Brownian bees in the infinite swarm limit

Julien Berestycki, Éric Brunet, James Nolen, Sarah Penington

June 12, 2020

Abstract

The Brownian bees model is a branching particle system with spatial selection. It is a system of $N$ particles which move as independent Brownian motions in $\mathbb{R}^d$ and independently branch at rate 1, and, crucially, at each branching event, the particle which is the furthest away from the origin is removed to keep the population size constant. In the present work we prove that as $N \to \infty$ the behaviour of the particle system is well approximated by the solution of a free boundary problem (which is the subject of a companion paper [BBNP20]), the hydrodynamic limit of the system. We then show that for this model the so-called selection principle holds, i.e. that as $N \to \infty$ the equilibrium density of the particle system converges to the steady state solution of the free boundary problem.

1 Introduction and main results

The Brownian bees model is a particular case of an $N$-particle branching Brownian motion ($N$-BBM for short) which is defined as follows. The system consists of $N$ particles with locations in $\mathbb{R}^d$ for some dimension $d$. Each particle moves independently according to a Brownian motion with diffusivity $\sqrt{2}$ and branches independently into two particles at rate one. Whenever a particle in the system branches, the particle in the system which is furthest (in Euclidean distance) from the origin is immediately removed from the system, so that there are exactly $N$ particles in the system at all times. Thus, the branching events arrive according to a Poisson process with rate $N$. The name Brownian bees, suggested by Jeremy Quastel, comes from the analogy with bees swarming around a hive; throughout the paper, we will refer to this process simply as $N$-BBM.

The particles can be labelled in a natural way, which will allow us to write the $N$-BBM as a càdlàg $(\mathbb{R}^d)^N$-valued process. Each of the $N$ particles carries a label from the set $\{1, \ldots, N\}$. Suppose at some time $\tau$ the particle labelled $k$ branches, and the particle with label $\ell$ is the furthest from the origin. Then at time $\tau$ the particle with label $\ell$ is removed from the system, and a new particle with label $\ell$ appears at the location of the particle with label $k$. For $k \in \{1, \ldots, N\}$, let $X_k^{(N)}(t)$ denote the location of the particle with label $k$ at time $t$. Then

$$X^{(N)}(t) = (X_1^{(N)}(t), \ldots, X_N^{(N)}(t))$$

is the vector of particle locations in the $N$-BBM at time $t$. This labelling is motivated by the following equivalent description of $N$-BBM in terms of jumping rather than branching: at rate $N$, the particle that is furthest from the origin jumps to the location of another particle chosen uniformly at random from all $N$ particles. (The particular choice of ordering of the particles here plays no essential role in our results.)
We shall prove two types of result about this interacting system of particles: results about the spatial distribution of particles at a fixed time \( t \) as the number of particles \( N \to \infty \), and results about the long-term behaviour of the particle system as time \( t \to \infty \) for a fixed large number of particles \( N \). As we show, there is a sense in which these limits commute. Showing this so-called selection principle holds is a major motivation of the present work and was originally conjectured by Nathanaël Berestycki. More precisely, he predicted that as \( N \to \infty \), the particles would localise in a ball of finite radius at large times.

Our first main result is a hydrodynamic limit for the distribution of particle locations at a fixed time \( t \). This limit involves the solution of the following free boundary problem: for a probability measure \( \mu_0 \) on \( \mathbb{R}^d \), find \( u(x, t) : \mathbb{R}^d \times (0, \infty) \to [0, \infty) \) and \( R_t : (0, \infty) \to [0, \infty] \) such that

\[
\begin{aligned}
\partial_t u &= \Delta u + u, &\text{for } t > 0 \text{ and } \|x\| < R_t, \\
u(x, t) &= 0, &\text{for } t > 0 \text{ and } \|x\| \geq R_t, \\
u(x, t) &\text{ is continuous on } \mathbb{R}^d \times (0, \infty), \\
\int_{\mathbb{R}^d} u(x, t) \, dx &= 1, &\text{for } t > 0, \\
u(\cdot, t) &\to \mu_0 &\text{weakly as } t \searrow 0.
\end{aligned}
\]

(1)

In the companion paper [BBNP20], we prove that (1) has a unique solution \((u, R)\), and that the function \( R_t \) is finite and continuous for \( t > 0 \).

For \( t \geq 0 \), we let

\[ M_t^{(N)} = \max_{i \in \{1, \ldots, N\}} \|X_i^{(N)}(t)\| \]

denote the maximum distance of a particle from the origin at time \( t \). For \( A \subseteq \mathbb{R}^d \) measurable, we let

\[ \mu^{(N)}(A, t) = \frac{1}{N} \left| \left\{ i \in \{1, \ldots, N\} : X_i^{(N)}(t) \in A \right\} \right| \]

denote the proportion of particles which are in the set \( A \) at time \( t \). In other words, \( \mu^{(N)}(dx, t) \) is the empirical measure of the particles at time \( t \), i.e.

\[ \mu^{(N)}(dx, t) = \frac{1}{N} \sum_{k=1}^N \delta_{X_k^{(N)}(t)}(dx). \]

We can now state our hydrodynamic limit result.

**Theorem 1.1.** Suppose that \( \mu_0 \) is a Borel probability measure on \( \mathbb{R}^d \), and that

- \( X_1^{(N)}(0), \ldots, X_N^{(N)}(0) \) are i.i.d. with distribution given by \( \mu_0 \), and
- \( (u, R) \) is the solution to (1) with initial condition \( \mu_0 \).

Then, for any \( t > 0 \) and any measurable \( A \subseteq \mathbb{R}^d \), almost surely,

\[ \mu^{(N)}(A, t) \to \int_A u(x, t) \, dx \quad \text{and} \quad M_t^{(N)} \to R_t \quad \text{as } N \to \infty \]

(this holds for any coupling of the processes \( X^{(N)}_{N \in \mathbb{N}} \)).

Note that Theorem 1.1 implies that for \( t > 0 \), almost surely \( \mu^{(N)}(dx, t) \to u(x, t) \, dx \) weakly as \( N \to \infty \).

Our second set of results concerns the long-term behaviour \( (t \to \infty) \) of the particle system for large \( N \). We can show that for large fixed \( N \), the particle system converges in distribution as \( t \to \infty \) to an invariant measure. For \( X \in (\mathbb{R}^d)^N \), we write \( \mathbb{P}_X \) to denote the probability measure under which \( (X^{(N)}(t), t \geq 0) \) is an \( N \)-BBM process with \( X^{(N)}(0) = X \).
Theorem 1.2. For $N$ sufficiently large, the process $(X^{(N)}(t), t \geq 0)$ has a unique invariant measure $\pi^{(N)}$, a probability measure on $(\mathbb{R}^d)^N$. For any $X \in (\mathbb{R}^d)^N$, under $P_X$, the law of $X^{(N)}(t)$ converges in total variation norm to $\pi^{(N)}$ as $t \to \infty$:

$$\lim_{t \to \infty} \sup_C |P_X (X^{(N)}(t) \in C) - \pi^{(N)}(C)| = 0,$$

where the supremum is over all Borel measurable sets $C \subseteq (\mathbb{R}^d)^N$.

For each $t \geq 0$, the empirical measure $\mu^{(N)}(\cdot, t)$ is a random element of $P(\mathbb{R}^d)$, the set of Borel probability measures on $\mathbb{R}^d$. Theorem 1.2 implies that as $t \to \infty$ the law of $\mu^{(N)}(\cdot, t)$ converges in total variation to the measure $\pi^{(N)} \circ H^{-1}$, where $H : (\mathbb{R}^d)^N \to P(\mathbb{R}^d)$ is the map defined by $H(x_1, \ldots, x_N) = \frac{1}{N} \sum_{i=1}^N \delta_{x_i}$ and $\pi^{(N)} \circ H^{-1}$ is the pushforward of $\pi^{(N)}$ under the map $H$. The law of $\mu^{(N)}(\cdot, t)$ and the measure $\pi^{(N)} \circ H^{-1}$, which are both probability measures on the Polish space $P(\mathbb{R}^d)$, do not depend on the particular ordering of particles used to define $X^{(N)}(t)$ as an $(\mathbb{R}^d)^N$-valued process.

We also obtain more explicit results about the long-term behaviour of the particle system. We let $U : \mathbb{R}^d \to \mathbb{R}$ denote the principal Dirichlet eigenfunction of $(-\Delta)$ in a spherical domain with radius uniquely chosen so that the eigenvalue is 1. That is, let $(U, R_\infty)$ denote the unique solution to

$$\begin{align*}
-\Delta U(x) &= U(x), \quad \|x\| < R_\infty, \\
U(x) &= 0, \quad \|x\| < R_\infty, \\
U(x) &= 0, \quad \|x\| \geq R_\infty,
\end{align*}$$

(2)

Then $(U, R_\infty)$ is a stationary solution to (1). In [BBNP20], we prove that any solution $(u(\cdot, t), R_t)$ of the free boundary problem (1) converges to the stationary solution $(U, R_\infty)$ as $t \to \infty$, and it turns out that this stationary solution also controls the long-term behaviour of the particle system for large $N$.

We shall use the following notation to denote a reasonable class of initial particle configurations. For $K > 0$ and $c \geq 0$, let

$$\Gamma(K, c) = \left\{ X \in (\mathbb{R}^d)^N : \frac{1}{N} \sum_{i=1}^N \{i : \|X_i\| < K\} \geq c \right\}.$$  

(3)

This is the set of particle configurations which put at least a fraction $c$ of the particles within distance $K$ of the origin. The following result shows that if $N$ is large, then at a large time $t$, the particles are approximately distributed according to $U$, and the largest particle distance from the origin is approximately $R_\infty$.

Theorem 1.3. Take $K > 0$ and $c \in (0,1]$. For $\epsilon > 0$, there exist $N_\epsilon < \infty$ and $T_\epsilon < \infty$ such that for $N \geq N_\epsilon$ and $t \geq T_\epsilon$, for an initial condition $X \in \Gamma(K, c)$ and $A \subseteq \mathbb{R}^d$ measurable,

$$P_X \left( \left| \mu^{(N)}(A, t) - \int_A U(x) \, dx \right| \geq \epsilon \right) < \epsilon$$

and

$$P_X \left( \left| M^{(N)}_t - R_\infty \right| \geq \epsilon \right) < \epsilon.$$

As a consequence of Theorems 1.2 and 1.3, for large $N$, under the invariant distribution $\pi^{(N)}$, the proportion of particles in a set $A$ is approximately $\int_A U(x) \, dx$ and the furthest particle distance from the origin is approximately $R_\infty$:

Theorem 1.4. For $\epsilon > 0$ and $A \subseteq \mathbb{R}^d$ measurable,

$$\pi^{(N)} \left( \left\{ X \in (\mathbb{R}^d)^N : \frac{1}{N} \sum_{i=1}^N \mathbb{1}_{\{X_i \in A\}} - \int_A U(x) \, dx \right\} \geq \epsilon \right) \to 0 \quad \text{as} \quad N \to \infty$$

(4)
and \( \pi^{(N)} \left( \{ \mathcal{X} \in (\mathbb{R}^d)^N : \max_{i \in \{1, \ldots, N\}} \| \mathcal{X}_i \| - R_\infty \geq \epsilon \} \right) \rightarrow 0 \) as \( N \rightarrow \infty \). \( \text{(5)} \)

The results in this article and in the companion article [BBNP20] can be summarised in the following informal diagram:

\[
\begin{array}{c}
\text{N-BBM, } \mu^{(N)}(dx, t) \rightarrow N \rightarrow \infty \rightarrow \text{Hydrodynamic limit } (u(x, t), R_t) \\
\text{Stationary distribution, } \pi^{(N)} \rightarrow N \rightarrow \infty \rightarrow \text{Stationary solution } (U(x), R_\infty)
\end{array}
\]

In [BBNP20], we deal with the right hand side of the diagram: well-posedness of the free boundary problem \((1)\) and the long-term behaviour of its solutions. In the present article, Theorem 1.1 gives rigorous meaning to the top of the diagram, Theorem 1.2 covers the left hand side, and Theorem 1.4 covers the bottom of the diagram.

### 1.1 Related works

The particle system we are considering is a particular case of a more general \( N \)-particle branching Brownian motion (\( N \)-BBM) with spatial selection, described as follows: The system consists of \( N \) particles moving in \( \mathbb{R}^d \) with locations \( \{ X_1^{(N)}(t), \ldots, X_N^{(N)}(t) \} \). Each particle moves independently according to a Brownian motion with diffusivity \( \sqrt{2} \) and branches independently into two particles at rate 1. Whenever a particle branches, however, the particle having least “fitness” or “score” (out of the entire ensemble) is instantly removed (killed), so that there are exactly \( N \) particles in the system at all times. The fitness of a particle is a function \( F(x) \) of its location \( x \in \mathbb{R}^d \), and as a result, the elimination of least-fit particles tends to push the ensemble toward regions of higher fitness. Variants of this stochastic process were first studied in one spatial dimension, beginning with work of Brunet, Derrida, Mueller, and Munier [BDMM06, BDMM07] on discrete-time processes, and the work of Maillard [Mai16] on the continuous-time model involving Brownian motions. In these works, the particle removed from the system is always the leftmost particle, which means that they could be described by a monotone fitness function (e.g. \( F(r) = r, r \in \mathbb{R} \)). The general multidimensional model which we have just described above was first studied by N. Berestycki and Zhao [BZ18]; specifically, they studied the particle system with fitness functions \( F(x) = \| x \| \) and \( F(x) = \lambda \cdot x \), both of which have the effect of pushing the ensemble of particles away from the origin. The Brownian bees model that we consider in this article corresponds to the fitness function \( F(x) = -\| x \| \).

In the setting of one spatial dimension and with monotone fitness function \( F(r) = r, r \in \mathbb{R} \), De Masi, Ferrari, Presutti, and Soprano-Loto [DMFPSL19a] determined the hydrodynamic limit of the particle system. For \( t > 0 \), define the measure

\[
\mu^{(N)}(dr, t) = \frac{1}{N} \sum_{k=1}^{N} \delta_{X_k^{(N)}(t)}(dr).
\]

De Masi et al. proved that if the initial particle locations \( X_1^{(N)}(0), \ldots, X_N^{(N)}(0) \) are i.i.d., with certain assumptions on the distribution of \( X_1^{(N)}(0) \), then the family of empirical measures \( \mu^{(N)}(dr, t) \) converges, as \( N \rightarrow \infty \), to a limit which can be identified with a solution \( u(r, t) \) to a free boundary problem:

\[
\begin{cases}
\partial_t u = \partial_r^2 u + u, & r > \gamma_t, \quad t > 0, \\
u(r, t) = 0, & r \leq \gamma_t, \quad t > 0, \\
\int_{\gamma_t}^{\infty} u(r, t) \, dr = 1, & t > 0,
\end{cases}
\]

(6)
where the free boundary at \( r = \gamma_t \in \mathbb{R} \) is related to \( u \) through the integral constraint. Global existence of solutions to this free boundary problem was proved by J. Berestycki, Brunet, and Penington [BBP19]. De Masi et al. also state that for fixed \( N \), the particle system (seen from the leftmost particle) converges in distribution as \( t \to \infty \) to an invariant measure \( \nu_N \), but they did not prove asymptotic results about the shape of the cloud of particles under \( \nu_N \) as \( N \to \infty \).

As discussed in Section 1.2 below, a related one-dimensional result plays a fundamental role in our work. We use some coupling ideas similar to those in the proof of the hydrodynamic limit result in [DMFPSL19a], but we obtain a more quantitative result for our particle system (see Proposition 1.5 below) which does not require the initial particle locations to be i.i.d. random variables. This, together with results about the long-term behaviour of the free boundary problem (1) from [BBNP20], allows us to control the long-term behaviour of the Brownian bees particle system for large \( N \).

Building on the approach of [DMFPSL19a], Beckman [Bec19] derived a similar hydrodynamic limit in the one-dimensional setting with symmetric fitness \( \mathcal{F}(r) = -|r| \), which coincides with our case if \( d = 1 \). Beckman also studied the long-term behaviour of the \( N \)-BBM in one dimension with a non-monotone fitness function of the form \( \mathcal{F}(r) = r + \psi(r) \), \( \psi \) being periodic, and proved existence of a stationary distribution in a certain moving reference frame. In earlier work, Durrett and Remenik [DR11] studied a related branching-selection model in which non-diffusing particles in \( \mathbb{R} \) are born at random locations but do not move during their lifetimes. They showed that the hydrodynamic limit of this particle system is given by a non-local free boundary problem.

A related model is the Fleming-Viot system studied by Burdzy, Holyst, and March [BHMM00]. In that model, particles diffuse within a bounded domain having fixed boundary; whenever a particle hits the boundary it is instantly killed and one of the internal particles simultaneously branches, preserving the total mass. As in our case, the stationary distribution for that system also converges to a stationary distribution in a certain moving reference frame. In earlier work, Durrett and Remenik [DR11] studied a related branching-selection model in which non-diffusing particles in \( \mathbb{R} \) are born at random locations but do not move during their lifetimes. They showed that the hydrodynamic limit of this particle system is given by a non-local free boundary problem.

In [AFGJ16], Asselah, Ferrari, Groisman, and Jonckheere considered a slightly different Fleming-Viot particle system. In their work, the \( N \) particles live on \( \{0, 1, 2, \ldots\} \), move independently as continuous-time sub-critical Galton-Watson processes, and are killed when they hit 0 (each time a particle is killed, one of the remaining \( N - 1 \) particles, chosen at random, branches). Recall that for a single sub-critical Galton-Watson process conditioned on non-extinction, there exists an infinite family of quasi-stationary distributions. (By contrast, observe that a diffusion on a bounded domain conditioned on not exiting the domain has a unique quasi-stationary distribution.) Asselah et al. showed that for each \( N \), the Fleming-Viot particle system has a unique invariant distribution, and that its stationary empirical distribution converges as \( N \to \infty \) to the minimal quasi-stationary distribution of the Galton-Watson process conditioned on non-extinction (which is the quasi-stationary distribution with the minimal expected time of extinction). This has been called the selection principle in the literature. It is reminiscent of the fact that the solution of the Fisher-KPP equation started from a fast decreasing initial condition converges to the minimal-velocity travelling wave (see in particular [GJ18] and the note [GJ13] of Groisman and Jonckheere). This principle is conjectured to hold in quite broad generality. For instance, for the one-dimensional \( N \)-BBM studied in [DMFPSL19a], it is conjectured that the unique invariant distribution of the system seen from the leftmost particle converges, as \( N \to \infty \), to the centred minimal-velocity travelling wave solution of (6) (which is given by \( \gamma_t = 2t, u(2t + r, t) = re^{-t} \mathbb{1}_{\{r \geq 0\}} \)).

Finally, we mention the very recent work [ABL20] of Addario-Berry, Lin, and Tendoron, in which a variant of the Brownian bees model with the following selection rule is considered: each time one of the \( N \) particles branches, the particle currently furthest away from the centre of mass of the cloud of particles is removed from the system. Addario-Berry et al. show that the
movement of the centre of mass, appropriately rescaled, converges to a Brownian motion.

### 1.2 One-dimensional results and outline of the article

The first step in the proofs of Theorems 1.1, 1.2, 1.3 and 1.4 is to control the proportion of particles within distance \( r \) of the origin at a fixed time \( t \), when the number of particles \( N \) is very large.

For \( r > 0 \), let \( B(r) = \{ x \in \mathbb{R}^d : \| x \| < r \} \) be the open ball of radius \( r \) centred at the origin. Suppose that \((u, R)\) solves (1) with some initial probability measure \( \mu_0 \), and let \( v : [0, \infty) \times (0, \infty) \to [0, 1] \) denote the mass of \( u \) within distance \( r \) of the origin at time \( t \):

\[
v(r, t) = \int_{B(r)} u(x, t) \, dx.
\]

Then \( r \mapsto v(r, t) \) is non-decreasing and \( v(r, t) = 1 \) for \( r \geq R_t \). Let \( v_0(r) = \mu_0(B(r)) \); then, by Lemma 6.2 of [BBNP20], \( v \) satisfies the following parabolic obstacle problem:

\[
\begin{align*}
0 &\leq v(r, t) \leq 1, & \text{for } t > 0, r \geq 0, \\
\partial_t v &= \partial_r^2 v - \frac{d-1}{r} \partial_r v + v, & \text{if } v(r, t) < 1, \\
v(0, t) &= 0, & \text{for } t > 0, \\
v(r, t) &\text{ is continuous on } [0, \infty) \times (0, \infty), \\
\partial_r v(r, t) &\text{ is continuous on } [0, \infty), & \text{for } t > 0, \\
v(\cdot, t) &\to v_0 & \text{in } L^1_{\text{loc}} \text{ as } t \searrow 0.
\end{align*}
\]

We prove in Theorem 2.1 of [BBNP20] that for any measurable \( v_0 : [0, \infty) \to [0, 1] \), (7) has a unique solution.

The following result is a hydrodynamic limit result for the distances of particles from the origin, and will be an important step in the proofs of Theorems 1.1 and 1.3. Introduce

\[
F^{(N)}(r, t) := \mu^{(N)}(B(r), t)
\]

as the proportion of particles within distance \( r \) of the origin at time \( t \). Then, Proposition 1.5 below says that for any initial configuration of particles, at a fixed time \( t \), the proportion \( F^{(N)}(r, t) \) is close to the solution \( v^{(N)}(r, t) \) of (7) with initial condition \( v_0 \) determined by the initial configuration of particles. The bound does not depend on the initial particle configuration; this will be crucial when the result is used in the proof of Theorem 1.3.

**Proposition 1.5.** There exists \( c_1 \in (0, 1) \) such that for \( N \) sufficiently large, for \( t > 0 \) and any \( X \in (\mathbb{R}^d)^N \),

\[
\mathbb{P}_X \left( \sup_{r \geq 0} \left| F^{(N)}(r, t) - v^{(N)}(r, t) \right| \geq e^{2t} N^{-c_1} \right) \leq e^{t} N^{-1-c_1},
\]

where \( v^{(N)} \) is the solution of (7) with \( v_0(r) = F^{(N)}(r, 0) \) \( \forall r \geq 0 \).

The next result uses Proposition 1.5 to get an upper bound on the largest distance from the origin which holds over a time interval of fixed length. This will then allow us to compare the particle system to a system in which particles are killed if they are further than a deterministic distance from the origin, which will enable us to prove the \( d \)-dimensional hydrodynamic limit in Theorem 1.1.

**Proposition 1.6.** There exists \( c_2 \in (0, 1) \) such that under the assumptions of Theorem 1.1, for any \( 0 < \eta < T \), for \( N \) sufficiently large (depending on \( \eta \) and \( T \)),

\[
\mathbb{P} \left( \exists t \in [\eta, T] : M^{(N)}_t > R_t + \eta \right) \leq N^{-1-c_2}.
\]
In the case where \( \mu_0 \) has compact support, the proof of Proposition 1.6 can easily be extended to bound the probability that there exists \( t \in (0, T] \) with \( M_t^{(N)} > R_t + \eta \). However, an upper bound on \( M_t^{(N)} \) in the time interval \( [\eta, T] \) (for an arbitrarily small \( \eta \)) is enough to allow us to prove Theorem 1.1.

Using Proposition 1.5 and results about the long-term behaviour of solutions to the obstacle problem (7) from the companion paper [BBNP20], we can also prove the following result about the long-term behaviour of particle distances from the origin when \( N \) is large. For \( r \geq 0 \), let

\[
V(r) = \int_{B(r)} U(x) \, dx,
\]

where \( U \) is defined in (2).

**Proposition 1.7.** Take \( K > 0 \) and \( c \in (0, 1] \). For \( \epsilon > 0 \), there exist \( N_\epsilon < \infty \) and \( T_\epsilon < \infty \) such that for \( N \geq N_\epsilon \) and \( t \geq T_\epsilon \), for an initial condition \( X \in \Gamma(K,c) \),

\[
\mathbb{P}(X \left( \sup_{r \geq 0} \left| F^{(N)}(r, t) - V(r) \right| \geq \epsilon \right) < \epsilon),
\]

\[
\mathbb{P}(X \left( \left| M_t^{(N)} - R_\infty \right| \geq \epsilon \right) < \epsilon),
\]

and

\[
\mathbb{P}(X \left( \sup_{s \in [0,1]} M_{t+s}^{(N)} > R_\infty + \epsilon \right) < \epsilon).
\]

Using (12), we can compare the particle system at large times to a system in which particles are killed if they are further than distance \( R_\infty + \epsilon \) from the origin. This, together with (10), will allow us to prove Theorem 1.3.

