Anisotropic particle production and azimuthal correlations in high-energy pA collisions

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We summarize some recent ideas relating to anisotropic particle production in high-energy collisions. Anisotropic gluon distributions lead to anisotropies of the single-particle azimuthal distribution and hence to disconnected contributions to multi-particle cumulants. When these dominate, the four-particle elliptic anisotropy $c_2\{4\}$ changes sign. On the other hand, connected diagrams for $m$-particle cumulants are found to quickly saturate with increasing $m$, a “coherence” quite unlike conventional “non-flow” contributions such as decays. Finally, we perform a first exploratory phenomenological analysis in order to estimate the amplitude $A$ of the $\cos(2\phi)$ anisotropy of the gluon distribution at small $x$, and we provide a qualitative prediction for the elliptic asymmetry from three-particle correlations, $c_2\{3\}$.

Contents

I. Introduction 2

II. Scattering of a charge off a semi-classical field 2

III. Multi-particle cumulants 3

A. Connected contributions to high-order cumulants 4
B.Disconnected contributions to the two- and four-particle cumulants 5
C. The three-particle cumulant $c_2\{3\}$ 6
D. BBGKY-like hierarchy of $m$-particle $c_2\{m\}$ cumulants 8
E. Odd-index two-particle cumulants, $c_1\{2\}$ and $c_3\{2\}$ 8

IV. Anisotropic gluon distribution at small $x$ 9

A. Classical McLerran-Venugopalan model 9
B. Quantum fluctuations and high-energy evolution 11

V. Application to phenomenology of proton-nucleus collisions 12

VI. Summary and Outlook 14

Acknowledgements 15

References 16
I. INTRODUCTION

The observation of large asymmetries, predominantly an elliptic \( \cos(2\phi) \) asymmetry, in the azimuthal distribution of particles produced in heavy-ion collisions has been one of the main indications for the formation of a “nearly perfect QCD liquid” \cite{1}. For particles with transverse momenta up to a few times \( \langle p_T \rangle \) this phenomenon is usually explained in terms of (nearly inviscid) hydrodynamic expansion of an asymmetric “fireball”; on the other hand, for high-\( p_T \) particles the asymmetry is thought to originate from energy loss of (mini-) jets along different paths through the hot and dense Quark-Gluon plasma \cite{2}.

More recently, substantial azimuthal asymmetries have also been observed in p+Pb collisions at the LHC \cite{3,4} and in d+Au collisions at RHIC \cite{5}. They are measured via multi-particle angular correlations (see below) and were found to extend over a long range in rapidity. By causality, the correlations must originate from the earliest times of the collision \cite{6}. The data shows that the asymmetries persist up to rather high transverse momenta, well beyond \( p_\perp \approx 1 \text{ GeV} \). In fact, a recent publication by the ATLAS collaboration shows that substantial “elliptic” (\( v_2 \)) asymmetries in p+Pb collisions at \( \sqrt{s} = 5 \text{ TeV} \) persist up to \( p_\perp = 10 \text{ GeV} \) \cite{7}. Final state energy loss is expected to be much less prominent in smaller systems created in p+p and p+A collisions; thus it appears reasonable to investigate if azimuthal asymmetries could originate from the instant of collision when (anti-) quarks and gluons are “liberated” from the wave functions of the colliding hadrons. Since semi-hard processes involve short-distance QCD dynamics, we believe that it is important to develop an understanding of possible origins of azimuthal asymmetries in perturbative QCD \cite{8,9,10,11,12,13,14,15,16,17,18,19}.

This paper is a write-up of the talks presented by the authors at the “Initial Stages 2014” conference in Napa, CA. It is not a comprehensive review but attempts to summarize and combine in one paper a few recent ideas for anisotropic particle production and correlations within short-distance, small-\( x \) QCD.

II. SCATTERING OF A CHARGE OFF A SEMI-CLASSICAL FIELD

In the eikonal approximation the S-matrix for scattering of a parton in the representation \( \mathcal{R} \) of color-SU\( (N_c) \) off the target is given by \cite{19}

\[
S_1(\mathbf{r}, \mathbf{b}) = \frac{1}{d_\mathcal{R}} \text{tr}_\mathcal{R} V^\dagger(\mathbf{x}) V(\mathbf{y}) \, , \tag{1}
\]

where \( \mathbf{r} \equiv \mathbf{x} - \mathbf{y} \) and \( \mathbf{b} \equiv \frac{1}{2}(\mathbf{x} + \mathbf{y}) \) are the dipole radius and the impact parameter respectively. We have implicitly assumed that the target field is written in covariant gauge so that the gauge links from \( \mathbf{x} \) to \( \mathbf{y} \) and back can be dropped. \( d_\mathcal{R} \) is the dimension of the representation \( \mathcal{R} \) and \( V(\mathbf{x}) \) denotes a light-like Wilson line describing the propagation of the parton in the field of the target

\[
V(\mathbf{x}) = \mathbb{P} \exp \left\{ ig \int dx^- A^a_+(x^-, \mathbf{x}) t^a_R \right\} . \tag{2}
\]

Below we shall write most expressions for a fundamental projectile charge, a quark or anti-quark. The S-matrix for an adjoint charge (gluon) can be obtained from group theory,

\[
S_A(\mathbf{r}) = \frac{N_c^2 |S_E(\mathbf{r})|^2 - 1}{N_c^2 - 1} . \tag{3}
\]

While \( S_E(\mathbf{r}) \) is complex (for \( N_c \geq 3 \) colors), \( S_A(\mathbf{r}) \) is manifestly real. As we shall see below, this implies that the single-particle azimuthal distribution for a quark may in general exhibit odd \( v_{2n+1} \) moments while that for a gluon only has non-zero even moments \( v_{2n} \).

Scattering to high transverse momentum corresponds to small \( |\mathbf{r}| \). This allows us to perform a gradient expansion of the vector potential \( A^+_+(x^-, \mathbf{x}) \) resulting in

\[
S_1(\mathbf{r}, \mathbf{b}) - 1 = \frac{(ig)^2}{2N_c} \text{tr} (\mathbf{r} \cdot \mathbf{E}(\mathbf{b}))^2 + \frac{1}{2} \left( \frac{(ig)^2}{2N_c} \text{tr} (\mathbf{r} \cdot \mathbf{E}(\mathbf{b})) \right)^2 + O(r^6) , \tag{4}
\]

if C-odd exchanges are dropped. The term of order \( r^4 \) will be used in the computation of \( c_2 \) below but is not important for our main point here. In covariant gauge the light-cone electric field of the target in Eq. (4) given by

\[
E^+(\mathbf{b}) = \int dx^- F^+ = -\partial^i \int dx^- A^+(x^-, \mathbf{b}) , \tag{5}
\]
The S-matrix for single parton scattering can be generalized to \( m \) particles,

\[
S_m(r_1, b_1, \ldots, r_m, b_m) - 1 = \left( \frac{ig}{2N_c} \right)^2 \prod_{i=1}^{m} \text{tr} \left( r_i \cdot E(b_i) \right)^2,
\]

where we wrote only the leading order in \( r \) to simplify the expression.

