On the Hochschild cohomology and the automorphism group of $U_q(\mathfrak{sl}_4^\pm)$

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Abstract

We compute the automorphism group of the $q$-enveloping algebra $U_q(\mathfrak{sl}_4^\pm)$ of the nilpotent Lie algebra of strictly upper triangular matrices of size 4. The result obtained gives a positive answer to a conjecture of Andruskiewitsch and Dumas. We also compute the derivations of this algebra and then show that the Hochschild cohomology group of degree 1 of this algebra is a free (left)-module of rank 3 (which is the rank of the Lie algebra $\mathfrak{sl}_4$) over the center of $U_q(\mathfrak{sl}_4^\pm)$.

Keywords: quantized enveloping algebra; automorphisms; derivations; Hochschild cohomology.

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Introduction

Let $K$ be a field, $\mathcal{L}$ a Lie algebra over the $K$ and $U(\mathcal{L})$ its enveloping algebra. The group $\text{Aut}_K U(\mathcal{L})$ of $K$-algebra automorphisms of $U(\mathcal{L})$ is still for the most part unknown (except in particular instances, e.g. $\dim \mathcal{L} \leq 2$). For example, if $\mathcal{L}$ is the two-dimensional abelian Lie algebra, then $U(\mathcal{L})$ is the polynomial algebra in two indeterminates $x_1$, $x_2$, whose group of automorphisms is generated by the elementary automorphisms of the form

$$x_i \mapsto \lambda x_i + f(x_j), \quad x_j \mapsto x_j \quad (i \neq j)$$

with $\lambda \in K^*$ and $f(x_j)$ a polynomial in the variable $x_j$ ([13], [25]). In contrast with this simple description, the conjecture that the polynomial algebra in three variables over $K$ has wild automorphisms (i.e. automorphisms not of the above type) has recently been settled (see [23]) assuming $K$ has characteristic 0. Another example is the enveloping algebra of $\mathfrak{sl}_2$, which is known to have wild automorphisms by a result of Joseph [14].

Pertaining more to what is studied in this paper is the enveloping algebra of the three-dimensional Heisenberg Lie algebra, which is given by generators $x$, $y$ and $z$, subject to the relations

$$[x, y] = z,$$  
$$[z, x] = 0 = [z, y].$$

This algebra can also be seen as the enveloping algebra of the Lie algebra $\mathfrak{sl}_3^\pm$ of strictly upper triangular matrices of size 3. The infinite dimensional simple quotients of $U(\mathfrak{sl}_3^\pm)$ are isomorphic to the first Weyl algebra $A_1(K)$, whose group of automorphisms was described by Dixmier in [10].

Yet, the full group of automorphisms of $U(\mathfrak{sl}_3^\pm)$ remains to be described, and Alev [11] proved the existence of wild automorphisms of this algebra.

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Unlike the classical scenario, quantum algebras are believed to possess less symmetry (see [12, 1.1]) and the group of automorphisms of several algebras of this kind has been computed successfully. Making use of a general result relating automorphisms and derivations of \( N \)-graded algebras, Alev and Chamard [2] described the automorphism group of quantum affine space, of the algebra of \( 2 \times 2 \) quantum matrices and of the quantized enveloping algebra \( U_q(\mathfrak{sl}_2) \). Also, in [2] the authors found the automorphism groups of the quantum Weyl algebra, the Weyl-Hayashi algebra, the quantum Heisenberg algebra \( U_q(\mathfrak{su}^+_2) \) (see also [3]) and of other related algebras. Here the methods used included describing the set of normal elements of the algebras involved and using appropriate filtrations to carry out computations. In [21], Rigal used the invariance under automorphisms of the set of height 1 prime ideals of quantum Weyl algebras to describe their automorphism group. Related methods were employed by Gómez-Torrecillas and Kaoutit [11] regarding the coordinate ring of quantum symplectic space, and by Lenagan and the first author [18] regarding the algebra of non-square quantum matrices. In all of these cases, the automorphism group of the algebras involved does not differ from the natural torus which acts diagonally on the generators by more than a finite group and perhaps a copy of \( \mathbb{Z} \).

In their paper [3], Andruskiewitsch and Dumas conjectured that, given a finite-dimensional complex simple Lie algebra \( \mathfrak{g} \) with triangular decomposition \( \mathfrak{g} = \mathfrak{g}^- \oplus \mathfrak{h} \oplus \mathfrak{g}^+ \), then \( \text{Aut}_\mathbb{K} U_q(\mathfrak{g}^+) \), the group of \( \mathbb{K} \)-algebra automorphisms of the quantized enveloping algebra of the nilpotent Lie algebra \( \mathfrak{g}^+ \), is isomorphic to the semi-direct product of the torus \((\mathbb{K}^*)^n \) (\( n \) being the rank of \( \mathfrak{g} \)) with the group of order 1, 2 or 3 generated by the diagram automorphism of \( \mathfrak{g}^+ \), see [3] Prob. 1. This conjecture holds for \( \mathfrak{g}^+ = \mathfrak{sl}^+_4 \) ([3, 4]) and recently the first author proved in [16] that it holds as well in the \( B_3 \) case, i.e., with \( \mathfrak{g}^+ = \mathfrak{so}^+_5 \).

In this paper we settle the conjecture of Andruskiewitsch and Dumas in the \( A_3 \) case, so that \( \mathfrak{g}^+ = \mathfrak{sl}^+_4 \) is the Lie algebra of strictly upper triangular matrices of size 4. We also compute the Lie algebra of derivations and the first Hochschild cohomology group of \( U_q(\mathfrak{sl}^+_4) \), which is shown to be a free module of rank 3 over the center of \( U_q(\mathfrak{sl}^+_4) \).

Let us briefly summarise what is done in the paper. There exist normal elements \( \Delta_1, \Delta_2 \) and \( \Delta_3 \) such that the center of \( U_q(\mathfrak{sl}^+_4) \) is the polynomial algebra in the variables \( z_1 = \Delta_1 \Delta_3 \) and \( z_2 = \Delta_2 \). Given an automorphism \( \phi \) of \( U_q(\mathfrak{sl}^+_4) \), our strategy is to show that, up to the diagram automorphism and the diagonal action of the torus \((\mathbb{K}^*)^3 \) on the Chevalley generators of \( U_q(\mathfrak{sl}^+_4) \), \( \phi \) fixes \( \Delta_1, \Delta_2 \) and \( \Delta_3 \). Then, by using degree arguments, we conclude that \( \phi \) is the identity.

The difficulty that arises is in showing that the central element \( \Delta_2 \) is fixed. Hence we use the methods of [2] and [17] and determine the derivations of \( U_q(\mathfrak{sl}^+_4) \). To do this, we first apply the deleting derivations algorithm of Cauchon [9] so that, after suitably localising, we can embed \( U_q(\mathfrak{sl}^+_4) \) in a quantum torus \( P(\Lambda) \). Extending a derivation \( D \) of \( U_q(\mathfrak{sl}^+_4) \) to \( P(\Lambda) \) we obtain, by a result of Osborn and Passman [20], a decomposition

\[ D = \text{ad}_x + \theta \]

with \( x \in P(\Lambda) \) and \( \theta \) a central derivation of \( P(\Lambda) \). Using a sort of restoring derivations algorithm, we finish by deducing that \( x \in U_q(\mathfrak{sl}^+_4) \) and that \( \theta \) sends each Chevalley generator of \( U_q(\mathfrak{sl}^+_4) \) to a multiple of itself by a central element of \( U_q(\mathfrak{sl}^+_4) \).

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1 Basic aspects of \( U_q(\mathfrak{sl}^+_4) \)

Let \( \mathbb{K} \) be a field of characteristic 0 and fix a parameter \( q \in \mathbb{K}^* \) which we assume is not a root of unity. Consider, for \( n \geq 2 \), the Lie algebra \( \mathfrak{sl}_n \) of \( n \times n \) matrices of trace 0 and its maximal nilpotent subalgebra \( \mathfrak{sl}_n^+ \) consisting of the strictly upper triangular matrices of size \( n \).

Throughout this paper \( \mathbb{N} \) is the set of nonnegative integers. For \( k \in \mathbb{N} \), the \( q \)-integer \([k] \) is defined by \([k] = \frac{q^k - q^{-k}}{q - q^{-1}} \) and we use the notation \( \hat{q} = q - q^{-1} \).
1.1 $q$-Serre relations

The algebra $U_q(\mathfrak{sl}_1^+)$ is the $q$-deformation of the universal enveloping algebra of the nilpotent Lie algebra $\mathfrak{sl}_1^+$. It is the unital associative $\mathbb{K}$-algebra with generators $e_1$, $e_2$ and $e_3$, subject to the quantum Serre relations:

\begin{align*}
e_1e_3 - e_3e_1 &= 0 \quad (1) \\
e_i^2e_j - (q + q^{-1})e_ie_je_i + e_je_i^2 &= 0 \quad \text{if } |i - j| = 1. \quad (2)
\end{align*}

1.2 Weight space decomposition

Let $Q = \mathbb{Z}^3$ be the free abelian group with canonical basis $\alpha_1$, $\alpha_2$, $\alpha_3$ and $Q^+ = \mathbb{N}^3$ be its submonoid. Since the quantum Serre relations are homogeneous in the given generators, there is a $Q^+$-grading on $U_q(\mathfrak{sl}_1^+)$ obtained by assigning to $e_i$ degree $\alpha_i$. We use the terminology weight instead of degree for this grading, and write $wt(u) = \beta$ if $u \in U_q(\mathfrak{sl}_1^+)$ has weight $\beta$.

