Parametric design of reduced-order functional observers for linear time-varying delay systems

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Abstract
This paper investigates the design of reduced-order functional observer for linear time-invariant systems (LTI) with time-varying delay. Firstly, the sufficient condition for the existence of reduced-order delay-free functional observers is obtained. Secondly, by solving generalized Sylvester matrix equations (GSEs), completely parameterized formulas of the coefficient matrices of reduced-order function observer are established. The free parameter matrices in the formulas provide the design degrees of the freedom and improve the convergence rate of the error system. Finally, two numerical and a practical examples are provided to illustrate the effectiveness of the proposed approach.

Keywords
Functional observer, reduced-order, delay-free, generalized Sylvester matrix equations

Introduction
For a dynamical system, the estimation of the state is essential in many fields, either to construct a control law or to monitor the system’s real-time operating status. Luenberger made a remarkable contribution to the observer,1–2 and has been attracted the attention of many researchers, such as fault detection,3 permanent magnet synchronous motor servo systems,4 DC motors,5 ball and beam systems,6 and complex dynamical networks,7 etc.

The functional observer greatly decreases the order and complexity of the observer. Therefore, it has attracted a large number of scholars to carry out related research. For LTI systems, functional observers and existence conditions were presented.8 The concept of functional observability/detectability and a minimum-order function observer were introduced.9 For linear time-varying (LTV) systems, Gu et al.10 investigated functional observers without transforming the systems into a particular form. By the efficient matrix factorization, the constructive procedures of the functional observers for the LTV system were provided.11 Further, Gu et al.12 constructed a functional observer for second-order linear time-varying systems under the framework of second-order systems. More related research can be found in the literature.13–19

Time-delay phenomena are frequently encountered in many practical systems. It has a significant effect on system performance and may cause instability, oscillation, and so on. Therefore, time-delay systems are also widely concerned, recent results can be found in the literature,20–27 and reference therein. According to the spectral decomposition techniques, Bhat and Koivo28 presented the theory of Luenberger observer for time-delay systems. Zheng et al.29 solved the design of the unknown input Luenberger observer for the linear time-delay systems. Moreover, observer design problem of descriptor time-delay systems was investigated and the existence condition of the observer was also given by Zheng and Bejarano30. Based on the matrix equation method, Wang et al.31 addressed the observer design problem for neutral delay systems. In the case when the state is not available, for the design problem of functional observers, some interesting results also have been obtained. The linear functional state observer was introduced for time-delay (TD) systems.32 Further, Trinh et al.33 designed the reduced-order functional observers for positive TD systems. Yuan et al.34 proposed a class of observer via virtue of the adaptive law. Based on a generalized coordinate change, the design problem of observers was extended

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to LTI systems with TD.\textsuperscript{35} Darouach presented the conditions for the existence of the $r$ th-order observers for systems with delays in state variables,\textsuperscript{36} then researched the discrete-time systems.\textsuperscript{37} However, the approaches mentioned above still yield a delay term in the observer. Delay-free state observers of linear systems with a constant TD have been considered.\textsuperscript{38,39} To the authors’ knowledge, only Trinh et al.\textsuperscript{40} provided the reduced-order delay-free observer with TD.

Different from existing methods, through the simple transformation, the design of the parameters of the delay-free observers for LTI systems with time-varying delay is converted into the solution of GSEs in the literature.\textsuperscript{41,42} The proposed method has many advantages. For example, the delay of the system is time-varying and there are no restrictions. Besides, the considered observer is reduced-order and delay-free, which can reduce the order of the observer and the complexity of implementation. Moreover, due to the existence of arbitrary parameters, it provides the degrees of freedom to optimize the performance of the observer. For the design of observers, we proposed both necessary and sufficient conditions. Then, we researched the order of delay-free observers and proposed the conditions for the existence of reduced-order function observers. With parametric methods, we can effectively solve the corresponding constraints.

The rest of this paper is arranged as follows. In Section 2, the problem statement is presented, and some notations and assumptions are provided. Section 3 introduces some preliminary results. Section 4 provides the generally parametric delay-free reduced-order observer. Section 5 provides two numerical and one practical example to verify the proposed approach. Finally, Section 6 concludes this paper.

### Problem statement

Consider the linear time-invariant (LTI) system with time-varying delay as follows

\[
\begin{align*}
\dot{x}(t) &= Ax(t) + A_d x(t - \tau(t)) + Bu(t), \\
y(t) &= Cx(t), \\
x(t) &= \phi(t), t \in [-\tau_u, 0),
\end{align*}
\]

where $x(t) \in \mathbb{R}^n$ is the state vector, $y(t) \in \mathbb{R}^m$ is the measured output vector and $u(t) \in \mathbb{R}^r$ is the control vector, $A, A_d, B$ and $C$ are known coefficient matrices, $\phi(t)$ is a continuous initial function. The delay $\tau(t)$ is time-varying in the time interval $(0, \tau_u]$, $\tau_u > 0$.

