GENERALIZED SCHRODINGER EQUATION AND CONSTRUCTIONS OF QUANTUM FIELD THEORY

A. V. STOYANOVSKY

ABSTRACT. The generalized Schrödinger equation deduced in the earlier papers is compared with conventional constructions of quantum field theory. In particular, it yields the usual Schrödinger equation of quantum field theory written without normal ordering. This leads to a definition of certain mathematical version of Feynman integral.

In the papers [2,3] a generalization of the Schrödinger equation to multidimensional variational problems has been deduced. In the present paper this equation is compared with constructions of quantum field theory. To make the paper more self-contained, first the derivation of the generalized Schrödinger equation from [3] is briefly recalled.

1. Derivation of the generalized Schrödinger equation. Consider the action functional of the form

\[ J = \int_D F(x^0, \ldots, x^n, z^1, \ldots, z^m, z^1_{x^0}, \ldots, z^m_{x^n}) \, dx^0 \ldots dx^n. \]

Here \( x^0, \ldots, x^n \) are the independent variables, \( z^1, \ldots, z^m \) are the dependent variables, \( z^i_{x^j} = \frac{\partial z^i}{\partial x^j} \), and integration goes over an \((n+1)\)-dimensional surface \( D \) in \( \mathbb{R}^{m+n+1} \) (the graph of the functions \( z^i(x) \)) with the boundary \( \partial D \). Suppose that for each \( n \)-dimensional surface \( C \) in \( \mathbb{R}^{m+n+1} \) given by the equations

\[ x^i = x^j(s^1, \ldots, s^n), \quad z^i = z^j(s^1, \ldots, s^n) \]

and sufficiently close to a fixed \( n \)-dimensional surface, there exists a unique \((n+1)\)-dimensional surface \( D \) with the boundary \( \partial D = C \) which is an extremal of the variational problem, i.e., the graph of a solution \( z(x) \) to the Euler–Lagrange equations. Denote by \( S(C) \) the integral (1) over the surface \( D \). Then one has the following well known formula for the variation of the functional \( S \):

\[ \delta S = \int_C \left( \sum p^i \delta z^i - \sum H^j \delta x^j \right) ds^1 \ldots ds^n, \]
or
\[
\frac{\delta S}{\delta z^i(s)} = p^i(s),
\]
\[
\frac{\delta S}{\delta x^j(s)} = -H^j(s),
\]
where
\[
p^i = \sum_l (-1)^l F^i_{z^l} \frac{\partial (x^0, \ldots, \hat{x}^l, \ldots, x^n)}{\partial (s^1, \ldots, s^n)},
\]
\[
H^j = \sum_{l \neq j} (-1)^l F^j_{z^l} \frac{\partial (x^0, \ldots, \hat{x}^j, \ldots, x^n)}{\partial (s^1, \ldots, s^n)}
\]
\[+ (-1)^j (F^j_{x^j} z^i_{x^j} - F) \frac{\partial (x^0, \ldots, \hat{x}^j, \ldots, x^n)}{\partial (s^1, \ldots, s^n)}.
\]

Here \(\frac{\partial (x^1, \ldots, x^n)}{\partial (s^1, \ldots, s^n)}\) is the Jacobian; the cap over a variable means that the variable is omitted; and the summation sign over the index \(i\) repeated twice is omitted.

The quantities \(p^i\) and \(H^j\) satisfy the relations
\[
p^i z^i_{sk} - H^j x^j_{sk} = 0, \quad k = 1, \ldots, n,
\]
and one more relation which we denote by
\[
\mathcal{H}(x^j(s), z^i(s), x^j_{sk}, z^i_{sk}, p^i(s), -H^j(s)) = 0.
\]

From these relations one can express (in general) the quantities \(H^j\) as functions of \(p^i\):
\[
H^j = H^j(x^l, z^i, x^l_{sk}, z^i_{sk}, p^i), \quad j = 0, \ldots, n.
\]

