A meromorphic quadratic differential on $\mathbb{CP}^1$ with two poles of finite order greater than 2, induces horizontal and vertical measured foliations comprising foliated strips and half-planes. This defines a map $\Phi$ from the space of such quadratic differentials to the space of pairs of such foliations. We describe a global parametrization of the space of these induced foliations, and determine the image of the map $\Phi$. To do so we consider an associated space of combinatorial objects on the surface, namely, metric graphs that are the leaf spaces of the foliations.

1. Introduction

A holomorphic quadratic differential on a Riemann surface has associated coordinate charts with transition maps that are half-translations ($z \mapsto \pm z + c$). This induces a singular-flat structure on the surface, namely, a flat metric with conical singularities, together with a pair (horizontal and vertical) of measured foliations. These structures have been useful in Teichmüller theory, and the study of the mapping class group of a surface (see [FLP12]). The correspondence between these analytical objects (the differentials) and their induced geometric structures is well-understood for a compact Riemann surface $X$ of genus $g \geq 2$. In particular, [HM79] proves that the induced map defines a homeomorphism between the space $Q(X)$ of holomorphic quadratic differentials, and the space of measured foliations $MF(X)$.

In recent work [GW] this has been extended to the case of a punctured surface $S$ of negative Euler characteristic, for the space of meromorphic quadratic differentials with higher order poles at the punctures. There, it was shown that the induced map to the space of foliations is not a bijective one, and the fibers were parametrized by some extra parameters at the poles (related to the “principal part” of the differential). This article arose from trying to understand these additional parameters in a more geometric way.

In this article we give a complete description of this correspondence for a Riemann sphere with two punctures, or in other words, a punctured plane. This is the last remaining non-compact surface of finite type to analyze; the planar case, as we shall describe later, was dealt with in the work of Au and Wan in [AW06].
It is a classical result that for meromorphic quadratic differentials on $\mathbb{CP}^1$ with up to three punctures with poles of finite order, the induced foliations comprises foliated half-planes and infinite strips (see §2). This permits a more combinatorial description of their leaf-spaces as finite-edge graphs, which we shall exploit.

**Planar case.** The case of holomorphic quadratic differentials on $\mathbb{C}$ with a single pole of finite order at $\infty$ was dealt with in [AW06]. In such a case, the quadratic differential is of the form

$$p(z)dz^2$$

where $p(z)$ is a polynomial of degree $n \geq 2$, and the leaf-spaces of the induced foliations are planar trees (see §3).

Indeed, such a polynomial quadratic differential has a pole of order $(n + 4)$ at infinity, and the induced measured foliation has a pole-singularity with $(n + 2)$ foliated half-planes in a cyclic order around infinity, each giving rise to an infinite ray in its leaf-space. By a conformal change of coordinates, it can be arranged that the polynomial is *monic*, namely, that the leading coefficient is 1, and *centered*, namely, that the zeroes of the polynomial have vanishing mean. The foliated half-planes are then asymptotic to a fixed set of directions, thus acquiring a labelling that is inherited by the infinite rays.

The result for this case can be summarized as follows (see §3 for details):

**Theorem (Au-Wan, [AW06]).** Let $Q_1(n) \cong \mathbb{C}^{n-1}$ be the space of monic centered polynomial quadratic differentials of degree $n$. Then the space $F(n)$ of the induced measured foliations admits a bijective correspondence with the space $T(n + 2)$ of planar metric trees with $(n + 2)$ labelled rays, and we have the parametrization

$$F(n) \cong T(n + 2) \cong \mathbb{R}^{n-1}.$$ 

Moreover, the map

$$\Phi_1 : Q_1(n) \to F(n) \times F(n)$$

that assigns to a polynomial quadratic differential its associated horizontal and vertical foliations, is a homeomorphism.

**Remark.** The space of measured foliations $F(n)$ decomposes into regions corresponding to the different combinatorial types of planar trees with labelled ends, and there is exactly a Catalan number of them. This is closely related to the classification of the trajectory-structure for polynomial vector fields on $\mathbb{C}$ (see [BD10], [Dia13], [DES] and the references therein). One of the differences is that a foliation induced by a quadratic differential is typically not orientable.
Punctured-plane case. Any holomorphic quadratic differential on $\mathbb{C}^*$ with poles of orders $n$ and $m \geq 3$ at $\infty$ and $0$ respectively, can be expressed as:

$$
\frac{p(z)}{z^m} dz^2
$$

where $p(z)$ is a monic polynomial of degree $n + m - 4$.

The space $Q(n, m) \cong \mathbb{C}^{n+m-4}$ of such differentials is then parametrized by the remaining coefficients of $p(z)$.

Let $F(n, m)$ be the space of measured foliations on the punctured plane with two pole-singularities corresponding to poles of orders $n$ and $m$, equipped with a labelling of the foliated half-planes around the poles.

To aid in visualizing, we shall henceforth uniformize $\mathbb{C}^*$ to an infinite Euclidean cylinder, with one end at 0 and the other end at $\infty$.

We shall prove:

**Theorem 1.1.** For $n, m \geq 3$, the space of measured foliations $F(n, m)$ on $\mathbb{C}^*$ with pole-singularities of orders $n$ and $m$ is homeomorphic to $\mathbb{R}^{n+m-4}$.

Moreover, the map

$$
\Phi : Q(n, m) \to F(n, m) \times F(n, m)
$$

that assigns to a quadratic differential its induced horizontal and vertical measured foliations defines a homeomorphism to the image

$$
F(n, m)^{(2)} = F(n, m) \times F(n, m) \setminus \Delta
$$

where $\Delta$ is the subspace comprising pairs of foliations both of which have zero transverse measure around the punctures.

See Lemma 4.2 for the reason that the image excludes the set $\Delta$. 
The leaf space of a measured foliation in $\mathcal{F}(n, m)$ is an embedded metric graph in $\mathbb{C}^*$ with $(n - 2)$ labelled infinite rays to $\infty$, and $(m - 2)$ labelled infinite rays to $0$. We shall denote the space of such metric graphs by $\mathcal{G}(n, m)$.

The proof of Theorem 1.1 divides into two parts - in §4 we use Au-Wan’s work to build a singular-flat structure with prescribed horizontal and vertical leaf-spaces in $\mathcal{G}(n, m)$. One difficulty is that the leaf-space is no longer a tree: whenever the foliation has positive transverse measure around the punctures, the corresponding graph in $\mathcal{G}(n, m)$ has a cycle. This we resolve by passing to the universal cover.

To complete the proof, we then parametrize the space of foliations $\mathcal{F}(n, m)$ in §5. Any such measured foliation is determined by its associated metric graph in $\mathcal{G}(n, m)$, together with the data of an integer “twist”. The necessity of this additional “twist” parameter is special to the two-pole case, and arises from the fact that the fundamental group of the punctured plane $\mathbb{C}^*$ is non-trivial, so the trajectories might have an integer winding number around the punctures. The parametrization of $\mathcal{F}(n, m)$ follows by first providing parameters for the space of graphs $\mathcal{G}(n, m)$, and then explaining how the infinitely many domains labelled by the integer twists assemble to produce a space homeomorphic to $\mathbb{R}^{n+m-4}$. This relies on some combinatorial arguments and parametrizations of some related spaces of graphs that we relegate to the Appendix.

For a punctured Riemann surface $X$ of higher topological complexity (e.g. a four-punctured sphere), the measured foliations induced by a holomorphic quadratic differential with higher order poles at the punctures, can be more complicated. Indeed, the leaf-spaces of the foliations when lifted to the universal cover are “augmented” $\mathbb{R}$-trees, and a parametrization of the space of such trees, and hence the space of such measured foliations, is given in [GW]. However, the moduli space of the underlying topological surface is no longer a point (as in the case for a punctured plane); a pair of horizontal and vertical measured foliations determines a complex structure that might not agree with that of $X$. In these cases, the corresponding induced map $\Phi$ from quadratic differentials on $X$ to pairs of measured foliations has a larger exceptional set that is more difficult to characterize.

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2. Preliminaries

Quadratic differentials and their induced geometry. A holomorphic quadratic differential \( q \) on a Riemann surface \( X \) is a holomorphic section of the symmetric square of canonical bundle \( K_X^2 \). Throughout this paper, the underlying Riemann surface \( X \) shall be either the complex plane \( \mathbb{C} \) or the punctured complex plane \( \mathbb{C}^* \). In either case the holomorphic quadratic differential can be expressed as \( q(z)dz^2 \) where \( q(z) \) is a globally defined holomorphic function.

