A Study of $U(N)$ Lattice Gauge Theory in 2-dimensions

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Abstract

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A Study of $U(N)$ Lattice Gauge Theory in 2-dimensions*

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Abstract

The $U(N)$ lattice gauge theory in 2-dimensions can be considered as the statistical mechanics of a Coulomb gas on a circle in a constant electric field. The large $N$ limit of this system is discussed and compared with exact answers for finite $N$. Near the fixed points of the renormalization group and especially in the critical region where one can define a continuum theory, computations in the thermodynamic limit ($N \to \infty$) are in remarkable agreement with those for finite and small $N$. However, in the intermediate coupling region the thermodynamic computation, unlike the one for finite $N$, shows a continuous phase transition. This transition seems to be a pathology of the infinite $N$ limit and in this simple model has no bearing on the physical continuum limit.

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Introduction

The $1/N$ expansion has proved very useful in studying the physical spectrum of $N$ component spin systems ($\sigma$-models) and several other tractable field theories with large dimension of internal symmetry group, in two dim.\textsuperscript{1} It is used on the assumption that the qualitative, and, in certain cases, even quantitative character of the spectrum is $N$ independent. Then taking the large $N$ limit enormously simplifies the computation since one can use a combination of a steepest descent and mean field approach in which each component of the spin interacts with the mean field of the other components.

For $SU(N)$ color gauge theories of the strong interaction, the $1/N$ expansion was introduced by ‘t Hooft\textsuperscript{2} who showed that in the Feynman graph expansion of the theory to each order in $g^2N$ (fixed as $N \to \infty$), the leading contribution comes from planar diagrams. Diagrams with holes (quark loops) and handles are suppressed by factors of $1/N$ and $1/N^2$ respectively. In this sense, the expansion in $1/N$ is analogous to the topological expansion of the dual model and as $N \to \infty$ the mesonic and gluonic states appear as free particle states. Witten\textsuperscript{3} has recently incorporated baryons as solitons into this scheme and has enumerated and emphasized the qualitative agreement between simple and basic aspects of hadronic spectroscopy and the $1/N$ expansion. However, the computational difficulties of planar QCD remain intractable in the diagrammatic language of its formulation, though some remarkable progress in the counting of planar diagrams for quartic and cubic vertices has been made by Brézin et al., using a WKB approach in $1/N$\textsuperscript{4}.

Now in the statistical mechanics of $N$ component spin systems the relation of the large $N$ limit to a mean field computation has received precise mathematical formulation by the introduction of a Lagrange multiplier as an auxiliary field. It is desirable to have an analogous formulation in the case of QCD, which avoids direct encounters with the diagram technique. Further, it is desirable to evaluate the relevance of the large $N$ limit. The question is very simple: to what extent and in which domains of coupling is a thermodynamic computation in which $N \to \infty$, relevant for the case of finite $N$.

We present in this paper a study of $U(N)$ lattice gauge theories in 2-dimensions to clarify these questions. Our method follows studies in the statistical theories of spectra and ref. (4).
In two dimensions, since plaquette variables are the independent ones, the partition function reduces to that of a random unitary matrix with $N^2$ degrees of freedom. This in turn is the partition function of a Coulomb gas on a circle in a constant electric field $E = 2\beta/N$. The problem of computing the free energy and other thermodynamic quantities as $N \to \infty$ reduces to the solution of a singular integral equation for the charge density, with a Cauchy kernel. This problem is always exactly soluble. The solution depends on the strength of the electric field $E$. For $0 \leq E \leq 1$ there is no gap in the charge distribution and the charge density is conjugate to the electric field. For $1 \leq E < \infty$ the charge distribution has a gap and the problem reduces to the Riemann-Hilbert problem for a simple arc in the plane. The appearance of a gap in the charge density signals a continuous phase transition at $E = 1$. An order parameter that measures the randomness of the system has a constant value in the ‘strong coupling’ phase $0 \leq E \leq 1$ and goes to zero for large $E$ in the spin wave phase. The string constant is computed and a simple renormalization group used to discuss the physically relevant continuum limit.

The Coulomb gas partition function is also a Toeplitz determinant involving Bessel functions. For finite $N$ it is obviously an analytic function of the temperature $\beta$. Taylor expansions around the fixed points $\beta = 0$ and $\beta = \infty$ indicate an excellent agreement with the computation for $N \to \infty$. To us, this agreement is far from self-evident. Near $\beta = \infty$, this can be seen to result from a scaling argument.

An interesting mathematical corollary is that the problem of evaluating Toeplitz determinant of order $N$, can always be mapped to the problem of a Coulomb gas on a circle, which in the $N = \infty$ limit is always exactly soluble. Toeplitz determinants are encountered in the Ising model and their evaluation is usually done by using Szegö’s theorem (c.f. ref. (11)).
I. Equivalence of $U(N)$ Gauge Theory in 2-dimensions to the Statistical Mechanics of a 1-dimensional Coulomb gas

The partition function of the $U(N)$ lattice gauge theory is defined by

$$Z(V, N, \beta) = \int d\mu \exp \left[ \beta \sum_P (U(P) - N) \right]$$

(1)

$d\mu$ is a real measure over the gauge group elements at each link; the summation over $P$ is over all oriented plaquettes and $U(P)$ is a product of group elements, in the fundamental representation, at the 4 oriented links of $P$. $U^+(P)$ would be the group element associated with the links in the opposite direction. \( \beta = \frac{1}{g_0^2} \) is the temperature and $V$ is the volume of the system in lattice units.

