Abstract. We revisit the construction of the eigenvectors of the single and double-row transfer matrices associated with the Zamolodchikov–Fateev model, within the algebraic Bethe ansatz method. The left and right eigenvectors are constructed using two different methods: the fusion technique and Tarasov’s construction. A simple explicit relation between the eigenvectors from the two Bethe ansätze is obtained. As a consequence, we obtain the Slavnov formula for the scalar product between on-shell and off-shell Tarasov–Bethe vectors.

Keywords: integrable spin chains and vertex models, quantum integrability (Bethe ansatz)
1. Introduction

The Zamolodchikov–Fateev (ZF) model [1] is a nineteen-vertex model that can be seen as a generalization of the six-vertex model. Indeed, while the latter is associated with the spin-$\frac{1}{2}$ XXZ chain, the former describes an integrable spin-1 XXZ chain. The ZF R-matrix is a solution of the Yang–Baxter equation, which is given by a $9 \times 9$ matrix with nineteen non-null entries. Starting from this R-matrix, one can use the quantum inverse scattering method [2–4] to construct single and double-row commuting transfer matrices, which are associated with closed and open spin chains, respectively.

The spectral problem for the transfer matrix can be solved by means of the algebraic Bethe ansatz. For the ZF model, the construction of the Bethe vectors of the single and double-row transfer matrices can be performed in two different ways. The first one is by means of the fusion technique [5–17], where the auxiliary space is still 2-dimensional, and therefore the Bethe vectors are obtained similarly to the six-vertex model case. We
shall refer to these as fusion-Bethe vectors. The second way is by means of Tarasov’s construction [18], which entails working with a 3-dimensional auxiliary space. This approach was originally developed for the Izergin–Korepin (IK) model [19], see also [21] for the double-row case, and was applied to the ZF model in [22, 23]. We shall refer to these as Tarasov–Bethe vectors.

The scalar product between on-shell and off-shell fusion-Bethe vectors, i.e. a formula of the type found in the celebrated paper by Slavnov [24], has been obtained for closed chains, see [25] for the rational case and [26] for the trigonometric case. For open chains, the Slavnov formula has been obtained only for spin $\frac{1}{2}$, see [27, 28] for the rational case and [29] for the trigonometric case. On the other hand, Slavnov-type formulas for Tarasov–Bethe vectors have not yet been found, to the best of our knowledge. This is probably due to the intricacy of the exchange relations arising from the Yang–Baxter and reflection algebras when the auxiliary space has dimension greater than two, as is the case in Tarasov’s construction.

The purpose of this note is to initiate the investigation of the Slavnov scalar products for Tarasov–Bethe vectors. Here we handle this problem by showing that the fusion-Bethe vectors and the Tarasov–Bethe vectors are simply related: the result for the closed chain is given by (4.40) and (4.41) and for the open chain it is given by (5.39) and (5.40). Thanks to these simple relations, we can easily obtain the Slavnov formula for the Tarasov–Bethe vectors, for both the closed (4.58) and open chains (5.54), without going through Yang–Baxter and reflection algebras in Tarasov’s construction.

This paper is organized as follows. In section 2 we recall the basics of the fusion technique. The fundamental and fused monodromy and transfer matrices are given in section 3. In section 4 the Bethe ansatz is implemented and the Slavnov formula is obtained for the single-row transfer matrix, while in section 5 these results are generalized to the double-row transfer matrix. Our concluding remarks are given in section 6.

2. Fusion for R-matrices and K-matrices

In this section we recall some basic ingredients of the fusion technique, by means of which we can construct integrable generalizations of the fundamental six-vertex model, for both single-row [5–14] and double-row [15–17] transfer matrices.

2.1. R-matrices

We start with the fundamental R-matrix

$$R^{(\frac{1}{2}, \frac{1}{2})}(u) = \begin{pmatrix} \sinh(u + \eta) & 0 & 0 & 0 \\ 0 & \sinh(u) & \sinh(\eta) & 0 \\ 0 & \sinh(\eta) & \sinh(u) & 0 \\ 0 & 0 & 0 & \sinh(u + \eta) \end{pmatrix},$$

(2.1)

4 This model is also a nineteen-vertex model, but it cannot be obtained by fusion from the fundamental six-vertex model. See, however [20].

5 For simplicity, we focus here on the case of diagonal boundary conditions.
A tale of two Bethe ansätze

which acts on $\mathbb{C}^2 \otimes \mathbb{C}^2$ and where $\eta$ is the free anisotropy parameter. This R-matrix has the parity symmetry

$$R(u) = \mathcal{P} R(u) \mathcal{P},$$

(2.2)

where $\mathcal{P}$ is the permutation matrix, and the unitarity property

$$R(u) R(-u) = \xi(u) \xi(-u) \mathbb{I} \otimes \mathbb{I},$$

(2.3)

with $\xi(u) = \xi^{(\frac{1}{2}, \frac{1}{2})}(u) = \sinh(u + \eta)$. It also has crossing symmetry

$$R_{12}(u) = V_1 R^{(\ell, k)}_{12}(-u - \rho) V_1 = V_2 R^{(\ell, k)}_{12}(-u - \rho) V_2,$$

(2.4)

with

$$V = V^{(\frac{1}{2}, \frac{1}{2})} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \rho = \rho^{(\frac{1}{2}, \frac{1}{2})} = \eta + i\pi.$$  

(2.5)

Hence, the matrix $M$ defined by

$$M = V^t V$$

(2.6)

is given by

$$M = M^{(\frac{1}{2}, \frac{1}{2})} = \mathbb{I}.$$  

(2.7)

By means of the fusion procedure, we can construct R-matrices $R^{(i, j)}(u)$ associated with $su(2)$ representations of spins $i$ and $j$, with $i, j \in \{\frac{1}{2}, 1\}$. These R-matrices, which map $\mathbb{C}^{2i+1} \otimes \mathbb{C}^{2j+1} \mapsto \mathbb{C}^{2i+1} \otimes \mathbb{C}^{2j+1}$, satisfy the various Yang–Baxter equations

$$R^{(i, j)}_{12} (u - v) R^{(i, k)}_{13} (u) R^{(j, k)}_{23} (v) = R^{(j, k)}_{23} (v) R^{(i, k)}_{13} (u) R^{(i, j)}_{12} (u - v).$$

(2.8)

Using the fusion procedure (following [13, 14]), we define the fused R-matrix

$$R^{(\frac{1}{2}, \frac{1}{2})}_{12(23)}(u) = \frac{1}{\sinh(u + \frac{\eta}{2})} F_{(23)} R^{(\frac{1}{2}, \frac{1}{2})}_{13} (u + \frac{\eta}{2}) R^{(\frac{1}{2}, \frac{1}{2})}_{12} (-u - \frac{\eta}{2}) E_{(23)},$$

(2.9)

$$\begin{pmatrix} \sinh(u + \frac{\eta}{2}) & 0 & 0 & 0 & 0 \\ 0 & \sinh(u - \frac{\eta}{2}) & 0 & \frac{1}{\sqrt{2}} \sinh(2\eta) & 0 \\ 0 & 0 & \sqrt{2} \sinh(\eta) & 0 & \sqrt{2} \sinh(\eta) \\ 0 & 0 & \frac{1}{\sqrt{2}} \sinh(2\eta) & \sinh(u + \frac{\eta}{2}) & 0 \\ 0 & 0 & 0 & 0 & \sinh(u + \frac{3\eta}{2}) \end{pmatrix}.$$  

(2.9)

which acts on $\mathbb{C}^2 \otimes \mathbb{C}^3$. Here

$$E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad F = E^t = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (2.10)$$

This R-matrix has the unitarity property (2.3) with $\xi(u) = \xi^{(\frac{1}{2}, \frac{1}{2})}(u) = \sinh(u + \frac{3\eta}{2})$. 

https://doi.org/10.1088/1742-5468/aab851  

4
A tale of two Bethe ansätze

We also define the fused R-matrix

\[ R^{(1, \frac{1}{2})}_{(12,3)}(u) = \frac{1}{\sinh(u + \frac{\eta}{2})} F_{(12)} R^{(\frac{1}{2}, \frac{1}{2})}_{13} (u + \frac{\eta}{2}) R^{(\frac{1}{2}, \frac{1}{2})}_{23} (u - \frac{\eta}{2}) E_{(12)}, \]

