Quantum complex scalar fields
and noncommutativity

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Abstract

In this work we analyze complex scalar fields using a new framework where the object of noncommutativity $\theta^{\mu\nu}$ represents independent degrees of freedom. In a first quantized formalism, $\theta^{\mu\nu}$ and its canonical momentum $\pi_{\mu\nu}$ are seen as operators living in some Hilbert space. This structure is compatible with the minimal canonical extension of the Doplicher-Fredenhagen-Roberts (DFR) algebra and is invariant under an extended Poincaré group of symmetry. In a second quantized formalism perspective, we present an explicit form for the extended Poincaré generators and the same algebra is generated via generalized Heisenberg relations. We also introduce a source term and construct the general solution for the complex scalar fields using the Green’s function technique.
# Introduction

Through the last years space-time noncommutativity has been a target of intense analysis. After the first published work by Snyder [1] a huge amount of papers has appeared in the literature. The connection with strings [2], gravity [3, 4, 5] and noncommutative field theories (NCFT) [6] brought attention to the subject.

The fundamental idea is that space-time may lose its standard properties at very high energy regimes. One of the approaches to study space-time at those regimes could be related with the noncommutativity of the coordinates [4, 5]. Other approaches to noncommutativity can also be given in a global way, generalizing some of the celebrated Connes ideas [7].

Most of the theories cited above pinpoint to the fact that at Planck scale, the space-time coordinates \( x^\mu \) have to be replaced by Hermitean operators \( x^\mu \) obeying the commutation relations

\[
[x^\mu, x^\nu] = i\theta^{\mu\nu}, \tag{1}
\]

where \( \theta^{\mu\nu} \) is considered as a constant antisymmetric matrix. Although it maintains translational invariance, Lorentz symmetry is broken [6] or correspondingly the rotation symmetry for non-relativistic theories. The violation of Lorentz invariance is problematic, among other facts, because it brings effects such as vacuum birefringence [8]. Other approaches are possible [9, 10, 11, 12, 13, 14], permitting to construct Lorentz invariant theories, by considering in some sense \( \theta^{\mu\nu} \) as independent degrees of freedom. These approaches are related to seminal works by Doplicher, Fredenhagen and Roberts (DFR) [15], which contain a blend of the principles of classical General Relativity and Quantum Mechanics. Their theory constrains localizability in a quantum spacetime, that has to be extended to include \( \theta^{\mu\nu} \) as an independent set of coordinates.

The structure of the DFR theory, besides equation (1) comprises

\[
[x^\mu, \theta^{\alpha\beta}] = 0 \quad \text{and} \quad [\theta^{\mu\nu}, \theta^{\alpha\beta}] = 0, \tag{2}
\]

with subsidiary quantum conditions

\[
\theta_{\mu\nu}\theta^{\mu\nu} = 0 \quad \text{and} \quad \left( \frac{1}{4} * \theta_{\mu\nu} \theta_{\mu\nu} \right)^2 = \lambda_P^8, \tag{3}
\]

where \( * \theta_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} \theta^{\rho\sigma} \) and \( \lambda_P \) is the Planck length.

The main motivation of DFR to study the relations (1) and (2) was the belief that a tentative of exact measurements involving space-time localization could confine photons due to gravitational fields. This phenomenon is directly related to (1) and (2) together with (3). In a somehow different perspective, other relevant results are obtained in [9, 10, 11, 12, 13, 14] relying on conditions (1) and (2). These authors use the value of \( \theta \) taken as a mean with some weigh function, generating Lorentz invariant theories and providing a connection with usual theories constructed in an ordinary \( D = 4 \) space-time.

