MINIMAL MODIFIED ENERGY CONTROL
FOR FRACTIONAL LINEAR CONTROL SYSTEMS
WITH THE CAPUTO DERIVATIVE

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Abstract. Fractional control systems with the Caputo derivative are considered. The modified controllability Gramian and the minimum energy optimal control problem are investigated. Construction of minimizing steering controls for the modified energy functional are proposed.

1. Introduction

The Fractional Calculus is an important Mathematical discipline [18, 19, 21]. Several recent books on the subject have been written, illustrating the usefulness of the theory in applications [8, 16, 20, 24, 26]. Different notions of fractional-order derivatives are available, including the Riemann–Liouville, the Grünwald–Letnikov, and Caputo, as well as the generalized functions approach. One of the youngest fractional derivative, formulated in 1967, is the Caputo derivative. The main advantage in using the Caputo derivative is that it avoids problems connected with initial conditions of fractional differential equations. Indeed, when an initial value problem is formulated for the Caputo derivative, initial conditions look like in the classical (non-fractional) case.

The state-space description of fractional order control systems is developed in [7, 12, 14, 15, 22, 23]. Controllability and observability of finite-dimensional fractional differential systems are investigated mainly in [17], for systems with Riemann–Liouville derivatives. For positive continuous-time linear systems, the reachability property has been worked out in [14]. Differently, here we deal with the controllability property of finite-dimensional linear fractional-order differential systems via the Caputo derivative. The novelty is the construction of control laws corresponding to the minimal modified energy, where we use a neutralizer of the singularity similarly to the one defined in [17].

The paper is organized as follows. In Section 2 we recall the main properties of the generalized Mittag–Leffler function and the fractional analogue for the exponential matrix. We also review some properties of fractional integrals and derivatives. In Section 3 we discuss the Gramian formula for fractional control systems and we prove a new result about the minimal modified energy steering control law (Theorem 3.1). Illustrative examples are discussed. In Section 4 we recall the classical conditions for controllability and propose the construction of a different control law for fractional control systems (Theorem 4.3).
2. Preliminaries

We investigate control systems with commensurate order \( \alpha \in (0, 1] \). For \( \alpha = 1 \) the classical situation is obtained; the novelty appears when \( \alpha \in (0, 1) \).

**Definition 2.1** ([10]). We denote by \( E_{\alpha, \beta} \) the two-parameter Mittag–Leffler function defined by the series expansion

\[
E_{\alpha, \beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha + \beta)}, \quad \alpha > 0, \; \beta > 0.
\]

When \( \beta = 1 \), we use the notation \( E_\alpha = E_{\alpha,1} \). Let \( A \in \mathbb{R}^{n \times n} \). We extend (2.1) to the matrix case as follows:

\[
E_{\alpha, \beta}(At^\alpha) = \sum_{k=0}^{\infty} A^k \frac{t^{k\alpha}}{\Gamma(k\alpha + \beta)}.
\]

**Definition 2.2** ([16]). Let \( A \in \mathbb{R}^{n \times n} \). By

\[
e_\alpha^A = t^{\alpha-1} \sum_{k=0}^{\infty} A^k \frac{t^{k\alpha}}{\Gamma(k+1)\alpha} = \sum_{k=0}^{\infty} A^k \frac{t^{(k+1)\alpha-1}}{\Gamma(k+1)\alpha} = t^{\alpha-1} E_{\alpha, \alpha}(At^\alpha)
\]

we denote the \( \alpha \)-exponential matrix function.

For \( \alpha = 1 \) we have \( E_1(At) = e_1^A = \exp(At) \), where \( \exp \) denotes the classical exponential matrix. We remark that the properties \( \exp(A+B)t = \exp(At)\exp(Bt) \) and \( \exp^{-1}(At) = \exp(-At) \), satisfied for square matrices \( A \) and \( B \), are not valid for functions \( e_\alpha^A \) and \( E_{\alpha, \alpha}(At^\alpha) \).

**Definition 2.3** ([16] [19] [21]). Let \( \varphi \in L_1([t_0, t_1], \mathbb{R}) \). The integrals

\[
I_{t_0+}^{\alpha}\varphi(t) = \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} \varphi(\tau)(t - \tau)^{\alpha-1}d\tau, \quad t > t_0,
\]

\[
I_{t_1-}^{\alpha}\varphi(t) = \frac{1}{\Gamma(\alpha)} \int_{t}^{t_1} \varphi(\tau)(t - \tau)^{\alpha-1}d\tau, \quad t < t_1,
\]

where \( \Gamma \) is the gamma function and \( \alpha > 0 \), are called, respectively, the left-sided and the right-sided fractional integrals of order \( \alpha \). Additionally, we define the identity operator \( I \) by \( I := I_{t_0+}^0 = I_{t_1-}^0 \).

