Prepotential
of
\(N = 2\) SU(2) Yang-Mills Gauge Theory
Coupled with a Massive Matter Multiplet

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Abstract

We discuss the \(N = 2\) SU(2) Yang-Mills theory coupled with a massive matter in the weak coupling. In particular, we obtain the instanton expansion of its prepotential. Instanton contributions in the mass-less limit are completely reproduced. We study also the double scaling limit of this massive theory and find that the prepotential with instanton corrections in the double scaling limit coincides with that of \(N = 2\) SU(2) Yang-Mills theory without matter.
1 Introduction

As is well-known, low energy properties of $N = 2$ supersymmetric Yang-Mills gauge theory are dominated by a holomorphic function (prepotential) and were recently studied by Seiberg and Witten \cite{1, 2}. In particular, they showed that the quantum moduli space was described by a kind of special geometry \cite{3} and identified the quantum moduli space of $N = 2$ SU(2) Yang-Mills theory without matter with the moduli space of a certain elliptic curve. Though they did not explicitly calculate the prepotential with instanton corrections, they qualitatively discussed the monopole and dyon masses, the metric on the quantum moduli space and a version of Olive-Montonen electric-magnetic duality and found that the strongly coupled vacuum turned out to be a weakly coupled theory of monopoles.

After the discovery of \cite{1, 2}, generalizations for another gauge groups \cite{4, 5, 6, 7, 8} and for $N = 2$ SU(2) Yang-Mills theory coupled with several mass-less matters \cite{9} have been discussed and the instanton expansion of the prepotential has been found in \cite{5}, but discussions on the massive versions of \cite{3, 4} do not have been established quantitatively so far. For this reason, we can not say that the structure of quantum moduli space have been understood in detail even for the case of $N = 2$ SU(2) Yang-Mills theory when massive matters are introduced. Thus we will study the quantum moduli spaces of $N = 2$ SU(2) Yang-Mills gauge theory coupled with several massive matters in this and subsequent papers \cite{10}. In particular, we will study the quantum moduli space of $N = 2$ SU(2) Yang-Mills theory coupled with a massive matter at weak coupling in this paper.

The paper organizes as follows. In the next section, we derive the Picard-Fuchs equation for massive $N_f = 1$ $N = 2$ SU(2) Yang-Mills theory \cite{2} and discuss the property of its solutions. It is noteworthy that the order of the differential equation is three in contrast with that of the mass-less theory whose order of the Picard-Fuchs equation is two. We also obtain the monodromy matrix near the weak coupling limit. In section 3, we derive the prepotential and its instanton expansion. The result coincides with \cite{2} if the matter is mass-less. Considerations on double scaling limit of the $N_f = 1$ massive theory are done in section 4. We will see that we can reproduce the instanton expansion of the $N_f = 0$ theory. Last section 5 is summary.

2 $N_f = 1$ Picard-Fuchs equation

Quantum moduli space of $N_f = 1$ $N = 2$ SU(2) Yang-Mills theory can be described by a kind of hyperelliptic curve

$$y^2 = (x^2 - u)^2 - \Lambda_1^3(x + m),$$  \hspace{1cm} (2.1)

\footnote{We mean “matters” as the hypermultiplets of quarks in the fundamental representation of the gauge group.}

\footnote{$N_f$ is the flavour index.}
and the meromorphic 1-form \( \lambda_1 \) which is given by

\[
\lambda_1 = \frac{\sqrt{2}xdx}{4\pi iy} \left[ \frac{x^2 - u}{2(x + m)} - 2x \right],
\]

where \( x, y \in \mathbb{C} \), \( u \) is the gauge invariant parameter, \( \Lambda_1 \) is a dynamical mass scale of this theory and \( m \) is the mass of the hypermultiplet \([8]\). Formulation by an elliptic curve can be found in \([2]\).

This curve has four branching points. In particular, in the weak coupling limit \((u \to \infty)\), they will be

\[
\begin{align*}
    x_1 &= -\sqrt{u} - \frac{i\Lambda_1^{3/2}}{2u^{1/4}} + \frac{i\Lambda_1^{3/2}m}{4u^{3/4}} + \cdots, \\
    x_2 &= -\sqrt{u} + \frac{i\Lambda_1^{3/2}}{2u^{1/4}} - \frac{i\Lambda_1^{3/2}m}{4u^{3/4}} + \cdots, \\
    x_3 &= \sqrt{u} - \frac{\Lambda_1^{3/2}}{2u^{1/4}} - \frac{\Lambda_1^{3/2}m}{4u^{3/4}} + \cdots, \\
    x_4 &= \sqrt{u} + \frac{\Lambda_1^{3/2}}{2u^{1/4}} + \frac{\Lambda_1^{3/2}m}{4u^{3/4}} + \cdots.
\end{align*}
\]

Since we can take the cuts to run from \( x_1 \) to \( x_2 \) and \( x_3 \) to \( x_4 \), we may identify this curve as a genus one Riemann surface, as shown in the following figure. We then identify \( \alpha \)-cycle as a loop going around the cut from \( x_4 \) to \( x_3 \) counter clockwise and \( \beta \)-cycle from \( x_3 \) to \( x_2 \). As is obvious from the figure, the intersection of these cycles is \( \alpha \cap \beta = 1 \).

