Abstract

Most environments where assignment mechanisms (possibly random) are used are such that participants have outside options. For instance private schools and private housing are options that participants in a public choice or public housing assignment problems may have. We postulate that, in cardinal mechanisms, chances inside the assignment process could favor agents with better outside options. By imposing a robustness to outside options condition, we conclude that, on the universal domain of cardinal preferences, any mechanism must be (interim) ordinal.

Previous title: “Cardinal Assignment Mechanisms: Money Matters More than It Should.” Caterina Calsamiglia acknowledges financial support by the ERC Starting Grant 638893. Francisco Martínez-Mora gratefully acknowledges funding support from Fundación La Caixa and Fundación Caja Navarra under contract LCF/PR/PR13/51080004. Antonio Miralles thanks the Ortygia Business School and acknowledges financial support from the Spanish Government R&D programs SEV-2015-0563 and ECO2017-83534P, and to the Catalan Government (SGR2017 0711.) We are also grateful to Jordi Massó, Clemens Puppe and to the audiences at both the 13th Matching in Practice workshop and the 2017 UECE Lisbon Meetings, for their comments and help.

Caterina Calsamiglia
icrema, Barcelona, Spain

Francisco Martínez-Mora
University of Leicester School of Business, Leicester, UK

Antonio Miralles
Universitat Autonoma de Barcelona and Barcelona GSE, Barcelona, Spain
1 Introduction

Centralized matching markets are used to assign vacancies in kindergarten, school, colleges, public housing or hospitals. The match is done through an assignment procedure because of the belief that income shall not determine access to such spots. However, the analysis of centralized mechanisms often ignores that participants in assignment problems often have outside options (mostly privately provided options) in case the obtained assignment is not good enough.

For instance, in the context of school choice under the Boston mechanism, Calsamiglia et al. (2020) find that the existence of private schools that are available only for richer families will decrease the probability of low and middle income families of entering the best schools in the public system. That is, the fact that the outside option differs across individuals introduces an inequality in the probability of assignment within the public system, even when preferences over public schools are identical.

Consider the following example that serves to illustrate the problem at hand. We have two public schools with one slot each (schools 1 and 2), one public school with three slots (school 3,) and five students. Students 1 and 2 have valuations $v = (1.2, 1, 0)$ where entries refer to, respectively, school 1, school 2 and school 3. Students 3, 4 and 5 have valuations $v' = (1.2, 1, 0.9999)$. Consider the following random assignment: probabilities $q = (0, 1/2, 1/2)$ for both students 1 and 2 and $q' = (1/3, 0, 2/3)$ for students 3,4, and 5. The aforementioned assignment is ex-ante both efficient and envy-free, indeed coinciding with a Competitive Equilibrium from Equal Incomes assignment à la Hylland and Zeckhauser (1979).

Suppose now that school 3 is instead a private school outside of the assignment process, that one can always obtain access to by paying a tuition fee. We argue that in this case this assignment may not be a convincingly fair assignment of probabilities anymore. One could imagine that agents 3 to 5 give more value to the outside option than students 1 and 2 entirely or partly because they have more wealth and therefore less marginal utility of money. This fact triggers an advantage that wealthier students have in accessing school 1, the best public school, as compared to poorer students. Wealthier students are better-equipped to bear the risk of ending up with the outside option.2

We use differences in wealth as a motivating example. However, results apply to any setup where some feature may unequally affect the valuation of the outside option but not of the objects in the assignment system. Private schools constitute one example, but also privileged access to a social network may give an agent a backup option when applying for public housing or college admission, allowing those who have a safety net to take riskier decisions with better possible outcomes.3

---

1 On a different topic, a similar effect is also noticed in Arnosti and Randolf (2021).
2 The lack of enough supply of public slots in this example could just be offset by the existence of other public schools of very low quality that everyone would like to avoid (e.g. ghetto schools.)
3 Another example. Consider a student who is talented enough for a scholarship at the school of arts. Yet she would prefer to attend the best public school in the area. Given that she has an extra outside option available, she could better bear the risks of applying for the most popular school than a not so talented
The outside option is a special object in this sense. Just as priorities are not exactly “school preferences” even if they could be mathematically modelled as such, the outside option may be conceptually different from an object with sufficiently many copies, even if there is apparently no difference between the two approaches from a purely methodological point of view. We claim that an assignment mechanism should avoid the distortions that an outside option may have on the assignment.