The rest of the article is laid out as follows. In Section 2, we recall results from [BBNP20] which will be used in this article. In Section 3, we define notation which will be used throughout the proofs. Then in Section 4, we prove Propositions 1.5 and 1.6, and in Section 5 we use Proposition 1.6 to prove Theorem 1.1. In Section 6, we prove Proposition 1.7 and use this to prove Theorem 1.3, and, finally, Theorems 1.2 and 1.4.

**Acknowledgements:** The work of JN was partially funded through grant DMS-1351653 from the US National Science Foundation. The authors wish to thank Louigi Addario-Berry, Erin Beckman, Nathanaël Berestycki and Pascal Maillard for stimulating discussions at various points of this project.

## 2 Results from [BBNP20]

In this section, we state some results from [BBNP20] which play a key role in the present work. The first one is Theorem 1.1 in [BBNP20], which says that the free boundary problem (1) has a unique solution, and that moreover the free boundary radius \( R_t \) is continuous.

**Theorem 2.1** (Theorem 1.1 in [BBNP20]). Let \( \mu_0 \) be a Borel probability measure on \( \mathbb{R}^d \). Then there exists a unique classical solution to problem (1). Furthermore,

- \( t \mapsto R_t \) is continuous (and finite) for \( t > 0 \).
- As \( t \downarrow 0 \), \( R_t \to R_0 := \inf \{ r > 0 : \mu_0(\mathcal{B}(r)) = 1 \} \in [0, \infty] \).
- For \( t > 0 \) and \( \|x\| < R_t \), \( u(x, t) > 0 \).
For a Borel probability measure \( \mu_0 \) on \( \mathbb{R}^d \), let \( (u, R) \) denote the solution of (1). Let \( (B_t)_{t \geq 0} \) denote a \( d \)-dimensional Brownian motion with diffusivity \( \sqrt{2} \), and for \( x \in \mathbb{R}^d \), write \( \mathbb{P}_x \) for the probability measure under which \( B_0 = x \). For \( t > 0 \), define a family of measures on \( \mathbb{R}^d \) according to

\[
\rho_t(x, A) = \mathbb{P}_x \left( B_t \in A, \|B_s\| < R_s \quad \forall s \in (0, t) \right)
\]

for all Borel sets \( A \subseteq \mathbb{R}^d \). Then \( \rho_t(x, dy) \) is absolutely continuous with respect to the Lebesgue measure, so it has a density. Abusing notation, we denote this density by \( \rho_t(x, y) \). Then by Proposition 6.1 in [BBNP20],

\[
u(y, t) = e^t \int_{\mathbb{R}^d} \mu_0(dx) \rho_t(x, y), \quad y \in \mathbb{R}^d, \quad t > 0.
\]

Define the cumulative distribution of the norm process \( \|B_t\| \) conditional on \( \|B_0\| = y \) as

\[
w(y, r, t) := \mathbb{P} \left( \|B_t\| < r \mid \|B_0\| = y \right).
\]

Then the function \( r \mapsto g(y, r, t) := \partial_r w(y, r, t) \) is the density of \( \|B_t\| \) conditional on \( \|B_0\| = y \); in other words, \( g \) is the transition density of the \( d \)-dimensional Bessel process with diffusivity \( \sqrt{2} \). The function

\[
G(y, r, t) := -\partial_y w(y, r, t)
\]

is the fundamental solution of the equation

\[
\partial_t G = \partial_y^2 G - \frac{d - 1}{r} \partial_r G, \quad G(y, 0, t) = 0, \quad G(y, r, 0) = \delta(r - y).
\]

(See Section 3 of [BBNP20] for more details on the properties of \( G \).) For \( t, r, y > 0 \), the fundamental solution \( G \) and transition density \( g \) are smooth functions of their arguments, and are related by

\[
\partial_r G = -\partial_y g.
\]

For a given initial condition \( v_0^0 \in L^\infty(0, \infty) \), we let

\[
v^t(r, t) = e^t \int_0^\infty dy G(y, r, t)v_0^0(y).
\]

This \( v^t \) is a solution to the linear problem

\[
\begin{cases}
\partial_t v^t = \partial_y^2 v^t - \frac{d - 1}{r} \partial_r v^t + v^t, & \text{for } t > 0, \ r \geq 0, \\
v^t(0, t) = 0, & \text{for } t > 0, \\
v^t(\cdot, t) \to v_0^0 & \text{in } L^1_{\text{loc}} \text{ as } t \searrow 0,
\end{cases}
\]

and it is the unique solution to (19) which is bounded on \( [0, \infty) \times [0, T] \) for each \( T > 0 \). In the particular case \( v_0^0(r) = 1_{\{y \leq r\}} \), we have \( v^t(r, t) = e^t w(y, r, t) \).

For \( t > 0 \) and \( m \in \mathbb{R} \), we define the operators \( G_t \) and \( C_m \) by letting

\[
G_t f(r) = \int_0^\infty dy G(y, r, t)f(y) \quad \text{and} \quad C_m f(r) = \min \left( f(r), m \right).
\]

In particular, \( v^t = e^t G_t v^0_0 \). By Lemma 3.1 in [BBNP20] we have that \( \|f\|_{L^\infty} \), and so for \( f, g \in L^\infty[0, \infty) \) and \( t > 0 \),

\[
\|G_t f - G_t g\|_{L^\infty} \leq \|f - g\|_{L^\infty}.
\]

Suppose \( v_0 : [0, \infty) \to [0, 1] \) is non-decreasing, and let \( v \) denote the solution of the obstacle problem (7) with initial condition \( v_0 \). For \( \delta > 0 \) and \( k \in \mathbb{N}_0 \), we let

\[
v^{k, \delta, -} = (e^{\delta} G_{\delta} C_{\delta - \delta})^k v_0, \quad v^{k, \delta, +} = (C_1 e^{\delta} G_{\delta})^k v_0.
\]

Then by Lemmas 4.3 and 4.4 in [BBNP20], we have the following result.
Lemma 2.2 (Lemmas 4.3 and 4.4 in [BBNP20]). For any $\delta > 0$ and $k \in \mathbb{N}_0$,
\[ v^{k,\delta,-}(r) \leq v(r, k\delta) \leq v^{k,\delta,+}(r) \quad \forall \, r \geq 0 \quad \text{and} \quad \|v^{k,\delta,+} - v^{k,\delta,-}\|_{L^\infty} \leq (e^{k\delta} + 1)(e^\delta - 1). \]

We shall also use the following result which was proved as part of Theorem 2.1 in [BBNP20].

The result says that the solution of (7) is continuous with respect to the initial condition in the following sense.

Lemma 2.3 (From Theorem 2.1 in [BBNP20]). Let $v$ and $\tilde{v}$ be the solutions to (7) corresponding to the initial conditions $v_0$ and $\tilde{v}_0$. Then for $t > 0$,
\[ \|v(\cdot, t) - \tilde{v}(\cdot, t)\|_{L^\infty} \leq e^t \|v_0 - \tilde{v}_0\|_{L^\infty}. \]

Recall the definition of $V$ in (9). The following result, which follows directly from Theorem 2.2 in [BBNP20], gives us control over how quickly the solution $v(\cdot, t)$ of the obstacle problem (7) converges to $V$.

Proposition 2.4 (From Theorem 2.2 in [BBNP20]). For $c \in (0, 1]$, $K > 0$ and $\epsilon > 0$, there exists $t_\epsilon = t_\epsilon(c, K) \in (0, \infty)$ such that the following holds. Suppose $v_0 : [0, \infty) \to [0, 1]$ is non-decreasing with $v_0(K) \geq c$, and let $v$ solve the obstacle problem (7) with initial condition $v_0$. For $t > 0$, let $R_t = \inf \{r \geq 0 : v(r, t) = 1\}$. Then for $t \geq t_\epsilon$,
\[ |v(r, t) - V(r)| < \epsilon \quad \forall \, r \geq 0 \quad \text{and} \quad |R_t - R_\infty| < \epsilon. \]

The final result from [BBNP20] which we need in this article is Proposition 5.10, which says that for large $K$ and small $c$, if an initial condition $v_0$ has mass at least $c$ within distance $K$ of the origin then the solution $v$ of (7) has mass at least $2c$ within distance $K - 1$ of the origin during a fixed time interval $[t_0, 2t_0]$.

Proposition 2.5 (Proposition 5.10 in [BBNP20]). There exist $t_0 > 1$ and $c_0 \in (0, 1/2)$ such that for all $c \in (0, c_0]$, all $K \geq 2$, and all $t_1 \in [t_0, 2t_0]$, for $v_0 : [0, \infty) \to [0, 1]$ measurable, the condition
\[ v_0(r) \geq c \mathbb{1}_{\{r \geq K\}} \quad \forall \, r \geq 0 \]
implies that
\[ v(r, t_1) \geq 2c \mathbb{1}_{\{r \geq K - 1\}} \quad \forall \, r \geq 0, \]
and
\[ v(r, nt_1) \geq \min(2c_0, 2^nc) \mathbb{1}_{\{r \geq \max(K - n, 1)\}}, \quad \forall \, r \geq 0, \quad n \in \mathbb{N}, \]
where $v(r, t)$ denotes the solution of (7) with initial condition $v_0$.

3 Notation

From now on, we let $(B_t)_{t \geq 0}$ denote a $d$-dimensional Brownian motion with diffusivity $\sqrt{d}$, and for $x \in \mathbb{R}^d$ we write $\mathbb{P}_x$ for the probability measure under which $B_0 = x$, and write $\mathbb{E}_x$ for the corresponding expectation.

The locations (positions) of a collection of $m$ particles in $\mathbb{R}^d$ are written as a vector $X \in (\mathbb{R}^d)^m$. The size of the vector (i.e. the number of particles in the collection) is written $|X|$ (i.e. $|X| = m$ for $X \in (\mathbb{R}^d)^m$). The individual locations in $X$ are written $X_k$ for $k \in \{1, \ldots, |X|\}$:
\[ X = (X_1, \ldots, X_m) \quad \text{with} \quad m = |X|. \]

We extend some set notation to vectors. Specifically, we write $\mathcal{X} \subseteq \mathcal{Y}$ to mean that all the particles in $\mathcal{X}$ are also in $\mathcal{Y}$:
\[ \mathcal{X} \subseteq \mathcal{Y} \quad \Leftrightarrow \quad \exists j : \{1, \ldots, |\mathcal{X}|\} \to \{1, \ldots, |\mathcal{Y}|\} \text{ injective such that } X_k = Y_{j(k)} \forall k. \]
We write $|A \cap \mathcal{X}|$ for the number of particles in $\mathcal{X}$ which lie in some set $A \subseteq \mathbb{R}^d$:

$$|A \cap \mathcal{X}| = \left| \left\{ k \in \{1, \ldots, |\mathcal{X}| \} : \mathcal{X}_k \in A \right\} \right|.$$ 

If $\mathcal{X}$ and $\mathcal{Y}$ are two vectors of particles with locations in $\mathbb{R}^d$, we write $\mathcal{X} \preceq \mathcal{Y}$ to mean that the vector $\mathcal{X}$ contains more particles than the vector $\mathcal{Y}$ in any ball centred on the origin:

$$\mathcal{X} \preceq \mathcal{Y} \iff |\mathcal{X} \cap B(r)| \geq |\mathcal{Y} \cap B(r)| \quad \text{for all } r > 0,$$

where we recall that $B(r)$ is the centred open ball of radius $r$:

$$B(r) = \left\{ x \in \mathbb{R}^d : \|x\| < r \right\}.$$ 

Notice that $\mathcal{X} \preceq \mathcal{Y}$ implies that $|\mathcal{X}| \geq |\mathcal{Y}|$.

The order in which the particle locations are written within a vector $\mathcal{X}$ is irrelevant for the operations $\subseteq$, $\cap$ and $\preceq$ described above.

It will be useful to compare the $N$-BBM to the standard $d$-dimensional binary branching Brownian motion (BBM) without selection, in which particles move independently in $\mathbb{R}^d$ according to Brownian motions with diffusivity $\sqrt{2}$, and branch into two particles at rate 1. The BBM may be labelled using the Ulam-Harris scheme (see [Jag89] and references therein). Let $\mathcal{U} = \bigcup_{n=1}^{\infty} \mathbb{N}^n$, and for $t \geq 0$, let $\mathcal{N}_t^+ \subset \mathcal{U}$ denote the set of Ulam-Harris labels of the particles in the BBM at time $t$. (If the BBM has $m$ particles at time 0 then the particles initially have labels 1, $\ldots$, $m$ and for each $t \geq 0$, $\mathcal{N}_t^+ \subset \mathcal{U}$ denote the set of Ulam-Harris labels of the particles in the BBM at time $t$.) The order in which the particle locations are written within a vector $\mathcal{X}$ is irrelevant for the operations $\subseteq$, $\cap$ and $\preceq$ described above.

In the proofs, we shall often use the following standard coupling between the BBM and the $N$-BBM: consider a standard $d$-dimensional binary BBM as described above. In addition to their spatial location, let each particle carry a colour attribute, either red or blue. When a blue particle branches, the two offspring particles are coloured blue, and simultaneously the blue particle furtherest from the origin turns red. When a red particle branches, the two offspring particles are coloured red. The system begins with $N$ blue particles at time 0. The set of blue particles is a realisation of standard BBM. Specifically, for $t \geq 0$, let $\mathcal{N}_t^{(N)} \subseteq \mathcal{N}_t^+$ denote the set of blue particles at time $t$. Then, $\mathcal{N}_t^{(N)}$ is always a set of size $N$, and there exists an enumeration $(u_k)_{k=1}^N$ of $\mathcal{N}_t^{(N)}$ such that $X^{(N)}(t) = (X_{u_1}(t), \ldots, X_{u_N}(t))$. Recall that we let $M_t^{(N)} = \max_{k \in \{1, \ldots, N\}} \|X_k^{(N)}(t)\|$, the maximum distance of a particle in the $N$-BBM from the origin at time $t$. Notice that for $t \geq 0$, almost surely

$$\mathcal{N}_t^{(N)} = \left\{ u \in \mathcal{N}_t^+ : \|X_u^{(N)}(s)\| \leq M_s^{(N)} \forall s \in [0, t] \right\}. \quad (24)$$

We usually write $\mathcal{X}$ for the initial configuration of the $N$-BBM and of the BBM. In some cases where we compare an $N$-BBM and a BBM with different initial conditions, we write $\mathcal{X}^+$ for the initial condition of the BBM. Expectations and laws started from an initial condition $\mathcal{X}$ are written $\mathbb{E}_\mathcal{X}$ and $\mathbb{P}_\mathcal{X}$ respectively. We also write $(\mathcal{F}_t)_{t \geq 0}$ for the natural filtration of $(X^{(N)}(t), t \geq 0)$, i.e. $\mathcal{F}_t = \sigma((X^{(N)}(s), s \leq t))$.

## 4 One-dimensional hydrodynamic limit results

In this section, we prove the hydrodynamic limit results about the distances of particles from the origin, Propositions 1.5 and 1.6.
For \( t \geq 0 \) and \( r > 0 \), recall the definition (8) of the cumulative distribution function

\[
F^{(N)}(r, t) = \mu^{(N)}(B(r), t) = \frac{1}{N} \left| X^{(N)}(t) \cap B(r) \right| = \frac{1}{N} \sum_{i=1}^{N} \mathbf{1}_{\{\|X_i^{(N)}(t)\| < r\}}
\]  

(25)

and introduce

\[
F^+(r, t) = \frac{1}{N} \left| X^+(t) \cap B(r) \right| = \frac{1}{N} \sum_{u \in \mathcal{N}_t^+} \mathbf{1}_{\{\|X_u^+(t)\| < r\}}.
\]

### 4.1 Proof of Proposition 1.5

#### 4.1.1 Upper bound for the proof of Proposition 1.5

Recall from (20) that for \( m \in \mathbb{R} \) and \( f : [0, \infty) \to \mathbb{R} \), we let \( C_m f(r) = \min(f(r), m) \). The following proposition will play a crucial role in the proof of Proposition 1.5; it says that the random function \( F^+ \) corresponding to the BBM stochastically dominates the random function \( F^{(N)} \) corresponding to the \( N \)-BBM, if both processes start from the same particle configuration.

**Proposition 4.1.** Suppose \( \mathcal{X} = (\mathcal{X}_1, \ldots, \mathcal{X}_N) \in (\mathbb{R}^d)^N \). There exists a coupling of the \( N \)-BBM \( X^{(N)}(t) \) started from \( \mathcal{X} \) and of the BBM \( X^+(t) \) also started from \( \mathcal{X} \) such that

\[
F^{(N)}(\cdot, t) \leq C_1 F^+(\cdot, t) \quad \forall t \geq 0.
\]

(Equivalently, \( X^+(t) \preceq X^{(N)}(t) \) for all \( t \geq 0 \).) In particular, if \( f : [0, \infty) \to \mathbb{R} \cup \{+\infty\} \) is measurable, then for all \( t \geq 0 \),

\[
\mathbb{P}_\mathcal{X} \left( \sup_{r \geq 0} \left( F^{(N)}(r, t) - f(r) \right) \geq 0 \right) \leq \mathbb{P}_\mathcal{X} \left( \sup_{r \geq 0} \left( C_1 F^+(r, t) - f(r) \right) \geq 0 \right).
\]

**Proof.** This is a direct property of the standard coupling between the \( N \)-BBM \( X^{(N)}(t) \) and the BBM \( X^+(t) \) as described in Section 3. Observe that with \( X^{(N)}(0) = \mathcal{X} = X^+(0) \), under the coupling described in Section 3, for \( t \geq 0 \) we have \( X^{(N)}(t) \subseteq X^+(t) \). It follows that \( F^{(N)}(r, t) \leq F^+(r, t) \) for all \( r > 0 \) and \( t \geq 0 \). Therefore, since \( F^{(N)}(r, t) \leq 1 \) also holds for all \( r > 0 \) and \( t \geq 0 \), we have

\[
F^{(N)}(r, t) \leq C_1 F^+(r, t), \quad r > 0, \quad t \geq 0,
\]

which completes the proof.

Recall the definition of the operator \( G_t \) from (20), and introduce

\[
v^f(r, t) = e^t G_t v_0^f(r) \quad \text{with} \quad v_0^f(r) = F^+(r, 0),
\]

(26)

the solution of (19) with initial condition determined by the initial configuration \( X^+(0) \) of the BBM.

**Lemma 4.2.** There exists \( N_0 < \infty \) such that for all \( N \geq N_0 \), all \( \mathcal{X} \in (\mathbb{R}^d)^m \) with \( m \leq N \), and all \( t > 0 \),

\[
\sup_{r > 0} \mathbb{P}_\mathcal{X} \left( \left| F^+(r, t) - v^f(r, t) \right| \geq N^{-1/5} \right) \leq 13 e^{4t} N^{-6/5}
\]

and

\[
\mathbb{P}_\mathcal{X} \left( N^{-1}(\|X^+(t)\| - e^t |\mathcal{X}|) \geq N^{-1/5} \right) \leq 13 e^{4t} N^{-6/5}.
\]
Proof. Recall that \((B_t)_{t \geq 0}\) is a \(d\)-dimensional Brownian motion with diffusivity \(\sqrt{2}\). We claim that for \(r > 0\) and \(t \geq 0\),

\[
v^\ell(r,t) = \frac{1}{N} \sum_{i=1}^{\mid\mathcal{X}\mid} e^t \mathbb{P}_{X_i}(\|B_t\| < r).
\]

(27)

Indeed, by the definition of \(G_t\) in (20),

\[
G_t F^+(r,0) = \frac{1}{N} \int_0^\infty G(y, r, t) \sum_{i=1}^{\mid\mathcal{X}\mid} \mathbb{1}_{\{\|\mathcal{X}_i\| < y\}} \, dy
\]

\[
= \frac{1}{N} \sum_{i=1}^{\mid\mathcal{X}\mid} \int_0^\infty G(y, r, t) \, dy
\]

\[
= \frac{1}{N} \sum_{i=1}^{\mid\mathcal{X}\mid} \int_0^r \partial_y g(y, r', t) \, dr' \, dy
\]

\[
= \frac{1}{N} \sum_{i=1}^{\mid\mathcal{X}\mid} \int_0^r g(\|\mathcal{X}_i\|, r', t) \, dr'
\]

\[
= \frac{1}{N} \sum_{i=1}^{\mid\mathcal{X}\mid} \mathbb{P}_{X_i}(\|B_t\| < r),
\]

where the third line follows by (16) and (17), and the last two lines follow since \(g(y, \cdot, t)\) is the density at time \(t\) of a \(d\)-dimensional Bessel process started at \(y\). This proves the claim (27).

In the BBM \(X^+\) started from \(\mathcal{X}\), for \(i \in \{1, \ldots, |\mathcal{X}|\}\), denote by \(\mathcal{N}^+,i\) the family of particles at time \(t\) descended from the \(i\)-th particle in \(\mathcal{X}\), i.e., recalling that particles are labelled according to the Ulam-Harris scheme, let

\[
\mathcal{N}^+,i = \{u \in \mathcal{N}^+ : u = (i, u_2, \ldots)\}.
\]

Then letting \(X^+,i(t) = (X^+_u(t))_{u \in \mathcal{N}^+,i}\), the processes \(X^+,i\) form a family of independent BBMs, and for each \(i\) the process \(X^+,i\) is started from a single particle at location \(\mathcal{X}_i\). Fix a time \(t > 0\) and write \(n_i = |\mathcal{N}^+,i|\) for the number of particles descended from \(\mathcal{X}_i\) at time \(t\), and

\[
n_i(r) = \left|\left\{u \in \mathcal{N}^+,i : X^+_u(t) \in B(r)\right\}\right|
\]

for the number of particles at time \(t\) descended from \(\mathcal{X}_i\) which lie within distance \(r\) of the origin. Then \(F^+(r, t) = \frac{1}{N} \sum_{i=1}^{\mid\mathcal{X}\mid} n_i(r)\) and, by the many-to-one lemma (see [HR17]) and (27),

\[
\frac{1}{N} \sum_{i=1}^{\mid\mathcal{X}\mid} \mathbb{E}_{\mathcal{X}}[n_i(r)] = \frac{1}{N} \sum_{i=1}^{\mid\mathcal{X}\mid} e^t \mathbb{P}_{X_i}(\|B_t\| < r) = v^\ell(r,t).
\]

Therefore

\[
\mathbb{E}_{\mathcal{X}}\left[\left(F^+(r, t) - v^\ell(r, t)\right)^4\right]
\]

\[
= \frac{1}{N^4} \mathbb{E}_{\mathcal{X}}\left[\left(\sum_{i=1}^{\mid\mathcal{X}\mid} (n_i(r) - \mathbb{E}_{\mathcal{X}}[n_i(r)])\right)^4\right]
\]

\[
= \frac{1}{N^4} \left(\sum_{i=1}^{\mid\mathcal{X}\mid} \mathbb{E}_{\mathcal{X}}\left((n_i(r) - \mathbb{E}_{\mathcal{X}}[n_i(r)])^4\right)\right) + 6 \sum_{i,j=1}^{\mid\mathcal{X}\mid} \text{Var}_{\mathcal{X}}(n_i(r)) \text{Var}_{\mathcal{X}}(n_j(r)).
\]
Note that
\[ \mathbb{E}_X \left[ (n_i(r) - \mathbb{E}_X[n_i(r)])^4 \right] \leq \mathbb{E}_X \left[ n_i(r)^4 + \mathbb{E}_X[n_i(r)]^4 \right] \leq 2 \mathbb{E}_X \left[ n_i(r)^4 \right] \leq 2 \mathbb{E}_X \left[ n_i^4 \right] \]  
(28)
and that
\[ \text{Var}_X(n_i(r)) \leq \mathbb{E}_X \left[ n_i(r)^2 \right] \leq \mathbb{E}_X \left[ n_i^2 \right]. \]  
(29)
Furthermore, since \( n_i \) has geometric distribution with parameter \( e^{-t} \),
\[ \mathbb{E}_X \left[ n_i^4 \right] \leq 24e^{4t} \quad \text{and} \quad \mathbb{E}_X \left[ n_i^2 \right] \leq 2e^{2t}. \]  
(30)
We now have that, since \( |X| \leq N \),
\[ \mathbb{E}_X \left[ (\mathbb{F}^+(r,t) - \mathbb{F}^\ell(r,t))^4 \right] \leq \frac{1}{N^4} (N48e^{4t} + 3N^24e^{4t}) \leq (48N^{-3} + 12N^{-2})e^{4t}. \]
By the same argument, since \( |X^+(t)| = \sum_{i=1}^{\lfloor |X| \rfloor} n_i \) and \( \sum_{i=1}^{\lfloor |X| \rfloor} \mathbb{E}_X[n_i] = e^t|X| \), we also have
\[ \mathbb{E}_X \left[ (N^{-1}(|X^+(t)| - e^t|X|))^4 \right] \leq (48N^{-3} + 12N^{-2})e^{4t}. \]
By Markov’s inequality, the stated results follow for \( N \) sufficiently large that
\[ N^{4/5}(48N^{-3} + 12N^{-2}) \leq 13N^{-6/5}. \]
Next, we upgrade the estimate in Lemma 4.2 to be uniform in \( r \), at the expense of slightly slower decay in \( N \). Recall the definition of \( \mathbb{F}^\ell \) in (26).

