MNĚV’S UNIVERSALITY THEOREM FOR SCHEMES

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ABSTRACT. We prove a scheme-theoretic version of Mněv’s Universality Theorem, suitable to be used in the proof of Murphy’s Law in Algebraic Geometry [Vak Main Thm. 1.1]. Somewhat more precisely, we show that any singularity type of finite type over $\mathbb{Z}$ appears on some incidence scheme of points and lines, subject to some particular further constraints.

This paper is dedicated to Joe Harris on the occasion of his birthday, with warmth and gratitude.

CONTENTS

1. Introduction 1
2. Structure of the construction 3
3. The configurations 6
4. Putting everything together 9
References 12

1. INTRODUCTION

1.1. Define an equivalence relation $\sim$ on pointed schemes generated by the following: if $(X, P) \rightarrow (Y, Q)$ is a smooth morphism of pointed schemes $(P \in X, Q \in Y)$ — i.e. a smooth morphism $\pi : X \rightarrow Y$ with $\pi(P) = Q$ — then $(X, P) \sim (Y, Q)$. We call equivalence classes singularity types, and we call pointed schemes singularities. We say that Murphy’s Law holds for a (moduli) scheme $M$ if every singularity type appearing on a finite type scheme over $\mathbb{Z}$ also appears on $M$. (This use of the phrase “Murphy’s Law” is from [Va §1], and earlier appeared informally in [HM, p. 18]. Folklore ascribes it to Mumford.)

1.2. Definition. Define an incidence scheme of points and lines in $\mathbb{P}_\mathbb{Z}^2$ as a locally closed subscheme of $(\mathbb{P}_\mathbb{Z}^2)^M \times (\mathbb{P}_\mathbb{Z}^2)^N = \{p_1, \ldots, p_M, l_1, \ldots, l_N\}$ parametrizing $M$ labeled points and $N$ labeled lines, satisfying the following conditions.

(i) $p_1 = [0, 0, 1], p_2 = [0, 1, 0], p_3 = [1, 0, 0], p_4 = [1, 1, 1]$. 
(ii) We are given some specified incidences: for each pair $(p_i, l_j)$, either $p_i$ is required to lie on $l_j$, or $p_i$ is required not to lie on $l_j$.

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(iii) The marked points are required to be distinct, and the marked lines are required to be distinct.
(iv) Given any two marked lines, there is a marked point required to be on both of them (necessarily unique, given (iii)).
(v) Each marked line contains at least three marked points.

The goal of this paper is to establish the following.

1.3. Mnëv’s Universality Theorem for Schemes. — The disjoint union of all incidence schemes (over all possible M, N, and data of the form described in (iii)) satisfies Murphy’s Law.

As an example: a reducible incidence scheme is given in Figure 1 of p. 276 (and Proposition 3.10) of [KT].

Theorem 1.3 appeared as [Va Thm. 3.1], as an essential step in proving [Va Main Thm. 1.1], which stated that many important moduli spaces satisfy Murphy’s Law. A number of readers of [Va] have pointed out to the second author that the references given in [Va] and elsewhere in the literature do not establish the precise statement of Theorem 1.3 and that it is not clear how to execute the glib parenthetical assertion (“The only subtlety ...,” [Va p. 577, 1. 3-5]) to extend Lafforgue’s argument [L Thm. 1.14] to obtain the desired result. (In particular, (iii) is the key problem. The issue is in ensuring that the points and lines which were not specified to be incident never become incident — complicated by the fact that every pairwise intersections of lines is marked, by (iv). On might hope to rectify this problem as in [L] by passing to a fibration, but then surjectivity of the alleged smooth cover is not clear.) This paper was written in order to fill a possible gap in [Va], or at least to clarify details of an important construction. Although no one familiar with Mnëv’s theorem in this context would doubt that Theorem 1.3 holds, we will see that some care is needed to rigorously establish it. In particular, our argument is characteristic-dependent.

1.4. Key features in the argument. The goal is to build a smooth cover of

\[ \text{Spec } \mathbb{Z}[x_1, \ldots, x_n]/(f_1, \ldots, f_r) \]

by (open subsets of) incidence schemes, by encoding the variables and relations in incidence relations. We build the relations by combining “atomic” calculations encoding equality, negation, addition, and multiplication. We point out new features of the argument we use, in order to ensure (iii) in particular. We perform each “atomic” calculation on a separate line of the plane, to avoid having too many important points on a single line, because points on a line must be shown to not overlap. We need various cases to deal with when the “variable” in question is “near” 0 or 1 (i.e. has value 0 or 1 at the geometric point Q of \( \mathbb{Z}[x_1, \ldots, x_n]/(f_1, \ldots, f_r) \), but is not required to have that value “near” Q). Furthermore, the “usual” construction of addition and multiplication runs into problems in characteristic 2 due to unintended coincidences of points, so some care is required in this case (see §4.16).

