Classifying semisymmetric cubic graphs of order $20p$

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Abstract: A simple graph is called semisymmetric if it is regular and edge transitive but not vertex transitive. In this paper we classify all connected cubic semisymmetric graphs of order $20p$, $p$ prime.

Key words: Edge-transitive graph, vertex-transitive graph, semisymmetric graph, order of a graph, classification of cubic semisymmetric graphs

1. Introduction

In this paper all graphs are finite, undirected and simple, i.e. without loops and multiple edges. A graph is called semisymmetric if it is regular and edge-transitive but not vertex-transitive.

The class of semisymmetric graphs was first studied by Folkman [9], who found several infinite families of such graphs and posed eight open problems.

An interesting research problem is to classify connected cubic semisymmetric graphs of various types of orders. In [9], Folkman proved that there are no semisymmetric graphs of order $2p$ or $2p^2$ for any prime $p$.

The classification of semisymmetric graphs of order $2pq$, where $p$ and $q$ are distinct primes, was given in [7].

For prime $p$, cubic semisymmetric graphs of order $2p^3$ were investigated in [17], in which the authors proved that there is no connected cubic semisymmetric graph of order $2p^3$ for any prime $p \neq 3$ and that for $p = 3$ the only such graph is the Gray graph.

Also connected cubic semisymmetric graphs of orders $4p^3$, $6p^2$, $6p^3$, $8p^2$, $8p^3$, $10p^3$, $18p^n$ ($n \geq 1$) have been classified in [1, 2, 8, 11, 13, 21].

In this paper we investigate connected cubic semisymmetric graphs of order $20p$ for all primes $p$. Note that for orders like $4p$, $6p$, $10p$ and $14p$ which are of the form $2qp$ for some fixed prime $q$, the problem of classifying such graphs follows from the general result of [7].

We prove that if $\Gamma$ is a connected cubic semisymmetric graph of order $20p$, $p$ prime, then $p = 11$ and $\Gamma$ is isomorphic to a known graph. We go beyond however and prove that there is no connected cubic $G$-semisymmetric graph of order $20p$, for any prime $p \neq 2, 11$. This will put us near the classification of all connected cubic $G$-semisymmetric graphs of order $20p$: if there is any such graph, then its order must be either

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2. Preliminaries

In this paper the symmetric and alternating groups of degree $n$, the dihedral group of order $2n$ and the cyclic group of order $n$ are respectively denoted by $S_n$, $A_n$, $D_{2n}$, $Z_n$. If $G$ is a group and $H \leq G$, then $\text{Aut}(G)$, $G'$, $Z(G)$, $C_G(H)$ and $N_G(H)$ denote respectively the group of automorphisms of $G$, the commutator subgroup of $G$, the center of $G$, the centralizer and the normalizer of $H$ in $G$. We also write $H \trianglelefteq G$ to denote $H$ is a characteristic subgroup of $G$. If $H \leq G$ and $K \unlhd G$, then $H \trianglelefteq G$. For a prime $p$ dividing the order of a finite group $G$, $O_p(G)$ will denote the largest normal $p$-subgroup of $G$. It is easy to verify that $O_p(G) \leq G$.

For a group $G$ and a nonempty set $\Omega$, an action of $G$ on $\Omega$ is a function $(g, \omega) \to g.\omega$ from $G \times \Omega$ to $\Omega$, where $1.\omega = \omega$ and $g.(h.\omega) = (gh)\omega$, for every $g, h \in G$ and every $\omega \in \Omega$. We write $g\omega$ instead of $g.\omega$, if there is no fear of ambiguity. For $\omega \in \Omega$, the stabilizer of $\omega$ in $G$ is defined as $G_\omega = \{g \in G : g\omega = \omega\}$. The action is called semiregular if the stabilizer of each element in $\Omega$ is trivial; it is called regular if it is semiregular and transitive.

For any two groups $G$ and $H$ and any homomorphism $\varphi : H \to \text{Aut}(G)$ the external semidirect product $G \rtimes \varphi H$ is defined as the group whose underlying set is the cartesian product $G \times H$ and whose binary operation is defined as $(g_1, h_1)(g_2, h_2) = (g_1\varphi(h_1)(g_2), h_1h_2)$. If $\varphi(h) = 1$ for each $h \in H$, then the semidirect product will coincide with the usual direct product. If $G = NK$ where $N \trianglelefteq G$, $K \unlhd G$ and $N \cap K = 1$, then $G$ is said to be the internal semidirect product of $N$ and $K$. These two concepts are in fact equivalent in the sense that there is some homomorphism $\varphi : K \to \text{Aut}(N)$ where $G \simeq N \rtimes \varphi K$.

The dihedral group $D_{2n}$ is defined as

$$D_{2n} = \langle a, b | a^n = b^2 = 1, b^{-1}ab = a^{-1} \rangle$$

So $D_{2n} = \{a^i | i = 0, \ldots, n - 1\} \cup \{ba^i | i = 0, \ldots, n - 1\}$. All the elements of the form $ba^i$ are of order 2.

Let $\Gamma$ be a graph. For two vertices $u$ and $v$, we write $u \sim v$ to denote $u$ is adjacent to $v$. If $u \sim v$, then each of the ordered pairs $(u, v)$ and $(v, u)$ is called an arc. The set of all vertices adjacent to a vertex $u$ is denoted by $\Gamma(u)$. The degree or valency of $u$ is $|\Gamma(u)|$. We call $\Gamma$ regular if all of its vertices have the same valency. The vertex set, the edge set, the arc set and the set of all automorphisms of $\Gamma$ are respectively denoted by $V(\Gamma)$, $E(\Gamma)$, $\text{Arc}(\Gamma)$ and $\text{Aut}(\Gamma)$. If $\Gamma$ is a graph and $N \trianglelefteq \text{Aut}(\Gamma)$, then $\Gamma_N$ will denote a simple undirected graph whose vertices are the orbits of $N$ in its action on $V(\Gamma)$, and where two vertices $Nu$ and $Nv$ are adjacent if and only if $u \sim nv$ in $\Gamma$, for some $n \in N$.

