Quantum-Non-Demolition Endoscopic Tomography

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We present a new indirect method to measure the quantum state of a single mode of the electromagnetic field in a cavity. Our proposal combines the idea of (endoscopic) probing and that of tomography in the sense that the signal field is coupled via a quantum-non-demolition Hamiltonian to a meter field on which then quantum state tomography is performed using balanced homodyne detection. This technique provides full information about the signal state. We also discuss the influence of the measurement of the meter on the signal field.

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I. INTRODUCTION: STATE MEASUREMENT

The question of how to measure the quantum state of a single mode of the electromagnetic field in a cavity has recently attracted a great deal of attention as it determines the feasibility of the measurement of the state of a genuine quantum system. Several proposals have been made in the last few years to answer this question. Among them, quantum-state endoscopy and quantum-optical homodyne state-tomography are two notable examples. The former proposal makes use of the idea of probing, that is doing endoscopy on the field state inside the cavity. On the other hand, optical homodyne tomography is a method for obtaining the Wigner function (or, more generally, the matrix elements of the density operator in some representation) of the electromagnetic field. It therefore consists of an ensemble of repeated measurements of one quadrature operator for different phases relative to the local oscillator of the homodyne detector. The major drawback of the first method is the low detection efficiency for atoms, whereas in the second one one has to couple the field out of the resonator.

In the present paper we propose to couple the field via a quantum-non-demolition (QND) interaction to a meter field on which we then perform tomography using a balanced homodyne detector. In this way we combine the idea of probing, that is doing endoscopy on the field without taking it out of the cavity, and the tool of tomography and arrive at the method of endoscopic quantum state tomography. In contrast to the method of quantum state tomography based on homodyne detection, the present technique does not couple the signal field out of the resonator.

II. OUR MODEL

Let us assume that the electromagnetic field we want to probe (the signal mode) is in a pure quantum state and neglect dissipation. We note, however, that our method applies as well to a field described by a density operator. In order to measure the signal field we couple it to a meter field. Both fields are then coupled to a pump field. This coupling leads us to a quantum-non-demolition Hamiltonian describing the interaction between the signal and the meter mode.

Our model starts from the Hamiltonian

$$\hat{H} = i\hbar \chi [\hat{a}_s^\dagger \hat{a}_p - \hat{a}_s^\dagger \hat{a}_m] + i\hbar \sigma [\hat{a}_m^\dagger \hat{a}_m - \hat{a}_m^\dagger \hat{a}_m],$$

(1)

where $\hat{a}_s$, $\hat{a}_m$, and $\hat{a}_p$ represent the annihilation (creation) operators of the signal, meter, and pump field, respectively. The parameters $\chi$ and $\sigma$ measure the coupling between the three fields, and the meter and signal field, respectively. When the pump field is highly excited we can describe it by a coherent state of amplitude $\alpha$ and phase $2\phi$, that is

$$\hat{a}_p \simeq \alpha e^{2i\phi}.$$  

(2)

If we substitute the coherent state approximation, Eq. (2), into the Hamiltonian, Eq. (1), after some algebra we obtain

$$\hat{H} = 2\hbar \sigma X_s(\phi + \pi/2) \cdot X_m(\phi),$$

(3)

where we have arranged the strength $\alpha$ of the pump field such that $\chi \alpha = \sigma$, and the quadrature operators are given by

$$X_j(\theta) \equiv \frac{1}{\sqrt{2}} \left( \hat{a}_j e^{-i\theta} + \hat{a}_j^\dagger e^{i\theta} \right)$$

(4)

of the signal ($j = s$) and the meter ($j = m$) mode at the angle $\theta$. 


It is particularly interesting to note that due to the special choice $\chi = \sigma$ we have been able to obtain an interaction between the signal and the meter which couples the quadrature operator $\hat{X}_m(\phi)$ of the meter at phase angle $\phi$ to the out-of-phase quadrature operator $\hat{X}_s(\phi + \pi/2)$ of the signal. The present Hamiltonian is a genuine QND Hamiltonian. In the next sections we shall see how such a Hamiltonian can be used to measure the quantum state of the signal field. We note that according to the QND Hamiltonian Eq. (3) a (homodyne) measurement of the meter at a fixed phase $\phi$ of the pump field yields information about the signal in the orthogonal quadrature. By varying the phase $\phi$ of the pump field we can probe all quadratures of the signal.

