CHARACTERIZATION OF NUCLEARITY FOR BEURLING-BJÖRCK SPACES

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Abstract. We characterize the nuclearity of the Beurling-Björck spaces $S^{(ω)}(R^d)$ and $S^{(η)}(R^d)$ in terms of the defining weight functions $ω$ and $η$.

1. Introduction

In recent works Boiti et al. [3, 4, 5] have investigated the nuclearity of the Beurling-Björck space $S^{(ω)}(R^d)$ (in our notation below). Their most general result [5, Theorem 3.3] establishes the nuclearity of this Fréchet space when $ω$ is a Braun-Meise-Taylor type weight function [6] (where non-quasianalyticity is relaxed to $ω(t) = o(t)$ and the condition $\log(t) = o(ω(t))$ from [6] is weakened to $\log(t) = O(ω(t))$).

The aim of this note is to improve and generalize [5, Theorem 3.3] by considerably weakening the set of hypotheses on the weight functions, providing a complete characterization of the nuclearity of these spaces (for radially increasing weight functions), and considering anisotropic spaces and the Roumieu case as well. Particularly, we shall show that the conditions ($β$) and ($δ$) from [5, Definition 2.1] play no role in deducing nuclearity.

Let us introduce some concepts in order to state our main result. A weight function on $R^d$ is simply a non-negative, measurable, and locally bounded function. We consider the following standard conditions [2, 6]:

- (α) There are $L, C > 0$ such that $ω(x+y) \leq L(ω(x)+ω(y))+\log C$, for all $x, y \in R^d$.
- (γ) There are $A, B > 0$ such that $A \log(1 + |x|) \leq ω(x) + \log B$, for all $x \in R^d$.

\{γ\} $\lim_{|x|→∞} \frac{ω(x)}{\log |x|} = \infty$.

A weight function $ω$ is called radially increasing if $ω(x) \leq ω(y)$ whenever $|x| \leq |y|$. Given a weight function $ω$ and a parameter $λ > 0$, we introduce the family of norms

$$\|φ\|_{ω,λ} = \sup_{x \in R^d} |φ(x)|e^{λω(x)}.$$
If $\eta$ is another weight function, we consider the Banach space $S^{\lambda}_{\eta,\omega}(\mathbb{R}^d)$ consisting of all $\varphi \in S'(\mathbb{R}^d)$ such that $\|\varphi\|_{S^\lambda_{\eta,\omega}} := \|\varphi\|_{\eta,\lambda} + \|\hat{\varphi}\|_{\omega,\lambda} < \infty$, where $\hat{\varphi}$ stands for the Fourier transform of $\varphi$. Finally, we define the Beurling-Björck spaces (of Beurling and Roumieu type) as

$$S^{(\omega)}_{(\eta)}(\mathbb{R}^d) = \lim_{\lambda \to +\infty} S^{\lambda}_{\eta,\omega}(\mathbb{R}^d) \quad \text{and} \quad S^{(\omega)}_{\{\eta\}}(\mathbb{R}^d) = \lim_{\lambda \to 0^+} S^{\lambda}_{\eta,\omega}(\mathbb{R}^d).$$

**Theorem 1.1.** Let $\omega$ and $\eta$ be weight functions satisfying (a).

(a) If $\omega$ and $\eta$ satisfy $(\gamma)$, then $S^{(\omega)}_{(\eta)}(\mathbb{R}^d)$ is nuclear. Conversely, if in addition $\omega$ and $\eta$ are radially increasing, then the nuclearity of $S^{(\omega)}_{(\eta)}(\mathbb{R}^d)$ implies that $\omega$ and $\eta$ satisfy $(\gamma)$ (provided that $S^{(\omega)}_{\{\eta\}}(\mathbb{R}^d) \neq \{0\}$).

(b) If $\omega$ and $\eta$ satisfy $\{\gamma\}$, then $S^{(\omega)}_{\{\eta\}}(\mathbb{R}^d)$ is nuclear. Conversely, if in addition $\omega$ and $\eta$ are radially increasing, then the nuclearity of $S^{(\omega)}_{\{\eta\}}(\mathbb{R}^d)$ implies that $\omega$ and $\eta$ satisfy $\{\gamma\}$ (provided that $S^{(\omega)}_{\{\eta\}}(\mathbb{R}^d) \neq \{0\}$).

Furthermore, we discuss the equivalence of the various definitions of Beurling-Björck type spaces given in the literature [11, 10, 5] but considered here under milder assumptions. In particular, we show that, if $\omega$ satisfies (a) and $(\gamma)$, our definition of $S^{(\omega)}_{\{\eta\}}(\mathbb{R}^d)$ coincides with the one employed in [5].

