Temporally Semidiscrete Approximation of a Dirichlet Boundary Control for a Fractional/Normal Evolution Equation with a Final Observation

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Abstract
Optimal Dirichlet boundary control for a fractional/normal evolution equation with a final observation is considered. The unique existence of the solution and the first-order optimality condition of the optimal control problem are derived. The convergence of a temporally semidiscrete approximation is rigorously established, where the control is not explicitly discretized and the state equation is discretized by a discontinuous Galerkin method in time. Numerical results are provided to verify the theoretical results.

Keywords Dirichlet boundary control · Fractional evolution equation · Discontinuous Galerkin method · Convergence

1 Introduction

There is an extensive literature on the numerical optimization with PDE constraints. So far, most of the literature focuses on the distributed control problems, and the works on the Dirichlet boundary control problems are rather limited. Compared with the distributed control problems, the Dirichlet boundary control problems are more challenging in the following senses. Firstly, the solution of the state equation of a Dirichlet boundary control problem is of significantly lower regularity than that of a distributed control problem, and this increases the difficulty in both theoretical and numerical analysis. Secondly, in the weak form of the solution of the state equation of a Dirichlet boundary control problem, the test function space is more regular than the trial function space, and hence the weak form is not appropriate

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for the discretization. Thirdly, since the normal derivative of the adjoint state occurs in the first-order optimality condition, the discrete first-order optimality condition will essentially involve the discrete normal derivative of the discrete adjoint state, and this increases the implementation difficulty.

We summarize the works on the parabolic Dirichlet boundary control problems briefly as follows. Using an integral representation formula derived by the semigroup theory (cf. [4, Section 4.12] and [34]), Lasiecka [33,35] analyzed spatial Galerkin approximations of an optimal Dirichlet boundary control problem and a time optimal Dirichlet boundary control problem for the parabolic equations. Kunisch and Vexler [32] analyzed constrained Dirichlet boundary control problems for a class of parabolic equations and derived the convergence of the PDAS strategy for two Dirichlet boundary control problems. Applying the Robin penalization method to a Dirichlet boundary control problem for a parabolic equation with a final observation, Belgacem et al. [5] obtained a penalized Robin boundary control problem. Gong et al. [18] analyzed a finite element approximation of a Dirichlet boundary control for a parabolic equation, where the state equation was discretized by the usual continuous piecewise linear element method in space and discretized by the dG(0) scheme in time. Recently, Gong and Li [19] improved the spatial accuracy derived in [18], using the maximal \( L^p \)-regularity theory. We note that, for the state equation with rough Dirichlet boundary data, [33,35] used the semigroup theory to define the solution whereas [18,19,32] used the transposition technique to define the solution (called the very weak solution).

For the numerical analysis of parabolic Neumann/Robin boundary control problems, we refer the reader to [1,8,31,48]. For the numerical analysis of other optimal control problems for parabolic equations, we refer the reader to [9,17,38,39,51–54] and the references therein. Although the spatial discretization is not considered in this paper, we would like to refer the reader to [6,14,15,36] for the numerical analysis of elliptic and parabolic equations with rough Dirichlet boundary data.

To our best knowledge, no convergence result is available for the Galerkin-type approximations of the Dirichlet boundary control problems governed by the parabolic equations with final observations. The fractional evolution equation is an extension of the normal evolution equation, widely used to describe the physical phenomena with memory effect [59]. Recently, Harbir et al. [3] studied an optimal distributed control problem for a space-time fractional diffusion equation. For the numerical analysis of the optimal distributed problems governed by the time fractional diffusion equations, we refer the reader to [20,29,42,64]. To our knowledge, no numerical analysis is available for the Dirichlet boundary control problems governed by the fractional evolution equations. Hence, this paper tries to analyze the Dirichlet boundary control problems for the fractional and normal evolution equations in a unified way.

In this paper, we establish the convergence of a temporally semidiscrete approximation of an abstract optimal control problem of the following form:

\[
\min_{u \in U_{ad}} J_\alpha(u, y) := \frac{1}{2} \| y(T) - y_d \|^2_Y + \frac{\nu}{2} \| u \|^2_{L^2(0,T;Z)},
\]

subject to the state equation

\[
(\mathcal{D}^\alpha_0 - A)y = \mathcal{R}h_0 u, \quad y(0) = 0.
\]

This approximation uses the variational discretization concept [21] and uses a discontinuous Galerkin method to discretize the state equation in time. This discontinuous Galerkin method is the famous dG(0) scheme for the normal evolution equation, and is equivalent to the well-known L1 scheme [43] with uniform temporal grids for the fractional evolution equations.
The derived numerical analysis is applied to a Dirichlet boundary control problem. We note that there are many works (see [23–26,28,40,45,47] and the references therein) devoted to the numerical analysis of the fractional diffusion equations with rough initial value and source term, but, to our knowledge, no numerical analysis is available for the fractional diffusion equation with rough Dirichlet boundary value. This paper also fills in this gap.

The rest of this paper is organized as follows. Section 2 establishes the convergence of a temporally semidiscrete approximation of an abstract optimal control problem. Section 3 applies the theory developed in the previous section to a Dirichlet boundary control problem. Section 4 performs three numerical experiments to confirm the theoretical results.

2 An Abstract Optimal Control Problem

2.1 Preliminaries

We will use the following conventions: for each linear vector space, the field of the scalars is \( \mathbb{C} \), unless otherwise stated; for a Hilbert space \( X \), we use \( \langle \cdot, \cdot \rangle_X \) to denote its inner product; for a Banach space \( B \), we use \( \langle \cdot, \cdot \rangle_B \) to denote a duality paring between \( B^* \) (the dual space of \( B \)) and \( B \); for a linear operator \( A \), \( \rho (A) \) denotes the resolvent set of \( A \) and \( R(z, A) \) denotes the inverse of \( z - A \) for each \( z \in \rho (A) \); for two Banach spaces \( B_1 \) and \( B_2 \), \( \mathcal{L}(B_1, B_2) \) is the set of all bounded linear operators from \( B_1 \) to \( B_2 \), and \( \mathcal{L}(B_1, B_1) \) is abbreviated to \( \mathcal{L}(B_1) \); \( I \) denotes the identity map; for a Lebesgue measurable subset \( D \), \( \mathcal{L}(B_1, B_2) \) set of all bounded linear operators from \( B_1 \) to \( B_2 \); and \( \rho(z) \) is the resolvent set of \( A \); for a function \( v \) defined on \( (0, T) \), by \( v(t^-) \) (or \( v(t^+) \)), \( 0 < t \leq T \), we mean the limit \( \lim_{s \to t^-} v(s) \) (or \( \lim_{s \to t^+} v(s) \)); the notation \( C_r \) means a positive constant, depending only on its subscript(s), and its value may differ at each occurrence; for any complex number \( z \), \( \Re z \) means its real part and \( \Im z \) means its imaginary part; \( i \) denotes the imaginary unit. In addition, for any \( 0 < \theta < \pi \), define

\[
\begin{align*}
\Sigma_\theta := \{ r e^{i\gamma} : r > 0, -\theta < \gamma < \theta \}, \\
\Gamma_\theta := \{ r e^{-i\theta} : r \geq 0 \} \cup \{ r e^{i\theta} : r > 0 \} \\
\Upsilon_\theta := \{ z \in \Gamma_\theta : -\pi \leq \Im z \leq \pi \},
\end{align*}
\]

where \( \Gamma_\theta \) and \( \Upsilon_\theta \) are so oriented that the negative real axis is to their left. For the integral \( \int_{\Gamma_\theta} v \, dz \) or \( \int_{\Upsilon_\theta} v \, dz \), if \( v \) has singularity or is not defined at the origin, then \( \Gamma_\theta \) or \( \Upsilon_\theta \) should be deformed so that the origin is to its left; for example, \( \Gamma_\theta \) is deformed to

\[
(\infty, \epsilon) e^{-i\theta} \cup \{ \epsilon e^{i\gamma} : -\theta \leq \gamma \leq \theta \} \cup [\epsilon, \infty) e^{i\theta},
\]

where \( 0 < \epsilon < \infty \).

2.1.1 Time Fractional Sobolev Spaces

Assume that \( -\infty < a < b < \infty \) and \( B \) is a Banach space. Define

\[
\begin{align*}
0H^1(a, b; B) := \{ v \in L^2(a, b; B) : v' \in L^2(a, b; B), \; v(a) = 0 \}, \\
0H^1(a, b; B) := \{ v \in L^2(a, b; B) : v' \in L^2(a, b; B), \; v(b) = 0 \},
\end{align*}
\]

and endow them with the two norms

\[
\| v \|_{0H^1(a, b; B)} := \| v' \|_{L^2(a, b; B)} \quad \forall v \in 0H^1(a, b; B),
\]

\( Springer \)
respectively, where \( v' \) is the first-order weak derivative of \( v \).

For each \( 0 < \gamma < 1 \), define

\[
0^\gamma H^1(a, b; \mathcal{B}) := (L^2(a, b; \mathcal{B}), 0^H^1(a, b; \mathcal{B}))_{\gamma, 2},
\]

\[
0^\gamma H^2(a, b; \mathcal{B}) := (L^2(a, b; \mathcal{B}), 0^H^1(a, b; \mathcal{B}))_{\gamma, 2},
\]

where \((\cdot, \cdot)_{\gamma, 2}\) means the interpolation space defined by the \( K \)-method (cf. [46, Chapter 1]).

For convenience, the spaces \( 0^\gamma H^1(a, b; \mathbb{C}) \) and \( 0^\gamma H^2(a, b; \mathbb{C}) \) will be abbreviated to \( 0^\gamma H^1(a, b) \) and \( 0^\gamma H^2(a, b) \), respectively.

### 2.1.2 Riemann–Liouville Fractional Calculus Operators

Assume that \(-\infty \leq a < b \leq \infty \) and \( \mathcal{B} \) is a Banach space. For any \( \gamma > 0 \), define

\[
\left(D_{a+}^\gamma v\right)(t) := \frac{1}{\Gamma(\gamma)} \int_a^t (t-s)^{\gamma-1} v(s) \, ds, \quad \text{a.e. } t \in (a, b),
\]

\[
\left(D_{b-}^\gamma v\right)(t) := \frac{1}{\Gamma(\gamma)} \int_t^b (s-t)^{\gamma-1} v(s) \, ds, \quad \text{a.e. } t \in (a, b),
\]

for all \( v \in L^1(a, b; \mathcal{B}) \), where \( \Gamma(\cdot) \) is the gamma function. In addition, let \( D_{a+}^0 \) and \( D_{b-}^0 \) be the identity operator on \( L^1(a, b; \mathcal{B}) \). For \( j - 1 < \gamma < j, j \in \mathbb{N}_{>0} \), define

\[
D_{a+}^\gamma v := D^j D_{a+}^{\gamma-j} v,
\]

\[
D_{b-}^\gamma v := (-D)^j D_{b-}^{\gamma-j} v,
\]

for all \( v \in L^1(a, b; \mathcal{B}) \), where \( D \) is the first-order differential operator in the distribution sense.

We summarize some properties of the above fractional calculus operators, which will be used later in this paper.

**Lemma 2.1** For any \( v \in 0^\gamma H^1(a, b) \) with \( 0 < \gamma < 1/2 \),

\[
\cos(\gamma \pi) \| D_{a+}^\gamma v \|_{L^2(a, b)}^2 \leq \| D_{a+}^\gamma v, D_{b-}^\gamma v \|_{(a, b)}^2 \leq \sec(\gamma \pi) \| D_{a+}^\gamma v \|_{L^2(a, b)}^2,
\]

\[
\cos(\gamma \pi) \| D_{b-}^\gamma v \|_{L^2(a, b)}^2 \leq \| D_{a+}^\gamma v, D_{b-}^\gamma v \|_{(a, b)}^2 \leq \sec(\gamma \pi) \| D_{b-}^\gamma v \|_{L^2(a, b)}^2.
\]

**Lemma 2.2** For any \( v \in 0^\gamma H^1(a, b) \) and \( w \in 0^\gamma H^2(a, b) \) with \( 0 < \gamma < 1 \),

\[
C_1 \| v \|_{0^\gamma H^1(a, b)} \leq \| D_{a+}^\gamma v \|_{L^2(a, b)} \leq C_2 \| v \|_{0^\gamma H^1(a, b)},
\]

\[
C_1 \| w \|_{0^\gamma H^2(a, b)} \leq \| D_{b-}^\gamma w \|_{L^2(a, b)} \leq C_2 \| w \|_{0^\gamma H^2(a, b)},
\]

where \( C_1 \) and \( C_2 \) are two positive constants depending only on \( \gamma \).

