Elliptic Genera and the Landau–Ginzburg Approach to N=2 Orbifolds

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We compute the elliptic genera of orbifolds associated with $N = 2$ super–conformal theories which admit a Landau-Ginzburg description. The identification of the elliptic genera of the macroscopic Landau-Ginzburg orbifolds with those of the corresponding microscopic $N = 2$ orbifolds further supports the conjectured identification of these theories. For $SU(N)$ Kazama-Suzuki models the orbifolds are associated with certain $\mathbb{Z}_p$ subgroups of the various coset factors. Based on our approach we also conjecture the existence of "E-type" variants of these theories, their elliptic genera and the corresponding Landau-Ginzburg potentials.

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1. Introduction

The proposal of an effective field theoretical description of \( N = 2 \) superconformal theories “à la Landau–Ginzburg” (LG) has recently received much further support. On the one hand, one is dealing with the now familiar \( N = 2 \) rational superconformal field theories, mainly obtained by a coset construction using WZW factors. In the following we will mainly concentrate on the Kazama–Suzuki (KS) coset models of the form \( SU(N)_k \times SO(2(N−1))_1/SU(N−1)_{k+1} \times U(1) \), which will be referred to as “\( SU(N) \) KS models”. A tentative classification of such theories is made possible by the knowledge of modular invariants for the various WZW factors. Unfortunately only the \( SU(2) \) \([1]\) and \( SU(3) \) \([2]\) cases are completely classified so far, and it remains a formidable task to go beyond them (see however \([3]\) for some alternative route). However, the orbifold procedure enables us to manufacture non–trivial modular invariant theories, by moding out some symmetry of the initial WZW theory. The simplest of these is the \( \mathbb{Z}_p \) orbifold of \( SU(N) \) WZW theories, for \( p \) a divisor of \( N \).

On the other hand, we have an effective description of such theories using some \( N = 2 \) superfields \( \Phi_i \) governed by the Landau-Ginzburg action

\[
S = \int d^2 x d^4 \theta \, \Phi_i \bar{\Phi}_i + \{ \int d^2 x d^2 \theta \, W(\Phi_i) + \text{c.c.} \},
\]

where \( W \) is some quasi–homogeneous polynomial potential of the graded fields \( \Phi_i \). In addition to the identification of central charges of both theories \([1]\) (linked to the degrees of \( W \) and of the fields \( \Phi_i \)), the “chiral rings” of the superconformal theories (the ring structure formed by the chiral primary fields under OPE) were also identified with those of the corresponding LG descriptions \([3]\) (the ground state ring \( \mathbb{C}[x_1, x_2, ...]/\nabla W \)). Recently E. Witten \([6]\) was able to compute yet another quantity in the LG framework for the \( SU(2) \) KS models of diagonal \( A \)-type modular invariant, and to compare it with the corresponding superconformal field theory results. This quantity is the elliptic genus, defined as a certain twisted boundary condition toroidal partition function

\[
Z_2(u|\tau) = \sum_l \text{Tr}_{\mathcal{R}_l} (-1)^F q^{H_L} e^{i\gamma_L J_{0,L}}, \tag{1.1}
\]

where \( F, H_L, J_{0,L} \) and \( \gamma_L \) denote respectively the total fermion number, the hamiltonian \( H_L = L_0 - \frac{c}{24} \), the zero mode of the \( U(1) \) symmetry generator and the associated charge of the left–moving Ramond states. The sum extends over the states with vanishing right–moving hamiltonian and \( U(1) \) charge, \( H_R = \gamma_R = 0 \), which are Ramond sector
representations of the $N = 2$ superalgebra containing a ground state of $H_L = 0$. In the following, we denote $u = \frac{2^L}{2\pi(k+N)}$ and $q = e^{2i\pi\tau}$ (for a general LG theory the degree of $W$ will appear in the expression for $u$ instead of $k + N$). The direct LG calculation of this quantity turns out to be a simple free field calculation: due to the topological invariance of (1.1)\[6\], one can set the potential piece of the action to zero, after integration over the bosonic upper components of the superfields, without altering the result. The actual calculation only involves sorting out the left/right moving bosonic and fermionic mode contributions, and finally gives rise to a product formula for the LG elliptic genus.

In the $N = 2$ superconformal framework, the elliptic genus is expressed as a sum over (twisted) Ramond characters. In \[7\], two of us established a general scheme for proving the identity between the elliptic genera in the two approaches, and were able to extend this to the $D$ and $E$–type modular invariant $SU(2)$ KS theories, as well as to the $A$–type invariants of the $SU(N)$ KS models. The proof is based on an elementary lemma on elliptic modular functions. The idea is to show that both elliptic genera behave similarly under the elliptic $z \rightarrow z + 1$, $z \rightarrow z + \tau$ and modular $\tau \rightarrow \tau + 1$ and $(z, \tau) \rightarrow (\frac{z}{\tau}, -\frac{1}{\tau})$ transformations, and share the same $q \rightarrow 0$ (i.e. $\tau \rightarrow i\infty$) limit. Consequently, the lemma states that their ratio is an elliptic modular form of weight zero, with limit 1 when $q \rightarrow 0$. Therefore, it is equal to 1 identically. Usually the first two transformations are easy to find, using the definition of the Ramond characters. The third one is obvious, due to the identity for the character with highest representation of weight $h$

$$\chi_h(z|\tau + 1) = e^{2i\pi(h-\frac{c}{24})}\chi_h(z|\tau),$$

and since we have $h = \frac{c}{24}$ for the Ramond states contributing to the elliptic genus. The only tricky point is the last transformation. We give now a general proof of modular covariance of the $N = 2$ superconformal elliptic genus. First, the elliptic genus (1.1) for a general $N = 2$ superconformal theory can be expressed as a particular limit of the modular covariant (twisted) Ramond sector partition function

$$\mathcal{Z}(z|\tau) = \sum_{h,\bar{h}} N_{h,\bar{h}} \chi_h(z|\tau) \chi^{\bar{h}}_h(z|\tau)$$

\[1.2\]

\equiv \phi(z, \tau, \bar{z}, \bar{\tau}),

$N_{h,\bar{h}} \in \mathbb{N}$, considered as a function of $z$ and $\bar{z}$ as independent variables, and with

$$\chi_h(z|\tau) = \text{tr}_{R_h}(q^{L_0} - \frac{c}{24} \tau e^{2i\pi\alpha J_0} (-1)^F_L)$$
where \( \alpha \) is a fixed factor which makes all \( U(1) \) charges (eigenvalues of \( J_0 \)) integer: for the \( SU(N) \) KS models, we have \( \alpha = N(k+N) \), and we have to substitute \( h \to \bar{h}, L \to R \) for right moving representations. Taking \( \bar{z} = 0 \) in (1.2), we end up with the elliptic genus
\[
\phi(z, \tau) = K(z|\tau) = \sum_{\bar{h}}' \chi_{\bar{h}}(z|\tau),
\]
where the sum extends over the same states as in eqn.(1.1). The modular covariance of the full partition function
\[
\mathcal{Z}(\frac{z}{\tau}-\frac{1}{\tau}) = e^{i\pi\beta(\frac{z^2}{\tau}-\frac{\bar{z}^2}{\bar{\tau}})}\mathcal{Z}(z|\tau)
\]
follows from the simple coset form of the modular transformations of the characters, explicitly factorizing a phase \( \exp(i\pi\beta(\frac{z^2}{\tau}-\frac{\bar{z}^2}{\bar{\tau}})) \), with \( \beta = N^2(N-1)k(k+N) \) for the \( SU(N) \) KS case. Taking \( \bar{z} = 0 \) in (1.4), we find that the elliptic genus of the \( N = 2 \) superconformal theory is a modular form of weight 0, namely
\[
K(\frac{z}{\tau}-\frac{1}{\tau}) = e^{i\pi\beta\frac{z^2}{\tau}}K(z|\tau).
\]
Thus, whenever we will be able to compute a LG elliptic genus, our first task will be to compute its elliptic and modular properties and to compare its \( q \to 0 \) limit to that of the corresponding \( N = 2 \) superconformal one.