In a recent series of works [16, 17, 18, 19] a new version of noncommutative quantum mechanics (NCQM) has been presented by one of us, where not only the coordinates
\(x^\mu\) and their canonical momenta \(p_\mu\) are considered as operators in a Hilbert space \(\mathcal{H}\), but also the objects of noncommutativity \(\theta^{\mu\nu}\) and their canonical conjugate momenta \(\pi_{\mu\nu}\). All these operators belong to the same algebra and have the same hierarchical level, introducing a minimal canonical extension of the DFR algebra. This enlargement of the usual set of Hilbert space operators allows the theory to be invariant under the rotation group \(SO(D)\), as showed in detail in Ref. [16, 19], when the treatment is a nonrelativistic one. Rotation invariance in a nonrelativistic theory, is fundamental if one intends to describe any physical system in a consistent way. In Ref. [17, 18], the corresponding relativistic treatment is presented, which permits to implement Poincaré invariance as a dynamical symmetry [20] in NCQM [21]. In the present work we essentially consider the ”second quantization” of the model discussed in Ref [17], showing that the extended Poincaré symmetry here is generated via generalized Heisenberg relations, giving the same algebra displayed in [17, 18].

The structure of this paper is organized as: after this introductory section, the minimal canonical extension of the DFR algebra is reviewed in Section 2. In Section 3, we introduce the new noncommutative charged Klein-Gordon theory in this \(D = 10, x + \theta\) space and analyze its symmetry structure, associated with the invariance of the action under some extended Poincaré (\(\mathcal{P}'\)) group. This symmetry structure is also displayed at the second quantization level, constructed via generalized Heisenberg relations. In Section 4 we expand the fields in a plane wave basis in order to solve the equations of motion using the Green’s functions formalism adapted for this new \((x + \theta)D = 4 + 6\) space. The conclusions and perspectives are reserved to the last section.

## 2 The minimal canonical extension of the DFR algebra

Besides (1), (2), the minimal canonical extension [17] of the DFR algebra[15] is given by

\[
\begin{align*}
[x^\mu, p_\nu] &= i\delta^\mu_\nu, \\
[p_\mu, p_\nu] &= 0, \\
[\theta^{\mu\nu}, \pi_{\rho\sigma}] &= i\delta^{\mu\nu}_{\rho\sigma}, \\
[\pi_{\mu\nu}, \pi_{\rho\sigma}] &= 0, \\
[p_\mu, \theta^{\rho\sigma}] &= 0, \\
[p_\mu, \pi_{\rho\sigma}] &= 0, \\
[x^\mu, \pi_{\rho\sigma}] &= -\frac{i}{2}\delta^{\mu\nu}_{\rho\sigma} p_\nu, 
\end{align*}
\]

(4)

where \(\delta^{\mu\nu}_{\rho\sigma} = \delta_{\rho}^{\mu}\delta_{\sigma}^{\nu} - \delta_{\rho}^{\nu}\delta_{\sigma}^{\mu}\). The relations above are consistent under all possible Jacobi identities. Notice that the ordinary form of the Lorentz generator, given by \(l^{\mu\nu} = x^\mu p^\nu - x^\nu p^\mu\), fails to close in an algebra if (1) is adopted, even if one considers \(\theta^{\mu\nu}\) as constant.
The relations (1), (2) and (4) allow us to utilize [22],

\[ M^{\mu \nu} = X^{\nu} p^{\mu} - X^{\nu} p^{\mu} - \theta^{\mu \sigma} \pi^{\nu}_{\sigma} + \theta^{\nu \sigma} \pi^{\mu}_{\sigma} \]  

as the generator of the Lorentz group, where [21]

\[ X^{\mu} = x^{\mu} + \frac{1}{2} \theta^{\mu \nu} p^{\nu} , \]  

and we see that the proper algebra is closed, i.e.,

\[ [M^{\mu \nu}, M^{\rho \sigma}] = i\eta^{\mu \sigma} M^{\rho \nu} - i\eta^{\mu \rho} M^{\sigma \nu} + i\eta^{\nu \rho} M^{\sigma \mu} . \]  

