The finite and large-$N$ behaviors of independent-value matrix models

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Abstract

We investigate the finite and large $N$ behaviors of independent-value $O(N)$-invariant matrix models. These are models defined with matrix-type fields and with no gradient term in their action. They are generically nonrenormalizable but can be handled by nonperturbative techniques. We find that the functional of any $O(N)$ matrix trace invariant may be expressed in terms of an $O(N)$-invariant measure. Based on this result, we prove that, in the limit that all interaction coupling constants go to zero, any interacting theory is continuously connected to a pseudo-free theory. This theory differs radically from the familiar free theory consisting in putting the coupling constants to zero in the initial action. The proof is given for generic finite-size matrix models, whereas, in the limiting case $N \to \infty$, we succeed in showing this behavior for restricted types of actions using a particular scaling of the parameters.

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1 Introduction

Large-$N$ expansion techniques are useful in the study of systems with an infinite number of degrees of freedom in quantum field theory [1]. In the case of many-body systems, this method has led to the well known Hartree-Fock-type approximations [2]. The approach also proves to be crucial when dealing with nonrenormalizable models in order to localize the main source of divergences and to find a way to extract these, for instance, in an infinite lattice regularization scheme [3, 4, 5, 6]. The models under discussion here have the property of infinite divisibility (defined below), which, as is well known [7], exhibits both Gaussian and Poisson behavior. The former (Gaussian) behavior typically applies to free models, while interacting models typically involve the latter (Poisson) behavior.

The specific kind of problems we will be dealing with in this work are of the nonrenormalizable type. Such models can occur more often than one may think. Indeed, on one hand, the criteria for a theory to be renormalizable are specific and express a fine balance between several ingredients of the theory [8]. On the other hand, one agrees with the following fact: Consider a scalar field theory in a Euclidean $D$-dimensional space characterized by a functional integral (in standard notation using a time-ordered product)

$$Z(J) = K \int [D\phi] e^{i \int J(x)\phi(x)d^Dx - \int \left( \frac{1}{2} \partial^\mu \phi(x) \partial_\mu \phi(x) + \frac{1}{2} m^2 \phi(x)^2 + \lambda V(\phi(x)) \right) d^Dx},$$

where $K$ is a normalization, $J$ is a real source field—we have chosen an imaginary multiplier to make closer contact with the language of characteristic functions, i.e., Fourier transforms of probability distributions—$m$ is the mass, and $V$ is the interaction with coupling $\lambda$. If we discard the gradient term $\partial^\mu \phi(x) \partial_\mu \phi(x)$ altogether, we reach a theory with propagator $1/m^2$ hence without ultraviolet (UV) momentum damping, which will lead to arbitrarily many divergences in a perturbative study of the expression. When there is no gradient term in the action, the field is statistically independent at each point $x$ of spacetime and, consequently, no excitation can spread from one point to another. Following earlier practice [3], we call these models independent-value models. Such models with no damping in momentum, would be generically nonrenormalizable at the perturbative level.

We should emphasize that models with no gradient in the kinetic term are not normally encountered in relativistic quantum field theory because of the kinematic behavior they possess. However, there are several models in statistical mechanics, especially in matrix models [9], for which the kinetic term is exactly of the modified kind. Thus such models are certainly worthwhile to discuss.

Closer to the interest of the present work has been an investigation of the large-$N$ behavior of independent-value vector models [3, 4]. In these contributions, $O(N)$-invariant Euclidean actions defined over vectors $\phi = (\phi_k)_{k=1}^N$ have been studied at finite $N$ and in the limit $N = \infty$. Using nonperturbative techniques, the authors showed, both at finite and infinite $N$, that the solutions of any interacting theory do not reduce to those of the free theory in the limit where the interaction coupling constant goes to zero. Moreover, using a nontraditional...
asymptotic dependence of $N$ on the several parameters of the interacting theories, the limit $N \to \infty$ does not lead to a conventional free solution. This new solution, which arose by continuity when the interaction is reduced to zero, is called the pseudo-free theory, and when it is different from the free theory, it must be considered as the theory about which to expand the interacting theory. As a result, the existence and use of the pseudo-free theory is highly important in remedying certain pathological features of some otherwise puzzling models [5].

In this paper, we extend the results obtained for vector models to $O(N)$-invariant matrix models. Both finite and infinite size $N \times N$ matrices are addressed. The formulation is focused on symmetric matrices but, without much additional work, our results can be extended to any real or complex square matrices. We prove as well that the interacting matrix theories are continuously connected to a suitable pseudo-free matrix theory. Our solution rests on the existence of a model-dependent, invariant measure which needs to be determined to fully specify the solution. Indeed, we can find the explicit and nontrivial expression for this invariant measure in the case of a finite-size matrix, whereas for the infinite-size matrix case, we succeed to find a nontrivial measure under some restrictions regarding the type of invariant models when $N = \infty$.

The paper is organized as follows. The next section reviews the previous work [3] which is our guiding thread from the vector case toward the matrix case, which is studied in Section 3. In the last paragraph of that section, we comment on how our results extend to real matrices (the step from real matrices to complex ones would be straightforward). Theorem 1 is one of our main results in this paper. Given a model, Section 4 investigates the explicit invariant measure as stated in Theorem 1 which is another important result of this work. In Section 5 we summarize our results as well as include comments on future projects.

2 Independent-value vector models: a review

This section undertakes a review of the results obtained for vectors and is largely based on [3].

**Finite-component vectors.** Consider a real vector field $\phi(x) = \{\phi_k(x), k = 1, \ldots, N\}$, $x \in \mathbb{R}^n$, with $n \geq 1$ and a model described by the $O(N)$-invariant Euclidean action built from $\phi$:

$$S_\lambda[\phi] = \int_{\mathbb{R}^n} \left( \frac{1}{2} m^2 |\phi(x)|^2 + \lambda V(|\phi(x)|^2) \right) dx,$$

where $|\phi(x)|^2 \equiv \sum_k \phi_k(x)^2$. $S_{\lambda=0}$ corresponds to the free theory. We call these models independent-value vector models, which have been studied in [3, 4]. Such models with no damping in momentum, would be generically nonrenormalizable at the perturbative level as already discussed in the first section.

The functional integral with a source $J(x) = \{J_k(x), k = 1, \ldots, N\}$ in the Euclidean formulation, related to the action $S$, Eq. (2), is given by

$$Z(J, \lambda) = \mathcal{N} \int \prod_{x,k} d\phi_k(x) \ e^{i \int (J \cdot \phi)(x) dx - S_\lambda[\phi]}.$$
Due to the lack of gradients and the rotation invariance under $\phi \to O\phi$, where $O \in O(N)$, we can write

$$Z(J, \lambda) = e^{-\int L_\lambda(|J(x)|) dx},$$  \hspace{1cm} (4)$$

where $L_\lambda$ is an $O(N)$-invariant function of the source $J$. From its construction, the characteristic function $Z(J, \lambda)$ is infinitely divisible, i.e., $Z(J, \lambda)^{1/m}$ is also a characteristic function for all positive integers $m$ \cite{7}. We next exploit that very fact.

