Quasi bi-slant submersions in contact geometry

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ABSTRACT
The aim of the paper is to introduce the concept of quasi bi-slant submersions from almost contact metric manifolds onto Riemannian manifolds as a generalization of semi-slant and hemi-slant submersions. We mainly focus on quasi bi-slant submersions from cosymplectic manifolds. We give some non-trivial examples and study the geometry of leaves of distributions which are involved in the definition of the submersion. Moreover, we find some conditions for such submersions to be integrable and totally geodesic.

RESUMEN
El objetivo de este artículo es introducir el concepto de submersiones cuasi bi-inclinadas desde variedades casi contacto métricas hacia variedades Riemannianas, como una generalización de submersiones semi-inclinadas y hemi-inclinadas. Principalmente nos enfocamos en submersiones cuasi bi-inclinadas desde variedades cosimplécticas. Damos algunos ejemplos no triviales y estudiamos la geometría de hojas de distribuciones que están involucradas en la definición de la submersión. Más aún, encontramos algunas condiciones para que estas submersiones sean integrables y totalmente geodesicas.

Keywords and Phrases: Riemannian submersion, semi-invariant submersion, bi-slant submersion, quasi bi-slant submersion, horizontal distribution.

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1 Introductions

In differential geometry, there are so many important applications of immersions and submersions both in mathematics and in physics. The properties of slant submersions became an interesting subject in differential geometry, both in complex geometry and in contact geometry.

In 1966 and 1967, the theory of Riemannian submersions was initiated by O'Neill [17] and Gray [11] respectively. Nowadays, Riemannian submersions are of great interest not only in mathematics, but also in theoretical physics, owing to their applications in the Yang-Mills theory, Kaluza-Klein theory, supergravity and superstring theories (see [7, 8, 10, 13, 14]). In 1976, the almost complex type of Riemannian submersions was studied by Watson [29]. He also introduced almost Hermitian submersions between almost Hermitian manifolds requiring that such Riemannian submersions are almost complex maps. In 1985, D. Chinea [9] extended the notion of almost Hermitian submersion to several kinds of sub-classes of almost contact manifolds. In [4] and [5], there are so many important and interesting results about Riemannian and almost Hermitian submersions. In 2010, B. Şahin introduced anti invariant submersions from almost Hermitian manifolds onto Riemannian manifolds [25]. Inspired by B. Şahin’s article, many geometers introduced several new types of Riemannian submersions in different ambient spaces such as semi-invariant submersion [21, 23], generic submersion [27], slant submersion [12, 22], hemi-slant submersion [28], semi-slant submersion [18], bi-slant submersion [26], quasi hemi-slant submersion [16], quasi bi-slant submersion [19, 20], conformal anti-invariant submersion [1], conformal slant submersion [2] and conformal semi-slant submersion [3, 15]. Also, these kinds of submersions were considered in different kinds of structures such as cosymplectic, Sasakian, Kenmotsu, nearly Kaehler, almost product, para-contact, etc. Recent developments in the theory of submersions can be found in the book [24].

Inspired from the good and interesting results of above studies, we introduce the notion of quasi bi-slant submersions from cosymplectic manifolds onto Riemannian manifolds.

The paper is organized as follows: In the second section, we gather some basic definitions related to quasi bi-slant Riemannian submersion. In the third section, we obtain some results on quasi bi-slant Riemannian submersions from a cosymplectic manifold onto a Riemannian manifold. We also study the geometry of the leaves of the distributions involved in the considered submersions and discuss their totally geodesicity. We obtain conditions for the fibres or the horizontal distribution to be totally geodesic. In the last section, we provide some examples for such submersions.
2 Preliminaries

An $n$–dimensional smooth manifold $M$ is said to have an almost contact structure, if there exist on $M$, a tensor field $\phi$ of type $(1, 1)$, a vector field $\xi$ and 1–form $\eta$ such that:

\begin{align}
\phi^2 & = -I + \eta \otimes \xi, \quad \phi \xi = 0, \quad \eta \circ \phi = 0, \\
\eta(\xi) & = 1.
\end{align}

There exists a Riemannian metric $g$ on an almost contact manifold $M$ satisfying the next conditions:

\begin{align}
g(\phi U, \phi V) & = g(U, V) - \eta(U)\eta(V), \\
g(U, \xi) & = \eta(U),
\end{align}

where $U, V$ are vector fields on $M$.

An almost contact structure $(\phi, \xi, \eta)$ is said to be normal if the almost complex structure $J$ on the product manifold $M \times \mathbb{R}$ is given by

\begin{equation}
J \left( U, \frac{d}{dt} \right) = \left( \phi U - \alpha \xi, \eta(U)\frac{d}{dt} \right)
\end{equation}

and $\alpha$ is the differentiable function on $M \times \mathbb{R}$ has no torsion, i.e., $J$ is integrable. The condition for normality in terms of $\phi$, $\xi$, and $\eta$ is $[\phi, \phi] + 2d\eta \otimes \xi = 0$ on $M$, where $[\phi, \phi]$ is the Nijenhuis tensor of $\phi$. Finally, the fundamental 2–form $\Phi$ is defined by $\Phi(U, V) = g(U, \phi V)$.

An almost contact metric manifold with almost contact structure $(\phi, \xi, \eta, g)$ is said to be cosymplectic if

\begin{equation}
(\nabla_U \phi)V = 0,
\end{equation}

for any $U, V$ on $M$.

It is both normal and closed and the structure equation of a cosymplectic manifold is given by

\begin{equation}
\nabla_U \xi = 0,
\end{equation}

for any $U$ on $M$, where $\nabla$ denotes the Riemannian connection of the metric $g$ on $M$.

**Example 2.1** ([6]). $\mathbb{R}^{2n+1}$ with Cartesian coordinates $(x_i, y_i, z)(i = 1, \ldots, n)$ and its usual contact form

$$
\eta = dz.
$$

The characteristic vector field $\xi$ is given by $\frac{\partial}{\partial z}$ and its Riemannian metric $g$ and tensor field $\phi$ are given by

\begin{equation}
g = \sum_{i=1}^{n} ((dx_i)^2 + (dy_i)^2) + (dz)^2, \quad \phi = \begin{pmatrix}
0 & \delta_{ij} & 0 \\
-\delta_{ij} & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}, \quad i = 1, \ldots, n.
\end{equation}
This gives a cosymplectic manifold on $\mathbb{R}^{2n+1}$. The vector fields $e_i = \frac{\partial}{\partial y_i}$, $e_{n+i} = \frac{\partial}{\partial x_i}$, $\xi$ form a $\phi$-basis for the cosymplectic structure.

