GENERALIZED TWISTED COHOM OBJECTS

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Abstract. A generalization of the concept of twisted internal coHom object in the category of conic quantum spaces (c.f. [2]) was outlined in [3]. The aim of this article is to develop in more detail this generalization.

1. Introduction

Given a couple $A, B$ of conic quantum spaces, i.e. $A, B \in \mathcal{C}A$ (the monoidal category of finitely generated graded algebras [4][5][2]), their symmetric twisted tensor products $A \circ \otimes B$ [1] [2] can also be seen as (2nd admissible) cotwist 2-cocycle twisting of the quantum space $\mathcal{C}A \otimes B$. Then, in the same sense as in [2], we can study, instead of the maps $A \rightarrow \mathcal{H} \circ B$ (which define a comma category whose initial objects are the proper internal coHom objects of $\mathcal{C}A$), certain subclasses of arrows $A \rightarrow (\mathcal{H} \circ B)_{\omega}$, where $\omega$ is a cotwist 2-cocycle defining a twist transformation on $\mathcal{H} \circ B$. The aim of this paper is to show that, in certain circumstances, these classes give rise for each pair $A, B \in \mathcal{C}A$ to a category $\Omega^{A,B}$ with initial object, namely $hom_{\Omega}^{\omega} [B, A]$, such that the disjoint union $\Omega = \bigcup_{A,B} \Omega^{A,B}$ has a semigroupoid structure together with a related embedding $\Omega \hookrightarrow \mathcal{C}A$ that preserves the involved (partial) products. Consequently, $(B, A) \mapsto hom_{\Omega}^{\omega} [B, A]$ defines an $\mathcal{C}A$-cobased category with an additional notion of evaluation given by arrows

$$A \rightarrow \left(hom_{\Omega}^{\omega} [B, A] \circ B\right)_{\omega}.$$ 

The categories $\Omega$, obtained in [2], are particular cases of the categories $\Omega$. In this way we generalize the idea of twisted coHom objects in the more general framework of twisting of quantum spaces. This setting enable us, in turn, to a better understanding of the results obtained in the mentioned paper.

This article is based on the contents of [2] and [3] and references therein, thus we shall frequently refer the reader to them. Notation and terminology also follow those papers.

2. The categories $\Omega$

In order to build up the categories $\Omega^{A,B}$, let us first make a couple of observations.

1. Consider on the category $\mathcal{C}Vct$ (of graded vector spaces) the monoid

$$\mathcal{V} \circ \mathcal{W} = \bigoplus_{n \in \mathbb{N}_0} (V_n \otimes W_n),$$

for $\mathcal{V} = \bigoplus_{n \in \mathbb{N}_0} V_n$ and $\mathcal{W} = \bigoplus_{n \in \mathbb{N}_0} W_n$. The arrows $\mathcal{V} \rightarrow \mathcal{W}$ in $\mathcal{C}Vct$ are homogeneous linear maps; i.e. its restrictions to each $V_n$ define maps $V_n \rightarrow W_n$. It is clear that the forgetful functor $\mathcal{F} : \mathcal{C}A \hookrightarrow \mathcal{C}Vct$ turns into a monoidal one, and $\mathcal{F}(A \circ B) = A = \bigoplus_{n \in \mathbb{N}_0} A_n$, holds for every twist transformation $\psi \in \mathbb{S}^2 [A_1].$

2. Let us construct the comma categories $(\mathcal{F}(A) \downarrow \mathcal{F}(\mathcal{C}A \circ \mathcal{B})))$, where the functor $\mathcal{F}(\mathcal{C}A \circ \mathcal{B})$ is the composition of $\mathcal{C}A \circ \mathcal{B}$ and $\mathcal{F}$. Its objects are pairs $(\varphi, \mathcal{H})$ where $\mathcal{H} \in \mathcal{C}A$ and $\varphi$ is an arrow in $\mathcal{C}Vct$,

$$\varphi : (\mathcal{F}(A) \downarrow \mathcal{F}(\mathcal{C}A \circ \mathcal{B})) = (\mathcal{F}(\mathcal{H}) \circ \mathcal{F}(B)).$$

