Existence Theory for a Fractional $q$-Integro-Difference Equation with $q$-Integral Boundary Conditions of Different Orders

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Abstract: In this paper, we study the existence of solutions for a new class of fractional $q$-integro-difference equations involving Riemann-Liouville $q$-derivatives and a $q$-integral of different orders, supplemented with boundary conditions containing $q$-integrals of different orders. The first existence result is obtained by means of Krasnoselskii’s fixed point theorem, while the second one relies on a Leray-Schauder nonlinear alternative. The uniqueness result is derived via the Banach contraction mapping principle. Finally, illustrative examples are presented to show the validity of the obtained results. The paper concludes with some interesting observations.

Keywords: $q$-integro-difference equation; boundary value problem; existence; fixed point

1. Introduction and Preliminaries

Fractional calculus, dealing with differential and integral operators of arbitrary order, serves as a powerful modelling tool for many real-world phenomena. An interesting feature of such operators is their nonlocal nature that accounts for the history of the phenomena involved in the fractional models. Motivated by the extensive applications of fractional calculus, many researchers turned to the theoretical development of fractional-order initial and boundary value problems. Now, the literature on the topic contains many interesting and important results on the existence and uniqueness of solutions, and other properties of solutions for fractional-order problems. The available material includes different types of derivatives such as Riemann-Liouville, Caputo, Hadamard, etc. and a variety of boundary conditions. For some recent works on the topic, for instance, see [1–8] and the references therein.

Fractional $q$-difference equations (fractional analogue of $q$-difference equations) also received significant attention. One can find preliminary work on the topic in [9], while some interesting details about initial and boundary value problems of $q$-difference and fractional $q$-difference equations can be found in the book [10].

In 2012, Ahmad et al. [11] discussed the existence and uniqueness of solutions for the nonlocal boundary value problem of fractional $q$-difference equations:

\[
\begin{align*}
\mathcal{D}_q^a x(t) &= f(t, x(t)), \quad 0 \leq t \leq 1, 1 < a < 2, 0 < q < 1, \\
\alpha_1 x(0) - \beta_1 \mathcal{D}_q x(0) &= \gamma_1 x(\eta_1), \quad \alpha_2 x(1) - \beta_2 \mathcal{D}_q x(1) = \gamma_2 x(\eta_2),
\end{align*}
\]

where $f \in C([0,1] \times \mathbb{R}, \mathbb{R}), \mathcal{D}_q^a$ is the fractional $q$-derivative of the Caputo type, and $\alpha_i, \beta_i, \gamma_i, \eta_i \in \mathbb{R}, i = 1, 2.$
In 2013, Zhou and Liu [12] applied Mönch’s fixed point theorem together with the technique of measure of weak noncompactness to investigate the existence of solutions for the following fractional $q$-difference equation with boundary conditions:

\[
\begin{cases}
\ cD_q^\alpha u(t) + f(t, u(t)) = 0, & 0 \leq t \leq 1, \ 0 < q < 1, \\
\ u(0) = (D_q^\alpha u)(0) = 0, \quad \gamma(D_q u)(1) + \beta(D_q^2 u)(1) = 0,
\end{cases}
\]

where $2 < \alpha \leq 3$, $\gamma, \beta \geq 0$ and $f : [0, 1] \times \mathbb{R} \to \mathbb{R}$ is a continuous function.

In 2014, Ahmad et al. [13] derived some existence results for a nonlinear fractional $q$-difference equation with four-point nonlocal integral boundary conditions given by

\[
\begin{cases}
\ cD_q^\beta (cD_q^\gamma + \lambda) u(t) = f(t, u(t)), & 0 \leq t \leq 1, \ 0 < q < 1, \ \lambda \in \mathbb{R}, \\
\ u(0) = aI_q^{\alpha-1}u(\eta), \quad u(1) = bI_q^{\beta-1}u(\sigma), \quad a, b \in \mathbb{R},
\end{cases}
\]

where $0 < \beta, \gamma \leq 1, 0 < \eta, \sigma < 1, \alpha > 2, f : [0, 1] \times \mathbb{R} \to \mathbb{R}$ is a continuous function and $I_q^\alpha$ denotes the Riemann-Liouville fractional $q$-integral of order $\alpha$.

Later, Niyom et al. [14] studied the following boundary value problem containing Riemann-Liouville fractional derivatives of different orders:

\[
\begin{cases}
\ (\lambda D^\alpha + (1 - \lambda) D^\beta) u(t) = f(t, u(t)), & t \in [0, T], \ 1 < \alpha, \beta < 2, \\
\ u(0) = 0, \quad \mu D^\alpha u(T) + (1 - \mu) D^{\alpha+1} u(T) = \gamma_3, \quad 0 < \gamma_1, \gamma_2 < \alpha - \beta,
\end{cases}
\]

where $D^\beta$ is the ordinary Riemann-Liouville fractional derivative of order $\beta \in \{\alpha, \beta, \gamma_1, \gamma_2\}$ such that $0 < \lambda \leq 1, 0 \leq \mu \leq 1, \gamma_3 \in \mathbb{R}$ and $f \in C([0, T] \times \mathbb{R}, \mathbb{R})$ for $T > 0$.

Some recent results on fractional $q$-difference equations equipped with different kinds of boundary conditions can be found in the papers [15–25].

Now, we recall some important results on fractional $q$-integro-difference equations. In [26], the authors studied a nonlinear four-point boundary value problem of nonlinear fractional $q$-integro-difference equations given by

\[
\begin{cases}
\ cD_q^\gamma (cD_q^\alpha + \lambda) x(t) = pf(t, x(t)) + kI_q^\beta g(t, x(t)), & 0 \leq t \leq 1, \ 0 < q < 1, \\
\ a_1 x(0) - \beta \left[ \left( f^{(1-q)} I_q^\alpha x(0) \right) \right]_{t=0} = \sigma_1 x(\eta_1), \quad a_2 x(1) + \beta_2 D_q g(1) = \sigma_2 x(\eta_2),
\end{cases}
\]

where $cD_q^\gamma$ and $cD_q^\alpha$ denote the fractional $q$-derivative of the Caputo type, $0 < \beta, \gamma \leq 1$, $I_q^\beta(.)$ represents a Riemann-Liouville fractional integral of order $\xi \in (0, 1)$, $f, g : [0, 1] \times \mathbb{R} \to \mathbb{R}$ are continuous functions, $\lambda \neq 0$ and $p, k, a_i, \beta_i, \eta_i \in \mathbb{R}, \eta_i \in (0, 1), i = 1, 2$. For some recent works on boundary value problems of fractional $q$-integro-difference equations, for instance, see [27–31].