1.5. Algebro-geometric history. Vershik’s “universality” philosophy (e.g. [Ve Sect. 7]) has led to a number of important constructions in many parts of mathematics. One of the
most famous is Mnëv’s Universality Theorem [M1, M2]. Lafforgue outlined a proof of a scheme-theoretic version in [L, Thm. 1.14]. Keel and Tevelev used this construction in [KT] (see §1.8 and Theorem 3.13 of that article). Another algebro-geometric application of Mnëv’s theorem (this time in its manifestation in the representation problem of matroids) was Belkale and Brosnan’s surprising counterexample to a conjecture of Kontsevich, [BB].

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2. Structure of the Construction

2.1. Strategy. Fix a singularity \((Y, Q)\) of finite type over \(\text{Spec } \mathbb{Z}\). We will show that there exists a point \(P\) of some incidence scheme \(X\) (i.e. some configuration of points and lines, as described in Definition [L2]), along with a smooth morphism \(\pi : (X, P) \to (Y, Q)\) of pointed schemes. Because smooth morphisms are open, it suffices to deal with the case where \(Q\) is a closed point of \(Y\). Then the residue field \(\kappa(Q)\) has finite characteristic \(p\). (The reduction to characteristic \(p\) is not important; it is done to allow us to construct a configuration over a fixed infinite field. Those interested only in the characteristic 0 version of this result will readily figure out how to replace \(\mathbb{F}_p\) with \(\mathbb{Q}\) or \(\mathbb{C}\).)

By replacing \(Y\) with an affine neighborhood of \(Q\), we may assume \(Y\) is affine, say \(Y = \text{Spec } \mathbb{Z}[x_1, \ldots, x_n]/(f_1, \ldots, f_r)\). The morphism

\[
\text{Spec } \mathbb{F}_p[x_1, \ldots, x_n]/(f_1, \ldots, f_r) \to \text{Spec } \mathbb{F}_p[x_1, \ldots, x_n]/(f_1, \ldots, f_r)
\]

is surjective by the Lying Over Theorem. Choose a pre-image \(\overline{Q} \in \text{Spec } \mathbb{F}_p[x_1, \ldots, x_n]/(f_1, \ldots, f_r)\) of \(Q\) — say the (closed) point \((x_1, \ldots, x_n) = (q_1, \ldots, q_n)\), where \(\overline{q} \in \mathbb{F}_p^n\).

We make the following constructions.

(a) We describe a configuration of points and lines over \(\mathbb{F}_p\), which is thus an \(\mathbb{F}_p\)-valued point \(\overline{P}\) of an incidence scheme \(X\).

(b) The incidence scheme \(X\) will be an open subscheme of an affine scheme \(X'\), and we construct (a finite number of) coordinates on \(X'\), which we name \(X_1, \ldots, X_n, Y_1, \ldots, Y_s\) subject only to the relations \(f_i(X_1, \ldots, X_n) = 0\) (1 \(\leq i \leq r\)). We thus have a smooth morphism \(\pi : X \to Y\) given by \(X_i \mapsto x_i\). Letting \(X_K = X \times_{\text{Spec } \mathbb{Z}} \text{Spec } K\) for \(K = \mathbb{F}_p\) and \(\mathbb{F}_p\), and similarly for \(Y_K\) and \(\pi_K\), we have a diagram:

\[
\begin{array}{ccc}
X_{\mathbb{F}_p} & \xrightarrow{\pi_{\mathbb{F}_p}} & X'_{\mathbb{F}_p} \\
\downarrow \pi_{\mathbb{F}_p} & & \downarrow \pi_{\mathbb{F}_p} \\
Y_{\mathbb{F}_p} & \xrightarrow{\pi_{\mathbb{F}_p}} & Y_{\mathbb{F}_p} \\
\end{array}
\]

In the course of the construction, we will not explicitly name the variables \(Y_j\), but whenever a free choice is made this corresponds to adding a new variable \(Y_j\).

(c) We will have \(\pi_{\mathbb{F}_p}(\overline{P}) = \overline{Q}\). Thus the image of \(\pi\) includes \(Q\).
2.2. Notation and variables for the incidence scheme.

