Poincaré Duality and Spin\textsuperscript{c} Structures for Complete Noncommutative Manifolds

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October 28, 2018

Abstract

We prove a noncompact Serre-Swan theorem characterising modules which are sections of vector bundles not necessarily trivial at infinity. We then identify the endomorphism algebras of the resulting modules. The endomorphism results continue to hold for the generalisation of these modules to noncommutative, nonunital algebras. Finally, we apply these results to not necessarily compact noncommutative spin manifolds, proving that Poincaré Duality implies the Morita equivalence of the ‘algebra of functions’ and the ‘Clifford algebra.’

AMS 2000 Subject Classification 46L08 and 46L89; also 46L80

1 Introduction

Part of the folklore of noncommutative geometry is that the Serre-Swan theorem generalises to noncompact spaces. It is also generally assumed (and sometimes stated) that this generalisation introduces essentially nothing new. In section 2, we show that there is a nontrivial characterisation of the sections of vector bundles tending to zero at infinity on locally compact spaces.

Let $C_0(X)$ be the continuous functions on $X$ which vanish at infinity. Then the main result of the next section is that a $C_0(X)$ module $E$ is isomorphic to the bundle of sections vanishing at infinity on some vector bundle $V \to X$ if and only if

\[ E \cong pC_0(X)^n, \quad p^* = p^2 = p \in M_n(C(X^c)), \]

where $X^c$ is some compactification of $X$. Dealing with general compactifications allows one to speak about bundles not trivial at infinity.

If $A$ is a nonunital $\sigma$-unital $C^*$-algebra, then we say that an $A$-module $E$ is $A_b$ finite projective if $E = pA^n$ for some projection $p \in M_n(A_b)$ where $A_b$ is some unitization of $A$ (we write $A_b$ in analogy with a subalgebra of bounded functions). We identify such modules as the noncommutative analogue of sections of vector bundles vanishing at infinity. With this level of generality we identify the compact endomorphism and the full endomorphism algebras of these modules. We find that

\[ \text{End}_A(E) = pM_n(A_b)p \quad \text{End}_A^0(E) = pM_n(A)p. \]

In particular, the compact endomorphism algebra of such a module is nonunital.
In the last section we apply these results to noncompact versions of Connes’ noncommutative spin manifolds. This requires generalisations of his axioms which are not presented here in full; see [1]. We describe the results and assumptions necessary for us to show that, with \( \mathcal{H}_\infty \) the right pre-C*-\( \mathcal{A} \)-module associated to the manifold,

\[
\Omega^*_D(\mathcal{A}) \cong \text{End}_A^\mathcal{A}(\mathcal{H}_\infty),
\]

\[
\Omega^*_D(\mathcal{A}_b) \cong \text{End}_A(\mathcal{H}_\infty).
\]

Here \( \mathcal{A}_b \) is a unitization of \( \mathcal{A} \) and the algebra \( \Omega^*_D(\mathcal{A}) \) is obtained by representing the universal differential algebra of \( \mathcal{A} \). Similar comments apply to \( \Omega^*_D(\mathcal{A}_b) \), but we can also obtain it as the ‘smooth strong’ closure of \( \Omega^*_D(\mathcal{A}) \), [1]. These results show that \( \mathcal{A} \) and \( \Omega^*_D(\mathcal{A}) \) are strongly Morita equivalent. From the description of spin\(^c\) structures on a manifold as Morita equivalence bimodules between the algebra of functions and the complex Clifford algebra, [8], and the fact that \( \Omega^*_D(\mathcal{A}) \) coincides with the Clifford algebra in the commutative case, we see that these isomorphisms provide a noncommutative definition of spin\(^c\) structure. This point of view is strengthened by the module \( \mathcal{H}_\infty \) coinciding with the smooth sections of the corresponding complex spinor bundle in the commutative case, [3, 10, 11].

2 The Nonunital Serre-Swan Theorem

The first result concerns modules finitely generated over the nonunital algebra \( A \), and projective over \( A^+ \). These have occasionally been touted as the analogues of sections of vector bundles vanishing at infinity. Essentially we characterise them in order to dispense with them.

