Algebraic differential equations of periods integrals

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Abstract. We explain that in the study of the asymptotic expansion at the origin of a period integral like \( \int_{\gamma} \omega/df \) or of a hermitian period like \( \int_{f=0} \rho, \omega/df \wedge \omega'/df \) the computation of the Bernstein polynomial of the "fresco" (filtered differential equation) associated to the pair of germs \((f, \omega)\) gives a better control than the computation of the Bernstein polynomial of the full Brieskorn module of the germ of \(f\) at the origin. Moreover, it is easier to compute as it has a better functoriality and smaller degree. We illustrate this in the case where \(f \in \mathbb{C}[x_0, \ldots, x_n]\) has \(n+2\) monomials and is not quasi-homogeneous, by giving an explicit simple algorithm to produce a multiple of the Bernstein polynomial when \(\omega\) is a monomial holomorphic volume form. Several concrete examples are given.

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1 Introduction

This article resumés and simplifies two unpublished papers, see [9] and [10] on the computation of the Bernstein polynomial associated to a period integral or to a hermitian period.

It begins with a short overview on geometric (a, b)-modules and frescos destined to the reader which is not familiar with the use of Brieskorn modules in the study of the singularities of a holomorphic function on a complex manifold.

In opposition with the second preprint cited above, we let aside the global point of view, that is to say the study of the global fresco associated to a period integral in the case of a proper holomorphic function on a complex manifold, because it uses more heavy tools and very often the local study presented here would be enough to obtain good informations, using a partition of unity.

The reader interested in this global setting may consult the first preprint mentioned above and also the preprint [8].

It is important to notice that we are dealing here with general singularities of a holomorphic function (not only the isolated singularity case as in the classical use of the Brieskorn module) and our illustration in the case of a (not quasi-homogeneous) polynomial in \( C[x_0, \ldots, x_n] \) with \( n + 2 \) monomials does not assume also that the singularity is isolated.

2 A short overview on (a, b)-modules and frescos

2.1 Why to use an \((a, b)\)-module structure instead of a differential system?

Note first that in "\((a, b)\)" \(a\) is the multiplication by the variable \(z\) and \(b\) is the primitive vanishing at \(z = 0\), so \(b(f)(z) := \int_0^z f(t) \, dt\) where \(f\) is, for instance, a holomorphic multivalued function with \(z = 0\) as an eventual ramification point. So we are working with the non commutative algebra \(\mathcal{A}\) generated by \(a\) and \(b\) with the
commutation relation \( a.b - b.a = b^2 \) which corresponds to the usual commutation relation \( \partial_z z - z \partial_z = 1 \) in the Weyl algebra \( \mathbb{C}(z, \partial_z) \).

Then why not to use the usual Weyl algebra?

The initial motivation comes from the study of germs of isolated singularities of holomorphic functions \((f, 0) : (\mathbb{C}^{n+1}, 0) \rightarrow (\mathbb{C}, 0)\) initiated at the end of the sixties by Milnor \([15]\), Brieskorn \([11]\), Deligne \([12]\), Malgrange, \([14]\), Varchenko \([19]\), Saito Kyoji \([17]\), Saito Morihiko \([18]\), ... and many others.

To my knowledge the first who introduced the "operator" \( \partial_z^{-1} \) was Kyoji Saito in the beginning of the eighties (see \([17]\)). The main reason comes from the fact that, looking at periods integrals of the type \( z \mapsto \int_{\gamma_z} \omega / df \) where \( \omega \in \Omega^{n+1}_{C^{n+1}, 0} \) is a germ of holomorphic volume form, \((f, 0) : (\mathbb{C}^{n+1}, 0) \rightarrow (\mathbb{C}, 0)\) a germ of holomorphic function with an isolated and \( \gamma_z, z \in H \) a horizontal family of \( n \)-cycles in the fibers \( \{f = z\} \) of \( f, H \rightarrow D^* \) been the universal cover of a small punctured disc \( D^* \) with center 0.

The derivation \( \partial_z \) of such an integral is given by the following formula

\[
\partial_z \left( \int_{\gamma_z} \omega / df \right) = \int_{\gamma_z} du / df \quad (1)
\]

where \( u \in \Omega^n_{C^{n+1}, 0} \) satisfies \( \omega = df \wedge u \). But in general it is not possible to find such a \( u \in \Omega^n_{C^{n+1}, 0} \) because writing \( \omega = g(x).dx \), the holomorphic germ \( g \) is not in the Jacobian ideal of \( f \). Nevertheless, as the coherent sheaf \( \Omega^{n+1} / df \wedge \Omega^n \) has support inside \( \{f = 0\} \) near 0, the Nullstellensatz gives a positive integer \( p \) such that \( f^p \) annihilates this sheaf near 0 and we may find \( u \in f^{-p}.\Omega^n_{C^{n+1}, 0} \) such that \( \omega = df \wedge u \) and \( (1) \) holds.

Thanks to the positivity theorem of Malgrange (see \([14]\)) we may look the formula \( (1) \) as follows:

\[
\int_0^z \left( \int_{\gamma_t} du / df \right) dt = \int_{\gamma_z} \omega / df \quad (2)
\]

where we begin with \( du \in \Omega^{n+1}_{C^{n+1}, 0} \) and where \( \omega \) is defined by \( \omega := df \wedge u \); the surjectivity of the de Rham differential \( d : \Omega^n \rightarrow \Omega^{n+1} \) shows that \( du \) is any germ in \( \Omega^{n+1}_{C^{n+1}, 0} \) and we may write

\[
b \left( \int_{\gamma_z} du / df \right) = \int_{\gamma_z} df \wedge du / df = \int_{\gamma_z} u \quad (3)
\]

so that the action of \( b \) only needs to solve the equation \( du = \omega \), and this is always possible with \( u \in \Omega^n_{C^{n+1}, 0} \) without introducing a denominator in \( f \) (so no denominator in \( z \) downstairs).

As \( a \) is given by (because the \( n \)-cycle \( \gamma_z \) is in the fiber \( f^{-1}(z) \)):

\[
a \left( \int_{\gamma_z} du / df \right) = z \left( \int_{\gamma_z} du / df \right) = \int_{\gamma_z} f . du / df \quad (4)
\]
we see that the \( \mathcal{A} \)-module structure on the quotient \(^3\) \( E_f := \Omega^{n+1}_{Cn+1,k} / d(Ker(df)_0^n) \) does not need to consider meromorphic \((n + 1)\)-differential forms with poles along the 0-fiber \( \{ f = 0 \} \).

Note that in the case of an isolated singularity for \( f \) we have the equality
\[
Ker(df)_0^n = df \wedge d(\Omega^{n-1})
\]
because the partial derivatives of \( f \) define a regular sequence. Also in this case we find that \( E_f / bE_f \) is equal to the finite dimensional vector space \( \mathcal{O}_{Cn+1,k} / J(f)_0 \) where \( J(f) \) is the Jacobian ideal of \( f \). So we find the classical ”Brieskorn lattice”.

But why is this presentation interesting if, at the end, we are compelled to introduce denominators in \( f \) (or in \( z \) working downstairs) to reach an ordinary differential system (or a differential equation ) ?

The answer comes from the following considerations:
If you keep a module structure over the algebra \( \mathcal{A} \) as a substitute for a differential system you have a richer structure (so a more precise informations) than a structure of module over the localized Weyl algebra \( \mathbb{C}(z, z^{-1}, \partial_z) \) associated to your differential system. This comes from the fact that the commutation relation \( a.b - b.a = b^2 \) is homogeneous of degree 2 in \( (a, b) \) and implies the existence of the decreasing sequence of two-sided ideals in \( \mathcal{A} \) given by \( b^m.\mathcal{A} = \mathcal{A}.b^m, \forall m \in \mathbb{N} \). So any \( \mathcal{A} \)-module \( E \) is endowed with a ”natural filtration” \( (b^m.E)_{m \in \mathbb{N}} \) by sub-\( \mathcal{A} \)-modules.

**EXERCISE.** Show that \( a.b^m = b^m.a + m.b^{m+1}, \forall m \in \mathbb{N} \) is consequence of the commutation relation corresponding to \( m = 1 \).

Note that this relation implies that \( a \) and \( b^m \) commute modulo \( b^{m+1}.\mathcal{A} \).

For instance, if you look at the ”natural action” of \( \mathcal{A} \) on \( \mathbb{C}[[z]] \) which is given by \( a(z^m) = z^{m+1} \) and \( b(z^m) = z^{m+1} / (m + 1) \), you will see that \( b^m.\mathbb{C}[[z]] = z^m.\mathbb{C}[[z]], m \in \mathbb{N} \) is the filtration by the valuation.

Another simple remark may also help to convince the reader that a module structure over \( \mathcal{A} \) is interesting:

**Lemma 2.1.1** Let \( E := \bigoplus_{j=1}^k \mathbb{C}[b].e_j \) be a free \( \mathbb{C}[b] \)-module with basis \( e_1, \ldots, e_k \) and let \( x_1, \ldots, x_k \) be any given collection of elements in \( E \). Then there exists an unique \( \mathcal{A} \)-module structure on \( E \) such that

a) The action of \( a \) is defined by \( a.e_j = x_j \) for each \( j \in [1, k] \).

b) The action of \( b \) is given by the \( \mathbb{C}[b] \)-structure of \( E \).

The proof of this lemma is easily deduced from the following formula which is an easy consequence of the exercice above:
\[
a.(S(b).e_j) = S(b).x_j + b^2.S'(b).e_j \quad \forall j \in [1, k]
\]

\(^3\)This quotient allows to define \( b[\omega] = [df \wedge u] \) independently on the choice of \( u \in \Omega^n_0 \) such that \( \omega = du \) because when \( \omega \) is in \( d(Ker(df))^n_0 \) the period integral is identically 0 near \( z = 0 \).
where $S'(b)$ is the "usual" derivative of the polynomial $S \in \mathbb{C}[b]$.

In fact, the presence of the filtration by the two-sided ideals $b^m.A$ of the algebra $A$ and the lemma above leads to the following considerations

- The "fundamental" operation in the action of $A$ is $b$!
- It seems convenient, as we are interested in the asymptotic expansions of the periods integrals $\int_{\gamma} \omega/df$ when $z \to 0$, to complete the algebra $A$ for the uniform structure defined by the filtration $b^m.A, m \in \mathbb{N}$.