To every $(\varphi, \mathcal{H}) \in (\mathcal{F}(A) \downarrow \mathcal{F}(\mathcal{C}A \circ \mathcal{B}))$, the surjection $\pi^\varphi : B^1_1 \otimes A_1 \rightarrow H^\varphi_1 : b^1 \otimes a_1 \mapsto h^\varphi_1$ can be related, being $h^\varphi_1$ the elements of $H_1$ defining the restriction of $\varphi$ to $A_1$, i.e. $\varphi(a_1) = h^\varphi_1 \otimes b_j$. This linear surjection gives rise to a functor

$$\mathfrak{f} : (\mathcal{F}(A) \downarrow \mathcal{F}(\mathcal{C}A \circ \mathcal{B})) \rightarrow \mathcal{C}A,$$

$$\langle \varphi, \mathcal{H} \rangle \mapsto H^\varphi = (H^\varphi_1, H^\varphi) ; \quad \alpha \mapsto \alpha|_{H^\varphi},$$

where $H^\varphi$ is the subalgebra of $H$ generated by $H^\varphi_1$ (an analogous functor is used in [2] to built up the categories $\mathcal{Y}^{A,B}$).

Using last functor we shall construct each $\Omega^{A,B}$ as a full subcategory of the corresponding comma category $(\mathcal{F}(A) \downarrow \mathcal{F}(\mathcal{C}A \circ \mathcal{B}))$. Given a couple of conic quantum spaces $A$ and $B$, consider a counital element

$$\omega : (|B^1_1 \otimes A_1 \otimes B^1_1| \otimes)^2 \otimes (|B^1_1 \otimes A_1 \otimes B^1_1| \otimes)^2$$

and define the restriction of $\omega$ to $H^\varphi$ by

$$\omega^\varphi : (|B^1_1 \otimes A_1 \otimes B^1_1| \otimes)^2 \otimes (|B^1_1 \otimes A_1 \otimes B^1_1| \otimes)^2$$

for each $(\varphi, \mathcal{H}) \in (\mathcal{F}(A) \downarrow \mathcal{F}(\mathcal{C}A \circ \mathcal{B})))$. Consequently, $(\mathcal{F}(A) \downarrow \mathcal{F}(\mathcal{C}A \circ \mathcal{B})))$ gives rise to a category $\Omega^{A,B}$ with initial object, namely $hom_{\Omega}^{\omega} [B, A]$.
of $3^2[|B_1^1 \otimes A_1 \otimes B_1|]$. Eventually, for $(\varphi, \mathcal{H}) \in \mathcal{H}_\mathcal{F}(A \downarrow \mathcal{F}(CA \circ B))$, we can translate $\omega$ to $[H_1^0 \otimes B_1]^\otimes$ through $\pi^\otimes : B_1^1 \otimes A_1 \rightarrow H_1^0$ in such a way that the diagram

$$
\begin{array}{ccc}
B_1 \otimes B_1^1 \otimes A_1 & \rightarrow & B_1 \otimes H_1^0 \\
\downarrow & & \downarrow \\
B_1 \otimes B_1^1 \otimes A_1 & \rightarrow & B_1 \otimes H_1^0
\end{array}
$$

be commutative, defining in this way a counital 2-cochain in $3^2[H_1^0 \otimes B_1]$. Last affirmation lies on the results given in Prop. 4 of [3], applied to the injection $H_1^0 \rightarrow B_1^1 \otimes A_1$. Then, if the resulting automorphism is admissible, we can define with it a twist transformation on $H^\omega \circ \mathcal{B} = \mathcal{F} \langle (\varphi, \mathcal{H}) \rangle \circ \mathcal{B}$.

**Definition 1.** For every pair $A, B \in \mathcal{C}$ and $\omega \in 3^2[B_1^1 \otimes A_1 \otimes B_1]$, we define $\Omega^{A,B}$ as the full subcategory of $(\mathcal{F}(A) \downarrow \mathcal{F}(CA \circ B))$ formed out by diagrams $(\varphi, \mathcal{H})$ such that $\omega$ defines a 2-cocycle twisting on $\mathcal{F} \langle (\varphi, \mathcal{H}) \rangle \circ \mathcal{B}$, and the homogeneous linear map $\varphi$ is a morphism of quantum spaces $A \rightarrow (\mathcal{F} \langle (\varphi, \mathcal{H}) \rangle \circ \mathcal{B})^\omega$.