The traditional (and only reasonable) approach is to construct a configuration of points and lines encoding this singularity, by encoding the “atomic” operations of equality, negation, addition, and multiplication. The most difficult desideratum is 1.2(iii).

Our incidence scheme will parametrize points and lines of the following form. In the course of this description we give names to the relevant types of points and lines, and give our chosen coordinates. We will later describe our particular point $P \in X_{\overline{F}}$.

The first type of points are $p_1$ through $p_4$ (see 1.2(i)). We call these anchor points. We interpret $P_2$ in the usual way: $p_2 p_3$ is the line at infinity, and $p_1$ is the origin; lines through $p_3$ are called horizontal. The first type of lines in our incidence scheme are the lines $p_i p_j$.

We call these anchor lines.

The next type of line, which we call variable-bearing lines, will be required to pass through $p_3 = [1, 0, 0]$ (they are “horizontal”), and not through $p_1, p_2, p_4$. Each variable-bearing line is parametrized by where it meets the $y$-axis (which is finite, as the lines do not pass through $p_2 = [0, 1, 0]$). Thus for each variable-bearing line $l_i$, we have a coordinate $y_i$. (By 1.2(iii), the $y_i$ are distinct, and not 0 or 1. These $y_i$ will be among the $Y_j$ of 2.1(b) above.)

Each variable-bearing line $l_i$ has a framing-type $Fr_i$, which is a size two subset of $\{-1, 0, 1\}$ if $p > 2$, and of $\{0, 1, j\}$ where $j$ is a chosen solution of $j^2 + j - 1 = 0$ (see 4.19 for more) if $p = 2$. Each variable-bearing line $l_i$ contains (in addition to $p_3$) the following three distinct marked points:

- two framing points $P_{i,s}$, where $s \in Fr_i$; and
- one variable-bearing point $V_i$.

The point $P_{i,s}$, we parametrize by its $x$-coordinate, which we confusingly name $y_{i,s}$ (because it will be one of the “free” variables $Y_i$ of 2.1(b)). The variable-bearing point $V_i$ we parametrize using the isomorphism $l_i \to \mathbb{P}^1$ obtained by sending $p_3$ to $\infty$, and $P_{i,s}$ to $s$ for $s \in Fr_i$. We denote this coordinate $x_i$. (In our construction, these coordinates will be either among those $X_i$ of 2.1(b) above, or will be determined by the other variables.) A variable-bearing line over $\mathbb{P}_p$ of framing-type $Fr_i$, whose variable-bearing point carries the variable $x_i = q \in \overline{F}$, we will call a $(Fr_i, q)$-line or a $(Fr_i, x_i)$-line.

We have a number of additional configurations of points and lines, called connecting configurations, which are required to contain a specified subset of the above-named points, and required to not contain the rest. These will add additional free variables (which, in keeping with 2.1(b) above, we call $Y_j$ for an appropriate $j$), and will (scheme-theoretically) impose a single constraint upon the $x$-variables:

- $x_a = x_b$ (an equality configuration)
- $x_a = -x_b$ (a negation configuration),
- $x_a + x_b = x_c$ (an addition configuration), or
- $x_a x_b = x_c$ (a multiplication configuration).
Finally, for each pair of above-named lines that do not have an above-named point contained in both, we have an additional marked point at their intersection (in order that (1.2)(iv) holds), which we call **bystander points**. In our construction, we never have more than two lines meeting at a point except at the previously-named points, and the only lines passing through the previously-named points are the ones specified above. The name “bystander points” reflects the fact that they play no further role, and no additional variables are needed to parametrize them.

2.3. **Reduction to four problems.**

We reduce Theorem 1.3 to four atomic (or perhaps “molecular”) problems.

We construct the expression for each \( f_i \) sequentially, starting with the variables \( x_i \) and the constant 1 (which we name \( x_0 \) to simplify notation later), and using negation of one term or addition or multiplication of two terms at each step. Somewhat more precisely, we make a finite sequence of intermediate expressions, where each expression is the negation, sum, or product of earlier (one or two) expressions. We assign new variables \( x_{n+1}, x_{n+2}, \ldots \) for each new intermediate expression. Those additional variables will come along with single equation — negation, addition, or multiplication — describing how \( x_k \) is obtained from its predecessor(s). Finally, for the variable \( x_a \) representing the final expression \( f_i \) (one for each \( f_i \)), we add the equation \( x_a = 0 \). These simple equations (which we call \( g_i \)) are equivalent to our original equations \( f_i = 0 \), so we have (canonically)

\[
(1) \quad \mathbb{Z}[x_1, \ldots, x_n]/(f_1, \ldots, f_r) \cong \mathbb{Z}[x_1, \ldots, x_n, x_{n+1}, \ldots, x_m]/(g_1, g_2, \ldots, g_{r'})
\]

where each \( g_i \) is of the form \( x_a - x_b, x_a + x_b, x_a + x_b - x_c, \text{ or } x_a x_b - x_c \).