Note that for the "obvious" action of $A$ on formal power series in $z$ this filtration is associated to the valuation in $z$ (see the remark following the exercise above).

This means that we shall work with the algebra

$$\hat{A} := \{ \sum_{\nu \geq 0} P_\nu(a).b^\nu, \ P_\nu \in \mathbb{C}[a] \ \forall \nu \in \mathbb{N} \}. \quad (5)$$

The initial idea of Kyoji Saito was to add some convergence conditions in order that such series acts on convergent (multivalued) series like

$$\sum_{r \in R, j \in [0,N]} \mathbb{C}\{z\}.z^r.(Log z)^j$$

where $R$ is a finite subset in $\mathbb{Q}$ and $N$ is a non negative integer, which are the kind of asymptotic expansions which are valid for our period integrals.

But thanks to the regularity of the Gauss-Manin connection, we don't loose any information by staying at the formal series level and this avoids a lot of painful (standard) estimates!

Remark also that the construction given in the lemma above is also valid for the algebra $\hat{A}$ and, moreover, that a module $E$ over $\hat{A}$ without $b$--torsion is of finite type over $\mathbb{C}[[b]]$ if and only if the vector space $E/b.E$ is finite dimensional.

So, our definition of an $(a, b)$-module will be

- An $(a, b)$-module is a $\hat{A}$--module which is a free and finite type module over the (commutative) sub-algebra $\mathbb{C}[[b]] \subset \hat{A}$.

**Examples.**

1. Let $(f, 0) : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ be a germ of holomorphic function with an isolated singularity. Let $\hat{\Omega}_{0}^{p}$ be the formal completion at the origin of $\Omega^{p}_{\mathbb{C}^{n+1}, 0}$.

   The quotient $\hat{E}_f := \hat{\Omega}_{0}^{n+1}/(df \wedge d(\hat{\Omega}_0^{p}))$ endowed with the actions of $a := \times f$ and $b := df \wedge d^{-1}$ is an $(a, b)$-module (Note that the absence of $b$--torsion is a theorem; see [16] or [3]).

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   2This is psychologically the most difficult fact to accept after a standard education in maths.
2. Let $E := \mathbb{C}[[b]]e_0 \oplus \mathbb{C}[[b]].e_1$ be the $A$-module defined by $a.e_0 := b.e_0$ and $a.e_1 = b.e_1 - b.e_0$. Then it is an easy exercise to show that $E$ is isomorphic to

$$\mathbb{C}[[z]] \oplus \mathbb{C}[[z]].Log z$$

where $a := xz$ and $b := \int_0^z$.

Determine the filtration $(b^m.E)_{m \in \mathbb{N}}$ in this example and compare it with the filtration by the $(a^m.E)_{m \in \mathbb{N}}$.

Compute the module over the Weyl algebra generated by $Log z$ and compare with $E$.

### 2.2 Geometric (a, b)-modules

The (a,b)-modules which appear in singularity theory of a function are special. They correspond to regular differential system and the notion of regularity is easy to define for an (a,b)-module:

First we say that the (a,b)-module $E$ has a simple pole when $a.E \subset b.E$. When it is the case, $-b^{-1}.a$ acts on the (finite dimensional) vector space $E/b.E$ and its minimal polynomial is called the Bernstein polynomial of $E$.

For a general (a,b)-module the saturation $\tilde{E}$ of $E$ by the action of $b^{-1}.a$ is not always a finite type $\mathbb{C}[[b]]$-module. When $\tilde{E}$ is of finite type over $\mathbb{C}[[b]]$, $\tilde{E}$ is an (a,b)-module (with simple pole) with the same rank over $\mathbb{C}[[b]]$ than the rank of $E$.

We say in this case that $E$ is regular. This is equivalent to the fact that $E$ can be embedded in an (a,b)-module having a simple pole.

Then we defined its Bernstein polynomial of a regular (a,b)-module $E$ as the Bernstein polynomial of its saturation $\tilde{E}$ by $b^{-1}.a$.

There is one more specific property for the (regular) (a,b)-modules coming from the singularity of a function $f$, which reflects the fact that the monodromy of $f$ is quasi-unipotent and the positivity theorem of Malgrange: the fact that the roots of the Bernstein polynomial are negative rational numbers (compare with the famous theorem of Kashiwara [13]). So we call geometric a regular (a,b)-module whose Bernstein polynomial have negative rational roots.

**Example.** In the previous example 2 the (a,b)-module has a simple pole and its Bernstein polynomial is, by definition, the minimal polynomial of the matrix

$$\begin{pmatrix}
-1 & -1 \\
0 & -1
\end{pmatrix}$$

so its Bernstein polynomial is $(\lambda + 1)^2$. Compare with the Bernstein type identity

$$((2\lambda + 1).\partial_z - z.\partial_z^2)(z^{\lambda+1}.Log z) = (\lambda + 1)^2.z^\lambda.Log z.$$
Proposition 2.2.1 Let $E$ be a geometric $(a,b)$-module and $F$ any sub-$\tilde{A}$-module in $E$. Then $F$ is a geometric $(a,b)$-module.

PROOF. From the regularity of $E$ we may assume that $E$ is a simple pole module (i.e. $a.E \subset b.E$). Then the Bernstein polynomial of $E$ is the minimal polynomial of the action of $-b^{-1}.a$ on the finite dimensional vector space $E/b.E$. As $F$ is a $\mathbb{C}[[b]]$ sub-module of $E$ which is free and finite rank on $\mathbb{C}[[b]]$, $F$ is also free and finite rank on $\mathbb{C}[[b]]$ and stable by $a$. So $F$ is an $(a,b)$-module. Its saturation by $b^{-1}.a$ is again contained in $E$ and so it is also free of finite type on $\mathbb{C}[[b]]$. This gives the regularity of $F$. The last point to prove is the fact that the Bernstein polynomial of $F$ has negative rational roots (i.e. $F$ is geometric). We shall argue by induction on the rank of $F$. In the rank 1 case let $e$ be a generator of $F$ over $\mathbb{C}[[b]]$ such that $a.e = \lambda.b.e$ (see the classification of rank 1 regular $(a,b)$-module in [B.93], lemma 2.4). Let $\nu$ in $\mathbb{N}$ be maximal such that $b^{-\nu}.e$ lies in $E$. Then $\mathbb{C}[[b]].b^{-\nu}.e = b^{-\nu}.F$ is a normal sub-module of $E$ and we have an exact sequence of simple poles $(a,b)$-modules

$$0 \to b^{-\nu}.F \to E \to Q \to 0$$

and also an exact sequence of $(-b^{-1}.a)$ finite dimensional vector spaces

$$0 \to \mathbb{C}.b^{-\nu}.e \to E/b.E \to Q/b.Q \to 0.$$ 

Then the minimal polynomial $B_E$ of the action of $-b^{-1}.a$ on $E/b.E$ is either equal to the minimal polynomial $B_Q$ of the action of $-b^{-1}.a$ on $Q/b.Q$, and in this case $-(\lambda - \nu)$ divides $B_Q = B_E$, or we have $B_E[x] = (x + (\lambda - \nu)).B_Q[x]$. In both cases, as $E$ is geometric, we obtain that $-(\lambda - \nu)$ is a negative rational number, and so is $-\lambda$.

The induction step follows easily by considering a rank 1 normal sub-module $G$ in $F$ and the quotient of $E$ by the normalization $\tilde{G}$ of $G$ in $E$. Then there exists an integer $\nu \geq 0$ such that $G = b^{\nu}.\tilde{G}$ and $\tilde{G} \cap F = G$. We conclude using the fact that a quotient of a geometric $(a,b)$-module by a normal rank 1 sub-module is again geometric already proved above. \qed

As we want to consider, in the non isolated singularity case, a sheaf of geometric $(a,b)$-module along the singular set $\{df = 0\}$ of the zero set $\{f = 0\}$ of a holomorphic function on a complex manifold $M$, we have to replace the completion at the origin upstairs used above in the isolated singularity case by a $f-$completion which is in fact the $z-$completion downstairs. This will not change seriously the considerations above, thanks to the following easy proposition which implies that any geometric $(a,b)$-module is in fact a module over the algebra $\hat{A} := \{\sum_{p,q \geq 0} c_{p,q}.a^p.b^q\}$ which contains both $\mathbb{C}[[b]]$ and $\mathbb{C}[[a]]$.

\[\text{3}G \subset E\text{ is a normal sub-module when } bx \in G\text{ for } x \in E\text{ implies } x \in G.\]

\[\text{4}So we consider the smallest normal sub-module of } E \text{ containing } G.\]
Proposition 2.2.2 Any geometric \((a,b)\)-module is complete for the decreasing filtration by the \(\mathbb{C}[a]–\)sub-modules (not stable by \(b\)) \((a^m.E)_m\in\mathbb{N}^+\).

Note that the hypothesis "geometric" insure that \(a\) is injective (this is not the case if we assume only the regularity).

The following elementary lemma will be useful.

Lemma 2.2.3 Let \(E\) be a regular \((a,b)\)-module and let \(F \subset E\) be a sub-\((a,b)\)-module. Assume that \(\lambda\) is a root of the Bernstein polynomial \(B_F\) of \(F\). Then there exists \(\lambda' \in \lambda + \mathbb{N}\) such that \(\lambda'\) is a root of the Bernstein polynomial \(B_E\) of \(E\). So, if \(E\) is geometric, \(F\) is also geometric.

Proof. By definition of the Bernstein polynomial, we may assume that \(E\) and \(F\) has simple poles. Moreover, by an induction on the rank of \(E\) we may assume that \(F\) and \(E\) have the same rank. Let \(k\) be this rank. Then our assumption gives that \(F\) admits a normal rank 1 sub-module \(G := \mathbb{C}[[b]].e_\mu\) with \(a.e_\mu = \mu.b.e_\mu\), with \(\mu = -\lambda+k-1\). Then \(\tilde{G} := \{x \in E \exists N \in \mathbb{N} / b^N.x \in G\}\) is a rank 1 normal submodule in \(E\) isomorphic to \(\mathbb{C}[[b]].e_{\mu-q}\) for some \(q \in \mathbb{N}\). So there exists a root \(\lambda'\) of \(B_E\) such that \(\mu - q = -\lambda' + k - 1\). This gives \(\lambda' = \lambda + q\) concluding the proof.