Given now a collection of cochains $\{\omega_{A,B}\}_{A,B \in \mathcal{C}} \subset 3^2[B_1^1 \otimes A_1 \otimes B_1]$, we name $\Omega$ the disjoint union of the categories $\Omega^{A,B}$ just defined. Clearly, $CA^\circ$ (see [2], or §1.2 of [3] for a brief review) is a category $\Omega$ with an associated collection given by identity maps.

Calling $\mathcal{H}CA^\circ$ the disjoint union of $(\mathcal{F}(A) \downarrow \mathcal{F}(CA \circ B))$, it follows that every $\Omega$ is a full subcategory of $\mathcal{H}CA^\circ$. On the other hand, let us observe that $\Omega^{CA^\circ}$ has a semigroupoid structure given by the functor

$$(\mathcal{H} \times \mathcal{G}) \mapsto (\langle I_B \circ \chi \rangle \varphi, \mathcal{H} \circ \mathcal{G}) : \alpha \times \beta \mapsto \alpha \circ \beta,$$

and $CA^\circ \subset \mathcal{H}CA^\circ$ is a sub-semigroupoid. In fact, this map is a partial product functor with domain

$$\bigvee_{A,B,C \in \mathcal{C}} \langle \mathcal{H}(A) \downarrow \mathcal{H}(CA \circ C) \rangle \times \langle \mathcal{H}(C) \downarrow \mathcal{H}(CA \circ B) \rangle$$

and codomain $\mathcal{H}CA^\circ$, such that

$$\langle \mathcal{H}(A) \downarrow \mathcal{H}(CA \circ C) \rangle \times \langle \mathcal{H}(C) \downarrow \mathcal{H}(CA \circ B) \rangle \rightarrow \langle \mathcal{H}(A) \downarrow \mathcal{H}(CA \circ B) \rangle$$

Its associativity comes from that of $\circ$, and the unit elements are given by the diagrams $(\ell_A, K)$, where $\ell_A$ is the homogeneous isomorphism $A \simeq \mathbb{K}[c] \otimes A$, such that $a \mapsto e^n \otimes a$ if $a \in A_n$. Nevertheless, for a generic collection $\{\omega_{A,B}\}_{A,B \in \mathcal{C}}$ of cochains, $\Omega$ fails to be a semigroupoid. Furthermore, in the generic case, each $\Omega^{A,B}$ fails to have initial objects. To address this problem, we shall consider particular cases.

3. The Semigroupoid Structure of $\Omega$

In what follows, all references to sections and theorems correspond to [3]. Recall the monics (c.f. §2.3.2)

$$(3.1) \quad j : \mathfrak{c}^\bullet[B_1^1] \times \mathfrak{c}^\bullet[A_1] \times \mathfrak{c}^\bullet[B_1] \rightarrow \mathfrak{c}^\bullet[B_1^1 \otimes A_1 \otimes B_1].$$

**Definition 2.** A collection $\{\omega_{A,B}\}_{A,B \in \mathcal{C}}$ is **factorizable** if there exists another collection

$$\{\psi_A\}_{A \in \mathcal{C}}, \quad \psi_A \in 3^2[A_1],$$

such that $\omega_{A,B} = j \left(\psi_B^i, \psi_A, I^{\otimes 2}\right)$. §

Since Eq. (3.1), such cochains $\omega_{A,B}$ are in $3^2[B_1^1 \otimes A_1 \otimes B_1]$. To give an example, in the TTP case with $\hat{\mathcal{F}}_{A,B} = id \otimes \sigma_B \otimes \sigma_A$, the cochain $\psi_A$ would be given by the assignment

$$(3.2) \quad a_{k_1} \ldots a_{k_r} \otimes a_{k_{r+1}} \ldots a_{k_{r+s}} \mapsto a_{k_1} \ldots a_{k_r} \otimes (\sigma_A^{-1})_{k_{r+1}}^{j_1} \ldots (\sigma_A^{-1})_{k_{r+s}}^{j_s} a_{j_1} \ldots a_{j_s}.$$

From the injection

$$(3.3) \quad 3^2[B_1^1] \times 3^2[A_1] \times 3^2[B_1] \rightarrow 3^2[B_1^1 \otimes A_1] \times 3^2[B_1],$$

we shall also regard $\omega_{A,B}$ as a cochain belonging to the latter set, depending on our convenience.

