Stochastic perturbation of the 
Lighthill–Whitham–Richards model via the 
method of stochastic characteristics

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Abstract
In this paper we apply the method of stochastic characteristics to a 
Lighthill–Whitham–Richards model. The stochastic perturbation can be seen as errors 
in measurement of the traffic density. For concrete examples we solve the equation 
perturbed by a standard Brownian motion and the geometric Brownian motion 
without drift.

Keywords: Method of stochastic characteristics; Lighthill–Whitham–Richards model; 
Explicit solutions

1 Introduction
Many traffic flow models go back to scalar conservation laws of generally non-linear type, 
i.e.

\[ u_t + f(u)_x = 0, \tag{1} \]

see e.g. [4]. There are many different results, making the area of scalar conservation laws 
to an area of very active research in mathematics, see e.g. [3, 5, 15, 16, 21, 23], to name just 
a few of those.

The function \( u \) in (1) describes the density of vehicles on a road and thus has values on 
the compact set \([0,1]\). A conservation law is derived under the assumption, that the time 
propagation of a mass on a certain interval is only affected by the flux at the boundary of 
the interval. Hence one often chooses \( f(u) = u \cdot v(u) \), where \( v \) is the Eulerian velocity of the 
traffic. There are many different models for traffic flow discussed in the literature where 
the most famous ones are Aw–Rascle–Zhang model [1, 2, 24] and Lighthill–Whitham– 
Richards model, which uses \( f(u) = u \cdot (1 – u) \), i.e. a velocity depending linearly on the 
density. The scalar conservation law (1) now reads

\[ u_t + (1 – 2u) \cdot u_x = 0. \tag{2} \]

The flux function in the Lighthill–Whitham–Richards model is in a relatively good agree-
ment with traffic measurements, see [19]. The main problem is that measurements show...
that data points are quite accurate for low and high densities but are noisy around the maximum point. The flux function is hence rather given by

\[ f(u) = u \cdot (1 - u) + H(u) \circ \frac{dM_t}{dt}, \]

where \( H \) is a function vanishing at \( u = 0 \) and \( u = 1 \) and \( M_t \) is a suitable nice enough stochastic process. Plugging this in the above conservation law (1) yields

\[ u_t + f(u)_x = u_t + (1 - 2u) \cdot u_x + H(u)_x \circ \frac{dM_t}{dt} = 0. \] (3)

The random perturbation is chosen in such a way that \( H(u)_x \) controls the dependence of the noise with respect to the density in space. The stochastic process itself give a temporal noise, in order to represent fluctuation in time. Of course with the use of a random field the model may be more general. However, the method of characteristics cannot be applied in the same way as in this article.

The choice of the Stratonovich integral is due to the existence of the chain rule in the same form as in the deterministic case. For Itô integral a correction term is needed, which leads in some cases to non-existing solutions.

General results for such stochastic conservation laws can be found in e.g. [3, 6–8, 10–14, 20]. The results base on Itô- and Stratonovich type noise in finite and infinite dimensional spaced as well as rough noise and going much further than the model studied in this short article. We hence can of course not give a comprehensive list and refer to the above works and their references.

Industrial application and especially the study of traffic flows are nowadays interested in shocks and nonlinearities. Such shocks occure if the corresponding characteristic intersect, see [4]. In the method of stochastic characteristics, which we treat in this article, this is exactly the same. Our focus in this paper, however, is to show how stochastically perturbed conservation laws behave until the time of shocks. The investigation of shocks would be an interesting study for the future.

In this manuscript, we use the method of stochastic characteristics to solve such equations for the stochastically perturbed Lighthill–Whitham–Richards model explicitly for different cases, where the driving process is given by a Brownian motion or a geometrical Brownian motion. This work can be seen as the starting point to the investigation of different stochastically perturbed hyperbolic equations in direct applications.

The main advantage of the applied method is that the solutions are given in an explicit expression. This makes tedious and highly involved numerical simulation unnecessary. The motivation of the method is given by traffic flow equations, however, it can be applied to other stochastically perturbed conservation laws.