Motivated by aforementioned works, in this paper, we study the following nonlinear fractional $q$-integro-difference equation

\[
(\lambda D_q^a + (1 - \lambda) D_q^\beta) u(t) = af(t, u(t)) + bI_q^\gamma g(t, u(t)), \quad t \in [0, 1], a, b \in \mathbb{R}^+,
\]

supplemented with $q$-integral boundary conditions

\[
u(0) = 0, \mu \int_0^1 (1 - qs)^{(\gamma_1-1)} I_q^{\gamma_1}(u(s)) ds + (1 - \mu) \int_0^1 (1 - qs)^{(\gamma_2-1)} I_q^{\gamma_2}(u(s)) ds = 0, \quad \gamma_1, \gamma_2 > 0,
\]

where $0 < q < 1, 1 < a, \beta < 2, 0 < \delta < 1, 0 < \lambda \leq 1, 0 \leq \mu \leq 1, a - \beta > 1$ and $D_q^a$ denotes the Riemann-Liouville fractional $q$-derivative of order $a$ and $f, g : [0, 1] \times \mathbb{R} \to \mathbb{R}$ are continuous functions. Notice that Equation (1) contains $q$-derivatives of fractional orders $a$ and $\beta$ and a fractional $q$-integral of orders $\delta$, while fractional $q$-integrals of orders $\gamma_1$ and $\gamma_2$ are involved in the boundary conditions (2).
We make use of Krasnoselskii’s fixed point theorem and a Leray-Schauder nonlinear alternative to prove the existence results, while the uniqueness result is proved via Banach contraction mapping principle for the given problem.

Let us first recall some necessary concepts and definitions about $q$-fractional calculus and fixed point theory.

Let $0 < q < 1$ be an arbitrary real number. For every $a \in \mathbb{R}$, the $q$-number $[a]_q$ is defined by $[a]_q = \frac{1 - a^q}{1 - q}$ [9]. In addition, the $q$-shifted factorial of real number $a$ is defined by $(a; q)_0 = 1$ and $(a; q)_n = \prod_{j=0}^{n-1} (1 - aq^j)$ for $n \in \mathbb{N} \cup \{\infty\}$. For $a, b \in \mathbb{R}$, the $q$-analogue of the power function $(a - b)^n$ with $n \in \mathbb{N}_0 := \{0, 1, 2, \ldots\}$ is given by

$$(a - b)^{q0} = 1, \quad (a - b)^{(n)} = \prod_{j=0}^{n-1} (a - bq^j).$$

In general, if $a$ is real number, then $(a - b)^{(a)} = a^a \prod_{j=0}^{\infty} \frac{a - bq^j}{a - bq^{a+j}}$ and $a^{(a)} = a^a$ when $b = 0$.

If $a > 0$ and $0 \leq a \leq b \leq t$, then $(t - b)^{(a)} \leq (t - a)^{(a)}$. The $q$-Gamma function $\Gamma_q(a)$ is defined as

$$\Gamma_q(a) = \frac{(1 - q)^{(a-1)}}{(1 - q)^a} - 1, \quad a \in \mathbb{R} \setminus \{0, -1, -2, \ldots\}$$

and satisfies the relation $\Gamma_q(a + 1) = [a]_q \Gamma_q(a)$ [9].

Let $\alpha \geq 0$ and $u : (0, \infty) \rightarrow \mathbb{R}$ be a continuous function. The Riemann-Liouville fractional $q$-integral for the function $u$ of order $\alpha$ is defined by $(I^\alpha_q u)(t) = u(t)$ and

$$(I^\alpha_q u)(t) = \frac{1}{\Gamma_q(a)} \int_0^t (t - qs)^{(a-1)} u(s) dq s, \quad \alpha > 0$$

for $t \in (0, \infty)$, provided that the right-hand side is pointwise defined on $(0, \infty)$ [9].

Recall that $I^\alpha_q I^\beta_q u(t) = I^{\alpha + \beta}_q u(t)$ for $\alpha, \beta \in \mathbb{R}^+$ [9] and

$$I^\alpha_q t^\beta = \frac{\Gamma_q(\beta + 1)}{\Gamma_q(\alpha + \beta + 1)} t^{\alpha + \beta}, \quad \beta \in (-1, \infty), \quad \alpha \geq 0, \quad t > 0.$$

If $f \equiv 1$, then $I^\alpha_q 1(t) = \frac{1}{\Gamma_q(\alpha + 1)} t^\alpha$ for all $t > 0$.

The Riemann-Liouville fractional $q$-derivative of order $\alpha > 0$ for a function $u : (0, \infty) \rightarrow \mathbb{R}$ is defined by [9]

$$D^\alpha_q u(t) = \frac{1}{\Gamma_q(n - \alpha)} \int_0^t \frac{u(s)}{(t - qs)^{a-n+1}} dq s, \quad n - 1 < \alpha < n.$$

Next, we state some fixed point theorems related to our work.

**Lemma 1.** Let $M$ be a closed, bounded, convex and nonempty subset of a Banach space $E$. Let $A$ and $B$ be operators mapping $M$ into $E$, such that

(i) $Ax + By \in M$, where $x, y \in M$;
(ii) $A$ is compact and continuous;
(iii) $B$ is a contraction mapping.

Then, there exists $z \in M$ such that $z = Az + Bz$ (Krasnoselskii’s fixed point theorem [32]).
Lemma 2. Let $X$ be a closed and convex subset of a Banach space $E$ and let $Y$ be an open subset of $X$ with $0 \in Y$. Then, a continuous compact map $H : Y \to X$ has a fixed point in $Y$ or there is a $y \in \partial Y$ and $\sigma \in (0, 1)$ such that $y = \sigma H(y)$, where $\partial Y$ is the boundary of $Y$ in $X$ (Nonlinear alternative for single-valued maps [33]).

2. Main Results

Let $E = C([0, 1], \mathbb{R})$ be the set of continuous functions defined on $[0, 1]$. The set $E$ is a Banach space with the following norm

$$||u||_E = \sup_{t \in [0,1]} |u(t)|, \quad u \in E.$$

Now, we prove the following lemma which characterizes the structure of solutions for boundary value problems (1) and (2).