The main purpose of this paper is to investigate the normal function $Kx$ observer. Then, we concentrate on the delay-free Luenberger-type functional observer as follows

\[
\begin{align*}
\dot{z}(t) &= Fz(t) + Hu(t) + Ly(t), \\
w(t) &= Mz(t) + Gy(t),
\end{align*}
\]

where $z(t) \in \mathbb{R}^r$ is the observer state vector, $w(t) \in \mathbb{R}^{l'}$ is the observer output vector. Finding the matrix parameters $F, H, L, M,$ and $G$ is the design purpose such that

\[
\lim_{t \to \infty} (Kx(t) - w(t)) = 0
\]

hold for the given matrix $K \in \mathbb{R}^{n \times r}$, initial values $x(0)$, $z(0)$, and control input $u(t)$.

This paper proposes the conditions for the existence of delay-free Luenberger-type functional observers for system (1), then presents a parametric design method to the observers under the following assumptions.

### Assumption 1

\[
\text{rank } Q_x = \text{rank} \begin{bmatrix} C \\ CA \\ \vdots \\ CA^r \end{bmatrix} = n. \tag{4}
\]

The observability index $\nu$ of the system (1) is defined as the least positive integer such that equation (4) holds.

### Assumption 2

$C = m$, rank $[C^T \ K^T] = m + l$.

### Preliminaries

Let $T \in \mathbb{R}^{n \times n}$, and define

\[
e(t) = z(t) - Tx(t), e(t) = w(t) - Kx(t). \tag{5}
\]

The following theorem is given by the above discussion.

### Theorem 1

Assume that the systems (1) and (2) are observable. Then, system (2) is a delay-free Luenberger-type functional observer for system (1) if and only if the matrix $F$ is Hurwitz, and there exists a matrix $T \in \mathbb{R}^{n \times n}$ satisfying

\[
TA_d = 0, \tag{6}
\]

\[
H = TB, \tag{7}
\]

\[
TA - FT = LC, \tag{8}
\]

Theorem 1. Assume that the systems (1) and (2) are observable. Then, system (2) is a delay-free Luenberger-type functional observer for system (1) if and only if the matrix $F$ is Hurwitz, and there exists a matrix $T \in \mathbb{R}^{n \times n}$ satisfying
and
\[ K = MT + GC. \]  

**Proof.** *(Sufficiency):* Combining equations (5), (1), and (2), then
\[
\begin{aligned}
\dot{x}(t) &= F_s(t) + (FT - TA + LC)x(t) \\
&\quad -T_A_d\dot{x}(t - \tau) + (H - TB)y(t), \\
\dot{y}(t) &= T_b\dot{b}(t), \quad t \in [-\tau, 0], \\
e(t) &= M_\alpha(t) + (MT + GC - K)x(t).
\end{aligned}
\] (10)

If \( F \) is Hurwitz, and equations (6)-(9) are satisfied, system (10) becomes
\[
\begin{aligned}
\dot{x}(t) &= F_s(t), \\
\dot{y}(t) &= T_b\dot{b}(t), \quad t \in [-\tau, 0], \\
e(t) &= M_\alpha(t).
\end{aligned}
\] (11)

This implies that for arbitrarily \( x(0), z(0) \), and control input \( u(t) \) there holds
\[
\lim_{t \to \infty} e(t) = 0.
\]

*(Necessity):* If \( F \) is not Hurwitz, then there exists an initial function \( \phi(t) \) such that \( \phi(t) \neq 0, t \in [-\tau, 0] \), which makes \( x \to 0 \) as \( t \to \infty \). If equation (7) is not satisfied, we can find a \( u(t) \) such that \( x(t) \to 0 \) and also \( e(t) \to 0 \) as \( t \to \infty \). If any of the equations (6), (8), and (9) is not satisfied, we can find a \( u(t) \) to generate a \( x(t) \) to make \( x \to 0 \) as \( t \to \infty \). The proof is completed.

Assume that rank \( A_d = q, q \leq n \), then there exists a rank decomposition of the form \( A_d = RS \), where \( R \in \mathbb{R}^{n \times q} \) and \( S \in \mathbb{R}^{q \times n} \) are two full rank constant matrices. Substituting the rank decomposition into (6) yields
\[ TRS = 0. \]

Due to \( S \) is an invertible matrix, \( TRS = 0 \) if and only if
\[ TR = 0. \]

We introduce some results of the solution to the following GSE
\[ \mathcal{E}V\mathcal{F} - AV = BW, \] (13)
where \( \mathcal{E}, A \in \mathbb{R}^{n \times q}, \mathcal{B} \in \mathbb{R}^{n \times r}, q + r > n, \) and \( \mathcal{F} \in \mathbb{C}^{p \times p} \) are the coefficient matrices, and the matrix \( \mathcal{F} \) may not be prescribed, where \( V \) and \( W \) are the undetermined matrices.

