SOME SPECIAL SOLUTIONS OF A NONLINEAR SYSTEM OF 4 ORDINARY DIFFERENTIAL EQUATIONS RECENTLY INTRODUCED TO INVESTIGATE THE EVOLUTION OF HUMAN RESPIRATORY VIRUS EPIDEMICS

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Abstract

A system of 4 nonlinearly-coupled Ordinary Differential Equations has been recently introduced to investigate the evolution of human respiratory virus epidemics. In this paper we point out that some explicit solutions of that system can be obtained by algebraic operations, provided the parameters of the model satisfy certain constraints.

1 Introduction

The following system of 4 nonlinearly-coupled Ordinary Differential Equations (ODEs) has been recently introduced to investigate the evolution of human respiratory virus epidemics \cite{1}:

\begin{align}
\dot{x}_1 &= -k_D \dot{x}_1 + \alpha k_R (\dot{x}_3 + \dot{x}_4) , \\
\dot{x}_2 &= k_B \dot{x}_1 + [k_B - k_D - f(\dot{x}_1, \dot{x}_2, \dot{x}_3, \dot{x}_4)] \dot{x}_2 + [k_B + (1 - \alpha) k_R] (\dot{x}_3 + \dot{x}_4) , \\
\dot{x}_3 &= f(\dot{x}_1, \dot{x}_2, \dot{x}_3, \dot{x}_4) \dot{x}_2 - (k_R + k_D + k_P) \dot{x}_3 , \\
\dot{x}_4 &= k_P \dot{x}_3 - (k_R + k_D + k_{DV}) \dot{x}_4 ,
\end{align}

where

\begin{equation}
f(\dot{x}_1, \dot{x}_2, \dot{x}_3, \dot{x}_4) = \frac{k_I (\dot{x}_3 + \beta \dot{x}_4)}{x_1 + \dot{x}_2 + \dot{x}_3 + \beta \dot{x}_4} .
\end{equation}

Notation. We maintained the original notation of \cite{1}, except for the following replacement of the 4 dependent variables \( I(t) \) (number of Immune hosts), \( S(t) \) (number of Susceptible hosts), \( A(t) \) (number of Asymptomatic and infectious hosts), \( C(t) \) (number of Symptomatic and infectious hosts) used there, and the use of a superimposed dot (instead of an appended prime) to denote differentiation with respect to the dependent variable \( t \) ("time"):

\begin{equation}
I(t) = \dot{x}_1(t) , \quad S(t) = \dot{x}_2(t) , \quad A(t) = \dot{x}_3(t) , \quad C(t) = \dot{x}_4(t) ; \quad \dot{x}(t) \equiv d\dot{x}(t)/dt .
\end{equation}
where of course now for the elimination of the parameter system, because it can be altogether eliminated from it via the following very simple change of dependent variables:

\[
\tilde{x}_n(t) = x_n(t) \exp(-k_D t), \quad n = 1, 2, 3, 4,
\]

implying of course

\[
\tilde{x}_n(0) = x_n(0), \quad n = 1, 2, 3, 4;
\]

indeed the system of ODEs satisfied by the 4 variables \( x_n(t) \) is then identical with the original system (I), except for the elimination of the parameter \( k_D \):

\[
\begin{align*}
\dot{x}_1 &= \alpha k_R (x_3 + x_4), \\
\dot{x}_2 &= k_B x_1 + [k_B - f(x_1, x_2, x_3, x_4)] x_2 + [k_B + (1 - \alpha) k_R] (x_3 + x_4), \\
\dot{x}_3 &= f(x_1, x_2, x_3, x_4) x_2 - (k_R + k_P) x_3, \\
\dot{x}_4 &= k_P x_3 - (k_R + k_D V) x_4,
\end{align*}
\]

where of course now

\[
f(x_1, x_2, x_3, x_4) = \frac{k_I (x_3 + \beta x_4)}{x_1 + x_2 + x_3 + \beta x_4},
\]

Hence hereafter we shall mainly deal with this, marginally simpler, system (I).

In the following Section 2 we investigate a very simple solution of this model (I), characterized by the fact that the 4 components \( x_n(t) \) of this solution all evolve proportionally to the same exponential function of time, \( \exp(\mu t) \), with \( \mu \) an appropriate parameter determined in terms of the parameters of the model; implying that the quantity \( f(x_1, x_2, x_3, x_4) \) is time-independent (see (4e)), hence that the system (I)—for this class of solutions—reduces to a linear system of 4 ODEs.

In the subsequent Section 3 we discuss the somewhat less simple solutions characterized by the presumably more interesting requirement that each of the 4 components \( x_n(t) \) of the solution be linear combinations—with time-independent coefficients—of 2 exponential functions of time, \( \exp(\mu_1 t) \) and \( \exp(\mu_2 t) \), and moreover that the quantity \( f(x_1, x_2, x_3, x_4) \), see (4c), be again time-independent. The related restrictions on the parameters of the model and the initial-data of this solution are also explicitly determined, up to algebraic operations.