Let $\Gamma_c$ and $\Gamma$ be two graphs. Then $\Gamma_c$ is said to be a covering graph for $\Gamma$ if there is a surjection $f : V(\Gamma_c) \to V(\Gamma)$ which preserves adjacency and for each $u \in V(\Gamma_c)$, the restricted function $f|_{\Gamma_c(u)} : \Gamma_c(u) \to \Gamma(f(u))$ is a one to one correspondence. $f$ is called a covering projection. Clearly, if $\Gamma$ is bipartite, then so is $\Gamma_c$. For each $u \in V(\Gamma)$, the fibre on $u$ is defined as $fib_u = f^{-1}(u)$. The following important set is a subgroup of $\text{Aut}(\Gamma_c)$ and is called the group of covering transformations for $f$:

$$\text{CT}(f) = \{\sigma \in \text{Aut}(\Gamma_c) | \forall u \in V(\Gamma), \sigma(fib_u) = fib_u\}$$

It is known that $K = \text{CT}(f)$ acts semisymmetrically on each fibre [14]. If this action is regular, then $\Gamma_c$ is said to be a regular $K$-cover of $\Gamma$.

Let $X \trianglelefteq \text{Aut}(\Gamma)$. Then $\Gamma$ is said to be $X$-vertex transitive, $X$-edge transitive or $X$-arc transitive if $X$ acts transitively on $V(\Gamma)$, $E(\Gamma)$ or $\text{Arc}(\Gamma)$ respectively. The graph $\Gamma$ is called $X$-semisymmetric if it is
regular and $X$-edge transitive but not $X$-vertex transitive. Also $\Gamma$ is called $X$-symmetric if it is $X$-vertex transitive and $X$-arc transitive. For $X = \text{Aut}(\Gamma)$, we omit $X$ and simply talk about $\Gamma$ being edge transitive, vertex transitive, symmetric or semisymmetric. As an example, $\Gamma = K_{3,3}$, the complete bipartite graph on 6 vertices, is not semisymmetric but it is $X$-semisymmetric for some $X \leq \text{Aut}(\Gamma)$.

An $X$-edge transitive but not $X$-vertex transitive graph is necessarily bipartite, where the two partites are the orbits of the action of $X$ on $V(\Gamma)$. If $\Gamma$ is regular, then the two partite sets have equal cardinality. So an $X$-semisymmetric graph is bipartite such that $X$ is transitive on each partite but $X$ carries no vertex from one partite set to the other.

According to [5], if there is a unique known cubic semisymmetric graph of order $n$, then it is denoted by $\text{Sn}$. The symmetric counterpart of $\text{Sn}$ is denoted by $\text{F}_n$ ([6]). There are only two symmetric cubic graphs of order 20 which are denoted by $\text{F}_20\text{A}$ and $\text{F}_20\text{B}$. Only $\text{F}_20\text{B}$ is bipartite ([6]).

Any minimal normal subgroup of a finite group, is the internal direct product of isomorphic copies of a simple group.

A finite group $G$ is called a $K_n$-group if its order has exactly $n$ distinct prime divisors, where $n \in \mathbb{N}$.

The following two results determine all simple $K_3$-groups and $K_4$-groups [3, 12, 19, 24].

Theorem 2.1
(i) If $G$ is a simple $K_3$-group, then $G$ is isomorphic to one of the following groups: $\text{A}_5$, $\text{A}_6$, $L_2(7)$, $L_2(2^3)$, $L_2(17)$, $L_3(3)$, $U_3(3)$, $U_4(2)$.

(ii) If $G$ is a simple $K_4$-group, then $G$ is isomorphic to one of the following groups:

1. $\text{A}_7$, $\text{A}_8$, $\text{A}_9$, $\text{A}_{10}$, $\text{M}_{11}$, $\text{M}_{12}$, $J_2$, $L_2(2^4)$, $L_2(5^2)$, $L_2(7^2)$, $L_2(3^4)$, $L_2(97)$, $L_2(3^5)$, $L_2(577)$, $L_3(2^3)$, $L_3(5)$, $L_3(7)$, $L_3(2^3)$, $L_3(17)$, $L_4(3)$, $U_3(2^2)$, $U_3(5)$, $U_3(7)$, $U_3(2^3)$, $U_3(3^2)$, $U_4(3)$, $U_5(2)$, $S_4(2)$, $S_4(5)$, $S_4(7)$, $S_4(2^3)$, $S_6(2)$, $O_8^+(2)$, $G_2(3)$, $S_3(2^5)$, $S_3(5)$, $3D_4(2)$, $2F_4(2)$;

2. $L_2(r)$ where $r$ is a prime, $r^2 - 1 = 2^a \cdot 3^b$, $s > 3$ is a prime, $a, b \in \mathbb{N}$;

3. $L_2(2^m)$ where $m$, $2^m - 1$, $\frac{2^m + 1}{3}$ are primes greater than 3;

4. $L_2(3^m)$ where $m$, $\frac{3^m + 1}{4}$ and $\frac{3^m - 1}{2}$ are odd primes.

Proposition 2.2 ([18], Theorem 9.1.2) Let $G$ be a finite group and $N \leq G$. If $|N|$ and $|\frac{G}{N}|$ are relatively prime, then $G$ has a subgroup $H$ such that $G = NH$ and $N \cap H = 1$ (therefore $G$ is the internal semidirect product of $N$ and $H$).

An immediate consequence of the following theorem of Burnside is that the order of every nonabelian simple group is divisible by at least 3 distinct primes.

Theorem 2.3 [20] For any two distinct primes $p$ and $q$ and any two nonnegative integers $a$ and $b$, every finite group of order $p^aq^b$ is solvable.

In the following theorem, the inverse of a pair $(a, b)$, is meant to be $(b, a)$. Also for each $i$, $A_i$, $B_i$, $C_i$ and $D_i$ are certain groups of order $i$ with known structures. We will not need their structures.
Theorem 2.4 [10] If $\Gamma$ is a connected cubic $X$-semisymmetric graph, then the order of the stabilizer of any vertex is of the form $2^r \cdot 3$ for some $0 \leq r \leq 7$. More precisely, if $\{u,v\}$ is any edge of $\Gamma$, then the pair $(X_u,X_v)$ can only be one of the following fifteen pairs or their inverses:

- $H_3, \{S_3, S_3\}$,
- $(D_{12}, D_{12})$,
- $(D_{12}, A_4)$,
- $(S_4, D_{24})$,
- $(S_4, S_3 \times D_8)$,
- $(A_4 \times Z_2, D_{12} \times Z_2)$,
- $(S_4 \times Z_2, D_8 \times S_3)$,
- $(S_4, S_4)$,
- $(S_4 \times Z_2, S_4 \times Z_2)$,
- $(A_{96}, B_{96})$,
- $(A_{192}, B_{192})$,
- $(C_{192}, D_{192})$,
- $(A_{384}, B_{384})$.

