PLANAR ALGEBRAS AND THE DECATEGORYIFICATION OF BORDERED KHOVANOV HOMOLOGY

LAWRENCE P. ROBERTS

ABSTRACT. We describe a generalization of the state sum approach to the Jones polynomial (using the conventions of Khovanov homology), and the decategorifications of the type A and type D structures in bordered Khovanov homology [15], to tangle diagrams in oriented planar surfaces. The result is a multi-linear module map that is an invariant of the tangle. These maps compose according to the rules for planar algebras as defined by V. Jones.

1. Introduction

Let $L$ be an oriented link diagram in an oriented sphere $S^2$, with $n_{\pm}(L)$ positive/negative crossings. In the study of Khovanov homology, [7] [3], there is a variant $J_L(q)$ of the Jones polynomial for $L$ which has the form

$$J_L(q) = (-1)^{n_-(L)}q^{n_+ - 2n_-(L)}\bar{J}_L(q)$$

where $\bar{J}_L(q)$ satisfies the following (unoriented) skein relations

$$\bar{J}_U(q) = q + q^{-1}$$
$$\bar{J}_{L\sqcup U}(q) = (q + q^{-1})\bar{J}_L$$
$$\bar{J}_L = \bar{J}_{L_0} - q\bar{J}_{L_1}$$

where $U$ denotes an unknot, $L \sqcup U$ is a link with an unlinked and unknotted component $U$, and $L_0$ and $L_1$ are the 0 and 1 resolutions at a crossing of $L$ (see definition [5] for our resolution conventions). As usual, $\bar{J}_L(q)$ can be given a state-sum representation as a sum over all the complete resolutions of the diagram $L$.

In [7], this polynomial arises as the decategorification of a homology theory. To any oriented link diagram, M. Khovanov associates a bigraded, free Abelian group $\langle \langle L; \mathbb{Z} \rangle \rangle^{*,*}$, whose generators are decorated resolutions of $L$, equipped with a $(1, 0)$ differential $\partial_{\text{Kh}}$, for which the homology $\text{Kh}^{*,*}(L, \mathbb{Z})$ is an invariant of the link $L$. Since the differential is $(1, 0)$ we can decompose $\langle L; \mathbb{Z} \rangle^{*,*}$ as a direct sum of chain complexes $\bigoplus_{j \in \mathbb{Z}} (C^{*,j}, \partial_{\text{Kh}})$ where $C^{*,j}$ is the subgroup of $\langle L; \mathbb{Z} \rangle^{*,*}$ whose second grading equals $j$. If we take the Euler characteristic $\chi(C^{*,j})q^j$ of each of these complexes, we recover the coefficients of $J_L(q)$:

$$J_L(q) = \sum_{j \in \mathbb{Z}} \chi(C^{*,j})q^j$$

In [13], [14] the author, inspired by bordered Floer homology, describes a similar construction for tangles. The formal algebraic structure of these tangle invariants mimics the formal structure of Ozsváth, Lipshitz, and Thurston’s description of bordered Floer homology, [10].
In particular, the construction in [14] takes a tangle diagram $T$ in a disc $D$, with a marked point $* \in \partial D$:

and associates to it a bigraded differential module $\langle \langle T \rangle \rangle$ over a differential bigraded algebra $\mathcal{B} \Gamma_n$, where $T$ has $2n$ endpoints on $\partial D$. Each differential is $(1, 0)$ in the respective bigradings, and the action $\langle \langle T \rangle \rangle \otimes \mathcal{B} \Gamma_n \rightarrow \langle \langle T \rangle \rangle$ preserves the bigradings on its source and target. The homotopy type of $\langle \langle T \rangle \rangle$ as an $A_{\infty}$-module is invariant under the three Reidemeister moves applied to swatches in the diagram $T$, and thus defines a tangle invariant.

In [15] the author describes the decategorifications of $\langle \langle T \rangle \rangle$ and $\langle [T] \rangle$. Like the Jones polynomial, these decategorifications arise as state-sums over all the complete resolutions of the diagram, but, suitably interpreted, yield a sum of module maps. Furthermore, the maps are built, as above, as multiples of maps which satisfy an unoriented skein relation. For tangles $T_1$ and $T_2$ which glue to form a link $L$, the composition of the maps for $\langle \langle T_1 \rangle \rangle$ and $\langle [T_2] \rangle$ is the map which multiplies by $J_L(q)$.

In this paper, we generalize these state-sum representations to tangle diagrams in oriented planar surfaces to obtain invariants for the corresponding tangles. In particular, we will generalize them to planar tangle diagrams, and associate a multi-linear map to any planar tangle diagram. These maps will compose, under the gluing of planar surfaces and tangles, to satisfy the composition laws for a planar algebra, [5]. Furthermore, they restrict to the decategorifications of the Khovanov homology, in the case that the tangle is a link diagram in the surface, and to the decategorifications of $\langle \langle T \rangle \rangle$ and $\langle [T] \rangle$ for tangles as above. Thus, by splitting a link diagram into pieces, and composing the resulting maps, we can recover the Jones polynomial. Furthermore, these maps seem to be a straightforward representation of a well known planar algebra – the Temperley-Lieb planar algebra, and we can construct finite dimensional representations of the Temperley-Lieb algebra and the braid group using these techniques.

The next section provides more detail on these topics, and a sketch of how we will proceed.
2. Detailed summary of results and constructions

We will define an invariant for certain tangles in $\Sigma \times [-1,1]$ where $\Sigma$ is a compact planar surface. To describe this invariant it will be convenient to embed the planar surface in an oriented sphere. We start with a definition which unites many aspects of this paper:

**Definition 1 (Disc configurations).** A disc configuration in an oriented two-sphere $S$ is a non-empty ordered tuple $D = (\tilde{D}_0; \tilde{D}_1, \ldots, \tilde{D}_m)$, $m \geq 0$, and a choice of points $\ast \tilde{D}_i \in S$ for each $i = 0, \ldots, m$, where

1. each $\tilde{D}_i$ is a closed disc embedded in $S$, and oriented from $S$,
2. $\ast \tilde{D}_i \in \partial \tilde{D}_i$ for each $i = 0, \ldots, m$,
3. $\tilde{D}_i \subset \text{int} \tilde{D}_0$ for each $i \geq 1$, and
4. the elements of $\{ \tilde{D}_i \mid i \geq 1 \}$ are pairwise disjoint

For each disc $\tilde{D}_i$ in a disc configuration, we let $\partial \tilde{D}_i$ be the oriented boundary of $\tilde{D}_i$, and $\tilde{D}_i = S \setminus (\text{int} \tilde{D}_i)$. The discs $\tilde{D}_i$ will be called inside discs, while the discs $\tilde{D}_i$ is an outside discs.

The surface associated to a disc configuration $D$ is

$$\Sigma_D = \tilde{D}_0 \setminus \left( \bigcup_{i=1}^{m} \text{int} \tilde{D}_i \right)$$
An example of a disc configuration is given in Figure 1. Besides providing a way to cap off the boundary of a planar surface \( \Sigma \), a disc configuration also imposes two structures on \( \Sigma \): the choice of a special boundary component \( \partial \hat{D}_0 \), and an ordering on the remaining boundary components.

Our main objects of study are certain tangle diagrams for a given disc configuration.

**Definition 2** (Tangles subordinate to \( \mathbb{D} \)). An (even) tangle diagram \( T \), subordinate to a given disc configuration \( \mathbb{D} \), is a tangle diagram in \( \Sigma_{\mathbb{D}} \) such that for each \( i = 0, \ldots, m \) the cardinality of \( Q_i = \partial \hat{D}_i \cap T \) equals \( 2n_i \) for \( n_i \in \{0\} \cup \mathbb{N} \). The signature \( \text{Sign}(T) \) is the tuple \( (n_0; n_1, \ldots, n_m) \).

As part of the tangle diagram structure, each \( Q_i \) is also equipped with the ordering inherited from the orientation of \( \partial \hat{D}_i \setminus \{ \ast \hat{D}_i \} \). See Figure 1 for an example of a tangle diagram, with the points of \( Q_0 \) enumerated according to their ordering.

The tangle diagram \( T \) may or may not be oriented. However, when \( T \) is oriented, \( n_\pm(T) \) will be the number of positive/negative crossings in \( T \) (with reference to the orientation on \( \Sigma_{\mathbb{D}} \)).

As usual a tangle diagram subordinate to \( \mathbb{D} \) defines a tangle in \( \Sigma_{\mathbb{D}} \times [-1,1] \) up to isotopy which fixes the points in \( \partial \Sigma_{\mathbb{D}} \times [-1,1] \), and any diagrams \( T_1 \) and \( T_2 \) which are related by sequences of Reidemeister moves performed in \( \text{int} \Sigma_{\mathbb{D}} \) will define isotopic tangles.

We can now summarize the main constructions of this paper. We will define modules \( \mathcal{I}_{2n} \) over \( \mathbb{Z}[q^{1/2}, q^{-1/2}] \) for each \( n \geq 0 \). If \( T \) is an oriented tangle diagram subordinate to \( \mathbb{D} = (\hat{D}_0; \hat{D}_1, \ldots, \hat{D}_m) \) where \( m \geq 1 \), with \( \text{Sign}(T) = (n_0; n_1, \ldots, n_m) \) we will construct a map

\[
Z_T : \mathcal{I}_{2n_1} \otimes \mathcal{I}_{2n_2} \otimes \cdots \otimes \mathcal{I}_{2n_m} \rightarrow \mathcal{I}_{2n_0}
\]

where the tensor products are taken over \( \mathbb{Z}[q^{1/2}, q^{-1/2}] \). \( Z_T \) will be called the partition map for \( T \), following the terminology of [5].

We will show that the maps \( Z_T \) have the following properties:

1. \( Z_T \) is invariant under Reidemeister moves performed in \( \text{int} \Sigma_{\mathbb{D}} \) and thus defines an isotopy invariant of the tangle in \( \Sigma_{\mathbb{D}} \times [-1,1] \)

2. [Temperley-Lieb] If \( T \) has a unlinked component embedded in \( \Sigma_{\mathbb{D}} \), and separated from the rest of the diagram, then

\[
Z_T = (q + q^{-1})Z_{T'}
\]

where \( T' \) is the tangle diagram found by removing this component,

3. [Conjugation] For \( p \in \mathbb{Z}[q^{1/2}, q^{-1/2}] \) let the conjugate \( p^* \) be the polynomial \( p(q^{-1}) \). Then \( \mathcal{I}_{2n} \) admits a conjugate linear involution \( \xi \mapsto \xi^* \). If \( T^* \) is the tangle diagram where each crossing is the mirror of that in \( T \), then

\[
Z_{T^*}(\xi^*) = Z_T(\xi)^*
\]
(4) [Normalization] If $T$ has signature $(0; 0)$ then $T$ comes from a link diagram $L$ in $\Sigma_D$, and $Z_T : \mathbb{Z}[q^{1/2}, q^{-1/2}] \to \mathbb{Z}[q^{1/2}, q^{-1/2}]$ is multiplication by the polynomial $J_L(q)$ described in equation [4].

(5) [Composition] The maps $Z_T$ satisfy compositional properties similar to those of a planar algebra (see section 2.1) as defined by Vaughan Jones in [3], [4].

Comment #1: Item 4 can be extended to include the decategorifications of bordered Khovanov homology defined in [15]. A tangle $T$ with signature $(n_0; 0)$ is a tangle in $\widehat{D}_0$ which misses $D_1$. It can be considered to be an inside tangle, in the terminology of bordered Khovanov homology ([14] and [13]). The map $Z_T$ equals the map $\mathbb{Z}[q^{1/2}, q^{-1/2}] \to I_{2n_0}$ defined in [15] using the decategorification of the type A invariants of [14]. When $T$ has signature $(0; n_1)$, $T$ defines a tangle in $\widehat{D}_1$ which can be considered as an outside tangle in the terminology of [13]. To outside tangles, bordered Khovanov homology assigns a type D structure ([10], [13]) as in Bordered Heegaard-Floer homology, whose decategorification is defined in [15]. A tangle $T$ defines a tangle in $\text{int} D_2$. Thus, the map $Z_T$ defined in this paper.

Comment #2: This items in this list are deeply related to the definition of a planar algebra, [4]. In particular, the assumption that there are maps $Z_P$ for each planar diagram which satisfy the composition formula means that $\{I_{2n}\mid n \geq 0\}$ satisfies the minimal requirements for an (unshaded) planar algebra. In fact, the identity annular tangle will have partition map equal to the identity, and the conjugation map also correspond to requirements on planar algebras. The simplest planar algebra is the Temperley-Lieb planar algebra. For a commutative ring $R$, let $\delta \in R$ be an element of the ring. Let $TL_{2n}$ be the free $R$ module generated by planar matchings of the points $1, \ldots, 2n$ on the boundary of a disc $D^2$. Given a planar diagram $P$ with signature $(n_0; n_1, \ldots, n_m)$, the map $Z_P^{TL}$ takes $TL_{2n_1} \otimes \cdots \otimes TL_{2n_m} \to TL_{2n_0}$. The image of the map on a basis element $a_1 \otimes \cdots \otimes a_m$ is found by taking the diagram for $P$ and filling in each $\widehat{D}_i$, $i \geq 1$, with the matching $a_i$ on the $2n_i$ points in $Q_i$. This result of this gluing is a planar diagram in $\widehat{D}_0$. Let $N$ be the number of circles in this diagram, and let $m$ be the matching found by deleting all the circles. Then $Z_T^{TL} : a_1 \otimes \cdots \otimes a_m \to \delta^N m$. Thus, assuming the Temperley-Lieb property means that the maps $Z_P$ found in this paper are a representation of the Temperley-Lieb planar algebra with $R = \mathbb{Z}[q^{1/2}, q^{-1/2}]$ and $\delta = q + q^{-1}$.