Now \( M^{\mu \nu} \) generates the expected symmetry transformations when acting on all the operators in Hilbert space. Namely, by defining the dynamical transformation of an arbitrary operator \( A \) in \( \mathcal{H} \) in such a way that \( \delta A = i[A, G] \), where

\[ G = \frac{1}{2} \omega^{\mu \nu} M^{\mu \nu} - a^{\mu} p^{\mu} + \frac{1}{2} b^{\mu \nu} \pi^{\mu \nu} , \]  

and \( \omega^{\mu \nu} = -\omega^{\nu \mu}, \ a^{\mu}, \ b^{\mu \nu} = -b^{\nu \mu} \) are infinitesimal parameters, it follows that

\[ \delta x^{\mu} = \omega^{\mu \nu} x^{\nu} + a^{\mu} + \frac{1}{2} b^{\mu \nu} p^{\nu} , \]  

\[ \delta X^{\mu} = \omega^{\mu \nu} X^{\nu} + a^{\mu} , \]  

\[ \delta p_{\mu} = \omega^{\mu \nu} p_{\nu} , \]  

\[ \delta \theta^{\mu \nu} = \omega^{\mu \nu} \theta^{\rho \nu} + \omega^{\nu \rho} \theta^{\mu \nu} + b^{\mu \nu} , \]  

\[ \delta \pi^{\mu \nu} = \omega^{\mu \rho} \pi^{\rho \nu} + \omega^{\nu \rho} \pi^{\mu \rho} , \]  

\[ \delta M^{\mu \nu} = \omega^{\mu \rho} M^{\rho \nu} + \omega^{\nu \rho} M^{\mu \rho} + a^{\mu} p^{\nu} - a^{\nu} p^{\mu} + b^{\mu \rho} \pi^{\rho \nu} + b^{\nu \rho} \pi^{\mu \rho} , \]  

generalizing the action of the Poincaré group \( \mathcal{P} \) in order to include \( \theta \) and \( \pi \) transformations. Let us refer to this group as \( \mathcal{P}' \). The \( \mathcal{P}' \) transformations close in an algebra. Actually,

\[ [\delta_2, \delta_1] A = \delta_3 A , \]  

and the parameters composition rule is given by

\[ \omega^{\mu \nu}_3 = \omega^{\mu \nu}_1 \alpha^{\alpha \beta}_{\mu \nu} - \omega^{\mu \nu}_2 \alpha^{\alpha \beta}_{\mu \nu} , \]  

\[ \alpha^{\mu \nu}_3 = \omega^{\mu \nu}_1 \alpha^{\mu \nu}_2 - \omega^{\mu \nu}_2 \alpha^{\mu \nu}_1 , \]  

\[ b^{\mu \nu}_3 = \omega^{\mu \nu}_1 b^{\rho \nu}_2 - \omega^{\mu \nu}_2 b^{\rho \nu}_1 - \omega^{\mu \nu}_1 b^{\mu \rho}_2 + \omega^{\mu \nu}_2 b^{\mu \rho}_1 . \]  

The symmetry structure displayed in (9) is discussed in details in [17].

Also in [17], it was studied how these symmetries could be dynamically implemented in a Lagrangian formalism. Theories that are invariant under \( \mathcal{P} \) and \( \mathcal{P}' \) were considered. The
The underlying point relies in the use of the Noether’s formalism adapted to such $x+\theta$ extended space. Moreover, this last cited work introduced possible NCQM actions constructed with the Casimir operators of $\mathcal{P}'$. As can be verified, if we define $M_{1}^{\mu\nu} = X^{\mu}p^{\nu} - X^{\nu}p^{\mu}$ and $M_{2}^{\mu\nu} = -\theta^{\mu\sigma}\pi^{\sigma\nu} + \theta^{\nu\sigma}\pi^{\sigma\mu}$, both satisfying (7), one can verify that four Casimir operators for $\mathcal{P}'$ can be constructed. Namely, the first two of such invariant operators are the usual one given by

