ORBITAL STABILITY OF PERIODIC WAVES FOR THE KLEIN-GORDON-SCHRÖDINGER SYSTEM

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Abstract. This article deals with the existence and orbital stability of a two-parameter family of periodic traveling-wave solutions for the Klein-Gordon-Schrödinger system with Yukawa interaction. The existence of such a family of periodic waves is deduced from the Implicit Function Theorem, and the orbital stability is obtained from arguments due to Benjamin, Bona, and Weinstein.

1. Introduction. In this work we establish the existence and orbital stability of periodic traveling-wave solutions associated with the following Klein-Gordon-Schrödinger system (KG-NLS henceforth):

\[
\begin{cases}
    i u_t + \frac{1}{2} \Delta u = -uv, \\
    v_{tt} - \Delta v + m^2 v = |u|^2,
\end{cases}
\]

where \(u = u(x, t)\) is a complex-valued function, \(v = v(x, t)\) is a real-valued function, \(m\) is a real constant, and \(\Delta\) stands for the Laplace operator.

System (1) is often referred to by physicists as the Klein-Gordon-Schrödinger system with Yukawa interaction. In the physical context, \(u\) represents a complex scalar nucleon field interacting with a real scalar meson field represented by \(v\), and \(m\) depends on the meson mass. Furthermore, systems similar to (1) have been used to describe the dynamics of coupled electrostatic upper-hybrid and ion-cyclotron waves in a uniform magnetoplasma (see [31], [35] and references therein).

The Cauchy problem associated with (1) has been studied in recent years. In the 3-dimensional case, Baillon and Chadam [7], using Strichartz and energy estimates, deduced the existence of global smooth solutions. Fukuda and Tsutsumi [13] discussed the initial-boundary value problem posed on a regular domain and, by appealing to Galerkin’s method, obtained the existence of global strong smooth solutions. Recently, Colliander, Holmer, and Tzirakis [12] have put forward a new

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method in order to establish global existence for a general class of dispersive equations. As an application of the method they proved the existence of global solutions in low regularity $L^2$-based Sobolev spaces (see also Pecher [28] and Tzirakis [32]). In the 1-dimensional case, by using Kato’s theory, Rabsztyn [29] established local well-posedness in the energy space $H^1(\mathbb{R}) \times H^1(\mathbb{R}) \times L^2(\mathbb{R})$. Global well-posedness is then deduced thanks to energy conservation. Most recently, by applying the the $I$-method combined with Strichartz-type estimates, Tzirakis [32] showed global well-posedness below the energy space.

The existence and orbital stability (in the energy space) of standing-wave solutions have also been considered in the literature. In the 3-dimensional case, Ohta [27] obtained the existence and stability of stationary states by using the variational approach introduced by Cazenave and Lions [11]. Later, Kikuchi and Ohta [21], [22] established the existence of standing waves, still applying a variational approach. They proved orbital stability if the frequency is sufficiently large [22] and orbital instability if the frequency is sufficiently small [21]. In the 1-dimensional case, Ohta [26] proved the existence and orbital stability of solitary-wave solutions of the form

$$u(x,t) = e^{i\mu t} \phi_c(x - \lambda t), \quad v(x,t) = \psi(x - \lambda t),$$

where

$$\phi_c(x) = \beta_2 + (\beta_3 - \beta_2)cn^2\left(\frac{\sqrt{\beta_3 - \beta_1}}{12} x; k\right), \quad k^2 = \frac{\beta_3 - \beta_2}{\beta_3 - \beta_1}.$$ (3)

The function $cn(\cdot; k)$ represents the Jacobian elliptic function of cnoidal type, $k$ is the elliptic modulus, and $\beta_1$, $\beta_2$, and $\beta_3$ are smooth functions depending on the parameter $c$. The approaches to obtain the stability/instability results were those developed by Grillakis et al. [16], [17] and Grillakis [18], [19].

In recent years, efforts have been made on the stability theory of periodic traveling-waves solutions and, apparently, it has gained high visibility in the work by Angulo, Bona, and Scialom [2], who, adapting the Grillakis et al. theory [18], established a complete stability theory of cnoidal-wave solutions for the Korteweg-de Vries (KdV) equation

$$u_t + uu_x + u_{xxx} = 0.$$ (4)

In addition, by exploring Benjamin, Bona, and Weinstein’s ideas (see [8], [9], [33]), Angulo [1] has studied the existence and orbital stability of periodic standing- and traveling-wave solutions for the cubic NLS (see also [14], [15]) and modified KdV equations. Recently, by using the Grillakis et al. [17] approach, Natali and Pastor [25] established the orbital stability/instability of periodic standing waves for the Klein–Gordon equation

$$u_{tt} - u_{xx} + f(|u|^2)u = 0,$$ (5)

when $f(v) = v$ and $f(v) = 1 - v$.

Next, for dispersive evolution equations in a general form

$$u_t + u^p u_x - Mu_x = 0,$$ (6)

Angulo and Natali in [4] have established a theory for the study of the nonlinear orbital stability of positive and periodic traveling-wave solutions of the form
In (6), \( p \geq 1 \) is an integer and \( M \) is a Fourier multiplier operator defined via the Fourier transform as
\[
\hat{M}g(k) = \beta(k)\hat{g}(k), \quad k \in \mathbb{Z},
\]
where the symbol \( \beta \) is measurable, locally bounded, and even.

The approach in [4] determined, for instance, the first proof of the nonlinear stability of a family of periodic traveling-wave solutions for the Benjamin-Ono equation
\[
\frac{\partial u}{\partial t} + uu_x - \mathcal{H}u_{xx} = 0,
\]
where \( \mathcal{H} \) represents the periodic Hilbert transform defined by
\[
\mathcal{H}f(x) = \frac{1}{L} \text{p.v.} \int_0^L \cotg\left(\frac{\pi(x-y)}{L}\right) f(y) \, dy.
\]

For complementary references see [3], [5], and [6].

Attention is now turned to describing our results. Our purpose is to consider the 1-dimensional case, and to study the existence and orbital (nonlinear) stability of periodic traveling-wave solutions for (1) of the form
\[
\begin{align*}
  u(x,t) &= e^{-i\omega t}e^{i c(x-ct)}\varphi_{\omega,c}(x-ct), \\
  v(x,t) &= \varphi_{\omega,c}(x-ct),
\end{align*}
\]
where \( \varphi_{\omega,c}, \varphi_{\omega,c}: \mathbb{R} \to \mathbb{R} \) are smooth periodic functions with the same fixed period \( L > 0 \) and \( \omega, c \) are real parameters to be determined later. It should be noted that the function
\[
\psi_{\omega,c}(\xi) = e^{ic\xi}\varphi_{\omega,c}(\xi)
\]
is not necessarily a periodic function (with period \( L \)). Thus, the relation
\[
\frac{2q\pi}{c} = L,
\]
for some \( q \in \mathbb{N} \), is assumed in the whole paper. This assures that \( \psi_{\omega,c} \) is periodic with period \( L \).

Substituting the waves (10) into (1), it follows that \( \varphi_{\omega,c} \) and \( \phi_{\omega,c} \) must satisfy the following nonlinear system of ordinary differential equations:
\[
\begin{aligned}
  2\left(\omega + \frac{c^2}{2}\right)\varphi_{\omega,c} + \varphi''_{\omega,c} &= -2\varphi_{\omega,c}\phi_{\omega,c}, \\
  (c^2 - 1)\phi''_{\omega,c} + \phi_{\omega,c} &= \varphi^2_{\omega,c},
\end{aligned}
\]
where, for simplicity, it is assumed that \( m^2 = 1 \). In view of the nonlinearity of (12), it is a hard task to get explicit representation of solutions. However, it is substantially simpler in the case \( c^2 = 1 \). Indeed, by assuming \( c^2 = 1 \) and defining \( -\sigma := \omega + 1/2 \) and \( \varphi_{\omega,1} := \varphi_\sigma \), (12) reads as
\[
\begin{aligned}
  \sigma \varphi_\sigma + \frac{1}{2} \varphi''_\sigma &= -\varphi_\sigma \phi_\sigma, \\
  \phi_\sigma &= \varphi^2_\sigma,
\end{aligned}
\]
which reduces to the ordinary differential equation,
\[
-\varphi''_\sigma + 2\sigma \varphi_\sigma - 2\sigma^3 = 0.
\]
(14)

For \( \sigma > \pi^2/L^2 \), an \( L \)-periodic solution of (14) is given in terms of the Jacobian elliptic function of \( \text{dnoidal} \) type, namely,
\[
\varphi_\sigma(\xi) = \beta dn(\beta \xi; k),
\]
where \( \beta \) and \( k \) are determined later.
where $dn(\cdot; k)$ denotes the dnoidal function, and $\beta$ is a parameter depending smoothly on $\sigma$.

At this point, it should be noted that the pair $\varphi = \varphi_\sigma$, $\phi = \varphi_\sigma^2$, where $\varphi_\sigma$ is defined in (15), gives us a solution of (12) with $c^2 = 1$.