Let us assume that $J = P \chi_\Delta(x)$, where we separate the vector part $P = \{P_k\} \in \mathbb{R}^N$ and the background space via $\chi_\Delta(x)$ the indicator function of $\Delta$, a compact subset of $\mathbb{R}^n$ with volume $\Delta$; precisely, if $x \in \Delta$, then $\chi_\Delta(x) = 1$, otherwise $\chi_\Delta(x) = 0$. Then, we have

$$Z(J, \lambda) = e^{-\Delta L_\lambda(|P|)} = \int_{\mathbb{R}^N} \cos(P \cdot u) d\mu_{\Delta,L_\lambda}(u),$$  \hspace{1cm} (5)$$

for some $O(N)$-invariant probability measure $d\mu_{\Delta,L_\lambda}(u)$ on $\mathbb{R}^N$. Hence, it is straightforward to obtain

$$L_\lambda(|P|) = \lim_{\Delta \to 0} \Delta^{-1} \left( \int_{\mathbb{R}^N} [1 - \cos(P \cdot u)] d\mu_{\Delta,L_\lambda}(u) \right).$$  \hspace{1cm} (6)$$

The most general form for such a limit is given by \cite{7} as

$$L_\lambda(|P|) = a|P|^2 + \int_{|u|>0} [1 - \cos(P \cdot u)] d\sigma_\lambda(u),$$  \hspace{1cm} (7)$$

where $a \geq 0$ and $d\sigma_\lambda$ is an $O(N)$-invariant nonnegative measure subject to the condition that

$$\int_{|u|>0} [|u|^2/(1 + |u|^2)] d\sigma_\lambda(u) < \infty.$$  \hspace{1cm} (8)$$

Interestingly, we note that:

- the Gaussian cases, $a > 0$ and $\sigma_\lambda \equiv 0$, yields all free theories with different masses;
- the Poisson cases, $a \equiv 0$ and $\sigma_\lambda \neq 0$, cover all interacting theories (i.e., nonfree theories).

While it is possible to consider both terms being nonzero, local powers of the Gaussian field or of the Poisson field are made in very different ways, as we shall soon see, and thus, for quantum field applications where local products are important, it is necessary to consider the Gaussian and Poisson cases separately. Finally, for the Poisson cases, given an interacting theory with coupling $\lambda$, the remaining task is to find the measure $\sigma_\lambda$ associated with it.

**Infinite-component vectors.** Assuming that we are dealing with an infinite component vector field $\phi(x) = \{\phi_k(x)\}_{k=1}^\infty$, the above analysis extends in the following way. Eq. (4) holds still with $L(|J|)$ an even $O(\infty)$-invariant function. Using again particular fields $J = P \chi_\Delta$, we can formulate (5) as

$$\int \cos(P \cdot u) d\mu_{\Delta,L_\lambda}(u) = \int_0^\infty e^{-b|P|^2} d\mu_{\Delta,L_\lambda}(b),$$  \hspace{1cm} (9)$$

where use has been made of the fact that every characteristic function with $O(\infty)$ invariance is a convex combination of Gaussians \cite{10}.

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We derive from (9) that

\[ L_\lambda(|P|) = \lim_{\Delta \to 0} \Delta^{-1} \left( \int_0^\infty [1 - e^{-b|P|^2}] d\mu_{\Delta,L_\lambda}(b) \right), \tag{10} \]

with the most general form given by

\[ L_\lambda(|P|) = a|P|^2 + \int_0^\infty [1 - e^{-b|P|^2}] d\sigma_\lambda(b), \tag{11} \]

where \(a \geq 0\), \(d\sigma_\lambda\) is a nonnegative measure on \((0, \infty)\) obeying the condition

\[ \int_0^\infty [b/(1 + b)] d\sigma_\lambda(b) < \infty. \tag{12} \]

The same remarks associated with free theories characterized by \(a > 0\) and \(\sigma_\lambda \equiv 0\) hold here. These provide \(N = \infty\) free solutions for vector models. Meanwhile, \(a \equiv 0\) and \(\sigma_\lambda \neq 0\) covers interacting theories.

In [3], as an illustration, for the model (2) and under suitable conditions, one can establish that

\[ N < \infty, \quad d\sigma_\lambda(u) = C e^{-\frac{1}{2}m^2 u^2 - \lambda V(u)} \frac{1}{|u|^N} d\vec{u}, \]

\[ N = \infty, \quad d\sigma_\lambda(b) = C e^{-\frac{1}{2}m^2 b - \lambda V(b)} \frac{1}{b} db, \tag{13} \]

for a suitable value of \(C\) in each case.

### 3 Independent-value matrix models

We now investigate analogs of the statements in Section 2 for real matrix models. The simple way to address this is to consider real symmetric matrices \(M(x) = \{M_{ab}(x) = M_{ba}(x)\}_{a,b=1,...,N}\), where \(x \in \mathbb{R}^n\). There exists an extension of the following discussion for non-symmetric matrices and even complex ones, however the basic ideas would remain the same. We will come back to this point later on.

Consider now an \(O(N)\)-invariant action which can be written as

\[ S_{\tilde{\lambda}}[M] = \int_{\mathbb{R}^n} \left[ \frac{1}{2} \mu^2 \text{tr}[M^2(x)] + V_{\tilde{\lambda}} \left( \{\text{tr}[ (M(x))^{2p} ] \}_p \right) \right] dx, \tag{14} \]

where \(\mu\) is the mass parameter and \(V_{\tilde{\lambda}}\) is an \(O(N)\)-invariant function of some of the invariants \(\text{tr}[(M(x))^{2p}]\), for \(p\) an integer such that \(2 \leq p \leq p_{\text{max}}\), and we can consider \(p_{\text{max}}\) finite or infinite. Furthermore, \(\tilde{\lambda} = (\lambda_2, \ldots, \lambda_q)\) which simply collects all coupling constants depending on the number of terms involved in \(V_{\lambda}\). A typical situation would be to consider an interaction of the form

\[ V_{\tilde{\lambda}} \left( \{\text{tr}[M^{2p}] \}_p \right) = \sum_{p=2}^{p_{\text{max}}} \lambda_p V_p(\text{tr}(M^{2p})) \tag{15} \]
where \( \vec{\lambda} = (\lambda_2, \ldots, \lambda_{p_{\max}}) \). Note that the expression \( \det[M(x)] \) and its powers \( \det[M^p(x)] = (\det[M(x)])^p \) are also \( O(N) \)-invariant functions that could be considered for the interaction term. For reasons of simplicity, we do not include such terms.

One notices that again the model is independently distributed at each spacetime point \( x \). Hence, we call these independent-value matrix models and write the corresponding characteristic functional integral for such a model as

\[
Z(J, \vec{\lambda}) = K \int \left[ \prod_{x;ab} dM_{ab}(x) \right] e^{i \int [\text{tr}(J(x)M(x))] dx - S_{\vec{\lambda}}[M]},
\]

where \( K \) is a normalization factor and \( J \) can be chosen as a symmetric matrix.

The following statement holds.

**Proposition 1**

\[
Z(J, \vec{\lambda}) = e^{-\int L_{\vec{\lambda}}[\{\text{tr}(J(x)^{2p})\}] dx},
\]

where \( L_{\vec{\lambda}} \) is a \( O(N) \)-invariant function of all possible invariants in \( \{\text{tr}(J(x)^{2p})\} \).

**Proof.** We start by slicing \( \mathbb{R}^n \) in \( \{\Delta_\ell\}_\ell \) where each \( \Delta_\ell \subset \mathbb{R}^n \) has a fixed finite volume \( \Delta \) and an indicator function \( \chi_{\Delta_\ell} \). The index \( \ell \) depends on the slicing but should typically run over an infinite discrete set because \( \mathbb{R}^n \) is non-compact. Then, we write the matrix field \( M(x) = \sum_\ell M_\ell(x) \), where \( M_\ell = M_\ell \chi_{\Delta_\ell} \), and \( M_\ell = (M_{\ell,ab}) \) is a symmetric matrix.

