Quantum Fourier states and gates: teleportation via rough entanglement

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Abstract
Quantum Fourier gates (QFG) constitute a complete family of quantum gates that result from an exact combination of the quantum Fourier transform (QFT) and the SWAP gate. Therefore, the Feynman Gate constitutes the simplest example of this family, while the Bell states are the simplest cases of entangled states derived from the family. Moreover, this new tool will allow us to demonstrate that teleportation is not something that happens exclusively thanks to maximally and non-maximally entangled states, but that it is also possible with an incomplete form of entanglement known as rough entanglement. Finally, other applications necessary for the quantum Internet are incorporated.

Keywords Entanglement · Entanglement swapping · Quantum secret sharing · Quantum Fourier transform · Teleportation

1 Introduction
During the last almost two and a half centuries, there have been innumerable interventions of Fourier's works (Jean-Baptiste Joseph Fourier, Auxerre, France, March 21, 1768, Paris, May 16, 1830) in fields of Science as diverse as Mathematics, Physics, Chemistry, Electronics, Bioengineering, and more recently Quantum Information Processing (QIP) (Nielsen and Chuang 2004). In the latter case, one application stands out above all others, known as phase estimation (Nielsen and Chuang 2004), which is of vital importance in order-finding (Nielsen and Chuang 2004) and factoring algorithms (Nielsen and Chuang 2004). The most famous quantum algorithm in the QIP arsenal is without a doubt Shor’s algorithm (Shor 1997), which consists of a hybrid configuration (classical-quantum) that is useful to factor an extremely large number into two other prime numbers in polynomial time.

Knowledge about the link between all QIP gates with QFT is relatively recent (Mastriani 2021a, 2021b, 2021c, 2021d), specifically, we refer to QFT as an underlying generator of the aforementioned gates. In this sense, the alternative expressions of the Feynman gate...
gates (Control-X, CNOT, or simply CX), Toffoli (Mastriani 2021a), and Hadamard from the QFT stand out, being the Hadamard gate the same QFT for the case of a single qubit (Mastriani 2021a, 2021b, 2021c, 2021d).

Moreover, in these works (Mastriani 2021a, 2021b, 2021c, 2021d) the spectral nature of the entanglement was established, where both the Bell states (Nielsen and Chuang 2004) and the Greenberger-Horne-Zeilinger (GHZ) type configurations (Nielsen and Chuang 2004) arise from appropriate combinations of the QFT. In this way, the dual nature of entanglement was revealed, where the spectral aspect was added to its well-known temporal facet. Taking into account both sides of the entanglement will allow the development of new and better Quantum Communications protocols (Cariolaro 2015), in particular, more efficient implementations of the quantum teleportation protocol (Bennett et al. 1993) with a strong projection on the future quantum Internet (Caleffi et al. 2020; Cacciapuoti et al. 2020a; Cacciapuoti et al. 2020b; Gyongyosi and Imre 2020, 2019a, 2019b), given that when we try to implement quantum key distribution protocols on the ground (Hiskett et al. 2006) through fiber-optic lines, we must use quantum repeaters at regular distances (Ruihong and Ying 2019). The problem with these repeaters is that the key to be distributed is exposed when passing through them, so an alternative to this serious security problem is quantum teleportation (Bennett et al. 1993). Hence the importance of testing better implementations of this protocol (Bennett et al. 1993) thanks to a better understanding of the inner springs of entanglement.

Finally, this work comes to fill interstitial and complementary spaces to those already mentioned regarding our knowledge about entanglement and quantum teleportation, given that as will be demonstrated in this work, the essential element for teleportation is not entanglement but states derived from the application of QFT, of which entanglement is only a particular case.

1.1 Quantum Fourier gates

As it was previously mentioned, quantum Fourier gates (QFG) constitute a family of quantum gates that result from an exact combination of the quantum Fourier transform (QFT) (Nielsen and Chuang 2004) and the SWAP gate (Nielsen and Chuang 2004). In their generic form, these gates will be represented as

\[
F_p^d = \text{SWAP}_{2^p\times2^p} \text{QFT}_{2^p\times2^p}^d \text{SWAP}_{2^p\times2^p},
\]

where \( F \) is the corresponding QFG, the subscript \( p \) indicates the number of qubits involved by the gate (although the subscript \( 2^p \times 2^p \) represents the dimension of the QFT and SWAP matrices), while the superscript \( d \) represents the degree of the gate, which is equivalent to the number of QFT blocks that the mentioned gate involves. A QFG can only have four possible degrees \((0, 1, 2, \text{ and } 3)\) regardless of the “number of QFT blocks” it includes. Specifically, “number of QFT blocks” \( \equiv d \pmod{4} \), i.e., the integers “number of QFT blocks” and \( d \) are said to be congruent modulo 4 if there is an integer \( k \) such that “number of QFT blocks” \(- d = k \times 4 \). Congruence modulo 4 is a congruence relation, where the parentheses mean that \( \pmod{4} \) applies to the entire equation, not just to the right-hand side (here \( d \)). As will be seen below, this is because the collection of four cascaded QFT blocks is equivalent to the identity matrix (Nielsen and Chuang 2004). Without losing generality, this property
of the QFT will be proved for the case of two qubits, where it is known that (Mastriani 2021a, 2021b, 2021c, 2021d):

\[
QFT_{2^2} \times QFT_{2^2} = \text{CNOT flipped} = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0
\end{bmatrix} = \text{SWAP}_{2^2} \times \text{CNOT} \times \text{SWAP}_{2^2}, \quad (2)
\]

with:

\[
QFT_{2^2} = \frac{1}{2} \begin{bmatrix}
1 & 1 & 1 & 1 \\
i & -1 & i & -i \\
1 & -1 & 1 & -1 \\
i & -i & 1 & i
\end{bmatrix}, \quad \text{with } i = \sqrt{-1}, \quad (3a)
\]

\[
\text{SWAP}_{2^2} = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0
\end{bmatrix}, \quad \text{and} \quad (3b)
\]

\[
\text{CNOT} = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0
\end{bmatrix}. \quad (3c)
\]

Then, being

\[
\text{SWAP}_{2^2} \times \text{SWAP}_{2^2} = \text{CNOT} \times \text{CNOT} = I_{2^2} = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}. \quad (4)
\]

since all quantum gates must be reversible (Nielsen and Chuang 2004) and taking into account Eq. (2), it turns out that,

\[
QFT_{2^2} \times QFT_{2^2} \times QFT_{2^2} \times QFT_{2^2} = \text{CNOT flipped} \times \text{CNOT flipped} = (\text{SWAP}_{2^2} \times \text{CNOT} \times \text{SWAP}_{2^2}) \times (\text{SWAP}_{2^2} \times \text{CNOT} \times \text{SWAP}_{2^2}) = I_{2^2}. \quad (5)
\]

Equation (5) indirectly proves the validity of the postulate “the number of QFT blocks” \( \equiv d \pmod{4} \).

From now on, it is necessary to define a generalized SWAP gate for \( p \) qubits, called \textit{swapping between equidistant qubits} (SBEQ) gate. Based on Fig. 1, \( p \) qubits will be numbered in ascending order, that is, from top to bottom of such figure, or what is similar, from 0 to \((p-1)\). Therefore, \( \forall p \) (even or odd) it turns out that the SBEQ gate performs swap operations between qubits equidistant from the center whose index is \((p-1)/2\), and in the correct order from the ends towards the mentioned center:
where the operator $\lfloor \cdot \rfloor$ returns the smallest integer greater than or equal to the specified numeric expression $\cdot$, meanwhile, Fig. 1 shows a generic representation of the SBEQ gate for $p$ qubits.

The domain of $k$ also represents the number of individual swaps that are applied within the SBEQ gate. For an odd $p$, the center qubit (i.e., $q[(p-1)/2]$) remains unchanged. A particular case of odd $p$ occurs for $p=1$, where the SBEQ gate implies only one SWAP gate, which leaves the only qubit intact since it is the identity matrix (Nielsen and Chuang 2004),

Equation (7) must be taken for what it is, i.e., an extreme (trivial) case, since, by definition, it makes no sense to speak of a SWAP gate if there are not at least two qubits that exchange their states (Nielsen and Chuang 2004).

Next, and without loss of generality, the four mentioned degrees of QFG will be exposed for some conspicuous cases regarding the number of qubits involved ($p$).

**Zero-degree QFG** This gate leaves the inputs as is regardless of the number of $q$ qubits involved, being zero the number of QFT blocks used.

For a generic number of qubits $p$, QFG will be,
Specifically, for \( p = 2 \), the result is found in Eq. (4) and that is repeated in Eq. (9) with the absence of QFT blocks,

\[
F_0^p = \text{SWAP}_{2^p \times 2^p} \text{QFT}^0_{2^p \times 2^p} \text{SWAP}_{2^p \times 2^p} = \text{SWAP}_{2^p \times 2^p} \text{SWAP}_{2^p \times 2^p} = I_{2^p \times 2^p}, \forall p. \tag{8}
\]

The importance of this gate will be appreciated in the next section.

**First-degree QFG** This gate implies a single QFT block between both SWAP gates,

\[
F_1^p = \text{SWAP}_{2^p \times 2^p} \text{QFT}^1_{2^p \times 2^p} \text{SWAP}_{2^p \times 2^p}, \forall p, \tag{9}
\]

where Eq. (10) constitutes a key piece of the teleportation protocol without entanglement that will be developed in a later section. Moreover, a relevant example of QFG for a single qubit is represented by the Hadamard matrix (Mastriani 2021a, 2021b, 2021c, 2021d),

\[
F_1^1 = \text{SWAP}_{2^1 \times 2^1} \text{QFT}^1_{2^1 \times 2^1} \text{SWAP}_{2^1 \times 2^1} = I_{2^1 \times 2^1} H I_{2^1 \times 2^1} = H, \tag{10}
\]

being \( \text{QFT}^1_{2^1 \times 2^1} = H \), i.e., the first-degree version of the QFG for a single qubit is the same Hadamard matrix (Mastriani 2021a, 2021b, 2021c, 2021d), since, as shown in Eq. (7), any application of the SWAP gate on any other gate for the case of a single qubit result in the same gate, since according to Eq. (7) it is the identity matrix.

**Second-degree QFG** This gate involves two QFT blocks between both SWAP gates, and as established in previous works (Mastriani 2021a, 2021b, 2021c, 2021d), it constitutes a fundamental piece in the generation of entanglement (Nielsen and Chuang 2004), which will be extensively exploited in a later section. For the case of a generic number of qubits \( p \), the superscript 2 means 2 QFT modules (matrices of \( 2^p \times 2^p \) elements each one), the gate is expressed as,

\[
F_2^p = \text{SWAP}_{2^p \times 2^p} \text{QFT}^2_{2^p \times 2^p} \text{SWAP}_{2^p \times 2^p}, \forall p. \tag{11}
\]

Next, examples of this gate are developed for the 4, 3, 2, and 1 qubit cases. The first case is for 4 qubits, resulting in the following gate,

\[
F_4^2 = \text{SWAP}_{2^2 \times 2^2} \text{QFT}^2_{2^2 \times 2^2} \text{SWAP}_{2^2 \times 2^2}, \tag{12}
\]

which can be seen in Fig. 2a. This gate will be essential in the implementation of the \( \text{GHZ}_4 \) state. The second case is for 3 qubits, where the gate of Fig. 2b is obtained,

\[
F_3^2 = \text{SWAP}_{2^2 \times 2^2} \text{QFT}^2_{2^2 \times 2^2} \text{SWAP}_{2^2 \times 2^2}. \tag{13}
\]

This gate will be essential in the implementation of the \( \text{GHZ}_3 \) state. The third example has to do with Fig. 2c, where the gate resulting ends up being the Feynman’s gate,

\[
F_2^2 = \text{SWAP}_{2^2 \times 2^2} \text{QFT}^2_{2^2 \times 2^2} \text{SWAP}_{2^2 \times 2^2} = \text{CNOT}. \tag{14}
\]

This case was extensively treated in previous works (Mastriani 2021a, 2021b, 2021c, 2021d), and as already mentioned, it is of wide application in both entanglement (Nielsen and Chuang 2004) and quantum teleportation (Bennett et al. 1993).
Finally, taking into account Eq. (7), and considering that multiplication of two Hadamard (Nielsen and Chuang 2004) matrices \( H \in \mathbb{C}^{2^1 \times 2^1} \) is the identity matrix,

\[
HH = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_{2^1 \times 2^1}, \tag{16}
\]

the corresponding QFG results,

\[
F_2^2 = SWAP_{2^1 \times 2^1} QFT_{2^1 \times 2^1} SWAP_{2^1 \times 2^1}. \tag{17}
\]

Remembering that \( QFT_{2^1 \times 2^1} = H \), (Mastriani 2021a, 2021b, 2021c, 2021d) and taking into account Eqs. (16) and (17) is redefined as,

\[
F_1^1 = SWAP_{2^1 \times 2^1} QFT_{2^1 \times 2^1} SWAP_{2^1 \times 2^1} = SWAP_{2^1 \times 2^1} QFT_{2^1 \times 2^1} QFT_{2^1 \times 2^1} SWAP_{2^1 \times 2^1} = I_{2^1 \times 2^1}.
\]

That is, Eq. (18) shows that \( F_1^1 \) is the identity matrix, which will be used in the section corresponding to quantum teleportation (Nielsen and Chuang 2004).