Now we can define periods \( a(u), a_D(u) \) of \( \lambda_1 \) by

\[
\begin{align*}
    a(u) &= \oint \lambda_1, \quad (2.4) \\
    a_D(u) &= \oint_{\beta} \lambda_1. \quad (2.5)
\end{align*}
\]

\( a(u) \) is identified with the scalar component of the \( N = 1 \) chiral multiplet and \( a_D(u) \) is its dual. We are interested in their evaluation, but it is not so easy to accomplish it exactly. So we take a method of Picard-Fuchs equation. It is given by

\[
\frac{d^3 \Pi_1}{du^3} + \frac{3\Delta_1(m) + \Delta_1'(m)(4m^2 - 3u)}{\Delta_1(m)(4m^2 - 3u)} \frac{d^2 \Pi_1}{du^2} - \frac{8[4(2m^2 - 3u)(4m^2 - 3u) + 3(3\Lambda_1^2m - 4u^2)]d\Pi_1}{\Delta_1(m)(4m^2 - 3u)} = 0,
\]

where

\[
\Delta_1(m) = 27\Lambda_1^6 + 256\Lambda_1^3m^3 - 288\Lambda_1^3mu - 256m^2u^2 + 256u^3, \quad (2.7)
\]

\( ^3 \)Note that normalization factor of our \( \lambda_1 \) is different from that of \([8]\).
and $\Pi_1 = \int_\gamma \lambda_1$, $\gamma$ is a suitable 1-cycle and $\Delta'_1 = d\Delta_1/du$. Note that $\Delta_1(m)$ is the discriminant of the curve \((2.1)\). It is easy to find that \((2.6)\) has no symmetry over the $u$-plane and the mass plays a role to break the symmetry. \((2.6)\) has obviously regular singular points which are solutions to $\Delta_1(m) = 0$ and $4m^2 - 3u = 0$. These singular points correspond to mass-less states. Since we are going to treat only weak coupling limit in this paper, we do not discuss the behaviour of the moduli space near these singular points, but they should be discussed elsewhere. We have checked that \((2.6)\) can be also obtained as a result of the double scaling limit of the Picard-Fuchs equation of the massive $N_f = 2$ theory.

In the case of mass-less limit ($m \to 0$), this third order differential equation reduces to the second one,

$$
\frac{d^3 \Pi_1}{du^3} - \frac{\Delta_1(0) - \Delta'_1(0)u}{\Delta_1(0)u} \frac{d^2 \Pi_1}{du^2} + \frac{64u}{\Delta_1(0)} \frac{d \Pi_1}{du} = 0,
$$

i.e,

$$
(27\Lambda^6_1 + 256u^3) \frac{d^2 \Pi_1}{du^2} + 64u \Pi_1 = 0,
$$

where we set the integration constant as 0 because it can be shown directly. This equation has already been obtained in \([9]\).

The mechanism of this reduction is explained as follows. When the matter is massive, $\lambda_1$ will acquire an extra simple pole corresponding to $x = -m$ in contrast with the mass-less case. At first sight, even if $m = 0$, $\lambda_1$ seems to have a pole at $x = 0$, but the locus of the pole can be canceled out between denominator and numerator of $\lambda_1$. Accordingly, in general, the number of poles of massive meromorphic 1-form is equal to that of the mass-less meromorphic 1-form plus 1. Since the differentiation reduces the order by 1 \([11]\), the reduction will require one step more when $\lambda_1$ is massive. Therefore the order of the differential equation which periods of $\lambda_1$ should satisfy will increase one more than that of mass-less $\lambda_1$, i.e, the order will be three and this observation is consistent with \((2.6)\) and \((2.9)\).

In order to get the solutions of \((2.8)\) near $u = \infty$, we take $z = 1/u$. After this change of variable, we use Frobenius’s method. Then we find that its indicial equation has three roots, i.e, 0, $-1/2$, $-1/2$ (double roots). The solution $\rho_0(z)$ corresponding to the index 0 is in fact trivial, i.e, it is a constant,

$$
\rho_0(z) = \epsilon.
$$

However, this constant $\epsilon$ may depend on $\Lambda_1$ or $m$ and will be determined below. Geometrically, $\epsilon$ corresponds to the residue contribution of the pole of the meromorphic 1-form. Of course, $\epsilon$ must vanish in the mass-less limit because the mass-less theory does not have such a pole. Thus $\epsilon$ is a function of the mass. On the other hand, there are two independent solutions corresponding to the index $-1/2$. One of them is

$$
\rho_1(z) = z^{-1/2} \sum_{i=0}^{\infty} a_i z^i,
$$

\[3\]
where the first several expansion coefficients \( a_i \) are given in appendix A. We find that \( a_n \) can be represented by a polynomial of \( \Lambda^{3i}m^j \) with \( 2n = 3i + j \), where \( i \) and \( j \) are some non-negative integers. The other solution behaves logarithmic. It is

\[
\rho_2(z) = \rho_1(z) \ln z + z^{-1/2} \sum_{i=1}^{\infty} b_i z^i,
\]

where the first several coefficients \( b_i \) are given in appendix B. Note that \( b_n \) can also be represented by a polynomial of \( \Lambda^{3i}m^j \), where \( i \) and \( j \) are some non-negative integer with \( 2n = 3i + j \). But in this time, \( i \) and \( j \) must move over all combinations.

