I describe a method of inferring the past of quantum observables given the initial state and the subsequent measurement results using Wigner quasi-probability representations. The method is proved to be compatible with logic for large subclasses of quantum systems, including those that involve incompatible observables and can still exhibit some quantum features, such as the uncertainty relation, measurement backaction, and entanglement.

II. LOGICAL INFERENCE

A. Classical

Consider first the classical inference problem. Cox’s theorem [15, 16] states that the only consistent method of assigning plausibilities to propositions is equivalent to the laws of probability, and the Bayes theorem in particular, if one assumes a set of desiderata that can be regarded as an extension of Aristotelian logic and summarized as follows [16]:

1. Representation of degrees of plausibility by real numbers.
2. Qualitative correspondence with common sense.
3. Consistency.

In the rest of the paper I shall take the view that Cox’s desiderata and logic are equivalent notions. Readers who wish to challenge this definition of logic are urged to read Refs. [15, 16], and until they have come up with a better definition, Cox’s theorem remains the most rigorous logical foundation available for statistical inference.

To be specific, consider the inference problem depicted in Fig. 1. Let \( x \) be a parameter that represents facts known at an initial time, \( \lambda \) be another parameter that represents facts known at a final time, and \( \lambda \) be the hidden variable whose value at the intermediate time is to be inferred. Suppose that the predictive probability function \( P_1(\lambda | x) \) and the retrodictive likelihood function \( P_2(y | \lambda, x) \) are given. This is a reasonable assumption, as causality implies that \( \lambda \) should depend on \( x \), \( y \) should depend on both \( \lambda \) and \( x \), and noise should introduce uncertainties to the dependencies. The posterior probability function of \( \lambda \) conditioned on \( x \) and \( y \) is

\[
P(\lambda | x, y) = \frac{P_2(y | \lambda, x)P_1(\lambda | x)}{\sum_{\lambda} P_2(y | \lambda, x)P_1(\lambda | x)},
\]

which is the logical assignment of plausibilities to values of \( \lambda \) given known facts \( x \) and \( y \). For example, the most
likely \( \lambda \) can be determined from \( P(\lambda|x, y) \) by finding the \( \lambda \) that maximizes the posterior and is called the maximum \textit{a posteriori} (MAP) estimate. In accordance with the terminology in estimation theory [13, 14, 18], I refer to the inference of a hidden variable at the intermediate time as smoothing.

**B. Quantum**

I now ask how one can logically infer the past of a quantum system. Suppose that the initial quantum state given prior facts represented by \( x \) is \( \rho_x \), and the positive operator-valued measure (POVM) for the measurement at the final time with outcome \( y \) is \( E(y|x) \), which, for generality, can be adaptive and depend on \( x \). Born’s rule gives

\[
P(y|x) = \operatorname{tr} E(y|x) \rho_x. \tag{2.2}
\]

To infer logically the value of quantum observables, denoted by \( \lambda \), at the intermediate time is to assign a non-negative posterior probability function to their possible values through the Bayes theorem. Define a pair of maps that transform \( \rho_x \) and \( E(y|x) \) to quasi-probability functions of \( \lambda \):

\[
W_1 \rho_x = W_1(\lambda|x), \tag{2.3}
\]
\[
W_2 E(y|x) = W_2(y|\lambda, x), \tag{2.4}
\]

and require that they obey Born’s rule in the following way:

\[
P(y|x) = \operatorname{tr} E(y|x) \rho_x = \sum_\lambda W_2(y|\lambda, x) W_1(\lambda|x). \tag{2.5}
\]

If \( W_2(y|\lambda, x) \) and \( W_1(\lambda|x) \) are both non-negative, the posterior function given by

\[
W(\lambda|x, y) = \frac{W_2(y|\lambda, x) W_1(\lambda|x)}{P(y|x)} \tag{2.6}
\]

is also non-negative and compatible with the laws of probability and Cox’s desiderata. Hence, the logicality of quantum smoothing and the non-negativity of quasi-probability representations are equivalent notions.

