Decision trees for regular factorial languages

Mikhail Moshkov

Computer, Electrical and Mathematical Sciences and Engineering Division and Computational Bioscience Research Center, King Abdullah University of Science and Technology (KAUST), Thuwal 23955-6900, Saudi Arabia

A B S T R A C T

In this paper, we study arbitrary regular factorial languages over a finite alphabet Σ. For the set of words \( L(n) \) of the length \( n \) belonging to a regular factorial language \( L \), we investigate the depth of decision trees solving the recognition and the membership problems deterministically and nondeterministically. In the case of recognition problem, for a given word from \( L(n) \), we should recognize it using queries each of which, for some \( i \in \{1, \ldots, n\} \), returns the \( i \)th letter of the word. In the case of membership problem, for a given word over the alphabet \( Σ \) of the length \( n \), we should recognize if it belongs to the set \( L(n) \) using the same queries. For a given problem and type of trees, instead of the minimum depth \( h(n) \) of a decision tree of the considered type solving the problem for \( L(n) \), we study the smoothed minimum depth \( H(n) = \max\{h(m) : m \leq n\} \). With the growth of \( n \), the smoothed minimum depth of decision trees solving the problem of recognition deterministically is either bounded from above by a constant, or grows as a logarithm, or linearly. For other cases (decision trees solving the problem of recognition nondeterministically, and decision trees solving the membership problem deterministically and nondeterministically), with the growth of \( n \), the smoothed minimum depth of decision trees is either bounded from above by a constant or grows linearly. As corollaries of the obtained results, we study joint behavior of smoothed minimum depths of decision trees for the considered four cases and describe five complexity classes of regular factorial languages. We also investigate the class of regular factorial languages over the alphabet \( \{0,1\} \) each of which is given by one forbidden word.

1. Introduction

In this paper, we study arbitrary regular factorial languages over a finite alphabet \( Σ \). A factorial language satisfies the following condition: if a word \( u_1u_2u_3 \) belongs to the language, then the word \( u \) also belongs to it. For the set of words \( L(n) \) of the length \( n \) belonging to a regular factorial language \( L \), we investigate the depth of decision trees solving the recognition and the membership problems deterministically and nondeterministically. In the case of recognition problem, for a given word from \( L(n) \), we should recognize it using queries each of which, for some \( i \in \{1, \ldots, n\} \), returns the \( i \)th letter of the word. In the case of membership problem, for a given word over the alphabet \( Σ \) of the length \( n \), we should recognize if it belongs to \( L(n) \) using the same queries.

For a given problem (problem of recognition or membership problem) and type of trees (solving the problem deterministically or nondeterministically), instead of the minimum depth \( h(n) \) of a decision tree of the considered type solving the problem for \( L(n) \), we study the smoothed minimum depth \( H(n) = \max\{h(m) : m \leq n\} \).

For an arbitrary regular factorial language, with the growth of \( n \), the smoothed minimum depth of decision trees solving the problem of recognition deterministically is either bounded from above by a constant, or grows as a logarithm, or linearly. These results follow immediately from more general, obtained in [1] for arbitrary regular languages.

For other cases (decision trees solving the problem of recognition nondeterministically, and decision trees solving the membership problem deterministically and nondeterministically), with the growth of \( n \), the smoothed minimum depth of decision trees is either bounded from above by a constant, or grows linearly. In the conference paper [2], a classification of arbitrary regular languages depending on the smoothed minimum depth of decision trees solving the problem of recognition nondeterministically was announced without proofs. In the present paper, we consider simpler classification for regular factorial languages with full proof. Results related to the decision trees solving the membership problem are new.

As corollaries of the obtained results, we study joint behavior of smoothed minimum depths of decision trees for the considered four cases and describe five complexity classes of regular factorial languages. We also investigate the class of regular factorial languages over the alphabet \( E = \{0,1\} \) each of which is given by one forbidden word.

A well-known approach to evaluate complexity of an infinite language \( L \) over a finite alphabet \( Σ \) is to study its so-called combinatorial
complexity (known also as counting function) \( f_2(n) \) that is the number of words of the length \( n \) in \( L \) [3,4]. The present paper proposes additional ways to evaluate the complexity of the language \( L \) based on the study how the depth of decision trees solving the recognition and the membership problems deterministically and nondeterministically depends on the length of words. This way is more complicated, but can give more detailed classification of languages. To show this, we compare languages generated by diagrams \( I_1 \) and \( I_2 \) depicted in Figs. 5 and 6. For both languages, the counting function grows linearly. For the first language, the minimum depth of decision trees solving the problem of recognition deterministically grows as a logarithm, but for the second language, the minimum depth of decision trees solving the problem of recognition deterministically grows linearly.

