Faithful actions of braid groups by twists along ADE-configurations of spherical objects

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Abstract. We prove that the actions of the generalized braid groups on enhanced triangulated categories, generated by spherical twist functors along ADE-configurations of \( \omega \)-spherical objects, are faithful for any \( \omega \neq 1 \).

1. Introduction

Spherical twists along spherical objects are a prominent type of autoequivalences of triangulated categories. The notion of a spherical twist functor was first introduced in [17] by P. Seidel and R. Thomas in connection with the Kontsevich’s homological mirror symmetry program. Their original motivation was to look at the autoequivalences of the derived category of coherent sheaves on a variety that would arise as counterparts of generalized Dehn twists via mirror symmetry. In a general setting, a spherical twist is a functor constructed in a particular way from a spherical object, an object of a triangulated category whose Ext algebra is the same as the cohomology of a sphere.

The theory of Seidel and Thomas received a lot of attention and developed rapidly in numerous works of other mathematicians. For instance, in [1] the notion of a spherical twist along a spherical functor was introduced, generalizing spherical twists along spherical objects. In [16], it was established that any triangulated autoequivalence is in fact a twist along some spherical functor. The theory of spherical twists constructed from spherical sequences was developed in [6]. Groups that can be generated by two spherical twists constructed from spherical sequences were described in [15]. Since the appearance of [17] in 2001, spherical twists have proved to be a useful tool in algebraic geometry and beyond. Among their applications are categorifications of Braid groups ([13]), Bridgeland stability conditions manifolds ([9]) and derived Picard groups ([19]).

Let \( \Gamma \) be a simply-laced Dynkin diagram. Seidel and Thomas showed that spherical twists along a so-called \( \Gamma \)-configuration of \( \omega \)-spherical objects satisfy braid relations of type \( \Gamma \) modulo natural isomorphisms, hence induce an action of an Artin group (generalized braid group) \( B_{\Gamma} \) on the triangulated category in question. In the same paper they showed that for \( \omega \geq 2 \) and \( \Gamma = A_n \) this action is faithful. By [6], their result can be also extended to the case \( \omega = 0 \). The paper [17] also provides an example when the action is not faithful for \( \omega = 1 \). Later several more partial results on faithfulness have appeared. In [2], C. Brav and H. Thomas proved that the braid group action is faithful for \( \omega = 2 \) and all \( \Gamma \). Recently Y. Qui and J. Woolf generalized their result to \( \omega \geq 2 \) for the derived category of a Ginzburg algebra ([12]).

In this paper we present a new method that enables us to generalize all of the existing faithfulness results, proving that the braid group action is faithful for any enhanced triangulated category, all \( \Gamma = A_n, D_n, E_6, E_7, E_8 \) and all \( \omega \neq 1 \), including the case \( \omega < 0 \), not covered before. Although originally Seidel and Thomas did not define \( \omega \)-spherical objects with negative \( \omega \), they are also worth considering (for instance, see [8], [3], [4], [5]).

The proofs we present for \( \omega \geq 2 \) and \( \omega \leq 0 \) follow the same general strategy at the beginning, but go their very different ways from some point. For the former we begin by simplifying the proof for \( \omega = 2 \) by Brav and Thomas [2], which then allows us to extend
it to arbitrary \( \omega \geq 2 \). After some preparation, this takes us only about two pages, and the proof we obtain for the case \( \omega = 2 \) in particular is shorter and more elementary than in the original exposition of \([2]\). Proving faithfulness in the case \( \omega \leq 0 \) turns out to be a much more difficult task. For that we will develop the theory of so-called two-term objects in a triangulated category with a configuration of spherical objects. This part also requires more sophisticated combinatorial arguments dealing with words in the generalized braid monoids.

Section 2 introduces basic notions crucial in the paper, i.e. we define spherical objects, spherical twist functors, configurations of spherical objects, etc. Section 3 contains the main result of the paper as well as the the key lemma on which its proof is based on and some useful technical observations. At the end of this section it is also explained how the key lemma implies the main result. The rest of the paper is mainly devoted to proving the key lemma. Section 4 provides the proof in the case \( \omega \geq 2 \). In Section 5 we outline the proof of the key lemma in the case \( \omega \leq 0 \) and then provide the details in the next three sections. In Section 6 we introduce two-term objects and study some of their properties that we use later. Section 7 explains the first step in the proof of the key lemma when \( \omega \leq 0 \), the factorization. Section 8 contains the second step, to which we refer as ”braiding”. This finishes the proof of the key lemma and hence the main result of the paper. The statement obtained in Section 8 deals with words in the generalized braid monoids and might be of interest not only in connection with braid group actions on triangulated categories. In the last section one application of the main result of this paper is presented. Namely, we show how it may be used to make the first yet crucial step towards the description of the derived Picard groups of representation-finite selfinjective algebras.

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2. Preliminaries

Throughout this paper \( \mathcal{D} \) is a triangulated category linear over a field \( k \) and with a fixed enhancement. For example, it can be an algebraic triangulated category in the sense of Keller (see \([11]\)), i.e. the stable category of some Frobenius category. In this case we are equipped with functorial cones of natural transformations of exact functors, and for every object \( X \) of \( \mathcal{D} \) there is the derived Hom-complex functor \( \text{RHom}(X, -) : \mathcal{D} \to D(k) \) and its right adjoint \( - \otimes X : D(k) \to \mathcal{D} \), where \( D(k) \) denotes the unbounded derived category of \( k \)-vector spaces.

Let \( \text{Hom}^k(A, B) \) denote \( \text{Hom}_{\mathcal{D}}(A, B[k]) \) and \( \text{Hom}^*(A, B) = \bigoplus_{t \in \mathbb{Z}} \text{Hom}^k(A, B) \). For elements \( g \in \text{Hom}^k(A, B) \) and \( f \in \text{Hom}^i(B, C) \), we will write \( fg \) for the element \( f[k]g \in \text{Hom}^{k+i}(A, C) \).

**Definition 2.1.** (Seidel, Thomas \([17]\)) Let \( \omega \in \mathbb{Z} \). An object \( P \in \mathcal{D} \) is called \( \omega \)-spherical if

(i) \( \text{dim}_k \text{Hom}^*(X, P) < \infty \) for any object \( X \in \mathcal{D} \).

(ii) \( \text{Hom}^*(P, P) \cong k[t]/(t^\omega) \) as graded \( k \)-algebras, where \( \text{deg}(t) = \omega \).

(iii) \( \text{Hom}^*(P, X) \times \text{Hom}^*(X, P) \xrightarrow{\delta} \text{Hom}^*(P, P)/(Id_P) \cong k \)

is a perfect pairing for any \( X \in \mathcal{D} \) (defined by the composition).

Fix some undirected graph \( \Gamma \). We will assume that \( \Gamma \) is an ADE Dynkin diagram, but part of our arguments can be transferred to more general cases. Let us represent the set \( \Gamma_0 \) of vertices of \( \Gamma \) as a disjoint union of sets \( V^0 \) and \( V^1 \) in such a way that all edges of \( \Gamma \) have one endpoint in \( V^0 \) and the second one in \( V^1 \). By \( N(A) \) for \( A \subseteq \Gamma_0 \) we denote the set all
neighbors of all vertices in \( A \) (i.e. \( N(A) \) is formed by \( j \in \Gamma_0 \) such that \( s_j s_k \neq s_k s_j \) for some \( k \in A \)).

Following Brav and Thomas (2), we now define a \( \Gamma \)-configuration of spherical objects.

**Definition 2.2.** A collection of \( \omega \)-spherical objects \( \{P_i\}_{i \in \Gamma} \) enumerated by vertices of \( \Gamma \) is a \( \Gamma \)-\textit{configuration} if for any \( i \neq j \)

1. \( \text{Hom}^* (P_i, P_j) \) is one-dimensional if \( i \in N(j) \);
2. \( \text{Hom}^* (P_i, P_j) = 0 \) if \( i \notin N(j) \).

Fix some integers \( \omega_0 \) and \( \omega_1 \) such that \( \omega_0 + \omega_1 = \omega \). To simplify our proofs we will assume that \( \omega_0, \omega_1 \leq 0 \) if \( \omega \leq 0 \) and \( \omega_0, \omega_1 \geq 1 \) if \( \omega \geq 2 \). For example, we can simply take \( \omega_0 = \omega_1 = \frac{\omega}{2} \) if \( \omega \) is even and \( \omega_0 = \frac{\omega+1}{2}, \omega_1 = \frac{\omega-1}{2} \) if \( \omega \) is odd. It follows from the definition of \( \omega \)-spherical object that after shifting the objects \( P_i \) in a \( \Gamma \)-configuration, one may assume that \( \text{Hom}^* (P_i, P_j) \) is concentrated in degree \( \omega_i \), where \( u \) is such that \( i \) belongs to \( V^u \). We always take the index \( u \) in the notation \( \omega_u, V^u \) modulo 2.

**Definition 2.3.** (Seidel, Thomas [17]) Let \( P \) be an \( \omega \)-spherical object. The \textit{spherical twist} functor \( t_P \) along \( P \) is defined by

\[
t_P(X) = \text{cone}(P \otimes \text{RHom}(P, X) \xrightarrow{\text{counit}} X)
\]

**Remark.** It is indeed a functor since we have functorial cones of natural transformations of exact functors.

**Definition 2.4.** The generalized braid group (Artin group) \( B_\Gamma \) is generated by \( s_i, i \in \Gamma_0 \) subject to the braid relations \( s_i s_j s_i = s_j s_i s_j \) for \( i, j \) adjacent in \( \Gamma \) and \( s_i s_j = s_j s_i \) for \( i, j \) not adjacent in \( \Gamma \). The braid monoid \( B_\Gamma^+ \) is a monoid given by the same generators and relations.

We are now going to recall some crucial facts about spherical twists and spherical objects (see [17]) that we will actively use throughout the paper.

1. If \( P \) is spherical, then \( t_P \) is an autoequivalence of \( \mathcal{D} \) with a quasi inverse \( t'_P \) defined by
   \[
t'_P(X) = \text{cone}(X \xrightarrow{\text{unit}} P \otimes \text{RHom}(P, X)^*)[-1],
\]
   where \( * \) is the usual duality on the category \( D(k) \).
2. If \( P \) is \( \omega \)-spherical, then \( t_P(P) = P[1 - \omega] \).
3. For \( \{P_i\}_{i \in \Gamma_0} \) forming a \( \Gamma \)-configuration, \( t_P \) satisfy braid relations of type \( \Gamma \) up to a natural isomorphism. In particular, for spherical \( P, Q \) not adjacent in their \( \Gamma \)-configuration (equivalently \( \text{Hom}^*(P, Q) = 0 \)), \( t_P(Q) = Q \). In other words, there is a group homomorphism
   \[
   F: B_\Gamma \to \text{Aut}(\mathcal{D})
   \]
   where \( \text{Aut}(\mathcal{D}) \) is a group of autoequivalences of \( \mathcal{D} \) modulo natural isomorphisms. For \( \alpha \in B_\Gamma \), we denote \( F(\alpha) \) by \( t_\alpha \).

3. **Main result**

We have just defined an action of the braid group \( B_\Gamma \) of type \( \Gamma \) on the category \( \mathcal{D} \). The main result of this paper says that this action is faithful for every \( \omega \neq 1 \). The rest of this paper is mainly devoted to proving this claim:

**Theorem 1.** If \( \Gamma \) is a simply-laced Dynkin diagram, \( \omega \neq 1 \) and \( \{P_i\}_{i \in \Gamma_0} \) is a \( \Gamma \)-configuration of \( \omega \)-spherical objects in an enhanced triangulated category \( \mathcal{D} \), then the action of \( B_\Gamma \) on \( \mathcal{D} \) generated by the spherical twists \( t_P \) is faithful.

As we have already mentioned in the Introduction, the action for \( \omega = 1 \) can be not faithful (see [17]).

Since the case \( |\Gamma_0| = 1 \) is clear, we will assume that \( \Gamma \) has at least two vertices. We will use the following auxiliary notation in our proof. For \( i \in V^u \) and \( j \in N(i) \), we fix some
nonzero element of \( \text{Hom}^{ω}(P_i, P_j) \) and denote it by \( γ_{i,j} \). We also introduce the morphisms \( ρ_{i,j}: P_j \to τ_iP_j \) and \( ξ_{i,j}: τ_iP_j \to P_i[1−ω_u] \) via the triangle

\[
P_i[−ω_u] \xrightarrow{γ_{i,j}} P_j \xrightarrow{ξ_{i,j}} τ_iP_j \xrightarrow{ρ_{i,j}} P_i[1−ω_u].
\]

We follow the same strategy to prove Theorem \( \#1 \) in the cases \( ω \leq 0 \) and \( ω \geq 2 \). Nevertheless, the proof of the case \( ω \geq 2 \) will be finished much earlier because it turns out to be somewhat easier. In fact, we begin by simplifying the proof for \( ω = 2 \) by Brav and Thomas \( \#2 \) and then adapt it for \( ω \geq 2 \). The proof we obtained for \( ω = 2 \) basing on the ideas of \( \#2 \) not only allows us to then extend it to arbitrary \( ω \geq 2 \), but also has the advantage of being shorter and easier then the original version of Brav and Thomas.

**Let \( Λ = \bigoplus_{i=1}^n P_i \). We will also write \( t_i \) instead of \( t_P \) for brevity.**

We define the minimal and the maximal nonzero degree of an object in \( \mathfrak{D} \):

**Definition 3.1.** Let \( T \in \mathfrak{D} \). We say that \( \text{min}(T) \) (respectively \( \text{max}(T) \)) is the minimal (respectively the maximal) nonzero degree of \( T \) if \( \text{Hom}^{k+ω}(\Lambda, T) = 0 \) for \( k \leq \text{min}(T) − 1 \) (respectively for \( k \geq \text{max}(T) + 1 \)) and \( \text{Hom}^{\min(T)+ω}(\Lambda, T) \) (respectively \( \text{Hom}^{\max(T)+ω}(\Lambda, T) \)) is nonzero.

**Definition 3.2.** Let \( T \) be an object of \( \mathfrak{D} \). We say that \( P_j \) with \( j \in Π_0 \) is a direct summand of \( T_r \) if there exists a nonzero \( f \in \text{Hom}^{r+ω}(P_j, T) \) such that \( fγ_{k,j} = 0 \) for any \( k \in N(j) \). A morphism \( f \) satisfying this condition will be referred to as long. In other words, a nonzero morphism \( f: P_j \to T \) is called long if the induced morphism \( \text{Hom}^*(P_k, f): \text{Hom}^*(P_k, P_j) \to \text{Hom}^*(P_k, T) \) is zero for any \( k \neq j \). We also say that \( P_j \) is a direct summand of \( T_{[a,b]} \) if \( P_j \) is a direct summand of \( T_c \) for some \( c \in [a, b] \).