In two space-time dimensions it is possible to treat plaquette variables as independent.\(^6\) A simple way to see this is to fix the axial gauge in $Z$: all time-like links are chosen to be one and all space-like links at one particular time ‘$t_0$’ are chosen to be one. Since the time links are trivial the space-like links at time ‘$t$’ can be expressed as a unique product of a time ordered string of independent plaquette variables, the expression for $Z$ in (1) becomes an integral over a single unitary matrix

$$Z = \left[ \int dU e^{\beta[TrU + TrU^+]} \right]^V e^{-2\beta NV}$$

Hence, the problem is reduced to the computation of the partition function of a random unitary matrix with $N^2$ real independent variables

$$z = \int dU e^{\beta(TrU + TrU^+)}$$

(2)

Integration over the group $U(N)$ is known.\(^7\) It is convenient to make a separation of variables into group invariant and non-invariant parameters. Let $R$ be the unitary matrix that brings $U$ to diagonal form

$$U = R^+ DR, \quad D_{ij} = \delta_{ij} e^{i\theta_j}$$

$R$ is arbitrary up to an invariant subgroup of $U(N)$, hence, after appropriate choice of gauge it depends on $N^2 - N$ parameters and

$$dU = C_N dR \prod_{i=1}^N \frac{d\theta_i}{2\pi} \prod_{i<j} |e^{i\theta_i} - e^{i\theta_j}|^2$$

(3)
Substituting (3) in (2) we get

\[
z = \frac{1}{N!} \int_0^{2\pi} \prod_{i=1}^N \frac{d\theta_i}{2\pi} e^{N^2 \left[ \frac{1}{N} \sum_{i=1}^N \cos \theta_i + 2 \sum_{i<j} \frac{1}{N^2} \log | e^{i\theta_i} - e^{i\theta_j} | \right]}
\]

The factor \(N!\) appeared by choice of \(C_N\) in (3) such that \(z = 1\) for \(\beta = 0\).

Now define an action

\[-S = \frac{\beta}{N} \sum_{i=1}^N \frac{1}{N} \cos \theta_i + \sum_{i<j} \frac{1}{N^2} \log \left| e^{i\theta_i} - e^{i\theta_j} \right|\]

then

\[z = \frac{1}{N!} \int_0^{2\pi} \prod_{i=1}^N \frac{d\theta_i}{2\pi} e^{-2SN^2}\]

The action \(S = o(1)\) in \(N\) provided \(\beta/N = E/2\) is a fixed parameter of the theory. If so, then for large \(N\) we can consider evaluating \(z\) by a WKB expansion in the small parameter \(1/N^2\).

The action \(S\) in (5) is essentially the electrostatic potential energy of \(N\) charges of strength \(1/N\) distributed on a unit circle in the plane, in the presence of a constant electric field \(E/2\) along the \(x\)-axis. The repulsive logarithmic interaction tends to distribute the charge uniformly about the circle and the electric field tends to drive all \(\theta_i \to 0\). There is another analogy with spin systems. Define a unit spin vector

\[
\vec{\sigma}_i = (\cos \theta_i, \sin \theta_i)
\]

then

\[-S = \frac{H}{2} \sum_{i=1}^N \frac{1}{N} \sigma_i^x + \sum_{i<j} \frac{1}{N^2} \log (1 - \vec{\sigma}_i \cdot \vec{\sigma}_j)\]

which is the potential energy of unit spins equally spaced along the \(x\)-axis in a constant magnetic field \(H/2 = \beta/N\). The logarithm represents a continuous version of an anti-ferromagnetic coupling. In the absence of \(H\), \(\langle \vec{\sigma}_n \rangle = \left( \cos \frac{2\pi n}{N}, \sin \frac{2\pi n}{N} \right)\). In the rest of this paper we shall only consider the electrostatic analogy.
II. WKB and Mean Field Calculation

If we define the free energy per degree of freedom \( f(E) \) by \( z = \exp(N^2 f(E)) \), then we have the WKB expansion in \( 1/N^2 \)

\[
f(E) = -2S_0 + \frac{S_1}{N^2} + 0 \left( \frac{1}{N^4} \right)
\]

(7)

\( S_0 \) is the minimum value of the potential energy \( S \) and \( S_1 \) is the result of gaussian fluctuations around the most probable configuration that minimizes \( S \). For the present, we shall be concerned with evaluating \( F(E) \) as \( N \to \infty \).

As we have noted, the point charges on the unit circle carry a charge \( = 1/N \). The interaction energy between two charges is \( 0 \left( \frac{1}{N^2} \right) \), which means that they are very weakly coupled. However, the interaction energy of any one charge in the mean field of the other \( N^2 - N - 1 \approx N^2 \) charges is \( 0(1) \). This motivates a mean field theory type computation in which one introduces a macroscopic charge density \( u(\theta) \geq 0 \), on the unit circle, such that \( u(\theta)d\theta/2\pi \) is the fraction of the number of charges between \( \theta \) and \( \theta + d\theta \). By definition

\[
\int_0^{2\pi} u(\theta) \frac{d\theta}{2\pi} = 1
\]

(8)

In this continuum limit as \( N \to \infty \) we have

\[
-S[u] = \frac{E}{2} \int_0^{2\pi} \frac{d\theta}{2\pi} u(\theta) \cos \theta + P \int_0^{2\pi} \frac{d\theta}{2\pi} \int_0^{2\pi} \frac{d\phi}{2\pi} u(\theta) \log |e^{i\theta} - e^{i\phi}| u(\phi)
\]

(9)

\( P \) stands for the Cauchy principal value.

