Symmetric quiver Hecke algebras and $R$-matrices of quantum affine algebras III

Seok-Jin Kang, Masaki Kashiwara, Myungho Kim and Se-jin Oh

ABSTRACT

Let $\mathcal{C}_g^0$ be the category of finite-dimensional integrable modules over the quantum affine algebra $U_q'(g)$ and let $R^{A_\infty}$-gmod denote the category of finite-dimensional graded modules over the quiver Hecke algebra of type $A_\infty$. In this paper, we investigate the relationship between the categories $\mathcal{C}_g^0(A_{N-1}^{(1)})$ and $\mathcal{C}_g^0(A_{N-1}^{(2)})$, by constructing the generalized quantum affine Schur–Weyl duality functors $F^{(t)}$ from $R^{A_\infty}$-gmod to $\mathcal{C}_g^0(A_{N-1}^{(t)})$ ($t = 1, 2$).

Introduction

The quiver Hecke algebra $R$, introduced independently by Khovanov–Lauda [18] and Rouquier [23], provides a categorification of the negative half $U_q'(g)$ of a quantum group $U_q(g)$. Moreover, its cyclotomic quotients $R^\Lambda$, depending on dominant integral weights $\Lambda$, also provide a categorification of the integrable highest weight modules $V_q^\Lambda(g)$ over $U_q(g)$ [11]. Recall that the cyclotomic quotients of an affine Hecke algebra give a categorification of integrable highest weight $U(A_{N-1}^{(1)})$-modules. Thus the quiver Hecke algebras play the role of affine Hecke algebras in the representation theory of all symmetrizable quantum groups.

In [4, 5, 8], Chari–Pressley, Cherednik and Ginzburg–Reshetikhin–Vasserot constructed the quantum affine Schur–Weyl duality functor that relates the category of finite-dimensional modules over an affine Hecke algebra and the category of finite-dimensional integrable $U_q'(A_{N-1}^{(1)})$-modules. In [12], the first three authors of this paper constructed a functor $F$ from the category of finite-dimensional graded modules over a symmetric quiver Hecke algebra $R$ to the category of finite-dimensional integrable modules over any quantum affine algebra $U_q'(g)$. Here, the quiver Hecke algebra $R$ is determined by a family of good $U_q'(g)$-modules. In this context, the quiver Hecke algebras can be thought of as a generalization of affine Hecke algebras, which gives the generalized quantum affine Schur–Weyl duality functor $F$.

The representation theory of quantum affine algebras has been extensively investigated with various approaches (see, for example, [2, 7, 9, 17, 21]). By the work of [12], we propose a new approach for studying the representations of quantum affine algebras through the representation theory of quiver Hecke algebras.

Let $\mathcal{C}_g^0$ denote the category of finite-dimensional integrable modules over the quantum affine algebra $U_q'(g)$, and let $R^{A_\infty}$-gmod denote the category of finite-dimensional graded modules over the quiver Hecke algebra of type $A_\infty$. The purpose of this paper is to investigate the relationship between the categories $\mathcal{C}_g^0(A_{N-1}^{(1)})$ and $\mathcal{C}_g^0(A_{N-1}^{(2)})$ (see Table 1) by constructing exact
functors $\mathcal{F}(t): R^A_{\infty}\text{-gmod} \to \mathcal{C}^0_{\tilde{A}^t_{N-1}} (t = 1, 2)$ (see also [9]).

To construct such functors, we first choose a family of good $U'_q(\mathfrak{g})$-modules and study the distribution of poles of normalized $R$-matrices between them. Then, by the general argument given in [12], we obtain the generalized quantum affine Schur–Weyl duality functor $\tilde{\mathcal{F}}: R\text{-gmod} \to \mathcal{C}^0_{\mathfrak{g}}$. In particular, it was shown in [12] that the family of good $U'_q(A^{(1)}_{N-1})$-modules $\{V(\varpi_1)_{q^{2s}} \mid s \in \mathbb{Z}\}$ gives the functor $\mathcal{F}(1)$.

In this paper, based on the results of [22] on the normalized $R$-matrices of $U'_q(A^{(2)}_{N-1})$-modules, we prove that the family of good $U'_q(A^{(2)}_{N-1})$-modules $\{V(\varpi_1)_{q^{2s}} \mid s \in \mathbb{Z}\}$ yields a quiver whose underlying graph is of type $A_\infty$, and then we construct the exact functor $\mathcal{F}(2): R^A_{\infty}\text{-gmod} \to \mathcal{C}^0_{\tilde{A}^{(2)}_{N-1}}$. Through the exact functors $\mathcal{F}(t) (t = 1, 2)$, one can observe that the categories $\mathcal{C}^0_{\tilde{A}^{(1)}_{N-1}}$ and $\mathcal{C}^0_{\tilde{A}^{(2)}_{N-1}}$ have many similar properties (for example, see Proposition 2.4 and Corollary 3.4). Note that some of these similarities have been already observed by Hernandez [9] by a different approach.

We prove that the functor $\mathcal{F}(2): R^A_{\infty}\text{-gmod} \to \mathcal{C}^0_{\tilde{A}^{(2)}_{N-1}}$ factors through a certain localization $T_N$. Furthermore, the induced functor $\tilde{\mathcal{F}}(2): T_N \to \mathcal{C}^0_{\tilde{A}^{(2)}_{N-1}}$ gives a ring isomorphism

$$K(T_N)/(q-1)K(T_N) \xrightarrow{\sim} K(\mathcal{C}^0_{\tilde{A}^{(2)}_{N-1}})$$

as in the case of the Grothendieck ring $K(\mathcal{C}^0_{\tilde{A}^{(1)}_{N-1}})$ in [12]. Hence we obtain the diagram

where $\tilde{\mathcal{F}}(t)$ gives a bijection between the simple modules (up to degree shift and isomorphism) in $T_N$ and the simple modules (up to isomorphism) in $\mathcal{C}^0_{\tilde{A}^{(t)}_{N-1}} (t = 1, 2)$. With this approach, we prove that the induced functors $\tilde{\mathcal{F}}(1)$ and $\tilde{\mathcal{F}}(2)$ give the correspondence between the simple modules in $\mathcal{C}^0_{\tilde{A}^{(1)}_{N-1}}$ and the simple modules in $\mathcal{C}^0_{\tilde{A}^{(2)}_{N-1}}$, which preserves their dimensions (Theorem 3.18).

Let us compare this with one of the results in [9]. First, Hernandez defined the twisted $q$-character morphism $\chi^\sigma_q$ from the Grothendieck ring of finite-dimensional integrable modules over a twisted quantum affine algebra to a certain polynomial ring. Note that it is an analogue of the $q$-character morphism $\chi_q$ for untwisted quantum affine algebras in [7]. Then he found a ring homomorphism from the codomain of $\chi^\sigma_q$ to the codomain of $\chi^\sigma_0$, which induces an isomorphism between the image of $\chi_q$ and that of $\chi^\sigma_0$. During its proof, he showed that this isomorphism sends the $q$-characters of Kirillov–Reshetikhin modules to the twisted $q$-characters of Kirillov–Reshetikhin modules. But it is not known whether the isomorphism sends the $q$-characters of simple modules to the twisted $q$-characters of simple modules or not. We expect that the isomorphism in [9, Theorem 4.15] coincides with ours in Corollary 3.19.
Since the results in [9] cover not only type $A$ but also other types, one may expect that there are similar correspondences between untwisted and twisted quantum affine algebras of other types. Another paper of ours [14] is initiated by this observation and provides a correspondence between certain subcategories of $\mathcal{C}_g^0$ over untwisted and twisted quantum affine algebras of type $A$ and $D$ through the representation theory of quiver Hecke algebras.

This paper is organized as follows. In Section 1, we briefly review the results of [12] on the generalized quantum affine Schur–Weyl duality functors. In Section 2, we compare the denominators of normalized $R$-matrices and the homomorphisms between fundamental representations over $U_q(A^{(2)}_{N-1})$. This comparison provides the main ingredients for the construction of exact functors $F(t)$ ($t = 1, 2$). In Section 3, we construct the exact functor $F(2): R_{A^\infty}\mathcal{g}\mathcal{m}\mathcal{d} \rightarrow \mathcal{C}_0^{A^{(2)}}_{N-1}$ and investigate the relationship between the categories $\mathcal{C}_0^{A^{(2)}}_{N-1}(t = 1, 2)$ via $R_{A^\infty}\mathcal{g}\mathcal{m}\mathcal{d}$.

Convention

1. All the algebras and rings in this paper are assumed to have a unit, and modules over them are unitary.
2. For a ring $A$, an $A$-module means a left $A$-module.
3. For a statement $P$, $\delta(P)$ is 1 if $P$ is true and 0 if $P$ is false.
4. For a ring $A$, we denote by $\text{Mod}(A)$ the category of $A$-modules. When $A$ is an algebra over a field $k$, we denote by $A$-mod the category of $A$-modules that are finite-dimensional over $k$.
5. For a ring $A$, we denote by $A^\times$ the set of invertible elements of $A$.
6. For an abelian category $\mathcal{C}$, we denote by $K(\mathcal{C})$ the Grothendieck group of $\mathcal{C}$.

1. Symmetric quiver Hecke algebras and quantum affine algebras

1.1. Cartan datum and quantum groups

In this subsection, we recall the definition of quantum groups. Let $I$ be an index set. A Cartan datum is a quintuple $(A, P, \Pi, \Pi^\vee, \Pi^\vee)$ consisting of

1. an integer-valued matrix $A = (a_{ij})_{i,j \in I}$, called the symmetrizable generalized Cartan matrix, which satisfies
   (i) $a_{ii} = 2 (i \in I)$, 
   (ii) $a_{ij} \leq 0 (i \neq j)$, 
   (iii) $a_{ij} = 0$ if $a_{ji} = 0$ ($i, j \in I$), 
   (iv) there exists a diagonal matrix $D = \text{diag}(s_i | i \in I)$ such that $DA$ is symmetric and $s_i$ are positive integers;
2. a free abelian group $P$, called the weight lattice;
3. $\Pi = \{\alpha_i \in P | i \in I\}$, called the set of simple roots;
4. $P^\vee := \text{Hom}(P, \mathbb{Z})$, called the co-weight lattice;
5. $\Pi^\vee = \{h_i | i \in I\} \subset P^\vee$, called the set of simple coroots,

satisfying the following properties:

1. $\langle h_i, \alpha_j \rangle = a_{ij}$ for all $i, j \in I$;
2. $\Pi$ is linearly independent;
3. for each $i \in I$, there exists $\Lambda_i \in P$ such that $\langle h_j, \Lambda_i \rangle = \delta_{ij}$ for all $j \in I$. 

Convention
We recall the definition of quiver Hecke algebras associated with a given Cartan datum

\[ V \].

Quiver Hecke algebras

\[
\begin{align*}
\text{defining relations:} \\
(\alpha_i, \alpha_j) = s_{ij} (i, j \in I) \quad \text{and} \quad \langle h_i, \lambda \rangle = \frac{2(\alpha_i, \lambda)}{\langle \alpha_i, \alpha_i \rangle} \quad \text{for any } \lambda \in h^* \quad \text{and} \quad i \in I. \\
\end{align*}
\]

Let \( q \) be an indeterminate. For each \( i \in I \), set \( q_i = q^{\delta_i} \).

**Definition 1.1.** The quantum group \( U_q(g) \) with a Cartan datum \((A, P, \Pi, P^\vee, \Pi^\vee)\) is the algebra over \( \mathbb{Q}(q) \) generated by \( e_i, f_i \ (i \in I) \) and \( q^h (h \in P^\vee) \) satisfying the following relations:

\[
\begin{align*}
q^0 &= 1, \quad q^h q^{h'} = q^{h+h'} \quad \text{for } h, h' \in P, \\
q^h e_i q^{-h} &= q^{(h, \alpha_i)} e_i, \quad q^h f_i q^{-h} = q^{-(h, \alpha_i)} f_i \quad \text{for } h \in P^\vee, i \in I, \\
e_i f_j - f_j e_i &= s_{ij} K_i K_j^{-1} \quad \text{where } K_i = q^{\delta_i}, \\
\sum_{r=0}^{1-a_{ij}} (-1)^r e_i^{(1-a_{ij}-r)} e_j^{(r)} &= 0 \quad \text{if } i \neq j.
\end{align*}
\]

Here, we set \([n]_i = (q^n - q^{-n})/(q - q^{-1}) \); \([n]_i = \prod_{k=1}^n [k]_i \); \( e_i^{(m)} = e_i^n/[n]_i! \) and \( f_i^{(n)} = f_i^n/[n]_i! \) for all \( n \in \mathbb{Z}_{\geq 0}, \ i \in I \).

1.2. Quiver Hecke algebras

We recall the definition of quiver Hecke algebras associated with a given Cartan datum \((A, P, \Pi, P^\vee, \Pi^\vee)\).

Let \( k \) be a commutative ring. For \( i, j \in I \) such that \( i \neq j \), let us take a family of polynomials \((Q_{ij})_{i,j\in I}\) in \( k[u,v] \) which are of the form

\[
Q_{ij}(u,v) = \delta(i \neq j) \sum_{(\alpha_i, \alpha_j) \in \mathbb{Z}_0^2} t_{i,j,p,q} u^p v^q
\]

with \( t_{i,j,p,q} \in k \), \( t_{i,j,p,q} = t_{j,i;p,q} \), and \( t_{i,j;-a_{ij},0} \in k^\times \). Thus we have \( Q_{ij}(u,v) = Q_{ji}(v,u) \).

We denote by \( \mathcal{S}_n = \langle s_1, \ldots, s_{n-1} \rangle \) the symmetric group on \( n \) letters, where \( s_i := (i, i+1) \) is the transposition of \( i \) and \( i+1 \). Then \( \mathcal{S}_n \) acts on \( I^n \) by place permutations.