In the current formalism, event averaging corresponds to averaging over the target ensemble, which is defined by the field-field correlator. Conventionally, in the McLerran-Venugopalan model \[20\] one uses

\[
\frac{g^2}{N_c} \langle E^a_i(b_1) E^b_j(b_2) \rangle = \frac{1}{N^2_c - 1} \delta^{ab} \delta_{ij} Q^2 \Delta(b_1 - b_2),
\]

where a general form of the impact parameter dependence of the correlator \( \Delta(b) \) with the Fourier image \( \hat{\Delta}(k) \) has been introduced. \( \Delta(b) \) exhibits a logarithmic divergence as \(|b| \to 0\) which is cut off by the dipole scale \( r \) since the gradient expansion assumes that the electric field is smooth over scales on the order of the size of the probe.

It should be clear that Eq. (7) averages over all fluctuations of the target fields, and hence is isotropic. On the other hand, for observables which are sensitive to the angular structure of the target fields, instead we integrate over target field ensembles subject to the constraint that the anisotropic contribution to the electric field point in a specific direction \( \hat{a} \) \[12, 21, 22\]:

\[
\frac{g^2}{N_c} \langle E^a_i(b_1) E^b_j(b_2) \rangle_{\hat{a}} = \frac{1}{N^2_c - 1} \delta^{ab} Q^2 \Delta(b_1 - b_2) \left( \delta_{ij} + 2A \left[ a_i \hat{a}_j - \frac{1}{2} \delta_{ij} \right] \right).
\]

That is, we divide the target ensembles into subclasses corresponding to a particular direction of \( \hat{a} \) in the vicinity of the point with the coordinates \( b \). The summation over all subclasses (integration with respect to all possible orientations \( \hat{a} \)) is performed after the observables (such as the \( m \)-particle cumulants) have been computed. In other words, the fluctuations from one configuration \( E(x) \) to another which spontaneously break 2D rotational symmetry constitute slow variables.

The transverse momentum distribution of scattered partons can now be written as\(^1\)

\[
(2\pi)^2 \frac{dN}{dk dk d\varphi_k} = \int d^2b \int d^2r e^{-ik \cdot \vec{r}} S(r, b) \}
\]

\[
= \int d^2b \int dr \, d\varphi_r \, e^{-ikr \cos(\varphi_k - \varphi_r)} S(r, \varphi_r, b).
\]

The S-matrix satisfies

\[
S(r, \varphi_r) = S^*(r, \varphi_r + \pi).
\]

Thus, its real part is even under \( \varphi_r \to \varphi_r + \pi \) (i.e. \( \vec{r} \to -\vec{r} \)) while its imaginary part is odd.

We can define various asymmetry moments \( v_n \) of the single-inclusive distribution through

\[
v_n(k_T) = \langle \cos n(\varphi_k - \varphi_\hat{a}) \rangle = \frac{1}{N} \int d\varphi_k 2\pi \left\langle \cos(n(\varphi_k - \varphi_\hat{a})) \frac{dN}{dy k_T dk_T d\varphi_k} \right\rangle,
\]

with the normalization

\[
N = \int d\varphi_k 2\pi \left\langle \frac{dN}{dy k_T dk_T d\varphi_k} \right\rangle = \frac{1}{\pi} \left\langle \frac{dN}{dk_T^2} \right\rangle.
\]

The brackets \( \langle \cdot \rangle \) indicate an average over all configurations \( E(x) \).

Even (odd) moments have positive (negative) parity under \( r \to -r \):

\[
\langle \cos 2n \varphi_k \rangle = + \langle \cos 2n(\varphi_k + \pi) \rangle, \quad \langle \cos(2n + 1) \varphi_k \rangle = - \langle \cos(2n + 1)(\varphi_k + \pi) \rangle.
\]

If \( S(r, \varphi_r) \) is independent of the orientation of the dipole then all \( v_n = 0 \). An angular dependence of its real part gives rise to non-zero parity even moments \( v_{2n+1} \); an angular dependence of its imaginary part produces odd moments \( v_{2n+1} \).

For a more detailed discussion of the \( p_T \)-dependence of single-particle \( v_1, v_2, v_3 \) we refer to Ref. \[21\]. We note that obtaining non-zero odd-index two-particle cumulants \( v_1\{2\}, v_3\{2\} \) as measured in the experiments is more subtle, see sec. \[11,12,13\] below.

\(^1\) Eq. (9) includes the “no scattering” contribution for transverse momentum exchange \( k = 0 \). It plays no role in our subsequent analysis since we are interested in finite \( k \) only.
III. MULTI-PARTICLE CUMULANTS

A. Connected contributions to high-order cumulants

We begin this section with the (fully) connected contributions from $\langle S_m \rangle$ to multi-particle cumulants. We show that these generate positive contributions to $c_2\{m\}$ and so would lead to complex harmonics $v_2\{m\}$ if the number $m$ of particles is a multiple of four \[22, 23\]. Hence, that $\langle S_m \rangle$ gives real $v_2\{m\}$ for all $m$ only if the presence of an azimuthal anisotropy at the single particle level would generate disconnected contributions.

Furthermore, we show that $|v_2\{m\}|$ beyond $m \simeq 4$ is only weakly dependent on $m$. This indicates a remarkable coherence of the connected “non-flow” contributions obtained from small-$x$ QCD. Together with their long-range correlation in rapidity, the properties are quite unlike “conventional” non-flow, for example, from resonance decays or fragmentation of jets.

We then proceed to discuss contributions from fully disconnected diagrams which arise if rotational symmetry of the single-particle distribution is broken. These contribute with opposite sign to $c_2\{2\}$ vs. $c_2\{4\}$. We also present a detailed derivation of the elliptic anisotropy from three-particle correlations, $c_2\{3\}$. This enables us to analyze a “BBGKY-like” hierarchy of $m$-particle correlations.

The $m$-th order cumulant of the elliptic anisotropy is given by

$$c_2\{m\} = \langle \exp [i \Delta(\varphi_1 + \varphi_2 + \cdots + \varphi_m - \varphi_{m+1} - \varphi_{m+2} - \cdots - \varphi_{2n})] \rangle_\varphi.$$  \hspace{1cm} (16)

The normalization in Eq. (16) is dominated by the disconnected contributions, corrections are suppressed by powers of $1/N_c^2$. Thus, after averaging with respect to the impact parameters $b_m$ the normalization at leading order in $N_c$ is

$$\langle S_m(r_1, \ldots, r_m) \rangle \approx \left( -\frac{Q_s^2}{4} \right)^m \prod_{i=1}^{m} \nu_i^2.$$  \hspace{1cm} (17)

Equation (16) involves all possible contractions that generate the fully connected diagrams. Altogether there are $(2m - 2)!!$ contractions:

$$\langle S_m(r_1, b_1, \ldots, r_m, b_m) - 1 \rangle_{\text{conn.}} = \left( -\frac{Q_s^2}{4} \right)^m \frac{1}{(N_c^2 - 1)^{m-1}} \Delta(b_1 - b_2) \Delta(b_2 - b_1) \cdots \Delta(b_{m-1} - b_m) \Delta(b_m - b_1) \prod_{i=1}^{m} \nu_i.$$

In what follows we adopt a Gaussian $\Delta(b) = \exp (-b^2/\sigma^2)$ so that

$$\frac{1}{S_\perp} \int d^2b \Delta(b) = \frac{\pi \sigma^2}{S_\perp} = \frac{S_\perp^2}{S_\perp} = \frac{1}{N_D}.$$  \hspace{1cm} (19)

Here $1/N_D$ is the ratio of the correlated area, $S_\perp^2$, to the area of the projectile, $S_\perp$ (the proton in p-A collisions), i.e. the inverse number of domains.