Before giving our definition, we recall the following definition:

**Definition 2.2** ([28]). Let $M$ be an almost Hermitian manifold with Hermitian metric $g_M$ and almost complex structure $J$, and let $N$ be a Riemannian manifold with Riemannian metric $g_N$. A Riemannian submersion $f : (M, g_M, J) \to (N, g_N)$ is called a hemi-slant submersion if the vertical distribution $\ker f^*$ of $f$ admits two orthogonal complementary distributions $D^0$ and $D^\perp$ such that $D^0$ is slant with angle $\theta$ and $D^\perp$ is anti-invariant, i.e., we have

$$\ker f^* = D^0 \oplus D^\perp.$$ 

In this case, the angle $\theta$ is called the hemi-slant angle of the submersion.

**Definition 2.3.** Let $(M, \phi, \xi, \eta, g_M)$ be an almost contact metric manifold and $(N, g_N)$ a Riemannian manifold. A Riemannian submersion

$$f : (M, \phi, \xi, \eta, g_M) \to (N, g_N),$$

is called a quasi bi-slant submersion if there exist four mutually orthogonal distributions $D, D_1, D_2$ and $<\xi>$ such that

(i) $\ker f^* = D \oplus D_1 \oplus D_2 \oplus <\xi>$,

(ii) $\phi(D) = D$ i.e., $D$ is invariant,

(iii) $\phi(D_1) \perp D_2$ and $\phi(D_2) \perp D_1$,

(iv) for any non-zero vector field $U \in (D_1)_p$, $p \in M$, the angle $\theta_1$ between $\phi U$ and $(D_1)_p$ is constant and independent of the choice of the point $p$ and $U$ in $(D_1)_p$,

(v) for any non-zero vector field $U \in (D_2)_q$, $q \in M$, the angle $\theta_2$ between $\phi U$ and $(D_2)_q$ is constant and independent of the choice of point $q$ and $U$ in $(D_2)_q$.

These angles $\theta_1$ and $\theta_2$ are called the slant angles of the submersion.

We easily observe that

(a) If $\dim D \neq 0$, $\dim D_1 = 0$ and $\dim D_2 = 0$, then $f$ is an invariant submersion.

(b) If $\dim D \neq 0$, $\dim D_1 \neq 0$, $0 < \theta_1 < \frac{\pi}{2}$ and $\dim D_2 = 0$, then $f$ is proper semi-slant submersion.

(c) If $\dim D = 0$, $\dim D_1 \neq 0$, $0 < \theta_1 < \frac{\pi}{2}$ and $\dim D_2 = 0$, then $f$ is slant submersion with slant angle $\theta_1$. 

(d) If \( \dim D = 0, \dim D_1 = 0 \) and \( \dim D_2 \neq 0, 0 < \theta_2 < \frac{\pi}{2} \), then \( f \) is slant submersion with slant angle \( \theta_2 \).

(e) If \( \dim D = 0, \dim D_1 \neq 0, \theta_1 = \frac{\pi}{2} \) and \( \dim D_2 = 0 \), then \( f \) is an anti-invariant submersion.

(f) If \( \dim D \neq 0, \dim D_1 \neq 0, \theta_1 = \frac{\pi}{2} \) and \( \dim D_2 = 0 \), then \( f \) is an semi-invariant submersion.

(g) If \( \dim D = 0, \dim D_1 \neq 0, 0 < \theta_1 < \frac{\pi}{2} \) and \( \dim D_2 \neq 0, \theta_2 = \frac{\pi}{2} \), then \( f \) is a hemi-slant submersion.

(h) If \( \dim D \neq 0, \dim D_1 \neq 0, 0 < \theta_1 < \frac{\pi}{2} \) and \( \dim D_2 \neq 0, 0 < \theta_2 < \frac{\pi}{2} \), then \( f \) is a bi-slant submersion.

(i) If \( \dim D \neq 0, \dim D_1 \neq 0, 0 < \theta_1 < \frac{\pi}{2} \) and \( \dim D_2 \neq 0, \theta_2 = \frac{\pi}{2} \), then we may call \( f \) is an quasi-hemi-slant submersion.

(j) If \( \dim D \neq 0, \dim D_1 \neq 0, 0 < \theta_1 < \frac{\pi}{2} \) and \( \dim D_2 \neq 0, 0 < \theta_2 < \frac{\pi}{2} \), then \( f \) is proper quasi bi-slant submersion.

Define O’Neill’s tensors \( \mathcal{T} \) and \( \mathcal{A} \) by

\[
\mathcal{A}_E F = \mathcal{H} \nabla_H E \nabla F + \nabla \nabla_H \mathcal{H} F, \tag{2.8}
\]

\[
\mathcal{T}_E F = \mathcal{H} \nabla_V E \nabla F + \nabla \nabla_V \mathcal{H} F, \tag{2.9}
\]

for any vector fields \( E, F \) on \( M \), where \( \nabla \) is the Levi-Civita connection of \( g_M \). It is easy to see that \( \mathcal{T}_E \) and \( \mathcal{A}_E \) are skew-symmetric operators on the tangent bundle of \( M \) reversing the vertical and the horizontal distributions.

From equations (2.8) and (2.9) we have

\[
\nabla_U V = \mathcal{T}_U V + \mathcal{A}_V \nabla_U V, \tag{2.10}
\]

\[
\nabla_U X = \mathcal{T}_U X + \mathcal{A}_X \nabla_U X, \tag{2.11}
\]

\[
\nabla_X U = \mathcal{A}_X U + \mathcal{A}_U \nabla_X U, \tag{2.12}
\]

\[
\nabla_X Y = \mathcal{H} \nabla_X Y + \mathcal{A}_X Y, \tag{2.13}
\]

for \( U, V \in \Gamma(\ker f_* ) \) and \( X, Y \in \Gamma(\ker f_* )^\perp \), where \( \mathcal{H} \nabla_X Y = \mathcal{A}_Y U \), if \( Y \) is basic. It is not difficult to observe that \( \mathcal{T} \) acts on the fibers as the second fundamental form, while \( \mathcal{A} \) acts on the horizontal distribution and measures the obstruction to the integrability of this distribution.

It is seen that for \( q \in M, U \in \mathcal{V}_q \) and \( X \in \mathcal{H}_q \) the linear operators

\[
\mathcal{A}_X, \mathcal{T}_U : T_q M \rightarrow T_q M
\]
are skew-symmetric, that is
\[ g_M(A_X E, F) = -g_M(E, A_X F) \quad \text{and} \quad g_M(T_U E, F) = -g_M(E, T_U F) \] (2.14)
for each \( E, F \in T_q M \). Since \( T_U \) is skew-symmetric, we observe that \( f \) has totally geodesic fibres if and only if \( T \equiv 0 \).

Let \((M, \phi, \xi, \eta, g_M)\) be a cosymplectic manifold, \((N, g_N)\) be a Riemannian manifold and \( f : M \to N \) a smooth map. Then the second fundamental form of \( f \) is given by
\[
(\nabla f^*)(Y, Z) = \nabla_Y f^* Z - f^*(\nabla_Y Z), \quad \text{for } Y, Z \in \Gamma(TM),
\] (2.15)
where we denote conveniently by \( \nabla \) the Levi-Civita connections of the metrics \( g_M \) and \( g_N \) and \( \nabla^f \) is the pullback connection.