## 2. The conditions $(\gamma)$ and $\{\gamma\}$

In this preliminary section, we study the connection between the conditions $(\gamma)$ and $\{\gamma\}$ and the equivalence of the various definitions of Beurling-Björck type spaces. Let $\omega$ and $\eta$ be two weight functions. Given parameters $k, l \in \mathbb{N}$ and $\lambda > 0$, we introduce the family of norms

$$\|\varphi\|_{S^k_{\eta,\omega}} := \max_{|\alpha| \leq k} \max_{|\beta| \leq l} \sup_{x \in \mathbb{R}^d} |x^\beta \varphi^{(\alpha)}(x) e^{\lambda \omega(x)}|.$$  

We define $\tilde{S}^{(\omega)}_{\eta,\omega}(\mathbb{R}^d)$ as the Fréchet space consisting of all $\varphi \in S(\mathbb{R}^d)$ such that

$$\|\varphi\|_{\tilde{S}^k_{\eta,\omega}} := \|\varphi\|_{\eta,k,\omega,\lambda} + \|\hat{\varphi}\|_{\omega,k,\omega,\lambda} < \infty, \quad \forall k \in \mathbb{N}.$$  

We set

$$\tilde{S}^{(\omega)}_{(\eta)}(\mathbb{R}^d) = \lim_{\lambda \to +\infty} \tilde{S}^{\lambda}_{\eta,\omega}(\mathbb{R}^d) \quad \text{and} \quad \tilde{S}^{(\omega)}_{\{\eta\}}(\mathbb{R}^d) = \lim_{\lambda \to 0^+} \tilde{S}^{\lambda}_{\eta,\omega}(\mathbb{R}^d).$$

We use $S^{(\omega)}_{(\eta)}(\mathbb{R}^d)$ as a common notation for $S^{(\omega)}_{(\eta)}(\mathbb{R}^d)$ and $S^{(\omega)}_{\{\eta\}}(\mathbb{R}^d)$; a similar convention will be used for other spaces. In accordance to this, $\{\gamma\}$ stands for $(\gamma)$ and $\{\gamma\}$.

Let us point out that the spaces $S^{(\omega)}_{(\eta)}(\mathbb{R}^d)$ might be trivial, due to uncertainty principles for Fourier transform pairs (cf. [12, 14, 13, 15]). On the other hand, a classical result of Gelfand and Shilov [9] implies that if there are $a > 0$ and $b > 0$ such that $\omega(x) = O(|x|^a)$ and $\eta(x) = O(|x|^b)$, then $S^{(\omega)}_{\{\eta\}}(\mathbb{R}^d) \neq \{0\}$ whenever $1/a + 1/b \geq 1$, while $S^{(\omega)}_{(\eta)}(\mathbb{R}^d) \neq \{0\}$ if $1/a + 1/b > 1$ holds. In particular, if one of the two weight
functions is $O(|x|)$ and the other one is polynomially bounded, then $S^{[\omega]}(\mathbb{R}^d)$ is always non-trivial. In this regard, we mention that condition $(\alpha)$ implies polynomial growth.

The following result is a generalization of [7, Theorem 3.3] and [10, Corollary 2.9] (see also [5, Theorem 2.3]).

**Theorem 2.1.** Let $\omega$ and $\eta$ be two weight functions satisfying $(\alpha)$. Suppose that $S^{[\omega]}(\mathbb{R}^d) \neq \{0\}$. The following statements are equivalent:

(i) $\omega$ and $\eta$ satisfy $[\gamma]$.

(ii) $S^{[\omega]}(\mathbb{R}^d) = \widetilde{S}^{[\omega]}(\mathbb{R}^d)$ as locally convex spaces.

(iii) $S^{[\omega]}(\mathbb{R}^d) = \{ \varphi \in S'(\mathbb{R}^d) | \forall \lambda > 0 (\exists \lambda > 0) \forall \alpha \in \mathbb{N}^d : \sup_{x \in \mathbb{R}^d} |x^\alpha \varphi(x)| e^{\lambda \eta(x)} < \infty \text{ and } \sup_{\xi \in \mathbb{R}^d} |\xi^\alpha \varphi(\xi)| e^{\lambda \omega(\xi)} < \infty \}.$

(iv) $S^{[\omega]}(\mathbb{R}^d) = \{ \varphi \in S'(\mathbb{R}^d) | \forall \lambda > 0 (\exists \lambda > 0) \forall \alpha \in \mathbb{N}^d : \int_{\mathbb{R}^d} |\varphi(\alpha)(x)| e^{\lambda \eta(x)} dx < \infty \text{ and } \int_{\mathbb{R}^d} |\varphi(\alpha)(\xi)| e^{\lambda \omega(\xi)} d\xi < \infty \}.$

(v) $S^{[\omega]}(\mathbb{R}^d) \subseteq S(\mathbb{R}^d)$.

Following [10], our proof of Theorem 2.1 is based on the mapping properties of the short-time Fourier transform (STFT). We fix the constants in the Fourier transform as

$$\mathcal{F}(\varphi)(\xi) = \tilde{\varphi}(\xi) = \int_{\mathbb{R}^d} \varphi(t) e^{-2\pi i t \cdot \xi} dt.$$

The STFT of $f \in L^2(\mathbb{R}^d)$ with respect to the window $\psi \in L^2(\mathbb{R}^d)$ is given by

$$V_\psi f(x, \xi) = \int_{\mathbb{R}^d} f(t) \overline{\psi(t-x)} e^{-2\pi i t \cdot \xi} dt, \quad (x, \xi) \in \mathbb{R}^{2d}.$$

The adjoint of $V_\psi : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^{2d})$ is given by the (weak) integral

$$V_\psi^* F(t) = \iint_{\mathbb{R}^{2d}} F(x, \xi) e^{2\pi i t \cdot \xi} \psi(t-x) dx d\xi.$$

A straightforward calculation shows that, whenever $(\chi, \psi)_{L^2} \neq 0$, then

$$\frac{1}{(\chi, \psi)_{L^2}} V_\psi^* \circ V_\psi = \text{id}_{L^2(\mathbb{R}^d)}.$$

