Abstract

Complex networks are everywhere. They appear for example in the form of biological networks, social networks, or computer networks and have been studied extensively. Efficient algorithms to solve problems on complex networks play a central role in today’s society. Algorithmic meta-theorems show that many problems can be solved efficiently. Since logic is a powerful tool to model problems, it has been used to obtain very general meta-theorems. In this work, we consider all problems definable in first-order logic and analyze which properties of complex networks allow them to be solved efficiently.

The mathematical tool to describe complex networks are random graph models. We define a property of random graph models called \( \alpha \)-power-law-boundedness. Roughly speaking, a random graph is \( \alpha \)-power-law-bounded if it does not admit strong clustering and its degree sequence is bounded by a power-law distribution with exponent at least \( \alpha \) (i.e. the fraction of vertices with degree \( k \) is roughly \( O(k^{-\alpha}) \)).

We solve the first-order model-checking problem (parameterized by the length of the formula) in almost linear FPT time on random graph models satisfying this property with \( \alpha \geq 3 \). This means in particular that one can solve every problem expressible in first-order logic in almost linear expected time on these random graph models. This includes for example preferential attachment graphs, Chung–Lu graphs, configuration graphs, and sparse Erdős–Rényi graphs. Our results match known hardness results and generalize previous tractability results on this topic.

1 Introduction

Complex networks, as they occur in society, biology and technology, play a central role in our everyday lives. Even though these networks occur in vastly different contexts, they are structured and evolve according to a common set of underlying principles. Over the last two decades, with the emergence of the field of network science, there has been an explosion in research to understand these fundamental laws. One well observed property is the small-world phenomenon, which means that distances between vertices are very small. This has been verified for the internet and many other networks [1,55]. Furthermore, many real networks tend to be clustered. They contain groups of vertices that are densely connected [66].

If two vertices share a common neighbor, then there is a high chance that there is also an edge between them. A network can be considered clustered if the ratio between the number of triangles and the number of paths with three vertices is non-vanishing. This is formalized by the clustering coefficient, which is high for many networks [72]. A third important property is a heavy tailed degree distribution. While most vertices have a low number of connections, there are a few hubs with a high degree. Experiments show that the degrees follow for example a power-law or log-normal distribution. In a power-law distribution, the

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fraction of vertices with degree $k$ is proportional to $k^{-\alpha}$ (usually with $\alpha$ between 2 and 3). This behavior makes complex networks highly inhomogeneous [64, 57, 10, 15].

One important goal of theoretical computer science has always been to explore what kinds of inputs allow or forbid us to construct efficient algorithms. In this context, algorithmic meta-theorems [51] are of particular interest. They are usually theorems stating that problems definable in a certain logic can be solved efficiently on graph classes that satisfy certain properties. Logic is a powerful tool to model problems and therefore has been used to obtain very general meta-theorems. A well-known example is Courcelle’s theorem [16], which states that every problem expressible in counting monadic second-order logic can be solved in linear time on graph classes with bounded treewidth. It has been further generalized to graph classes with bounded cliquewidth [17]. To obtain results for larger graph classes one has to consider weaker logics. The languages of relational database systems are based on first-order logic. In this logic, one is allowed to quantify over vertices and to test equality and adjacency of vertices. With $k$ existential quantifiers, one may ask for the existence of a fixed graph with $k$ vertices ($k$-subgraph isomorphism), a problem relevant to motif-counting [56, 28]. On the other hand, connectivity properties cannot be expressed in first-order logic. We define for every graph class $G$ the parameterized first-order model-checking problem $p$-MC(FO, $G$) [44].

$$p$-MC(FO, $G$)$$

**Input:** A graph $G \in G$ and a first-order sentence $\varphi$

**Parameter:** The number of symbols in $\varphi$, denoted by $|\varphi|$.

**Problem:** Does $\varphi$ hold on $G$ (i.e., $G \models \varphi$)?

The aim is to show for a given graph class $G$ that $p$-MC(FO, $G$) is fixed parameter tractable (FPT), i.e., can be decided in time $f(|\varphi|)n^{O(1)}$ for some function $f$ (see for example [18] for an introduction to fixed parameter tractability). Since input graphs may be large, a linear dependence on $n$ is desirable. If one is successful, then every problem expressible in first-order logic can be solved on $G$ in linear time.

For the class of all graphs $\mathcal{G}$, $p$-MC(FO, $\mathcal{G}$) is AW[$\ast$]-complete [25] and therefore most likely not fpt. Over time, tractability of $p$-MC(FO, $\mathcal{G}$) has been shown for more and more sparse graph classes $G$: bounded vertex degree [68], forbidden minors [31], bounded local treewidth [23], and further generalizations [44, 29, 67]. Grohe, Kreutzer and Siebertz prove that $p$-MC(FO, $G$) can be solved in almost linear FPT time $f(|\varphi|, \varepsilon)n^{1+\varepsilon}$ for all $\varepsilon > 0$ if $G$ is a nowhere dense graph class. On the other hand if $G$ is a monotone somewhere dense graph class, $p$-MC(FO, $G$) is AW[$\ast$]-hard [15]. Nowhere dense graph classes were introduced by Nešetřil and Ossona de Mendez as those graph classes where for every $r \in \mathbb{N}$ the size of all $r$-shallow clique minors of all graphs in the graph class is bounded by a function of $r$ (Section 4.3). A graph class is somewhere dense if it is not nowhere dense. The tractability of the model-checking problem on monotone graph classes is completely characterized with a dichotomy between nowhere dense and somewhere dense graph classes. These very general results come at a cost: Frick and Grohe showed that the dependence of the run time on $\varphi$ is non-elementary [37]. We want to transfer this rich algorithmic theory to complex networks.

But what is the right abstraction to describe complex networks? Network scientists observed that the chaotic and unordered structure of real networks can by captured using randomness. There is a vast body of research using random processes to create graphs that mimic the fundamental properties of complex networks. The most prominent ones are the preferential attachment model [3, 63], Chung–Lu model [12, 13], configuration model [59, 58], Kleinberg model [49, 50], hyperbolic graph model [52], and random intersection graph model [46, 65]. All these are random models. It has been thoroughly analyzed how well they predict various properties of complex networks [42].

When it comes to algorithmic meta-theorems on random graph models “even the most basic questions are wide open,” as Grohe puts it [44]. By analyzing which models of complex
networks and which values of the model-parameters allow for efficient algorithms, we aim to develop an understanding how the different properties of complex networks control their algorithmic tractability.

In this work we show for a wide range of models, including the well known preferential attachment model, that one can solve the parameterized first-order model-checking problem in almost linear FPT time. This means in particular that one can solve every problem expressible in first-order logic efficiently on these models. Our original goal was to obtain efficient algorithms only for preferential attachment graphs, but we found an abstraction that transfers these results to many other random graph models. Roughly speaking, the following two criteria are sufficient for efficiently solving first-order definable problems on a random graph model:

- The model needs to be unclustered. In particular the expected number of triangles needs to be subpolynomial.
- For every $k$, the fraction of vertices with degree $k$ is roughly $O(k^{-3})$. In other words, the degree sequence needs to be bounded by a power-law distribution with exponent 3 or higher.

Models satisfying these properties include sparse Erdős–Rényi graphs, preferential attachment graphs as well as certain Chung–Lu and configuration graphs. On the other hand, the Kleinberg model, the hyperbolic random graph model, or the random intersection graph model do not satisfy these properties. Our results generalize previous results [43, 22] and match known hardness results: The model-checking problem has been proven to be hard on power-law distributions with exponent smaller than 3 [27]. We therefore identify the threshold for tractability to be a power-law coefficient of 3. It is also a big open question whether the model-checking problem can also be solved on clustered random graph models, especially since real networks tend to be clustered. Furthermore, significant engineering challenges need to be overcome to make our algorithms applicable in practice.

1.1 Average Case Complexity

Average-case complexity analyzes the typical run time of algorithms on random instances (see [7] for a survey), based on the idea that a worst-case analysis often is too pessimistic as for many problems hard instances occur rarely in the real world. Since models of complex networks are probability distributions over graphs, we analyze the run time of algorithms under average-case complexity. However, there are multiple notions and one needs to be careful which one to choose.

Assume a random graph model is asymptotically almost surely (a.a.s.) nowhere dense, i.e., a random graph from the model with $n$ vertices belongs with probability $1 - \delta(n)$ to a nowhere dense graph class, where $\lim_{n \to \infty} \delta(n) = 0$ (Section 4.3). Then the first-order model-checking problem can be efficiently solved with a probability converging to one [45]. However, with probability $\delta(n)$ the run time can be arbitrarily high and the rate of convergence of $\delta(n)$ to zero can be arbitrarily slow. These two missing bounds are undesirable from an algorithmic standpoint and the field of average-case complexity has established a theory on how the run time needs to be bounded with respect to the fraction of inputs that lead to this run time.

This is formalized by the well-established notion of average polynomial run time, introduced by Levin [53]. An algorithm has average polynomial run time with respect to a random graph model if there is an $\varepsilon > 0$ and a polynomial $p$ such that for every $n, t$ the probability that the algorithm runs longer than $t$ steps on an input of size $n$ is at most $p(n)/t^\varepsilon$. This means there is a polynomial trade-off between run time and fraction of inputs. This notion has been widely studied [7, 2] and is considered from a complexity theoretic standpoint the right notion of polynomial run time on random inputs. It is closed under invoking polynomial subroutines.
expected linear $\Rightarrow$ expected polynomial $\Rightarrow$ average polynomial $\Rightarrow$ a.a.s. polynomial polynomial

With this in mind we can present our notion of algorithmic tractability. A labeled graph is a graph where every vertex can have (multiple) labels. First-order formulas can have unary predicates for each type of label. These predicates test whether a vertex has a label of a certain type. We define $\mathcal{G}$ to be the class of all graphs, and $\mathcal{G}_l$ to be the class of all vertex-labeled graphs. A function $L: \mathcal{G} \rightarrow \mathcal{G}_l$ is an $l$-labeling function for $l \in \mathbb{N}$ if for every $G \in \mathcal{G}$, $L(G)$ is a labeling of $G$ with up to $l$ classes of labels (see Section 3 for details). Furthermore, a random graph model is a sequence $\mathcal{G} = (\mathcal{G}_n)_{n \in \mathbb{N}}$, where $\mathcal{G}_n$ is a probability distribution over unlabeled simple graphs with $n$ vertices.

**Definition 1.6** We say $p$-MC(FO, $\mathcal{G}_l$) can be decided on a random graph model $(\mathcal{G}_n)_{n \in \mathbb{N}}$ in expected time $f(|\varphi|, n)$ if there exists a deterministic algorithm $\mathcal{A}$ which decides $p$-MC(FO, $\mathcal{G}_l$) on input $G$, $\varphi$ in time $t_{\mathcal{A}}(G, \varphi)$ and if for all $n \in \mathbb{N}$, all first-order sentences $\varphi$ and all $l$-labeling functions $L$, $E_{G \sim \mathcal{G}_n}[t_{\mathcal{A}}(L(G), \varphi)] \leq f(|\varphi|, n)$. We say $p$-MC(FO, $\mathcal{G}_l$) on a random graph model can be decided in expected FPT time if it can be decided in expected time $g(|\varphi|)n^{O(1)}$ for some function $g$.

In particular, this definition implies efficient average run time according to Levin’s notion (which is closed under polynomial subroutines). We choose to include labels into our notion of average-case hardness for two reasons: First, it makes our algorithmic results stronger, as the expected run time is small, even in the presence of an adversary that labels the vertices of the graph. Secondly, it matches known hardness results that require adversary labeling.

### 1.2 Previous Work

There have been efforts to transfer the results for classical graph classes to random graph models by showing that a graph sampled from some random graph model belongs with high probability to a certain algorithmically tractable graph class. For most random graph models the treewidth is polynomial in the size of the graph [10] [5]. Therefore, people have considered more permissive graph measures than treewidth, such as low degree [43], or bounded expansion [22] [32]. Demaine et al. showed that some Chung–Lu and configuration graphs have bounded expansion and provided empirical evidence that some real-world networks do too [22]. However, this technique is still limited, as many random graph models (such as the preferential attachment model [22] [26]) are not known to be contained in any of the well-known tractable graph classes.

The previous tractability results presented in this section all use the following technique: Assume we have a formula $\varphi$ and sample a graph of size $n$ from a random graph model. If the sampled graph belongs to the tractable graph class, an efficient model-checking algorithm for the graph class can solve the instance in FPT time. If the graph does not belong the graph class, the naïve model-checking algorithm can still solve the instance in time $O(n^{l|\varphi|})$. Assume we can show that the second case only happens with probability $\delta(n)$ converging to zero faster than any polynomial. Then $\delta(n)O(n^{l|\varphi|})$ converges to zero and the expected run time remains bounded by an FPT function.

Let $p(n)$ be a function with $p(n) = O(n^c/n)$ for all $c > 0$. Grohe showed that one can solve $p$-MC(FO, $\mathcal{G}_l$) on Erdős–Rényi graphs $G(n, p(n))$ in expected time $f(|\varphi|, \varepsilon)n^{l+\varepsilon}$ for every $\varepsilon > 0$ [43]. This result was obtained by showing that with high probability the maximum degree of the random graph model is $O(n^c)$ for every $c > 0$ and then using a
model-checking algorithm for low degree graphs. Later Demaine et al. and Farrell et al. showed that certain Chung–Lu and configuration graphs whose degrees follow a power-law distribution with exponent $\alpha > 3$ \cite{22} as well as certain random intersection graphs \cite{32} belong with high probability to a graph class with bounded expansion. While they do not mention it explicitly, the previous argument implies that one can solve $p$-MC($\text{FO}, \mathcal{G}_b$) in expected time $f(|\varphi|)n$ on these random graph models.

There further exist some average-case hardness results for the model-checking problem. It has been shown that one cannot decide $p$-MC($\text{FO}, \mathcal{G}_b$) on Erdős–Rényi graphs $G(n, 1/2)$ or $G(n, p(n))$ with $p(n) = n^\varepsilon/n$ for some $0 < \varepsilon < 1$, $\varepsilon \in \mathbb{Q}$, in expected FPT time (unless AW[\varepsilon] \subseteq \text{FPT(poly)} \cite{22}). The same holds for Chung–Lu graphs with exponent $2.5 < \alpha < 3$, $\alpha \in \mathbb{Q}$. These hardness results fundamentally require the adversary labeling of Definition 4.6. It is a big open question whether they can be transferred to model-checking without labels.

Another thing to keep in mind when considering logic and random graphs \cite{69} are zero-one laws. They state that in many Erdős–Rényi graphs every first-order formula holds in the limit either with probability zero or one \cite{69, 41, 31}. Not all random graph models satisfy this model first. A Chung–Lu graph with exponent $2.5 < \alpha < 3$, $\alpha \in \mathbb{Q}$. These hardness results fundamentally require the adversary labeling of Definition 4.6. It is a big open question whether they can be transferred to model-checking without labels.

2 Our Results

We define a property called $\alpha$-power-law-boundedness. This property depends on a parameter $\alpha$ and captures many unclustered random graph models for which the fraction of vertices with expected degree $d \in \mathbb{N}$ is roughly $O(d^{-\alpha})$. Our main contribution is solving the model-checking problem efficiently on all $\alpha$-power-law-bounded random graph models with $\alpha \geq 3$. This includes preferential attachment graphs, Chung–Lu graphs, Erdős–Rényi graphs and other random graph models. Note that graphs do not need to have a power-law degree distribution to be $\alpha$-power-law-bounded. Our results hold for arbitrary labelings of the random graph and are based on a novel decomposition technique for local regions of random graphs. While all previous algorithms work by placing the random graph model with high probability in a sparse graph class, our technique also works for some a.a.s. somewhere dense random graphs (e.g. preferential attachment graphs \cite{26}).

2.1 Power-Law-Boundedness

We start by formalizing our property. Since it generalizes the Chung–Lu model, we define this model first. A Chung–Lu graph with exponent $\alpha$ and vertices $v_1, \ldots, v_n$ is defined such that two vertices $v_i$ and $v_j$ are adjacent with probability $\Theta(w_i w_j/n)$ where $w_i = (n/i)^{1/(\alpha - 1)} \cite{12}$. Furthermore all edges are independent, which means that the probability that a set of edges occurs equals the product over the probabilities of each individual edge. In our model the probability of a set of edges can be a certain factor larger than the product of the individual probabilities, which allows edges to be moderately dependent.

**Definition 2.1.** Let $\alpha > 2$. We say a random graph model $(G_n)_{n \in \mathbb{N}}$ is $\alpha$-power-law-bounded if for every $n \in \mathbb{N}$ there exists an ordering $v_1, \ldots, v_n$ of $V(G_n)$ such that for all $E \subseteq \binom{\{v_1, \ldots, v_n\}}{2}$

$$\Pr [E \subseteq E(G_n)] \leq \prod_{v_i, v_j \in E} \frac{(n/i)^{1/(\alpha - 1)}(n/j)^{1/(\alpha - 1)}}{n} \cdot \begin{cases} 2^{O(|E|^2)} & \text{if } \alpha > 3 \\ \log(n)^{O(|E|^2)} & \text{if } \alpha = 3 \\ O(n^2 |E|^2) & \text{for every } \varepsilon > 0 \text{ if } \alpha < 3. \end{cases}$$

The probability that a set of edges $E$ occurs may be up to a factor $2^{O(|E|^2)}$ or $\log(n)^{O(|E|^2)}$ or $O(n^2 |E|^2)$ (depending on $\alpha$) larger than the probability in the corresponding Chung–Lu graph. For conditional probabilities this means the following: The probability bound for an
edge under the condition that some set of \( l \) edges is already present may be up to a factor \( 2^{O(l)} \) or \( \log(n)^{O(l)} \) or \( O(n^\varepsilon) \) larger than the unconditional probability. This lets power-law-bounded random graphs capture moderate dependence between edges. The factor undergoes a phase transition at \( \alpha = 3 \). The smaller factor \( 2^{O(l)} \) for \( \alpha > 3 \) was chosen to guarantee linear FPT run time of our model-checking algorithm (Theorem 8.5) if \( \alpha > 3 \). The slightly larger factor of \( \log(n)^{O(l^2)} \) for \( \alpha = 3 \) was chosen to capture preferential attachment graphs while still maintaining a quasilinear FPT run time of our algorithm.

The parameter \( \alpha \) of an \( \alpha \)-power-law-bounded random graph model controls the degree distribution. Note that if a graph class is \( \alpha \)-power-law-bounded it is also \( \alpha' \)-power-law-bounded for all \( 2 < \alpha' < \alpha \). It can be easily seen that a vertex \( v_i \) has expected degree at most \( O(n^\varepsilon)(n/i)^{1/(\alpha-1)} \) for every \( \varepsilon > 0 \). This means the expected degree sequence of an \( \alpha \)-power-law-bounded random graph model is not power-law distributed with exponent smaller than \( \alpha \). The gap is often tight: For example, Chung–Lu graphs with a power-law degree distribution exponent \( \alpha \) are \( \alpha \)-power-law-bounded and preferential attachment graphs have a power-law degree distribution with exponent 3 and are 3-power-law-bounded. For the interesting case \( \alpha = 3 \), the inequality in Definition 2.1 simplifies to

\[
\Pr[E \subseteq E(G_n)] \leq \log(n)^{O(l^2)} \prod_{v_i,v_j \in E} \frac{1}{\sqrt{|ij|}}.
\]

2.2 Model Checking

We now present our model-checking algorithm for \( \alpha \)-power-law-bounded graphs. We express its run time relative to the term

\[
\tilde{d}_n(n) = \begin{cases} 
O(1) & \alpha > 3 \\
\log(n)^{O(1)} & \alpha = 3 \\
O(n^{3-\alpha}) & \alpha < 3.
\end{cases}
\]

This term is related to an established property of degree distributions, namely the second order average degree \( [12] \). If a random graph with \( n \) vertices has expected degrees \( w_1, \ldots, w_n \), then the second order average degree is defined as \( \sum_{i=1}^{n} w_i^2 / \sum_{k=1}^{n} w_k \). In graphs with a power-law degree distribution \( \alpha \) we have \( w_i = \Theta((n/i)^{1/(\alpha-1)}) \). The second order average degree then equals \( \Theta\left(\sum_{i=1}^{n} (n/i)^{2/(\alpha-1)} / \sum_{k=1}^{n} (n/k)^{1/(\alpha-1)}\right) \). For \( \alpha > 3 \), this term is constant, for \( \alpha = 3 \) it is logarithmic, and for \( \alpha < 3 \) it is polynomial in \( n \) \([12]\). Thus, we can interpret \( \tilde{d}_n(n) \) as an estimate of the second order average degree. We prove that the model-checking problem can be solved efficiently if \( \tilde{d}_n(n) \) is small.

**Theorem 8.5** There exists a function \( f \) such that one can solve \( p\text{-MC}(\text{FO}, \mathcal{G}_n) \) on every \( \alpha \)-power-law-bounded random graph model in expected time \( \tilde{d}_n(n)^{f(|\varphi|)}n \).

The term \( \tilde{d}_n(n) \) naturally arises in our proofs and is not a consequence of how we defined the multiplicative factor (i.e., \( 2^{O(l^2)}, \log(n)^{O(l^2)}, O(n^\varepsilon)|E|^2 \) in Definition 2.1). In fact the dependence goes the other way: We defined the factor for each \( \alpha \) as large as possible such that it does not dominate the run time of the algorithm. Next we specify exactly those values of \( \alpha \) where the previous theorem leads to FPT run times. (In the third case \( \varepsilon > 0 \) can be chosen arbitrarily small since we require \( \alpha \) to be arbitrarily close to 3.)

**Theorem 8.6** Let \( (G_n)_{n \in \mathbb{N}} \) be a random graph model. There exists a function \( f \) such that one can solve \( p\text{-MC}(\text{FO}, \mathcal{G}_n) \) in expected time

- \( f(|\varphi|)n \) if \( (G_n)_{n \in \mathbb{N}} \) is \( \alpha \)-power-law-bounded for some \( \alpha > 3 \),
- \( \log(n)^{f(|\varphi|)}n \) if \( (G_n)_{n \in \mathbb{N}} \) is \( \alpha \)-power-law-bounded for \( \alpha = 3 \),
- \( f(|\varphi|, \varepsilon)n^{1+\varepsilon} \) for all \( \varepsilon > 0 \) if \( (G_n)_{n \in \mathbb{N}} \) is \( \alpha \)-power-law-bounded for every \( 2 < \alpha < 3 \).
This solves the model-checking problem efficiently on a wide range of random graph models. These tractability results are matched by previous intractability results. (Note that Proposition 2.2 does not contradict Proposition 2.2.)

Proposition 2.2 ([27] and Lemma 10.3). For every $2 < \alpha < 3$ there exists an $\alpha$-power-law-bounded random graph model $(G_n)_{n \in \mathbb{N}}$ such that one cannot solve $p$-MC(FO, $\Theta_0$) on $(G_n)_{n \in \mathbb{N}}$ in expected FPT time unless AW[*] $\subseteq$ FPT/poly.