**Remark.** Let \( T \in \mathfrak{D} \) and \( i \in V^u \). If \( \text{Hom}^{r+ω}(P_i, T) \neq 0 \) for some \( r \leq \text{min}(T) − ω_{u+1} − 1 \), then \( P_i \) is a direct summand of \( T_r \). Indeed, any composition of the form \( P_j \to P_i[ω_{u+1}] \to T[r + ω + ω_{u+1}] \) is zero by the definition of minimal nonzero degree. Moreover, if \( \text{Hom}^{\min(T)+ω}(P_j, T) = 0 \) for every \( j \in N(i) \) and \( \text{Hom}^{\min(T)+ω−ω_{u+1}}(P_i, T) \neq 0 \), then \( P_i \) is a direct summand of \( T_{\min(T)−ω_{u+1}} \), because in this case any composition of the form \( P_j \to P_i[ω_{u+1}] \to T[\min(T) + ω] \) is zero too. Analogously, if \( \text{Hom}^{r+ω}(P_i, T) \neq 0 \) for some \( i \in V^u \) and \( r \geq \max(T) − ω_{u+1} + 1 \), then \( P_i \) is a direct summand of \( T_r \).

We denote \( t_α(Λ) \) by \( T_α \). Let \( \text{min}(α) \) and \( \text{max}(α) \) denote the minimal and the maximal nonzero degrees of \( T_α \) respectively. Following \( \#2 \), we will deduce the faithfulness of the braid group action from the injectivity of the induced monoid homomorphism \( B_Γ^+ \to \text{Aut}(\mathfrak{D}) \). In turn, to prove the injectivity of the aforementioned monoid homomorphism, we require a tool that would allow us to find a leftmost factor of the reduced expression of \( α \in B_Γ^+ \) using only information about \( T_α \). This tool is presented in the following key lemma.

**Lemma 1.** Let \( α \in B_Γ^+ \), \( α \neq 1 \). For \( u \in \{0, 1\} \) let \( I_{u,α} = \text{max}(α)−ω_u−1 \) in the case \( ω \leq 0 \) and \( I_{u,α} = \text{max}(α)−ω_{u+1}+1 \) in the case \( ω \geq 2 \). Then for any \( u \in \{0, 1\} \) and any \( j \in V^u \) such that the corresponding object \( P_j \) is a direct summand of \( T_{I_{u,α}} \), the word \( α \) can be written as

\[
α = s_jα'
\]

for some \( α' \in B_Γ^+ \) with \( l(α) = l(α') + 1 \).

Before proving Lemma \( \#1 \) we are going to show how it easily implies our main result. However, first we are going to prove some rather technical facts regarding the way spherical twists affect the minimal and the maximal degree of an object. These statements will be required in our considerations throughout the paper. In this technical part we will consider the cases \( ω \leq 0 \) and \( ω \geq 2 \) separately as in the major part of our proofs. Although here these two cases can be unified, we are not going to do so to avoid giving a feeling that the case \( ω \geq 2 \) is more complicated than it really is.
Lemma 2. Let $\omega \leq 0$, $T$ be an object of $\mathcal{O}$ and $m$ be its minimal nonzero degree.

1) If $\operatorname{Hom}^r(P_i, t_i^{-1}T) \neq 0$, then $m \leq r - 1$.

2) For $k \in V^u$, $r \leq m - \omega_{u+1}$ and $i \neq k$, $P_k$ is a direct summand of $(t_i^{-1}T)_r$ if and only if $P_k$ is a direct summand of $T_r$.

3) The minimal nonzero degree of $t_i T$ belongs to $[m - 1 + \omega, m]$ and $P_k$ can be a direct summand of $T_r$ with $r < m$ only if $k = i$.

Proof. 1) As spherical twists are autoequivalences,

$$0 \neq \operatorname{Hom}^r(P_i, t_i^{-1}T) \cong \operatorname{Hom}^r(t_i P_i, T) = \operatorname{Hom}^r(P_i[1 - \omega], T) \cong \operatorname{Hom}^{r-1+\omega}(P_i, T).$$

Hence the minimal nonzero degree $m$ of $T$ is not greater than $r - 1$.

2) Suppose that $k \notin N(i)$. Since $t_i$ is an autoequivalence and $t_i P_k = P_k$ in this case, we have an isomorphism $t_i : \operatorname{Hom}^r(P_k, t_i^{-1}T) \cong \operatorname{Hom}^r(P_k, T)$. It is sufficient to show that $f : P_k \rightarrow t_i^{-1}T[r + \omega]$ is long if and only if $t_i f$ is long. Pick some $l \in N(k)$. If $l \notin N(i)$, then $t_l P_l = P_l$ and we have $\operatorname{Im} \operatorname{Hom}^r(P_l, t_l f) = t_l \operatorname{Im} \operatorname{Hom}^r(P_l, f)$. Since $t_i$ is an autoequivalence, we have $\operatorname{Hom}^r(P_l, t_l f) = 0$ if and only if $\operatorname{Hom}^r(P_l, f) = 0$.

If $l \in N(i)$, then $i \in V^u$ and we have a triangle

$$P_k \xrightarrow{\rho_{i,k}} t_i P_k \xrightarrow{t_i f_{l,k}} P_l[1 - \omega_u].$$

Since $\operatorname{Hom}^r(P_l, P_k) = 0$, we have $\gamma_{l,k} = g_{i,l}$ for some $g \in \operatorname{Hom}^u(t_l P_l, P_k)$. Hence, if $(t_l f)_{l,k} \neq 0$, then $f(t_i^{-1}g) \neq 0$, and $\operatorname{Hom}^r(P_l, f) \neq 0$. On the other hand, if $f_{l,k} \neq 0$, then the composition

$$P_l \xrightarrow{\rho_{i,l}} t_l P_l \xrightarrow{t_l(f_{l,k})} T[r + \omega + \omega_{u+1}]$$

is nonzero, because $r + \omega - 1 \leq m + \omega_u - 1 < m$, and hence

$$\operatorname{Hom}^{r+\omega+\omega_{u+1}}(P_l[1 - \omega_u], T) \cong \operatorname{Hom}^{r+2\omega-1}(P_l, T) = 0.$$  

Then we have $\operatorname{Hom}^r(P_l, t_l f) = 0$. Thus, $f$ is long if and only if $t_i f$ is long.

Suppose now that $k \in N(i)$. Consider the triangle

$$(1) \quad P_k \xrightarrow{\rho_{i,k}} t_i P_k \xrightarrow{t_i f_{l,k}} P_l[1 - \omega_u].$$

For any nonzero $f : P_k \rightarrow t_i^{-1}T[r + \omega]$, one has $t_i f \rho_{i,k} \neq 0$, because $r + \omega_{u+1} - 1 \leq m - 1$, the minimal nonzero degree of $T$ is $m$, and hence $\operatorname{Hom}^r(P_l, P_k[1 - \omega_{u+1}], T) = 0$. Since $(t_i f)_{l,k} \neq 0$, it remains to show that $(t_i f)_{l,k} \neq 0$ for $l \in N(k) \setminus \{i\}$ if $f$ is long. Indeed, suppose it is nonzero and apply $t_i^{-1}$. Since $l \notin N(i)$, we get a nonzero morphism $P_l \rightarrow P_k$.

Pick now a long morphism $f' : P_k \rightarrow T[r + \omega]$. Then $f' \gamma_{l,k}[-\omega_{u+1}] = 0$ by the definition of a long morphism, and hence $f' = (t_l f) \rho_{i,k}$ for some $f : P_k \rightarrow t_i^{-1}T[r + \omega]$. We have $0 = f \gamma_{l,k} : P_l \rightarrow t_i^{-1}T[r + \omega + \omega_{u+1}]$, because otherwise the minimal nonzero degree of $T$ would be greater than $r + \omega + \omega_{u+1} - 1 \leq m + \omega - 1 < m$, by the first assertion of the current lemma. Pick some $l \in N(k) \setminus \{i\}$. Since $\operatorname{Hom}^r(P_l, P_i) = 0$, the morphism $t_l f_{l,k} : P_l \rightarrow t_l P_k[\omega_{u+1}]$ factors through $\rho_{i,k}[\omega_{u+1}]$, and hence equals $\rho_{i,k} \gamma_{l,k}$ modulo a nonzero scalar. Since $(t_l f)_{l,k}$ is long, one has $(t_l f)_{l,k} \gamma_{l,k} = 0$.

Applying $t_i^{-1}$, one sees that $f \gamma_{l,k} = 0$ as well. Thus, there exists a long morphism from $P_k$ to $T[r + \omega]$ if and only if there exists a long morphism from $P_k$ to $t_i^{-1}T[r + \omega]$.

3) First we show that the minimal nonzero degree of $t_i T$ is not greater than $m$. There exists some $k \in \Gamma_0$ such that $P_k$ is a direct summand of $T_m$. If $k \neq i$ and the minimal nonzero degree of $t_i T$ is greater than $m$, then $P_k$ is a direct summand of $(t_i T)_m$ by the second assertion of this lemma and we get a contradiction. If $k = i$, then $\operatorname{Hom}^{m+\omega}(P_i, T) \neq 0$ and the minimal nonzero degree of $t_i T$ is not greater than $m + \omega - 1 < m$ by the first assertion of this lemma. Thus, the minimal nonzero degree of $t_i T$ is not greater than $m$.

Now suppose $P_k$ is a direct summand of $(t_i T)_r$ for $r < m$, $k \neq i$. If $k \notin N(i)$, then $0 \neq \operatorname{Hom}^{r+\omega}(P_k, t_i T) \cong \operatorname{Hom}^{r+\omega}(P_k, T)$, which contradicts the minimal nonzero
Lemma 3. Let $\omega \geq 2$, $T$ be an object of $\mathcal{D}$ and $h$ be its maximal nonzero degree.

1) If $\text{Hom}^r(P_i, t_i^{-1}T) \neq 0$, then $h \geq r - 1$.

2) Suppose that $k \in \Gamma_0 \setminus \{\{i\} \cup N(i)\}$. Then $\text{Hom}^{r+i}(P_k, t_i^{-1}T) \neq 0$ if and only if $\text{Hom}^{r+i}(P_k, T) \neq 0$.

3) Suppose that $r \geq h - \omega_u + 2$ and $k \in V_u$, $k \neq i$. If $\text{Hom}^{r+i}(P_k, t_i^{-1}T) \neq 0$, then $\text{Hom}^{r+i}(P_k, T) \neq 0$.

4) Suppose that $r \geq h - \omega_u + 1$ and $k \in V_u$, $k \neq i$. If $\text{Hom}^{r+i}(P_k, T) \neq 0$, then $\text{Hom}^{r+i}(P_k, t_i^{-1}T) \neq 0$.

5) The maximal nonzero degree of $t_iT$ is not less than $h$.

Proof. 1) Just as in Lemma 2, we get $0 \neq \text{Hom}^{r-1+i}(P_i, T)$, and hence $h \geq r - 1$.

2) The required assertion is clear, because in the case $k \notin \{i\} \cup N(i)$ one has $\text{Hom}^{r+i}(P_k, t_i^{-1}T) = \text{Hom}^{r+i}(t_i^{-1}P_k, t_i^{-1}T) \cong \text{Hom}^{r+i}(P_k, T)$.

3) The case $k \notin N(i)$ is already considered. Suppose now that $k \in N(i)$. Consider the triangle \([1]\). For any nonzero $f: P_k \to t_i^{-1}T[r + \omega]$, one has $0 \neq (t_i f)\rho_{i,k}: P_k \to T[r + \omega]$, because $r + \omega_u + 1 \geq h + 1$. The required assertion now follows.

4) Due to already proven assertions, it remains to consider the case $k \in N(i)$. Consider the triangle \([1]\) and pick a nonzero morphism $f': P_k \to T[r + \omega]$. Since $r + \omega_u + 1 \geq h + 1$, one has $f'\rho_{i,k}t_i^{-1}T[r + \omega] = 0$, and hence $f' = (t_i f)\rho_{i,k}$ for some nonzero $f: P_k \to t_i^{-1}T[r + \omega]$. This finishes the proof.

5) There exists some $k \in \Gamma_0$ such that $\text{Hom}^{h+i}(P_k, T) \neq 0$. If $k \neq i$ and the maximal nonzero degree of $t_iT$ is less than $h$, then $\text{Hom}^{h+i}(P_k, t_iT) \neq 0$ by the previous assertions of this lemma, hence we get a contradiction. If $k = i$, then $\text{Hom}^{h+i}(P_i, T) \neq 0$ and the maximal nonzero degree of $t_iT$ is not less than $h + \omega - 1 > h$ by the first assertion of this lemma. Thus, the required assertion is valid in all cases.

Now we are ready to deduce Theorem 1 from Lemma 1.

Proof of Theorem 1. According to [2] Proposition 2.3, a group homomorphism $B_T \to G$ is injective if and only if the induced monoid homomorphism $B_T^+ \to B \to G$ is injective. Hence, in our case it is sufficient to show that $B_T^+ \to \text{Aut}(\mathcal{D})$ is injective. Assume that it is not. Choose two words $\alpha, \beta$ with the smallest sum of lengths $l(\alpha) + l(\beta)$ among all pairs of words with coinciding images in $\text{Aut}(\mathcal{D})$ and $\alpha \neq \beta$. In particular, $t_\alpha(\Lambda) = T_\alpha \cong T_\beta = t_\beta(\Lambda)$ in $\mathcal{D}$. Thus, for any $u \in \{0, 1\}$, $I_{u,\alpha} = I_{u,\beta}$ and $P_t(i \in V_u)$ is a direct summand of $(T_\alpha)_{t_{i-u,\alpha}}$ if and only if it is a direct summand of $(T_\beta)_{t_{i-u,\beta}}$. First assume that one of $\alpha$ and $\beta$ is $1$, say $\alpha$. Then $\min(\alpha) = 0$ if $\omega \leq 0$ and $\max(\alpha) = 0$ if $\omega \geq 2$. But since $\beta \neq 1$, $l(\beta) \geq 1$ and Lemmas 2 and 3 imply that $\min(\beta) < 0$ if $\omega \leq 0$ and $\max(\beta) > 0$ if $\omega \geq 2$. Thus we may assume that $\alpha \neq 1$, $\beta \neq 1$.

By Lemma 1 there exists $i \in \Gamma_0$ such that $\alpha = s_i\alpha'$ and $\beta = s_i\beta'$.
Obviously, the images of $\alpha'$ and $\beta'$ also coincide in $\text{Aut}(\mathfrak{O})$, and since $l(\alpha') + l(\beta') = l(\alpha) + l(\beta) - 2 < l(\alpha) + l(\beta)$ we get $\alpha' = \beta'$. But then $\alpha = s_i \alpha' = s_i \beta' = \beta$, which contradicts the assumption that $\alpha \neq \beta$. \hfill $\square$

4. The proof of Lemma 1 for $\omega \geq 2$

In this section we assume that $\omega \geq 2$ and prove Lemma 1 in this case. Denote $\max(\alpha)$ by $h_\alpha$. Observe that if $\omega \geq 2$, then $P_j$ with $j \in V^u$ is a direct summand of $(T_\alpha)_{[h_\alpha - \omega_{u+1} + 1, h_\alpha]}$ if and only if $\text{Hom}^{r+\omega}(P_j, T_\alpha) \neq 0$ for some $r \in [h_\alpha - \omega_{u+1} + 1, h_\alpha]$, due to the remark right after the definition of a direct summand. In fact, we used the notion of a direct summand in the form it was first provided only to unify the two cases $\omega \leq 0$ and $\omega \geq 2$ in the statement of Lemma 1 and the deduction of Theorem 1 from it.