Lemma 3. Let $h \in C([0, 1], \mathbb{R})$ and

$$\Delta := \frac{\mu \Gamma_q(\alpha)}{\Gamma_q(\alpha + \gamma_1)} + \frac{(1 - \mu) \Gamma_q(\alpha)}{\Gamma_q(\alpha + \gamma_2)} \neq 0. \quad (3)$$

The function $u$ is a solution for the fractional $q$-difference boundary value problem

$$\begin{cases} (\lambda D^a_q + (1 - \lambda) D^\beta_q) u(t) = h(t), & t \in [0, 1], \\ u(0) = 0, \mu \int_0^1 \frac{(1 - qs)^{(\gamma_1 - 1)}}{\Gamma_q(\gamma_1)} u(s)dqs + (1 - \mu) \int_0^1 \frac{(1 - qs)^{(\gamma_2 - 1)}}{\Gamma_q(\gamma_2)} u(s)dqs = 0, \end{cases} \quad (4)$$

if and only if $u$ is a solution for the fractional $q$-integral equation

$$u(t) = \frac{(\lambda - 1)}{\lambda \Gamma_q(\alpha - \beta)} \int_0^t (t - qs)^{(\alpha - \beta - 1)} u(s)dqs + \frac{1}{\lambda \Gamma_q(\alpha)} \int_0^t (t - qs)^{(\alpha - 1)} h(s)dqs$$

$$+ \frac{\mu}{\Delta} \left[ - \frac{\mu (\lambda - 1)}{\lambda \Gamma_q(\alpha - \beta + \gamma_1)} \int_0^1 (1 - qs)^{(\alpha - \beta + \gamma_1 - 1)} u(s)dqs - \frac{\mu}{\lambda \Gamma_q(\alpha + \gamma_1)} \int_0^1 (1 - qs)^{(\alpha + \gamma_1 - 1)} h(s)dqs \right.$$

$$- \frac{\mu}{\lambda \Gamma_q(\alpha - \beta + \gamma_2)} \int_0^1 (1 - qs)^{(\alpha - \beta + \gamma_2 - 1)} u(s)dqs$$

$$\left. - \frac{(1 - \mu)(\lambda - 1)}{\lambda \Gamma_q(\alpha - \beta + \gamma_2)} \int_0^1 (1 - qs)^{(\alpha - \beta + \gamma_2 - 1)} h(s)dqs \right]. \quad (5)$$

Proof. Let $u$ be a solution of the $q$-fractional boundary value problem (4). Then, we have

$$D^a_q u(t) = \frac{\lambda - 1}{\lambda} D^\beta_q u(t) + \frac{1}{\lambda} h(t).$$

Taking the Riemann-Liouville fractional $q$-integral of order $\alpha$ to both sides of the above equation, we get

$$u(t) = \frac{\lambda - 1}{\lambda} I^a_q D^\beta_q u(t) + \frac{1}{\lambda} I^\alpha_q h(t) + c_1 t^{\alpha - 1} + c_2 t^{\alpha - 2},$$

where $c_1, c_2 \in \mathbb{R}$ are arbitrary constants. Since $1 < \alpha < 2$, it follows from the first boundary condition that $c_2 = 0$. Thus,

$$u(t) = \frac{\lambda - 1}{\lambda \Gamma_q(\alpha - \beta)} \int_0^t (t - qs)^{(\alpha - \beta - 1)} u(s)dqs + \frac{1}{\lambda \Gamma_q(\alpha)} \int_0^t (t - qs)^{(\alpha - 1)} h(s)dqs + c_1 t^{\alpha - 1}. \quad (6)$$
On the other hand, if \( \sigma \in \{ \gamma_1, \gamma_2 \} \), then we have

\[
I_q^\sigma u(t) = \frac{\lambda - 1}{\lambda \Gamma_q(a - \beta + \gamma)} \int_0^t (t - qs)^{(a - \beta + \gamma - 1)}u(s)d_q s + \frac{1}{\lambda \Gamma_q(a + \gamma)} \int_0^t (t - qs)^{(a + \gamma - 1)}h(s)d_q s + c_1 \Gamma_q(a) \Gamma_q(a + \gamma) \mu^{a + \gamma - 1}.
\]

Now, by using the second boundary value condition and substituting the values \( \sigma \in \{ \gamma_1, \gamma_2 \} \) into the above expression, we obtain

\[
\mu(\lambda - 1) \int_0^1 (1 - qs)^{(a - \beta + \gamma_1 - 1)}u(s)d_q s + \frac{\mu}{\lambda \Gamma_q(a + \gamma)} \int_0^1 (1 - qs)^{(a + \gamma_1 - 1)}h(s)d_q s + c_1 \frac{\mu \Gamma_q(a)}{\Gamma_q(a + \gamma_1)} + (1 - \mu)(\lambda - 1) \frac{\mu}{\lambda \Gamma_q(a - \beta + \gamma_2)} \int_0^1 (1 - qs)^{(a - \beta + \gamma_2 - 1)}u(s)d_q s + (1 - \mu) \frac{\rho}{\lambda \Gamma_q(a + \gamma_2)} \int_0^1 (1 - qs)^{(a + \gamma_2 - 1)}h(s)d_q s + c_1 (1 - \mu) \Gamma_q(a) \Gamma_q(a + \gamma_2) = 0.
\]

Solving the above equation for \( c_1 \), we find that

\[
c_1 = \frac{1}{\lambda} \left( - \frac{\mu(\lambda - 1)}{\lambda \Gamma_q(a - \beta + \gamma_1)} \int_0^1 (1 - qs)^{(a - \beta + \gamma_1 - 1)}u(s)d_q s - \frac{\mu}{\lambda \Gamma_q(a + \gamma)} \int_0^1 (1 - qs)^{(a + \gamma_1 - 1)}h(s)d_q s - (1 - \mu)(\lambda - 1) \frac{\mu}{\lambda \Gamma_q(a - \beta + \gamma_2)} \int_0^1 (1 - qs)^{(a - \beta + \gamma_2 - 1)}u(s)d_q s - (1 - \mu) \frac{\rho}{\lambda \Gamma_q(a + \gamma_2)} \int_0^1 (1 - qs)^{(a + \gamma_2 - 1)}h(s)d_q s \right),
\]

where \( \Delta \) is defined in (3).