For \( n \in \mathbb{Z}_{\geq 0} \) and \( \beta \in Q^+ \) such that \( |\beta| = n \), we set

\[
I^\beta = \{ \nu = (\nu_1, \ldots, \nu_n) \in I^n : \alpha_{\nu_1} + \cdots + \alpha_{\nu_n} = \beta \}.
\]

**Definition 1.2.** For \( \beta \in Q^+ \) with \( |\beta| = n \), the Khovanov–Lauda–Rouquier algebra \( R(\beta) \) at \( \beta \) associated with a Cartan datum \((A, P, \Pi, P^\vee, \Pi^\vee)\) and a matrix \((Q_{ij})_{i,j\in I}\) is the \( k \)-algebra generated by the elements \( \{e(\nu)\}_{\nu \in I^\beta}, \ \{x_k\}_{1 \leq k \leq n}, \ \{\tau_m\}_{1 \leq m \leq n-1} \) satisfying the following defining relations:

\[
\begin{align*}
e(\nu)e(\nu') &= \delta_{\nu, \nu'} e(\nu), \\
\sum_{\nu' \in I^\beta} e(\nu) &= 1, \\
x_k x_m &= x_m x_k, \quad x_k e(\nu) = e(\nu) x_k, \\
\end{align*}
\]
\[
\tau_m e(\nu) = e(s_m(\nu))\tau_m, \quad \tau_k \tau_m = \tau_m \tau_k \quad \text{if} \ |k - m| > 1,
\]
\[
\tau_k^2 e(\nu) = Q_{\nu_k, \nu_{k+1}}(x_k, x_{k+1}) e(\nu),
\]
\[
(\tau_k x_m - x_{s_k(m)} \tau_k) e(\nu) = \begin{cases} -e(\nu) & \text{if} \ m = k, \nu_k = \nu_{k+1}, \\ e(\nu) & \text{if} \ m = k + 1, \nu_k = \nu_{k+1}, \\ 0 & \text{otherwise}, \end{cases}
\]
\[
(\tau_{k+1} \tau_k \tau_{k+1} - \tau_k \tau_{k+1} \tau_k) e(\nu) = \begin{cases} Q_{\nu_k, \nu_{k+1}}(x_k, x_{k+1}) - Q_{\nu_k, \nu_{k+1}}(x_{k+2}, x_{k+1}) e(\nu) & \text{if} \ \nu_k = \nu_{k+2}, \\ 0 & \text{otherwise}. \end{cases}
\]

The above relations become homogeneous by assigning
\[
\deg e(\nu) = 0, \quad \deg x_k e(\nu) = (\alpha_{\nu_k}, \alpha_{\nu_k}), \quad \deg \tau_l e(\nu) = -(\alpha_{\nu_l}, \alpha_{\nu_{l+1}}),
\]
and hence \( R(\beta) \) is \( \mathbb{Z} \)-gradable.

For an element \( w \) of the symmetric group \( S_n \), let us choose a reduced expression \( w = s_{i_1} \cdots s_{i_l} \), and set
\[
\tau_w = \tau_{i_1} \cdots \tau_{i_l}.
\]

In general, it depends on the choice of reduced expressions of \( w \). Then we have
\[
R(\beta) = \bigoplus_{\nu \in I^\beta, \ w \in S_n} k[x_1, \ldots, x_n] e(\nu) \tau_w.
\]

For a graded \( R(\beta) \)-module \( M = \bigoplus_{k \in \mathbb{Z}} M_k \), we define \( qM = \bigoplus_{k \in \mathbb{Z}} (qM)_k \), where
\[
(qM)_k = M_{k-1} \quad (k \in \mathbb{Z}).
\]
We call \( q \) the grade shift functor on the category of graded \( R(\beta) \)-modules.

For \( \beta, \gamma \in \mathbb{Q}^+ \) with \( |\beta| = m, |\gamma| = n \), set
\[
e(\beta, \gamma) = \sum_{\nu \in I^m + I^n, \ (\nu_1, \ldots, \nu_m) \in I^\beta, \ (\nu_{m+1}, \ldots, \nu_{m+n}) \in I^\gamma} e(\nu) \in R(\beta + \gamma).
\]

Then \( e(\beta, \gamma) \) is an idempotent. Let
\[
R(\beta) \otimes R(\gamma) \longrightarrow e(\beta, \gamma) R(\beta + \gamma) e(\beta, \gamma)
\] (1.2)
be the \( k \)-algebra homomorphism given by
\[
e(\mu) \otimes e(\nu) \longmapsto e(\mu \ast \nu) \quad (\mu \in I^\beta),
\]
\[
x_k \otimes 1 \longmapsto x_k e(\beta, \gamma) \quad (1 \leq k \leq m),
\]
\[
1 \otimes x_k \longmapsto x_{m+k} e(\beta, \gamma) \quad (1 \leq k \leq n),
\]
\[
\tau_k \otimes 1 \longmapsto \tau_k e(\beta, \gamma) \quad (1 \leq k < m),
\]
\[
1 \otimes \tau_k \longmapsto \tau_{m+k} e(\beta, \gamma) \quad (1 \leq k < n),
\]
where \( \mu \ast \nu \) is the concatenation of \( \mu \) and \( \nu \); that is, \( \mu \ast \nu = (\mu_1, \ldots, \mu_m, \nu_1, \ldots, \nu_n) \).

For a \( R(\beta) \)-module \( M \) and a \( R(\gamma) \)-module \( N \), we define the convolution product \( M \circ N \) by
\[
M \circ N := R(\beta + \gamma) e(\beta, \gamma) \underset{R(\beta) \otimes R(\gamma)}{\otimes} (M \otimes N).
\]
1.3. \textit{R-matrices with spectral parameters}

For $|\beta| = n$ and $1 \leq a < n$, we define $\varphi_a \in R(\beta)$ by

$$\varphi_a e(\nu) = \begin{cases} (\tau_a x_a - x_a \tau_a) e(\nu) & \text{if } \nu_a = \nu_{a+1}, \\ \tau_a e(\nu) & \text{otherwise}. \end{cases} \quad (1.3)$$

They are called the \textit{intertwiners}. Since $\{\varphi_k\}_{1 \leq k \leq n-1}$ satisfies the braid relation, we have a well-defined element $\varphi_w \in R(\beta)$ for each $w \in S_n$.

For $m, n \in \mathbb{Z}_{\geq 0}$, we set

$$S_{m,n} := \{ w \in S_{m+n} : w(i) < w(i+1) \text{ for any } i \neq m \}.$$

For example,

$$w[m,n](k) = \begin{cases} k + n & \text{if } 1 \leq k \leq m, \\ k - m & \text{if } m < k \leq m + n, \end{cases}$$

is an element in $S_{m,n}$.

Let $\beta, \gamma \in \mathbb{Q}^+$ with $|\beta| = m$, $|\gamma| = n$, and let $M$ be an $R(\beta)$-module and $N$ an $R(\gamma)$-module. Then the map

$$M \otimes N \longrightarrow q^{2(\beta,\gamma)} M \circ N$$

given by

$$u \otimes v \longmapsto \varphi_{w[m,n]}(v \otimes u)$$

is an $R(\beta,\gamma)$-module homomorphism by [12, Lemma 1.3.1], and it extends to an $R(\beta+\gamma)$-module homomorphism

$$R_{M,N} : M \circ N \longrightarrow q^{2(\beta,\gamma)} M \circ N,$$ \quad (1.4)

where the symmetric bilinear form $(\cdot, \cdot)_n$ on $\mathbb{Q}$ is given by $(\alpha_i, \alpha_j)_n = \delta_{ij}$.

\textbf{Definition 1.3.} A quiver Hecke algebra $R(\beta)$ is symmetric if $Q_{i,j}(u,v)$ is a polynomial in $k[u-v]$ for all $i, j \in \text{supp } \beta$. Here $\text{supp } \beta = \{ i \in I : n_i \neq 0 \text{ for } \beta = \sum_{i \in I} n_i \alpha_i \}$.

From now on, we assume that quiver Hecke algebras are symmetric. Let $z$ be an indeterminate that is homogeneous of degree 2, and let $\psi_z$ be the algebra homomorphism

$$\psi_z : R(\beta) \longrightarrow k[z] \otimes R(\beta)$$

given by

$$\psi_z(x_k) = x_k + z, \quad \psi_z(\tau_k) = \tau_k, \quad \psi_z(e(\nu)) = e(\nu).$$

For an $R(\beta)$-module $M$, we denote by $M_z$ the $(k[z] \otimes R(\beta))$-module $k[z] \otimes M$ with the action of $R(\beta)$ twisted by $\psi_z$. Namely,

$$e(\nu)(a \otimes u) = a \otimes e(\nu)u,$$

$$x_k(a \otimes u) = (za) \otimes u + a \otimes (x_k u),$$

$$\tau_k(a \otimes u) = a \otimes (\tau_k u)$$ \quad (1.5)

for $\nu \in I^\beta$, $a \in k[z]$ and $u \in M$. For $u \in M$, we sometimes denote by $u_z$ the corresponding element $1 \otimes u$ of the $R(\beta)$-module $M_z$. 

For a non-zero $R(\beta)$-module $M$ and a non-zero $R(\gamma)$-module $N$,
let $s$ be the order of zeroes of $R_{M_z,N_z} : M_z \circ N_{z'} \longrightarrow q^{(\beta,\gamma)-2(\beta,\gamma)_n} N_{z'} \circ M_z$;
that is, the largest non-negative integer such that the image of $R_{M_z,N_z}$
is contained in $(z' - z)^s q^{(\beta,\gamma)-2(\beta,\gamma)_n} N_{z'} \circ M_z$. \hfill (1.6)

Note that \cite[Proposition 1.4.4(iii)]{12} shows that such an $s$ exists and $s \leq (\beta,\gamma)_n$.

**Definition 1.4.** For a non-zero $R(\beta)$-module $M$ and a non-zero $R(\gamma)$-module $N$, we set
\[ d(M, N) := (\beta, \gamma) - 2(\beta, \gamma)_n + 2s, \]
and define
\[ r_{M,N} : M \circ N \longrightarrow q^{d(M,N)} N \circ M \]
by
\[ r_{M,N} = ((z' - z)^{-s} R_{M_z,N_z})|_{z = z'} = 0. \]

By \cite[Proposition 1.4.4(ii)]{12}, the morphism $r_{M,N}$ does not vanish if $M$ and $N$ are non-zero.
For $\beta_1, \ldots, \beta_t \in \mathbb{Q}^+$, a sequence of $R(\beta_k)$-modules $M_k$ ($k = 1, \ldots, t$) and $w \in \mathcal{S}_t$, we set
\[ d = \sum d(M_i, M_j), \]
where the summation ranges over the set
\[ \{ (i, j) : 1 \leq i < j \leq t, w(i) > w(j) \}. \]
We define
\[ r_{M_1, \ldots, M_t}^w = r_{\{M_s\}_{1 \leq s \leq t}}^w : M_1 \circ \cdots \circ M_t \longrightarrow q^d M_{w(1)} \circ \cdots \circ M_{w(t)} \] \hfill (1.7)
by induction on the length of $w$ as follows:

\[ r_{\{M_s\}_{1 \leq s \leq t}}^w = \begin{cases} 
\text{id}_{M_1} \circ \cdots \circ M_t & \text{if } w = e, \\
r_{\{M_k\}_{1 \leq k \leq t}}^{w|_k} \circ (M_1 \circ \cdots \circ M_{k-1}) & \text{if } w(k) = w(k+1).
\end{cases} \]

Then it does not depend on the choice of $k$ and $r_{M_1, \ldots, M_t}^w$ is well-defined, because the homomorphisms $r_{M,N}$ satisfy the Yang–Baxter equation \cite[Subsection 1.4]{12}.

Similarly, we define
\[ R_{M_1, \ldots, M_t}^w : M_1 \circ \cdots \circ M_t \longrightarrow q^b M_{w(1)} \circ \cdots \circ M_{w(t)}, \] \hfill (1.8)
where $b = \sum_{1 \leq k < k' \leq t, (\beta_k, \beta_{k'}) - 2(\beta_k, \beta_{k'})_n}$.

We set
\[ r_{M_1, \ldots, M_t} := r_{M_1, \ldots, M_t}^w \quad \text{and} \quad R_{M_1, \ldots, M_t} := R_{M_1, \ldots, M_t}^w, \]
\hfill (1.9)
where $w_t$ is the longest element of $\mathcal{S}_t$.

### 1.4. Quantum affine algebras

In this subsection, we briefly review the representation theory of quantum affine algebras
following \cite{1, 17}. When concerned with quantum affine algebras, we take the algebraic closure of $\mathbb{C}(q)$ in $\mathbb{C}(q^{1/m})$ as the base field $k$.

We choose $0 \in I$ as the leftmost vertices in the tables in \cite[pages 54, 55]{10} except $A_{2n}^{(2)}$-case in which case we take the longest simple root as $\alpha_0$. Set $I_0 = I \setminus \{0\}$. 

The weight lattice $P$ is given by

$$ P = \bigoplus_{\lambda \in \Lambda} \mathbb{Z} \lambda, $$

and the simple roots are given by

$$ \alpha_i = \sum_{j \in I} a_{ij} \lambda_j + \delta(i = 0) \delta. $$

The weight $\delta$ is called the imaginary root. There exist $d_i \in \mathbb{Z}_{>0}$ such that

$$ \delta = \sum_{i \in I} d_i \alpha_i. $$

Note that $d_i = 1$ for $i = 0$. The simple coroots $\delta_i \in P^\vee := \text{Hom}_\mathbb{Z}(P, \mathbb{Z})$ are given by

$$ \langle \delta_i, \lambda \rangle = \delta_{ij}, \quad \langle \delta_i, \delta \rangle = 0. $$

Hence we have $\langle h_i, \alpha_j \rangle = \delta_{ij}$.