Remark that if \(E\) is geometric (so \(F\) is also geometric) we have \(\lambda' < 0\); for instance if \(\lambda\) is in \([-1,0]\) then \(\lambda' = \lambda\).

2.3 Frescos

We have seen that the \((a,b)\)-module structure may be an interesting way to study the differential system associated to periods integrals for a germ of holomorphic function \((f,0) : (\mathbb{C}^{n+1},0) \rightarrow (\mathbb{C},0)\). In fact, the Brieskorn module, or the \((a,b)\)-module \(\tilde{E}_f\) defined above (say in the isolated singularity case to be simple) gives in fact a filtered version of the differential system satisfied by all the periods integrals associated to the germ \((f,0)\).

But if we are interested by the periods corresponding to a specific holomorphic differential form, it is clear that such a differential system, that is to say the all \((a,b)\)-module \(\tilde{E}_f\) does not give very precise informations. In term of differential system, we would prefer to have a specific differential equation satisfied by the periods \(\int_{\gamma} \omega/df\) for our choice of \(\omega\) than the differential system satisfied by all periods, so associated to all choices of \(\omega \in \Omega^{n+1}_0\). The analog of the differential equation in term of \((a,b)\)-modules is the notion of "fresco". A fresco is a, by definition, a geometric \((a,b)\)-module which is generated, as a \(\tilde{A}\)–module, by one generator. For instance, in the previous situation, we shall consider the fresco given by \(\tilde{A}.[\omega] \subset \tilde{E}_f\) and we shall call it the fresco of the pair \((f,\omega)\) at the origin. The following structure theorem describes in a very simple way such a fresco (see [7]).
**Theorem 2.3.1** Any rank k fresco $F$ is isomorphic (as an $\hat{A}$–module) to a quotient $\hat{A}/\hat{A}.\Pi$ where $\Pi \in \hat{A}$ has the following form

$$\Pi := (a - \lambda_1.b).S_1^{-1}.(a - \lambda_2.b).S_2^{-1} \ldots S_{k-1}^{-1}.(a - \lambda_k.b)$$

(6)

where the numbers $-(\lambda_j + j - k)$ are the roots of the Bernstein polynomial of $F$ and where $S_j$ are in $\mathbb{C}[b]$ and satisfy $S_j(0) = 1$ (so $S_j$ is invertible in $\mathbb{C}[[b]]$).

Note that the initial form in $(a,b)$ of $\Pi$ is $P_F := (a - \lambda_1.b) \ldots (a - \lambda_k.b)$. It is called the Bernstein element of the fresco $F$. It does not depend of the choice of the generator of $F$ over $\hat{A}$ (choice which determines $\Pi$) and is related to the Bernstein polynomial $B_F$ of $F$ by the relation in the ring $\hat{A}[b^{-1}]$:

$$(-b)^k.B_F(-b^{-1}.a) = P_F, \quad \text{where} \quad k := rk(F).$$

(7)

In the case of a fresco $F$ the Bernstein polynomial $B_F$ is equal to the characteristic polynomial of the action of $-b^{-1}.a$ on $\hat{F}/b.\hat{F}$ where $\hat{F}$ is the saturation of $F$ by $b^{-1}.a$. This makes the computation of the Bernstein polynomial of a fresco easier than for a general geometric $(a,b)$-module, for instance by the use of the following remark:

If $0 \to F \to G \to H \to 0$ is an exact sequence of frescos we have the relation $P_G = P_F.P_H$ (product in $\mathcal{A}$) between the Bernstein elements and this gives the relation( see [B.09b])

$$B_G(x) = B_F(x + rk(H)).B_H(x)$$

between the Bernstein polynomials.

Our next proposition will be useful in the section 4.

**Proposition 2.3.2** Let $F$ be a rank k fresco with generator e. Assume that $Q \in \hat{A}$ has the following properties:

i) The initial form $Q$ of $Q$ in $(a,b)$ has degree $d$.

ii) $Q[e] = 0$ in $F$.

Then $Q$ is a left multiple in $\mathcal{A}$ of $P_F$, the Bernstein element of $F$. If moreover we have $d = k$, then $Q$ is the Bernstein element of $F$ up to a non zero multiplicative constant.

**Proof.** Using the structure theorem of [7] recalled above (see 2.3.1) we have an isomorphism $F \simeq \hat{A}/\hat{A}.\Pi$ where the initial form in $(a,b)$ $P$ of $\Pi$ is the Bernstein element of $F$. As $F$ is a $\hat{A}$–module (see the proposition 2.2.2) we have also an

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5 Recall that we consider the rank on $\mathbb{C}[[b]] \subset \hat{A}$. 

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isomorphism $F \simeq \hat{A}/\hat{A}.\Pi$ and our hypothesis $ii$) implies that there exists $Z \in \hat{A}$ such that

$$Q = Z.\Pi.$$  

This gives $Q = \text{in}(Z).P$ where $\text{in}(Z)$ is the initial form in $(a,b)$ of $Z$. This already implies that $d \geq k$ and that $\text{in}(Z)$ is of degree $d - k$. In the case $d = k$ we have $\text{In}(Z) = \zeta \in \mathbb{C}^*$ and $Q = \zeta.P$. Note that $Z$ is invertible in $\hat{A}$ in this case. ■

### 2.4 A general existence theorem

Now consider a germ $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ such that $\{f = 0\}$ is reduced. Let $\hat{\Omega}^\bullet$, the formal $f-$completion of the sheaf of holomorphic differential forms on $(\mathbb{C}^{n+1}, 0)$ and let $\hat{\text{Ker}} df^\bullet$ be the kernel of the map

$$\wedge df : \hat{\Omega}^\bullet \to \hat{\Omega}^{\bullet + 1}.$$  

Then for any $p \geq 0$ the $p-$th cohomology sheaf of the complex $(\hat{\text{Ker}} df^\bullet, d^\bullet)$ has a natural structure of left $\hat{A}-$module, where the action of $a$ is given by multiplication by $f$ and the action of $b$ is (locally) given by $df \wedge d^{-1}$.

The following result is known (see [B.06] [B-S.07] and [B.08])

**Theorem 2.4.1** For each integer $p$ the germ at $0$, denoted by $E^p$, of the $p-$th cohomology sheaf of the complex $(\hat{\text{Ker}} df^\bullet, d^\bullet)$ satisfies the following properties:

1. We have in $E^p$ the commutation relation $a.b - b.a = b^2$.
2. $E^p$ is $b-$separated and $b-$complete (so also $a-$complete). Then it is a $\hat{A}-$module (and also a $\hat{A}-$module).
3. There exists an integer $m \geq 1$ such that $a^m.E^p \subset b.E^p$.
4. We have $B(E^p) = A(E^p) = \hat{A}(E^p)$ and there exists an integer $N \geq 1$ such that $a^N.A(E^p) = 0$ and $b^{2N}.B(E^p) = 0$.
5. The quotient $E^p/B(E^p)$ is a geometric $(a,b)$-module.

Recall that $B(E)$ is the $b-$torsion in $E$, $\hat{A}(E)$ the $a-$torsion of $E$ and $A(E)$ the $\mathbb{C}[b]-$module generated by $\hat{A}(E)$ in $E$.

We shall mainly use this result in the case $p = n + 1$ to obtain that for any class $[\omega] \in E^{n+1}$ the $\hat{A}-$module $\hat{A}.[\omega] \subset E^{n+1}/B(E^{n+1})$ is a fresco, result which is a direct consequence of the proposition 2.2.1 and property v) of the previous theorem.
Definition 2.4.2 We assume that the non constant holomorphic germ \( \tilde{f} \) fixed as above. For any germ \( \omega \in \hat{\Omega}_0^{n+1} \) we shall denote by \( F_{[\omega]} \) the fresco generated by \( [\omega] \) in the geometric \((a,b)\)-module \( E^{n+1}/B(E^{n+1}) \). So we have \( \tilde{A},[\omega] \subset E^{n+1}/B(E^{n+1}) \). We shall denote \( B_{[\omega]} \in \mathbb{C}[x] \) and \( P_{[\omega]} \in A \) respectively the Bernstein polynomial and the Bernstein element of the fresco \( F_{[\omega]} \).

We shall study several examples in section 4.

3 Mellin transform of hermitian periods

3.1 Hermitian periods

We shall give here the basic idea showing how the roots of the Bernstein polynomial of the fresco associated to \((f, \omega)\) controls the singular terms in the asymptotic expansion of a (local) hermitian period. First recall that a (local) hermitian period in the case we are interested with, is, by definition, a fiber-integral of the type

\[
z \mapsto \varphi(z) := \frac{1}{(2i\pi)^n} \int_{f=z} \rho.\omega/\bar{\omega}'/df
\]

where \( f : (\mathbb{C}^{n+1},0) \to (\mathbb{C},0) \) is a germ of holomorphic function, \( \omega,\omega' \) are germs of holomorphic volume forms and \( \rho \in \mathcal{C}_c(\mathbb{C}^{n+1}) \) has a small enough compact support and satisfies \( \rho \equiv 1 \) near 0. We are interested in the singular (i.e. non \( \mathcal{C}_\infty \)) terms in the asymptotic expansion of the function \( \varphi \) at \( z = 0 \). It is a classical result (see [2]) that this study is equivalent to the study of the polar part of the complex Mellin transform of \( \varphi \) which is given by

\[
F_{\omega,\omega'}^h(\lambda) := \frac{1}{\Gamma(\lambda + 1)} \cdot \int_X \rho.|f|^{2\lambda}.\bar{f}^h.\omega \wedge \bar{\omega}'
\]  

where \( h \) a integer in \( \mathbb{Z} \) and \( \lambda \) is a complex number with \( \Re(\lambda) \gg 1 \).