In this paper, we address orbifolds within the LG approach [8], in particular those associated with the \( SU(N) \) KS theories. In the first section we compute the LG elliptic genus for the \( D \)-type \( \mathbb{Z}_2 \) orbifolds of the \( SU(2) \) KS models, giving rise to yet another identity between Jacobi theta functions. In the second section, we address the \( SU(3) \) case in detail: in addition to the expected \( \mathbb{Z}_3 \) orbifold, we can also carry out a \( \mathbb{Z}_2 \) orbifold corresponding to the \( SU(2) \) factor of the KS coset model. Modular invariance tells us also that some \( E_{6,7,8} \) theories of the \( SU(2) \) factor should also exist, and we give a natural conjecture for their elliptic genera and LG potentials. Next we turn to the general \( \mathbb{Z}_p \) orbifold case of \( SU(N) \) KS models. Although no direct LG description exists for the orbifolds in general, the elliptic genera can be computed by twisted field techniques, starting from the LG theory for \( SU(N) \) KS models. In the last section we analyze the elliptic genus of orbifolds of general LG theories. We end with conclusions, remarks and open questions.

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1 The notation \((h, \bar{h})\) for the left–right representations involved is abusive, as \( h = \bar{h} = \frac{c}{24} \) for all the remaining states, after one takes \( \bar{z} \to 0 \). What distinguishes them is actually their \( U(1) \) charge, and the sum in (1.3) is over the “chiral” Ramond states, obtained from the spinless \((h = \bar{h})\) chiral states of the Neveu–Schwarz sector by spectral flow. For instance in the \( SU(2) \) A, D, E cases, these states are indexed by the Coxeter exponents (shifted by \(-1\)) of the corresponding Lie algebra.
2. The $SU(2)$ case: $D$ series as $\mathbb{Z}_2$ orbifolds.

The $N = 2$ LG potentials for the $A_{k+1}$–type $SU(2)$ KS models (i.e. the cosets $SU(2) / U(1) \times U(1)$) read

$$W^{(2,A)}_{k+2}(\Phi) = \frac{\Phi^{k+2}}{k+2}. \quad (2.1)$$

Whenever $k$ is even, say $k = 2n$, the action has a $\Phi \rightarrow -\Phi$ symmetry. Modding out by this symmetry amounts to performing the change of variable $\Phi_1 = \Phi^2$ in the path integral\footnote{See appendix A of ref.\cite{9} for a complete argument.}. The resulting Jacobian, $\Phi_1^{-1/2}$, can be put back into the action by introducing a second field $\Phi_2$, finally leading to the effective potential\footnote{See appendix A of ref.\cite{9} for a complete argument.}

$$W^{(2,D)}_{2n+2}(\Phi_1, \Phi_2) = \frac{\Phi_1^{n+1}}{2(n+1)} + \frac{1}{2} \Phi_1 \Phi_2^2. \quad (2.2)$$

In \cite{7}, we performed the direct free field calculation of the elliptic genus starting from the latter LG potential. The main point here is that the calculation can be done directly using the initial potential (2.1). We proceed as for the calculation of the one loop orbifold partition functions of free fields. The 'R'–symmetry of the fields which preserves the action after integration over the upper component $F$ of the superfield $\Phi = \phi + \theta_+ \psi_+ + \theta_- \psi_- + \theta_+ \theta_- F$ is

$$\phi \rightarrow e^{2i\pi u} \phi$$
$$\psi_+ \rightarrow e^{2i\pi u} \psi_+$$
$$\psi_- \rightarrow e^{-2i\pi (k+1) u} \psi_-.$$

The additional $\mathbb{Z}_2$ symmetry allowed by the potential (2.1) for $k = 2n$ translates into the possibility of an extra simultaneous change

$$\phi \rightarrow -\phi \quad \psi_\pm \rightarrow -\psi_\pm.$$

Including this possibility in the free field toroidal calculation of \cite{11} gives rise to four possible sectors (PP), (AP), (PA), (AA), according to the sign chosen in the 1 or $\tau$ directions of the torus (e.g. the (AP) sector corresponds to $\Phi(z + 1) \rightarrow -\Phi(z)$ and $\Phi(z + \tau) \rightarrow \Phi(z)$). The (PP) contribution is just that of the non–twisted superfield, and coincides with the $A$–type theory result\footnote{See appendix A of ref.\cite{9} for a complete argument.}
where we identify the contributions of left moving fermion and boson modes of the free superfield. Here and in the following, $\Theta_j(u|\tau)$, $j = 1, 2, 3, 4$, denote the Jacobi theta functions. The effect of the twist is clear: it amounts to replacing $n$ by $n - \frac{1}{2}$ (the modes of the components of the superfield $\phi_n$ get twisted, i.e. changed into $\phi_{n+\frac{1}{2}}, \phi_{n-\frac{1}{2}}$ for antiperiodicity in the 1 direction) or $u \to u - \frac{1}{2}$ (twist in the $\tau$ direction). This gives

$$Z_{AP}(u|\tau) = \frac{\Theta_4((k+1)u|\tau)}{\Theta_3(u|\tau)},$$

$$Z_{PA}(u|\tau) = \frac{\Theta_2((k+1)u|\tau)}{\Theta_2(u|\tau)},$$

$$Z_{AA}(u|\tau) = \frac{\Theta_3((k+1)u|\tau)}{\Theta_3(u|\tau)}.$$

We get the final expression for the orbifold elliptic genus

$$Z^{(D)}_{2p+2}(u|\tau) = \frac{1}{2} (Z_{PP} + Z_{AP} + Z_{PA} + Z_{AA})$$

$$= \frac{1}{2} \sum_{j=1}^{4} \frac{\Theta_j((k+1)u|\tau)}{\Theta_j(u|\tau)}.$$

(2.3)

A few remarks are in order. The above expression is a modular form of weight zero, with simple $u \to u + 2$ and $u \to u + 2\tau$ transformations

$$Z^{(D)}_{2p+2}(u + 2|\tau) = Z^{(D)}_{2p+2}(u|\tau),$$

$$Z^{(D)}_{2p+2}(u + 2\tau|\tau) = e^{-4i\pi k^2(u+\tau)} Z^{(D)}_{2p+2}(u|\tau),$$

$$Z^{(D)}_{2p+2}(u|\tau + 1) = Z^{(D)}_{2p+2}(u|\tau),$$

$$Z^{(D)}_{2p+2}\left(\frac{u}{\tau}, -\frac{1}{\tau}\right) = e^{i\pi k^2} Z^{(D)}_{2p+2}(u|\tau).$$

Note that this is true only for even $k$’s, otherwise some phases would appear. Moreover, its $q \to 0$ ($\tau \to i\infty$) limit is easily derived, using the limits

$$\Theta_1(u|\tau) \to 2q^{\frac{1}{8}} \sin \pi u$$

$$\Theta_2(u|\tau) \to 2q^{\frac{1}{8}} \cos \pi u$$

$$\Theta_3(u|\tau) \to \Theta_3(0|i\infty) = 1$$

$$\Theta_4(u|\tau) \to \Theta_4(0|i\infty) = 1,$$

so that

$$\lim_{q \to 0} Z^{(D)}_{2p+2}(u|\tau) = \frac{1}{2} \left( \frac{\sin \pi (k+1) u}{\sin \pi u} + \frac{\cos \pi (k+1) u}{\cos \pi u} + 2 \right)$$

$$= \frac{\sin \pi k u \sin \pi (\frac{k}{2} + 2) u}{\sin \pi k u \sin 2\pi u}.$$
Finally we use the lemma of appendix A of ref.[7] on elliptic modular functions, to conclude that the expression (2.3) is identical to the D series elliptic genus

\[ Z_{2p+2}^{(D)}(u|\tau) = \frac{\Theta_1(ku|\tau)\Theta_1((k+4)u/2|\tau)}{\Theta_1(ku/2|\tau)\Theta_1(2u|\tau)}, \] (2.4)

as computed in [7] from the second potential (2.2), because the ratio of both expressions is an elliptic modular form of weight zero, whose \( q \to 0 \) limit equals one. Therefore, this ratio equals one identically.