The most probable charge distribution is a stationary point of (9) subject to the normalization (8). Hence, this distribution satisfies the following singular integral equation

\[
\frac{E}{2} \cos \phi + P \int_0^{2\pi} \frac{d\theta}{2\pi} u(\theta) \log |e^{i\theta} - e^{i\phi}| = \lambda
\]

\( \lambda \) is a lagrange multiplier corresponding to (8). It represents the average potential at \( \phi \) due to all the other charges, and it is \( \phi \) independent. Taking \( \phi \) derivatives we have

\[
E \sin \phi = -P \int_0^{2\pi} \frac{d\theta}{2\pi} u(\theta) \cot \left( \frac{\theta - \phi}{2} \right)
\]

(10)
which states that the tangential forces on the charge at $\phi$ due to all the other charges balance the force of the electric field.

**III. Solution of Integral Equation**

Let us begin by assuming that the function $u(\theta)$ is smooth in the sense that $u'(\theta)$ is continuous for $0 < \theta < 2\pi$. The solution to (10) can be immediately written down if we recall the Hilbert transform formula for the circle:

If $\Phi(Z)$ is analytic for $|Z| \leq 1$, then on the unit circle we have

$$\Phi(e^{i\theta}) = \Phi(Z = 0) + \frac{P}{2\pi i} \int_0^{2\pi} d\theta \Phi(e^{i\theta}) \cot \left( \frac{\theta - \phi}{2} \right)$$

The choice $\Phi(Z) = 1 + EZ$ gives the correctly normalized solution

$$u(\theta) = 1 + E \cos \theta \quad (11)$$

The condition $u(\theta) \geq 0$ for $0 \leq \theta \leq 2\pi$, however, requires $0 \leq E \leq 1$.

When $E = 0$, $u(\theta) = 1$ as expected; turning on the electric field along the $x$-direction diminishes the density around $\theta = \pi$ and produces a crouching of charge near $\theta = 0$. Also $u(\theta) = u(-\theta)$. Note that $u(\theta)$ is smooth and never vanishes for $E < 1$. At $E = 1$ the distribution hits a zero at the single point $\theta = \pi$. This gives us a hint that as $E$ starts exceeding the critical value 1, there would start appearing a gap in the charge distribution around $\theta = \pi$ which would increase as a continuous and monotonic function of $E$, $1 < E < \infty$. In the following we shall establish this picture by solving the integral equation (10) with the boundary condition that $u(\theta) = 0$ outside the interval $-\alpha < \theta < \alpha$. For $\alpha \neq \pi$ we do not assume $u(\theta)$ to be smooth at the end points $\theta = \alpha$. The integral equation now becomes

$$E \sin \phi = -P \int_{-\alpha}^\alpha d\theta \frac{2\pi}{2\pi} u(\theta) \cot \left( \frac{\theta - \phi}{2} \right)$$

$$\int_{-\alpha}^\alpha \frac{d\theta}{2\pi} u(\theta) = 1 \quad (12)$$

To solve (12) we introduce complex notation $t = e^{i\theta}$ and $t_0 = e^{i\phi}$. (12) becomes

$$E(t_0 - t_0^{-1}) = \frac{P}{\pi i} \int_{t_1}^{t_2} dt \frac{t + t_0}{t - t_0} u(t)$$
Now the important point is that the normalization condition enables us to write the integral equation in the form

$$E(t_0 - t_0^{-1}) = \frac{P}{\pi i} \int_{t_1}^{t_2} \frac{dt}{t - t_0} 2u(t) - 1$$

The kernel is Cauchy type and we have adapted the general formalism to solve such equations to our problem. One starts out by considering the function

$$\Phi(z) = \frac{1}{2\pi i} \int_{t_1}^{t_2} \frac{dt}{t - z} u(t)$$

analytic in the plane cut by the \((t_1, t_2)\). The direction of the arc is assumed counterclockwise. Denote the boundary values of this function as \(z\) approaches a point \(t\) on the arc \((t \neq t_1, t_2)\), from the left and right by \(\Phi^+(t)\) and \(\Phi^-(t)\) respectively. Then by the Plemelj formulae\(^8\) for the boundary values of analytic functions we have

$$\Phi^+(t) - \Phi^-(t) = 2u(t)$$

$$\Phi^+(t) + \Phi^-(t) = \frac{P}{\pi i} \int_{t_1}^{t_2} \frac{d\xi}{\xi} u(\xi) \frac{\xi + t}{\xi - t} = E(t - t^{-1})$$

Equation (16) specifies the Reimann-Hilbert problem for the arc \((t_1, t_2)\): to find an analytic function in the cut plane whose boundary values satisfy (16). In our case we have an additional condition to satisfy at \(z = \infty\) which follows from the normalization of \(u(t)\),

$$\Phi(\infty) = -1$$

The discontinuity formula (15) (a reflection of Gauss’ law in electro-statics) implies that a solution to the Reimann-Hilbert problem solves the integral equation (13).