We observe a phase transition in tractability at power-law exponent $\alpha = 3$. Also the runtime of our algorithm cannot be linear in $n$ for $\alpha \leq 3$ as a 3-power-law-bounded random graph can have for example $n \log(n)$ edges in expectation. We discuss the algorithmic implications of our result for some well-known random graph models in Section 10.

2.3 Structure

Many algorithmic results are based on structural decompositions. For example, bidimensionality theory introduced by Demaine et al. [20] [21] is based on the grid minor theorem, which is itself based on a structural decomposition into a clique-sum of almost-embeddable graphs developed by Robertson and Seymour [60]. The model-checking algorithm for graph classes with bounded expansion by Dvořák, Král, and Thomas [20] relies on a structural decomposition of bounded expansion graph classes by Nešetřil and Ossona de Mendez called low tree-depth colorings [62]. Our algorithms are based on a structural decomposition of $\alpha$-power-law-bounded random graph models.

All algorithms prior to this work rely on showing that a certain graph model is with high probability contained in a certain well-known tractable graph class (for example bounded expansion) and then use the structural decompositions [62] of said graph class. However, these decompositions were not originally designed with random graphs in mind and therefore may not provide the optimal level of abstraction for random graphs. Our algorithms are based on a specially defined structural decomposition. This direct approach helps us capture random graph models that could otherwise not be captured such as the a.a.s. somewhere dense preferential attachment model. By focusing on $\alpha$-power-law-bounded random graph models, we obtain structural decompositions for a wide range of models.

We observe that $\alpha$-power-law-bounded random graphs have mostly an extremely sparse structure with the exception of a part whose size is bounded by the second order average degree. However, this denser part can be separated well from the remaining graph. We show that local regions consist of a core part, bounded in size by the second order average degree, to which trees and graphs of constant size are attached by a constant number of edges. This decomposition is similar to so called protrusion decompositions, which have been used by Bodlaender et al. to obtain meta-theorems on kernelization [6]. Our structural decomposition is valid for all graphs that fit into the framework of $\alpha$-power-law-boundedness, such as preferential attachment graphs or Chung–Lu graphs. We define an approximation of the second order average degree of the degree distribution as $d_\alpha(n) = 2$ for $\alpha > 3$, $d_\alpha(n) = \log(n)$ for $\alpha = 3$ and $d_\alpha(n) = n^{3-\alpha}$ for $\alpha < 3$ (similarly to $\tilde{d}_\alpha(n)$ without O-notation).

Theorem 9.5. Let $(G_n)_{n \in \mathbb{N}}$ be an $\alpha$-power-law-bounded random graph model. There exist constants $c, r_0$ such that for every $r \geq r_0$ a.a.s. for every $r$-neighborhood $H$ of $G_n$ one can partition $V(H)$ into three (possibly empty) sets $X, Y, Z$ with the following properties.

- $|X| \leq \tilde{d}_\alpha(n)^{cr^2}$.
- Every connected component of $H[Y]$ has size at most $cr$ and at most $c$ neighbors in $X$.
- Every connected component of $H[Z]$ is a tree with at most one edge to $H[X \cup Y]$.

Removing a few vertices makes the local neighborhoods even sparser:
Corollary 9.4. Let \((G_n)_{n \in \mathbb{N}}\) be an \(\alpha\)-power-law-bounded random graph model. There exist constants \(c, r_0\) such that for every \(r \geq r_0\) a.a.s. one can remove \(d_\alpha(n)^{cr^2}\) vertices from \(G_n\) such that every \(r\)-neighborhood has treewidth at most 26.

Corollary 9.4 is a consequence of Theorem 9.3 from Section 9. Further structural results that may be interesting beyond the purpose of model-checking can be found in Section 9.

We now discuss how we use the decomposition of Theorem 9.5 for our algorithms and why decompositions similar to Corollary 9.4 are not sufficient for our purposes.

## 3 Techniques

A first building block of our algorithm is Gaifman’s locality theorem \cite{Gaifman}. It implies that in order to solve the first-order model-checking problem on a graph, it is sufficient to solve the problem on all \(r\)-neighborhoods of the graph for some small \(r\). We can therefore restrict ourselves to the model-checking problem on the neighborhoods of random graphs. With this in mind, we want to obtain structural decompositions of these neighborhoods.

One important thing to note is that a decomposition according to Corollary 9.4 is not sufficient. Let us focus on the interesting case \(\alpha = 3\) where efficient model-checking is still possible. Corollary 9.4 then states that the removal of polylogarithmically many vertices yields neighborhoods with treewidth at most 26. While we could easily solve the model-checking problem on graphs with treewidth at most 26 via Courcelle’s theorem \cite{Courcelle}, we cannot solve it on graphs where we need to remove a set \(X\) of \(\log(n)\) vertices to obtain a treewidth of at most 26. Every vertex not in \(X\) may have an arbitrary subset of \(X\) as neighborhood. Since there are \(2^{\left|X\right|} = n\) possible neighborhoods, we can encode a large complicated structure into this graph by stating that two vertices \(i, j \in \mathbb{N}\) are adjacent if and only if there is a vertex whose neighborhood in \(X\) represents a binary encoding of the edge \(ij\) (omitting some details). Because of this, the model-checking problem on this graph class is as hard as on general graphs. We need the additional requirement that \(X\) is only loosely connected to the remaining graph. The decomposition in Theorem 9.5 fulfills this requirement. Every component of \(H \setminus X\) has at most a constant number of neighbors in \(X\).

Let us assume we have decompositions of the neighborhoods of a graph according to Theorem 9.5 where the sets \(X\) are chosen as small as possible. We can now use a variant of the Feferman–Vaught theorem \cite{Feferman-Vaught} for each \(r\)-neighborhood to prune the protrusions and thereby construct a smaller graph that satisfies the same (short) first-order formulas as the original graph. We call this smaller graph the *kernel*. The size of this kernel will be some function of \(|X|\). We then use the brute-force model-checking algorithm on the kernel.

For the first steps of the algorithm (decomposition into neighborhoods, kernelization using Feferman–Vaught) one can easily show that they always take \(\text{FPT}\) time. However, the run time of the last step requires a careful analysis. One can check a formula \(\phi\) on a graph of size \(x\) in time \(O(x^{\left|\phi\right|})\) by brute force. Thus, checking the formula on the kernel of all \(n\) many \(r\)-neighborhoods of a random graph takes expected time at most \(n \sum_{x=1}^{n} p_x O(x^{\left|\phi\right|})\), where \(p_x\) is the probability that the kernelization procedure on an \(r\)-neighborhood of a random graph yields a kernel of size \(x\). In order to guarantee a run time of the form \(\log(n)^{f(|\phi|)}\) for some function \(f\), \(p_x\) should be of order \(\log(n)^{f(|\phi|)} x^{-\left|\phi\right|}\).

Earlier, we discussed that the size of the kernel will be some function of \(|X|\) and that we choose \(X\) as small as possible. It is therefore sufficient to bound the probability that the set \(X\) of the decomposition of a neighborhood exceeds a certain size. Parameterizing the decomposition by two values (denoted by \(b\) and \(\mu\) later on) gives us enough control to guarantee such a bound on \(p_x\). A large part of this work is devoted to proving a good trade-off between the size of the set \(X\) of the decomposition and the probability that \(X\) is of minimal size. Furthermore, computing the set \(X\) is computationally hard, so the whole procedure has to work without knowing the set \(X\), but only its existence.

Our proofs are structured as follows. First, we show in Section 5 that \(\alpha\)-power-law-bounded random graph models have the following structure with high probability: They
can be partitioned into sets $A$, $B$, $C$, where $A \cup B$ is small, $B \cup C$ is sparse and $A$ and $C$ locally share only few edges. This is done by characterizing this structure by a collection of small forbidden edge-sets and then excluding these edge-sets using the union bound and Definition 2.1. Then in Section 8 we show that the partition into $A$, $B$, $C$ implies the protrusion decomposition of Theorem 9.3. In Section 7 we partially recover the protrusion decomposition from a given input, and use it to kernelize each $r$-neighborhood into an equivalent smaller graph. At last, in Section 8 we combine Gaifman’s locality theorem with the previous algorithms and probability bounds to obtain our algorithm and bound its run time. Some proofs are quite tedious, but the nature of this problem seems to stop us from using simpler methods. Furthermore, in Section 9 give a simpler presentation of our results and in Section 10 we discuss the algorithmic implications of our results for various random graph models.

4 Notations and Definitions

4.1 Graph Notation

We use common graph theory notation [23]. The length of a path equals its number of edges. The distance between to vertices $u$ and $v$ (dist$(u, v)$) equals the length of a shortest path between $u$ and $v$. For a vertex $v$ let $N^G_r(v)$ be the set of vertices which have in $G$ distance at most $r$ to $v$. The radius of a graph is the minimum among all maximum distances from one vertex to all other vertices. An $r$-neighborhood in $G$ is an induced subgraph of $G$ with radius at most $r$. The order of a graph is $|G| = |V(G)|$. The size of a graph is $|G| = |V(G) + E(G)|$. The edge-excess of a graph $G$ is $|E(G)| - |V(G)|$.

In this work we obtain results for labeled graphs [14]. A labeled graph is a tuple $G = (V(G), E(G), P_1(G), \ldots, P_l(G))$ with $P_i(G) \subseteq V(G)$. We call $P_1(G), \ldots, P_l(G)$ the labels of $G$. We say a vertex $v$ is labeled with label $P_i(G)$ if $v \in P_i(G)$. A vertex may have multiple labels. We say the unlabeled simple graph $G' = (V(G), E(G))$ is the underlying graph of $G$ and $G$ is a labeling of $G'$. All notion for graphs extends to labeled graphs as expected. The union of two labeled graphs $G$ and $H$, $(G \cup H)$, is obtained by setting $V(G \cup H) = V(G) \cup V(H)$, $E(G \cup H) = E(G) \cup E(H)$ and for each label $P_i(G \cup H) = P_i(G) \cup P_i(H)$.

For a graph class $\mathcal{G}$, we define $\mathcal{G}_\ell$ to be the class of all labelings of $\mathcal{G}$. We define $\mathcal{G}$ to be the class of all simple graphs and $\mathcal{G}_\ell$ to be the class of all labeled simple graphs.

4.2 Probabilities and Random Graph Models

We denote probabilities by Pr[$\ast$] and expectation by E[$\ast$]. We consider a random graph model to be a sequence of probability distributions. For every $n \in \mathbb{N}$ a random graph model describes a probability distribution on unlabeled simple graphs with $n$ vertices. In order to speak of probability distributions over graphs we fix a sequence of vertices $(v_i)_{i \geq 1}$ and require that a graph with $n$ vertices has the vertex set $\{v_1, \ldots, v_n\}$. A random graph model is a sequence $\mathcal{G} = (\mathcal{G}_n)_{n \in \mathbb{N}}$, where $\mathcal{G}_n$ is a probability distribution over all unlabeled simple graphs $G$ with $V(G) = \{v_1, \ldots, v_n\}$. Even though some random processes naturally lead to graphs with multi-edges or self-loops, we interpret them as simple graphs by removing all self-loops and replacing multiple edges with one single edge. In slight abuse of notation, we also write $\mathcal{G}_n$ for the random variable which is distributed according to $\mathcal{G}_n$. This way, we can lift graph notation to notion for random variables of graphs: For example edge sets and neighborhoods of a random graph $\mathcal{G}_n$ are represented by random variables $E(\mathcal{G}_n)$ and $N^{\mathcal{G}_n}_v(v)$.

4.3 Sparsity

At first, we define nowhere and somewhere density as a property of graph classes and then lift the notation to random graph models. There are various equivalent definitions and we
use the most common definition based on shallow topological minors.

**Definition 4.1** (Shallow topological minor [62]). A graph \( H \) is an \( r \)-shallow topological minor of \( G \) if a graph obtained from \( H \) by subdividing every edge up to \( 2r \) times is isomorphic to a subgraph of \( G \). The set of all \( r \)-shallow topological minors of a graph \( G \) is denoted by \( G \not\nearrow r \). We define the maximum clique size over all shallow topological minors of \( G \) as

\[
\omega(G \not\nearrow r) = \max_{H \in G \not\nearrow r} \omega(H).
\]

**Definition 4.2** (Nowhere dense [61]). A graph class \( G \) is nowhere dense if there exists a function \( f \), such that for all \( r \in \mathbb{N} \) and all \( G \in G \), \( \omega(G \not\nearrow r) \leq f(r) \).

**Definition 4.3** (Somewhere dense [61]). A graph class \( G \) is somewhere dense if for all functions \( f \) there exists an \( r \in \mathbb{N} \) and a \( G \in G \), such that \( \omega(G \not\nearrow r) > f(r) \).

Observe that a graph class is somewhere dense if and only if it is not nowhere dense. We lift these notions to random graph models using the following two definitions.

**Definition 4.4** (a.a.s. nowhere dense). A random graph model \( G \) is a.a.s. nowhere dense if there exists a function \( f \) such that for all \( r \in \mathbb{N} \)

\[
\lim_{n \to \infty} \Pr[\omega(G_n \not\nearrow r) \leq f(r)] = 1.
\]

**Definition 4.5** (a.a.s. somewhere dense). A random graph model \( G \) is a.a.s. somewhere dense if for all functions \( f \) there is an \( r \in \mathbb{N} \) such that

\[
\lim_{n \to \infty} \Pr[\omega(G_n \not\nearrow r) > f(r)] = 1.
\]

While for graph classes the concepts are complementary, a random graph model can both be neither a.a.s. somewhere dense nor a.a.s. nowhere dense (e.g., if the random graph model is either the empty or the complete graph, both with a probability of \( 1/2 \)).

### 4.4 First-Order Logic

We consider only first-order logic over labeled graphs. We interpret a labeled graph \( G = (V, E, P_1, \ldots, P_l) \), as a structure with universe \( V \) and signature \( (E, P_1, \ldots, P_l) \). The binary relation \( E \) expresses adjacency between vertices and the unary relations \( P_1, \ldots, P_l \) indicate the labels of the vertices. Other structures can be easily converted into labeled graphs. We write \( \varphi(x_1, \ldots, x_k) \) to indicate that a formula \( \varphi \) has free variables \( x_1, \ldots, x_k \). The quantifier rank of a formula is the maximum nesting depth of quantifiers in the formula. Two labeled graphs \( G_1, G_2 \) with the same signature are \( q \)-equivalent \( (G_1 \equiv_q G_2) \) if for every first-order sentence \( \varphi \) with quantifier rank at most \( q \) and matching signature holds \( G_1 \models \varphi \) if and only if \( G_2 \models \varphi \). Furthermore, \(|\varphi|\) is the number of symbols in \( \varphi \). There exists a simple algorithm which decides whether \( G \models \varphi \) in time \( O(|G|^{|\varphi|}) \).

### 4.5 Model-Checking

With all definitions in place, we can now properly restate the model-checking problem and what it means to solve it efficiently on a random graph model. The model-checking problem on labeled graphs is defined as follows.

| p-MC(FO, Gb) |
|----------------|
| **Input:** A graph \( G \in Gb \) and a first-order sentence \( \varphi \) |
| **Parameter:** \(|\varphi|\) |
| **Problem:** \( G \models \varphi \)? |

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Under worst-case complexity, \( p\text{-MC}(\mathcal{F}_b) \) is \( \text{AW}[\ast]\text{-complete} \) \( [25] \) (and \( \text{PSPACE}\text{-complete when unparameterized} \) \( [70] \)). We want average case algorithms for \( p\text{-MC}(\mathcal{F}_b) \) to be efficient for all possible labelings of a random graph model. A function \( L : \mathcal{G} \rightarrow \mathcal{G}_b \) is called a \( l\)\text{-labeling function for \( l \in \mathbb{N} \) if for every \( G \in \mathcal{G} \), \( L(G) \) is a labeling of \( G \) with up to \( l \) labels.

**Definition 4.6.** We say \( p\text{-MC}(\mathcal{F}_b) \) can be decided on a random graph model \( (\mathcal{G}_n)_{n \in \mathbb{N}} \) in expected time \( f(|\varphi|, n) \) if there exists a deterministic algorithm \( A \) which decides \( p\text{-MC}(\mathcal{F}_b) \) on input \( G, \varphi \) in time \( t_A(G, \varphi) \) and if for all \( n \in \mathbb{N} \), all first-order sentences \( \varphi \) and all \( |\varphi|\text{-labeling functions} L, E_{G \sim \mathcal{G}_n} |t_A(L(G), \varphi)| \leq f(|\varphi|, n) \). We say \( p\text{-MC}(\mathcal{F}_b) \) on a random graph model can be decided in \textit{expected FPT time} if it can be decided in expected time \( g(|\varphi|)n^{O(1)} \) for some function \( g \).

### 5 Structure Theorem for Power-Law-Bounded Random Graph Models

The goal of this section is to partition \( \alpha\text{-power-law-bounded random graph models. We show in Theorem 5.10 that their vertices can with high probability be partitioned into sets} \ A, B, C \text{ with the following properties: The sets} A \text{ and} B \text{ are small, the graph} G[B \cup C] \text{ is locally almost a tree, i.e., has locally only a small edge-excess, and the set} B \text{ almost separates} A \text{ from} C \text{, i.e., every neighborhood in} G[C] \text{ has only a small number of edges to} A \text{. We call} (A, B, C) \text{ an} b-r-\mu\text{-partition. We state the formal definition.}

**Definition 5.1** \((b-r-\mu\text{-partition})\). Let \( b, r, \mu \in \mathbb{N}^+ \). Let \( G \) be a graph. A tuple \((A, B, C)\) is called a \( b-r-\mu\text{-partition of} \ G \) if

1. the sets \( A, B, C \) are pairwise disjoint and their union is \( V(G) \),
2. \(|A| \leq b \) and \(|B| \leq b^r \),
3. every 40\( r \mu \)-neighborhood in \( G[B \cup C] \) has an edge-excess of at most \( \mu^2 \), and
4. for every 20\( r \mu \)-neighborhood in \( G[C] \) there are at most \( \mu \) edges incident to both the neighborhood and to \( A \).

A graph for which an \( b-r-\mu\text{-partition exists} \) is called \( b-r-\mu\text{-partitionable.}

In summary, \( B \) and \( C \) are well behaved and the large set \( C \) is almost separated from \( A \). Note that the properties of an \( b-r-\mu\text{-partition depend on three parameters} b, r, \mu \). The results of this section imply that our random graphs are asymptotically almost surely \( b-r-\mu\text{-partitionable for} b = \tilde{d}_{\alpha}(n)^{O(1)} \) and constant \( r, \mu \). It therefore helps to assume that \( b \) is a slowly growing function in \( n \), such as \( \log(n) \) and \( r, \mu \) are constants. Higher values of \( \mu \) boost the probability of a random graph being \( b-r-\mu\text{-partitionable. The parameter} \mu \) is therefore crucial for the design of efficient algorithms.

For an \( \alpha\text{-power-law-bounded random graph model} \ (\mathcal{G}_n)_{n \in \mathbb{N}} \), we always assume the vertices of \( \mathcal{G}_n \) to be \( v_1, \ldots, v_n \), ordered as in Definition 2.1. We will choose \( A = \{v_1, \ldots, v_b\}, B = \{v_{b+1}, \ldots, v_{b^r}\}, C = \{v_{b^r+1}, \ldots, v_n\} \) and show that the probability is low that \((A, B, C)\) does not form a \( b-r-\mu\text{-partition. We do this in two steps: In Section 5.1 we define} \mathcal{H}_n(b, r, \mu) \text{ to be a set of graphs over the vertex set} \{v_1, \ldots, v_n\} \text{. We show that if} (A, B, C) \text{ is not a} b-r-\mu\text{-partition then the complete edge-set of some graph in} \mathcal{H}_n(b, r, \mu) \text{ is present in the graph. In Section 5.2 we bound the probability of the edge-set of any graph from} \mathcal{H}_n(b, r, \mu) \text{ being present in the random graph model.}

At last, in Lemma 5.12 Section 5.3 we bound the sum of the expected sizes of all \( r\text{-neighborhoods in an} \alpha\text{-power-law-bounded graph class. This is needed to bound the expected run time of an algorithm that iterates over all} r\text{-neighborhoods of a graph.}
5.1 Forbidden Edge-Sets Characterization

Here we show that if a graph is not $b$-$r$-$\mu$-partionable then it contains some forbidden edge-set.

Definition 5.2. Let $G$ be a graph and $\mathcal{H}$ be a set of graphs over $V(G)$. We say $\mathcal{H} \subseteq G$ if for some $H \in \mathcal{H}$, $E(H) \subseteq E(G)$.

Definition 5.3. Let $b, r, \mu, n \in \mathbb{N}^+$. We define $\mathcal{H}_n(b, r, \mu)$ to be the set of

- all graphs with vertex set $V \subseteq \{v_{b+1}, \ldots, v_n\}$ such that $|V| \leq 200b\mu^3$, all vertices have degree at least two, and the graph has an edge-excess of $\mu^2$, and
- all graphs $(V_1 \cup V_2, E)$ such that $V_1 \subseteq \{v_1, \ldots, v_b\}$, $V_2 \subseteq \{v_{b+1}, \ldots, v_n\}$, $|V_1 \cup V_2| \leq 25r\mu^2$, $|V_1| \leq \mu$, all vertices in $V_2$ have degree at least two, and the summed degree of $V_2$ is $2|V_2| - 2 + \mu$.

Lemma 5.4. Let $b, r, \mu, n \in \mathbb{N}^+$. If a graph $G$ with vertex set $\{v_1, \ldots, v_n\}$ is not $b$-$r$-$\mu$-partionable, then $\mathcal{H}_n(b, r, \mu) \subseteq G$.

Proof. Assume a graph $G$ is not $b$-$r$-$\mu$-partionable. Then the tuple $(A, B, C)$ with $A = \{v_1, \ldots, v_b\}$, $B = \{v_{b+1}, \ldots, v_{n-1}\}$, $C = \{v_n\}$ is not a $b$-$r$-$\mu$-partition of $G$. This means $(A, B, C)$ either does not satisfy Property 3 or 4 of Definition 5.1.

Assume now $(A, B, C)$ does not satisfy Property 3. Then there is a 40$r$-neighborhood in $G[B \cup C]$ with an edge-excess of at least $\mu^2$. Let $T$ be a breadth-first-search tree of this neighborhood of depth 40$r$ with a root $v$. There are $\mu^2 + 1$ extra edges in this neighborhood which are not contained in $T$. Let $H$ be the graph constructed by the following procedure: We induce $G$ on all vertices which are either an endpoint of the $\mu^2 + 1$ extra edges or lie on the unique path in $T$ from such an endpoint to the root $v$. Then we iteratively remove all vertices with degree one. Every vertex in $H$ has degree at least two. In $T$, each path starting at $v$ has length at most 40$r$, and there are at most $(\mu^2 + 1)$ extra edges. Therefore, $H$ consists of at most $2(\mu^2 + 1)(40r + 1)$ vertices. Furthermore, $H$ contains $\mu^2$ more edges than vertices. This means $G[B \cup C]$ contains a subgraph with an edge-excess of $\mu^2$ and $2(\mu^2 + 1)(40r + 1) \leq 200\mu^3$ vertices that all have degree at least two. Such a graph is contained in $\mathcal{H}_n(b, r, \mu)$.