We next consider the case $c^2 \neq 1$, with $c^2 \approx 1$. Particular calculations are given to the case $c \approx 1$, since that for $c \approx -1$ can be similarly dealt with. The main idea is quite simple: fix $\sigma_0 > \pi^2/L^2$; once the solution $\varphi_{\sigma_0} = \varphi_{\omega_0}$, where $\varphi_{\omega_0}$ is defined in (15), is obtained, the Implicit Function Theorem is then applied to extend the range of parameters $(\omega, c)$ to a small ball in $\mathbb{R}^2$ centered at $(\omega_0, 1)$. This requires a detailed spectral description of the operator arising in the linearization of (1) around the traveling wave (10) at $(\omega, c) = (\omega_0, 1)$. Furthermore, since $\sigma_0 > \pi^2/L^2$ is arbitrarily fixed, the parameters can be extended to an even large set, say, $\mathcal{O}$. As a consequence, a smooth curve

$$(\omega, c) \in \mathcal{O} \mapsto (\varphi_{\omega,c}, \phi_{\omega,c}) \in H^{m}_{\text{per},e}([0, L]) \times H^{m}_{\text{per},e}([0, L]), \quad m \geq 1,$$

of solutions for (12) is obtained. Here $H^{m}_{\text{per},e}([0, L]) \subset H^{m}_{\text{per}}([0, L])$ denotes the closed subspace constituted by even periodic functions.

Before proceeding, it should be pointed out that the quantities

$${\cal E}(u, v, w) = \frac{1}{2} \int_0^L \left[ |u_x|^2 + w^2 + v_x^2 + v^2 - 2v|u|^2 \right] dx,$$

$${\cal F}(u, v, w) = \int_0^L |u|^2 dx,$$

and

$${\cal G}(u, v, w) = -\int_0^L v_x w + \text{Im} \int_0^L \pi u_x dx,$$

where it is written $w = v_t$, are conserved quantities by the flow of (1).

With the periodic waves given in (16) in hand, their nonlinear stability is at issue. We just consider periodic perturbations with the same period as the underlying wave.

As mentioned above, the abstract Grillakis et al. theory can be applied to obtain, in a successfully way, the nonlinear stability for a wide class equations. Meanwhile, it is not clear how to apply it in the present case. Indeed, it must be observed that solutions in (16) are critical points of the functional $\mathcal{R} := \mathcal{E} - \omega \mathcal{F} - c \mathcal{G}$, that is,

$$\delta \mathcal{R}(\psi_{\omega,c}, \phi_{\omega,c}, -c \phi'_{\omega,c}) = 0,$$

where $\delta \mathcal{R}$ stands for the Fréchet derivative of $\mathcal{R}$ and $\psi_{\omega,c}$ is given in (11). Moreover, as is well known, the conclusion in the Stability/Instability Theorem of [17] is obtained from the exact number of negative eigenvalues of the linearized operator $\delta^2 \mathcal{R}$ and the exact number of positive eigenvalues of the Hessian $d''(\omega, c)$, where $d$ is the real-valued function given by

$$d(\omega, c) = \mathcal{R}(\psi_{\omega,c}, \phi_{\omega,c}, -c \phi'_{\omega,c}).$$

However, it seems difficult to get the number of positive eigenvalues of $d''(\omega, c)$ because the parameter $c$ is abduced from the Implicit Function Theorem. In order to overcome these difficulties, at least for $c$ near $\pm 1$ with $|c| < 1$, we employ the pioneering ideas of Benjamin, Bona, and Weinstein, whose analysis do not require this type of information (see [8], [9], [33]).
Next, we describe the main step in [8], [9], [33] applied to our case. We consider perturbations of the periodic wave \((\varphi_{\omega,c}, \phi_{\omega,c}, -c\varphi_{\omega,c})\) by defining
\[
\begin{aligned}
\zeta(x, t) &= e^{i\theta}(T_c u)(x + x_0, t) - \varphi_{\omega,c}(x), \\
\eta(x, t) &= v(x + x_0, t) - \phi_{\omega,c}, \\
\nu(x, t) &= w(x + x_0, t) + c\varphi'_{\omega,c}(x),
\end{aligned}
\]
where
\[(T_c u)(x, t) = e^{-ic(x-ct)}u(x, t).
\]

On \(X := H^1_{\text{per}}([0, L]) \times H^1_{\text{per}}([0, L]) \times L^2_{\text{per}}([0, L])\), define the functional \(\mathcal{R}\) by
\[
\begin{aligned}
\mathcal{R}(u(t), v(t), w(t)) &= \mathcal{E}(u(t), v(t), w(t)) - c\mathcal{G}(u(t), v(t), w(t)) - \omega\mathcal{F}(u(t), v(t), w(t))
\end{aligned}
\]
where \(\mathcal{E}, \mathcal{F},\) and \(\mathcal{G}\) are defined in (17), (18), and (19), respectively. The key point is to show that
\[
\begin{aligned}
\Delta \mathcal{R}(t) &= \mathcal{R}(u(t), v(t), w(t)) - \mathcal{R}(\Phi_{\omega,c}, \Gamma_{\omega,c}, \Lambda_{\omega,c}) \\
&\geq C_1||\zeta||^2_1 - C_2||\eta||^2_3 - C_3||\eta||^2_4,
\end{aligned}
\]
where \(\Phi_{\omega,c}(x) = e^{icx}\varphi_{\omega,c}(x), \Gamma_{\omega,c}(x) = \phi_{\omega,c}(x), \Lambda_{\omega,c}(x) = -c\varphi'_{\omega,c},\) and \(C_j > 0, j = 1, 2, 3.\)

In order to obtain inequality (22) it is necessary to analyze the non-positive spectrum (see Lemma 3.3) of the operators
\[
L_{\omega,c} = -\frac{d^2}{dx^2} + 2\alpha - 2\phi_{\omega,c} - 4\varphi_{\omega,c} \circ K_c^{-1} \circ \varphi_{\omega,c} \quad \text{and} \quad L_{\omega,c}^+ = -\frac{d^2}{dx^2} + 2\alpha - 2\phi_{\omega,c},
\]
where \([\varphi_{\omega,c} \circ K_c^{-1} \circ \varphi_{\omega,c}](f) := \varphi_{\omega,c}K_c^{-1}(\varphi_{\omega,c}f)\) and \(K_c^{-1}\) is defined by the Fourier transform as
\[
\hat{K}_c^{-1}f(k) = \frac{1}{(1 - c^2k^2 + i\hat{f}(k)), \quad k \in \mathbb{Z}.
\]

The paper is organized as follows. Section 2 is concerned with the existence of periodic traveling waves for (1) of the form (10). The nonlinear stability of the aforementioned waves is presented in Section 3.

**Notation and Well-Posedness Results.** For \(s \in \mathbb{R},\) the Sobolev space \(H^s_{\text{per}} = H^s_{\text{per}}([0, L])\) is the set of all periodic distributions such that
\[
||f||_{H^s_{\text{per}}}^2 := ||f||_s^2 \equiv L \sum_{k=-\infty}^{+\infty} (1 + |k|^2)^s |\hat{f}(k)|^2 < \infty,
\]
where \(\hat{f}\) is the Fourier transform of \(f.\) To simplify, if no confusion is caused, we denote \(\|\cdot\|_a\) by \(\|\cdot\|\). The notation \(f \perp g\) means that \(f\) is orthogonal to \(g\) with respect to the \(L^2_{\text{per}}-\)inner product. The symbols \(sn(\cdot;k), dn(\cdot;k),\) and \(cn(\cdot;k)\) represent the Jacobian elliptic functions of snoidal, dnodial, and cnoidal type, respectively. The quantities \(\Re(z)\) and \(\Im(z)\) are the real and imaginary parts of the complex number \(z.\) We use \(c \approx 1\) to mean that \(c\) is sufficiently close to 1. Moreover, \(c \approx 1^- (c \approx 1^+)\) means \(c \approx 1\) and \(c < 1\) (\(c > 1\) and \(-1 < c)).

The question about local well-posedness in the space \(H^1_{\text{per}}([0, L]) \times H^1_{\text{per}}([0, L]) \times L^2_{\text{per}}([0, L])\) associated with system (1) can be established by a direct application of classical semigroup theory and density arguments, which are similar to the proof of the continuous case (see [29]). Moreover, one can obtain global well-posedness...
just by taking the advantage that (1) conserves the quantities $F$ and $E$. In fact, we consider the Cauchy problem
\[
\begin{align*}
\begin{cases}
   iu_t + \frac{1}{2} \Delta u = -uv, \\
v_{tt} - \Delta v + m^2 v = |u|^2, \\
u(x, 0) = u_0(x), \\v(x, 0) = v_0(x), \\v_t(x, 0) = w_0(x).
\end{cases}
\end{align*}
\]

(24)

It is clear that (24) can be written in the abstract form
\[
\begin{align*}
\begin{cases}
   \frac{dU(t)}{dt} = -iAU(t) + G(U(t)) \\
U(0) = U_0(x),
\end{cases}
\end{align*}
\]

(25)

where $U = \begin{pmatrix} u \\ v \\ w \end{pmatrix}$, $w = v_t$, $U_0 = \begin{pmatrix} u_0 \\ v_0 \\ w_0 \end{pmatrix}$,

\[A = \begin{pmatrix}
\frac{1}{2}(-\Delta + 1) & 0 & 0 \\
0 & 0 & i \\
0 & i(\Delta - 1) & 0
\end{pmatrix}\]

and $G(U) = \begin{pmatrix} iuv + \frac{i}{2}u \\ 0 \\ \mu v + |u|^2 \end{pmatrix}$ with $\mu = 1 - m^2$.

