Edge Growth in Graph Cubes

Matt DeVos†  Stéphan Thomassé‡

Abstract

We show that for every connected graph $G$ of diameter $\geq 3$, the graph $G^3$ has average degree $\geq \frac{7}{4}\delta(G)$. We also provide an example showing that this bound is best possible. This resolves a question of Hegarty [3].

1 Introduction

Throughout the paper, we only consider simple graphs. Let $G$ be a graph. We denote by $v(G)$, $e(G)$ its number of vertices, edges respectively, and let $\delta(G)$ denote the minimum degree of $G$. The $k^{th}$-power of $G$, denoted by $G^k$, has vertex set $V(G)$ and edges the pair of vertices at distance at most $k$ in $G$. If $G$ is connected, the diameter of $G$ is the maximum distance between a pair of vertices of $G$, or, equivalently, the smallest integer $k$ so that $G^k$ is a clique.

Consider a generating set $A$ of a finite (multiplicative) group and suppose that $1 \in A$ and $g \in A \Rightarrow g^{-1} \in A$. Numerous important questions in Number Theory and Group Theory concern the increase in size from $|A|$ to $|A^k|$. Such problems can be phrased naturally in terms of Cayley graphs. If $G$ is the (simple) Cayley graph generated by $A$, then $G^k$ is generated by $A^k$ and the sizes of the sets $A$ and $A^k$ are given by the degrees of these (regular) graphs. Thus the growth of the set $A^k$ can be studied in terms of the number of additional edges in the graph $G^k$. For instance, the following result is an easy corollary of a famous theorem of Cauchy and Davenport.

Theorem 1.1 (Cauchy-Davenport). If $G$ is a connected Cayley graph on a group of prime order with diameter $< k$ then $e(G^k) \geq ke(G)$.

This research was done at the Graph Coloring workshop at the Technion, Haifa Israel.

†Supported in part by an NSERC Discovery Grant (Canada) and a Sloan Fellowship.

‡Université Montpellier 2 - CNRS, LIRMM 161 rue Ada, 34392 Montpellier Cedex, France thomasse@lirmm.fr
Inspired by this connection, Hegarty considered the more general problem of how many extra edges are formed when we move from a graph $G$ to the $k^{th}$ power of $G$. Although little can be said for graphs in general, the problem is interesting for connected regular graphs with a diameter constraint. Perhaps surprisingly, even for this class of graphs, there does not exist a positive constant $c$ so that $e(G^2) \geq (1 + c)e(G)$. In contrast to this, the following holds for the third power:

**Theorem 1.2** (Hegarty). There exists a positive constant $c$ so that every connected regular graph of diameter $\geq 3$ satisfies $e(G^3) \geq (1 + c)e(G)$.

Hegarty proved this for $c = 0.087$ and this was subsequently improved by Pokrovskiy [5] who showed that the same result holds with $c = \frac{1}{6}$ (Pokrovskiy also established some results for higher powers of $G$). These authors both raised the question of the best possible value of $c$. We settle this problem in the following theorem.

**Theorem 1.3.** If $G$ is a connected graph with diameter $\geq 3$, then $e(G^3) \geq \frac{7}{8}\delta(G)v(G)$.

In particular, when $G$ is regular, this shows that $c$ can be chosen to be $\frac{3}{4}$. To see that this is best possible, we construct a family of regular graphs defined as follows. The graph $G_k$ is obtained from the disjoint union of the graphs $H_1, H_2, \ldots, H_5$ by adding all possible edges between vertices in $H_i$ and $H_{i+1}$ for $1 \leq i \leq 4$, where the graphs $H_1$ and $H_5$ are copies of $K_{2k+1}$, the graphs $H_2$ and $H_4$ are copies of $K_{2k}$ minus a perfect matching, and $H_3$ is a single vertex. It follows that $G_k$ is $4k$-regular with $8k + 3$ vertices so $e(G_k) = \frac{1}{2}(8k + 3)(4k) = 16k^2 + 6k$. Its cube $G_k^3$ has $4k + 1$ vertices of degree $8k + 2$ and $4k + 2$ vertices of degree $6k + 1$ so it satisfies $e(G_k^3) = \frac{1}{2}(4k + 1)(8k + 2) + \frac{1}{2}(4k + 2)(6k + 1) = 28k^2 + 16k + 2$. The family of graphs $\{G_k\}_{k \in \mathbb{N}}$ hence shows that the constant $\frac{7}{8}$ in Theorem 1.3 is best possible.

![Figure 1: The graph $G_k$](image)

There are a number of interesting related problems for directed graphs. Here we highlight a rather basic conjecture, which, if true, would resolve a special case of the Caccetta-Häggkvist conjecture.

**Conjecture 1.4.** If $D$ is an orientation of a simple graph and every vertex of $D$ has indegree and outdegree equal to $d$ then $e(D^2) \geq 2e(D)$. 

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2 Proof

For a set of vertices $X$ we let $N(X)$ denote the closed neighbourhood of $X$, i.e. $N(X)$ is the union of $X$ and the set of vertices with a neighbour in $X$. For a nonnegative integer $k$ we let $N^k(X)$ denote the set of vertices at distance $\leq k$ from a point in $X$. For a vertex $v$ we simplify this notation by $N(v) = N(\{v\})$ and $N^k(v) = N^k(\{v\})$. Note that the degree of a $v$ in $G^3$ satisfies $\deg_{G^3}(v) = |N^3(v)| - 1$.

