CHARACTERIZING THE NUMBER OF COLOURED $m$-ARY PARTITIONS MODULO $m$, WITH AND WITHOUT GAPS

I. P. GOULDEN AND PAVEL SHULDINER

Abstract. In a pair of recent papers, Andrews, Fraenkel and Sellers provide a complete characterization for the number of $m$-ary partitions modulo $m$, with and without gaps. In this paper we extend these results to the case of coloured $m$-ary partitions, with and without gaps. Our method of proof is different, giving explicit expansions for the generating functions modulo $m$.

1. Introduction

An $m$-ary partition is an integer partition in which each part is a nonnegative integer power of a fixed integer $m \geq 2$. An $m$-ary partition without gaps is an $m$-ary partition in which $m^j$ must occur as a part whenever $m^{j+1}$ occurs as a part, for every nonnegative integer $j$.

Recently, Andrews, Fraenkel and Sellers [AFS15] found an explicit expression that characterizes the number of $m$-ary partitions of a nonnegative integer $n$ modulo $m$; remarkably, this expression depended only on the coefficients in the base $m$ representation of $n$. Subsequently Andrews, Fraenkel and Sellers [AFS16] followed this up with a similar result for the number of $m$-ary partitions without gaps, of a nonnegative integer $n$ modulo $m$; again, they were able to obtain a (more complicated) explicit expression, and again this expression depended only on the coefficients in the base $m$ representation of $n$. See also Edgar [E16] and Ekhad and Zeilberger [EZ15] for more on these results.

The study of congruences for integer partition numbers has a long history, starting with the work of Ramanujan (see, e.g., [R19]). For the special case of $m$-ary partitions, a number of authors have studied congruence properties, including Churchhouse [C69] for $m = 2$, Rødseth [R70] for $m$ a prime, and Andrews [A71] for arbitrary positive integers $m \geq 2$. The numbers of $m$-ary partitions without gaps had been previously considered by Bessenrodt, Olsson and Sellers [BOS13] for $m = 2$.

In this note, we consider $m$-ary partitions, with and without gaps, in which the parts are coloured. To specify the number of colours for parts of each size, we let $k = (k_0, k_1, \ldots)$ for positive integers $k_0, k_1, \ldots$, and say that an $m$-ary partition is $k$-coloured when there are $k_j$ colours for the part $m^j$, for $j \geq 0$. This means that there are $k_j$ different kinds of parts of the same size $m^j$. Let $b_m^{(k)}(n)$ denote the number of $k$-coloured $m$-ary partitions of $n$, and let $a_m^{(k)}(n)$ denote the number of $k$-coloured $m$-ary partitions of $n$ without gaps. For the latter, some part $m^j$ of any colour must occur as a part whenever some part $m^{j+1}$ of any colour (not necessarily the same colour) occurs as a part, for every nonnegative integer $j$.

We extend the results of Andrews, Fraenkel and Sellers in [AFS15] and [AFS16] to the case of $k$-coloured $m$-ary partitions, where $m$ is relatively prime to $(k_0 - 1)!$ and to $k_j!$ for $j \geq 1$. Our method of proof is different, giving explicit expansions for the generating functions modulo $m$. These expansions depend on the following simple result.

Proposition 1.1. For positive integers $m, a$ with $m$ relatively prime to $(a-1)!$, we have

$$(1-q)^{-a} \equiv (1-q^m)^{-1} \sum_{\ell=0}^{m-1} \binom{a-1+\ell}{a-1} q^\ell \pmod{m}.$$
**Proof.** From the binomial theorem we have
\[
(1 - q)^{-a} = \sum_{\ell=0}^{\infty} \binom{a-1+\ell}{a-1} q^{\ell}.
\]
Now using the falling factorial notation \((a - 1 + \ell)_{a-1} = (a - 1 + \ell)(a - 2 + \ell) \cdots (1 + \ell)\) we have
\[
\binom{a-1+\ell}{a-1} = ((a-1)!)^{-1} (a - 1 + \ell)_{a-1}.
\]
But
\[
(a - 1 + \ell + m)_{a-1} \equiv (a - 1 + \ell)_{a-1} \pmod{m},
\]
for any integer \(\ell\), and \(((a-1)!)^{-1}\) exists in \(\mathbb{Z}_m\) since \(m\) is relatively prime to \((a-1)!\), which gives
\[
\binom{a-1+\ell+m}{a-1} \equiv \binom{a-1+\ell}{a-1} \pmod{m},
\]
and the result follows. \(\square\)

2. **Coloured \(m\)-ary partitions**

In this section we consider the following generating function for the numbers \(b_m^{(k)}(n)\) of \(k\)-coloured \(m\)-ary partitions:
\[
B_m^{(k)}(q) = \sum_{n=0}^{\infty} b_m^{(k)}(n)q^n = \prod_{j=0}^{\infty} \left(1 - q^{m^j}\right)^{-k_j}.
\]
The following result gives an explicit expansion for \(B_m^{(k)}(q) \pmod{m}\).