A holomorphic quadratic differential induces a singular-flat metric and horizontal and vertical foliations on the underlying Riemann surface that we now describe. For an account of what follows, see [Str84] or [Gar87].

**Definition 2.1 (Singular-flat metric).** A holomorphic quadratic differential induces a conformal metric locally of the form \(|q(z)|dz^2\), which is a flat Euclidean metric with cone-type singularities at the zeroes, where a zero of order \( n \) has a cone-angle of \((n + 2)\pi\).

**Definition 2.2 (Horizontal and vertical foliations).** A holomorphic quadratic differential on \( X = \mathbb{C} \) or \( \mathbb{C}^* \) determines a bilinear form \( q : T_xX \otimes T_xX \to \mathbb{C} \) at any point \( x \in X \) away from the poles. Away from the zeroes, there is a unique (un-oriented) horizontal direction \( v \) where \( q(v, v) \in \mathbb{R}^+ \). Integral curves of this line field on \( X \) determine the horizontal foliation on \( X \). Similarly, away from the zeroes, there is a unique (un-oriented) vertical direction \( h \) where \( q(h, h) \in i\mathbb{R}^+ \). Integral curves of this line field on \( X \) determine the vertical foliation on \( \mathbb{C} \).

**Remark.** The terminology arises from the fact that for the quadratic differential \( dz^2 \) on any subset of \( \mathbb{C} \) (equipped with the coordinate \( z \)), the horizontal and vertical foliations are exactly the foliations by horizontal and vertical lines.

**Definition 2.3 (Pole and prong singularities).** At the zero of order \( k \), horizontal (and vertical) foliation has a \((k + 2)\)-prong singularity. This can be thought of as the pullback of the horizontal foliation on \( \mathbb{C} \) by the local change of coordinates \( z \mapsto \xi = z^{k/2+1} \) (a branched cover at the zero) that transforms the quadratic differential to \( d\xi^2 \) (up to a constant multiplicative factor) on the complex plane with coordinate \( \xi \).

Similarly, at a pole of order \( k \geq 3 \), the induced foliation has a pole-singularity of order \( k \). (See Figure 2.)

**Definition 2.4 (Transverse measure).** The horizontal (resp. vertical) foliation induced by a holomorphic quadratic differential is equipped with a transverse measure, that is, any arc transverse to the foliation acquires a measure that is invariant under transverse homotopy of the arc. Namely, the transverse
measure of such an arc \( \gamma \) transverse to the horizontal foliation is
\[
\tau_h = \int_{\gamma} |\Im(\sqrt{q}(z))| dz
\]
assuming \( \gamma \) is contained in a coordinate chart, and and similarly the transverse measure \( \tau_v \) of an arc transverse to the vertical foliation is given by the integral of the real part \( |\Re(\sqrt{q}(z))| \). In general one adds such distances along a cover of the arc comprising of coordinate charts; this is well-defined as the above integrals are preserved under change of coordinates.

**Global structure of the induced foliation.** The measured foliation induced by a holomorphic quadratic differential on \( X \) decomposes into flat foliated Euclidean regions that are attached to each other by isometric identifications along their boundaries. Resulting edges on the boundary between singularities are called critical trajectories for the foliation; note they have zero transverse measure.

Indeed, the following is a consequence of the Three Pole Theorem, (see Theorem 3.6 of [Jen58]):

**Theorem 2.5 (Jenkins).** For \( X = \mathbb{C} \) or \( \mathbb{C}^* \), the horizontal foliation induced by a holomorphic quadratic differential on \( X \) can be decomposed into the following types of Euclidean regions

1. half-planes isometric to \( \{ z \in \mathbb{C} \mid \text{Im}(z) > 0 \} \)
2. horizontal strips isometric to \( \{ z \in \mathbb{C} \mid -1 < \text{Im}(z) < 1 \} \)
3. a ring domain isometric to a finite-length cylinder (which can only occur for the punctured plane).

In the first two cases, the horizontal lines and vertical lines are the leaves of the horizontal and vertical foliations. In the last case, these are the meridional circles and longitudinal segments respectively.

**Remark.** In particular, for \( X = \mathbb{C} \) or \( \mathbb{C}^* \), the induced foliations have no recurrent trajectories; all trajectories either close up and form a ring domain or end.
Figure 3. The metric graph associated to an induced foliation on $\mathbb{C}$ has finite-edges corresponding to strips, and infinite rays corresponding to the half-planes. See Definition 2.6.

up in poles. For a general Riemann surface, there is an additional possibility of a spiral domain in which each horizontal leaf is dense (see the Basic Structure Theorem in [Jen58]). After the work of Hubbard and Masur in [HM79], these trajectory structures precisely coincide with those of measured foliations on surfaces, as explained in the work of Thurston (see [FLP12]).

We shall record this structure in terms of a dual structure defined as follows:

**Definition 2.6** (Metric graph of a foliation). The leaf-space $G$ of a horizontal foliation of a holomorphic quadratic differential on $X = \mathbb{C}$ or $\mathbb{C}^*$ with finite order poles at 0 and $\infty$ is defined as

$$G := X/\sim$$

where $x \sim y$ if $x, y$ lie on the same horizontal leaf, or on horizontal leaves that are incident on a common zero. The graph $G$ acquires a metric from the transverse measure. By the theorem above, $G$ has finitely many vertices and edges, where the finite-length edges of $G$ are the leaf-spaces of the strips and the (possible) ring-domain, and infinite-length edges are rays that are the leaf-spaces of the half-planes. (See Figure 3.) We shall consider $G$ to be embedded on $X$, with each infinite ray transverse to the horizontal leaves in the corresponding half-plane.

Moreover, the number of infinite rays incident at a puncture is determined by the order of the pole:

**Lemma 2.7.** A holomorphic quadratic differential $q$ with a pole at a puncture of order $n \geq 3$ has exactly $n - 2$ half-planes in its induced singular-flat metric, surrounding the puncture in a cyclic order. Moreover, if the quadratic differential is of the form

$$q = \left(1 + \frac{\alpha_{n-1}}{z^{n-1}} + \frac{\alpha_{n-2}}{z^{n-2}} + \cdots \right) dz^2$$

with the pole at the origin, then the horizontal and vertical directions on each half-plane are asymptotic to a fixed set of directions at the pole.
Proof. It is easy to verify that the quadratic differential $dz^2$ has a pole of order 4 at infinity and the induced Euclidean metric has two Euclidean half-planes surrounding the pole. Also, the differential $\frac{1}{z}dz^2$ has a pole of order 3 at infinity, and one Euclidean half-plane, since a branched double covering $w = z^2 \mapsto z$ pulls it back to $dw^2$.

Any other pole is a branched covering of one of these two examples, ramified at the pole. For example, if the differential is $\frac{1}{z^n}dz^2$ where $n$ is even, then the coordinate change $z \mapsto \xi = z^{(2-n)/2}$ transforms this to $d\xi^2$, and this coordinate change is a branched cover of degree $(n-2)/2$ that pulls back two half-planes to $(n-2)$ half-planes. For $n$ odd, the coordinate change $z \mapsto \xi = z^{2-n}$ transforms this to $\frac{1}{2}d\xi^2$, and this coordinate change is a branched cover of degree $(n-2)$ that pulls back one half-plane to $(n-2)$ half-planes.

It is easy to check that these branched covers then pull-back the horizontal directions on the $\xi$-plane to directions at angles

$$\left\{ \frac{2\pi \cdot k}{n-2} \mid k = 1, 2 \ldots n-2 \right\},$$

and the vertical directions to ones at directions bisecting the above angles.

These remain the asymptotic directions when the quadratic differential has some lower-order perturbations as in [2].

In this paper we assume that the quadratic differentials on $\mathbb{C}^*$ are normalized (by a conformal change of coordinates) to be of the form (2) at the poles.

From the Lemma above, we can define:

**Definition 2.8 (Labelling).** Let $H_1, H_2, \ldots H_{n-2}$ be the half-planes appearing around a pole of order $n \geq 3$ for the horizontal foliation of a quadratic differential normalized to have the form (2). These half-planes then acquire a labelling by $\{1, 2, \ldots n-2\}$ depending on the angle the vertical directions on the half-plane are asymptotic to: namely, the label is $k$ if the asymptotic vertical direction is $\frac{2\pi}{n-2} \cdot (k + \frac{1}{2})$. This labelling is acquired by the infinite rays in the leaf-space $G$ corresponding to these half-planes.

3. **Planar case: The work of Au-Wan**

Any holomorphic quadratic differential on $\mathbb{C}$ with a pole of finite order at infinity, can be expressed as $p(z)dz^2$ where $p(z)$ is a polynomial.