The operator $A$ defined on $X = H^1_{\text{per}}([0, L]) \times H^1_{\text{per}}([0, L]) \times L^2_{\text{per}}([0, L])$ with domain $D(A) := H^2_{\text{per}}([0, L]) \times H^2_{\text{per}}([0, L]) \times H^1_{\text{per}}([0, L])$ is clearly self-adjoint. Thus, from Stone’s Theorem, $-iA$ is a strongly continuous one-parameter group of unitary operators on $X$. Since $G$ is a locally Lipschitz perturbation on $X$ such that $G$ maps $D(A)$ into itself and
\[||A(G(U) - G(V))|| \leq C(||U||, ||AU||, ||V||, ||AV||)||AU - AV||,\]

it follows that for $U_0 \in D(A)$ there is $T > 0$ and a unique strongly continuous differentiable solution $U(t) \in D(A)$, $t \in (-T, T)$.

Now, since the embedding $H^r_{\text{per}}([0, L]) \hookrightarrow H^s_{\text{per}}([0, L])$ is compact for $r > s \geq 0$, by using standard density results combined with the arguments stated above, we establish the following result.

**Theorem 1.1.** The Cauchy problem (24) is locally well-posed for data $(u_0, v_0, w_0) \in X$ in the mild sense. More precisely, there are $T' > 0$ and a unique pair solution $(u, v)$ such that

\[u \in C((-T', T'); H^1_{\text{per}}([0, L])\]

and

\[v \in C((-T', T'); H^1_{\text{per}}([0, L]) \cap C^1((-T', T'); L^2_{\text{per}}([0, L]))\]

satisfying $u(x, 0) = u_0(x)$, $v(x, 0) = v_0(x)$ and $w(x, 0) = v_t(x, 0) = w_0(x)$. Moreover, for every $0 < T_1 < T'$ the mapping

\[(u_0, v_0, w_0) \mapsto (u, v, w),\]

is continuous from $X \mapsto C((-T_1, T_1); X)$. 
We discuss about the global well-posedness associated with problem (24). Indeed, let 
\( U_0 = (u_0, v_0, w_0) \). Since \( \mathcal{F} \) and \( \mathcal{E} \) are conserved quantities, from the Gagliardo-
Nirenberg inequality, we obtain
\[
\frac{1}{2} \int_0^L \left[ |u_x|^2 + w^2 + v_x^2 + v^2 \right] dx \leq \mathcal{E}(u_0, v_0, w_0) + \int_0^L v|u|^2 dx \\
\leq M + \frac{1}{2} \int_0^L |v|^2 dx + \frac{1}{2} \int_0^L |u|^4 dx \\
\leq M + \frac{1}{2} ||v||^2 + \frac{1}{4} ||u_x||^2,
\]
(26)
where \( M_1 = M + C ||u_0||^6 \). Thus, from (26), we obtain the uniform bound
\[
\int_0^L \left[ \frac{1}{2} |u_x|^2 + w^2 + v_x^2 \right] dx \leq 2M_1.
\]
(27)
Next, by using (27) and since \( H^1_{per}([0, L]) \rightarrow L^4_{per}([0, L]) \), we can proceed in a similar way, as made in inequality (26), in order to deduce,
\[
\int_0^L \left[ |u_x|^2 + \frac{1}{2} v^2 + w^2 + v_x^2 \right] dx \leq M_2,
\]
(28)
where \( M_2 = M_2(||u_0||, ||v_0||, ||w_0||) \). These a priori estimates allow us to establish the following theorem.

**Theorem 1.2.** Let \((u(t), v(t))\) be the pair solution in Theorem 1.1. Then, it can be extended to any interval of time.

2. **Existence of periodic-wave solutions.** The aim in this section is to show the existence of a two-parameter smooth branch of periodic-wave solutions for (12) of the form (16). Our idea is first to consider \( c^2 = 1 \) and then to apply the Implicit Function Theorem to extend the parameter \( c \) such that \( c^2 \approx 1 \). Without loss of generality, we consider \( m^2 = 1 \).

2.1. **Existence of Dnoidal waves for (14).** We start by taking \( c = 1 \) in (12) (when \( c = -1 \) the procedure is similar). Then, we immediately obtain the equation
\[
-\dot{\varphi}_\sigma'' + 2\sigma \dot{\varphi}_\sigma - 2\varphi_\sigma^3 = 0,
\]
(29)
where \(-\sigma = \omega + 1/2\) and \( \dot{\varphi}_\sigma = \varphi_\omega,1 \). An \( L \)-periodic solution for (29) is given by (see e.g., [1], [4], [5], [6] or [24])
\[
\varphi_\sigma(\xi) = \beta dn(\beta \xi; k),
\]
(30)
where \( \beta = 2K(k)/L \) and \( \sigma = [2K(k)^2(2 - k^2)]/L^2 \). Here, \( k \in (0, 1) \) represents the elliptic modulus and \( K = K(k) \) is the complete elliptic integral of the first kind (see e.g., [10]).

The existence of a smooth curve of dnoidal-wave solutions for (29) depending on the parameter \( \sigma \) is established in the following proposition.
Proposition 1. Let $L > 0$ be fixed and $\sigma \in \left(\frac{\pi^2}{L^2}, +\infty\right)$. Then, the dnoidal wave $\tilde{\varphi}_\sigma$ given by (30) has fundamental period $L$, satisfies (29), and the modulus $k = k(\sigma)$ satisfies $dk/d\sigma > 0$. Moreover, the mapping

$$\sigma \in \left(\frac{\pi^2}{L^2}, +\infty\right) \mapsto \tilde{\varphi}_\sigma \in H^m_{\text{per}}([0, L])$$

is a smooth curve.

Proof. The proof can be found in [1] or [6] (see also [24]).

Next result gives us a smooth curve (depending on $\omega$) of dnoidal-wave solutions for (12) with $c = 1$.

Corollary 1. Let $L > 0$ be a fixed number. Then, for $\omega \in (-\infty, -\pi^2/L^2 - 1/2)$ and $\sigma = \sigma(\omega) := -\omega - 1/2$, the curve

$$\omega \in (-\infty, -\pi^2/L^2 - 1/2) \mapsto (\tilde{\varphi}_\omega, \tilde{\varphi}_\omega^2) \in H^m_{\text{per}}([0, L]) \times H^m_{\text{per}}([0, L])$$

where $\tilde{\varphi}_\omega = \tilde{\varphi}_{\sigma(\omega)}$, is a smooth branch of $L$-periodic solutions for (12) with $c = 1$. Moreover,

$$\frac{d}{d\omega} \int_0^L \tilde{\varphi}_\omega^2(x) dx < 0.$$  \hfill (31)

Proof. The first part follows immediately from Proposition 1. Then, it remains to show (31). In fact, since $\beta = 2K(k)/L$ and $\int_0^L dn^2(x; k) dx = E(k)$, where $E$ denotes the complete elliptic integral of the second kind (see e.g., [10]), we have

$$\frac{d}{d\omega} \left( \int_0^L \tilde{\varphi}_\omega^2(x) dx \right) = \frac{d}{d\sigma} \left( \int_0^L \tilde{\varphi}_{\sigma(\omega)}^2(x) dx \right) \frac{d\sigma}{d\omega} = -\frac{4L}{K(k)E(k)} \frac{dk}{d\sigma} < 0,$$

where in the last inequality we used Proposition 1 and the fact that the function $k \in (0, 1) \mapsto K(k)E(k)$ is a strictly increasing function. This argument completes the proof.

2.2. Existence of periodic waves for (12). In this subsection, we show the existence of a smooth branch of periodic wave of (12) for $c \approx 1$ (if $c \approx -1$ the procedure is similar) such that these solutions bifurcate the dnoidal solutions found in the last subsection.

In fact, we start by analyzing the periodic eigenvalue problem,

$$\begin{aligned}
L_\sigma f := \left( -\frac{d^2}{dx^2} + 2\sigma - 6\tilde{\varphi}_\sigma^2 \right) f &= \lambda f, \\
f(0) = f(L), \quad f'(0) = f'(L),
\end{aligned}$$

where for $\sigma > \pi^2/L^2$, $\tilde{\varphi}_\sigma$ is the dnoidal wave in Proposition 1. The proof of the next theorem can be found in [1] or [6] (see also [24]).

Theorem 2.1. Let $\tilde{\varphi}_\sigma$ be the dnoidal wave solution in Proposition 1. Then, the operator $L_\sigma$ in (32) defined in $L^2_{\text{per}}([0, L])$ with domain $H^2_{\text{per}}([0, L])$ has exactly one negative eigenvalues, which is simple; zero is a simple eigenvalue (with eigenfunction $\tilde{\varphi}_\sigma'$). Moreover, the remainder of the spectrum is constituted by a discrete set of eigenvalues bounded away from zero.
Remark 1. In order to show Theorem 2.1, one observes that the periodic eigenvalue problem (32) is equivalent (under a suitable transformation) to the periodic eigenvalue problem associated with the Lamé operator

\[ L_{\text{Lame}} = -\frac{d^2}{dx^2} + 6k^2 \text{sn}^2(x; k), \]

posed on the interval \([0, 2K]\).

