Attempting to remove infinites from divergent series: Hardy will hardly help.

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Abstract

The consequences of adopting other definitions of the concepts of sum and convergence of a series are discussed in the light of historical and epistemological contexts. We show that some divergent series appearing in the context of renormalization methods cannot be assigned finite values in a form consistent with Hardy’s axioms without at the same time equating one to zero, thus destroying the mathematical building. We show that if the replacements for the concept of sum of a series are required to be associative, to be invariant under finite permutations of the terms and dilution, further restrictions emerge. We finally discuss the epistemological costs of accepting these practices in the name of instrumentalism.

Keywords: divergent series, Hardy, Cesaro, Feynman

1 Introduction

The possibility of assigning a finite value to divergent series has recently made it to the news¹ in a way that is unusual for science or mathematics news. Indeed, this news and even some related Youtube videos seem to lie halfway between joke and serious stuff, but in the end it turns out that these contributions are intended (by their authors) to be serious.

It is then compelling to assess the scientific relevance of these methods and in particular the issue of (logical) consistency of these methods in relation to the body of mathematical knowledge, as well as their epistemological implications.

In Section 2 we display the mathematical background of the problem. Section 3 contains the statement of the main results of this work along with their

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¹See the New York edition of New York Times, page D6 of February 4, 2014, on-line at http://www.nytimes.com/2014/02/04/science/in-the-end-it-all-adds-up-to.html?smid=fb-share&_r=1
proofs. Further, we discuss historical and epistemological issues in Section 4 while Section 5 is devoted to conclusions.

2 Background on Series

2.1 Sum of a Series

We follow here standard textbooks [1, p. 383-4] reviewing the definition of series and some of its basic properties, focusing on what will be relevant for the coming Sections.

Consider a sequence \(\{a_0, a_1, a_2, \ldots\}\) and also its sequence of partial sums \(S_N = \sum_{k=0}^{N} a_k\). This sequence of partial sums is called an infinite series, or simply series.

Definition 1. If there exists a number \(s\) such that \(s = \lim_{N \to \infty} S_N\) we say that the series \(\sum_{k=0}^{\infty} a_k\) is convergent with sum \(s\) and write \(\sum_{k=0}^{\infty} a_k = s\). Otherwise, we say that the series is divergent.

We may further distinguish among divergent series those where the \(\lim_{N \to \infty} S_N\) is \(+\infty\) or \(-\infty\) on the one hand (we may call them divergent to infinity) and those where this limit does not exist.

Definition 2. A convergent series \(\sum_{k=0}^{\infty} a_k\) is called absolutely convergent if the series \(\sum_{k=0}^{\infty} |a_k|\) is convergent. Otherwise, it is called conditionally convergent.

2.1.1 Basic Properties

We follow an approach inspired on the posthumous book by Hardy [7] on divergent series. For convergent series, the following properties can be demonstrated as theorems:

(A) For any real \(k\), \(\sum_{n} a_n = s \Rightarrow \sum_{n} (ka_n) = ks\).

(B) \(\sum_{n} a_n = s\) and \(\sum_{n} b_n = r\) \(\Rightarrow \sum_{n} (a_n + b_n) = s + r\).

(C) \(\sum_{n=0}^{\infty} a_n = s \Leftrightarrow \sum_{n=1}^{\infty} a_n = s - a_0\).

The proofs are a direct application of the properties of the limit of a sequence (in this case the sequence of partial sums).
Remark 1. Properties (A) and (B) are recognised as linearity and (C) is called stability. They extend the natural properties of sums for the sequence of partial sums $S_N$ all the way through the limit.

An immediate generalisation of (C) to finitely many operations is the following

**Corollary 1.** For any positive integer $N$,

$$\sum_{n=0}^{N} a_n = s \iff \sum_{n=N}^{N-1} a_n = s - \sum_{k=0}^{N-1} a_k.$$  

We list here other natural properties of convergent series extrapolated from finite sums via the limit properties for the sequence of partial sums.

**Corollary 2.**

(a) **Associativity** If the series $\sum_n a_n$ has a (finite or infinite) sum, then the series $\sum_k b_k$ obtained via $b_k = a_{2k} + a_{2k+1}$ for some or all non-negative integers $k$, has the same sum.