We have the following integration by parts formula for fractional integrals.

**Proposition 2.1** ([16]). Let \( \alpha > 0 \) and \( 1/p + 1/q \leq 1 + \alpha, \; p \geq 1, \; q \geq 1 \), with \( p \neq 1 \) and \( q \neq 1 \) in the case \( 1/p + 1/q = 1 + \alpha \). Then, for \( \varphi \in L_p([t_0, t_1], \mathbb{R}) \) and \( \psi \in L_q([t_0, t_1], \mathbb{R}) \), the following equality holds:

\[
\int_{t_0}^{t_1} \varphi(\tau)I_{t_0+}^{\alpha}\psi(\tau)d\tau = \int_{t_0}^{t_1} \psi(\tau)I_{t_1-}^{\alpha}\varphi(\tau)d\tau.
\]

**Definition 2.4** ([16] [19]). Let \( \varphi \) be defined on the interval \([t_0, t_1] \). The left-sided Riemann–Liouville derivative of order \( \alpha \) with lower limit \( t_0 \) is defined by

\[
D_{t_0+}^{\alpha}\varphi(t) = \frac{1}{\Gamma(n-\alpha)} \left( \frac{d}{dt} \right)^n \int_{t_0}^{t} \varphi(\tau)(t - \tau)^{n-\alpha-1}d\tau,
\]

where \( n \) is the natural number satisfying \( n = [\alpha] + 1 \) (\( [\alpha] \) denotes the integer part of \( \alpha \)). Similarly, the right-sided Riemann–Liouville derivative of order \( \alpha \) with upper limit \( t_1 \) is defined by

\[
D_{t_1-}^{\alpha}\varphi(t) = \frac{1}{\Gamma(n-\alpha)} \left( -\frac{d}{dt} \right)^n \int_{t}^{t_1} \varphi(\tau)(\tau - t)^{n-\alpha-1}d\tau.
\]
The next proposition is based on [21] Corollary 2, p. 46 and is particularly useful for our purposes.

**Proposition 2.2** (Integration by parts). Let \( f \in L^p_t(L^q) \) and \( g \in L^q_t(L^p) \) with \( 1/p + 1/q \leq 1 + \alpha \). The following formula holds:

\[
\int_{t_0}^{t_1} f(t) D^\alpha_{t_0+} g(t) dt = \int_{t_0}^{t_1} g(t) D^\alpha_{t_0+} f(t) dt + 0 < \alpha < 1.
\]

**Proof.** Denote \( D^\alpha_{t_0+} f(t) = \varphi(t) \) and \( D^\alpha_{t_0+} g(t) = \psi(t) \). The equality (2.5) follows from (2.2) since \( I^\alpha_{t_0+} D^\alpha_{t_0+} f(t) = f(t) \) is valid for \( f \in L^p_t(L^q) \) (cf. [21]). \( \Box \)

In the description of fractional control systems we use the notion of Caputo derivative, which is the preferred fractional derivative among Engineers.

**Definition 2.5** ([16]). The left- and right-sided Caputo fractional derivatives of order \( \alpha \geq 0 \) on \([0, t_1] \), denoted by \( C D^\alpha_{t_0+} \varphi(t) \) and \( C D^\alpha_{t_0-} \varphi(t) \), respectively, are defined via the Riemann–Liouville fractional derivatives (2.3) and (2.4) by

\[
C D^\alpha_{t_0+} \varphi(t) = D^\alpha_{t_0+} \left( \varphi(t) - \sum_{k=0}^{n-1} \frac{\varphi^{(k)}(t_0)(t - t_0)^k}{k!} \right),
\]

and

\[
C D^\alpha_{t_0-} \varphi(t) = D^\alpha_{t_0-} \left( \varphi(t) - \sum_{k=0}^{n-1} \frac{\varphi^{(k)}(t_0)(t - t_0)^k}{k!} \right),
\]

where \( n = [\alpha] + 1 \) for \( \alpha \notin \mathbb{N}_0 \) and \( n = \alpha \) for \( \alpha \in \mathbb{N}_0 \).

When \( 0 < \alpha < 1 \) the relations (2.6) and (2.7) take the following form:

\[
C D^\alpha_{t_0+} \varphi(t) = D^\alpha_{t_0+} \left( \varphi(t) - \varphi(t_0) \right), \quad C D^\alpha_{t_0-} \varphi(t) = D^\alpha_{t_0-} \left( \varphi(t) - \varphi(t_1) \right).
\]

Let \( AC[t_0, t_1] \) be the space of functions that are absolutely continuous on \([t_0, t_1] \) and \( AC^n[t_0, t_1] \) denote the space of functions \( \varphi \) that have continuous derivatives up to order \( n-1 \) on \([t_0, t_1] \) and such that \( \varphi^{(n-1)} \in AC[a, b] \).