We now construct our configuration over \( \overline{\mathbb{F}_p} \). Via (1), we interpret \( \overline{Q} \) as a geometric point of \( \text{Spec} \mathbb{Z}[x_1, \ldots, x_m]/(g_1, g_2, \ldots, g_{r'}) \), and we let \( q_i \in \overline{\mathbb{F}_p} \) be coordinates of \( x_i \) for all \( i \). For each \( i \in \{1, \ldots, m\} \), we choose two distinct \( q's \) in \( \{-1, 0, 1\} \) or \( \{0, 1, j\} \) (according to whether \( p > 2 \) or \( p = 2 \)) distinct from \( q_i \); these will be the framing-type \( \text{Fr}_i \) of \( x_i \). We place (generally chosen) variable-bearing lines \( l_i \) one for each \( x_i \) for \( i \in \{1, \ldots, m\} \), with framing points (corresponding to the framing-type \( \text{Fr}_i \)) chosen generally on \( l_i \), then with variable-bearing points chosen so that the coordinate of the variable point for the line \( l_i \) is \( q_i \).

We then sequentially do the following for each simple equation \( g_j \). For each \( g_j \) involving variables \( x_a, x_b \) (and possibly \( x_c \)), we place a corresponding configuration joining variable-bearing points for those variables and enforcing (scheme-theoretically) the equation \( g_j \). We will do this in such a way that the connecting configuration passes through no points or lines it is not supposed to. We will of course do this by a general position argument.

We are thus reduced to the following four problems. Suppose we are given a configuration of points and lines in the plane, including the anchor points \( p_i \), and the anchor lines \( p_i p_j \) (\( 1 \leq i < j \leq 4 \)) (and hence implicitly a point of some incidence scheme). Note that this incidence scheme is quasiaffine, say \( U \subset \text{Spec} A \):
The non-vertical lines (those non-anchor lines not containing \( p_2 = [0,1,0] \)) \( y = mx + b \) are parametrized by \( m \) and \( b \);

- vertical lines (those non-anchor lines passing through \( p_2 \)) \( x = a \) are parametrized by \( a \);
- those points \((x, y) = (a, b)\) not on the line at infinity are parametrized by \( a \) and \( b \);
- and those non-anchor points \([1, c, 0]\) on the line at \( \infty \) are parametrized by \( c \).

The conditions of \([1.2]\) are clearly locally closed.

**Equality problem.** If we have two variable-bearing lines \( l_a \) and \( l_b \) with coordinates \( x_a \) and \( x_b \), we must show that we may superimpose an equality configuration (i.e. add more points and lines), where except for the framing and variable-bearing points on these two lines \( l_a \) and \( l_b \), no point of the additional configuration lies on any pre-existing lines, and no line in the additional configuration passes through any pre-existing points (including pre-existing bystander points — pairwise intersections of pre-existing lines). Furthermore, the addition of this configuration must add only open conditions for added free variables and \( x_a, x_b \) and enforce exactly (scheme-theoretically) the equation \( x_a = x_b \).

More precisely, we desire that the morphism from the new incidence scheme to the old one is of the following form:

\[
\begin{array}{ccc}
\mathbb{U}' \overset{\text{open}}{\to} \text{Spec } A[y_1, \ldots, y_N]/(x_a - x_b) \\
\downarrow \downarrow \\
\mathbb{U} \overset{\text{open}}{\to} \text{Spec } A
\end{array}
\]

(for some value of \( N \)).

**Negation problem:** the same problem, except with \( x_a = -x_b \) replacing \( x_a = x_b \).

**Addition problem:** the analogous problem, except \( x_c = x_a + x_b \).

**Multiplication problem:** the analogous problem, except \( x_c = x_a x_b \).

### 3. The Configurations

We now describe the configurations needed to make this work.