**Lemma 4.3.** There exists \( N_0 < \infty \) such that for all \( N \geq N_0 \), all \( \mathcal{X} \in (\mathbb{R}^d)^m \) with \( m \leq N \), and all \( t > 0 \) such that \( e^t|\mathcal{X}|N^{-1} \geq 1 - N^{-1/10} \),
\[ \mathbb{P}_X \left( \sup_{r \geq 0} \left| C_1 \mathbb{F}^+(r,t) - C_1 \mathbb{F}^\ell(r,t) \right| \geq 3N^{-1/10} \right) \leq 13e^{4t}N^{-11/10}. \]

**Proof.** Fix \( t > 0 \) such that \( e^t|\mathcal{X}|N^{-1} \geq 1 - N^{-1/10} \). For \( k \in \mathbb{N} \) with \( k \leq \lfloor |N^{1/10}| \rfloor \), let
\[ r_k = \inf \left\{ y \geq 0 : \mathbb{F}^\ell(y,t) \geq \frac{k}{\lfloor N^{1/10} \rfloor} \right\}. \]
Recall the formula for \( \mathbb{F}^\ell(\cdot,t) \) in (27) in the proof of Lemma 4.2. Note that \( \mathbb{F}^+(\cdot,t) \) and \( \mathbb{F}^\ell(\cdot,t) \) are non-decreasing, that \( \mathbb{F}^\ell(\cdot,t) \) is continuous and that \( (r_k) \) for \( k = 0, 1, \ldots, \lfloor |N^{1/10}| \rfloor \) is an increasing sequence with \( r_0 = 0 \). As \( r \to \infty \), \( \mathbb{F}^\ell(r,t) \to e^tN^{-1}|\mathcal{X}| \geq 1 - N^{-1/10} \), so \( r_k < \infty \) for \( k \leq \lfloor N^{1/10} \rfloor - 2 \).

Suppose that for every \( k \in \{1, \ldots, \lfloor N^{1/10} \rfloor - 2\} \),
\[ \left| \mathbb{F}^+(r_k,t) - \mathbb{F}^\ell(r_k,t) \right| \leq N^{-1/5}. \]  
(31)
Then, for \( r \geq 0 \), if \( r \in [r_k, r_{k+1}] \) for some \( k \in \{0, \ldots, \lfloor N^{1/10} \rfloor - 3\} \) we have
\[ \frac{k}{\lfloor N^{1/10} \rfloor} - N^{-1/5} \leq \mathbb{F}^+(r_k,t) \leq \mathbb{F}^+(r,t) \leq \mathbb{F}^+(r_{k+1},t) \leq N^{-1/5} + \frac{k + 1}{\lfloor N^{1/10} \rfloor} \]
and
\[ \mathbb{F}^\ell(r,t) \in \left[ \frac{k}{\lfloor N^{1/10} \rfloor}, \frac{k + 1}{\lfloor N^{1/10} \rfloor} \right]. \]
If $N$ is large enough that $\frac{N^{1/10} - 2}{N^{1/10}} + N^{-1/5} \leq 1$, then for all $r \leq r_{[N^{1/10}]} - 2$ we have $C_1 F^+(r, t) = F^+(r, t)$ and $C_1 v^f(r, t) = v^f(r, t)$. Thus,

$$\left| C_1 F^+(r, t) - C_1 v^f(r, t) \right| \leq N^{-1/5} + (\lfloor N^{1/10} \rfloor)^{-1} < 3N^{-1/10}$$

for $N$ sufficiently large.

If instead $r \geq r_k$ where $k = \lfloor N^{1/10} \rfloor - 2$, then

$$v^f(r, t) \geq \frac{k}{\lfloor N^{1/10} \rfloor} \geq 1 - 2(\lfloor N^{1/10} \rfloor)^{-1},$$

and by (31),

$$F^+(r, t) \geq F^+(r_k, t) \geq 1 - N^{-1/5} - 2(\lfloor N^{1/10} \rfloor)^{-1}.$$

Hence

$$\left| C_1 F^+(r, t) - C_1 v^f(r, t) \right| < 3N^{-1/10}$$

for $N$ sufficiently large. So for $N$ sufficiently large, if (31) holds for each $k \in \{1, \ldots, \lfloor N^{1/10} \rfloor - 2\}$ then

$$\sup_{r \geq 0} \left| C_1 F^+(r, t) - C_1 v^f(r, t) \right| < 3N^{-1/10}.$$

Now by a union bound and Lemma 4.2, for $N$ sufficiently large,

$$\mathbb{P}_X \left( \sup_{r \geq 0} \left| C_1 F^+(r, t) - C_1 v^f(r, t) \right| \geq 3N^{-1/10} \right)$$

$$\leq \mathbb{P}_X \left( \exists k \in \{1, \ldots, \lfloor N^{1/10} \rfloor - 2\} : \left| F^+(r_k, t) - v^f(r_k, t) \right| \geq N^{-1/5} \right)$$

$$\leq N^{1/10} \cdot 13e^{4t}N^{-6/5}$$

$$= 13e^{4t}N^{-11/10},$$

which completes the proof. \qed

**Corollary 4.4.** There exists $N_0 < \infty$ such that for all $N \geq N_0$, for $X \in (\mathbb{R}^d)^N$ and $t > 0$,

$$\mathbb{P}_X \left( \sup_{r \geq 0} \left( F^{(N)}(r, t) - C_1 e^t G_t F^{(N)}(r, 0) \right) \geq 3N^{-1/10} \right) \leq 13e^{4t} N^{-11/10}.$$  

**Proof.** This follows immediately from Lemma 4.3 and Proposition 4.1, and the definition of $v^f$ in (26). \qed

As in (22), for $\delta > 0$ and $k \in \mathbb{N}_0$, let

$$v^{k, \delta, +} = (C_1 e^\delta G_\delta)^k F^{(N)}(\cdot, 0).$$

We now apply Corollary 4.4 repeatedly on successive timesteps to prove the following result.

**Proposition 4.5.** There exists $N_0 < \infty$ such that for all $N \geq N_0$, $\delta > 0$, $K \in \mathbb{N}$ and $X \in (\mathbb{R}^d)^N$,

$$\mathbb{P}_X \left( \sup_{r \geq 0} \left( F^{(N)}(r, K\delta) - v^{K\delta, +}(r) \right) \geq 3Ke^{K\delta}N^{-1/10} \right) \leq 13Ke^{4\delta} N^{-11/10}.$$
Proof. For \( k \in \{1, \ldots, K\} \) and \( r \geq 0 \), we can write
\[
F^{(N)}(r, k\delta) - v^{k, \delta, +}(r) = F^{(N)}(r, k\delta) - C_1e^{\delta}G_\delta F^{(N)}(r, (k-1)\delta) \\
+ C_1e^{\delta}G_\delta F^{(N)}(r, (k-1)\delta) - C_1e^{\delta}G_\delta v^{k-1, \delta, +}(r).
\]
(32)

To control (32), we define the events
\[
E_k := \left\{ \sup_{r \geq 0} \left( F^{(N)}(r, k\delta) - C_1e^{\delta}G_\delta F^{(N)}(r, (k-1)\delta) \right) < 3N^{-1/10} \right\}
\]
and
\[
E_* = \bigcap_{k=1}^{K} E_k.
\]

Then on the event \( E_* \), for each \( k \in \{1, \ldots, K\} \) we have by (32) that for \( r \geq 0 \),
\[
F^{(N)}(r, k\delta) - v^{k, \delta, +}(r) < 3N^{-1/10} + C_1e^{\delta}G_\delta F^{(N)}(r, (k-1)\delta) - C_1e^{\delta}G_\delta v^{k-1, \delta, +}(r).
\]

Note that since \( G \geq 0 \) and \( \int_{0}^{\infty} G(y, r, t) \mathrm{d}y \leq 1 \) by (14) and (15), we have that \( G_\delta f - G_\delta g \leq \max(0, \sup_{y \geq 0} (f(r) - g(r))) \) for any \( f, g : [0, \infty) \to \mathbb{R} \). Moreover, \( C_1f(r) - C_1g(r) \leq \max(0, f(r) - g(r)) \) for any \( f, g : [0, \infty) \to \mathbb{R} \). Therefore for \( r \geq 0 \),
\[
C_1e^{\delta}G_\delta f(r) - C_1e^{\delta}G_\delta g(r) \leq e^{\delta} \max(0, \sup_{r \geq 0} (f(r) - g(r))).
\]

It follows that
\[
F^{(N)}(r, k\delta) - v^{k, \delta, +}(r) < 3N^{-1/10} + e^{\delta} \max \left(0, \sup_{y \geq 0} \left( F^{(N)}(y, (k-1)\delta) - v^{k-1, \delta, +}(y) \right) \right)
\]
also holds on the event \( E_* \). By iterating this argument, it follows that for \( k \in \{1, \ldots, K\} \),
\[
\sup_{r \geq 0} \left( F^{(N)}(r, k\delta) - v^{k, \delta, +}(r) \right) < 3ke^{k\delta}N^{-1/10}
\]
holds on \( E_* \).

To estimate \( \mathbb{P}_X(E_*) \), we use a union bound and Corollary 4.4 with \( t = \delta \). Specifically,
\[
\mathbb{P}_X(E_*) = \mathbb{P}_X \left( \bigcup_{k=1}^{K} E_k^c \right) \leq \sum_{k=1}^{K} \mathbb{P}_X(E_k^c).
\]

By the Markov property, for \( k \in \{1, \ldots, K\} \),
\[
\mathbb{P}_X(E_k^c) = \mathbb{E}_X \left[ \mathbb{P}_X \left( E_k^c \mid F_{(k-1)\delta} \right) \right] = \mathbb{E}_X \left[ H \left( X^{(N)}((k-1)\delta) \right) \right]
\]
where \( H : (\mathbb{R}^d)^N \to \mathbb{R} \) is defined by
\[
H(X') = \mathbb{P}_X \left( \sup_{r \geq 0} \left( F^{(N)}(r, \delta) - C_1e^{\delta}G_\delta F^{(N)}(r, 0) \right) \geq 3N^{-1/10} \right).
\]

Therefore, by Corollary 4.4, for \( N \geq N_0 \),
\[
\mathbb{P}_X(E_*) \leq \sum_{k=1}^{K} 13e^{4\delta}N^{-11/10} = 13Ke^{4\delta}N^{-11/10}.
\]

The result follows. 

\(\square\)
4.1.2 Lower bound for the proof of Proposition 1.5

We begin by proving that under a suitable coupling, the random function $F^{(N)}$ for the $N$-BBM stochastically dominates the random function $F^+$ for the BBM with an initial condition consisting of less than $N$ particles. This result is very similar to the lower bound in Theorem 5.1 of [DMFPSL19b].

Recall from our definition of the $\leq$ notation in (23) in Section 3 that

$$X^{(N)}(t) \leq X^+(t) \iff F^{(N)}(\cdot, t) \geq F^+(\cdot, t).$$

Moreover, the relation $\mathcal{X} \leq \mathcal{X}^+$ is not affected by the ordering of the points in the vectors $\mathcal{X}$ and $\mathcal{X}^+$.

**Proposition 4.6.** Suppose $\mathcal{X} = (X_1, \ldots, X_N) \in (\mathbb{R}^d)^N$ and $\mathcal{X}^+ = (X_1^+ \ldots, X_m^+) \in (\mathbb{R}^d)^m$ with $m \leq N$ and such that $\mathcal{X} \leq \mathcal{X}^+$. There exists a coupling of the $N$-BBM $X^{(N)}(t)$ started from $\mathcal{X}$ and of the BBM $X^+(t)$ started from $\mathcal{X}^+$ such that, for $t \geq 0$,

$$X^{(N)}(t) \leq X^+(t) \quad \text{if} \quad |X^+(t)| \leq N.$$  

In particular, under that coupling, $F^{(N)}(\cdot, t) \geq F^+(\cdot, t)$ if $|X^+(t)| \leq N$. Then if $f : [0, \infty) \to \mathbb{R}$ is measurable, for $t \geq 0$,

$$\mathbb{P}_\mathcal{X} \left( \inf_{r \geq 0} \left( F^{(N)}(r, t) - f(r) \right) > 0 \right) \geq \mathbb{P}_{\mathcal{X}^+} \left( \inf_{r \geq 0} \left( F^+(r, t) - f(r) \right) > 0, \quad |X^+(t)| \leq N \right).$$  

**Proof.** The coupling of the processes $X^{(N)}$ and $X^+$ is similar in spirit to the lower bound in Section 5.4 of [DMFPSL19b].

Let $\tau^+_\ell$ for $\ell \in \mathbb{N}$ denote the successive branch times of the BBM process:

$$\tau^+_\ell = \inf \left\{ t \geq 0 : |X^+(t)| = m + \ell \right\}.$$

By induction on $|\mathcal{X}^+|$, we claim that it is sufficient to find a coupling of the processes $X^{(N)}$ and $X^+$ on the same probability space with $X^{(N)}(0) = \mathcal{X}$ and $X^+(0) = \mathcal{X}^+$ such that

$$X^{(N)}(t) \leq X^+(t), \quad \forall \ t \in \begin{cases} [0, \tau^+_m], & \text{if} \ m < N \\ [0, \tau^+_N), & \text{if} \ m = N \end{cases}$$  

(34)

holds almost surely. Indeed, assume that (34) holds. If $m = N$, then the proposition is proved.

If $m < N$, then $X^{(N)}(\tau^+_m) \leq X^+(\tau^+_m)$, and the construction of the coupling can be repeated up to time $\tau^+_m$ using $X^{(N)}(\tau^+_m)$ and $X^+(\tau^+_m)$ as the new initial particle configurations, by the strong Markov property. By induction, the property $X^{(N)}(t) \leq X^+(t)$ holds for $t \in [0, \tau^+_{N-m+1})$, where $\tau^+_{N-m+1}$ is the first time at which there are $N+1$ particles in the BBM $X^+$, and the proposition is proved.

We now show that (34) holds. Set $\tau_0 = 0$, and let $(\tau_i)_{i=1}^\infty$ be the arrival times in a Poisson process with rate $N$, so that $(\tau_{i+1} - \tau_i)_{i \geq 0}$ is a family of independent $\text{Exp}(N)$ random variables. These $\tau_i$ for $i \geq 1$ will define the branch times for the $N$-BBM process $X^{(N)}$. The coupling will ensure that $\tau^+_i = \tau_p$ for some $p \in \mathbb{N}$, where we recall that $\tau^+_i$ is the time of the first branching event in the BBM $X^+$.

We now construct the motion of the particles for $t \in (0, \tau_1)$. Given $x, x^+ \in \mathbb{R}^d$ with $\|x\| \leq \|x^+\|$, we say that $(B, B^+)$ are a pair of spherically-ordered Brownian motions starting from $(x, x^+)$ if $B$ and $B^+$ are Brownian motions in $\mathbb{R}^d$ (with diffusivity $\sqrt{2}$), starting from $B_0 = x$ and $B^+_0 = x^+$ and such that, with probability one, $\|B_t\| \leq \|B^+_t\|$ holds for all $t \geq 0$. There are multiple ways to construct such a pair. For example, $B$ and $B^+$ might evolve as independent Brownian motions in $\mathbb{R}^d$ up to the first time $T$ at which $\|B_T\| = \|B^+_T\|$; after that time they are coupled in such a way that $\|B_t\| = \|B^+_t\|$ for all $t \geq T$, for example by taking $B_t - B_T = \Theta(B^+_t -$
$B_+^\tau$) with $\Theta : \mathbb{R}^d \to \mathbb{R}^d$ being an orthogonal transformation such that $\Theta B_+^\tau = B_T$. Alternatively, one could use the skew-product decomposition of Brownian motion (see, for example, the proof of Proposition 2.10 in [BZ18]), driving the radial components of $B$ and $B^+$ by the same Bessel process.

Since the condition $X \preceq X^+$ is invariant under permutation of the indices of points in $X$ and $X^+$, it suffices to assume that the vectors $X$ and $X^+$ are ordered in such a way that

$$\|X_k\| \leq \|X_k^+\|, \quad \forall k \in \{1, \ldots, m\},$$

where we recall that $m = |X^+| \leq N$. (For example, order the vectors $X$ and $X^+$ so that the points with the lowest indices are the points closest to the origin.)

Then, for $t \in (0, \tau_1)$, we define $(X_k(t), X_k^+(t))$ for $k \in \{1, \ldots, m\}$ to be $m$ independent pairs of spherically-ordered Brownian motions starting from $(X_k, X_k^+)$, and for $k \in \{m+1, \ldots, N\}$, the $X_k(t)$ are defined to be independent Brownian motions starting from $X_k$. Hence, for $t \in (0, \tau_1)$, we have $\|X_k(t)\| \leq \|X_k^+(t)\|$ for $k \in \{1, \ldots, m\}$, which in turn implies that

$$X^{(N)}(t) \preceq X^+(t) \quad \forall t < \tau_1.$$

We now describe the first branching event at time $\tau_1$. Defining $X^{(N)}(\tau_1-) = (X_1^{(N)}(\tau_1-), \ldots, X_M^{(N)}(\tau_1-)) = \lim_{t \nearrow \tau_1} X^{(N)}(t)$, and similarly defining $X^+(\tau_1-) = \lim_{t \nearrow \tau_1} X^+(t)$, we have

$$\|X_k^{(N)}(\tau_1-)\| \leq \|X_k^+(\tau_1-)\|, \quad \forall k \in \{1, \ldots, m\}.$$

Let $j_1$, a random variable uniformly distributed on $\{1, \ldots, N\}$, be the index of the branching particle in the $N$-BBM at time $\tau_1$. Let $k_1$ denote the index of the particle in $X^{(N)}(\tau_1-)$ with maximal distance from the origin. If $j_1 \neq k_1$, then at time $\tau_1$, the $N$-BBM branches but the BBM does not. The particle in $X^{(N)}$ with index $j_1$ is duplicated; the particle in $X^{(N)}$ with index $k_1$ is eliminated. More precisely, let

$$X_k^{(N)}(\tau_1) = X_k^{(N)}(\tau_1-) \quad \text{for} \quad k \neq k_1, \quad X_{k_1}^{(N)}(\tau_1) = X_{k_1}^{(N)}(\tau_1-), \quad \text{and} \quad X^+(\tau_1) = X^+(\tau_1-).$$

Then for $k \in \{1, \ldots, m\}$,

$$\|X_k^{(N)}(\tau_1)\| \leq \|X_k^+(\tau_1)\| \leq \|X_k^+(\tau_1-)\| = \|X_k^+(\tau_1-\tau_1)\|,$$

and, in particular, $X^{(N)}(\tau_1) \preceq X^+(\tau_1)$. The construction is then repeated to extend the definition from time $\tau_1$ to $\tau_2$: take new pairs of spherically-ordered Brownian motions to determine the motion of particles up to time $\tau_2$, pick the branching particle $j_2$ in the $N$-BBM at random, and so on until time $\tau_{i^+}$, where $i^+ = \inf\{i \geq 1 : j_i \leq m\}$. This time $\tau_{i^+} = \tau_1^{i^+}$ is a branching time for both the $N$-BBM and the BBM. Observe that $\tau_1^{i^+} \sim \exp(m)$.

In the construction of the coupling so far we have that for $t < \tau_1^{i^+}$,

$$\|X_k^{(N)}(t)\| \leq \|X_k^+(t)\| \quad \forall k \in \{1, \ldots, m\},$$

and so, in particular, $X^{(N)}(t) \preceq X^+(t)$. At time $\tau_1^{i^+} = \tau_{i^+}$, the particles with index $j_{i^+} \in \{1, \ldots, m\}$ from both $X^{(N)}$ and $X^+$ branch, and the particle in $X^{(N)}$ of maximal distance from the origin is removed. More precisely, if $k_{i^+}$ is the index of the particle in $X^{(N)}(\tau_1^{i^+}-)$ with maximal distance from the origin, we let

$$X_k^{(N)}(\tau_1^{i^+}) = X_k^{(N)}(\tau_1^{i^+}-) \quad \text{for} \quad k \neq k_{i^+}, \quad X_{k_{i^+}}^{(N)}(\tau_1^{i^+}) = X_{k_{i^+}}^{(N)}(\tau_1^{i^+}-), \quad \text{and} \quad X^+(\tau_1^{i^+}) = \left( X_1^{(N)}(\tau_1^{i^+}-), \ldots, X_{k_{i^+}}^{(N)}(\tau_1^{i^+}-), X_{k_{i^+}}^{(N)}(\tau_1^{i^+}-), X_{k_{i^+}}^{(N)}(\tau_1^{i^+}-), \ldots, X_m^{(N)}(\tau_1^{i^+}-) \right).$$

Suppose $m < N$. Then $\|X_k^{(N)}(\tau_1^{i^+})\| \leq \|X_k^+(\tau_1^{i^+}-)\|$ for $k \neq k_{i^+}$, and also $\|X_{k_{i^+}}^{(N)}(\tau_1^{i^+})\| = \|X_{k_{i^+}}^+(\tau_1^{i^+}-)\|$. Moreover, if $k_{i^+} \leq m$ then $\|X_{k_{i^+}}^{(N)}(\tau_1^{i^+})\| \leq \|X_{k_{i^+}}^+(\tau_1^{i^+}-)\| \leq \|X_{k_{i^+}}^+(\tau_1^{i^+}-)\|$. It follows that $X^{(N)}(\tau_1^{i^+}) \preceq X^+(\tau_1^{i^+})$ if $m < N$. This concludes the coupling construction to achieve (34), and the proof of proposition is now complete.
We now use Proposition 4.6 to prove a lower bound on $F^{(N)}(\cdot, \delta)$.