Similarly we introduce \( J(x) = \sum_\ell J_\ell(x) \) and \( J_\ell = J_\ell \chi_{\Delta_\ell} \), where \( J_\ell \) is a symmetric matrix as well. The functional integral can be re-expressed as

\[
Z(J, \vec{\lambda}) = \lim_{\Delta \to 0} \int \left[ \prod_{\ell,a,b} dM_{\ell,ab} \right] e^{\Delta [i \text{tr}(J_\ell M_\ell) - S_{\ell,\vec{\lambda}}[M_\ell]]} = \lim_{\Delta \to 0} \prod_\ell \int \left[ \prod_{a,b} dM_{ab} \right] e^{\Delta [i \text{tr}(J_\ell M_\ell) - S_{\ell,\vec{\lambda}}[M_\ell]]}.
\]

To rely on a well-defined continuum limit with \( O(N) \)-invariance of the matrix \( J \) (as \( Z \) in (16) could be), the infinite product must be of the form

\[
Z(J, \vec{\lambda}) = \lim_{\Delta \to 0} \prod_\ell [1 - \Delta C_{\Delta_\ell,\vec{\lambda}}[\{\text{tr}(J_\ell^2)\}] = \lim_{\Delta \to 0} e^{-\Delta \sum_\ell L_{\ell,\vec{\lambda}}[\{\text{tr}(J_\ell^2)\}] = e^{-\int L_{\vec{\lambda}}[\{\text{tr}(J(x)^2)\}] dx} \]

where \( C_{\Delta,\vec{\lambda}}[\{\text{tr}(J_\ell^2)\}] \) is so defined such that \( L_{\ell,\vec{\lambda}}[\{\text{tr}(J_\ell^2)\}] \) is independent of \( \Delta \).

Assuming that \( J = J \chi_{\Delta}, \Delta \subset \mathbb{R}^n \) with volume \( \Delta \), we simply rewrite

\[
Z(J, \vec{\lambda}) = e^{-\Delta L_{\vec{\lambda}}[\{\text{tr}(J_\ell^2)\}] = e^{-\int L_{\vec{\lambda}}[\{\text{tr}(J(x)^2)\}] dx} \]

On the other hand, a direct reduction from (16) leads to

\[
Z(J, \vec{\lambda}) = e^{-\Delta L_{\vec{\lambda}}[\{\text{tr}(J_\ell^2)\}] = \int_{S_N} e^{i \text{tr}(JU)} d\mu_{\Delta,\vec{\lambda}}(U),
\]

where \( U \) is a real symmetric matrix and \( d\mu_{\Delta,\vec{\lambda}}(U) \) is an invariant probability measure on the space \( S_N \) of real \( N \times N \) symmetric matrices. Note that we do not need to integrate over the entire space of matrices \( M_N(\mathbb{R}) \), because \( J \) is symmetric and the trace will necessarily yield a reduction on \( S_N \) provided that the measure \( \nu(\cdot) \) on \( M_N(\mathbb{R}) \) is factorized so that \( d\nu(\cdot) = d\nu'(\cdot) d\nu''(\cdot) \), where \( d\nu'' \) is a probability measure on the space of anti-symmetric matrices.
Theorem 1 Let $J$ be a symmetric matrix of order $N$ (finite or infinite) and let $d\mu_{\Delta,\lambda}$ be an invariant probability measure on the set of real symmetric matrices of order $N$ such that the following limit converges

$$\lim_{\Delta \to 0} \Delta^{-1} \int_{\text{tr}(U^2) > \Delta} f(U) d\mu_{\Delta,\lambda}(U) = \int_{\text{tr}(U^2) > 0} f(U) d\sigma_{\lambda}(U),$$

for suitable functions $f$ on $S_N$, which are independent of $\Delta$. Then any invariant $O(N)$-invariant function built over $J$ with some couplings $\lambda = \{\lambda_i\}_i$ satisfies

$$L_{N,\lambda}(\{\text{tr}[J^{2p}]\}_p) = a \text{tr}[J^2] + \int_{\text{tr}(U^2) > 0} (1 - e^{i \text{tr}(J U)}) d\sigma_{\lambda}(U),$$

where $a \geq 0$ and $d\sigma_{\lambda}(U)$ is a nonnegative invariant measure over $S_N$ such that

$$\int_{\text{tr}(U^2) > 0} \text{tr}(U^2)/(1 + \text{tr}(U^2)) d\sigma_{\lambda}(U) < \infty.$$

In the case of infinite size symmetric matrices $J$, or $N \to \infty$ limit, the corresponding $O(\infty)$-invariant function reads

$$L_{\infty,\lambda}(\{\text{tr}[J^{2p}]\}_p) = a \text{tr}[J^2] + \int_0^\infty (1 - e^{-b \text{tr}[J^2]}) d\sigma_{\lambda}(b),$$

where $a \geq 0$ and $d\sigma_{\lambda}(b)$ is a non negative measure over $\mathbb{R}$ such that

$$\int_0^\infty b/(1 + b) d\sigma_{\lambda}(b) < \infty.$$

Proof. From (21), we have

$$L_{N,\lambda}(\{\text{tr}[J^{2p}]\}_p) = \lim_{\Delta \to 0} \Delta^{-1} \left( \int_{S_N} (1 - e^{i \text{tr}(J U)}) d\mu_{\Delta,\lambda}(U) \right).$$

Case $N$ finite. We start by decomposing $\int_{S_N}$ as $\int_{\text{tr}(U^2) > \Delta} + \int_{\text{tr}(U^2) \leq \Delta}$ and evaluate

$$A_{\leq \Delta} = \int_{\text{tr}(U^2) \leq \Delta} (1 - e^{i \text{tr}(J U)}) d\mu_{\Delta,\lambda}(U) = -\sum_{k=1}^\infty \frac{i^{2k}}{(2k)!} \int_{\text{tr}(\tilde{U}^2) \leq \Delta} [\text{tr}(j\tilde{U})]^{2k} d\mu_{\Delta,\lambda}(\tilde{U}),$$

where we used the fact that the measure is invariant in order to cancel all odd powers in $\text{tr}(J U)$, then we diagonalize $J = O j O^t$ (with $O \in O(N)$ and $j$ diagonal) and introduce $\tilde{U} = O^t U O$ such that $d\mu_{\Delta,\lambda}(\tilde{U}) = d\mu_{\Delta,\lambda}(U)$. We concentrate on the first two terms of the series, which are

$$\text{tr}(j\tilde{U}) = \sum_{a=1}^N j_{aa} \tilde{U}_{aa}, \quad [\text{tr}(j\tilde{U})]^2 = \sum_{a=1}^N j_{aa}^2 \tilde{U}_{aa}^2 + \sum_{a \neq b} j_{aa} \tilde{U}_{aa} j_{bb} \tilde{U}_{bb}. $$

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The first term and the very last sum \( \sum_{a \neq b} \) including cross terms vanish because of the \( O(N) \)-symmetry. The next relevant term of the series is

\[
\text{tr}[(j\tilde{U})^4] = \sum_a j_{aa}^4 \tilde{U}_{aa}^4 + 6 \sum_{a \neq b} j_{aa}^2 \tilde{U}_{aa}^2 \tilde{U}_{bb}^2 \tilde{U}_{bb}^2 + \ldots ,
\]