Averaging with respect to the impact parameter and angular variables leads to

$$c_2\{m\} = \frac{m!!(m-2)!!}{m 2^m} \left[ \frac{1}{N_D(N_c^2 - 1)} \right]^{m-1}, \hspace{1cm} (m \geq 2 \text{ and even} ).$$  \hspace{1cm} (20)

The azimuthal harmonics are now readily obtained as \[23\]:

$$(v_2\{m\})^m = \frac{(-1)^{m+1}}{m \beta_m} \left( \frac{1}{N_D(N_c^2 - 1)} \right)^{m-1}, \hspace{1cm} (m \geq 2 \text{ and even} ),$$  \hspace{1cm} (21)

with

$$\beta_m = 2 \sum_{k=1}^{\infty} \left( \frac{2}{j_{0,k}} \right)^n,$$

where $j_{0,k}$ is the $k$-th zero of Bessel function $J_0(x)$. Details on transforming the cumulants, $c_2\{m\}$, to the harmonics, $v_2\{m\}$, can be found in Ref. \[24\]. Eq. (21) also remains true for gluons scattering off the target owing to the cancelation of Casimir factors in normalized observables.
The absolute values of the harmonics are approximately equal at large \( m \), quickly approaching the limit
\[
\lim_{m \to \infty} |v_2(m)| = \frac{1}{N_D} \frac{j_{0,1}}{2(N_c^2 - 1)}.
\] (23)

Although we were unable to prove this rigorously, we believe that this result holds for any short range correlation \( \Delta(b) \). The fact that the disconnected “non-flow” small-\( x \) diagrams at large \( m \geq 4 \) are of the “wrong sign” and approximately independent of \( m \) could perhaps be used to distinguish them from conventional effects.

The other interesting point here is that the fully connected diagrams give positive cumulants of any order and thus, every second \( v_2(m) \) is complex, starting from \( m = 4 \):
\[
(v_2(4))^4 = -c_2(4) = -\left(\frac{1}{4} \left[ \frac{1}{N_D(N_c^2 - 1)} \right] \right)^3 < 0,
\] (24)

This is also illustrated in Fig. 1. A possible resolution consists in an azimuthal anisotropy of the single dipole S-matrix [12, 21]. This generates “flow-like” disconnected contributions to the cumulants [22] which we discuss next.

### B. Disconnected contributions to the two- and four-particle cumulants

Equation (7) corresponds to averaging over all possible configurations of \( \vec{E}(\vec{b}) \) and is isotropic. However, as we shall demonstrate in section [14] for any particular configuration the S-matrix does exhibit an angular dependence, see e.g. Fig. 2. In order to account for this anisotropy we instead perform the average according to (6).

The first thing to compute is the angular distribution for scattering of a single dipole, for fixed \( \hat{a} \). Using the leading term in Eq. (1) and Eq. (6), and performing a Fourier transform to momentum space, as well as an average over the impact parameter, one arrives at
\[
\left( \frac{1}{\pi} \frac{dN}{dk^2} \right)^{-1} \frac{dN}{dk} = 1 - 2A + 4A(\hat{k} \cdot \hat{a})^2.
\] (25)

Consequently, the elliptic harmonic of the single-particle distribution is given by
\[
v_2 = \left\langle e^{2i(\phi_k - \phi_a)} \right\rangle_{\hat{a}} = A.
\] (26)
It is straightforward to generalize the computation of the connected diagrams from above to include the single-particle anisotropy. The 2- and 4-particle cumulants turn out to be

\[ c_2\{2\} \equiv (v_2\{2\})^2 = \frac{1}{N_D} \left( A^2 + \frac{1}{4(N_c^2 - 1)} \right), \]

\[ c_2\{4\} \equiv -(v_2\{4\})^4 = -\frac{1}{N_D^2} \left( A^4 - \frac{1}{4(N_c^2 - 1)} \right). \]

(27)

(28)

The detailed derivation can be found in Ref. [22].

Before presenting the result for the 3-particle cumulant \( c_2\{3\} \) we first examine the results \( c_2\{2\} \). The first term in (27) is the square of the single-particle \( v_2 \); it is scaled by \( 1/N_D \) since both particles must scatter from the same domain to exhibit a correlation. The second contribution corresponds to genuine non-factorizable two-particle correlations, as discussed above. Both contributions are positive; nonetheless Eq. (27) reveals the existence of two distinct regimes. For \( A \gg \frac{1}{N_c} \) the ellipticity is mainly due to the asymmetry of the single-particle distribution induced by the \( \vec{E} \)-field domains. In the opposite limit \( A \ll \frac{1}{N_c} \), \( c_2\{2\} \) is mainly due to genuine, non-factorizable two-particle correlations.

On the other hand, the fourth order cumulant \( c_2\{4\} \) changes sign as a function of \( A \). Furthermore, the magnitude of the fully connected contribution relative to \( v_2\{1\}^4 \) is \( \sim 1/(A^4 N_c^6) \). Hence, parametrically \( c_2\{4\} \) crosses zero when \( A \sim 1/N_c^{3/2} \). Thus, the presence of both connected and disconnected contributions built from the QCD dipole – \( E \)-field interaction \( \sim \text{tr} (\mathbf{r} \cdot \mathbf{E})^2 \) can in principle describe a change of sign of \( c_2\{4\} \) as seen in experiment\(^2\).

We did not manage to derive the general form of \( c_2\{m\} \) for arbitrary \( m \), if both connected and disconnected contributions are included. However, when the single particle contribution dominates, one obtains

\[ v_2\{m\} = \frac{A}{N_D^{1-1/m}}. \]

(29)

Consequently, in this case, too, the higher order harmonics are approximately equal to each other, \( v_2\{m\} \approx \frac{A}{N_D} \), for sufficiently large \( m \). We illustrated this in Fig. [2] (right).

### C. The three-particle cumulant \( c_2\{3\} \)

In this section we calculate the quadrupole anisotropy from 3-particle correlations \[25\].

\[ v_3\{3\} = c_2\{3\} = \langle \exp[2i(\varphi_1 + \varphi_2 - 2\varphi_3)] \rangle. \]

(30)

This cumulant is again defined in such a way as to be invariant under a simultaneous rotation of all particle transverse momenta by the same angle.