This leads us to define:

**Definition 3.1 (Polynomial quadratic differentials).** Let $n \geq 2$. Consider the space

$$Q_1(n) = \{(z^n + a_1z^{n-2} + \cdots + a_{n-2}z + a_{n-1})dz^2 \mid a_i \in \mathbb{C} \text{ for } i = 1, \ldots n-1\}.$$
where note that the polynomials appearing are *monic*, that is, having the leading (first) coefficient equal to one, and *centered*, that is, having the mean of the zeroes of the polynomial (namely, the second coefficient) being equal to zero. The identification with complex affine space is via the remaining \((n - 1)\) coefficients, which can be arbitrary complex numbers.

**Remark.** Note that any polynomial quadratic differential can be made monic and centered by a conformal change of variables \(z \mapsto \alpha z + \beta\). However the space \(Q_1(n)\) is different from the space of such differentials up to conformal equivalence; namely, \(Q_1(n)\) is a branched cover over the latter space, with finite ramification. As an example, the quadratic polynomial differentials \((z^2 + a)dz^2\) and \((z^2 - a)dz^2\) in \(Q_1(2)\) are conformally related by the change of coordinates \(z \mapsto iz\). For a further discussion of this, see [BH88].

3.1. **Parametrizing \(F(n)\).** The structure of an induced foliation \(F \in F(n)\) has half-planes and strips (see Theorem 2.5). Moreover, by Lemma 2.7 there are exactly \((n + 2)\) half-planes arranged around the pole at infinity, which acquire a labelling from its asymptotic directions (see Definition 2.8). The metric graph for such a foliation (see Definition 2.6) is then a planar metric tree with \((n + 2)\) labelled infinite rays.

Following the work of Mulase-Penkava in [MP98], any such tree is obtained by a “metric expansion” of a \((n + 2)\)-pronged tree, defined as follows.

**Definition 3.2** (Metric expansion). For an integer \(k \geq 4\), let \(G(k)\) be a \(k\)-pronged star, that is, a planar graph with a single vertex and \(k\) infinite edges (rays) from it, that are labelled in a cyclic order. A metric expansion of \(G(k)\) is a new graph obtained by replacing the vertex by a tree (with each new vertex of degree greater than two) that connects with the rest of the graph. The new edges are equipped with edge-lengths.

The space of such metric trees \(\mathcal{T}(k) = \{\text{metric expansions of } G(k) \text{ with labelled ends}\}\) can be given a topology by declaring two such trees to be close if they differ by a small change of edge-lengths, or a composition of collapsing or expanding of short edges (also called Whitehead moves).

The following result is Theorem 3.3 of [MP98]:

**Theorem 3.3** (Mulase-Penkava). The space of metric expansions \(\mathcal{T}(k)\) is homeomorphic to \(\mathbb{R}^{k-3}\).

**Remark.** It is easy to check that a generic tree in \(\mathcal{T}(k)\) is trivalent at each vertex, and has exactly \(k - 3\) edges of finite length. These (non-negative) lengths form parameters that parametrize a subset of \(\mathcal{T}(k)\) corresponding to a fixed combinatorial type; there are Catalan number of types that are obtained by Whitehead
Figure 4. The different combinatorial types of metric trees in $\mathcal{F}(5)$ form $\mathbb{R}^2$ and parametrize the space of foliations in $\mathcal{F}(3)$ (see Theorem 3.3 and Proposition 3.4).

As a consequence, we have:

**Proposition 3.4.** The space of foliations $\mathcal{F}(n)$ is homeomorphic to $\mathbb{R}^{n-1}$.

**Proof.** We have seen before that we have a map $\Psi : \mathcal{F}(n) \to \mathcal{T}(n+2)$ that assigns to a foliation its leaf-space. It is not difficult to construct an inverse map: given a planar metric tree in $\mathcal{T}(n+2)$, we arrange half-planes and strips in the pattern prescribed by the tree, and identify by isometries of the boundaries (which are bi-infinite lines). Note that the strip widths are prescribed by the edge-lengths of the tree. Then proposition then follows from Theorem 3.3. □

3.2. **Prescribing horizontal and vertical trees.** Given a pair of foliations $(F_v, F_h)$ in $\mathcal{F}(n)$ (or equivalently, a pair of metric trees $H, V \in \mathcal{T}(n+2)$ - see Proposition 3.3) one can construct a holomorphic quadratic differential which has these as its vertical and horizontal foliations.

Such a quadratic differential is obtained by attaching Euclidean half-planes and bi-infinite strips to each other by isometries (half-translations) on their boundaries; the standard differential $dz^2$ on each piece then descends to a well-defined holomorphic quadratic differential on the resulting surface.
The proof of this is carried out in [AW06] (see Theorem 4.1 of that paper), where they show, in particular:

**Theorem 3.5 (Au-Wan).** Given two properly embedded planar metric trees \( V, H \) in \( \mathbb{C} \) and a bijection \( f \) between the complementary regions of \( V \) with the infinite rays of \( H \), there is a unique quadratic differential metric on \( \mathbb{C} \) with induced horizontal and vertical foliations that have leaf-spaces \( V \) and \( H \) respectively. Moreover, the arrangement of their foliated half-planes induces the prescribed bijection \( f \).

**Remarks.** 1. Note that in the above theorem, we do not require that the trees have finitely many edges; we shall use this subsequently in this paper when we apply it to the universal cover of \( \mathbb{C}^* \).
2. The uniqueness above is clarified in Theorem 4.2 of [AW06]: in our set-up, if there are homeomorphisms \( F, G : \mathbb{C} \to \mathbb{C} \) that restrict to isometries of \( V \) and \( H \) respectively, then the quadratic differential that realizes \((V, H, f)\) is identical to the one that realizes \((V, H, G \circ f \circ F^{-1})\).

**Example.** In the case when \( V \) is an \((n + 2)\)-pronged star, and \( H \) is a metric tree, such a singular-flat surface (and hence a quadratic differential) is obtained by attaching \((n + 2)\) half planes to \( H \) by isometries along the boundaries, one for each complementary region of the tree \( H \).

4. **Proof of Theorem 1.1: Image of \( \Phi \)**

Recall the space \( Q(n, m) \) of meromorphic quadratic differentials with two poles of orders \( n, m \geq 3 \), admits a map \( \Phi \) to the space \( \mathcal{F}(n, m) \times \mathcal{F}(n, m) \) of pairs of measured foliations that each has two pole-singularities of orders \( n - 2 \) and \( m - 2 \) respectively. Namely, the map \( \Phi \) assigns to such a quadratic differential the induced horizontal and vertical measured foliations.

In this section, we shall describe the image of \( \Phi \).

Throughout, we shall use the following terminology:

**Definition 4.1.** The transverse measure of a foliation \( F \) in \( \mathcal{F}(n, m) \) shall refer to the transverse measure of a loop around the puncture(s), unless otherwise specified.

We observe:

**Lemma 4.2.** The transverse measures \( \tau_h, \tau_v \) of the induced horizontal and vertical measured foliations cannot both be zero.

**Proof.** Let \( \tau \) be the Euclidean circumference of the cylinder \( \mathbb{C}^* \), which is a positive real number. Then \( \tau^2 = \tau_h^2 + \tau_v^2 \) (see Definition 2.4) and the lemma follows. \( \Box \)

Let \( \mathcal{F}(n, m)^{(2)} \) be the space of pairs of foliations that do not both have zero transverse measure. In this section, we shall prove that the above lemma is the only obstruction (see Proposition 4.9).
4.1. The induced metric graphs. The metric graph $G$ for a measured foliation in $\mathcal{F}(n, m)$ (see Definition 2.6) has $n - 2$ infinite rays to $\infty$ and $m - 2$ infinite rays to 0.

Recall from Definition 2.8 that these infinite rays acquire a labelling according to the directions in which they are incident at the poles.

Here, the horizontal (or vertical) foliation could have infinite strips between (the same or different) poles. Recall from Definition 2.6 that the finite-length edges of the metric graph $G$ of the horizontal (resp. vertical) foliation correspond to the foliated horizontal (resp. vertical) strips or ring-domains in the singular-flat metric. The length $l(e)$ of such an edge $e$ is the transverse measure of the foliation, across the strip or ring-domain.

We first observe:

**Lemma 4.3.** For a foliation with positive transverse measure $\tau$, there are no ring domains, and there is at least one strip of positive transverse measure from 0 to $\infty$. The metric graph $G$ has a unique cycle $L$ such the cylinder $C^*$ deform-retracts to $L$, and $\sum_{e \in L} l(e) = \tau$.