As explained above, the following lemma is equivalent to Lemma 1 in the case $\omega \geq 2$:

**Lemma 4.** If $\text{Hom}^{r+\omega}(P_j, T_\alpha) \neq 0$ for some $r \in [h_\alpha - \omega_{u+1} + 1, h_\alpha]$, then $\alpha$ is left-divisible by $s_j$.

**Proof.** We argue by contradiction. Take $\alpha \in B^+_1$ not satisfying the required condition and of minimal length. It is clear that $l(\alpha) > 0$, and hence $\alpha$ can be presented as $\alpha = s_i \beta$ for some $i \in \Gamma_0$ and $\beta \in B^+_2$ with $l(\beta) < l(\alpha)$. In particular, the statement of the lemma is true for $\beta$ and all its right factors. Without loss of generality we may assume that $i \in V^0$.

Since the assertion of the lemma fails for $\alpha$, there exists some $j \in V^u$ ($u = 0, 1$) such that $\text{Hom}^{h+\omega}(P_j, T_\alpha) \neq 0$ for some $h \in [h_\alpha - \omega_{u+1} + 1, h_\alpha]$, but $\alpha$ is not left-divisible by $s_j$. It is clear that $j \neq i$. We may assume without loss of generality that if $\text{Hom}^{r+\omega}(P_k, T_\alpha) \neq 0$ for some $k \in V^u$ and $r > h$, then $\alpha$ is left-divisible by $s_k$.

1) First observe that $\beta = s_j \gamma$ for some $\gamma \in B^+_1$ with $l(\gamma) = l(\alpha) - 2$. Indeed, we have $\text{Hom}^{h+\omega}(P_j, T_\beta) \neq 0$ and $h \in [h_\beta - \omega_{u+1} + 1, h_\beta]$ by Lemma 3. Then $\beta$ is left-divisible by $s_j$ because the assertion of the lemma holds for $\beta$.

2) Now we show that $j \in N(i) \subset V^1$ and, in particular, $h \in [h_\alpha - \omega + 1, h_\alpha]$. Indeed, this easily follows from the previous claim, because in the case $j \notin N(i)$ one has $\alpha = s_j s_j \gamma = s_j s_i \gamma$ which contradicts the choice of $j$.

3) Note that

$$\text{Hom}^{h-\omega_1+\omega}(P_i, T_\gamma) \cong \text{Hom}^{h+\omega}(t_it_j P_i[\omega_1 - 1], t_it_j T_\gamma) = \text{Hom}^{h+\omega}(P_j, T_\alpha) \neq 0.$$ 

4) The next step is to establish that $\text{Hom}^{r+\omega}(P_k, T_\gamma) = 0$ for any $k \in V^1$ and $r > h$. Suppose for a contradiction that $\text{Hom}^{r+\omega}(P_k, T_\gamma) \neq 0$ for some $k \in V^1$ and $r > h$. According to Lemma 3 we have $h_\gamma \leq h_\beta \leq h_\alpha$ and $r \geq h + 1 \geq h_\alpha - \omega + 2$. Then $\text{Hom}^{r+\omega}(P_k, T_\alpha) \neq 0$, also by Lemma 3. Thus $s_k$ divides $\alpha$ on the left by the choice of $h$, and hence we have $\alpha = s_k \alpha'$ for some $\alpha' \in B^+_1$ with $l(\alpha') = l(\alpha) - 1$. Note that Lemma 3 again implies that $\text{Hom}^{h+\omega}(P_j, T_\alpha') \neq 0$ and $h \in [h_{\alpha'} - \omega_1 + 1, h_{\alpha'}]$. Since the assertion of Lemma 1 is valid for $\alpha'$, $s_j$ divides $\alpha'$ on the left. Since $s_j$ and $s_k$ commute, $s_j$ also divides $\alpha$ on the left which contradicts the choice of $j$.

5) Let $\theta_0 = \gamma$ and $\Delta_0 = \emptyset$. We repeat the following procedure. If at some moment $h_{\theta_p}$ is greater than $h$, we define $\theta_p = \theta$ and $\Delta = \Delta_p$. If $h_{\theta_p}$ is greater than $h$, we pick some $k$ such that $\text{Hom}^{h_{\theta_p}+\omega}(P_k, T_{\theta_p}) \neq 0$ and $\Delta_{p+1} = \Delta_p \cup \{k\}$. We have $\theta_{p+1} \in B^+_1$ and $l(\theta_p + 1) = l(\theta_p) - 1$, because the assertion of the lemma holds for $\theta_p$. Now recall that $\text{Hom}^{r+\omega}(P_l, T_{\theta_p}) = 0$ for any $r > h$ and $l \in V^1$. Since $h \geq h_\alpha - \omega + 1 \geq h_{\theta_p} - \omega + 1$ for any $p \geq 0$ by Lemma 3, we have $\text{Hom}^{r+\omega}(P_l, T_{\theta_p}) = 0$ for any $r > h$, $l \in V^1$ and $p \geq 0$ by the same lemma and induction on $p$. Therefore $\Delta_p \subset V^0$ for any $p \geq 0$. Note also that $k \in \Delta_p$, $\theta_p \neq \theta$...
Proof.

Lemma 5. with $k \geq T_j$. It is clear that $P_{\alpha}$ the statement of Lemma 1 is true for $\beta \omega$ will be true for any integer $\alpha$ equal to $\omega$. Thus, as before, $\Hom_{\omega}^{\alpha}(P_{\alpha}, T_{\beta})$.

7) If $i \in \Delta$, then $s_i$ divides $\gamma$ on the left. In this case $s_i s_j s_i = s_j s_i s_i$ divides $\gamma$ on the left, and hence so does $s_j$ which contradicts the choice of $j$.

6) If $\alpha \not\in \Delta$, then $\Hom_{\omega}^{\alpha+1+\omega}(P_{\alpha}, T_{\beta}) \neq 0$ by Lemma 3. Since the assertion of the lemma is true for $\theta$, $s_i$ divides $\theta$ on the left. Then $\gamma$ is again left-divisible by $s_i$ and we get a contradiction just as before.

□

Remark. Note that in the case $\omega = 2$ the number $h$ appearing in the proof is automatically equal to $h_\alpha$. This allows to omit the steps 4-6 in the proof and make it even shorter.

5. Outline of the proof of Lemma 4 for $\omega \leq 0$

In this section we give a general plan of our proof of Lemma 4 in the case $\omega \leq 0$. Until the end of the paper we will assume that we are in this situation though some statements will be true for any integer $\omega$. From here on we denote $\min(\alpha)$ by $m_\alpha$.

Similarly to the case $\omega \geq 2$, we argue by contradiction, and the beginning of this proof is the same, i.e. we again pick $\alpha \in B^+_\omega$ not satisfying the assertion of Lemma 4 and of minimal length.

As before, $\alpha$ can be presented as $\alpha = s_i \beta$ for some $i \in \Gamma_0$ and $\beta \in B^+_\omega$ with $l(\beta) < l(\alpha)$ and the statement of Lemma 4 is true for $\beta$ and all its right factors. Without loss of generality we may assume that $i \in V^0$. Since Lemma 4 fails for $\alpha$, there exists some $j \in V^u$ ($u = 0, 1$) such that $P_j$ is a direct summand of $(T_{\alpha})_{m_\alpha, m_\alpha - \omega_{u+1}}$ while $\alpha$ is not divisible by $s_j$ on the left. It is clear that $j \neq i$. Let $m \in [m_\alpha, m_\alpha - \omega_{u+1}]$ be the minimal degree such that $P_j$ is a direct summand of $(T_{\alpha})_{m}$. We may assume without loss of generality that if $r < m$ and $P_k$ with $k \in V^u$ is a direct summand of $(T_{\alpha})_{r}$, then $\alpha$ is divisible by $s_k$ on the left.

Lemma 5. 1) $j \in N(i) \subset V^1$.

2) If $P_k$ with $k \in V^1$ is a direct summand of $(T_{\beta})_r$ with $r \leq m$, then $r = m$ and $k \in N(i)$.

3) If $P_k$ is a direct summand of $(T_{\beta})_{m}$ for some $k \in V^1 \setminus \{j\}$, then $s_l$ does not divide $\alpha$ on the left for any $l \neq i$.

4) $\Hom^r(P_k, T_{\beta}) = 0$ for $r \leq m + \omega_0$.

Proof. 1) Suppose that $j \not\in N(i)$. It follows from Lemma 2 that $P_j$ is a direct summand of $(T_{\beta})_{m}$ and $m \in [m_{\beta}, m_{\beta} - \omega_{u+1}]$. Since the statement of Lemma 4 holds for $\beta$, we have $\alpha = s_{i+j} \beta = s_{i+j} \beta'$ for some $\beta' \in B^+_\omega$ with $l(\beta') = l(\alpha) - 2$. Thus, $j$ does not give a contradiction to the statement of Lemma 4 that contradicts the choice of $j$. Thus, $j \in N(i) \subset V^1$ and, in particular, we have $m \in [m_\alpha, m_\alpha - \omega_0]$.

2) Pick some $k \in V^1$ such that $P_k$ is a direct summand of $(T_{\beta})_r$ with $r \leq m$. Then $P_k$ is a direct summand of $(T_{\alpha})_r$ by Lemma 2. If $r < m$, then $s_k$ divides $\alpha$ on the left by the definition of $m$. If $r = m$ but $k \not\in N(i)$, then $s_k$ divides $\alpha$ on the left by the argument from the proof of the first issue. In any case, we have $\alpha = s_k \alpha'$ for some $\alpha' \in B^+_\omega$ with $l(\alpha') = l(\alpha) - 1$. Note that Lemma 2 implies again that $P_j$ is a direct summand of $(T_{\alpha'})_m$ and $m \in [m_{\alpha'}, m_{\alpha'} - \omega_0]$. Since the assertion of Lemma 4 is valid
for $\alpha'$, $s_j$ divides $\alpha'$ on the left. Since $s_j$ and $s_k$ commute, $s_j$ divides also $\alpha$ on the left that contradicts the choice of $j$.

3) Suppose that $s_l$ divides $\alpha$ on the left for some $l \neq i$. Then $s_l$ commutes with at least one of the elements $s_j$ and $s_k$. The argument from the proof of the second item shows that $s_j$ or $s_k$ divides $\alpha$ on the left. Since $s_k$ and $s_j$ commute, the same argument shows that $s_j$ divides $\alpha$ on the left in any case. This contradicts the choice of $j$.

4) Suppose that $\text{Hom}(P_1, T_\beta) \neq 0$ for some $r \leq m + \omega_0$. By Lemma 2, one has $m_\alpha \leq r - 1 \leq m + \omega_0 - 1$. But in this case $m \notin [m_\alpha, m_\alpha - \omega_0]$ and we have a contradiction. □

Set $s_A = \prod_{k \in A} s_k$, $t_A = \prod_{k \in A} t_k$ for $A \subseteq V^u$ with some $u \in \{0, 1\}$. Define $\sigma_u$ for $u \in \mathbb{Z}$ by $\sigma_0 = m$ and $\sigma_{u+1} = \sigma_u + 1 - \omega_{u+1}$. In other words, $\sigma_{2u} = m + u(2 - \omega)$, $\sigma_{2u+1} = m + u(2 - \omega) + 1 - \omega_{u+1}$. Let

$$\Delta_0 = \emptyset, \quad \Delta_0 = \{i\}, \quad \Delta_1 = \{j \in V^1 : P_j \text{ is a direct summand of } (T_\beta)_m\}.$$

If $|\Delta_1| \geq 2$, then $\alpha$ cannot be written as $\alpha = s_k \alpha'$ with $l(\alpha') = l(\alpha) - 1$ and $k \in \Gamma_0 \setminus \{i\}$ by Lemma 5. Hence, we need to show that either $|\Delta_1| = 1$ and $\alpha$ is left-divisible by $s_j$ where $j$ is the unique element of $\Delta_1$ or $|\Delta_1| = 2$ and $\alpha$ is left-divisible by $s_k$ with some $k \neq i$. With this end in view, we will employ the following scheme:

Step I: **Factorization.** First we are going to construct a presentation for $\alpha$ of a particular form.

We start with the presentation $\alpha = s_\Delta \beta$ obtained earlier. Set $\chi(0) = 1$. We continue the process inductively and obtain a presentation of the form $\alpha = s_{\Delta_0} s_{\Delta_1} \ldots s_{\Delta_u} \beta$ satisfying the following conditions: for $1 \leq u \leq q$:

1. $\alpha = s_\Delta \beta$ for some $\beta_u \in B_1^+$ with $l(\beta_u) = l(\alpha) - \sum_{u=0}^{\Delta_u}$
2. $\Delta_{u-2} \subset \Delta_u \supset N(\Delta_{u-1})$.
3. $P_j$ is not a direct summand of $(T_{\beta_u})_{|\sigma_{u-3+1}, \sigma_{u-1}}$ for any $l \in V^u$.
4. The minimal nonzero degree of $T_{\beta_u}$ is not smaller than $\sigma_{u-2} + 1$.
5. For any $l \in \Delta_u$, $P_j$ is a direct summand of $(t_l T_{\beta_u})_{|\sigma_{u-3+1}, \sigma_{u-1}}$.
6. $\chi(0) = 0$ if $\chi(0) = 0$

Moreover, $y \in \Delta_{q-1}$, $s_y$ divides $\beta$ on the left and

$$\chi_{q+1}(y) = \sum_{t \in N(y) \cap \Delta_n} \chi_q(t) \chi_{q-1}(y) = 0.$$

Note that the sets $\Delta_0$ and $\Delta_1$ defined earlier satisfy the required conditions. One has $\Delta_1 \subseteq N(i) = N(\Delta_0)$ by Lemma 2. The minimal nonzero degree of $T_{\beta_0}$ is not smaller than $m_{\alpha} \geq m + \omega_0 = \sigma_{-1} + 1$ by Lemma 2. It follows from the same lemma and the fact that $P_j$ is a direct summand of $(T_{\beta})_m$ that $P_j$ is a direct summand of $(t \Delta_{\{j\}} T_{\beta})_m = (t_j T_{\beta_1})_m$ for any $j \in \Delta_1$. Moreover, for any $r \leq m = \sigma_0$, $P_k$ with $k \in V^1 \setminus \Delta_1$ is not a direct summand of $(T_{\beta})_r$ by Lemma 3 and hence is not a direct summand of $(T_{\beta_1})_r$ by Lemma 2 and $P_k$ with $k \in \Delta_1$ cannot be a direct summand of $(T_{\beta_1})_r$, because

$$\text{Hom}^{r+w}(P_k, T_{\beta_1}) \cong \text{Hom}^{r + 2w - 1}(t \Delta, P_k[r - 1], T_\beta) = \text{Hom}^{r + 2w - 1}(P_k, T_\beta) = 0$$

due to the inequality $r + w - 1 \leq m + \omega - 1 \leq m_\beta + \omega - 1 < m_\beta$. Finally, we clearly have $\chi(0) = 0$ for any $j \in \Delta_1$.