Proposition 2.5 [17] Let $\Gamma$ be a connected cubic $X$-semisymmetric graph for some $X \leq \text{Aut}(\Gamma)$; then either $\Gamma \cong K_{3,3}$, the complete bipartite graph on 6 vertices, or $X$ acts faithfully on each of the bipartition sets of $\Gamma$.

Theorem 2.6 [15] Let $\Gamma$ be a connected cubic $X$-semisymmetric graph. Let $\{U,W\}$ be a bipartition for $\Gamma$ and assume $N \leq X$. If the actions of $N$ on both $U$ and $W$ are intransitive, then $N$ acts semiregularly on both $U$ and $W$. $\Gamma N$ is $\frac{X}{N}$-semisymmetric, and $\Gamma$ is a regular $N$-covering of $\Gamma_N$.

This theorem has a nice result. For every normal subgroup $N \leq X$ either $N$ is transitive on at least one partite set or it is intransitive on both partite sets. In the former case, the order of $N$ is divisible by $|U| = |W|$. In the latter case, according to Theorem 2.6, the induced action of $N$ on both $U$ and $W$ is semiregular and hence the order of $N$ divides $|U| = |W|$. So we have the following handy corollary.

Corollary 2.7 If $\Gamma$ is a connected cubic $X$-semisymmetric graph with $\{U,W\}$ as a bipartition and $N \leq X$, then either $|N|$ divides $|U|$ or $|U|$ divides $|N|$.

Following [10](see also [16]) the coset graph $C(G;H_0,H_1)$ of a group $G$ with respect to finite subgroups $H_0$ and $H_1$ is a bipartite graph with $\{H_0g|g \in G\}$ and $\{H_1g|g \in G\}$ as its bipartition sets of vertices where $H_0g$ is adjacent to $H_1g'$ whenever $H_0g \cap H_1g' \neq \emptyset$. The following proposition may be extracted from [10]:

Proposition 2.8 Let $G$ be a finite group and $H_0, H_1 \leq G$. The coset graph $C(G;H_0,H_1)$ has the following properties:

(i) $C(G;H_0,H_1)$ is regular of valency $d$ if and only if $H_0 \cap H_1$ has index $d$ in both $H_0$ and $H_1$.

(ii) $C(G;H_0,H_1)$ is connected if and only if $G = \langle H_0, H_1 \rangle$.

(iii) $G$ acts on $C(G;H_0,H_1)$ by right multiplication. Moreover this action is faithful if and only if $\text{Core}_G(H_0 \cap H_1) = 1$.

(iv) In the case when the action of $G$ is faithful, the coset graph $C(G;H_0,H_1)$ is $G$-semisymmetric.

Proposition 2.9 [16] Let $\Gamma$ be a regular graph and $G \leq \text{Aut}(\Gamma)$. If $\Gamma$ is $G$-semisymmetric, then $\Gamma$ is isomorphic to the coset graph $C(G;G_u,G_v)$ where $u$ and $v$ are adjacent vertices.

3. Main Results

Our goal in this paper is to fully classify connected cubic semisymmetric graphs of order $20p$. We also derive a very restrictive necessary condition for the existence of connected cubic $G$-semisymmetric graphs of order $20p$. We prove the following important result. Part (i) is a full classification whereas part (ii) is only a necessary condition.
Theorem 3.1 Let $p$ be a prime.

(i) If $\Gamma$ is a connected cubic semisymmetric graph of order $20p$, then $p = 11$ and $\Gamma \simeq S_{220}$. 

(ii) If $\Gamma$ is a connected cubic $G$-semisymmetric graph of order $20p$ for some $G \leq \text{Aut}(\Gamma)$, then $p = 2$ or 11.

To prove the main theorem, we need some lemmas.

Lemma 3.2 The only simple $K_4$-groups whose orders are of the form $2^i \cdot 3 \cdot 5 \cdot p$ for some prime $p > 5$ and some $1 \leq i \leq 8$, are the following three projective special linear groups: $L_2(2^4)$, $L_2(11)$ and $L_2(31)$.

Proof Considering the powers of primes, there is no possibility for such a group in sub-item (4) of item (ii) of Theorem 2.1. By inspecting orders of groups in sub-item (1), the only group of the desired form is $L_2(2^4)$. As for sub-item (3), let $L_2(2^m)$ be a group of order $2^i \cdot 3 \cdot 5 \cdot p$; then

$$2^m \cdot 3 \cdot (2^m - 1) \cdot \left(\frac{2^m + 1}{3}\right) = 2^i \cdot 3 \cdot 5 \cdot p$$

where $m$, $2^m - 1$ and $\frac{2^m + 1}{3}$ are all primes according to Theorem 2.1. This equation has no answer as neither $2^m - 1$ nor $\frac{2^m + 1}{3}$ could be equal to 5. Finally consider groups $L_2(r)$ in sub-item (2). If for odd prime $r$ and for prime $s > 3$, we have $r^2 - 1 = 2^a \cdot 3^b \cdot s$ and

$$2^{a-1} \cdot 3^b \cdot s \cdot r = 2^i \cdot 3 \cdot 5 \cdot p,$$

then $b = 1$, $a - 1 = i$ and either $s = 5$ or $r = 5$. The equality $r = 5$ is not possible, since $L_2(5)$ is not a $K_4$-group. Also if $s = 5$, then the equation $r^2 - 1 = 2^a \cdot 3 \cdot 5$ gives us only two solutions $r = 11, 31$ when $a$ spans integers 2, 3, . . . , 9.

Lemma 3.3 Let $p > 11$ be a prime and $p \neq 17, 31$. If $\Gamma$ is a connected cubic $G$-semisymmetric graph of order $20p$, then $G$ has a normal Sylow $p$-subgroup.

Proof Take $\{U, W\}$ to be a bipartition for $\Gamma$. Then $|U| = |W| = 10p$. For $u \in U$ according to Theorem 2.4, $|G_u| = 2^r \cdot 3$ for some $0 \leq r \leq 7$. Due to transitivity of $G$ on $U$, the equality $|G : G_u| = |U|$ holds which yields $|G| = 2^{r+1} \cdot 3 \cdot 5 \cdot p$. If $G$ does not have a normal Sylow $p$-subgroup, then $O_p(G) = 1$. We derive a contradiction out of this.