We should mention a recent paper [5] in which similar results were obtained for languages over the alphabet \( E \) that are subword-closed: if a word \( \omega = \omega_1 \omega_2 \cdots \omega_m \omega_m + 1 \) belongs to the language, then the word \( \omega_1 \cdots \omega_m \) also belongs to it.

It is clear that each subword-closed language is a factorial language. Moreover, each subword-closed language over a finite alphabet is a regular language [6]. One can show that the language \( L(00) \) over the alphabet \( E \) given by one forbidden word \( 00 \) is a regular factorial language, which is not subword-closed. Therefore the class of subword-closed languages over the alphabet \( E \) is a proper subclass of the class of regular factorial languages over the alphabet \( E \).

The main difference between the present paper and [5] is that, in the latter paper, we do not assume that the subword-closed languages are given by deterministic finite automata. Instead of this, we describe simple criteria (based on the presence in the language of words of special types) for the behavior of the minimum depths of decision trees solving the problem of recognition deterministically and nondeterministically. Differently formulated criteria for the behavior of the minimum depth of decision trees solving the recognition problem require very different proofs. One more difference is that in [5] we directly consider the minimum depth of decision trees.

The rest of the paper is organized as follows. In Section 2, we consider main notions, in Section 3 – main results, and in Section 4 – two corollaries of these results.

2. Main notions
In this section, we discuss the notions related to regular factorial languages and decision trees solving problems of recognition and membership for these languages.

2.1. Regular factorial languages
Let \( \omega = \{0, 1, 2, \ldots \} \) be the set of nonnegative integers and \( \Sigma \) be a finite alphabet with at least two letters. By \( \Sigma^* \), we denote the set of all finite words over the alphabet \( \Sigma \), including the empty word \( \lambda \). A word \( \omega \in \Sigma^* \) is called a factor of a word \( u \in \Sigma^* \) if \( \omega \subseteq u \). A language \( L \subseteq \Sigma^* \) is called factorial if it contains all factors of its words. A word \( \omega \in \Sigma^* \) is called a minimal forbidden word for \( L \) if \( \omega \notin L \) and all proper factors of \( \omega \) belong to \( L \). We denote by \( M(L) \) the set of minimal forbidden words for \( L \). It is known [7] that a factorial language \( L \) is regular if and only if the language \( M(L) \) is regular. In particular, a factorial language \( L \) with a finite set of minimal forbidden words \( M(L) \) is regular. In this paper, we study arbitrary nonempty regular factorial languages.

It is well known that each regular language can be represented by a deterministic finite automaton (DFA) [8]. As in [8], we will consider not only complete DFA with total transition function but also partial DFA with partial transition function. Such DFA can be represented by its transition diagram (diagram for short) [9].

A diagram over the alphabet \( \Sigma \) is a triple \( I = (G, q_0, Q) \), where \( G \) is a finite directed graph, possibly with multiple edges and loops, in which each edge is labeled with a letter from \( \Sigma \) and edges leaving each node are labeled with pairwise different letters, \( q_0 \) is a node of \( G \) called starting, and \( Q \) is a nonempty set of the graph \( G \) nodes called final.

A path of the diagram \( I \) is an arbitrary sequence \( \xi = v_1 \cdots v_{\ell} v_{\ell + 1} \) of nodes and edges of \( G \) such that the edge \( \delta_i \) leaves the node \( v_i \) and enters the node \( v_{i+1} \) for \( i = 1, \ldots, \ell \). We now define a word \( w(\xi) \) from \( \Sigma^* \) in the following way: if \( m = 0 \), then \( w(\xi) = \lambda \). Let \( m > 0 \) and let \( \delta_i \) be the letter attached to the edge \( \delta_j, j = 1, \ldots, m \). Then \( w(\xi) = \delta_1 \cdots \delta_m \). We say that the path \( \xi \) generates the word \( w(\xi) \). Note that different paths which start in the same node generate different words.

We denote by \( \Xi(I) \) the set of all paths of the diagram \( I \) each of which starts in the node \( q_0 \) and finishes in a node from \( Q \). Let \( L_I = \{ w(\xi) : \xi \in \Xi(I) \} \).

We say that the diagram \( I \) generates the language \( L_I \). It is well known that \( L_I \) is a regular language.

The diagram \( I \) is called complete over the alphabet \( \Sigma \) if exactly \( |\Sigma| \) edges leave each node of \( G \). Note that these edges are labeled with pairwise different letters from \( \Sigma \). Such diagram corresponds to a complete DFA \([8]\). The diagram \( I \) is called reduced if, for each node of \( G \), there exists a path in \( \Xi(I) \), which contains this node. Such diagram corresponds to a reduced DFA \([8]\). It is known \([8]\) that, for each regular language over the alphabet \( \Sigma \), there exists a complete over the alphabet \( \Sigma \) diagram, which generates this language. Therefore, for each nonempty regular language, there exists a reduced diagram, which generates this language.