**Lemma 2.3** Assume that \( v \in 0^\gamma H^{1/2}(a, b) \) and \( w \in 0^\gamma H^{1/2}(a, b) \) with \( 0 < \gamma < 1 \). Then

\[
\langle D_{a+}^\gamma v, w \rangle_{0^\gamma H^{1/2}(a, b)} = \langle D_{b-}^\gamma w, v \rangle_{0^\gamma H^{1/2}(a, b)}.
\]

If \( D_{a+}^\gamma v \in L^{2/(1+\gamma)}(a, b) \), then

\[
\langle D_{a+}^\gamma v, w \rangle_{0^\gamma H^{1/2}(a, b)} = \langle D_{a+}^\gamma v, w \rangle_{(a, b)}.
\]
If \( D_{b-}^\gamma w \in L^{2/(1+\gamma)}(a, b) \), then
\[
\langle D_{b-}^\gamma w, v \rangle_{0, H^{\gamma/2}(a,b)} = \langle D_{b-}^\gamma w, v \rangle_{(a,b)}.
\]

(7)

For the proof of Lemma 2.1, we refer the reader to [12]. For the proof of Lemma 2.2, we refer the reader to [47]. Since the proof of Lemma 2.3 is a standard density argument by Lemmas 2.1 and 2.2, it is omitted here.

2.1.3 Definitions of \( \mathcal{A} \) and \( \mathcal{A}^* \)

Assume that \( X \) and \( Y \) are two Hilbert spaces such that \( X \) is continuously embedded into \( Y \) and \( X \) is dense in \( Y \). We will regard \( Y \) as a subspace of \( X^* \), the dual space of \( X \), in the sense that
\[
\langle v, w \rangle_X := (w, v)_Y \quad \text{for all } v \in Y \text{ and } w \in X.
\]

We will also identify \( Y \) as the dual space of \( Y \) in the conventional sense that
\[
\langle v, w \rangle_Y := (w, v)_Y \quad \text{for all } v, w \in Y.
\]

Let \( \mathcal{A} : X \to Y \) be a bounded linear operator satisfying that
\[
\begin{cases}
\rho(\mathcal{A}) \supset \Sigma_{\omega_0} \cup \{0\}, \\
||R(z, \mathcal{A})||_{L(Y)} \leq \frac{\mathcal{M}_0}{1 + |z|} \quad \forall z \in \Sigma_{\omega_0}, \\
c_0||v||_X \leq ||\mathcal{A}v||_Y \leq c_1||v||_X \quad \forall v \in X, \\
\Re(\mathcal{A}v, v)_Y \geq 0 \quad \forall v \in X,
\end{cases}
\]

(8a, 8b, 8c, 8d)

where \( \pi/2 < \omega_0 < \pi \), \( c_0, c_1 \) and \( \mathcal{M}_0 \) are four positive constants. Let \( \mathcal{A}^* : Y \to X^* \) be the usual dual operator of \( \mathcal{A} \). Note that [63, Theorem 1.19] implies
\[
\rho(\mathcal{A}^*) \supset \Sigma_{\omega_0} \cup \{0\}.
\]

(9)

We assume that the restriction of \( \mathcal{A}^* \) on \( X \) is a bounded linear operator from \( X \) to \( Y \), which admits a bounded inverse and satisfies that
\[
c_0||v||_X \leq ||\mathcal{A}^*v||_Y \leq c_1||v||_X \quad \text{for all } v \in X.
\]

(10)

By definition, it is evident that
\[
\langle \mathcal{A}v, w \rangle_Y = (v, \mathcal{A}^*w)_Y \quad \text{for all } v, w \in X,
\]
\[
\langle \mathcal{A}^*v, w \rangle_X = (\mathcal{A}w, v)_Y \quad \text{for all } v \in Y \text{ and } w \in X.
\]

(11, 12)

Moreover, by (8b), an elementary calculation yields
\[
||R(z, \mathcal{A}^*)||_{L(Y)} \leq \frac{\mathcal{M}_0}{1 + |z|} \quad \text{for all } z \in \Sigma_{\omega_0}.
\]

(13)

By the transposition technique (cf. [44, Chapter 2, Section 6]), \( \mathcal{A} \) can be extended as a bounded linear operator from \( Y \) to \( X^* \) by
\[
\langle \mathcal{A}v, w \rangle_X := (\mathcal{A}^*w, v)_Y \quad \text{for all } v \in Y \text{ and } w \in X.
\]

(14)

Since
\[
||\mathcal{A}||_{L(Y, X^*)} = \sup_{0 \neq v \in Y} \frac{||\mathcal{A}v||_{X^*}}{||v||_{Y}} = \sup_{0 \neq v \in Y} \sup_{0 \neq w \in X} \frac{\langle \mathcal{A}v, w \rangle_X}{||v||_{Y}||w||_{X}}
\]

(15)
\begin{equation}
\sup_{0 \neq v \in Y} \sup_{0 \neq w \in X} \frac{(A^* w, v)_Y}{\|v\|_Y \|w\|_X} \quad \text{(by (14))},
\end{equation}

by (10) we obtain
\begin{equation}
\|A\|_{\mathcal{L}(Y, X^*)} \leq c_1.
\end{equation}

For each $0 \leq \theta \leq 1$, let $[X^*, Y]_\theta$ and $[X, Y]_\theta$ be the interpolation spaces defined by the famous complex interpolation method (cf. [46, Chapter 2]). By (8c), (17) and [46, Theorem 2.6], we obtain
\begin{equation}
\|A\|_{\mathcal{L}([X, Y]_\theta, [X^*, Y]_{1-\theta})} \leq c_1 \quad \text{for all } 0 \leq \theta \leq 1.
\end{equation}

Similarly,
\begin{equation}
\|A^*\|_{\mathcal{L}([X, Y]_\theta, [X^*, Y]_{1-\theta})} \leq c_1 \quad \text{for all } 0 \leq \theta \leq 1.
\end{equation}

**Remark 2.1** For any $0 < \theta < 1$, we have that (cf. [46, pp. 55])
\begin{align*}
[X^*, Y]_\theta &= ([X, Y]_\theta)^* \quad \text{with equivalent norms,} \\
[X, Y]_\theta &= ([X^*, Y]_\theta)^* \quad \text{with equivalent norms.}
\end{align*}

**Lemma 2.4** Assume that $0 \leq \theta \leq 1$ and $z \in \Sigma_{\omega_0}$. Then
\begin{align}
\|R(z, A)\|_{\mathcal{L}(Y, [X, Y]_\theta)} &\leq \frac{C_{\omega_0, \mathcal{M}_0}}{1 + |z|^\theta}, \\
\|R(z, A^*)\|_{\mathcal{L}(Y, [X, Y]_\theta)} &\leq \frac{C_{\omega_0, \mathcal{M}_0}}{1 + |z|^\theta}, \\
\|R(z, A)\|_{\mathcal{L}([X^*, Y]_\theta, Y)} &\leq \frac{C_{\omega_0, \mathcal{M}_0}}{1 + |z|^\theta}, \\
\|R(z, A^*)\|_{\mathcal{L}([X^*, Y]_\theta, Y)} &\leq \frac{C_{\omega_0, \mathcal{M}_0}}{1 + |z|^\theta}.
\end{align}

Moreover, for any $0 \leq \epsilon \leq 1$,
\begin{equation}
\|R(z, A)\|_{\mathcal{L}([X^*, Y]_{1-(1-\epsilon)\theta}, Y)} \leq \frac{C_{\omega_0, \mathcal{M}_0, \epsilon, \theta}}{1 + |z|^\theta}.
\end{equation}

**Proof** A straightforward computation gives
\begin{equation}
\|AR(z, A)\|_{\mathcal{L}(Y)} = \|z R(z, A) - I\|_{\mathcal{L}(Y)} \leq |z| \|R(z, A)\|_{\mathcal{L}(Y)} + 1 \leq 1 + \mathcal{M}_0,
\end{equation}

by (8b), so that (8c) implies
\begin{equation}
\|R(z, A)\|_{\mathcal{L}(Y, X)} \leq \frac{1 + \mathcal{M}_0}{c_0}.
\end{equation}

By this estimate and (8b), using [46, Theorem 2.6] yields (20). Estimate (21) can be proved analogously.

Then let us prove (22). We first consider the following problem: seek $v \in Y$ such that
\begin{equation}
((\overline{z} - A^*) \varphi, v)_Y = (g, \varphi)_X \quad \forall \varphi \in X,
\end{equation}

where $g \in X^*$ is arbitrary but fixed. By (10) and the fact that $X$ is continuously embedded into $Y$, we conclude that $((\overline{z} - A^*) \cdot, \cdot)_Y$ is a continuous bilinear form on $X \times Y$. Inserting
\(\theta = 0\) into (21) implies that, for any \(v \in Y \setminus \{0\},\)
\[
\sup_{\varphi \in X \setminus \{0\}} \frac{|(\overline{z} - A^* \varphi, v)_Y|}{\|\varphi\|_X} \geq \frac{\|v\|_Y^2}{\|R(\overline{z}, A^*)v\|_X} \geq C_{c_0, M_0} \|v\|_Y.
\]
Since \(z \in \rho(A^*)\), it is evident that, for any \(\varphi \in X \setminus \{0\},\)
\[
\sup_{v \in Y} |(\overline{z} - A^* \varphi, v)_Y| > 0.
\]
Consequently, the Babuska-Lax-Milgram theorem yields that problem (25) admits a unique solution \(v \in Y\) and \(\|v\|_Y \leq C_{c_0, M_0} \|g\|_{X^*}\). Since (14) and (25) imply \(v = R(z, A)g\), this indicates that
\[
\|R(z, A)g\|_Y \leq C_{c_0, M_0} \|g\|_{X^*} \quad \forall g \in X^*,
\]
and hence
\[
\|R(z, A)\|_{L(X^*, Y)} \leq C_{c_0, M_0}.
\] (26)
By (8b) and (26), using [46, Theorem 2.6] yields (22). Estimate (23) is derived similarly.
Finally, let us prove (24). Inserting \(\theta = \epsilon\) and \(\theta = 0\) into (20) and (22) respectively yields
\[
\|R(z, A)\|_{L(Y, [X, Y]_{1-\epsilon})} \leq \frac{C_{c_0, M_0}}{1 + |\epsilon|^\epsilon}, \quad \|R(z, A)\|_{L(Y^*, Y)} \leq C_{c_0, M_0}.
\]
Using [46, Theorem 2.6] then gives
\[
\|R(z, A)\|_{L([X^*, Y_{1-\epsilon}]; [Y, X]_{1-\epsilon})} \leq \frac{C_{c_0, M_0}}{1 + |\epsilon|^\epsilon}.
\]
Hence, by the fact that (cf. [7])
\[
[Y, [X, Y]_{1-\epsilon}]_\theta = [X, Y]_{1-(1-\epsilon)\theta}
\]
with equivalent norms,
we readily obtain (24). This completes the proof. \(\square\)

**Remark 2.2** For any \(z \in \Sigma_{c_0},\)
\[
\|R(z, A)\|_{L(X^*)} = \|z^{-1}(z - A + A)R(z, A)\|_{L(X^*)}
\]
\[
= \|I + AR(z, A)\|_{L(X^*)}/|z|
\]
\[
\leq \frac{1 + \|AR(z, A)\|_{L(X^*)}}{|z|}
\]
\[
\leq \frac{1 + c_1 \|R(z, A)\|_{L(X^*, Y)}}{|\epsilon|} \quad \text{(by inserting } \theta = 1 \text{ into (18)})
\]
\[
\leq \frac{C_{c_0, c_1, M_0}}{|\epsilon|} \quad \text{(by inserting } \theta = 0 \text{ into (22))}.
\]
Also, we have
\[
\|R(z, A)\|_{L(X^*)} = \|A^{-1}R(z, A)A\|_{L(X^*)}
\]
\[
\leq c_1 \|A^{-1}R(z, A)\|_{L(X^*, Y)} \quad \text{(by inserting } \theta = 1 \text{ into (18)})
\]
\[
\leq C_{c_1, M_0} \|R(z, A)\|_{L(X^*, Y)} \quad \text{(by (8b)})
\]
\[
\leq C_{c_0, c_1, M_0} \quad \text{(by inserting } \theta = 0 \text{ into (22))}.
\]
Consequently,

\[ \| R(z, A) \|_{L(X^*)} \leq \frac{C_{c_0, c_1, M_0}}{1 + |z|} \quad \forall z \in \Sigma_{\omega_0}. \]  

(27)

**Remark 2.3** Assume \( z \in \rho(A) \). By definition, \( A : X \rightarrow Y \) is the restriction of \( A : Y \rightarrow X^* \) on \( X \), and hence \( z - A : X \rightarrow Y \) is the restriction of \( z - A : Y \rightarrow X^* \) on \( X \). From the facts that \( z - A : X \rightarrow Y \) has a bounded inverse by assumption and that we have proved that \( z - A : Y \rightarrow X^* \) has a bounded inverse, it is evident that \( R(z, A) : Y \rightarrow X \) is the restriction of \( R(z, A) : X^* \rightarrow Y \) on \( Y \).