Suppose $G$ has a normal subgroup $M$ of order 10. Due to its order, $M$ is intransitive on the partite sets and according to Theorem 2.6, the quotient graph $\Gamma_M$ is $G_M$-semisymmetric with a bipartition $\{U_M, W_M\}$ where $|U_M| = |W_M| = p$ and $|G_M| = 2^r \cdot 3 \cdot p$.

Let $\frac{K}{M}$ be a minimal normal subgroup of $G_M$. If $\frac{K}{M}$ is unsolvable, it must be simple of order $2^i \cdot 3 \cdot p$ for some $i$. So $\frac{K}{M} \simeq A_5$ or $L_2(7)$. However these are not possible since $p > 11$. Now if $\frac{K}{M}$ is solvable and hence elementary abelian, then by Corollary 2.7, its order must be $p$ implying $|K| = 10p$. The Sylow $p$-subgroup of $K$ is normal and hence characteristic in $K$. Therefore it is normal in $G$, contradicting the assumption that $O_p(G) = 1$. So $O_p(G) = 1$ implies that $G$ does not have a normal subgroup of order 10.

Next, let $N \simeq T^k$ be a minimal normal subgroup of $G$, where $T$ is simple. If $T$ is nonabelian, then $k = 1$ and $N = T$ since the powers of 3 and 5 in $|G|$ equal 1. According to Corollary 2.7, either $|N|$ divides $|U| = 10p$ or $10p$ divides $|N|$.
If \(|N|\) divides \(10p\), then \(|N| = 2 \cdot 5 \cdot p\), since \(|N|\) should be divisible by at least three distinct primes (Theorem 2.3). But there is no simple \(K_3\)-group of order \(2 \cdot 5 \cdot p\) according to part (i) of Theorem 2.1. So \(10p\) divides \(|N|\). Since the order of every simple \(K_3\)-group is divisible by 3, \(N\) must be a simple \(K_4\)-group whose order is of the form \(2^i \cdot 3 \cdot 5 \cdot p\). According to Lemma 3.2, \(N \simeq L_2(2^i); L_2(11)\) or \(L_2(31)\) corresponding to \(p = 11, 17, 31\) respectively. But these cases are ruled out in the statement of the Lemma.

Now suppose \(T\) is abelian and hence \(N\) would be elementary abelian. It follows from Corollary 2.7, that \(|N|\) divides \(10p\) and so \(|N| = 2, 5\) or \(p\). Certainly \(|N| = p\) contradicts the assumption on \(O_p(G)\). In the remaining two cases \(\Gamma_N\) would itself be a connected cubic \(\frac{G}{N}\)-semisymmetric graph of order \(\frac{20p}{|N|}\). Take \(\{U_N, W_N\}\) to be the bipartition for \(\Gamma_N\). Also let \(\frac{M}{N}\) be a minimal normal subgroup of \(\frac{G}{N}\).

If \(N \simeq Z_2\), then \(|\frac{G}{N}| = 2^i \cdot 3 \cdot 5 \cdot p\) and \(|U_N| = |W_N| = 5p\). If \(\frac{M}{N}\) is unsolvable, then it must be a simple \(K_4\)-group whose order is of the form \(2^i \cdot 3 \cdot 5 \cdot p\). It follows from Lemma 3.2, that \(p = 17, 11\) or \(31\) which are ruled out by our assumption on \(p\). On the other hand if \(\frac{M}{N}\) is solvable, then its order should divide \(|U_N| = 5p\) and hence \(|\frac{G}{N}| = 5\) or \(p\). If \(|\frac{M}{N}| = 5\), then \(|M| = 10\) which is not possible (as we showed at the beginning of the proof), and if \(|\frac{M}{N}| = p\), then \(|M| = 2p\) which contradicts our assumption on \(O_p(G)\) since a Sylow \(p\)-subgroup of \(M\) would be characteristic in \(M\) and so would be normal in \(G\).

Now if \(N \simeq Z_5\), then \(|\frac{G}{N}| = 2^{i+1} \cdot 3 \cdot p\) and \(|U_N| = |W_N| = 2p\). In this case if \(\frac{M}{N}\) is unsolvable, it would be a simple group of order \(2^i \cdot 3 \cdot p\) for some \(i\) and hence according to Theorem 2.1, \(\frac{M}{N}\) \(\simeq A_5\) or \(L_2(7)\) implying \(p = 5\) or \(7\). This is in contradiction to our assumption on \(p\). On the other hand if \(\frac{M}{N}\) is solvable, then like before, we conclude that \(|\frac{M}{N}| = 2\) or \(p\) which again lead to contradictions as in the previous case. □

**Lemma 3.4** Let \(p > 11\) be a prime and \(p \neq 17, 31\). Suppose \(\Gamma\) is a connected cubic \(G\)-semisymmetric graph of order \(20p\). Let \(M\) be the Sylow \(p\)-subgroup of \(G\). If \(\frac{G}{M} \simeq H\), then

1. For each vertex \(u\) the stabilizer \(G_u\) is isomorphic to a subgroup of \(H\).
2. \(G \simeq M \rtimes \phi H\) for some homomorphism \(\phi : H \to \text{Aut}(M)\).

**Proof** For each vertex \(u\) of \(\Gamma\), \(MG_u \leq G\). Therefore \(G_u \simeq \frac{G_u}{MG_u} \simeq \frac{MG_u}{M} \leq \frac{G}{M}\). This proves (1). Now since obviously the orders of \(M\) and \(\frac{G}{M}\) are coprime, it follows from Proposition 2.2, that \(G = MK\) for some subgroup \(K \leq G\) where \(M \cap K = 1\). So \(G\) is the internal semidirect product of \(M\) and \(K\) and hence it is isomorphic to the external semidirect product of \(M\) and \(K\); i.e. \(G \simeq M \rtimes \psi K\) for some \(\psi : K \to \text{Aut}(M)\). Since \(H \simeq \frac{G}{M} = \frac{MK}{M} \simeq \frac{K}{M \cap K} \simeq K\), we can write \(G \simeq M \rtimes \phi H\) for some \(\phi : H \to \text{Aut}(M)\). □

**Lemma 3.5** Let \(p > 11\) be a prime and \(p \neq 17, 31\). If \(\Gamma\) is a connected cubic \(G\)-semisymmetric graph of order \(20p\) and if \(M\) is the Sylow \(p\)-subgroup of \(G\), then \(\frac{G}{M}\) cannot be isomorphic to \(A_5\).

