Existence And Uniqueness Of Stationary Solution Of Nonlinear Stochastic Differential Equation With Memory.

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1 Introduction.

In this paper a stochastic differential equation (SDE) with infinite memory is considered. The drift coefficient of the equation is a nonlinear functional of the past history of the solution. Sufficient conditions for existence and uniqueness of stationary solution are given. This work is motivated by recent papers [1] and [2] where stochastically forced nonlinear equations of hydrodynamics were considered and it was shown how the infinite-dimensional stochastic Markovian dynamics related to these equations can be reduced to finite-dimensional stochastic dynamics. The corresponding finite-dimensional systems are however essentially non-Markovian.

So, the important problem of existence and uniqueness of stationary solutions for stochastic hydrodynamical equations is tightly related to existence and uniqueness of stationary solutions of SDEs with infinite memory. Some results in the area were established in [3]. In the first part of this paper some necessary notions are introduced and the main result is stated. A proof of the main result is given in the second part. We combine the approach of [3] with an interesting method for establishing the desired uniqueness suggested in [1] and [2] for the problems considered therein.

The equation under consideration is

$$dX(t) = a(\pi_t X)dt + dW(t).$$

Here $W(t), t \in \mathbb{R}$ is standard $d-$dimensional Wiener process (i.e. a Gaussian $\mathbb{R}^d$-valued stochastic process with continuous trajectories defined on the whole real line $\mathbb{R}$ with independent and stationary increments, $W(0) = 0$, $\mathbb{E}W(t) = 0$, $\mathbb{E}(W(t))^2 = |t|$, $t \in \mathbb{R}$), $\pi_t$ is a map from the space $C$ of $\mathbb{R}^d$-valued continuous functions defined on $\mathbb{R}$ to the space $C_-$ of continuous functions defined on $\mathbb{R}_- = (-\infty, 0]$:

$$\pi_t X(s) = X(s + t), \quad s \in \mathbb{R}_-.$$

This map gives the past history of a continuous process up to time $t \in \mathbb{R}$. From now on suppose $a(\cdot) : C_- \to \mathbb{R}^d$ to be a continuous functional with respect to metric

$$\rho_-(f, g) = \sum_{n=1}^{\infty} 2^{-n}(\|f - g\|_n \wedge 1), \quad f, g \in C_-,$$

which defines LU-topology on the space $C_-$. Here $\|h\|_n = \max_{-n \leq t \leq 0} |h(t)|$, and $|\cdot|$ denotes the Euclidean norm.

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For a stochastic process $X$ and a set $A \subset \mathbb{R}$ the $\sigma$-algebra generated by r.v.’s $X(s), s \in A$ will be denoted by $\sigma_A(X)$ and the $\sigma$-algebra generated by r.v.’s $X(s) - X(t), s, t \in A$ will be denoted by $\sigma_A(dX)$.

Consider the space $\Omega = C \times C$ with the metric analogous to the metric $\rho_-$ defined above. A probability measure $P$ on the space $\Omega$ with Borel $\sigma$-algebra $B$ is said to define a solution to the equation (1) on $\mathbb{R}$ if the following three conditions are fulfilled with respect to the measure $P$:

1. The projection $W : C \times C \to C$, $\omega = (\omega_1, \omega_2) \mapsto \omega_2$, is a standard $d$-dimensional Wiener process.
2. For any $t \in \mathbb{R}$
   \[ \sigma_{(\infty,t]}(X) \lor \sigma_{(\infty,t]}(dW) \quad \text{is independent of} \quad \sigma_{[t,\infty]}(dW). \]
   Here and further $X : C \times C \to C$, $\omega = (\omega_1, \omega_2) \mapsto \omega_1$.
3. If $s < t$ then
   \[ X(t) - X(s) \overset{\text{a.s.}}{=} \int_s^t a(\pi_\theta X)d\theta + W(t) - W(s). \]

If in addition the distribution of the process
\[ (X, dW) \equiv (X(t), -\infty < t < \infty, W(v) - W(u), -\infty < u < v < \infty) \]
does not change under time shifts then the measure $P$ is said to define a stationary solution.

Let’s state the main result.