**Comparison between Toffoli (\( T_p \)) and \( F_p^2 \) gates** Next, a presentation by the opposition will take place between both gates to establish the main logical differences between them, that is, between \( F_p^2 \) and a known gate like Toffoli (Nielsen and Chuang 2004) for the case in which the inputs are computational basis states (Nielsen and Chuang 2004) (CBS) \( \{ |0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, |1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \} \), where \( |0\rangle \) and \( |1\rangle \) are the north and south poles of the Bloch sphere (Nielsen and Chuang 2004), respectively.

The aforementioned comparison will take place concerning Fig. 3 for which \( \land, \lor, \vee \) represent the logical gates AND (logical conjunction), OR (logical disjunction), and XOR (exclusive OR, that is, \( A\lor B = \bar{A} \land B \lor A \land \bar{B} \)), respectively.

Figure 3a shows the Toffoli gate for 4 qubits where the q[3] output is \( q[3] \lor (q[0] \land q[1] \land q[2]) \) while in the case of the other outputs they are equal to the
respective inputs. However, Fig. 3b shows the logical outputs of $F^2_4$, where the respective outputs are: $q[0]$, $q[0] \lor q[1]$, $q[2] \lor (q[0] \lor q[1])$, and $q[3] \lor (q[0] \lor q[1] \lor q[2])$. Figure 3c and d shows both cases for 3 qubits.

In a general case, we will have,

**Toffoli** ($T_{i+1}$):

\[
q[k] = q[k], \forall k \in [0,i-1], \text{ and}
\]

\[
q[i] = q[i] \lor \left( \bigwedge_{j=0}^{i-1} q[j] \right), \text{ for } k = i, \text{ and}
\]

\[
QFG(F^2_{i+1})
\]

\[
q[0] = q[0], \text{ and}
\]

\[
q[i] = q[i] \lor \left( \bigvee_{j=0}^{i-1} q[j] \right), \forall k \in [1,i].
\]

**Third-degree QFG** At first, this gate returns complementary results to those obtained with the first-degree gate seen above. This will become apparent in the next section. It implies the use of three QFT blocks in the middle of both SWAP gates,

\[
F^3_p = \text{SWAP}_{2^p \times 2^p} \ QFT^3_{2^p \times 2^p} \ SWAP_{2^p \times 2^p}, \forall p.
\]

For degrees greater than 3, the following equivalences arise:
and in this way, coincidences between gates that differ by 4 degrees begin to manifest independently of the number of qubits \( p \), that is,

\[
F^0_p = F^4_p = F^8_p = \ldots = F^{2+4k}_p, 
\]

(23a)

\[
F^1_p = F^5_p = F^9_p = \ldots = F^{2+4k}_p, 
\]

(23b)

\[
F^2_p = F^6_p = F^{10}_p = \ldots = F^{2+4k}_p, \quad \text{and} 
\]

(23c)

\[
F^3_p = F^7_p = F^{11}_p = \ldots = F^{3+4k}_p. 
\]

(23d)

In general, it turns out that

\[
F^d_p = F^{d+4k}_p, 
\]

(24)

where, in all cases, “the number of QFT blocks” \( -d = k \times 4/k \in \mathbb{Z} \), however, this is valid only for values of \( k \geq 0 \), i.e., \( k \in \mathbb{N}_0 \) (natural with zero).

1.2 Quantum Fourier states

In the same way that there are four families of quantum Fourier gates (QFG), there are four families of quantum Fourier states (QFS). This is because QFGs are the central engine in the generation of QFSs. In general terms, the QFS will depend on the degree of the QFG used and this in turn on the number of qubits involved, although for the particular case of dealing with two qubits, that dependence will extend to the type of CBS that will constitute the input qubits of each configuration, i.e., spin-up \( |0\rangle \) or spin-down \( |1\rangle \). Therefore, since these states directly depend on their corresponding QFG, there will only be four possible degrees for them.

From now on, and as a consequence of the nomenclature adopted for the QGS, we will generically represent the QFS as follows: \( |F\rangle^d_p \), where \( d \) is the degree of the QFG gate, i.e., the number of QFT blocks it contains, while \( p \) is the number of qubits involved.

Finally, the gates of Eqs. (11), and (18), and their respective equivalences will be fundamental in obtaining the preliminary conclusions that we will arrive at the end of this section.

Zero-degree QFS In the generic case of \( p \) qubits, for all inputs equal to \( |0\rangle \), and considering Eq. (8) and that \( \bigotimes \) is the Kronecker’s product (Nielsen and Chuang 2004), it results,

\[
|F\rangle^0_p = (H \otimes I_{2^{p-1}}} |0\rangle \otimes \otimes = I_{2^{p-1}}} (H \otimes I_{2^{p-1}}} |0\rangle \otimes \otimes = F^0_p (F^1_p \otimes F^0_{p-1}) |0\rangle \otimes \otimes = |+\rangle |0\rangle \otimes \otimes^{-1}, 
\]

(25)

where \( |+\rangle = H |0\rangle = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \), \( |-\rangle = H |1\rangle = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix} \). While in the particular case of two qubits, and considering Eq. (9), with \( |F\rangle^2_1 = |F\rangle^0_1 \), we will have four states
where $a$ is the phase bit and $b$ is the parity bit. In consequence, these four states in terms of phase and parity bits, will be,

$$|X^a+b⟩ = (H \otimes I_{2^1 \times 2^1}) |ab⟩ = I_{2^2 \times 2^2} (H \otimes I_{2^1 \times 2^1}) |ab⟩ = F_0^0 (F_1^1 \otimes F_2^2) |ab⟩ = |F_{ab}⟩^0,$$

where $X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ is the inverter gate (Nielsen and Chuang 2004). Table 1 shows the $|F_{ab}⟩^0$ states in terms of phase and parity bits, while Fig. 4 represents the $|F_{ab}⟩^0$ states thanks to three different representations, where the last one is implemented exclusively in terms of QFGs.

Table 1 $|F_{ab}⟩^0$ states in terms of phase ($a$) and parity ($b$) bits

| $a$ | $b$ | $|F_{ab}⟩^0$ |
|-----|-----|------------|
| 0   | 0   | $|+0⟩$     |
| 1   | 0   | $|−0⟩$     |
| 0   | 1   | $|+1⟩$     |
| 1   | 1   | $|−1⟩$     |

Fig. 4 $|F_{ab}⟩^0$ states via different representations, where the last one is exclusively in terms of QFGs

Table 2 $|F_{ab}⟩^0$ states density matrices in terms of phase and parity bits

| Phase bit | Parity bit |
|-----------|------------|
| 0         | 1          |

1
QFGs. Finally, Table 2 shows the density matrix of the four $\left| F_{ab}\right\rangle_2^0$ states in terms of phase and parity bits, where the density matrix is equal to $\left| F_{ab}\right\rangle_2^0 \langle F_{ab}\right|_2^0$.

First-degree QFS In the generic case of $p$ qubits, and for all its inputs equal to $|0\rangle$, it turns out

$$|F\rangle_p = F_p^1 \left( H \otimes I_{2^{p-1} \times 2^{p-1}} \right) |0\rangle^{\otimes p} = F_p^1 \left(F_1^1 \otimes F_0^{p-1}\right) |0\rangle^{\otimes p}. \quad (27)$$

In the particular case of two qubits, we will have

$$|F\rangle_2^1 = F_2^1 \left( H \otimes I_{2^1 \times 2^1} \right) |ab\rangle = F_2^1 \left(F_1^1 \otimes F_1^1\right) |ab\rangle = \left[ a \ a \ (-i)^a(-1)^b (1 + i)/2 \ (i)^a(-1)^b (1 - i)/2 \right]^T \sqrt{2} \quad (28)$$

where $\bar{a}$ is the inverse of $a$, i.e., if $a = 0$, then $\bar{a} = 1$, and vice versa, $\left(\cdot\right)^T$ means transpose of $(\cdot)$, and $i = \sqrt{-1}$. Figure 5 represents the four $\left| F_{ab}\right\rangle_2^1$ states thanks to two different representations, where the last one is implemented exclusively in terms of QFGs, while Table 3 shows the $\left| F_{ab}\right\rangle_2^1$ states in terms of phase and parity bits.

As we can see both in Eq. (28) and in Table 3, these states do not constitute a maximally-entangled pair (Nielsen and Chuang 2004) or even a non-maximally-entangled pair (Adhikari et al. 2010; Roy and Ghosh 2017; Campbell and Paternostro 2010; Koniorczyk and Bužek 2005). The density matrices for the four cases of $\left| F_{ab}\right\rangle_2^1$ (that is, according to the phase and parity bits) can be seen in Table 4, where the complexity increases concerning the case of $\left| F_{ab}\right\rangle_2^0$ because they are complex matrices with real and imaginary parts.

The density matrices arise from the external products $\left| F_{ab}\right\rangle_2^1 \langle F_{ab}\right|_2^1$, and they are a fundamental witness element that confirms that it is not about maximally or non-maximally entangled states. This is reflected by the elements that are occupied in those arrays, where even in the case of non-maximally-entangled states (Nielsen and Chuang 2004), the elements occupied are the same as in the case of maximally-entangled states (Adhikari et al. 2010; Roy and Ghosh 2017; Campbell and Paternostro 2010; Koniorczyk and Bužek 2005). This will be seen in detail in the subsection called Preliminary Conclusions, at the end of this section.
As we will see in the next section, even if it is not any form of known entanglement, the \( |F_{ab}\rangle_2 \) states will give rise to valid forms of teleportation (Bennett et al. 1993) where this characteristic (that is, not being some traditional form of entanglement) will not condition its performance at all, when these states are used in the context of quantum communications (Cariolaro 2015), particularly in the future quantum Internet (Caleffi et al. 2020; Cacciapuoti et al. 2020a; Cacciapuoti et al. 2020b; Gyongyosi and Imre 2020, 2019a, 2019b).

\textit{Second-degree QFS} In the generic case of \( p \) qubits, and for all its inputs equal to \( |0\rangle \), it turns out

\begin{table}[h]
\centering
\caption{\( |F_{ab}\rangle_2 \) states density matrices in terms of phase and parity bits}
\begin{tabular}{|ccc|}
\hline
\text{Phase bit} & \text{Parity bit} & \text{Density matrix} \\
\hline
\text{Real part} & \text{Imaginary part} \\
\hline
0 & 0 & \includegraphics[width=\textwidth]{density_matrix_00} \\
0 & 1 & \includegraphics[width=\textwidth]{density_matrix_01} \\
1 & 0 & \includegraphics[width=\textwidth]{density_matrix_10} \\
1 & 1 & \includegraphics[width=\textwidth]{density_matrix_11} \\
\hline
\end{tabular}
\end{table}
In the particular case of two qubits, and considering Eqs. (3c) and (15), we will have

\[ |F_2^2 \rangle = F_2^p (H \otimes I_{2^p-1}) |0 \rangle^{\otimes p} = F_2^p (F_1^1 \otimes F_0^{p-1}) |0 \rangle^{\otimes p}. \]  

(29)

In the particular case of two qubits, and considering Eqs. (3c) and (15), we will have

\[ |\beta_{ab} \rangle = \text{CNOT} (H \otimes I_{2^p-1}) |ab \rangle = F_2^2 (F_1^1 \otimes F_1^2) |ab \rangle = |F_{ab} \rangle_2^2 = (|0 \rangle + (-1)^{a+b} |1 \rangle) / \sqrt{2}. \]  

(30)

These are the famous Bell states (Nielsen and Chuang 2004), which are a particular case of the Fourier states, that is, the Bell states are the second-degree Fourier states. Figure 6 represents the four $|F_{ab} \rangle_2^2$ states thanks to three different representations, where the

| $a$ | $b$ | $|F_{ab} \rangle_2^2$ |
|-----|-----|---------------|
| 0   | 0   | $\begin{bmatrix} 1 & 0 & 0 & 1 \end{bmatrix}^T / \sqrt{2} = |\beta_{00}\rangle$ |
| 1   | 0   | $\begin{bmatrix} 1 & 0 & 0 & -1 \end{bmatrix}^T / \sqrt{2} = |\beta_{10}\rangle$ |
| 0   | 1   | $\begin{bmatrix} 0 & 1 & 1 & 0 \end{bmatrix}^T / \sqrt{2} = |\beta_{01}\rangle$ |
| 1   | 1   | $\begin{bmatrix} 0 & 1 & -1 & 0 \end{bmatrix}^T / \sqrt{2} = |\beta_{11}\rangle$ |

| Phase bit | Parity bit |
|-----------|------------|
| 0         | 0          |
| 0         | 1          |
| 1         | 0          |
| 1         | 1          |
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last one is implemented exclusively in terms of QFGs, while Table 5 shows the $|F_{ab}\rangle_2^2$ states in terms of phase and parity bits.

The density matrices for the four cases of $|F_{ab}\rangle_2^2$ (that is, according to the phase and parity bits) can be seen in Table 6, which arise from the external products $|F_{ab}\rangle_2^2\langle F_{ab}|_2^2$. These four matrices represent, like no other tool, the four maximally-entangled states (Nielsen and Chuang 2004). The positions of the elements occupied in these matrices, as well as the signs of their values, are extremely relevant to be able to identify the type of Bell state to which it refers. These positions are the border that separates the Fourier states of the first degree and the second degree, and the fundamental reason to affirm that the first-degree Fourier states are not even non-maximally entangled states.