The subsequent Section 4 outlines the analogous treatments when the solution being identified is the sum of 3, or 4, exponentials.

A final Section 5 concludes the paper, by mentioning its applicative relevance and possible further developments of the approach used in this paper.

## 2 A very simple solution

The right-hand sides of the 4 ODEs (I) are all homogeneous of degree 1 in the 4 dependent variables \( x_n(t) \). This implies a well-known (see for instance [2]) consequence, which can be stated as the following

**Proposition 2.1.** The system of 4 ODEs (I) features the simple explicit solution

\[
x_n(t) = x_n(0) \exp(\mu t), \quad n = 1, 2, 3, 4,
\]
where \( x_n (0) \) are clearly the 4 initial values of the 4 dependent variables \( x_n (t) \) and \( \mu \) is an a priori arbitrary time-independent parameter, provided these 5 quantities—i.e., \( x_n (0) \) and \( \mu \), together with the 7 parameters of the model (4)—satisfy (as it were, a posteriori) the following 4 algebraic equations:

\[
\mu x_1 (0) = \alpha k_R [x_3 (0) + x_4 (0)] ,
\]

\[
\mu x_2 (0) = k_B x_1 (0) + [k_B - f (0)] x_2 (0) + [k_B + (1 - \alpha) k_R] [x_3 (0) + x_4 (0)] ,
\]

\[
\mu x_3 (0) = f (0) x_2 (0) - (k_R + k_P) x_3 (0) ,
\]

\[
\mu x_4 (0) = k_P x_3 (0) - (k_R + k_{DV}) x_4 (0) ,
\]

where of course (see (4e) and (5))

\[
f (x_1, x_2, x_3, x_4) \equiv f (0) = \frac{k_I [x_3 (0) + \beta x_4 (0)]}{x_1 (0) + x_2 (0) + x_3 (0) + \beta x_4 (0)} . \]

The validity of this Proposition 2.1 can be easily verified by inserting the solution (5) in the system (4) and by then taking advantage of the conditions (3).

Somewhat less trivial is to ascertain which are the constraints on the 4 initial data \( x_n (0) \) and on the parameter \( \mu \)—by solving the system of algebraic equations (6)—when we consider the model (4) for an arbitrary assignment of its 7 parameters \( k_R, k_B, k_P, k_{DV}, k_I, \alpha, \beta \). Remarkably, as we show below, this turns out to be explicitly doable by purely algebraic operations.

Since all the 5 eqs. (6) are invariant under a common rescaling of the 4 initial data \( x_n (0) \), it is convenient to assume that one of them—say \( x_4 (0) \)—can be arbitrarily assigned, and to focus on the ratios of the other 3 to that one, hence on the 3 quantities

\[
r_m = x_m (0) / x_4 (0) , \quad x_m (0) = r_m x_4 (0) , \quad m = 1, 2, 3 ;
\]

thereby replacing the 5 eqs. (6) with the following 5 equations:

\[
\mu r_1 = \alpha k_R (r_3 + 1) ,
\]

\[
\mu r_2 = k_B r_1 + [k_B - F (r_1, r_2, r_3)] r_2 + [k_B + (1 - \alpha) k_R] (r_3 + 1) ,
\]

\[
\mu r_3 = F (r_1, r_2, r_3) r_2 - (k_R + k_P) r_3 ,
\]

\[
\mu = k_P r_3 - (k_R + k_{DV}) ,
\]

where of course (above and hereafter)

\[
F (r_1, r_2, r_3) = k_I (r_3 + \beta) / (r_1 + r_2 + r_3 + \beta) .
\]

It is now convenient—in order to get rid of the nonlinear function \( F (r_1, r_2, r_3) \)—to sum the 2 eqs. (8d) and (8e), getting thereby

\[
\mu (r_2 + r_3) = k_B + (1 - \alpha) k_R + k_B r_1 + k_B r_2 + (k_B - k_P - \alpha k_R) r_3 .
\]

The 3 eqs. (8a), (8d) and (8e) constitute now a system of 3 linear algebraic equations for the 3 unknowns \( r_1, r_2, r_3 \), which can be easily solved. Indeed from (8a) we get

\[
r_3 = (\mu + k_{DV} + k_R) / k_P ;
\]

then from (8a) and (10a) we get

\[
r_1 = \alpha k_R (\mu + k_{DV} + k_P + k_R) / (\mu k_P) ;
\]

and then from (9), (10a) and (10b) we get

\[
r_2 = \frac{- \mu + k_{DV}}{k_P} - \frac{\mu - k_B + k_{DV}}{\mu - k_B} - \frac{[(1 + \alpha) \mu + \alpha (k_{DV} + k_P)] k_R + \alpha (k_R)^2}{\mu k_P} .
\]
Note that these are *explicit* expressions of the 3 parameters $r_m$ in terms of the 6 parameters $k_R$, $k_B$, $k_P$, $k_{DV}$, $k_I$, $\alpha$ of the system (4), and moreover of the parameter $\mu$ featured by the solution (5) (where of course now $x_m(0) = r_m x_4(0)$ for $m = 1, 2, 3$, with $x_4(0)$ remaining as a free parameter).

