SHUFFLE BIALGEBRAS

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Abstract. The goal of our work is to study the spaces of primitive elements of the Hopf algebras associated to the permutahedra and the associahedra. We introduce the notion of shuffle bialgebras, and compute the subspaces of primitive elements associated to these algebras. These spaces of primitive elements have natural structure of some type of algebras which we describe in terms of generators and relations. Applying these results we are able to compute primitive elements of other combinatorial Hopf algebras, and describe the algebraic theories associated to them.

Introduction

The aim of this paper is to compute the subspaces of primitive elements of some combinatorial Hopf algebras and to describe them as free objects for some new types of algebras.

The main examples of combinatorial Hopf algebras studied are the Malvenuto-Reutenauer Hopf algebra (see [26]), the algebra spanned by the faces of the permutahedra (see [7], [4]), the algebra of functions between finite sets, and the algebra of planar rooted trees (see [23]). In all cases we describe them as free objects for some algebraic theories, and compute the subspaces of their primitive elements. Although the primitive elements of the Malvenuto-Reutenauer algebra and of the algebra of planar binary rooted trees have been previously computed in [2], [3] and [12], our description has the advantage of showing them as free objects for some algebraic theories described in terms of generators and relations. These results give in each case a Cartier-Milnor-Moore type theorem.

In the general case, there does not exist a standard method to compute the space of primitive elements of a non-cocommutative coalgebra. The examples studied in this paper have one point in common: they are equipped with an associative product ×, called the concatenation product, which verifies a nonunital infinitesimal relation with the coproduct. Nonunital infinitesimal bialgebras were introduced in [24], where we proved that any connected nonunital infinitesimal bialgebra is isomorphic to the cofree coalgebra spanned by the space of its primitive elements. This result is the main tool used in the present work to compute the primitive elements.

We deal first with shuffle algebras, whose free objects are closely related to the Malvenuto-Reutenauer and Solomon-Tits Hopf algebras. Shuffle algebras are a particular case of monoids in the category of \( S \)-modules, as described in [29] and [20], where the operations do not preserve the action of the symmetric group. Afterwards, we extend our results to other algebraic structures: the preshuffle algebras.

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and the grafting algebras, the last ones are given by the underlying spaces of non-symmetric operads. The way we study them is largely inspired by the treatment given by J.-L. Loday to the so-called triples of operads, see [22]. Let us describe briefly the method employed:

1. Given a linear algebraic theory \( \mathcal{T} \), we introduce the notion of \( \mathcal{T} \) bialgebra, in such a way that any free \( \mathcal{T} \) algebra has a natural structure of \( \mathcal{T} \) bialgebra.
2. In a second step, we identify the theory \( \mathcal{T} \) with the space of all the operations of the theory, and compute a basis for a subspace \( \text{Prim}_\mathcal{T} \) of \( \mathcal{T} \), such that the space of primitive elements of any \( \mathcal{T} \) bialgebra is closed under the action of the elements of \( \text{Prim}_\mathcal{T} \).
3. Afterwards, we prove that any free \( \mathcal{T} \) algebra \( \mathcal{T}(X) \) is isomorphic, as a coalgebra, to the cofree coalgebra spanned by the space \( \text{Prim}_\mathcal{T}(X) \), generated by the operations of \( \text{Prim}_\mathcal{T} \) on the elements of \( X \). Applying results proved in [24], we get that \( \text{Prim}_\mathcal{T}(X) \) is the space of primitive elements of \( \mathcal{T}(X) \).
4. Finally, we describe the algebraic theory associated to \( \text{Prim}_\mathcal{T} \) in terms of generators and relations; and prove that the category of connected \( \mathcal{T} \) bialgebras is equivalent to the category of \( \text{Prim}_\mathcal{T} \) algebras.

It is quite easy to compute the theory \( \text{Prim}_{\text{sh}} \) when \( \mathcal{T} \) is the theory of shuffle algebras. The other examples follow easily from this case.

The paper is organised as follows:
The first section of the paper recalls some contructions on planar rooted trees and on permutations, needed in the following sections.
In Section 2 we give the definition of a shuffle algebra, describe the free objects for this theory in terms of permutations, and give the main examples. Shuffle bialgebras are introduced in Section 3. We also show that there exist natural functors between the categories of graded infinitesimal bialgebras and the category of shuffle bialgebras.
In Section 4 we compute the primitive elements of a shuffle bialgebra, and prove that they are given by some new algebraic objects called \( \text{Prim}_{\text{sh}} \) algebras. We prove a Cartier-Milnor-Moore Theorem in this context, showing that the category of connected shuffle bialgebras is equivalent to the category of \( \text{Prim}_{\text{sh}} \) algebras.
The next Section is devoted to introduce the relationship between the algebraic theories introduced in the paper and the operads of dendriform, infinitesimal and 2–associative algebras. In Section 6 we show that any coproduct on a free shuffle algebra gives rise to a boundary map, and show that the classical boundary map of the permutohedra is an example of this construction.
In Section 7 the notion of preshuffle bialgebras is introduced. As a particular case of preshuffle algebras, we introduce grafting algebras, and prove that the free objects for this theory are given by the spaces spanned by planar coloured trees. In fact, the notion of grafting algebra coincides with non symmetric algebraic operad, however since we study them as algebras we keep the name of grafting algebra to designate them. From? the definition of \( \text{Prim}_{\text{sh}} \) algebras, we compute the suspaces of primitive elements of preshuffle bialgebras and grafting bialgebras, and describe any connected preshuffle (respectively, grafting) bialgebra as an enveloping algebra over its primitive part. A variation of Cartier-Milnor-Moore Theorem is proved in this context, showing that the category of connected preshuffle (respectively,
grafting) bialgebras is equivalent to the category of $\text{Prim}_{\text{psb}}$ (respectively, $\text{Prim}_{\text{gr}}$) algebras.

The last section of the paper contains some applications of the computations of primitive elements made for preshuffle, shuffle and grafting bialgebras to some good triples of operads (see [22]).

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1. **Preliminaries**

We introduce here some definitions and notations that are used in the paper.

Let $K$ be a field, $\otimes$ denotes the tensor product of vector spaces over $K$. Given a graded $K$-vector space $A$, $A_+$ is the space $A \oplus K$ equipped with the canonical maps $K \rightarrow A_+ \rightarrow K$.

Given a graded vector space $A = \bigoplus_{n \geq 0} A_n$, we denote the degree of an homogeneous element $x \in A_n$ by $|x| = n$.

For any set $X$, we denote by $K[X]$ the vector space spanned by $X$.

Given a $K$-vector space $V$, the tensor space over $V$ is the graded vector space $T(V) := \bigoplus_{n \geq 0} V \otimes^n$. The reduced tensor space $\overline{T}(V)$ over $V$ is the subspace $\bigoplus_{n \geq 1} V \otimes^n$.

The space $\overline{T}(V)$ with the concatenation product, given by:

$$(v_1 \otimes \cdots \otimes v_n) \cdot (w_1 \otimes \cdots \otimes w_m) := v_1 \otimes \cdots \otimes v_n \otimes w_1 \otimes \cdots \otimes w_m,$$

for $v_1, \ldots, v_n, w_1, \ldots, w_m \in V$, is the free associative algebra spanned by $V$.

This product is extended to $T(V)$ in the unique way such that the unit $1_K$ of the field $K$ becomes the unit for the concatenation product.

**Coalgebras.** A coalgebra $C$ over $K$ is a vector space, equipped with a coproduct $\Delta : C \rightarrow C \otimes C$, which is coassociative.

We use Sweedler’s notation, and denotes $\Delta(x) = \sum x_{(1)} \otimes x_{(2)}$, for $x \in C$.

A coalgebra $C$ is counital if there exists a linear map $\epsilon : C \rightarrow K$ such that $(\epsilon \otimes \text{Id}_C) \circ \Delta = \text{Id}_C = (\text{Id}_C \otimes \epsilon) \circ \Delta$, where we identify $K \otimes C$ and $C \otimes K$ with $C$, via the canonical isomorphism.

For a counital coalgebra $(C, \Delta, \epsilon)$, we define the reduced coproduct $\overline{\Delta} := \Delta - \text{Id}_C \otimes \epsilon - \epsilon \otimes \text{Id}_C$ on $\ker(\epsilon)$. Note that $\overline{\Delta} : \ker(\epsilon) \rightarrow \ker(\epsilon) \otimes \ker(\epsilon)$ is coassociative too.

Let $C = \bigoplus_{n \geq 0} C_n$ be a graded $K$-vector space. A graded coassociative coproduct on $C$ is a coassociative coproduct $\Delta$ such that $\Delta(A_n) \subseteq \bigoplus_{i=0}^n A_i \otimes A_{n-i}$.

Given a coassociative coproduct $\Delta$ on $C$ and $r \geq 1$, we denote by $\Delta^r : C \rightarrow C^{\otimes r+1}$ the homomorphism defined recursively as $\Delta^1 := \Delta$ and $\Delta^{r+1} := (\Delta^r \otimes \text{Id}_C) \circ \Delta$, for $r \geq 1$. 

Most of the coalgebras we deal with in the paper are not counital. Given such a coalgebra \((C, \Delta)\) we define an associate counital coalgebra \((C_+, \Delta_+)\), where \(C_+ := K \oplus C\) and
\[
\Delta_+(x) = \begin{cases} 
1 \otimes x + x \otimes 1 + \Delta(x), & \text{for } x \in C \\
x1K \otimes 1K, & \text{for } x \in K.
\end{cases}
\]

The space \(C_+\) with the coproduct \(\Delta_+\) and the counit \(\epsilon\) given by:
\[
\epsilon(x) := \begin{cases} 
0, & \text{for } x \in C, \\
x, & \text{for } x \in K,
\end{cases}
\]
is a counital coassociative coalgebra.

Let \(V\) be a vector space, the deconcatenation coproduct on \(T(V)\) is given by:
\[
\Delta_c(v_1 \otimes \cdots \otimes v_n) := \sum_{i=1}^{n-1} (v_1 \otimes \cdots \otimes v_i) \otimes (v_{i+1} \otimes \cdots \otimes v_n).
\]
The graded space \(\overline{T}(V) = T(V)_+\) equipped with the coproduct \(\overline{\Delta}\) is a coassociative coalgebra.

1.1. **Definition.** Let \(C = \bigoplus_{n \geq 1} C_n\) be a positively graded \(K\)-vector space, equipped with a coassociative coproduct \(\Delta\). An element \(x \in C\) is called primitive if \(\Delta(x) = 0\). The subspace of primitive elements of \(C\) is denoted by \(\text{Prim}(C)\).

1.2. **Definition.** Let \((C, \Delta)\) be a coassociative coalgebra, and let \(F_pC\) be the following filtration on \(C\):
\[
F_1C := \text{Prim}(C) \\
F_pC := \{ x \in C \mid \Delta(x) \in F_{p-1}C \otimes F_{p-1}C\}.
\]
Following the definition of D. Quillen (see [31]), we say that \(C\) is connected if \(C = \bigcup_{p \geq 1} F_pC\).

The definition of primitive element for the counital coalgebra \(C_+\) becomes \(x \in \text{Prim}(C_+)\) if \(\Delta_+(x) = x \otimes 1 + 1 \otimes x\). In this case, \(\text{Prim}(C_+) = \text{Prim}(C)\).

The main purpose of this work is to study bialgebra structures on spaces spanned by (coloured) functions between finite sets, permutations and trees. The rest of this section is devoted to introduce definitions and elementary results on these objects.

**Permutations and shuffles.** Let \(S_n\) be the group of permutations on \(n\) elements. A permutation \(\sigma\) is denoted by its image \((\sigma(1), \ldots, \sigma(n))\).

The element \(1_n = (1, 2, \ldots, n)\) denotes the identity of \(S_n\). The set \(S_\infty := \bigcup_{n \geq 1} S_n\) is the graded set of all permutations.
1.3. Definition. Given $1 \leq r \leq n$, a composition $\underline{n}$ of $n$ of length $r$ is a family of positive integers $(n_1, \ldots, n_r)$ such that $\sum_{i=1}^{r} n_i = n$. The number $r$ is called the length of the composition $\underline{n}$.

For any composition $\underline{n} = (n_1, \ldots, n_r)$ of $n$, there exists a homomorphism $S_{n_1} \times \cdots \times S_{n_r} \to S_n$ given by $(\sigma_1, \ldots, \sigma_r) \mapsto \sigma_1 \times \cdots \times \sigma_r$, where

$$(\sigma_1 \times \cdots \times \sigma_r)(i) := \sigma_k(i - n_1 - \cdots - n_{k-1} + n_1 + \cdots + n_k),$$

for $n_1 + \cdots + n_{k-1} < i \leq n_1 + \cdots + n_k$. For any composition $\underline{n}$, $S_n$ denotes the subgroup of $S_n$ which is the image of $S_{n_1} \times \cdots \times S_{n_r}$ under this embedding.

The operation $\times : S_n \times S_m \to S_{n+m}$ defined previously is an associative product on $S_\infty$, called the concatenation.

In [26], C. Malvenuto and C. Reutenauer introduce the following definition.

1.4. Definition. A permutation $\sigma \in S_n$ is irreducible if $\sigma \notin \bigcup_{i=1}^{n-1} S_i \times S_{n-i}$. We denote by $Irr_{S_n}$ the set of irreducible permutations of $S_n$.

Note that the graded vector space $K[S_\infty] := \bigoplus_{n \geq 1} K[S_n]$, where $K[S_n]$ denotes the vector space spanned by the set of permutations, equipped with the concatenation product, is the free associative product generated by $\bigcup_{n \geq 1} Irr_{S_n}$.

1.5. Definition. (1) A subgroup $W$ of $S_n$ is called a parabolic standard subgroup if $W = S_{n_1} \times \cdots \times S_{n_r}$, for some composition $(n_1, \ldots, n_r)$ of $n$.

(2) Given a composition $\underline{n} = (n_1, \ldots, n_r)$ of $n$, a $(n_1, \ldots, n_r)$-shuffle, or $\underline{n}$-shuffle, is an element $\sigma$ of $S_n$ such that:

$$\sigma^{-1}(n_1 + \cdots + n_{k-1} + 1) < \cdots < \sigma^{-1}(n_1 + \cdots + n_k),$$

for $1 \leq k \leq r - 1$.

The set of all $(n_1, \ldots, n_r)$-shuffles is denoted indistinctly $Sh(n_1, \ldots, n_r)$ or $Sh(\underline{n})$.

Note that

(i) $Sh(n_1, \ldots, n_r, 0, n_{r+1}, \ldots, n_r) = Sh(\underline{n})$, for any composition $\underline{n} = (n_1, \ldots, n_r)$ and any $0 \leq i \leq r$,

(ii) $Sh(n) = \{1^n\}$,

(iii) $Sh(\underline{1}) = Sh(1, \ldots, 1) = S_n$, for $n \geq 1$.

Given positive integers $n, m$, the permutation $\epsilon_{n,m} := (n+1, \ldots, n+m, 1, \ldots, n)$ belongs to $Sh(n, m)$.

The following results about Coxeter groups are well-known. For the first assertion see for instance [33], the second one is proved, in a more general context, in [6].

1.6. Proposition. (1) Given a permutation $\sigma \in S_n$ and an integer $0 \leq i \leq n$ there exists unique elements $\sigma_{(i)}^1 \in S_i$, $\sigma_{n-i}^{n-1} \in S_{n-i}$ and $\gamma \in Sh(i, n-i)$ such that $\sigma = (\sigma_{(i)}^1 \times \sigma_{n-i}^{n-1}) \cdot \gamma$.

(2) Given compositions $\underline{n}$ of $n$ and $Sh(\underline{m})$ of $m$, we have that:

$$(Sh(\underline{n}) \times Sh(\underline{m})) \cdot Sh(n, m) = Sh(\underline{n} \cup \underline{m}),$$
where \( \mathbf{n} \cup \mathbf{m} := (n_1, \ldots, n_r, m_1, \ldots, m_p) \).

**Coxeter poset of the symmetric group.** The results stated in the previous section imply that, for any composition \( \mathbf{n} \) of \( n \), and any right coset \( S_{n_1 \times \cdots \times n_r} \cdot \sigma \), there exists a unique element \( \delta \in \text{Sh}(\mathbf{n}) \) such that \( S_{n_1 \times \cdots \times n_r} \cdot \sigma = S_{n_1 \times \cdots \times n_r} \cdot \delta \).

1.7. **Definition.** For \( n \geq 1 \), the Coxeter poset \( \mathcal{P}_n \) of \( S_n \) is the set of cosets:
\[
\mathcal{P}_n := \{ S_{n_1 \times \cdots \times n_r} \cdot \delta \mid \text{with } \mathbf{n} = (n_1, \ldots, n_r) \text{ a composition of } n \text{ and } \delta \in \text{Sh}(\mathbf{n}) \},
\]
ordered by the inclusion relation.

The maximal element of \( \mathcal{P}_n \) is the unique coset modulo \( S_n \), that is \( S_n \cdot 1_n \) which will be denoted by \( \xi_n \), and the minimal elements are the cosets \( S_{1 \times \cdots \times 1} \cdot \sigma \) which are denoted simply by \( \sigma \), for \( \sigma \in S_n \).

Note that \( \mathcal{P}_n \) is the disjoint union of the subsets
\[
\mathcal{P}_n^r := \{ S_{n_1 \times \cdots \times n_r} \cdot \delta \mid \delta \in \text{Sh}(\mathbf{n}) \}, \text{ for } 1 \leq r \leq n.
\]

**Functions on finite sets.** Given positive integers \( n \) and \( r \), let \( \mathcal{F}_n^r \) be the set of all maps \( f : \{1, \ldots, n\} \rightarrow \{1, \ldots, r\} \). For \( f \in \mathcal{F}_n^r \), we denote it by its image \( (f(1), \ldots, f(n)) \). The constant function \( (1, \ldots, 1) \in \mathcal{F}_n^1 \) is denoted by \( \xi_n \). For \( n \geq 1 \), let \( \mathcal{F}_n := \bigcup_{r=1}^{n} \mathcal{F}_n^r \).

For any \( n, m, r, k \), there exists an embedding \( \mathcal{F}_n^r \times \mathcal{F}_m^k \hookrightarrow \mathcal{F}_{n+m}^{r+k} \), given by
\[
f \times g := (f(1), \ldots, f(n), g(1) + r, \ldots, g(m) + r), \text{ for } f \in \mathcal{F}_n^r \text{ and } g \in \mathcal{F}_m^k.
\]

1.8. **Remark.**

(1) The set of permutations \( S_n \) is a subset of \( \mathcal{F}_n^n \) and the embedding \( \mathcal{F}_n^r \times \mathcal{F}_m^k \hookrightarrow \mathcal{F}_{n+m}^{r+k} \), restricted to \( \mathcal{F}_n^r \times \mathcal{F}_m^k \), coincides with the concatenation \( \times \), previously defined.

(2) For any element \( f \in \mathcal{F}_n^r \) there exists a unique non-decreasing function \( f^\uparrow \in \mathcal{F}_n^r \) and a unique permutation \( \sigma_f \in \text{Sh}(n_1, \ldots, n_r) \) such that
\[
f = f^\uparrow \cdot \sigma_f,
\]
where \( n_i = |f^{-1}(i)| \) for \( 1 \leq i \leq r \), and \( \cdot \) denotes the composition of functions.

(3) For \( n \geq 1 \) and \( 1 \leq r \leq n \), there exists a natural bijection between \( \mathcal{P}_n^r \) and the set of all surjective maps from \( \{1, \ldots, n\} \) to \( \{1, \ldots, r\} \). Given a composition \( \mathbf{n} = (n_1, \ldots, n_r) \) of \( n \) and \( \delta \in \text{Sh}(\mathbf{n}) \), the element \( S_{n_1 \times \cdots \times n_r} \cdot \delta \) maps to the function \( \xi_{\mathbf{n}} \cdot \delta \), where:
\[
\xi_{\mathbf{n}}(j) := (\xi_{n_1} \times \cdots \times \xi_{n_r})(j) = k,
\]
for \( n_1 + \cdots + n_{k-1} < j \leq n_1 + \cdots + n_k \).

For example, the element \( S_{2,3,1} \cdot (3,1,2,4,6,5) \) maps to the function \( (2,1,1,2,3,2) \in \mathcal{F}_6^3 \).

For \( n, r \geq 1 \), we may identify \( \mathcal{P}_n^r \) with the subset \( \{ f \in \mathcal{F}_n \mid f \text{ is surjective} \} \) of \( \mathcal{F}_n \). The map \( \times : \mathcal{F}_n \times \mathcal{F}_m \rightarrow \mathcal{F}_{n+m} \) restricts to \( \times : \mathcal{P}_n \times \mathcal{P}_m \rightarrow \mathcal{P}_{n+m} \).
From now on, we shall use Remark 1.8 and denote the elements of $P_n$ as surjective maps by their image $f = (f(1), \ldots, f(n))$.

Let $K_n$ be the set of all maps $f \in P_n$ verifying the following condition:

if $f(i) = f(j)$, for some $i < j$, then $f(k) \leq f(i)$ for all $i \leq k \leq j$.

Is is immediate to check that $f \times g \in K_{n+m}$, for any $f \in K_n$ and $g \in K_m$.

We extend the definition of irreducible permutation to $\bigcup_{n \geq 1} F_n$ as follows:

1.9. Definition. An element $f \in F_n$ is called irreducible if $f \notin \bigcup_{i=1}^{n-1} F_i \times F_{n-1}$.

The set of irreducible elements of $F_n$ is denoted $\text{Irr}_{F_n}$. In a similar way, the sets of irreducible elements of $P_n$ and $K_n$ are the sets $\text{Irr}_{P_n} := P_n \cap \text{Irr}_{F_n}$ and $\text{Irr}_{K_n} := K_n \cap \text{Irr}_{F_n}$, respectively.

Again, the graded space $K[F_\infty] := \bigoplus_{n \geq 1} K[F_n]$ equipped with the concatenation product is the free associative algebra spanned by $\bigcup_{n \geq 1} \text{Irr}_{F_n}$. Analogous results hold for the spaces $K[P_\infty] := \bigoplus_{n \geq 1} K[P_n]$ and $K[K_\infty] := \bigoplus_{n \geq 1} K[K_n]$.

Planar rooted trees

1.10. Definition. A planar rooted tree is a non-empty oriented connected planar graph such that any vertex has at least two input edges and one output edge, equipped with a final vertex called the root. For $n \geq 2$, a planar $n$-ary tree is a planar rooted tree such that any vertex has exactly $n$ input edges.

Note that in a planar tree the set of input edges of any vertex is totally ordered. All trees we deal with are reduced planar rooted ones. From now on, we shall use the term planar tree instead of planar rooted tree.

1.11. Notation. (1) We denote by $Y_m$ the set of planar binary trees with $m+1$ leaves, and by $T_m$ the set of all planar trees with $m+1$ leaves. Clearly, $Y_m$ is a subset of $T_m$.

(2) Given a tree $t \in T_m$, we denote by $v(t)$ the number of internal vertices of $t$ and by $|t| = m$ the degree of $t$. The set $T_m$ is the disjoint union $\bigcup_{r=1}^m T^r_m$, where $T^r_m$ is the set of planar rooted trees with $m+1$ leaves and $r$ internal vertices.

Let $t$ be an element of $T_m$, the leaves of $t$ are numbered from left to right, beginning with 0 up to $m$. We denote by $c_m$ the unique element of $T_m$, which has $m+1$ leaves and only one vertex (the corolla). For any $t \in Y_m$, the number of internal vertices of $t$ is $v(t) = m$. 

Let $X = \bigcup_{n \geq 1} X_n$ be a positively graded set. The set $T_{m,X}$ is the set of planar binary trees with the internal vertices coloured by the elements of $X$ in such a way that any vertex with $k$ input edges is coloured by an element of $X_{k-1}$.

1.12. **Definition.** Given coloured trees $t$ and $w$, for any $0 \leq i \leq |w|$, define $t \circ_i w$ to be the coloured tree obtained by attaching the root of $t$ to the $i$-th leaf of $w$.

For instance

\[ \begin{array}{c}
\text{\downarrow} \quad \triangleright \quad \circ_2 \quad \triangleright \quad \downarrow \\
\triangleright \quad \downarrow \\
\triangleright \quad \downarrow
\end{array} = \begin{array}{c}
\text{\downarrow} \quad \triangleright \quad \downarrow \\
\downarrow \\
\triangleright \quad \downarrow
\end{array}. \]

1.13. **Notation.**

1. Given coloured trees $t^0, t^1, \ldots, t^{|w|}$ and $w$, we denote by $(t^0, \ldots, t^{|w|}) \circ w$ the tree obtained as follows:

$$(t^0, \ldots, t^{|w|}) \circ w := t^0 \circ_0 (t^1 \circ_1 (\ldots t^{|w|-1} \circ_{|w|-1} (t^{|w|} \circ_{|w|} w))).$$

2. Given two coloured trees $t$ and $w$ and $x \in X_1$, we denote by $t \lor_x w$ the tree obtained by joining the roots of $t$ and $w$ to a new root, coloured by $x$. More generally, we denote by $\lor_x (t^0, \ldots, t^r)$ the tree $(t^0, \ldots, t^r) \circ (c_r, x)$, for $x \in X_r$.

Any coloured tree $t$ may be written in a unique way as $t = \lor_x (t^0, \ldots, t^r)$, with $|t| = \sum_{i=0}^{r} |t^i| + r - 1$ and $x \in X_r$.

Recall that the number of elements of the set of planar binary rooted trees with $n+1$-leaves is the Catalan number $c_n = \frac{(2n)!}{n!(n+1)!}$ (see [4]). The number of planar rooted trees with $n+1$ leaves is called the super Catalan number $C_m$, and it is given by the following recursive formula:

$$C_m = C_{m-1} + 2 \sum_{i=1}^{n-1} C_i C_{n-(i+1)}.$$ 

2. **Shuffle algebras**

Our goal is to describe the spaces spanned by coloured permutations and coloured elements of $P_\infty$ as free objects for some type of algebraic structure.

In order to achieve our task we introduce the notion of shuffle algebras.

2.1. **Definition.** A *shuffle algebra* over $K$ is a graded $K$-vector space $A = \bigoplus_{n \geq 0} A_n$ equipped with linear maps

$$\bullet_\gamma : A_n \otimes_K A_m \rightarrow A, \text{ for } \gamma \in Sh(n,m),$$

verifying that:

$$x \bullet_\gamma (y \bullet_\delta z) = (x \bullet_\sigma y) \bullet_\lambda z,$$

whenever $(1_n \times \delta) \cdot \gamma = (\sigma \times 1_r) \cdot \lambda \in Sh(n,m,r)$. 

Let $\mathcal{S}$–$\text{Mod}$ be the category whose objects are infinite families $M = \{M(n)\}_{n \geq 0}$ of $K$-modules, such that each $M(n)$ is a right $K[S_n]$-module, for $n \geq 1$. A homomorphism $f$ from $M$ to $N$ in $\mathcal{S}$–$\text{Mod}$ is a family of linear maps $f(n) : M(n) \rightarrow N(n)$ such that each $f(n)$ is a morphism of $K[S_n]$-modules. The category $\mathcal{S}$–$\text{Mod}$ is endowed with a symmetric monoidal structure $\otimes_\mathcal{S}$ given by:

$$(M \otimes_\mathcal{S} N)(n) = \bigoplus_{i=0}^{n} (M(i) \otimes N(n-i)) \otimes_{K[S_i \times S_{n-i}]} K[S_n],$$

where $M(i) \otimes N(n-i)$ has the natural structure of right $K[S_i \times S_{n-i}]$-module.

By Proposition 1.6 the tensor product $(M(i) \otimes N(n-i)) \otimes_{K[S_i \times S_{n-i}]} K[S_n]$ is isomorphic to $M(i) \otimes N(n-i) \otimes K[Sh(i, n-i)]$.

