The Carleman regularization technique in the modelling of the plane \( E \)-polarized electromagnetic wave scattering by a flat system of impedance strips

George I. Koshovy | Andrew G. Koshovy

1 | INTRODUCTION

The theory and applications of regular and singular integral equations is an important subject within applied mathematics [1–4]. Integral equations and their systems are used as mathematical models in various physical situations; besides, they occur as reformulations of other complicated mathematical problems.

In particular, the plane wave scattering problems can be reformulated in the form of singular integral equation systems [5–8]. First-kind singular integral equations are usually classified as ill-conditioned [7]. This means that small changes in input data, usually associated with the right-hand part, can lead to much larger changes in their solution. Then, reformulation of scattering problems in the form of a second-kind integral equations with smooth or integrable kernels, such as Fredholm second-kind integral equations, allows one to obtain correct solutions.

The plane \( E \)-polarized electromagnetic wave scattering by a finite flat grating made of perfectly electrically conducting (PEC) strips is one of the simplest wave scattering problems. In this case, the three-dimensional (3D) physical model is reduced to the 2D outer Dirichlet problem (or the first boundary-value problem) for the Helmholtz equation. This problem has been studied in many publications using the correct and efficient mathematical models (see e.g., the article by Kvach and Sologub [9] and the book by Panasyuk et al. [10]).

In particular, the analytical method of the Carleman regularization was used successfully in the analysis of scattering from a prefractal flat PEC strip grating in the article by Koshovy [11]. The Carleman technique gives good results for smooth curvilinear PEC strips as well [12, 13].

In the case of impedance strips (IS), the scattering problem is reduced to the 2D outer problem with the mixed-type boundary condition (or the third boundary-value problem) for the Helmholtz equation, which is more general and important for practical applications. The third problem involves, as the limit cases, two other problems, the first and second ones, to which the plane \( H \)-polarized wave scattering problem is reformulated [14]. Thus, it is interesting and important to examine this 2D outer mixed problem in detail.

Our work deals with the general mathematical models of the plane \( E \)-polarized wave scattering by a finite flat IS grating. In particular, we consider the wave scattering by a sparsely filled IS grating. In such a case, the grating has electrically narrow flat IS, and the corresponding system of integral equations has an explicit asymptotic solution. The presented asymptotic solution, based on the use of the Carleman regularization technique, makes it possible to obtain simple expressions of integral scattering characteristics. Thus, we prove that the Carleman regularization is an efficient technique that brings useful and practically important results.
THE PLANE E-POLARIZED WAVE SCATTERING BY FLAT IS GRATING

The mathematical formulation of the plane wave scattering by a zero-thickness strip grating is well known and widely used [5–7]. We will repeat it for a grating with finite number of IS, located within a single plane (finite flat grating).

2.1 Formulation of the scattering problem

Assume that the plane linearly polarized electromagnetic wave is incident upon a finite grating consisting of zero-thickness flat strips with parallel edges. In Figure 1, we depict the simplest case of such grating, which consists of two unequal flat strips. The left strip width is 2a, whereas the right strip width is 2b, and the distance between their median lines is 2d. Thus, the transverse cross-section of the simplest flat grating consists of two straight segments.

In the case of an arbitrary finite flat strip grating, its transverse cross-section consists of the corresponding number of straight segments.

Because the strips have parallel edges, we introduce the Cartesian coordinates placed along the y-axis. Then, the Maxwell equations can be separated into two independent partial differential equations (DE) and reduced to the 2D homogeneous Helmholtz equation [5]:

\[
\left( \Delta + k^2 \right) u(x, z) = 0. \tag{1}
\]

where \( k \) is the wave number and \( u(x,z) \) is an unknown function of two variables that determines an electromagnetic field in the presence of strips. On the segments of the x-axis, which correspond to the transverse cross-section of flat strip grating, the impedance boundary condition is requested [7].

In the case of the plane E-polarized wave scattering, we denote the single non-zero component of the electric field as \( E_y = u(x, z) \). If this component is found, the magnetic field can be determined from the expression \( \mathbf{H} = (-u_z', 0, u_y') : i k \chi \), where \( i \) is the imaginary unit and \( \chi \) is the wave resistance of the medium around the strips. Then, the impedance boundary condition is:

\[
ku + i \gamma \cdot u_z' = 0. \tag{2}
\]

Here, we use the dimensionless impedance parameter \( \gamma = w/\chi \), where \( w \) is the strip surface impedance. In the case of zero value for the strip impedance, or \( \gamma = 0 \), the third (mixed) boundary-value problem for the homogeneous Helmholtz equation turns to the first one.