**Proof** Suppose on the contrary, that \(\frac{G}{M} \simeq A_5\). Then for any vertex \(u\) from \(|G : G_u| = 10p\) we obtain \(|G_u| = 6\) and hence \(G_u \simeq Z_6\) or \(S_3\). By Lemma 3.4, \(G_u \leq A_5\). Since \(A_5\) does not have elements of order 6, we conclude that \(G_u \simeq S_3\). Also according to Lemma 3.4, \(G \simeq M \rtimes \phi A_5\). There are only two possibilities for the kernel of \(\phi : A_5 \to \text{Aut}(M)\).
(a) If $\ker(\varphi) = 1$, then $\mathcal{A}_5$ is isomorphic to a subgroup of $\text{Aut}(M) \cong \text{Aut}(\mathbb{Z}_p) \cong \mathbb{Z}_{p-1}$ which is obviously not the case.

(b) If $\ker(\varphi) = \mathcal{A}_5$, then $\varphi$ is the trivial homomorphism and so $G \cong M \times \mathcal{A}_5$. Since $\Gamma$ is $G$-semisymmetric, according to Proposition 2.9, $\Gamma$ is isomorphic to $C(G; G_u, G_v)$ where $u$ and $v$ are two adjacent vertices in $\Gamma$. As $\Gamma$ is connected, according to Proposition 2.8, we must have $G = (G_u, G_v)$. In view of $G_u \cong G_v \cong S_3$, this means that $M \times \mathcal{A}_5$ is generated by two of its subgroups, say $H$ and $K$, both isomorphic to $S_3$. Now for each element $(m, a) \in H$ we have $(m, a)^6 = 1$ which means $m^6 = 1$ in $M$. As $|M| = p > 31$, we conclude $m = 1$. Therefore the first component of each element of $H$ (and similarly for $K$) equals 1. Consequently the first component of each element in $M \times \mathcal{A}_5 = \langle H, K \rangle$ equals 1 which is a contradiction. $\square$

Consider a semidirect product $\mathbb{Z}_p \rtimes_{\varphi} S_5$ where $\varphi: S_5 \to \text{Aut}(\mathbb{Z}_p)$ is a homomorphism. Let $H \leq \mathbb{Z}_p \rtimes_{\varphi} S_5$ and $H \cong D_{12}$ or $\mathcal{A}_4$. We call $H$ of type A if all the elements of $H$ have their second component in $\mathcal{A}_5$. We also call $H$ of type D if there is at least one element in $H$ whose second component is not in $\mathcal{A}_5$. Also for any $x \in \mathbb{Z}_p$ and any $g, h \in S_5$ we define two subsets $R_{x, g, h}, S_{x, g, h} \subset \mathbb{Z}_p \rtimes_{\varphi} S_5$ as follows:

$R_{x, g, h} = \{(1, 1), (x, g), (1, g^2), (x, g^3), (1, g^4), (x, g^5), (1, h), (x, hg), (1, hg^2), (x, hg^3), (1, hg^4), (x, hg^5)\}$

and

$S_{x, g, h} = \{(1, 1), (x, g), (1, g^2), (x, g^3), (1, g^4), (x, g^5), (x, h), (1, hg), (x, hg^2), (1, hg^3), (x, hg^4), (1, hg^5)\}$.

As we will see later, these two subsets are sometimes subgroups of $\mathbb{Z}_p \rtimes_{\varphi} S_5$.

The group $D_{12} = \langle a, b | a^6 = b^2 = 1, b^{-1}ab = a^{-1} \rangle$ has exactly three Sylow 2-subgroups, all isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$, which are listed below:

$P_1 = \{1, a^3, b, ba^3\}, P_2 = \{1, a^3, ba, ba^4\}, P_3 = \{1, a^3, ba^2, ba^5\}$.

**Lemma 3.6** Let $p > 3$ be a prime and let $\varphi: S_5 \to \text{Aut}(\mathbb{Z}_p)$ be a homomorphism where $\ker(\varphi) = \mathcal{A}_5$. Let $H \leq \mathbb{Z}_p \rtimes_{\varphi} S_5$.

(i) If $H \cong \mathcal{A}_4$ then $H$ is of type A and if $H \cong D_{12}$ then $H$ is of type D.

(ii) Moreover if $H \cong D_{12}$, then there are some $x \in \mathbb{Z}_p$, some $g, g' \notin \mathcal{A}_5$ and some $h \in \mathcal{A}_5$ where $H = R_{x, g, h}$ or $H = S_{x, g, g'}$.

**Proof** The image of $\varphi$ is isomorphic to $\frac{\mathcal{A}_5}{\mathcal{A}_5} \cong \mathbb{Z}_2$. So there is some $F \in \text{Aut}(\mathbb{Z}_p)$ of order 2 for which $\varphi(x) = 1$ for all $x \in \mathcal{A}_5$ and $\varphi(x) = F$ for any $x \notin \mathcal{A}_5$. For any two elements $(x, g)$ and $(y, h)$ from $\mathbb{Z}_p \rtimes_{\varphi} S_5$ the multiplication $(x, g)(y, h)$ equals $(xy, gh)$ if $g \notin \mathcal{A}_5$ and equals $(xF(y), gh)$ if $g \notin \mathcal{A}_5$. It is easy to see that for any positive integer $n$ if $g \in \mathcal{A}_5$, then $(x, g)^n = (x^n, g^n)$ for all $x \in \mathbb{Z}_p$, and if $g \notin \mathcal{A}_5$, then $(x, g)^{2n} = (x^nF(x^n), g^{2n})$ and $(x, g)^{2n+1} = (x^{n+1}F(x^n), g^{2n+1})$ for all $x \in \mathbb{Z}_p$.

Now to prove part (i), it suffices to prove that for a subgroup $H \leq \mathbb{Z}_p \rtimes_{\varphi} S_5$ which is isomorphic to either $D_{12}$ or $\mathcal{A}_4$, both of the following statements are true:

- if $H$ is of type A, then $H \cong \mathcal{A}_4$
- if $H$ is of type D, then $H \cong D_{12}$
Let $H \leq \mathbb{Z}_p \rtimes_{φ} S_5$ and $H \simeq D_{12}$ or $A_4$. If $K := \{(x, g) \in H | g \in A_5\}$ then $K \leq H$. For the homomorphism $f : K \to \mathbb{Z}_p$ defined by $f(x, g) = x$, the isomorphism $\frac{K}{\ker(f)} \simeq \text{Im}(f)$ implies that $|\frac{K}{\ker(f)}|$ divides both 12 and $p$ and hence $K = \ker(f)$. Therefore for each $(x, g) \in H$ if $g \in A_5$, then $x = 1$.