Assume now $(A, B, C)$ does not satisfy Property 4. Then $G[C]$ contains a 20$r$-neighborhood such that there are $\mu$ edges going from this 20$r$-neighborhood to $A$. Let these edges be $u_1w_1, \ldots, u_\mu w_\mu$ with $u_i \in A$ and $w_i \in C$. Let $T$ be a breadth-first-search tree of depth 20$r$ of this 20$r$-neighborhood with root $v$. Let $V_1 = \{u_1, \ldots, u_\mu\}$ and let $V_2$ be the set of vertices that lie for each $w_i$ on the unique path in $T$ of length at most 20$r$ from $w_i$ to the root $v$, including $w_i$ and $v$. Let $H$ be the graph with vertex set $V_1 \cup V_2$ and all edges from $T[V_2]$, as well as all edges between $V_1$ and $V_2$ in $G$. Notice that $|V_1| \leq \mu$ and $|V_2| \leq (20r + 1)\mu$. Also $H[V_2]$ forms a tree with $\mu$ outgoing edges to $V_1$. Therefore, the vertices in $V_2$ have in $H$ a summed degree of $2|V_2| - 2 + \mu$. They also have degree at least two in $H$. This means $G$ contains a subgraph $(V_1 \cup V_2, E)$ such that $V_1 \subseteq A$, $V_2 \subseteq C$, $|V_1 + V_2| \leq 25\mu^2$, $|V_1| \leq \mu$, $|E| \leq 25\mu^2$, and the vertices in $V_2$ have degree at least two and a summed degree of $2|V_2| - 2 + \mu$. Such a graph is contained in $\mathcal{H}_n(b, r, \mu)$.

5.2 Bounding Probabilities of Edge-Sets

In this section we bound for an $\alpha$-power-law-bounded random graph model $(\mathcal{G}_n)_{n \in \mathbb{N}}$ the probability that $\mathcal{H}_n(b, r, \mu) \subseteq \mathcal{G}_n$, thereby bounding the probability that $\mathcal{G}_n$ is not $b$-$r$-$\mu$-partionable.
Definition 5.5. Let $\mathcal{H}$ be a set of graphs over the vertex set $\{v_1, v_2, \ldots\}$, and let $E$ be a set of edges over the same vertex set. We define

$$p_\alpha(E, n) = \tilde{d}_\alpha(n)|E|^2 \prod_{\{i,j\} \in E} \frac{(n/i)^{1/(\alpha-1)}(n/j)^{1/(\alpha-1)}}{n^{1/2}}$$

$$p_\alpha(\mathcal{H}, n) = \sum_{H \in \mathcal{H}} p_\alpha(E(H), n).$$

Lemma 5.6. Let $(\mathcal{G}_n)_{n \in \mathbb{N}}$ be an $\alpha$-power-law-bounded random graph model. Let $(\mathcal{H}_n)_{n \in \mathbb{N}}$ be a sequence of sets of graphs over the vertex set $\{v_1, v_2, \ldots\}$. Then $\Pr[\mathcal{H}_n \subseteq \mathcal{G}_n] \leq p_\alpha(\mathcal{H}_n, n)$.

Proof. Using the union bound and Definition 2.1 we see

$$\Pr[\mathcal{H}_n \subseteq \mathcal{G}_n] \leq \sum_{H \in \mathcal{H}_n} \Pr[E(H) \subseteq E(\mathcal{G}_n)] \leq \sum_{H \in \mathcal{H}_n} p_\alpha(H, n) = p_\alpha(\mathcal{H}_n, n).$$

It is therefore sufficient to bound $p_\alpha(\mathcal{H}_n(b, r, \mu))$. The following two Lemmas prove some technicalities we need to do so.

Lemma 5.7. Let $\alpha \geq 2$. For $n \in \mathbb{N}^+$

$$\sum_{i=1}^{n} \frac{(n/i)^{1/(\alpha-1)}}{n^{1/2}} \leq \tilde{d}_\alpha(n)\sqrt{n}, \quad \sum_{i=1}^{n} \frac{(n/i)^{2/(\alpha-1)}}{n} \leq \tilde{d}_\alpha(n)^2.$$

For $b, n \in \mathbb{N}^+$, $\delta \in \mathbb{N}$

$$\sum_{i=1}^{b} \frac{(n/i)^{\delta/(\alpha-1)}}{n^{\delta/2}} \leq \tilde{d}_\alpha(n)^\delta b.$$

For $b, n, \delta \in \mathbb{N}^+$, $\delta \geq 2$

$$\sum_{i=b+1}^{n} \frac{(n/i)^{\delta/(\alpha-1)}}{n^{\delta/2}} \leq \tilde{d}_\alpha(n)^\delta b^{1-\delta/2}.$$

Proof. We define $\gamma = 1/(\alpha - 1)$ and

$$\rho_\gamma(n) = \begin{cases} O(1) & \gamma < 1/2 \\ \log(n)^{O(1)} & \gamma = 1/2 \\ O(n^{\gamma-1/2}) & \gamma > 1/2. \end{cases}$$

For $\alpha > 3$ we have $\gamma < 1/2$ and thus $\rho_\gamma(n) = \tilde{d}_\alpha(n) = O(1)$. Similarly, for $\alpha = 3$ holds $\gamma = 1/2$ and $\rho_\gamma(n) = \tilde{d}_\alpha(n) = \log(n)^{O(1)}$. For $2 \leq \alpha < 3$ we have $\gamma - 1/2 \leq 3 - \alpha$. Therefore $\rho_\gamma(n) \leq \tilde{d}_\alpha(n)$ for all values of $\alpha \geq 2$. It is now sufficient to show

$$\sum_{i=1}^{n} \frac{(n/i)^\gamma}{n^{1/2}} = \rho_\gamma(\sqrt{n}), \quad \sum_{i=1}^{n} \frac{(n/i)^{2\gamma}}{n} = \rho_\gamma(n)^2,$$

$$\sum_{i=1}^{b} \frac{(n/i)^{\delta_1}}{n^{\delta/2}} \leq \rho_\gamma(n)^\delta b,$$

and for $\delta \geq 2$

$$\sum_{i=b+1}^{n} \frac{(n/i)^{\delta_1}}{n^{\delta/2}} \leq \rho_\gamma(n)^\delta b^{1-\delta/2}.$$
We bound with $\gamma < 1$
\[
\sum_{i=1}^{n} \frac{(n/i)^\gamma}{n^{1/2}} \leq n^{\gamma - 1/2} \int_0^n \frac{1}{t^{\gamma}} dt = n^{\gamma - 1/2}n^{1-\gamma}/(1-\gamma) \leq \rho_\gamma(n)\sqrt{n}
\]
and
\[
\sum_{i=1}^{n} \frac{(n/i)^{2\gamma}}{n} = n^{2\gamma - 1} + n^{2\gamma - 1} \sum_{i=2}^{n} \frac{1}{i^{2\gamma}} \leq O(n^{2\gamma - 1}) \int_1^n \frac{1}{t^{2\gamma}} dt \leq \rho_\gamma(n)^2.
\]

(1)

We further bound
\[
\sum_{i=1}^{b} \frac{(n/i)^{\delta\gamma}}{n^{\delta/2}} \leq n^{\delta(\gamma - 1/2)b} \leq \rho_\gamma(n)^{\delta b^{1-\delta/2}}.
\]

To prove the last bound, we make a case distinction over $\delta$ and $\gamma$. At first, assume $\delta = 2$. Then
\[
\sum_{i=b+1}^{n} \frac{(n/i)^{\delta\gamma}}{n^{\delta/2}} \leq \rho_\gamma(n)^{\delta} = \rho_\gamma(n)^{\delta b^{1-\delta/2}}.
\]

Assume now that $\delta \geq 3$. We have for $\gamma \leq 1/2$
\[
\sum_{i=b+1}^{n} \frac{(n/i)^{\delta\gamma}}{n^{\delta/2}} \leq \sum_{i=b+1}^{n} \frac{(n/i)^{\delta\gamma}}{n^{\delta/2}} \leq \int_b^n \frac{1}{t^{\delta/2}} dt \leq O(1)b^{1-\delta/2} \leq \rho_\gamma(n)^{\delta b^{1-\delta/2}}
\]

and for $\gamma > 1/2$
\[
\sum_{i=b+1}^{n} \frac{(n/i)^{\delta\gamma}}{n^{\delta/2}} \leq n^{\delta(\gamma - 1/2)} \int_b^n \frac{1}{t^{\delta\gamma}} dt \leq O(n^{\delta(\gamma - 1/2)})b^{1-\delta/2} \leq \rho_\gamma(n)^{\delta b^{1-\delta/2}}.
\]

\[\square\]

**Lemma 5.8.** Let $b_1, b_2, l, k, n \in \mathbb{N}^+$ with $b_1 \leq b_2$. Let $\mathcal{L}_n(b_1, b_2, l, k)$ be the set of all graphs $(V_1 \cup V_2, E)$ such that $V_1 \subseteq \{v_1, \ldots, v_{b_1}\}$, $V_2 \subseteq \{v_{b_2+1}, \ldots, v_n\}$, $|V_1 \cup V_2| \leq l$, $|E| \leq l$, $|V_1| \leq k$, all vertices in $V_2$ have degree at least two, and the summed degree of $V_2$ is $2|V_2| - 2 + k$. Then $p_\alpha(\mathcal{L}_n(b_1, b_2, l, k), n) \leq \tilde{d}_\alpha(n)^{O(\delta)}b_1^{1-k/2}b_2^{1-k/2}$.

**Proof.** We can partition the set $\mathcal{L}_n(b_1, b_2, l, k)$ into at most $2^l$ many isomorphism classes. Let $\mathcal{L}' \subseteq \mathcal{L}_n(b_1, b_2, l, k)$ be the isomorphism class which maximizes $p_\alpha(\mathcal{L}', n)$. We have that $p_\alpha(\mathcal{L}_n(b_1, b_2, l, k), n) \leq 2^l p_\alpha(\mathcal{L}', n)$. We fix a representative $H = (V_1 \cup V_2, E) \in \mathcal{L}'$.

Let now $\gamma = |V_1|$ and $\Gamma = |V_2|$. We order the sets $V_1$ and $V_2$ such that we can speak of the first, second, etc. vertex in each set. Let $F$ be the set of all sequences of integers $(x_1, \ldots, x_\gamma, y_1, \ldots, y_{\Gamma})$ without duplicates and with $1 \leq x_i \leq b_1$ and $b_2 + 1 \leq y_i \leq n$. For a sequence $f \in F$ let $f(H)$ be the homomorphism of $H$ where the $i$th vertex from $V_1$ is assigned to $v_{x_i}$ (for $1 \leq i \leq \gamma$) and the $i$th vertex from $V_2$ is assigned to $v_{y_i}$ (for $1 \leq i \leq \Gamma$). Then $\mathcal{L}' = \bigcup_{f \in F} f(H)$.

The vertices in $V_1$ and $V_2$ have a degree sequence $\delta_1, \ldots, \delta_\gamma, \Delta_1, \ldots, \Delta_\Gamma$. We fix a sequence $f = (x_1, \ldots, x_\gamma, y_1, \ldots, y_{\Gamma}) \in F$. Then by Definition 5.5
\[
p_\alpha(E(f(H)), n) = \tilde{d}_\alpha(n)^{\delta_1 + \cdots + \delta_\gamma + \Delta_1 + \cdots + \Delta_\Gamma} \frac{\Gamma}{\prod_{i=1}^{\gamma} (n/x_i)^{\delta_i/(\alpha - 1)} \prod_{i=1}^{\Gamma} (n/y_i)^{\Delta_i/(\alpha - 1)}} \frac{\Gamma}{n^{\Delta_\Gamma/2}}.
\]

(2)

Observe that
\[
\Delta_i \geq 2 \text{ for } 1 \leq i \leq \Gamma;
\]
\[
\delta_1 + \cdots + \delta_\gamma + \Delta_1 + \cdots + \Delta_\Gamma \leq 2l,
\]
\[\text{(3)}\]
\[\text{(4)}\]
\[
\sum_{i=1}^{r} (1 - \Delta_i/2) = \Gamma - \frac{1}{2} \sum_{i=1}^{r} \Delta_i = \Gamma - (\Gamma - 1 + k/2) = 1 - k/2.
\]

(5)

We enumerate all sequences in \( F \), and use (2), (3), (4), (5), and Lemma 5.7 to bound
\[
p_\alpha(\mathcal{L}_n(b_1, b_2, l, k, n)) \leq 2^{2^2} p_\alpha(\mathcal{L}', n) = 2^{2^2} \sum_{f \in F} p_\alpha(E(f(H)), n)
\]
\[
\leq 2^{2^2} b_1 \cdots b_1 \sum_{x_i = l}^{n} \sum_{y_i = b_2}^{n} \tilde{d}_\alpha(n)^{O(i^2)}
\]
\[
\prod_{i=1}^{\gamma} \frac{\left(\frac{n}{x_i}\right)^{\Delta_i/(\alpha - 1)}}{n^{\Delta_i/2}} \prod_{i=1}^{\gamma} \frac{\left(\frac{n}{y_i}\right)^{\Delta_i/(\alpha - 1)}}{n^{\Delta_i/2}}
\]
\[
\leq \tilde{d}_\alpha(n)^{O(i^2)} \sum_{x_i = l}^{n} \frac{\left(\frac{n}{x_i}\right)^{\Delta_i/(\alpha - 1)}}{n^{\Delta_i/2}} \cdots \sum_{y_i = b_2}^{n} \frac{\left(\frac{n}{y_i}\right)^{\Delta_i/(\alpha - 1)}}{n^{\Delta_i/2}}
\]
\[
\leq \tilde{d}_\alpha(n)^{O(i^2)} \prod_{i=1}^{\gamma} \frac{n}{b_1} \prod_{i=1}^{\gamma} \frac{n}{b_2} \Delta_i/2 \leq \tilde{d}_\alpha(n)^{O(i^2)} b_1 b_2^{1-k/2}.
\]

\[\square\]

Lemma 5.9. Let \( b, r, \mu, n \in \mathbb{N}^+ \) with \( \mu \geq 5 \). Then
\[
p_\alpha(\mathcal{H}_n(b, r, \mu, n)) \leq \tilde{d}_\alpha(n)^{O(n^{\mu^2})} b_1 - \mu^2/10.
\]

Proof. We compare the definition of \( \mathcal{H}_n(b, r, \mu) \) and \( \mathcal{L}_n(b_1, b_2, l, k) \) and see that
\[
\mathcal{H}_n(b, r, \mu) \subseteq \mathcal{L}_n(1, b, 200r \mu^3 + \mu^2, 2 \mu^2 + 2) \cup \mathcal{L}_n(b, 25r \mu^2, \mu).
\]

Using Lemma 5.8, and the union bound we compute
\[
p_\alpha(\mathcal{H}_n(b, r, \mu, n))
\]
\[
\leq \tilde{d}_\alpha(n)^{O(n^{\mu^2})} b_1 - \mu^2 + \tilde{d}_\alpha(n)^{O(n^{\mu^2})} b_1^2 (\mu^2 - \mu^2/2 + 1)
\]
\[
\leq \tilde{d}_\alpha(n)^{O(n^{\mu^2})} (b_1 - \mu^2 + b_1^2)(\mu^2 - \mu^2/2 + 1)
\]
\[
\leq \tilde{d}_\alpha(n)^{O(n^{\mu^2})} b_1 - \mu^2/2 + 2 \mu
\]
\[
\leq \tilde{d}_\alpha(n)^{O(n^{\mu^2})} b_1^{\mu^2/2 + 2 \mu}
\]
\[
\leq \tilde{d}_\alpha(n)^{O(n^{\mu^2})} b_1^{\mu^2/10}.
\]

\[\square\]

Theorem 5.10. Let \( (\mathcal{G}_n)_{n \in \mathbb{N}} \) be an \( \alpha \)-power-law-bounded random graph model and let \( b, r, \mu, n \in \mathbb{N}^+ \) with \( \mu \geq 5 \). The probability that \( \mathcal{G}_n \) is not \( b-r-\mu \)-partionable is at most \( \tilde{d}_\alpha(n)^{O(n^{\mu^2})} b_1 - \mu^2/10 \).

Proof. Combining Lemma 5.4, 5.6, and 5.9 \[\square\]
5.3 Expected Neighborhood Sizes

In this section we bound the sum of the expected sizes of all $r$-neighborhoods of a graph from an $\alpha$-power-law-bounded random graph model under the condition that $b \in \mathbb{N}$ is the minimal value such that a graph is $b$-$r$-$\mu$-partitionable. We start with the simpler condition that $\mathcal{H}_n \subseteq \mathcal{G}_n$ for some set of graphs $\mathcal{H}_n$ and then lift this result using Lemma 5.4 from the previous section.

Lemma 5.11. Let $(\mathcal{G}_n)_{n \in \mathbb{N}}$ be an $\alpha$-power-law-bounded random graph model. Let $r \in \mathbb{N}^+$ and let $(\mathcal{H}_n)_{n \in \mathbb{N}}$ be a sequence of sets of graphs where every graph has size at most $h \geq 1$. Then

$$E[\sum_{v \in V(\mathcal{G}_n)} \|\mathcal{G}_n[N^{\mathcal{G}_n}_r(v)]\| \mid \mathcal{H}_n \subseteq \mathcal{G}_n] \Pr[\mathcal{H}_n \subseteq \mathcal{G}_n] \leq h^{O(r)}\tilde{d}_\alpha(n)^{O(r^2+h^2)}np_\alpha(\mathcal{H}_n, n).$$

Proof. We fix an $n$. Let $Q$ be the set of all paths of length at most $r+1$ over $V(\mathcal{G}_n)$. Then by linearity of expectation

$$E[\sum_{v \in V(\mathcal{G}_n)} \|\mathcal{G}_n[N^{\mathcal{G}_n}_r(v)]\| \mid \mathcal{H}_n \subseteq \mathcal{G}_n] = 2 \sum_{Q \in \mathcal{Q}} \Pr[E(Q) \subseteq E(\mathcal{G}_n)].$$

We use this observation and the union bound to compute

$$E[\sum_{v \in V(\mathcal{G}_n)} \|\mathcal{G}_n[N^{\mathcal{G}_n}_r(v)]\| \mid \mathcal{H}_n \subseteq \mathcal{G}_n] \Pr[\mathcal{H}_n \subseteq \mathcal{G}_n] \leq \sum_{H \in \mathcal{H}_n} \sum_{v \in V(\mathcal{G}_n)} \|\mathcal{G}_n[N^{\mathcal{G}_n}_r(v)]\| \mid E(H) \subseteq E(\mathcal{G}_n)] \Pr[E(H) \subseteq E(\mathcal{G}_n)]$$

$$\leq 2 \sum_{H \in \mathcal{H}_n} \Pr[E(Q) \subseteq E(\mathcal{G}_n), E(H) \subseteq E(\mathcal{G}_n)]$$

$$\leq 2 \sum_{H \in \mathcal{H}_n} \Pr[E(Q) \subseteq E(\mathcal{G}_n), E(H) \subseteq E(\mathcal{G}_n)]$$

$$\leq 2 \sum_{H \in \mathcal{H}_n} \sum_{Q \in \mathcal{Q}} p_\alpha(E(H) \cup (E(Q) \setminus E(H)), n)$$

$$\leq \tilde{d}_\alpha(n)^{O(r^2+h^2)} \sum_{H \in \mathcal{H}_n} p_\alpha(E(H), n) \sum_{Q \in \mathcal{Q}} p_\alpha(E(Q) \setminus E(H), n).$$

We fix a graph $H \in \mathcal{H}_n$. We want to find a good bound for $p_\alpha(E(Q) \setminus E(H), n)$ for every $Q \in \mathcal{Q}$. Let $Q \in \mathcal{Q}$ be a path. We assume the vertices $V(Q) = \{w_1, \ldots, w_q\}$ with $q \leq r + 2$ to be ordered such that edges are only between consecutive vertices. Let $s = (s_1, \ldots, s_{q-1}) \in \{0, 1\}^{q-1}$ be the unique bit-string with

$$\{(w_i, w_{i+1}) \mid s_i = 1, 1 \leq i < q\} = E(Q) \setminus E(H).$$

This means $s$ describes which edges of $Q$ are not present in $H$. Let $Q' = (V(Q), E(Q) \setminus E(H))$. The degree sequence of $Q'$ is $\delta^*_1, \ldots, \delta^*_q$ with $\delta^*_i = s_{i-1} + s_i$ (we assume $s_0 = s_q = 0$).