We recall that a differentiable map \( f \) between two Riemannian manifolds is totally geodesic if
\[
(\nabla f^*)(Y, Z) = 0, \quad \text{for all } Y, Z \in \Gamma(TM).
\]
A totally geodesic map is that it maps every geodesic in the total space into a geodesic in the base space in proportion to arc lengths.

### 3 Quasi bi-slant submersions

Let \( f \) be quasi bi-slant submersion from an almost contact metric manifold \((M, \phi, \xi, \eta, g_M)\) onto a Riemannian manifold \((N, g_N)\). Then, we have
\[
TM = \ker f_* \oplus (\ker f_*)^\perp. \tag{3.1}
\]
Now, for any vector field \( U \in \Gamma(\ker f_*) \), we put
\[
U = PU + QU + RU + \eta(U)\xi, \tag{3.2}
\]
where \( P, Q \) and \( R \) are projection morphisms of \( \ker f_* \) onto \( D, D_1 \) and \( D_2 \), respectively. For any \( U \in \Gamma(\ker f_*) \), we set
\[
\phi U = \psi U + \omega U, \tag{3.3}
\]
where \( \psi U \in \Gamma(\ker f_*) \) and \( \omega U \in \Gamma(\ker f_*)^\perp \).

Now, let \( U_1, U_2 \) and \( U_3 \) be vector fields in \( D, D_1 \) and \( D_2 \) respectively. Since \( D \) is invariant, i.e. \( \phi D = D \), we get \( \omega U_1 = 0 \). For any \( U_2 \in \Gamma(D_1) \) we get \( \omega U_2 \in \Gamma(\omega D_1) \) and for any \( U_3 \in \Gamma(D_2) \) we get \( \omega U_3 \in \Gamma(\omega D_2) \), hence \( \omega U_2 \oplus \omega U_3 \in \Gamma(\omega D_1 \oplus \omega D_2) \subseteq \Gamma(\ker f_*)^\perp \).

From equations (3.2) and (3.3), we have
\[
\phi U = \phi(PU) + \phi(QU) + \phi(RU), \quad \psi(PU) + \omega(QU) + \omega(RU) + \omega(RU).
\]
Since $\phi D = D$, we get $\omega PU = 0$.

Hence above equation reduces to

$$
\phi U = \psi PU + \psi QU + \omega QU + \psi RU + \omega RU.
$$

Thus we have the following decomposition according to equation (3.4)

$$
\phi(\ker f^*) = (\psi D) \oplus (\psi D_1 \oplus \psi D_2) \oplus (\omega D_1 \oplus \omega D_2),
$$

where $\oplus$ denotes orthogonal direct sum.

Further, let $U \in \Gamma(D_1)$ and $V \in \Gamma(D_2)$. Then

$$
g_M(U, V) = 0.
$$

From Definition 2.3 (iii), we have

$$
g_M(\phi U, V) = g_M(U, \phi V) = 0.
$$

Now, consider

$$
g_M(\psi U, V) = g_M(\phi U - \omega U, V) = g_M(\phi U, V) = 0.
$$

Similarly, we have

$$
g_M(U, \psi V) = 0.
$$

Let $W \in \Gamma(D)$ and $U \in \Gamma(D_1)$. Then we have

$$
g_M(\psi U, W) = g_M(\phi U - \omega U, W) = g_M(\phi U, W) = -g(U, \phi W) = 0,
$$

as $D$ is invariant, i.e., $\phi W \in \Gamma(D)$.

Similarly, for $W \in \Gamma(D)$ and $V \in \Gamma(D_2)$, we obtain

$$
g_M(\psi V, W) = 0,
$$

From above equations, we have

$$
g_M(\psi U, \psi V) = 0,
$$

and

$$
g_M(\omega U, \omega V) = 0,
$$

for all $U \in \Gamma(D_1)$ and $V \in \Gamma(D_2)$.

So, we can write

$$
\psi D_1 \cap \psi D_2 = \{0\}, \quad \omega D_1 \cap \omega D_2 = \{0\}.
$$

If $\theta_2 = \frac{\pi}{2}$, then $\psi R = 0$ and $D_2$ is anti-invariant, i.e., $\phi(D_2) \subseteq (\ker f^*)_\perp$.
We also have
\[ \phi(\ker f) = \psi D \oplus \psi D_1 \oplus \omega D_1 \oplus \omega D_2. \]  
\hspace{1cm} (3.6)

Since \( \omega D_1 \subseteq (\ker f)^\perp, \omega D_2 \subseteq (\ker f)^\perp\). So we can write
\[ (\ker f)^\perp = \omega D_1 \oplus \omega D_2 \oplus V, \]
where \( V \) is invariant and orthogonal complement of \( (\omega D_1 \oplus \omega D_2) \) in \( (\ker f)^\perp \).

Also for any non-zero vector field \( W \in \Gamma((\ker f)^\perp) \), we have
\[ \phi W = BW + CW, \]  
\hspace{1cm} (3.7)

where \( BW \in \Gamma(\ker f) \) and \( CW \in \Gamma(V) \).

**Lemma 3.1.** Let \( f \) be a quasi bi-slant submersion from an almost contact metric manifold \((M, \phi, \xi, \eta, g_M)\) onto a Riemannian manifold \((N, g_N)\). Then, we have
\[ \psi^2 U + B \omega U = -U + \eta(U) \xi, \quad \omega \psi U + C \omega U = 0, \]
\[ \omega BW + C^2 W = -W, \quad \psi BW + BC W = 0, \]
for all \( U \in \Gamma(\ker f) \) and \( W \in \Gamma((\ker f)^\perp) \).

**Lemma 3.2.** Let \( f \) be a quasi bi-slant submersion from an almost contact metric manifold \((M, \phi, \xi, \eta, g_M)\) onto a Riemannian manifold \((N, g_N)\). Then, we have
\[ (i) \quad \psi^2 U = -\left(\cos^2 \theta_1\right) U, \]
\[ (ii) \quad g_M(\psi U, \psi V) = \cos^2 \theta_1 g_M(U, V), \]
\[ (iii) \quad g_M(\omega U, \omega V) = \sin^2 \theta_1 g_M(U, V), \]
for all \( U, V \in \Gamma(D_1) \).

