Concept and Attribute Reduction Based on Rectangle Theory of Formal Concept

Jianqin Zhou\textsuperscript{1}, Sichun Yang\textsuperscript{1}, Xifeng Wang\textsuperscript{1} and Wanquan Liu\textsuperscript{2} \textsuperscript{†}

\textsuperscript{1}Department of Computer Science, Anhui University of Technology, Ma’anshan 243002, China
\textsuperscript{2}School of Intelligent Systems Engineering, Sun Yat-sen University, Shenzhen 518000, China

Based on rectangle theory of formal concept and set covering theory, the concept reduction preserving binary relations is investigated in this paper. It is known that there are three types of formal concepts: core concepts, relative necessary concepts and unnecessary concepts. First, we present the new judgment results for relative necessary concepts and unnecessary concepts. Second, we derive the bounds for both the maximum number of relative necessary concepts and the maximum number of unnecessary concepts and it is a difficult problem as either in concept reduction preserving binary relations or attribute reduction of decision formal contexts, the computation of formal contexts from formal concepts is a challenging problem. Third, based on rectangle theory of formal concept, a fast algorithm for reducing attributes while preserving the extensions for a set of formal concepts is proposed using the extension bit-array technique, which allows multiple context cells to be processed by a single 32-bit or 64-bit operator. Technically, the new algorithm could store both formal context and extent of a concept as bit-arrays, and we can use bit-operations to process set operations "or" as well as "and". One more merit is that the new algorithm does not need to consider other concepts in the concept lattice, thus the algorithm is explicit to understand and fast. Experiments demonstrate that the new algorithm is effective in the computation of attribute reductions.

\textit{Keywords}: Formal Concept Analysis; Concept lattice; Concept reduction; Attribute reduction
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1 Introduction

Concept lattice is a key tool for information analysis and processing. The mathematical basis of concept lattice is the lattice theory, the visualization tool is the Hasse graph, and the research methods are abstract algebra, discrete mathematics, data structure

\textsuperscript{1}The authors are supported by NSF grant of Anhui Province(No.180805MF178), China.
\textsuperscript{†}Corresponding author. Email: liuwq63@mail.sysu.edu.cn
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and algorithm analysis, fuzzy set [28], rough set [17], granular computing [29], etc. So far, formal concept analysis has been widely used in information retrieval [4, 12], knowledge discovery [13], association analysis [24], recommendation system [32] and other fields [16, 18, 23, 25].

When we decide to study a certain kind of concepts, we need to first consider how to find out all the concepts from the specific data. This problem is called concept lattice construction [1, 2, 14, 15, 20, 21]. Second, in order to better analyze data and save storage space, it is necessary to reduce concept lattice [3, 8, 11, 22, 26, 30]. Furthermore, the nodes of concept lattice can infer from each other, and based on this, one can extract rules [9, 12, 19].

Based factorization and attribute reduction, the concept reduction while preserving binary relations is proposed in [3, 26]. The significance of a concept in a given formal context may be different and they play different roles in the concept reduction. These concepts can be classified into three types: the core concepts, the relatively necessary concepts, and the unnecessary concepts. The judgment theorems for three types of concepts have been investigated. However, the judgment theorems for relative necessary concepts and unnecessary concepts in [3, 26] are very complicated, and difficult to implemented.

One can visualise a formal concept in a table of 0 and 1 as a closed rectangle of 1s, where the rows and columns are not necessarily contiguous [1]. Based on rectangle theory of formal concept and set covering theory, we will present the new judgment theorems for relative necessary concepts and unnecessary concepts in this paper. Also, we deduce the bounds for both the maximum number of relative necessary concepts and the maximum number of unnecessary concepts. From our results, it is known that both relatively necessary concepts and unnecessary concepts are exponential time problems. However, by using the proposed judgment theorems, it is easier to develop a program for computing both relative necessary concepts and unnecessary concepts than that of [26].

Based on the discernibility matrix and Boolean function, a method to attribute reduction is proposed while preserving the extensions in concept lattices [30, 31]. Similarly, two kinds of attribute reduction are proposed in the decision formal context based on the maximal rules [12]. Note that the attribute reduction in both [30, 31] and [12] can be simplified to reducing attributes while preserving the extensions for a set of formal concepts. A fast algorithm is proposed in this paper for reducing attributes while preserving the extensions for a set of formal concepts. As the new algorithm store both formal context and extension of a concept as a bit-array, and we can use bit-operations to process set operation "or" as well as "and". Further more, there is no need to consider other formal concepts, thus the algorithm is very fast.