Moreover, our criteria along this paper shall in some cases affect objects, other than the outside options, for which income has an important influence on preferences. Take an example from the allocation of infants to kindergartens. Some of the kindergartens are public hence highly subsidized. Some others are private with an agreement with public institutions. These are not subsidized, or at least not as much as public institutions. Finally, there are outside options, namely other day-care choices out of the centralized allocation process. While our paper is focused on the latter, the same concerns we postulate here should also apply to the private schools that are participating in the allocation process. Well-off families bear less risk in applying to the best and most demanded public kindergartens, since being allocated to a private kindergarten in case of rejection is affordable to them.

We postulate the following no-regret, robustness to outside options (RTOO) condition: that no agent may prefer a different (interim) assignment offered by the mechanism other than the one assigned, should the valuation for her outside option marginally increase.

For the reader who thinks that a robust mechanism should simply assign the same interim probabilities to any two agent’s types differing only in the valuation for the outside option, we indeed show (Lemma 1) that such a normative requirement is a consequence of our robustness property. One can think of this simplified approach and ours as equivalent.

Also, one might think that our robustness notion is equivalent to robustness to a small noise in cardinal preferences. This is not entirely the case. We do not allow for any deviation in cardinal preferences, just for deviations compatible with an increase in the valuation for the outside option. Even under such a restriction we obtain a strong implication as seen below.

The main result (Theorem 1) is the prescription of ordinality: in the universal domain of vNM preferences, we must restrict attention to (interim) ordinal mechanisms, where only ordinal preferences are taken into account. Moreover,

Footnote 3 (continued)

student. Differences in talent jointly with selective schools may also generate disruptions in the assignment of public slots. Any other discriminatory criterion, given maybe by other socioeconomics (religion, ethnicity etc.) that give some agents higher access to reservation objects, may trigger a similar normative requirement.

4 We thank a referee who pointed out this observation.

5 Seemingly, the reader could wonder why the mechanism should not be robust to any modification of the valuation for the outside option, in any quantity and direction. (We thank a referee for this observation.) Such a more stringent condition would lead to identical conclusions as in this paper. We consider our no-regret condition as stated in our paper as a minimal condition regarding outside options that leads to ordinality.
ordinal bayesian incentive compatibility is another consequence of our robustness requirement.

When we assess the relevance of such a recommendation, one has to pay attention to plausible restrictions in the domain of preferences over lotteries. In an extreme case in which all objects are acceptable (i.e. better than the outside option) for every agent and there are sufficient copies of objects, our robustness condition is innocuous and there is full scope for cardinal mechanisms. In environments in which three or more objects are acceptable for every agent, there is still scope for taking cardinal preferences into account.

There has been some literature stressing the fact that outside options shall be studied in depth in assignment problems: for instance Kesten and Kurino (2016) and Pycia and Ünver (2017). According to this new strand, outside options are more than an object with infinite capacity. However, we are not aware of other papers considering this fact as the source for a robustness concept in cardinal mechanisms.

A significant array of papers (Hylland and Zeckhauser 1979; Miralles 2008; Abdulkadiroglu et al. 2011 and Abdulkadiroglu et al. 2015; Pycia 2014; Ashlagi and Shi 2016; Featherstone and Niederle 2016; Kim 2017 for voting schemes; He et al. 2018; Miralles and Pycia 2020) has stressed the importance of taking cardinal utilities into account. Empirically, and particularly the school choice case, the seminal paper by Black (1999) evidences that parents have cardinal preferences for the schools that can be expressed in monetary terms as willingness to pay through the residential market.