Next, we introduce two additional function spaces. Given a parameter $\lambda > 0$, we define $\mathcal{K}_\omega^\lambda(\mathbb{R}^d)$ as the Fréchet space consisting of all $\varphi \in C^\infty(\mathbb{R}^d)$ such that $\|\varphi\|_{\omega, k, \lambda} := \|\varphi\|_{\omega, 0, \lambda} < \infty$ for all $k \in \mathbb{N}$ and set

$$\mathcal{K}_\omega((\mathbb{R}^d) = \lim_{\lambda \to \infty} \mathcal{K}_\omega^\lambda(\mathbb{R}^d) \quad \text{and} \quad \mathcal{K}_\omega(\mathbb{R}^d) = \lim_{\lambda \to 0^+} \mathcal{K}_\omega^\lambda(\mathbb{R}^d).$$

Given a parameter $\lambda > 0$, we define $C_\omega^\lambda(\mathbb{R}^d)$ as the Banach space consisting of all $\varphi \in C(\mathbb{R}^d)$ such that $\|\varphi\|_{\omega, \lambda} < \infty$ and set

$$C_\omega((\mathbb{R}^d) = \lim_{\lambda \to \infty} C_\omega^\lambda(\mathbb{R}^d) \quad \text{and} \quad C_\omega(\mathbb{R}^d) = \lim_{\lambda \to 0^+} C_\omega^\lambda(\mathbb{R}^d).$$

We need the following extension of [10, Theorem 2.7]. We write $\bar{f}(t) = f(-t).$
Proposition 2.2. Let \( \omega \) and \( \eta \) be weight functions satisfying (\( \alpha \)) and [\( \gamma \)]. Define the weight \( (\eta \oplus \omega)(x, \xi) = \eta(x) + \omega(\xi) \) for \( (x, \xi) \in \mathbb{R}^{2d} \). Fix a window \( \psi \in S_{[\eta]}^{[\omega]}(\mathbb{R}^d) \).

(a) The linear mappings

\[
V_\psi : \overline{\mathcal{S}}_{[\eta]}^{[\omega]}(\mathbb{R}^d) \to C_{[\eta \oplus \omega]}(\mathbb{R}^{2d}) \quad \text{and} \quad V_\psi^* : C_{[\eta \oplus \omega]}(\mathbb{R}^{2d}) \to \overline{\mathcal{S}}_{[\eta]}^{[\omega]}(\mathbb{R}^d)
\]

are continuous.

(b) The linear mappings

\[
V_\psi : \mathcal{S}_{[\eta]}^{[\omega]}(\mathbb{R}^d) \to \mathcal{K}_{[\eta \oplus \omega]}(\mathbb{R}^{2d}) \quad \text{and} \quad V_\psi^* : \mathcal{K}_{[\eta \oplus \omega]}(\mathbb{R}^{2d}) \to \mathcal{S}_{[\eta]}^{[\omega]}(\mathbb{R}^d)
\]

are continuous.

Proof. It suffices to show that \( V_\psi : \overline{\mathcal{S}}_{[\eta]}^{[\omega]}(\mathbb{R}^d) \to \mathcal{K}_{[\eta \oplus \omega]}(\mathbb{R}^{2d}) \), \( V_\psi : \mathcal{S}_{[\eta]}^{[\omega]}(\mathbb{R}^d) \to C_{[\eta \oplus \omega]}(\mathbb{R}^{2d}) \), and \( V_\psi^* : \mathcal{K}_{[\eta \oplus \omega]}(\mathbb{R}^{2d}) \to \mathcal{S}_{[\eta]}^{[\omega]}(\mathbb{R}^d) \) are continuous. Indeed, the continuity of \( V_\psi^* : \mathcal{K}_{[\eta \oplus \omega]}(\mathbb{R}^{2d}) \to \mathcal{S}_{[\eta]}^{[\omega]}(\mathbb{R}^d) \) and \( V_\psi : \mathcal{S}_{[\eta]}^{[\omega]}(\mathbb{R}^d) \to C_{[\eta \oplus \omega]}(\mathbb{R}^{2d}) \) would be immediate consequences, whereas, in view of (2.1), we could then always factor \( V_\psi \) on \( \mathcal{S}^{[\omega]}_{[\eta]}(\mathbb{R}^d) \) as a composition of continuous mappings,

\[
V_\psi : \mathcal{S}^{[\omega]}_{[\eta]}(\mathbb{R}^d) \xrightarrow{\psi} C_{[\eta \oplus \omega]}(\mathbb{R}^{2d}) \xrightarrow{\overline{\mathcal{S}}_{[\eta]}^{[\omega]}} \mathcal{S}^{[\omega]}_{[\eta]}(\mathbb{R}^d) \xrightarrow{\mathcal{K}_{[\eta \oplus \omega]}} \mathcal{K}_{[\eta \oplus \omega]}(\mathbb{R}^{2d}),
\]

where, when \( \psi \neq 0 \), the window \( \chi \) is chosen such that \( \chi \in \mathcal{S}^{[\omega]}_{[\eta]}(\mathbb{R}^d) \) and \( (\psi, \chi)_{L^2} = 1 \).

(The relation (2.2) actually yields \( \mathcal{S}^{[\omega]}_{[\eta]}(\mathbb{R}^d) = \overline{\mathcal{S}}_{[\eta]}^{[\omega]}(\mathbb{R}^d) \).)