Let \( I \) be a regular factorial language and \( I = (G, q_0, Q) \) be a reduced diagram that generates the language \( L \). Since the language \( L \) is factorial, we can assume additionally that each node of the graph \( G \) is final — it will not change the language generated by \( I \) since with each word the language \( L \) contains each prefix of this word. The diagram \( I \) will be called \( \ell \)-reduced if it is reduced and each node of the graph \( G \) is final. Further we will assume that a considered regular factorial language \( L \) is nonempty and it is given by an \( \ell \)-reduced diagram, which generates this language.

We will not consider nondeterministic finite automata (NFA) to represent regular factorial languages since the study of NFA is essentially more complicated task.

2.2. Decision trees for recognition and membership problems
Let \( L \) be a regular factorial language over the alphabet \( \Sigma \). For any natural \( n \), denote \( L(n) = L \cap \Sigma^n \), where \( \Sigma^n \) is the set of words over the alphabet \( \Sigma \), which length is equal to \( n \). We consider two problems related to the set \( L(n) \). The problem of recognition: for a given word from \( L(n) \), we should recognize it using attributes (queries) \( l_0^0, \ldots, l_n^0 \), where \( l_i^j \), \( i \in \{1, \ldots, n\}, \) is a function from \( \Sigma^n \) to \( \Sigma \) such that \( l_i^0(a_1 \cdots a_n) = a_i \) for any word \( a_1 \cdots a_n \in \Sigma^n \). The problem of membership: for a given word from \( \Sigma^n \), we should recognize if this word belongs to the set \( L(n) \) using the same attributes. To solve these problems, we use decision trees over \( L(n) \).

A decision tree over \( L(n) \) is a marked finite directed tree with root, which has the following properties:

- The root and the edges leaving the root are not labeled.
- Each node, which is not the root nor terminal node, is labeled with an attribute from the set \( \{l_0^0, \ldots, l_n^0\} \).
- Each edge leaving a node, which is not a root, is labeled with a letter from the alphabet \( \Sigma \).

A decision tree over \( L(n) \) is called deterministic if it satisfies the following conditions:

- Exactly one edge leaves the root.
- For any node, which is not the root nor terminal node, the edges leaving this node are labeled with pairwise different letters.
Let $\Gamma$ be a decision tree over $L(n)$. A complete path in $\Gamma$ is any sequence $\xi = v_0, v_1, \ldots, v_r$, of nodes and edges of $\Gamma$ such that $v_0$ is the root, $v_{r+1}$ is a terminal node, and $v_i$ is the initial and $v_{i+1}$ the terminal node of the edge $e_i$ for $i = 0, \ldots, m$. We define a subset $\Sigma(n, \xi)$ of the set $\Sigma^*$ in the following way: if $m = 0$, then $\Sigma(n, \xi) = \Sigma^*$. Let $m > 0$, the attribute $a_i^0$ be attached to the node $v_i$, and $b_i$ be the letter attached to the edge $e_i$, $j = 1, \ldots, m$. Then $\Sigma(n, \xi) = \{a_1 \ldots a_m \in \Sigma^* : a_1 = b_1, \ldots, a_m = b_m\}$.

Let $L(n) \neq \emptyset$. We say that a decision tree $\Gamma$ over $L(n)$ solves the problem of recognition for $L(n)$ nondeterministically if $\Gamma$ satisfies the following conditions:

- Each terminal node of $\Gamma$ is labeled with a word from $L(n)$.
- For any word $w \in L(n)$, there exists a complete path $\xi$ in the tree $\Gamma$ such that $w \in \Sigma(n, \xi)$.
- For any word $w \in L(n)$ and for any complete path $\zeta$ in the tree $\Gamma$ such that $w \in \Sigma(n, \zeta)$, the terminal node of the path $\zeta$ is labeled with the word $w$.

We say that a decision tree $\Gamma$ over $L(n)$ solves the problem of recognition for $L(n)$ deterministically if $\Gamma$ is a deterministic decision tree, which solves the problem of recognition for $L(n)$ nondeterministically.

Examples of decision trees illustrating the considered notions are presented in Fig. 1.

We say that a decision tree $\Gamma$ over $L(n)$ solves the problem of membership for $L(n)$ nondeterministically if $\Gamma$ satisfies the following conditions:

- Each terminal node of $\Gamma$ is labeled with a number from the set $\{0, 1\}$.
- For any word $w \in \Sigma^*$, there exists a complete path $\xi$ in the tree $\Gamma$ such that $w \in \Sigma(n, \xi)$.
- For any word $w \in \Sigma^*$ and for any complete path $\zeta$ in the tree $\Gamma$ such that $w \in \Sigma(n, \zeta)$, the terminal node of the path $\zeta$ is labeled with the number 1 if $w \in L(n)$ and with the number 0, otherwise.