**Proposition 2.3** ([16]). Let \( \alpha \geq 0, n = [\alpha] + 1 \) if \( \alpha \notin \mathbb{N}_0 \) and \( n = \alpha \) if \( \alpha \in \mathbb{N}_0 \). If \( \varphi \in AC^n[t_0, t_1] \), then the Caputo fractional derivatives \( C D^\alpha_{t_0+} \varphi(t) \) and \( C D^\alpha_{t_0-} \varphi(t) \) exist almost everywhere on \([t_0, t_1] \). Moreover,

(a) If \( \alpha \notin \mathbb{N}_0 \), then

\[
C D^\alpha_{t_0+} \varphi(t) = \frac{1}{\Gamma(n - \alpha)} \int_{t_0}^{t} \varphi^{(n)}(\tau)(t - \tau)^{n-\alpha-1} d\tau = I^{n-\alpha}_{t_0+} \left( \varphi^{(n)}(\cdot) \right)(t),
\]

\[
C D^\alpha_{t_0-} \varphi(t) = (-1)^n \frac{1}{\Gamma(n - \alpha)} \int_{t}^{t_1} \varphi^{(n)}(\tau)(\tau - t)^{n-\alpha-1} d\tau = (-1)^n I^{n-\alpha}_{t_0-} \left( \varphi^{(n)}(\cdot) \right)(t).
\]

(b) If \( \alpha = n \in \mathbb{N}_0 \), then \( C D^\alpha_{t_0+} \varphi(t) = \varphi^{(n)}(t) \) and \( C D^\alpha_{t_0-} \varphi(t) = (-1)^n \varphi^{(n)}(t) \).

For a function \( x : [0, T] \to \mathbb{R}^n \) we use similar notation as in the classical case:

\[
C D^\alpha_{t_0+} x(t) = C D^\alpha_{t_0+} \left( \begin{array}{c} x_1(t) \\ \vdots \\ x_n(t) \end{array} \right) = \left( \begin{array}{c} C D^\alpha_{t_0+} x_1(t) \\ \vdots \\ C D^\alpha_{t_0+} x_n(t) \end{array} \right).
\]

Such situation, when for each component we use the same fractional order \( \alpha \) of differentiation (in the Riemann–Liouville or Caputo sense), is known in the literature as the fractional derivative with commensurate order [7] [11] [22].

**Proposition 2.4.** For \( \alpha > 0 \) the following holds:

(i) \( C D^\alpha_{t_0+} E_\alpha(\Lambda (t-t_0)^\alpha) = \Lambda E_\alpha(\Lambda (t-t_0)^\alpha) \);
Proof. (i) Directly from the definition of the classical Mittag–Leffler function and from the formula of the Caputo derivative of a power function, one has:

\[ CD_{t_0}^\alpha(t-t_0)^\beta = \frac{\Gamma(\beta + 1)}{\Gamma(\beta - \alpha + 1)}(t-t_0)^{\beta - \alpha}, \quad \beta \neq 0. \]

Since the Caputo derivative of a constant function is zero,

\[ CD_{t_0}^\alpha, E_\alpha(t-t_0)^\alpha = CD_{t_0}^\alpha, \sum_{k=0}^\infty A_k \frac{(t-t_0)^{k\alpha}}{\Gamma((k+1)\alpha)} = \sum_{k=1}^\infty A_k \frac{(t-t_0)^{(k-1)\alpha}}{\Gamma((k+1)\alpha)} = AE_\alpha(t-t_0)^\alpha. \]

(ii) As the formula for the Riemann–Liouville derivative of a power function is the same as for the Caputo derivative, we have:

\[ D_{t_0}^\alpha e^{A(t-t_0)\alpha} = D_{t_0}^\alpha, \sum_{k=0}^\infty A_k \frac{(t-t_0)^{k\alpha}}{\Gamma((k+1)\alpha)} = \sum_{k=1}^\infty A_k \frac{(t-t_0)^{(k-1)\alpha}}{\Gamma((k+1)\alpha)} = AE_\alpha(t-t_0)^\alpha, \]

where we used the fact that \( \lim_{\alpha \to 0} \frac{1}{\Gamma(\alpha)} = 0. \)

(iii) Using the formulas

\[ D_{T-}^\alpha (T-\tau)^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta - \alpha)}(T-\tau)^{\beta - \alpha - 1}, \quad \lim_{\beta \to \alpha} D_{T-}^\beta (T-\tau)^{\alpha - 1} = 0 \]