(30)

where the dots include cross terms of odd power which should vanish by symmetry as well. Notice that the \( 2p \)-order term \( \text{tr}(j\tilde{U})^{2p} \) in the expansion, cumulates to a power of \( 2p \) in the variable \( \tilde{U}_{aa} \). For simplicity, we write \( \text{tr}(j\tilde{U})^{2p} = O(\tilde{U}^{2p}) \). We then re-express (28) as

\[
A_{\leq \Delta} = \frac{1}{2} \sum_{a=1}^{N} j_{aa}^2 \int_{\text{tr}(U^2) \leq \Delta} \tilde{U}_{aa}^2 d\mu_{\Delta, \bar{\chi}}(U)
\]

\[
\quad - \frac{j^4}{4!} \left( \sum_{a=1}^{N} j_{aa}^4 \int_{\text{tr}(U^2) \leq \Delta} \tilde{U}_{aa}^4 d\mu_{\Delta, \bar{\chi}}(U) + \sum_{a \neq b} j_{aa}^2 j_{bb}^2 \int_{\text{tr}(U^2) \leq \Delta} \tilde{U}_{aa}^2 \tilde{U}_{bb}^2 d\mu_{\Delta, \bar{\chi}}(U) \right)
\]

\[
\quad + \int_{\text{tr}(U^2) \leq \Delta} O(\tilde{U}^6) d\mu_{\Delta, \bar{\chi}}(U)
\]

\[
= \frac{\Delta}{2} \sum_{a=1}^{N} j_{aa}^2 \int_{\text{tr}(U'^2) \leq 1} \frac{\text{tr}(U'^2)}{N} d\mu_{\Delta, \bar{\chi}}(\sqrt{\Delta} U') + \Delta^2 \int_{\text{tr}(U'^2) \leq 1} O(U'^4) d\mu_{\Delta, \bar{\chi}}(\sqrt{\Delta} U')
\]

\[
+ \Delta^3 \int_{\text{tr}(U'^2) \leq 1} O(U'^6) d\mu_{\Delta, \bar{\chi}}(\sqrt{\Delta} U') ,
\]

(31)

where we used the fact that \( \langle \tilde{U}_{aa}^2 \rangle = \langle \tilde{U}^2 \rangle / N \) and then changed variables so that \( U' = \tilde{U} / \sqrt{\Delta} \).

We now choose \( a = \langle \text{tr}(U^2) \rangle / N \geq 0 \). Note that the measure should be chosen such that \( a \) neither depends on \( N \) nor on \( \Delta \). We can now substitute (31) in (27), then using (22), it is direct to recover (23). The condition (24) ensures the finiteness of the integral, namely:

\[
\int_{\text{tr}(U'^2) > 0} (1 - e^{\text{tr}(J U)}) d\sigma_{\bar{\chi}} < \infty \iff \int_{\text{tr}(U^2) > 0} \text{tr}(U'^2) / (1 + \text{tr}(U^2)) d\sigma_{\bar{\chi}}(U) < \infty .
\]

(32)

**Case** \( N = \infty \). Let us turn now to the case \( N = \infty \) and show (24). The expression (27) is again valid. We use the eigenvalue decomposition of \( J = O \hat{J} O^t \), change variable \( U \to \hat{U} = O \hat{O} U \) and write \( \text{tr}(J \hat{U}) = \sum_a j_{aa} \hat{U}_{aa} \) so that, introducing \( r^2 = \text{tr}(\hat{U}^2) = \sum_{ab} \hat{U}_{ab}^2 \), we set

\[
\hat{U}_{11} = r \cos \theta_1 , \quad \text{and for } a \geq 2 , \quad \hat{U}_{aa} = r \left( \prod_{l=1}^{a-1} \sin \theta_l \right) \cos \theta_a ,
\]

(33)

and, with \( N^* = N(N + 1) / 2 \), we re-express (24) in spherical coordinates as

\[
\lim_{N \to \infty} \int_{S_N} e^{i \text{tr}(J U)} d\mu_{\Delta, N, \bar{\chi}}(U) =
\]

\[
\lim_{N \to \infty} \int_{S_N} e^{i r \sum_a j_{aa} \left( \prod_{l=1}^{a-1} \sin \theta_l \right) \cos \theta_a} \omega_{\Delta, N, \bar{\chi}}(r) r^{N^*-1} dr \left[ \prod_{l=1}^{N} (\sin \theta_l)^{N^*-l+1} d\theta_l \right] d\Omega_{N^*-N+2}
\]

(34)
At large $N$, $N^* - (l + 1) \geq N^2 - (N + 1) \sim N^2$ is large too, for all $l = 1, \ldots, N$. Again, using a steepest-descent technique, one obtains, for each $\theta_l$ variable, a saddle point $\theta_{l,*} = \pi/2$. Then, we change variable $\Theta_l = \theta_l - \theta_{l,*}$ such that

$$
\lim_{N \to \infty} \int_{[-\pi/2, \pi/2]^N} e^{tr \sum_a \hat{J} \hat{T}_a \Theta_a e^{-\sum_a (N^* - (a+1)) \Theta_a^2}} \prod_l d\Theta_l = \lim_{N \to \infty} k_N e^{-\sum_a \frac{1}{2(N^* - (a+1))} \hat{J}^2 a^2 r^2} 
$$

for some constant $k_N$. Substituting this result in (34) and considering that the integration with $d\Omega_{N^* - (N+2)}$ contributes to most to an overall factor, we get at large $N$

$$
\int S_N e^{i \text{tr}(\mathcal{J}U)} d\mu_{N^*, \bar{\lambda}}(U) \simeq K_N \int_0^\infty e^{-\sum_{a} \frac{1}{2(N^* - (a+1))} \hat{J}^2 a^2 r^2} w_{N^*, \bar{\lambda}}(r) r^{N^*-1} dr ,
$$

where $K_N$ includes all constants depending on $N$. We change variable as $r \to \sqrt{2bN^*}$ such that, after taking the limit $N \to \infty$, and provided that $d\mu_{N^*, \bar{\lambda}}$ is chosen in a class of probability measures such that this limit converges to a nonnegative invariant measure $d\mu_{\Delta, \bar{\lambda}}$, one obtains

$$
\lim_{N \to \infty} \int S_N e^{i \text{tr}(\mathcal{J}U)} d\mu_{N, \bar{\lambda}}(U) = \int_0^\infty e^{-b \text{tr}(\mathcal{J}^2)} d\mu_{\Delta, \bar{\lambda}}(b) .
$$

