Effectivesurface-tensioninthenoise-reducedvotermodel

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Abstract - The role of memory is crucial in determining the properties of many dynamical processes in statistical physics. We show that the simple addition of memory, in the form of noise reduction, modifies the overall scaling behavior of the voter model, introducing an effective surface tension analogous to that recently observed in memory-based models of social dynamics. The numerical results for low-dimensional lattices show a scaling behavior in good agreement with usual Cahn-Allen curvature-driven coarsening, even though slower preasymptotic regimes may be observed depending on the memory properties. Simple arguments and a mean-field analysis provide an explanation for the observed behavior that clarifies the origin of surface tension and the mechanism underlying the coarsening process.

The theory of phase-ordering kinetics [1] describes the scaling behavior of domain coarsening phenomena occurring in nonequilibrium statistical physics, with applications ranging from social sciences [2] to ecology [3]. The way in which a system orders starting from a disordered phase follows different paths depending on generic properties (e.g., symmetries, the type of interactions, etc.) that allow to identify specific universality classes of nonequilibrium phase-ordering dynamics. In particular, the important family of “kinetic Ising models” with $Z_2$-symmetry and no strict conservation of the order parameter displays two different ordering behaviors, depending on whether a surface tension exists or not. If this is the case (as in Glauber dynamics, for example), coarsening is curvature-driven [4]: the domain length scale grows following a power law, $\ell(t) \sim t^{1/2}$ in any dimension [1]. Other models, such as the voter model [5], exhibit domain coarsening without surface tension: dynamics is driven by interfacial noise [4]. In this case $\ell(t)$ grows as $t^{1/2}$ in one dimension, logarithmically in $d = 2$ and it does not diverge for $d > 2$ [6]. It is also possible to define a family of kinetic Ising models interpolating between zero-temperature Glauber dynamics (TOGD) and the voter model (VM) [4,7,8]: in these models, the transition rates depend on two parameters measuring the strength of bulk and interfacial noise. When bulk noise is present the coarsening is curvature-driven. When only interfacial noise exists, coarsening is voter-like for a broad class of models, either characterized by global conservation of magnetization or by nonconserving $Z_2$-symmetric dynamical rules [4].

In this letter, we introduce a simple model, consisting of a voter-like dynamics modified by a noise-reduction procedure. The main motivation of this work consists in understanding the mechanism governing the ordering process of some recently proposed models of social dynamics, such as the Naming Game [9,10] and the Wang-Minett model of bilingualism [11,12]. In both these models, the pairwise interaction rules are reminiscent of the voter dynamics, but the update follows a two-steps process involving memory effects. Surprisingly, on $d$-dimensional lattices, these models undergo a domain coarsening dynamics following approximately the same temporal laws of curvature-driven models with non-conserved order parameter. In order to elucidate the origin of this effective curvature-driven coarsening, we define a minimal model, that we call “noise-reduced voter model” (NRVM), in which a noise-reduction prescription is added to the usual interaction rule of the VM, introducing a memory effect. As will be clear below, the introduction of a noise-reduction prescription does not give rise to bulk noise in the spin dynamics. Instead, it turns out that the additional ingredient gives rise to an effective,
memory-induced, surface tension, that generates in low dimension a curvature-driven coarsening process.

Noise reduction has been used in the study of surface growth [13] and diffusion-limited aggregation [14] as a method to speed up the approach to asymptotic scaling. It amounts to attaching to each site of the system a counter recording how many times a particle has touched the site. Only when the counter reaches a given threshold value does the site become active and the particle sticks to it. We introduce the noise reduction in the framework of the VM in the following way: each site $i$ is endowed with a spin variable, $s_i$, as well as with a pair of counters $C_i^+$ and $C_i^-$. At each time step, a site $i$ and one of its neighbors $j$ are selected at random to interact. The counter associated with the spin value of the selected neighbor is increased by one: $C_j^i \rightarrow C_j^i + 1$. At odds with the normal VM dynamics, the spin variable $s_i$ is not necessarily modified upon interaction. It is updated only when one of the counters attains an a priori fixed threshold value $r$ (e.g. if the positive counter reaches the value $C_i^+ = r$ then $s_i$ is set to $+1$). In this way the interaction between spins is indirect, being mediated by the dynamics of counters. Two possible update rules for the counters are considered: when the spin at site $i$ flips, only the counter that has reached $r$ is set to zero (model 1), or instead both counters are reset (model 2).