#### 3.1. Building blocks for the building blocks: five configurations.

The building blocks we use are shown in Figures 2–5. The figures follow certain conventions. (See Figure 1 for a legend.) Lines that appear horizontal are indeed so — they are required to pass through \( p_3 = [1,0,0] \). The horizontal lines often have (at least) three labeled points, which suggest an isomorphism with \( \mathbb{P}^1 \). The dashed lines (and marked points thereon) are those that are in the configuration before we begin. The points and lines marked with a box are added next, and involve free choices (two coordinates for each boxed point, one for the each boxed horizontal line). The remaining points and lines are then determined. The triangle indicates the “goal” of the construction, if interpreted...
as constructing midpoint, addition, multiplication, and so forth (which is admittedly not our point of view).

\[ (P_1, V; P_2, p_3) = (P_1', V'; P_2', p_3) \]

(where \((\cdot, \cdot; \cdot, \cdot)\) throughout the paper means cross-ratio, or moduli point in \(M_{0,4}\)), so if \(l\) and \(l'\) are lines of the same framing-type, with \((P_1, P_2)\) and \((P_1', P_2')\) the corresponding framing points and \(V\) and \(V'\) the variable-bearing points, then the coordinates of \(V\) and \(V'\) are the same (scheme-theoretically). Note that we are adding three free variables (two for the point, one for the line), plus an open condition to ensure no unintended incidences with preexisting points and lines.

\[ (A, M; B, p_3) = (A', M'; B', p_3) \]

\[ (A, M; B, p_3) = (B, M; A, p_3) \]

\[ = 1/(A, M; B, p_3) \]

so \((A, M; B, p_3)\) is either 1 or \(-1\). For \(p \neq 2\) and \(A \neq B\), it is straightforward to verify that \(M \neq p_3\) so \((A, M; B, p_3) \neq 1\). Thus \((A, M; B, p_3) = -1\), so \(M\) is the “midpoint” of \(AB\). (More precisely: given any isomorphism of \(l\) with \(\mathbb{P}^1\) identifying \(p_3\) with \(\infty\), the coordinate of \(M\) is the average of the coordinates of \(A\) and \(B\). In classical language, \(M\) is the harmonic conjugate of \(p_3\) with respect to \(A\) and \(B\).)
The \textit{generic addition} configuration (Figure 3) deals with addition $x_a + x_b$ in the “generic” case where $x_a$, $x_b$, and $x_a + x_b$ are distinct from 0 and 1, and the framing-type of their lines are all $\{0, 1\}$. Given two lines $l_a$ and $l_b$ with variables $x_a$, $x_b$, with framing points $(P_{a,0}, P_{a,1})$ and $(P_{b,0}, P_{b,1})$ on $l_a$ and $l_b$ respectively, we choose a general horizontal line $l'$ and a general point $X$, and superimpose the construction shown in Figure 3. If the line $l'$ is given the framing-type $Fr = \{0, 1\}$ with framing points $P'_0$ and $P'_1$, the reader will readily verify that the coordinate of $V'$ is $x' = x_a + x_b$, and that this equation is precisely what is (scheme-theoretically) enforced by the configuration.

![Figure 3. Midpoint](image)

The \textit{generic multiplication} configuration (Figure 4) constructs/enforces multiplication $x_c = x_a x_b$ in the “generic” case where $x_a$, $x_b$, and $x_a x_b$ are distinct from 0 and 1, and the framing-type of their lines are all $\{0, 1\}$. As with the “generic addition” case, given two lines $l_a$ and $l_b$ with variables $x_a$, $x_b$, with framing points $(P_{a,0}, P_{a,1})$ and $(P_{b,0}, P_{b,1})$ on $l_a$ and $l_b$ respectively, we choose a general horizontal line $l'$ and a general point $X$, and superimpose the construction shown in Figure 4. If the line $l'$ is given the framing-type $Fr = \{0, 1\}$ with framing points $P'_0$ and $P'_1$, the reader will readily verify that the coordinate $x'$ of $V'$ is $x' = x_a x_b$, and that this equation is precisely what is (scheme-theoretically) enforced by the configuration.

![Figure 4. Generic addition](image)

The main part of the argument is that

\[
x_a = (P_{a,0}, V_a; P_{a,1}, p_3) = (P'_0, V; P'_1, p_3)
\]

and

\[
(P'_0, V'; V, p_3) = (P_{b,0}, V_b; P_{b,1}, p_3) = x_b
\]

yield

\[
x' = (P'_0, V'; P'_1, p_3) = (P'_0, V'; V, p_3)(P'_0, V; P'_1, p_3) = x_a x_b.
\]
We remark that we are parallel-shifting the point $V_b$ from $l_b$ to $l'$ to avoid accidental overlaps of points in our later argument.