Technically, suppose that one column of context or the extension of a concept is (1, 1, 1, 1, 1, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0), it is stored as 255 in our algorithm, where the first bit means $2^0$, the second bit means $2^1$, the third bit means $2^2$ and so on. Applying our new algorithm to mushroom data from [6], 19 columns of formal context can be reduced while preserving the extensions for 512 formal concepts. Experiments demonstrate that the new algorithm is much efficient in the computation of attribute reductions.

The remainder of this paper is organized as follows. In Section 2 some basic def-
 initiations and lemmas are reviewed. In Section 3, concept reduction preserving binary relations is discussed, we present the new judgment theorems for relative necessary concepts and unnecessary concepts. In Section 4, based on rectangle theory of formal concept, a fast algorithm for reducing attributes while preserving the extensions for a set of formal concepts is proposed by using extension bit-array technique. Finally, in Section 5, we conclude the paper with a summary and an outlook for future work.

2 Basic notions and properties

For the convenience of discussion, some basic notions and properties of formal concepts related to this paper will be reviewed. We first present the formal context and its operators as follows.

Definition 1. A triplet \((U, A, I)\) is called a formal context, where \(U = \{x_1, x_2, \ldots, x_m\}\), \(A = \{a_1, a_2, \ldots, a_n\}\), and \(I \subseteq U \times A\) is a binary relation between \(U\) and \(A\). Here each \(x_i (i \leq m)\) is called an object, and each \(a_j (j \leq n)\) is called an attribute. \(xIa\) or \((x, a) \in I\) indicates that an object \(x \in U\) has the attribute \(a \in A\).

Definition 2. Let \((U, A, I)\) be a formal context. For any \(X \subseteq U\) and \(B \subseteq A\), two operations are defined respectively:

\[ ^* : P(U) \rightarrow P(A), \quad X^* = \{m \in A | \forall g \in X, (g, m) \in I\} \]

\[ ^* : P(A) \rightarrow P(U), \quad B^* = \{g \in U | \forall m \in B, (g, m) \in I\} \]

The following are the definitions of formal concepts and concept lattices.

Definition 3. Let \((U, A, I)\) be a formal context. For any \(X \subseteq U\) and \(B \subseteq A\), if \(X^* = B\) and \(B^* = X\), then the pair \((X, B)\) is called a formal concept, where \(X\) and \(B\) are called the extension and the intension of \((X, B)\), respectively. For concepts \((X_1, B_1), (X_2, B_2)\), where \(X_1, X_2 \subseteq U, B_1, B_2 \subseteq A\), one can define the partial order as follows:

\[(X_1, B_1) \leq (X_2, B_2) \iff X_1 \subseteq X_2 \iff B_2 \subseteq B_1 \]

Furthermore, we have the following definitions:

\[(X_1, B_1) \land (X_2, B_2) = (X_1 \cap X_2, (B_1 \cup B_2)^*)\] or \((X_1 \cap X_2, (X_1 \cap X_2)^*)\)

\[(X_1, B_1) \lor (X_2, B_2) = ((X_1 \cup X_2)^*, B_1 \cap B_2)\] or \(((B_1 \cap B_2)^*, B_1 \cap B_2)\)

Thus, all formal concepts from \((U, A, I)\) form a complete lattice, and it is defined as a concept lattice and denoted by \(L(U, A, I)\).