**Theorem 1.** Let the drift coefficient $a(\cdot)$ satisfy the following conditions:
1. There exist such constants $K > 0, \lambda > 0$ that the estimate
   \[ |a(x_-) - a(y_-)| \leq K \int_{-\infty}^0 e^{\lambda t} |x_-(t) - y_-(t)| dt \]
   is fulfilled whenever $x_-, y_- \in C_-$, $x_-(0) = y_-(0)$ and the integral in the right-hand side is finite.
2. There exist such constants $C_1 \geq 0$ and $C_2 > 0$ that
   \[ (a(x_-), x_-(0)) \leq C_1 - C_2 |x_-(0)|^2, \quad x_- \in C_. \]
3. There exist such a constant $C_3 > 0$ that
   \[ |a(x_-)| \leq C_3 |x_-|, \quad x_- \in C_. \]

Then there exist a probabilistic measure $P$ on the space $C \times C$ which defines a stationary solution of the equation (1). Such measure is unique in the class of measures for which almost every realization $X$ possesses the following property:
\[ |X(t)| \leq K' e^{\lambda'|t|}, \quad t \leq 0. \]

Here $K' \in \mathbb{R}$ and $\lambda' \in (0, \lambda)$ are some constants depending on the realization $X$. 

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2 Proof of the main result.

First, let us prove the existence of the stationary solution using the Krylov–Bogolyubov approach.

A probabilistic law in $C \times C$ is said to define a solution of Cauchy problem for the equation (denoted $M$) with initial data $x_- \in C_-$ if the following conditions are satisfied: $W$ is a standard Wiener process; for every $t \in \mathbb{R}$ the relation (2) is true; the equality (3) is fulfilled for $s = 0$ and every $t > 0$; $X(t) = x_-(t)$ for any $t < 0$. Existence theorem for solutions of Cauchy problem is proved in [3].

Let $P_0$ denote such a law for the initial data identically equal to zero and $P_s$ denote the time $s$-shift of this distribution i.e. a solution of the Cauchy problem subject to zero initial data defined on the set $(-\infty, \pi]$, $s \in \mathbb{R}$. Formally $P_s = P_0 \theta_s^{-1}$ where $\theta_s(f, g) = (\tilde{f}, \tilde{g})$, $\tilde{f}(t) = f(t - s)$, $\tilde{g}(t) = g(t - s) - g(-s)$.

Since the function $P_s(E)$ is measurable with respect to $s$ for all $E \in \mathcal{B}$ (see [3]), for $T > 0$ one can define a propability measure

$$Q_T(\cdot) = \frac{1}{T} \int_{-T}^{0} P_s(\cdot) ds$$

on the space $(\Omega, \mathcal{B})$. We will show that the family of measures $\{Q_T\}$ is tight.

Theorem 12.3 of the book [4] implies that in order to prove tightness of a family of measures $\{Q_T\}$ it is sufficient to verify that corresponding one-dimensional distributions constitute a tight family and there exist a nondecreasing continuous function and constants $\gamma \geq 0, \alpha > 1, z_0 > 0$ such that for all $z \in (0, z_0)$, $t_1, t_2 \in \mathbb{R}$ and $s \geq 0$ the following estimate holds:

$$Q_T\{|X(t_2) - X(t_1)| \geq z\} \leq z^{-\gamma}|F(t_2) - F(t_1)|^{\alpha}.$$

Introduce stopping time $\tau_r(t) = t \wedge \tau_r$ where $\tau_r = \inf\{t : |X(t)| \geq r\}$. Then for $T > 0$ the Ito formula implies that the equality

$$X^2(\tau_r(T)) = 2 \int_{0}^{\tau_r(T)} a(\pi \theta(X), X(\theta)) d\theta + 2 \int_{0}^{\tau_r(T)} (X(\theta), dW(\theta)) + \tau_r(T)$$

holds $P_0$-a.s. Taking expectations of both sides, passing to limit $r \to \infty$, using the regularity of the solution ($\tau_r \to \infty$ a.s. for $r \to \infty$) and finiteness of second-order moments of the solution established in [3] and the inequality (5), one can obtain that for all $S \in \mathbb{R}$

$$\mathbb{E}_{P_0}[X(T)]^2 \leq (2C_1 + 1)T - 2C_2 \int_{0}^{T} \mathbb{E}_{P_0}|X(\theta)|^2 d\theta \leq (2C_1 + 1)T - 2C_2 \int_{-T}^{S} \mathbb{E}_{P_\eta}|X(S)|^2 d\eta.$$