**Lemma 3.3.** Let \( f \) be a quasi bi-slant submersion from a contact metric manifold \((M, \phi, \xi, \eta, g_M)\) onto a Riemannian manifold \((N, g_N)\). Then, we have
\[ (i) \quad \psi^2 W = -\left(\cos^2 \theta_2\right) W, \]
\[ (ii) \quad g_M(\psi W, \psi Z) = \cos^2 \theta_2 g_M(W, Z), \]
\[ (iii) \quad g_M(\omega W, \omega Z) = \sin^2 \theta_2 g_M(W, Z), \]
for all \( W, Z \in \Gamma(D_2) \).
Lemma 3.4. Let \( f \) be a quasi bi-slant submersion from a cosymplectic manifold \((M, \phi, \xi, \eta, g_M)\) onto a Riemannian manifold \((N, g_N)\). Then, we have

\[
\begin{align*}
\nabla_U \psi V + T_U \omega V & = \psi \nabla_U V + B T_U V, \quad (3.8) \\
T_U \psi V + H \nabla_U \omega V & = \omega \psi \nabla_U V + C T_U V, \quad (3.9) \\
V \nabla_X BY + A_X CY & = \psi A_X Y + B H \nabla_X Y, \quad (3.10) \\
A_X BY + H \nabla_X CY & = \omega A_X Y + C H \nabla_X Y, \quad (3.11) \\
V \nabla_U BX + T_U CX & = \psi T_U X + B H \nabla_U X, \quad (3.12) \\
T_U BX + H \nabla_U CX & = \omega T_U X + C H \nabla_U X, \quad (3.13) \\
V \nabla_Y \psi U + A_Y \omega U & = B A_Y U + \psi \nabla_Y U, \quad (3.14) \\
A_Y \psi U + H \nabla_Y \omega U & = C A_Y U + \omega \nabla_Y U, \quad (3.15)
\end{align*}
\]

for any \( U, V \in \Gamma(ker f^*) \) and \( X, Y \in \Gamma(ker f^*)^\perp \).

Now, we define

\[
\begin{align*}
(\nabla_U \psi) V & = \nabla_U \psi V - \psi \nabla_U V, \quad (3.16) \\
(\nabla_U \omega) V & = H \nabla_U \omega V - \omega \nabla_U V, \quad (3.17) \\
(\nabla_X C) Y & = H \nabla_X CY - C H \nabla_X Y, \quad (3.18) \\
(\nabla_X B) Y & = \nabla_X BY - B H \nabla_X Y, \quad (3.19)
\end{align*}
\]

for any \( U, V \in \Gamma(ker f^*) \) and \( X, Y \in \Gamma(ker f^*)^\perp \).

Lemma 3.5. Let \( f \) be a quasi bi-slant submersion from a cosymplectic manifold \((M, \phi, \xi, \eta, g_M)\) onto a Riemannian manifold \((N, g_N)\). Then, we have

\[
\begin{align*}
(\nabla_U \psi) V & = B T_U V - T_U \omega V, \\
(\nabla_U \omega) V & = C T_U V - T_U \psi V, \\
(\nabla_X C) Y & = \omega A_X Y - A_X BY, \\
(\nabla_X B) Y & = \psi A_X Y - A_X CY,
\end{align*}
\]

for any vectors \( U, V \in \Gamma(ker f^*) \) and \( X, Y \in \Gamma(ker f^*)^\perp \).

The proofs of above Lemmas follow from straightforward computations, so we omit them.

If the tensors \( \psi \) and \( \omega \) are parallel with respect to the linear connection \( \nabla \) on \( M \) respectively, then

\[
B T_U V = T_U \omega V,
\]

and

\[
C T_U V = T_U \psi V,
\]

for any \( U, V \in \Gamma(TM) \).
Lemma 3.6. Let $f$ be a quasi bi-slant submersion from a cosymplectic manifold $(M,\phi,\xi,\eta,g_M)$ onto a Riemannian manifold $(N,g_N)$. Then, we have

(i) $g_M(\nabla_X Y, \xi) = 0$ for all $X,Y \in \Gamma(D \oplus D_1 \oplus D_2)$,

(ii) $g_M([X,Y], \xi) = 0$ for all $X,Y \in \Gamma(D \oplus D_1 \oplus D_2)$.

Proof. Let $X,Y \in \Gamma(D \oplus D_1 \oplus D_2)$, consider

$$\nabla_X \{g_M(Y, \xi)\} = (\nabla_X g_M)(Y, \xi) + g_M(\nabla_X Y, \xi) + g_M(Y, \nabla_X \xi).$$

Since $X$ and $Y$ are orthogonal to $\xi$ i.e. $g_M(\nabla_X Y, \xi) = -g_M(Y, \nabla_X \xi)$,

using equation (2.7) and the property that metric tensor is $\nabla$-parallel, we have both results of this lemma.

Theorem 3.7. Let $f$ be a proper quasi bi-slant submersion from a cosymplectic manifold $(M,\phi,\xi,\eta,g_M)$ onto a Riemannian manifold $(N,g_N)$. Then, the invariant distribution $D$ is integrable if and only if

$$g_M(\mathcal{T}_V \psi U - \mathcal{T}_U \psi V, \omega QW + \omega RW) = g_M(V \nabla_U \psi V - V \nabla_V \psi U, \psi QW + \psi RW),$$

for $U,V \in \Gamma(D)$ and $W \in \Gamma(D_1 \oplus D_2)$.

Proof. For $U,V \in \Gamma(D)$, and $W \in \Gamma(D_1 \oplus D_2)$, using equations (2.1)-(2.4), (2.6), (2.10), (3.2), (3.3) and Lemma 3.6 we have

$$g_M([U,V],W) = g_M(\nabla_U \phi V, \phi W) + \eta(W)\eta(\nabla_U V) - g_M(\nabla_V \phi U, \phi W) - \eta(W)\eta(\nabla_V U),$$

$$= g_M(\nabla_U \psi V, \phi W) - g_M(\nabla_V \psi U, \phi W),$$

$$= g_M(\mathcal{T}_U \psi V - \mathcal{T}_V \psi U, \omega QW + \omega RW) - g_M(V \nabla_U \psi V - V \nabla_V \psi U, \psi QW + \psi RW),$$

which completes the proof.

Theorem 3.8. Let $f$ be a proper quasi bi-slant submersion from a cosymplectic manifold $(M,\phi,\xi,\eta,g_M)$ onto a Riemannian manifold $(N,g_N)$. Then, the slant distribution $D_1$ is integrable if and only if

$$g_M(\mathcal{T}_W \omega Z - \mathcal{T}_Z \omega W, X) = g_M(\mathcal{T}_W \omega Z - \mathcal{T}_Z \omega W, \phi PX + \psi RX)$$

$$+ g_M(\mathcal{H} \nabla_W \omega Z - \mathcal{H} \nabla_Z \omega W, \omega RX),$$