2.1. Compositions. Suppose $T$ and $R$ are (unoriented) tangle diagrams subordinate to disc configurations $D_T$ and $D_R$ with signatures $\text{Sign}(T) = (n_0; n_1, \ldots, n_m)$ and $\text{Sign}(R) = (n'_0; n'_1, \ldots, n'_m)$ such that $n_0 = n'_i$ for some $i \in \{1, \ldots, m'\}$. Then we can glue $T$ and $R$ along $\partial \widehat{D}_i$ and $\partial \widehat{D}_0$ to obtain a new tangle diagram $R \circ_i T$. More specifically, we remove $\text{int} \widehat{D}_0$ from $S_T$, the sphere containing $T$, and $\text{int} \widehat{D}'_i$ from $S_R$, and glue the resulting discs $S_T \backslash \text{int} \widehat{D}_0$ and $S_R \backslash \text{int} \widehat{D}'_i$ along their boundaries, so that $\ast \widehat{D}_0$ is glued to $\ast \widehat{D}'_i$, and each point in $Q_0$ is glued to the corresponding point of $Q'_i$, while preserving the orderings.

The tangle $R \circ_i T$ is subordinate to the disc configuration $D_R \circ_i D_T$ with discs

$$(\widehat{D}_0, \widehat{D}_1, \ldots, \widehat{D}_{i-1}, \widehat{D}_i, \ldots, \widehat{D}_m, \widehat{D}_{i+1}, \ldots, \widehat{D}_{m'})$$
and the inherited marked points. It has the same orderings on the sets \(Q_k\) for all discs in \(\mathbb{D}_R \circ_i \mathbb{D}_T\). Furthermore,

\[
S(R \circ_i T) = (n_0'; n_1', \ldots, n_{i-1}', n_1, \ldots, n_m, n_{i+1}', \ldots, n_m')
\]

When \(T\) and \(R\) are oriented tangle diagrams, we will additionally require that there is an orientation of \(R \circ_i T\) which restricts to the original orientations on \(T\) and \(R\). Otherwise, \(R \circ_i T\) will be undefined.

Following the construction of planar algebras, \[5\], \[6\], we define a composition \(Z_R \circ_i Z_T\) of \(Z_R\) and \(Z_T\) along the \(i\)th entry in the tensor product defining the domain of \(Z_R\). In particular, if \(a_i \in \mathbb{I}_{2n_i'}\) and \(b_j \in \mathbb{I}_{2n_j}\), then

\[
Z_{R \circ_i T}(a_1 \otimes \cdots \otimes a_{i-1} \otimes b_1 \otimes \cdots \otimes b_{m'} \otimes a_{i+1} \otimes \cdots \otimes a_m) = Z_R(a_1 \otimes \cdots \otimes a_{i-1} \otimes Z_T(b_1 \otimes \cdots \otimes b_{m'}) \otimes a_{i+1} \otimes \cdots \otimes a_m)
\]

Then the partition maps \(Z_T\) will be shown to satisfy the defining property of a planar algebra:

**Theorem 3.** Let \(T\) and \(R\) be tangle diagrams so that \(R \circ_i T\) is defined. Then the partition map \(Z_{R \circ_i T}\) equals the map \(Z_R \circ_i Z_T\).

**2.2. Strategy and Planar algebras.** We will now go into more detail about how the maps \(Z_T\) are constructed. Let \(T\) be a oriented tangle diagram subordinate to \(\mathbb{D}_T\) with signature \(\text{Sign}(T) = (n_0; n_1, \ldots, n_m)\). Let \(\text{cr}(T)\) be the set of crossings in \(T\).

Our strategy is to define \(Z_P\) for planar diagrams – those diagrams which can be thought of as a submanifold of \(\Sigma_D\):

**Definition 4 (Planar diagram).** A planar diagram \(P\) subordinate to a disc configuration \(\mathbb{D}\) is a tangle diagram with no crossings: \(\text{cr}(P) = \emptyset\)

For an example of a planar diagram, see Figure 1. Each planar diagram, being a tangle diagram, has all the structures we have discussed, including the ordering on the sets \(Q_i\). Defining \(Z_P\) for planar diagrams occupies most of the rest of the paper; however, we can see the full construction if we make a few assumptions about these maps.

For the remainder of this section, assume we have already constructed \(Z_P\) for each planar diagram and \(Z_P\) satisfies the Temperley-Lieb property (4), the Normalization property (4), and the Composition property as described in the conclusion of Theorem 3 above. We will show how to use these to construct \(Z_T\) for each tangle diagram \(T\). The connection with planar diagrams comes through resolutions:

**Definition 5.** A resolution \(\rho\) of \(T\) is a map \(\rho : \text{cr}(T) \to \{0, 1\}\). For each resolution, \(\rho\), there is a planar diagram, \(\rho(T)\), called a resolution diagram, obtained by locally replacing (disjoint) neighborhoods of the crossings of \(T\) using the following rule for each crossing \(c \in \text{cr}(T)\):
The set of resolutions will be denoted \( \text{RES}(T) \). The (unshifted) homological dimension of \( \rho \in \text{RES}(T) \) is

\[
h(\rho) = \sum_{c \in \text{CR}(T)} \rho(c)
\]

For a tangle diagram, each resolution diagram \( \rho(T) \) is a planar diagram. Note that even if \( T \) is oriented, its resolution planar diagrams will be unoriented. W can use resolutions to define the partition map for a tangle diagram:

**Definition 6.** Let \( T \) be an oriented tangle diagram subordinate to a disc configuration \( D \) with signature \( \text{SIGN}(T) = (n_0; n_1, \ldots, n_m) \). The partition map

\[
Z_T : I_{2n_1} \otimes I_{2n_2} \otimes \cdots \otimes I_{2n_m} \rightarrow I_{2n_0}
\]

is defined to be the map

\[
Z_T = (-1)^{n_T} q^{(n_+ - 2n_-)(T)} \tilde{Z}_T
\]

where

\[
\tilde{Z}_T = \sum_{\rho \in \text{RES}(T)} (-q)^{h(\rho)} Z_{\rho(T)}
\]

\( \tilde{Z}_T \) is called the unshifted partition map for \( T \). While \( \tilde{Z}_T \) is not an isotopy invariant of the tangle, oriented or unoriented, it does satisfy a skein relation:

**Proposition 7** (Unshifted skein relations). Let \( T \) be an unoriented tangle diagram. Suppose \( c \in \text{CR}(T) \), and let \( T_0 \) and \( T_1 \) be the tangle diagrams obtained from performing the 0 and 1 resolutions locally at \( c \). Then

\[
\tilde{Z}_T = \tilde{Z}_{T_0} - q \cdot \tilde{Z}_{T_1}
\]

**Proof:** Take the summation in Definition 6 and group the terms into those with \( \rho(c) = 0 \) and those with \( \rho(c) = 1 \). Let the former set be \( S_0 \) and the latter be \( S_1 \). Then \( \tilde{Z}_{T_0} \) is exactly the sum of the terms in \( S_0 \). \( T_1 \) has one fewer crossing than \( T \), and we let \( \rho' \) be the restriction of \( \rho \) to the crossings in \( T_1 \). Then \( h(\rho) = 1 + h(\rho') \), since \( h(\rho) \) counts the 1-resolution at \( c \), while \( h(\rho') \) does not. Consequently, so the terms in \( S_1 \) are \(-q\) times the terms in \( \tilde{Z}_{T_1} \), from which the skein relation follows. ♦

It immediately follows that \( Z_T \) will satisfy properties 2 and 3 above, since \( Z_{\rho(T)} \) satisfies them for each resolution \( \rho(T) \). For example, if there is a circle component \( C \) of \( T \) which is embedded in \( \Sigma_D \) and geometrically split from the rest of the diagram, then \( C \) will be present in each \( \rho(T) \). Thus if \( \rho(T') \) is the diagram without this circle, \( Z_{\rho(T')} = (q + q^{-1}) Z_{\rho(T)} \) for each resolution. Since \( Z_T \) is just a sum of these maps, multiplied by \((-1)^n q^m\) factors, we see that \( Z_T = (q + q^{-1}) Z_{T'} \), where \( T' \) is the tangle found by deleting \( C \) from \( T \).

Similarly, if \( T \) has signature \((0; 0)\), then \( \rho(T) \) consists of closed circles embedded in \( \Sigma_D \).
By assumption, each map $Z_{\rho(T)}$ is just multiplication by $J_{\rho(T)}(q)$. Using the skein relations in the introduction, $J_{\rho(T)}(q) = (q + q^{-1})^{n_C(\rho)}$ where $n_C(\rho)$ is the number of circles in $\rho(T)$. Substituting into equation (2) yields that $Z_T$ is multiplication by the polynomial

$$(-1)^{n_-(T)} q^{(n_+ - 2n_-)(T)} \sum_{\rho \in \text{RES}(T)} (-q)^{h(\rho)} (q + q^{-1})^{n_C(\rho)}$$

This formula, however, is the state sum representation of $J_T(q)$ for Khovanov’s version of the Jones polynomial.

It is somewhat more involved to see that conclusion of theorem 3 follows immediately from the same result for planar diagrams. Nevertheless,

**Proposition 8.** Let $T$ and $R$ be oriented tangles subordinate to $\mathbb{D}_T$ and $\mathbb{D}_R$ such that $R \circ_i T$ is defined as an oriented tangle. If the partition maps $Z_P$ for planar diagrams satisfy the composition property, then

$$Z_{R \circ_i T} = Z_R \circ_i Z_T$$

**Proof of proposition:** First, $\text{CR}(R \circ_i T) = \text{CR}(R) \sqcup \text{CR}(T)$, and $n_{\pm}(R \circ_i T) = n_{\pm}(R) + n_{\pm}(T)$ since the gluing preserves orientations on $\Sigma_{\mathbb{D}_R}$ and $\Sigma_{\mathbb{D}_T}$. Consequently $(n_+ - 2n_-)(R) + (n_+ - 2n_-)(T) = (n_+ - 2n_-)(R \circ_i T)$. Therefore, it is enough to verify that $\bar{Z}_{R \circ_i T} = \bar{Z}_R \circ_i \bar{Z}_T$ since if this is true, then

$$Z_{R \circ_i T} = \left((-1)^{n_-(R \circ_i T)} q^{(n_+ - 2n_-)(R \circ_i T)} \bar{Z}_{R \circ_i T}\right) \circ_i \left((-1)^{n_-(T)} q^{(n_+ - 2n_-)(T)} \bar{Z}_T\right)$$

$$= Z_R \circ_i Z_T$$

Furthermore, $\text{RES}(R \circ_i T)$ is in one-to-one correspondence with $\text{RES}(R) \times \text{RES}(T)$ since the choices at the crossings are made independently for the two tangles. If $\rho \in \text{RES}(R \circ_i T)$ corresponds to $(\rho_2, \rho_1)$ then $\rho(R \circ_i T) = \rho_2(R) \circ_i \rho_1(T)$ and $h(\rho) = h(\rho_1) + h(\rho_2)$. Since we are assuming that $Z_P(R \circ_i T) = Z_{\rho_2(R)} \circ_i Z_{\rho_1(T)}$, formula (2) and these observations imply

$$\bar{Z}_R \circ_i \bar{Z}_T = \left(\sum_{\rho_2 \in \text{RES}(R)} (-q)^{h(\rho_2)} Z_{\rho_2(R)}\right) \circ_i \left(\sum_{\rho_1 \in \text{RES}(T)} (-q)^{h(\rho_1)} Z_{\rho_1(T)}\right)$$

$$= \sum_{(\rho_2, \rho_1) \in \text{RES}(R) \times \text{RES}(T)} (-q)^{h(\rho_2) + h(\rho_1)} Z_{\rho_2(R)} \circ_i Z_{\rho_1(T)}$$

$$= \sum_{\rho \in \text{RES}(R \circ_i T)} (-q)^{h(\rho)} Z_{\rho(R \circ_i T)}$$

$$= \bar{Z}_{R \circ_i T}$$

This completes the calculation. ♦

We can also verify the conjugation identity, Item 3, above, if we assume that $Z_P(\xi^*) = Z_P(\xi)^*$ for any planar diagram. Note that the mirror $P^* = P$ since $P$ has no crossings. Recall that conjugation on $\mathbb{Z}[q^{1/2}, q^{-1/2}]$ amounts to $p(q) \longrightarrow p(q^{-1})$. On the modules $\mathcal{I}_{2n}$,
if $\xi = p(q)\nu$ then $\xi^* = p(q^{-1})\nu^*$. This is enough to prove the next proposition, subject to our assumption about $Z_P$.