**Lemma 4.7.** There exists $N_0 < \infty$ such that for all $N \geq N_0$, for all $\mathcal{X} \in (\mathbb{R}^d)^N$ and for all $\delta \in (0, 1)$,

$$\mathbb{P}_{\mathcal{X}} \left( \inf_{r \geq 0} \left( F^{(N)}(r, \delta) - e^\delta G_\delta C_{e-\delta-N^{-1/5}} F^{(N)}(r, 0) \right) \leq -4N^{-1/10} \right) \leq 26e^{45} N^{-11/10}.$$  

**Proof.** Take $\delta \in (0, 1)$ and $\mathcal{X} \in (\mathbb{R}^d)^N$. For $r \geq 0$, let $f_\mathcal{X}(r) = N^{-1}|\mathcal{X} \cap E(r)|$; note that if $X^{(N)}(0) = \mathcal{X}$ then $F^{(N)}(r, 0) = f_\mathcal{X}(r)$. Let $\mathcal{X}^+ \subset \mathcal{X}$ consist of the $|N(e^{-\delta} - N^{-1/5})|$ particles in $\mathcal{X}$ which are closest to the origin. Let $(X^+\{t\}, t \geq 0)$ be the BBM started from $X^+(0) = \mathcal{X}^+$. Observe that $F^+(r, 0) = C_{|N(e^{-\delta} - N^{-1/5})|} f_\mathcal{X}(r)$, so that

$$C_{e^{-\delta}-N^{-1/5}} f_\mathcal{X}(r) \geq N^{-1} \leq F^+(r, 0) \leq C_{e^{-\delta}-N^{-1/5}} f_\mathcal{X}(r) \quad \forall r \geq 0. \quad (35)$$

Let $R_1$ be the event

$$R_1 = \left\{ |X^+(\delta)| \geq N \right\}.$$

Since $|\mathcal{X}^+| \leq N(e^{-\delta} - N^{-1/5})$, one has $N^{-1}e^{\delta}|\mathcal{X}^+| - 1 \leq -N^{-1/5} e^{\delta} \leq -N^{-1/5}$, and Lemma 4.2 implies that for $N$ sufficiently large,

$$\mathbb{P}_{\mathcal{X}^+} \left( R_1^c \right) \leq \mathbb{P}_{\mathcal{X}^+} \left( N^{-1} \left( |X^+(\delta)| - e^{\delta}|\mathcal{X}^+| \right) \geq N^{-1/5} \right) \leq 13e^{45} N^{-6/5}.$$

Let $R_2$ be the event

$$R_2 = \left\{ \sup_{r \geq 0} \left| C_1 F^+(r, \delta) - C_1 v^\delta(r, \delta) \right| < 3N^{-1/10} \right\},$$

where, as in (26) in Section 4.1.1, we let $v^\delta(r, \delta) = e^{\delta} G_\delta F^+(r, 0)$, since $|\mathcal{X}^+| > N(e^{-\delta} - N^{-1/5}) - 1$, one has $e^{\delta} N^{-1/5} \geq 1 - e^{\delta} N^{-1/5} - e^{\delta} N^{-1} \geq 1 - N^{-1/10}$ if $N$ is sufficiently large that $e(N^{-1/5} + N^{-1}) \leq N^{-1/10}$. Then, by Lemma 4.3, we know that for $N$ sufficiently large,

$$\mathbb{P}_{\mathcal{X}^+} \left( R_2 \right) \leq 13e^{45} N^{-11/10}.$$

Since $e^{\delta} N^{-1}|\mathcal{X}^+| < 1$ we have $v^\delta(\cdot, \delta) < 1$ (by (27) in the proof of Lemma 4.2). By (35), we therefore have that for $r \geq 0$,

$$C_1 v^\delta(r, \delta) = \inf_{r \geq 0} \left( F^+(r, \delta) - e^\delta G_\delta C_{e^{-\delta}-N^{-1/5}} f_\mathcal{X}(r) \right) \geq -3N^{-1/10}.$$

On the event $R_1$ we also have $F^+(\cdot, \delta) \leq N^{-1}|X^+(\delta)| \leq 1$ and hence $C_1 F^+(\cdot, \delta) = F^+(\cdot, \delta)$. This shows that on the event $R_1 \cap R_2$, we have both $|X^+(\delta)| \leq N$ and

$$\inf_{r \geq 0} \left( F^+(r, \delta) - e^\delta G_\delta C_{e^{-\delta}-N^{-1/5}} f_\mathcal{X}(r) \right) \geq \inf_{r \geq 0} \left( C_1 F^+(r, \delta) - C_1 v^\delta(r, \delta) \right) \geq -\delta N^{-1} \geq -3N^{-1/10} - e^\delta N^{-1}.$$

Note that since $\mathcal{X}^+ \subset \mathcal{X}$ we have $\mathcal{X} \leq \mathcal{X}^+$. Therefore, by Proposition 4.6, and taking $N$ sufficiently large that $eN^{-1} \leq N^{-1/10}$,

$$\mathbb{P}_{\mathcal{X}} \left( \inf_{r \geq 0} \left( F^{(N)}(r, \delta) - e^\delta G_\delta C_{e^{-\delta}-N^{-1/5}} f_\mathcal{X}(r) \right) > -4N^{-1/10} \right) \geq \mathbb{P}_{\mathcal{X}^+} \left( R_1 \cap R_2 \right) \geq - \mathbb{P}_{\mathcal{X}^+} \left( R_1^c \right) - \mathbb{P}_{\mathcal{X}^+} \left( R_2^c \right) \geq 1 - 13e^{45} N^{-6/5} - 13e^{45} N^{-11/10},$$

which completes the proof, since $f_{\mathcal{X}}(\cdot) = F^{(N)}(\cdot, 0)$ if $X^{(N)}(0) = \mathcal{X}$. \qed
As in (22), for \( k \in \mathbb{N}_0 \) and \( \delta > 0 \), let
\[
v^{k,\delta,-} = (e^{\delta G_{\delta} C_{e^{-\delta}}} )^k F^{(N)} (\cdot, 0).
\]
By applying Lemma 4.7 repeatedly, we can prove the following lower bound, which, together with the upper bound in Proposition 4.5, will allow us to prove Proposition 1.5.

**Proposition 4.8.** There exists \( N_0 < \infty \) such for all \( N \geq N_0 \), for all \( \delta \in (0, 1) \) and \( K \in \mathbb{N} \),
\[
\mathbb{P} \cdot \left( \inf_{r \geq 0} \left( F^{(N)}(r, K \delta) - v^{K, \delta,-}(r) \right) \leq -5K e^{K \delta N^{-1/10}} \right) \leq 26Ke^{4 \delta N^{-11/10}}.
\]

**Proof.** For \( k \in \{1, \ldots, K\} \) and \( r \geq 0 \), we can write
\[
F^{(N)}(r, k\delta) - v^{k, \delta,-}(r) = F^{(N)}(r, k\delta) - e^{\delta G_{\delta} C_{e^{-\delta} - N^{-1/5}} F^{(N)}(r, (k-1)\delta)}
\]
\[
+ e^{\delta G_{\delta} C_{e^{-\delta} - N^{-1/5}} F^{(N)}(r, (k-1)\delta)} - e^{\delta G_{\delta} C_{e^{-\delta}} F^{(N)}(r, (k-1)\delta)}
\]
\[
+ e^{\delta G_{\delta} C_{e^{-\delta}} F^{(N)}(r, (k-1)\delta)} - e^{\delta G_{\delta} C_{e^{-\delta}} v^{k-1, \delta,-}(r)}.
\]
(36)

For the second line on the right hand side of (36), note that \( \| C_{e^{-\delta} - N^{-1/5}} f - C_{e^{-\delta}} f \|_{L_\infty} \leq N^{-1/5} \) for any \( f : [0, \infty) \rightarrow \mathbb{R} \), and so, using (21),
\[
\left\| e^{\delta G_{\delta} C_{e^{-\delta} - N^{-1/5}} F^{(N)}(\cdot, (k-1)\delta)} - e^{\delta G_{\delta} C_{e^{-\delta}} F^{(N)}(\cdot, (k-1)\delta)} \right\|_{L_\infty} \leq e^{\delta N^{-1/5}}.
\]
(37)

For the third line on the right hand side of (36), observe that \( C_m f(r) - C_{m+1} g(r) \geq \min \{ 0, f(r) - g(r) \} \) for any \( m \in (0, 1) \) and any \( f, g : [0, \infty) \rightarrow \mathbb{R} \). Then, since \( \min \{ 0, f(r) - g(r) \} \leq 0 \), we have that for any \( \delta > 0 \), \( m \in (0, 1) \) and \( r \geq 0 \),
\[
G_{\delta} C_m f(r) - G_{\delta} C_{m+1} g(r) \geq G_{\delta} \min \left( 0, f(r) - g(r) \right) \geq \inf_{y \geq 0} \left( f(y) - g(y) \right),
\]
where we used from (14) and (15) that \( G \geq 0 \) and \( \int_0^\infty G(y, r, t) \, dy \leq 1 \). It follows that
\[
\inf_{r \geq 0} \left( G_{\delta} C_{e^{-\delta}} F^{(N)}(r, (k-1)\delta) - G_{\delta} C_{e^{-\delta}} v^{k-1, \delta,-}(r) \right)
\]
\[
\geq \min \left( 0, \inf_{y \geq 0} \left( F^{(N)}(y, (k-1)\delta) - v^{k-1, \delta,-}(y) \right) \right).
\]
(38)

To control the first line of the right hand side of (36), for \( k \in \mathbb{N} \), define the event
\[
E_k := \left\{ \inf_{r \geq 0} \left( F^{(N)}(r, k\delta) - e^{\delta G_{\delta} C_{e^{-\delta} - N^{-1/5}} F^{(N)}(r, (k-1)\delta)} \right) > -4N^{-1/10} \right\}
\]
and let
\[
E_* = \bigcap_{k=1}^K E_k.
\]
Then on the event \( E_* \), using (37) and (38), we have for each \( k \in \{1, \ldots, K\} \), for all \( r \geq 0 \),
\[
F^{(N)}(r, k\delta) - v^{k, \delta,-}(r) > -4N^{-1/10} - e^{\delta N^{-1/5}}
\]
\[
+ e^{\delta \min \left( 0, \inf_{y \geq 0} \left( F^{(N)}(y, (k-1)\delta) - v^{k-1, \delta,-}(y) \right) \right)}.
\]

By iterating this bound, it follows that on the event \( E_* \), for \( k \in \{1, \ldots, K\} \),
\[
\inf_{r \geq 0} \left( F^{(N)}(r, k\delta) - v^{k, \delta,-}(r) \right) > -(4N^{-1/10} + e^{\delta N^{-1/5}}) k e^{k \delta}.
\]
(39)

19
To estimate $\mathbb{P}_X(E^c_N)$, we use a union bound and Lemma 4.7. Specifically,

$$ \mathbb{P}_X(E^c_N) = \mathbb{P}_X \left( \bigcup_{k=1}^{K} E^c_k \right) \leq \sum_{k=1}^{K} \mathbb{P}_X(E^c_k). $$

By the Markov property, for $k \in \{1, \ldots, K\}$,

$$ \mathbb{P}_X(E^c_k) = \mathbb{E}_X \left[ \mathbb{P}_X \left( E^c_k \mid \mathcal{F}_{(k-1)\delta} \right) \right] = \mathbb{E}_X \left[ H \left( X^{(N)}((k-1)\delta) \right) \right], $$

where $H : (\mathbb{R}^d)^N \to \mathbb{R}$ is defined by

$$ H(X) = \mathbb{P}_X \left( \inf_{r \geq 0} \left( F^{(N)}(r, \delta) - \epsilon \delta G_0 \epsilon \delta - N^{-1/5} F^{(N)}(r, 0) \right) \leq -4N^{-1/10} \right). $$

Therefore, by Lemma 4.7, for $N \geq N_0$,

$$ \mathbb{P}_X(E^c_N) \leq \sum_{k=1}^{K} 26e^{4\delta} N^{-11/10} = 26K e^{4\delta} N^{-11/10}. $$

Taking $N$ sufficiently large that $eN^{-1/5} \leq N^{-1/10}$, the result follows by (39).

**4.1.3 Combining the upper and lower bounds for the proof of Proposition 1.5**

We can now complete the proof of Proposition 1.5. Let $v^{(N)}$ denote the solution of (7) with initial condition $v_0(r) = F^{(N)}(r, 0)$ for $r \geq 0$. By Lemma 2.2 we have that for $\delta > 0$, $k \in \mathbb{N}_0$ and $r \geq 0$,

$$ v^{k,\delta,-}(r) = (e^{\delta} G_0 e^{-\delta})^k F^{(N)}(r, 0) \leq v^{(N)}(r, k\delta) \leq (C_1 e^{\delta} G_0)^k F^{(N)}(r, 0) = v^{k,\delta,+}(r) $$

and

$$ \| v^{k,\delta,+} - v^{k,\delta,-} \|_{L^\infty} \leq (e^{k\delta} + 1)(e^{\delta} - 1). $$

Therefore, for $N$ sufficiently large, for $X \in (\mathbb{R}^d)^N$, $\delta \in (0, 1)$ and $K \in \mathbb{N}$, by Proposition 4.5,

$$ \mathbb{P}_X \left( \sup_{r \geq 0} \left( F^{(N)}(r, K\delta) - v^{(N)}(r, K\delta) \right) \geq 3Ke^{K\delta} N^{-1/10} + (e^{K\delta} + 1)(e^{\delta} - 1) \right) \leq 13Ke^{4\delta} N^{-11/10} $$

and by Proposition 4.8,

$$ \mathbb{P}_X \left( \inf_{r \geq 0} \left( F^{(N)}(r, K\delta) - v^{(N)}(r, K\delta) \right) \leq -5Ke^{K\delta} N^{-1/10} - (e^{K\delta} + 1)(e^{\delta} - 1) \right) \leq 26Ke^{4\delta} N^{-11/10}. $$

It follows that for $N$ sufficiently large, for $X \in (\mathbb{R}^d)^N$, $\delta \in (0, 1)$ and $K \in \mathbb{N},

$$ \mathbb{P}_X \left( \sup_{r \geq 0} \left| F^{(N)}(r, K\delta) - v^{(N)}(r, K\delta) \right| \geq 5Ke^{K\delta} N^{-1/10} + (e^{K\delta} + 1)(e^{\delta} - 1) \right) \leq 39Ke^{4\delta} N^{-11/10}. $$

Take $t > 0$, and let $K = \lceil N^{1/20} t \rceil$ and $\delta = t/K$. Then $\delta \leq N^{-1/20}$ and so for $N$ sufficiently large (not depending on $t$),

$$ 5Ke^{K\delta} N^{-1/10} + (e^{K\delta} + 1)(e^{\delta} - 1) \leq 5(N^{1/20} t + 1)e^{t} N^{-1/10} + (e^{t} + 1)(e^{N^{-1/20}} - 1) $$

$$ \leq 5(t + 1)e^{t} N^{-1/20} + 4e^{t} N^{-1/20} $$

$$ \leq 9e^{2t} N^{-1/20}. $$

Also, for $N$ sufficiently large (still not depending on $t$), $39Ke^{4\delta} N^{-11/10} \leq 40(t + 1)N^{-21/20}$. Therefore for $N$ sufficiently large, for $X \in (\mathbb{R}^d)^N$ and $t > 0$,

$$ \mathbb{P}_X \left( \sup_{r \geq 0} \left| F^{(N)}(r, t) - v^{(N)}(r, t) \right| \geq 9e^{2t} N^{-1/20} \right) \leq 40e^{t} N^{-21/20}. $$

This completes the proof of Proposition 1.5.
4.2 Proof of Proposition 1.6

We begin with the following lemma, which is a consequence of Proposition 1.5 and a concentration estimate for $F^{(N)}(\cdot, 0)$ in the case where $X_i^{(N)}(0), \ldots, X_N^{(N)}(0)$ are i.i.d. with some fixed distribution $\mu_0$. This lemma will be used later to argue that at a fixed time $t$, $F^{(N)}(R_t, t)$ is close to 1, where $(v, R)$ is the solution of the free boundary problem (1) with initial condition $\mu_0$.

**Lemma 4.9.** There exists a constant $c_3 \in (0, 1)$ such that the following holds. Suppose $X_i^{(N)}(0), \ldots, X_N^{(N)}(0)$ are i.i.d. with distribution given by $\mu_0$. Let $v$ denote the solution of (7) with initial condition $v_0(r) = \mu_0(B(r))$. Then for $N$ sufficiently large, for $t \geq 0$,

$$
P \left( \|F^{(N)}(\cdot, t) - v(\cdot, t)\|_{L^\infty} \geq e^{2tN^{-c_3}} \right) \leq e^t N^{-1-c_3}.
$$

The difference between Lemma 4.9 and Proposition 1.5 is that the initial condition for $v(\cdot, t)$ in Lemma 4.9 is given by $v_0(r) = \mu_0(B(r))$, whereas the initial condition for $v^{(N)}(\cdot, t)$ in Proposition 1.5 is given by $v_0(r) = F^{(N)}(r, 0)$.

**Proof.** Recall from Proposition 1.5 that there is a constant $c_1 > 0$ such that for $N$ large enough, for $t \geq 0$,

$$
P \left( \|F^{(N)}(\cdot, t) - v^{(N)}(\cdot, t)\|_{L^\infty} \geq e^{2tN^{-c_1}} \right) \leq e^t N^{-1-c_1},
$$

where $v^{(N)}$ is the solution of (7) with initial condition $v_0^{(N)}(r) = F^{(N)}(r, 0)$. By Lemma 2.3,

$$
\|v^{(N)}(\cdot, t) - v(\cdot, t)\|_{L^\infty} \leq e^t \|F^{(N)}(\cdot, 0) - \mu_0(B(\cdot))\|_{L^\infty},
$$

(40)

The function $v_0(r) = \mu_0(B(r))$ is the cumulative distribution function for each of the real-valued random variables $\|X_i^{(N)}(0)\|, i = 1, \ldots, N$, which are independent. Therefore, it follows immediately from Corollary 1 and Comment 2(iii) of [Mas90] (which is a sharp, quantitative version of the Glivenko-Cantelli theorem), that

$$
P \left( \|F^{(N)}(\cdot, 0) - \mu_0(B(\cdot))\|_{L^\infty} > \epsilon \right) \leq 2e^{-2N\epsilon^2}
$$

(41)

holds for all $\epsilon > 0$ and $N \geq 1$.

For $\epsilon > 0$ to be chosen, let $E$ be the event

$$
E = \left\{ \|F^{(N)}(\cdot, 0) - \mu_0(B(\cdot))\|_{L^\infty} \leq \epsilon \right\}.
$$

Then, for $c_3 > 0$ to be determined, $N$ large enough for Proposition 1.5 to hold, and $t \geq 0$,

$$
P \left( \|F^{(N)}(\cdot, t) - v(\cdot, t)\|_{L^\infty} \geq e^{2tN^{-c_3}} \right)
$$

$$
\leq P \left( \|F^{(N)}(\cdot, t) - v^{(N)}(\cdot, t)\|_{L^\infty} + \|v^{(N)}(\cdot, t) - v(\cdot, t)\|_{L^\infty} \geq e^{2tN^{-c_3}} \right)
$$

$$
\leq P \left( \|F^{(N)}(\cdot, t) - v^{(N)}(\cdot, t)\|_{L^\infty} + \|v^{(N)}(\cdot, t) - v(\cdot, t)\|_{L^\infty} \geq e^{2tN^{-c_3}}, E \right) + P(E^c)
$$

$$
\leq P \left( \|F^{(N)}(\cdot, t) - v^{(N)}(\cdot, t)\|_{L^\infty} + \epsilon^t \geq e^{2tN^{-c_3}}, E \right) + 2e^{-2N\epsilon^2},
$$

by (40) and (41). Choose $c_3 \in (0, \min(c_1, 1/2))$ and $\epsilon = \frac{1}{2} N^{-c_3}$. Then, for $N$ large enough, one has $e^{2tN^{-c_3}} - \epsilon^t \geq e^{2tN^{-c_3}}$ for all $t \geq 0$ (it is sufficient for the inequality to hold at $t = 0$). Therefore

$$
P \left( \|F^{(N)}(\cdot, t) - v(\cdot, t)\|_{L^\infty} \geq e^{2tN^{-c_3}} \right) \leq e^t N^{-1-c_1} + 2e^{-\frac{1}{2} N^{1-2c_3}}.
$$

Since $1 - 2c_3 > 0$, we may take $N$ larger if necessary so that $e^t N^{-1-c_1} + 2e^{-\frac{1}{2} N^{1-2c_3}} \leq e^t N^{-1-c_3}$ for all $t \geq 0$ (it is sufficient for the inequality to hold at $t = 0$), and the proof is complete.
Now consider an $N$-BBM $X^{(N)}$ started from an initial condition $\mathcal{X} \in (\mathbb{R}^d)^N$. Recall the coupling in Section 3 between $X^{(N)}$ and $X^+$, where $X^+(t) = (X_u^+(t))_{u \in \mathcal{N}_t^+}$ is the vector of particle locations at time $t$ in a standard BBM started from the same initial condition $\mathcal{X}$, such that under the coupling, for all $t \geq 0$, $X^{(N)}(t) \subseteq X^+(t)$. Recall also from (24) that under the coupling, for $t \geq 0$, almost surely

$$\{X_k^{(N)}(t)\}_{k=1}^N = \{X_u^+(t)\}_{u \in \mathcal{N}_t^+}, \quad \text{where} \quad \mathcal{N}_t^+ = \left\{ u \in \mathcal{N}_t^+ : \|X_u^+(s)\| \leq M_s^{(N)}(t) \forall s \in [0, t] \right\},$$

and $M_s^{(N)} = \max_{k \in \{1, \ldots, N\}} \|X_k^{(N)}(s)\|$ is the maximum distance of a particle in the $N$-BBM from the origin at time $s$.

For Lemmas 4.10 to 4.12, we also introduce two quantities. For any particle $u \in \mathcal{N}_t^+$ in the BBM at time $t$, let $B_u(s)$ be the displacement of that particle at time $s \in [0, t]$ from its location at time 0:

$$B_u(s) = X_u^+(s) - X_u^+(0).$$

For $\epsilon > 0$ and $r > 0$, let $Z_\epsilon(r)$ denote the number of particles in the BBM at time $\epsilon$ which started (at time 0) from a particle in $B(r)$:

$$Z_\epsilon(r) = \left| \left\{ u \in \mathcal{N}_\epsilon^+ : X_u^+(0) \in B(r) \right\} \right|.$$

**Lemma 4.10.** For $\epsilon, r > 0$, if the following two conditions hold:

- $Z_\epsilon(r) > N$,
- $\max_{u \in \mathcal{N}_\epsilon^+} \sup_{s \in [0, 2\epsilon]} \|B_u(s)\| \leq \frac{1}{3} \epsilon^{1/3},$

then

$$M_s^{(N)} \leq r + \epsilon^{1/3} \forall s \in [\epsilon, 2\epsilon].$$

**Proof.** Assume the hypotheses hold. Then we must have

$$\exists s^* \in [\epsilon, 2\epsilon] : \quad M_s^{(N)} \leq r + \frac{1}{3} \epsilon^{1/3}.$$

Indeed, if we had $M_s^{(N)} > r + \frac{1}{3} \epsilon^{1/3} \forall s \in [0, \epsilon]$, then by (42) and using that $\|B_u(s)\| = \|X_u^+(s) - X_u^+(0)\| \leq \frac{1}{3} \epsilon^{1/3}$,

$$\mathcal{N}_\epsilon^+ \supseteq \left\{ u \in \mathcal{N}_\epsilon^+ : \|X_u^+(s)\| \leq r + \frac{1}{3} \epsilon^{1/3} \forall s \in [0, \epsilon] \right\}$$

$$\supseteq \left\{ u \in \mathcal{N}_\epsilon^+ : \|X_u^+(0)\| < r \right\},$$

which is impossible as the set on the right hand side has size $Z_\epsilon(r)$ and we assumed that $Z_\epsilon(r) > N$.

Then, for any $s \in [s^*, 2\epsilon]$, for any particle $u \in \mathcal{N}_s^{(N)}$, its ancestor at time $s^*$ must have been within distance $M_s^{(N)}$ of the origin, i.e. $\|X_u^+(s^*)\| \leq M_s^{(N)}$. Hence

$$M_s^{(N)} \leq M_s^{(N)} + \max_{u \in \mathcal{N}_s^+} \|B_u(s) - B_u(s^*)\| \leq \left( r + \frac{1}{3} \epsilon^{1/3} \right) + 2 \cdot \frac{1}{3} \epsilon^{1/3},$$

which completes the proof.  