We shall assume, for simplicity, that for any \( \rho \equiv 0 \) near the origin, such an integral has never a pole at the point \( \lambda = \lambda_0 \) for any \( \omega,\omega' \in \Omega^{n+1}(X) \) and for any \( h \in \mathbb{Z} \). This is the case, for instance, when \( \exp(2i\pi.\lambda_0) \) is not an eigenvalue of the local monodromy of \( f \) at any point \( x \) near by 0 and distinct of the origin.

Then denote by \( P(\lambda_0, F_{\omega,\omega'}^h(\lambda)) \) the polar part of the Laurent expansion at the point \( \lambda = \lambda_0 \) of this meromorphic function, which is independent of \( \rho \) by our assumption. Then first remark that if \( \omega = du \) where \( u \in \Omega^n(X) \) satisfies \( df \wedge u = 0 \) then Stokes formula gives for \( \Re(\lambda) \gg 1 \)

\[
0 = \int_X d(\rho.|f|^{2\lambda}.\bar{f}^h.u \wedge \bar{\omega}') = \int_X \rho.|f|^{2\lambda}.\bar{f}^h.\omega \wedge \bar{\omega}' + \int_X |f|^{2\lambda}.\bar{f}^h.d\rho \wedge u \wedge \bar{\omega}'
\]

\(^6\)The existence of the Bernstein polynomial of \( f \) at the origin is enough to obtain the asymptotic expansion of this integral to the complex plane; see for instance [11].
showing, thanks to the analytic extension and our hypothesis, that
\[ P(\lambda_0, F_{\omega, \omega'}^h(\lambda)) = 0. \]
This implies that \( P(\lambda_0, F_{\omega, \omega'}^h(\lambda)) \) only depends (under our assumption) on the class induced by \( \omega \) in the quotient \( E_{f, 0} := \Omega_{X, 0}^{n+1}/d(\text{Ker}(df))_0 \).

Assume now that \( (a - (\lambda_0 - 1).b)[\omega] = 0 \) in \( E_{f, 0} \). Note that this means that \(-\lambda_0 + 1\) is a root of the Bernstein polynomial of the Fresco \( \tilde{A}.[\omega] \subset \tilde{E}_{f, 0} \) associated to \((f, \omega)\). Writing \( \omega = du \) near 0 and shrinking the support of \( \rho \) if necessary we obtain, applying the previous computation to \( \alpha := f.du - (\lambda_0 - 1).df \wedge u \)
\[ P(\lambda_0, F_{\alpha, \omega'}^h(\lambda)) = 0. \]
But we have
\[ F_{f.du, \omega'}^h(\lambda) = \int_X \rho |f|^{2(\lambda+1)} j^{h-1}.du \wedge \bar{\omega}' \]
and Stokes formula, as above, implies that
\[ P(\lambda_0, F_{f.du, \omega'}^h(\lambda)) = -P(\lambda_0, (\lambda + 1).F_{df \wedge u, \omega'}^h(\lambda)) \]
\[ = - (\lambda - \lambda_0).P(\lambda_0, F_{df \wedge u, \omega'}^h(\lambda)) + (\lambda_0 - 1).P(\lambda_0, F_{df \wedge u, \omega'}^h(\lambda)) \]
and then
\[ (\lambda - \lambda_0).P(\lambda_0, F_{df \wedge u, \omega'}^h(\lambda)) = 0 \quad (10) \]
that is to say that the meromorphic extension of \( F_{df \wedge u, \omega'}^h(\lambda) \) has at most a simple pole at \( \lambda = \lambda_0 \) for any \( h \) and any \( \omega' \).
This will be the core of our next result.

### 3.2 The main result.

In the sequel we fix a positive integer \( q \) and a class \( \xi \in \mathbb{Q}/\mathbb{Z} \). We shall denote \( \mathcal{P}_{\xi, q}^{n+1} \) the quotient of the space of meromorphic functions on \( \mathbb{C} \) with poles contained in the open set \( \{\Re(\lambda) < 0\} \) and of order at most \( n + 1 \) by the subspace of meromorphic functions having poles of order at most \( q - 1 \) at the points \( \xi + \mathbb{Z} \).

This quotient is in a natural way a \( \mathbb{C}[\lambda] \)–module as multiplication by a polynomial preserves the meromorphy and does not increase the order of poles.

An other operation on this quotient is the shift operator, denoted \( Sh \), which is induced on \( \mathcal{P}_{\xi, q}^{n+1} \) by the map \( F(\lambda) \mapsto F(\lambda+1) \) which translate by \(-1\) the localization of poles.

Remark that any formal power series in \( \mathbb{C}[[Sh]] \) will act on \( \mathcal{P}_{\xi, q}^{n+1} \).

We consider \( \hat{f} : (\mathbb{C}^{n+1}, 0) \rightarrow (\mathbb{C}, 0) \) a germ of non constant holomorphic function and a Milnor representative \( f : X \rightarrow D \) of this germ. We assume in the sequel the following hypothesis :\(^7\)

\(^7\)In fact we could restrict ourself to the space of meromorphic functions having poles located on a finite union of subsets of the form \( \alpha + \mathbb{Z} \) where \( \alpha \) is in a finite set of rational numbers.
• For a given \( \xi \in \mathbb{Q}/\mathbb{Z} \) and a \( q \in \mathbb{N}^* \) the meromorphic extension of

\[
\frac{1}{\Gamma(\lambda + 1)} \cdot \int_X |f|^{2\lambda} \cdot \tilde{f}^h \cdot \varphi \quad (H(\xi, q))
\]

has poles of order at most \( q - 1 \) at points of \( \xi + \mathbb{Z} \) for any \( h \in \mathbb{Z} \) and any differential form \( \varphi \in \mathcal{C}_c^\infty(X \setminus \{0\})^{n+1,n+1} \).

For \( q = 1 \) this says that for any \( \varphi \in \mathcal{C}_c^\infty(X) \) which is identically 0 near the origin there is no pole for \( \lambda \in \xi + \mathbb{Z} \) for any \( h \in \mathbb{Z} \). This is true, for instance, if the monodromy of \( f \) (acting on the reduced cohomology of the Milnor fiber) at each point \( x \neq 0 \) nearby 0 does not admit the eigenvalue \( \exp(2i\pi, \xi) \).

Now fix \( \rho \in \mathcal{C}_c^\infty(X) \) such that \( \rho \equiv 1 \) in a neighbourhood of the origin and two \((n + 1)-\)holomorphic forms \( \omega \) and \( \omega' \) on \( X \). Then define, for \( h \in \mathbb{Z} \), a meromorphic function on \( \mathbb{C} \) as the meromorphic extension of the holomorphic function defined for \( \Re(\lambda) \gg 1 \) by

\[
F_h^{\omega,\omega'}(\lambda) := \frac{1}{\Gamma(\lambda + 1)} \cdot \int_X |f|^{2\lambda} \cdot \tilde{f}^h \cdot \rho \cdot \omega \wedge \bar{\omega}'
\]

(8)

Note that our hypothesis \( (H(\xi, q)) \) implies that the class induced in \( \mathcal{P}_{n+1}^\xi \) by \( F_h^{\omega,\omega'}(\lambda) \) is independent of the choice of such a \( \rho \).

**Lemma 3.2.1** Under the hypothesis \( H(\xi, q) \) we have for each \( h \in \mathbb{Z} \) a sesqui-linear map \( \Omega_n^{n+1} \times \Omega_0^{n+1} \rightarrow \mathcal{P}_{n+1}^{\xi,q} \) given by \( (\omega, \omega') \rightarrow [F_h^{\omega,\omega'}] \in \mathcal{P}_{n+1}^{\xi,q} \). It is independent of the choice of \( \rho \) and vanishes when \( \omega \) (resp. \( \omega' \)), is in \( d(Ker df^n_0) \) where

\[
Ker df^n_0 := Ker [\wedge df : \Omega_0^n \rightarrow \Omega_0^{n+1}] \]

So it induces, for any given \( h \in \mathbb{Z} \), a sesqui-linear map

\[
\Phi_h^{\xi,q} : E_0 \times E_0 \rightarrow \mathcal{P}_{n+1}^{\xi,q}
\]

where \( E_0 := \Omega_0^{n+1}/d(Ker df^n_0) \).

**Proof.** For any given two germs \( \omega, \omega' \in \Omega_0^{n+1} \) we can find a Milnor representative \( f : X \rightarrow D \) of \( \tilde{f} \) such that \( \omega \) and \( \omega' \) have representative on \( X \). Then we may choose \( \rho \in \mathcal{C}_c^\infty(X) \) which is identically equal to 1 near 0 and define \( \Phi_h^{\xi,q}[\omega, \omega'] \) as the class in \( \mathcal{P}_{n+1}^{\xi,q} \) of the meromorphic extension of the function defined by (8).

This is independent of the choice of \( \rho \) because of the hypothesis \( H(\xi, q) \), and so independent of all choices for the given \( f, \omega, \omega' \) and \( h \). The sesqui-linearity is obvious. Let us show that if we have \( \omega = du \) with \( u \in \Gamma(X, Ker df^n) \) then we have \( \Phi_h^{\xi,q}[\omega, \omega'] = 0 \).

Write for \( \Re(\lambda) \gg 1 \)

\[
d(|f|^{2\lambda} \cdot \tilde{f}^h \cdot \rho \cdot u \wedge \bar{\omega}') = |f|^{2\lambda} \cdot \tilde{f}^h \cdot d\rho \wedge u \wedge \bar{\omega}' + |f|^{2\lambda} \cdot \tilde{f}^h \cdot \rho \cdot \omega \wedge \bar{\omega}'.
\]
As \( \phi := \varphi \wedge \omega \wedge \bar{\omega}' \) is \( C^\infty \) and has compact support in \( X \setminus \{0\} \) the hypothesis \( H(\xi, q) \) allows to conclude using meromorphic extension and Stokes’ formula. 

Note that if \( \omega' = dv \) with \( v \in d(Ker df^\omega_\xi) \) we have the same conclusion.

Now we shall use the action of \( a := x \cdot f \) and \( b := df \wedge d\omega^{-1} \) on \( E_0 \) and we shall denote by \( \tilde{E}_0 \) the \( b \)-completion of \( E_0 \) modulo its \( b \)-torsion. As we have seen in the previous section \( \tilde{E}_0 \) is a geometric \((a, b)\)-module.