Thus (27) becomes,

$$
L_{\Delta, \bar{\lambda}}(\{\text{tr}[\mathcal{J}^2]\p\}) = \lim_{\Delta \to 0} \Delta^{-1} \left( \int_0^\infty (1 - e^{-b \text{tr}(\mathcal{J}^2)}) d\mu_{\Delta, \bar{\lambda}}(b) \right) ,
$$

where $S_{\Delta}$ is the set of symmetric matrices with infinite size. The rest is very similar to the finite case. Decomposing $\int S_{\Delta}$ as $\int_{b>\Delta} + \int_{b<\Delta}$ and expand the sector $b \leq \Delta$ as

$$
A_{\leq \Delta}^\infty = \int_{b<\Delta} (1 - e^{-b \text{tr}(\mathcal{J}^2)}) d\mu_{\Delta, \bar{\lambda}}(b) = -\sum_{k=1}^\infty \frac{(-1)^k}{k!} [\text{tr}(\mathcal{J}^2)]^k \int_{b<\Delta} b^k d\mu_{\Delta, \bar{\lambda}}(b)
$$

$$
= [\text{tr}(\mathcal{J}^2)] \int_{b<\Delta} b d\mu_{\Delta, \bar{\lambda}}(b) + \int_{b<\Delta} O(b^2) \ d\mu_{\Delta, \bar{\lambda}}(b) ,
$$

where $O(b^2)$ includes all remaining terms in the expansion. We perform a change of variables such that $b \to b' = b/\Delta$, and obtain

$$
A_{\leq \Delta}^\infty = \Delta \text{tr}(\mathcal{J}^2) \int_{b'\leq 1} b' d\mu_{\Delta, \bar{\lambda}}(\Delta b') + \Delta^2 \int_{b'\leq 1} O(b^2) \ d\mu_{\Delta, \bar{\lambda}}(\Delta b') .
$$

Re-injecting (40) in (38), recalling that the limit (22) holds, one can easily identify an $a$ and finally reach the result (25). The condition (26) is equivalent to the convergence of the term $\int_0^\infty (1 - e^{-b \text{tr}(\mathcal{J}^2)}) d\sigma_{\bar{\lambda}}(b) < \infty$. This ends the proof of the theorem.

Note that the order in which we have taken the limits $\lim_{\Delta \to 0}$ and $\lim_{N \to \infty}$ does not matter for obtaining $L_{\Delta, \bar{\lambda}}$, namely

$$
L_{\Delta, \bar{\lambda}} = \lim_{\Delta \to 0} \lim_{N \to \infty} \mathcal{F}_{N, \Delta} = \lim_{N \to \infty} \lim_{\Delta \to 0} \mathcal{F}_{N, \Delta} = \lim_{N \to \infty} L_{N, \bar{\lambda}} ,
$$

(41)
where $\mathcal{F}_{N,\Delta} = \Delta^{-1} \int_{\mathcal{S}_N} (1 - e^{i \text{tr}(\mathcal{J}U)}) d\mu_{\Delta,\lambda}(U)$. This displays the fact that the function $L_{\infty,\lambda}$ is continuously connected to $L_{N,\lambda}$.

**Case of a non-symmetric matrix.** Consider $M$ the initial matrix field now to be real but non symmetric. The source function $J$ should be non symmetric and real as well. In any case, we use a singular value decomposition for matrix part of $J$ as $\mathcal{J} = V_1 \Sigma V_2^\dagger$ where $\Sigma$ is a real nonnegative entry matrix, and where $V_{1,2} \in O(N)$. But $\Sigma$ contains in fact the non-negative square root of the eigenvalues of $\mathcal{J}^t \mathcal{J}$ or $\mathcal{J} \mathcal{J}^t$, that is

$\text{tr}(\Sigma^2) = \text{tr}(\mathcal{J}^t \mathcal{J})$ (42)

which allows to perform all the above analysis. One recovers the analog of (23) for a generic matrix $\mathcal{J}$ as

$L_{N,\lambda}(\{\text{tr}[\mathcal{J}^t \mathcal{J}]^p\}_p) = a \text{tr}[\mathcal{J}^t \mathcal{J}] + \int_{\text{tr}(U^t U) > 0} (1 - e^{i \text{tr}(\mathcal{J}U)}) d\sigma_\lambda(U), \quad (43)$

where $a \geq 0$ and $d\sigma_\lambda(U)$ is a non negative invariant measure over the space of square matrices such that

$\int_{\mathcal{M}_N} \text{tr}(U^t U)/(1 + \text{tr}(U^t U)) d\sigma_\lambda(U) < \infty. \quad (44)$

A similar analysis works for the case $N = \infty$. The formulation for complex matrices can be inferred in the same manner.

### 4 Applications: Finding the measure

Although we can address the case of non symmetric matrices, for simplicity, we will perform the analysis for only symmetric matrices. The aim of this section is to provide an explicit formula for the measures $d\sigma_\lambda(U)$ and $d\sigma_\lambda(b)$ such that one may infer the other possible terms that one can introduce in the initial model.

#### 4.1 Finite size matrices

In the previous section, we have establish that, for independent value matrix models,

$Z(J, \lambda) = e^{- \int L_{\lambda}[\{\text{tr}(J(x)^2p)\}_p] dx}, \quad (45)$

where $L_{\lambda}[\{\text{tr}(J(x)^2p)\}_p]$ is of the most general form given by the $O(N)$-invariant quantity

$L_{N,\lambda}(\{\text{tr}[\mathcal{J}^2p(x)]\}_p) = a \text{tr}[\mathcal{J}^2(x)] + \int_{\text{tr}(U^2) > 0} (1 - e^{i \text{tr}(\mathcal{J}U)}) d\sigma_\lambda(U), \quad (46)$

with $d\sigma_\lambda(U)$ an $O(N)$-invariant nonnegative measure over $\mathcal{S}_N$. In the remaining, the analysis will be restricted to the generic situation such that $d\sigma_\lambda(U) = C^2(U) dU$. Note that implicitly $C^2(U)$ should be a function of the basic invariants $\{\text{tr}[U^{2p}(x)]\}_p$ and $\lambda$. We are also interested only in the case $a = 0$, hence in an interacting theory.
The use of \( C(U) \) gives us a useful representation of the field as follows. Let \( A(x, U) \) and \( A^\dagger(x, U) \) be the annihilation and creation operators in the ordinary sense such that they satisfy the commutation relation

\[
[A(x, U), A^\dagger(y, V)] = \delta_{\mathbb{R}^n}(x - y)\delta_{\mathcal{S}_N}(U - V),
\]

with a self-explanatory notation. We assume that there is a vacuum state \( |0\rangle \) so that

\[
A(x, U)|0\rangle = 0.
\]

Out of these initial operators, we built two new ones, namely

\[
B(x, U) = A(x, U) + C(U), \quad B^\dagger(x, U) = A^\dagger(x, U) + C(U),
\]

obeying the same relation \([47]\) and \( B(x, U)|0\rangle = C(U)|0\rangle \). From the \( B \)'s, we express the matrix field operator as

\[
M(x) = \int_{\mathcal{S}_N} B^\dagger(x, U)B(x, U)\,dU, \quad M_{ab}(x) = \int_{\mathcal{S}_N} B^\dagger(x, U)U_{ab}B(x, U)\,dU.
\]

As we will soon learn,

\[
\left\langle 0 \left| e^{i \int \text{tr}[J(x) \cdot M(x)]\,dx} \right| 0 \right\rangle = e^{-\int \left[ f[1-e^{\text{tr}[J(x) \cdot U]}]C^2(U)dU \right]}\,dx,
\]

and our task is to determine the connection between \( C^2(U) \) and the model action functional. The bilinear representation of our basic operators means that local products arise from an operator product expansion and not by Wick ordering. As a consequence, we have

\[
M_{ab}(x)M_{cd}(y) = \int_{\mathcal{S}_N \times \mathcal{S}_N} B^\dagger(x, U)B^\dagger(y, V)U_{ab}V_{cd}B(x, U)B(y, V)\,dU\,dV
\]

\[
+ \delta_{\mathbb{R}^n}(x - y)\int_{\mathcal{S}_N} B^\dagger(x, U)U_{ab}U_{cd}B(y, U)\,dU,
\]

\[
= :M_{ab}(x)M_{cd}(y): + \delta_{\mathbb{R}^n}(x - y)\int_{\mathcal{S}_N} B^\dagger(x, U)U_{ab}U_{cd}B(y, U)\,dU.
\]

We define (\( R \) stands for “renormalized”)

\[
M_{R,abcd}^2 = \beta \int_{\mathcal{S}_N} B^\dagger(x, U)U_{ab}U_{cd}B(y, U)\,dU,
\]

where \( \beta \) has the dimension of \( L^{-n} = \text{dimension of} \ \delta_{\mathbb{R}^n}(0) \). For simplicity, hereafter, we choose the numerical value \( \beta = 1 \). Higher order products can be computed as well and, using the same prescription, will lead to \( M_{R,a_1b_1...a_nb_n}^n \).