For $r = 1$ both models coincide with the usual voter dynamics, whose behavior is well known in any dimension [6] and is characterized by the absence of surface-tension [4]. For $r > 1$, one expects an effective surface-tension to arise: the repeated interaction between spins is direct, being mediated by the dynamics of counters. The joint probability that a neighboring site of opposite spin is chosen and the counter $C_j^{-s_i} = r-1$. On a d-dimensional lattice, the probability to choose a neighbor of $s_i$ is defined by the equation $W(s_i) = \frac{1}{2} \left[ 1 - \frac{1}{r} \sum_{j \in V(i)} s_j \right]$, where $V(i)$ is the neighborhood of $i$. In model 1, the dynamics of the two counters at a site $i$ are decoupled, thus the probability $p_n^{\pm s_i}(t)$ that the counter $C_i^{\pm s_i}$ is equal to a value $n$ at time $t$ is defined by the equation

$$\frac{\partial}{\partial t} p_n^{\pm s_i}(t) = - \left[ p_n^{\pm s_i}(t) - p_{n-1}^{\pm s_i}(t) \right] W(\mp s_i), \quad (1)$$

with periodic boundary conditions on the variable $n$. The solution of this equation requires the knowledge of the probability $W(s_i)$ at all times, i.e. of the whole dynamics, but formally solving the equation in the Fourier space, we can write

$$p_{n-1}^{\pm s_i}(t) = \frac{1}{r} \left\{ 1 + \sum_{k=1}^{r-1} \cos \left( \frac{2\pi k}{r} \right) + \sin \left( \frac{2\pi k}{r} \right) W_i^{\mp}(t) \right\} e^{-\frac{1}{r} \cos \left( \frac{2\pi k}{r} \right) W_i^{\mp}(t)}, \quad (2)$$

where $W_i^{\mp}(t) = \int_0^t W(\pm s_i(t')) dt'$. Note that the sum in the r.h.s. of eq. (2) tends to 0 when the integral is much larger than $r$, while it tends to $-1$ when it is much smaller. For a spin in the bulk of a domain $W(-s_i) = 1$, while $W(s_i) = 0$. Hence, in the former case the integral $W_i^{\mp}(t)$ becomes 1.0, while in the latter case it becomes 0.0. The growth of a domain is determined by the time it takes to reach a certain threshold $r_m$, and is given by $t_m = \int_0^{r_m} \frac{1}{W_i^{\mp}(t)} dt$. The linear trend is clear even if the data refer to a single realization. Time is rescaled by $\tau = t \sqrt{N}$, where $N$ is the total number of sites.

In the inset, the average time $\tau$ needed to dissolve the droplet is plotted as a function of $r$ for both models: $\tau$ scales linearly for model 1 and with an effective exponent 1.1 for model 2.

Fig. 1: Illustration of the presence of surface tension in the two-dimensional voter model with noise reduction. The initial condition is a configuration ($L = 500$) with a droplet of radius $R_0 = 200$ surrounded by a sea of opposite spins. We have monitored the normalized area $\pi R(t)^2/L^2$ as a function of time for model 1. This quantity decreases linearly for $r > 1$, as expected in the presence of surface-tension [1]. Note that the linear trend is clear even if the data refer to a single realization. Time is rescaled by $\tau = t \sqrt{N}$, where $N$ is the total number of sites.

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of size $\frac{1}{\theta} / \tan^2 \frac{1}{\theta}$ with only the spin $\dot{n}$, which is one of the crucial features of the voter model.