**Definition 1.** \(^{41}\) \( s\mathcal{E} - A \) and \( \mathcal{B} \) are \( \mathcal{F} \)-left coprime with rank \( \alpha \) if
\[
\begin{aligned}
\text{rank } [s\mathcal{E} - A \mathcal{B}] &\leq \alpha, \forall s \in \mathbb{C},
\end{aligned}
\]
and
\[
\begin{aligned}
\text{rank } [s\mathcal{E} - A \mathcal{B}] = \alpha, \forall s \in \text{eig}([\mathcal{F}]).
\end{aligned}
\]

When the rank condition is met, there exists right coprime factorization (RCF) as follows
\[
(s\mathcal{E} - A)^{-1}\mathcal{B} = N(s)D^{-1}(s).
\] (16)

Denote \( D(s) = [d_{ij}]_{\alpha \times \beta} \) and
\[
\sigma = \max\{\text{deg}(d_{ij}), i = 1, 2, \ldots, \alpha, j = 1, 2, \ldots, \beta\},
\]
then
\[
\begin{aligned}
N(s) &= \sum_{i=0}^{\sigma} N_is^i, N_i \in \mathbb{R}^{q \times \beta},
\\
D(s) &= \sum_{i=0}^{\sigma} D_is^i, D_i \in \mathbb{R}^{r \times \beta}.
\end{aligned}
\] (17)

Based on above the discussion, we obtain the lemma as follows.

**Lemma 1.** \(^{41}\) Let \( \mathcal{E}, A \in \mathbb{R}^{n \times q}, \mathcal{B} \in \mathbb{R}^{n \times r}, q + r > n, \mathcal{F} \in \mathbb{C}^{p \times p} \) and rank conditions in Definition 1 hold. Moreover, \( N(s) \in \mathbb{R}^{q \times \beta}[s] \) and \( D(s) \in \mathbb{R}^{r \times \beta}[s] \) satisfy (16). Then all the solutions \( V \in \mathbb{C}^{q \times p} \) and \( W \in \mathbb{C}^{r \times p} \) to the equation (13) are given by
\[
\begin{aligned}
V &= N_1Z + N_{\infty}Z\mathcal{F} + \cdots + N_{\sigma}Z\mathcal{F}^\sigma \\
&= \sum_{i=0}^{\sigma} N_is^iZ^i, \\
W &= D_1Z + D_{\infty}Z\mathcal{F} + \cdots + D_{\sigma}Z\mathcal{F}^\sigma \\
&= \sum_{i=0}^{\sigma} D_is^iZ^i,
\end{aligned}
\] (18)

where \( Z \in \mathbb{C}^{p \times p} \) is an arbitrary parameter matrix.

**Main results**

Now introduce the change of variable
\[ \dot{x}(t) = Px(t). \] (19)

where
A S

B of the system

Lemma 2. matrices, respectively. By the equation (19), the system can be inverted the following partitioned form

\[
\begin{cases}
\dot{x}(t) = \tilde{A}x(t) + \tilde{A}_d (x(t) - \tau(t)) + \tilde{B}u(t), \\
y(t) = Cx(t),
\end{cases}
\]

where

\[
\begin{align*}
\tilde{x}(t) &= \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \\
A &= PAP^{-1} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \\
\tilde{A}_d &= PRSP^{-1} = RS = \begin{bmatrix} \tilde{R}_1 \\ \tilde{R}_2 \end{bmatrix}, \\
\tilde{B} &= PB = \begin{bmatrix} \tilde{B}_1 \\ \tilde{B}_2 \end{bmatrix}, \\
\tilde{C} &= CP^{-1} = \begin{bmatrix} I_{m} \\ 0_{m \times (n-m)} \end{bmatrix}.
\end{align*}
\]

and \( C_0 \) is any \((n - m) \times n\) matrix such that the matrix \( P \) is full rank. By the equation (19), the system can be inverted the following partitioned form

\[
\begin{cases}
\dot{x}(t) = \tilde{A}x(t) + \tilde{A}_d (x(t) - \tau(t)) + \tilde{B}u(t), \\
y(t) = Cx(t),
\end{cases}
\]

Theorem 2. Assume that the system (1) and (2) are observable. Then (2) is a delay-free Luenberger-type functional observer for the transformed system (20) if and only if the matrix \( F \) is Hurwitz, and there exists a functional observer for the transformed system (20). The following Theorem is given to describe the parametric coefficient matrices of the delay-free Luenberger-type functional observer (2) can be simplified as

\[
\begin{cases}
\dot{z}(t) = Fz(t) + (\tilde{T}_1 \tilde{B}_1 + \tilde{T}_2 \tilde{B}_2)u(t) \\
u(t) = Mz(t) + (\tilde{K}_1 - MT_1)y(t),
\end{cases}
\]

where \( \tilde{K} = KP^{-1} = \begin{bmatrix} \tilde{K}_1 \\ \tilde{K}_2 \end{bmatrix} \), \( \tilde{K}_1 \in \mathbb{R}^{m \times m} \), \( \tilde{K}_2 \in \mathbb{R}^{m \times (n-m)} \).

Proof. Due to all results may be simply demonstrated by replacement, omit the proof of this theorem.

Remark 1. From the transformation (19), the full state of the original system can be obtained from the \( m \) output together with the \( (n - m) \) state variables of the observer. When \( K_2 = 0 \), the function \( Kx(t) = \tilde{K}x(t) = K_3y(t) \). This means that it is not necessary to establish the observer (29), we can obtain the function \( Kx(t) \) by linear combination of output \( y(t) \). The following assumption is made without losing generality.

\[
\text{Assumption 3.} \quad K_2 \neq 0, \quad \text{rank} \quad K_2 = l.
\]

In the following subsections, we propose the observer design method.