Since we would like to get \( a(u) \) and \( a_D(u) \), let us consider whether we can express them as linear combinations of \( \rho_0, \rho_1 \) and \( \rho_2 \). First, in order to see an asymptotic behaviour of \( a(u) \) near \( u = \infty \), we must calculate the lower order expansion of the integral (2.4). This is done in appendix C. Making a comparison \( \rho_1 \) and \( \rho_0 \) with (C4), we can see that

\[
a(u) = n\rho_0(z) + \frac{\rho_1(z)}{\sqrt{2}},
\]

where we identified as

\[
\rho_0(z) = -\frac{\sqrt{2}}{4} m.
\]

It is easy to find that \( a(u) \) can be expressed by a hypergeometric function in the mass-less limit [9]. \( a_D(u) \) can be written as a linear combination of \( \rho_0, \rho_1 \) and \( \rho_2 \) by comparison it with (C7),

\[
a_D(u) = A \rho_2(z) + B \rho_1(z) + n' \rho_0(z),
\]

where

\[
A = -\frac{i3\sqrt{2}}{4\pi},
\]

\[
B = \frac{i\sqrt{2}}{4\pi c}
\]

and \( c = -6 + 8 \ln 2 - i\pi - 6 \ln \Lambda_1 \).

From these explicit expressions for the periods, we can easily find that the monodromy matrix near \( u = \infty \) acts to the three objects as

\[
\begin{pmatrix}
    a_D \\
    a \\
    \epsilon
\end{pmatrix}
\rightarrow
\begin{pmatrix}
    -1 & 3 & 2n' - 3n \\
    0 & -1 & 2n \\
    0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
    a_D \\
    a \\
    \epsilon
\end{pmatrix}.
\]

Note that the monodromy matrix is now quantized by the winding number \( n \) and \( n' \). Since these winding numbers are arbitrary, we may say that there are “many” monodromy matrices near \( u = \infty \). This observation will be valued even for near the regular singular points.
Let us try to construct the prepotential $F_1$ which is a solution to the following differential equation
\[
  a_D(u) = \frac{dF_1}{da},
\]
but we use a new variable $\tilde{a} = a - n\epsilon$ for convenience. For that purpose, first, we must express $u$ as a series of $\tilde{a}$. We can easily get it from (2.13), i.e,
\[
  u = 2\tilde{a}^2 + \frac{\Lambda_3^3 m}{16\tilde{a}^2} - \frac{3\Lambda_4^6}{2048\tilde{a}^4} + \frac{5\Lambda_6^m}{4096\tilde{a}^6} - \frac{7\Lambda_9^0 m}{65536\tilde{a}^8} + \frac{1}{a_{10}^0} \left( \frac{153\Lambda_1^{12}}{67108864} + \frac{9\Lambda_9^0 m^3}{131072} \right) 
  - \frac{715\Lambda_1^{12} m^2}{67108864\tilde{a}^{12}} + \frac{1}{\tilde{a}^{14}} \left( \frac{1131\Lambda_1^{15} m}{2147483648} + \frac{1469\Lambda_1^{12} m^4}{268435456} \right) 
  - \frac{1}{\tilde{a}^{16}} \left( \frac{1155\Lambda_1^{18}}{137438953472} + \frac{2625\Lambda_1^{15} m^3}{2147483648} \right) 
  - \frac{1}{\tilde{a}^{18}} \left( \frac{667879\Lambda_1^{18} m^2}{109951162776} + \frac{148819\Lambda_1^{15} m^5}{17179869184} \right) + \cdots
\]
(3.2)

Inserting (3.2) into (2.15) and integrating it over $\tilde{a}$, we can obtain the prepotential
\[
  F_1 = \frac{i\tilde{a}^2}{\pi} \left[ \frac{3}{4} \ln \left( \frac{\tilde{a}}{\Lambda_1} \right)^2 + \frac{3}{4} \left( -1 + \frac{c}{3} + \ln 2 \right) - \frac{\sqrt{2\pi}}{4i\tilde{a}} n' m - \frac{m^2}{4\tilde{a}^2} \ln \tilde{a} + \sum_{i=2}^{\infty} \tilde{F}_1^i \tilde{a}^{-2i} \right],
\]
(3.3)

where the first several coefficients $\tilde{F}_1^i$ are recorded in appendix D. Note that $\tilde{F}_1^i$ is expressed again by a polynomial of $\Lambda_1^3 m^j$ with $2n = 3i + j$. We can find that our result (3.3) coincide with \cite{9} in the mass-less limit. It is interesting to note that (3.3) has a curious term proportional to $(\ln \tilde{a})/\tilde{a}^2$ in the brackets. However, we can find that
\[
  \tilde{F}_s^1 = \left( \tilde{a} - \frac{m}{\sqrt{2}} \right)^2 \ln \left( \tilde{a} - \frac{m}{\sqrt{2}} \right) + \left( \tilde{a} + \frac{m}{\sqrt{2}} \right)^2 \ln \left( \tilde{a} + \frac{m}{\sqrt{2}} \right) 
  = 2\tilde{a}^2 \ln \tilde{a} + m^2 \ln \tilde{a} + \frac{3}{2} m^2 - \frac{m^4}{24\tilde{a}^2} - \frac{m^4}{240\tilde{a}^4} - \cdots
\]
(3.4)