If \( W_1(\lambda|x) \) and \( W_2(y|\lambda, x) \) are restricted to be non-negative, it is known that no single pair of maps \( W_1 \) and \( W_2 \) can explain all predictions of quantum mechanics [19, 20]. This is known as contextuality; without this property, quantum mechanics would be equivalent to a classical hidden variable model. The key point here is that many large and important subclasses of quantum systems, such as the odd-dimensional stabilizer quantum computation model [21, 22] and the linear Gaussian model for canonical observables [23–25], turn out to be perfectly described by non-negative \( W_1(\lambda|x) \) and \( W_2(y|\lambda, x) \) if they are chosen to be the appropriate discrete or continuous Wigner representations of \( \rho_x \) and \( E(y|x) \). These models generally involve incompressible observables and can still exhibit some quantum features, such as the uncertainty relation, measurement backaction, and entanglement.

To prove the logicality of quantum smoothing for the aforementioned subclasses explicitly, I write Born’s rule in the following way:

\[
P(y|x) = \operatorname{tr} E(y|x, t_j) \rho_x(t_j) \tag{2.7}
\]

for any time \( t_j, j = 0, 1, \ldots, J \), where

\[
\rho_x(t_j) \equiv U_x(t_j) \cdots U_x(t_2) U_x(t_1) \rho_x(t_0),
\]

\[
E(y|x, t_j) \equiv U_x^* (t_{j+1}) \cdots U_x^* (t_{j-1}) U_x^* (t_j) E(y|x, t_j),
\]

\[
\rho_x(t_0) \text{ is the initial density operator, } E(y|x, t_j) \text{ is the final POVM, and } U_x(t_j) \text{ denotes a unitary operation with unitary operator } U_x(t_j):
\]

\[
U_x(t_j) \rho \equiv U_x(t_j) \rho U_x^* (t_j), \tag{2.10}
\]
\[
U_x^* (t_j) E \equiv U_x^* (t_j) EU_x(t_j). \tag{2.11}
\]

The unitary operations can also model open-system evolution and sequential and adaptive measurements if the Hilbert space is suitably dilated [26, 27]. If I choose \( W_1 \) to be a Wigner representation of \( \rho_x(t_j) \):

\[
W_1(\lambda, t_j|x) = W_1 \rho_x(t_j), \tag{2.12}
\]

and \( W_2 \) to be \( W_1 \) multiplied by a normalization factor \( \mathcal{N} \):

\[
W_2(y|\lambda, x, t_j) = W_2 E(y|x, t_j) = \mathcal{N} W_1 E(y|x, t_j), \tag{2.13}
\]

a fundamental property of the Wigner representation [19] can be used to give

\[
\operatorname{tr} E(y|x, t_j) \rho_x(t_j) = \sum_\lambda W_2(y|\lambda, x, t_j) W_1(\lambda, t_j|x), \tag{2.14}
\]
making the hidden-variable model consistent with Born’s rule. For continuous variables, $\sum_\lambda$ should be replaced by an integral.

For quantum systems with odd dimensions, Gross [21, 22] showed that a pure state is a stabilizer state if and only if the natural analog of the Wigner representation for odd dimensions is non-negative. Restricting $\rho_c(t_0)$ to stabilizer states and $U_c(t_j)$ to Clifford operations, which transform stabilizer states to stabilizer states, $W_1(\lambda, t_j|x)$ is non-negative at any time. If $E(y|x, t_j)$ is a stabilizer-state projection, $W_2(y|\lambda, x, t_j)$ is non-negative as well, since $U^*_c$ is also a Clifford operation if $U_c$ is one. Hence, the odd-dimensional stabilizer model, which consists of stabilizer states, Clifford operations, and stabilizer-state projections, can always be represented by non-negative $W_1$ and $W_2$ at any time $t_j$ and permits logical smoothing inference.

Similarly, for canonical observables, such as the continuous positions and momenta of harmonic oscillators, it is well known that a state with a Gaussian Wigner representation remains Gaussian if the Hamiltonian is quadratic with respect to the canonical observables [23–25]. If $E(y|x, t_j)$ is a projective measurement with respect to the canonical observables, and the unitary operations are restricted to those with quadratic Hamiltonians, $W_2(y|\lambda, x, t_j)$ is also Gaussian. The linear Gaussian model, which consists of Gaussian states, quadratic Hamiltonians, and canonical-observable measurements, can therefore be represented by non-negative and Gaussian $W_1$ and $W_2$ at all times and also permits logical smoothing inference.