(2.11)

which acts on \( \mathbb{C}^3 \otimes \mathbb{C}^2 \). The R-matrices (2.9) and (2.11) are related by

\[ R^{(1, \frac{1}{2})}(u) = \mathcal{P}^{(\frac{1}{2}, 1)} R^{(\frac{1}{2}, 1)}(u) \mathcal{P}^{(1, \frac{1}{2})}, \]

(2.12)

where

\[
\mathcal{P}^{(\frac{1}{2}, 1)} = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0
\end{pmatrix}
\]

(2.13)

is a permutation matrix that maps \( \mathbb{C}^2 \otimes \mathbb{C}^3 \) to \( \mathbb{C}^3 \otimes \mathbb{C}^2 \), and \( \mathcal{P}^{(1, \frac{1}{2})} = \left( \mathcal{P}^{(\frac{1}{2}, 1)} \right)^{-1} \). Finally, we define the fused R-matrix

\[
R^{(1, 1)}_{(12,34)}(u) = F_{(12)} R^{(\frac{1}{2}, 1)}_{(1,34)} (u + \frac{\eta}{2}) R^{(\frac{1}{2}, 1)}_{2(34)} (u - \frac{\eta}{2}) E_{(12)}
\]

\[
= \begin{pmatrix}
a(u) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & b(u) & 0 & c(u) & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & f(u) & 0 & d(u) & 0 & h(u) & 0 & 0 & 0 \\
0 & c(u) & 0 & b(u) & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \tilde{d}(u) & 0 & c(u) & 0 & \tilde{d}(u) & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & b(u) & 0 & c(u) & 0 & 0 \\
0 & 0 & h(u) & 0 & d(u) & 0 & f(u) & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & c(u) & 0 & b(u) & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a(u)
\end{pmatrix},
\]

(2.14)

with

\[
a(u) = \sinh(u + \eta) \sinh(u + 2\eta), \quad b(u) = \sinh(u) \sinh(u + \eta),
\]
\[
d(u) = 2 \sinh(u) \sinh(\eta), \quad \tilde{d}(u) = 2 \sinh(u) \sinh(\eta) \cosh^2(\eta),
\]
\[
c(u) = \sinh(2\eta) \sinh(u + \eta), \quad f(u) = \sinh(u) \sinh(u - \eta),
\]
\[
e(u) = \sinh(u) \sinh(u + \eta) + \sinh(2\eta) \sinh(\eta), \quad h(u) = \sinh(2\eta) \sinh(\eta),
\]

(2.15)

which acts on \( \mathbb{C}^3 \otimes \mathbb{C}^3 \). This R-matrix has the parity symmetry (2.2) and the unitarity property (2.3) with \( \xi(u) = \xi^{(1,1)}(u) = \sinh(u + 2\eta) \sinh(u + \eta) \). Although it does not have the crossing symmetry (2.4), it is related to a symmetric R-matrix

\[
\tilde{R}^{(1,1)}(u) = \left[ \tilde{R}^{(1,1)}(u) \right]^{11}_{22}
\]

by a constant gauge transformation

\[
\tilde{R}^{(1,1)}(u) = C_1^{-1} C_2^{-1} R^{(1,1)}(u) C_1 C_2, \quad C = \text{diag}(\cosh(\eta), \cosh(\eta), 1);
\]

(2.16)

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and this symmetric R-matrix \( R \) does have crossing symmetry, with
\[
V = V^{(1,1)} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \rho = \rho^{(1,1)} = \eta + i\pi, \tag{2.17}
\]
and therefore
\[
M = M^{(1,1)} = I. \tag{2.18}
\]
The R-matrix (2.16) was firstly obtained by Zamolodchikov and Fateev [1].

### 2.2. K-matrices

We now construct corresponding K-matrices \( K^{(i)}(u) \), which map \( \mathbb{C}^{2i+1} \rightarrow \mathbb{C}^{2i+1} \). For \( K^{-}(u) \), the boundary Yang–Baxter equations are
\[
R_{12}^{(j,j)}(u-v) K_{1}^{(i)}(u) R_{21}^{(j,j)}(u+v) K_{2}^{-}(j)(v) = K_{2}^{-}(j)(v) R_{12}^{(j,j)}(u+v) K_{1}^{(i)}(u) R_{21}^{(j,j)}(u-v), \tag{2.19}
\]
where
\[
R_{21}^{(j,j)}(u) = P^{(j,j)}(u) R_{12}^{(j,j)}(u) P^{(j,j)}. \tag{2.20}
\]

For the fundamental \( K^{-} \)-matrix, we restrict for simplicity to the diagonal solution
\[
K^{-}\left(\frac{1}{2}\right)(u) = \text{diag} \left( \sinh(u+\xi^-), -\sinh(u-\xi^-) \right), \tag{2.21}
\]
where \( \xi^- \) is an arbitrary boundary parameter. For \( K^+ \), we take
\[
K^{+}\left(\frac{1}{2}\right)(u) = K^{-}\left(\frac{1}{2}\right)\left(-u - \rho^{(1,1)}\right)|_{\xi^-\rightarrow\xi^+}
= \text{diag} \left( \sinh(u+\eta - \xi^+), -\sinh(u+\eta + \xi^+) \right), \tag{2.22}
\]
where \( \xi^+ \) is another arbitrary boundary parameter.

The fused \( K^{-} \)-matrix is given by (following [17])
\[
K^{-}\left(1\right)(u) = \frac{1}{2\sinh(u + \frac{\eta}{2})} F_{12}^{\left(1\right)} K_{1}^{-}\left(\frac{1}{2}\right)\left(u + \frac{\eta}{2}\right) R_{12}^{\left(1\right)}\left(2u\right) K_{2}^{-}\left(\frac{1}{2}\right)\left(u - \frac{\eta}{2}\right) P_{12} E_{12}
= \text{diag} \left( k_{1}^{-}(u), k_{2}^{-}(u), k_{3}^{-}(u) \right), \tag{2.23}
\]
where
\[
k_{1}^{-}(u) = \cosh(u + \frac{\eta}{2}) \sinh(u + \frac{\eta}{2} + \xi^-) \sinh(u - \frac{\eta}{2} + \xi^-),
k_{2}^{-}(u) = -\cosh(u + \frac{\eta}{2}) \sinh(u - \frac{\eta}{2} + \xi^-) \sinh(u - \frac{\eta}{2} - \xi^-),
k_{3}^{-}(u) = \cosh(u + \frac{\eta}{2}) \sinh(u + \frac{\eta}{2} - \xi^-) \sinh(u - \frac{\eta}{2} - \xi^-). \tag{2.24}
\]

For \( K^+ \), we similarly take
\[
K^{+}\left(1\right)(u) = K^{-}\left(1\right)\left(-u - \rho^{(1,1)}\right)|_{\xi^-\rightarrow\xi^+}. \tag{2.25}
\]
3. Monodromy and transfer matrices

Using the $R$ and $K$ matrices introduced in the previous section, we can obtain families of commuting operators, i.e. transfer matrices.

3.1. Single-row

Indeed, the single-row monodromy matrix for the R-matrix (2.9) is defined as

$$ T_a^{(1/2,1)}(u) = R_{aN}^{(1/2,1)}(u) \cdots R_{a1}^{(1/2,1)}(u), $$

and the associated transfer matrix by

$$ t^{(1/2,1)}(u) = \text{tr}_a T_a^{(1/2,1)}(u). $$

Similarly, for the R-matrix (2.14)

$$ T_a^{(1,1)}(u) = R_{aN}^{(1,1)}(u) \cdots R_{a1}^{(1,1)}(u), $$

and

$$ t^{(1,1)}(u) = \text{tr}_a T_a^{(1,1)}(u). $$

The Yang–Baxter equation (2.8) imply

$$ R_{12}^{(1/2,1)}(u - v) T_1^{(1/2,1)}(u) T_2^{(1/2,1)}(v) = T_2^{(1/2,1)}(v) T_1^{(1/2,1)}(u) R_{12}^{(1/2,1)}(u - v) $$

and

$$ R_{12}^{(1,1)}(u - v) T_1^{(1,1)}(u) T_2^{(1,1)}(v) = T_2^{(1,1)}(v) T_1^{(1,1)}(u) R_{12}^{(1,1)}(u - v), $$

as well as

$$ [t^{(1/2,1)}(u), t^{(1/2,1)}(v)] = 0, \quad [t^{(1,1)}(u), t^{(1,1)}(v)] = 0, $$

and

$$ [t^{(1/2,1)}(u), t^{(1,1)}(v)] = 0. $$

The latter relation implies that $t^{(1/2,1)}(u)$ and $t^{(1,1)}(v)$ can be diagonalized simultaneously. In addition, one obtains with the help of (2.14) the important relation

$$ T_{(12)}^{(1,1)}(u) = F_{(12)} T_1^{(1/2,1)}(u + \frac{\eta}{2}) T_2^{(1/2,1)}(u - \frac{\eta}{2}) E_{(12)}, $$

which will be used in section 4.3.