Thus, we have the outer mixed boundary-value problem for Equation (1). This 2D problem has a unique solution provided the radiation condition at infinity (according to Sommerfeld):

\[
\lim_{r \to \infty} \sqrt{r} \left( u_r - i u \right) = 0 \tag{3}
\]

and conditions at the strip edges (segment end points), in the Meixner form, are met [5].

A powerful approach to examining outer boundary-value problems for elliptic-type partial differential equations is the integral equation technique [4]. Therefore, at the first step, this technique will be used to reduce this formulated problem to a 1D problem for the mixed integral-DE (IDE).

2.2 Reformulation of the boundary-value problem

To reduce the considered outer mixed boundary-value problem to a 1D problem, we represent the sought for function in the form:

\[
u(x, z) = \exp\left[i k (q_1 x + q_2 z)\right] - \frac{i}{4} \int_{S} \left[ \psi(x') \cdot H_0^{(1)}(kR) \right. \\
+ \left. \phi(x') \cdot \frac{\partial}{\partial z} H_0^{(1)}(kR) \right] \cdot dx'. \tag{4}
\]

Here, \( q = (q_1, q_2) \) is the direction-of-propagation unit vector of the plane wave (Figure 1), the domain of integration \( S \) is a set of segments, which are the cross-sections of the strips, \( H_0^{(1)}(z) \) is the first Hankel function of zero order, and \( R = \sqrt{(x-x')^2 + z^2} \). Two unknown functions, \( \psi(x) \) and \( \phi(x) \), are the jumps of the sought-for function of two variables and its partial derivative with respect to \( z \) on the segments, i.e.

\[\psi(x) = u_2'(x, +o) - u_2'(x, -o), \quad \phi(x) = u(x, +o) - u(x, -o)\]

(see, for instance, Honl et al. [5]).

Equation (4) satisfies both Equation (1) and the radiation condition at infinity (3), owing to the properties of the first Hankel function of zero order. Then, the impedance boundary condition (2) on the set of segments \( S \) can be used twice, as in the preceding paper [15]. The first use yields the relationship between unknown functions \( k u_2(x) = -i \gamma \psi(x) \). It is important to simplify the scattered part of Equation (4) to the form:

\[4iu(x, z) = \int_{S} \left[ \psi(x') \cdot \left( 1 - \frac{i \gamma}{k} \cdot \frac{\partial}{\partial z} \right) H_0^{(1)}(kR) \right] dx'. \tag{5}
\]
Then, we use the condition (2) next to develop a modified version of the integral equation technique, originally presented in the preceding paper [15]. Namely, using the impedance boundary condition (2) and substituting Equations (4) and (5), we obtain the following mixed differential-integral equation:

\[
\begin{align*}
\int_s \left[ \frac{\varphi^2}{k} \frac{\partial^2}{\partial z^2} H_0^{(1)}(kR) + k H_0^{(1)}(kR) \right] \cdot \psi(x') dx' \\
= -4ik(1 - \gamma \cdot q_2) e^{q_2 kx}.
\end{align*}
\]

(6)

Owing to the Helmholtz equation, we have the relationship for the Hankel function:

\[
\frac{\partial^2}{\partial z^2} H_0^{(1)}(kR) = - \left( \frac{\partial^2}{\partial x^2} + k^2 \right) H_0^{(1)}(kR).
\]

(7)

It allows passing to the limit \( z \to 0 \) in Equation (6) and then separating the differential and the integral parts in it. As a result, we arrive at the ordinary inhomogeneous DE of the second order:

\[
\varphi^2 \cdot Y''(x) + k^2 (\varphi^2 - 1) \cdot Y(x) = 4ik^2 (1 - \gamma \cdot q_2) \cdot \exp(ikq_2 x).
\]

(8)

Thus, because of the previously explained modification of the integral equation technique, the obtained mixed IDE system (8) with Equation (9) can be regarded as a general full-wave 1D mathematical model of the plane \( E \)-polarized EM wave scattering. Still, this model is inconvenient for the use with a direct numerical method because it has a general form. Therefore, to have a more convenient model, it is necessary to examine the ordinary inhomogeneous DE of the second order (8). Of course, it is easy to find its general solution as in the preceding paper [16]. However, in the case of the plane \( E \)-polarized electromagnetic wave scattering by a flat strip grating, this way is not optimal, in particular, because it is unstable for small values of dimensionless parameter \( \gamma \). Thus, a new treatment should be developed that is different from the plane \( H \)-polarized wave scattering analysed in the articles by Koshovy [17] and by Karpenko et al. [18]. In this case, the perturbation method as an intermediate step, and the analytical method of the Carleman regularization as the principal step, are preferable.