The function \(h(z) = [(z - t_1)(z - t_2)]^{1/2}\) clearly solves the homogeneous problem \(\phi^+(t) + \phi^-(t) = 0\) up to an entire function. This choice of the square root function vanishing at the end points leads to \(u(t_1) = u(t_2) = 0\). Now write

$$\Phi(z) = h(z)H(z)$$
substitution in (16) gives

\[ H^+(t) - H^-(t) = \frac{E(t - t^{-1})}{\sqrt{(t-t_1)(t-t_2)}} \]  

(18)

(By the square root in the discontinuity formula for \( H(z) \) we will mean
\[ h^+(t) = \exp \frac{i\theta}{2} \left[ \sin^2 \frac{\alpha}{2} - \sin^2 \frac{\theta}{2} \right]^{1/2} \].) From (17) we know that \( H(\infty) = 0 \). Hence \( H(z) \) is uniquely given by the Plemelj formulae

\[ H(z) = \frac{1}{2\pi i} \int_{t_1}^{t_2} \frac{E(t - t^{-1})}{\sqrt{(t-t_1)(t-t_2)}} \frac{dt}{t - z} \]

This integral can be evaluated by standard methods. Note

\[ H(z) = \frac{1}{4\pi i} \int_{\Gamma} \frac{E(t - t^{-1})}{\sqrt{(t-t_1)(t-t_2)}} \frac{dt}{t - z} \]

where \( \Gamma \) encloses the cut \((t_1, t_2)\) and \( z \) lies outside it. This is always possible since \( z \notin (t_1, t_2) \). \( H(z) \) has poles at \( t = 0 \) and \( t = z \) with residues \( E/2z \) and \( E(z - z^{-1})/2[(z - t_1)(z - t_2)]^{1/2} \) respectively. The contribution from infinity is \( E/2 \) where we have taken the branch \( \sqrt{(z-t_1)(z-t_2)} \to -z \) as \( z \to \infty \). Hence

\[ H(z) = \frac{E}{2} \left[ \frac{z - z^{-1}}{\sqrt{(z-t_1)(z-t_2)}} + 1 + \frac{1}{2} \right] \]

and

\[ \Phi(z) = \frac{E}{2} \left( z + \frac{1}{z} \right) + \frac{E}{2} \left( \frac{1}{z} + 1 \right) \sqrt{(z-t_1)(z-t_2)} \]

\( \Phi(\infty) = -1 \) implies

\[ \frac{1 - \cos \alpha}{2} = \frac{1}{E} = \sin^2 \frac{\alpha}{2}, \quad 1 \leq E < \infty \]

From the discontinuity formula (15) we get

\[ u(t) = \frac{E}{2} \left( \frac{1}{t} + 1 \right) h^+(t) \]
\[ u(\theta) = 2E \cos \frac{\theta}{2} \sqrt{E^{-1} - \sin^2 \frac{\theta}{2}} \geq 0 \]

We summarize the computation

\[ u(\theta) = \begin{cases} 
1 + E \cos \theta & 0 \leq E \leq 1 \\
2E \cos \frac{\theta}{2} \sqrt{E^{-1} - \sin^2 \frac{\theta}{2}} & 1 \leq E < \infty 
\end{cases} \tag{19a} \]

\[ u(\theta) = \begin{cases} 
1 + E \cos \theta & 0 \leq E \leq 1 \\
2E \cos \frac{\theta}{2} \sqrt{E^{-1} - \sin^2 \frac{\theta}{2}} & 1 \leq E < \infty 
\end{cases} \tag{19b} \]

Our expectation for the appearance of a gap at \( \theta = \pi \) beyond \( E = 1 \) is confirmed; also the size of the gap \( 2\alpha \) is a continuous and monotonic function of \( E \): \( 1/E = \sin^2 \alpha /2 \).

**IV. Computation of Thermodynamic Quantities in the \( N \rightarrow \infty \) Limit**

The appearance of the gap at \( E = 1 \) is reflected in the behavior of the thermodynamic functions. From (7) and (9) we have for the free energy per degree of freedom

\[ f(E) = \begin{cases} 
\frac{E^2}{4} & 0 \leq E \leq 1 \\
E - \frac{1}{2} \log E - \frac{3}{4} & 1 \leq E < \infty 
\end{cases} \tag{20a} \]

\[ f(E) = \begin{cases} 
\frac{E^2}{4} & 0 \leq E \leq 1 \\
E - \frac{1}{2} \log E - \frac{3}{4} & 1 \leq E < \infty 
\end{cases} \tag{20b} \]

\( f(E) \) and its first two derivatives are continuous at \( E = 1 \), indicating a continuous phase transition at the point \( E = 1 \).

A useful order parameter\(^9\) that characterizes the phases is

\[ R = \sum_{i=1}^{N} \frac{1}{N} \sin^2 \theta_i = \frac{1}{N} \text{Tr} \left( \frac{U - U^+}{2i} \right)^2 \tag{21} \]

In the continuum approximation as \( N \rightarrow \infty \) we have

\[ R(E) = \lim_{N \rightarrow \infty} \left( \frac{1}{N} \sum_{i=1}^{N} \sin^2 \theta_i \right) = \int_{0}^{2\pi} \frac{d\theta}{2\pi} u(\theta) \sin^2 \theta \tag{22} \]

Therefore

\[ R(E) = \begin{cases} 
1/2 & 0 \leq E \leq 1 \\
\frac{1}{E} - \frac{1}{2E^2} & 1 \leq E < \infty 
\end{cases} \tag{23} \]
$R$ represents the average value of the sum of the squares of the $y$-co-ordinates of the charges. $R(E)$ and $R'(E)$ are continuous at $E = 1$. In a sense, $R$ is a disorder parameter because it is a measure of the randomness of the system. In the phase $0 \leq E \leq 1$ where its expectation value is a constant, the distribution of eigenvalues of the unitary matrix is truly random and the eigenvalues go over their entire range. We shall call this phase the random phase (RP). In the phase $1 \leq E < \infty$ we see that for $E \gg 1$, $\langle R \rangle \approx 0$; the random distribution has continuously disappeared and the eigenvalues are confined to a small region around $U = 1$. Also the eigenvalue density (196) for $E \gg 1$ is written in scaled form

$$ u(\theta) \approx 2\sqrt{E} \sqrt{1 - \frac{(\sqrt{E}\theta)^2}{4}}, \quad |\theta| \leq \frac{2}{\sqrt{E}} $$

We call this phase the familiar spin wave phase (SWP). We are aware that for $E \approx 1$, the term SWP is certainly not very appropriate.