Moreover, the associativity and symmetry of $\otimes_\mathcal{S}$ are given by the isomorphisms:

1. $a_{MNR} : (M \otimes_\mathcal{S} N) \otimes_\mathcal{S} R \rightarrow M \otimes_\mathcal{S} (N \otimes_\mathcal{S} R)$, with

$$a_{MNR}(x \otimes y \otimes \sigma) \otimes z \otimes \delta := x \otimes (y \otimes z \otimes \gamma) \otimes \tau,$$

whenever $(\sigma \times 1_r) \cdot \delta = (1_n \times \gamma) \cdot \tau$ in $Sh(m, n, r)$, for $x \in M(m)$, $y \in N(n)$ and $z \in R(r)$.

2. $c_{MN} : M \otimes_\mathcal{S} N \rightarrow N \otimes_\mathcal{S} M$, with

$$c_{MN}(x \otimes y \otimes \sigma) := y \otimes x \otimes (\epsilon_{n,m} \cdot \sigma),$$

where $\epsilon_{n,m} = (n + 1, \ldots, n+m, 1, \ldots, n) \in Sh(n, m)$, for $x \in M(m)$ and $y \in N(n)$.

Let $(M, \circ)$ be a monoid in $(\mathcal{S}$–$\text{Mod}, \otimes_\mathcal{S})$, the space $M = \bigoplus_{n \geq 0} M(n)$ has a natural structure of shuffle algebra, given by:

$$x \bullet_\gamma y := \circ(x \otimes y \otimes \gamma),$$

for $x \in M(n)$ and $y \in M(m)$. The associativity of $\circ$ implies that the products $\bullet_\gamma$ fulfill the conditions of Definition 2.1.

In [29] and [20], an associative monoid in $(\mathcal{S}$–$\text{Mod}, \otimes_\mathcal{S})$ is called a twisted associative algebra or an $\text{As}$-algebra in the category $\mathcal{S}$–$\text{Mod}$.

Given an associative graded algebra $(A = \bigoplus_{n \geq 0} A_n, \bullet)$, consider $\overline{A} = \{A_n \otimes K[S_n]\}_{n \geq 0}$.

The $\mathcal{S}$-module $\overline{A}$ has a natural structure of monoid in $(\mathcal{S}$–$\text{Mod}, \otimes_\mathcal{S})$, given by:

$$\circ((x, \sigma) \otimes (y, \tau) \otimes \gamma) := (x \bullet y) \otimes (\sigma \times \tau) \cdot \gamma \in A_{n+m} \otimes K[S_{n+m}],$$

for $x \in A_n$, $y \in A_m$, $\sigma \in S_n$, $\tau \in S_m$ and $\gamma \in Sh(n,m)$.

2.2. Examples. a) The tensor space For any vector space $V$ the tensor space $\mathcal{T}(V) := \bigoplus_{n \geq 1} V \otimes^n$, with the products $\bullet_\gamma$ given by:

$$(v_1 \otimes \cdots \otimes v_n) \bullet_\gamma (v_{n+1} \otimes \cdots \otimes v_{n+m}) := v_{\gamma(1)} \otimes \cdots \otimes v_{\gamma(n+m)},$$

for $v_1, \ldots, v_{n+m} \in V$, is a shuffle algebra.

b) Free shuffle algebras. Define, on the graded vector space $K[S_\infty] := \bigoplus_{n \geq 1} K[S_n]$, the operations $\bullet_\gamma : K[S_n] \otimes K[S_m] \rightarrow K[S_{n+m}]$ as follows:

$$\sigma \bullet_\gamma \tau := (\sigma \times \tau) \cdot \gamma,$$
for $\sigma \in S_n$, $\tau \in S_m$ and $\gamma \in Sh(n,m)$. It is immediate to check that $(K[S_{\infty}], \bullet)$ is a shuffle algebra.

In general, for any set $E$, the space $K[S_{\infty}, E] := \bigoplus_{n \geq 1} K[S_n \times E^n]$ equipped with the operations

$$(\sigma, x_1, \ldots, x_n) \bullet (\tau, x_{n+1}, \ldots, x_{n+m}) := (\sigma \bullet \psi, x_{\gamma(1)}, \ldots, x_{\gamma(n+m)})$$

is a shuffle algebra.

Note that the map $E \mapsto K[S_{\infty}, E]$ maps $e \in E$ to the element $(\{1\}; e) \in S_1 \times E$. So, the degree of any element $e \in E$ is one.

In general, given a positively graded set $X = \bigcup_{n \geq 1} X_n$, consider the vector space $K[\mathcal{F}_{\infty}, X]$ spanned by the elements $(f, x_1, \ldots, x_r) \in \mathcal{F}_n^r \times X^r$ such that $|x_i| = |f^{-1}(i)|$ for $1 \leq i \leq r$, with the operations given by:

$$(f; x_1, \ldots, x_r) \bullet (g; y_1, \ldots, y_k) := ((f \times g) \cdot \gamma; x_1, \ldots, x_r, y_1, \ldots, y_k),$$

for $(f; x_1, \ldots, x_r) \in \mathcal{F}_n X$, $(g; y_1, \ldots, y_k) \in \mathcal{F}_m X$ and $\gamma \in Sh(n, m)$, is a shuffle algebra.

The subspace $K[\mathcal{P}_{\infty}, X]$ of $K[\mathcal{F}_{\infty}, X]$ is closed under the products $\bullet$. So, $K[\mathcal{P}_{\infty}, X]$ has also a natural structure of shuffle algebra.

2.2.1. **Proposition.** Given a positively graded set $X$, the algebra $(K[\mathcal{P}_{\infty}, X], \bullet)$ is the free shuffle algebra spanned by $X$.

**Proof.** From the definition of shuffle algebra and Proposition 2.1.4 one has that any element in the free shuffle algebra spanned by $X$ is a sum of elements $x$, with

$$x = x_1 \bullet_{\gamma_1} (x_2 \bullet_{\gamma_2} \ldots (x_{k-1} \bullet_{\gamma_{k-1}} x_k)),$$

for unique elements $x_i \in X$ and unique shuffles $\gamma_i$, for $1 \leq i \leq k$. Let $\psi$ be the homomorphism from the free shuffle algebra spanned by $X$ to the space $K[\mathcal{P}_{\infty}, X]$, such that:

$$\psi(x_1 \bullet_{\gamma_1} (x_2 \bullet_{\gamma_2} \ldots (x_{k-1} \bullet_{\gamma_{k-1}} x_k))) := (\xi_{n_1} \cdot \gamma; x_1, \ldots, x_k),$$

where

1. $n_i = |x_i|$, for $1 \leq i \leq k$,
2. $\gamma = (1_{n_1+\ldots+n_k-2} \times \gamma_{k-1}) \cdots (1_{n_1} \times \gamma_2) \cdot \gamma_1$.

Conversely, let $f : \{1, \ldots, n\} \rightarrow \{1, \ldots, r\}$ be a surjective map, and let $n_i := |f^{-1}(i)|$, for $1 \leq i \leq r$. There exists a unique permutation $\gamma \in Sh(n)$ such that

$f = \xi_{n_1} \cdot \gamma$.

Moreover, there exist unique permutations $\gamma_i \in Sh(n_i, n_i + 1 + \cdots + n_k)$ such that:

$$\gamma = (1_{n_1+\ldots+n_k-2} \times \gamma_k) \cdots (1_{n_1} \times \gamma_2) \cdot \gamma_1.$$ 

The inverse of $\psi$ is given by:

$$\psi^{-1}(f; x_1, \ldots, x_k) = x_1 \bullet_{\gamma_1} (x_2 \bullet_{\gamma_2} \ldots (x_{k-1} \bullet_{\gamma_{k-1}} x_k)).$$

Clearly, if $E$ is concentrated in degree 0, the free shuffle algebra spanned by $E$ is $K[S_{\infty}, E]$. 

\[ \diamond \]
c) Non-unital infinitesimal bialgebras. Suppose that \((A, \cdot)\) is a graded \(K\)-algebra, equipped with a coassociative coproduct \(\Delta : A \otimes A \to A\) such that:

\[
\Delta(x \cdot y) = \sum x \cdot y_{(1)} \otimes y_{(2)} + \sum x_{(1)} \otimes (x \cdot y_{(2)}) + x \otimes y, \quad \text{for } x, y \in A,
\]

where \(\Delta(z) = \sum z_{(1)} \otimes z_{(2)}\), for \(z \in A\). The triple \((A, \cdot, \Delta)\) is called a nonunital infinitesimal bialgebra (see [27] and [28]).

It is easy to see that the reduced tensor space \(\overline{T}(V)\), equipped with the concatenation product and the deconcatenation coproduct, is a graded unital infinitesimal bialgebra which is denoted \(\overline{T}(V)\).

2.2.2. Remark. Given a permutation \(\gamma \in Sh(n, m)\) there exists unique integers \(n_1, \ldots, n_r\) and \(m_1, \ldots, m_r\), such that:

\[
\gamma = (1, \ldots, n_1, n + 1, \ldots, n + m_1, n_1 + 1, \ldots, n_1 + n_2, \ldots, m_1 + \cdots + m_{r-1} + 1, \ldots, m),
\]

where \(\sum_{i=1}^r n_i = n\) and \(\sum_{j=1}^r m_j = m\), \(n_i \geq 0\), \(n_i \geq 1\) for \(i > 2\), \(m_j \geq 1\) for \(j < r\), and \(m_r \geq 0\).

Let \(A = \bigoplus_{n \geq 1} A_n\) be a positively graded nonunital infinitesimal bialgebra. The map \(\Delta_{n_1, \ldots, n_r} : A_n \to A_{n_1} \otimes \cdots \otimes A_{n_r}\) is given by the composition of \(\Delta_{n}^{-1}\) with the projection \(p_{n_1 \ldots n_r} : A^n \to A_{n_1} \otimes \cdots \otimes A_{n_r}\).

For any \(x \in A_n\), let \(\Delta_{n_1, \ldots, n_r}(x) = \sum x_{(1)}^{n_1} \otimes \cdots \otimes x_{(r)}^{n_r}\).

The proof of the following result is given for a general case in Theorem 7.9.

2.2.3. Lemma. Let \((\bigoplus_{n \geq 1} A_n, \cdot, \Delta)\) be a graded nonunital infinitesimal bialgebra. The graded space \(A\), equipped with the operations:

\[
x \cdot y = \sum_{|x_{(i)}|=1} x_{(1)}^{n_1} \cdot y_{(1)}^{m_1} \cdot x_{(2)}^{n_2} \cdot \cdots \cdot y_{(r)}^{m_r}, \quad \text{for } x \in A_n, y \in A_n, \text{ and } \gamma \in Sh(n, m),
\]

where \(n_1, \ldots, n_r\) and \(m_1, \ldots, m_r\) are the integers which determine \(\gamma\), as pointed out in Remark 2.2.2 is a shuffle algebra.

d) The algebra of parking functions. (see [27] and [28]) Let \(PF_n\) be the subset of all functions \(f \in F_n^n\) which may be written as a composition \(f = f^1 \cdot \sigma\), with \(f_1 \in F_n^n\) such that \(f^1(i) \leq i\) for all \(1 \leq i \leq n\), and \(\sigma \in S_n\). Such a function is called a parking function.

Applying Remark 1.3, we get that for any parking function \(f \in PF_n\) there exist unique elements \(f^\dagger \in PF_n\) and \(\sigma \in Sh(r_1, \ldots, r_n)\) such that \(f^\dagger\) is a non-decreasing parking function and \(f = f^\dagger \cdot \sigma\), where \(r_i = |f^{-1}(i)|\).

Define the concatenation map \(\times : PF_n \times PF_m \to PF_{n+m}\) as the restriction of the concatenation product \(F_n^n \times F_m^m \to F_{n+m}^{n+m}\), that is:

\[
f \times g := (f(1), \ldots, f(n), g(1) + n, \ldots, g(m) + n).
\]

Note that \(f \times g\) is also a parking function. Moreover, for any functions \(f \in PF_n\), \(g \in PF_m\) and \(\gamma \in Sh(n, m)\), the product \(f \circ \gamma = (f \times g) \cdot \gamma\) belongs to \(PF_{n+m}\). Let \(\text{PQSym}_n\) denote the \(K\)-vector space spanned by the set \(PF_n\) for \(n \geq 1\), the space
spanned by all parking functions $\text{PQSym} := \bigoplus_{n \geq 1} \text{PQSym}_n$ is a shuffle subalgebra of $K[F_\infty]$.

Following 2.2.3 of [28], for a parking function $f \in PF_n$, an integer $b \in \{0, 1, \ldots, n\}$ is called a breakpoint of $f$ if $|\{i \mid F(i) \leq b\}| = b$.

I. Gessel defined a primitive parking function as an element $f \in PF_n$ such that its unique breakpoints are the trivial ones: 0 and $n$. Let $PPF_n$ be the subset of prime parking functions of $PF_n$. It is immediate to check that $f \in PF_n$ if its associated non-decreasing parking function cannot be written as a concatenation of parking functions of smaller degree. Note that the definition of breakpoint implies that if for any parking function $f \in P_n$ and any permutation $\sigma \in S_n$ the sets of breakpoints of $f$ and of $f \cdot \sigma$ are the same. So, the subset $PPF_n$ is invariant under the right action of $S_n$.

2.2.4. Remark. (see 2.2.3 of [28]) A element in $PPF_n$ is a parking function which cannot be described as $f \cdot \sigma$, $g$ for some $f \in PF_k$, $g \in PF_{n-k}$ and $\gamma \in Sh(k, n-k)$.

Remark 2.2.4 implies the following result.

2.2.5. Lemma. The shuffle algebra $\text{PQSym}$ is the free shuffle algebra spanned by the set $PPF := \bigcup_{n \geq 1} PPF_n$ of all prime parking functions.

Proof. As is pointed out in [28], a parking function $f \in PF_n$ has a breakpoint at $0 < b < n$ if and only if there exist unique functions $f_1 \in PF_b$, $f_2 \in PF_{n-b}$ and a shuffle $\sigma \in Sh(b, n-b)$ (not necessarily unique) such that $f = (f_1 \times f_2) \cdot \sigma$. By a recursive argument on the number of breakpoints of $f$, it is immediate to check that the set $PPF := \bigcup_{n \geq 1} PPF_n$ spans $\text{PQSym}$ as a shuffle algebra.

To see that $\text{PQSym}$ is free as a shuffle algebra, it suffices to note that for any function $f \in PF_n$ with breakpoints $0 < b_1 < \cdots < b_r < n$, there exist unique elements $f_1 \in PPF_{b_1}, f_2 \in PPF_{b_2-b_1}, \ldots, f_{r+1} \in PPF_{n-b_r}$ such that

$$f = (f_1 \times f_2 \times \cdots \times f_{r+1}) \cdot \sigma, \text{ with } \sigma \in Sh(b_1, b_2 - b_1, \ldots, n - b_r).$$

Note that the group $S_n$ acts on the right on the set $PPF_n$, for $n \geq 1$. So, $PPF = \{K[PPF_n]\}_{n \geq 1}$ is an object in the category $S\text{-Mod}$. Applying Lemma 2.2.5, it is immediate to check that $\text{PQSym} = T_\otimes(PPF) = \bigoplus_{n \geq 1} PPF_\otimes^n$ in the category $S\text{-Mod}$, which means that as a shuffle algebra $\text{PQSym}$ is the free monoid spanned by $PPF$ in the monoidal category $(S\text{-Mod}, \otimes\_S)$.

2.3. Definition. Given graded spaces $V = \bigoplus_{n \geq 0} V_n$ and $W = \bigoplus_{m \geq 0} W_m$ there exist two ways to obtain the product of both spaces:

(1) The Hadamard product of $V$ and $W$, denoted by $V \otimes_H W$, is the graded vector space such that $(V \otimes_H W)_n := V_n \otimes W_n$, for $n \geq 0$. 

2.4. **Remark.** Let $0 \leq r \leq n + m$ be an integer and let $\gamma$ be a $(n,m)$-shuffle. There exist a unique non negative integer $0 \leq n_1 \leq r$ and permutations $\gamma_{(1)}^r \in S_r$ and $\gamma_{(2)}^{n+m-r} \in S_{n+m-r}$ such that $\gamma = (1_{n_1} \times 1_{n-n_1} \times 1_{m-m_1}) \cdot (\gamma_{(1)}^r \times \gamma_{(2)}^{n+m-r})$, where $n_1 := |\gamma^{-1}([1, \ldots, n]) \cap \{1, \ldots, r\}|$ and $m_1 := r - n_1$. Moreover, the permutation $\gamma_{(1)}^r$ belongs to $Sh(n_1, m_1)$ and $\gamma_{(2)}^{n+m-r}$ belongs to $Sh(n - n_1, m - m_1)$.

The proof of the following result is immediate.

2.5. **Lemma.** Let $(A, \bullet)$ and $(B, \circ)$ be two shuffle algebras.

1. The Hadamard product $A \otimes B$ has a natural structure of shuffle algebra, given by the operations:

   $$(x \otimes y) \bullet (x' \otimes y') := (x \bullet x') \otimes (y \circ y'),$$

   for $x \in A_n, y \in B_n, x' \in A_m, y' \in B_m$ and $\gamma \in Sh(n, m)$.

2. The tensor product $A \otimes B$ has a natural structure of shuffle algebra, given by the operations:

   $$\begin{align*}
   (x \otimes y) \bullet (x' \otimes y') := & \begin{cases} 
   (x \bullet_{\gamma_{(1)}^{n+m}} x') \otimes (y \circ_{\gamma_{(2)}^{n+m}} y'), & \text{for } n = (n + m)_1, \\
   0, & \text{otherwise,}
   \end{cases} \\
   \end{align*}$$

   where $x \in A_n, x' \in A_{n'}, y \in A_m, y' \in A_{m'}, \gamma_{(1)}^{n+m} \in Sh(n, n')$ and $\gamma_{(2)}^{n+m} \in Sh(m, m')$ are the permutations defined in Remark 2.4 and $(n + n')_1 := |\gamma^{-1}([1, \ldots, n + m] \cap \{1, \ldots, n + m\})|$.

For any shuffle algebra $(A, \bullet)$, define the products $\bullet_0$ and $\bullet_{top}$ as follows:

- $x \bullet_0 y := x \bullet_{1_{n+m}} y$,
- $x \bullet_{top} y := y \bullet_{1_{n+m}} x$,

for $x \in A_n$ and $y \in A_m$.

2.6. **Remark.** The products $\bullet_0$ and $\bullet_{top}$ are associative.

Moreover, Proposition 1.6 implies the following result.

2.7. **Lemma.** Let $(A, \bullet)$ be a shuffle algebra. The product $\ast : A \otimes A \to A$ defined as:

$$x \ast y := \sum_{\gamma \in Sh(n,m)} x \bullet_{\gamma} y,$$

for $x \in A_n$ and $y \in A_m$,

is associative.
3. Shuffle bialgebras.

We study coproducts on shuffle algebras that turn them into Hopf algebras.

3.1. Definition. Let \((A, \bullet, \Delta)\) be a positively graded shuffle algebra, such that \(A\) is equipped with a graded coassociative coproduct \(\Delta\). We say that \((A, \bullet, \Delta)\) is a shuffle bialgebra if it verifies:

\[
\Delta(x \bullet y) = \sum_{r=1}^{n+m-1} \left( \sum (x(1) \bullet \gamma_{r} y(1)) \otimes (x(2) \bullet \gamma_{n+m-r} y(2)) \right),
\]

where \(\gamma_{r}\) and \(\gamma_{n+m-r}\) are defined in Remark 2.3, the second sum is taken over all \(|x(1)| = n_1\) and \(|y(1)| = m_1\), and we fix

\[
x(1) \bullet \gamma_{r} y(1) := \begin{cases} x, & \text{for } n_1 = n \\ y, & \text{for } n_1 = 0 \end{cases},
\]

\[
x(2) \bullet \gamma_{n+m-r} y(2) := \begin{cases} x, & \text{for } n_1 = 0 \\ y, & \text{for } n_1 = n \end{cases}.
\]

3.2. Proposition. Let \((A, \bullet, \Delta_A)\) and \((B, \circ, \Delta_B)\) be shuffle bialgebras. The Hadamard product \(A \otimes_B \otimes H\) with the operations \(\bullet\gamma\) given in Lemma 2.5 and the coproduct given by:

\[
\Delta_{A \otimes_B \otimes H}(x \otimes y) = \sum_{|x(1)| = |y(1)|} (x(1) \otimes y(1)) \otimes (x(2) \otimes y(2)),
\]

is a shuffle bialgebra.

Proof. Let \(x \in A_n, y \in B_n, z \in A_m, w \in B_m\) and \(\gamma \in Sh(n,m)\).

For \(1 \leq r, s \leq n + m\), we have that \(|x_{r(1)}^{n_1} \bullet \gamma_{r} z_{r(2)}^{m_1}| = r\) and \(|y_{r(1)}^{k_1} \bullet \gamma_{r} w_{r(2)}^{l_1}| = s\), for \(n_1 + m_1 = r\) and \(k_1 + l_1 = s\). So, \(|x_{r(1)}^{n_1} \bullet \gamma_{r} z_{r(2)}^{m_1}| = |y_{r(1)}^{k_1} \bullet \gamma_{r} w_{r(2)}^{l_1}|\) if, and only if \(r = s\), which implies that \(|x_{r(1)}^{n_1}| = n_1 = |y_{r(1)}^{k_1}|\) and \(|z_{r(2)}^{m_1}| = m_1 = |w_{r(2)}^{l_1}|\).

The argument above implies the result. \(\Diamond\)

The proof of the following Lemma is straightforward.

3.3. Lemma. (1) Let \((A, \bullet, \Delta)\) be a shuffle bialgebra. The relationship between \(\Delta\) and the associative products \(\bullet_0\) and \(\bullet_{top}\) defined in the previous subsection, is given by the following equalities:

\[
\Delta(x \bullet_0 y) = \sum (x \bullet_0 y(1)) \otimes y(2) + \sum x(1) \otimes (x(2) \bullet_0 y) + x \otimes y,
\]

\[
\Delta(x \bullet_{top} y) = \sum (x \bullet_{top} y(1)) \otimes y(2) + \sum x(1) \otimes (x(2) \bullet_{top} y) + x \otimes y,
\]

for \(x, y \in A\).
(2) Let \((A, \bullet, \Delta)\) be a shuffle bialgebra and let \(*\) be the associative product defined on \(A_+ = A \oplus K\) by:

\[
x * y = \begin{cases}
x \bullet y, & \text{for } x \in A_n \text{ and } y \in A_m \\
yx, & \text{if } x \in K \text{ or } y \in K.
\end{cases}
\]

Then \((A_+, *, \Delta_+)\) is a Hopf in the usual sense, which means that:

\[
\Delta_+(x * y) = \sum (x(1) * y(1)) \otimes (x(2) * y(2)), \quad \text{for } x, y \in A_+,
\]

where \(\Delta_+(\lambda) = \lambda 1_K \otimes 1_K\), for \(\lambda \in K\), and \(\Delta_+(x) := x \otimes 1_K + 1_K \otimes x + \Delta(x)\), for \(x \in A\).

3.4. Corollary. (1) If \((A, \bullet, \Delta)\) is a shuffle bialgebra, then \((A, \bullet_0, \Delta)\) and \((A, \bullet_{top}, \Delta)\) are nonunital infinitesimal bialgebras.

(2) If \((A, \bullet, \Delta)\) is a shuffle bialgebra, then \((A_+, *, \Delta_+)\) is a Hopf algebra.

The previous result implies that there exists two functors, \(H_0\) and \(H_{top}\), from the category of shuffle bialgebras to the category of graded nonunital infinitesimal bialgebras.

We prove that all the examples of shuffle algebras given in the previous Section can be equipped with a structure of shuffle bialgebra.

3.5. Examples. a) The Malvenuto-Reutenauer bialgebra (see [26]) On the vector space \(K[S\infty]\), let \(\Delta_{MR}\) be the unique coproduct such that:

\[
\Delta_{MR}(\sigma) := \sum_{r=1}^{n-1} \sigma^{r}_{(1)} \otimes \sigma^{n-r}_{(2)},
\]

for \(\sigma \in S_n\), where \(\sigma = \delta_r \cdot (\sigma^{r}_{(1)} \times \sigma^{n-r}_{(2)}), \) with \(\delta_r^{-1} \in Sh(r, n-r), \) for \(1 \leq r \leq n - 1\).

3.5.1. Proposition. The shuffle algebra \((K[S\infty], \bullet_\gamma)\), equipped with the coproduct \(\Delta_{MR}\) is a shuffle bialgebra.

Proof. Let \(\gamma\) be a \((n,m)\)-shuffle, and let \(0 \leq r \leq n + m\) be an integer.

From Remark 2.4, we have that there exists \(0 \leq n_1, m_1 \leq r\) such that:

\[
\gamma = (1_{n_1} \times \epsilon_{n-n_1,m_1} \times 1_{m-m_1}) \cdot (\gamma^{r}_{(1)} \times \gamma^{n+m-r}_{(2)}).
\]

Suppose that, for \(\sigma \in S_n\) and \(\tau \in S_m\),

\[
\sigma = \alpha \cdot (\sigma^{n_1}_{(1)} \times \sigma^{n_1}_{(2)}) \quad \text{and} \quad \tau = \beta \cdot (\tau^{m_1}_{(1)} \times \tau^{m_1}_{(2)}).
\]

Note that

\[
(\sigma^{n_1}_{(1)} \times \sigma^{n_1}_{(2)}) \cdot \epsilon_{n-n_1,m_1} = (\epsilon^{m_1}_{(1)} \times \sigma^{n_1}_{(2)}).
\]

So, the following equality holds:

\[
\sigma \bullet_\gamma \tau =
\]

\[
(\alpha \times \beta) \cdot (\sigma^{n_1}_{(1)} \times \epsilon_{n-n_1,m_1}) \cdot (\tau^{m_1}_{(1)} \times \sigma^{n_1}_{(2)} \times \tau^{m_1}_{(2)}) \cdot (\gamma^{r}_{(1)} \times \gamma^{n+m-r}_{(2)}) =
\]

\[
(\alpha \times \beta) \cdot (1_{n_1} \times \epsilon_{n-n_1,m_1} \times 1_{m-m_1}) \cdot ((\sigma^{n_1}_{(1)} \bullet_\gamma \tau^{m_1}_{(1)} \times \gamma^{r}_{(1)} \times \tau^{m_1}_{(2)})) \cdot (\sigma^{n_1}_{(1)} \bullet_\gamma \tau^{m_1}_{(2)} \cdot \gamma^{n+m-r}_{(2)} \gamma^{r}_{(1)} \times \tau^{m_1}_{(2)})).
\]
To end the proof it suffices to note that \((\alpha \times \beta) \cdot (1_{n_1} \times \epsilon_{m_1, n-n_1} \times 1_{m-m_1})\) belongs to \(Sh(r, n+m-r)\). \(\diamondsuit\)

Let \(E\) be a set, the coproduct of \(K[S_\infty]\) extends to \(K[S_\infty, E]\) as follows:

\[
\Delta_{MR}(\sigma, e_1, \ldots, e_n) = \sum_{i=1}^{n-1} (\sigma_{(1)}^i, e_{\delta_i(1)}, \ldots, e_{\delta_i(i)}) \otimes (\sigma_{(2)}^{n-i}, e_{\delta_i(i+1)}, \ldots, e_{\delta_i(n)}),
\]

where \(\sigma = \delta_i \cdot (\sigma_{(1)}^i \times \sigma_{(2)}^{n-i})\), with \(\delta_i \in Sh(i, n-i)\) for \(1 \leq i \leq n-1\).

Using the Proposition above, is not difficult to check that \((K[S_\infty, E], \bullet, \Delta_{MR})\) is a shuffle bialgebra.

b) Nonunital infinitesimal bialgebras. Let \((A, \cdot, \Delta)\) be a graded nonunital infinitesimal bialgebra.