Our remaining task in order to get the special solution (5) of the system (4) is to ascertain the permitted values of the parameter $\mu$, as implied by inserting the following expression of $F(r_1, r_2, r_3)$ (obtained by inserting the 3 expressions (10) of $r_1, r_2, r_3$ in (9)),

$$F(r_1, r_2, r_3) = \frac{-k_I (\mu - k_B)(\mu + k_{DV} + \beta k_P + k_R)}{k_P [(1 - \beta)(\mu - k_B) + k_{DV}]} ,$$ (11)

into any one of the 2 eqs. (8a) or (8b). This yields the following *algebraic* solvable *equation* of degree 4 (hence *explicitly solvable*) for the quantity $\mu$:

$$\sum_{k=0}^{4} (c_k \mu^k) = 0 ,$$ (12a)

with the following definitions of the 5 parameters $c_k$:

$$c_4 = k_I - k_P + \beta k_P ,$$ (12b)

$$c_3 = 2 k_{DV} k_I - 2 k_{DV} k_P + k_I k_P - (k_P)^2 + 2 k_I k_R - 2 k_P k_R + \alpha k_I k_R - k_B [k_I - (1 - \beta) k_P] + \beta k_P (k_{DV} + k_I + k_P + 2 k_R) ,$$ (12c)

$$c_2 = (k_I - k_P) [(k_{DV})^2 + k_R (k_P + k_R)] + \alpha k_I k_R (k_P + 2 k_R) + \beta k_P [(k_I + k_R)(k_P + k_R) + \alpha k_I k_R]$$

$$+ k_{DV} \{2 + \beta\} k_I k_P + 2 k_I (k_R + \alpha k_R) - k_P [(2 - \beta) k_P + (3 - \beta) k_R] + k_B \{(1 - \beta) k_P (k_P + 2 k_R) + k_{DV} (-2 k_I + k_P - \beta k_P) - k_I (k_P + 2 k_R + \alpha k_R + \beta k_P)\} ,$$ (12d)

$$c_1 = (k_{DV} + k_R) [k_{DV} k_P (k_I - k_P - k_R) + \alpha k_I k_R (k_{DV} + k_P + k_R)]$$

$$+ \beta k_I k_P [k_{DV} k_P + \alpha k_R (k_{DV} + k_P + k_R)] + k_B \{-(k_{DV} + k_R) [k_{DV} k_I + (k_I - k_P)(k_P + k_R)]$$

$$- \alpha k_I k_R (2 k_{DV} + k_P + 2 k_R)$$

$$- \beta k_P [(k_I + k_R)(k_P + k_R) + k_{DV} (k_I + k_P + k_R) + \alpha k_I k_R]\} ,$$ (12e)

$$c_0 = - \alpha k_B k_I k_R (k_{DV} + k_P + k_R) (k_{DV} + k_R + \beta k_P) .$$ (12f)

**Remark 2.2.** For completeness let us mention that the results reported just above require the validity of the following *inequalities*:

$$\mu \neq 0 , \ k_P \neq 0 , \ (1 - \beta)(k_B - \mu) - k_{DV} \neq 0 .$$ (13)

Some of these expressions of the 5 parameters $c_k$—see (12)—are rather cumbersome (albeit quite explicit), featuring the 7 *a priori arbitrary* parameters $k_R$, $k_B$, $k_P$, $k_{DV}$, $k_I$, $\alpha$, $\beta$ characterizing the system (4); and of course much more complicated are the—in principle easily available—*explicit* expressions of the 4 roots $\mu_n$ ($n = 1, 2, 3, 4$) of the fourth-degree equation (12a). We do not consider useful to report these formulas in this paper; since analogous—much more practical—formulas can be easily obtained from eq. (12a) in *applicative* contexts, whenever the 7 *a priori arbitrary* parameters $k_R$, $k_B$, $k_P$, $k_{DV}$, $k_I$, $\alpha$, $\beta$ have been assigned specific numerical values, entailing, via the explicit expressions of the parameters $c_k$ written above (see (12)), the corresponding numerical values of these parameters $c_k$, to be then inserted in (12a) before the standard task of solving this *quartic* equation is performed.