![Figure 5. Generic multiplication](image)

4. Putting everything together

We now put the atomic configurations together in various ways in order to solve the four problems of §2.3. We begin with the case $p \neq 2$, leaving the case $p = 2$ until §4.16.

4.1. Relabeling. Before we start, we note that it will be convenient to use the same framing points but a different framing-type to change the value of the variable “carried” by the line. For example, a $(\{0, 1\}, q)$-line may be interpreted as a $(\{0, -1\}, -q)$-line (as $(0, 1; q, \infty) = (0, -1; -q, \infty)$).

4.2. Initial framing. Before we start, we “construct $-1$ on the x-axis”. More precisely, on the x-axis, we have identified the points $0 := [0, 0, 1] = p_1$ and $1 := [1, 0, 1] = p_2p_4 \cap p_1p_3$. We use the midpoint construction (Figure 3) to construct $-1 := [-1, 0, 1]$ as well (using $M = 0, B = 1, A = -1$).

We now construct equality, negation, addition, and multiplication.

4.3. Equality: enforcing $x_a = x_b$. We enforce equality $x_a = x_b$ as follows.

4.4. First case: same framing-type. Suppose first that two variables $x_a$ and $x_b$ are of same framing-type $\{s_1, s_2\}$. Then after a general choice of horizontal line $l'$, we parallel shift (Figure 2) $(P_{a,s_1}, V_a, P_{a,s_2})$ onto $(P'_{1}, V', P'_{2})$ on $l'$, using a generally chosen $X$. By similarly shifting $(P_{b,s_1}, V_b, P_{b,s_2})$ onto $(P'_{1}, V'', P'_{2})$ on $l'$, and requiring $V' = V''$, we enforce the equality $x_a = x_b$. The reader will verify that with the general choice of point $X$ and line $l'$ in Figure 2 the newly constructed points will miss any finite number of previously constructed points and lines (except for those in the Figure); and the newly constructed lines will miss any finite number of previously constructed points (except for those in the Figure, and of course $p_3$) — we will have no “unintended coincidences”. This can be readily checked in all later constructions (an essential point in the entire strategy!), but for concision’s sake we will not constantly repeat this.
4.5. Second case: different framing-type. Next, suppose that \( x_a \) and \( x_b \) have different framing-type, say \( \{s_{a,1}, s_{a,2}\} \) and \( \{s_{b,1}, s_{b,2}\} \) respectively (two distinct subsets of \( \{-1, 0, 1\} \)). Then \( q_a = q_b \) is not in \( \{-1, 0, 1\} \). We apply parallel shift (Figure 2) to move \( x_a \) to a generally chosen horizontal line \( l' \). We then parallel shift the points \(-1, 0, 1\) on the \( x\)-axis to \( l' \), so we have marked points on \( l' \) that can be identified (with the obvious isomorphism to \( \mathbb{P}^1 \)) with \( \infty = p_2, -1, 0, 1 \), and \( q_a = q_b \). Then (using the subset \( \{s_{b,1}, s_{b,2}\} \) of the marked points on \( l' \)) \( l' \) and \( l_b \) have same framing-type and we can apply previous construction.

We remark that in this and later constructions, we can take \( x_a \) or \( x_b \) (or, later, \( x_c \)) to be the constants 0 or 1, by treating the \( x\)-axis as a variable-bearing line. For example, to take \( x_a = 1 \), treat the \( x\)-axis as a \( (\{-1, 0\}, 1) \)-line.

4.6. Remark: choosing framing-type freely. The argument of §4.5 shows that given a variable “carried by” a variable-bearing line, we can change the framing-type of the line it “lives on”, at the cost of moving it to another generally chosen horizontal line (so long as the value of the variable does not lie in the new framing-type of course). From now on, given a variable, we freely choose a framing-type to suit our purposes at the time.

4.7. Negation: enforcing \( x_b = -x_a \). We now explain how to enforce \( x_b = -x_a \). Suppose \( x_a \) is carried on a line with framing-type \( \{s_1, s_2\} \), and \( x_b \) is carried on a line with framing-type \( \{-s_1, -s_2\} \) (possible as \( q_a = -q_b \) — here we use Remark 4.6). We enforce \( x_b = -x_a \) by adding the equality configuration (first case, §4.4), except interpreting line \( l_b \) as an \( \{\{s_1, s_2\}, -q_b\} \)-line (the relabel construction, §4.1).