3.5.2. Lemma. The associated shuffle algebra \((A, \cdot, \gamma)\), with the coproduct \(\Delta\) is a shuffle bialgebra.

Proof. Let \(\gamma \in Sh(n, m)\) be the permutation given by the sequences \(n_1, \ldots, n_r\) and \(m_1, \ldots, m_r\), as described in Remark 2.2.4. For any \(1 \leq s \leq n+m-1\), note that the decomposition

\[
\gamma = (1_{p_1} \times \epsilon_{n-p_1, q_1} \times 1_{m-q_1}) \cdot (\gamma_{(1)}^s \times \gamma_{(2)}^{n+m-s})
\]

of Remark 2.4 is such that there exists \(1 \leq k \leq r\), and \(1 \leq n_k' \leq n_k\) or \(1 \leq m_k' \leq m_k\) with:

\[
p_1 = \begin{cases} 
n_1 + \cdots + n_k', & \text{if } s = \sum_{i=1}^{k-1} (n_i + m_i) + n_k' \\\nn_1 + \cdots + n_k, & \text{if } s = \sum_{i=1}^{k-1} (n_i + m_i) + n_k + m_k'. \end{cases}
\]

In the first case \(\gamma_{(1)}^s\) is given by the sequence \(n_1, \ldots, n_k-1, n_k'\) and \(m_1, \ldots, m_k-1, 0\), while in the second one \(\gamma_{(1)}^s\) is given by the sequence \(n_1, \ldots, n_k\) and \(m_1, \ldots, m_k-1, m_k'\).

Given elements \(x \in A_n, y \in A_m\), the coassociativity of \(\Delta\) and the relation between \(\cdot\) and \(\Delta\) state that:

\[
\Delta(x \cdot y) = \sum \Delta(x_{(1)}^{n_1} \cdot y_{(1)}^{m_1} \cdots x_{(r)}^{n_r} \cdot y_{(r)}^{m_r}) =
\]

\[
\sum_{1 \leq k \leq r} \sum_{1 \leq n_k' \leq n_k} \left( x_{(1)}^{n_1} \cdot y_{(1)}^{m_1} \cdots x_{(k)}^{n_k'} \cdot y_{(k)}^{m_k} \cdot y_{(r)}^{m_r} \right) + \sum_{1 \leq m_k' \leq m_k} \left( x_{(1)}^{n_1} \cdots x_{(k)}^{n_k} \cdot y_{(k+1)}^{m_k'} \cdot y_{(k+1)}^{m_{k+1}} \cdots y_{(r)}^{m_r} \right).
\]

But \((x_{(1)} \cdot \gamma_{(1)}^s y_{(1)}) \otimes (x_{(2)} \cdot \gamma_{(2)}^{n+m-s} y_{(2)}) =
\]

\[
\begin{cases} 
(x_{(1)}^{n_1} \cdot y_{(1)}^{m_1} \cdots x_{(k)}^{n_k'}) \otimes (x_{(k+1)}^{n_k} \cdot y_{(k)}^{m_k} \cdots y_{(r)}^{m_r}), & \text{for } s = \sum_{i=1}^{k-1} (n_i + m_i) + n_k' \\
(x_{(1)}^{n_1} \cdots x_{(k)}^{n_k} \cdot y_{(k)}^{m_k}) \otimes (y_{(k+1)}^{m_k} \cdots x_{(k+1)}^{m_{k+1}} \cdots y_{(r)}^{m_r}), & \text{for } s = \sum_{i=1}^{k-1} (n_i + m_i) + n_k + m_k',
\end{cases}
\]

\[
\end{cases}
\]
which implies the result. \(\Diamond\)

Let \(G : Gr \rightarrow Sh\) be the functor which assigns to any graded nonunital infinitesimal bialgebra \((A, \cdot, \Delta)\) the shuffle bialgebra \((A, \bullet, \Delta)\). It is easy to check that the compositions \(H_0 \circ G\) and \(H_{top} \circ G\) are the identity functor, where \(H_0\) and \(H_{top}\) are the functors defined in Corollary 3.4.

c) The free shuffle algebra over a graded set. Let \(X\) be a positively graded set, and let \(\Theta : K[X] \rightarrow K[X] \otimes K[X]\), be a graded coassociative coproduct on \(K[X]\).

The coproduct \(\Delta_\Theta\) on \(K[\mathcal{P}_\infty, X]\) is defined as follows: given \(f = \xi_\mathbb{N} \cdot \sigma\), with \(\mathbb{N} = (n_1, \ldots, n_r)\), \(\sigma \in Sh(\mathbb{N})\), and elements \(x_1, \ldots, x_r \in X\), with \(x_i \in X_{n_i}\):

\[
\Delta_\Theta(f; x_1, \ldots, x_r) := \sum_{i=0}^n \left( \sum_{|x_{j(1)}|=m_j} (\xi_{\mathbb{N}} \cdot \sigma_{(1)}; x_{1(1)}, \ldots, x_{r(1)}) \otimes (\xi_{\mathbb{N}} \cdot \sigma_{(2)}; x_{1(2)}, \ldots, x_{r(2)}) \right),
\]

where

1. \(\sigma = \delta_i \cdot (\sigma_{(1)}^{n_1} \otimes \sigma_{(2)}^{n_2})\), with \(\delta_i \in Sh(i, n-i)\),
2. for each \(1 \leq i \leq n-1\),
   \[m_j := |\delta^{-1}_i{1, \ldots, i} \cap \{n_1 + \cdots + n_{j-1} + 1, \ldots, n_1 + \cdots + n_j}\|,
3. \(\mathbb{M}^i := (m_1^i, \ldots, m_n^i)\) and \(\mathbb{N} - \mathbb{M}^i := (n_1 - m_1^i, \ldots, n_r - m_r^i)\),
4. \(\Theta(x_j) = \sum x_{j(1)} \otimes x_{j(2)}, \) for \(1 \leq j \leq r\).

For example, suppose that \(X_n = \{\xi_n\}\), for \(n \geq 1\), if \(\Theta\) is the unique coassociative coproduct on \(K[X]\) such that \(\Theta(\xi_n) = \sum_{i=0}^n \xi_i \otimes \xi_{n-i}\), then for \(f = (2, 3, 3, 5, 4, 1, 4, 3)\), we get that:

\[
\Delta_\Theta(f) = (1) \otimes (2, 2, 5, 3, 1, 3, 2) + (1, 2) \otimes (2, 4, 3, 1, 3, 2) + (1, 2, 2) \otimes (4, 3, 1, 3, 2) + (1, 2, 2, 3) \otimes (3, 1, 3, 2) + (1, 2, 2, 4, 3) \otimes (1, 3, 2) + (2, 3, 3, 5, 4, 1) \otimes (2, 1) + (2, 3, 3, 5, 4, 1, 4) \otimes (1).
\]

3.5.3. Proposition. For any positively graded set \(X\) and any coassociative coproduct \(\Theta\) defined on \(K[X]\), the coproduct \(\Delta_\Theta\) defines a shuffle bialgebra structure on \((K[\mathcal{P}_\infty, X], \bullet)\).

Proof. Let \(f = \xi_{\mathbb{N}} \cdot \sigma \in \mathcal{P}_r^r, g = \xi_{\mathbb{M}} \cdot \tau \in \mathcal{P}_m^m, x_1, \ldots, x_r, y_1, \ldots, y_k \in X\) and \(\gamma \in Sh(n, m)\), be such that \(\sigma \in Sh(\mathbb{N}), \tau \in Sh(\mathbb{M}), |x_i| = n_i\) and \(|y_j| = m_j\), for \(1 \leq i \leq r\) and \(1 \leq j \leq k\).

Suppose that, for \(0 \leq r \leq n + m, \gamma = \delta_r \cdot (\gamma_{(1)}^{n_1} \times \gamma_{(2)}^{n_1+r}), with\)

\[
\delta_r = 1_{n_1} \times \epsilon_{n - n_1, m_1} \times 1_{m - m_1} \in Sh(r, n + m - r).
\]

In Example a) we prove that:

\[
\sigma \cdot \gamma = (\alpha \times \beta) \cdot \delta_r \cdot ((\sigma_{(1)}^{n_1} \cdot \epsilon_{(1)}^{n_1} \times \tau_{(1)}^{m_1}) \times (\sigma_{(2)}^{n_1} \cdot \epsilon_{(2)}^{n_1} \times \gamma_{(2)}^{n_1+r} \times \epsilon_{(2)}^{m_1})).
\]
with $\alpha \in \text{Sh}(n_1, n - n_1)$ and $\beta \in \text{Sh}(m_1, m - m_1)$. So,

$$\sigma \ast \gamma \tau = \left((\alpha \times \beta) \cdot \delta_1\right) \cdot ((\sigma \ast \gamma \tau)^{(1)}_1 \times (\sigma \ast \gamma \tau)^{(m-r)}_2).$$

We have that:

$$\Delta_\phi((f; x_1, \ldots, x_r) \ast (g; y_1, \ldots, y_k)) = \sum_i \left( \sum_{|x(i)_1| = l_j, |y(i)_1| = r_j} (\xi_{l, h} \cdot (\sigma \ast \gamma \tau)^{(1)}_i \cdot x(i)_1, \ldots, x(i)_r, y_1, \ldots, y_k) \otimes (\xi_{n-1, m-h} \cdot (\sigma \ast \gamma \tau)^{(m-r)}_{n-i} \cdot x(i)_2, \ldots, x(i)_2, y_1, \ldots, y_k) \right),$$

where $l_j := |\alpha| \cap \{n_1 + \cdots + n_j, n_1 + \cdots + n_j \}$, $h_j := |\beta| \cap \{m_1 + \cdots + m_j, m_1 + \cdots + m_j \}$, $1 = (l_1, \ldots, l_r)$ and $h = (h_1, \ldots, h_k)$.

So, we get that

$$\Delta_\phi((f; x_1, \ldots, x_r) \ast (g; y_1, \ldots, y_k)) = \sum_i (f; x_1, \ldots, x_r)_1 \cdot g(i) \cdot y_1, \ldots, y_k) \otimes (f; x_1, \ldots, x_r)_2 \cdot (n-m-i) \cdot (g; y_1, \ldots, y_k) \otimes (g; y_1, \ldots, y_k) \otimes (n-m-i) \cdot (g; y_1, \ldots, y_k) \otimes (n-m-i) \cdot (g; y_1, \ldots, y_k).$$

### d) Monoids in $(\mathbb{S}\text{-Mod}, \otimes)$. For an $\mathbb{S}$-module $M$, a coproduct on $M$ is a family of homomorphisms of $K[S_n]$-modules $\Omega : M(n) \rightarrow \bigoplus_{i=0}^{n} M(i) \otimes M(n-i) \otimes K[\text{Sh}(i, n-i)]$, for each $n \geq 0$. For $x \in M(n)$, we have that

$$\Omega(x) = \sum_{\sigma \in \text{Sh}(i, n-i)} x^{(1)}_1 \otimes x^{(2)}_2 \otimes \sigma.$$ 

The coproduct $\Omega$ is coassociative if for any $\sigma \in \text{Sh}(n, m + r)$, $\tau \in \text{Sh}(m, r)$, $\delta \in \text{Sh}(n + m, r)$ and $\omega \in \text{Sh}(m, n)$, such that $(1_n \times \tau) \cdot \sigma = (\omega \times 1_r) \cdot \delta$, it verifies the equality:

$$\sum x^{(1)}_1 \otimes (x^{(2)}_2)^{(1)} \otimes (x^{(2)}_2)^{(2)} = \sum (x^{(1)}_1)^{(1)} \otimes (x^{(1)}_1)^{(2)} \otimes (x^{(2)}_2)^{(1)} \otimes (x^{(2)}_2)^{(2)}.$$

A monoid $(M, \circ)$ in the category $(\mathbb{S}\text{-Mod}, \otimes)$ is a bialgebra if it is equipped with a coassociative coproduct verifying the condition:

$$\Omega(\circ(x \otimes y \otimes \gamma)) = \sum (x^{(1)}_1 \otimes y^{(1)}_1 \otimes \alpha_1) \otimes (x^{(2)}_2 \otimes y^{(2)}_2 \otimes \alpha_2) \otimes \rho, \quad (\ast)$$

for $x \in M(n)$, $y \in M(m)$ and $\gamma \in \text{Sh}(n, m)$, where

$$(1_n \times \epsilon_{m_1, n_2} \times 1_{m_2}) \cdot (\delta \times \tau) \cdot \gamma = (\alpha_1 \times \alpha_2) \cdot \rho \text{ in } \text{Sh}(n_1, m_1, n_2, m_2),$$

with $\alpha_1 \in \text{Sh}(n_1, m_1)$, $\alpha_2 \in \text{Sh}(n_2, m_2)$ and $\rho \in \text{Sh}(r, n + m - r)$, with $r = n_1 + m_1$.

Note that if $(M, \circ)$ is an algebra in $(\mathbb{S}\text{-Mod}, \otimes)$, then $\bigoplus M(n)$ is a shuffle algebra with the products $x \ast \gamma = \circ(x \otimes y \otimes \gamma)$. However, even if $(M, \circ, \Omega)$ is a bialgebra in $(\mathbb{S} - \text{Mod,} \otimes)$, the space $\bigoplus M(n)$ with $\ast$ and $\Omega$ is not necessarily a shuffle bialgebra. But it is possible to obtain two shuffle bialgebras from it, as we describe above.
3.5.4. **Proposition.** Let \( (M, \circ, \Omega) \) be a bialgebra in \((\mathbb{S} - \text{Mod}, \otimes_{\mathbb{S}})\), then:

1. The shuffle algebra \( (M = \bigoplus_{n \geq 0} M(n), \bullet) \) and the coproduct \( \Omega_0 \) given by:

\[
\Omega_0(x) := \sum_{i=0}^{n} x_{(1)}^{i} \otimes x_{(2)}^{n-i},
\]

where \( 1_n \) is considered as a \((i, n - i)\)-shuffle for \( 0 \leq i \leq n \), is a shuffle bialgebra.

2. The shuffle algebra \( (M = \bigoplus_{n \geq 0} M(n), \ast) \) and the coproduct \( \Omega_{\text{top}} \) given by:

\[
\Omega_{\text{top}}(x) := \sum_{i=0}^{n} x_{(2)}^{i} \otimes x_{(1)}^{n-i},
\]

where \( \varepsilon_{i,n-i} \) is considered as a \((i, n - i)\)-shuffle for \( 0 \leq i \leq n \), is a shuffle bialgebra.

**Proof.** For elements \( x \in M(n) \) and \( y \in M(m) \), a shuffle \( \gamma \in \text{Sh}(n,m) \) and an integer \( 0 \leq r \leq n + m \), we have that:

\[
\gamma = (1_{n_1} \times \varepsilon_{n-n_1,m_1} \times 1_{m-m_1}) \cdot (\gamma_r^\top \times \gamma_{n+m-r}^\bot),
\]

for \( n_1 = |\gamma^{-1}\{1, \ldots, r\} \cap \{1, \ldots, n\}| \) and \( m_1 = r - n_1 \).

If \( \delta = n \) and \( \tau = m \), then by formula (*) we get that:

\[
(1_{n_1} \times \varepsilon_{n-m-n_1,1} \times 1_{m-m_1}) \cdot \gamma =
(1_{n_1} \times \varepsilon_{n-n_1,m_1} \times 1_{m-m_1}) \cdot (\gamma_r^\top \times \gamma_{n+m-r}^\bot) =
(\gamma_r^\top \times \gamma_{n+m-r}^\bot) \cdot 1_{n+m},
\]

which implies the first statement.

Suppose that \( \delta = \varepsilon_{n_1(n-n_1)} \) and \( \tau = \varepsilon_{m_1(m-m_1)} \). We have that \( \varepsilon_{r(n+m-r)} =
(1_{n_1} \times \varepsilon_{m_1(n-n_1) \times 1_{m-m_1}} \cdot (\varepsilon_{n-n_1} \times \varepsilon_{m_1(m-m_1)} \cdot (1_{n_1} \times \varepsilon_{n-m-n_1} \times 1_{m-m_1}).

Moreover, if \( \gamma = (1_{n_1} \times \varepsilon_{n-m-n_1} \times 1_{m-m_1}) \cdot (\gamma_r^\top \times \gamma_{n+m-r}^\bot) \), then
\[
\varepsilon_{r,n+m-r} \cdot (\gamma_r^\top \times \gamma_{n+m-r}^\bot) = (\gamma_r^\top \times \gamma_{n+m-r}^\bot) \cdot \varepsilon_{n+m-r,n}.
\]

So, the formula (*) implies that \( (x \ast y)^{(r,n+m-r)} \ast y_{(1)}^{\gamma_1 \gamma_{(1)}} = x_{(1)}^{\gamma_1 \gamma_{(1)}} (x \ast y)^{(r,n+m-r)} y_{(1)}^{\gamma_{(1)}} \) and \( (x \ast y)^{(r,n+m-r)} \ast y_{(2)}^{\gamma_2 \gamma_{(2)}} = x_{(2)}^{\gamma_2 \gamma_{(2)}} (x \ast y)^{(r,n+m-r)} y_{(2)}^{\gamma_{(2)}} \). We may conclude that:

\[
\Omega_{\text{top}}(x \ast y) = \sum_{r=0}^{n+m} (x \ast y)^{(r,n+m-r)} \otimes (x \ast y)^{(r,n+m-r)} =
\sum_{r=0}^{n+m} x_{(2)}^{\varepsilon_{r,n-m-n_1} \gamma_{(1)}} \otimes y_{(2)}^{\gamma_{(1)}} y_{(1)}^{\gamma_{(1)}} y_{(1)}^{\gamma_{(1)}}
\]

which ends the proof.

\( \diamondsuit \)

**e) The bialgebra of parking functions.** (see [28]) Given a function \( f \in \mathcal{F}_n^k \), there exist a unique non-decreasing function \( f^\uparrow \in \mathcal{F}_n^k \), and a unique permutation \( \sigma \in \text{Sh}(n) \) such that

\[
f = f^\uparrow \cdot \sigma,
\]

where \( n_i := |f^{-1}(k_i)| \) for \( \text{Im}(f) = \{ k_1 < \cdots < k_r \} \).

In [28], J.-C. Novelli and J.-Y. Thibon define a graded map
Park : \bigcup_{n \geq 1} \mathcal{F}_n \rightarrow \bigcup_{n \geq 1} PF_n. We give a different description of Park, but it is easy to check that it coincides with the one defined in [28]. Let \( f^\uparrow \in \mathcal{F}_n \) be a non-decreasing function, the parking function Park\((f^\uparrow)\) is defined as follows:

\[
\text{Park}(f^\uparrow)(j) := \begin{cases} 
1, & \text{for } j = 1, \\
\min\{\text{Park}(f^\uparrow)(j-1)\} + f^\uparrow(j) - f^\uparrow(j-1), j, & \text{for } j > 1.
\end{cases}
\]

For \( f = f^\uparrow \cdot \sigma \), define:

\[
\text{Park}(f) := \text{Park}(f^\uparrow) \cdot \sigma.
\]

3.5.5. **Remark.**

(1) Let \( f \in PF_n \) be a parking function. It is easy to check that:

(a) \( f(i) = f(j) \) if, and only if \( \text{Park}(f)(i) = \text{Park}(f)(j) \).
(b) \( f(i) < f(j) \) if, and only if \( \text{Park}(f)(i) < \text{Park}(f)(j) \)

for \( 1 \leq i, j \leq n \).

(2) Let \( f, g \) be a pair of parking functions,

\[
\text{Park}(f \times g) = \text{Park}(f) \times \text{Park}(g).
\]

(3) If \( f \in PF_n \) is a parking function and \( \gamma \in S_n \) is a permutation, then

\[
\text{Park}(f \cdot \gamma) = \text{Park}(f) \cdot \gamma.
\]

Let \( \text{PQSym} = \bigoplus_{n \geq 1} PF_n \) be the graded space of all parking functions. The coproduct on \( \text{PQSym} \) is defined as follows (see [28]).

For \( f \in PF_n \) and \( 0 \leq r \leq n \),

\[
\Delta_{\text{PQSym}}(f) := \sum_{r=0}^{n} \text{Park}(f_1^r) \otimes \text{Park}(f_2^{n-r}),
\]

for \( f_1^r := (f(1), \ldots, f(r)) \) and \( f_2^{n-r} := (f(r+1), \ldots, f(n)) \).

3.5.6. **Proposition.** The shuffle algebra \( (\text{PQSym}, \bullet, \gamma) \) equipped with the coproduct \( \Delta_{\text{PQSym}} \) is a shuffle bialgebra.

**Proof.** Let \( f \in PF_m, g \in PF_n \) be parking functions, and let \( \gamma \) be a \( (n,m) \)-shuffle. For \( 0 \leq r \leq n \), we want to check that:

\[
\text{Park}\((f \bullet \gamma, g)_1^r\) \otimes \text{Park}\((f \bullet \gamma, g)_2^{n+m-r}\) = \\
(\text{Park}(f_1^{n_1}) \bullet_{(1)} \text{Park}(g_1^{m_1})) \otimes (\text{Park}(f_2^{n_2}) \bullet_{(2)} \text{Park}(g_2^{m_2})),
\]

where \( \gamma = (1_{n_1}, \epsilon_{m_1, n_1, 1} \times 1_{m_2}) \cdot (\gamma_{(1)}^r \times \gamma_{(2)}^{n+m-r}) \) is the decomposition described in Remark 2.3.

Computing \((f \times g) \cdot (1_{n_1, \epsilon_{m_1, n_1, 1} \times 1_{m_2}})\), we get that:

\[
(f \bullet \gamma, g)_1^r = (f_1^{n_1} \times g_1^{m_1}[n_1]) \cdot \gamma_{(1)}^r
\]

and

\[
(f \bullet \gamma, g)_2^{n-r} = (f_2^{n_2} \times g_2^{m_2}[m_2]) \cdot \gamma_{(2)}^{n+m-r},
\]

where \( f_1^{n_1} \times g_1^{m_1}[n_1] = (f(1), \ldots, f(n_1), g(1) + n, \ldots, g(m_2) + n) \).

By the Remark 3.5.5, we get that

\[
\text{Park}\((f \bullet \gamma, g)_1^r\) = \text{Park}(f_1^{n_1}) \bullet_{(1)} \text{Park}(g_1^{m_1}),
\]

\[
\text{Park}\((f \bullet \gamma, g)_2^{n+m-r}\) = \text{Park}(f_2^{n_2}) \bullet_{(2)} \text{Park}(g_2^{m_2}).
\]
which ends the proof. ◊

4. Primitive elements of shuffle bialgebras

In this section we use the notations and definitions given in the preliminaries about coalgebras. We recall some results proved in [24] that we need to study primitive elements in shuffle algebras.

Following [24], let \((H, \cdot, \Delta)\) be a triple such that \((H, \Delta)\) is a connected coassociative coalgebra and \((H, \cdot)\) is an associative algebra. Define the \(K\)-linear map \(e \in \text{End}_K(H)\) as follows:
\[
e(x) := x - \sum x^{(1)} \cdot x^{(2)} + \cdots + (-1)^{r+1} \sum \cdots + \sum_{r \geq 1} (-1)^{r+1} r \circ \Delta^r(x),
\]
where \(\Delta^r(x) = \sum x^{(1)} \otimes \cdots \otimes x^{(r)}\).

4.1. Proposition. (see Proposition 2.5 of [24]) Any connected infinitesimal bialgebra \((H, \cdot, \Delta)\) verifies that:
1. the image of \(e\) is \(\text{Prim}(H)\),
2. the restriction \(e|_{\text{Prim}(H)} = \text{Id}_{\text{Prim}(H)}\), and
3. \(e(x \cdot y) = 0\) for all \(x, y \in \text{Ker}(\epsilon)\).
4. any element \(x\) of \(\text{Ker}(\epsilon)\) verifies that
\[
x = e(x) + \sum e(x^{(1)}) \cdot e(x^{(2)}) + \cdots + \sum \cdots \sum e(x^{(1)}) \cdots e(x^{(n)}) + \cdots,
\]
where \(\Delta^n(x) = \sum x^{(1)} \otimes \cdots \otimes x^{(n)}\).

4.2. Theorem. (see Theorem 2.6 of [24]) Any connected infinitesimal bialgebra \(H\) is isomorphic to \((\mathcal{T}(\text{Prim}(H)) := (\bigoplus_{n \geq 1} \text{Prim}(H)^{\otimes n}, \nu, \Delta), where \(\nu\) is the concatenation product and \(\Delta\) is the deconcatenation coproduct.

This section is devoted to compute the subspace of primitive elements of a shuffle bialgebra, and describe it as an algebra for certain type of algebraic structure.

Let \((A, \bullet)\) be a shuffle algebra over \(K\). For any pair of positive integers, the permutations \(1_{n+m}\) and \(e_{nm} := (n + 1, \ldots, m + n, 1, \ldots, n)\) belong to \(\text{Sh}(n, m)\). Given elements \(x \in A_n\) and \(y \in A_m\), we shall keep the notations \(x \bullet_0 y\) for the element \(x \bullet_{1_{n+m}} y\), and \(y \bullet_{\text{top}} x\) for the element \(x \bullet_{e_{nm}} y\), in order to simplify notation. Recall that both operations are associative.

The binary operation \(\{-, -\} : A \otimes A \rightarrow A\) is given by the formula:
\[
\{x, y\} := x \bullet_{\text{top}} y - x \bullet_0 y, \text{ for } x, y \in A.
\]
We want to prove that the subspace of primitive elements of a shuffle bialgebra is closed under the operations \{-, -\} and \(\cdot_\gamma\), for \(\gamma \in Sh(n, m) \setminus \{1_{n+m}, \epsilon_{nm}\}\), with \(n, m \geq 1\).

4.3. Proposition. Let \((A = \bigoplus_{n \geq 1} A_n, \cdot_\gamma, \Delta)\) be a shuffle bialgebra. If the homogeneous elements \(x \in A_n\) and \(y \in A_m\) belong to \(\text{Prim}(A)\), then \(x \cdot_\gamma y\) and \(\{x, y\}\) belong to \(\text{Prim}(A)\), for any \(\gamma \in Sh(n, m) \setminus \{1_{n+m}, \epsilon_{nm}\}\).

Proof. Note that:

\[
\Delta(x \cdot_{\text{top}} y) = x \otimes y = \Delta(x \cdot_0 y), \quad \text{for } x, y \in \text{Prim}(A),
\]
which implies that \(\Delta(\{x, y\}) = 0\), whenever \(x, y \in \text{Prim}(A)\).

For any permutation \(\gamma \in Sh(n, m) \setminus \{1_{n+m}, \epsilon_{nm}\}\), the coproduct verifies that:

\[
\Delta(x \cdot_\gamma y) = \sum_r \left(\sum_l (x(1) \cdot_{\gamma(l)} y(1)) \otimes (x(2) \cdot_{\gamma(l)^{-1}+m-r} y(2))\right),
\]
where \(\gamma = (\gamma_1 \times \gamma_2^{n+m-r}) \cdot (1_n \times \epsilon_{(n-m)1_m} \times 1_{m-m})\).

Since \(\sum x(1) \otimes x(2) = 0\) and \(\sum y(1) \otimes y(2) = 0\), we get that \((x(1) \cdot_{\gamma(1)} y(1)) \otimes (x(2) \cdot_{\gamma(2)^{-1}+m-r} y(2))\) if, and only if, \(r = n = n\) and therefore \(\gamma = 1_{n+m}\), or \(r = m = m\) and therefore \(\gamma = \epsilon_{nm}\).

But \(\gamma \notin \{1_{n+m}, \epsilon_{nm}\}\), so \(\Delta(x \cdot_\gamma y) = 0\). \(\Box\)

In order to get a nice expression for the elements of the subspace of primitive elements of a shuffle bialgebra, let us introduce \((q + 1)\)-ary operations \(B_q^\gamma\).