Lemma 1. For any \(X_1, X_2, X \subseteq U, B_1, B_2, B \subseteq A\), here \((U, A, I)\) is a formal context, the following statements hold:

1. \(X_1 \subseteq X_2 \Rightarrow X_1^* \subseteq X_2^*\)
2. \(X \subseteq X^{**}\), \(B \subseteq B^{**}\)
3. \(X^* = X^{***}, \quad B^* = B^{***}\)
4. \(X \subseteq B^* \Leftrightarrow B \subseteq X^*\)
5. \((X_1 \cup X_2)^* = X_1^* \cup X_2^*, \quad (B_1 \cup B_2)^* = B_1^* \cap B_2^*\)
6. \((X_1 \cap X_2)^* \supseteq X_1^* \cup X_2^*, \quad (B_1 \cap B_2)^* \supseteq B_1^* \cup B_2^*\)

A formal context is typically represented by a table of 0 and 1, with 1s meaning binary relations between objects (rows) and attributes (columns). A simple example of a formal context is presented as follows:

The formal concepts in Table 1 can be calculated as given in the following Table 2.
Table 1: Formal context \((G, M, I)\)

| \(G\) | \(a_1\) | \(a_2\) | \(a_3\) | \(a_4\) | \(a_5\) |
|-------|---------|---------|---------|---------|---------|
| 1     | 0       | 1       | 1       | 0       | 0       |
| 2     | 1       | 1       | 0       | 0       | 0       |
| 3     | 1       | 0       | 0       | 0       | 0       |
| 4     | 0       | 0       | 0       | 0       | 1       |
| 5     | 0       | 0       | 0       | 1       | 1       |
| 6     | 0       | 0       | 1       | 1       | 1       |
| 7     | 1       | 1       | 1       | 0       | 0       |

Table 2: Formal concepts in Table 1

\[ C_0 = (\{1,2,3,4,5,6,7\}, \emptyset) \]
\[ C_4 = (\{2,3,7\}, \{a_1\}) \]
\[ C_3 = (\{1,2,7\}, \{a_2\}) \]
\[ C_2 = (\{1,6,7\}, \{a_3\}) \]
\[ C_1 = (\{4,5,6\}, \{a_5\}) \]
\[ C_5 = (\{2,7\}, \{a_1, a_2\}) \]
\[ C_6 = (\{1,7\}, \{a_2, a_3\}) \]
\[ C_7 = (\{5,6\}, \{a_4, a_5\}) \]
\[ C_9 = (\{7\}, \{a_1, a_2, a_3\}) \]
\[ C_8 = (\{6\}, \{a_3, a_4, a_5\}) \]
\[ C_{10} = (\emptyset, \{a_1, a_2, a_3, a_4, a_5\}) \]

In fact, one can visualise a formal concept in a table of 0 and 1 as a closed rectangle of 1s, where the rows and columns are not necessarily contiguous. Suppose we define the cell of the \(i\)th row and \(j\)th column as \((i,j)\). Thus in Table 1, \((5,4), (5,5), (6,4)\) and \((6,5)\) form the concept \(C_7\), and \(C_7\) is a rectangle of height 2 and width 2. Similarly \((6,3), (6,4)\) and \((6,5)\) form the concept \(C_8\), and \(C_8\) is a rectangle of height 1 and width 3. \((1,3), (6,3)\) and \((7,3)\) form the concept \(C_2\), and \(C_2\) is a rectangle of height 3 and width 1, here \((1,3)\) and \((6,3)\) are not contiguous.

On the other hand, a formal concept can be obtained by applying the \(^*\) operator to a set of attributes to get its extension, and then applying the \(^*\) operator to the extension to get the intension. From the context in Table 1, \(\{a_4\}^* = \{5,6\}\) and \(\{5,6\}^* = \{a_4, a_5\}\). So \((\{5,6\}, \{a_4, a_5\})\) is concept \(C_7\) in Table 2. From the rectangle theory of a formal concept, \(C_7\) is the biggest rectangle to cover \((5,4), (5,5), (6,4)\) and \((6,5)\) simultaneously.

Based on the rectangle theory of a formal concept, next we will discuss the concept reduction preserving binary relations.

### 3 Concept reduction preserving binary relations

We first present the definition of concept reduction preserving binary relations as follows.

**Definition 4.** [3][26] Let \((G, M, I)\) be a formal context. \(F \subseteq L(G, M, I)\) is called a consistent concept set preserving binary relations if

\[
I = \bigcup_{(A_i, B_i) \in F} A_i \times B_i
\]

Based on the rectangle theory of a formal concept, next we will discuss the concept reduction preserving binary relations.
Further, $\mathcal{F} \subseteq L(G, M, I)$ is called a concept reduction set preserving binary relations if for any $(A, B) \in \mathcal{F}$
\[ I \neq \bigcup_{(A_i, B_i) \in \mathcal{F} \setminus \{(A, B)\}} A_i \times B_i \]

It follows from Definitions 4 that there exists at least one concept reduction set preserving binary relations from $L(G, M, I)$ [3].