Let $c = \sum_{i \in I} c_i h_i$ be a unique element such that $c_i \in \mathbb{Z}_{>0}$ and

$$ \mathbb{Z} c = \left\{ h \in \bigoplus_{i \in I} \mathbb{Z} h_i ; \langle h, \alpha_i \rangle = 0 \text{ for any } i \in I \right\}. $$

Let us take a $\mathbb{Q}$-valued symmetric bilinear form $(\cdot, \cdot)$ on $P$ such that

$$ \langle h_i, \lambda \rangle = \frac{2(\alpha_i, \lambda)}{\langle \alpha_i, \alpha_i \rangle} \quad \text{and} \quad \langle \delta, \lambda \rangle = \langle c, \lambda \rangle \text{ for any } \lambda \in P. $$

Let $q$ be an indeterminate. For each $i \in I$, set $q_i = q^{(\alpha_i, \alpha_i)/2}$.

Let us denote by $U_q(\mathfrak{g})$ the quantum group associated with the affine Cartan datum $(A, P, \Pi, P^\vee, \Pi^\vee)$. We denote by $U_q'(\mathfrak{g})$ the subalgebra of $U_q(\mathfrak{g})$ generated by $e_i, f_i, K_i^{\pm 1}$ ($i = 0, 1, \ldots, n$). We call $U_q'(\mathfrak{g})$ the quantum affine algebra associated with $(A, P, \Pi, P^\vee, \Pi^\vee)$.

The algebra $U_q'(\mathfrak{g})$ has a Hopf algebra structure with the coproduct:

$$ \Delta(K_i) = K_i \otimes K_i, \quad \Delta(e_i) = e_i \otimes K_i^{-1} + 1 \otimes e_i, \quad \Delta(f_i) = f_i \otimes 1 + K_i \otimes f_i. \quad (1.10) $$

Set

$$ P_{\text{cl}} = P/\mathbb{Z} \delta $$

and call it the classical weight lattice. Let $\text{cl} : P \to P_{\text{cl}}$ be the projection. Then $P_{\text{cl}} = \bigoplus_{\lambda \in \Lambda} \mathbb{Z} \lambda(\lambda_i)$. Set $P_{\text{cl}}^0 = \{ \lambda \in P_{\text{cl}} : \langle c, \lambda \rangle = 0 \} \subset P_{\text{cl}}$.

A $U_q'(\mathfrak{g})$-module $M$ is called an integrable module if

1. $M$ has a weight space decomposition

$$ M = \bigoplus_{\lambda \in P_{\text{cl}}} M_\lambda, $$

where $M_\lambda = \{ u \in M ; K_i u = q_i^{(h_i, \lambda)} u \text{ for all } i \in I \}$;

2. the actions of $e_i$ and $f_i$ on $M$ are locally nilpotent for any $i \in I$.

Let us denote by $\mathcal{C}_g$ the abelian tensor category of finite-dimensional integrable $U_q'(\mathfrak{g})$-modules.

If $M$ is a simple module in $\mathcal{C}_g$, then there exists a non-zero vector $u \in M$ of weight $\lambda \in P_{\text{cl}}^0$ such that $\lambda$ is dominant (that is, $\langle h_i, \lambda \rangle \geq 0$ for any $i \in I_0$) and all the weights of $M$ lie in $\lambda - \sum_{i \in I_0} \mathbb{Z}_{\geq 0} \alpha_i$. We say that $\lambda$ is the dominant extremal weight of $M$ and $u$ is a dominant extremal vector of $M$. Note that a dominant extremal vector of $M$ is unique up to a constant multiple.
Let \( M \) be an integrable \( U'_q(\mathfrak{g}) \)-module. Then the \textit{affinization} \( M_{\text{aff}} \) of \( M \) is the \( P \)-graded \( U'_q(\mathfrak{g}) \)-module

\[
M_{\text{aff}} = \bigoplus_{\lambda \in P} (M_{\text{aff}})_{\lambda} \quad \text{with} \quad (M_{\text{aff}})_{\lambda} = M_{\text{cl}(\lambda)}.
\]

Let us denote by \( \text{cl}: M_{\text{aff}} \to M \) the canonical \( k \)-linear homomorphism. The actions

\[
e_i: (M_{\text{aff}})_{\lambda} \to (M_{\text{aff}})_{\lambda + \alpha_i} \quad \text{and} \quad f_i: (M_{\text{aff}})_{\lambda} \to (M_{\text{aff}})_{\lambda - \alpha_i}
\]

are defined in a way that they commute with \( \text{cl}: M_{\text{aff}} \to M \).

We denote by \( z_M: M_{\text{aff}} \to M_{\text{aff}} \) the \( U'_q(\mathfrak{g}) \)-module automorphism of weight \( \delta \) defined by

\[
(M_{\text{aff}})_{\lambda} \sim M_{\text{cl}(\lambda)} \sim (M_{\text{aff}})_{\lambda + \delta}.
\]

For \( x \in k^x \), we define

\[
M_x := M_{\text{aff}}/(z_M - x)M_{\text{aff}}.
\]

We embed \( P_{\text{cl}} \) into \( P \) by \( \iota: P_{\text{cl}} \to P \) which is given by \( \iota(\text{cl}(\Lambda_i)) = \Lambda_i \). For \( u \in M_{\Lambda} (\lambda \in P_{\text{cl}}) \), let us denote by \( u_{\iota}(\in M_{\text{cl}(\iota)}(\lambda)) \) the element such that \( \iota(u_{\iota}) = u \). With this notation, we have

\[
e_i(u_{\iota}) = z_0^{\lambda_i,0}(e_iu_{\iota}), \quad f_i(u_{\iota}) = z^{-\lambda_i,0}(f_iu_{\iota}), \quad K_i(u_{\iota}) = (K_iu_{\iota}).
\]

Then we have \( M_{\text{aff}} \cong k[z, z^{-1}] \otimes M \).

Let \( M \) be an integrable \( U'_q(\mathfrak{g}) \)-module. A weight vector \( u \in M_{\Lambda} (\lambda \in P) \) is called an extremal vector if there exists a family of vectors \( \{u_w\}_{w \in W} \) satisfying the following properties:

\[
u_w = u \quad \text{for} \quad w = e,
\]

\[
\text{if} \quad \langle h_i, w\lambda \rangle > 0, \quad \text{then} \quad e_iu_w = 0 \quad \text{and} \quad f_i^{\langle h_i, w\lambda \rangle}u_w = u_{s_iw},
\]

\[
\text{if} \quad \langle h_i, w\lambda \rangle \leq 0, \quad \text{then} \quad f_iu_w = 0 \quad \text{and} \quad e_i^{\langle h_i, w\lambda \rangle}u_w = u_{s_iw}.
\]

If such \( \{w_w\}_{w \in W} \) exists, then it is unique and \( u_w \) has weight \( w\lambda \).

For \( \lambda \in P \), let us denote by \( W(\lambda) \) the \( U'_q(\mathfrak{g}) \)-module generated by \( u_{\lambda} \) with the defining relation that \( u_{\lambda} \) is an extremal vector of weight \( \lambda \) (see [16]). This is in fact a set of infinitely many linear relations on \( u_{\lambda} \).

Set \( \varpi_i = \gcd(c_0, c_i)^{-1} (c_0\Lambda_i - c_i\Lambda_0) \in P \) for \( i = 1, 2, \ldots, n \). Then \( \{\text{cl}(\varpi_i)\}_{i=1,2,\ldots,n} \) forms a basis of \( P_{\text{cl}}^0 \). We call \( \varpi_i \) a level 0 fundamental weight. As shown in [17], for each \( i = 1, 2, \ldots, n \), there exists a \( U'_q(\mathfrak{g}) \)-module automorphism \( z_i: W(\varpi_i) \to W(\varpi_i) \) which sends \( u_{\varpi_i} \) to \( u_{\varpi_i + d_i \delta} \), where \( d_i \in \mathbb{Z}_{>0} \) denotes the generator of the free abelian group \( \{m \in \mathbb{Z}: \varpi_i + m\delta \in W(\varpi_i)\} \).

We define the \( U'_q(\mathfrak{g}) \)-module \( V(\varpi_i) \) by

\[
V(\varpi_i) = W(\varpi_i)/(z_i - 1)W(\varpi_i).
\]

We call \( V(\varpi_i) \) the fundamental representation of \( U'_q(\mathfrak{g}) \) of weight \( \varpi_i \). We have \( V(\varpi_i)_{\text{aff}} \cong k[z_i^{1/\delta_i}] \otimes_{k[z_i]} W(\varpi_i) \).

If a \( U'_q(\mathfrak{g}) \)-module \( M \in \mathcal{C}_g \) has a bar involution, a crystal basis with simple crystal graph and a lower global basis, then we say that \( M \) is a good module. For the precise definition, see [17, Section 8]. For example, the fundamental representation \( V(\varpi_i) \) is a good \( U'_q(\mathfrak{g}) \)-module. Every good module is a simple \( U'_q(\mathfrak{g}) \)-module.

### 1.5 Generalized quantum affine Schur–Weyl duality functors

In this subsection, we recall the construction of the generalized quantum affine Schur–Weyl duality functor [12].

Let \( U'_q(\mathfrak{g}) \) be a quantum affine algebra over \( k \) and let \( \{V_s\}_{s \in S} \) be a family of good \( U'_q(\mathfrak{g}) \)-modules. For each \( s \in S \), let \( \lambda_s \) be a dominant extremal weight of \( V_s \) and let \( v_s \) be a dominant extremal weight vector in \( V_s \) of weight \( \lambda_s \).

Assume that we have an index set \( J \) and two maps \( X: J \to k^x \), \( S: J \to S \).
For each $i$ and $j$ in $J$, we have a $U'_q(\mathfrak{g})$-module homomorphism
\[ R^\text{norm}_{V_{S(i)}, V_{S(j)}}(z_i, z_j) : (V_{S(i)\text{aff}} \otimes (V_{S(j)\text{aff}}) \overset{k(z_i, z_j)}{\otimes} \mathbb{k}[z_i^\pm 1, z_j^\pm 1] \]
which sends $v_{S(i)} \otimes v_{S(j)}$ to $v_{S(i)} \otimes v_{S(i)}$. Here, $z_i := z_{V_{S(i)}}$ denotes the $U'_q(\mathfrak{g})$-module automorphism on $(V_{S(i)\text{aff}})$ of weight $\delta$. We denote by $d_{V_{S(i)}, V_{S(j)}}(z_j/z_i)$ the denominator of $R^\text{norm}_{V_{S(i)}, V_{S(j)}}(z_i, z_j)$, which is the monic polynomial in $z_j/z_i$ of the smallest degree such that
\[ d_{V_{S(i)}, V_{S(j)}}(z_j/z_i) R^\text{norm}_{V_{S(i)}, V_{S(j)}}(z_i, z_j)((V_{S(i)\text{aff}}) \otimes (V_{S(j)\text{aff}})) \subset (V_{S(j)\text{aff}}) \otimes (V_{S(i)\text{aff}}). \]

We define a quiver $\Gamma^J$ associated with the datum $(J, X, S)$ as follows:

1. We take $J$ as the set of vertices;
2. We put $d_{ij}$ many arrows from $i$ to $j$, where $d_{ij}$ denotes the order of the zero of $d_{V_{S(i)}, V_{S(j)}}(z_j/z_i)$ at $z_j/z_i = X(j)/X(i)$. (1.11)

We define a symmetric Cartan matrix $A^J = (a^J_{ij})_{i,j \in J}$ by
\[ a^J_{ij} = \begin{cases} 2 & \text{if } i = j, \\ -d_{ij} - d_{ji} & \text{if } i \neq j. \end{cases} \] (1.12)

Set
\[ P_{ij}(u, v) = (u - v)^{d_{ij}} c_{ij}(u, v), \] (1.13)
where $\{c_{ij}(u, v)\}_{i,j \in J}$ is a family of functions in $\mathbb{k}[u, v]$ satisfying
\[ c_{ii}(u, v) = 1, \quad c_{ij}(u, v)c_{ji}(v, u) = 1. \] (1.14)

Let $\{\alpha_i : i \in J\}$ be the set of simple roots corresponding to the Cartan matrix $A^J$ and $Q^J = \sum_{i \in J} \mathbb{Z}_{\geq 0} \alpha_i$ be the corresponding positive root lattice.

Let $R^J(\beta)$ ($\beta \in Q^J_+$) be the quiver Hecke algebra associated with the Cartan matrix $A^J$ and the parameter
\[ Q_{ij}(u, v) = \delta(i \neq j) P_{ij}(u, v) P_{ji}(v, u) = \delta(i \neq j)(u - v)^{d_{ij}}(v - u)^{d_{ji}} \quad (i, j \in J). \] (1.15)

For each $\nu = (\nu_1, \ldots, \nu_n) \in J^\beta$, let
\[ \hat{O}_{\nu} = \mathbb{k}[X_1 - X(\nu_1), \ldots, X_n - X(\nu_n)] \]
be the completion of the local ring $O_{\nu}$ of $\mathbb{T}^n$ at $X(\nu) := (X(\nu_1), \ldots, X(\nu_n))$. Set
\[ V_{\nu} = (V_{S(\nu_1)})_{\text{aff}} \otimes \cdots \otimes (V_{S(\nu_n)})_{\text{aff}}. \]

Then $V_{\nu}$ is a $(\mathbb{k}[X_1^{\pm 1}, \ldots, X_n^{\pm 1}] \otimes U'_q(\mathfrak{g}))$-module, where $X_k = z_{V_{S(\nu_k)}}$. We define
\[ \hat{V}_\nu := \hat{O}_{\nu} \otimes_{\mathbb{k}[X_1^{\pm 1}, \ldots, X_n^{\pm 1}]} V_{\nu}, \quad \hat{V}^\otimes \beta := \bigoplus_{\nu \in J^\beta} \hat{V}_\nu e(\nu). \] (1.16)

The following theorem is one of the main results of [12].