Our main result concerning the existence of periodic waves is the following.

**Theorem 2.2.** Let \( L > 0 \) be a fixed number. Then there is an open set \( O \subset \mathbb{R}^2 \) containing the semi-line

\[ A = \left\{ (\omega, 1) \in \mathbb{R}^2; \omega < -\frac{\pi^2}{L^2} - \frac{1}{2} \right\}, \]

and \( L \)-periodic functions \( \varphi := \varphi_{\omega,c} \) and \( \phi := \phi_{\omega,c} \) such that for all \((\omega, c) \in O\) the pair \((\varphi, \phi)\) is a periodic solution of system (12). Moreover,

(i) the map

\[ (\omega, c) \in O \rightarrow (\varphi_{\omega,c}, \phi_{\omega,c}) \in H^2_{\text{per},e}([0,L]) \times H^2_{\text{per},e}([0,L]), \]

is smooth.

(ii) The pair \((\varphi_{\omega,c}, \phi_{\omega,c})\) converges to \((\tilde{\varphi}_\omega, \tilde{\phi}_\omega^2)\), as \(c \to 1\), uniformly for \(\xi \in [0,L]\). In particular,

\[ (\varphi_{\omega,1}, \phi_{\omega,1}) \equiv (\tilde{\varphi}_\omega, \tilde{\phi}_\omega^2), \]

where \(\tilde{\varphi}_\omega\) is given in Corollary 1.

**Proof.** Let \( \omega_0 < -\pi^2/L^2 - 1/2 \) be fixed and let \( \tilde{\varphi}_{\omega_0} \) be the corresponding dnoidal wave in Corollary 1. Consider the product space \( Y_e = H^2_{\text{per},e}([0,L]) \times H^2_{\text{per},e}([0,L]) \) and define the map

\[ \Upsilon : (-\infty, -\frac{\pi^2}{L^2} - \frac{1}{2}) \times \mathbb{R} \times Y_e \to L^2_{\text{per},e}([0,L]) \times L^2_{\text{per},e}([0,L]) \]

by

\[ \Upsilon(\omega, c, \varphi, \phi) = \left( -\varphi'' + 2\left( -\omega - \frac{c^2}{2} \right) \varphi - 2\varphi\phi, (c^2 - 1)\phi'' + \phi - \varphi^2 \right). \]

So, from Corollary 1 we see that \( \Upsilon(\omega_0, 1, \tilde{\varphi}_{\omega_0}, \tilde{\phi}_{\omega_0}^2) = (0,0) \). Moreover, a straightforward calculation shows us that

\[ \Upsilon_{(\varphi, \phi)}(\omega, c, \varphi, \phi) = \begin{pmatrix} -\frac{d^2}{dx^2} + 2\left( -\omega - \frac{c^2}{2} \right) \varphi - 2\varphi\phi, -2\varphi \\ -2\varphi & (c^2 - 1)\frac{d^2}{dx^2} + 1 \end{pmatrix}, \]

where \( \Upsilon_{(\varphi, \phi)} \) denotes the Fréchet derivative of \( \Upsilon \) with respect to \((\varphi, \phi)\). Thus, by defining \( D = \Upsilon_{(\varphi, \phi)}(\omega_0, 1, \tilde{\varphi}_{\omega_0}, \tilde{\phi}_{\omega_0}^2) \), we obtain

\[ D = \begin{pmatrix} -\frac{d^2}{dx^2} + 2\left( -\omega_0 - \frac{1}{2} \right) - 2\tilde{\varphi}_{\omega_0}^2 & -2\tilde{\varphi}_{\omega_0} \\ -2\tilde{\varphi}_{\omega_0} & 1 \end{pmatrix}. \]
Let us show that the kernel of \( D \) (as an operator on \( L^2_{\text{per}} \times L^2_{\text{per}} \)) is generated by \((\tilde{\phi}_{\omega_0}, (\tilde{\phi}_{\omega_0}'))\). Indeed, let \((f, g) \neq 0\) such that \( D(f, g) = (0, 0) \), that is,

\[
\begin{cases}
-\frac{d^2}{dx^2} + 2\left(-\omega_0 - \frac{1}{2}\right) - 2\tilde{\phi}_{\omega_0}^2 f - 2\tilde{\phi}_{\omega_0} g = 0, \\
-2\tilde{\phi}_{\omega_0} f + g = 0.
\end{cases}
\] (34)

Substituting the second equation in (34) in the first one, we immediately obtain \( L_{\sigma_0} f = 0 \), where \( \sigma_0 = -\omega_0 - 1/2 \) and \( L_{\sigma_0} \) is defined in (32). Hence, from Theorem 2.1, we have \( f = \theta \phi'_{\omega_0} \), for some constant \( \theta \neq 0 \). Moreover, from (34), \( g = \theta (\tilde{\phi}_{\omega_0}') \). This proves our assertion. Thus, since \( \tilde{\phi}_{\omega_0} \) is an even function, it follows that \((\tilde{\phi}_{\omega_0}, (\tilde{\phi}_{\omega_0}'))\) does not belong to \( Y_c \) and so \( D \) is injective (as an operator on \( L^2_{\text{per,e}} \times L^2_{\text{per,e}} \)).

Now, let us prove that, with domain \( H^2_{\text{per,e}} \times H^2_{\text{per,e}} \), \( D \) is also surjective on \( L^2_{\text{per,e}} \times L^2_{\text{per,e}} \). Indeed, \( D \) is clearly a self-adjoint operator. Thus \( \text{spec}(D) = \text{spec}_{\text{disc}}(D) \cup \text{spec}_{\text{ess}}(D) \), where \( \text{spec}(D) \) denotes the spectrum of \( D \), and \( \text{spec}_{\text{disc}}(D) \) and \( \text{spec}_{\text{ess}}(D) \) denote, respectively, the discrete and essential spectra of \( D \). Since \( H^2_{\text{per,e}}([0, L]) \) is compactly embedded in \( L^2_{\text{per,e}}([0, L]) \), the operator \( D \) has compact resolvent. Consequently, \( \text{spec}_{\text{ess}}(D) = \emptyset \) and \( \text{spec}(D) = \text{spec}_{\text{disc}}(D) \) consists of isolated eigenvalues with finite algebraic multiplicities (see e.g., [20, Section III.6]). Therefore, since \( D \) is one-to-one, it follows that \( 0 \) is not an eigenvalue of \( D \), and so it does not belong to \( \text{spec}(D) \). This means that \( 0 \in \rho(D) \), where \( \rho(D) \) is used to denote the resolvent set of \( D \), and so, by definition, \( D \) is surjective.

The arguments above imply that \( D^{-1} \) exists and, moreover, is a bounded linear operator. Consequently, since \( \Upsilon \) and \( \Upsilon(\phi, \phi) \) are clearly smooth maps on their domains, from the Implicit Function Theorem there are an open set \( \mathcal{B} \) containing \((\omega_0, 1)\) and a unique smooth function \( F : \mathcal{B} \to Y_c \),

\[
F(\omega, c) = (\phi_{\omega,c}, \phi_{\omega,c}),
\]
such that \( F(\omega_0, 1) = (\tilde{\phi}_{\omega_0}, (\tilde{\phi}_{\omega_0}')) \) and \( \Upsilon(\omega, c, F(\omega, c)) = (0, 0) \) for all \((\omega, c) \in \mathcal{B}\). In addition, since \( \omega_0 < -\pi^2 / L^2 - 1/2 \) is arbitrarily fixed and \( F \) is unique, we can extend it to an open set, say \( \mathcal{O} \), containing \( \mathcal{A} \), such that \( \Upsilon(\omega, c, F(\omega, c)) = (0, 0) \), for all \((\omega, c) \in \mathcal{O} \) and

\[
F(\omega, 1) = (\tilde{\phi}_{\omega}, (\tilde{\phi}_{\omega}')), \quad \omega < -\frac{\pi^2}{L^2} - \frac{1}{2}.
\]

This completes the proof of the theorem. \( \square \)

**Remark 2.** A result similar to that obtained in Theorem 2.2 can be established if one considers \( c = -1 \). In this case, it is possible to show the existence of an open set \( \mathcal{O} \), containing

\[
\mathcal{A} = \left\{ (\omega, -1) \in \mathbb{R}^2; \omega < -\frac{\pi^2}{L^2} - \frac{1}{2} \right\}
\]

and another smooth branch \((\omega, c) \in \mathcal{O} \mapsto (\phi_{\omega,c}, \phi_{\omega,c})\) of solutions for (12).