We define

$$W(\delta) = \begin{cases} V(\mathcal{G}_n) & \delta = 2 \\ V(H) & \delta < 2 \end{cases} \quad \text{and} \quad X(\delta) = \begin{cases} \{1, \ldots, n\} & \delta = 2 \\ \{i \mid v_i \in V(H)\} & \delta < 2. \end{cases}$$

If $\delta^*_1 = 0$ then $w_1 \in V(H) = W(\delta^*_1 + 1)$. If $\delta^*_q = 0$ then $w_q \in V(H) = W(\delta^*_q + 1)$. If $\delta^*_i \in \{0, 1\}$
then $w_i \in V(H) = W(\delta^*_i)$ for $2 \leq i \leq q - 1$. Therefore
\[
\sum_{Q \in \mathcal{Q}} p_\alpha(E(Q) \setminus E(H)) \\
\leq \sum_{q=1}^{r+2} \sum_{s \in \{0,1\}^{q-1}} \sum_{w_1 \in W(\delta^*_1)} \sum_{w_2 \in W(\delta^*_2)} \cdots \sum_{w_{q-1} \in W(\delta^*_{q-1})} \sum_{w_q \in W(\delta^*_q+1)} p_\alpha(\{(w_i, w_{i+1}) \mid s_i = 1, 1 \leq i < q\}, n)
\]
\[
= \sum_{q=1}^{r+2} \sum_{s \in \{0,1\}^{q-1}} \sum_{x_1 \in X(\delta^*_1)} \sum_{x_2 \in X(\delta^*_2)} \cdots \sum_{x_{q-1} \in X(\delta^*_{q-1})} \sum_{x_q \in X(\delta^*_q+1)} \tilde{d}_\alpha(n)^{O(\tau^2)} \prod_{i=1}^{q} \frac{(n/x_i)^{\delta^*_i/(\alpha-1)}}{n^{\delta^*_i/2}}
\]
\[
= \tilde{d}_\alpha(n)^{O(\tau^2)} \sum_{q=1}^{r+2} \sum_{s \in \{0,1\}^{q-1}} \sum_{x_1 \in X(\delta^*_1)} \sum_{x_2 \in X(\delta^*_2)} \cdots \sum_{x_{q-1} \in X(\delta^*_{q-1})} \sum_{x_q \in X(\delta^*_q+1)} \frac{(n/x_1)^{\delta^*_1/(\alpha-1)}}{n^{\delta^*_1/2}} \frac{(n/x_2)^{\delta^*_2/(\alpha-1)}}{n^{\delta^*_2/2}} \cdots \frac{(n/x_q)^{\delta^*_q/(\alpha-1)}}{n^{\delta^*_q/2}}.
\]

The bound of (7) depends on the degree sequence $\delta^*_1, \ldots, \delta^*_q$. Remember that $\delta^*_i, \delta^*_q \in \{0,1\}$ and $\delta^*_i \in \{0,1,2\}$ for $1 < i < q$. The following five bounds follow from Lemma 5.7:
\[
\sum_{x \in X(0)} \frac{(n/x)^{0/(\alpha-1)}}{n^{0/2}} \leq h \leq \tilde{d}_\alpha(n)^2 h
\]
\[
\sum_{x \in X(1)} \frac{(n/x)^{1/(\alpha-1)}}{n^{1/2}} \leq \tilde{d}_\alpha(n)h \leq \tilde{d}_\alpha(n)^2 h
\]
\[
\sum_{x \in X(2)} \frac{(n/x)^{2/(\alpha-1)}}{n^{2/2}} \leq \tilde{d}_\alpha(n)^2 \leq \tilde{d}_\alpha(n)^2 h
\]
\[
\sum_{x \in X(1)} \frac{(n/x)^{0/(\alpha-1)}}{n^{0/2}} \leq h \leq \tilde{d}_\alpha(n)^2 h
\]
\[
\sum_{x \in X(2)} \frac{(n/x)^{1/(\alpha-1)}}{n^{1/2}} \leq \tilde{d}_\alpha(n)\sqrt{n} \leq \tilde{d}_\alpha(n)^2 h^2 \sqrt{n}
\]

These five bounds can be used to bound the inner $q$ sums of (7). This yields
\[
\sum_{Q \in \mathcal{Q}} p_\alpha(E(Q) \setminus E(H)) \leq \tilde{d}_\alpha(n)^{O(\tau^2)} \sum_{q=1}^{r+2} \sum_{s \in \{0,1\}^{q-1}} \tilde{d}_\alpha(n)^2 h^q \sqrt{n} \sqrt{n} \leq h^{O(\tau)} \tilde{d}_\alpha(n)^{O(\tau^2)} n.
\]

At last, we combine (6) and (8) and get
\[
E[\sum_{v \in V(G_n)} \|G_n[N^G_{\tau}(v)] \| \mid H_n \subseteq G_n] \Pr[H \subseteq G_n]
\leq \tilde{d}_\alpha(n)^{O(\tau^2 + h^2)} \sum_{H \in \mathcal{H}^n} p_\alpha(E(H), n) \sum_{Q \in \mathcal{Q}} p_\alpha(E(Q) \setminus E(H), n)
\leq p_\alpha(H_n, n) h^{O(\tau)} \tilde{d}_\alpha(n)^{O(\tau^2 + h^2)} n.
\]

Lemma 5.12. Let \((\mathcal{G}_n)_{n \in \mathbb{N}}\) be an \(\alpha\)-power-law-bounded random graph model. Let \(r, \mu, n \in \mathbb{N}^+\) with \(\mu \geq 5\). Let \(A_b\) be the event that \(b \in \mathbb{N}^+\) is the minimal value such that \(\mathcal{G}_n\) is \(b\)-\(r\)-\(\mu\)-partitionable. Then

\[
E\left[ \sum_{v \in V(\mathcal{G}_n)} \| \mathcal{G}_n[N^G_{\mathcal{G}_n}(v)] \| \mid A_b \right] \Pr[A_b] \leq (r\mu)^O(r) \tilde{d}_n(n)O(\mu^2 r^2)b^{-\mu^2/10n}.
\]

Proof. We start with a general observation about conditional expected values. Let \(X\) be a non-negative random variable and \(A \subseteq B\) be events. Then

\[
E[X \mid A] \Pr[A] \leq E[X \mid B] \Pr[B].
\]

(9)

Assume \(b \geq 3\). Let \(G\) be a graph with \(V(G) = \{v_1, \ldots, v_n\}\). If \(b \in \mathbb{N}^+\) is the minimal value such that \(G\) is \(b\)-\(r\)-\(\mu\)-partitionable then \(G\) is not \((b-1)\)-\(r\)-\(\mu\)-partitionable. Then by Lemma 5.11 and 5.9 imply

\[
E\left[ \sum_{v \in V(\mathcal{G}_n)} \| \mathcal{G}_n[N^G_{\mathcal{G}_n}(v)] \| \mid A_b \right] \Pr[A_b] \leq E\left[ \sum_{v \in V(\mathcal{G}_n)} \| \mathcal{G}_n[N^G_{\mathcal{G}_n}(v)] \| \mid \mathcal{H}_n(b-1, r, \mu) \subseteq \mathcal{G}_n \right] \Pr[\mathcal{H}_n(b-1, r, \mu) \subseteq \mathcal{G}_n].
\]

Every subgraph in \(\mathcal{H}_n(b-1, r, \mu)\) has by Definition 5.3 size at most \(200r\mu^3\). Also for \(b \geq 3\) we have \((b-1)^{-1} \leq b^{-1/2}\). Lemma 5.11 and 5.9 imply

\[
E\left[ \sum_{v \in V(\mathcal{G}_n)} \| \mathcal{G}_n[N^G_{\mathcal{G}_n}(v)] \| \mid A_b \right] \Pr[A_b] \leq E\left[ \sum_{v \in V(\mathcal{G}_n)} \| \mathcal{G}_n[N^G_{\mathcal{G}_n}(v)] \| \mid \mathcal{H}_n(b-1, r, \mu) \subseteq \mathcal{G}_n \right] \Pr[\mathcal{H}_n(b-1, r, \mu) \subseteq \mathcal{G}_n] \leq (200r\mu^3)^O(r) \tilde{d}_n(n)O(\mu^2 r^2)n\Pr(\mathcal{H}_n(b-1, r, \mu), n) \leq (r\mu)^O(r) \tilde{d}_n(n)O(\mu^2 r^2)(b-1)^{-\mu^2/5} \leq (r\mu)^O(r) \tilde{d}_n(n)O(\mu^2 r^2)b^{-\mu^2/10n}.
\]

Assume \(b \leq 2\). By (9) and Lemma 5.11 with \(\mathcal{H}_n = \{0\}\)

\[
E\left[ \sum_{v \in V(\mathcal{G}_n)} \| \mathcal{G}_n[N^G_{\mathcal{G}_n}(v)] \| \mid A_b \right] \Pr[A_b] \leq E\left[ \sum_{v \in V(\mathcal{G}_n)} \| \mathcal{G}_n[N^G_{\mathcal{G}_n}(v)] \| \right] \leq \tilde{d}_n(n)O(\mu^2 r^2)n \leq (r\mu)^O(r) \tilde{d}_n(n)O(\mu^2 r^2)b^{-\mu^2/10n}.
\]

\[
\square
\]

6 Protrusion Decompositions of Neighborhoods

In this section, we show that local neighborhoods of power-law-bounded graph classes are likely to have the following nice structure: They consist of a (small) core graph to which so called protrusions are attached. Protrusions are (possibly large) subgraphs with small treewidth and boundary. The boundary of a subgraph is the size of its neighborhood in the remaining graph. Protrusions were introduced by Bodlaender et al. for very general kernelization results in graph classes with bounded genus [5].

Earlier, (Theorem 5.10) we showed that \(\alpha\)-power-law-bounded random graph models are (for certain values of \(\alpha, b, r, \mu\) likely to be \(b\)-\(r\)-\(\mu\)-partitionable. It is therefore sufficient
to show that \(r\)-neighborhoods of \(b\)-\(r\)-\(\mu\)-partitionable graphs have such a nice protrusion structure.

However, in general it is not easy to find protrusions in a graph [18]. As we later need to be able to find them, we define special protrusion decompositions, called \(b\)-\(r\)-\(\mu\)-local-protrusion-partitions in which (most of) the protrusions can be efficiently identified. The main and only result of this section is the following theorem.

**Theorem 6.15.** Let \(b, r, \mu \in \mathbb{N}^+\) and let \(G\) be a \(b\)-\(r\)-\(\mu\)-partitionable graph. Let \(G^r\) be an \(r\)-neighborhood in \(G\). Then \(G^r\) is \(O(\mu^{17} r^3 b)\)-\(r\)-\(\mu\)-locally-protrusion-partitionable.

It remains to define what a \(b\)-\(r\)-\(\mu\)-local-protrusion-partition of a graph \(G^r\) with radius at most \(r\) is. The definition has to strike the right balance: It needs to be permissive enough such that neighborhoods of power-law-bounded graph classes are likely to have this structure and it needs to be restrictive enough to admit efficient algorithms. Informally speaking, a \(b\)-\(r\)-\(\mu\)-local-protrusion-partition of a graph \(G^r\) is a partition \((X, Y, Z)\) of the vertices of \(G^r\) such that \(X\) has small size and the connected components of \(G^r[Y \cup Z]\) are protrusions. In order to be able to efficiently identify the protrusions, we further require that the components of \(G^r[Y]\) have bounded size and the components of \(G^r[Z]\) are trees. This is formalized in the following definition.

**Definition 6.1 (\(b\)-\(r\)-\(\mu\)-local-protrusion-partition).** Let \(b, r, \mu \in \mathbb{N}^+\). Let \(G\) be a graph with radius at most \(r\). A tuple \((X, Y, Z)\) is called an \(b\)-\(r\)-\(\mu\)-local-protrusion-partition of \(G^r\) if

1. the sets \(X, Y, Z\) are pairwise disjoint and their union is \(V(G^r)\).
2. \(|X| \leq b^r\),
3. every connected component of \(G^r[Y]\) has size at most \(r \mu^7\) and at most \(\mu\) neighbors in \(X\),
4. every connected component of \(G^r[Z]\) is a tree with at most one edge to \(G^r[X \cup Y]\).
5. For a subgraph \(H\) of \(G^r[Y \cup Z]\) we say \(N^{G^r}(V(H)) \cap X\) is the boundary of \(H\). The connected components of \(G^r[Y]\) may have at most \(b^r\) distinct boundaries, i.e., \(|\{N^{G^r}(V(H)) \cap X \mid H\text{ connected component of }G^r[Y \cup Z]\}| \leq b^r\).

A graph for which an \(b\)-\(r\)-\(\mu\)-local-protrusion-partition exists is called \(b\)-\(r\)-\(\mu\)-locally-protrusion-partitionable.

Property 3 and 4 enforce that the components of \(G^r[Y \cup Z]\) are protrusions. Later, we will transform \(b\)-\(r\)-\(\mu\)-local-protrusion-partitions into equivalent graphs of bounded size by replacing the protrusions with small graphs. Thus, Property 2 and 5 are there to ensure the resulting kernelized graph will have size roughly \(b^r\) (without Property 5 we could only guarantee a size of roughly \(b^{2r}\)).

To simplify our proofs, we fix some notation which will be valid for this whole section. Let a graph \(G\) and \(b, r, \mu \in \mathbb{N}^+\) be fixed. We further assume \(G\) to be \(b\)-\(r\)-\(\mu\)-partitionable and we fix a \(b\)-\(r\)-\(\mu\)-partition \((A, B, C)\) of \(G\). Let further \(G^r\) be an \(r\)-neighborhood in \(G\) and let \(A^r = A \cap V(G^r)\), \(B^r = B \cap V(G^r)\), \(C^r = C \cap V(G^r)\).

The \(O(\mu^{17} r^3 b)\)-\(r\)-\(\mu\)-locally-protrusion-partition \((X, Y, Z)\) of \(G^r\) will be created by building \(X\) from \(A^r \cup B^r\) and some vertices from \(C^r\). The remaining vertices from \(C^r\) will be split into the sets \(Y\) and \(Z\). The remainder of this section will describe how this procedure happens in detail.

### 6.1 Neighborhoods of \(b\)-\(r\)-\(\mu\)-Partitionable Graphs

We will start with the straightforward result that Properties 3 and 4 of a \(b\)-\(r\)-\(\mu\)-partition (Definition 5.1) can be transferred to neighborhoods.
Lemma 6.2. Every $40\mu r$-neighborhood in $G[B^r \cup C^r]$ has an edge-excess of at most $\mu^2$, and every $20\mu r$-neighborhood in $G[C^r]$ has at most $\mu$ edges to $A^r$.

Proof. Since $C^r \subseteq C$, an $r$-neighborhood in $G[C^r]$ is a connected subgraph of an $r$-neighborhood in $G[C]$. Since $G$ is $b$-r-$\mu$-partitionable, a $20\mu r$-neighborhood in $G[C]$ has at most $\mu$ edges to $A$ and $A^r \subseteq A$. Therefore, a $20\mu r$-neighborhood of $G[C^r]$ has at most $\mu$ edges to $A^r$.

Similarly, a $40\mu r$-neighborhood in $G[B^r \cup C^r]$ is a connected subgraph of a $40\mu r$-neighborhood in $G[B \cup C]$. If a connected graph has an edge-excess of at most $\mu^2$, then so does every connected subgraph. Since $G$ is $b$-r-$\mu$-partitionable, an $40\mu r$-neighborhood in $G[B \cup C]$ has an edge-excess of at most $\mu^2$, which bounds the excess of every $40\mu r$-neighborhood in $G[B^r \cup C^r]$. \hfill \qed

6.2 Ties

The vertices from $A^r$ and $B^r$ will all be put into the set $X$ of a $O(\mu^{17}r^3b)$-r-$O(\mu)$-local-protrusion-partition. The situation for the $C^r$ vertices is more complicated. In this subsection we define so called ties, which we use in the next subsection to distribute the vertices $C^r$ to the sets $X$, $Y$, and $Z$ of a $b$-r-$\mu$-local-protrusion-partition.

Definition 6.3 (Tie). Let $W \subseteq B^r \cup C^r$. We say $(u_1, u_2, v)$ is a $W$-tie if $u_1, u_2 \in W$ and $v$ lies on a walk $p$ with the following properties: Every inner vertex of $p$ is contained in $C^r$ and has at least two neighbors in $C^r$. We define for 1-way to choose $u_1, u_2$, and $v$ contained only as endpoints of $p$; and $p$ is contained in a $20\mu r$-neighborhood in $G[B^r \cup C^r]$. We further say $V(p)$ is a walk set of $(u_1, u_2, v)$.

Ties are triples of vertices that are connected by a walk with certain properties. In the following two lemmas, we bound the size of their walk sets, as well as the number of ties. We need this later to prove the size constraints of a $b$-r-$\mu$-local-protrusion-partition.

Lemma 6.4. A walk set of a tie has at most size $130r^2\mu^3$.

Proof. Let the walk set of a tie be the vertices on a walk $p$. By definition, $p$ is contained in a $20\mu r$-neighborhood in $G[B^r \cup C^r]$. Let $T$ be a breadth-first-search spanning tree of such a neighborhood. According to Lemma 6.2, every $40\mu r$-neighborhood in $G[B^r \cup C^r]$ has an edge-excess of at most $\mu^2$. A tree has an edge-excess of at most $\mu^2$. Therefore, there are at most $\mu^2 + 1$ edges in $p$ which are not contained in $T$. Also, every path in $T$ contains at most $2 \cdot 20\mu r + 1$ vertices. Thus, $p$ contains at most $(2 \cdot 20\mu r + 1) (\mu^2 + 2) \leq 130r^2\mu^3$ vertices. \hfill \qed

Next we take the first step of counting the vertices of $C^r$, by showing that for $W \subseteq B^r \cup C^r$, the number of $W$-ties in $G^r$ is quadratic in $|W|$. Note that this does not directly lead to a bound for $|C^r|$ since it might be that $|W| > |C^r|$.

Lemma 6.5. Let $W \subseteq B^r \cup C^r$. There are at most $390r^2\mu^5 |W|^2$ $W$-ties in $G^r$.

Proof. We fix $u_1, u_2 \in W$. Let $X_{u_1, u_2} = \{(u_1, u_2, v) \mid v \in B^r \cup C^r, (u_1, u_2, v)$ is a $W$-tie$\}$ be the set of all $W$-ties for fixed endpoints $u_1, u_2$. There are exactly $|W|^2$ ways to choose $u_1, u_2$, thus, it is sufficient to show that $|X_{u_1, u_2}| \leq 130r^3\mu^3 (\mu^2 + 2) \leq 390r^2\mu^5$.

Assume for contradiction $|X_{u_1, u_2}| > 130r^3\mu^3 (\mu^2 + 2)$. For every $x \in X_{u_1, u_2}$ let $V(x)$ be a walk set of $x$. The size of a walk set of a tie is at most $130r^3\mu^3$ (Lemma 6.4). Let $l = \mu^2 + 3$. By a pigeonhole argument, one can choose $l$-many $W$-ties $x_1, \ldots, x_l \in X_{u_1, u_2}$ such that $V(x_i) \setminus V(x_1) \cup \cdots \cup V(x_{i-1}) \neq \emptyset$ for $1 \leq i \leq l$. We define for $1 \leq i \leq l$ a graph $G_i = G[V(x_1) \cup \cdots \cup V(x_l)]$. We show by induction that $G_i$ has an edge-excess of at least $l - 2$.

By Definition 6.3 the graphs $G[V(x_i)]$ are connected and all vertices except for $u_1, u_2$ have degree at least two in $G[V(x_i)]$. That means $G_1 = G[V(x_1)]$ has an edge-excess of at least $-1$. It also means that every vertex in the non-empty set $V(G_i) \setminus V(G_{i-1})$ has degree
at least two in $G_i$. Since $G_i$ is connected, there is at least one edge between $V(G_i) \setminus V(G_{i-1})$ and $V(G_{i-1})$ in $G_i$. If every vertex in $V(G_i) \setminus V(G_{i-1})$ has degree exactly two in $G_i$, there are at least two edges between $V(G_i) \setminus V(G_{i-1})$ and $V(G_{i-1})$ in $G_i$. Thus, in the step from $G_{i-1}$ to $G_i$ the number of added edges is at least one greater than the number of added vertices. The edge-excess increases by one.

The walk set of each $W$-tie in $X_{u_1,u_2}$ contains $u_1$ and is contained in a $20\mu r$-neighborhood in $G[B^r \cup C']$. This means $G_i$ is contained in the $(2\cdot 20\mu r)$-neighborhood in $u_1$ in $G[B^r \cup C']$. The graph $G_i$ has an edge-excess of at least $l - 2 = \mu^2 + 1$ and according to Lemma 6.2, this is a contradiction.

6.3 Partitioning $C'$ into $C_A^r$, $C_B^r$, $C_Y^r$, and $C_Z^r$

We use the notion of ties (Definition 6.3) to partition the set $C'$. We distinguish vertices connected to $A'$, vertices connected to $B'$ (but not to $A'$), those which are connected to neither but lie on a tie, and the rest. We set

- $C_A^r = N(A') \cap C'$,
- $C_B^r = (N(B') \setminus N(A')) \cap C'$,
- $C_Y^r = \{v \mid v \in C' \setminus (C_A^r \cup C_B^r)\}$ and there exist $u_1,u_2 \in C_A^r \cup C_B^r$ such that $(u_1,u_2,v)$ is a $(C_A^r \cup C_B^r)$-tie,
- $C_Z^r = C' \setminus (C_A^r \cup C_B^r \cup C_Y^r)$.

We will show that the four previously defined sets have desirable structural properties. By Definition 6.1, we know that $|C_A^r| \leq a$ and $|C_B^r| \leq a^\mu$. We use the previously defined ties to show that $G[C_Z^r]$ is a forest and to bound the size of components of $G[C_A^r \cup C_B^r \cup C_Y^r]$. We then use these properties to construct a $b-r$-$\mu$-local-protrusion-partition. We start with two auxiliary lemmas.

Lemma 6.6. In $G[C']$, every vertex has distance at most $2r$ to a vertex in $C_A^r \cup C_B^r$.

Proof. We fix a vertex $v \in C'$. The graph $G'$ has radius at most $r$, thus, $v$ has in $G'$ distance at most $2r$ to $C_A^r \cup C_B^r$. Vertices in $C' \setminus (C_A^r \cup C_B^r)$ are in $G'$ only adjacent to other vertices from $C'$. This means, the distance from $v$ to the nearest vertex in $C_A^r \cup C_B^r$ in $G[C']$ the same as in $G'$.

Lemma 6.7. A connected component of $G[C']$ with at most $l \in \mathbb{N}$ vertices from $C_A^r \cup C_B^r$ is contained in a $5rl$-neighborhood in $G[C']$.

Proof. Let $G^*$ be a connected component of $G[C']$. By Lemma 6.6, every vertex from $(C_A^r \cup C_B^r) \cap V(G^*)$ has distance at most $2r$ from $(C_A^r \cup C_B^r) \cap V(G^*)$ in $G[C']$. Therefore, $G^*$ is contained in a $(4r + 1)l$-neighborhood in $G[C']$. We have $(4r + 1)l \leq 5rl$.

6.4 Components of $G[C_Z^r]$ are Trees

In this section we show the somewhat surprising property that if you take away all vertices that are connected to $A \cup B$ and those that lie on a tie, you are left with a forest.

Lemma 6.8. Each connected component of $G[C_Z^r]$ is a tree and has at most one outgoing edge in $G'$.

Proof. We consider a connected component $H$ of $G[C_Z^r]$. Assume for contradiction that either $H$ is not a tree, or has more than one outgoing edge in $G'$. Then there has to exist a walk $p$ in $G'$ whose inner vertices are in $V(H)$, whose endpoints are in $V(C^r) \setminus C_Z^r$, and every inner vertex of $p$ has at least two different neighbors in $p$. We pick an arbitrary inner vertex $v \in V(H)$ from $p$. In this proof, we will successively construct walks $p'$, $p''$ and $p^*$.
with endpoints \((w_1', w_2')\), \((w_1'', w_2'')\) and \((w_1', w_2')\) which contain \(v\). The final walk \(p^*\) will be such that \((w_1', w_2', v)\) is a \((C_A \cup C_B')\)-tie. This means by definition that \(v \in C_Y\) and therefore \(v \not\in C_Z\) (a contradiction).