III. ENTANGLED STATE

The aim of the present section is to calculate the combined state $|\Psi\rangle$ of signal and meter achieved after some interaction time according to the QND interaction Hamiltonian, Eq. (3).

When we couple for an interaction time $t$ the signal and meter mode prepared initially in the states $|\psi_s\rangle$ and $|\psi_m\rangle$ we find the quantum state

$$|\Psi(t)\rangle = \exp(-i\hat{H}t/\hbar)|\psi_m\rangle|\psi_s\rangle$$

$$= \exp[-2i\sigma t\hat{X}_s(\phi + \pi/2)\hat{X}_m(\phi)]|\psi_m\rangle|\psi_s\rangle$$

for the combined system.

We expand the initial signal state in quadrature states $|X_s(\phi + \pi/2)\rangle$,

$$|\psi_s\rangle = \int_{-\infty}^{\infty} dX_s |X_s;\phi + \pi/2\rangle |X_s(\phi + \pi/2)\rangle .$$

(6)

We stress that this representation and, in particular, the wave function $|\psi_s\rangle$ depend crucially on the angle $\theta_s$.

Combining Eqs. (3) and (4), we may rewrite the combined state as

$$|\Psi(t)\rangle = \int_{-\infty}^{\infty} dX_s |\psi_s(X_s;\phi + \pi/2)\rangle |X_s(\phi + \pi/2)\rangle$$

$$\times \exp[-2i\sigma t\hat{X}_s(\phi)\hat{X}_m(\phi)]|\psi_m\rangle .$$

(7)

Expanding $|\psi_m\rangle$ in quadrature states $|X_m(\theta)\rangle$ of the meter at the angle $\theta$, that is

$$|\psi_m\rangle = \int_{-\infty}^{\infty} dX_m \psi_m(X_m;\theta) |X_m(\theta)\rangle ,$$

(8)

where $\psi_m(X_m;\theta) \equiv \langle X_m(\theta) |\psi_m\rangle$ denotes the wave function of the meter state at the angle $\theta$, it is straightforward to find

$$\exp[-i(2\sigma t X_s)\hat{X}_m(\phi)] |\psi_m\rangle$$

$$= \int dX_m \exp[-i\gamma(X_s, X_m; \theta - \phi)]$$

$$\times \psi_m[X_m - 2\sigma t X_s \sin(\theta - \phi); \theta] |X_m(\theta)\rangle ,$$

(9)

where

$$\gamma(X_s, X_m; \theta - \phi) \equiv (\sigma t X_s)^2 \sin[2(\theta - \phi)]$$

$$+ 2\sigma t X_s X_m \cos(\theta - \phi) .$$

(10)

Hence the combined quantum state reads

$$|\Psi(t)\rangle = \int_{-\infty}^{\infty} dX_s \int_{-\infty}^{\infty} dX_m \psi_s(X_s;\phi + \pi/2)$$

$$\times \psi_m[X_m - 2\sigma t X_s \sin(\theta - \phi); \theta]$$

$$\times \exp[-i\gamma(X_s, X_m; \theta - \phi)]$$

$$\times |X_s(\phi + \pi/2)\rangle |X_m(\theta)\rangle .$$

(11)

We note that due to the coupling between the meter and the signal via the Hamiltonian Eq. (3), the meter wave function $\psi_m(X_m;\theta)$ at the angle $\theta$ gets shifted by an amount $\delta X_m \equiv 2\sigma \tau X_s \sin(\theta - \phi)$.

IV. EFFECT OF THE METER MEASUREMENT ON THE SIGNAL STATE

In the present section we shall show how a measurement of the meter influences the state of the signal. Let us first consider an arbitrary quadrature state of phase angle $\theta$.