The next lemma gives the behavior of the maps \( \Phi^{\xi, q}_{\xi, q} \) for \( (\xi, q) \) fixed, under the actions of \( a \) and \( b \) on \( E_0 \). As a corollary, we shall obtain that the maps \( \Phi^{\xi, q}_{\xi, q} \) extend to the \( b \)-completion of \( E_0 \) and pass to the quotient by its \( b \)-torsion so gives sesqui-linear maps on \( \tilde{E}_0 \) which satisfy the same properties.

**Lemma 3.2.2** Under the hypothesis \( H(\xi, q) \) we have, for each \( h \in \mathbb{Z} \), the following relations, where \( Sh : \mathcal{P}_{n+1} \to \mathcal{P}_{n+1} \) is the shift operator by \(-1\) defined above:

1. \( \Phi^{\xi, q}_h[a.\omega, \omega'](\lambda) = Sh(\lambda.\Phi^{\xi, q}_{\xi, q}[\omega, \omega'](\lambda)) = (\lambda + 1).Sh(\Phi^{\xi, q}_{h-1}[\omega, \omega'](\lambda)) \);
2. \( \Phi^{\xi, q}_h[(a + (\lambda + 1).b)\omega, \omega'](\lambda) = 0 \);
3. \( \Phi^{\xi, q}_h[b.\omega, \omega'](\lambda) = Sh(\Phi^{\xi, q}_{h-1}[\omega, \omega']) \);
4. For any \( \xi_1 \in \mathbb{C} \) we have \( \Phi^{\xi, q}_h[(a + \xi_1.b)(\omega), \omega'](\lambda) = Sh((\lambda + \xi_1).\Phi^{\xi, q}_{h-1}[\omega, \omega'](\lambda)) \).

**Proof.** First note that the formulas ii) and iii) use the \( \mathbb{C}[\lambda] \)-linear extension of \( \Phi^{\xi, q}_h \) to \( (\mathbb{C}[\lambda] \otimes \mathcal{E}_0) \times E_0 \) given by

\[
(P(\lambda) \otimes \omega, \omega') \mapsto [P(\lambda), \Phi^{\xi, q}_h[\omega, \omega'](\lambda)]
\]

using the natural action of \( \mathbb{C}[\lambda] \) on \( \mathcal{P}_{n+1}^{\xi, q} \).

The formula i) is a reformulation of the obvious following formula, as \( a.\omega := f.\omega, \)

\[
\frac{1}{\Gamma(\lambda + 1)}.\int_X |f|^{2\lambda} \cdot \bar{f}^{\omega}.\rho.f.\omega \wedge \bar{\omega}' = \frac{\lambda + 1}{\Gamma(\lambda + 2)}.\int_X |f|^{2(\lambda + 1)} \cdot \bar{f}^{\omega-1}.\rho.\omega \wedge \bar{\omega}'.
\]

and the fact that

\[
Sh\left(\frac{\lambda}{\Gamma(\lambda + 1)}.\int_X |f|^{2\lambda} \cdot \bar{f}^{\omega-1}.\rho.\omega \wedge \bar{\omega}'\right) = \frac{\lambda + 1}{\Gamma(\lambda + 2)}.\int_X |f|^{2(\lambda + 1)} \cdot \bar{f}^{\omega-1}.\rho.\omega \wedge \bar{\omega}'.
\]

To prove the formula ii) write for \( \Re(\lambda) \gg 1 \), if \( u \in \Omega_0^\ast \) satisfies \( du = \omega \)

\[
d(|f|^{2\lambda} \cdot \bar{f}^{\omega}.\rho.f.u \wedge \omega') = (\lambda + 1).|f|^{2\lambda} \cdot \bar{f}^{\omega}.\rho.d\omega \wedge \omega' + |f|^{2\lambda} \cdot \bar{f}^{\omega}.\rho.f.d\omega \wedge \omega' + |f|^{2\lambda} \cdot \bar{f}^{\omega}.d\rho \wedge f.u \wedge \omega'.
\]

and the last term of the second handside, after integration and multiplication by \( \frac{1}{\Gamma(\lambda + 1)} \), is sent to 0 by \( \Phi^{\xi, q}_h \), thanks to our hypothesis \( H(\xi, q). \) As the left handside will give 0 after integration by the Stokes’ formula, the conclusion follows from the equalities \( a.\omega = f.\omega \) and \( b.\omega = df \wedge u. \)

The formulas iii) and iv) are direct consequences of i) and ii).
Remarks.

1. The formula iii) implies that the maps $\Phi_{h}^{\xi,q}$ vanish on the $b-$torsion of $E_0$ as the shift operator is injective on $P_{n+1}^{\xi,q}.$

2. The formulas i) and iii) of the lemma above show that the action of $a$ and $b$ shift the poles of $-1$ and also the integer $h$ by $-1.$ So, if we act on $E_0$ by a formal power serie in $(a,b),$ the maps $\Phi_{h}^{\xi,q}$ extends in a natural way to the formal completion in $b$ of $E_0$ modulo its $b-$torsion. This completion is a geometric $(a,b)$-module thanks to the theorem 2.3.1 and also a $\hat{\mathcal{A}}$-module thanks to the proposition 2.2.2.

3. In the formula i) the factor $\lambda$ does not change the orders of poles involved as we assume that in $P_{n+1}^{\xi,q}$ the poles are in the open set $\{\Re(\lambda) < 0\}. \Phi_{h}^{\xi,q}[(a - \lambda,b)\omega, \omega']$ up to replace $\xi_0$ by $\xi_0 - 1$ and $h_0$ by $h_0 + 1.$

Corollary 3.2.3 Assume that the hypothesis $H(\xi,q)$ is satisfied for $\tilde{f},$ and that $\omega$ and $\omega'$ are given in $\Omega_{n+1}^{\xi,q}.$ Let $q + d$ be the maximal order of pole for $\Phi_{h}^{\xi,q}[\omega, \omega'](\lambda)$ at a point in $\xi + Z$ for any $h \in Z.$

i) Let $\xi_0$ be the maximal point in $\xi + Z$ where such a pole occurs for some $h_0.$ Then, for any $S \in \mathbb{C}[[b]]$ such that $S(0) = 1,$ we have again a pole of order $q + d$ at $\xi_0$ for $\Phi_{h_0}^{\xi,q}[S(b)\omega, \omega'].$

ii) In the same situation than in i) let $\xi_1 \neq \xi_0.$ Then we shall have a pole of order $q + d$ at $\xi_0$ for $\Phi_{h_0+1}^{\xi,q}[(a + \xi_1,b)(\omega), \omega'].$ Moreover, $\xi_0 - 1$ is maximal in $\xi + Z$ such that for some $h \in Z$ there exists an order $q + d$ pole for $\Phi_{h}^{\xi,q}[(a + \xi_1,b)(\omega), \omega'].$

iii) In the same situation than in i) assume that $d \geq 1.$ Then we shall have a pole of order $q + d - 1$ at $\xi_0$ for $\Phi_{h_0+1}^{\xi,q}[(a + \xi_0,b)(\omega), \omega'].$ Moreover, $\xi_0$ is still maximal in $\xi + Z$ in order that for some $h \in Z$ there exists an order $q + d - 1$ pole for $\Phi_{h}^{\xi,q}[(a + \xi_0,b)(\omega), \omega'].$

Proof. To prove i) it is enough to remark that for any $k \geq 1$ the pole of $\Phi_{h}^{\xi,q}[b^k(\omega), \omega']$ at $\xi_0$ is of order at most $q + d - 1$ for any $h \in Z$ by the maximality of $\xi_0$ and the formula iii) of the lemma 3.2.2.

The proof of ii) and iii) are direct consequences of the formula iv) of the same lemma 3.2.2.

Remark. Note that the conclusion of i) in the previous lemma would be the same if $S$ would have been a formal power series in $a$ an $b$ with constant term equal to 1 using the formulas i) and iii) of the lemma 3.2.2. As a consequence, it shows that
the maximal point in $[\xi]$ where the order of the pole is $q+d$ for some $h \in \mathbb{Z}$ is independent of the choice of the generator $\omega$ of the fresco

$$F_\omega := \tilde{A}\omega \subset E^{n+1}/B(E^{n+1}).$$

We shall note $k$ (resp. $k'$) the rank (over $\mathbb{C}[b]$) of $F_\omega$ (resp. of $F_{\omega'}$).

**Theorem 3.2.4** Let $\tilde{f} : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ be a non constant holomorphic germ. Fix $[\xi] \in \mathbb{Q}/\mathbb{Z}$ and a positive integer $q$. Assume that the hypothesis $H(\xi, q)$ holds for $\tilde{f}$ and consider $\omega$ and $\omega'$ two germs of $(n+1)$–holomorphic forms. Let $q+d$ be the maximal order of pole for $\Phi_{h_0}^{\xi,q}[\omega, \omega'](\lambda)$ at a point in $\xi + \mathbb{Z}$ for any $h \in \mathbb{Z}$.

Assume that there exists some $h_0 \in \mathbb{Z}$ and a pole of order $q+d$ at $\xi_0 \in [\xi]$ for $\Phi_{h_0}^{\xi,q}[\omega, \omega']$, and choose $\xi_0$ maximal in $[\xi]$ such that this happens (for some $h$). Then $\xi_0$ is a root of the Bernstein polynomial of $F_\omega$ and there exists at least $d$ roots of the Bernstein polynomial of $F_\omega$ in $(\xi_0 + N) \cap [\xi_0, 0]$. Moreover, let $\xi_1 = \xi_0 + h_1$, with $h_1 \in \mathbb{Z}$, be maximal such that $\Phi_{h_1}^{\xi_1,q}[\omega, \omega']$ has a pole of order $q+d$ at $\xi_1$; then $\xi_1$ is a root of the Bernstein polynomial of $F_{\omega'}$ and there exists at least $d$ roots of the Bernstein polynomial of $F_{\omega'}$ in $(\xi_1 + N) \cap [\xi_1, 0]$.