for all $W,Z \in \Gamma(D_1)$ and $X \in \Gamma(D \oplus D_2)$. 
Proof. For all $W, Z \in \Gamma(D_1)$ and $X \in \Gamma(D \oplus D_2)$, we have
\[
g_M([W, Z], X) = g_M(\nabla_W Z, X) - g_M(\nabla_Z W, X).
\]
Using equations (2.1)–(2.4), (2.6), (2.7), (2.11), (3.2), (3.3) and Lemma 3.2 we have
\[
g_M([W, Z], X) = g_M(\nabla_W \phi Z, \phi X) - g_M(\nabla_Z \phi W, \phi X)
\]
\[
= \cos^2 \theta_1 g_M(\nabla_W Z, X) - \cos^2 \theta_1 g_M(\nabla_Z W, X) - g_M(\mathcal{T}_W \omega Z - \mathcal{T}_Z \omega W, X)
\]
\[
+ g_M(\mathcal{H}_W \omega Z - \mathcal{H}_Z \omega W, \phi P X + \psi R X + \omega R X)
\]
\[
- g_M(\mathcal{H}_Z \omega W + \mathcal{T}_Z \omega W, \phi P X + \psi R X + \omega R X).
\]
Now, we obtain
\[
\sin^2 \theta_1 g_M([W, Z], X) = g_M(\mathcal{T}_W \omega Z - \mathcal{T}_Z \omega W, \phi P X + \psi R X + \omega R X)
\]
\[
- g_M(\mathcal{H}_W \omega Z - \mathcal{H}_Z \omega W, \phi P X + \psi R X + \omega R X),
\]
which completes the proof.

Theorem 3.9. Let $f$ be a proper quasi bi-slant submersion from a cosymplectic manifold $(M, \phi, \xi, \eta, g_M)$ onto a Riemannian manifold $(N, g_N)$. Then, the slant distribution $D_2$ is integrable if and only if
\[
g_M(\mathcal{T}_U \omega \psi V - \mathcal{T}_V \omega \psi U, Y) = g_M(\mathcal{H}_U \omega V - \mathcal{H}_V \omega U, \omega Q Y)
\]
\[
+ g_M(\mathcal{T}_U \omega V - \mathcal{T}_V \omega U, \phi P Y + \psi Q Y),
\]
(3.22)
for all $U, V \in \Gamma(D_2)$ and $Y \in \Gamma(D \oplus D_1)$.

Proof. For all $U, V \in \Gamma(D_2)$ and $Y \in \Gamma(D \oplus D_1)$, using equations (2.1)–(2.4), (2.6), (2.7), (3.3) and Lemma 3.6 we have
\[
g_M([U, V], Y) = g_M(\nabla_U \psi V, \phi Y) + g_M(\nabla_U \omega V, \phi Y) - g_M(\nabla_V \psi U, \phi Y) - g_M(\nabla_V \omega U, \phi Y).
\]
From equations (2.9), (3.2) and Lemma 3.3 we have
\[
g_M([U, V], Y) = \cos^2 \theta_2 g_M([U, V], Y) + g_M(\mathcal{H}_U \omega V - \mathcal{H}_V \omega U, \omega Q Y)
\]
\[
+ g_M(\mathcal{T}_U \omega V - \mathcal{T}_V \omega U, \phi P Y + \psi Q Y) - g_M(\mathcal{T}_U \omega \psi V - \mathcal{T}_V \omega \psi U, Y).
\]
Now, we have
\[
\sin^2 \theta_2 g_M([U, V], Y) = g_M(\mathcal{T}_U \omega V - \mathcal{T}_V \omega U, \phi P Y + \psi Q Y) - g_M(\mathcal{T}_U \omega \psi V - \mathcal{T}_V \omega \psi U, Y)
\]
\[
+ g_M(\mathcal{H}_U \omega V - \mathcal{H}_V \omega U, \omega Q Y),
\]
which the proof follows from the above equations.
Theorem 3.10. Let $f$ be a proper quasi bi-slant submersion from a cosymplectic manifold $(M, \phi, \xi, \eta, g_M)$ onto a Riemannian manifold $(N, g_N)$. Then the horizontal distribution $(\ker f^\ast)^\perp$ defines a totally geodesic foliation on $M$ if and only if

$$g_M(A_UV, PW + \cos^2 \theta_1 QW + \cos^2 \theta_2 RW) = g_M(H\nabla_U V, \omega \psi PW + \omega \psi QW + \omega \psi RW) - g_M(A_U BV + H\nabla_U CV, \omega W), \quad (3.23)$$

for all $U, V \in \Gamma(\ker f^\ast)^\perp$ and $W \in \Gamma(\ker f^\ast)$.

Proof. For $U, V \in \Gamma(\ker f^\ast)^\perp$ and $W \in \Gamma(\ker f^\ast)$, we have

$$g_M(\nabla_U V, W) = g_M(\nabla_U V, PW + QW + RW + \eta(W)\xi).$$

Using equations (2.1)–(2.4), (2.6), (2.7), (2.12), (2.13), (3.2), (3.3), (3.7) and Lemmas 3.2 and 3.3 we have

$$g_M(\nabla_U V, W) = g_M(A_U V, PW + \cos^2 \theta_1 QW + \cos^2 \theta_2 RW)
- g_M(H\nabla_U V, \omega \psi PW + \omega \psi QW + \omega \psi RW)
+ g_M(A_U BV + H\nabla_U CV, \omega PW + QW + RW).$$

Taking into account $\omega PW + \omega QW + \omega RW = \omega W$ and $\omega PW = 0$ in the above, one obtains

$$g_M(\nabla_U V, W) = g_M(A_U V, PW + \cos^2 \theta_1 QW + \cos^2 \theta_2 RW)
- g_M(H\nabla_U V, \omega \psi PW + \omega \psi QW + \omega \psi RW)
+ g_M(A_U BV + H\nabla_U CV, \omega W). \quad \square$$

Theorem 3.11. Let $f$ be a proper quasi bi-slant submersion from a cosymplectic manifold $(M, \phi, \xi, \eta, g_M)$ onto a Riemannian manifold $(N, g_N)$. Then the vertical distribution $(\ker f^\ast)$ defines a totally geodesic foliation on $M$ if and only if

$$g_M(T_X PY + \cos^2 \theta_1 T_X QY + \cos^2 \theta_2 T_X RY, U) = g_M(H\nabla_X \omega \psi PY + H\nabla_X \omega \psi QY + H\nabla_X \omega \psi RY, U) + g_M(T_X \omega Y, BU) + g_M(H\nabla_X \omega Y, CU), \quad (3.24)$$

for all $X, Y \in \Gamma(\ker f^\ast)$ and $U \in \Gamma(\ker f^\ast)^\perp$.