**Theorem 1.5.** The space $\hat{V}^\otimes \beta$ is a $(U'_q(\mathfrak{g}), R^J(\beta))$-bimodule.

For each $\beta \in Q^J_+$, we construct the functor
\[ F_\beta : \text{Mod}(R^J(\beta)) \longrightarrow \text{Mod}(U'_q(\mathfrak{g})) \] (1.17)
defined by
\[ F_\beta(M) := \hat{V}^\otimes \beta \otimes_{R^J(\beta)} M, \] (1.18)
where $M$ is an $R^J(\beta)$-module.
Set
\[ \mathcal{F} := \bigoplus_{\beta \in \mathbb{Q}^+_J} \mathcal{F}_\beta : \bigoplus_{\beta \in \mathbb{Q}^+_J} \text{Mod}(R^J(\beta)) \longrightarrow \text{Mod}(U'_q(\mathfrak{g})). \]

**Theorem 1.6** [12]. If the Cartan matrix \( A^J \) associated with \( R^J \) is of finite type \( A, D \) or \( E \), then the functor \( \mathcal{F}_\beta \) is exact for every \( \beta \in \mathbb{Q}^+_J \).

For each \( i \in J \), let \( L(i) \) be the one-dimensional \( R^J(\alpha_i) \)-module generated by a non-zero vector \( u(i) \) with relation \( x_1 u(i) = 0 \) and \( e(j)u(i) = \delta(j = i)u(i) \) for \( j \in J \). The space \( L(i)_z := k[z] \otimes L(i) \) admits an \( R^J(\alpha_i) \)-module structure as follows:
\[ x_1(a \otimes u(i)) = (za) \otimes u(i), \quad e(j)(a \otimes u(i)) = \delta_{j,i}(a \otimes u(i)). \]
Note that it is isomorphic to \( R^J(\alpha_i) \) as a left \( R^J(\alpha_i) \)-module.

By the construction in [12], we have the following proposition.

**Proposition 1.7.** (i) For any \( i \in J \), we have
\[ \mathcal{F}(L(i)_z) \simeq k[[z]] \otimes_{k[[z^\pm 1]]} (V_{S(i)}(i))_{\text{aff}}, \quad (1.19) \]
where \( k[z^\pm 1] \rightarrow k[[z]] \) is given by \( z_{V_{S(i)}} \mapsto X(i)(1 + z) \).

(ii) For \( i, j \in J \), let
\[ \phi = R_{L(i), L(j), s} : L(i)_z \circ L(j)_z' \longrightarrow L(j)_z' \circ L(i)_z. \]
That is, let \( \phi \) be the \( R^J(\alpha_i + \alpha_j) \)-module homomorphism given by
\[ \phi(u(i)_z \otimes u(j)_z') = \varphi_1(u(j)_z' \otimes u(i)_z), \quad (1.20) \]
where \( \varphi_1 \) is the intertwiner in (1.3). Then we have
\[ \mathcal{F}(\phi) = (X_i/X(i) - X_j/X(j))^{d_{i,j}}c_{i,j}(X_i/X(i) - 1, X_j/X(j) - 1)R_{V_{S(i)}, V_{S(j)}}^{\text{norm}} \]
as a \( U'_q(\mathfrak{g}) \)-module homomorphism
\[ \tilde{\varphi}_{T^2, (X(i), X(j))} \otimes_{k[X_i^\pm 1, X_j^\pm 1]} ((V_{S(i)}(i))_{\text{aff}} \otimes (V_{S(j)}(j))_{\text{aff}}) \]
\[ \longrightarrow \tilde{\varphi}_{T^2, (X(j), X(i))} \otimes_{k[X_j^\pm 1, X_i^\pm 1]} ((V_{S(j)}(j))_{\text{aff}} \otimes (V_{S(i)}(i))_{\text{aff}}), \]
where \( X_i = z_{V(S(i))} \) and \( X_j = z_{V(S(j))} \).

Recall that \( \mathcal{C}_g \) denotes the category of finite-dimensional integrable \( U'_q(\mathfrak{g}) \)-modules.

**Theorem 1.8.** The functor \( \mathcal{F} \) induces a tensor functor
\[ \mathcal{F} : \bigoplus_{\beta \in \mathbb{Q}^+_J} R^J(\beta)_{-\text{mod}} \longrightarrow \mathcal{C}_g. \]
Namely, \( \mathcal{F} \) sends finite-dimensional graded \( R^J(\beta) \)-modules to \( U'_q(\mathfrak{g}) \)-modules in \( \mathcal{C}_g \), and there exist canonical \( U'_q(\mathfrak{g}) \)-module isomorphisms
\[ \mathcal{F}(R^J(0)) \simeq k, \quad \mathcal{F}(M_1 \circ M_2) \simeq \mathcal{F}(M_1) \otimes \mathcal{F}(M_2) \]
for \( M_1 \in R^J(\beta_1)_{-\text{mod}} \) and \( M_2 \in R^J(\beta_2)_{-\text{mod}} \) such that the diagrams in [12, A.1.2] are commutative.
2. Comparison of denominators in untwisted cases and twisted cases

2.1. Denominators of normalized $R$-matrices for $U'_q(A^{(1)}_{N-1})$ and $U'_q(A^{(2)}_{N-1})$

In this subsection, we recall the denominators of normalized $R$-matrices for quantum affine algebras of type A. In Table 1, we list the Dynkin diagrams with an enumeration of vertices by simple roots and list the corresponding fundamental weights.

By [6], for $g = A^{(1)}_{N-1}$ ($N \geq 1$), $1 \leq k, l \leq N - 1$, we have

$$d_{V(\varpi_k), V(\varpi_l)}(z) = \prod_{s=1}^{\min(k, l, N-k, N-l)} (z - (-q)^{|k-l|+2s}).$$  \hfill (2.1)

We recall the denominators of normalized $R$-matrices between fundamental representations of type $A^{(2)}_{N-1}$, given in [22].

**Theorem 2.1.** For $g = A^{(2)}_{N-1}$ ($N \geq 3$), $1 \leq k, l \leq \lfloor N/2 \rfloor$, we have

$$d_{k,l}(z) = \prod_{s=1}^{\min(k, l)} (z - (-q)^{|k-l|+2s})(z + q^N(-q)^{-k+l+2s}).$$  \hfill (2.2)

**Remark 1.** Even though our enumeration of vertices of the Dynkin diagram of type $A^{(2)}_{2n}$ is different from the one in [10], for each $i = 1, \ldots, n$ the corresponding fundamental representations $V(\varpi_i)$ are isomorphic to each other, since the corresponding fundamental weights are conjugate to each other under the Weyl group action (see [17, Subsection 5.2]).

| Type       | Dynkin diagram | Fundamental weights |
|------------|----------------|---------------------|
| $A^{(1)}_n$ ($n \geq 2$) | $\alpha_0 \rightarrow \alpha_1 \rightarrow \alpha_2 \rightarrow \cdots \rightarrow \alpha_{n-1} \rightarrow \alpha_n$ | $\varpi_i = \Lambda_i - \Lambda_0$ ($1 \leq i \leq n$) |
| $A^{(2)}_2$  | $\alpha_0 \rightarrow \alpha_1$ | $\varpi_1 = 2\Lambda_1 - \Lambda_0$ |
| $A^{(2)}_3$  | $\alpha_0 \rightarrow \alpha_1 \rightarrow \alpha_2 \rightarrow \alpha_3$ | $\varpi_1 = \Lambda_1 - \Lambda_0$, $\varpi_2 = \Lambda_2 - 2\Lambda_0$ |
| $A^{(2)}_{2n-1}$ ($n \geq 3$) | $\alpha_0 \rightarrow \alpha_1 \rightarrow \alpha_2 \rightarrow \cdots \rightarrow \alpha_{n-1} \rightarrow \alpha_n$ | $\varpi_i = \Lambda_i - \Lambda_0$ ($1 \leq i \leq n$), $\varpi_i = \Lambda_1 - 2\Lambda_0$ ($2 \leq i \leq n - 1$) |
| $A^{(2)}_{2n}$ ($n \geq 2$) | $\alpha_0 \rightarrow \alpha_1 \rightarrow \alpha_2 \rightarrow \cdots \rightarrow \alpha_{n-1} \rightarrow \alpha_n$ | $\varpi_i = \Lambda_i - \Lambda_0$ ($i = 1, \ldots, n - 1$), $\varpi_n = 2\Lambda_n - \Lambda_0$ |
For the sake of notational simplicity, we denote the Dynkin diagram of type $D_{3}^{(2)}$ in [10] by $\tilde{A}_{3}^{(2)}$ throughout this paper.

2.2. The quiver isomorphism

For each quantum affine algebra $U_{q}(\mathfrak{g})$, we define a quiver $\mathcal{I}(\mathfrak{g})$ as follows:

1. we take the set of equivalence classes $I_{\mathfrak{g}} := (I_{0} \times k^{\times}) / \sim$ as the set of vertices, where the equivalence relation is given by $(i, x) \sim (j, y)$ if and only if $V(\varpi_{i})_{x} \cong V(\varpi_{j})_{y}$;
2. we put $d$ many arrows from $(i, x)$ to $(j, y)$, where $d$ denotes the order of the zero of $d_{V(\varpi_{i}), V(\varpi_{j})}(z_{V(\varpi_{i})}/z_{V(\varpi_{j})})$ at $z_{V(\varpi_{i})}/z_{V(\varpi_{j})} = y/x$.

Note that $(i, x)$ and $(j, y)$ are linked by at least one arrow in $\mathcal{I}(\mathfrak{g})$ if and only if the tensor product $V(\varpi_{i})_{x} \otimes V(\varpi_{j})_{y}$ is reducible [1, Corollary 2.4].

Let $\mathcal{I}_{0}(\mathfrak{g})$ be a connected component of $\mathcal{I}(\mathfrak{g})$. Note that a connected component of $\mathcal{I}(\mathfrak{g})$ is unique up to a spectral parameter and hence $\mathcal{I}_{0}(\mathfrak{g})$ is uniquely determined up to a quiver isomorphism. For example, one can take

\[
\mathcal{I}_{0}(A_{1}^{(1)}) := \{(i, (-q)^{p}) \in \{1, \ldots, n\} \times k^{\times} ; \; p \equiv i + 1 \mod 2\},
\]

\[
\mathcal{I}_{0}(A_{2n-1}^{(2)}) := \{(i, \pm(q)^{p}) \in \{1, \ldots, n\} \times k^{\times} ; \; i \in \{1, \ldots, n\}, \; p \equiv i + 1 \mod 2\},
\]

\[
\mathcal{I}_{0}(A_{2n}^{(2)}) := \{(i, (-q)^{p}) \in \{1, \ldots, n\} \times k^{\times} ; \; p \in \mathbb{Z}\}.
\]

Remark that we have $V(\varpi_{i})_{x} \cong V(\varpi_{i})_{-x}$ in the $A_{2n-1}^{(2)}$-case.

Let $\mathcal{C}_{0}^{\mathfrak{g}}$ be the smallest full subcategory of $\mathcal{C}_{\mathfrak{g}}$ stable under taking subquotients, extensions and tensor products, and containing $\{V(\varpi_{i})_{x} ; (i, x) \in \mathcal{I}_{0}(\mathfrak{g})\}$.

Define a map

\[
\pi_{N-1}^{(2)} : \hat{I}_{A_{N-1}^{(1)}} \longrightarrow \hat{I}_{A_{N-1}^{(2)}}
\]

by

\[
\pi_{N-1}^{(2)}(i, x) = \begin{cases} (i, x) & \text{if } 1 \leqslant i \leqslant \lfloor N/2 \rfloor, \\ (N - i, (-1)^{N-1}x) & \text{if } \lfloor N/2 \rfloor < i \leqslant N - 1. \end{cases}
\]

When there is no fear of confusion, then we just write $\pi^{(2)}$ instead of $\pi_{N-1}^{(2)}$.

By (2.1) and (2.2), we have the following proposition.

**Proposition 2.2.** The map $\pi^{(2)}$ induces quiver isomorphisms

\[
\mathcal{I}(A_{N-1}^{(1)}) \overset{\sim}{\longrightarrow} \mathcal{I}(A_{N-1}^{(2)}) \quad \text{and} \quad \mathcal{I}_{0}(A_{N-1}^{(1)}) \overset{\sim}{\longrightarrow} \mathcal{I}_{0}(A_{N-1}^{(2)}).
\]

To avoid confusion, we use the notation $V^{(t)}(\varpi_{i})$ for the fundamental representation of weight $\varpi_{i}$ of $U_{q}(A_{N-1}^{(t)})$ $(t = 1, 2)$. We also use the following notation: for $(i, x) \in \mathcal{I}(A_{N-1}^{(2)})$, set $V^{(t)}(i, x) := V^{(t)}(\varpi_{i})_{x}$ $(t = 1, 2)$. We write $V(i, x)$ instead of $V^{(t)}(i, x)$ when there is no fear of confusion.

We record the following propositions here for later use.

**Proposition 2.3 [9, Theorem 4.15].** For all $(i, x) \in \mathcal{I}(A_{N-1}^{(1)})$, we have

\[
\dim_{k} V^{(1)}(i, x) = \dim_{k} V^{(2)}(\pi^{(2)}(i, x)).
\]
Proposition 2.4 [3, Theorem 6.1]. Let \(\mathfrak{g} = A^{(1)}_{N-1}, 1 \leq i, j, k \leq N - 1, x, y, z \in k^\times\). Then
\[
\text{Hom}_{U'_q(\mathfrak{g})}(V^{(1)}(\varpi_i)_x \otimes V^{(1)}(\varpi_j)_y, V^{(1)}(\varpi_k)_z) \neq 0
\]
if and only if one of the following conditions holds:
(i) \(i + j < N, k = i + j, x/z = (-q)^{-j}, y/z = (-q)^i\);
(ii) \(i + j > N, k = i + j - N, x/z = (-q)^{-N+j}, y/z = (-q)^{-N-i}\).