Recall from the definition of $\Gamma$ in (3) that for $r > 0$ and $\delta \in [0, 1)$, $X \in \Gamma(r, 1 - \delta)$ means that at least a fraction $1 - \delta$ of the $N$ particles of the vector $\mathcal{X}$ are in $B(r)$, and, in particular, for $t \geq 0$,

$$X^{(N)}(t) \in \Gamma(r, 1 - \delta) \quad \Leftrightarrow \quad F^{(N)}(r, t) \geq 1 - \delta. \quad (43)$$
Lemma 4.11. There exists a constant $c_4 > 0$ such that for $r > 0$ and $\epsilon \in (0, 1)$, if $X \in \Gamma(r, 1 - \frac{1}{4}\epsilon)$ then

$$
P_X \left( Z_r(r) \leq N \right) \leq e^{-c_4 \epsilon N}.
$$

Proof. Note that since each particle in the BBM branches independently at rate 1, and since we assumed that the number $|X \cap B(r)|$ of initial particles in $B(r)$ is at least $N(1 - \frac{1}{4}\epsilon)$, we have $Z_r(r) \geq N(1 - \frac{1}{4}\epsilon) + \xi$

where $\xi$ is a Poisson random variable with mean $N(1 - \frac{1}{4}\epsilon)\epsilon$. Hence, for any $c > 0$, by Markov’s inequality,

$$
P_X \left( Z_r(r) \leq N \right) \leq P(\xi \leq \frac{1}{4}\epsilon N)
= P \left( e^{-c\epsilon} \geq e^{-\frac{1}{4}\epsilon N} \right)
\leq e^{\frac{1}{4}\epsilon N} E \left[ e^{-c\xi} \right]
= e^{\frac{1}{4}\epsilon N + N(1 - \frac{1}{4}\epsilon)(e^{-c}-1)}
\leq e^{N(1 - \frac{1}{4}\epsilon)c(e^{1/2} + e^{-c} - 1)},
$$

where we used $\frac{1}{4}\epsilon \leq \frac{1}{2}(1 - \frac{1}{4}\epsilon)$ in the last line. Fixing $c > 0$ sufficiently small that $\frac{1}{4}\epsilon + e^{-c} - 1 = -\epsilon < 0$, it follows that

$$
P_X \left( Z_r(r) \leq N \right) \leq e^{-c\epsilon N(1 - \frac{1}{4}\epsilon)} \leq e^{-\frac{1}{4}\epsilon N}. \quad \square
$$

Lemma 4.12. Let $\epsilon = N^{-b}$ for some $b \in (0, 1/2)$. Then for $N$ sufficiently large, for $r > 0$, if $X \in \Gamma(r, 1 - \frac{1}{4}\epsilon)$ then

$$
P_X \left( \sup_{s \in [r, 2r]} M_s^{(N)} > r + \epsilon^{1/3} \right) \leq e^{-\epsilon^{-1/4}}.
$$

Proof. Take $X \in \Gamma(r, 1 - \frac{1}{4}\epsilon)$. By Lemma 4.10, observe that

$$
P_X \left( \sup_{s \in [r, 2r]} M_s^{(N)} > r + \epsilon^{1/3} \right) \leq \mathbb{P}_X \left( Z_r(r) \leq N \right)
+ \mathbb{P}_X \left( \exists u \in N_2^+: \sup_{s \in [0,2r]} \|B_u(s)\| > \frac{1}{2} \epsilon^{1/3} \right). \tag{44}
$$

By Lemma 4.11, the first term on the right hand side is bounded by $e^{-c_4 \epsilon N}$. We focus on the second term. By the many-to-one lemma, recalling that we let $(B_s)_{s \geq 0}$ denote a $d$-dimensional Brownian motion with diffusivity $\sqrt{2}$,

$$
P_X \left( \exists u \in N_2^+: \sup_{s \in [0,2r]} \|B_u(s)\| > \frac{1}{2} \epsilon^{1/3} \right) \leq Ne^{2r} \mathbb{P}_0 \left( \sup_{s \in [0,2r]} \|B_s\| > \frac{1}{2} \epsilon^{1/3} \right).
$$

Letting $\xi_{1,s}, \ldots, \xi_{d,s}$ denote the $d$ coordinates of $B_s$, which are themselves independent one-dimensional Brownian motions,

$$
P_0 \left( \sup_{s \in [0,2r]} \|B_s\| > \frac{1}{2} \epsilon^{1/3} \right) = P_0 \left( \sup_{s \in [0,2r]} (\xi_{1,s}^2 + \cdots + \xi_{d,s}^2) > \frac{1}{9} \epsilon^{2/3} \right)
\leq P_0 \left( \sup_{s \in [0,2r]} \xi_{1,s}^2 > \frac{1}{9d} \epsilon^{2/3} \text{ or } \cdots \text{ or } \sup_{s \in [0,2r]} \xi_{d,s}^2 > \frac{1}{9d} \epsilon^{2/3} \right)
\leq d P_0 \left( \xi_{1,s} > \frac{1}{3\sqrt{d}} \epsilon^{1/3} \right)
\leq 4d \exp \left( -\frac{\epsilon^{-1/3}}{72d} \right), \tag{45}
$$

23
where the fourth line follows by the reflection principle, and the last line by a Gaussian tail bound.

By (44) we now have that
\[
\mathbb{P}(\sup_{s \in [\epsilon,2\epsilon]} M^{(N)}_s > r + \epsilon^{1/3}) \leq e^{-c\epsilon r N} + Ne^{2\epsilon} \cdot 4d \exp\left(-\frac{\epsilon^{-1/3}}{72d}\right) \leq e^{-\epsilon^{-1/4}}
\]
for \(N\) sufficiently large, since \(\epsilon = N^{-b}\) and \(b \in (0,1/2)\).

We can now complete the proof of Proposition 1.6. Recall that we assume that the \(N\)-BBM is started from \(N\) i.i.d. particles with distribution given by some \(\mu_0\), and that \((u,R)\) denotes the solution to the free boundary problem (1) with initial condition \(\mu_0\). As in Lemma 4.9, let \(v\) denote the solution of (7) with initial condition \(v_0(r) = \mu_0(B(r))\), and recall from Section 1.2 that \(v(r,t) = \int_{B(r)} u(x,t) \text{d}x\) and so, in particular, \(v(R_\epsilon,t) = 1\) for \(t > 0\).

Take \(c_3 \in (0,1)\) as in Lemma 4.9, and let \(\epsilon = N^{-c_3/2}\). Then for \(T > 0\), by a union bound,
\[
\mathbb{P}\left(\exists t \in [2\epsilon,T] : M^{(N)}_t > R_\epsilon([\epsilon/\epsilon] - 1 + \epsilon^{1/3})\right) \\
\leq \sum_{k=1}^{[T/\epsilon]-1} \mathbb{P}\left(\sup_{s \in [\epsilon,2\epsilon]} M^{(N)}_{ek+s} > R_{ek} + \epsilon^{1/3}\right) \\
\leq \sum_{k=1}^{[T/\epsilon]-1} \left(\mathbb{P}(E_{ek}^c) + \mathbb{P}\left(E_{ek} : \sup_{s \in [\epsilon,2\epsilon]} M^{(N)}_{ek+s} > R_{ek} + \epsilon^{1/3}\right)\right).
\]
The above is of course valid for any choice of events \(E_t\) for each \(t > 0\) but, in order to use Lemma 4.12, we let \(E_t = \{X^{(N)}(t) \notin \Gamma(R_\epsilon,1 - \frac{1}{4}\epsilon)\}\). Then for \(N\) sufficiently large that \(\frac{1}{4}\epsilon > e^{2T}N^{-c_3}\), recalling the meaning of \(\Gamma\) in (43) and since \(v(R_\epsilon,t) = 1\), for \(t \in (0,T]\),
\[
\mathbb{P}(E_t^c) = \mathbb{P}(X^{(N)}(t) \notin \Gamma(R_\epsilon,1 - \frac{1}{4}\epsilon)) = \mathbb{P}(F^{(N)}(R_\epsilon,t) < 1 - \frac{1}{4}\epsilon) \\
\leq \mathbb{P}\left(\sup_{r \geq 0} |F^{(N)}(r,t) - v(r,t)| \geq e^{2T}N^{-c_3}\right) \\
\leq e^{T}N^{-1-c_3}
\]
by Lemma 4.9. For \(N\) sufficiently large, for \(k \in \mathbb{N}\), by Lemma 4.12 and the Markov property at time \(\epsilon k\) we have
\[
\mathbb{P}\left(E_{ek} : \sup_{s \in [\epsilon,2\epsilon]} M^{(N)}_{ek+s} > R_{ek} + \epsilon^{1/3}\right) \leq e^{-\epsilon^{-1/4}}.
\]
Therefore
\[
\mathbb{P}\left(\exists t \in [2\epsilon,T] : M^{(N)}_t > R_\epsilon([\epsilon/\epsilon] - 1 + \epsilon^{1/3})\right) \leq \mathbb{P}(E_t^c) \leq e^{T}N^{-1-c_3} + e^{-\epsilon^{-1/4}} \leq N^{-1-\frac{1}{8}c_3}
\]
for \(N\) sufficiently large. For \(\eta \in (0,T]\), since \((R_t)_{t \in [\eta,T]}\) is continuous (by Theorem 2.1), for \(N\) sufficiently large,
\[
R_\epsilon([\epsilon/\epsilon] - 1) + \epsilon^{1/3} \leq R_\epsilon + \eta \quad \forall t \in [\eta,T].
\]
(46)
Therefore for \(N\) sufficiently large,
\[
\mathbb{P}\left(\exists t \in [\eta,T] : M^{(N)}_t > R_\epsilon + \eta\right) \leq N^{-1-\frac{1}{8}c_3},
\]
which completes the proof of Proposition 1.6.
5 Proof of Theorem 1.1: Hydrodynamic limit result for $u$

First we notice that it is sufficient to prove that there exists $c_5 > 0$ such that for any $t > 0$, $A \subseteq \mathbb{R}^d$ measurable and $\delta > 0$, for $N$ sufficiently large,

$$\mathbb{P}\left(\mu^{(N)}(A, t) - \int_A u(x, t) \, dx \geq \delta\right) \leq N^{-1-c_5}. \quad (47)$$

Indeed, since $\mu^{(N)}(A, t) + \mu^{(N)}(\mathbb{R}^d \setminus A, t) = 1$ and $\int_A u(x, t) \, dx + \int_{\mathbb{R}^d \setminus A} u(x, t) \, dx = \int_{\mathbb{R}^d} u(x, t) \, dx = 1$,

$$\mathbb{P}\left(\mu^{(N)}(A, t) - \int_A u(x, t) \, dx \leq -\delta\right) = \mathbb{P}\left(\mu^{(N)}(\mathbb{R}^d \setminus A, t) - \int_{\mathbb{R}^d \setminus A} u(x, t) \, dx \geq \delta\right). \quad (48)$$

Hence it follows from (47) that for $t > 0, A \subseteq \mathbb{R}^d$ measurable and $\delta > 0$, for $N$ sufficiently large,

$$\mathbb{P}\left(\left|\mu^{(N)}(A, t) - \int_A u(x, t) \, dx\right| \geq \delta\right) \leq 2N^{-1-c_5},$$

and so by Borel-Cantelli, $\mu^{(N)}(A, t) \rightarrow \int_A u(x, t) \, dx$ almost surely as $N \rightarrow \infty$. Moreover, for $t > 0$ and $\delta > 0$, let $\delta' = 1 - \int_{B(R_t - \delta)} u(x, t) \, dx > 0$ by Theorem 2.1. Then

$$\mathbb{P}\left(M_{t}^{(N)} < R_t - \delta\right) = \mathbb{P}\left(\mu^{(N)}(B(R_t - \delta), t) = 1\right)$$

$$= \mathbb{P}\left(\mu^{(N)}(B(R_t - \delta), t) - \int_{B(R_t - \delta)} u(x, t) \, dx \geq \delta'\right) \leq N^{-1-c_5}$$

for $N$ sufficiently large, by (47). Also, by Proposition 1.6 with $\eta = \min(\delta, t)$, for $N$ sufficiently large,

$$\mathbb{P}\left(M_{t}^{(N)} > R_t + \delta\right) \leq N^{-1-c_2}.$$ 

Therefore, by Borel-Cantelli, $M_{t}^{(N)} \rightarrow R_t$ almost surely as $N \rightarrow \infty$.

It now remains to prove (47). Let $(X^+(t), t \geq 0)$ be a BBM with the same initial particle distribution as the BBM, i.e. such that $X^+(0) = (X^+_0(0))_{i=1}^N$ where $(X^+_0(0))_{i=1}^N$ are i.i.d. with distribution given by $\mu_0$. Recall the coupling described in Section 3 between the N-BBM $X^{(N)}$ and the BBM $X^+$ such that under the coupling, for all $t \geq 0$,

$$X^{(N)}(t) \subseteq X^+(t).$$

Take $t > 0$ and $\eta \in (0, t)$. We let $C_{\eta,t}$ denote the set of locations of particles in the BBM (without killing) at time $t$ whose ancestors at times $s \in [\eta, t]$ were always within distance $R_s + \eta$ of the origin:

$$C_{\eta,t} = \left\{ X^+_u(t) : u \in N^+_t, \|X^+_u(s)\| \leq R_s + \eta \, \forall s \in [\eta, t] \right\}.$$ 

Notice that if $M_{s}^{(N)} \leq R_s + \eta \, \forall s \in [\eta, t]$, then, by (24), almost surely

$$X^{(N)}(t) \subseteq C_{\eta,t}. $$

Therefore, for $A \subseteq \mathbb{R}^d$ measurable and $\delta > 0$,

$$\mathbb{P}\left(\mu^{(N)}(A, t) - \int_A u(x, t) \, dx \geq \delta\right) \leq \mathbb{P}\left(\exists s \in [\eta, t] : M_{s}^{(N)} > R_s + \eta\right)$$

$$+ \mathbb{P}\left(\frac{1}{N} |C_{\eta,t} \cap A| - \int_A u(x, t) \, dx \geq \delta\right) \leq N^{-1-c_2} + \mathbb{P}\left(\frac{1}{N} |C_{\eta,t} \cap A| - \int_A u(x, t) \, dx \geq \delta\right) \quad (49)$$

for $N$ sufficiently large (depending on $\eta$ and $t$) by Proposition 1.6. We now focus on the second term on the right hand side.
Lemma 5.1. For any \( t > 0 \) and \( A \subseteq \mathbb{R}^d \) measurable,

\[
\lim_{\eta \searrow 0} \mathbb{E} \left[ \frac{1}{N} \mathcal{C}_{\eta,t} \cap A \right] = \int_A u(x,t) \, dx \quad \text{uniformly in } N.
\]

**Proof.** We claim that for \( y \in \mathbb{R}^d \),

\[
\mathbb{P}_y(B_t \in A, \| B_s \| \leq R_s \ \forall s \in (0,t]) = \mathbb{P}_y(B_t \in A, \| B_s \| < R_s \ \forall s \in (0,t)).
\]

(In words: the probability that the Brownian motion touches the moving boundary \( R \) at a positive time without crossing it is zero.) We shall begin by showing that the lemma follows from (50), and then prove the claim (50).

For \( \eta \in (0,t) \), by the many-to-one lemma, and since at time 0 the BBM consists of \( N \) particles with locations which are random variables with distribution \( \mu_0 \),

\[
\mathbb{E} \left[ \mathcal{C}_{\eta,t} \cap A \right] = N e^t \int_{\mathbb{R}^d} \mu_0(dy) \mathbb{P}_y \left( B_t \in A, \| B_s \| \leq R_s + \eta \ \forall s \in [\eta,t] \right).
\]

By dominated convergence, and uniformly in \( N \),

\[
\lim_{\eta \searrow 0} \mathbb{E} \left[ \frac{1}{N} \mathcal{C}_{\eta,t} \cap A \right] = e^t \int_{\mathbb{R}^d} \mu_0(dy) \mathbb{P}_y \left( B_t \in A, \| B_s \| \leq R_s \ \forall s \in (0,t) \right)
\]

\[
= e^t \int_{\mathbb{R}^d} \mu_0(dy) \mathbb{P}_y \left( B_t \in A, \| B_s \| < R_s \ \forall s \in (0,t) \right)
\]

\[
= \int_A u(x,t) \, dx,
\]

where the second equality holds by (50) and the last equality follows from (13).

It remains to prove (50): we shall use the following claim. Suppose \( (\xi_s)_{s \in [0,t]} \) is a continuous path in \( \mathbb{R}^d \) with \( \xi_0 = y \) and \( \xi_t = 0 \), and suppose \( e \in \mathbb{R}^d \) is a unit vector. We claim that there are at most two values of \( r \in \mathbb{R} \) such that

\[
\min_{s \in (0,t)} (R_s - \| \xi_s + \frac{r}{2} e \|) = 0,
\]

which we write to mean that the min exists in \( (0,t) \) and is equal to 0; in other words, \( R_s \geq \| \xi_s + \frac{r}{2} e \| \ \forall s \in (0,t) \) and \( \exists s \in (0,t) \) such that \( R_s = \| \xi_s + \frac{r}{2} e \| \). Indeed, we have that

\[
\| \xi_s + \frac{r}{2} e \|^2 = \| \xi_s - (\xi_s \cdot e) e \|^2 + (\xi_s \cdot e + \frac{r}{2})^2,
\]

and so if (51) holds, then the inequality \( R_s \geq \| \xi_s + \frac{r}{2} e \| \) implies that, for each \( s \in (0,t) \),

\[
-\frac{r}{2} \left( (R_s^2 - \| \xi_s - (\xi_s \cdot e) e \|^2)^{1/2} + \xi_s \cdot e \right) \leq r \leq \frac{r}{2} \left( (R_s^2 - \| \xi_s - (\xi_s \cdot e) e \|^2)^{1/2} - \xi_s \cdot e \right).
\]

Moreover, for any value of \( s \in (0,t) \) such that \( R_s = \| \xi_s + \frac{r}{2} e \| \), one of the two inequalities in (52) must be an equality. Therefore

\[
\inf_{s \in (0,t)} \left( \frac{r}{2} \left( (R_s^2 - \| \xi_s - (\xi_s \cdot e) e \|^2)^{1/2} - \xi_s \cdot e \right) \right),
\]

\[
\sup_{s \in (0,t)} \left( -\frac{r}{2} \left( (R_s^2 - \| \xi_s - (\xi_s \cdot e) e \|^2)^{1/2} + \xi_s \cdot e \right) \right),
\]

which establishes the claim that (51) holds for at most two values of \( r \).

Now for \( y \in \mathbb{R}^d \), under the probability measure \( \mathbb{P}_y \), let \( (\xi_s)_{s \in [0,t]} \) denote a \( d \)-dimensional Brownian bridge with diffusivity \( \sqrt{2} \) from \( y \) to \( 0 \) in time \( t \). Then

\[
\mathbb{P}_y \left( \{ \| B_s \| \leq R_s \ \forall s \in (0,t] \} \cap \{ \exists s \in (0,t) : \| B_s \| = R_s \} \right)
\]
Recall that we let $F$ or Lemma 5.2. by Fubini’s theorem, and where $\Phi$ given by $s$ that $\min_{s \in (0,t)} (R_s - \|\xi_s + \frac{z}{t}\|) = 0$

\[ = \mathbb{E}_y \left[ \frac{1}{N} \right] \left[ \|B_s\| \leq R_s \forall s \in (0,t) \right] \cap \left\{ \exists s \in (0,t) : \|B_s\| = R_s \right\} \right] \left| B_t \right| \right] \]

\[ = \int_{\mathbb{R}^d} dz \Phi_t(y - z) \mathbb{P}_y \left[ \min_{s \in (0,t)} (R_s - \|\xi_s + \frac{z}{t}\|) = 0 \right] \]

\[ = \mathbb{E}_y \left[ \int_{\mathbb{R}^d} dz \Phi_t(y - z) \mathbb{I}_{\{\min_{s \in (0,t)} (R_s - \|\xi_s + \frac{z}{t}\|) = 0 \}} \right] \]

by Fubini’s theorem, and where $\Phi_t(x) = (4\pi t)^{-d/2} e^{-\|x\|^2/(4t)}$ is the heat kernel. By (51) we have that $\min_{s \in (0,t)} (R_s - \|\xi_s + \frac{z}{t}\|) \neq 0$ for almost every $z$, and (50) follows.

\[ \square \]

**Lemma 5.2.** For $N$ sufficiently large, for any $A \subseteq \mathbb{R}^d$ measurable and any $0 < \eta < t$,

\[ \mathbb{E} \left[ \left( \frac{1}{N} |C_{\eta,t} \cap A| - \mathbb{E} \left[ \frac{1}{N} |C_{\eta,t} \cap A| \right] \right)^4 \right] \leq 13e^{4t}N^{-2}. \]

**Proof.** Recall that we let $X^+(0) = (X^+_i(0))_{i=1}^N$, where $(X^+_i(0))_{i=1}^N$ are i.i.d. with distribution given by $\mu_0$. As in the proof of Lemma 4.2, denote by $X^{+,i}$ the family of particles descended from the $i$-th particle in the initial configuration $X^+(0)$. The $X^{+,i}$ form a family of independent BBMs, and for each $i$ the process $X^{+,i}$ is started from a single particle at location $X^+_i(0)$. Fix $0 < \eta < t$, write $n_i = |X^{+,i}(t)|$ for the number of particles descended from $X^+_i(0)$ at time $t$ and introduce $n_{i,A}$ as the number of particles in $C_{\eta,t} \cap A$ which are descendants of particle $X^+_i(0)$:

\[ n_{i,A} = |C_{\eta,t} \cap A \cap X^{+,i}(t)|. \]

Then $|C_{\eta,t} \cap A| = \sum_{i=1}^N n_{i,A}$ and $(n_{i,A})_{i=1}^N$ are i.i.d., so

\[ \mathbb{E} \left[ \left( |C_{\eta,t} \cap A| - \mathbb{E} \left[ |C_{\eta,t} \cap A| \right] \right)^4 \right] = \mathbb{E} \left[ \left( \sum_{i=1}^N (n_{i,A} - \mathbb{E}[n_{i,A}]) \right)^4 \right] = \sum_{i=1}^N \mathbb{E} \left[ (n_{i,A} - \mathbb{E}[n_{i,A}])^4 \right] + 6 \sum_{i,j=1}^N \text{Var}(n_{i,A}) \text{Var}(n_{j,A}). \]

By the same argument as in (28), (29) and (30) in the proof of Lemma 4.2,

\[ \mathbb{E} \left[ (n_{i,A} - \mathbb{E}[n_{i,A}])^4 \right] \leq 2 \mathbb{E} [n_i^4] \leq 48e^{4t} \quad \text{and} \quad \text{Var}(n_{i,A}) \leq \mathbb{E} [n_i^2] \leq 2e^{2t}. \]

Therefore

\[ \mathbb{E} \left[ \left( |C_{\eta,t} \cap A| - \mathbb{E} \left[ |C_{\eta,t} \cap A| \right] \right)^4 \right] \leq N \cdot 48e^{4t} + 3N(N - 1)(2e^{2t})^2 \leq 13e^{4t}N^2 \]

for $N$ sufficiently large.

\[ \square \]

We can now conclude; for fixed $t > 0$, $A \subseteq \mathbb{R}^d$ measurable and $\delta > 0$, let $\eta > 0$ be sufficiently small that, by Lemma 5.1,

\[ \left| \frac{1}{N} \mathbb{E} \left[ |C_{\eta,t} \cap A| \right] - \int_A u(x,t) \, dx \right| < \frac{\delta}{2} \quad \forall N \in \mathbb{N}. \]
Then for $N$ sufficiently large,
\[
\mathbb{P}
\left(\frac{1}{N}|\mathcal{C}_{n,t} \cap A| - \int_A u(x,t) \, dx \geq \delta\right) \leq \mathbb{P}\left(\frac{1}{N}|\mathcal{C}_{n,t} \cap A| - \frac{1}{N} \mathbb{E}\left[|\mathcal{C}_{n,t} \cap A|\right] > \frac{\delta}{2}\right)
\leq \frac{16}{\delta^2} \cdot 13e^{4t} N^{-2},
\]
by Lemma 5.2 and Markov’s inequality. By (49), it follows that for $N$ sufficiently large,
\[
\mathbb{P}\left(\mu^{(N)}(A,t) - \int_A u(x,t) \, dx \geq \delta\right) \leq N^{-1-\epsilon_2} + 16\delta^{-4} \cdot 13e^{4t} N^{-2}
\leq N^{-1-\frac{1}{2}\epsilon_2},
\]
for $N$ large enough, which establishes (47) and completes the proof of Theorem 1.1.