3. The \( SU(3) \) case.

**Orbifolds and more for the \( SU(2) \) factor.**

The \( A \)-type LG potentials for the \( SU(3) \) KS models are generated by the function [11]

\[-\log(1-t\Phi_1 + t^2\Phi_2) = \sum_{m \geq 0} t^m W_m^{(3)}(\Phi_1, \Phi_2) = \sum_{m \geq 0} \frac{t^m}{m} T_m(\Phi_2/\Phi_1) \frac{\Phi_2^m}{\Phi_1^m}, \] (3.1)

where the index \( m \) stands for \( k+3 \), \( k \) the level of the \( SU(3) \) factor of the KS model, and \( T_m \) are the Chebyshev polynomials of the first kind \( T_m(2\cos \theta) = 2\cos m\theta \). These polynomials have interesting parity properties: the \( T_{2m} \) are even, while the \( T_{2m+1} \) are odd. Therefore, when \( k+3 \) is even, the potential is a function of \( \Phi_1^2 \) and \( \Phi_2 \) only, say

\[ W_{k+3}^{(3)}(\Phi_1, \Phi_2) = \Pi_{k+3}(\Phi_1^2, \Phi_2) \]

This fact allows moding out by the extra \( \mathbb{Z}_2 \) symmetry already encountered in the \( SU(2) \) case of previous section. Setting \( \theta_1 = \Phi_1^2 \) and \( \theta_2 = \Phi_2 \), and putting back the Jacobian \( \theta_1^{-1/2} \) into the action by introducing a third superfield \( \theta_3 \), we get the \( \mathbb{Z}_2 \) orbifold potential

\[ W_{k+3}^{(3,D)}(\theta_1, \theta_2, \theta_3) = \Pi_{k+3}(\theta_1, \theta_2) + \frac{1}{2} \theta_1 \theta_3^2. \] (3.2)

Following the lines of ref.[7], we can perform the direct LG calculation of the elliptic genus for this potential. Note that it is quasi–homogeneous of degree \( k + 3 \) for a grading of the fields where \( \theta_{1,2,3} \) have respective degrees \( 2, 2, \frac{k+1}{2} \). The \( U(1) \) transformation of the
bosonic and fermionic components of the fields \( \theta_i = \alpha_i + \theta_+ \beta_i^1 + \theta_- \beta_i^- \) which preserves the action reads
\[
\begin{align*}
\alpha_1 & \rightarrow e^{2i\pi(2u)}\alpha_1 \\
\alpha_2 & \rightarrow e^{2i\pi(2u)}\alpha_2 \\
\alpha_3 & \rightarrow e^{2i\pi(\frac{k+1}{2}u)}\alpha_3 \\
\beta_1^+ & \rightarrow e^{2i\pi(2u)}\beta_1^+ \\
\beta_2^+ & \rightarrow e^{2i\pi(2u)}\beta_2^+ \\
\beta_3^+ & \rightarrow e^{2i\pi(\frac{k+1}{2}u)}\beta_3^+ \\
\beta_1^- & \rightarrow e^{-2i\pi(k+1)u}\beta_1^- \\
\beta_2^- & \rightarrow e^{-2i\pi(k+1)u}\beta_2^- \\
\beta_3^- & \rightarrow e^{-2i\pi(\frac{k+1}{2}+2)u}\beta_3^- 
\end{align*}
\]

and the elliptic genus reads
\[
Z_3^{(D)}(u|\tau) = \frac{\Theta_1((\frac{k+1}{2}+2)u|\tau)}{\Theta_1((\frac{k+1}{2})u|\tau)} \left( \frac{\Theta_1((k+1)u|\tau)}{\Theta_1(2u|\tau)} \right)^2. \tag{3.3}
\]

Comparing this with the \( A \)-type elliptic genus
\[
Z_3^{(A)}(u|\tau) = \frac{\Theta_1((k+1)u|\tau)}{\Theta_1(u|\tau)} \frac{\Theta_1((k+2)u|\tau)}{\Theta_1(2u|\tau)},
\]
we see that the \( \mathbb{Z}_2 \) orbifold has just replaced the “\( SU(2)_{k+1} \) factor” (with even level \( k+1 \)) of the \( A \)-type elliptic genus, \( \Theta_1((k+2)u|\tau)/\Theta_1(u|\tau) \), by the \( \mathbb{Z}_2 \) orbifold elliptic genus \((2.3)\) or equivalently \((2.3)\), with \( k \to k+1 \). This suggests that, although we do not know the corresponding LG potentials, a similar mechanism should take place in the exceptional \( E_{6,7,8} \) cases. We conjecture that the corresponding elliptic genera are obtained by replacing the “\( SU(2)_{k+1} \) factor” for \( k+1 = 10, 16, 28 \) by the corresponding \( E_{6,7,8} \) elliptic genus \([7]\), namely
\[
\begin{align*}
Z_3^{(E_6)}(u|\tau) &= \frac{\Theta_1(10u|\tau)}{\Theta_1(2u|\tau)} \frac{\Theta_1(9u|\tau)}{\Theta_1(3u|\tau)} \frac{\Theta_1(8u|\tau)}{\Theta_1(4u|\tau)} \\
Z_3^{(E_7)}(u|\tau) &= \frac{\Theta_1(16u|\tau)}{\Theta_1(2u|\tau)} \frac{\Theta_1(14u|\tau)}{\Theta_1(4u|\tau)} \frac{\Theta_1(12u|\tau)}{\Theta_1(6u|\tau)} \\
Z_3^{(E_8)}(u|\tau) &= \frac{\Theta_1(28u|\tau)}{\Theta_1(2u|\tau)} \frac{\Theta_1(24u|\tau)}{\Theta_1(6u|\tau)} \frac{\Theta_1(20u|\tau)}{\Theta_1(10u|\tau)}. \tag{3.4}
\end{align*}
\]
The corresponding potentials, if they exist, should be decorations of the \( E_{6,7,8} \) potentials involving an extra field of dimension 2. A possibility is given by the expressions for the
(flat, massive) perturbations of these theories \cite{12,13}, which all involve a “decoration” by a perturbation parameter of dimension two. The guess we make here is inspired by the fact that the $SU(N)$ LG potentials for the KS models can be obtained from the $SU(2)$ perturbed one, by a suitable substitution of the dimensionful perturbation parameters for LG by superfields \cite{9}. Moreover, the $D$ case treated above exhibits the same phenomenon: the flat perturbations of the potential for the $D_{n+2}$ series found in \cite{12,9} are such that when retaining only the dimension 2 perturbation parameter (denoted by $t_{2n}$ in ref.\cite{9}), it has exactly the form (3.2) with the substitution $\theta_1 \rightarrow x$, $\theta_3 \rightarrow y$ and $\theta_2 \rightarrow t_{2n}$. So we propose the following candidates, obtained by substituting $t_{10,16,28} \rightarrow z$, a third superfield, in the expressions of the corresponding E–type perturbed potentials. We get

\[
W_{3}^{(E_6)}(x, y, z) = \frac{x^3}{3} + \frac{y^4}{4} - xy^2z + \frac{y^2}{2}z^3 + \frac{xz^4}{12}.
\]

\[
W_{3}^{(E_7)}(x, y, z) = \frac{x^3}{3} + xy^3 - 3x^2yz + 4x^2z^3 - 3xyz^4 + xz^6 + \frac{z^9}{6}.
\]

\[
W_{3}^{(E_8)}(x, y, z) = \frac{x^3}{3} + \frac{y^5}{5} - xy^3z + x^2z^4 - \frac{6}{5}y^3z^6 - \frac{19}{15}xyz^7 + \frac{28}{15}y^2z^9
+ \frac{11}{45}xz^{10} - \frac{82}{75}y^{12} + \frac{103}{450}z^{15}.
\] (3.5)

We checked that the LG elliptic genus for these potentials indeed reproduces the above conjectures (3.4).