Thus, it is enough to show that if we have sets $\Delta_0, \ldots, \Delta_p$ such that the properties above are satisfied for any $1 \leq u \leq p$, then we either can construct $\Delta_{p+1}$ in such a way that the properties above will be satisfied for $u = p + 1$ or find $y \in \Delta_{p-1}$ such
that $s_y$ divides $\beta_p$ on the left and $\sum_{t \in N(y) \cap \Delta_p} \chi_p(t) = \chi_{p-1}(y)$. We introduce all the necessary technical tools and discuss this step in detail in Section [7].

Step II: **Braiding.** Once a presentation for $\alpha$ of the form $s_{\Delta_1} s_{\Delta_2} \ldots s_{\Delta_i} s_y \beta$ is obtained, it remains to show that either $\Delta_1 = \{j\}$ for some $j$ such that $s_j$ divides $s_{\Delta_0} s_{\Delta_1} \ldots s_{\Delta_p} s_y$ on the left or $|\Delta_1| \geq 2$ and at least one of $s_k$ with $k \in \Gamma_0 \setminus \{i\}$ divides $s_{\Delta_0} s_{\Delta_1} \ldots s_{\Delta_p} s_y$ on the left. Note that if $\Delta_1 = \{j\}$, then $\chi_2(i) = 0$, and hence our presentation is of the form $\alpha = s_i s_j s_i \beta = s_j s_i s_j \beta$. Thus, at this point it is enough to consider the case $|\Delta_1| \geq 2$ and show that some $k \in \Gamma_0 \setminus \{i\}$ can be pulled to the very left of the subword $s_{\Delta_0} s_{\Delta_1} \ldots s_{\Delta_p} s_y$, applying a sequence of braid and commutator relations. This step is discussed in Section [8].

6. **TWO-TERM OBJECTS**

In this section we introduce the notion of the a two-term object in $\mathfrak{D}$ and prove some facts about them. These facts will be required to fulfill the factorization process announced above.

**Definition 6.1.** An object $X$ of a triangulated category $\mathfrak{D}$ with a fixed $\Gamma$-configuration of $\omega$-spherical objects $\{P_j\}_{j \in \Gamma_u}$ is called two-term if there exists a triangle

$$X[-1] \xrightarrow{\beta_X} \bigoplus_{j \in V^u} P^x_j [-\omega_u] \xrightarrow{\varphi_X} \bigoplus_{k \in V^{u+1}} P^x_k \xrightarrow{\alpha_X} X$$

in $\mathfrak{D}$ for some $u \in \{0,1\}$ and some $x_j \geq 0$ ($j \in \Gamma_0$). A two-term object $X$ is called right-proper if $f \varphi_X \neq 0$ for any split epimorphism $f: \bigoplus_{k \in V^{u+1}} P^x_k \to P_l$ with $l \in V^{u+1}$ and is called left-proper if $\varphi_X g[-\omega_u] \neq 0$ for any split monomorphism $g: P_l \to \bigoplus_{j \in V^u} P^x_j$ with $l \in V^u$. For a two-term object $X$ we also define $\text{lsupp}(X) = \{j \in V^u \mid x_j \neq 0\}$ and $\text{rsupp}(X) = \{k \in V^{u+1} \mid x_k \neq 0\}$.

For example, $P_l$ is a left-proper two-term object with $\alpha_{P_l} = \text{id}_{P_l}$ and $\beta_{P_l} = \varphi_{P_l} = 0$ for any $l \in \Gamma_0$. Moreover, $\text{lsupp}(P_l) = \emptyset$ and $\text{rsupp}(P_l) = \{l\}$. It is not difficult to show that any two-term object $X$ can be represented in the form $X = X' \oplus \bigoplus_{k \in V^{u+1}} P^x_k$, where $X'$ is a right-proper two-term object.

**Remark.** The notion of a two-term object in some sense generalizes the notion of a two-term partial tilting complex to the setting of triangulated categories with a $\Gamma$-configuration of spherical objects.

**Lemma 6.** Let $X$ be a two-term object as in the definition above.

1) The following three conditions are equivalent:

- $X$ is right-proper;
- if $l \in V^{u+1}$ and $g: P_l \to \bigoplus_{k \in V^{u+1}} P^x_k$ is not a split monomorphism, then $\alpha_X g = 0$;
- $\dim_k \text{Hom}^*(X, P_l) = \sum_{k \in V^{u+1}} x_k$ for any $l \in V^{u+1}$.

2) The following three conditions are equivalent:

- $X$ is left-proper;
- if $l \in V^u$ and $f: \bigoplus_{j \in V^u} P^x_j \to P_l$ is not a split epimorphism, then $f[-\omega_u] \beta_X = 0$;
- $\dim_k \text{Hom}^*(X, P_l) = \sum_{k \in V^{u+1}} x_k$ for any $l \in V^u$.

**Proof.** We will prove only the first assertion, the second one can be deduced by dual arguments.
Fix some \( l \in V^{u+1} \). One has
\[
\dim_k \text{Hom}^*(X, P_l) = \dim_k \text{Hom}^* \left( \bigoplus_{j \in V^u} P^x_j, P_l \right) + \dim_k \text{Hom}^* \left( \bigoplus_{k \in V^{u+1}} P^x_k, P_l \right) - 2 \dim_k \text{Im} \varphi_X = \sum_{k \in N(l)} x_k + 2(x_l - \dim_k \text{Im} \varphi_X),
\]
where \( \varphi_X^* \colon \text{Hom}^* \left( \bigoplus_{k \in V^{u+1}} P^x_k, P_l \right) \to \text{Hom}^* \left( \bigoplus_{j \in V^u} P^x_j, P_l \right) \) is the map induced by \( \varphi_X \).

Note that the set \( \text{Hom}^* \left( \bigoplus_{k \in V^{u+1}} P^x_k, P_l \right) = \text{Hom}^* (P^x_l, P_l) \) is a direct sum of two spaces of dimension \( x_l \) of which the first is annihilated by \( \varphi_X \) and the second is formed by split epimorphisms. Thus, \( \dim_k \text{Im} \varphi_X \leq x_l \) and the equality holds for all \( l \in V^{u+1} \) precisely when \( X \) is right-proper. Thus, \( X \) is right-proper if and only if \( \dim_k \text{Hom}^*(X, P_l) = \sum_{k \in N(l)} x_k \) for any \( l \in V^{u+1} \). Note that \( \dim_k \text{Hom}^*(X, P_l) = \dim_k \text{Hom}^*(P_l, X) \) because \( P_l \) is spherical. Analogous arguments show that \( \dim_k \text{Hom}^*(P_l, X) = \sum_{k \in N(l)} x_k \) for any \( l \in V^{u+1} \) if and only if \( \alpha_X g = 0 \) for any \( g \colon P_l \to \bigoplus_{k \in V^{u+1}} P^x_k \) that is not a split monomorphism.

The first crucial fact about the set of two-term objects is that it is stable under certain autoequivalences of \( \mathcal{D} \). For \( \Delta \subseteq V^u \) let us set \( t_{\Delta}^* = t_\Delta [\omega_{u+1} - 1] \) and \( t_{\Delta}^- = t_{\Delta}^- [1 - \omega_{u+1}] \).

**Lemma 7.** Let \( X \) be as above.

1) If \( X \) is right-proper and \( \text{lsupp}(X) \subseteq \Delta \subseteq V^{u+1} \), then \( t_{\Delta}^* X \) is a left-proper two-term object with the defining triangle of the form
\[
t_{\Delta}^* X [-1] \xrightarrow{\beta_{\Delta}^X} \bigoplus_{k \in V^{u+1}} P^x_k [-\omega_{u+1}] \xrightarrow{\varphi_{\Delta}^X} \bigoplus_{j \in V^u} P^x_j \xrightarrow{\alpha_{\Delta}^X} t_{\Delta}^* X,
\]
where \( x_k' = \sum_{j \in N(k)} x_j - x_k \) for \( k \in \Delta \) and \( x_k' = 0 \) for \( k \in V^{u+1} \setminus \Delta \).

2) If \( X \) is left-proper and \( \text{lsupp}(X) \subseteq \Delta \subseteq V^u \), then \( t_{\Delta}^- X \) is a right-proper two-term object with the defining triangle of the form
\[
t_{\Delta}^- X [-1] \xrightarrow{\beta_{\Delta}^- X} \bigoplus_{k \in V^{u+1}} P^x_k [-\omega_{u+1}] \xrightarrow{\varphi_{\Delta}^- X} \bigoplus_{j \in V^u} P^x_j \xrightarrow{\alpha_{\Delta}^- X} t_{\Delta}^- X,
\]
where \( x_j' = \sum_{k \in N(j)} x_k - x_j \) for \( j \in \Delta \) and \( x_j' = 0 \) for \( j \in V^u \setminus \Delta \).

**Proof.** We will prove only the first assertion, the second one can be deduced by dual arguments.

Let us fix some \( l \in V^{u+1} \). By Lemma \( \Box \) \( \text{Hom}^*(P_l, X) \) has dimension \( \sum_{k \in N(l)} x_k \), and hence is generated by \( x_l \) compositions \( P_l \to \bigoplus_{k \in V^{u+1}} P^x_k \xrightarrow{\alpha_X} X \), where the first arrow ranges over \( x_l \) linearly independent direct inclusions, and \( x_l' \) maps \( P_l \to X[\omega - 1] \) that after composition with \( \beta_X [\omega] \) give \( x_l' \) linearly independent maps from \( P_l \) to \( \bigoplus_{j \in V^u} P^x_j [\omega_{u+1}] \) annihilated by \( \varphi_X [\omega] \). Taking all these morphisms for all \( l \) together we get a map
\[
\bigoplus_{k \in V^{u+1}} P^x_k [\omega_{u} - 1] \oplus \bigoplus_{k \in V^{u+1}} P^x_k [-\omega_{u+1}] \xrightarrow{\left( \alpha_X [\omega_{u} - 1], \gamma \right)} X[\omega_{u} - 1]
\]
whose cone is isomorphic to $t^+_\Delta X$. Applying the octahedral axiom to the composition 
$$(\alpha_X[\omega_u-1] \gamma) \circ \begin{pmatrix} id_Z \\ 0 \end{pmatrix} = \alpha_X[\omega_u-1],$$ 
where $Z$ denotes $\bigoplus_{k \in V^{u+1}} P^x_k[\omega_u-1]$, we get the commutative diagram whose rows and columns are triangles:
\[
\begin{array}{ccc}
\bigoplus_{k \in V^{u+1}} P^x_k[\omega_u-1] & \xrightarrow{\bigoplus_{k \in V^{u+1}} P^x_k[\omega_u-1]} & \bigoplus_{k \in V^{u+1}} P^x_k[-\omega_u+1] \\
\bigoplus_{k \in V^{u+1}} P^x_k[\omega_u-1] & \xrightarrow{\bigoplus_{k \in V^{u+1}} P^x_k[\omega_u-1]} & \bigoplus_{k \in V^{u+1}} P^x_k[-\omega_u+1] \\
\bigoplus_{k \in V^{u+1}} P^x_k[\omega_u-1] & \xrightarrow{\bigoplus_{k \in V^{u+1}} P^x_k[\omega_u-1]} & \bigoplus_{k \in V^{u+1}} P^x_k[-\omega_u+1]
\end{array}
\]

Hence, $t^+_\Delta X$ indeed has the required form. A direct inclusion $g: P_l \to \bigoplus_{k \in V^{u+1}} P^x_k$ such that the composition
\[
P_l[-\omega_u+1] \xrightarrow{g[-\omega_u+1]} \bigoplus_{k \in V^{u+1}} P^x_k[-\omega_u+1] \xrightarrow{\bigoplus_{k \in V^{u+1}} P^x_k[-\omega_u+1]} \bigoplus_{k \in V^{u+1}} P^x_k
\]
is zero would give a linear dependence between $x^l_j$ components of the morphism $P^x_j[-\omega_u+1] \to \bigoplus_{j \in V^u} P^x_j$ which are linearly independent by our construction. Thus, $t^+_\Delta X$ is left-proper. 

Next we need to study the behaviour of some relations between two-term objects with respect to autoequivalences $t^+_\Delta$.

**Definition 6.2.** Let 
\[
X = \mathrm{cone} \left( \bigoplus_{j \in V^u} P^y_j[-\omega_u] \xrightarrow{\varphi X} \bigoplus_{k \in V^{u+1}} P^x_k \right) \quad \text{and} \quad Y = \mathrm{cone} \left( \bigoplus_{j \in V^u} P^y_j[-\omega_u] \xrightarrow{\varphi Y} \bigoplus_{k \in V^{u+1}} P^y_k \right)
\]
be two-term objects. We will call $X$ a two-term subobject of $Y$ if there exist split monomorphisms $\iota_u: \bigoplus_{j \in V^u} P^y_j \to \bigoplus_{j \in V^u} P^y_j$ and $\iota_u: \bigoplus_{k \in V^{u+1}} P^x_k \to \bigoplus_{k \in V^{u+1}} P^x_k$ such that $\iota_u \varphi X = \varphi Y \iota_u [-\omega_u]$. The two-term subobject $X$ of $Y$ is called trivial if either $X = 0$ or both of the maps $\iota_u, \iota_u : \iota$ are isomorphisms. Otherwise $X$ is called a nontrivial two-term subobject of $Y$.

We will say that a morphism $f: X \to Y[\omega]$ is a right socle morphism if it can be presented in the form $f = \alpha_Y[\omega] f'$ for some $f': X \to \bigoplus_{k \in V^{u+1}} P^y_k[\omega]$ such that for any split epimorphism $g: \bigoplus_{k \in V^{u+1}} P^y_k \to P_l$ with $l \in V^{u+1}$ the morphism $g[\omega] f' \alpha_X: \bigoplus_{k \in V^{u+1}} P^y_k \to P_l[\omega]$ is not a split epimorphism anymore.

**Remark.** The second condition in the definition of a right socle morphism is valid automatically if $X$ is right-proper or $\omega \neq 0$. Moreover, if $\omega \neq 0$ and $X$ is left-proper, then any morphism of the form $X \xrightarrow{f} Y[\omega]$ is automatically right socle. It follows from the fact that $\mathrm{Hom}_\Delta(P_k, P_l[1+\omega_u]) = 0$ for any $k \in V^{u+1}$, $l \in V^u$ and $g[1-\omega_u] \iota X[1] = 0$ for any $g: \bigoplus_{j \in V^u} P^y_j \to \bigoplus_{j \in V^u} P^y_j[\omega]$. In fact, the definition of a right socle morphism is introduced to cover the case $\omega = 0$ which nevertheless is of special interest for us in view of an application to the derived Picard groups of algebras. Most of the assertions about right socle morphisms we provide below are trivial for $\omega \neq 0$.

**Lemma 8.** Let $X$ and $Y$ be as above. Suppose also that both $X$ and $Y$ are right-proper. If $X$ is a nontrivial two-term subobject of $Y$, then $t^+_\Delta X$ is a non-trivial two-term subobject of $t^+_\Delta Y$ for any $\text{rsupp}(Y) \subseteq \Delta \subseteq V^{u+1}$. 

