A Priori Error Estimates of a Discontinuous Galerkin Finite Element Method for the Kelvin-Voigt Viscoelastic Fluid Motion Equations

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Abstract

This paper applies a discontinuous Galerkin finite element method to the Kelvin-Voigt viscoelastic fluid motion equations when the forcing function is in $L^\infty(L^2)$-space. Optimal a priori error estimates in $L^\infty(L^2)$-norm for the velocity and in $L^\infty(L^2)$-norm for the pressure approximations for the semi-discrete discontinuous Galerkin method are derived here. The main ingredients for establishing the error estimates are the standard elliptic duality argument and a modified version of the Sobolev-Stokes operator defined on appropriate broken Sobolev spaces. Further, under the smallness assumption on the data, it has been proved that these estimates are valid uniformly in time. Then, a first-order accurate backward Euler method is employed to discretize the semi-discrete discontinuous Galerkin Kelvin-Voigt formulation completely. The fully discrete optimal error estimates for the velocity and pressure are established. Finally, using the numerical experiments, theoretical results are verified. It is worth highlighting here that the error results in this article for the discontinuous Galerkin method applied to the Kelvin-Voigt model using finite element analysis are the first attempt in this direction.

Key Words. Kelvin-Voigt viscoelastic fluid model, semidiscrete discontinuous Galerkin approximations, uniform in time optimal error estimates, backward Euler method, numerical examples.

1 Introduction

Consider the following system of partial differential equations arising in the Kelvin-Voigt model of viscoelastic fluid flow

\[ \frac{\partial \textbf{u}}{\partial t} + \textbf{u} \cdot \nabla \textbf{u} - \kappa \Delta \textbf{u}_t - \nu \Delta \textbf{u} + \nabla p = \textbf{f}(x,t), \quad x \in \Omega, \quad t > 0, \]

(1.1)

and incompressibility condition

\[ \nabla \cdot \textbf{u} = 0, \quad x \in \Omega, \quad t > 0, \]

(1.2)

with initial and boundary conditions

\[ \textbf{u}(x,0) = \textbf{u}_0 \quad \text{in} \ \Omega, \quad \textbf{u} = 0, \quad \text{on} \ \partial \Omega, \quad t \geq 0, \]

(1.3)

where, $\Omega$ is a bounded convex polygonal or polyhedral domain in $\mathbb{R}^d$, $d = 2, 3$ with boundary $\partial \Omega$. Here, $\nu$ is the coefficient of kinematic viscosity, $\kappa$ is the retardation time or the time of relaxation of deformations, $\textbf{u} = (u_1, u_2)$ (or $(u_1, u_2, u_3)$) is the fluid velocity, $p$ is the pressure and $\textbf{f}$ is the external force. The Kelvin-Voigt model was introduced by Oskolkov [25] to represent the dynamics of the viscoelastic fluid motion. Later, Cao et al. in [9] viewed it as a smooth and inviscid regularization of the Navier-Stokes model. One may refer to [7], [8], [12] and literature therein for more detailed physical description and applications of the model. Using the proof techniques of Ladyzenskaya presented in [21], Oskolkov and his collaborators carried out a considerable amount of work in

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and literature referred to therein, related to the global existence of a unique "almost" classical solution of the Kelvin-Voigt system of equations \((1.1)-(1.3)\) with varying assumptions on the right-hand side force function \(f\).

There is significant literature devoted to the numerical approximations of the problem \((1.1)-(1.3)\). In [2], the regularity estimates for the weak solution of \((1.1)-(1.3)\) with the right-hand side function \(f = 0\) and the exponential decay property for the weak solution have been established. The authors defined a semidiscrete Galerkin finite element formulation for the Kelvin-Voigt model and have obtained optimal error estimates for the velocity in \(L^\infty(L^2)\) and \(L^\infty(H^1_0)\)-norms and pressure in \(L^\infty(L^2)\)-norm with the initial data \(u_0 \in H^2 \cap H^1_0\), which again preserve the exponential decay property. They have further extended the analysis in [3] by applying a first-order accurate backward Euler method and a second-order backward difference scheme to the semidiscrete model and have established fully discrete optimal error estimates for both velocity and pressure approximations. The finite element analysis of \((1.1)-(1.3)\) with right-hand side \(f \neq 0\) can be found in [4] and literature referred in.

In [28], Pani et al. have derived \(L^\infty(L^2)\) and \(L^\infty(H^1_0)\)-norms optimal error estimates for the spectral Galerkin and modified nonlinear Galerkin methods applied to \((1.1)-(1.3)\). The results related to the spectral Galerkin method are an improvement over the Oskolkov work in [26] as the constants in error bounds do not depend on time. Zang et al. in [38] have applied a pressure projection method based on the differences of two local Gauss integrations and have derived semidiscrete optimal error estimates for the lowest equal-order finite element space pairs. Then, they have employed a backward Euler method for the time discretization and have discussed the stability and convergence analysis for the fully discrete approximations. For more developments of numerical methods applied to the model \((1.1)-(1.3)\) and their finite element analysis, one may refer to [2], [29], [37] and the literature mentioned therein.

As can be seen, the literature is confined to the finite element analysis for the continuous Galerkin methods applied to the problem \((1.1)-(1.3)\). The work dedicated to the finite element analysis for discontinuous Galerkin (DG) methods is still missing from the literature. In recent years, discontinuous Galerkin finite element methods (DGFEM) have been proven to be powerful and popular computational methods for the numerical solution of partial differential equations. The DGFEMs possess many vital properties: They are element-wise conservative and readily parallelizable compared to the continuous Galerkin method. Furthermore, they compensate the continuous Galerkin methods that fail to support nonuniform higher-order local approximation over the mesh for solutions whose smoothness exhibits variation over the computational domain and for unstructured grids. Due to these properties, the DGFEM has received significant recognition from theoretical and practical perspectives. DG methods, introduced in [20, 31], have been applied to the Euler and Navier-Stokes equations as early as in [6, 23]. Later on, Girault et al. in [17] have presented the first rigorous finite element analysis for the DG method with nonoverlapping domain decomposition and approximations of order \(k = 1, 2, 3\) for the steady state Stokes and Navier-Stokes system of equations. The authors have derived a uniform discrete inf-sup condition for the discrete discontinuous spaces and established optimal energy error estimates for the velocity and \(L^2\)-estimates for the pressure for the proposed DG method. As an extension to the analysis in [17], the authors in [33] have established an improved inf-sup condition and discussed several numerical methods along with their numerical convergence rates. Kaya et al. in [20] have extended the work to the time-dependent Navier-Stokes equations by applying a linear subgrid-scale eddy viscosity method combined with discontinuous Galerkin approximations.

They have obtained optimal semi-discrete error estimates of the velocity and pressure with reasonable dependence on Reynolds number. They have further applied first and second-order accurate schemes in the time direction and derived fully discrete optimal velocity error estimates. In [16], Girault et al. have employed a projection method to decouple the velocity and pressure with a discontinuous Galerkin method for the time-dependent incompressible Navier-Stokes equations and established the optimal error estimates for the velocity and suboptimal for pressure. A few more notable efforts that deal with DG methods for incompressible NSEs can be found in [10, 11, 13, 19] and literature therein. And for the references related to the numerical methods and their convergence rates for the DG methods applied to the Navier-Stokes equations, one may refer to [24, 30, 51].

In all the literature mentioned above, for the steady and transient Navier-Stokes equations, the error bounds for the finite element approximations have been shown to be optimal only for the energy norm for the velocity. However, the optimal convergence rate has been numerically achieved for both energy and \(L^2\)-norms. The authors in [17] have provided a hint about the proof of \(L^2\)-norm error estimate of velocity for the SIPG (symmetric interior penalty Galerkin) method to the steady Navier-Stokes model only. But, there is no detailed analysis available in the literature. Furthermore, no rigorous analysis exists for the optimal \(L^2\)-norm error estimate for \(t > 0\) to the discontinuous velocity approximations for time dependent Navier-Stokes problem. And, to the best of
our knowledge, there is hardly any literature dedicated to the finite element analysis of DG methods applied to the Kelvin-Voigt equations of motion. This article can be considered as the first attempt in this direction. The article mainly focuses on deriving semidiscrete and fully discrete optimal error estimates for the symmetric interior penalty discontinuous Galerkin method applied to the problem \( \text{(1.1)-(1.3)} \) as the non-symmetric interior penalty discontinuous Galerkin method is known to provide suboptimal error estimates due to its dependence on the degree of polynomial approximation \( (33) \). The main ingredients in achieving the goals of the article are as follows:

1. Based on the analysis of Heywood and Rannacher \((18)\) developed for the non-conforming finite element method applied to the time-dependent Navier-Stokes equations, we introduce an \( L^2 \)-projection \( P_h \) onto an appropriate DG finite element space with the help of the approximation operator \( R_h \) (see Section \( 5 \)) and aim at deriving the approximation properties for \( P_h \).

2. Next, we define a modified Sobolev-Stokes projection \( S_h \) \((4)\) for broken Sobolev spaces, which plays an essential role in deriving the semidiscrete error estimates related to the DG method. Then, we concentrate on establishing the estimates for \( S_h \) in which the approximation properties of \( P_h \) play an vital role. Although we apply the ideas of \( 4 \), there are differences and analytical difficulties due to the DG formulation and difference in finite element spaces. In particular, the analysis of the nonlinear term in DG formulation needs a special kind of attention.

3. With the help of the modified Sobolev-Stokes projection \( S_h \) and duality arguments, we achieve optimal \textit{a priori} error estimates for the semidiscrete discontinuous Galerkin approximations to the velocity in \( L^\infty(L^2) \)-norm and pressure in \( L^\infty(L^2) \)-norm. These estimates are further shown to be uniform under the smallness assumption on the data.

4. We then apply the backward Euler scheme to the semidiscrete discontinuous Kelvin-Voigt model and establish optimal error estimates for fully discrete velocity and pressure.

5. Finally, we provide numerical examples and analyze the outcomes to verify the theoretical results.

This article is divided into the following sections: The functional spaces with notations required for the problem analysis, the discontinuous weak formulation, and some basic assumptions are presented in Section \( 2 \). The semidiscrete discontinuous Galerkin formulation, derivation of \textit{a priori} bounds of the discrete solution, and some trace inequalities are dealt with in Section \( 3 \). The modified Sobolev-Stokes operator and its properties and optimal \textit{a priori} error estimates for the velocity are represented in Section \( 4 \). The modified Sobolev-Stokes projection \( S_h \) and duality arguments, we achieve optimal \textit{a priori} error estimates for the semidiscrete discontinuous Galerkin approximations to the velocity in \( L^\infty(L^2) \)-norm and pressure in \( L^\infty(L^2) \)-norm. These estimates are further shown to be uniform under the smallness assumption on the data. Finally, the main contributions of the article are summarized in Section \( 8 \).

2 Preliminaries and Weak Formulations

In the rest of the paper, we denote by bold face letters the \( \mathbb{R}^2 \)-valued function spaces such as \( H^1_0(\Omega) = (H^1(\Omega))^2 \), \( L^2(\Omega) = (L^2(\Omega))^2 \) etc. For finite element analysis, we use the standard Sobolev spaces \( W^{l,r}(\Omega) \) \((\mathbb{H})\), which is defined for any nonnegative integer \( l \) and \( r \geq 1 \), as follows:

\[
W^{l,r}(\Omega) = \{ \phi \in L^r(\Omega) : \forall |m| \leq l, \ \partial^m \phi \in L^r(\Omega) \},
\]

where \( m \) is a two (or a three) dimensional multi-index, \( |m| \) is the sum of its components and \( \partial^m \phi \) are the partial derivatives of \( \phi \) of order \( |m| \). The norm in \( W^{l,r}(\Omega) \) is denoted by \( \| \cdot \|_{l,r,\Omega} \) and the seminorm by \( | \cdot |_{l,r,\Omega} \). The \( L^2 \) inner-product is denoted by \( \langle \cdot, \cdot \rangle \). For the Hilbert space \( H^1(\Omega) := W^{1,2}(\Omega) \), the norm is denoted by \( \| \cdot \|_{l,\Omega} \) or \( | \cdot |_l \). The space \( H^1_0(\Omega) \) is equipped with the norm

\[
\| \nabla v \| = \left( \sum_{i,j=1}^2 (\partial_j v_i, \partial_i v_j) \right)^{1/2} = \left( \sum_{i=1}^2 (\nabla v_i, \nabla v_i) \right)^{1/2}.
\]
Let \( H^m(\Omega)/\mathbb{R} \) be the quotient space with norm \( \|\phi\|_{H^m(\Omega)/\mathbb{R}} = \inf_{c \in \mathbb{R}} \|\phi + c\|_m \). For \( m = 0 \), it is denoted by \( L^2(\Omega)/\mathbb{R} \).

And for any Banach space \( X \), let \( L^p(0,T;X) \) denote the space of measurable \( X \)-valued functions \( \phi \) on \((0,T)\) such that
\[
\int_0^T \|\phi(t)\|_X^p \, dt < \infty \text{ if } 1 \leq p < \infty, \quad \text{and} \quad \operatorname{ess\, sup}_{0 < t < T} \|\phi(t)\|_X < \infty \text{ if } p = \infty.
\]

The dual space of \( H^m(\Omega) \) is denoted by \( H^{-m}(\Omega) \) with norm defined as
\[
\|\phi\|_{-m} := \sup \left\{ \frac{\langle \phi, \psi \rangle}{\|\psi\|_m} : \psi \in H^m(\Omega), \|\psi\|_m \neq 0 \right\}.
\]

From future analysis point of view, we further define the following divergence free spaces:
\[
\mathbf{J}_1 = \{ \mathbf{w} \in H^1_0(\Omega) : \nabla \cdot \mathbf{w} = 0 \} \quad \text{and} \quad \mathbf{J} = \{ \mathbf{w} \in L^2(\Omega) : \nabla \cdot \mathbf{w} = 0, \mathbf{w} \cdot \mathbf{n}|_{\partial \Omega} = 0 \text{ holds weakly} \},
\]

where \( \mathbf{n} \) is the outward normal to the boundary \( \partial \Omega \) and \( \mathbf{w} \cdot \mathbf{n}|_{\partial \Omega} = 0 \) should be understood in the sense of trace in \( H^{-1/2}(\partial \Omega) \).

Throughout the paper, we make the following assumptions:

\( (A1) \). The initial velocity \( \mathbf{u}_0 \) and the external force \( \mathbf{f} \) satisfy, for some positive constant \( M_0 \) and for time \( T \) with \( 0 < T < \infty \), \( \mathbf{u}_0 \in H^2(\Omega) \cap \mathbf{J}_1, \mathbf{f}, \mathbf{f}_t \in L^\infty(0,T;L^2(\Omega)) \) with \( \|\mathbf{u}_0\|_2 \leq M_0 \) and \( \sup_{0 < t < T} \{ \|\mathbf{f}(\cdot,t)\|, \|\mathbf{f}_t(\cdot,t)\| \} \leq M_0 \).

\( (A2) \). For \( \mathbf{g} \in L^2(\Omega) \), let the unique solutions \( \mathbf{v} \in \mathbf{J}_1, \mathbf{q} \in L^2(\Omega)/\mathbb{R} \) of the steady state Stokes problem
\[
-\Delta \mathbf{v} + \nabla \mathbf{q} = \mathbf{g},
\]
\[
\nabla \cdot \mathbf{v} = 0 \text{ in } \Omega, \quad \mathbf{v}|_{\partial \Omega} = 0,
\]

satisfy
\[
\|\mathbf{v}\|_2 + \|\mathbf{q}\|_{H^1(\Omega)/\mathbb{R}} \leq C\|\mathbf{g}\|.
\]

Now the weak formulation of \( 1.1, 1.3 \) is as follows: Find a pair \( (\mathbf{u}(t), p(t)) \in H^1_0(\Omega) \times L^2(\Omega)/\mathbb{R}, t > 0 \), such that
\begin{align}
(1.1) & \quad (\mathbf{u}_t, \phi) + \kappa (\nabla \mathbf{u}_t, \nabla \phi) + \nu (\nabla \mathbf{u}, \nabla \phi) + (\mathbf{u} \cdot \nabla \mathbf{u}, \phi) - (p, \nabla \cdot \phi) = (\mathbf{f}, \phi) \quad \forall \phi \in H^1_0(\Omega), \\
(1.2) & \quad (\nabla \cdot \mathbf{u}, \mathbf{q}) = 0 \quad \forall \mathbf{q} \in L^2(\Omega)/\mathbb{R}, \\
(1.3) & \quad (\mathbf{u}(0), \phi) = (\mathbf{u}_0, \phi) \quad \forall \phi \in H^1_0(\Omega).
\end{align}

Now, we recall the following Sobolev inequalities which will be useful to estimate the nonlinear terms.

**Lemma 2.1 (35).** For any open set \( \Omega \subset \mathbb{R}^2 \) and \( \mathbf{v} \in H^1_0(\Omega) \)
\[
\|\mathbf{v}\|_{L^4(\Omega)} \leq 2^{1/4}\|\mathbf{v}\|^{1/2}\|\nabla \mathbf{v}\|^{1/2}.
\]

In addition, when \( \Omega \) is bounded and \( \mathbf{v} \in H^2(\Omega) \), the following estimate holds:
\[
\|\mathbf{v}\|_{L^\infty(\Omega)} \leq C\|\mathbf{v}\|^{1/2}\|\Delta \mathbf{v}\|^{1/2}.
\]

In Lemma 2.2 below, we state the regularity estimates of weak solution pair \( (\mathbf{u}, p) \) of \( 2.1, 2.3 \), which will be used in the subsequent error analysis.