It is obvious that
\[ \mathcal{O}(G, M, I) = \{(g^{**}, g^*)|g \in G\} \]
is a consistent concept set preserving binary relations.
\[ \mathcal{A}(G, M, I) = \{(m^*, m^{**})|m \in M\} \]
is also a consistent concept set preserving binary relations [3].

For example, in Table 1, $\{c_1, c_2, c_3, c_4, c_7\}$ is a concept reduction set preserving binary relations for formal concepts in Table 1. In other words, $\{c_1, c_4, c_5, c_6, c_7, c_8\}$ is also a concept reduction set preserving binary relations, thus the concept reduction set is not unique.

Based on Definition 4, the significance of a concept in a given formal context may be different and they play different roles in the concept reduction. These concepts can be classified into three types: the core concepts, the relatively necessary concepts, and the unnecessary concepts. Next we provide the characteristics of these types of concepts.

**Definition 5.** [3, 26] Let $(G, M, I)$ be a formal context. $F_i \subseteq L(G, M, I)$, where $i \in \tau$, are all concept reductions of $L(G, M, I)$, then the concepts in $L(G, M, I)$ are classified into three types:

1. Core concept set: $L = \bigcap_{i \in \tau} F_i$
2. Relatively necessary concept set: $M = \bigcup_{i \in \tau} F_i - \bigcap_{i \in \tau} F_i$
3. Unnecessary concept set: $N = L(G, M, I) - \bigcup_{i \in \tau} F_i$

For example, in Table 1 (5, 4) is only covered by concept $c_7$, so $c_7$ is a core concept. It is easy to show that $c_1$ and $c_4$ are also core concepts. Thus, $c_2, c_3, c_5, c_6$ and $c_8$ are relatively necessary concepts. Later we will explain why $c_9$ is an unnecessary concept.

Based on the rectangle theory of formal concept, the following judgment theorem of the core concept is given.

**Theorem 1.** [3, 26] Let $(G, M, I)$ be a formal context. For concept $(A, B) \in L(G, M, I)$, if $(g, m) \in (A, B)$ is only covered by $(A, B)$, then concept $(A, B)$ is a core concept.

In what follows, our focus is the discussions of relatively necessary concept and unnecessary concept. First we present the following lemma.

**Lemma 2.** Let $(G, M, I)$ be a formal context. For concept $(A, B) \in L(G, M, I)$, if there exists a set $A' \subseteq G \setminus A$ such that $B = \bigcup_{g \in A'} g^*$, or there exists a set $B' \subseteq M \setminus B$ such that $A = \bigcup_{m \in B'} m^*$, then $(A, B)$ is an unnecessary concept, and $(A, B)$ is called a side covered concept.
Proof. Without loss of generality, we assume that \( B = \bigcup_{g \in A'} g^* \).

For any \((x, y) \in (A, B)\), suppose that \( y \in g^* \). As \( g \notin A \), so \((g, y) \notin (A, B)\).

For any \((C, D)\), such that \((g, y) \in (C, D)\), by Lemma 1(1),

\[ g \in C \Rightarrow D \subseteq g^* \Rightarrow D \subseteq B \Rightarrow A \subseteq C \Rightarrow (x, y) \in (C, D) \]

Suppose that \( F \subseteq L(G, M, I) \) is a consistent concept set, then \( F \setminus \{(A, B)\} \) is still a consistent concept set. Thus \((A, B)\) is an unnecessary concept. \( \square \)

For example, in Table 1, \( \{a_1, a_2, a_3\} = \{1\}^* \cup \{2\}^* \), so \( C_9 = (\{7\}, \{a_1, a_2, a_3\}) \) is an unnecessary concept.

If we visualise a formal concept as a rectangle, then the following lemma is obvious.

**Lemma 3.** Let \((G, M, I)\) be a formal context. Concept \((A, B) \in L(G, M, I)\) must be unnecessary if \((A, B)\) is covered by some core concepts, where cover means for any \((g, m) \in (A, B)\), \((g, m)\) is also in a core concept.