It follows immediately that if $H$ is of type A, then the first component of each element of $H$ equals 1 and hence $H$ is isomorphic to a subgroup of $A_5$. As $A_5$ has no element of order 6, $H$ cannot be isomorphic to $D_{12}$ and so $H \cong A_4$.

Now suppose $H$ is of type D. For two arbitrary elements $(x, g), (y, h) \in H$ with $g, h \notin A_5$, we have $(xF(x), g^2) = (x, g)^2 \in H$ and $(yF(x), hg) = (y, h)(x, g) \in H$. Since $g^2$ and $hg$ are in $A_5$, the first components must equal 1; i.e. $xF(x) = 1$ and $yF(x) = 1$ which imply $x = y$. In other words, for any pair of elements $(x, g) \in H$ and $(y, h) \in H$ with $g, h \notin A_5$ we must have $x = y$. There are always elements in $H$ whose second component lies in $A_5$ and hence their first component is 1. Therefore we can write

$$H = \{(x, g_1), (x, g_2), \ldots, (x, g_n), (1, h_1), \ldots, (1, h_m)\}$$

(3.1)

where $n + m = 12$ and where $g_1, \ldots, g_n \notin A_5$ and $h_1, \ldots, h_m \in A_5$. It also follows that for this specific $x$, $F(x) = x^{-1}$. Let

$$\overline{H} = \{(1, h_1), \ldots, (1, h_m)\}, H_1 = \{h_1, \ldots, h_m\};$$

then $H_1 \cong \overline{H}, \overline{H} \leq H$ and $H_1 \leq A_5$. Multiplying all the elements of $H$ from equation 3.1, by $(x, g_i)$ for an arbitrary $t$, we again obtain $H$. Therefore

$$H = \{(1, g_1), (1, g_2), \ldots, (1, g_n), (x, g_1h_1), \ldots, (x, g_th_m)\}.$$  

(3.2)

Comparing the equalities 3.1 and 3.2 and by taking into account that $g_ig_j \in A_5$ for $i = 1, \ldots, n$ and $g_ih_j \notin A_5$ for $j = 1, \ldots, m$, it follows that

$$\{g_1h_1, \ldots, g_th_m\} = \{g_1, \ldots, g_n\}.$$

Therefore $m = n = 6$ and so $|\overline{H}| = 6$. Since $A_4$ does not have a subgroup of order 6, we conclude that $H \cong D_{12}$.

We now proceed to prove part (ii). So let $D_{12} \simeq H \leq \mathbb{Z}_p \rtimes_{φ} S_5$. According to part (i) $H$ is of type D. We continue to use the notations invented in the proof of part (i). The group $D_{12}$ has only two subgroups of order 6, namely $Z_6$ and $S_3$. Since $\overline{H} \leq H$ and $|\overline{H}| = 6$, we have $\overline{H} \cong Z_6$ or $S_3$. Since $\overline{H} \cong H_1 \leq A_5$ and $A_5$ does not have elements of order 6, it follows that $\overline{H}$ cannot be isomorphic to $Z_6$ and hence $\overline{H} \cong S_3$. Also as $H \cong D_{12}$, we can write $H = \{a^i | i = 0, \ldots, 5\} \cup \{ba^i | i = 0, \ldots, 5\}$. As $\overline{H} \cong S_3$ does not have any element of order 6, we must have $a \in H - \overline{H}$; i.e. $a = (x, g)$ for some $g \notin A_5$ (see equation 3.1). As for $b$, there are two possible cases; either $b = (1, h) \in \overline{H}$ or $b = (x, g') \in H - \overline{H}$.

If $b = (1, h)$, $h \in A_5$, then

$$H = \{(x, g^i) | i = 0, \ldots, 5\} \cup \{(1, h)(x, g^i) | i = 0, \ldots, 5\} = \{(1, 1), (x, g), (1, g^2), (x, g^3), (1, g^4), (x, g^5), (1, h)(x, h), (1, h^2), (x, h^3), (1, h^4), (x, h^5)\} = R_{x,g,h}.$$  

Also if $b = (x, g')$, $g' \notin A_5$, then

$$H = \{(x, g') | i = 0, \ldots, 5\} \cup \{(1, h)(x, g') | i = 0, \ldots, 5\} = \{(1, 1), (x, g'), (1, g'^2), (x, g'^3), (1, g'^4), (x, g'^5)\} = R_{x,g',h}.$$
\[ H = \{(x, g)^{i}|i = 0, \ldots, 5\} \cup \{(x, g')(x, g)^{i}|i = 0, \ldots, 5\} = \{(1, 1), (x, g), (1, g^2), (x, g^3), (1, g^4), (x, g^5), (x, g'), (1, g'g^2), (x, g'g^3), (1, g'g^4), (x, g'g^5)\}, \]

\[ S_{x, g, g'}. \]

\[ \tag*{\square} \]

**Lemma 3.7** Let \( p > 3 \) be a prime. A semidirect product \( \mathbb{Z}_p \rtimes \phi \mathbb{S}_5 \) does not have two subgroups \( U \) and \( V \) with all the following properties:

1. \( (U, V) \cong (D_{12}, D_{12}) \) or \( (D_{12}, A_4) \); and
2. \( \mathbb{Z}_p \rtimes \phi \mathbb{S}_5 = (U, V) \); and
3. \( U \cap V \) is a common Sylow 2-subgroup of both \( U \) and \( V \).

**Proof** Let \( \phi : \mathbb{S}_5 \rightarrow \text{Aut}(\mathbb{Z}_p) \) be a homomorphism. The kernel of \( \phi \) could not be identity since otherwise \( \mathbb{S}_5 \)
would be isomorphic to a subgroup of \( \text{Aut}(\mathbb{Z}_p) \cong \mathbb{Z}_{p-1} \) which is impossible. On the other hand if \( \ker(\phi) = \mathbb{S}_5 \),
then \( \phi \) is the trivial homomorphism and so \( \mathbb{Z}_p \rtimes \phi \mathbb{S}_5 = \mathbb{Z}_p \times \mathbb{S}_5 \). For any two subgroups \( U, V \leq \mathbb{Z}_p \times \mathbb{S}_5 \) both
of order 12, the equality \((x, a)^{12} = 1\) holds for each \((x, a) \in U \cup V\). This implies \( x^{12} = 1 \) and hence \( x = 1 \),
since \( x \in \mathbb{Z}_p \). Consequently the equality \( \mathbb{Z}_p \times \mathbb{S}_5 = (U, V) \) cannot hold.