(b1) **Commutativity** If the series $\sum_n a_n$ has a (finite or infinite) sum then the series $\sum_n b_n$ obtained via $(b_{2k}, b_{2k+1}) = P(a_{2k}, a_{2k+1})$ for some or all nonnegative integers $k$, has the same sum ($P$ is the nontrivial permutation of 2 elements).

(b2) For series having the commutativity property, finite compositions of permutations of order up to $N$ (where $N$ a positive integer) do not alter the sum of the series.

(c) **Dilution** If the series with elements $a_0, a_1, a_2, \cdots$ has a (finite or infinite) sum then the series with elements $a_0, 0, a_1, 0, a_2, 0, \cdots$ i.e., inserting a zero between some or all pairs of elements in the original sequence, has the same sum.

**Proof.** For associativity, collecting up to $N$ terms corresponds to picking a subsequence from the original sequence of partial sums, having thus the same limit. For (b1) every other partial sum coincides with the original ones. For (b2), the partial sums of the new series coincides with the original one every $N$ steps. In between, they differ at most in a finite sum of terms that goes to zero for $k \to \infty$. Hence, both sequences of partial sums have the same limit. As for dilution, if one takes a convergent sequence $\{S_N\}$ and duplicates its terms: $S_1, S_1, S_2, S_2, \cdots$, the new sequence has the same limit as the original one. Hence, dilution does not alter the sum of the series. \[\square\]

The above properties can be arbitrarily (but finitely) combined, without altering the sum of a series. However, more drastic rearrangements need not preserve the sum unless the series is absolutely convergent. In fact, any rearrangement of an absolutely convergent series produces a new series with the same sum as the original one. However, invariance in front of arbitrary rearrangements of terms does not hold for conditionally convergent series. This is the content of Riemann rearrangement theorem [1, p. 413].
Theorem (Riemann). Let \( \sigma(n) \) be an injective function of the positive integers and \( K \) some real number. Suppose that \( \{a_1, a_2, a_3, \cdots \} \) is a sequence of real numbers, and that \( \sum_{n=1}^{\infty} a_n \) is conditionally convergent. Then there exists a rearrangement \( \sigma(n) \) of the sequence such that \( \sum_{n=1}^{\infty} a_{\sigma(n)} = K \). The sum can also be rearranged to diverge to \( \pm \infty \) or to fail to approach any limit.

Remark 2. A classical example of rearrangement is the alternating series \( a_n = \left\{ \frac{(-1)^{n+1}}{n} \right\}, n \geq 1 \). This series sums to \( s = \ln 2 \). Create a new series by dilution adding one zero before each element of the series and dividing by 2, i.e., \( \{0, a_1/2, 0, a_2/2, \cdots \} \). Combine then this series and the original one as in property (B). The new series has sum \( s = \frac{1}{2} \ln 2 \). However, after disregarding the intermediate zeroes, the combined series is a rearrangement of the original one, where the negative terms (for \( n = 2k \)) appear every third element instead of every other element. The new series adds two positive numbers for every negative contribution, thus subtracting the negative contributions in a different way from that in the original series.

Remark 3. All conditionally convergent series can be decomposed into two monotonic series: one with the positive terms, diverging to \( +\infty \) and another with the negative terms, diverging to \( -\infty \). What Riemann's Rearrangement Theorem teaches us is that if one wishes to interpret the sum of such a series pictorially as the outcome from “cancellation of both infinities”, then there is actually not one way to do it, but infinitely many, depending of the order in which the elements of the two participant series are picked up. However, each possible result is the unique limit of a specific sequence of partial sums.

2.2 Signification

Mathematics can be regarded as a process of successive abstractions originating in real-life situations. Thus, natural numbers abstract the process of counting objects and involve the abstraction of the concept of addition as well. Instead of saying: one goat and another goat makes two goats and another goat makes three goats . . . and the same for all indivisible objects, we say \( 1 + 1 + 1 \ldots \) abstracting away all objects and dressing the final answer with them again (I counted \( n \) goats). We increase our ability of counting moving to objects with more elaborated properties (e.g., divisible objects or portions of objects, debts, missing objects), producing the realms of integers, rationals, etc. The properties of addition are extended (i.e., preserving its original properties) to these higher levels of abstraction. The operations of abstraction and its inverse dressing relate mathematics to the material world. We call them more precisely abstraction and signification.