From Eq. (30) it is clear that the third particle requires a “\( v_4\)-like” structure or else \( v_2\{3\} \) would be zero. Such a contribution \( \sim \cos(4\varphi) \) can be obtained from the expansion of the S-matrix to second order in \( \text{tr} (\mathbf{r} \cdot \mathbf{E})^2 \), see Eq. (4). This leads to the three-dipole S-matrix

\[ \langle S_3 \rangle - 1 = \frac{1}{2} \left( \frac{ig}{2N_c} \right)^4 \langle \text{tr} (\mathbf{r}_1 \cdot \mathbf{E}(\mathbf{b}_1))^2 \text{tr} (\mathbf{r}_2 \cdot \mathbf{E}(\mathbf{b}_2))^2 \text{tr} (\mathbf{r}_3 \cdot \mathbf{E}(\mathbf{b}_3))^2 \rangle. \]

(31)

In this case, the most general decomposition of \( c_2\{3\} \) is given by

\[ c_2\{3\} = \langle \exp(2i(\varphi_1 + \varphi_2 - 2\varphi_3))^{\text{disc.}} + \langle \exp(-4i\varphi_3) \rangle \langle \exp(2i(\varphi_1 + \varphi_2))^{\text{conn.}} + 2\langle \exp(2i\varphi_1) \rangle \langle \exp(2i(\varphi_2 - 2\varphi_3)) \rangle^{\text{conn.}} + 2\langle \exp(2i\varphi_1) \rangle \langle \exp(2i(\varphi_2 - 2\varphi_3)) \rangle^{\text{conn.}} + 2\langle \exp(2i\varphi_1) \rangle \langle \exp(2i(\varphi_2 - 2\varphi_3)) \rangle^{\text{conn.}}. \]

(32)

Although we have computed all of the above terms here we shall focus on the fully disconnected contribution \( \sim A^4 \) as well as on those connected contributions which are of the same order when \( A = O(N_c^{-1}) \). The second, third and the last term in (32) then do not contribute.

\(^2\) Our result probably does not provide a quantitative explanation of the \( p_T \)-integrated data for \( c_2\{4\} \) which is dominated by particles with low transverse momenta. Also, the relation of the anisotropy amplitude \( A \) and the multiplicity is presently not clear.
The overall normalization implicit in Eq. (32) will be approximated by the angular average of the fully disconnected diagram. It is given by

\[ N = -\frac{1}{4\pi} r_1 r_2 r_3 Q_s^6. \]  

(33)

For the fully disconnected contribution we have

\[ \left( \frac{ig^2}{2N_c} \right)^4 \int \frac{d\varphi_{i'}}{2\pi} \int \frac{d\varphi_{i''}}{2\pi} \left\langle \frac{1}{\hat{a}} \left[ \frac{1}{\hat{a}'} \left[ \left\langle \varphi_1 \right\rangle \left\langle \varphi_2 \right\rangle \left\langle \varphi_3 \right\rangle \right] \Delta(b_i - b_j) \right] \right\rangle = \frac{1}{4\pi^2} r_1 r_2 r_3 Q_s^6 (1 - A + 2A(\hat{b}_1 \cdot \hat{a})^2)(1 - A + 2A(\hat{b}_2 \cdot \hat{a})^2)(1 - A + 2A(\hat{b}_3 \cdot \hat{a})^2) \Delta(\tilde{b}_1 - \tilde{b}_2) \Delta(\tilde{b}_1 - \tilde{b}_3), \]  

(34)

As in ref. [22] here we employed \( C(a, a') = 2\pi \delta(a - a')\Delta(\tilde{b}_1 - \tilde{b}_2) \) and \( C(a, a'') = 2\pi \delta(a - a'')\Delta(\tilde{b}_1 - \tilde{b}_3) \) with \( \Delta(\tilde{b}_1 - \tilde{b}_j) = \exp(-|\tilde{b}_1 - \tilde{b}_j|/\xi^2) \). Averaging over the impact parameters results in

\[ \int \frac{d^2b_1}{S_1} \frac{d^2b_2}{S_1} \frac{d^2b_3}{S_1} \Delta(\tilde{b}_1 - \tilde{b}_2) \Delta(\tilde{b}_1 - \tilde{b}_3) = \frac{\pi \xi^2 \pi \xi^2}{S_1 S_1} \equiv \left( \frac{1}{N_D} \right)^2, \]  

(35)

with \( N_D \) the number of E-field domains in the target nucleus.

Multiplying Eq. (34) by \( \exp(2i(\varphi_1 + \varphi_2 - 2\varphi_3)) \) and averaging over the azimuthal angles leads to the disconnected (single particle factorizable) contribution to \( c_2 \{3\} \); in momentum space,

\[ \langle \exp(2i(\varphi_1 + \varphi_2 - 2\varphi_3)) \rangle_{\text{disc}} = \frac{1}{N_D^2} \frac{1}{4k_3^2} \frac{Q_s^2 A^4}{2}. \]  

(36)

Because we have expanded in the numerator of this cumulant the S-matrix for the third dipole to second order, we obtain that \( c_2 \{3\} \sim 1/k_3^2 \) drops at high momentum with the square of the \( p_T \) of the third particle.

The connected and disconnected parts of the fourth term in Eq. (32) are

\[ \left( \frac{ig^2}{4N_c} \right)^4 \left\langle \frac{1}{\hat{a}} \left[ \frac{1}{\hat{a}'} \left[ \left\langle \varphi_1 \right\rangle \left\langle \varphi_2 \right\rangle \left\langle \varphi_3 \right\rangle \right] \Delta(\tilde{b}_1 - \tilde{b}_3) \right] \right\rangle = \frac{1}{4\pi^2} r_1 r_2 r_3 Q_s^4 \Delta^2(\tilde{b}_1 - \tilde{b}_3) \left[ \cos(\varphi_2 - \varphi_3) + 2A(2\cos(\varphi_2 - \varphi_a) \cos(\varphi_3 - \varphi_a) - \cos(\varphi_2 - \varphi_3))^2 \right], \]  

(37)

\[ \left( \frac{ig^2}{4N_c} \right)^4 \left\langle \frac{1}{\hat{a}} \left[ \frac{1}{\hat{a}'} \left[ \left\langle \varphi_1 \right\rangle \left\langle \varphi_2 \right\rangle \left\langle \varphi_3 \right\rangle \right] \Delta(\tilde{b}_1 - \tilde{b}_3) \right] \right\rangle = \frac{1}{4\pi^2} r_1 r_2 r_3 Q_s^4 \Delta(\tilde{b}_1 - \tilde{b}_3) \left( 1 - A + 2A(\hat{b}_1 \cdot \hat{a})^2 \right) \left( 1 - A + 2A(\hat{b}_3 \cdot \hat{a})^2 \right), \]  

(38)

respectively. Averaging over impact parameters generates a factor of \( (1/2N_D^2) \). We may now calculate the Fourier transform and sum over the 4 contractions of the amplitudes/conjugate amplitudes of the dipoles 1 to 3. This leads to

\[ 2\langle \exp(2i\varphi_1) \rangle\langle \exp(2i\varphi_2) \rangle\langle \exp(2i(\varphi_2 - \varphi_3)) \rangle_{\text{conn}} = \frac{1}{N_D^2} \frac{1}{4k_3^2} \frac{Q_s^2 A^2}{2(2N_c^2 - 1)}. \]  