V. The Wilson Loop

An important order parameter of the original $U(N)$ theory is the Wilson loop for a closed curve $C$,

$$ W[C] = \frac{1}{Z_N} \int d\mu \text{Tr} \left( \prod_{\ell} U_{\ell} \right) e^{\beta(\text{Tr}U + \text{Tr}U^+ - N)} $$

Once more, because in two dimensions plaquette variables are the independent variables, we have for a simple curve

$$ W(C) = \omega^A $$

$$ \omega = \frac{1}{Z_N} \int dU(\text{Tr}U)e^{\beta(\text{Tr}U + \text{Tr}U^+)} $$

$A$ is the area enclosed by the curve in lattice units. In the diagonal representation

$$ \omega = \left\langle \frac{1}{N} \sum_{i=1}^{N} \cos \theta_i \right\rangle $$

which in the continuum approximation of the large $N$ limit becomes

$$ = \lim_{N \to \infty} \left\langle \frac{1}{N} \sum_{i=1}^{N} \cos \theta_i \right\rangle = \int_0^{2\pi} \frac{d\theta}{2\pi} u(\theta) \cos \theta. $$
Using (19) we have

\[ W_A = \begin{cases} \left( \frac{E}{2} \right)^A & 0 \leq E \leq 1 \\ \left( 1 - \frac{1}{2E} \right)^A & 1 \leq E < \infty \end{cases} \] (28)

The string constant in lattice units is defined by

\[ W_A = e^{-A \frac{1}{4\pi\alpha_0}} \]

\[ \alpha_0 = \begin{cases} -\frac{1}{4\pi \log E/2} & 0 \leq E \leq 1 \\ -\frac{1}{4\pi \log \left( 1 - \frac{1}{2E} \right)} & 1 \leq E < \infty \end{cases} \] (29)

it is the exact analogue of the correlation ‘length’ in spin systems. Up until now we have worked from the statistical mechanics point of view in which the basic unit of length is the lattice spacing ‘a’. In order to make contact with field theory\(^{10}\) where the basic unit of length is usually set by the inverse renormalized mass we go over to laboratory units in which the string constant has the dimension of area

\[ \alpha(a; E) = a^2 \alpha_0(E) \] (30)

This physical quantity is a renormalization group invariant i.e., if \( a \to \lambda a, \lambda > 0 \)

\[ \alpha(\lambda a; E) = \alpha(a; R_\lambda(E)) \] (31)

\( R_\lambda(E) \) is the renormalized coupling. Then from (29) we have

\[ R_\lambda(E) = \begin{cases} \frac{1}{2} \left( \frac{E}{2} \right)^{1/\lambda^2} & 0 \leq E \leq 1 \\ \frac{1}{2 \left[ 1 - \left( 1 - \frac{1}{2E} \right)^{1/\lambda^2} \right]} & 1 \leq E < \infty \end{cases} \] (32)

Now consider moving towards short distances, e.g., \( \lambda = \frac{1}{2} < 1 \). The recursion formulae which follows from (32) have two trivial fixed points \( E^* = 0 \) and
$E^* = \infty$ and the flow is away from $E^* = 0$ towards $E^* = \infty$. Note that $E = 1$ is not a fixed point. At the fixed points we have

\[
\alpha_0(E^* = 0) = 0
\]

\[
\alpha_0(E^* = \infty) = \infty
\]

Hence $E^* = \infty$ is the critical point of the statistical mechanics problem for fixed $a$. In the neighborhood of the critical point $E^* = \infty$, the lattice constant is an irrelevant parameter and it is possible to define a euclidean invariant theory characterized by the adjustable parameter

\[
\alpha_R = \alpha(a \to 0, E \to E^*)
\]

From (29) we have for $\alpha_R$ near the critical point

\[
\alpha_R = \frac{Ea^2}{2\pi} \equiv \frac{E_R}{2\pi}
\]

In the field theory notation $\frac{E}{2} = \frac{\beta}{N} = \frac{1}{g_0^2 N}$, hence

\[
\alpha_R = \frac{1}{\pi N(g_0^2/a^2)}
\]

The renormalized parameter $\alpha_R$ ranges from 0 to $\infty$, in contrast to the bare parameter $\alpha_0$ which ranges from $1/2\pi$ to $\infty$. This agrees with ’t Hooft’s computation\(^2\) of the same quantity in the Yang-Mills version of the theory. In fact, for the interaction energy of static quarks in the fundamental representation, we have

\[
E(x_1, x_2) = \frac{1}{4\pi \alpha_R} |x_1 - x_2|
\]

$x_1$ and $x_2$ are the locations of the quarks in one dimension.
VI. Partition Function Near Critical Point