We say that a decision tree $\Gamma$ over $L(n)$ solves the problem of membership for $L(n)$ deterministically if $\Gamma$ is a deterministic decision tree which solves the problem of membership for $L(n)$ nondeterministically.

Let $\Gamma$ be a decision tree over $L(n)$. We denote by $h(\Gamma)$ the maximum number of nodes in a complete path in $\Gamma$ that are not the root nor terminal node. The value $h(\Gamma)$ is called the depth of the decision tree $\Gamma$.

We denote by $h_{\text{ld}}^L(n)$ ($h_{\text{rd}}^L(n)$) the minimum depth of a decision tree over $L(n)$, which solves the problem of recognition for $L(n)$ nondeterministically (deterministically). If $L(n) = \emptyset$, then $h_{\text{ld}}^L(n) = h_{\text{rd}}^L(n) = 0$.

We denote by $h_{\text{sm}}^L(n)$ ($h_{\text{ms}}^L(n)$) the minimum depth of a decision tree over $L(n)$, which solves the problem of membership for $L(n)$ nondeterministically (deterministically). If $L(n) = \emptyset$, then $h_{\text{sm}}^L(n) = h_{\text{ms}}^L(n) = 0$.

3. Bounds on decision tree depth

Let $L$ be a nonempty factorial regular language. In this section, we consider the behavior of four functions $H_{\text{ld}}^L(n)$, $H_{\text{rd}}^L(n)$, $H_{\text{sm}}^L(n)$, and $H_{\text{ms}}^L(n)$ defined on the set $\omega \setminus \{0\}$ and with values from $\omega$. For any natural $n$,

- $H_{\text{ld}}^L(n) = \max\{h_{\text{ld}}^L(m) : 1 \leq m \leq n\}$,
- $H_{\text{rd}}^L(n) = \max\{h_{\text{rd}}^L(m) : 1 \leq m \leq n\}$,
- $H_{\text{sm}}^L(n) = \max\{h_{\text{sm}}^L(m) : 1 \leq m \leq n\}$,
- $H_{\text{ms}}^L(n) = \max\{h_{\text{ms}}^L(m) : 1 \leq m \leq n\}$.

For any pair $bc \in \{ra, rd, ma, md\}$, the function $H_{\text{bc}}^L(n)$ is a smoothed analog of the function $h_{\text{bc}}^L(n)$.

3.1. Decision trees solving recognition problem deterministically

Let $L = (G, q_0, Q)$ be a f-reduced diagram over the alphabet $\Sigma$. A path of the diagram $L$ is called a cycle of the diagram $L$ if there is at least one edge in this path, and the first node of this path is equal to the last node of this path. A cycle of the diagram $L$ is called elementary if nodes of this cycle, with the exception of the last node, are pairwise different.

The diagram $L$ is called simple if every two different elementary cycles of the diagram $L$ do not have common nodes. Let $L$ be a simple diagram and $\xi$ be a path of the diagram $L$. The number of different elementary cycles of the diagram $L$, which have common nodes with $\xi$, is denoted by $c(\xi)$ and is called the cyclic length of the path $\xi$. The value $c(L) = \max\{c(\xi) : \xi \in \Xi(L)\}$ is called the cyclic length of the diagram $L$.

Let $I$ be a simple diagram, $C$ be an elementary cycle of the diagram $L$, and $v$ be a node of the cycle $C$. Beginning with the node $v$, the cycle $C$ generates an infinite periodic word over the alphabet $\Sigma$. This word will be denoted by $W(I, C, v)$. We denote by $r(I, C, v)$ the minimum period of the word $W(I, C, v)$. The diagram $L$ is called dependent if there exist two different elementary cycles $C_1$ and $C_2$ of the diagram $L$, nodes $v_1$ and $v_2$ of the cycles $C_1$ and $C_2$, respectively, and a path $x$ of the diagram $L$ from $v_1$ to $v_2$, which satisfy the following conditions: $W(I, C_1, v_1) = W(I, C_2, v_2)$ and the length of the path $x$ is a number divisible by $r(I, C_1, v_1)$. If the diagram $L$ is not dependent, then it is called independent. Next theorem follows immediately from Theorem 2.1 [1], which is a similar statement that holds for all regular languages.

**Theorem 1.** Let $L$ be a nonempty regular factorial language over the alphabet $\Sigma$ and $I$ be a f-reduced diagram, which generates the language $L$. Then the following statements hold:

(a) If $I$ is an independent simple diagram and $c(I) \leq 1$, then $H_{\text{rd}}^L(n) = O(1)$.
(b) If $I$ is an independent simple diagram and $c(I) \geq 2$, then $H_{\text{rd}}^L(n) = \Theta(n)$.
(c) If $I$ is not independent simple diagram, then $H_{\text{rd}}^L(n) = \Theta(n)$.