**Theorem 2.1.** If \(m\) is relatively prime to \((k_0 - 1)!\) and to \(k_j!\) for \(j \geq 1\), then we have
\[
B_m^{(k)}(q) \equiv \left(\sum_{\ell_0=0}^{m-1} \binom{k_0 - 1 + \ell_0}{k_0 - 1} q^{\ell_0}\right) \prod_{j=1}^{\infty} \left(\sum_{\ell_j=0}^{m-1} \binom{k_j + \ell_j}{k_j} q^{\ell_j m^j}\right) \pmod{m}.
\]

**Proof.** Consider the finite product
\[
P_i = \prod_{j=0}^{i} \left(1 - q^{m^j}\right)^{-k_j}, \quad i \geq 0.
\]
We prove that
\[
P_i \equiv \left(\sum_{\ell_0=0}^{m-1} \binom{k_0 - 1 + \ell_0}{k_0 - 1} q^{\ell_0}\right) \left(1 - q^{m^{i+1}}\right)^{-k_{i+1}} \prod_{j=1}^{i} \left(\sum_{\ell_j=0}^{m-1} \binom{k_j + \ell_j}{k_j} q^{\ell_j m^j}\right) \pmod{m},
\]
by induction on \(i\). As a base case, the result for \(i = 0\) follows immediately from Proposition 1.1 with \(a = k_0\). Now assume that (2) holds for some choice of \(i \geq 0\), and we obtain
\[
P_{i+1} = \prod_{j=0}^{i+1} \left(1 - q^{m^j}\right)^{-k_j} = \left(1 - q^{m^{i+1}}\right)^{-k_{i+1}} P_i
\]
\[
\equiv \left(\sum_{\ell_0=0}^{m-1} \binom{k_0 - 1 + \ell_0}{k_0 - 1} q^{\ell_0}\right) \left(1 - q^{m^{i+1}}\right)^{-k_{i+1}} \prod_{j=1}^{i} \left(\sum_{\ell_j=0}^{m-1} \binom{k_j + \ell_j}{k_j} q^{\ell_j m^j}\right) \pmod{m},
\]
where the second last equivalence follows from the induction hypothesis, and the last equivalence follows from Proposition 1.1 with \(a = k_{i+1} + 1\), \(q = q^{m^{i+1}}\).
This completes the proof of [2] by induction on $i$, and the result follows immediately since

$$B^{(k)}_m(q) = \lim_{i \to \infty} P_i.$$

Now we give the explicit expression for the coefficients modulo $m$ that follows from the above expansion of the generating function $B^{(k)}_m(q)$.

**Corollary 2.2.** For $n \geq 0$, suppose that the base $m$ representation of $n$ is given by

$$n = d_0 + d_1 m + \ldots + d_t m^t, \quad 0 \leq t.$$

If $m$ is relatively prime to $(k_0 - 1)!$ and to $k_j!$ for $j \geq 1$, then we have

$$b^{(k)}_m(n) \equiv \left( k_0 - 1 + d_0 \right) \prod_{j=1}^{t} \binom{k_j + d_j}{k_j} \pmod{m}.$$

**Proof.** In the expansion of the series $B^{(k)}_m(q)$ given in Theorem 2.1, the monomial $q^n$ arises uniquely with the specializations $\ell_j = d_j$, $j = 0, \ldots, t$ and $\ell_j = 0$, $j \geq t$. But with these specializations, we have $(b_j, \ell_j) = (k_j, q^m) = 1$, and the result follows immediately. \qed

Specializing the expression given in Corollary 2.2 to the case $k_j = 1$ for $j \geq 0$ provides an alternative proof to Andrews, Fraenkel and Sellers’ characterization of $m$–ary partitions modulo $m$, which was given as Theorem 1 of [AFS13].