**Proof.** A ring domain would necessarily separate the cylinder, and hence it suffices to prove that there is a strip between the poles.

If not, let $U_0$ be the union of the $(m - 2)$ half-planes and strips incident on the pole 0. Since $U_0$ is disjoint from the pole at $\infty$ by assumption, its boundary goes around the circumference of the cylinder. But then a loop $\gamma$ around the circumference is homotopic to a concatenation of critical trajectories, and hence has transverse measure 0. This contradicts our assumption that $\tau$ is positive.

Now, the transverse measure of a loop $\gamma$ around the circumference is positive; a representative in its homotopy class realizing the minimum transverse measure crosses a (non-empty) collection of strips and the corresponding finite-length edges in the metric graph $G$ form a cycle $L$ in the graph.
Figure 6. Example of a foliation in \( F(3,3) \) with zero transverse measure and its corresponding metric graph.

By collapsing along the leaves the cylinder \( C^* \) deform-retracts to the metric graph \( G \). Since \( \pi_1(C^*) = \mathbb{Z} \), the cycle \( L \) must be the unique cycle in \( G \). See Figure 5. \( \square \)

In contrast to this, we have:

**Lemma 4.4.** For a foliation with zero transverse measure, there is a single ring domain whose core curve is homotopic to the punctures. The metric graph \( G \) for the foliation is a tree embedded on the cylinder, with a (possibly degenerate) edge corresponding to the leaf space of the ring-domain.

**Proof.** When the transverse measure \( \tau \) vanishes, a loop around the circumference of the cylinder is homotopic to either a leaf of the foliation, or a concatenation of saddle-connections. Hence the foliation has a ring domain, which may be degenerate in the latter case. (See Figure 6.)

Thus the ring domain corresponds to a (possibly degenerate) edge in the metric graph \( G \).

**Claim.** There is a unique ring domain.

**Proof of Claim.** Let \( R_0 \) be the union of all ring domains in \( C^* \). Then any component of the complement of \( R_0 \) comprises half-planes and strips, and hence cannot have bounded diameter. Then observe:

(a) If a ring domain has a core curve not homotopic to the circumference of the cylinder, then the core curve is contractible and one of the complementary regions of the ring-domain is a bounded disk.

(b) If there are two parallel ring domains with core curves homotopic to the circumference, then the portion of the cylinder \( C^* \) between them would be bounded. There cannot be two ring domains separated by a cycle of critical trajectories, since this would contradict the assumption \( \tau = 0 \).

These complete the proof of the claim. \( \square \).
Figure 7. The strips in a foliation with positive transverse measure may twist around the cylinder.

This unique ring domain then separates the cylinder into two foliated half-cylinders (which are simply-connected), each of which have half-planes and strips whose metric graph (that it deform-retracts to) is a tree. These metric trees emanate from the endpoints of the edge corresponding to the ring domain, and the entire metric graph is also a tree.

Examples of metric graphs in \( G(n, m) \) for the positive and zero transverse measures are provided in Figures 5 and 6.

In what follows, we shall use:

**Definition 4.5 (Space of metric graphs).** The space \( G(n, m) \) shall be the space of metric graphs of measured foliations (say horizontal) induced by quadratic differentials in \( Q(n, m) \).

We postpone a parametrization of the spaces of metric graphs \( G(n, m) \) to a later section.

4.2. **Determining the foliation.** In contrast to the planar case discussed in §3, the metric graph of a foliation \( F \in \mathcal{F}(n, m) \) does not determine the foliation completely. The ambiguity is that of the “twist” of the leaves across the cylinder, and is present only when \( F \) has positive transverse measure. This is resolved by having an additional discrete parameter associated with \( F \), as in the following definition. (See Figure 7.) We shall later refine it to a continuous parameter in §5.

**Definition 4.6 (Topological twist).** Let \( \gamma \subset F \) be an infinite leaf between the poles (this exists by Lemma 4.3). The (topological) twist of the foliation \( F \) is an integer \( j(F) \in \mathbb{Z} \) that is the number of left Dehn twists one needs to apply to the cylinder so that the image of a fixed longitude of the cylinder \( C^* \) (say, the positive real axis) is in the homotopy class of \( \gamma \).

**Remarks.** 1. The twist is well-defined (that is, independent of the choice of \( \gamma \)) because the leaves of a foliation do not intersect.
2. By “homotopy class of \( \gamma \)” we allow a homotopy that is compactly supported. By definition, a left (respectively, right) Dehn twist about the core of the cylinder...
increases (respectively, decreases) the twist parameter by one.

3. For a foliation $F$ with zero transverse measure (that is, with a ring-domain), the above twist parameter is ill-defined. As we shall see in Proposition 5.5, such a foliation is reached as a limit as the topological twist parameter diverges.

We can then prove:

**Lemma 4.7.** A foliation $F \in \mathcal{F}(n,m)$ is uniquely determined by its metric graph $G \in \mathcal{G}(n,m)$ and, in the case it has positive transverse measure, its twist parameter $j(F) \in \mathbb{Z}$.

**Proof.** Assume $F$ has positive transverse measure. One can achieve a zero twist parameter by assembling half-planes and strips in the pattern of the edges of $G$ where the widths of the strips are determined by the finite edge-lengths, and one of the strips is along the chosen longitude of the cylinder (e.g. the one corresponding to the real axis on the punctured plane). Positive and negative twist parameters are then achieved by left and right Dehn-twists about the core curve of the cylinder. The case that $F$ has zero transverse measure is similar, and the proof is left to the reader.

We also note:

**Lemma 4.8.** The map $\Phi: Q(n,m) \rightarrow \mathcal{F}(n,m) \times \mathcal{F}(n,m)$ is injective.

**Proof.** Consider a pair of induced measured foliations $F_h, F_v$ with leaf-spaces $V$ and $H$ respectively. We want to show that the inducing quadratic differential is uniquely determined by this pair.

We first show that the metric is indeed determined uniquely by the pair.

By Theorem 2.5, the horizontal foliation $F_h$ determines a decomposition of the quadratic-differential metric into Euclidean half-planes and strips that are identified by isometries along their boundaries. ($F_h$ is then the foliation by horizontal lines on the half-planes and strips.)

Since the boundaries are bi-infinite lines, there is an ambiguity of translation when choosing such an isometric identification. However, as we shall now see, different translations in such an identification result in different transverse measures for the vertical foliation.

To see this, consider an embedded line $L$ obtained after identifying two such boundary components, in the final singular-flat surface $S$. Assume first that the two components $S_1$ and $S_2$ that $L$ divides $S$ into, has at least one singularity each, of the horizontal foliation. Choose such a pair $p_1, p_2$. Since singularities of the quadratic differential metric determine singularities of both the horizontal and vertical foliations, these two singularities will determine a pair of vertices $v_1, v_2$ (which could also coincide) on the leaf-space of the resulting vertical foliation on $S$. 
Since the desired vertical foliation $\mathcal{F}_v$, and hence its leaf-space $H$ is fixed, we know the desired transverse measure (of the vertical foliation) between this pair of vertices $v_1, v_2$. There is a unique choice of a translation in the identification of the two sides of $L$, that realizes this transverse measure: clearly, translating one way or the other in the identification of the two sides of $L$ would increase or decrease this quantity. (Note that such a horizontal translation does not change the transverse measure with respect to $\mathcal{F}_h$ between $p_1$ and $p_2$.)

The only other case is when one of the complementary components $S_1, S_2$ of the embedded line $L$ has no singularity: this foliated region, say $S_1$, must then necessarily be a half-plane. However, the singular-flat metric determined by attaching the half-plane $S_1$ to $S_2$ by a boundary isometry is independent of the choice of the translation.

Altogether, we have proved the singular-flat metric is uniquely determined by the pair $\mathcal{F}_h, \mathcal{F}_v$ of foliations.

The associated quadratic differential is then uniquely determined by equipp- ing each of the domains in the decomposition with the quadratic differential $dz^2$ in the natural coordinates on each; since the identifications are by translations or half-translations (i.e., of the form $z \mapsto \pm z + c$, depending on the orientation of the bi-infinite lines being identified) there is a well-defined quadratic differential on the resulting singular-flat surface.

□

Remark. The analogue of the above lemma for induced measured foliations on compact surfaces of genus greater than one is Theorem 3.1 of [GM91].

4.3. Prescribing horizontal and vertical foliations. Our goal is to prove:

**Proposition 4.9.** Let $F_v, F_h \in \mathcal{F}(m, n)$. Then, there exists a holomorphic quadratic differential $\Phi$ defined on $\mathbb{C}^*$ such that the vertical and horizontal foliations are $F_v$ and $F_h$ respectively.