3.2. Decision trees solving recognition problem nondeterministically

Let $L$ be a nonempty regular factorial language over the alphabet $\Sigma$. For any natural $n$, we define a parameter $T_L(n)$ of the language $L$.

- If $L(n) = \emptyset$, then $T_L(n) = 0$.
- If $L(n) \neq \emptyset$, denote $L(w, J) = \{b_1 \ldots b_k \in L(n) : b_j = a_j, j \in J\}$ for any word $w \in \Sigma^*$.

Let $L(n) \neq \emptyset$. Then $T_L(n, w) = \min\{|J| : J \subseteq \{1, \ldots, n\}, |L(w, J)| = 1\}$.

Note that, for any word $w \in L(n)$, $M_L(n, w)$ is the minimum number of letters of the word $w$, which allow us to distinguish it from all other words belonging to $L(n)$.
Lemma 2. Let L be a nonempty regular factorial language over the alphabet Σ. Then \( h_{L}^0(n) = T_0(n) \) for any natural n.

Proof. First, we prove that \( h_{L}^0(n) \geq T_0(n) \). Let \( \Gamma \) be a decision tree over \( L(n) \), which solves the problem of recognition for \( L(n) \) nondeterministically and for which \( h(\Gamma) = h_{L}^0(n) \). Let \( w \) be a word from \( L(n) \) for which \( T_0(n) = M_w(n, w) \). Then the decision tree \( \Gamma \) contains a complete path \( \xi \) such that \( w \in \Sigma(\xi) \) and the terminal node of the path \( \xi \) is labeled with the word \( w \). It is clear that \( M_w(n, w) = T_0(n) \). Therefore \( m \geq M_w(n, w) = T_0(n) \). It is clear that \( h(\Gamma) \geq m \). Thus, \( h_{L}^0(n) = h(\Gamma) \geq m \geq M_w(n, w) = T_0(n) \).

We now prove that \( h_{L}^0(n) \leq T_0(n) \). One can see that, for each \( w \in L(n) \), we can construct a complete path \( \xi_w \) which satisfies the following conditions: the number of nodes in \( \xi_w \) that are not the root nor terminal node and \( 2^i, 2^i + 1 \) be attributes attached to these nodes. Denote \( J = \{ t_1, \ldots, t_n \} \). Then \( L(w, J) = \{ w \} \).


\[
\begin{align*}
\text{Theorem 3.} & \quad \text{Let } L \text{ be a nonempty regular factorial language over the alphabet } \Sigma \text{ and } I = (G, q_0, Q) \text{ be a } \omega \text{-reduced diagram, which generates the language } L. \text{ Then the following statements hold:} \\
& (a) \quad \text{If } L \text{ is a decision tree, then } H_{L}^0(n) = O(1). \\
& (b) \quad \text{If } L \text{ is not a decision tree, then } H_{L}^0(n) = \Theta(n).
\end{align*}
\]

Proof. (a) Let \( I \) be an independent simple diagram and \( c(I) \leq 1 \). By Theorem 1, \( H_{L}^0(n) = O(1) \). It is clear that \( H_{L}^0(n) = O(1) \). Therefore \( H_{L}^0(n) = O(1) \).

(b) Let \( I \) be an independent simple diagram and \( c(I) \geq 2 \). Let \( n \) be a natural number. If \( L(n) = \emptyset \), then \( T_0(n) = 0 \). Let \( L(n) \neq \emptyset \). Denote by \( d \) the number of nodes in the graph \( G \) in the proof of Lemma 4.5 [1], it was proved that \( M_w(n, w) \leq d(4d + 1) \) for any word \( w \in L(n) \). Therefore \( T_0(n) \leq d(4d + 1) \). Thus, by Lemma 2, \( H_{L}^0(n) \leq d(4d + 1) \) for any natural \( n \) and \( H_{L}^0(n) = O(1) \).

Proof. (a) Let \( I \) be a \( \omega \)-reduced diagram and \( C_1, C_2 \) be different elementary cycles of the diagram \( I \), which have a common node \( v \). Since \( I \) is a \( \omega \)-reduced diagram, it contains a path \( \xi \) from the node \( q_0 \) to the node \( v \), and \( v \) is a final node. Let the length of the path \( \xi \) be equal to \( a \), the length of the cycle \( C_1 \) be equal to \( b \), and the length of the cycle \( C_2 \) be equal to \( c \). Let a be the word generated by the path \( \xi \), \( \beta \) be the word generated by a path from \( v \) to \( v \) obtained by the passage \( \epsilon \) times along the cycle \( C_2 \), and \( \gamma \) be the word generated by a path from \( v \) to \( v \) obtained by the passage \( \epsilon \) times along the cycle \( C_2 \). The words \( \beta \) and \( \gamma \) are different and they have the same length \( b \).