2 Prelimiaries

We look at some examples for the Lighthill–Whitham–Richards model for different initial conditions. Due to the underlying model the initial condition describes the density of our traffic problem at time \( t = 0 \) and at position \( x \in [0, 1] \).
Lemma 2.1 Consider the following partial differential equation on \([0,1]\)

\[
\begin{align*}
 du &= -(1-2u) \cdot u_x \, dt, \\
 u(x,0) &= g(x),
\end{align*}
\]

where \(g(x)\) is a smooth function. Let \((\xi_t, \eta_t)\) be the solutions to the so called characteristic equations given by

\[
\begin{align*}
 d\xi_t &= (1-2\eta_t) \, dt, \\
 \xi_0(x) &= x, \\
 d\eta_t &= 0 \, dt, \\
 \eta_0(x) &= g(x).
\end{align*}
\]

Hence we obtain

\[
\begin{align*}
 \eta_t(x) &= g(x), \\
 \xi_t(x) &= x + \int_0^t 1 - 2g(s) \, ds = x + t - 2g(x)t. 
\end{align*}
\]

Then the solution to (4) is given by \(u(x,t) = g(\xi_t^{-1}(x))\), where \(\xi_t^{-1}\) denotes the inverse function of \(\xi_t\).

Proof The proof is an direct consequence of Chap. 3 in [9], in particular [9, § 3.2, Theorem 2] with an application of the inverse mapping theorem [18, Chapter XIV, Theorem 1.2].

Example 2.2 Consider the following partial differential equation (PDE) for the Lighthill–Whitham–Richards model on \([0,1]\)

\[
 du = -(1-2u) \cdot u_x \, dt, \quad u(x,0) = g(x).
\]

Due to Lemma 2.1 we obtain with the initial condition \(g(x) = 1 - x\):

\[
\begin{align*}
 \xi_t(x) &= x - t + 2xt, \\
 \eta_t(x) &= 1 - x, \\
 \text{hence} \quad \xi_t^{-1}(x) &= \frac{x + t}{1 + 2t}. 
\end{align*}
\]

Thus the solution of the above PDE (6) is given by

\[
 u(x,t) = \frac{1-x+t}{1+2t}. 
\]

If we change the initial condition to be \(g(x) = 1 - x^2\), we obtain

\[
\begin{align*}
 \xi_t(x) &= x - t + 2x^2t, \\
 \eta_t(x) &= 1 - x^2, \\
 \text{and hence} \quad \xi_t^{-1}(x) &= \frac{\sqrt{1 + 8t^2 + 8tx} - 1}{4t}.
\end{align*}
\]

Thus the corresponding solution is equal to

\[
 u(x,t) = \begin{cases} 
 1 - \frac{(\sqrt{1 + 8t^2 + 8tx} - 1)^2}{16t^2}, & \text{for } t \neq 0, \\
 1 - x^2, & \text{for } t = 0.
\end{cases}
\]

One can easily verify that (10) solves indeed the PDE (6).
The main advantage of this method is the precise expression of a solution to a PDE – provided that the corresponding initial condition \( g(x) \) and coefficient functions are explicitly given. Due to this fact and for a better comparison between the deterministic and stochastic case we present a collection of solutions in Appendix A.

Along the characteristics the solution remains constant. In the case of the traffic problem and under the considered initial conditions the characteristics never cross each other which means that no shocks appear and hence the solutions are global. As written in the introduction we will study the perturbed case (3) for \( H(u) \neq 0 \) and for \( M_t \) to be the standard Brownian motion (Bm) as well as the so called geometric Brownian motion defined in the following way.

**Definition 2.3** A stochastic process \( S_t, t \geq 0 \), is said to be a **geometric Brownian motion** (gBm), if it satisfies

\[
dS_t = \mu S_t \, dt + \sigma S_t \, dW_t = S_t \, dW_t,
\]

where \( W_t \) is a Brownian motion. Hence the geometric Brownian motion without drift is given by

\[
S_t := \exp \left( -\frac{t}{2} + W_t \right).
\]

**Definition 2.4** Let \( W_t \) be a standard one-dimensional Brownian motion on a complete separable probability space \((\Omega, \mathcal{F}, P, \mathcal{F}_t)\), with right-continuous filtration \((\mathcal{F}_t)_{t \geq 0}\). Then we define for any smooth function \( H(x, u, p, t) \), \( x, u \in [0,1] \), \( p \) bounded, \( t \in [0,T] \), for \( 0 < T < \infty \) the following integral expression

\[
\int_0^t F(x, u, p, s) \, ds := \begin{cases} 
\int_0^t (1 - 2u)p \, ds + \int_0^t H(x, u, p, s) \, dW_s, & \text{for Bm,} \\
\int_0^t (1 - 2u)p \, ds + \int_0^t H(x, u, p, s) \, dS_s, & \text{for gBm.}
\end{cases}
\]

The integrals are given in the sense of Stratonovich.