1.3 PBW basis

Several authors have constructed PBW bases for quantized enveloping algebras (e.g. [26], [24], [22]). It will be convenient for us to use the following construction:

\begin{align*}
X_1 &= e_1, & X_2 &= e_1e_2 - q^{-1}e_2e_1, \\
X_4 &= e_2, & X_5 &= e_2e_3 - q^{-1}e_3e_2, \\
X_6 &= e_3, & X_3 &= e_1X_5 - q^{-1}X_5e_1.
\end{align*}

Then, the set of monomials $\{X_i^{b_1} \cdots X_6^{b_6} \mid b_i \in \mathbb{N}\}$ is a linear basis of $U_q(\mathfrak{sl}_1^+)$. Notice that all $X_i$ are weight vectors.

1.4 Ring theoretical properties of $U_q(\mathfrak{sl}_1^+)$

It was seen in [22] (see also [6] I.6.10 and references therein) that $U_q(\mathfrak{sl}_1^+)$ is an iterated skew polynomial ring. In terms of the PBW basis described above, we have

\begin{align*}
U_q(\mathfrak{sl}_1^+) &= \mathbb{K}[X_1][X_2; \tau_2][X_3; \tau_3][X_4; \tau_4][X_5; \tau_5][X_6; \tau_6, \delta_2, \delta_3, \delta_4, \delta_5, \delta_6], \quad (3)
\end{align*}

with $\tau_i$ a $\mathbb{K}$-algebra automorphism and $\delta_i$ a $\mathbb{K}$-linear (left) $\tau_i$-derivation of the appropriate subalgebra. Thus $U_q(\mathfrak{sl}_1^+)$ is a Noetherian domain.

So that we can easily compute in $U_q(\mathfrak{sl}_1^+)$, and also because this information will be needed in Section 4.4, we specify these automorphisms and skew-derivations below by giving their values on the $X_j$ ($\delta_i(X_j) = 0$ unless otherwise specified):

\begin{align*}
\tau_2(X_1) &= q^{-1}X_1, \\
\tau_3(X_1) &= q^{-1}X_1, & \tau_3(X_2) &= q^{-1}X_2, \\
\tau_4(X_1) &= qX_1, & \tau_4(X_2) &= q^{-1}X_2, & \tau_4(X_3) &= X_3, & \delta_4(X_1) &= -qX_2 \\
\tau_5(X_1) &= qX_1, & \tau_5(X_2) &= X_2, & \tau_5(X_3) &= q^{-1}X_3, \\
\tau_5(X_4) &= q^{-1}X_4, & \delta_5(X_1) &= -qX_3, & \delta_5(X_2) &= -qX_3X_4 \\
\tau_6(X_1) &= X_1, & \tau_6(X_2) &= qX_2, & \tau_6(X_3) &= q^{-1}X_3 \\
\tau_6(X_4) &= qX_4, & \tau_6(X_5) &= q^{-1}X_5, & \delta_6(X_2) &= -qX_3, & \delta_6(X_4) &= -qX_5 \\
\end{align*}

Furthermore, for $4 \leq i \leq 6$, $\tau_i \circ \delta_i = q^{-2}\delta_i \circ \tau_i$, so the theory of deleting derivations of [9] applies to $U_q(\mathfrak{sl}_1^+)$. In particular, as shown in [22], all prime ideals of $U_q(\mathfrak{sl}_1^+)$ are completely prime.
1.5 Normal elements and the center

The elements $a, b \in U_q(\mathfrak{sl}_3^+)$ are said to $q$-commute if there is an integer $\lambda$ such that $ab = q^\lambda ba$. If $u$ $q$-commutes with the generators $e_i$ of $U_q(\mathfrak{sl}_3^+)$ then we say that $u$ is $q$-central. Clearly, $q$-central elements are normal and Caldero [3] and by Caldero [7, 8] of $\lambda$, $\Delta_1 = \Delta_2 = \Delta_3$. Take $\phi = \theta_1^\lambda \theta_2^\lambda \theta_3^\lambda$ with $\theta_j = z_j^\gamma z_j^\beta$ central. Since $
abla = \Delta_1$, $\lambda_2 = 0$, we would have obtained an analogous statement with $\Delta_1$ replaced by $\Delta_2$. Conversely, it is clear that all elements of $\Delta_i Z_q(\mathfrak{sl}_3^+)$ are $q$-central, for $c \in \mathbb{N}$ and $i \in \{1, 3\}$, so we have established the following:

**Lemma 1.2.** Let $u \in U_q(\mathfrak{sl}_3^+)$ be normal. Then there exists a central element $z$, a nonnegative integer $c$ and $i \in \{1, 3\}$ such that $u = \Delta_i^c z$.

In terms of the PBW basis we are using, the $\Delta_i$ are given by the formulae (see [7 Sec. 4] or [19] Sec. 4.1) but notice that we have ordered the PBW basis elements differently):

\begin{align*}
\Delta_1 &= X_3, \\
\Delta_2 &= X_2 X_5 - q X_3 X_4, \\
\Delta_3 &= q^2 X_1 X_4 X_6 - q q X_2 X_6 - q q X_1 X_5 + q^2 X_3.
\end{align*}

2 The automorphism group of $U_q(\mathfrak{sl}_3^+)$

In this section we compute the group of algebra automorphisms of $U_q(\mathfrak{sl}_3^+)$ and confirm the conjecture of Andruskiewitsch and Dumas [3] for this case. Let $\text{Aut}_q U_q(\mathfrak{sl}_3^+)$ denote this group. We shall show that $\text{Aut}_q U_q(\mathfrak{sl}_3^+)$ is the semi-direct product of the 3-torus $(\mathbb{K}^*)^3$ and the group of order two generated by the diagram automorphism of $U_q(\mathfrak{sl}_3^+)$. Let $\mathcal{H} = (\mathbb{K}^*)^3$. Each $\lambda = (\lambda_1, \lambda_2, \lambda_3) \in \mathcal{H}$ determines an algebra automorphism $\phi_\lambda$ of $U_q(\mathfrak{sl}_3^+)$ with $\phi_\lambda(e_i) = \lambda_i e_i$ for $i = 1, 2, 3$, with inverse $\phi_\lambda^{-1} = \phi_{-\lambda}$. Hence we think of $\mathcal{H}$ as a subgroup of $\text{Aut}_q U_q(\mathfrak{sl}_3^+)$ via this correspondence. There is also a diagram automorphism $\eta$ of $U_q(\mathfrak{sl}_3^+)$ arising from the symmetry of the Dynkin diagram of type $A$, and defined on the generators by
\( \eta(e_i) = e_{i-1} \). Notice that \( \eta^2 \) is the identity morphism and that, up to nonzero scalars, \( \eta \) permutes \( \Delta_1 \) and \( \Delta_3 \), and fixes \( \Delta_2 \). Finally, as is to be expected,
\[
\eta \circ \phi(\lambda_1, \lambda_2, \lambda_3) \circ \eta^{-1} = \phi(\lambda_3, \lambda_2, \lambda_1).
\] (7)

2.1 An \( \mathbb{N} \)-grading on \( U_q(\mathfrak{sl}_2^+) \)

In addition to the weight space decomposition of Section 1.2 \( U_q(\mathfrak{sl}_2^+) \) has an \( \mathbb{N} \)-grading induced by the monoid homomorphism \( a \circ \omega_1 + b \circ \omega_2 + c \circ \omega_3 \mapsto a + b + c \), from \( Q^+ \) to \( \mathbb{N} \). Let
\[
U_q(\mathfrak{sl}_2^+) = \bigoplus_{i \in \mathbb{N}} U_i
\] (8)
be the corresponding decomposition, with \( U_i \) the subspace of homogeneous elements of degree \( i \). In particular, \( U_0 = \mathbb{K} \) and \( U_1 \) is the 3-dimensional space spanned by the generators \( e_1, e_2, e_3 \). For \( t \in \mathbb{N} \) set \( U_{\leq t} = \bigoplus_{i \leq t} U_i \) and define \( U_{\geq t} \) similarly.

We say that the nonzero element \( u \in U_q(\mathfrak{sl}_2^+) \) has degree \( t \), and write \( \deg(u) = t \), if \( u \in U_{\leq t} \setminus U_{\leq t-1} \) (using the convention that \( U_{\leq -1} = \{0\} \)). In such a case, if \( u = \sum_{0 \leq i \leq t} u_i \) with \( u_i \in U_i \) and \( u_t \neq 0 \), we set \( \bar{u} = u_t \). By definition, \( \bar{u} \neq 0 \), \( \bar{u} \bar{v} \) and \( \deg(uv) = \deg(u) + \deg(v) \) for \( u, v \neq 0 \), as \( U_q(\mathfrak{sl}_2^+) \) is a domain.

The hypotheses of [18, Prop. 3.2] can be slightly weakened to yield, with essentially the same proof, the following proposition.