### 2.1.4 Definitions of \( E_\alpha \) and \( E^*_\alpha \)

For any \( 0 < \alpha \leq 1 \) and \( t > 0 \), define

\[ E_\alpha(t) := \frac{1}{2\pi i} \int_{\Gamma_{\omega_0}} e^{tz} R(z^\alpha, A) \, dz, \]  

(28)

\[ E^*_\alpha(t) := \frac{1}{-2\pi i} \int_{\Gamma_{\omega_0}} e^{tz} R(z^{\alpha}, A^*) \, d\overline{z}. \]  

(29)

Here we recall that \( \pi/2 < \omega_0 < \pi \). For any \( v \in X^* \) and \( w \in Y \), by the definitions of \( A \) and \( A^* \) and Lemma 2.4 we have that, for any \( z \in \Gamma_{\omega_0} \setminus \{0\} \),

\[ (w, R(z^\alpha, A)v)_Y = \langle (z^\alpha - A^*)R(z^{\alpha}, A^*)w, R(z^\alpha, A)v \rangle_Y \]

\[ = \langle (z^\alpha - A)R(z^\alpha, A)v, R(z^{\alpha}, A^*)w \rangle_X \]

\[ = \langle v, R(z^{\alpha}, A^*)w \rangle_X, \]

and hence from (28) and (29) we obtain that, for any \( t > 0 \),

\[ (w, E_\alpha(t)v)_Y = \frac{1}{-2\pi i} \int_{\Gamma_{\omega_0}} e^{tz} (w, R(z^\alpha, A)v)_Y \, d\overline{z} \]

\[ = \frac{1}{-2\pi i} \int_{\Gamma_{\omega_0}} e^{tz} \langle v, R(z^{\alpha}, A^*)w \rangle_X \, d\overline{z} \]

\[ = \langle v, E^*_\alpha(t)w \rangle_X. \]  

(30)

Furthermore, by Lemma 2.4, a routine calculation (cf. [30, Lemma 3.2]) yields the following lemma.

**Lemma 2.5** Assume that \( 0 < \alpha \leq 1 \), \( 0 \leq \theta \leq 1 \), \( t > 0 \), and \( G = E_\alpha \) or \( E^*_\alpha \). Then

\[ \| G(t) \|_{L(Y, \{X, Y\}_0)} \leq C_{c_0, c_1, M_0} t^{\theta \alpha - 1}. \]  

(31)

\[ \| G(t) \|_{L(\{X^*, Y\}_0, Y)} \leq C_{c_0, c_1, M_0} t^{\theta \alpha - 1}. \]  

(32)

\[ \| G'(t) \|_{L(Y, \{X, Y\}_0)} \leq C_{c_0, c_1, M_0} t^{\theta \alpha - 2}. \]  

(33)

\[ \| G'(t) \|_{L(\{X^*, Y\}_0, Y)} \leq C_{c_0, c_1, M_0} t^{\theta \alpha - 2}. \]  

(34)

Moreover, for any \( 0 \leq \epsilon \leq 1 \),

\[ E_\alpha \in C((0, \infty); \mathcal{L}(\{X^*, Y\}_0, [X, Y]_{1-(1-\epsilon)\theta})). \]  

(35)

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\[ \| E_\alpha(t) \|_{L_t([X^*, Y]_{\theta})} \leq C_{c_0, c_0, M_0, \epsilon, \theta} t^{\epsilon \theta - 1}. \] (36)

**Remark 2.4** By (27) we have
\[ \| E_\alpha(t) \|_{L_t([X^*])} \leq C_{c_0, c_1, c_0, M_0} t^{\alpha - 1}, \quad \forall t > 0. \]

### 2.1.5 Solutions of the fractional evolution equations

Following the mild solution theory of fractional/normal evolution equations in [45, 50, 58], we introduce the following mild solutions. Assume that \( 0 < \alpha \leq 1 \) and \( 0 < T < \infty \). For any
\[ g \in L^1(0, T; [X^*, Y]_{\theta}) \quad \text{with} \quad 0 \leq \theta \leq 1, \]
we call
\[ (S_\alpha g)(t) := \int_0^t E_\alpha(t - s)g(s) \, ds, \quad \text{a.e.} \quad 0 < t \leq T, \] (37)
the mild solution to the following fractional evolution equation:
\[ (D_{0+}^\alpha - A)w = g, \quad w(0) = 0. \] (38)

For any \( v \in [X^*, Y]_{\theta}, 0 \leq \theta \leq 1 \), we call
\[ (S_\alpha(v\delta_0))(t) := E_\alpha(t)v, \quad 0 < t \leq T, \] (39)
the mild solution to (38) with \( g = v\delta_0 \), where \( \delta_0 \) is the Dirac measure in time concentrated at \( t = 0 \). Symmetrically, for any
\[ g \in L^1(0, T; [X^*, Y]_{\theta}) \quad \text{with} \quad 0 \leq \theta \leq 1, \]
we call
\[ (S_{\alpha}^* g)(t) := \int_t^T E_{\alpha}^*(s - t)g(s) \, ds, \quad \text{a.e.} \quad 0 < t < T, \] (40)
the mild solution to the following backward fractional evolution equation:
\[ (D_{T-}^\alpha - A^*)w = g, \quad w(T) = 0. \] (41)

For any \( v \in [X^*, Y]_{\theta}, 0 \leq \theta \leq 1 \), we call
\[ (S_{\alpha}^*(v\delta_T))(t) := E_{\alpha}^*(T - t)v, \quad 0 < t \leq T, \] (42)
the mild solution to equation (41) with \( g = v\delta_T \), where \( \delta_T \) is the Dirac measure in time concentrated at \( t = T \).

**Lemma 2.6** Assume that \( 0 < \alpha, \theta \leq 1 \) and \( q > 1/(\theta \alpha) \). Then
\[ S_\alpha \in L(L^2(0, T; [X^*, Y]_{\theta}), L^2(0, T; Y)), \] (43)
\[ S_\alpha^* \in L(L^q(0, T; [X^*, Y]_{\theta}), C([0, T]; Y)). \] (44)

Moreover, for any \( g \in L^q(0, T; [X^*, Y]_{\theta}) \) and \( v \in Y \),
\[ (v, (S_\alpha g)(T))_Y = \int_0^T \langle g(t), (S_\alpha^*(v\delta_T))(t) \rangle_X \, dt. \] (45)
Proof By (32) and (37), a routine argument (cf. [10, Theorem 2.6]) yields (43) and (44). Note that (37) and (44) imply \((S_\alpha g)(T) \in Y\) and
\[
(S_\alpha g)(T) = \int_0^T E_\alpha(T - t)g(t) \, dt,
\]
and hence
\[
(v, (S_\alpha g)(T))_Y = \left( v, \int_0^T E_\alpha(T - t)g(t) \, dt \right)_Y = \int_0^T \left( v, E_\alpha(T - t)g(t) \right)_Y \, dt
\]
\[
= \int_0^T \left( g(t), E_\alpha^*(T - t)v \right)_X \, dt \quad \text{(by (30))}
\]
\[
= \int_0^T \left( g(t), (S_\alpha^*(v\delta_T))(t) \right)_X \, dt \quad \text{(by (42)).}
\]
This proves (45) and hence this lemma. \(\square\)

Lemma 2.7 For any \(0 < \theta \leq 1\),
\[
\lim_{\alpha \to 1-} \|S_\alpha - S_1\|_{L(\mathbb{L}^\infty(0,T;[X^*, Y]_0), L^\infty(0,T;Y))} = 0.
\]  \hfill (46)

Proof Since (37) implies
\[
\|S_\alpha - S_1\|_{L(\mathbb{L}^\infty(0,T;[X^*, Y]_0), L^\infty(0,T;Y))} \leq \|E_\alpha - E_1\|_{L^1(0,T;\mathbb{L}([X^*, Y]_0, Y))},
\]
it suffices to prove that
\[
\lim_{\alpha \to 1-} \|E_\alpha - E_1\|_{L^1(0,T;\mathbb{L}([X^*, Y]_0, Y))} = 0.
\]  \hfill (47)

To this end, note that
\[
R(z, \mathcal{A}) - R(z^\alpha, \mathcal{A}) = (z^\alpha - z)R(z, \mathcal{A})R(z^\alpha, \mathcal{A}) \quad \text{for all } z \in \Sigma_{a00}.
\]

Then a straightforward calculation gives, by (28) and Lemma 2.4, that
\[
\|E_\alpha - E_1\|_{L^1(0,T;\mathbb{L}([X^*, Y]_0, Y))} \leq C_{c_0, M_0} \int_0^\infty e^{\cos\omega_0} \frac{|r e^{i\omega_0} - (r e^{i\omega_0})^\alpha|}{(1 + r)(1 + r^{\theta\alpha})} \, dr
\]
for all \(t > 0\). It follows that
\[
\|E_\alpha - E_1\|_{L^1(0,T;\mathbb{L}([X^*, Y]_0, Y))} \leq C_{c_0, a_00, M_0} \int_0^\infty \frac{|r e^{i\omega_0} - (r e^{i\omega_0})^\alpha|}{r(1 + r)(1 + r^{\theta\alpha})} \, dr.
\]

Therefore, Lebesgue’s dominated convergence theorem yields (47) and thus concludes the proof. \(\square\)

Lemma 2.8 Assume that \(0 < \alpha, \theta \leq 1\). Then for any \(g \in C([0,T]; [X^*, Y]_0)\) we have
\[
(D_{0+}^\alpha - \mathcal{A})S_\alpha g = g
\]  \hfill (48)
and
\[
\|D_{0+}^\alpha S_\alpha g\|_{C([0,T];[X^*, Y]_{(1-\epsilon)\theta})} + \|A S_\alpha g\|_{C([0,T];[X^*, Y]_{(1-\epsilon)\theta})}
\]
\[
\leq c\|g\|_{C([0,T];[X^*, Y]_0)},
\]  \hfill (49)
where \(0 < \epsilon < 1\) and \(c\) is a positive constant independent of \(g\).
**Proof** Since a complete rigorous proof of this lemma is tedious but standard (cf. [58]), we only present briefly the key ingredients of the proof.