**Theorem 1.** If a category $\Omega$ is associated to a factorizable collection, then $\Omega$ is a sub-semigroupoid of $\mathcal{H}CA^\circ$.

**Proof.** Consider the quantum spaces $A$, $B$ and $C$, and diagrams $(\varphi, \mathcal{H}) \in \Omega^{A,B}$ and $(\psi, \mathcal{G}) \in \Omega^{B,C}$, with associated linear spaces (via the functor $\mathcal{F}$)

$$H_1^\varphi = \text{span} \left[ h^{n,m}_{i,j=1} \right] \quad \text{and} \quad G_1^\psi = \text{span} \left[ g^{m,p}_{i,j=1} \right].$$
We are denoting by \( \dim A_1 = n \), \( \dim B_1 = m \) and \( \dim C_1 = p \) the dimensions of the generator spaces defining \( \mathcal{A}, \mathcal{B}, \) and \( \mathcal{C}, \) respectively. We must show that \( \langle (I_H \circ \chi) \varphi, \mathcal{H} \circ \mathcal{G} \rangle \) (see Eq. (2.3)) is an object of \( \Omega^{A,C} \), and that the objects \( \langle \ell_A, \mathcal{K} \rangle \) are in \( \Omega^c \). That means the quantum space \( \mathfrak{S} \langle (I_H \circ \chi) \varphi, \mathcal{H} \circ \mathcal{G} \rangle \), generated by

\[
\text{span} \left[ \sum_{j=1}^{n} h^j_i \otimes g^j_k \right]_{i,k=1}^{n,p} \subset \mathcal{H}^i \otimes \mathcal{G}^i \subset \mathcal{H}^p \otimes \mathcal{G}^p,
\]

is such that \( (I_H \circ \chi) \varphi \) defines an arrow \( \mathcal{A} \to \langle \mathfrak{S} \langle (I_H \circ \chi) \varphi, \mathcal{H} \circ \mathcal{G} \rangle \rangle \) in \( \mathcal{C} \). To this end, let us introduce some notation.

Denote by \( h \) and \( g \) the matrices with entries \( h^j_i \in \mathcal{H}_1 \) and \( g^j_i \in \mathcal{G}_1 \), and by \( a, b \) and \( c \) the vectors whose components are \( a_i \in A_1, b_i \in B_1 \) and \( c_i \in C_1 \). Since \( \langle \varphi, \mathcal{H} \rangle \) and \( \langle \psi, \mathcal{G} \rangle \) are elements of \( \Omega^c \), \( \omega_{AB} \) and \( \omega_{BC} \) defines cochains in \( 3^2 \mathcal{H}_1 \times 3^2 \mathcal{B}_1 \) and \( 3^2 \mathcal{G}_1 \times 3^2 \mathcal{C}_1 \) (see Eq. (3.3), respectively. The latter are given by

(3.4)
\[
\omega_{AB} (h)_{r,s} \otimes (b)_{r,s} = [\psi_A]_{r,s} \cdot (h)_{r,s} \cdot [\psi_B^{-1}]_{r,s} \otimes (b)_{r,s}
\]

and

(3.5)
\[
\omega_{BC} (g)_{r,s} \otimes (c)_{r,s} = [\psi_B]_{r,s} \cdot (g)_{r,s} \cdot [\psi_C^{-1}]_{r,s} \otimes (c)_{r,s}
\]

where the symbols \( h_{r,s} = h^r \otimes h^s \) and \( [\psi_A]_{r,s} \cdot (h)_{r,s} \cdot [\psi_B^{-1}]_{r,s} \otimes (b)_{r,s} \) denote elements of the form

\[
h^i_1 \cdots h^i_r \otimes h^l_1 \cdots h^l_s \in (\mathcal{H}^i)^{\otimes r} \otimes (\mathcal{H}^l)^{\otimes s}
\]

and

\[
(\psi_A)_{m_1 \cdots m_r, r_1 \cdots r_s, k_1 \cdots k_s} h^i_1 \cdots h^i_r \otimes h^l_1 \cdots h^l_s \in (\mathcal{H}^i)^{p_1 \cdots p_r} \otimes (\mathcal{H}^l)^{q_1 \cdots q_s}
\]

respectively. Now, \( \langle (I_H \circ \chi) \varphi, \mathcal{H} \circ \mathcal{G} \rangle \) in \( \Omega^{A,C} \) if and only if \( (I_H \circ \chi) \varphi \) defines the mentioned arrow in \( \mathcal{C} \), with \( \omega \) given by

\[
\omega_{AC} \left( (h \otimes g)_{r,s} \otimes (c)_{r,s} \right) = \omega_{A} \left( (h \otimes g)_{r,s} \otimes (c)_{r,s} \right)
\]