As a natural extension of the Bell state $|\psi_0\rangle$, we can refer to the Greenberger-Horne-Zeilinger states of 3 and 4 qubits, i.e., $|GHZ_3\rangle$ and $|GHZ_4\rangle$, which are implemented in Figs. 7 and 8, respectively, where the $|GHZ_3\rangle$ is known as $|F_{000}\rangle_3^2$, or $|F\rangle_3^2$, and the three subscript zeros correspond to the three spin-up (or $|0\rangle$) inputs of $|GHZ_3\rangle$ configuration.

Figure 7a are all completely equivalents, where the last one is implemented exclusively in terms of QFGs, however, this does not mean at all that the nesting of two Feynman gates in the first case of this figure is equivalent to the $F_3^2$ gate. For this case, the same procedure has been followed as in the previous cases, that is, it is a process of approximation to the final equivalence through individual equivalences. Taking into account Eq. (9), $|GHZ_3\rangle$ in terms of QFGs results,

$$|GHZ_3\rangle = \left[ \begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right] \sqrt{2}$$

$$= (SWAP \otimes I_{2^3 \times 2^3}) (I_{2^3 \times 2^3} \otimes CNOT) (SWAP \otimes I_{2^3 \times 2^3}) (CNOT \otimes I_{2^3 \times 2^3}) (H \otimes I_{2^3 \times 2^3}) |000\rangle$$

$$= F_3^2 (F_1^1 \otimes F_2^0) |000\rangle = |F_{000}\rangle_3^2 = |F\rangle_3^2,$$

$$\quad (31)$$

### Fig. 7 $|GHZ_3\rangle \equiv |F_{000}\rangle_3^2 = |F\rangle_3^2$ state, where a represents equivalences between gates, while, b shows a 3D implementation of its density matrix
Finally, Fig. 7b represents the density matrix of $|\psi_3\rangle^2$, state, that is $|\psi_3\rangle^2\langle\psi_3|$, which is very similar to that of the Bell state $|\psi_{00}\rangle = |\psi_0\rangle^2$ of Table 6 though stretched from all four ends.

Figure 8a shows four equivalences of $|\text{GHZ}_4\rangle$, where the last one is implemented exclusively in terms of QFGs, while Fig. 8b represents the density matrix of the $|\psi_4\rangle^2$ state, that is $|\psi_4\rangle^2\langle\psi_4|$, with the same similarity considerations as those expressed in the previous case regarding the density matrix of the Bell state $|\psi_{00}\rangle$. For this particular case, that is, $|\text{GHZ}_4\rangle \equiv |F_{0000}\rangle^2 = |F_4^0\rangle^2$, and considering that $F_3^0 = I_{2\times2^3}$, we will have,

\[
|\text{GHZ}_4\rangle = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}^T / \sqrt{2}
= (\text{SWAP } \otimes I_{2\times2^2})(I_{2\times2^2} \otimes \text{SWAP } \otimes I_{2\times2^1})(I_{2\times2^2} \otimes \text{CNOT })(I_{2\times2^2} \otimes \text{SWAP } \otimes I_{2\times2^1}) (\text{SWAP } \otimes I_{2\times2^2}) \\
(\text{SWAP } \otimes I_{2\times2^2})(I_{2\times2^2} \otimes \text{CNOT } \otimes I_{2\times2^1})(\text{SWAP } \otimes I_{2\times2^2})(CNOT \otimes I_{2\times2^2})(H \otimes I_{2\times2^3})(0000) \\
= F_4^0(F_1^0 \otimes F_3^0)(0000) = |F_{0000}\rangle^2 = |F_4^0\rangle^2.
\]

(32)

Finally, at this point, similar considerations are made to those clarified for the case of $|\text{GHZ}_3\rangle$ in that the nesting of CNOT gates at the beginning of Fig. 8a is not equivalent to the $F_4^0$ gate, although all implementations of this figure are, i.e., they all have output to $|\text{GHZ}_4\rangle \equiv |F_{0000}\rangle^2 = |F_4^0\rangle^2$.

Third-degree QFS In the generic case of $p$ qubits, for all its inputs equal to $|0\rangle$, and taking into account Eq. (8), it turns out
In the particular case of two qubits, we will have

\[
|F_{ab}^3\rangle = F_p^3 (H \otimes I_{2^{p-1} \times 2^{p-1}})|0\rangle^{\otimes p} = F_p^3 \left(F_1^1 \otimes F_{p-1}^0\right)|0\rangle^{\otimes p}.
\]  

(33)

In the particular case of two qubits, we will have

\[
\begin{align*}
|F_{ab}^3\rangle & = F_2^3 (H \otimes I_{2^2 \times 2^2})|ab\rangle = F_2^3 \left(F_1^1 \otimes F_{2}^0\right)|ab\rangle \\
& = \left[ a^a b^b (1-i)/2 (1+i)/2 \right]/\sqrt{2}.
\end{align*}
\]  

(34)

Figure 9 represents the four $|F_{ab}^3\rangle$ states thanks to two different representations, where the last one is implemented exclusively in terms of QFGs, while Table 7 shows the $|F_{ab}^3\rangle$ states in terms of phase ($a$) and parity ($b$) bits. As in the case of Tables 3 and 4 for $|F_{ab}^1\rangle$, Table 7 shows that these states also do not constitute a maximally-entangled pair (Nielsen and Chuang 2004) or even a non-maximally-entangled pair (Adhikari et al. 2010; Roy and Ghosh 2017; Campbell and Paternostro 2010; Koniorczyk and Bužek 2005). The density matrices for the four cases of $|F_{ab}^3\rangle$ (that is, according to the phase and parity bits) can be seen in Table 8, where their complexity is similar to that of the state $|F_{ab}^1\rangle$, because they are complex matrices with real and imaginary parts. This density matrix is $|F_{ab}^3\rangle^\dagger F_{ab}^3\rangle$ and its elements occupy practically the same positions as those of the density matrix of the state $|F_{ab}^1\rangle$. Moreover, as can be seen, the third-degree states $|F_{ab}^3\rangle$ are complex conjugates of those of the first-degree states $|F_{ab}^1\rangle$.

**Non-maximally entangled states** There are several versions to represent non-maximally entangled states (Adhikari et al. 2010; Roy and Ghosh 2017; Campbell and Paternostro 2010; Koniorczyk and Bužek 2005), so we will choose one (Adhikari et al. 2010), by which we will replace the Hadamard matrix with another gate, for example,

\[
\sqrt{X} = \sqrt{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}} = \begin{bmatrix} 0.8536 + 0.3536i & 0.1464 - 0.3536i \\ 0.1464 - 0.3536i & 0.8536 + 0.3536i \end{bmatrix} = \begin{bmatrix} u & v \\ v & u \end{bmatrix},
\]  

(35)

where $u=0.8536+0.3536i$, and $v=0.1464-0.3536i$. Based on Fig. 10, the resulting state of this two-qubit configuration for the non-maximally entangled case will be,
Table 8 \(|F_{ab}\rangle^3\) states density matrices in terms of phase and parity bits

| Phase bit | Parity bit | Density matrix |
|-----------|------------|----------------|
| 0         | 0          | ![Image](real_part00.png) ![Image](imaginary_part00.png) |
| 0         | 1          | ![Image](real_part01.png) ![Image](imaginary_part01.png) |
| 1         | 0          | ![Image](real_part10.png) ![Image](imaginary_part10.png) |
| 1         | 1          | ![Image](real_part11.png) ![Image](imaginary_part11.png) |

Fig. 10 \(|\gamma_{ab}\rangle\) states via three representations, whereas the last one is exclusively in terms of QFGs
The two first implementations of Fig. 10 show Eq. (36) depending on the phase and parity bits, while Table 9 shows the four resulting states. The corresponding four density matrices of the states $|y_{ab}\rangle$ are $|y_{ab}\rangle\langle y_{ab}|$, and they can be seen in Fig. 11. The positions that the elements occupy in the density matrices are similar to the maximally-entangled case, only the values change according to the matrix chosen to replace the Hadamard matrix ($H$).

Pauli’s matrices (Nielsen and Chuang 2004) can be expressed in terms of the so-called Hadamard rotation gates (Gruska 1999) or the general unitary operator $U(\theta, \varphi, \lambda)$ as follows:

$$U(\theta, \varphi, \lambda) = \begin{bmatrix} \cos(\theta/2) - e^{i\lambda}\sin(\varphi \theta/2) \\ e^{i\varphi\tan(\theta/2)}e^{i(\lambda + \varphi) }\cos(\theta/2) \\ \end{bmatrix}$$

Table 9 $|y_{ab}\rangle$ states in terms of phase ($a$) and parity ($b$) bits

| $a$ | $b$ | $|y_{ab}\rangle$ |
|-----|-----|------------------|
| 0   | 0   | $\begin{bmatrix} u & 0 & 0 & v \end{bmatrix}^T = |y_{00}\rangle$ |
| 1   | 0   | $\begin{bmatrix} v & 0 & 0 & u \end{bmatrix}^T = |y_{10}\rangle$ |
| 0   | 1   | $\begin{bmatrix} 0 & u & v & 0 \end{bmatrix}^T = |y_{01}\rangle$ |
| 1   | 1   | $\begin{bmatrix} 0 & v & u & 0 \end{bmatrix}^T = |y_{11}\rangle$ |

$$|y_{ab}\rangle = CNOT\left(\sqrt{X} \otimes I_{2^1\times2^1}\right) |ab\rangle.$$ (36)
where $I$ is a $2 \times 2$ identity matrix, $i = \sqrt{-1}$, $H_I = H = U(\pi / 2, 0, 0)$ of Eq. (3), while

$$
H_{III} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = U(\pi / 2, 0, 0), H_{III} = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} = U(5\pi / 2, \pi, 0), \text{ and } H_{IV} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = U(\pi / 2, \pi, \pi).
$$

Therefore, we rewrite Eq. (36) taking into account Eq. (37b),

$$
|f_{ab}\rangle = CNOT\left(\sqrt{X} \otimes I_{2^2} \otimes 2^1 \otimes 2^1\right) |ab\rangle = CNOT\left(\sqrt{H_I H_IV} \otimes F^2 \right) |ab\rangle,
$$

where $H_I = F_1^1$, and $H_{IV}$ is $H_I$ after a left-to-right flipping procedure (Koniorczyk and Bužek 2005).

Preliminary conclusions As a result of this section, it is possible to arrive at a series of preliminary conclusions about the most outstanding distinctive characteristics of Fourier states:

(1) Bell states are particular cases of Fourier states, it is just that, one more case,

(2) Previous works have shown that entanglement is also a spectral phenomenon (Mastriani 2021a, 2021b, 2021c, 2021d), as well as a temporal one (Nielsen and Chuang 2004),

(3) The density matrices in Tables 4 and 8 (for $|F_{ab}\rangle_2$ and $|F_{ab}\rangle_3$, respectively) show that they are different entanglements, that is to say, they are not typical entanglements, not even non-maximally entanglements, but rather rough entanglements. This designation is due to it being a rustic, unfinished, incomplete entanglement that is missing a Fourier layer,

(4) Bell states are a special case of Fourier states. Also, the $|GHZ_n\rangle$ states are particular cases of the Fourier states. Both the Bell and $|GHZ_n\rangle$ states can be represented by Fourier gates,

(5) The Fourier states preserve the same Modular Arithmetic as for the case of the QFGs, that is, there are coincidences between states that differ by 4 degrees, and this manifests independently of the number of qubits $p$, and it is for this reason that we say that there are only four Fourier states. Then,

$$
|F_p^0\rangle = |F_p^4\rangle = |F_p^8\rangle = \ldots = |F_p^{0+4k}\rangle, \quad (40a)
$$

$$
|F_p^1\rangle = |F_p^5\rangle = |F_p^9\rangle = \ldots = |F_p^{1+4k}\rangle, \quad (40b)
$$

$$
|F_p^2\rangle = |F_p^6\rangle = |F_p^{10}\rangle = \ldots = |F_p^{2+4k}\rangle, \quad (40c)
$$
In general, it turns out that
\[ |F\rangle_p^d = |F\rangle_p^{d+4k}, \] (41)
where, in all cases, “number of QFT blocks” \(d = k \times 4 \quad k \in \mathbb{Z}\), however, this is valid only for values of \(k \geq 0\), i.e., \(k \in \mathbb{N}_0\) (natural with zero).

(6) Fourier states of degree 1 are the complex conjugates of those of degree 3 and vice versa.

(7) All configurations of Fig. 7a are completely similar in their results, i.e., \(|GHZ_3\rangle \equiv |F_{000}\rangle_3^2 = |F\rangle_3^2\). Something similar happens between all the configurations of Fig. 8a but for four qubits, which result in \(|GHZ_4\rangle \equiv |F_{000}\rangle_4^2 = |F\rangle_4^2\), however, when we exclusively introduce CBS to the inputs of the configurations of Fig. 12 \(|a\rangle, |b\rangle\) and \(|c\rangle\) for 3 qubits, and \(|a\rangle, |b\rangle, |c\rangle,\) and \(|d\rangle\) for 4 qubits), the equivalencies break down for some combinations of those inputs. An example of these differences for the 3-qubit case is \(|a\rangle = |0\rangle, |b\rangle = |1\rangle,\) and \(|c\rangle = |0\rangle\) where the outputs are \{|011\} for \(F_3^2\) and \{|010\}\) for the nesting of CNOTs in Fig. 12a, while for the 4-qubit case, one of the differences
is \((|a⟩ = |0⟩, |b⟩ = |1⟩, |c⟩ = |0⟩, \text{and} |d⟩ = |0⟩\)) where the outputs are \(|0111⟩\) for \(F_{4}^{2}\) and \(|0100⟩\) for the nested CNOTs of Fig. 12b.