(cf., e.g., [21]), we get:

\[ D_{T-}^\alpha S(T-\tau) = D_{T-}^\alpha \left( I \frac{1}{\Gamma(\alpha)}(T-\tau)^{\alpha - 1} + A \frac{(T-\tau)^{2\alpha - 1}}{\Gamma(2\alpha)} + \cdots \right) \]

\[ = A \frac{(T-\tau)^{\alpha - 1}}{\Gamma(\alpha)} + A^2 \frac{(T-\tau)^{2\alpha - 1}}{\Gamma(2\alpha)} + \cdots \]

\[ = (T-\tau)^{\alpha - 1} A \sum_{k=0}^\infty A_k \frac{(T-\tau)^{k\alpha}}{\Gamma((k+1)\alpha)} \]

\[ = AS(T-\tau). \]

\[ \square \]

Proposition 2.5. For \( \alpha > 0 \) the following relation holds:

\[ E_\alpha(t-t_0)^\alpha = I + \int_{t_0}^t A e^{A(t-\tau)} d\tau. \]

Proof. Follows by direct calculation of the integral:

\[ \int_{t_0}^t A e^{A(t-\tau)} d\tau = \int_{t_0}^t \sum_{k=0}^\infty A^{k+1} \frac{(t-\tau)^{(k+1)\alpha - 1}}{\Gamma((k+1)\alpha)} d\tau \]

\[ = \sum_{k=1}^\infty A^k \frac{(t-t_0)^{k\alpha}}{\Gamma(k\alpha + 1)} = E_\alpha(A(t-t_0)^\alpha) - I. \]

\[ \square \]

Since \( e^{At} = t^{\alpha - 1}E_{\alpha,\alpha}(At^\alpha) \) and each Mittag–Leffler function \( E_{\alpha,\alpha}(az^\alpha), \alpha > 0, \) is an entire function on the complex plane, we can state the following:

Proposition 2.6. Let \( \alpha > 0. \) There is a uniquely determined function \( g(t) = t^{1-\alpha}G(t) \) such that \( e^{At}g(t) = E_{\alpha,\alpha}(At^\alpha)G(t) = I, \) for \( t \neq 0, \) and \( \lim_{t \to 0} e^{At}g(t) = I. \)
Lemma 2.1. Let \( \alpha > 0 \) and \( \psi(t) \in \mathbb{L}^p_{2+}(L_q) \), where \( 1/p + 1/q \leq 1 + \alpha \) for \( p \) such that all components of \( S(T - t) \) belong to \( \mathbb{L}^p_{2-}(L_p) \). Then,
\[
\int_0^T S(T - \tau)D_{0+}^\alpha \psi(\tau)d\tau = \int_0^T AS(T - \tau)\psi(\tau)d\tau.
\]

Proof. Taking into account \((2.5)\) and item (iii) of Proposition \(2.3\) we have
\[
\int_0^T S(T - \tau)D_{0+}^\alpha \psi(\tau)d\tau = \int_0^T \psi(\tau)D_{0-}^\alpha S(T - \tau)d\tau = \int_0^T AS(T - \tau)\psi(\tau)d\tau.
\]
\(\square\)

3. Control systems and the Gramian

We consider the following linear time-invariant control system of order \( \alpha \in (0, 1] \), denoted by \( \Sigma \):
\[
C D_{0+}^\alpha x(t) = Ax(t) + Bu(t), \quad y = Cx(t),
\]
where \( x(t) \in \mathbb{R}^n \), \( u(t) \in \mathbb{R}^m \), matrix \( A \in M_{n \times n}(\mathbb{R}) \), \( B \in M_{n \times m}(\mathbb{R}) \), \( C \in M_{p \times n}(\mathbb{R}) \), and \( C D_{0+}^\alpha \) indicates the fractional Caputo derivative of commensurate order \( \alpha \).

The forward trajectory of the system \( \Sigma \), starting at \( t_0 = 0 \) and evaluated at \( t \geq 0 \), is the solution of the initial value problem \( C D_{0+}^\alpha x(t) = Ax(t) + Bu(t), \ x(0) = a \in \mathbb{R}^n \)[10]:
\[
\gamma(t, a, u) = \left( I + \int_0^t S(\tau)Ad\tau \right) a + \int_0^t S(t - \tau)Bu(\tau)d\tau,
\]
where \( S(t) = e_{\alpha t}^A \). Moreover, we can represent \((3.8)\) in the following way:
\[
\gamma(t, a, u) = S_0(t)a + \int_0^t S(t - \tau)Bu(\tau)d\tau,
\]

where \( S_0(t) = E_{\alpha}(At) = I + \int_0^t S(\tau)Ad\tau \). The formula for the forward trajectory can be obtained using the Laplace transform [14]. Taking into account the output of \( \Sigma \), the forward output trajectory is then defined by values evaluated at \( t \geq 0 \):
\[
\eta(t, a, u) = C\gamma(t, a, u) = C \left( I + \int_0^t S(\tau)Ad\tau \right) a + C \int_0^t S(t - \tau)Bu(\tau)d\tau.
\]