The process of abstraction is also known as idealization and in physics is historically linked with Galileo Galilei and his discussion of free fall. [5] pp. 205 (for an English translation see [6] pp. 170) Insight on the process of signification can also be traced back to Galileo’s words announcing that mathematics is the
language of the universe \cite{4}, thus recognising mathematics as belonging to the realm of the material world.

Hence, when addressing issues of the material world and its sciences, mathematical objects retain a specific signification. Any new mathematical object introduced along the investigation is related by abstraction and signification on one hand to the material world, on the other hand to mathematics where the object becomes context independent by the very process of abstraction. Thus, when counting apples we use the same addition than when we count goats, nodes in a vibrating string or smiling faces. The reciprocally inverse processes of abstraction and signification lie in the foundations of any attempt to understand the material world with mathematical tools, and cannot be disrupted nor neglected in any of its parts.

2.3 Minimal Requirements

Let us suppose that we want to resign Definitions 1 and 2 and produce another definition related to the sequence \( \{a_0, a_1, a_2, \ldots\} \). We require however to keep as much as possible of the original properties of series summation. We raise then properties (A-C) to the status of axioms, while replacing the series symbol by something new, since that symbol received its meaning in Definition 1 which we are now resigning. Inspired by Hardy in \cite{7}, we assume hereafter that axioms (A-C) hold for the new object \( \sum_n a_n \). Each new method of assigning values to infinite sequences should provide its own definition (as well as signification) for this object. Corollary 1 now reads: \( \sum_{n=0} a_n = s \Leftrightarrow \sum_{n=N} a_n = s - \sum_{k=0}^{N-1} a_k \).

The last term of the RHS is the plain sum of a finite set of numbers.

We will distinguish those methods that sum convergent series and series diverging to infinity in the usual way, namely,

Definition 3 (see p. 10 in \cite{7}). (a) A method \( Y \) assigning a finite value to a series is called regular if this value coincides with the standard sum in the case of standard convergent series.

(b) A regular method \( Y \) is called totally regular if series diverging to \( \pm \infty \) with the standard definition also diverge to \( \pm \infty \) in \( Y \).

Axioms (A-C) may be regarded as a minimal compromise. Any method which is not linear and stable cannot be seriously considered as an alternative to the sum of a series (it would be either not linear or not finitely related to ordinary sum). Also, in Hardy’s view regularity is a minimal requirement: whichever method not complying with standard results for standard convergent problems cannot be seriously considered as an extension of the concept of sum. Already in Section 1.4 of Hardy’s book it is noted that the method \( \mathcal{E} \) of analytic extension to assign values for divergent series is not regular. We will comment on this method in Subsection 3.3.

\footnote{The symbol \( Y \) is inspired in the Cyrillic word указ meaning decree or edict (formally: imposition).}
3 Statement of Results

We state now the main results of this work:

Theorem 1. Any method $Y$ assigning a finite number to $\sum_n 1$ (proposed replacement for $1 + 1 + 1 + 1 + \cdots$) is (i) not totally regular, (ii) not regular and (iii) contradictory.

Theorem 2. Any method $Y$ assigning a finite number to $\sum_n n$ (proposed replacement for $1 + 2 + 3 + 4 + 5 + \cdots$) is (i) not totally regular, (ii) not regular and (iii) contradictory.

By contradictory we mean that incompatible statements corresponding to $r = Y(\{a_n\}) = s$ for different real numbers $r$ and $s$ can be proved in this context.

It goes without saying that we are speaking about well-posed methods $Y$ where the assigned values for $\sum_n a_n$ are unique, whenever the sequence belongs to the domain of the method. Also, the definition of regularity naturally assumes that the sequences associated to all convergent series belong to the domain of whichever method $Y$ is under consideration (even non-regular ones).