According to Eq. (11) the conditioned state

$$|\psi_s^{(c)}\rangle = \frac{1}{\sqrt{W(X_m)}} |X_m(\theta)\rangle |\Psi(t)\rangle$$

(12)

of the signal given that our quadrature measurement at angle $\theta$ has provided the value $X_m$ reads

$$|\psi_s^{(c)}\rangle = \int_{-\infty}^{\infty} dX_s |\psi_s(X_s;\phi + \pi/2)f(X_s|X_m)\rangle |X_s(\phi + \pi/2)\rangle ,$$

(13)

where the “filter function” $f$ is given by

$$f(X_s|X_m) = \frac{1}{\sqrt{W(X_m)}} \psi_m[X_m - 2\sigma t X_s \sin(\theta - \phi); \theta]$$

$$\times \exp[-i\gamma(X_s, X_m; \theta - \phi)] .$$

(14)

The normalization condition directly yields the probability $W(X_m)$ of finding the meter variable $X_m$, that is

$$W(X_m) = \int_{-\infty}^{\infty} dX_m |\psi_m[X_m - 2\sigma t X_s \sin(\theta - \phi); \theta]|^2 .$$

(15)
\[ W(X_m) = \int_{-\infty}^{\infty} dX_s \psi_s(X_s; \phi + \pi/2)^2 \]

\[ \times |\psi_m[X_m - 2\sigma_t X_s \sin(\theta - \phi); \theta]|^2 . \quad (15) \]

Equation (13) clearly shows how the measurement of the meter influences the quantum state of the signal: The filter function determined by the wave function of the meter selects those parts of the signal wave function that are entangled with the corresponding parts in the meter. To study this in more detail we now calculate the Wigner function

\[ W_s^{(c)}(X_s, P_s|X_m) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dY e^{ip_sY} \psi_s(X_s - Y/2|X_m) \psi_{\phi}^*(X_s + Y/2) \quad (16) \]

of the signal state conditioned on the measured meter value \( X_m \). Substituting the state \( |\psi_s^{(c)}\rangle \), Eq. (13), into this expression we arrive at

\[ W_s^{(c)}(X_s, P_s|X_m) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dY e^{ip_sY} \psi_s(X_s - Y/2) \]

\[ \times \psi_s^*(X_s + Y/2)f(X_s - Y/2|X_m)f^*(X_s + Y/2|X_m) . \quad (17) \]

The integral may be expressed as the convolution

\[ W_s^{(c)}(X_s, P_s|X_m) = \int_{-\infty}^{\infty} dP' W_s(X_s, P_s - P') W_f(X_s, P'|X_m) \quad (18) \]

between the Wigner function of the original signal state

\[ W_s(X_s, P_s) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dY e^{ip_s Y} \psi_s(X_s - Y/2) \psi_s^*(X_s + Y/2) , \quad (19) \]

and the Wigner function

\[ W_f(X_s, P_s|X_m) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dP' e^{ip_s Y} f(X_s - Y/2|X_m)f^*(X_s + Y/2|X_m) \quad (20) \]

of the “filter” provided by the measurement on the meter.

V. SPECIAL EXAMPLES

A. In phase measurement

If we take the angle \( \theta = \phi \), the state \( |\Psi\rangle \), Eq. (11), of the complete system reduces to

\[ |\Psi(t)\rangle = \int_{-\infty}^{\infty} dX_s \int_{-\infty}^{\infty} dX_m \psi_s(X_s; \phi + \pi/2)\psi_m(X_m; \phi) \]

\[ \times \exp(-i2\sigma_t X_s X_m)|X_s(\phi + \pi/2); X_m(\phi)\rangle . \quad (21) \]

In this case the meter wave function is not shifted. Nevertheless, the states of signal and meter are still entangled. Since the shift \( \delta X_m \) vanishes, the probability

\[ W(X_m) = |\psi_m[X_m]|^2 \int_{-\infty}^{\infty} dX_s |\psi_s(X_s)|^2 = |\psi_m|^2 , \quad (22) \]

of finding the meter variable \( X_m \) following from Eq. (14) for \( \theta = \phi \) is identical to the initial probability of the meter, that is

\[ W(X_m) = |\psi_m[X_m]|^2 . \quad (23) \]