3.2. Double-row

In order to construct double-row objects, one needs to introduce ‘reflected’ single-row monodromy matrices

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A tale of two Bethe ansätze

\[ \widehat{T}_{a}^{(\frac{1}{2},1)}(u) = R_{a_1}^{(1)}(u) \ldots R_{a_N}^{(\frac{1}{2},1)}(u), \]

and similarly

\[ \widehat{T}_{a}^{{(1,1)}}(u) = R_{a_1}^{(1,1)}(u) \ldots R_{a_N}^{(1,1)}(u). \]

The corresponding double-row monodromy matrices are then defined as follows

\[
U_{a}^{(\frac{1}{2},1)}(u) = T_{a}^{(\frac{1}{2},1)}(u) K_{a}^{(-\frac{1}{2})}(u) \widehat{T}_{a}^{(\frac{1}{2},1)}(u),
\]

\[
U_{a}^{(1,1)}(u) = T_{a}^{(1,1)}(u) K_{a}^{(-1)}(u) \widehat{T}_{a}^{(1,1)}(u).
\]

They obey the boundary Yang–Baxter equation (2.19), in particular

\[
R_{12}^{(\frac{1}{2},\frac{1}{2})}(u-v) U_{12}^{(\frac{1}{2},1)}(u) R_{12}^{(\frac{1}{2},\frac{1}{2})}(u+v) U_{2}^{(\frac{1}{2},1)}(v) = U_{2}^{(\frac{1}{2},1)}(v) R_{12}^{(\frac{1}{2},\frac{1}{2})}(u+v) U_{12}^{(\frac{1}{2},1)}(u) R_{12}^{(\frac{1}{2},\frac{1}{2})}(u-v),
\]

and

\[
R_{12}^{(1,1)}(u-v) U_{12}^{(1,1)}(u) R_{12}^{(1,1)}(u+v) U_{2}^{(1,1)}(v) = U_{2}^{(1,1)}(v) R_{12}^{(1,1)}(u+v) U_{12}^{(1,1)}(u) R_{12}^{(1,1)}(u-v).
\]

Moreover, they are related by

\[
U_{(12)}^{(1,1)}(u) = \frac{1}{2 \sinh(u + \frac{\eta}{2})} F_{(12)} U_{1}^{(\frac{1}{2},1)}(u + \frac{\eta}{2}) R_{12}^{(\frac{1}{2},\frac{1}{2})}(2u) U_{2}^{(\frac{1}{2},1)}(u - \frac{\eta}{2}) P_{12} E_{(12)},
\]

which can be obtained with the help of (2.23). This relation, which is the double-row version of (3.9), will be exploited in section 5.3.

Finally, the corresponding transfer matrices are given by

\[
\tau^{(\frac{1}{2},1)}(u) = \text{tr}_{a} K_{a}^{(\frac{1}{2})}(u) U_{a}^{(\frac{1}{2},1)}(u),
\]

\[
\tau^{(1,1)}(u) = \text{tr}_{a} K_{a}^{(1)}(u) U_{a}^{(1,1)}(u).
\]

One can show that these transfer matrices obey

\[
[\tau^{(\frac{1}{2},1)}(u), \tau^{(\frac{1}{2},1)}(v)] = 0, \quad [\tau^{(1,1)}(u), \tau^{(1,1)}(v)] = 0,
\]

as well as

\[
[\tau^{(\frac{1}{2},1)}(u), \tau^{(1,1)}(v)] = 0.
\]

Similarly to the single-row case, the latter relation implies that \( \tau^{(\frac{1}{2},1)}(u) \) and \( \tau^{(1,1)}(v) \) can be diagonalized simultaneously.

4. Bethe ansatz: single-row transfer matrix

In this section we obtain the eigenvectors of the single-row transfer matrices (3.2) and (3.4).

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4.1. Bethe vectors from fusion

We start by applying the Bethe ansatz to (3.2), and we refer to [5–14] for more details. For that, we use the 2-dimensional auxiliary space representation, i.e. we set

\[ T_a^{(\frac{1}{2}, 1)}(u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}, \]

in which each entry is an operator acting on the quantum space \((\mathbb{C}^3)^\otimes N\). These operators satisfy exchange relations dictated by (3.5).

We also introduce the reference state vector

\[ |0\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \otimes^N \]

as well as its dual

\[ \langle 0 | = (1 \ 0 \ldots 0) \otimes^N, \]

such that \( \langle 0 | 0 \rangle = 1 \). The action of the monodromy operators on (4.2) and (4.3) is given by

\[ A(u)|0\rangle = \lambda_1(u)|0\rangle, \quad D(u)|0\rangle = \lambda_2(u)|0\rangle, \quad C(u)|0\rangle = 0, \]

\[ \langle 0 | A(u) = \langle 0 | \lambda_1(u), \quad \langle 0 | D(u) = \langle 0 | \lambda_2(u), \quad \langle 0 | B(u) = 0, \]

where

\[ \lambda_1(u) = \sinh^N \left( u + \frac{3\eta}{2} \right), \quad \lambda_2(u) = \sinh^N \left( u - \frac{\eta}{2} \right). \]

The transfer matrix (3.2) is given by

\[ t^{(\frac{1}{2}, 1)}(u) = A(u) + D(u). \]

The Bethe vectors are given by

\[ |\phi_m(u_1, \ldots, u_m)\rangle = B(u_1) \ldots B(u_m)|0\rangle \]

and

\[ \langle \phi_m(u_1, \ldots, u_m)| = \langle 0 | C(u_1) \ldots C(u_m). \]

In fact, using the Yang–Baxter algebra (3.5) one can show that

\[ t^{(\frac{1}{2}, 1)}(u)|\phi_m(u_1, \ldots, u_m)\rangle = \lambda(u, u_1, \ldots, u_m)|\phi_m(u_1, \ldots, u_m)\rangle \]

\[ + \sum_{j=1}^{m} \sinh(\eta) F(u, u_j) \frac{E_j(u_j)}{Q_j(u_j)} B(u)|\phi_{m-1}(u_1, \ldots, \hat{u}_j, \ldots, u_m)\rangle \]

and

\[ \langle \phi_m(u_1, \ldots, u_m)| t^{(\frac{1}{2}, 1)}(u) = \lambda(u, u_1, \ldots, u_m) \langle \phi_m(u_1, \ldots, u_m)|. \]

\[ \text{https://doi.org/10.1088/1742-5468/aab851} \]
A tale of two Bethe ansätze

\[ \langle \phi_m(u_1, \ldots, u_m) | t^{\left( \frac{1}{2}, 1 \right)}(u) = \langle \phi_m(u_1, \ldots, u_m) | \lambda(u, u_1, \ldots, u_m) \]
\[ + \sum_{j=1}^{m} \langle \phi_m(u_1, \ldots, \hat{u}_j, \ldots, u_m) | C(u) E_j(u_j) Q_j(u_j) F(u, u_j) \sinh(\eta), \]

where

\[ \lambda(u, u_1, \ldots, u_m) = \lambda_1(u) \frac{Q(u - \eta)}{Q(u)} + \lambda_2(u) \frac{Q(u + \eta)}{Q(u)}, \]

(4.12)

\[ E_j(u) = \lambda_1(u) Q_j(u) - \lambda_2(u) Q_j(u + \eta), \]

(4.13)

and

\[ F(u, v) = \frac{1}{\sinh(u - v)}. \]

(4.14)

In the above formulae we have introduced the Baxter Q-polynomial

\[ Q(u) = \prod_{i=1}^{m} \sinh(u - u_i), \]

(4.15)

which is indeed a Laurent polynomial in \( z \equiv e^u \), as well as the indexed Baxter Q-polynomial

\[ Q_j(u) = \prod_{i \neq j} \sinh(u - u_i). \]

(4.16)

Let us also introduce the two-index polynomial which we will use later

\[ Q_{j,k}(u) = \prod_{i \neq j,k} \sinh(u - u_i). \]

(4.17)

The equations

\[ E_j(u_j) = 0 \quad \text{for} \quad j = 1, \ldots, m, \]

(4.18)

are the conditions for the ‘unwanted’ terms in the off-shell equations (4.10) and (4.11) to vanish; and therefore, for the Bethe vectors (4.8) and (4.9) to be eigenvectors of the transfer matrix \( t^{\left( \frac{1}{2}, 1 \right)}(u) \), with corresponding eigenvalue \( \lambda(u, u_1, \ldots, u_m) \) given by (4.12).