**Remark 2.3.** Let us finally mention that clearly, by setting

$$\mu = k_D ,$$ (14)
one is looking—see (3a) and (5)—at the equilibrium solution
\[ \ddot{x}_n(t) = \ddot{x}_n, \quad \dot{x}_n(t) = 0, \quad n = 1, 2, 3, 4, \]  
(15)
of the original pandemic system (11), as given by the formulas (implied via (7) by (10))
\[ \ddot{x}_1 = \ddot{x}_4 \left[ ak_R(k_D + k_{DV} + k_P + k_R)/(k_D k_P) \right], \]  
(16a)
\[ \ddot{x}_2 = \ddot{x}_4 \left\{ -\frac{k_D + k_{DV}}{k_P} - \frac{k_D - k_B + k_{DV}}{k_D - k_B} \right\}, \]  
(16b)
\[ \ddot{x}_3 = \ddot{x}_4 \left[ (1 + \alpha)k_D + \alpha (k_{DV} + k_P) \right]k_R + \alpha (k_R)^2 \right\}, \]  
(16c)
\[ \ddot{x}_4 = \ddot{x}_4(k_D + k_{DV} + k_R)/k_P, \]  
(16d)
where \( \ddot{x}_4 \) is of course an arbitrary parameter.

**Remark 2.4.** Note that throughout this paper we assume that the 4 roots \( \mu_n \) of the quartic algebraic equation (12a) are all different among themselves.

To conclude this Section 2, let us mention that the special solutions (3) are not very interesting in applicative contexts, since they imply that the 4 dependent variables \( x_n(t) \) all evolve in the same, very simple, manner. But fortunately—as shown below—it is also possible to identify other—presumably more interesting—explicit solutions of the system of nonlinear ODEs (4).

### 3 A less simple solution: the linear combination of 2 exponentials

In this Section 3 we investigate the following class of solutions of the system of ODEs (4):
\[ x_n(t) = a_{n1} \exp(\mu_1 t) + a_{n2} \exp(\mu_2 t), \quad n = 1, 2, 3, 4, \]  
(17a)
where \( \mu_1 \) and \( \mu_2 \) are 2 different roots of eq. (12); while corresponding values for the 8 time-independent parameters \( a_{n1} \) and \( a_{n2} \) are obtained below.

**Remark 3.1.** Since there are 4 (assumedly different) solutions \( \mu \) of the fourth-order algebraic eq. (12a), there are \( 4 \cdot 3/2 = 6 \) different assignments of the pair of values \( \mu_1, \mu_2 \). Note moreover that, even if the 7 parameters \( k_R, k_B, k_P, k_{DV}, k_1, \alpha, \beta \) of the system of ODEs (4) are all real numbers (as is of course the case in the pandemics case), the 4 solutions \( \mu_n \) of the fourth-order algebraic eq. (12a) need not be real numbers; but if the 7 parameters of the system of ODEs (4) are all real numbers, then non-real solutions of the algebraic eq. (12a) must be present in complex conjugate pairs.

**Remark 3.2.** Note that we are again assuming, throughout this Section 3, that the quantity \( f(x_1, x_2, x_3, x_4) \) in the system (4) is time-independent, hence equal to its value at the initial time \( t = 0 \) (see (6c)); although this property, which was obvious in the treatment of Section 2 (see (4c) and (5))—and which is essential to justify the existence of the subclass of solutions (17a)—is now instead far from obvious: indeed conditions for it to hold—involving the initial data of these solutions, and also 1 constraint on the parameters of the system (4) shall have to be ascertained, see below.

The 4 eqs. (17a) involve of course the following 4 relations among the 8 parameters \( a_{n1} \) and \( a_{n2} \) and the 4 initial data \( x_n(0) \):
\[ x_n(0) = a_{n1} + a_{n2}, \quad n = 1, 2, 3, 4. \]  
(17b)
The assumption (see Remark 3.2) that the function \( f(x_1, x_2, x_3, x_4) \) be time-independent implies that the system (4) is again a linear system of 4 ODEs with time-independent parameters; hence each of the 2 exponential functions in the right-hand side of the ansatz (17a) must satisfy (as it were, separately) the system (4). Therefore each of the 2 sets of 4 parameters \( a_{n1} \) and \( a_{n2} \) must satisfy the same requirements—see the 4 eqs. (10)—satisfied by the initial data \( x_n(0) \) in the treatment of the previous Section 2; namely there must now hold the 8 relations
\[ \mu_1 a_1 + \mu_2 a_2 = ak_R(a_3 + a_4) , \quad \ell = 1, 2, \]  
(18a)
\[ \mu \alpha_{2\ell} = k_B a_{1\ell} + [k_B - f(0)] a_{2\ell} + [k_B + (1 - \alpha) k_R] [a_{3\ell} + a_{4\ell}] , \quad \ell = 1, 2 , \]  
\[ \mu \alpha_{3\ell} = f(0) a_{2\ell} - (k_R + k_P) a_{3\ell} , \quad \ell = 1, 2 , \]  
\[ \mu \alpha_{4\ell} = k_P a_{3\ell} - (k_R + k_{DV}) a_{4\ell} , \quad \ell = 1, 2 ; \]  

where of course we again set the relations (17a), hence their insertion in the definition (21b) implies the following expression of parameters system (4) as follows (see (10)):