Any quantity $q_{n_1} \propto n_1$ diverges and $p_{r^{-1}} \rightarrow 1/r$, while in the latter the integral $\mathcal{W}_{r}^{1} (t)$ is finite, so that $p_{r^{-1}} \rightarrow 0$. As a consequence the effect of varying $r \gg 1$ can be reabsorbed in a temporal rescaling $t \rightarrow t/r$ (see the inset of fig. 1). In order to extend the analysis to model 2, one should consider the equation for the joint probability $q_{n_1}^{1} (t)$ that the counters $C_{r}^{1}$ assume some values $n_+$ and $n_-$, since the counters are now coupled. We will consider such approach later for the simpler case of a mean-field model with $r = 2$. However, we find numerically that in $d = 2$ the scaling is not far from the one for model 1. In $d = 3$, changing $r$ in simulations apparently gives more than a simple rescaling of time (see below), but this is probably just a presymptotic effect.

Another remark arises considering the master equation for the probability distribution $P(S, t)$ of having a certain $d$-dimensional configuration $S = \{ s_i \}$ at the time $t$ in model 1,

$$\frac{d}{dt} P(S, t) = \sum_{s_i} \left[ W(-s_i) P(S, t) p_{r^{-1}}^{s_i} (t) \right],$$

where $P(S', t)$ is the probability of the configuration $S$ with only the spin $s_i$ flipped. A similar equation holds for model 2. From eq. (3), we can write the evolution equation for the average quantities, such as the local magnetization $\langle s_i \rangle$

$$\frac{d}{dt} \langle s_i \rangle \propto \langle p_{r^{-1}}^{s_i} (t) \nabla^2 s_i \rangle,$$

in which $p_{r^{-1}}^{s_i} (t)$ evolves according to eq. (1). Equation (4) is similar to the analogous equation for the usual voter model [6], but the presence of the counter probability $p_{r^{-1}}^{s_i}$ breaks down the conservation of the average magnetization, which is one of the crucial features of the voter model on regular lattices.

The presence of an effective surface tension is expected to induce a curvature-driven coarsening, resulting in a phase-ordering process with scaling properties belonging to the Cahn-Allen universality class [1]. This has been checked directly by studying numerically the coarsening dynamics in $d = 2$ and $d = 3$ starting from a fully disordered random configuration with zero magnetization. The temporal evolution of the density of interfaces, $n_A(t)$, reported in fig. 2, shows in general a power-law decay $n_A(t) \propto t^{-\beta}$ for all cases. For model 1, after the trivial initial transient, the effective exponent sets to about 0.45 in $d = 2$ and above 0.45 in $d = 3$ (see table below), while in the latter the integral $\mathcal{W}_{r}^{1} (t)$ is finite, so that $p_{r^{-1}} \rightarrow 0$. As a consequence the effect of varying $r \gg 1$ can be reabsorbed in a temporal rescaling $t \rightarrow t/r$ (see the inset of fig. 1). In order to extend the analysis to model 2, one should consider the equation for the joint probability $q_{n_1}^{1} (t)$ that the counters $C_{r}^{1}$ assume some values $n_+$ and $n_-$, since the counters are now coupled. We will consider such approach later for the simpler case of a mean-field model with $r = 2$. However, we find numerically that in $d = 2$ the scaling is not far from the one for model 1. In $d = 3$, changing $r$ in simulations apparently gives more than a simple rescaling of time (see below), but this is probably just a presymptotic effect.

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$$\frac{d}{dt} P(S, t) = \sum_{s_i} \left[ W(-s_i) P(S, t) p_{r^{-1}}^{s_i} (t) \right] - W(s_i) P(S, t) p_{r^{-1}}^{s_i} (t)],$$

where $P(S', t)$ is the probability of the configuration $S$ with only the spin $s_i$ flipped. A similar equation holds for model 2. From eq. (3), we can write the evolution equation for the average quantities, such as the local magnetization $\langle s_i \rangle$

$$\frac{d}{dt} \langle s_i \rangle \propto \langle p_{r^{-1}}^{s_i} (t) \nabla^2 s_i \rangle,$$

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asymptotic regime, although present numerical evidence is not clear-cut in this respect.