(39)

The two factors from the fifth term of Eq. (32) each have the form of Eq. (37). Averaging over impact parameters, performing the Fourier transform, and summing over the 8 contractions of the amplitudes/conjugate amplitudes of the dipoles 1 to 3 leads to

\[ \langle \exp(2i(\varphi_1 - \varphi_3)) \rangle_{\text{conn}} \langle \exp(2i(\varphi_2 - \varphi_3)) \rangle_{\text{conn}} = \frac{1}{N_D^4} \frac{1}{4k_3^2} \frac{1}{16(N_c^2 - 1)^2}. \]  

(40)

Finally, for \( A \sim 1/N_c \) we have that

\[ c_2 \{3\} = (v_2 \{3\})^3 = \frac{1}{N_D^2} \frac{1}{4k_3^2} \left( \frac{1}{2} + \frac{A^2}{2(N_c^2 - 1)} + \frac{1}{16(N_c^2 - 1)^2} \right). \]  

(41)

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3 In principle one should Fourier transform first to momentum space. At high \( p_T \) this transform is simply \( \text{F.T.} \{r\} \sim i\mathbf{k}/k^2 \).
As already indicated above, we find that $c_2\{3\} \sim 1/k_2^3$ at high transverse momentum. This is due to the fact that in the numerator we expanded the S-matrix to order $r_3^2$ while we only require terms of order $r_3^4$ in the normalization. However, expression (21) for the S-matrix relies again on the gradient expansion of the dipole operator. In sec. [V] below we shall see that the exact S-matrix (obtained numerically) does appear to include a $\cos(4\varphi)$ harmonic even at order $r^2$, indicating the presence of corrections to the gradient expansion.

This provides another way for a “$v_4$-like” structure at order $r_3^2$. In this case $c_2\{3\}$ is given by

$$c_2\{3\} = \langle \exp(2i(\varphi_1 + \varphi_2 - \varphi_3))\rangle^{\text{dis.}} + \langle \exp(4i\varphi_3)\rangle + \langle \exp(2i(\varphi_1 + \varphi_2))\rangle^{\text{conn.}} + \langle \exp(2i(\varphi_2 - \varphi_3))\rangle^{\text{conn.}} .$$

(42)

So far we have not yet computed the diagrams involving a contraction of the third particle with either of the other particles. On the other hand, it is easy to write down the contributions from the first two terms in Eq. (42).

When the third particle is disconnected its S-matrix is given by the S-matrix for a single dipole and we may decompose its real part into a Fourier series,

$$S_1(r_3) - 1 = N(r_3) \left( 1 + \sum_{n=1}^{\infty} A_{2n} \cos(2n(\varphi_r - \psi)) \right) .$$

(43)

The function $N(r_3) = -r_3^2 Q_s^2/4$ (at small $r_3$) is the isotropic part of the S-matrix and $\psi$ is the “event plane” angle.

The only term in this series relevant for $c_2\{3\}$ is that for $n = 2$. The average over $\langle \exp(-4i\varphi_3)\rangle$ will contribute with

$$\langle \exp(-4i\varphi_3) \rangle = A_4 \int_0^{2\pi} d\varphi_3 \frac{1}{2\pi} e^{-4i(\varphi_3 - \varphi_a)} \cos(4(\varphi_3 - \varphi_a)) = \frac{A_4}{2} .$$

(44)

The integrand entering the average over the azimuthal angle for the particles 1 and 2 when they are connected or disconnected has the same form as in equations (37) and (38), respectively. In both diagrams the average over the impact parameters will generate a factor of $1/N_D^2$. The overall normalization factor is given by Eq. (33) as before.

After computing the averages over the azimuthal angles for all particles we have that $c_2\{3\}$ is now given by

$$c_2\{3\} \sim \frac{1}{N_D^2} \frac{A_4 A^2}{8} .$$

(45)

We assume that $A_4$ is of order $A^2$ and drop contributions beyond order $A^4 \sim N_e^{-4}$. It is evident that if the S-matrix exhibits a $\cos(4\varphi)$ dependence at order $r^4$ then $c_2\{3\} \rightarrow \text{const}$ at high transverse momentum. Hence, the behavior of this cumulant at high $p_T$ could provide interesting information about the angular structure of the dipole S-matrix.

D. BBGKY-like hierarchy of $m$-particle $c_2\{m\}$ cumulants

In the previous sections we have shown that all $c_2\{m\}$ eventually are dominated by the fully disconnected contribution proportional to the single-particle elliptic anisotropy $A$ to the $m$-th power. This occurs in different stages. The four-particle cumulant $c_2\{4\}$ factorizes when (parametrically) $A > N_e^{-3/2}$. On the other hand, the two-particle cumulant $c_2\{2\}$ requires a stronger E-field anisotropy of order $A > N_e^{-1}$. These correlators thus satisfy a BBGKY-like hierarchy. On the other hand, in the previous section we have seen that the factorization of $c_2\{3\}$ does not occur at some intermediate value of $A$ but, again, for $A > N_e^{-1}$, just like for $c_2\{2\}$. This correlation function thus represents an exception to the hierarchy. In sec. [V] below we shall attempt to go beyond parametric estimates of the connected vs. disconnected contributions by performing a phenomenological comparison to data.

E. Odd-index two-particle cumulants, $c_4\{2\}$ and $c_4\{2\}$

In sec. [II] we argued that the angular distribution for a scattered fundamental charge gives rise to odd parity moments $v_1$ and $v_3$. Their dependence on $p_T$ has been discussed and compared to measured two-particle $v_1\{2\}$, $v_3\{2\}$ in Ref. [21]. We would like to point out here that this issue requires more theoretical investigation, for the following reason.

The two particle correlation function summed over $qg$, $qg$, $gq$ and $qq$ channels is $C$-even if one assumes quark—anti-quark symmetry of the projectile wave function at small $x$. Indeed the two-particle S-matrix

$$S_2 \propto \langle \text{tr} V^\dagger(x_1) V(y_1) + \text{tr} V(x_1) V^\dagger(y_1) \rangle \langle \text{tr} V^\dagger(x_2) V(y_2) + \text{tr} V(x_2) V^\dagger(y_2) \rangle$$

(46)
is real, and so has even cumulants only.

Therefore, obtaining non-zero $c_1\{2\}$ and $c_3\{2\}$ may require to account for (at least) one additional soft rescattering of the (anti-) quarks besides their hard scattering from the target shockwave. In his talk at this conference Schlichting showed that classical Yang-Mills evolution of the liberated gluons in the forward light cone immediately leads to non-zero $v_3\{2\}$ at time $\tau = 0.1$ fm \cite{26}. If such rescattering is soft then the $p_T$-distribution of $v_3$ and $v_3$ shown in Ref. \cite{21} should be mostly preserved. Either way, this clearly is an interesting problem which requires more theoretical analysis.

IV. ANISOTROPIC GLUON DISTRIBUTION AT SMALL $x$

The main goal of this section is to compute scattering of a dipole off a large nucleus to demonstrate its non-trivial angular dependence \cite{27}. We shall first consider the classical MV model and then proceed to resum quantum fluctuations with large longitudinal phase space via the JIMWLK evolution equation.