**Step 1.** Define

$$
\eta(t) := \frac{1}{2\pi i} \int_{\Gamma_{0}} e^{i\alpha - 1} R(z^\alpha, A) \, dz, \quad t > 0.
$$

A straightforward computation gives that, for any $0 < t \leq T$,

$$
\eta(t) = \int_{\Gamma_{0}} e^{i\alpha - 1} (z^\alpha - A + A) R(z^\alpha, A) \, dz
= \frac{1}{2\pi i} \int_{\Gamma_{0}} e^{i\alpha - 1} I \, dz + \frac{1}{2\pi i} \int_{\Gamma_{0}} e^{i\alpha - 1} A R(z^\alpha, A) \, dz
= I + \frac{1}{2\pi i} A \int_{\Gamma_{0}} e^{i\alpha - 1} R(z^\alpha, A) \, dz.
$$

Hence, we conclude from (18) and Lemma 2.4 the following properties:

$$
\begin{align*}
\eta &\in C([0, T]; \mathcal{L}([X^*, Y^*]) \cap C^1((0, T]; \mathcal{L}([X^*, Y^*])); \\
\eta(0) &= I; \\
\eta'(t) &= \frac{1}{2\pi i} A \int_{\Gamma_{0}} e^{i\alpha - 1} R(z^\alpha, A) \, dz, \quad t > 0; \\
\|\eta'(t)\|_{\mathcal{L}([X^*, Y^*])} &\leq C_{c, 1, \omega, 0, \alpha} t^{\alpha - 1}, \quad t > 0.
\end{align*}
$$

**Step 2.** By the theory of Laplace transform, from (37) we obtain that (cf. [30, Section 3] and [50, Section 1])

$$
(D_{0+}^{\alpha - 1} S_{\alpha} g)(t) = \int_{0}^{t} \eta(t - s) g(s) \, ds, \quad 0 < t \leq T.
$$

Hence, by the properties of $\eta$ presented in Step 1,

$$
\begin{align*}
(D_{0+}^{\alpha} S_{\alpha} g)(t) &= \frac{d}{dt} (D_{0+}^{\alpha - 1} S_{\alpha} g)(t) = \frac{d}{dt} \int_{0}^{t} \eta(t - s) g(s) \, ds \\
&= g(t) + \int_{0}^{t} \eta'(t - s) g(s) \, ds \\
&= g(t) + A \int_{0}^{t} \frac{1}{2\pi i} \int_{\Gamma_{0}} e^{i(t-s)\alpha} R(z^\alpha, A) \, dz \, g(s) \, ds \\
&= g(t) + A \int_{0}^{t} E_{\alpha} (t-s) g(s) \, ds \quad \text{(by (28))} \\
&= g(t) + A (S_{\alpha} g)(t) \quad \text{(by (37))}
\end{align*}
$$

for each $0 \leq t \leq T$. This implies equality (48).

**Step 3.** For convenience, we will use $c$ to denote a positive constant, whose value is independent of $g$ but may differ at each occurrence. A routine calculation gives, by (37), (35) and (36), that

$$
\|S_{\alpha} g\|_{C([0, T]; [X, Y]_{1-\omega})} \leq c \|g\|_{C([0, T]; [X^*, Y^*])},
$$

so that (18) implies

$$
\|AS_{\alpha} g\|_{C([0, T]; [X^*, Y^*]_{1-\omega})} \leq c \|g\|_{C([0, T]; [X^*, Y^*])}.
$$

(52)
Since \([X^*, Y]_0\) is continuously embedded into \([X^*, Y]_{(1-\epsilon)0}\), we have
\[
\|g\|_{C([0,T];[X^*, Y]_{(1-\epsilon)0})} \leq C \|g\|_{C([0,T];[X^*, Y]_0)}.
\] (53)
Combining (48), (52) and (53) proves (49) and thus concludes the proof. \(\square\)

**Remark 2.5** We present a brief proof of (51) as follows, where we refer the reader to [30, Section 3] and [50, Section 1] for the relevant techniques. Extend \(g\) to \((T, \infty)\) by zero and define
\[
w_0(t) := \int_0^t E_\alpha (t-s)g(s) \, ds, \quad 0 \leq t < \infty,
\]
\[
w_1(t) := \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} w_0(s) \, ds, \quad 0 \leq t < \infty,
\]
\[
w_2(t) := \int_0^t \eta(t-s)g(s) \, ds, \quad 0 \leq t < \infty.
\]
Let \(\hat{w}_0, \hat{w}_1\) and \(\hat{w}_2\) be the Laplace transforms of \(w_0, w_1\) and \(w_2\), respectively. A direct calculation gives that, for any \(z \in \mathbb{C}\) with \(|\Re z| > 0\),
\[
\hat{w}_0(z) = R(z^\alpha, \mathcal{A})\hat{g}(z),
\]
\[
\hat{w}_1(z) = z^{\alpha-1}\hat{w}_0(z),
\]
\[
\hat{w}_2(z) = z^{\alpha-1}R(z^\alpha, \mathcal{A})\hat{g}(z).
\]
It follows that
\[
\hat{w}_1(z) = \hat{w}_2(z), \quad |\Re z| > 0.
\]
Hence, we conclude from the uniqueness of the Laplace transform (cf. [2, Theorem 1.7.3]) that
\[
w_1(t) = w_2(t), \quad t \geq 0.
\]
The desired equality (51) then follows from the fact that
\[
(D_{0+}^{\alpha-1}S_\alpha g)(t) = w_1(t), \quad 0 \leq t \leq T,
\]
\[
\int_0^t \eta(t-s)g(s) \, ds = w_2(t), \quad 0 \leq t \leq T.
\]

### 2.2 Continuous Problem

Let \(Z\) be a Hilbert space and let \(U_{ad} \subset L^\infty(0, T; Z)\) be a convex, bounded and closed subset of \(L^2(0, T; Z)\). We consider the following abstract optimal control problem:
\[
\min_{u \in U_{ad}} \mathcal{J}_\alpha(u) := \frac{1}{2} \| (S_\alpha \mathcal{R}_{\theta_0}u)(T) - y_d \|_Y^2 + \frac{\nu}{2} \|u\|_{L^2(0, T; Z)}^2.
\] (54)
where \(0 < \alpha \leq 1, y_d \in Y, \nu > 0\) is a regularization parameter, and \(\mathcal{R}_{\theta_0} : Z \to [X^*, Y]_{\theta_0}\) is a bounded linear operator for some \(0 < \theta_0 \leq 1\).

Define \(\mathcal{R}^*_\theta : [X, Y]_{\theta_0} \to Z\) by
\[
(\mathcal{R}^*_\theta v, w)_Z := \overline{\langle \mathcal{R}_{\theta_0}w, v \rangle}_{[X, Y]_{\theta_0}}
\] (55)
for all $v \in [X, Y]_{\theta_0}$ and $w \in Z$. Here we recall that
\[ [X^*, Y]_{\theta_0} = ([X, Y]_{\theta_0})^* \] with equivalent norms.

Assume that $\max\{1/(\theta_0\alpha), 2\} < q < \infty$. By (44), $(S_\alpha R_{\theta_0})(T)$ is a bounded linear operator from $L^q(0, T; Z)$ to $Y$. Clearly, $J_\alpha$ in (54) is a strictly convex functional on $L^q(0, T; Z)$, and $U_{ad}$ is a convex, bounded and closed subset of $L^q(0, T; Z)$.

Theorem 2.1 Problem (54) admits a unique solution $u \in U_{ad}$, and the following first-order optimality condition holds:
\[
\begin{alignat}{2}
&y := S_\alpha R_{\theta_0} u, \\
&p := S_\alpha^0((y(T) - y_0)\delta_T), \\
&\forall \int_0^T (R_{\theta_0} p(t) + vu(t), v(t) - u(t))_Z \, dt \ge 0 \text{ for all } v \in U_{ad}.
\end{alignat}
\]

Remark 2.6 By (32), (37) and (56a),
\[ \|y(T)\|_Y \le C_{c_0, \theta_0, M_0, T}(\theta_0\alpha)^{-1} \|R_{\theta_0}\|_{\mathcal{L}(Z, [X^*, Y]_{\theta_0})}\|u\|_{L^\infty(0, T; Z)}. \]

2.3 Temporally Discrete Problem

Let $J > 1$ be an integer and define $t_j := j\tau$ for each $j = 0, 1, 2, \ldots, J$, where $\tau := T/J$. For each Banach space $X$, define
\[ W_T(X) := \{V \in L^\infty(0, T; X) : V \text{ is constant on } (t_{j-1}, t_j) \text{ } \forall 1 \le j \le J\}. \]

For any $0 < \alpha < 1$ and $g \in L^1(0, T; X^*)$, define $S_{\alpha, \tau} g \in W_T(Y)$ and $S_{\alpha, \tau}^* g \in W_T(Y)$, respectively, by that
\[
\begin{alignat}{2}
&\int_0^T \langle (D_{0+}^g - A)S_{\alpha, \tau} g, V \rangle_X \, dt = \int_0^T \langle g, V \rangle_X \, dt, \\
&\int_0^T \langle (D_{T-}^g - A^*)S_{\alpha, \tau}^* g, V \rangle_X \, dt = \int_0^T \langle g, V \rangle_X \, dt,
\end{alignat}
\]
for all $V \in W_T(X)$. For any $g \in L^1(0, T; X^*)$, define $S_{1, \tau} g \in W_T(Y)$ and $S_{1, \tau}^* g \in W_T(Y)$, respectively, by
\[
\begin{alignat}{2}
&\langle (S_{1, \tau} g)(0+), V(0+) \rangle_X + \sum_{j=1}^{J-1} \langle (S_{1, \tau} g)(t_j) - (S_{1, \tau} g)(t_{j-1}), V(t_{j+}) \rangle_X, \\
&- \int_0^T \langle AS_{1, \tau} g, V \rangle_X \, dt = \int_0^T \langle g, V \rangle_X \, dt,
\end{alignat}
\]
\[
\begin{alignat}{2}
&\langle (S_{1, \tau}^* g)(T-), V(T-) \rangle_X + \sum_{j=1}^{J-1} \langle (S_{1, \tau}^* g)(t_j) - (S_{1, \tau}^* g)(t_{j-1}), V(t_{j+}) \rangle_X, \\
&- \int_0^T \langle A^*S_{1, \tau}^* g, V \rangle_X \, dt = \int_0^T \langle g, V \rangle_X \, dt,
\end{alignat}
\]
for all $V \in W_T(X)$. We will present some properties of $S_{\alpha, \tau}$ and $S_{\alpha, \tau}^*, 0 < \alpha \le 1$, in Sect. 2.4.
Remark 2.7 Scheme (60) is a famous discontinuous Galerkin method for parabolic equations (cf. [11]), and this scheme is a variant of the backward Euler difference scheme.

Remark 2.8 We note that the idea of using the Galerkin methods to discretize the time fractional calculus operators was firstly developed by McLean and Mustapha [49,55–57]. The L1 scheme [43,60] is widely used for the discretizations of the fractional diffusion equations. Jin et al. [27, Remark 3] discovered that the L1 scheme is equivalent to discretization (58) with uniform temporal grids. For the numerical analysis of discretization (58) with nonuniform temporal grids, we refer the reader to [40,41].

Using the variational discretization concept proposed in [21], we consider the following temporally discrete problem:

\[
\begin{align*}
\min_{U \in U_{ad}} J_{\alpha,\tau}(U) := & \frac{1}{2} \| (S_{\alpha,\tau} R_{\theta_0} U)(T-\cdot) - y_d \|_Y^2 + \frac{v}{2} \| U \|_{L^2(0,T;Z)}^2.
\end{align*}
\] (62)

Note that (72) implies that \((S_{\alpha,\tau} R_{\theta_0} \cdot)(T-\cdot)\) is a bounded linear operator from \(L^2(0, T; Z)\) to \(Y\). In addition, \(U_{ad}\) is a convex, bounded and closed subset of \(L^2(0, T; Z)\). Hence, applying [62, Theorems 2.14 and 2.21] to problem (62) yields the following theorem, by Lemma 2.9.