Suppose that \( \psi \in \mathcal{S}^{[\omega]}_{[\eta]}(\mathbb{R}^d) \), so that \( \lambda_0 > 0 \) is fixed in the Roumieu case but can be taken as large as needed in the Beurling case. Let \( A \) and \( B = B_A \) be the constants occurring in (\( \gamma \)) (in the Roumieu case, \( A \) can be taken as large as needed due to \( \{ \gamma \} \)). Furthermore, we assume that all constants occurring in (\( \alpha \)) and [\( \gamma \)] are the same for both \( \omega \) and \( \eta \). We may also assume that \( \lambda_0 - k/A > 0 \). We first consider \( V_\psi \). Let \( \lambda < (\lambda_0 - k/A)/L \) be arbitrary. For all \( k \in \mathbb{N} \) and \( \varphi \in \mathcal{S}^{[\lambda L + k/A]}_{[\eta \omega]}(\mathbb{R}^d) \), it holds that

\[
\max_{|\alpha| + |\beta| \leq k} \sup_{x, \xi \in \mathbb{R}^{2d}} |\partial^\beta_x \partial^\alpha_\xi V_\psi \varphi(x, \xi)| e^{\lambda \eta(x)} \\
\leq (2\pi)^k \max_{|\alpha| \in \mathbb{N}} \sup_{x \in \mathbb{R}^d} e^{\lambda \eta(x)} \int_{\mathbb{R}^d} |\varphi(t)| (1 + |t|)^k |\psi^{(\alpha)}(x - t)| dt \\
\leq (2\pi)^k \|\psi\|_{\eta, k, \lambda_0} \|\varphi\|_{\eta, \lambda L + k/A} \sup_{x \in \mathbb{R}^d} \int_{\mathbb{R}^d} e^{\lambda (\eta(x) - L \eta(t))} (1 + |t|)^k e^{-\frac{k}{\lambda} \eta(t)} e^{-\lambda_0 \eta(x - t)} dt \\
\leq (2\pi)^k B \pi C \|\psi\|_{\eta, k, \lambda_0} \|\varphi\|_{\eta, \lambda L + k/A} \int_{\mathbb{R}^d} e^{-(\lambda_0 - \lambda L) \eta(y)} dy
\]
and
\[
\max_{|\alpha + \beta| \leq k} \sup_{(x,\xi) \in \mathbb{R}^{2d}} |\partial^\alpha_x \partial^\beta_\xi V_\psi(x,\xi) e^{\lambda \omega(\xi)}| = \max_{|\alpha + \beta| \leq k} \sup_{(x,\xi) \in \mathbb{R}^{2d}} |\partial^\alpha_x \partial^\beta_\xi V_{\hat{\psi}}(x, -\xi) e^{\lambda \omega(\xi)}|
\]
\[
\leq (2\pi)^k \max_{|\beta| \leq \xi} e^{\lambda \omega(\xi)} \int_{\mathbb{R}^d} \left|\hat{\psi}(t)(1 + |t|)^k \hat{\varphi}(\beta)(\xi - t)\right| dt
\]
\[
\leq (2\pi)^kB^\frac{\omega}{\lambda} C^\lambda \|\hat{\varphi}\|_{\omega,k\lambda} \|\hat{\psi}\|_{\omega,\lambda_0} \int_{\mathbb{R}^d} e^{-(\lambda_0 - \lambda \omega \xi)}(t) dt.
\]
These inequalities imply the continuity of \(V_\psi : \widetilde{S}^{[\omega]}_{[\beta]}(\mathbb{R}^d) \to \mathcal{K}_{[\eta \equiv \omega]}(\mathbb{R}^{2d})\). Taking \(k = 0\) in the above norm bounds, we also obtain that \(V_\psi : \widetilde{S}^{[\omega]}_{[\beta]}(\mathbb{R}^d) \to C_{[\eta \equiv \omega]}(\mathbb{R}^{2d})\) is continuous.