However, the preference for mechanisms eliciting cardinal preferences is not so clear-cut. A recent paper by Carroll (2018) is most related to ours. Carroll elaborates on the literature of robust mechanism design (see for instance the seminal paper by Bergemann and Morris 2005) applied to Social Choice Correspondences. He summarizes the debate and postulates that, in environments with uncertainty about the own cardinal preferences, simple mechanisms eliciting information on ordinal preferences only shall be preferable. In contrast with Carroll’s model, we allow cardinal preferences to vary in a way that alter ordinal preferences, yet only when compatible with an increase in the valuation of the outside option. Moreover, our conclusions do not depend upon a social planner wishing to implement an ordinal social choice correspondence (e.g. ex-post Pareto-optimality, ordinal efficiency, etc.)

In a similar trend of literature, Hylland (1980) and more recently Dutta et al. (2007) have shown that the only strategy-proof cardinal decision scheme satisfying a weak unanimity property is the random dictatorship, hence eliminating cardinality. Quite recently, Ehlers et al. (2020) establish that under some continuity criteria, incentive-compatible cardinal mechanisms are ordinal. The alternative approach in our paper is appealing since our model starts from a pure cardinal approach with no bias in favor of ordinality. As argued in Carroll (2018), Ehlers et al. gives some initial advantage to ordinal mechanisms since continuity criteria are not required around indifferences over sure objects.

To sum up, the present paper constitutes a warning. By taking outside options seriously, we must reckon with insurmountable trade-offs between cardinal efficiency and robustness with respect to outside options. Restrictions in the domain of ordinal preferences are necessary in order to make cardinality compatible to
Random assignments and outside options

We hope that this contribution will foster further research on this important matter.

Section 2 introduces the model and robustness concepts. Section 3 extracts properties of the mechanisms as direct implications. Section 4 concludes.

2 Notation and definitions

Since the theoretical body of this paper considers incentives rather than, say, feasibility or envy-freeness, it is better, for the sake of simplicity, to restrict attention to single-agent mechanisms.6

The representative agent is to be assigned to exactly one of a set of objects $S \cup \{o\}$ where $o$ is her outside option. Notation for objects include $s, s'$. We use $-s$ for all the objects that are not $s$.

Our representative agent has vNM valuations for the objects $v = (v^s)_{s \in S \cup \{o\}} \in \mathcal{V} \subset \mathbb{R}^{|S|+1}$. Abusing notation, we sometimes use $(v^{-s}, v^s)$ for a vector of valuations containing a vector $v^{-s}$ for objects other than $s$ and $v^s$ for object $s$. The agent knows her type $v$.

Considering that preferences over lotteries are invariant to affine transformations of vNM valuations, we can readily assume that $\mathcal{V}$ contains all affine transformations of itself:7

$$v \in \mathcal{V} \implies \forall \alpha > 0, \beta \in \mathbb{R}, \alpha v + \beta 1_{|S|+1} \in \mathcal{V}$$

We say that the domain (of cardinal preferences) is universal if $\mathcal{V} = \mathbb{R}^{|S|+1}$.

A (direct interim) assignment mechanism $Q$ is a function $Q = \mathcal{V} \rightarrow \Delta(S \cup \{o\})$, where each $Q(v)$ is a probability vector over $S \cup \{o\}$. We restrict attention to mechanisms that are invariant to affine transformations expressing the same preferences over lotteries:

$$Q(\alpha v + \beta 1_{|S|+1}) = Q(v)$$

for every $v \in \mathcal{V}$ and every $\alpha > 0, \beta \in \mathbb{R}$.

We also assume that $Q$ assigns zero probability for objects (weakly) less-preferred than the outside option. These objects are regarded as unacceptable for the agent.