A. Classical McLerran-Venugopalan model

In the MV model \cite{20} the large-$x$ valence partons are viewed as random, recoilless color charges $\rho^a(x)$ described by the effective action

$$S_{\text{eff}}[\rho^a] = \int dx^- d^2x \left \langle \frac{\rho^a(x^-,x) \rho^a(x^-,x)}{2\mu^2} \right \rangle$$

with $\mu^2 \sim g^2 A^{1/3}$ proportional to the thickness of a nucleus; here $A$ denotes the number of nucleons in the nucleus. The variance of color charge fluctuations determines the average saturation scale $Q_s^2 \sim g^4 \mu^2$ \cite{28}. The Weizs"acker-Williams fields generated by $\rho^a(x)$ are pure gauges; in covariant gauge,

$$A^\mu(x^-,x) = -\delta^\mu + \frac{g}{\sqrt{2}} \rho^a(x^-,x) .$$

Using Eq. \cite{48} in Eqs. \cite{1} and \cite{2} we can compute the S-matrix for each configuration of the target fields and extract its Fourier harmonics.

It is rather evident that the random distribution of color charges $\rho^a(x)$ would generate azimuthally anisotropic soft fields. Less trivially, we shall show that the angular structure of the target electric fields does not fluctuate randomly on arbitrarily short scales, i.e. that it is characterized by a finite correlation length $\sim 1/Q_s$ in the transverse plane. This fact is related to the saturation of the gluon distribution from highly occupied classical fields at momentum scales $k_T < Q_s$ \cite{25}; over distances $> 1/Q_s$ the soft, classical color fields become “smooth”. This is why the number of domains $N_D$ introduced in previous sections is finite.

The most crucial aspect, however, is the following. The angular structure will obviously fluctuate from one configuration $\rho^a(x)$ of valence charges to the next. Averaging over these fluctuations like in Eq. \cite{7} would obviously project onto the isotropic part of the gluon distribution. Instead, we point out that the angular fluctuations of $\rho^a(x)$ are slow variables, i.e. that they should be averaged over only after the $m$-particle cumulants have been computed.

For a general configuration of the sources, the S-matrix for a fundamental charge is complex. The real (imaginary) part corresponds to $C$-even ($C$-odd) interactions \cite{29}:

$$1 - D_\rho(r) \equiv \text{Re} S_\rho(r) = \frac{1}{2N_c} \text{tr} \left [ V^\dagger(x) V(y) + V^\dagger(y) V(x) \right ] ,$$

$$O_\rho(r) \equiv \text{Im} S_\rho(r) = \frac{-i}{2N_c} \text{tr} \left [ V^\dagger(x) V(y) - V^\dagger(y) V(x) \right ] .$$

We use Monte-Carlo techniques on a lattice describing the longitudinal and transverse coordinates to generate the random configurations $\rho^a(x^-,x)$. The number of sites in the longitudinal direction is taken to be $N_\perp = 100$, while $N_\perp = 1024$ for either of the transverse directions. We fix the parameters of the lattice such that $g^2 \mu a = 0.05$, where $a \equiv L/N_\perp$ denotes the transverse lattice spacing. Defining the saturation scale from

$$\langle S_\rho \rangle (r = \sqrt{2}/Q_s) \equiv e^{-1/2}$$

we determined numerically that $Q_s \approx 0.7125g^2\mu$. Further details of the numerical implementation can be found in Refs. \cite{27} \cite{30}.
The azimuthal amplitudes can be extracted by expanding the real and imaginary parts of the S-matrix in a Fourier series:

\[ D_{\rho}(r) = N(r) \left( 1 + \sum_{n=1}^{\infty} A_{2n}'(r) \cos(2n\varphi) \right), \]

\[ O_{\rho}(r) = N(r) \sum_{n=0}^{\infty} A_{2n+1}'(r) \cos[(2n+1)\varphi]. \]

Here, the function \( N(r) \) denotes the isotropic part of the dipole S-matrix. As already mentioned above each amplitude \( A_n' \) contains a random phase which fluctuates from configuration to configuration. To discard this phase we define \( A_n = \frac{\pi}{2} |A_n'| \); this arises due to

\[ \int \frac{d\psi}{2\pi} |\cos n\psi| = \frac{2}{\pi}. \]

Then, averaging over \( 10^4 \) configurations we finally obtain \( \langle A_1 \rangle, \cdots, \langle A_4 \rangle \) as well as the variances of \( A_1 \) and \( A_2 \), presented in Fig. 2.

Our results show that, as expected, the biggest amplitude is the quadrupole; at \( r \lesssim 1/Q_s \) the amplitude \( \langle A_2 \rangle \sim 20\% \). As we argue in the next section, such values are in the range of the asymmetries relevant for phenomenology of high-multiplicity p+Pb collisions at LHC energies. We stress, however, that in this calculation we did not attempt to bias the configurations towards “high multiplicities”, which requires a dedicated investigation. The function \( \langle A_2 \rangle(r) \) is almost independent of \( r \) for \( r < 1/Q_s \) which justifies our treatment in the previous section where \( \mathcal{A} \equiv \langle A_2 \rangle \) has been treated as constant. Figure 2 shows furthermore that the variance \( \sqrt{\langle \delta A_2^2 \rangle} \) is similar in magnitude to the mean value \( \langle A_2 \rangle \). This points at rather large fluctuations of \( A_2 \) for different configurations.

Figure 3 shows the same amplitudes as the previous figure but for \( E \)-fields which have been “smeared” over an area \( \pi r^2 \) set by the size of the dipole. Comparing Figs. 2 and 3, one sees that “smearing” has a negligible effect for \( r \lesssim 1/Q_s \) while the anisotropy amplitudes at large \( r \) are suppressed. This behavior shows the correlation over finite transverse distance scales of the angular structure of the \( E(x) \) configurations.

Reference [27] showed that the MV-model amplitude \( \langle A_2 \rangle(r) \) matches the distribution of linearly polarized gluons (for an unpolarized target) \( h_1^{1g}(x, k^2) \) introduced in TMD factorization [31, 32].

\[ \delta^{ij} f_i^g(x, k^2) + \left( \hat{k}^i \hat{k}^j - \frac{1}{2} \delta^{ij} \right) h_1^{1g}(x, k^2). \]

Within the framework of the MV model, the result for \( h_1^{1g}(x, r) \) derived analytically in Ref. [32],

\[ h_1^{1g}(x, r^2) \propto \frac{1}{r^2 Q_s^2} \left[ 1 - \exp \left( -\frac{r^2 Q_s^2}{4} \right) \right], \]
is in good agreement with our numerical results at small values of \( r \lesssim 2Q_s^{-1} \).

Figure 2 also shows a non-zero amplitude of the \( \cos(4\varphi) \) angular component. It appears to be essentially constant at small \( r \) unlike the \( \sim r^2 \) behavior expected from the second term in Eq. (4) once scaled by \( N(r) \sim (rQ_s)^2 \) at small \( r \). This may be due to corrections to the gradient expansion which was used to derive Eq. (4). Such a term would provide another contribution to the hexadecupole \( v_4 \)-like asymmetry, c.f. previous section.