### 3. Coloured $m$-ary partitions without gaps

In this section we consider the following generating function for the numbers $c^{(k)}_m(n)$ of $k$-coloured $m$-ary partitions without gaps:

$$C^{(k)}_m(q) = 1 + \sum_{n=0}^{\infty} c^{(k)}_m(n) q^n = 1 + \sum_{i=0}^{\infty} \prod_{j=0}^{i} \left( (1 - q^{m^j})^{-k_j} - 1 \right).$$

The following result gives an explicit expansion for $C^{(k)}_m(q)$ modulo $m$.

**Theorem 3.1.** If $m$ is relatively prime to $(k_0 - 1)!$ and to $k_j!$ for $j \geq 1$, then we have

$$C^{(k)}_m(q) \equiv 1 + \left( \sum_{\ell_0=0}^{m} \binom{k_0 - 1 + \ell_0}{k_0 - 1} q^{\ell_0} \right) \sum_{i=0}^{\infty} \left( 1 - q^{m^i+1} \right)^{-1} \prod_{j=1}^{i} \left( \sum_{\ell_j=0}^{m-1} \binom{k_j + \ell_j}{k_j} - 1 \right) q^{\ell_j m^i} \pmod{m}.$$

**Proof.** Consider the finite product

$$R_i = \prod_{j=0}^{i} \left( (1 - q^{m^j})^{-k_j} - 1 \right), \quad i \geq 0.$$

We prove that

$$R_i \equiv \left( \sum_{\ell_0=0}^{m} \binom{k_0 - 1 + \ell_0}{k_0 - 1} q^{\ell_0} \right) \left( 1 - q^{m^{i+1}} \right)^{-1} \prod_{j=1}^{i} \left( \sum_{\ell_j=0}^{m-1} \binom{k_j + \ell_j}{k_j} - 1 \right) q^{\ell_j m^i} \pmod{m},$$

rail
by induction on \(i\). As a base case, the result for \(i = 0\) follows immediately from Proposition 1.1 with \(a = k_0\). Now assume that (3) holds for some choice of \(i \geq 0\), and we obtain

\[
R_{i+1} = \prod_{j=0}^{i+1} \left( \left( 1 - q^{m^j} \right)^{k_j} - 1 \right) = \left( 1 - q^{m^{i+1}} \right)^{k_{i+1}} - 1 \right) R_i
\]

\[
\equiv \left( \sum_{\ell_0=1}^{m} \left( \frac{k_0 - 1 + \ell_0}{k_0 - 1} \right) q^{\ell_0} \right) \left( 1 - q^{m^{i+1}} \right)^{-k_{i+1}-1} - \left( 1 - q^{m^{i+1}} \right)^{-1} \right) \prod_{j=1}^{i} \left( \left( \frac{k_j + \ell_j}{k_j} \right) - 1 \right) q^{\ell_j m^j} (\text{mod } m)
\]

\[
= \left( \sum_{\ell_0=1}^{m} \left( \frac{k_0 - 1 + \ell_0}{k_0 - 1} \right) q^{\ell_0} \right) \left( 1 - q^{m^{i+2}} \right)^{-1} \prod_{j=1}^{i+1} \left( \left( \frac{k_j + \ell_j}{k_j} \right) - 1 \right) q^{\ell_j m^j} (\text{mod } m),
\]

where the second last equivalence follows from the induction hypothesis, and the last equivalence follows from Proposition 1.1 with \(a = k_{i+1} + 1, q = q^{m^{i+1}} \) and \(a = 1, q = q^{m^{i+1}} \).

This completes the proof of (3) by induction on \(i\), and the result follows immediately since

\[
C_m^{(k)}(q) = 1 + \sum_{i=0}^{\infty} R_i.
\]

\[\square\]

**Corollary 3.2.** For \(n \geq 1\), suppose that \(n\) is divisible by \(m\), with base \(m\) representation given by

\[
n = d_sm^s + \ldots + d_t m^t,
\]

where \(1 \leq d_s \leq m - 1\), and \(0 \leq d_{s+1}, \ldots, d_t \leq m - 1\). If \(m\) is relatively prime to \((k_0 - 1)!\) and to \(k_s\) for \(j \geq 1\), then for \(0 \leq d_0 \leq m - 1\) we have

\[
c_m^{(k)}(n - d_0) \equiv \left( \frac{k_0 - 1 - d_0}{k_0 - 1} \right) \left( \varepsilon_s + (-1)^{s-1} \left\{ \left( \frac{k_s + d_s - 1}{k_s} \right) - 1 \right\} \right) \prod_{j=s+1}^{i} \left( \left( \frac{k_j + d_j}{k_j} \right) - 1 \right) (\text{mod } m),
\]

where \(\varepsilon_s = 0\) if \(s\) is even, and \(\varepsilon_s = 1\) if \(s\) is odd.