These polynomial equations are known as the Bethe ansatz equations, and we shall refer to \( E_j(u) \) as the Bethe ansatz polynomials. As usual, the notation \( \hat{u}_j \) means that the rapidity \( u_j \) is absent from a given function or operator argument.

Finally, let us suppose that \( \{ u_1, \ldots, u_m \} \) satisfy the Bethe equation (4.18) (i.e. they are on-shell rapidities) and that there is no restriction on \( \{ v_1, \ldots, v_m \} \) (i.e. they are off-shell rapidities). Then, the scalar product \( \langle \phi_m(u_1, \ldots, u_m) | \phi_m(v_1, \ldots, v_m) \rangle \) is given by the determinant formula [24, 26]

\[ \langle \phi_m(u_1, \ldots, u_m) | \phi_m(v_1, \ldots, v_m) \rangle = \prod_{i=1}^{m} \lambda_2(u_i) \frac{\det_m \left( \frac{\partial}{\partial u_i} \lambda(v_j, u_1, \ldots, u_m) \right)}{\det_m(F(v_i, u_j))}, \]

(4.19)

https://doi.org/10.1088/1742-5468/aab851
A tale of two Bethe ansätze

which is known as the Slavnov formula. Using

\[
\frac{\partial}{\partial u_i} \lambda(v_j, u_1, \ldots, u_m) = \sinh(\eta)F(u_i, v_j)F(v_j, u_i) \frac{E_i(v_j)}{Q_i(v_j)},
\]

we can rewrite the Slavnov formula as

\[
\langle \phi_m(u_1, \ldots, u_m)|\phi_m(v_1, \ldots, v_m) \rangle = \prod_{i=1}^m \lambda_2(u_i) \frac{\det_m \left( \sinh(\eta)F(u_i, v_j)F(v_j, u_i) \frac{E_i(v_j)}{Q_i(v_j)} \right)}{\det_m (F(v_i, u_j))}.
\]

(4.21)

Although the Slavnov formula is frequently written in the form (4.19), which was introduced in [30], it seems more suitable for generalization to rewrite the Slavnov formula as in (4.21) in terms of the Bethe ansatz and Baxter polynomials instead of the derivative of the transfer matrix eigenvalue, as it will become clear in section 4.4. The form (4.21) is closer to the expression presented by Slavnov in his pioneering paper [24].

4.2. Bethe vectors from Tarasov’s construction

We now apply the Bethe ansatz to (3.4), see [18, 22]. Here, we use the 3-dimensional auxiliary space representation, i.e. we set

\[
T^{(1,1)}_a(u) = \left( \begin{array}{ccc} A_1(u) & B_1(u) & B_2(u) \\ C_1(u) & A_2(u) & B_3(u) \\ C_2(u) & C_3(u) & A_3(u) \end{array} \right),
\]

(4.22)

where each entry acts on the quantum space \((\mathbb{C}^3)^\otimes N\). These operators satisfy exchange relations dictated by (3.6). The action of the monodromy operators (4.22) on the reference state (4.2, 4.3) is given by

\[
A_j(u)|0\rangle = \Lambda_j(u)|0\rangle, \quad C_j(u)|0\rangle = 0,
\]

(4.23)

\[
\langle 0|A_j(u) = \langle 0|\Lambda_j(u), \quad \langle 0|B_j(u) = 0,
\]

(4.24)

for \(j = 1, 2, 3\) and where

\[
\Lambda_1(u) = (\sinh(u + \eta) \sinh(u + 2\eta))^N, \quad \Lambda_2(u) = (\sinh(u) \sinh(u + \eta))^N,
\]

\[
\Lambda_3(u) = (\sinh(u) \sinh(u - \eta))^N.
\]

(4.25)

The transfer matrix (3.4) is given by

\[
t^{(1,1)}(u) = A_1(u) + A_2(u) + A_3(u).
\]

(4.26)

The Bethe vector is constructed by means of the recursion relation

\[
|\psi_m(u_1, \ldots, u_m)\rangle = B_1(u_1)|\psi_{m-1}(u_2, \ldots, u_m)\rangle - B_2(u_1) \sum_{i=2}^m \gamma_i^{(m)}(u_1, \ldots, u_m)|\psi_{m-2}(u_2, \ldots, \hat{u}_i, \ldots, u_m)\rangle,
\]

(4.27)

where

https://doi.org/10.1088/1742-5468/aab851
A tale of two Bethe ansätze

\[ \gamma^{(m)}_{i}(u_1, \ldots, u_m) = 2 \sinh(\eta) \Lambda(u_i) F(u_1, u_i + \eta) \frac{Q_{1,i}(u_i - 2\eta)}{Q_{1,i}(u_i)} \prod_{j=2, j<i}^{m} \Omega(u_i, u_j), \]  

(4.28)

and

\[ \Omega(u, v) = \frac{\sinh(u - v - \eta) \sinh(u - v + 2\eta)}{\sinh(u - v - 2\eta) \sinh(u - v + \eta)}. \]  

(4.29)

The dual Bethe vector is given by

\[ \langle \psi_{m}(u_1, \ldots, u_m) | = \langle \psi_{m-1}(u_2, \ldots, u_m) | C_1(u_1) \]

\[ - \sum_{i=2}^{m} \tilde{\gamma}^{(m)}_{i}(u_1, \ldots, u_m) \langle \psi_{m-2}(u_2, \ldots, \hat{u}_i, \ldots, u_m) | C_2(u_1), \]  

(4.30)

where

\[ \tilde{\gamma}^{(m)}_{i}(u_1, \ldots, u_m) = \sinh(2\eta) \cosh(\eta) \Lambda(u_i) F(u_1, u_i + \eta) \frac{Q_{1,i}(u_i - 2\eta)}{Q_{1,i}(u_i)} \prod_{j=2, j<i}^{m} \Omega(u_i, u_j). \]  

(4.31)

Note that the initial conditions

\[ |\psi_0\rangle = |0\rangle, \quad \langle \psi_0 | = \langle 0 |, \]  

(4.32)

are assumed for (4.27) and (4.30).

Using the Yang–Baxter algebra (3.6) one can show that the Tarasov–Bethe vectors (4.27) and (4.30) satisfy the off-shell equations

\[ t^{(1,1)}(u) | \psi_{m}(u_1, \ldots, u_m) \rangle = \Lambda(u, u_1, \ldots, u_m) | \psi_{m}(u_1, \ldots, u_m) \rangle \]

\[ + \sum_{j=1}^{m} \sinh(2\eta) F(u, u_j) \frac{E_j(u_j) Q_j(u_j - 2\eta)}{Q_j(u_j) Q_j(u_j - \eta)} \prod_{p<j}^{m} \Omega(u_j, u_p) B_1(u) | \psi_{m-1}(u_1, \ldots, \hat{u}_j, \ldots, u_m) \rangle \]

\[ + \sum_{j=1}^{m} 2 \sinh(\eta) F(u, u_j + \eta) \frac{E_j(u_j) Q_j(u_j - 2\eta)}{Q_j(u_j) Q_j(u_j - \eta)} \prod_{p<j}^{m} \Omega(u_j, u_p) B_3(u) | \psi_{m-1}(u_1, \ldots, \hat{u}_j, \ldots, u_m) \rangle \]

\[ + \sum_{j<k}^{m} H^{(m)}_{jk}(u, u_1, \ldots, u_m) B_2(u) | \psi_{m-2}(u_1, \ldots, \hat{u}_j, \ldots, \hat{u}_k, \ldots, u_m) \rangle, \]  