**Proposition 9.** Suppose $T$ is a tangle diagram subordinate to $D$, and let $T^*$ be the planar tangle diagram subordinate to $D$ which has the mirror of the crossings in $T$. Then $Z_{T^*}(\xi^*) = (Z_T(\xi))^*$

**Proof:** For each $\rho \in \text{RES}(T)$, let $\rho^* \in \text{RES}(T^*)$ be the resolution with $\rho^*(c) = 1 - \rho(c)$. Then $h(\rho^*) = n - h(\rho)$ where $n$ is the number of crossings, and $\rho(T) = \rho^*(T^*)$ as planar diagrams. Thus,

$$
\bar{Z}_{T^*}(\xi^*) = \sum_{\rho^* \in \text{RES}(T^*)} (-q)^{h(\rho^*)} Z_{\rho(T^*)}(\xi^*)
$$

$$
= (-q)^n \sum_{\rho \in \text{RES}(T)} (-q^{-1})^{h(\rho)} Z_{\rho(T)}(\xi^*)
$$

$$
= (-q)^n \sum_{\rho \in \text{RES}(T)} (-q^{-1})^{h(\rho)} (Z_{\rho(T)}(\xi))^*
$$

$$
= (-q)^n (\bar{Z}_T(\xi))^*
$$

However, $n_{\pm}(T^*) = n - n_{\pm}(T)$. So

$$
(-1)^{n_{-}(T^*)}q^{n_{+}(T^*)-2n_{-}(T^*)} = (-1)^n (-1)^{n_{-}(T)}q^{n_{-}(T)-2n_{+}(T)+2n_{-}(T)} = (-q)^{-n} (-1)^{n_{-}(T)}q^{-(n_{+}-2n_{-})(T)}
$$

Thus,

$$
Z_{T^*}(\xi^*) = (-1)^{n_{-}(T^*)}q^{(n_{+}+2n_{-})(T^*)} \bar{Z}_{T^*}(\xi^*)
$$

$$
= (-q)^{-n} (-1)^{n_{-}(T)}q^{-(n_{+}+2n_{-})(T)} (-q)^n (\bar{Z}_T(\xi))^*
$$

$$
= (-1)^{n_{-}(T)}q^{-(n_{+}+2n_{-})(T)} (\bar{Z}_T(\xi))^*
$$

$$
= (-1)^{n_{-}(T)}q^{(n_{+}+2n_{-})(T)} Z_T(\xi)
$$

$$
= (Z_T(\xi))^*
$$

From which the proposition follows. $\diamond$

2.3. **Outline of the rest of the paper:** It remains to construct $Z_P$ for a planar diagram, and verify that $Z_P$ satisfies the Temperley-Lieb, conjugation, normalization, and composition properties assumed above. In section 3 we will describe the main tool for constructing $Z_P$. In section 4 we will construct $Z_P$ and prove its main properties. We will then turn to proving the invariance of $Z_T$ under Reidemeister moves applied to the tangle $T$. Finally, we describe the representations of the braid groups obtained from this construction. Examples are present in most sections.

**Comments about the equivalence of tangle diagrams:** We will consider various objects, such as tangle diagrams and planar tangles, subordinate to a disc configuration $D$. In each case, these will be considered up to equivalence. As these equivalences are all defined in the same
manner, we mention the technical details here, and omit their mention in other contexts. We start with the basic structure:

**Definition 10.** Let $\mathbb{D} = (\overrightarrow{D_0}, \overrightarrow{D_1}, \ldots, \overrightarrow{D_m})$ be a disc configuration in an oriented two-sphere $S$ and $\mathbb{D}' = (\overrightarrow{D'_0}, \overrightarrow{D'_1}, \ldots, \overrightarrow{D'_m})$ be a disc configuration in the sphere $S'$. Then $\mathbb{D}$ is equivalent to $\mathbb{D}'$ if $m = m'$, and there is an orientation preserving diffeomorphism $\phi : S \rightarrow S'$ such that

1. $\phi$ maps $\overrightarrow{D_i}$ diffeomorphically to $\overrightarrow{D'_i}$
2. $\phi(\ast\overrightarrow{D_i}) = \ast\overrightarrow{D'_i}$

It follows that an equivalence $\phi$ induces a diffeomorphism of $\overrightarrow{D_i}$ with $\overrightarrow{D'_i}$, and an orientation preserving diffeomorphism of $\partial\overrightarrow{D_i}$ with $\partial\overrightarrow{D'_i}$, for each $i$.

For each of the objects subordinate to $\mathbb{D}$, equivalence will mean that there is an equivalence of the underlying disc configurations which preserve the additional data. Thus, a tangle diagram $T$ subordinate to $\mathbb{D}$ and a tangle diagram $T'$ subordinate to $\mathbb{D}'$ are equivalent if there is an equivalence $\phi$ of $\mathbb{D}$ with $\mathbb{D}'$ which maps the diagram $T$ to the diagram for $T'$. Among other things, this requires $\psi$ to restrict to an order preserving bijection $Q_i \leftrightarrow Q'_i$ for each $i = 0, \ldots, m$.

### 3. Decorated, Multiply Cleaved Links

In this section we describe the workhorse objects needed to define partition functions $Z_P$ for a planar diagram $P$. Suppose $P$ is subordinate to $\mathbb{D}$. The main idea will be to choose planar matchings of the points $Q_i$ in $\partial\overrightarrow{D_i}$ and use these in $\overrightarrow{D_0}$ and $\overrightarrow{D_i}$, for $i \geq 1$, to close the planar tangle diagram into a planar link diagram. We will retain the information from $\mathbb{D}$, however. We start by making this precise.

#### 3.1. Definitions and general setup.

**Definition 11.** A decorated, multiply cleaved link $(M, \sigma)$ subordinate to a disc configuration $\mathbb{D}$ in an oriented two-sphere $S$ is a (possibly empty) embedded submanifold $\text{Circles}(M) \subset S$, consisting of finitely many circles, and a decoration map $\sigma : \text{Circles}(M) \rightarrow \{+,-\}$, such that

1. $\text{Circles}(M)$ intersects the boundary $\partial\overrightarrow{D_i}$ of each disc $\overrightarrow{D_i}$ transversely, and away from $\ast\overrightarrow{D_i}$, and
2. $\text{Circles}(M)$ has no component contained in $\text{int}\overrightarrow{D_0}$, or in $\text{int}\overrightarrow{D_i}$ for $i = 1, \ldots, m$.

For an example of a multiply cleaved link, see the left diagram in Figure 2.

Because of items 1 and 2, $\overrightarrow{m_i} = \overrightarrow{D_i} \cap \text{Circles}(M)$, $i = 1, \ldots, m$, (and $\overrightarrow{m_0} = \overrightarrow{D_0} \cap \text{Circles}(M)$), are each planar matchings on the points $Q_i$ inside $\overrightarrow{D_i}$ (or $\overrightarrow{D_0}$), where

**Definition 12.** Let $D$ be a closed, two-dimensional disc, and $Q_D$ be an even number of points in the boundary of $D$. A planar matching $m$ of $Q_D$ is a disjoint collection of arcs
Figure 2. The left diagram is a decorated, multiply cleaved link \((M, \sigma)\) with \(\mathcal{P}(M, \sigma)\) equal to the planar diagram in Figure 1. The inside and outside matchings are shown as dashed lines. The 2\textsuperscript{nd} boundary \(\partial_2(M, \sigma)\) is shown on the right. In the notation of section 3.2 \(\partial_2(M, \sigma) = ((24), (24), -+).\)

whose boundary is exactly \(Q_D\). If \(D\) is oriented and \(Q_D\) is equipped with a linear ordering compatible with the orientation on \(\partial D\), the matching will be called inside and be denoted \(\hat{m}\). If \(D\) is oriented and \(Q_D\) is equipped with the ordering compatible with \(-\partial D\) then the matching is called outside and will be denoted \(\overline{m}\).

For a multiply cleaved link \((M, \sigma)\), and our orientation conventions, each \(\hat{m}_i\) is an inside matching, while \(\overline{m}_0\) is an outside matching.

**Definition 13.** Let \((M, \sigma)\) be a decorated, multiply cleaved link.

(1) \(\text{CutCircs}(M)\) is the subset of \(\text{Circles}(M)\) which intersect \(\partial \overline{D}\) for some \(\overline{D} \in \text{Discs}(M)\). These will be called cut circles. The cardinality of \(\text{CutCircs}(M)\) will be denoted \(k_M\).

(2) \(\text{FreeCircs}(M)\) is the subset of \(\text{Circles}(M)\) which do not intersect \(\partial \overline{D}\) for each \(\overline{D} \in \text{Discs}(M)\). These will be called free circles.

(3) \(\text{Circles}(M, \sigma)_\pm\) is the subset of \(\text{Circles}(M)\) mapped to \(\pm\) by \(\sigma\)

Each multiply cleaved link is built on top of a planar diagram:

**Definition 14.** For a decorated, multiply cleaved link \((M, \sigma)\) subordinate to \(D\), the associated planar diagram \(\mathcal{P}(M)\) is the planar diagram \(\text{Circles}(M) \cap \Sigma_D\).
For example, given the multiply cleaved link \((M, \sigma)\) in Figure 2, \(\mathcal{P}(M, \sigma)\) is the planar tangle in Figure 1. Equivalences of \(M\) induce equivalences of \(\mathcal{P}(M)\). Furthermore, \(\mathcal{P}(M)\) is subordinate to \(\mathcal{D}\) and inherits the orderings and marked points of \(M\). We use the associated tangle to import all the notions defined for tangle diagrams to the diagrams of multiply cleaved links.

**Definition 15.** Let \((M, \sigma)\) be a decorated, multiply cleaved link. The signature \(\text{SIGN}(M)\) of \((M, \sigma)\) is the signature \(\text{SIGN}(\mathcal{P}(M))\). The set of equivalence classes of decorated, multiply cleaved links \((M, \sigma)\) with signature \((n_0n_1, \ldots, n_m)\) will be denoted \(\mathcal{MCL}(n_0n_1, \ldots, n_m)\).

**Definition 16.** Let \(P\) be a planar diagram subordinate to \(\mathcal{D}\). Then
\[
\mathcal{M}(P) = \{ (M, \sigma) \in \mathcal{MCL}(\text{SIGN}(P)) \mid \mathcal{P}(M) = P \}
\]

Given \(P\) we can obtain all multiply cleaved links \((M, \sigma)\) in \(\mathcal{M}(P)\) by choosing an outside matching \(\tilde{m}_0\) of \(Q_0\), and inside matchings \(\tilde{m}_i\) of \(Q_i\) for \(i \geq 1\), and gluing these into the discs \(\tilde{D}_0, \tilde{D}_i, i \geq 1\), respectively. We then choose a decoration for each of the resulting circles.

The signature and the associated tangle do not depend on the decoration map. The decoration, however, returns in the weight of the link:

**Definition 17.** The weight \(W(M, \sigma) \in \mathbb{Z}[q^{1/2}, q^{-1/2}]\) of a decorated, multiply cleaved link \((M, \sigma)\) is the quantity
\[
W(M, \sigma) = \prod_{C \in \text{CIRCLES}(M, \sigma)_+} q^{1 - \frac{1}{2} N_M(C)} \cdot \prod_{C \in \text{CIRCLES}(M, \sigma)_-} q^{-1 + \frac{1}{2} N_M(C)}
\]

where \(N_M(C)\) is the number of circles \(\partial \tilde{D}_i, i \geq 0\), which non-trivially intersect the circle \(C \in \text{CIRCLES}(M)\). If \(\text{CIRCLES}(M) = \emptyset\) then \(W(M, \sigma) = 1\).

**Definition 18.** Let \((M, \sigma)\) be a decorated, multiply cleaved link. The conjugate \((M, \sigma)^*\) is the decorated, multiply cleaved link \((M, \sigma')\) where \(\sigma'(C) = \pm \sigma(C)\).

The conjugate of \((M, \sigma)\) is obtained by fixing the discs and circles of \(M\) and flipping the decoration assigned by \(\sigma\) on each circle in \(\text{CIRCLES}(M)\). The following proposition is obvious from the definitions.

**Proposition 19.** Let \((M, \sigma)\) be a decorated multiply cleaved link. Then \(\text{SIGN}(M, \sigma)^* = \text{SIGN}(M, \sigma)\), and \(W((M, \sigma)^*) = W(M, \sigma)^* \in \mathbb{Z}[q^{\pm 1/2}]\).

3.2. **A special case: cleaved links.** As an important special case, consider those decorated, multiply cleaved links \((M, \sigma)\) subordinate to a minimal disc configuration \(\mathcal{D}\):

**Definition 20.** A decorated, cleaved link \((L, \sigma)\) in an oriented two-dimensional sphere \(S\) is a decorated, multiply cleaved link subordinate to a disc configuration \(\mathcal{D} = (\tilde{D}_0)\) containing only one disk, such that \(\text{FREECIRCS}(L) = \emptyset\).

The second condition in the definition implies that \(\text{CIRCLES}(M) = \text{CUTCIRCS}(M)\), and thus each circle intersects \(\partial \tilde{D}_0\). For a cleaved link, we will usually omit the subscript on
The signature of a decorated, cleaved link is determined by one number: $n_L$ is half the cardinality of $Q_L$. Due to their importance, we will use a different notation for the equivalence classes:

**Definition 21.** $\mathcal{CL}_n$ is the set of equivalence classes of decorated, cleaved links $(L, \sigma)$ with $n_L = n$.