For system \( \Sigma \) we define the notion of controllability in the standard manner:

Definition 3.6. Let \( T > 0 \). The system \( \Sigma \) is controllable on \([0, T]\) if for any \( a \in \mathbb{R}^n \) and \( b \in \mathbb{R}^m \) there is a control \( u(\cdot) \) defined on \([0, T]\) which steers the initial state \( \gamma(0, a, u) = a \) to the final state \( \gamma(T, a, u) = b \).

Following [17, 25] we denote by
\[
Q_T = \int_0^T S(T - t)BB^*S^*(T - t)(T - t)^{2(1-\alpha)}dt
\]
the controllability Gramian of fractional order \( \alpha \), on the time interval \([0, T]\), corresponding to the system \( \Sigma \). As in the classical case [25], \( Q_T \) is symmetric and nonnegative definite. The term \((T - t)^{2(1-\alpha)}\) under the integral is called in [17] a neutralizer of the singularity at \( t = T \). It is needed in order to ensure the convergence of the integral.

Let \( T > 0 \). By \( L^2([0, T], \mathbb{R}^m) \) we denote the set of functions \( \varphi: [0, T] \to \mathbb{R}^m \) such that \( \tilde{\varphi} \) defined by \( \tilde{\varphi}(t) = (T - t)^{\alpha-1}\varphi(t) \) is square integrable on \([0, T]\).

Theorem 3.1. Let \( T > 0 \) and \( Q_T \) be nonsingular. Then,
\( (a) \) for any states \( a, b \in \mathbb{R}^n \) the control law
\begin{equation}
\mathfrak{u}(t) = -(T - t)^{2(1 - \alpha)} B^* S^*(T - t) Q_T^{-1} f_T(a, b), \quad t \in [0, T),
\end{equation}
where
\[
f_T(a, b) = \left( I + \int_0^T S(t)Adt \right) a - b = -b + S_0(T)a
\]
and \( \mathfrak{u}(T) = 0 \), drives point \( a \) to point \( b \) in time \( T \);
\( (b) \) among all possible controls from \( L^2_\alpha([0, T], \mathbb{R}^n) \) driving \( a \) to \( b \) in time \( T \),
the control \( \mathfrak{u} \) defined by (3.9) minimizes the integral
\begin{equation}
\int_0^T |(T - t)^{\alpha - 1} u(t)|^2 dt.
\end{equation}

Moreover,
\[
\int_0^T |(T - t)^{\alpha - 1} \mathfrak{u}(t)|^2 dt = \langle Q_T^{-1} f_T(a, b), f_T(a, b) \rangle >,
\]
where \( \langle \cdot, \cdot \rangle \) denotes the inner product.

**Proof.** (a) From the expression of \( \mathfrak{u} \) we have that
\[
\gamma(T, a, \mathfrak{u}) = f_T(a, b) + b - \left( \int_0^T (T - t)^{2(1 - \alpha)} S(T - t) B B^* S^*(T - t) dt \right) Q_T^{-1} f_T(a, b)
\]
\[
= f_T(a, b) + b - Q_T Q_T^{-1} f_T(a, b) = b.
\]
Moreover, \( \lim_{t \to T^-} \mathfrak{u}(t) = \mathfrak{u}(T) \) since \( \lim_{t \to T^-} (T - t)^{2(1 - \alpha)} B^* S^*(T - t) Q_T^{-1} f_T(a, b) \).

(b) We notice that for \( h(t) = |(T - t)|^{2(1 - \alpha)} \)
\[
\int_0^T |(T - t)^{\alpha - 1} \mathfrak{u}(t)|^2 dt = \int_0^T |(T - t)^{\alpha - 1} B^* S^*(T - t) Q_T^{-1} f_T(a, b)|^2 dt
\]
\[
= \int_0^T h(t) \langle B^* S^*(T - t) Q_T^{-1} f_T(a, b), B^* S^*(T - t) Q_T^{-1} f_T(a, b) \rangle dt
\]
\[
= \left\langle \int_0^T h(t) S(T - t) B B^* S^*(T - t) dt, Q_T^{-1} f_T(a, b) \right\rangle
\]
\[
= \langle Q_T Q_T^{-1} f_T(a, b), Q_T^{-1} f_T(a, b) \rangle = \langle f_T(a, b), Q_T^{-1} f_T(a, b) \rangle.
\]