Dividing both parts of this inequality by $T$ and considering the last summand, one can obtain that for some positive constant $M$ and for all $S > 0$

$$\lim_{T \to \infty} \sup_{t \in [-S, S]} \mathbb{E}_{Q_\tau}|X(t)|^2 \leq M.$$
Now let us estimate increments of the process $X$.

$$Q_T \{|X(t_2) - X(t_1)| > z \} \leq Q_T \{|W(t_2) - W(t_1)| > z/2 \} + Q_T \left\{ \int_{t_1}^{t_2} a(\pi_{\theta} X) d\theta > z/2 \right\} \leq 16z^{-4}E_{Q_T}|W(t_2) - W(t_1)|^4 + 4z^{-2}E_{Q_T} \left( \int_{t_1}^{t_2} a(\pi_{\theta} X) d\theta \right)^2. \quad (9)$$

The next inequality is a consequence of the Fubini theorem, elementary inequality $|xy| \leq (x^2 + y^2)/2$, well-known expression for moments of Gaussian distribution and relations (6) and (9):

$$Q_T \{|X(t_2) - X(t_1)| > z \} \leq 48z^{-4}|t_2 - t_1|^2 + 4C_3^2z^{-2}M|t_2 - t_1|^2. \quad (10)$$

Tightness of the family of projections of measures $Q_T$ on the first component $\{Q_T X \}$ and hence the desired tightness of the family $\{Q_T\}$ is implied now by (8) and (10).

Lemma 1. For any $\delta > 0$ the following estimate is true $P-$a.s.

$$\lim_{t \to -\infty} |X_t| \cdot |t|^{1/2+\delta} = 0.$$ 

Proof. An estimate for measure $P$, analogous to the estimate (10), implies that for any $s \in \mathbb{R}$

$$P \left\{ \max_{t \in [s, s+1]} |X(t)| > Kz \right\} \leq P\{X(s) > z\} + P\{X(s+1) > z\} + C(z^{-2} + z^{-4})$$

for sufficiently large constants $K, C > 0$. Using this inequality and Chebyshev inequality and uniform in $t \in \mathbb{R}$ boundedness of the second-order moment of $X(t)$ one obtains that for all $\delta_0 \in (0, \delta)$ the series

$$\sum_{n=0}^{\infty} P \left\{ \max_{t \in [-n-1,-n]} |X(t)| > Kn^{1/2+\delta_0} \right\}$$

is convergent and the lemma follows from the Borel–Kantelly lemma.

In particular, Lemma 4 implies that the trajectories of the process $X$ satisfy the condition (4) $P$-a.s.

Now we turn to the proof of uniqueness. Consider an arbitrary measure $P$ which defines a stationary solution of the equation (1). Suppose also that the realizations of the process $X$ satisfy condition (7) $P$-a.s. Introduce a space $C_+$ of $\mathbb{R}^d$-valued continuous functions defined on $\mathbb{R}_+ = [0, \infty)$. For $x_\in C_-$ we denote $P_{x_-}$ the measure on $\Omega_+ = C_+ \times C_+$ which defines a solution of Cauchy problem with the initial data $x_-$. $P_{x_-}$ is a conditional distribution of the measure $P$ conditioned on $X_- = x_-$. 

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Lemma 2. Condition 1 of Theorem 1 implies that there exists a set \( A \subset C_\cdot \) such that \( P(\pi_0X \in A) = 1 \) and if \( x_-, y_- \in A \) \( x_- = y_-(0) \) then the measures \( P_{x_-} \) and \( P_{y_-} \) are equivalent.

Proof. Consider \( x_-, y_- \in C_\cdot \) such that each of these functions admits an exponential estimate like \( \| \| \). To prove that \( P_{y_-} \) is absolutely continuous with respect to \( P_{x_-} \), we use the Girsanov theorem and verify the Novikov condition (see, e.g., [5, Chapter 8]). The same reasoning will be valid for interchanged \( x_- \) and \( y_- \).