Proposition 2.5 [22, Theorem 3.5, Theorem 3.9]. For \(1 \leq i, j \leq \lfloor N/2 \rfloor\) such that \(i + j \leq [N/2]\), there exists an exact sequence
\[
0 \rightarrow V^{(2)}(\varpi_{i+j}) \rightarrow V^{(2)}(\varpi_j)(-q)^i \otimes V^{(2)}(\varpi_i)(-q)^{-j} \rightarrow V^{(2)}(\varpi_i)(-q)^{-j} \otimes V^{(2)}(\varpi_j)(-q)^i \rightarrow 0.
\]
Assume that \(N - 1 = 2n (n \geq 2)\). Then there exists an exact sequence of \(U'_q(A^{(2)}_{2n})\)-modules
\[
0 \rightarrow V^{(2)}(\varpi_n) \rightarrow V^{(2)}(\varpi_n)(-q)^{-1} \otimes V^{(2)}(\varpi_n)(-q) \rightarrow 0.
\]

3. The functor \(\mathcal{F}^{(2)}\)

3.1. Symmetric quiver Hecke algebra of type \(A_\infty\)

Let \(V = V^{(2)}(\varpi_1)\) be the fundamental representation of \(U'_q(A^{(2)}_{N-1})\) with extremal weight \(\varpi_1\). Let \(\mathcal{S} = \{V\}, J = \mathbb{Z}\) and let \(X : J \rightarrow k^\times\) be the map given by \(X(j) = q^{2j}\). Then we have
\[
d_{ij} = \delta(j = i + 1) \quad \text{for} \quad i, j \in J. \tag{3.1}
\]
For \(i, j \in J\), we have
\[
(\alpha_i, \alpha_j) = \begin{cases}
-1 & \text{if } i - j = \pm 1, \\
2 & \text{if } i = j, \\
0 & \text{otherwise},
\end{cases}
\]
and
\[
P_{ij}(u, v) = (u - v)^{\delta(j = i + 1)} c_{i,j}(u, v),
\]
and
\[
Q_{ij}(u, v) = \begin{cases}
\pm(u - v) & \text{if } j = i \pm 1, \\
0 & \text{if } i = j, \\
1 & \text{otherwise}.
\end{cases}
\]
The family \(\{c_{i,j}(u, v)\}_{i,j \in J}\) will be given later in (3.15).

Therefore the corresponding quiver Hecke algebra \(R\) is of type \(A_\infty\).

We take
\[
P_J = \bigoplus_{a \in \mathbb{Z}} \mathbb{Z} \epsilon_a
\]
as the weight lattice with \((\epsilon_a, \epsilon_b) = \delta_{a,b}\). The root lattice \(Q_J = \bigoplus_{i \in J} \mathbb{Z} \alpha_i\) is embedded into \(P_J\) by \(\alpha_i = \epsilon_i - \epsilon_{i+1}\). We write \(Q^+_J\) for \(\bigoplus_{i \in J} \mathbb{Z}_{\geq 0} \alpha_i\).

Recall that the functor
\[
\mathcal{F}^{(2)} : \bigoplus_{\beta \in Q_J^+} \text{Mod}_{gr}(R(\beta)) \rightarrow \text{Mod}(U'_q(A^{(2)}_{N-1}))
\]
defined in (1.18) is exact (Theorem 1.6).
3.2. Segments

A pair of integers \((a, b)\) such that \(a \leq b\) is called a segment. The length of \((a, b)\) is \(b - a + 1\). A multisegment is a finite sequence of segments.

For a segment \((a, b)\) of length \(\ell\), we define a graded one-dimensional \(R(\epsilon_a - \epsilon_{b+1})\)-module \(L(a, b) = \text{ku}(a, b)\) in \(R(\epsilon_a - \epsilon_{b+1})\)-mod which is generated by a vector \(u(a, b)\) of degree 0 with the action of \(R(\epsilon_a - \epsilon_{b+1})\) given by

\[
x_m u(a, b) = 0, \quad \tau_k u(a, b) = 0, \quad e(\nu) u(a, b) = \begin{cases} u(a, b) & \text{if } \nu = (a, a + 1, \ldots, b), \\ 0 & \text{otherwise}. \end{cases} \tag{3.2}
\]

We understand that \(L(a, a - 1)\) is the one-dimensional module over \(R(0) = \text{k}\) and the length of \((a, a - 1)\) is 0. When \(a = b\), we use the notation \(L(a)\) instead of \(L(a, a)\).

We give a total order on the set of segments as follows:

\[
(a_1, b_1) > (a_2, b_2) \quad \text{if} \quad a_1 > a_2 \quad \text{or} \quad a_1 = a_2 \quad \text{and} \quad b_1 > b_2.
\]

Then we have the following proposition.

**Proposition 3.1** [15, Theorem 4.8, Theorem 5.1; 12, Proposition 4.2.7].

(i) Let \(M\) be a finite-dimensional simple graded \(R(\ell)\)-module. Then there exists a unique pair of a multisegment \(((a_1, b_1), \ldots, (a_t, b_t))\) and an integer \(c\) such that

\[
\begin{align*}
(a) \quad & (a_k, b_k) \geq (a_{k+1}, b_{k+1}) \quad \text{for} \quad 1 \leq k \leq t - 1; \\
(b) \quad & \sum_{k=1}^t \ell_k = \ell, \quad \text{where} \quad \ell_k := b_k - a_k + 1; \\
(c) \quad & \text{If} \quad \text{hd}(L(a_1, b_1) \circ \cdots \circ L(a_t, b_t)) \quad \text{satisfies} \quad (a) \quad \text{and} \quad (b), \quad \text{then} \quad M \cong q'\text{hd}(L(a_1, b_1) \circ \cdots \circ L(a_t, b_t)) \quad \text{is simple and} \quad \text{hd}(L(a_1, b_1) \circ \cdots \circ L(a_t, b_t)) \quad \text{is isomorphic to} \quad \text{Im}(r_{L(a_1, b_1), \ldots, L(a_t, b_t)}) \quad \text{up to a grade shift.}
\end{align*}
\]

(ii) Conversely, if a multisegment \(((a_1, b_1), \ldots, (a_t, b_t))\) satisfies the condition (a) above, then we say that it is an ordered multisegment. We call the ordered multisegment \(((a_k, b_k))_{1 \leq k \leq t}\) in Proposition 3.1(i) the multisegment associated with \(M\).

**Proposition 3.2** [12, Proposition 4.2.3]. For \(a \leq b\) and \(a' \leq b'\), set \(\ell = b - a + 1, \ell' = b' - a' + 1, \beta = \epsilon_a - \epsilon_{b+1}\) and \(\beta' = \epsilon_{a'} - \epsilon_{b'+1}\).

(i) If \(a' = a\) and \(b' = b\), then we have \(r_{L(a, b), L(a, b)} = \text{id}_{L(a, b) \circ L(a, b)}\).

(ii) If \(a \leq a' \leq b \leq b'\), then there exists a non-zero homomorphism

\[
r_{L(a, b), L(a', b')} : L(a, b) \circ L(a', b') \longrightarrow q^{\delta_{a, a'} + \delta_{b, b'} - 2} L(a', b') \circ L(a, b).
\]

(b) Unless \(a \leq a' \leq b \leq b'\), there exists a non-zero homomorphism

\[
g := r_{L(a, b), L(a', b')} : L(a, b) \circ L(a', b') \longrightarrow q^{(\beta, \beta')} L(a', b') \circ L(a, b).
\]

(iii) If \(a \leq a' \leq b' \leq b\), then \(L(a, b) \circ L(a', b')\) is simple and

\[
L(a, b) \circ L(a', b') \cong q^{\delta_{a, a'} - \delta_{b, b'}} L(a', b') \circ L(a, b).
\]

(iv) If \(b' < a - 1\), then \(L(a, b) \circ L(a', b')\) is simple and

\[
g : L(a, b) \circ L(a', b') \longrightarrow L(a', b') \circ L(a, b).
\]

(v) If \(a' < a \leq b' < b\), then we have the following exact sequence:

\[
0 \longrightarrow qL(a', b) \circ L(a, b') \longrightarrow L(a, b) \circ L(a', b') \xrightarrow{g} L(a', b') \circ L(a, b) \longrightarrow q^{-1} L(a', b) \circ L(a, b') \longrightarrow 0.
\]
Moreover, the image of $g$ coincides with the head of $L(a, b) \circ L(a', b')$ and the socle of $L(a', b') \circ L(a, b)$.

(vi) If $a = b' + 1$, then we have an exact sequence

$$0 \longrightarrow gL(a', b) \longrightarrow L(a, b) \circ L(a', b') \xrightarrow{2} q^{-1}L(a', b') \circ L(a, b) \longrightarrow q^{-1}L(a', b) \longrightarrow 0.$$

Moreover, the image of $g$ coincides with the head of $L(a, b) \circ L(a', b')$ and the socle of $q^{-1}L(a', b') \circ L(a, b)$.

3.3. Properties of the functor $\mathcal{F}^{(2)}$

For $k > \lfloor N/2 \rfloor + 1$ or $k < 0$, $V^{(2)}(\varpi_k)$ is understood to be zero, and the modules $V^{(2)}(\varpi_0)$ and $V^{(2)}(\varpi_{\lfloor N/2 \rfloor + 1})$ are understood to be the trivial representation.

Proposition 3.3. Let $(a, b)$ be a segment with length $\ell := b - a + 1$. Then we have

$$\mathcal{F}^{(2)}(L(a, b)) \simeq V((\pi^{(2)}(\ell, (-q)^{a+b}))).$$

Proof. We will show our assertion by induction on $\ell$. In the course of the proof, we omit the grading of modules over quiver Hecke algebras. When $\ell = 1$, we have $\mathcal{F}^{(2)}(L(a)) \simeq V(-q)^{2b}$ by Proposition 1.7(i).

Assume that $2 \leq \ell \leq N$. Consider the following exact sequence in $R(\ell)$-mod,

$$0 \longrightarrow L(a, b) \longrightarrow L(b) \circ L(a, b - 1) \xrightarrow{r_{(a,b),L(a',b')}} L(a, b - 1) \circ L(b) \longrightarrow L(a, b) \longrightarrow 0,$$

given in Proposition 3.2(vi). Here, we write

$$r_{(a,b),(a',b')}: L(a, b) \circ L(a', b') \longrightarrow L(a', b') \circ L(a, b)$$

for $r_{L(a,b),L(a',b')}$. Applying the exact functor $\mathcal{F}^{(2)}$ and using the induction hypothesis, we obtain an exact sequence

$$0 \longrightarrow \mathcal{F}^{(2)}(L(a, b)) \longrightarrow V(-q)^{2b} \otimes V(\pi^{(2)}(\ell - 1, (-q)^{a+b-1}))$$

$$\xrightarrow{\mathcal{F}^{(2)}(r_{(a,b),L(a',b')})} V(\pi^{(2)}(\ell - 1, (-q)^{a+b-1})) \otimes V(-q)^{2b} \longrightarrow \mathcal{F}^{(2)}(L(a, b)) \longrightarrow 0.$$ 

If $\mathcal{F}^{(2)}(r_{(a,b),L(a',b')})$ vanishes, then we have

$$V(-q)^{2b} \otimes V(\pi^{(2)}(\ell - 1, (-q)^{a+b-1})) \simeq V(\pi^{(2)}(\ell - 1, (-q)^{a+b-1})) \otimes V(-q)^{2b}$$

because they are both isomorphic to $\mathcal{F}^{(2)}(L(a, b))$. Hence $V(-q)^{2b} \otimes V(\pi^{(2)}(\ell - 1, (-q)^{a+b-1}))$ is simple, which is a contradiction. Therefore, we have $\mathcal{F}^{(2)}(r_{(a,b),L(a',b')}) \neq 0$.

On the other hand, by Proposition 2.5 we have an exact sequence

$$0 \longrightarrow V(\pi^{(2)}(\ell, (-q)^{a+b})) \longrightarrow V(-q)^{2b} \otimes V(\pi^{(2)}(\ell - 1, (-q)^{a+b-1}))$$

$$\xrightarrow{h} V(\pi^{(2)}(\ell - 1, (-q)^{a+b-1})) \otimes V(-q)^{2b} \longrightarrow V(\pi^{(2)}(\ell, (-q)^{a+b})) \longrightarrow 0$$

such that $h$ is non-zero. Here $V(\pi^{(2)}(N, (-q)^{a+b}))$ is understood to be the trivial representation. Since $2b > a + b - 1$, [12, Theorem 2.2.1] implies that the module $V(-q)^{2b} \otimes V(\pi^{(2)}(\ell - 1, (-q)^{a+b-1}))$ is generated by the tensor product of dominant extremal weight vectors. Hence we obtain

$$\text{Hom}_{U_{\mathcal{A}N^{(2)}_{\ell-1}}}(V(-q)^{2b} \otimes V(\pi^{(2)}(\ell - 1, (-q)^{a+b-1})), V(\pi^{(2)}(\ell, (-q)^{a+b})) \otimes V(-q)^{2b}) = kh.$$

Thus $\mathcal{F}^{(2)}(r_{(a,b),L(a',b')})$ is equal to $h$ up to a constant multiple and hence $\mathcal{F}^{(2)}(L(a, b))$ is isomorphic to $V(\pi^{(2)}(\ell, (-q)^{a+b}))$. Thus we have proved the proposition when $\ell \leq N$. 