These 6 parameters — be time-independent — are defined by the eq. (17b), which are necessary in order to comply with the requirement—essential for our treatment—that the quantity \( h(t) \), related to the quantity \( f(x_1, x_2, x_3, x_4) \), see (10), by the simple relation

\[ f(x_1, x_2, x_3, x_4) = f(0) = \frac{k_1 [x_3(0) + \beta x_4(0)]}{x_1(0) + x_2(0) + x_3(0) + \beta x_4(0)} ; \]  

but now with the 4 initial data \( x_n(0) \) related to the 2 parameters \( a_{n1} \) and \( a_{n2} \) by the eq. (17b).

We can therefore now proceed in close analogy to the treatment of the previous Section 2, introducing 6 parameters \( b_{m\ell} \) via the following position:

\[ a_{m\ell} = b_{m\ell} a_{4\ell} , \quad b_{m\ell} = a_{m\ell}/a_{4\ell} , \quad m = 1, 2, 3 , \quad \ell = 1, 2 . \]  

These 6 parameters \( b_{m\ell} \) \( (m = 1, 2, 3 , \quad \ell = 1, 2) \) are then explicitly expressed in terms of the parameters of the system (4) as follows (see (10)):

\[ b_{1\ell} = \alpha k_R (\mu_\ell + k_{DV} + k_P + k_R)/(k_P \mu_\ell) , \quad \ell = 1, 2 , \]  
\[ b_{2\ell} = -\frac{\mu_\ell + k_{DV}}{k_P} - \frac{\mu_\ell - k_B + k_{DV}}{\mu_\ell - k_B} \]  
\[ - \frac{[(1 + \alpha) \mu_\ell + \alpha (k_{DV} + k_P)] k_R + \alpha (k_R)^2}{k_P \mu_\ell} , \quad \ell = 1, 2 , \]  
\[ b_{3\ell} = (\mu_\ell + k_{DV} + k_R)/k_P , \quad \ell = 1, 2 ; \]  

with \( a_{41} \) and \( a_{42} \) remaining as 2 free parameters.

To complete the treatment of this case, it is necessary to identify the constraints on the parameters of the system (4) and on the parameters of the solution under present consideration, see (17a), which are necessary in order to comply with the requirement—essential for our treatment—that the quantity \( h(t) \), related to the quantity \( f(x_1, x_2, x_3, x_4) \), see (10), by the simple relation

\[ f(x_1, x_2, x_3, x_4) = f(0) = \frac{k_1 [x_3(0) + \beta x_4(0)]}{x_1(0) + x_2(0) + x_3(0) + \beta x_4(0)} ; \]  

— hence reading as follows,

\[ h(t) = \frac{x_3(t) + \beta x_4(t)}{x_1(t) + x_2(t) + x_3(t) + \beta x_4(t)} ; \]  

— be time-independent: therefore given in terms of the initial data as follows:

\[ h(t) = h(0) = \frac{x_3(0) + \beta x_4(0)}{x_1(0) + x_2(0) + x_3(0) + \beta x_4(0)} . \]  

To fulfill this task, we now note that the solutions \( x_n(t) \) under consideration in this Section 3 are defined by the relations (17a), hence their insertion in the definition (21c) implies the following expression of \( h(t) \):

\[ h(t) = \frac{a_{31} + \beta a_{41} + (a_{32} + \beta a_{42}) \exp[(\mu_2 - \mu_1) t]}{a_{11} + a_{21} + a_{31} + \beta a_{41} + (a_{12} + a_{22} + a_{32} + \beta a_{42}) \exp[(\mu_2 - \mu_1) t]} . \]  

It is therefore easily seen that the requirement that this expression of \( h(t) \) be time-independent implies that the 8 parameters \( a_{n\ell} \) \( (n = 1, 2, 3, 4 ; \quad \ell = 1, 2) \) satisfy the following single constraint on the 8 parameters \( a_{n\ell} \):

\[ (a_{12} + a_{22}) (a_{31} + \beta a_{41}) - (a_{11} + a_{21}) (a_{32} + \beta a_{42}) = 0 ; \]  

entailing then that

\[ h(t) = h(0) = \frac{a_{31} + \beta a_{41}}{a_{11} + a_{21} + a_{31} + \beta a_{41}} = \frac{a_{32} + \beta a_{42}}{a_{12} + a_{22} + a_{32} + \beta a_{42}} . \]
By inserting the formulas (19) in (22) we then get, for the 6 parameters $b_{ml}$ $(m = 1, 2, 3; \ell = 1, 2)$, the following single constraint:

$$(b_{12} + b_{22}) (b_{31} + \beta) - (b_{11} + b_{21}) (b_{32} + \beta) = 0 ;$$

and, via (20), we finally get the following single constraint on the 7 parameters of the system (17a):

$$(1 - \beta) (k_B)^2 - k_B k_{DV} + k_{DV} (k_{DV} + \beta k_P + k_R + \mu_1)$$

$$+ [k_{DV} + (1 - \beta) \mu_1] \mu_2 - (1 - \beta) k_B (\mu_1 + \mu_2) = 0 ,$$

provided there hold the following inequalities:

$$k_P \neq 0 , \quad k_B \neq \mu_1 , \quad k_B \neq \mu_2 , \quad \mu_1 \neq \mu_2 .$$