Let us take another control \( u \) for which \( (T - t)^{\alpha - 1} u(t) \) is square integrable on \([0, T]\) and \( \gamma(T, a, u) = b \). Then,
\[
\int_0^T (T - t)^{2(\alpha - 1)} \langle u(t), \mathfrak{u}(t) \rangle dt
\]
\[
= - \int_0^T h(t) \left\langle u(t), (T - t)^{2(1 - \alpha)} B^* S^*(T - t) Q_T^{-1} f_T(a, b) \right\rangle dt
\]
\[
= - \int_0^T \langle u(t), B^* S^*(T - t) Q_T^{-1} f_T(a, b) \rangle dt = \langle f_T(a, b), Q_T^{-1} f_T(a, b) \rangle.
\]

Hence,
\[
\int_0^T (T - t)^{2(1 - \alpha)} \langle u(t), \mathfrak{u}(t) \rangle dt = \int_0^T (T - t)^{2(1 - \alpha)} \langle \mathfrak{u}(t), \mathfrak{u}(t) \rangle dt
\]
Let us take $u$, we obtain that is a linear operator from the space $L_0^2([0,T],\mathbb{R}^m)$ into $\mathbb{R}^n$. Hence the classical result can be replaced by the following proposition.

Proposition 3.7 (cf. \cite{23}). If any state $b \in \mathbb{R}^n$ is attainable from $a = 0$, then the matrix $Q_T$ is nonsingular for any arbitrary $T > 0$.

Example 3.1. Let $\Sigma$ be the following system evaluated on $\mathbb{R}^2$:
\[
\Sigma : \begin{cases}
C D_0^{0.5} x_1(t) = x_2(t), \\
C D_0^{0.5} x_2(t) = u(t).
\end{cases}
\]

Let us take $a = (1,0)^*$ and $b = (0,0)^*$. Since $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, we obtain that $S(t) = \begin{pmatrix} \frac{1}{\sqrt{2\pi t}} & 1 \\ 0 & \frac{1}{\sqrt{2\pi t}} \end{pmatrix}$ while $S_0(t) = \begin{pmatrix} 1 & 2\sqrt{t} \\ 0 & 1 \end{pmatrix}$. Hence, the formula for the solution with the initial condition $\gamma(0,a,u) = a$ is
\[
\gamma(t,a,u) = \begin{pmatrix} 1 & 2\sqrt{t} \\ 0 & 1 \end{pmatrix} a + \int_0^t \begin{pmatrix} 1 & 0 \\ \sqrt{\pi(t-\tau)} & \sqrt{\pi(t-\tau)} \end{pmatrix} Bu(\tau)d\tau.
\]

Let us take $u(t) \equiv 1$. Then, for the given $a$, $\gamma(t,a,u) = \begin{pmatrix} 1 + t & 2\sqrt{t} \\ 0 & 1 \end{pmatrix}$. From the last expression we see that using constant $u(\cdot) \equiv 1$ for $t > 0$ we are not able to steer the given initial point $a$ to the origin.

Let now $f_T(a,b) = S_0(T)a - b = a$. The Gramian has the form
\[
Q_T = \begin{pmatrix}
\frac{T^2}{2} & \frac{2T^{3/2}}{\sqrt{\pi}} \\
\frac{2T^{3/2}}{\sqrt{\pi}} & \frac{T^2}{2}
\end{pmatrix}
\]
and the control
\[
\overline{m}(t) = -\frac{18(T-t)}{T^2} + \frac{12\sqrt{T-t}}{T^{3/2}}
\]
drives $a$ to $b$ with the modified energy
\[
m = \int_0^T |(T-t)^{-0.5} \overline{m}(t)|^2 dt = \frac{18}{T^2}.
\]

Example 3.2. Let $\alpha \in (0,1)$. Consider the following fractional system $\Sigma$ on $\mathbb{R}^2$:
\[
\Sigma : \begin{cases}
C D_0^\alpha x_1(t) = x_2(t), \\
C D_0^\alpha x_2(t) = -x_1(t) + u(t).
\end{cases}
\]
The matrix $A$ is now skew-symmetric, and thus
\[
A^0 = I, \quad A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad A^2 = -I.
\]
Hence, $A^k = I$ if $k = 0,4,8,\ldots$; $A^k = A$ if $k = 1,5,9,\ldots$; $A^k = -I$ if $k = 2,6,10,\ldots$; and $A^k = -A$ if $k = 3,7,11,\ldots$. Moreover,

$$S(t) = t^{\alpha-1} \left( \frac{1}{\Gamma(\alpha)} + A \frac{t^\alpha}{\Gamma(2\alpha)} - I \frac{t^{2\alpha}}{\Gamma(3\alpha)} - A \frac{t^{3\alpha}}{\Gamma(4\alpha)} + \cdots \right),$$

$$= I \left( \frac{t^{\alpha-1}}{\Gamma(\alpha)} - \frac{t^{3\alpha-1}}{\Gamma(3\alpha)} \right) + A \left( \frac{t^{2\alpha-1}}{\Gamma(2\alpha)} - \frac{t^{4\alpha-1}}{\Gamma(4\alpha)} + \cdots \right).$$