Now assume that \( \ell = N + 1 \). Then \( \mathcal{F}^{(2)}(L(a, b - 1)) \simeq \mathcal{F}^{(2)}(L(a - 1, b)) \simeq \mathbf{k} \). Applying \( \mathcal{F}^{(2)} \) to the epimorphism \( L(a, b - 1) \odot L(b) \to L(a, b) \), \( \mathcal{F}(L(a, b)) \) is a quotient of \( V(-q)^{2n} \). Similarly, applying \( \mathcal{F}^{(2)} \) to the epimorphism \( L(a) \odot L(a + 1, b) \to L(a, b) \), \( \mathcal{F}(L(a, b)) \) is a quotient of \( V(-q)^{2n} \). Since \( V(-q)^{2n} \) and \( V(-q)^{2n} \) are simple modules and they are not isomorphic to each other, we conclude that \( \mathcal{F}^{(2)}(L(a, b)) \) vanishes.

For \( \ell > N + 1 \), \( \mathcal{F}^{(2)}(L(a, b)) \) vanishes since it is a quotient of
\[
\mathcal{F}^{(2)}(L(a, a + N)) \odot \mathcal{F}^{(2)}(L(a + N + 1, b)) \simeq 0 \odot \mathcal{F}^{(2)}(L(a + N + 1, b)) \simeq 0.
\]

**Corollary 3.4.** If one of conditions (i), (ii) in Proposition 2.4 holds, then we have
\[
\text{Hom}_{U_q(A^2_{N-1})}(V(\pi(2)(i, x)) \otimes V(\pi(2)(j, y)), V(\pi(2)(k, z))) \neq 0.
\]

**Proof.** Note that, in the both cases, \((i, x)\) and \((j, y)\) are linked by an arrow in \( \mathcal{E}(A^2_{N-1}) \).
Thus we have
\[
V(\pi(2)(i, x)) \otimes V(\pi(2)(j, y)) \ncong V(\pi(2)(j, y)) \otimes V(\pi(2)(i, x)).
\] (3.6)

Indeed, if these tensor products were isomorphic, then by [13, Corollary 3.9] they would be
simple, which is a contradiction.

Assume that condition (i) holds. We may assume that \( x = (-q)^{1-i}, y = (-q)^{i+1} \) and \( z = (-q)^{i+j+1} \). Take the segments \((a, b) := (1, j)\) and \((a', b') := (1 - i, 0)\). Applying \( \mathcal{F}^{(2)} \) to the exact sequence in Proposition 3.2(vi), we obtain a surjective homomorphism
\[
V(\pi(2)(i, (-q)^{a+b}')) \otimes V(\pi(2)(j, (-q)^{a+b})) \to V(\pi(2)(i + j, (-q)^{a+b})).
\]

Assume that condition (ii) holds. We may assume that \( x = (-q)^{i+1}, y = (-q)^{2N-j+1} \) and \( z = (-q)^{N+i-j+1} \). Take the segments \((a, b) := (N - j + 1, N)\) and \((a', b') := (1, i)\). Applying \( \mathcal{F}^{(2)} \) to the exact sequence in Proposition 3.2(v), we obtain a surjective homomorphism
\[
V(\pi(2)(i, (-q)^{a+b}')) \otimes V(\pi(2)(j, (-q)^{a+b})) \to V(\pi(2)(N, (-q)^{a+b})) \otimes V(\pi(2)(i + j - N, (-q)^{a+b})).
\]

Since \( V(\pi(2)(N, (-q)^{a+b})) \) is isomorphic to the trivial representation, we obtain the desired result. \( \square \)

**Lemma 3.5.** Assume that two segments \((a, b)\) and \((a', b')\) satisfy \((a, b) \geq (a', b')\). Set \( \ell = b - a + 1, \ell' = b' - a' + 1, x = (-q)^{a+b} \) and \( x' = (-q)^{a'+b'} \). Then the following statements hold:

(i) \( x'/x \) is not a zero of the denominator \( d_{V^{(2)}(\pi, \nu), V^{(2)}(\pi, \nu)}(z'/z) \) of the normalized \( R \)-matrix \( R_{V^{(2)}(\pi, \nu), V^{(2)}(\pi, \nu)}(z, z') \);

(ii) the homomorphism
\[
\mathcal{F}^{(2)}(r_{L(a, b), L(a', b')}) : V(\pi(2)(\ell, x)) \otimes V(\pi(2)(\ell', x')) \to V(\pi(2)(\ell', x')) \otimes V(\pi(2)(\ell, x))
\]
is a non-zero constant multiple of the normalized \( R \)-matrix \( R_{V^{(2)}(\pi, \nu), V^{(2)}(\pi, \nu)}(x, x') \).

**Proof.** Because \((b' - a' + 1) - (b - a + 1) \geq (a' + b') - (a + b)\), there is no arrow in \( \mathcal{E}(A^1_{N-1}) \) from \((\ell, x)\) to \((\ell', x')\), and hence so is for \( \pi(2)(\ell, x) \) and \( \pi(2)(\ell', x') \) in \( \mathcal{E}(A^2_{N-1}) \), by Proposition 2.2. Thus we have (i).

By (i) and [12, Theorem 2.2.1], the module \( V(\pi(2)(\ell, x)) \otimes V(\pi(2)(\ell', x')) \) is generated by the tensor product \( v_\ell \otimes v_{\ell'} \) of dominant extremal weight vectors. Thus any non-zero homomorphism from \( V(\pi(2)(\ell, x)) \otimes V(\pi(2)(\ell', x')) \) to \( V(\pi(2)(\ell', x')) \otimes V(\pi(2)(\ell, x)) \) is a constant multiple of the normalized \( R \)-matrix. Hence it is enough to show that \( \mathcal{F}^{(2)}(r_{L(a, b), L(a', b')}) \) does not vanish.
We may therefore assume that \( r := r_{(a,b), (a', b')} \) is not an isomorphism. Equivalently, \((\ell, x)\) and \((\ell', x')\) are linked by an arrow in \( \mathcal{S}_0(A^{(1)}_{N-1}) \). Since \( \mathcal{S}_0(A^{(2)}_{N-1}) \) is isomorphic to \( \mathcal{S}_0(A^{(1)}_{N-1}) \) as a quiver, \( \pi(\ell, x) \) and \( \pi(\ell', x') \) are linked. Hence, \( V(\pi(\ell, x)) \otimes V(\pi(\ell', x')) \) is not simple.

On the other hand, since \((\ell, x)\) and \((\ell', x')\) are linked, we have \( a' < a \leq b' < b \) or \( a = b' + 1 \). Applying \( \mathcal{F}^{(2)} \) to the exact sequences (v) or (vi) in Proposition 3.2, we obtain an exact sequence:

\[
0 \rightarrow V(\pi(\ell, 1, (q)\alpha' + b)) \otimes V(\pi(\ell, 2, (q)\alpha' + b)) \rightarrow V(\pi(\ell, 1, (q)\alpha' + b)) \otimes V(\pi(\ell, 2, (q)\alpha' + b)) \rightarrow 0,
\]

where \( \ell_1 = b - a' + 1 \) and \( \ell_2 = b' - a + 1 \).

Since \((\ell_1, (q)\alpha' + b)\) and \((\ell_2, (q)\alpha' + b)\) are not linked in \( \mathcal{S}_0(A^{(1)}_{N-1}) \), \( \pi(\ell_1, (q)\alpha' + b) \) and \( \pi(\ell_2, (q)\alpha' + b) \) are not linked in \( \mathcal{S}_0(A^{(2)}_{N-1}) \) either. It follows that

\[
V(\pi(\ell, 1, (q)\alpha' + b)) \otimes V(\pi(\ell, 2, (q)\alpha' + b)) \not\cong V(\pi(\ell, 1, (q)\alpha' + b)) \otimes V(\pi(\ell, 2, (q)\alpha' + b)),
\]

because the left-hand side is simple but the right-hand side is not. Therefore, \( \mathcal{F}^{(2)}(r) \) does not vanish, as desired.

**Theorem 3.6.** Let \( M \) be a finite-dimensional simple graded \( R(\ell) \)-module and

\[
((a_1, b_1), \ldots, (a_r, b_r))
\]

be the multisegment associated with \( M \). Set \( \ell_k = b_k - a_k + 1 \).

(i) If \( \ell_k > N \) for some \( 1 \leq k \leq r \), then \( \mathcal{F}^{(2)}(M) \cong 0 \).

(ii) If \( \ell_k \leq N \) for all \( 1 \leq k \leq r \), then \( \mathcal{F}^{(2)}(M) \) is simple.

**Proof.** (i) follows from Proposition 3.3.

By Proposition 3.1, we have \( M \cong \text{Im} r_{(a_1, b_1), \ldots, (a_r, b_r)} \). On the other hand, if \( \ell_k \leq N \) for all \( 1 \leq k \leq r \), then by the above lemma, we know that \( \mathcal{F}^{(2)}(r_{(a_1, b_1), \ldots, (a_r, b_r)}) \) is a constant multiple of a composition of normalized \( R \)-matrices. It follows that \( \text{Im} \mathcal{F}^{(2)}(r_{(a_1, b_1), \ldots, (a_r, b_r)}) \) is simple. Hence we conclude that

\[
\mathcal{F}^{(2)}(M) \cong \mathcal{F}^{(2)}(\text{Im} r_{(a_1, b_1), \ldots, (a_r, b_r)}) \cong \text{Im} \mathcal{F}^{(2)}(r_{(a_1, b_1), \ldots, (a_r, b_r)})
\]

is simple, as desired.

### 3.4. Quotient of the category \( \text{R-gmod} \)

We will recall the quotient category of \( \text{R-gmod} \) introduced in [12, Subsection 4.4]. Set \( \mathcal{A}_\alpha = R(\alpha) \)-gmod and set \( \mathcal{A} = \bigoplus_{\alpha \in Q^+_j} \mathcal{A}_\alpha \). Similarly, we define \( \mathcal{A}^{\text{big}}_{\alpha} \) and \( \mathcal{A}^{\text{big}} \) by \( \mathcal{A}^{\text{big}}_{\alpha} = \text{Mod}_{\mathcal{S}}(R(\alpha)) \) and \( \mathcal{A}^{\text{big}} = \bigoplus_{\alpha \in Q^+_j} \mathcal{A}^{\text{big}}_{\alpha} \). Then we have a functor \( \mathcal{F}^{(2)} = \bigoplus_{\alpha \in Q^+_j} \mathcal{F}^{(2)}_{\alpha} : \mathcal{A}^{\text{big}} \rightarrow \text{Mod}(U'_q(A^{(2)}_{N-1})) \), where \( \mathcal{F}^{(2)}_{\alpha} \) is the functor given in (1.18).

Let \( \mathcal{S}_N \) be the smallest Serre subcategory of \( \mathcal{A} \) (see [12, Appendix B.1]) such that

1. \( \mathcal{S}_N \) contains \( L(a, a + N) \) for any \( a \in \mathbb{Z} \);
2. \( \mathcal{S}_N \) contains \( L(a, b) \) if \( b \geq a + N \).

Note that \( \mathcal{S}_N \) contains \( L(a, b) \) if \( b \geq a + N \).
Let us denote by $\mathcal{A}/\mathcal{S}_N$ the quotient category of $\mathcal{A}$ by $\mathcal{S}_N$ and denote by $Q: \mathcal{A} \to \mathcal{A}/\mathcal{S}_N$ the canonical functor. Since $\mathcal{F}^{(2)}$ sends $\mathcal{S}_N$ to $0$, the functor $\mathcal{F}^{(2)}: \mathcal{A} \to U'_q(A^{(2)}_{N-1})$-mod factors through $Q$ by [12, Theorem B.1.1 (v)]:

\[
\begin{array}{ccc}
\mathcal{A} & \xrightarrow{Q} & \mathcal{A}/\mathcal{S}_N \\
\mathcal{F}^{(2)} & \downarrow & \mathcal{F}^{(2)'} \\
 & U'_q(A^{(2)}_{N-1}) \text{-mod} & 
\end{array}
\]

That is, there exists a unique functor $\mathcal{F}^{(2)'}: \mathcal{A}/\mathcal{S}_N \to U'_q(A^{(2)}_{N-1})$-mod up to an isomorphism such that the above diagram quasi-commutes.

Note that $\mathcal{A}$ and $\mathcal{A}/\mathcal{S}_N$ are tensor categories with the convolution as tensor products. The module $R(0) \cong k$ is a unit object. Note also that $Q := qR(0)$ is an invertible central object of $\mathcal{A}/\mathcal{S}_N$ and $X \mapsto Q \circ X \cong X \circ Q$ coincides with the grade shift functor. Moreover, the functors $Q$, $\mathcal{F}^{(2)}$ and $\mathcal{F}^{(2)'}$ are tensor functors.

Similarly, we define $\mathcal{S}_N^{\text{big}}$ as the smallest Serre subcategory of $\mathcal{A}^{\text{big}}$ such that

\begin{enumerate}
\item $\mathcal{S}_N^{\text{big}}$ contains $L(a, a + N)$;
\item $X \circ Y, Y \circ X \in \mathcal{S}_N^{\text{big}}$ for all $X \in \mathcal{A}^{\text{big}}, Y \in \mathcal{S}_N^{\text{big}}$; (3.8)
\item $\mathcal{S}_N^{\text{big}}$ is stable under (not necessarily finite) direct sums.
\end{enumerate}

Then we can easily see that $\mathcal{S}_N^{\text{big}} \cap \mathcal{A} = \mathcal{S}_N$ and hence the functor $\mathcal{A}/\mathcal{S}_N \to \mathcal{A}^{\text{big}}/\mathcal{S}_N^{\text{big}}$ is fully faithful.

The following proposition is proved in [12, Subsection 4.4], and its corollary below can be proved similarly to [12, Corollary 4.4.2].