(8) The only Fourier states that do not represent some kind of entanglement are those of zero-degree, i.e., without QFT blocks, which shows the correspondence between entanglement and Fourier (Mastriani 2021a, 2021b, 2021c, 2021d).

(9) In the cases of maximally, and non-maximally entangled states, as well as rough entanglement, Fig. 13 represents the positions of non-zero elements in their density matrices as gray tiles. Figure 13a corresponds to \(|β_{00}⟩ = |F_{00}⟩_{2}^{2}, |β_{10}⟩ = |F_{10}⟩_{2}^{2}, |γ_{00}⟩, \text{and} |γ_{10}⟩\).

Fig. 13b represents \(|β_{01}⟩ = |F_{01}⟩_{2}^{2}, |β_{11}⟩ = |F_{11}⟩_{2}^{2}, \text{and} |γ_{10}⟩\). Fig. 13c contains the gray tiles of \(|F_{00}⟩_{2}^{1}, |F_{01}⟩_{2}^{1}, |F_{00}⟩_{2}^{3} \text{and} |F_{01}⟩_{2}^{3}\), while Fig. 13d shows \(|F_{10}⟩_{2}^{1}, |F_{11}⟩_{2}^{1}, |F_{10}⟩_{2}^{3}, \text{and} |F_{11}⟩_{2}^{3}\). The same locations are occupied by nonzero elements for the equivalent cases of maximally, and non-maximally entangled states, i.e., for the same combination of phase and parity bits. However, for the case of rough entanglement (for both \(|F_{ab}⟩_{2}^{1}\) and \(|F_{ab}⟩_{2}^{3}\) the positions occupied by non-zero elements in their density matrices are completely different (regardless of whether some elements are complex), from those of Fig. 13a, and b. The marked difference between the tiles occupied by the cases of maximally entangled states and non-maximally entangled states on one hand, and rough entanglement on the other hand indicates that we are in the presence of another entanglement type. A simple visual inspection of Fig. 13a–d tell us that rough entanglement is a very different case from previously known entanglements. However, as we will see in the next section, this rustic form of entanglement will allow the successful teleportation of various types of qubits.

A few examples of the respective density matrices highlight everything previously expressed, that is, based on Eq. (30) we have a pair of cases of maximally-entangled states, such that for \(|a⟩ = |0⟩ \text{and} |b⟩ = |0⟩\), the resulting Bell state is:

\[
|β_{00}⟩ = (|00⟩ + |11⟩)/\sqrt{2}
= \begin{bmatrix}0 & 0 & 1 \\1 & 0 & 0\end{bmatrix}/\sqrt{2},
\]

(42)

and its corresponding density matrix is,

\[
|β_{00}⟩⟨β_{00}| = (|00⟩ + |11⟩)/\sqrt{2}⟨00| + ⟨11|)/\sqrt{2}
= (|00⟩⟨00| + |11⟩⟨00| + |00⟩⟨11| + |11⟩⟨11|)/2
= \begin{bmatrix}1 & 0 & 0 & 1 \\0 & 0 & 0 & 0 \end{bmatrix}/2.
\]

(43)

This matrix represents one of the most conspicuous examples that respond to the characteristics of Fig. 13a, while for \(|a⟩ = |0⟩ \text{and} |b⟩ = |1⟩\), the corresponding Bell state turns out to be,
\[ |\beta_{01}\rangle = \frac{(|01\rangle + |10\rangle)}{\sqrt{2}} \]
\[ = \left[ \begin{array}{cccc} 0 & 0 & 1 & 1 \end{array} \right]^T / \sqrt{2}, \]  

(44)

and its density matrix is,
\[ |\beta_{01}\rangle \langle \beta_{01}| = \frac{(|01\rangle + |10\rangle)}{\sqrt{2}} \frac{\langle 01| + \langle 10|}{\sqrt{2}} \]
\[ = \frac{1}{2} \left[ \begin{array}{cccc} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{array} \right]. \]  

(45)

Matrix of Eq. (45) exactly represents one of the cases that confirm the layout of Fig. 13b. Something similar happens with two examples of non-maximally entangled states. Based on Eqs. (35), and (36) and for \( |a\rangle = |0\rangle \) and, \( |b\rangle = |0\rangle \), the resulting state is,
\[ |\gamma_{00}\rangle = u|00\rangle + v|11\rangle = (0.8536 + 0.3536i)|00\rangle + (0.1464 - 0.3536i)|11\rangle \]
\[ = [ u \ 0 \ 0 \ v ]^T = \left[ \begin{array}{cccc} 0.8536 & + & 0.3536i & 0 & 0 & 0 \end{array} \right]^{T}, \]  

such that,
\[ (0.8536 + 0.3536i)\text{conj}(0.8536 + 0.3536i) + (0.1464 - 0.3536i)\text{conj}(0.1464 - 0.3536i) = (0.8536 + 0.3536i)(0.8536 - 0.3536i) + (0.1464 - 0.3536i)(0.1464 + 0.3536i) = 1, \]  

(47)

where \( (0.8536 + 0.3536i) \neq (0.1464 - 0.3536i) \), which is why it is not a case of maximally entangled states and \( \text{conj}(\cdot) \) means complex conjugate of \( \cdot \). So, the resulting density matrix is,
\[ |\gamma_{00}\rangle \langle \gamma_{00}| = \left[ \begin{array}{cccc} (0.8536 + 0.3536i)|00\rangle + (0.1464 - 0.3536i)|11\rangle \\ (0.8536 - 0.3536i)|00\rangle + (0.1464 + 0.3536i)|11\rangle \end{array} \right] \]
\[ = (0.8536 + 0.3536i)(0.8536 - 0.3536i)|00\rangle \langle 00| \]
\[ + (0.1464 - 0.3536i)(0.1464 + 0.3536i)|11\rangle \langle 11| \]
\[ + (0.8536 + 0.3536i)(0.1464 + 0.3536i)|00\rangle \langle 11| \]
\[ + (0.1464 - 0.3536i)(0.8536 - 0.3536i)|11\rangle \langle 00| \]
\[ = \left[ \begin{array}{cccc} 0.8536 & 0 & 0 & 0.3536i \\ 0 & 0 & 0 & 0.3536i \\ 0 & 0 & 0 & 0 \\ -0.3536i & 0 & 0 & 0.1464 \end{array} \right]. \]  

(48)

As we can see in Eq. (48), the non-zero elements of this matrix occupy the same four corners as in the case \( |\beta_{00}\rangle \) of Eq. (43), although with some values expressed in imaginary numbers. Whereas for \( |a\rangle = |0\rangle \) and, \( |b\rangle = |1\rangle \), the resulting state is,
\[ |\gamma_{01}\rangle = (0.8536 + 0.3536i)|01\rangle + (0.1464 - 0.3536i)|10\rangle \]
\[ = \begin{bmatrix} 0 & 0.8536 + 0.3536i & 0.1464 - 0.3536i & 0 \end{bmatrix}^T, \] (49)

and its density matrix is,
\[ |\gamma_{01}\rangle\langle\gamma_{01}| = \begin{bmatrix} (0.8536 + 0.3536i)|01\rangle + (0.1464 - 0.3536i)|10\rangle \\ (0.8536 - 0.3536i)(0.8536 - 0.3536i)|01\rangle \langle 01| + (0.1464 + 0.3536i)(0.1464 + 0.3536i)|10\rangle \langle 10| + (0.8536 + 0.3536i)(0.1464 + 0.3536i)|01\rangle \langle 10| + (0.1464 - 0.3536i)(0.1464 - 0.3536i)|10\rangle \langle 10| \end{bmatrix} \]
\[ = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0.8536 & 0.3536i & 0 \\ 0 & 0 & 0 & 0.1464 - 0.3536i \\ 0 & 0 & 0 & 0 \end{bmatrix}. \] (50)

In this matrix, it can be seen that only the four central tiles are different from zero, as is the case of the Bell state \(|\beta_{01}\rangle\), although with some imaginary values.

Finally, both the first-degree Fourier states \(|F_{00}\rangle^1_2\) of Eq. (28), like their equivalents in Eq. (34) of third-degree, that is, \(|F_{00}\rangle^3_2\), can be represented with a few cases. For example, for \(|F_{00}\rangle^1_2\) where \(|a\rangle = |0\rangle\) and \(|b\rangle = |0\rangle\), the corresponding Fourier state is,
\[ |F_{00}\rangle^1_2 = \begin{bmatrix} 1 & 0 & (1 + i)/2 & (1 - i)/2 \end{bmatrix}^T / \sqrt{2} \]
\[ = [|00\rangle + |10\rangle(1 + i)/2 + |11\rangle(1 - i)/2] / \sqrt{2} \]
\[ = \begin{bmatrix} 0.5 & 0.25 - 0.25i & 0.25 + 0.25i \\ 0 & 0 & 0 \\ 0.25 + 0.25i & 0.25 & 0 \\ 0.25 - 0.25i & 0 - 0.25i & 0.25 \end{bmatrix}, \] (51)

and its density matrix results,
\[ |F_{00}\rangle^1_2\langle F_{00}|^1_2 = \begin{bmatrix} 0.5 & 0.25 - 0.25i & 0.25 + 0.25i \\ 0 & 0 & 0 \\ 0.25 + 0.25i & 0.25 & 0 \\ 0.25 - 0.25i & 0 - 0.25i & 0.25 \end{bmatrix}, \] (52)

which has a layout corresponding to the case of Fig. 13c. Something similar happens for \(|a\rangle = |0\rangle\) and \(|b\rangle = |1\rangle\), where the resulting state is,
\[ |F_{01}\rangle^1_2 = \begin{bmatrix} 1 & 0 & -(1 + i)/2 & -(1 - i)/2 \end{bmatrix}^T / \sqrt{2} \]
\[ = [|00\rangle - |10\rangle(1 + i)/2 - |11\rangle(1 - i)/2] / \sqrt{2} \]
\[ = \begin{bmatrix} 0.5 & 0 & 0.25 - 0.25i & 0.25 + 0.25i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0.25 + 0.25i & 0 \\ 0 & 0 & 0.25 - 0.25i & 0.25 \end{bmatrix}, \] (53)

and its density matrix occupies the same positions as in the previous example.
Instead, something very different happens for $|a\rangle = |1\rangle$ and $|b\rangle = |0\rangle$, where the resulting state is,

$$
|F_{10}\rangle_2^1 = \left[ 0 1 \frac{(1-i)/2}{\sqrt{2}} \frac{(1+i)/2}{\sqrt{2}} \right]^T
= |01\rangle + |10\rangle \frac{(1-i)/2}{\sqrt{2}} + |11\rangle \frac{(1+i)/2}{\sqrt{2}}.
$$

(55)

with the following density matrix,

$$
|F_{10}\rangle_2^1 |F_{10}\rangle_2^1 = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0.5 & 0.25 + 0.25i & 0.25 \\
0 & 0.25 - 0.25i & 0.25 & -0.25i \\
0 & 0.25 + 0.25i & 0.25i & 0.25
\end{bmatrix},
$$

(56)

which has the same non-zero elements indicated in Fig. 13d.

Finally, quantum configuration simulation platforms such as Quirk (Simulator 2022) present the density matrices flipped in both directions simultaneously, that is, left-to-right, and up-to-down. Aspects of this type must be taken into account when implementing any of the circuits exposed in this work. On the other hand, this same platform presents QFT modules that already have SWAP (for two qubits) and SEBQ (for more than 2 qubits) gates incorporated, both at the input and the output of the QFT modules, therefore it is not necessary to introduce them in the simulation since doing so would lead to incorrect outcomes.

1.3 Quantum teleportation via rough entanglement

Quantum teleportation (Bennett et al. 1993) is the first protocol created in the field of quantum communications (Cariolaro 2015), which also has a true projection on the future quantum
Internet (Caleffi et al. 2020; Cacciapuoti et al. 2020a, 2020b; Gyongyosi and Imre 2020, 2019a, 2019b). In the context of quantum cryptography (Kumar et al. 2021), fiber optic cabling for terrestrial implementations of quantum key distribution (QKD) protocols (Kumar et al. 2021) requires quantum repeaters every certain number of kilometers (Mehic et al. 2020), which in turn requires a large amount of quantum memory. The problem is that the key is exposed in its passage through them. There are currently two well-defined lines of research, the first has to do with the development of quantum repeaters that do not require quantum memory, at least not that much, and the second is to replace the same quantum repeaters with some type of implementation based on quantum teleportation (Bennett et al. 1993).

Taking Fig. 14 as an initial reference, this section develops both the theoretical deductions and the implementations in a simulator and an IBM Q Experience (IBM Quantum Experience 2022) 5-qubit physical machine called Lima, of the quantum teleportation protocol (Bennett et al. 1993) having as a source of pairs to the three cases studied in the previous section, that is, maximally-entangled, non-maximally entangled, and rough entangled states.