6 Proof of Proposition 1.7 and of Theorems 1.2, 1.3 and 1.4

We begin by proving Proposition 1.7. We shall first describe heuristically how the proof works. Recall from (3) that for $K > 0$ and $c \in (0,1],$
\[
\Gamma(K, c) = \left\{\mathcal{X} \in (\mathbb{R}^d)^N : \frac{1}{N} \sum_{i \in \mathcal{X}} |\{i : \|X_i\| < K\}| \geq c \right\}
\]
is the set of “good” initial particle configurations that have at least a fraction $c$ of particles within distance $K$ of the origin (the dependence on $N$ is implicit). Recall from (25) that $F^{(N)}(\cdot, t)$ is the empirical cumulative distribution of the particles at time $t$. Let us explain how we go about proving that for $\epsilon > 0$ there exist $N_\epsilon, T_\epsilon$ such that
\[
\mathbb{P}_\mathcal{X}\left(\sup_{r \geq 0} |F^{(N)}(r,t) - V(r)| \geq \epsilon\right) < \epsilon
\]
holds for any $N \geq N_\epsilon, t \geq T_\epsilon$ and $\mathcal{X} \in \Gamma(K, c)$.

Let $K_0 = 3$, and take $c_0$ as in Proposition 2.5 and $t_\epsilon = t_\epsilon(c_0, K_0)$ as in Proposition 2.4. We are going to show that there exists a time $t_a = t_a(c,K)$ independent of $N$ such that, for $t$ large enough,
\[
\mathcal{X} \in \Gamma(K, c) \implies X^{(N)}(t_a) \in \Gamma(K_0, c_0) \text{ with high probability}
\implies X^{(N)}(t - t_a) \in \Gamma(K_0, c_0) \text{ with high probability}
\implies F^{(N)}(\cdot, t) \text{ is close to } V \text{ with high probability.}
\]

For the first step, we use Proposition 2.5 to choose $t_a$ such that $v^{(N)}(r, t_a) \geq 2c_0 I_{\{r \geq K_0\}}$ for all $r \geq 0$, where $v^{(N)}$ denotes the solution of the obstacle problem (7) with initial condition $v_0(r) = F^{(N)}(r, 0) = N^{-1}|\mathcal{X} \cap B(r)|$. Then Proposition 1.5 tells us that $X^{(N)}(t_a) \in \Gamma(K_0, c_0)$ with high probability for $N$ large enough.

The second step will be provided by Proposition 6.2 below, with Proposition 6.1 as an intermediate result.

For the third step, let $\tilde{v}$ denote the solution of the obstacle problem (7) with initial condition $\tilde{v}_0(r) = F^{(N)}(r, t - t_\epsilon)$. From the second step, $X^{(N)}(t - t_\epsilon) \in \Gamma(K_0, c_0)$ with high probability, which is equivalent to $\tilde{v}_0(K_0) \geq c_0$. Proposition 2.4 then gives us that $\tilde{v}(\cdot, t_\epsilon)$ is close to $V(\cdot)$. Furthermore, Proposition 1.5, together with the Markov property at time $t - t_\epsilon$, implies that $F^{(N)}(\cdot, t)$ is close to $\tilde{v}(\cdot, t_\epsilon)$, which in turn is close to $V(\cdot)$.

We start by proving the two propositions needed for the second step, and then we prove Proposition 1.7. The first proposition implies that if $X^{(N)}(0) \in \Gamma(m, c)$ for some $m$ and $c$, then, for $j$ large and $t$ not too large, with high probability $X^{(N)}(t) \in \Gamma(m + j, c)$. 28
Proposition 6.1. For any $c \in (0,1)$, any $N \in \mathbb{N}$, any $t \geq 0$ and any $m > 0$, if $X \in \Gamma(m,c)$, then
\[ \mathbb{P}_X (X^{(N)}(t) \notin \Gamma(m+j,c)) \leq 4dN e^t e^{-j^2/(36d t)} \text{ for all } j > 0. \]

Proof. The argument, in particular the final Gaussian tail estimate, is very similar to the one used in the proof of Lemma 4.12. Recall the coupling in Section 3 between the $N$-BBM $X^{(N)}$ and the BBM $X^+$ such that under the coupling, $X^{(N)}(0) = X^+(0)$ and for $t \geq 0$, almost surely
\[ \{X^{(N)}(t)\}_{k=1}^N = \{X^+(t)\}_{u \in \mathcal{N}^+_i} \text{, where } \mathcal{N}_i^+ = \{u \in \mathcal{N}_i^+ : \|X_u^+(s)\| \leq M^+_i(s) \forall s \in [0,t]\}. \]

Fix $t \geq 0$ and $j > 0$, and define the event
\[ E = \left\{ \max_{u \in \mathcal{N}_i^+} \sup_{s \in [0,t]} \|X_u^+(s) - X_u^+(0)\| < \frac{1}{3}j \right\}. \]

We claim that on the event $E$, if $X^{(N)}(0) \in \Gamma(m,c)$ for some $c \in (0,1)$ and $m > 0$, then $X^{(N)}(t) \in \Gamma(m+j,c)$. The proof of the claim is split into two cases, depending on the stopping time
\[ \tau = \inf \{s \geq 0 : M_s^+ \leq m + \frac{1}{j}j \}. \]

We first consider the case $\tau > t$, meaning that $M_s^+ > m + \frac{1}{j}j \forall s \leq t$. By (53), on the event $E \cap \{ \tau > t \}$ we have
\[ X^{(N)}(t) \supseteq \left\{ X_u^+(t) : u \in \mathcal{N}_i^+, \|X_u^+(s)\| < m + \frac{1}{j}j \forall s \in [0,t] \right\} \supseteq \left\{ X_u^+(t) : u \in \mathcal{N}_i^+, \|X_u^+(0)\| < m \right\}, \]

where we used $\|X_u^+(s) - X_u^+(0)\| < \frac{1}{j}j$ in the last line. Now using that $\|X_u^+(t) - X_u^+(0)\| < j$ on the event $E$, it follows that if $X^{(N)}(0) \in \Gamma(m,c)$ then $X^{(N)}(t) \in \Gamma(m+j,c)$.

We now consider the second case, $\tau \leq t$. Take $u \in \mathcal{N}_i^+$. On the event $E \cap \{ \tau \leq t \}$, $\|X_u^+(\tau)\| \leq M_u^+ \leq m + \frac{1}{j}j$, and by the triangle inequality,
\[ \|X_u^+(t)\| \leq \|X_u^+(t) - X_u^+(0)\| + \|X_u^+(\tau) - X_u^+(0)\| + \|X_u^+(\tau)\| < m + j \]

by the definition of the event $E$. By (53), this shows that $\|X^{(N)}(k)\| < m + j \forall k \in \{1, \ldots, N\}$, and, in particular, $X^{(N)}(t) \in \Gamma(m+j,c)$. This completes the proof of the claim.

Hence for $X \in \Gamma(m,c)$,
\[ \mathbb{P}_X (X^{(N)}(t) \notin \Gamma(m+j,c)) \leq \mathbb{P}_X (E^c) \leq N e^t \mathbb{P}_0 \left( \sup_{s \in [0,t]} \|B_s\| \geq \frac{1}{3}j \right) \]

by the many-to-one lemma. Letting $\xi_{1,s}, \ldots, \xi_{d,s}$ denote the coordinates of $B_s$, we have
\[ \mathbb{P}_X (X^{(N)}(t) \notin \Gamma(m+j,c)) \leq N e^t \mathbb{P}_0 \left( \sup_{s \in [0,t]} \xi_{1,s}^2 \geq \frac{1}{36}j^2 \text{ or } \ldots \text{ or } \sup_{s \in [0,t]} \xi_{d,s}^2 \geq \frac{1}{36}j^2 \right) \]
\[ \leq N e^t \cdot d \mathbb{P}_0 \left( \sup_{s \in [0,t]} |\xi_{1,s}| \geq \frac{1}{3\sqrt{d}j} \right) \]
\[ \leq N e^t \cdot 4d \mathbb{P}_0 \left( |\xi_{1,t}| \geq \frac{1}{3\sqrt{d}j} \right) \]
\[ \leq 4dN e^t e^{-j^2/(36dt)}, \]

where the third inequality follows by the reflection principle and the fourth by a Gaussian tail estimate. \qed
We now show that for \( c_0 \in (0, 1) \) and \( K_0 > 0 \) appropriately chosen, if \( N \) and \( t \) are sufficiently large, if \( X^{(N)}(0) \in \Gamma(K_0, c_0) \) then \( X^{(N)}(t) \in \Gamma(K_0, c_0) \) with high probability.

**Proposition 6.2.** Take \( t_0 > 1 \) and \( c_0 > 0 \) as in Proposition 2.5, and fix \( K_0 = 3 \). For \( \epsilon > 0 \), there exist \( N'_\epsilon < \infty \) and \( t'_\epsilon < \infty \) such that for \( N \geq N'_\epsilon \), the following holds. For \( t \geq t'_\epsilon \),

\[
\inf_{X \in \Gamma(K_0, c_0)} \mathbb{P}_X \left( X^{(N)}(t) \in \Gamma(K_0, c_0) \right) \geq 1 - \epsilon. \tag{54}
\]

Furthermore, for \( t_1 \in [t_0, 2t_0] \) and any \( K > 0 \),

\[
\sup_{X \in \Gamma(K_0, c_0)} \mathbb{E}_X \left[ \inf\{ n \geq 1 : X^{(N)}(nt_1) \in \Gamma(K_0, c_0) \} \right] < \infty. \tag{55}
\]

Note that for any initial condition \( X \in (\mathbb{R}^d)^N \), we have that \( X \in \Gamma(K_0, c_0) \) where \( K = \max_{i \leq N} \| X_i \| + 1 \), and so it follows immediately from (55) that for \( N \) sufficiently large, for \( X \in (\mathbb{R}^d)^N \) and \( t_1 \in [t_0, 2t_0] \),

\[
\mathbb{P}_X \left( \inf\{ n \geq 1 : X^{(N)}(nt_1) \in \Gamma(K_0, c_0) \} < \infty \right) = 1. \tag{56}
\]

**Proof.** The proof uses Propositions 1.5, 6.1 and 2.5 to establish a coupling with a Markov chain. Take \( \delta \in (0, 1/16) \) sufficiently small that \( \frac{1}{1+15\delta} \geq 1 - \frac{1}{4} \epsilon \). Suppose \( N'_\epsilon \) is sufficiently large that for \( N \geq N'_\epsilon \) we have

\[
4dN \exp(2t_0) e^{-(\log N)^{2/3}/(144d\delta)} < \frac{1}{2^\delta} \quad \text{and} \quad \frac{(\log N)^2}{144d\delta} > \log 2. \tag{57}
\]

Also, recalling the definition of \( c_1 \) in Proposition 1.5, suppose that \( N'_\epsilon \) is sufficiently large that for \( N \geq N'_\epsilon \), Proposition 1.5 holds,

\[
e^{2t_0} N^{-c_1} \leq c_0 \quad \text{and} \quad e^{2t_0} N^{-1-c_1} \leq \delta 2^{-(\log N)^{2/3}-1}. \tag{58}
\]

Take \( N \geq N'_\epsilon \) and \( t_1 \in [t_0, 2t_0] \). We first show that

\[
X \in \Gamma(m, c_0) \implies \mathbb{P}_X \left( X^{(N)}(t_1) \notin \Gamma(m + j - 1, c_0) \right) \leq \delta 2^{-j} \quad \forall j, m \in \mathbb{N}_0 \text{ with } m \geq K_0. \tag{59}
\]

Take \( m \in \mathbb{N} \) with \( m \geq K_0 \) and suppose \( X \in \Gamma(m, c_0) \). We first assume that \( j \in \mathbb{N} \) with \( j \geq (\log N)^{2/3} + 1 \). Then by Proposition 6.1, the fact that \( t_1 \in [t_0, 2t_0] \) and then by (57) we have

\[
\mathbb{P}_X \left( X^{(N)}(t_1) \notin \Gamma(m + j - 1, c_0) \right) \leq 4dN \exp(2t_0) e^{-(j-1)^2/(172d\delta)} \leq \frac{1}{2^\delta} e^{-(j-1)^2/(144d\delta)} \leq \delta 2^{-j}.
\]

We now consider the case \( j \leq (\log N)^{2/3} + 1 \). By Proposition 2.5, for \( v^{(N)} \) the solution of (7) with \( v_0(r) = N^{-1}|X \cap B(r)| \), we have that \( v^{(N)}(m - 1, t_1) \geq 2c_0 \). Therefore by Proposition 1.5, since, by (58), \( e^{2t_1} N^{-c_1} \leq c_0 \),

\[
\mathbb{P}_X \left( X^{(N)}(t_1) \notin \Gamma(m - 1, c_0) \right) \leq \mathbb{P}_X \left( \left| F^{(N)}(m - 1, t_1) - v^{(N)}(m - 1, t_1) \right| \geq c_0 \right) \leq e^{t_1} N^{-1-c_1}.
\]

In particular, this and the condition (58) imply that for \( j \in \mathbb{N}_0 \) with \( j \leq (\log N)^{2/3} + 1 \),

\[
\mathbb{P}_X \left( X^{(N)}(t_1) \notin \Gamma(m + j - 1, c_0) \right) \leq \mathbb{P}_X \left( X^{(N)}(t_1) \notin \Gamma(m - 1, c_0) \right) \leq e^{t_1} N^{-1-c_1} \leq 2^{-j},
\]

and (59) is proved.

Let us define the sequence of random variables \( (\theta_n)_{n=0}^{\infty} \) by

\[
\theta_n = \min \left\{ i \in \mathbb{N}_0 : X^{(N)}(nt_1) \notin \Gamma(K_0 + i, c_0) \right\} \quad \text{and} \quad \theta_n = \min \left\{ i \in \mathbb{N}_0 : F^{(N)}(K_0 + i, nt_1) \geq c_0 \right\}.
\]
Although $\theta_n$ itself is not a Markovian process, (59) and the Markov property applied to $X^{(N)}$ implies that for all $X \in (\mathbb{R}^d)^N$ and $n, i, j \in \mathbb{N}_0$,
\[
\theta_n \leq i \implies \mathbb{P}_X (\theta_{n+1} \geq i + j \mid \mathcal{F}_{nt_1}) \leq \delta^{-j}.
\] (60)

Define a Markov chain $(Y_n)_{n=0}^\infty$ on $\mathbb{N}_0$ as follows. For $n \in \mathbb{N}_0$ and $i, j \in \mathbb{N}_0$, let
\[
\mathbb{P} (Y_{n+1} = j \mid Y_n = i) = p_{i,j},
\]
where
\[
p_{0,j} = \begin{cases} 1 - \delta & \text{if } j = 0 \\ 2^{-j} & \text{if } j \geq 1 \end{cases}
\text{ and, for } i \geq 1, \quad p_{i,i+j} = \begin{cases} 1 - 2\delta & \text{if } j = -1 \\ 2^{-j} & \text{if } j \geq 0. \end{cases}
\]

Suppose for $K > 0$ that $X \in \Gamma(K, c_0)$. Then by (60), conditional on $X^{(N)}(0) = X$ and $Y_0 = \max(0, [K - K_0])$, we can couple $(X^{(N)}(nt_1))_{n=0}^\infty$ and $(Y_n)_{n=0}^\infty$ in such a way that almost surely, $\theta_n \leq Y_n$ holds for each $n \in \mathbb{N}_0$, which means that

$$X^{(N)}(nt_1) \in \Gamma(K_0 + Y_n, c_0).$$

For $j \in \mathbb{N}_0$, introduce $m_j \geq 1$ as the expected number of steps needed for $Y_n$ to reach zero starting from $Y_0 = j$:

$$m_j := \mathbb{E} [\inf \{n \geq 1 : Y_n = 0 \} \mid Y_0 = j].$$

Then for $n \in \mathbb{N}_0$ and $X \in \Gamma(K, c_0)$, the coupling implies

$$\mathbb{P}_X (X^{(N)}(nt_1) \notin \Gamma(K_0, c_0)) = \mathbb{P}_X (\theta_n > 0) \leq \mathbb{P} (Y_n \neq 0 \mid Y_0 = [K - K_0] \lor 0)$$ (61)

and
\[
\mathbb{E}_X [\inf \{n \geq 1 : X^{(N)}(nt_1) \in \Gamma(K_0, c_0)\}] \leq m_{[K - K_0] \lor 0},
\]

Note also that
\[
m_0 = 1 - \delta + \sum_{j=1}^{\infty} \delta^{2-j} m_j. \quad (63)
\]

We now bound $m_j$ for $j \in \mathbb{N}$. Let $(A_i)_{i=1}^\infty$ be i.i.d. with $A_i \sim \text{Bernoulli}(2\delta)$, and let $(G_i)_{i=1}^\infty$ be i.i.d. geometric random variables with $\mathbb{P}(G_1 = k) = 2^{-k-1}$ for $k \in \mathbb{N}_0$. Then for $j \in \mathbb{N},$
\[
m_j = \mathbb{E} \left[ \inf \left\{ n \geq 1 : \sum_{i=1}^{n} (A_i G_i - (1 - A_i)) \leq 0 \right\} \right]
= \mathbb{E} \left[ \inf \left\{ n \geq 1 : \sum_{i=1}^{n} A_i (G_i + 1) \leq n - j \right\} \right]
\leq 1 + \sum_{k=1}^{\infty} \mathbb{P} \left( \sum_{i=1}^{k} A_i (G_i + 1) > k - j \right),
\]
(64)
since for a random variable $Z$ taking values in $\mathbb{N}_0$, we have that $\mathbb{E}[Z] = \sum_{k=0}^{\infty} \mathbb{P}(Z > k)$, and since for $k \geq 1,$
\[
\mathbb{P} \left( \inf \left\{ n \geq 1 : \sum_{i=1}^{n} A_i (G_i + 1) \leq n - j \right\} > k \right) \leq \mathbb{P} \left( \sum_{i=1}^{k} A_i (G_i + 1) > k - j \right).
\]

For $k \in \mathbb{N}$ and $\lambda > 0$, by Markov’s inequality,
\[
\mathbb{P} \left( \sum_{i=1}^{k} A_i (G_i + 1) > k - j \right) \leq e^{-\lambda(k-j)} \mathbb{E} \left[ e^{\lambda \sum_{i=1}^{k} A_i (G_i + 1)} \right] \leq e^{-\lambda(k-j)} \mathbb{E} \left[ e^{\lambda \sum_{i=1}^{k} A_i} \right].
\]

31
\[ e^{-\lambda(k-j)} \mathbb{E} \left[ e^{\lambda A_1(G_1+1)} \right]^k. \]

For \( \lambda \in (0, \log 2) \),
\[ \mathbb{E} \left[ e^{\lambda A_1(G_1+1)} \right] = 2\delta \frac{1-Pe^{\lambda}}{1-\frac{1}{2}e^{\lambda}} + 1 - 2\delta. \]

Hence letting \( \lambda = \log(3/2) \),
\[ \mathbb{E} \left[ e^{\log(3/2) A_1(G_1+1)} \right] = 1 + 4\delta < \frac{5}{4} \]

since we chose \( \delta < \frac{1}{16} \). It follows that for \( k \in \mathbb{N} \),
\[ \mathbb{P} \left( \sum_{i=1}^{k} A_i(G_i + 1) > k - j \right) \leq \left( \frac{2}{3} \right)^j \left( \frac{5}{6} \right). \]

Hence by (64),
\[ m_j \leq 1 + \left( \frac{2}{3} \right)^j \sum_{k=1}^{\infty} \left( \frac{5}{6} \right) = 1 + 5 \left( \frac{2}{3} \right)^j. \]

Therefore by (63),
\[ m_0 \leq 1 - \delta + \sum_{j=1}^{\infty} \delta 2^{-j} \left[ 1 + 5 \left( \frac{2}{3} \right)^j \right] = 1 + 15\delta. \]

It follows that \( (Y_n)_{n=0}^{\infty} \) is positive recurrent, and since it is also irreducible and aperiodic, by convergence to equilibrium for Markov chains we have that as \( n \to \infty \),
\[ \mathbb{P}(Y_n = 0 | Y_0 = 0) \to \frac{1}{m_0} \geq \frac{1}{1 + 15\delta}. \]

Since \( \frac{1}{1 + 15\delta} \geq 1 - \frac{1}{1+\epsilon} \), there exists \( n_{\epsilon} < \infty \) such that for \( n \geq n_{\epsilon} \),
\[ \mathbb{P}(Y_n \neq 0 | Y_0 = 0) < \epsilon, \]

and so by (61), for \( \mathcal{X} \in \Gamma(K_0, c_0) \) and \( n \geq n_{\epsilon} \),
\[ \mathbb{P}_\mathcal{X} \left( X^{(\mathcal{X})}(nt_1) \notin \Gamma(K_0, c_0) \right) \leq \epsilon. \]

Let \( t_{\epsilon}' = \max(n_{\epsilon} t_0, 2t_0) \). Then for \( t \geq t_{\epsilon}' \), we have that \( \lfloor t/t_0 \rfloor \geq n_{\epsilon} \) and \( t/\lfloor t/t_0 \rfloor \in [t_0, 2t_0] \).

Therefore (54) follows from (66) with \( t_1 = t/\lfloor t/t_0 \rfloor \) and \( n = \lfloor t/t_0 \rfloor \).

Finally, note that by (62) and (65), for \( K > 0 \), if \( \mathcal{X} \in \Gamma(K, c_0) \) then
\[ \mathbb{E}_\mathcal{X} \left[ \inf\{n \geq 1 : X^{(\mathcal{X})}(nt_1) \in \Gamma(K_0, c_0)\} \right] \leq 1 + 5 \left( \frac{2}{3} \right)^{\lceil K - K_0 \rceil} V_0, \]

which establishes (55) and completes the proof. \( \square \)

**Proof of Proposition 1.7.** Take \( t_0 \) and \( c_0 \) as in Proposition 2.5, and fix \( K_0 = 3 \). Take \( \epsilon > 0 \) and take \( N_\epsilon \) and \( t_{\epsilon}' \) as in Proposition 6.2. Take \( t_{\epsilon} = t_{\epsilon}(c_0, K_0) \) as defined in Proposition 2.4. Take \( c \in (0,c_0) \) and \( K \geq K_0 \), and let
\[ L = \lceil K - K_0 \rceil + 1 + \left\lceil \frac{\log(c_0/c)}{\log 2} \right\rceil. \]

(67)
Recall the definition of $c_1$ in Proposition 1.5, and suppose $N \geq N'_1$ is sufficiently large that Proposition 1.5 holds and that

$$e^{2L_0}N^{-c_1} \leq c_0, \quad e^{L_0}N^{-1-c_1} \leq \epsilon, \quad e^{2L_0}N^{-c_1} < \epsilon \quad \text{and} \quad e^{L_0}N^{-1-c_1} < \epsilon.$$  

Take $X \in \Gamma(K,c)$, and let $v^{(N)}$ denote the solution of (7) with initial condition $v_0(r) = N^{-1}|X \cap B(r)|$, which satisfies $v_0(r) \geq c\mathbb{I}_{\{r \geq K\}}$. By Proposition 2.5, recalling the definition of $L$ in (67), we have the lower bound

$$v^{(N)}(r, L_0) \geq 2c_0\mathbb{I}_{\{r \geq K_0\}}, \quad \forall \; r \geq 0. \quad (68)$$

(The time $L_0$ is the same as the time $t_0$ mentioned in the outline at the start of Section 6.)

Now we compare $F^{(N)}(\cdot, L_0)$ with $v^{(N)}(\cdot, L_0)$, to show that $F^{(N)}(K_0, L_0) \geq c_0$ with high probability. By (68) and by Proposition 1.5, since $N$ is sufficiently large that $e^{2L_0}N^{-c_1} \leq c_0$ and $e^{L_0}N^{-1-c_1} \leq \epsilon$, we now have that

$$\mathbb{P}_X( F^{(N)}(K_0, L_0) \leq c_0 ) \leq \mathbb{P}_X \left( \sup_{r \geq 0} |F^{(N)}(r, L_0) - v^{(N)}(r, L_0)| \geq c_0 \right) \leq \epsilon. \quad (69)$$

Take $t \geq t'_c + L_0$. Then

$$\mathbb{P}_X( F^{(N)}(K_0, t) < c_0 ) \leq \mathbb{P}_X( F^{(N)}(K_0, L_0) \leq c_0 ) + \mathbb{P}_X( F^{(N)}(K_0, L_0) \geq c_0, F^{(N)}(K_0, t) < c_0 ) \leq \epsilon + \mathbb{E}_X \left[ \mathbb{P}(X^{(N)}(L_0) \notin \Gamma(K_0,c_0)) \mathbb{I}_{\{X^{(N)}(L_0) \in \Gamma(K_0,c_0)\}} \right] \leq 2\epsilon, \quad (70)$$

where the second inequality follows by (69) and the last inequality follows by (54) in Proposition 6.2.