(4.33)

and

\[ | \psi_{m}(u_1, \ldots, u_m) \rangle t^{(1,1)}(u) = \langle \psi_{m}(u_1, \ldots, u_m) | \Lambda(u, u_1, \ldots, u_m) \]

\[ + \sum_{j=1}^{m} \langle \psi_{m-1}(u_1, \ldots, \hat{u}_j, \ldots, u_m) | C_1(u) \prod_{p<j}^{m} \Omega(u_j, u_p) \frac{E_j(u_j) Q_j(u_j - 2\eta)}{Q_j(u_j) Q_j(u_j - \eta)} F(u, u_j) \sinh(2\eta) \]

\[ + \sum_{j=1}^{m} \langle \psi_{m-1}(u_1, \ldots, \hat{u}_j, \ldots, u_m) | C_3(u) \prod_{p<j}^{m} \Omega(u_j, u_p) \frac{E_j(u_j) Q_j(u_j - 2\eta)}{Q_j(u_j) Q_j(u_j - \eta)} \sinh(2\eta) \cosh(\eta) F(u, u_j + \eta) \]

\[ + \sum_{j<k}^{m} \langle \psi_{m-2}(u_1, \ldots, \hat{u}_j, \ldots, \hat{u}_k, \ldots, u_m) | C_2(u) H^{(m)}_{jk}(u, u_1, \ldots, u_m) \cosh^2(\eta), \]  

(4.34)

where

https://doi.org/10.1088/1742-5468/aab851
A tale of two Bethe ansätze

\[ \Lambda(u, u_1, \ldots, u_m) = \Lambda_1(u) \frac{Q(u - 2\eta) - 2}{Q(u)} + \Lambda_2(u) \frac{Q(u - 2\eta) - 2}{Q(u)} + \Lambda_3(u) \frac{Q(u + \eta)}{Q(u - \eta)}. \]

\[ E_j(u) = \Lambda_1(u) Q_j(u - \eta) - \Lambda_2(u) Q_j(u + \eta), \]

and

\[ H^{(m)}_{jk}(u, u_1, \ldots, u_m) \]

\[ = 4 \cosh(\eta) \sinh^2(\eta) \frac{Q_j(u_j - 2\eta) Q_j(u_k - 2\eta)}{Q_j(u_j - \eta) Q_j(u_k - \eta)} \prod_{p<j} \Omega(u_j, u_p) \prod_{q<k; q\neq j} \Omega(u_k, u_q) \]

\[ \times \left\{ F(u, u_j) F(u, u_k) F(u, u_k + \eta) F(u, u_k - \eta) E_j(u_j) E_j(u_k) + \sinh(\eta) F(u, u_k) F(u, u_k + \eta) F(u, u_k) F(u, u_k + \eta) Q_j(u_j + \eta) \Lambda_2(u_j) E_j(u_k) + \sinh(\eta) F(u, u_j) F(u, u_j + \eta) F(u, u_k) F(u, u_k - \eta) Q_j(u_k + \eta) \Lambda_2(u_k) E_j(u_j) \right\}. \]

The set of polynomial equations

\[ E_j(u_j) = 0 \quad \text{for} \quad j = 1, \ldots, m, \]

are the Bethe ansatz equations from Tarasov’s construction. We remark that the unwanted terms have been written explicitly in terms of the Bethe polynomials; in this way, it is clear that all unwanted terms disappear on-shell.

4.3. Relating the two closed-chain Bethe ansätze

The Bethe equations from fusion (4.18) and from Tarasov’s construction (4.38) are equivalent. Indeed, since the factors in the Bethe polynomials \( E_j(u) \) (4.13) and \( E_j(u) \) (4.36) are related by

\[ \frac{\Lambda_1(u)}{\Lambda_2(u)} = \frac{\lambda_1(u + \frac{\eta}{2})}{\lambda_2(u + \frac{\eta}{2})}, \]

the solutions of the two sets of Bethe equations are related by \( \frac{\eta}{2} \) shifts.

We claim that the relation between the off-shell fusion-Bethe vectors (4.8) and (4.9), and the off-shell Tarasov–Bethe vectors (4.27) and (4.30) is given by

\[ |\psi_m(u_1, \ldots, u_m)| = s(u_1, \ldots, u_m) \phi_m \left( u_1 + \frac{\eta}{2}, \ldots, u_m + \frac{\eta}{2} \right), \]

and

\[ \langle \psi_m(u_1, \ldots, u_m) \rangle = \left\langle \phi_m \left( u_1 + \frac{\eta}{2}, \ldots, u_m + \frac{\eta}{2} \right) s(u_1, \ldots, u_m) \cosh^m(\eta) \right\rangle, \]

where

\[ s(u_p, \ldots, u_m) = 2^{m-(p-1)} \prod_{i=p}^m \lambda_1 \left( u_i - \frac{\eta}{2} \right) \prod_{j,k=p: j<k}^m \frac{\sinh(u_j - u_k - 2\eta)}{\sinh(u_j - u_k - \eta)}. \]

We prove (4.40) by induction; the proof of (4.41) is similar. The cases \( m = 1 \) and \( m = 2 \) follow from brute force computation. We first note that the representations (4.1) and

https://doi.org/10.1088/1742-5468/aab851
A tale of two Bethe ansätze

(4.22) are connected by means of the relation (3.9), which gives us in particular (see e.g. [25])

\[
B_1(u) = \frac{1}{\sqrt{2}} \left( A \left( u + \frac{\eta}{2} \right) B \left( u - \frac{\eta}{2} \right) + B \left( u + \frac{\eta}{2} \right) A \left( u - \frac{\eta}{2} \right) \right) = \sqrt{2} B \left( u + \frac{\eta}{2} \right) A \left( u - \frac{\eta}{2} \right),
\]

\[
B_2(u) = B \left( u + \frac{\eta}{2} \right) B \left( u - \frac{\eta}{2} \right),
\]

and then use commutation relations from (3.5), the action (4.4) and the identity

\[
\lambda_1(u) = \lambda_1 \left( u + \frac{\eta}{2} \right) \lambda_1 \left( u - \frac{\eta}{2} \right).
\]

Next, let us compute \( |\psi_{m+1}(u_1, \ldots, u_{m+1})\rangle \) supposing (4.40) is valid. By (4.27) we have

\[
|\psi_{m+1}(u_1, \ldots, u_{m+1})\rangle = B_1(u_1)|\psi_m(u_2, \ldots, u_{m+1})\rangle
- B_2(u_1) \sum_{i=2}^{m+1} \gamma_i^{(m+1)}(u_1, \ldots, u_{m+1})|\psi_{m-1}(u_2, \ldots, \hat{u}_i, \ldots, u_{m+1})\rangle,
\]

and, by means of (4.40), we obtain

\[
|\psi_{m+1}(u_1, \ldots, u_{m+1})\rangle = s(u_2, \ldots, u_{m+1}) B_1(u_1)|\phi_m(\tilde{u}_2, \ldots, \tilde{u}_{m+1})\rangle
- B_2(u_1) \sum_{i=2}^{m+1} \gamma_i^{(m+1)}(u_1, \ldots, u_{m+1}) s(u_2, \ldots, \hat{u}_i, \ldots, u_{m+1})|\phi_{m-1}(\tilde{u}_2, \ldots, \hat{u}_i, \ldots, \tilde{u}_{m+1})\rangle,
\]

where we have introduced

\[
s(u_p, \ldots, \hat{u}_i, \ldots, u_m) = 2 \prod_{i=p,i\neq l}^{m} \lambda_1 \left( u_i - \frac{\eta}{2} \right) \prod_{j=k=p,j<k,j\neq l,k\neq i}^{m} \frac{\sinh(u_j - u_k - 2\eta)}{\sinh(u_j - u_k - \eta)}
\]

and the notation

\[
\tilde{u} = u + \frac{\eta}{2}.
\]

Using (4.43) we express \( B_1(u_1) \) and \( B_2(u_1) \) in terms of \( A(u_1) \) and \( B(u_1) \), and compute

\[
B_1(u_1)|\phi_m(\tilde{u}_2, \ldots, \tilde{u}_{m+1})\rangle = \sqrt{2} B \tilde{u}_1 A \left( u_1 - \frac{\eta}{2} \right)|\phi_m(\tilde{u}_2, \ldots, \tilde{u}_{m+1})\rangle,
\]

\[
B_2(u_1)|\phi_{m-1}(\tilde{u}_2, \ldots, \hat{u}_i, \ldots, \tilde{u}_{m+1})\rangle = B \tilde{u}_1 B \left( u_1 - \frac{\eta}{2} \right)|\phi_{m-1}(\tilde{u}_2, \ldots, \hat{u}_i, \ldots, \tilde{u}_{m+1})\rangle
= B \left( u_1 - \frac{\eta}{2} \right)|\phi_{m}(\tilde{u}_1, \tilde{u}_2, \ldots, \tilde{u}_i, \ldots, \tilde{u}_{m+1})\rangle.
\]