To understand the dynamical interplay between the voter dynamics and the nonequilibrium surface-tension, it is useful to perform a mean-field analysis, considering the NRVM on the complete graph. Following ref. [15], we derive a master equation for the probability \( P(m,t) \) that the magnetization takes the value \( m \) at time \( t \). In a mean-field framework, single-site counters are well represented by the average values \( C^\pm \) over the whole system. We call \( p^\pm(t) \) the mean-field probabilities that the average counter value \( C^\pm \) is equal to \( n \) at time \( t \). In a temporal step, the magnetization can change of a value \( \pm 2/N \) or remain constant. The corresponding probabilities appearing in the master equation are

\[
Prob\left\{ m \rightarrow m \pm \frac{2}{N} \right\} = \frac{1 + m}{2} - \frac{1 - m}{2} p^\pm_{r-1}(t).
\] (5)

Writing down the master equation, using eq. (5), expanding the expressions up to the second order in \( 1/N \) (with \( N \to \infty \)), and neglecting higher-order terms, we get

\[
\frac{\partial}{\partial t} P(m,t) = -\frac{1}{2N} \left[ p^+_r(t) - p^-_r(t) \right] \frac{\partial}{\partial m} \left[ (1 - m^2)P(m,t) \right] + \frac{1}{2N^2} \left[ p^+_r(t) + p^-_r(t) \right] \frac{\partial^2}{\partial m^2} \left[ (1 - m^2)P(m,t) \right].
\] (6)

Two other equations for the counter probabilities \( p^\pm(t) \) are required to close the equation for \( P(m,t) \). For model 1, the equations have the same form of eq. (1), while in the case of model 2 one should start from the evolution equation for the mean-field joint probability \( q_{n,n}(t) \), noting that \( p^\pm_{r-1}(t) = \sum_{n \neq r} q_{n,n}(t) \). The solution of the coupled set of equations is, in general, beyond reach, but it is instructive to look at eq. (6): apart from the counters dependent prefactors, the second term is the one present in the equation for the VM [15], while the first is instead the one found for TGGD [16], expressing the existence of a drift, due to the imbalance between the counters. The interplay of the two terms reflects the competition between curvature-driven and noise-driven coarsening.

The rate equations for the mean-field magnetization \( m(t) \) can be similarly derived; let us elucidate the simple cases for \( r = 2 \). In model 1, the following system of rate equations holds:

\[
\dot{m}(t) = \frac{1 - m(t)}{2} \left[ p^+_1(t) - p^-_1(t) \right],
\] (7)

\[
\dot{p}^+_1(t) = \frac{1 + m(t)}{2} \left[ 1 - 2p^+_1(t) \right].
\] (8)

The case of model 2 is more complicate, since we need to compute the four coupled evolution equations for \( q_{n,n}(t) \). The corresponding system of rate equations reads

\[
\dot{m}(t) = \frac{1 - m(t)}{2} \left[ q_{1,0}(t) - q_{0,1}(t) \right],
\] (9)

\[
\dot{q}_{0,0}(t) = -q_{0,0}(t) + \frac{1 + m(t)}{2} q_{1,0}(t) + \frac{1 - m(t)}{2} q_{0,1}(t) + q_{1,1}(t),
\] (10)

\[
\dot{q}_{1,0}(t) = -q_{1,0}(t) + \frac{1 + m(t)}{2} q_{0,0}(t),
\] (11)

\[
\dot{q}_{0,1}(t) = -q_{0,1}(t) + \frac{1 - m(t)}{2} q_{0,0}(t),
\] (12)

\[
\dot{q}_{1,1}(t) = -q_{1,1}(t) + \frac{1 + m(t)}{2} q_{0,1}(t) + \frac{1 - m(t)}{2} q_{1,0}(t).
\] (13)