Hence, up to an overall phase \( \mu_m \) determined by the meter wave function \( \psi(X_m) = |\psi(X_m)|\exp[\mu_m(X_m)] \), we find from Eq. (14) the filter function \( f(X_s|X_m) = \exp(-i2\sigma_t X_s X_m) \), and from Eq. (13) the conditioned signal state

\[ |\psi_s^{(c)}\rangle = \int_{-\infty}^{\infty} dX_s \psi_s(X_s) \exp(-i2\sigma_t X_s X_m)|X_s(\phi + \pi/2)\rangle . \quad (24) \]

Note that the measurement of the meter has indeed changed the state of the system but did not alter the probability

\[ W(X_s) = |\langle X_s|\psi_s^{(c)}\rangle|^2 = |\psi_s(X_s) \exp(-i2\sigma_t X_s X_m)|^2 \]

\[ = |\psi_s(X_s)|^2 \quad (25) \]

of finding the signal variable \( X_s \). Hence, the measurement has left untouched the shape of the original state (in the Wigner function representation) but has moved it along the momentum axis by an amount of \( 2\sigma_t X_m \). Consequently, the measurement did not change the probability distribution in the conjugate variable, namely the \( X_s \) variable. We note, however, that in this way we cannot gain information about the signal since according to Eqs. (22) and (23) the probability distribution \( W(X_m) \) of measuring the variable \( X_m \) is identical to the original distribution.

This finding is actually a rather general result. In fact, it can be rigorously shown [1] that a (QND) measurement which does not change the probability density of the observable which is being measured on a single quantum system gives no information about the measured observable.
Turning now to the case of $\theta = \phi + \pi/2$, we see that the shift $\delta X_m = 2\sigma t X_s$ in the meter wave function is maximal and according to Eq. (18) the phase $\gamma$ vanishes. Hence, the combined state

$$\langle \Psi(t) \rangle = \int_{-\infty}^{\infty} dX_s \int_{-\infty}^{\infty} dX_m \psi_s(X_s) \psi_m(X_m - 2\sigma t X_s) \psi_m(X_m + 2\sigma t X_s)$$

(26)

is an entangled state in which the entanglement between the meter and signal is due to the shift of the meter. In contrast to the discussion of Sec. V A we can now deduce properties of the signal from the shift of the meter wave function. Unfortunately, we cannot simultaneously keep the probability distribution $W(X_m) = |\psi_s(X_s)|^2$ of the original signal state invariant, in accordance with the discussion at the end of Sec. V A. Indeed, we find from Eqs. (13) or (24) the conditional state

$$\langle \tilde{\psi}_s(\phi) \rangle = \frac{1}{\sqrt{\tilde{W}(X_m)}} \int_{-\infty}^{\infty} dX_s \psi_s(X_s) \psi_m(X_m - 2\sigma t X_s)$$

(27)

of the system given the meter measurement at phase $\phi + \pi/2$ has provided the value $X_m$. The probability

$$\tilde{W}(X_m) = \int_{-\infty}^{\infty} dX_s |\psi_s(X_s)|^2 |\psi_m(X_m - 2\sigma t X_s)|^2$$

(28)

of finding the meter value $X_m$ following from Eq. (13) is now a convolution of the system and the meter function.

VI. SPECIAL MEASUREMENTS

A. Weak measurements

If $\psi_m$ is broad compared to $\psi_s$ we can evaluate $\psi_m$ at some characteristic value of $X_s$, such as $\langle X_s \rangle$. As a consequence the conditional state, Eq. (27), is simply given by

$$|\tilde{\psi}_s(\phi)\rangle \equiv \int_{-\infty}^{\infty} dX_s \psi_s(X_s) |X_s\rangle \ .$$

(29)

The probability

$$\tilde{W}(X_m) \equiv |\psi_m(X_m + 2\sigma t X_s)|^2$$

(30)

reduces to the original meter probability shifted by an amount $2\sigma t X_s$. Hence, when this shift $2\sigma t X_s$ is larger than the width of $W_m(X_m) = |\psi_m(X_m)|^2$, we can learn about $\langle X_s \rangle$. As seen from Eq. (29), in this case the state of the signal mode does not change appreciably.