Consider the sequence of numbers \( n_j = a + ibc, i = 1, 2, \ldots \). Let \( i \in \omega \setminus \{ 0 \} \). Let \( L(n_j) \) contains the word \( a^j \) and the words \( a^j \beta^{i(j-1)} \) for \( j = 0, \ldots, i \). It is easy to show that \( M_{n_j}(v, \beta^{i(j-1)} \gamma) = \gamma \) to distinguish the word \( a^j \beta^{i(j-1)} \gamma \). Therefore \( T_0(n_j) \leq i \). By Lemma 2, \( H_{L}^0(n_j) \geq i = (n_j - a)/(bc) \). Let \( n \geq n_j \) and let \( I \) be the maximum natural number such that \( n \geq n_j \). Evidently, \( n - n_j \leq bc \). Hence \( H_{L}^0(n) \geq H_{L}^0(n_j) \geq (n - b - c - a)/(bc) \). Therefore \( H_{L}^0(n) \geq n/(2bc) \) for large enough \( n \).

(b) If \( |L(n)| \leq \omega \) and \( I \) be a complete path in \( I \) such that \( w \in \Sigma(n, \xi) \). Then the terminal node of \( \xi \) is labeled with the number 1. Beginning with the first letter, we divide the word \( w \) into \( [n/\ell] \) blocks with \( \ell \) letters in each and the suffix of the length \( n - 1 \). Let us assume that the number of nodes labeled with attributes \( \xi \) is less than \( n \). Then there is a block such that queries (attributes) attached to nodes of \( \xi \) does not ask letters from the block. We replace this block in the word \( w \) with the word \( w_0 \) and denote by \( w' \) the obtained word. It is clear that \( w' \not\in L \) and \( w' \in \Sigma(n, \xi) \), but this is impossible since the terminal node of the path \( \xi \) is labeled with the number 1. Therefore the depth of \( I \) is greater than or equal to \( \delta(n/\ell) \). Thus, \( H_{L}^0(n) \geq (\delta(n/\ell) - \ell) \). It is easy to construct a decision tree over \( L(n) \) that solves the problem of membership for \( L(n) \) deterministically and has the depth equals to \( \delta(n/\ell) \). Therefore \( h_{L}^0(n) \leq n \). Thus, \( H_{L}^0(n) = \Theta(n) \) and \( H_{L}^0(n) = \Theta(n) \).
4. Corollaries

In this section, we consider two corollaries of Theorems 1, 3, and 4.

4.1. Joint behavior of functions $H^{ir}_{L_i}$, $H^{ms}_{L_i}$, and $H^{md}_{L_i}$

In this section, we assume that each regular factorial language over the alphabet $\Sigma$ is given by a f-reduced diagram $I$, which generates the considered language denoted by $L_I$. To study all possible types of joint behavior of functions $H^{ir}_{L_i}$, $H^{ms}_{L_i}$, and $H^{md}_{L_i}$, we consider five classes of regular factorial languages $F_1, \ldots, F_5$ described in the columns 2-4 of Table 1. In particular, $F_1$ consists of all regular factorial languages $L_I$ for which the diagram $I$ is an independent simple diagram and $c(I) = 0$. It is easy to show that the complexity classes $F_1, \ldots, F_5$ are pairwise disjoint, and each regular factorial language $L_I$ belongs to one of these classes. The behavior of functions $H^{ir}_{L_i}$, $H^{ms}_{L_i}$, and $H^{md}_{L_i}$ for languages from these classes is described in the last four columns of Table 1. For each class, the results considered in Table 1 for the functions $H^{ir}_{L_i}$ and $H^{ms}_{L_i}$ follow directly from Theorems 1 and 3.

We now consider the behavior of the functions $H^{md}_{L_i}$ and $H^{ms}_{L_i}$ for each of the classes $F_1, \ldots, F_5$. Let $I = (G, q_0, Q)$ be a f-reduced diagram over the alphabet $\Sigma$, which generates a regular factorial language.

Let $L_I \in F_1$. Since $c(I) = 0$, $G$ is a directed acyclic graph, and the language $L_I$ is finite. Using Theorem 4 we obtain $H^{md}_{L_i}(n) = O(1)$ and $H^{ms}_{L_i}(n) = O(1)$.

Let $L_I \in F_2$. Since $c(I) = 1$, $G$ is a graph containing a cycle, and the language $L_I$ is infinite. By Lemma 4.2 [1], $|L_I(n)| = O(n^\omega)$, which is factorial. Therefore $L_I \in F_2$.