**Lemma 2.2 (4).** Let the assumptions \( (A1) \) and \( (A2) \) be satisfied. Then, there exists a positive constant \( C = C(\kappa, \nu, \alpha, C_2, M_0) \), such that, for \( t > 0 \), the following estimates hold true:
\[
\sup_{0 < t < \infty} \left\{ \|\mathbf{u}(t)\|_2 + \|p(t)\|_{H^1(\Omega)/\mathbb{R}} + \|\mathbf{u}_t(t)\|_2 + \|p_t(t)\|_{H^1(\Omega)/\mathbb{R}} \right\} \leq C,
\]
\[
e^{-2\alpha t} \int_0^t e^{2\alpha s} \left( \|\mathbf{u}(s)\|_2^2 + \|\mathbf{u}_s(s)\|_2^2 + \|p(s)\|_{H^1(\Omega)/\mathbb{R}}^2 + \|p_s(s)\|_{H^1(\Omega)/\mathbb{R}}^2 \right) ds \leq C.
\]
Next, for defining the DG formulation, we consider a quasiuniform family of triangulations $\mathcal{T}_h$ of $\Omega$, consisting of triangles of maximum diameter $h$. Let $\Gamma_h$ denotes the set of all edges of $\mathcal{T}_h$ and $n_e$ represents a unit normal to each edge $e \in \Gamma_h$. For an edge $e$ on the boundary $\partial \Omega$, $n_e$ is taken to be the unit outward vector normal to $\partial \Omega$ and for an edge $e$ shared by two elements $T_m$ and $T_n$ of $\mathcal{T}_h$, a unit normal vector $n_e$ is directed from $T_m$ to $T_n$. The jump $[\phi]$ and the average $\{\phi\}$ of a function $\phi$ on an edge $e \in \Gamma_h$ are defined by

$$
[\phi] = (\phi|_{T_m})|_e - (\phi|_{T_n})|_e,
\{\phi\} = \frac{1}{2}(\phi|_{T_m})|_e + \frac{1}{2}(\phi|_{T_n})|_e,
$$

and on an edge $e \in \partial \Omega$ coincide with the value of $\phi$ on $e$, in both the cases.

We further need the following discontinuous spaces

$$
\mathbf{V} = \{ w \in L^2(\Omega) : w|_T \in W^{2,4/3}(T), \ \forall T \in \mathcal{T}_h \},
M = \{ q \in L^2(\Omega)/\mathbb{R} : q|_T \in W^{1,4/3}(T), \ \forall T \in \mathcal{T}_h \}.
$$

equipped with the "broken" norms

$$
\|v\|_e = (\|\nabla v\|^2 + J_0(v,v))^{1/2} \ \forall v \in \mathbf{V},
\|q\|_{L^2/\mathbb{R}} = \|q\|_{L^2(\Omega)/\mathbb{R}} \ \forall q \in M,
$$

where,

$$
\|v\|_l = \left( \sum_{T \in \mathcal{T}_h} \|v\|^2_T \right)^{1/2}, \ l \geq 0.
$$

The jump term $J_0$ that appeared on the right hand side of (2.3) is defined as

$$
J_0(v,w) = \sum_{e \in \Gamma_h} \frac{\sigma_e}{|e|} \int_e [v] \cdot [w] \ ds,
$$

where $|e|$ denotes the measure of the edge $e$ and $\sigma_e > 0$ is penalty parameter defined for each edge $e$.

In the course of our analysis, we frequently use the following bounds for the $L^p$-norm of functions in $\mathbf{V}$ (17):

$$
\|v\|_{L^p(\Omega)} \leq \gamma \|v\|_e, \ \forall v \in \mathbf{V}, \ 2 \leq p < \infty,
$$

where $\gamma = C(p)$ is a positive constant that depends on $p$.

Below, we state some standard trace and inverse inequalities on the discontinuous space $\mathbf{V}$.

**Lemma 2.3** (14). For every triangle $T$ in $\mathcal{T}_h$, the following inequalities hold true

$$
\|v\|_{L^2(\mathcal{T})} \leq C(h_T^{-1/2}\|v\|_{L^2(T)} + h_T^{1/2}\|\nabla v\|_{L^2(T)}) \ \forall e \in \partial T, \ \forall v \in \mathbf{V},
\|\nabla v\|_{L^2(\mathcal{T})} \leq C(h_T^{-1/2}\|\nabla v\|_{L^2(T)} + h_T^{1/2}\|\nabla^2 v\|_{L^2(T)}) \ \forall e \in \partial T, \ \forall v \in \mathbf{V},
\|v\|_{L^4(\mathcal{T})} \leq C h_T^{-3/4}(\|v\|_{L^2(T)} + h_T\|\nabla v\|_{L^2(T)}) \ \forall e \in \partial T, \ v \in \mathbf{V},
$$

where $h_T > 0$ is the diameter of triangle $T$.

In order to write the discontinuous Galerkin formulation of (1.1)-(1.3), we define the bilinear forms $a : \mathbf{V} \times \mathbf{V} \to \mathbb{R}$ and $b : \mathbf{V} \times M \to \mathbb{R}$ as follows:

$$
a(w,v) = \sum_{T \in \mathcal{T}_h} \int_T \nabla w \cdot \nabla v \ dt - \sum_{e \in \Gamma_h} \int_e \{\nabla w\} n_e \cdot [v] \ ds + \epsilon \sum_{e \in \Gamma_h} \int_e \{\nabla v\} n_e \cdot [w] \ ds,
$$

$$
b(v,q) = - \sum_{T \in \mathcal{T}_h} \int_T q \nabla \cdot v \ dt + \sum_{e \in \Gamma_h} \int_e \{q\} [v] \cdot n_e \ ds,
$$

Note that, here $\epsilon = \pm 1$. If $\epsilon = -1$, then the DG formulation is known as SIPG and if $\epsilon = +1$, then it is called as NIPG.
This article is dedicated to the derivation of optimal error estimates for the SIPG case. Therefore, throughout the paper, we make the following assumption:

**(A3)**. The SIPG DG formulation is considered with \( \epsilon = -1 \) and the penalty parameter \( \sigma_\epsilon \) is bounded below by sufficiently large \( \sigma_0 > 0 \).

We further define the following trilinear form \( c(\cdot, \cdot, \cdot) \) for the nonlinear term present in the system (1.1)-(1.3):

\[
c^w(u, z, \rho) = \sum_{T \in \mathcal{T}_h} \left( \int_T (u \cdot \nabla z) \cdot \rho \, dT + \int_{\partial T} |\{u\} \cdot n_T| (z^{int} - z^{ext}) \cdot \rho^{int} \, ds \right)
\]

\[
+ \frac{1}{2} \sum_{T \in \mathcal{T}_h} \int_T (\nabla \cdot u)z \cdot \rho \, dT - \frac{1}{2} \sum_{e \in \Gamma_h} \int_e |u| \cdot n_e \{z \cdot \rho\} \, ds, \quad \forall u, z, \rho \in V,
\]

where

\[
\partial T_\epsilon = \{ x \in \partial T : \{w\} \cdot n_T < 0 \}.
\]

The superscript \( w \) denotes the dependence of \( \partial T_\epsilon \) on \( w \) and the superscript \(\text{int} \) ( respectively \(\text{ext} \) ) refers to the trace of the function on a side of \( T \) coming from the interior (respectively exterior) of \( T \) on that side. The exterior trace is considered to be zero for a side of \( T \) belongs to \( \partial \Omega \). The first two terms in the definition of \( c(\cdot, \cdot, \cdot) \) were introduced in [22] for solving transport problems; the last term is taken to ensure the positivity property of \( c \) (2.11). Using the definition of \( b(\cdot, \cdot, \cdot) \), the trilinear form can be presented as

\[
c^w(u, z, \rho) = \sum_{T \in \mathcal{T}_h} \left( \int_T (u \cdot \nabla z) \cdot \rho \, dT + \int_{\partial T} |\{u\} \cdot n_T| (z^{int} - z^{ext}) \cdot \rho^{int} \, ds \right) - \frac{1}{2} b(u, z, \rho).
\]

The above definition of trilinear form is motivated from the Lesaint-Raviart upwinding scheme (see [22]) and is introduced in [17] for Navier-Stokes system of equations. It can be easily verified that for \( u, z, \rho \in H^1_0(\Omega) \), the trilinear form \( c \) satisfies

\[
c(u; z, \rho) = \int_\Omega (u \cdot \nabla z) \cdot \rho \, dT + \frac{1}{2} \int_\Omega (\nabla \cdot u)z \cdot \rho \, dT, \quad .
\]

The superscript \( w \) is dropped in (2.9) since the integral on \( \partial T_\epsilon \) disappears.

Using the integration by parts formula, the trilinear form can be reduced to

\[
c^\epsilon(u, z, \rho) = - \sum_{T \in \mathcal{T}_h} \left( \int_T (u \cdot \nabla \rho) \cdot z \, dT + \frac{1}{2} \int_T (\nabla \cdot u)z \cdot \rho \, dT \right) + \frac{1}{2} \sum_{e \in \Gamma_h} \int_e |u| \cdot n_e \{z \cdot \rho\} \, ds
\]

\[
- \sum_{T \in \mathcal{T}_h} \int_{\partial T_\epsilon} |\{u\} \cdot n_T| z^{ext} \cdot (\rho^{int} - \rho^{ext}) \, ds + \int_{\Gamma_+} |u \cdot n| z \cdot \rho \, ds, \quad u, z, \rho \in V,
\]

where \( \Gamma_+ \) is the subset of \( \partial \Omega \) with \( u \cdot n > 0 \). For a proof, please refer to [17]. Using a particular choice \( z = \rho \) in (2.11), we arrive at

\[
c^\epsilon(u, z, z) \geq 0, \quad u, z \in V.
\]

Now, the DG formulation of (1.1)-(1.3) is as follows: Find the pair \((u(t), p(t)) \in V \times M, \; t > 0, \) such that

\[
\begin{align*}
(u(t), \phi) + \kappa (a(u(t), \phi) + J_0(u(t), \phi)) + \nu (a(u(t), \phi) + J_0(u(t), \phi)) + c^\epsilon(u(t), u(t), \phi) + \nu (a(u(t), \phi) + J_0(u(t), \phi)) \\
&\quad + c^\epsilon(u(t), u(t), \phi) + b(\phi, p(t)) = (f(t), \phi) \quad \forall \phi \in V, \\
b(u(t), q) = 0 \quad \forall q \in M, \\
(u(0), \phi) = (u_0, \phi) \quad \forall \phi \in V.
\end{align*}
\]

The consistency proof of (2.12)-(2.14) can be done following the similar analysis as adopted in [20] Lemma 3.2] for the DG formulation of NSE.
3 Semidiscrete Discontinuous Galerkin Formulation

Consider $V_h$ and $M_h$ be the finite-dimensional discontinuous subspaces of $V$ and $M$, respectively, defined for any positive integer $k \geq 1$ as follows:

$$V_h = \{ v \in L^2(\Omega) : \forall T \in T_h, v \in (P_k(T))^2 \},$$

$$M_h = \{ q \in L^2(\Omega)/\mathbb{R} : \forall T \in T_h, q \in P_{k-1}(T) \}.$$ 

Let there exist a projection operator $R_h \in \mathcal{L}(H^1(\Omega); V_h)$, a linear map from $H^1(\Omega)$ to $V_h$, with the properties stated below.

**Lemma 3.1** ([17]). For $V_h$, there exists an operator $R_h \in \mathcal{L}(H^1(\Omega); V_h)$, such that for any $T \in T_h$,

$$\forall \nu \in H^2(\Omega) \cap H_0^1(\Omega), \| R_h(\nu) - \nu \|_2 \leq C h |\nu|_2,$$

$$\forall s \in [1, 2], \forall \nu \in H^s(\Omega), \| \nu - R_h(\nu) \|_{L^2(T)} \leq C h^s |\nu|_{s, T},$$

where $\Delta_T$ is a suitable macro-element containing $T$.

Further, let there exist an operator $r_h \in \mathcal{L}(L^2(\Omega)/\mathbb{R}; M_h)$ (see [17]), such that, for any $T \in T_h$, the following properties are satisfied:

$$\forall q \in L^2(\Omega)/\mathbb{R}, \forall z_h \in P_{k-1}(T), \int_T z_h(r_h(q) - q) dT = 0,$$

$$\forall s \in [0, k], \forall \nu \in H^s(\Omega) \cap L^2(\Omega)/\mathbb{R}, \| q - r_h(q) \|_{L^2(T)} \leq C h^s |\nu|_{s, T}.$$

Moreover, for the operator $R_h$, we have (see [17])

$$\forall \nu \in H_0^1(\Omega), \forall q \in M_h, b(R_h(\nu) - \nu, q) = 0,$$

and the stability property (see [16]): there exists a constant $C$, independent of $h$, such that

$$\forall \nu \in H_0^1(\Omega), \| R_h \nu \|_e \leq C |\nu|_1.$$ 

We define below the semidiscrete discontinuous Galerkin approximations for the equations (2.1)-(2.3) as follows: For $t > 0$, find $(u_h(t), p_h(t)) \in V_h \times M_h$ such that

$$(u_{ht}(t), \phi_h) + k(a(u_{ht}(t), \phi_h) + J_0(u_{ht}(t), \phi_h)) + \nu(a(u_h(t), \phi_h) + J_0(u_h(t), \phi_h))$$

$$+ c u_h^\nu(u_h(t), u_h(t), \phi_h) + b(\phi_h, p_h(t)) = (f(t), \phi_h), \forall \phi_h \in V_h$$

$$b(u_h(t), q_h) = 0, \forall q_h \in M_h$$

$$u_h(0, \phi_h) = u_0, \forall \phi_h \in V_h.$$ 

The discrete discontinuous space $J_h$, which is analogous to $J_1$, is defined as

$$J_h = \{ v_h \in V_h : \forall q_h \in M_h, b(v_h, q_h) = 0 \}.$$ 

The equivalent formulation of (3.7)-(3.9) on $J_h$ is as follows: find $u_h(t) \in J_h$, such that for $t > 0$

$$(u_{ht}(t), \phi_h) + k(a(u_{ht}(t), \phi_h) + J_0(u_{ht}(t), \phi_h)) + \nu(a(u_h(t), \phi_h) + J_0(u_h(t), \phi_h))$$

$$+ c u_h^\nu(u_h, u_h, \phi_h) = (f, \phi_h), \forall \phi_h \in J_h.$$ 

The next two lemmas state the boundedness properties and the coercivity of the bilinear form $(a + J_0)\langle \cdot, \cdot \rangle$, respectively.

**Lemma 3.2** ([32]). There exists a constant $C_1 > 0$, independent of $h$, such that, for all $v_h, w_h \in V_h$,

$$|a(v_h, w_h) + J_0(v_h, w_h)| \leq C_1 \| v_h \|_e \| w_h \|_e.$$
Lemma 3.3 ([36]). Under the assumption of (A3) there exists a constant \( C_2 > 0 \), independent of \( h \), such that
\[
\forall \mathbf{v}_h \in V_h, \quad a(\mathbf{v}_h, \mathbf{v}_h) + J_0(\mathbf{v}_h, \mathbf{v}_h) \geq C_2 \| \mathbf{v}_h \|_{\epsilon}^2.
\]

Similar to the continuous case, we have a discrete inf-sup condition for the pair of discontinuous spaces \((\tilde{V}_h, M_h)\), where
\[
\tilde{V}_h = \{ \mathbf{v}_h \in V_h : \forall \mathbf{e} \in \Gamma_h, \quad \int_{\mathbf{e}} q_h \cdot [\mathbf{v}_h] \, ds = 0, \quad \forall q_h \in \mathbb{P}_{k-1}(e)^2 \},
\]
which will be an essential tool for deriving the pressure estimates.

Lemma 3.4 ([17]). There exists a constant \( \beta^* > 0 \), independent of \( h \), such that
\[
\inf_{p_h \in M_h} \sup_{\mathbf{v}_h \in \tilde{V}_h} \frac{b(\mathbf{v}_h, p_h)}{\| \mathbf{v}_h \|_{\epsilon} \| p_h \|_0} \geq \beta^*.
\]

Analogous to the first estimate in Lemma 2.1, we have the following estimate for the elements in the discrete space \( V_h \).

Lemma 3.5 ([16]). When \( \Omega \) is convex, there exists a positive constant \( C \), independent of \( h \), such that
\[
\| \mathbf{v}_h \|_{L^4(\Omega)} \leq C \| \mathbf{v}_h \|_{1/2}^{1/2} \| \mathbf{v}_h \|_{\epsilon}^{1/2}, \quad \forall \mathbf{v}_h \in V_h.
\]

The following estimates of the trilinear form \( e(\cdot, \cdot, \cdot) \) will be useful for our error analysis.

Lemma 3.6 ([16]). (i) Assume that \( \mathbf{u} \in W^{1,4}(\Omega) \). There exists a positive constant \( C \) independent of \( h \) such that
\[
|e(\mathbf{v}_h, \mathbf{u}, w_h)| \leq C \| \mathbf{v}_h \| \| \mathbf{u} \|_{W^{1,4}(\Omega)} \| w_h \|_{\epsilon}, \quad \forall \mathbf{v}_h, w_h \in J_h.
\]

(ii) For any \( \mathbf{v} \in V, \mathbf{v}_h, w_h \) and \( z_h \) in \( V_h \), we have the following estimate:
\[
|e^\gamma(\mathbf{v}_h, w_h, z_h)| \leq C \| \mathbf{v}_h \| \| w_h \| \| z_h \|_{\epsilon}.
\]

In Lemma 3.7 we present a couple of trace inequalities along with inverse inequalities for the discrete discontinuous space \( V_h \), similar to those stated in Lemma 2.3 for the space \( V \).

Lemma 3.7 ([14]). For every element \( T \) in \( \mathcal{T}_h \), the following inequalities hold
\[
\begin{align*}
\| \mathbf{v}_h \|_{L^2(\Omega)} & \leq Ch_T^{-1/2} \| \mathbf{v}_h \|_{L^2(T)} \quad \forall \mathbf{v}_h \in V_h, \\
\| \nabla \mathbf{v}_h \|_{L^2(\epsilon)} & \leq Ch_T^{-1/2} \| \nabla \mathbf{v}_h \|_{L^2(T)} \quad \forall \mathbf{v}_h \in V_h, \\
\| \nabla \mathbf{v}_h \|_{L^2(T)} & \leq Ch_T^{-1} \| \mathbf{v}_h \|_{L^2(T)} \quad \forall \mathbf{v}_h \in V_h, \\
\| \mathbf{v}_h \|_{L^4(T)} & \leq Ch_T^{-1/2} \| \mathbf{v}_h \|_{L^2(T)} \quad \forall \mathbf{v}_h \in V_h,
\end{align*}
\]
where \( C \) is a constant independent of \( h_T \) and \( \mathbf{v}_h \).