Next, we treat \(V_\psi^*\). Let \(\lambda < \lambda_0 / L\) be arbitrary. For all \(k \in \mathbb{N}\) and \(\Phi \in C^{\lambda L + \frac{k}{\lambda}}_{[\eta \equiv \omega]}(\mathbb{R}^{2d})\) it holds that
\[
\|V_\psi^* \Phi\|_{\eta,k\lambda} \leq (2\pi)^k \max_{|\alpha| \leq k} e^{\lambda \omega(t)} \sum_{\beta \leq \alpha} \left(\frac{\alpha}{\beta}\right) \int_{\mathbb{R}^{2d}} |\Phi(x,\xi)|(1 + |\xi|)^k |\psi(\beta)(t - x)| dxd\xi
\]
\[
\leq (4\pi)^k \|\psi\|_{\eta,k\lambda,\lambda_0} \|\Phi\|_{\eta \equiv \omega,\lambda L + \frac{k}{\lambda}} \int_{\mathbb{R}^d} (1 + |\xi|) e^{-(\frac{\lambda}{L} + \lambda \omega)(\xi)} e^{\lambda \eta(t) - L \eta(x)} e^{-\lambda \eta(t - x)} dxd\xi
\]
\[
\leq (4\pi)^k B^\frac{\omega}{\lambda} C^\lambda \|\psi\|_{\eta,k\lambda,\lambda_0} \|\Phi\|_{\eta \equiv \omega,\lambda L + \frac{k}{\lambda}} \int_{\mathbb{R}^d} e^{-\lambda \omega(\xi) - (\lambda_0 - \lambda \eta)(\eta) y} dy d\xi
\]
and
\[
\|\mathcal{F}(V_\psi^* \Phi)\|_{\omega,k\lambda} = \max_{|\alpha| \leq k} e^{\lambda \omega(t)} \left|\partial^\alpha_t \int_{\mathbb{R}^d} \Phi(x,\xi) e^{2\pi i \xi \cdot x} e^{-2\pi i t \cdot \hat{\psi}(t - \xi)} dxd\xi\right|
\]
\[
\leq (4\pi)^k B^\frac{\omega}{\lambda} C^\lambda \|\psi\|_{\omega,k\lambda,\lambda_0} \|\Phi\|_{\omega \equiv \omega,\lambda L + \frac{k}{\lambda}} \int_{\mathbb{R}^d} e^{-\lambda \omega(\xi) - (\lambda_0 - \lambda \eta)(\eta) y} dxd\xi.
\]
Since \(\|\cdot\|_{\eta,k\lambda,\lambda_0} \leq B^\frac{\omega}{\lambda} \|\cdot\|_{\omega,k\lambda,\lambda_0 + \frac{k}{\lambda}}\) and \(\|\cdot\|_{\omega,k\lambda,\lambda_0} \leq B^\frac{\omega}{\lambda} \|\cdot\|_{\omega,k\lambda,\lambda_0 + \frac{k}{\lambda}}\) for all \(\lambda > 0\) and \(k \in \mathbb{N}\), the above inequalities show the continuity of \(V_\psi^*\).

In order to be able to apply Proposition 2.2, we show the ensuing simple lemma.

Lemma 2.3. Let \(\omega\) and \(\eta\) be weight functions satisfying (\(\alpha\)). If \(\mathcal{S}^{[\omega]}_{[\beta]}(\mathbb{R}^d) \neq \emptyset\), then also \(\widetilde{S}^{[\omega]}_{[\beta]}(\mathbb{R}^d) \neq \emptyset\).

Proof. Let \(\varphi \in \mathcal{S}^{[\omega]}_{[\beta]}(\mathbb{R}^d)\) \(\setminus\{0\}\). Pick \(\psi, \chi \in \mathcal{D}(\mathbb{R}^d)\) such that \(\int_{\mathbb{R}^d} \varphi(x) \chi(-x) dx = 1\) and \(\int_{\mathbb{R}^d} \psi(x) dx = 1\). Then, \(\varphi_0 = (\varphi * \chi) \mathcal{F}^{-1}(\psi) \in \widetilde{S}^{[\omega]}_{[\beta]}(\mathbb{R}^d)\) and \(\varphi_0 \neq 0\) (as \(\varphi_0(0) = 1\)).

Proof of Theorem 2.1
(i) \(\Rightarrow\) (ii) In view of Lemma 2.3 this follows from Proposition 2.2 and the reconstruction formula (2.1).

(ii) \(\Rightarrow\) (iii) Trivial.

(iii) \(\Rightarrow\) (iv) and (iv) \(\Rightarrow\) (v) These implications follow from the fact that \(\mathcal{S}(\mathbb{R}^d)\) consists precisely of all those \(\varphi \in \mathcal{S}'(\mathbb{R})\) such that
\[
\sup_{x \in \mathbb{R}^d} |x^\alpha \varphi(x)| < \infty \quad \text{and} \quad \sup_{\xi \in \mathbb{R}^d} |\xi^\alpha \hat{\varphi}(\xi)| < \infty
\]
for all $\alpha \in \mathbb{N}^d$ (see e.g. [7 Corollary 2.2]).

$(v) \Rightarrow (i)$ Since the Fourier transform is an isomorphism from $S_{[\alpha]}^{[\omega]}(\mathbb{R}^d)$ onto $S_{[\alpha]}^{[\omega]}(\mathbb{R}^d)$ and from $S(\mathbb{R}^d)$ onto itself, it is enough to show that $\eta$ satisfies $[\gamma]$. We start by constructing $\varphi_0 \in S_{[\alpha]}^{[\omega]}(\mathbb{R}^d)$ such that $\varphi_0(j) = \delta_{j,0}$ for all $j \in \mathbb{Z}^d$. Choose $\psi \in S_{[\alpha]}^{[\omega]}(\mathbb{R}^d)$ such that $\psi(0) = 1$. Set 

$$\chi(x) = \int_{[-\frac{1}{2}, \frac{1}{2}]^d} e^{-2\pi i x \cdot t} dt, \quad x \in \mathbb{R}^d.$$ 

Then, $\chi(j) = \delta_{j,0}$ for all $j \in \mathbb{Z}^d$. Hence, $\varphi_0 = \psi \chi$ satisfies all requirements. Let $(\lambda_j)_{j \in \mathbb{Z}^d}$ be an arbitrary multi-indexed sequence of positive numbers such that $\lambda_j \to \infty$ as $|j| \to \infty$ ($(\lambda_j)_{j \in \mathbb{Z}^d} = (\lambda)_{j \in \mathbb{Z}^d}$ for $\lambda > 0$ in the Roumieu case). Consider 

$$\varphi = \sum_{j \in \mathbb{Z}^d} \frac{e^{-\lambda_j \eta(j)}}{(1 + |j|)^{d+1}} \varphi_0(\cdot, -j) \in S_{[\alpha]}^{[\omega]}(\mathbb{R}^d).$$ 