We are now in position to seek a relationship between \( C^2(U) \) and the model. Consider the action of a model given by \([44]\), \( C^2(U) \) is such that

\[
\left\langle 0 \left| e^{i \int \text{tr}[J(x) \cdot M(x)]\,dx} \right| 0 \right\rangle = K \int e^{\int \left[ i\text{tr}[J(x)M(x)] - \left( \frac{1}{2}\mu^2 \text{tr}[M^2(x)] + V_x(\text{tr}[M^2(x)]) \right) \right]}\,dM
\]

\[\]

\[10\]
with $K$ a constant normalization factor chosen so that the entire expression reduces to unity if $J \equiv 0$. Modifying the left hand side of the above equation, where $W$ is chosen so the left side is unity when $J \equiv 0$, we consider

$$W \left< 0 \left| e \int \left( i \text{tr} [J(x) - M(x)] \chi_{\Delta}(x) V'(|\text{tr}[M^{2p}(x)]|) \right) dx \right| 0 \right>,$$

in which the interaction, in the right hand side of (54), picks a term of the form $\chi_{\Delta} V'(|\text{tr}[M^{2p}(x)]|)$ and $\chi_{\Delta}(x)$ keeps its previous meaning as an indicator function. We can determine as well how $C^2(U)$ gets modified under such a transformation. Using the $R$-product prescription for all local products, one evaluates

$$W \left< 0 \left| e \int \left( i \text{tr} [J(x) - M(x)] - \chi_{\Delta}(x) V'(|\text{tr}[M^{2p}(x)]|) \right) dx \right| 0 \right> =$$

$$W \left< 0 \left| e \left\{ \int_S B(x,U) \left[ i \text{tr} [J(x) - M(x)] - \chi_{\Delta}(x) V'(|\text{tr}[M^{2p}(x)]|) \right] B(x,U) dU \right\} dx \right| 0 \right> =$$

$$W \left< 0 \left| e \left\{ \int_S B(x,U) \left[ i \text{tr} [J(x) - M(x)] - \chi_{\Delta}(x) V'(|\text{tr}[M^{2p}(x)]|) \right] B(x,U) dU \right\} dx \right| 0 \right> =$$

$$e^{- \int dx \left[ \int \left( 1 - e^{i \text{tr} [J(x, U)]} - e^{-\chi_{\Delta} V'(|\text{tr}[M^{2p}(x)]|)} \right) C^2(U) dU \right]}.$$

(56)



(note the change in the line (56) for $R$-product which gives sense to the local products. To obtain the final form of this equation, there is an intermediate step in the above calculation which can be explained by a simple exercise using canonical (for the harmonic oscillator for instance) coherent states expectation values (in the usual notation such that $[a, a^\dagger] = 1$, $b_n = 1/\sqrt{n!}$)

$$\langle z | e^{i \alpha a} | z \rangle = \langle z | e^{i \alpha z} \rangle = e^{-|z|^2} \sum_n \langle n | b_n z^n b_m (e^{i \alpha z})^m | m \rangle = e^{z^* (e^{i \alpha} - 1) z} = \langle z | e^{a^\dagger (e^{i \alpha} - 1) a} | z \rangle,$$

(58)

using the fact that $a | z \rangle = | z \rangle$; incidentally, an extension of this argument may be used to verify (51). Now, coming back to (57), we let $\Delta_* \to \infty$ expand to cover all $\mathbb{R}^n$, and thus

$$\lim_{\Delta \to \infty} W e^{- \int dx \left[ \int \left( 1 - e^{i \text{tr} [J(x, U)]} - e^{-\chi_{\Delta} V'(|\text{tr}[M^{2p}(x)]|)} \right) C^2(U) dU \right]} =$$

$$\lim_{\Delta \to \infty} W e^{- \int dx \left[ \int \left( e^{\chi_{\Delta} V'(|\text{tr}[M^{2p}(x)]|)} - e^{i \text{tr} [J(x, U)]} \right) e^{-V'(|\text{tr}[M^{2p}(x)]|) C^2(U) dU \right]} =$$

$$e^{- \int dx \left[ \int \left( 1 - \text{tr} [J(x, U)] \right) e^{-V'(|\text{tr}[M^{2p}(x)]|) C^2(U) dU \right]}.$$

(59)

for the right normalization factor $W$, which is simply the denominator in (57). Therefore, we find the effect on the measure on the matrix space given by

$$S \to S + \int V'(|\text{tr}[M^{2p}(x)]|) dx, \quad C^2(U) \to e^{-V'(|\text{tr}[M^{2p}]|) C^2(U)}.$$
As a special choice, the potential $V'(\{\text{tr}[M^{2p}(x)]\}) = -V(\{\text{tr}[M^{2p}(x)]\})$, simply cancels the original nonlinear interaction leaving only the mass term such that

$$Z_{PF}(J, \mu) = K \int e^{\int \left[ \text{tr}[\mathcal{J}(x)\cdot M(x)] - \frac{\beta}{2} \mu^2 \text{tr}[M^2_R(x)] \right] dx} dM. \quad (61)$$

As expected, this is not the characteristic functional of the free-theory because of the $R$-multiplication prescription on the “quadratic term” $\text{tr}[M^2_R(x)]$. Instead, (61) represents the pseudo-free matrix theory, i.e., the model continuously connected to the interacting theories. In order to characterize the pseudo-free model, let us investigate a special change in the matrix measure. From the same procedure used above, we obtain

$$Z_{PF}(J, \mu) = e^{-\int \left[ \{1 - e^{\text{tr}[\mathcal{J}(x) \cdot U]}\} - \frac{1}{2} \mu^2 \text{tr}[U^2] \right] C_0^2(U) dU} \quad (62)$$

Note that, for any constant $\alpha$, from (61) one learns that

$$Z_{PF}(\alpha J, \alpha \mu) = Z_{PF}(J, \mu). \quad (63)$$

Performing the same scale transformation, using now (62) leads to

$$Z_{PF}(\alpha J, \alpha \mu) = e^{-\int \left[ \{1 - e^{\text{tr}[\mathcal{J}(x) \cdot (\alpha U)]}\} - \frac{1}{2} \mu^2 \text{tr}[\alpha^2 U^2] \right] C_0^2(\alpha U) d(\alpha U)} \quad (64)$$

which can be scale invariant if and only if

$$C_0^2(\alpha U) d(\alpha U) = C_0^2(U) dU, \quad \text{hence} \quad C_0^2(\alpha) = \Gamma |\alpha|^{-N^*}, \quad (65)$$

for some constant $\Gamma$ and, as before, $N^* = N(N + 1)/2$. One should keep in mind that we assume that there exists a positive integer $p_{\text{max}}$ such that $C_0^2(U) = C_0^2(\{\text{tr}[U^{2p}]\}_{p=1}^{p_{\text{max}}})$. At this point, (63) implies that

$$C_0^2(U) = C_0^2(\{\frac{\text{tr}[U^{2p}]}{\alpha^{2p}}\}_{p=1}^{p_{\text{max}}}). \quad (66)$$