**Parametric approach of \( \mu \)-th order delay-free observer**

The following Theorem is given to describe the parametric forms of the delay-free Luenberger-type function \( Kx \) observer.

Theorem 3. Given the system (1) and the delay-free Luenberger-type functional observer (2) meeting Assumption 1-2, and an arbitrary Hurwitz matrix \( F \in \mathbb{R}^{n \times n} \). Let matrix \( \tilde{N}(s) \) and \( \tilde{D}(s) \) satisfy equation (16), the following parametric coefficient matrices of \( \mu \)-th order delay-free observer are proposed.
\[
\begin{aligned}
L &= T_1 A_{11} + T_2 A_{21} - FT_1, \\
G &= K L - M T_1, \\
H &= T_1 B_1 + T_2 B_2, \\
M &= K_2 T_2^{-1} (T_2 T_2^{-1})^{-1},
\end{aligned}
\]  
\tag{30}

where
\[
\begin{aligned}
T_2 &= \sum_{i=0}^{\sigma} F^T Z_i N_i^T, \\
T_1 &= \sum_{i=0}^{\sigma} F^T D_i^T.
\end{aligned}
\tag{31}
\]

and
\[
T = TP = \begin{bmatrix} T_1 C & T_2 C_0 \end{bmatrix},
\tag{32}
\]
satisfying the following constraints

\textbf{Constraint 1.}
\[
\text{rank} \begin{bmatrix} R_1 \\ -T_2 R_2 \end{bmatrix} = \text{rank} \tilde{R}_1,
\tag{33}
\]
and

\textbf{Constraint 2.}
\[
\text{rank} \begin{bmatrix} \tilde{T}_2 \\ \tilde{K}_2 \end{bmatrix} = \text{rank} \tilde{T}_2,
\tag{34}
\]

with \(Z \in \mathbb{R}^{m \times \mu}\) is an arbitrary parameter matrix.

\textbf{Proof.} The proof includes two steps.

\textbf{Step 1. Parametric Expression for the Matrix} \(T\).

According to equation (27), first-order homogeneous GSE is given as
\[
A_{12}^T T_2^T - T_2^T F T = - A_{12}^T T_1^T.
\tag{35}
\]

Then the polynomial matrices of the above GSE are
\[
\begin{aligned}
sE - A &= sI - A_{22}^T, \\
B &= - A_{12}^T.
\end{aligned}
\tag{36}
\]

Assume that the rank condition (15) is satisfied, there exist. Therefore, according to equation (18), the parametric solutions are presented in equations (31) and (32).

\textbf{Step 2. Parametric Expressions for the Matrices} \(L, G, H, \text{and} \ M\).

The matrices \(T_1\) and \(M\) are obtained through equations (26) and (28) from Theorem 2. As a consequence, matrices \(T_1\) and \(M\) meet equations (26) and (28) if and only if Constraints 1 and 2 are satisfied, respectively.

We obtain the parametric gain matrices \(L, G, H, \text{and} \ M\) as (30). The proof is finished.

\textbf{Remark 2.} A major result was firstly presented by D. G. Luenberger\(^2\) since for any completely observable system \(v - 1 \leq n - m\) and for most systems \(v - 1\) is much less than \(n - m\), observing a scalar linear functional of the state may be far simpler than the entire state vector, that is, the upper bound for the order of a scalar linear functional observer is \(v - 1\). When there is \(l\)-dimensional linear function to be estimated, the order of multifunctional observer is \(\mu = (\mu_1 + \cdots + \mu_i)\), where \(\mu_i\) is the order of the \(i\)-th scalar functional observer. Based on the above discussion, a reduced-order observer of order less than \((v - 1)\) can be found, hence the whole order will be less than \(k(v - 1)\). Note that \(l(v - 1) < (n - m)\), otherwise, \(\mu = (n - m)\) is the well-known minimum-order state observer. We expect the order of the designed delay-free observer to be less than the upper bound \(\min \{l(v - 1), (n - m)\}\).

\textbf{Remark 3.} The parameter matrix \(Z\) provides the degrees of freedom. Use the degrees of freedom to meet Constraints 1 and 2. The matrix \(Z\) can provide the feasibility and convenience to design the reduced-order delay-free Luenberger-type functional observer.

\textbf{Design algorithm}

We proposed the following algorithm to find the \(\mu\)-th order delay-free observer for the system (1)

\textbf{Step 1. Choose a non-singular matrix} \(P\).

Choose an adjustable matrix \(C_0\) to ensure matrix \(P\) be invertible.

\textbf{Step 2. Calculate the matrices} \(N(s)\) and \(D(s)\).

We obtain the following particular solutions through equation (16)
\[
\begin{aligned}
N(s) &= \text{adj}(sI_n - A_{22}^T A_{22}) A_{22}, \\
D(s) &= \det(sI_n - A_{12}^T A_{22}) I_n.
\end{aligned}
\]

\textbf{Step 3. Design the structure of matrix} \(F\).