Based on these definitions we are able to apply the so called method of stochastic characteristics to the PDEs as (4) but perturbed by Brownian motion respectively geometric Brownian motion. Since we consider partial differential equations with perturbations by (geometric) Brownian motions we get an \( \omega \) - dependence in the solutions. The idea of the method is nearly the same as before: now we fix \( \omega \in \Omega \) and transform a stochastic partial differential equation (SPDE) into a system of stochastic differential equations (SDEs), solve it and determine the solution to the original SPDE by using stopping times. Hence the precisely determined solutions are given for almost all \( \omega \) and all space and time variables \((x, t)\) up to a certain stopping time denoted by \( \sigma(x) \). In contrast to the deterministic case we will introduce in the following the method of stochastic characteristics in a more detailed way. Based on Definition 2.4 a perturbed Lighthill–Whitham–Richards model (3) is equivalent to the Cauchy problem

\[
\begin{align*}
\frac{du}{dt} &= F(x, u, u_t, \sigma(t)), \\
u &= g \quad \text{on } \Gamma := [x \in [0,1] \times [0,T]|x = (x_1, t), t = 0].
\end{align*}
\]
Therefore the solution to equation (12) is denoted by $u(x, t, \omega)$, but for short notation we only write $u(x, t)$. Suppose $u$ is a solution to (12) and at least one-times continuously differentiable with respect to space and time for fixed $\omega \in \Omega$. Furthermore, we assume that there exists a curve $\xi(r)$ which maps the point $r \in \Gamma$ to a point of a neighborhood in $\Gamma'$ at time $s$. Additionally, we assume $\xi_0(x) = x$ for all $x \in [0, 1]$ as the initial condition. Due to these assumptions we consider and define the following functions, now for fixed $\omega, r \in [0, 1]$ and $s \in [0, T]$:

\[
\begin{align*}
(\xi_t, r, \omega,s), \\
\eta_t(r, \omega) &:= u(\xi_t(r, \omega), s), \\
\chi_t(r, \omega) &:= u_0(\xi_t(r, \omega), s).
\end{align*}
\]

(13)

In the next step we combine (12) with equations (13) and obtain

\[
\frac{d}{dt} \left[ u(\xi_t(r), t) - u(\xi_0(r), 0) - \int_0^t F(\xi_s(r), \eta_s(r), \chi_s(r), \omega) \, ds \right] = 0.
\]

By similar calculations as in [9, § 3.2.1, equation (11)] we get

\[
\begin{align*}
d\xi_t &= -F_{\chi_t}(\xi_t, \eta_t, \chi_t, \omega) \, dt, \\
d\eta_t &= F(\xi_t, \eta_t, \chi_t, \omega) - \chi_t \cdot F_{\chi_t}(\xi_t, \eta_t, \chi_t, \omega) \, dt, \\
d\chi_t &= F_{\chi_t}(\xi_t, \eta_t, \chi_t, \omega) \, dt + F_{\eta_t}(\xi_t, \eta_t, \chi_t, \omega) \, d\xi_t.
\end{align*}
\]

The above stochastic differential equations (SCE) are called *stochastic characteristic equations*, for a more detailed description and proofs, see also [17]. Given a point $x \in [0, 1]$ and assuming that there exist unique solutions to (SCE) starting from $x$ at time $t = 0$, these solutions solve the corresponding integral equation with initial function $g$:

\[
\begin{align*}
\xi_t(x) &= x - \int_0^t F_{\chi_s}(\xi_s(x), \eta_s(x), \chi_s(x), \omega) \, ds, \\
\eta_t(x) &= g(x) - \int_0^t \chi_s \cdot F_{\chi_s}(\xi_s(x), \eta_s(x), \chi_s(x), \omega) \, ds + \int_0^t F(\xi_s(x), \eta_s(x), \chi_s(x), \omega) \, ds, \\
\chi_t(x) &= g_s(x) + \int_0^t F_{\chi_s}(\xi_s(x), \eta_s(x), \chi_s(x), \omega) \, ds + \int_0^t F_{\eta_s}(\xi_s(x), \eta_s(x), \chi_s(x), \omega) \, ds \chi_s.
\end{align*}
\]