Thus we may rewrite (3.3) as
\[
  F_1 = \frac{i\tilde{a}^2}{\pi} \left[ \frac{3}{4} \ln \left( \frac{\tilde{a}}{\Lambda_1} \right)^2 + \frac{3}{4} \left( -1 + \frac{c}{3} + \ln 2 \right) - \frac{\sqrt{2\pi}}{4i\tilde{a}} n' m - \frac{1}{4\tilde{a}^2} \tilde{F}_s^1 
  + \frac{1}{2} \ln \tilde{a} + \frac{3m^2}{8\tilde{a}^2} + \sum_{i=2}^{\infty} \tilde{F}_1^i \tilde{a}^{-2i} \right],
\]
(3.5)

where $\tilde{F}_1^i$ is $\Lambda_1$ dependent part of $F_1^i$.

\footnote{After the completion of this work, Prof.S.K.Yang pointed out this series expansion.}
4 Double scaling limit

In this section, we examine the double scaling limit of the massive $N_f = 1$ theory discussed in previous sections.

To begin with, let us discuss the Picard-Fuchs equation. Since the $N_f = 1$ curve turns to the $N_f = 0$ curve in the double scaling limit ($m \to \infty, \Lambda_1 \to 0, m\Lambda_1^3 = \Lambda_0^4$ fixed, where $\Lambda_0$ is a dynamical parameter of the $N_f = 0$ theory), in other words, $N = 2 SU(2)$ Yang-Mills theory is considered as a low energy theory of the massive $N_f = 1$ theory [2], we may expect that (2.6) reduces to the Picard-Fuchs equation of the $N_f = 0$ theory. In fact, we find that (2.6) in the double scaling limit is given by

$$\frac{d^3 \Pi_1}{du^3} + \frac{3 \cdot 256m^2(\Lambda_0^4 - u^2) - 2 \cdot 256m^2u \cdot 4m^2}{256m^2(\Lambda_0^4 - u^2) \cdot 4m^2} \frac{d^2 \Pi_1}{du^2} - \frac{32 \cdot 8m^4}{256m^2(\Lambda_0^4 - u^2) \cdot 4m^2} \frac{d \Pi_1}{du} = 0,$$

(4.1)

i.e,

$$\frac{d^2 \Pi_1}{du^2} - \frac{1}{4(\Lambda_0^4 - u^2)} \Pi_1 = \text{constant}. \quad (4.2)$$

(4.2) shows the global $Z_2$ symmetry over the $u$-plane. At first sight, this can be seen as the Picard-Fuchs equation of $N = 2 SU(2)$ Yang-Mills theory without matter [3, 12]. However, we can not say that $\Pi_1$ also reduces to that of $N_f = 0$ Picard-Fuchs equation because there is no reason why the relation

$$\Pi_1(u, m, \Lambda_1) \overset{\text{double scaling limit}}{\longrightarrow} \Pi_0(u, \Lambda_0)$$

(4.3)

should hold, where $\Pi_0$ is a period integral of $N_f = 0$ theory. So we can not insist on that the integration constant in the right hand side of (4.2) must be 0.

More precisely speaking, the meromorphic 1-form $\lambda_1$ in the double scaling limit will behave as

$$\lambda_1 \rightarrow \frac{\sqrt{2x}dx}{4\pi i\tilde{y}} \left[ \frac{x^2 - u}{2m} \left( 1 - \frac{x}{m} + \frac{x^2}{m^2} - \cdots \right) - 2x \right]$$

$$= \frac{\sqrt{2x}(x^2 - u)dx}{8\pi im\tilde{y}} \left( 1 - \frac{x}{m} + \frac{x^2}{m^2} - \cdots \right) - \frac{\sqrt{2x^2}dx}{2\pi i\tilde{y}}, \quad (4.4)$$

where $\tilde{y}^2 = (x^2 - u)^2 - \Lambda_0^4$ is the curve for the $N_f = 0$ theory. The first term in the last expression is an “extra” 1-form which depends on the mass $m$ while the second is nothing other than the meromorphic 1-form of the $N_f = 0$ theory. Therefore, naively speaking, the solutions to (2.6) consist of the contributions originated from this extra 1-form and $\lambda_0$, in the double scaling limit. In fact, (2.10) and (2.12) diverges to infinity in the double scaling limit as is easy to find. Accordingly, when we discuss the low energy limit of $N_f = 1$ theory, we must carefully treat (4.3). Though $\rho_0, \rho_1$ and $\rho_2$ are indeed solutions to (2.6) for finite $\Lambda_1$ and $m$, nothing gives an assurance for that they constitute fundamental solutions even for infinitely large $\Lambda_1$ or $m$. Recall that we have obtained the solutions to
assuming that $\Lambda$ and $m$ are finite. This fact will be reflected on the solutions to the Picard-Fuchs equation. Consequently, there must be a gap between $\Pi_1$ in the double scaling limit and $\Pi_0$. This gap will appear as divergence due to the mass.

