PARAMETER-ROBUST UZAWA-TYPE ITERATIVE METHODS FOR DOUBLE SADDLE POINT PROBLEMS ARISING IN BIOT’S CONSOLIDATION AND MULTIPLE-NETWORK Poroelasticity Models

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Abstract. This work is concerned with the iterative solution of systems of quasi-static multiple-network poroelasticity (MPET) equations describing flow in elastic porous media that is permeated by single or multiple fluid networks. Here the focus is on a three-field formulation of the problem in which the displacement field of the elastic matrix and, additionally, one velocity field and one pressure field for each of the \( n \geq 1 \) fluid networks are the unknown physical quantities. Generalizing Biot’s model of consolidation, which is obtained for \( n = 1 \), the MPET equations for \( n \geq 1 \) exhibit a double saddle point structure.

The proposed approach is based on a framework of augmenting and splitting this three-by-three block system in such a way that the resulting block Gauss-Seidel preconditioner defines a fully decoupled iterative scheme for the flux-, pressure-, and displacement fields. In this manner, one obtains an augmented Lagrangian Uzawa-type method, the analysis of which is the main contribution of this work. The parameter-robust uniform linear convergence of this fixed-point iteration is proved by showing that its rate of contraction is strictly less than one independent of all physical and discretization parameters.

The theoretical results are confirmed by a series of numerical tests that compare the new fully decoupled scheme to the partially decoupled fixed-stress split iterative method, which decouples only flow—the flux and pressure fields remain coupled in this case—from the mechanics problem. We further test the performance of the block triangular preconditioner defining the new scheme when used to accelerate the GMRES algorithm.

1. Introduction

In this paper we propose and analyze stationary iterative methods for solving the equations of multiple network poroelastic theory (MPET) which describe flow in deformable porous media. The latter is modeled as an elastic solid matrix comprising \( n \geq 1 \) superimposed fluid networks with possibly vastly varying characteristic length scales and hydraulic conductivities, see e.g., [43] and the references therein.

Dual-porosity/dual-permeability models have been proposed and studied in geomechanical context, see, e.g., [7, 6], providing a generalization of Biot’s consolidation model which is obtained for \( n = 1 \), see [11, 12]. Over the last decade, the MPET equations have gradually gained attention as a tool for modeling flow across scales and networks in soft tissue. Biological multicompartmental poroelasticity models can be used to embed more specific medical models, e.g., to describe water transport in the cerebral environment and explore the pathogenesis of acute and chronic hydrocephalus [42], or to study effects of obstructing cerebrospinal fluid (CSF) transport and to demonstrate the impact of aqueductal stenosis and fourth ventricle outlet obstruction (FVOO) [45, 44], or to find medical indications of oedema formation [19].

Very recently, the MPET model has also been used in order to gain a better understanding of the processes involved with the mechanisms behind Alzheimer’s disease (AD), the most common form of dementia [24]. Most prominently, the so-called amyloid hypothesis states that the accumulation of neurotoxic amyloid-\( \beta \) (A\( \beta \)) into parenchymal senile plaques or within the walls of arteries is a basic cause of this disease. In [23] a partial validation of a four-network poroelastic model for metabolic waste clearance is presented in a qualitative way, i.e., by showing a qualitative agreement of the cerebral blood flow (CBF) data obtained from arterial spin labeling (ASL) images and the corresponding model output.
in different regions of the brain. Although the authors conclude that there is a need for more experimental and clinical data to optimize the boundary conditions and parameters used in numerical modeling, they also stress the potential of MPET modeling as a testing bed for hypotheses and new theories in neuroscience research.

Regarding the numerical solution of the MPET equations mainly two different approaches have been investigated in the last couple of years. The first one has been proposed in [34] and uses a mixed finite element formulation based on introducing an additional total pressure variable. Energy estimates for the continuous solutions and a priori error estimates for a family of compatible semidiscretizations demonstrate that this formulation is robust for nearly incompressible materials, small storage coefficients, and small or vanishing transfer between networks.

The second approach is based on a generalization of the classical three-field formulation of Biot’s model and accommodates explicitly Darcy’s law for each fluid network. This formulation enforces the exact conservation of mass at the price of including additionally \( n \) vector fields for the Darcy velocities (fluxes). A parameter-robust stability analysis of this flux-based MPET model has been presented in [26] along with fully parameter-robust norm-equivalent preconditioners. Following [25, 29], the authors propose in [26] a family of strongly conservative locking-free discretizations for the MPET model and establish the related optimal error estimates for the stationary problems arising from implicit time discretization by the backward Euler method. The results also cover the case of vanishing storage coefficients.

Various works can be found on discretizations and efficient iterative solvers and preconditioning techniques for the quasi-static Biot model addressing two-field, see, e.g. [13, 1], three-field, see, e.g., [38, 28, 33, 25], and four-field formulations, see, e.g., [32, 5].

Two of the most popular and probably also most efficient iterative schemes for solving the equation of poroelasticity are the so-called undrained split and fixed-stress split iterative methods, which, contrary to the drained split and the fixed-strain split methods, are unconditionally stable, see [30]. The first convergence analysis of the former methods has been presented in [37] for the quasi-static Biot system. Subsequent refined results focus mostly on variants of the fixed-stress method addressing multirate fixed-stress split iterative schemes [2], fully discrete iterative coupling of flow and geomechanics [3], heterogenous media and linearized Biot’s equations [15], two-grid fixed-stress schemes for heterogeneous media [21], or space-time finite element approximations of the quasi-static Biot system [8]. A strategy for optimizing the stabilization parameter in the fixed-stress split iterative method for the Biot problem in two-field formulation has been presented in [41].

Very recently, the fixed-stress method has also been used successfully in combination with Anderson acceleration for the solution of non-linear poromechanics problems [16]. Moreover, monolithic and splitting based solution schemes have been considered and analyzed for solving quasi-static thermo-poroelasticity problems with nonlinear convective transport [18]. The latter work focuses on the analysis of fully and partially decoupled schemes for heat, mechanics and flow applied to the linearized problem obtained via the so-called L-scheme. All previously mentioned works, in presence of flux and pressure unknowns, solve the flow equations implicitly, i.e., as a coupled subsystem, a strategy which we will not pursue in this paper.

A desirable property of preconditioners, except their uniformity with respect to discretizations parameters, is their robustness regarding potentially large variations of the physical parameters. This task can be studied in the framework of operator preconditioning on the level of the continuous model, cf. [36]. Targeting Biot’s consolidation model this parameter-robustness has been established in [33] for the total-pressure based formulation and in [25] for the classical three-field formulation based on displacement, Darcy velocity and fluid pressure fields. Both approaches have been generalized to the MPET model, see [34, 26].

One potential advantage of the approach presented in [26] is exact mass conservation, whereas the presence of \( n \) fluxes and \( n \) associated pressures makes the system in general more difficult and also more time-consuming to solve. The fixed-stress split iterative method has recently been generalized to be applicable not only to the Biot \((n = 1)\) but also to the more general MPET \((n \geq 1)\) systems in [27]. The paper presents a fully parameter-robust convergence analysis and determines a close to optimal acceleration parameter.
However, in the conservative approach the block of $n$ unknown fluxes (of $d$ scalar quantities each) couples to a block of $n$ pressure unknowns creating a subsystem with $n(d+1)$ scalar quantities of interest as compared to the $(n(d+1)+d)$ unknown scalar functions in the whole system. Hence, considering the above-mentioned four-network model ($n = 4$) in three space dimensions ($d = 3$), for example, this results in a flux-pressure subsystem with approximately 16/19 of the size of the whole system. This explains why a further decoupling of the flux from the pressure block of unknowns in an iterative method is of particular interest in this approach.

The goal of the present paper is to propose and analyze a class of fully decoupled iterative schemes, which contrary to the fixed-stress split iterative method also decouple the flux-pressure subsystem. In this respect, it can be seen as a continuation of the analysis presented in [27].

As already mentioned, the target problem in this paper is a three-by-three block system with a double saddle point. The abstract canonical form of the operator (matrix) of the related operator equation can be represented in the form

\[
\begin{bmatrix}
A_1 & 0 & B_1^T \\
0 & A_2 & B_2^T \\
B_1 & B_2 & -C
\end{bmatrix}
\]

with $A_1$ and $A_2$ being symmetric positive definite (SPD) operators and $C$ being a symmetric positive semidefinite (SPSD) operator. The operator (1.1) defines a double saddle point problem and can be rearranged in such a way that it takes the form

\[
\begin{bmatrix}
A_1 & B_1^T & 0 \\
B_1 & -C & B_2 \\
0 & B_2^T & A_2
\end{bmatrix}
\]

and thus fits the definition of a multiple saddle point operator as given in [40] where block-diagonal Schur complement preconditioners for multiple saddle point problems of block tridiagonal form are analyzed. We will use a combined augmentation and splitting technique to construct in a block Gauss-Seidel framework fully decoupled augmented Lagrangian Uzawa-type methods for linear systems with an operator (matrix) of the canonical form (1.1). Although our methodical approach to construct preconditioners is similar to the one taken in the recent works [9, 10], see also [46], there are also major differences. First of all, the double saddle point problems considered in [9, 10] are generated by operators of the canonical form

\[
\begin{bmatrix}
A_1 & B_1^T & B_2^T \\
B_1 & 0 & 0 \\
B_2 & 0 & -C
\end{bmatrix}
\]

with $A_1$ being SPD and $C$ being SPSD. It can easily be seen that the operators (1.1) and (1.3) are of a different form in the sense that they cannot be transferred one into the other by permutations of rows and columns. The second main difference is that the analysis in [9, 10] uses arguments from classical linear algebra whereas our convergence proofs use techniques from functional analysis aiming at quantitative bounds that may be useful when applying the proposed iterative methods at the level of finite element approximations of the continuous problems.