Proposition 2.1. Let \( A = \bigoplus_{i \in \mathbb{N}} A_i \) be an \( \mathbb{N} \)-graded \( \mathbb{K} \)-algebra with \( A_0 = \mathbb{K} \) which is generated as an algebra by \( A_1 = \mathbb{K} x_1 \oplus \cdots \oplus \mathbb{K} x_n \). Assume that for each \( i \in \{1, \ldots, n\} \) there exist \( 0 \neq a \in A \) and a scalar \( q_i a \neq 1 \) such that \( x_i a = q_i a x_i \). Then, given an algebra automorphism \( \sigma \) of \( A \) and a nonzero homogeneous element \( x \) of degree \( d \), there exist \( y_d \in A_d \setminus \{0\} \) and \( y_{>d} \in A_{>d+1} \) so that
\[ \sigma(x) = y_d + y_{>d}. \]

The algebra \( U_q(\mathfrak{sl}_2^+) \), endowed with the grading just defined, satisfies the conditions of the above proposition. Indeed, the quantum Serre relations involving \( i \) and \( i+1 \) are equivalent to
\[
e_i (e_i e_{i+1} - q^{-1} e_{i+1} e_i) = q (e_i e_{i+1} - q^{-1} e_{i+1} e_i) e_i \]
\[
e_{i+1} (e_i e_{i+1} - q^{-1} e_{i+1} e_i) = q^{-1} (e_i e_{i+1} - q^{-1} e_{i+1} e_i) e_{i+1}.
\] (9) (10)
Thus we have an analogue of [18, Cor. 3.3]:

Corollary 2.2. Let \( \sigma \in \text{Aut}_\mathbb{K} U_q(\mathfrak{sl}_2^+) \) and \( x \in U_d \setminus \{0\} \). Then \( \sigma(x) = y_d + y_{>d} \), for some \( y_d \in U_d \setminus \{0\} \) and \( y_{>d} \in U_{>d+1} \).

2.2 Invariance of the normal elements

Proposition 2.3. Given \( \sigma \in \text{Aut}_\mathbb{K} U_q(\mathfrak{sl}_2^+) \), there exist \( \epsilon \in \{0,1\} \) and nonzero scalars \( \mu_1 \) and \( \mu_3 \) such that \( \eta^\epsilon \circ \sigma(\Delta_i) = \mu_i \Delta_i \) for \( i = 1,3 \).

Proof. Since \( \Delta_1 \) is normal, so is \( \sigma(\Delta_1) \). By Lemma 2.2 there exist \( \epsilon \in \{1,3\} \), \( c \in \mathbb{N} \) and a central element \( z \) such that \( \sigma(\Delta_1) = \Delta_1^c z \). Furthermore, \( c \geq 1 \) as \( \Delta_1 \) is not central. It follows from Corollary 2.2 that \( c = 1 \), as \( \deg(\Delta_j) = 3 \) for \( j = 1,3 \). Thus,
\[
\sigma(\Delta_1) = \Delta_1 z.
\] (11)
If we repeat the argument above replacing \( \Delta_1 \) by \( \Delta_i \) and \( \sigma \) by its inverse, apply \( \sigma^{-1} \) to equation (11) and compute degrees, we find that \( z \) is a (nonzero) scalar. This same result can be reached by noticing that \( \Delta_1 \) generates a (completely) prime ideal of \( U_q(\mathfrak{sl}_2^+) \), and so the normal element \( \sigma(\Delta_1) \) must also generate such an ideal. This, as well, implies that \( z \in \mathbb{K}^* \). Similarly, \( \sigma(\Delta_3) \) is a nonzero scalar multiple of \( \Delta_3 \) for some \( j \in \{1,3\} \) with \( j \neq i \). If \( i = 1 \) and \( j = 3 \), we take \( \epsilon = 0 \); if \( i = 3 \) and \( j = 1 \), we take \( \epsilon = 1 \). In either case, as \( \eta \) interchanges \( \Delta_1 \) and \( \Delta_3 \), \( \eta^\epsilon \circ \sigma \) fixes \( \Delta_1 \) and \( \Delta_3 \) up to scalars.

\[ \square \]
We have as a corollary of Proposition 2.3 that any algebra automorphism of $U_q(\mathfrak{sl}_3^\perp)$ acts on the central element $z_1 = \Delta_1 \Delta_3$ as multiplication by a scalar. Since the center of $U_q(\mathfrak{sl}_3^\perp)$ is $\mathbb{K}[z_1, z_2]$ with $z_2 = \Delta_2$ and any $\sigma \in \text{Aut}_K U_q(\mathfrak{sl}_3^\perp)$ induces an automorphism of this polynomial algebra, it is not hard to see that $(\sigma(\Delta_2)) = \lambda \Delta_2 + p(z_1)$ with $\lambda \in \mathbb{K}^*$ and $p(z_1)$ a polynomial in $z_1$ with zero constant term (by Corollary 2.2). Unfortunately, this is not quite sufficient. In fact, if – as we claim – $\text{Aut}_K U_q(\mathfrak{sl}_3^\perp)$ is the semi-direct product of $\mathcal{H}$ and the order 2 group generated by $\eta$, it must be that $p(z_1) = 0$. Our next result, preceded by a preparatory lemma, provides this step.

Lemma 2.4. For any $\sigma \in \text{Aut}_K U_q(\mathfrak{sl}_3^\perp)$ there exist $\epsilon \in \{0, 1\}$ and $\lambda \in \mathcal{H}$ such that

$$(\phi_{\lambda} \circ \eta^\epsilon \circ \sigma - \text{Id}) (U_1) \subseteq U_{\geq 2}.$$  \hfill (12)

Proof. By Proposition 2.3 $\eta^\epsilon \circ \sigma(\Delta_1) = t \Delta_1$, for some $t \in \mathbb{K}^*$. Let $\psi = \eta^\epsilon \circ \sigma$. By Corollary 2.2 there exist $u_1 \in U_1 \setminus \{0\}$ and $u_{>1} \in U_{>2}$ such that $\psi(e_1) = u_1 + u_{>1}$. If now we apply $\psi$ to the relation $e_1 \Delta_1 = q \Delta_1 e_1$ and equate the homogeneous terms of degree 4, we obtain $u_1 \Delta_1 = q \Delta_1 u_1$. As $u_1$ is a linear combination of $e_1$, $e_2$ and $e_3$, Theorem 1.4(a) implies that $u_1 = \lambda_1 e_1$ for some $\lambda_1 \in \mathbb{K}^*$. Analogously, $\psi(e_i) = \lambda_i e_i + w_i$ for $\lambda_i \in \mathbb{K}^*$ and $w_i \in U_{>2}$, $i = 2, 3$. Let $\tilde{\lambda} = (\lambda_1^{-1}, \lambda_2^{-1}, \lambda_3^{-1})$. Then $(\tilde{\phi}_{\lambda} \circ \psi - \text{Id}) (U_1) \subseteq U_{>2}$, since $\tilde{\phi}_{\lambda} (U_{>=q} 2) \subseteq U_{>2}$. \hfill \Box

Theorem 2.5. Let $\sigma$ be an algebra automorphism of $U_q(\mathfrak{sl}_3^\perp)$. Then there is a nonzero scalar $\mu_2 \in \mathbb{K}^*$ such that $\sigma(\Delta_2) = \mu_2 \Delta_2$.

Proof. Since the statement of the theorem is valid for the automorphisms $\eta$ and $\phi_{\lambda}$, $\tilde{\lambda} \in \mathcal{H}$, we can assume by the previous lemma that $(\sigma - \text{Id}) (U_1) \subseteq U_{>2}$. Thus, by [2 Lem. 1.4.2], there exist $d_l \in D(U_q(\mathfrak{sl}_3^\perp))$, $l \geq 0$, such that

$$\sigma(\Delta_2) = \sum_{l \geq 0} d_l(\Delta_2),$$  \hfill (13)

where $D(U_q(\mathfrak{sl}_3^\perp))$ is the $\mathbb{K}$-subalgebra of $\text{End}_K U_q(\mathfrak{sl}_3^\perp)$ generated by the $\mathbb{K}$-derivations of $U_q(\mathfrak{sl}_3^\perp)$. Furthermore, $d_0(\Delta_2) = \Delta_2$ and $d_l(\Delta_2)$ is the homogeneous component of $\sigma(\Delta_2)$ of degree $l + 4$, as $\Delta_2$ is homogeneous of degree 4.

In Section 3 it will be shown (see Theorem 2.8) that $\delta(\Delta_2)$ is in the ideal of $U_q(\mathfrak{sl}_3^\perp)$ generated by $\Delta_2$, for any derivation $\delta$ of $U_q(\mathfrak{sl}_3^\perp)$, and this will be done independently of Theorem 2.6. Therefore, $d(\Delta_2) \in \langle \Delta_2 \rangle$ for all $d \in D(U_q(\mathfrak{sl}_3^\perp))$ and thus $\sigma(\Delta_2) \in \langle \Delta_2 \rangle$, by [13]. This same reasoning applies to $\sigma^{-1}$, so that $(\sigma(\Delta_2)) = \langle \Delta_2 \rangle$. Since $\Delta_2$ is central, it is then obvious that there exists a unit $\mu_2 \in U_q(\mathfrak{sl}_3^\perp)$ such that $\sigma(\Delta_2) = \mu_2 \Delta_2$. However, the set of units of $U_q(\mathfrak{sl}_3^\perp)$ is precisely $\mathbb{K}^*$, so that $\mu_2 \in \mathbb{K}^*$, as desired. \hfill \Box

2.3 Determination of $\text{Aut}_K U_q(\mathfrak{sl}_3^\perp)$

We are now ready to compute the group of algebra automorphisms of $U_q(\mathfrak{sl}_3^\perp)$.

Proposition 2.6. Let $\psi$ be an algebra automorphism of $U_q(\mathfrak{sl}_3^\perp)$ with the property that $(\psi - \text{Id}) (U_1) \subseteq U_{>2}$. Then $\psi$ is the identity morphism.