$$C_{1} = p^{2}$$

and $C_{2} = s^{2}$, where $s_{\mu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}M_{1}^{\nu\rho}p^{\sigma}$ is the Pauli-Lubanski vector. The last two are defined as $C_{3} = \pi^{2}$ and $C_{4} = M_{2}^{\mu\nu}\pi_{\mu\nu}$. The important point to be stressed here is that the usual Casimir operators for the Poincaré group are kept by the theory, which does not destroy the usual classification scheme for the elementary particles. The same scheme can also be extended to fermionic fields [18]. Furthermore, it was shown that a corresponding classical underlying theory can also be constructed, as the one given in Ref. [19].

### 3 The action and symmetry relations

An important point is that, due to (1), the operator $x^{\mu}$ can not be used to label a possible basis in $\mathcal{H}$. However, as the components of $X^{\mu}$ commute, as can be verified from (4) and (6), their eigenvalues can be used for such purpose. From now on let us denote by $x$ and $\theta$ the eigenvalues of $X$ and $\theta$. In [17] we have considered these points with some detail and have proposed a way for constructing actions representing possible field theories in this extended $x + \theta$ space-time. One of such actions, generalized in order to permit the scalar fields to be complex, is given by

$$S = -\int d^{4}x d^{6}\theta \left\{ \partial^{\mu}\phi^{*}\partial_{\mu}\phi + \frac{\lambda^{2}}{4}\partial^{\mu\nu}\phi^{*}\partial_{\mu\nu}\phi + m^{2}\phi^{*}\phi \right\} , \quad (12)$$

where $\lambda$ is a parameter with dimension of length, as the Planck length, which is introduced due to dimensional reasons. Here we are also suppressing a possible factor $\Omega(\theta)$ in the measure, which is a scalar weight function, used in Refs. [9]-[14], in a noncommutative gauge theory context, to make the connection between the $D = 4 + 6$ and the $D = 4$ formalisms. Also $\Box = \partial^{\mu}\partial_{\mu}$, with $\partial_{\mu} = \frac{\partial}{\partial x^{\mu}}$ and $\Box_{\theta} = \frac{1}{2}\partial^{\mu\nu}\partial_{\mu\nu}$, where $\partial_{\mu\nu} = \frac{\partial}{\partial \theta^{\mu\nu}}$ and $\eta^{\mu\nu} = diag(-1,1,1,1)$.

The corresponding Euler-Lagrange equation reads

$$\frac{\delta S}{\delta \phi} = (\Box + \lambda^{2}\Box_{\theta} - m^{2})\phi^{*}
= 0 , \quad (13)$$

with a similar equation of motion for $\phi$. The action (12) is invariant under the transformation

$$\delta \phi = -(a^{\mu} + \omega_{\mu}^{\nu}x^{\nu})\partial_{\mu}\phi - \frac{1}{2}(b^{\mu\nu} + 2\omega^{\mu}_{\rho}\theta^{\rho\nu})\partial_{\mu\nu}\phi , \quad (14)$$
besides the phase transformation
\[ \delta\phi = -i\alpha \phi , \] (15)
with similar expressions for \( \phi^* \), obtained from (14) and (15) by complex conjugation. We observe that (9c) closes in an algebra, as in (10), with the same composition rule defined in (11). That equation defines how a complex scalar field transforms in the \( x + \theta \) space under \( \mathcal{P}' \). The transformation subalgebra generated by (9d) is of course Abelian, although it could be directly generalized to a more general setting.