For quantum systems that do not admit non-negative quasi-probability representations, the smoothing inference necessarily violates Cox’s desiderata and can lead to paradoxes. One example is shown in Ref. [14] for Hardy’s paradox [28] using a discrete Wigner representation. It must be emphasized that negative quasi-probabilities do not resolve any paradox; they simply mean that the inference according to the hidden-variable model is illogical. For illustrative and pedagogical purposes it is still useful to report the quasi-probability functions for experiments, as they are more obvious indicators of non-classicality than density matrices and POVMs. For a more sensible estimate than the weak value, one can choose

$$\lambda_{x,y}^{\text{MAP}} \equiv \arg \max_\lambda W(\lambda|x, y) \quad (2.15)$$

as the MAP quasi-estimate, which is still one of the possible values of $\lambda$ and will never become anomalously large or complex, thus avoiding some of the counter-intuitive features of the weak value. In the case of Hardy’s paradox, Ref. [14] shows that $\lambda_{x,y}^{\text{MAP}}$ indeed reproduces the most likely paths suggested by classical reasoning, even though they still lead to logical contradictions.

### C. Weak value as an estimate

In the weak-value approach, an additional weak measurement [26, 27, 29, 30] is made before the final measurement. To model that scenario, it is important not to confuse the weak measurement outcome with the hidden variable to be inferred. The weak measurement outcome can be grouped with either $x$ or $y$, and the Bayesian protocol, if it exists, will give us a posterior distribution of $\lambda$ for any number of trials. From the perspective of Cox’s theorem, any method that deviates from the Bayesian approach is illogical [16], and if we view the weak value simply as an inference method that combines data in a special way to produce an estimate of the observable, it should not surprise us that it can produce non-sense, unless it happens to agree with the Bayesian approach. No amount of heuristic reasoning can save the weak value from illogicality otherwise.

In the context of the quasi-probability framework, one can ask whether the weak value corresponds to an estimate arising from a posterior quasi-probability function. The answer was provided by Johansen and Luis [31], who showed that the weak value is equivalent to the conditional average using what they called the $S$ distributions, which can be negative or even complex. It is currently unknown under what situation the $S$ distributions are non-negative, so the logical foundation of the weak value remains questionable.

### D. Logical inference versus decision theory

Besides the logical interpretation, one can also justify Bayesian inference in a more utilitarian manner using decision theory [16, 32], which shows that Bayesian inference can minimize the expected error between an estimate and the true value of a hidden variable. In this paper, I do not attach any decision-theoretic significance to the quantum inference, as the past of a quantum observable is usually not available for error evaluation because of the no-cloning theorem. Any apparent illogicality that arises from negative quasi-probabilities exists only in the mind and can be attributed to the wrong model being used for a quantum system; the experimenter only has access to prior facts and measurement outcomes, which obey Born’s rule and hence the laws of probability and logic.

This kind of mental exercise is not entirely philosophical however. The smoothing inference provides an alternative way of thinking about when a quantum system follows classical logic internally and is therefore simulable by a classical computer. This question is central not only to quantum computation and quantum simulation [33], but also to the implementation of quantum estimation and control algorithms [13, 14, 27, 34].

The method presented here is naturally extensible to the inference of classical signals coupled to quantum systems, a problem studied extensively in Refs. [13, 14, 34].
In that case, it must be emphasized that, although the quasi-probability functions can be used as an intermediate and often convenient step, the end result is always consistent with both logic and decision theory. The hybrid smoothing method is known to be optimal and superior to conventional prediction or filtering methods for certain quantum waveform estimation problems [35–38], analogous to the classical case [18, 39], whereas classical parameter estimation based on the weak-value approach often turns out to be suboptimal [40–43].