Next, let us directly examine the above observations, focusing on the periods $a(u)$ and $a_D(u)$. As for the residue part of the periods, i.e., $\rho_0$, it moves to infinity due to $m \rightarrow \infty$. However, as is discussed in [2], the quark with large mass can be integrated out, so we can eliminate the residue dependence of the periods. We can see that among the expansion coefficients $a_n$ ($n > 1$) of $a(u)$, only the coefficients of even degree survive in the double scaling limit and those of the odd degrees vanish. Accordingly, we may then say that $a(u)$ converges to a solution to (4.2). In other words, $a(u)$ is not affected by the contributions from the extra 1-form. This fact suggests that $a(u)$ has a nice property which is valid under the double scaling. It is easy to check that the period $a(u)$ in the double scaling limit can be again expressed as a hypergeometric function.

In order to see the behaviour of $a_D(u)$ in the double scaling limit, we must rewrite it as a series of $a(u)$ and then take the double scaling limit. We can easily see that the expansion coefficients $b_n$’s diverge to infinity in the double scaling limit. This means that the contributions for $a_D(u)$ from the extra 1-form are non-trivial. Since it is hard to see the difference of it with the period over $\beta$-cycle of the $N_f = 0$ theory in this situation, we should first arrange $a_D(u)$ with some “good” variable. For that purpose, we can take $a(u)$ as the good variable. From (3.2), $a_D(u)$ will be expanded as

$$a_D(u) = n'n'\epsilon + \sqrt{2}a[B - A \ln 2 - 2A \ln a] + \frac{A}{\sqrt{2}} \left[ \frac{m^2}{3a} + \frac{m}{72a^3}(-3\Lambda_1^3 + 2m^3) \right. \right.$$ 

$$+ \left. \frac{1}{46080a^5}(45\Lambda_1^6 + 256m^6) + \frac{m^2}{86016a^7}(-105\Lambda_1^6 + 128m^6) + \cdots \right], \quad (4.5)$$

where $\epsilon, A$ and $B$ are given in (2.14) and (2.16). Note that each coefficient of $a^{-2i+1}$ ($i > 0$) consists of a finite part and a “divergent” part and the latter is always proportional to $m^{2l}$, $l \in \mathbb{N}$. However, since the heavy quark with large mass must be integrated out as have been mentioned above, it would be enough to consider the finite parts. The divergence due to the large mass can be eliminated in that sense. Accordingly, if we extract only finite contributions we can arrive at a correct answer to get the periods $a_D(u)$ of the $N_f = 0$ theory. The reader may ask how to deal with the constants $A$ and $B$ in (4.5) under the double scaling. These constants should be replaced with that of the $N_f = 0$ theory. This is because the initial conditions for the Picard-Fuchs equation of the $N_f = 1$ theory are different from that of the $N_f = 0$ theory. In this way, $\Pi_1$ reduces to $\Pi_0$ and under this situation the integration constant in (4.2) will be 0. Then (4.2) is nothing other than the Picard-Fuchs equation of the $N_f = 0$ theory. Of course, in this time we must change $\Pi_1$ to $\Pi_0$ in (4.2).

Let us examine the double scaling limit of the prepotential. The procedure to do it consists of two steps. The first one is to take the double scaling limit of the first four terms in the brackets in (3.3). The second one is to consider the double scaling limit of the coefficients of $a^{-2i}$.
In order to accomplish the first step, we use a trick. Recall that we can add or minus infinity related to the mass because of the reason described before. Thus,

\[
\frac{3}{4} \ln \left( \frac{a}{\Lambda_1} \right)^2 + \frac{3}{4} C = \frac{3}{4} \ln \left( \frac{a}{\Lambda_1} \right)^2 + \frac{3}{4} C + \ln \left[ \left( \frac{a}{m} \right)^{1/2} \cdot e^{D/2 - 3C/4} \right] = \ln \left( \frac{a}{\Lambda_0} \right)^2 + \frac{D}{2} \tag{4.6}
\]

where \( C = -1 + \ln 2 + c/3 \), \( D = -6 + 6 \ln 2 \), and \( \tilde{a} \) is now replaced with \( a \). Note that this trick essentially corresponds to the replacement of the “initial” conditions described above. Since the remaining two terms can be integrated out, we have dropped them here.