**Proof.** Let $k$ be the rank (as a $\mathbb{C}[b]$–module) of the fresco $F_\omega$. Then, as recalled in section 2 theorem 2.3.1 there exists an element $\Pi \in \tilde{A}$ which generates the left ideal in $\tilde{A}$ which is the annihilator of $[\omega]$ in the $\tilde{A}$–module given by the geometric $(a,b)$-module $E^{n+1}/B(E^{n+1})$. This element can be chosen of the form

$$\Pi = (a - \lambda_1.b).S_1^{-1} \ldots (a - \lambda_k.b).S_k^{-1}$$

where the $S_j, j \in [1, k]$ are in $\mathbb{C}[b]$ and satisfy $S_j(0) = 1, \forall j$, and where the Bernstein element of the fresco $F_\omega$ is $P_\omega := (a - \lambda_1.b) \ldots (a - \lambda_k.b)$. Recall that the roots of the Bernstein polynomial of $F_\omega$ are then equal to $-\lambda_j - j + k$ for $j \in [1, k]$.

Now using inductively the assertions i) ii) and iii) of the corollary 3.2.3 we see that we have at least $d$ occurrences of $\xi_0 + N$ in the set of the roots $\{-(\lambda_j + j - k), j \in [1, k]\}$ of the Bernstein polynomial of $F_\omega$. The maximality of $\xi_0$ implies that $\xi_0$ is one of them.

The second statement is easily deduced of the first one by complex conjugation; as the conjugate (in the sense $F(\lambda) \mapsto F(\overline{\lambda})$) of

$$\frac{1}{\Gamma(\lambda + 1)} \int_X |f|^{2\lambda}.\tilde{f}.\rho.\omega \wedge \overline{\omega'}$$

is given by

$$\frac{1}{\Gamma(\lambda + h + 1)} \int_X |f|^{2(\lambda + h)}.\tilde{f}.\rho.\omega' \wedge \overline{\omega} =$$

$$\frac{1}{\Gamma(\mu - h + 1)} \int_X |f|^{2\mu}.\tilde{f}.\rho.\omega' \wedge \overline{\omega}. \quad (2)$$
with $\mu := \lambda + h$. Let now $h = h_1$.

Remark that the meromorphic extension of (1) has no pole for $\Re(\lambda) \geq 0$ of for $\Re(\lambda + h) \geq 0$ using the negativity of the roots of the Bernstein polynomial of $f$ and the fact that we may use either the Bernstein identity of $f$ to make the meromorphic extension or its conjugate.

We have $\Gamma(\lambda + h_1 + 1) = (\lambda + 1) \ldots (\lambda + h_1)\Gamma(\lambda + 1)$ and as the integral (1) has no pole for $\Re(\lambda + h_1) \geq 0$ and for $\Re(\lambda) \geq 0$, so, to replace $\Gamma(\lambda + 1)$ by $\Gamma(\lambda + h_1 + 1)$ in (1) does not change the orders of poles; so we can replace $\Gamma(\mu - h_1 + 1)$ by $\Gamma(\mu + 1)$ in (2) without changing the order of poles and apply the previous result to the pair $(\omega', \omega)$. ■

Remark. We may consider also the case of the $(a,b)$-module $E^p$ which appears in the theorem 2.3.1 by considering the Hermitian periods of the form

$$\frac{1}{\Gamma(\lambda + 1)} \int_X |f|^{2\lambda} \bar{f}^h \rho. K^{n+1-p} \wedge \omega \wedge \bar{\omega}'$$

where $K$ is a Kähler form near the origin in $\mathbb{C}^{n+1}$, $\rho \in \mathcal{C}^\infty_c(X)$ is identically equal to 1 near 0 and $\omega, \omega'$ are in $\Omega^p_0$. Then, using the same line of proof we obtain an analogous result in this case.

4 The case of a polynomial with $(n + 1)$ variables and $(n + 2)$ monomials

The purpose of this section is to give a general algorithm in order to obtain an "estimate" of the Bernstein polynomial of the fresco associated to $(f, \omega)$ for any polynomial $f \in \mathbb{C}[x_0, \ldots, x_n]$ with $(n + 2)$ monomials and any monomial volume form $\omega = x^\beta dx$ where $\beta$ is in $\mathbb{N}^{n+1}$ (we exclude the quasi-homogeneous case which is obvious). Using the results of the previous sections we obtain a rather precise information of the exponents of the asymptotic expansions of the period integrals $\int_{\gamma_z} \omega/df$ where $(\gamma_z)_{z \in H}$ is an horizontal family of $n-$cycles in the fibers of $f$. This gives also rather precise informations on the poles of the meromorphic extensions of the Mellin transform of the hermitian periods $\int_{f-x} \rho. \omega/df \wedge \omega'/df$:

$$\frac{1}{\Gamma(\lambda + 1)} \int_{\mathbb{C}^{n+1}} |f|^{2\lambda} \bar{f}^h \rho. \omega \wedge \bar{\omega}'$$

where $\rho \in \mathcal{C}^\infty_c(\mathbb{C}^{n+1})$ satisfies $\rho \equiv 1$ near 0 and where $\omega, \omega'$ are monomial volume forms. We shall illustrate the result by several examples.
4.1 Our setting

We consider a polynomial \( f \in \mathbb{C}[x_0, \ldots, x_n] \) which is the sum of \( n+2 \) monomials
\[
  f = \sum_{j=1}^{n+2} m_j
\]
where \( m_j := \sigma_j x^{\alpha_j} \), with \( \sigma_j \in \mathbb{C}^* \) and \( \alpha_j \in \mathbb{N}^{n+1} \) are not 0. Define the matrix
with \((n+1)\) lines and \((n+2)\) columns \( M = (\alpha_{i,j}) \) and let \( \tilde{M} \) be the square
\((n+2, n+2)\) matrix obtained from \( M \) by adding a first line equal to \((1, \ldots, 1)\).

We shall assume the following conditions:

(C1) \( \alpha_1, \ldots, \alpha_{n+1} \) is a \( \mathbb{Q} \)-basis of \( \mathbb{Q}^{n+1} \).

(C2) The rank of \( \tilde{M} \) is \( n + 2 \).

Remarks.

1. Only the condition (C2) is restrictive on \( f \): when (C2) is fulfilled the condition
(C1) may always be satisfied without changing \( f \) by a suitable ordering of the
\( n+2 \) monomials.

2. The condition (C2) is equivalent to the fact that \( f \) is not quasi-homogeneous.

A diagonal linear change of variables allows to reduce the study to the case where

\[
f(x) = \sum_{j=1}^{n+1} x^{\alpha_j} + \lambda x^{\alpha_{n+2}}
\]  \hspace{1cm} (a)

for some \( \lambda \in \mathbb{C}^* \). So in what follows, we shall assume that \( m_j = x^{\alpha_j} \) for \( j \in [1, n+1] \)
and \( m_{n+2} = \lambda x^{\alpha_{n+2}} \) where \( \lambda \in \mathbb{C}^* \) is a parameter.

Then we shall write, using our hypothesis (C1):

\[
  \alpha_{n+2} = \sum_{j=1}^{n+1} \rho_j \cdot \alpha_j \quad \text{where} \quad \rho_j \in \mathbb{Q}.
\]  \hspace{1cm} (b)

where \( \rho_j \) are rational numbers. We define

\[
  H := \{ j \in [1, n+1] / \rho_j = 0 \};
\]
\[
  J_+ := \{ j \in [1, n+1] / \rho_j > 0 \};
\]
\[
  J_- := \{ j \in [1, n+1] / \rho_j < 0 \}.
\]

Let \( |r| \) be the smallest positive integer such that \( |r| \cdot \rho_j := p_j \) is an integer for each
\( j \in [1, n+1] \). Write now the relation above as

\[
  |r| \cdot \alpha_{n+2} + \sum_{j \in J_-} (-p_j) \cdot \alpha_j = \sum_{j \in J_+} p_j \cdot \alpha_j.
\]  \hspace{1cm} (c)
Now define $d + h$ and $d$ as respectively the supremum and infimum of the two numbers $|r| + \sum_{j \in J_-} (-p_j)$ and $\sum_{j \in J_+} p_j$.

Then $d$ and $h$ are positive:
The positivity of $d$ is consequence of the fact that $|r| \geq 1$ and that at least one $p_j$ is positive.
The positivity of $h$ is consequence of the fact that the equality of these two integers would imply that the first line in $\tilde{M}$ satisfies the same linear relation ($b$) than all the other lines in $\tilde{M}$, contradicting our hypothesis (C2).

Now the relation above gives the relation between the monomials $(m_j)_{j \in [1, n+2]}$:
\[
m_n^{[r]} \prod_{j \in J_-} m_j^{-p_j} = \lambda |r|, \prod_{j \in J_+} m_j^{p_j}
\]
and we shall write it
\[
m^\Delta = \lambda^r \cdot m^\delta
\]
where $\Delta$ and $\delta$ are in $\mathbb{N}^{n+2}$ of respective length $d + h$ and $d$.
Remark that $\Delta_h$ and $\delta_h$ are zero for each $h \in H$.
Note that the relation (e) defines the sign of $r$ which is equal to $\pm |r|$ and so $r$ is in $\mathbb{Z}^*$. We shall also use the following observation later on:

**Lemma 4.1.1** The $h$–th element of the first column of the matrix $\tilde{M}^{-1}$ is zero if and only if $h$ is in $H$.

**Proof.** The co-factor of the element $(1, h)$ in $\tilde{M}$ is the $(n + 1, n + 1)$ determinant of the matrix with columns $\alpha_1, \ldots, \hat{\alpha}_h, \ldots, \alpha_{n+1}$. This matrix has rank at most $n$ if and only if $\alpha_{n+2}$ is a linear combination of $\alpha_1, \ldots, \hat{\alpha}_h, \ldots, \alpha_{n+1}$. This is the case if and only if $\rho_h = 0$, thanks to our hypothesis (C1). 

4.2 The result

Let $\Omega^p$ be the $\mathbb{C}[x_0, \ldots, x_n]$–module of algebraic $p$–differential forms on $\mathbb{C}^{n+1}$ and fix a polynomial $f \in \mathbb{C}[x_0, \ldots, x_n]$ with $(n + 2)$ monomials
\[
f := \sum_{j=1}^{n+2} m_j
\]
where $m_j := x^{\alpha_j}$ for $j \in [1, n + 1]$ and $m_{n+2} := \lambda x^{\alpha_{n+1}}$ satisfying the conditions (C1) and (C2) above.