Using the notation

$$\sin_a t = \sum_{k=0}^{\infty} (-1)^k \frac{t^{2(k+1)\alpha-1}}{\Gamma[2(k+1)\alpha]}, \quad \cos_a t = \sum_{k=0}^{\infty} (-1)^k \frac{t^{2(k+1)\alpha-1}}{\Gamma[2(k+1)\alpha]},$$

we can write $S(t) = \begin{pmatrix} \cos_a t & \sin_a t \\ -\sin_a t & \cos_a t \end{pmatrix}$. As $B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, we have

$$Q_{T} = \int_{0}^{T} (T - t)^{2(1-\alpha)} M_{\alpha}(t) dt$$

with

$$M_{\alpha}(t) = \begin{pmatrix} \sin^2_{\alpha}(T-t) & \sin_{\alpha}(T-t) \cos_{\alpha}(T-t) \\ \sin_{\alpha}(T-t) \cos_{\alpha}(T-t) & \cos^2_{\alpha}(T-t) \end{pmatrix}.$$

To get an exact formula for $Q_{T}$ and $Q_{T}^{-1}$ is difficult. We can, however, easily obtain approximations with the desired precision for both matrices, and an approximate formula for the optimal control. Let us consider a concrete situation. Let $\alpha = \frac{1}{2}$.

Then $\sin_{\frac{1}{2}} t = e^{-t}$ and

$$\cos_{\frac{1}{2}} t = \frac{1}{\sqrt{\pi}t} \left( 1 - \sum_{k=1}^{+\infty} \frac{2k2^{2k-1}}{\prod_{i=1}^{k}(2i-1)} \right) = \frac{1}{\sqrt{\pi}t}(1 - 2t + \cdots).$$

We choose to approximate $\cos_{\frac{1}{2}} t$ by functions

$$c_{L}(t) = \frac{1}{\sqrt{\pi}t} \left( 1 - \sum_{k=1}^{L} \frac{2k2^{2k-1}}{\prod_{i=1}^{k}(2i-1)} \right), \quad L = 1,2,\ldots$$

In this way we can approximate $Q_{T}$ by

$$\int_{0}^{T} (T - t) \begin{pmatrix} e^{-2(T-t)} & c_{L}(T-t)e^{-(T-t)} \\ c_{L}(T-t)e^{-(T-t)} & c^2_{L}(t) \end{pmatrix} dt.$$
Proposition 4.8. Let \( \text{rank} B = n \) and \( B^+ \) be such that \( BB^+ = I \). Let \( g(\cdot) \) be the matrix function defined by Proposition 2.6. Then the control

\[
\hat{u}(t) = \frac{1}{T} B^+ g(T - t) (b - S_0(T)a), \quad t \in \lbrack 0, T \rbrack,
\]

transfers \( a \) to \( b \) in time \( T > 0 \).

Proof. The proof follows by a direct calculation:

\[
\gamma(T, a, \hat{u}) = S_0(T)a + \frac{1}{T} \int_0^T S(T - t)BB^+ g(T - t) (b - S_0(T)a) \, dt = b.
\]

\( \square \)

Example 4.3. Let \( \Sigma \) be the following fractional system evaluated on \( \mathbb{R} \):

\[
\Sigma : \quad \frac{d}{dt} D_{0+}^\alpha x(t) = u(t),
\]

where \( \alpha \in (0, 1] \). Let us take \( a, b \in \mathbb{R}, \, T > 0 \). Then,

\[
S(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}, \quad S_0(t) = 1, \quad B = B^+ = 1,
\]

and

\[
\gamma(T, a, u) = a + \frac{1}{T} \int_0^T (T - t)^{\alpha-1} u(t) \, dt.
\]

Consider, accordingly to Proposition 2.6, \( g(t) = t^{1-\alpha} \Gamma(\alpha) \). From Proposition 4.8

\[
\hat{u}(t) = \frac{\Gamma(\alpha)}{T} (T - t)^{1-\alpha} (b - a)
\]

transfers \( a \) to \( b \): \( \gamma(T, a, \hat{u}) = b \) with the modified energy (3.10) of value

\[
m = \frac{\Gamma^2(\alpha)(b-a)^2}{T}.
\]

It is also the minimum energy as \( \hat{u}(t) = \overline{\pi}(t) \).