**Lemma 1** Let \( A \) be nonunital. Then \( E \) is a finitely generated (right) \( A \)-module which is projective over \( A^+ \) if and only if \( E \) is of the form \( pA^n \), where \( p = p^2 \in M_n(\mathcal{A}) \).

**Proof** Since \( E \) is a finitely generated and projective \( A^+ \)-module,

\[
E \cong p(A^+)^n,
\]

where \( p \in M_n(\mathcal{A}) \). However, \( E \) is finitely generated over \( A \), so for some \( M \) there exists \( e_1, \ldots, e_M \in E \) such that for all \( e \in E \) there are \( a_1, \ldots, a_M \in A \) with

\[
e = e_1a_1 + \cdots + e_Ma_M.
\]

So in fact, as \( A \) is an ideal in \( A^+ \), every \( e \in E \) is an element of \( pA^n \). In particular, the elements

\[
\begin{pmatrix}
1 \\
0 \\
\vdots \\
0
\end{pmatrix}
= \begin{pmatrix}
p_{11} \\
p_{21} \\
\vdots \\
p_{n1}
\end{pmatrix}
, \quad
\begin{pmatrix}
0 \\
0 \\
\vdots \\
1
\end{pmatrix}
= \begin{pmatrix}
p_{1n} \\
p_{2n} \\
\vdots \\
p_{nn}
\end{pmatrix}
,\]

are contained in \( E \), and so \( p \in M_n(\mathcal{A}) \).

Conversely, if \( E = pA^n = p(A^+)^n \), \( p \in M_n(\mathcal{A}) \), then \( E \) is projective over \( A^+ \) since

\[
E \oplus (1 - p)(A^+)^n = (A^+)^n
\]

is a free \( A^+ \)-module. It is finitely generated over \( A \), because the column vectors \( p_j = (p_{ij}) \), \( j = 1, \ldots, n \), above provide a system of generators. □
To continue the investigation, we note that if \( i : A \hookrightarrow A_b \) is a unitization of \( A \), we can define the pull-back of a right \( A_b \)-module \( E \) by

\[
i^*E = E|_A := Ei(A) = \{ei(a) : e \in E, a \in A\},
\]

with the obvious right action of \( A \) on \( E|_A \). Note that as an \( A_b \)-module, \( E|_A \) is a submodule of \( E \). In fact, the pullback works for any embedding \( i : A \hookrightarrow A_b \) of \( A \) as an ideal in \( A_b \), even if it is not closed.

If \( i : X \hookrightarrow X^c \) is an embedding of the locally compact Hausdorff space \( X \) in the compact Hausdorff space \( X^c \) (automatically as a dense open subset), then we may define a unitization of \( C_0(X) \) as follows. Define, \[i_*: C_0(X) \hookrightarrow C(X^c)\]

by

\[
(i_*f)(y) = \begin{cases} 
0 & \text{if } y \notin i(X) \\
 f(x) & \text{if } y = i(x), \ x \in X.
\end{cases}
\]

**Example** Let \( 1 \to X^c \) be the trivial line bundle over \( X^c \), where \( X^c \) is a compactification of the locally compact noncompact Hausdorff space \( X \). The space of sections of this line bundle is \( E = C(X^c) \), and this is a right \( C(X^c) \) module. We have the unitization map, \( i : C_0(X) \hookrightarrow C(X^c) \), and so we can pull \( E \) back to \( C_0(X) \). We find

\[
i^*E = i^*C(X^c) = C(X^c)i(C_0(X)) = C_0(X).
\]

Thus we obtain the space of sections vanishing at infinity.

**Example** Let \( X \) be as above, and consider the embedding \( i : C_c(X) \hookrightarrow C_0(X) \), where \( C_c(X) \) is the algebra of compactly supported functions. If \( E = \Gamma_0(X,V) \) is the \( C_0(X) \)-module of sections vanishing at infinity of the vector bundle \( V \), then we can pull it back to \( C_c(X) \) as above. Then

\[
i^*E = \Gamma_c(X,V)
\]

and we obtain the compactly supported sections. In this case the resulting module is only a pre-C*-module, because \( C_c(X) \) is not a C*-algebra.