We define $df^p : \Omega^p \to \Omega^{p+1}$ the $\mathbb{C}[x_0, \ldots, x_n]$–linear map given by exterior product by $df$ and we note $\text{Ker}(df)^p$ its kernel.
Let \( E_f := \Omega^{n+1}/d(Ker(df))^n \) endowed with its natural structure of module over the \( \mathbb{C} \)-algebra \( \mathcal{A} := \mathbb{C}(a, b) \) where the variables \( a \) and \( b \) satisfy the commutation relation \( a.b - b.a = b^2 \). Recall that on \( E_f \) the action of \( a \) is the multiplication by \( f \) and the action of \( b \) is given by \( df \wedge d^{-1} \) where \( d \) is the de Rham differential (which is surjective on \( \Omega^{n+1} \)).

We extend this structure of (left) \( \mathcal{A} \)-module to \( E_f[\lambda] := E_f \otimes_{\mathbb{C}} \mathbb{C}[\lambda] \) by asking that \( a \) and \( b \) are \( \mathbb{C}[\lambda] \)-linear.

**Lemma 4.2.1** For any monomial \( x^\beta \) the image of \( \mathbb{C}[m_1, \ldots, m_{n+2}]x^\beta \) via the map defined by
\[
 m^n \mapsto \lambda^{n+2} \prod_{i=0}^{n} x_i^{\sum_{j=1}^{n+2} \alpha_{i,j} \eta_j} \quad \text{where} \quad \eta \in \mathbb{N}^{n+2},
\]
is a sub-\( \mathcal{A} \)-module of \( E_f[\lambda] \).

**Proof.** We want to prove that this image is stable by the action of \( a \) and \( b \).

The stability by \( a \) is obvious because \( a \) is given by the multiplication by \( \sum_{j=1}^{n+2} m_j \).

Let \( \gamma \in \mathbb{N}^{n+1} \) and compute \( b \) using a primitive in \( x_i, i \in [0, n] \):
\[
b(x^{\gamma+\beta}\cdot dx) = \frac{1}{\gamma_i + \beta_i + 1} x_i x^{\gamma+\beta} \frac{\partial f}{\partial x_i} \cdot dx.
\]

Remark now that \( x_i \frac{\partial f}{\partial x_i} \cdot dx = \sum_{j=1}^{n+2} \alpha_{i,j} m_j \) so the previous computation gives with \( \gamma_i := \sum_{j=1}^{n+2} \alpha_{i,j} \eta_j \) when \( \eta \) is in \( \mathbb{N}^{n+2} \),
\[
\Gamma_i(\eta, \beta) b(m^n.x^\beta \cdot dx) = \sum_{j=1}^{n+2} \alpha_{i,j} m_j m^n.x^\beta \cdot dx \quad \forall i \in [0, n],
\]
where \( \Gamma_i(\eta, \beta) := 1 + \beta_i + \sum_{j=1}^{n+2} \alpha_{i,j} \eta_j \).

Note that we have also
\[
a(m^n.x^\beta \cdot dx) = \sum_{j=1}^{n+2} m_j m^n.x^\beta \cdot dx
\]
\( \text{(@n+1)} \)

The formulas (\( \text{@i} \)) for \( i \in [0, n+1] \) are enough to conclude the proof. \( \blacksquare \)

**Corollary 4.2.2** Fix \( \beta \in \mathbb{N}^{n+1} \). For each \( \eta \in \mathbb{N}^{n+2} \) there exists an element \( P_{\beta, \eta}(a, b) \) in \( \mathcal{A} \), homogeneous of degree \( q := |\eta| := \sum_{j=1}^{n+2} \eta_j \) in \( a, b \) such that:

1. There exists \( c(\beta, \eta) \in \mathbb{Q}^* \) such that \( P_{\beta, \eta}(a, b)[x^\beta \cdot dx] = c(\beta, \eta).m^n.x^\beta \cdot dx \) in \( E_f \).

2. Assuming that \( \eta \) satisfies \( \eta_j = 0 \) for each \( j \in H \), there exists rational numbers (depending on \( \beta \) and \( \eta \)) \( r_1, \ldots, r_q \) such that \( P_{\beta, \eta}(a, b) = \prod_{h=1}^{q} (a - r_h b) \) in \( \mathcal{A} \).
Proof. Let first show by induction on $q \geq 0$ that such a $P_{\beta, \eta}(a, b)$ satisfying 1) exists. As for $q = 0$ the assertion is clear with $P \equiv 1$, assume that our assertion is proved for any $\eta$ with $|\eta| = q - 1$ with $q \geq 1$. Then it is enough to prove the assertion for $m_j, m^\eta$ for each $j \in [1, n + 2]$ and each $\eta$ with $|\eta| = q - 1$.

Then consider the equations (\@i) for $i \in [0, n + 1]$ as a square $Q$-linear system of size $(n + 2, n + 2)$ with unknown the elements $m_j, m^\eta, x^\beta$ in $E_f$. The matrix of this system is in $Gl(Q, n + 2)$ thanks to our hypothesis (C2), and so there exists rationals numbers $u_j$ and $v_j$ such that we have, for each $j \in [1, n + 2]$

$$m_j, m^\eta, x^\beta = (u_j, a + v_j, b)(m^\eta, x^\beta).$$

With our induction hypothesis this gives that $P_{\beta, \eta+1}(a, b) := (u_j, a + v_j, b)P_{\beta, \eta}(a, b)$ satisfies 1) without the fact that this homogeneous element in $(a, b)$ of degree $q$ is monic in $a$ when $\eta_h = 0$ for any $h \in H$. To show this last point it is enough to prove that $u_j \neq 0$ when $j \notin H$. Assuming this fact in our induction, the induction step is given by the lemma 4.1.1.

**Theorem 4.2.3** Assume that $f \in \mathbb{C}[x_0, \ldots, x_n]$ has $(n + 2)$ monomials and satisfies the conditions (C1) and (C2) described above. Let $d$ and $h$ the positive integers defined after (c) and $r \in \mathbb{Z}^*$ defined in (c) and (e). For each $\beta \in \mathbb{N}^{n+1}$ there exists homogeneous elements $P_{d+h}$ and $P_d$ of respective degrees $d + h$ and $d$ which are products of homogeneous factors of degree 1 of the form $a - \xi b$ where $\xi \in \mathbb{Q}$ such that

$$\left( P_{d+h}(a, b) - c \cdot \lambda \cdot P_d(a, b) \right)[x^\beta] = 0 \text{ in } E_f[\lambda] \text{ where } c \in \mathbb{Q}^*$$

Proof. Applying the previous corollary to both sides of the equality in $E_f[\lambda]$ $m^\Delta, x^\beta = \lambda' \cdot m^\delta, x^\beta$ deduced from (e) and the corollary 4.2.2 allows to conclude because we know that $\Delta_h = 0$ and $\delta_h = 0$ for each $h \in H$. This theorem has the following corollary.

**Corollary 4.2.4** Let $f \in \mathbb{C}[x_0, \ldots, x_n]$ as in the previous theorem and choose any monomial $x^\beta$. Define for any horizontal family of $n$-cycles $(\gamma_s)_{s \in S}$ over a simply connected open set $S$ in $\mathbb{C}^*$ avoiding the critical values of $f$ and having 0 in its boundary, the period integral

$$\varphi_{\beta}(s) := \int_{\gamma_s} x^\beta / df$$

Then $\varphi_{\beta}$ is solution on $S$ of the differential equation (which is regular singular at 0) obtained from (11) by letting $a = \times s$ and $b := \int_0^s$.  

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Proof. Thanks to the proposition 2.3.2 the Bernstein element of the fresco generated by \([x^\beta \cdot dx]\) in \(E_f \otimes_\Lambda A\) is a left multiple of \(P_d(a, b)\) as \(c\) and \(\lambda\) are not 0. This is enough to conclude. ■

This shows that the computation of \(P_d(a, b)\) gives rather precise informations on the asymptotic expansion at 0 of such a period integral.

We leave the corresponding statement for the poles of the Mellin transform of the hermitian period integrals corresponding to such an \(f\) and monomials volume forms \(\omega\) and \(\omega'\) to the reader. Of course, by conjugaison, the Bernstein polynomial of the fresco \((f, \omega')\) also gives constraints on the possible poles of this Mellin transform, as in the theorem 3.2.4.

4.3 Examples

The control of the Bernstein polynomial of a fresco will use the theorem 4.2.3 and the proposition 2.3.2.

4.3.1 \(f_\lambda := x^5 + y^5 + z^5 + \lambda x y z^2\)

We assume that \(\lambda\) is a non zero complex number. Then 0 is the only singular point of the hypersurface \(\{f = 0\}\): as on the set \(\Sigma := \{df = 0\} \subset \mathbb{C}^3\) we have \(f(x, y, z) = \frac{1}{5} \lambda x y z\), we easily deduced that \(\Sigma \cap \{f = 0\} = \{0\}\).

Now using the method explained above we obtain, after some elementary computations, for each monomial form \(\omega\) below a degree 4 polynomial multiple of the Bernstein polynomial of the fresco \(F_\omega\). Note that such a fresco has rank at most equal to 4 and if the rank is equal to 4 then we obtain the Bernstein polynomial itself.