4.4. Definition. Let \((A, \cdot_\gamma)\) be a shuffle algebra over \(K\). For \(n \geq 1\), let \(x \in A_n\), \(y_1 \in A_{m_1}, \ldots, y_q \in A_{m_q}\) be elements of \(A\), and let \(\gamma \in Sh(n, m)\) be such that \(\gamma^{-1}(1) < \gamma^{-1}(n+m)\) and \(\gamma^{-1}(n+m_1+\cdots+m_{q-1})+1 < \gamma^{-1}(n)\), where \(m = \sum_{i=1}^q m_i\).

The \((q + 1)\)-ary operation \(B_q^\gamma : A^{q+1} \to A\) is defined by the following formula:

\[
B_q^\gamma(x; y_1, \ldots, y_q) := x \cdot_\gamma (y_1 \bullet_0 y_2 \bullet_0 \cdots \bullet_0 y_q).
\]

Note that, in the case \(n = 1\), the conditions on \(\gamma\) imply that \(\gamma \notin \{1_{n+m}, \epsilon_{nm}\}\). So, \(B_q^\gamma(x, y)\) is simply \(x \cdot_\gamma y\).

4.5. Remark. Let \(x \in A_n, y \in A_m, y_i \in A_{m_i}\) for \(1 \leq i \leq q\), and \(z \in A_r\).

(1) For \(q = 1\) the following equalities are immediate to check:

\[
B_1^\gamma(B_1^\gamma(x; y); z) = \begin{cases} B_2^\gamma(x; y, z), & \text{for } \sigma = 1_{m+r} \\ B_1^\gamma(x; B_1^\gamma(y; z)), & \text{for } \sigma \notin \{1_{m+r}, \epsilon_{mr}\} \\ B_1^\gamma(x; \{z, y\}) + B_2^\gamma(x; z, y), & \text{for } \sigma = \epsilon_{mr}, \end{cases}
\]

where \((\tau \times 1_r) \cdot \gamma = \delta \cdot (1_n \times \sigma) \cdot \delta\).

(2) Given a permutation \(\tau \in Sh(n, m)\) such that \(\tau^{-1}(1) < \tau^{-1}(n+m)\) and \(\tau^{-1}(n+m_1+\cdots+m_{q-1})+1 < \tau^{-1}(n)\), we get that:

\[
B_q^\tau(x; y_1, \ldots, y_q) = B_{q-1}^{\tau_1}(B_1^\tau(x; y_1); y_2, \ldots, y_q) = B_1^{\tau_{q-1}}(\ldots B_1^{\tau_2}(x; y_1); \ldots; y_{q-1}); y_q),
\]
for $\tau = (\sigma_1 \times 1_{m_2 + \ldots + m_q}) \cdot \tau_1$ and $\tau_j = (\sigma_{j+1} \times 1_{m_{j+1} + \ldots + m_q}) \cdot \tau_{j+1}$, where $1 \leq j \leq q-2$.

(3) Let $\sigma \in Sh(n, m_1 + \cdots + m_r)$, $\sigma \neq 1_{n+m_1+\cdots+m_r}$, then there exist integers $0 \leq j \leq k+1 \leq r+1$ such that:

$$\sigma^{-1}(n + m_1 + \cdots + m_{j-1}) + 1 < \sigma^{-1}(1) \leq \sigma^{-1}(n + m_1 + \cdots + m_j)$$

$$\sigma^{-1}(n + m_1 + \cdots + m_{k-1} + 1) < \sigma^{-1}(n)\sigma^{-1}(n + m_1 + \cdots + m_k + 1).$$

If $j = k+1$, then $\sigma = \epsilon_{n(m_1+\cdots+m_{j-1})} \times 1_{m_j+\cdots+m_r}$. In this case,

$$x \circ_\sigma (y_1 \circ_0 \cdots \circ_0 y_r) =$$

$$\sum_{i=1}^{k} y_1 \circ_0 \cdots \circ_0 y_{i-1} \circ_0 (\{y_i;x\} \circ_{\sigma_i} (y_{i+1} \circ_0 \cdots \circ_0 y_k)) \circ_0 y_{k+1} \circ_0 \cdots \circ_0 y_r,$$

where $\sigma_i = 1_{m_i} \times \epsilon_{n(m_{i+1}+\cdots+m_k)}$, for $1 \leq i \leq k$.

If $j \leq k$, then

$$\sigma = (\epsilon_{n(m_1+\cdots+m_{j-1})} \times 1_{m_j+\cdots+m_r}) \cdot (1_{m_1+\cdots+m_{j-1}} \times \hat{\sigma} \times 1_{m_{k+1}+\cdots+m_r}),$$

where $\hat{\sigma} \in Sh(n; m_i + \cdots + m_k)$. The permutation

$$\sigma_i := (1_{m_i} \times \epsilon_{n(m_{i+1}+\cdots+m_{j-1})} \times 1_{m_j+\cdots+m_k}) \cdot (1_{m_i+\cdots+m_j} \times \hat{\sigma},$$

belongs to $Sh(n, m_i + \cdots + m_k)$ for $1 \leq i \leq j$, and we have that:

$$x \circ_\sigma (y_1 \circ_0 \cdots \circ_0 y_r) =$$

$$\sum_{i=1}^{j-1} y_1 \circ_0 \cdots \circ_0 y_{i-1} \circ_0 B^\sigma_{k-i}(\{y_i;x\}; y_{i+1}, \ldots, y_k) \circ_0 y_{k+1} \circ_0 \cdots \circ_0 y_r +$$

$$y_1 \circ_0 \cdots \circ_0 y_{j-1} \circ_0 B^\sigma_{k-j+1}(x; y_j, \ldots, y_k) \circ_0 y_{k+1} \circ_0 \cdots \circ_0 y_r.$$

Point (2) of Remark 4.5 states that given primitive elements $x, y_1, \ldots, y_q$ in a shuffle bialgebra $A$, the element $B^\sigma_q(x; y_1, \ldots, y_q)$ is primitive too.

The following Proposition describes the relationships between the operations $B^\gamma_q$ and $\{-, -\}$.

4.6. **Proposition.** Let $x \in A_m, y \in A_m$, $z_i \in A_{r_i}$ for $1 \leq i \leq q$ and $w \in A_n$ be elements of a shuffle algebra $A$. The operations $B^\gamma_q$ and $\{-, -\}$ defined above verify
the following equalities:

1. \( \{x, \{y, w\}\} = \{(x, y), w\} + B^1_{\gamma}((x, y); y) - B_{\gamma}^{1 \times \epsilon_m}(x, w) \).
2. \( \{x; B^1_\gamma(y; w)\} = B_\gamma^1(y; x, w) + B^1_\gamma((x, y); w) \),
   where \( \gamma := (\epsilon_m \times 1_s) \cdot (1_n \times \gamma) \) and \( \tilde{\gamma} = 1_n \times \gamma \).
3. \( \{B^1_\gamma(x; y), w\} = B_\gamma^2((x; y); w) + B^{1 \times 1}(x; y, w) \),
   where \( \gamma := (1_m \times \epsilon_m) \cdot (\gamma \times 1_s) \).
4. Let \( \gamma \in Sh(n + m, r) \setminus \{1_{n+m+r}, \epsilon_{(n+m)r}\} \), and \( \tau \in Sh(n, m) \setminus \{1_{n+m}, \epsilon_{nm}\} \),
   be permutations such that \( (\tau \times 1_r) \cdot \gamma = (1_n \times \sigma) \cdot \delta \),
   with \( \delta \in Sh(n, m + r) \) and \( \sigma \in Sh(m, r) \), where \( r := \sum_{i=1}^{q} r_i \).

a) If \( \sigma = 1_{m+r} \), then
   \[
   B^\delta_q(B^1_\gamma(x; y); z_1, \ldots, z_q) = B^\delta_{q+1}(x; y, z_1, \ldots, z_q).
   \]
b) If \( \sigma \neq 1_{m+r} \), then there exist integers \( 0 \leq j \leq q \) and \( 1 \leq k \leq q \), defined in the same way
   that in point (4) of Remark 4.5, and we get that: i) For \( j = k + 1 \),
   \[
   B^\delta_q(B^1_\gamma(x; y); z_1, \ldots, z_q) = \sum_{i=1}^{k} B^\delta_{q-k+i}(x; z_1, \ldots, z_{i-1}, B^{\sigma_i}_{k-i}(\{z_i, y\}; z_{i+1}, \ldots, z_k), z_{k+1}, \ldots, z_q) + B^\delta_{q+1}(x; z_1, \ldots, z_k, y, z_{k+1}, \ldots, z_q), l
   \]
   where \( \sigma_i := 1_{1_r} \times \epsilon_{(r_{i+1}+\cdots+r_k)m} \), for \( 1 \leq i \leq k \).

ii) For \( j \leq k \),
   \[
   B^\delta_q(B^1_\gamma(x; y); z_1, \ldots, z_q) = \sum_{i=1}^{j-1} B^\delta_{q-k+i}(x; z_1, \ldots, z_{i-1}, B^{\sigma_i}_{k-i}(\{z_i, y\}; z_{i+1}, \ldots, z_k), z_{k+1}, \ldots, z_m) + B^\delta_{q-k+j}(x; z_1, \ldots, z_{j-1}, B^{\sigma_j}_{k-j}(y; z_j, \ldots, z_k), z_{k+1}, \ldots, z_m),
   \]
   where \( \sigma_i \) is the permutation defined in Point (4) of Remark 4.5 for \( 1 \leq i \leq j \).

Proof. First, note that if \( \gamma \notin \{1_{n+m+r}, \epsilon_{(n+m)r}\} \) and \( \tau \notin \{1_{n+m}, \epsilon_{nm}\} \), then \( \delta \notin \{1_{n+m+r}, \epsilon_{(n+m)r}\} \).

Points (1), (2) and (3) are easily verified, while point (4) is a straightforward consequence of Remark 4.5. \( \diamond \)

4.7. Definition. A \( \text{Prim}_{sh} \) algebra is a graded vector space \( V \) equipped with operations \( B^\gamma : V_n \otimes V_m \to V \), for \( \gamma \in Sh(n, m) \setminus \{1_{n+m}, \epsilon_{nm}\} \), and a binary operation \{−, −\} which verify the following relations:

1. \( \{x, \{y, w\}\} = \{(x, y), w\} - B^{1 \times \epsilon_m}(x, w; y) + B_{\gamma}^{1 \times \epsilon_m}(x, w) \).
2. \( \{x; B^\gamma(y; w)\} = B^\gamma(y; x, w) + B^\gamma((x, y); w) \),
   where \( \gamma := (\epsilon_m \times 1_s) \cdot (1_n \times \gamma) \) and \( \tilde{\gamma} = 1_n \times \gamma \);
(3) \( \{B^\gamma(x; y), w\} = B2\{x; w\}; y) + B^\gamma \times 1_\gamma(x; \{y, w\}) \),
where \( \gamma := (1_m \times e_{sn}) \cdot (\gamma \times 1_x) \);
(4) \( B^\gamma(B^\delta(x; y); w) = B^\gamma(x; B^\delta(y; w)) \), whenever \( (\delta \times 1_x) \cdot \gamma = (1_n \times \sigma) \cdot \tau \),
with \( \sigma \neq 1_{m+s} \);
for \( x \in \mathcal{A}_m \), \( y \in \mathcal{A}_n \) and \( w \in \mathcal{A}_s \).

Proposition [1.4.6] states that there exists a functor from the category Sh-alg of shuffle algebras to the category of \( \mathcal{P}_{rim} \) algebras. Given a shuffle bialgebra \( \mathcal{A} \), the subspace of primitive elements \( \text{Prim}(\mathcal{A}) \) is a \( \mathcal{P}_{rim} \) subalgebra of \( \mathcal{A} \).

Let \( (\mathcal{A}, \bullet, \Delta) \) be a shuffle bialgebra, recall that \( (\mathcal{A}_+, \bullet_0, \Delta_+) \) is a unital infinitesimal bialgebra.

Given a positively graded set \( X \), we use Proposition 4.1 and Theorem 4.2 to describe the free shuffle algebra \( K[\mathcal{P}_\infty, X] \) in terms of its primitive elements. We look at the \( K \)-linear map \( e : K[\mathcal{P}_\infty, X] \rightarrow \text{Prim}(K[\mathcal{P}_\infty, X]) \).

For any \( x \in X_n \), the map \( e(\xi_n, x) = \)
\[ (\xi_n; x) - \sum (\xi_{n_1, n_2}; x_1, x_2) + \cdots + (-1)^{r+1} \sum (\xi_{n_1, \ldots, n_r}; x_1, \ldots, x_r) + \ldots, \]
gives a bijection between \( X \) and the subspace \( e(X) \) of \( \text{Prim}(K[\mathcal{P}_\infty, X]) \). In order to simplify notation, we denote by \( \overline{\xi} \) the image under \( e \) of the element \( (\xi_n, x) \).

Let \( \mathcal{P}_{rim}(X) \) be the subspace of \( K[\mathcal{P}_\infty, X] \) spanned by the set \( e(X) \) with the operations \( B^\gamma \) and \( \{\cdot, \cdot\} \), and let \( \mathcal{P}_{rim}(X)*^{\infty} \) be the space spanned by all the elements of the form \( z_1 \bullet_0 z_2 \bullet_0 \cdots \bullet_0 z_n \), with \( z_j \in \mathcal{P}_{rim}(X) \), for \( 1 \leq j \leq n \).

Note that Proposition 4.3 implies that any homogeneous element in \( \mathcal{P}_{rim}(X) \) is a sum of elements of type \( B^\gamma(x; y_1, \ldots, y_q) \), with \( x = \{\ldots \{x_1, x_2\}, x_3\}, \ldots, x_n\} \), for \( x_1, \ldots, x_n \in X, y_1, \ldots, y_q \in \mathcal{P}_{rim}(X), q \geq 0, n \geq 1 \) and \( |x| + \sum |y_j| = n \).

4.8. Proposition. Let \( X \) be a positively graded set, equipped with a coassociative graded coproduct \( \Theta \) on \( K[X] \). Any element \( z \in K[\mathcal{P}_\infty, X] \) may be written as a sum \( z = \sum_k z_1^k \bullet_0 z_2^k \bullet_0 \cdots \bullet_0 z_r^k \), with \( z_j^k \in \mathcal{P}_{rim}(X) \).

Proof. Clearly, if \( x \in X_1 \) then \( \overline{x} = e(\xi_1, x) \in \mathcal{P}_{rim}(X) \). If \( x \in X_n \), for \( n \geq 2 \), then
\[ (\xi_n, x) = \overline{x} + \sum (\xi_{n_1}; x_1) \bullet_0 \overline{x_{(2)}}, \]
with \( \overline{x} \) and \( \overline{x_{(2)}} \) in \( e(X) \subseteq \mathcal{P}_{rim}(X) \). By a recursive argument
\[ (\xi_{n_1}; x_1) = \sum_k \overline{y}_1^k \bullet_0 \overline{y}_2^k \bullet_0 \cdots \bullet_0 \overline{y}_r^k, \]
with \( y_l^k \in X_{m^L} \), for \( 1 \leq l \leq r_k \).
So, any element of the form \( (\xi_n; x) \) belongs to \( \bigoplus_{n \geq 1} \mathcal{P}_{rim}(X)^*^{\infty} \).

Since \( (K[\mathcal{P}_\infty, X], \bullet) \) is the free shuffle algebra spanned by all the elements \( (\xi_n; x) \), with \( x \in X_n \), one has that any homogeneous element \( y \in K[\mathcal{P}_\infty, X] \) may be written in a unique way as \( y = \sum \gamma(\xi_n; x_l) \bullet_{n_l} y_l^l \), with \( x_l \in X_{n_l}, y_l^l \in K[\mathcal{P}_\infty, X] \) such that \( |y_l^l| < n \) and \( \gamma_l \in \text{Sh}(n_l, n - n_l) \).
We have proved yet that
\[(\xi_n; x) = \sum_{\ell} x_1^\ell \cdot x_0 \cdot x_{r-\ell},\]
so, we only need to check that an element
\[y = \mathcal{F} \cdot \gamma \left( z_1 \cdot z_2 \cdot z_3 \cdots z_q \right),\]
with \(x \in X_n\) and \(z_i \in \text{Prim}_{sh}(X)\), belongs to \(\oplus_{n \geq 1} \text{Prim}_{sh}(X)^{\bullet n}\).

If \(\gamma = 1_{[y]}\), then \(y = x \cdot z_1 \cdot z_2 \cdot z_3 \cdots z_q\), with \(x\) and \(z_j \in \text{Prim}_{sh}(X)\).

If \(\gamma \neq 1_{[y]}\), then there exist \(1 \leq j \leq q + 1\) and \(1 \leq k \leq q\) such that
\[
\gamma(n + r_1 + \cdots + r_{j-1}) \leq \gamma(1) - 1 < \gamma(n + r_1 + \cdots + r_j)
\]
\[
\gamma(n + r_1 + \cdots + r_{j-1} + 1) < \gamma(n) < \gamma(n + r_1 + \cdots + r_k + 1),
\]
where \(|z_i| = r_j\).

If \(j = k + 1\), then \(\gamma = \epsilon_{n(r_1+\cdots+r_k)} \times 1_{r_{k+1}+\cdots+r_q}\), and we have that:
\[
y = x \cdot \gamma (z_1 \cdot z_2 \cdot z_3 \cdots z_q) = B_{k-1}^{\gamma_1}(\{z_1, x\}; z_2, \ldots, z_k) \cdot z_{k+1} \cdot z_{q+1} +
\]
\[
z_1 \cdot z_2 \cdots z_{k-1} \cdot z_k (\{x; z_{k+1}, \ldots, z_q\}) \cdot z_{k+1} \cdot z_{q+1} +
\]
\[
z_1 \cdot z_2 \cdots z_{k-1} \cdot z_k (\{x; z_{k+1}, \ldots, z_q\}) \cdot z_{k+1} \cdot z_{q+1} +
\]
where \(\gamma_i = 1_{r_i} \times \epsilon_{n(r_i+\cdots+r_k)}\).

If \(j \leq k\), then
\[
y = x \cdot \gamma (z_1 \cdot z_2 \cdots z_q) = B_{k-1}^{\gamma_1}(\{z_1, x\}; z_2, \ldots, z_k) \cdot z_{k+1} \cdot z_{q+1} +
\]
\[
z_1 \cdot z_2 \cdots z_{k-1} \cdot z_k (\{x; z_{k+1}, \ldots, z_q\}) \cdot z_{k+1} \cdot z_{q+1} +
\]
where the permutations \(\gamma_i\) are defined in Proposition 4.6.

Let \(X\) be a set, we denote by
\[
\{X\}_0 := \{\{\{x_1, x_2, \ldots, x_n\} \cdot x_1 \in X, 1 \leq i \leq n\} : n \geq 1\}
\]
which is a subset of the free \(\text{Prim}_{sh}\) algebra spanned by \(X\). Let \(\{X\} = \bigcup_{n \geq 0} \{X\}_n\)
be the set defined recursively as follows:
\[
\{X\}_1 := \{X\}_0 \bigcup \{B_1^\gamma(x; y), y \in \{X\}_0\},
\]
\[
\{X\}_2 := \{X\}_1 \bigcup \{B_1^\gamma(x; y), y \in \{X\}_0 \text{ and } 0 \leq x \in \{X\}_1 \bigcup\}
\]
\[
\{B_2^\gamma(x; y_1, y_2), x \in \{X\}_0 \text{ and } 0 \leq y_j \in \{X\}_1, \text{ for } j = 1, 2\},
\]
\[
\{X\}_n := \{X\}_{n-1} \bigcup
\]
\[
(\bigcup_{m \geq 1} \{B_m^\gamma(x; y_1, \ldots, y_m), x \in \{X\}_0 \text{ and } 0 \leq y_j \in \{X\}_{n-1}, \text{ for } 1 \leq j \leq m\}).
\]
Note that \(\{X\}_n \subseteq \{X\}_{n+1}\), and that Proposition 4.6 and the Remark 4.5 imply
that the free \(\text{Prim}_{sh}\) algebra over \(X\) is the vector space spanned by \(\{X\}\).
4.9. **Proposition.** Let $X$ be a positively graded set. The subspace $\mathcal{Prim}_{sh}(X)$ is the subspace of primitive elements of $K[\mathcal{P}_\infty, X]$. Moreover, it is the free $\mathcal{Prim}_{sh}$ algebra spanned by $X$.

**Proof.** Note first that it suffices to prove the result for the case where $X = \bigcup_{n \geq 1} X_n$ with $X_n$ finite, for all $n \geq 1$. For the general case, $X$ is a limit of graded sets which verify this condition, and the result follows.

Proposition 4.3 states that $\mathcal{Prim}_{sh}(X) \subseteq \text{Prim}(K[\mathcal{P}_\infty, X])$, while Proposition 4.8 implies that $K[\mathcal{P}_\infty, X] = T(\mathcal{Prim}_{sh}(X))$ as a vector space. From Theorem 4.2 one has that $K[\mathcal{P}_\infty, X] = T(\text{Prim}(K[\mathcal{P}_\infty, X]))$, so $\mathcal{Prim}_{sh}(X) = \text{Prim}(K[\mathcal{P}_\infty, X])$.

For the second assertion note that, since $K[\mathcal{P}_\infty, X]$ is the free associative algebra over the set $\text{Irr}_{\mathcal{P}, X} = \displaystyle \bigcup_{n \geq 1} \text{Irr}_{\mathcal{P}_n, X}$ of irreducible elements of $\mathcal{P}_\infty$ coloured with elements of $X$, the previous assertion states that $\mathcal{Prim}_{sh}(X)$ is linearly spanned by the set $\text{Irr}_{\mathcal{P}, X}$. From Proposition 4.6 we know that $\mathcal{Prim}_{sh}(X)$ is a $\mathcal{Prim}_{sh}$ algebra which contains $e(X)$. To see that it is free, it suffices to define a bijective map $\beta : \text{Irr}_{\mathcal{P}, X} \rightarrow \{X\}$, where $\{X\}$ is the set defined above. On $X \subset \text{Irr}_{\mathcal{P}, X}$, $\beta$ coincides with the identity map. Let $y = (\xi_n, x) \cdot y_1 \in \text{Irr}_{\mathcal{P}, X}$, with $x \in X_n$ and $n \geq 1$. We define $\beta(y)$ as follows:

- If $y_1 \in \text{Irr}_{\mathcal{P}, X}$ and $\gamma \neq \epsilon_{nm_1}$, then $\beta(y) := B^1_T(x; \beta(y_1))$.
- If $y_1 \in \text{Irr}_{\mathcal{P}, X}$ and $\gamma = \epsilon_{nm_1}$, then $\beta(y) := \left\{ \begin{array}{ll} \{y_1\}, & \text{for } \beta(y_1) \in \{X\}_0, \\ B^\tau_Q(w, x; \gamma_1, \ldots, \gamma_q), & \text{for } \beta(y_1) = B^\tau_Q(w; \gamma_1, \ldots, \gamma_q), \end{array} \right.$

where $|y_1| = m_1$, $w \in \{X\}_0$, $|w| = s$, $z_j \in \{X\}$ for $1 \leq j \leq q$, $\sum j |z_j| = r$ and $\tau := \tau \times 1_n \cdot (1_s \times \epsilon_{nr})$.

Suppose that $y_1 = t_1 \cdot \cdots \cdot t_p$, with $t_i \in \text{Irr}_{\mathcal{P}, X}$ and $p > 1$. The fact that $y$ is irreducible, implies that $\gamma(n + h_1 + \cdots + h_{p-1} + 1) < \gamma(n)$, for $|t_i| = h_i$.

- If $\gamma(1) < \gamma(n + h_1)$, then $\beta(y) := B^\tau_Q(x; \beta(t_1), \ldots, \beta(t_p))$.
- If $\gamma(n + h_1) < \gamma(1) - 1$, then $\beta(y) := \left\{ \begin{array}{ll} B^\tau_Q((-1)^{\tau_1} (\beta(t_1), x); \beta(t_2), \ldots, \beta(t_p)), & \text{for } \beta(t_1) \in \{X\}_0 \\ B^\tau_Q(w, x; \gamma_1, \ldots, \gamma_q, \beta(t_2), \ldots, \beta(t_p)), & \text{for } \beta(t_1) = B^\tau_Q(w; \gamma_1, \ldots, \gamma_q), \end{array} \right.$

where $\gamma = \gamma \cdot (\epsilon_{1 \times h_1 + 1 \times h_2 + \cdots + h_p})$ and $\tau := \gamma \cdot (1_n \times \tau \times 1_h \cdot (\epsilon_s - 1 \times h_s))$.

It is not difficult to check that $\beta$ is bijective, which implies that $\mathcal{Prim}_{sh}(X)$ is isomorphic to the free $\mathcal{Prim}_{sh}$ algebra spanned by $X$. $\diamond$

The following result is a straightforward consequence of Theorem 4.2 and the previous results.
4.10. **Proposition.** Let $X$ be a positively graded set, such that $K[X]$ is equipped with a coassociative graded coproduct $\Theta$. The unital infinitesimal bialgebra $K[P_\infty,X]_+$ is isomorphic to $T^{fc}(\text{Prim}_{sh}(X))$, where $\text{Prim}_{sh}(X)$ is the free $\text{Prim}_{sh}$ algebra spanned by $X$.

We want to prove the equivalence between the categories of connected shuffle bialgebras and $\text{Prim}_{sh}$ algebras. More precisely, given a $\text{Prim}_{sh}$ algebra $(V,B^\gamma,\{-,-\})$ and an homogeneous basis $X$ of the underlying vector space $V$, let $U_{sh}(V)$ be the shuffle bialgebra obtained by taking the quotient of the free shuffle algebra $K[P,X]$ by the ideal (as a shuffle algebra) spanned by the set:

$$\{B^\gamma(x;y) - B^\gamma(x;y), \{x,z\} - [x,z]\},$$

with $x \in X_n$, $y \in X_m$, $z \in X_r$ and $\gamma \in Sh(n,m)$ such that $\gamma(1) < \gamma(n+m_1)$ and $\gamma(n + m_1 + \cdots + m_{n-1} + 1) < \gamma(n)$, where $B^\gamma$ and $\{-,-\}$ denote the operations associated to the shuffle algebra $K[P,X]$.

4.11. **Theorem.**

a) Let $(H,\circ,\Delta)$ be a connected shuffle bialgebra, then $H$ is isomorphic to $U_{sh}(\text{Prim}(H))$, where $\text{Prim}(H)$ is the $\text{Prim}_{sh}$ algebra of primitive elements of $H$.

b) Let $(V,\overline{B^\gamma},\{-,-\})$ be a $\text{Prim}_{sh}$ algebra, then $V$ is isomorphic to $\text{Prim}(U_{sh}(V))$.

**Proof.**

a) Let $X$ be a basis of the vector space Prim$(H)$. Define $\varphi : Sh(X) \longrightarrow H$ as follows:

$$\varphi(x_1 \circ_{\gamma_1} (x_2 \circ_{\gamma_2} \cdots (x_{n-1} \circ_{\gamma_{n-1}} x_n)))) := x_1 \circ_{\gamma_1} (x_2 \circ_{\gamma_2} \cdots (x_{n-1} \circ_{\gamma_{n-1}} x_n)))$$

where $x_i \in X$ for $1 \leq i \leq n$. Note that $\varphi(B^\gamma(x;y)) = \varphi(x \circ_{\top} y) = \overline{B^\gamma}(x;y)$, and $\varphi([x,y]) = \varphi(x \circ_{\top} y - x \circ_{\circ} y) = [x;y]$, so $\varphi$ factorizes through $U_{sh}(\text{Prim}(H))$. Moreover, it is immediate to check that $\varphi$ is a bialgebra homomorphism. Applying Theorem 4.2, the inverse morphism is given by

$$\varphi^{-1}(x) = cl(e(x)) + \sum cl(x_{(1)} \bullet_0 x_2) + \cdots + \sum cl(x_1 \bullet_0 \cdots \bullet_0 x_n),$$

where $cl$ denotes the class of the element in $U_{sh}(V)$.

b) It is clear that $V \subseteq \text{Prim}(U_{sh}(V))$. If $X$ is a basis of $V$, then Proposition 4.10 implies that $\text{Prim}(Sh(X)) = \text{Prim}_{sh}(X)$. So, the primitive elements of $U_{sh}(V)$ are generated by $X$ under the operations $B^\gamma$ and $\{-,-\}$, which are precisely the elements of $V$ in the quotient.

\[ \diamond \]

5. **Boundary map on the free shuffle bialgebra**

Let $X$ be a positively graded set, and let $\Theta : K[X] \longrightarrow K[X] \otimes K[X]$ be a coassociative graded coproduct on $K[X]$.