By means of (3.5) and the action (4.4), we obtain

\[
A(u)|\phi_m(u_1, \ldots, u_m)\rangle = \lambda_1(u) \frac{Q(u - \eta)}{Q(u)}|\phi_m(u_1, \ldots, u_m)\rangle
+ \sum_{j=1}^{m} \frac{\sinh(\eta)}{\sinh(u - u_j)} \lambda_1(u_j) \frac{Q_j(u_j - \eta)}{Q_j(u_j)} B(u)|\phi_{m-1}(u_1, \ldots, \hat{u}_j, \ldots, u_m)\rangle,
\]

which leads to

https://doi.org/10.1088/1742-5468/aab851
A tale of two Bethe ansätze

\[ B_1(u_1)\left| \phi_m(\tilde{u}_2, \ldots, \tilde{u}_{m+1}) \right\rangle = \sqrt{2} \lambda_1 \left( u_1 - \eta \right) \prod_{j=2}^{m+1} \frac{\sinh(u_1 - u_j - 2\eta)}{\sinh(u_1 - u_j - \eta)} \left| \phi_{m+1}(\tilde{u}_1, \tilde{u}_2, \ldots, \tilde{u}_{m+1}) \right\rangle \]

\[ + \sqrt{2} \sum_{j=2}^{m+1} \frac{\sinh(\eta)\lambda_1(\tilde{u}_j)}{\sinh(u_1 - u_j - \eta)} \prod_{k=2, k \neq j}^{m+1} \frac{\sinh(u_k - u_j + \eta)}{\sinh(u_k - u_j)} B \left( u_1 - \frac{\eta}{2} \right) \langle \psi_m(\tilde{u}_1, \tilde{u}_2, \ldots, \tilde{u}_{j-1}, \tilde{u}_{j+1}, \ldots, \tilde{u}_{m+1}) \right\rangle. \]

Using (4.50) and (4.52) we obtain

\[ \langle \psi_{m+1}(u_1, \ldots, u_{m+1}) \rangle \]

\[ = s(u_2, \ldots, u_{m+1}) \sqrt{2} \lambda_1 \left( u_1 - \frac{\eta}{2} \right) \prod_{j=2}^{m+1} \frac{\sinh(u_1 - u_j - 2\eta)}{\sinh(u_1 - u_j - \eta)} \left| \phi_{m+1}(\tilde{u}_1, \tilde{u}_2, \ldots, \tilde{u}_{m+1}) \right\rangle \]

\[ + \sum_{j=2}^{m+1} \left\{ s(u_2, \ldots, u_{m+1}) \sqrt{2} \frac{\sinh(\eta)\lambda_1(\tilde{u}_j)}{\sinh(u_1 - u_j - \eta)} \prod_{k=2, k \neq j}^{m+1} \frac{\sinh(u_k - u_j + \eta)}{\sinh(u_k - u_j)} \right. \]

\[ - \gamma_j^{(m+1)}(u_1, \ldots, u_{m+1}) s(u_2, \ldots, \hat{u}_j, \ldots, u_{m+1}) \}

\[ \times B \left( u_1 - \frac{\eta}{2} \right) \langle \phi_m(\hat{\tilde{u}}_1, \hat{\tilde{u}}_2, \ldots, \hat{\tilde{u}}_{m+1}) \rangle. \]

We can check that

\[ s(u_1, \ldots, u_{m+1}) = s(u_2, \ldots, u_{m+1}) \sqrt{2} \lambda_1 \left( u_1 - \frac{\eta}{2} \right) \prod_{j=2}^{m+1} \frac{\sinh(u_1 - u_j - 2\eta)}{\sinh(u_1 - u_j - \eta)}, \]

and, thanks to (4.44),

\[ s(u_2, \ldots, u_{m+1}) \sqrt{2} \frac{\sinh(\eta)\lambda_1(u_j + \frac{\eta}{2})}{\sinh(u_1 - u_j - \eta)} \prod_{k=2, k \neq j}^{m+1} \frac{\sinh(u_k - u_j + \eta)}{\sinh(u_k - u_j)} \]

\[ = \gamma_j^{(m+1)}(u_1, \ldots, u_{m+1}) s(u_2, \ldots, \hat{u}_j, \ldots, u_{m+1}), \]

which implies

\[ \langle \psi_{m+1}(u_1, \ldots, u_{m+1}) \rangle = s(u_1, \ldots, u_{m+1}) \langle \phi_{m+1}(\tilde{u}_1, \tilde{u}_2, \ldots, \tilde{u}_{m+1}) \rangle, \]

and therefore completes the inductive proof of (4.40).

\[ \square \]

4.4. Scalar products for Tarasov–Bethe vectors

Combining the scalar product for the fusion-Bethe vectors (4.19) with the relations between Tarasov–Bethe vectors and fusion-Bethe vectors (4.40) and (4.41), it is now straightforward to compute the scalar product between the off-shell state \( \langle \psi_m(v_1, \ldots, v_m) \rangle \) (4.30) and the on-shell state \( \langle \psi_m(v_1, \ldots, v_m) \rangle \) (4.27)

\[ \langle \psi_m(v_1, \ldots, v_m) | \psi_m(v_1, \ldots, v_m) \rangle = \cosh^m(\eta) s(v_1, \ldots, v_m) s(v_1, \ldots, v_m) \]

\[ \times \prod_{i=1}^{m} \lambda_2 \left( v_i + \eta \right) \frac{\det_m \left( \begin{array}{c} \pi_1(v_1 + \frac{\eta}{2}, u_1 + \frac{\eta}{2}, \ldots, u_m + \frac{\eta}{2}) \end{array} \right) \det_m (F(v_i, u_j)) \right), \]

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which is in terms of the quantities $\lambda$ and $\lambda_2$ arising from fusion. We can rewrite (4.57) in a similar way as in (4.21)

$$
\langle \psi_m(u_1, \ldots, u_m) | \psi_m(v_1, \ldots, v_m) \rangle = \prod_{j<k}^{m} \frac{\sinh(u_j - u_k - 2\eta) \sinh(v_j - v_k - 2\eta)}{\sinh(u_j - u_k - \eta) \sinh(v_j - v_k - \eta)} \times \prod_{i=1}^{m} \Lambda_2(u_i) \frac{\operatorname{det}_m \left( \frac{\sinh(2\eta) F(u_i, v_j) F(v_j, u_i) \tilde{E}_j(v_j)}{\det_m (F(v_j, u_i))} \right)}{\det_m (F(v_j, u_i))},
$$

(4.58)

which is instead in terms of quantities arising from Tarasov’s construction, namely, the Bethe ansatz polynomial (4.38) and the Baxter polynomial (4.15). The Slavnov formula written in (4.21) or (4.58) therefore seems more ‘universal’ than the formula given in terms of the eigenvalue of the transfer matrix.

5. Bethe ansatz: double-row transfer matrix

In this section we obtain the eigenvectors of the double-row transfer matrices (3.16) and (3.17).