The idea of the proof is to reduce to the planar case that is dealt with in the work of Au-Wan (Theorem 3.5).

To illustrate, we start with the following observation:

**Example.** If we have a singular-flat metric $S$ on the cylinder induced by some quadratic differential in $Q(n, m)$, and suppose both have positive transverse measures. Then by Lemma 4.3 there is a horizontal strip going across the cylinder. Cutting along a horizontal leaf $\gamma$ in this horizontal strip yields a singular-flat metric on a bi-infinite strip with two boundary components that are two copies of $\gamma$. Attaching a Euclidean half-plane to these boundary components then results in a singular flat metric $S'$ on the plane, realizing a pair of planar metric trees $V', H'$. 
This suggests the following procedure for finding a singular-flat metric realizing a given pair of metric graphs \(V, H \in \mathcal{G}(m, n)\): namely, suitably modify the pair of graphs \(V, H\) to obtain a pair of planar metric trees \(V', H'\), realize the corresponding singular-flat metric on \(C\) using Au-Wan’s theorem, and obtain the desired metric back on the cylinder by suitable operations.

The modifications to obtain the trees are also easy to see: the metric graph \(V'\) of the horizontal foliation is obtained from \(V\) by deleting a point on the cycle and completing the graph by attaching Euclidean rays to the resulting pair of boundary points. (See Figure 8.) Moreover, the horizontal leaf \(\gamma\) determines a bi-infinite line \(\Gamma\) in the graph \(H\) cutting along which yields \(H'\).

This motivating example, disregards the twist parameter, however. To incorporate that in our construction, we shall need to pass to the universal cover in the actual proof below.

**Proof of Proposition 4.9.** Let \(V, H\) be the metric graphs for the measured foliations \(F_h, F_v\) respectively.

Lift the two foliations to the universal cover \(\hat{\mathbb{C}}\), via the covering map \(\pi : \mathbb{C} \to \hat{\mathbb{C}}\) defined by \(\pi(z) = \exp(2\pi i z)\), and let \(V'\) and \(H'\) be the metric trees for the lifted foliations. Note that the lifted foliations are invariant by the the action of \(\pi_1(\hat{\mathbb{C}}) = \mathbb{Z}\) generated by the translation \(T = z \mapsto z + i\tau\) where \(\tau\) is the circumference of the cylinder in the desired singular-flat metric (equal to \((\tau_h^2 + \tau_v^2)^{1/2}\)).

In both the lifts there is a tiling of the complex plane by fundamental domains that are strips. We fix a labelling of these strips by the integers – namely, fix a strip to the 0-th fundamental domain, and then the \(i\)-th fundamental domain is its image under an \(i\)-th power of the translation \(T\).

In what follows we first describe the planar metric trees \(V', H'\) and a bijection \(f\) between the complementary regions of \(V'\), and the infinite rays of \(H'\), so that
Figure 9. The lift of a foliation to the universal cover and its metric graph in the case of positive transverse measure. Note that the index of the labels of the complementary regions on either side depends on the twist parameter.

we can apply Theorem 3.5 of Au-Wan.

**Case 1: Both foliations have positive transverse measures \( \tau_h \) and \( \tau_v \).**

In this case both \( V', H' \) are invariant by the action of \( \mathbb{Z} \) on \( \mathbb{C} \) by translations (Recall that the graphs \( V, H \) have cycles of lengths \( \tau_h \) and \( \tau_v \) respectively.) The twist parameters shall play a role in specifying the bijection \( f \), as we shall now describe.

Let \( j_h, j_v \in \mathbb{Z} \) denote the twist parameters for the horizontal and vertical foliations (c.f. Lemma 4.7).

Let \( \{\alpha_1, \alpha_2, \ldots \alpha_{n-2}\} \) and \( \{\beta_1, \beta_2, \ldots \beta_{m-2}\} \) be the labels of the complementary regions of \( V \) in the cylinder, that are asymptotic to the two ends, at \( \infty \) and \( 0 \), respectively. On the universal cover, the complementary regions of \( V' \) acquire a labelling by the index set \( \{\alpha_i^j, \beta_k^i \mid i \in \mathbb{Z}, 1 \leq j \leq n - 2, 1 \leq k \leq m - 2\} \) where the lift of \( \alpha_j \) to the \( i \)-th copy of the fundamental domain is labelled \( \alpha_i^j \), and the lift of \( \beta_j \) to the \( (i + j_h) \)-th copy of the fundamental domain is labelled \( \beta^i_j \). (The shift by \( j_h \) in the labelling of the \( \alpha \) and \( \beta \) sides is a reflection of the fact the foliation \( F_h \) has \( j_h \) topological twists around the cylinder.) See Figure 9.

Similarly, let \( \{a_1, a_2, \ldots a_{n-2}\} \) and \( \{b_1, b_2, \ldots b_{m-2}\} \) be the labels of the infinite rays of \( H \) in the cylinder, that are asymptotic to the two ends, respectively. On the universal cover, we obtain a labelling of the infinite rays of \( H' \) by the index set
Figure 10. The lift of a foliation to the universal cover and its metric graph in the case of zero transverse measure.

\[ \{a_i^j, b_k^i \mid i \in \mathbb{Z}, 1 \leq j \leq n - 2, 1 \leq k \leq m - 2\} \] where the labels of on the \( i \)-th copy of the fundamental domain differ by a shift of \( j \).

Then the desired bijection \( f \) is the one that assigns the complementary region of \( V' \) labelled \( \alpha_i^j \) or \( \beta_i^k \) to the ray of \( H' \) labelled \( a_i^j \) or \( b_k^i \), respectively. (This is for each \( j \in \{1, 2, \ldots, n - 2\} \) and \( k \in \{1, 2, \ldots, m - 2\} \) and \( i \in \mathbb{Z} \).)

**Case 2: One of the foliations, say \( F_v \), has zero transverse measure.**

In this case, we label the complementary regions of \( V' \) by labels in \( \{\alpha_i^j, \beta_i^k \mid i \in \mathbb{Z}, 1 \leq j \leq n - 2, 1 \leq k \leq m - 2\} \), exactly as in Case 1.

The metric graph \( H \) for the foliation \( F_v \), however, is already a tree in \( \mathbb{C}^* \) (see Lemma 4.4). Indeed, \( H \) has a distinguished finite-length (but possibly zero length) edge \( e \) that corresponds to the ring-domain, and a pair of trees \( T_0 \) and \( T_\infty \) at the two endpoints, that are the metric graphs for the foliated half-cylinders that are neighborhoods of 0 and \( \infty \) respectively.

We can describe \( H' \) to be the planar metric tree obtained by attaching infinitely many copies of \( T_0 \) and \( T_\infty \), indexed by the integers, to the endpoints of a distinguished edge \( e \). Namely, if \( v_0 \) and \( v_\infty \) are the two endpoints of the edge \( e \), we attach trees \( \{T_i^0 \mid i \in \mathbb{Z}\} \) so that each has root \( v_0 \), and \( T_0^i \) is followed by \( T_0^{i+1} \) in a clockwise order around the vertex \( v_0 \). Similarly, we attach trees \( \{T_i^\infty \mid i \in \mathbb{Z}\} \) so that each has root \( v_\infty \), and \( T_\infty^i \) is followed by \( T_\infty^{i+1} \) in a clockwise order around the vertex \( v_\infty \).

This is indeed the leaf space of the lift of the foliation \( F_v \) to the universal cover. Note that the ring domain lifts to a single foliated strip. (See Figure 10.)

The infinite rays in \( T_\infty \) and \( T_0 \), asymptotic to \( \infty \) and \( 0 \) respectively, are labelled by \( \{a_1, a_2, \ldots a_{n-2}\} \) and \( \{b_1, b_2, \ldots b_{m-2}\} \) respectively. This induces a labelling of the
infinite rays in $T_i^\infty$ and $T_i^0$ for each $i \in \mathbb{Z}$, by the index set $\{a^i_j, b^i_k | 1 \leq j \leq n-2, 1 \leq k \leq m-2\}$.

The bijection $f$ between the complementary regions of $V'$ to the infinite rays of $H'$ is then again given by the corresponding map of labels $a^i_j \mapsto a^i_j$ and $b^i_j \mapsto b^i_j$.