(25a)

**Remark 3.3.** Let us recall that there are in general 6 different versions of the constraint (25a) due to the 6 different possible selections of the 2 roots $\mu_1$ and $\mu_2$ (see Remark 3.1); and that the simplicity of this formula (25a) as providing a constraint on the 7 parameters of the system (17a) is somewhat misleading, due to the (explicit but) quite complicated dependence on these parameters of the solutions $\mu_1$ and $\mu_2$ of the fourth-degree algebraic equation (12a). However—as already mentioned above—all these complicated relations (including those yielding the initial data of the class of solutions considered in this Section 3) become much more easily managed whenever any 6 of the 7 a priori arbitrary parameters featured by the system (17a) are assigned specific numerical values, so that the remaining task left is to ascertain the values of the 7-th parameter implied by the constraint (25a) (as well as those characterizing the initial data $x_n(0)$ of the class of solutions considered in this Section 3), thereby identifying the corresponding class of systems (17a) featuring the simple explicit solutions of type (17a). ■

4 Solutions which are the linear superposition of 3 or 4 exponentials

In this Section 4 we treat the subclass of solutions of the system (17a) whose time-evolution is a linear superposition of 3 or 4 exponentials.

**Remark 4.1.** Hence, throughout this Section 4, we assume that the quantity $f(x_1, x_2, x_3, x_4)$ in (17a) is time-independent; although this property is—as in Section 3: see for instance Remark 3.2—far from obvious: indeed conditions for it to hold—involving the initial data of these subclass of solutions, as well as constraints on the parameters of the system (17a)—shall have to be ascertained, see below. ■

4.1 Solutions which are the linear superposition of 3 exponentials

In this Subsection 4.1 we investigate the following class of solutions of the system of ODEs (17a):

$$x_n(t) = a_{n1} \exp(\mu_1 t) + a_{n2} \exp(\mu_2 t) + a_{n3} \exp(\mu_3 t) , \quad n = 1, 2, 3, 4 ,$$

(26a)

where $\mu_1, \mu_2, \mu_3$ are 3 different roots of the 4th-degree algebraic eq. (12a); while corresponding values for the 12 time-independent parameters $a_{n1}, a_{n2}, a_{n3}$ are obtained below.

Of course these formulas (26a) imply the following relations among the 4 initial data $x_n(0)$ and the 12 parameters $a_{nj}$ $(n = 1, 2, 3, 4; j = 1, 2, 3)$:

$$x_n(0) = a_{n1} + a_{n2} + a_{n3} , \quad n = 1, 2, 3, 4 .$$

(26b)

**Remark 4.1.1.** Clearly symbols such as $x_n, \mu_{n \ell}, \ a_{n \ell}$ need not have the same significance nor the same values when appearing in different Sections or Subsections of this paper. But of course the statements made in Remark 3.1 concerning the possibility that not all the 4 roots of the fourth-order algebraic eq. (12a) be real numbers are generally valid. ■

**Remark 4.1.2.** Since there are 4 (assumedly different) solutions $\mu$ of the fourth-order algebraic eq. (12a), there are 4 different selections—from the quartet of solutions $\mu_n$ of eq. (12a)—of the trio of values $\mu_1, \mu_2, \mu_3$ in the ansatz (26a). ■
Let us now proceed in close analogy to the treatment provided in Section 3. Again we assume that the quantity \( f \) in the right-hand sides of the ODEs \((11)\) and \((12)\) is a time-independent parameter, up to identifying below conditions on the parameters of the system \((1)\) and on the initial data of the solution \((26)\) under consideration which are sufficient to guarantee—as it were, a posteriori—that this be the case; hence that the system of ODEs \((1)\) be equivalent to a system of 4 linear ODEs, featuring independent solutions \( a \exp(\mu t) \) each depending exponentially on the independent variable \( t \) (which can therefore be added without losing the property to satisfy the system of ODEs \((1)\)).