Let us now compute the corresponding quantity in the framework of the $N = 2$ superconformal KS model. The $A$ elliptic genus is by definition a sum over Ramond characters

\[
K_3(z|\tau) = \sum_{(\lambda_1, \lambda_2) \in P^{(3)}_k} \chi_{\lambda_1, \lambda_1+2\lambda_2}^{(\lambda_1, \lambda_2)}(z|\tau),
\] (3.6)

where $P^{(3)}_k$ denotes the Weyl alcôve at level $k$ for $SU(3)$ (“integrable weights”, i.e. such that $\lambda_i \geq 0$ and $\sum \lambda_i \leq k$), and the various indices on the Ramond character of the KS coset denote from bottom to top respectively the $SU(2)$ weight, $U(1)$ charge (chosen to be an integer) and $SU(3)$ weight. The $\mathbb{Z}_2$ orbifold is obtained by restricting the allowed $SU(2)$ weights $\lambda_1$ to the set $\text{Exp}(D) = \{0, 2, 4, \ldots, k + 1\} \cup \{\frac{k+1}{2}\}$ corresponding to the allowed weights of the $SU(2)$ $D$ series. Note that the weight $(\lambda_1 = k + 1, \lambda_2)$ is not in the

\[\text{see appendix B of \cite{12} for a recapitulation of these perturbed potentials. In this reference, the dimension 2 parameter is denoted by } t_{10,16,28} \text{ for } E_{6,7,8} \text{ respectively.}\]
Weyl alcôve, but the Ramond character actually vanishes at this point, so we can include it in the summation. So we have the $\mathbb{Z}_2$ orbifold elliptic genus

$$K_3^{(D)}(z|\tau) = \sum_{\lambda_1 \in \text{Exp}(D), \lambda_2 \leq k - \lambda_1} \chi^{(\lambda_1, \lambda_2)}_{\lambda_1, \lambda_1 + 2\lambda_2}(z|\tau). \quad (3.7)$$

We want to prove that this expression coincides with that of the LG approach, eqn.(3.3). Using the explicit form of the characters and their modular transformations, one can show that this function is a modular form of weight zero with simple $z \to z + 1, z \to z + \tau$ transformations

$$K_3^{(D)}(z + 1|\tau) = K_3^{(D)}(z|\tau)$$
$$K_3^{(D)}(z + \tau|\tau) = e^{-18\pi k (k+3)(2z + \tau)} K_3^{(D)}(z|\tau)$$
$$K_3^{(D)}(z|\tau + 1) = K_3^{(D)}(z|\tau)$$
$$K_3^{(D)}\left(\frac{z}{\tau} - \frac{1}{\tau}\right) = e^{18\pi k (k+3)} K_3^{(D)}(z|\tau),$$

and with $q \to 0$ limit

$$\lim_{q \to 0} K_3^{(D)}(z|\tau) = x^{-k} \sum_{\lambda_1 \in \text{Exp}(D), \lambda_2 \leq k - \lambda_1} x^{\lambda_1 + 2\lambda_2}$$

$$= x^{-k} \left(1 - x^k + 1\right) \left(1 - x^{\frac{k+1}{2}} + 2\right) \frac{(1 - x^{k+1})^2}{(1 - x^2)^2 (1 - x^{\frac{k+1}{2}})},$$

where we set $x = e^{6i\pi z}$. These transformations and limit coincide with those of the LG result (3.3) provided we take $u = 3z$. Hence, using the lemma mentioned above on elliptic modular functions, we conclude that

$$Z_3^{(D)}(u = 3z|\tau) = K_3^{(D)}(z|\tau).$$

The same calculation for the $E_{6,7,8}$ modular invariant theories is tedious, but confirms the above conjectures (where again we take $u = 3z$). It involves in particular a restriction of the sum over $\lambda_1$ to the sets of shifted exponents of $E_{6,7,8}$, namely

$$\text{Exp}(E_6) = \{0, 3, 4, 6, 7, 10\}$$
$$\text{Exp}(E_7) = \{0, 4, 6, 8, 10, 12, 16\}$$
$$\text{Exp}(E_8) = \{0, 6, 10, 12, 16, 18, 22, 28\},$$

which in particular yields the same $q \to 0$ limit as the above conjecture (3.4), up to $u = 3z$. The last exponent $k + 1 = 10, 16, 28$ is again outside of the Weyl alcôve, but the characters
vanish there, so we can safely include them in the summation. The only delicate point is
the modular properties of the corresponding sums $K_3^{(E)}$. However, thanks to the general
proof we gave in the introduction, the modular covariance eqn. (1.5) is automatically
ensured, with $\beta = 18k(k + 3)$, so the various transformations agree perfectly with those
of (3.4), up to $u = 3z$. This is very strong evidence for the conjecture to actually be
ture. It would be nice to derive our candidates for the LG potentials directly from the
$A$ potential $W^{(3)}_{k+3}$, although even in the $SU(2)$ case no direct link is known between the
$E$–type potentials and the $A$ ones.

$\mathbb{Z}_3$ orbifold of the $SU(3)$ factor: the $D$ series.

The LG theory for the $A$–type $SU(3)$ KS model admits also a $\mathbb{Z}_3$ orbifold of the
$SU(3)_k$ factor, whenever $k$ is a multiple of 3. The extra $\mathbb{Z}_3$ symmetry of the potential
$W^{(3)}_{k+3}$ of eqn.(3.1) for $3|k$ is readily seen to be the simultaneous change

$$\Phi_1 \rightarrow e^{2i\pi l/3} \Phi_1, \quad \Phi_2 \rightarrow e^{4i\pi l/3} \Phi_2,$$

where $l \in \mathbb{Z}_3$. So, we have to deal with an extra $\mathbb{Z}_3$ twist in our free field calculation for
the LG elliptic genus, giving rise to nine possible sectors according to the choice of phases
in the 1 and $\tau$ directions of the torus. We choose to index these sectors by a fractional
number $a \in [−\frac{1}{3}, \frac{1}{3})$, where in this case $a \in \{−\frac{1}{3}, 0, \frac{1}{3}\}$, and to call $f_{a,b}(u|\tau)$ the elliptic
genus of the corresponding sector $(a,b) \in \{−\frac{1}{3}, 0, \frac{1}{3}\}^2$. The untwisted sector $a = b = 0$ has
the $A$–type elliptic genus

$$f_{0,0}(u|\tau) = \frac{\Theta_1((k + 2)u|\tau)}{\Theta_1(u|\tau)} \times \frac{\Theta_1((k + 1)u|\tau)}{\Theta_1(2u|\tau)},$$

where the two factors gather the respective contributions of the modes of $\Phi_1$ and $\Phi_2$. The elliptic
genus $f_{a,b}(u|\tau)$ for the $(a,b)$ twisted sector is easily obtained by a change $n \rightarrow n+a$
(resp. $n \rightarrow n+2a$) in the modes of the components of $\Phi_1$ (resp. $\Phi_2$) for the 1 direction,
and by the change $u \rightarrow u + b$ for the $\tau$ direction. For convenience, let us define

$$\Theta_{a,b}(u|\tau) = \sum_{n \in \mathbb{Z}} q^{\frac{1}{2}(n+\frac{1}{2}+a)^2} e^{2i\pi(n+\frac{1}{2}+a)(u+\frac{1}{2}+b)}$$

which is proportional to $\Theta_1(u+a\tau+b|\tau)$, satisfies $\Theta_{a+1,b} = \Theta_{a,b}$, and which for $−1 < a \leq 0$
is equal to

$$q^{\frac{1}{2}(a+\frac{1}{2})^2} e^{2i\pi(a+\frac{1}{2})(u+\frac{1}{2}+b)} (1-q^{-a}e^{-2i\pi(u+b)}) \prod_{n \geq 1} (1-q^n)(1-q^{n+a}e^{2i\pi(u+b)})(1-q^{n-a}e^{-2i\pi(u+b)}),$$
We recover the standard Jacobi theta functions for the choices
\[ \Theta_1 \equiv \Theta_{0,0} \quad \Theta_2 \equiv \Theta_{0,-\frac{1}{2}} \]
\[ \Theta_3 \equiv \Theta_{\frac{1}{2},-\frac{1}{2}} \quad \Theta_4 \equiv \Theta_{-\frac{1}{2},0}. \]

Let us also denote by \([x]\) the unique element of \(x + \mathbb{Z}\) in the interval \([-\frac{1}{2}, \frac{1}{2})\). Then, the \((a, b)\) sector elliptic genus reads (for \(k\) a multiple of 3)
\[
f_{a,b}(u|\tau) = \frac{\Theta_{[(k+2)a],[k+2b]}((k+2)u|\tau)\Theta_{[(k+1)a],[k+1b]}((k+1)u|\tau)}{\Theta_{a,b}(u|\tau)\Theta_{[2a],[2b]}(2u|\tau)}
= \frac{\Theta_{-a,-b}((k+2)u|\tau)\Theta_{a,b}((k+1)u|\tau)}{\Theta_{a,b}(u|\tau)\Theta_{-a,-b}(2u|\tau)},
\]
where we used the fact that \([-x] = -[x]\) and \([3x] = 0\) for \(x \in \{0, \pm \frac{1}{3}\}\). The total \(\mathbb{Z}_3\) orbifold elliptic genus is
\[
Z_3^{(p)}(u|\tau) = \frac{1}{3} \sum_{a,b \in \{0, \pm \frac{1}{3}\}} f_{a,b}(u|\tau).
\]