Substituting the value of \( c_1 \) in (6), we get the solution (5). Conversely, it is clear that \( u \) is a solution for the fractional \( q \)-difference Equation (4) whenever \( u \) is a solution for the fractional \( q \)-integral Equation (5). This completes the proof. \( \Box \)
In relation to the problems (1) and (2), we introduce an operator \( T : E \to E \) by
\[
(Tu)(t) = \frac{(\lambda - 1)}{\Delta q_q(a - \beta)} \int_0^t (t - qs)^{(a-\beta-1)}u(s)dqs + \frac{a}{\Delta q_q(a)} \int_0^t (t - qs)^{(a-1)}f(s, u(s))dqs
\]
\[+ \frac{b}{\Delta q_q(a + \delta)} \int_0^t (t - qs)^{(a+\delta-1)}g(s, u(s))dqs
\]
\[+ \frac{\mu(\lambda - 1)}{\Delta q_q(a + \beta + \gamma_1)} \int_0^t (1 - qs)^{(a-\beta+\gamma_1-1)}u(s)dqs
\]
\[- \frac{\mu(\lambda - 1)}{\Delta q_q(a + \beta + \gamma_1)} \int_0^1 (1 - qs)^{(a+\gamma_1-1)}f(s, u(s))dqs
\]
\[- \frac{\mu(\lambda - 1)}{\Delta q_q(a + \beta + \gamma_1)} \int_0^1 (1 - qs)^{(a+\gamma_1-1)}g(s, u(s))dqs
\]
\[\land (1 - \mu) \int_0^1 (1 - qs)^{(a+\gamma_1-1)}u(s)dqs
\]
\[- \frac{\mu(\lambda - 1)}{\Delta q_q(a + \beta + \gamma_1)} \int_0^1 (1 - qs)^{(a+\gamma_1-1)}f(s, u(s))dqs
\]
\[- \frac{\mu(\lambda - 1)}{\Delta q_q(a + \beta + \gamma_1)} \int_0^1 (1 - qs)^{(a+\gamma_1-1)}g(s, u(s))dqs
\]
where \( u \in E \) and \( t \in [0, 1] \). In the sequel, we set
\[
\Lambda_0 := \frac{|\lambda - 1|}{\Delta q_q(a - \beta + 1)} + \frac{\mu|\lambda - 1|}{\Delta q_q(a - \beta + \gamma_1 + 1)} + \frac{(1 - \mu)|\lambda - 1|}{\Delta q_q(a - \beta + \gamma_1 + 1)}
\]
\[
\Lambda_1 := \frac{a}{\Delta q_q(a + 1)} + \frac{a\mu}{\Delta q_q(a + \gamma_1 + 1)} + \frac{b(1 - \mu)}{\Delta q_q(a + \gamma_1 + 1)}
\]
\[
\Lambda_2 := \frac{b}{\Delta q_q(a + \delta + 1)} + \frac{b\mu}{\Delta q_q(a + \delta + \gamma_1 + 1)} + \frac{b(1 - \mu)}{\Delta q_q(a + \delta + \gamma_1 + 1)}
\]

Now, we are ready to present our main results. The first existence result is based on Krasnoselskii’s fixed point theorem.

**Theorem 1.** Suppose that \( f, g : [0, 1] \times \mathbb{R} \to \mathbb{R} \) are continuous functions satisfying the following conditions:

(i) there exists a positive constant \( L \) such that for each \( u, v \in \mathbb{R} \),
\[
|f(t, u) - f(t, v)| \leq L|u - v|, \quad t \in [0, 1];
\]

(ii) For each \( u \in \mathbb{R} \), there exists a continuous function \( m \) on \([0, 1]\) such that
\[
|g(t, u)| \leq m(t), \quad t \in [0, 1].
\]

If \( \Lambda_0 + \Lambda_1 < 1 \), then the fractional \( q \)-integro-difference Equation (1) with \( q \)-integral boundary conditions (2) has at least one solution on \([0, 1]\), where \( \Lambda_0 \) and \( \Lambda_1 \) are defined by (8).

**Proof.** Let \( \|m\| = \sup_{t \in [0, 1]} |m(t)| \). Define \( B_r := \{ u \in E : \|u\| \leq r \} \) with
\[
r \geq \frac{\|m\|\Lambda_2 + K\Lambda_1}{1 - (\Lambda_0 + \Lambda_1)},
\]
where \( K := \sup_{t \in [0, 1]} |f(t, 0)| \) and \( \Lambda_1 \) and \( \Lambda_2 \) are given by (8). Clearly, \( B_r \) is a closed, bounded, convex and nonempty subset of Banach space \( E \). We consider the operator \( T : E \to E \) as (7). By Lemma 3,
it is obvious that the fixed point of $T$ is the solution of problems (1) and (2). Now, for each $t \in [0,1]$, we define two operators from $B_r$ to $E$ as follows:

$$T_1u(t) = \frac{(\lambda - 1)}{\lambda \Gamma_q(\alpha - \beta)} \int_0^t (t - q)\alpha\beta u(s)ds + \frac{a}{\lambda \Gamma_q(\alpha)} \int_0^t (t - q)\alpha\beta f(s,u(s))ds$$

$$+ \frac{\mu - 1}{\Delta \left(-\frac{\mu(\lambda - 1)}{\lambda \Gamma_q(\alpha - \beta + \gamma_1 - 1)} + \int_0^1 (1 - q)\alpha\beta u(s)ds\right)$$

$$- \frac{a\mu}{\lambda \Gamma_q(\alpha + \gamma_1 - 1)} \int_0^1 (1 - q)\alpha\beta u(s)ds$$

$$- \frac{(1 - \mu)(\lambda - 1)}{\lambda \Gamma_q(\alpha - \beta + \gamma_2 - 1)} \int_0^1 (1 - q)\alpha\beta u(s)ds$$

$$- \frac{a(1 - \mu)}{\lambda \Gamma_q(\alpha + \gamma_2 - 1)} \int_0^1 (1 - q)\alpha\beta u(s)ds\right].$$

and

$$T_2u(t) = \frac{b}{\lambda \Gamma_q(\alpha + \delta)} \int_0^t (t - q)\alpha\beta g(s,u(s))ds$$

$$+ \frac{\mu - 1}{\Delta \left(-\frac{\mu(\lambda - 1)}{\lambda \Gamma_q(\alpha + \delta + \gamma_1 - 1)} + \int_0^1 (1 - q)\alpha\beta g(s,u(s))ds\right)$$

$$- \frac{a\mu}{\lambda \Gamma_q(\alpha + \delta + \gamma_2 - 1)} \int_0^1 (1 - q)\alpha\beta g(s,u(s))ds\right].$$