Let us assume that the solutions $(\xi_t(x), \eta_t(x), \chi_t(x))$ exist up to a stopping time $T(x)$. As mentioned above we have to work on different stopping times based on the following definition of an explosion time.

**Definition 2.5** Let $X_t, t \in [0, \tau)$, be a local process. The stopping time $\tau$ is called *terminal time* of the local process $X_t$. If

\[
\lim_{t \to \tau} |X_t| = \infty,
\]

then $\tau$ is called *explosion time*. 
In the case of our solutions \((\xi_t, \eta_t, \chi_t)\) this yields to the following definitions of stopping times.

**Definition 2.6** Let \(T(x)\) be the infimum of all explosion times of the solutions \((\xi_t, \eta_t, \chi_t)\).
Then we define for all \(x, y \in [0,1]\) the stopping times

\[
\tau_{\text{inv}}(x) := \inf \{ t > 0 | \det D\xi_t(x) = 0 \},
\]

\[
\tau(x) := \tau_{\text{inv}}(x) \wedge T(x),
\]

\[
\sigma(y) := \inf \{ t > 0 | y \notin \xi_t \{ x \in [0,1] | \tau(x) > t \} \},
\]

where \(D\xi_t\) denotes the Jacobian matrix.

Now let the inverse process \(\xi_t^{-1}\) of \(\xi_t\) exist up to some stopping time \(\sigma(x)\). Then we define for almost all \(\omega\) and for all \((x, t)\) with \(t < \sigma(x, \omega)\) the solution

\[
u(x, t) := \eta_t(\xi_t^{-1}(x)). \tag{14}\]

Detailed derivations and introductions can be found in [22, Chap. 3]. Now we are able to solve different SPDEs concerning the Lighthill–Whitham–Richards model by using the method of stochastic characteristics.

### 3 Application & representation

In particular industrial applications the equations are much more involved than in the examples this played in here. Of course to use our method the characteristics have to computed explicitly. After that standard ODE or SDE methods can be used, if needed.

Based on the flow rate function \(H(u)\) and the continuity equation the most natural choice of the drift term is \(H(u) = u - u^2\). In a first step we perturb the Lighthill–Whitham–Richards model by a standard Brownian motion. Hence we consider

\[
\begin{cases}
\text{d}u = -(1 - 2u) \cdot u_x \, \text{d}t - (1 - 2u) \cdot u_x \circ \text{d}W_t, \\
u(x, 0) = g(x).
\end{cases} \tag{15}
\]

By using direct computation one can show that the corresponding stochastic characteristic equations are given for almost all \(\omega\) and all \((x, t)\) up to a stopping time \(\sigma(x)\) by

\[
\begin{cases}
\text{d}\xi_t = (1 - 2\eta_t) \, \text{d}t + (1 - 2\eta_t) \circ \text{d}W_t, \\
\xi_0(x) = x, \\
\end{cases}
\quad \text{and} \quad
\begin{cases}
\text{d}\eta_t = 0 \, \text{d}t + 0 \circ \text{d}W_t, \\
\eta_0(x) = g(x).
\end{cases} \tag{16}
\]

Due to the linearity in the space derivative \(u_x\) the solution \(\eta_t(x) = g(x)\) is always valid. Therefore we receive the solution

\[
\xi_t(x) = x + (1 - 2g(x))(t + W_t).
\]

At this point we compare the characteristics in the deterministic case with the corresponding perturbed one, see Fig. 1. As the initial condition we use here \(g(x) = 1 - x\).
In this case of \( g(x) = 1 - x \) there exists obviously a process \( \xi_t^{-1} \), such that the inverse property is fulfilled for almost all \( \omega \) and all \((x, t)\) up to stopping time \( \sigma(x) \), i.e.