III. DISCRETE WIGNER REPRESENTATION OF QUBITS

Odd-dimensional systems possess a natural and unique definition of the discrete Wigner function [21, 22], but many equally qualified definitions can exist for even dimensions [44]. Here I consider the ones proposed by Feynman [45] and Wootters and co-workers [44, 46, 47]. Consider first a qubit, which can describe the path of a quantum particle in a two-arm interferometer, the polarization of a photon, or the spin of an electron for example. Define the observables of interest as

\[ \hat{q} \equiv \frac{1}{2} (I - \sigma_Z), \quad \hat{p} \equiv \frac{1}{2} (I - \sigma_X), \quad \hat{r} \equiv \frac{1}{2} (I - \sigma_Y), \]

(3.1)

where \( \sigma_Z, \sigma_X, \) and \( \sigma_Y \) are Pauli matrices and \( I \) is the identity matrix. Define eigenstates of \( \hat{q}, \hat{p}, \) and \( \hat{r} \) as \( |q=0\rangle = |0\rangle, \hat{q}|1\rangle = |1\rangle, \hat{p}|+\rangle = |+\rangle, \hat{p}|-\rangle = |-\rangle, \hat{r}|i\rangle = |i\rangle, \) and \( \hat{r}|-i\rangle = |-i\rangle \). The discrete Wigner function proposed by Feynman [45] and Wootters [46] with respect to the eigenvalues of \( \hat{q} \) and \( \hat{p} \) is

\[ W_1 \rho_x = \frac{1}{2} \text{tr} A(q,p) \rho_x = W_1(q,p|x), \]

(3.2)

where

\[ A(q,p) = \frac{1}{2} \left[ (-1)^q \sigma_Z + (-1)^p \sigma_X + (-1)^{q+p} \sigma_Y + I \right]. \]

(3.3)

In the phase-space matrix format,

\[ W_1(q,p|x) = \begin{pmatrix} W_1(0,1|x) & W_1(1,1|x) \\ W_1(0,0|x) & W_1(1,0|x) \end{pmatrix}. \]

(3.4)

For example,

\[ W_1|0\rangle\langle 0| = \begin{pmatrix} 0.5 & 0 \\ 0 & 0.5 \end{pmatrix}, \quad W_1|1\rangle\langle 1| = \begin{pmatrix} 0 & 0.5 \\ 0.5 & 0 \end{pmatrix}, \]

(3.5)

\[ W_1|+\rangle\langle +| = \begin{pmatrix} 0 & 0 \\ 0.5 & 0.5 \end{pmatrix}, \quad W_1|-\rangle\langle -| = \begin{pmatrix} 0.5 & 0 \\ 0 & 0 \end{pmatrix}, \]

(3.6)

\[ W_1|i\rangle\langle i| = \begin{pmatrix} 0 & 0.5 \\ 0.5 & 0 \end{pmatrix}, \quad W_1|-i\rangle\langle -i| = \begin{pmatrix} 0.5 & 0 \\ 0 & 0.5 \end{pmatrix}. \]

(3.7)

These are plotted in Fig. 2. Note that the value of the \( \hat{r} \) observable can also be inferred by defining

\[ r = (q + p) \mod 2. \]

(3.8)

Note also how the uncertainty relations among the three spin components are observed in phase space.

FIG. 2. A discrete Wigner representation of some qubit states.

For two qubits, one of the possible definitions of the Wigner functions proposed by Gibbons et al. [44, 47] is

\[ W_2 \rho_x = \frac{1}{4} \text{tr} A(q_1,q_2,p_1,p_2) \rho_x, \]

(3.9)

\[ A(q_1,q_2,p_1,p_2) = \frac{1}{2} \left[ (-1)^{q_2} \sigma_Z + (-1)^{p_2} \sigma_X + (-1)^{q_2+p_2} \sigma_Y + I \right] \otimes \]

\[ \frac{1}{2} \left[ (-1)^{q_1} \sigma_Z + (-1)^{p_1} \sigma_X + (-1)^{q_1+p_1} \sigma_Y + I \right]. \]

(3.10)

Remarkably, the function stays non-negative for many separable states, as shown in Fig. 3, as well as some entangled states, as shown in Fig. 4. Unlike the case of odd dimensions, the correspondence between non-negative Wigner representations and the stabilizer model is not as strong for even dimensions [48, 49]. For two qubits,
For the measurement, the map on the POVM can be chosen as
\[ W_2 E(y|x) = NW_1 E(y|x), \]  
(3.11)  
with \( N \) being the dimension, such that a fundamental property of the Wigner representation [44, 46] makes the hidden-variable model agree with Born’s rule according to Eq. (2.5). Figs. 2, 3, and 4 then also depict the Wigner representations of projective measurements, normalization notwithstanding. Systems with initial states, state transitions, and measurements within this set of states with non-negative Wigner representations naturally permit logical smoothing inference.