**Examples:** The elements of $\mathcal{CL}_2$ are depicted in Figure 3.

Each equivalence class of decorated, cleaved links in $\mathcal{CL}_n$ can be uniquely described by the following combinatorial data:

1. Two permutations $\hat{m}_L$ and $\overrightarrow{m}_L$ of the set $\{2, 4, \ldots, 2n\}$
2. A ordered $k_L$-tuple from $\{+, -\}^{k_L}$

Given $(L, \sigma)$ we obtain $\hat{m}_L$ from the planar matching $L \cap \overleftarrow{D}_L$ of $Q_L$ in $\overleftarrow{D}_L$. Using the ordering of $Q_L$, we know that the $i^{th}$ term in the permutation is $2j$ if the arc starting at the point labeled as $p_{2i-1}$ ends at the point $p_{2j}$. Similarly, $L \cap \overrightarrow{D}_i$ determines a permutation of $\{2, 4, \ldots, 2n\}$ by looking at the endpoints of the arcs in $\overrightarrow{D}_L$, starting at the odd numbered points in numerical order. Not all permutations of $\{2, 4, \ldots, 2n\}$ correspond to planar matchings, but if we are given two which do, we can glue them to obtain a cleaved link. The decoration $\sigma$ on $L$ determines an element in $\{+, -\}^{k_L}$ once we order the circles in $\text{CutCircs}(L)$. We order the circles using the smallest point in $Q_L$ that lies on the circle: so $C < C'$ if, and only if, the smallest subscript of any $p_j \in Q_L \cap C$ is less than the smallest subscript of any $p_j \in Q_L \cap C'$.

For example, the decorated, cleaved links in Figure 3 can be described with this convention. The decorated cleaved link $B_{-+}$ is specified by $\hat{m}_L = (42)$ and $\overrightarrow{m}_L = (42)$. The $-$ decoration occurs on the circle through the point 1, while the second circle, which is nested inside the other circle in the diagram, is decorated with a $+$.

### 3.3. Gluing decorated, multiply cleaved links

Suppose $(M, \sigma)$ is a multiply cleaved link subordinate to $\mathbb{D}$ with signature $(n_0; n_1, \ldots, n_m)$ and $(N, \nu)$ is a decorated multiply cleaved link with signature $(n'_0; n'_1, \ldots, n'_{m'})$. We will need a gluing $c_i$ for multiply cleaved links. Although it will be necessary for $n_0 = n'_0$, as for tangle diagrams, the presence of planar matching in the discs $\overleftarrow{D}_0$ and $\overrightarrow{D}_i$ will require a slightly different approach. In fact, we will enhance the condition $n_0 = n'_0$ using the $i^{th}$-boundary of $(M, \sigma)$.

The $i^{th}$-boundary map

$$\partial_i : M\mathcal{CL}(\text{Sign}(M)) \rightarrow \mathcal{CL}_{n_i}$$

is the map which takes $(M, \sigma)$ to the cleaved link found by ignoring all the information in $\mathbb{D}$ except the disc $\overleftarrow{D}_i$ and its marked point, and using only the circles intersecting $\overrightarrow{D}_i$. The $2^{nd}$ boundary of the multiply cleaved link on the left of Figure 2 is the diagram on the right of that figure. More specifically, $\partial_i(M, \sigma)$ is the decorated cleaved link in the same sphere as $M$, with
The twelve generators of \( I_4 \) grouped based on their inside and outside matchings. There are two generators of type \( A \) and \( D \), and four of type \( B \) and \( C \), as determined by the choice of decorations.

\( \partial_i(M, \sigma) \) is subordinate to \( \hat{D}_i \),

\( *\partial_i(M, \sigma) = *\hat{D}_i \)

\( \text{CIRCLES}(\partial_i(M, \sigma)) \) is the subset of \( \text{CUTCIRCS}(M, \sigma) \) which intersect \( \hat{D}_i \),

\( \sigma_{\partial_i(M, \sigma)} \) is the restriction of \( \sigma \) to \( \text{CIRCLES}(\partial_i(M, \sigma)) \)

The map \( \partial_i \) respects equivalences of multiply cleaved links. Furthermore, \( Q_i(M, \sigma) = Q_i \) as ordered sets, and \( \partial_i((M, \sigma)^*) = (\partial_i(M, \sigma))^* \).

Using the maps \( \partial_i \) we can refine the definition of the set of multiply cleaved links associated to a planar diagram \( \mathcal{M}(P) \).

**Definition 22.** Let \( P \) be a planar diagram subordinate to \( \mathbb{D} \) with signature \((n_0; n_1, \ldots, n_m)\). Let \((L_i, \sigma_i) \in \mathcal{CL}_{n_i} \). Then

\[
\mathcal{M}_P((L_0, \sigma_0); (L_1, \sigma_1), \ldots, (L_m, \sigma_m)) = \{ (M, \sigma) \in \mathcal{M}(P) \mid \partial_i(M, \sigma) = (L_i, \sigma_i) \}
\]

We will usually shorten the notation to \( \mathcal{M}_P(L_0; L_1, \ldots, L_m) \) with the understanding that there are decoration maps \( \sigma_i \) on each of the cleaved links.

Our goal is to define a map

\[
\circ_i : \mathcal{M}_R(L'_0; L'_1, \ldots, L'_{i-1}, L, L'_{i+1}, \ldots, L'_m) \times \mathcal{M}_P(L; L_1, \ldots, L_m) \rightarrow \mathcal{M}_{R \circ_i P}(L'_0; L'_1, \ldots, L'_{i-1}, L_i, \ldots, L_m, L'_{i+1}, \ldots, L'_m)
\]
so that the following proposition is true:

**Proposition 23.** Let $P, R$ be planar diagrams subordinate to $D_P$ and $D_R$ with signatures $(n_0; n_1, \ldots, n_m)$ and $(n'_0; n'_1, \ldots, n'_m)$ respectively. Suppose $n_0 = n'_0 = n$. Let $(L_j, \sigma_j) \in \mathcal{C}L_{n_j}$ for $j \neq 0$ and $(L_j', \sigma'_j) \in \mathcal{C}L_{n'_j}$ for $j \neq i$. Then $\circ_i$ induces a bijection

$$\Psi : \bigcup_{L \in \mathcal{C}L_n} (\mathcal{M}_R(L'_0; L'_1, \ldots, L'_{i-1}, L, L'_{i+1}, \ldots, L'_{m'})) \times \mathcal{M}_P(L; L_1, \ldots, L_m) \rightarrow \mathcal{M}_{R\circ_i P}(L'_0; L'_1, \ldots, L'_{i-1}, L_1, \ldots, L_m, L'_{i+1}, \ldots, L'_{m'})$$

such that $W((M_R, \sigma_R) \circ_i (M_P, \sigma_P)) = W(M_R, \sigma_R) \cdot W(M_P, \sigma_P)$.

This proposition immediately implies

**Corollary 24.** Let $P, R$ be planar diagrams subordinate to $D_P$ and $D_R$ with signatures $(n_0; n_1, \ldots, n_m)$ and $(n'_0; n'_1, \ldots, n'_m)$ with $n_0 = n'_0$. Then $\circ_i$ induces a weight preserving bijection $\mathcal{M}(R) \times_{\partial_0, \partial_0} \mathcal{M}(P) \rightarrow \mathcal{M}(R \circ_i P)$.

by taking the union over all choices of $L_i$ and $L'_i$. These propositions are the heart of proving that the partition maps $Z_P$ satisfy the composition relations of a planar algebra.

**Gluing multiply cleaved links:** We will use the notation from section 2.1. Suppose the multiply cleaved link $(M, \sigma)$ is subordinate to $D = (\overline{D}_0; \overline{D}_1, \ldots, \overline{D}_m)$ and $(N, \nu)$ subordinate to $D' = (\overline{D}'_0; \overline{D}'_1, \ldots, \overline{D}'_m)$, with $\partial_0(M, \sigma) = \partial_i(N, \nu)$. $(N, \nu) \circ_i (M, \sigma)$ is the (equivalence class) of decorated, multiply cleaved links subordinate to $D \circ_i D'$ found by gluing the disc $\overline{D}_0$ to the disc $\overline{D}'_i$ so that $\ast_{\overline{D}_i}$ is glued to $\ast_{\overline{D}'_i}$, and each point in $Q_0$ is glued to the correspondingly ordered point in $Q'_i$. Thus, $(N, \nu) \circ_i (M, \sigma)$ intersected with $\overline{D}_0$ equals $M \cap \overline{D}_0$, while intersected with $\overline{D}'_i$ it equals $N \cap \overline{D}'_i$. The new decoration map is obtained by restriction. There is no ambiguity in the assigned decorations since $\partial_0(M, \sigma) = \partial_i(N, \nu)$ implies that the circle through the $i^{th}$ point of $Q_0(M)$ and the circle through the $i^{th}$ point of $Q'_i(N)$ receive the same decoration under $\sigma$ and $\nu$.

Let $C$ be the image of $\partial \overline{D}_0$ and $\partial \overline{D}'_i$ in the sphere resulting from gluing $\overline{D}_0$ to $\overline{D}'_i$.

We can define a cleaved link $(L, \eta)$ for the disc configuration $(\overline{D}_0)$ with circles $\{U \in \text{CIRCLES}(N \circ_i M)|U \cap C \neq \emptyset\}$ equipped with the restricted decoration map. The gluing is defined so that $(L, \eta) = \partial_0(M, \sigma) = \partial'_i(N, \nu)$. With this definition we can turn to proving proposition 23.

**Proof of proposition 23:** We first show that the map

$$\circ_i : \mathcal{M}_R(L'_0; L'_1, \ldots, L'_{i-1}, L, L'_{i+1}, \ldots, L'_{m'}) \times \mathcal{M}_P(L; L_1, \ldots, L_m) \rightarrow \mathcal{M}_{R\circ_i P}(L'_0; L'_1, \ldots, L'_{i-1}, L_1, \ldots, L_m, L'_{i+1}, \ldots, L'_{m'})$$

is well-defined. This follows from the following two lemmas:

**Lemma 25.** If $\Psi(M, \sigma) = P$ and $\Psi(N, \nu) = R$ then $\Psi(N \circ_i M) = R \circ_i P$.
Proof: The lemma follows from the observation that $\Sigma_{D'} \cup \partial D$ is $\Sigma_D$ glued to $\Sigma_{D'}$ where we glue the $j^{th}$ boundary of $\Sigma_{D'}$ to the $i^{th}$ boundary of $\Sigma_{D}$. Since the gluing map comes from the gluing of multiply cleaved links, the map takes $Q_0(M) = Q_0(P)$ to $Q'_i(N) = Q'_i(R)$. Taking and decomposing the intersection of $N \circ_i M$ with $\Sigma_{D'} \cup \partial D$ proves the result. ◁

Lemma 26. Let $(M, \sigma)$ and $(N, \nu)$ be as above. Then

$$\partial_j(N \circ_i M) = \left\{ \begin{array}{ll}
\partial_j(N, \nu) & 0 \leq j \leq i - 1 \\
\partial_{j-i+1}(M, \sigma) & i \leq j \leq m + i - 1 \\
\partial_{j-m+1}(N, \nu) & m + i \leq j \leq m + m' - 1
\end{array} \right.$$ 

This lemma confirms the appearance and order of the $L_j$ and $L'_k$ in the statement of $\mathcal{M}_{R_0, P}$.

Proof of lemma: For $i \leq j \leq m + i - 1$ the $j^{th}$ boundary $\partial_j(N \circ_i M)$ is the cleaved link formed from the circles which intersect $\partial D_j$ in $\mathbb{D}$. Due to the requirement that $\partial_j(N, \nu) = \partial_0(M, \sigma)$, such circles must come from $\text{CIRCLES}(M)$. This immediately implies that $\partial_j(N \circ_i M) = \partial_{j-i+1}(M, \sigma)$, where the indices correspond to the insertion of the discs from $\mathbb{D}$ after the $(i - 1)^{st}$ disc of $\mathbb{D}'$. The same argument applies when $0 \leq j \leq i - 1$ or $m + i \leq j \leq m + m' - 1$. For these ranges, the boundary $\partial_j(N \circ_i M)$ corresponds to all the circles which intersect one of the discs $D'_k$ for $k = 0, \ldots, i - 1, i + 1, \ldots, m'$ and thus equals $\partial_k(N, \nu)$. ◁

We now verify that $\circ_i$ is weight preserving:

Lemma 27. Let $(M, \sigma) \in \mathcal{M}_P(L_0; L_1, \ldots, L_m)$ and $(N, \nu) \in \mathcal{M}_R(L'_0; L'_1, \ldots, L'_m)$, with $L_0 = L'_k$. Then

$$W(N \circ_i M) = W(N, \nu) \cdot W(M, \sigma)$$

in $\mathbb{Z}[q^{1/2}, q^{-1/2}]$.

Proof of lemma: Circles in $\text{CIRCLES}(N \circ_i M)$ which do not intersect $C = \partial D_0$ contribute to either $W(N, \nu)$ or $W(M, \sigma)$, but not to both. For such a circle which contributes to $W(M, \sigma)$, the contribution $q^k$ is determined by the decoration and the number of discs it intersects in the image of $\overrightarrow{D}_0$. These are the same as in $N \circ_i M$, and thus such a circle makes the same contribution to $W(N \circ_i M)$. The same argument applies to circles which do not intersect $C$ and lie entirely in the image of $\overrightarrow{D}'_i$: they will contribute the same factor to $W(N, \nu)$ and $W(N \circ_i M)$. Now let $K$ be a circle which intersects $C$. Suppose $K$ intersects $\overrightarrow{A}$ discs of $R \circ_i T$ in $\overrightarrow{D}_0$ and $\overrightarrow{A}$ discs of $R \circ_i T$ in $\overrightarrow{D}'_i$. Then $K$ intersects a total of $\overrightarrow{A} + \overrightarrow{A}$ discs in the disc configuration for $R \circ_i T$. On the other hand, the corresponding circle in $(M, \sigma)$ intersects only $\overrightarrow{A} + 1$ discs: the same $\overrightarrow{A}$ discs in $\overrightarrow{D}_0$ and the disc $\overrightarrow{D}_0$. Likewise, the circle intersects $\overrightarrow{A} + 1$ discs of $(N, \nu)$. However,

$$q^{\pm(1 - \frac{1}{2}(\overrightarrow{A} + 1))} = q^{\pm(1 - \frac{1}{2}(\overrightarrow{A} + 1))} = q^{\pm(1 - \frac{1}{2}(\overrightarrow{A} + 1))}$$

The left term is the contribution $K$ makes to $W(N \circ_i M)$ while the right term is the product of the contributions of $K$ to $W(M, \sigma)$ and $W(N, \nu)$. Thus the contribution of $K$ is the same in both $W(N \circ_i M)$ and $W(N, \nu) \cdot W(M, \sigma)$. Since all circles in $\text{CIRCLES}(M)$ make the same contribution to both $W(N \circ_i M)$ and $W(N, \nu) \cdot W(M, \sigma)$, these must be equal in
Finally, we verify that the map $\circ_i$ induces the bijection $\Psi$ in the statement of proposition 23.