The partition function $z$ in (6) can be written by expanding the cosine and the log in their respective power series. The action is

$$-2S = B \sum_{i=1}^{N} \frac{\cos \theta_i}{N} + \sum_{i<j} \frac{1}{N^2} \log 2[1 - \cos(\theta_i - \theta_j)]$$

$$= B \sum_{i=1}^{N} \frac{1}{N} \left[1 - \frac{\theta_i^2}{2!} + \frac{\theta_i^4}{4!} \cdots \right] +$$

$$\sum_{i<j} \frac{1}{N^2} \log(\theta_i - \theta_j)^2 - \sum_{i<j} \frac{1}{N^2} \frac{(\theta_i - \theta_j)^2}{3.4}$$

Now performing a scale transformation

$$\theta_i \to \sqrt{E} \theta_i = \xi_i$$

we get

$$-2S = -\frac{1}{N} \sum_{i=1}^{N} \xi_i^2 / 2 + \frac{1}{N^2} \sum_{i<j} \log |\xi_i - \xi_j|^2$$

$$+ E - \frac{N(N-1)}{2N^2} \log E + \sum_{n=1}^{\infty} \left(\frac{1}{E}\right)^{\theta(n)} [\xi]$$

$\theta(n)$ stands for the coefficient of $1/E^n$, e.g.,

$$\theta^{(1)} = \frac{1}{N} \sum_{i} \xi_i^4 - \frac{1}{N^2} \sum_{i<j} (\xi_i - \xi_j)^2 / 12$$

$$\theta^{(2)} = -\frac{1}{N} \sum_{i} \xi_i^6 - \frac{1}{N^2} \sum_{i<j} (\xi_i - \xi_j)^4 / 3.4.5.6 \cdots \text{etc.}$$

After scaling the measure factor we have

$$z = e^{N^2(E - \frac{1}{2} \log E)} \frac{1}{N!} \int_{-\infty}^{+\infty} \prod_{i} d\xi_i \prod \theta(\sqrt{E} \pi - |\xi_i|)$$

$$\times e^{N^2} \cdot e^{N^2} \sum_{n=1}^{\infty} \frac{\theta(n)}{E^n} (\xi)$$
\[
V = -\frac{1}{N} \sum_{i=1}^{N} \frac{\xi_i^2}{2} + \frac{1}{N^2} \sum_{i<j} \log |\xi_i - \xi_j|^2
\]

(33)

The \( \theta \) function in (33) is a cut-off that reflects the compactness of the gauge group.

So far, (33) is exact without approximation. In the large \( N \) limit, one can once more consider a WKB expansion in \( 1/N^2 \). Since this is the same problem as before except for a change of scale, we have, for the distribution of eigenvalues,

\[
\tilde{u}(\xi) = \frac{1}{\sqrt{E}} u(\xi/\sqrt{E}) = 2 \cos \frac{\xi}{2\sqrt{E}} \sqrt{1 - E \sin^2 \frac{\xi}{2\sqrt{E}}}
\]

(34)

The function \( u \) above is given by (19b), since we are, ultimately, interested in large values of \( E \). Expanding, we have

\[
\tilde{u}(\xi) = \sqrt{4 - \xi^2} + \sum_{n=1}^{\infty} \frac{C_n(\xi)}{E^n}, \quad |\xi| \leq 2, \quad E \gg 1
\]

This means that the contribution of the terms involving the operators \( \theta^{(n)}(\xi) \) in (33), to the leading term in the WKB expansion in \( 1/N^2 \), are negligible near critical point i.e., as \( E \to \infty \) (N.B. the large \( E \) limit is taken after the large \( N \) limit). Further, the cut-off in (33) is automatically respected, and we have

\[
\int_{-\infty}^{\infty} \prod_{i} d\xi_i 2\pi e^{N^2 V} (35)
\]

We now recall that if \( H \) is a \( N \times N \) hermetian matrix with \( N^2 \) independent components, the integration measure over \( H \) is defined by\(^7\)

\[
dH = \prod_{i=1}^{N} dH_{ii} \prod_{i<j} dH_{ij}^R dH_{ij}^I, \quad H_{ij} = H_{ij}^R + iH_{ij}^I
\]

(36)

Once more one can perform a separation of variables to extract those which are left invariant by a unitary transformation. Let \( W \) be the unitary matrix that diagonalizes \( H \),

\[
H = W^+ h W, \quad h_{ij} = \delta_{ij} \xi_j
\]

then

\[
dH \propto dW, \quad \prod_{i=1}^{N} d\xi_i \prod_{i<j} |\xi_i - \xi_j|^2
\]
and (35) becomes

\[ z \propto e^{N^2(E - \log E/2)} \int dHe^{-\frac{N}{2}TrH^2} \]  

(37)

Therefore, the partition function of the original gauge theory near critical point is

\[ Z = e^{-2N\beta V} z^V \]

\[ \propto e^{-N^2V \log E/2} \left[ \int dHe^{-\frac{N}{2}TrH^2} \right]^V \]

\[ \propto \left[ \int \frac{dH}{(\sqrt{E})^{N^2}} e^{-\frac{N}{2}TrH^2} \right]^V \]  

(38)

Expression (38) is what one expects to be proportional to the partition function of the Yang-Mills version of the theory, which is defined in a spacetime volume \( V \) by

\[ Z_{YM} \propto \int \prod_{x,t} dA_0 dA_1 \delta(A_0(x,t)) \delta(A_1(x,t_0)) e^{-\frac{N}{2g^2} \int_V dx dt F_{\mu \nu}^2} \]

(39)

