935, Birkhäuser, Basel. Translated into English by H.I. Blocher, 1981, Birkhäuser, Boston Inc. https://archive.org/stream/EmmyNoether1882-1935/
Lévy, Maurice 1890 ‘L’hydrodynamique moderne et l’hypothèse des actions a distance.’ Revue Générale des Sciences pures et appliquées, 23, 721–728. http://gallica.bnf.fr/ark:/12148/bpt6k213928x/f723.image.r=maurice%20levy%20hydrodynamique%20moderne%20a%20distance.langEN

Lichtenstein, Leon 1927 ‘Über einige Existenzprobleme der Hydrodynamik. Zweite Abhandlung. Nichthomogene, unzusammenhängende, reibunglose Flüssigkeiten.’ Mathematische Zeitschrift, 26, 1, 196–233. http://gdz.sub.uni-goettingen.de/dms/load/img/?PPN=GDZPPN002369230&IDDOC=82659

Lichtenstein, Leon 1929 Grundlagen der Hydrodynamik, Die Grundzüge der mathematischen Wissenschaften. Band XXX, Julius Springer, Berlin.

Majda A.J. & Bertozzi, A.V. 2002 Vorticity and Incompressible Flow, Cambridge University Press.

Meleshko, V.V. & Aref, H. 2007 ‘A bibliography of vortex dynamics 1858–1956.’ Advances in Applied Mechanics, 41, 197–292.

Moffatt, H.K. 1978 Magnetic Field Generation in Electrically Conducting Fluids, Cambridge University Press, Cambridge, London, New York, Melbourne.

Monna, A.F. 1973 ‘Hermann Hankel’ Nieuw Arch. voor Wisk., 21, 64–87.

Moutarde, F., Alimi, J.-M., Bouchet, F., Pellat, R., Ramani, A. 1991 ‘Precollapse scale invariance in gravitational instability.’ Astrophys. J., 382, 377–381.

Newcomb, W. A. 1967 ‘Exchange invariance in fluid systems.’ Proc. Symp. Appl. Math., 18, 152–161.

Noether, Emmy 1918 ‘Invariante Variationserprobungen.’ Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen, mathematisch-physikalische Klasse 1918, 235–257. http://gdz.sub.uni-goettingen.de/dms/load/img/?PPN=GDZPPN00250510X&IDDOC=63716, Translated into English by M.A. Tavel, 1971 ‘Invariant Variation Problem.’ Transport Theory and Statistical Physics, 1, 3, 183–207. http://arxiv.org/pdf/physics/0503066.pdf

Okitani, K. 2014 ‘An Elementary Account of Vorticity and Related Equations: Towards understanding dynamics of localised vortices in nonlocal interaction, Cambridge University Press, to appear.

Oswatitsch, K. 1959 Strömungsmechanik, Band VIII/1 in Handbuch der Physik, edited by S. Flügge, Springer.

Padhye, N. & Morrison, P.J. 1996 ‘Fluid element relabeling symmetry.’ Phys. Lett. A, 219, 287–292.

Parkinson, E.M. 2008 ‘Stokes, George Gabriel’, in Dictionary of Scientific Biography. Encyclopedia.com. http://www.encyclopedia.com/doc/1G2-2830904174.html

Pohle, F.V. 1951 ‘The Lagrangian equations of hydrodynamics: solutions which are analytic functions of the time.’ Thesis, New York University, January 1951.

Poincaré, Henri 1893 Théorie des tourbillons, Gauthier-Villars, Paris. https://archive.org/stream/thoriedestourb00poin#page/n5/mode/2up

Prandtl, Ludwig 1905 ‘Über Flüssigkeitsbewegung bei sehr kleiner Reibung.’ in Verhandlungen des III. internationalen Mathematiker-Kongresses in Heidelberg vom 8. bis 13. August 1904, edited by A. Krazer, B.G. Teubner, Leipzig, 484–491. https://archive.org/stream/verhandlungende00krazgoog#page/n504/mode/2up, Translated into English 1928 ‘On the motion of fluids of very small viscosity.’ NACA, Tech. Memo., N. 452 http://ntrs.nasa.gov/archive/nasa_casi.ntrs.nasa.gov/19930090813_1993090813.pdf

Procès-Verbaux des Séances de l’Académie, Tome V, 1812–1815, 1915, Académie des Sciences, Institut de France. http://gallica.bnf.fr/ark:/12148/bpt6k32981/f7.image

Ramsey, Arthur Stanley 1913 A Treatise On Hydromechanics. Part II, Hydromechanics G. Bell and Sons, LTD, London. http://ebook.lib.hku.hk/CADAL/B31396288V2

Risser, M.R. 1925 ‘Études sur la théorie des ondes par émission.’ Thèse présentée a la faculté des sciences de Paris, Gauthier-Villars et Cie, Éditeurs, Paris. http://archive.numdam.org/ARCHIVE/these/these1925__53_/these1925__53__1_0/These_1925__53__1_0.pdf

Rose, H.A. & Sulem, P.-L. 1978 ‘Fully developed turbulence and statistical mechanics.’ J. Phys. Paris, 39, 441–484.

Salmon, R. 1988 ‘Hamiltonian fluid mechanics.’ Ann. Rev. Fluid Mech., 20, 225–256.

Serfati, P. 1995 ‘Équation d’Euler et holomorphies à faible régularité spatiale.’ C. R. Acad. Sci. Sér. I, Math., 320, 2, 175–180.