By the condition $(H_1)$, we have that $|f(t,u(t))| \leq |f(t,u(t)) - f(t,0)| + |f(t,0)| \leq L\|u\| + K \leq Lr + K$ for any $u \in \mathbb{R}$ and $t \in [0,1]$. Thus, for any $u, v \in B_r$ and $t \in [0,1]$, it follows by means of (8) and (9) that

$$|T_1u(t) + T_2v(t)| \leq \frac{|\lambda - 1|}{\lambda \Gamma_q(\alpha - \beta + 1)} \|u\| + \frac{a}{\lambda \Gamma_q(\alpha + 1)} \left(L\|u\| + K\right)$$

$$+ \frac{\mu |\lambda - 1|}{\lambda |\lambda - 1| \Gamma_q(\alpha - \beta + \gamma_1 + 1)} \|u\| + \frac{a\mu}{\lambda |\lambda - 1| \Gamma_q(\alpha + \gamma_1 + 1)} \left(L\|u\| + K\right)$$

$$+ \frac{b(1 - \mu)}{\lambda \Gamma_q(\alpha + \delta + \gamma_2 + 1)} \|m\| + \frac{b(1 - \mu)}{\lambda \Gamma_q(\alpha + \delta + \gamma_2 + 1)} \|m\|$$

which implies that $\|T_1u + T_2v\| \leq r$ and so $T_1u + T_2v \in B_r$ for all $u, v \in B_r$.

Now, we prove that $T_2$ is continuous. Let $\{u_n\}_{n \geq 1}$ be a sequence in $B_r$ such that $u_n \to u$. Then, for each $t \in [0,1]$, we have

$$|T_2u_n(t) - T_2u(t)| \leq \frac{b}{\lambda \Gamma_q(\alpha + \delta + 1)} |g(s,u_n(s)) - g(s,u(s))|$$

$$+ \frac{b\mu}{\lambda |\lambda - 1| \Gamma_q(\alpha + \delta + \gamma_1 + 1)} |g(s,u_n(s)) - g(s,u(s))|$$

$$+ \frac{b(1 - \mu)}{\lambda |\lambda - 1| \Gamma_q(\alpha + \delta + \gamma_2 + 1)} |g(s,u_n(s)) - g(s,u(s))|. $$
Since \( g \) is continuous, we get \( \| T_2 u_n - T_2 u \| \to 0 \) as \( u_n \to u \). In consequence, it follows that the operator \( T_2 \) is continuous on \( B_r \).

In the next step, we show that the operator \( T_2 \) is compact. Let us first show that \( T_2 \) is uniformly bounded. For each \( u \in B_r \) and \( t \in [0, 1] \), we have

\[
|T_2 u(t)| \leq \frac{b t^{a+\delta}}{\lambda \Gamma_0(\alpha + \delta + 1)} |g(s, u(s))| + \frac{b \mu t^{a-1}}{\lambda |\Delta \Gamma_0(\alpha + \delta + \gamma_1 + 1)} |g(s, u(s))| \\
+ \frac{b(1 - \mu) t^{a-1}}{\lambda |\Delta \Gamma_0(\alpha + \delta + \gamma_2 + 1)} |g(s, u(s))| \\
\leq \|m\| \left\{ \frac{b}{\lambda \Gamma_0(\alpha + \delta + 1)} + \frac{b \mu}{\lambda |\Delta \Gamma_0(\alpha + \delta + \gamma_1 + 1)} + \frac{b(1 - \mu)}{\lambda |\Delta \Gamma_0(\alpha + \delta + \gamma_2 + 1)} \right\},
\]

which implies that \( \| T_2 u \| \leq \Lambda_2 \| m \| \).

In order to establish the equicontinuity of the operator \( T_2 \), we assume that \( t_1, t_2 \in [0, 1] \) such that \( t_2 > t_1 \). We will show that \( T_2 \) maps bounded sets into equicontinuous sets. For each \( u \in B_r \), we have

\[
|T_2 u(t_2) - T_2 u(t_1)| \leq \frac{b}{\lambda \Gamma_0(\alpha + \delta + 1)} \int_{t_1}^{t_2} [(t_2 - qs)^{(\alpha+\delta-1)} - (t_1 - qs)^{(\alpha+\delta-1)}] |g(s, u(s))| ds \\
+ \frac{b}{\lambda \Gamma_0(\alpha + \delta + \gamma_1 + 1)} \int_{t_1}^{t_2} (t_2 - qs)^{(\alpha+\delta+\gamma_1-1)} |g(s, u(s))| ds \\
+ \frac{b(1 - \mu)}{\lambda \Gamma_0(\alpha + \delta + \gamma_2 + 1)} \int_{t_1}^{t_2} (1 - qs)^{(\alpha+\delta+\gamma_2-1)} |g(s, u(s))| ds \\
\leq \|m\| \left\{ \frac{2b(t_2 - t_1)^{\alpha+\delta} + b[t_2^{\alpha+\delta} - t_1^{\alpha+\delta}]}{\lambda \Gamma_0(\alpha + \delta + 1)} + \frac{b \mu (t_2^{\alpha-1} - t_1^{\alpha-1})}{\lambda |\Delta \Gamma_0(\alpha + \delta + \gamma_1 + 1)} \\
+ \frac{b(1 - \mu) (t_2^{\alpha-1} - t_1^{\alpha-1})}{\lambda |\Delta \Gamma_0(\alpha + \delta + \gamma_2 + 1)} \right\}.
\]

Observe that the right-hand side of the above inequality is independent of \( u \in B_r \) and tends to zero as \( t_1 \to t_2 \). This shows that \( T_2 \) is equicontinuous. Therefore, the operator \( T_2 \) is relatively compact on \( B_r \) and the Arzelá-Ascoli theorem implies that \( T_2 \) is completely continuous and so \( T_2 \) is compact operator on \( B_r \).