### Table 3: Formal context \((G, M, I_3)\)

| \(G\) | \(a_1\) | \(a_2\) | \(a_3\) | \(a_4\) | \(a_5\) |
|-------|--------|--------|--------|--------|--------|
| 1     | 1      | 1      | 1      | 0      | 0      |
| 2     | 1      | 1      | 1      | 0      | 0      |
| 3     | 1      | 1      | 1      | 0      | 0      |
| 4     | 0      | 1      | 1      | 1      | 1      |
| 5     | 0      | 1      | 1      | 1      | 1      |
| 6     | 0      | 1      | 1      | 1      | 1      |

For example, in Table 3, there are concepts \(C_1 = (\{4, 5, 6\}, \{a_2, a_3, a_4, a_5\})\), \(C_2 = (\{1, 2, 3\}, \{a_1, a_2, a_3\})\) and \(C_3 = (\{1, 2, 3, 4, 5, 6\}, \{a_2, a_3\})\). \((1, a_1)\) is only covered by \(C_2\), so \(C_2\) is a core concept. \((4, a_5)\) is only covered by \(C_1\), so \(C_1\) is a core concept. If we visualise a formal concept as a rectangle, \(C_3\) is covered by \(C_2\) and \(C_1\), so \(C_3\) is an unnecessary concept.

Based on Lemma 2 and Lemma 3, we have the following judgment theorem of unnecessary concept.

**Theorem 2.** Let \((G, M, I)\) be a formal context. For \((A, B) \in L(G, M, I)\) is not a core concept, \((A, B)\) is an unnecessary concept if and only if \((A, B)\) is covered by some core concepts, or covered by some side covered concepts, or covered by both some core concepts, and some side covered concepts, where cover means for any \((g, m) \in (A, B)\), \((g, m)\) is also in a core concept, or in a side covered concept.

Proof. \(\Leftarrow\) It follows immediately from Lemma 2 and Lemma 3.

\(\Rightarrow\) Let

\[ G' = G \setminus A, \mathcal{O}'(G, M, I) = \{(g^{**}, g^*)|g \in G'\} \]

\[ M' = M \setminus B, \mathcal{A}'(G, M, I) = \{(m^*, m^{**})|m \in M'\} \]

\[ \mathcal{F} = \mathcal{O}'(G, M, I) \cup \mathcal{A}'(G, M, I) \]
\((A, B) \in L(G, M, I)\) is not a core concept, thus for each \((g, m) \in (A, B)\), \((g, m)\) is covered by at least two concepts. Therefore, all core concepts are in \(\mathcal{F}\).

\((A, B)\) is an unnecessary concept, thus

\[ I = \bigcup_{(A_i, B_i) \in \mathcal{F}} A_i \times B_i \]

Therefore, \((A, B)\) is covered by some core concepts, or covered by some side covered concepts, or covered by both some core concepts, and some side covered concepts. \(\Box\)

We also have the following judgment theorem of relatively necessary concept.

**Theorem 3.** Let \((G, M, I)\) be a formal context. For \((A, B) \in L(G, M, I)\) is not a core concept, \((A, B)\) is a relatively necessary concept if and only if

\[ I \neq \bigcup_{(A_i, B_i) \in \mathcal{F}} A_i \times B_i \]

where

\[ G' = G \setminus A, \mathcal{O}'(G, M, I) = \{(g^{**}, g^*)|g \in G'\} \]
\[ M' = M \setminus B, \mathcal{A}'(G, M, I) = \{(m^*, m^{**})|m \in M'\} \]
\[ \mathcal{F} = \mathcal{O}'(G, M, I) \cup \mathcal{A}'(G, M, I) \]

It should be noted that with our simple judgment theorems, it is easy to develop a program for computing both relative necessary concepts and unnecessary concepts.

| \(G\) | \(a_1\) | \(a_2\) | \(a_3\) | \(a_4\) | \(a_5\) |
|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 0 | 0 |
| 2 | 1 | 1 | 1 | 0 | 0 |
| 3 | 1 | 1 | 1 | 0 | 0 |
| 4 | 0 | 1 | 1 | 1 | 1 |
| 5 | 0 | 1 | 1 | 1 | 1 |
| 6 | 0 | 1 | 1 | 1 | 1 |
| 7 | 1 | 1 | 0 | 0 | 0 |
| 8 | 1 | 0 | 1 | 0 | 0 |