3.1 Proofs

Proof of Theorem 1 That the method $Y$ is not totally regular is immediate since otherwise it should assign the value $+\infty$ to the proposed expression. By (C), $\sum_n 1 \equiv Y(\{1,1,1,\cdots\})$ has the same value as $0 + 1 + 1 + 1 + \cdots$ (namely $Y(\{0,1,1,\cdots\})$) and hence the term-wise difference of both objects by (B) satisfies: $1 + 0 + 0 + 0 + \cdots \equiv Y(\{1,0,0,0,\cdots\}) = 0$. Since $1 + 0 + 0 + 0 + \cdots$ is a convergent series with sum 1 with the standard definition, we conclude that the method $Y$ is not regular. As for contradiction, using (A) we obtain in the same way: $-1 + 0 + 0 + 0 + \cdots \equiv Y(\{-1,0,0,0,\cdots\}) = 0$ thus establishing by (C) that $1 = Y(\{0,0,0,0,\cdots\}) = -1$ since both those numbers can be assigned as value for $0 + 0 + 0 + 0 + \cdots$, which by (C) belongs to the domain of the method $Y$. This result is contradictory with the whole body of mathematics since obviously $1 \neq -1$.

Proof of Theorem 2 That the method $Y$ is not totally regular is immediate since otherwise it should assign the value $+\infty$ to the proposed expression. As for regularity and consistency, we will prove that $Y(\{1,1,1,\cdots\})$ belongs to the domain of $Y$ and compute its value. Let $s$ be the value of $1+2+3+4+\cdots$, i.e., $Y(\{1,2,3,4,\cdots\}) = s$. Then by (C), $Y(\{0,1,2,3,4,\cdots\}) = s + 0 = s$ and also by (A), $Y((-0,1,-2,-3,-4,\cdots)) = -s$. Hence, we obtain by (B): $1 + 1 + 1 + \cdots \equiv Y(\{1,1,1,\cdots\}) = 0$. Hence, this expression belongs to the domain of the method $Y$ having the value zero and the previous theorem applies. Moreover, repeated application of (C) on $1+1+1+\cdots \equiv Y(\{1,1,1,\cdots\}) = 0$, results in $1 + 1 + 1 + \cdots \equiv Y(\{1,1,1,\cdots\}) = -N$ for any nonnegative integer $N$. This result provides an alternative proof of contradiction in Theorem 1.
3.2 Associativity, Commutativity and Dilution

Let us now consider \( \sum_{n=0}^k (-1)^n \). Any method assigning a value \( s \) to it should comply \( s = 1 - s \) and hence \( s = 1/2 \). There exist many totally regular methods for the purpose, the most famous of which is probably the Cesaro sum, defined as the limit of the the sequence of successive averages of partial sums, i.e., letting \( Z_n = \frac{1}{n} \sum_{k=0}^{n-1} S_k \) for \( n \geq 1 \), (being \( S_k \) the partial sum of the first \( k \) elements of the original series up to \( k \)) we have \( Z_n = \left( \frac{1}{2} \right) + \left( \frac{1}{n} \right) \), where \( c = 0 \) or 1 and the Cesaro sum of \( \sum_{n=0}^k (-1)^n \) is \( \lim_{n \to \infty} Z_n = 1/2 \).

**Lemma 1.** Cesaro sums fulfill none of the properties associativity, commutativity and dilution.

*Proof.* For associativity just note that summing the elements of \( \sum_{n=0}^k (-1)^n \) pairwise we obtain either \( 0+0+0+\cdots \) or \( 1+0+0+\cdots \) (starting the association in the first or in the second element of the original series), both having different Cesaro sums and both different from 1/2. For commutativity, permute the elements \( (a_n, a_{n+1}) \) for all odd \( n \), obtaining \( 1+1+1+\cdots \) whose Cesaro sum is unity. For dilution, insert one zero only after each positive element, obtaining the Cesaro sum 2/3. \( \square \)