**Constructing \(p'\):** Let \(w_1, w_2\) be the endpoints of \(p\). Since \(C_A \cup C_B\) separates \(C_Z\) from \(A' \cup B'\) in \(G'\), we know that \(w_1, w_2 \in C_A' \cup C_A \cup C_B\). If all \(w_i \in C_A' \cup C_B\), we set \(p' = p\). If any \(w_i \in C_Y\), then, by definition, \(w_i\) lies on a \((C_A \cup C_B')\)-tie walk \(p_i\). By definition the walk \(p_i\) contains no vertex from \(V(H)\), since this would imply that said vertex is in \(C_Y\). We modify \(p\) into \(p'\) as follows: At every endpoint \(w_i \in C_Y\), we extend \(p\) by traversing \(p_i\) in an arbitrary direction until we reach an endpoint \(w_i' \in C_A' \cup C_B'\) and then iteratively removing vertices with degree one that might have been introduced. Now \(p'\) is a walk from \(w_1'\) to \(w_2'\), that goes over \(v\) and where every inner vertex of \(p'\) has at least two different neighbors in \(p'\). Furthermore, \(w_1', w_2' \in C_A' \cup C_B'\). However, \(p'\) is still not necessarily a tie-walk, since it is not guaranteed to be contained in a \(20\mu r\)-neighborhood in \(G[B' \cup C']\).

**Constructing \(p''\):** We construct a sub-walk \(p''\) of \(p'\) by starting at \(v\) and traversing \(p'\) in both directions until we either reach an endpoint in \(C_A' \cup C_B'\) or a vertex with distance exactly \(2r + 1\) in \(G[C']\) to \(v\). The walk \(p''\) contains \(v\) and every vertex on \(p''\) has distance at most \(2r + 1\) in \(G[C']\) from \(v\). The endpoints \(w_1'', w_2''\) of \(p''\) are either in \(C_A' \cup C_B'\) or have distance exactly \(2r + 1\) in \(G[C']\) from \(v\). Every inner vertex of \(p''\) has at least two different neighbors in \(p''\).

**Constructing \(p^*\):** At last, we extend \(p''\) into \(p^*\) as follows: If \(w_1'' \in C_A' \cup C_B'\), we set \(w_1^* = w_1''\). Otherwise, by Lemma 6.6, there exists a vertex \(w_1^* \in C_A' \cup C_B'\) with distance at most \(2r\) in \(G[C']\) from \(w_1''\). Let \(q_i\) be the shortest path from \(w_1^*\) to \(w_i''\) in \(G[C']\). The vertex \(v\) has in \(G[C]\) distance exactly \(2r + 1\) from \(w_i^*\), thus, \(v\) is not contained in \(q_i\). We traverse \(p''\) from \(v\) in both directions. While traversing in direction of \(w_i^*\), as soon as we reach a vertex from \(q_i\), we continue traversing \(q_i\) until we reach \(w_i^*\). The walk \(p^*\) contains \(v\), and every inner vertex has at least two neighbors on \(p^*\). Also, every vertex has distance at most \(4r + 1\) in \(G[C']\) from \(v\). The endpoints \(w_1^*, w_2^*\) are contained in \(C_A' \cup C_B'\). This means that \((w_1^*, w_2^*, v)\) is a \((C_A \cup B')\)-tie.

### 6.5 Connected Components of \(G[C']\)

In this subsection we will speak only about connected components of \(G[C']\). While their number is unbounded we show that inside a component the number of vertices that are not from \(C_Z\) will be bounded. For every component, we first show that if it has few vertices from \(C_Z\), it has few vertices from \(C_A'\) (Lemma 6.3), and that a component with few edges to \(B'\) has few vertices from \(C_A' \cup C_B' \cup C_Y\) (Lemma 6.4).

**Lemma 6.9.** A connected component of \(G[C']\) with \(l \in \mathbb{N}\) vertices from \(C_B'\) contains at most \((l + 1)\mu\) vertices from \(C_A'\).

**Proof.** Let \(G^*\) be a connected component of \(G[C']\) and let \(C_A = C_A' \cap V(G^*), C_B = C_B' \cap V(G^*), C_Y = C_Y' \cap V(G^*), C_Z = C_Z' \cap V(G^*)\). We show that \(|C_A| \leq (|C_B| + 1)\mu\) in two steps: At first we show that if \(|C_A| > (|C_B| + 1)\mu\) then there exists a connected subgraph \(H\) of \(G^*\) which contains at least \(\mu + 1\) vertices from \(C_A\) and at most one vertex from \(C_B\). Second, we show that such a subgraph \(H\) cannot exist.

Assume that \(|C_A| > (|C_B| + 1)\mu\). If \(C_B' = \emptyset\) we set \(H = G^*\). Then \(H\) contains at least \(\mu + 1\) vertices from \(C_A'\) and no vertex from \(C_B'\). If \(C_B' \neq \emptyset\) we proceed as follows. For every \(v \in C_B'\) we define \(A(v)\) to be the set of all vertices from \(C_A\) that are reachable from \(v\) in \(G[C_A' \cup C_Y' \cup C_Z' \cup \{v\}]\). Since \(G^*\) is connected and \(C_B' \neq \emptyset\), for all \(v \in C_B\) exists \(v \in C_B'\) with \(w \in A(v)\). This means \(|C_A'| \leq \sum_{v \in C_B'} |A(v)|\). Since \(|C_A'| > (|C_B'| + 1)\mu\), there exists \(v \in C_B'\) with \(|A(v)| > \mu\). Let \(H\) be the connected component of \(v\) in \(G[C_A' \cup C_Y' \cup C_Z' \cup \{v\}]\). The graph \(H\) is connected and contains exactly one vertex from \(C_B'\). Since \(|A(v)| > \mu\), it also contains at least \(\mu + 1\) vertices from \(C_A'\).

We now show that such a graph \(H\) cannot exist. Let \(H'\) be a connected subgraph of \(H\) which contains exactly \(\mu + 1\) vertices from \(C_A'\) and at most one vertex from \(C_B'\) (we can
construct $H'$ by taking a spanning tree of $H$ and iteratively removing leaves until we have exactly $\mu + 1$ vertices from $C'_A$). The graph $H'$ contains $\mu + 1$ vertices from $C'_A$ and at most one vertex from $C'_B$. According to Lemma 6.11, $H'$ is contained in a $5r(\mu + 2)$-neighborhood in $G'[C']$. Furthermore $H'$ has by construction at least $\mu + 1$ edges to $A'$. Since $G$ is an $b$-$r$-$\mu$-partition every $20\mu r$-neighborhood in $G'[C']$ has, by Lemma 6.2 at most $\mu$ edges to $A'$. This is a contradiction, so $H$ cannot exist.

Lemma 6.10. A connected component of $G'[C']$ with $l \in \mathbb{N}$ edges to $B'$ contains at most $1600 \mu^2 r(l + 1)^2$ vertices from $C'_A \cup C'_B \cup C'_Y$.

Proof. Let $G^*$ be a connected component of $G'[C']$ and let $C'_A = C'_r \cap V(G^*), C'_B = C'_r \cap V(G^*), C'_Y = C'_r \cap V(G^*)$. Since $G^*$ has $l$ edges to $B'$ we have $|C'_B| \leq l$. According to Lemma 6.9, $|C'_A| \leq (l + 1)\mu$. Let $v \in C'_Y$. By definition, there exists a $(C'_A \cup C'_B)$-tie $x = (u_1, u_2, v)$. Since $G^*$ is a connected component, $x$ is also a $(C'_A \cup C'_B)$-tie. By Lemma 6.5 $G^*$ contains at most $390\mu^2(\mu C'_A \cup C'_B)^2 \leq 390\mu^2(l + (l + 1)\mu)^2$ many $(C'_A \cup C'_B)$-ties, which also bounds the number of vertices in $C'_Y$. We add up the bounds for the number of vertices from $C'_B, C'_A$, and $C'_Y$ and get $l + (l + 1)\mu + 390\mu^2(l + (l + 1)\mu)^2 \leq 1600 \mu^2 r(l + 1)^2$.

For components that only have one edge to $B'$ we can directly say how many edges to $A'$ it has.

Lemma 6.11. A connected component of $G'[C']$ with at most one edge to $B'$ has at most $\mu$ edges to $A'$.

Proof. Let $G^*$ be a connected component of $G'[C']$ with at most one edge to $B'$ and therefore at most one vertex from $C'_B$. According to Lemma 6.9, it contains at most $2\mu$ vertices from $C'_A$. By Lemma 6.11, $G^*$ is contained in a $5r(2\mu + 1)$-neighborhood in $G'[C']$. By Lemma 6.2, every $20\mu r$-neighborhood can only have at most $\mu$ edges to $A'$.

6.6 Connected Components of $G'[C']$ With More Than One Edge to $B'$

In this subsection we want to look at components that have more than one edge to $B'$. We start with a helping lemma, that states that for every vertex there is a close vertex from $C'_B$ (or none at all).

Lemma 6.12. Let $v \in C'_r$. If a vertex $u \in C'_B$ with $u \neq v$ is reachable from $v$ in $G'[C']$ then there also is a vertex $w \in C'_B$ with $w \neq v$ that has in $G'[C']$ distance at most $17\mu r$ from $v$.

Proof. We can assume that the shortest path from $u \in C'_B$ to $v$ in $G'[C']$ has length at least $17\mu r$ (otherwise let $w = u$). We pick vertices $x_1, \ldots, x_{\mu + 2}$ along this path, such that $x_i$ has distance $5r$ from $v$ in $G'[C']$. Therefore, $x_i$ has distance at least $5r$ from $x_j$ in $G'[C']$ for $i \neq j$. For every $x_i$ there exists a vertex $s_i \in C'_A \cup C'_B$ with distance at most $2r$ in $G'[C']$ from $x_i$ (Lemma 6.6). Since the vertices $x_i$ are spaced sufficiently far apart, we have $s_i \neq s_j$, and $v \neq s_i$ for $i \neq j$. Each vertex $s_i$ has in $G'[C']$ distance at most $5r + 2r \leq 5r(\mu + 2) + 2r \leq 17\mu r$ from $v$. If $s_i \in C'_B$ for some $i$ we set $w = s_i$, and there is a path in $G'[C']$ from $v$ to $w$ of length at most $17\mu r$. Assume now $s_i \in C'_A$ for all $i \leq \mu + 2$. The vertices $s_i$ are contained in the $17\mu r$-neighborhood of $v$ in $G'[C']$ and each vertex $s_i$ has one edge to $A'$. In total, there are at least $\mu + 2$ edges to $A'$. According to Lemma 6.2, every $20\mu r$-neighborhood in $G'[C']$ has at most $\mu$ edges to $A$. This is a contradiction.

As stated earlier ties are our tool of choice that we use to count vertices. We will establish this in the following lemma that shows that for every edge a component has to $B'$ one introduces more ties. This will in turn bound the number of vertices in components with more than one edge to $B'$.
Lemma 6.13. Let $G^*$ be a connected component of $G[C^r]$ with $l \geq 2$ edges to $B^r$. There are at least $l$ many $B^r$-ties of the form $(u_1, u_2, v)$ with $v \in V(G^*)$.

Proof. Let $u_1v$ be an edge between $B^r$ and $G^*$ with $u_1 \in B^r$ and $v \in V(G^*)$. Since $l \geq 2$, there has to be another edge $u_2w$ between $B^r$ and $G^*$ with $u_2 \in B^r$ and $w \in V(G^*)$. If $w = v$, it follows $u_2 \neq u_1$ and $(u_1, u_2, v)$ is a $B^r$-tie. Otherwise, $w \neq v$ and since $G^*$ is a connected component, $w$ is reachable from $v$ in $G^*$. According to Lemma 6.12 there also is a vertex $w' \in C_B^r \cap V(G^*)$ with $w' \neq v$ which has in $G^*$ distance at most $17\mu r$ from $v$. Since $w' \in C_B^r$, $w'$ also has a neighbor $u_2 \in B^r$. There is a path from $u_1$ to $u_2$ which contains $v$, whose inner vertices are contained in $G^*$, and which has length at most $17\mu r + 2$. This means $(u_1, u_2, v)$ is a $B^r$-tie.

For each of the $l$ edges between $G^*$ and $B^r$ we can use the technique above to construct a $B^r$-tie. The first and third entry of the tuple correspond to an edge between $G^*$ and $B^r$ and thus no two edges create the same tie.

With the next lemma we show that a connected component in $G[C^r]$ with many edges to $B^r$ has many paths with certain properties and then show that only $b\mu^{17r^3} b^{4\mu}$ many vertices from $C^r_A \cup C^r_B \cup C^r_Y$ are in a connected component of $G[C^r]$ with more than one edge to $B^r$.

Lemma 6.14. The number of vertices in $C^r_A \cup C^r_B \cup C^r_Y$ which are in a connected component of $G[C^r]$ with more than one edge to $B^r$ is at most $O(\mu^{17r^3} b^{4\mu})$.

Proof. Let $H_1, \ldots, H_m$ be the connected components of $G[C^r]$ with more than one edge to $B^r$. Let $k_i$ be the number of vertices in $H_i$ which are from $C^r_A \cup C^r_B \cup C^r_Y$. Let $l_i$ be the number of edges to $B^r$ in $H_i$. Let $k = \sum_{i=1}^m k_i$ and $l = \sum_{i=1}^m l_i$. At first, we show that $k \leq 6400\mu r^7 l^2$. Then, we show that $l \leq 390r\mu^6 b^{2\mu}$. Together, this yields $k = O(\mu^{17r^3} b^{4\mu})$.

According to Lemma 6.10 each connected component $H_i$ contains at most $1600\mu^7 r(l_i + 1)^2$ vertices from $C^r_A \cup C^r_B \cup C^r_Y$. We bound $k \leq \sum_{i=1}^m 1600\mu^7 r(l_i + 1)^2 \leq 6400\mu^7 r(\sum_{i=1}^m l_i)^2 = 6400\mu^7 r l^2$. By Lemma 6.13 for $1 \leq i \leq m$ there are at least $l_i$ many $B^r$-ties $(u_1, u_2, v)$ with $v \in V(H_i)$, so in total, there are at least $l$ many $B^r$-ties. With Lemma 6.3 and $|B^r| \leq a^\mu$, we bound $l \leq 390r\mu^6 |B^r|^2 \leq 390r\mu^5 b^{2\mu}$.

6.7 Protrusion Decomposition

Having analyzed the structure of $G[C^r]$ we can finally show that for every $b$-r-$\mu$-partitional graph $G$, every $r$-neighborhood $G^*$ is $O(\mu^{17r^3} b)$-r-$O(\mu)$-locally-protrusion-partitionable.

Theorem 6.15. Let $b, r, \mu \in \mathbb{N}^+$ and let $G$ be a $b$-r-$\mu$-partitional graph. Let $G^*$ be an $r$-neighborhood in $G$. Then $G^*$ is $O(\mu^{17r^3} b)$-r-$O(\mu)$-locally-protrusion-partitionable.

Proof. Let $A^r$, $B^r$, $C^r$, $C^r_A$, $C^r_B$, $C^r_Y$, $C^r_Z$ be as defined earlier. We need to define sets $(X, Y, Z)$ and show all the properties of Definition 6.1. We define $X$ to be the union of $A^r$, $B^r$ and all vertices from $C^r_A \cup C^r_B \cup C^r_Y$ which are in a connected component of $G[C^r]$ with more than one edge to $B^r$. Since $(A, B, C)$ is an $b$-r-$\mu$-partition, we know that $|A^r| \leq b$ and $|B^r| \leq b^\mu$. Lemma 6.13 bounds the number of vertices from $C^r_A \cup C^r_B \cup C^r_Y$ which are in a connected component of $G[C^r]$ with more than one edge to $B^r$ by at most $O(\mu^{17r^3} b^{4\mu})$. This implies $|X| = O(\mu^{17r^3} b)^O(\mu)$ (Property 2).

We define $Y$ to be the vertices from $C^r_A \cup C^r_B \cup C^r_Y$ which are in a connected component of $G[Y]$ with at most one edge to $B^r$. Each connected component of $G[Y]$ is contained in a connected component of $G[C^r]$ with at most one edge to $B^r$. Thus, by Lemma 6.10 connected components of $G[Y]$ have size at most $1600\mu^7 r(1+1)^2 = O(\mu^7 r)$. Every connected component of $G[Y]$ has at most one edge to $B^r$ and by Lemma 6.11 at most $l\mu$ edges to $A^r$. By construction, every edge from it to $X$ goes either to $A^r$ or $B^r$. This means it has at most $\mu + 1$ neighbors in $X$ (Property 3). Since $|A^r| \leq b$ and $|B^r| \leq b^\mu$ there are at most $b^O(\mu)$ (choose at most $\mu$ of $b$ and at most one of $b^\mu$ vertices) possible sets of boundaries in $X$. This satisfies Property 5.
We define $Z = C_Z$. According to Lemma 6.8, every connected component of $G[Z]$ is a tree and has at most one edge to $X \cup Y$ (Property 1). Finally, the sets $X, Y, Z$ are pairwise disjoint and their union is $V(G^*)$ (Property 3).

7 Compressing Neighborhoods

Earlier (Theorem 5.10, Theorem 6.15) we showed that neighborhoods of $\alpha$-power-law-bounded random graph models are (for certain values of $\alpha$, $b$, $r$, $\mu$) likely to be $b$-$r$-locally-protrusion-partitionable (Definition 6.1). This means these neighborhoods have the following nice structure: They consist of a (small) core graph to which protrusions are attached. Remember that protrusions are (possibly large) subgraphs with small treewidth and boundary. Theorem 7.9.

In this section, we replace these protrusions by subgraphs with bounded size that retain the same boundary. This yields a small graph which is $q$-equivalent to the original graph. The same technique has been used for obtaining small kernels in larger graph classes, e.g., in graphs that exclude a fixed minor [35]. The main result of this section is the following theorem.

Theorem 7.9. There exists an algorithm that takes $q, r, \mu \in \mathbb{N}^+$ and a connected labeled graph $G$ with radius at most $r$ and at most $q$ labels as input, runs in time at most $f(q, r, \mu)|G|$ for some function $f(q, r, \mu)$, and computes a labeled graph $G^* \equiv_g G$. If $G$ is $b$-$r$-$\mu$-locally-protrusion-partitionable for some $b \in \mathbb{N}^+$ then $|G^*| \leq f(q, r, \mu)b^\mu$.

This kernelization procedure and its run time bound is independent in $b$ but the size of the output kernel is not: If $b$ is small, then the output is small. The result is obtained by replacing protrusions with the help of the Feferman–Vaught theorem [47]. However, in order to replace the protrusions, one first has to identify them. The main complication in this section lies in partitioning a graph such that the relevant protrusions can be easily identified. It is crucial that we obtain the size bound $|G^*| \leq f(q, r, \mu)b^\mu$ in Theorem 7.9. Weaker bounds are easier to obtain but would not be sufficient for our purposes.

7.1 Protrusion replacement and the Feferman–Vaught Theorem

In this subsection we obtain a suitable protrusion replacement procedure (Lemma 7.4). We use a variant of the Feferman–Vaught theorem [47] to replace a protrusion by a $q$-equivalent boundaryed graph of minimal size. This size depends only on $q$ and the size of the boundary. The original Feferman–Vaught theorem states that the validity of FO-formulas on the disjoint union or Cartesian product of two graphs is uniquely determined by the value of FO-formulas on the individual graphs. Makowsky adjusted the theorem for algorithmic use [54] in the context of MSO model-checking. The following proposition contains the Feferman–Vaught theorem in a very accessible form. There is also a nice and short proof in [44]. The notation is borrowed from [44], too. At first, we need to define so called $q$-types.

Definition 7.1 ([44]). Let $G$ be a labeled graph and $\bar{v} = (v_1, \ldots, v_k) \in V(G)^k$, for some nonnegative integer $k$. The first-order $q$-type of $\bar{v}$ in $G$ is the set $tp^q_{FO}(G, \bar{v})$ of all first-order formulas $\psi(x_1, \ldots, x_k)$ of rank at most $q$ such that $G \models \psi(v_1, \ldots, v_k)$.

A $q$-type could be an infinite set, but one can reduce them to a finite set by syntactically normalizing formulas, so that there are only finitely many normalized formulas of fixed quantifier rank and with a fixed set of free variables. These finitely many formulas can be enumerated. For a tuple $\bar{u} = (u_1, \ldots, u_k)$, we write $\{\bar{u}\}$ for the set $\{u_1, \ldots, u_k\}$. The following is a variant of the Feferman–Vaught theorem [47].

Proposition 7.2 ([44 Lemma 2.3]). Let $G, H$ be labeled graphs and $\bar{u} \in V(G)^k$, such that $V(G) \cap V(H) = \{\bar{u}\}$. Then for all $q \geq 0$, $tp^q_{FO}(G \cup H, \bar{u})$ is determined by $tp^q_{FO}(G, \bar{u})$ and $tp^q_{FO}(H, \bar{u})$.  

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We use this proposition in the following two lemmas to introduce a q-type preserving protrusion replacement procedure.

**Lemma 7.3.** Let $H$ be a connected labeled graph with treewidth at most $t$, at most $q$ labels, and $\bar{u} \in V(H)^k$ for some $k$. One can find in time $h(q,t,k)|H|$ a connected labeled graph $H'$ with $\{\bar{u}\} \subseteq V(H') \subseteq V(H)$, such that $|H'| \leq h(q,t,k)$ and $\text{tp}_q^\text{FO}(H,\bar{u}) = \text{tp}_q^\text{FO}(H',\bar{u})$, for some function $h(q,t,k)$.

**Proof.** The q-type $\text{tp}_q^\text{FO}(H,\bar{u})$ can be represented by a set of normalized FO-formulas with quantifier rank at most $q$ and $k$ free variables. The number and length of these representing formulas can be bounded by a function of $q$ and $k$. Courcelle’s theorem states that for a graph $H$ (with treewidth at most $t$) and a formula $\psi$ (with quantifier rank at most $q$ and $k$ free variables) one can decide whether $H \models \psi(\bar{u})$ in time $g(q,t,k)|H|$, for some function $g(q,t,k)$. This lets us efficiently compute the q-type $\text{tp}_q^\text{FO}(H,\bar{u})$ by checking all representing formulas.

We now have to find a small graph $H'$ with the same q-type as $H$. We enumerate all connected graphs whose vertex set is a superset of $\{\bar{u}\}$ and which are labeled using the same labels as $H$ in ascending order by vertex count. For each graph, we compute the q-type. We finish as soon as we find a graph $H'$ with $\text{tp}_q^\text{FO}(H,\bar{u}) = \text{tp}_q^\text{FO}(H',\bar{u})$. Such a graph $H'$ exists. Each q-type of a $k$-tuple is represented by a subset of normalized formulas of rank at most $q$ and at most $k$ free variables. This bounds the number of different q-types of $k$-tuples by a function of $q$ and $k$. Thus, the size of $H'$ can also be bounded by a function of $q$ and $k$. Since $|H'| \leq |H|$, we can rename the vertices of $H'$ such that $V(H') \subseteq V(H)$.