In both cases, we now have a pair of metric trees $V'$ and $H'$ on $\mathbb{C}$, and a bijection $f$ between the complementary regions of $V'$, and the infinite rays of $H'$. By Theorem 3.5 there is a singular-flat metric on $\mathbb{C}$ with horizontal and vertical foliations having metric graphs $V', H'$, which induces the prescribed bijection $f$.

By construction, the lift of the foliations $F_h$ and $F_v$ on $\mathbb{C}$ are invariant under the action of $\mathbb{Z}$ acting by translations $\{T^i | i \in \mathbb{Z}\}$, so their corresponding metric graphs $V', H'$ are identical. Moreover, by construction the bijection $f$ is invariant under the action, that is, if $t_V$ is the relabelling of complementary regions of $V'$ and $t_H$ is the relabelling of infinite rays of $H'$ induced by the translation $T$ in the deck-group, then $t_H \circ f \circ t_V^{-1} = f$. (Note that $t_V$ relabels $a^i_j, b^i_j$ by $a^{i-1}_j, b^{i-1}_j$ respectively, and $t_H$ relabels $a^i_j, b^i_j$ by $a^{i-1}_j, b^{i-1}_j$ respectively, for each $i \in \mathbb{Z}$.)

By the uniqueness part of Au-Wan’s theorem (see Remark 2 following Theorem 3.5), the singular-flat metric obtained on $\mathbb{C}$ is invariant under the same group of deck-translations. The singular-flat metric thus passes to the quotient cylinder, and we obtain the desired singular-flat metric on $\mathbb{C}^*$, realizing the foliations $F_h$ and $F_v$.

\[\square\]

5. Parametrizing $\mathcal{F}(n, m)$

In this section, we shall parametrize the space of measured foliations $\mathcal{F}(n, m)$ on a cylinder, where $n, m \geq 3$ and the foliations have pole-singularities of orders $(n-2)$ and $(m-2)$ at the two ends of the cylinder (at $\infty$ and 0 respectively).

As in the planar case (see §3), it is useful to have a parametrization of the space of the metric graphs $\mathcal{G}(n, m)$ for these measured foliations.

For this, we consider the associated space of planar metric graphs $\mathcal{G}(k, \tau)$ for $\tau \in \mathbb{R}_{\geq 0}$ that we define in the Appendix. Each graph in $\mathcal{G}(k, \tau)$ is obtained by attaching trees to an embedded loop of length $\tau$ with $(m-2)$, and identifying a pair of graphs in $\mathcal{G}(n-2, \tau)$ and $\mathcal{G}(m-2, \tau)$ results in a graph in $\mathcal{G}(n, m)$ (see Figure 11).

We defer the parametrizations of the space $\mathcal{G}(n, \tau)$ and some relevant subspaces to the Appendix.

In what follows it will be useful to consider cases when the transverse measure $\tau$ is positive, and zero, separately.
There is an additional real “twist” parameter when identifying the two graphs in $\mathcal{G}(n-2, \tau)$ and $\mathcal{G}(m-2, \tau)$ (Lemma 5.2).

Let $F_0(n, m)$ denote the space of foliations with zero transverse measure, and $F_\tau(n, m)$ for those with a positive transverse measure $\tau$.

Let $F_+(n, m)$ be the subspace of $F(n, m)$ comprising those with positive transverse measure. For a foliation $F \in F_+(n, m)$, we also define the following:

**Definition 5.1 (Continuous twist parameter).** Let the transverse measure of $F$ be $\tau > 0$. As discussed above the metric graph of $F$ is obtained by identifying the cycles of a pair of graphs in $\mathcal{G}(n-2, 0)$ and $\mathcal{G}(m-2, 0)$ respectively. These graphs have labelled rays with labels in $\{1, 2, \ldots, n-2\}$ and $\{1', 2', \ldots, (m-2)\}'$ respectively, that are asymptotic to the two ends of the cylinder. We define the (continuous) twist parameter of $F$ to be

$$\text{twist}(F) = j \cdot \tau + l_0$$

where $j(F) \in \mathbb{Z}$ is the topological twist (see Definition 4.6) and $l_0$ is the length of the path $\gamma_0$ in the identified cycle $L$, between the endpoints of the roots of the rays labelled 1 and $1'$. Here $\gamma_0$ is chosen so that one turns left when one travels from the ray 1 to the root and then follows $\gamma_0$.

**Positive transverse measure.**

**Lemma 5.2.** Let $\tau > 0$. The subset

$$\mathcal{F}_\tau(n, m) = \{ F \in \mathcal{F}(n, m) \mid \text{the transverse measure is } \tau \} \cong \mathbb{R}^{n+m-5}.$$

**Proof.** By the argument of Lemma 4.3 the metric graph $G$ has a non-trivial cycle $L$. Moreover, it has $n-2$ rays to 0 and $m-2$ rays to $\infty$. 
Such a graph is obtained by identifying graphs in $\mathcal{G}(n-2, \tau)$ and $\mathcal{G}(m-2, \tau)$ along their respective cycles. (See Figure 11 and the Appendix).

The identification of circles has another real parameter associated with it, namely, the continuous twist parameter (see Definition 5.1).

Conversely, given such a pair of graphs identified along a cycle together with a (real-valued) twist data, one can construct a foliation in $\mathcal{F}_\tau(n, m)$ by gluing in strips and half-planes in the pattern determined by the resulting graph (where the strip-widths are equal to the lengths of the edges) and then performing a suitable Dehn twist to realize the integer number of topological twists.

Applying Proposition A.4, we know that $\mathcal{G}(n-2, \tau) \times \mathcal{G}(m-2, \tau) \cong \mathbb{R}^{n-m-6}$.

Together with the additional twist parameter, we obtain a total space $\mathbb{R}^{n+m-5}$ as claimed.

Varying the transverse measure $\tau \in \mathbb{R}_+$, we immediately have:

**Corollary 5.3.** The subset

$$\mathcal{F}_+(n, m) = \{ F \in \mathcal{F}(n, m) \mid \text{the transverse measure } \tau \text{ is positive} \} \cong \mathbb{R}^{n+m-4}.$$  

**Zero transverse measure.** Recall that a foliation with a zero transverse measure has a (possibly degenerate) ring domain.

**Lemma 5.4.** The subset

$$\mathcal{F}_0(n, m) = \{ F \in \mathcal{F}(n, m) \mid \text{the transverse measure of } F \text{ is zero} \} \cong \mathbb{R}^{n+m-6} \times \mathbb{R}_{\geq 0}$$  

where the second factor is the height of the ring domain.

**Proof.** A foliation in $\mathcal{F}_0(n, m)$ can be viewed as a ring domain with two foliated half-cylinders on either side. The metric graphs of the foliations on these half-cylinders are obtained by rooted metric expansions of a $(n-2)$ rooted tree, and $(m-2)$ rooted tree respectively. In particular, the metric graph $G$ obtained by identifying 1-vertex graphs in $\mathcal{G}(m-2, 0)$ and $\mathcal{G}(n-2, 0)$.

Conversely, given any metric graph $G$ in $\mathcal{G}(m-2, 0)$ or $\mathcal{G}(n-2, 0)$, it is easy to construct a foliation on a half-cylinder whose metric graph is $G$. Hence by applying Lemma [A.6] the parameter space of the two foliated half-cylinders is homeomorphic to $\mathcal{G}(m-2, 0) \times \mathcal{G}(n-2, 0)$ which is exactly $\mathbb{R}^{n-3} \times \mathbb{R}^{m-3}$.

As mentioned, the additional non-negative real parameter ($\mathbb{R}_{\geq 0}$) records the height of the ring domain. □

**Remark.** For the zero transverse measure case, there is no strip going across the cylinder, and the twist parameter is absent.

**Combining the cases.** We can put the cases of the transverse measure $\tau$ positive, and zero, together, and finally obtain:

**Proposition 5.5.** The total space of foliations $\mathcal{F}(n, m)$ homeomorphic to $\mathbb{R}^{n+m-4}$. 
Figure 12. The different integer twists form wedge-shaped regions that tile the space of foliations and accumulate on a ray that corresponds to the subset of foliations with ring-domains.

Proof. Consider the closed half plane $H = \{(x, y) \mid x \in \mathbb{R}, y \in \mathbb{R}_{\geq 0}\}$.

Note that the quotient space $H/\sim$ where $(x, 0) \sim (-x, 0)$ is homeomorphic to $\mathbb{R}^2$, and we denote this by $[\mathbb{R}^2]$.

We shall describe a fibration

$$\pi : \mathcal{F}(n, m) \to [\mathbb{R}^2]$$

(4)

with each fiber homeomorphic to $\mathbb{R}^{n+m-6}$.