**Lemma 2.** Any commutative method \( Y \) assigning a (finite) value to \( \sum_{n=0}^k (-1)^n \) is contradictory.

*Proof.* Whichever commutative method \( Y \) is adopted, (C) forces \( \sum_{n=0}^k (-1)^n = r = 1 - r \) and hence \( r = 1/2 \). However, by permuting the elements pairwise, from \( \sum_{n=0}^k (-1)^n \equiv Y(\{1,-1,1,-1,\cdots\}) = r \) we obtain \( \sum_{n=0}^{n+1} (-1)^{n+1} \equiv Y(\{-1,1,-1,1,\cdots\}) = r \). Also, the first expression by (A) and multiplication by \( -1 \) leads to \( \sum_{n=0}^k (-1)^{n+1} \equiv Y(\{-1,-1,1,1,\cdots\}) = -r \). We obtain the contradiction \( r = Y(\{-1,1,-1,1,\cdots\}) = -r \) with \( r = 1/2 \). \( \square \)

3.3 Euler’s Continuation Method \( \mathcal{E} [7] \)

In Hardy’s account of possible alternatives to standard summation, Abel’s method and Euler’s continuation method \( \mathcal{E} \) are highlighted [7] p. 7 (among others). We may say that Euler glimpsed a possible approach in a moment in history where the concept of convergence was not fully developed (see next Section) while Abel formulated the idea with exhaustive precision. In fact, Abel’s second theorem [16] p. 3 can be stated as:

**Theorem** (Abel). If \( \sum_n a_n = s \) and \( \lim_{x \to 1^-} \sum_n a_n x^n = S \), then \( s = S \).

In other words, if the series is convergent, with finite sum \( s \), and if the power series has a (finite) limit \( S \), then both numbers coincide.

Hardy’s account of \( \mathcal{E} \) is that if \( \sum_n a_n z^n \) defines an analytic function \( f(z) \) in some region of the complex plane such that the function is properly defined along a path from that region up to \( z = 1 \), then \( \sum_n a_n = f(1) \). \( \mathcal{E} \) is indeed less precise than Abel’s theorem. Let us consider in which ways the assumptions of
Abel’s theorem may fail in the context of $E$. Either $\{a_n\}$ is a divergent series or $\lim_{z \to 1} \sum_n a_n z^n$ does not exist (now with $z \in \mathbb{C}$).

For the first case, there are examples of power series having exactly the same shape in different regions of the complex plane, while defining different functions. Then there is no uniquely defined “$f(z_0)$”. For an example with the functions $f(z) = \pm \frac{1+z^2}{1-z^2}$ and $z_0 = 2i$ see [7, p.16]. Another example could be the geometric series for $f(z) = (1-z)^{-1}$ contrasting the series inspired in Riemann’s zeta function: $g(z) = \sum_{n \geq 1} n^{1-z}$. Both series have disjoint domains of convergence in the complex plane. For $z = 1$ both series -in the context of $E$ or $Y()$- could be taken to represent $1 + 1 + 1 + \cdots$ but the second function gives a finite number while the first diverges.

For the second case, Sierpinski [15] gives an example of a convergent series $\sum_n a_n$ and a power series derived from it (defining a function $f(z)$) such that while the power series has a limit for $x \to 1^-$ along the real axis (fully compatible with Abel’s theorem), the limit does not exist along arbitrary paths $z \to 1$ in the complex plane. This has more than anecdotic value, since Sierpinski series combined with $E$ could be used to “destroy convergence” in any convergent series. Let $r = \sum a_n$ be the sum of Sierpinski’s series. Let $\sum b_n$ be a convergent series and consider the expressions $r - r + \sum b_n$ and $f(z) - r + \sum b_n z^n$. Using Abel’s method, both expressions have the same value, namely $\sum b_n$. Using $E$, the first expression still yields the same value, while the second one diverges.

Clearly, $E$ cannot be used as a tool in this context without further and accurate specification. On the contrary, following Borel [16] we may say that Abel’s approach is exhaustive and there is no room for improvement. Different attempts to prove the converse of Abel’s theorem after adding adequate additional hypotheses, have originated the branch of mathematics called Tauberian theorems.