For example, in Table 4, there are concepts \(C_1 = (\{4, 5, 6\}, \{a_2, a_3, a_4, a_5\})\), \(C_2 = (\{1, 2, 3\}, \{a_1, a_2, a_3\})\) and \(C_3 = (\{1, 2, 3, 4, 5, 6\}, \{a_2, a_3\})\). \(\{a_1, a_2, a_3\} = \{7\}^* \cup \{8\}^*\), so \(C_2 = (\{1, 2, 3\}, \{a_1, a_2, a_3\})\) is a side covered concept. \((4, a_5)\) is only covered by \(C_1\), so \(C_1\) is a core concept. If we visualise a formal concept as a rectangle, \(C_3\) is covered by \(C_2\) and \(C_1\), so \(C_3\) is an unnecessary concept.

In Table 4 consider concept \(C_4 = (\{1, 2, 3, 7\}, \{a_1, a_2\})\). It is easy to show that \(C_4\) is not a core concept. Further, \(\mathcal{O}'(G, M, I) = \{(g^{**}, g^*)|g \in G'\} = \{c_1, c_5\}\), where \(C_1 = \)
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\((\{4, 5, 6\}, \{a_2, a_3, a_4, a_5\})\), \(C_5 = (\{1, 2, 3, 8\}, \{a_1, a_3\})\). \(A'(G, M, I) = \{(m^*, m^{**})|m \in M'\} = \{c_1, c_6\}\), where \(C_6 = (\{1, 2, 3, 4, 5, 6, 8\}, \{a_3\})\). As

\[ F = O'(G, M, I) \cup A'(G, M, I) = \{c_1, c_5, c_6\} \]

\(C_4\) is not covered by \(F\), thus \(C_4\) is a relatively necessary concept.

To further discuss the characteristics of concepts of each type, we consider the maximum number of concepts of each type. As a core concept is in \(O(G, M, I) \cap A(G, M, I)\), the following lemma is straightforward.

**Lemma 4.** Let \((G, M, I)\) be a formal context, and \(m = |G|, n = |M|\). Then the maximum number of core concepts is not greater than \(\min(m, n)\).

**Table 5: Formal context \((G, M, I_5)\)**

| \(G\) | \(a_1\) | \(a_2\) | \(a_3\) | \(a_4\) | \(a_5\) |
|---|---|---|---|---|---|
| 1 | 1 | 1 | 0 | 0 | 0 |
| 2 | 1 | 0 | 1 | 0 | 0 |
| 3 | 1 | 0 | 0 | 1 | 0 |
| 4 | 1 | 0 | 0 | 0 | 1 |
| 5 | 0 | 1 | 1 | 0 | 0 |
| 6 | 0 | 1 | 0 | 1 | 0 |
| 7 | 0 | 1 | 0 | 0 | 1 |
| 8 | 0 | 0 | 1 | 1 | 0 |
| 9 | 0 | 0 | 1 | 0 | 1 |
| 10 | 0 | 0 | 0 | 1 | 1 |

For example, in Table\[7\] there are 5 concepts in \(A(G, M, I)\), and there are 10 concepts in \(O(G, M, I)\), but there is neither core concepts nor unnecessary concepts. The number of relatively necessary concepts is

\[ 10 + 5 = \binom{5}{2} + 5 = \binom{|M|}{\lfloor|M|/2\rfloor} + |M| \]

Inspired by this example, we have the following lemma.

**Lemma 5.** Let \((G, M, I)\) be a formal context, and \(n = |M|\). Suppose \(|G| > \binom{n}{\lfloor n/2 \rfloor}\), then the maximum number of relatively necessary concepts is not less than

\[ \binom{n}{\lfloor n/2 \rfloor} + n \]

where \(\lfloor x \rfloor\) is the largest number that is less than or equal to \(x\).

**Proof.** Note that \(\binom{n}{k} = \binom{n}{n-k}\), and \(\binom{n}{\lfloor n/2 \rfloor}\) is the maximum, where \(1 \leq k \leq n\).