A mechanism is ordinal if it only responds to the ordinal component of agent’s preferences.

Note that, in an environment with outside options, the probabilities assigned by the mechanism might be modified (adapted), acknowledging that no agent shall be forced into an unacceptable assignment, even if preferences have been misreported. Objects valued weakly below the outside option would drop their assignment probabilities to zero, and such dropped probabilities would be added to the outside option. Accordingly, given $v', v \in \mathcal{V}$, $Q(v; v')$ is an adaptation of $Q(v)$ to $v'$ if for

---

6 We thank a referee and the associate editor for pointing this simplification of the model.

7 In what follows, $1_{|S|+1}$ is a vector of ones with dimension $|S| + 1$. 

 Springer
all \( s \in S \) we have \( \tilde{Q}^s(v;v') = Q^s(v) \) if \( v^s > v'^o \) and \( \tilde{Q}^s(v;v') = 0 \) otherwise, whereas
\[
\tilde{Q}^o(v;v') = 1 - \sum_{s' \in S} \tilde{Q}^s(v;v').
\]**

A mechanism is **Bayesian Incentive Compatible** for each pair of valuation vectors \( v, v' \in \mathcal{V} \) we have \( Q(v) \cdot v \geq \tilde{Q}(v';v) \cdot v \).

Our model focuses on the sensitivity of the mechanism to changes in the valuation for the outside option. The implicit assumption of this model is that the background variable that could render the assignment unfair (e.g. income, connections and extracurricular talent) does not alter agent’s preferences over lotteries that do not include the outside option as a possible outcome.

With that in mind, we are ready to introduce our notion of robustness. This notion is one of no regret. It says that once \( v \) is reported to the mechanism, the agent would not have preferred to declare otherwise should her outside option become marginally higher valued.

**Definition 1** For \( \varepsilon > 0 \), a mechanism \( Q \) is \( \varepsilon \)-**Robust–To–Outside–Options** (or just \( \varepsilon \)-RTOO) if for every pair of vNM vectors \( v = (v^o, v^o) \), \( v' \in \mathcal{V} \) and every \( \tilde{v}^o \in [v^o, v^o + \varepsilon] \) we have
\[
\tilde{Q}(v;(v^o, \tilde{v}^o)) \cdot (v^o, \tilde{v}^o) \geq \tilde{Q}(v';(v^o, \tilde{v}^o)) \cdot (v^o, \tilde{v}^o).
\]

We conclude this section with an important Lemma. It states that \( \varepsilon \)-RTOO brings Bayesian Incentive Compatibility and a second property: any agent is indifferent between reporting her valuation vector and a vector with a modified valuation for the outside option, as long as ordinal preferences are preserved.

**Lemma 1** Let the domain \( \mathcal{V} \) be universal. If a mechanism \( Q \) is \( \varepsilon \)-RTOO, then

1. \( Q \) is Bayesian Incentive Compatible.

2. For any valuation vector \( v^o \) for objects other than \( o \), and any two vNM vectors \( v' = (v^o, \tilde{v}^o) \) and \( v = (v^o, v^o) \) such that agent’s ordinal preferences are the same in both cases, we have: a) \( Q^o(v) = Q^o(v') \), b) \( Q(v) \cdot v' = Q(v') \cdot v' \) and c) \( Q(v) \cdot v = Q(v') \cdot v \).

**Proof**

1. \( BIC \) is an obvious consequence of \( \tilde{v}^o = v^o \) being an admissible value, in the previous definition.

2. For a type \( v = (v^o, v^o) \) with strict associated ordinal preferences, consider a \( \tilde{v}^o \in (v^o, v^o + \varepsilon) \), such that \( (v^o, \tilde{v}^o) \) keeps (strict) ordinal preferences unchanged with respect to \( (v^o, v^o) \). To shorten notation, denote \( v' = (v^o, \tilde{v}^o) \). Notice that, since ordinal preferences remain unvaried, we have \( \tilde{Q}(v;v') = Q(v) \) and \( \tilde{Q}(v';v) = Q(v') \).