Substituting (4) into (6,7) or into (8), one obtains the generalized Hamilton–Jacobi equations:
\[
\frac{\delta S}{\delta z^i(s)} z^i_{sk} + \frac{\delta S}{\delta x^j(s)} x^j_{sk} = 0, \quad k = 1, \ldots, n,
\]
\[
\mathcal{H}\left(x^j, z^i, x^j_{sk}, z^i_{sk}, \frac{\delta S}{\delta z^i(s)}, \frac{\delta S}{\delta x^j(s)}\right) = 0,
\]
or
\[
\frac{\delta S}{\delta x^j(s)} + H^j\left(x^l, z^i, x^l_{sk}, z^i_{sk}, \frac{\delta S}{\delta z^i(s)}\right) = 0, \quad j = 0, \ldots, n.
\]

The first \(n\) equations in the system (9) correspond to the fact that the functional \(S(C)\) is independent on the parameterization of the surface.
C. For a theory of the generalized Hamilton–Jacobi equations, see [3] or [1].

Assume that functions (8) are polynomials in the variables $p^i$. Let us make the following substitution in the generalized Hamilton–Jacobi equations:

$$
\frac{\delta S}{\delta x^j(s)} \rightarrow -i\hbar \frac{\delta}{\delta x^j(s)},
$$

$$
\frac{\delta S}{\delta z^i(s)} = p^i \rightarrow -i\hbar \frac{\delta}{\delta z^i(s)}.
$$

Here $i$ is the imaginary unit, $\hbar$ is a very small constant (the Plank constant). We obtain the system of linear variational differential equations which can be naturally called the generalized Schrödinger equations:

$$
-i\hbar \frac{\delta \Psi}{\delta x^j(s)} + H^j \left( x^j, z^i, x^j_{,k}, z^i_{,k}, -i\hbar \frac{\delta}{\delta z^i(s)} \right) \Psi = 0, \quad j = 0, \ldots, n,
$$

or

$$
\frac{\delta \Psi}{\delta z^i(s)} z^i_{,k} + \frac{\delta \Psi}{\delta x^j(s)} x^j_{,k} = 0, \quad k = 1, \ldots, n,
$$

$$
\mathcal{H} \left( x^j, z^i, x^j_{,k}, z^i_{,k}, -i\hbar \frac{\delta}{\delta z^i(s)}, -i\hbar \frac{\delta}{\delta x^j(s)} \right) \Psi = 0
$$

provided that the left-hand side of equation (7) is also a polynomial in $p^i$ and $H^j$. Here $\Psi = \Psi(C)$ is the unknown complex-valued functional of the surface $C$ (2). The first $n$ equations in the system (13) mean that the value $\Psi(C)$ is independent on the parameterization of the surface $C$.

2. Comparison with quantum field theory. Since the value $\Psi(C)$ is independent of the parameterization of the surface $C$, we can choose a particular parameterization. Put $s^1 = x^1, \ldots, s^n = x^n$. The generalized Schrödinger equation becomes a single equation, which we choose to be the first equation from the system (12) corresponding to $j = 0$. This equation can be easily computed from (5):

$$
-i\hbar \frac{\delta \Psi}{\delta x^0(x)} + H \left( x^0(x), x, z^i(x), \frac{\partial x^0}{\partial x}, \frac{\partial z^i}{\partial x}, -i\hbar \frac{\delta}{\delta z^i(x)} \right) \Psi = 0,
$$
where \( \mathbf{x} = (x^1, \ldots, x^n) \), \( \frac{\partial z^i}{\partial \mathbf{x}} = (\frac{\partial z^i_1}{\partial x^1}, \ldots, \frac{\partial z^i_n}{\partial x^n}) \), and

\[
H = H \left( x^0, \mathbf{x}, z^i, \frac{\partial x^0}{\partial \mathbf{x}}, \frac{\partial z^i}{\partial \mathbf{x}}, \mathbf{p} \right) = \sum_i p^i z^i_{x^0} - F(x^0, \mathbf{x}, z^i, z^i_{x^0}, z^i_{x^j}),
\]

\[
z^i_{x^j} = \frac{\partial z^i}{\partial x^j} - z^i_{x^0} \frac{\partial x^0}{\partial x^j}, \quad j = 1, \ldots, n,
\]

\[
p^i = F_{z^i_{x^0}} - \sum_{j=1}^n F_{z^i_{x^j}} \frac{\partial x^0}{\partial x^j} = \frac{\partial F}{\partial z^i_{x^0}}.
\]