Next, we state the regularity bounds for \( \mathbf{u}_h \) which will be used in deriving the existence and uniqueness of the discrete solution and fully discrete error estimates.

Lemma 3.8. Let the assumptions (A1)-(A3) be satisfied and let \( 0 < \alpha < \frac{\kappa}{2(\gamma + \alpha C_2 M_0)} \). Then, there exists a positive constant \( C = C(\kappa, \nu, \alpha, C_2, M_0) \), such that, for each \( t > 0 \), the semidiscrete discontinuous Galerkin solution \( \mathbf{u}_h(t) \), satisfies the following estimates:
\[
\sup_{0 < t < \infty} \| \mathbf{u}_h(t) \| + e^{-2\alpha t} \int_0^t e^{2\alpha s} \left( \| \mathbf{u}_h(s) \|_{\epsilon}^2 + \| \mathbf{u}_h(t) \|_{\epsilon}^2 \right) \, ds \leq C,
\]
\[
\sup_{0 < t < \infty} (\| \mathbf{u}_h(t) \|_{L^1(\Omega)} + \| \mathbf{u}_h(t) \|_{\epsilon}) + e^{-2\alpha t} \int_0^t e^{2\alpha s} \left( \| \mathbf{u}_h(t) \|_{L^1(\Omega)}^2 + \| \mathbf{u}_h(t) \|_{\epsilon}^2 \right) \, ds \leq C,
\]

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where

\[ (3.17) \quad \| u_{tt} \|_{L^\infty} = \sup \left\{ \frac{\langle u_{tt}, \phi_h \rangle}{\| \phi_h \|_\varepsilon}, \phi_h \in V_h, \phi_h \neq 0 \right\}. \]

Moreover,

\[ \limsup_{t \to \infty} \| u_h(t) \|_\varepsilon \leq \frac{\gamma \| \mathbf{f} \|_{L^\infty(L^2(\Omega))}}{C_2 \nu}, \]

Proof. Choose \( \phi_h = u_h \) in (3.10) and apply the coercivity result, positivity of \( c(\cdot, \cdot) \) (2.11), estimate (2.5), Cauchy-Schwarz’s inequality and Young’s inequality to obtain

\[ (3.18) \quad \frac{1}{2} \frac{d}{dt} \left( \| u_h \|^2 + \kappa(a(u_h, u_h) + J_0(u_h, u_h)) + \nu C_2 \| u_h \|_\varepsilon^2 \right) \leq \nu C_2 \| u_h \|_\varepsilon^2 + \frac{\gamma^2}{2 \nu C_2} \| \mathbf{f} \|^2. \]

A multiplication of (3.18) by \( e^{2t} \), an integration from 0 to \( t \), and a use of estimate (2.5), Lemmas 3.2 and 3.3, lead to

\[ e^{2t} \left( \| u_h(t) \|^2 + C_2 \kappa \| u_h(t) \|_\varepsilon^2 \right) + (\nu C_2 - 2\alpha(\gamma + \kappa C_1)) \int_0^t e^{2s} \| u_h(s) \|_\varepsilon^2 \, ds \leq \| u_h(0) \|^2 \]

\[ + C_1 \kappa \| u_h(0) \|_\varepsilon + C \int_0^t e^{2s} \| \mathbf{f}(s) \|_\varepsilon^2 \, ds. \]

Again, multiply (3.19) by \( e^{-2t} \), use the fact that

\[ e^{-2t} \int_0^t e^{2s} \, ds = \frac{1}{2\alpha}(1 - e^{-2t}) \]

and choose \( 0 < \alpha < \frac{\nu C_2}{2(\gamma + \kappa C_1)} \) to obtain

\[ (3.20) \quad \| u_h(t) \|^2 + \| u_h(t) \|_\varepsilon^2 + e^{-2\alpha t} \int_0^t e^{2s} \| u_h(s) \|_\varepsilon^2 \, ds \leq C. \]

Again, multiply (3.18) by \( e^{2t} \), integrate from 0 to \( t \), and a use of Lemma 3.2 implies

\[ e^{2t} \left( \| u_h(t) \|^2 + \kappa(a(u_h(t), u_h(t)) + J_0(u_h(t), u_h(t))) + \nu C_2 \int_0^t e^{2s} \| u_h(s) \|_\varepsilon^2 \, ds \right) \leq (\| u_h(0) \|^2 + \kappa C_1 \| u_h(0) \|_\varepsilon^2) \]

\[ + 2\alpha \int_0^t e^{2s} \left( \| u_h(s) \|^2 + \kappa(a(u_h(s), u_h(s)) + J_0(u_h(s), u_h(s))) \right) \, ds + (e^{2t} - 1) \frac{\gamma^2 \| \mathbf{f} \|^2_{L^\infty(L^2(\Omega))}}{2\alpha \nu C_2}. \]

Multiply the above inequality by \( e^{-2t} \), take limit supremum as \( t \to \infty \) and noting that,

\[ \nu C_2 \limsup_{t \to \infty} e^{-2t} \int_0^t e^{2s} \| u_h(s) \|_\varepsilon^2 \, ds = \frac{\nu C_2}{2\alpha} \limsup_{t \to \infty} \| u_h(t) \|_\varepsilon^2, \]

we arrive at

\[ (3.21) \quad \nu C_2 \limsup_{t \to \infty} \| u_h(t) \|_\varepsilon^2 \leq \frac{\gamma^2 \| \mathbf{f} \|^2_{L^\infty(L^2(\Omega))}}{2\alpha \nu C_2}. \]

Next, differentiating (3.10) with respect to \( t \), we obtain

\[ (u_{tt} + \phi_h) + \kappa(a(u_{tt}, \phi_h) + J_0(u_{tt}, \phi_h)) + \nu(a(u_{tt}, \phi_h) + J_0(u_{tt}, \phi_h)) + \nu \mathbf{u}_{tt}(u_{tt}, \phi_h) + \nu \mathbf{u}_{tt}(u_{tt}, \phi_h) = (f, \phi_h), \quad \forall \phi_h \in J_h. \]

(3.22)

Substitute $\phi_h = u_{ht}$ in (3.22), apply Lemma 3.3 and the fact that $c_{un}(u_h, u_{ht}, u_{ht}) \geq 0$ from (2.11), we obtain

\begin{equation}
\frac{d}{dt} \left( \|u_{ht}\|^2 + \kappa (a(u_{ht}, u_{ht}) + J_0(u_{ht}, u_{ht})) \right) + \nu C_2 \|u_{ht}\|^2 \leq -2c_{un}(u_h, u_h, u_{ht}) + C \|f_t\|^2.
\end{equation}

On expanding the nonlinear term in (3.23), we find that

\begin{equation*}
c_{un}(u_{ht}, u_h, u_{ht}) = \sum_{T \in T_h} \int_T (u_{ht} \cdot \nabla u_{ht}) \cdot u_{ht} dt + \sum_{T \in T_h} \int_{\partial T^-} |(u_{ht}) \cdot n_T|(u_{ht}^n - u_{ht}^e)^t \cdot u_{ht}^e ds
\end{equation*}

\begin{equation*}
+ \frac{1}{2} \sum_{T \in T_h} \int_T (\nabla \cdot u_{ht}) u_h \cdot u_{ht} dt - \frac{1}{2} \sum_{e \in F_h} \int_e [u_{ht}] \cdot n_e \{u_h \cdot u_{ht}\} ds
\end{equation*}

\begin{equation*}
= A_1 + A_2 + A_3 + A_4.
\end{equation*}

Using Hölder’s inequality, Young’s inequality and Lemma 3.5, we can bound $A_1$ as follows:

\begin{equation}
|A_1| \leq \|u_{ht}\|_{L^2(\Omega)} \|\nabla u_{ht}\| \|u_{ht}\|_{L^2(\Omega)} \leq C \|u_{ht}\| \|u_{ht}\| \|u_h\| \leq \frac{\nu C_2}{8} \|u_{ht}\|^2 + C \|u_{ht}\|^2 \|u_h\|^2.
\end{equation}

An application of Hölder’s, Young’s, trace inequalities and Lemma 3.5 yield

\begin{equation}
|A_2| \leq \sum_{T \in T_h} \|u_{ht}\|_{L^2(\partial T)} \|u_{ht}\|_{L^2(\partial T)} \|u_{ht}\|_{L^2(\partial T)} \leq C \sum_{T \in T_h} \|u_{ht}\|_{L^2(\Omega)} \frac{1}{|T|^{1/2}} \|u_{ht}\|_{L^2(\Omega)} \|u_{ht}\|_{L^2(\Omega)}
\end{equation}

\begin{equation*}
\leq C \|u_{ht}\|_{L^2(\Omega)} \|u_{ht}\| \|u_{ht}\|_{L^2(\Omega)} \leq \frac{\nu C_2}{8} \|u_{ht}\|^2 + C \|u_{ht}\|^2 \|u_h\|^2
\end{equation*}

The term $A_3$ can be bounded using Hölder’s, Young’s inequalities and Lemma 3.5 as

\begin{equation}
|A_3| \leq \frac{1}{2} \|\nabla u_{ht}\| \|u_{ht}\|_{L^2(\Omega)} \|u_{ht}\|_{L^2(\Omega)} \leq C \|u_{ht}\| \|u_{ht}\| \|u_h\| \|u_{ht}\| \|u_{ht}\| \|u_{ht}\| \|u_h\|^2 \|u_{ht}\| \|u_{ht}\|
\end{equation}

\begin{equation*}
\leq \frac{\nu C_2}{8} \|u_{ht}\|^2 + C \|u_{ht}\|^2 \|u_h\|^2 \|u_{ht}\|^2.
\end{equation*}

Similar to $A_2$ and $A_3$, we can bound $A_4$ as follows

\begin{equation}
|A_4| \leq \frac{\nu C_2}{8} \|u_{ht}\|^2 + C \|u_{ht}\|^2 \|u_h\|^2 \|u_{ht}\|^2.
\end{equation}

Collecting the bounds form (3.24)–(3.27) and applying in (3.23), we arrive at

\begin{equation}
\frac{d}{dt} \left( \|u_{ht}\|^2 + \kappa (a(u_{ht}, u_{ht}) + J_0(u_{ht}, u_{ht})) \right) + \nu C_2 \|u_{ht}\|^2 \leq C \|u_{ht}\| \|u_{ht}\| (1 + \|u_h\|^2) + C \|f_t\|^2.
\end{equation}

On multiplying (3.28) by $e^{2\alpha t}$, integrating from 0 to $t$, and using Lemmas 3.2 and 3.3, we observe that

\begin{equation*}
e^{2\alpha t}(\|u_{ht}(t)\|^2 + \kappa C_2 \|u_{ht}(t)\|^2) + (\nu C_2 - 2\alpha (\gamma + \kappa C_1)) \int_0^t e^{2\alpha s} \|u_{ht}(s)\|^2 ds \leq (\|u_{ht}(0)\|^2 + C_1 \kappa \|u_{ht}(0)\|^2) + C \int_0^t e^{2\alpha s} \|f_t(s)\|^2 ds.
\end{equation*}

Choosing $0 < \alpha < \frac{\nu C_2}{2(\gamma + \kappa C_1)}$, applying the assumption (A1), (3.28), Gronwall’s lemma and after a final multiplication by $e^{-2\alpha t}$, we obtain the estimate as

\begin{equation}
\|u_{ht}(t)\|^2 + \|u_{ht}(t)\|^2 + e^{-2\alpha t} \int_0^t e^{2\alpha s} \|u_{ht}(s)\|^2 ds \leq C.
\end{equation}

Now, we substitute $\phi_h = u_{htt}$ in (3.22) and obtain

\begin{equation}
\|u_{htt}\|^2 + C_2 \kappa \|u_{htt}\|^2 = -\nu (a(u_{ht}, u_{ht}) + J_0(u_{ht}, u_{ht})) - c_{un}(u_h, u_h, u_{ht}) - c_{un}(u_h, u_h, u_{ht}) - (f_t, u_{ht}).
\end{equation}
Drop the first term on the left hand side of (3.30) as it is positive. Apply (3.12), Lemma 3.2 and Young’s inequality to obtain
\[ C_2 \kappa \| u_{htt} \|_2^2 \leq C \left( \| u_{ht} \|_2^2 + \| u_{ht} \|_2^2 \| u_h \|_2^2 + \| f_t \|^2 \right). \]

A use of (3.20), (3.29) in (3.31) yield
\[ \| u_{htt} \|_2 \leq C. \]

Finally, we write (3.22) as follows:
\[
(u_{htt}, \phi_h) = -\kappa (a(u_{htt}, \phi_h) + J_0(u_{htt}, \phi_h)) - \nu (a(u_{htt}, \phi_h)) - c^{u_h}(u_{htt}, u_h, \phi_h) - \mathcal{C}(h, M), \quad \forall \quad \phi_h \in J_h.
\]

An application of Lemma 3.2 and (3.12) leads to
\[
(u_{htt}, \phi_h) \leq C(\kappa, \nu)(\| u_{htt} \|_2 + \| u_{htt} \|_2 + \| u_{htt} \|_2 \| u_h \|_2 + \| f_t \|) \| \phi_h \|_2.
\]

Using (3.17), the estimates in (3.29), (3.32) and the assumption (A1) in (3.33), we obtain
\[
\| u_{ht} \|_{-1,h} \leq C(\kappa, \nu)(\| u_{ht} \|_2 + \| u_{ht} \|_2 + \| f_t \|) \leq C.
\]

After taking square of (3.32) and (3.34), add the resulting two inequalities, multiply by $e^{2\alpha t}$ and integrate with respect to time from 0 to $t$. Then multiply by $e^{-2\alpha t}$ to arrive at
\[
e^{-2\alpha t} \int_0^t e^{2\alpha s}(\| u_{htt} \|_2)^2 \| u_{htt} \|_2^2 ds \leq C.
\]

A combination of (3.20), (3.21), (3.29), (3.32), (3.34) and (3.35) completes the proof of Lemma 3.8.

Now, the existence and uniqueness of the semidiscrete discontinuous Galerkin Kelvin-Voigt model (3.7)-(3.9) (or (3.10)) can be proved following the analysis in [20] and using the results in (2.11), Lemmas 3.3, 3.4, 3.8.

For deriving the optimal error estimates for semidiscrete discontinuous velocity and pressure approximations, we work on the weakly divergence free spaces. Therefore, there is a need to define an approximation operator on $J_1 \cap H^2(\Omega)$.

**Lemma 3.9.** For every $v \in J_1 \cap H^2(\Omega)$, there exists an approximation $i_h v \in J_h$, such that, the following approximation property hold true
\[
\| v - i_h v \|_2 \leq C h | v |_2.
\]

**Proof.** From Lemma 3.4 there exists an operator $R_h : H^1_0(\Omega) \rightarrow V_h$ satisfying $\| R_h v \|_2 \leq C | v |_1$ (see (3.6)) and
\[
b(v - R_h v, q_h) = 0, \quad \forall \quad v \in H^1_0(\Omega), \quad q_h \in M_h.
\]

We observe that $b(R_h v, q_h) = 0$. This implies that $R_h v \in J_h$. We now restrict $R_h$ on $J_1 \cap H^2(\Omega)$ and call it as $i_h$. Therefore, $i_h$ is an approximation operator from $J_1 \cap H^2(\Omega)$ to $J_h$. Using (3.4), we obtain
\[
\| v - i_h v \|_2 = \| v - R_h v \|_2 \leq C h | v |_2.
\]

This completes the rest of the proof.

We also define a projection $P_h : L^2(\Omega) \rightarrow J_h$, such that, for every $v \in L^2(\Omega)$, $(v - P_h v, v_h) = 0, \quad \forall v_h \in J_h$.

The projection $P_h$ satisfies the following properties
\[
\| v - P_h v \|_2 + h \| v - P_h v \|_2 \leq C h \| v \|_2, \quad \forall \quad v \in J_1 \cap H^2(\Omega).
\]

The energy norm estimates in (3.36) can be easily obtained using the definition of $P_h$ and the estimates in Lemma 3.9 and the $L^\infty(L^2)$-norm estimate can be derived using elliptic duality argument and energy norm estimate of $P_h$.

In Theorem 3.1 we provide the main contribution of the article related to the semidiscrete error estimates.
Theorem 3.1. Let the assumptions (A1)-(A3) be satisfied and let \( 0 < \alpha < \min \left\{ \frac{\nu C_5}{2(\gamma + \kappa C_1)}, \frac{\nu}{2\kappa} \right\} \). Further, let the discrete initial velocity \( \mathbf{u}_h(0) \in \mathbf{J}_h \) with \( \mathbf{u}_h(0) = P_h \mathbf{u}_0 \). Then, there exists a positive constant \( C = C(\kappa, \nu, \alpha, C_2, M_0) \), independent of \( h \), such that

\[
\|(u - u_h)(t)\| + h\|(u - u_h)(t)\|_e \leq C e^{C_1 h^2},
\]

where \( C \) is a positive constant independent of \( h \).

Theorem 3.2. Let the assumptions (A1)-(A3) be satisfied and let \( 0 < \alpha < \min \left\{ \frac{\nu C_5}{2(\gamma + \kappa C_1)}, \frac{\nu}{2\kappa} \right\} \). Further, let the discrete initial velocity \( \mathbf{u}_h(0) \in \mathbf{J}_h \) with \( \mathbf{u}_h(0) = P_h \mathbf{u}_0 \). Then, there exists a positive constant \( C = C(\kappa, \nu, \alpha, C_2, M_0) \), independent of \( h \), such that

\[
\|(p - p_h)(t)\| \leq C e^{C_1 h},
\]

where \( C \) is a positive constant independent of \( h \).