**Definition 1** Let \( A \) be a \( \sigma \)-unital C*-algebra. If \( i : A \hookrightarrow A_b \) is a unitization, we say that an \( A \)-module is \( A_b \) finite projective if it is of the form \( i^*E \) for some finite projective \( A_b \)-module \( E \).

With this definition we are ready to state the main result of this section.

**Theorem 1 (Nonunital Serre-Swan)** Let \( X \) be a locally compact Hausdorff space, \( A = C_0(X) \), and \( A_b = C(X^c) \) for some compactification \( X^c \) of \( X \). Then a right \( A \)-module \( E \) is of the form \( E = pA^n \), \( p \in M_{n}(A_b) \) a projection, if and only if \( E = \Gamma_0(X,V|_X) \), where \( V \to X^c \) is a vector bundle and \( \Gamma_0 \) denotes the sections vanishing at infinity.

**Proof** Suppose that \( E = \Gamma_0(X,V|_X) \), where \( V \to X^c \) is a vector bundle. Then for some \( n \) and projection \( p \in M_{n}(A_b) \),

\[
\Gamma(X^c,V) \cong p(A_b)^n,
\]

\[
(\text{13})
\]
by the Serre-Swan theorem, [12]. Note that as $X \subset X^c$ is dense and open, and rank $p$ is locally constant, $p \not\in M_n(A)$. Setting $i : A \hookrightarrow A_b$ to be the unitization, we have

$$i^* \Gamma(X^c, V) = pA^n = \Gamma_0(X, V|_X).$$  \hspace{1cm} (14)

Conversely, let $E = pA^n$ with $p \in M_n(A_b)$ a projection. Then we can define a finite projective $A_b$-module, $\tilde{E} = p(A_b)^n$, with the obvious right action of $A_b$. By the Serre-Swan theorem, there is a vector bundle $V \to X^c$ such that

$$\tilde{E} \cong p(A_b)^n = \Gamma(X^c, V).$$ \hspace{1cm} (15)

Employing the pull-back by the unitization map as above immediately shows that

$$i^* \tilde{E} = E = \Gamma_0(X, V|_X).$$ \hspace{1cm} (16)

\[\square\]

**Corollary 1** With $A$ and $A_b$ as above, the $A$-module $E$ is isomorphic to $\Gamma_0(X, V)$, for $V \to X$ trivial at infinity, if and only if $E \cong pA^n$ for some $p \in M_n(A^+)$.

In fact, since a bundle trivial at infinity will extend to any compactification (trivially), there must be $p \in M_n(A_b)$ such that $\Gamma_0(X, V) = pA^n$, for any unitization $A_b$. This does not contradict the corollary. Excision in $K$-theory, [3], tells us that the compactly supported $K$-theory (the usual definition in the nonunital/noncompact case) is independent of any compactification. However, the nonunital Serre-Swan theorem is telling us that to get a handle on the actual vector bundles, and not just the resulting cohomology theory, we need to take account of all compactifications/unitizations simultaneously. Fortunately, the existence of a maximal compactification ensures the existence of a maximal compactification to which a given bundle extends. Note that any notion of bounded cohomology (the natural dual of singular homology defined using finite chains) in $K$-theory will require this notion.

Also, it is now clear why we are not interested in modules of the form $pA^n$, where $A$ is nonunital and $p \in M_n(A)$. These do not correspond to sections vanishing at infinity in the commutative case; they do not even characterise a vector bundle on $X^+$, since we do not have locally constant rank at infinity. It would be interesting to see what, if any, geometric content these modules have in the commutative case.

**Example** If $X$ is the interior of a manifold with boundary $\overline{X}$, then the sections of the tangent bundle of $X$ vanishing on the boundary is certainly an example of the above phenomena. This seems somewhat trivial as we have a compact space to which the vector bundle extends, and so we can realise it in the usual compact Serre-Swan fashion.