Of course, the reader interested by more monomials can easily complete this list, where \(|\) means “divides”:

- \(\omega = dx \wedge dy \wedge dz\) \(B_1(\xi) \mid (\xi + \frac{7}{10})(\xi + \frac{4}{5})(\xi + \frac{6}{5})\).
- \(\omega = x \cdot dx \wedge dy \wedge dz\) \(B_x(\xi) \mid (\xi + \frac{9}{10})(\xi + 1)(\xi + \frac{6}{5})(\xi + \frac{7}{5})\).
- \(\omega = z \cdot dx \wedge dy \wedge dz\) \(B_z(\xi) \mid (\xi + 1)^3(x + \frac{3}{2})\).
- \(\omega = z^2 \cdot dx \wedge dy \wedge dz\) \(B_{z^2}(\xi) \mid (\xi + \frac{6}{5})^2(\xi + \frac{13}{10})(\xi + \frac{6}{5})\).
- \(\omega = x y \cdot dx \wedge dy \wedge dz\) \(B_{x y}(\xi) \mid (\xi + \frac{11}{10})(\xi + \frac{6}{5})^2(\xi + \frac{8}{5})\).
- \(\omega = x^2 \cdot dx \wedge dy \wedge dz\) \(B_{x^2}(\xi) \mid (\xi + \frac{9}{5})(\xi + \frac{6}{5})^2(\xi + \frac{11}{10})\).
- \(\omega = x z \cdot dx \wedge dy \wedge dz\) \(B_{x z}(\xi) \mid (\xi + \frac{6}{5})^2(\xi + \frac{7}{5})(\xi + \frac{17}{10})\).
• \( \omega = x.y.z.dx \wedge dy \wedge dz \quad B_{x.y.z}(\xi) \mid (\xi + \frac{7}{17})(\xi + \frac{8}{17})(\xi + \frac{10}{17}) \) etc ...

Note that in this example the differential forms corresponding to degree 2 monomials in \( x, y, z \) are global holomorphic 3–forms on the fibers of the family of compact surfaces given, for \( \lambda \) fixed, by

\[ X_\lambda := \{(s, (x, y, z, t)) \in \mathbb{C} \times \mathbb{P}_3(\mathbb{C}) / s.t^5 = x^5 + y^5 + z^5 + \lambda.x.y.z.t \}, \quad \pi_\lambda((s, (x, y, z, t))) = s. \]

As, moreover, the map \( \pi_\lambda \) has no singular point at infinity, the affine computation controls also the global case for these forms.

Remark that the global computation for these forms gives the same frescos than in the affine case here because \( f_\lambda \) has an isolated singularity at the origin.

4.3.2 \( f = x.y^3 + y.z^3 + z.x^3 + \lambda.x.y.z \)

The singularity of the hypersurface \( \{f = 0\} \) is the origin:

It is easy to see that any monomial of \( f \) is a linear combination of \( x, y, z \) \( \frac{\partial f}{\partial x}, y, \frac{\partial f}{\partial y}, z \), so that each monomial in \( f \) has to vanish on the singular set of \( \{f = 0\} \). Then this implies easily our claim.

Again using the theorem n+2 and the proposition 2.3.2 allows, after some elementary computations, to find for each monomial form \( \omega \) below a degree 3 polynomial dividing the Bernstein polynomial of the fresco \( F_\omega \).

• \( \omega = dx \wedge dy \wedge dz \quad B_1(\xi) \mid (\xi + 1)^3. \)
• \( \omega = x.dx \wedge dy \wedge dz \quad B_x(\xi) \mid (\xi + \frac{9}{17})(\xi + \frac{9}{17})(\xi + \frac{11}{17}). \)
• \( \omega = x^2.dx \wedge dy \wedge dz \quad B_{x^2}(\xi) \mid (\xi + \frac{9}{17})(\xi + \frac{11}{17})(\xi + \frac{15}{17}). \)
• \( \omega = x.y.dx \wedge dy \wedge dz \quad B_{x.y}(\xi) \mid (\xi + \frac{11}{17})(\xi + \frac{15}{17})(\xi + \frac{15}{17}). \)
• \( \omega = x.y.z.dx \wedge dy \wedge dz \quad B_{x.y.z}(\xi) \mid (\xi + 2)^3. \)
• \( \omega = x^7.dx \wedge dy \wedge dz \quad B_{x^7}(\xi) \mid (\xi + 5)(\xi + 3).(\xi + 2). \)

4.3.3 \( f := x.y^2.z^3 + y.z^2.t^3 + z.t^2.x^3 + t.x^2.y^3 + \lambda.x.y.z.t \)

In this case the singularity is not isolated: the singular of \( \{f = 0\} \) is the union of the lines \( \{x = y = z = 0\}, \{y = z = t = 0\}, \{z = t = x = 0\}, \{t = x = y = 0\} \). The estimate for the Bernstein polynomial associated to the monomial 1 (so of the fresco \( F_\omega \) with \( \omega := dx \wedge dy \wedge dz \wedge dt \) is \( B_1(\xi) \mid (\xi + 1)^3 \). So we may have a maximal unipotent monodromy.
4.3.4 \[ f := x.y^2 + x^2.y + z.t^3 + t.z^3 + \lambda.x.y.z.t \]

Again we assume that \( \lambda \) is a non-zero complex number. The hypersurface \( \{ f = 0 \} \) has an isolated singularity at the origin:

If \( \Sigma := \{ \text{df} = 0 \} \subset \mathbb{C}^4 \) we have on \( \Sigma \) the relations \( x.y^2 = x^2.y = \frac{-1}{3}.\lambda.x.y.z.t \) and \( z.t^3 = z^3.t = \frac{-1}{4}.\lambda.x.y.z.t \). So on \( \Sigma \cap \{ f = 0 \} \) we have \( x.y = 0 = z.t \) and this implies that \( \Sigma \cap \{ f = 0 \} = \{ 0 \} \).

Now we shall use again the theorem 4.2.3 and the proposition 2.3.2 in order to give a polynomial of degree 12 which divides the Bernstein polynomial of the fresco \( F_\omega \) for \( \omega := dx \wedge dy \wedge dz \wedge dt \). The reader interested by another holomorphic monomial form can follow the same line to obtain an analogous result.

The relation between the monomials of \( f \) is

\[ \lambda^{12}(x.y^2)^4(y.x^2)^4(z.t^3)^3(z^3)^3 = (\lambda.x.y.z.t)^{12}. \]

So to compute the initial form in \( (a, b) \) of the polynomial in \( A \) constructed in the theorem 4.2.3 annihilating \( [\omega] \) in \( E^4/B(E^4) \), it is enough to compute the homogeneous in \( (a, b) \) polynomial \( P \) of degree 12 satisfying in \( E^4 \) the relation \( P.[\omega] = [(\lambda.x.y.z.t)^{12}.\omega] \).

Note \( m_1, \ldots, m_4 \) the first monomials in \( f \) and \( m := \lambda.x.y.z.t. \) Then we have in \( E^4 \) the equality for any integer \( k \geq 0 \) (where \( \omega \) is omitted)

\[
\begin{align*}
\bullet m_1.m^k &= \frac{1}{3}.((k + 1).b[m^k] - m^{k+1}) \\
\bullet m_2.m^k &= \frac{1}{3}.((k + 1).b[m^k] - m^{k+1}) \\
\bullet m_3.m^k &= \frac{1}{4}.((k + 1).b[m^k] - m^{k+1}) \\
\bullet m_4.m^k &= \frac{1}{4}.((k + 1).b[m^k] - m^{k+1})
\end{align*}
\]

and so we obtain

\[
\left(a - \frac{7}{6}(k + 1).b\right)[m^k] = -\frac{1}{6}m^{k+1}.
\]

Then the initial form of the polynomial annihilating \( [\omega] \) is equal to the product ordered from left to right by decreasing \( k \)

\[
\prod_{k=0}^{11} \left(a - \frac{7}{6}(k + 1).b\right)[m^k].
\]

This gives the following estimate for the Bernstein polynomial

\[
B(\xi)\left| \prod_{k=0}^{11} (\xi + \frac{k + 7}{6}) \right.
\]

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References

[1] Barlet, D. *Développements asymptotiques des fonctions obtenues par intégration sur les fibres*, Inv. Math. vol. 68 (1982), p. 129-174.

[2] Barlet, D. et Maire, H.M. *Développements asymptotiques, transformation de Mellin complexe et intégration dans les fibres*, in Sem. P. Lelong, Lecture Notes, vol. 1295 Springer Verlag (1987), p. 11-23.

[3] Barlet, D. *Sur certaines singularités non isolées d’hypersurfaces I*, Bull. Soc. Math. France 134 fasc.2 (2006), p. 173-200.

[4] Barlet, D. and Saito, M. *Brieskorn modules and Gauss-Manin systems for non isolated hypersurface singularities* Bull. of London Math. Soc. (2007).

[5] Barlet, D. *Sur certaines singularités d’hypersurfaces II*, Journal of Algebraic Geometry 17 (2008), p.199-254.

[6] Barlet,D. *Sur les fonctions à lieu singulier de dimension 1*, Bull. Soc. math. France 137 (4), (2009), p. 587-612.

[7] Barlet,D. *Périodes évanescents et (a,b)-modules monogènes*, Bollettino U.M.I. (9) II (2009) p.651-697.

[8] Barlet, D. *A finiteness theorem for S-relative formal Brieskorn module*, math. arXiv 1207.4013, math.AG and math.CV.

[9] Barlet, D. *Algebraic differential equations associated to some polynomials*, math. arXiv:1305.6778, math.AG and math.CV.

[10] Barlet, D. *A note on some fiber-integrals* math. arXiv:1512.07062, math.CV, math.AG:

[11] Brieskorn, E. *Die Monodromie der isolierten Singularitäten von Hyperflächen*, Manuscripta Math. 2 (1970), pp.103-161.

[12] Deligne, P. *Equations différentielles à points singuliers réguliers*, L-N 163 (1970) Springer.

[13] Kashiwara,M. *b-function and holonomic systems, rationality of roots of b-functions*, Invent. Math. 38 (1976) p. 33-53.

[14] Malgrange, B. *Intégrale asymptotique et monodromie*, Ann. Sc. ENS t.7 (1974), pp.405-430.

[15] Milnor, J. *Singular Points of Complex Hypersurfaces*, Ann. of Math. Studies 61 (1968) Princeton.

[16] Sebastiani, M. *Preuve d’une conjecture de Brieskorn*, Manuscripta Math. 2 (1970), pp. 301-308.
[17] Saito, Kyoji *Period mapping associated to a primitive form*, Publ. Res. Inst. Math. Sci. 19 (1983), no. 3, pp.1231-1264

[18] Saito, Morihiko *On the structure of Brieskorn lattice*, Ann. Inst. Fourier (Grenoble) 39 (1989), no. 1, pp. 27-72.

[19] Varchenko, A. N. *Asymptotic behavior of holomorphic forms determines a mixed Hodge structure*, (Russian) Dokl. Akad. Nauk SSSR 255 (1980), no. 5, pp. 1035-1038.