Since $S_{[\eta]}^{[\omega]}(\mathbb{R}^d) \subseteq S(\mathbb{R}^d)$, there is $C > 0$ such that 

$$\frac{e^{-\lambda_j \eta(j)}}{(1 + |j|)^{d+1}} = |\varphi(j)| \leq \frac{C}{(1 + |j|)^{d+2}}$$ 

for all $j \in \mathbb{Z}^d$. Hence, 

$$\log(1 + |j|) \leq \lambda_j \eta(j) + \log C$$ 

for all $j \in \mathbb{Z}^d$. As $\eta$ satisfies $(\alpha)$ and $(\lambda_j)_{j \in \mathbb{Z}^d}$ is arbitrary, the latter inequality is equivalent to $[\gamma]$.

$(i) \Rightarrow (iv)$ Let us denote the space in the right-hand side of $(iv)$ by $S_{[\alpha]}^{[\omega],1}(\mathbb{R}^d)$. Since we already showed that $(i) \Rightarrow (ii)$ and we have that $S_{[\eta]}^{[\omega],1}(\mathbb{R}^d) \subseteq S_{[\alpha]}^{[\omega],1}(\mathbb{R}^d)$, it suffices to show that $S_{[\alpha]}^{[\omega],1}(\mathbb{R}^d) \subseteq S_{[\alpha]}^{[\omega]}(\mathbb{R}^d)$.

By Proposition 2.2(a), Lemma 2.3 and the reconstruction formula (2.1), it suffices to show that $V_\psi(\varphi) \in C_{[\eta, \omega]}(\mathbb{R}^{2d})$ for all $\varphi \in S_{[\alpha]}^{[\omega],1}(\mathbb{R}^d)$, where $\psi \in S_{[\eta]}^{[\omega]}(\mathbb{R}^d)$ is a fixed non-zero window. But the latter can be shown by using the same method employed in the first part of the proof of Proposition 2.2. 

\[\square\]

3. Proof of Theorem 1.1

Our proof of Theorem 1.1 is based on Proposition 2.2(b) and the next two auxiliary results.

Proposition 3.1. Let $\eta$ be a weight function satisfying $(\alpha)$ and $[\gamma]$. Then, $\mathcal{K}_{[\eta]}(\mathbb{R}^d)$ is nuclear.

Proof. We present two different proofs:

$(i)$ The first one is based on a classical result of Gelfand and Shilov [8, p. 181]. The nuclearity of $\mathcal{K}_{[\eta]}(\mathbb{R}^d)$ is a particular case of this result, as the increasing sequence of
weight functions \((e^{n\eta})_{n\in\mathbb{N}}\) satisfies the so-called \((P)\) and \((N)\) conditions because of \((\gamma)\). For the Roumieu case, note that
\[
K_{\{\eta\}}(\mathbb{R}^d) = \lim_{n\in\mathbb{Z}_+, k\geq n} \frac{1}{(1 + |x|)^\eta} \|\varphi\|_{L^\infty(\mathbb{R}^d)}
\]
as locally convex spaces. The above mentioned result implies that, for each \(n\in\mathbb{Z}_+\), the Fréchet space \(K_{\{\eta\}}(\mathbb{R}^d)\) is nuclear, as the increasing sequence of weight functions \((e^{n\eta})_{k\geq n}\) satisfies the conditions \((P)\) and \((N)\) because of \(\{\gamma\}\). The result now follows from the fact that the inductive limit of a countable spectrum of nuclear spaces is again nuclear [18].