A more genuine seeming example is any (finite dimensional) manifold which does not have finitely generated cohomology groups. It is easy to construct a vector bundle $V$ on such a space which is not trivial outside any compact set. An example of such a space would be $\mathbb{R}^n$ with the balls centred at integer coordinates of radius $\frac{1}{4}$ deleted. It is easy to construct bundles on this space which are not trivial outside any compact set.

We now drop the assumption that $A$ is commutative, and describe the endomorphism algebras of $A_b$ finite projective $A$-modules. By results in [4, 5], finite projective $A_b$-modules $E$ can be regarded as $C^*$-modules once an $A_b$-Hermitian form is chosen. This form automatically
restricts to the $A_b$-submodule $E|_A$, since $A$ is an ideal in $A_b$. It is the endomorphism algebras of these modules that we now describe. This is relevant for the noncommutative geometry of nonunital algebras. \[3\).

**Proposition 1** Let $E|_A$ be an $A_b$-finite projective $A$-module. Then, as a $C^*$ $A$-module, we have

\[ \text{End}_{A_b}(E|_A) = pM_n(A)p \quad \text{End}_A(E|_A) = pM_n(A_b)p, \]

where $\text{End}_{A_b}(E|_A)$ denotes the compact endomorphisms of $E|_A$ and $\text{End}_A(E|_A)$ denotes the $C^*$-algebra of adjointable operators on $E|_A$.

**Proof** The $A_b$-module $E$ is of the form $E = p(A_b)^n$. Writing $M(B)$ for the multiplier algebra of a $C^*$-algebra $B$, it is then standard that, \[4, 6\],

\[ \text{End}_{A_b}(E) \cong M(\text{End}_{A_b}(E|_A)) = \text{End}_{A_b}(E|_A) \cong pM_n(A_b)p. \]

These equalities follow from $\text{End}_{A_b}(E)$ being unital, which is itself a standard result.

To prove the above assertions, we begin with a variant on a standard isomorphism, \[4\]. Denoting the $A$-finite rank operators on $E|_A$ by $\text{End}_{A_b}^0(E|_A)$, define $\theta : \text{End}_{A_b}^0(E|_A) \rightarrow pM_n(A_b)p$ by setting $\theta([p\xi](p\eta))$ to be the matrix with $i, j$-th entry $\sum_{k,l} p_{ik} \xi_k \eta_l^* p_{lj}$. One checks that this is a $*$-homomorphism and an isometry and has dense range. As such, $\theta$ extends to a $*$-isomorphism of $\text{End}_{A_b}^0(E|_A) \cong pM_n(A_b)p$.

As $i^*E$ is a right $A_b$-submodule of $E$, we see that

\[ \text{End}_{A_b}(i^*E) = \text{End}_A(i^*E). \]

Also, $i^*E$ is a left $\text{End}_{A_b}(E)$-submodule, as is easily checked. Consequently

\[ \text{End}_A(i^*E) = \text{End}_{A_b}(E) = pM_n(A_b)p. \]

\[\square\]

Note that in \[5\], Appendix A], there is a different approach, and elements of $\text{End}_{A_b}^0(E|_A)$ are called endomorphisms carried by $A$. Modules of the above type and their endomorphism algebras are used to describe elements of relative $K$-theory.

### 3 Complete Noncommutative Manifolds

In \[3, 10, 4\], closed noncommutative spin manifolds were defined as those spectral triples satisfying certain additional axioms. Moreover, these were shown to reduce to ordinary closed spin manifolds in the commutative case.

In \[11\], this is generalised to complete (i.e. not necessarily compact but geodesically complete and without boundary) spin manifolds. The extension of the noncompact axioms to the noncommutative case makes sense, and the resulting objects are called complete noncommutative spin manifolds.

Note that this extension is nontrivial in several regards, there being algebraic, analytic and homological problems. The chief algebraic problem is the appropriate characterisation of the
sections of vector bundles on noncompact spaces. This was addressed in the previous section, and in this section we apply these results to complete noncommutative manifolds.