Given an element $x = x_1 \bullet_{\gamma_1} (x_2 \bullet_{\gamma_2} \cdots (x_r \bullet_{\gamma_{r-1}} x_n))$ in the free shuffle algebra $Sh(X)$, with $x_i \in X$ for $1 \leq i \leq r$, the weight of $x$ is $r$. We denote the weight of an element $x$ by $w(x)$. Note that the elements of weight $r$ in $Sh(X)$ correspond to the elements of the subspace $\bigoplus_{n \geq 1} K[P^r_{n,X}]$.

Define the linear map $\partial_{\Theta} : Sh(X) \longrightarrow Sh(X)$ as the unique linear homomorphism such that:
(1) For $x \in X_n$, 
\[ \partial_\Theta(x) := \sum_{\sigma} \sum_{\gamma} (-1)^{n_1} \text{sgn}(\sigma) x(1) \otimes x(2), \]
where $\Theta(x) = \sum x(1) \otimes x(2)$, and $|x(1)| = n_1$. 

(2) For $x, y \in \Sh(X)$, with $w(x) = r$,
\[ \partial_\Theta(x \bullet y) := \partial_\Theta(x) \bullet y - (-1)^r x \bullet \partial_\Theta(y). \]

Note that $\partial_\Theta(\mathcal{P}_{n,X}^r) \subseteq \mathcal{P}_{n,X}^{r-1}$. 

5.1. Lemma. The homomorphism $\partial_\Theta$ is a boundary map, that is $\partial_\Theta^2 = \partial_\Theta \circ \partial_\Theta = 0$. 

Proof. Let $x \in X_n$, we have that:
\[ \partial_\Theta^2(x) = \sum_{0 < n_1 < n} \sum_{0 < n_2 < n-n_1} (-1)^{2n_1+n_2} \text{sgn}(\gamma) \text{sgn}(\sigma)(x(1) \bullet x(2)) \bullet x(3) + \]
\[ \sum_{\delta \in \Sh(n_2,n-n_1)} (-1)^{2n_1+n_2+1} \text{sgn}(\delta) \text{sgn}(\tau)x(1) \bullet x(2) \bullet x(3)), \]
where $\Theta^3(x) = \sum x(1) \otimes x(2) \otimes x(3)$ with $|x(i)| = n_i$ for $i = 1, 2$. So,
\[ \partial_\Theta^2(x) = \sum_{n_1+n_2+n_3 = n} \sum_{\alpha \in \Sh(n_1,n_2,n_3)} (-1)^{n_2} \text{sgn}(\gamma) \text{sgn}(\sigma)(x(1) \bullet x(2)) \bullet x(3)) + \]
\[ \sum_{n_1+n_2+n_3 = n} \sum_{\alpha \in \Sh(n_1,n_2,n_3)} \text{sgn}(\alpha)((-1)^{n_2} + (-1)^{n_2+1})(x(1) \bullet x(2)) \bullet x(3)) = 0, \]
where $\alpha = (\sigma \times 1_{n_3}) \gamma = (1_{n_1} \times \tau). $ for $\alpha \in Sh(n_1,n_2,n_3)$ and $|x(i)| = n_i$. 

Suppose that $x = y \otimes z$, with $y \in X$ and $z \in K[\mathcal{P}_{n-1,X}^r]$. Applying the definition of $\partial_\Theta$ and a recursive argument, we have that:
\[ \partial_\Theta^2(x) = \partial_\Theta(\partial_\Theta(y) \otimes z - (-1)^{1}y \otimes \partial_\Theta(z)) = \]
\[ (-1)^{w(\partial_\Theta(y))} \partial_\Theta(y) \otimes \partial_\Theta(z) - (-1)^1 \partial_\Theta(y) \otimes \partial_\Theta(z) = 0. \]

In the case that $X = \{\xi_i\}_{n \geq 1}$, we have that $\Sh(X) = K[\mathcal{P}_\infty]$. If we consider the coassociative coproduct $\Theta$ on $K[X]$ given by:
\[ \Theta(\xi_n) := \sum_{i=1}^{n-1} \xi_i \otimes \xi_{n-i}, \]
then the boundary map $\partial_\Theta$ coincides with the boundary map of the permutahedron (see [25]).
6. Relations with dendriform and 2-associative algebras

In the previous sections we have defined functors:

(1) $H_{\text{inf-gr}}$ from the category of graded unital infinitesimal bialgebras to the category of shuffle bialgebras,

(2) two functors from the category of shuffle algebras into the category of associative algebras, in such a way that the restrictions of both functors to the category of shuffle bialgebras, $H_0$ and $H_L$, have their images contained in the subcategory of unital infinitesimal bialgebras,

(3) from the category of shuffle algebras to the category of associative algebras, in such a way that the image under this functor of a shuffle bialgebra gives an associative bialgebra. We denote the restriction of this functor to the category of shuffle bialgebras by $H_{\text{sh-as}}$.

In [1], M. Aguiar constructs functors relating infinitesimal bialgebras (see [18]) to dendriform algebras (see [21]) and brace algebras (see [19] and [10]). We want to include shuffle bialgebras in his framework.

6.1. Definition. A dendriform algebra over $K$ is a vector space $D$ equipped with two bilinear maps $\succ, \prec: D \otimes D \rightarrow D$ which verify the following relations:

(1) $x \succ (y \succ z) = (x \succ y + x \prec y) \succ z$,

(2) $x \succ (y \prec z) = (x \succ y) \prec z$,

(3) $x \prec (y \succ z + y \prec z) = (x \prec y) \prec z$,

for $x, y, z \in D$.

Note that any dendriform algebra $D$ has a natural structure of associative algebra with the product $\ast$, defined by: $x \ast y = x \succ y + x \prec y$.

For nonnegative integers $n, m$, let $\text{Sh}^\succ(n, m)$ and $\text{Sh}^\prec(n, m)$ be the following subsets of $\text{Sh}(n, m)$:

a) $\text{Sh}^\succ(n, m) := \{ \sigma \in \text{Sh}(n, m) \mid \sigma(n + m) = n + m \}$,

b) $\text{Sh}^\prec(n, m) := \{ \sigma \in \text{Sh}(n, m) \mid \sigma(n + m) = n \}$.

It is immediate to check that $\text{Sh}(n, m)$ is the disjoint union of $\text{Sh}^\succ(n, m)$ and $\text{Sh}^\prec(n, m)$.

Let $(A, \bullet)$ be a shuffle algebra, Define on $A$ the operations $\succ$ and $\prec$ as follows:

(1) $x \succ y := \sum_{\gamma \in \text{Sh}^\succ(n, m)} x \bullet_{\gamma} y$,

(2) $x \prec y := \sum_{\gamma \in \text{Sh}^\prec(n, m)} x \bullet_{\gamma} y$,

for $x \in A_n$ and $y \in A_m$. Note that the associative product $\ast$ defined in Lemma 2.7 is the sum of $\succ$ and $\prec$.

6.2. Lemma. Let $(A, \bullet)$ be a shuffle algebra, then $(A, \succ, \prec)$ is a dendriform bialgebra. Moreover, if $(A, \bullet, \Delta)$ is a shuffle bialgebra, then $\succ, \prec$ and $\Delta$ verify the
following equalities:
\[
\Delta(x \triangleright y) = \sum (x_{(1)} \ast y_{(1)}) \otimes (x_{(2)} \triangleright y_{(2)}) + \sum x_{(1)} \otimes (x_{(2)} \triangleright y) + \sum y_{(1)} \otimes (x \triangleright y_{(2)}) + \sum (x \ast y_{(1)}) \otimes y_{(2)} + x \otimes y,
\]
\[
\Delta(x \prec y) = \sum (x_{(1)} \ast y_{(1)}) \otimes (x_{(2)} \prec y_{(2)}) + \sum y_{(1)} \otimes (x \prec y_{(2)}) + \sum (x \ast y_{(1)}) \otimes y_{(2)} + \sum x_{(1)} \otimes (x_{(2)} \prec y) + y \otimes x,
\]
for all \(x, y \in A\).

**Proof.** To prove that \((A, \triangleright, \prec)\) is a dendriform algebra, it suffices to note that the equality:
\[
(1_n \times Sh(m, r)) \cdot Sh(n, m + r) = (Sh(n, m) \times 1_r) \cdot Sh(n + m, r),
\]
is the sum of the following three equalities:
\[
(1_n \times Sh^r(m, r)) \cdot Sh^r(n, m + r) = (Sh(n, m) \times 1_r) \cdot Sh^r(n + m, r)
\]
\[
(1_n \times Sh^r(m, r)) \cdot Sh^r(n, m + r) = (Sh(n, m) \times 1_r) \cdot Sh^r(n + m, r),
\]
\[
(1_n \times Sh(m, r)) \cdot Sh^r(n, m + r) = (Sh(n, m) \times 1_r) \cdot Sh^r(n + m, r).
\]

On the other hand given \(1 \leq r \leq n + m\), if \(\gamma = (1_{n_1} \times \epsilon_{n-n_1,m_1} \times 1_{m-m_1}) \cdot (\gamma^{r}_{(1)} \times \gamma^{n+m-r}_{(2)}) \in Sh(n, m)\) is the decomposition given in Remark 2.4 with \(\gamma^{r}_{(1)} \in Sh(n_1, m_1), \gamma^{n+m-r}_{(2)} \in Sh(n-n_1, m-m_1)\) and \(r = n_1 + m_1\), then it is easy to prove that \(\gamma \in Sh^r(n, m)\) if, and only if \(\gamma^{n+m-r}_{(2)} \in Sh^r(n-r_1, m+r_1-r)\), which implies the formulas for the coproduct.

The Lemma above states that any shuffle bialgebra has a natural structure of dendriform bialgebra, as defined in [32].

6.3. **Definition.** (see [24]) Let \(X\) be a \(K\)-vector space equipped with two associative products \(*\) and \(\cdot\), and a coassociative coproduct \(\Delta\), such that:

- (1) \((X, *, \Delta)\) is a bialgebra over \(K\),
- (2) \((X, \cdot, \Delta)\) is an infinitesimal unital bialgebra.

Then \((X, *, \cdot, \Delta)\) is called a 2-associative bialgebra.

The following statement is a consequence of Corollary 3.4. If \((A, \bullet, \Delta)\) is a shuffle bialgebra, then \(A_+ = K \oplus A\) with the products:

\[
(1) \quad x \ast y = \sum_{\gamma \in Sh(n, m)} x \bullet y, \quad \text{for } x \in A_n, \ y \in A_m,
\]
\[
x, \quad \text{for } y = 1_K, \quad \text{and}
\]
\[
y, \quad \text{for } x = 1_K,
\]
\[
(2) \quad x \cdot y = \left\{ \begin{array}{ll}
 x \bullet_{1+n+m} y, & \text{for } x \in A_n, \ y \in A_m, \\
x, & \text{for } y = 1_K, \\
y, & \text{for } x = 1_K.
\end{array} \right.
\]

is a 2-associative bialgebra.

In previous work, we prove that:
(1) The subspace of primitive elements of a dendriform bialgebra has natural structure of brace algebra. Moreover, there exists an equivalence between the category of connected dendriform bialgebras and the category of brace algebras (see [32]).

(2) The subspace of primitive elements of a 2-associative bialgebra has a natural structure of non-differential $B_\infty$ algebra (see [24]). As in the previous case, the category of connected 2-associative bialgebras is equivalent to the category of non-differential $B_\infty$ algebras.

A non-differential $B_\infty$ algebra $B$ is a $K$-vector space equipped with linear maps $B_{n,m}: B \otimes (n+m) \rightarrow B$, verifying certain relations (see [34]). In particular, any brace algebra $(B, M_{1,n})$ is a non-differential $B_\infty$ algebra such that $B_{1,m} = M_{1,m}$, for $m \geq 1$, and $B_{n,m} = 0$, for $n \neq 1$.

The previous results show that the associative bialgebra associated to any shuffle bialgebra has structures of both dendriform bialgebra and 2-associative bialgebra. The subspace of primitive elements of a shuffle bialgebra is a brace algebra, as well as a non-differential $B_\infty$ algebra. However, these structures are not always isomorphic as non-differential $B_\infty$ algebras, even if they give the same associative bialgebra structure. For instance, using the formulas given in [24], the subspace of primitive elements of the Malvenuto-Reutenauer Hopf algebra $K[S_\infty]$ has a structure of non-differential $B_\infty$ algebra which is not isomorphic to the brace algebra structure given in [32].

7. Preshuffle and grafting bialgebras

We introduce the notions of preshuffle algebras, related to leveled trees, and of a particular type of preshuffle algebras called grafting algebras, related to trees. Shuffles are related to monoids in the category $(\mathcal{S} \text{-Mod}, \otimes)$, where we do not ask for a compatibility relation between the operations $\cdot$ and the action of the symmetric group. In a similar way, grafting algebras are related to non-symmetric operads (see Ma): a grafting algebra structure on $A = \bigoplus A_n$ is equivalent to a non-symmetric operad $P$ with $P(n) = A_{n-1}$.

7.1. Definition. (1) A preshuffle algebra over $K$ is a graded vector space $A = \bigoplus A_n$ equipped with linear maps $\bullet_i: A \otimes A_m \rightarrow A$, for $0 \leq i \leq m$ and $m \geq 0$,

verifying:

$$(x \bullet_i y) \bullet_j z = x \bullet_{i+j} (y \bullet_i z), \text{ for } 0 \leq i \leq |y| \text{ and } 0 \leq j \leq |z|.$$ 

(2) A grafting algebra is a preshuffle algebra $(A, \bullet_i)$ such that the operations $\bullet_i$ verify the following additional conditions:

$x \bullet_i (y \bullet_j z) = y \bullet_{j+|z|} (x \bullet_i z), \text{ for } 0 \leq i < j,$

for any elements $x, y, z \in A$.

The relations verified by a preshuffle algebra may be pictured as follows:
It is immediate to see that the following equality holds:

\[ \omega_{i} \]

Moreover, note that the element \( \text{shuffle algebra} \) is a preshuffle algebra, with the operations \( \Delta \). We say that \( (A, \bullet, \Delta) \) is equipped with a graded coassociative coproduct \( \Delta \). We say that \( (A, \bullet, \Delta) \) is a \textit{shuffle bialgebra} if it verifies:

\[ \omega_{i}^{n,m} := \epsilon_{n,i} \times 1_{m-i} \in Sh(n, m). \]

It is immediate to see that the following equality holds:

\[ (1_{n} \times \omega_{j}^{m,r}) \cdot \omega_{i}^{n,m+r} = (\omega_{i,j}^{n,m} \times 1_{r}) \cdot \omega_{j}^{n+m,r}. \]

Moreover, note that the element \( \omega_{m}^{n,m} = \epsilon_{n,m} \).

Note that, since the permutation \( \omega_{i}^{n,m} \) belongs to \( Sh(n, m) \), for \( 0 \leq i \leq m \), any shuffle algebra is a preshuffle algebra, with the operations

\[ x \bullet_{i} y := x \bullet_{i}^{n,m} y, \text{ for } x \in A_{n}, y \in A_{m} \text{ and } 0 \leq i \leq m. \]

7.2. Definition. (1) Let \( (A, \bullet, \Delta) \) be a positively graded preshuffle algebra, such that \( A \) is equipped with a graded coassociative coproduct \( \Delta \). We say that \( (A, \bullet_{i}, \Delta) \) is a \textit{preshuffle bialgebra} if it verifies:

\[ \Delta(x \bullet_{0} y) = \sum x^{(1)} \otimes (x^{(2)} \bullet_{0} y) + x \otimes y + \sum (x \bullet_{0} y^{(1)}) \otimes y^{(2)}. \]

\[ \Delta(x \bullet_{i} y) = \sum_{|y^{(1)}| \leq i} y^{(1)} \otimes (x \bullet_{i-|y^{(1)}|} y^{(2)}) + \sum_{|y^{(1)}| \geq i} (x \bullet_{i} y^{(1)}) \otimes y^{(2)}, \]

for \( 0 \leq i \leq |y| \).

\[ \Delta(x \bullet_{|y|} y) = \sum y^{(1)} \otimes (x \bullet_{|y|} y^{(2)}) + y \otimes x + \sum (x^{(1)} \bullet_{|y|} y) \otimes y^{(2)}. \]

(2) A \textit{grafting bialgebra} is a preshuffle bialgebra \( (A, \bullet_{i}, \Delta) \) such that \( (A, \bullet_{i}) \) is a grafting algebra.

It is immediate to check that any shuffle bialgebra is a preshuffle bialgebra.

7.3. Examples. a) The free preshuffle algebra. Let \( X = \bigcup_{n \geq 1} X_{n} \) be a positively graded set. Let \( K[F_{\infty}, X] := \bigoplus_{n \geq 1} K[F_{n,X}] \), where \( F_{n,X} \) is the set of pairs \( (f; x_{1}, \ldots, x_{r}) \) with \( f \) a map from \( \{1, \ldots, n\} \) to \( \{1, \ldots, r\} \), and \( x_{i} \in X_{n_{i}} \), for \( n_{i} := |f^{-1}(i)| \). The operations \( \bullet_{i} \) are defined, using the shuffle algebra structure of \( K[F_{\infty}, X] \) defined in example b) of [23], as follows:

\[ (f; x_{1}, \ldots, x_{r}) \bullet_{i} (g; y_{1}, \ldots, y_{k}) := ((g(1)+r, \ldots, g(i)+r, f(1), \ldots, f(n), g(i+1)+r, \ldots, g(m)+r); x_{1}, \ldots, x_{r}, y_{1}, \ldots, y_{k}). \]
The subspaces \( K[K_\infty, X] := \bigoplus_{n \geq 1} K[K_n, X] \) and \( K[P_\infty, X] := \bigoplus_{n \geq 1} K[P_n, X] \) are closed under the operations \( \bullet_i \), so they are sub-preshuffle algebras of \( K[F_\infty, X] \).

**Theorem.** For any graded set \( X = \bigcup_{n \geq 1} X_n \), the space \( K[K_\infty, X] \) with the operations \( \bullet_i \) described above is the free preshuffle algebra spanned by \( X \).

**Proof.** Any element of \( x \in X_n \) is identified with the pair \((\xi_n; x)\). The result follows easily using that any element \( z \) in the free preshuffle algebra spanned by \( X \) is of the form \( x \bullet_i y \), with \( x \in X_r \) for some \( 1 \leq r \) and \( y \) an element of the free preshuffle algebra such that \( |y| < |z| \).

Moreover, given a coassociative coproduct \( \Theta : K[X] \to K[X] \otimes K[X] \), the coproduct \( \Delta_\Theta : K[P_\infty, X] \to K[P_\infty, X] \otimes K[P_\infty, X] \), defined in Example c) of 3.5, restricts to \( K[K_\infty, X] \). So, any free preshuffle algebra is a preshuffle bialgebra.

Note that the free preshuffle algebra spanned by one element of degree one is just the space \( K[S_\infty] := \bigoplus_{n \geq 1} K[S_n] \), equipped with the products

\[
\sigma \bullet_i \tau := (\tau(1) + n, \ldots, \tau(i) + n, \sigma(1), \ldots, \sigma(n), \tau(i + 1) + n, \ldots, \tau(m) + n),
\]

for \( \sigma \in S_n \), \( \tau \in S_m \) and \( 0 \leq i \leq m \). Its associated Hopf algebra is the Malvenuto-Reutenauer bialgebra.

**b) Infinitesimal bialgebras** Given a graded nonunital infinitesimal bialgebra \((A, \cdot, \Delta)\), example b) of 3.5 shows that there exists a natural way to define a shuffle bialgebra structure on \( A \), where the coproduct is \( \Delta \) and the operations \( \bullet \) are constructed using \( \cdot \) and \( \Delta \). It is easy to see that the preshuffle algebra structure on \( A \), given by \( \bullet_i = \bullet_{\omega_{n,m}} \) is in fact a grafting algebra. So, any graded nonunital infinitesimal bialgebra gives rise to a grafting bialgebra.

**c) The algebra of planar trees.** The graded vector space \( K[T_\infty] \) spanned by the set of planar trees \( T_\infty := \bigcup_{n \geq 1} T_n \), with the products \( \circ_i \) described in Definition 1.12, is a grafting algebra. Moreover, the subspace \( K[Y_\infty] \), spanned by the set of planar binary trees, is a grafting subalgebra of \( K[T_\infty] \).

For any graded set \( X = \bigcup_{n \geq 1} X_n \), let \( T_{n,X} \) be the set of planar rooted trees with \( n + 1 \) leaves and the internal vertices coloured by the elements of \( X \), in such a way that a vertex with \( r + 1 \) inputs is coloured by an element \( x \) of \( X_r \). The grafting structure of \( K[T_\infty] \) induces a grafting algebra structure on the space \( K[T_\infty, X] := \bigoplus_{n \geq 1} K[T_{n,X}] \) in an obvious way.

To describe the free grafting algebra spanned by a graded set \( X \), we need the following result. Its proof is straightforward.
7.4. Lemma. Let $A$ be a grafting algebra. Let $x_0, \ldots, x_q, y, z$ be elements of $A$ with $|x_i| = n_i$, $|y| = m$ and $|z| = r$. For any family of integers $0 \leq i_0 < \cdots < i_r \leq r$ and $0 \leq j \leq n_q$ we have that:

$$x_0 \circ_{i_0} (x_1 \ldots (x_q \circ_{i_q-1} ((y \circ_{j} x_q) \circ_{i_q} z))) = (-1)^{m(n_0 + \cdots + n_q - 1)} y \circ_{j} (x_0 \circ_{i_0} (\ldots (x_q \circ_{i_q} z))),$$

where $i = j + \sum_{l=0}^{q-1} n_l + i_q 1$.

7.5. Theorem. For any graded set $X$, the vector space $K[T_{\infty},X]$, equipped with the linear maps $\circ_i$, for $i \geq 0$, is the free grafting algebra spanned by $X$.

Proof. Let $\text{Graft}(X)$ denote the free grafting algebra spanned by $X$. For any element $x \in X_k$, the tree $\epsilon_k$ with its vertex coloured by $x$ is denoted by $(\epsilon_k, x)$. Let $\iota : X \to K[T_{\infty},X]$ be the map which sends an element $x \in X_k$ to $(\epsilon_k, x)$.

From the definition of grafting it is immediate to check that for any $z$ in $\text{Graft}(X) \setminus X$ there exist elements $z_1, \ldots, z_r$ in $\text{Graft}(X)$ and $x \in X_n$ such that:

$$z = z_1 \circ_{i_1} (z_2 \circ_{i_2} (\ldots (z_r \circ_{i_r} x))).$$

Moreover the integer $r$, the elements $z_1, \ldots, z_r$ and the collection $i_1, \ldots, i_r$ are unique.

The homomorphism $\kappa : \text{Graft}(X) \to K[T_{\infty},X]$ is defined by induction on the number of elements of $X$ which appear in $z$, this number is denoted by $o(z)$. If $o(z) = 1$, then $\kappa(z) := \iota(z)$. If $z = z_1 \circ_{i_1} (z_2 \circ_{i_2} (\ldots (z_r \circ_{i_r} x)))$, with $r > 0$, then $o(z_i) < o(z)$ for all $1 \leq i \leq r$. Define

$$\kappa(z) := (t^0, \ldots, t^n) \circ \iota(x),$$

where $t^j = \begin{cases} \kappa(z_{i_j}) & \text{if } j = i_j, \\ \emptyset & \text{if } j \notin \{i_1, \ldots, i_r\}. \end{cases}$

Lemma 7.4 implies that $\kappa$ is a homomorphism of grafting algebras.

Conversely, since any planar rooted tree $t$ with the vertices coloured with the elements of $X$, may be written in a unique way as $(t^0, \ldots, t^n) \circ (\epsilon_n, x)$, with $x \in X_n$, the inverse of $\kappa$ is defined by the conditions:

(1) $\kappa^{-1}(\epsilon_n, x) := x$,
(2) $\kappa^{-1}(t) := \kappa^{-1}(t_{i_1}) \circ_{i_1} (\ldots (\kappa^{-1}(t_{i_k} \circ_{i_k} x)))$, where $t_{i_1}, \ldots, t_{i_k}$ are the trees in $\{t^0, \ldots, t^n\}$ which are different from $\emptyset$.

7.6. Corollary. The free grafting algebra on one generator is $(K[Y_{\infty}], \circ_1)$, while $(K[T_{\infty}], \circ_1)$ is the free grafting algebra on the graded set $\{\epsilon_n\}_{n \geq 1}$, which has exactly one element of degree $n$: the tree $\epsilon_n$, with $n + 1$ leaves and a unique vertex.

On the graded vector space $K[Y_{\infty}]$, spanned by all planar binary rooted trees, the coproduct $\Delta_{PR}$ is defined as the unique counital coproduct such that:

$$\Delta_{PR}(x) = 0,$$

$$\Delta_{PR}(t \lor w) := \sum t_{(1)} \otimes (t_2 \lor w) + t \otimes (| \lor w) + (t \lor |) \otimes w + \sum (t \lor w_{(1)}) \otimes w_{(2)}.$$

The vector space $K[Y_{\infty}]$, with the products $\circ_i$ and $\Delta_{PR}$ is a grafting bialgebra.
Let $X = \bigcup_{n \geq 1} X_n$ be a graded set. If $\Theta$ is a graded coassociative coproduct on $K[X]$, then there exists a coproduct $\Delta_\Theta$ on $K[T_{\infty,X}]$ given by:

$$\Delta_\Theta(\varepsilon_n, x) := \sum_{1 \leq i \leq n-1} \sum_{|c_{i(1)}| = i} (\varepsilon_i, x_{(1)}) \otimes (\varepsilon_{n-i}, x_{(2)}), \quad \text{for } x \in X_n \text{ and } n \geq 1,$$

$$\Delta_\Theta((t^0, \ldots, t^n) \circ (\varepsilon_n, x)) := \sum_{0 \leq i \leq n} \sum_{|c_{i(1)}| = |t^0| + \cdots + |t^{i-1}|} (t^0, \ldots, t^i) \circ (\varepsilon_{i}, x_{(1)}) \otimes (t^{i+1}, \ldots, t^n) \circ (\varepsilon_{n-i}, x_{(2)}),$$

where $|x_{(1)}| = n_j$, $\Delta_\Theta(t^i) = \sum t^i_{(1)} \otimes t^i_{(2)}$, and $\Theta(x) := \sum x_{(1)} \otimes x_{(2)}$.

It is not difficult to check that, for any $\Theta$, the space $K[T_{\infty}]$ equipped with the operations $\circ_i$ and $\Delta_\Theta$ is a grafting bialgebra.

d) The space of Hochschild cochains. (see [9]) Let $A$ be a unital $K$-algebra, and let $C^*(A) := \bigoplus_{n \geq 0} \text{Hom}_K(A^\otimes n, A)$ be the space of Hochschild cochains on $A$.