The only remaining possibility is to have \( \ker(\phi) = A_5 \). We assume there are subgroups \( U, V \) with the
desired properties and reach a contradiction. So \( U \cong D_{12} \) and hence according to Lemma 3.6, there are some
\( x \in \mathbb{Z}_p \), some \( g, k \notin A_5 \) and some \( h \in A_5 \) where \( U = R_{x, g, h} \) or \( U = S_{x, g, k} \). If \( U = R_{x, g, h} \), then all the Sylow
2-subgroups of \( U \) are as follows:

\[ RP^1_{x, g, h} = \{(1, 1), (x, g^3), (1, h), (x, hg^5)\}, \]
\[ RP^2_{x, g, h} = \{(1, 1), (x, g^3), (x, hg), (1, hg^4)\}, \]
\[ RP^3_{x, g, h} = \{(1, 1), (x, g^3), (1, hg^2), (x, hg^5)\} \]

and if \( U = S_{x, g, k} \), then all the Sylow 2-subgroups of \( U \) are as follows:

\[ SP^1_{x, g, k} = \{(1, 1), (x, g^3), (x, k), (1, kg^3)\}, \]
\[ SP^2_{x, g, k} = \{(1, 1), (x, g^3), (1, kg), (x, kg^4)\}, \]
\[ SP^3_{x, g, k} = \{(1, 1), (x, g^3), (x, kg^2), (1, kg^5)\}. \]

For some \( i \) either \( RP^i_{x, g, h} \) or \( SP^i_{x, g, k} \) must also be a Sylow 2-subgroup of \( V \). If \( V \cong A_4 \), then according to
Lemma 3.6, it is of type A and hence the first components of all the elements of each of its Sylow 2-subgroups
equal 1. However there are elements in \( RP^i_{x, g, h} \) and in \( SP^i_{x, g, k} \) whose first components are equal to \( x \). So if
\( V \cong A_4 \), then \( x = 1 \). Every element of \( \langle U, V \rangle \) is an alternating product of elements from \( U \) and \( V \). Since
in the semidirect product we have \( (1, t)(1, s) = (1, ts) \) for any \( t, s \in \mathbb{S}_5 \), it follows that the first component of
every element from \( \langle U, V \rangle \) is 1 and hence \( \langle U, V \rangle \neq \mathbb{Z}_p \rtimes \phi \mathbb{S}_5 \).

On the other hand if \( V \cong D_{12} \), then according to Lemma 3.6, either \( V = R_{y, g', h'} \) or \( V = S_{y, g', k'} \) for
some \( y \in \mathbb{Z}_p \), some \( g', h' \notin A_5 \) and some \( h' \in A_5 \). Again all the sylow 2-subgroups of \( V \) are known. The first
component of each element from any sylow 2-subgroup of \( U \) is 1 or \( x \) and the first component of each element
from any sylow 2-subgroup of \( V \) is 1 or \( y \). Since \( U \) and \( V \) have at least one common Sylow 2-subgroup
(namely \( U \cap V \)), we must have \( x = y \).
Now define \( W = (\{1\} \times \mathbb{A}_3) \cup (\{x\} \times (\mathbb{S}_5 - \mathbb{A}_5)) \). It is easy to check that \( W \leq \mathbb{Z}_p \rtimes \mathbb{S}_5 \). Obviously \( U \cup V \subset W \), and so \( \langle U, V \rangle \leq W \). Therefore \( \langle U, V \rangle \neq \mathbb{Z}_p \rtimes \mathbb{S}_5 \). \( \square \)

**Proof of Theorem 3.1.** We first make a general discussion on \( G \)-semisymmetric graphs. Let \( \Gamma \) be a connected cubic \( G \)-semisymmetric graph of order \( n \). Then \( \Gamma \) is regular and bipartite. Moreover it is \( G \)-edge-transitive and hence edge-transitive. Now if \( \Gamma \) is not vertex-transitive, then by definition it is semisymmetric cubic of order \( n \). On the other hand if \( \Gamma \) is vertex-transitive, then it is symmetric cubic of order \( n \), since according to [22] a cubic vertex- and edge-transitive graph is necessarily symmetric. Therefore \( \Gamma \) is either a bipartite cubic symmetric graph of order \( n \) or it is a cubic semisymmetric graph of order \( n \).

We now set off to prove part (ii) of Theorem 3.1. For \( p = 3, 5, 7, 17, 31 \) there is no connected cubic semisymmetric graph of order \( 20p \) according to [5]. Also for \( p = 5, 7, 17 \) no connected cubic symmetric graph of order \( 20p \) exists according to [6]. As for \( p = 31 \), according to [6] there exists only one connected cubic symmetric graph of order \( 20p \) which is not bipartite. Therefore we conclude that for \( p = 3, 5, 7, 17, 31 \) there is no connected cubic \( G \)-semisymmetric graph of order \( 20p \).

Now let \( p > 11 \) be a prime such that \( p \neq 17, 31 \). Suppose on the contrary that \( \Gamma \) is a connected cubic \( G \)-semisymmetric graph of order \( 20p \) for some \( G \leq \text{Aut}(\Gamma) \). Let \( \{U, W\} \) be the bipartition for \( \Gamma \). Then \( |U| = |W| = 10p \) and \( |G| = 2^{r+1} \cdot 3 \cdot 5 \cdot p \) for some \( 0 \leq r \leq 7 \). If \( M \) is a Sylow \( p \)-subgroup of \( G \), then according to Lemma 3.3, \( M \leq G \). Due to its order, \( M \) is intransitive on both \( U \) and \( W \) and so according to Theorem 2.6, \( \Gamma_M \) is a connected cubic \( G_M \)-semisymmetric graph of order \( 20 \) with the bipartition \( \{U_M, W_M\} \), where \( G_M \simeq \frac{G}{M} \) and \( |U_M| = |W_M| = 10 \). According to the general discussion we just made, \( \Gamma_M \) is either a bipartite cubic symmetric graph or a cubic semisymmetric graph of order 20. By [5] there is no semisymmetric cubic graph of order 20 and by [6] there is only one bipartite symmetric cubic graph of order 20, namely \( \text{F}20\text{B} \). Therefore \( \Gamma_M \simeq \text{F}20\text{B} \).