Associated with those symmetry transformations, we can define the conserved currents [17]

\[
j^\mu = \frac{\partial L}{\partial \partial_\mu \phi} \delta\phi + \frac{\partial L}{\partial \partial_\mu \phi^*} \delta\phi^* + \mathcal{L} \delta x^\mu ,
\]
\[
j^{\mu\nu} = \frac{\partial L}{\partial \partial_{\mu\nu} \phi} \delta\phi + \frac{\partial L}{\partial \partial_{\mu\nu} \phi^*} \delta\phi^* + \mathcal{L} \delta\theta^{\mu\nu} . \] (16)

Actually, by using (9c) and (9d), as well as (9b) and (9d), we can show, after some algebra, that

\[
\partial_\mu j^\mu + \partial_{\mu\nu} j^{\mu\nu} = -\frac{\delta S}{\delta\phi} \delta\phi - \frac{\delta S}{\delta\phi^*} \delta\phi^* . \] (17)

Similar calculations can be found, for instance, in [17]. The expressions above allow us to derive a specific charge

\[ Q = - \int d^3x d^6\theta j^0 , \] (18)
for each kind of conserved symmetry encoded in (9c) and (9d), since

\[ \dot{Q} = \int d^3x d^6\theta (\partial_i j^i + \frac{1}{2} \partial_{\mu\nu} j^{\mu\nu}) \] (19)
vanishes as a consequence of the divergence theorem in this \( x + \theta \) extended space. Let us consider each specific symmetry in (9c) and (9d). For usual \( x \)-translations, we can write

\[ j^0 = j^0_\mu a^\mu , \] permitting to define the total momentum

\[ P_\mu = - \int d^3x d^6\theta j^0_\mu = \int d^3x d^6\theta (\dot{\phi}^* \partial_\mu \phi + \dot{\phi} \partial_\mu \phi^* + \mathcal{L} \delta^0_\mu) . \] (20)

For \( \theta \)-translations, we can write that \( j^0 = j^0_{\mu\nu} b^{\mu\nu} \), giving

\[ P_{\mu\nu} = - \int d^3x d^6\theta j^0_{\mu\nu} = \frac{1}{2} \int d^3x d^6\theta (\dot{\phi}^* \partial_{\mu\nu} \phi + \dot{\phi} \partial_{\mu\nu} \phi^* ) . \] (21)
In a similar way we define the Lorentz charge. By using the operator

$$\Delta_{\mu\nu} = x_{[\mu} \partial_{\nu]} + \theta_{[\mu} \theta_{\nu]\alpha} \, ,$$

and defining $$j^0 = j^0_{\mu\nu} \omega^{\mu\nu}$$, we can write

$$M_{\mu\nu} = -\int d^3 x d^6 \theta \bar{j}^0_{\mu\nu}$$

$$= \int d^3 x d^6 \theta (\dot{\phi}^* \Delta_{\nu\mu} \phi + \dot{\phi} \Delta_{\nu\mu} \phi^* - \mathcal{L} \delta^0_{[\mu} x_{\nu]} \, .$$

(23)

At last, for the symmetry given by (9d), we get the $$U(1)$$ charge as

$$Q = i \int d^3 x d^6 \theta (\dot{\phi}^* \phi - \dot{\phi} \phi^*) \, .$$

(24)

Now let us show that these charges generate the appropriate field transformations (and dynamics) in a quantum scenario, as generalized Heisenberg relations. To start the quantization of such theory, we can define as usual the field momenta

$$\pi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \dot{\phi}^* \, ,$$

$$\pi^* = \frac{\partial \mathcal{L}}{\partial \dot{\phi}^*} = \dot{\phi} \, ,$$

(25)

satisfying the non vanishing equal time commutators (in what follows the commutators are to be understood as equal time commutators)

$$[\pi(x, \theta), \phi(x', \theta')] = -i \delta^3(x - x') \delta^6(\theta - \theta') \, ,$$

$$[\pi^*(x, \theta), \phi^*(x', \theta')] = -i \delta^3(x - x') \delta^6(\theta - \theta') \, .$$

(26)