Due to fluctuations of the saturation momentum \( Q_s \) in impact parameter space, every particular configuration of semi-classical small-\( x \) fields contains a \( C \)-odd component and \( O(r) \) as defined in Eq. (50) is non-zero. This results in non-zero odd-index amplitudes \( A_{1,3} \), see Fig. 2. The figure also shows that the expectation values of the odd amplitudes are significantly smaller than \( A_2 \); as expected, they vanish as \( r \to 0 \):

\[
iO(r) \sim i \alpha_s r \cdot \nabla_b (1 - D(r, b)) \simeq i \alpha_s r^3 Q_s^2 Q_c \cos \varphi_r \left[ 1 - \frac{r^2}{4} \left( \frac{Q_r^2 \cos^2 \varphi_r}{3} + Q_s^2 \right) \right]. \tag{57}
\]

The expression on the r.h.s. corresponds again to a gradient expansion in powers of \( r \), assuming a generic spectrum of fluctuations of \( Q_s(b) \) cut off at \( Q_c \).

The presence of odd harmonics does not indicate that the expectation value of the \( C \)-odd part of the \( S \)-matrix is non-zero. Indeed, the average of the odderon \( O(r) \) over the \( C \)-even ensemble generated by the action is zero. However, the product of \( O(r) \) with another \( C \)-odd operator, which effectively arises due to our dropping of the phases of the amplitudes \( A'_n \), is even under \( C \)-conjugation and its expectation value is not zero.

B. Quantum fluctuations and high-energy evolution

In the previous subsection, within the framework of the classical MV model, we showed that scattering of a dipole from the soft fields sourced by a particular configuration \( \rho^c(x) \) of valence charges is not isotropic, and that the amplitudes of the azimuthal anisotropy are quite significant. In this section we consider how these amplitudes are affected by small-\( x \) / high energy evolution. This corresponds to a resummation of (nearly) boost-invariant quantum fluctuations to the classical field.

The evolution of the elliptic anisotropy with rapidity was first addressed by Kovner and Lublinsky in Ref. [12]. They solved the BK evolution equation for the dipole scattering amplitude \( 1 - S(r) \) as a function of dipole size and orientation. They found that the anisotropy decays exponentially with \( Y = \ln(x_0/x) \). Their solution, however, was based on the assumption that the impact parameter space is homogeneous. As explained in the previous section, even at the level of the initial condition (given by the MV model), the azimuthal anisotropy of \( S(r, b) \) arises due to fluctuations of the soft fields in the transverse impact parameter plane. Hence, in this subsection we describe solutions of JIMWLK evolution which account for fluctuations of the light-like electric Wilson lines in \( b \)-space.

Going beyond the classical theory, quantum gluon emissions which are enhanced by a large longitudinal phase space \( Y = \log x_0/x \) are resummed by the so-called JIMWLK functional renormalization group evolution. It modifies the ensemble of electric Wilson lines over which observables are averaged thereby resumming corrections to all orders.
in $\alpha_s Y$. Evolution over a step $\Delta Y$ in rapidity opens up phase space for radiation of gluons and modifies the classical action (47). The evolution can be formulated in terms of a “random walk” in the space of Wilson lines $V(x)$ [35–36]:

$$\partial_y V(x) = V(x) \frac{i}{\pi} \int d^2 u \frac{(x-u)^i \eta^j(u)}{(x-u)^2} - \frac{i}{\pi} \int d^2 v V(v) \frac{(x-v)^i \eta^j(v)}{(x-v)^2} V^j(v)V(x).$$  (58)

The Gaussian white noise $\eta^i = \eta^i_{(t)}$ satisfies $\langle \eta^i_{(t)}(x) \rangle = 0$ and

$$\langle \eta^i_{(t)}(x) \eta^j_{(t)}(y) \rangle = \alpha_s \delta^{ab} \delta_{ij} \langle \eta^{(2)}(x-y) \rangle.  \quad \text{(59)}$$

The so-called “left-right symmetric” form of Eq. (58) was introduced in Ref. [37]. We solve Eq. (58) numerically assuming a fixed but small coupling $\alpha_s = 0.1$; for such coupling the speed of evolution is at least roughly comparable to more realistic running coupling evolution.

Once an ensemble of Wilson lines on the transverse lattice has evolved to rapidity $Y$, we can again compute the dipole scattering amplitude $S_Y(r, b)$, its azimuthal Fourier decomposition and the corresponding saturation scale $Q_s(Y)$ using Eq. (61). It is important to note here that even though we consider a target of infinite transverse extent (periodic boundary conditions), that the evolution equation is solved on a transverse lattice which does allow for impact parameter dependent fluctuations.

![Figure 4](image_url)

**FIG. 4:** JIMWLK evolution of $\langle A_2 \rangle(r)$ and $\langle A_4 \rangle(r)$ (left) resp. of $\langle A_1 \rangle(r)$ and $\langle A_3 \rangle(r)$ (right). The lower order harmonics correspond to the upper sets of curves. Figure from Ref. [27].

In Fig. 4 (left) we show the evolution of $\langle A_2 \rangle(r)$ and $\langle A_4 \rangle(r)$ with $Y$. As already mentioned above, mean-field evolution of the dipole was shown to wash out initial elliptic anisotropies rather quickly [12]. On the other hand, here we only observe a relatively slow decrease of $\langle A_2 \rangle(r)$ with $Y$. This is rather intuitive since both the initial anisotropies at $Y = 0$, as well as those of the evolved JIMWLK configurations are generated by fluctuations of the hard “valence charges” in the transverse impact parameter plane. Furthermore, we observe that those harmonics which are initially small, i.e. $\langle A_1 \rangle(r)$, $\langle A_3 \rangle(r)$ and $\langle A_4 \rangle(r)$, in fact increase with $Y$ at small $r$. The harmonics also display universal behavior at very large $r$. The evolution of the amplitudes with $Y$ at fixed $r Q_s(Y)$ is shown in Fig. 4.

Thus, we conclude that the anisotropies are not washed out by high energy evolution and that they might be essential to describe short distance azimuthal asymmetries observed at LHC. An initial phenomenological analysis is presented in the next section.

**V. APPLICATION TO PHENOMENOLOGY OF PROTON-NUCLEUS COLLISIONS**

In this section we present a first phenomenological comparison of the measured $v_2\{2\}$ and $v_2\{4\}$ at high transverse momentum to some of the expectations from above. Our analysis is certainly not definitive but preliminary and qualitative. Our main goals are:

- to check if the magnitudes of $v_2\{2\}$ and $v_2\{4\}$ can be reproduced for “reasonable” values of $N_D$, the number of $E$-field domains, and of $A$, the $E$-field $\cos(2\phi)$ anisotropy amplitude;

-
FIG. 5: JIMWLK evolution of $\langle A_{1,2,3,4} \rangle$ at fixed $rQ_s(Y)$.