3. **Stability of periodic-wave solutions.** The orbital stability theory of the periodic waves determined in Section 2 is presented in this section. Before proceeding, we explain our notion of orbital stability.
Definition 3.1. Let $\Phi(x) = e^{icx} \varphi(x)$, $\Lambda(x) = \phi(x)$ and $\Gamma(x) = -c\phi'(x)$, where $(\varphi, \phi)$ is a solution of (12). We say that the orbit generated by $(\Phi, \Gamma, \Lambda)$, namely,

$$\Omega(\Phi, \Gamma, \Lambda) = \left\{ \left\{ e^{it}\Phi(\cdot + x_0), \Gamma(\cdot + x_0), \Lambda(\cdot + x_0) \right\}; \ (\theta, x_0) \in [0, 2\pi) \times \mathbb{R} \right\} ,$$

is orbitally stable in $X = H^1_{per}([0, L]) \times H^1_{per}([0, L]) \times L^2_{per}([0, L])$ by the flow generated by system (1), if for every $\varepsilon > 0$ there is $\delta = \delta(\varepsilon) > 0$ such that for $(u_0, v_0, w_0) \in X$ satisfying

$$||(u_0, v_0, w_0) - (\Phi, \Gamma, \Lambda)||_X < \delta,$$

the global solution $(u, v, w)$ of (1) with initial data $(u(0), v(0), w(0)) = (u_0, v_0, w_0)$ satisfies $(u, v, w) \in \mathcal{C}(\mathbb{R}; X)$ and

$$\inf_{x_0 \in \mathbb{R}, \theta \in [0, 2\pi)} ||(e^{it}u(\cdot + x_0, t), v(\cdot + x_0, t), w(\cdot + x_0, t)) - (\Phi, \Gamma, \Lambda)||_X < \varepsilon, \quad (35)$$

for all $t \in \mathbb{R}$. Otherwise, we say that $\Omega(\Phi, \Gamma, \Lambda)$ is orbitally unstable in $X$.

Our main result concerning the orbital stability is as follows,

Theorem 3.2. Let $L > 0$ be fixed and $(\omega, c) \in \mathcal{O}$. Consider $(\varphi_{\omega,c}, \phi_{\omega,c})$ the corresponding periodic traveling wave in Theorem 2.2. Let

$$\Phi_{\omega,c}(x) = e^{icx} \varphi_{\omega,c}(x), \quad \Gamma_{\omega,c}(x) = \phi_{\omega,c}(x) \quad \text{and} \quad \Lambda_{\omega,c}(x) = -c\phi'_{\omega,c}(x).$$

Then for any $c \approx 1^{-}$ and $\omega \in \mathbb{R}$ such that $(\omega, c) \in \mathcal{O}$, the orbit generated by $(\Phi_{\omega,c}, \Gamma_{\omega,c}, \Lambda_{\omega,c})$ is orbitally stable in $H^1_{per}([0, L]) \times H^1_{per}([0, L]) \times L^2_{per}([0, L])$.

Note that Theorem 3.2 gives us a stability result just for $c \approx 1^{-}$. We need this condition because we want to make the norm $\left\| K_t^{1/2}(\cdot) \right\|$ (see (41) below) equivalent to that of $H^1_{per}$.

Before proving Theorem 3.2, we remind the reader that, as we have already pointed out in the introduction, we do not know how to use the Grillakis et al. theory. Thus, we use the ideas of Benjamin, Bona, and Weinstein [8], [9], [33] (see also [3]).

Firstly, for $t \geq 0$, define

$$\Omega_t(x_0, \theta) = ||e^{it}(T_c u)'(\cdot + x_0, t) - \varphi'_{\omega,c}|^2_{L^2_{per}} + 2\alpha||e^{it}(T_c u)(\cdot + x_0, t) - \varphi_{\omega,c}|^2_{L^2_{per}} \quad (36)$$

where $\alpha = -\omega - c^2/2 > -\omega - 1/2 > 0$ and $T_c$ denotes the bounded linear operator given by

$$(T_c u)(x, t) = e^{-ic(x-ct)} u(x, t).$$

Therefore, the deviation of the solution $u(\cdot, t)$ from the orbit generated by $\Phi$ is measured by

$$\rho_\alpha(u(\cdot, t), \varphi_{\omega,c})^2 = \inf_{x_0 \in [0, L], \theta \in [0, 2\pi]} \Omega_t(x_0, \theta). \quad (37)$$

Hence, since $\Omega_t$ is a continuous function on $[0, L] \times [0, 2\pi]$, the infimum in (37) is attained at $(x_0, \theta_0) = (x_0(t), \theta_0(t))$ (see e.g., [9]).

Proof of Theorem 3.2. We consider perturbation of the periodic wave $(\varphi_{\omega,c}, \phi_{\omega,c}, -c\phi'_{\omega,c})$ by defining

$$\zeta(x, t) = e^{it_0}(T_c u)(x + x_0, t) - \varphi_{\omega,c}(x),$$

$$\eta(x, t) = v(x + x_0, t) - \phi_{\omega,c},$$

$$\nu(x, t) = w(x + x_0, t) + c\phi'_{\omega,c}(x). \quad (38)$$
Then, by using the property that the infimum of $\Omega_t$ is attained at $(x_0, \theta_0) = (x_0(t), \theta_0(t))$, we obtain from (38) that $P(x, t) = \text{Re}(\zeta(x, t))$ and $Q(x, t) = \text{Im}(\zeta(x, t))$ must satisfy the following compatibility relations:

\[
\begin{cases}
\int_0^L Q(x, t)\phi_{\omega,c}(x)\phi_{\omega,c}(x)dx = 0, \\
\int_0^L P(x, t)(\phi_{\omega,c}(x)\phi_{\omega,c}(x))'dx = 0.
\end{cases}
\] (39)

Next, we define the continuous functional $\mathcal{R}$ defined on $H^1_{\text{per}}([0, L]) \times H^1_{\text{per}}([0, L]) \times L^2_{\text{per}}([0, L])$ by

$\mathcal{R}(u(t), v(t), w(t)) = \mathcal{E}(u(t), v(t), w(t)) - c\mathcal{G}(u(t), v(t), w(t)) - \omega\mathcal{F}(u(t), v(t), w(t))$, where $\mathcal{E}, \mathcal{F},$ and $\mathcal{G}$ are defined in (17), (18), and (19), respectively. Thus, from (38) and (12), we have

\[
\Delta \mathcal{R}(t) := \mathcal{R}(u(t), v(t), w(t)) - \mathcal{R}(\Phi_{\omega,c}, \Gamma_{\omega,c}, \Lambda_{\omega,c})
\]

\[
= \mathcal{R}(\Phi_{\omega,c} + e^{i\omega t}\zeta, \phi_{\omega,c} + \eta, -c\phi'_{\omega,c} + \nu) - \mathcal{R}(\Phi_{\omega,c}, \Gamma_{\omega,c}, \Lambda_{\omega,c})
\]

\[
= \frac{1}{2} \int_0^L (P^2_x + Q^2_x) - c^2(P^2 + Q^2) - 2\omega(P^2 + Q^2) - 2\phi_{\omega,c}(P^2 + Q^2)dx
\]

\[
+ \frac{1}{2} \int_0^L \eta^2 + \nu^2 + 2c\eta'\nu + \eta^2 - 4\eta\phi_{\omega,c}P - \eta(P^2 + Q^2)dx
\]

\[
= \frac{1}{2}\langle L_{\omega,c}P, P \rangle + \frac{1}{2}\langle L^+_{\omega,c}Q, Q \rangle + \frac{1}{2}||\eta'||^2
\]

\[
+ \frac{1}{2} \int_0^L \left[\mathcal{K}^{-1/2}_c \eta - 2\mathcal{K}^{-1/2}_c(\phi_{\omega,c}P) - \mathcal{K}^{-1/2}_c(P^2 + Q^2)\right]^2 dx
\]

\[
- \frac{1}{2} \int_0^L \left(\mathcal{K}^{-1/2}_c(P^2 + Q^2)\right)^2 + 4\mathcal{K}^{-1/2}_c(\phi_{\omega,c}P)\mathcal{K}^{-1/2}_c(P^2 + Q^2)dx,
\] (40)

where the operator $\mathcal{K}^{-1}_c$ is defined as a Fourier multiplier by

\[
\mathcal{K}^{-1}_c f(k) = \frac{1}{(1 - c^2)k^2 + 1} \hat{f}(k), \quad k \in \mathbb{Z}.
\]

Note that $\mathcal{K}^{-1}_c$ is the inverse of the positive operator $\mathcal{K}_c : H^2_{\text{per}}([0, L]) \to H^2_{\text{per}}([0, L])$ given by

\[
\mathcal{K}_c = -(1 - c^2)\partial_x^2 + 1.
\] (41)

Moreover, $\mathcal{K}^{1/2}_c$ and $\mathcal{K}^{-1/2}_c$ are the positive square roots of $\mathcal{K}^{-1}_c$ and $\mathcal{K}_c$, respectively. The operators $L_{\omega,c}$ and $L^+_{\omega,c}$ are defined by

\[
L_{\omega,c} = -\frac{d^2}{dx^2} + 2\alpha - 2\phi_{\omega,c} - 4\phi_{\omega,c} \circ \mathcal{K}^{-1}_c \circ \phi_{\omega,c} \quad \text{and} \quad L^+_{\omega,c} = -\frac{d^2}{dx^2} + 2\alpha - 2\phi_{\omega,c},
\] (42)

where by definition $[\phi_{\omega,c} \circ \mathcal{K}^{-1}_c \circ \phi_{\omega,c}](f) := \phi_{\omega,c} \mathcal{K}^{-1}_c(\phi_{\omega,c} f)$. 

Now, we recall the papers due to Benjamin [8] and Bona [9], to establish “good” bounds for \( \triangle R \). In order to estimate \( \langle L_{\omega,c} P, P \rangle \) and \( \langle L_{\omega,c}^+, Q, Q \rangle \), it is necessary some technical results concerning the spectra of the linear operators \( L_{\omega,c} \) and \( L_{\omega,c}^+ \).