Proof. By the hypothesis on $\psi$, there exist $u_i \in U_{\geq (\deg(X_i) + 1)}$ such that

$$\psi(X_i) = X_i + u_i$$

for all $1 \leq i \leq 6$. Also, by Proposition 2.3 and Theorem 2.3, we know that $\psi(\Delta_j) = \Delta_j$ for $j = 1, 2, 3$. In particular, $u_1 = 0$ as $\Delta_1 = X_3$. Define, for $1 \leq i \leq 6$, $d_i = \deg(\psi(X_i))$. It is enough to prove that $d_1 = d_4 = d_6 = 1$ as $X_1 = e_1$, $X_4 = e_2$ and $X_6 = e_3$ generate $U_q(\mathfrak{sl}_3^\perp)$ as an algebra. Let us assume, by way of contradiction, that this is not the case. Thus $d_1 + d_4 + d_6 > 3$.

Notice that by Corollary 2.2, $d_i \geq \deg(X_i)$ for all $i$. Looking at the expression (5) of $\Delta_2$ in the PBW basis and using the fact that $\psi$ fixes $\Delta_2$, we can conclude that

$$d_2 + d_5 = d_3 + d_4 = 3 + d_4.$$  \hfill (14)
Also, since $X_2$ is a linear combination of $X_1X_4$ and $X_4X_1$, we have $2 \leq d_2 \leq d_1 + d_4$ and similarly $2 \leq d_5 \leq d_4 + d_6$. Therefore,

\begin{align*}
    d_1 + d_4 + d_6 &\geq \max\{d_2 + d_6, d_1 + d_5\} \quad \text{and} \\
    d_1 + d_4 + d_6 &> 3 = d_3.
\end{align*}

(15) \hspace{1cm} (16)

Since $\psi$ fixes the degree 3 element $\Delta_3$, the inequality in (15) cannot be strict, by (6). Hence either $d_1 + d_4 + d_6 = d_2 + d_6$ or $d_1 + d_4 + d_6 = d_1 + d_5$. These cases are symmetric and we can assume without loss of generality that $d_1 + d_4 + d_6 = d_2 + d_6$. Thus, using (14), $d_1 + d_4 = d_2 + d_4 - d_5$ and $d_1 + d_5 = 3$. Since $d_1 \geq 1$ and $d_5 \geq 2$, it must be $d_1 = 1$ and $d_5 = 2$. In other words, $u_1 = 0 = u_5$ and $\psi$ fixes $X_1$ and $X_5$.

Now we apply $\psi$ to the defining equation (3) of $\Delta_2$ to obtain

$$u_2X_5 = qX_3u_4$$

(17)

similarly, the relation $X_5X_4 = q^{-1}X_4X_5$ yields

$$X_5u_4 = q^{-1}u_4X_5$$

(18)

after applying $\psi$; finally, $\psi$ applied to equation (6) gives

$$\hat{q}(X_1X_4u_6 + X_1u_4X_6 + X_1u_4u_6) = q(X_2u_6 + u_2X_6 + u_2u_6).$$

(19)

By (17), $u_2 = 0 \iff u_4 = 0$ and if this occurs then $\hat{q}X_1X_4u_6 = qX_2u_6$, on account of (15). If $u_6 \neq 0$ the latter implies $\hat{q}X_1X_4 = qX_2$, which is false as the $X_i$ form a PBW basis. Thus $u_6 = 0$ and $d_1 + d_4 + d_6 = 3$, contradicting our assumption. Hence $u_4, u_2 \neq 0$. Likewise, if $u_6 = 0$ then (19) implies $\hat{q}X_1u_4 = qu_2$ and then by (17) followed by (13) we get $\hat{q}X_1X_5u_4 = qX_3u_4$, which is again a contradiction as $u_4 \neq 0$. Hence $d_2 = \deg(u_2) \geq 3$, $d_4 = \deg(u_4) \geq 2$ and $d_6 = \deg(u_6) \geq 2$.

To obtain the final contradiction, we just have to look at the degrees occurring in (19). Indeed, $\deg(X_1X_4u_6) = 2 + d_6 < 1 + d_4 + d_6 = \deg(X_1u_4u_6)$; similarly, $\deg(X_1u_4X_6) < \deg(X_1u_4u_6)$, $\deg(X_2u_6) < \deg(u_2u_6)$ and $\deg(u_2X_6) < \deg(u_2u_6)$. Therefore we must have $\deg(X_1u_4u_6) = \deg(u_2u_6)$ and, using the notation introduced in section 2.1

$$\hat{q}X_1u_4u_6 = q\bar{u}_2\bar{u}_6,$$

(20)

so that $\hat{q}X_1u_4 = q\bar{u}_2$. Multiplying this equation on the right by $X_5$, using relations $\bar{u}_2X_5 = qX_3\bar{u}_4$ and $\bar{u}_4X_5 = qX_3\bar{u}_4$, arising from (17) and (18), respectively, we obtain the equality $\hat{q}X_1X_5u_4 = qX_3\bar{u}_4$, which leads to the contradiction $\hat{q}X_1X_5 = qX_3$. The contradiction was derived from the assumption that $d_1 + d_4 + d_6 > 3$. Consequently $d_1 = d_4 = d_6 = 1$ and $\psi$ is the identity on $U_q(\mathfrak{sl}_3\mathfrak{l})$.

At last, we prove our main result of this section, which gives a positive answer to the conjecture of Andruskiewitsch and Dumas [3] for $U_q(\mathfrak{sl}_3\mathfrak{l})$.

**Theorem 2.7.** $\text{Aut}_K U_q(\mathfrak{sl}_3\mathfrak{l})$ is isomorphic to the semi-direct product of the 3-torus $\mathcal{H}$ and the group of order 2 generated by the diagram automorphism $\eta$ of $U_q(\mathfrak{sl}_3\mathfrak{l})$.

**Proof.** Let $\sigma \in \text{Aut}_K U_q(\mathfrak{sl}_3\mathfrak{l})$. By Lemma 2.4 and Proposition 2.6 there exist $\epsilon \in \{0, 1\}$ and $\lambda \in \mathcal{H}$ such that $\phi^{\lambda} \circ \eta^{-\epsilon} \circ \sigma$ is the identity on $U_q(\mathfrak{sl}_3\mathfrak{l})$. Thus,

$$\sigma = \eta^{-\epsilon} \circ \phi^{\mu},$$

(21)

where $\mu = \lambda^{-1}$. Furthermore, the above expression is easily seen to be unique, so the theorem follows from (7).
3 Derivations of $U_q(\mathfrak{sl}_1^+)$

The aim of this section is to describe the Lie algebra of $\mathbb{K}$-derivations of $U_q(\mathfrak{sl}_1^+)$. In particular, we show that the Hochschild cohomology group of degree 1 of $U_q(\mathfrak{sl}_1^+)$ is a free module of rank 3 over the center of $U_q(\mathfrak{sl}_1^+)$. Our method consists of using previous results of Osborn and Passman, \cite{Osborn_Passman}, on the theory of deleting derivations of a certain quantum torus (in which $U_q(\mathfrak{sl}_1^+)$ embeds) to transfer information on the derivations of the algebra of quantum matrices and of some related algebras.

3.1 The deleting derivations algorithm in $U_q(\mathfrak{sl}_1^+)$

It follows from Section 1.4 that the theory of deleting derivations (see \cite{Osborn_Passman}) can be applied to the iterated Ore extension $R := U_q(\mathfrak{sl}_1^+) = \mathbb{K}[X_1, \ldots, X_6; \tau_6, \delta_6]$. The corresponding deleting derivations algorithm constructs, for each $r \in \{6, 5, 4, 3, 2\}$, a family $(X_i^{(r)})_{i \in \{1, \ldots, 6\}}$ of elements of $\text{Frac}(U_q(\mathfrak{sl}_1^+))$, defined as follows (see \cite{Osborn_Passman} Sec. 3.2):

1. $X_1^{(6)} = X_1, X_2^{(6)} = X_2 - q\hat{q}^{-1}X_3X_6^{-1}, X_3^{(6)} = X_3, X_4^{(6)} = X_4 - q\hat{q}^{-1}X_5X_6^{-1}, X_5^{(6)} = X_5$ and $X_6^{(6)} = X_6$.

In order to simplify the notations, we set $Y_i := X_i^{(6)}$ for all $i \in \{1, \ldots, 6\}$.

2. $X_1^{(5)} = Y_1 - q\hat{q}^{-1}Y_3Y_5^{-1}, X_2^{(5)} = Y_2 - qY_3Y_4Y_5^{-1}, X_3^{(5)} = Y_3, X_4^{(5)} = Y_4, X_5^{(5)} = Y_5$ and $X_6^{(5)} = Y_6$.

In order to simplify the notations, we set $Z_i := X_i^{(5)}$ for all $i \in \{1, \ldots, 6\}$.

3. $X_1^{(4)} = Z_1 - q\hat{q}^{-1}Z_2Z_4^{-1}, X_2^{(4)} = Z_2, X_3^{(4)} = Z_3, X_4^{(4)} = Z_4, X_5^{(4)} = Z_5$ and $X_6^{(4)} = Z_6$.

In order to simplify the notations, we set $T_i := X_i^{(4)}$ for all $i \in \{1, \ldots, 6\}$.

4. For all $r \in \{2, 3\}$ and $i \in \{1, \ldots, 6\}$, $X_i^{(r)} = T_i$.

As in \cite{Osborn_Passman}, for all $r \in \{6, 5, 4, 3, 2\}$, we denote by $R^{(r)}$ the subalgebra of $\text{Frac}(R)$ generated by the elements $X_i^{(r)}$ for $i \in \{1, \ldots, 6\}$. Also, we denote by $\overline{R}$ the subalgebra of $\text{Frac}(R)$ generated by the indeterminates obtained at the end of this algorithm, that is, $\overline{R} = R^{(2)}$ is the subalgebra of $\text{Frac}(R)$ generated by the $T_i$, for each $i \in \{1, \ldots, 6\}$. Finally, by convention, we set $R^{(7)} := R$.