\((ii)\) Next, we give a proof that simultaneously applies to the Beurling and Roumieu case and only makes use of the fact that \(\mathcal{S}(\mathbb{R}^d)\) is nuclear. Our argument adapts an idea of Hasumi [11]. Fix a non-negative function \(\chi \in \mathcal{D}(\mathbb{R}^d)\) such that \(\int_{\mathbb{R}^d} \chi(y)dy = 1\) and for each \(\lambda > 0\) let
\[
\Psi_{\lambda}(x) = \exp\left(\lambda L \int_{\mathbb{R}^d} \chi(y)\eta(x + y)dy\right).
\]
It is clear from the assumption \((\alpha)\) that \(\eta\) should have at most polynomial growth. So, we fix \(q > 0\) such that \((1 + |x|)^{-q}\eta(x)\) is bounded. We obtain that there are positive constants \(c_\lambda, C_\lambda, C_{\lambda, \beta},\) and \(C_{\lambda, \beta, \gamma}\) such that
\begin{equation}
(3.1) \quad c_\lambda \exp(\lambda \eta(x)) \leq \Psi_{\lambda}(x) \leq C_\lambda \exp(L^n \lambda \eta(x)), \quad |\Psi_{\lambda}(\beta)(x)| \leq C_{\lambda, \beta}(1 + |x|)^q|x|^{\beta} \Psi_{\lambda}(x),
\end{equation}
and
\begin{equation}
(3.2) \quad \left|\left(\frac{\Psi_{\lambda_1}}{\Psi_{\lambda_2}}\right)^{(\beta)}(x)\right| \leq C_{\lambda_1, \lambda_2, \beta}(1 + |x|)^q|x|^{\beta},
\end{equation}
for each \(\beta \in \mathbb{N}^d, \lambda_1 \leq \lambda_2.\) Let \(X_\lambda = \Psi_{\lambda}^{-1}\mathcal{S}(\mathbb{R}^d)\) and topologize each of these spaces in such a way that the multiplier mappings \(M_{\Psi_{\lambda}} : X_\lambda \rightarrow \mathcal{S}(\mathbb{R}^d) : \varphi \mapsto \Psi_{\lambda} \cdot \varphi\) are isomorphisms. The bounds \((3.2)\) guarantee that the inclusion mappings \(X_{\lambda_2} \rightarrow X_{\lambda_1}\) are continuous whenever \(\lambda_1 \leq \lambda_2.\) If \(A\) is a constant such that \((\gamma)\) holds for \(\eta,\) then the inequalities \((3.1)\) clearly yield
\[
\max_{|\beta| \leq k} \sup_{x \in \mathbb{R}^d} (1 + |x|^k)|\Psi_{\lambda}(\varphi)^{(\beta)}(x)| \leq b_{k, \lambda, A} \|\varphi\|_{\eta, k, \lambda, L^2(1 + qk/k/A)}
\]
and
\[
\|\varphi\|_{\eta, k, \lambda} \leq \frac{1}{c_\lambda} \max_{|\beta| \leq k} \|\Psi_{\lambda}(\varphi)^{(\beta)}\|_{L^\infty(\mathbb{R}^d)}
\leq \frac{1}{c_\lambda} \sum_{\nu < \beta} \left(\frac{\beta}{\nu}\right) \left(\Psi_{\lambda}^{(\beta - \nu)}(\varphi)^{(\nu)}\right)_{L^\infty(\mathbb{R}^d)}
\leq b'_{k, \lambda} \left(\max_{|\beta| \leq k} \|\Psi_{\lambda}(\varphi)^{(\beta)}\|_{L^\infty(\mathbb{R}^d)} + \max_{|\beta| \leq k-1} \|\Psi_{\lambda}(\varphi)^{(\beta)}\|_{L^\infty(\mathbb{R}^d)}\right)
\leq b_{k, \lambda} \max_{|\beta| \leq k} \|\Psi_{\lambda}(\varphi)^{(\beta)}\|_{L^\infty(\mathbb{R}^d)},
\]
for some positive constants $B_{k,A}, b_{k,A}$ and $b_{k,A}$. This gives, as locally convex spaces,

$$\mathcal{K}_{(\eta)}(\mathbb{R}^d) = \lim_{n \to n_+} X_n$$

and the continuity of the inclusion $X_\lambda \to \mathcal{K}_{(\eta)}^\lambda(\mathbb{R}^d)$. If in addition $\{\gamma\}$ holds, we can choose $A$ arbitrarily large above. Consequently, the inclusion $\mathcal{K}_{(\eta)}^{L^2,\lambda+\varepsilon}(\mathbb{R}^d) \to X_\lambda$ is continuous as well for any arbitrary $\varepsilon > 0$, whence we infer the topological equality

$$\mathcal{K}_{(\eta)}(\mathbb{R}^d) = \lim_{n \to n_+} X_{1/n}.$$ 

The claimed nuclearity of $\mathcal{K}_{(\eta)}(\mathbb{R}^d)$ and $\mathcal{K}_{(\eta)}(\mathbb{R}^d)$ therefore follows from that of $\mathcal{S}(\mathbb{R}^d)$ and the well-known stability of this property under projective and (countable) inductive limits [18].

The next result is essentially due to Petzsche [17]. Given a multi-indexed sequence $a = (a_j)_{j \in \mathbb{Z}^d}$ of positive numbers, we define $l^r(a) = l^r(\mathbb{Z}^d; a)$, $r \in \{1, \infty\}$, as the Banach space consisting of all $c = (c_j)_{j \in \mathbb{Z}^d} \in \mathbb{C}^{\mathbb{Z}^d}$ such that $\|c\|_r(a) := \|(c_j a_j)_{j \in \mathbb{Z}^d}\|_r < \infty$.

**Proposition 3.2 ([17] Satz 3.5 and Satz 3.6).**

(a) Let $A = (a_n)_{n \in \mathbb{N}} = (a_{n,j})_{n \in \mathbb{N}, j \in \mathbb{Z}^d}$ be a matrix of positive numbers such that $a_{n,j} \leq a_{n+1,j}$ for all $n \in \mathbb{N}, j \in \mathbb{Z}^d$. Consider the Köthe echelon spaces $\lambda^r(A) := \lim_{\gamma \to \gamma_+} l^r(a_n)$, $r \in \{1, \infty\}$. Let $E$ be a nuclear locally convex Hausdorff space and suppose that there are continuous linear mappings $T : \lambda^1(A) \to E$ and $S : E \to \lambda^\infty(A)$ such that $S \circ T = \iota$, where $\iota : \lambda^1(A) \to \lambda^\infty(A)$ denotes the natural embedding. Then, $\lambda^1(A)$ is nuclear.

(b) Let $A = (a_n)_{n \in \mathbb{N}} = (a_{n,j})_{n \in \mathbb{N}, j \in \mathbb{Z}^d}$ be a matrix of positive numbers such that $a_{n+1,j} \leq a_{n,j}$ for all $n \in \mathbb{N}, j \in \mathbb{Z}^d$. Consider the Köthe co-echelon spaces $\lambda^r\{A\} := \lim_{\gamma \to \gamma_+} l^r(a_n)$, $r \in \{1, \infty\}$. Let $E$ be a locally convex Hausdorff space such that its strong dual $E'_b$ is nuclear and suppose that there are continuous linear mappings $T : \lambda^1\{A\} \to E$ and $S : E \to \lambda^\infty\{A\}$ such that $S \circ T = \iota$, where $\iota : \lambda^1\{A\} \to \lambda^\infty\{A\}$ denotes the natural embedding. Then, $\lambda^1\{A\}$ is nuclear.