Although the theoretical deductions of the three cases will be carried out using generic qubits, both the simulations and the implementations on the 5-qubit physical machine will take as an example the teleportation of computational basis states (CBS), i.e., \{0\}, \{1\}, given that being orthogonal they notably facilitate the comparison of the outcomes between the different cases of entanglement. In addition, with this type of state, it is easier to assess the internal traceability of the states (timeline) through the protocol and thus better compare the outcomes. These states with \(|+\rangle = H|0\rangle\) and \(|-\rangle = H|1\rangle\), \(|R\rangle = SH|0\rangle\), and \(|L\rangle = SH|1\rangle\) are essential in quantum communications (Cariolaro 2015), in general, and QKD (Kumar et al. 2021), in particular, such that if \(Z\) is the phase gate (Nielsen and Chuang 2004), then \(S = \sqrt{\frac{1}{2}} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}\).

Based on Fig. 14, the generic state to be teleported is:

\[|\psi\rangle = \alpha|0\rangle + \beta|1\rangle, \tag{57}\]

such that,

\[\alpha^2 + \beta^2 = 1.\]

Then for \(t_0\), the 3-qubits combined state is:

\[|\psi_{t_0}\rangle = |\psi_{00}\rangle = (\alpha|0\rangle + \beta|1\rangle)|00\rangle = \alpha|000\rangle + \beta|100\rangle. \tag{58}\]

The states of Eqs. (57), and (58) are common to the following three deductions.

**Maximally-entangled states** In this case, the pair-source module of Fig. 14 distributes a Bell state of type \(|\beta_{00}\rangle\), like that of Eq. (42). Therefore, at time \(t_1\) of this figure, we will have,

\[|\psi_{t_1}\rangle = |\psi\rangle|\beta_{00}\rangle = (\alpha|0\rangle + \beta|1\rangle)|00\rangle + \frac{\alpha}{\sqrt{2}} |11\rangle \tag{59}\]

In \(t_2\), a CNOT gate is applied between qubits q[0], and q[1].

\[|\psi_{t_2}\rangle = \frac{\alpha}{\sqrt{2}} |00\rangle + \frac{\beta}{\sqrt{2}} |11\rangle + \frac{\alpha}{\sqrt{2}} |01\rangle + \frac{\beta}{\sqrt{2}} |10\rangle \tag{60}\]

In \(t_3\), a Hadamard (H) gate is applied in qubit q[0],
Quantum Fourier states and gates: teleportation via rough

\[ |\psi_{t_0}\rangle = \frac{1}{2} |000\rangle + \frac{1}{2} |100\rangle + \frac{1}{2} |010\rangle - \frac{\beta}{2} |110\rangle + \frac{\alpha}{2} |011\rangle + \frac{\beta}{2} |111\rangle + \frac{\beta}{2} |001\rangle - \frac{\beta}{2} |101\rangle \]

\[ = \frac{|00\rangle}{2} (\alpha|0\rangle + \beta|1\rangle) + \frac{|01\rangle}{2} (\alpha|1\rangle + \beta|0\rangle) + \frac{|10\rangle}{2} (\alpha|0\rangle - \beta|1\rangle) + \frac{|11\rangle}{2} (\alpha|1\rangle - \beta|0\rangle) \]

\[ = \left\{ \frac{|00\rangle}{2} X^0 Z^0 |\psi\rangle \right\} + \left\{ \frac{|01\rangle}{2} X^1 Z^0 |\psi\rangle \right\} + \left\{ \frac{|10\rangle}{2} X^0 Z^1 |\psi\rangle \right\} + \left\{ \frac{|11\rangle}{2} X^1 Z^1 |\psi\rangle \right\} . \tag{61} \]

All the terms of the last row of Eq. (61) have the same probability, that is, 25%, since the four bases \{\{00\}, \{01\}, \{10\}, \{11\}\} are equiprobable at the output of the Bell-State-Measurement (BSM) module (Nielsen and Chuang 2004), which is between the qubits q[0] and q[1] and is composed of the gates CNOT, H and the two quantum measurement blocks. This happens at \( t_3 \), where the exponents of the \( X \) and \( Z \) matrices are the classical bits of disambiguation needed to reconstruct the teleported state on Bob’s (receiver) side. As a direct consequence of this, Bob must apply a gate \( X \) at \( t_5 \), and a gate \( Z \) at \( t_6 \), if the respective disambiguation bits have a value equal to 1. Thus, the rebuilt outcome is obtained in \( t_7 \) after the quantum measurement in q[2].

Now, if \( |\psi\rangle = \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = |0\rangle \), and replacing \( \alpha \) and \( \beta \) in Eq. (61) at time \( t_5 \), we will have,

\[ |\psi_{t_5}\rangle = \frac{1}{2} |000\rangle + \frac{1}{2} |100\rangle + \frac{1}{2} |011\rangle + \frac{1}{2} |111\rangle \]

\[ = \frac{|00\rangle}{2} |0\rangle + \frac{|01\rangle}{2} |1\rangle + \frac{|10\rangle}{2} |0\rangle + \frac{|11\rangle}{2} |1\rangle \]

\[ = \left\{ \frac{|00\rangle}{2} X^0 Z^0 |0\rangle \right\} + \left\{ \frac{|01\rangle}{2} X^1 Z^0 |0\rangle \right\} + \left\{ \frac{|10\rangle}{2} X^0 Z^1 |0\rangle \right\} + \left\{ \frac{|11\rangle}{2} X^1 Z^1 |0\rangle \right\} . \tag{62} \]

From Eq. (62), for \( |0\rangle \), the sum of the probabilities is \( 25\% + 25\% + 25\% + 25\% = 100\% \), while for \( |1\rangle \), the sum of the probabilities is \( 0\% \).

Instead, if \( |\psi\rangle = \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = |1\rangle \), replacing \( \alpha \) and \( \beta \) in Eq. (61) at time \( t_5 \), and taking into account that \( Z|0\rangle = |0\rangle \), but \( Z|1\rangle = -|1\rangle \), it turns out that,

\[ |\psi_{t_5}\rangle = \frac{1}{2} |010\rangle - \frac{1}{2} |110\rangle + \frac{1}{2} |001\rangle - \frac{\beta}{2} |101\rangle \]

\[ = \frac{|00\rangle}{2} |1\rangle + \frac{|01\rangle}{2} |0\rangle + \frac{|10\rangle}{2} (-|1\rangle) + \frac{|11\rangle}{2} (-|0\rangle) \]

\[ = \left\{ \frac{|00\rangle}{2} X^0 Z^0 |1\rangle \right\} + \left\{ \frac{|01\rangle}{2} X^1 Z^0 |1\rangle \right\} + \left\{ \frac{|10\rangle}{2} X^0 Z^1 |1\rangle \right\} + \left\{ \frac{|11\rangle}{2} X^1 Z^1 |1\rangle \right\} . \tag{63} \]

From Eq. (63), for \( |0\rangle \), the sum of the probabilities is \( 0\% \), while for \( |1\rangle \), the sum of the probabilities is \( 25\% + 25\% + 25\% + 25\% = 100\% \).

**Non-maximally-entangled states** Given the state \(|\gamma_{00}\rangle\) of Eq. (46) but in a general way, i.e., with generic \( u \) and \( v \), and considering that for instant \( t_0 \) this case is the same as the previous one, at \( t_1 \), it results,

\[ |\psi_{t_1}\rangle = |\psi\rangle|\gamma_{00}\rangle = (\alpha|0\rangle + \beta|1\rangle)(u|00\rangle + v|11\rangle) = \alpha u|000\rangle + \beta u|100\rangle + \alpha v|011\rangle + \beta v|111\rangle , \tag{64} \]

while at time \( t_2 \) a CNOT gate is applied between the qubits q[0] and q[1]:
\[ |\psi_{t_5} \rangle = \alpha u|000 \rangle + \beta u|100 \rangle + \alpha v|011 \rangle + \beta v|101 \rangle. \] (65)

At instant \( t_5 \), a Hadamard gate (\( H \)) is applied in qubit \( q[0] \),

\[
\begin{align*}
|\psi_{t_5} \rangle &= \frac{\alpha u|000 \rangle + \beta u|100 \rangle + \alpha v|011 \rangle + \beta v|101 \rangle}{\sqrt{2}} \\
&= \left\{ \frac{|00 \rangle}{\sqrt{2}} (\alpha u|0 \rangle + \beta v|1 \rangle) + \frac{|01 \rangle}{\sqrt{2}} (\alpha v|0 \rangle + \beta u|1 \rangle) + \frac{|10 \rangle}{\sqrt{2}} (\alpha u|0 \rangle - \beta v|1 \rangle) + \frac{|11 \rangle}{\sqrt{2}} (\alpha v|0 \rangle - \beta u|1 \rangle) \right\} \\
&= \left\{ \frac{|00 \rangle}{\sqrt{2}} X^0 Z^0 |\psi\rangle \right\} + \left\{ \frac{|01 \rangle}{\sqrt{2}} X^1 Z^0 |\psi\rangle \right\} + \left\{ \frac{|10 \rangle}{\sqrt{2}} X^0 Z^1 |\psi\rangle \right\} + \left\{ \frac{|11 \rangle}{\sqrt{2}} X^1 Z^1 |\psi\rangle \right\},
\end{align*}
\]

(66)

where \( |\psi\rangle = \alpha u|0 \rangle + \beta v|1 \rangle \). The last line of Eq. (66) shows that the four terms are equiprobable for the state \( |\psi\rangle \), with a 25% probability for each of the four bases. However, since \( u \) and \( v \) are generally different, as seen in Eqs. (35), and (46), this causes an imbalance in the probability distribution concerning the four bases, due to the crossing between the coefficients \( \alpha \), \( \beta \), \( u \), and \( v \). In practice, this type of entanglement does not facilitate the teleportation of any type of state \( |\psi\rangle \), this being another reason why we resort to the CBS teleportation as an example, in this way, it is possible to compare the outcomes for the three cases of entanglement. As far as times \( t_4 \), \( t_5 \), \( t_6 \), and \( t_7 \) are concerned, a similar description of the case of maximally-entangled states takes place.

As for the previous case, if \( |\psi\rangle = \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = |0 \rangle \), we can now replace \( \alpha \) and \( \beta \) in Eq. (66) at time \( t_3 \), however, we must take into account Eqs. (35), (36), and (46), in which case we consider that \( \sqrt[4]{X} = HTH \), where \( T = \sqrt[4]{Z} = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\pi/2} \end{bmatrix} \). This replacement will make it possible to implement this particular case of entanglement on the selected IBM Q (IBM Quantum Experience 2022) machine since the aforementioned platform does not have the gate \( \sqrt[4]{X} \) in its arsenal. Therefore, we will have,

\[
\begin{align*}
|\psi_{t_3} \rangle &= \frac{u|000 \rangle + u|100 \rangle + v|011 \rangle + v|111 \rangle}{\sqrt{2}} \\
&= \left\{ \frac{|00 \rangle}{\sqrt{2}} u|0 \rangle + \frac{|01 \rangle}{\sqrt{2}} v|1 \rangle \right\} + \left\{ \frac{|10 \rangle}{\sqrt{2}} u|0 \rangle + \frac{|11 \rangle}{\sqrt{2}} v|1 \rangle \right\} \\
&= \left\{ \frac{|00 \rangle}{\sqrt{2}} X^0 Z^0 u|0 \rangle \right\} + \left\{ \frac{|01 \rangle}{\sqrt{2}} X^1 Z^0 v|0 \rangle \right\} + \left\{ \frac{|10 \rangle}{\sqrt{2}} X^0 Z^1 u|0 \rangle \right\} + \left\{ \frac{|11 \rangle}{\sqrt{2}} X^1 Z^1 v|0 \rangle \right\},
\end{align*}
\]

(67)

The very particular distribution of probabilities of the last line of Eq. (67) arises from:\n\( u \, \text{conj}(u)/2 = (0.8536 + 0.3536i) \, \text{conj}(0.8536 + 0.3536i)/2 = 0.4268 \to 42.68\% \), and \\
\( v \, \text{conj}(v)/2 = (0.1464 - 0.3536i) \, \text{conj}(0.1464 - 0.3536i)/2 = 0.732 \to 7.32\%. \)

From Eq. (67), for \( |0 \rangle \), the sum of the probabilities is \( 42.68\% + 7.32\% + 42.68\% + 7.32\% = 100\% \), while for \( |1 \rangle \), the sum of the probabilities results in 0%.

Instead, if \( |\psi\rangle = \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = |1 \rangle \), replacing \( \alpha \) and \( \beta \) in Eq. (66) at time \( t_3 \), and bearing in mind again that \( Z|0 \rangle = |0 \rangle \), but \( Z|1 \rangle = -|1 \rangle \), it turns out that,
\[ |\psi_0\rangle = \frac{u|010\rangle - u|110\rangle + v|001\rangle - v|101\rangle}{\sqrt{2}} \]
\[ = \frac{|00\rangle + |11\rangle}{\sqrt{2}} + \frac{|10\rangle}{\sqrt{2}}(-v|1\rangle) + \frac{|11\rangle}{\sqrt{2}}(-u|0\rangle) \]
\[ = \left\{ \begin{array}{c}
\frac{|00\rangle}{\sqrt{2}} X^{0} Z^{0}|v|1\rangle \quad \text{7.32\%} \\
\frac{|01\rangle}{\sqrt{2}} X^{1} Z^{0}|u|1\rangle \quad \text{42.68\%} \\
\frac{|10\rangle}{\sqrt{2}} X^{1} Z^{1}|v|1\rangle \quad \text{7.32\%} \\
\frac{|11\rangle}{\sqrt{2}} X^{1} Z^{1}|u|1\rangle \quad \text{42.68\%}
\end{array} \right\}.
\]

(68)

From Eq. (68), for |0\>, the sum of the probabilities is 0%, while for |1\>, the sum of the probabilities is 7.32% + 42.68% + 7.32% + 42.68% = 100%, that is, an identical result to the case of maximally-entangled states, although for a very particular state to be teleported as is the case of |0\>.