**Lemma 28.** There is a map

$$\Phi : \mathcal{M}(R_0; P(L_0'; L_1', \ldots, L_{i-1}', L_1, \ldots, L_m, L_{i+1}', \ldots, L_m'))$$

$$\to \bigcup_{L \in \mathcal{L}_n} \left( \mathcal{M}(L_0'; L_1', \ldots, L_{i-1}', L, L_{i+1}', \ldots, L_m') \times \mathcal{M}(L; L_1, \ldots, L_m) \right)$$

which is an inverse to $\Psi$. 

**Proof of lemma:** Let $(D, \eta) \in \mathcal{M}(R \circ_i P)(L_0'; L_1', \ldots, L_{i-1}', L_1, \ldots, L_m, L_{i+1}', \ldots, L_m')$ subordinate to $\mathcal{D}' \circ_i \mathcal{D}$. Let $\widehat{B}$ be the image of $\widehat{D}_0$ after the gluing and let $\overline{B}$ be the image of $\overline{D}_i$. Let $\mathcal{D}_1$ be the disc configuration $(\overline{B}; \overline{D}_1, \ldots, \overline{D}_m)$ in the same sphere, with the marked points inherited from $\overline{D}_i$ and $*_{\overline{D}_i}$. Now we define a new multiply cleaved link $M$ in $S$ by taking $\text{Circles}(M) = \{U \in \text{Circles}(D) | U \cap \overline{B} \neq \emptyset\}$ and the decoration $\sigma$ found by restricting $\eta$ to the subset $\text{Circles}(M) \subset \text{Circles}(D)$. This defines a multiply cleaved link which, given the identification of $\overline{B}$ with $\overline{D}_0$, is a decorated, multiply cleaved link in $\mathcal{M}(P)$. Due to lemma 26, $(M, \sigma) \in \mathcal{M}_P(L; L_1, \ldots, L_m)$ where $L = \partial_0(M, \sigma)$. Similarly, if we consider the disc configuration $(\overline{D}_0', \overline{D}_1', \ldots, \overline{D}_i', \overline{B}, \overline{D}_i, \ldots, \overline{D}_m')$ and the subset of $\text{Circles}(D)$ defined by $\{U \in \text{Circles}(D) | U \cap \overline{B} \neq \emptyset\}$ we obtain a decorated multiply cleaved link $N$ in $\mathcal{M}_R(L_0'; L_1', \ldots, L_m')$. Furthermore, $\partial_i(N, \nu) = \partial_0(M, \sigma)$ since both are equal to the cleaved link for the disc configuration $(\overline{B})$ with circles $\{U \in \text{Circles}(M) | U \cap C \neq \emptyset\}$. Thus $L_i = L$, and $(N, M)$ is an element of

$$\left( \mathcal{M}_R(L_0'; L_1', \ldots, L_{i-1}', L, L_{i+1}', \ldots, L_m') \times \mathcal{M}_P(L; L_1, \ldots, L_m) \right)$$

If we define $\Phi(D, \eta) = (N, M)$ we have a well-defined map as required in the lemma, since we have no control over $L$. That this is an inverse of the map $\Psi$ follows almost immediately from the definitions of these maps. \(\diamondsuit\)

This completes the proof of proposition 23. \(\diamondsuit\)

4. Maps from planar diagrams

As in definition 4, let $P$ be a planar diagram subordinate to $\mathcal{D}$ with signature $\text{Sign}(P) = (n_0; n_1, \ldots, n_m)$ and $m \geq 1$. We will define modules $\mathcal{I}_{2n}$ for $n \geq 0$, and a map

$$Z_P : \mathcal{I}_{2n_1} \otimes \mathcal{I}_{2n_2} \otimes \cdots \otimes \mathcal{I}_{2n_m} \to \mathcal{I}_{2n_0}$$

where the tensor products are taken over $\mathbb{Z}[q^{1/2}, q^{-1/2}]$.

**Modules:** We start by describing the modules $\mathcal{I}_{2n}$:

**Definition 29.** For each $n \geq 0$, $\mathcal{I}_{2n}$ is the free $\mathbb{Z}[q^{1/2}, q^{-1/2}]$-module generated by the elements of $\mathcal{C}_n$. The generator corresponding to $(L, \sigma) \in \mathcal{C}_n$ will be denoted $I_{(L, \sigma)}$. 

$$\mathbb{Z}[q^{1/2}, q^{-1/2}] \diamondsuit$$
Examples: When \( n = 0 \), \( \mathcal{CL}_0 = \{ e \} \) where \( e \) is the equivalence class of the cleaved link in \( S^2 \subset \mathbb{R}^3 \), oriented as the boundary of the unit ball, with \( \overrightarrow{D} = \{(x,y,z) \in S^2 \mid z \geq 0 \} \) and \( * = (1,0,0) \), and \( \text{CIRCLES}(e) = \emptyset \). Then \( \mathcal{I}_0 = \mathbb{Z}[q^{1/2}, q^{-1/2}] \mathcal{I}_0 \cong \mathbb{Z}[q^{1/2}, q^{-1/2}] \) where \( \mathcal{I}_0 \) is the generator corresponding to \( e \). See [15] for more details.

The generators of \( \mathcal{I}_2 \) are cleaved links whose circles intersect its equator exactly twice. Consequently \( \text{CutCircs}(L) = \{C\} \). \( C \) can be decorated with either a + or a −. Then \( \mathcal{I}_2 \) has two corresponding generators, which we will write \( \mathcal{I}_+ \) and \( \mathcal{I}_- \). Thus, \( \mathcal{I}_2 \cong \mathbb{Z}[q^{1/2}, q^{-1/2}] \mathcal{I}_+ \oplus \mathbb{Z}[q^{1/2}, q^{-1/2}] \mathcal{I}_- \).

For \( \mathcal{I}_4 \), there are twelve generators, depicted and labeled in Figure 3.

Maps: For the planar diagram \( P \) (with \( m \geq 1 \)) we define the partition function \( Z_P \) by defining it on the generators of \( \mathcal{I}_{2n_1} \otimes \mathcal{I}_{2n_2} \otimes \cdots \otimes \mathcal{I}_{2n_m} \) and linearly extending to the tensor product. For a generator

\[
\xi = I_{(L_1, \sigma_1)} \otimes I_{(L_2, \sigma_2)} \otimes \cdots \otimes I_{(L_m, \sigma_m)}
\]

with \( (L_i, \sigma_i) \in \mathcal{CL}_{n_i} \) let

\[
Z_P(\xi) = \sum_{(M, \sigma) \in \mathcal{M}(P)} W(M, \sigma) I_{\partial_0(M, \sigma)}
\]

Recall that \( \mathcal{I}_0 = \mathbb{Z}[q^{1/2}, q^{-1/2}] \), so this definition applies even when some of the disks \( \overrightarrow{D}_i \), \( i \geq 1 \), or \( \overrightarrow{D}_0 \), do not need to intersect \( P \).

It is not obvious from the description how \( Z_P \) depends upon the location of the marked points \( \ast_{\overrightarrow{D}_i} \) for \( i \geq 1 \). However, the location of the marked points determines the identifications with the generators of \( \mathcal{I}_{2n_i} \) for each \( i \). Changing the location of the marked points can change these identifications, and thus the map. An example below illustrates this phenomenon.

Examples: Suppose that \( P \) is planar diagram with

1. \( \text{DISCS}(P) = \{\overrightarrow{D}\} \),
2. \( \text{Q}_{\overrightarrow{D}} = \{p_1, p_2\} \),
3. \( \text{FREECIRC}(P) = \emptyset \), and
4. \( \text{ARCS}(T) = \{\gamma\} \) where \( \gamma \) has boundary equal to \( Q_{\overrightarrow{D}} \).

There are two possible planar diagrams compatible with this data, depending upon whether \( P \subset \overrightarrow{D} \) or \( P \subset \overrightarrow{D} \).

Case i: when \( P \subset \overrightarrow{D} \) In this case \( \overrightarrow{D}_0 = \overrightarrow{D} \). Since \( n_P = 1 \) we will have a map \( Z_P : \mathbb{Z}[q^{1/2}, q^{-1/2}] \longrightarrow \mathcal{I}_2 \) once we choose a disc \( \overrightarrow{D}_1 \subset \overrightarrow{D}_0 \) which does not intersect \( P \). This is depicted as
The multiply cleaved links \((M, \sigma)\) in \(\mathcal{M}(P)\) are those obtained from the diagram for \(P\) by adding outside matching \(\bar{m}\) of \(Q_0\) in \(\overrightarrow{D}_0\). Since \(n_D = 1\) there is only one possibility: \(\bar{m}\) must consist of a single arc joining \(p_1\) to \(p_2\). This creates a single cut circle \(C\) which can be decorated as either \(\sigma(C) = +\) or \(\sigma(C) = -\). Thus the \((M, \sigma)\) in \(\mathcal{M}(P)\) are the elements of \(\mathcal{C}\mathcal{L}_1\). Then \(\partial_1(L_{\pm}) = e\) where \(e\) is the unique cleaved link in \(\mathcal{C}\mathcal{L}_0\), and \(\partial_0(L_{\pm}) = L_{\pm}\). Thus the map \(Z_P\) is a map \(I_2 \to \mathbb{Z}[q^{\pm 1/2}]\). The multiply cleaved links \((M, \sigma)\) in \(\mathcal{M}(P)\) are as in case i, since there is only one way to fill in \(P\) using an matching inside \(\overrightarrow{D}_1\). As before, this implies that \(L_{\pm}\) are the only decorated multiply cleaved links with associated tangle \(P\) and \(\partial_1(L_{\pm}) = L_{\pm}\) and \(\partial_0(L_{\pm}) = e\) we will have

\[
Z_P(I_{+}) = W(L_{+})I_0 = q^{1/2} \in \mathbb{Z}[q^{1/2}, q^{-1/2}]
\]

Likewise,

\[
Z_P(I_{-}) = W(L_{-})I_0 = q^{-1/2}
\]

As another example, consider the planar diagram \(P\) with

1. \(\mathbb{D} = (\overrightarrow{D}_0; \overrightarrow{D}_1)\),
2. \(Q_{\overrightarrow{D}_0} = \{p_1, p_2\}\), and \(Q_{\overrightarrow{D}_1} = \{q_1, q_2\}\)
(3) \text{FREECIRCS}(P) = \emptyset, \text{ and}

(4) \text{ARCS}(P) = \{\gamma_1, \gamma_2\} \text{ where } \gamma_i \text{ joins } p_i \text{ to } q_i \text{ in the annulus } \hat{D}_0 \setminus (\text{int } \hat{D}_1).

To find the multiply cleaved links in \( \mathcal{M}(P) \), we will first pick an (inside) planar matching of \( Q_1 \) in \( \hat{D}_1 \), and another (outside) planar matching of \( Q_0 \) in \( \hat{D}_0 \). Each of these matching must consist of a single arc. When we glue these to \( P \) we obtain \( (M, \sigma) \) with a single cut circle, which is then decorated with either a + or a −. We denote these two decorated, multiply cleaved diagrams by \( M_+ \) and \( M_- \).

Then \( \partial_i(M_{\pm}) = L_\pm \) for both \( i = 0, 1 \). The definition of \( Z_P \) should give a map \( \mathcal{I}_2 \to \mathcal{I}_2 \) which is the extension of \( I_+ \to I_+, I_- \to I_- \). Thus, we obtain the identity map on \( \mathcal{I}_2 \).

As out last example, suppose we have a planar diagram \( P \) with the following data:

1. \( \mathcal{D} = (\hat{D}_0), \)
2. A choice of \( * \) in \( \partial \hat{D}_0 \)
3. \( Q_0 = \{p_1, p_2, p_3, p_4\} \), ordered according to our convention,
4. \( \text{FREECIRCS}(P) = \emptyset, \text{ and} \)
5. \( \text{ARCS}(P) = \{\gamma_1, \gamma_2\} \), where \( \gamma_i \) is properly embedded in \( \hat{D}_0 \) with \( \gamma_1 \) joining \( p_1 \) to \( p_4 \), and \( \gamma_2 \) joining \( p_3 \) to \( p_2 \).

For this planar diagram the map \( Z_P \) should be a map \( \mathbb{Z}[q^{1/2}, q^{-1/2}] \to \mathcal{I}_4 \).