**Lemma 3.3.** Let \( L > 0 \) be fixed. Then, for \( c \approx 1^- \), the self-adjoint operators \( L_{\omega,c} \) and \( L_{\omega,c}^+ \) in \( (42) \) defined in \( L_{\text{per}}^2([0,L]) \) have the following spectral properties.

(i) \( L_{\omega,c} \) has a unique negative eigenvalue \( \lambda_{\omega,c} \), which is simple with associated eigenfunction \( \chi_{\omega,c} \).

(ii) \( L_{\omega,c} \) has a simple eigenvalue at zero with eigenfunction \( \frac{d}{dx} \varphi_{\omega,c} \).

(iii) \( L_{\omega,c}^+ \) is a non-negative operator. Zero is a simple eigenvalue with eigenfunction \( \varphi_{\omega,c} \).

Moreover, the remainder of the spectra of \( L_{\omega,c} \) and \( L_{\omega,c}^+ \) are constituted by a discrete set of eigenvalues bounded away from zero.

**Proof.** First of all we observe that from Weyl’s essential spectrum theorem, all operators we study here have only point spectrum.

(i) Note that for \( c \approx 1^- \),

\[
\langle L_{\omega,c} \varphi_{\omega,c}, \varphi_{\omega,c} \rangle = -4 \int_0^L \varphi_{\omega,c}^2 K^{-1}_c(\varphi_{\omega,c})^2 dx = -4 \int_0^L \varphi_{\omega,c} \phi_{\omega,c} dx < 0,
\]

and therefore \( L_{\omega,c} \) possesses a negative eigenvalue, say \( \lambda_{\omega,c} \). On the other hand, for any \( f \in H_{\text{per}}^2([0,L]) \), we have

\[
\langle L_{\omega,c} f, f \rangle \geq \langle L_{\sigma} f, f \rangle + 2 \int_0^L (\tilde{\varphi}_{\sigma}^2 - \varphi_{\omega,c}) f^2 dx + 4 \int_0^L [\tilde{\varphi}_{\sigma} f^2 - \varphi_{\omega,c} f K^{-1}_c(\varphi_{\omega,c} f)] dx,
\]

where \( \sigma = -\omega - 1/2 \) and \( L_{\sigma} \) is defined in \( (32) \). Since \( \varphi_{\omega,c} \to \tilde{\varphi}_{\sigma} \), as \( c \to 1^- \), uniformly in \( \xi \in [0,L] \), and \( K^{-1}_c \) is a bounded operator, it is clear that the last two terms in \( (43) \) converge to 0, as \( c \to 1^- \), for any \( f \) such that \( \|f\| = 1 \). Hence, for any \( c \approx 1^- \) and \( \varepsilon > 0 \) small enough

\[
\langle L_{\omega,c} f, f \rangle \geq \langle L_{\sigma} f, f \rangle - \varepsilon,
\]

for all \( f \) such that \( \|f\| = 1 \).

Let \( \chi_{\sigma} \) be the eigenfunction associated with the unique negative eigenvalue of \( L_{\sigma} \) (see Theorem 2.1). Then, if \( f \perp \chi_{\sigma} \), it follows from Theorem 2.1 that \( \langle L_{\sigma} f, f \rangle \geq 0 \). Therefore, if \( \lambda_1 \) denotes the second eigenvalue of \( L_{\omega,c} \) from the min–max principle (see e.g., [30, Theorem XIII.1]), we obtain

\[
\lambda_1 = \max_{\chi} \min_{f \perp \chi, \|f\|=1} \langle L_{\omega,c} f, f \rangle \geq \min_{f \perp \chi_{\sigma}, \|f\|=1} \langle L_{\omega,c} f, f \rangle \geq -\varepsilon.
\]

Since \( \varepsilon > 0 \) is arbitrary, this proves (i).

(ii) From \( (12) \) and the definition of \( K^{-1}_c \) we have that \( L_{\omega,c} (\frac{d}{dx} \varphi_{\omega,c}) = 0 \). If we repeat the same steps as in (i), using the min–max principle we get, for \( c \approx 1^- \), that zero is a simple eigenvalue of \( L_{\omega,c} \).

(iii) Since \( L_{\omega,c}^+ \varphi_{\omega,c} = 0 \) and \( \varphi_{\omega,c} > 0 \), it follows from Floquet theory (see e.g., [23]) that zero is the first eigenvalue of \( L_{\omega,c}^+ \), and it is simple. This completes the proof of the lemma.

The next lemma was proved by Weinstein [34] for a general kind of operators. In our case, it reads as follows.
Lemma 3.4. Let $\varphi_{\omega,c}$ be as in Theorem 2.2. Let $\mathcal{L}_{\omega,c}$ be the self-adjoint operator defined in (42). We define
\[-\infty < a := \min_{\chi} \langle \mathcal{L}_{\omega,c} \chi, \chi \rangle; \quad ||\chi|| = 1 \quad \text{and} \quad \langle \chi, \varphi_{\omega,c} \rangle = 0 \rangle.
Assuming that $\langle \chi_{\omega,c}, \varphi_{\omega,c} \rangle \not= 0$ and $\varphi_{\omega,c} \in \{ \ker(\mathcal{L}_{\omega,c}) \}^\perp$, where $\chi_{\omega,c}$ is the eigenfunction associated with the negative eigenvalue of $\mathcal{L}_{\omega,c}$ given in Lemma 3.3. Then, if
\[\langle \mathcal{L}_{\omega,c}^{-1} \varphi_{\omega,c}, \varphi_{\omega,c} \rangle \leq 0,\]
it follows that $a \geq 0$.

Proof. See Lemma E.1 in [34].

Lemma 3.5. Let $\varphi_{\omega,c}$ be as in Theorem 2.2. Then, for $c \approx 1^-$, we have
(i) $\inf \{ \langle \mathcal{L}_{\omega,c} f, f \rangle; \quad ||f|| = 1, \quad \langle f, \varphi_{\omega,c} \rangle = 0 \} \equiv a_0 = 0$.
(ii) $\inf \{ \langle \mathcal{L}_{\omega,c} f, f \rangle; \quad ||f|| = 1, \quad \langle f, \varphi_{\omega,c} \rangle = 0, \quad \langle f, \langle \varphi_{\omega,c} \varphi_{\omega,c} \rangle \rangle = 0 \} \equiv b_0 > 0$.

Proof. (i) By letting $\tilde{f} = \frac{\varphi_{\omega,c}}{||\varphi_{\omega,c}||}$, we have that $||\tilde{f}|| = 1$ and $\langle \tilde{f}, \varphi_{\omega,c} \rangle = 0$. Moreover, since $\langle \mathcal{L}_{\omega,c} \tilde{f}, \tilde{f} \rangle = 0$ (see Lemma 3.3), we conclude that $a_0 \leq 0$. Let us show that $a_0 \geq 0$. To do this, we use Lemma 3.4 (see also Angulo et al. [3]), by showing first that the infimum is attained. Let $\{ \psi_j \} \in H^1_{\per}(\{0,L\})$ with $||\psi_j|| = 1$, $\langle \psi_j, \varphi_{\omega,c} \rangle = 0$ and $\lim_{j \to \infty} \langle \mathcal{L}_{\omega,c} \psi_j, \psi_j \rangle = a_0$. From this last fact, we deduce that there is a subsequence of $\{ \psi_j \}$, which we still denote by $\{ \psi_j \}$ such that $\psi_j \to \psi$ weakly in $H^1_{\per}(\{0,L\})$. Compactness arguments allow us to conclude that $\psi_j \to \psi$ in $L^2_{\per}(\{0,L\})$, as $j \to \infty$, and so $||\psi|| = 1$ and $\psi \perp \varphi_{\omega,c}$. On the other hand, since $||\psi'||^2 \leq \lim \inf ||\psi'||^2$ and $\mathcal{K}^{-1}(\varphi_{\omega,c} \psi_j) \to \mathcal{K}^{-1}(\varphi_{\omega,c} \psi)$ in $L^2_{\per}(\{0,L\})$, as $j \to \infty$, we have
\[a_0 \leq \langle \mathcal{L}_{\omega,c} \psi, \psi \rangle \leq \lim \inf \langle \mathcal{L}_{\omega,c} \psi_j, \psi_j \rangle = a_0.

Therefore, the infimum is attained at $\psi$.

The next step is to show that $\langle \mathcal{L}_{\omega,c}^{-1} \varphi_{\omega,c}, \varphi_{\omega,c} \rangle \leq 0$. In view of (12) and Theorem 2.2, $\mathcal{L}_{\omega,c} \left( \frac{d}{dx} \varphi_{\omega,c} \right) = 2 \varphi_{\omega,c}$. Moreover, from Corollary 1, it follows that $\langle \frac{d}{dx} \varphi_{\omega,c}, \varphi_{\omega,c} \rangle < 0$ and so, we get $\langle \frac{d}{dx} \varphi_{\omega,c}, \varphi_{\omega,c} \rangle < 0$, for $c \approx 1^-$. Hence, from Lemma 3.4, we obtain $a_0 \geq 0$.