Namely, for a foliation $F \in \mathcal{F}(n, m)$ we define $\pi(F) = (x, y) \in H$ as follows.

The coordinate $y$ is the transverse measure $\tau$ of the foliation. The coordinate $x$ of $\pi(F)$ is the (continuous) twist parameter of $F$ (see Definition 5.1).

Note that at height $y$, the twist parameter in the interval $[0, y)$ corresponds to zero integer twist, and so on.

This divides the upper half-plane into wedge-shaped region

$$V_j = \{j \cdot y \leq x \leq (j + 1) \cdot y\}$$

bounded between two rays from the origin.

The region $V_j$ thus represents all the foliations with twist parameter $j$.

Note that $\bigcup_{j \in \mathbb{Z}} V_j$ is a tiling of the interior of $H$. (See Figure 12)

In the case when the transverse measure $\tau = 0$, the absolute value of the $x$ coordinate equals $a$, the transverse measure across the ring domain. Recall that we identify points $(x, 0)$ and $(-x, 0)$ on the real axis, so this is well-defined.
As $j \to \pm \infty$, the wedges $V_j$ accumulate on the positive or negative ray (respectively) on the real axis $\partial H$.

In particular, if the total twist $x$ is kept fixed, and $\tau \to 0$, then the foliations converge to a ring domain of length $x$. (note that this implies that integer twists tend to infinity).

This describes the phenomenon that foliations converge to one with a ring domain, as we twist more and more, and decrease the transverse measure at the appropriate rate so that the foliations converge. Note that one can converge to such a foliation both by positive or negative twists; this results in the identification of the positive and negative rays as described above.

This completes the description of the map $\pi$.

To examine the fiber $\pi^{-1}(x, y)$, consider first the case when $y = 0$. Then, the fiber is the subset of foliations that have a ring-domain of transverse measure $|x|$. By the proof of Lemma 5.4, this subset is homeomorphic to $\mathbb{R}^{n+m-6}$.

For the case $y > 0$, note that by the proof of Lemma 5.2, the set of foliations that have total transverse measure fixed to be $y$ is homeomorphic to $\mathbb{R}^{n+m-6}$.

This completes the description of the fibration (4), with a total space homeomorphic to $\mathbb{R}^{n+m-4}$.

5.1. Proof of Theorem 1.1. By Proposition 4.9 and Lemma 4.8, we can now conclude that for any pair of measured foliations $(F_h, F_v) \in \mathcal{F}(n, m)^{(2)}$, we have a unique quadratic differential in $Q(n, m)$ whose horizontal and vertical foliations are precisely $F_h$ and $F_v$ respectively.

We have also shown $\mathcal{F}(n, m) \cong \mathbb{R}^{n+m-4}$ in Proposition 5.5.

The domain of the map $\Phi$ in Theorem 1.1 is $Q(n, m) \cong \mathbb{C}^{n+m-4}$ (see §1).

Thus, the map $\Phi$ is a continuous bijection from a Euclidean space to another, and by the Invariance of Domain, is a homeomorphism to its image.

This completes the proof of Theorem 1.1. □

Note that the product space $\mathcal{F}(n, m) \times \mathcal{F}(n, m)$ is homeomorphic to $\mathbb{R}^{2n+2m-8}$. We conclude with an independent verification that the subspace $\mathcal{F}(n, m)^{(2)}$ is also homeomorphic to $\mathbb{R}^{2n+2m-8}$.

Lemma 5.6. The subset $\mathcal{F}(n, m)^{(2)}$ comprising foliations that do not both have transverse measure $0$ is homeomorphic to $\mathbb{R}^{2n+2m-8}$.

Proof. Let the transverse measures be $(\tau_h, \tau_v)$. For any fixed $\tau^2 = \tau_h^2 + \tau_v^2$, we get that $\tau_h \in [0, \tau]$ (and $\tau_v$ is then determined). For $\tau_h = 0$ or $\tau_h = \tau$, the possible horizontal and vertical foliations are parametrized by $\mathbb{R}^{n+m-6} \times \mathbb{R}_{\geq 0}$ (for the one having zero transverse measure) and $\mathbb{R}^{n+m-5}$ (for the one having transverse measure $\tau$) to give a parameter space of $\mathbb{R}^{2n+2m-11} \times \mathbb{R}_{\geq 0}$. For any $\tau_h \in (0, \tau)$, each of the foliations have a fixed transverse measure, and hence by Lemma 5.2 we
get a parameter space of $\mathbb{R}^{2n+2m-10}$. Together, this parametrizes $\mathbb{R}^{2n+2m-9}$. The above discussion was for a fixed $\tau$, which can take any value in $\mathbb{R}^+$. So the total parameter space is $\mathbb{R}^{2n+2m-8}$, as claimed. □

Appendix A. Spaces of metric graphs

We record the following combinatorial facts in the spirit of the result of Mulase-Penkava parametrization of planar metric trees (see Theorem 3.3 and the remark following it). Some of these results might be well-known, though we were unable to locate this in the literature. For a closely related setting, see [BHV01].

The results in this section have been used in §5 to parametrize the space of foliations $\mathcal{F}(n,m)$ by understanding the spaces of their corresponding metric graphs $\mathcal{G}(n,m)$.

We begin by parametrizing some associated spaces of metric graphs of certain simple types. Throughout, we fix an integer $k \geq 1$ and a real $\tau > 0$ and we consider a collection of graphs $\mathcal{G}(k, \tau)$ of the following form:

- each graph is planar,
- each graph has a unique cycle $L$ of total length $\tau$,
- each graph has $k$ infinite rays asymptotic to $k$ fixed directions at infinity, that are labelled $\{1, 2, \ldots, k\}$ in a cyclic order. In what follows, $\sigma$ will be the cyclic shift $\{1 \mapsto 2, 2 \mapsto 3, \ldots, k-1 \mapsto k, k \mapsto 1\}$.

As a warmup we begin with:
Lemma A.1. Suppose each of the $k$ infinite rays are incident on the cycle $L$. Then the space $\mathcal{G}_c(k, \tau) \subset \mathcal{G}(k, \tau)$ of such “simple” metric graphs is homeomorphic to a closed simplex $\Delta^{k-1}$.

Proof. The endpoints of the rays form an ordered collection of points on $L$. They determine $k$ intervals of lengths $a_1, a_2, \ldots, a_k$ such that the sum is $\tau$. This is a closed simplex. (See Figures 14 and 20) \[\square\]

In general, the infinite rays may not have endpoints on the cycle $L$, but instead on other vertices that form a tree attached to the cycle $L$. We define:

Definition A.2 (Root, Split, Zip). The root of a labelled ray is the unique vertex on the cycle $L$ that is connected by a path to the ray. (See Figure 15)

Figure 15. The root of rays labelled $1$ and $2$ is the white vertex, and the root of the rays labelled $3$ and $4$ is the black vertex.

For $G \in \mathcal{G}(k, \tau)$, splitting the root of the ray labelled $1$ in $G$ results in a graph $G' \in \mathcal{G}(k+1, \tau)$ obtained by duplicating the ray and the path to the root (See Figure 16). The opposite process is called zipping roots.

An opposite extreme case to that of Lemma A.1 is the following:
Figure 16. Zipping the roots of the rays labelled 1 and 2 is the opposite of splitting the roots (see above), and it decreases the number of rays.

**Lemma A.3.** Let \( k \geq 2 \). Suppose \( L \) has a single vertex \( p \). Fix a labelling of the rays. Then the space of such “uni-rooted” metric graphs \( \mathcal{G}_u(k, \tau) \subset \mathcal{G}(k, \tau) \) is homeomorphic to \( \mathbb{R}^{k-2} \times \mathbb{R}_{\geq 0} \).

**Proof.** We fix a labelling of a graph with \( k \) rays and replace a cycle \( L \) with a finite edge. It will give us a star of valence \( k + 1 \) with \( k \) infinite rays and one finite edge. Every graph is a metric expansion of this standard graph. Using Mulase-Penkava we get a parametrization of \( \mathbb{R}^{k-2} \) for a \( k + 1 \) pronged star. The additional label is needed for the finite edge of the star and it is used for the edge to the root. The total space of “rooted” metric graphs is then homeomorphic to \( \mathbb{R}^{k-2} \times \mathbb{R}_{\geq 0} \) where the non-negative parameter \( t \) is the length of the distinguished edge. (See Figure 20.)

Finally, we can put these together to obtain:

**Proposition A.4.** Fix \( k \geq 2 \) and \( \tau > 0 \) as above. The space of all possible labelled metric graphs \( \mathcal{G}(k, \tau) \) is homeomorphic to \( \mathbb{R}^{k-1} \).