Finally, we prove that the operator \( T_1 \) is a contraction. For any \( u, v \in B_r \) and \( t \in [0, 1] \), we obtain

\[
|T_1 u(t) - T_1 v(t)| \leq \frac{|\lambda - 1|}{\lambda \Gamma_0(\alpha - \beta + 1)} |u(s) - v(s)| + \frac{a}{\lambda \Gamma_0(\alpha + 1)} L |u(s) - v(s)| \\
+ \frac{|\lambda - 1|}{\lambda |\Delta \Gamma_0(\alpha - \beta + \gamma_1 + 1)} |u(s) - v(s)| + \frac{a \mu}{\lambda |\Delta \Gamma_0(\alpha + \gamma_1 + 1)} L |u(s) - v(s)| \\
+ \frac{(1 - \mu) |\lambda - 1|}{\lambda |\Delta \Gamma_0(\alpha - \beta + \gamma_2 + 1)} |u(s) - v(s)| + \frac{a(1 - \mu)}{\lambda |\Delta \Gamma_0(\alpha + \gamma_2 + 1)} L |u(s) - v(s)| \\
\leq (\Lambda_0 + L \Lambda_1) \| u - v \|.
\]

Since \( \Lambda_0 + L \Lambda_1 < 1 \), \( T_1 \) is a contraction. Thus, all the assumptions of Lemma 1 are satisfied. Therefore, the fractional \( q \)-integro-difference Equation (1) with \( q \)-integral boundary conditions (2) has at least one solution on \( [0, 1] \) and the proof is completed. \( \square \)

In the following result, we prove the existence of solutions for the problem (1) and (2) by means of a Leray-Schauder nonlinear alternative.
Theorem 2. Let \( f : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R} \) be a continuous function satisfying the conditions:

(H3) there exist continuous nondecreasing functions \( \psi_1, \psi_2 : [0, \infty) \rightarrow (0, \infty) \) and functions \( \phi_1, \phi_2 \in C([0, 1], \mathbb{R}^+) \) such that \( |f(t, u)| \leq \phi_1(t)\psi_1(|u|) \) and \( |g(t, u)| \leq \phi_2(t)\psi_2(|u|) \) for each \( (t, u) \in [0, 1] \times \mathbb{R} \);

(H4) there exists a constant \( \Xi > 0 \) such that

\[
\frac{(1 - \Lambda_0)\Xi}{\Lambda_1\|\psi_1(\Xi)\| + \Lambda_2\|\psi_2(\Xi)\|} > 1, \quad \Lambda_0 < 1,
\]

where \( \Lambda_0, \Lambda_1, \Lambda_2 \) are defined by (8).

Then, the fractional q-integro-difference Equation (1) with q-integral boundary conditions (2) has at least one solution on \([0, 1]\).

Proof. We verify the hypothesis of a Leray-Schauder nonlinear alternative (Lemma 2) in several steps. Let us first show that the operator \( T \), defined by (7), maps bounded sets (balls) into bounded sets in \( E \). For a positive number \( R \), let \( B_R = \{ u \in E : \| u \| \leq R \} \) be a bounded ball in \( E \). Then, for \( t \in [0, 1] \), we have

\[
|T u(t)| \leq \frac{|\lambda - 1|}{\lambda\Gamma_q(\alpha - \beta)} \int_0^t (t - s)^{(\alpha - \beta - 1)}\| u \| d_qs + \frac{a}{\lambda\Gamma_q(\alpha)} \int_0^t (t - s)^{(\alpha - 1)}\| \phi_1(\| u \|) \| d_qs
\]

where \( \lambda, \mu, a, \beta, \gamma_1, \gamma_2 \) are defined by (8).

Therefore,

\[
\| Tu \| \leq \lambda_0\| u \| + \Lambda_1\| \phi_1(\| u \|) \| + \Lambda_2\| \phi_2(\| u \|) \|
\]

Secondly, we show that \( T \) maps bounded sets into equicontinuous sets of \( E \). Let \( t_1, t_2 \in [0, 1] \) with \( t_1 < t_2 \) and \( u \in B_R \). Then, we have

\[
|T u(t_2) - T u(t_1)| \leq \frac{|\lambda - 1|R}{\lambda\Gamma_q(\alpha - \beta)} \int_0^{t_1} [(t_2 - s)^{(\alpha - \beta - 1)} - (t_1 - s)^{(\alpha - \beta - 1)}] d_qs + \frac{a\| \phi_1(\| u \|) \|}{\lambda\Gamma_q(\alpha)} \int_0^{t_1} [(t_2 - s)^{(\alpha - 1)} - (t_1 - s)^{(\alpha - 1)}] d_qs
\]

where \( \lambda, \mu, a, \beta, \gamma_1, \gamma_2 \) are defined by (8).
Let $f : U : \text{Banach contraction mapping principle}\ [34]$. a nonlinear alternative of the Leray-Schauder type (Lemma 2) that the operator and observe that the operator $T$ be a solution of

$$
\int_0^{t_1} (t_2 - qs)^{(a+\delta-1)} - (t_1 - qs)^{(a+\delta-1)}dt_2s + \int_{t_1}^{t_2} (t_2 - qs)^{(a+\delta-1)}dt_2s
$$

$$
\lambda \Gamma_1\left[(t_2 - t_1)^{a-\beta} + |t_2^{a-\beta} - t_1^{a-\beta}| + a\frac{|\phi_1| |\psi_1(R)|}{\lambda \Gamma_1(\alpha + \gamma_1 + 1)}\right]
$$

Thus, the Arzelà-Ascoli theorem applies and hence $T : E \to E$ is completely continuous.

In the last step, we show that all solutions to the equation $u = \theta Tu$ are bounded for $\theta \in [0, 1]$. For that, let $u$ be a solution of $u = \theta Tu$ for $\theta \in [0, 1]$. Then, for $t \in [0, 1]$, we apply the strategy used in the first step to obtain

$$
\|u\| \leq \Lambda_0\|u\| + \Lambda_1\|\phi_1\|\psi_1(\|u\|) + \Lambda_2\|\phi_2\|\psi_1(\|u\|).
$$

Consequently, we have

$$
\frac{(1 - \Lambda_0)\|u\|}{\Lambda_1\|\phi_1\|\psi_1(\|u\|) + \Lambda_2\|\phi_2\|\psi_1(\|u\|)} \leq 1.
$$

By the condition $(H_4)$, we can find a positive number $\Xi$ such that $\|u\| \neq \Xi$. Introduce a set

$$
U = \{u \in E : \|u\| < \Xi\},
$$

and observe that the operator $T : \overline{U} \to E$ is continuous and completely continuous. With this choice of $U$, we cannot find $u \in \partial U$ satisfying the relation $u = \theta Tx$ for some $\theta \in (0, 1)$. Therefore, it follows by a nonlinear alternative of the Leray-Schauder type (Lemma 2) that the operator $T$ has a fixed point in $\overline{U}$. Thus, there exists a solution of problems (1) and (2) on $[0, 1]$. The proof is complete. 