The remainder of the paper is organized as follow: In Section 2, we first formulate the MPET problem, introduce the notation and transform the problem into a coupled system with a double saddle point operator of the form (1.2). Based on this notation we then recall the fixed-stress split iterative method in a block Gauss-Seidel framework. It follows the construction of a new class of fully decoupled iterative Uzawa-type methods, which requires an additional augmentation step. The section ends with summarizing some preliminary and auxiliary results that are used in the convergence analysis of the new class of methods presented in Section 3. The numerical tests in Section 4 serve the assessment of the performance of the iterative methods and preconditioners developed in this paper comparing them also to the fixed-stress split iterative method analyzed in [27].
2. Iterative coupling methods for the MPET problem

2.1. The MPET system - formulation and notation. Consider the quasi-static MPET equations in a bounded Lipschitz domain $\Omega \subset \mathbb{R}^d, d = 2, 3$:

\begin{align*}
(2.1a) & \quad \mathbf{v}_i = -K_i \nabla p_i \quad \text{in} \quad \Omega \times (0, T), \ i = 1, \ldots, n, \\
(2.1b) & \quad -\alpha_i \text{div} \mathbf{u} - \text{div} \mathbf{v}_i - c_p \dot{p}_i - \sum_{j \neq i} \beta_{ij} (p_i - p_j) = g_i \quad \text{in} \quad \Omega \times (0, T), \ i = 1, \ldots, n, \\
(2.1c) & \quad -\text{div} \sigma + \sum_{i=1}^n \alpha_i \nabla p_i = \mathbf{f} \quad \text{in} \quad \Omega \times (0, T).
\end{align*}

The effective stress and strain tensors are given by

\begin{equation}
(2.2) \quad \sigma = 2\mu \varepsilon(\mathbf{u}) + \lambda \text{div}(\mathbf{u}) \mathbf{I} \quad \text{and} \quad \varepsilon(\mathbf{u}) = \frac{1}{2} (\nabla \mathbf{u} + (\nabla \mathbf{u})^T),
\end{equation}

respectively whereas the Lamé parameters $\lambda$ and $\mu$ are defined via the modulus of elasticity $E$ and the Poisson ratio $\nu \in [0, 1/2]$ as follows:

\[ \lambda := \frac{\nu E}{(1 + \nu)(1 - 2\nu)}, \quad \mu := \frac{E}{2(1 + \nu)}. \]

In (2.1), $\alpha_i$ denote the Biot-Willis coefficients, $K_i$ the hydraulic conductivity tensors, $c_p$ the constrained specific storage coefficients, $f$ the body force density whereas $g_i$ represent the fluid extractions or injections, see e.g. [39] and the references therein. The parameters $\beta_{ij} = \beta_{ji}, i \neq j$ couple the network pressures and are called network transfer coefficients.

By substituting the expression for the stress tensor from (2.2) in (2.1c) the MPET system takes the form:

\begin{align*}
(2.3a) & \quad \mathbf{v}_i + K_i \nabla p_i = \mathbf{0}, \ i = 1, \ldots, n, \\
(2.3b) & \quad -\text{div} \mathbf{v}_i - c_p \dot{p}_i - \sum_{j=1}^n \beta_{ij} (p_i - p_j) - \alpha_i \text{div} \mathbf{u} = g_i, \ i = 1, \ldots, n, \\
(2.3c) & \quad \sum_{i=1}^n \alpha_i \nabla p_i - 2\mu \varepsilon(\mathbf{u}) - \lambda \nabla \varepsilon(\mathbf{u}) = \mathbf{f}.
\end{align*}

After imposing proper boundary and initial conditions, see [26], and using the backward Euler method for time discretization, one has to solve a static problem of the form

\begin{align*}
(2.4a) & \quad K_i^{-1} \mathbf{v}_i^k + \nabla p_i^k = \mathbf{0}, \ i = 1, \ldots, n, \\
(2.4b) & \quad -\alpha_i \varepsilon(\mathbf{u}^k) - \tau \varepsilon(\mathbf{v}_i^k) - c_p p_i^k - \tau \sum_{j=1}^n \beta_{ij} (p_i^k - p_j^k) = g_i^k, \ i = 1, \ldots, n, \\
(2.4c) & \quad -2\mu \varepsilon(\mathbf{u}^k) - \lambda \nabla \varepsilon(\mathbf{u}^k) + \sum_{i=1}^n \alpha_i \nabla p_i^k = \mathbf{f}^k,
\end{align*}

in each time step, i.e., at every time moment $t_k = t_{k-1} + \tau, k = 1, 2, \ldots$. Here, $\mathbf{u}^k, \mathbf{v}_i^k, p_i^k$ are approximations of $\mathbf{u}, \mathbf{v}_i, p_i$ at $t = t_k$ and $\mathbf{f}^k = \mathbf{f}(x, t_k)$, $g_i^k = -\tau g_i(x, t_k) - \alpha_i \varepsilon(\mathbf{u}^{k-1}) - c_p p_i^{k-1}$ for $i = 1, \ldots, n$. After dividing (2.4) by $2\mu$,
denoting
\[\frac{\lambda}{2\mu} \rightarrow \lambda, \frac{\alpha_i}{2\mu} \rightarrow \alpha_i, \frac{f^k}{2\mu} \rightarrow f^k, \frac{\tau}{2\mu} \rightarrow \tau, \frac{c_p}{2\mu} \rightarrow c_p, \frac{q^k}{2\mu} \rightarrow q^k, \quad i = 1, \ldots, n,\]
and further introducing the new variables
\[v_i := \frac{\tau}{\alpha_i} v^k_i, \quad p_i := \alpha_i p^k_i, \quad u := u^k, \quad f := f^k, \quad g_i := \frac{q^k}{\alpha_i}, \quad i = 1, \ldots, n,\]
system (2.4) can be presented in the form
\[(2.5a) \quad \tau^{-1} K_i^{-1} \alpha_i^2 v_i + \nabla p_i = 0, \quad i = 1, \ldots, n,\]
\[(2.5b) \quad -\text{div} u - \text{div} v_i - \frac{c_p}{\alpha_i} p_i + \sum_{j=1}^{n} \left( -\frac{\tau \beta_{ij}}{\alpha_i} p_i + \frac{\tau \beta_{ij}}{\alpha_i \alpha_j} p_j \right) = g_i, \quad i = 1, \ldots, n,\]
\[(2.5c) \quad -\text{div} \epsilon(u) - \lambda \nabla \text{div} u + \sum_{i=1}^{n} \nabla p_i = f,\]
where we have also multiplied (2.4a) by \(\alpha_i\) and (2.4b) by \(\alpha_i^{-1}\). Using the parameter substitutions
\[R_i^{-1} := \tau^{-1} K_i^{-1} \alpha_i^2, \quad \alpha_{p_i} := \frac{c_p}{\alpha_i^2}, \quad \beta_{ii} := \sum_{j=1}^{n} \beta_{ij}, \quad \alpha_{ij} := \frac{\tau \beta_{ij}}{\alpha_i \alpha_j}, \quad \tilde{\alpha}_{ii} := -\alpha_{p_i} - \alpha_{ii}\]
for \(i, j = 1, \ldots, n\), we further rewrite system (2.5) as
\[(2.6) \quad A \begin{bmatrix} v_1^T, \ldots, v_n^T, p_1, \ldots, p_n, u^T \end{bmatrix}^T = \begin{bmatrix} 0^T, \ldots, 0^T, g_1, \ldots, g_n, f^T \end{bmatrix}^T,\]
where \(A\) in (2.6) is given by
\[(2.7) \quad A := \begin{bmatrix} R_1^{-1} I & 0 & \ldots & 0 & \nabla & 0 & \ldots & 0 & 0 & 0 \\ 0 & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \ddots & 0 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \ldots & 0 & R_n^{-1} I & 0 & \ldots & 0 & \nabla & 0 & \nabla \\ 0 & \ldots & 0 & \tilde{\alpha}_{11} I & \alpha_{12} I & \ldots & \alpha_{1n} I & -\text{div} & \text{div} & \text{div} \\ 0 & \ldots & 0 & \alpha_{21} I & \ddots & \alpha_{2n} I & \vdots & \vdots & \vdots & \vdots \\ \vdots & \ldots & 0 & \ldots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \ldots & 0 & -\text{div} \alpha_{n1} I & \alpha_{n2} I & \ldots & \tilde{\alpha}_{nn} I & -\text{div} & \text{div} & \text{div} \\ 0 & \ldots & 0 & \nabla & \ldots & \nabla & \vdots & -\text{div} \epsilon - \lambda \nabla \text{div} \end{bmatrix}.
\]
For the scaled parameters, we make the non-restrictive assumptions
\[(2.8) \quad \lambda > 0, \quad R_1^{-1}, \ldots, R_n^{-1} > 0, \quad \alpha_{p_1}, \ldots, \alpha_{p_n} \geq 0, \quad \alpha_{ij} \geq 0, \quad i, j = 1, \ldots, n.\]
In what follows we will also make use of the notation \(v^T := (v_1^T, \ldots, v_n^T), \quad z^T := (z_1^T, \ldots, z_n^T), \quad p^T := (p_1, \ldots, p_n), \quad q^T := (q_1, \ldots, q_n)\) where \(v, z \in V = V_1 \times \cdots \times V_n, \quad p, q \in P = P_1 \times \cdots \times P_n\) and \(U = \{u \in H^1(\Omega)^d : u = 0 \text{ on } \Gamma_{u,D}\}, \quad V_i = \{v_i \in H(\text{div}, \Omega) : v_i \cdot n = 0 \text{ on } \Gamma_{p_i,N}\}, \quad P_1 = L^2(\Omega), \quad P_k = L^2(\Omega) \text{ if } \Gamma_{u,D} = \Gamma = \partial \Omega.\)
Further, we define
\[
A_v := \begin{bmatrix} R_1^{-1}I & 0 & \cdots & 0 \\ 0 & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & R_n^{-1}I \end{bmatrix}, \quad B_v := \begin{bmatrix} -\text{div} & 0 & \cdots & 0 \\ 0 & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & -\text{div} \end{bmatrix}, \quad B_u := \begin{bmatrix} -\text{div} & 0 & \cdots & 0 \\ 0 & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & -\text{div} \end{bmatrix}, \quad -C := \begin{bmatrix} \tilde{\alpha}_{11}I & \alpha_{12}I & \cdots & \alpha_{1n}I \\ \alpha_{21}I & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ \alpha_{n1}I & \alpha_{n2}I & \cdots & \tilde{\alpha}_{nn}I \end{bmatrix}
\]
and \(A_u := -\text{div}\epsilon - \lambda \nabla \text{div}\).