We thus obtain—in analogy to the 8 relations \((15)\)—the following 12 relations:

\[
\mu_j a_{1j} = \alpha k_R (a_{3j} + a_{4j}) , \quad j = 1, 2, 3 ,
\]

\[
\mu_j a_{2j} = k_B a_{1j} + [k_B - f (0)] a_{2j} + [k_B + (1 - \alpha) k_R] (a_{3j} + a_{4j}) , \quad j = 1, 2, 3 ,
\]

\[
\mu_j a_{3j} = f (0) a_{2j} - (k_R + k_P) a_{3j} , \quad j = 1, 2, 3 ,
\]

\[
\mu_j a_{4j} = k_P a_{3j} - (k_R + k_{DV}) a_{4j} , \quad j = 1, 2, 3 .
\]

Next we set (in analogy to \((19)\))

\[
a_{m_j} = b_{m_j} a_{4j} , \quad b_{m_j} = a_{m_j} / a_{4j} , \quad m = 1, 2, 3 , \quad j = 1, 2, 3 ,
\]

getting thereby the following 9 relations:

\[
b_{1j} = \alpha k_R (\mu_j + k_{DV} + k_P + k_R) / (k_P \mu_j) , \quad j = 1, 2, 3 ,
\]

\[
b_{2j} = \frac{- \mu_j + k_{DV}}{k_P} - \frac{- \mu_j - k_B + k_{DV}}{\mu_j - k_B}
\]

\[
- \frac{[1 + \alpha] \mu_j + \alpha (k_{DV} + k_P) k_R + \alpha (k_R)^2 }{k_P \mu_j} , \quad j = 1, 2, 3 ,
\]

\[
b_{3j} = (\mu_j + k_{DV} + k_R) / k_P , \quad j = 1, 2, 3 ;
\]

with \(a_{41}, a_{42}, a_{43}\) remaining as 3 free parameters.

We must now investigate the restrictions on the parameters \(a_{mj}\) implied by the requirement that the quantity \(h(t)\) be time-independent. The analogous formula to \((21d)\) now reads as follows:

\[
h(t) = numh(t) / denh(t) ,
\]

\[
numh(t) = a_{31} + \beta a_{41} + (a_{32} + \beta a_{42}) \exp [(\mu_2 - \mu_1) t]
\]

\[
+ (a_{33} + \beta a_{43}) \exp [(\mu_3 - \mu_1) t] ,
\]

\[
denh(t) = a_{11} + a_{21} + a_{31} + \beta a_{41}
\]

\[
+ (a_{12} + a_{22} + a_{32} + \beta a_{42}) \exp [(\mu_2 - \mu_1) t]
\]

\[
+ (a_{13} + a_{23} + a_{33} + \beta a_{43}) \exp [(\mu_3 - \mu_1) t] .
\]

It is then easily seen that the condition \((22)\) is now replaced by the following 2 restrictions:

\[
(a_{11} + a_{21}) (a_{3k} + \beta a_{4k}) = (a_{1k} + a_{2k}) (a_{31} + \beta a_{41}) , \quad k = 2, 3 .
\]

Hence, after the change of parameters \((28)\), we get the following 2 constraints,

\[
(b_{11} + b_{21}) (b_{3k} + \beta) = (b_{1k} + b_{2k}) (b_{31} + \beta) , \quad k = 2, 3 ,
\]

on the 9 parameters \(b_{mj}\) (\(m = 1, 2, 3 ; \ j = 1, 2, 3\)). Which, via the expressions \((29)\) of these 9 parameters, entail the following 2 constraints on the original 7 parameters of the system \((1)\):

\[
(1 - \beta) (k_B)^2 + (k_{DV})^2 + (1 - \beta) \mu_1 \mu_2 + k_{DV} (\beta k_P + k_R + \mu_1 + \mu_2)
\]

\[
- k_B [k_{DV} + (1 - \beta) (\mu_1 + \mu_2)] = 0 ,
\]
\[ k_{DV} (k_{DV} + \beta k_P + k_R + \mu_1) + [k_{DV} + (1 - \beta) \mu_1] \mu_3 \\
- (1 - \beta) k_B (\mu_1 + \mu_3) + k_B [(1 - \beta) k_B - k_{DV}] = 0. \quad (33b) \]

While of course the initial data \( x_q (0) \) of the solution under consideration in this **Subsection 4.1** are explicitly given by the formulas \( 26b \) with \( 28 \) and \( 29 \).

### 4.2 Solutions which are the linear superposition of 4 exponentials

The treatment in this **Subsection 4.2** is quite terse, since it is quite analogous to that provided above in **Subsection 4.1**; hence we only report the key formulas which play an analogous role to the key formulas in **Subsection 4.1**.