**Lemma 7.4.** Let $G,H$ be labeled graphs and $\bar{u} \in V(G)^k$ for some $k$, such that $V(G) \cap V(H) = \{\bar{u}\}$. Let $H$ be connected with treewidth at most $t$ and at most $q$ labels. One can find in time $h(q,t,k)|H|$ a connected labeled graph $H'$ such that $|H'| \leq h(q,t,k)$, $V(G) \cap V(H') = \{\bar{u}\}$, and $G \cup H \equiv_q G \cup H'$, for some function $h(q,t,k)$.

**Proof.** We use Lemma 7.3 to construct a connected labeled graph $H'$ such that $|H'| \leq h(q,t,k)$, $\text{tp}_q^\text{FO}(H,\bar{u}) = \text{tp}_q^\text{FO}(H',\bar{u})$, and $V(G) \cap V(H') = \{\bar{u}\}$. According to Proposition 7.2, $\text{tp}_q^\text{FO}(G \cup H,\bar{u})$ is determined by $\text{tp}_q^\text{FO}(G,\bar{u})$ and $\text{tp}_q^\text{FO}(H,\bar{u})$. Therefore, $\text{tp}_q^\text{FO}(G \cup H,\bar{u}) = \text{tp}_q^\text{FO}(G \cup H',\bar{u})$, which implies $G \cup H \equiv_q G \cup H'$.

### 7.2 Reduction Rules

Let $G$ be an $r$-$\mu$-locally-protrusion-partitionable graph. We want to construct a graph which is $q$-equivalent to $G$ and small if $b$ is small. We know there exists an $r$-$\mu$-locally-protrusion-partition $(X,Y,Z)$ of $G$, but it is non-trivial to compute it. In Lemma 7.5 and 7.6, we identify $Z$ and parts of $Y$. In Lemma 7.8 and Theorem 7.9, we replace parts using the protrusion-replace technique from Lemma 7.4.

**Lemma 7.5.** There exists an algorithm that takes $r,\mu \in \mathbb{N}^+$ and a graph $G$ with radius at most $r$ as input, runs in time $O(|G|)$, and computes a set $Z \subseteq V(G)$ with the following property: If $G$ is $r$-$\mu$-locally-protrusion-partitionable for some $b \in \mathbb{N}^+$ then there exists an $b$-$r$-$\mu$-local-protrusion-partition $(X,Y,Z)$ of $G$.

**Proof.** We construct $Z$ iteratively. At first $Z$ is empty. Then in each step we add a vertex $v \in V(G)$ to $Z$ if it is a degree-one vertex of $G[V(G) \setminus Z]$. We repeat this until there are no degree-one vertices in $G[V(G) \setminus Z]$. This can be done in $O(|G|)$ steps. The set $Z$ satisfies Property 3 of Definition 6.1. Assume that $G$ is $b$-$r$-$\mu$-locally-protrusion-partitionable. We need to show that $X,Y$ exist such that $(X,Y,Z)$ is an $b$-$r$-$\mu$-local-protrusion-partition of $G$. We consider an arbitrary $b$-$r$-$\mu$-local-protrusion-partition $(X',Y',Z')$ of $G$. Every connected component of $G[Z']$ is a tree with at most one edge to $V(G) \setminus Z'$. The set $Z$ was constructed such that $Z' \subseteq Z$. We set $X = X' \setminus Z$ and $Y = Y' \setminus Z$. The sets $X, Y, Z$ are disjoint and
their union is \(V(G)\) (Property 1). Since \(X \subseteq X'\) and \(Y \subseteq Y'\), the tuple \((X, Y, Z)\) satisfies Properties 2 and 3.

Let \(H\) be a connected component of \(G[Y]\). By definition of \(Z\), there exists a unique component \(H'\) of \(G[Y]\) such that \(H = H' \cap Z\). Furthermore, \(X \subseteq X'\). This means the number of distinct boundaries of \(G[Y]\) in \(X\) is not larger than the number of distinct boundaries of \(G[Y]\) in \(X'\). Since \((X', Y', Z')\) satisfies Property 5, \((X, Y, Z)\) satisfies Property 5 as well.

**Definition 7.6** (Heavy boundary). Let \((X, Y, Z)\) be an \(b-r-\mu\)-local-protrusion-partition of a graph \(G\). We call a set \(S \subseteq X\) a heavy boundary if there exist more than \(r \mu^7 + \mu\) connected components in \(G[Y]\) whose boundary in \(X\) is exactly \(S\).

**Lemma 7.7.** There exists an algorithm that takes \(r, \mu \in \mathbb{N}^+\) and a graph \(G\) with radius at most \(r\) as input, runs in time \(O(|G|)\), and computes sets \(Z, P \subseteq V(G), S \subseteq 2^{V(G)}\) with the following properties: If \(G\) is \(b-r-\mu\)-locally-protrusion-partionable for some \(b \in \mathbb{N}^+\) then there exists an \(b-r-\mu\)-local-protrusion-partition \((X, Y, Z)\) of \(G\). The connected components of \(G[Y]\) with a heavy boundary are connected components of \(G[P]\). The set \(S\) contains subsets of \(X\) of size at most \(\mu\) and \(|S| \leq \min(2b^6, |G|)\). Every heavy boundary of \((X, Y, Z)\) is contained in \(S\).

**Proof.** Let \(b \in \mathbb{N}^+\) such that \(G\) is \(b-r-\mu\)-locally-protrusion-partionable. We use Lemma 7.1 to construct in time \(O(|G|)\) a set \(Z\) such that there exists an \(b-r-\mu\)-local-protrusion-partition \((X, Y, Z)\) of \(G\).

Let \(P\) be the set of all vertices with degree at most \(r \mu^7 + \mu\) in \(V(G)\). The set \(P\) can be computed in \(O(|G|)\). A vertex \(v \in Y\) is contained in a connected component of \(G[Y]\) of size at most \(r \mu^7\) with at most \(\mu\) neighbors in \(X\) (Property 5). Therefore \(v\) has degree at most \(r \mu^7 + \mu\) in \(G[V(G) \setminus Z]\), which implies \(Y \subseteq P\). Let \(H\) be a connected component of \(G[Y]\) with the heavy boundary \(S\). There are more than \(r \mu^7 + \mu\) connected components in \(G[Y]\) with boundary \(S\). Therefore, every vertex in \(S\) has degree more than \(r \mu^7 + \mu\) in \(G[V(G) \setminus Z]\) and thus \(S \cap P = \emptyset\). This and \(V(H) \subseteq P\) imply that \(H\) is a connected component of \(G[P]\), i.e., the connected components of \(G[Y]\) with a heavy boundary are connected components of \(G[P]\).

Let \(S\) be the set of all subsets of \(V(G) \setminus (P \cup Z)\) with size at most \(\mu\) which are the boundary of some connected component of \(G[P]\) in \(V(G) \setminus (P \cup Z)\). The set \(S\) can be computed in time \(O(|G|)\). Note that \(|S| \leq |G|\). Let \(S \in S\). Since \(S \cap (P \cup Z) = \emptyset\) and \(Y \subseteq P\), we have \(S \subseteq X\). For every heavy boundary of \((X, Y, Z)\), there exists a connected component of \(G[P]\) with this boundary in \(V(G) \setminus (P \cup Z)\). Boundaries of \((X, Y, Z)\) have by definition (Property 5) size at most \(\mu\). Thus, every heavy boundary of \((X, Y, Z)\) is contained in \(S\).

Since \(X \subseteq P\) and \(P \cap Z = \emptyset\), connected components of \(G[P]\) are either connected components of \(G[Y]\) or contain a vertex from \(X\). Since \(|X| \leq b^\mu\), there are at most \(b^\mu\) connected components of \(G[P]\) that are not connected component of \(G[Y]\). Components of \(G[Y]\) have by definition (Property 5) at most \(b^\mu\) distinct boundaries in \(V(G) \setminus (P \cup Z)\). The remaining at most \(b^\mu\) many connected components of \(G[P]\) that contain a vertex from \(X\) have at most \(b^\mu\) boundaries in \(V(G) \setminus (P \cup Z)\). Together, this gives \(|S| \leq 2b^\mu\),

**Lemma 7.8.** There exists an algorithm that takes \(q, r, \mu \in \mathbb{N}^+\) and a connected labeled graph \(G\) with radius at most \(r\) and at most \(q\) labels as input, runs in time at most \(f(q, r, \mu)|G|\) for some function \(f(q, r, \mu)\), and computes a connected labeled graph \(G^* \equiv G\) and a set \(Z^* \subseteq V(G^*)\). In \(G^*\), every connected component of \(G^*[Z^*]\) is a tree and has at most one neighbor in \(V(G^*) \setminus Z^*\). If \(G\) is \(b-r-\mu\)-locally-protrusion-partitionable for some \(b \in \mathbb{N}^+\) then \(|V(G^*) \setminus Z^*| \leq f(q, r, \mu)\min(b^\mu, |G|)\).

**Proof.** Let \(b \in \mathbb{N}^+\) such that \(G\) is \(b-r-\mu\)-locally-protrusion-partitionable. We use Lemma 7.7 to compute sets \(Z, P \subseteq V(G), S \subseteq 2^{V(G)}\). There exists an \(b-r-\mu\)-local-protrusion-partition \((X, Y, Z)\) of \(G\). The connected components of \(G[Y]\) with a heavy boundary are connected components of \(G[P]\). The set \(S\) contains subsets of \(X\) of size at most \(\mu\) and
$|S| \leq \min(2b^\mu, |G|)$. Every heavy boundary of $(X, Y, Z)$ is contained in $S$. For every $W \subseteq V(G) \setminus Z$, we define $Z(W) \subseteq Z$ to be the vertices that are reachable from $W$ in $G[W \cup Z]$.

For every $S \in \mathcal{S}$ we do the following: We compute the set $P_S$ of all vertices which are contained in a connected component of $G[P]$ with size at most $r\mu^7$ and boundary $S$ in $V(G) \setminus (P \cup Z)$. We compute $H_S = G[S \cup P_S \cup Z(P_S)]$ and $G_S = G[V(G) \setminus (P_S \cup Z(P_S))]$. Notice that $G = H_S \cup G_S$ and $V(H_S) \cap V(G_S) = S$. Furthermore, $|S| \leq \mu$ and if we remove $S$ from $H_S$, the remaining graph consists of connected components of size at most $r\mu^7$ to which trees are attached. This means $H_S$ has treewidth at most $t = \mu + r\mu^7$. Also, $H_S$ has at most $q$ labels. Lemma 7.4 lets us construct in time at most $h(q, t, \mu)|H_S|$ a graph $H'_S$ such that $|H'_S| \leq h(q, t, \mu), V(G_S) \cap V(H'_S) = S$, and $G \equiv G_S \cup H'_S$. We now replace $H_S$ with $H'_S$.

This replacement procedure gives us graphs $\hat{G} = G[V(G) \setminus \bigcup_{S \in \mathcal{S}} (P_S \cup Z(P_S))]$ and $\hat{H} = \bigcup_{S \in \mathcal{S}} H'_S$ with $G \equiv \hat{G} \cup \hat{H}$. We set $G^* = \hat{G} \cup \hat{H}$. Notice that $G^*$ is connected, since the described construction preserves connectivity.

We now bound the run time of this procedure. The underlying graph of $G$ can be extracted in time $q||G||$. Constructing $Z$, $P$, and $S$ by Lemma 7.4 takes time $O(||G||)$. The graphs $G$, and $\{H_S \mid S \in \mathcal{S}\}$ can be constructed in time $O(\sum_{S \in \mathcal{S}} ||H_S||)$. Constructing $\hat{H}$ takes time $h(q, t, \mu)\sum_{S \in \mathcal{S}} ||H_S||$.

For every vertex $v \in P \cup Z$ there exists at most one $S \in \mathcal{S}$ such that $v \in H_S$. Also $|S| \leq |G|$. Furthermore, since a graph $H_S$ has treewidth at most $t$, $|H_S| \leq t|G|$. Therefore,

$$\sum_{S \in \mathcal{S}} ||H_S|| \leq t \sum_{S \in \mathcal{S}} |S| = t \sum_{S \in \mathcal{S}} |S| + |H_S \cap (P \cup Z)| \leq t\mu|S| + t|P \cup Z| \leq t\mu|G| + t|G|.$$ 

In total, the whole algorithm runs in $t(q, r, \mu)||G||$, for some function $f(q, r, \mu)$.

We proceed to show that $|V(G^*) \setminus Z|$ is small. Note that

$$|V(G^*) \setminus Z| \leq |X| + |V(\hat{G}) \cap Y| + |\hat{H}|.$$  

(10)

We know that $|X| \leq b^\mu$. Furthermore, with $|S| \leq \min(2b^\mu, |G|)$

$$|\hat{H}| \leq \sum_{S \in \mathcal{S}} |H'_S| \leq \min(2b^\mu, |G|)h(q, t, \mu).$$  

(11)

We further bound the number of vertices from $Y$ in $\hat{G}$. Let $\mathcal{X}$ be the set of all boundaries in $X$ of connected components of $G[Y]$. Let $S \in \mathcal{X}$ be a boundary, we define $Y_S$ to be the vertices of all connected components of $G[Y]$ which have $S$ as their boundary. We distinguish between $S$ being a heavy or non-heavy boundary: Assume $S$ is heavy. Then $S \in \mathcal{S}$. The connected components of $G[Y]$ with a heavy boundary are connected components of $G[P]$ and have size at most $r\mu^7$. This means $Y_S \subseteq P_S$. The graph $\hat{G}$ was defined such that $P_S \cap V(\hat{G}) = \emptyset$, and we have $|Y_S \cap V(\hat{G})| = 0$. Assume now $S$ is non-heavy. Thus, $Y_S$ consists of at most $r\mu^7 + \mu$ connected components of $G[Y]$ of size at most $r\mu^7$. This means $|Y_S| \leq (r\mu^7 + \mu)r\mu^7$.

In total, for every $S \in \mathcal{X}$, $|V(\hat{G}) \cap Y_S| \leq (r\mu^7 + \mu)r\mu^7$. Note that $Y = \bigcup_{S \in \mathcal{X}} Y_S$ and $|\mathcal{X}| \leq b^\mu$ (Property 5). We can therefore bound

$$|V(\hat{G}) \cap Y| \leq \sum_{S \in \mathcal{X}} |V(\hat{G}) \cap Y_S| \leq b^\mu(r\mu^7 + \mu)r\mu^7.$$  

(12)

Combining (10), (11), (12), and $|X| \leq b^\mu$ yields

$$|V(G^*) \setminus Z| \leq b^\mu + b^\nu(r\mu^7 + \mu)r\mu^7 + 2\mu^\mu h(q, t, \mu),$$

which can be bounded by $f(q, r, \mu)b^\mu$ for some function $f(q, r, \mu)$. Furthermore, combining $|V(G^*) \setminus Z| \leq |\hat{G}| + |\hat{H}|$, (11), and $|\hat{G}| \leq |G|$ yields $|V(G^*) \setminus Z| \leq f(q, r, \mu)||G||$ for some function $f(q, r, \mu)$.
Let \( Z^* = Z \cap V(G^*) \). At last, we need to show that in \( G^* \), the connected components of \( G^*[Z^*] \) are trees with at most one neighbor in \( V(G^*) \setminus Z^* \). This follows from that fact that in \( G \) the connected components of \( G[Z] \) are trees with at most one neighbor in \( X \cup Y \), and the graph \( G^* \) introduces no new edges to vertices from \( Z \).

**Theorem 7.9.** There exists an algorithm that takes \( q, r, \mu \in \mathbb{N}^+ \) and a connected labeled graph \( G \) with radius at most \( r \) and at most \( q \) labels as input, runs in time at most \( f(q, r, \mu)||G|| \) for some function \( f(q, r, \mu) \), and computes a labeled graph \( G^* \equiv_q G \). If \( G \) is \( b\)-\( r\)-\( \mu \)-locally-protrusion-partitionable for some \( b \in \mathbb{N}^+ \) then \( ||G^*|| \leq f(q, r, \mu)b^\mu \).

**Proof.** Let \( b \in \mathbb{N}^+ \) such that \( G \) is \( b\)-\( r\)-\( \mu \)-locally-protrusion-partitionable. We use the algorithm of Lemma 7.8 to compute a connected graph \( G' \equiv_q G \) and a set \( Z \subseteq V(G') \). Let \( \bar{Z} = V(G') \setminus Z \). We have \( \vert \bar{Z} \vert \leq g(q, r, \mu) \min(b^\mu, \vert G \vert) \), for some function \( g(q, r, \mu) \). Also, in \( G' \), every connected component of \( G'[Z] \) is a tree and has at most one neighbor in \( \bar{Z} \). Since \( G' \equiv_q G \), \( G' \) also has at most \( q \) labels. If \( \bar{Z} \) is empty, then \( G' \) is a tree with at most \( q \) labels and we use Lemma 8.1 to construct in time \( h(q, 1, 1)||G'|| \) a graph \( G^* \) with \( ||G^*|| \leq h(q, 1, 1) \) and \( G^* \equiv_q G' \equiv_q G \). We therefore assume \( \bar{Z} \neq \emptyset \).

For every \( v \in \bar{Z} \) we do the following: We define \( Z_v \) to be the set of vertices which are contained in a connected component of \( G'[Z] \) which has \( v \) as its only neighbor. We also define the graph \( H_v = G'[\{v\} \cup Z_v] \), which is a tree with at most \( q \) labels and intersects \( G'[Z] \) only in \( v \). We use Lemma 7.4 to construct in time \( h(q, 1, 1)||H_v|| \) a graph \( H'_v \) with \( \vert H'_v \vert \leq h(q, 1, 1) \) and \( G \equiv_q G'[V(G') \setminus Z_v] \cup H'_v \). We replace the subgraph \( H_v \) of \( G' \) with \( H'_v \).

This gives us a graph \( G^* = G'[V(G') \setminus \bigcup_{v \in Z} Z_v] \cup \bigcup_{v \in Z} H'_v \) with \( G^* \equiv_q G' \equiv_q G \). Since \( \bar{Z} \neq \emptyset \) and \( G' \) is connected \( \bigcup_{v \in Z} \bar{Z}_v = \bar{Z} \). We bound with \( \vert \bar{Z} \vert \leq g(q, r, \mu)b^\mu \) and \( v \in V(H'_v) \)

\[
\vert G^* \vert = \sum_{v \in \bar{Z}} \vert H'_v \vert \leq g(q, r, \mu)b^\mu h(q, 1, 1) \leq f(q, r, \mu)b^\mu,
\]

for some function \( f(q, r, \mu) \geq g(q, r, \mu)h(q, 1, 1) \). Notice that the graphs \( \{ H_v \mid v \in \bar{Z} \} \) are disjoint and their union is \( G^* \). The time needed to construct \( H \) therefore is at most

\[
\sum_{v \in \bar{Z}} h(q, 1, 1)||H_v|| \leq h(q, 1, 1)||G'|| \leq h(q, 1, 1)g(q, r, \mu)||G|| \leq f(q, r, \mu)||G||.
\]

As in Lemma 7.8, \( f \) can be chosen such that the algorithm runs in time \( f(q, r, \mu)||G|| \). \( \square \)

**8 Model-Checking**

In this section, we finally obtain the main result of this paper, namely that for certain values of \( \alpha \) one can perform model-checking on power-law-bounded random graph models in efficient expected time.

An important tool in this section is Gaifman’s locality theorem [35]. It states that first-order formulas can express only local properties of graphs. It is a well established tool for the design of model-checking algorithms (e.g. [13, 24]). We use it to reduce the model-checking problem on a graph to the model-checking problem on neighborhoods of said graph (Lemma 8.2). This technique is described well by Grohe [13] section 5).

To illustrate our approach, consider the following thought experiment: Let \( X \) be a non-negative random variable with \( \Pr[X = b] = \Theta(b^{-10}) \) for all \( b \in \mathbb{N} \). Assume an algorithm that gets an integer \( b \in \mathbb{N} \) as input and runs in time \( t(b) \). Its expected run time on input \( X \) is \( \sum_{b \in \mathbb{N}} \Theta(b^{-10})t(X) \). If \( t(b) = b^8 \) then the expected run time is infinite. If \( t(b) = b^8 \) then the expected run time is \( \Theta(1) \). Thus, small polynomial differences in the run time can have a huge impact on the expected run time. We notice that the run time on an input has to grow slower than the inverse of the probability that the input occurs.

Let us fix a formula \( \varphi \) and let \( r \) and \( \mu \) be constants depending on \( \varphi \). In this section we provide a model-checking algorithm whose run time on a graph \( G \) depends on the minimal
value \( b \in \mathbb{N} \) such that \( G \) is \( b\cdot r \cdot \mu\)-partitionable. This means, we need to solve the model-checking problem on \( b\cdot r \cdot \mu\)-partitionable graphs faster than the inverse of the probability that \( b \) is minimal.

Section 5 states that a graph from power-law-bounded graph classes is for some \( b \) not \( b\cdot r \cdot \mu\)-partitionable with probability approximately \( b^{-\mu^2} \) (we ignore the terms in \( r \), \( \mu \) and \( \bar{d}_a(n) \)) for now. Thus, the probability that a value \( b \) is minimal is approximately \( b^{-\mu^2} \).

Let \( G \) be a graph and \( a \) be the minimal value such that \( G \) is \( b\cdot r \cdot \mu\)-partitionable. In Section 4 we showed that all its \( r\)-neighborhoods are \( O(\mu^{17} r^3 b^{-} r \cdot O(\mu))\)-locally-protrusion-partitionable. The kernelization result from Section 4 states that such \( r\)-neighborhoods can be converted in linear time into \( |\phi|\)-equivalent graphs of size approximately \( b^\mu \) (we again ignore the factors independent of \( b \) for now). This means, using the naive model-checking algorithm, one can decide for an \( r\)-neighborhood \( G' \) of \( G \) whether \( G' \models \phi \) in time approximately \( |G||\phi|^2 \). Thus, one can perform model-checking on all \( r\)-neighborhoods of \( G \) in time approximately \( b^\mu |v|^2 \sum_v \|N^G_r(v)\| \). Using Gaifman’s locality theorem, this (more or less) yields the answer to the model checking problem in the whole graph.

Let \( G \) be a graph from a power-law-bounded random graph model. In summary, we have for every \( b \in \mathbb{N} \):

- \( b \in \mathbb{N} \) is the minimal value such that a graph is \( b\cdot r \cdot \mu\)-partitionable with probability approximately \( b^{-\mu^2} \).
- If \( b \in \mathbb{N} \) is the minimal value such that \( G \) is \( b\cdot r \cdot \mu\)-partitionable then we can decide whether \( G \models \phi \) in time approximately \( b^\mu |v|^2 \sum_v \|N^G_r(v)\| \).

In this example one may choose \( \mu = |\phi|^2 \) such that the run time grows slower than the inverse of the probability. We changed some numbers in these examples to simplify our arguments. Thus, in reality, \( \mu \) needs to be chosen slightly differently.