---

8 In an extreme case in which the final assignment could be enforced and no adaptation were allowed, our results (Lemma 1 and Theorem 1) would still follow.
\(\varepsilon\)-RTOO implies \(Q(v) \cdot v' \geq Q(v') \cdot v'\), and at the same time Bayesian Incentive Compatibility implies \(Q(v) \cdot v' \leq Q(v') \cdot v'\). We obtain \(Q(v) \cdot v' = Q(v') \cdot v'\).

Consider any value \(\tilde{\tilde{v}}^o \in (\tilde{\tilde{v}}^o, v^o + \varepsilon)\) such that \(v'' = (v^o, \tilde{\tilde{v}}^o)\) preserves the same ordinal preferences as in \(v\) and \(v'\). Along the lines of the previous paragraph, we know that

\[
Q(v) \cdot v'' = Q(v') \cdot v''
\]

Subtracting one equation from the other, we obtain \([Q(v) - Q(v')] \cdot v'' = 0\). Subtracting from this equation the already proven equation \([Q(v) - Q(v')] \cdot v' = 0\), we reach \(Q(v'')(\tilde{\tilde{v}}^o - v^o) = Q(v')(\tilde{\tilde{v}}^o - v^o)\), thus \(Q(v) = Q(v')\).

This result easily extends to cases in which \(\tilde{\tilde{v}}^o - v^o \geq \varepsilon\). There is \(\gamma < \varepsilon\) and a natural number \(n \in \mathbb{N}\) such that \(\tilde{\tilde{v}}^o - v^o = n\gamma\). Let \(v^{(j)} = (v^o, v^o + nj\gamma), j = 1, \ldots, n\). As already shown, we know \(Q(v) = Q(v') = \cdots = Q(v^{(j)}), \) since the difference between valuations for the outside option is less than \(\varepsilon\) in each step. Since \(v' = v^{(j)}\), this concludes the proof of (2a).

As for (2b) and (2c). By Bayesian Incentive Compatibility, we have

\[
Q(v) \cdot v \geq Q(v') \cdot v
\]

\[
Q(v) \cdot v' \leq Q(v') \cdot v'
\]

Should one or both of these inequalities be strict, by subtracting one from the other we would obtain \([Q(v) - Q(v')] \cdot (v^o - v^o) > 0\), implying \(Q(v) < Q(v')\). A contradiction with 2a). We conclude that both inequalities must bind, as we aimed. \(\Box\)

### 3 Robustness and ordinality

We are ready to state the main result of this paper: robustness to outside options implies ordinality. Moreover, any two agent’s types sharing identical ordinal preferences over objects up to some position in the ranking obtain the same probabilities for those objects. Finally we also find the necessity of Ordinal (FOSD) Bayesian Incentive Compatibility, which is defined as \(\sum_{v^o' \geq v^o} Q'(v) \geq \sum_{v^o' \geq v^o} Q''(v';v)\) for every \(s \in S \cup \{o\}\) and \(v, v' \in V\).

**Theorem 1** Let the domain \(V\) be universal. For any \(\varepsilon > 0\), if a mechanism \(Q\) is \(\varepsilon\)-RTOO then it is ordinal. Moreover, the mechanism is Ordinal Bayesian Incentive Compatible.