Proof. For all $X, Y \in \Gamma(\ker f^\ast)$ and $U \in \Gamma(\ker f^\ast)^\perp$, by using equations (2.1)–(2.4), (2.6) and (2.7) we have

$$g_M(\nabla_X Y, U) = g_M(\nabla_X \phi PY, \phi U) + g_M(\nabla_X \phi QY, \phi U) + g_M(\nabla_X \phi RY, \phi U).$$
Taking into account of (2.10), (2.11), (3.2), (3.3), (3.7) and Lemmas 3.2 and 3.3 we have

\[ g_M(\nabla_X Y, U) = g_M(T_X P Y, U) + \cos^2 \theta_1 g_M(T_X Q Y, U) + \cos^2 \theta_2 g_M(T_X R Y, U) \]

\[ - g_M(H \nabla_X \omega \psi P Y + H \nabla_X \omega \psi Q Y + H \nabla_X \omega \psi R Y, U) \]

\[ + g_M(\nabla_X \omega P Y + \nabla_X \omega Q Y + \nabla_X \omega R Y, \phi U). \]

Since \( \omega P Y + \omega Q Y + \omega R Y = \omega Y \) and \( \omega P Y = 0 \), we derive

\[ g_M(\nabla_X Y, U) = g_M(T_X P Y + \cos^2 \theta_1 T_X Q Y + \cos^2 \theta_2 T_X R Y, U) \]

\[ - g_M(H \nabla_X \omega \psi P Y + H \nabla_X \omega \psi Q Y + H \nabla_X \omega \psi R Y, U) \]

\[ + g_M(T_X \omega Y, BU) + g_M(H \nabla_X \omega Y, CU), \]

which completes the proof.

From Theorems 3.10 and 3.11 we also have the following decomposition results.

**Theorem 3.12.** Let \( f \) be a proper quasi bi-slant submersion from a cosymplectic manifold \((M, \phi, \xi, \eta, g_M)\) onto a Riemannian manifold \((N, g_N)\). Then, the total space is locally a product manifold of the form \( M_{ker f} \times M_{(ker f)^\perp} \), where \( M_{ker f} \) and \( M_{(ker f)^\perp} \) are leaves of \( ker f \) and \( (ker f)^\perp \) respectively if and only if

\[ g_M(A_U V, P Y + \cos^2 \theta_1 Q Y + \cos^2 \theta_2 R Y) = g_M(H \nabla_U V, \omega \psi P Y + \omega \psi Q Y + \omega \psi R Y) \]

\[ + g_M(A_U BV + H \nabla_U CV, \omega Y), \]

and

\[ g_M(T_X Y + \cos^2 \theta_1 T_X Q Y + \cos^2 \theta_2 T_X R Y, U) = g_M(H \nabla_X \omega \psi P Y + H \nabla_X \omega \psi Q Y + H \nabla_X \omega \psi R Y, U) \]

\[ + g_M(T_X \omega Y, BU) + g_M(H \nabla_X \omega Y, CU), \]

for all \( X, Y \in \Gamma(ker f) \) and \( U, V \in \Gamma(ker f^\perp) \).

**Theorem 3.13.** Let \( f \) be a proper quasi bi-slant submersion from a cosymplectic manifold \((M, \phi, \xi, \eta, g_M)\) onto a Riemannian manifold \((N, g_N)\). Then the distribution \( D \) defines a totally geodesic foliation if and only if

\[ g_M(T_U \phi PV, \omega Q W + \omega R W) = - g_M(V \nabla_U \phi PV, \psi Q W + \psi R W), \] (3.25)

and

\[ g_M(T_U \phi PV, CY) = - g_M(V \nabla_U \phi PV, BY), \] (3.26)

for all \( U, V \in \Gamma(D), W \in \Gamma(D_1 \oplus D_2) \) and \( Y \in \Gamma(ker f^\perp) \).
Proof. For all $U, V \in \Gamma(D)$, $W \in \Gamma(D_1 \oplus D_2)$ and $Y \in \Gamma(\ker f_\ast)^\perp$, using equations (2.1)–(2.4), (2.6), (2.7), (3.2), (3.3) and Lemma 3.6 we have

\[
\begin{align*}
\langle \nabla_U V, W \rangle & = \langle \nabla_U \phi V, \phi W \rangle, \\
& = \langle \nabla_U \phi PV, \phi QW + \psi RW \rangle, \\
& = \langle \nabla_U \phi PV, \phi QW + \psi RW \rangle + \langle \nabla_U \phi PV, \psi QW + \psi RW \rangle.
\end{align*}
\]

Now, again using equations (2.10), (3.2), (3.3) and (3.7) we have

\[
\begin{align*}
\langle \nabla_U V, Y \rangle & = \langle \nabla_U \phi V, \phi Y \rangle, \\
& = \langle \nabla_U \phi PV, By + CY \rangle, \\
& = \langle \nabla_U \phi PV, By \rangle + \langle \nabla_U \phi PV, CY \rangle,
\end{align*}
\]

which completes the proof. □

Theorem 3.14. Let $f$ be a proper quasi bi-slant submersion from a cosymplectic manifold $(M, \phi, \xi, \eta, g_M)$ onto a Riemannian manifold $(N, g_N)$. Then the distribution $D_1$ defines a totally geodesic foliation if and only if

\[
\langle T_W \omega \psi Z, U \rangle = \langle T_W \omega QZ, \phi PU + \psi RU \rangle + \langle H \nabla_W \omega QZ, \omega RU \rangle,
\]

and

\[
\langle H \nabla_W \omega \psi Z, Y \rangle = \langle H \nabla_W \omega Z, CY \rangle + \langle T_W \omega Z, BY \rangle,
\]

for all $W, Z \in \Gamma(D_1), U \in \Gamma(D \oplus D_2)$ and $Y \in \Gamma(\ker f_\ast)^\perp$.

Proof. For all $W, Z \in \Gamma(D_1)$, $U \in \Gamma(D \oplus D_2)$ and $Y \in \Gamma(\ker f_\ast)^\perp$, using equations (2.1)–(2.4), (2.6), (2.7), (2.11), (3.2), (3.3) and Lemma 3.2, we have

\[
\begin{align*}
\langle \nabla_W Z, U \rangle & = \langle \nabla_W \phi Z, \phi U \rangle \\
& = \langle \nabla_W \psi Z, \phi U \rangle + \langle \nabla_W \omega Z, \phi U \rangle, \\
& = \cos^2 \theta_1 \langle \nabla_W Z, U \rangle - \langle T_W \omega \psi Z, U \rangle \\
& \quad + \langle T_W \omega QZ, \phi PU + \psi RU \rangle + \langle H \nabla_W \omega QZ, \omega RU \rangle.
\end{align*}
\]

Now, we obtain

\[
\sin^2 \theta_1 \langle \nabla_W Z, U \rangle = -\langle T_W \omega \psi Z, U \rangle + \langle T_W \omega QZ, \phi PU + \psi RU \rangle + \langle H \nabla_W \omega QZ, \omega RZ \rangle
\]

Next, from equations (2.1)–(2.4), (2.6), (2.7), (2.12), (3.3), (3.7) and Lemma 3.2, we have

\[
\begin{align*}
\langle \nabla_W Z, Y \rangle & = \langle \nabla_W \phi Z, \phi Y \rangle, \\
& = \langle \nabla_W \psi Z, \phi Y \rangle + \langle \nabla_W \omega Z, \phi Y \rangle, \\
& = \cos^2 \theta_1 \langle \nabla_W Z, Y \rangle - \langle H \nabla_W \phi Z, Y \rangle \\
& \quad + \langle H \nabla_W \omega Z, CY \rangle + \langle T_W \omega Z, BY \rangle.
\end{align*}
\]
Now, we arrive
\[ \sin^2 \theta_1 g_M(\nabla_W Z, Y) = -g_M(\mathcal{H}\nabla_W \omega Z, Y) + g_M(\mathcal{H}\nabla_W \omega Z, CY) + g_M(\mathcal{T}_W \omega Z, BY), \]
which completes the proof.