IV. LOGICAL SMOOTHING AND COMPLEX WEAK VALUE

It is not difficult to construct examples where the weak value does not make sense, while the Wigner functions provide a logical path for smoothing inference. Consider an initial state given by
\[
\begin{align*}
\rho_{x=0} &= |0\rangle \langle 0|, \\
\rho_{x=1} &= |1\rangle \langle 1|,
\end{align*}
\]  
(4.1, 4.2)  
and a final \( \hat{r} \) measurement given by
\[
\begin{align*}
E(y = i) &= |i\rangle \langle i|, \\
E(y = -i) &= |-i\rangle \langle -i|,
\end{align*}
\]  
(4.5, 4.6)  
\[
\begin{align*}
W_1(q,p|0) &= \begin{pmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{pmatrix}, \\
W_1(q,p|1) &= \begin{pmatrix} 0 & 0.5 \\ 0.5 & 0 \end{pmatrix},
\end{align*}
\]  
(4.3, 4.4)  
\[
\begin{align*}
W_2(i|q,p) &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \\
W_2(-i|q,p) &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.
\end{align*}
\]  
(4.7, 4.8)
$W_1$ and $W_2$ are consistent with Born’s rule and remain non-negative for all $x$ and $y$, enabling logical inference of the past values of $q$, $p$, and $r = (q + p) \mod 2$. Suppose $x = 0$ and $y = i$. The smoothing quasi-probability function becomes

$$W(q, p|x = 0, y = i) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

which infers with certainty that $q = p = r = 0$. This result concerning the three incompatible observables follows from a logical inference protocol but seems to be at odds with the uncertainty principle. The apparent conflict can be resolved by realizing that, under the quasi-probability framework, the uncertainty relation holds only for the predictive part $W_1$, and a smoothing inference can violate the principle even if it is logical. The weak value of $\hat{p}$, on the other hand, is

$$\langle i|\hat{p}|0\rangle = 1 + i \frac{1}{2},$$

which is complex and makes no sense as an estimator. One could take the real part or the absolute value of the weak value to make it look more reasonable, but again there is no logical foundation that justifies such heuristic operations.

This example can also be studied using the approach of consistent histories [3]. The coherence functional is the central quantity in this approach and for the present example with $x = 0$ and $y = i$ is defined as $C(p, p')$ and given by

$$C(0, 0) = \text{tr}|i\rangle\langle i+\rangle\langle +|0\rangle\langle 0+\rangle\langle +|i\rangle = \frac{1}{4},$$

$$C(0, 1) = \text{tr}|i\rangle\langle i+\rangle\langle +|0\rangle\langle 0−\rangle\langle −|i\rangle = −\frac{i}{4},$$

$$C(1, 0) = \text{tr}|i\rangle\langle i−\rangle\langle −|0\rangle\langle 0+\rangle\langle +|i\rangle = \frac{i}{4},$$

$$C(1, 1) = \text{tr}|i\rangle\langle i−\rangle\langle −|0\rangle\langle 0−\rangle\langle −|i\rangle = \frac{1}{4}. $$

Although the off-diagonal components $C(0, 1)$ and $C(1, 0)$ are not zero, they are imaginary and still satisfy the “weak consistency condition” defined in Ref. [3]. The general relation between the consistent histories approach and the quasi-probability approach here is an interesting open problem but beyond the scope of this paper.

**V. THE AHARONOV-ALBERT-VAIDMAN GEDANKEN EXPERIMENT**

To demonstrate the limitations of the quasi-probability approach, consider the original experiment proposed by AAV [5], as depicted in Fig. 5. A spin-$1/2$ particle is known to be in a pure state $|\psi\rangle$ at time $t_1$. A weak $\hat{q}$ measurement is then performed and can be modeled by

![Flowchart](image)