(ii) Since $\{ \langle \mathcal{L}_{\omega,c} f, f \rangle; \quad ||f|| = 1, \quad \langle f, \varphi_{\omega,c} \rangle = 0, \quad \langle f, \langle \varphi_{\omega,c} \varphi_{\omega,c} \rangle \rangle = 0 \}$ is a subset of $\{ \langle \mathcal{L}_{\omega,c} f, f \rangle; \quad ||f|| = 1, \quad \langle f, \varphi_{\omega,c} \rangle = 0 \}$, we have from (i) that $b_0 \geq 0$. By using similar arguments as in part (i) we conclude that the infimum defined in (ii) is attained at a function $\kappa$. We suppose by contradiction that $b_0 = 0$ and let us consider the following minimization problem:

\[
\text{Minimize} \quad J(f) = \langle \mathcal{L}_{\omega,c} f, f \rangle
\]
subject to

\[
\begin{align*}
V_1(f) &= 1, \\
V_2(f) &= V_3(f) = 0,
\end{align*}
\]
where $V_1(g) = ||g||^2$, $V_2(g) = \langle g, \varphi_{\omega,c} \rangle$, and $V_3(g) = \langle g, \langle \varphi_{\omega,c} \varphi_{\omega,c} \rangle \rangle$.

Since $\kappa$ solves problem (45), from Lagrange’s Multiplier Theorem there are $m, n, p \in \mathbb{R}$ such that
\[\mathcal{L}_{\omega,c} \kappa = m \kappa + n \varphi_{\omega,c} + p \langle \varphi_{\omega,c} \varphi_{\omega,c} \rangle \].
But, \( \langle L_{\omega,c}\kappa, \kappa \rangle = b_0 = 0 \) immediately implies \( m = 0 \). Next, from the self-adjointness of \( L_{\omega,c} \), we see that \( 0 = \langle L_{\omega,c}\varphi_{\omega,c}', \kappa \rangle = p\langle \varphi_{\omega,c}', (\varphi_{\omega,c}\phi_{\omega,c})' \rangle \). On the other hand, as \( c \to 1^- \),

\[
\int_0^L \varphi_{\omega,c}(\varphi_{\omega,c}\phi_{\omega,c})' \omega,c \to \int_0^L (\varphi''_o \phi_{\omega,c})^2 d\omega \neq 0,
\]

which implies \( p = 0 \). These facts allow us to conclude that \( L_{\omega,c}\kappa = n\varphi_{\omega,c} \). In (i), we found that \( L_{\omega,c} (\varphi_{\omega,c}') = 2\varphi_{\omega,c} \). Then, by letting \( v_{\omega,c} = \frac{d}{dx}\varphi_{\omega,c} \) we get \( L_{\omega,c}(2\kappa - n\varphi_{\omega,c}) = 0 \). Therefore, from Lemma 3.3, there is \( d_0 \in \mathbb{R} \) such that \( 2\kappa - n\varphi_{\omega,c} = d_0\varphi_{\omega,c}' \). Since \( \langle (2\kappa - n\varphi_{\omega,c}), \varphi_{\omega,c} \rangle = 0 \) and \( \kappa \perp \varphi_{\omega,c} \) we obtain \( n\langle v_{\omega,c}, \varphi_{\omega,c} \rangle = 0 \). In addition, from Corollary 1, we have for \( c \approx 1^- \),

\[
\langle v_{\omega,c}, \varphi_{\omega,c} \rangle = \frac{1}{2} \frac{d}{dx} \int_0^L \varphi_{\omega,c}^2(x) d\omega < 0.
\]

Hence \( n = 0 \) and \( L_{\omega,c}\kappa = 0 \), which implies \( \kappa = \mu \varphi_{\omega,c}' \) for some \( \mu \neq 0 \). Since \( \kappa \perp (\varphi_{\omega,c}\phi_{\omega,c})' \), we get, as \( c \to 1^- \),

\[
\langle (\varphi_{\omega,c}\phi_{\omega,c})', \kappa \rangle = \mu \langle (\varphi_{\omega,c}\phi_{\omega,c})', \varphi_{\omega,c} \rangle \to 3\mu \int_0^L (\varphi''_o \phi_{\omega,c})^2 d\omega = 0,
\]

which is a contradiction. This completes the proof of the lemma. \( \square \)

**Lemma 3.6.** Consider \( c \approx 1^- \) as in Lemma 3.3. If

\[
L_{\omega,c}^+ = -\frac{d^2}{dx^2} + 2\alpha - 2\varphi_{\omega,c},
\]

then

\[
\inf \{ \langle L_{\omega,c}^+, f, f \rangle : \|f\| = 1, \langle f, (\varphi_{\omega,c}\phi_{\omega,c}) \rangle = 0 \} \equiv a_1 > 0.
\]

**Proof.** From Lemma 3.3 it follows that \( a_1 \geq 0 \). If \( a_1 = 0 \), then by following similar ideas as in the proof of Lemma 3.5, we have that the infimum is attained at a function \( \kappa^* \neq 0 \), and there are \( q, r \in \mathbb{R} \) such that

\[
L_{\omega,c}^+\kappa^* = q\kappa^* + r\varphi_{\omega,c}\phi_{\omega,c}.
\]

Since \( \langle L_{\omega,c}^+\kappa, \kappa \rangle = 0 \), we deduce from (47) that \( q = 0 \). On the other hand, taking the inner product of (47) with \( \varphi_{\omega,c} \), we obtain

\[
0 = \langle L_{\omega,c}^+, \varphi_{\omega,c}, \kappa^* \rangle = r \int_0^L \varphi_{\omega,c} \phi_{\omega,c} d\omega,
\]

which implies that \( r = 0 \). Thus, because zero is a simple eigenvalue of \( L_{\omega,c}^+ \) (see Lemma 3.3), it follows that \( \kappa^* = \epsilon \varphi_{\omega,c} \) for some constant \( \epsilon \neq 0 \). Therefore,

\[
0 = \langle \kappa^*, \varphi_{\omega,c}\phi_{\omega,c} \rangle = \epsilon \int_0^L \varphi_{\omega,c}^2 \phi_{\omega,c} d\omega \neq 0,
\]

which is a contradiction. \( \square \)

Now, we turn back to the proof of Theorem 3.2. From (39) and Lemma 3.6, there is \( C_1 > 0 \) such that

\[
\langle L_{\omega,c}^+Q, Q \rangle \geq C_1 \|Q\|_1^2,
\]

where \( Q(x, t) = Im(\zeta(x, t)) \). To estimate \( \langle L_{\omega,c}P, P \rangle \) from below, suppose without loss of generality that \( \|\varphi_{\omega,c}\| = 1 \). We define

\[
P_\perp = P - P_\|, \quad \text{where} \quad P_\| = \langle P, \varphi_{\omega,c} \rangle \varphi_{\omega,c}.
\]
From (12) and (39), it follows that
\[ \langle P_\perp, \varphi_{\omega,c} \rangle = \langle P, \varphi_{\omega,c} \rangle \langle \varphi'_{\omega,c}, -\varphi_{\omega,c} \rangle \]
\[ = \langle P, \varphi_{\omega,c} \rangle \{ \alpha \langle \varphi'_{\omega,c}, \varphi_{\omega,c} \rangle + \frac{1}{2} \langle \varphi''_{\omega,c}, \varphi'_{\omega,c} \rangle \} = 0. \]

Further, since \( \langle P_\perp, \varphi_{\omega,c} \rangle = 0 \), Lemma 3.5 yields
\[ \langle L_{\omega,c} P_\perp, P_\perp \rangle \geq C_2 ||P_\perp||^2, \]
for some \( C_2 > 0 \).

Assuming first that \( ||u_0|| = ||\varphi_{\omega,c}|| = 1 \). Since \( \mathcal{F} \) is a conserved quantity, we obtain \( ||u(t)||^2 = 1 \) for all \( t \). Hence, because (1) is invariant by translations and rotations, we obtain \( \langle P, \varphi_{\omega,c} \rangle = -||\zeta||^2/2 \). Thus, there are positive constants \( C_3 \) and \( C_4 \) such that
\[ \langle L_{\omega,c} P_\perp, P_\perp \rangle \geq C_3 ||P||^2 - C_4 ||\zeta||^4_{1,\alpha}, \]
where \( ||f||^2_{1,\alpha} = ||f'||^2 + 2\alpha ||f||^2 \). Since \( \langle L_{\omega,c} \varphi_{\omega,c}, \varphi_{\omega,c} \rangle < 0 \) it follows that
\[ \langle L_{\omega,c} P_\perp, P_\perp \rangle \geq -C_5 ||\zeta||^4_{1,\alpha}, \]
for some \( C_5 > 0 \). Moreover, from the Cauchy-Schwartz inequality,
\[ \langle L_{\omega,c} P_\perp, P_\perp \rangle \geq -C_6 ||\zeta||^3_{1,\alpha}, \]
for some \( C_6 > 0 \). Therefore, (49), (50), and (51), yield
\[ \langle L_{\omega,c} P_\perp, P_\perp \rangle \geq D_1 ||P||^2_{1,\alpha} - D_2 ||\zeta||^3_{1,\alpha} - D_3 ||\zeta||^4_{1,\alpha}, \]
where \( D_i > 0, i = 1, 2, 3 \).