**Theorem 3.15.** Let \( f \) be a proper quasi bi-slant submersion from a cosymplectic manifold \((M, \phi, \xi, \eta, g_M)\) onto a Riemannian manifold \((N, g_N)\). Then the distribution \( D_2 \) defines a totally geodesic foliation if and only if

\[ g_M(\mathcal{T}_U \omega V, W) = g_M(\mathcal{T}_U Q V, \phi PW + \psi RW) + g_M(\mathcal{H}\nabla_U \omega Q V, \omega RW), \quad (3.29) \]

and

\[ g_M(\mathcal{H}\nabla_U \omega V, Y) = g_M(\mathcal{H}\nabla_U \omega V, CY) + g_M(\mathcal{T}_U \omega V, BY), \quad (3.30) \]

for all \( U, V \in \Gamma(D_2), W \in \Gamma(D \oplus D_1) \) and \( Y \in \Gamma(\ker f^*)^\perp \).

**Proof.** For all \( U, V \in \Gamma(D_2), W \in \Gamma(D \oplus D_1) \) and \( Y \in \Gamma(\ker f^*)^\perp \), by using equations (2.1)–(2.4), (2.6), (2.7), (2.10), (3.3) and from Lemma 3.2 and Lemma 3.6, we have

\[ g_M(\nabla_U V, W) = g_M(\nabla_U \psi V, \phi W) + g_M(\nabla_U \omega V, \phi W), \]

\[ = \cos^2 \theta_2 g_M(\nabla_U V, W) - g_M(\mathcal{T}_U \omega V, W) \]

\[ + g_M(\mathcal{T}_U Q V, \phi PW + \psi RW) + g_M(\mathcal{H}\nabla_U \omega Q V, \omega RW). \]

Now, we get

\[ \sin^2 \theta_1 g_M(\nabla_U V, W) = -g_M(\mathcal{T}_U \omega V, W) + g_M(\mathcal{T}_U Q V, \phi PW + \psi RW) + g_M(\mathcal{H}\nabla_U \omega Q V, \omega RW). \]

Next, from equations (2.1)–(2.4), (2.6), (2.7), (2.12), (3.2) (3.3), (3.7) and Lemma 3.2, we have

\[ g_M(\nabla_U V, Y) = g_M(\nabla_U \psi V, \phi Y) + g_M(\nabla_U \omega V, \phi Y), \]

\[ = \cos^2 \theta_2 g_M(\nabla_U V, Y) - g_M(\mathcal{H}\nabla_U \omega V, Y) \]

\[ + g_M(\mathcal{H}\nabla_U \omega V, CY) + g_M(\mathcal{T}_U \omega V, BY). \]

Now, we obtain

\[ \sin^2 \theta_1 g_M(\nabla_U V, Y) = -g_M(\mathcal{H}\nabla_U \omega V, Y) + g_M(\mathcal{H}\nabla_U \omega V, CY) + g_M(\mathcal{T}_U \omega V, BY), \]

which completes the proof.

We recall that a differentiable map \( f \) between two Riemannian manifolds is totally geodesic if

\[ (\nabla f_*)(Y, Z) = 0, \text{ for all } Y, Z \in \Gamma(TM). \]

A totally geodesic map is that it maps every geodesic in the total space into a geodesic in the base space in proportion to arc lengths.
Theorem 3.16. Let $f$ be a proper quasi bi-slant submersion from a cosymplectic manifold $(M, \phi, \xi, \eta, g_M)$ onto a Riemannian manifold $(N, g_N)$. Then the map $f$ is totally geodesic if and only if

$$
g_M(\mathcal{H}\nabla_U \omega \psi QV + \mathcal{H}\nabla_U \omega \psi RV - \cos^2 \theta_1 \mathcal{T}_U QV - \cos^2 \theta_2 \mathcal{T}_U RV, W)
$$

$$= g_M(\mathcal{V}\nabla_U \phi PV + \mathcal{T}_U \omega RV + \mathcal{T}_U \omega RV, BW) + g_M(\mathcal{T}_U \phi PV + \mathcal{H}\nabla_U \omega QV + \mathcal{H}\nabla_U \omega RV, CW),$$

and

$$
g_M(\mathcal{H}\nabla_W \omega \psi QU + \mathcal{H}\nabla_W \omega \psi RU - \cos^2 \theta_1 A_W QU - \cos^2 \theta_2 A_W RU, Z)
$$

$$= g_M(\mathcal{V}\nabla_W \phi PU + A_W \omega QU + A_W \omega RU, BZ) + g_M(\mathcal{A}_W \phi PU + \mathcal{H}\nabla_W \omega QU + \mathcal{H}\nabla_W \omega RU, CZ),$$

for all $U, V \in \Gamma(\ker f^*)$ and $W, Z \in \Gamma(\ker f^*)_\perp$.

Proof. For all $U, V \in \Gamma(\ker f^*)$ and $W, Z \in \Gamma(\ker f^*)_\perp$, making use of (2.1)–(2.4), (2.6), (2.7), (2.10), (2.11), (3.2), (3.3), (3.7) and from Lemma 3.2 and 3.3, we derive

$$
g_M(\nabla_U V, W) = g_M(\nabla_U \phi V, \phi W)
$$

$$= g_M(\nabla_U \phi PV, \phi W) + g_M(\nabla_U \phi QV, \phi W) + g_M(\nabla_U \phi RV, \phi W),
$$

$$= g_M(\nabla_U \phi PV, \phi W) + g_M(\nabla_U \psi QV, \phi W) + g_M(\nabla_U \psi RV, \phi W)
$$

$$+ g_M(\nabla_U \omega QV, \phi W) + g_M(\nabla_U \omega RV, \phi W),
$$

$$= g_M(\mathcal{V}\nabla_U \phi PV + \mathcal{T}_U \omega QV + \mathcal{T}_U \omega RV, W)
$$

$$+ g_M(\mathcal{T}_U \phi PV + \mathcal{H}\nabla_U \omega QV + \mathcal{H}\nabla_U \omega RV, CW)
$$

$$+ g_M(\cos^2 \theta_1 \mathcal{T}_U QV + \cos^2 \theta_2 \mathcal{T}_U RV - \mathcal{H}\nabla_U \omega \psi QV - \mathcal{H}\nabla_U \omega \psi RV, W),$$