The Kraus operator [26, 27]:

$$K(\delta z) = \frac{1}{(2\pi\delta t)^{1/4}} \exp\left[-\frac{(\delta z - \hat{q}\delta t)^2}{4\delta t}\right]$$

$$\approx \frac{1}{(2\pi\delta t)^{1/4}} \exp\left(-\frac{\delta z^2}{4\delta t}\right)\left(1 + \frac{\delta z}{2\hat{q}} - \frac{\delta t}{8\hat{q}^2}\right),$$

where $\delta t$ characterizes the measurement strength, assumed to be unitless and $\ll 1$. The time after the weak measurement and before the final measurement is denoted as $t_2$. The final measurement is an $p$ measurement:

$$E(\xi = +) = |+\rangle\langle +|,$$

$$E(\xi = -) = |--\rangle\langle --|,$$

such that Born’s rule is given by

$$P(\xi, \delta z|\psi) = \text{tr}|\xi\rangle\langle \xi|K(\delta z)|\psi\rangle\langle \psi|K^\dagger(\delta z).$$

It is important here not to confuse the measurement outcome $\delta z$ with the hidden observable it is measuring. The weak-value approach calculates the average of $\delta z/\delta t$ for many trials and claims that it is an estimate of $q$, but such an approach cannot be justified unless it happens to agree with a Bayesian estimate. The approach would work for an analogous classical problem because there exists a likelihood function describing the weak measurement such that averaging the outcomes over many trials is akin to evaluating $\int_{-\infty}^{\infty} d(\delta z)P_2(\delta z|q)(\delta z/\delta t) = q$, which gives the true $q$. For the quantum problem, however, it is not obvious how such a non-negative likelihood function can be defined to justify the weak value as an estimate, unless the Kraus operator $K(\delta z)$ happens to commute with all the other operators in the problem.

Let’s focus on the case $|\psi\rangle = |−\rangle$ with the final result $\xi = +$, which gives an infinite weak value [5]. The $\xi = +$ final result is possible because of the small backaction
noise introduced by the weak measurement to $p$. Consider the predictive Wigner functions $W_1(\lambda, t_1|\psi)$ and $W_1(\lambda, t_1|\psi, \delta z)$ before and after the weak measurement:

$$W_1(\lambda, t_1|-) \equiv W_1|-|-\rangle\langle-| = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix},$$

$$W_1(\lambda, t_1|-, \delta z) \equiv W_1 \frac{K(\delta z)|-\rangle\langle-|K^\dagger(\delta z)}{\text{tr}(\text{numerator})} \propto \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} + \frac{\delta z}{2} \begin{pmatrix} 0.5 & 1.5 \\ -0.5 & 0.5 \end{pmatrix} + \frac{\delta t}{8} \begin{pmatrix} 0.5 & 0.5 \\ -0.5 & -0.5 \end{pmatrix},$$

and the following retrodictive quasi-likelihood functions:

$$W_2(\lambda, \delta z|\lambda, t_1) \equiv W_2K^\dagger(\delta z)\langle+|K(\delta z) \propto \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} + \frac{\delta z}{2} \begin{pmatrix} -0.5 & 0.5 \\ 0.5 & -0.5 \end{pmatrix} + \frac{\delta t}{8} \begin{pmatrix} 0.5 & 0.5 \\ -0.5 & -0.5 \end{pmatrix},$$

$$W_2(+|\lambda, t_2) \equiv W_2|+\rangle\langle+| = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}.$$

The smoothing quasi-probability functions $W(\lambda, t_{1,2}|\psi, \delta z, \xi)$ at times $t_1$ and $t_2$ are then

$$W(\lambda, t_1|-, \delta z, +) \approx \begin{pmatrix} 1/2 - 2\delta z/\delta t & 1/2 + 2\delta z/\delta t \\ 0 & 0 \end{pmatrix},$$

$$W(\lambda, t_2|-, \delta z, +) \approx \begin{pmatrix} 0 & 0 \\ 1/2 - 2\delta z/\delta t & 1/2 + 2\delta z/\delta t \end{pmatrix}.$$

All the predictive, retrodictive, and smoothing quasi-probability functions are plotted in Fig. 6.

There are a few interesting observations to be made:

- Both $W_1(\lambda, t_2|-, \delta z)$ and $W_2(+, \delta z|\lambda, t_1)$ become negative when $\delta z \neq 0$, rendering the inference illogical. The smoothing quasi-probability functions still agree with common sense about $p$ however, as they infer that $p$ must be 1 at $t_1$ because of the initial state and 0 at $t_2$ because of the final measurement outcome.