Then system (2.6) can be rewritten as
\[
(2.9) \quad A \begin{pmatrix} v \\ p \\ u \end{pmatrix} = \begin{bmatrix} A_v & B_v^T & 0 \\ B_v & -C & B_u \\ 0 & B_u^T & A_u \end{bmatrix} \begin{pmatrix} v \\ p \\ u \end{pmatrix} = \begin{pmatrix} 0 \\ g \\ f \end{pmatrix}.
\]

From now on, we will use the same symbols for denoting operators and their corresponding coefficient matrices. Additionally, let us introduce
\[
\Lambda_1 := \begin{bmatrix} \alpha_{11} & -\alpha_{12} & \cdots & -\alpha_{1n} \\ -\alpha_{21} & \alpha_{22} & \cdots & -\alpha_{2n} \\ \vdots & \ddots & \ddots & \vdots \\ -\alpha_{n1} & -\alpha_{n2} & \cdots & \alpha_{nn} \end{bmatrix}, \quad \Lambda_2 := \begin{bmatrix} \alpha_{p1} & 0 & \cdots & 0 \\ 0 & \alpha_{p2} & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & \alpha_{pn} \end{bmatrix},
\]
i.e. \(C = \Lambda_1 + \Lambda_2\). Further, denote \(R_i^{-1} := \max\{R_i^{-1} : i = 1, 2, \cdots, n\}, \lambda_0 := \max\{1, \lambda\},
\[
\Lambda_3 := \begin{pmatrix} R & 0 & \cdots & 0 \\ 0 & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & R \end{pmatrix}, \quad \Lambda_4 := \begin{pmatrix} \frac{1}{\lambda_0} & \cdots & \frac{1}{\lambda_0} \\ \vdots & \ddots & \vdots \\ \frac{1}{\lambda_0} & \cdots & \frac{1}{\lambda_0} \end{pmatrix},
\]
and also, for any block vector \(z\) and vector \(u\)
\[
\text{Div}z := \begin{pmatrix} \text{div}z_1 \\ \vdots \\ \text{div}z_n \end{pmatrix}, \quad \text{Div}u := \begin{pmatrix} \text{div}u \\ \vdots \\ \text{div}u \end{pmatrix}.
\]

2.2. The fixed-stress split iterative method revisited. For any operator \(\Lambda_L : P \rightarrow P^*, A\) can be decomposed as follows:
\[
(2.10) \quad A = \begin{bmatrix} A_v & B_v^T & 0 \\ B_v & -C - \Lambda_L & 0 \\ 0 & B_u^T & A_u \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \Lambda_L & B_u \\ 0 & 0 & 0 \end{bmatrix}.
\]

Applying the block Gauss-Seidel method to the above system, we obtain
\[
(2.11) \quad \begin{bmatrix} A_v & B_v^T & 0 \\ B_v & -C - \Lambda_L & 0 \\ 0 & B_u^T & A_u \end{bmatrix} \begin{pmatrix} v^{k+1} \\ p^{k+1} \\ u^{k+1} \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & \Lambda_L & B_u \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} v^k \\ p^k \\ u^k \end{pmatrix} = \begin{pmatrix} 0 \\ g \\ f \end{pmatrix}.
\]
or, equivalently,

\[
\begin{bmatrix}
A_v & B_v^T MB_v & 0 \\
B_v & -C - \Lambda L & 0 \\
0 & B_u^T A_u & A_u
\end{bmatrix}
\begin{bmatrix}
v^{k+1} \\
p^{k+1} \\
w^{k+1}
\end{bmatrix} =
\begin{bmatrix}
0 \\
g \\
f
\end{bmatrix} -
\begin{bmatrix}
0 & 0 & 0 \\
0 & \Lambda L & B_u \\
0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
v^k \\
p^k \\
w^k
\end{bmatrix}
\]

which is (a block variant of) the fixed-stress method. In [27] a parameter-robust convergence analysis of this method has been presented for the choice

\[
\Lambda_L = L \begin{bmatrix}
I & I & \ldots & I \\
I & \ddots & & I \\
\vdots & & \ddots & \vdots \\
I & I & \ldots & I
\end{bmatrix}
\]

where

\[
L \geq \frac{1}{\lambda + c_K^2},
\]

and \(c_K\) is the constant in the estimate

\[
\|\epsilon(w)\| \geq c_K \|\text{div}w\| \quad \text{for all } w \in U.
\]

Note that (2.14) holds true for example for \(c_K = 1/\sqrt{d}\) where \(d\) is the space dimension.

2.3. Uzawa-type methods in block Gauss-Seidel framework. Now for any positive definite operator \(M : P^* \to P\), we consider the equivalent augmented MPET system

\[
\begin{bmatrix}
A_v + B_v^T MB_v & B_v^T - B_v^T MC & B_v^T MB_u \\
-B_v & C & -B_u \\
0 & B_u^T & A_u
\end{bmatrix}
\begin{bmatrix}
v \\
p \\
u
\end{bmatrix} =
\begin{bmatrix}
B_v^T M g \\
g \\
f
\end{bmatrix}.
\]

Further, for any positive definite operator \(S : P \to P^*\), we decompose \(\hat{A}\) in the form

\[
\hat{A} =
\begin{bmatrix}
A_v + B_v^T MB_v & 0 & 0 \\
-B_v & S & 0 \\
0 & B_u^T & A_u
\end{bmatrix}
+ \begin{bmatrix}
0 & B_v^T - B_v^T MC & B_v^T MB_u \\
0 & -S + C & -B_u \\
0 & 0 & 0
\end{bmatrix}.
\]

Next, applying the block Gauss-Seidel method to the above system yields

\[
\begin{bmatrix}
A_v + B_v^T MB_v & 0 & 0 \\
-B_v & S & 0 \\
0 & B_u^T & A_u
\end{bmatrix}
\begin{bmatrix}
v^{k+1} \\
p^{k+1} \\
w^{k+1}
\end{bmatrix} =
\begin{bmatrix}
0 & B_v^T - B_v^T MC & B_v^T MB_u \\
0 & -S + C & -B_u \\
0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
v^k \\
p^k \\
w^k
\end{bmatrix} =
\begin{bmatrix}
B_v^T M g \\
g \\
f
\end{bmatrix},
\]

namely

\[
\begin{bmatrix}
A_v + B_v^T MB_v & 0 & 0 \\
-B_v & S & 0 \\
0 & B_u^T & A_u
\end{bmatrix}
\begin{bmatrix}
v^{k+1} \\
p^{k+1} \\
w^{k+1}
\end{bmatrix} =
\begin{bmatrix}
B_v^T M g \\
g \\
f
\end{bmatrix} -
\begin{bmatrix}
0 & B_v^T - B_v^T MC & B_v^T MB_u \\
0 & -S + C & -B_u \\
0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
v^k \\
p^k \\
w^k
\end{bmatrix}.
\]

System (2.18) can be expressed in terms of bilinear forms as follows:
Algorithm 1 Fully decoupled iterative scheme for flux-pressure-displacement formulation of the MPET system

Step a: Given \( p^k \) and \( u^k \), we solve first for \( v^{k+1} \),

\[
(A, v^{k+1}, z) + (M(\text{Div}v^{k+1}, \text{Div}z) = -(Mg, \text{Div}z) + (p^k, \text{Div}z) - (M(\Lambda_1 + \Lambda_2)p^k, \text{Div}z) - (M(\text{Div}u^k, \text{Div}z).
\]

Step b: Given \( u^k \) and \( v^{k+1} \), we solve for \( p^{k+1} \),

\[
(Sp^{k+1}, q) = -(g, q) + (Sp^k, q) - ((\Lambda_1 + \Lambda_2)p^k, q) - (\text{Div}u^k, q) - (\text{Div}v^{k+1}, q).
\]

Step c: Given \( p^{k+1} \) and \( v^{k+1} \), we solve for \( u^{k+1} \),

\[
(e(u^{k+1}), e(w)) + \lambda(divu^{k+1}, divw) = (f, w) + (p^{k+1}, \text{Div}w).
\]

2.4. Preliminary results. We first present a result from linear algebra which will be useful in the proof of Theorem 8 in Section 3.

Lemma 1. For any \( a > 0, b > 0 \), denote \( e = (1, \ldots, 1)^T \) and \( (aI_{n \times n} + b ee^T)^{-1} = (b_{ij})_{n \times n} \). Then we have that

\[
0 < \sum_{i=1}^{n} \sum_{j=1}^{n} b_{ij} = \frac{n}{(a + nb)}.
\]

Proof. In the manner of the proof of Lemma 1 in [26] and by using the Sherman-Morrison-Woodbury formula, the proof is obvious. \( \square \)

Next, let us recall some well known results, see [17, 14].