Let $L_I \in F_3$. Then $L_I \neq \Sigma^*$. Using Theorem 4 we obtain $H^{md}_{L_i}(n) = \Theta(n)$ and $H^{ms}_{L_i}(n) = \Theta(n)$.

Let $L_I \in F_4$. Since $I$ is an independent simple diagram, $G$ is a graph containing a cycle, and the language $L_I$ is finite. We know that $L_I \neq \Sigma^*$. Using Theorem 4 we obtain $H^{md}_{L_i}(n) = \Theta(n)$ and $H^{ms}_{L_i}(n) = \Theta(n)$.

Let $L_I \in F_5$. Then $L_I \neq \Sigma^*$. Using Theorem 4 we obtain $H^{md}_{L_i}(n) = O(1)$ and $H^{ms}_{L_i}(n) = O(1)$.

Now we show that the classes $F_1, \ldots, F_5$ are nonempty. For simplicity, we assume that $\Sigma = E$, where $E = \{0, 1\}$. It is easy to generalize the considered examples to the case of an arbitrary finite alphabet $\Sigma$ with at least two letters. In the examples of diagrams, the starting node is labeled with the symbol $\lambda$, and all nodes are final.

Denote by $I_1$ the diagram over the alphabet $E$ depicted in Fig. 3. One can show that $I_1$ is an independent simple f-reduced diagram and $c(I_1) = 0$. This diagram generates the language $L_{I_1} = \{\lambda, 0\}$, which is factorial. Therefore $L_{I_1} \in F_1$.

Denote by $I_2$ the diagram over the alphabet $E$ depicted in Fig. 4. One can show that $I_2$ is an independent simple f-reduced diagram and $c(I_2) = 1$. This diagram generates the language $L_{I_2} = \{0^i : i \in \omega\}$, which is factorial. Therefore $L_{I_1} \in F_2$.

Denote by $I_3$ the diagram over the alphabet $E$ depicted in Fig. 5. One can show that $I_3$ is an independent simple f-reduced diagram and $c(I_3) = 2$. This diagram generates the language $L_{I_3} = \{0^i1^j : i, j \in \omega\}$, which is factorial. Therefore $L_{I_3} \in F_3$.

Denote by $I_4$ the diagram over the alphabet $E$ depicted in Fig. 6. One can show that $I_4$ is a f-reduced diagram that is not simple. This diagram generates the language $L_{I_4} = E^*$, which is factorial. It is clear that $L_{I_4} \neq E^*$. Therefore $L_{I_4} \in F_4$.

Denote by $I_5$ the diagram over the alphabet $E$ depicted in Fig. 7. One can show that $I_5$ is a f-reduced diagram that is not simple. This diagram generates the language $L_{I_5} = E^*$, which is factorial. It is clear that $L_{I_5} \neq E^*$. Therefore $L_{I_5} \in F_5$.

A regular factorial language $L$ can have different f-reduced diagrams, which generate it. However, for each of such diagrams $I$, the language $L_I = L$ will belong to the same complexity class. Let us assume the contrary: there exist a regular factorial language $L$ and two f-reduced diagrams $I_1$ and $I_2$, which generate it and for which languages $L_{I_1}$ and $L_{I_2}$ belong to different complexity classes. Then, for some pair $bc \in \{rd, ra, md, ma\}$, the functions $H^{md}_{L_{I_1}}$ and $H^{md}_{L_{I_2}}$ have different behavior, but this is impossible since $H^{md}_{L_{I_1}}(n) = H^{md}_{L_{I_2}}(n)$ for any natural $n$.\[\]
4.2. Languages over alphabet \( \{ 0, 1 \} \) given by one forbidden word

Let \( E = \{ 0, 1 \}, \ a \in E^*, \ \text{and} \ a \neq \lambda \). We denote by \( L(a) \) the language over the alphabet \( E \), which consists of all words from \( E^* \) that do not contain \( a \) as a factor. This is a regular factorial language with \( MF(L(a)) = \{ a \} \). The following theorem indicates for each nonempty word \( a \in E^* \) the complexity class \( F_i \) to which the language \( L(a) \) belongs.

**Theorem 5.** Let \( a \in E^* \) and \( a \neq \lambda \).

(a) If \( a \in \{0, 1\} \), then \( L(a) \in F_2 \).

(b) If \( a \in \{01, 10\} \), then \( L(a) \in F_4 \).

(c) If \( a \notin \{0, 01, 10\} \), then \( L(a) \in F_3 \).

We now describe a f-reduced diagram \( I(a) \) that generates the language \( L(a) \) for a nonempty word \( a \in E^* \). Let \( a = a_1 \ldots a_n \), \( a_0 = \lambda \), and \( a_i = a_{i-1} \ldots a_0 \) for \( i = 1, \ldots, n \). The set \( P(a) = \{a_0, a_1, \ldots, a_n\} \) is the set of all proper prefixes of the word \( a \). Then \( I(a) = (G, q_0, Q) \), where the set of nodes of the graph \( G \) is equal to \( P(a) \), \( q_0 = a_0 \), and \( Q = P(a) \).