**Proof**

1. **Ordinality** It is enough if we study the set of vNM types that (strictly) prefer object 1 to object 2 to object 3 and so on. Their ordinal preferences differ only in the position the outside option \(o\) occupies in the agent’s ranking. We prove the...
following induction argument: if for the set of valuations in which \( o \) occupies the \( n \)-th position in the agents’ ranking, all of them obtain same interim probabilities namely \( \bar{Q}^1, \ldots, \bar{Q}^{n-1} \) for the objects ranked above \( o \), then for the set of types for which \( o \) occupies the \( (n+1) \)-th position we have that all of them obtain the same probabilities \( \bar{Q}^1, \ldots, \bar{Q}^{n-1}, \bar{Q}^n \) for the objects ranked above \( o \). Proof of the induction argument Take a valuation vector \( v \) where \( o \) occupies the \( (n+1) \)-th position, and its assigned interim probabilities \( Q^1 (v), \ldots, Q^n (v), Q^o (v) \). All objects (weakly) less-preferred than the outside option obtain zero probability by assumption of the model. Define a valuation vector \( v' \) as \( v' = v, \) if \( s \neq o \), and \( v' = v - \gamma \), where \( \gamma \) is an arbitrarily small positive number. Take the valuation vector \( v'' \) defined as \( v'' = v' - v' \), an affine transformation of \( v' \), which receives the same individual random allocation as \( v' \) does. Then \( Q(v') = Q(v) \). By Lemma 1 we have \( Q(v) \cdot v' = Q(v) \cdot v' \). Given that \( v'' \) is obtained by subtracting the same number from each of the components of \( v' \), \( Q(v') \cdot v'' = Q(v) \cdot v'' \). This is true for every \( \gamma \) we choose. Since \( v'' \) can be arbitrarily close to \( v'' \), Bayesian Incentive Compatibility (implied by Lemma 1 as well) imposes that

\[
\bar{Q}^1 v'' + \cdots + \bar{Q}^{n-1} v'' + \bar{Q}^n (v) v'' \leq Q^1 (v) v'' + \cdots + Q^n (v) v''
\]

Now, for all valuations such that \( o \) occupies the \( n \)-th position, the assignment of interim probabilities \( \bar{Q}^1, \ldots, \bar{Q}^{n-1}, \bar{Q}^n \) must be preferable to the adaptation \( Q^1 (v), \ldots, Q^n (v), Q^o (v) \) for acceptable objects and \( Q^{n+1} (v) + Q^o (v) \) for the outside option. This implies the FOSD condition \( \sum_{i=1}^n \bar{Q}^i \geq \sum_{i=1}^n Q^i (v), n = 1, \ldots, n - 1 \). Together with the former BIC inequality we must have \( (\bar{Q}^1, \ldots, \bar{Q}^{n-1}) = (Q^1 (v), \ldots, Q^n (v)) \). Note that this is true for every valuation vector \( v \) such that \( o \) occupies the \( (n+1) \)-th position. It is easy to conclude (by Bayesian Incentive Compatibility) that we must also have \( Q^o (v) = \bar{Q}^n \) for all \( v \) such that \( o \) occupies the \( (n+1) \)-th position, since

\[
Q^o (v) = Q^o (v) = 1 - \bar{Q}^1 - \cdots - \bar{Q}^{n-1}
\]

which is invariant in \( v \). The induction argument is closed by noticing that the initial condition trivially holds for \( n = 1 \): all valuation vectors such that the outside option is right below object 1 must obtain the same interim probability for object 1.

2. Ordinal Bayesian incentive compatibility This is immediate by Bayesian Incentive Compatibility (implied by Lemma 1) since the assignment of interim probabilities is ordinal.

Comments on the scope of Theorem 1

Our main result is hopelessly restrictive on the potential use of information on cardinal preferences, and thus on the possibilities of achieving ex-ante efficiency goals. It is consequently sound to explore ways to escape from the “ordinal trap”. We might imagine plausible constraints in the domain of ordinal preferences. If there is a linear ordering of elements of \( S \) such that the outside option cannot be ranked ahead of the first three elements in the ordering, then for such an ordering \( \varepsilon \)-RTOO
Random assignments and outside options

admits cardinality. Environments in which ordinal preferences are highly correlated among agents might justify such assumption. Also, one might account for possible gaps in the domain of vNM preferences that break the argument in the previous proof. Gaps (e.g. a finite domain of preferences) give some room to cardinality.