There are two outside matchings \( \hat{m}_1 \) and \( \hat{m}_2 \) which can be used in \( \hat{D}_0 \) to construct the multiply, cleaved links with associated tangle \( P \). \( \hat{m}_1 \) joins \( p_1 \) to \( p_2 \) and \( p_3 \) to \( p_4 \), while \( \hat{m}_2 \) joins \( p_1 \leftrightarrow p_4, p_3 \leftrightarrow p_2 \). This is depicted as

If we use \( \hat{m}_1 \), and decorate the resulting circle, we obtain the generators \( A_{\pm} \) from Figure \( 3 \) \( A_+ \) contributes \( q^{-1/2}I_{A_+} = q^{1/2}I_{A_+} \) to the image of \( 1 \in \mathbb{Z}[q^{\pm 1/2}] \) while \( A_- \) contributes \( q^{-1+1/2}I_{A_-} = q^{-1/2}I_{A_-} \). If we use \( \hat{m}_2 \), and decorate the resulting circles, we obtain the generators \( B_{\pm} \) from Figure \( 3 \) \( B_{++} \) contributes \( q^{1-1/2}q^{1-1/2}I_{B_{++}} = qI_{B_{++}} \), while \( B_{+-} \) contributes \( q^{1-1/2}q^{1-1/2}I_{B_{+-}} = I_{B_{+-}} \). Likewise, \( B_{-+} \) contributes \( q^{1+1/2}q^{1+1/2}I_{B_{+}} = I_{B_{+}} \) and \( B_{--} \) contributes \( q^{-1+1/2}q^{-1+1/2}I_{B_{--}} = q^{-1}I_{B_{--}} \). Thus the map \( Z_P \) is the linear extension of

\[
1 \to q^{1/2}I_{A_+} + q^{-1/2}I_{A_-} + qI_{B_{++}} + I_{B_{++}} + I_{B_{+-}} + q^{-1}I_{B_{--}}
\]

Now suppose we move \( * \) to sit between \( p_1 \) and \( p_2 \), then, according to our convention, we must relabel \( Q_0 \) as \( \{q_1, q_2, q_3, q_4\} \) where \( q_i = p_{i+1} \). Then \( P' \) will be the matching \( q_1 \leftrightarrow q_2 \)
and $q_3 \rightarrow q_4$, while $\overrightarrow{m}_1$ becomes $q_1 \leftrightarrow q_4$ and $q_2 \leftrightarrow q_3$. Using $\overrightarrow{m}_1$ now gives a generator of type $D_{\pm}$ while using $\overrightarrow{m}_2$ gives a generator of type $C_{\pm \pm}$. Indeed, $Z_{P'}$ with the new choice of marked point is the map determined by

$$1 \rightarrow q^{1/2}I_{D_{+}} + q^{-1/2}I_{D_{-}} + qI_{C_{++}} + I_{C_{+-}} + I_{C_{-+}} + q^{-1}I_{C_{--}}$$

5. Properties of the partition maps for planar diagrams

We showed in section 2.2 that our results for the partition map of a tangle diagram, with the exception of Reidemeister invariance, would follow from verifying the Temperley-Lieb, Normalization, Conjugation, and Composition properties for the partition maps of planar diagrams. We now verify those properties for planar diagrams.

5.1. Removing free circles and the Temperley-Lieb property. First we show that the maps above have the Temperley-Lieb property:

**Proposition 30.** Let $P$ be a planar diagram and $C \in \text{FREECIRCS}(P)$. Let $P'$ be the planar diagram which is the same as $P$ except that $\text{FREECIRCS}(P') = \text{FREECIRCS}(P) \setminus \{C\}$. Then $Z_P = (q + q^{-1})Z_{P'}$.

**Proof:** There is a 2 : 1 map $M(P) \rightarrow M(P')$ which takes $(M, \sigma)$ to $(M', \sigma')$ with the same configuration of discs, marked points, and cut circles, except $\text{FREECIRCS}(M', \sigma') = \text{FREECIRCS}(M, \sigma) \setminus \{C\}$, and $\sigma'$ is the restriction of $\sigma$ to the remaining circles. The pre-image of $(M', \sigma')$ under this map consists of $\{M'_+, M'_-\}$ where we have added $C$ back and $\sigma(C) = +$ or $\sigma(C) = -$, respectively. By partitioning $M(P)$ into $M_+(P) \sqcup M_-(P)$, based on the value of $\sigma(C)$, we obtain

$$Z_P(\xi) = \sum_{(M, \sigma) \in M_+(P)} W(M, \sigma)I_{\partial_1(M, \sigma)} + \sum_{(M, \sigma) \in M_-(P)} W(M, \sigma)I_{\partial_1(M, \sigma)}$$

(4)

Since $\partial_1(M, \sigma)$ depends only on the cut circles, for each $i$, we know that $\partial_1(M'_+, \sigma') = \partial_1(M', \sigma')$. In addition, $W(M'_+, \sigma') = q^{\pm 1}W(M', \sigma')$ since the contribution of $C$ to the weight is $q^{1-0} = q$ when $\sigma(C) = +$ and $q^{-1+0} = q^{-1}$ if $\sigma(C) = -$. Thus, when we express each of the sums in equation (4) as a sum over the $(M', \sigma')$ we obtain

$$Z_{P'}(\xi) = \sum_{(M', \sigma') \in M(P')} (q + q^{-1})W(M')I_{\partial_1(M', \sigma')}$$

$$= (q + q^{-1})Z_{P'}(\xi)$$

from which the result follows. \hfill \Box

5.2. Removing discs with $n_i = 0$ and the normalization property.

**Proposition 31.** Suppose $P$ is a planar diagram subordinate to $D = (\overrightarrow{D}_0; \overrightarrow{D}_1, \ldots, \overrightarrow{D}_m)$ with $m \geq 2$. Suppose $\overrightarrow{D}_i$ has $n_i = 0$ for $i \geq 1$. Let $P'$ be planar diagram determined by $P$ subordinate to $D' = D \setminus \{D_i\}$, with the inherited order. Then $Z_{P'} = Z_P$. 
This proposition allows us to introduce and remove discs with \( n_i(P) = 0 \) in any manner we like.

**Proof:** The only place where \( \widetilde{D}_i \) contributes is in the tensor product \( \mathcal{I}_{2n_1} \otimes \cdots \otimes \mathcal{I}_{2n_m} \). There it corresponds to a factor of \( \mathcal{I}_0 = \mathbb{Z}[q^{1/2}, q^{-1/2}] \). Since the tensor products for \( P \) and \( P' \) are over \( \mathbb{Z}[q^{1/2}, q^{-1/2}] \), they are naturally isomorphic. The image of each generator is determined by the configuration of circles, in particular the cut circles, and as these are the same in the two diagrams, the partition maps \( Z_P \) and \( Z_{P'} \) are identical, under the isomorphism.

For a planar diagram \( P \) with neither arc nor circle components (which is just a disc configuration) both \( \mathcal{I}_{2n_1} \otimes \cdots \otimes \mathcal{I}_{2n_m} \) and \( \mathcal{I}_{2n_1} \) will be isomorphic to \( \mathbb{Z}[q^{1/2}, q^{-1/2}] \). In this case, \( Z_P \) will be the identity map \( \mathbb{Z}[q^{1/2}, q^{-1/2}] \rightarrow \mathbb{Z}[q^{1/2}, q^{-1/2}] \). In particular, for each disc \( \widetilde{D}_i \) we have \( n_i = 0 \), and we can remove all but two of them without changing the map. It is then straightforward to see that the map is the identity.

We now verify the normalization property:

**Proposition 32.** Let \( P \) be a planar diagram subordinate to \( \mathcal{D} = (\widetilde{D}_0; \widetilde{D}_1) \) with \( \text{SIGN}(P) = (0; 0) \). Then \( Z_P \) is the map \( \mathbb{Z}[q^{1/2}, q^{-1/2}] \rightarrow \mathbb{Z}[q^{1/2}, q^{-1/2}] \) which multiplies \( p(q) \) by \( J_P(q) = (q + q^{-1})^N \) where \( N \) is the number of circles in \( P \).

**Proof:** A planar diagram with signature \( (0; 0) \) has only free circles, and thus the Temperley-Lieb property and the preceding result implies that \( Z_P \) is multiplication by \( (q+q^{-1})^N \). This is identical to \( J_P(q) \), from the skein relations defining \( J_P(q) \). \( \checkmark \)

More details about normalization: In the introduction, we mentioned that in \([15]\) there is a distinction between inside tangles and outside tangles. This distinction also occurs for inside planar diagrams and outside planar diagrams. Suppose an oriented sphere \( S \) is composed of two closed discs, \( \widetilde{D} \) and \( \overline{D} \), glued along their boundaries. Let \( E \) be the image of their boundaries in \( S \), oriented as the boundary of \( \overline{D} \). In addition suppose we have chosen a marked point \( * \in E \). An inside planar tangle \( P \) is a planar tangle in \( \overline{D} \) with boundary \( Q_{\overline{D}} \subset E \), ordered as above. Thus an inside planar tangle corresponds to a planar diagram for the inside disc configuration \((\overline{D})\). An outside planar tangle is a planar tangle in \( \overline{D} \) – thus the ordering on the boundary points of the tangle is opposite that induced from \( \partial \overline{D} \). In \([15]\) we associate a map \([P] : \mathbb{Z}[q^{1/2}, q^{-1/2}] \rightarrow \mathcal{I}_{2n_p} \) to an inside tangle and a map \([[P]] : \mathcal{I}_{2n_p} \rightarrow \mathbb{Z}[q^{1/2}, q^{-1/2}] \) to an outside tangle.

The definition of the partition maps \( Z_P \) above does not apply to inside and outside tangles since there are too few discs. However, we can finesse this difficulty to obtain partition maps. Suppose \( P \) is an inside tangle. Let \( \overline{D}_1 \) be any disc in the interior of \( \overline{D} \) which does not intersect \( P \). Then \( P \) also defines a planar diagram for the disc configuration \((\overline{D}; \overline{D}_1)\), which has a partition map \( Z_P : \mathbb{Z}[q^{1/2}, q^{-1/2}] \rightarrow \mathcal{I}_{2n_p} \). For an outside planar tangle in \( \overline{D} \), we introduce a disc \( \overline{D}_0 \) in the interior of \( \overline{D} \) which does not intersect \( P \), and then consider
$P$ as a planar diagram for the configuration $(\overline{D}; \overline{D})$. For this configuration $Z_P$ is a map $\mathcal{I}_{2n} \rightarrow \mathbb{Z}[q^{1/2}, q^{-1/2}]$.

In keeping with the results of proposition 31, we can “erase” the trivial discs we introduced:

**Proposition 33.** Let $P$ be an inside, planar tangle (as in [15]). Then the map $\llbracket P \rrbracket$ equals the partition map $Z_P$, computed from the disc configuration $(\overline{D}; \overline{D})$ as described in the preceding paragraph. If $P'$ is an outside, planar tangle, then $\llbracket P' \rrbracket$ equals the partition map $Z_{P'}$, computed using the disc configuration $(\overline{D}; \overline{D})$, described above.

**Proof:** We prove this for the case of an inside planar tangle, as the outside case is similar. For the disc configuration $(\overline{D}; \overline{D})$, if $1$ is a generator for $\mathcal{I}_{2n_1} = \mathcal{I}_0$, then we can compute $Z_P(1)$ as the sum of the terms found by the following steps. First, choose a planar matching on $Q^D_\partial$ and an embedding of that matching in $\overline{D}$. Second, assign $\pm$ to each circle in the result of gluing the matching to $P$. The decorated, multiply cleaved circles obtained by these choices exhaust all the $(\mathcal{M}, \sigma)$ in $\mathcal{M}(P)$. We then add the terms $W(\mathcal{M}, \sigma)I_{\partial_0(\mathcal{M}, \sigma)}$.

According to Section 5 of [15], we can think of $\llbracket P \rrbracket$ as the map taking 1 to the result of the sum

$$\sum_{(L, \sigma) \in \mathbb{C}_{n_0}(P)} \left( \sum_{(\rho, \overline{m}, s) \in \partial(\rho, \overline{m}, s) = (L, \sigma)} (-1)^{h(\rho)} q^{I(\rho, \overline{m}, s)} \right) I_{(L, \sigma)}$$

where $\rho$ is a choice of resolution for all the crossings, $\overline{m}$ is a planar matching of $Q^D_\partial$ in $\overline{D}$, and $s$ is an assignment of $\pm$ to each circle in the resolved diagram. Since $P$ is a planar diagram, $P$ has no crossings and there is only one resolution diagram $\rho(P) = P$. The boundary $\partial(\rho, \overline{m}, s)$ in the language of [15] is identical to $\partial_0(\mathcal{M}, s)$ where $\mathcal{M}$ is the result of gluing $\overline{m}$ to $P$. Therefore $h(\rho) = 0$. Thus, the sum defining $\llbracket P \rrbracket$ occurs over the same choices as the sum defining $Z_P(1)$. All that remains is to show that $(-1)^{h(\rho)} q^{I(\rho, \overline{m}, s)} = W(\mathcal{M}, s)$.

Since $\rho$ is trivial, $h(\rho) = 0$, as the value is the sum of values depending on the choice of resolution at each crossing. Furthermore,

$$I(\rho, \overline{m}, s) = h(\rho) + \sum_{C \in \text{FREECIRC}(\rho(P \# \overline{m}))} s(C) + \frac{1}{2} \sum_{C \in \text{CUTCIRC}(\rho(P \# \overline{m}))} s(C) + (n_+(T) - n_-(T))$$

We know that $h(\rho) = 0$, and since there are no crossings $n_+(P) = n_-(P) = 0$. The remainder implies that $q^{I(\rho, \overline{m}, s)}$ is a product over $\text{CIRCLES}(\mathcal{M})$, counting each $\pm$ free circle as $q^{\pm 1}$, just as $W(\mathcal{M}, s)$ would, and each cut circle as $q^{\pm 1/2}$, just as $W(\mathcal{M}, s)$ would since each cut circle only intersects one disc $\overline{D}$ for the configuration $(\overline{D}; \overline{D})$. ◇

### 5.3. The partition maps and the conjugation property.

We consider the effect of conjugation $(\mathcal{M}, \sigma) \rightarrow (\mathcal{M}, \sigma^*)$ on the partition map. First, we extend conjugation to an automorphism $p \rightarrow p^*$ of $\mathbb{Z}[q^{1/2}, q^{-1/2}]$. 