\[
\xi_t^{-1}(x) = \frac{x + t + W_t}{1 + 2t + 2W_t}.
\]

The solution to the considered SPDE (15) is given for almost all \( \omega \) and all \((x, t)\) up to stopping time \( \sigma(x) \) by

\[
u(x, t) = \frac{1 - x + t + W_t}{1 + 2t + 2W_t},
\]

which looks similar to the deterministic solution (8). Due to the explicit expression of the solution we are able to visualize a sample path easily, see Fig. 2. As introduced in Definition 2.6 the stopping time can be determined explicitly in this example by

\[
\sigma(x) = \inf\left\{ t > 0 \left| \frac{x + t + W_t}{1 + 2t + 2W_t} \notin [0, 1] \right\} \right\} \wedge \inf\{ t > 0 | 1 + 2t + 2W_t = 0 \}
\]

\[
= \inf\left\{ t > 0 \left| \frac{x + t + W_t}{1 + 2t + 2W_t} \notin [0, 1] \right\} \right\} \wedge \infty.
\]
The perturbation by a geometric Brownian motion as given in Definition 2.3 is in this case straightforward. According to Definition 2.4 we practically can replace the Brownian motion \( W_t \) by \( \exp(-t/2 + W_t) \) – 1. Let us consider

\[
\begin{cases}
    du = -(1 - 2u) \cdot u_x \ dt - (1 - 2u) \cdot u_x \circ d[\exp(-t/2 + W_t)], \\
    u(x, 0) = 1 - x^2.
\end{cases}

(18)
\]

By an application of the method of stochastic characteristics we finally get the precise solution for almost all \( \omega \) and \( (x, t) \) up to a stopping time \( \sigma(x) \) by

\[
u(x, t) = \begin{cases}
  1 - \frac{\sqrt{8(t + e^{(-t/2 + W_t)} - 1)}(t + x + e^{(-t/2 + W_t)} - 1) + 1}{4(e^{-t/2 + W_t} + t - 1)} \cdot \frac{1}{e^{(-t/2 + W_t)}} \in [0, 1] \ , & \text{if } t \neq 0, \\
  1 - x^2, & \text{if } t = 0,
\end{cases}
\]

(19)

where we can use the classical l'Hospital argument. The corresponding stopping time is equal to

\[
\sigma(x) = \inf \left\{ t > 0 \left| \frac{\sqrt{8(e^{(-t/2 + W_t)} + t - 1)(t + x + e^{(-t/2 + W_t)} - 1) + 1 - 1}}{4(e^{(-t/2 + W_t)} + t - 1)} \notin [0, 1] \right. \right\}
\]

\[
\land \inf \left\{ t > 0 \left| \sqrt{8(t + e^{(-t/2 + W_t)} - 1)(t + x + e^{(-t/2 + W_t)} - 1) + 1} = 0 \right. \right\}
\]

\[
\inf \left\{ t > 0 \left| \frac{\sqrt{8(e^{(-t/2 + W_t)} + t - 1)(t + x + e^{(-t/2 + W_t)} - 1) + 1 - 1}}{4(e^{(-t/2 + W_t)} + t - 1)} \notin [0, 1] \right. \right\} \land \infty.
\]

In Fig. 3 we display one sample path with initial condition \( 1 - x^2 \) perturbed by the term \( - (1 - 2u) \cdot u_x \circ d[\exp(-t/2 + W_t)] \).

Due to this approach we have to verify that the equations (17) as well as (19) really solve the underlying problems. For the sake of simplicity these necessary but lengthy calculation can be found in the Appendix C for equation (17).

![Figure 3](image-url) Sample path of the Lighthill–Whitham–Richards model with initial condition \( 1 - x^2 \) perturbed by the term \( - (1 - 2u) \cdot u_x \circ d[\exp(-t/2 + W_t)] \). Here we took a sample path without doubling points due to stochasticity.
Figure 4  Sample path of the Lighthill–Whitham–Richards model with initial condition $1 - x$ perturbed by the term $-(1 - 2x)\cdot u_x \circ dW_t$. The solution is just defined on a small time interval.