\( \bar{F}_{\mu \nu} \) is the field strength; \( g \) is the dimensional coupling constant. We have fixed the generalized axial gauge

\[ A_0(x,t) = 0 \]
\[ A_1(x,t_0) = 0 \]

In this gauge one can express \( A_1 \) uniquely in terms of the field strength

\[ A_1(x,t) = \int_{t_0}^{t} F_{01}(x,t') dt' \]  

(40)

Making this change of variable in (39)

\[ Z_{YM} \propto \int \prod_{x,t} dF(x,t) e^{-\frac{1}{2g^2} \int_V dx dt Tr F^2(x,t)} \]

At this stage we can manipulate using a cut-off to write

\[ Z_{YM} \propto \left( \int dF e^{-\frac{2}{2g^2} Tr F^2} \right)^V \]
$dF$ is the integration measure of the Hermitian matrix $F$. Introducing the dimensionless coupling $g_0^2 = g^2 a^2$, which vanishes as $a \to 0$, and the dimensionless matrix $H = 2a^2 F / \sqrt{E}$; $E = 2/g_0^2 N$ a fixed number as $g_0 \to 0$ and $N \to \infty$, we have

$$Z_{YM} \propto \left( \int \frac{dH}{(\sqrt{E})^{N^2}} e^{-\frac{N}{2} \text{Tr} H^2} \right)^V$$

(41)

Hence the formal equivalence of (38) and (39) is established.

Using entirely similar arguments it is easy to show that in the thermodynamic limit of large $N$, near the critical point, the Wilson loop is given by

$$W[C] = \left[ 1 - \frac{1}{2E} \left\langle \frac{1}{N} \sum_{i=1}^{N} \xi_i^2 \right\rangle \right]^A$$

(42)

$$\left\langle \frac{1}{N} \sum_{i=1}^{N} \xi_i^2 \right\rangle = \frac{\int \prod_i \frac{d\xi_i}{2\pi} \prod_{i<j} |\xi_i - \xi_j|^2 \sum_{i=1}^{N} \frac{\xi_i^2}{N e} - \frac{\alpha^2}{2} \sum \frac{\xi_i^2}{N}}{\int \prod_i \frac{d\xi_i}{2} \prod_{i<j} |\xi_i - \xi_j|^2 e^{-\frac{\alpha^2}{2} \sum \frac{\xi_i^2}{N}}}$$

(43)

(N.B. to leading order in WKB $\left\langle \frac{1}{N} \sum_{i=1}^{N} \xi_i^2 \right\rangle = 1$, in agreement with 28). (42) can be rewritten as an integral over a random Hermitian matrix $H$:

$$W[C] = \left[ 1 - \frac{1}{2E} \left\langle \frac{1}{N} \text{Tr} H^2 \right\rangle \right]^A$$

(44)

$$\left\langle \frac{1}{N} \text{Tr} H^2 \right\rangle = \frac{\int dH \left( \frac{1}{N} \text{Tr} H^2 \right) e^{-\frac{N}{2} \text{Tr} H^2}}{\int dH e^{-\frac{N}{2} \text{Tr} H^2}}$$

(44) is exactly the expression for the Wilson loop in the Yang-Mills theory. The steps are similar to those which led from (39) to (41).
VII. Exact Computation of Partition Function for Fixed $\beta$ and $N$

So far, we have studied the $U(N)$ gauge theory in the limit $N \to \infty$, holding the ratio $E = 2\beta/N$ fixed. We now wish to compare these results with computations for finite $N$.

The partition function $z$ can be expressed as a Toeplitz determinant. The trick lies in expressing the repulsive measure factor in (4) as a product of Vandermonde determinants

$$
\prod_{i<j} (e^{i\theta_i} - e^{i\theta_j}) \prod_{i<j} (e^{-i\theta_i} - e^{-i\theta_j}) = \det V \det V^* 
$$

$$
V_{\ell m} = e^{im\theta_{\ell}} \quad (45)
$$

$$
\det V = \sum_{J_1 \cdots J_N} \epsilon_{J_1 \cdots J_N} V_{J_1 J_N} \cdots V_{J_N J_1}
$$

Substituting (45) in (4) we have

$$
z = \frac{1}{N!} \sum_{J_1 \cdots J_N} \epsilon_{J_1 \cdots J_N} \epsilon_{K_1 \cdots K_N} \int_0^{2\pi} \frac{d\theta}{2\pi} e^{i \sum_{\ell} [(j_\ell - k_\ell)\theta_\ell + 2\beta \cos \theta_\ell]}
$$

$$
= \det M \quad (46)
$$

$$
M_{ij} = I_{i-j}(2\beta) = \int_0^{2\pi} \frac{d\theta}{2\pi} e^{i(k-j)\theta + 2\beta \cos \theta}
$$

$I_n(2\beta)$ is a Bessel function with imaginary argument$^{11}$.

It is clear that as long as $N$ is finite, $z$ is an analytic function of $\beta$. It is also known that the $U(N)$ gauge theory with finite $N$ has only 2 trivial fixed points of the renormalization group$^6$, $\beta = 0$ and $\beta = \infty$. As one goes to short distances, the coupling constant renormalizes to larger values and $\beta = \infty$ is a critical point near which one can define a continuum theory.

From formula (26) we have, for the Wilson loop,

$$
W[C] = \omega^A
$$

$$
\omega(\beta, N) = \frac{1}{2N} \frac{\partial}{\partial \beta} \log z \quad (47)
$$

It is instructive to compute $\omega(\beta, N)$ near the fixed points $\beta = 0$ and $\beta = \infty$. 