B. Out of phase measurement

In the present section we show that it is possible to perform tomography on the meter mode to obtain information about the signal state. To this end, we rewrite Eq. (28)

$$\tilde{W}(X_m) = \int_{-\infty}^{\infty} dX_s |\psi_s(X_s)|^2 |\psi_m(X_m - 2\sigma t X_s)|^2$$

(31)

which gives the marginal distribution of the meter (probability distribution of the results of the measurements of $X_m$) in the case of out of phase measurements. Let us assume that the meter wave function is extremely narrow, that is the meter is initially in a highly squeezed state, for example a squeezed vacuum $|0, r\rangle$, where $r$ is the (real) squeezing parameter. Then, according to Eq. (31), the marginal distribution $\tilde{W}(X_m)$ is given by a convolution of the modulus square of the signal wave function with a narrow Gaussian

$$|\psi_m(X_m - 2\sigma t X_s)|^2 = \frac{1}{\sqrt{\pi} \cosh r} \int_{-\infty}^{\infty} dx \exp \left\{ \frac{1 + \tanh r}{1 - \tanh r} (X_m - 2\sigma t X_s)^2 \right\} \ .$$

(32)

Now, if the squeezing parameter $r$ is large enough, the Gaussian (24) approaches a delta function in the meter and signal variables

$$|\psi_m(X_m - 2\sigma t X_s)|^2 \rightarrow \frac{1}{2\sigma t} \delta \left( X_s - \frac{X_m}{2\sigma t} \right) \ ,$$

(33)

and Eq. (31) reduces to

$$\tilde{W}(X_m) \equiv \frac{1}{2\sigma t} \int_{-\infty}^{\infty} dX_s |\psi_s(X_s)|^2 \delta \left( X_s - \frac{X_m}{2\sigma t} \right) \ .$$

(34a)

$$= \frac{1}{2\sigma t} \left| \psi_s \left( \frac{X_m}{2\sigma t} \right) \right|^2 = \frac{1}{2\sigma t} W \left( \frac{X_m}{2\sigma t} \right) \. \tag{34b}$$

Hence, by measuring the probability distribution $\tilde{W}(X_m)$ of the outcomes of the meter variable $X_m$ (for example via balanced homodyne detection performed on the meter field) we indirectly obtain the probability distribution $W(X_s)$, up to a rescaling given by the factor $2\sigma t$. However, from Eq. (27) it is clear that in this case the signal wave function is changed, and therefore we need to prepare the signal field again in the same state after each measurement. This is what is usually done in quantum optical tomography [4].

The advantage of the present scheme is that we perform an indirect measurement: We do not detect the signal mode outside the cavity (that is, we do not have to take the signal field outside the cavity), but we couple it to a meter field which is successively detected,
thus overcoming the smearing effect introduced by the direct detection of the signal [4]. Moreover, there is no need of a smoothing procedure, since we are interested in the marginal probability distribution $W(X_s)$ which is directly related to $\tilde{W}(X_m)$ through Eq. (34). In order to probe the full state of the signal field, however, we would need to measure the probability distribution $\tilde{W}(X_m)$ for various values of the phase [4].

VII. CONCLUSIONS

To summarize, we have suggested a method to measure the quadrature probability distribution (or, more generally, the full quantum state) of a single mode of the electromagnetic field inside a cavity. It is based on indirect homodyne measurements performed on a meter field which is coupled to the signal field via a QND interaction Hamiltonian. We have named this procedure “endoscopic tomography” because (i) it does not require (in contrast to Ref. [4]) to take the field out of the cavity, just as in “quantum state endoscopy” [3], where a beam of two-level atoms is used as a probe; (ii) tomographic measurements performed (by balanced homodyne detection) on the meter mode allow us to reconstruct the marginal probability distribution of the signal variable or even the full quantum state.

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