The strategy now is just to generalize the usual field theory and rewrite the charges (20-24) by eliminating the time derivatives of the fields in favor of the field momenta. After that we use (26) to dynamically generate the symmetry operations. In this spirit, accordingly to (20) and (25), the spatial translation is generated by

$$P_i = \int d^3 x d^6 \theta (\pi \partial_i \phi + \pi^* \partial_i \phi^*) \, ,$$

(27)

and it is trivial to verify, by using (26), that

$$[P_i, \mathcal{Y}(x, \theta)] = -i \partial_i \mathcal{Y}(x, \theta) \, ,$$

(28)

where $$\mathcal{Y}$$ represents $$\phi, \phi^*, \pi$$ or $$\pi^*$$. The dynamics is generated by

$$P_0 = \int d^3 x d^6 \theta (\pi^* \pi + \partial^i \phi^* \partial_i \phi + \frac{\lambda^2}{4} \partial^{\mu\nu} \phi^* \partial_{\mu\nu} \phi + m^2 \phi^* \phi)$$

(29)
accordingly to (20) and (25). At this stage it is convenient to assume that classically 
\[ \partial_{\mu\nu} \phi^* \partial_{\mu\nu} \phi \geq 0 \] 
to assure that the Hamiltonian \( H = P_0 \) is positive definite. By using the fundamental commutators (26), the equations of motion (13) and the definitions (25), it is possible to prove the Heisenberg relation

\[ [P_0, \mathcal{Y}(x, \theta)] = -i \partial_0 \mathcal{Y}(x, \theta). \]  

(30)

The \( \theta \)-translations, accordingly to (21) and (25), are generated by

\[ P_{\mu\nu} = \int d^3x d^6\theta (\pi \partial_{\mu\nu} \phi + \pi^* \partial_{\mu\nu} \phi^*), \]  

(31)

and one obtains trivially by (26) that

\[ [P_{\mu\nu}, \mathcal{Y}(x, \theta)] = -i \partial_{\mu\nu} \mathcal{Y}(x, \theta). \]  

(32)

Lorentz transformations are generated by (23) in a similar way. The spatial rotations generator is given by

\[ M_{ij} = \int d^3x d^6\theta \left( \pi \Delta_{ji} \phi + \pi^* \Delta_{ji} \phi^* \right), \]  

(33)

while the boosts are generated by

\[ M_{0i} = \frac{1}{2} \int d^3x d^6\theta \left\{ \pi^* \pi x_i - x_0 (\pi \partial_i \phi + \pi^* \partial_i \phi^*) + \pi (2\theta_i^\gamma \partial_0 \gamma - x_0 \partial_i) \right\} \]  

\[ + \left( \partial_j \phi^* \partial_j \phi + \frac{\lambda^2}{4} \partial_{\mu\nu} \phi^* \partial_{\mu\nu} \phi + m^2 \phi^* \phi \right) x_i . \]  

(34)

As can be verified in a direct way for (33) and in a little more indirect way for (34),

\[ [M_{\mu\nu}, \mathcal{Y}(x, \theta)] = i \Delta_{\mu\nu} \mathcal{Y}(x, \theta), \]  

(35)

for any dynamical quantity \( \mathcal{Y} \), where \( \Delta_{\mu\nu} \) has been defined in (22). At last we can rewrite (24) as

\[ Q = i \int d^3x d^6\theta (\pi \phi - \pi^* \phi^*) , \]  

(36)

generating (9d) and its conjugate, and similar expressions for \( \pi \) and \( \pi^* \). So, the \( P' \) and (global) gauge transformations can be generated by the action of the operator

\[ G = \frac{1}{2} \omega_{\mu\nu} M^{\mu\nu} - a^\mu P_\mu + \frac{1}{2} b^{\mu\nu} P_{\mu\nu} - \alpha Q \]  

(37)

over the complex fields and their momenta, by using the canonical commutation relations (26). In this way the \( P' \) and gauge transformations are generated as generalized Heisenberg relations. This is a new result that shows the consistence of the above formalism. Furthermore, there are also four Casimir operators defined with the operators given
above, with the same form as those previously defined at a first quantized perspective. So, the structure displayed above is very similar to the usual one found in ordinary quantum complex scalar fields. We can go one step further, by expanding the fields and momenta in modes, giving as well some order prescription, to define the relevant Fock space, spectrum, Green’s functions and all the basic structure related to free bosonic fields. In what follows we consider some of these issues and postpone others for a forthcoming work [23].