- The marginal smoothing distributions with respect to $q$ can be written as

$$W(q, t_1|-, \delta z, +) \approx W(q, t_2|-, \delta z, +) \approx \begin{pmatrix} 1/2 - 2\delta z/\delta t & 1/2 + 2\delta z/\delta t \end{pmatrix}.$$  

Even though the negativity of Wigner functions is limited, one sees here that the smoothing quasi-probabilities that result from them can have arbitrarily negative values if $|\delta z/\delta t| > 1/4$.

- The MAP quasi-estimate of $q$ is

$$q^{\text{MAP}} = \begin{cases} 0, & \delta z < 0, \\ 1, & \delta z > 0, \\ \text{ambiguous}, & \delta z = 0. \end{cases}$$

This makes sense, as the outcome $\delta z$ of a $q$ measurement, however noisy, should constitute evidence that should persuade the observer one way or the other. Because of the negative quasi-probabilities, the quasi-estimate should not be taken seriously as a logical estimator, but it is at least more sensible than the infinite weak value or, say, the naive conditional average:

$$\tilde{q} \equiv \sum q W(q|-, \delta z, +) \approx \frac{1}{2} + \frac{2\delta z}{\delta t}.$$ 

$\tilde{q}$ exceeds the range $[0, 1]$ when a smoothing quasi-probability becomes negative, that is, when $|\delta z/\delta t| > 1/4$. Since $\delta z \in (-\infty, \infty)$ and is typically

\begin{align*}
\begin{array}{c}
W_1(\lambda, t_1|-) \\
W_1(\lambda, t_2|-, \delta z) \\
W_2(+, \delta z|\lambda, t_1) \\
W_2(+|\lambda, t_2) \\
W(\lambda, t_1|-, \delta z, +) \\
W(\lambda, t_2|-, \delta z, +)
\end{array}
\end{align*}
on the order of $\sqrt{\delta t}$, the magnitude of $\bar{q}$ can become extremely large, like the weak value. This anomaly is simply another manifestation of the illogicality that arises from the negative quasi-probabilities.

VI. OTHER RELATED WORK

The Bayesian inference of an intermediate quantum projective measurement outcome was first considered by Watanabe [51] and Aharonov, Bergmann, and Lebowitz [52]. Yanagisawa first introduced the term quantum smoothing and applied it to quantum non-demolition (QND) observables, which are compatible observables in the Heisenberg picture [53]. Refs. [13, 14, 34] focus on the smoothing inference of classical stochastic waveforms coupled to quantum systems under continuous measurements, while more recent papers by Dressel, Agarwal, and Jordan [54, 55] and Gammelmark, Julsgaard, and Mølmer [56] extend the theory to the inference of weak measurement outcomes. These results can be regarded as sharing the same foundation, as projective or weak measurement outcomes and classical random variables can all be modeled as QND observables in a suitably dilated Hilbert space [57]. A collection of QND observables that commute with each other at all times of interest are called a quantum-mechanics-free subsystem in Ref. [58] to emphasize that they have no quantum feature. It is easy to show that QND observables always have consistent histories [3].

The inference of QND observables is always compatible with decision theory, since they can be measured without any backaction and compared with the estimates for error evaluation, but the theory is not as general as the one proposed here and in Refs. [13, 14], as quasi-probability distributions generally involve incompatible observables. Another recent work by Chantasri, Dressel, and Jordan [59] also proposes a phase-space approach to the quantum smoothing problem, but their method is based on path integrals and its connection with more well known and useful quasi-probability functions is unclear.

VII. CONCLUSION

The Wigner representations are currently some of the best tools for finding classical models of quantum systems [17, 60], and negative Wigner quasi-probability is known to be a necessary resource for quantum computation [61–63]. By equating the logicality of quantum smoothing and the non-negativity of quasi-probability representations, I have also made a connection between the quantum smoothing inference problem and the notion of contextuality [19, 20]. These connections suggest that the smoothing method based on logical inference and Wigner functions is a pretty good, if not the best, attempt at reconciling logic and quantum mechanics when one tries to infer the past of a quantum system, and further progress along these lines will benefit multiple areas of quantum information processing and quantum foundations.

ACKNOWLEDGMENTS

Inspiring and helpful discussions with Joshua Combes, Christopher Ferrie, Justin Dressel, and George Knee are gratefully acknowledged. This work is supported by the Singapore National Research Foundation under NRF Grant No. NRF-NRFF2011-07.

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