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Proof. It is clear that \( \text{rsupp}(X) \subseteq \text{rsupp}(Y) \), and hence both objects \( t^+_\Delta X \) and \( t^+_\Delta Y \) are two-term by Lemma \( \text{[7]} \). Recall that due to the proof of Lemma \( \text{[7]} \) one has

\[
t^+_\Delta X = \text{cone} \left( \bigoplus_{k \in V^+} P^{x_k}_k [-\omega_{u+1}] \xrightarrow{\varphi^+_\Delta X} \bigoplus_{j \in V^u} P^{x_j}_j \right)
\]

where \( x'_l = 0 \) for any \( l \in V^{u+1} \setminus \Delta \) and for any \( l \in \Delta \) the components of the map \( \varphi^+_\Delta X |_{P^{x'_l}} \) constitute the basis of \( \text{Ker Hom}_\Delta(P, \varphi_X) \). The morphism \( \varphi^+_\Delta Y \) satisfies analogous conditions.

Since \( \varphi_Y[\omega_u] \mu \varphi^+_\Delta X = 0 \), there exists a map \( \iota'_u : \bigoplus_{k \in V^u+1} P^{x'_k}_k \to \bigoplus_{k \in V^u+1} P^{y'_k}_k \) such that

\[
\iota'_u \varphi^+_\Delta X = \varphi^+_\Delta Y \iota'_u[-\omega_{u+1}].
\]

If \( \iota'_u \) is not a split monomorphism, then for some \( l \in V^{u+1} \) there exists a direct inclusion \( g : P_l \to \bigoplus_{k \in V^u+1} P^{x'_k}_k \) such that \( \iota'_u g \) is not a direct inclusion. In this case we have \( \iota_u \varphi^+_\Delta X g = \varphi^+_\Delta Y \iota'_u g = 0 \), and hence \( \varphi^+_\Delta X g = 0 \). This contradicts \( t^+_\Delta X \) being left-proper, which has been proved in Lemma \( \text{[7]} \). The obtained contradiction implies that \( \iota'_u \) is a split monomorphism, and hence we are done. \( \square \)

Lemma 9. Let \( X \) and \( Y \) be as above. Then any right socle morphism \( f : X \to Y[\omega] \) factors through some right socle morphism \( X' \to Y[\omega] \), where \( X' \) is a two-term object such that \( \text{rsupp}(X') \subseteq \text{rsupp}(Y) \).

Proof. Let \( f = \alpha Y[\omega]f' : X \to Y[\omega] \) be a decomposition of the right socle morphism \( f \) and set \( \Delta^x = \text{rsupp}(X) \setminus \text{rsupp}(Y) \). By \( \iota \) we denote the direct inclusion \( \bigoplus_{k \in \Delta^x} P^{x_k}_k \to \bigoplus_{k \in V^{u+1}} P^{x_k}_k \) and by \( \pi \) the split epimorphism \( \bigoplus_{k \in V^{u+1}} P^{x_k}_k \to \bigoplus_{k \in V^{u+1} \setminus \Delta^x} P^{x_k}_k \). Since

\[
\text{Hom}_\Delta \left( \bigoplus_{k \in \Delta^x} P^{x_k}_k, \bigoplus_{k \in V^{u+1}} P^{y_k}_k \right) = 0,
\]

one has \( f' \alpha_X \iota = 0 \), and hence \( f' \) factors through some morphism \( f'' : X' \to \bigoplus_{k \in V^{u+1}} P^{y_k}_k \), where \( X' = \text{cone}(\alpha_X \iota) \). Note that by the octahedral axiom one has

\[
X' \cong \text{cone} \left( \bigoplus_{j \in V^u} P^{x_j}_j [-\omega_u] \xrightarrow{\pi \varphi_X} \bigoplus_{k \in V^{u+1} \setminus \Delta^x} P^{x_k}_k \right),
\]

i.e. \( X' \) is a two-term object of the required form.
Lemma 10. If $X$ is right-proper and $l \in V^{u+1}$, then any right socle morphism from $P_l$ to $X[\omega]$ is zero.

Proof. Let $f = \alpha_X[\omega]f' : P_l \to X$ be a right socle morphism. It follows from the definition of a right socle morphism that $f'$ is not a split monomorphism. Then one has $f = \alpha_X[\omega]f' = 0$ by Lemma 6.

Lemma 11. Let $X$ and $Y$ be as above. Suppose that both $X$ and $Y$ are right-proper and $\text{rsupp}(X) \subseteq \text{rsupp}(Y)$. Then for any right socle $f : X \to Y[\omega]$ and any $\text{rsupp}(Y) \subseteq \Delta \subseteq V^{u+1}$, the morphism $t_\Delta^+f : t_\Delta^+X \to t_\Delta^+Y[\omega]$ is right socle too.

Proof. To simplify the notation, we consider here only the case $\omega = 0$. Other cases are obvious due to the remark after the definition of a right socle morphism. Note that $\omega_u = \omega_{u+1} = 0$ by our assumption. Due to the proof of Lemma 7 there is a triangle

$$t_\Delta^+X[-1] \xrightarrow{\beta_{\Delta X}[\gamma]} \bigoplus_{k \in V^{u+1}} P_{k}^X[-1] \oplus \bigoplus_{k \in V^{u+1}} P_{k}^Y \xrightarrow{\alpha_X \gamma} X[-1] \xrightarrow{\alpha_{\Delta X} \beta_X} t_\Delta^+X$$

emerging from the definition of the spherical twist. One has an analogous triangle for $Y$ too. Observe that since $f$ factors through $\alpha_Y$, we have $\beta_Y[1]f = 0$. Then $t_\Delta^+f$ satisfies the condition $(t_\Delta^+f)\alpha_{\Delta X} \beta_X = \alpha_{\Delta X} \beta_Y f[-1] = 0$.

Hence $t_\Delta^+f$ factors through $(\psi[1]_{\beta_{\Delta X}[1]})$, i.e. $t_\Delta^+f = (\beta_{\Delta Y}[1]f')$. Since $\beta_{\Delta Y}[1]f'$ is annihilated by $\varphi_{\Delta Y}[1]$ and $t_\Delta^+Y$ is left-proper, all components of $\beta_{\Delta Y}[1]f'$ are not isomorphisms, i.e. all morphisms

$$\bigoplus_{k \in V^{u+1}} P_{k}^X \oplus \bigoplus_{k \in V^{u+1}} P_{k}^Y \to P_l[1]$$

constituting $\beta_{\Delta Y}[1]f'$ are not split epimorphisms. Note that $\beta_{\Delta X}[1]$ annihilates all morphisms of the form $\bigoplus_{k \in V^{u+1}} P_{k}^X \to P_l[1]$ with $l \in V^{u+1}$ that are not split epimorphisms by the left-properness of $t_\Delta^+X$. 

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Since $\text{Hom}_\Delta \left( \bigoplus_{k \in V^{+1}} P^k_\Delta, \bigoplus_{k \in V^{+1}} P^k_\Delta \right) = 0$, we have $\beta \Delta \oplus_\Delta \{1\} = 0$, and hence $\Delta \oplus \Delta \oplus_\Delta \{1\} = 0$. Suppose now that $g \Delta \oplus_\Delta \{1\}$ is a split epimorphism for some $g: \bigoplus_{j \in V^u} P^{y_j}_j \to P_j[0]$. Then $g \Delta \oplus_\Delta \{1\} \beta \Delta \neq 0$, and hence also $\Delta \oplus_\Delta \{1\} \beta \Delta \neq 0$. On the other hand, we have proved earlier that $\alpha \Delta \oplus_\Delta \{1\} \beta \Delta \neq 0$. Then $f'' \alpha \Delta \oplus_\Delta \{1\} \beta \Delta$ factors as $\varphi \Delta \oplus_\Delta \{1\} \theta$ for some morphism $\theta: \Delta \to \bigoplus_{k \in V^{+1}} P^k_\Delta$.

Since $\text{Hom}_\Delta \left( \bigoplus_{k \in V^{+1}} P^k_\Delta \{1\}, \bigoplus_{k \in V^{+1}} P^k_\Delta \right) = 0$, $\theta$ factors through $\beta \Delta$, and hence we have $f'' \alpha \Delta \oplus_\Delta \{1\} \beta \Delta = \varphi \Delta \oplus_\Delta \{1\} \beta \Delta$ for some $f''$: $\bigoplus_{j \in V^u} P^{y_j}_j \to \bigoplus_{k \in V^{+1}} P^k_\Delta$. Since all components of $\varphi \Delta \oplus_\Delta \{1\} \beta \Delta$ are not isomorphisms, the morphism $g(f'' \alpha \Delta \oplus_\Delta \{1\} \beta \Delta - \varphi \Delta \oplus_\Delta \{1\} \beta \Delta)$ is a split epimorphism, and hence $(f'' \alpha \Delta \oplus_\Delta \{1\} \beta \Delta - \varphi \Delta \oplus_\Delta \{1\} \beta \Delta)$ cannot be zero. We have a contradiction, and this finishes the proof of the lemma. \hfill \Box

7. FACTORIZATION

We return to the context of Lemma 1. Until the end of the paper, we assume that $\omega \leq 0$. Recall that $\sigma_u = m + u(2 - \omega), \sigma_u + 1 = m + u(2 - \omega) + 1 - \omega$. Now suppose that we have sets $\Delta_0, \ldots, \Delta_p$ for some $p \geq 1$ such that $\Delta_0 = \{i\}$ for some $i \in V^0$. Let us recall that the numbers $\chi_u(k) (0 \leq u \leq p, k \in \Delta_u)$ are defined inductively by the equalities $\chi_0(i) = 1$ and $\chi_u(k) := \sum_{t \in N(k)} \chi_{u-1}(t) - \chi_{u-2}(k)$ for $u \geq 1$, where we set for convenience $\chi_u(t) = 0$ if $\nabla < 0$ or $t \notin \Delta_u$.

Suppose also that the following conditions hold for $1 \leq u \leq p$:

1. $\alpha = s_1 \ldots s_u \beta_u$ for some $\beta_u \in B^+_u$ with $l(\beta_u) = l(\alpha) - \sum_{v=0}^u |\Delta_v|$
2. $\Delta_{u-2} \subseteq \Delta_u \subseteq N(\Delta_{u-1})$, where we set $\Delta_{-1} = \emptyset$ for convenience.
3. $P_l$ is a direct summand of $(T_{\beta_u})|_{\sigma_u-1, \sigma_u-1}$ for any $l \in V^u$.
4. The minimal nonzero degree of $T_{\beta_u}$ is not smaller than $\sigma_{u-2} + 1$.
5. For any $l \in \Delta_u, P_l$ is a direct summand of $(T_l|_{\sigma_u-1, \sigma_u-1})$.
6. $\chi_u(k) > 0$ for any $k \in \Delta_u$.

We want to continue the process and construct $\Delta_{u+1}$ in such a way that the conditions [1] and [2] are fulfilled for $u = p + 1$ and, whenever the condition [6] is valid for $u = p + 1$, then so are the conditions [3] and [5]. To this purpose, define

$C_u := \Delta_1 \ldots \Delta_u \beta_u = (\Delta_1 \ldots \Delta_u P_1) \sigma_u - m + \omega_u - \omega_0$.

The first crucial fact that we will need is that $C_u$ is a two-term object of a certain form.

**Lemma 12.** For any $1 \leq u \leq p$ there exists a triangle of the form

$C_u \to \bigoplus_{j \in \Delta_{u-1}} P^{\chi_{u-1}(j)}_{j} [-\omega_{u-1}] \to \bigoplus_{k \in \Delta_u} P^k_{\omega(k)} \to C_u$.

**Proof.** Since $P_l$ is a left-proper two-term object, the required triangle can be obtained by iterated application of Lemma 1. \hfill \Box

The next proposition elaborates on the properties of the objects $C_u$ that will allow us to find direct summands of $(T_{\beta_u})|_{\sigma_{u-2}+1, \sigma_u}$.

**Proposition 1.** For any $1 \leq u \leq p$, the two-term object $C_u$ satisfies the following two conditions:

1) $\text{Hom}^r(C_u, T_{\beta_u}) = 0$ for any $r \leq \omega_u + \sigma_u$.
2) For any nontrivial two-term subobject $C'$ of $C_u$, there exists a nonzero morphism $f: C' \to T_{\beta_u}[r]$ with $\omega_u + \sigma_{u-2} + 1 \leq r \leq \omega_u + \sigma_u$ such that $f[\omega]g = 0$ for any right socle morphism $g: C'' \to C'[\omega]$.

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Proof. We proceed by induction on $u$ with the base case $u = 0$. Since $C_0 = P_1$, $P_1$ does not have nontrivial two-term subobjects and $\text{Hom}^r(P_i, T_B) = 0$ for any $r \leq m + \omega_0$ by Lemma 3 there is nothing left to prove in the base case. Now we prove 1) and 2) for $u$, assuming they are true for $u - 1$. The first property of $C_u$ is clear, because

$$\text{Hom}^r(C_u, T_{\beta_u}) = \text{Hom}^r(t_{\Delta_u} C_{u-1}, t_{\Delta_u}^{-1} T_{\beta_{u-1}}) \cong \text{Hom}^{r+\omega_{u+1}}(C_{u-1}, T_{\beta_{u-1}}).$$

Now we turn to the second property. Suppose that $C'$ is a nontrivial two-term subobject of $C_u$. If $C'$ is right-proper, then $t_{\Delta_u}^+ C'$ is a nontrivial two-term subobject of $C_{u-1}$ by Lemma 5. By the induction hypothesis, there is a nonzero morphism $f: t_{\Delta_u}^+ C' \to T_{\beta_{u-1}}[r]$ with $\omega_{u-1} + \sigma_{u-3} + 1 \leq r \leq \omega_{u-1} + \sigma_{u-1}$ such that $f[\omega]g = 0$ for any right socle morphism $g: C'' \to t_{\Delta_u}^+ C'[\omega]$. Let us prove that the morphism $t_{\Delta_u} f: C' \to T_{\beta_u} [r + 1 - \omega_{u-1}]$ satisfies the required properties.

Suppose for a contradiction that there exists a right socle morphism $g: C'' \to C'[\omega]$ such that $(t_{\Delta_u} f)[\omega]g \neq 0$. Then applying Lemmas 3 and 13 we can find such a $g$ with right-proper $C''$ satisfying the condition $\text{rssp}(C'') \subseteq \text{rssp}(C')$. Then $t_{\Delta_u}^+ g: t_{\Delta_u}^+ C'' \to t_{\Delta_u}^+ C'[\omega]$ is a right socle by Lemma 13 and satisfies the condition $f[\omega](t_{\Delta_u}^+ g) = t_{\Delta_u}^+ ((t_{\Delta_u} f)[\omega]g) \neq 0$ which is impossible. This shows that $t_{\Delta_u}^+ f$ is indeed the required morphism.

It remains to consider the case when $C'$ is not right-proper. In this case we may assume that $C' = P_l$ for some $l \in \Delta_u$. Let $a$ be the minimal integer such that $P_l$ is a direct summand of $(t_{\beta_{u}})_a$. Note that $a \in [\sigma_{u-3} + 1, \sigma_{u-1}]$ by the properties (3), (5) of $\Delta_u$. Picking some long morphism $f: P_l \to t_l T_{\beta_u}[a + \omega]$ and applying $t_{l}^{-1}[1 - \omega]$, we get a nonzero morphism $t_{l}^{-1} f[1 - \omega]: P_l \to T_{\beta_u}[a + 1]$. Note that $a + 1 \in [\sigma_{u-3} + 2, \sigma_{u-1} + 1] = [\omega_u + \sigma_{u-2} + 1, \omega_u + \sigma_u]$.