This result has again some particularly simple modular properties thanks to the various transformations of \(\Theta_{a,b}\)
\[
\Theta_{a,b}(u+1|\tau) = e^{2i\pi(a+\frac{1}{2})} \Theta_{a,b}(u|\tau)
\]
\[
\Theta_{a,b}(u+\tau|\tau) = -e^{-i\pi(\tau+2a+2b)} \Theta_{a,b}(u|\tau)
\]
\[
\Theta_{a,b}(u|\tau+1) = e^{i\pi(\frac{1}{4}-a^2)} \Theta_{a,a+b}(u|\tau)
\]
\[
\Theta_{a,b}\left(\frac{u}{\tau} - \frac{1}{\tau}\right) = (i\tau)^{1/2} e^{2i\pi(a-\frac{1}{3})(b+\frac{1}{3})} e^{i\pi \frac{u^2}{\tau}} \Theta_{b,-a}(u|\tau).
\]

To get the transformations of \(f_{a,b}\), we still have to put the resulting indices of \(\Theta\) back into the interval \([-\frac{1}{2}, \frac{1}{2})\), at the possible cost of some phases. In the \(\mathbb{Z}_3\) case here, we have \(-[x] = [-x]\), therefore no phase appears in the \((z, \tau) \rightarrow (\frac{z}{\tau}, -\frac{1}{\tau})\) transformation, but in the \(\tau \rightarrow \tau + 1\) transformation, we have to replace \(a + b\) by \([a + b] = a + b - m\), introducing a phase \(e^{2i\pi(a+\frac{1}{3})m}\). Remarkably, these phases cancel each other in \(f_{a,b}\), which transforms as
\[
f_{a,b}(u+3|\tau) = f_{a,b}(u|\tau)
\]
\[
f_{a,b}(u+3\tau|\tau) = e^{-6i\pi(k+3)(3\tau+2u)} f_{a,b}(u|\tau)
\]
\[
f_{a,b}(u|\tau+1) = f_{a,a+b}(u|\tau)
\]
\[
f_{a,b}\left(\frac{u}{\tau} - \frac{1}{\tau}\right) = e^{2i\pi(k+3)\frac{u^2}{\tau}} f_{b,-a}(u|\tau).
\]
Moreover, the $q \to 0$ limit of the LG orbifold elliptic genus is easily derived, using the behaviors
\[
\Theta_{a,b}(u|\tau) \to q^{\frac{1}{2}(a+\frac{1}{2})^2} e^{2i\pi (a+\frac{1}{2})(u+b+\frac{1}{2})} \quad \text{for } a < 0 \\
\to -2q^{\frac{1}{2}} \sin \pi (u+b) \quad \text{for } a = 0 \\
\to q^{\frac{1}{2}(a+\frac{1}{2})^2-a} e^{2i\pi (a-\frac{1}{2})(u+b+\frac{1}{2})} \quad \text{for } a > 0,
\]
(3.11)
giving
\[
f_{a,b}(u|\tau) \to 1 \quad \text{for } a \neq 0 \\
\to e^{-2i\pi ku} g(u+b) \quad \text{for } a = 0,
\]
with
\[
g(u) = \frac{(1-e^{2i\pi (k+2)u})(1-e^{2i\pi (k+1)u})}{(1-e^{2i\pi u})(1-e^{4i\pi u})},
\]
so that the $q \to 0$ of the elliptic genus finally reads
\[
\lim_{q \to 0} Z_3^{(D)}(u|\tau) = \frac{1}{3}(e^{-2i\pi ku}[g(u) + g(u + \frac{1}{3}) + g(u - \frac{1}{3})] + 6).
\]

We now compare the LG orbifold elliptic genus to that of the $D$ series of the $SU(3)$ KS theories, corresponding to the $\mathbb{Z}_3$ orbifold of the $SU(3)$ factor. The elliptic genus $K_3^{(D)}(z|\tau)$ is obtained by restricting the sum of (3.10) to the $SU(3)$ weights at level $k$ with triality zero ($\lambda_1 - \lambda_2 = 0 \mod 3$), and with a triplication of the center of the Weyl alcôve $(\frac{k}{3}, \frac{k}{3})$, the fixed point of the $\mathbb{Z}_3$ transformation. It is again a straightforward exercise to derive the modular and elliptic properties of the resulting sum, and we find that they match exactly (3.10). Finally, the $q \to 0$ limit reads
\[
x^{-k} \sum_{(\lambda_1, \lambda_2) \in P_k^{(3)}} x^{\lambda_1+2\lambda_2} = \frac{1}{3}(x^{-k}[h(x) + h(\omega x) + h(\omega^2 x)] + 6),
\]
with $x = e^{6i\pi z}$, $\omega = e^{\frac{2i\pi}{3}}$, and
\[
h(x) = \frac{(1-x^{k+1})(1-x^{k+2})}{(1-x)(1-x^2)} = g(u = 3z).
\]

Thanks to our lemma on elliptic modular functions, we conclude that
\[
Z_3^{(D)}(u = 3z|\tau) = K_3^{(D)}(z|\tau).
\]
4. The $\mathbb{Z}_p$ orbifolds of the $SU(N)$ KS model.

The LG potentials for the $A$–type $SU(N)$ KS models are generated by \[11\]

$$- \log(1 - t\Phi_1 + t^2\Phi_2 - \cdots + (-t)^{N-1}\Phi_{N-1}) = \sum_{m \geq 0} t^m W_m^{(N)}(\Phi_1, \ldots, \Phi_{N-1}),$$

where $m$ stands for $k + N$, $k$ the level of the $SU(N)$ factor. We can form the $\mathbb{Z}_p$ orbifold of the LG theory at levels $k$ such that $p | k + N$. In this case, the potential $W_k^{(N)}$ is invariant under the simultaneous change

$$\Phi_j \rightarrow e^{\frac{2\pi i j}{p}} \Phi_j \quad j = 1, \ldots, N - 1,$$

where $l \in \mathbb{Z}_p$. The corresponding twists introduce $p^2$ sectors $(a, b)$, with $a, b \in \Sigma_p$, and

$$\Sigma_p = \{0, \pm \frac{1}{p}, \pm \frac{2}{p}, \ldots, \pm \frac{p-1}{2p}\} \quad \text{for } p \text{ odd}$$

$$= \{-\frac{1}{2}, 0, \pm \frac{1}{p}, \pm \frac{2}{p}, \ldots, \pm \frac{p-2}{2p}\} \quad \text{for } p \text{ even.}$$

These correspond to the values of the twist of the LG free fields in the 1 and $\tau$ directions of the torus. Let $f_{a,b}(u|\tau)$ denote again the elliptic genus for the $(a, b)$ sector, $a, b \in \Sigma_p$. The untwisted elliptic genus is that of the $A$ series computed in [7]:

$$f_{0,0}(u|\tau) = \prod_{j=1}^{N-1} \frac{\Theta_1((k + N - j)u|\tau)}{\Theta_1(ju|\tau)},$$

where the $j$-th factor in the product gathers the contributions of the various modes of the superfield $\Phi_j$. The effect of the twists $(a, b)$ is easily identified as a shift of the modes $n \rightarrow n + a$ and of the $u$ variable $u \rightarrow u + b$, up to a global phase, and we find

$$f_{a,b}(u|\tau) = \prod_{j=1}^{N-1} \frac{\Theta_{[(k+N-j)a],[j]}((k+N-j)u|\tau)}{\Theta_{[ja],[jb]}(ju|\tau)}$$

$$= \prod_{j=1}^{N-1} \frac{\Theta_{[-ja],[-jb]}((k+N-j)u|\tau)}{\Theta_{[ja],[jb]}(ju|\tau)}.$$