The Novikov condition can be written as follows:

\[
\mathbb{E}_{P_{x_-}} \exp \left\{ \frac{1}{2} \int_0^\infty \left| a(\pi_tX) - a(\pi_t(y\hat{X})) \right|^2 \, dt \right\} < \infty \tag{11}
\]

where

\[
y\hat{X}(s) = \begin{cases} y_-(s), & s < 0 \\ X(s), & s \geq 0. \end{cases}
\]

The condition \( \| \| \) implies that for some constants \( K' > 0 \) \( \lambda' \in (0, \lambda) \) the inequality \( \| x_-(s) - y_-\hat{X}(s) \| \leq K'e^{\lambda'|s|} \) is fulfilled, and the condition \( \| \| \) implies that

\[
\left| a(\pi_tX) - a(\pi_t(y\hat{X})) \right| \leq K \int_0^\infty |x_-(s) - y_-\hat{X}(s)| e^{\lambda(s-t)} \, ds \\
\leq KK' e^{-\lambda} \int_0^\infty e^{(\lambda'-\lambda)s} \, ds \leq Le^{-\lambda t}
\]

for some constant \( L > 0 \). So, in (11) the expectation of a bounded random variable is taken. The lemma is proved.

Lemma 3 implies the following result:

Lemma 3. For \( P\{X(0) \in \cdot\} \)-almost every \( l \in \mathbb{R}^d \) the measure \( P\{\pi_0^+X \in \cdot \mid X(0) = l\} \) has a component which is equivalent to the measure \( P\{\pi_0^+X \in \cdot \mid \pi_0X \equiv l\} \). Here \( \pi_0^+X(s) = X(s + t), s \in \mathbb{R}^+ \).

In the same way the following lemma can be proved.

Lemma 4. There exists a set \( A_\cdot \subset C_- \) such that \( P\{\pi_0X \in A_\cdot \} = 1 \) and if \( x_- \in A_\cdot \) then the projection of the measure \( P_{x_\cdot} \) on the space \( \Omega_{[0,T]} \) of \( \mathbb{R}^d \times \mathbb{R}^d \)-valued continuous functions defined on finite segment is equivalent to the distribution of the standard Wiener process in \( \mathbb{R}^{2d} \) emitted from \( (x_- (0), 0) \).

Let \( R_{T,x_-} (\cdot) \) be the restriction of this measure on sets of the form \( \{ z \in \Omega_{[0,T]} \mid z(T) \in B \} \), \( B \in \mathcal{B}(\mathbb{R}^{2d}) \). Since restrictions of equivalent measures on \( \sigma \)-subalgebra are equivalent, the measure \( R_{T,x_-} (\cdot) \) is equivalent to a non-degenerate Gaussian measure in \( \mathbb{R}^{2d} \), and hence to the Lebesgue measure. This fact and stationarity of the process \( X \) imply the following result.

Lemma 5. For any \( t \in \mathbb{R} \) the measure \( P\{X(t) \in \cdot\} \) has a component which is equivalent to the Lebesgue measure.

The proof of uniqueness of stationary solution of the equation (4) given here is based on the lemmas above. It is a modification of reasoning from (4).
Suppose there are two different ergodic measures $P^{(1)}$ and $P^{(2)}$ defining stationary solutions. There exists a bounded functional $F$ such that

$$F_1 = E_{P^{(1)}} F((X, W)) \neq E_{P^{(2)}} F((X, W)) = F_2,$$

and for some $S > 0$ $x(s) = y(s)$, $s \in [0, S]$ implies $F(x) = F(y)$.

Then there exist sets $B_1, B_2 \in \mathcal{B}$ such that $P^{(i)}(B_i) = 1$ and

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T F(\theta_s(X, W)) ds = F_i$$
on $B_i$, $i = 1, 2$.

Lemmas 3 and 5 imply that

$$P^{(2)}(B_1) = \int_{\mathbb{R}^d} P^{(2)}(B_1 | X(0) = l) P^{(2)}(X(0) \in dl) > 0.$$

So, $P^{(2)}(B_1 \cap B_2) > 0$ and $B_1 \cap B_2 \neq \emptyset$, which contradicts the assumption $F_1 \neq F_2$.

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