The automorphism group of \( \text{F}20\text{B} \) has 240 elements ([6]) and \( G_M \) is isomorphic to a subgroup of \( \text{Aut}(\text{F}20\text{B}) \) of order \( |G_M| = 2^{r+1} \cdot 3 \cdot 5 \). The equality is not possible since \( G_M \) is not transitive on \( V(\text{F}20\text{B}) \) whereas \( \text{Aut}(\text{F}20\text{B}) \) is. So \( |G_M| < 240 \) and hence \( 1 \leq r + 1 \leq 3 \). Also \( G_M \) is transitive on both \( U_M \) and \( W_M \) and according to Proposition 2.5, the action of \( G_M \) on each of \( U_M \) and \( W_M \) is faithful. Therefore \( G_M \) is a transitive permutation group of degree 10. Transitive permutation groups of degree 10 have been completely classified in [4]. There are 45 such groups up to isomorphism which are denoted \( T1, T2, \ldots, T45 \) in [4] and the only ones whose orders are of the form \( 2^i \cdot 3 \cdot 5 \) for \( 1 \leq i \leq 3 \), are \( T7 \simeq \mathbb{A}_5 \) of order 60, and \( T11, T12 \) and \( T13 \simeq \mathbb{S}_5 \) of order 120.

First note that \( G_M \simeq T7 \) is not possible according to Lemma 3.5. Next, we argue that \( G_M \) could not be isomorphic to \( T11 \) or \( T12 \).

In [4] all the transitive groups of degree 10 are defined with a set of generating permutations on ten points. If

\[
a = (1, 2, 3, 4, 5), \quad b = (6, 7, 8, 9, 10), \quad e = (1, 5)(2, 3), \quad f = (6, 10)(7, 8), \\
g = (1, 2), \quad h = (6, 7) \quad \text{and} \quad i = (1, 6)(2, 7)(3, 8)(4, 9)(5, 10),
\]

then \( T11 = \langle a, b, ef, i \rangle \) and \( T12 = \langle a, b, ef, ghi \rangle \). Using the GAP software ([23]) it is easy to verify that \( H = \langle i \rangle \) of order 2 is a normal subgroup of \( T11 \).

If \( G_M \simeq T11 \), then according to Theorem 2.6, the quotient graph of \( \Gamma_M \) with respect to \( H \) which we denote by \( (\Gamma_M)_H \), would be \( R \)-semisymmetric of order 10, where \( R \simeq \frac{T11}{H} \). This implies that \( R \) is transitive
on each partite set and by Proposition 2.5, \( R \) would be a transitive permutation group of degree 5. Again according to [4] the only transitive permutation group of degree 5 and of order 60 is \( \mathbb{A}_5 \). So we should have \( R \cong \mathbb{A}_5 \). Now the stabilizer of any vertex of \((\Gamma_M)_H\) under the action of \( R \) has \( |R|/5 = 12 \) points and the only subgroup of \( \mathbb{A}_5 \) of order 12 is isomorphic to \( \mathbb{A}_4 \). So for an edge \( \{u, w\} \) of the cubic \( R \)-semisymmetric graph \((\Gamma_M)_H\), we have \((R_u, R_w) = (\mathbb{A}_4, \mathbb{A}_4)\) which is not possible according to Theorem 2.4. Therefore the assumption that \( G_M \cong T_{11} \), leads to a contradiction.

Now suppose \( G_M \cong T_{12} \). Calculated by GAP, the stabilizer of 1 under \( T_{12} \) is

\[
(T_{12})_1 = \langle (2,4)(3,5)(7,9)(8,10), (3,5,4)(8,10,9) \rangle
\]

Again, using GAP one finds out that this group is nonabelian of order 12 which has the following group as a normal subgroup:

\[
\langle (2,3)(4,5)(7,8)(9,10), (2,4)(3,5)(7,9)(8,10) \rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2
\]

There are only 3 nonabelian groups of order 12 up to isomorphism: \( \mathbb{A}_4 \), \( D_{12} \) and the dicyclic group of order 12. Among these, only \( \mathbb{A}_4 \) has a normal subgroup of order 4. So \((T_{12})_i \cong (T_{12})_1 \cong \mathbb{A}_4\) for any \( i = 1, 2, \ldots 10 \). However this is impossible by Theorem 2.4.

Finally suppose \( G_M \cong \frac{G}{M} \cong T_{13} \). Since \( M \cong \mathbb{Z}_p \), by Lemma 3.4, \( G \cong \mathbb{Z}_p \rtimes S_5 \) for some homomorphism \( \phi : S_5 \rightarrow \text{Aut}(\mathbb{Z}_p) \). From \( |G : G_u| = 10p \) we have \( |G_u| = 12 \) for any vertex \( u \). So if \( \{u, v\} \) is a fixed edge of \( \Gamma \), then it follows from Theorem 2.4, that \((G_u, G_v) \cong (D_{12}, D_{12})\) or \((D_{12}, \mathbb{A}_4)\). Of course \((G_u, G_v) \cong (\mathbb{A}_4, D_{12})\) is nothing new, since then we can change the roles of \( u \) and \( v \).

Also \( \Gamma \cong C(G; G_u, G_v) \) by Proposition 2.9. Now it follows from part \((ii)\) of Proposition 2.8, that \( G = \langle G_u, G_v \rangle \) and from part \((i)\) of the same Proposition that \( |G_u \cap G_v| = 4 \); i.e. \( G_u \cap G_v \) is a common Sylow 2-subgroup of \( G_u \) and \( G_v \). But the existence of \( G_u \) and \( G_v \) with all these properties contradicts Lemma 3.7.

Since every case for \( G_M \) is contradictory, part \((ii)\) follows.

Next, we turn to part \((i)\) of Theorem 3.1. For \( p \neq 2, 11 \) there is no connected cubic semisymmetric graph of order \( 20p \) according to part \((ii)\). Also there is no such graph of order \( 20 \times 2 \) according to [5] and by the same reference, there is only one connected cubic semisymmetric graph of order \( 20 \times 11 \), namely \( S_{220} \).

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