The total $\mathbb{Z}_p$ orbifold elliptic genus reads

$$Z_N^{(p)}(u|\tau) = \frac{1}{p} \sum_{a,b \in \Sigma_p} f_{a,b}(u|\tau). \quad (4.1)$$
To verify that this is correct, we need to check that we get the same modular transformations and $q \to 0$ limit as can be derived directly from the character expression for the elliptic genus. This will give us conditions on what $p$'s are allowed (in addition to the obvious condition $p|(k+N)$). Let us start with the $q \to 0$ limit: we derive this for the general LG case in the next chapter, and find an expression that differs from the $q \to 0$ limit of the characters by a phase $(A_3)_{a,b}$, which must therefore vanish (modulo 1) for all $a, b \in \Sigma_p$. This phase is given (modulo 1) by

$$(A_3)_{a,b} = \sum_{[b_j] \neq -\frac{1}{2}} (\sum_{[a_j] \neq 0} (\frac{1}{2} - a_j) + \sum_{[a_j] = 0} b_j)$$

where the sums are over $j = 1, \ldots, N - 1$. It is in fact simpler to work with the difference $(A_3)_{a,b} - (A_3)_{a,0}$ which we will denote by $B_{a,b}$: it is given by

$$B_{a,b} = \sum_{j=1}^{N-1} (\delta_{[a_j]} - \delta_{[b_j],-\frac{1}{2}})(b-a)j.$$ 

We shall now show that this vanishes for all $a, b$ if and only if $p$ is a divisor of $N$ or of $N - 1$ (meaning that $\mathbb{Z}_p$ is a subgroup of either the center of $G = SU(N)$ or of $H = SU(N-1)$). First, let us take $a = 0$ and $b = \frac{1}{p}$: for this case we get

$$B_{0,\frac{1}{p}} = \sum_{j=1}^{N-1} (1 - \delta_{[\frac{j}{p}],-\frac{1}{2}})(\frac{1}{p})j = \frac{1}{p} \sum_{j=1}^{N-1} j$$

and this must be an integer. Since the contribution of $j$'s for which $[\frac{j}{p}] = -\frac{1}{2}$ to the sum is a multiple of $\frac{1}{2}$ (a half integer), we find that the unrestricted sum $\frac{1}{p} \sum_{j=1}^{N-1} j$ must be a half integer as well, so that $p$ must be a divisor of $2 \sum_{j=1}^{N-1} j = N(N - 1)$. Let us now take $b = \frac{1}{p}$ with a general $a$: we get

$$B_{a,\frac{1}{p}} = \sum_{j=1}^{N-1} (\delta_{[a_j]} - \delta_{[\frac{a_j}{p}],-\frac{1}{2}})(\frac{1}{p} - a)j$$

which must be an integer for all $a$. We notice that once again the contribution of terms with $[\frac{j}{p}] = -\frac{1}{2}$ to the sum is a half integer, and so the expression $\frac{1}{p} \sum_{j=1}^{N-1} j \delta_{[a_j]}$ must also be a half integer, meaning that $p$ must divide $2 \sum_{j=1}^{N-1} j \delta_{[a_j]}$ for all $a$. Since we found that $p|N(N-1)$, we can write $p = qr$ where $q|N$ and $r|(N-1)$, and $q$ and $r$ have no common
factors since \( N \) and \( N - 1 \) are relatively prime. Let us now take \( a = \frac{1}{q} \); we get that \( p \) must divide \( 2(q + 2q + 3q + \ldots + (N - q)) = \frac{N(N-q)}{q} \), but since \( r \) and \( N \) have no common factors this implies that \( r \) divides \( N - q \) and hence also that \( r \) divides \( q - 1 \) (since \( r | (N - 1) \)). Taking \( a = \frac{1}{r} \) similarly gives that \( p \) must divide \( 2(r + 2r + 3r + \ldots + (N - 1)) = \frac{(N-1)(N-1+r)}{r} \), so that \( q \) must divide \( N - 1 + r \) and hence \( q \) must divide \( r - 1 \) (since \( q | N \)). Since we found that \( q | r - 1 \) and also \( r | q - 1 \) either \( r \) or \( q \) must equal 1, so that \( p \) must divide either \( N \) or \( N - 1 \) as claimed above. It is easy to check that when this is satisfied, all the phases \((A_3)_{a,b}\) vanish, due to cancellations between the contributions of the fields \( \Phi_j \) and \( \Phi_{N-j} \).

The modular and elliptic transformations of the functions \( f_{a,b} \) are derived from the properties of \( \Theta_{a,b} \) \((3.9)\). We find that

\[
\begin{align*}
f_{a,b}(u + N|\tau) &= e^{2i\pi NA_{N}(a,b)}f_{a,b}(u|\tau) \\
f_{a,b}(u + N\tau|\tau) &= e^{2i\pi NB_{N}(a,b)}e^{-i\pi kN(N-1)(k+N)(N\tau+2u)}f_{a,b}(u|\tau) \\
f_{a,b}(u|\tau + 1) &= e^{2i\pi C_{N}(a,b)}f_{a,a+b}(u|\tau) \\
f_{a,b}\left(\frac{u}{\tau} - \frac{1}{\tau}\right) &= (i\tau)^{1/2}e^{i\pi k(N-1)(k+N)^2}e^{2i\pi D_{N}(a,b)}f_{b,-a}(u|\tau),
\end{align*}
\]

where the phases \( A_N, B_N, C_N, D_N \) combine the effect of the phases in the transformations of the \( \Theta \)'s \((3.9)\) and the necessity of putting back the indices of the transformed \( \Theta \)'s into the interval \([-\frac{1}{2}, \frac{1}{2}]\). The latter occurs only for the third and fourth transformations, and gives contributions due to

\[
\tau \to \tau + 1 : \Theta_{[a],[a]+[b]}(u|\tau) = e^{2i\pi ([a]+\frac{1}{2})([a]+[b]-[[a]+[b]])\Theta_{[a],[a]+[b]}(u|\tau) \\
\frac{u}{\tau} \to \left(\frac{u}{\tau} - \frac{1}{\tau}\right) : \Theta_{[b],[−a]}(u|\tau) = e^{2i\pi \delta_{(a),−\frac{1}{2}([b]+\frac{1}{2})}}\Theta_{[b],[−a]}(u|\tau).
\]

We get by a simple calculation (with equalities satisfied modulo 1)

\[
A_N(a,b) = \sum_{j=1}^{N-1} [-ja] - [ja] = \sum_{j=1}^{N-1} (-2ja) = -aN(N - 1) \quad (4.2)
\]

and

\[
B_N(a,b) = \sum_{j=1}^{N-1} [jb] - [-jb] = -A_N(b,a).
\]
so that both phases obviously vanish whenever $p|(N(N-1))$. The other other phases come out to be

$$C_N(a, b) = \sum_{j=1}^{N-1} ([-ja] + [-jb] - [-j(a+b)](\lfloor ja \rfloor + \lfloor jb \rfloor)(\lfloor ja \rfloor + \lfloor jb \rfloor) - \lfloor ja \rfloor - \lfloor jb \rfloor)$$

$$= \sum_{\lfloor ja \rfloor \neq -\frac{1}{2}} [ja]([ja] + [-j(a+b)] - [jb] - [-jb]) + \frac{1}{2}([ja] + [-j(a+b)] - [jb] - [-jb] - 2[ja])$$

$$= \sum_{[aj]= -\frac{1}{2}} (a_j - \frac{1}{2}) - \sum_{[(a+b)j]=-\frac{1}{2}} (a_j - \frac{1}{2}),$$

and

$$D_N(a, b) = \sum_{j=1}^{N-1} ([ja] - \frac{1}{2})([jb] + \frac{1}{2}) - ([ja] - \frac{1}{2})([jb] + \frac{1}{2}) + \delta_{[ja], -\frac{1}{2}}(\lfloor ja \rfloor - \lfloor jb \rfloor)$$

$$= \sum_{\lfloor ja \rfloor \neq -\frac{1}{2}} [-ja](1 + [-jb] + [jb]) + \frac{1}{2}([jb] - [-jb])$$

$$= \sum_{\lfloor ja \rfloor, [jb] \neq -\frac{1}{2}} ([jb] - [ja]) = (b - a) \sum_{\lfloor ja \rfloor, [jb] \neq -\frac{1}{2}} j$$

and it is easy to check that all of these phases vanish whenever $p$ is a divisor of $N$ or of $N - 1$, again because of cancellations between the contributions of $\Phi_j$ and $\Phi_{N-j}$.