Recall from \cite{Osborn_Passman} Thé. 3.2.1 that, for all $r \in \{6, 5, 4, 3, 2\}$, $R^{(r)}$ can be presented as an iterated Ore extension over $\mathbb{K}$, with the generators $X_i^{(r)}$ adjoined in lexicographic order. Thus the ring $R^{(r)}$ is a Noetherian domain. Observe in particular that we have (with some abuse of notation):

$$R^{(6)} = \mathbb{K}[Y_1][Y_2; \tau_2][Y_3; \tau_3][Y_4; \tau_4][Y_5; \tau_5][Y_6; \tau_6],$$  

$$R^{(5)} = \mathbb{K}[Z_1][Z_2; \tau_2][Z_3; \tau_3][Z_4; \tau_4][Z_5; \tau_5][Z_6; \tau_6],$$  

$$R^{(4)} = R^{(3)} = R^{(2)} = \mathbb{K}[T_1][T_2; \tau_2][T_3; \tau_3][T_4; \tau_4][T_5; \tau_5][T_6; \tau_6].$$  

Let $N \in \mathbb{N}^*$ and let $\Lambda = (\Lambda_{i,j})$ be a multiplicatively antisymmetric $N \times N$ matrix over $\mathbb{K}^*$, that is, $\Lambda_{i,i} = 1$ and $\Lambda_{i,j} = -\Lambda_{j,i}$ for all $i, j \in \{1, \ldots, N\}$. We denote by $\mathbb{K}_\Lambda[T_1, \ldots, T_N]$ the corresponding quantum affine space; that is, the $\mathbb{K}$-algebra generated by the $N$ indeterminates $T_1, \ldots, T_N$ subject to the relations $T_iT_j = \Lambda_{i,j}T_jT_i$ for all $i, j \in \{1, \ldots, N\}$. Next, we denote by $P(\Lambda)$ the quantum torus associated to the quantum affine space $\mathbb{K}_\Lambda[T_1, \ldots, T_N]$, which is the
localisation of $\mathbb{K}_A[T_1, \ldots, T_N]$ with respect to the multiplicative system generated by the $T_i$. For 
$\gamma = (\gamma_1, \ldots, \gamma_N) \in \mathbb{Z}^N$, set $T^\gamma := T_1^{\gamma_1} \cdots T_N^{\gamma_N}$. Note that the monomials $(T^\gamma)_{\gamma \in \mathbb{Z}^N}$ form a PBW basis of $P(A)$.

It follows from [9] Prop. 3.2.1 that $\overline{R}$ is a quantum affine space over $\mathbb{K}$ in the indeterminates 
$T_1, \ldots, T_N$. We denote by $P(A)$ the corresponding quantum torus. In the present case, the matrix 
that defines the quantum affine space $\overline{R}$ is the following:

$$
\Lambda = \begin{pmatrix}
1 & q & q^{-1} & q^{-1} & 1 \\
q^{-1} & 1 & q & 1 & q^{-1} \\
q^{-1} & q^{-1} & 1 & 1 & q \\
q & q^{-1} & 1 & 1 & q^{-1} \\
1 & q & q^{-1} & q^{-1} & 1
\end{pmatrix}
$$

For all $r \in \{6, 5, 4, 3, 2\}$, we denote by $S_r$ the multiplicative system generated by the indeterminates $T_i$ with $i \geq r$. Since $T_i = X_i^{(r)}$ for all $i \geq r$, $S_r$ is a multiplicative system of regular 
elements of $R^{(r)}$. Moreover, the $T_i$ with $i \geq r$ are normal in $R^{(r)}$. Hence $S_r$ is an Ore set in $R^{(r)}$ 
and one can form the localisation:

$$A_r := R^{(r)} S_r^{-1}.$$

Clearly, the family $\left((X_1^{(r)})^{\gamma_1}(X_2^{(r)})^{\gamma_2} \cdots (X_6^{(r)})^{\gamma_6}\right)$, with $\gamma_i \in \mathbb{N}$ if $i < r$ and $\gamma_i \in \mathbb{Z}$ otherwise, is a 
PBW basis of $A_r$. Further, recall from [9] Thé. 3.2.1 that $\Sigma_r := \{T_r^k \mid k \in \mathbb{N}\}$ is an Ore set in both $R^{(r)}$ and $R^{(r+1)}$, and that 
$R^{(r)} \Sigma_r^{-1} = R^{(r+1)} \Sigma_r^{-1}.$

Hence we get the following result.

**Lemma 3.1.** For all $r \in \{6, 5, 4, 3, 2\}$, we have $A_r = A_{r+1} \Sigma_r^{-1}$ with the convention that $A_7 := R = U_q(\mathfrak{sl}_1^\mathbb{C})$.

Now, observe that $T_1$ is a normal element in $A_2$, so that one can form the Ore localisation 
$A_1 := A_2 \Sigma_1^{-1}$, where $\Sigma_1$ is the multiplicative system generated by $T_1$. Naturally, $A_1$ is the quantum 
torus associated to the quantum affine space $\overline{R}$. Hence we also denote $A_1$ by $P(A)$, and we deduce 
from Lemma 3.1 the following tower of algebras:

$$A_7 = R \subset A_6 = A_7 \Sigma_6^{-1} \subset A_5 = A_6 \Sigma_5^{-1} \subset A_4 = A_5 \Sigma_4^{-1}$$

$$\subset A_3 = A_4 \Sigma_3^{-1} \subset A_2 = A_3 \Sigma_2^{-1} \subset A_1 := P(A).$$

### 3.2 Action of the deleting derivations algorithm on the normal elements

Observe that the formulas expressing the $Y_i$ in terms of the $X_i$ can be rewritten in order to express 
the $X_i$ in terms of the $Y_i$. In particular, one can easily check that:

$X_1 = Y_1, \quad X_2 = Y_2 + q\bar{q}^{-1}Y_5Y^{-1}_6, \quad X_3 = Y_3, \quad X_4 = Y_4 + q\bar{q}^{-1}Y_5Y^{-1}_6, \quad X_5 = Y_5 \text{ and } X_6 = Y_6.$

In a similar manner, one can express the $Y_i$ in terms of the $Z_i$, and the $Z_i$ in terms of the $T_i$. 
More precisely, we have:

$Y_1 = Z_1 + q\bar{q}^{-1}Z_3Z_5^{-1}, \quad Y_2 = Z_2 + qZ_3Z_4Z_5^{-1}, \quad Y_3 = Z_3, \quad Y_4 = Z_4, \quad Y_5 = Z_5 \text{ and } Y_6 = Z_6$

and

$Z_1 = T_1 + q\bar{q}^{-1}T_2T_4^{-1}, \quad Z_2 = T_2, \quad Z_3 = T_3, \quad Z_4 = T_4, \quad Z_5 = T_5 \text{ and } Z_6 = T_6.$

Using these formulas, one can express the three normal elements $\Delta_1, \Delta_2$ and $\Delta_3$ defined in 
Section 15 in terms of the $Y_i$, or in terms of the $Z_i$, or in terms of the $T_i$. Indeed, straightforward 
computations lead to the following results.

**Lemma 3.2.**

1. $\Delta_1 = X_3 = Y_3 = Z_3 = T_3.$

2. $\Delta_2 = X_2X_5 - qX_3X_4 = Y_2Y_5 - qY_3Y_4 = Z_2Z_5 = T_2T_5.$
\[
\Delta_3 = q^2X_1X_4X_6 - q\hat{q}X_2X_6 - q\hat{q}X_1X_5 + q^2X_3 \\
= q^2Y_1Y_4Y_6 - q\hat{q}Y_2Y_6 \\
= q^2Z_1Z_4Z_6 - q\hat{q}Z_2Z_6 \\
= q^2T_1T_4T_6
\]

### 3.3 Centers of the algebras \(A_i\)

First, recall that the center of \(U_q(\mathfrak{sl}_1^+) = A_7\) has been computed by Alev and Dumas [3] and by Caldero [7, 8], who have shown that this is the polynomial algebra \(K[z_1, z_2]\), where \(z_1 = \Delta_1\Delta_3\) and \(z_2 = \Delta_2\).

On the other hand, the center of the quantum torus \(A_4 = P(\Lambda)\) is easy to compute. Indeed, it is well known (see for instance [12]) that it is a Laurent polynomial ring over \(K\), and that it is generated by the monomials \(T_1^{\gamma_1}T_2^{\gamma_2} \cdots T_6^{\gamma_6}\), with \(\gamma_i \in \mathbb{Z}\), that are central. Easy computations show that such a monomial is central if and only if \(\gamma_1 = \gamma_4 = \gamma_6 = \gamma_3\) and \(\gamma_2 = \gamma_5\). Hence, we deduce from Lemma 3.2 that the center of \(P(\Lambda)\) is the Laurent polynomial ring over \(K\) generated by \(z_1\) and \(z_2\), that is:

\[
Z(P(\Lambda)) = Z(A_4) = K[z_1^\pm, z_2^\pm].
\]

It will be convenient to denote by \(\mathcal{F}\) the set of all \(\gamma \in \mathbb{Z}^6\) such that \(T^\gamma \in Z(P(\Lambda))\), that is:

\[
\mathcal{F} = \{\gamma \in \mathbb{Z}^6 \mid \gamma_1 = \gamma_4 = \gamma_6 = \gamma_3\text{ and }\gamma_2 = \gamma_5\}.
\]  

In the sequel we will also need to know the center of \(A_4\). Recall that \(A_4\) is the localisation of the quantum affine space \(R^{(4)} = \hat{R}\) at the multiplicative system generated by \(T_4, T_5\) and \(T_6\). In particular, the monomials \((T_1^{\gamma_1}T_2^{\gamma_2} \cdots T_6^{\gamma_6})\), with \(\gamma_i \in \mathbb{N}\) if \(i \leq 3\) and \(\gamma_i \in \mathbb{Z}\) otherwise, form a linear basis of \(A_4\). The argument used above to compute the center of \(P(\Lambda)\) also works for \(A_4\), with the additional restrictions that \(\gamma_i \geq 0\) for \(i \leq 3\). So we have the following result.