Lemma 2. There exists a constant \( \beta_s > 0 \) such that:

\[
\inf_{(q_1, \ldots, q_n) \in \mathbb{R}^n} \sup_{u \in U} \frac{(\text{div}u, \sum_{i=1}^{n} q_i)}{\|u\|_1 \|\sum_{i=1}^{n} q_i\|} \geq \beta_s
\]

Lemma 3. There exists a constant \( \beta_d > 0 \) such that:

\[
\inf_{q \in \mathbb{R}^n} \sup_{u \in U} \frac{(\text{div}v, q)}{\|v\|_{\text{div}} \|q\|} \geq \beta_d, \quad i = 1, \ldots, n.
\]

Our task will be to study the errors

\begin{align*}
  e^k_u &= u^k - u \in U, \quad (2.25a) \\
  e^k_{v_i} &= v^k_i - v_i \in V_i, \quad i = 1, \ldots, n, \quad (2.25b) \\
  e^k_{p_i} &= p^k_i - p_i \in P_i, \quad i = 1, \ldots, n, \quad (2.25c)
\end{align*}

of the \( k \)-th iterates \( u^k, v^k, p^k_i, \ i = 1, \ldots, n \), generated by Algorithm 1. For that reason, we consider the following error equations

\begin{align*}
  (A, e^{k+1}_u, z) - (e^k_p, \text{Div}z) + (M(\text{Div}e^{k+1}_u, \text{Div}z) = -(Mg, \text{Div}z) + (e^k_p, \text{Div}z) - (M(\Lambda_1 + \Lambda_2)e^k_p, \text{Div}z) = 0, \quad (2.26a) \\
  (Se^{k+1}_p, q) - (Se^k_p, q) + (\text{Div}e^{k+1}_u, q) + (\text{Div}e^k_p, q) + ((\Lambda_1 + \Lambda_2)e^k_p, q) = 0, \quad (2.26b) \\
  (e(e^{k+1}_u, e(w)) + \lambda(\text{div}e^{k+1}_u, \text{div}w) - (e^k_p, \text{Div}w) = 0 \quad (2.26c)
\end{align*}
where the error block-vectors $e_u^k$ and $e_p^k$ are given by $(e_u^k)^T = ((e_u^k)^T, \ldots, (e_u^k)^T)$, $(e_p^k)^T = (e_p^k, \ldots, e_p^k)$.

To complete the design of Algorithm 1, we need to specify $M$ and $S$. By Lemma 3 we have that for all $e_p^{k+1} \in P$, there exists $\psi_i \in V_i$ such that $\text{div} \psi_i = e_p^{k+1}$ and $\|\psi_i\|_{\text{div}} \leq \beta^{-1}_d \|e_p^k\|$ for all $i = 1, \ldots, n$, i.e., $\text{Div} e_p = e_p^{k+1}$ and $\|\psi\|_{\text{div}} \leq \beta^{-1}_d \|e_p^{k+1}\|$. Setting $q = S^{-1} e_p^{k+1}$ in (2.26b) and $z = \psi$ in (2.26a), from $\text{Div} \psi = e_p^{k+1}$ it follows that

\begin{align}
(2.27a) & \quad (A_v e_v^{k+1}, \psi) - (e_p^k, e_p^{k+1}) + (M \text{Div} e_u^{k+1}, e_v^{k+1}) + (M \text{Div} e_v^{k+1}, e_p^{k+1}) + (M (\Lambda_1 + \Lambda_2) e_p^k, e_p^{k+1}) = 0, \\
(2.27b) & \quad (e_p^{k+1}, e_p^{k+1}) - (e_p^k, e_p^{k+1}) + (S^{-1} \text{Div} e_u, e_v^{k+1}) + (S^{-1} \text{Div} e_v, e_p^{k+1}) + (S^{-1} (\Lambda_1 + \Lambda_2) e_p^k, e_p^{k+1}) = 0.
\end{align}

Subtracting (2.27a) from (2.27b) yields

$$
\|e_p^{k+1}\|^2 = (A_v e_v^{k+1}, \psi) - ((S^{-1} - M) (\text{Div} e_u + \text{Div} e_v^{k+1} + (\Lambda_1 + \Lambda_2) e_p^k), e_p^{k+1}),
$$

implying

$$
\|e_p^{k+1}\|^2 \leq \|A_v \| e_v^{k+1}\| A_v \| \| \psi \| + \|(S^{-1} - M) (\text{Div} e_u + \text{Div} e_v^{k+1} + (\Lambda_1 + \Lambda_2) e_p^k)\| \| e_p^{k+1}\| \\
\leq \sqrt{R^{-1}} \| A_v \| e_v^{k+1}\| \| \psi \| + \|(S^{-1} - M) (\text{Div} e_u + \text{Div} e_v^{k+1} + (\Lambda_1 + \Lambda_2) e_p^k)\| \| e_p^{k+1}\| \\
\leq \beta^{-1}_d \sqrt{R^{-1}} \| A_v \| e_v^{k+1}\| \| e_p^{k+1}\| + \|(S^{-1} - M) (\text{Div} e_u + \text{Div} e_v^{k+1} + (\Lambda_1 + \Lambda_2) e_p^k)\| \| e_p^{k+1}\|.
$$

We conclude that

$$
(2.28) \quad \|e_p^{k+1}\| \leq \beta^{-1}_d \sqrt{R^{-1}} \| A_v \| e_v^{k+1}\| + \|(S^{-1} - M) (\text{Div} e_u + \text{Div} e_v^{k+1} + (\Lambda_1 + \Lambda_2) e_p^k)\|.
$$

Estimate (2.28) suggests to choose $S = M^{-1}$ if we want to minimize the upper bound for $\|e_p^{k+1}\|$. This results in the following statement.

**Lemma 4.** Consider Algorithm 1 and let $S = M^{-1}$, then we have

$$
(2.29) \quad \| A_v \| e_v^{k+1}\|^2 \geq R \beta^{-2}_d \| e_p^{k+1}\|^2 = \beta^{-2}_d \| A_v \| e_v^{k+1}\|^2.
$$

The relationship $S = M^{-1}$ reduces our design task to the determination of either $S$ or $M$. In the remainder of this paper we will analyze and numerically test Algorithm 1 for the specific choice

$$
(2.30) \quad S := \Lambda_1 + \Lambda_2 + L_1 \Lambda_4 + L_2 \Lambda_4,
$$

where $L_1$ and $L_2$ are scalar parameters which are to be determined later.

3. **Convergence Theory of Uzawa-Type Algorithms for MPET**

This section is devoted to the convergence analysis of Algorithm 1. Our aim is to establish a uniform bound on the convergence rate, i.e., a bound which is independent of any model and discretization parameters.

We start with deriving some useful auxiliary results presented in the following two lemmas which afterwards will help us to establish a parameter-robust upper bound on the pressure error in a weighted norm.

**Lemma 5.** Consider Algorithm 1 with $S$ as defined in (2.30), the errors $e_u^k$, $e_v^k$ and $e_p^k$ defined in (2.25) satisfy the following estimate:

$$
\begin{align}
&\frac{1}{2} \| e_u^{k+1}\|^2 + \frac{\lambda}{2} \| e_v^{k+1}\|^2 + \| A_v \| e_v^{k+1}\|^2 + \| (\Lambda_1 + \Lambda_2) e_p^k \|^2 + \frac{L_1}{2} \| A_v \| e_p^k \|^2 + \frac{L_2}{2} \| A_v \| e_p^k \|^2 \\
\leq & \frac{L_1}{2} \| A_v \| e_p^k \|^2 + \frac{L_2}{2} \| A_v \| e_p^k \|^2 + \left( \frac{\lambda_0}{2(c_K + \lambda)} - \frac{L_2}{2} - \frac{L_1 R \lambda_0}{2n} \right) \| A_v \| e_p^k \|^2.
\end{align}
$$

(3.1)
Proof. By setting \( q = \text{MDive} e^{k+1}_v \) in (2.26b) and \( z = e^{k+1}_v \) in (2.26a) we obtain

\[
(A_v e^{k+1}_v, e^{k+1}_v) - (e^{k}_v, \text{Dive} e^{k+1}_v) + (\text{MDive} u, \text{Dive} e^{k+1}_v) + (\text{MDive} v, \text{Dive} e^{k+1}_v) + (M(L_1 + L_2) e^k_p, \text{Dive} e^{k+1}_v) = 0,
\]

\[
(e^{k+1}_p, \text{Dive} e^{k+1}_v) = (e^{k}_p, \text{Dive} e^{k+1}_v) - (\text{MDive} u, \text{Dive} e^{k+1}_v) - (\text{MDive} v, \text{Dive} e^{k+1}_v) - (M(L_1 + L_2) e^k_p, \text{Dive} e^{k+1}_v)
\]

from where it immediately follows that

\[
(e^{k+1}_p, \text{Dive} e^{k+1}_v) = (A_v e^{k+1}_v, e^{k+1}_v).
\]

Choosing \( q = e^{k+1}_p \) in (2.26b) and \( w = e^{k+1}_u \) in (2.26c) yields

\[
\begin{align*}
(\epsilon(e^{k+1}_u), (e^{k+1}_u)) + \lambda(\text{div} e^{k+1}_u, e^{k+1}_u) - (e^{k+1}_p, \text{div} e^{k+1}_u) &= 0, \\
(S e^{k+1}_p, e^{k+1}_u) &= (S e^{k}_p, e^{k+1}_u) - (\text{Dive} e^{k}_u, e^{k+1}_u) - (\text{Dive} e^{k}_v, e^{k+1}_u) - ((L_1 + L_2) e^k_p, e^{k+1}_u) + (\text{div} e^{k+1}_p, e^{k+1}_u).
\end{align*}
\]

Next, summing (3.3a) and (3.3b) and applying (3.1) it follows that

\[
\begin{align*}
||\epsilon(e^{k+1}_u)||^2 + \lambda||\text{div} e^{k+1}_u||^2 + ||S \epsilon e^{k+1}||^2 - ((L_1 A_3 + L_2 A_4) e^k_v, e^{k+1}_v) &= (\text{div} e^{k+1}_u, e^{k+1}_u) - ||A_{\frac{1}{2}} e^{k+1}_v||^2.
\end{align*}
\]