In addition, to prove that one recovers complete noncompact manifolds in the commutative case, one requires that the nonunital algebra of functions is local, in that it possesses a dense subalgebra which has local units. We will not need any particular details of these algebras, other than knowing that the constructions and definitions we present make sense for these algebras; see [11]. Also, a unital algebra is necessarily local.

We prove two main results. The first is that the (pre) Hilbert space appearing in the definitions provides a (pre) strong Morita equivalence between $A$ and $\Omega^\ast_D(A)$, the analogues of the algebra of functions and Clifford algebra respectively. The second result is a corollary of the first, and states that $\Omega^\ast_D(A)$ is finite projective over $A$ as a left or right module. This shows that it behaves like the sections of a bundle. We begin by introducing the basic objects of noncommutative geometry.

Definition 2 A spectral triple $(A, \mathcal{H}, D)$ is given by

1) A representation $\pi : A \to \mathcal{B}(\mathcal{H})$ of a local algebra $A$ on the Hilbert space $\mathcal{H}$.

2) A closed, (unbounded) self-adjoint operator $D : \text{dom}D \to \mathcal{H}$ such that $[D, a]$ extends to a bounded operator on $\mathcal{H}$ for all $a \in A$ and $a(1 + D^2)^{-\frac{1}{2}}$ is compact for all $a \in A$.

The triple is said to be even if there is an operator $\Gamma = \Gamma^\ast$ such that $\Gamma^2 = 1$, $[\Gamma, a] = 0$ for all $a \in A$ and $\Gamma D + D \Gamma = 0$ (i.e. $\Gamma$ is a $\mathbb{Z}_2$-grading such that $D$ is odd and $A$ is even.) Otherwise the triple is called odd.

Since $A$ is represented on Hilbert space we may unambiguously speak about the norm on $A$, and the norm closure $\overline{A} = A$.

We recall the basic setup, and refer to [10] for the details. Given a spectral triple $(A, \mathcal{H}, D)$, we say that it is a complete (noncommutative) spin manifold if it satisfies the axioms in [11]. There are several of these axioms and their consequences that we shall require.

1) There is a unitization $A_b$ of $A$ such that the smooth domain of $D$,

$$\mathcal{H}_\infty = \bigcap_{m \geq 1} \text{dom}D^m,$$

is an $A_b$ finite projective (right) $A$-module, and $A$ is a Fréchet algebra for the seminorms provided by

$$q_n(a) = \| \delta^n(a) \|, \quad \delta(a) = [\|D\|, a].$$

One can show that $A_b$ is also complete for the strong topology determined by these seminorms, and is in fact characterised as the strong closure of $A$ with respect to these seminorms.

2) The first order condition holds. This means that $\mathcal{H}_\infty$ is in fact an $A$-bimodule, or equivalently an $A \otimes A^{op}$ module, and

$$[a, b^{op}] = 0 \quad [[D, a], b^{op}] = 0, \quad \forall a, b \in A.$$
This ensures that the representation of the universal differential algebra of $A$, $Ω^*(A)$, given by
\[ \pi(aδ(b_1)⋯δ(b_k)) = a[D, b_1]⋯[D, b_k], \quad (24) \]
is contained in the commutant of $A^{op}$, $[3, 10, 11]$. As the above conditions are symmetric in $A$ and $A^{op}$, a similar ‘opified’ statement holds also. Moreover, all the above comments hold for $A_b$. One can also show that $Ω^*_D(A)$ is contained in the smooth domain of the derivation $δ$.

3) Most importantly, the spectral triple satisfies Poincaré Duality. The $K$-homology class defined by the above spectral triple $\mu \in K(A \otimes A^{op})$, provides us with a map
\[ \cap \mu : K_*(A) \to K_*(A) \]
We say that $(A, H, D)$ satisfies Poincaré Duality if this map is an isomorphism. Here $K_*(A)$ is $K$-homology with compact supports. This group cannot be defined for all smooth, nonunital algebras, and imposes a tight restriction on the algebras under consideration, $[11]$.

We have written the map as a cap product which is not strictly true in the noncommutative case, but the Kasparov product gives us a map more properly called a slant map, and this is how we compute it; see $[3, 11]$. For information on $K$-homology and the various products, see $[5]$.