The space $C^*(A)[1] := \bigoplus_{n \geq 0} \text{Hom}_K(A^\otimes (n+1), A)$ is a grafting algebra with the operations $\bullet_i$ defined as follows:

$$g \bullet_i f := f \circ (\text{id}_A^\otimes (i-1) \times g \times \text{id}_A^\otimes (n-i)),$$

for $g \in C^n(A, A)$ and $f \in C^n(A, A)$.

Consider on $C^*(A)[1]$ the following coproduct:

$$\Delta(f) := \sum_{i=1}^{n-1} f_{(1)}^i \otimes f_{(2)}^{n-i+1}, \quad \text{for } f \in C^n(A, A),$$

where

(1) $f_{(1)}^i(x_1, \ldots, x_i) := f(x_1, \ldots, x_i, 1_A, \ldots, 1_A) \in C^i(A, A)$

(2) $f_{(2)}^{n-i+1}(x_1, \ldots, x_{n+1-i}) := f(1_A, \ldots, 1_A, x_1, \ldots, x_{n+1-i}) \in C^{n+1-i}(A, A)$.

It is easy to see that $(C^*(A)[1], \bullet_i, \Delta)$ is a grafting bialgebra.

e) The underlying space of an algebraic operad. Let $K$ be a field of characteristic 0, and let $P$ be a $K$-linear operad as described in [17]. Consider the graded $K$-vector space $P[1] := \bigoplus_{n \geq 0} P(n+1)$ equipped with the maps:

$$\lambda \bullet_i \nu := \gamma_{1, \ldots, 1, n, 1, \ldots, 1} (\nu \otimes 1 \otimes \cdots \otimes 1 \otimes \lambda \otimes 1 \otimes \cdots \otimes 1),$$

where $1 \in P(1) = P[1]_0$ is the identity operation, and $\lambda \in P(m)$ is at the $i+1$-th place. It is easy to check that $P[1]$ with these products is a grafting algebra over $K$.

As an example of grafting bialgebra consider the grafting algebra associated to the operad $A_\infty$.
The grafting structure of $\text{As}[1] = \bigoplus_{n \geq 0} K[S_{n+1}]$ is given by the operations:

$$(\sigma \bullet_i \tau) = (\tau_i^{(1)} \times \sigma \times \tau_i^{(2)} \cdot \delta^n_i),$$

where $\tau = (\tilde{\tau}_i^{(1)} \times \lambda_i \times \tilde{\tau}_i^{(2)} \cdot \delta)$ with $\delta \in \text{Sh}(i, 1, m - i - 1)$, $\tilde{\tau}_i^{(1)} \in S_i$, $\tilde{\tau}_i^{(2)} \cdot \delta^{-1} \in S_{m-i-1}$, and

$$\delta_i^n(k) :=
\begin{cases}
\delta(k), & \text{for } \delta(k) \leq i \text{ and } k < \delta^{-1}(i + 1), \\
\delta(k) + n - 1, & \text{for } \delta(k) > i \text{ and } k < \delta^{-1}(i + 1), \\
i + r + 1, & \text{for } k = \delta^{-1}(i + 1) + r \text{ and } 0 \leq r < n, \\
\delta(k - n + 1), & \text{for } \delta(k) \leq i \text{ and } k > \delta^{-1}(i + 1) + n - 1, \\
\delta(k - n + 1) + n - 1, & \text{for } \delta(k) > i \text{ and } k > \delta^{-1}(i + 1) + n - 1.
\end{cases}$$

In fact, $\sigma \bullet_i \tau$ is obtained by replacing $i+1$ in the image of $\tau$ by $(\sigma(1)+i, \ldots, \sigma(n)+i)$. For instance,

$$(2, 4, 1, 3) \bullet (1, 3, 2, 5, 4) = (1, 6, 3, 5, 2, 4, 8, 7).$$

To define a coproduct on $\text{As}[1]$, let $\gamma \in S_{m+1}$ be a permutation, for an integer $0 \leq i \leq m$, there exists unique decompositions:

$$\gamma = (\tilde{\gamma}_i^{(1)} \times \tilde{\gamma}_i^{(2)} \cdot \delta) = (\tilde{\gamma}_i^{(1)} \times \tilde{\gamma}_i^{(3)} \cdot \epsilon),$$

where $\tilde{\gamma}_i^{(j)} \in S_j$, for $i = 1, 2$, $\delta^{-1} \in \text{Sh}(i+1, m-i)$ and $\epsilon \in \text{Sh}(i, m-i+1)$. Define

$$\Delta_{\text{As}}(\gamma) := \sum_{i=0}^{m} \tilde{\gamma}^{i+1}_{(1)} \otimes \tilde{\gamma}^{m-i}_{(2)}.$$

7.6.1. Proposition. The space $\text{As}[1] = \bigoplus_{n \geq 1} K[S_{n+1}]$, equipped with the operations $\bullet_i$ and the coproduct $\Delta_{\text{As}}$ is a grafting bialgebra.

Proof. We know that $(\text{As}[1], \bullet_i)$ is a grafting algebra. To check that $\Delta$ is coassociative, it suffices to note that, for $\gamma \in S_{m+1}$, we have that:

$$(\Delta_{\text{As}} \otimes id_{\text{As}[1]}) \circ \Delta_{\text{As}}(\gamma) = \sum_{i+j+k=m} \tilde{\gamma}^{i+1}_{(1)} \otimes \tilde{\gamma}^{j+1}_{(2)} \otimes \tilde{\gamma}^{k+1}_{(3)} = (id_{\text{As}[1]} \otimes \Delta_{\text{As}}) \circ \Delta_{\text{As}}(\gamma),$$

where, for each compositions $(i, j, k)$ of $m$, the following equalities hold:

$$\gamma = (\tilde{\gamma}^{i+1}_{(1)} \times \tilde{\gamma}^{j}_{(2)} \times \tilde{\gamma}^{k}_{(3)}) \cdot \delta_1 = (\tilde{\gamma}^{i}_{(1)} \times \tilde{\gamma}^{j+1}_{(2)} \times \tilde{\gamma}^{k}_{(3)}) \cdot \delta_2 = (\tilde{\gamma}^{i}_{(1)} \times \tilde{\gamma}^{j}_{(2)} \times \tilde{\gamma}^{k+1}_{(3)}) \cdot \delta_3,$$

with $\tilde{\gamma}^{x}_{(l)} \in S_{p}$ for $l = 1, 2, 3$, $\delta_1 \in \text{Sh}(i+1, j, k)$, $\delta_2 \in \text{Sh}(i, j+1, k)$ and $\delta_3 \in \text{Sh}(i, j, k+1)$.

To prove the relationship between $\Delta_{\text{As}}$ and the operations $\bullet_i$, note that for any $\gamma \in S_n$ and any $0 \leq i \leq n$, there exist unique order preserving bijections $\varphi^{i}_{(1)} : \{1, \ldots, i\} \rightarrow \gamma^{-1}({1, \ldots, i})$ and $\varphi^{n-i}_{(2)} : \{1, \ldots, n-i\} \rightarrow \gamma^{-1}({i+1, \ldots, n})$. The permutations $\tilde{\gamma}^{i}_{(1)}$ and $\tilde{\gamma}^{n-i}_{(2)}$ are given by the formulas:

$$\tilde{\gamma}^{i}_{(1)} = (\gamma(\varphi^{i}_{(1)}(1)), \ldots, \gamma(\varphi^{i}_{(1)}(i)))$$
$$\tilde{\gamma}^{n-i}_{(2)} = (\gamma(\varphi^{n-i}_{(2)}(1)), \ldots, \gamma(\varphi^{n-i}_{(2)}(n-i))).$$
Using the formulas above, it is easily seen that, for \( \sigma \in S_{n+1} \), \( \tau \in S_{m+1} \) and \( 0 \leq j \leq n + m \), we have that:

\[
(\sigma \bullet \tau)^{i+j}_1 = \begin{cases} 
\frac{\sigma^j_1}{i+1}, & \text{for } 0 \leq j < i \\
\frac{\sigma^{j-i}_1 \bullet z^{i+1}_1}{i+1}, & \text{for } i \leq j \leq i + n \\
\sigma \bullet \cdot z^{i-n+1}_1, & \text{for } i + n < j \leq n + m.
\end{cases}
\]

\[
(\sigma \bullet \tau)^{n+m-j+1}_2 = \begin{cases} 
\sigma \bullet \cdot z^{m-j+1}_2, & \text{for } 0 \leq j < i \\
\frac{\sigma^{n+i-j+1} \bullet z^{m-j+1}_2}{i+1}, & \text{for } i \leq j \leq i + n \\
z^{m+n-j+1}_2, & \text{for } i + n < j \leq n + m,
\end{cases}
\]

which ends the proof.

\[\Box\]

The following result is an extension of Lemma 2.5 to preshuffle algebras, its proof is straightforward.

7.7. Lemma. Let \( (A, \bullet) \) and \( (B, o_j) \) be preshuffle (respectively grafting) algebras.

\begin{enumerate}
\item The Hadamard product \( A \otimes_B \) has a natural structure of preshuffle (respectively grafting) algebra, given by the operations:
\[
(x \otimes y) \bullet (x' \otimes y') := (x \bullet x') \otimes (y \circ_j y'),
\]
for \( x \in A_n, y \in B_n, x' \in A_m, y' \in B_m \) and \( 0 \leq i \leq m \).
\item The tensor product \( A \otimes B \) has a natural structure of preshuffle (respectively grafting) algebra, given by the operations:
\[
(x \otimes y) \bullet_i (x' \otimes y') := \begin{cases} 
(x \bullet_i x') \otimes (y \circ_i |x'| y'), & \text{for } i = |x'|, \\
0, & \text{otherwise}.
\end{cases}
\]
\end{enumerate}

7.8. Proposition. Let \( (A, \bullet, \Delta_A) \) and \( (B, o_j, \Delta_B) \) be two preshuffle (respectively grafting) bialgebras. The Hadamard product \( A \otimes_B \) with the operations \( \bullet_i \) given in Definition 7.7 and the coproduct given by:

\[
\Delta_{A \otimes_B} = \sum_{|x| = |y|} (x(1) \otimes y(1)) \otimes (x(2) \otimes y(2)),
\]

where \( \Delta_A(x) = \sum x(1) \otimes x(2) \) and \( \Delta_B(y) = \sum y(1) \otimes y(2) \), is a preshuffle (respectively grafting) bialgebra.

\[\text{Proof.}\] Let \( x \in A_n, y \in B_n, x' \in A_m \) and \( y' \in B_m \). Since \( |x(1)| < |y| < |x \bullet x'(1)| \) and \( |y(1)| < |x| < |y \circ y'(1)| \), for all \( x(1) \) and \( y(1) \), we have that:

\[
\Delta_{A \otimes_B}((x \bullet x') \otimes (y \circ y')) = \sum_{|x(1)| = |y(1)|} (x(1) \otimes y(1)) \otimes ((x(2) \bullet x') \otimes (y(2) \circ y')) +
\]

\[
(x \otimes y) \otimes (x' \otimes y') + \sum_{|x(1)| = |y(1)|} ((x \bullet x'(1)) \otimes (y \circ y'(1))) \otimes (x'(2) \otimes y'(2)) =
\]

\[
\sum ((x \otimes y)(1) \otimes ((x \otimes y)(2) \bullet (x' \otimes y'))(2) + (x \otimes y) \otimes (x' \otimes y') +
\]

\[+ \sum ((x \otimes y)(1) \bullet (x' \otimes y')(1)) \otimes (x' \otimes y')(2).\]
For $0 < i < m$, note that $|(x \bullet x')_{(1)}| = |(y \circ y')_{(1)}|$ only in the following cases:

1. $(x \bullet x')_{(1)} = x_{(1)}', (y \circ y')_{(1)} = y'_{(1)}$ and $|x_{(1)}'| = |y'_{(1)}| \leq i$,
2. $(x \bullet x')_{(1)} = x_{(1)} \bullet x'_{(1)}$, $(y \circ y')_{(1)} = y_{(1)} \circ y'_{(1)}$, $|x_{(1)}'| = |y'_{(1)}| = i$, and $|x_{(1)}| = |y'_{(1)}|$

3. $(x \bullet x')_{(1)} = x \bullet x'_{(1)}$, $(y \circ y')_{(1)} = y \circ y'_{(1)}$, and $|x_{(1)}'| = |y'_{(1)}| \geq i$.

The equalities above imply that $\Delta_{A \otimes B}((x \bullet x') \otimes (y \circ y')) =$

$$\sum_{|(x' \otimes y')_{(1)}| \leq i} (x' \otimes y')_{(1)} \otimes ((x \otimes y) \circ (x' \otimes y')_{(2)}) + \sum_{|(x' \otimes y')_{(1)}| = i} ((x \otimes y) \circ_i (x' \otimes y')_{(1)}) \otimes ((x \otimes y)_{(2)} \bullet (x' \otimes y')_{(2)}) + \sum_{|(x' \otimes y')_{(1)}| \geq i} ((x \otimes y) \circ_i (x' \otimes y')_{(1)}) \otimes (x' \otimes y')_{(2)},$$

which ends the proof for $0 < i < m$. The result for $i = m$ is obtained in the same way.

Let $(A, \bullet, \Delta)$ be a grafting bialgebra, we want to show that there exist a natural way of defining operations $\cdot \gamma$ on $A$ is such a way that $(A, \bullet, \Delta)$ is a shuffle bialgebra.

1. Given a composition $(n_1, \ldots, n_p)$ of $n$, we denote by $\Delta_{n_1, \ldots, n_p}$ the composition $\pi_{n_1} \cdots \pi_{n_p} \circ \Delta^p$, where $\pi_{n_i} \cdots \pi_{n_p}$ is the projection from $A^\otimes p$ to $A_{n_1} \otimes \cdots \otimes A_{n_p}$.

2. Let $\gamma$ be an $(n,m)$-shuffle. There exist unique compositions $(n_1, \ldots, n_r)$ of $n$ and $(m_1, \ldots, m_{r+1})$ of $m$ such that $m_1 \geq 0$, $m_{r+1} \geq 0$, $m_i \geq 1$ for $2 \leq i \leq r$, and $n_j \geq 1$ for $1 \leq j \leq r$, such that

$$\gamma = (n+1, \ldots, n+1, m_1, \ldots, n, n+m_1+1, \ldots, n, n+m_1+1, \ldots, n, n+m_1+1),$$

that is

$$\gamma(j) = \begin{cases} j + n - \sum_{i=1}^{k} n_i, & \text{for } 0 < j - \sum_{i=1}^{k} n_i + m_i \leq m_{k+1}, \text{ with } 0 \leq k \leq r \\ j - \sum_{i=1}^{k} m_i, & \text{for } m_{k+1} < j - \sum_{i=1}^{k} n_i + m_i \leq m_{k+1} + n_{k+1}, \text{ with } 1 \leq k \leq r. \end{cases}$$

For instance, if $\gamma = (1, 3, 4, 2, 5, 6) \in Sh(2, 4)$, then $(m_1, m_2, m_3) = (0, 2, 2)$, and $(n_1, n_2) = (1, 1)$.

Given elements $x \in A_n$ and $y \in A_m$, define the element $x \bullet \gamma y \in A_{n+m}$ as follows:

$$x \bullet \gamma y := \sum x_{(1)}^{n_1} \bullet_{m_1} \cdots \otimes x_{(r-1)}^{n_{r-1}} \bullet_{m_1+\cdots+m_{r-1}} (x_{(r)}^{n_r} \bullet_{m_1+\cdots+m_r} y)), $$

where $\Delta_{n_1, \ldots, n_p}(x) = \sum x_{(1)}^{n_1} \otimes \cdots \otimes x_{(p)}^{n_p}$.

7.9. Theorem. Let $(A, \bullet, \Delta)$ be a grafting bialgebra. The graded space $A$ equipped with the operations $\cdot \gamma$ defined above for any shuffle $\gamma$, is a shuffle bialgebra.

Proof. Let $x \in A_n$, $y \in A_m$ and $z \in A_r$ be homogeneous elements of $A$, and let $\gamma \in Sh(n, m + r)$, $\delta \in Sh(m, r)$, $\lambda \in Sh(n + m, r)$ and $\sigma \in Sh(n, m)$ be such that

$$(1_n \times \delta) \cdot \gamma = (1_r \times \delta) \cdot \lambda.$$

We want to verify that $x \bullet \gamma (y \bullet \delta z) = (x \bullet \gamma y) \bullet \lambda z.$
Let $\gamma$ be the permutation given by the integers $(n_1, \ldots, n_p) \vdash n$ and $(h_1, \ldots, h_{p+1}) \vdash m + r$. We proceed by a recursive argument on $p$.

If $p = 1$, then $\gamma = \omega_{n,m+r}$.

Suppose that $\delta \in Sh(m, r)$ is given by integers $(m_1, \ldots, m_q) \vdash m$ and $(r_1, \ldots, r_{q+1}) \vdash r$, we have to consider two different cases.

**a)** If there exists $0 \leq k \leq q$ such that $0 < h - \sum_{i=1}^{k} r_i + m_i < r_{k+1}$, then

$$(1_n \times \delta) \cdot \gamma = (\sigma \times 1_r) \cdot \lambda,$$

where $\sigma = \omega_{m_1 + \ldots + m_k}$ and $\lambda$ is the $(n + m, r)$-shuffle associated to the compositions $(m_1, \ldots, m_k, n, m_{k+1}, \ldots, m_q)$ of $n + m$ and $(r_1, \ldots, r_k, h_k, r_{k+1} - h_k, r_{k+2}, \ldots, r_{q+1})$ of $r$, with $h := h - \sum_{i=1}^{k} r_i + m_i$.

Applying the properties of a grafting algebra, we get that:

$$x \bullet_y (y \bullet_z z) =
\begin{cases}
  x \bullet_y (y_{(1)} \cdot r_1 \cdot \ldots \cdot (y_{(q)} \cdot r_1 + \ldots + r_q \cdot z)) = \\
  y_{(1)} \cdot r_1 \cdot \ldots \cdot y_{(k)} \cdot r_1 + \ldots + r_k \cdot (x \bullet_y \sum_{i=1}^{k} m_i \cdot (y_{(k+1)} \cdot r_1 + \ldots + r_k + \ldots \cdot (y_{(q)} \cdot r_1 + \ldots + r_q \cdot z))))
\end{cases}.$$  

Since $(x \bullet_{m_1 + \ldots + m_k} y)_{(j)} = \begin{cases} y_{m_j}, & \text{for } 1 \leq j \leq k, \\
  x, & \text{for } j = k + 1, \\
  y_{m_{j-1}}, & \text{for } j < k + 1,
\end{cases}$

we get the result.

**b)** If there exists $0 \leq k \leq q$ such that $0 < h - \sum_{i=1}^{k-1} (r_i + m_i) + r_k < m_k$, then

$$\sigma = \omega_{n,m} \cdot n_{m,i} + h_k,$$

with $h := h - \sum_{i=1}^{k-1} (r_i + m_i) + r_k$, and $\lambda$ is the $(n + m, r)$-shuffle associated to the compositions $(m_1, \ldots, m_{k-1}, m_k + n, \ldots, m_p)$ of $n + m$ and $(r_1, \ldots, r_{p+1})$ of $r$.

We have that

$$\begin{align*}
  (1) & (x \bullet_{m_1 + \ldots + m_{k-1} + h_k} y)_{(j)} = y_{m_j}, & \text{for } j \neq k, \\
  (2) & (x \bullet_{m_1 + \ldots + m_{k-1} + h_k} y)_{(k)} = x \bullet_h y_{(k)}.
\end{align*}$$

In this case, using the properties of grafting algebras, it is immediate to check that:

$$x \bullet_\gamma (y \bullet_\delta z) = (x \bullet_{m_1 + \ldots + m_{k-1} + h_k} y) \bullet_\lambda z.$$

For $p > 1$, note that if $\gamma$ is the $(n, m + r)$-shuffle associated to the compositions $(n_1, \ldots, n_p) \vdash n$ and $(h_1, \ldots, h_{p+1}) \vdash m + r$, then

$$x \bullet_\gamma (y \bullet_\delta z) = x_{(1)}^{n_1} \bullet_h (x_{(2)}^{n-n_1} \bullet_\gamma (y \bullet_\delta z),$$

where $\tilde{\gamma}$ is the $(n - n_1, m + r)$-shuffle associated to the compositions $(n_2, \ldots, n_p)$ of $n - n_1$ and $(h_1 + h_2, \ldots, h_{p+1})$ of $m + r$.

We get that:

$$x \bullet_\gamma (y \bullet_\delta z) = x_{(1)}^{n_1} \bullet_h ((x_{(2)}^{n-n_1} \bullet_\gamma y) \bullet_\lambda z) = (x_{(1)}^{n_1} \bullet_{h_1} (x_{(2)}^{n-n_1} \bullet_\gamma y)) \bullet_\lambda z,$$

where $(1_{n-n_1} \times \delta) \cdot \tilde{\gamma} = (\delta \times 1_r) \cdot \tilde{\gamma}$, and
\[(1_{n+m-r} \times \lambda) \cdot \omega_{k_1}^{n_1,n+m-r-n_1} = (\omega_{k_1}^{n_1,n+m-r-n_1} \times 1_r) \cdot \lambda.\]

So, we have that
\[x \bullet \gamma (y \bullet z) = (x \bullet y) \bullet z,\]
with \(\sigma := (1_{n_1} \times \delta) \cdot \omega_{k_1}^{n_1,n+m-r-n_1}\) and \((1_n \times \delta) \cdot \gamma = (\sigma \times 1_r) \cdot \lambda\), which ends the proof.

**Primitive elements of preshuffle bialgebras.**

Since any shuffle bialgebra is a preshuffle algebra, we look for the operations obtained by compositions and linear combinations of the primitive operations \((-,-\)} and \(B^7\), introduced in Section 4, which can be defined in terms of the multiplications \(\bullet\) of a preshuffle algebra.

Let \((A, \bullet_i)\) be a preshuffle algebra, and let \(x \in A_m, y \in A_n\) and \(z \in A_r\) be elements of \(A\). Note that \(\{x, y\} = x \bullet \top y - x \bullet y\) and \(B^{m+n}(-, z) = x \bullet y\) are defined for all \(1 \leq i \leq m - 1\). But also the element
\[B^{1_n \times \omega_{m-r}^{m-r}}(\{x, z\}; y) = z \bullet_{i+n} (x \bullet 0 y) - x \bullet 0 (z \bullet y)\]
may be defined in \(A\) for \(1 \leq i \leq m\). In a similar way, the element
\[B^{1_n \times \omega_{m-n}^{n \geq 2}}(\{x_1, z\}; x_2, \ldots, x_q; y) = z \bullet_{n+i} (x_1 \bullet x_2 \bullet \ldots \bullet x_q \bullet 0 y) - x_1 \bullet 0 (z \bullet x_2 \bullet \ldots \bullet x_q \bullet 0 y),\]
where \(|x_i| = n_i, n = \sum_{i=1}^{q} n_i\) and \(n_{\geq k} := \sum_{i=k}^{q} n_i\), may be defined on \(A\).

7.10. Definition. Let \((A, \bullet_i)\) be a preshuffle algebra over \(K\). For \(q \geq 0\) and \(1 \leq p \leq n\), the \(q + 2\)-ary operation \(L_q^p : A^\otimes q \otimes A \to A\) is defined by the following formulas:
\[L_q^p(x_1, \ldots, x_q; y; z) :=\]
\[z \bullet p y, \quad \text{for } q = 0 \text{ and } 0 < p < |y|;\]
\[\{y, z\} = y \bullet \top z - y \bullet 0 z, \quad \text{for } q = 0 \text{ and } p = |y|;\]
\[z \bullet_{p+n} (x_1 \bullet \ldots \bullet x_q \bullet 0 y) - x_1 \bullet 0 (z \bullet_{p+n} x_2 \bullet \ldots \bullet x_q \bullet 0 y), \quad \text{for } q \geq 1,\]
where \((x_1, \ldots, x_q; y; z)\) denotes the element \(x_1 \otimes \cdots \otimes x_q \otimes y \otimes z \in A^\otimes (q+2)\),
\[n_k := |x_k|, n_{\geq k} := \sum_{i=k}^{q} n_i\] and \(n = n_{\geq 1}\), for \(1 \leq k \leq q\).

The following result implies that the subspace of primitive elements of a preshuffle bialgebra is closed under the \(q + 1\)-ary operations \(L_q^p\), its proof is similar to Proposition 13 one.

7.11. Proposition. Let \((A = \bigoplus_{k \geq 1} A_k, \bullet, \Delta)\) be a preshuffle bialgebra. If the elements \(x, y, z\) belong to \(\text{Prim}(A)\), then \(L_q^p(x_1, \ldots, x_q; y; z)\) belongs to \(\text{Prim}(A)\), for any \(1 \leq p \leq |x_q|\).
Note that, in fact, \( \Delta(L^0_q(x; y)) = 0 \) for all \( y \in A \) and \( x \in \text{Prim}(A) \) and \( 0 < i < |x| \).

In order to describe the relationship verified by the new operations \( L^p_q \), we need the following Lemma.

7.12. **Lemma.** Let \( x_1, \ldots, x_q, y, z \) be elements of a preshuffle algebra \( A \). With the same notations that in Definition 7.10, the product \( \bullet_0 \) and the operations \( L^p_q \) defined above verify the following equalities:

1) For \( j < |y| \),

\[
L^j_q(x_1, \ldots, x_q; y \bullet_0 w) = \sum_{k=0}^q L_k^{j+n} (x_1, \ldots, x_k; L^j_{q-k}(x_{k+1}, \ldots, x_q; y; w); z),
\]

and

\[
L_{|y|}^j_q(x_1, \ldots, x_q; y \bullet_0 w) = \sum_{k=0}^q L_k^{|y|+n} (x_1, \ldots, x_k; L_{|y|}^j_q(x_{k+1}, \ldots, x_q; y; w); z) + L_{|y|}^j_q(x_1, \ldots, x_q; y; z) \bullet_0 w,
\]

2) For \( 1 \leq j \leq |z| \), \( L^j_q(x_1, \ldots, x_q, z \bullet_0 y; w) = L^j_q(x_1, \ldots, x_q; z; w) \bullet_0 y \), and

\[
L_{|y|+z}^j_q(x_1, \ldots, x_q; z \bullet_0 y; w) = \begin{cases} L^j_1(z, y; w) + z \bullet_0 L^j_0(y; z), & \text{for } q = 0, \\ L_{q+1}^j(x_1, \ldots, x_q; z; y; w), & \text{for } q \geq 1, \end{cases}
\]

3) \( L^0_q(x_1, \ldots, x_{q-1}, z \bullet_0 x_q; y; w) = \begin{cases} L^j_{q+1}(x_1, \ldots, x_{q-1}, z, x_q; y; w), & \text{for } q \geq 2, \\ L^2_2(z, x_1; y; w) + z \bullet_0 L^1_1(x_1; y; w), & \text{for } q = 1. \end{cases} \)

**Proof.** The formulas are straightforward to check. We prove for instance the last one, the other ones may be obtained similarly.

For \( q \geq 2 \), the result is obvious.

For \( q = 1 \), we have that:

\[
L^1_1(z \bullet_0 x; y; w) = w \bullet_{j+n+r} (z \bullet_0 x \bullet_0 y) - (z \bullet_0 x) \bullet_0 (w \bullet_j y) = (w \bullet_{j+n+r} (z \bullet_0 x \bullet_0 y) - z \bullet_0 (w \bullet_{j+n} (x \bullet_0 y))) + (z \bullet_0 (w \bullet_{j+n} (x \bullet_0 y)) - (z \bullet_0 x) \bullet_0 (w \bullet_j y)) = L^2_2(z, x; y; w) + z \bullet_0 L^1_1(x; y; w).
\]

We introduce some notation, in order to prove the relations satisfied by the operations \( L^p_q \).