In order to simplify (3.4) we first rewrite \( ||S \epsilon e^{k+1}||^2 - ((L_1 A_3 + L_2 A_4) e^k_v, e^{k+1}_v) \), that is

\[
\begin{align*}
||S \epsilon e^{k+1}||^2 - ((L_1 A_3 + L_2 A_4) e^k_v, e^{k+1}_v) &= ||(A_1 + A_2) \epsilon e^{k+1}||^2 + L_1 (||A_{\frac{1}{2}} e^{k+1}||^2 - ||A_{\frac{1}{2}} e^{k}||^2) + L_2 (||A_{\frac{1}{2}} e^{k}||^2 - ||A_{\frac{1}{2}} e^{k+1}||^2) \\
&\geq ||(A_1 + A_2) \epsilon e^{k+1}||^2 + L_1 ||A_{\frac{1}{2}} e^{k}||^2 + L_2 ||A_{\frac{1}{2}} e^{k+1}||^2 \\
&\geq L_1 ||A_{\frac{1}{2}} e^{k}||^2 - L_2 ||A_{\frac{1}{2}} e^{k+1}||^2 + L_2 ||A_{\frac{1}{2}} e^{k+1}||^2.
\end{align*}
\]

Second, we estimate \( (\text{div} e^{k+1}_u, e^{k+1}_u) \). By setting \( w = e^{k+1}_u - e^k_u \) in (2.26c) we get

\[
\begin{align*}
(e^{k+1}_p, \text{Dive} e^{k+1}_u, e^{k+1}_u) &= (e^{k+1}_p, e^{k+1}_u) + \lambda(\text{div} e^{k+1}_u, e^{k+1}_u) - \frac{1}{2}(||e^{k+1}_u - e^k_u||^2 + \lambda||\text{div} e^{k+1}_u||^2) \leq \frac{1}{2}(||e^{k+1}_u - e^k_u||^2 + \lambda||\text{div} e^{k+1}_u||^2).
\end{align*}
\]

In order to estimate the right-hand side of (3.6), we subtract the \( k \)-th error from the \( (k+1) \)-th error and choose \( w = e^{k+1}_u - e^k_u \) in (2.26c) and herewith obtaining

\[
\begin{align*}
||e^{k+1}_u - e^k_u||^2 + \lambda||\text{div} e^{k+1}_u - e^k_u||^2 &= (\sum_{i=1}^n (e^{k+1}_p - e^k_p), \text{div} e^{k+1}_u - e^k_u).
\end{align*}
\]

Applying Cauchy’s inequality further yields

\[
\begin{align*}
||e^{k+1}_u - e^k_u||^2 + \lambda||\text{div} e^{k+1}_u - e^k_u||^2 &= (\sum_{i=1}^n (e^{k+1}_p - e^k_p), \text{div} e^{k+1}_u - e^k_u) \leq \sum_{i=1}^n (e^{k+1}_p - e^k_p)||\text{div} e^{k+1}_u - e^k_u||
\end{align*}
\]

(3.7)

Noting that

\[
(c^2 + \lambda)||\text{div} w||^2 \leq ||e(w)||^2 + \lambda||\text{div} w||^2,
\]

(3.8)
which follows from (2.14), we directly obtain
\[(c_K^2 + \lambda)\|\text{div}(e_u^{k+1} - e_u^k)\|^2 \leq \sqrt{\lambda_0}\|\Lambda_4^\frac{1}{2}(e_p^{k+1} - e_p^k)\| \cdot \|\text{div}(e_u^{k+1} - e_u^k)\|,\]
from (3.7). The latter estimate implies
\[\|\text{div}(e_u^{k+1} - e_u^k)\| \leq \frac{\sqrt{\lambda_0}}{c_K^2 + \lambda} \|\Lambda_4^\frac{1}{2}(e_p^{k+1} - e_p^k)\|.\]
By using the above inequality in (3.7), it follows that
\[\|e(e_u^{k+1} - e_u^k)\|^2 + \lambda \|\text{div}(e_u^{k+1} - e_u^k)\|^2 \leq \frac{\lambda_0}{c_K^2 + \lambda} \|\Lambda_4^\frac{1}{2}(e_p^{k+1} - e_p^k)\|^2.\]
Now, combining (3.6) and (3.9) yields
\[(3.10) \quad \left(\epsilon_p^{k+1}, \text{div}(e_u^{k+1} - e_u^k)\right) \leq \frac{\lambda_0}{2(c_K^2 + \lambda)} \|\Lambda_4^\frac{1}{2}(e_p^{k+1} - e_p^k)\|^2 + \frac{1}{2} \|\epsilon(e_u^{k+1})\|^2 + \lambda \|\text{div}e_u^{k+1}\|^2.\]
Finally, inserting (3.5) and (3.10) in (3.4) we get
\[
\|\epsilon(e_u^{k+1})\|^2 + \lambda \|\text{div}e_u^{k+1}\|^2 + \|\Lambda_4 \epsilon_p^{k+1}\|^2 + \frac{L_1}{2} \|\Lambda_4 \epsilon_p^{k+1}\|^2 + \frac{L_2}{2} \|\Lambda_4 \epsilon_p^{k+1}\|^2
\leq \frac{\lambda_0}{2(c_K^2 + \lambda)} \|\Lambda_4^\frac{1}{2}(e_p^{k+1} - e_p^k)\|^2 + \frac{1}{2} \|\epsilon(e_u^{k+1})\|^2 + \lambda \|\text{div}e_u^{k+1}\|^2
\leq \frac{L_1}{2} \|\Lambda_4 e_p^{k+1}\|^2 + \frac{L_2}{2} \|\Lambda_4 e_p^{k+1}\|^2 + \frac{L_1 R \lambda_0}{2n} \|\Lambda_4^\frac{1}{2}(e_p^{k+1} - e_p^k)\|^2.
\]
which shows (3.1).

The next lemma provides a preliminary estimate for the pressure errors.

**Lemma 6.** Consider Algorithm 1 with $S$ as in (2.30). Then the errors $\epsilon_p^k$ defined in (2.25) satisfy
\[
\frac{\lambda_0}{2(\beta_s^2 + \lambda)} \|\Lambda_4^\frac{1}{2}(e_p^{k+1})\|^2 + \beta_s^2 \|\Lambda_4^\frac{1}{2}(e_p^{k+1})\|^2 + \|\Lambda_4 \epsilon_p^{k+1}\|^2 + \frac{L_1}{2} \|\Lambda_4 \epsilon_p^{k+1}\|^2 + \frac{L_2}{2} \|\Lambda_4 \epsilon_p^{k+1}\|^2
\leq \frac{L_1}{2} \|\Lambda_4 e_p^{k+1}\|^2 + \frac{L_2}{2} \|\Lambda_4 e_p^{k+1}\|^2 + \frac{L_1 R \lambda_0}{2n} \|\Lambda_4^\frac{1}{2}(e_p^{k+1} - e_p^k)\|^2.
\]

**Proof.** From Lemma 2 it follows that for all $\sum_{i=1}^n e_{p_i}^{k+1} \in P_i$ there exists $w_0 \in U$ such that $\text{div}w_0 = \frac{1}{\sqrt{\lambda_0}} \sum_{i=1}^n e_{p_i}^{k+1}$ and $\|w_0\|_1 \leq \beta_s^{-1} \frac{1}{\sqrt{\lambda_0}} \sum_{i=1}^n e_{p_i}^{k+1} = \beta_s^{-1} \|\Lambda_4^\frac{1}{2}(e_p^{k+1})\|$, also,
\[
\text{div}w_0 = \left( \frac{1}{\sqrt{\lambda_0}} \sum_{i=1}^n e_{p_i}^{k+1} \right) = \sqrt{\lambda_0} \Lambda_4 e_p^{k+1}.
\]
Taking $w = w_0$ in (2.26c), it follows that
\[
\sqrt{\lambda_0} \|\Lambda_4^\frac{1}{2}(e_p^{k+1})\|^2 = (\epsilon(e_u^{k+1}), \epsilon(w_0)) + \lambda (\text{div}e_u^{k+1}, \text{div}w_0) \leq (\|\epsilon(e_u^{k+1})\|^2 + \lambda \|\text{div}e_u^{k+1}\|^2)^{\frac{1}{2}} \cdot (\|\epsilon(w_0)\|^2 + \lambda \|\text{div}w_0\|^2)^{\frac{1}{2}} \leq (\|\epsilon(e_u^{k+1})\|^2 + \lambda \|\text{div}e_u^{k+1}\|^2)^{\frac{1}{2}} \cdot (\beta_s^{-2} \|\Lambda_4^\frac{1}{2}(e_p^{k+1})\|^2 + \lambda \|\Lambda_4^\frac{1}{2}(e_p^{k+1})\|^2)^{\frac{1}{2}} \leq (\|\epsilon(e_u^{k+1})\|^2 + \lambda \|\text{div}e_u^{k+1}\|^2)^{\frac{1}{2}} \cdot (\beta_s^{-2} + \lambda)^{\frac{1}{2}} \|\Lambda_4^\frac{1}{2}(e_p^{k+1})\|.
and, therefore,

\begin{equation}
\frac{\lambda_0}{\beta_s^2 + \lambda} \|A_0^{1/2}e_p^{k+1}\|^2 \leq \|\epsilon(e_u^{k+1})\|^2 + \lambda \|\text{div}e_u^{k+1}\|^2.
\end{equation}

Using (3.11) and (2.29) in (3.1), we arrive at

\begin{equation}
\frac{\lambda_0}{2(\beta_s^2 + \lambda)} \|A_0^{1/2}e_p^{k+1}\|^2 + \beta_d^2 \|A_0^{1/2}e_p^{k}\|^2 + \|(A_1 + A_2)^{1/2}e_p^{k+1}\|^2 + \frac{L_1}{2} \|A_0^{1/2}e_p^{k+1}\|^2 + \frac{L_2}{2} \|A_0^{1/2}e_p^{k+1}\|^2
\end{equation}

\begin{equation}
\leq \frac{L_1}{2} \|A_0^{1/2}e_p^{k}\|^2 + \frac{L_2}{2} \|A_0^{1/2}e_p^{k}\|^2 + \left(\frac{\lambda_0}{2(c_R^2 + \lambda)} - \frac{L_2}{2} - \frac{L_1 R\lambda_0}{2n}\right) \|A_0^{1/2}(e_p^{k+1} - e_p^{k})\|^2.
\end{equation}

The following two theorems present the main convergence results for Algorithm 1.