Our aim for the rest of this section is to show that $Ω^*_D(A)$ and $A$ are strongly Morita equivalent. To do this, we recall that sections of the spinor bundle form an irreducible representation of the Clifford algebra in the sense that it is irreducible (in the usual sense of the word irreducible) fibrewise. The following concept is useful in dealing with this situation.

**Definition 3** Let $E$ be a right $C^*$ $A$-module. Then we say that the representation $\pi : B \to End_A(E)$ is $A$-irreducible if the only operators in $End_A(E)$ commuting with $B$ are the scalars.

**Remark** This works just as well for pre-$C^*$-modules and pre-$C^*$-algebras.

**Example** Let $E = \Gamma(X, V)$ be the sections of a vector bundle $V$ over the closed Riemannian spin manifold $X$, and suppose that $E$ is a Clifford bundle. That is, $E$ admits a (fibrewise) representation of the sections of the Clifford algebra of the cotangent bundle of $X$. Then $E$ is a $C^\infty(X)$ pre-$C^*$-module, and the representation
\[ Cliff(T^*X) \to End_{C^\infty(X)}(E) \]
is irreducible if it is so fibrewise. For if $B : E \to E$ is $C^\infty(X)$-linear, $B \in End(\Gamma(X, V)) = \Gamma(End(X, V))$. Thus $B$ is a smooth matrix-valued function on $X$, and if it is to commute with $Cliff(T^*X)$, it must do so fibrewise. Hence the representation of $Cliff(T^*X)$ is $C^\infty(X)$ irreducible if and only if the representation of $Cliff(T^*_xX)$ is irreducible on $E_x$ for all $x \in X$.

To compare this notion of irreducible with others we have the following two results.
Lemma 2 The representation $\pi : B \to \text{End}_A(E)$ is irreducible if and only if $B$ has no invariant complementable submodules in $E$.

Proof Suppose that $V \subseteq E$ is a complementable submodule with $BV \subseteq V$. Then, since complementable submodules are precisely those which are the images of projections in $\text{End}_A(E)$, there exists a projection $Q \in \text{End}_A(E)$ such that $QE = V$ and $Q$ commutes with $B$.

Conversely, suppose there exists $T \in \text{End}_A(E)$ such that $Tb = bT$ for all $b \in B$. Then $\text{Image}T$ is a complementable submodule, being the image of an adjointable map, and $B\text{Image}T \subseteq \text{Image}T$. \[\square\]

Corollary 2 The representation $\pi : B \to \text{End}_A(E)$ is irreducible if and only if there do not exist projections $P, Q \in \text{End}_A(E)$ such that $E = PE \oplus QE$ and $P, Q$ commute with $B$.

With these tools in hand we can prove our main technical result. The hypotheses apparently make the result somewhat obvious and perhaps trivial, but the proof shows that there is much more at work.

Theorem 2 Suppose that $(A, \mathcal{H}, \mathcal{D})$ is a noncommutative spin manifold, and that the only bounded operators commuting with $A$ and $\mathcal{D}$ on $\mathcal{H}$ are scalars. Then, regarding $\mathcal{H}_\infty$ as a pre-$C^*$-$A$-module, the representation of $\Omega^*_\mathcal{D}(A_b)$ is $A$-irreducible.

Proof Suppose not. We begin by writing $\mathcal{H}_\infty = pA^n$ for some projection $p \in M_n(A_b)$. Then there exists $e = e^* = e^2 \in pM_n(A_b)p$ such that

1) $[e, p] = 0$, since $e \in pM_n(A_b)p$, and

2) $[\omega, e] = 0$ for all $\omega \in \Omega^*_\mathcal{D}(A_b)$, by hypothesis.