### 3 | THE CARLEMAN REGULARIZATION TECHNIQUE

First, let us examine the ordinary inhomogeneous DE of the second-order equation (8) using the method of a small parameter. Here, the small parameter will be dimensionless impedance parameter \( \gamma \), and we try to integrate Equation (8) using the power series approach.

#### 3.1 | The Perturbation method

A powerful method in applied mathematics is the method of the small parameter, or the perturbation method. For a zero value of strip impedance, or if \( \gamma = 0 \), DE (8) is simplified to the relation \( Y(x) = -4i \cdot \exp(ikq_2 x) \). Thus, we have not a DE with respect to \( Y(x) \), but just an explicit solution. Considering linear integral transform (9) and substituting it into the left-hand part of the derived relation, we obtain well-known logarithmically singular IE of the first order [11–13]:

\[
\int_\psi(x') \cdot H_0^{(1)}(k|x-x'|) dx' = -4i \cdot \exp(ikq_2 x).
\]

(10)

One can see the principal difference between complicated mixed IDE (8) and (9) and this simple log-singular IE. It can be anticipated that the power series approach to the ordinary DE (8) examination will be successful. Here, we start from the assumption that \( |\gamma| = << 1 \) and introduce a usual power (with respect to \( \gamma \)) series, \( Y(x) = \sum_{n=0}^{\infty} Y_n(x) \gamma^n \). Then, we rewrite the mentioned DE as \( \varphi^2 \cdot [Y''(x) + k^2 \cdot Y(x)] - k^2 \cdot Y(x) = 4ik^2 (1 - \gamma \cdot q_2) \cdot \exp(ikq_2 x) \). Substituting this power series into that equation and equating the coefficients at the powers of \( \gamma \), we obtain the sequence of relationships:

\[
Y_0(x) = -4i \cdot \exp(ikq_2 x),
Y_1(x) = 4iq_2 \cdot \exp(ikq_2 x) = q_1 \cdot Y_0(x),
Y_2(x) = Y_{n-2} \cdot Y_n(x) = Y_{n-2} \cdot Y_{n+2}(x) + k^2 \cdot Y_{n+2}(x) \quad \text{for all} \quad n \geq 2.
\]

To find the simplest expression of \( Y_n(x) \) for all \( n \geq 2 \), depending on \( Y_0(x) \), the basic method of mathematical induction can be used. The basic method of mathematical induction can be used. Because of the helix relationship of the sequence turns it into the identity \( k^2 (-1)^n q_2^2 \cdot Y_n(x) = (-1)^n q_2^2 \cdot Y_{n-2}(x) \cdot Y_{n+2}(x) \), here \( Y_0(x) = -k^2 q_2^2 \cdot Y_0(x) \). Substituting these expressions into the latter relationship of the sequence turns it into the identity \( k^2 (-1)^n q_2^2 \cdot Y_n(x) = (-1)^n q_2^2 \cdot Y_{n-2}(x) \cdot Y_{n+2}(x), \) because \( q_1^2 + q_2^2 = 1 \) (and \( -1^2 = (-1)^n \). Thus, the formula

\[
Y_n(x) = (1)^n q_2^2 \cdot Y_0(x) \quad \text{is proved for all} \quad n \geq 1 \quad \text{and the power series for the solution of DE (8)} \quad \text{is simplified to the ordinary geometrical progression sum}, \quad Y(x) = Y_0(x) \cdot \sum_{n=0}^{\infty} (-1)^n q_2^2 \gamma^n. \]