4 Error estimates for velocity

This section deals with the optimal estimates of the velocity error \( \mathbf{e} = \mathbf{u} - \mathbf{u}_h \) in \( L^2 \) and energy-norms for \( t > 0 \). We split the error into two parts, \( \mathbf{e} = \xi + \eta \), where \( \xi = \mathbf{u} - \mathbf{v}_h \) represents the error inherent due to the DG finite element approximation of (1.1) by a linearized Kelvin-Voigt problem, and \( \eta = \mathbf{v}_h - \mathbf{u}_h \) represents the error caused by the presence of the nonlinearity in problem (1.1). The linearized equation to be satisfied by the auxiliary function \( \mathbf{v}_h \) is:

\[
(\mathbf{v}_{ht}, \phi_h) + \kappa (a(\mathbf{v}_{ht}, \phi_h) + J_0(\mathbf{v}_{ht}, \phi_h)) + \nu (a(\mathbf{v}_h, \phi_h) + J_0(\mathbf{v}_h, \phi_h)) = (f, \phi_h) - c^\alpha (u, u, \phi_h) \quad \forall \phi_h \in \mathbf{J}_h.
\]

(4.1)

Below we derive some estimates of \( \xi \). Subtracting (4.1) from (2.12), the equation in \( \xi \) can be written as

\[
(\xi_t, \phi_h) + \kappa (a(\xi_t, \phi_h) + J_0(\xi_t, \phi_h)) + \nu (a(\xi, \phi_h) + J_0(\xi, \phi_h)) = -b(\phi_h, p), \quad \phi_h \in \mathbf{J}_h.
\]

(4.2)

For deriving the optimal error estimates of \( \xi \) in \( L^2 \) and energy-norms for \( t > 0 \), we introduce, as in [3], the following modified Sobolev-Stokes’s projection \( \mathbf{S}_h \mathbf{u} : [0, \infty) \to \mathbf{J}_h \) satisfying

\[
\kappa (a(\mathbf{u}_t - \mathbf{S}_h \mathbf{u}_t, \phi_h) + J_0(\mathbf{u}_t - \mathbf{S}_h \mathbf{u}_t, \phi_h)) + \nu (a(\mathbf{u} - \mathbf{S}_h \mathbf{u}, \phi_h) + J_0(\mathbf{u} - \mathbf{S}_h \mathbf{u}, \phi_h)) = -b(\phi_h, p) \quad \forall \phi_h \in \mathbf{J}_h,
\]

(4.3)

where \( \mathbf{S}_h \mathbf{u}(0) = P_h \mathbf{u}_0 \). In other words, given \( (\mathbf{u}, p) \), find \( \mathbf{S}_h \mathbf{u} : [0, \infty) \to \mathbf{J}_h \) satisfying (4.3). With \( \mathbf{S}_h \mathbf{u} \) defined as above, we now split \( \xi \) as

\[
\xi = \mathbf{u} - \mathbf{S}_h \mathbf{u} + \mathbf{S}_h \mathbf{u} - \mathbf{v}_h =: \zeta + \rho.
\]

Using (4.3), we find the equation in \( \zeta \) to be

\[
\kappa (a(\zeta_t, \phi_h) + J_0(\zeta_t, \phi_h)) + \nu (a(\zeta, \phi_h) + J_0(\zeta, \phi_h)) = -b(\phi_h, p) \quad \forall \phi_h \in \mathbf{J}_h,
\]

(4.4)

Firstly, we will focus on deriving the estimates of \( \zeta \). Next, we will establish the estimates of \( \rho \). Combining these estimates will result in the estimates of \( \xi \).

Lemma 4.1. Let the assumptions (A1)-(A3) hold true and let \( 0 < \alpha < \min \left\{ \frac{\nu C_5}{2(\gamma + \kappa C_1)}, \frac{\nu}{2\kappa} \right\} \). Then, there exists a positive constant \( C = C(\kappa, \nu, \alpha, C_2, M_0) \), such that, for \( t > 0 \), \( \zeta \) satisfies the following estimates:

\[
\|\zeta(t)\|^2 + h^2 \|\zeta(t)\|_e^2 + e^{-2\alpha t} \int_0^t e^{2\alpha s} \left( \|\zeta(s)\|^2 + h^2 \|\zeta(s)\|_e^2 + h^2 \|\zeta(s)\|_e^2 \right) ds \leq C h^4,
\]

(4.5)

\[
e^{-2\alpha t} \int_0^t e^{2\alpha s} \|\zeta(s)\|^2 ds \leq C h^4.
\]

(4.6)
Proof. Set \( \phi_h = P_h \zeta = \zeta - (u - P_h u) \) in (4.3), use the definition of space \( J_h \) and obtain
\[
\kappa (a(P_h \zeta, P_h \zeta) + J_0(P_h \zeta, P_h \zeta)) + \nu(a(P_h \zeta, P_h \zeta) + J_0(P_h \zeta, P_h \zeta)) = -\kappa(a(u - P_h u_t, P_h \zeta)) \\
+ J_0(u - P_h u_t, P_h \zeta) - \nu(a(u - P_h u, P_h \zeta) + J_0(u - P_h u, P_h \zeta)) - b(P_h \zeta, p - r_h(p)).
\]
(4.7)

Rewrite the first term on the right hand side of (4.7) as
\[
\kappa a(u_t - P_h u_t, P_h \zeta) = \kappa \sum_{T \in T_h} \int_T \nabla(u_t - P_h u_t) : \nabla(P_h \zeta) \, dT \\
- \kappa \sum_{e \in \Gamma_h} \int_e \{\nabla(u_t - P_h u_t)\}_e \cdot [P_h \zeta] \, ds - \kappa \sum_{e \in \Gamma_h} \int_e \{\nabla(P_h \zeta)\}_e \cdot [u_t - P_h u_t] \, ds
\]
(4.8)

A use of Cauchy-Schwarz’s, Young’s inequalities and (3.36) yield
\[
|I_1| \leq \kappa \sum_{T \in T_h} \|\nabla(u_t - P_h u_t)\|_{L^2(T)} \|\nabla(P_h \zeta)\|_{L^2(T)} \leq \frac{C_2 \nu}{24} \|P_h \zeta\|_e^2 + Ch^2 |u_t|_{e,2}.
\]

If the edge \( e \) belongs to the element \( T \), using Lemma 2.3 and Cauchy-Schwarz’s inequality, we bound \( I_2 \) as
\[
\left| \int_e \{\nabla(u_t - P_h u_t)\}_e \cdot [P_h \zeta] \, ds \right| \leq C(\|\nabla(u_t - P_h u_t)\|_{L^2(T)} + h_T \|\nabla^2(u_t - P_h u_t)\|_{L^2(T)}) \frac{1}{|e|^{1/2}} \|P_h \zeta\|_{L^2(e)}.
\]

By introducing the standard Lagrange interpolant \( L_h(u) \) of degree 1 (17) and using the triangle inequality, inverse inequality of Lemma 3.7, we obtain
\[
\|\nabla^2(u_t - P_h u_t)\|_{L^2(T)} \leq \|\nabla^2(u_t - L_h(u_t))\|_{L^2(T)} + \|\nabla^2(L_h(u_t) - P_h \zeta)\|_{L^2(T)} \leq \|\nabla^2(u_t - L_h(u_t))\|_{L^2(T)} + Ch_T^{-1} \|\nabla(L_h(u_t) - P_h \zeta)\|_{L^2(T)}.
\]
(4.9)

A use of the Young inequality, (3.36), (4.9) and the approximation properties of \( L_h \) (17) leads to
\[
|I_2| \leq C h J_0(P_h \zeta, P_h \zeta)^{1/2} |u_t|_2 \leq \frac{C_2 \nu}{24} \|P_h \zeta\|_e^2 + Ch^2 |u_t|_2.
\]

Furthermore, a use of Cauchy-Schwarz’s, Young’s inequalities, (3.36) and Lemma 3.7 yield
\[
|I_3| \leq C \kappa \left( \sum_{e \in \Gamma_h} \frac{|e|}{\sigma_e} \|\nabla(P_h \zeta)\|_{L^2(e)}^2 \right)^{1/2} \left( \sum_{e \in \Gamma_h} \frac{\sigma_e}{|e|} \|u_t - P_h u_t\|_{L^2(e)}^2 \right)^{1/2} \leq \frac{C_2 \nu}{24} \|P_h \zeta\|_e^2 + Ch^2 |u_t|_2.
\]

By a virtue of the Cauchy-Schwarz, Young inequalities and (3.36), the jump term of the first term on the right hand side of (4.7) is bounded as follows:
\[
\kappa J_0(u_t - P_h u_t, P_h \zeta) \leq \kappa J_0(u_t - P_h u_t, u_t - P_h u_t)^{1/2} J_0(P_h \zeta, P_h \zeta)^{1/2} \leq \frac{C_2 \nu}{24} \|P_h \zeta\|_e^2 + Ch^2 |u_t|_2.
\]
(4.10)

Owing to 3.3, the term involving the pressure in (4.7) is reduced to
\[
b(P_h \zeta, p - r_h p) = \sum_{e \in \Gamma_h} \int_e \{p - r_h p\} [P_h \zeta] \cdot n_e \, ds.
\]

Apply the Cauchy-Schwarz inequality, the result of 3.4 and Lemma 2.3 to arrive at
\[
|b(P_h \zeta, p - r_h p)| \leq C \sum_{T \in T_h} \left( \|p - r_h p\|_{L^2(T)} + h_T \|\nabla(p - r_h p)\|_{L^2(T)} \right) J_0(P_h \zeta, P_h \zeta)^{1/2} \leq \frac{C_2 \nu}{6} \|P_h \zeta\|_e^2 + Ch^2 |p|_1.
\]
(4.11)
Similarly, by replacing \( u_t - P_h u_t \) with \( u - P_h u \), and using the analysis involved in bounding \( I_1, I_2 \) and \( I_3 \) and (4.10), we estimate the second term on the right hand side of (4.12) as

\[
(4.12) \quad \nu |a(u - P_h u, P_h \zeta) + J_0(u - P_h u, P_h \zeta)| \leq \frac{C_2 \nu}{6} \|P_h \zeta\|^2 + Ch^2 \|u\|^2.
\]

Apply (4.5)-(4.12) and the bound of Lemma 5.3 in (4.17). Then, multiply the resulting equation by \( e^{2\alpha t} \), integrate from 0 to \( t \), use Lemmas 3.2, 3.3 and observe that \( P_h \zeta(0) = 0 \), we obtain

\[
(4.13) \quad \kappa C_2 e^{2\alpha t} \|P_h \zeta\|^2 + (C_2 \nu - 2\alpha \kappa C_1) \int_0^t e^{2\alpha s} \|P_h \zeta(s)\|^2 ds \leq Ch^2 \int_0^t e^{2\alpha s} \left( |u_s(s)|^2 + |u(s)|^2 + |p(s)|^2 \right) ds.
\]

Multiply (4.13) by \( e^{-2\alpha t} \) and use the regularity estimates of \( u \) and \( p \) from Lemma 2.2 to complete the energy norm estimates of \( P_h \zeta \) as

\[
(4.14) \quad \|P_h \zeta(t)\|^2 + e^{-2\alpha t} \int_0^t e^{2\alpha s} \|P_h \zeta(s)\|^2 ds \leq Ch^2.
\]

Since \( \zeta = u - P_h u + P_h \zeta \), using the triangle inequality and the bounds in (3.36), (4.14), we arrive at

\[
(4.15) \quad \| \zeta(t) \|^2 + e^{-2\alpha t} \int_0^t e^{2\alpha s} \| \zeta(s) \|^2 ds \leq Ch^2.
\]

To derive the estimates of \( \zeta_t \) in energy norm, we substitute \( \phi_h = P_h \zeta_t \) in (4.14). Then, apply Cauchy-Schwarz’s, Young’s inequalities, Lemma 3.3, (3.36) and (4.15) to the resulting equation and arrive at

\[
(4.16) \quad \|P_h \zeta_t(t)\|^2 \leq Ch^2.
\]

A multiplication of (4.16) by \( e^{2\alpha t} \), an integration from 0 to \( t \) with respect to time, and then again a multiplication by \( e^{-2\alpha t} \) lead to

\[
(4.17) \quad e^{-2\alpha t} \int_0^t e^{2\alpha s} \|P_h \zeta_t(s)\|^2 ds \leq Ch^2.
\]

A use of the triangle inequality and bounds of (3.36), (4.16), (4.17) yields

\[
(4.18) \quad \| \zeta_t(t) \|^2 + e^{-2\alpha t} \int_0^t e^{2\alpha s} \| \zeta_t(s) \|^2 ds \leq Ch^2.
\]

For obtaining the \( L^2 \)-norm estimates of \( \zeta \), we apply the Aubin-Nitsche duality argument. Let \( \{ w, q \} \) be the pair of unique solution of the steady state \( \zeta \) system stated as

\[
(4.19) \quad -\nu \Delta w + \nabla q = \zeta \quad \text{in} \Omega, \quad \nabla \cdot w = 0 \quad \text{in} \Omega, \quad w|_{\partial \Omega} = 0.
\]

The above solution pair satisfies the following regularity results:

\[
(4.20) \quad \| w \|_2 + \| q \|_1 \leq C \| \zeta \|.
\]

Forming \( L^2 \) inner product between (4.19) and \( \zeta \), and applying the regularity estimates of \( w \) and \( q \), we arrive at

\[
\| \zeta \|^2 = \nu \sum_{T \in T_h} \int_T \nabla w : \nabla \zeta \, dT - \nu \sum_{T \in T_h} \int_{\partial T} (\nabla w n_T) : \zeta \, ds
\]

\[
- \sum_{T \in T_h} \int_T q \nabla \cdot \zeta \, dT + \sum_{T \in T_h} \int_{\partial T} q n_T \cdot \zeta \, ds
\]

\[
= \nu \sum_{T \in T_h} \int_T \nabla \zeta : \nabla w \, dT - \nu \sum_{e \in E_h} \int_e \{ \nabla w \} \cdot |\zeta| \, ds + b(\zeta, q).
\]
Using (4.4) with $\phi_h$ replaced by $P_h w$, and observing that on each interior edge $[w] \cdot n_e = 0$, we obtain
\[
\|\zeta(t)\|^2 = \nu \sum_{T \in T_h} \int_T \nabla \zeta : \nabla (w - P_h w) \, dT - \nu \sum_{e \in \Gamma_h} \int_e \{\nabla (w - P_h w)\} n_e \cdot [\zeta] \, ds \\
+ \nu \sum_{e \in \Gamma_h} \int_e \{\nabla \zeta\} n_e \cdot [P_h w - w] \, ds + \nu J_0(\zeta, w - P_h w) \\
+ \kappa (a(\zeta, w - P_h w) + J_0(\zeta, w - P_h w)) + b(\zeta, q) - b(P_h w - w, p - r_h(p)) \\
- \kappa (a(\zeta, w) + J_0(\zeta, w)).
\] (4.21)

Again, form an $L^2$ inner product between (4.19) and $\zeta_t$. Then, using integration by parts and $[w] \cdot n_e = 0$ on each interior edge, we rewrite the last term on the right-hand side of (4.21) and obtain
\[
\|\zeta(t)\|^2 = \nu \sum_{T \in T_h} \int_T \nabla \zeta : \nabla (w - P_h w) \, dT - \nu \sum_{e \in \Gamma_h} \int_e \{\nabla (w - P_h w)\} n_e \cdot [\zeta] \, ds \\
+ \nu \sum_{e \in \Gamma_h} \int_e \{\nabla \zeta\} n_e \cdot [P_h w - w] \, ds + \nu J_0(\zeta, w - P_h w) \\
+ \kappa (a(\zeta, w - P_h w) + J_0(\zeta, w - P_h w)) + b(\zeta, q) - b(P_h w - w, p - r_h(p)) \\
+ \frac{\kappa}{\nu} b(\zeta_t, q) - \frac{\kappa}{\nu} b(\zeta, \zeta_t).
\] (4.22)

We follow the similar steps as used in bounding the first term on the right hand side of (4.7) to estimate the first, second, third, fourth and fifth terms in the right hand side of (4.22). Then, we apply (3.36) and (4.20) to arrive at
\[
\left| \nu \sum_{T \in T_h} \int_T \nabla \zeta : \nabla (w - P_h w) \, dT - \nu \sum_{e \in \Gamma_h} \int_e \{\nabla (w - P_h w)\} n_e \cdot [\zeta] \, ds \\
+ \nu \sum_{e \in \Gamma_h} \int_e \{\nabla \zeta\} n_e \cdot [P_h w - w] \, ds + \nu J_0(\zeta, w - P_h w) + \kappa (a(\zeta, w - P_h w) + J_0(\zeta, w - P_h w)) \right| \\
\leq Ch^2 \|w\|_2^2 \|\zeta\|_e + Ch^2 \|u\|_2^2 \|w\|_2 + Ch^2 \|\zeta_t\|_e + Ch^2 \|u_t\|_2 \|w\|_2 \\
\leq \frac{1}{8} \|\zeta\|^2 + Ch^2 (\|\zeta\|^2 + \|\zeta_t\|^2) + Ch^2 (\|u\|^2 + \|u_t\|^2).
\] (4.23)

We can handle the sixth term on the right-hand side of (4.22) as
\[
b(\zeta, q) = b(\zeta - P_h u + S_h u, q) + b(P_h \zeta, q) = b(u - P_h u, q) + b(P_h \zeta, q - r_h(q)) \\
= - \sum_{T \in T_h} \int_T q \nabla \cdot (u - P_h u) \, dT + \sum_{e \in \Gamma_h} \int_e \{q - r_h(q)\} n_e \cdot (u - P_h u) \, ds + b(P_h \zeta, q - r_h(q)).
\] (4.24)

Furthermore, using integration by parts formula to the first term on the right hand side of (4.24) and noting that $q$ is continuous, we arrive at
\[
b(\zeta, q) = \sum_{T \in T_h} \int_T \nabla q \cdot (u - P_h u) \, dT + b(P_h \zeta, q - r_h(q)).
\]