For the formal context \((G, M, I)\), let each object (corresponding to a row in the table) have \(\lfloor n/2 \rfloor\) attributes. According to combinatorial mathematics, there are \(\binom{n}{\lfloor n/2 \rfloor}\)
different objects. The $k$th object forms a concept $(\{k\}^*, \{k\}^*)$, where $\{k\}^* = \{k\}$, which is not covered by

$$\mathcal{F} = \mathcal{O}'(G,M,I) \cup \mathcal{A}'(G,M,I)$$

where

$$\mathcal{O}'(G,M,I) = \{(g^{**}, g^*) | g \in G \setminus \{k\}\}$$

$$\mathcal{A}'(G,M,I) = \{(m^*, m^{**}) | m \in M \setminus \{k\}\}$$

Thus $(\{k\}^*, \{k\}^*)$ of the $k$th object is relatively necessary.

In the table of formal context $(G,M,I)$, if there exist two same columns, which means the values of the two positions are always the same. This is a contradiction by combinatorial mathematics, thus there are $n$ different columns. Similarly, one can prove that $(\{j\}^*, \{j\}^{**})$, where $\{j\}^{**} = \{j\}$ and $1 \leq j \leq n$, of the $j$th attribute is relatively necessary.

Therefore, there are $(\begin{bmatrix}n
\end{bmatrix} + n$ relatively necessary concepts.

\[\square\]

Table 6: Formal context $(G, M, I_6)$

|   | $a_1$ | $a_2$ | $a_3$ | $a_4$ |
|---|---|---|---|---|
| 1 | 1 | 1 | 0 | 0 |
| 2 | 1 | 0 | 1 | 0 |
| 3 | 1 | 0 | 0 | 1 |
| 4 | 0 | 1 | 1 | 0 |
| 5 | 0 | 1 | 0 | 1 |
| 6 | 0 | 0 | 1 | 1 |
| 7 | 1 | 1 | 1 | 0 |
| 8 | 1 | 1 | 0 | 1 |
| 9 | 1 | 0 | 1 | 1 |
| 10| 0 | 1 | 1 | 1 |

In Table 6 let $C_1 = ((1, 7), \{a_1, a_2\})$, $C_2 = ((2, 7), \{a_1, a_3\})$, $C_7 = ((7), \{a_1, a_2, a_3\})$, then $C_7$ is covered by $C_1$ and $C_2$, thus $C_7$ is an unnecessary concept. In fact, if we add one more object with 3 attributes, then this object (or row) must form an unnecessary concept. Thus we have the following lemma.

**Lemma 6.** Let $(G, M, I)$ be a formal context, and $n = |M|$. Suppose $|G| > \left(\begin{bmatrix}n
\end{bmatrix} + \begin{bmatrix}n
\end{bmatrix} + 1\right)$, then the maximum number of unnecessary concepts is not less than

$$\begin{bmatrix}n
\end{bmatrix}$$

where $[x]$ is the largest number that is less than or equal to $x$.

From the discussion above, we know that both relatively necessary concepts and unnecessary concepts are exponential time problems.
4 An algorithm for attribute reduction

Based on the discernibility matrix and Boolean function, a method of attribute reduction is proposed while preserving the extensions in concept lattices in [30, 31]. Similarly, two kinds of attribute reduction are proposed in the decision formal context based on the maximal rules [12], and an approach to computing attribute reductions is proposed based on the discernibility matrix and Boolean function. However, the approaches in [12, 30, 31] are very difficult to implemented.

Note that the attribute reduction in both [30, 31] and [12] can be simplified to reducing attributes while preserving the extensions for a set of formal concepts. We also know that for any formal concept \((X, B)\) of \(L(U, A, I)\), \(X^* = B\) and \(B^* = X\).

Based on this explicit idea, a fast algorithm for reducing attributes while preserving the extensions for a set of formal concepts is proposed in this paper. Technically, by checking \(X^{**} = X\) to determine whether to delete a column of formal context or not.