4 Plane waves and Green’s functions

In order to explore a little more the framework described in the last sections, let us rewrite the generalized charged Klein-Gordon action (12) with source terms as

\[
S = - \int d^4x d^6\theta \left\{ \partial^\mu \phi^* \partial_\mu \phi + \frac{\lambda^2}{4} \partial^{\mu\nu} \phi^* \partial_{\mu\nu} \phi + m^2 \phi^* \phi + J^* \phi + J \phi^* \right\}.
\]

The corresponding equations of motion are

\[
(\Box + \lambda^2 \Box_\theta - m^2) \phi(x, \theta) = J(x, \theta)
\]

as well as its complex conjugate one. We have the following formal solution

\[
\phi(x, \theta) = \phi_{J=0}(x, \theta) + \phi_J(x, \theta)
\]

where, clearly, \(\phi_{J=0}(x, \theta)\) is the source free solution and \(\phi_J(x, \theta)\) is the solution with \(J \neq 0\). The Green’s function for (39) satisfies

\[
(\Box + \lambda^2 \Box_\theta - m^2) G(x - x', \theta - \theta') = \delta^4(x - x') \delta^6(\theta - \theta'),
\]

where \(\delta^4(x - x')\) and \(\delta^6(\theta - \theta')\) are the Dirac’s delta functions

\[
\delta^4(x - x') = \frac{1}{(2\pi)^4} \int d^4K_{(1)} e^{iK_{(1)} \cdot (x - x')},
\]

\[
\delta^6(\theta - \theta') = \frac{1}{(2\pi)^6} \int d^6K_{(2)} e^{iK_{(2)} \cdot (\theta - \theta')},
\]

Now let us define

\[
X = (x^\mu, \frac{1}{\lambda} \theta^{\mu\nu})
\]

and

\[
K = (K^\mu_{(1)}, \lambda K^{\mu\nu}_{(2)}),
\]

where \(\lambda\) is a parameter that carries the dimension of length, as said before. From (44) and (45) we write that \(K \cdot X = K_{(1)}^\mu x^\mu + \frac{1}{2} K_{(2)}^{\mu\nu} \theta^{\mu\nu}\). The factor \(\frac{1}{2}\) is introduced in order to eliminate repeated terms. In what follows it will also be considered that \(d^{10} K = d^4 K_{(1)} d^6 K_{(2)}\) and \(d^{10} X = d^4 x d^6 \theta\).
So, from (39) and (41) we formally have that
\[ \phi_J(X) = \int d^{10}X' G(X - X') J(X') . \] (46)

To derive an explicit form for the Green’s function, let us expand \( G(X - X') \) in terms of plane waves. Hence, we can write that,
\[ G(X - X') = \frac{1}{(2\pi)^{10}} \int d^{10}K \tilde{G}(K) e^{iK \cdot (X - X')} . \] (47)

Now, from (41), (42), (43) and (47) we obtain that,
\[ (\Box + \lambda^2 \Box_\theta - m^2) \int \frac{d^{10}K}{(2\pi)^{10}} \tilde{G}(K) e^{iK \cdot (x - x')} = \int \frac{d^{10}x}{(2\pi)^{10}} e^{iK \cdot (x - x')} \] (48)
giving the solution for \( \tilde{G}(K) \) as
\[ \tilde{G}(K) = -\frac{1}{K^2 + m^2} \] (49)

where, from (45), \( K^2 = K^{(1)}_\mu K^{\mu} (1) + \frac{\lambda^2}{2} K^{(2)}_{\mu\nu} K^{\mu\nu} (2) \).