**Proof.** First note that we have

\[
n - d_0 = m - d_0 + (m - 1)m^1 + \ldots + (m - 1)m^{s-1} + (d_s - 1)m^s + d_{s+1}m^{s+1} + \ldots + d_t m^t.
\]

Now consider the following specializations: \(\ell_0 = m - d_0, \ell_j = m - 1, j = 1, \ldots, s - 1, \ell_s = d_s - 1, \ell_j = d_j, j = s + 1, \ldots, t, \) and \(\ell_j = 0, j > t\). Then, in the expansion of the series \(C_m^{(k)}(q)\) given in Theorem 3.1, the monomial \(q^n\) arises once for each \(i \geq 0\), in particular with the above specializations truncated to \(\ell_0, \ldots, \ell_i\).

But with these specializations we have

- for \(j = 0\):

\[
\left( \frac{k_j + \ell_j}{k_j} \right) - 1 = \left( \frac{k_0 - 1 + m - d_0}{k_0 - 1} \right) = \left( \frac{k_0 - 1 - d_0}{k_0 - 1} \right), \quad \text{from (1)},
\]

- for \(j = 1, \ldots, s - 1\):

\[
\left( \frac{k_j + \ell_j}{k_j} \right) - 1 = \left( \frac{k_j - 1}{k_j} \right) - 1 = 0 - 1 = -1,
\]

and

\[
\sum_{i=0}^{s-1} \prod_{j=1}^{i} \left( \left( \frac{k_j + \ell_j}{k_j} \right) - 1 \right) = \sum_{i=0}^{s-1} (-1)^i = \varepsilon_s,
\]

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• for $j = s$:
  \[
  \binom{k_j + \ell_j}{k_j} - 1 = \binom{k_s + d_s - 1}{k_s} - 1,
  \]

• for $j = s + 1, \ldots, t$:
  \[
  \binom{k_j + \ell_j}{k_j} - 1 = \binom{k_j + d_j}{k_j} - 1,
  \]

• for $j > t$:
  \[
  \binom{k_j + \ell_j}{k_j} - 1 = \binom{k_j}{k_j} - 1 = 1 - 1 = 0.
  \]

The result follows straightforwardly from Theorem 3.1.

Specializing the expression given in Corollary 3.2 to the case $k_j = 1$ for $j \geq 0$ provides an alternative proof to Andrews, Fraenkel and Sellers’ characterization of $m$–ary partitions modulo $m$ without gaps, which was given as Theorem 2.1 of [AFS16].

References

A71. G. E. Andrews, *Congruence properties of the m-ary partition function*, J. Number Theory 3 (1971), 104–110.

AFS15. George E. Andrews, Aviezri S. Fraenkel, James A. Sellers, *Characterizing the Number of m-ary Partitions Modulo m*, American Mathematical Monthly 122 (2015), 880–885.

AFS16. George E. Andrews, Aviezri S. Fraenkel, James A. Sellers, *m-ary partitions with no gaps: A characterization modulo m*, Discrete Mathematics 339 (2016), 283–287.

BOS13. C. Bessenrodt, J. B. Olsson, J. A. Sellers, *Unique path partitions: characterization and congruences*, Annals Comb. 17 (2013), 591–602.

C69. R. F. Churchhouse, *Congruence properties of the binary partition function*, Proc. Cambridge Philos. Soc. 66 (1969), 371–376.

E16. Tom Edgar, *The distribution of the number of parts of m-ary partitions modulo m*, Rocky Mountain J. Math. (to appear), arXiv 1603.00085 math.CO

EZ15. Shalosh B. Ekhad and Doron Zeilberger, *Computerizing the Andrews-Fraenkel-Sellers Proofs on the Number of m-ary partitions mod m (and doing MUCH more!)*, arXiv 1511.06791 math.CO

R19. S. Ramanujan, *Some properties of $p(n)$, the number of partitions of $n$*, Proc. Cambridge Philos. Soc. 19 (1919), 207–210.

R70. Ø. Rødseth, *Some arithmetical properties of m-ary partitions*, Proc. Cambridge Philos. Soc. 68 (1970), 447–453.

Dept. of Combinatorics and Optimization, University of Waterloo, Canada

E-mail address: ipgoulde@uwaterloo.ca, pavel.shuldiner@gmail.com