Denote a Hurwitz matrix \(F\), to make the error system stable, the eigenvalues of the matrix are required to be on the left half \(S\)-plane, that is,
\[
\lambda_i(F) \in \mathbb{C}^-, i = 1, 2, \cdots, \mu.
\]

One first begins with the case \(\mu = 1\).
Step 4. Calculate parameter matrices $T_1$ and $T_2$.

Obtain the matrices $T_1$ and $T_2$ from equation (31), check Constrains 1 and 2 of Theorem 3. If there exits a parameter matrix $Z$ satisfying Constrains 1 and 2, go to Step 5. If there exits a parameter matrix $Z$ not satisfying Constrains 1 and 2, go back to Step 3, one proceeds to the case $\mu + 1$.

Step 5. Compute the coefficient matrices of observer.

Matrices $M$, $L$, $G$, and $H$ can be obtained through the formulas (30).

Example
Based on three examples verify the proposed approach.

Example I
Consider system (1) with the coefficient matrices as follows.
\[
A = \begin{bmatrix}
-2 & 0 & 2 & 1 & 0 \\
0 & -3 & 0 & 1 & 0 \\
1 & -1 & -5 & 0 & 2 \\
0 & 0 & 0 & -4 & 0 \\
0 & 0 & 0 & 0 & -1
\end{bmatrix},
\]
\[
A_d = \begin{bmatrix}
-0.6 & 1 & 0.5 & -0.3 & -0.7 \\
0.6 & -1 & -0.5 & 0.3 & 0.7 \\
2.7 & -4 & -1.5 & 1.1 & 1.9 \\
-1.8 & 3 & 1.5 & -0.9 & -2.1 \\
1.35 & 2.5 & 1.5 & -0.8 & -2.2
\end{bmatrix},
\]
\[
B^T = [1 \ 2 \ 1 \ 1 \ 2], C = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 2 & 0 & 0 & 0
\end{bmatrix}.
\]

We design a reduced-order delay-free Luenberger-type observer and the scalar function is given by
\[
K = \begin{bmatrix}
0 & 0 & -0.5 & -3 & 1 \\
0 & 0 & 0.5 & 0 & -1
\end{bmatrix}.
\]

In the following, the observer gain matrices can be calculated. Choose the non-singular matrix as
\[
P = \begin{bmatrix}
C \\
C_0
\end{bmatrix} = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
1 & 2 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{bmatrix}
\]
and we can get the transformed system as follows
\[
\begin{align*}
\dot{x}_1(t) &= \begin{bmatrix} A_{11} & A_{12} \end{bmatrix} \dot{x}_1(t) + \begin{bmatrix} B_1 \end{bmatrix} u, \\
\dot{x}_2(t) &= \begin{bmatrix} A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} \dot{x}_2(t) \\
\end{bmatrix} + \begin{bmatrix} B_2 \end{bmatrix} \begin{bmatrix} R_1 \\
R_2
\end{bmatrix} + \begin{bmatrix} S_1 \end{bmatrix} \begin{bmatrix} x_1(t) \\
\end{bmatrix} + \begin{bmatrix} S_2 \end{bmatrix} \begin{bmatrix} x_2(t) \end{bmatrix}, \\
y(t) &= \begin{bmatrix} J_2 \end{bmatrix} \begin{bmatrix} 0_{2 \times 3} \\
\end{bmatrix} \begin{bmatrix} x_1(t) \\
\end{bmatrix} + \begin{bmatrix} x_2(t) \end{bmatrix},
\end{align*}
\]
where
\[
\begin{align*}
\bar{A}_{11} &= \begin{bmatrix} -2 & 0 \\
1 & -3 
\end{bmatrix}, \bar{A}_{12} = \begin{bmatrix} 2 & 1 \\
2 & 3 
\end{bmatrix}, \\
\bar{A}_{21} &= \begin{bmatrix} 1.5 & -0.5 \\
0 & 0 
\end{bmatrix}, \bar{A}_{22} = \begin{bmatrix} -5 & 2 \\
0 & -4 
\end{bmatrix}, \\
\bar{R}_1 &= \begin{bmatrix} 1 & -1 \\
-1 & 1 
\end{bmatrix}, \bar{R}_2 = \begin{bmatrix} 7 & 2 \\
3 & -3 
\end{bmatrix}, \\
\bar{S}_1 &= \begin{bmatrix} -0.5 & 0.2 \\
0.6 & -0.3 
\end{bmatrix}, \bar{S}_2 = \begin{bmatrix} 0.1 & -0.1 \\
-0.4 & 0.2 
\end{bmatrix}, \\
B_1 &= \begin{bmatrix} 1 \\
5 
\end{bmatrix}, B_2 = \begin{bmatrix} 1 \\
2 
\end{bmatrix},
\end{align*}
\]
and the linear functional $\bar{K}$ is given by
\[
\bar{K} = \begin{bmatrix} 0 & 0 & -0.5 & -3 & 1 \\
0 & 0 & 0.5 & 0 & -1
\end{bmatrix}.
\]

Then, we can infer that RCF (16) holds, and
\[
\mathcal{N}(s) = \begin{bmatrix} 0 & s + 1 \\
1 & 0 
\end{bmatrix}, \quad \mathcal{D}(s) = \begin{bmatrix} -\frac{s}{2} - 2 & \left(\frac{s + 15}{2}\right)(s + 5) - \frac{s}{2} \\
\frac{s}{2} + 2 & \frac{s}{2} - \left(\frac{s + 15}{2}\right)(s + 5)
\end{bmatrix}.
\]