Proof of Theorem 1.3: Let $G$ be a connected graph with minimum degree $\delta$ and diameter $\geq 3$. We say that a path is geodesic if it is a shortest path between its endpoints. A vertex $v$ is doubling if $\deg_{G^3}(v) \geq 2\delta$. We let $Z$ be the set of doubling vertices in $G$. We now prove a sequence of claims.

(1) If $v$ is an internal vertex in a geodesic path of length 3, then $v$ is doubling.

To see this, suppose that our geodesic path has vertex sequence $u,v,v',u'$. Now $N(u) \cap N(u') = \emptyset$ and $N(u) \cup N(u') \subseteq N^3(v)$ so $v$ is doubling.

Now let $X_1,X_2,\ldots,X_m$ be the vertex sets of the components of $G - Z$.

(2) If $v$ and $v'$ both belong to the same $X_i$, for some $1 \leq i \leq m$, then $N^2(v) = N^2(v')$.

Since $G[X_i]$ is connected, it suffices to prove that $N^2(v) \subseteq N^2(v')$ when $v,v'$ are adjacent. In this case, suppose that $u \in N^2(v)$. Then there is a path of length 3 from $v'$ to $u$ which has $v$ as an internal vertex. By (1) this path cannot be geodesic, so there must be a path of length at most 2 from $v'$ to $u$, i.e. $u \in N^2(v')$.

Next, define a relation $\sim$ on $\{X_1,\ldots,X_m\}$ by the rule that $X_i \sim X_j$ if $N(X_i) \cap N(X_j) \neq \emptyset$.

(3) If $X_i \sim X_j$, $v \in X_i$ and $v' \in X_j$, then $N^2(v) = N^2(v')$.

In light of (2), it suffices to prove this in the case that $N(v) \cap N(v') \neq \emptyset$. To see this, suppose (for a contradiction) that $u \in N^2(v) \setminus N^2(v')$. Then we have $N(u) \cap N(v') = \emptyset$ and $N(u) \cup N(v') \subseteq N^3(v)$ so $v$ is doubling, which is contradictory.

(4) $\sim$ is an equivalence relation.

To check that $\sim$ is transitive, suppose that $X_i \sim X_j \sim X_k$ and choose $v \in X_i$ and $v' \in X_k$. It follows from (3) that $N^2(v) = N^2(v')$ but then $v$ and $v'$ have a common neighbour, hence $N(X_i) \cap N(X_k) \neq \emptyset$.

Let $\{Y_1,Y_2,\ldots,Y_{\ell}\}$ be the set of unions of equivalence classes of $\sim$.

(5) The subgraph of $G^2$ induced by $N(Y_i)$ is a clique for every $1 \leq i \leq \ell$. 

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Let \( v, v' \in N(Y_i) \). If one of \( v, v' \) is in \( Y_i \) then it follows from (3) that \( v, v' \) are adjacent in \( G^2 \). In the remaining case, choose \( u \in Y_i \) adjacent to \( v \). Since \( v' \in N^2(u) \) there is a path of length \( \leq 3 \) from \( v \) to \( v' \) which has \( u \) as an internal vertex. It now follows from (1) that \( v \) and \( v' \) are distance \( \leq 2 \) in \( G \), so they are adjacent in \( G^2 \).

Let \( y_i = |Y_i| \) for every \( 1 \leq i \leq \ell \).

(6) \( \text{deg}_{G^3}(v) \geq \delta + y_i \) for every \( v \in Y_i \).

Claim (5) shows that \( N(Y_i) \) induces a clique in \( G^2 \). Since \( G \) has diameter \( \geq 3 \) the graph \( G^2 \) is not a clique. Hence there must exist a vertex \( u \in N^2(Y_i) \setminus N(Y_i) \). Now \( N(u) \cap Y_i \subseteq N^3(v) \) which gives us \( \text{deg}_{G^3}(v) \geq \delta + y_i \) as desired.

Set \( y = y_1 + y_2 + \ldots + y_\ell \) and set \( z = |Z| \).

(7) \( z \geq \delta \ell - y \)

First note that \( \delta \leq |N(Y_i)| = |Y_i| + |N(Y_i) \cap Z| \) so \( |N(Y_i) \cap Z| \geq \delta - y_i \). Next, observe that \( N(Y_i) \cap N(Y_j) = \emptyset \) whenever \( i \neq j \). This gives us \( z = |Z| \geq \sum_{i=1}^{\ell} |N(Y_i) \cap Z| \geq \sum_{i=1}^{\ell} (\delta - y_i) = \delta \ell - y \) as desired.

We now have the tools to complete the proof. Combining the fact that every vertex in \( Z \) has degree at least \( 2\delta \) in \( G^3 \) with (6), gives us the following inequality (here we use Cauchy-Schwarz and (7) in getting to the third line)

\[
\sum_{v \in V(G)} \text{deg}_{G^3}(v) - \frac{7}{4}\delta v(G) \geq 2\delta z + \sum_{i=1}^{\ell} y_i (\delta + y_i) - \frac{7}{4}\delta (z + y) \\
= \frac{1}{4}\delta z - \frac{3}{4}\delta y + \sum_{i=1}^{\ell} y_i^2 \\
\geq \frac{1}{4}\delta (\delta \ell - y) - \frac{3}{4}\delta y + \frac{y^2}{\ell} \\
= \left( \frac{\delta \sqrt{\ell}}{2} - \frac{y}{\sqrt{\ell}} \right)^2 \\
\geq 0.
\]

This shows that \( G^3 \) has average degree \( \geq \frac{7}{4}\delta \), thus completing the proof.  \( \square \)

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