For reader’s convenience we add some other examples in Appendix B with precise expressions of solutions and different choices of $H(u)$, but which may not rigorously fit the Lighthill–Whitham–Richards model.

4 Conclusions and discussion
The method of stochastic characteristics can be used effectively to solve a stochastically perturbed Lighthill–Whitham–Richards model. The solutions are explicitly given up to a stopping time in closed form. Numerical simulations based on these models can hence been implemented straightforward. However one has to be careful, that the intersection of characteristics due to stochastic perturbation can lead to solutions which are only defined on a smaller time interval than the non-perturbed ones. On the other hand, it may be also possible, that the stochastic perturbations increase the time interval where solutions are defined. An example for a solution which is ill-defined due to intersecting characteristics can be seen in Fig. 4.

A collection for different examples of stochastic perturbations can be found in Appendix B. Note that with the considered perturbations measurement errors can be modelled effectively. This could be of high interest for more complicated traffic flow models.

Appendix A: Collection of examples in the deterministic case
For reader’s convenience the authors itemize the corresponding solutions to the deterministic Lighthill–Whitham–Richards model (4) for different initial functions $g(x)$. Based on the model a couple of initial conditions are possible apart from Example 2.2 with $g(x) = 1 - x$ and $g(x) = 1 - x^2$. The opposite to the above case is $g(x) = x$, i.e. the road at position $x = 0$ has empty density but at $x = 1$ there is e.g. a tailback or a red light, hence the initial density is maximal. We also want to consider a quadratic form by $g(x) = x - x^2$, which coincide with the behaviour of the drift part. Analogous calculations yield the following solutions (see Table 1).

Appendix B: Collection of examples in the stochastic case
Analogously to the observation in Appendix A we specify the solutions to the perturbed Lighthill–Whitham–Richards model for different choices of the initial function $g(x)$ as
Table 1  Solution for the Lighthill-Whitham-Richards model for different \( g \)

| \( g(x) = x \) | \( u(x, t) \approx \frac{1}{2(1-2t)} + \frac{1}{t}, t \neq \frac{1}{2} \) |
| \( g(x) = x - x^2 \) | \( u(x, t) \approx \frac{\sqrt{-4t^3 + 8x \sqrt{t} + 4x^2 + W_t^2 + 4tx^2 - 4x - 2t + 2x}}{2(1 - 4t + 4t^2 + W_t^2)} \):

well as for different diffusion terms \( H(u) \). Taking into account that these might not model the original traffic flow problem perfectly, the approach of the method of stochastic characteristics will give explicit solutions. Firstly we perturb the equation by standard Brownian motion.

- The solution to the equation

\[
\begin{align*}
\frac{du}{dt} &= -(1 - 2u) \cdot u_x \, dt + u_x \cdot dW_t, \\
u(x, 0) &= 1 - x^2,
\end{align*}
\]

is given for almost all \( \omega \) and all \((x, t)\) up to a certain stopping time by

\[
u(x, t) = 1 - \left( \frac{\sqrt{8t(W_t + t + x)} + 1 - 1}{16t^2} \right).
\]

- The solution to the equation

\[
\begin{align*}
\frac{du}{dt} &= -(1 - 2u) \cdot u_x \, dt + u \cdot dW_t, \\
u(x, 0) &= x,
\end{align*}
\]

is given for almost all \( \omega \) and all \((x, t)\) up to a certain stopping time by

\[
u(x, t) = \frac{t - x}{2 \int_0^t \exp(W_s) \, ds - 1}.
\]

- The solution to the equation

\[
\begin{align*}
\frac{du}{dt} &= -(1 - 2u) \cdot u_x \, dt + \sqrt{u - u^2} \cdot u_x \cdot dW_t, \\
u(x, 0) &= x,
\end{align*}
\]

is given for almost all \( \omega \) and all \((x, t)\) up to a certain stopping time by

\[
u(x, t) = \frac{W_t \sqrt{W_t^2 + 4t^2 - 4t - 4x^2 + 4x + W_t^2 + 4t^2 - 4xt - 2t + 2x}}{2(1 - 4t + 4t^2 + W_t^2)}.
\]

Replacing the standard Brownian motion by the geometric Brownian motion without drift we are able to determine also explicit solutions to different SPDEs.