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A Taylor series expansion of the function

\[ F_N(\beta) = z(\beta, N) = \det(I_{i-j}(2\beta))_{N \times N} \]  

was done on a computer around the fixed points \( \beta = 0 \) and \( \beta = \infty \) for \( N = 2, 3, 4, 5 \). This calculation was done by Mark Sweeny. The result are

(i) Taylor series around \( \beta = 0 \):

\[ F_2(\beta) = 1 + \beta^2 + \frac{\beta^4}{2} + \frac{5}{36} \beta^6 + \frac{7}{288} \beta^8 + \cdots \]
\[ F_3(\beta) = 1 + \beta^2 + \frac{\beta^4}{2} + \frac{\beta^6}{6} + \frac{23}{576} \beta^8 + \cdots \]  
\[ F_4(\beta) = 1 + \beta^2 + \frac{\beta^4}{2} + \frac{\beta^6}{6} + \frac{1}{24} \beta^8 + \frac{119}{14400} \beta^{10} + \cdots \]
\[ F_5(\beta) = 1 + \beta^2 + \frac{\beta^4}{2} + \frac{\beta^6}{6} + \frac{1}{24} \beta^8 + \frac{1}{120} \beta^{10} + \cdots \]  

We see that

\[ F_N(\beta) \sim e^{\beta^2} \]  

is a fairly accurate fit that grows better with increase in \( N \). From (50) we can compute

\[ \omega(\beta, N) \simeq \frac{\beta}{N} = \frac{E}{2} \]  

(ii) Taylor series around \( \beta = \infty \):

Defining

\[ F_N(\infty) = \left( \frac{e^{2\beta}}{\sqrt{4\pi\beta}} \right)^N f_N(x), \quad x = \beta^{-1} \]  

The Taylor expansion of \( f_N(x) \) around \( x = 0 \) is

\[ f_2(x) = 4x - 7x^2 + \frac{61}{8} x^3 - \frac{151}{32} x^4 + \frac{771}{512} x^5 \cdots \]
\[ f_3(x) = 128x^3 - 912x^4 + 3297x^5 \]  
\[ f_4(x) = 49152x^6 - 860160x^7 + 7452672x^8 + \cdots \]  

Note that the powers of the leading terms are given by \( N(N-1)/2 \). This leads to

\[ \omega(\beta, N) \simeq 1 - \frac{1}{22} N \beta^2 + O \left( \frac{1}{\beta^2} \right) \]  

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\[ \approx 1 - \frac{1}{2E}, \quad \beta \gg 1 \]  \hspace{1cm} (53)

The small corrections are computable from (52). We note that this result is also inferred from the exponential factor that scales \( z \) in (33). We do not know a simple explanation for the linear fit in (51).

We see that the finite \( N \) results (51) and (53) near the fixed points exactly match the corresponding expressions (28) computed in the thermodynamic limit of \( N \to \infty \).

\section*{VIII. Conclusion and Discussion}

The analytic computation done in the large \( N \) limit holding \( E = 2\beta/N \) fixed is in very good agreement with the computation for finite \( N \), especially near the fixed points and it is important to emphasize this agreement in the critical region which is the physically relevant one. The difference is that while the computation for finite \( N \) involves analytic interpolation between weak and strong coupling the thermodynamic computation shows non-analytic behavior at \( E = 1 \). Certainly this non-analyticity is a pathology of the \( N = \infty \) limit. We do not know whether the qualitative agreement of the two methods and even the anomaly at \( E = 1 \), is a feature of the theory in higher dimensions which is vastly more complicated. It is desirable to express the summation of the planar diagrams in higher dimensions as a coupled set of integral equations in the mean field approximation.

We end with a final comment that the expression of the partition function as a determinant implies that this theory has a fermionic representation.

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[7] Integration over matrices has been extensively discussed in Nuclear Physics. See, e.g., F.J. Dyson, J. Math. Phys. 3, (1962) reprinted in ‘Statistical Theories of Spectra: Fluctuations’ C. Porter (Academic Press, 1965); ‘Random Matrices’, M.L. Mehta (Academic Press, 1967).

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[9] Suggested by L. Kadanoff.

[10] See, e.g., K.G. Wilson and J. Kogut, Phys. Reports 12C (1974). L.P. Kadanoff, Rev. Mod. Phys. 49, (1977).

[11] After this work was well begun we received a preprint on $U(N)$ lattice gauge theories by I. Bars and F. Green (IAS preprint) which also contains formula (40). In our notation they have computed the determinant for small $E = \frac{2\beta}{N}$. We remark that in this limit the determinant can be exactly computed using the classical theorem of Szegő on Toeplitz determinant: For large $N$, fixed $\beta$

$$\log \det I_{i-j}(2\beta) \sim N g_0 + \sum_{n=1}^{\infty} n g_n g_{-n} = \beta^2$$
\[ g_n = \int_{0}^{2\pi} \frac{d\theta}{2\pi} e^{in\theta} \log(e^{2\beta \cos \theta}) \]

(See McCoy and Wu, ‘The Two Dimensional Ising Model [Harvard Univ. Press], for an extensive discussion).

Other recent papers that discuss \( U(N) \) lattice theories: T. Eguchi (Chicago preprint EFI 79/22); D. Weingarten (Indiana preprint).

[12] I thank Mark Sweeny for responding with interest to do the computer calculation.

[13] While this work was going on and after the solutions of the integral equations were obtained we learnt of a similar work by D. J. Gross and E. Witten (Princeton preprint)