In our final result, the uniqueness of solutions for the given problem is shown with the aid of a Banach contraction mapping principle [34].

**Theorem 3.** Let $f : [0, 1] \times \mathbb{R} \to \mathbb{R}$ be a function satisfying the assumption $(H_1)$. In addition, assume that the function $g : [0, 1] \times \mathbb{R} \to \mathbb{R}$ satisfies the condition

$$(H_5) \text{ there exists a positive constant } M \text{ such that, for each } u, v \in \mathbb{R},$$

$$
|g(t, u) - g(t, v)| \leq M|u - v|, \quad t \in [0, 1].
$$
Then, the fractional q-integro-difference Equation (1) with q-integral boundary conditions (2) has a unique solution on $[0, 1]$, provided that $\Lambda_0 + LA_1 + MA_2 < 1$, where $\Lambda_0, \Lambda_1, \Lambda_2$ are defined by (8).

**Proof.** By a Banach contraction mapping principle, we will show that the operator $T : E \rightarrow E$ defined by (7) has a unique fixed point which corresponds to the unique solution of problems (1) and (2). Setting $\sup_{t\in[0,1]} |f(t,0)| = K < \infty$ and $\sup_{t\in[0,1]} |g(t,0)| = N < \infty$ and selecting

$$r \geq \frac{NA_2 + KA_1}{1 - (\Lambda_0 + LA_1 + MA_2)},$$

we show that $TB_r \subset B_r$, where $B_r = \{ u \in E : \|u\| \leq r \}$. For any $u \in B_r$, following the arguments used in the proof of Theorem 1, one can obtain

$$\|Tu\| \leq (\Lambda_0 + LA_1 + MA_2)r + A_2N + A_1K < r,$$

which implies that $TB_r \subset B_r$. For any $t \in [0, 1]$ and any $u, v \in \mathbb{R}$, we obtain

$$\| (Tu) - (Tv) \| \leq \frac{|\lambda - 1|}{\lambda t} \int_0^t (t - qs)^{\alpha - \beta - 1} |u(s) - v(s)|dq$$

$$+ \frac{a}{\Gamma_q(\alpha + \delta)} \int_0^t (t - qs)^{\alpha - 1} \int_0^1 (1 - qs)^{\beta - 1} |f_s(u(s)) - f_s(v(s))|dq$$

$$+ \frac{b}{\Gamma_q(\alpha + \beta + \gamma_1)} \int_0^t (t - qs)^{\alpha - \delta + \gamma_1 - 1} \int_0^1 (1 - qs)^{\beta + 1} |u(s) - v(s)|dq$$

$$+ \int_0^1 (1 - qs)^{\alpha + \gamma_1 - 1} f_s(u(s)) - f_s(v(s))|dq,\beta$$

$$+ \frac{\alpha}{\Gamma_q(\alpha + \beta + \gamma_1)} \int_0^1 (1 - qs)^{\alpha + \gamma_1 - 1} |f_s(u(s)) - f_s(v(s))|dq$$

$$+ \int_0^1 (1 - qs)^{\alpha + \beta + \gamma_1 - 1} |u(s) - v(s)|dq$$

$$+ \frac{b(1 - \mu)}{\Gamma_q(\alpha + \beta + \gamma_2)} \int_0^1 (1 - qs)^{\alpha + \gamma_2 - 1} |f_s(u(s)) - f_s(v(s))|dq$$

$$+ \frac{b(1 - \mu)}{\Gamma_q(\alpha + \beta + \gamma_2)} \int_0^1 (1 - qs)^{\alpha + \gamma_2 - 1} |g_s(u(s)) - g_s(v(s))|dq$$

$$\leq (\Lambda_0 + LA_1 + MA_2) \| u - v \|.$$

As $\Lambda_0 + LA_1 + MA_2 < 1$, therefore $T$ is a contraction. Hence, we deduce by the conclusion of the Banach contraction mapping principle that the operator $T$ has a unique fixed point, which is the unique solution of problems (1) and (2). The proof is completed. $\Box$

3. Examples

I. Illustration of Theorem 1

**Example 1.** Consider the fractional q-integro-difference equation

$$(0.9D_{0.5}^{0.5} + (1 - 0.9)D_{0.5}^{0.5})u(t) = 0.2 \frac{|90t \sin(u(t))|}{100(|\sin(u(t))| + 1)} + 0.35 \frac{|\sin t|^{3}(t)}{(2 + t)^{3}(1 + |u^3(t)|)},$$

(11)
subject to q-integral boundary conditions

\[
    u(0) = 0, \quad 0.1 \int_0^1 \frac{(1 - qs)^{(0.1-1)}}{\Gamma_q(0.1)} u(s)dqs + (1 - 0.1) \int_0^1 \frac{(1 - qs)^{(0.1-1)}}{\Gamma_q(0.1)} u(s)dqs = 0. \tag{12}
\]

Here, \( \alpha = 1.5, q = 0.5, \beta = 1.01, a = 0.2, b = 0.3, \delta = 0.35, \lambda = 0.9, \mu = 0.1, \gamma_1 = \gamma_2 = 0.1, t \in [0,1] \) and \( f, g : [0,1] \times \mathbb{R} \rightarrow \mathbb{R} \) are

\[
f(t, u) = \frac{|90t \sin u|}{100(|\sin u| + 1)}, \quad g(t, u) = \frac{\sin t |u|^2}{(2 + t)^2(1 + |u|^2)}.
\]

For each \( u, v \in \mathbb{R} \), notice that \( |f(t, u) - f(t, v)| \leq L|u - v| \) with \( L = 0.9 > 0 \). On the other hand, there exists a continuous function \( m(t) = \frac{1}{(2 + t)^3} \) on \( [0,1] \) such that \( |g(t, u)| \leq m(t) \) for all \( u \in \mathbb{R} \). In addition, we have \( \|m\| = \sup_{t \in [0,1]} m(t) = 0.125 \). Using the given values, it is found that \( \Delta = 0.9935 \) and \( \Lambda_0 + \Lambda_1 = 0.5567 < 1 \). Clearly, all the assumptions of Theorem 1 are satisfied. Therefore, the conclusion of Theorem 1 implies that the fractional q-integro-difference Equation (11) with q-integral boundary conditions (12) has at least one solution on \( [0,1] \).