Linear stability analysis indicates that the two models present different stationary solutions. For model 2, there...
are only two possible stable stationary solutions, \( \{ m = -1, q_{0,0} = 1/2, q_{1,0} = 0, q_{0,1} = 1/2, q_{1,1} = 0 \} \), and \( \{ m = +1, q_{0,0} = 1/2, q_{1,0} = 1/2, q_{0,1} = 0, q_{1,1} = 0 \} \), i.e. the system completely orders into a ferromagnetic phase. The solution with \( m = 0 \) is instead unstable. Hence, in the model with double counter update, the mean-field dynamics is completely governed by the drift term. On the contrary, in the model 1, fluctuations play a relevant role. The possible stationary states are \( p_{\pm} = 1/2 \) for all possible values of \( m \), and \( m = 1 \) and \( p_{\pm} = 1/2 \) for any value of \( p_{\pm} \leq 1/2 \) (or the symmetric case \( m = -1, p_{\pm} = 1/2, p_{\pm} \leq 1/2 \)). The existence of multiple stable stationary states points to the existence of a transition depending on the initial value of the magnetization \( m(0) \). Indeed, solving numerically eqs. (7)-(8) we find

\[
\overline{m} = \begin{cases} 
\pm m(0)/m^* & \text{for } |m(0)| < m^*, \\
\pm 1 & \text{for } |m(0)| \geq m^*,
\end{cases}
\]  

while the difference \( p_{+} - p_{-} \) is 0 below the transition and finite for \( |m(0)| > m^* \) (fig. 4). The transition point is \( m^* \approx 0.6382 \ldots \).

If one performs simulations on the complete graph, the sharp transition is actually smeared out by fluctuations. Nevertheless, we can observe a rather abrupt crossover between a regime where \( p_{+} - p_{-} \) is exponentially small (for \( |m(0)| \leq 0.22 \ldots \)) to a regime with finite \( p_{+} - p_{-} \), whereas asymptotically \( |m| \) is equal to 1 in both cases. The shift in the position of this transition is due to the diffusive contribution of the second term in l.h.s. of eq. (6), that is neglected in the rate equations. For small \( |m(0)| \) the fully ordered state is not reached because the initial condition is not sufficiently asymmetric and the drift, which decreases during the evolution (both \( p_{+} \) and \( p_{-} \) tend asymptotically to 1/2), is not strong enough to fully order the system. If \( |m(0)| \) is large enough instead, the counter imbalance is sufficient to drive the system to full order, keeping an asymptotic nonvanishing drift. For \( r > 2 \) the general phenomenology remains similar, with the difference that the counters probabilities may exhibit oscillations that in turn induce a nonmonotonic behavior of the magnetization.

In conclusion, we have introduced a noise-reduced voter model, in which memory effects induce an effective surface tension. While in finite dimension this implies that the coarsening process is always curvature-driven, on a fully connected graph it is possible to observe a crossover between voter-like and zero temperature Glauber behavior. We have provided an explanation of the mechanism generating surface tension, and numerical results supporting the thesis that the NRVM ordering process belongs to the Cahn-Allen universality class of coarsening with non-conserved order parameter [1]. Further investigations on model 2 are required in order to clarify the asymptotic value of the coarsening exponent for large \( r \). The present analysis is consistent with recent results obtained by Castelló et al. [12], showing that the existence of an intermediate state (here due to the introduction of memory) is relevant in modifying the scaling properties of the VM. In order to give further insight on more realistic situations, such as those involving memory-driven nonequilibrium ordering dynamics on social networks, it would be interesting to study the behavior of the model on systems with heterogeneous topologies. We expect that the competition of the two types of kinetic mechanisms, i.e. curvature-driven coarsening and interfacial noise, may give rise to nontrivial ordering phenomena.

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