We write

\[\mathcal{D} = e\mathcal{D}e + (1 - e)\mathcal{D}(1 - e) + e\mathcal{D}(1 - e) + (1 - e)e\mathcal{D}e\]

\[= e\mathcal{D}e + (1 - e)\mathcal{D}(1 - e) + B\]

\[= \mathcal{D}e + B.\]

Using 2) we have that for all $a \in A_b$

\[[\mathcal{D}, a] = e[\mathcal{D}, a]e + (1 - e)[\mathcal{D}, a](1 - e) = [\mathcal{D}e, a]\]

so $[B, a] = 0$. In other words, $B = B^*$ is $A_b$-linear and so can easily be seen to be bounded (just check its behaviour on a generating set). Consequently, the map

\[t \mapsto \mathcal{D}e + tB\]

provides us with an operator homotopy from $\mathcal{D}$ to $\mathcal{D}e$. It is easy to check that this homotopy preserves the $K$-homology class of $\mu$, [3]. Hence we may write

\[\mu = [(A, \mathcal{H}, \mathcal{D})] = [(A, e\mathcal{H}, e\mathcal{D}e)] + [(A, (1 - e)\mathcal{H}, (1 - e)\mathcal{D}(1 - e))]\]

\[\in K^*(A \otimes A^{\text{op}}).\]
The next step is to show that \([e] = [Id_H - e] = [p - e] = [p] - [e]\) in the \(K\)-theory of \(A_b\). To do this we will construct an explicit Murray-von Neumann equivalence. First we note that

\[
[D, e] = [B, e] = (1 - e)D e - e D(1 - e),
\]

and using \(e^2 = e\), \((de)e = (1 - e)(de)\) et cetera, we compute

\[
D de = D(1 - e)D e - D e D(1 - e) = [D, (1 - e)](1 - e)[D, e] - [D, e][D, (1 - e)] = -e[D, e] + (1 - e)[D, e][D, e] = -(2e - 1)dede.
\]

In a completely analogous fashion we find that

\[
ded = (2e - 1)dede.
\]

This shows that \(D de = -de D\) and so \(D ded = dede D\). Hence \([D, e][D, e]\) commutes with all \(a \in A\) and \(D\), and so must be a scalar. Moreover, \((de)^* = -(de)\) so

\[
-dede = B^2 \geq 0
\]

is a positive real number. Suppose first that \(m = 0\). Then \(de = 0\) so \(e\) is a scalar, and so the representation of \(\Omega_D^*(A_b)\) is \(A\) irreducible. So suppose that \(m > 0\), and set \(B' = \frac{1}{\sqrt{m}}B\) so that

\[
B'^* B' = B' B'^* = Id.
\]

Since \(B'e = (1 - e)B'\) we see that \(B'e\) provides a partial isometry implementing a Murray-von Neumann equivalence

\[
(B'e)^* (B'e) = e \quad (B'e)(B'e)^* = (1 - e).
\]

Now we have established that

\[
\mu = 2[(A, eH, eD e)] \in K^*(A \otimes A^{op}),
\]

from which we may conclude that \(\mu\) does not satisfy Poincaré Duality, since for any generator \(q \in K_*(A), q \cap [(A, eH, eD e)]\) can not be in the image of \(\cap \mu\). Hence we have a contradiction, and the representation of \(\Omega_D^*(A_b)\) must be irreducible. \(\square\)

Before we can prove the Morita equivalence of \(A\) and \(\Omega_D^*(A),\) we need a few results to help us identify our various algebras precisely. Recall that an ideal \(I\) in a \(C^*\)-algebra \(A\) is called essential if the intersection of \(I\) with every other ideal in \(A\) is nonzero. In turn this is equivalent to \(aI = a = \{0\} \Rightarrow a = 0.\)

**Lemma 3** Suppose that \(I\) is an ideal in the unital \(C^*\)-algebra \(A\). Then \(M_n(I)\) acts irreducibly on \(A^n\) if and only if \(I\) is an essential ideal of \(A.\)