For the expansion coefficients \( \tilde{\mathcal{F}}_i \), we can easily find that they will then be,

\[
\begin{align*}
\tilde{\mathcal{F}}_2 & \rightarrow -\frac{\Lambda_0^4}{64}, \\
\tilde{\mathcal{F}}_3 & \rightarrow 0, \\
\tilde{\mathcal{F}}_4 & \rightarrow -\frac{5\Lambda_0^8}{32768}, \\
\tilde{\mathcal{F}}_5 & \rightarrow 0, \\
\tilde{\mathcal{F}}_6 & \rightarrow -\frac{3\Lambda_0^{12}}{524288}, \\
\tilde{\mathcal{F}}_7 & \rightarrow 0, \\
\tilde{\mathcal{F}}_8 & \rightarrow -\frac{1469\Lambda_0^{16}}{4294967296}, \\
\tilde{\mathcal{F}}_9 & \rightarrow 0. \tag{4.7}
\end{align*}
\]

Then we can get the following “renormalized” prepotential \( \tilde{\mathcal{F}}_1 \) in the double scaling limit

\[
\tilde{\mathcal{F}}_1 = \frac{ia^2}{\pi} \left[ \ln \left( \frac{a}{\Lambda_0} \right)^2 + \frac{D}{2} - \frac{\Lambda_0^4}{2^6 a^4} - \frac{5\Lambda_0^8}{2^9 a^8} - \frac{3\Lambda_0^{12}}{2^{19} a^{12}} - \frac{1469\Lambda_0^{16}}{2^{32} a^{16}} - \cdots \right]. \tag{4.8}
\]

This agrees with the result of [5].

5 Summary

We have studied the moduli space of \( N = 2 \, SU(2) \) Yang-Mills theory coupled with a matter multiplet at weak coupling. In particular, we have determined its prepotential and monodromy matrix. For general values of \( \Lambda_1 \) and \( m \), we have established that the

\footnote{The number “\( D \)” is different from that of [5], but it seems to be their mistake although it is not so important.}
two periods of the meromorphic 1-form can be written as

\[ a(u) = -\frac{\sqrt{2}}{4} nm + \frac{1}{2}\sqrt{2}u \left[ 1 + \sum_{i=2}^{\infty} a_i(\Lambda_1^3, m) u^{-i} \right], \]

\[ a_D(u) = -\frac{\sqrt{2}}{4} n'm + \frac{3i}{2\pi} \tilde{a}(u) \ln \left( \frac{u}{\Lambda_1^3} \right) + \sqrt{u} \sum_{i=0}^{\infty} a_{D_i}(\Lambda_1^3, m) u^{-i}, \quad (5.1) \]

where \( a_i(\Lambda_1^3, m) \) and \( a_{D_i}(\Lambda_1^3, m) \) are homogeneous polynomials of order \( 2i \), instead of the formulae noted in \[2, 9\]. And we have proposed the exact expression for the prepotential as in \( \text{(3.3)} \). This prepotential has a curious term, as we have already seen, so it should be examined by some field theoretical method. The coefficients of instanton expansion in the mass-less limit completely coincide with \[9\]. On the other hand, we have succeeded in constructing the \( N_f = 0 \) theory as a low energy theory of the massive \( N_f = 1 \) theory and have found that we can recover the instanton expansion of the prepotential of the \( N_f = 0 \) theory.

Finally, we give some comments. Since the massive \( N_f = 1 \) theory can be considered as a low energy theory of the massive \( N_f = 2 \) theory \[2\], all our results will be expected to be reproduced from it. In addition to this, it will be interesting to reconstruct our results in the languages of integrable systems such as Whitham hierarchy and so on \[12, 13\]. The discussions in this paper should be compared with those approaches, but such considerations unfortunately are not proceeded at present.

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Appendix A Expansion coefficients \( a_i \)

The first several coefficients of \( \rho_1 \) are

\[ a_0 = 1, \]
\[ a_1 = 0, \]
\[ a_2 = -\frac{\Lambda_1^3 m}{16}, \]
\[ a_3 = \frac{3\Lambda_1^6}{1024}, \]
\[ a_4 = -\frac{15\Lambda_1^6 m^2}{1024}, \]
\[ a_5 = \frac{35\Lambda_1^6 m}{16384}, \]
\[
\begin{align*}
a_6 &= -\frac{105\Lambda_1^6}{4194304}(3\Lambda_1^3 + 256m^3), \\
a_7 &= \frac{3465\Lambda_1^{12}m^2}{2097152}, \\
a_8 &= -\frac{3003\Lambda_1^{12}m}{67108864}(3\Lambda_1^3 + 80m^3), \\
a_9 &= \frac{15015\Lambda_1^{15}}{4294967296}(\Lambda_1^3 + 384m^3). \\
\end{align*}
\]