Shnirelman, A. 2012 ‘On the analyticity of particle trajectories in the ideal incompressible fluid.’ submitted to Global and Stochastic Analysis, 2.

Smithies, F. 1997 Cauchy and the creation of complex function theory, Cambridge University Press. Sommerfeld, A. 1950 Mechanics of Deformable Bodies, Academic Press, Inc., New York.

Stoker, James Johnston 1957 Water waves: The Mathematical Theory with Applications, Wiley-Interscience. https://archive.org/stream/waterwaves00stok#page/n9/mode/2up

Stokes, George Gabriel 1845, [printed in 1847], ‘On the theories of the internal friction of fluids in motion and of the equilibrium and motion of elastic solids.’ Transactions of the Cambridge Philosophical Society, 8, 287–319. http://archive.org/stream/transactionsofca08camb#page/287/mode/1up
Stokes, George Gabriel 1846, [printed in 1847], ‘Report on the recent researches in Hydrodynamics.’ Report of the British Association for the Advancement of Science, 1–20. Report of the sixteenth Meeting of the British association held at Southampton in September 1846. [http://archive.org/stream/reportofbritish46brit#page/n47/mode/2up]

Stokes, George Gabriel 1848 ‘Notes on Hydrodynamics. IV Demonstration of a fundamental theorem.’ The Cambridge and Dublin Mathematical Journal, Vol. III, 209–219. [https://archive.org/stream/streemathphyss02stokrich#page/n51/mode/2up]

Stokes, George Gabriel 1883 ‘Notes on Hydrodynamics. There is also Demonstration of a fundamental theorem.’ Mathematical and Physical Papers by George Gabriel Stokes, reprinted from the original journals and transactions, with additional notes by the Author, Cambridge University Press, 2, 36–50. [https://archive.org/stream/mathphyspapers02stokrich#page/n51/mode/2up]

Stokes, George Gabriel & Thomson, William (Lord Kelvin) 1846–1869 & 1870–1903. Vols. 1 and 2 of The Correspondence Between Sir George Gabriel Stokes and Sir William Thomson, Baron Kelvin of Largs, edited with an Introduction by D.B. Wilson, Cambridge University Press.

Strutt, John William (Lord Rayleigh) 1904 ‘Sir George Gabriel Stokes, Bart. 1819-1903.’ Proceedings of the Royal Society of London, 75, 199–216. [http://rspl.royalsocietypublishing.org/content/75/19.full.pdf]

Stuart, J.T. & Tabor, M. 1990 ‘The Lagrangian picture of Fluid Motion.’ Phil. Trans. R. Soc. A, 333, 263–271.

Thomson, William (Lord Kelvin) 1869 ‘On vortex motion.’ Transactions of the Royal Society of Edinburgh, 25, 217–260. [https://archive.org/stream/transactionsofroyal#page/n245/mode/2up]

Toschi, F. & Bodenschatz, E. 2009 ‘Lagrangian properties of particles in turbulence.’ Annu. Rev. Fluid Mech., 41, 375–404.

Truesdell, C. 1954a The Kinematics of Vorticity, Indiana University Science Series no. 19. Indiana University Press (Bloomington).

Truesdell, C. 1954b ‘Rational fluid mechanics, 1657–1765.’ In Euler, Opera omnia, ser. 2, 12 (Lausanne), IX–CXXV.

Vallon, Claude-Alphonse 1868 La vie et les travaux du baron Cauchy: membre de l’Academie des sciences, Vols. 1 and 2, Gauthier-Villars. [https://archive.org/details/lavieetlestrava00valsgoog]

Vishik, M.M. 1989 ‘Magnetic field generation by the motion of a highly conducting fluid.’ Geophys. Astro. Fluid, 48, 151–167.

Weber, H.M., 1868 ‘Über eine Transformation der hydrodynamischen Gleichungen.’ Journal für die reine und angewandte Mathematik (Crelle), Berlin, 68, 286–292. [http://gdz.sub.uni-goettingen.de/dms/load/img/?PPN=GDZPPN00215353X]

Wood, A. 2003 ‘George Gabriel Stokes 1819–1903’ in Physicists of Ireland, Passion and Precision McCartney, M. & Whitaker, A., eds., pp. 85–94, Institute of Physics Publishing (Bristol & Philadelphia).

Yakubovich, E.I., & Zenkovich, D.A. 2001 ‘Matrix approach to Lagrangian fluid dynamics.’ J.Fluid Mech., 443, 1, 167–196.

von Zahn, Wilhelm 1874 ‘Einige Worte zum Andenken an Hermann Hankel.’ Mathematische Annalen, 7, 4, 583–590. [http://gdz.sub.uni-goettingen.de/dms/load/img/?PPN=PPN236581684_0007&DMDID=dmidxg69]

Zakharov, V.E. & Kuznetsov, E.A., 1997 ‘Hamiltonian formalism for nonlinear waves.’ Phys.-Usp., 40, 11, 1087–1116. Translated from Usp. Fiz. Nauk, 167 (11), 1137–1167, in Russian.

Zheligovsky, V. & Frisch, U. 2013 ‘Time-analyticity of Lagrangian particle trajectories in ideal fluid flow.’ Submitted to J.Fluid Mech. [http://arxiv.org/abs/1312.6320]