**Theorem 2.2** Problem (62) admits a unique solution \(U \in U_{ad}\), and the following first-order optimality condition holds:

\[
\begin{align*}
Y := & \quad S_{\alpha,\tau} R_{\theta_0} U, \\
P := & \quad S^*_a,\tau ((Y(T-\cdot) - y_d) \delta_T), \\
\Re \int_0^T \left( R^*_0 P(t) + vU(t), V(t) - U(t) \right)_Z \, dt \geq 0 \quad \text{for all } V \in U_{ad},
\end{align*}
\] (63a–63c)

where

\[
\delta_T := \begin{cases} 
0 & \text{if } 0 < t < T - \tau, \\
\tau^{-1} & \text{if } T - \tau < t < T.
\end{cases}
\] (64)

The main result of this section is the following theorem.

**Theorem 2.3** Assume that \(0 < \alpha \leq 1\). Let \(u\) and \(y\) be defined in Theorem 2.1, and let \(U\) and \(Y\) be defined in Theorem 2.2. Then

\[
\begin{align*}
\| (y - Y)(T-\cdot) \|_Y + & \sqrt{v} \| u - U \|_{L^2(0,T;Z)} \leq C_{\varepsilon,\theta_0,\omega_0,\mathcal{M}_0,T} \tau^{\theta_0 \alpha/2} \times \\
& \sqrt{\varepsilon(\alpha, \theta_0, J)} \left( \sqrt{\varepsilon(\alpha, \theta_0, J)} \| R_{\theta_0} \|_{\mathcal{L}(Z,[X^*,Y]_{[\theta_0]})} \| u \|_{L^\infty(0,T;Z)} \tau^{\theta_0 \alpha/2} \\
& + \left( \frac{1}{\theta_0 \alpha} \| R_{\theta_0} \|_{\mathcal{L}(Z,[X^*,Y]_{[\theta_0]})} \| u \|_{L^\infty(0,T;Z)} + \| y_d \| Y \right)^{1/2} \right)^{1/2} \times \| u - U \|^{1/2}_{L^\infty(0,T;Z)} \| R^*_0 \|_{\mathcal{L}(Z,[X,Y]_{[\theta_0]},Z)}^{1/2},
\end{align*}
\] (65)

where

\[
\varepsilon(\alpha, \theta_0, J) := \begin{cases} 
\frac{1}{\theta_0 \alpha} + \frac{1-J^{\alpha}}{1-\theta_0 \alpha} & \text{if } \theta_0 \alpha \neq 1, \\
\ln J & \text{if } \theta_0 \alpha = 1.
\end{cases}
\] (66)
Proof} Since the idea of the proof is standard (cf. [22, Theorem 3.4]), we only provide a brief proof. Let us first prove that
\[
\| (y - Y)(T^-) \|_Y + \sqrt{v} \| u - U \|_L^2(0, T; Z)
\leq 2 \|(S_\alpha R_{\delta_0} u)(T) - (S_{\alpha, \tau} R_{\delta_0} u)(T^-) \|_Y + 2 \| u - U \|_L^{1/2}(0, T; Z) \| R_{\delta_0}^\alpha \|_L^1(X, Y_{\delta_0}, Z)
\times \| S_\alpha^* ((y(T) - y_d) \hat{\delta}_T) - S_{\alpha, \tau}^* ((y(T) - y_d) \hat{\delta}_T) \|_L^{1/2}(0, T; [X, Y_{\delta_0}]).
\] (67)

By (56c) and (63c), we have
\[
\Re \int_0^T (R_{\delta_0}^\alpha S_\alpha^* ((y(T) - y_d) \hat{\delta}_T) + v u, U - u)_Z \, dt \geq 0,
\]
so that
\[
v \| u - U \|_L^2(0, T; Z) \leq I_1 + I_2,
\] (68)

where
\[
I_1 := \Re \int_0^T (R_{\delta_0}^\alpha S_{\alpha}((y(T) - y_d) \hat{\delta}_T) - R_{\delta_0}^\alpha S_{\alpha, \tau}((y(T) - y_d) \hat{\delta}_T), U - u)_Z \, dt,
\]
\[
I_2 := \Re \int_0^T (R_{\delta_0}^\alpha S_{\alpha, \tau}((y(T) - Y(T-)) \hat{\delta}_T) + v U, U - U)_Z \, dt.
\]

It is clear that
\[
I_1 \leq \| u - U \|_L^\infty(0, T; Z) \| R_{\delta_0}^\alpha \|_L((X, Y_{\delta_0}, Z)
\times \| S_{\alpha}^* ((y(T) - y_d) \hat{\delta}_T) - S_{\alpha, \tau}^* ((y(T) - y_d) \hat{\delta}_T) \|_L^1(0, T; [X, Y_{\delta_0}]).
\]

We also have
\[
I_2 = \Re \int_0^T ((y(T) - Y(T-)) \hat{\delta}_T, S_{\alpha, \tau} R_{\delta_0}(U - u))_Y \, dt \quad \text{by (55) and (76)}
\]
\[
= \Re ((S_\alpha R_{\delta_0} u)(T) - (S_{\alpha, \tau} R_{\delta_0} U)(T-), (S_{\alpha, \tau} R_{\delta_0} (U - u))(T))_Y \quad \text{by (64)}
\]
\[
= \Re ((S_\alpha R_{\delta_0} u)(T) - (S_{\alpha, \tau} R_{\delta_0} U)(T-), (S_{\alpha, \tau} R_{\delta_0} (U - u))(T))_Y
\]
\[
- \| (S_{\alpha, \tau} R_{\delta_0} u)(T-)(u - U) \|_Y^2
\]
\[
\leq \frac{1}{2} \| (S_\alpha R_{\delta_0} u)(T) - (S_{\alpha, \tau} R_{\delta_0} U)(T) \|_Y^2
\]
\[
- \frac{1}{2} \| (S_{\alpha, \tau} R_{\delta_0} u)(T-)(u - U) \|_Y^2.
\]

Combining (68) and the above estimates of $I_1$ and $I_2$ gives
\[
\| (S_\alpha R_{\delta_0} u)(T) - (S_{\alpha, \tau} R_{\delta_0} U)(T-) \|_Y + \sqrt{v} \| u - U \|_L^2(0, T; Z)
\leq 2 \|(S_\alpha R_{\delta_0} u)(T) - (S_{\alpha, \tau} R_{\delta_0} U)(T-) \|_Y + 2 \| u - U \|_L^{1/2}(0, T; Z) \| R_{\delta_0}^\alpha \|_L^1(X, Y_{\delta_0}, Z)
\times \| S_{\alpha}^* ((y(T) - y_d) \hat{\delta}_T) - S_{\alpha, \tau}^* ((y(T) - y_d) \hat{\delta}_T) \|_L^{1/2}(0, T; [X, Y_{\delta_0}]),
\]

which, together with (56a) and (63a), implies (67).
Then, by the symmetric version of (79),
\[
\|S_{\alpha}^{*}(y(T) - y_d)\delta_{T}) - S_{\alpha,\tau}^{*}(y(T) - y_d)\delta_{T})\|_{L^{1}(0,T; X,V_{\alpha,\theta}, J)}
\leq C_{c_0,\alpha,\theta, J} \tau^{\theta_0\alpha} \epsilon(\alpha, \theta, J) \|y(T) - y_d\|_{Y},
\]
so that (57) implies
\[
\|S_{\alpha}^{*}(y(T) - y_d)\delta_{T}) - S_{\alpha,\tau}^{*}(y(T) - y_d)\delta_{T})\|_{L^{1}(0,T; [X,Y]_{\theta_0})}
\leq C_{c_0,\alpha,\theta, J} \tau^{\theta_0\alpha} \epsilon(\alpha, \theta, J) \|y(T) - y_d\|_{Y}.
\]
Moreover, we obtain from (78) that
\[
\|((S_{\alpha} \circ \mathcal{R}_{\theta_0}) u)(T) - (S_{\alpha,\tau} \circ \mathcal{R}_{\theta_0}) u)(T-))\|_{Y}
\leq C_{c_0,\alpha,\theta, J} \tau^{\theta_0\alpha} \epsilon(\alpha, \theta, J).
\]
Finally, combining (67), (69) and (70) gives (65) and thus concludes the proof. □

### 2.4 Properties of $S_{\alpha,\tau}$

Assume that $0 < \alpha \leq 1$ and $g \in L^{1}(0, T; X^{*})$. Define $\{W_j\}_{j=1}^{J} \subset Y$ by that, for any $1 \leq k \leq J$,
\[
b_1 W_k + \sum_{j=1}^{k-1} (b_{k-j+1} - 2b_{k-j} + b_{k-j-1})W_j - \tau^{\alpha} A W_k = \tau^{\alpha-1} \int_{k-1}^{k} g(t) \, dt \tag{71}
\]
in $X^{*}$, where $b_j := j^{1-\alpha} / \Gamma(2 - \alpha)$ for each $j \in \mathbb{N}$. A straightforward computation yields that (cf. [27, Remark 3])
\[
(S_{\alpha,\tau} g)(t_j) = W_j \quad \forall 1 \leq j \leq J.
\]
Hence, we conclude from (71) and lemma 2.4 that, for any $0 \leq \beta \leq 1$,
\[
S_{\alpha,\tau} \in \mathcal{L}(L^{1}(0, T; [X^{*}, Y]_{1-\beta}), L^{\infty}(0, T; [X, Y]_{\beta})) \tag{72}
\]
and
\[
\lim_{\alpha \to 1} \|S_{\alpha,\tau} - S_{1,\tau}\|_{\mathcal{L}(L^{1}(0, T; [X^{*}, Y]_{1-\beta}), L^{\infty}(0, T; [X, Y]_{\beta}))} = 0. \tag{73}
\]
Symmetrically, for any $0 \leq \beta \leq 1$ we have that
\[
S_{\alpha,\tau}^{*} \in \mathcal{L}(L^{1}(0, T; [X^{*}, Y]_{1-\beta}), L^{\infty}(0, T; [X, Y]_{\beta})) \tag{74}
\]
and
\[
\lim_{\alpha \to 1} \|S_{\alpha,\tau}^{*} - S_{1,\tau}^{*}\|_{\mathcal{L}(L^{1}(0, T; [X^{*}, Y]_{1-\beta}), L^{\infty}(0, T; [X, Y]_{\beta}))} = 0. \tag{75}
\]

**Lemma 2.9** Assume that $0 < \alpha \leq 1$. For any $f \in L^{1}(0, T; X^{*})$ and $g \in L^{1}(0, T; Y)$,
\[
\int_{0}^{T} (g, S_{\alpha,\tau} f)_{Y} \, dt = \int_{0}^{T} (f, S_{\alpha,\tau}^{*} g)_{X} \, dt. \tag{76}
\]
Proof Assume that $0 < \alpha < 1$. By (72) we have $S_{\alpha, \tau} f \in W_T(Y)$ and by (74) we have $S_{\alpha, \tau} g \in W_T(X)$. Then (59) and the density of $X$ in $Y$ yield that

$$
\int_0^T \left( (D_{T-}^\alpha - A^*) S_{\alpha, \tau}^* g, V \right)_Y dt = \int_0^T (g, V)_Y dt \tag{77}
$$

for all $V \in W_T(Y)$. Hence,

$$
\int_0^T (g, S_{\alpha, \tau} f)_Y dt = \int_0^T \left( (D_{T-}^\alpha - A^*) S_{\alpha, \tau}^* g, S_{\alpha, \tau} f \right)_Y dt = \int_0^T \langle (D_{T-}^\alpha - A) S_{\alpha, \tau} f, S_{\alpha, \tau} g \rangle_X dt \quad \text{(by Lemma 2.3 and (14))}
$$

$$
= \int_0^T \langle f, S_{\alpha, \tau}^* g \rangle_X dt \quad \text{(by (58)).}
$$

This proves (76) for $0 < \alpha < 1$. For the proof of (76) with $\alpha = 1$, we refer the reader to [61, Chapter 12]. \hfill \Box