Now note that for any configuration $X \in \Gamma(K_0, c_0)$, letting $\bar{v}$ denote the solution of (7) with initial condition $v_0(r) = N^{-1}|X \cap B(r)|$, we have by Proposition 2.4 that $\sup_{r \geq 0} |\bar{v}(r, t_{e}) - V(r)| < \epsilon$. Hence

$$\mathbb{P}_X \left( \sup_{r \geq 0} |F^{(N)}(r, t_{e}) - V(r)| \geq 2\epsilon \right) \leq \mathbb{P}_X \left( \sup_{r \geq 0} |F^{(N)}(r, t_{e}) - \bar{v}(r, t_{e})| \geq \epsilon \right) \leq \epsilon, \quad (71)$$

by Proposition 1.5, since $N$ is large enough that $e^{2L_0}N^{-c_1} \leq \epsilon$ and $e^{L_0}N^{-1-c_1} \leq \epsilon$.

Hence for $t \geq t'_c + L_0 + t_{e}$ and $X \in \Gamma(K,c)$,

$$\mathbb{P}_X \left( \sup_{r \geq 0} |F^{(N)}(r, t) - V(r)| \geq 2\epsilon \right) \leq \mathbb{P}_X \left( \sup_{r \geq 0} |F^{(N)}(r, t) - V(r)| \geq 2\epsilon, F^{(N)}(K_0, t - t_{e}) \geq c_0 \right)$$

$$\quad \quad + \mathbb{P}_X \left( F^{(N)}(K_0, t - t_{e}) < c_0 \right) \leq \epsilon + 2\epsilon, \quad (72)$$

using (70) for the second term and (71) with the Markov property and $\tilde{X} := X^{(N)}(t - t_{e})$ for the first term. (Indeed, $F^{(N)}(K_0, t - t_{e}) \geq c_0$ is equivalent to $X^{(N)}(t - t_{e}) \in \Gamma(K_0,c_0)$.) This concludes the proof of (10), the first statement of Proposition 1.7. We now turn to proving (12), the third statement.

Assume that $t \geq t'_c + L_0 + t_{e} + 1$ and $X \in \Gamma(K,c)$, and introduce $\lambda = N^{-c_1/3}$. For any family $(E_k)_{k=0}^{\infty}$ of events, we have using (70) that

$$\mathbb{P}_X \left( \sup_{s \in [0,1]} M^N_{t+s} > R_{\infty} + 2\epsilon \right) \leq 2\epsilon + \sum_{k=0}^{\lfloor 1/\lambda \rfloor} \mathbb{P}_X \left( E_k, \sup_{s \in [\lambda, 2\lambda]} M^N_{t+(k-1)\lambda+s} > R_{\infty} + 2\epsilon \right)$$

33
\[ E_k = \{ F^{(N)}(R_\infty + \epsilon, t + (k-1)\lambda) \geq 1 - N^{-c_1/2} \}. \]

For \( \bar{X} \in \Gamma(K_0, c_0) \), let \( \bar{v} \) denote the solution of (7) with initial condition
\[ v_0(r) = N^{-1} |\bar{X} \cap B(r)| \]
and let \( \bar{R}_t = \inf\{ r \geq 0 : \bar{v}(r,t) = 1 \} \) for \( t > 0 \). Then by Proposition 2.4, for \( t \geq t_\epsilon \), \( |\bar{R}_t - R_\infty| < \epsilon \) and so \( \bar{v}(R_\infty + \epsilon, t) = 1 \). Hence by Proposition 1.5, for \( k \in \{0, \ldots, [1/\lambda]\} \),
\[ \mathbb{P}_X \left( F^{(N)}(R_\infty + \epsilon, t_\epsilon + k\lambda) \leq 1 - e^{2(t_\epsilon + 1)N^{-c_1}} \right) \leq e^{t_\epsilon + 1 N^{-1-c_1}}. \]

For \( N \) sufficiently large that \( e^{2(t_\epsilon + 1)N^{-c_1}} < N^{-c_1/2} \), by the Markov property at time \( t - t_\epsilon - \lambda \) this implies that
\[ \mathbb{P}_X \left( E_k \right) \leq e^{t_\epsilon + 1 N^{-1-c_1}}. \]

By Lemma 4.12 with \( b = c_1/3 \), for \( N \) sufficiently large, for \( t' \geq 0 \),
\[ \mathbb{P}_X \left( F^{(N)}(R_\infty + \epsilon, t') \geq 1 - \frac{1}{4} \lambda, \sup_{s \in [\lambda, 2\lambda]} M^{(N)}_{t+s} > R_\infty + \epsilon + \lambda^{1/3} \right) \leq e^{-\lambda^{1/4}}. \]

For \( N \) sufficiently large that \( \frac{1}{4} \lambda \geq N^{-c_1/2} \) and \( \lambda^{1/3} = N^{-c_1/9} < \epsilon \), choosing \( t' = t + (k-1)\lambda \), this implies that
\[ \mathbb{P}_X \left( E_k, \sup_{s \in [\lambda, 2\lambda]} M^{(N)}_{t+(k-1)\lambda+s} > R_\infty + 2\epsilon \right) \leq e^{-\lambda^{1/4}} = e^{-N^{-c_1/12}}. \]

By (73), it follows that for \( N \) sufficiently large,
\[ \mathbb{P}_X \left( \sup_{s \in [0,1]} M^{(N)}_{t+s} > R_\infty + 2\epsilon \right) \leq 2\epsilon + (N^{-c_1/3} + 1)(e^{-N^{-c_1/12}} + e^{t_\epsilon + 1 N^{-1-c_1}}) \leq 3\epsilon, \]
which concludes the proof of (12) in Proposition 1.7. It remains to show (11).

Recall from (9) that \( V \) is strictly increasing and continuous on \([0, R_\infty]\), with \( V(0) = 0 \) and \( V(R_\infty) = 1 \). If \( u \in \mathcal{N}^{+}_{c_1/2} \), \( \|X^+_u(s)\| \leq R_\infty + \epsilon \) \( \forall s \in [\epsilon, \epsilon^{-1/2}] \).

The statement (11) now follows from (74) and (12), which completes the proof.

The following lemma is the main remaining step in the proof of Theorem 1.3.

**Lemma 6.3.** For \( \epsilon > 0 \), let
\[ C_\epsilon = \left\{ X^+_u(\epsilon^{-1/2}) : u \in \mathcal{N}^{+}_{c_1/2}, \|X^+_u(s)\| \leq R_\infty + \epsilon \right\}. \]

Take \( \delta > 0 \). Then for \( \epsilon > 0 \) sufficiently small, for \( N \) sufficiently large, the following holds: if the initial particle configuration \( X \in (\mathbb{R}^d)^N \) satisfies
\[ \sup_{r \geq 0} \left| N^{-1} |X \cap B(r)| - V(r) \right| \leq \epsilon, \]
then for any \( A \subseteq \mathbb{R}^d \) measurable,
\[ \mathbb{P}_X \left( \frac{1}{N} |C_\epsilon \cap A| - \int_A U(x) \, dx \geq \delta \right) < \frac{1}{7} \delta. \]
Proof. First, we estimate the mean of $\frac{1}{N}|C_\epsilon \cap A|$. By the many-to-one lemma,

$$
\mathbb{E}_X \left[ \frac{1}{N} |C_\epsilon \cap A| \right] = \frac{1}{N} e^{-\epsilon/2} \sum_{k=1}^N \mathbb{P}_X \left( B_{\epsilon^{-1/2}} \in A, \|B_{s}\| \leq R_\infty + \epsilon \forall s \in [\epsilon, \epsilon^{-1/2}] \right)
$$

$$
= \frac{1}{N} e^{-\epsilon/2} \sum_{k=1}^N \mathbb{E}_X \left[ \mathbb{P}_X \left( B_{\epsilon^{-1/2}} \in A, \|B_{s}\| \leq R_\infty + \epsilon \forall s \in [0, \epsilon^{-1/2} - \epsilon] \right) \right]
$$

$$
= e^{-\epsilon/2} \int_A w(y, \epsilon^{-1/2} - \epsilon) \, dy,
$$

where $w(y, s)$ is the unique solution to

$$
\begin{cases}
    \partial_t w = \Delta w & s > 0, \|y\| < R_\infty + \epsilon, \\
    w(y, s) = 0 & s \geq 0, \|y\| \geq R_\infty + \epsilon, \\
    w(y, 0) = (\Phi_\epsilon * \mu_X)(y) & \|y\| < R_\infty + \epsilon,
\end{cases}
$$

(75)

where $\Phi_\epsilon(y)$ is the heat kernel and $\mu_X(dy)$ is the empirical measure on $\mathbb{R}^d$ determined by the points $\{X_k\}_{k=1}^N$:

$$
\mu_X(dy) = \frac{1}{N} \sum_{k=1}^N \delta_{X_k}, \quad \Phi_\epsilon(y) = (4\pi\epsilon)^{-d/2} e^{-\frac{|y|^2}{4\epsilon}}.
$$

The function $w(x, s)$ can be expanded as a series in terms of the Dirichlet eigenfunctions of the Laplacian on $\mathcal{B}(R_\infty + \epsilon)$ (see Theorem 7.1.3 of [Eva10]):

$$
w(x, s) = \sum_{k=1}^\infty a_k e^{-s \lambda_k^0} U_k^0(x), \quad s \geq 0, \|x\| \leq R_\infty + \epsilon,
$$

(77)

where the partial sums converge weakly in $L^2([0, T]; H^1_0)$. Here $\{(U_k^s(x), \lambda_k^s)\}_{k \geq 1}$ denote the Dirichlet eigenfunctions and eigenvalues for $-\Delta$ on the ball $\{|x| \leq R_\infty + \epsilon\}$:

$$
-\Delta U_k^s = \lambda_k^s U_k^s, \quad \text{for } \|x\| < R_\infty + \epsilon,
$$

$$
U_k^s(x) = 0, \quad \text{for } \|x\| \geq R_\infty + \epsilon.
$$

By scaling we have

$$
\lambda_k^s = \left( \frac{R_\infty}{R_\infty + \epsilon} \right)^2 \lambda_k^0 \quad \text{and} \quad U_k^s(x) = \left( \frac{R_\infty}{R_\infty + \epsilon} \right)^{d/2} U_k^0 \left( x \frac{R_\infty}{R_\infty + \epsilon} \right).
$$

The eigenvalues satisfy $\lambda_1^0 = 1 < \lambda_2^0 \leq \ldots$ and the eigenfunctions $\{U_k^s\}_{k=1}^\infty$ form an orthonormal basis in $L^2(\mathcal{B}(R_\infty + \epsilon))$. Furthermore, $U_k^0(x) = \|U\|^{-1}_L U(x)$, see (2).

Define $\tilde{w}(y, s) = \sum_{k=2}^\infty a_k e^{-s \lambda_k^0} U_k^s(y)$. Thus,

$$
\tilde{w}(y, s) = a_1 e^{-s \lambda_1^0} U_1^s(y) + \tilde{w}(y, s),
$$

(78)

and $\tilde{w}(\cdot, s)$ is orthogonal to $U_1^s$ in $L^2$ for all $s \geq 0$. Observe that

$$
\|\tilde{w}(\cdot, 0)\|_{L^2}^2 \leq \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} \Phi_\epsilon(y - x) \mu_X(dx) \right)^2 \, dy
$$

$$
\leq \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \Phi_\epsilon(y - x)^2 \mu_X(dx) \, dy
$$

$$
= \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} \Phi_\epsilon(y - x)^2 \, dy \right) \mu_X(dx) = (8\pi\epsilon)^{-d/2},
$$

35
where the second line follows by Jensen’s inequality. Therefore, for $s \geq 0$,
\[
\|\tilde{w}(\cdot, s)\|^2_{L^2} \leq e^{-2\lambda_s^2 s}\|\tilde{w}(\cdot, 0)\|^2_{L^2} \leq e^{-2\lambda_s^2 s}\|w(\cdot, 0)\|^2_{L^2} \leq e^{-2\lambda_s^2 (8\pi\epsilon)^{-d/2}}.
\]
(79)

Now we estimate $a_1$. Since $\|U'_1\|_{L^2} = \|U'_0\|_{L^2} = 1$ and since $U'_1$ is spherically symmetric, writing $e$ for a unit vector, we have
\[
a_1 = \int_{B(R, \epsilon)} U'_1(re) w(x, 0) \, dx = \int_0^{R + \epsilon} U'_1(re) \left( \int_{\partial B(r)} (\Phi \ast \mu)(y) \, d\mathcal{S}(y) \right) \, dr
\]
\[-\int_0^{R + \epsilon} \partial_r U'_1(re) \left( \int_{\partial B(r)} (\Phi \ast \mu)(y) \, dy \right) \, dr \tag{80}
\]
by integration by parts. Also, for $r \geq 0$,
\[
-\partial_r U'_1(re) = -\left( \frac{R}{R + \epsilon} \right)^{d/2+1} \|U\|_{L^2}^{-1} \partial_r U \left( \frac{R \epsilon e}{R + \epsilon} \right). \tag{81}
\]
In particular, since $U(re)$ is non-increasing in $r$ for $r \geq 0$, $-\partial_r U'_1(re) \geq 0$.

Now recall that we assume $|N^{-1}X \cap B(r)| - V(r) < \epsilon$ for all $r \geq 0$, and so
\[
\sup_{r \geq 0} |\mu_X(B(r)) - V(r)| < \epsilon. \tag{82}
\]

Then for $r \geq 0$,
\[
\int_{B(r)} \int_{\|x\| < r + \epsilon^{1/3}} \Phi \ast \mu_X(dx) \, dy \leq \mu_X(B(r + \epsilon^{1/3})) \leq V(r + \epsilon^{1/3}) + \epsilon,
\]
where the first inequality follows since $\int_{\mathbb{R}^d} \Phi \ast \mu_X(dx) \, dy = 1$. Furthermore, writing $y = x + z$,
\[
\int_{B(r)} \int_{\|z\| \geq r + \epsilon^{1/3}} \Phi \ast \mu_X(dx) \, dy = \int_{\|z\| \geq r + \epsilon^{1/3}} \int_{\mathbb{R}^d} \mathbf{1}_{\{\|x + z\| < r\}} \Phi \ast \mu_X(dx) \, dz
\]
\[\leq \int_{\|z\| \geq r + \epsilon^{1/3}} \int_{\|y\| > \epsilon^{1/3}} \Phi \ast \mu_X(dx) \, dz
\]
\[\leq \int_{\|y\| > \epsilon^{1/3}} \Phi \ast \mu_X(dx) \, dz
\]
\[\leq 2d e^{-\epsilon^{-1/3} / (4d)},
\]
where we used $\mu_X(\mathbb{R}^d) = 1$ in the third line and a Gaussian tail bound in the last line. This implies that
\[
\int_{B(r)} (\Phi \ast \mu)(y) \, dy \leq V(r + \epsilon^{1/3}) + \epsilon + 2d e^{-\epsilon^{-1/3} / (4d)}.
\]

Therefore, by (80) and (81),
\[
a_1 \leq -\int_0^{R + \epsilon} \left( \frac{R}{R + \epsilon} \right)^{d/2+1} \|U\|_{L^2}^{-1} \partial_r U \left( \frac{R \epsilon e}{R + \epsilon} \right) \left( V(r + \epsilon^{1/3}) + \epsilon + 2d e^{-\epsilon^{-1/3} / (4d)} \right) \, dr
\]
\[\leq \|U\|_{L^2}^{-1} \int_0^{R + \epsilon} \left( -\partial_r U(re) + \epsilon \|\partial_r^2 U\|_{\infty} \right) \left( V(r) + \epsilon^{1/3} \|V'\|_{\infty} + \epsilon + 2d e^{-\epsilon^{-1/3} / (4d)} \right) \, dr
\]
\[\leq -\|U\|_{L^2}^{-1} \int_0^{R + \epsilon} \partial_r U(re)V(r) \, dr + O(\epsilon^{1/3}).
\]

Note that by integration by parts,
\[
-\int_0^{R + \epsilon} \partial_r U(re)V(r) \, dr = \int_0^{R + \epsilon} U(re)V'(r) \, dr = \|U\|_{L^2}^2,
\]
36
since $V(r) = \int_{B(r)} U(y) \, dy$. Hence for $\delta > 0$, for $\epsilon$ sufficiently small, we have that

$$a_1 \leq \|U\|_{L^2}(1 + \frac{1}{4} \delta). \quad (83)$$

By (78), we have now shown that for $\epsilon$ sufficiently small, for $s \geq 0$ and $\|x\| \leq R_\infty + \epsilon$,

$$w(x, s) \leq (1 + \frac{1}{4} \delta) e^{-s \frac{R_\infty}{R_\infty + \epsilon}} \cdot \frac{R_\infty}{R_\infty + \epsilon} U \left( \frac{xR_\infty}{R_\infty + \epsilon} \right) + \tilde{w}(x, s), \quad (84)$$

where $\tilde{w}$ satisfies (79). Then

$$\int_A w(x, s) \, dx \leq (1 + \frac{1}{4} \delta + O(\epsilon)) e^{-s \frac{R_\infty}{R_\infty + \epsilon}} \cdot \frac{R_\infty}{R_\infty + \epsilon} U \left( \frac{xR_\infty}{R_\infty + \epsilon} \right) + \int_{B(R_\infty + \epsilon)} |\tilde{w}(x, s)| \, dx. \quad (85)$$

By Jensen’s inequality and then by (79),

$$\int_{B(R_\infty + \epsilon)} |\tilde{w}(x, s)| \, dx \leq |B(R_\infty + \epsilon)|^{1/2} \|\tilde{w}(\cdot, s)\|_{L^2} \leq |B(R_\infty + \epsilon)|^{1/2} e^{-\lambda_2 s (8\pi \epsilon)^{-d/4}}.$$

Take $c \in (0, \lambda_2^0 - 1)$, and suppose $\epsilon$ is sufficiently small that $\lambda_2^0 = (\frac{R_\infty}{R_\infty + \epsilon})^2 \lambda_2^0 > 1 + c$. Then

$$\int_{B(R_\infty + \epsilon)} |\tilde{w}(x, s)| \, dx \leq |B(R_\infty + \epsilon)|^{1/2} e^{-((1+c)s (8\pi \epsilon)^{-d/4}}. \quad (86)$$

By (85) and (75), it follows that

$$\mathbb{E}_X \left[ \frac{1}{N} |C_\epsilon \cap A| \right] \leq (1 + \frac{1}{4} \delta + O(\epsilon)) e^{-s \frac{R_\infty}{R_\infty + \epsilon}} \cdot \frac{R_\infty}{R_\infty + \epsilon} U \left( \frac{xR_\infty}{R_\infty + \epsilon} \right) + \int_A U(x) \, dx + O(\epsilon).$$

Therefore, for $\delta > 0$, for $\epsilon > 0$ sufficiently small, if $X \in (\mathbb{R}^d)^N$ satisfies (82) then for $A \subseteq \mathbb{R}^d$ measurable,

$$\mathbb{E}_X \left[ \frac{1}{N} |C_\epsilon \cap A| \right] \leq \int_A U(x) \, dx + \frac{1}{2} \delta.$$

By the same argument as for Lemma 5.2, for $N$ sufficiently large, for $A \subseteq \mathbb{R}^d$ measurable and $\epsilon > 0$,

$$\mathbb{E}_X \left[ \left( \frac{1}{N} |C_\epsilon \cap A| - \mathbb{E}_X \left[ \frac{1}{N} |C_\epsilon \cap A| \right] \right)^4 \right] \leq 13 e^{4\epsilon - 1/2} N^{-2}. \quad (87)$$

So for $\delta > 0$, for $\epsilon > 0$ sufficiently small and $N$ sufficiently large, if $X \in (\mathbb{R}^d)^N$ satisfies (82) then for $A \subseteq \mathbb{R}^d$ measurable,

$$\mathbb{P}_X \left( \frac{1}{N} |C_\epsilon \cap A| - \int_A U(x) \, dx \geq \delta \right) \leq \mathbb{P}_X \left( \left| \frac{1}{N} |C_\epsilon \cap A| - \mathbb{E}_X \left[ \frac{1}{N} |C_\epsilon \cap A| \right] \right| \geq \frac{1}{4} \delta \right) \leq \frac{1}{2} \delta.$$

by (87) and Markov’s inequality. The result follows by taking $N$ sufficiently large.

The following result is an immediate consequence of Lemma 6.3 and the coupling between the BBM $X^+(t)$ and the $N$-BBM $X^{(N)}(t)$ described in Section 3.
Corollary 6.4. Take $\delta > 0$. Then for $\epsilon > 0$ sufficiently small, for $N$ sufficiently large, the following holds: if the initial particle configuration $\mathcal{X} \in (\mathbb{R}^d)^N$ satisfies
\[ \sup_{r \geq 0} |N^{-1}|\mathcal{X} \cap \mathcal{B}(r)| - V(r)| \leq \epsilon, \]
then for any $A \subseteq \mathbb{R}^d$ measurable,
\[ \mathbb{P}_\mathcal{X} \left( M_s^{(N)} \leq R_\infty + \epsilon \forall s \in [0, \epsilon^{-1/2}], \mu^{(N)}(A, \epsilon^{-1/2}) - \int_A U(x) \, dx \geq \delta \right) < \frac{1}{2} \delta. \]

Proof. Under the coupling between the BBM $X^+$ and the $N$-BBM $X^{(N)}$ described in Section 3, we have that for $\epsilon > 0$, if $M_s^{(N)} \leq R_\infty + \epsilon \forall s \in [0, \epsilon^{-1/2}]$ then by (24), almost surely
\[ \left\{ X_k^{(N)}(\epsilon^{-1/2}) : k \in \{1, \ldots, N\} \right\} \subseteq \left\{ X_u^{+}(\epsilon^{-1/2}) : u \in \mathcal{N}^{+}_{\epsilon^{-1/2}}, \|X_u^{+}(s)\| \leq R_\infty + \epsilon \forall s \in [0, \epsilon^{-1/2}] \right\} \subseteq \mathcal{C}_\epsilon, \]
where $\mathcal{C}_\epsilon$ is defined in the statement of Lemma 6.3. Hence for any $A \subseteq \mathbb{R}^d$ measurable,
\[ \mu^{(N)}(A, \epsilon^{-1/2}) \leq \frac{1}{2} |\mathcal{C}_\epsilon \cap A|. \] The result follows by Lemma 6.3.

We can now use Proposition 1.7 and Corollary 6.4 to prove Theorem 1.3.

Proof of Theorem 1.3. Take $K > 0, c \in (0, 1]$ and $\delta > 0$. As the second statement of the theorem was already proved in Proposition 1.7, it remains to prove that for $N$ and $t$ large enough, for $\mathcal{X} \in \Gamma(K, c)$ and $A \subseteq \mathbb{R}^d$ measurable,
\[ \mathbb{P}_\mathcal{X} \left( \left| \mu^{(N)}(A, t) - \int_A U(x) \, dx \right| \geq \delta \right) < \delta. \]

Take $\epsilon > 0$ sufficiently small that Corollary 6.4 holds and $[\epsilon^{-1/2}] \epsilon + \epsilon = \epsilon < \frac{1}{2} \delta$, and take $N_\epsilon, T_\epsilon$ as defined in Proposition 1.7. Take $N \geq N_\epsilon$ sufficiently large that Corollary 6.4 holds, and take $t \geq T_\epsilon + \epsilon^{-1/2}$. Let $t_0 = t - \epsilon^{-1/2}$.