As the new algorithm would store both formal context and extension of a concept as a bit-array, and we can use bit-operations to process set operations "or" as well as "and". What is more, there is no need to consider other formal concepts in \(L(U, A, I)\), thus the algorithm, given below with analysis, is very fast.

\[
\text{Check whether to reduce the jth column} \\
\text{Input: formal context (U, A, I), without jth column,} \\
\text{and a set of concepts (X_i, B_i), 0 \leq i < k} \\
\text{Output: true if extensions of the concepts not changed,} \\
\text{false otherwise} \\
\begin{align*}
1 & \text{for } i \leftarrow k - 1 \text{ downto } 0 \text{ do} \\
2 & \quad \text{compute } X_i^* \text{ in formal context (U, A, I)} \\
3 & \quad \text{compute } X_i^{**} \text{ in formal context (U, A, I)} \\
4 & \quad \text{if } X_i^{**} \neq X_i \text{ return false} \\
5 & \quad \text{end for} \\
6 & \text{return true}
\end{align*}
\]

Line 2 – Construct an intension of \(X\), which could be different from \(B\). It is implemented in C language as the following.

```c
memset(Row, 0, sizeof(Row)); // clear the Row, which is used to store the intension
for(q=0; q<n; q++) {
    for(int k = (m - 1)/32; k >= 0; k--) {
        // check whether the qth column contains the ith extension column
        if((contextCol[k][q] & extensionCol[k][i]) != extensionCol[k][i]) break;
    }
    // the qth column contains the ith extension column, so to change the intension
    if(k<0) Row[q] = (1<<(q%32));
}
```

Note that here \(m = |U|, n = |A|\), and both formal context contextCol and extension of a concept extensionCol are stored as bit-arrays (in the form of 32-bit unsigned integers). Without processing each object in the formal context and extension of a
concept, the bit and operation is used when check whether the $q$th column of formal context contains the $i$th extension column.

First call the procedure to check whether to reduce the first column. If return true, then let the first column of the formal context $(U, A, I)$ be zero, otherwise there is no change. Second call the procedure to check whether to reduce the second column. If return true, then let the second column of the formal context $(U, A, I)$ be zero, otherwise there is no change. Keep this process going and at the end of this process, we get an attribute reduction preserving the extensions for a set of formal concepts.

Suppose that first call the procedure to check whether to reduce the second column, second call the procedure to check whether to reduce the third column. Finally call the procedure to check whether to reduce the first column. At the end of this process, we may get a different attribute reduction preserving the same set of formal concepts.

Thus by calling the procedure with different starting $i$th column, where $1 \leq i \leq n$, we can get all attribute reductions preserving the same set of formal concepts.

Some experiments are done to compute attribute reductions with the new algorithm. The experiment results are given in Table 7. Here mushroom data and nursery data are from [6]. The experiments are carried out using a laptop computer with an Intel Core i5-2450M 2.50 GHz processor and 8GB of RAM.

|                    | Mushroom | Nursery |
|--------------------|----------|---------|
| $|G| \times |M|$     | 8, 124 x 115 | 12, 960 x 30 |
| Number of concepts | 512      | 512     |
| Time in seconds    | 4.218    | 0.582   |
| Number of reduced columns | 19    | 2       |

In Table 7, In-Close4 algorithm from [2] is used to compute all the formal concepts for mushroom data. From which, we take the first 512 concepts. With our new algorithm, 19 columns of formal context can be reduced while preserving the extensions for 512 formal concepts. Experiments demonstrate that the new algorithm is much effective in the computation of attribute reductions.
5 Conclusions and future work

A pair \((X, B)\) is called a formal concept if \(X^* = B\) and \(B^* = X\), which is a frequently used definition. With another perspective, based on rectangle theory of formal concept and set covering theory, the concept reduction preserving binary relations was discussed. It is known that there are three types of formal concepts: core concepts, relative necessary concepts and unnecessary concepts. We presented the new judgment theorems for relative necessary concepts and unnecessary concepts. Also, we gave bounds for both the maximum number of relative necessary concepts and the maximum number of unnecessary concepts. From our results, it is known that both relatively necessary concepts and unnecessary concepts are exponential time problems.

As the attribute reduction approaches in both [30] and [12] can be simplified to reducing attributes while preserving the extensions for a set of formal concepts, a fast algorithm was proposed for reducing attributes while preserving the extensions for a set of formal concepts. Experiments demonstrated that the new algorithm is much effective in the computation of attribute reductions.

Object oriented concept lattice is a more extensive concept lattice [27]. So, it is more difficult to reduce concepts or attributes in object oriented concept lattices. In the future, we will apply the data structure and technique in these algorithms to object oriented concept lattices, attribute oriented concept lattices and so on.

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