Next step, we determine $\mu$. First, let $\mu = 1$, $F = -1$ and the parameter matrix $Z$ be
\[
Z = \begin{bmatrix} z_{11} \\
z_{21}
\end{bmatrix}.
\]
Combining matrix $Z$ with equation (31), then yields
\[
\bar{T}_1 = \begin{bmatrix} -\frac{3z_{11}}{2} - \frac{5z_{21}}{4} & \frac{3z_{11}}{2} - \frac{3z_{21}}{4} \\
\end{bmatrix}, \quad \bar{T}_2 = \begin{bmatrix} 0 & z_{11} & z_{21}
\end{bmatrix}.
It is obvious that Constraints 2 is not satisfied, go back, let $\mu = 2$,

$$F = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix},$$  \hspace{1cm} (42)$$

and the parameter matrix $Z$ can be represented as

$$Z = \begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix}.$$  \hspace{1cm} (43)$$

Substituting equations (42)–(43) into (31), we have

$$\dot{\hat{T}}_1 = \begin{bmatrix} -\frac{3z_{11}}{2} & -\frac{3z_{12}}{8} \\ -z_{12} & \frac{z_{11}}{8} \\ -\frac{z_{21}}{2} & z_{22} \end{bmatrix},$$  \hspace{1cm} (44)$$

$$\dot{\hat{T}}_2 = \begin{bmatrix} 0 & z_{11} \\ z_{12} & z_{22} \end{bmatrix}.$$  \hspace{1cm} (45)$$

Let

$$\bar{T}_1 = \begin{bmatrix} -1.5 & 1.5 \\ -1.375 & 0.125 \end{bmatrix}, \bar{T}_2 = \begin{bmatrix} 0 & 1 \\ -0.5 & 1 \end{bmatrix},$$

that is, Constraints 1 and 2 hold.

Further, the second-order delay-free Luenberger-type observer can be obtained by Theorem 2

$$\begin{cases} \dot{\hat{x}}(t) = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \hat{x}(t) + \begin{bmatrix} 7 \\ -0.25 \end{bmatrix} u(t) \\ + \begin{bmatrix} 3 \\ -3 \end{bmatrix} \hat{y}(t) \\ w(t) = \begin{bmatrix} -2 & 1 \\ -1 & -1 \end{bmatrix} \hat{x}(t) + \begin{bmatrix} -1.625 & 2.875 \\ -2.875 & 1.625 \end{bmatrix} \hat{y}(t) \end{cases}$$  \hspace{1cm} (46)$$

Due to the upper bound is $\min \{|(\nu - 1), (n - m)| \}$ = 3, however, there exists a second-order delay-free Luenberger-type observer.

In order to compare with the method in Trinh et al.,\textsuperscript{40} we give the following initial conditions

$$x(0) = \begin{bmatrix} 0 \\ 8 \\ 13 \end{bmatrix}, \quad \dot{x}(0) = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix},$$

$$u(t) = \sin (5t), \quad \forall t \in \mathbb{R}^+$$

Figure 1 shows the comparison of the output among the estimated state, Trinh et al.,\textsuperscript{40} and the proposed method. The estimation errors are plotted in Figure 2.

Under the same initial condition, comparison of Trinh et al.,\textsuperscript{40} and the proposed method, the proposed observer has better performance.

**Example 2**

Consider the system with the following coefficient matrices

$$A = \begin{bmatrix} -1 & 0 & 0 & 1 \\ 0 & -5 & 3 & 4 \\ 1 & 1 & -8 & 3 \\ -4 & 0 & 2 & -6 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 2 \end{bmatrix},$$

$$A_d = \begin{bmatrix} -0.02 & -0.02 & -0.1 & -0.02 \\ -3.53 & -5.03 & -1.9 & -1.72 \\ 1.99 & 2.99 & -0.55 & 1.51 \\ -0.28 & -0.48 & 0.7 & -0.42 \end{bmatrix},$$

$$C = [1 \\ 0 \\ 0 \\ 0],$$

and the scalar function is given by

$$K = [-1 \\ -4 \\ -4 \\ 10].$$
We choose the non-singular matrix as
\[
P = \begin{bmatrix} C \\ C_0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}
\]
and the transformed system can be obtained as
\[
\frac{\dot{x}(t)}{C_{22}} = \begin{bmatrix} -1 & 0 & 0 & 1 \\ 4 & -5 & 6 & -9 \\ 0.5 & 0.5 & -8 & 3 \\ -4 & 0 & 2 & -6 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 5 \\ 1 \\ 2 \end{bmatrix} u
\]
\[
y(t) = [1 \ 0 \ 0 \ 0] x(t)
\]
where
\[
\bar{R} = \begin{bmatrix} \bar{R}_1 \\ \bar{R}_2 \end{bmatrix} = \begin{bmatrix} 0 & -0.1 \\ -15 & -5.4 \\ 5 & -0.05 \\ -1 & 0.6 \end{bmatrix},
\]
\[
\bar{S} = [\bar{S}_1, \bar{S}_2] = \begin{bmatrix} 0.1 & 0.3 & -0.1 & 0.3 \\ 0.1 & 0.1 & 1 & -0.2 \end{bmatrix},
\]
and the linear functional \( \bar{K} \) is given as
\[
\bar{K} = \begin{bmatrix} 1 & -2 & -4 & 10 \end{bmatrix}.
\]