It converges if inequality \( |q_2 \gamma| < 1 \) holds. Here, \( q_2 \) is the second component of the direction-of-propagation unit vector (Figure 1), defined by the formula, \( q_2 = \sin \varphi_0 \), where \( \varphi_0 \) is the angle of the plane wave incidence on the grating. Therefore, oblique and near-to-the-grazing angles are preferable for the inequality to be satisfied. Using the well-known sum of the geometrical progression, in the result we obtain a log-singular IE of the first order:
\[ f_{\psi}(x') \cdot H_0^{(1)}(k|x-x'|)dx' = -4i \cdot \exp(ikq_x \cdot x) \cdot (1 + q_y)^{-1}. \] (11)

It is not difficult to check that the right-hand side function satisfies the inhomogeneous ordinary DE of the second order (8), because after substitution and simplification we obtain the identity: \[ [-\gamma^2 \cdot (-q_1^2 + 1) + 1] \cdot \exp(ikq_x \cdot x) \equiv (1 - \gamma^2 \cdot q_2^2) \cdot \exp(ikq_x \cdot x). \]

The difference between the singular IE (10) and the singular IE (11) is on the right-hand side functions only, so the analytical method of the Carleman regularization can be used successfully, as in the case of prefractal PEC strip gratings [11]. Now, we will consider this technique in more detail for the case of the simplest flat grating of two IS, shown in Figure 1.

### 3.2 Analytical method of the Carleman regularization

To obtain a correct and convenient mathematical model of the plane \( E \)-polarized wave scattering from the two-IS grating, we will use the simplest transformations of the IE system (11).

First, we assign concrete values to the strip locations (Figure 1) will use the simplest transformations of the IE system (11).

The known constants in the right-hand part functions of the simplest flat grating of two IS, shown in Figure 1.

The systems of the first-kind log-singular IE are ill-conditioned, and therefore their direct discretizations are not convergent. Still, this system is convenient for further transformation, using the Carleman regularization, to the set of the Fredholm second-kind IE. The formal scheme of regularization (or semi-inversion) method is deceptively simple: see, for instance, the article by Nosich [19].

Let us represent the first-kind system of singular IE (12) in the simplest form:

\[ f_{\psi}^{-1} j_k(t) \ln|\tau - t| (dt = \pi \cdot F_k(\tau)). \] (15)

Then, assuming that the right-hand part of Equation (15) is known, the Carleman inversion formula [2] yields an explicit expression for the sought-for function:

\[ j_k(x) = \frac{1}{\pi \sqrt{1 - x^2}} \left[ \int_{\pi}^{1} \frac{1 - \tau^2}{\tau - x} F_\kappa(\tau) \frac{d\tau}{\ln 2} \right]. \] (16)

In our case, functions \( F_\kappa(\tau)(\kappa = 1, 2) \) contain additional integral terms depending on \( j_{1,2}(\tau); \) however, with smooth kernels, see Equations (12) and (15):

\[ F_\kappa(\tau) = \left[ c_\kappa \cdot \exp(iq_1 \alpha_\kappa x) \cdot \sum_{m=1}^{2} \int_{-1}^{1} j_m(t) R_{\kappa m}(\tau, t) dt \right]/\pi. \] (17)

Therefore, after substituting these expressions into Equation (16) and changing the order of integration, we obtain the system of the second-kind IE of the form [2, 11]:

\[ j_k(x) + \sum_{m=1}^{2} \int_{-1}^{1} j_m(t) R_{\kappa m}(x, t) dt = g_k(x), \quad |x| \leq 1, \quad \kappa = 1, 2. \] (18)

The kernel functions of new system, as well as the right-hand part functions, are determined through the kernel functions and the right-hand part functions of the initial system (12), with the aid of the Carleman formula (16). Here we present the expression for the right-hand part functions:

\[ g_k(x) = \frac{c_\kappa}{\pi \sqrt{1 - x^2}} \left[ \int_{-1}^{1} \frac{1 - \tau^2}{\tau - x} \exp(iq_1 \alpha_\kappa x) d\tau \right] - \frac{1}{\ln 2} \int_{-1}^{1} \frac{1}{\tau - x} \exp(iq_1 \alpha_\kappa x) d\tau. \] (19)

The Cauchy-type integral of the right-hand part (understood in the principal value sense) was examined in the article.
by Koshov [20]. The second improper integral in the right-hand part of Equation (19) can easily be calculated in the following way: \( \int_{-1}^{1} \exp(iq_{1}a_{c}r) \frac{dr}{\sqrt{1-r^{2}}} = 2 \int_{0}^{1} \cos \frac{q_{1}a_{c}r}{\sqrt{1-r^{2}}} dr = \pi \cdot i \cdot j_{0}(q_{1}a_{c}) \). The kernel functions of the system can be treated in similar way. Properly speaking, the Fredholm-type second-kind IE system represents the final output of the Carleman regularization technique in the case of the simplest two-IS flat grating. It is easy to prove that this technique can be extended to flat gratings of any finite number of IS, in particular, prefractal flat IS gratings [11]. Using the second-kind IE system with smooth kernel functions, it is possible to derive mathematically grounded asymptotic solutions to the considered scattering problems.