Here \( \otimes \) is denoting matrix contraction between \( h \) and \( g \). It follows from straightforward calculations that, if \( \omega_{AC} \) is well defined on \( \mathfrak{S} \langle (I_H \circ \chi) \varphi, \mathcal{H} \circ \mathcal{G} \rangle \circ \mathcal{C} \), then \( (I_H \circ \chi) \varphi \) is a homogeneous linear map \( A \to \mathcal{H} \otimes \mathcal{G} \otimes \mathcal{C} \) defining the wanted morphism. So, let us first show that. To this end, extend \( \omega_{AC} \) to \( \mathcal{H}^p \circ \mathcal{G}^o \circ \mathcal{C} \) by putting \( \omega_{AC} (h)_{r,s} \otimes (g)_{r,s} \otimes (c)_{r,s} \) equal to

\[
[\psi_A]_{r,s} \cdot (h)_{r,s} \otimes [\psi_B^{-1}]_{r,s} \otimes (c)_{r,s} =
\]

\[
[\psi_A]_{r,s} \cdot (h)_{r,s} \otimes [\psi_B^{-1}]_{r,s} \otimes (g)_{r,s} \otimes (c)_{r,s}.
\]

From the last expression, and recalling that \( \omega_{AB} \) and \( \omega_{BC} \) (given in (3.4) and (3.5)) are admissible, it follows that \( \omega_{AC} \) is admissible for \( \mathcal{H}^p \circ \mathcal{G}^o \circ \mathcal{C} \), and also for the subspace \( \mathfrak{S} \langle (I_H \circ \chi) \varphi, \mathcal{H} \circ \mathcal{G} \rangle \circ \mathcal{C} \). Then, \( \langle (I_H \circ \chi) \varphi, \mathcal{H} \circ \mathcal{G} \rangle \) in \( \Omega^{A,C} \).

Finally, we have to show the units \( \langle \ell_A, \mathcal{K} \rangle \) are objects of the corresponding categories \( \Omega^A \). We know that \( \ell_A \) is a homogeneous linear map such that \( \ell_A (a) = e^a \otimes a \), if \( a \in A_n \). In particular, we can write \( \ell_A (a_i) = e \delta_i^j \otimes a_j \). Then, the cochain \( \omega_{AC} \) for \( \langle \ell_A, \mathcal{K} \rangle \) is given by

\[
\omega_{AC} (k)_{r,s} \otimes (a)_{r,s} = \omega_{A} (k)_{r,s} \cdot (a)_{r,s}
\]

with

\[
k_{r,s} = e^r \delta_{i_1}^{r_1} \otimes \delta_{i_r}^{r_r} \otimes e^s \delta_{k_1}^{l_1} \otimes \delta_{k_s}^{l_s}.
\]

Accordingly \( \omega_{AC} \) for \( \langle \ell_A, \mathcal{K} \rangle \) is the identity map, and \( \mathcal{K} \circ \mathcal{A} \) is \( \mathcal{A} \).

The following result is immediate.

**Proposition 1.** Let us call \( \circ \Omega \) the partial product associated to above mentioned semigroupoid structure of \( \Omega \). The embedding \( \mathfrak{P}^\Omega : \Omega \to \mathcal{C} : \mathfrak{P} : \mathfrak{P}^\Omega \) preserves the respective (partial) products and units, i.e.

\[
\mathfrak{P}^\Omega \circ \Omega = \mathfrak{P} (\mathfrak{P}^\Omega \times \mathfrak{P}^\Omega) \quad \text{and} \quad \mathfrak{P}^\Omega \langle \ell_A, \mathcal{K} \rangle = \mathcal{K}.
\]
For $\Omega^{A,B}$ to have initial objects we need an additional condition on $\omega_{A,B}$.