For an initial particle configuration $\mathcal{X} \in \Gamma(K, c)$ and $A \subseteq \mathbb{R}^d$ measurable, we have by a union bound that
\[
\mathbb{P}_\mathcal{X} \left( \mu^{(N)}(A, t) - \int_A U(x) \, dx \geq \delta \right) \\
\leq \mathbb{P}_\mathcal{X} \left( M_s^{(N)} \leq R_\infty + \epsilon \forall s \in [t_0, t], \sup_{r \geq 0} |F^{(N)}(r, t_0) - V(r)| \leq \epsilon, \mu^{(N)}(A, t) - \int_A U(x) \, dx \geq \delta \right) \\
\quad + \mathbb{P}_\mathcal{X} \left( \sup_{s \in [t_0, t]} M_s^{(N)} > R_\infty + \epsilon \right) \quad + \mathbb{P}_\mathcal{X} \left( \sup_{r \geq 0} |F^{(N)}(r, t_0) - V(r)| \geq \epsilon \right) \\
\leq \frac{1}{2} \delta + |t - t_0| \epsilon + \epsilon,
\]
by (12) and (10) in Proposition 1.7 for the last two terms (since $N \geq N_\epsilon$ and $t_0 = t - \epsilon^{-1/2} \geq T_\epsilon$), and by Corollary 6.4 and the Markov property at time $t_0$ for the first term. Therefore, since $|t - t_0| \epsilon + \epsilon = |\epsilon^{-1/2}| \epsilon + \epsilon < \frac{1}{2} \delta$, it follows that for any $A \subseteq \mathbb{R}^d$ measurable,
\[ \mathbb{P}_\mathcal{X} \left( \mu^{(N)}(A, t) - \int_A U(x) \, dx \geq \delta \right) \leq \delta.
\]
As in (48), since $\mu^{(N)}(A, t) + \mu^{(N)}(\mathbb{R}^d \setminus A, t) = 1$ and $\int_A U(x) \, dx + \int_{\mathbb{R}^d \setminus A} U(x) \, dx = 1$, we have that
\[ \mathbb{P}_\mathcal{X} \left( \mu^{(N)}(A, t) - \int_A U(x) \, dx \leq -\delta \right) = \mathbb{P}_\mathcal{X} \left( \mu^{(N)}(\mathbb{R}^d \setminus A, t) - \int_{\mathbb{R}^d \setminus A} U(x) \, dx \geq \delta \right) \leq \delta,
\]
which completes the proof. \( \square \)
It remains to prove Theorems 1.2 and 1.4, which will follow easily from the following proposition.

**Proposition 6.5.** Take $t_0 > 1$ as in Proposition 6.2. For $N$ sufficiently large, for any $t_1 \in (0, t_0]$, the Markov chain $(X(t_1 n))_{n=0}^{\infty}$ is a positive recurrent strongly aperiodic Harris chain.

**Remark:** This proposition will be used in the proof of Theorem 1.2 in combination with Theorems 6.1 and 4.1 of [AN78], which say that a positive recurrent strongly aperiodic Harris chain admits a unique invariant probability measure, and that the distribution of the state of the Harris chain after $n$ steps converges to that invariant probability measure as $n \to \infty$.

**Proof.** For $n \in \mathbb{N}_0$, let $Y_n = X^{(N)}(t_1 n)$. We use a similar strategy to the proof of Proposition 3.1 in [DR11]. By [AN78], to show that $(Y_n)_{n=0}^{\infty}$ is a recurrent strongly aperiodic Harris chain, it suffices to show that there exists a set $\Lambda \subseteq (\mathbb{R}^d)^N$ such that

1. $\mathbb{P}_X(\tau_\Lambda < \infty) = 1 \forall \Lambda \in (\mathbb{R}^d)^N$, where $\tau_\Lambda = \inf \{n \geq 1 : Y_n \in \Lambda\}$.
2. There exist $\epsilon > 0$ and a probability measure $q$ on $\Lambda$ such that $\mathbb{P}_X(Y_1 \in C) \geq \epsilon q(C)$ for any $\Lambda \subseteq \Lambda$ and $C \subseteq \Lambda$.

To furthermore prove that the Harris chain is positive recurrent, we also need to show that

3. $\sup_{\Lambda \in \Lambda'} \mathbb{E}_\Lambda[\tau_\Lambda] < \infty$.

We prove the proposition with the set $\Lambda = (\mathcal{B}(R_\infty + 1))^{\otimes N}$.

We start with the third point: showing that $\sup_{\Lambda \in \Lambda'} \mathbb{E}_\Lambda[\tau_\Lambda] < \infty$.

Take $K_0 = 3$ and $c_0 > 0$ as in Proposition 6.2; let $\Lambda' = \Gamma(K_0 \vee (R_\infty + 1), c_0) \supset \Lambda$. By (11) in Proposition 1.7, there exist $n_1, N_1 < \infty$ such that for $N \geq N_1$, for $\Lambda \in \Lambda'$,

$$\mathbb{P}_X\left(M_{n_1 t_1}^{(N)} \geq R_\infty + 1\right) \leq \frac{1}{2}.$$

Hence

$$\sup_{\Lambda \in \Lambda'} \mathbb{P}_\Lambda(Y_{n_1} \notin \Lambda) \leq \frac{1}{2}.$$ \hspace{1cm} (88)

Let $\tau_{\Lambda'} = \inf \{n \geq 1 : Y_n \in \Lambda'\}$. Note that letting $t_2 = [t_0/t_1]t_1$, we have that

$$\tau_{\Lambda'} \leq \left\lceil \frac{t_0}{t_1} \right\rceil \inf \{n \geq 1 : X^{(N)}(nt_2) \in \Gamma(K_0 \vee (R_\infty + 1), c_0)\}.$$

Hence, since $t_2 \in [t_0, 2t_0]$, if $N$ is sufficiently large then by (56) and (55) in Proposition 6.2,

$$\mathbb{P}_\Lambda(\tau_{\Lambda'} < \infty) = 1 \forall \Lambda \in (\mathbb{R}^d)^N \text{ and } \sup_{\Lambda \in \Lambda'} \mathbb{E}_\Lambda[\tau_{\Lambda'}] < \infty.$$ \hspace{1cm} (89)

Let $\tau(0) = 0$ and for $k \in \mathbb{N}$, let

$$\tau(k) = \inf \{n \geq 1 + \tau(k - 1) : Y_n \in \Lambda'\}.$$

Then by (89), $(\tau(k))_{k=1}^{\infty}$ form an increasing sequence of almost surely finite times such that $Y_{\tau(k)} \in \Lambda'$. Notice that $\tau(1) = \tau_{\Lambda'}$ and that

$$\tau_{\Lambda} \leq \tau(k^*) + n_1 \text{ where } k^* = \inf \{k \geq 1 : Y_{\tau(k) + n_1} \in \Lambda\}.$$

It is therefore sufficient to show that $\sup_{\Lambda \in \Lambda} \mathbb{E}_\Lambda[\tau(k^*)] < \infty$ to establish that $\sup_{\Lambda \in \Lambda} \mathbb{E}_\Lambda[\tau_{\Lambda}] < \infty$. 

39
Write \((\mathcal{F}_n)_{n=0}^\infty\) for the natural filtration of the Markov chain \((Y_n)_{n=0}^\infty\). Notice that for \(k \geq n_1\), the event \(\{k^* > k - n_1\} = \{Y_{\tau(j)+n_1} \notin \Lambda \ \forall 1 \leq j \leq k - n_1\}\) is measurable in \(\mathcal{F}_{\tau(k)}\) since \(\tau(k - n_1) + n_1 \leq \tau(k)\). Therefore by the strong Markov property, for \(k \geq n_1\) and \(X \in \Lambda',\)

\[
\mathbb{P}_X(k^* > k) = \mathbb{P}_X(Y_{\tau(j)+n_1} \notin \Lambda \ \forall 1 \leq j \leq k) \leq \mathbb{P}_X(k^* > k - n_1, Y_{\tau(k)+n_1} \notin \Lambda) = \mathbb{E}_X\left[\mathbb{I}_{(k^* > k-n_1)} \mathbb{P}_Y(Y_{n_1} \notin \Lambda)\right] \leq \frac{1}{2} \mathbb{P}_X(k^* > k - n_1),
\]

where we used (88) in the last step. Then, by an induction argument, for \(k \in \mathbb{N}\) and \(X \in \Lambda',\)

\[
\mathbb{P}_X(k^* > k) \leq 2^{-\lceil k/n_1 \rceil}.
\]

In particular, \(k^*\) is almost surely finite. For \(X \in \Lambda',\)

\[
\mathbb{E}_X[\tau(k^*)] = \sum_{k=1}^\infty \mathbb{E}_X[\tau(k) \mathbb{I}_{\{k^*=k\}}] = \sum_{k=1}^\infty \sum_{\ell=1}^k \mathbb{E}_X[(\tau(\ell) - \tau(\ell - 1)) \mathbb{I}_{\{k^*=k\}}]
\]

\[
= \sum_{\ell=1}^\infty \mathbb{E}_X[(\tau(\ell) - \tau(\ell - 1)) \mathbb{I}_{\{k^*\geq\ell\}}].
\]

Then, for \(X \in \Lambda'\) and \(\ell \geq 1\), by (90) and the strong Markov property,

\[
\mathbb{E}_X[(\tau(\ell) - \tau(\ell - 1)) \mathbb{I}_{\{k^*\geq\ell\}}] \leq \mathbb{E}_X[(\tau(\ell) - \tau(\ell - 1)) \mathbb{I}_{\{k^*\geq\ell-1-n_1\}}]
\]

\[
= \mathbb{E}_X[\mathbb{E}_Y[\tau(\ell) - \tau(\ell - 1) \mid \mathcal{F}_{\tau(\ell-1)}] \mathbb{I}_{\{k^*\geq\ell-1-n_1\}}] \leq \sup_{Y \in \Lambda'} \mathbb{E}_Y[\tau(1)] \mathbb{P}_X(k^* > \ell - 1 - n_1) \leq 2^{-\lceil (\ell-1-n_1)/n_1 \rceil} \sup_{Y \in \Lambda'} \mathbb{E}_Y[\tau(1)].
\]

By substituting into (91), we get that \(\sup_{X \in \Lambda', Y \in \Lambda'} \mathbb{E}_X[\tau(k^*)] < \infty\), since we have that \(\tau(1) = \tau_{\Lambda'}\) and, from (89), that \(\sup_{Y \in \Lambda'} \mathbb{E}_Y[\tau_{\Lambda'}] < \infty\). This implies that \(\sup_{X \in \Lambda', Y \in \Lambda'} \mathbb{E}_X[\tau_{\Lambda}] < \infty\) and, in particular, \(\mathbb{P}_X[\tau_{\Lambda} < \infty] = 1\) for \(X \in \Lambda'\),

\[
\mathbb{P}_X(\tau_{\Lambda} < \infty) = 1.
\]

Finally, it remains to prove the second of the three points at the beginning of the proof. Note that conditional on the event that none of the particles in the \(N\)-BBM branch in the time interval \([0, t_1]\), the \(N\) particles move according to independent Brownian motions. Therefore, for \(X = (X_1, \ldots, X_N) \in \Lambda\) and \(C \subseteq \Lambda,\)

\[
\mathbb{P}_X(Y_1 \in C) \geq e^{-t_1 N} \int_C \frac{1}{(4\pi t_1)^{d/2}} e^{-\frac{1}{4t_1} \|X_i - y_i\|^2} \, dy_1 \ldots dy_N \geq e^{-t_1 N (4\pi t_1)^{-d/2} e^{-N (R_{\infty} + 1)^2 / t_1} \text{Leb}(C)},
\]

where \(\text{Leb}(\cdot)\) is the Lebesgue measure on \((\mathbb{R}^d)^N\). Indeed, for \(y \in \Lambda, \|X_i - y_i\|^2 \leq 4(R_{\infty} + 1)^2 \forall i \in \{1, \ldots, N\}\). Let

\[
e = e^{-t_1 N (4\pi t_1)^{-d/2} e^{-N (R_{\infty} + 1)^2 / t_1} \text{Leb}(\Lambda)},
\]

and define a probability measure \(q\) on \(\Lambda\) by letting \(q(C) = \text{Leb}(C) / \text{Leb}(\Lambda)\) for \(C \subseteq \Lambda\). Then for \(X \in \Lambda\) and \(C \subseteq \Lambda, \mathbb{P}_X(Y_1 \in C) \geq \epsilon q(C)\). The result follows.
Proof of Theorem 1.2. By Proposition 6.5, and by Theorems 6.1 and 4.1 in [AN78], for $N$ sufficiently large, for $t_1 \in (0, t_0]$, $(X^{(N)}(nt_1))_{n=0}^{\infty}$ has a unique invariant measure $\pi_{t_1}^{(N)}$ which is a probability measure on $(\mathbb{R}^d)^N$, and for any $X \in (\mathbb{R}^d)^N$, the law of $X^{(N)}(nt_1)$ under $P_X$ converges as $n \to \infty$ to $\pi_{t_1}^{(N)}$ in total variation norm. In particular, if $C \subseteq (\mathbb{R}^d)^N$ is measurable,

$$P_X(X^{(N)}(t_1n) \in C) \to \pi_{t_1}^{(N)}(C) \quad \text{as } n \to \infty. \quad (92)$$

Fix $N$ large enough for Proposition 6.5 to hold. We begin by showing that

$$\pi_{t_1}^{(N)} = \pi_{t_0}^{(N)} =: \pi^{(N)} \quad \forall t_1 \in (0, t_0]. \quad (93)$$

Take $X \in (\mathbb{R}^d)^N$, and $C \subseteq (\mathbb{R}^d)^N$ a closed set. Take $\delta > 0$. For $\epsilon > 0$, let

$$C' = \{ Y \in (\mathbb{R}^d)^N : \inf_{Z \in C} \| Y - Z \| < \epsilon \}. \quad \text{(97)}$$

Here $\| Y - Z \|$ denotes the Euclidean norm of $Y - Z$ regarded as a vector in $\mathbb{R}^d N$. Then

$$\pi_{t_0}^{(N)}(C') \to \pi_{t_0}^{(N)}(C) \quad \text{as } \epsilon \to 0. \quad \text{(98)}$$

Take $\epsilon > 0$ sufficiently small that

$$\pi_{t_0}^{(N)}(C') < \pi_{t_0}^{(N)}(C) + \frac{1}{3}\delta. \quad \text{(99)}$$

It is easy to see that if $t_1 > 0$ is small enough, then

$$P_{X'}(X^{(N)}(s) \not\in C') < \frac{1}{3}\delta \quad \forall X' \in C, \ s \in [0, t_1]. \quad (100)$$

Indeed, the event that no particle branches on the time interval $[0, t_1]$ has probability $e^{-N t_1}$, which can be arbitrarily close to 1 if $t_1$ is small enough. Conditioned on this event, the random process $Y(s) = X^{(N)}(s) - X^{(N)}(0)$ is a Brownian motion in $\mathbb{R}^d N$. In particular, $Y(s)$ is almost surely continuous on $[0, t_1]$, with $Y_0 = 0$, and the law of $Y(s)$ does not depend on $X^{(N)}(0)$ or $t_1$. Then $P_{X'}(X^{(N)}(s) \in C') \geq e^{-N t_1} P(Y(s) < \epsilon \ \forall s \in [0, t_1])$, which can be made arbitrarily close to 1 by taking $t_1$ sufficiently small.

When choosing $t_1$ small enough for (100), we furthermore require that $t_0/t_1 \in \mathbb{N}$. It is then clear from (92) than $\pi_{t_1}^{(N)} = \pi_{t_0}^{(N)}$. Take $n_0 \in \mathbb{N}$ sufficiently large that for $n \geq n_0$,

$$P_X(X^{(N)}(t_1n) \in C') \leq \pi_{t_0}^{(N)}(C') + \frac{1}{3}\delta \leq \pi_{t_0}^{(N)}(C) + \frac{4}{3}\delta. \quad \text{(101)}$$

For $t \geq t_1n_0$ we have

$$P_X(X^{(N)}(t) \in C) \leq P_X(X^{(N)}([t/t_1]t_1) \in C') + P_X(X^{(N)}(t) \in C, X^{(N)}([t/t_1]t_1) \not\in C') \leq \pi_{t_0}^{(N)}(C) + \frac{4}{3}\delta + \pi_{t_0}^{(N)}(C, X^{(N)}([t/t_1]t_1) \not\in C') \leq \pi_{t_0}^{(N)}(C) + \delta. \quad \text{(102)}$$

by the Markov property at time $t$ and (100). Since $\delta > 0$ was arbitrary, it follows that

$$\limsup_{t \to \infty} P_X(X^{(N)}(t) \in C) \leq \pi_{t_0}^{(N)}(C). \quad \text{(103)}$$

Comparing (103) and (92), we see that $\pi_{t_1}^{(N)}(C) \leq \pi_{t_0}^{(N)}(C)$ for all closed sets $C$ and all $t_1 \in (0, t_0]$. Hence, by the Portmanteau theorem, $\pi_{t_1}^{(N)} = \pi_{t_0}^{(N)}$ for all $t_1 \in (0, t_0]$. This proves (93), so we now write $\pi^{(N)}$ for the unique invariant measure of the process $(X^{(N)}(t), t \geq 0)$.

By the Markov property, we have that for any $X_0 \in (\mathbb{R}^d)^N, D \subseteq (\mathbb{R}^d)^N$ and $t > s > 0$,

$$P_{X_0}(X^{(N)}(t) \in D) = \int_{(\mathbb{R}^d)^N} P_{X_0}(X^{(N)}(s) \in dX) P_X(X^{(N)}(t-s) \in D)$$

and $$\pi^{(N)}(D) = \int_{(\mathbb{R}^d)^N} \pi^{(N)}(dX) P_X(X^{(N)}(t-s) \in D).$$

41
By taking the difference between these two equations,
\[
\left| \mathbb{P}_{X_0} \left( X^{(N)}(t) \in D \right) - \pi^{(N)}(D) \right| \leq \int_{\mathbb{R}^d} \left| \mathbb{P}_{X_0} \left( X^{(N)}(s) \in \mathcal{A} \right) - \pi^{(N)}(\mathcal{A}) \right| \mathbb{P}_X \left( X^{(N)}(t-s) \in D \right) \, d\mathcal{A} \\
\leq \int_{\mathbb{R}^d} \left| \mathbb{P}_{X_0} \left( X^{(N)}(s) \in \mathcal{A} \right) - \pi^{(N)}(\mathcal{A}) \right|, \tag{96}
\]
where the right hand side is the total variation norm of the difference between \( \pi^{(N)} \) and the law of \( X^{(N)}(s) \) under \( \mathbb{P}_{X_0} \).

Now choose \( s = [t/t_0]t_0 \) and let \( t \to \infty \). Since the law of \( X^{(N)}(nt_0) \) under \( \mathbb{P}_{X_0} \) converges to \( \pi^{(N)} \) as \( n \to \infty \) in total variation norm, the right hand side of (96) converges to zero as \( t \to \infty \), and the result follows.

**Proof of Theorem 1.4.** Take \( \epsilon > 0 \) and \( A \subseteq \mathbb{R}^d \) measurable. Let
\[
D_\epsilon = \left\{ \mathcal{X} \in (\mathbb{R}^d)^N : \frac{1}{N} \sum_{i=1}^{N} 1_{\{\mathcal{X}_i \in A\}} - \int_{A} U(x) \, dx \geq \epsilon \text{ or } \max_{1 \leq i \leq N} \|\mathcal{X}_i\| - R_\infty \geq \epsilon \right\}.
\]
Take the initial condition \( \mathcal{X} = (0, \ldots, 0) \in (\mathbb{R}^d)^N \). By Theorem 1.3, for any \( \delta \in (0, \epsilon) \), there exist \( N_\delta, T_\delta < \infty \) such that for \( N \geq N_\delta \) and \( t \geq T_\delta \),
\[
\mathbb{P}_X \left( X^{(N)}(t) \in D_\epsilon \right) \leq \mathbb{P}_X \left( X^{(N)}(t) \in D_\delta \right) < 2\delta.
\]
But by Theorem 1.2,
\[
\mathbb{P}_X \left( X^{(N)}(t) \in D_\epsilon \right) \to \pi^{(N)}(D_\epsilon) \quad \text{as } t \to \infty.
\]
It follows that \( \pi^{(N)}(D_\epsilon) \leq 2\delta \) for \( N \geq N_\delta \), and so \( \lim_{N \to \infty} \pi^{(N)}(D_\epsilon) = 0 \), which completes the proof.

**References**

[ABL20] Louigi Addario-Berry, Jessica Lin, and Thomas Tendor. Barycentric Brownian Bees. *arXiv preprint arXiv:2006.04743*, 2020.

[AFGJ16] Amine Asselah, Pablo Ferrari, Pablo Groisman, and Matthieu Jonckheere. Fleming-Viot selects the minimal quasi-stationary distribution: the Galton-Watson case. *Ann. Inst. Henri Poincaré Probab. Stat.*, 52(2):647–668, 2016.

[AN78] Krishna Athreya and Peter Ney. A new approach to the limit theory of recurrent Markov chains. *Transactions of the American Mathematical Society*, 245:493–501, 1978.

[BBNP20] Julien Berestycki, Éric Brunet, James Nolen, and Sarah Penington. A free boundary problem arising from branching Brownian motion with selection. *arXiv preprint, arXiv:2005.09384*, 2020.

[BBP19] Julien Berestycki, Éric Brunet, and Sarah Penington. Global existence for a free boundary problem of Fisher–KPP type. *Nonlinearity*, 32(10):3912–3939, 2019.

[BDMM06] Éric Brunet, Bernard Derrida, Alfred Mueller, and Stéphane Munier. Noisy traveling waves: effect of selection on genealogies. *Europhys. Lett.*, 76(1):1–7, 2006.

[BDMM07] Éric Brunet, Bernard Derrida, Alfred Mueller, and Stéphane Munier. Effect of selection on ancestry: an exactly soluble case and its phenomenological generalization. *Phys. Rev. E (3)*, 76(4):041104, 20, 2007.
Erin Beckman. *Asymptotic Behavior of Certain Branching Processes*. PhD thesis, Duke University, 2019.

Krzysztof Burdzy, Robert Holyst, and Peter March. A Fleming-Viot particle representation of the Dirichlet Laplacian. *Comm. Math. Phys.*, 214(3):679–703, 2000.

Nathanaël Berestycki and Lee Zhuo Zhao. The shape of multidimensional Brunet-Derrida particle systems. *Ann. Appl. Probab.*, 28(2):651–687, 2018.

Pierre Collet, Servet Martínez, and Jaime San Martín. *Quasi-stationary distributions*. Probability and its Applications. Springer, Heidelberg, 2013. Markov chains, diffusions and dynamical systems.

Anna De Masi, Pablo Ferrari, Errico Presutti, and Nahuel Soprano-Loto. Hydrodynamics of the $N$-BBM process. In *Stochastic dynamics out of equilibrium*, volume 282 of *Springer Proc. Math. Stat.*, pages 523–549. Springer, 2019.

Anna De Masi, Pablo Ferrari, Errico Presutti, and Nahuel Soprano-Loto. Nonlocal branching Brownian motions with annihilation and free boundary problems. *Electronic Journal of Probability*, 24, 2019.

Rick Durrett and Daniel Remenik. Brunet-Derrida particle systems, free boundary problems and Wiener-Hopf equations. *Ann. Probab.*, 39(6):2043–2078, 2011.

Lawrence Evans. *Partial differential equations*, volume 19 of *Graduate Studies in Mathematics*. American Mathematical Society, second edition, 2010.

Pablo Groisman and Matthieu Jonckheere. Front propagation and quasi-stationary distributions: the same selection principle? *arXiv preprint arXiv:1304.4847*, 2013.

Pablo Groisman and Matthieu Jonckheere. Front propagation and quasi-stationary distributions for one-dimensional Lévy processes. *Electronic Comm. Probab.*, 23(93):1–11, 2018.

Simon Harris and Matthew Roberts. The many-to-few lemma and multiple spines. *Ann. Inst. Henri Poincaré Probab. Stat.*, 53(1):226–242, 2017.

Peter Jagers. General branching processes as Markov fields. *Stochastic Process. Appl.*, 32(2):183–212, 1989.

Pascal Maillard. Speed and fluctuations of $N$-particle branching Brownian motion with spatial selection. *Probab. Theory Related Fields*, 166(3-4):1061–1173, 2016.

P. Massart. The tight constant in the Dvoretzky-Kiefer-Wolfowitz inequality. *Ann. Probab.*, 18(3):1269–1283, 1990.