**Appendix B Expansion coefficients \( b_i \)**

The first several coefficients of \( \rho_2 \) are

\[
\begin{align*}
b_1 &= \frac{1}{3}m^2, \\
b_2 &= \frac{m}{72}(3\Lambda_1^3 + 4m^3), \\
b_3 &= \frac{1}{23040}(-45\Lambda_1^6 + 480\Lambda_1^3m^3 + 512m^6), \\
b_4 &= \frac{m^2}{21504}(-21\Lambda_1^6 + 224\Lambda_1^3m^3 + 256m^6), \\
b_5 &= \frac{1}{276480}(-120\Lambda_1^9 + 1575\Lambda_1^6m^3 + 1920\Lambda_1^3m^6 + 2048m^9), \\
b_6 &= \frac{1}{415236096}(9801\Lambda_1^{12} - 937728\Lambda_1^9m^3 + 1419264\Lambda_1^6m^6 \\
&\quad + 2162688\Lambda_1^3m^9 + 2097152m^{12}), \\
b_7 &= \frac{m^2}{5725224960}(323505\Lambda_1^{12} + 13453440\Lambda_1^9m^3 + 15375360\Lambda_1^6m^6 \\
&\quad + 23855104\Lambda_1^3m^9 + 20971520m^{12}), \\
b_8 &= \frac{m}{3019898880}(45837\Lambda_1^{15} - 5636520\Lambda_1^{12}m^3 + 4392960\Lambda_1^9m^6 \\
&\quad + 7028736\Lambda_1^6m^9 + 10485760\Lambda_1^3m^{12} + 8388608m^{15}), \\
b_9 &= \frac{1}{13799729922048}(-9689337\Lambda_1^{18} + 3483611712\Lambda_1^{15}m^3 \\
&\quad + 16467010560\Lambda_1^{12}m^6 + 16728391680\Lambda_1^9m^9 + 29198647296\Lambda_1^6m^{12} \\
&\quad + 41070624768\Lambda_1^3m^{15} + 30064771072m^{18}).
\end{align*}
\]

**Appendix C Lower order expansion of the periods**

In this appendix, we show that the lower order expansion of the period integral (2.4) in detail as an example. However, we must treat (2.4) carefully because this \( \alpha \)-cycle is defined to be an usual homology basis. Recall that the meromorphic 1-form was constructed
under the assumption such that the asymptotic behaviour of \( a(u) \) at \( u = \infty \) was to be
\( a(u) \sim \sqrt{u/2} \) even if the theory was massive \([8]\).
Therefore even if we evaluate the period (2.4) by direct calculation, we cannot obtain the correct contribution from the pole.
When the cycle may cross the pole, then the integration must pick up the residue of the pole. However, since the \( \alpha \)-cycle in (2.4), as we have stated above, avoids the pole, we must study the case such that the cycle deforms from \( \alpha \) to \( \alpha' \) which enclose the pole and the two branching points \( x_3 \) and \( x_4 \) as shown in the following figure 2. As is easily seen from this figure, the direction of \( \alpha' \) is the same as that of \( \delta \) which enclose only the pole.

However, taking into account of an effect for topological deformation, we can find that the \( \alpha \)-cycle can be identified with the loop \( \alpha'' \) in the figure. Namely, if the \( \alpha \)-cycle should move on the another covering of this \( x \)-plane and back onto the original one, it will enclose \( x_1 \) and \( x_2 \), i.e., another cut. But in this time, the directions of \( \alpha \) and \( \alpha'' \) will be different. Therefore when \( \alpha'' \) across the pole, the direction of \( \alpha''' \) which is a deformation of \( \alpha'' \) and that of \( \delta \) will be different. This fact causes to change the sign of the residue.

The reader may ask that the sign of the integral over \( \alpha'' \) should be reflected. Of course it is right. But since we usually use the convention such that the overall sign of \( a(u) \) without the residue contribution, for example, to be +, e.g., \( a(u) \sim +\sqrt{u/2} \) in \( N = 2 \) \( SU(2) \) Yang-Mills theory, we should change the sign of the residue instead of that of the integral in order to preserve the convention.

From these discussions, the true expression for the massive period \( a(u) \) should be defined by

\[
a(u) := \oint_{\tilde{\alpha}} \lambda_1 = \oint_{\alpha} \lambda_1 + \oint_{\delta} \lambda_1 = \oint_{\alpha} \lambda_1 + 2\pi in \cdot \text{Res} (\lambda_1)|_{x=-m},
\]

where \( \tilde{\alpha} \) is a certain member of the family of \( \alpha \)-cycle which may include the pole and the cut inside the loop, \( \delta \) is a small loop around the pole and \( n = 0, \pm 1 \). If \( \tilde{\alpha} \) avoids the pole, then \( \tilde{\alpha} = \alpha \) and \( n = 0 \). If \( \tilde{\alpha} \) enclose the pole and the directions of \( \tilde{\alpha} \) and \( \delta \) coincide, \( n = +1 \). If the directions are different while \( \tilde{\alpha} \) enclose the pole, then \( n = -1 \). To clarify, we should further comment on the number \( n \). We have treated only the case such that \( \tilde{\alpha} \) winds once around the pole, but we may also allow the case such that it winds several times around the pole. In this time, \( n \) can be interpreted as winding number and will be \( n \in \mathbb{Z} \).