Lemma 2.10 Assume that $0 < \alpha, \theta \leq 1$. For any $g \in L^\infty(0, T; [X^*, Y])$ we have

$$
\max_{1 \leq j \leq J} \| (S_{\alpha} g)(t_j) - (S_{\alpha, \tau} g)(t_j^-) \|_Y \leq C_{0, \theta, \alpha, \tau} \varepsilon(\alpha, \theta, J) \tau^\theta \| g \|_{L^\infty(0, T; [X^*, Y])}, \tag{78}
$$

and for any $v \in Y$ we have

$$
\| S_{\alpha} (v\delta_0) - S_{\alpha, \tau} (v\delta_0) \|_{L^1(0, T; [X, Y])} \leq C_{0, \theta, \alpha, \tau} \varepsilon(\alpha, \theta, J) \tau^{\theta} \| v \|_Y, \tag{79}
$$

where $\varepsilon(\cdot, \cdot, \cdot)$ is defined by (66) and

$$
\tilde{\delta}_0(t) := \begin{cases} 
I_t^{-1} & \text{if } 0 < t < t_1, \\
0 & \text{if } t_1 < t < T.
\end{cases} \tag{80}
$$

The main task of the rest of this subsection is to prove Lemma 2.10. Firstly, we summarize some auxiliary results in [42, section 3]. Assume that $0 < \alpha < 1$. For any $z \in \mathbb{C} \setminus \{0\}$ with $-\pi \leq \Im z \leq \pi$, define

$$
\psi_\alpha(z) := (e^z - 1)^2 \sum_{k=-\infty}^\infty (z + 2k\pi i)^{\alpha - 2}. \tag{81}
$$

There exists $\pi/2 < \omega^* \leq \omega_0$, depending only on $\omega_0$, such that

$$
e^{-z} \psi_\alpha(z) \in \Sigma_{\omega_0} \quad \text{for all } z \in \Sigma_{\omega^*} \text{ with } -\pi \leq \Im z \leq \pi \tag{82}
$$

and that, for any $z \in \Sigma_{\omega^*} \setminus \{0\}$,

$$
|e^{-z} \psi_\alpha(z)| \geq C_{\omega_0} |z|^{\alpha}, \tag{83}
$$

$$
|\psi_\alpha(z) - z^\alpha| \leq C_{\omega_0} |z|^{\alpha + 1}. \tag{84}
$$

Define

$$
\mathcal{E}_\alpha(t) := \tau^{-1} \mathcal{E}_{[t/\tau]}, \quad t > 0, \tag{85}
$$

where $\lfloor \cdot \rfloor$ is the floor function and

$$
\mathcal{E}_j := \frac{1}{2\pi i} \int_{T_{\omega^*}} e^{jz} R(\tau^{\alpha} e^{-z} \psi_\alpha(z), A) dz, \quad j \in \mathbb{N}. \tag{86}
$$
Following the proof of [42, Lemma 3.5], we obtain that, for any $g \in L^1(0,T;Y)$,

$$\langle S_{\alpha}, g \rangle(t_j -) = \int_0^{t_j} \mathcal{E}_{\alpha}(t_j - t)g(t) \, dt \quad \forall 1 \leq j \leq J. \quad (87)$$

Since (22), (82), (83), (85) and (86) imply \(\|\mathcal{E}_{\alpha}\|_{L^\infty(0,T;\mathcal{L}(X^*,Y))} < \infty\), from (72) and the fact that \(L^1(0,T;X^*)\) is dense in \(L^1(0,T;X^*)\) we conclude that (87) holds for all \(g \in L^1(0,T;X^*)\).

**Lemma 2.11** For any \(0 < \alpha < 1, 0 \leq \theta \leq 1\) and \(z \in \mathcal{Y}_{\alpha^*} \setminus \{0\},\)

$$\|e^z R(\tau^{-\alpha}z^\alpha, A) - R(\tau^{-\alpha}e^{-z}\psi_\alpha(z), A)\|_{\mathcal{L}(Y;[X,Y]_\theta)} \leq \frac{C_{C_0,0,M_0}|z|}{1 + (\tau^{-\alpha}|z|^\alpha)^{\theta}}, \quad (88)$$

$$\|e^z R(\tau^{-\alpha}z^\alpha, A) - R(\tau^{-\alpha}e^{-z}\psi_\alpha(z), A)\|_{\mathcal{L}(X^*,[X,Y]_\theta)} \leq \frac{C_{C_0,0,M_0}|z|}{1 + (\tau^{-\alpha}|z|^\alpha)^{\theta}}. \quad (89)$$

**Proof** A straightforward computation gives

$$e^z R(\tau^{-\alpha}z^\alpha, A) - R(\tau^{-\alpha}e^{-z}\psi_\alpha(z), A) = (\tau^{-\alpha}(\psi_\alpha(z) - z^\alpha) + (1 - e^z)A)R(\tau^{-\alpha}z^\alpha, A)R(\tau^{-\alpha}e^{-z}\psi_\alpha(z), A) = \mathbb{I}_1 + \mathbb{I}_2,$$

where

$$\mathbb{I}_1 := \tau^{-\alpha}(\psi_\alpha(z) - z^\alpha)R(\tau^{-\alpha}z^\alpha, A)R(\tau^{-\alpha}e^{-z}\psi_\alpha(z), A),$$

$$\mathbb{I}_2 := (1 - e^z)AR(\tau^{-\alpha}z^\alpha, A)R(\tau^{-\alpha}e^{-z}\psi_\alpha(z), A).$$

We conclude from (20), (82) and (83) that, for any \(0 \leq \beta \leq 1,\)

$$\|R(\tau^{-\alpha}z^\alpha, A)\|_{\mathcal{L}(Y;[X,Y]_\theta)} \leq C_{M_0}(1 + (\tau^{-\alpha}|z|^\alpha)^\beta)^{-1}, \quad (90)$$

$$\|R(\tau^{-\alpha}e^{-z}\psi_\alpha(z), A)\|_{\mathcal{L}(Y;[X,Y]_\theta)} \leq C_{C_0,0,M_0}(1 + (\tau^{-\alpha}|z|^\alpha)^\beta)^{-1}. \quad (91)$$

For \(\mathbb{I}_1\) we have, by (84), (90) and (91),

$$\|\mathbb{I}_1\|_{\mathcal{L}(Y;[X,Y]_\theta)} \leq C_{0,0} \tau^{-\alpha}|z|^\alpha + \|R(\tau^{-\alpha}z^\alpha, A)\|_{\mathcal{L}(Y;[X,Y]_\theta)} \|R(\tau^{-\alpha}e^{-z}\psi_\alpha(z), A)\|_{\mathcal{L}(Y)} \leq \frac{C_{C_0,0,M_0}|z|}{1 + (\tau^{-\alpha}|z|^\alpha)^{\theta}}.$$

Since

$$\|AR(\tau^{-\alpha}z^\alpha, A)R(\tau^{-\alpha}e^{-z}\psi_\alpha(z), A)\|_{\mathcal{L}(Y;[X,Y]_\theta)} = \|\tau^{-\alpha}z^\alpha R(\tau^{-\alpha}z^\alpha, A) - I\|\tau^{-\alpha}e^{-z}\psi_\alpha(z), A)\|_{\mathcal{L}(Y;[X,Y]_\theta)} \leq \|\tau^{-\alpha}z^\alpha R(\tau^{-\alpha}z^\alpha, A)\|_{\mathcal{L}(Y;[X,Y]_\theta)} \|R(\tau^{-\alpha}e^{-z}\psi_\alpha(z), A)\|_{\mathcal{L}(Y)} + \|R(\tau^{-\alpha}e^{-z}\psi_\alpha(z), A)\|_{\mathcal{L}(Y)} \leq \frac{C_{C_0,0,M_0}}{1 + (\tau^{-\alpha}|z|^\alpha)^{\theta}} \quad \text{(by (90) and (91))},$$

we obtain

$$\|\mathbb{I}_2\|_{\mathcal{L}(Y;[X,Y]_\theta)} \leq \frac{C_{C_0,0,M_0}|z|}{1 + (\tau^{-\alpha}|z|^\alpha)^{\theta}}.$$

Combining the above estimates of \(\mathbb{I}_1\) and \(\mathbb{I}_2\) proves (88). Since (89) can be derived analogously, this completes the proof. \(\Box\)
Lemma 2.12 Assume that $0 < \alpha < 1$ and $0 \leq \theta \leq 1$. Then

$$
\max_{1 \leq j \leq J} j^{2-\theta \alpha} \|E_\alpha(t_j) - E_\alpha(t_j^-)\|_{\mathcal{L}(Y, \{X, Y\}_0)} \leq C_{c_0, c_0, \mathcal{M}_0} \tau^{\theta \alpha - 1},
$$

(92)

$$
\max_{1 \leq j \leq J} j^{2-\theta \alpha} \|E_\alpha(t_j) - E_\alpha(t_j^-)\|_{\mathcal{L}([X^*, Y]_0, Y)} \leq C_{c_0, c_0, \mathcal{M}_0} \tau^{\theta \alpha - 1}.
$$

(93)

**Proof** For each $1 \leq j \leq J$, inserting $t = t_j$ into (28) yields

$$
E_\alpha(t_j) = \frac{1}{2\pi i} \int_{\Gamma_{\omega^*}} e^{jz} R(z^\alpha, A) \, dz = \frac{1}{2\pi i} \int_{\Gamma_{\omega^*}} e^{jz} R(\tau^{-\alpha} z^\alpha, A) \, dz,
$$

and so we conclude from (85) and (86) that

$$
E_\alpha(t_j) - E_\alpha(t_j^-) = \Pi_1 + \Pi_2,
$$

where

$$
\Pi_1 := \frac{1}{2\pi i} \int_{\Gamma_{\omega^*} \setminus \gamma_{\omega^*}} e^{jz} R(\tau^{-\alpha} z^\alpha, A) \, dz,
$$

$$
\Pi_2 := \frac{1}{2\pi i} \int_{\gamma_{\omega^*}} e^{(j-1)z}(e^{z} R(\tau^{-\alpha} z^\alpha, A) - R(\tau^{-\alpha} e^{-z} \psi_\alpha(z), A)) \, dz.
$$

A straightforward computation gives

$$
\|\Pi_1\|_{\mathcal{L}(Y, \{X, Y\}_0)} \leq C_{c_0, \mathcal{M}_0} (1 + \tau^{-\alpha} r^\alpha)^{-1} \int_0^\infty \frac{e^{j\cos \omega^* r}}{\sin \omega^*} \, dr
$$

(by (20))

$$
< C_{c_0, \mathcal{M}_0} \tau^{\theta \alpha - 1} \int_0^\infty \frac{e^{j\cos \omega^* r}}{\sin \omega^*} \, dr
$$

$$
< C_{c_0, \mathcal{M}_0} \tau^{\theta \alpha - 1} \int_0^\infty e^{j\cos \omega^* r} \, dr
$$

$$
< C_{c_0, c_0, \mathcal{M}_0} \tau^{\theta \alpha - 1} j^{-1} e^{j\pi \cot \omega^*}
$$

and

$$
\|\Pi_2\|_{\mathcal{L}(Y, \{X, Y\}_0)}
$$

$$
< C_{c_0, c_0, \mathcal{M}_0} \tau^{-1} \int_0^\pi \frac{e^{(j-1)\cos \omega^* r}}{\sin \omega^*} \, dr
$$

(by (88))

$$
< C_{c_0, c_0, \mathcal{M}_0} \tau^{\theta \alpha - 1} \int_0^\pi e^{(j-1)\cos \omega^* r} \, dr
$$

$$
< C_{c_0, c_0, \mathcal{M}_0} \tau^{\theta \alpha - 1} j^{\theta \alpha - 2}.
$$

Together the above estimates of $\Pi_1$ and $\Pi_2$ yields (92). Since (93) can be proved analogously by (22) and (89), this completes the proof. \qed