4 Conclusions

We have proposed a desirable property an assignment mechanism should accomplish in the presence of outside options unequally accessible due to income, or talent, or socioeconomic differences. The deduced recipe for universal domains of preferences is clear: use ordinal assignment rules that ignore the cardinality of agents’ preferences. This suggestion should mainly be taken into account in environments in which the outside option (e.g. private schooling and alternatives) could possibly rank high in agents’ ordinal preferences. Unequal access to outside options constitutes a relevant issue in paradigmatic examples of assignment problems such as school choice.

References

Abdulkadiroglu A, Che YK, Yasuda Y (2011) Resolving conflicting preferences in school choice: the Boston mechanism reconsidered. Am Econ Rev 101:399–410
Abdulkadiroglu A, Che YK, Yasuda Y (2015) Expanding “Choice” in School Choice. Am Econ J Microecon 7:1–42
Ashlagi I, Shi P (2016) Optimal allocation without money: an engineering approach. Manage Sci 62:1078–1097
Arnosti N, Randolf T (2021) Parallel lotteries: insights from Alaskan hunting permit allocation (unpublished manuscript)
Bergemann D, Morris S (2005) Robust mechanism design. Econometrica 73:1771–1813
Black SE (1999) Do better schools matter? Parental valuation of elementary education. Q J Econ 114:577–599
Calsamiglia C, Martínez-Mora F, Miralles A (2020) School choice design, risk aversion, and cardinal segregation. Econ J (forthcoming; published online)
Carroll G (2018) On mechanisms eliciting ordinal preferences. Theor Econ 13:1275–1318
Dutta B, Peters H, Sen A (2007) Strategy-proof cardinal decision schemes. Soc Choice Welf 28:163–179
Ehlers L, Majumdar D, Mishra D, Sen A (2020) Continuity and incentive compatibility in cardinal mechanisms. J Math Econ 88:31–41
Featherstone CR, Niederle M (2016) Boston versus deferred acceptance in an interim setting: An experimental investigation. Games Econ Behav 100:353–375
He Y, Miralles A, Pycia M, Yan J (2018) A pseudomarket approach to allocation with priorities. Am Econ J Microecon 10:272–314
Hylland A (1980) Strategy-proofness of voting procedures with lotteries as outcomes and infinite sets of strategies (unpublished manuscript)

Notice that two elements ahead of \( a \) are not enough. Notice, along the lines of the proof of Lemma 1, that \( Q(\nu) \) must be invariant to \( \nu' \) as long as ordinal preferences are maintained and no matter how low the valuation of objects worse than \( a \) is. Fix any two \( \nu, \nu' \) consistent with those ordinal preferences. There are affine transformations \( \hat{\nu}, \hat{\nu}' \) of the former vectors such that \( \hat{\nu} \) and \( \hat{\nu}' \) differ only, if any, on the valuation for the outside option. Hence these types receive equal probability bundles.
Hylland A, Zeckhauser R (1979) A pseudomarket approach to allocation with priorities. J Political Econ. 87:293–314
Kesten O, Kurino M (2016) Do outside options matter in matching? A new perspective on the trade-offs in student assignment (unpublished manuscript)
Kim S (2017) Ordinal versus cardinal voting rules: a mechanism design approach. Games Econ Behav 104:350–371
Miralles A (2008) School choice: the case for the Boston mechanism (unpublished manuscript)
Miralles A, Pycia M (2020) Foundations of pseudomarkets: Walrasian equilibria for discrete resources (unpublished manuscript)
Pycia M (2014) The cost of ordinality (unpublished manuscript)
Pycia M, Ünver U (2017) Outside options in neutral allocation of discrete resources (unpublished manuscript)

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.