Substituting (49) in (47) we obtain
\[ G(x - x', \theta - \theta') = \frac{1}{(2\pi)^{10}} \int d^9K \int dK^0 \frac{1}{(K^0)^2 - \omega^2} e^{iK \cdot (x - x')} \] (50)
where the “frequency” in the \((x + \theta)\) space is defined to be
\[ \omega = \omega(\vec{K} (1), K(2)) = \sqrt{\vec{K}^2 (1) + \frac{\lambda^2}{2} K_{(2)}^{\mu\nu} K^{\mu\nu} (2) - m^2} \] (51)
which can be understood as the dispersion relation in this \( D = 4 + 6 \) space. We can see also, from (50), that there are two poles \( K^0 = \pm \omega \) in this framework. Of course we can construct an analogous solution for \( \phi_J^*(x, \theta) \).

In general, the poles of the Green’s function can be interpreted as masses for the stable particles described by the theory. We can see directly from equation (51) that the plane waves in the \((x + \theta)\) space establish the interaction between the currents in this space and have energy given by \( \omega(\vec{K} (1), K(2)) \) since \( \omega^2 = \vec{K}^2 (1) + \frac{\lambda^2}{2} K_{(2)}^{\mu\nu} K^{\mu\nu} (2) + m^2 = K^2 (1,2) + m^2 \), where \( K^2 (1,2) = \vec{K}^2 (1) + \frac{\lambda^2}{2} K^2 (2) \). So, one can say that the plane waves that mediate the interaction describe the propagation of particles in a \((x + \theta)\) space-time with a mass equal to \( m \). We ask if we can use the Cauchy residue theorem in this new space to investigate the contributions of the poles in (50). Accordingly to the point described in section 3, we can assume that the Hamiltonian is positive definite and it is directly related to the hypothesis that \( K^2 (1,2) = -m^2 < 0 \). However if the observables are constrained to a four dimensional space-time, due to some kind of compactification, the physical mass can have contributions from the noncommutative sector. This point is left for a forthcoming work.
[23], when we will consider the Fock space structure of the theory and possibles schemes for compactification.

For completeness, let us note that substituting (46) and (50) into (38), we arrive at the effective action

$$S_{\text{eff}} = - \int d^4x \, d^6\theta \, d^4x' \, d^6\theta' \, J^*(X) \int \frac{d^9K}{(2\pi)^{10}} \int dK^0 \frac{1}{(K^0)^2 - \omega^2 + i\varepsilon} e^{iK \cdot (X - X')} J(X') ,$$

which could be obtained, in a functional formalism, after integrating out the fields.

5 Conclusions and perspectives

In this work we have considered complex scalar fields using a new framework where the object of noncommutativity $\theta^{\mu\nu}$ represents independent degrees of freedom. We have started from a first quantized formalism, where $\theta^{\mu\nu}$ and its canonical momentum $\pi^{\mu\nu}$ are considered as operators living in some Hilbert space. This structure, which is compatible with the minimal canonical extension of the Doplicher-Fredenhagen-Roberts (DFR) algebra, is also invariant under an extended Poincaré group of symmetry, but keeping, among others, the usual Casimir invariant operators. After that, in a second quantized formalism perspective, we succeed in presenting an explicit form for the extended Poincaré generators and the same algebra of the first quantized description has been generated via generalized Heisenberg relations. This is a fundamental point because the usual Casimir operators for the Poincaré group are proven to be kept, permitting to maintain the usual classification scheme for the elementary particles. We also have introduced source terms in order to construct the general solution for the complex scalar fields using the Green’s function technique. The next step in this program is to construct the mode expansion in order to represent the fields in terms of annihilation and creation operators, acting on some Fock space to be properly defined. Also possible compactifications schemes will also be considered. These point are under study and will published elsewhere [23].

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