**Rough-entangled states** From Eq. (51) we obtain \(|F_{00}\rangle^\dagger\), then at time \(t_1\), we have,
\[ |\psi_{t_1}\rangle = |\psi\rangle |F_{00}\rangle^\dagger = (\alpha|0\rangle + \beta|1\rangle) \left( \frac{|00\rangle + |10\rangle(1 + i)/2 + |11\rangle(1 - i)/2}{\sqrt{2}} \right) \]
\[ = \frac{\alpha}{\sqrt{2}}|000\rangle + \frac{\beta}{\sqrt{2}}|100\rangle + \frac{\alpha}{\sqrt{2}}\left( \frac{1 + i}{2} \right)|010\rangle + \frac{\beta}{\sqrt{2}}\left( \frac{1 + i}{2} \right)|110\rangle \]
\[ + \frac{\alpha}{\sqrt{2}}\left( \frac{1 - i}{2} \right)|011\rangle + \frac{\beta}{\sqrt{2}}\left( \frac{1 - i}{2} \right)|101\rangle,
\]
while at time \(t_2\) a CNOT gate is applied between qubits q[0] and q[1],
\[ |\psi_{t_2}\rangle = \frac{\alpha}{\sqrt{2}}|000\rangle + \frac{\beta}{\sqrt{2}}|110\rangle + \frac{\alpha}{\sqrt{2}}\left( \frac{1 + i}{2} \right)|010\rangle + \frac{\beta}{\sqrt{2}}\left( \frac{1 + i}{2} \right)|100\rangle \]
\[ + \frac{\alpha}{\sqrt{2}}\left( \frac{1 - i}{2} \right)|011\rangle + \frac{\beta}{\sqrt{2}}\left( \frac{1 - i}{2} \right)|101\rangle.
\]
(70)

At time \(t_3\), a Hadamard (H) gate is applied in qubit q[0],
\[ |\psi_{t_3}\rangle = \frac{\alpha}{2}|000\rangle + \frac{\beta}{2}|100\rangle + \frac{\beta}{2}|010\rangle - \frac{\beta}{2}|110\rangle + \frac{\alpha}{2}\left( \frac{1 + i}{2} \right)|010\rangle + \frac{\alpha}{2}\left( \frac{1 + i}{2} \right)|100\rangle + \frac{\alpha}{2}\left( \frac{1 - i}{2} \right)|011\rangle + \frac{\beta}{2}\left( \frac{1 - i}{2} \right)|001\rangle \]
\[ - \frac{\beta}{2}\left( \frac{1 + i}{2} \right)|011\rangle + \frac{\alpha}{2}\left( \frac{1 - i}{2} \right)|111\rangle + \frac{\alpha}{2}\left( \frac{1 - i}{2} \right)|001\rangle + \frac{\beta}{2}\left( \frac{1 - i}{2} \right)|001\rangle \]
\[ = \frac{\alpha}{2}\left( \frac{1}{2} \right)\left( \begin{array}{c}
|00\rangle \left\{ \begin{array}{c}
\frac{|0\rangle}{2}\left[ \alpha - \beta \frac{(1 + i)}{2} \right] \quad \text{12.5\%} \\
\frac{|1\rangle}{2}\left[ \alpha + \beta \frac{(1 + i)}{2} \right] \quad \text{12.5\%}
\end{array} \right\} + \\
|01\rangle \left\{ \begin{array}{c}
\frac{|0\rangle}{2}\left[ \alpha + \beta \frac{(1 + i)}{2} \right] \quad \text{12.5\%} \\
\frac{|1\rangle}{2}\left[ \alpha - \beta \frac{(1 + i)}{2} \right] \quad \text{12.5\%}
\end{array} \right\} + \\
|10\rangle \left\{ \begin{array}{c}
\frac{|0\rangle}{2}\left[ \alpha - \beta \frac{(1 - i)}{2} \right] \quad \text{12.5\%} \\
\frac{|1\rangle}{2}\left[ \alpha + \beta \frac{(1 - i)}{2} \right] \quad \text{12.5\%}
\end{array} \right\} + \\
|11\rangle \left\{ \begin{array}{c}
\frac{|0\rangle}{2}\left[ \alpha + \beta \frac{(1 - i)}{2} \right] \quad \text{12.5\%} \\
\frac{|1\rangle}{2}\left[ \alpha - \beta \frac{(1 - i)}{2} \right] \quad \text{12.5\%}
\end{array} \right\}
\end{array} \right\}
\]

(71)

The eight final terms of Eq. (71) show us a distribution of probabilities quite fragmented due to the intervention of complex coefficients because of a single intervention of the QFT. However, for each of the four bases the associated final probability is 25%. Similar
considerations to the previous cases take place at times \( t_4, t_5, t_6, \) and \( t_7 \). Moreover, very similar results would be obtained using \( |F_{00}\rangle^3_2 \) instead of \( |F_{00}\rangle^3_2 \).

As for the two previous cases, if \( |\psi\rangle = \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = |0\rangle \), we can replace \( \alpha \) and \( \beta \) in Eq. (71) at time \( t_3 \), where we get,

\[
|\psi_2\rangle = |0\rangle \left\{ \frac{|0\rangle}{2} + |10\rangle \frac{|0\rangle}{2} + |01\rangle \right\} \left[ \frac{|0\rangle (1 + i)}{2} + \frac{|1\rangle (1 - i)}{2} \right] \\
+ |11\rangle \left\{ \frac{|0\rangle (1 + i)}{2} + \frac{|1\rangle (1 - i)}{2} \right\} = |00\rangle X^0 Z^0 \frac{|0\rangle}{2} + |01\rangle X^1 Z^0 \left[ \frac{|1\rangle (1 + i)}{2} + \frac{|0\rangle (1 - i)}{2} \right] \\
+ |10\rangle X^0 Z^1 \frac{|0\rangle}{2} + |11\rangle X^1 Z^1 \left[ \frac{|1\rangle (1 + i)}{2} + \frac{|0\rangle (1 - i)}{2} \right] = |00\rangle \left\{ \frac{|0\rangle Z^0 |0\rangle}{2} \right\} \\
+ |01\rangle \left\{ X^1 Z^0 \frac{|1\rangle (1 + i)}{4} + \frac{|0\rangle Z^0 |0\rangle (1 - i)}{4} \right\} \\
+ |10\rangle \left\{ X^0 Z^1 \frac{|0\rangle}{2} + \frac{|1\rangle Z^1 |0\rangle}{2} \right\} + |11\rangle \left\{ X^1 Z^1 \frac{|1\rangle (1 + i)}{4} + \frac{|0\rangle Z^1 |0\rangle (1 - i)}{4} \right\}.
\]

The probabilities of the last terms of Eq. (72) arise from:

\[
\frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4} \rightarrow 25\%, \quad \text{while} \quad \frac{1}{4^2} \sqrt{2} \cdot \frac{1}{4^2} \sqrt{2} = \frac{4}{32} = \frac{1}{8} \rightarrow 12.5%.
\]

Then, from Eq. (72), for \( |0\rangle \), the sum of the probabilities is 25% + 12.5% + 25% + 12.5% = 75%, while for \( |1\rangle \), the sum of the probabilities is 12.5% + 12.5% = 25%.

Instead, if \( |\psi\rangle = \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = |1\rangle \), replacing \( \alpha \) and \( \beta \) in Eq. (71) at time \( t_3 \), and taking into account again that \( Z^0 |0\rangle = |0\rangle \), but \( Z^1 |1\rangle = -|1\rangle \), it turns out that,

\[
|\psi_3\rangle = |00\rangle \left\{ \frac{|0\rangle (1 + i)}{2} + \frac{|1\rangle (1 - i)}{2} \right\} + |10\rangle \left\{ -\frac{|0\rangle (1 + i)}{2} - \frac{|1\rangle (1 - i)}{2} \right\} \\
+ |01\rangle \left\{ \frac{|0\rangle}{2} - \frac{|1\rangle}{2} \right\} = |00\rangle X^0 Z^0 \left[ \frac{|0\rangle (1 + i)}{2} + \frac{|1\rangle (1 - i)}{2} \right] \\
+ |01\rangle X^1 Z^0 \left[ \frac{|0\rangle}{2} + \frac{|1\rangle}{2} \right] + |10\rangle X^0 Z^1 \left[ \frac{|1\rangle (1 + i)}{2} \right] = |00\rangle \left\{ \frac{|0\rangle Z^0 |0\rangle (1 + i)}{4} \right\} + |01\rangle \left\{ X^1 Z^0 \frac{|1\rangle (1 - i)}{4} \right\} \\
+ |10\rangle \left\{ X^0 Z^1 \frac{|0\rangle (1 + i)}{4} \right\} + |11\rangle \left\{ X^1 Z^1 \frac{|1\rangle (1 - i)}{4} \right\}.
\]

Therefore, from Eq. (73), for \( |0\rangle \), the sum of the probabilities is 12.5% + 12.5% = 25%, while for \( |1\rangle \), the sum of the probabilities is 25% + 12.5% + 25% + 12.5% = 75%.

From these two examples, that is, when the state to be teleported is a CBS, we see that the outcomes have a probability of 75% for the teleported state and 25% for its counterpart. As will be seen below in the implementations that will take place on both platforms of IBM Q (IBM Quantum Experience 2022), i.e., simulator and Lima 5-qubit physical machine, this will not constitute a problem at all in terms of the discrimination of both states after the process of obtaining the outcome from the quantum measurement, given that the difference between both probabilities is twice the smallest of them.

*Results in the IBM Q (IBM Quantum Experience 2022) simulator:* Teleportations of both CBSs are carried out for all three types of entanglement on this platform. However,
before starting, a point of fundamental importance must be clarified to understand the implementations and conveniently contrast these implementations with the theoretical deductions of the previous sections. This point consists of the following: in the abscissa axis of the histograms obtained both in the simulator and the IBM Q (IBM Quantum Experience 2022) physical machines, the qubits are always shown in the following order: q[2]
q[1]q[0], where q[2] is found at the bottom of Fig. 14, and q[0] at the top of it, while in the theoretical derivations of Eqs. (61), (62), (63), (66), (67), (68), and (71), (72), (73), the order is exactly the opposite, i.e., q[0]q[1]q[2]. However, for both criteria, q[2] is the qubit under analysis, while q[1] and q[0] constitute the base present in the Bell State Measurement (BSM) module. With this important point clear, the simulations begin in Fig. 15, where figures (a) for $|0\rangle$, and (b) for $|1\rangle$ show the histograms resulting from working with maximally-entangled states, while figures (c) for $|0\rangle$, and (d) for $|1\rangle$ contain the histograms relative to the non-maximally entangled states. Finally, the figures (e) for $|0\rangle$, and (f) for $|1\rangle$ represent the histograms when teleportation is carried out using rough entangled states.

**Fig. 17** Histograms of the teleportations for the three types of entanglement. The top row corresponds to maximally entangled states, the middle row to non-maximally entangled states, and the bottom row corresponds to rough-entangled states. The left column results from teleporting $|\psi\rangle = |0\rangle$, while the right column results from teleporting $|\psi\rangle = |1\rangle$.
Both in figures (a) and (c) the percentages are 100% for $|0\rangle$ and 0% for $|1\rangle$, while in figures (b) and (d) the opposite occurs, as was deduced theoretically. Finally, in the figures (e, and f) corresponding to $|F_{00}\rangle$, the four measurement bases are involved as predicted in the theoretical deduction with 75% for $|0\rangle$ and 25% for $|1\rangle$ for the first case, and the opposite for the second one.

**Results in the IBM Q (IBM Quantum Experience 2022) Lima 5-qubits processor** For these implementations, we will resort to the simplified version of the quantum teleportation protocol of, Fig. 16 i.e., the one without quantum measurement modules in the qubits $q[0]$ and $q[1]$. This is because the physical machines of IBM Q (IBM Quantum Experience 2022), as in the case of Lima, do not allow quantum measurement modules in intermediate instances of the quantum circuit, so we resort to the simplified version of the protocol shown in Fig. 16.

In Fig. 17, figures (a) for $|0\rangle$, and (b) for $|1\rangle$ show the histograms resulting from working with maximally-entangled states, while figures (c) for $|0\rangle$, and (d) for $|1\rangle$ contain the histograms relative to the non-maximally entangled states. The decoherence present in every physical machine, of which Lima is no exception, makes non-zero probabilities appear where it does not correspond, both when we work with maximally entangled states and in the case of non-maximally entangled states. Both in figures (a) and (c) the percentages are not 100% for $|0\rangle$ and are not 0% for $|1\rangle$, while in figures (b) and (d) the opposite occurs. This contrasts both with what was deduced theoretically and with the results obtained in the simulator. On the other hand, figures (e) for $|0\rangle$, and (f) for $|1\rangle$ represent the histograms when teleportation is carried out using rough entangled states. In these figures, corresponding to $|F_{00}\rangle$, the four measurement bases are involved as
predicted in the theoretical deduction, and although the results are not exactly 75% for $|0\rangle$ and 25% for $|1\rangle$ for the first case, and the opposite for the second. However, as will be seen in the next section, the case of rough entangled states has the smallest absolute value of the three types of entanglement.