Given an even integer $q \in \mathbb{N}$, we introduce the set $\mathfrak{P}_q$ of nontrivial partitions $q$ and the set $\mathfrak{P}^*_{q}$ of nontrivial partition of $N^* + q$, such that

$$\mathcal{P}_{q,I} \in \mathfrak{P}_q, \quad 1 \leq I \leq q, \quad \mathcal{P}_{q,I} = (N_1, \ldots, N_I),$$

$$q = \sum_{i \in I} 2p_i N_i, \quad 1 \leq N_i \leq q, \quad 1 \leq p_i \leq p_{\text{max}},$$

$$\mathcal{P}_{q,I}^* \in \mathfrak{P}^*_{q}, \quad 1 \leq I \leq N^* + q, \quad \mathcal{P}_{q,I}^* = (N_1, \ldots, N_I),$$

$$N^* + q = \sum_{i \in I} 2p_i N_i, \quad 1 \leq N_i \leq N^* + q, \quad 1 \leq p_i \leq p_{\text{max}}. \quad (67)$$

Then, one writes a general solution for (66) as

$$C_0^2(\{\text{tr}[U^{2p}]\}_{p=1}^{p_{\text{max}}}) = \sum_{q=0}^{q_{\text{max}}} \sum_{q \text{ even, } \mathfrak{P}_q \subset \mathfrak{P}^*_{q}} g_{\mathfrak{P}_q \subset \mathfrak{P}^*_{q}} \frac{\sum_{\mathcal{P}_{q,I} \in \mathfrak{P}_q} g_{\{N_1\}}^{(1)} \prod_{i \in I} (\text{tr}[U^{2p_i}]) N_i}{\sum_{\mathcal{P}_{q,I}^* \in \mathfrak{P}^*_{q}} g_{\{N_j\}}^{(2)} \prod_{j \in I} (\text{tr}[U^{2p_j}]) N_j} \quad (68)$$
where $q_{\text{max}}$ is an arbitrary finite even integer (the case $q_{\text{max}} = \infty$ might lead to convergence issues that we shall avoid), $g(\mathcal{A}_q; \mathcal{A}_q')$ are positive constants which contain some dimensional normalization coming from the $R$-regularization of the two-point function, $g^{(1,2)}_{\{N_1\}}$ are also constants which can be chosen without dimension; the sum $\sum_{\mathcal{A}_q \subset \mathcal{P}_q, \mathcal{A}_q' \subset \mathcal{P}_q'}$ is performed over all subsets $\mathcal{A}_q$ of $\mathcal{P}_q$ and $\mathcal{A}_q'$ of $\mathcal{P}_q'$; in the ratios, the sums are performed over elements of these subsets $\mathcal{A}_q$ and $\mathcal{A}_q'$ consisting in partitions themselves. Note that this solution may not be the most general one nor is it unique. However, it provides a wide class of solutions materializing the fact that each ratio in (68) should scale as $\alpha^{N^*}$ should scale as $\alpha^{N^*}$ after mapping $U \rightarrow U/\alpha$.

As an illustration, this solution includes the following kind of terms, assuming that

\begin{equation}
\frac{1}{(\text{tr}[U^2])^{\frac{N^*}{2}}}, \quad \frac{1}{(\text{tr}[U^4])^{\frac{N^*}{2}}} \quad \frac{1}{(\text{tr}[U^2])^{N_1}(\text{tr}[U^4])^{\frac{(N^*-2N_1)}{4}}}, \quad \frac{1}{(\text{tr}[U^2])^{N_1}(\text{tr}[U^6])^{\frac{(N^*-2N_1)}{6}}},
\end{equation}

\begin{equation}
\frac{1}{g_1^{(2)}(\text{tr}[U^2])^{\frac{N^*}{2}} + g_2^{(2)}(\text{tr}[U^6])^{\frac{N^*}{2}}}, \quad \frac{g_1^{(1)}(\text{tr}[U^2])^{\frac{N^*+x^*}{2}} + g_2^{(1)}(\text{tr}[U^4])^{\frac{N^*+x^*}{2}}}{g_1^{(2)}(\text{tr}[U^2])^{\frac{N^*+x^*}{2}} + g_2^{(2)}(\text{tr}[U^6])^{N_1}(\text{tr}[U^6])^{\frac{(N^*+y^*-2N_1)}{6}}}.
\end{equation}

In summary, the functional integral of an $O(N)$-invariant matrix model is given by

\begin{equation}
Z(J, \lambda) = K \int e^{i \int \text{tr}[J(x)M(x)]dx} - \int \left\{ \frac{1}{2} \mu^2 \text{tr}[M^2(x)] + V(x) \left\{ \text{tr}[M^2(x)] \right\}_p \right\} dx.
\end{equation}

When the theory is genuinely interacting, we have $\lambda \neq 0$, and (70) reduces to

\begin{equation}
Z(J, \lambda) = e^{-\int \left\{ f_1 - e^{\text{tr}[J(x)U]} \right\} e^{-\frac{1}{2} \mu^2 \text{tr}[U^2] V(x) \left\{ \text{tr}[U^2] \right\}_p C_p \left\{ \text{tr}[U^2] \right\}_p \right\} dx}.
\end{equation}

In the limit that the coupling constants all vanish, $\lambda = 0$, this generating functional yields the pseudo-free theory,

\begin{equation}
Z_{PF}(J, \mu) = e^{-\int \left\{ f_1 - e^{\text{tr}[J(x)U]} \right\} dx},
\end{equation}

which differs significantly from the generating functional for the free theory,

\begin{equation}
Z_F(J, \mu) = e^{-\frac{1}{2\mu^2} \int \text{tr}[J^2(x)]dx},
\end{equation}

obtained from (70) by formally putting $\lambda = 0$ and computing the remaining functional integral as a traditional Gaussian functional integral. The interacting theory provides a continuous perturbation of the pseudo-free theory and a discontinuous perturbation of the free theory. This also shows that the results for the $O(N)$-invariant, finite-component, vector case [3] basically extend to the $O(N)$-invariant (symmetric) matrix models with finite size. Since we did not use in the above calculation any feature about the symmetric property of the matrices, one can reasonably infer that similar results hold for general square matrices, according to the discussion in Section [3] in the paragraph: Case of a non-symmetric matrix.
4.2 Infinite size matrices

The task now is to determine the characteristic functional of an infinite size independent-
value matrix model from a limit of those with finite size (we recall that the limits defining
$L_{\infty,\lambda}$ commute, i.e., specifically, \(11\) holds). For this limit to hold, one must pay attention
to the parameters \(V_N, \mu_N\) and the family of arbitrary couplings \(g_{(\gamma)}, g_{(\gamma)}^{(1)}, \) and \(g_{(\gamma)}^{(2)}\).