4 Discussion

4.1 Historical Digression

Extending on Hardy’s account, it is to be noted that the modern concept of limit was established by Cauchy around 1821. However, he could not solve the question of uniform convergence. In fact, it is said [10] that this issue worried Cauchy to the point of never publishing the second volume of his course of analysis, nor consenting to a reedition of the first. He eventually allowed the publication of the lecture notes of his classes by his friend and student Moigno in 1840[11]. Again according to Lakatos, the distinction between point convergence and uniform convergence was unraveled by Seidel in 1847[10], thus completing the approach of Cauchy. The modern way of regarding limits and convergence could be said to originate around 1847.
4.2 Epistemological Issues

The idea of substituting a definition with another one is not free from consequences. Definitions in mathematics may look arbitrary at a first glance but they are always motivated. Fundamentally, (a) they satisfy the need of filling a vacancy of content in critical places where precision is needed (however, since many textbooks present definitions without discussing the process for producing them, the epistemological requirements remain usually obscure) and (b) they are explicitly forbidden to be contradictory or logically inconsistent with the previously existing body of mathematics on which they rest. In addition, when dealing with the mathematisation of natural sciences, definitions carry a signification, which is the support for using that particular piece of mathematics in that particular science.

While we appreciate the exploration work around concepts that has been done over the years, we do not substitute a meaningful and established concept with something that is inequivalent to it in the common domain of application. Again, when understanding natural sciences, such substitution would disrupt the signification chain. In simple words, we do not replace a meaningful content with a meaningless one. This would be to depart from rationality, something that is positively rejected by mathematics as a whole as well as by science in general and by a large part of society.

Along the presentation, we cared to put limits to the possible relation between \( Y (\cdot) \) and ordinary sums. In the light of the proven Theorems, it is verified that such relation is feeble or absent. Hence, the very inspirational root of these techniques becomes divorced from its results and effects. As stated above, the alternative of resigning one of axioms (A-C) also destroys any possible relation to ordinary sums.

Replacements that assign \( \sum n = a \) or \( \sum 1 = b \) (with \( a, b \) real numbers) destroy the basis of mathematics, making it the same to have one goat that having a million goats. We must emphasize that regularity is a necessary condition to preserve signification but it is not sufficient. Whatever replacement we attempt must provide a rationale for the method, preserving signification within mathematics (in the chain of abstractions it belongs) and in relation to natural sciences. For the case of series, signification is further destroyed along with properties such as association, permutation and dilution. The alternative of using one or another definition depending of the matter under study simply destroys the role mathematics as a whole. Instead of having eternal and pure relations accessible by reason alone [13], it will turn mathematics to be dependent of the context of use.

4.2.1 The Epistemology of Success

The issue of assigning a finite value to divergent series with methods that are not regular and are contradictory under Hardy’s axioms is not only material of newspaper notes, discussion blogs or Youtube videos. It has actually reached the surface of society from stuff published as scientific material. We support
this statement by commenting on a couple of references. This issue is not just a feature of these two citations, but the standard procedure of a community: just read the references in [2] to find a large amount of practitioners of this community.

In [12] we encounter an attempt to justify the use of the Riemann’s Zeta function. The authors refer to Hardy’s book for the actual method. They use axioms A and B and the zeta function to write equality between $\sum_{n=1}^{\infty} n$ and $-1/12$ (see their eq. (2.20)). The conclusion is evident: the method does not comply with Hardy’s axioms. Furthermore, the result is false since to reach their conclusion the authors disregard a divergent contribution. Hence, the equal sign does not relate identical quantities as it should. The correct expression would be

$$\bigtriangleup\sum_{n=1}^{\infty} n \equiv Y(\{1, 2, 3, 4, \cdots\}) = -\frac{1}{12}$$

where $Y$ must be understood as the method based on Riemann’s Zeta function. Here, Theorem 2 applies.

Our second example, concerning Theorem 1, is the book [2] where on page 167 we read “The analytic continuation method converts a manifestly infinite series into a finite result” exemplifying with $\bigtriangleup\sum_{n=1}^{\infty} 1 \equiv Y(\{1, 1, 1, \cdots\}) = 1/2$ (our notation, the authors use standard series notation) using the same procedure as [12]. On p. 165 this expression was given the value $-1/2$, probably a typo. Needless to say, the authors do not use the symbol $Y$ but they refer to the method as a “formal procedure” which leaves the matter in the ambiguities of language. If by formal we read belonging to or constituting the form or essence of a thing, we strongly disagree, since essence is the result of an abstracting (usually analytic) procedure [8]. However, if “formal” is intended as in its second accepted meaning: following or according with established form, custom, or rule, we agree, observing that such social agreements are not a part of science.