- to verify that the connected contributions to the two- and four-particle cumulants indeed describe the splitting between $v_2\{2\}$ and $v_2\{4\}$ observed experimentally at semi-hard $p_T$;
- to estimate the relative magnitudes of connected vs. disconnected contributions to the two-, three-, and four-particle cumulants, i.e. how far the respective cumulants are from the factorization limit (dominance of fully disconnected diagrams);
- to make a prediction for $v_2\{3\}$ in pA collisions at the LHC.

In order to fix $A$ and $N_D$, we shall use the CMS $v_2(p_T)$ data from 2- and 4-particle correlations in p+Pb collision at 5 TeV. We focus on the highest multiplicity events. Equations (27) and (28) provide the theoretical expectations for the cumulants $c_2\{2\}$ and $c_2\{4\}$ which we repeat here for convenience:

\[
c_2\{2\} \equiv (v_2\{2\})^2 = \frac{1}{N_D} \left( A^2 + \frac{1}{4(N_c^2 - 1)} \right),
\]

\[
c_2\{4\} \equiv -(v_2\{4\})^4 = -\frac{1}{N_D^3} \left( A^4 - \frac{1}{4(N_c^2 - 1)^3} \right).
\]

Note that these expressions do not include subleading corrections in $N_c^{-2}$ which we defer to a future analysis. Strictly,
Eqs. (60,61) apply only for $A = O(N_c^{-1})$ and $A = O(N_c^{-3/2})$, respectively. Furthermore, the transverse momenta of all particles are assumed to far exceed the saturation scale.

Depending on the value of $A$ there are two different regimes, see left panel of Fig. [6] for small values of $A$ there are strong genuine (non-factorizable) correlations and so the connected diagrams are important; for large values of $A$, however, the cumulants approach the factorization limit where they are dominated by the fully disconnected diagram and where genuine correlations are suppressed.

In Fig. [6] (right panel) we plot the $N_D$-independent ratio $(v_2(4))^{4/3}/(v_2(2))^2$ as a function of $A$. The dash-dotted line corresponds to Eqs. (27,28) while the straight horizontal lines represent that same ratio obtained from the two highest $p_T$ data points for $v_2(2)$ and $v_2(4)$ shown in Fig. [7]. As one can see the high-$p_T$ data allows two regimes of $A$: one around $A \sim 0.2$ and another for $0.35 \lesssim A \lesssim 0.7$. From Fig. [6] we see that the first solution is in the regime where strong correlation effects are present while the second one is close to the factorization limit.

The comparison of Eqs. (27,28) to the CMS data in the high $p_T$ region is shown in Fig. [7]. The values of $A = 0.20$ and $A = 0.53$ employed in the figure correspond to the two possible solutions mentioned in the previous paragraph. $N_D$ was fixed so as to reproduce the correct magnitudes of $v_2(2)$ and $v_2(4)$. Since both set of parameters, for small and large $A$, are able to describe the data with comparable quality we must conclude that our analysis is not sufficient to determine $A$ and $N_D$ uniquely from the high $p_T$ data alone. If the data down to about $p_T = 1$ GeV is included in the analysis then the model would prefer smaller values $A \simeq 0.2$ [21].

Figure 8 shows the predictions for $v_2(3)$ obtained from both expressions

\[ c_2(3) = (v_2(3))^3 \sim \frac{1}{N_D^2} \frac{A^2}{4} \left( \frac{A^4}{2} + \frac{A^2}{2(N_c^2 - 1)} + \frac{1}{16(N_c^2 - 1)^2} \right), \]  
\[ c_2(3) = \frac{1}{N_D^2} \frac{A^4 A^2}{8} \]  

derived above, using the same values for $A$ and $N_D$ as deduced from $v_2(2)$ and $v_2(4)$. Recall that (62) results from an expansion of the dipole S-matrix to second order in $\text{tr}(r \cdot E)^2$ while (63) arises if the S-matrix exhibits a $\cos(4\varphi)$ component already at order $r^2$. For Fig. 8 we assumed that $A_4 = A^2$ to avoid introducing additional parameters. The figure shows that despite the uncertainty in $A$ and $N_D$ that $v_2(3)$ does not vary too widely. We expect $v_2(3) \approx 2 - 4\%$ for semi-hard transverse momenta.

VI. SUMMARY AND OUTLOOK

Our goal here was to provide a summary and overview of recent ideas regarding anisotropic particle production at semi-hard transverse momenta in high-energy collisions. The basic point is that azimuthally anisotropic correlations should occur essentially due to an anisotropic small-$x$ gluon distribution.

The McLerran-Venugopalan model for the gluon distribution of dense hadrons or nuclei integrates out the fast dynamics of the large-$x$ degrees of freedom and replaces them by “frozen” sources for the small-$x$ semi-classical fields.
We point out that each such configuration exhibits azimuthal anisotropies with a finite transverse correlation length, and that the angular structure of these configurations is a slow variable, too.

As a consequence, before one averages over the random angular structure of the source, the single-particle distribution due to scattering of a projectile parton off such a target is anisotropic. This gives rise to contributions to multi-particle correlations from disconnected diagrams [12, 21, 22]. By analogy to the BBGKY hierarchy the disconnected contributions dominate the $m$-particle correlation functions in the limit of large anisotropy $A$ of the gluon distribution of the target. More specifically, they have been shown [22] to lead to a sign flip of the four-particle “elliptic” cumulant $c_2\{4\}$.

On the other hand, the connected contributions to the cumulants from small-$x$ dynamics exhibit a rather unexpected coherence in that $c_2\{m\}$ depends weakly on the order $m$ of the cumulant, for sufficiently large $m$ [23], quickly approaching a constant as $1/m \to 0$. Also, unlike conventional “non-flow” contributions, the connected diagrams from the CGC (coherent small-$x$ QCD dynamics) are long range in rapidity [8, 11].

Much work is still needed before we might claim to understand the data. From the point of view of phenomenology one should, for example, compute subleading in $N_c^{-2}$ and $Q_s^2/p_T^2$ corrections to the cumulants in order to improve the analysis performed in sec. V. It would be very interesting, too, to resum the time evolution of the gluon fields in the forward light cone [26] in order to understand over what range of $p_T$ final-state interactions are important. Also, this approach could clarify the time scale over which odd-index cumulants like $c_1\{m\}$ and $c_3\{m\}$ develop. As a last point, let us mention that the effect of the multiplicity bias employed in experiments on the anisotropy of the gluon field configurations is poorly understood at present.

An interesting theoretical issue is the relationship of the azimuthal cumulants to the gluon distributions introduced (via specific operator-level relations) in TMD factorization [31]. Within the MV model at least, it has been shown by explicit computation that $A(r)$, which corresponds to $v_2\{1\}(p_T)$ at high $p_T$, coincides with the distribution of linearly polarized gluons $h_{TQ}^{1,9}(r)$ [32]. It remains to be seen if this relation still applies after resummation of small-$x$ quantum corrections. Azimuthal correlations in high-energy p+p and p+A collisions could provide much insight into non-trivial QCD dynamics.

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4 From the point of view of conventional “non-flow” expectations.
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