From Cauchy-Schwarz’s, Young’s inequalities, (3.4), (3.36), (4.20) and Lemma 2.3 we obtain
\[
|b(\zeta, q)| \leq \left| Ch^2 |q|_1 |u|_2 \right| - \sum_{T \in T_h} \int_T \{\nabla \cdot P_h \zeta\} (q - r_h(q)) \, dT + \sum_{e \in \Gamma_h} \int_e \{q - r_h(q)\} |P_h \zeta| \cdot n_e \, ds \\
\leq Ch^2 \|u\|_2 \|\zeta\| + Ch \|q\|_1 \|P_h \zeta\|_e \leq \frac{1}{8} \|\zeta\|^2 + Ch^2 (|u\|^2 + \|P_h \zeta\|^2).
\] (4.25)
Similar to (4.28), using (3.4), (3.36), (4.20) and Lemma 2.3 the 8th term on the right hand side of (4.22) can be bounded as follows:

$$|b(\zeta, q)| \leq \left| Ch^2 |q|_1 |\mathbf{u}|_2 - \sum_{T \in \mathcal{T}_h} \int_T (\nabla \cdot \mathbf{P}_h \zeta_i(q(r_h(q))) dT + \sum_{e \in \Gamma_h} \int_{e} \{q(r_h(q))\} |\mathbf{P}_h \zeta_i| \cdot n_e \, ds \right|$$

(4.26)

$$\leq Ch^2 |\mathbf{u}|_2 |\zeta|_\infty + Ch |q|_1 |\mathbf{P}_h \zeta|_\infty \leq \frac{1}{8} |\zeta|_2^2 + Ch^2 (h^2 |\mathbf{u}|_2^2 + |\mathbf{P}_h \zeta|_2^2).$$

Apply Cauchy-Schwarz's inequality and (3.4), (3.36), (4.20) to arrive at

$$|b(\mathbf{P}_h \mathbf{w} - \mathbf{w}, p - r_h(p))| \leq Ch^2 |p|_1 |\mathbf{w}|_2 \leq \frac{1}{8} |\zeta|_2^2 + Ch^4 |p|_1^2.$$  

A use of (4.28) and (4.26)-(4.27) in (4.22) leads to

$$\frac{1}{2} \|\zeta(t)\|_2^2 + \frac{\kappa}{2} \int_0^t \|\zeta(t)\|^2 \leq Ch^2 (|\zeta|_2^2 + |\zeta_t|_2^2 + |\mathbf{P}_h \zeta|_2^2 + |\mathbf{P}_h \zeta_t|_2^2) + Ch^4 (|\mathbf{u}|_2^2 + |\mathbf{P}_h \zeta|_2^2 + |p|_1^2).$$

(4.29)

A multiplication of (4.29) by $e^{2\alpha t}$ and an integration of the resulting equation with respect to time from 0 to $t$ yield

$$e^{2\alpha t} \|\zeta(t)\|^2 + \left(\frac{\nu - 2\kappa \alpha}{\nu}\right) \int_0^t e^{2\alpha s} \|\zeta(s)\|^2 \, ds \leq Ch^4 |\mathbf{u}|_2^2$$

$$+ Ch \int_0^t e^{2\alpha s} (|\zeta(s)|_2^2 + |\zeta_t(s)|_2^2 + |\mathbf{P}_h \zeta(s)|_2^2 + |\mathbf{P}_h \zeta_t(s)|_2^2) \, ds + Ch^4 \int_0^t e^{2\alpha s} (|\mathbf{u}(s)|_2^2 + |\mathbf{u}_t(s)|_2^2 + |p(s)|_1^2) \, ds.$$  

Multiply (4.29) by $e^{-2\alpha t}$ and use (4.11), (4.13), (4.17), (4.18) with Lemma 2.2 to arrive at

$$\|\zeta(t)\|^2 + e^{-2\alpha t} \int_0^t e^{2\alpha s} \|\zeta(s)\|^2 \, ds \leq Ch^4.$$  

(4.30)

A combination of (4.15), (4.18) and (4.30) completes the proof of (4.10) in Lemma 4.1

Following the similar steps as involved in proving the $L^2$ estimate of $\zeta$ in (4.5), we arrive at the $L^2$ estimate (4.30) involving $\zeta_t$. Only difference is in the dual problem, where the right hand side is changed to $\zeta_t$. With the resulting $L^2$ estimate of $\zeta_t$, we conclude the proof of Lemma 4.1.

Below, in Lemma 4.2 we derive the bounds of $\mathbf{p}$.

**Lemma 4.2.** Let the assumptions (A1)-(A3) be satisfied and let $0 < \alpha < \min \left\{ \frac{\nu C_2}{2(\gamma + \kappa C_1)}, \frac{\nu}{2\kappa} \right\}$. Then, there exists a positive constant $C = C(\kappa, \nu, \alpha, C_2, M_0)$, such that, for each $t > 0$, the following estimates hold true:

$$\|\mathbf{p}(t)\|^2 + h^2 \|\mathbf{p}(t)\|_\infty^2 + e^{-2\alpha t} \int_0^t e^{2\alpha s} (\|\mathbf{p}(s)\|^2 + h^2 \|\mathbf{p}(s)\|_\infty^2) \, ds \leq Ch^4.$$  

**Proof.** Subtract (4.1) from (4.2) and write the equation in $\mathbf{p}$ as

$$(\mathbf{p}_t, \phi_h) + \kappa (\mathbf{p}, \phi_h) + J_0(\mathbf{p}_t, \phi_h) + \nu (\mathbf{p}, \phi_h) + J_0(\mathbf{p}, \phi_h) = - (\zeta_t, \phi_h), \quad \forall \phi_h \in J_h.$$  

Substitute $\phi_h = \mathbf{p}$ in the above equation and use Lemma 3.3 to obtain

$$\frac{1}{2} \frac{d}{dt} (\|\mathbf{p}\|^2 + \kappa (\mathbf{p}, \mathbf{p}) + J_0(\mathbf{p}, \mathbf{p})) + \nu C_2 \|\mathbf{p}\|_\infty^2 \leq - (\zeta_t, \mathbf{p}).$$  

(4.31)

Multiply (4.31) by $e^{2\alpha t}$, integrate the resulting inequality with respect to time from 0 to $t$, and use Lemmas 3.3 and 2.2 $\mathbf{p}(0) = 0$ to arrive at

$$e^{2\alpha t} (\|\mathbf{p}\|^2 + \kappa C_2 \|\mathbf{p}\|_\infty^2) + (\nu C_2 - 2\alpha (\gamma + \kappa C_1)) \int_0^t e^{2\alpha s} \|\mathbf{p}(s)\|^2 \, ds \leq C \int_0^t e^{2\alpha s} \|\zeta(s)\|^2 \, ds.$$  

(4.32)
A multiplication of (4.32) by $e^{-2\alpha t}$ and a use of (4.6) in the resulting inequality yield

\[(4.33) \quad \|\rho(t)\|^2 + \kappa C_2\|\rho(t)\|^2 + e^{-2\alpha t}\int_0^t e^{2\alpha s}\|\rho(s)\|^2 ds \leq Ch^4.\]

An application of (3.15) and (4.33) leads to

\[(4.34) \quad h^2\|\rho(t)\|^2 + e^{-2\alpha t}\int_0^t e^{2\alpha s}\|\rho(s)\|^2 ds \leq Ch^4.\]

A combination of (4.33) and (4.34) concludes the proof of Lemma 4.2.

Since $\xi = \zeta + (S_h u - v_h) = \zeta + \rho$, we now apply Lemmas 3.1 and 4.2 along with the triangle inequality to obtain the following estimates of $\xi$:

\[(4.35) \quad \|\xi(t)\|^2 + h^2\|\xi(t)\|^2 + e^{-2\alpha t}\int_0^t e^{2\alpha s}\|\xi(s)\|^2 ds \leq Ch^4, \quad 0 \leq t \leq T.\]

We are now left with the estimates of $\eta = v_h - u_h$.

**Lemma 4.3.** Let the assumptions (A1)-(A3) be satisfied and let $0 < \alpha < \min \left\{ \frac{\nu C_2}{2(\gamma + \kappa C_1)}, \frac{\nu}{2\kappa} \right\}$. Further, let $v_h(t) \in J_h$ be a solution of (4.1) corresponding to the initial value $v_h(0) = P_h u_0$. Then, there exists a positive constant $C = C(\kappa, \nu, \alpha, C_2, M_0)$, independent of $h$, such that, the following estimates hold true:

\[
\|\eta(t)\|^2 + \|\eta(t)\|^2 + e^{-2\alpha t}\int_0^t e^{2\alpha s}\|\eta(s)\|^2 ds \leq Ce^{Ch^4}.
\]

**Proof.** From (3.10) and (4.1), we observe that

\[(\eta, \phi_h) + \kappa (a(\eta, \phi_h) + J_0(\eta, \phi_h)) + \nu(a(\eta, \phi_h) + J_0(\eta, \phi_h)) = c^u_h(u_h, u_h, \phi_h) - c^u(u, u, \phi_h) \quad \text{for } \phi_h \in J_h.
\]

Substitute $\phi_h = \eta$ and use Lemma 3.3 to arrive at

\[(4.36) \quad \frac{1}{2} \frac{d}{dt} (\|\eta\|^2 + \kappa (a(\eta, \eta) + J_0(\eta, \eta))) + \nu C_2 \|\eta\|^2 \leq c^u_h(u_h, u_h, \eta) - c^u(u, u, \eta).
\]

Since $u$ is continuous, we can rewrite the nonlinear term as

\[c^u(u, u, \eta) = c^u_h(u, u, \eta).
\]

For the sake of simplicity, we drop the superscript in the nonlinear terms unless there is no confusion. We now rewrite the nonlinear terms as follows:

\[(4.37) \quad c(u_h, u_h, \eta) - c(u, u, \eta) = -c(u, \xi, \eta) + c(\xi, \xi, \eta) - c(\xi, u, \eta) + c(\eta, \xi, \eta) - c(\eta, u, \eta) - c(u_h, \eta, \eta).
\]

Note that, the last term is non-negative due to (2.11) and is therefore dropped. We find that

\[(4.38) \quad c(u_h, u_h, \eta) - c(u, u, \eta) \leq -c(u, \xi, \eta) + c(\xi, \xi, \eta) - c(\xi, u, \eta) + c(\eta, \xi, \eta) - c(\eta, u, \eta).
\]

Apply (3.11), Lemma 2.1 and Young’s inequality to obtain

\[(4.39) \quad |c(\eta, u, \eta)| \leq C\|\eta\|\|u\|_{2}\|\eta\|_{\varepsilon} \leq \frac{C_2\nu}{10}\|\eta\|^2 + C\|u\|^2\|\eta\|^2.
\]
We bound the second and fourth nonlinear terms on the right hand side of (4.38) using the Cauchy-Schwarz, Young inequalities, 2.10 and Lemmas 2.23 and 3.7 as follows

\[
|c(\xi, \eta)| = \left| \sum_{T \in T_h} \int_T (\xi \cdot \nabla \xi) \cdot \eta \, dT + \sum_{T \in T_h} \int_{\partial T_-} |\{\xi\} \cdot n_T| (\xi^{int} - \xi^{ext}) \cdot \eta^{int} \, ds + \frac{1}{2} \sum_{T \in T_h} \int_T (\nabla \cdot \xi) \xi \cdot \eta \, dT \\
- \frac{1}{2} \sum_{e \in \Gamma_h} \int_e |\{\xi\} \cdot n_e (\xi \cdot \eta) \, ds \right|
\]

\[
\leq \sum_{T \in T_h} \|\xi\|_{L^4(T)} \|\nabla \xi\|_{L^2(T)} \|\eta\|_{L^4(T)} + C \sum_{T \in T_h} h_T^{-3/4} (\|\xi\|_{L^2(T)} + h_T \|\nabla \xi\|_{L^2(T)}) |e|^{1/2} |e|^{-1/2} \|\xi\|_{L^2(e)} \|\eta\|_{L^4(T)}
\]

\[
+ C \sum_{T \in T_h} \|\nabla \xi\|_{L^2(T)} \|\xi\|_{L^4(T)} \|\eta\|_{L^4(T)} + C \sum_{T \in T_h} |e|^{1/4} |e|^{-1/2} \|\xi\|_{L^2(e)} h_T^{-3/4} (\|\xi\|_{L^2(T)} + h_T \|\nabla \xi\|_{L^2(T)}) \|\eta\|_{L^4(T)}
\]

(4.40)

\[
\leq \frac{C_2 \nu}{16} \|\eta\|_{L^2}^2 + C h^{-2} (\|\xi\|^2 + h^2 \|\xi\|_{L^2}^2) \|\xi\|^2
\]

and

\[
|c(\eta, \xi, \eta)| = \left| \sum_{T \in T_h} \int_T (\eta \cdot \nabla \eta) \cdot \xi \, dT + \sum_{T \in T_h} \int_{\partial T_-} |\{\eta\} \cdot n_T| (\xi^{int} - \xi^{ext}) \cdot \eta^{int} \, ds + \frac{1}{2} \sum_{T \in T_h} \int_T (\nabla \cdot \eta) \xi \cdot \eta \, dT \\
- \frac{1}{2} \sum_{e \in \Gamma_h} \int_e |\{\eta\} \cdot n_e (\xi \cdot \eta) \, ds \right|
\]

\[
\leq \sum_{T \in T_h} \|\eta\|_{L^4(T)} \|\nabla \xi\|_{L^2(T)} \|\eta\|_{L^4(T)} + C \sum_{T \in T_h} \|\eta\|_{L^4(T)} \frac{1}{|e|^{1/2}} \|\xi\|_{L^2(e)} \|\eta\|_{L^4(T)}
\]

\[
+ C \sum_{T \in T_h} \|\nabla \eta\|_{L^2(T)} \|\xi\|_{L^4(T)} \|\eta\|_{L^4(T)} + C \sum_{T \in T_h} h_T^{-3/4} \|\eta\|_{L^2(T)} h_T^{-1/2} (\|\xi\|_{L^2(T)} + h_T \|\nabla \xi\|_{L^2(T)}) h_T^{-1/4} \|\eta\|_{L^4(T)}
\]

(4.41)

\[
\leq \frac{C_2 \nu}{16} \|\eta\|_{L^2}^2 + C h^{-4} (\|\xi\|^2 + h^2 \|\xi\|_{L^2}^2) \|\eta\|^2
\]

By applying (2.10), the first nonlinear term on the right hand side of (4.38) can be written as

\[
c(u; \xi, \eta) = - \sum_{T \in T_h} \int_T (u \cdot \nabla \eta) \cdot \xi \, dT - \frac{1}{2} \sum_{T \in T_h} \int_T (\nabla \cdot u) \xi \cdot \eta \, dT
\]

\[
+ \frac{1}{2} \sum_{e \in \Gamma_h} \int_e |u| \cdot n_e (\xi \cdot \eta) \, ds - \sum_{T \in T_h} \int_{\partial T_-} |u| \cdot n_T |\xi^{ext} \cdot (\eta^{int} - \eta^{ext})| \, ds
\]

\[
= I_1 + I_2 + I_3 + I_4 + I_5, \text{ (say)}
\]

Note that, \( u \) is continuous. This will lead to \( I_3 = I_5 = 0 \). An application of Cauchy-Schwarz’s, Young’s inequalities, Lemma 2.21 and (2.5) yield

\[
|I_1| + |I_2| \leq C \|u\|_{L^\infty(\Omega)} \|\nabla \eta\|_{L^2(\Omega)} \|\xi\| + C \|\nabla u\|_{L^4(\Omega)} \|\xi\| \|\eta\|_{L^4(\Omega)}
\]

(4.42)

\[
\leq C \|u\|_2 \|\xi\| \|\eta\|_{L^2} \leq C \|u\|_2^2 \|\xi\|^2 + \frac{C_2 \nu}{16} \|\eta\|^2
\]

A use of Lemmas 2.21 and 2.23 and Young’s inequality leads to

\[
|I_4| \leq C \|u\|_{L^\infty(\Omega)} \sum_{T \in T_h} \|\xi\|_{L^2(\partial T)} |e|^{1/2} |e|^{-1/2} \|\eta\|_{L^2(\partial T)}
\]

\[
\leq C \|u\|_2 \sum_{T \in T_h} |e|^{1/2} h_T^{-1/2} (\|\xi\|_{L^2(T)} + h_T \|\nabla \xi\|_{L^2(T)}) \frac{1}{|e|^{1/2}} \|\eta\|_{L^2(\partial T)}
\]