II. Illustration of Theorem 2

Example 2. We consider the fractional q-integro-difference equation

\[
(0.9D^1_{0.5} + (1 - 0.9)D^0_{0.5})u(t) = \frac{0.2}{\sqrt{256 + t^2}} \left( \sin u(t) + \frac{|u(t)|}{1 + |u(t)|} \right) + 0.35 \frac{0.35}{4 + t} \left( \frac{1}{2} + \frac{|\arcsin u(t)|}{1 + |\arcsin u(t)|} \right) \tag{13}
\]

supplemented with q-integral boundary conditions

\[
u(0) = 0, \quad 0.1 \int_0^1 \frac{(1 - qs)^{(0.1-1)}}{\Gamma_q(0.1)} u(s)dqs + (1 - 0.1) \int_0^1 \frac{(1 - qs)^{(0.1-1)}}{\Gamma_q(0.1)} u(s)dqs = 0, \tag{14}
\]

where \( \alpha = 1.5, q = 0.5, \beta = 1.01, a = 0.2, b = 0.3, \delta = 0.35, \lambda = 0.9, \mu = 0.1, \gamma_1 = \gamma_2 = 0.1, t \in [0,1] \) and

\[
f(t, u(t)) = \frac{1}{\sqrt{256 + t^2}} \left( \sin u(t) + \frac{|u(t)|}{1 + |u(t)|} \right), \quad g(t, u(t)) = \frac{1}{4 + t} \left( \frac{1}{2} + \frac{|\arcsin u(t)|}{1 + |\arcsin u(t)|} \right).
\]

Obviously,

\[
|f(t, u(t))| \leq \frac{1}{\sqrt{256 + t^2}} (1 + \|u\|), \quad |g(t, u(t))| \leq \frac{1}{4 + t} (1 + \|u\|),
\]

with \( \phi_1(t) = \frac{1}{\sqrt{256 + t^2}}, \phi_2(t) = \frac{1}{4 + t} \) and \( \phi_1(\|u\|) = \phi_2(\|u\|) = 1 + \|u\| \). Note that \( \|\phi_1\| = 0.0625 \), \( \|\phi_2\| = \frac{1}{4} = 0.25 \) and \( \phi_1(\Xi) = \phi_2(\Xi) = 1 + \Xi \). Using the given data, we find that \( \Delta = 0.9935 \), \( \Lambda_0 = 0.2414 < 1 \), \( \Lambda_1 = 0.3504 \), and \( \Lambda_2 = 0.4685 \). Then, by condition (H_4), we get \( \Xi > 0.22438 \). Thus, all the assumptions of Theorem 2 are satisfied. Therefore, by Theorem 2, problems (13) and (14) have at least one solution on \( [0,1] \).

III. Illustration of Theorem 3

Example 3. Let us consider the fractional q-integro-difference equation

\[
(0.9D^1_{0.5} + (1 - 0.9)D^0_{0.5})u(t) = 0.2 \frac{0.9t |\sin \left( \frac{\pi}{6} (t) \right) | |u(t)|}{5 + |u(t)|} + 0.35 \frac{0.35}{4 + t} \frac{80 \cos (\pi t) |u(t)|}{100(7 + |u(t)|)} \tag{15}
\]
with \(q\)-integral boundary conditions

\[
u(0) = 0, \quad 0.1 \int_0^1 \frac{(1 - qs) (0.1 - 1)}{\Gamma_q(0.1)} u(s) dqs + (1 - 0.1) \int_0^1 \frac{(1 - qs) (0.1 - 1)}{\Gamma_q(0.1)} u(s) dqs = 0,
\]

(16)

where \(\alpha = 1.5, q = 0.5, \beta = 1.01, a = 0.2, b = 0.3, \delta = 0.35, \lambda = 0.9, \mu = 0.1, \gamma_1 = \gamma_2 = 0.1, t \in [0, 1] \) and

\[f(t,u(t)) = \frac{0.9|\sin(\frac{2}{5} t)||u(t)||}{5 + |u(t)|}, \quad g(t,u(t)) = \frac{8|\cos(\pi t)||u(t)||}{100(7 + |u(t)|)}.
\]

Then, \(L = 9/10, M = 8/10\) as

\[|f(t,u(t)) - f(t,v(t))| \leq \frac{9}{10}(|u(t) - v(t)|), \quad |g(t,u(t)) - g(t,v(t))| \leq \frac{8}{10}(|u(t) - v(t)|).
\]

With the given data, it is found that \(\Delta = 0.9935, \Lambda_0 + \Lambda_1 + \Lambda_2 = 0.9315 < 1\). Clearly, the assumptions of Theorem 3 hold. Thus, by the conclusion of Theorem 3, problems (15) and (16) have a unique solution \([0, 1]\).

4. Conclusions

We have derived some new existence and uniqueness results for a nonlinear fractional \(q\)-integro-difference equation equipped with \(q\)-integral boundary conditions. The obtained results significantly contribute to the literature on boundary value problems of fractional \(q\)-integro-difference equations and yield several new results as special cases. Some of these results are listed below.

(a) By letting \(\lambda = 1/2\) in the results of this paper, we obtain the ones for a nonlinear fractional \(q\)-integro-difference equation of the form:

\[
\left(D_q^a + D_q^b\right) u(t) = 2a f(t, u(t)) + 2b t_q^\delta g(t, u(t)), \quad t \in [0, 1], a, b \in \mathbb{R}^+.
\]

(b) For \(\mu = 1/2\), our results correspond to the following boundary conditions:

\[
u(0) = 0, \quad \int_0^1 \left[ \frac{1 - qs}{\Gamma_q(\gamma_1)} + \frac{1 - qs}{\Gamma_q(\gamma_2)} \right] u(s) dqs = 0, \quad \gamma_1, \gamma_2 > 0.
\]

(c) Our results with \(a = 0\) and \(b = 0\) correspond to the ones with purely integral nonlinearity and purely non-integral nonlinearity, respectively.

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