Next step, we determine \( \mu \). First, let \( \mu = 1 \), \( F = -1 \) and the parameter matrix \( Z = z_1 \). Combining equation (48) with (31), then
\[
\bar{T}_1 = \begin{bmatrix} z_1 & 8z_1 & 25z_1 \end{bmatrix} \]
\[
\bar{T}_2 = 92z_1.
\]

It is obvious that Constraints 2 is not satisfied, go back, let \( \mu = 2 \),
\[
F = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix},
\]
and
\[
Z = [z_1, z_2].
\]

Substituting equations (50) and (51) into (31), we have
\[
\bar{T}_1 = \begin{bmatrix} 92z_1 \\ 33z_2 \end{bmatrix} \]
\[
\bar{T}_2 = \begin{bmatrix} z_1 & 8z_1 & 25z_1 \\ z_2 & 6z_2 & 15z_2 \end{bmatrix}.
\]

Let
\[
Z = [2 \ 2],
\]
then, we get
\[
\bar{T}_1 = \begin{bmatrix} 184 \\ 66 \end{bmatrix} \]
\[
\bar{T}_2 = \begin{bmatrix} 2 & 16 & 50 \\ 2 \end{bmatrix}.
\]
that is, Constraints 1 and 2 hold.

Then, according to Theorem 2, we can obtain the second-order delay-free Luenberger-type observer as follows

\[
\mathcal{N}(s) = \begin{bmatrix} 1 \\ 2s + 10 \\ s^2 + 13s + 37 \end{bmatrix},
\]
\[
\mathcal{D}(s) = s^3 + 19s^2 + 109s + 183.
\]
\[
\begin{aligned}
z(t) &= \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} z(t) + \begin{bmatrix} -184 \\ -40 \end{bmatrix} y(t) \\
&\quad + \begin{bmatrix} 310 \\ 148 \end{bmatrix} u(t), \\
w(t) &= \begin{bmatrix} 2 & -3 \end{bmatrix} z(t) - 169 y(t).
\end{aligned}
\] (54)

Due to the upper bound is \((\nu - 1) = 3\), there exists a second-order delay-free Luenberger-type observer.

**Remark 4.** The method proposed by Trinh et al. needs to satisfy some suitable constrains. [30] Unfortunately, under the condition that \(m \leq q\) and rank \(R_1 = m\), the second-order observer does not exist, that is, the problem of estimating the function \(Kx\) cannot be solved by Trinh et al. [40] Unlike Trinh et al.’s method, the proposed approach in this paper solves the problem and provides the free parameter matrix \(Z\) in the observer design.

We choose the following initial conditions
\[
\begin{aligned}
x(0) &= \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}, \\
z(0) &= \begin{bmatrix} 0 \\ 0 \end{bmatrix}.
\end{aligned}
\]
\[\tau(t) = 1 + e^{-t}, \quad \forall t \in \mathbb{R}^+\]
and \(u(t) = \sin(5t)\), meanwhile, Figures 3 and 4 are the simulation results.

Figures 3 and 4 show the output of the observer and the estimated state, and the estimation errors, respectively. From Figures 3 and 4, it is clear that the proposed observer can track the estimated state quickly and accurately.

**Example 3**

Consider the system consisting of a diesel engine and turbo compressor shown in Figure 5. [43] Due to the existence of intake-to-exhaust transport delay, the system becomes a typical nonlinear system model with time-delay. The linearized model was presented by literature. [44] The mathematical model of this system can be written in the form of system (1)

\[
A = \begin{bmatrix} -27 & 3.6 & 6 \\ 9.6 & -12.5 & 0 \\ 0 & 9 & -5 \end{bmatrix},
A_d = \begin{bmatrix} -6.0417 & 0 & 0 \\ 2 & 0 & 0 \\ 0.4167 & 0 & 0 \end{bmatrix}
\]
\[
B = \begin{bmatrix} 0.26 & 0 \\ -0.9 & -0.8 \\ 0 & 0.18 \end{bmatrix},
C = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}.
\] (55)

The state vector \(x(t)\) and the control input vector \(u(t)\) can be written as
\[
x(t) = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix},
\]
\[
u(t) = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix},
\] (56)
with the variables \(x_1, x_2, x_3\) and \(u_1, u_2\) represent intake manifold pressure, exhaust manifold pressure, compressor power, control inputs for valve opening and the turbine, respectively.