(4.43)

\[
\leq C \|u\|_2^2 (\|\xi\|^2 + h^2 \|\xi\|_{L^2}^2) + \frac{C_2 \nu}{16} \|\eta\|^2.
\]
Finally for the third term on the right hand side of (4.38), we first note that
\[
\int_T (\nabla \cdot u) v \cdot w \, dT = \int_T \nabla \cdot (v \otimes u) \cdot w \, dT - \int_T (u \cdot \nabla v) \cdot w \, dT
\]
\[
= - \int_T (u \cdot \nabla w) \cdot v \, dT + \int_{\partial T} (u^{\text{int}} \cdot n_T) v^{\text{int}} \cdot w \, ds - \int_T (u \cdot \nabla v) \cdot w \, dT,
\]
by applying $(\nabla \cdot u)v = \nabla \cdot (uv) - u \cdot \nabla v$ and based on it, we rewrite the third term as
\[
e(\xi, u, \eta) = \frac{1}{2} \sum_{T \in T_h} \int_T (\xi \cdot \nabla u) \cdot \eta \, dT - \frac{1}{2} \sum_{T \in T_h} \int_T (\xi \cdot \nabla \eta) \cdot u \, dT
\]
\[
- \sum_{T \in T_h} \int_{\partial T^-} (\eta T^{\text{int}} - u^{\text{ext}}) \cdot \eta^{\text{int}} \, ds - \frac{1}{2} \sum_{e \in \Gamma_h} \int_e (\xi \cdot n_e) u \cdot \eta \, ds
\]
\[
+ \frac{1}{2} \sum_{T \in T_h} \int_{\partial T} (\xi^{\text{int}} \cdot n_T) u^{\text{int}} \cdot \eta^{\text{int}} \, ds
\]
\[
= I_8 + I_7 + I_8 + I_9 + I_{10}.
\]
The term $I_8$ is zero due to the continuity of $u$. A use of (2.5), Young’s inequality and Lemma 2.1, bounds $I_6$ and $I_7$ as
\[
|I_6| \leq ||\xi|| ||\nabla u||_{L^2(\Omega)} ||\eta||_{L^4(\Omega)} \leq \frac{C_{2 \nu}}{16} ||\eta||^2 + C ||u||^2 \xi^2,
\]
\[
|I_7| \leq ||\xi|| ||\nabla \eta|| ||u||_{L^\infty(\Gamma)} \leq \frac{C_{2 \nu}}{16} ||\eta||^2 + C ||u||^2 \xi^2.
\]
Next, sum the integral in $I_{10}$ over all $T$ and consider the contribution of this sum to one interior edge $e$. Let us assume that $e$ is shared by two triangles $T_r$ and $T_s$, with exterior normal $n_r$ and $n_s$. Then, we arrive at
\[
\int_e (\xi|_{T_r} \cdot n_r) u|_{T_r} \cdot \eta|_{T_r} \, ds + \int_e (\xi|_{T_s} \cdot n_s) u|_{T_s} \cdot \eta|_{T_s} \, ds = \int_e (\xi \cdot n_e) u \cdot \eta \, ds.
\]
An application of Lemma 2.3 leads to
\[
I_9 + I_{10} = \frac{1}{2} \sum_{e \in \Gamma_h} \int_e (\xi \cdot n_e) u \cdot \eta \, ds \leq C ||u||_{L^2(\Omega)} \sum_{e \in \Gamma_h} \frac{1}{|e|^{1/2}} ||\eta||_{L^2(e)} |e|^{1/2} ||\xi||_{L^2(e)}
\]
\[
\leq C ||u||^2 \sum_{T \in T_h} ||\xi||_{L^2(T)} \leq C_{2 \nu} ||\eta||^2 + C ||u||^2 ||\xi||^2 + h^2 ||\xi||_e^2.
\]
Substitute (4.39)–(4.40) in (4.38), and thereby in (4.36), and multiply the resulting inequality by $e^{2\alpha t}$ to obtain
\[
\frac{1}{2} \frac{d}{dt} (e^{2\alpha t} (||\eta||^2 + \kappa (a(\eta, \eta) + J_0(\eta, \eta)))) + \left( \frac{\nu C_2}{2} - 2\alpha(\gamma + \kappa C_1) \right) e^{2\alpha t} ||\eta||^2
\]
\[
\leq C(||u||^2 + h^{-2} ||\xi||^2 + h^{-2} ||\xi||^2) e^{2\alpha t} ||\eta||^2 + \kappa ||\eta||^2 + C e^{2\alpha t} (||u||^2 + h^{-2} ||\xi||^2) (||\xi||^2 + h^{-2} ||\xi||_e^2).
\]
An integration of (4.41) with respect to time from 0 to $t$, a use of $\eta(0) = 0$, Lemmas 3.3 and 3.2, Gronwall’s inequality, (3.35) and Lemma 2.2 leads to the following estimates of $\eta$
\[
e^{2\alpha t} (||\eta(t)||^2 + \kappa C_2 ||\eta(t)||^2) + \nu C_2 \int_0^t e^{2\alpha s} ||\eta||^2 ds \leq Ce^{Ct} h^4 e^{2\alpha t}.
\]
Multiply the resulting inequality by $e^{-2\alpha t}$ to arrive at the desired estimates. \hfill \Box

**Proof of Theorem 5.4.** A use of $e = \xi + \eta$, triangle’s inequality, the estimates in (4.35) and Lemma 1.3 completes the proof. \hfill \Box
Remark 4.1. Under the following smallness assumption on the data

\begin{equation}
N = \sup_{v_h, u_h, \in \mathcal{V}_h} \frac{c(v_h, u_h, w_h)}{\|w_h\|^2 \|v_h\|} \quad \text{and} \quad \frac{N \gamma}{C_2^2 \nu^2} \|f\| < 1,
\end{equation}

the bounds of Theorem 3.7 can be shown to be uniform in time, that is,

\[ \|(u - u_h)(t)\| + h\|(u - u_h)(t)\| \leq C h^2, \]

where the constant C is independent of time t and h.

Firstly, rewrite the nonlinear terms as follows:

\begin{equation}
c^{u_h}(u_h, u_h, \eta) - c^{u_h}(u, u, \eta) = -c^{u_h}(u, \xi, \eta) + c^{u_h}(\xi, \eta, \eta) - c^{u_h}(\xi, u, \eta) - c^{u_h}(\eta, u_h, \eta) - c^{u_h}(v_h, \eta, \eta).
\end{equation}

The last nonlinear term on the right hand side of (4.49) is positive due to the positivity property (2.11). Using the proof of the Lemma 4.3, the first three nonlinear terms on the right hand side of (4.49) can be bounded as

\[ |c^{u_h}(u, \xi, \eta) + c^{u_h}(\xi, u, \eta) + c^{u_h}(\xi, \eta, \eta)| \leq C(\|u\|^2 + ch^{-1}\|\xi\|\|\eta\| + h\|\xi\|\|\eta\|), \]

For the fourth nonlinear term on the right hand side of (4.49), we use the smallness condition and find that

\[ c^{u_h}(\eta, u_h, \eta) \leq N\|\eta\|^2 \|u_h\|, \]

Now, in order to achieve the uniform in time velocity error estimates, we modify the proof of Lemma 4.3 as follows: From (4.36) and using (4.35), Lemma 2.2, we arrive at

\begin{equation}
dt(\|\eta\|^2 + \kappa (a(\eta, \eta) + J_0(\eta, \eta))) + 2(\nu C_2 - N\|u_h\|\|\eta\|) \|\eta\| \leq C h^2 \|\eta\|.
\end{equation}

After multiplying (4.50) by \(e^{2\alpha t}\), integrate with respect to time from 0 to t. Then, multiply the resulting equation by \(e^{-2\alpha t}\) and use Lemma 3.2 to find that

\[ \|\eta\|^2 + \kappa (a(\eta, \eta) + J_0(\eta, \eta)) + 2e^{-2\alpha t} \int_0^t e^{2\alpha s}(\nu C_2 - N\|u_h\|)\|\eta(s)\|^2 ds \leq C h^2 \|\eta\|.
\]

\[ + 2e^{-2\alpha t} \int_0^t e^{2\alpha s}(\|\eta(s)\|^2 + \kappa (a(\eta(s), \eta(s)) + J_0(\eta(s), \eta(s)))) ds + C h^2 e^{-2\alpha t} \int_0^t e^{2\alpha s}\|\eta(s)\| ds.
\]

Letting \(t \to \infty\), apply the L’Hospital rule and use Lemma 3.8 to arrive at

\[ \frac{1}{\alpha} \left( \nu C_2 - \frac{N \gamma}{C_2^2 \nu^2} \|f\|_{L^\infty(0, \infty; L^2(\Omega))} \right) \limsup_{t \to \infty} \|\eta(t)\|^2 \leq \frac{C h^2}{\alpha} \limsup_{t \to \infty} \|\eta(t)\|.
\]

Owing to the smallness condition (4.48), we arrive at

\[ \limsup_{t \to \infty} \|\eta(t)\| \leq C h^2.
\]

Hence,

\begin{equation}
\limsup_{t \to \infty} \|\eta(t)\| \leq C h^2.
\end{equation}

Combining the estimates of (4.35) and (4.51), we find that

\[ \limsup_{t \to \infty} \|u(t) - u_h(t)\| \leq C h^2. \]
5 Error estimates for pressure

This section presents the derivation of error estimates for the semidiscrete discontinuous Galerkin approximation of the pressure. We begin by proving a lemma which is crucial for establishing these error estimates.

**Lemma 5.1.** Let the assumptions (A1)-(A3) be satisfied and let $0 < \alpha < \min \left\{ \frac{\nu C_2}{2(\gamma + \alpha C_1)}, \frac{\nu}{2\alpha} \right\}$. Then, the error $e = u - u_h$ in approximating the velocity satisfies for $t > 0$

$$
\| e_t(t) \| + \kappa \| e_t(t) \|_2 \leq C e^{C_1 h}.
$$

**Proof.** Subtract \((5.10)\) from \((5.12)\) to write the equation in error $e = u - u_h$ as

$$
(e_t, \phi_h) + \alpha (a(e_t, \phi_h) + J_0(e_t, \phi_h)) + \nu (a(e, \phi_h) + J_0(e, \phi_h)) + c^u(u, u, \phi_h) - c^u(u, u_h, \phi_h) + b(\phi_h) = 0 \quad \forall \phi_h \in J_h.
$$

Choose $\phi_h = P_h e_t = e_t - (u_t - P_h u_t)$ in \((5.1)\) and use Lemma \(5.3\) to observe that

$$
\| e_t \|^2 + \kappa C_2 \| P_h e_t \|_2^2 \leq (e_t, e_t - P_h u_t) + \alpha (a(P_h u_t - u_t, P_h e_t) + J_0(P_h u_t - u_t, P_h e_t)) \leq \kappa C_2 \| P_h e_t \|_2^2 + C \| e \|_2^2 + C \| e \|_2^2.
$$

Using the continuity of $u$, Lemmas \(2.2\), \(2.3\) and \(2.1\) we arrive at

$$
|c(e, u, P_h e_t)| \leq \sum_{T \in T_h} \left| \int_T (e \cdot \nabla u - P_h e_t) \cdot P_h e_t \, dT \right| + \frac{1}{2} \sum_{T \in T_h} \left| \int_T (\nabla \cdot e) u \cdot P_h e_t \, dT \right| - \frac{1}{2} \sum_{e \in \Gamma_h} \int_e |e| \cdot n_e \{ u \cdot P_h e_t \} \, ds \leq \sum_{T \in T_h} \| e \|_{L^4(T)} \| \nabla u \|_{L^4(T)} \| P_h e_t \|_{L^2(T)} + C \| u \|_{L^\infty(\Omega)} \| \nabla e \|_{L^2(\Omega)} \| P_h e_t \|_{L^2(T)} \leq \frac{\kappa C_2}{12} \| P_h e_t \|_2^2 + C \| e \|_2^2.
$$

Similarly, using the steps involved in obtaining estimates in \((5.5)\), we bound

$$
|c(e, u, P_h e_t)| \leq \frac{\kappa C_2}{12} \| P_h e_t \|_2^2 + C \| e \|_2^2.
$$

Using \((5.4)-(5.6)\) in \((5.3)\), we arrive at

$$
|c(u_h, u_h, P_h e_t) - (u, u, P_h e_t)| \leq \frac{\kappa C_2}{4} \| P_h e_t \|_2^2 + C \| e \|_2^2.
$$

Applying the Cauchy-Schwarz and Young inequalities, definition of space $J_h$, \((3.1)\) and \((3.36)\), we bound the second, third and sixth terms on the right hand side of \((5.7)\) as

$$
\kappa |a(P_h u_t - u_t, P_h e_t) + J_0(P_h u_t - u_t, P_h e_t)| \leq \frac{C_2 \kappa}{12} \| P_h e_t \|_2^2 + C h^2 \| u_t \|_2^2,
$$

$$
\nu |a(e, P_h e_t) + J_0(e, P_h e_t)| \leq \frac{C_2 \kappa}{12} \| P_h e_t \|_2^2 + C \| e \|_2^2 + Ch^2 \| u \|_2^2,
$$

$$
|b(P_h e_t, p)| = |b(P_h e_t, p - r_h(p))| \leq \frac{C_2 \kappa}{12} \| P_h e_t \|_2^2 + Ch^2 \| p \|_1^2.
$$
Using the triangle inequality and (3.4), we arrive at
\[ t \in \mathbb{R} \]
In this section, we employ the backward Euler method for temporal discretization of the semidiscrete discontinuous
\[ (5.15) \]
\[ B^{\infty} = \frac{1}{\Delta t} (\psi^n - \psi^{n-1}), \quad n \geq 0. \]
Finally, a use of triangle inequality, (3.36), (5.11), Lemma 2.2 and Theorem 3.1 yield
\[ \|e_t\|^2 + K \|e_t\|^2 \leq C h^2 e^{CT}. \]
This completes the rest of the proof.

**Proof of Theorem 3.2.** A use of (2.12), (2.13), (3.7) and (3.8) yield
\[ (u_{ht} - u_{t}, v_h) + \kappa (a(u_{ht} - u_{t}, v_h) + J_0(u_{ht} - u_{t}, v_h)) + \nu (a(u_h - u, v_h) + J_0(u_h - u, v_h)) \]
\[ + c(u_h, u_h, v_h) - c(u, u, v_h) - b(v_h, p - r_h(p)) = -b(v_h, p - r_h(p)), \quad \forall v_h \in V_h. \]
Using the inf-sup condition stated in Lemma 3.4, there exists \( v_h \in V_h \) such that
\[ (5.13) \quad b(v_h, p - r_h(p)) = -\|p_h - r_h(p)\|^2, \quad \|v_h\| \leq \frac{1}{\beta_0} \|p_h - r_h(p)\|. \]
A combination of (5.12) and (5.13) leads to
\[ (5.14) \quad \|p_h - r_h(p)\|^2 = (u_{ht} - u_{t}, v_h) + \kappa (a(u_{ht} - u_{t}, v_h) + J_0(u_{ht} - u_{t}, v_h)) \]
\[ + \nu (a(u_h - u, v_h) + J_0(u_h - u, v_h)) \]
\[ + c(u_h, u_h, v_h) - c(u, u, v_h) - b(v_h, p - r_h(p)). \]
Since \( u \) is continuous, the superscripts in nonlinear terms of (5.14) are dropped. Now, following the analysis used in
\[ (5.15) \quad \|p - p_h\|^2 = (\|u_{ht} - u_t\|^2 + \|u_{ht} - u_t\|^2 + \|u_h - u_h\|^2 + \|u_h - u\|^2 + h^2(\|u_t^2 + |p|^2 + |e_t^2|)). \]
Using the triangle inequality and (5.14), we arrive at
An application of Lemma 3.4 and Theorem 3.1 in (5.15) leads to the desired pressure error estimate. This completes the proof.

**6 Fully discrete approximation and error estimates**

In this section, we employ the backward Euler method for temporal discretization of the semidiscrete discontinuous
\[ (6.1) \quad b(U^n, u_h) = 0, \quad \forall q_h \in M_h, \]
\[ (6.2) \quad b(U^n, v_h) + \kappa (a(U^n, v_h) + J_0(U^n, v_h)) + \nu (a(U^n, v_h) + J_0(U^n, v_h)) \]
\[ + c(U^n - 1, U^n, v_h) + b(v_h, P^n) = (f^n, v_h), \quad \forall v_h \in V_h, \]
\[ \tilde{\bar{\psi}}^n = \frac{1}{\Delta t} (\psi^n - \psi^{n-1}), \quad n \geq 0. \]
Now, the backward Euler approximations for (3.7)-(3.9) is defined as follows: Given \( U^0 \), find \( \{U^n\}_{n \geq 1} \in V_h \) and \( \{P^n\}_{n \geq 1} \in M_h \), such that
\[ (6.1) \quad b(U^n, v_h) + \kappa (a(U^n, v_h) + J_0(U^n, v_h)) + \nu (a(U^n, v_h) + J_0(U^n, v_h)) \]
\[ + c(U^n - 1, U^n, v_h) + b(v_h, P^n) = (f^n, v_h), \quad \forall v_h \in V_h, \]
\[ \tilde{\bar{\psi}}^n = \frac{1}{\Delta t} (\psi^n - \psi^{n-1}), \quad n \geq 0. \]
where $U^0 = u_h(0) = P_hu_0$.

An equivalent formulation of (6.1)-(6.2) is defined as follows: For $v_h \in J_h$, we seek $\{U^n\}_{n \geq 1} \in J_h$, such that

$$(\partial_t U^n, v_h) + \kappa (a(\partial_t U^n, v_h) + J_0(\partial_t U^n, v_h)) + \nu (a(U^n, v_h) + J_0(U^n, v_h))$$
$$+ c^{U^n-1}(U^{n-1}, U^n, v_h) = (F^n, v_h),$$

(6.3)

where $U^0 = u_h(0) = P_hu_0$.

Next in Lemma 6.1, we present a priori estimates of backward Euler solution $U^n$ of (6.3).

**Lemma 6.1.** Let the assumptions (A1)-(A3) be satisfied and let $0 < \alpha < \frac{\nu C_2}{2(1 + \kappa C_1)}$. Then, the solution $\{U^n\}_{n \geq 1}$ of (6.3) satisfies the following estimates:

$$\| U^n \|^2 + \| U^n \|^2 + e^{-2\alpha \Delta t} \sum_{n=1}^{M} e^{2\alpha \Delta t} \| U^n \|^2 \leq C, \quad n = 0, 1, \ldots, M,$$

where $C$ depends on the given data.