4.8. Addition: enforcing \( x_a + x_b = x_c \).

4.9. First case: “(general) + (general) = (general)”. Suppose \( q_a, q_b, q_c \) are all distinct from 0 and 1. We apply parallel shifts to move the three relevant variable-bearing lines onto generally chosen lines, and then superimpose the “generic addition” configuration of Figure 4. (The parallel shifts are to guarantee no unintended coincidences.)

4.10. Second case: “1 + (general) = (general)”. Suppose next that \( q_a = 1 \), and \( q_b \) and \( q_c \) are neither 0 nor 1. Then \( q_c \neq -1 \) (or else \( q_b \) would be 0). The equation we wish to enforce may be rewritten as \( -x_c + x_b = -x_a \), and \( -x_c, x_b, \) and \( -x_a \) are all distinct from 0 and 1. (Here we use \( p \neq 2 \), as we require \( -1 \neq 1 \).) We thus accomplish our goal by applying the negation configuration to \( x_a \) and \( x_c \), then applying the first case of the addition construction, §4.9.

4.11. Third case: “0 + (general) = (general)”. Suppose that \( q_a = 0 \), and \( q_b \) and \( q_c \) are not in \( \{-1, 0, 1, 2\} \). We take the framing-sets on \( l_a \) and \( l_c \) to be \( \{-1, 1\} \) (using Remark 4.6). As in §4.1, we interpret/relabel the \( \{0, 1\} \)-line \( l_a \) as a \( \{0, 1\}, x'_a \)-line (where \( x'_a = (x_a + 1)/2 \)) and the \( \{-1, 1\} \)-line as a \( \{0, 1\}, x'_c \)-line (where \( x'_c = (x_c + 1)/2 \)). We take the framing-set \( \{0, 1\} \) on \( l_b \). We parallel shift \( x_b \) onto a general horizontal line \( l'_b \), then use the midpoint construction (Figure 3) to construct the midpoint of \( V_b \) and \( P_{b,0} \) on \( l'_b \), so we have constructed the variable \( x_{b/2} \), which we name \( x'_b \). The equation we wish to enforce, \( x_a + x_b = x_c \), is algebraically equivalent to \( x_a' + x_b' = x_c' \), and the values of \( x'_a \), \( x'_b \), and \( x'_c \)
are all distinct from 0 and 1, so we can apply the construction of the first case of addition, \[4.9\]

4.12. **Fourth case: everything else.** We begin by adding two extra free variables \(s\) and \(t\) on two generally chosen horizontal lines. More precisely, for \(s\), we pick a generally chosen horizontal line \(l_s\), and three generally chosen points \(P_{i,0}, P_{i,1}\), and \(V_i\) on it, and define \(s = (P_{i,0}, P_{i,1}, V_i, P_3)\), so \(l_s\) is a \([\{0, 1\}, s]\)-line. We do the same for \(t\). Using the previous cases of addition, we successively construct \(x_a + s, x_b + t, (x_a + s) + (x_b + t), s + t,\) and \(x_c + (s + t)\). (Because \(s\) and \(t\) were generally chosen, one of the three previous cases can always be used.) Then we impose the equation
\[
(x_a + s) + (x_b + t) = x_c + (s + t)
\]
(using the third case of addition, \[4.11\] twice). Thus we have scheme-theoretically enforced \(x_a + x_b = x_c\) as desired.

4.13. **Multiplication: enforcing** \(x_a x_b = x_c\). As with addition, we deal with a “sufficiently general” case first, and then deal with arbitrary cases by translating by a general value.

4.14. **First case:** \(
("(general) \times (general) = (general)"
\). Suppose \(q_a, q_b, q_c \neq 0, 1\). We parallel shift all variables \(x_a, x_b, x_c\) to generally chosen lines \(l'_a, l'_b,\) and \(l'_c\) (to avoid later unintended incidences), and then superimpose the generic multiplication configuration to impose \(x_c = x_a x_b\) (where \(l'_a, l'_b,\) and \(l'_c\) here correspond to \(l_a, l_b,\) and \(l'\) in Figure 5).

4.15. **Second case: everything else.** To enforce \(x_a x_b = x_c\), we proceed as follows. We add two extra free variables \(u\) and \(v\) as in \[4.12\]. We then use the addition constructions of \[4.9, 4.12\] to construct \(x_a + u\) and \(x_b + v\) (on generally chosen horizontal lines). We use the construction of \[4.14\] to construct \((x_a + u)(x_b + v), uv, (x_a + u)v,\) and \((x_b + v)u\) (each on generally chosen lines). Finally, we use the addition constructions (several times) to enforce
\[
(x_a + u)(x_b + v) + uv = x_c + (x_a + u)v + (x_b + v)u.
\]
The result then follows from the algebraic identity
\[
(a + c)(b + d) + cd = ab + (a + c)d + (b + d)c.
\]

4.16. **Characteristic 2.**

As the above constructions at several points use \(-1 \neq 1\), the case \(p = 2\) requires a variant strategy.