An algebraic condition equivalent to controllability for fractional linear control systems has been derived in [13] and cited again in [7, 17]. According to Theorem 3.1 and Proposition 3.7, we can state a similar formulation as in [25]:

Theorem 4.2. The following conditions are equivalent:

(a) An arbitrary state \( b \in \mathbb{R}^n \) is attainable from 0.
(b) System \( \Sigma \) is controllable.
(c) System \( \Sigma \) is controllable at a given time \( T > 0 \).
(d) Matrix \( Q_T \) is nonsingular for some \( T > 0 \).
(e) Matrix \( Q_T \) is nonsingular for an arbitrary \( T > 0 \).
(f) \( \text{rank} [A|B] = \text{rank} [B, AB, \ldots, A^{n-1}B] = n \).

If the rank condition is satisfied, then the control \( \overline{\pi}(\cdot) \) given by (3.9) steers \( a \) to \( b \) at time \( T \). Our goal now is to find another formula for the steering control by using the matrix \( [A|B] \) instead of the controllability matrix \( Q_T \). It is a classical result that if \( \text{rank} [A|B] = n \), then there exists a matrix \( K \in M(mn, n) \) such that \( [A|B] K = I \in M(n, n) \) or, equivalently, there are matrices \( K_1, K_2, \ldots, K_n \in M(m, n) \) such that \( BK_1 + ABK_2 + \cdots + A^{n-1}BK_n = I \).

For the next construction it is more convenient to use the notion of Riemann–Liouville derivative. We begin by introducing a notation for compositions of the Riemann–Liouville derivatives with the same order \( \alpha, \alpha \in (0, 1) \). Let \( R_{0+}^{\alpha, 0} \psi(t) = \psi(t) \). Then for \( j \in \mathbb{N} \), recursively, we put \( R_{0+}^{\alpha, j+1} \psi(t) := D_{0+}^{\alpha} \left( R_{0+}^{\alpha, j} \psi(t) \right) \).
Theorem 4.3. Let rank $|A|B| = n$ and $\alpha \in (0, 1)$. Let $p$ be such that $S(T - t) \in I^\alpha_{p-}(L_p)$ and $\varphi$ be a real function given on $[0, T]$ such that

(i) $\int_0^T \varphi(t)dt = 1$;

(ii) $R^\alpha_{0+} \psi(t) \in I^\alpha_{p-}(L_q)$ for $j = 0, \ldots, n - 1$, where $\psi(t) = g(t)(b - S_0(T)a) \varphi(t)$ and $S(T - t)g(t) = I, t \in [0, T]$, for $1/p + 1/q \leq 1 + \alpha$.

Then the control $\hat{u}(t) = K_1 \psi(t) + K_2 D^\alpha_{0+} \psi(t) + \cdots + K_n R_n^\alpha_{0+} \psi(t), t \in [0, T],$ transfers $a$ to $b$ at time $T \geq 0$.

Proof. Using $j - 1$ times the formula (2.5) of integration by parts and Lemma 2.1,

$$\int_0^T S(T - t)BK_j R^\alpha_{0+} j - 1 \psi(t)dt = \int_0^T S(T - t)A^j - 1 BK_j \psi(t)dt$$

for $j = 1, \ldots, n$. Then,

$$\int_0^T S(T - t)B\hat{u}(t)dt = \int_0^T S(T - t)B\psi(t)dt$$

and $\gamma(T, a, \hat{u}) = S_0(T)a + \int_0^T S(T - t)g(t)(b - S_0(T)a) \varphi(t)dt = b$. □

5. Conclusion

The concept of fractional (i.e., non-integer) derivative and integral is increasingly being recognized as a good tool to model the behavior of complex systems in various fields of science and engineering [2 3 4 6 9 20 23]. This is particularly true in the areas of control theory and control engineering, where fractional order controllers and plants provide confirmed evidence of better performances when compared with the best integer order controllers [1 5 7 8 11 12 14 15 17]. In this work we consider linear time-invariant control systems of fractional order $\alpha, \alpha \in (0, 1)$, in the sense of Caputo. The controllability question for such systems is addressed. Main results give sufficient conditions assuring the existence of controls that drive the system between any two desired states. Explicit formulas for such controls are given. A control law is obtained that minimizes the modified energy $\int_0^T |(T - t)\alpha - 3 u(t)|^2dt$. Our results are reduced to the classical ones when the fractional order of differentiation $\alpha$ tends to one.

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