The general $q \to 0$ limit is worked out in the next section. As an example we can look at the case of $p = N$ with $N$ prime, for which we get (using (3.11))

$$\lim_{q \to 0} Z_N^{(p)}(u|\tau) = \frac{1}{p} (x^{-k(N-1)/2} \sum_{j=0}^{p-1} h_N^{(k)}(\omega^j x) + p(p - 1)),$$

where $x = e^{2i\pi u}$, $\omega = e^{2\pi i/p}$ (there are only $p$ functions $f_{a,b}$ which have a limit different from 1, namely those with $a = 0$), and

$$h_N^{(k)}(x) = \lim_{q \to 0} x^{k(N-1)/2} f_{0,0}(u|\tau) = \prod_{j=1}^{N-1} \frac{1 - x^{k+N-j}}{1 - x^j}.$$

The Poincaré polynomial of the corresponding “chiral ring” $\mathbb{C}[x_1, \ldots, x_{N-1}]/\nabla W_{k+N}$, namely

$$P_N^{(p)}(x) = \sum \text{ring elements} x^\text{degree},$$

(4.3)
is equal to the \( q \to 0 \) limit of the elliptic genus (for \( x \) defined as above), up to a constant power of \( x \). This polynomial must satisfy a duality symmetry of the form

\[
P(x) = x^m P\left(\frac{1}{x}\right),
\]

in any \( N = 2 \) theory \cite{14} \cite{15} with \( m = d\hat{c} \) where \( d \) is the degree of the Landau-Ginzburg potential, and in our case \( m = k(N - 1) \). This symmetry is assured in our case whenever we have a modular invariant theory, since the \( x \to \frac{1}{x} \) transformation is exactly the square of the \( S \) modular transformation.

5. Orbifolds of general Landau-Ginzburg theories

Let us consider now a general Landau-Ginzburg model, with a quasi-homogenous potential including fields of degrees (dimensions) \( d_i \) (for \( i = 1, \ldots, n \)) and a potential of degree \( d \) (we choose a normalization in which the degrees are all integers). Classically, this theory has a \( \mathbb{Z}_d \) symmetry which takes the field \( \Phi_i \) to \( e^{2i\pi\omega\frac{d_i}{d}}\Phi_i \) where \( \omega \) is a member of \( \mathbb{Z}_d \). Obviously there is also a \( \mathbb{Z}_p \) symmetry for any \( p \) which is a divisor of \( d \). We shall now try to orbifold the theory by such a \( \mathbb{Z}_p \) factor - this means allowing the fields to be twisted by any member of the symmetry group when going around both non-trivial cycles of the torus. The computation of the elliptic genus is reduced, as in \cite{3}, to a free field computation which is similar to the one performed in the previous sections. The contribution to the elliptic genus of the sector in which we have a twist \( a \) in one direction and a twist \( b \) in the other (meaning \( \Phi_i \) transforms to \( e^{2i\pi ad_i}\Phi_i \) around one cycle and to \( e^{2i\pi bd_i}\Phi_i \) around the other) comes out to be, up to a constant phase related to the choice of the relative fermion numbers of the vacua in the different sectors,

\[
f_{a,b}(z|\tau) = \prod_j \frac{\Theta[-ad_j][bd_j][(d - d_j)z|\tau]}{\Theta[ad_j][bd_j][d_jz|\tau]}.
\]

(5.1)

The total elliptic genus will then be

\[
K(z|\tau) = \frac{1}{p} \sum_{a,b=[0],[\frac{1}{p}],\ldots,[\frac{p-1}{p}]} e^{2i\pi\Phi_{a,b}} f_{a,b}(z|\tau).
\]

(5.2)

The phases \( \Phi_{a,b} \) will be determined later by the requirements of modular invariance and a correct \( q \to 0 \) limit. For now let us compute the modular transformations of \( f_{a,b} \) : they come out, using the transformations of the Theta functions (3.9), to be

\[
f_{a,b}(u + 1|\tau) = (-1)^{nd} e^{-4i\pi a} \sum d_j f_{a,b}(u|\tau)
\]

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\[ f_{a,b}(N(z + 1)|\tau) = (-1)^{Nnd}e^{-4N\pi a}\sum d_j f_{a,b}(Nz|\tau) \]

\[ f_{a,b}(u + \tau|\tau) = e^{-i\pi d^2(\tau + 2u)}(-1)^{nd} f_{a,b}(u|\tau) \]

where as usual \( \hat{c} = \sum_j (1 - \frac{2d_j}{d}) \), and

\[ f_{a,b}(N(z + \tau)|\tau) = e^{-i\pi d^2N^2\hat{c}(\tau + 2z)}(-1)^{nNd} f_{a,b}(Nz|\tau). \]

We see that if we wish \( u \) to be identified with \( Nz \) (as we did for the \( SU(N) \) cases), modular invariance demands that \( Nnd \) be even, and that \( p \) be a divisor of \( 2N \sum d_j \).

For the \( \tau \) transformations we have:

\[ f_{a,b}(u|\tau + 1) = e^{2i\pi(A_1)_{a,b}} f_{a,a+b}(u|\tau) \]

where in the general orbifold case there is a phase \( (A_1)_{a,b} \) which arises from cases in which \( [ad_j] + [bd_j] \neq [(a + b)d_j] \) or \( [-ad_j] + [-bd_j] \neq [-(a + b)d_j] \), and which comes out to be (up to integer shifts)

\[ (A_1)_{a,b} = \sum_{[bd_j] = -\frac{1}{2}} (ad_j - \frac{1}{2}) - \sum_{[(a+b)d_j] = -\frac{1}{2}} (ad_j - \frac{1}{2}) \]

so that the condition we get for modular invariance of \( K(z|\tau) \) defined above is that \( (A_1)_{a,b} = \Phi_{a,a+b} - \Phi_{a,b} \) (mod 1) for all \( a, b \) (since \( f_{a,b} \) satisfies \( f_{a+1,b} = f_{a,b+1} = f_{a,b} \)). Note that for odd \( r \) \( (A_1)_{a,b} \) vanishes identically.

For the \( S \) transformation we have:

\[ f_{a,b}(\frac{u}{\tau} - \frac{1}{\tau}) = e^{i\pi d^2\hat{c}N^2} e^{2i\pi(A_2)_{a,b}} f_{b,-a}(u|\tau) \]

where in this case the phase \( (A_2)_{a,b} \) arises both from the \( \Theta_{a,b} \) transformation and from corrections in case \(-[a] \neq [-a]\). It comes out to be

\[ (A_2)_{a,b} = \sum_j (([ad_j] - \frac{1}{2})([-bd_j] + \frac{1}{2}) - ([ad_j] - \frac{1}{2})([bd_j] + \frac{1}{2}) + \delta_{[ad_j], -\frac{1}{2}}([ad_j] - [bd_j])) \]

which turns out to be equal (modulo 1) to

\[ (A_2)_{a,b} = \sum_{[ad_j] \neq -\frac{1}{2}, [bd_j] \neq -\frac{1}{2}} (b - a)d_j. \]
This phase should also satisfy \((A_2)_{a,b} = \Phi_{b,-a} - \Phi_{a,b} \pmod{1}\) for all \(a,b\) for the elliptic genus to be modular invariant. Note, that if we act twice with this transformation we get

\[ f_{a,b}(-u|\tau) = e^{2i\pi((A_2)_{a,b} + (A_2)_{b,-a})} f_{-a,-b}(u|\tau) \]

and this transformation is exactly the duality transformation of the Poincaré polynomial \((x \to \frac{1}{x})\) related to the charge conjugation symmetry of \(N = 2\) theories. We see that this polynomial is automatically self-dual whenever our theory is modular invariant. The inverse of this statement is not necessarily correct.