Instead of \( 26 \) we now have

\[
x_n (t) = \sum_{q=1}^{4} [a_{nq} \exp (\mu_q t)] , \quad n = 1, 2, 3, 4 , \quad (34a)\]

\[
x_n (0) = \sum_{q=1}^{4} (a_{nq}) , \quad n = 1, 2, 3, 4 . \quad (34b)\]

In place of the 12 equations \( 27 \) we now have the following 16 equations:

\[ \mu_q a_{1q} = \alpha k_R (a_{3q} + a_{4q}) , \quad q = 1, 2, 3, 4 , \quad (35a) \]

\[ \mu_q a_{2q} = k_B a_{1q} + [k_B - f (0)] a_{2q} + [k_B + (1 - \alpha) k_R] (a_{3q} + a_{4q}) , \quad q = 1, 2, 3, 4 , \quad (35b) \]

\[ \mu_q a_{3q} = f (0) a_{2q} - (k_R + k_P) a_{3q} , \quad q = 1, 2, 3, 4 , \quad (35c) \]

\[ \mu_q a_{4q} = k_P a_{3q} - (k_R + k_{DV}) a_{4q} , \quad q = 1, 2, 3, 4 . \quad (35d) \]

Likewise, in place of the 9 equations \( 28 \) we now write the 12 relations

\[ a_{jq} = b_{jq} a_{4j} \quad j = 1, 2, 3 , \quad q = 1, 2, 3, 4 , \quad (36) \]

getting thereby from \( 35 \), the following 12 relations (analogous to \( 29 \)):

\[ b_{1q} = \alpha k_R (\mu_q + k_{DV} + k_P + k_R)/(k_P \mu_q) , \quad q = 1, 2, 3, 4 , \quad (37a) \]

\[ b_{2q} = \frac{- \mu_q + k_{DV}}{k_P} - \frac{\mu_q - k_B + k_{DV}}{\mu_q - k_B} \]

\[ - \frac{[(1 + \alpha) \mu_q + \alpha (k_{DV} + k_P)] k_R + \alpha (k_R)^2}{k_P \mu_q} , \quad q = 1, 2, 3, 4 , \quad (37b) \]

\[ b_{3q} = (\mu_q + k_{DV} + k_P)/k_P , \quad q = 1, 2, 3, 4 ; \quad (37c) \]

with \( a_{41}, a_{42}, a_{43}, a_{44} \) remaining as 4 free parameters.

Next come the constraints on the parameters of the system \( 34 \) needed in order that the more general solution \( 34a \), when inserted in the definition \( 44 \) of the function \( f (t) \), hence now reading

\[
f (t) = \frac{k_1 \sum_{q=1}^{4} [(a_{3q} + \beta a_{4q}) \exp (\mu_q t)]}{\sum_{q=1}^{4} [(a_{1q} + a_{2q} + a_{3q} + \beta a_{4q}) \exp (\mu_q t)]} , \quad (38a)\]

—or, equivalently, see \( 36 \)—

\[
f (t) = \frac{k_1 \sum_{q=1}^{4} [(b_{3q} + 1) \exp (\mu_q t)]}{\sum_{q=1}^{4} [(b_{1q} + b_{2q} + b_{3q} + \beta) \exp (\mu_q t)]} , \quad (38b)\]
be *time-independent*.

And since it is easily seen that

\[
f(t) = \frac{k_f (b_{31} + 1) \varphi(t)}{(b_{11} + b_{21} + b_{31} + \beta)},
\]

(39a)

with

\[
\varphi(t) = \frac{1 + \sum_{q=2}^{q=4} \left\{ \left( \frac{b_{3q} + 1}{b_{31} + 1} \right) \exp \left[ (\mu_q - \mu_1) t \right] \right\}}{1 + \sum_{q=2}^{q=4} \left\{ \left( \frac{b_{1q} + b_{2q} + b_{3q} + \beta}{b_{11} + b_{21} + b_{31} + \beta} \right) \exp \left[ (\mu_q - \mu_1) t \right] \right\}},
\]

(39b)

the requirement that \( f(t) \) be *time-independent* amounts to the following 3 constraints:

\[
\frac{b_{3q} + 1}{b_{31} + 1} = \frac{b_{1q} + b_{2q} + b_{3q} + \beta}{b_{11} + b_{21} + b_{31} + \beta}, \quad q = 2, 3, 4,
\]

(40)

which clearly entail—via the expressions (29)—3 corresponding *constraints* on the parameters of the original model (4).

5 Concluding remarks

In this paper we have identified certain solutions of the pandemic model introduced in the paper [1]; these solutions—and the constraints on the parameters of the model required for their validity—are all identified by *algebraic* equations which can be *explicitly* solved; we did not report the corresponding explicit formulas because they are so complicated to be hardly useful when written for *a priori arbitrary* assignments of the parameters of the pandemic model, while they can instead be easily managed for any *specific numerical* assignment of these parameters. We therefore leave the utilization of these findings to the interested pandemics experts.

Additional solutions—more special but perhaps displaying more interesting evolutions—correspond to the special cases in which the algebraic quartic-equation (12) features 4 roots \( \mu_n \) which are *not* all different among themselves. This case shall perhaps be eventually treated in a separate paper by ourselves or others.

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