**Lemma 3.3.**

1. \(Z(A_4) = Z(A_7) = K[z_1, z_2]\).

2. \(Z(A_1) = K[z_1^\pm, z_2^\pm]\).

### 3.4 Derivations of \(U_q(\mathfrak{sl}_4^+)\)

Our aim in this section is to investigate the Lie algebra of \(K\)-derivations of \(U_q(\mathfrak{sl}_4^+)\), which we denote by \(\text{Der}(U_q(\mathfrak{sl}_4^+))\).

Let \(D\) be a derivation of \(U_q(\mathfrak{sl}_4^+) = A_7\). It follows from Lemma 3.1 that \(D\) extends (uniquely) to a derivation of each of the algebras in the tower

\[
A_7 \subseteq A_6 \subseteq \cdots \subseteq A_2 \subseteq A_1 = P(\Lambda)
\]

In particular, \(D\) extends to a derivation of the quantum torus \(P(\Lambda)\). So it follows from [20, Cor. 2.3] that \(D\) can be written as

\[
D = \text{ad}_x + \theta,
\]

where \(x \in P(\Lambda)\) and, in the terminology of [20], \(\theta\) is a central derivation of \(P(\Lambda)\), that is, \(\theta(T_i) = \mu_i T_i\) with \(\mu_i \in Z(P(\Lambda)) = K[z_1^\pm, z_2^\pm]\).

Since the monomials \((T^\gamma)_{\gamma \in \mathbb{Z}^6}\) form a PBW basis of \(P(\Lambda)\), one can write:

\[
x = \sum_{\gamma \in \mathcal{E}} c_\gamma T^\gamma,
\]

where \(\mathcal{E}\) is a finite subset of \(\mathbb{Z}^6\) and \(c_\gamma \in K\). Moreover, since \(\text{ad}_x = \text{ad}_{x+z}\) for all \(z \in Z(P(\Lambda))\), it can be assumed that no monomial \(T^\gamma\), with \(\gamma \in \mathcal{E}\), belongs to \(Z(P(\Lambda))\), i.e., one can assume
that $\mathcal{E} \cap \mathcal{F} = \emptyset$. Furthermore, by Lemmas 3.2 and 3.3 we can write, for each $i \in \{1, \ldots, 6\}$, $\mu_i$ as follows:

$$\mu_i = \sum_{\gamma \in \mathcal{F}} \mu_{i, \gamma} T^\gamma,$$

where $\mu_{i, \gamma} \in K$.

**Lemma 3.4.** For all $i \in \{1, 2, 3, 4\}$, we have $x \in A_i$.

**Proof.** We prove this lemma by induction on $i$. The case $i = 1$ is trivial. Hence we assume that $x \in A_{i-1}$ for some $2 \leq i \leq 4$.

It follows that

$$x = \sum_{\gamma \in \mathcal{E}} c_\gamma T^\gamma,$$

where $\mathcal{E}$ is a finite subset of $\{\gamma \in \mathbb{Z}^6 \mid \gamma_1 \geq 0, \ldots, \gamma_{i-2} \geq 0\}$ with $\mathcal{E} \cap \mathcal{F} = \emptyset$. We need to prove that $\gamma_{i-1} \geq 0$.

Let $j \in \{1, \ldots, 6\}$ with $j \neq i - 1$. As we have previously observed, $D$ extends uniquely to a derivation of $A_i$. Hence, since $T_j \in A_i$, we must have $D(T_j) \in A_i$, that is:

$$x T_j - T_j x + \mu_j T_j \in A_i. \quad (28)$$

We set

$$x_+ := \sum_{\gamma \in \mathcal{E}, \gamma_{i-1} \geq 0} c_\gamma T^\gamma,$$

and

$$x_- := \sum_{\gamma \in \mathcal{E}, \gamma_{i-1} < 0} c_\gamma T^\gamma. \quad (29)$$

We shall prove that $x_- = 0$.

First, we deduce from (28) that

$$u := x_- T_j - T_j x_- + \mu_j T_j \in A_i.$$

Next, using the commutation relations between the $T_k$, we get

$$u = \sum_{\gamma \in \mathcal{E}, \gamma_{i-1} < 0} c'_{j, \gamma} c_\gamma T^{\gamma+\varepsilon_j} + \sum_{\gamma \in \mathcal{F}} \mu'_{j, \gamma} T^{\gamma+\varepsilon_j} \quad (30)$$

where $\varepsilon_j$ denotes the $j$-th element of the canonical basis of $\mathbb{Z}^6$, $\mu'_{j, \gamma} = q^\bullet \mu_{j, \gamma}$ for some integer $\bullet$, and $c'_{j, \gamma} \in K$ is defined by

$$x_- T_j - T_j x_- = \sum_{\gamma \in \mathcal{E}, \gamma_{i-1} < 0} c'_{j, \gamma} c_\gamma T^{\gamma+\varepsilon_j}.$$

Observe that since we assume that $\mathcal{E} \cap \mathcal{F} = \emptyset$, we have:

for all $\gamma \in \mathcal{E}$ and all $\gamma' \in \mathcal{F}$, $\gamma + \varepsilon_j \neq \gamma' + \varepsilon_j$.

Hence, (30) gives the expression of $u$ in the PBW basis of $P(\Lambda)$.

On the other hand, since $u$ belongs to $A_i$, we get that:

$$u = \sum_{\gamma \in \mathcal{E}'} x_\gamma T^\gamma,$$

where $\mathcal{E}'$ is a finite subset of $\{\gamma \in \mathbb{Z}^6 \mid \gamma_1 \geq 0, \ldots, \gamma_{i-1} \geq 0\}$. Comparing the two expressions of $u$ in the PBW basis of $P(\Lambda)$ leads to $c'_{j, \gamma} c_\gamma = 0$ for all $\gamma \in \mathcal{E}$ such that $\gamma_{i-1} < 0$, as $j \neq i - 1$.

Hence, we have

$$x_- T_j - T_j x_- = \sum_{\gamma \in \mathcal{E}, \gamma_{i-1} < 0} c'_{j, \gamma} c_\gamma T^{\gamma+\varepsilon_j} = 0,$$
for all \( j \neq i - 1 \). In other words, \( x_- \) commutes with those \( T_j \) such that \( j \neq i - 1 \).

Now, recall from Lemma 3.2 that \( z_1 = \Delta_1 \Delta_3 = q^2 T_1 T_4 T_6 T_3 \) and \( z_2 = \Delta_2 = T_2 T_5 \) are central in \( P(\Lambda) \), so that \( x_- \) commutes with those \( T_j \) such that \( j \neq i - 1 \), and with \( T_1 T_4 T_6 T_3 \) and \( T_2 T_5 \).

Naturally this implies that \( x_- \) also commutes with \( T_{i-1} \), so that \( x_- \in Z(P(\Lambda)) \). Thus one can write \( x_- \) as follows:

\[
x_- = \sum_{\gamma \in \mathcal{F}} d_\gamma T^\gamma.
\]  

(31)

As \( \mathcal{E} \cap \mathcal{F} = \emptyset \), it follows from (29) and (31) that \( x_- = 0 \), so that \( x = x_+ \in A_i \), as desired. \( \square \)

In particular, it follows from Lemma 3.4 that \( x \in A_4 \). Since the derivation \( D \) of \( U_q(\mathfrak{sl}_1^\dagger) \) extends to a derivation of \( A_4 \), we must have \( D(T_i) \in A_4 \) for all \( i \in \{1, \ldots, 6\} \). Hence

\[
D(T_i) = xT_i - T_i x + \mu_i T_i \in A_4.
\]

Since \( x \in A_4 \), this implies that \( \mu_i T_i \in A_4 \) for all \( i \in \{1, \ldots, 6\} \). On the other hand, recall that \( \mu_i \) is central in \( P(\Lambda) \) and can be written as:

\[
\mu_i = \sum_{\gamma \in \mathcal{F}} \mu_i,\gamma T^\gamma;
\]

where \( \mathcal{F} \) is given by (27). Hence we get

\[
\mu_i T_i = \sum_{\gamma \in \mathcal{F}} \mu'_{i,\gamma} T^{\gamma + \epsilon_i} = \sum_{\gamma = (\gamma_1, \gamma_2) \in \mathbb{Z}^2} \mu'_{i,\gamma} T_{1}^{\gamma_1 + \delta_1} T_{2}^{\gamma_2 + \delta_2} T_{3}^{\gamma_3} T_{4}^{\gamma_4 + \delta_4} T_{5}^{\gamma_5 + \delta_5} T_{6}^{\gamma_6} \in A_4,
\]

where \( \mu'_{i,\gamma} = q^\bullet \mu_i,\gamma \) for some integer \( \bullet \).