**Analysis of the results** Table 10 shows that the theoretical predictions were carried out due to the deduction of Eqs. (62), (63), (67), (68), (72), and (73), for the three types of entanglement when the state to be teleported is a CBS, fit satisfactorily with the experimental results obtained both in the simulator and in the 5-qubit Lima processor of IBM Q (IBM Quantum Experience 2022). Table 11 represents the correspondence between the qubit to be teleported, the obtained outcome, and the post-processing required by the rough entanglement case to be useful in the context of the future quantum Internet (Caleffi et al. 2020; Cacciapuoti et al. 2020a, 2020b; Gyongyosi and Imre 2020, 2019a, 2019b). This post-processing does not represent any reduction in the performance of the teleportation protocol, since its application does not imply the use of any type of special technology to achieve it, since it must be carried out after the quantum measurement of the qubit q[2], both in the configuration of Figs. 14 and 16, that is, once the wave function has collapsed, or what is the same, in the classical world.

Notwithstanding what has been said, the absolute outcome error, i.e., the difference between the theoretical values and those obtained in the 5-qubit Lima processor of IBM Q (IBM Quantum Experience 2022), will always be lower in the case of working with rough entangled states. For example, if the qubit to be teleported is $|\psi\rangle = |0\rangle$, for all three entanglement types, that absolute outcome error is:

Maximally-entangled states

$$\Delta_0 = |P_{0,\text{theoretical}} - P_{0,\text{Lima}}| = 1 - 0.9101 = 0.0899,$$

(74)

Non-maximally-entangled states

$$\Delta_0 = |P_{0,\text{theoretical}} - P_{0,\text{Lima}}| = 1 - 0.8808 = 0.1192,$$

(75)

Rough-entangled states

$$\Delta_0 = |P_{0,\text{theoretical}} - P_{0,\text{Lima}}| = 0.75 - 0.6972 = 0.0528.$$

(76)

From Eqs. (74), (75), (76), It follows that the case of non-maximally-entangled states has the highest absolute outcome error, while rough entangled states have the lowest one. As we have seen, the compilation of probabilities, the terms of Eq. (71) to reconstruct the outcome, is more rustic for rough entangled states, however, the difference in probabilities is greater than the smallest probability, i.e., 75–25% = 50% > 25%. For this reason, although both possible outcomes are not orthogonal to each other (as in the other two cases), there is enough discrimination between them to be distinguished.

When choosing to teleport computational basis states (CBS) $\{ |0\rangle, |1\rangle \}$, which constitute the mutually orthogonal pair par excellence, it is possible to evaluate decoherence introduced by the platform that hosts the experiment better than with any other qubits. Moreover, the use of CBS as qubits to be teleported makes it easier to compare the three entanglement cases than any other pair of qubits.

All implementations of this study are available in both Quirk (Simulator 2022) and IBM Q (IBM Quantum Experience 2022) in the Data Availability Statement section.

Finally, similar results to those obtained in this section would be obtained using $|F_{00}\rangle^3_2$ instead of $|F_{00}\rangle^1_2$, the reason why the same implementations with $|F_{00}\rangle^3_2$ are not repeated.
1.4 Other applications

In this section, five of the most conspicuous cases of quantum-Fourier-gates (QFG) application have been selected to be developed, which are:

- Quantum stretching,
- Entanglement levels,
- Entanglement parallelization,
- Quantum secret sharing (Joy et al. 2020) (QSS) for quantum cryptography (Kumar et al. 2021), and
- Quantum repeaters (Ruihong and Ying 2019) for QKD (Mehic, et al. 2020), and the future quantum Internet (Caleffi et al. 2020; Cacciapuoti et al. 2020a, 2020b; Gyongyosi and Imre 2020, 2019a, 2019b).

These techniques will have a great projection on quantum communication and cryptography, and their development here constitutes only a small part of the universe of QFG applications. Moreover, without loss of generality, for the first three techniques it was decided to explain them for configurations of no more than four qubits, while for the last two, we will resort to the same criterion, but for three, and two qubits, respectively. This is why a greater number of qubits would cause a great increase in the number, as well as in the size of the associated figures necessary to explain them, which would inappropriately extend the dimension of this work.

Quantum stretching It specifically consists of a detailed analysis of the dimensional transition between entanglement configurations for a consecutive number of qubits, where the mentioned transition is regulated by the subscript of the $F_2^k$ gate that accompanies the Hadamard gate (H). That is, by changing the subscript of the $F_2^k$ gate, the degree of stretching is changed, $k$ being the stretching index. Then, Eqs. (77), (78), (79, (80) show the mentioned transition for incremental values of the subscript $k$, which can be seen in detail in the second lines of each equation, where the stretching is manifested as the inclusion of numerous zeros in the middle of the ones that are found at the ends. In other words, the coefficients involved are the same, it is simply stretching of the respective vectors, which is modeled by a simple change in a parameter of the $F_2^k$ gate, which unites the entire family of entangled particles.

\[
|F_000\rangle_3 = |GHZ_3\rangle = F_3^2(H \otimes I_{2^2 \times 2^2})|000\rangle = (|000\rangle + |111\rangle)/\sqrt{2}
\]

\[
= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}^T / \sqrt{2}
\]  

(79)
In Fig. 18, the four cases of Eqs. (77), (78), (79), (80) are represented using a correlative transition that goes from \(\langle +\rangle\) to \(\langle GHZ_4\rangle\), passing through \(\langle GHZ_2\rangle\) and \(\langle GHZ_3\rangle\).

The behavior of Eqs. (77), (78), (79), (80) respecting to the Hadamard gate (H), can be reproduced with any other gate, e.g., that of Eq. (35), i.e., \(\sqrt{X}\) gate, for the state \(\langle \gamma_{00}\rangle\) = \([ u \ 0 \ 0 \ v ]\), with \(u = 0.8536 + 0.3536i\), and \(v = 0.1464 - 0.3536i\), of Table 9. In consequence, the following set of equations arises,

\[
\langle F_{0000}\rangle_4 = \langle GHZ_4\rangle = F_4^2(H \otimes I_{2^4})\langle 0000\rangle = (\langle 0000\rangle + \langle 1111\rangle)/\sqrt{2}
\]

\[
= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}^T / \sqrt{2}
\]

(80)

In Fig. 18, the four cases of Eqs. (77), (78), (79), (80) are represented using a correlative transition that goes from \(\langle +\rangle\) to \(\langle GHZ_4\rangle\), passing through \(\langle GHZ_2\rangle\) and \(\langle GHZ_3\rangle\).

The behavior of Eqs. (77), (78), (79), (80) respecting to the Hadamard gate (H), can be reproduced with any other gate, e.g., that of Eq. (35), i.e., \(\sqrt{X}\) gate, for the state \(\langle \gamma_{00}\rangle\) = \([ u \ 0 \ 0 \ v ]\), with \(u = 0.8536 + 0.3536i\), and \(v = 0.1464 - 0.3536i\), of Table 9. In consequence, the following set of equations arises,

\[
\langle \gamma_0\rangle = F_2^2(\sqrt{X})\langle 0\rangle = u\langle 0\rangle + v\langle 1\rangle
\]

\[
= \begin{bmatrix} u & v \end{bmatrix}^T
\]

(81)

\[
\langle \gamma_{00}\rangle = F_2^2(\sqrt{X} \otimes I_{2^2})\langle 00\rangle = u\langle 00\rangle + v\langle 11\rangle
\]

\[
= \begin{bmatrix} u & 0 & 0 & v \end{bmatrix}^T
\]

(82)

\[
\langle \gamma_{000}\rangle = F_3^2(\sqrt{X} \otimes I_{2^3})\langle 000\rangle = u\langle 000\rangle + v\langle 111\rangle
\]

\[
= \begin{bmatrix} u & 0 & 0 & 0 & 0 & v \end{bmatrix}^T
\]

(83)

\[
\langle \gamma_{0000}\rangle = F_4^2(\sqrt{X} \otimes I_{2^4})\langle 0000\rangle = u\langle 0000\rangle + v\langle 1111\rangle
\]

\[
= \begin{bmatrix} u & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & v \end{bmatrix}^T
\]

(84)

Thus, stretching analysis works for both maximally and non-maximally entangled states.
Entanglement levels

Another case of quantum stretching occurs when working also with a gate $F_k^2$ but in a configuration in which its subscript $k$ is not increased, but rather the position of the Hadamard gate (H) at the input of gate $F_k^2$ changes. For example, without losing generality, if we work with the gate of four qubits as $F_4^2$, the sequence of Eqs. (81), (82), (83), (84) shows us that, although we work with the same gates, the results are different according to the location of the Hadamard gate (H). This simple process of shifting the Hadamard gate (H) gives rise to a stretching identical to the previous case.

\[
|000\rangle|F_0\rangle_1^2 = |000\rangle|+\rangle = F_4^2(I_{2^3\times2^3} \otimes H)|0000\rangle \\
= |000\rangle[1 \ 1]^T/\sqrt{2} \tag{85}
\]

\[
|00\rangle|F_{00}\rangle_2^2 = |00\rangle|\beta_{00}\rangle = F_4^2(I_{2^2\times2^2} \otimes H \otimes I_{2^2\times2^2})|0000\rangle \\
= |00\rangle[1 \ 0 \ 0 \ 1]^T/\sqrt{2} \tag{86}
\]

\[
|0\rangle|F_{000}\rangle_3^2 = |0\rangle|GHZ_3\rangle = F_4^2(I_{2^2\times2^2} \otimes H \otimes I_{2^2\times2^2})|0000\rangle \\
= |0\rangle[1 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 1]^T/\sqrt{2} \tag{87}
\]

\[
|F_{0000}\rangle_4^2 = |GHZ_4\rangle = F_4^2(H \otimes I_{2^3\times2^3})|0000\rangle \\
= [1 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 1]^T/\sqrt{2} \tag{88}
\]
In Fig. 19, the transition from $|+\rangle$ to $|GHZ_4\rangle$, passing through $|GHZ_2\rangle$ and $|GHZ_3\rangle$, based on the migration of the Hadamard gate (H) becomes noticeable. This technique can only be carried out with an $F^2_k$-type gate, that is, a QFGS, and not with nested Feynman gates (CNOT). Finally, as in the previous case, if we replace the Hadamard matrix (H) by $\sqrt[4]{X}$, a stretching of the resulting states will be obtained as the gate $\sqrt[4]{X}$ migrates from the lower to the upper qubit. This is how Eqs. (89), (90), (91), (92) arise.

\[
|000\rangle|\gamma_0\rangle = |000\rangle \sqrt[4]{X} |0\rangle = F^2_4 \left( I_{2^3 \times 2^3} \otimes \sqrt[4]{X} \right) |000\rangle
= [u \ v]^T
\]  

(89)

\[
|00\rangle|\gamma_{00}\rangle = F^2_4 \left( I_{2^2 \times 2^2} \otimes \sqrt[4]{X} \otimes I_{2^1 \times 2^1} \right) |0000\rangle
= |00\rangle [u \ 0 \ 0 \ v]^T
\]  

(90)

\[
|0\rangle|\gamma_{000}\rangle = F^2_4 \left( I_{2^1 \times 2^3} \otimes \sqrt[4]{X} \otimes I_{2^2 \times 2^2} \right) |0000\rangle
= |0\rangle [u \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ v]^T
\]  

(91)

\[
|\gamma_{0000}\rangle = F^2_4 \left( \sqrt[4]{X} \otimes I_{2^1 \times 2^3} \right) |0000\rangle
= [u \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ v]^T
\]  

(92)

**Fig. 20**  Entanglement parallelization: Based on a $F^1_4$ gate, each output pair $\{c_i, t_i\}$ ($\forall i \in [0,3]$) shares a state $|\rho_{00}\rangle$ among them, but they are completely uncorrelated with the members of the other pairs.
Thanks to Eqs. (81), (82), (83), (84), as well as Eqs. (89), (90), (91), (92), it is possible to see that this technique works equally well for both maximally and non-maximally entangled states, respectively.

**Entanglement parallelization** Based on gates of $F^1_k$-type $\forall k$, mutually independent parallel sources of entangled particles can be constructed. For example, without losing generality, for the case of four qubits, we resort to the $F^4_4$ gate, which, together with four *CNOT* gates located in each of its outputs, generates four pairs of entangled particles, i.e., four pairs of the *control-target* type $\{c_i, t_i\}$ ($\forall i \in [0, 3]$), where each of these pairs shares a Bell state of type $|\psi_{\text{GHZ}}\rangle$. However, they are completely uncorrelated with the members of the other pairs. In Fig. 20, it is possible to identify this configuration. The density matrices between elements of the same pair (control-target), that is, between $c_i$ and $t_i$ ($\forall i \in [0, 3]$) is that of Eq. (43), and which we repeat here.
while for any other combination of two outputs such that \( \forall j \neq i \), the density matrices are:

\[
DM_{c_i, t_j} = \frac{1}{4} \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix},
\]

The same situation occurs among groups of three (\(|GHZ_3\rangle\)) or four (\(|GHZ_4\rangle\)) particles associated with a gate \( F^1_2 \), where Fig. 21 represents both configurations. In the case of Fig. 21a, the top three qubits constitute a \(|GHZ_3\rangle\) state, as do the bottom three qubits, however, the middle two qubits are completely decorrelated, i.e., they are independent. Based on Fig. 7b, we can see the three grids (density matrices) in Fig. 21a here too, an
element in white means a value equal to zero, while an element in gray represents a value other than zero. Something similar happens in Fig. 21b, where the top four qubits form a $|\text{GHZ}_4\rangle$ state, as do the bottom four qubits, while the middle two qubits have zero correlation. In Fig. 21, the combination of each output of the gate $F_1^2$ with its corresponding $CNOT$ gates gives rise to two parallel sources of entangled pairs of three (a) and four (b) qubits.