**Proof.** Diagram for \( k = 2 \) is shown below.

Figure 17. Metric graphs in \( \mathcal{G}(2, \tau) \).
Figure 18. Building a graph in $\mathcal{G}_0(k, \tau)$ (See Claim 1).

In general, the proof is by induction on $k$:

Recall from Lemma A.1 that the “simple” metric graphs $\mathcal{G}_c(k, \tau)$ is a simplex parametrized by the edge-lengths $a_1, a_2, \ldots, a_k$ between the endpoints of the rays labelled $\{1, 2, \ldots, k\}$.

We first partition the space into “slices”

$$\mathcal{G}(k, \tau) = \bigsqcup_{a \in [0, \tau]} \mathcal{G}_a(k, \tau)$$

where $\mathcal{G}_a(k, \tau)$ is the space of labelled metric graphs where the roots of the rays with labels 1 and 2 (see Definition A.2) are the endpoints of an edge in $L$ of length $a$.

Note that:

- The intersection of $\mathcal{G}_a(k, \tau)$ with the simplex of “simple” metric graphs $\mathcal{G}_c(k, \tau)$ (Lemma A.1) is the set of simple graphs for which the parameter $a_1$ equals $a$.
- $\mathcal{G}_\tau(k, \tau)$ comprises graphs where all the roots coincide, since $\tau$ is the length of the cycle, and this space is thus the same as $\mathcal{G}_u(k, \tau) \equiv \mathbb{R}^{k-2} \times \mathbb{R}_{\geq 0}$ is the space of uni-rooted metric graphs as defined in Lemma A.3, for the cyclic labelling that differs by a single cyclic shift $\sigma$.

The Proposition shall follow from the following two claims:

Claim 1. The space $\mathcal{G}_0(k, \tau)$ is homeomorphic to $\mathbb{R}^{k-2} \times \mathbb{R}_{\geq 0}$.

Proof of claim. This is where the inductive hypothesis is used: any labelled metric graph in $\mathcal{G}_0(k, \tau)$ is obtained by the following process:

1. take a metric graph in $\mathcal{G}(k-1, \tau)$ with labels that are cyclic permutations of $\{2, 3, \ldots, k\}$,
2. add an infinite ray labelled 1 to the root $p$ of the ray labelled 2, and finally
Figure 19. Collapsing an edge in the proof of Claim 2. This, when followed by “zipping the roots” (see Figure 16), reduces a graph in $G_a(k, \tau)$ to one in $G(k-1, \tau-a)$.

(3) insert an edge of finite (possibly zero) length between $p$ and a new vertex introduced on $L$ (which coincides with $p$ if the length of the new edge is zero).

Conversely, by collapsing a finite edge from the root of the ray labelled 1, and then removing that ray, results in a graph in $G(k-1, \tau)$.

By the inductive hypothesis, we know that the set of possibilities in the first step is homeomorphic to $\mathbb{R}^{k-2}$. This, together with the choice of non-negative length in the final step yields a total space homeomorphic to $\mathbb{R}^{k-2} \times \mathbb{R}_{\geq 0}$. (See Figure 20 for the case $k = 3$.) □

Claim 2. For each $a \in (0, \tau)$ the space $G_a(k, \tau)$ is homeomorphic to $\mathbb{R}^{k-2}$.

Proof of claim. We use the inductive hypothesis again. Collapsing the edge in $L$ of length $a$ between the roots of the rays labelled 1 and 2, and then zipping the roots of the two rays (see Definition A.2) we obtain a graph in $G(k-1, \tau-a)$ where the zipped up ray is relabelled 2, so the labellings are all cyclic permutations of $\{2, 3, \ldots, k\}$.

Conversely, any graph in $G_a(k, \tau)$ is obtained by taking such a graph in $G(k-1, \tau-a)$, splitting the root of the ray labelled 2, introducing a label 1 on the resulting new ray, and inserting an edge of length $a$ separating the roots of the rays labelled 1 and 2 (see Definition A.2). By the inductive hypothesis, we know that $G(k-1, \tau-a)$ is homeomorphic to $\mathbb{R}^{k-2}$, and we are done. See Figure 19 □

The two closed “half-spaces” $G_0(k, \tau)$ and $G_{\tau}(k, \tau)$ together with the sections $G_a(k, \tau)$ together constitute $G(k, \tau)$, as depicted in Figure 20.
Figure 20. The space of metric graphs $\mathcal{G}(3, \tau) \cong \mathbb{R}^2$, comprises sections where $a = 0$, $a \in (0, \tau)$, and $a = \tau$ respectively from top to bottom, where $a$ is the length of the edge in $L$ between the roots of the rays with labels 1 and 2. The top half plane corresponds to $\mathcal{G}_0(3, \tau) \cong \mathbb{R} \times \mathbb{R}_{\geq 0}$ and consists of graphs with edge length $a = 0$. It is parameterized by $k - 2$ of the remaining $k - 1$ edges in $L$ (the sum of their lengths must be $\tau$) and the parameter $t$ which is the length of the finite-length edge that is not in $L$. (See Claim 1.) The middle strip corresponds to the set of $\mathcal{G}_a(3, \tau) \cong \mathbb{R}$ for each $a \in (0, \tau)$. (See Claim 2.) It contains the simplex $\mathcal{G}_c(3, \tau)$. (See Lemma A.1.) The bottom half plane corresponds to $\mathcal{G}_u(3, \tau) \cong \mathbb{R} \times \mathbb{R}_{\geq 0}$, consisting of graphs with edge length $a = \tau$, the set of uni-rooted graphs. It is parameterized by the finite edge lengths in the metric expansions of the $k + 1$ star with $k$ edges infinite and one finite-length edge of length $t$ to the root. (See Lemma A.3.) Finally, $\mathcal{G}(3, 0)$ can be seen as the limit of $\mathcal{G}(3, \tau)$ as $\tau \to 0$. The simplex $\mathcal{G}_c(3, \tau)$ collapses to a point, and the central strip collapses to a single copy of $\mathbb{R}$ to which the two subspaces homeomorphic to $\mathbb{R} \times \mathbb{R}_{\geq 0}$ are attached, yielding $\mathcal{G}(3, 0) \cong \mathbb{R}^2$. (See Lemma A.6.)
Observe that the two closed “half-spaces” \( \mathcal{G}_0(k, \tau) \) and \( \mathcal{G}_\tau(k, \tau) \) both homeomorphic to \( \mathbb{R}^{k-2} \times \mathbb{R}_{\geq 0} \) are attached with either boundaries of the “slab”

\[
\bigcup_{\mathcal{G}_a(k, \tau) \in (0, \tau)} \mathcal{G}_a(k, \tau) \cong \mathbb{R}^{k-2} \times (0, \tau)
\]

to yield a space homeomorphic to \( \mathbb{R}^{k-1} \), as desired. (Figure 20 describes the case \( k = 3 \).) \( \square \)

Finally, we can extend the definition of \( \mathcal{G}(k, \tau) \) to the case that the transverse measure is zero, namely:

**Definition A.5.** The space \( \mathcal{G}(k, 0) \) is the space of planar metric trees with labelled rays such that

1. there is a distinguished vertex \( p \) (the root) which can be thought of as the collapsed cycle \( L \) for graphs in \( \mathcal{G}(k, \tau) \) when \( \tau = 0 \), and
2. There are \( k \) infinite rays with labels that is a cyclic permutation of \( \{1, 2, \ldots, k\} \).

The space \( \mathcal{G}(k, 0) \) can be thought of as being obtained as a limit of the spaces \( \mathcal{G}(k, \tau) \) as \( \tau \to 0 \). In particular, the closure of the central “slab” that arises in the proof of Proposition A.4 (see (5)) collapses to a single copy of \( \mathbb{R}^{k-2} \) to which the two subspaces homeomorphic to \( \mathbb{R}^{k-2} \times \mathbb{R}_{\geq 0} \) are attached (c.f. the caption of Figure 20). Thus, we obtain:

**Lemma A.6.** The space \( \mathcal{G}(k, 0) \) is homeomorphic to \( \mathbb{R}^{k-1} \).

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Department of Mathematics and Computer Science, Bronx Community College of the City University of New York, Bronx, NY 10453 USA.

E-mail address: kealey.dias@bcc.cuny.edu

Department of Mathematics, Indian Institute of Science, Bangalore 560012, India.

E-mail address: subhojoy@iisc.ac.in

Department of Mathematics, University of California, Davis, CA 95616, USA.

E-mail address: mtrnkova@math.ucdavis.edu