Lemma 2.13 For any $0 < \alpha < 1$ and $0 < \theta \leq 1$,

$$
\|E_\alpha - E_\alpha\|_{L^1([0, T; \mathcal{L}(Y, \{X, Y\}_0))} \leq C_{c_0, c_0, \mathcal{M}_0} \left(\frac{1}{\theta^\alpha} + \frac{1 - \theta^\alpha}{1 - \theta^\alpha}\right)^{\tau^{\theta \alpha}},
$$

(94)

$$
\|E_\alpha - E_\alpha\|_{L^1([0, T; \mathcal{L}([X^*, Y]_0, Y))} \leq C_{c_0, c_0, \mathcal{M}_0} \left(\frac{1}{\theta^\alpha} + \frac{1 - \theta^\alpha}{1 - \theta^\alpha}\right)^{\tau^{\theta \alpha}}.
$$

(95)
Proof By (31) we have
\[ \| E_\alpha - E_\alpha(t_1) \|_{L^1(0,t_1;\mathcal{L}(Y,[X,Y]))} < C_{c_0,\omega_0,\mathcal{M}0} \tau^{\theta\alpha} / (\theta\alpha), \]  
and a straightforward calculation gives
\[ \sum_{j=2}^{J} \| E_\alpha - E_\alpha(t_j) \|_{L^1(t_{j-1},t_j;\mathcal{L}(Y,[X,Y]))} \]
\[ \leq \tau \| E'_\alpha \|_{L^1(t_1,T;\mathcal{L}(Y,[X,Y]))} \leq C_{c_0,\omega_0,\mathcal{M}0} \tau \int_{t_1}^{T} \tau^{\theta\alpha-2} \text{d}t \quad \text{(by (33))} \]
\[ = C_{c_0,\omega_0,\mathcal{M}0} \tau^{\theta\alpha} (1 - J^{\theta\alpha-1}) / (1 - \theta\alpha). \]  

It follows that
\[ \sum_{j=1}^{J} \| E_\alpha - E_\alpha(t_j) \|_{L^1(t_{j-1},t_j;\mathcal{L}(Y,[X,Y]))} \]
\[ \leq C_{c_0,\omega_0,\mathcal{M}0} \left( \frac{1}{\theta\alpha} + \frac{1 - J^{\theta\alpha-1}}{1 - \theta\alpha} \right) \tau^{\theta\alpha}. \]  

In addition, by (92),
\[ \sum_{j=1}^{J} \tau \| E_\alpha(t_j) - E_\alpha(t_j-) \|_{\mathcal{L}(Y,[X,Y])} \leq C_{c_0,\omega_0,\mathcal{M}0} \tau^{\theta\alpha} \sum_{j=1}^{J} J^{\theta\alpha-2} \]
\[ \leq C_{c_0,\omega_0,\mathcal{M}0} \tau^{\theta\alpha} (1 - J^{\theta\alpha-1}) / (1 - \theta\alpha). \]  

Consequently,
\[ \| E_\alpha - E_\alpha \|_{L^1(0,T;\mathcal{L}(Y,[X,Y]))} \]
\[ \leq \sum_{j=1}^{J} \left( \| E_\alpha - E_\alpha(t_j) \|_{L^1(t_{j-1},t_j;\mathcal{L}(Y,[X,Y]))} \right. \]
\[ + \tau \| E_\alpha(t_j) - E_\alpha(t_j-) \|_{L^1(t_{j-1},t_j;\mathcal{L}(Y,[X,Y]))} \right) \]
\[ \leq C_{c_0,\omega_0,\mathcal{M}0} \left( \frac{1}{\theta\alpha} + \frac{1 - J^{\theta\alpha-1}}{1 - \theta\alpha} \right) \tau^{\theta\alpha}, \]  
which proves (94). Since (95) can be proved analogously by (32), (34) and (93), this completes the proof. \( \square \)

Finally, we are in a position to prove Lemma 2.10 as follows.

**Proof of Lemma 2.10** Let us first consider the case \( 0 < \alpha < 1 \). By virtue of (37) and (87) we have
\[ \max_{1 \leq j \leq J} \| (S_\alpha g)(t_j) - (S_\alpha,\tau g)(t_j-) \|_{\mathcal{Y}} \]
\[ \leq \| E - E_\alpha \|_{L^1(0,T;\mathcal{L}([X^*,Y],[X,Y])))} \| g \|_{L^\infty(0,T;([X^*,Y])))}, \]  
so that (78) follows from (95). Combining (87) and (80) gives that
\[ (S_{\alpha,\tau}(\hat{v}\delta_0))(t_j-) = E_\alpha(t_j-) v, \quad 1 \leq j \leq J, \]
and so (39) implies
\[ \|S_\alpha(v_\delta_0) - S_\alpha,\tau(v_\delta_0)\|_{L^1(0,T;[X,Y]_b)} \leq \|E - E\|_{L^1(0,T;[Y,Y]_b)}\|v\|_{Y}. \]

Therefore, inequality (94) yields (79).

Then let us consider the case \( \alpha = 1 \). Since \([X^*, Y]_\theta\) is continuously embedded into \( X^* \), inserting \( \beta = 1 \) into (73) yields
\[ \lim_{\alpha \to 1^-} \|S_\alpha,\tau - S_1,\tau\|_{L^\infty(0,T;Y)} = 0. \]

Also, Lemma 2.7 implies
\[ \lim_{\alpha \to 1^-} \|S_\alpha - S_1\|_{L^\infty(0,T;Y)} = 0. \]

Hence, passing to the limit \( \alpha \to 1^- \) in (79) proves that (79) holds with \( \alpha = 1 \). A simple modification of the proof of (47) gives
\[ \lim_{\alpha \to 1^-} \|E_\alpha - E_1\|_{L^1(0,T;[Y,Y]_b)} = 0, \]
so that (39) implies
\[ \lim_{\alpha \to 1^-} \|S_\alpha(v_\delta_0) - S_1(v_\delta_0)\|_{L^1(0,T;[X,Y]_b)} = 0. \]

Moreover, (73) yields
\[ \lim_{\alpha \to 1^-} \|S_\alpha,\tau(v_\delta_0) - S_1,\tau(v_\delta_0)\|_{L^1(0,T;[X,Y]_b)} = 0. \]

Therefore, passing to the limit \( \alpha \to 1^- \) in (79) yields that (79) holds with \( \alpha = 1 \). This completes the proof of Lemma 2.10. \( \square \)

3 A Dirichlet Boundary Control Problem

In this section, each linear vector space is real. Assume that \( 0 < \alpha \leq 1, 0 < T < \infty \), and \( \Omega \subset \mathbb{R}^d \) \((d = 2, 3)\) is a bounded convex polygonal domain with a boundary \( \partial \Omega \). Define
\[ U_{ad} := \{ v \in L^2(0,T;L^2(\partial \Omega)) : u_* \leq v(x,t) \leq u^* \text{ for a.e. } (x,t) \in \partial \Omega \times (0,T) \}, \]
where \( u_* \) and \( u^* \) are two given constants. For any \( y \in C(\{0,T\};L^2(\Omega)) \) and \( u \in L^2(0,T;L^2(\partial \Omega)) \), define
\[ J_\alpha(u, y) := \frac{1}{2} \|y(T) - y_d\|^2_{L^2(\Omega)} + \frac{\nu}{2} \|u\|^2_{L^2(0,T;L^2(\partial \Omega))}. \quad (98) \]

where \( y_d \in L^2(\Omega) \) and \( \nu > 0 \) is a regularization parameter. We are concerned with the following optimal Dirichlet boundary control problem:

Minimize \( J_\alpha(u, y) \) subject to \( u \in U_{ad} \) and
\[ \begin{cases} \frac{\partial u}{\partial t} - \Delta y = 0 & \text{in } \Omega \times (0,T) \\ y = u & \text{on } \partial \Omega \times (0,T) \\ y(\cdot, 0) = 0 & \text{in } \Omega. \end{cases} \quad (99) \]
Here, $\partial_0^{\alpha}$, a fractional partial differential operator, is the scalar-valued version of $D_0^\alpha$ with respect to the time variable $t$.

To apply the theory in the previous section to problem (99), we will use the following settings:

$$A := \Delta; \quad A^* := \Delta; \quad X := H^1_0(\Omega) \cap H^2(\Omega); \quad Y := L^2(\Omega); \quad Z := L^2(\partial\Omega);$$

the operator $\mathcal{R}_{\theta_0} : Z \to [X^*, Y]_{\theta_0}, 0 < \theta_0 < 1/4$, is defined by that

$$\langle \mathcal{R}_{\theta_0} w, v \rangle_{(X, Y)_{\theta_0}} := -\langle w, \partial_n v \rangle_{\partial\Omega}$$

for all $w \in Z$ and $v \in [X, Y]_{\theta_0}$. By the well-known trace inequality (cf. [16, Theorem 1.5.1.2])

$$\|v\|_Z \leq C_{\epsilon, \Omega} \|v\|_{[X, Y]_{3/4-\epsilon}} \text{ for all } v \in [X, Y]_{3/4-\epsilon} \text{ with } 0 < \epsilon \leq 3/4,$$

we readily conclude that, for any $0 < \theta_0 < 1/4$,

$$\|\mathcal{R}_{\theta_0}\|_{\mathcal{L}(Z; [X^*, Y]_{\theta_0})} \leq C_{\theta_0, \Omega}. \quad (102)$$

Let $u$ and $y$ be defined in Theorem 2.1, and let $U$ and $Y$ be defined in Theorem 2.2. A straightforward calculation gives, by (65), that

$$\|(y - Y)(T - \cdot)\|_Y + \sqrt{v} \|u - U\|_{L^2(0, T; Z)} \leq C \tau^{\theta_0 \alpha/2}, \quad (103)$$

where $0 < \theta_0 < 1/4$ is arbitrary and $C$ is a positive constant independent of $v$ and $\tau$.

**Remark 3.1** For the case $\alpha = 1$, [33–37] used

$$-A \int_0^t E_1(t-s)(-A)^{-1}\mathcal{R}_{\theta_0}u(s) \, ds, \quad 0 \leq t \leq T,$$

as the solution of the state equation of problem (99) with $u \in L^2(0, T; Z)$. It is evident that the above solution is exactly $S_1\mathcal{R}_{\theta_0}u$. We note that the numerical results in [33–37] all require that $\tau \leq ch^2$, where $h$ is the spatial mesh size. Our error estimate (103) indicates that the restriction $\tau \leq ch^2$ is unnecessary.

It remains to prove that $S_\alpha \mathcal{R}_{\theta_0}u$ is a sensible solution to the state equation of problem (99) for each $u \in L^2(0, T; Z)$. To this end, we first introduce the very weak solution concept of the state equation, following the idea in [44, Chapter 6]. For any $0 < \alpha < 1$ and $g \in L^2(0, T; Y)$, there exists a unique $w \in H^{\alpha}(0, T; Y) \cap L^2(0, T; X)$ such that (cf. [40,47])

$$(D_{T-}^{\alpha} - A)w = g$$

and

$$\|w\|_{H^{\alpha}(0, T; Y)} + \|w\|_{L^2(0, T; X)} \leq C_\alpha \|g\|_{L^2(0, T; L^2(\Omega))}.$$

For $\alpha = 1$ the above results are standard (cf. [13]). Hence, we define the very weak solution $y \in L^2(0, T; Y)$ to the state equation of problem (99) with $u \in L^2(0, T; Z)$ by

$$\int_0^T (y, (D_{T-}^{\alpha} - A)\varphi) \, dt = -\langle u, \partial_n \varphi \rangle_{\partial\Omega \times (0, T)}$$

for all $\varphi \in H^\alpha(0, T; Y) \cap L^2(0, T; X)$.

Then we will prove that, for any $0 < \theta_0 < 1/4$ and $u \in L^2(0, T; Z)$, $S_\alpha \mathcal{R}_{\theta_0}u$ is identical to the very weak solution to the state equation of problem (99), and hence the application of the theory in the previous section to problem (99) is reasonable.
Lemma 3.1 Assume that $0 < \alpha \leq 1$ and $0 < \theta_0 < 1/4$. Then $S_\alpha \mathcal{R}_{\theta_0} u$ is the very weak solution to the state equation of problem (99) for each $u \in L^2(0, T; Z)$.