**Proof.** Substitute $v_h = U^n$ in (6.3). Then, using

$$(\partial_t U^n, U^n) = \frac{1}{\Delta t} (U^n - U^{n-1}, U^n) \geq \frac{1}{2\Delta t} (\| U^n \|^2 - \| U^{n-1} \|^2) = \frac{1}{2} \partial_t \| U^n \|^2,$$

$$a(\partial_t U^n, U^n) = \frac{1}{\Delta t} (a(U^n, U^n) - \frac{1}{\Delta t} a(U^{n-1}, U^{n-1}) + \Delta t a(\partial_t U^n, \partial_t U^n)) \geq \frac{1}{2} \partial_t a(U^n, U^n),$$

(2.11) and Lemma 3.3 in the resulting equation and then the Cauchy-Schwarz inequality, we arrive at

$$(\partial_t U^n, U^n) + \kappa (a(U^n, U^n) + J_0(U^n, U^n)) + 2\nu C_2 \| U^n \|^2 \leq \| F^n \| \| U^n \|.$$  

(6.4)

Note that,

$$\sum_{n=1}^{M} \Delta t e^{2\alpha \Delta t} \| U^n \|^2 = e^{2\alpha \Delta t} \| U^n \|^2 - \sum_{n=1}^{M-1} e^{2\alpha \Delta t} (e^{2\alpha \Delta t} - 1) \| U^n \|^2 - e^{2\alpha \Delta t} \| U^0 \|^2.$$  

(6.5)

Multiply (6.4) by $\Delta t e^{2\alpha \Delta t}$, sum over $n = 1$ to $m$, and use (2.5), (5.5) and Lemmas 3.2, 3.3 to obtain

$$e^{2\alpha \Delta t} \| U^n \|^2 + C_2 \kappa e^{2\alpha \Delta t} \| U^n \|^2 + \left( \nu C_2 - (\gamma + \kappa C_1) \frac{2\alpha \Delta t - 1}{\Delta t} \right) \Delta t \sum_{n=1}^{m} e^{2\alpha \Delta t} \| U^n \|^2 \leq e^{2\alpha \Delta t} \| U^0 \|^2 + \frac{\gamma C_1 e^{2\alpha \Delta t} \| U^n \|^2 + 2\gamma^2 e^{2\alpha \Delta t} \| F^n \|^2.}$$

(6.6)

With $\alpha$ as $0 < \alpha < \frac{\nu C_2}{2(1 + \kappa C_1)}$ we have

$$1 + \frac{\nu C_2 \Delta t}{\gamma + \kappa C_1} \geq e^{2\alpha \Delta t}.$$  

On multiplying (6.6) by $e^{-2\alpha \Delta t}$ and applying assumption (A1), we establish our desired estimates. 

Below, we focus on the derivation of error estimates for the backward Euler method.

We denote $e_n = U^n - u_h(t_n) = U^n - u^n_h$, $n \in \mathbb{N}$, $1 < n \leq M$. Now, consider the semidiscrete formulation (3.10) at $t = t_n$ and subtract it from (6.3) to arrive at

$$(\partial_t e_n, \phi_h) + \kappa (a(\partial_t e_n, \phi_h) + J_0(\partial_t e_n, \phi_h)) + \nu (a(e_n, \phi_h) + J_0(e_n, \phi_h)) = (u^n_h, \phi_h) - (\partial_t u^n_h, \phi_h) + \kappa (a(u^n_{ht}, \phi_h) + J_0(u^n_{ht}, \phi_h)) - \kappa (a(\partial_t u^n_h, \phi_h) + J_0(\partial_t u^n_h, \phi_h)) + c^{U^n_h}(u^n_h, u^n_h, \phi_h)$$
$$- e^{U^n-1}(U^{n-1}, U^n, \phi_h),$$

(6.7)
Lemma 6.2. Under the assumptions of Theorem 4.7, there exists a positive constant $C = C(\kappa, \nu, \alpha, C_2, M_0)$, independent of $h$ and $\Delta t$, such that, the following estimates hold true:

$$
(\|e_n\| + C_2 \kappa \|e_n\|_\varepsilon) + \left(\nu C_2 e^{-2\alpha t M} \Delta t \sum_{n=1}^{M} e^{2\alpha t_n} \|e_n\|_\varepsilon^2\right)^{1/2} \leq C e^{CT} \Delta t.
$$

Proof. Choose $\phi_h = e_n$ in (6.7) and use the facts

$$
(\delta_t e_n, e_n) = \frac{1}{\Delta t} (e_n - e_{n-1}, e_n) \geq \frac{1}{2 \Delta t} (\|e_n\|^2 - \|e_{n-1}\|^2) = \frac{1}{2} \delta_t \|e_n\|^2,
$$

$$
a(\delta_t e_n, e_n) = \frac{1}{2} \left( \frac{1}{\Delta t} a(e_n, e_n) - \frac{1}{\Delta t} a(e_{n-1}, e_{n-1}) + \Delta t a(\delta_t e_n, \delta_t e_n) \right) \geq \frac{1}{2} \delta_t a(e_n, e_n)
$$

and Lemma 3.3 to arrive at

$$
\delta_t (\|e_n\|^2 + \kappa (a(e_n, e_n) + J_0(e_n, e_n))) + 2 C_2 \nu \|e_n\|_\varepsilon^2 \leq 2 (u^{n}_{h^0}, e_n) - 2 (\delta_t u^{n}_{h^0}, e_n) + 2 \kappa (a(u^{n}_{h^0}, e_n) + J_0(u^{n}_{h^0}, e_n)) - 2 \kappa (a(\delta_t u^{n}_{h^0}, e_n) + J_0(\delta_t u^{n}_{h^0}, e_n))
$$

\begin{equation}
+ 2 c^{n}_{h^0}(u^{n}_{h^0}, u^{n}_{h^0}, e_n) - 2 c^{n-1}_{h^0}(U^{n-1}, U^{n}, e_n).
\end{equation}

For the nonlinear terms in (6.8), we note that

$$
c^{n}_{h^0}(u^{n}_{h^0}, u^{n}_{h^0}, e_n) - c^{n-1}_{h^0}(U^{n-1}, U^{n}, e_n) = c^{n}_{h^0}(u^{n}_{h^0}, u^{n}_{h^0}, e_n) - c^{n}_{h^0}(U^{n-1}, U^{n}, e_n)
$$

\begin{equation}
+ c^{n}_{h^0}(U^{n-1}, U^{n}, e_n) - c^{n-1}_{h^0}(U^{n-1}, U^{n}, e_n).
\end{equation}

After dropping the superscripts for the first two nonlinear terms on the right hand side of (6.9), rewrite them as

$$
c(n_{h^0}, u^{n}_{h^0}, e_n) = -c(U^{n-1}, U^{n}, e_n) + c(e_n, u^{n}_{h^0}, e_n) - c(e_{n-1}, u^{n}_{h^0}, e_n) + c(u^{n}_{h^0} - u^{n-1}_{h^0}, u^{n}_{h^0} - u^{n}_{h^0}, e_n) + c(u^{n}_{h^0} - u^{n-1}_{h^0}, u^{n}_{h^0}, e_n).
$$

The first term on the right hand side of (6.10) is positive due to (2.11). The second term on the right hand side of (6.10) can be bounded following the similar steps as involved in estimating $c(\eta, \xi, \eta)$ in Lemma 4.3. A use of Cauchy-Schwarz’s, Young’s inequalities and Theorem 3.1 yield

$$
|c(e_{n-1}, u^{n} - u^{n}_{h^0}, e_n)| \leq C \|e_{n-1}\| \|e_n\|_\varepsilon \leq \frac{C_2 \nu}{8} \|e_n\|_\varepsilon^2 + C \|e_{n-1}\|^2.
$$

Apply (3.1), the Cauchy-Schwarz and Young inequalities and Lemmas 2.1 2.2 to bound the third term as

$$
|c(e_{n-1}, u^{n}, e_n)| \leq C \|e_{n-1}\| \|u^{n}\|_\varepsilon \|e_n\|_\varepsilon \leq \frac{C_2 \nu}{8} \|e_n\|_\varepsilon^2 + C \|e_{n-1}\|^2.
$$

A use of $u^{n}_{h^0} - u^{n-1}_{h^0} = \int_{t_{n-1}}^{t_n} u_{h^0}(t) \, dt$ and Hölder’s, Young’s inequalities, (2.5), Lemmas 2.3 3.7 Theorem 3.1 in the fourth term on the right hand side of (6.10) leads to

$$
|c(u^{n}_{h^0} - u^{n-1}_{h^0}, u^{n}_{h^0} - u^{n}, e_n)| \leq \int_{t_{n-1}}^{t_n} \sum_{T \in T_h} \|u_{h^0}\|_{L^1(T)} \|\nabla (u^{n}_{h^0} - u^{n})\|_{L^2(T)} \|e_n\|_{L^1(T)} \, dt
$$

\begin{equation}
+ \int_{t_{n-1}}^{t_n} \sum_{e \in \Gamma_h} \|u_{h^0}\|_{L^1(e)} \|u^{n}_{h^0} - u^{n}\|_{L^2(e)} \|e_n\|_{L^1(e)} \, dt + C \int_{t_{n-1}}^{t_n} \sum_{T \in T_h} \|\nabla u_{h^0}\|_{L^2(T)} \|u^{n}_{h^0} - u^{n}\|_{L^1(T)} \|e_n\|_{L^1(T)} \, dt
$$

\begin{equation}
+ C \int_{t_{n-1}}^{t_n} \sum_{e \in \Gamma_h} \|u_{h^0}\|_{L^1(e)} \|u^{n}_{h^0} - u^{n}\|_{L^1(e)} \|e_n\|_{L^1(e)} \, dt
\end{equation}

\begin{equation}
\leq \frac{C_2 \nu}{8} \|e_n\|_\varepsilon^2 + C \Delta t \|u_{h^0}\|_{L^2(t_{n-1}, t_n)\varepsilon}^2.
\end{equation}
Using the form of $c(\cdot, \cdot)$ observed in (2.8), (2.5) and Lemmas 2.1, 2.7 along with the fact that $u^n$ has zero jump, the last nonlinear term $c(u^n_h - u^n_{h-1}, u^n, e_n)$ on the right hand side of (6.10) can be bounded as

$$|c(u^n_h - u^n_{h-1}, u^n, e_n)| \leq \Delta t^{1/2} \| u_{ht} \|_{L^2(t_{n-1}, t_n; L^2(\Omega))} \| \nabla u^n \|_{L^t(\Omega)} \| e_n \|_{L^t(\Omega)} + C \Delta t^{1/2} \| u_{ht} \|_{L^2(t_{n-1}, t_n; \mathbb{R})} \| u^n \|_{L^\infty([0, T] \times \Omega)} \| e_n \|
$$

$$+ C \int_{t_{n-1}}^{t_n} \sum_{T \in \mathcal{T}_h} \| u^n \|_{L^\infty([0, T] \times \mathbb{R})} \| e^n \|_{L^2(\mathbb{R})} \left( |(u_{ht})_T| \| e^n \|_{L^2(\mathbb{R})} + |(u_{ht})_{TT}^T| \| e^n \|_{L^2(\mathbb{R})} \right) dt,$$

(6.14)

$$\leq C \Delta t \| u_{ht} \|_{L^2(t_{n-1}, t_n; \mathbb{R})}^2.$$

Using a result in [15, Proposition 4.10], we observe that

$$|c(u^n_h(U^{n-1}, U^n, e_n) - c(U^{n-1}, U^n, e_n))| \leq C \| u^n_h - U^{n-1} \|_{L^t(\Omega)} \| U^n \|_{\mathbb{R}} \| e_n \|_{\mathbb{R}}.$$

An application of the triangle inequality, Lemmas 3.5, 3.8, 6.1 and Cauchy-Schwarz’s inequality leads to

$$|c(\tilde{u}^n_h, \phi_h) - c(U^n, e_n)| \leq C \| u^n_h - U^{n-1} \|_{L^t(\Omega)} \| U^n \|_{\mathbb{R}} \| e_n \|_{\mathbb{R}}.
$$

(6.15)

Now to bound the first term on the right hand side of (6.8), we use the Taylor series expansion and observe that

$$\sum_{n=1}^m C_2 \| e_n \|_{\mathbb{R}}^2 + C \| e_n \|_{\mathbb{R}}^2 \leq C \Delta t \| u_{ht} \|_{L^2(t_{n-1}, t_n; \mathbb{R})}^2.$$

(6.16)

From (6.10) and using Cauchy-Schwarz’s inequality, we obtain

$$2(u^n_{ht}, e_n) - 2(\tilde{u}^n_h, e_n) \leq C \Delta t^{1/2} \left( \int_{t_{n-1}}^{t_n} \| u_{ht}^n(t) \|_{-1,h}^2 dt \right)^{1/2} \| e_n \|_{\mathbb{R}}.$$

(6.17)

Similarly, following the steps involved in bounding (6.17), we arrive at

$$2(\kappa(u^n_{ht}, e_n) + J_0(u^n_{ht}, e_n) - a(\tilde{u}^n_h, e_n) - J_0(\tilde{u}^n_h, e_n)) \leq C \Delta t^{1/2} \left( \int_{t_{n-1}}^{t_n} \| u_{ht}^n(t) \|_{-1,h}^2 dt \right)^{1/2} \| e_n \|_{\mathbb{R}}.$$

(6.18)

Apply (6.9)–(6.18) in (6.8), multiply the resulting inequality by $\Delta t e^{2\alpha n \Delta t}$ and sum over $1 \leq n \leq m \leq M$, where $T = M \Delta t$. Then, using

$$\sum_{n=1}^m \Delta t e^{2\alpha n \Delta t} \| e_n \|_{\mathbb{R}}^2 = e^{2\alpha M \Delta t} \| e_m \|_{\mathbb{R}}^2 - \sum_{n=1}^{m-1} e^{2\alpha n \Delta t} (e^{2\alpha \Delta t} - 1) \| e_n \|_{\mathbb{R}}^2,$$

$$\sum_{n=1}^m \Delta t e^{2\alpha n \Delta t} \| e_n \|_{\mathbb{R}}^2 = e^{2\alpha M \Delta t} (a + J_0)(e_m, e_m) - \sum_{n=1}^{m-1} e^{2\alpha \Delta t} (e^{2\alpha \Delta t} - 1) a + J_0(e_n, e_n),$$

the energy estimate follows.
and Lemmas 3.2, 3.3 we arrive at

\[ e^{2\alpha n \Delta t} \left( \| e_m \|^2 + C_2 \kappa \| e_m \|_\varepsilon^2 \right) + C_2 \nu \Delta t \sum_{n=1}^{m} e^{2\alpha n \Delta t} \| e_n \|_\varepsilon^2 \leq \sum_{n=1}^{m-1} e^{2\alpha n \Delta t} (e^{2\alpha \Delta t} - 1) (\| e_n \|^2 + C_1 \kappa \| e_n \|_\varepsilon^2) \]

\[ + \Delta t \sum_{n=1}^{m} e^{2\alpha n \Delta t} (C + C \| e_n-1 \|_\varepsilon^2) \| e_n-1 \|^2 + C \Delta t^2 \sum_{n=1}^{m} e^{2\alpha n \Delta t} \int_{t_{n-1}}^{t_n} \| u_{ht}(t) \|_\varepsilon^2 \, dt \]

\[ + C \Delta t^2 \sum_{n=1}^{m} e^{2\alpha n \Delta t} \int_{t_{n-1}}^{t_n} (\| u_{h_{tt}}(t) \|_\varepsilon^2_{-1,h} + \| u_{h_{tt}}(t) \|_\varepsilon^2) \, dt. \]

(6.19)

Using \( e^{2\alpha \Delta t} - 1 \leq C \Delta t \), the first term on right hand side of (6.19) can be merged with the second term on right hand side of (6.19). Apply Lemma 3.3 to observe that

\[ \sum_{n=1}^{m} e^{2\alpha n \Delta t} \int_{t_{n-1}}^{t_n} \| u_{ht}(t) \|_\varepsilon^2 \, dt = \sum_{n=1}^{m} \int_{t_{n-1}}^{t_n} e^{2\alpha (t-n) \Delta t} e^{2\alpha \Delta t} \| u_{ht}(t) \|_\varepsilon^2 \, dt \leq e^{2\alpha \Delta t} \int_{0}^{t_m} e^{2\alpha \Delta t} \| u_{h_{tt}}(t) \|_\varepsilon^2 \, dt \]

\[ \leq C e^{2\alpha (m+1) \Delta t}. \]

(6.20)

The last term on the right hand side of (6.19) can be bounded in a similar manner as in (6.20) by \( C \Delta t^2 e^{2\alpha (m+1) \Delta t} \). An application of (6.20) in (6.19) leads to

\[ e^{2\alpha \Delta t} \left( \| e_m \|^2 + C_2 \kappa \| e_m \|_\varepsilon^2 \right) + C_2 \nu \Delta t \sum_{n=1}^{m} e^{2\alpha n \Delta t} \| e_n \|_\varepsilon^2 \leq C \Delta t \sum_{n=1}^{m} e^{2\alpha (n-1) \Delta t} (1 + \| e_{n-1} \|_\varepsilon^2) (\| e_{n-1} \|^2 + \| e_{n-1} \|_\varepsilon^2) \]

\[ + C \Delta t^2 e^{2\alpha (m+1) \Delta t}. \]

An application of discrete Gronwall’s lemma completes the rest of the proof. \( \square \)

**Theorem 6.1.** Under the assumptions of the Theorem 3.1 and Lemma 6.2, the following holds true:

\[ \| u(t_n) - U^n \| \leq C e^{C \Delta t (h^2 + \Delta t)}, \]

\[ \| u(t_n) - U^n \|_\varepsilon \leq C e^{C \Delta t (h + \Delta t)}. \]

**Proof.** A combination of Theorem 3.1 and Lemma 6.2 leads to the desired result. \( \square \)

**Lemma 6.3.** Under the assumptions of Theorem 3.1 and Lemma 6.2, there exists a positive constant \( C = C(\kappa, \nu, \alpha, C_2, M_0) \), independent of \( h \) and \( \Delta t \), such that, the error \( e_n = U^n - u^n_h \), satisfies

\[ \| \partial_t e_n \| + C_2 \kappa \| \partial_t e_n \|_\varepsilon \leq C \Delta t. \]

**Proof.** Rewrite the non-linear terms in (6.11) as

\[ c^u_h(u^n_h, u^n_h, \phi_h) - c^{U_{n-1}}(U^{n-1}, U^n, \phi_h) = -c^u_h(u^n_h - u^{n-1}_h, e_n, \phi_h) + c^u_h(u^n_h - u^{n-1}_h, u^n_h - u^n, \phi_h) \]

\[ + c^u_h(u^n_h - u^{n-1}_h, u^n, \phi_h) - c^u_h(e_{n-1}, e_n, \phi_h) - c^u_h(e_{n-1}, u^n_h, \phi_h) \]

\[ + c^u_h(U^{n-1}, U^n, \phi_h) - c^{U_{n-1}}(U^{n-1}, U^n, \phi_h). \]

(6.21)