In the finite case, we start with the solution (71) above for \(L_{\infty,\lambda}(\{\text{tr}[J^{2p}(x)]\})\) in the interacting theory (i.e., \(a = 0\) in \(10\)). Let us again restrict to a small sector \(\Delta_*\) of \(\mathbb{R}^n\) and consider the part of that functional independent of the positions as

\[
L_{N,\lambda}(\{\text{tr}[J^{2p}]\}) = \int [1 - e^{\gamma[JU]}] e^{-\frac{1}{2} \mu_N^2 \text{tr}[U^2] - V_{N,\lambda}(\text{tr}[U^{2p}])} C^2_{0,N}(\{\text{tr}[U^{2p}]\}) dU .
\]

Due to the intricate expansion of the solution \(C^2_{0,N}(\{\text{tr}[U^{2p}]\})\) in terms of several matrix invariants, the \(N = \infty\) limit becomes difficult to track.

To proceed further, it is useful to restrict attention to a more narrow range of models. Thus, let us restrict the analysis to a potential of the form

\[
V_{N,\lambda}(\{\text{tr}[U^{2p}]\}) = V_{N,\lambda}(\text{tr}[U^2]) .
\]

Also some limitation on the general solution for \(C^2_{0,N}(U)\) is in order, such as

\[
C^2_{0,N}(\{\text{tr}[U^{2p}]\}) = C^2_{0,N}(\text{tr}[U^2]) = g_N \frac{1}{(\text{tr}[U^2])^2} ,
\]

which is not unreasonable since within the functional integral expressions of the form \([\text{tr}(U^2)]^p\)
dominate, and do so greatly for large \(N\), over homogeneous expressions such as \([\text{tr}(U^2)p]\).

In particular, let us consider

\[
L_{N,\lambda}(\{\text{tr}[J^2]\}) = g_N \int [1 - e^{\gamma[JU]}] e^{-\frac{1}{2} \mu_N^2 \text{tr}[U^2] - V_{N,\lambda}(\text{tr}[U^2])} \frac{1}{(\text{tr}[U^2])^2} dU .
\]

Following similar procedures to those in Section 3, we proceed by diagonalizing \(J=OjO^t\)
and introducing \(r^2 = \text{tr}[U^2]\) and spherical coordinates \((33)\), such that

\[
\lim_{N \to \infty} L_{N,\lambda}(\{\text{tr}[J^2]\}) = \lim_{N \to \infty} g_N \int \left\{ [1 - e^{\gamma \sum_{a} M_{a} a \left( \prod_{l=1}^{N-1} \sin \theta_{l} \right) \cos \theta_{a}}] ] \prod_{l=1}^{N} (\sin \theta_{l})^{N_\ast -(l+1)} d\theta_{l} d\Omega_{N^\ast -(N+2)} \right\} .
\]

Using steepest descent techniques for integrating \(\theta\) as done in Section 3, introducing

\[
r \to \sqrt{2bN^\ast}, \quad \mu_N = \frac{1}{N^\ast} \mu , \quad V_{N,\lambda}(2bN^\ast) = V_{\lambda}(b) ,
\]

and using \(g_N\) to neutralize the contributions depending on \(N\) coming from the integration
and the change of variables, we have

\[
L_{\infty,\lambda}(\{\text{tr}[J^2]\}) = g \int_{\infty}^{\infty} [1 - e^{\gamma[J^2]}] e^{-\frac{1}{2} \mu^2 b - V_{\lambda}(b)} db .
\]
Setting $\lambda \to 0$ in this expression yields the pseudo-free theory and, now returning to the full position space, one obtains

$$-\ln[Z_{PF}(J)] = \int L_{PF}(\text{tr}[J^2]) \, dx = g \int \left\{ \int_0^\infty [1 - e^{-b\text{tr}[J^2(x)]}] e^{-\frac{1}{2} \mu^2 b} \, db \right\} \, dx$$

which should be compared with the free theory

$$-\ln[Z_F(J)] = \int L_F(\text{tr}[J^2(x)]) \, dx = \int \frac{2}{\mu^2} \text{tr}[J^2(x)] \, dx . \quad (81)$$

It is interesting to observe that if the factor $g = 1$ then for weak values of the source $J(x)$ and/or large values of the mass parameter $\mu$—or more precisely, $\text{tr}[J^2(x)]/\mu^2 \ll 1$—the pseudo-free theory essentially agrees with the free theory. It may be argued that this fact can be used to fix the value of $g$. However, the functional form of the pseudo-free and free theories remains manifestly different over the entire range of the strength of the source, which includes both large values as well as small values. Thus one should not read too much into the similarity of the behavior for a limited range of parameters. For example, any discussion of “perturbation” about some form of a “free theory” would necessarily involve large and small source values and would therefore explore the fundamental differences between the pseudo-free and free theories.

Alternatively, one notices that all previous developments leading to (81) can be reached using the field definition $M(x) = \int_{S_\infty} B(x, U) U B^\dagger(x, U) \, d\rho(U)$ for infinite size matrices. This makes sense if one introduces a weight function $\rho(U)$ so that the measure $d\rho(U)$ is well defined on an infinite dimensional space. The operator $B$ is defined as $B(x, U) = A(x, U) + 1$ with $[A(x, U), A^\dagger(y, V)] = \delta_{R^n}(x - y) \delta_{S_\infty}(U - V)$, where $\delta_{S_\infty}(U - V)$ is understood in the distribution sense with respect to the measure $d\rho(U)$, namely

$$\int \delta_{S_\infty}(U - V) \, d\rho(U) = 1 . \quad (83)$$

The notion of $R$-ordering extends in the present setting as well. The rest of the analysis naturally follows and leads to (81). The same conclusion can be inferred: the interacting theory is continuous connected to the pseudo-free theory (81).

5 Conclusion

We have investigated independent-value matrix models and extended results obtained in the vector situation [3, 4] to both finite and infinite size matrices. We first find the general formula of a functional over matrix invariants in terms of an invariant measure. The determination of that measure amounts to specify invariant actions. As another interesting result in such invariant models, we find that the interacting theory is again continuously connected with the so-called pseudo-free theory and not with the free theory. This has been proved for finite size $N$ matrix models with an $O(N)$ invariant action which incorporates invariants of
any order $\{\text{tr}[M^{2p}]\}_p$. The case $N = \infty$ is more peculiar but we succeed to prove the similar result for $O(\infty)$ invariant matrix models equipped with an interaction as a general function of a unique basic invariant $\text{tr}[M^2]$. A way perhaps to extend our developments to other types of matrix invariants such as at least $\text{tr}[M^4]$ would be to apply resolvent methods as used in the framework of statistical mechanics in random matrices models [9]. This aspect deserves to be analyzed.

This study is part of a larger program which advocates that the expansion in perturbation theory should not be performed around the free theory but the one that is continuously connected to the interacting theories, namely the pseudo-free theory, when that theory differs from the free theory. Concerning this, the vector case has been resolved, and the matrix case, although not totally resolved, seems to support similar conclusions according to the present work. Then one may naturally ask how the present formalism might be led further by discussing the case of independent-value, multi-index, tensor models. We point out that, recently, a basis of unitary invariants has been highlighted in the framework of colored tensor models [11, 12, 13, 14]. Tensor invariants can be traced back for years [15] but have been rediscovered after analyzing of the $1/N$ expansion of colored random tensors [13]. Their Gaussianity at large $N$ proves to be universal [12, 13]. We expect that, using a different method by introducing a position space $x$ attached to these tensors, and several ingredients of this work, we might find in the tensor situation another behavior at large $N$ when the theory is fully in interaction.

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