**Theorem 7.** Consider Algorithm 1. For any \( \theta > 0 \) and \( L_2 \geq \frac{\lambda_0}{(c_R^2 + \lambda)(1 + \frac{\beta_d^2 R\lambda_0}{n})} \), \( L_1 = \theta \beta_d^2 L_2 \), the errors \( e_p^{k} \) defined in (2.25) satisfy the estimate:

\begin{equation}
\|e_p^{k+1}\|_{P_0}^2 \leq \text{rate}^2(\lambda, R, \theta)\|e_p^{k}\|_{P_0}^2
\end{equation}

with

\[ \text{rate}^2(\lambda, R, \theta) = \frac{1}{\frac{\lambda_0}{(c_R^2 + \lambda)(1 + \frac{\beta_d^2 R\lambda_0}{n})} + 1} \geq \min \left\{ \frac{\lambda_0}{\beta_s^2 + \lambda}, \lambda_0 \right\} L_2^{-1} \]

and

\begin{equation}
\|e_p^{k+1}\|_{P_0} = \|A_0^{1/2}e_p^{k+1}\|^2 + \beta_d^2 \|A_0^{1/2}e_p^{k}\|^2 + \|(A_1 + A_2)^{1/2}e_p^{k+1}\|^2.
\end{equation}

1. For \( \theta = \theta_a := \beta_d^{-2} \) and \( L_2 = \frac{\lambda_0}{(c_R^2 + \lambda)(1 + \frac{\beta_d^2 R\lambda_0}{n})} \), we obtain the convergence factor under the norm \( \| \cdot \|_{P_0} \) estimated by

\[
\text{rate}^2(\lambda, R) \leq \frac{1}{\frac{\lambda_0}{(c_R^2 + \lambda)(1 + \frac{\beta_d^2 R\lambda_0}{n})} + 1} \leq \max \left\{ \frac{1}{c_0 + 1}, \frac{1}{c_0 c_R^2 + 1}, \frac{1}{2} \right\}, \text{ where } c_0 = \min \left\{ \frac{\lambda_0}{\beta_s^2 + \lambda}, \frac{\lambda_0}{2 \beta_d^2} \right\}.
\]

Here for any \( x \in P \)

\[
\|x\|_{P_0} := \|A_0^{1/2}x\|^2 + \|A_0^{1/2}x\|^2 + \|(A_1 + A_2)^{1/2}x\|^2.
\]

2. For the best choice \( \theta = \theta_a := \frac{2(\beta_d^{-2} + \lambda)}{\lambda_0} \) and \( L_2 = \frac{\lambda_0}{(c_R^2 + \lambda)(1 + \frac{2 \beta_d^2 R\lambda_0}{n})} \), the errors \( e_p^{k} \) satisfy the estimate

\[
\|e_p^{k+1}\|_{P_0} \leq \text{rate}^2(\lambda, R) \leq \frac{1}{\frac{\lambda_0}{(c_R^2 + \lambda)(1 + \frac{2 \beta_d^2 R\lambda_0}{n})} + 1} \leq \max \left\{ \frac{\beta_s^2}{c_R^2 + \beta_s^2}, \frac{1}{2} \right\},
\]

where

\begin{equation}
\|e_p^{k+1}\|_{P_0} = \|A_0^{1/2}e_p^{k+1}\|^2 + \frac{2(\beta_d^{-2} + \lambda)\beta_d^2}{\lambda_0} \|A_0^{1/2}e_p^{k}\|^2 + \|(A_1 + A_2)^{1/2}e_p^{k+1}\|^2.
\end{equation}

**Proof.** In view of the estimate presented in Lemma 6, we want to find \( L_2 \) and \( L_1 \) subject to the condition

\begin{equation}
\frac{\lambda_0}{2(c_R^2 + \lambda)} - \frac{L_2}{2} - \frac{L_1 R\lambda_0}{2n} \leq 0.
\end{equation}
For any $\theta > 0$, we rewrite (3.12) as
\begin{equation}
\frac{\lambda_0}{2(\beta_2^2 + \lambda)} \| A_{\beta}^2 e_p^{k+1} \|^2 + \theta^{-1} \theta^2 \| A_{\beta}^2 e_p^k \|^2 + \frac{L_1}{2} \| (\Lambda_1 + \Lambda_2) e_p^k \|^2 + \frac{L_2}{2} \| e_p^k \|^2 \leq \frac{L_1}{\theta^2 \beta_2^2} \| A_{\beta} e_p^k \|^2 + \frac{L_2}{2} \| A_{\beta} e_p^k \|^2
\end{equation}

(3.17) namely,
\begin{equation}
\left( \frac{\lambda_0}{2(\beta_2^2 + \lambda)} + \frac{L_2}{2} \right) \| A_{\beta}^2 e_p^{k+1} \|^2 + \left( \theta^{-1} + \frac{L_1}{2 \theta \beta_2^2} \right) \theta^2 \| A_{\beta}^2 e_p^k \|^2 + \| (\Lambda_1 + \Lambda_2) e_p^k \|^2 \leq \frac{L_1}{\theta^2 \beta_2^2} \| A_{\beta}^2 e_p^k \|^2 + \frac{L_2}{2} \| A_{\beta} e_p^k \|^2.
\end{equation}

(3.18) Then, for $L_2 \leq 1$ we obtain
\begin{equation}
\min \left\{ \frac{\lambda_0}{2(\beta_2^2 + \lambda)} + \frac{L_2}{2} \theta^{-1} + \frac{L_1}{2 \theta \beta_2^2} \right\} \left( \| A_{\beta}^2 e_p^{k+1} \|^2 + \theta^2 \| A_{\beta}^2 e_p^k \|^2 + \| (\Lambda_1 + \Lambda_2) e_p^k \|^2 \right)
\end{equation}

(3.19) ≤ max \left\{ \frac{L_1}{2 \beta_2^2} \frac{L_2}{2} \right\} \left( \| A_{\beta}^2 e_p^k \|^2 + \| A_{\beta} e_p^k \|^2 \right).

Now, choose $L_1 = \theta \beta_2^2 L_2$. Then, condition (3.16) becomes
\begin{equation}
\frac{\lambda_0}{2(c_K^2 + \lambda)} - \frac{L_2}{2} \theta \beta_2^2 L_2 R \lambda_0 \leq 0 \text{ or } L_2 \geq \frac{\lambda_0}{\theta \beta_2^2 R \lambda_0 + 2n}
\end{equation}

(3.20) and we can simplify (3.19) as follows
\begin{equation}
\min \left\{ \frac{\lambda_0}{2(\beta_2^2 + \lambda)} + \frac{L_2}{2} \theta^{-1} + \frac{L_2}{2} \right\} \left( \| A_{\beta}^2 e_p^{k+1} \|^2 + \theta^2 \| A_{\beta}^2 e_p^k \|^2 + \| (\Lambda_1 + \Lambda_2) e_p^k \|^2 \right)
\end{equation}

(3.21) ≤ \frac{L_2}{2} \left( \theta \beta_2^2 \| A_{\beta}^2 e_p^k \|^2 + \| A_{\beta} e_p^k \|^2 \right),
which shows (3.13). Statements 1. and 2. are direct consequences of (3.13) for the particular choices of $\theta$ in the corresponding norms.

\[\square\]

**Theorem 8.** Consider Algorithm 1 with $S$ as introduced in (2.30). Then the errors $e_u^k$ and $e_v^k$ defined in (2.25) satisfy the estimates:
\begin{equation}
\| e_u^k \|_{U} \leq C_u [\text{rate}(\lambda, R)]^k, \quad \| e_v^k \|_{V_v} \leq C_v [\text{rate}(\lambda, R)]^k
\end{equation}

(3.22) where
\begin{equation}
\| e_v^k \|_{V_v} = (A_v e_v^k, e_v^k) + (S^{-1} \text{Div} e_v^k, \text{Div} e_v^k), \quad \| u \|_{U}^2 = \| e(u) \|^2 + \lambda \| \text{div} u \|^2
\end{equation}

(3.23) and the constants $C_u$ and $C_v$ are independent of the model parameters and the time step size.
Proof. First, we estimate \( \|e^{k+1}_u\|^2_U \). By setting \( w = e^{k+1}_u \) in (2.26c), applying Cauchy’s inequality and using (3.8) we obtain

\[
\|e^{k+1}_u\|^2 + \lambda \|\text{div} e^{k+1}_u\|^2 = \left( \sum_{i=1}^n e^{k+1}_{p_i}, \text{div} e^{k+1}_u \right) \leq \left\| \sum_{i=1}^n e^{k+1}_{p_i} \right\| \cdot \|\text{div} e^{k+1}_u\| = \sqrt{\lambda_0} \|A_1^g e^{k+1}_p\| \cdot \|\text{div} e^{k+1}_u\|
\]

\[
\leq \sqrt{\lambda_0} \|A_1^g e^{k+1}_p\| \cdot \sqrt{\frac{1}{c_K^2 + \lambda} (\|e^{k+1}_u\|^2 + \lambda \|\text{div} e^{k+1}_u\|^2)},
\]

or, equivalently,

\[
\|e^{k+1}_u\|^2_U \leq \frac{\lambda_0}{c_K^2 + \lambda} \|A_1^g e^{k+1}_p\|^2 \leq \frac{\lambda_0}{c_K^2 + \lambda} \|e^{k+1}_p\|^2_{P_\ast}.
\]