This section is structured as follows: In Section 8.1 we introduce the concept of Gaifman locality. Then, in Section 8.2 we use Gaifman locality and the kernelization result from Section 4 to solve the model-checking in \( b\cdot r \cdot \mu\)-partitionable graphs. At last, in Section 8.3 we prove our main result by showing that the run time of this algorithm grows slower than the inverse of the probability that \( b \) is minimal.

### 8.1 Locality

In this section, we present a well-known technique which reduces the model-checking problem to local regions. Lemma 8.2 gives a slightly different version of what can be found in the literature [44]. Without this modification we would only be able to prove expected polynomial time of our model-checking algorithm instead of expected linear time.

A formula \( \omega(x) \) is called \( r\)-local if \( G \models \omega(v) \) if and only if \( G[N^G_r(v)] \models \omega(v) \) for all labeled graphs \( G \) and all \( v \in V(G) \). Let \( \text{dist}_{G,r}(x_1, x_2) \) be the first-order formula denoting that the distance between \( x_1 \) and \( x_2 \) is greater than \( r \). Let \( \omega \) be an \( r\)-local formula. A basic local sentence is a sentence of the form

\[
\exists x_1 \ldots \exists x_s \left( \bigwedge_{i \neq j} \text{dist}_{G,r}(x_i, x_j) \land \bigwedge_i \omega(x_i) \right).
\]

**Proposition 8.1** (Gaifman’s locality theorem [38, 44]). Every first-order sentence is equivalent to a boolean combination of basic local sentences. Furthermore, there is an algorithm that computes a boolean combination of basic local sentences equivalent to a given first-order sentence.

The following lemma uses Gaifman locality (Proposition 8.1) to reduce model-checking in graphs to model-checking in neighborhoods of graphs. The proof is similar to [44, Lemma 4.9].
Lemma 8.2. Let $g$ be a function such that for every graph $G$ with at most $r$ labels, every $r$-neighborhood $H$ of $G$, every $v \in V(H)$, and every first-order formula $\varphi(x)$ with $|\varphi| \leq r$ one can decide whether $H \models \varphi(v)$ in time $g(v, G, r)$.

There exists a function $\rho$ such that for every first-order sentence $\varphi$ and every labeled graph $G$ with at most $|\varphi|$ labels one can decide whether $G \models \varphi$ in time at most $O(|G|^3 + \rho(|\varphi|))$.

Proof. We can reduce the first-order sentence $\varphi$ to a boolean combination of basic local sentences $\Psi$ with Proposition 8.1. We will independently evaluate each basic local sentence $\psi \in \Psi$ in the graph and use the result to determine whether $\varphi$ is satisfied. Let

$$\psi = \exists x_1 \ldots \exists x_s \left( \bigwedge_{i \neq j} \text{dist} _{2r}(x_i, x_j) \land \bigwedge_i \omega(x_i) \right)$$

be a basic local sentence, where $\omega$ is $r$-local. Let $G$ be a graph with at most $|\varphi|$ labels and $v \in V(G)$. We have $G \models \omega(v)$ if and only if $G[N_r(v)] \models \omega(v)$ for $v \in V$. We compute for all $v \in V$ whether $G \models \omega(v)$. By our assumption, this can be done in time

$$\sum_{v \in V(G)} g(v, G, r + |\omega| + |\varphi|).$$

Let now $W$ be the set of all $v \in V$ such that $G \models \omega(v)$. A set of vertices is called an $r$-scattered set if the $r$-neighborhoods of all pairs of vertices in this set are disjoint. Notice that $G \models \psi$ if and only if there exists an $r$-scattered set of cardinality $s$ which is a subset of $W$. Therefore, all left to do is to find out whether there is an $r$-scattered set $S \subseteq W$ with $|S| \geq s$.

In time $O(|G|^3)$ we do the following: Construct a graph $H$ that consists of all nodes that have distance at most $r$ from $W$, and construct the connected components of $H$. For each component $H'$ of $H$ pick a vertex $v \in V(H')$ and perform a breadth-first-search in $H'$, starting at $v$. This way, we either find out that the radius of $H'$ is larger than $12rs$ or that the diameter of $H'$ is at most $12rs$.

Let us consider two cases. First, we verified that there is a component of $H$ whose diameter is at least $12rs$. Then this component must contain a shortest path $p$ of length $12rs$. We constructed $H$ such that the $r$-neighborhoods of every vertex $u$ on $p$ contains a vertex from $W$. Since there are at least $s$ nodes on $p$ whose $r$-neighborhoods are disjoint and each of the neighborhoods contains a vertex from $W$, we know that $W$ contains an $r$-scattered set of size at least $s$.

The second case we have to consider is that we verified that all components of $H$ have a radius at most $12rs$. Note that $W \subseteq V(H)$. For $u, v \in W$ from different components of $H$, the distance between $u$ and $v$ in $G$ is at least $2r$. Hence, maximal cardinality $r$-scattered subsets of $W$ of the components of $H$ form together a maximal cardinality $r$-scattered subset of $W$ in $G$. A component $H'$ of $H$ contains an $r$-scattered subset of $W$ of size $l$ if $H' \models \psi_l$ with

$$\psi_l = \exists x_1 \ldots \exists x_l \left( \bigwedge_{i \neq j} \text{dist} _{2r}(x_i, x_j) \land \bigwedge_i \omega(x_i) \right),$$

which we can evaluate in time $g(v, G, 12rs + |\psi_l| + |\varphi|)$ for some $v \in V(H')$. We need to check whether $H' \models \psi_l$ for every component $H'$ of $H$ and $l \in \{1, \ldots, s\}$. In that way we can compute the maximal size of an $r$-scattered subset of $W$ in $H$ and therefore in $G$.

The complete procedure has to be repeated for each $\psi \in \Psi$. Note that $r$, $s$, $|\psi_l|$, and $|\psi|$ depend only on $\varphi$. This means we can choose $\rho$ such that all this can be done in time

$$O(|G|^3) + \rho(|\varphi|) \sum_{v \in V(G)} g(v, G, \rho(|\varphi|)).$$

$\square$

8.2 Model-Checking in $b$-$r$-$\mu$-Locally-Protrusion-Partitionable Graphs

We use the kernelization result of Theorem 7.29 to construct a model-checking algorithm for neighborhoods.
Lemma 8.3. There is a function $f(r, \mu)$ such that for every $r, \mu \in \mathbb{N}^+$, every graph $G$ with at most $r$ labels, every $r$-neighborhood $H$ of $G$, every $v \in V(H)$, and every first-order formula $\varphi(x)$ with $|\varphi| \leq r$ one can decide whether $H \models \varphi(v)$ in time $f(r, \mu)O(|\varphi|\|G[N_G^G(v)]\|)$, where $b \in \mathbb{N}^+$ be the minimal value such that $G$ is $b$-$r$-$\mu$-partitionable.

Proof. We construct a graph $H'$ by adding another label to $H$ that identifies $v$ and construct a sentence $\varphi'$ with $|\varphi'| = O(|\varphi|)$ such that $H' \models \varphi'$ if and only if $H \models \varphi(v)$. Let $b \in \mathbb{N}^+$ be the minimal value such that $G$ is $b$-$r$-$\mu$-partitionable. According to Theorem 6.16 $H'$ is $O(\mu^{17}b^{-r})$-locally-protrusion-partitionable. We use Theorem 7.9 to construct in time $f'(r, \mu)\|H'\|$ a graph $H^*$ with $H^* \equiv_{|\varphi'|} H'$ and $|H^*| \leq f'(r, \mu)O(\mu^2)$, for some function $f'$. On this smaller structure we can perform the naive model-checking algorithm in time $O(|H^*|) = O(f'(r, \mu)|\varphi|O(\mu^2))$. Furthermore, the radius of $H'$ is at most $r$, thus $\|H'\| \leq \|G[N_G^G(v)]\|$. We choose $f(r, \mu)$ accordingly.

Lemma 8.4. Let $\mu \in \mathbb{N}^+$. There exist functions $\rho$ and $f$ such that for every first-order sentence $\varphi$ and every labeled graph $G$ with at most $|\varphi|$ labels one can decide whether $G \models \varphi$ in time $f(\rho(|\varphi|), \mu)\|G\|\sum_{v \in V(G)} \|G[N_G^G(v)]\|$, where $b \in \mathbb{N}^+$ is the minimal value such that $G$ is $b$-$\rho(r)$-$\mu$-partitionable.

Proof. By Lemma 8.2 and 8.3 there exist functions $\rho'$ and $f'$ such that one can decide whether $G \models \varphi$ in time

$$O(||G||) + \rho'(|\varphi|) \sum_{v \in V(G)} f'(\rho'(|\varphi|), \mu)O(\mu^2(|\varphi|))\|G[N_G^G(v)]\|,$$

where $b \in \mathbb{N}^+$ is the minimal value such that $G$ is $b$-$\rho'(r)$-$\mu$-partitionable. We choose $f$ and $\rho$ sufficiently large.

8.3 Model-Checking in Power-Law-Bounded Random Graph Models

In this section we show that the algorithm from Lemma 8.3 has efficient expected run time on power-law-bounded random graph models. Our analysis is based upon two results we established earlier: First, the run time of the algorithm in Lemma 8.4 depends on the minimal value $b$ such that the input graph is $b$-$r$-$\mu$-partitionable. If it is $b$-$r$-$\mu$-partitionable for a small $b$ the run time is fast. Secondly, Theorem 5.8 bounds for our random graphs the probability that $b \in \mathbb{N}$ is the minimal value such that a graph is an $b$-$r$-$\mu$-partitionable. For bigger $b$ it is more and more unlikely that $b$ is minimal. In order to have an efficient expected run time on our random graphs, the run time of the algorithm needs to grow asymptotically slower in $b$ than the inverse of the probability that $b$ is minimal. In Theorem 8.5 we show that this is the case.

The run time of the algorithm from Lemma 8.4 depends not only on $b$ but also on the sum of the sizes of all neighborhoods in a graph, which might be quadratic in the worst case. In order to get almost linear expected run time, we bound the expectation of this value in Lemma 5.12. We can now prove our main result.

Theorem 8.5. There exists a function $f$ such that one can solve $\alpha$-MC(FO, $\mathcal{G}_\mu$) on every $\alpha$-power-law-bounded random graph model in expected time $d_u(n)\rho(|\varphi|)n$. 

Proof. Let $(\mathcal{G}_n)_{n \in \mathbb{N}}$ be an $\alpha$-power-law-bounded random graph model and $\varphi$ be a first-order formula. We fix a $|\varphi|$-labeling function $L$ and $n \in \mathbb{N}$. We consider labeled graphs with vertices $V(\mathcal{G}_n)$ whose underlying graph is distributed according to $\mathcal{G}_n$, and analyze the expected run time of the model-checking algorithm from Lemma 8.4 on these graphs.

Let $\rho$ be the function from Lemma 8.4 and let $r = \rho(|\varphi|)$ and $\mu = \rho(|\varphi|)^2 + 100$. For every graph $G$ there exists a value $b \in \mathbb{N}^+$ such that $G$ is $b$-$r$-$\mu$-partitionable (i.e., by setting $b = |V(G)|$, $\mu = V(G)$). Let $A_b$ be the event that $b \in \mathbb{N}^+$ is the minimal value such that $\mathcal{G}_n$
Proof. Assume that there exists a function $O_{\text{run time}}$. Note that for $\sum_{b=1}^{\infty} \Pr[A_b]$ the expected run time of the algorithm is exactly $\sum_{b=1}^{\infty} \Pr[A_b]$.

We use Lemma 8.4 and 5.12 to bound

$$\sum_{b=1}^{\infty} \Pr[A_b] \leq \sum_{b=1}^{\infty} E[R | A_b] \Pr[A_b]$$

for algorithmic purposes. In this section we substitute the parameters $b$ obtained decompositions depending on parameters $\alpha$. Let $\epsilon > 0. With \epsilon = \epsilon / f(|\varphi|)$, the algorithm runs for all $n \in N$ in expected time $c'(\epsilon)f'(|\varphi|)^{n^{1+\epsilon}}$. We set $f(x, \epsilon) = c'(\epsilon)f'(x)$. The algorithm runs for all $n \in N$ in expected time $f(|\varphi|, \epsilon)^{n^{1+\epsilon}}$.

Theorem 8.6. Let $(G_n)_{n \in N}$ be a random graph model. There exists a function $f$ such that one can solve $p\text{-MC}(FO, \mathcal{G}_b)$ in expected time

- $f(|\varphi|)n$ if $(G_n)_{n \in N}$ is $\alpha$-power-law-bounded for some $\alpha > 3$,
- $\log(n)f(|\varphi|)n$ if $(G_n)_{n \in N}$ is $\alpha$-power-law-bounded for $\alpha = 3$,
- $f(|\varphi|, \epsilon)n^{1+\epsilon}$ for all $\epsilon > 0$ if $(G_n)_{n \in N}$ is $\alpha$-power-law-bounded for every $2 < \alpha < 3$.

Proof. Assume that $(G_n)_{n \in N}$ is $\alpha$-power-law-bounded for some $\alpha > 3$. By Theorem 8.5 there exists a function $f'$ such that $p\text{-MC}(FO, \mathcal{G}_b)$ can be solved on $(G_n)_{n \in N}$ in expected time $O(1)f'(|\varphi|)$.

Let $\epsilon > 0. With \epsilon = \epsilon / f'(|\varphi|)$, the algorithm runs for all $n \in N$ in expected time $c'(\epsilon)f'(|\varphi|)^{n^{1+\epsilon}}$. We set $f(x, \epsilon) = c'(\epsilon)f'(x)$. The algorithm runs for all $n \in N$ in expected time $f(|\varphi|, \epsilon)^{n^{1+\epsilon}}$.

9 Asymptotic Structural Properties

In Section 5 and 6 we analyzed the structure of $\alpha$-power-law-bounded random graphs. We obtained decompositions depending on parameters $b, r$ and $\mu$. These parameters are needed for algorithmic purposes. In this section we substitute the parameters $b$ and $\mu$, which leads to structural results in a more accessible form.

We observe that $\alpha$-power-law-bounded random graphs have mostly an extremely sparse structure, with the exception of a part whose size is bounded by the second order average degree of the degree distribution. This denser part can be separated well from the remaining
graph. We show that local regions admit a protrusion decomposition consisting of a core part, bounded in size by the second order average degree, to which trees and graphs of constant size are attached. At first, we define a function \( d_\alpha(n) \) similarly to \( \hat{d}_\alpha(n) \) without \( O \)-notation.

**Definition 9.1.** We define

\[
\hat{d}_\alpha(n) = \begin{cases} 
2 & \alpha > 3 \\
\log(n) & \alpha = 3 \\
\frac{n}{3^{\alpha-3}} & \alpha < 3.
\end{cases}
\]

We use \( \hat{d}_\alpha(n) \) to obtain a good bound on the minimal value \( b \) such that \( \alpha \)-power-law-bounded graphs are \( b \)-\( r \)-\( \mu \)-partitionable. We fix \( \mu = 5 \) to have one free variable less.

**Lemma 9.2.** Let \( (\mathcal{G}_n)_{n \in \mathbb{N}} \) be an \( \alpha \)-power-law-bounded random graph model. There exist constants \( c, r_0 \) such that for every \( r \geq r_0 \), \( (\mathcal{G}_n)_{n \in \mathbb{N}} \) is a.a.s. \( d_\alpha(n)^{n/2} \)-\( r \)-\( \mu \)-partitionable.

**Proof.** Assume \( (\mathcal{G}_n)_{n \in \mathbb{N}} \) is \( \alpha \)-power-law-bounded. By Theorem 5.10, the probability that \( (\mathcal{G}_n)_{n \in \mathbb{N}} \) is not \( b \)-\( r \)-\( \mu \)-partitionable is bounded by at most \( \hat{d}_\alpha(n)^{O(\mu^2 r^2)} b^{-\mu^2 / 10} \). Let \( \mu = 5 \), \( b = \hat{d}_\alpha(n)^{n/2} \). We bound the probability of not being \( d_\alpha(n)^{n/2} \)-\( r \)-\( \mu \)-partitionable by at most \( \hat{d}_\alpha(n)^{O(\mu^2)} \hat{d}_\alpha(n)^{-\mu^2 / 2} \). We set \( c \) large enough such that the probability converges to zero with \( n \).

Substituting the definition of a \( b \)-\( r \)-\( \mu \)-partition into Lemma 9.2 yields the following self-sufficient theorem.

**Theorem 9.3.** Let \( (\mathcal{G}_n)_{n \in \mathbb{N}} \) be an \( \alpha \)-power-law-bounded random graph model. There exist constants \( c, r_0 \) such that for every \( r \geq r_0 \) a.a.s. one can partition \( V(\mathcal{G}_n) \) into three (possibly empty) sets \( A, B, C \) with the following properties.

- \( |A|, |B| \leq \hat{d}_\alpha(n)^{n/2} \).
- Every \( r \)-neighborhood in \( \mathcal{G}_n[A \cup C] \) has at most 25 more edges than vertices.
- Every \( r \)-neighborhood in \( \mathcal{G}_n[C] \) has at most 5 edges to \( A \).

**Proof.** Direct consequence of Lemma 9.2.

Therefore, one can remove a few vertices to make the graph extremely sparse, as observed by the following corollary. This corollary might not have algorithmic consequences by itself, but sheds a lot of light on the structure of such graphs.

**Corollary 9.4.** Let \( (\mathcal{G}_n)_{n \in \mathbb{N}} \) be an \( \alpha \)-power-law-bounded random graph model. There exist constants \( c, r_0 \) such that for every \( r \geq r_0 \) a.a.s. one can remove \( \hat{d}_\alpha(n)^{n/2} \) vertices from \( \mathcal{G}_n \) such that every \( r \)-neighborhood has treewidth at most 26.

In Section 5 we analyze the local structure of \( \alpha \)-power-law-bounded graphs. We observe that local regions consist of a core part, bounded in size by the second order average degree, to which trees and graphs of constant size are attached. We obtain a self-contained theorem.

**Theorem 9.5.** Let \( (\mathcal{G}_n)_{n \in \mathbb{N}} \) be an \( \alpha \)-power-law-bounded random graph model. There exist constants \( c, r_0 \) such that for every \( r \geq r_0 \) a.a.s. for every \( r \)-neighborhood \( H \) of \( \mathcal{G}_n \) one can partition \( V(H) \) into three (possibly empty) sets \( X, Y, Z \) with the following properties.

- \( |X| \leq \hat{d}_\alpha(n)^{n/2} \).
- Every connected component of \( H[Y] \) has size at most \( cr \) and at most \( c \) neighbors in \( X \).
- Every connected component of \( H[Z] \) is a tree with at most one edge to \( H[X \cup Y] \).
Proof. By Lemma 9.2 \( G_n \) is a.a.s. \( \hat{d}_n(n)^{c'i^2-r}t_5 \)-partitionable for some constant \( c' \). Thus, by Theorem (0.13) every \( r \)-neighborhood of \( G_n \) is \( O(r^3\hat{d}_n(n)^{c'i^2-r}t_5) \)-r-\( O(5) \)-partitionable. We refer to Definition (6.1) and choose \( c \) large enough such that this statement holds. \( \square \)

Using Theorem 9.5 we can make statements about the structural sparsity of a random graph model. Note that locally bounded treewidth implies nowhere density [61]. The first corollary is based on the fact that \( X \) has a.a.s. constant size if \( \alpha > 3 \).

**Corollary 9.6.** Let \((G_n)_{n \in \mathbb{N}}\) be an \( \alpha \)-power-law-bounded random graph model with \( \alpha > 3 \). Then \((G_n)_{n \in \mathbb{N}}\) has a.a.s. locally bounded treewidth.

**Corollary 9.7.** Let \((G_n)_{n \in \mathbb{N}}\) be an \( \alpha \)-power-law-bounded random graph model. There exist constants \( c, r_0 \) such that for every \( r \geq r_0 \) a.a.s. the size of the largest \( r \)-subdivided clique in \( G_n \) is at most \( d_n(n)^{c'i^2} \).

## 10 Implications for Various Graph Models

A wide range of unclustered random graph models are \( \alpha \)-power-law-bounded. In this section, we show that certain Erdős–Rényi graphs, preferential attachment graphs, configuration graphs and Chung–Lu graphs are \( \alpha \)-power-law-bounded and discuss what implications this has for the tractability of the model-checking problem on these graph models. We also discuss the connections to clustered random graph models, which currently do not fit into our framework. For convenience, we restate the definition of \( \alpha \)-power-law-boundedness.

**Definition 2.1.** Let \( \alpha > 2 \). We say a random graph model \((G_n)_{n \in \mathbb{N}}\) is \( \alpha \)-power-law-bounded if for every \( n \in \mathbb{N} \) there exists an ordering \( v_1, \ldots, v_n \) of \( V(G_n) \) such that for all \( E \subseteq \binom{\{v_1, \ldots, v_n\}}{2} \)

\[
\Pr[E \subseteq E(G_n)] \leq \prod_{v_i v_j \in E} \frac{(n/i)^{1/(\alpha-1)}(n/j)^{1/(\alpha-1)}}{n} \cdot \begin{cases} 2^{O(|E|^2)} & \text{if } \alpha > 3 \\ \log(n)O(|E|^2) & \text{if } \alpha = 3 \\ O(n^\epsilon|E|^2) \text{ for every } \epsilon > 0 & \text{if } \alpha < 3. \end{cases}
\]

### 10.1 Preferential Attachment Model

The maybe best-known model proposed to mimic the features observed in complex networks are preferential attachment graphs introduced by Barabási and Albert [3, 63]. They have been studied in great detail (see for example [71]). These random graphs are created by a process that iteratively adds new vertices and randomly connects them to already existing ones, where the attachment probability is proportional to the current degree of a vertex. The model depends on a constant \( m \) which is the number of edges that are inserted per vertex. The random graph with \( n \) vertices and parameter \( m \) is denoted by \( G_n^m \).

The preferential attachment process exhibits small world behavior [24] and has been widely recognized as a reasonable explanation of the heavy tailed degree distribution of complex networks [8].

Recent efficient model-checking algorithms on random graph models only worked on random graph models that asymptotically almost surely (a.a.s.) are nowhere dense [13, 22]. It is known that preferential attachment graphs are not a.a.s. nowhere dense [22] and even a.a.s. somewhere dense [20], thus previous techniques do not work.

Nevertheless, we are able to solve the model-checking problem efficiently on these graphs. Usually, the parameter \( m \) of the model is considered to be constant. We obtain efficient algorithms even if we allow \( m \) to be a function of the size of the network. For a function \( m(n) : \mathbb{N} \to \mathbb{N} \) we define \((G_n^m)_{n \in \mathbb{N}}\) be the corresponding preferential attachment model. The following lemma follows directly from [28].
Lemma 10.1 ([28], Lemma 10). Let \( m : \mathbb{N} \to \mathbb{N} \). The preferential attachment model \((G_n^{m(n)})_{n \in \mathbb{N}}\) is

- 3-power-law-bounded if \( m(n) = \log(n)^{O(1)} \),
- \( \alpha \)-power-law-bounded for every \( 2 < \alpha < 3 \) if \( m(n) = O(n^\varepsilon) \) for every \( \varepsilon > 0 \).