- The solution to the equation

\[
\begin{align*}
\frac{du}{dt} &= -(1 - 2u) \cdot u_x \, dt + u_x \cdot d[\exp(-t/2 + W_t)], \\
u(x, 0) &= x.
\end{align*}
\]
is given for almost all $\omega$ and all $(x,t)$ up to a certain stopping time by
\[
u(x,t) = \frac{1 - x + t - \exp(-t/2 + W_t)}{2t - 1}.
\]

- The solution to the equation
\[
\begin{cases}
    d\nu = -(1-2\nu) \cdot \nu_t \, dt + \nu \circ d[\exp(-t/2 + W_t)], \\
u(x,0) = x,
\end{cases}
\]

is given for almost all $\omega$ and all $(x,t)$ up to a certain stopping time by
\[
u(x,t) = \frac{e(x-t)}{e - 2 \int_0^t \exp(\exp(-s/2 + W_s)) \, ds}.
\]

- The solution to the equation
\[
\begin{cases}
    d\nu = -(1-2\nu) \cdot \nu_t \, dt + \sqrt{\nu - \nu^2} \cdot \nu_x \circ d[\exp(-t/2 + W_t)], \\
u(x,0) = x,
\end{cases}
\]

is given for almost all $\omega$ and all $(x,t)$ up to a certain stopping time by
\[
u(x,t) = \exp(-t/2 + 2W_t) - 1
\cdot (\sqrt{4t^2 - 4t - 4x^2 + 4x + \exp(-t + 2W_t)} - 2 \exp(-t/2 + W_t) + 1
+ 4t^2 - 4xt - 2t + 2x + \exp(-t + 2W_t) - 2 \exp(-t/2 + W_t) + 1)
\cdot (2(4t^2 - 4t + \exp(-t + 2W_t) - 2 \exp(-t/2 + W_t) + 2))^{-1}
\]

Formally all given solutions need a verification, similarly to the proofs in Appendix C. But this should not be part of this manuscript.

**Appendix C: Calculation and proofs**

**Claim** (17) solves the stochastic partial differential equation (15).

**Proof** In a first step we determine the partial derivatives $\nu_t$ and $\nu_x$ by using $\frac{dW_t}{dt} = \dot{W}_t$. We obtain
\[
\frac{du}{dt} = \frac{d}{dt} \left[ \frac{1 - x + t + W_t}{1 + 2t + 2W_t} \right] = \frac{(1 + 2t + 2W_t)(1 + \dot{W}_t) - (1 - x + t + W_t)(2 + 2\dot{W}_t)}{(1 + 2t + 2W_t)^2}
\]
and
\[
\frac{du}{dx} = \frac{d}{dx} \left[ \frac{1 - x + t + W_t}{1 + 2t + 2W_t} \right] = -\frac{1}{(1 + 2t + 2W_t)}.
\]
Finally we have to verify that \( u_t + (1 - 2u)u_x + (1 - 2u)u_x W_t = 0. \)

\[
\begin{align*}
  u_t + (1 - 2u)u_x + (1 - 2u)u_x W_t &= \frac{(1 + 2t + 2W_t^2)(1 + W_t^2) - (1 - x + t + W_t^2)(2 + 2W_t)}{(1 + 2t + 2W_t^2)^2} \\
  &\quad + \frac{(1 - 2x)}{(1 + 2t + 2W_t^2)^2} \frac{(1 - 2x) W_t}{(1 + 2t + 2W_t^2)^2} \\
  &= \frac{1 + W_t + 2t + 2W_t + 2W_t W_t - (2 + 2W_t - 2x - 2x W_t + 2t + 2t W_t + 2W_t + 2W_t W_t)}{(1 + 2t + 2W_t^2)^2} \\
  &\quad + \frac{1 - 2x + W_t - 2x W_t}{(1 + 2t + 2W_t)^2} \\
  &= \frac{W_t - 2W_t + 2 - 2x + W_t}{(1 + 2t + 2W_t^2)^2} \\
  &= 0
\end{align*}
\]

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Abbreviations
Bm, Brownian motion; gBm, geometric Brownian motion.

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Authors’ contributions
WB was involved in simulation and writing. NM was involved in exact computation and writing. All authors read and approved the final manuscript.

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