That is, \( H \) is the Legendre transform of the Lagrangian \( F \) with respect to the variables \( z^i_{x^0} \). Equation (14) looks approximately like the Tomonaga–Schwinger equation \([4]\), with three differences: a) it has a mathematical sense, unlike the Tomonaga–Schwinger equation \([5]\); b) instead of Hilbert space of states, we have the space of functionals of functions \( z^i(\mathbf{x}) \) for any spacelike surface \( x^0 = x^0(\mathbf{x}) \); c) there are no normal orderings. Given any functional \( \Psi_0(z^i(\mathbf{x})) \) for a fixed spacelike surface \( x^0 = x^0(\mathbf{x}) \), equation (14) describes the evolution of the functional \( \Psi_0 \) as the spacelike surface varies. We conjecture that equation (14) is integrable, i.e., the result of evolution exists (for conventional Lagrangians) and depends only on the initial and final spacelike surfaces and not on the concrete way of evolution of the surface.

The following definition is motivated in part by the geometric picture of excitations propagating along surfaces from \([3]\).

**Definition.** Assume we are given: 1) two spacelike surfaces \( C_0: x^0 = x^0(\mathbf{x}) \) and \( C_1 \) bounding a domain \( \mathcal{D} \) in \( \mathbb{R}^{n+1} \); 2) a functional \( \Psi_0(z^i(\mathbf{x})) \) on the space of functions \( z^i(\mathbf{x}) \) on the first surface \( C_0 \); and 3) functions \( z^i = z^i_1(\mathbf{x}) \) on the second surface \( C_1 \). Then a mathematical version of the Feynman integral

\[
\int e^{i \int_{x^0 \mathbf{x}} F_{\mathbf{x}} dx} \Psi_0(z^i(x^0(\mathbf{x}), \mathbf{x})) \prod Dz^i(x),
\]

taken over all functions \( z^i(x) \) on the closed domain \( \mathcal{D} \) whose restriction to the second boundary surface \( C_1 \) coincides with the given functions \( z^i_1 \) (and restriction to \( C_0 \) is arbitrary), is defined as follows. Take the functional \( \Psi_0 \) as the initial value corresponding to the first surface \( C_0 \), and consider the evolution of this functional given by the generalized Schrödinger equation (14). We obtain a functional \( \Psi \) of a spacelike surface and of functions \( z^i \) on it. Take the value of this functional at the surface \( C_1 \) and the functions \( z^i_1 \). This is the desired Feynman integral.
Considering evolution of flat spacelike surfaces \( x^0(\mathbf{x}) = \text{const} = t \), we arrive at the evolution equation for a functional \( \Psi(t, z^i(\mathbf{x})) \):

\[
(17) \quad i\hbar \frac{\partial \Psi}{\partial t} = \int H \left( t, \mathbf{x}, z^i(\mathbf{x}), \frac{\partial z^i}{\partial \mathbf{x}}, -i\hbar \frac{\delta}{\delta z^i(\mathbf{x})} \right) \Psi \, d^n\mathbf{x}.
\]

This equation differs from the usual quantum field theory Schrödinger equation by absence of normal orderings; but it has a mathematical sense.

It is natural to ask whether the operator of evolution of the functional \( \Psi \) from \( t = -\infty \) to \( t = \infty \) described by equation (17) exists for conventional Lagrangians of quantum field theory.

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Moscow Center for Continuous Mathematical Education, Bolshoj Vlasievskij per., 11, Moscow, 119002, Russia