Let us evaluate (C1). First, note that

\[
\oint_{\alpha} \lambda_1 = 2 \int_{x_4}^{x_3} \lambda_1.
\]

In the right hand side, the factor 2 is required because the integral over \( \alpha \)-cycle contains an integral from \( x_4 \) to \( x_3 \) and from \( x_3 \) to \( x_4 \) on the other side of the cut. In order to
calculate (C2), we introduce a new variable $t$ such as $x = \sqrt{ut}$. Then (C2) will be

$$\int_\alpha \lambda_1 = \frac{2\sqrt{2}}{4\pi} \int_{x_4/\sqrt{u}}^{x_3/\sqrt{u}} \frac{utdt}{\sqrt{u^2(t^2 - 1)^2 - \Lambda_1^2(\sqrt{ut} + m)}} \left[ \frac{u(t^2 - 1)}{2(\sqrt{ut} + m)} - 2\sqrt{ut} \right]$$

$$= \frac{i}{4\pi} \sqrt{2u} \int_{x_4/\sqrt{u}}^{x_3/\sqrt{u}} \frac{3\sqrt{ut^3} + 4mt^2 + \sqrt{ut}}{\sqrt{ut} + m}$$

$$\times \left[ \frac{1}{t^2 - 1} + \frac{\Lambda_1^3 m}{2u^2(t^2 - 1)^3} + \frac{3\Lambda_1^6 t^2}{8u^3(t^2 - 1)^5} + \cdots \right] dt$$

$$= \frac{1}{2} \sqrt{2u} \left( 1 - \frac{\Lambda_1^3 m}{16u^2} + \cdots \right). \quad (C3)$$

Taking the contribution from the pole into account, we can arrive at

$$a(u) = -\frac{\sqrt{2}}{4} nm + \frac{1}{2} \sqrt{2u} \left( 1 - \frac{\Lambda_1^3 m}{16u^2} + \cdots \right). \quad (C4)$$

On the other hand, the integration over $\beta$-cycle is not well-defined, as well. This can be seen by evaluating its lower order expansion. (2.5) will be

$$\int_\beta \lambda_1 = \frac{i}{4\pi} \sqrt{2u} \left( 3 \ln u + 8 \ln 2 - i \pi - 6 - 6 \ln \Lambda_1 + \frac{i m \pi}{\sqrt{u}} + \cdots \right). \quad (C5)$$

At first sight, this integration seems to be $a_D(u)$ with the contribution from the pole. In fact, this observation is not wrong. However, (C3) does not contain the possibilities such that the $\beta$-cycle does not enclose the pole, for example. In other words, the topological deformation of $\beta$-cycle as in the case of $\alpha$-cycle is (partially) ignored. Since the pole merely contributes as only constant term, (C5) will be well-defined as the period over the $\beta$-cycle avoiding the pole, if the constant term is extracted. Therefore the true definition of $a_D(u)$ will be

$$a_D(u) := \oint_\beta \lambda_1$$

$$= \oint_\beta \lambda_1 + \oint_\delta \lambda_1$$

$$= \oint_\beta \lambda_1 + 2\pi i n' \cdot \text{Res} (\lambda_1) |_{x=-m}, \quad (C6)$$

where $\tilde{\beta}$ is a certain member of $\beta$-cycle which may enclose the pole and the cut, $\beta$ in (C6) means now a loop avoiding the pole and $n' = 0, \pm 1$. If the loop $\tilde{\beta}$ winds around the pole several times, then $n' \in \mathbb{Z}$. In this way we can arrive at

$$a_D(u) = -\frac{\sqrt{2}}{4} n'm + \frac{i}{4\pi} \sqrt{2u} \left( 3 \ln u + 8 \ln 2 - i \pi - 6 - 6 \ln \Lambda_1 + \cdots \right). \quad (C7)$$
Appendix D Expansion coefficients $\mathcal{F}_i^1$

First several coefficients of the prepotential (3.3) are listed below.

\[
\begin{align*}
\mathcal{F}_2^1 &= -\frac{1}{64}\Lambda_1^3 m + \frac{m^4}{96}, \\
\mathcal{F}_3^1 &= \frac{3\Lambda_1^6}{16384} + \frac{m^6}{960}, \\
\mathcal{F}_4^1 &= -\frac{5\Lambda_1^5 m^2}{32768} + \frac{m^8}{5376}, \\
\mathcal{F}_5^1 &= -\frac{7\Lambda_1^9 m}{786432} + \frac{m^{10}}{23040}, \\
\mathcal{F}_6^1 &= -\frac{153\Lambda_1^{12}}{1073741824} - \frac{3\Lambda_1^9 m^3}{524288} + \frac{m^{12}}{84480}, \\
\mathcal{F}_7^1 &= \frac{715\Lambda_1^{12} m^2}{1073741824} + \frac{m^{14}}{279552}, \\
\mathcal{F}_8^1 &= -\frac{1131\Lambda_1^{15} m}{4294967296} - \frac{1469\Lambda_1^{12} m^4}{4294967296} + \frac{m^{16}}{860160}, \\
\mathcal{F}_9^1 &= \frac{385\Lambda_1^{18}}{1099511627776} + \frac{525\Lambda_1^{15} m^2}{8589934592} + \frac{m^{18}}{2506752}.
\end{align*}
\] (D1)

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