5.1. Bethe vectors from fusion

We firstly apply the Bethe ansatz to (3.16), see [4, 15]. For that, we use the 2-dimensional auxiliary space representation, i.e. we set

$$
U_a^{(1,2)}(u) = \left( \begin{array}{cc} \mathcal{A}(u) & \mathcal{B}(u) \\ \mathcal{C}(u) & \mathcal{D}(u) \end{array} \right) + \frac{\sinh(2\eta)}{\sinh(2u+\eta)} \mathcal{A}(u)
$$

(5.1)

in which each entry is an operator acting on the quantum space $(\mathbb{C}^3)^\otimes N$. These operators satisfy exchange relations dictated by (3.13). The action of the double-row monodromy operators on the reference states (4.2,4.3) is given by

$$
\mathcal{A}(u)|0\rangle = \delta_1(u)|0\rangle, \quad \mathcal{D}(u)|0\rangle = \delta_2(u)|0\rangle, \quad \mathcal{C}(u)|0\rangle = 0,
$$

(5.2)

$$
\langle 0|\mathcal{A}(u) = \langle 0|\delta_1(u), \quad \langle 0|\mathcal{D}(u) = \langle 0|\delta_2(u), \quad \langle 0|\mathcal{B}(u) = 0,
$$

(5.3)

where

$$
\delta_1(u) = \sinh(u + \xi^-) \sinh^{2N} \left( u + \frac{3\eta}{2} \right),
$$

$$
\delta_2(u) = \frac{\sinh(2u) \sinh(\xi^- - u - \eta)}{\sinh(2u + \eta)} \sinh^{2N} \left( u - \frac{\eta}{2} \right).
$$

(5.4)

The transfer matrix (3.16) is given by

$$
\tau^{(1,2)}(u) = \frac{\sinh(2(u + \eta))) \sinh(u - \xi^+)}{\sinh(2u + \eta)} \mathcal{A}(u) - \sinh(u + \eta + \xi^+) \mathcal{D}(u).
$$

(5.5)
The fusion-Bethe vectors are given by

$$ |\Phi_m(u_1, \ldots, u_m)\rangle = \mathcal{B}(u_1) \cdots \mathcal{B}(u_m)|0\rangle, \quad (5.6) $$

and

$$ \langle \Phi_m(u_1, \ldots, u_m) | = \langle 0 | \mathcal{C}(u_1) \cdots \mathcal{C}(u_m). \quad (5.7) $$

In fact, using the reflection algebra (3.13) we can show that

$$ \tau(\frac{1}{2},1)(u) |\Phi_m(u_1, \ldots, u_m)\rangle = \delta(u, u_1, \ldots, u_m) \langle \Phi_m(u_1, \ldots, u_m) | $$

$$ + \sum_{j=1}^{m} \sinh(\eta) \sinh(2(u + \eta)) \mathcal{F}(u, u_j) \mathcal{E}_j(u_j) \frac{\mathcal{Q}(u_j)}{\mathcal{Q}(u)} \mathcal{B}(u) |\Phi_{m-1}(u_1, \ldots, \hat{u}_j, \ldots, u_m)\rangle, $$

and

$$ \langle \Phi_m(u_1, \ldots, u_m) | \tau(\frac{1}{2},1)(u) = \langle \Phi_m(u_1, \ldots, u_m) | \delta(u, u_1, \ldots, u_m) $$

$$ + \sum_{j=1}^{m} \langle \Phi_m(u_1, \ldots, \hat{u}_j, \ldots, u_m) | \mathcal{C}(u_j) \mathcal{E}_j(u_j) \frac{\mathcal{F}(u, u_j)}{\mathcal{Q}_j(u_j)} \sinh(2(u + \eta)) \sinh(\eta), $$

where

$$ \delta(u, u_1, \ldots, u_m) = \frac{\sinh(2(u + \eta)) \sinh(u - \xi^+)}{\sinh(2u + \eta)} \delta_1(u) \frac{\mathcal{Q}(u - \eta)}{\mathcal{Q}(u)} $$

$$ - \sinh(u + \eta + \xi^+) \delta_2(u) \frac{\mathcal{Q}(u + \eta)}{\mathcal{Q}(u)}, $$

$$ \mathcal{E}_j(u) = \sinh(2u) \sinh(u - \xi^+) \delta_1(u) \mathcal{Q}_j(u - \eta) $$

$$ + \sinh(2u + \eta) \sinh(u + \eta + \xi^+) \delta_2(u) \mathcal{Q}_j(u + \eta), $$

and

$$ \mathcal{F}(u, v) = \frac{1}{\sinh(u - v) \sinh(u + v + \eta)}. $$

In the above formulae we have introduced the double-row Baxter $Q$-polynomial,

$$ \mathcal{Q}(u) = \prod_{i=1}^{m} \sinh(u - u_i) \sinh(u + u_i + \eta), $$

and the indexed double-row Baxter $Q$-polynomial,

$$ \mathcal{Q}_j(u) = \prod_{i \neq j}^{m} \sinh(u - u_i) \sinh(u + u_i + \eta). $$

The set of polynomial equations

$$ \mathcal{E}_j(u_j) = 0 \quad \text{for} \quad j = 1, \ldots, m, $$

are known as the Bethe ansatz equations for the double-row transfer matrix. Again, we will refer to $\mathcal{E}_j(u)$ as the Bethe ansatz polynomials for the double-row transfer matrix.
Finally, let us suppose that \( \{u_1, \ldots, u_m\} \) satisfy the Bethe equation (5.15) (i.e. they are on-shell rapidities) and that there is no restriction on \( \{v_1, \ldots, v_m\} \) (i.e. they are off-shell rapidities). Then, the scalar product \( \langle \Phi_m(u_1, \ldots, u_m)|\Phi_m(v_1, \ldots, v_m) \rangle \) is given by the compact formula [29],

\[
\langle \Phi_m(u_1, \ldots, u_m)|\Phi_m(v_1, \ldots, v_m) \rangle = \prod_{i=1}^{m} \frac{\delta_2(u_i)}{\sinh(2(v_i + \eta))} \prod_{j<i} \frac{\sinh(u_i + u_j + 2\eta)}{\sinh(u_i + u_j)} \times \frac{\det_m \left( \frac{\partial}{\partial u_i} \delta(v_j, u_1, \ldots, u_m) \right)}{\det_m (\mathcal{F}(v_i, u_j))},
\]

which is the Slavnov formula for the double-row transfer matrix. Using

\[
\frac{\partial}{\partial u_i} \delta(v_j, u_1, \ldots, u_m) = \sinh(\eta) \sinh(2u_i + \eta) \frac{\sinh(2(v_j + \eta))}{\sinh(2v_j + \eta)} \mathcal{F}(u_i, v_j) \mathcal{F}(v_j, u_i) \frac{\mathcal{E}_i(v_j)}{\mathcal{Q}_i(v_j)}
\]

we can rewrite the Slavnov formula as

\[
\langle \Phi_m(u_1, \ldots, u_m)|\Phi_m(v_1, \ldots, v_m) \rangle = \prod_{i=1}^{m} \frac{\sinh(2u_i + \eta)\delta_2(u_i)}{\sinh(2v_i + \eta)\sinh(u_i - \xi^+)} \prod_{j<i} \frac{\sinh(u_i + u_j + 2\eta)}{\sinh(u_i + u_j)} \times \frac{\det_m \left( \frac{\sinh(\eta)\mathcal{F}(u_i, v_j)\mathcal{F}(v_j, u_i)\mathcal{E}_i(v_j)}{\mathcal{Q}_i(v_j)} \right)}{\det_m (\mathcal{F}(v_i, u_j))},
\]

which is given in terms of the double-row Bethe ansatz and Baxter polynomials, similarly to the single-row case.

### 5.2. Bethe vectors from Tarasov’s construction

We now apply the Bethe ansatz to (3.17), see [21, 23]. Here, we use the 3-dimensional auxiliary space representation, i.e. we set

\[
U_a^{(1,1)}(u) = \left( \begin{array}{ccc}
A_1(u) & B_1(u) & B_2(u) \\
C_1(u) & A_2(u) + \frac{\sinh(2\eta)}{\sinh(2(u+\eta))} A_1(u) & B_3(u) \\
C_2(u) & C_3(u) & A_3(u) + \frac{\sinh(\eta)\sinh(2\eta)}{\sinh(2(u+\eta))\sinh(2u+\eta)} A_1(u) + \frac{\sinh(2\eta)}{\sinh(2\eta)} A_2(u) \\
\end{array} \right),
\]

where each entry acts on the quantum space \( (\mathbb{C}^3)^{\otimes N} \). These operators satisfy exchange relations dictated by (3.14). The action of the monodromy operators (5.19) on the reference state (4.2) and (4.3) is given by

\[
\langle 0|A_j(u) = \Delta_j(u)|0\rangle, \quad \langle 0|C_j(u) = 0, \quad \langle 0|B_j(u) = 0,
\]

for \( j = 1, 2, 3 \) and where
The dual Bethe vector is given by
\[
\Psi_m(u_1, \ldots, u_m) = B_1(u_1) |\Psi_{m-1}(u_2, \ldots, u_m)\rangle 
- B_2(u_1) \sum_{i=2}^m \Gamma_i^{(m)}(u_1, \ldots, u_m) |\Psi_{m-2}(u_2, \ldots, \hat{u}_i, \ldots, u_m)\rangle,
\]
where
\[
\Gamma_i^{(m)}(u_1, \ldots, u_m) = 2 \sinh(\eta) \frac{\tilde{Q}_{1,i}(u_i - 2\eta)}{Q_{1,i}(u_i)} \prod_{j=2,j<i}^m \Omega(u_i, u_j) \times \left( \frac{\sinh(2u_i)}{\sinh(u_i - u_i - \eta) \sinh(2(u_i + \eta))} \Delta_1(u_i) - \frac{1}{\sinh(u_1 + u_i + \eta) \tilde{Q}_{1,i}(u_i - \eta)} \Delta_2(u_i) \right),
\]
and where we have introduced a new Baxter double-row Q-polynomial
\[
\tilde{Q}(u) = \prod_{i=1}^m \sinh(u - u_i) \sinh(u + u_i + 2\eta),
\]
\[
\tilde{Q}_j(u) = \prod_{i \neq j}^m \sinh(u - u_i) \sinh(u + u_i + 2\eta),
\]
\[
\tilde{Q}_{j,k}(u) = \prod_{i \neq j,k}^m \sinh(u - u_i) \sinh(u + u_i + 2\eta).
\]