**Proof.** (a) This follows from an inspection of the second part of the proof of [17, Satz 3.5]; the conditions stated there are not necessary for this part of the proof.

(b) By transposing, we obtain continuous linear mappings $T' : E'_b \to (\lambda^1\{A\})'_b$ and $S' : (\lambda^\infty\{A\})'_b \to E'_b$ such that $T' \circ S' = \iota'$. Consider the matrix $A^o = (1/a_n)_{n \in \mathbb{N}}$ and the natural continuous embeddings $\iota_1 : \lambda^1(A^o) \to (\lambda^\infty\{A\})'_b$ (the continuity of $\iota_1$ follows from the fact that $\lambda^\infty\{A\}$ is a regular (LB)-space [II, p. 81]) and $\iota_2 : (\lambda^1\{A\})'_b \to \lambda^\infty(A^o)$. Then, we have that $(\iota_2 \circ T') \circ (S' \circ \iota_1) = \tau$, where $\tau : \lambda^1(A^o) \to \lambda^\infty(A^o)$ denotes the natural embedding. Since $E'_b$ is nuclear, part (a) yields that $\lambda^1(A^o)$ is nuclear, which in turn implies the nuclearity of $\lambda^1\{A\}$ [II, Proposition 15, p. 75].

We are now ready to prove Theorem [II].
Proof of Theorem 3.4. We first suppose that $\omega$ and $\eta$ satisfy $[\gamma]$. W.l.o.g. we may assume that $S_{[n]}(\mathbb{R}^d) \neq \{0\}$. In view of Lemma 2.3, Proposition 2.2(b) and the reconstruction formula (2.1) imply that $S_{[n]}(\mathbb{R}^d)$ is isomorphic to a (complemented) subspace of $K_{[n]}(\mathbb{R}^{2d})$. The latter space is nuclear by Proposition 3.1. The result now follows from the fact that nuclearity is inherited to subspaces [18].

Next, we suppose that $\omega$ and $\eta$ are radially increasing and that $S_{[n]}(\mathbb{R}^d)$ is nuclear and non-trivial. Since the Fourier transform is a topological isomorphism from $S_{[n]}(\mathbb{R}^d)$ onto $S_{[n]}(\mathbb{R}^d)$, it is enough to show that $\eta$ satisfies $[\gamma]$. Set $A_{(\eta)} = (e^{i\eta(j)})_{n \in \mathbb{N}, j \in \mathbb{Z}^d}$ and $A_{(\eta)} = (e^{i\eta(j)})_{n \in \mathbb{Z}^+ + j \in \mathbb{Z}^d}$. By [1], Proposition 15, p. 75, $\lambda^1[A_{[n]}]$ is nuclear if and only if

$$\exists \lambda > 0 \left( \forall \lambda > 0 : \sum_{j \in \mathbb{Z}^d} e^{-\lambda \eta(j)} < \infty. \right)$$

As $\eta$ is radially increasing and satisfies $(\alpha)$, the above condition is equivalent to $[\gamma]$. Hence, it suffices to show that $\lambda^1[A_{[n]}]$ is nuclear. To this end, we employ Proposition 3.2 with $A = A_{[n]}$ and $E = S_{[n]}(\mathbb{R}^d)$ (in the Roumieu case we use the well-known fact that the strong dual of a nuclear $(DF)$-space [18] is nuclear). We start by constructing $\varphi_0 \in S_{[n]}(\mathbb{R}^d)$ such that

$$\int_{[0,1]^d} \varphi_0(j + x)dx = \delta_{j,0}, \quad j \in \mathbb{Z}^d.$$ 

By Lemma 2.3 there is $\varphi \in S_{[n]}(\mathbb{R}^d)$ such that $\varphi(0) = 1$. Set $\chi(x) = \frac{1}{2^d} \int_{[-1,1]^d} e^{-2\pi i x \cdot t}dt$, $x \in \mathbb{R}^d$. Then, $\chi(j/2) = \delta_{j,0}$ for all $j \in \mathbb{Z}^d$. Hence, $\psi = \varphi \chi \in S_{[n]}(\mathbb{R}^d)$ and $\psi(j/2) = \delta_{j,0}$ for all $j \in \mathbb{Z}^d$. Then, $\varphi_0 = (-1)^d \partial^d \cdots \partial^d \psi$ satisfies all requirements. The linear mappings

$$T : \lambda^1[A_{[n]}] \rightarrow S_{[n]}(\mathbb{R}^d), \quad T((c_j)_{j \in \mathbb{Z}^d}) = \sum_{j \in \mathbb{Z}^d} c_j \varphi_0(\cdot - j)$$

and

$$S : S_{[n]}(\mathbb{R}^d) \rightarrow \lambda^\infty[A_{[n]}], \quad S(\varphi) = \left( \int_{[0,1]^d} \varphi(x + j)dx \right)_{j \in \mathbb{Z}^d}$$

are continuous. Moreover, by (3.3), we have that $S \circ T = \iota$. 

\[ \square \]

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