**Proposition 3.7** [12, Proposition 4.4.1].

(i) If an object $X$ is simple in $\mathcal{A}/\mathcal{S}_N$, then there exists a simple object $M$ in $\mathcal{A}$ satisfying the following conditions:

(a) $Q(M) \cong X$;

(b) $b_k - a_k + 1 \leq N$ for $1 \leq k \leq r$, where $((a_1, b_1), \ldots, (a_r, b_r))$ is the multisegment associated with $M$.

(ii) Let $((a_1, b_1), \ldots, (a_r, b_r))$ be the multisegment associated with a simple object $M$ in $\mathcal{A}$. If $b_k - a_k + 1 \leq N$ for $1 \leq k \leq r$, then $Q(M)$ is simple in $\mathcal{A}/\mathcal{S}_N$.

**Corollary 3.8.** The functor $\mathcal{F}^{(2)'}: \mathcal{A}/\mathcal{S}_N \to U'_q(A^{(2)}_{N-1})$-mod sends simple objects in $\mathcal{A}/\mathcal{S}_N$ to simple objects in $U'_q(A^{(2)}_{N-1})$-mod.

3.5. The categories $T'_N$ and $T_N$

In this section, we recall the categories $T'_N$ and $T_N$ introduced (and denoted by $T'_J$ and $T_J$, respectively) in [12, Subsection 4.5].

**Definition 3.9.** Let $S$ be the automorphism of $P_J := \bigoplus_{a \in \mathbb{Z}} \mathbb{Z} \epsilon_a$ given by $S(\epsilon_a) = \epsilon_{a+N}$. We define the bilinear form $B$ on $P_J$ by

\[
B(x, y) = -\sum_{k>0} (S^k x, y) \quad \text{for } x, y \in P_J.
\] (3.9)
Definition 3.10. We define the new tensor product $\star : \mathcal{A}^{\big\big}/\mathcal{S}_N^{\big\big} \times \mathcal{A}^{\big\big}/\mathcal{S}_N^{\big\big} \to \mathcal{A}^{\big\big}/\mathcal{S}_N^{\big\big}$ by

$$X \star Y = q^{B(\alpha, \beta)} X \circ Y \simeq Q^\otimes B(\alpha, \beta) \circ X \circ Y,$$

where $X \in (\mathcal{A}^{\big\big}/\mathcal{S}_N^{\big\big})_\alpha$, $Y \in (\mathcal{A}^{\big\big}/\mathcal{S}_N^{\big\big})_\beta$ and $Q = q\mathbf{1}$.

Then $\mathcal{A}^{\big\big}/\mathcal{S}_N^{\big\big}$ as well as $\mathcal{A}/\mathcal{S}_N$ is endowed with a new structure of tensor category by $\star$, as shown in [12, Appendix A.8].

Set

$$L_a := L(a, a + N - 1) \quad \text{and} \quad u_a := u(a, a + N - 1) \in L_a \quad \text{for} \ a \in \mathbb{Z}. \quad (3.10)$$

For $a, j \in \mathbb{Z}$, set

$$f_{a,j}(z) := (-1)^{\delta_{j,a+N}} z^{-\delta(a \leq j < a+N-1)-\delta_{j,a+N}} \in k[z^{\pm 1}]. \quad (3.11)$$

Theorem 3.11 [12, Theorem 4.5.8]. The following statements hold.

(i) The $L_a$ is a central object in $\mathcal{A}/\mathcal{S}_N$ (see [12, Appendix A.3]); that is,

(a) $f_{a,j}(z) R_{L_a,L(a(j))}$ induces an isomorphism in $\mathcal{A}/\mathcal{S}_N$

$$R_a(X) : L_a \star X \xrightarrow{\simeq} X \star L_a$$

functorial in $X \in \mathcal{A}/\mathcal{S}_N$;

(b) the diagram

$$\begin{array}{c}
L_a \star X \star Y \xrightarrow{R_a(X) \star Y} X \star L_a \star Y \\
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
X \star Y \star L_a \\
\end{array}$$

is commutative in $\mathcal{A}/\mathcal{S}_N$ for any $X, Y \in \mathcal{A}/\mathcal{S}_N$.

(ii) The isomorphism $R_a(L_a) : L_a \star L_a \xrightarrow{\simeq} L_a \star L_a$ coincides with $\text{id}_{L_a \star L_a}$ in $\mathcal{A}/\mathcal{S}_N$.

(iii) For $a, b \in \mathbb{Z}$, the isomorphisms

$$R_{a}(L_b) : L_a \star L_b \xrightarrow{\simeq} L_b \star L_a \quad \text{and} \quad R_{b}(L_a) : L_b \star L_a \xrightarrow{\simeq} L_a \star L_b$$

in $\mathcal{A}/\mathcal{S}_N$ are the inverses to each other.

By the preceding theorem, $\{(L_a, R_a)\}_{a \in J}$ forms a commuting family of central objects in $(\mathcal{A}/\mathcal{S}_N, \star)$ ([12, Appendix A.4]). Following [12, Appendix A.6], we localize $(\mathcal{A}/\mathcal{S}_N, \star)$ by this commuting family. Let us denote by $T_N$ the resulting category $(\mathcal{A}/\mathcal{S}_N)[L_a^{-1} \mid a \in J]$. Let $T : \mathcal{A}/\mathcal{S}_N \to T_N$ be the projection functor. We denote by $T_N$ the tensor category $(\mathcal{A}/\mathcal{S}_N)[L_a \simeq 1 \mid a \in J]$ and by $\Xi : T_N \to T_N$ the canonical functor (see [12, Appendix A.7; 12, Remark 4.5.9]). Thus we have a chain of tensor functors

$$\mathcal{A} \xrightarrow{Q} \mathcal{A}/\mathcal{S}_N \xrightarrow{T} T_N := (\mathcal{A}/\mathcal{S}_N)[L_a^{-1} \mid a \in J] \xrightarrow{\Xi} T_N := (\mathcal{A}/\mathcal{S}_N)[L_a \simeq 1 \mid a \in J].$$

The categories $T_N$ and $T_N$ are rigid tensor categories; that is, every object has a right dual and a left dual [12, Theorem 4.6.3]. In the rest of this section, we will show that the functor $\mathcal{F}^{(2)'}$ factors through the category $T_N$. First, we need the following lemma.
Lemma 3.12. For $b \in J$, set $V_k = V^{(2)}(\varpi_1)_{q^{-2k}}$ $(1 \leq k \leq N)$, $W = V_N \otimes V_{N-1} \otimes \cdots \otimes V_1$, and choose an epimorphism $\varphi : W \to k$ in $U'_q(A_{N-1}^{(2)})$-mod. Let

$$R_{V_{N-1},V_1}^{\text{norm}} : W \otimes V_z \to V_z \otimes W$$

be the $R$-matrix obtained by the composition of normalized $R$-matrices

$$V_N \otimes \cdots \otimes V_1 \otimes V_z \xrightarrow{R_{V_1,V_z}^{\text{norm}}} V_N \otimes \cdots \otimes V_2 \otimes V_z \otimes V_1 \xrightarrow{R_{V_{N-1},V_1}^{\text{norm}}} V_z \otimes V_N \otimes \cdots \otimes V_1,$$

and let $g(z) = q^N((z - q^{-2N})(z + q^{-N-2})/(z - q^{-2})(z + q^{-N}))$.

Then we have a commutative diagram

$$\begin{array}{ccc}
W \otimes V_z & \xrightarrow{g(z)} & V_z \otimes W \\
\varphi \otimes V_z & \downarrow & V_z \otimes \varphi \\
k \otimes V_z & \xrightarrow{g(z)} & V_z \otimes k
\end{array} \quad (3.12)$$

Proof. Let $a_{11}(z)$ be the function satisfying $R_{V_1,V_z}^{\text{univ}}(V^{(2)}(\varpi_1),V^{(1)}(\varpi_1)) = a_{11}(z)R_{V_{N-1},V_1}^{\text{norm}}(V^{(2)}(\varpi_1),V^{(2)}(\varpi_1))$, where $R_{V_1,V_z}^{\text{univ}}(V^{(2)}(\varpi_1),V^{(2)}(\varpi_1))$ denotes the universal $R$-matrix between $V^{(2)}(\varpi_1)$ and $V^{(2)}(\varpi_1)$ (see [1, Appendix A]). Then the diagram (3.12) is commutative if $g(z) = \prod_{k=1}^{N} a_{11}(q^{2k}z)^{-1}$.

By [22, (4.11)], we have

$$a_{11}(z) = q^{[N + 2'][[N - 2]' - [0][2N]}/[2][2N - 2]], \quad (3.13)$$

where $$(z;q)_\infty = \prod_{n=0}^{\infty} (1 - q^n z), \quad [a] = ((-q)^{a} z; p^2)^{\infty}, \quad [a]' = (-q^a z; p^2)^{\infty} \text{ and } p^* = -q^N.$$ It follows that

$$\prod_{k=1}^{N} a_{11}(q^{2k}z) = q^N \prod_{k=1}^{N} \frac{[N + 2 + 2k']^2[N - 2 + 2k']^2}{[N + 2k']^2[N + 2k][2N - 2 + 2k]} = q^{-N} \frac{[3N + 2][N']^2[2][4N]}{[N + 2][N + 2N][2N]} = q^{-N} \frac{(z + q^{-N})(z - q^{-2})}{(z + q^{-N-2})(z - q^{-2N})},$$

as desired. □

The proof of the following lemma is straightforward.

Lemma 3.13. Let \{\psi_k(z) : k \in \mathbb{Z}\} \subset \k[[z]]^\times be a family of power series such that

$$\psi_a(0)\psi_{-a-N+1}(0) = 1 \quad (a \in \mathbb{Z}), \quad \prod_{k=1}^{0} \psi_k(0) = 1. \quad (3.14)$$

Set

$$\begin{align*}
\phi_0(z) & := 1, \\
\phi_k(z) & := \psi_{-k}(z)^{-1} \quad (1 \leq k \leq N - 2), \\
\phi_{N-1}(z) & := \psi_0(z) \prod_{k=1}^{N-2} \phi_k(z)^{-1}, \\
\phi_k(z) & := \begin{cases} \\
\frac{\psi_{k+N+1}(z)}{\psi_{k+N}(z)} & \phi_{k-N}(z) \quad (k \geq N), \\
\frac{\psi_k(z)}{\psi_{k+1}(z)} & \phi_{k-N}(z) \quad (k \leq -1).
\end{cases}
\end{align*}$$
Then \( \{ \phi_k(z) : k \in \mathbb{Z} \} \) satisfies

\[
\phi_a(0) \phi_{-a}(0) = 1 \quad (a \in \mathbb{Z}),
\]

\[
\prod_{k=a}^{a+N-1} \phi_k(z) = \psi_a(z) \quad (a \in \mathbb{Z}).
\]

Now we make a special choice of \( c_{ij} \) in Subsection 1.5 as follows: For \( a \in \mathbb{Z} \), set

\[
\psi_a(z) := (-1)^{\delta(1-N \leq a \leq -1)}(z^2 \delta(a=0) - \delta(a=1-N))g(q^{2(-a-N)}(z + 1))^{-1}
\]

\[
= (-1)^{\delta(1-N \leq a \leq -1)} \frac{(z + 1 + q^{2a+N})(z + 1 - q^{2a+2N-2})\delta(a\neq1-N)}{(z + 1 + q^{2a+N-2})(z + 1 - q^{2a})\delta(a\neq0)} \in k[[z]]^\times.
\]

Then it is straightforward to show that \( \{ \psi_a(z) \}_{a \in \mathbb{Z}} \) satisfy the conditions (3.14). Define \( \{ \phi_a(z) \}_{a \in \mathbb{Z}} \) as in Lemma 3.13. Finally, set

\[
c_{i,j}(u, v) := \frac{\phi_{i-j}(v)}{\phi_{j-i}(u)} \phi_{j-i}(0).
\]

Note that \( \{ c_{i,j}(u, v) \}_{i, j \in \mathbb{Z}} \) satisfy the condition (1.14) by the construction.

**Theorem 3.14.** If we choose \( \{ c_{i,j}(u, v) \}_{i, j \in \mathbb{Z}} \) as above, then the diagram [12, (A.7.1)] is commutative for the functor \( \mathcal{F}^{(2)'}/A/\mathcal{S}_N \to U_q'(A_{N-1}^{(1)}) \)-mod and the commuting family of central objects \( \{ (L_a, R_a) \}_{a \in \mathbb{Z}} \). That is, the diagram

\[
\begin{array}{ccc}
\mathcal{F}^{(2)'}(L_a \star L_a) & \xrightarrow{\sim} & \mathcal{F}^{(2)'}(L_a) \otimes \mathcal{F}^{(2)'}(M) \\
\downarrow & & \downarrow \mathcal{F}^{(2)'}(R_a(M)) \\
\mathcal{F}^{(2)'}(M \star L_a) & \xrightarrow{\sim} & \mathcal{F}^{(2)'}(M) \otimes \mathcal{F}^{(2)'}(L_a)
\end{array}
\]

\[
\mathcal{F}^{(2)'}(M \star L_a) \xrightarrow{\sim} \mathcal{F}^{(2)'}(M) \otimes \mathcal{F}^{(2)'}(L_a)
\]

is commutative for any isomorphism \( g_a : \mathcal{F}^{(2)'}(L_a) \xrightarrow{\sim} k \).