Assume now that \( i \neq 2 \). Then, since the monomials \( T^\gamma \), with \( \gamma \in \mathbb{N}^3 \times \mathbb{Z}^3 \), form a PBW basis of \( A_4 \), we get that \( \mu'_{i,\gamma} = 0 \) if either \( \gamma_1 < 0 \) or \( \gamma_2 < 0 \). Hence \( \mu_i \) can be written as follows:

\[
\mu_i = \sum_{\gamma = (\gamma_1, \gamma_2) \in \mathbb{N}^2} c_{i,\gamma} T_{1}^{\gamma_1} T_{2}^{\gamma_2} T_{3}^{\gamma_3} T_{4}^{\gamma_4} T_{5}^{\gamma_5} T_{6}^{\gamma_6}.
\]

In other words, \( \mu_i \in \mathbb{K}[z_1, z_2] \subseteq U_q(\mathfrak{sl}_1^\dagger) \) since \( z_1 = \Delta_1 \Delta_3 = q^2 T_1 T_4 T_6 T_3 \) and \( z_2 = \Delta_2 = T_2 T_5 \) by Lemma 3.2.

Finally, assume that \( i = 2 \). One cannot yet prove that \( \mu_2 \in U_q(\mathfrak{sl}_1^\dagger) = A_7 \). However, one can prove the following weaker result: \( \mu_2 z_2 \in \mathbb{K}[z_1, z_2] \subseteq U_q(\mathfrak{sl}_1^\dagger) \). Indeed, we already know that \( \mu_2 T_2 \in A_4 \). Hence, it follows from Lemma 3.2 that \( \mu_2 z_2 = \mu_2 T_2 T_5 \in A_4 \). Further, \( \mu_2 z_2 \) is central in \( P(\Lambda) \supset A_4 \), so that \( \mu_2 z_2 \in Z(A_4) = \mathbb{K}[z_1, z_2] \), as desired.

To sum up, we have just proved the following result.

**Corollary 3.5.**

1. \( \mu_2 z_2 \in Z(A_4) = \mathbb{K}[z_1, z_2] \subseteq U_q(\mathfrak{sl}_1^\dagger) \).

2. For all \( i \neq 2 \), \( \mu_i \in \mathbb{K}[z_1, z_2] \subseteq U_q(\mathfrak{sl}_1^\dagger) \).

We now have to deal with localisation at elements which are not normal. We do this in three steps.

First, recall from Lemma 3.4 that \( A_4 = A_5 \Sigma_4^{-1} \), where \( \Sigma_4 \) is the multiplicative system generated by \( T_4 = Z_4 \). Recall also that the monomials \( Z_{1}^{\gamma_1} \ldots Z_{6}^{\gamma_6} \), with \( \gamma = (\gamma_1, \ldots, \gamma_6) \in \mathbb{N}^4 \times \mathbb{Z}^2 \), form a PBW basis of \( A_5 \). Of course, this implies that the monomials \( Z_{1}^{\gamma_1} \ldots Z_{6}^{\gamma_6} \), with \( \gamma \in \mathbb{N}^3 \times \mathbb{Z}^3 \), form a PBW basis of \( A_4 \). In order to simplify the notation we set, as usual,

\[
Z^\gamma := Z_{1}^{\gamma_1} Z_{2}^{\gamma_2} \ldots Z_{6}^{\gamma_6}
\]

for all \( \gamma \in \mathbb{N}^3 \times \mathbb{Z}^3 \).
Corollary 3.6. \( \mu_2 Z_2 \in A_5 \).

Proof. We know that \( \mu_2 z_2 \in Z(A_4) = Z(A_5) \), so that \( \mu_2 z_2 \in A_5 \). Now the result follows from the facts that \( z_2 = Z_2 Z_5 \) (Lemma 3.2) and that \( Z_5 \) is invertible in \( A_5 \).

We are now able to prove that \( x \in A_5 \).

Lemma 3.7. 1. \( x \in A_5 \).

2. \( \mu_2 = \mu_1 + \mu_4 \in Z_q(\mathfrak{sl}_4^+), \) where \( Z_q(\mathfrak{sl}_4^+) \) still denotes the center of \( U_q(\mathfrak{sl}_4^+) \).

3. \( D(Z_i) = \text{ad}_x(Z_i) + \mu_i Z_i \) for all \( i \in \{1, \ldots, 6\} \).

Proof. We proceed in three steps.

• Step 1: We prove that \( x \in A_5 \).

It follows from Lemma 3.4 that \( x \) belongs to \( A_4 \), so that \( x \) can be written as follows:

\[ x = \sum_{\gamma \in E} c_\gamma Z^\gamma, \]

where \( E \subseteq \mathbb{N}^3 \times \mathbb{Z}^3 \).

We set

\[ x_+ := \sum_{\gamma \in E, \gamma_4 \geq 0} c_\gamma Z^\gamma, \]

and

\[ x_- := \sum_{\gamma \in E, \gamma_4 < 0} c_\gamma Z^\gamma. \]

Assume that \( x_- \neq 0 \).

We denote by \( B \) the subalgebra of \( A_4 \) generated by the \( Z_j \) with \( j \neq 4, 5^{-1} \) and \( 6^{-1} \). Since \( Z_4 \) \( q \)-commutes with \( Z_5 \) and \( Z_6 \) in \( A_4 \), it is easy to check that \( A_4 \) is a free left \( B \)-module with basis \( (Z_4^a)_{a \in \mathbb{Z}} \), so that one can write:

\[ x_- = \sum_{a=0}^{a_0} b_a Z_4^a \]

with \( a_0 < 0, b_a \in B \) and \( b_{a_0} \neq 0 \). (Observe that this makes sense since we are assuming that \( x_- \neq 0 \).)

As \( D \) extends to a derivation of \( A_5 \), we have \( D(Z_1) \in A_5 \). Recalling from Section 3.2 that \( Z_1 = T_1 + q q^{-1} T_2 T_4^{-1} \), this leads to:

\[ x_- Z_1 - Z_1 x_- + \mu_1 Z_1 + q q^{-1}(\mu_2 - \mu_1 - \mu_4) Z_2 Z_4^{-1} \in A_5. \]

Since \( \mu_1 \in U_q(\mathfrak{sl}_4^+) \) \( \subset A_5 \) by Corollary 3.6 and \( Z_1 \in A_5 \), we get

\[ x_- Z_1 - Z_1 x_- + q q^{-1}(\mu_2 - \mu_1 - \mu_4) Z_2 Z_4^{-1} \in A_5. \] (32)

Then, multiplying this expression by \( Z_4 \) (on the right) yields

\[ (x_- Z_1 - Z_1 x_-) Z_4 + q q^{-1}(\mu_2 - \mu_1 - \mu_4) Z_2 Z_4^{-1} \in A_5. \]

Since \( \mu_1 \) and \( \mu_4 \) belong to \( U_q(\mathfrak{sl}_4^+) \) \( \subset A_5 \) and \( \mu_2 Z_2 \in A_5 \) by Corollary 3.6 this leads to

\[ u := (x_- Z_1 - Z_1 x_-) Z_4 \in A_5, \]

that is:

\[ u = \sum_{a=0}^{a_0} b_a Z_4^a Z_1 Z_4 - \sum_{a=0}^{a_0} Z_1 b_a Z_4^{a+1} \in A_5. \]

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Now, an easy induction shows that
\[ Z_4^{-k}Z_1 = q^{-k}Z_1Z_4^{-k} + q[k]Z_2Z_4^{-k-1} \]
for every positive integer \( k \). Hence we have
\[
 u = \sum_{a=a_0}^{a-1} (q^a b_a Z_1 - Z_1 b_a) Z_4^{a+1} + \sum_{a=a_0}^{a-1} q[-a] b_a Z_2 Z_4^a \in A_5.
\]

Since \( A_5 \) is a free left \( B \)-module with basis \( (Z_4^a)_{a \in \mathbb{N}} \) and \( u \in A_5 \), one can write
\[
 u = \sum_{a=0}^{k} u_a Z_4^a
\]
with \( k \in \mathbb{N} \) and \( u_a \in B \). Comparison of these two expressions of \( u \) in the basis of \( A_4 \) (viewed as a left \( B \)-module) shows that we must have \( b_{a_0} = 0 \), a contradiction. Hence, \( x_- = 0 \) and \( x = x_+ \in A_5 \), as desired.

- **Step 2:** We prove that \( \mu_2 = \mu_1 + \mu_4 \).

Since \( x_- = 0 \), we deduce from (32) that
\[
 (\mu_2 - \mu_1 - \mu_4) Z_2 Z_4^{-1} \in A_5,
\]
that is
\[
 (\mu_2 - \mu_1 - \mu_4) Z_2 \in A_5 Z_4.
\]

Multiplying this by \( Z_5 \) on the right leads to
\[
 (\mu_2 - \mu_1 - \mu_4) z_2 \in A_5 Z_4,
\]
since \( z_2 = Z_2 Z_5 \) by Lemma 3.3 and \( Z_4 Z_5 = q^{-1} Z_5 Z_4 \). We set \( z := (\mu_2 - \mu_1 - \mu_4) z_2 \) and \( J := A_5 Z_4 \), so that \( z \in J \).

It follows from Corollary 3.6 that \( \mu_1, \mu_4 \in \mathbb{K}[z_1, z_2] \) and \( \mu_2 z_2 \in \mathbb{K}[z_1, z_2] \). Hence \( z \in \mathbb{K}[z_1, z_2] \).