Next, taking account of (2.1)–(2.4), (2.6), (2.7), (2.10), (2.12), (2.13), (3.2), (3.3), (3.7) and from Lemma 3.2 and 3.3, we have

$$
g_M(\nabla_W U, Z) = g_M(\phi \nabla_W U, \phi Z)
$$

$$= g_M(\nabla_W \phi U, \phi Z),
$$

$$= g_M(\nabla_W \phi PU, \phi Z) + g_M(\nabla_W \phi QU, \phi Z) + g_M(\nabla_W \phi RU, \phi Z),
$$

$$= g_M(\nabla_W \phi PU, \phi Z) + g_M(\nabla_W \psi QU, \phi Z) + g_M(\nabla_W \psi RU, \phi Z)
$$

$$+ g_M(\nabla_W \omega QU, \phi Z) + g_M(\nabla_W \omega RU, \phi Z),
$$

$$= g_M(\mathcal{V}\nabla_W \phi PU + A_W \omega QU + A_W \omega RU, BZ)
$$

$$+ g_M(A_W \phi PU + \mathcal{H}\nabla_W \omega QU + \mathcal{H}\nabla_W \omega RU, CZ)
$$

$$+ g_M(\cos^2 \theta_1 A_W QU + \cos^2 \theta_2 A_W RU - \mathcal{H}\nabla_W \omega \psi QU - \mathcal{H}\nabla_W \omega \psi RU, Z),$$

which completes the proof. \qed
4 Examples

In this section, we are going to give some non-trivial examples. We will use the notation mentioned in Example 2.1.

Example 4.1. Define a map

\[ \pi : \mathbb{R}^{15} \to \mathbb{R}^6 \]

\[ \pi(x_1, x_2, \ldots, x_7, y_1, y_2, \ldots, y_7, z) = (x_2 \cos \theta_1 - y_3 \sin \theta_1, y_2, x_4 \sin \theta_2 - y_5 \cos \theta_2, x_5, x_7, y_7), \]

which is a quasi bi-slant submersion such that

\[ X_1 = \frac{\partial}{\partial x_1}, \quad X_2 = \frac{\partial}{\partial y_1}, \quad X_3 = \frac{\partial}{\partial x_2} \sin \theta_1 + \frac{\partial}{\partial y_3} \cos \theta_1, \quad X_4 = \frac{\partial}{\partial x_3}, \]

\[ X_5 = \frac{\partial}{\partial x_4} \cos \theta_2 + \frac{\partial}{\partial y_5} \sin \theta_2, \quad X_6 = \frac{\partial}{\partial y_4}, \quad X_7 = \frac{\partial}{\partial x_6}, \quad X_8 = \frac{\partial}{\partial y_6}, \]

\[ X_9 = \xi = \frac{\partial}{\partial z}, \]

\[ (\ker \pi^*) = (D \oplus D_1 \oplus D_2 \oplus \langle \xi \rangle), \]

where

\[ D = \left\langle X_1 = \frac{\partial}{\partial x_1}, X_2 = \frac{\partial}{\partial y_1}, X_7 = \frac{\partial}{\partial x_6}, X_8 = \frac{\partial}{\partial y_6} \right\rangle, \]

\[ D_1 = \left\langle X_3 = \frac{\partial}{\partial x_2} \sin \theta_1 + \frac{\partial}{\partial y_3} \cos \theta_1, X_4 = \frac{\partial}{\partial x_3} \right\rangle, \]

\[ D_2 = \left\langle X_5 = \frac{\partial}{\partial x_4} \cos \theta_2 + \frac{\partial}{\partial y_5} \sin \theta_2, X_6 = \frac{\partial}{\partial y_4} \right\rangle, \]

\[ \langle \xi \rangle = \left\langle X_9 = \frac{\partial}{\partial z} \right\rangle, \]

and

\[ (\ker \pi^*)^\perp = \left\langle \frac{\partial}{\partial x_2} \cos \theta_1 - \frac{\partial}{\partial y_3} \sin \theta_1, \frac{\partial}{\partial y_2}, \frac{\partial}{\partial x_4} \sin \theta_2 - \frac{\partial}{\partial y_5} \cos \theta_2, \frac{\partial}{\partial x_5}, \frac{\partial}{\partial x_7}, \frac{\partial}{\partial y_7} \right\rangle, \]

with bi-slant angles \( \theta_1 \) and \( \theta_2 \). Thus the above example verifies the Lemmas 3.1, 3.2, 3.3 and 3.6.

Example 4.2. Define a map

\[ \pi : \mathbb{R}^{13} \to \mathbb{R}^6 \]

\[ \pi(x_1, x_2, \ldots, x_6, y_1, y_2, \ldots, y_6, z) = \left( \frac{x_1 - x_2}{\sqrt{2}}, y_1, \sqrt{3} x_4 - \frac{x_5}{2}, y_5, x_6, y_6 \right), \]

which is a quasi bi-slant submersion such that

\[ X_1 = \frac{1}{\sqrt{2}} \left( \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} \right), \quad X_2 = \frac{\partial}{\partial y_1}, \quad X_3 = \frac{\partial}{\partial x_3}, \quad X_4 = \frac{\partial}{\partial y_3}, \]

\[ X_5 = \frac{1}{2} \left( \frac{\partial}{\partial x_4} + \sqrt{3} \frac{\partial}{\partial x_5} \right), \quad X_6 = \frac{\partial}{\partial y_4}, \]
\[ X_7 = \xi = \frac{\partial}{\partial z}, \]

\[ (\ker \pi_*) = (D \oplus D_1 \oplus D_2 \oplus \langle \xi \rangle), \]

where

\[ D = \left\langle X_3 = \frac{\partial}{\partial x_3}, X_4 = \frac{\partial}{\partial y_3} \right\rangle, \]

\[ D_1 = \left\langle X_1 = \frac{1}{\sqrt{2}} \left( \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} \right), X_2 = \frac{\partial}{\partial y_2} \right\rangle, \]

\[ D_2 = \left\langle X_5 = \frac{1}{2} \left( \frac{\partial}{\partial x_4} + \sqrt{3} \frac{\partial}{\partial x_5} \right), X_6 = \frac{\partial}{\partial y_4} \right\rangle, \]

\[ \langle \xi \rangle = \left\langle X_7 = \frac{\partial}{\partial z} \right\rangle, \]

and

\[ (\ker \pi_*)^\perp = \left\langle \frac{\partial}{\partial y_1}, \frac{1}{\sqrt{2}} \left( \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} \right), \frac{1}{2} \left( \sqrt{3} \frac{\partial}{\partial x_4} - \frac{\partial}{\partial x_5} \right), \frac{\partial}{\partial y_5}, \frac{\partial}{\partial x_6}, \frac{\partial}{\partial y_6} \right\rangle, \]

with bi-slant angles \( \theta_1 = \frac{\pi}{4} \) and \( \theta_2 = \frac{\pi}{3} \). Therefore, the above example verifies the Lemmas 3.1, 3.2, 3.3 and 3.6.
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