We will focus our attention on the plane \( E \)-polarized wave scattering by a flat grating of two electrically narrow IS. In this case, the second kind IE system (18) has an explicit asymptotic solution. The procedure of building that solution is similar to the case of the plane \( H \)-polarized wave scattering [11, 17, 18].

4 | THE PLANE WAVE SCATTERING BY SPARSELY FILLED GRATING

The asymptotic mathematical model of a sparsely filled grating appears under the assumptions that the strips are electrically narrow (\( a_{c} < < 1 \)) and sufficiently wide-spaced from each other (\( \delta = O(1) \)).

4.1 | Asymptotic solution of scattering problem

Using the asymptotic expressions for the kernel functions and the right-hand part functions of the second-kind IE system (18), we arrive at explicit solutions in the main term, \( j_{k0}(x) = j_{k0} : (\pi \sqrt{1-x^{2}}) \). Here, the constants \( j_{k0} \) are determined from the system of two linear algebraic equations:

\[
\begin{pmatrix}
C + \ln \frac{a_{1}}{4i} & \frac{\pi}{2i} H_{1}^{(1)}(2\delta) \\
\frac{\pi}{2i} H_{1}^{(1)}(2\delta) & C + \ln \frac{a_{2}}{4i}
\end{pmatrix}
\begin{pmatrix}
j_{10} \\
j_{20}
\end{pmatrix}
= \frac{-2\pi}{1 + \gamma \cdot q_{2}} \begin{pmatrix} e^{-iq_{1} \delta} & e^{iq_{1} \delta} \end{pmatrix}.
\]

(20)

One can see full agreement of these expressions with the case of the plane \( E \)-polarized wave scattering by the PEC strip grating, corresponding to the creator of self-similar fractal, presented in the article by Koshov [11]. The difference is seen only in the right-hand part coefficients: here, they contain the divisor, \( (1 + \gamma \cdot q_{2}) \), which depends on dimensionless impedance parameter \( \gamma \) and the angle of the plane wave incidence. Owing to this divisor, it is easy to examine the solution dependence on impedance \( \gamma \).

The determinant (denoted as \( \Delta \)) of the system (20), if the strips are equal (\( a_{1} = a_{2} \)), was studied in the article by Koshov [11]. Under these earlier assumptions, it is different from zero. In the case of \( a_{1} = a_{2} = O(a_{1}) \), the determinant is close to that examined for \( a_{1} = a_{2} \). So, the Cramer rule can be used for finding the unknown constants, \( j_{k0} \) (\( k = 1, 2 \)):

\[
j_{k0} = \frac{-2\pi}{1 + \gamma \cdot q_{2}} \frac{1}{\Delta} \begin{pmatrix} e^{-iq_{1} \delta} \left( C + \ln \frac{a_{c}}{4i} \right) \\
 e^{iq_{1} \delta} \end{pmatrix}.
\]

(21)

Thus, we have completely solved, in the main-term approximation, the problem of the plane \( E \)-polarized electromagnetic wave scattering by a flat grating of two electrically narrow IS.

To find the second term, that is, the first correction to the obtained approximation, we should follow Koshov [15]. However, because we are interested in the integral scattering characteristics, instead we derive the scattered field in the far zone, in the same main-term asymptotic approximation.