Equation (93) shows that the outputs of affine control-target pairs of identical subscript share a Bell state of type $|\beta_00\rangle$, while Eq. (94) tells us that, apart from the previous relationship, the outputs are completely independent. This feature makes the configuration of Fig. 20 particularly useful for performing four independent and simultaneous teleportations like those in Fig. 22. This is a natural path from entanglement parallelization to hyper-teleportation. Consequently, this setting controls various spurious effects such as cross-channeling.

**Quantum secret sharing** (Joy et al. 2020) (QSS) This protocol constitutes a true central tool inside the quantum cryptography (Kumar et al. 2021) toolbox, and can be interpreted as teleportation for the case of working with sources of 3, 4, and more entangled photons at the same time. The basic scheme of the QSS protocol (Mastriani 2021a) can be seen in Fig. 23 for the case of working with entangled states of the $|\text{GHZ}_3\rangle$ type.

Figure 24 represents a parallel QSS scheme based on two independent sources of $|\text{GHZ}_3\rangle$ states, for the simultaneous transmission of two different states, $|\psi_A\rangle$ and $|\psi_B\rangle$, thanks to a configuration that uses only one $F_1^2$ gate. Thanks to $F_1^2$ we can access the first bidirectional QSS protocol in the literature, with a particular projection on the transmission of secure information in the future quantum Internet (Caleffi et al. 2020; Cacciapuoti et al. 2020a, 2020b; Gyongyosi and Imre 2020, 2019a, 2019b).

**Quantum repeaters** Fiber optic cabling for terrestrial implementations of QKD requires quantum repeaters every certain number of kilometers (Mehic, et al. 2020), which in turn requires a large amount of quantum memory. The problem is that the key is exposed in its passage through them. There are currently two well-defined lines of research, the first has

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**Fig. 24** Parallel quantum secret sharing is based on two independent sources of $|\text{GHZ}_3\rangle$ states, for the simultaneous transmission of two different states, $|\psi_A\rangle$ and $|\psi_B\rangle$, thanks to a configuration that uses only one $F_1^2$ gate.
to do with the development of quantum repeaters that do not require quantum memory, at least not that much, and the second is to replace the same quantum repeaters with some type of implementation based on quantum teleportation (Bennett et al. 1993). It is imperative to solve this problem to implement the future quantum Internet (Caleffi et al. 2020; Cacciapuoti et al. 2020a, 2020b; Gyongyosi and Imre 2020, 2019a, 2019b).

At this point, two types of quantum repeaters can be identified, those which use:

- Entanglement swapping (Behera et al. 2019) (transitivity), or
- cascading teleportations (forward).

In previous works, the virtues of quantum Fourier gates (QFG) in the implementation of quantum repeaters based on entanglement swapping were shown (Mastriani 2021a, 2021b, 2021c, 2021d). Therefore, here quantum repeater based on entanglement swapping with the intervention of entanglement parallelization will be implemented.

In Fig. 25a, a classical implementation of a quantum repeater based on entanglement swapping is shown. Links of these structures are regularly used in such a way that a final entanglement is obtained between the first and last link of the chain, which considerably increases the range of the entanglement. For this reason, we speak of transitivity, that is, within the chain of quantum repeaters if A and B are entangled, and B and C are also entangled, eventually A and C will be. Instead, Fig. 25b shows the same type of quantum repeater, although the pair of entangled pairs are replaced with a configuration of two
sources of entangled particles (independent and parallel) based on the gate $F_2^1$, which puts in evidence another application of entanglement parallelization.

As we have mentioned before, this type of repeater exposes the key that crosses them in a QKD (Kumar et al. 2021) context. For this reason, it is advisable to try other forms of quantum repeaters based on teleportations (Bennett et al. 1993). In this case, a cascade of teleportations will increase the range of the broadcast. Because of this, the configurations of Fig. 26, which correspond to quantum repeaters based on cascaded quantum teleportations, are proposed. The first uses a single quantum repeater highlighted in orange, see Fig. 26a, while the second uses two quantum repeaters in cascade, that is, the state received by one is teleported to the next, as can be seen in Fig. 26b. In the latter case, the second quantum repeater is highlighted in light blue. In both cases of Fig. 26, blocks like those of Fig. 6 and Table 5 are used to generate the states $|\psi_{00}\rangle$.

### 2 Discussion

Next, we discuss the salient points of this study.

#### 2.1 Methodology

Based on the experience acquired in previous studies (Mastriani 2021a, 2021b, 2021c, 2021d), where a new facet about the intrinsic nature of entanglement was evidenced, which turned out to be entirely dependent on QFT blocks, it was decided to explore a series of modifications in the QFT blocks dosage when forming entanglement on a physical
platform of the IBM Q Experience program (IBM Quantum Experience 2022) known as Lima, which has 5 qubits.

2.2 Novelty that was achieved in this research

As a consequence of the methodology used in this study, other entanglement options arose, both due to lack of QFT blocks and excess of them, which led to new possibilities of this extraordinary phenomenon called entanglement with dramatic consequences in quantum communications, as is the case of the entanglement parallelization technique, which allows the simultaneous coexistence of parallel transmission circuits of several qubits at the same time from a single source of disjoint sets of entangled particles, that is to say, this last option allows:

1. A more efficient version of the so-called quantum repeaters, fundamental in quantum communications and cryptography, by joining two branches of a quantum network through a single source of entangled photons, instead of two as in the traditional case,
2. Multi-qubit teleportation, i.e., the simultaneous teleportation of several qubits from a single entangled particle source, and
3. Multi-quantum-secret-sharing transmission, i.e., the parallelization of this powerful quantum cryptography tool.

These possibilities are unfeasible with the traditional forms of entanglement, i.e., maximally, and non-maximally.

2.3 Configurations resulting from the methodology

In Fig. 1 a more generic version of the SWAP gate was developed, grouping the latter into blocks, to switch equidistant qubits simultaneously. These blocks were used in Fig. 2 for the cases of three and four qubits, and thus form gates of the $F_p^2$-type. In Fig. 3, a logical comparison was carried out between the Toffoli-type gates and the $F_p^2$-type gates, for the cases of four and three qubits. In Fig. 4, the most elementary form of the family of proposed states was analyzed, i.e., the $|F_{ab}\rangle^0_2$ case with null participation of QFT blocks in its constitution. In Fig. 5, the case of the $|F_{ab}\rangle^1_2$ state was exposed, which implies the use of a single QFT block to obtain it. Figure 6 corresponds to the traditional cases known as Bell states, for which the resulting states $|F_{ab}\rangle^2_2$ imply the use of two QFT blocks for their constitution. Figure 7a shows the use of two QFT blocks for the case of three qubits, which gives rise to states of the $|F_{000}\rangle^2_3$-type. An extension of this last case is shown in Fig. 8a, which implies the use of two QFT blocks but for four qubits, generating states of the $|F_{0000}\rangle^2_4$-type. Figure 9 shows the generation of states of the $|F_{ab}\rangle^3_2$-type using three QFT blocks in its constitution. Figure 10 shows a particular case of non-maximally entangled states thanks to two QFT blocks for the case of two qubits. Figure 12 highlights the logical difference between a cascade-splitting type configuration and an $F^2$ gate for the cases of three and four qubits. Figure 14 represents the original version of the quantum
teleportation protocol where the distributable pair is made up of roughly entangled particles. Figure 16 shows a simplified version of the quantum teleportation protocol due to the inability of the physical machines of the IBM Q Experience program to implement the if–then-else statement. Figure 18 shows the sequence of states $|F_{01}\rangle^2$ to $|F_{0000}_4\rangle^2$ from the smallest to the largest number of qubits, for the case of two QFT blocks used in the constitution of the states. Figure 19 shows how the displacement (from bottom to top) of a Hadamard gate (H) in the input qubits allows obtaining the same four states of Fig. 18. Figure 20 represents the maximum exponent of this technology, which it can only be carried out through the so-called roughly entangled states. We refer to the entanglement parallelization technique, which, as we have mentioned before, allows the simultaneous coexistence of parallel transmission circuits of several qubits at the same time from a single source of disjoint sets of entangled particles. Figure 21 shows the density matrices for two disjoint groups of entangled particles of three and four qubits each. In both schemes, the output density matrices of 4-by-4 elements clearly demonstrate that both output sets are totally independent, i.e., they are not correlated at all. Figure 22 depicts the entanglement parallelization technique in action, which allows for the simultaneous teleportation of four independent qubits from a single source of four disjoint pairs of entangled particles. Figure 23 shows the implementation of the quantum secret sharing (QSS) protocol from a state of the type $|F_{000}_3\rangle^2 = |\text{GHZ}_3\rangle$, while Fig. 24 shows the second direct application of the entanglement parallelization technique to produce simultaneous transmissions of two independent states $|\psi_A\rangle$ and $|\psi_B\rangle$ based on the double and simultaneous use of the QSS protocol. Figure 25 shows the application of the entanglement swapping technique for the construction of quantum repeaters, where the first is based on the traditional technique, i.e., thanks to two independent sources of entangled photons, while the second uses a single source generating two disjoint sets of entangled pairs based on the parallelization entanglement technique. Finally, Fig. 26 shows the use of type $F^2_2$ gates, which involve two QFT blocks, for the generation of entangled pairs as part of a quantum repeater cascade in order to extend the range of teleportation of an arbitrary state $|\psi\rangle$.

### 3 Results

The above configurations yielded the following results: Table 2 shows the density matrices of states of the $|F_{ab}\rangle^0_2$-type, while Table 4 shows the density matrices of states of the $|F_{ab}\rangle^1_2$-type. Table 6 represents the density matrices of states of the $|F_{ab}\rangle^2_2$-type, while Table 8 shows the density matrices of states of the $|F_{ab}\rangle^3_2$-type. Figures 7b and 8b show the density matrices of the states $|F_{000}_3\rangle^2$ and $|F_{0000}_4\rangle^2$, respectively. Figure 11 represents the density matrices for a particular case of the non-maximally entangled states according to the inputs and using two QFT blocks of two qubits each. Figure 13 shows the density matrices for the four groups of four maximally and non-maximally entangled states depending on the number of QFT blocks involved. Figure 15 shows the resulting histograms of the teleportations for the three types of entanglement: maximally, non-maximally, and roughly entanglement, by performing measurements on the three output qubits of the protocol, while Fig. 17 represents a configuration similar but measuring only on Bob’s qubit, i.e., q[2]. Table 10 shows the results obtained for the three types of entanglement in its as well as theoretical version on the two IBM Q platforms used, i.e.,
the QASM simulator and the 5-qubit Lima physical machine. Finally, Table 11 represents post-processing work on the outcomes resulting from the teleportation of the state of Fig. 17 using a roughly entangled particle distribution.

### 4 Conclusions

Another form of entanglement different from those already known and which produces maximally and non-maximally entangled states were presented in this study. The expression rough has to do with the lack of a second layer or block of QFT.

Both the Bell states and the $|\text{GHZ}_n\rangle$ states ($\forall n$) are particular cases of the quantum Fourier states. Figures 6, 7, 8 demonstrate this. In general, all forms of entanglement are derived from Fourier. In Fig. 2, the Feynman or CNOT gate (Nielsen and Chuang 2004) is expressed as a particular case of quantum Fourier gates.

Quantum teleportation, as well as its projection on the future quantum Internet, exclusively rests on Fourier (the first qubitizer), since all forms of entanglement are just tools in the Fourier toolbox. That is, entanglement is part of something bigger, the best version of it, but that is all, one more member.

The remarkable performance demonstrated by the applications of this technology in the last two sections of this study evidences its projection on QKD (Kumar et al. 2021), the future quantum Internet (Caleffi et al. 2020; Cacciauoti et al. 2020a, 2020b; Gyongyosi and Imre 2020, 2019a, 2019b), as well as fundamental applications in quantum communications in the presence of noise (Sharma 2016, 2014, 2307; Sharma and Bhardwaj 2022; Sharma and Banerjee 2020, 2018; Sharma et al. 2021, 2018; Sharma and Panchariya 2015; Sharma and Sharma 2014).

This work fills in part of the incomplete knowledge that is present in the literature about entanglement, given that the specialized literature only mentions two forms of entangled states: maximally and non-maximally. In such a way that by filling in the empty spaces about the intrinsic nature of this formidable and little-understood phenomenon of physics called quantum entanglement, new and more efficient communication protocols, and quantum cryptography can be developed. Finally, the intervention of the QFT is widely justified for two reasons:

1. Understanding that the Feynman’s Gate (CNOT) is not an elementary gate, but is derived from another, that is even more fundamental and is called QFT, which fully impacts our knowledge of the canonical nature of the gates that make up Quantum Information Processing, and
2. The QFT is a gate itself, which gives rise to an unprecedented type of entangled state based on rough entanglement, with enormous projections in future quantum communications and cryptography protocols.

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**Author contributions** MM conceived the idea and fully developed the theory, developed the experiments on the simulator and the optical table, wrote the complete manuscript, prepared figures, and reviewed the manuscript.

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**Declarations**

**Conflict of interest** The author declares no competing interests.

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