Thus, it remains to show that $(t_{l}^{-1} f)[1]g = 0$ for any right socle morphism $g: C'' \to P_l[\omega]$. Due to Lemma 9 we may assume that $C''$ has the form

$$C'' = \text{cone} \left( \bigoplus_{j \in \nu^{x+1}} P_j^x \left[ -\omega_{u+1} \right] \xrightarrow{\varphi_{l''}^c} P_l^x \right)$$

for some integers $x, j, x$. Note now that if $C'' = P_l$, then $tg: P_l[1 - \omega] \to P_l[1]$ is a right socle morphism, and hence factors through $P_j[1 - \omega_{u+1}]$ for $j \in \nu(l)$. Then $f[1](tg) = 0$ by the definition of a long morphism. Hence, we may assume that $C''$ is right-proper. Then $tg$ is a morphism from $\text{cone} \left( P_l^x[1 - \omega] \xrightarrow{\varphi_{l''}^c[1 - \omega_{u+1}]} \bigoplus_{j \in \nu^{x+1}} P_j^x[1 - \omega_{u+1}] \right)$ to $P_l[1]$ by Lemma 4.

Note that the composition

$$\bigoplus_{j \in \nu^{x+1}} P_j^x \left[ -\omega_{u+1} \right] \xrightarrow{\alpha_{l''}^c \varphi_{l''}^c[1 - \omega_{u+1}]} t_l C''[-1] \xrightarrow{tg[1]} P_l \xrightarrow{t_l} t_l T_{\beta_u}[a + \omega]$$

is zero by the definition of a long morphism. Hence $f(t_{l} g)[1]$ factors through some morphism $\theta: P_l^y[1 - \omega] \to t_l T_{\beta_u}[a + \omega]$. Since $a + \omega - 1 < a$, $P_l$ is not a direct summand of $(t_{\beta_{u}})_a + \omega_{u-1}$.

If $f[1](tg) \neq 0$, then $tg$ is also nonzero, so for some $j \in \nu(l)$ such that $P_j \theta[\omega + \omega_u - 1] \in \nu[\omega_u + \gamma_{l \beta_u}]$, $t_l T_{\beta_u}[a + 2 \omega + \omega_u - 1]$ is nonzero, where $\iota: P_l \to P_l^y$ is some split monomorphism. Note that by property (4) and the choice of $a$, the minimal nonzero degree of $t_l T_{\beta_u}$ is not smaller than $\min\{a, \sigma_{u-2} + 1\}$ by Lemma 2. On the other hand, if $f[1](tg) = 0$, then the minimal nonzero degree of $t_l T_{\beta_u}$ does not exceed $a + \omega + \omega_u - 1 < a$. Then we have

$$\sigma_{u-2} + 1 \leq a + \omega + \omega_u - 1 \leq \sigma_{u-1} + \omega + \omega_u - 1 = \sigma_{u-2} + \omega,$$

i.e. $\omega > 0$, a contradiction. This shows that $f[1](tg) = 0$, and hence $(t_{l}^{-1} f)[1]g = 0$, as required. □
Let us now construct $\Delta_{p+1} \subseteq V_{p+1}$. We will do this in the following way. Set $\Delta^0_{p+1} = \emptyset$. Suppose that we have defined the set $\Delta^i_{p+1}$. Choose a pair $l, a$ with $l \in V_{p+1} \setminus \Delta^i_{p+1}$, $a \in \mathbb{Z}$ such that $P_l$ is a direct summand of $(t_{\Delta^i_{p+1}} \bigtriangledown)^a$, and with $a$ the minimal possible among all such pairs. We define $\Delta^{i+1}_{p+1} = \Delta^i_{p+1} \cup \{l\}$ and continue the process if $a \in [\sigma_{p-2} + 1, \sigma_p]$. If either such an integer $a$ does not exist or $a > \sigma_p$, then we terminate the process, defining $\Delta_{p+1} := \Delta^i_{p+1}$. Now we are ready to prove the factorization theorem.

**Theorem 2.** The $\Delta_{p+1}$ constructed as described above satisfies the following conditions:

1. $\alpha = s_{\Delta_0} \ldots s_{\Delta_{p+1}, \beta_{p+1}}$ for some $\beta_{p+1} \in B^+_T$ with $l(\beta_{p+1}) = l(\alpha) - \sum_{i=0}^{p+1} |\Delta_i|$.
2. $\Delta_{p-2} \subseteq \Delta_p \subseteq N(\Delta_p)$.

Moreover, if $\chi_{p+1}(l) > 0$ for any $l \in \Delta_{p+1}$, then the following conditions are satisfied as well:

3. $P_l$ is not a direct summand of $(T_{\beta_{p+1}})[\sigma_{p-2} + 1, \sigma_p]$ for any $l \in V_{p+1}$.
4. The minimal nonzero degree of $T_{\beta_{p+1}}$ is not smaller than $\sigma_{p-1} + 1$.
5. For any $l \in \Delta_{p+1}$, $P_l$ is a direct summand of $(t_l T_{\beta_{p+1}})[\sigma_{p-2} + 1, \sigma_p]$.

**Proof.** The first condition can be obtained applying Lemma 4 to the words $\beta_p, s_{\Delta_{p+1}} \beta_p, \ldots, s_{\Delta_{p+1}} \beta_p$, all of which are of strictly smaller length than $\alpha$. Indeed, if $\Delta^{i+1}_{p+1} = \Delta^i_{p+1} \cup \{l\}$, then $P_l$ is a direct summand of $(t_{\Delta^i_{p+1}} \bigtriangledown)^a$ for $a \leq \sigma_p$ and it is sufficient to prove that the minimal nonzero degree of $(t_{\Delta^i_{p+1}} \bigtriangledown)^a$ is not smaller than $a + \omega \leq \sigma_{p-1} + 1$. If this is not the case, there exists some $b \leq \min(a - 1, \sigma_{p-1})$ and $k \in \Gamma_0$ such that $P_k$ is a direct summand of $(t_{\Delta^i_{p+1}} \bigtriangledown)^b$. Observe that $P_k$ cannot belong to $V_p$ by the conditions 5 and 4 with $u = p$ and Lemma 2 and cannot belong to $V_{p+1} \setminus \Delta^i_{p+1}$. On the other hand, for $k \in \Delta^i_{p+1}$, one has

$$\text{Hom}^{b+\omega}(P_k, t_{\Delta^i_{p+1}}^{-1} \bigtriangledown \bigtriangledown) \cong \text{Hom}^{b+\omega}(t_{\Delta^i_{p+1}} \bigtriangledown P_k, T_{\beta_p}) \cong \text{Hom}^{b+2}\omega^{-1}(P_k, T_{\beta_p}) = 0,$$

because the minimal nonzero degree of $T_{\beta_p}$ is not smaller than $\sigma_{p-2} + 1$, $\omega = 1$, and $b + \omega - 1 \leq \sigma_{p-2} + 1$.

Let us now prove the second condition. Suppose first that there exists some $l \in \Delta_{p+1} \setminus N(\Delta_p)$. Then it is clear that $l \notin N(\Delta_u)$ for all $0 \leq u \leq p$, in particular, $l \neq i$. Hence we have $\alpha = s_{\Delta_0} \ldots s_{\Delta_p} s_{\Delta_p}^{-1} \beta_p = s_i \gamma$ for some $\gamma \in B^+_T$ with $l(\gamma) = l(\alpha) - 1$. Then the assertion of Lemma 4 is valid for $\alpha$, a contradiction.

Suppose now that $l \notin \Delta_{p+1}$ for some $l \in \Delta_{p-1}$. By construction, this means that $P_l$ is not a direct summand of $(t_{\Delta^i_{p+1}} \bigtriangledown)^a$. Let $\pi$ denote the split epimorphism $\bigoplus_{l \in \Delta_{p+1}} P_l^{\chi_{p-1}(l)} \to P_l^{\chi_{p-1}(l)}$. The octahedral axiom applied to the composition $\pi[1 - \omega_{p+1}] \circ \beta_{\psi_p}[1]$ gives us the following diagram:

$$
\begin{array}{ccccccccc}
\bigoplus_{k \in \Delta_p} P_k^{\chi_l(k)} & \longrightarrow & C' & \longrightarrow & \bigoplus_{l \in \Delta_{p-1} \setminus \{l\}} P_l^{\chi_{p-1}(l)}[1 - \omega_{p+1}] \\
\bigoplus_{k \in \Delta_p} P_k^{\chi_l(k)} & \longrightarrow & C_p & \longrightarrow & \bigoplus_{l \in \Delta_{p-1}} P_l^{\chi_{p-1}(l)}[1 - \omega_{p+1}] \\
& & \downarrow & & \downarrow \pi[1 - \omega_{p+1}] & & \downarrow \pi[1 - \omega_{p+1}] \\
& & P_l^{\chi_{p-1}(l)}[1 - \omega_{p+1}] & & P_l^{\chi_{p-1}(l)}[1 - \omega_{p+1}] & & \end{array}
$$

Let $\psi$ denote the morphism from $P_l^{\chi_{p-1}(l)}[-\omega_{p+1}]$ to $C'$ arising from this diagram. Since $\chi_{p-1}(l) > 0$ by our assumptions, $C'$ is a nontrivial two-term subobject of $C_p$ and Proposition 3 can be applied. For some $\omega_p + \sigma_{p-2} + 1 \leq r \leq \omega_p + \sigma_p$, we have a nonzero morphism $f : C' \to T_{\beta_p}[r]$ such that $f[\omega]g = 0$ for any right socle morphism $g : C'' \to C'[\omega]$. Since
Hom\(\psi(C_p,T_{\beta_p})\) = 0 by Proposition 4 the morphism \(f\psi\) is nonzero for some component \(\psi' : P_l[-\omega_{p+1}] \to C'\) of the map \(\psi\). We are going to prove that \(t_{\Delta_{p+1}}^{-1}(f\psi)\mid_{\omega_{p+1}} : P_l \to t_{\Delta_{p+1}}^{-1}T_{\beta_p}r[\omega_{p+1}]\) is long. Since \(r - \omega_p + \sigma_{p-2} + 1, \sigma_p\), this contradicts the assumption that \(P_l\) is not a direct summand of \((t_{\Delta_{p+1}}^{-1}T_{\beta_p})\mid_{\sigma_{p-2}+1, \sigma_p}\).

Let us pick some \(r \in N(l)\). Since \(P_r[1 - \omega_p]\) is a right-proper two-term object with \(\alpha_{P_r[1 - \omega_p]} = \varphi_{P_r[1 - \omega_p]} = 0\) and \(\beta_{P_r[1 - \omega_p]} = id_{P_r[\omega_p]}\), we have

\[
t^\perp_{\Delta_{p+1}} P_r[1 - \omega_p] \cong \text{cone} \left( \bigoplus_{j \in (N(r) \cap \Delta_{p+1})} P_j[-\omega_{p+1}] \to P_r \right)
\]

by Lemma 7. Note that the morphism \(\psi'\mid_{\omega_p} : t^\perp_{\Delta_{p+1}} \gamma_{r,l}[1 - \omega_p]\) is right socle. Indeed, as the diagram above shows, \(\psi'\) factors through \(\bigoplus k \in \Delta_p P^\chi_p(k)\) and the map

\[
P_r \xrightarrow{\alpha_{t^\perp_{\Delta_{p+1}} P_r[1 - \omega_p]}} t^\perp_{\Delta_{p+1}} P_r[1 - \omega_p] \xrightarrow{t^\perp_{\Delta_{p+1}} \gamma_{r,l}[1 - \omega_p]} P_l[\omega_p] \to \bigoplus k \in \Delta_p P^\chi_p(k)\]

is not a split monomorphism, because it factors through \(P_l[\omega_p]\). Then \((f\psi')\mid_{\omega_p} : t^\perp_{\Delta_{p+1}} \gamma_{r,l}[1 - \omega_p]\) is a long morphism.

Suppose now that \(\chi_{p+1}(l) > 0\) for all \(l \in \Delta_{p+1}\). We define \(C_{p+1} : = t^\perp_{\Delta_{p+1}} C_p\). Then we have Hom\(\psi(C_{p+1},T_{\beta_{p+1}})\) = 0 for \(r \leq \omega_{p+1} + \sigma_{p+1}\) (see the proof of the first part of Proposition 4). Due to Lemma 2 we have

\[
C_{p+1} = \text{cone} \left( \bigoplus j \in \Delta_p P^\chi_p(j)[-\omega_p] \xrightarrow{\varphi_{p+1}} \bigoplus k \in \Delta_{p+1} P^\chi_p(k) \right)
\]

for some morphism \(\varphi_{p+1}\). We prove the third property by contradiction. If \(P_l\) with \(l \in \Delta_{p+1}\) is a direct summand of \((T_{\beta_{p+1}})\)_a for some \(a \leq \sigma_{p+1}\), then there exists a morphism from \(\bigoplus j \in \Delta_p P^\chi_p(j)\) to \(T_{\beta_{p+1}}[a + \omega]\) annihilated by \(\varphi_{p+1}\). This gives a nonzero morphism from \(C_{p+1}\) to \(T_{\beta_{p+1}}[a + \omega]\), which is impossible since \(a + \omega \leq \omega + \sigma_p < \omega_{p+1} + \sigma_{p+1}\). Observe that the minimal nonzero degree of \(T_{\beta_{p+1}}\) is not smaller than \(\sigma_{p-2} + 1\) and \(P_l\) with \(l \in V_p \setminus \Delta_{p+1}\) cannot be a direct summand of \((T_{\beta_{p+1}})[\sigma_{p-2}, \sigma_{p+1}]\) by the construction of \(\Delta_{p+1}\). This finishes the proof of the third property. The forth property follows from the third one, the property (3) with \(u = p\) and Lemma 2.

It remains to prove the fifth property. Let \(l \in \Delta_{p+1}\). There is some \(c\) such that \(l \not\in \Delta_{p+1}\) and \(l \in \Delta_{c+1}\). Then \(P_l\) is a direct summand of \((t_{\Delta_{p+1}}^{-1}T_{\beta_p})_a\) for some \(a \in [\sigma_{p-2} + 1, \sigma_p]\). Choose the minimal such a number \(a\). It follows from the construction of \(\Delta_{p+1}\), the property (3) with \(u = p\) and Lemma 2 that the minimal nonzero degree of \(t_{\Delta_{p+1}}^{-1}T_{\beta_p}\) is not smaller than \(\text{min}(a, \sigma_p + 1)\) (see the beginning of this proof). Then \(P_l\) is a direct summand of \((t_{\Delta_{p+1}}^{-1}T_{\beta_p})_a\) = \((t_lT_{\beta_p})_a\) by Lemma 2 and we are done.