In defense of such procedures it is usually said that the theories using them are among the most precise and successful in Physics. This argumentation claims, then, that questions of unicity of results, backward compatibility of a method with standard convergent series, or its relation to sums (let alone signification) are uninteresting. The value is assigned because in such a way one obtains the “correct” result. Hence they adhere to a (false) epistemology that Dirac called instrumentalism [9, page 185] and we plainly call the epistemology of success.

Hitting (what is claimed to be) the right answer is not equivalent to using the right method. One may hit a correct answer with a wrong method just by chance, by misunderstanding, or even by adaptation to the known answer, etc. Paraphrasing Feyerabend’s everything goes: an idea may be welcome as a starting point without deeper considerations (within ethical limits, of course). However, for that idea to be called scientific it has to comply with the scientific method. Moreover, it has to comply with rationality. The attitude described by something is right because it gives the correct answer is dangerous in many levels. The mathematical attitude is actually the opposite (and logically inequivalent to it): If it gives the wrong answer, either the assumptions or the method in...
use are incorrect\cite{14}. This holds also for natural sciences, where in addition we have, through signification, a safe and independent method to distinguish wrong answers from right answers. Note that independency is crucial. Natural science makes predictions that can be tested independently of the theory involved. If verified, they give continued support to the theory, while if refuted they indicate where and why to correct it. As a contrast, a theory making predictions that can only be tested within itself obtains at best internal support for being consistent, but it never speaks about Nature since predictions are not independently testable. In any case, having the right answer is not a certificate of correctness (there may be an error somewhere else) whereas having any wrong answer is a certificate of incorrectness (the error is “there”).

It is worth to keep in mind the attitude taken by the founding fathers of Quantum Electrodynamics

The shell game that we play [...] is technically called ‘renormalization’. But no matter how clever the word, it is still what I would call a dippy process! Having to resort to such hocus-pocus has prevented us from proving that the theory of quantum electrodynamics is mathematically self-consistent. It’s surprising that the theory still hasn’t been proved self-consistent one way or the other by now; I suspect that renormalization is not mathematically legitimate. Richard Feynman, 1985\cite{3}

I must say that I am very dissatisfied with the situation, because this so called good theory does involve neglecting infinities which appear in its equations, neglecting them in an arbitrary way. This is just not sensible mathematics. Sensible mathematics involves neglecting a quantity when it turns out to be small - not neglecting it just because it is infinitely great and you do not want it! Paul Dirac. \cite[page 195]{9}

5 Concluding Remarks

Definitions are not arbitrary, any extension of an established operation (such as addition) needs to preserve the properties of the operation when applied to objects in the original domain of definition (regularity). We have shown in this sense that some methods proposed as extensions for the sum of convergent series (standard definition) fail on this regard. An important example of this failure is the method based on Riemann’s Zeta function that has been associated to renormalization procedures in some branches of physics. Moreover, regularity is not enough, the extension needs to preserve all the mathematical building-blocks it rests on.

It is important to indicate that attempts to justify renormalization such as \cite{12} ought to be considered scientific attempts on the ground of Popper’s demarcationism\cite{14} since by connecting to mathematical subjects as the problem of infinite series they offer the rationale behind their procedures of scientific
enquire. The result of the examination indicates that such replacements must be rejected.

The decision to expose the issues concerning the mathematical foundations of these matters to the general public should also be commented. The understanding of scientific matters is not reserved to an elite of practitioners that guard the “truth” of the subject as priests of a cult. Opening science to the scrutiny of the general public, including scientists outside the paradigm is simply correct, as discussed by Lakatos [10].

The following thesis should be considered: the more than 50 years that this matter has stayed without resolution is a demonstration that in closed elitist communities the interest of (return for) the community may very well have priority over the public (humane) interest.

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