A use of (3.12) yield

\[ \| e^n_h(u^{n-1}_h, e_n, \phi_h) + e^n_h(e_{n-1}, e_n, \phi_h) + e^n_h(e_{n-1}, u^n_h, \phi_h) \| \]

\[ \leq C \| u^{n-1}_h \|_\varepsilon \| e_n \|_\varepsilon \| \phi_h \|_\varepsilon + C \| e_{n-1} \|_\varepsilon \| e_n \|_\varepsilon \| \phi_h \|_\varepsilon + C \| e_{n-1} \|_\varepsilon \| u^n_h \|_\varepsilon \| \phi_h \|_\varepsilon. \]

(6.22)

Similar to Lemma 6.2, using (3.11), Theorem 3.1 and Lemma 6.2, we bound the nonlinear terms as follows:

\[ \| c^u_h(u^n_h - u^{n-1}_h, u^n_h, \phi_h) + e^n_h(u^n_h - u^{n-1}_h, u^n, \phi_h) \| \leq C \| u^n_h - u^{n-1}_h \|_\varepsilon \| \phi_h \|_\varepsilon. \]

(6.23)
Further, using the similar steps as in Lemma 6.2 and the estimates of Theorem 3.1 we arrive at

\[
|c_{u_h}^n(U^{n-1}, U^n, \phi_h) - c_{u_h}^{U^n}(U^{n-1}, U^n, \phi_h)|
\leq \|u_h^n - U^{n-1}\|_{L^1(\Omega)}\|\textbf{U}^n - \textbf{u}^n\|_{\|\phi_h\|_{L^1(\Omega)}}
\leq \|u_h^n - U^{n-1}\|_{L^1(\Omega)}\|\textbf{e}_n\|_{\|\phi_h\|} + \|u_h^n - U^{n-1}\|_{L^1(\Omega)}\|\textbf{u}^n - \textbf{u}^n\|_{\|\phi_h\|}
\]

(6.24)

A use of Lemma 6.2 leads to

\[
|a(\textbf{e}_n, \phi_h) + J_0(\textbf{e}_n, \phi_h)| \leq C_1\|\textbf{e}_n\|_{\|\phi_h\|}.
\]

Applying Cauchy-Schwarz’ and Young’s inequalities to arrive at

\[
(u_h^n, \phi_h) - (\textbf{u}_h^n, \phi_h) \leq C\Delta t^{1/2}\left(\int_{t_{n-1}}^{t_n} \|\textbf{u}_{hTT}(s)\|_{L^1_h}^2 ds\right)^{1/2} \|\phi_h\|_\varepsilon
\]

(6.26)

Following the similar analysis as in (6.26), we obtain

\[
\kappa (a(u_h^n, \phi_h) + J_0(u_h^n, \phi_h) - a(\textbf{u}_h^n, \phi_h) - J_0(\textbf{u}_h^n, \phi_h)) \leq C \left(\sup_{0 < t < \infty} \|\textbf{u}_{hTT}(t)\|_{L^1_h}^2\right)\left(\int_{t_{n-1}}^{t_n} 1 ds\right)^{1/2} \Delta t^{1/2} \|\phi_h\|_\varepsilon.
\]

(6.27)

Substitute \(\phi_h = \textbf{e}_n\) in (6.7) and use (6.21)–(6.27) with \(\phi_h\) replaced by \(\textbf{e}_n\). Then, applying Young’s inequality and estimates from Lemmas 3.3, 3.8, 6.2 to arrive at the desired result. This completes the rest of the proof. □

**Lemma 6.4.** Under the assumptions of Theorem 3.1 and Lemma 6.2, there exists a positive constant \(C = C(\kappa, \nu, \alpha, \epsilon, M_0)\), independent of \(h\) and \(\Delta t\), such that for \(n = 1, 2, \cdots, M\)

\[
\|P^n - p^n_h\| \leq C\Delta t.
\]

**Proof.** Consider (6.7) at \(t = t_n\) and subtract it from (6.1) to arrive at

\[
b(\phi_h, P^n - p^n_h) = (u_h^n, \phi_h) - (\textbf{u}_h^n, \phi_h) + \kappa (a(u_h^n, \phi_h) + J_0(u_h^n, \phi_h) - a(\textbf{u}_h^n, \phi_h) - J_0(\textbf{u}_h^n, \phi_h)) - (\textbf{e}_n, \phi_h) - \kappa (a(\textbf{e}_n, \phi_h) + J_0(\textbf{e}_n, \phi_h) - \nu(a(\textbf{e}_n, \phi_h) + J_0(\textbf{e}_n, \phi_h))
\]

(6.28)

Following the steps involved in the proof of Lemma 6.3 and applying Lemma 6.4, we obtain

\[
\|P^n - p^n_h\| \leq C(||\textbf{e}_n|| + \kappa ||\textbf{e}_n||_\varepsilon + ||\textbf{e}_n||_\varepsilon + \Delta t).
\]

(6.29)

Finally, an application of the Lemmas 6.2 and 6.3 concludes the proof. □

A combination of Lemma 6.4 and Theorem 3.2 lead to the following fully discrete pressure error estimates.

**Theorem 6.2.** Under the assumptions of Theorem 3.2 and Lemma 6.4, the following hold true:

\[
\|p(t_n) - P^n\| \leq C(h + \Delta t).
\]
7 Numerical experiments

In this section, we present a couple of numerical experiments to verify the theoretical results stated in the Theorems 6.1 and 6.2. The domain $\Omega$ is considered as $[0,1] \times [0,1]$. We use the mixed finite element spaces $P_1 - P_0$ and $P_1 - P_1$ for the space discretization and a first order accurate backward Euler method for the time discretization. The time interval is chosen as $[0,1]$ with final time $T = 1$.

Example 7.1. In our first example, the right hand side function $f$ is chosen in such a way that the exact solution is

$$ u = (2x^2(x-1)^2y(y-1)(2y-1)\cos t, -2x(x-1)(2x-1)y^2(y-1)^2\cos t), \quad p = 2(x - y)\cos t. $$

Table 1 shows the errors and the convergence rates for the mixed finite element space $P_1 - P_0$ with kinematic viscosities $\nu = 1$ and $1/10$, respectively. And Figure 1 represents the velocity and pressure errors for $P_1 - P_0$ element with $\nu = 1$ and $1/10$. In Table 3, we choose $P_1 - P_1$ mixed finite element space with $\nu = 1$. Note that, we choose a constant penalty parameter $\sigma_e = 10$ and retardation $\kappa = 10^{-2}$ for Tables 1-3. In Table 4, we have represented the errors and the convergence rates for the backward Euler method applied to continuous Galerkin finite element method with $\nu = 1$ and $\kappa = 10^{-2}$. It can be observed that the numerical results represented in Tables 1-3 and Figure 1 validate our theoretical findings in Theorems 6.1 and 6.2. Further, the results in Tables 3 and 4 represent that the discontinuous Galerkin finite element method works well for equal order element $P_1 - P_1$, whereas continuous Galerkin finite element method fails to approximate the exact solution.

Table 1: Numerical errors and convergence rates, for $P_1 - P_0$ ($\nu=1, \kappa = 10^{-2}, \sigma_e = 10$) for Example 7.1 with $\Delta t = O(h^2)$.

| $h$  | $\|u(T) - U^M\|_e$ Rate | $\|u(T) - U^M\|_{L^2(\Omega)}$ Rate | $\|p(T) - P^M\|_{L^2(\Omega)}$ Rate |
|------|--------------------------|-----------------------------------|-----------------------------------|
| 1/4  | 3.3927 x 10^{-2}         | 2.4193 x 10^{-3}                  | 1.0667 x 10^{-2}                  |
| 1/8  | 1.4413 x 10^{-2}         | 1.2350                            | 7.0680 x 10^{-3}                  |
| 1/16 | 6.1566 x 10^{-3}         | 1.2271                            | 4.1982 x 10^{-3}                  |
| 1/32 | 2.7335 x 10^{-3}         | 1.1713                            | 2.3113 x 10^{-3}                  |
| 1/64 | 1.2676 x 10^{-3}         | 1.1086                            | 1.2137 x 10^{-3}                  |

Table 2: Numerical errors and convergence rates, for $P_1 - P_0$ ($\nu = 1/10, \kappa = 10^{-2}, \sigma_e = 10$) for Example 7.1 with $\Delta t = O(h^2)$.

| $h$  | $\|u(T) - U^M\|_e$ Rate | $\|u(T) - U^M\|_{L^2(\Omega)}$ Rate | $\|p(T) - P^M\|_{L^2(\Omega)}$ Rate |
|------|--------------------------|-----------------------------------|-----------------------------------|
| 1/4  | 1.5785 x 10^{-1}         | 1.2098 x 10^{-2}                  | 1.0790 x 10^{-2}                  |
| 1/8  | 6.6764 x 10^{-2}         | 1.2414                            | 7.4719 x 10^{-3}                  |
| 1/16 | 2.9235 x 10^{-2}         | 1.1913                            | 4.3042 x 10^{-3}                  |
| 1/32 | 1.3373 x 10^{-2}         | 1.1283                            | 2.3030 x 10^{-3}                  |
| 1/64 | 6.3422 x 10^{-3}         | 1.0762                            | 1.1906 x 10^{-3}                  |

Example 7.2. In this example, we take the forcing term $f$ such that the solution of the problem to be

$$ u = (e^l(-\cos(2\pi x)\sin(2\pi y) + \sin(2\pi y)), e^l\sin(2\pi x)\cos(2\pi y) - \sin(2\pi x)), \quad p = e^l(2\pi(\cos(2\pi y) - \cos(2\pi x))). $$

In Tables 5 and 6, we have shown the errors and convergence rates for the mixed finite element space $P_1 - P_0$ with viscosities $\nu = 1$ and $1/10$, respectively. And Figure 2 represents the velocity and pressure errors for $P_1 - P_0$
Table 3: Numerical errors and convergence rates, for \( P_1 - P_0 \) (\( \nu = 1 \), \( \kappa = 10^{-2} \), \( \sigma_e = 10 \)) for Example 7.1 with \( \Delta t = O(h^2) \).

| \( h \)   | \( \| u(T) - U^M \|_\varepsilon \) | Rate | \( \| u(T) - U^M \|_{L^2(\Omega)} \) | Rate | \( \| p(T) - P^M \|_{L^2(\Omega)} \) | Rate |
|----------|---------------------------------|------|---------------------------------|------|---------------------------------|------|
| 1/4      | \( 1.0869 \times 10^{-2} \)     |       | \( 7.9834 \times 10^{-4} \)     | 0.8226 | \( 2.6782 \times 10^{-2} \)     |       |
| 1/8      | \( 6.1457 \times 10^{-3} \)     | 0.8226 | \( 4.4474 \times 10^{-4} \)     | 0.8440 | \( 2.1336 \times 10^{-2} \)     | 0.3279 |
| 1/16     | \( 2.6823 \times 10^{-3} \)     | 1.1961 | \( 1.5730 \times 10^{-4} \)     | 1.4993 | \( 1.2302 \times 10^{-2} \)     | 0.7943 |
| 1/32     | \( 1.1742 \times 10^{-3} \)     | 1.1916 | \( 4.4715 \times 10^{-5} \)     | 1.8147 | \( 6.5318 \times 10^{-3} \)     | 0.9133 |
| 1/64     | \( 5.5135 \times 10^{-4} \)     | 1.0907 | \( 1.1724 \times 10^{-5} \)     | 1.9312 | \( 3.3487 \times 10^{-3} \)     | 0.9638 |

For the cases \( \nu = 1 \) and \( \nu = 1/10 \), we have chosen \( \kappa = 10^{-2} \) and \( \kappa = 10^{-3} \), respectively. Table 7 depicts the results for the mixed space \( P_1 - P_1 \) with \( \nu = 1 \) and \( \kappa = 10^{-2} \). We choose a constant penalty parameter \( \sigma_e = 10 \) for Tables 5-7. For continuous finite element method, the errors and the convergence rates for the case \( P_1 - P_1 \) with \( \nu = 1 \) and \( \kappa = 10^{-2} \) have shown in Table 8 for Example 7.2. Here again, the numerical outcomes verify the derived theoretical results. Also, it can be concluded that the discontinuous Galerkin finite element method works well for equal order elements in comparison to the continuous Galerkin method.
Table 4: Numerical errors and convergence rates for continuous finite element method using $P_1-P_1$ ($\nu = 1$, $\kappa = 10^{-2}$) for Example 7.1 with $\Delta t = O(h^2)$.

| $h$  | $\|u(T) - U^M\|_{H^1(\Omega)}$ Rate | $\|u(T) - U^M\|_{L^2(\Omega)}$ Rate | $\|p(T) - P^M\|_{L^2(\Omega)}$ Rate |
|------|------------------------------------|------------------------------------|------------------------------------|
| 1/4  | $1.8439 \times 10^{-2}$            | $1.9654 \times 10^{-3}$            | $0.20708$                          |
| 1/8  | $8.4605 \times 10^{-3}$           | $1.61972 \times 10^{-4}$           | $0.11661$                          |
| 1/16 | $3.8693 \times 10^{-3}$           | $1.5829 \times 10^{-4}$            | $0.07076$                          |
| 1/32 | $1.8594 \times 10^{-3}$           | $3.9421 \times 10^{-5}$            | $0.05129$                          |
| 1/64 | $9.1443 \times 10^{-4}$           | $9.8055 \times 10^{-6}$            | $0.04443$                          |

Table 5: Numerical errors and convergence rates, for $P_1-P_0$ ($\nu = 1$, $\kappa = 10^{-2}$, $\sigma_e = 10$) for Example 7.2 with $\Delta t = O(h^2)$.

| $h$  | $\|u(T) - U^M\|_e$ Rate | $\|u(T) - U^M\|_{L^2(\Omega)}$ Rate | $\|p(T) - P^M\|_{L^2(\Omega)}$ Rate |
|------|------------------------|------------------------------------|------------------------------------|
| 1/4  | $9.17093$              | $0.72430$                           | $9.37006$                          |
| 1/8  | $4.82057$              | $0.31886$                           | $6.19123$                          |
| 1/16 | $2.21524$              | $0.10630$                           | $4.15326$                          |
| 1/32 | $1.03043$              | $0.02973$                           | $2.38552$                          |
| 1/64 | $0.50189$              | $0.00771$                           | $1.25309$                          |

Table 6: Numerical errors and convergence rates, for $P_1-P_0$ ($\nu = 1/10$, $\kappa = 10^{-3}$, $\sigma_e = 10$) for Example 7.2 with $\Delta t = O(h^2)$.

| $h$  | $\|u(T) - U^M\|_e$ Rate | $\|u(T) - U^M\|_{L^2(\Omega)}$ Rate | $\|p(T) - P^M\|_{L^2(\Omega)}$ Rate |
|------|------------------------|------------------------------------|------------------------------------|
| 1/4  | $11.91970$             | $1.20830$                           | $5.89632$                          |
| 1/8  | $6.27597$              | $0.39185$                           | $2.34282$                          |
| 1/16 | $2.98342$              | $0.10691$                           | $0.83518$                          |
| 1/32 | $1.45801$              | $0.02704$                           | $0.33008$                          |
| 1/64 | $0.72815$              | $0.00673$                           | $0.14996$                          |

Table 7: Numerical errors and convergence rates, for $P_1-P_1$ ($\nu = 1$, $\kappa = 10^{-2}$, $\sigma_e = 10$) for Example 7.2 with $\Delta t = O(h^2)$.

| $h$  | $\|u(T) - U^M\|_e$ Rate | $\|u(T) - U^M\|_{L^2(\Omega)}$ Rate | $\|p(T) - P^M\|_{L^2(\Omega)}$ Rate |
|------|------------------------|------------------------------------|------------------------------------|
| 1/4  | $9.68528$              | $0.84240$                           | $20.72070$                         |
| 1/8  | $5.42842$              | $0.45434$                           | $18.31680$                         |
| 1/16 | $2.26422$              | $0.15667$                           | $10.42180$                         |
| 1/32 | $0.94989$              | $0.04369$                           | $5.44668$                          |
| 1/64 | $0.43939$              | $0.01130$                           | $2.76205$                          |

Example 7.3. (2D Lid Driven Cavity Flow Benchmark Problem). In this example, we consider a benchmark problem related to a lid driven cavity flow on a unit square with zero body forces. Further, no slip boundary
conditions are considered everywhere, except non zero velocity \((u_1, u_2) = (1, 0)\) on the upper part of boundary, that is, the lid of the cavity is moving horizontally with a prescribed velocity. For numerical experiments, we have chosen lines \((0.5, y)\) and \((x, 0.5)\). We choose \(P_1 - P_0\) mixed finite element space for the space discretization. In Figure 3 we have presented the comparison between unsteady backward Euler and steady state velocities, whereas Figure 4 depicts the comparison of backward Euler and steady state pressure for different values of viscosity \(\nu = \{1/100, 1/300, 1/600\}\), final time \(T = 75\), \(h = 1/32\) and \(\Delta t = \mathcal{O}(h^2)\) with \(\sigma_e = 40\) and \(\kappa = 0.1 \times \nu\). From the graphs, it is observed that the Kelvin-Voigt solutions converge to its steady state solutions for large time.
8 Summary

In this paper, we have applied the symmetric interior penalty discontinuous Galerkin method to the Kelvin-Voigt equations of motion represented by (1.1)-(1.3). We have defined the semidiscrete discontinuous Galerkin formulation to (1.1)-(1.3) and have derived a priori bounds to the velocity approximation. In order to establish error estimates, we have introduced the $L^2$-projection $P_h$ and a modified Sobolev-Stokes projection $S_h$ on appropriate DG spaces and proved the approximation properties. Note that, the bounds of $P_h$ play a crucial role in deriving the bounds for $S_h$. Then, by using regularity estimates for the weak solution and duality arguments along with the approximation properties of $P_h$ and $S_h$, we have obtained optimal error estimates for the velocity in $L^\infty(L^2)$ and pressure in $L^\infty(L^2)$-norms. Moreover, under the smallness assumption on the data, we have shown that the semidiscrete error estimates are uniform in time. Further, we have employed a backward Euler method for full discretization and have achieved optimal convergence rates for the approximate solution. Finally, we have conducted the numerical experiments and have shown that the outcomes verify the theoretical results.

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