4.2 | Scattered field in the far zone

Once the plane wave scattering problem has been solved mathematically, consider the field scattered by a flat IS grating with the aid of Formula (5) in the polar coordinates. Using the large-argument asymptotic expressions for the Hankel function and its partial derivative [5], the scattered field in the far zone \( (r \to \infty) \) can be represented as

\[
\varphi(r, \phi) \approx \frac{\exp(ikr)}{\sqrt{kr}} \Psi(\phi),
\]

(22)

\[
\Psi(\phi) = \sqrt{\frac{i}{2}} \cdot \tilde{\Psi}(k \cos \phi)(1 + \gamma \sin \phi).
\]

Here, \( \tilde{\Psi}(k \cos \phi) \) is the symmetrical integral Fourier transform of function \( \Psi(x) \) [21], defined on two segments: \([-d-a, d + a], [d-b, d+b] \) (Figure 1). In the case of a sparse grating, the asymptotic model of the plane \( E \)-polarized wave scattering allows us to find the expression:

\[
\sqrt{2\pi} \cdot \tilde{\Psi}(k \cos \phi) \approx \exp(i\delta \cos \phi) j_{10} + \exp(-i\delta \cos \phi) j_{20}.
\]

(23)

Because of Formula (21) and the second expression in Equation (22), the explicit dependence of the far field on the strip impedance can be separated. This dependence is given by the function, which does not depend on the strip grating geometrical parameters,
\[ F(\phi) = (1 + \gamma \sin \phi) \cdot (1 + \gamma \cdot q_2)^{-1}. \] (24)

The dependences of function \(|F(\phi)|\) on the polar angle, calculated at the fixed complex values of the dimensionless impedance parameter \(\gamma = (1 + i)R\), where \(R = 0.2\) (red line); \(R = 0.5\) (blue dots), \(R = 1\) (black line), are presented in Figure 2. Here, the plots in Figure 2a correspond to the plane wave incidence angle \(\varphi_0 = \pi/6\) (inclined incidence), and those in Figure 2b correspond to \(\varphi_0 = \pi/60\) (near-grazing incidence). One can see the symmetry of the curves with respect to the direction \(\varphi = \pi/2\) and asymmetry with respect to the direction \(\varphi = 0\), which is greater for larger values of dimensionless parameter \(R\).

Finally, Figure 3 presents far-field angular scattering patterns \(\Psi(\varphi)\) for the simplest case of the grating made of two IS (Figure 1) with different widths, \(a\) and \(b\). The dimensionless small electric-size parameters are assumed to be \(a_1 = ka = \pi/32\) and \(a_2 = kb = \pi/40\), which entails \(2a = \lambda/32\) and \(2b = \lambda/40\), where \(\lambda\) is the wavelength. The width of the whole grating is assumed to be equal to the wavelength \(\lambda\). Therefore, the distance between strip edges, \(2|d-a-b| = 311/32\lambda\), is close to \(\lambda\).

The curves in Figure 3 correspond to the angular scattering patterns of the PEC strip grating (red curves, symmetrical with respect to \(\varphi = 0\)) and two IS gratings, calculated using Equations (18)–(20) for \(R = 0.5\) (blue dotted curves) and \(R = 1\) (black curves). The angles of the plane wave incidence on the grating are the same as in Figure 2, \(\varphi_0 = \pi/6\) for the inclined incidence and \(\varphi_0 = \pi/60\) for the near-grazing incidence.

5 | CONCLUSIONS

The mathematical model of the plane \(E\)-polarized wave scattering by a finite flat IS system in the form of the first-kind logarithmic-singular integral equations can be treated similarly to the mathematical model of scattering by the PEC strip system. This is because of the use of the analytical method of the Carleman regularization, which recasts the considered problem to Fredholm second-kind integral equations. Here, it has been proved that the perturbation method, with respect to the dimensionless impedance parameter, is an efficient instrument for obtaining the explicit solution in the form of the asymptotic series in the powers of a small parameter, which is the electric width of the widest strip.

Thus, we have demonstrated that the Carleman method of analytical regularization, based on the explicit inversion of the logarithmically singular integral operator, is a convenient instrument for deriving small electric-size asymptotics in the plane \(E\)-polarized wave scattering from two parallel coplanar IS.

A similar approach can be successfully used for mathematical modelling of the plane \(E\)-polarized electromagnetic (EM) wave scattering by a finite multilevel coplanar system of flat IS. The plane \(H\)-polarized EM wave scattering problem has been considered in the conference paper [22]. Such modelling will be promising for the creation of 2D domains with given effective electric and magnetic properties by embedding a lot of small impedance segments into a homogeneous medium, in the sense of Andriychuk et al. [23].

ACKNOWLEDGEMENT

The authors would like to thank Professor A. I. Nosich for many valuable and helpful discussions.