If \(\chi_{p+1}(l) = 0\) for some \(l \in \Delta_{p+1}\), we can write \(\alpha = s_{\Delta_0} \cdots s_{\Delta_p}s_l\bar{\beta}\) and get the required presentation for \(\alpha\), as we have announced in the previous section. In this case we terminate the process of factorization (step I) and go to braiding (step II), which is described in the next section. Otherwise condition (6) is also satisfied for \(u = p + 1\) and write \(\alpha = s_{\Delta_0} \cdots s_{\Delta_p}s_{\Delta_{p+1}}\beta_{p+1}\) and continue the process, applying the arguments above to \(u = p + 2\) instead of \(u = p + 1\).
8. Mesh braiding in $B^+_{\Gamma}$

The goal of this section is to show that any word of the form $s_{\Delta_0} \cdots s_{\Delta_p}s_i$ as obtained in the previous section is left-divisible by some $s_j$ in $B^+_{\Gamma}$ with $j \neq i$. As we explained in Section 5, this finishes the proof of Lemma 1.

Let $\mathbb{Z}\Gamma$ be a directed graph whose vertices are pairs $(n, j)$ for every $n \in \mathbb{Z}, j \in V^n$. The arrows of $\mathbb{Z}\Gamma$ are of the form $((n, j) \rightarrow (n + 1, i))$ for every edge $(i, j)$ of $\Gamma$. The graph $\mathbb{Z}\Gamma$ is essentially the stable translation quiver associated to $\Gamma$, but with vertices enumerated differently. It is convenient to draw $\mathbb{Z}\Gamma$ in such a way that vertices with the same first coordinate form one “vertical slice”. Set $p_1(a) = n, p_2(a) = j$ for $a = (n, j) \in (\mathbb{Z}\Gamma)_0$. Let $\Theta_n$ denote the set $p_1^{-1}(n) = \{(n, j) \in (\mathbb{Z}\Gamma)_0 \mid j \in V^n\} \subset (\mathbb{Z}\Gamma)_0$.

Now any word $\gamma$ representing an element of $B^+_{\Gamma}$ can be depicted as some set of vertices $\Lambda_\gamma$ of $\mathbb{Z}\Gamma$ in the following way. First let us choose a presentation $\gamma = s_{\Sigma_0} \cdots s_{\Sigma_p}$, where $\Sigma_0, \ldots, \Sigma_p$ are such that $\Sigma_k \subseteq V^k$ and the convention $s_{\emptyset} = 1$ is used. Now we can define $\Lambda_\gamma = \{(k, j) \mid k \geq 0, j \in \Sigma_k\} \subset (\mathbb{Z}\Gamma)_0$. On the other hand, to any finite set $\Lambda \subset (\mathbb{Z}\Gamma)_0$ one can assign a word $\gamma_\Lambda$ in $B^+_{\Gamma}$. More precisely, set $\gamma_\Lambda = \prod_{k=-\infty}^{+\infty} s_{\Sigma_k}$, where $\Sigma_k = \Lambda \cap \Theta_k$. It is easy to see that $\gamma_{\Lambda_\gamma} = \gamma$.

**Example.** Let $\Gamma = D_4$ with vertices enumerated in such a way that the vertex of degree three is 2. Let $\gamma = s_2(s_1 s_3 s_4) s_2(s_1 s_3 s_4) s_2 s_4$. Then we have the following set $\Lambda_\gamma$:

![Figure 1](image1.png)

On the other hand, there are infinitely many sets $\Lambda$ such that $\gamma_\Lambda = \gamma$. For instance, the following set of vertices of $(\mathbb{Z}\Gamma)_0$ also corresponds to the word $\gamma$:

![Figure 2](image2.png)

Now fix some finite $\Lambda \subset (\mathbb{Z}\Gamma)_0$.

**Definition 8.1.** Let $a, b \in \Lambda$. We say that there is a *generalized mesh* starting at $a$ and ending at $b$ if $p_2(a) = p_2(b) = t, p_1(a) < p_1(b)$ and $(k, t) \notin \Lambda$ for every $k$ with $p_1(a) < k < p_1(b)$. In this case we set $\text{mesh}_\Lambda(a, b) = \{c \in \Lambda \mid p_2(c) \in N(t), p_1(a) < p_1(c) < p_1(b)\}$. 

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We will sometimes refer to $mesh_{\Lambda}(a, b)$ as a generalized mesh also.

For every $b \in \Lambda$, let $\tau_{\Lambda}(b)$ denote the element of $\Lambda$ such that there is a generalized mesh starting at $\tau_{\Lambda}(b)$ and ending at $b$. If there is no such an element in $\Lambda$, we say that $\tau_{\Lambda}(b)$ is an “imaginary” vertex ($-\infty, p_{2}(b)$). In this case we also say that there is an infinite mesh starting at $\tau_{\Lambda}(b)$ and ending at $b$ and set $mesh_{\Lambda}(\tau_{\Lambda}(b), b) = \{ c \in \Lambda \mid p_{2}(c) \in N(p_{2}(b)), p_{1}(c) < p_{1}(b) \}.$

Now suppose there is a non-negative integer assigned to each element of $\Lambda$ and each of the imaginary vertices at minus infinity, i.e. consider $\Lambda$ together with a function $\theta : \Lambda \cup (-\infty \times \Gamma_{0}) \rightarrow \mathbb{Z}$.

Definition 8.2. We say that $(\Lambda, \theta)$ satisfies mesh relations if

$$\theta(b) + \theta_{\tau_{\Lambda}(b)} = \sum_{c \in mesh_{\Lambda}(\tau_{\Lambda}(b), b)} \theta(c)$$

for every $b \in \Lambda$.

The following statement follows immediately from definitions:

Lemma 13. Let $\gamma$ be a word of the form $s_{\Delta_{0}} \ldots s_{\Delta_{p}} s_{i}$ as described in the previous section. For every $(n, j) \in \Lambda_{\gamma}$ set $\theta(n, j) = \chi_{n}(j)$, $\theta(-\infty, i) = -1$, $\theta(-\infty, j) = 0$ for every $j \neq i$. Then $(\Lambda_{\gamma}, \theta)$ satisfies mesh relations.

Example: The word $s_{2}(s_{1}s_{3}s_{4})s_{2}(s_{1}s_{3}s_{4})s_{2}s_{4}$ in $B_{D_{4}}^{+}$ as in the previous example is depicted below together with the function $\theta = \chi$.

![Figure 3](image)

It is natural to ask which finite subsets of $(\mathbb{Z}\Gamma)_{0}$ correspond to the words equal in the braid monoid $B_{D_{4}}^{+}$.

First let $a = (n, j) \in \Lambda$ be such that $b = (n + 2, j) \notin \Lambda$ (respectively $b = (n - 2, j) \notin \Lambda$) and $(n+1, k) \notin \Lambda$ (respectively $(n-1, k) \notin \Lambda$) for every $k \in N(j)$. Set $\overline{\Lambda} = (\Lambda \setminus \{a\}) \cup \{b\}$. In addition, let $\overline{\theta}(b) = \theta(a)$ and $\overline{\theta}|_{\overline{\Lambda} \setminus \{b\}} = \theta|_{\overline{\Lambda} \setminus \{a\}}$. We refer to this procedure as commutation.

![Figure 4](image)

The following statement is clear:
**Lemma 14.** Commutation does not change the corresponding word in $B_T$, i.e. $\gamma_{\Lambda} = \gamma_{\overline{\Lambda}}$. In addition, if $(\Lambda, \theta)$ satisfies mesh relations, then so does $(\overline{\Lambda}, \overline{\theta})$.

Now braid relations in $B_T$ can be also depicted in terms of vertices of $Z\Gamma$. We say that $a, b, c \in (Z\Gamma)_0$ form a braid if $a = (n, j)$, $b = (n + 1, k)$, $c = (n + 2, j)$ for some $n \in \mathbb{Z}$ and some $j, k \in \Gamma_0$ such that $k \in N(j)$. Now let $a, b, c \in \Lambda$ as above form a braid and suppose in addition that $(n + 1, t) \notin \Lambda$ for any $t \in N(j)$ \{ $\{ j \}$ \} and $d = (n + 3, k) \notin \Lambda$. Set $\overline{\Lambda} = (\Lambda \setminus \{ a \}) \cup \{ d \}$, $\overline{\theta}_{|\overline{\Lambda}(b,c,d)} = \theta_{|\Lambda(a,b,c)}$, $\overline{\theta}(b) = \theta(c), \overline{\theta}(c) = \theta(b), \overline{\theta}(d) = \theta(a)$. We refer to this procedure as braiding.

![Figure 5](image)

**Lemma 15.** Braiding does not change the corresponding word in $B_T$, i.e. $\gamma_{\Lambda} = \gamma_{\overline{\Lambda}}$. Moreover, if $(\Lambda, \theta)$ satisfies mesh relations, then so does $(\overline{\Lambda}, \overline{\theta})$.

**Proof.** First we establish that $\gamma_{\Lambda} = \gamma_{\overline{\Lambda}}$. It is sufficient to show that $s_{\Sigma_n} s_{\Sigma_{n+1}} s_{\Sigma_{n+2}} s_{\Sigma_{n+3}} = s_{\overline{\Sigma}_n} s_{\overline{\Sigma}_{n+1}} s_{\overline{\Sigma}_{n+2}} s_{\overline{\Sigma}_{n+3}}$ where $\Sigma_p = p_2(\Theta_p \cap \Lambda)$ and $\overline{\Sigma}_p = p_2(\Theta_p \cap \overline{\Lambda})$. Observe that $\Sigma_n = \overline{\Sigma}_n = \Sigma_n \setminus \{ j \}$, $\Sigma_{n+1} = \Sigma_{n+1} \cup \Sigma_{n+2} \cup \Sigma_{n+3}$ and $\Sigma_{n+3} = \Sigma_{n+3} \cup \{ k \}$. Hence

$$s_{\Sigma_n} s_{\Sigma_{n+1}} s_{\Sigma_{n+2}} s_{\Sigma_{n+3}} = s_{\Sigma_n} s_{\Sigma_{n+1}} s_{\Sigma_{n+2}} s_{\Sigma_{n+3}} = s_{\overline{\Sigma}_n} s_{\overline{\Sigma}_{n+1}} s_{\overline{\Sigma}_{n+2}} s_{\overline{\Sigma}_{n+3}} = s_{\overline{\Sigma}_n} s_{\overline{\Sigma}_{n+1}} s_{\overline{\Sigma}_{n+2}} s_{\overline{\Sigma}_{n+3}}$$

Now we will show that braiding respects mesh relations. Since $\overline{\theta}$ differs from $\theta$ only on $a, b, c$ and $d$, it is sufficient to check the relations only for generalized meshes these four vertices take part in. There are two types of such generalized meshes: those that have one of $a, b, c$ and $d$ as starting and/or ending vertices and those that do not.

1) Note that $\tau_{\overline{\Lambda}}(b) = \tau_{\Lambda}(b)$ and $mesh_{\overline{\Lambda}}(\tau_{\overline{\Lambda}}(b), b) = mesh_{\Lambda}(\tau_{\Lambda}(b), b)$ \{ $\{ a \}$ \}. We have

$$\theta_{\tau_{\overline{\Lambda}}(b)} + \theta(b) = \sum_{x \in mesh_{\Lambda}(\tau_{\Lambda}(b), b)} \theta(x) = \sum_{x \in mesh_{\overline{\Lambda}}(\tau_{\overline{\Lambda}}(b), b)} \theta(x) + \theta(a)$$

and $\theta(a) + \theta(c) = \theta(b)$, because $a, b, c$ form a generalized mesh in $\Lambda$ starting at $a$ and ending at $c$. Then

$$\overline{\theta}_{\tau_{\overline{\Lambda}}(b)} + \overline{\theta}(b) = \theta_{\tau_{\Lambda}(b)} + \theta(c) = \sum_{x \in mesh_{\Lambda}(\tau_{\Lambda}(b), b)} \theta(x).$$

The remaining cases are either analogous to the one just discussed or obvious.

2) Now consider some generalized mesh $mesh_{\overline{\Lambda}}(x, y)$ in $\overline{\Lambda}$ such that $x, y \notin \{ a, b, c, d \}$. If $b \in mesh_{\overline{\Lambda}}(x, y)$, then $d \in mesh_{\overline{\Lambda}}(x, y)$ and vise versa. The corresponding mesh relation remains valid after braiding, because $d \notin \Lambda$ and $\overline{\theta}(b) + \overline{\theta}(d) = \theta(c) + \theta(a) = \theta(b)$. If $c \in mesh_{\overline{\Lambda}}(x, y)$, then $a \in mesh_{\overline{\Lambda}}(x, y)$ and vise versa. The corresponding mesh relation remains valid after braiding, because $\overline{\theta}(c) = \theta(b) = \theta(c) + \theta(a)$.

\[\square\]
The main result of this section is the following theorem that in view of the facts we have already proved implies Lemma 11.

**Theorem 3.** Let $(\Lambda, \theta)$ satisfying mesh relations be such that $\theta$ is non-negative on $\Lambda$, vanishing on precisely one vertex of $\Lambda$, $\theta(-\infty, i) = -1$ for some $i \in \Gamma_0$ and $\theta(-\infty, k) = 0$ for all $k \in \Gamma_0 \setminus \{i\}$. Then $\gamma = \gamma_\Lambda$ is left-divisible by some $s_j$ in $B^+_\Lambda$ with $j \neq i$.

To prove this theorem we will need the following lemma.

**Lemma 16.** Let $(\Lambda, \theta)$ be as in Theorem 3 and $a \in \Lambda$ such that $\theta(a) = 0$, $\tau_\Lambda(a) \neq -\infty$. Then there exists a sequence of commutations and braidings turning $(\Lambda, \theta)$ into some $(\Lambda', \theta')$ such that

$$\{|x \in \Lambda' \mid p_1(x) \leq p_1(b)\}| < \{|x \in \Lambda \mid p_1(x) \leq p_1(a)\}|,$$

where $b$ is the unique vertex of $\Lambda'$ on which $\theta'$ vanishes.

Let us first show how Lemma 16 implies Theorem 3.

**Proof of Theorem 3** It is sufficient to show that applying a sequence of commutations and braidings we can transform $\Lambda$ into some $\Lambda'$ such that $\Theta_n \cap \Lambda' \neq \emptyset$, where $n$ is the smallest integer such that $\Theta_n \cap \Lambda' \neq \emptyset$. Note that commutation and braiding do not change the multiset $\{\theta(x)\}_{x \in \Lambda}$ of values of $\theta$.

Applying Lemma 16 several times, one can obtain a pair $(\Lambda'', \theta'')$ satisfying mesh relations such that $\gamma_{\Lambda''} = \gamma_\Lambda$ and $\theta''$ vanishes on (and only on) a vertex with the smallest first coordinate among all vertices of $\Lambda''$. Let $\theta''(x) = 0$. Then $\gamma_{\Lambda''} = \gamma_\Lambda$ is divisible by $s_{p_2(x)}$ on the left. It remains to ascertain that $p_2(x) \neq i$. Indeed, since $x$ is a vertex with the smallest $p_1(x)$ among all vertices of $\Lambda''$, there is an infinite generalized mesh ending at $x$ and starting at $(-\infty, j)$ for some $j \in \Gamma_0$. Since $(\Lambda'', \theta'')$ satisfies infinite mesh relations, $0 = \theta''(-\infty, j) + \theta''(x) = \theta''(-\infty, j)$. We immediately see that $j \neq i$, since $\theta''(-\infty, i) = -1$. □

Now we prove Lemma 16.