The scalar function is given by
\[
K = \begin{bmatrix} 1 & -1 & 0 \end{bmatrix}.
\]
We choose the nonsingular matrix as
\[
P = \begin{bmatrix} C \\ C_0 \end{bmatrix} = \begin{bmatrix} C_0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 2 \end{bmatrix},
\]
and the following system is obtained based on the equation (21)
\[
\begin{cases}
\hat{x}(t) = \begin{bmatrix} -12.5 & 9.6 & 0 \\ -8.4 & -27 & 6 \\ -6 & 19.2 & -5 \end{bmatrix} \hat{x}(t) + \begin{bmatrix} -0.9 & -0.8 \\ 0.26 & 0 \\ -1.8 & -1.42 \end{bmatrix} u \\
+ \hat{R}\hat{x}(t - \tau(t)), \\
y(t) = [1, 0, 0] \hat{x}(t),
\end{cases}
\]
where
\[
\hat{R} = \begin{bmatrix} \hat{R}_1 \\ \hat{R}_2 \end{bmatrix} = \begin{bmatrix} 2 \\ -6.0417 \\ 4.4167 \end{bmatrix},
\]
\[
S = \begin{bmatrix} S_1 & S_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix},
\]
and the linear functional \( \hat{K} \) is given as
\[
\hat{K} = \begin{bmatrix} -1 & 1 & 0 \end{bmatrix}.
\]
Then, we can verify that RCF (16) is correct, and
\[
\begin{align*}
\mathcal{N}(s) &= \begin{bmatrix} 9.6s + 48 \\ 57.6 \end{bmatrix}, \\
\mathcal{D}(s) &= s^3 + 32s + 19.8.
\end{align*}
\] (58)

Next step, we determine \( \mu \). First, let \( \mu = 1 \), \( F = -1 \) and the parameter matrix \( Z = z_1 \). Combining equation (58) with (31), then
\[
T_1 = 11.2z_1, T_2 = \begin{bmatrix} 38.4z_1 & 57.6z_1 \end{bmatrix}.
\] (59)

Obviously, Constraint 2 does not satisfy, go back, let \( \mu = 2 \),
\[
F = \begin{bmatrix} -3 & 0 \\ 0 & -4 \end{bmatrix},
\]
and
\[
Z = \begin{bmatrix} z_1 & z_2 \end{bmatrix}.
\] (61)

Substituting equations (60) and (61) into (31), we have
\[
\hat{T}_1 = \begin{bmatrix} -67.2z_1 \\ -92.2z_2 \end{bmatrix}, \quad \hat{T}_2 = \begin{bmatrix} 19.2z_1 & 57.6z_1 \\ 9.6z_2 & 57.6z_2 \end{bmatrix}.
\] (62)

Let
\[
\hat{Z} = \begin{bmatrix} 0.2 & 0.2 \end{bmatrix},
\]
then, we get
\[
\hat{Z}_1 = \begin{bmatrix} -13.44 \\ -18.44 \end{bmatrix}, \quad \hat{Z}_2 = \begin{bmatrix} 3.84 & 11.52 \\ 1.92 & 11.52 \end{bmatrix},
\]
that is, Constraints 1 and 2 hold.

Then, according to Theorem 2, we obtain the following second-order delay-free Luenberger-type observer
\[
\begin{cases}
\dot{z}(t) = \begin{bmatrix} -3 & 0 \\ 0 & -4 \end{bmatrix} z(t) + \begin{bmatrix} 26.3040 \\ 71.4920 \end{bmatrix} y(t) \\
+ \begin{bmatrix} -7.6416 & -5.6064 \\ -3.6048 & -1.6064 \end{bmatrix} u(t), \\
\dot{w}(t) = \begin{bmatrix} 0.5208 & -0.5208 \end{bmatrix} [z(t) - 3.6042y(t)].
\end{cases}
\] (64)

Due to the upper bound is \( (\nu - 1) = 2 \), there exists a second-order delay-free Luenberger-type observer. We choose the \( x(0), z(0) \) as
\[
\begin{cases}
x(0) = [0 & 2 & 4]^T, \\
z(0) = [0 & 0 & 0]^T,
\end{cases}
\]
\( \tau(t) = 1.5 \sin(t - \sin(0.9t)), \forall t \in \mathbb{R}^+, \) and \( u_1(t) = -\sin(5t), u_2(t) = -3 \sin t, \) meanwhile, Figures 6 and 7 plot the simulation results.

Figure 6 shows the output of the observer and the estimated state, and Figure 7 shows the estimation...
errors. From Figures 6 and 7, it is clear that the proposed observer can track the estimated state quickly and accurately.

Conclusions
In this work, the problem of delay-free function observer design has been investigated for LTI systems with time-varying delay. First, the proposed observers are reduced-order and delay-free which makes engineering practicability attractive in terms of cost economy. Then, the condition for the existence and the generally parameterized expression of the design observers have been provided. Furthermore, with the parametric approach, the parameters matrix $Z$ provides degrees of freedom. With the parameters, we can optimize the observer parameters to achieve the expected performance. Finally, based on three examples demonstrate the feasibility of the approach.

Declaration of conflicting interests
The author(s) declared no potential conflicts of interest with respect to the research, authorship, and/or publication of this article.

Funding
The author(s) disclosed receipt of the following financial support for the research, authorship, and/or publication of this article: This research was supported by the Science Center Program of the National Natural Science Foundation of China under grant No. 62188101, and also by the Major Program of the National Natural Science Foundation of China (61690210, 61690212).

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