4.17. **Addition and multiplication: general cases** \([4.9, 4.14]\). We begin by noting that the general cases of addition and multiplication, given in \[4.9\] and \[4.14\] respectively, work as before (where \(q_a, q_b,\) and \(q_c\) are all distinct from \(\{0, 1\}\), and the framing-type is taken to be \(\{0, 1\}\) in all cases).

4.18. **Relabeling** \([4.4],\) and the first case of equality \([4.4]\). Relabeling \([4.1]\) works as before. Equality in the case of same framing-type \([4.4]\) does as well.
4.19. Initial framing. In analogy with the initial framing of §4.2 before we begin the construction, we construct \( j \) and \( 1 - j = j^2 \) on the x-axis as follows. More precisely, we will add whose points on the x-axis which we label \( j \) and \( k \), as well as configurations forcing the coordinates to satisfy \( j^2 + j - 1 = 0 \), and \( k = j^2 \). (We then hereafter call the point \( k \) by the name \( j^2 \).) It is important to note that this construction of \( j \) is étale over \( \text{Spec} \mathbb{Z} \) away from \( \{(5)\} \), and in particular at 2; thus this choice will not affect the singularity type.

We construct these points as follows. Choose \( j \in \mathbb{F}_4 \setminus \mathbb{F}_2 \), and place a marked point at \( j \) on the x-axis. Construct the product of \( j \) with \( j \) by parallel shifting \( j \) separately onto two generally chosen horizontal lines, and then using the construction of §4.17 i.e. §4.14 (possible as \( j \) and \( j^2 \) are distinct from 0 and 1). Then construct \( 1 - j \) using the relabeling trick of §4.1 (§4.18): parallel shift 0, 1, and \( j \) to a generally chosen line, then reinterpret the \( \{(0,1), j\} \)-line as a \( \{(1,0), 1 - j\} \)-line, and parallel-shift it back to the x-axis. Finally, we use the equality configuration (the “same framing-type” case, §4.18 = §4.4) to enforce \( j^2 = 1 - j \).

4.20. Equality in general (§4.5), and freely choosing framing-type (Remark 4.6). Now that we have constructed \( j \), the second case of the equality construction works (with \( \{-1,0,1\} \) replaced by \( \{0,1,j\} \)), and we may choose framing-type freely on lines as observed in Remark 4.6.

4.21. Addition, second case: “1 + (general) = (general)”, cf. §4.10. Suppose \( q_a = 1 \), and \( q_b \) and \( q_c \) are not in \( \{0,1,j\} \). Then use the general case of multiplication (§4.17) i.e. §4.14 to construct (on separate general horizontal lines) \( q_a' = q_b/j \) and \( q_c' = q_c/j \). By considering the \( \{0,j\} \)-line as a \( \{(0,1), x_a/j\} \)-line (§4.18 i.e. §4.1), construct (on a general horizontal line, using parallel shift) \( x_a' = x_a/j \). Then impose \( x_a' + x_b' = x_c' \) using the general case of addition (§4.17).

4.22. Addition, third case: “0 + (general) = (general)”, cf. §4.11. Suppose \( q_a = 0 \), \( q_b \notin \{0,1,j,j^2\} \), and \( q_c \notin \{1,j\} \). Then construct \( x_a' = (x_a - 1)/j^2 \) (on a general horizontal line of framing-type \( \{0,1\} \)) by considering the \( \{1,j\}, x_a \)-line (carrying the variable \( x_a \)) as a \( \{0,1\}, (x_a - 1)/j^2 \)-line (as \( j - 1 = j^2 \)). Similarly, construct \( x_c' = (x_c - 1)/j^2 \). Using the general multiplication construction (§4.17 i.e. §4.14) twice, construct \( x_b' = x_b/j^2 \) (by way of the intermediate value of \( x_b/j \)). Then impose \( x_a' + x_b' = x_c' \) (using the general addition construction of §4.17 i.e. §4.9), and note that this is algebraically equivalent to \( x_a + x_b = x_c \).

4.23. Addition and multiplication, final cases: everything else (§4.12 and §4.15). These now work as before.

4.24. Negation (§4.7). Finally, negation can be imposed by constructing the configuration imposing \( x_a + x_b = 0 \) (using the final case of addition).

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