---
Definition 34. For \( p \in \mathbb{Z}[q^{1/2}, q^{-1/2}] \), the conjugate of \( p \) is the element \( p^* \in \mathbb{Z}[q^{1/2}, q^{-1/2}] \) given by \( p^*(q) = p(q^{-1}) \).

Then we will extend the conjugate map \( (L, \sigma) \rightarrow (L, \sigma)^* \) on \( \mathcal{CL}_n \) to a map on \( \mathcal{I}_{2n} \), retaining the notation \( \xi \rightarrow \xi^* \), by

\[
\xi = \sum_i p_i(q) I_{(L_i, \sigma_i)} \rightarrow \xi^* = \sum_i p_i^*(q) I_{(L_i, \sigma_i)^*} = \sum_i p_i(q^{-1}) I_{(L_i, \sigma_i)^*}
\]

For \( \mathcal{I}_0 = \mathbb{Z}[q^{1/2}, q^{-1/2}] \) this is identical to conjugate map \( p^*(q) = p(q^{-1}) \) in \( \mathbb{Z}[q^{1/2}, q^{-1/2}] \).

Finally, we extend to tensor products \( \mathcal{I}_{2n_1} \otimes \cdots \otimes \mathcal{I}_{2n_m} \) by

\[
\sum_i p_i(q) I_{(L_{i,1}, \sigma_{i,1})} \otimes \cdots \otimes I_{(L_{i,m}, \sigma_{i,m})} \rightarrow \sum_i p_i^*(q) I_{(L_{i,1}, \sigma_{i,1})^*} \otimes \cdots \otimes I_{(L_{i,m}, \sigma_{i,m})^*}
\]

Proposition 35. Let \( P \) be a planar diagram with signature \( \text{SIGN}(P) = (n_0; n_1, \ldots, n_m) \), and \( \xi \in \mathcal{I}_{2n_1} \otimes \cdots \otimes \mathcal{I}_{2n_m} \), then \( Z_P(\xi^*) = (Z_P(\xi))^* \).

Proof: Let \( \xi = I_{(L_1, \sigma_1)} \otimes \cdots \otimes I_{(L_m, \sigma_m)} \) be a generator of \( \mathcal{I}_{2n_1} \otimes \cdots \otimes \mathcal{I}_{2n_m} \) and \( (M, \sigma) \) be a decorated multiply cleaved link in \( \mathcal{M}(P) \) with \( \partial_i(M, \sigma) = (L_i, \sigma_i) \) for \( i \geq 0 \). Then \( (M, \sigma)^* \) has \( \mathcal{P}(M^*) = T \). Thus conjugation is a bijection on \( \mathcal{M}(P) \). In addition, \( \partial_i(M, \sigma)^* = (L_i, \sigma_i)^* \) for \( i \geq 0 \). Furthermore, \( W((M, \sigma)^*) = W(M, \sigma)^* \) in \( \mathbb{Z}[q^{1/2}, q^{-1/2}] \). Thus,

\[
Z_P(\xi^*) = \sum_{(M, \sigma) \in \mathcal{M}(P)} W(M, \sigma) I_{\partial_1(M, \sigma)}
\]

\[
= \sum_{(M', \sigma') \in \mathcal{M}(P)} W((M', \sigma')^*) I_{\partial_1((M', \sigma')^*)}
\]

\[
= \sum_{(M', \sigma') \in \mathcal{M}(P)} W(M', \sigma')^* I_{\partial_1(M', \sigma')}^*
\]

\[
= (Z_P(\xi))^*
\]

\[
\square
\]

5.4. Composition of the partition maps for planar diagrams. Let \( P \) and \( R \) be planar diagram with signatures \( \text{SIGN}(P) = (n_0; n_1, \ldots, n_m) \) and \( \text{SIGN}(R) = (n'_0; n'_1, \ldots, n'_m) \), with \( n_0 = n'_0 \). Then we can form the planar diagram \( R \circ_i P \). Recall that \( Z_R \circ_i Z_P \) is defined as for planar algebras, [5, 6]: if \( a_j \in \mathcal{I}_{2n_j} \) for \( j \geq 1, j \neq i \) and \( b_j \in \mathcal{I}_{2n_j} \) for \( j \geq 1 \) then

\[
Z_{R_{0,i}P}(a_1 \otimes \cdots \otimes a_{i-1} \otimes b_1 \otimes \cdots \otimes b_m \otimes a_{i+1} \otimes \cdots \otimes a_m')
\]

\[
= Z_R(a_1 \otimes \cdots \otimes a_{i-1} \otimes Z_P(b_1 \otimes \cdots \otimes b_m) \otimes a_{i+1} \otimes \cdots \otimes a_m')
\]

We wish to show that

Theorem 36. Let \( T \) and \( P \) be planar diagrams as above. The partition map \( Z_{R_{0,i}P} \) equals the map \( Z_R \circ_i Z_P \)
Proof: As both maps are linear over $\mathbb{Z}[q^{1/2}, q^{-1/2}]$ it suffices to check the result on generators

$$\xi = I_{L_1} \otimes \cdots \otimes I_{L_{i-1}} \otimes I_{L_1} \otimes \cdots \otimes I_{L_m} \otimes I_{L_{i+1}} \otimes \cdots \otimes I_{L_m'}$$

of the tensor product that is their common domain:

$$\mathcal{I}_{2n_1} \otimes \cdots \otimes \mathcal{I}_{2n_{i-1}} \otimes \mathcal{I}_{2n_1} \otimes \cdots \otimes \mathcal{I}_{2n_m} \otimes \mathcal{I}_{2n_{i+1}} \otimes \cdots \otimes \mathcal{I}_{2n_m},$$

Thus, we will show that $Z_{R \circ_i P}(\xi)$ equals $(Z_R \circ_i Z_P)(\xi)$.

(5) $(Z_R \circ_i Z_P)(\xi) = Z_R(I_{L_1} \otimes \cdots \otimes I_{L_{i-1}} \otimes Z_P(I_{L_1} \otimes \cdots \otimes I_{L_m}) \otimes I_{L_{i+1}} \otimes \cdots \otimes I_{L_m'})$

The definition of $Z_P$ implies that

$$Z_P(I_{L_1} \otimes \cdots \otimes I_{L_m}) = \sum_{L \in \mathcal{C}_{n_0}} \left( \sum_{(M, \sigma) \in \mathcal{M}(L; \underbrace{L, \ldots, L}_m)} W(M, \sigma) \right) I_L$$

Substituting into equation (5) and using the linearity of the tensor products and $Z_R$ gives

(6) $(Z_R \circ_i Z_P)(\xi) = \sum_{L \in \mathcal{C}_{n_0}} \sum_{(M, \sigma) \in \mathcal{M}(L; \underbrace{L_1, \ldots, L}_m)} W(M, \sigma) Z_R(I_{L_1} \otimes \cdots \otimes I_{L_{i-1}} \otimes I_L \otimes I_{L_{i+1}} \otimes \cdots \otimes I_{L_m'})$

Similarly,

$$Z_R(I_{L_1} \otimes \cdots \otimes I_{L_{i-1}} \otimes I_L \otimes I_{L_{i+1}} \otimes \cdots \otimes I_{L_m'}) = \sum_{L' \in \mathcal{C}_{n_0}} \left( \sum_{(N, \nu) \in \mathcal{M}(L'; \underbrace{L_1', \ldots, L_1, \ldots, L_m'}_{m'})} W(N, \nu) \right) I_{L'}$$

Therefore, $(Z_R \circ_i Z_P)(\xi) = \sum_{L' \in \mathcal{C}_{n_0}} W_{L'} I_{L'}$ where

(7) $W_{L'} = \sum_{L \in \mathcal{C}_{n_0}} \left( \sum_{((N, \nu), (M, \sigma)) \in \mathcal{M}(L; L_1', \ldots, L_1, \ldots, L_m') \times \mathcal{M}(L; L_1, \ldots, L_m)} W(N, \nu) \cdot W(M, \sigma) \right)$

However, by proposition 23 $W(N, \nu) \cdot W(M, \sigma) = W(N \circ_i M)$, and $\circ_i$ induces a bijection

$$\bigcup_{L \in \mathcal{C}_n} (\mathcal{M}_R(L_1', \ldots, L_1, \ldots, L_m') \times \mathcal{M}_P(L_1, \ldots, L_m)) \rightarrow \mathcal{M}_{R \circ_i P}(L_1', \ldots, L_{i-1}, L_1, \ldots, L_m, L_{i+1}', \ldots, L_m')$$

As the double sum in equation (7) is a sum over the elements of this union, we conclude that

$$W_{L'} = \sum_{(D, \eta) \in \mathcal{M}_{R \circ_i P}(L_1', \ldots, L_{i-1}, L_1, \ldots, L_m, L_{i+1}', \ldots, L_m')} W(D, \eta)$$
\[(Z_R \circ_i Z_P)(\xi) = \sum_{L' \in \mathcal{L}_{i,0}'} \left( \sum_{(D, \eta) \in \mathcal{M}_{R_0,p}(L'; L_1', \ldots, L_{i-1}', L_{i+1}, \ldots, L_m', L_{i+1}', \ldots, L_m')} W(D, \eta) \right) I_{L'} \]

The right hand side is equal to \(Z_{R_0,p}(\xi)\), so we have verified equality on each generator.

6. Invariance of \(Z_T\) for tangles

In this section we will show that

**Theorem 37.** Let \(T_a\) and \(T_b\) be oriented tangle diagrams, subordinate to a disc configuration \(\mathbb{D}\), which differ by a Reidemeister I, II, or III move in the interior of \(\Sigma_\mathbb{D}\). Then the partition maps for \(T_a\) and \(T_b\) are equal: \(Z_{T_a} = Z_{T_b}\).

Due to this result, \(Z_T\) is an invariant of the tangle represented by the diagrams, and does not just depend on the diagram. Recall from definition 6 that \(Z_T = (-1)^{n_-(T)} q^{(n_+ - 2n_-)(T)} \tilde{Z}_T\) and that, by proposition 7, satisfies the skein relation

\[\tilde{Z}_T = \tilde{Z}_{T_0} - q \cdot \tilde{Z}_{T_1}\]

as \(\mathbb{Z}[q^{1/2}, q^{-1/2}]\)-linear maps., and that \(\tilde{Z}_T\) has the Temperley-Lieb property.

With these facts in hand, the proof of theorem 37 is more or less identical to the proof that the Jones polynomial \(J_L(q)\), described in the introduction, is an invariant of a link \(L\). We provide a brief account here.

**Proof:** If \(T_a\) and \(T_b\) be tangle diagrams as in the statement of the theorem. Then there is a closed disc \(B \subset \text{int} \Sigma_\mathbb{D}\) where \(T_a\) and \(T_b\) differ by a Reidemeister I, II, or III move, which is minimal for this property. Furthermore, in the complement of \(\text{int} B\), \(T_a\) and \(T_b\) are the same, a tangle \(T\) subordinate to the disc configuration \(\mathbb{D}' = (\mathbb{D}, B)\). Thus, \(T_a = T \circ_{m+1} R_0\) and \(T_b = T \circ_{m+1} R_1\), where \(R_0\) and \(R_1\) are local models for one of the Reidemeister moves. Since \(Z_{T_a} = Z_T \circ_{m+1} Z_{R_0}\) and \(Z_{T_b} = Z_T \circ_{m+1} Z_{R_0}\), it suffices to show that \(Z_{R_0}\) and \(Z_{R_1}\) are identical; that is, the maps for the swatches interchanged by the move are the same. We have reduced the problem to showing that \(Z_{R_0} = Z_{R_1}\) by local calculation.

**Local Calculations:** Let \(T_{\pm}\) be the local diagram for an RI move: a disc \(\mathring{D}_0\) with \(n_0 = 1\) and a single positive/negative crossing on a single arc. It does not matter where the marked point is located. We apply the skein relation to \(T_+\) to get

\[\tilde{Z}_{T_+} = \tilde{Z}_{T_0} - q \tilde{Z}_{T_1} = ((q + q^{-1}) \tilde{Z}_{T_1} - q \tilde{Z}_{T_1}) = q^{-1} \tilde{Z}_{T_1}\]

where \(T_0\) is the planar diagram for the 0-resolution: a single arc and a free circle, while \(T_1\) is the diagram for the 1 resolution: a single planar arc. In the second equality, we use the Temperley-Lieb property to replace the free circle from \(T_0\) with a factor of \(q + q^{-1}\).
The resulting diagram is isotopic in \( B \) to the diagram \( T_1 \). Since there is only one positive crossing \( n_+(T_+) = 1 + n_+(T_1) \) while \( n_-(T_+) = n_-(T_1) = 0 \). Thus
\[
Z_{T_+} = (-1)^nq^{n_+(T_+)}\tilde{Z}_{T_+} = q^{1+n_+(T_1)}q^{-1}\tilde{Z}_{T_1} = q^{-n_+(T_1)}\tilde{Z}_{T_1} = Z_{T_1}
\]
which verifies the RI invariance of the partition maps. For \( T_- \) the roles of \( T_1 \) and \( T_0 \) switch: \( T_0 \) is now the single arc, and \( T_1 \) has the free circle. Thus
\[
Z_{T_-} = (-1)^nq^{-2}(Z_{T_0} - qZ_{T_1}) = -q^{-2}(Z_{T_0} - (q + q^{-1})qZ_{T_1}) = -q^{-2}(-q^2Z_{T_0}) = Z_{T_0}
\]
which again verifies the invariance under the first Reidemeister move.