In order to estimate \( \|e^{k+1}_v\|^2_U \), we set \( z = e^{k+1}_v \) in (2.26a) and apply the Cauchy inequality to derive

\[
(A_v e^{k+1}_v, e^{k+1}_v) + (S^{-1} \text{Div} e^{k+1}_v, \text{Div} e^{k+1}_v) = (e^{k+1}_p, \text{Div} e^{k+1}_v) - (S^{-1} \text{Div} e^{k+1}_u, \text{Div} e^{k+1}_v) - (S^{-1}(\Lambda_1 + \Lambda_2) e^{k+1}_p, \text{Div} e^{k+1}_v)
\]

\[
= (S^{-1}(\Lambda_1 + \Lambda_2) e^{k+1}_p, \text{Div} e^{k+1}_v) \leq (S^{-1}(\Lambda_1 + \Lambda_2) e^{k+1}_p, \text{Div} e^{k+1}_v)
\]

\[
(3.25)
\]

From the definition of \( S \), see (2.30), that of \( \| \cdot \|_{P_\ast} \), see (3.15), and noting that \( L_1 = \theta \beta_2^2 L_2 \), see Theorem 6, we have

\[
(S^{-1}(\Lambda_1 + \Lambda_2) e^{k+1}_p, \text{Div} e^{k+1}_v) \leq ((\Lambda_1 + \Lambda_2)^{-1}(L_1 A_3 + L_2 A_4) e^{k+1}_p, (L_1 A_3 + L_2 A_4) e^{k+1}_p)
\]

\[
= ((L_1 A_3 + L_2 A_4) e^{k+1}_p, e^{k+1}_p) \leq L_2 \|e^{k+1}_p\|^2_{P_\ast}.
\]

Then (3.25) can be rewritten in the form

\[
(3.27)
\]

Again, from the definition of \( S \), and observing that \( L_1 A_3 + L_2 A_4 = (L_1 R I_{n \times n} + \frac{K e}{\lambda_0} e e^T) \), then by choosing \( a = L_1 R \) and \( b = \frac{L_2}{\lambda_0} \) in Lemma 1, it follows that

\[
(3.28)
\]

\[
(3.29)
\]

Therefore, from (3.24), we have

\[
\|e^{k}_v\|_{V_\ast} = (A_v e^{k+1}_v, e^{k+1}_v) \leq L_2 \|e^{k+1}_p\|^2_{P_\ast} + (c_K^2 + \lambda) \|\text{div} e^{k+1}_u, \text{div} e^{k+1}_u\| \leq L_2 \|e^{k+1}_p\|^2_{P_\ast} + \lambda \|e^{k+1}_p\|^2_{P_\ast} = \left( L_2 + \frac{\lambda_0}{c_K^2 + \lambda} \right) \|e^{k+1}_p\|^2_{P_\ast},
\]

which completes the proof. \( \square \)

**Remark 9.** Note that for the particular choice of \( S \) and \( M \) that we have studied in this section, the block triangular matrix on the left-hand side of (2.18) provides a field of values equivalent preconditioner with equivalence constants independent of any model and discretization parameters.
4. Numerical results

In the following, we consider three widely used numerical test settings to demonstrate the effectiveness and the accuracy of the proposed Uzawa-type iterative schemes for the MPET model.

First, numerical results validating the theoretical estimates for Algorithm 1 are presented for the single network problem, i.e., the Biot model, in Table 1. In the second and third tests, the performance of Algorithm 1 is compared with the preconditioned GMRES algorithm and the fixed-stress algorithm as proposed in [27], cf. (2.12), for the two-network and four-network MPET problems. The block Gauss-Seidel preconditioner that we used to accelerate the GMRES method equals the lower block triangular matrix in the left-hand side of (2.18) where $M = S^{-1}$ and $S$ is given in (2.30).

All the numerical results in this section have been conducted on the FEniCS computing platform, see e.g. [4, 35]. In all test cases the set-up is as follows:
- The domain $\Omega \in \mathbb{R}^2$ is the unit square which throughout the discretization has been split into $2N^2$ congruent right-angled triangles;
- The discretization setting is the same as in [27, 26], i.e., we have used discontinuous piecewise constant elements, lowest-order Raviart-Thomas elements and Brezzi-Douglas-Marini elements to approximate the pressures, the fluxes and the displacement respectively;
- In all experiments we have set $L_2 = \frac{\lambda_0}{(c_0^2+\lambda)(1+\frac{\lambda}{n_0})}$, $L_1 = \frac{2(\beta_2^2+\lambda)}{\lambda_0}L_2$ and $\beta_2 = 0.18$, see [20], in Algorithm 1.
- The stopping criterion of the iterative process is an achieved residual reduction by a factor $10^8$ in the combined norm resulting from the norms defined in (3.15) and (3.23).

4.1. The Biot’s consolidation model. Consider system (2.1) for $n = 1$, i.e., a system for which only one pressure and one flux exists, where for $(x, y) \in \Omega$

\[
g = R_1 \left( \frac{\partial \phi_2}{\partial x} + \frac{\partial \phi_2}{\partial y} \right) - \alpha_{p_1}(\phi_2 - 1),
\]

\[
\phi_1 = (x-1)^2(y-1)^2x^2y^2, \quad \phi_2 = 900(x-1)^2(y-1)^2x^2y^2
\]

and

\[
f = \begin{pmatrix}
-(2y^3-3y^2+y)(12x^2-12x+2) - (x-1)^2x^2(12y-6) + 900(y-1)^2y^2(4x^3-6x^2+2x) \\
(2x^3-3x^2+x)(12y^2-12y+2) + (y-1)^2y^2(12y-6) + 900(x-1)^2x^2(4y^3-6y^2+2y)
\end{pmatrix}.
\]

Experiments over a wide-range of input parameters $\alpha_{p_1}, \lambda, R_1^{-1}$ have been run with Algorithm 1 and are presented in Table 1. Clearly, in all test cases the number of iterations required to achieve the prescribed solution accuracy is bounded by a constant that is independent of all model and discretization parameters.

4.2. The Biot-Barenblatt model. In the next test, system (2.1) is considered for $n = 2$ where the problem setting is as per the cantilever bracket benchmark problem in [22]. We denote the bottom, right, top and left parts of $\Gamma = \partial \Omega$ by $\Gamma_1$, $\Gamma_2$, $\Gamma_3$ and $\Gamma_4$ and, also, we impose $u = 0$ on $\Gamma_4$, $(\sigma - p_1 I - p_2 I)n = (0, 0)^T$ on $\Gamma_1 \cup \Gamma_2$, $(\sigma - p_1 I - p_2 I)n = (0, -1)^T$ on $\Gamma_3$, $p_1 = 2$ on $\Gamma$ and $p_2 = 20$ on $\Gamma$. Further, we set $f = 0$, $g_1 = 0$ and $g_2 = 0$. Table 2 shows the reference values of the model parameters as given in [31].

Tables 3–5 present a comparison between the preconditioned GMRES algorithm, the fixed-stress split algorithm as presented in [27] with a tuning parameter $L = 1/(1 + \lambda)$ and Algorithm 1. As can be seen, from Tables 3 and 5 for $\lambda$ being sufficiently large, the Uzawa-type method shows similar convergence behaviour as the preconditioned GMRES and fixed-stress methods.

Furthermore, all the numerical results included in Tables 3–5 demonstrate the robust performance of the Uzawa-type algorithm with respect to mesh refinements and variation of the hydraulic conductivities $K_1$ and $K_2$, and $\lambda$. 

\[
\text{PARAMETER-ROBUST UZAWA-TYPE METHODS FOR MULTIPLE-NETWORK PORELASTICITY MODELS} \ 15
\]
### Table 1. Number of iterations for solving the Biot problem with Algorithm 1

| $h$ | $\alpha_p$ | $\lambda$ | $1E0$ | $1E2$ | $1E3$ | $1E4$ | $1E8$ | $1E16$ |
|-----|-------------|------------|-------|-------|-------|-------|-------|-------|
|     |             | $1E0$     | 4     | 9     | 10    | 11    | 11    | 11    |
|     |             | $1E3$     | 1     | 2     | 2     | 2     | 2     | 2     |
|     |             | $1E6$     | 1     | 1     | 1     | 1     | 1     | 1     |
| $1E-4$ |             | $1E0$     | 4     | 14    | 21    | 26    | 29    | 29    |
|     |             | $1E3$     | 1     | 2     | 2     | 2     | 3     | 3     |
|     |             | $1E6$     | 1     | 1     | 1     | 1     | 1     | 1     |
| $1E-8$ |             | $1E0$     | 4     | 14    | 21    | 26    | 29    | 29    |
|     |             | $1E3$     | 1     | 2     | 2     | 2     | 3     | 3     |
|     |             | $1E6$     | 1     | 1     | 1     | 1     | 1     | 1     |
| $0$ |             | $1E0$     | 4     | 14    | 21    | 26    | 29    | 29    |
|     |             | $1E3$     | 1     | 2     | 2     | 2     | 3     | 3     |
|     |             | $1E6$     | 1     | 1     | 1     | 1     | 1     | 1     |
| $1$ |             | $1E0$     | 4     | 9     | 10    | 11    | 11    | 11    |
|     |             | $1E3$     | 1     | 2     | 2     | 2     | 2     | 2     |
|     |             | $1E6$     | 1     | 1     | 1     | 1     | 1     | 1     |
| $1E-4$ |             | $1E0$     | 4     | 12    | 21    | 26    | 29    | 29    |
|     |             | $1E3$     | 1     | 2     | 2     | 2     | 3     | 3     |
|     |             | $1E6$     | 1     | 1     | 1     | 1     | 1     | 1     |
| $1E-8$ |             | $1E0$     | 4     | 14    | 21    | 26    | 29    | 29    |
|     |             | $1E3$     | 1     | 2     | 2     | 2     | 3     | 3     |
|     |             | $1E6$     | 1     | 1     | 1     | 1     | 1     | 1     |
| $0$ |             | $1E0$     | 4     | 14    | 21    | 26    | 29    | 29    |
|     |             | $1E3$     | 1     | 2     | 2     | 2     | 3     | 3     |
|     |             | $1E6$     | 1     | 1     | 1     | 1     | 1     | 1     |

4.3. The four-network model. Lastly, we consider system (2.1) for $n = 4$. The test setting is analogous to the previous example, i.e., $\partial \Omega = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4$ with $\Gamma_1$, $\Gamma_2$, $\Gamma_3$, $\Gamma_4$ denoting the bottom, right, top and left boundaries respectively, $u = 0$ on $\Gamma_4$, $(\sigma - p_1 I - p_2 I - p_3 I - p_4 I) n = (0, 0)^T$ on $\Gamma_1 \cup \Gamma_2$, $(\sigma - p_1 I - p_2 I - p_3 I - p_4 I) n = (0, -1)^T$ on $\Gamma_3$, $p_1 = 2$ on $\Gamma$, $p_2 = 20$ on $\Gamma$, $p_3 = 30$ on $\Gamma$, $p_4 = 40$ on $\Gamma$. All the right-hand sides have been chosen to be zero. The reference values of the parameters are taken from [44] and are presented in Table 6.