**Proof of Lemma 16** Let $a_0 = a = (n, j)$. Since there is only one vertex of $\Lambda$ on which $\theta$ vanishes and $\tau_\Lambda(a_0) \in \Lambda$, $\theta \tau_\Lambda(a_0) + \theta(a_0) = \theta \tau_\Lambda(a_0) > 0$. Hence $\text{mesh}_\Lambda(\tau_\Lambda(a_0), a_0) \neq \emptyset$. Let $a_1 = (n_1, t) \in \text{mesh}_\Lambda(\tau_\Lambda(a_0), a_0)$ be any vertex with the smallest first coordinate among vertices of $\text{mesh}_\Lambda(\tau_\Lambda(a_0), a_0)$. Then, possibly applying several commutations, one can assume that $\tau_\Lambda(a_0) = (n_1 - 1, j)$. If $|\text{mesh}_\Lambda(\tau_\Lambda(a_0), a_0)| = 1$, then, possibly applying several commutations, one can assume that $\tau_\Lambda(a_0), a_1, a_0$ form a braid. Then we can apply braiding to the triple $(\tau_\Lambda(a_0), a_1, a_0)$ after some commutations as described below and we are done. Suppose now that $q_0 = |\text{mesh}_\Lambda(\tau_\Lambda(a_0), a_0)| > 1$.

Note that $q_1 = |\text{mesh}_\Lambda(\tau_\Lambda(a_1), a_1)| > 0$, since $\tau_\Lambda(a_0) \in \text{mesh}_\Lambda(\tau_\Lambda(a_1), a_1)$. Suppose that $q_1 > 1$. Let $a_2 = (n_2, p) \in \text{mesh}_\Lambda(\tau_\Lambda(a_1), a_1)$ be any vertex with the smallest first coordinate among vertices of $\text{mesh}_\Lambda(\tau_\Lambda(a_1), a_1) \setminus \{\tau_\Lambda(a_0)\}$. Then, possibly applying several commutations, one can assume that $\tau_\Lambda(a_1) = (n_2 - 1, t)$.

**Figure 6**
Continue in the same fashion to obtain a sequence of elements \(a_0, \ldots, a_k\) of \(\Lambda\) such that \(\tau_\Lambda(a_m) \in mesh_\Lambda(\tau_\Lambda(a_{m+1}), a_{m+1})\), \(a_{m+1} \in mesh_\Lambda(\tau_\Lambda(a_m), a_m)\) for every \(m = 0, \ldots, k - 1\) and \(\tau_\Lambda(a_k) \neq -\infty\). Let us set \(q_m := |mesh_\Lambda(\tau_\Lambda(a_k), a_k)|\) for \(m = 0, \ldots, k\). We claim that if the sequence \(a_0, \ldots, a_k\) is maximal with respect to inclusion among sequences satisfying this property, then \(q_k = 1\). Indeed, if \(q_k > 1\), then there is a vertex \(a_{k+1}\) with the smallest first coordinate among vertices of \(mesh_\Lambda(\tau_\Lambda(a_k), a_k)\). Because the sequence we consider is maximal, \(\tau_\Lambda(a_{k+1}) = (-\infty, p_2(a_{k+1}))\). Since \(\theta(-\infty, p_2(a_{k+1}))\) is either 0 or -1, we have \(\theta(a_{k+1}) \geq \theta(a_k)\). Then \(\theta(a_k) \geq \theta(a_{k-1})\), since 
\[
\theta(a_k) + \theta(\tau_\Lambda(a_k)) = \theta(a_{k+1}) + \theta(\tau_\Lambda(a_{k+1}) + \sum_{y \in mesh_\Lambda(\tau_\Lambda(a_k), a_k) \setminus \{a_{k+1}, \tau_\Lambda(a_{k-1})\}} \theta(y)
\]
Continuing in the same way we get \(\theta(a_1) \geq \theta(a_0)\). On the other hand, \(\theta(\tau_\Lambda(a_0)) = \theta(\tau_\Lambda(a_0) + \theta(a_0) > \theta(a_1)\), because \(q_0 > 1\), a contradiction.

Take a sequence \(a_0, \ldots, a_k\) maximal with respect to inclusion and satisfying the properties described above with the smallest corresponding sequence of integers \((q_0, \ldots, q_k)\) in the lexicographic order. Denote such a \((q_0, \ldots, q_k)\) by \(seq(\Lambda)\). If \(seq(\Lambda) = (1)\), there is nothing to prove, as we remarked earlier. Now suppose that the statement is known for all \((N', \theta')\) with \(seq(N') < seq(\Lambda)\). We will show that there exists a sequence of commutations and braidings that turns \((\Lambda, \theta)\) into \((\Lambda, \theta')\) such that \(seq(\Lambda) < seq(\Lambda)\) in the lexicographic order. Observe that commutations do not change \(seq(\Lambda)\).

Since \(q_k = 1\), \(mesh_\Lambda(\tau_\Lambda(a_k), a_k) = \{\tau_\Lambda(a_{k-1})\}\) and, possibly applying several commutations, one can assume that \(\tau_\Lambda(a_k), \tau_\Lambda(a_{k-1}), a_k\) form a braid. Let \(\tau_\Lambda(a_k) = (n, j), \tau_\Lambda(a_{k-1}) = (n+1, l), a_k = (n+2, j)\). We already have \((n+1, t) \notin \Lambda\) for every \(t \in N(j) \setminus \{l\}\).

To perform a braiding on \(\tau_\Lambda(a_k), \tau_\Lambda(a_{k-1}), a_k\), we also need to have \((n+3, j) \notin \Lambda\) and \((n+2, v) \notin \Lambda\) for every \(v \in N(l) \setminus \{j\}\). To this end we first apply a sequence of commutations shifting all vertices \(x\) of \(\Lambda\) with \(p_1(x) \geq n+2, x \neq a_k\), to the right. More precisely, the resulting set \(\Lambda\) is \(\Lambda_1 \cup \Lambda_2\), where \(\Lambda_1 = \{x \in \Lambda \mid p_1(x) \leq n + 1\} \cup \{a_k\}\), \(\Lambda_2 = \{(q + 2, r) \mid (q, r) \in \Lambda \setminus \{a_k\}, q \geq n + 2\}\) and \(\theta|_{\Lambda_1} = \theta|_{\Lambda_1}, \theta(q + 2, r) = \theta(q, r)\) for \((q + 2, r) \in \Lambda_2\). It is clear that such a transformation can be obtained consequently applying commutations to all vertices of \(\Lambda \setminus \Lambda_1\) starting with those with the largest first coordinate.

\[\text{Figure 7. } \Lambda\]

\[\text{Figure 8. } \Lambda\]
Now let \((\tilde{\Lambda}, \tilde{\theta})\) be a pair obtained by braiding \((\tau_\Lambda(a_k), \tau_\Lambda(a_{k-1}), a_k)\) in \(\overline{\Lambda}\). Clearly, \(q_0, \ldots, q_{k-2}\) remain unchanged. However, the mesh ending at \(a_{k-1}\) now starts at \(\tau_\tilde{\Lambda}(a_{k-1}) = (n + 3, l)\), and hence obviously does not contain the vertex \(a_k = (n + 2, j)\). We see that \(|mesh_\tilde{\Lambda}(\tau_\tilde{\Lambda}(a_{k-1}), a_{k-1})| = q_{k-1} - 1\). Since any sequence that starts with \(q_0, \ldots, q_{k-1} - 1\) is smaller in the lexicographic order than the sequence \((q_1, \ldots, q_{k-1}, q_k)\), we have \(seq(\tilde{\Lambda}) < seq(\Lambda)\).

\[ \begin{array}{c}
\tau_\Lambda(a_{k-1}) \\
\bullet \\
\tau_\Lambda(a_{k-1}) \\
\bullet \\
\tau_\Lambda(a_{k-2}) \\
\bullet \\
\end{array} \]

\[ \overline{\Lambda} \]

**Example.** Consider \(\Lambda_\gamma\) as in the previous example.

![Figure 10](image10.png)

**Figure 10.** The first application of Lemma 16 requires just one braiding.

![Figure 11](image11.png)

**Figure 11.** Now the sequence of meshes starting at 0 is \((2, 1)\).

![Figure 12](image12.png)

**Figure 12.** The second application of Lemma 16 begins with a sequence of commutations.

![Figure 13](image13.png)

**Figure 13.** Now three vertices can be braided and the sequence of meshes starting at 0 is \((1)\) again.
Remark. One can prove the assertion converse to Theorem 3. Namely, if $(\Lambda, \theta)$ satisfying mesh relations is such that $\theta$ is non-negative on $\Lambda$, $\theta(-\infty, i) = -1$ for some $i \in \Gamma_0$, $\theta(-\infty, k) = 0$ for all $k \in \Gamma_0 \setminus \{i\}$ and $\gamma_\Lambda$ is left-divisible by some $s_j$ in $B_\Gamma^+$ with $j \neq i$, then $\theta$ vanishes on some vertex of $\Lambda$. Indeed, in this case $(\Lambda, \theta)$ can be transformed using commutations and braidings into $(\widetilde{\Lambda}, \widetilde{\theta})$ such that there is a vertex $x = (n, j) \in \widetilde{\Lambda}$ with $\tau_{\widetilde{\Lambda}}(x) = (-\infty, j)$ and $\text{mesh}_{\widetilde{\Lambda}}(\tau_{\widetilde{\Lambda}}(x), x) = \emptyset$. Then the mesh relation corresponding to $x$ implies $\widetilde{\theta}(x) = 0$ and the assertion follows from the fact that commutations and braidings do not change the multiset of values of $\theta$.

9. AN APPLICATION TO DERIVED PICARD GROUPS

For this section we assume that $k$ is algebraically closed. All modules in this section are assumed to be left unless explicitly stated otherwise. Let $\mathcal{D}_\Gamma = D^b(\text{mod} -\Lambda_\Gamma)$, the bounded derived category of finitely generated $\Lambda_\Gamma$-modules, where $\Gamma$ is a simply laced Dynkin diagram and $\Lambda_\Gamma$ is the trivial extension algebra of the $\Gamma$ diagram with alternating orientation. In other words, $\Lambda_\Gamma = kQ_\Gamma/I_\Gamma$, where $Q_\Gamma$ is one of the following quivers:

![Figure 15. $\Gamma = A_n$](image)

![Figure 16. $\Gamma = D_n$](image)

![Figure 17. $\Gamma = E_n$ ($n = 6, 7, 8$)](image)
The ideal \( I_{\Gamma} \) of \( kQ_{\Gamma} \) is generated by all paths of length greater or equal to 2, except for the paths of length 2 starting and ending at the same vertex, and the differences of any two paths of length two starting at the same vertex.

The \( k \)-algebra \( \Lambda_{\Gamma} \) is finite-dimensional and symmetric. Let \( e_i \) be the idempotent associated with the vertex \( i \) of the quiver \( Q_{\Gamma} \). Denote by \( P_i = \Lambda_{\Gamma} e_i \) the corresponding indecomposable projective module. Note that \( P_i \) is a 0-spherical object of \( \mathcal{D}_{\Gamma} \). Indeed, the first condition is satisfied automatically. Since \( P_i \) is projective, \( \text{Ext}^1_{\Lambda_{\Gamma}}(P_i, -) \) vanishes for every \( m \neq 0 \), and hence the second condition simply means that \( \text{End}_{\Lambda_{\Gamma}}(P_i) \cong k[t]/(t^2) \). Finally, since \( \Lambda_{\Gamma} \) is symmetric, the last condition is satisfied automatically as well due to an isomorphism of functors \( \text{Hom}(P_i, -) \cong \text{Hom}(-, P_i)^* \). To sum up, it is now clear that \( \{P_i\}_{i \in (Q_{\Gamma})_0} \) is a \( \Gamma \)-configuration of 0-spherical objects of \( \mathcal{D}_{\Gamma} \).

**Remark.** In this less general setting we could have defined spherical twists in the following way:

**Definition 9.1.** (Grant, [7]) The spherical twist functor along \( P_i \) is

\[
t_{P_i} : \mathcal{D}_{\Gamma} \rightarrow \mathcal{D}_{\Gamma}
\]

\[
t_{P_i}(-) = \text{cone}(\Lambda_{\Gamma} e_i \otimes_k e_i \Lambda_{\Gamma} \xrightarrow{m} \Lambda_{\Gamma}) \otimes_{\Lambda_{\Gamma}} -
\]

where \( m \) is the multiplication map \( m(ae_i \otimes e_i b) = ab \) of \( \Lambda_{\Gamma} \)-\( \Lambda_{\Gamma} \) bimodules.

It is easy to check that on objects it coincides with our original definition, which is

\[
t_{P_i}(X) = \text{cone}(P_i \otimes_k \text{Hom}(P_i, X) \xrightarrow{ev} X)
\]

with \( ev \) the obvious evaluation map. Indeed, since \( \text{cone}(\Lambda_{\Gamma} e_i \otimes_k e_i \Lambda_{\Gamma} \xrightarrow{m} \Lambda_{\Gamma}) \) is a two-term complex of right-projective bimodules,

\[
\text{cone}(\Lambda_{\Gamma} e_i \otimes_k e_i \Lambda_{\Gamma} \xrightarrow{m} \Lambda_{\Gamma}) \otimes_{\Lambda_{\Gamma}} X \cong \text{cone}((\Lambda_{\Gamma} e_i \otimes_k e_i \Lambda_{\Gamma} \otimes_{\Lambda_{\Gamma}} X) \rightarrow (\Lambda_{\Gamma} \otimes_{\Lambda_{\Gamma}} X))
\]

\[
\cong \text{cone}(P_i \otimes_k (e_i \Lambda_{\Gamma} \otimes_{\Lambda_{\Gamma}} X) \rightarrow X) \cong \text{cone}(P_i \otimes_k \text{Hom}(P_i, X) \xrightarrow{ev} X).
\]

**Definition 9.2.** (R. Rouquier, A. Zimmermann, [14]) Let \( A \) be a finite-dimensional algebra. The derived Picard group \( \text{TrPic}(A) \) of \( A \) is the group of isomorphism classes of objects of the derived category of \( A \otimes A^{\text{op}} \)-modules, invertible under \( \otimes_{A} \). Equivalently, \( \text{TrPic}(A) \) is the group of standard autoequivalences of \( D^b(\text{mod} -A) \) modulo natural isomorphisms.

It is known (see, for example, [19]) that in the case \( \Gamma = A_n \) the subgroup of \( \text{TrPic}(\Lambda_{\Gamma}) \) generated by the spherical twist functors \( t_{P_i} \) is isomorphic to the braid group \( B_{A_n} \) on \( (n+1) \) strands. The next corollary of Theorem 1 generalizes this result.

**Corollary 1.** The subgroup of the derived Picard group \( \text{TrPic}(\Lambda_{\Gamma}) \) of \( \Lambda_{\Gamma} \) generated by the spherical twist functors \( t_{P_i} \) is isomorphic to the generalized braid group \( B_{\Gamma} \) of type \( \Gamma \).

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