Proof We only prove the case $0 < \alpha < 1$, the proof of the case $\alpha = 1$ being easier. Assume first that $u \in C([0, T]; Z)$. By Lemma 2.8 we have

$$(D_{0+}^\alpha - A)S_\alpha \mathcal{R}_{\theta_0} u = \mathcal{R}_{\theta_0} u$$

and

$$D_{0+}^\alpha S_\alpha \mathcal{R}_{\theta_0} u, \mathcal{A}S_\alpha \mathcal{R}_{\theta_0} u \in C([0, T]; X^*). \tag{104}$$

Hence, for any $\varphi \in 0H^\alpha(0, T; Y) \cap L^2(0, T; X)$ we have

$$\int_0^T \langle (D_{0+}^\alpha - A)S_\alpha \mathcal{R}_{\theta_0} u, \varphi \rangle_X \, dt = \int_0^T \langle \mathcal{R}_{\theta_0} u, \varphi \rangle_X \, dt. \tag{105}$$

Because (104) implies $D_{0+}^\alpha S_\alpha \mathcal{R}_{\theta_0} u \in L^2(0, T; X^*)$, by [47, Lemma 3.4] we have

$$S_\alpha \mathcal{R}_{\theta_0} u \in 0H^\alpha(0, T; X^*).$$

Also, (104) and (26) imply

$$S_\alpha \mathcal{R}_{\theta_0} u \in L^2(0, T; Y).$$

Consequently, by Lemma 2.3 we have

$$\int_0^T \langle D_{0+}^\alpha S_\alpha \mathcal{R}_{\theta_0} u, \varphi \rangle_X \, dt = \int_0^T \langle S_\alpha \mathcal{R}_{\theta_0} u, \mathcal{A}^\ast \varphi \rangle_Y \, dt, \tag{106}$$

and it is evident by (14) that

$$\int_0^T \langle -\mathcal{A}S_\alpha \mathcal{R}_{\theta_0} u, \varphi \rangle_X \, dt = \int_0^T \langle S_\alpha \mathcal{R}_{\theta_0} u, -\mathcal{A}^\ast \varphi \rangle_Y \, dt.$$

Combining (105) and the above two equations gives

$$\int_0^T \langle S_\alpha \mathcal{R}_{\theta_0} u, (D_T^{\alpha-} - \mathcal{A}^\ast) \varphi \rangle_Y \, dt = \int_0^T \langle \mathcal{R}_{\theta_0} u, \varphi \rangle_X \, dt.$$

The arbitrariness of $\varphi \in 0H^\alpha(0, T; Y) \cap L^2(0, T; X)$ proves that $S_\alpha \mathcal{R}_{\theta_0} u$ is indeed the very weak solution. The general case $u \in L^2(0, T; Z)$ then follows from a standard density argument by

$$S_\alpha \mathcal{R}_{\theta_0} \in \mathcal{L}(L^2(0, T; Z), L^2(0, T; Y)),$$

which is a direct consequence of (43) and the fact $\mathcal{R}_{\theta_0} \in \mathcal{L}(Z, [X^*, Y]_{\theta_0})$. This completes the proof. □

Remark 3.2 It is well-known that there exists an orthonormal basis $\{\phi_n : n \in \mathbb{N}\} \subset H^1_0(\Omega) \cap H^2(\Omega)$ of $L^2(\Omega)$ such that

$$-\Delta \phi_n = \lambda_n \phi_n,$$

where $\{\lambda_n : n \in \mathbb{N}\}$ is a nondecreasing sequence of positive numbers with limit $+\infty$. Let $\dot{H}^2(\Omega)$ be the space $H^1_0(\Omega) \cap H^2(\Omega)$ endowed with the norm

$$\|v\|_{\dot{H}^2(\Omega)} := \|\Delta v\|_{L^2(\Omega)} \quad \forall v \in \dot{H}^2(\Omega).$$
Let $\mathcal{H}^{-2}(\Omega)$ be the dual space of $\mathcal{H}^2(\Omega)$. Assume that 

$$f \in _0^1H^\gamma(0, T; \mathcal{H}^{-2}(\Omega)) \text{ and } g \in _0^1H^\gamma(0, T; L^2(\Omega)) \cap L^2(0, T; \mathcal{H}^2(\Omega)),$$

where $0 < \gamma < 1$. Then, by definition, $f$ admits a decomposition

$$f = \sum_{n=0}^{\infty} f_n \phi_n \text{ with } \sum_{n=0}^{\infty} \lambda_n^{-2} \|f_n\|_{0H^\gamma(0, T)}^2 < \infty,$$

and $g$ admits a decomposition

$$g = \sum_{n=0}^{\infty} g_n \phi_n \text{ with } \sum_{n=0}^{\infty} \left(\|g_n\|_{0H^\gamma(0, T)}^2 + \lambda_n^2 \|g_n\|_{L^2(0, T)}^2\right) < \infty.$$

By Lemma 2.2 we also have

$$\sum_{n=0}^{\infty} \lambda_n^{-2} \|D_{0+}^\gamma f_n\|_{L^2(0, T)}^2 < \infty, \quad \sum_{n=0}^{\infty} \|D_{T-}^\gamma g_n\|_{L^2(0, T)}^2 < \infty.$$

Since Lemma 2.3 implies, for any $n \in \mathbb{N}$,

$$\int_0^T D_{0+}^\gamma f_n \times g_n \, dt = \int_0^T f_n \times D_{T-}^\gamma g_n \, dt,$$

we conclude from (107), (108) and (109) that

$$\int_0^T \sum_{n=0}^{\infty} D_{0+}^\gamma f_n \times g_n \, dt = \int_0^T \sum_{n=0}^{\infty} f_n \times D_{T-}^\gamma g_n \, dt.$$

It follows that

$$\int_0^T (D_{0+}^\gamma f, g)_{\mathcal{H}^2(\Omega)} \, dt = \int_0^T (f, D_{T-}^\gamma g)_{L^2(\Omega)} \, dt,$$

where $(\cdot, \cdot)_{L^2(\Omega)}$ is the inner product in $L^2(\Omega)$. This proves equality (106).

### 4 Numerical Results

This section performs three numerical experiments in two-dimensional space to verify the theoretical results. We will use the following settings: $\Omega := (0, 1) \times (0, 1); T = 0.1; X, Y, Z, R_{\theta_0}$ and $U_{ad}$ are defined as in Sect. 3. Here we recall that $0 < \theta_0 < 1/4$ is arbitrary.

**Experiment 1** Define

$$g(t) := \begin{cases} 1 & \text{if } 0 < t < 2/3T, \\ 3 & \text{if } 2/3T < t < T, \end{cases}$$

$$v(x, y) := \begin{cases} y^{-1/2} & \text{if } (x, y) \in \{(0, y) : 0 < y < 1\}, \\ 0 & \text{if } (x, y) \in \partial \Omega \setminus \{(0, y) : 0 < y < 1\}. \end{cases}$$

To approximate $S_{\alpha}R_{\theta_0}(gv)$, we use discretization (58)($0 < \alpha < 1$) or (60)($\alpha = 1$) in time and use the continuous piecewise linear element method in space. Let $U^M$ be the corresponding numerical approximation with time step $\tau = T/2^M$ and spatial mesh size $h = 2^{-9}$. As $gv$ is
Table 1 \( \| \cdot \| \) means the norm \( \| \cdot \|_{L^\infty(0,T;L^2(\Omega))} \)

| \( M \) | \( U^M - U^{14} \) | \( \ln \frac{\| U^{M-2} - U^{14} \|}{\| U^M - U^{14} \|} / (2 \ln 2) \) | \( U^M - U^{14} \) | \( \ln \frac{\| U^{M-2} - U^{14} \|}{\| U^M - U^{14} \|} / (2 \ln 2) \) |
|---|---|---|---|---|
| 4 | 6.93e–1 | – | 4.34e–1 | – |
| 6 | 6.21e–1 | 0.08 | 3.24e–1 | 0.21 |
| 8 | 5.47e–1 | 0.09 | 2.27e–1 | 0.26 |
| 10 | 4.77e–1 | 0.10 | 1.62e–1 | 0.24 |

Table 2 \( \| \cdot \| \) means the norm \( \| \cdot \|_{L^1(0,T;H^1_0(\Omega))} \)

| \( M \) | \( U^M - U^{13} \) | \( \ln \frac{\| U^{M-1} - U^{13} \|}{\| U^M - U^{13} \|} / \ln 2 \) | \( U^M - U^{13} \) | \( \ln \frac{\| U^{M-1} - U^{13} \|}{\| U^M - U^{13} \|} / \ln 2 \) |
|---|---|---|---|---|
| 4 | 1.71e–0 | – | 4.24e–1 | – |
| 5 | 1.46e–0 | 0.23 | 3.01e–1 | 0.49 |
| 6 | 1.23e–0 | 0.25 | 2.10e–1 | 0.52 |
| 7 | 1.02e–0 | 0.27 | 1.45e–1 | 0.54 |

Table 3 \( \| \cdot \| \) means the norm \( \| \cdot \|_{L^2(0,T;L^2(\partial \Omega))} \)

| \( M \) | \( U^M - U^{13} \) | \( \ln \frac{\| U^{M-1} - U^{13} \|}{\| U^M - U^{13} \|} / \ln 2 \) | \( U^M - U^{13} \) | \( \ln \frac{\| U^{M-1} - U^{13} \|}{\| U^M - U^{13} \|} / \ln 2 \) |
|---|---|---|---|---|
| 4 | 3.90e–2 | – | 1.76e–2 | – |
| 5 | 2.27e–2 | 0.78 | 1.03e–2 | 0.78 |
| 6 | 1.46e–2 | 0.63 | 5.97e–3 | 0.78 |
| 7 | 1.12e–2 | 0.38 | 3.45e–3 | 0.79 |

close to possessing the \( L^\infty(0,T;L^2(\partial \Omega)) \)-regularity, estimates (78) and (102) predict that \( \| U^M - U^{13} \|_{L^\infty(0,T;L^2(\Omega))} \) is close to \( O(\tau^{0.125}) \) for \( \alpha = 0.5 \) and close to \( O(\tau^{0.25}) \) for \( \alpha = 1 \), and this is confirmed by the numerical results in Table 1.

**Experiment 2** Define

\[
    v(x, y) := x^{-1/2} \quad \text{for all } (x, y) \in \Omega.
\]

To approximate \( S_\alpha(v\delta_0) \), we use discretization \( 58)(0 < \alpha < 1) \) or \( 60)(\alpha = 1) \) in time and use the continuous piecewise linear element method in space. Let \( U^M \) be the corresponding numerical approximation with time step \( \tau = T/2^M \) and spatial mesh size \( h = 2^{-9} \). Table 2 illustrates that \( \| U^M - U^{13} \|_{L^1(0,T;H^1_0(\Omega))} \) is close to \( O(\tau^{0.25}) \) for \( \alpha = 0.5 \) and close to \( O(\tau^{0.5}) \) for \( \alpha = 1 \), which agrees well with estimate (79). Here we note that \( v \) is close to possessing the \( L^2(\Omega) \)-regularity.
Experiment 3 Let $\nu := 1$, $u_* := 0$, $u^* := 1$ and $y_d(x, y) := 1$ for all $(x, y) \in \Omega$.

To approximate problem (99), we will use the temporal discretization in Sect. 2.3 and the continuous piecewise linear element method in space to discretize the state equation in time and space, respectively; see [18] for the implementation details. The numerical solution is calculated by [22, Algorithm 3.2]. Let $U^M$ be the corresponding numerical solution with time step $\tau = T/2^M$ and spatial mesh size $h = 2^{-8}$. Error estimate (103) implies that $\|U^M - U^{13}\|_{L^2(0, T; L^2(\partial\Omega))}$ is close to $O(\tau^{0.0625})$ for $\alpha = 0.5$ and close to $O(\tau^{0.125})$ for $\alpha = 1$; however, the numerical results in Table 3 demonstrate significantly higher convergence rates.

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