**Theorem 2.** If $\Omega$ is associated to a factorizable collection given by $\{\psi_A\}_{A \in CA}$ such that each $i\psi_A = i\psi$ is 2nd $A$-admissible, then each $\Omega^{A,B}$ have initial object

$$\text{hom}_\Omega^\Omega [B, A] = B_{i\psi} \triangleright A_{i\psi} = (B \triangleright A)_{i(i\psi,i\psi)}.$$  

In particular, $\text{hom}_\Omega^\Omega [K, A] = A_{i\psi}$ and $\text{hom}_\Omega^\Omega [K, K] = K$; thus,

$$\text{hom}_\Omega^\Omega [B, A] = \text{hom}_\Omega^\Omega [K, B] \triangleright \text{hom}_\Omega^\Omega [K, A].$$

Before going to the proof, let us make some remarks. Since $\psi_A \in \mathbb{Z}^2 [A_1]$, there exists a primitive $\theta \in \mathbb{P}^1 [A_1]$ such that $\psi_A = \partial \theta \omega_A$, $\psi_A = \partial \theta \omega_A$ and $\psi_A = \partial \theta \omega_A$ (see §3.2.1, §3.3.1 and §4.2.1, respectively). In addition, if $I = \bigoplus_{n \geq 1} I_n$ is the graded ideal related to $A$, we have from §3.2.2 that (provided $\psi$ is admissible)

$$I_{i\psi,n} = \theta(I_n), \quad I_{i\psi,n} = \theta(I_n) = \theta^n(I_n).$$

**Proof. (of theorem)** We shall show $B_{i\psi} \triangleright A_{i\psi}$ defines an object of $\Omega^{A,B}$ and then that is initial.

Let us note that, given $\psi, \varphi \in \mathbb{Z}^2$, with $\psi = \partial \theta$ and $\varphi = \partial \chi$, it follows that

$$j(i\psi, i\varphi) = j((\partial \theta^{-1}), (\partial (\chi^{-1})) = j((\partial \theta^{-1}, \chi^{-1})) =$$

$$= \partial(j((\theta^{-1}, \chi^{-1})) = i(j(\psi, \varphi)).$$

Also recall, if $\psi$ is (2nd) $A$-admissible, then $\psi$ is (2nd) $A_{i\psi}$-admissible (see Prop. 14 of §3.3.1).

By **Theor. 16** of §4.2.2, using the 2nd $A$-admissibility of $i\psi_A = i\psi$,

$$(B_{i\psi} \triangleright A_{i\psi}) \circ B = (B \triangleright A)_{i(i\psi,i\psi)} \circ B = (B \triangleright A)_{i(i\psi,i\psi)} \circ B$$

$$= (((B \triangleright A)_{i(i\psi,i\psi)})_\omega) = ((B \triangleright A) \circ B)_{i\omega},$$

because $\omega = \omega_{A,B} = j(\psi_{B,\psi}, \psi_A, I^{\otimes 2})$. Then,

$$((B_{i\psi} \triangleright A_{i\psi}) \circ B)_{i\omega} = (((B \triangleright A) \circ B)_{i\omega})_\omega = (B \triangleright A) \circ B,$$

and consequently the map $\delta : A \to (B \triangleright A) \circ B : a_i \mapsto z_i^I \otimes b_j$ (with $z_i^I = b^I \otimes a_i$), defining the coevaluation of the proper coHom object $\text{hom}_\Omega^\Omega [B, A] = B \triangleright A$, gives also a morphism $\delta : A \to ((B_{i\psi} \triangleright A_{i\psi}) \circ B)_{i\omega}$. Moreover, since the equality $\exists (\delta, B_{i\psi} \triangleright A_{i\psi}) = B_{i\psi} \triangleright A_{i\psi}$, the pair $(\delta, B_{i\psi} \triangleright A_{i\psi})$ is in $\Omega^{A,B}$. Let us show such a pair is an initial object.