7.13. **Notation.** Let \((A, \bullet)\) be a preshuffle algebra over \( K \) and let \( x = (x_1, \ldots, x_n) \), \( y, z = (z_1, \ldots, z_m) \), \( t \) and \( w \) be a collection of elements in \( A \). Given nonnegative integers \( 0 \leq j \leq |y| \), \( 0 \leq k \leq |w| \) and \( 1 \leq l \leq m \) we define:

**a)** for a partition \( p = \{p_1, \ldots, p_m\} \) of \( m \) with \( p_i \geq 0 \) for \( 1 \leq i \leq m \), the element \( L^p_q(x, y, z) \) as follows:

(1) if \( m = 1 \), then \( L^p_q(x, y, z_1) := L^p_q(x_1, \ldots, x_n; y; z) \).
(2) if \( m > 1 \), then
\[
L^j_{\mathfrak{L}}(x, y, z) := L^j_{p_1}((x_1, \ldots, x_{p_1}; L^j_{\mathfrak{L}}((x_{k_1+1}, \ldots, x_n); y; (z_2, \ldots, z_m)); z_1),
\]
where \( p_1 := (p_2, \ldots, p_m) \) and \( n > j := \sum_{i=j+1}^{q} \left| x_i \right| \).

b) let \( m_1 := \sum_{i=1}^{m} \left| z_i \right| \), for a partition \( q = (q_1, \ldots, q_{m+1}) \) of \( n \), and an integer \( 0 < k \leq n_{q_1} + \left| y \right| + m_1 \), the element
\[
L^j_{\mathfrak{L}}(x, y, z, t) := L^j_{q_1}(x_1, \ldots, x_{q_1}; L^j_{\mathfrak{L}}(x^{j_1}, y, z); t),
\]
where \( q_1 := (q_2, \ldots, q_{m+1}) \) and \( x^{j_1} := (x_{q_1+1}, \ldots, x_n) \).

c) for a partition \( \mathfrak{z} = (r_1, \ldots, r_{l+1}) \) of \( n \), the element
\[
L^j_{\mathfrak{L}}(x, y, z, w, t) := L^{j+k+n_{q_1}+m_1}_{\mathfrak{L}}(x_1, \ldots, x_{r_1}, L^{|y|}(x^{r_1}, y, z; \mathfrak{z}); z_{l+1}, \ldots, z_m; w; t),
\]
where \( \mathfrak{z} := (r_2, \ldots, r_1) \), \( x^{r_1} := (x_{r_1+1}, \ldots, x_n) \) and \( z^{\mathfrak{z}} := (z_1, \ldots, z_l) \).

7.14. **Theorem.** Let \( (A, \bullet) \) be a preshuffle algebra over \( K \). Given elements \( x_1, \ldots, x_n, y, z_1, \ldots, z_m, w, t \) of \( A \), the operations \( L^j_n \) verify the following relations:

a) \( L^j_n(x_1, \ldots, x_n; y; L^k_m(w; t)) = \sum_{r=0}^{n} L^{j+k+n_{q_1}+m_1}_{r}((x_1, \ldots, x_r; L^j_{\mathfrak{L}}(x^{r_1}, y, z); t)) + \delta_{j+y}(\sum_{r=0}^{n} L^{j+k+n_{q_1}+m_1}_{r}(x_1, \ldots, x_r; L^j_{\mathfrak{L}}(x^{r_1}, y, z); t); w), \)

where \( \delta_{pq} := \begin{cases} 1, & \text{for } p = q, \\ 0, & \text{otherwise.} \end{cases} \)

b) For \( m \geq 1 \),
\[
L^j_n(x_1, \ldots, x_m; y; L^k_m(z_1, \ldots, z_m; w; t)) = \sum_{p} L^j_{p}(x, y, z', t) - L^{j+n_{q_1}+m_1}_{\mathfrak{L}}(x_1, \ldots, x_{p_1}; L^j_{\mathfrak{L}}(x^{p_1}, y, z'; t); z_1) + \delta_{j+y}(\sum_{q=1}^{m} L^k_{\mathfrak{L}}(x, y, z, w, t) - \sum_{l=2}^{m} \left( \sum_{q=1}^{l-1} L^{|y|+n_{q_1}+m_1}_{r}((x_1, \ldots, x_{r_1}; L^k_{\mathfrak{L}}(x^{r_1}, y, z; t); z_1) + L^k_{n+m+1}(x_1, \ldots, x_n, y, z_1, \ldots, z_m; w; t) - \sum_{s=0}^{n} L^{|y|+n_{q_1}+m_1}_{s}(x_1, \ldots, x_s; L^k_{n+m-s}(\ldots, x_n, y, z_2, \ldots, z_m; w; t); z_1)), \)
\]

where the first sum is taken over all partitions \( p = (p_1, \ldots, p_{m+1}) \), the second one is taken over all partitions \( q = (q_1, \ldots, q_{m+1}) \), and the third one over all \( r = (r_1, \ldots, r_{l+1}) \) of \( n \), \( z' := (z_1, \ldots, z_m, w) \) and \( y^s := (y_{s+1}, \ldots, y_h) \) for any partition \( y = (y_1, \ldots, y_h) \).
If Proposition.

7.16. We show it applying a recursive argument on $x$ and $y$, with $x = x_1, \ldots, x_n$ and $y = y_{n+1}$, as stated in Theorem 7.14. The other cases are obtained by recursive arguments on $n$ and $m$, applying Lemma 7.12 and the following formula:

$$I_{n+1}^i(x_1, \ldots, x_{n+1}; y; w) = L_n^i(x_1, \ldots, x_n; x_{n+1} \bullet_0 y; w),$$

for $q \geq 1$.

7.15. Definition. A $Prim_{psh}$ algebra is a graded vector space $V$ equipped with operations $L_i^n : V^\otimes n \otimes V \otimes V \rightarrow V$, for $n \geq 0$ and $1 \leq j \leq m$, which verify the relations of the Theorem 7.14.

Note that the Theorem 7.14 implies that the free $Prim_{psh}$ algebra over a set $X$ is linearly spanned by elements $x = L_0^n(x_1, \ldots, x_n; y; z)$, where $x_1, \ldots, x_n, y$ are elements in the free algebra, and $z \in X$.

Theorem 7.14 states that there exists a functor from the category $\text{Presh}$ of preshuffle algebras to the category of $Prim_{psh}$ algebras. Given a preshuffle bialgebra $H$, the subspace of primitive elements $\text{Prim}(H)$ is a $Prim_{psh}$ subalgebra of $H$.

Applying the same arguments that in Section 4 for shuffle algebras, we study the structure of free preshuffle algebras.

Let $X$ be a positively graded set, since the triple $(K[\mathcal{K}_\infty, X]_+, \bullet_0, \Delta_{\Theta+})$ is a connected unital infinitesimal bialgebra, the map

$$e(\xi_n; x) \mapsto (\xi_n; x) - \sum (\xi_{n_1}, n_2; x_{(1)}, x_{(2)}) + \cdots + (-1)^{r+1} \sum (\xi_{n_1}, \ldots, n_r; x_{(1)}, \ldots, x_{(r)}) + \cdots$$

of the space spanned by all the elements

$$\sum (\xi_{n_1}, \ldots, n_r; x_{(1)}, \ldots, x_{(r)}) + \cdots$$

and the subset $e(X)$ of $\text{Prim}(K[\mathcal{K}_\infty, X])$.

We denote by $\mathcal{P}rim_{psh}(X)$ the subspace of $K[\mathcal{K}_\infty, X]$ spanned by the set $e(X)$ with the operations $L_n^i$, and by $\mathcal{P}rim_{psh}(X)^{\otimes n}$ the space spanned by all the elements of the form $z_1 \bullet_0 \cdots \bullet_0 z_n$, with each $z_j \in \mathcal{P}rim_{psh}(X)$, for $1 \leq j \leq n$. Note that Theorem 7.14 states that any element $w$ in $\mathcal{P}rim_{psh}(X)$ is a sum of elements of type $L_n^i(x_1, \ldots, x_n; y; t)$, with $x_1, \ldots, x_n, y \in \mathcal{P}rim_{psh}(X)$ and $t \in e(X)$.

7.16. Proposition. Let $X$ be a positively graded set, equipped with a coassociative graded coproduct $\Theta$ on $K[X]$. Any element $z$ in $K[\mathcal{K}_\infty, X]$ belongs to $\bigoplus_{n \geq 1} \mathcal{P}rim_{psh}(X)^{\otimes n}$.

Proof. We only need to check that an element

$$z = e(\xi_n; x) \bullet_j (z_1 \bullet_0 z_2 \bullet_0 \cdots \bullet_0 z_r),$$

with $x \in X_n$ and $z_i \in \mathcal{P}rim_{psh}(X)$, belongs to $\bigoplus_{n \geq 1} \mathcal{P}rim_{psh}(X)^{\otimes n}$.

We show it applying a recursive argument on $r$.

If $r = 0$, then $z = e(\xi_n; x)$ belongs to $e(X)$, and the result is obvious.

If $r = 1$ and $0 < j < |z_1|$, then $z = L_0^j(z_1; e(\xi_n; x))$ belongs to $\mathcal{P}rim_{psh}(X)$.

If $r = 1$ and $j = |z_1|$, then $z = L_0^{|z_1|}(z_1; e(\xi_n; x)) + z_1 \bullet_0 e(\xi_n; x)$ belongs to $\mathcal{P}rim_{psh}(X) \bigoplus \mathcal{P}rim_{psh}(X)^{\otimes 2}$.
Suppose that \( r \geq 2 \). If \( 0 < j \leq |z_1| + \cdots + |z_r| \), then there exists \( 1 \leq k \leq r \) such that \( |z_1| + \cdots + |z_{k-1}| < j \leq |z_1| + \cdots + |z_k| \), and

\[
z = (e(\xi_n;x) \cdot_j (z_1 \cdot_0 \cdots \cdot_0 z_k)) \cdot_0 z_{k+1} \cdot_0 \cdots \cdot_0 z_r.
\]

Clearly, if \( k < r \) the result follows immediately by recursive hypothesis. If \( k = r \), then

\[
z = L_{r-1}^{1-m}(z_1; \ldots ; z_{r-1}; e(\xi_n;x)) + z_1 \cdot_0 (e(\xi_n;x) \cdot_{i-|z_i|} (z_2 \cdot_0 \cdots \cdot_0 z_r)),
\]

where \( m = |z_1| + \cdots + |z_{r-1}| \). But \( L_{r-1}^{1-m}(z_1; \ldots ; z_{r-1}; e(\xi_n;x)) \in \text{Prim}_{psh}(X) \) and, by recursive hypothesis, \( e(\xi_n;x) \cdot_{j-|z_j|} (z_2 \cdot_0 \cdots \cdot_0 z_r) \in \bigoplus_{n \geq 1} \text{Prim}_{psh}(X)^{\circ n} \).

So, \( z \in \bigoplus_{n \geq 1} \text{Prim}_{psh}(X)^{\circ n} \). \( \diamond \)

7.17. **Proposition.** Let \( X \) be a positively graded set. The subspace \( \text{Prim}_{psh}(X) \) is the subspace of primitive elements of \( K[K_\infty,X] \). Moreover, it is the free \( \text{Prim}_{psh} \) algebra spanned by \( X \).

**Proof.** As for shuffle algebras, it suffices to prove the result for the case where \( X = \bigcup_{n \geq 1} X_n \) with \( X_n \) finite, for all \( n \geq 1 \).

Proposition 7.11 states that \( \text{Prim}_{psh}(X) \subseteq \text{Prim}(K[K_\infty,X]) \), while Proposition 7.16 implies that \( K[K_\infty,X] = \overline{T}(\text{Prim}_{psh}(X)) \) as a vector space. From Theorem 4.2 one has that \( K[K_\infty,X] = \overline{T}(\text{Prim}(K[K_\infty,X])) \), so \( \text{Prim}_{psh}(X) = \text{Prim}(K[K_\infty,X]) \).

For the second point, we know that the dimension of the subspace \( \text{Prim}_{psh}(X)_n \) of homogeneous elements of degree \( n \) of \( \text{Prim}_{psh}(X) \) is \( |\text{Irr}_{K_{\infty,X}}| \).

An element \( (f;x_1, \ldots; x_r) \in K_{n,X} \) is irreducible if, and only if, \( f = \xi_{n_1, \ldots, n_r} \cdot \sigma \), with \( \sigma \in Sh^{-1}(n_1, \ldots, n_r) \cap \text{Irr}_{S_n} \) and \( |x_i| = n_i \).

It is easily seen that if \( \sigma \in \text{Irr}_{S_n}, \tau \in S_m \) and \( 1 \leq i \leq n \), then \( \tau \cdot_i \sigma \in \text{Irr}_{S_n+m} \).

Moreover, let \( \sigma = \sigma_1 \times \cdots \times \sigma_k \), with \( \sigma_i \in \text{Irr}_{S_{n_i}} \) and \( k \geq 2 \). The permutation \( \tau \cdot_j \sigma \in \text{Irr}_{S_{n+m}} \), for any \( \sum_{i=1}^{k-1} n_i < j \leq \sum_{i=1}^k n_i \).

We need to check that the dimension of the homogeneous subspace of degree \( n \) of the free \( \text{Prim}_{psh} \) algebra spanned by \( X \) is precisely \( |\text{Irr}_{K_{\infty,X}}| \).

Let \( P_{psh}(X)_n \) denotes the subspace of degree \( n \) of the free \( \text{Prim}_{psh} \) algebra spanned by \( X \). Note that \( P_{psh}(X)_n \) is the vector space with basis

\[
\mathbb{B}_n := X_n \bigcup \{ L_q^\ast(x_1, \ldots; x_q; y; z) \mid q \geq 0, 1 \leq j \leq |y|, x_1, \ldots, x_q, y \in \bigcup_{k=1}^{n-1} \mathbb{B}_k, z \in X \text{ and } \sum_{i=1}^q |x_i| + |y| + |z| = n \}.
\]

The map \( \alpha : \bigcup_{n \geq 1} \mathbb{B}_n \rightarrow \bigcup_{n \geq 1} \text{Irr}_{K_{\infty,X}} \) is defined as follows:
(1) $\alpha(x) := (\xi_\alpha; x)$, for $x \in X$, 
(2) $\alpha(L^j_q(x; y; z)) := (\xi_{|z|}; z) \bullet_j \alpha(y)$, for $1 \leq j \leq |y|$, 
(3) $\alpha(L_q(x_1, \ldots, x_q; y; z)) := (\xi_{|z|}; z) \bullet_{j+m} \left( \alpha(x_1) \bullet_0 \cdots \bullet_0 \alpha(x_q) \bullet_0 \alpha_{|y|}(y) \right)$, 
for $1 \leq j \leq |y|$, where $m = \sum_{i=1}^q |x_i|$.

Conversely, let $(f; x_1, \ldots, x_r) \in K_{n,X}$ be an irreducible element, where $f = \xi_{n_1,\ldots,n_r} \cdot \sigma$. One has that $\sigma^{-1}(i) = \sigma^{-1}(1) + i - 1$, for $1 \leq i \leq n_1$, with $\sigma^{-1}(1) > 1$ and $f = \xi_{n_1} \bullet_{n_1-1} \left( \xi_{n_2,\ldots,n_r} \cdot \sigma' \right)$, for a unique $\sigma' \in S_{n-n_1}$.

There exists a unique decomposition

$$
(\xi_{n_2,\ldots,n_r} \cdot \sigma'; x_1, \ldots, x_r) = (g_1; x_2, \ldots, x_{j_1}) \bullet_0 \cdots \bullet_0 (g_m; x_{j_m-1}+1, \ldots, x_r),
$$

with $(g_i; x_{j_i-1}+1, \ldots, x_{j_i})$ irreducible.

By a recursive argument we suppose that $\alpha^{-1}(g_i; x_{j_i-1}+1, \ldots, x_{j_i})$ is a well-defined element in $\bigcup_{i=1}^{n-1} \mathbb{B}_i$, for $1 \leq i \leq m$. Since $(f; x_1, \ldots, x_r) \in Irr_{K_{n,X}}$, one has that $\sum_{i=1}^{m-1} |g_i| < \sigma^{-1}(1)$. So, we define

$$
\alpha^{-1}(f; x_1, \ldots, x_r) = L_{m-1}^s(\alpha^{-1}(g_1; x_2, \ldots, x_{j_1}), \ldots, \alpha^{-1}(g_{m-1}; x_{j_{m-1}}-1, \ldots, x_{j_{m-1}}); \alpha^{-1}(g_m; x_{j_m-1}+1, \ldots, x_r); x_1),
$$

where $s = \sigma^{-1}(1) - \sum_{i=1}^{m-1} |g_i|$.

Clearly, the map $\alpha^{-1}$ is the inverse of $\alpha$. \hfill \Box

The following result is a straightforward consequence of Theorem 4.22 and the previous results.

7.18. Proposition. Let $X$ be a positively graded set, such that $K[X]$ is equipped with a coassociative graded coproduct $\Theta$. The unital infinitesimal bialgebra $K[K_{\infty,X}]_+$ is isomorphic to $T^{\infty}(Prim_{psh}(X))$, where $Prim_{psh}(X)$ is the free $Prim_{psh}$ algebra spanned by $X$.

We want to prove the equivalence between the categories of connected preshuffle bialgebras and $Prim_{psh}$ algebras. More precisely, given a $Prim_{psh}$ algebra $(V, \overline{T}_n)$ and an homogeneous basis $X$ of the underlying vector space $V$, let $U_{psh}(V)$ be the preshuffle bialgebra obtained by taking the quotient of the free preshuffle algebra $K[K, X]$ by the ideal (as a preshuffle algebra) spanned all the elements:

$$
L^j_q(x_1, \ldots, x_q; y; z) - \overline{T}_q^j(x_1, \ldots, x_q; y; z),
$$

with $x_1, \ldots, x_q, y, z \in X$, $q \geq 0$ and $1 \leq i \leq |y|$, where $L^j_q$ denotes the operations associated to the preshuffle algebra $K[K, X]$.

The proof of the following result is similar to the proof of Theorem 4.11

7.19. Theorem. a) Let $(\mathcal{H}, \circ, \Delta)$ be a connected preshuffle bialgebra, then $\mathcal{H}$ is isomorphic to $U_{psh}(Prim(H))$, where $Prim(H)$ is the $Prim_{psh}$ algebra of primitive elements of $H$.

b) Let $(V, \overline{T}_n)$ be a $Prim_{psh}$ algebra, then $V$ is isomorphic to $Prim(U_{psh}(V))$. 

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Primitive elements of grafting bialgebras.

Let \((A, \bullet, \Delta)\) be a grafting bialgebra.

By Proposition 7.11, the elements \(L_p^k(x_1, \ldots, x_n; y; z)\) are primitive, for \(1 \leq p < |y|\), whenever the elements \(x_1, \ldots, x_n, y, z\) belong to \(\text{Prim}(A)\). But an easy calculation shows that \(L_p^k(x_1, \ldots, x_n; y; z) = 0\) for any \(x_1, \ldots, x_n, y, z \in A\) and \(n \geq 1\). Using this fact, we introduce the following definition.

7.20. Definition. A \(\text{Prim}_{\text{gr}}\) algebra over \(K\) is a graded vector space \(V\) equipped with a family of binary operations \(\{-,-\}: V \otimes V \rightarrow V\) and \(\bullet_p: V \otimes V_n \rightarrow V\), for \(1 \leq p < n\), such that:

1. \(\{\{x, y\}, z\} = \{x, \{y, z\}\} + y \bullet_{|x|}\{x, z\},\) for \(x, y, z \in V\).
2. \(\{x \bullet_p y, z\} = x \bullet_p \{y, z\}\),
3. \(\{x, y \bullet_p z\} = y \bullet_{|x|+p}\{x, z\}\),
4. \(\{x, y\} \bullet_p z = y \bullet_{|x|+p}(x \bullet_p z) - x \bullet_p (y \bullet_p z),\) for \(1 \leq p < |z|\),
5. \(x \bullet_p (y \bullet_q z) = x \bullet_{p+q} (y \bullet_q z),\) if \(1 \leq p < q < |z|\),
6. \(x \bullet_p (y \bullet_q z) = y \bullet_{|x|+q}(x \bullet_p z),\) if \(1 \leq p < q < |z|\),

for \(x, y, z \in V\).

Clearly, any grafting bialgebra \((A, \bullet, \Delta)\) has a natural structure of \(\text{Prim}_{\text{gr}}\) algebra, such that \(\text{Prim}(A)\) is a \(\text{Prim}_{\text{gr}}\) subalgebra of \(A\).

For any positively graded set \(X\) and any coassociative coproduct \(\Theta\) on \(K[X]\), there exists a natural extension of the coproduct to a coassociative coproduct \(\Delta_\Theta\) such that \((\text{Graft}(X), \bullet, \Delta_\Theta)\) is a grafting bialgebra. Moreover, the vector space \(K[T_{\infty}, X]\) equipped with the associative product \(\circ_0\) and \(\Delta_{\Theta, +}\) on \(\text{Prim}(K[T_{\infty}, X]_+))\).

Let \(\text{Prim}_{\text{gr}}(X)\) the subspace of \(K[T_{\infty}, X]\) spanned by \(e(X)\) with the operations \(\{-,-\}\) and \(\circ_p\).

7.21. Proposition. Let \(X\) be a positively graded set, equipped with a coassociative graded coproduct \(\Theta\) on \(K[X]\). Any element \(z\) in \(K[T_{\infty}, X]\) may be written as a sum \(z = \sum_k z^k_{x_1}o_0 z^k_{x_2}o_0 \cdots o_0 z^k_{x_n},\) with \(z^k_i \in \text{Prim}_{\text{gr}}(X)\).

Proof. The space \(K[T_{\infty}, X]\) is a quotient of \(K[K_{\infty}, X]\), let \(\Pi: K[K_{\infty}, X] \rightarrow K[T_{\infty}, X]\). Let \(e_{\Theta}\) (respectively, \(e_{\tilde{\Theta}}\)) denotes the projection of \(K[K_{\infty}, X]\) (respectively, \(K[T_{\infty}, X]\)) into its primitive part. The set \(\Pi^{-1}(e_{\Theta}(x))\) has a unique element, for any \(x \in X\); which implies that the restriction of \(\Pi\) to \(e_{\Theta}(X)\) is an injective map, whose image is \(e_{\tilde{\Theta}}(X)\).

Moreover, since \(\Pi\) sends the product \(\bullet_p\) to \(\circ_p\), for \(p \geq 0\), we have that \(\Pi(\text{Prim}_{\text{gr}}(X)) \subseteq \Pi(\text{Prim}_{\text{gr}}(X))\).

Let \(z \in K[T_{\infty}, X]\), there exist at least one element \(\tilde{z} \in K[K_{\infty}, X]\) such that \(\Pi(\tilde{z}) = z\).

We know that \(\tilde{z} = \sum_k \tilde{z}_{x_1}k \bullet_0 \tilde{z}_{x_2}k \cdots \tilde{z}_{x_n}k\), with \(\tilde{z}_{x}^k_i \in \text{Prim}_{\text{gr}}(X)\).

So, \(z = \sum_k \Pi(\tilde{z}_{x_1}^k) \circ_0 \Pi(\tilde{z}_{x_2}^k) \cdots \circ_0 \Pi(\tilde{z}_{x_n}^k),\) with \(\Pi(\tilde{z}_{x}^k_i) \in \text{Prim}_{\text{gr}}(X)\).

7.22. Proposition. Let \(X\) be a positively graded set, equipped with an associative coproduct \(\Theta\) on \(K[X]\). The subspace \(\text{Prim}_{\text{gr}}(X)\) is the subspace of primitive elements of \(K[T_{\infty}, X]\). Moreover, it is the free \(\text{Prim}_{\text{gr}}\) algebra spanned by \(X\).
Proof. The proof of the first assertion is identical to the ones given for preshuffle and shuffle algebras. The unique point to see is that \( \text{Prim}_{gr}(X) \) is the free \( \text{Prim}_{gr} \) algebra spanned by \( X \).

To prove the second one, we may suppose that the set \( X_n \) is finite, for all \( n \geq 1 \).

Since the algebra \( (\text{Prim}(K[T_\infty,X]), \bullet_0) \) is free on the set \( \{ t = \bigvee_x (|t|, t^1, \ldots, t^r) \} \), the dimension of the subspace of homogeneous elements of degree \( n \) of \( \text{Prim}(K[T_\infty,X]) \) is the number of trees of the form \( t = \bigvee_x (|t|, t^1, \ldots, t^r) \), where \( x \in X_r \) and \( |t| = \sum_{1 \leq j \leq r} |t^j| + r - 1 \).

Let \( \{ X \} \) be the set of all elements of the form \( z = \{ x_1, \ldots, \{ x_{k-1}, x_k \} \} \), with \( k \geq 1 \) and \( x_i \in X \). From Definition 7.20 we have that the elements of \( X \) and the elements of type:

\[
z = x_1 \cdot i_1 \cdot \cdots \cdot i_{r-2} \cdot (x_{r-1} \cdot i_r \cdot w (w)),
\]

with \( i_1 > \cdots > i_r, 1 \leq i_j < |x_{j+1}| + \cdots + |x_r| + |w| \), \( x_j \in X \), and \( w \in \{ X \} \), are a basis of the free \( \text{Prim}_{gr} \) algebra over \( X \) as a vector space.

Define a map \( \gamma \) from the basis described above to the set \( \{ t = \bigvee_x (|t|, t^1, \ldots, t^r) \} \) with \( x \in X_r \) and \( |t| = \sum_{1 \leq j \leq r} |t^j| + r - 1 \), as follows:

\[
\gamma(x) := (c_n, x), \text{ for } x \in X
\]

\[
\gamma(\{ x_1, \ldots, \{ x_{k-1}, x_k \} \}) := \bigvee_{x_1} (|\gamma(\{ x_2, \ldots, \{ x_{k-1}, x_k \} \})|), \text{ for } x_1, \ldots, x_k \in X
\]

\[
\gamma(x_1 \cdot i_1 \cdot \cdots \cdot i_{r-2} \cdot (x_{r-1} \cdot i_r \cdot w (w))) := (c_{n_1}, x_1) \circ_i \cdot \cdots \cdot (c_{n_r}, x_r) \circ_i \cdot \gamma(w))
\]

for \( |x_i| = n_i \).

Clearly, \( \gamma \) is a graded bijection, which sends elements of degree \( n \) of the basis to trees of the same degree. So, \( \text{Prim}_{gr}(X) \) is a quotient of the free \( \text{Prim}_{gr} \) algebra over \( X \) such that both space have the same dimension on each degree, which implies they are isomorphic. \( \square \)

Again, we have a natural equivalence between the categories of connected grafting bialgebras and \( \mathcal{P}_{Gr} \) algebras. As in previous cases we define, for any \( \mathcal{P}_{Gr} \) algebra \( (V, [-,-], \circ_p) \) and an homogeneous basis \( X \) of \( V \), the universal grafting enveloppping algebra \( \mathcal{U}_{Gr}(V) \) as the quotient of the free grafting algebra \( K[T_\infty,X] \) by the ideal spanned by the elements:

\{
x, y \} - [x, y] \text{ and } x \bullet_p y - x \circ_p y, \text{ for } x, y \in X \text{ and } 1 \leq p < |y|, \text{ where } \{-,-\} \text{ and } \bullet_p \text{ denote the operations associated to the grafting algebra } K[T_\infty,X].
\}

The proof of the following result is similar to the proof given for shuffle and preshuffle bialgebras.

7.23. Theorem. a) Let \((H, \circ_i, \Delta)\) be a connected grafting bialgebra, then \( H \) is isomorphic to \( \mathcal{U}_{gr}(\text{Prim}(H)) \), where \( \text{Prim}(H) \) is the \( \mathcal{P}_{Gr} \) algebra of primitive elements of \( H \).

b) Let \((V, [-,-], \bullet_p)\) be a \( \mathcal{P}_{Gr} \) algebra, then \( V \) is isomorphic to \( \text{Prim}(\mathcal{U}_{Gr}(V)) \).