According to Lemma 10.1 and Theorem 8.6 one can therefore solve the model-checking problem efficiently on preferential attachment graphs.

Corollary 10.2. Let \( m : \mathbb{N} \to \mathbb{N} \). There exists a function \( f \) such that one can solve \( p\text{-MC}(\text{FO}, \mathcal{G}_n) \) on the preferential attachment model \((G_n^{m(n)})_{n \in \mathbb{N}}\) in expected time

- \( \log(n)^{f(|\varphi|)}n \) if \( m(n) = \log(n)^{O(1)} \),
- \( f(|\varphi|, \varepsilon)n^{1+\varepsilon} \) for every \( \varepsilon > 0 \) if \( m(n) = O(n^\varepsilon) \) for every \( \varepsilon > 0 \).

10.2 Chung–Lu Model

The Chung–Lu model has been proposed to generate random graphs that fit a certain degree sequence and has been studied extensively [12, 13, 14]. We completely characterize the tractability of the model-checking problem on Chung–Lu graphs based on the power-law exponent \( \alpha \) (Corollary 10.4). Previous tractability results were obtained for a non-standard variant of the model and did not cover the case \( \alpha = 3 \).

Let \( W = (w_1, \ldots, w_n) \) be a sequence of positive weights with \( \max_{i=1}^n w_i^2 \leq \sum_{k=1}^n w_k \). The Chung–Lu random graph to \( W \) is a random graph \( G_n \) with vertices \( v_1, \ldots, v_n \) such that each edge \( v_i v_j \) with \( 1 \leq i, j \leq n \) occurs in \( G_n \) independently with probability \( w_i w_j / \sum_{k=1}^n w_k \).

Often, the weights are chosen according to a power-law distribution. Let \( \alpha > 2 \). We say \((G_n)_{n \in \mathbb{N}}\) is the Chung–Lu random graph model with exponent \( \alpha \) if for every \( n \in \mathbb{N} \), \( G_n \) is the Chung–Lu random graph to \( W_n = \{w_1, \ldots, w_n\} \) with \( w_i = c \cdot (n/i)^{1/(\alpha-1)} \) where \( c \) is a constant depending on \( \alpha \) [12]. This model nicely matches our concept of \( \alpha \)-power-law-boundedness.

Lemma 10.3. Let \( \alpha > 2 \). The Chung–Lu random graph model with exponent \( \alpha \) is \( \alpha \)-power-law-bounded.

Proof. One can easily verify that \( \sum_{k=1}^n w_k = \Theta(n) \) for all \( \alpha > 2 \). Thus, the probability of an edge \( v_i v_j \) in a Chung–Lu graph with exponent \( \alpha > 2 \) of size \( n \) is

\[
\frac{w_i w_j / \sum_{k=1}^n w_k }{n} = \frac{(n/i)^{1/(\alpha-1)}(n/j)^{1/(\alpha-1)}}{\Theta(n)}.
\]

All edges are independent of each other, therefore the probability that an edge set \( E \) is contained is the product of the probabilities of the individual edges. This yields

\[
\Pr[E \subseteq E(G_n)] \leq 2^{O(|E|)} \prod_{v_i v_j \in E} \frac{(n/i)^{1/(\alpha-1)}(n/j)^{1/(\alpha-1)}}{n}.
\]

We can combine Lemma 10.3, Theorem 8.6 and [27] to characterize the tractability of the labeled model-checking problem on Chung–Lu graphs.

Corollary 10.4. Let \( \mathcal{G} \) be the Chung–Lu random graph model with exponent \( \alpha \). There exists a function \( f \) such that one can solve \( p\text{-MC}(\text{FO}, \mathcal{G}_n) \) on \( \mathcal{G} \) in expected time

- \( f(|\varphi|)n \) if \( \alpha > 3 \),
- \( \log(n)^{f(|\varphi|)}n \) if \( \alpha = 3 \).
Furthermore, if $2.5 \leq \alpha < 3$, $\alpha \in \mathbb{Q}$ then one cannot solve p-MC(FO, $\mathcal{G}_{lb}$) on $\mathcal{G}$ in expected FPT time unless $\text{AW}[s] \subseteq \text{FPT/poly}$.

Previously, the model-checking problem has been known to be tractable on Chung–Lu graphs with exponent $\alpha > 3$, and hard on Chung–Lu graphs with exponent $2.5 \leq \alpha < 3$. The important case $\alpha = 3$ was open. Furthermore, the previous tractability result assumes the maximum expected degree of a Chung–Lu graph with exponent $\alpha$ to be at most $O(n^{1/\alpha})$, while in the canonical definition of Chung–Lu graphs (stated above) it is $\Theta(n^{1/(\alpha-1)})$. Our results hold for the canonical definition. The missing case $\alpha < 2.5$ is still open. We believe it can be proven to be hard with similar techniques as for $2.5 \leq \alpha < 3$.

The second order average degree $d$ of a Chung–Lu graph with weights $w_1, \ldots, w_n$ is defined as $\frac{\sum_{i=1}^{n} w_i^2}{\sum_{k=1}^{n} w_k}$. After substituting the maximum degree $m = \Omega(n^{1/(\alpha-1)})$ in [12] one can see for the Chung–Lu graph with exponent $\alpha$ that

$$d = \begin{cases} \Omega(1) & \alpha > 3 \\ \Omega(\log(n)) & \alpha = 3 \\ \Omega(n^{(3-\alpha)/(\alpha-1)}) & \alpha < 3. \end{cases}$$

We can further bound the run time of the model-checking problem in terms of $d$.

**Lemma 10.5.** There exist a function $f$ such that one can solve p-MC(FO, $\mathcal{G}_{lb}$) on Chung–Lu graphs with exponent $\alpha$ in expected time $(c_\alpha d)^{f(|\varphi|)} n$, where $d$ is the second order average degree and $c_\alpha$ is a constant depending on $\alpha$.

**Proof.** According to Theorem 8.5 one can solve p-MC(FO, $\mathcal{G}_{lb}$) on the Chung–Lu graph with exponent $\alpha$ in expected time

$$\begin{align*} \mu_\alpha |\varphi| n & \quad \alpha > 3 \\ \log(n) \mu_\alpha |\varphi| n & \quad \alpha = 3 \\ \mu_\alpha n^{(3-\alpha)/(\alpha-1)} |\varphi| n & \quad \alpha < 3. \end{align*}$$

where $f$ is some function and $\mu_\alpha$ is a constant depending on $\alpha$. On the other hand, we have

$$d = \begin{cases} \lambda_\alpha & \alpha > 3 \\ \lambda_\alpha \log(n) & \alpha = 3 \\ \lambda_\alpha n^{(3-\alpha)/(\alpha-1)} & \alpha < 3 \end{cases}$$

for another constant $\lambda_\alpha$ depending on $\alpha$. Thus, one can solve p-MC(FO, $\mathcal{G}_{lb}$) in expected time

$$((\mu_\alpha / \lambda_\alpha d)^{\max(2, \mu_3) f(|\varphi|)} n).$$

The result follows by setting $c_\alpha = \mu_\alpha / \lambda_\alpha$ and $f(|\varphi|) = \max(2, \mu_3) f(|\varphi|)$. \qed

### 10.3 Configuration Model

The configuration model has been proposed to generate random multigraphs whose degrees are fixed [50, 53, 4]. We solve the model-checking problem on configuration graphs with a power-law exponent 3 (Corollary 10.7). Previously, this was only known for those configuration graphs with an exponent strictly larger than 3 [22].

Let $W = (w_1, \ldots, w_n)$ be a degree sequence of a multigraph (i.e., a sequence of positive integers whose sum is even). The configuration model constructs a random multigraph with $n$ vertices whose degree sequence is exactly $W$ as follows [53]: Let $v_1, \ldots, v_n$ be the vertices of the graph. We form a set $L$ of $w_i$ many distinct copies of $v_i$ for $1 \leq i \leq n$. We call the copies of a node $v_i$ in $L$ the stubs of $v_i$. We then construct a random perfect matching on $L$. This describes a multigraph on $v_1, \ldots, v_n$ where the number of edges between two vertices
equals the number of edges between their stubs. The degree sequence of this multigraph is exactly $W$. Since we only consider simple graphs in this work, we turn to the so called erased model. Here self-loops are removed and multi-edges are replaced with single edges. As self-loops can be expressed by labels, this is no real limitation for the model-checking problem. Let $G_n$ be the probability distribution over simple graphs with $n$ vertices defined by this process. We say $G_n$ is the random configuration graph corresponding to $W$.

This defines a random graph with a fixed number of vertices. In order to define a random graph model we need to define configuration graphs of arbitrary size. Let $(w_i(n))_{i \in \mathbb{N}}$ be a sequence of functions such that all $n \in \mathbb{N}$, $(w_1(n), \ldots, w_n(n))$ is a degree sequence of a multigraph. For $n \in \mathbb{N}$ let $G_n$ be the random configuration graph corresponding to the degree sequence $(w_1(n), \ldots, w_n(n))$. We then say $(G_n)_{n \in \mathbb{N}}$ is the random configuration graph model corresponding to $(w_i(n))_{i \in \mathbb{N}}$. For technical reasons, our definition differs slightly from the original one by Molloy and Reed [59].

**Lemma 10.6.** Let $(G_n)_{n \in \mathbb{N}}$ be a random configuration graph model with corresponding sequence $(w_i(n))_{i \in \mathbb{N}}$. Assume there exists a function $p(n)$ with $p(n) = O(n^\varepsilon)$ for all $\varepsilon > 0$ such that for all $i, n \in \mathbb{N}$, $w_i(n) \leq p(n) \sqrt{n/i}$ and $\sum_{k=1}^{n} w_k(n) \geq n/p(n)$. Then $(G_n)_{n \in \mathbb{N}}$ is $3$-power-law-bounded.

**Proof.** We consider the configuration model with weight sequence $(w_1(n), \ldots, w_n(n))$ and vertices $v_1, \ldots, v_n$. Let $E \subseteq \left(\{v_1, \ldots, v_n\}\right)^2$. By Definition 2.1 it suffices to show that for every $\varepsilon > 0$

$$\Pr[E \subseteq E(G_n)] \leq O(n^{\varepsilon}) |E|^2 \prod_{v_i,v_j \in E} \frac{1}{\sqrt{|v_j|}}.$$ 

We can assume $|E| \leq n^{1/4}$, since for $|E| > n^{1/4}$ and every $\varepsilon > 0$ trivially holds

$$\Pr[E \subseteq E(G_n)] \leq 1 = O(n^{\varepsilon}) |E|^2 \prod_{v_i,v_j \in E} \frac{1}{|v_j|}.$$ 

As described in [71 Lemma 7.6], the perfect matching of the stubs in the configuration model can also be generated by a so-called adaptive pairing scheme, where unmatched stubs are taken one-by-one and matched uniformly to the remaining unmatched stubs. Assume at most $l$ stubs have been matched already in such a scheme. We fix $i, j \in \mathbb{N}$ with $i, j \leq n$ and $i \neq j$. The probability that a fixed stub of $v_i$ is matched with some stub of $v_j$ is at most $w_j(n)/(\sum_{k=1}^{n} w_k(n)) - 1 - l$. By applying the union bound to a pairing scheme which matches the $w_i(n)$ many stubs of $v_i$ we obtain

$$\Pr[v_i,v_j \in E(G_n)] \leq \frac{w_i(n)w_j(n)}{(\sum_{k=1}^{n} w_k(n)) - 1 - l}.$$ 

Let $d$ be the maximum of $w_1(n), \ldots, w_n(n)$. We consider an adaptive pairing scheme which iteratively matches the stubs of the vertices in $E$ and obtain

$$\Pr[E \subseteq E(G_n)] \leq \prod_{v_i,v_j \in E} \frac{w_i(n)w_j(n)}{(\sum_{k=1}^{n} w_k(n)) - 1 - 2|E|d}.$$ 

Since $d \leq p(n)\sqrt{n}$, $|E| \leq n^{1/4}$ and $\sum_{k=1}^{n} w_k(n) \geq n/p(n)$ we can further bound

$$\frac{w_i(n)w_j(n)}{(\sum_{k=1}^{n} w_k(n)) - 1 - 2|E|d} = O(p(n)) \frac{w_i(n)w_j(n)}{n} = O(p(n)^3) \frac{1}{\sqrt{|v_j|}}.$$ 

The final result follows from the fact that $p(n) = O(n^\varepsilon)$ for all $\varepsilon > 0$. \hfill \Box

Now the previous lemma together with Theorem 5.6 yields an efficient model-checking algorithm for configuration graphs.
Corollary 10.7. Let $(G_n)_{n \in \mathbb{N}}$ be a random configuration graph model with corresponding sequence $(w_i(n))_{i \in \mathbb{N}}$. Assume there exists a function $p(n)$ with $p(n) = O(n^\varepsilon)$ for all $\varepsilon > 0$ such that for all $i, n \in \mathbb{N}$, $w_i(n) \leq p(n) \sqrt{n/i}$ and $\sum_{k=1}^n w_k(n) \geq n/p(n)$.

Then there exists a function $f$ such that one can decide $p$-MC($\Phi_{lb}$) on $(G_n)_{n \in \mathbb{N}}$ in expected time $f((\varphi),\varepsilon)n^{1+\varepsilon}$ for every $\varepsilon > 0$.

10.4 Erdős–Rényi Model

One of the earliest and most intensively studied random graphs is the Erdős–Rényi model [9, 30]. We say $G(n,p(n))$ is a random graph with $n$ vertices where each pair of vertices is connected independently uniformly at random with probability $p(n)$. Many properties of Erdős–Rényi graphs are well studied, including but not limited to, threshold phenomena, the sizes of components, diameter, and length of paths [9]. We classify sparse Erdős–Rényi graphs with respect to $\alpha$-power-law-boundedness.

Lemma 10.8. Erdős–Rényi graphs $G(n,p(n))$ are

- $\alpha$-power-law-bounded for every $2 < \alpha$ if $p(n) = O(1/n)$,
- $3$-power-bounded if $p(n) = \log(n)^{O(1)}/n$,
- $\alpha$-power-law-bounded for every $2 < \alpha < 3$ if $p(n) = O(n^\varepsilon/n)$ for every $\varepsilon > 0$.

Proof. The probability of a set of edges $E$ to exist in $G(n,p(n))$ is

$$\Pr[E \subseteq E(G(n,p(n))] = p(n)^{|E|} \leq (np(n))^{|E|} \prod_{v_i,v_j \in E} \frac{(n/i)^{(1-\varepsilon)(n/j)^{(1-\varepsilon)}}}{n},$$

since $(n/i)^{(1-\varepsilon)} \geq 1$ for all $1 \leq i \leq n$. The rest follows from Definition 2.1.

Using the previous Lemma 10.8 and Theorem 5.6 we obtain a fine grained picture over the tractability of the model-checking problem on sparse Erdős–Rényi graphs.

Corollary 10.9. There exists a function $f$ such that one can solve $p$-MC($\Phi_{lb}$) on $G(n,p(n))$ in expected time

- $f((\varphi))n$ if $p(n) = O(1/n)$,
- $\log(n)^{f((\varphi))}n$ if $p(n) = \log(n)^{O(1)}/n$,
- $f((\varphi),\varepsilon)n^{1+\varepsilon}$ for every $\varepsilon > 0$ if $p(n) = O(n^\varepsilon/n)$ for every $\varepsilon > 0$.

The third case has been shown previously by Grohe [43]. Furthermore, under reasonable assumptions ($AW[*] \not\subseteq$ FPT/poly) we know that $p$-MC($\Phi_{lb}$) cannot be decided in expected FPT time on denser Erdős–Rényi graphs with $p(n) = n^\delta/n$ for some $0 < \delta < 1$, $\delta \in Q$ [27].

10.5 Clustered Models

$\alpha$-power-law-bounded random graphs tend to capture unclustered random graphs. One can show that for the algorithmically tractable values of $\alpha$ close to or larger than three the expected number of triangles is subpolynomial (via union bound over all embeddings as in Lemma 5.8). Random models with non-vanishing clustering coefficient, such as the Kleinberg model [49, 50], the hyperbolic random graph model [52, 11], or the random intersection graph model [16, 65] generally have a high expected number of triangles. This means these models are not $\alpha$-power-law-bounded for interesting values of $\alpha$ close to three (they may be for smaller $\alpha$). We shall prove a stronger statement for the random intersection graph model which is defined as follows.
**Definition 10.10** (Random Intersection Graph Model, [32]). Fix a positive constant \( \delta \). Let \( E \) be a random bipartite graph on parts of sizes \( n \) and \([n^\delta]\) with each edge present independently with probability \( n^{-(1+\delta)/2} \). Let \( V \) (the vertices) denote the part of size \( n \) and \( A \) (the attributes) the part of size \([n^\delta]\). The associated random intersection graph \( G(n, \delta) \) is defined on the vertices \( V \): two vertices are connected in \( G \) if they share (are in \( B \) both adjacent to) at least one attribute in \( A \).

It has been shown that \((G(n, \delta))_{n \in \mathbb{N}}\) has a.a.s. bounded expansion [32] if and only if \( \delta > 1 \). Furthermore, if \( \delta > 1 \), then one can solve \( p\text{-MC(F O, } \mathcal{G}_\theta) \) in expected time \( f(|\varphi|)n \) [32]. We now argue that intersection graphs nevertheless do not fit into our framework of \( \alpha \)-power-law-boundedness.

**Lemma 10.11.** \((G(n, \delta))_{n \in \mathbb{N}}\) is not \( \alpha \)-power-law-bounded for all values of \( \delta \) and \( \alpha \).

**Proof.** Assume \( \delta \) and \( \alpha \) such that \((G(n, \delta))_{n \in \mathbb{N}}\) is \( \alpha \)-power-law-bounded. For a fixed \( n \) let the vertices of \( G(n, \delta) \) be \( v_1, \ldots, v_n \), ordered as in Definition 2.1.

If a fixed set of \( k \) vertices shares a common attribute then these vertices form a clique. The probability that this happens is at least \( n^{-(1+\delta)/2} \geq n^{-ck} \) for some constant \( c \). Let \( E = \{(v_{n-k}, \ldots, v_n)\} \) be the set of all edges between the last \( k \) vertices in the ordering. By the previous argument,

\[
\Pr[E \subseteq E(G(n, \delta))] \geq n^{-ck}.
\]

By Definition 2.1 there exists a term \( p(n) \) with \( p(n) = O(n^\varepsilon |E|^2) \) for every \( \varepsilon > 0 \) such that

\[
\Pr[E \subseteq E(G(n, \delta))] \leq p(n) \prod_{v_i, v_j \in E} \frac{(n/i)^{1/(\alpha-1)}(n/j)^{1/(\alpha-1)}}{n}.
\]

By the definition of \( p(n) \), there exists a monotone function \( f \) such that \( p(n) \leq f(|E|) |E|^2 \) for every \( \varepsilon > 0 \). By setting \( \varepsilon = 1/|E|^2 \), we obtain \( p(n) \leq f(|E|^2) |E|^2 n \). We consider only edges between the last \( k \) vertices, and for \( n \geq 2k \) holds \((n/(n-k))^{1/(\alpha-1)}(n/(n-k))^{1/(\alpha-1)} \leq 4 \). By assuming \( n \geq 2k \) we obtain

\[
\Pr[E \subseteq E(G(n, \delta))] \leq f(|E|^2) |E|^2 n \prod_{v_i, v_j \in E} \frac{4}{n} \leq 4k^2 f(k^4 k^4 n^{-\binom{k}{2} + 1}).
\]

Together, this yields

\[
n^{-ck} \leq \Pr[E \subseteq E(G(n, \delta))] \leq 4k^2 f(k^4 k^4 n^{-\binom{k}{2} + 1}).
\]

We choose \( k \) large enough such that \( ck < \binom{k}{2} - 1 \). Then the previous bound yields a contradiction for sufficiently large \( n \).

\( \square \)

**11 Conclusion**

We define \( \alpha \)-power-law-bounded random graphs which generalize many unclustered random graphs models. We provide a structural decomposition of neighborhoods of these graphs and use it to obtain a meta-algorithm for deciding first-order properties in the the preferential attachment-, Erdős–Rényi-, Chung–Lu- and configuration random graph model.

There are various factors to consider when evaluating the practical implications of this result. The degree distribution of most real world networks is similar to a power-law distribution with exponent between two and three [15], but our algorithm is only fast for exponents at least three. This leaves many real world networks where our algorithm is slow. However, it has been shown that the model-checking problem (with labels) becomes hard on these graphs if we assume independently distributed edges [27].
So far, we do not know whether the model-checking problem is hard or tractable on clustered random graphs. If a random graph model is 3-power-law-bounded then one can show that the expected number of triangles is polylogarithmic (via union bound of all possible embeddings of a triangle). Therefore, random models with clustering, such as the Kleinberg model [49], the hyperbolic random graph model [52, 11], or the random intersection graph model [46], which have a high number of triangles currently do not fit into our framework (see Section [10.5] for a proof that random intersection graphs are not \( \alpha \)-power-law-bounded for any \( \alpha \)). This is unfortunate, since clustering is a key aspect of real networks [72]. In the future, we hope to extend our results to clustered random graph models. We observe that some clustered random graph models can be expressed as first-order transductions of \( \alpha \)-power-law-bounded random graph models. For example the random intersection graph model is a transduction of a sparse Erdős–Rényi graph. We believe this connection can be used to transfer tractability results to clustered random graphs. If we can efficiently compute for a clustered random graph model \( G \) a pre-image of a transduction that is distributed like an \( \alpha \)-power-law-bounded random graph then we can efficiently solve \( p\text{-MC}(\varphi, G_{lb}) \) on \( G \). The same idea is currently being considered for solving the model checking problem for transductions of sparse graph classes (e.g. structurally bounded expansion classes) [39].

In our algorithm, we use Gaifman’s locality theorem to reduce our problem to \( r \)-neighborhoods of the input graph. In this construction the value of \( r \) can be exponential in the length of the formula [38]. On the other hand, the small world property states that the radius of real networks is rather small. This means, even for short formulas our neighborhood-based approach may practically be working on the whole graph instead of neighborhoods. It would be interesting to analyze for which values of \( r \) practical protrusion decompositions according to Theorem [9.5] exist in the real world.

At last, a big problem with all parameterized model-checking algorithms is their large run time dependence on the length of the formula. Grohe and Frick showed that already on trees every first-order model-checking algorithm takes worst-case time at least \( f(|\varphi|)n \) where \( f \) is a non-elementary tower function [37]. So far, it is unclear whether this also holds in the average-case setting. The results presented in this paper have a non-elementary dependence on the length of the formula. We are curious whether one can find average-case model-checking algorithms with elementary expected FPT run time. In summary, many more obstacles need to be overcome to obtain a truly practical general purpose meta-algorithm for complex networks.

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