Suppose $A$ and $B$ have related dimensions $\dim A_1 = n$ and $\dim B_1 = m$, and ideals $I$ and $J$, respectively. Let us consider the vector space

$$D_1 = \text{span} \left[ z_i^I \right]_{i,j=1}^{n,m}$$

and the linear map $\delta_1 : a_i \mapsto z_i^J \otimes b_j$. Under the identification $b^I \otimes a_i = z_i^I$, the cochain $\omega_{A,B} = \omega$ defines a counital 2-cocycle in $\mathbb{C}^2 [D_1 \otimes B_1]$, and $\delta_1$ can be extended to an algebra homomorphism

$$\delta_1^\otimes : A_1^\otimes \to (D_1^\otimes \otimes B_1^\otimes)_\omega.$$

Analogous calculations that enable us to arrive at Eq. (3.4) of [3], show that

$$\delta_1^\otimes (a_r) = [\theta_A]_r \cdot z_r \cdot \left[ \theta_B^{-1} \right]_r \otimes b_r,$$

(where we are using notation of previous theorem), or in coordinates,

$$\delta_1^\otimes (a_{i_1} \ldots a_{i_r}) = [\theta_A]_{j_1 \ldots j_r} \cdot z_{j_1}^I \ldots z_{j_r}^I \cdot \left( \theta_B^{-1} \right)_{k_1 \ldots k_r} \otimes b_{k_1} \ldots b_{k_r}.$$
Now, it is clear that a pair \( \langle \varphi, \mathcal{H} \rangle \in \mathcal{H} \mathcal{CA}^\circ \) is in \( \Omega^{A, B} \) if and only if there exist elements \( h_i^j \in H_1 \subset \mathfrak{g} \langle \varphi, \mathcal{H} \rangle \) satisfying relations (4.2) (replacing \( z_r \) by \( h_r \)). But the elements of (4.2) span precisely the space (from Eq. (1.2))

\[
\theta^B_i (J^\varphi_r) \otimes \theta_A (I^\psi_r) = J^\varphi_r \otimes I^\psi_r,
\]

which generates algebraically the ideal related to \( B_i^\varphi \otimes A_i^\psi \). Then, the function \( z^j_i \mapsto h^j_i \) can be extended to an arrow \( B_i^\varphi \otimes A_i^\psi \to \mathfrak{g} \langle \varphi, \mathcal{H} \rangle \). But this is the unique arrow in \( \Omega^{A, B} \) that can be defined between these objects, that is to say, \( (\delta, B_i^\varphi \otimes A_i^\psi) \) is an initial object of \( \Omega^{A, B} \).

Finally, since the cochains of \( C^* \mathbb[k] \) are always (2nd) \( K \)-admissible, in particular the primitive ones \( P^1 \mathbb[k] \), it follows that any twisting of \( K \) is isomorphic to \( K \). Then, the last claim of the theorem follows immediately from the first one.

Note that the 2nd admissibility condition for each \( i^\psi A \) replaces the automorphism property of the related \( \sigma_A \) of the TTP case. Such a condition is immediate in the TTP case, because the properties defining a twisting map imply the related 2-cocycles are anti-bicharacters (c.f. §2.2.2 and §4.1.2).

Now, a couple of immediate corollaries.

**Corollary 1.** The gauge equivalence (see §3.3) \( \text{hom}^{\Omega} [B, A] \sim \text{hom} [B, A] \) is valid for all conic quantum spaces \( B, A \). \( \blacksquare \)

**Corollary 2.** In the category of quadratic quantum spaces \( QA \) (and any \( CA^m \)), the initial objects of \( \Omega^{A, B} \) are isomorphic to

\[
\text{hom}^{\Omega} [B, A] = (B^i)^3 \bullet A^\varphi = (B^i)^3 \bullet A_i^\psi = (B^i \bullet A)_i (i^\psi, i^\varphi).
\]

Under the hypothesis mentioned in previous theorem, and from the semigroupoid structure of \( \Omega \) compatible with the monoid in \( CA \), we have the announced result:

**Theorem 3.** The assignment \( (B, A) \mapsto \text{hom}^{\Omega} [B, A] \) define an \( CA \)-cobased category with arrows

\[
\text{hom}^{\Omega} [C, A] \to \text{hom}^{\Omega} [B, A] \circ \text{hom}^{\Omega} [C, B],
\]

the cocomposition, and for \( \text{end}^{\Omega} [A] \) the counit epimorphism

\[
\text{end}^{\Omega} [A] \to \mathcal{K} / z^j_i \mapsto \delta^j_i e,
\]

and the monomorphic comultiplication

\[
\text{end}^{\Omega} [A] \to \text{end}^{\Omega} [A] \circ \text{end}^{\Omega} [A] / z^j_i \mapsto z^k_i \otimes z^j_k.
\]

**References**

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