We need to prove that \( z = 0 \). Let us write
\[
 z = \sum_{i,j \in \mathbb{N}} a_{i,j} z_1^i z_2^j,
\]
with \( a_{i,j} \in \mathbb{K} \) equal to zero except for a finite number of them. Since \( z_2 = Z_2 Z_5 \) and \( z_1 = q^{-1} q Z_5 Z_1 Z_4 - q q Z_5 Z_2 Z_6 \) (see Lemma 3.2), we get that \( z_1 + q q Z_3 Z_2 Z_6 \in J \). Then, using the fact that \( z_1 \) and \( z_2 \) are central, we obtain that
\[
 z - \sum_{i,j \in \mathbb{N}} q^* (-q q)^i a_{i,j} Z_2^{i+j} Z_3^i Z_5^j Z_6^i \in J,
\]
where \( \bullet \) denotes, as usual, an integer. Since we have already proved that \( z \in J \), this forces
\[
 \sum_{i,j \in \mathbb{N}} q^* (-q q)^i a_{i,j} Z_2^{i+j} Z_3^i Z_5^j Z_6^i \in J. \tag{33}
\]

However, since \( Z_4 q \)-commutes with \( Z_5 \) and \( Z_6 \), every element of \( J \) can be written as
\[
 \sum_{\gamma \in \mathbb{N}^4 \times \mathbb{Z}^2_{\gamma_4 > 0}} c_\gamma Z_1^{\gamma_1} \ldots Z_6^{\gamma_6} \tag{34}
\]
in the PBW basis of \( A_5 \). Identifying the two expressions \([53]\) and \([54]\) leads to \( a_{i,j} = 0 \) for all \( i, j \), so that \( z = 0 \). Thus we have proved that \((\mu_2 - \mu_1 - \mu_4)z_2 = 0 \). Since \( z_2 \neq 0 \), we get \( \mu_2 = \mu_1 + \mu_4 \), as desired. Observe that, since \( \mu_1 \) and \( \mu_4 \) belong to \( Z_q(\mathfrak{sl}_4^\dagger) \) by Corollary \([55]\) this implies that \( \mu_2 \) also belongs to \( Z_q(\mathfrak{sl}_4^\dagger) \).

- **Step 3:** We prove that \( D(Z_i) = \text{ad}_x(Z_i) + \mu_i Z_i \) for all \( i \in \{1, \ldots, 6\} \).

  If \( i > 1 \), this is trivial since \( Z_i = T_i \) and we already know that \( D(T_i) = \text{ad}_x(T_i) + \mu_i T_i \).

  Next, recall that \( Z_1 = T_1 + q\hat{q}^{-1}T_2T_4^{-1} \). Hence, we have
  \[
  D(Z_1) = \text{ad}_x(Z_1) + \mu_1 T_1 + q\hat{q}^{-1}(\mu_2 - \mu_4)T_2T_4^{-1}.
  \]

Since \( \mu_2 = \mu_1 + \mu_4 \), this implies that
  \[
  D(Z_1) = \text{ad}_x(Z_1) + \mu_1 T_1 + q\hat{q}^{-1}\mu_1 T_2T_4^{-1} = \text{ad}_x(Z_1) + \mu_1 Z_1,
  \]

as desired.

We are now able to prove that \( D(z_2) \) belongs to the ideal of \( U_q(\mathfrak{sl}_4^\dagger) \) generated by \( z_2 = \Delta_2 \).

This result is crucial in order to compute the automorphism group of \( U_q(\mathfrak{sl}_4^\dagger) \) (see Theorem \([26]\)).

**Theorem 3.8.** Let \( D \in \text{Der}(U_q(\mathfrak{sl}_4^\dagger)) \). Then there exists \( z \in Z_q(\mathfrak{sl}_4^\dagger) \) such that \( D(z_2) = z z_2 \).

**Proof.** Let \( D \in \text{Der}(U_q(\mathfrak{sl}_4^\dagger)) \). Since \( z_2 = \Delta_2 = Z_2Z_5 \in A_5 \) by Lemma \([52]\), we deduce from Lemma \([54]\) that \( D(z_2) = \text{ad}_x(z_2) + (\mu_2 + \mu_5)z_2 \) with \( x \in A_5 \) and \( \mu_2, \mu_5 \in Z_q(\mathfrak{sl}_4^\dagger) \). Now the result easily follows from the centrality of \( z_2 \) in \( A_5 \).

Having completed the proof of Theorem \([26]\) and thus described the automorphism group of \( U_q(\mathfrak{sl}_4^\dagger) \), we proceed to obtain a complete description of \( \text{Der}(U_q(\mathfrak{sl}_4^\dagger)) \).

Using arguments similar to those in the proof of Lemma \([54]\), one can prove the following two results.

**Lemma 3.9.**

1. \( x \in A_6 \).
2. \( \mu_3 = \mu_1 + \mu_5 \).
3. \( \mu_2 + \mu_5 = \mu_3 + \mu_4 \).
4. \( D(Y_i) = \text{ad}_x(Y_i) + \mu_i Y_i \) for all \( i \in \{1, \ldots, 6\} \).

And also:

**Lemma 3.10.**

1. \( x \in A_7 = U_q(\mathfrak{sl}_4^\dagger) \).
2. \( \mu_3 = \mu_2 + \mu_6 \).
3. \( \mu_5 = \mu_4 + \mu_6 \).
4. \( D(X_i) = \text{ad}_x(X_i) + \mu_i X_i \) for all \( i \in \{1, \ldots, 6\} \).

It is easy to check that we can define three derivations \( D_1, D_4 \) and \( D_6 \) of \( U_q(\mathfrak{sl}_4^\dagger) \) by setting:

\[
\begin{align*}
D_1(X_1) &= X_1 & D_1(X_2) &= X_2 & D_1(X_3) &= X_3 \\
D_4(X_2) &= X_2 & D_4(X_3) &= X_3 & D_4(X_4) &= X_4 & D_4(X_5) &= X_5 \\
D_6(X_3) &= X_3 & D_6(X_5) &= X_5 & D_6(X_6) &= X_6
\end{align*}
\]

and \( D_i(X_j) = 0 \) otherwise.

Then it follows from Lemmas \([54]\), \([55]\) and \([56]\) that any derivation \( D \) of \( U_q(\mathfrak{sl}_4^\dagger) \) can be written as follows:

\[ D = \text{ad}_x + \mu_1 D_1 + \mu_4 D_4 + \mu_6 D_6, \]

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with \( x \in U_q(\mathfrak{sl}_1^+) \) and \( \mu_1, \mu_4, \mu_6 \in Z_q(\mathfrak{sl}_1^+) \).

Recall that the Hochschild cohomology group in degree 1 of \( U_q(\mathfrak{sl}_1^+) \), denoted by \( HH^1(U_q(\mathfrak{sl}_1^+)) \), is defined by:

\[
HH^1(U_q(\mathfrak{sl}_1^+)) := \text{Der}(U_q(\mathfrak{sl}_1^+))/\text{InnDer}(U_q(\mathfrak{sl}_1^+)),
\]

where \( \text{InnDer}(U_q(\mathfrak{sl}_1^+)) := \{ \text{ad}_x \mid x \in U_q(\mathfrak{sl}_1^+) \} \) is the Lie algebra of inner derivations of \( U_q(\mathfrak{sl}_1^+) \).

It is well known that \( HH^1(U_q(\mathfrak{sl}_1^+)) \) is a module over \( HH^0(U_q(\mathfrak{sl}_1^+)) := Z_q(\mathfrak{sl}_1^+) \). Our final result makes this latter structure precise.

**Theorem 3.11.**

1. Every derivation \( D \) of \( U_q(\mathfrak{sl}_1^+) \) can be uniquely written as follows:

\[
D = \text{ad}_x + \mu_1 D_1 + \mu_4 D_4 + \mu_6 D_6,
\]
with \( \text{ad}_x \in \text{InnDer}(U_q(\mathfrak{sl}_1^+)) \) and \( \mu_1, \mu_4, \mu_6 \in Z_q(\mathfrak{sl}_1^+) \).

2. \( HH^1(U_q(\mathfrak{sl}_1^+)) \) is a free \( Z_q(\mathfrak{sl}_1^+) \)-module of rank 3 with basis \( (D_1, D_4, D_6) \).

**Proof.** It just remains to prove that, if \( x \in U_q(\mathfrak{sl}_1^+) \) and \( \mu_1, \mu_4, \mu_6 \in Z_q(\mathfrak{sl}_1^+) \) with \( \text{ad}_x + \mu_1 D_1 + \mu_4 D_4 + \mu_6 D_6 = 0 \), then \( \mu_1 = \mu_4 = \mu_6 = 0 \) and \( \text{ad}_x = 0 \). Set \( \theta := \mu_1 D_1 + \mu_4 D_4 + \mu_6 D_6 \), so that \( \text{ad}_x + \theta = 0 \). Since \( \theta \) is a derivation of \( U_q(\mathfrak{sl}_1^+), \theta \) uniquely extends to a derivation \( \tilde{\theta} \) of the quantum torus \( P(\Lambda) \). Naturally, we still have \( \text{ad}_x + \tilde{\theta} = 0 \). Further, straightforward computations show that:

\[
\begin{align*}
\tilde{\theta}(T_1) &= \mu_1 T_1 \\
\tilde{\theta}(T_2) &= (\mu_1 + \mu_4) T_2 \\
\tilde{\theta}(T_3) &= (\mu_1 + \mu_4 + \mu_6) T_3 \\
\tilde{\theta}(T_4) &= \mu_2 T_4 \\
\tilde{\theta}(T_5) &= (\mu_4 + \mu_6) T_5 \\
\tilde{\theta}(T_6) &= \mu_6 T_6
\end{align*}
\]

Hence \( \tilde{\theta} \) is a central derivation of \( P(\Lambda) \), in the terminology of [20]. Thus we deduce from [20, Cor. 2.3] that \( \text{ad}_x = 0 = \theta \). Evaluating \( \theta \) on \( X_1, X_4 \) and \( X_6 \) leads to \( \mu_1 = \mu_4 = \mu_6 = 0 \), as desired.

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