ORCID

George I. Koshovy \(\text{https://orcid.org/0000-0001-7991-8591}\)

REFERENCES

1. Muskhelishvili, N.I.: Singular Integral Equations. Noordhoff Publ, Groningen (1953)
2. Gakhov, F.D.: Boundary-Value Problems. Pergamon Press, Oxford (1977)
3. Lifanov, I.K.: Singular Integral Equations and Discrete Vortices. VCP VB, Utrecht (1996)
4. Atkinson, K.E.: The Numerical Solution of Integral Equations of the Second Kind. Cambridge University Press (1997)
5. Horn, H., Maue, A.W., Westpfahl, K.: Theorie der Beugung. Springer Verlag, Berlin (1961)
6. Born, M., Wolf, E.: Principles of Optics. Pergamon Press, Oxford, pp. 513–548 (1968)
7. Felsen, L.B., Marcuvitz, N.: Radiation and Scattering of Waves. Prentice Hall Inc, Englewood Cliffs (1973)
8. Colton, D., Kress, R.: Integral Equation Methods in Scattering Theory. John Wiley and Sons, New York (1983)
9. Kvasch, N.V., Sologub, V.G.: Scattering of a plane E-polarized wave by a finite number of strips located within a single plane. Radio Eng. Electron. Phys. 27(10), 2031–2034 (1982)
10. Panasyuk, V.V., Savruk, M.P., Nazarchuk, Z.T.: The Singular Integral Equations Technique in Two-Dimensional Diffraction Problems. Naukova Dumka, Kyiv (1984)
11. Koshovy, G.I.: Systems approach to investigating pre-fractal diffraction gratings. Telecommun. Radio Eng. 71(11), 487–500 (2012)
12. Koshovy, G.I.: Vekua-Carleman method and longwave asymptotic in scattering problem by curvilinear strips. In: Proc. International Seminar/Workshop Direct and Inverse Problems of Electromagnetic and Acoustic Wave Theory (DIPED-2003), Lviv, Ukraine, pp. 69–73 (2003)
13. Koshovy, G.I., Koshovy, A.G.: On interaction between E-polarized electromagnetic wave and the CSA pre-fractals. In: Proc. International Seminar/Workshop Direct and Inverse Problems of Electromagnetic and Acoustic Wave Theory (DIPED-2006), Lviv, Ukraine, pp. 47–50 (2006)
14. Koshovy, G.I.: Scattering of H-polarized waves by pre-fractal diffraction gratings. Telecom. Rad. Eng. 71(11), 961–973 (2012)
15. Koshovy, G.I.: Mathematical models of acoustic wave scattering by impedance strip. Proc. of International Seminar/Workshop on Direct and Inverse Problems Electromagnetic and Acoustic Wave Theory (DIPED-2017), Dnipro, Ukraine (2017)
16. Koshovy, G.I.: Rigorous asymptotic models of wave scattering by finite flat gratings of electrically narrow impedance strips. In: Proc. of International Conference on Mathematical Methods in Electromagnetic Theory (MMET-2018), Kyiv, Ukraine, pp. 70–74 (2018)
17. Koshovy, G.I.: The plane H-polarized electromagnetic wave scattering by pre-fractal grating of impedance strips. Int. J. Microw. Wirel. Technol. 12(10), 269–275 (2020)
18. Karpenko, V.I., Koshovy, G.I., Logginov, Y.F.: Mathematical models of the plane sonic wave scattering by pre-fractal flat impedance strips system. Telecom. Rad. Eng. 79(15), 1301–1314 (2020)
19. Nosich, A.I.: The method of analytical regularization in wave-scattering and eigenvalue problems: foundations and review of solutions. IEEE Antennas Propag. Mag. 41(3), 34–49 (1999)
20. Koshovy, G.I.: Electromagnetic wave scattering by pre-fractal structures of cylindrical strips. Telecom. Rad. Eng. 67(14), 1225–1238 (2008)
21. Akhiezer, N.I.: Lectures on Integral Transforms. AMS, Providence (1988)
22. Ahupova, O.O., Koshovy, G.I.: On EM wave scattering by coplanar system of flat impedance strips. In: Proc. IEEE 40th Int. Conf. Electr. Nanotechnol. (ELNANO-2020), pp. 34–38 (2020)
23. Andriychuk, M.I., Indratno, S.W.A.G., Ramm, A.G.: Electromagnetic wave scattering by a small impedance particle: theory and modeling. Opt. Commun. 285, 1684–1691 (2012)