The main aim of the numerical experiments discussed in this subsection is again the comparison between the three algorithms, namely the preconditioned GMRES algorithm, the fixed-stress split algorithm with $L = 1/(1 + \lambda)$ and the fully decoupling Algorithm 1.

Tables 7 shows that Algorithm 1 exhibits a similar convergence behaviour to the preconditioned GMRES method and the fixed-stress split iterative scheme over a wide-range of parameters as tabulated.

Moreover, the presented numerical results demonstrate the robustness of the newly proposed algorithm with respect to large variations of the coefficients $K_3$, $K = K_1 = K_2 = K_4$ and $\lambda$ and the mesh parameter $h$. 
Table 2. Reference values of model parameters for the Barenblatt model.

| parameter | value | unit       |
|-----------|-------|------------|
| $\lambda$ | 4.2   | MPa        |
| $\mu$     | 2.4   | MPa        |
| $c_{p1}$  | 54    | (GPa)$^{-1}$ |
| $c_{p2}$  | 14    | (GPa)$^{-1}$ |
| $\alpha_1$ | 0.95 |            |
| $\alpha_2$ | 0.12 |            |
| $\beta$   | 5     | $10^{-10}$kg/(m·s) |
| $K_1$     | 6.18  | $10^{-15}$m$^2$   |
| $K_2$     | 27.2  | $10^{-15}$m$^2$   |

Table 3. Number of preconditioned GMRES, fixed-stress split and augmented Uzawa-type iterations for residual reduction by a factor $10^8$ in the norm induced by the preconditioner when solving the Barenblatt problem.

| $h$   | $\beta$ | $K_2$ | $K_2 \cdot 10^2$ | $K_2 \cdot 10^4$ | $K_2 \cdot 10^6$ |
|-------|---------|-------|------------------|------------------|------------------|
| 5E-10 | $K_1 \cdot 10^{-2}$ | 9     | 8                | 13               | 10 8 13          |
| 1     | $K_1 \cdot 10^{-1}$ | 9     | 8                | 13               | 10 8 13          |
| 6E-8  | $K_1 \cdot 10^{-2}$ | 9     | 8                | 13               | 10 8 13          |
| 1     | $K_1 \cdot 10^{-1}$ | 9     | 8                | 13               | 10 8 13          |
| 5E-10 | $K_1 \cdot 10^{-2}$ | 9     | 8                | 13               | 10 8 13          |
| 1     | $K_1 \cdot 10^{-1}$ | 9     | 8                | 13               | 10 8 13          |
| 6E-8  | $K_1 \cdot 10^{-2}$ | 9     | 8                | 13               | 10 8 13          |
| 1     | $K_1 \cdot 10^{-1}$ | 9     | 8                | 13               | 10 8 13          |

5. Concluding Remarks

The main contribution of this manuscript is the development of an augmented Lagrangian Uzawa algorithm for MPET systems which fully decouples the fluid velocity, fluid pressure and solid displacement fields, contrary to the fixed-stress iterative scheme which decouples only the flow from the mechanics problem.

We prove the uniform parameter-robust convergence of the proposed method where, crucial for the analysis, is the introduction of special parameter-dependent norms. All performed numerical tests confirm the robustness and efficiency of the new fully decoupled iterative scheme.

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Table 4. Number of preconditioned GMRES, fixed-stress split and augmented Uzawa-type iterations for residual reduction by a factor $10^8$ in the norm induced by the preconditioner when solving the Barenblatt problem where we have redefined $\lambda := 0.01 \cdot \lambda$.

| $h$  | $\beta$ | $K_2$ | $K_2 \cdot 10^2$ | $K_2 \cdot 10^4$ | $K_2 \cdot 10^6$ |
|------|---------|-------|------------------|------------------|------------------|
| 1E-10|         |       |                  |                  |                  |
| 1    | $K_1 \cdot 10^{-2}$ | 12   | 11              | 13               | 11              |
|      | $K_1 \cdot 10^{-1}$ | 12   | 11              | 13               | 11              |
|      | $K_1$       | 12   | 11              | 13               | 11              |
| 1E-8 |         |       |                  |                  |                  |
| 1    | $K_1 \cdot 10^{-2}$ | 12   | 11              | 13               | 11              |
|      | $K_1 \cdot 10^{-1}$ | 12   | 11              | 13               | 11              |
|      | $K_1$       | 12   | 11              | 13               | 11              |
| 1E-10|         |       |                  |                  |                  |
| 1    | $K_1 \cdot 10^{-2}$ | 11   | 10              | 13               | 10              |
|      | $K_1 \cdot 10^{-1}$ | 11   | 10              | 13               | 10              |
|      | $K_1$       | 11   | 10              | 13               | 10              |
| 32   |         |       |                  |                  |                  |
| 1E-8 | $K_1 \cdot 10^{-2}$ | 11   | 10              | 13               | 10              |
|      | $K_1 \cdot 10^{-1}$ | 11   | 10              | 13               | 10              |
|      | $K_1$       | 11   | 10              | 13               | 10              |

Table 5. Number of preconditioned GMRES, fixed-stress split and augmented Uzawa-type iterations for residual reduction by a factor $10^8$ in the norm induced by the preconditioner when solving the Barenblatt problem where we have redefined $\lambda := 100 \cdot \lambda$.

| $h$  | $\beta$ | $K_2$ | $K_2 \cdot 10^2$ | $K_2 \cdot 10^4$ | $K_2 \cdot 10^6$ |
|------|---------|-------|------------------|------------------|------------------|
| 1E-10|         |       |                  |                  |                  |
| 1    | $K_1 \cdot 10^{-2}$ | 4    | 2               | 5               | 2               |
|      | $K_1 \cdot 10^{-1}$ | 4    | 2               | 5               | 2               |
|      | $K_1$       | 4    | 2               | 5               | 2               |
| 1E-8 |         |       |                  |                  |                  |
| 1    | $K_1 \cdot 10^{-2}$ | 4    | 2               | 5               | 2               |
|      | $K_1 \cdot 10^{-1}$ | 4    | 2               | 5               | 2               |
|      | $K_1$       | 4    | 2               | 5               | 2               |
| 1E-10|         |       |                  |                  |                  |
| 1    | $K_1 \cdot 10^{-2}$ | 3    | 2               | 5               | 2               |
|      | $K_1 \cdot 10^{-1}$ | 3    | 2               | 5               | 2               |
|      | $K_1$       | 3    | 2               | 5               | 2               |
| 32   |         |       |                  |                  |                  |
| 1E-8 | $K_1 \cdot 10^{-2}$ | 3    | 2               | 5               | 2               |
|      | $K_1 \cdot 10^{-1}$ | 3    | 2               | 5               | 2               |
|      | $K_1$       | 3    | 2               | 5               | 2               |

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Table 6. Reference values of model parameters for the four-network MPET model.

| parameter       | value   | unit       |
|-----------------|---------|------------|
| $\lambda$       | 505     | Nm$^{-2}$  |
| $\mu$           | 216     | Nm$^{-2}$  |
| $c_{p_1} = c_{p_2} = c_{p_3} = c_{p_4}$ | $4.5 \cdot 10^{-10}$ | m$^2$N$^{-1}$ |
| $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4$ | 0.99 |        |
| $\beta_{12} = \beta_{24}$ | $1.5 \cdot 10^{-19}$ | m$^2$N$^{-1}$s$^{-1}$ |
| $\beta_{32}$   | $2.0 \cdot 10^{-19}$ | m$^2$N$^{-1}$s$^{-1}$ |
| $\beta_{34}$   | $1.0 \cdot 10^{-13}$ | m$^2$N$^{-1}$s$^{-1}$ |
| $K_1 = K_2 = K_4 = K$ | $(1.0 \cdot 10^{-10})/(2.67 \cdot 10^{-3})$ | m$^2$/Nsm$^{-2}$ |
| $K_3$          | $(1.4 \cdot 10^{-14})/(8.9 \cdot 10^{-4})$ | m$^2$/Nsm$^{-2}$ |

Table 7. Number of preconditioned GMRES, fixed-stress split and augmented Uzawa-type iterations for residual reduction by a factor $10^8$ in the norm induced by the preconditioner when solving the four-network MPET problem.

| $h$ | $K_3 \cdot 10^2$ | $K_3$ | $K_3 \cdot 10^4$ | $K_3 \cdot 10^6$ |
|-----|-----------------|-------|-----------------|-----------------|
| 1/16| 12 10 16        | 15 10 16 | 13 10 16        | 13 10 16        |
| 1/32| 12 10 16        | 15 10 16 | 16 10 16        | 18 10 16        |

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