On the other hand, the last term in (40), is easily estimated as
\[ -\int_0^L \left( K_{c^{-1/2}}(P^2 + Q^2) \right)^2 + 4K_{c^{-1/2}}(\varphi_{\omega,c} P)K_{c^{-1/2}}(P^2 + Q^2) \right) dx \geq -C_7 ||\zeta||^3_{1,\alpha} - C_8 ||\zeta||^4_{1,\alpha}, \]
where \( C_7, C_8 > 0 \).

As a consequence of (40), (48), (52), and (53), we obtain
\[ \triangle \mathcal{R}(t) \geq D_4 ||\zeta||^2_{1,\alpha} - D_5 ||\zeta||^3_{1,\alpha} - D_6 ||\zeta||^4_{1,\alpha}, \]
where \( D_j > 0, j = 4, 5, 6 \). By using standard arguments (see e.g., [9] or [2]), for any \( \varepsilon > 0 \), there is \( \delta > 0 \) such that if \( ||u_0 - \Phi_{\omega,c}||_{1} < \delta, ||v_0 - \Gamma_{\omega,c}||_{1} < \delta, \) and \( ||w_0 - \Lambda_{\omega,c}|| < \delta \), then
\[ \rho_\alpha(u(\cdot,t), \varphi_{\omega,c})^2 = ||\zeta(t)||^2_{1,\alpha} < \varepsilon, \quad \text{for all } t \geq 0. \]

Further, it follows from (40) and the analysis already completed for \( \zeta \) that
\[ \frac{1}{2} \int_0^L \left[ K_{c^{-1/2}}(\eta^2 - 2K^{-1/2}(\varphi_{\omega,c} P) - K_{c^{-1/2}}(P^2 + Q^2)) dx \right]^2 dx \leq \varepsilon \]
and
\[ \frac{1}{2} ||c\eta' + \nu|| \leq \varepsilon. \]

Thus, from the equivalence of the norms \( ||K_{c^{-1/2}}\eta|| \) and \( ||\eta||_1 \), we infer from (56) that
\[ ||\eta(t)||_1 \leq \varepsilon. \]
Moreover, from (57) and (58), we deduce
\[ ||\nu(t)|| \leq \varepsilon. \]
Therefore, the desired inequality (35) follows from (55), (58) and (59). This argument proves that \((\Phi_{\omega,c}, \Gamma_{\omega,c}, \Lambda_{\omega,c})\) is stable relative to small perturbations which preserves the \(L^2_{per}\)-norm of \(\Phi_{\omega,c}\). The general case (that for \(\|u_0\| \neq \|\varphi_{\omega,c}\|\)) follows from the continuous dependence of the function \(\varphi_{\omega,c}\) with respect to \(\omega\).

**Remark 3.** From the proof of Theorem 3.2, we see that one can establish the stability of the periodic waves given in Remark 2 for \(c \approx -1^+\).

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**REFERENCES**

[1] J. Angulo Pava, *Nonlinear stability of periodic traveling wave solutions to the Schrödinger and modified Korteweg-de Vries equations*, J. Differential Equations, **235** (2007), 1–30.

[2] J. Angulo Pava, J. L. Bona, and M. Scialom, *Stability of cnoidal waves*, Adv. Differential Equations, **11** (2006), 1321–1374.

[3] J. Angulo Pava, C. Matheus and D. Pilod, *Global well-posedness and non-linear stability of periodic traveling waves for a Schrödinger-Benjamin-Ono system*, Commun. Pure Appl. Anal., **8** (2009), 815–844.

[4] J. Angulo Pava and F. Natali, *Positivity properties of the Fourier transform and stability of periodic travelling-wave solutions*, SIAM J. Math. Anal., **40** (2008), 1123–1151.

[5] J. Angulo Pava and F. Natali, *Stability and instability of periodic travelling wave solutions for the critical Korteweg-de Vries and non-linear Schrödinger equations*, Physica D, **238** (2009), 603–621.

[6] J. Angulo Pava and A. Pastor Ferreira, *Stability of periodic optical solitons for a nonlinear Schrödinger system*, Proc. Roy. Soc. Edinburgh Sect. A, **139** (2009), 927–959.

[7] J.-B. Baillon and J. M. Chadam, *The Cauchy problem for the coupled Schrödinger-Klein-Gordon equations*, in “Contemporary Developments in Continuum Mechanics and Partial Differential Equations,” Proc. Internat. Sympos., Inst. Mat., Univ. Fed. Rio de Janeiro, Rio de Janeiro, (1978), 37–44.

[8] T. B. Benjamin, *The stability of solitary waves*, Proc. Roy. Soc. London Ser. A, **338** (1972), 153–183.

[9] J. L. Bona, *On the stability theory of solitary waves*, Proc. Roy. Soc. London Ser. A, **344** (1975), 363–374.

[10] P. F. Byrd and M. D. Friedman, “Handbook of Elliptic Integrals for Engineers and Scientists,” 2nd edition, Springer-Verlag, New York-Heidelberg, 1971.

[11] T. Cazenave and P.-L. Lions, *Orbital stability of standing waves for some Schrödinger equations*, Comm. Math. Phys., **85** (1982), 549–561.

[12] J. Colliander, J. Holmer and N. Tzirakis, *Low regularity global well-posedness for the Zakharov and Klein-Gordon-Schrödinger systems*, Trans. Amer. Math. Soc., **360** (2008), 4619–4638.

[13] I. Fukuda and M. Tsutsumi, *On coupled Klein-Gordon-Schrödinger equations. II*, J. Math. Anal. Appl., **66** (1978), 358–378.

[14] T. Gallay and M. Hărăguş, *Orbital Stability of periodic waves for the nonlinear Schrödinger equation*, J. Differential Equations, **234** (2007), 514–581.

[15] T. Gallay and M. Hărăguş, *Orbital Stability of periodic waves for the nonlinear Schrödinger equation*, J. Dynam. Differential Equations, **19** (2007), 825–865.

[16] M. Grillakis, J. Shatah and W. Strauss, *Stability theory of solitary waves in the presence of symmetry. I*, J. Funct. Anal., **74** (1987), 160–197.

[17] M. Grillakis, J. Shatah and W. Strauss, *Stability theory of solitary waves in the presence of symmetry. II*, J. Funct. Anal., **94** (1990), 308–348.

[18] M. Grillakis, *Linearized instability for nonlinear Schrödinger and Klein-Gordon equations*, Comm. Pure Appl. Math., **41** (1988), 747–774.

[19] M. Grillakis, *Analysis of the linearization around a critical point of an infinite-dimensional Hamiltonian system*, Comm. Pure Appl. Math., **43** (1990), 299–333.

[20] T. Kato, “Perturbation Theory for Linear Operators,” reprint of the 1980 edition, Classics in Mathematics, Springer-Verlag, Berlin, 1995.
[21] H. Kikuchi and M. Ohta, *Instability of standing waves for the Klein-Gordon-Schrödinger system*, Hokkaido Math. J., **37** (2008), 735–748.

[22] H. Kikuchi and M. Ohta, *Stability of standing waves for the Klein-Gordon-Schrödinger system*, J. Math. Anal. Appl., **365** (2010), 109–114.

[23] W. Magnus and S. Winkler, “Hill’s Equation,” corrected reprint of the 1966 edition, Dover Publications, Inc., New York, 1979.

[24] F. Natali and A. Pastor Ferreira, *Stability properties of periodic standing waves for the Klein-Gordon-Schrödinger system*, Commun. Pure Appl. Anal., **9** (2010), 413–430.

[25] F. Natali and A. Pastor Ferreira, *Stability and instability of periodic standing wave solutions for some Klein-Gordon equations*, J. Math. Anal. Appl., **347** (2008), 428–441.

[26] M. Ohta, *Stability of solitary waves for coupled Klein-Gordon-Schrödinger equations in one space dimension*, Variational Problems and Related Topics, (Japanese) (Kyoto, 1998), S/=urikaisekikenkyūsho Kōkyūroku, **1076** (1999), 83–92.

[27] M. Ohta, *Stability of stationary states for the coupled Klein-Gordon-Schrödinger equations*, Nonlinear Anal., **27** (1996), 455–461.

[28] H. Pecher, *Global solutions of the Klein-Gordon-Schrödinger system with rough data*, Differential Integral Equations, **17** (2004), 179–214.

[29] S. Rabsztyn, *On the Cauchy problem for the coupled Schrödinger-Klein-Gordon equations in one space dimension*, J. Math. Phys., **25** (1984), 1262–1265.

[30] M. Reed and B. Simon, “Methods of Modern Mathematical Physics,” IV, Analysis of Operators, Academic Press, New York-London, 1978.

[31] X.-Y. Tang and W. Ding, *The general Klein-Gordon-Schrödinger system: Modulational instability and exact solutions*, Phys. Scr., **77** (2008), 1–8.

[32] N. Tsirakis, *The Cauchy problem for the Klein-Gordon-Schrödinger system in low dimensions below the energy space*, Comm. Partial Differential Equations, **30** (2005), 605–641.

[33] M. I. Weinstein, *Lyapunov stability of ground states of nonlinear dispersive evolution equations*, Comm. Pure Appl. Math., **39** (1986), 51–67.

[34] M. I. Weinstein, *Modulational stability of ground states of nonlinear Schrödinger equations*, SIAM J. Math. Anal., **16** (1985), 472–491.

[35] M. Y. Yu and P. K. Shukla, *On the formation of upper-hybrid solitons*, Plasma Phys., **19** (1977), 889–893.

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