Let us now look at the \(q \to 0\) limit of \(f_{a,b}\). The contribution of each term of the form

\[ \frac{\Theta_{[-ad_j][-bd_j]}((d-d_j)z|\tau)}{\Theta_{[ad_j][bd_j]}(d_jz|\tau)} \]

to this limit is one of the following:

(i) if \([ad_j] = -\frac{1}{2}\) the contribution is 1.
(ii) if \(-\frac{1}{2} < [ad_j] < 0\) the contribution is \(e^{2i\pi(-\frac{1}{2} - [ad_j])(dz+1+[bd_j]+[-bd_j])}\).
(iii) if \(0 < [ad_j] < \frac{1}{2}\) the contribution is \(e^{2i\pi(\frac{1}{2} - [ad_j])(dz+1+[bd_j]+[-bd_j])}\).
(iv) if \([ad_j] = 0\) the contribution is

\[ e^{-i\pi((d-2d_j)z+[-bd_j]-[bd_j])} \frac{1 - e^{2i\pi(z+b)(d-d_j)}}{1 - e^{2i\pi(z+b)d_j}}. \]

The total contribution is therefore

\[ P_{a,b}(x) = x^d \sum_{[ad_j] \neq 0} \left( \frac{1}{2} \text{sign}([ad_j]) - [ad_j] \right) (1 + [bd_j] + [-bd_j]) + \sum_{[ad_j] = 0} (d_j - \frac{1}{2}d) \sum_{[bd_j] \neq -\frac{1}{2}} \frac{1 - (xe^{2i\pi b})^{d-d_j}}{1 - (xe^{2i\pi b})^{d_j}} \]

where \(x = e^{2i\pi z}\). This expression was derived by a different method (and for \(p = d\)) in \(\text{[8]}\). Our expression differs from the result there and from the direct free field computation (assuming zero fermion number for the vacuum in all sectors) by a phase \((A_3)_{a,b}\), which equals (modulo 1) to

\[ (A_3)_{a,b} = \sum_{[ad_j] \neq 0} \frac{1}{2} \text{sign}([ad_j]) - [ad_j] (1 + [bd_j] + [-bd_j]) + \sum_{[ad_j] = 0, [bd_j] \neq -\frac{1}{2}} bd_j \]

or

\[ (A_3)_{a,b} = \sum_{[bd_j] \neq -\frac{1}{2}} \left( \sum_{[ad_j] \neq 0} \frac{1}{2} - ad_j \right) + \sum_{[ad_j] = 0} bd_j. \]
The only thing left in the computation is the determination of the phases \(\Phi_{a,b}\). For the case of \(\frac{SU(N)}{\mathbb{Z}_p}\) where \(p\) is a divisor of \(N\) or of \(N - 1\) we checked in previous chapters that all phase-corrections vanish and therefore we can choose \(\Phi_{a,b} = 0\) identically and get the correct elliptic genus.

In the general case, it is interesting to note that if we choose

\[
K(z|\tau) = \frac{1}{p} \sum_{a,b=[0],(1\ldots,\frac{p-1}{p}]} e^{-2\pi p a} \sum_j d_j \prod_j \frac{\Theta_{-[ad_j]-[bd_j]}((d - d_j)z|\tau)}{\Theta_{|ad_j||bd_j|}(d_j z|\tau)}
\]

(5.3)

we get no phases in the modular transformations (meaning the expression is modular invariant), but we do get a phase of \(\sum_{[ad_j]|\neq 0}(\frac{1}{2} - 2ad_j) + \sum_{[ad_j]|=0} bd_j\) in the \(q \to 0\) limit of the \((a,b)\) sector. If we want no phases in the \(q \to 0\) limit, the phases \(\Phi_{a,b}\) are uniquely determined to cancel the phases \((A_3)_{a,b}\), and we then get conditions on the dimensions of the fields and on \(p\) that our theory must satisfy for the elliptic genus to be modular invariant. Otherwise, there are many possible solutions \(\Phi_{a,b}\) to the modular invariance conditions, giving different \(q \to 0\) limits, so that some condition must be externally imposed on the phases in this limit, such as the one we got from the characters in the \(SU(N)\) case.

6. Discussion and remarks

In the present paper we discussed orbifolds associated with \(N = 2\) \(G/H\) models which admit a LG description. Most of our discussion is related to the \(SU(N)\) KS models. We proved that a consistent orbifold theory is obtained whenever we mode out by a \(\mathbb{Z}_p\) symmetry, with \(p\) a divisor of \(k + N\) and also of either \(N\) or \(N - 1\). This means that \(\mathbb{Z}_p\) is associated with a modular invariant of either the \(SU(N)\) or the \(SU(N - 1)\) factors of the \(SU(N)\) KS coset. We calculated the elliptic genera associated with these orbifolds using the LG potential of the \(N = 2\) superconformal theory and proved them to be equal to the elliptic genera of orbifolds of the \(N = 2\) SCFT. In our approach we use the fact that the elliptic genus is modular covariant. Thus, whenever we can compute the elliptic genus starting from the LG theory, it is enough to check its modular properties and compare the \(q \to 0\) limit to that of the corresponding orbifold theory. If they match, the elliptic genera match as well. This gives further support to the identification of the macroscopic LG theory with the microscopic \(N = 2\) theory. The orbifold theory, in general, does not admit a LG description. Nevertheless, the LG potential of the underlying \(N = 2\) SCFT encodes
the information about the various orbifold theories obtained by moding with respect to the appropriate $\mathbb{Z}_p$ symmetry groups.

For example, in the $SU(3)$ KS case we can construct not only the $\mathbb{Z}_3$ orbifold associated with the $SU(3)$ factor in the coset, but also a $\mathbb{Z}_2$ orbifold associated with the $SU(2)$ factor of this coset. Our approach leads us to conjecture that also some $E_{6,7,8}$ theories associated with the corresponding $E$ modular invariants of the $SU(2)$ factor should exist. We gave a "natural" conjecture for their elliptic genera and LG potentials based on the similarity between the perturbed $SU(2)$ potential and the unperturbed $SU(3)$ potential. It would be extremely interesting to establish whether one can understand these theories also in terms of an orbifold construction. The obvious suggestion that comes to mind is that (if at all) they can be associated with the finite non-abelian (tetra, octa and icoso–edral) subgroups of $SU(2)$. Similar constructions can be employed for higher $SU(N)$ KS theories taking advantage of the exceptional modular invariants. Again, it would be tantalizing to conjecture that those may be associated with some non-abelian orbifolds.

All these questions are intimately tied to the general question of the relationship between LG theories and CFT’s. It is a challenging question to investigate whether the LG theories can account for all CFT’s via some definite procedure. The orbifold approach, as demonstrated in this paper, allows for the construction of new CFT’s. It is an open problem to investigate whether by allowing also for non–abelian orbifolds we exhaust all modular invariants associated with CFT’s. Another, perhaps related, question is to try to understand embeddings directly via the LG approach.

Although we were mainly interested in the $SU(N)$ KS theories, we did investigate the elliptic genus associated with an orbifold of a general LG theory. In the general case it is not clear exactly what the phase of the $q \to 0$ limit of the elliptic genus should be, but for any definite prescription for this phase we get conditions that must be satisfied for a $\mathbb{Z}_p$ orbifold to exist on the quantum level.

It would be extremely interesting to generalize our approach for the construction of the elliptic genus also to other KS theories for which the LG description is unknown (and may very well not exist). In many cases the Poincaré polynomial, which is intimately related to the elliptic genus, is known, and seems similar (but not equal) to a polynomial connected to a LG theory of degree $k + h^v$ [15] [14] (where $h^v$ is the second Casimir invariant of the group $G$). Thus we may conjecture that these theories could be given by some orbifold of a LG theory of this degree. Obviously for a $\mathbb{Z}_p$ orbifold to exist in this case $p$ must be a divisor of $k + h^v$, and based on our results here it seems that $\mathbb{Z}_p$ must also be a subgroup
of either the center of $G$ or of the center of $H$ for modular invariance to be satisfied. In particular it would be important to clarify the relationship between the elliptic genera of $N = 2 \, G/H$ theories associated with a given $G$ and different choices for $H$.

Finally, we would like to raise yet another open problem concerning the identification of the macroscopic $N = 2$ theory and the microscopic $N = 2$ theories. We have proven that the elliptic genus and therefore the Ramond characters are encoded within the LG potential. For a complete identification we would like also to be able to get information pertaining to the characters of the NS sector directly from the LG potential. The ultimate identification would be to get all characters of the microscopic $N = 2$ theory directly from the corresponding LG potential.

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**Note added.**

While this work was completed, there appeared on the hep-th net a paper [16], which slightly overlaps with some of our superconformal elliptic genus results (SU(2) D series).
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