**Proof.** By the same argument in [12, Theorem 4.6.5], it is enough to show that

\[
f_{a,j}(z)g(q^{2(j-a-N)}(z + 1)) \prod_{a \leq k \leq a+N-1} P_{k,j}(0, z) = 1
\]

for all \( a, j \in \mathbb{Z} \). Recall that \( P_{k,j}(0, z) = c_{k,j}(0, z)(-z)^{\delta(j=k+1)} \) and

\[
f_{a,j}(z) = (-1)^{\delta_{j,a+N}}z^{-\delta(a \leq j < a+N-1) - \delta_{j,a+N}}.
\]

Hence it amounts to showing that

\[
\prod_{k=a}^{a+N-1} c_{k,j}(0, z) = (-1)^{\delta(0 \leq a \leq a+N-1)}(z^2 \delta(j=a) - \delta(j=a+N))g(q^{2(j-a-N)}(z + 1))^{-1}
\]

for all \( a, j \in \mathbb{Z} \). Since \( c_{i+1,j+1}(u, v) = c_{i,j}(u, v) \) for all \( i, j \in \mathbb{Z} \), we have only to show that

\[
\prod_{k=a}^{a+N-1} c_{k,0}(0, z) = (-1)^{\delta_{a+1 \leq a+N-1}}(z^2 \delta(a=0) - \delta(a=a+N-1))g(q^{2(-a-N)}(z + 1))^{-1}
\]

for all \( a \in \mathbb{Z} \).

Since \( c_{k,0}(0, z) = \phi_k(z) \), we obtain the desired result.
Hence, [12, Proposition A.7.3] implies that the functor $\mathcal{F}^{(2)}': A/S_N \to U_q'(A_{N-1}^{(2)})$-mod factors through $T_N$. Consequently, we obtain a tensor functor $\tilde{\mathcal{F}}^{(2)}: T_N \to U_q'(A_{N-1}^{(2)})$-mod such that the following diagram quasi-commutes:

\[
\begin{array}{ccc}
A & \overset{\Phi}{\longrightarrow} & A/S_N \\
\downarrow_{\mathcal{F}^{(2)}} & & \downarrow_{\mathcal{F}^{(2)'} \sim} \\
T_N & \overset{\varphi}{\longrightarrow} & T_N' \\
\downarrow_{\mathcal{F}^{(2)}} & & \downarrow_{\mathcal{F}^{(2)}} \\
U_q'(A_{N-1}^{(2)})\text{-mod} & & \\
\end{array}
\] (3.17)

Moreover, by [12, Proposition A.7.2], we obtain the following proposition.

**Proposition 3.15.** The functor $\tilde{\mathcal{F}}^{(2)}$ is exact.

Recall that $\mathcal{C}_0$ is the smallest full subcategory of $\mathcal{C}_0$ stable under taking subquotients, extensions and tensor products, and containing $\{V((i)x) : (i, x) \in \mathcal{J}_0(\mathcal{E})\}$. By Propositions 3.1 and 3.3, the images of the functors $\mathcal{F}^{(2)}$, $\mathcal{F}^{(2)'}$ and $\tilde{\mathcal{F}}^{(2)}$ are inside the category $\mathcal{C}_0$.

Let us denote by $\mathcal{Irr}(T_N)$ the set of the isomorphism classes of simple objects in $T_N$. Define an equivalence relation $\sim$ on $\mathcal{Irr}(T_N)$ by $X \sim Y$ if and only if $X \simeq q^c Y$ in $T_N$ for some integer $c$. Let $\mathcal{Irr}(T_N)_{q=1}$ be a set of representatives of elements in $\mathcal{Irr}(T_N)/\sim$. Then the set $\mathcal{Irr}(T_N)_{q=1}$ is isomorphic to the set of ordered multisegments

\[((a_1, b_1), \ldots, (a_r, b_r))\]

satisfying

\[b_k - a_k + 1 < N \quad \text{for any } 1 \leq k \leq r.\] (3.18)

Since the proofs of the following proposition and theorem are similar to the ones in [12, Subsection 4.7], we omit them.

**Proposition 3.16.** The functor $\tilde{\mathcal{F}}^{(2)}: T_N \to \mathcal{C}_0_{A_{N-1}^{(2)}}$ induces a bijection between $\mathcal{Irr}(T_N)_{q=1}$ and $\mathcal{Irr}(\mathcal{C}_0_{A_{N-1}^{(2)}})$, the set of isomorphism classes of simple objects in $\mathcal{C}_0_{A_{N-1}^{(2)}}$.

**Theorem 3.17.** The functor $\tilde{\mathcal{F}}^{(2)}: T_N \to \mathcal{C}_0_{A_{N-1}^{(2)}}$ induces a ring isomorphism

\[\phi_{\tilde{\mathcal{F}}^{(2)}}: K(T_N)/(q-1)K(T_N) \xrightarrow{\sim} K(\mathcal{C}_0_{A_{N-1}^{(2)}}).\]

Recall that in [12, Subsection 4.6], we obtained a functor $\tilde{\mathcal{F}}^{(1)}: T_N \to \mathcal{C}_0_{A_{N-1}^{(2)}}$, where $\tilde{\mathcal{F}}^{(1)}$, $T_N$ and $\mathcal{C}_0_{A_{N-1}^{(2)}}$ were denoted by $\tilde{\mathcal{F}}$, $T_J$ and $\mathcal{C}_J$, respectively. The functor $\tilde{\mathcal{F}}^{(1)}$ also induces a ring isomorphism

\[\phi_{\tilde{\mathcal{F}}^{(1)}}: K(T_N)/(q-1)K(T_N) \xrightarrow{\sim} K(\mathcal{C}_0_{A_{N-1}^{(2)}}).\]

**Theorem 3.18.** Let $M$ be a simple object in $T_N$. Then we have

\[\dim_k \tilde{\mathcal{F}}^{(1)}(M) = \dim_k \tilde{\mathcal{F}}^{(2)}(M).\] (3.19)

**Proof.** By Proposition 2.3 and Proposition 3.3, we know that (3.19) holds when $M$ is a one-dimensional module corresponding to a segment.

Note that the assignment

\[(a, b) \mapsto \alpha_a + \alpha_{a+1} + \cdots + \alpha_b\]
gives a bijection between the set of segments and the set of positive roots of type $A_\infty$. Under this bijection, the order $>$ on the set of segments induces a convex order $\succ$ on the set of positive roots; that is, we have $\alpha < \alpha + \beta < \beta$ if $\alpha, \beta, \alpha + \beta$ are positive roots and $\alpha < \beta$. It is not difficult to see that \( \{ L(a, b) : (a, b) \text{ is a segment} \} \) is the cuspidal system corresponding to the above convex order in the sense of [19, Definition 3.2]. For $\gamma \in \mathbb{Q}^+_\infty$, we denote by $\text{KP}(\gamma)$ the set of ordered multisegments such that the sum of the corresponding roots is equal to $\gamma$.

Let $((a_1, b_1), (a_2, b_2), \ldots, (a_r, b_r))$ be the ordered multisegment associated with a simple object $M$ in $\mathcal{T}_N$. Then, by [20, Theorem 3.1] (cf. [19, Theorem 3.15 (iv)]), every composition factor of $\text{rad}(L(a_1, b_1) \circ \cdots \circ L(a_r, b_r))$ has an associated multisegment

\[(a'_1, b'_1), \ldots, (a'_s, b'_s),\]
satisfying $(\alpha_{a'_1} + \cdots + \alpha_{a'_s}) + \cdots + (\alpha_{a'_1} + \cdots + \alpha_{a'_s}) \in \text{KP}(\gamma)$, where $\gamma = (\alpha_{a_1} + \cdots + \alpha_{a_1}) + \cdots + (\alpha_{a_s} + \cdots + \alpha_{a_s})$ and $((a'_1, b'_1), \ldots, (a'_s, b'_s)) \prec ((a_1, b_1), \ldots, (a_r, b_r))$. Here $\prec$ denotes the bi-lexicographic partial order on $\text{KP}(\gamma)$ given in [20, Section 3] (cf. [19, Subsection 3.2]). In particular, if $((a_1, b_1), \ldots, (a_r, b_r))$ is a minimal element in $\text{KP}(\gamma)$ with respect to $\prec$, then $M \simeq L(a_1, b_1) \circ \cdots \circ L(a_r, b_r)$. Thus we obtain (3.19) in this case.

Now, by induction on $\prec$, we may assume that

\[\dim_k \tilde{F}^{(1)}(\text{rad}(L(a_1, b_1) \circ \cdots \circ L(a_r, b_r))) = \dim_k \tilde{F}^{(2)}(\text{rad}(L(a_1, b_1) \circ \cdots \circ L(a_r, b_r))).\]

It follows that

\[
\dim_k \tilde{F}^{(1)}(M) = \prod_{k=1}^r \dim_k \tilde{F}^{(1)}(L(a_k, b_k)) - \dim_k \tilde{F}^{(1)}(\text{rad}(L(a_1, b_1) \circ \cdots \circ L(a_r, b_r))) \\
= \prod_{k=1}^r \dim_k \tilde{F}^{(2)}(L(a_k, b_k)) - \dim_k \tilde{F}^{(2)}(\text{rad}(L(a_1, b_1) \circ \cdots \circ L(a_r, b_r))) \\
= \dim_k \tilde{F}^{(2)}(M),
\]

as desired. \( \square \)

Set

\[\phi^{(2)} := \phi_{\tilde{F}^{(2)}} \circ \phi_{\tilde{F}^{(1)}}^{-1} : K(E_{A_{N-1}^{(1)}}) \cong K(E_{A_{N-1}^{(2)}}).\]

**COROLLARY 3.19.** The ring isomorphism $\phi^{(2)}$ induces a bijection between $\text{Irr}(E_{A_{N-1}^{(1)}})$ and $\text{Irr}(E_{A_{N-1}^{(2)}})$, which preserves the dimensions.

**References**

1. T. Akasaka and M. Kashiwara, ‘Finite-dimensional representations of quantum affine algebras’, Publ. RIMS, Kyoto Univ. 33 (1997) 839–867.
2. V. Chari and A. Pressley, ‘Quantum affine algebras and their representations’, Representations of groups (Banff, AB, 1994), Conference Proceedings, Canadian Mathematical Society 16 (American Mathematical Society, Providence, RI, 1995) 59–78.
3. V. Chari and A. Pressley, ‘Yangians, integrable quantum systems and Dorey’s rule’, Comm. Math. Phys. 181 (1996) 265–302.
4. V. Chari and A. Pressley, ‘Quantum affine algebras and affine Hecke algebras’, Pacific J. Math. 174 (1996) 295–326.
5. I. V. Cherednik, ‘A new interpretation of Gelfand–Tzetlin bases’, Duke Math. J. 54 (1987) 563–577.
6. E. Date and M. Okado, ‘Calculation of excitation spectra of the spin model related with the vector representation of the quantized affine algebra of type $A_1^{(1)}$’, Internat. J. Modern Phys. A 9 (1994) 399–417.
7. E. Frenkel and N. Yu. Reshetikhin, ‘The $q$-characters of representations of quantum affine algebras and deformations of $W$-algebras. Recent developments in quantum affine algebras and related topics’, Contemp. Math. 248 (1999) 163–205.
8. V. Ginzburg, N. Reshetikhin and E. Vasserot, ‘Quantum groups and flag varieties’, *Contemp. Math.* 175 (1994) 101–130.

9. D. Hernandez, ‘Kirillov–Reshetikhin conjecture: the general case’, *Int. Math. Res. Not.* 2010 (2010) 149–193.

10. V. Kac, *Infinite-dimensional Lie algebras*, 3rd edn (Cambridge University Press, Cambridge, 1990).

11. S.-J. Kang and M. Kashiwara, ‘Categorification of highest weight modules via Khovanov–Lauda–Rouquier algebras’, *Invent. Math.* 190 (2012) 699–742.

12. S.-J. Kang, M. Kashiwara and M. Kim, ‘Symmetric quiver Hecke algebras and $R$-matrices of quantum affine algebras’, Preprint, 2013, arXiv:1304.0323 [math.RT].

13. S.-J. Kang, M. Kashiwara, M. Kim and S.-J. Oh, ‘Simplicity of heads and socles of tensor products’, *Compos. Math.*, Preprint, 2014, arXiv:1404.4125v2.

14. S.-J. Kang, M. Kashiwara, M. Kim and S.-J. Oh, ‘Symmetric quiver Hecke algebras and $R$-matrices of quantum affine algebras IV’, Preprint, 2015, arXiv:1502.07415 [math.RT].

15. S.-J. Kang and E. Park, ‘Irreducible modules over Khovanov–Lauda–Rouquier algebras of type $A_n$ and semistandard tableaux’, *J. Algebra* 339 (2011) 223–251.

16. M. Kashiwara, ‘Crystal bases of modified quantized enveloping algebra’, *Duke Math. J.* 73 (1994) 383–413.

17. M. Kashiwara, ‘On level zero representations of quantum affine algebras’, *Duke. Math. J.* 112 (2002) 117–175.

18. M. Khovanov and A. Lauda, ‘A diagrammatic approach to categorification of quantum groups I’, *Represent. Theory* 13 (2009) 309–347.

19. A. Kleshchev, ‘Representation theory and cohomology of Khovanov–Lauda–Rouquier algebras’, *Conference Proceedings of the program ‘Modular Representation Theory of Finite and $p$-Adic Groups’ at the National University of Singapore*, Preprint, 2014, arXiv:1401.6151v1 [math.RT].

20. P. J. McNamara, ‘Finite-dimensional representations of Khovanov–Lauda–Rouquier algebras I: finite type’, *J. reine angew. Math.* , Preprint, 2012, arXiv:1207.5860 [math.RT].

21. H. Nakajima, ‘Quiver varieties and $t$-analogue of $q$-characters of quantum affine algebras’, *Ann. of Math.* 160 (2004) 1057–1097.

22. S.-J. Oh, ‘The denominators of normalized $R$-matrices of types $A_{2n−1}^{(2)}$, $A_{2n}^{(2)}$, $B_n^{(1)}$ and $D_{n+1}^{(2)}$’, Preprint, 2014, arXiv:1404.6715v3 [math.QA].

23. R. Rouquier, ‘2-Kac–Moody algebras’, Preprint, 2008, arXiv:0812.5023v1 [math.RT].