8. Applications.

I. Basis of primitive elements for the Malvenuto-Reutenauer algebra and for the bialgebra $K[P_\infty]$.

In [8] and [2] the authors describe different basis for the subspace of primitive elements of the Malvenuto-Reutenauer bialgebra. We construct another one using about preshuffle bialgebras. We may extend this basis to a basis of the subspace of primitive elements of $K[P_\infty]$.

1) The Malvenuto-Reutenauer bialgebra.

We know that the dimension of the subspace of primitive elements of degree $n$ of $K[S_\infty]$ is the number $|\text{Irr}_{S_n}|$ of irreducible permutations of $S_n$. Using Proposition 4.10 we associate to any $\sigma \in \text{Irr}_{S_n}$ a primitive element $E_\sigma$ in $K[S_\infty]$.

If $\sigma = (1)$, then $E_{(1)} := (1)$.

Let $\sigma \in \text{Irr}_{S_n}$ with $n \geq 1$, there exist a family of irreducible permutations $\sigma_1, \ldots, \sigma_r$ and a shuffle $\delta \in \text{Sh}(1, n-1)$ such that $\sigma = ((1) \times \sigma_1 \times \cdots \times \sigma_r) \cdot \delta$. The integer $r$ and the permutations $\sigma_1, \ldots, \sigma_r, \delta$ are unique.

For example, $\sigma = (5, 1, 4, 6, 3, 2) = ((1) \times (4, 3, 5, 2, 1)) \cdot (2, 1, 3, 4, 5, 6)$, and $\sigma' = (3, 4, 2, 5, 1) = ((1) \times (2, 3, 1) \times (1)) \cdot (2, 3, 4, 5, 1)$.

Suppose that $|\sigma_i| = n_i$, with $\sum_{i=1}^r n_i = n - 1$. If $\sigma^{-1}(1) \leq n_1 + \cdots + n_{r-1} + 1$, then

$\sigma = (((1) \times \sigma_1 \times \cdots \times \sigma_{r-1}) \cdot \delta') \cdot \sigma_r$, which is impossible because $\sigma$ is irreducible.

So, $\sigma^{-1}(1) - 1 - n_1 - \cdots - n_{r-1} > 0$.

Define $E_{\sigma}$ as the following primitive element,

$$E_{\sigma} := L_{r-1}^{\sigma^{-1}(1)-1-n_1-\cdots-n_{r-1}}(E_{\sigma_1}, \ldots, E_{\sigma_{r-1}}; E_{\sigma_r}; (1)),$$

where the operations $L_i^j$ are the operations introduced of Definition 7.10.

From Proposition 7.7, we get that the set $\{E_{\sigma}\}_{\sigma \in \text{Irr}_{S_n}}$ is a basis of the subspace of primitive elements of the Malvenuto-Reutenauer bialgebra. For instance, one has that:

$E_{(2,1)} = (2,1) - (1,2),$

$E_{(3,1,2)} = L_0^1(E_{(2,1)}; (1)) = (3,1,2) - (2,1,3),$

$E_{(3,4,2,5,7,1,6)} = L_2^1(E_{(2,3,1)}, E_{(1)}; E_{(2,1)}; (1)) = (3,4,2,5,7,1,6) - (2,3,1,5,7,4,6) - (3,4,2,5,6,1,7) + (2,3,1,5,6,4,7) - (2,4,3,5,7,1,6) + (1,3,2,5,7,4,6) + (2,4,3,5,6,1,7) - (1,3,2,5,6,4,7).$

2) The bialgebra $K[P_\infty]$. As a shuffle algebra $K[P_\infty]$ is the free shuffle algebra spanned by the family $\{\xi_n\}_{n \geq 1}$. So, any coassociative coproduct $\Theta$ on the vector space spanned by $\{\xi_n\}_{n \geq 1}$ gives a shuffle bialgebra structure on $K[P_\infty]$. Note that

$$\Theta^n(\xi_n) = \sum_{n_1+\cdots+n_r} c_{n_1\ldots n_r} \xi_{n_1} \otimes \cdots \otimes \xi_{n_r},$$

where $(n_1, \ldots, n_r)$ is a composition of $n$, and $c_{n_1\ldots n_r} \in K$. 
So,
\[ E_{ξ_n} := \sum_{r=1}^{n} (-1)^{r-1} \sum_{n_1 + \cdots + n_r = n} c_{n_1, \ldots, n_r} ξ_{n_1} \cdot \cdots \cdot ξ_{n_r}. \]

Given an irreducible element \( f \in \text{Irr}_{P_∞} \), we associate to it a primitive element \( E_θ(f) \) in \( (K[P_∞], Δ_θ) \) as follows. Let \( n_1 = |f^{-1}(1)| \).

If \( n_1 = n \), then \( f = ξ_n \) and \( E_θ(f) \) is defined above.

If \( n_1 < n \), then \( f = ξ_{n_1} \bullet_γ f_1 \), with \( γ \neq 1_{n_1} \). There exist a unique family \( g_1, \ldots, g_r \) of irreducible elements, such that \( f_1 = g_1 \bullet_0 \cdots \bullet_0 g_r \). Let \( |g_j| = m_j \), for \( 1 \leq j \leq r \).

If \( f \in \text{Irr}_{P_∞} \), it is immediate that \( γ(n_1) > γ(n_1 + m_1 + \cdots + m_{r-1} + 1) \).

There exists \( 0 \leq k \leq r - 1 \) such that \( γ(n_1 + m_1 + \cdots + m_k + 1) < γ(1) \).

1. If \( k = 0 \), then \( E_θ(f) := E_θ(ξ_{n_1}) \bullet_γ (E_θ(g_1) \bullet_0 \cdots \bullet_0 E_θ(g_r)). \)

2. If \( k \geq 1 \), then
\[ E_θ(f) := \{ E_θ(g_1), E_θ(ξ_{n_1}) \} \bullet_γ (E_θ(g_2) \bullet_0 \cdots \bullet_0 E_θ(g_r), \]
where \( γ = (c_{n_1,m_1} \times 1_{m_2+\cdots+m_{r-1}}) \cdot \tilde{γ} \), with \( \tilde{γ} = 1_{m_1} \times α \) and
\( α(n_1) > α(n_1 + m_2 + \cdots + m_{r-1} + 1) \).

Applying Proposition \( 7.17 \) we get that the family \( \{ E_θ(f) \}_{f \in \text{Irr}_{P_∞}} \) is a basis of the space of primitive elements of \( (K[P_∞], Δ_θ) \).

For example, let \( Θ(ξ_n) = \sum_{i=1}^{n-1} ξ_i \otimes ξ_{n-i} \), for \( n \geq 1 \), and let
\[ f = (3, 2, 4, 1, 6, 4, 1, 5, 5) = ξ_2 \bullet_{(3,4,5,1,6,7,2,8,9)} ((2, 1) \bullet_0 (1, 3, 1, 2, 2)). \]

We get that
\[ E_θ(ξ_2) = (1, 1) - (1, 2), \]
\[ E_θ(2, 1) = (2, 1) - (1, 2), \]
\[ E_θ(2, 1, 1) = \{ E_θ(ξ), E_θ(ξ_2) \} = (2, 1, 1) - (3, 1, 2) - (1, 2, 2) + (1, 2, 3), \]
\[ E_θ(1, 3, 1, 2, 2) = E_θ(ξ_2) \bullet_{(1,3,2,4,5)} E_θ(2, 1, 1) = (3, 1, 2, 2) - (1, 4, 1, 2, 3) - (1, 2, 1, 3, 3) + (1, 2, 1, 3, 4) - (1, 4, 2, 3, 3) + (1, 5, 2, 3, 4) + (1, 3, 2, 4, 4) - (1, 3, 2, 4, 5), \]
\[ E_θ(3, 2, 4, 1, 6, 4, 1, 5, 5) = \{ E_θ(2, 1), E_θ(ξ_2) \} \bullet_{(1,2,5,3,6,7,4,8,9)} E_θ(1, 3, 1, 2, 2). \]

II. Some triples of operads

Note that preshuffle algebras, shuffle algebras and grafting algebras are not described by classical linear operads. This Section is devoted to describe some good triples of \( K \)-linear operads, following the definition of \( [22] \), where the co-operad is always the co-associative operad.

1) Duplicial bialgebras.

This example is studied in \( [22] \), we include it since the results may be obtained easily from our computation of primitive elements.

8.1. Definition. A duplicial algebra over \( K \) is a vector space \( A \) equipped with two bilinear maps /\( /, \backslash : A \otimes A \to A \), verifying the following relations:
\[ x/(y/z) = (x/y)/z \]
\[ x/(y\backslash z) = (x/y)\backslash z \]
\[ x\backslash(y\backslash z) = (x\backslash y)\backslash z, \]
for \( x, y, z \in A \).

Note that for any grafting algebra \( (A, \bullet) \), the space \( A \) with the products:
\[
  x/y := x \bullet y \quad \text{and} \quad x\backslash y := y \bullet |y| x
\]
is a duplicial algebra.

It is not difficult to verify (see [30] or [22]) that the free duplicial algebra spanned by a set \( E \) is the space of planar binary rooted trees \( K[Y_\infty, E] \), with the vertices coloured by the elements of \( E \). We denote it by \( \text{Dup}(E) \).

Let \( (A, /, \backslash) \) be a duplicial algebra, a coassociative coproduct on \( A \) is admissible for the duplicial structure if
\[
\Delta(x/y) = \sum x_{(1)} \otimes (x_{(2)}/y) + \sum (x/y_1) \otimes y_{(2)} + x \otimes y,
\]
\[
\Delta(x\backslash y) = \sum x_{(1)} \otimes (x\backslash y_{(2)}) + \sum (x\backslash y_1) \otimes y_{(2)} + x \otimes y,
\]
for \( x, y \in A \).

An duplicial bialgebra is an duplicial algebra \( A \) equipped with an admissible coproduct.

Note that any grafting bialgebra is a duplicial bialgebra. In particular, for any set \( E \), the free duplicial bialgebra \( \text{Dup}(E) \) is a duplicial bialgebra.

Clearly, the unique operation of \( \text{Prim}_{gr} \), which may be defined in any duplicial algebra is the product \( \{-, -\} \), which does not verify any relation.

8.2. Definition. A magmatic algebra over \( K \) is a vector space \( M \) equipped with a bilinear map \( M \otimes M \rightarrow M \).

There exists a functor \( F_{\text{Dup-Mag}} \) from the category of duplicial algebras to the category of magmatic algebras, which maps \( (A, /, \backslash) \mapsto (A, \{-, -\}) \). If \( (A, /, \backslash, \Delta) \) is a duplicial bialgebra, then \( \text{Prim}(A, \{-, -\}) \) is a magmatic subalgebra of \( (A, \{-, -\}) \).

For any set \( E \), let \( \{E, E\} \) denote the subspace of the free duplicial algebra \( \text{Dup}(E) \) spanned by the elements of \( E \) under the operation \( \{-, -\} \) and let \( \overline{T}\{E, E\} \) be the subspace of \( \text{Dup}(E) \) spanned by the elements of the form \( z = z_1/\ldots/z_n \), with \( n \geq 1 \) and \( z_i \in \{E, E\} \) for \( 1 \leq i \leq n \).

8.3. Proposition. The space \( \overline{T}\{E, E\} \) is isomorphic to \( \text{Dup}(E) \).

Proof. We need to prove that any planar binary tree \( t \) with the vertices coloured with elements of \( E \) is a finite sum \( \sum z_i^1/\ldots/z_n^1 \), with \( z_i^1 \in \{E, E\} \). Note that it suffices to prove the result for the trees of the form \( t = | \lor e \ t' \), with \( e \in E \) and \( t' \in \text{Dup}(E) \).

In order to simplify notation we denote \( e \) the tree \( (e_1, e) \), for \( e \in E \).

If \( |t| = 1 \), then \( t = e \in \{E, E\} \).

If \( |t| = 2 \), then \( t = e \lor f = \{e, f\} + e/f \in \overline{T}\{E, E\} \).

The proof follows from the following assertions:

1. Let \( t = w_1/\ldots/w_r/z \), with \( w_i, z \in \{E, E\} \). We have that:
\[
  t = w_1/\ldots/w_{r-1}/\{w_r, z\} + w_1/\ldots/w_r/z \in \overline{T}\{E, E\}.
\]

A recursive argument on \( m \) proves that if \( w_1/\ldots/w_r/z_1/\ldots/z_{m-1} \in \overline{T}\{E, E\} \), then \( w_1/\ldots/w_r/z_1/\ldots/z_m \in \overline{T}\{E, E\} \), with the elements \( w_i \) and \( z_j \) in \( \{E, E\} \).
Again, we prove the result for any finite set $E$

**Proof.**

$E$ is the subspace of homogeneous elements of degree $n$

For $m = 2$, using the first formula, we get that:

$w_1/\ldots/w_r\{z_1/z_2\} = -w_1/\ldots/w_{r-1}/w_r\{z_1, z_2\} + w_1/\ldots/w_r\{z_1\} z_2$.

For $w_1, \ldots, w_r, z_1, z_2 \in \{E, E\}$. So, $w_1/\ldots/w_r\{z_1/z_2\} \in T\{E, E\}$.

For $m \geq 3$, a recursive argument and the formulas above imply that

$w_1/\ldots/w_r\{z_1/z_2/\ldots/z_m\} \in T\{E, E\}$.

Using that for any duplicial bialgebra $(A, /, \Delta)$, the triple $(A_+, /, \Delta_+)$ is a unital infinitesimal bialgebra, we get that for any set $E$, the subspace $\{E, E\}$ is equal to Prim$(Dup(E))$, and that, as coalgebras $T^f_c(\{E, E\})$ and $Dup(E)$ are isomorphic.

To end the example we only need to prove that $(\{E, E\}, \{-, -\})$ is the free magmatic algebra spanned by $E$, denoted by Mag$(E)$.

**8.4. Proposition.** For any set $E$, the subspace of primitive elements of Dup$(E)$, equipped with the binary product $\{-, -\}$ is the free magmatic algebra Mag$(E)$ over $E$.

**Proof.** Again, we prove the result for any finite set $E$, showing that the dimension of the subspaces of homogeneous elements of degree $n$ of Prim$(Dup(E))$ and Mag$(E)$ are the same, for all $n$.

We know (see ??) that the dimension of the space of homogeneous elements of degree $n$ of Mag$(E)$ is $c_{n-1}|E|^n$, where $c_{n-1}$ is the Catalan number, which counts the number of planar binary rooted trees with $n$ leaves.

On the other hand, we know that (Dup$(E), /)$ is the free associative algebra spanned by the trees of type $t = \mid \vee e t'$, with $e \in E$ and $|t'| = |t| - 1$. Since, we also have that $Dup(E)$ is isomorphic as a coalgebra to $T^f_c$(Prim$(Dup(E))$), we may assert that the dimension of the subspace of homogeneous elements of Prim$(Dup(E))$ of degree $n$ is the number of trees of type $\mid \vee e t'$, where $t'$ is a planar binary rooted tree with $n$ leaves and its $n - 1$ internal vertices coloured with elements of $E$. So, we have that $dim_k(Prim(Dup(E))) = c_{n-1}|E|^n$, which ends the proof.

As in the previous cases, given a magmatic algebra $(M, \cdot)$ we consider the free duplicial algebra Dup$(E)$ over the underlying vector space $M$. We define the universal duplicial enveloping algebra $U_{d}(M)$ of $(M, \cdot)$ as the quotient of $Dup(E)$ by the ideal spanned by the elements of the form $\{x, y\} - x \cdot y$, for $x, y \in M$; where $\{-, -\}$ is the magmatic product defined on $Dup(E)$.

The proof of a Cartier-Milnor-Moore type theorem for connected duplicial bialgebras follows using the previous result in the same way that we proved it for the cases of preshuffle and shuffle bialgebras. We just state it.

**8.5. Theorem. a)** Let $(A, /, \setminus, \Delta)$ be a connected duplicial bialgebra, then $A$ is isomorphic to $U_{d}(Prim(A))$, where Prim$(A)$ is the magmatic algebra of primitive elements of $A$.

**b)** Let $(M, \cdot)$ be a magmatic algebra, then $M$ is isomorphic to $Prim(U_{d}(M))$. 

2) The 2-infinitesimal nonunital bialgebra

Recall that a 2-ass algebra is simply a vector space equipped with two associative products \( \cdot \) and \( \circ \). Let \( E \) be a set, define \( \mathcal{T}_{n,E} \) as the set of all planar rooted trees with \( n \) leaves, with the leaves coloured by the elements of \( E \). Consider the vector space \( K[\mathcal{T}(E)] \) spanned by the graded set \( \mathcal{T}(E) := \bigcup_{n \geq 1} \mathcal{T}_{n,E} \). In [24], we equipped the tensor space \( T(K[\mathcal{T}(E)]) \) with two associative products \( \cdot \) and \( \circ \), and proved that \((T(K[\mathcal{T}(E)]), \cdot, \circ)\) is the free 2-ass algebra spanned by \( E \), we denote it by 2-ass(\( E \)). We just give a brief description of the 2-ass structure of 2-ass(\( E \)):

(1) For \( t = \bigvee(t_1, \ldots, t_r) \in \mathcal{T}_{n,E} \) and \( w = \bigvee(w_1, \ldots, w_k) \in \mathcal{T}_{m,E} \),

\[
t \cdot w := \bigvee(t_1, \ldots, t_r, w_1, \ldots, w_k),
\]

and \( t \circ w := t \otimes w \).

(2) For \( x = t^1 \otimes \cdots \otimes t^r \) and \( y = w_1 \otimes \cdots \otimes w_k \), with \( t_1, \ldots, t_r, w_1, \ldots, w_k \in \mathcal{T}(X) \),

\[
x \cdot y := (\bigvee(t_1, \ldots, t^r)) \vee (\bigvee(w_1, \ldots, w_k)),
\]

and \( x \circ y := t^1 \otimes \cdots \otimes t^r \otimes w_1 \otimes \cdots \otimes w_k \).

8.6. **Definition.** A 2-infinitesimal nonunital bialgebra is a 2-ass algebra \( (A, \cdot, \circ) \) equipped with a coassociative coproduct \( \Delta \), such that the triples \((A_+, \cdot, \Delta_+)\) and \((A_+, \circ, \Delta_+)\) are infinitesimal unital bialgebras. That means that \( \Delta \) verifies the following relations:

\[
\Delta(x \cdot y) = \sum (x \cdot y(1)) \otimes y(2) + \sum x(1) \otimes (x(2) \cdot y) + x \otimes y,
\]

\[
\Delta(x \circ y) = \sum (x \circ y(1)) \otimes y(2) + \sum x(1) \otimes (x(2) \circ y) + x \otimes y,
\]

for \( x, y \in A \), where \( \Delta(x) = \sum x(1) \otimes x(2) \).

For any preshuffle algebra \( (A, \bullet) \), the triple \( (A, \bullet_0, \bullet_L) \) is a 2-ass algebra; and if \( (A, \bullet, \Delta) \) is a preshuffle bialgebra, then \( (A, \bullet_0, \bullet_L, \Delta) \) is a 2-infinitesimal nonunital bialgebra.

Let us describe the 2-infinitesimal nonunital bialgebra structure of 2-ass(\( E \)), for any set \( E \). We define \( \Delta \) recursively as follows:

(1) \( \Delta(\emptyset, e) := 0 \), for all \( e \in E \),

(2) \[
\Delta(\bigvee(t^1, \ldots, t^r)) := \sum_{i=1}^{r} \bigvee(t^1, \ldots, t^{i-1}, \tilde{t}^1_{(i)} \otimes \bigvee(\tilde{t}^1_{(2)}, t^{i+1}, \ldots, t^r) + \tilde{t}^1_{(2)} \otimes \bigvee(t^1, \ldots, t^{i-1}) \otimes \tilde{t}^r, \]

\[
\bigvee(\tilde{t}^1_{(2)}, t^{i+1}, \ldots, t^r) + \sum_{j=2}^{r-2} \bigvee(t^1, \ldots, t^j) \otimes \bigvee(t^{j+1}, \ldots, t^r) + \bigvee(t^1, \ldots, t^{r-1}) \otimes \tilde{w},
\]

where \( \tilde{w} := \begin{cases} w, & \text{for } w = (\emptyset, e) \\ w^1 \otimes \cdots \otimes w^p, & \text{for } w = \bigvee(w^1, \ldots, w^p). \end{cases} \)
we know that both elements belong to $T$. In the recursive argument states that any $\tilde{\tau}$ can be written as a sum of elements of the form $t^1 \otimes \cdots \otimes t^n$.

\[ \Delta(t^1 \otimes \cdots \otimes t^n) := \sum_{i=1}^{r} (t^1 \otimes \cdots \otimes t^{i-1} \otimes t_i^{(1)}) \otimes (t_i^{(2)} \otimes t^{i+1} \otimes \cdots \otimes t^n) + \sum_{j=1}^{r-1} (t^1 \otimes \cdots \otimes t^j) \otimes (t^{j+1} \otimes \cdots \otimes t^n). \]

If we look at the structure of $\mathcal{P}_{\text{prim}}$ algebra of $A$, given in Definition 8.10, the operations which are defined using only the products $\bullet_0$ and $\bullet_L$ are the $n+2$-ary products $L_{n+1}^{\mid y \mid}(x_1, \ldots, x_n; y; z)$, for $n \geq 1$. Note that they do not verify any relationship, which leads us to the following definition.

8.7. Definition. A $\text{Mag}(\infty)$ algebra over $K$ is a vector space $M$, equipped with $n$-linear maps $\mu_n : M^{\otimes n} \rightarrow M$, for $n \geq 2$.

Let $(A, \cdot, \circ, \Delta)$ be a 2-ass algebra, define $\mu_n : A^{\otimes n} \rightarrow A$ be the operations defined as follows:

\[ \mu_2(x_1, x_2) := x_1 \cdot x_2 - x_1 \circ x_2, \]
\[ \mu_n(x_1, \ldots, x_n) := (x_1 \cdot (\cdots (x_{n-2} \cdot x_{n-1}))) \circ x_n - x_1 \cdot ((x_2 \cdot (\cdots (x_{n-2} \cdot x_{n-1}))) \circ x_n), \]

for $x_1, \ldots, x_n \in A$ and $n \geq 2$.

Clearly, $(A, \mu_n)$ is a $\text{Mag}(\infty)$ algebra.

8.8. Proposition. Let $(A, \cdot, \circ, \Delta)$ be a 2-infinitesimal nonunital bialgebra, the subspace $\text{Prim}(A)$ of primitive elements of $A$ is closed under the products $\mu_n$, for $n \geq 2$.

Proof. The result is a straightforward consequence of Proposition 8.11 it suffices to note that $\mu_n(x_1, \ldots, x_n)$ coincides with $L_{n+1}^{\mid y \mid}(x_1, \ldots, x_{n-2}; x_{n-1}; x_n)$, for $x \bullet_0 y := x \cdot y$ and $x \bullet_L y := y \circ x$.

For a set $E$, let 2-ass$(E)$ be the free 2-ass algebra spanned by $E$, let $M(E)$ be the subspace of 2-ass$(E)$ spanned by the elements of $E$ under the operations $\mu_n$ defined above, and let $T(M(E)) = \bigoplus_{n \geq 1} M(E)^{\otimes n}$ the tensor space over $M(E)$ equipped with the deconcatenation coproduct.

8.9. Proposition. Given a set $E$, the coalgebra 2-ass$(E)$ is isomorphic to $T(M(E))$.

Proof. It suffices to prove that any homogeneous element of degree $n$ of 2-ass$(E)$ belongs to $T(M(E))$ for all $n$. We proceed by induction on the degree of $n$. For $n = 1$ and $n = 2$ the result is immediate to check.

For a tree $t = \bigvee(t^1, \ldots, t^n) = t^1 \cdot \cdots \cdot t^n$ the result is immediate because, a recursive argument states that any $\tilde{t}^n$ belongs to $T(M(E))$.

Suppose then that $t = t^1 \otimes \cdots \otimes t^n$, since $|t^1 \otimes \cdots \otimes t^n| < |t|$ and $|t^1| < |t|$, we know that both elements belong to $T(M(E))$. So, we may restrict ourselves to prove that an element of the form $x \otimes (z^1 \cdots z^m)$ is in $T(M(E))$, for $x \in T(M(E))$ and $z^j \in M(E)$. 
Again, we proceed by a recursive argument on $m$. Note first that:

$$\mu_2(z^1, \mu_2(z^2, \ldots, \mu_2(z_{m-1}, z_m))) = z^1 \cdots z^m - z^1 \cdots z^{m-2} (z^{m-1} \circ z^m) - z^1 \cdots z^{m-3} (z^{m-2} \circ \mu_2(z_{m-1}, z_m)) - \cdots - z^1 \circ \mu_2(z^2, \ldots, \mu_2(z_{m-1}, z_m))),$$

But, using the recursive hypothesis and the arguments above, we get that:

$$x \otimes (z^1 \cdots z^k \cdot (z^{k+1} \circ \mu_2(z^{k+2}, \ldots, \mu_2(z_{m-1}, z_m))) \in T(M(E)),$$

for $0 \leq k < m - 2$, which implies that $x \otimes (z^1 \cdots z^m)$ is in $T(M(E))$. ◊

For any 2-infinite-simal nonunital bialgebra $(A, \cdot, \circ, \Delta)$, the triple $(A_+, \cdot, \Delta_+)$ is a unital infinite-simal bialgebra. So, we have that for any set $E$, the subspace $M(E)$ is the subspace of primitive elements of $2\text{-ass}(E)$. But, since as a vector space $2 - \text{ass}(E)$ is just $T(K[T(E)])$, the dimension of the homogeneous elements of degree $n$ of $\text{Prim}(2\text{-ass}(E))$ is just $C_{n-1}|E|^n$, where $C_{n-1}$ is the super Catalan number. So, for any finite set $E$, the dimension of $M(E)_n$ is $C_{n-1}|E|^n$.

8.10. Proposition. For any set $E$, the subspace of primitive elements of $2\text{-ass}(E)$, equipped with the $n$-ary products $\mu_n$, is the free Mag\((\infty)\) algebra over $E$.

Proof. It suffices to note that for any finite set $E$, the subspace of homogeneous elements of degree $n$ of the free Mag\((\infty)\) algebra spanned by $E$ is the number of planar rooted trees with $n$ leaves, with the leaves coloured by the elements of $E$, which is precisely $C_{n-1}|E|^n$ (for a more detailed study of this algebra we refer to [16]). ◊

As in the previous cases, given a Mag\((\infty)\) algebra $(M, \mu_n)$ we consider the free $2 - \text{ass}$ algebra spanned by $M$. We define the universal $2 - \text{ass}$ enveloping algebra $\mathcal{U}_{2\text{-ass}}(M)$ of $(M, \mu_n)$ as the quotient of $2\text{-ass}(M)$ by the ideal spanned by the elements of the form $\overline{\mu_n(x_1, \ldots, x_n)} - \mu_n(x_1, \ldots, x_n)$, for $x_1, \ldots, x_n \in M$; where $\overline{\mu_n}$ are the magmatic products defined on $2\text{-ass}(M)$.

The proof of a Cartier-Milnor-Moore type theorem for connected duplicial bialgebras follows as in previous cases.

8.11. Theorem. a) Let $(A, \cdot, \circ, \Delta)$ be a connected 2-infinite-simal nonunital bialgebra, then $A$ is isomorphic to $\mathcal{U}_{2\text{-ass}}(\text{Prim}(A))$, where Prim$(A)$ is the Mag\((\infty)\) algebra of primitive elements of $A$.

b) Let $(M, \cdot)$ be a Mag\((\infty)\) algebra, then $M$ is isomorphic to $\text{Prim}(\mathcal{U}_{2\text{-ass}}(M))$.

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