For \( i = 0, \ldots, n-2 \), an edge leaves the node \( a_i \) and enters the node \( a_{i+1} \). This edge is labeled with the letter \( a_{i+1} \). For \( i = 0, \ldots, n-1 \), an edge leaves the node \( a_i \) and enters the node \( a_{i+1} \) such that \( a_j \) is the longest suffix of the word \( a_i \), \( a_{i+1} = 0 \) if \( a_{i+1} = 1 \) and \( a_{i+1} = 1 \) if \( a_{i+1} = 0 \). This edge is labeled with the letter \( a_{i+1} \). It is easy to show that \( I(a) \) is a f-reduced diagram over the alphabet \( E \). From Theorem 10 [7] it follows that the diagram \( I(a) \) generates the language \( L(a) \).

Let \( a \in E^* \setminus \{ \lambda \} \) and \( a = a_1 \ldots a_n \). We denote by \( \hat{a} \) the word \( a_1 \ldots a_n \). It is easy to prove the following statement.

**Lemma 6.** Let \( a \in E^* \) and \( a \neq \lambda \). Then \( H^b_{L(a)}(n) = H^b_{L(\hat{a})}(n) \) for any pair \( b \in \{ rd, ra, md, ma \} \) and any natural \( n \).

**Lemma 7.** Let \( a \in E^* \setminus \{ \lambda \}, b \in E^* \), and \( L(a) \in F_4 \). Then \( L(\hat{a}b) \in F_3 \).

**Proof.** Since \( L(a) \in F_4 \), \( H^d_{L(a)}(n) = \Theta(n) \) and \( H^e_{L(a)}(n) = \Theta(n) \). One can show that \( L(\hat{a}) \subseteq L(a) \). Using this fact it is not difficult to prove that \( H^d_{L(\hat{a})}(n) \leq H^d_{L(a)}(n) \) and \( H^e_{L(\hat{a})}(n) \leq H^e_{L(a)}(n) \) for any natural \( n \). From here and from Theorems 1 and 3 it follows that \( H^d_{L(\hat{a})}(n) = \Theta(n) \) and \( H^e_{L(\hat{a})}(n) = \Theta(n) \).

Since \( a \hat{a} \notin L(a) \), \( L(\hat{a}b) \neq E^* \). The diagram \( I(\hat{a}b) \) contains at least one circle formed by the edge that leaves and enters the node \( \hat{a} \) and is labeled with the letter \( \hat{a} \), where \( \hat{a} \) is the first letter of the word \( a \). Therefore the language \( L(\hat{a}b) \) is infinite. By Theorem 4, \( H^d_{L(\hat{a}b)}(n) = \Theta(n) \) and \( H^e_{L(\hat{a}b)}(n) = \Theta(n) \). Thus, \( L(\hat{a}b) \in F_3 \). □

**Proof of Theorem 5.** In each figure depicting a diagram \( I(a) \), \( a \in E^* \setminus \{ \lambda \} \), we label each node with a corresponding prefix of the word \( a \).

(a) The diagram \( I(00) \) is depicted in Fig. 8. This is an independent simple f-reduced diagram where \( c(I(00)) = 1 \). Therefore \( L(00) \in F_2 \). By Lemma 6, \( L(1) \in F_2 \).

(b) The diagram \( I(01) \) is depicted in Fig. 9. This is an independent simple f-reduced diagram where \( c(I(01)) = 2 \). Therefore \( L(01) \in F_3 \). By Lemma 6, \( L(10) \in F_3 \).

(c) The diagram \( I(00) \) is depicted in Fig. 10. This is not a simple diagram. It is clear that \( L(00) \neq E^* \). Therefore \( L(00) \notin F_4 \). By Lemma 6, \( L(11) \in F_4 \). Using Lemma 7 we obtain that \( L(000), L(001), L(100), L(111) \in F_4 \).

The diagram \( I(01) \) is depicted in Fig. 11. This is not a simple diagram. It is clear that \( L(01) \neq E^* \). Therefore \( L(01) \notin F_4 \). By Lemma 6, \( L(10) \in F_4 \).

The diagram \( I(011) \) is depicted in Fig. 12. This is not a simple diagram. It is clear that \( L(011) \neq E^* \). Therefore \( L(011) \notin F_4 \). By Lemma 6, \( L(100) \in F_4 \).

We proved that, for any word \( a \in E^* \) of the length three, \( L(a) \in F_4 \). Using Lemma 7 we obtain that, for any word \( a \in E^* \) of the length greater than or equal to four, \( L(a) \in F_4 \). □

**Declaration of competing interest**

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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