**Proof** Suppose that \(M_n(I)\) acts on \(A^n\), and that \(I\) is not essential. Then there exists \(a \in A\) such that \(aI = \{0\} = Ia,\) and so \(a\) commutes with every \(b \in I,\) contradicting irreducibility. Conversely, suppose that \(I\) is essential in \(A.\) Then \(M_n(I)\) is essential in \(M_n(A),\) and as
$M_n(A)$ acts irreducibly on $A^n$, we may suppose without loss of generality that $n = 1$. Now suppose that $I$ does not act irreducibly on $A$. Then there exists a projection $p \in A$ such that $A = pA \oplus (1-p)A$ as an $I$-module, and $pb = bp$ for all $b \in I$. As $I \cdot I \subseteq I$, $I$ must be contained in one of the two halves. Suppose that it is $pA$. Then $pb = bp = b$, and so $(1-p)b = 0 = b(1-p)$, contradicting $I$ essential. Similarly, if $I \subseteq (1-p)A$ we have $(1-p)b = b$ for all $b \in I$ and so $pb = bp = 0$, and we again obtain a contradiction. □

**Corollary 3** If $A$ is unital, $E = pA^n$ is finite projective, and $I$ is an ideal in $A$, the representation of $pM_n(I)p$ is irreducible if and only if $I$ is an essential ideal in $A$.

**Proof** This follows from the above by replacing $a \in M_n(A)$ by $pap$. □

We can now state our main result.

**Theorem 3** If $(A, \mathcal{H}, \mathcal{D}, e, J)$ is a spin manifold, with $\mathcal{H}_\infty = pA^n$, then $\Omega^*_D(A_b) \cong pM_n(A_b)p$. Furthermore, $\Omega^*_D(A) = pM_n(A)p = End^0_{A}(\mathcal{H}_\infty)$, so $A$ and $\Omega^*_D(A)$ are strongly Morita equivalent.

**Proof** As $\Omega^*_D(A_b)$ is unital and acts irreducibly on $\mathcal{H}_\infty$, 

$$\Omega^*_D(A_b) \cong pM_n(B)p$$  \hspace{1cm} (26)

where $B \subseteq A_b$ is a unital subalgebra. This follows because

1) $A_b$ is unital and so is its own multiplier algebra, and

2) $\Omega^*_D(A_b)$ acts irreducibly, and so must comprise a full matrix algebra.

However, $\Omega^*_D(A_b)$ is also an $A_b$ bimodule, so $A_b \Omega^*_D(A_b) \subseteq \Omega^*_D(A_b)$. Thus $B$ must be an ideal, and as it is unital, $B = A_b$.

The same argument applies to $\Omega^*_D(A)$ except that now $B$ can be a proper ideal of $A_b$. By the above results, it must be an essential ideal, and since $\Omega^*_D(A)$ is also an $A$-bimodule, we have $\Omega^*_D(A) \cong pM_n(A)p$.

As two algebras $A, B$ are (pre) strongly Morita equivalent if and only if $B \cong End^0_{A}(E)$ for some (pre) $C^*$ $A$-module $E$, we have shown that $A$ and $\Omega^*_D(A)$ are strongly Morita equivalent, with $\mathcal{H}_\infty$ providing an equivalence bimodule. □

**Corollary 4** With $(A, \mathcal{H}, \mathcal{D})$ as above, $\Omega^*_D(A_b)$ is finite projective (left or right) $A_b$-module, while $\Omega^*_D(A)$ is an $A_b$-finite projective module.

From what we have shown about the module $pA^n$ we may also conclude that $pA^n$ provides a strong Morita equivalence between $A_b$ and $\Omega^*_D(A_b)$. Thus in noncommutative (spin) geometry the algebra $\Omega^*(A)$ plays a rôle strongly analogous to that of the Clifford algebra in the commutative case. In particular, any triple $(A, \mathcal{H}, \mathcal{D})$ satisfying the axioms listed above provides a noncommutative analogue of a spin$^c$ structure, $\mathbb{S}$.

A further problem is determining when a noncommutative Riemannian geometry, as discussed in $\mathbb{S}$, has a spin$^c$ structure. This amounts to identifying a noncommutative analogue of the (first few) Stieffel-Whitney classes. Other problems then arise, such as the compatibility of the two structures and the relation to spin/Real structures. This will be the subject of a future paper.
4 Acknowledgements

I would like to thank Steven Lord for drawing this problem to my attention, and for several interesting conversations on the subject. I would also like to thank Alan Carey for his support and interest.

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