Given an oriented local picture for an RII move, with two crossings, one of the crossings will be a positive crossing and the other will be a negative crossing. Thus \( n_+ = n_- = 1 \) and \( Z_T = -q^{-1}\tilde{Z}_T \). We will label the four resolution diagrams as \( 00, 01, 10, \) and \( 11 \) where the we order the crossings as \( c_1 \) and \( c_2 \) so that the 01 resolution \( T_{01} \) has one free circle, and the 10 resolution \( T_{10} \) is identical to the local picture after the RII move. Finally \( T_{00} \) is isotopic to \( T_{11} \). Then,
\[
\tilde{Z}_T = \tilde{Z}_{T_{00}} - q\tilde{Z}_{T_{11}} = \tilde{Z}_{T_{00}} - q(\tilde{Z}_{T_{10}} - q\tilde{Z}_{T_{11}})
\]
\[
= (1 + q^2)\tilde{Z}_{T_{00}} - q(q + q^{-1})\tilde{Z}_{T_{10}} - q\tilde{Z}_{T_{11}}
\]
\[
= -q\tilde{Z}_{T_{10}}
\]
Thus \( Z_T = -q^{-1}\tilde{Z}_T = -q^{-1}(-q\tilde{Z}_{T_{10}}) = \tilde{Z}_{T_{10}} = Z_{T_{10}} \) where we use that \( T_{10} \) has no crossings to assert the final equality. Since \( T_{10} \) is the diagram after the RII move, the map \( Z_T \) is invariant under RII moves.

We now compute the partition map for the following two tangles, whose replacement by each other implements a Reidemeister III move (the same argument works for all RII moves), following the usual argument for invariance. In each diagram \( T_a \) and \( T_b \), we resolve a crossing which gives \( \tilde{Z}_{T_a} = \tilde{Z}_C - q\tilde{Z}_D \) while \( \tilde{Z}_{T_b} = \tilde{Z}_C - q\tilde{Z}_E \). In both \( D \) and \( E \), there is a smaller disc where we can implement an RII move to get three independent strands. By the argument for invariance under RII moves, we know that \( \tilde{Z}_E = \tilde{Z}_D \) since each is \(-q\tilde{Z}_F \) where \( F \) is the diagram with three independent strands. Thus, \( \tilde{Z}_{T_a} = \tilde{Z}_{T_b} \). For any orientation of the strands \( T_a \) and \( T_b \) will have the same number of positive and negative crossings, so we can conclude that \( Z_{T_a} = Z_{T_b} \). \( \diamond \)

In [13] it is shown that mutation of an inside tangle does not affect the map \( \langle [T] \rangle \). The argument there applies also to our partition maps:

**Proposition 38.** Let \( T' \) be a mutant of \( T \) by a mutation in \( \text{int} \Sigma_D \). Then \( Z_{T'} = Z_T \).

### 7. Representations of braid groups

Let \( B_{2n} \) be the braid group on \( 2n \)-strands. Each diagram \( D_\sigma \) for \( \sigma \in B_{2n} \) can be thought of as a tangle diagram subordinate to two discs. If we think of the diagram \( D_\sigma \) as being
in \([0,1] \times [0,1]\), with \(2n\) points on \([0,1] \times \{0\}\) and \([0,1] \times \{1\}\), we can glue \((0,x)\) to \((1,x)\) to obtain a diagram in an annulus. We glue two discs into the annulus to obtain a sphere. The points on \([0,1] \times \{0\}\) are labeled \(p_1, \ldots, p_{2n}\) in the increasing \(x\) direction, and likewise for the points on \([0,1] \times \{1\}\). Let \(D_0\) be the disc glued to the image \([0,1] \times \{0\}\), and \(D_1\) be the disc glued to the image of \([0,1] \times \{1\}\). For \(*D_0\) we use the image of \((0,0) \sim (1,0)\), while for \(*D_1\) we use the image of \((0,1) \sim (1,1)\). Then \(D_1 \in I_T\) and \(D_0 \in O_T\) (in keeping with the view that braids go down the page).

This construction allows us to associate a map \(Z_{\sigma} : I_{2n} \rightarrow I_{2n}\). Furthermore, if \(\sigma\) and \(\sigma'\) are braids in \(B_{2n}\) then \(\sigma' \circ_1 \sigma\) is the braid bound by stacking \(\sigma\) on top of \(\sigma'\) (i.e. identifying \([0,1] \times \{0\}\) for \(\sigma\) with \([0,1] \times \{1\}\) for \(\sigma'\) using \((x,0) \sim (x,1)\)'). The composition formula shows that \(Z_{\sigma' \circ \sigma} = Z_{\sigma'} Z_{\sigma}\) and thus the maps \(Z_{\sigma}\) are a representation of \(B_{2n}\).

**Examples:** (1) Suppose that \(\sigma\) is the trivial braid on \(2n\) strands. The minimal planar tangle diagram for \(\sigma\) has no crossings and is thus a planar diagram. If \(\overrightarrow{m}\) and \(\overleftarrow{m}\) are the crossingless matchings used to complete the boundary of \(\sigma\) to get a multiply cleaved link \(L\), and we equip this with a decoration \(s\), then \(\partial_1(\overrightarrow{m} \# \sigma \# \overleftarrow{m})\) is the cleaved link \(\overrightarrow{m} \# \overleftarrow{m}\), with the decoration from \(s\). Thus, if \(I_{(L,s)} \in I_{2n}\) is the generator with \(L = \overrightarrow{m} \# \sigma \# \overleftarrow{m}\) with this decoration, then only \(M = \overrightarrow{m} \# \sigma \# \overleftarrow{m}\) has \(\mathfrak{P}(M) = (L,s)\). However, \(\partial_0(\overrightarrow{m} \# \sigma \# \overleftarrow{m})\) is the same cleaved link \(\overrightarrow{m} \# \overleftarrow{m}\) with the same decoration. Thus \(Z_{\sigma}(I_{(L,s)}) = I_{(L,s)}\), from which we can conclude that \(Z_{\sigma}\) is the identity map.

(2) We consider the representation of \(B_2\). Let \(\sigma \in B_2\) correspond the the braid diagram with a single right-handed crossing, and both strands oriented down the page. Then \(n_-(\sigma) = 0\) and \(n_+(\sigma) = 1\), so \(Z_{\sigma} = (-1)^0q^{1-2^0}Z_{\sigma}\). We use the skein relation to expand \(\overrightarrow{Z_{\sigma}} = \overrightarrow{Z_1} - q\overrightarrow{Z_T}\) where \(T\) is the tangle consisting of a cup and a cap, see Figure 4. The latter map can be found from composing the map \(I_2 \rightarrow \mathbb{Z}[q^{1/2}, q^{-1/2}]\) for a single cup, with the map for a single cap, which is dual to the former map. These both occur in the annulus, but as one of the circles does not intersect either the cup or cap, that circle can be erased. We computed the maps for the resulting disk tangle in the examples in section.
Thus,
\[ Z_{T'} : \begin{cases} I_+ \xrightarrow{\text{cup}} q^{1/2} \xrightarrow{\text{cap}} q^{1/2} (q^{1/2} I_+ + q^{-1/2} I_-) = q I_+ + I_- \\ I_- \xrightarrow{\text{cup}} q^{-1/2} \xrightarrow{\text{cap}} q^{-1/2} (q^{1/2} I_+ + q^{-1/2} I_-) = I_+ + q^{-1} I_- \end{cases} \]

Converting to matrix notation in the basis \{I_+, I_-\}, we have
\[ \tilde{Z}_\sigma = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - q \begin{bmatrix} q & 1 \\ 1 & q^{-1} \end{bmatrix} \]
\[ = \begin{bmatrix} 1 - q^2 & -q \\ -q & 0 \end{bmatrix} \]

Thus,
\[ Z_\sigma = q \begin{bmatrix} 1 - q^2 & -q \\ -q & 0 \end{bmatrix} = \begin{bmatrix} q - q^3 & -q^2 \\ -q^2 & 0 \end{bmatrix} \]

We can use this computation to calculate \( Z_{\sigma^{-1}} \). Since \( \sigma^{-1} \) is represented by the mirror diagram subordinate to the same disc configuration, we have \( Z_{\sigma^{-1}}(\xi) = (Z_\sigma(\xi^*))^* \) thus,
\[ Z_{\sigma^{-1}}(I_+) = (Z_\sigma(I_-))^* = (-q^2 I_+)^* = -q^{-2} I_- \]

while
\[ Z_{\sigma^{-1}}(I_-) = (Z_\sigma(I_+))^* = ((q - q^3) I_+ - q^2 I_-)^* = (q^{-1} - q^{-3}) I_- - q^{-2} I_+ \]

In matrix notation, this yields
\[ Z_{\sigma^{-1}} = \begin{bmatrix} 0 & -q^{-2} \\ -q^2 & q^{-1} - q^{-3} \end{bmatrix} \]

It can then be checked directly that \( Z_\sigma Z_{\sigma^{-1}} = I = Z_{\sigma^{-1}} Z_\sigma \), as required by the Markov moves on braid diagrams.

**Note:** Above we computed the partition map \( \mathcal{I}_2 \rightarrow \mathcal{I}_2 \) for the planar tangle in Figure 4. In matrix form, this map is
\[ \begin{bmatrix} q & 1 \\ 1 & q^{-1} \end{bmatrix} \]

If we stack two copies of this tangle, we obtain a tangle with a free circle. If we delete the free circle we obtain another copy of the planar tangle. We illustrate the influence of the free circle by computing the square of the matrix:
\[ \begin{bmatrix} q & 1 \\ 1 & q^{-1} \end{bmatrix}^2 = \begin{bmatrix} q & 1 \\ 1 & q^{-1} \end{bmatrix} \begin{bmatrix} q & 1 \\ 1 & q^{-1} \end{bmatrix} \]
\[ = \begin{bmatrix} q^2 + 1 & q + q^{-1} \\ q + q^{-1} & 1 + q^{-2} \end{bmatrix} \]
\[ = (q + q^{-1}) \begin{bmatrix} q & 1 \\ 1 & q^{-1} \end{bmatrix} \]

This conforms with the requirements of the Temperley-Lieb property.
8. Comments on planar algebra properties

In [6] several distinctions are made between different types of planar algebras. We briefly mention where the maps $Z_P$, for $P$ a planar diagram, fit within these distinctions.

The planar algebra is a unital, spherical, $*$-planar algebra. It is unital since each planar diagram for a disc configuration ($\leftarrow - D_0$) gives rise to a map $Z_P$ which satisfies the compositional requirement in [6]. It is spherical since $\dim \mathcal{I}_0 = 1$ (over $\mathbb{Z}[q^{1/2}, q^{-1/2}]$) and the maps are invariant under isotopies. Finally, the conjugation $\xi \ast \xi^*$ acts correctly for it to be a $*$-planar algebra.

Finally, it is non-degenerate. In fact, the pairing depicted defined by the diagram in Figure 5 is a sum of hyperbolic pairs.

**Definition 39.** Let $(L, \sigma) \in \mathcal{CL}_n$. Then the dual $(\overline{L}, \overline{\sigma})$ of $(L, \sigma)$ is the cleaved link obtained by changing the orientation on the sphere, but fixing the orientation on the points in $E \cap L$.

Thus, $(\overline{L}, \overline{\sigma})$ comes from switching which disc is playing the role of the inside disc.

**Definition 40.** Let $L, L' \in \mathcal{CL}_n$. Then $\langle L, L' \rangle$ is the image of $L \otimes L'$ under the map $Z_P : \mathcal{I}_{2n} \otimes \mathcal{I}_{2n} \rightarrow \mathbb{Z}[q^{1/2}, q^{-1/2}]$ for the diagram $P$ in Figure 5.

**Proposition 41.** Let $L, L' \in \mathcal{CL}_n$, then

\[
\langle L, L' \rangle = \begin{cases} 
0 & L' \neq \overline{L} \\
1 & L' = \overline{L}
\end{cases}
\]

**Proof:** To obtain the diagrams in $\mathcal{M}(P)$ we pick two inside matchings $\overleftarrow{m}_1$ and $\overleftarrow{m}_2$ to fill in the discs $\overleftarrow{D}_i$. Let $\partial_1(P \circ_1 \overleftarrow{m}_1 \circ_2 \overleftarrow{m}_2) = \overleftarrow{m}_1 \circ \overrightarrow{m}_2$ where $\overrightarrow{m}_2$ is the dual of $\overleftarrow{m}_2$, i.e. the same matching considered as an outside matching. On the other hand, $\partial_2(P \circ_1 \overrightarrow{m}_1 \circ_2 \overrightarrow{m}_2) = \overrightarrow{m}_1 \# \overleftarrow{m}_2$. These boundaries are dual, so only dual cleaved links can have non-zero pairing. To see that these pair to give 1, note that any circle, with any decoration, in the multiply
cleaved link necessarily intersects both inside discs. Thus the total weight will be multiples of $q^{1-1/2} = 1$ and $q^{-1+1/2} = 1$.

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