The dual Bethe vector is given by
\[
\langle \Psi_m(u_1, \ldots, u_m) \rangle = \langle \Psi_{m-1}(u_2, \ldots, u_m) | C_1(u_1) 
- \sum_{i=2}^m \tilde{\Gamma}_i^{(m)}(u_1, \ldots, u_m) \langle \Psi_{m-2}(u_2, \ldots, \hat{u}_i, \ldots, u_m) | C_2(u_1) \rangle,
\]
\]

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where
\[ \tilde{\Gamma}^{(m)}_i(u_1, \ldots, u_m) = \sinh(2\eta) \cosh(\eta) \frac{Q_{1,i}(u_i - 2\eta)}{Q_{1,i}(u_i)} \prod_{j=2,j<i}^m \Omega(u_i, u_j) \times \left( \frac{\sinh(2u_i)}{\sinh(u_i - u_i - \eta) \sinh(2(u_i + \eta))} \Delta_1(u_i) - \frac{1}{\sinh(u_i + u_i + \eta)} \frac{Q_{1,i}(u_i + \eta)}{Q_{1,i}(u_i - \eta)} \Delta_2(u_i) \right). \]

Notice that the initial conditions
\[ |\Psi_0\rangle = |0\rangle, \quad \langle \Psi_0 | = \langle 0 | \]
are assumed for (5.24) and (5.29).

Using the reflection algebra (3.14) one can show that the vectors (5.24) and (5.29) satisfy the off-shell equations
\[
\tau^{(1,1)}(u)|\Psi_m(u_1, \ldots, u_m)\rangle = \Delta(u, u_1, \ldots, u_m)|\Psi_m(u_1, \ldots, u_m)\rangle
\]
\[ + \sum_{j=1}^m G_1(u, u_j) \frac{\tilde{E}_j(u_j) Q_{j}(u_j - 2\eta)}{Q_{j}(u_j)} \prod_{p<j}^m \Omega(u_j, u_p) B_1(u)|\Psi_{m-1}(u_1, \ldots, \hat{u}_j, \ldots, u_m)\rangle \]
\[ + \sum_{j=1}^m G_2(u, u_j) \frac{\tilde{E}_j(u_j) Q_{j}(u_j - 2\eta)}{Q_{j}(u_j)} \prod_{p<j}^m \Omega(u_j, u_p) B_3(u)|\Psi_{m-1}(u_1, \ldots, \hat{u}_j, \ldots, u_m)\rangle \]
\[ + \sum_{j<k} \mathcal{H}_{jk}^{(m)}(u, u_1, \ldots, u_m) B_2(u)|\Psi_{m-2}(u_1, \ldots, \hat{u}_j, \ldots, \hat{u}_k, \ldots, u_m)\rangle, \]
and
\[ \langle \Psi_m(u_1, \ldots, u_m)|\tau^{(1,1)}(u)\rangle = \langle \Psi_m(u_1, \ldots, u_m)|\Delta(u, u_1, \ldots, u_m) \]
\[ + \sum_{j=1}^m \langle \Psi_{m-1}(u_1, \ldots, \hat{u}_j, \ldots, u_m)|C_1(u) \prod_{p<j}^m \Omega(u_j, u_p) \frac{\tilde{E}_j(u_j) Q_{j}(u_j - 2\eta)}{Q_{j}(u_j)} G_1(u, u_j) \]
\[ + \sum_{j=1}^m \langle \Psi_{m-1}(u_1, \ldots, \hat{u}_j, \ldots, u_m)|C_3(u) \prod_{p<j}^m \Omega(u_j, u_p) \frac{\tilde{E}_j(u_j) Q_{j}(u_j - 2\eta)}{Q_{j}(u_j)} G_2(u, u_j) \cosh^2(\eta) \]
\[ + \sum_{j<k} \langle \Psi_{m-2}(u_1, \ldots, \hat{u}_j, \ldots, \hat{u}_k, \ldots, u_m)|C_2(u) \mathcal{H}_{jk}^{(m)}(u, u_1, \ldots, u_m) \cosh^2(\eta), \]

where
\[
\Delta(u, u_1, \ldots, u_m) = - \frac{\sinh(2u + 3\eta) \sinh\left(u - \frac{\eta}{2} - \xi^+\right) \sinh\left(u + \frac{\eta}{2} - \xi^+\right)}{2 \sinh\left(u + \frac{\eta}{2}\right)} \Delta_1(u) \frac{Q(u - 2\eta)}{Q(u)}
\]
\[ + \frac{\cosh\left(u + \frac{\eta}{2}\right) \sinh(2(u + \eta)) \sinh\left(u - \frac{\eta}{2} - \xi^+\right) \sinh\left(u + \frac{3\eta}{2} + \xi^+\right)}{\sinh(2u)} \Delta_2(u) \frac{Q(u - 2\eta) Q(u + \eta)}{Q(u) Q(u - \eta)}
\]
\[ - \frac{\cosh\left(u + \frac{\eta}{2}\right) \sinh\left(u + \frac{\eta}{2} + \xi^+\right) \sinh\left(u + \frac{3\eta}{2} + \xi^+\right)}{\sinh(2u)} \Delta_3(u) \frac{Q(u + \eta)}{Q(u - \eta)}.
\]
A tale of two Bethe ansätze

\( \tilde{E}_j(u) = \sinh(2u) \sinh \left( u + \frac{\eta}{2} - \xi^+ \right) \Delta_1(u) \tilde{Q}_j(u - \eta) \\
+ \sinh(2(u + \eta)) \sinh \left( u + \frac{3\eta}{2} + \xi^+ \right) \Delta_2(u) \tilde{Q}_j(u + \eta), \) (5.35)

\[ \mathcal{H}^{(m)}_{jk}(u, u_1, \ldots, u_m) = \frac{\tilde{Q}_j(u_j - 2\eta) \tilde{Q}_{j,k}(u_k - 2\eta)}{\tilde{Q}_j(u_j - \eta) \tilde{Q}_{j,k}(u_j - \eta)} \prod_{p<j} \Omega(u_j, u_p) \prod_{q<k, q\neq j} \Omega(u_k, u_q) \times \{ \beta_1(u, u_j, u_k) \tilde{E}_j(u_j) \tilde{E}_k(u_k) + \beta_2(u, u_k, u_j) \tilde{Q}_{j,k}(u_j + \eta) \Lambda_2(u_j) \tilde{E}_k(u_k) \\
+ \beta_2(u, u_j, u_k) \tilde{Q}_{j,k}(u_k + \eta) \Lambda_2(u_k) \tilde{E}_j(u_j) \}, \] (5.36)

and the functions \( \mathcal{G}_1(u, u_j), \mathcal{G}_2(u, u_j), \beta_1(u, u_j, u_k) \) and \( \beta_2(u, u_j, u_k) \) are given in the appendix.

The set of polynomial equations

\[ \tilde{E}_j(u_j) = 0 \quad \text{for} \quad j = 1, \ldots, m, \] (5.37)

are the Bethe ansatz equations for the double-row transfer matrix from Tarasov’s construction. We remark that the unwanted terms have been written explicitly in terms of the Bethe polynomials; in this way, it is clear that all unwanted terms disappear on-shell.

5.3. Relating the two open-chain Bethe ansätze

As in the periodic case, the open-chain Bethe equations from fusion (5.15) and from Tarasov’s construction (5.37) are equivalent. Indeed, since the factors in the Bethe polynomials \( \tilde{E}_j(u) \) (5.11) and \( \tilde{E}_j(u) \) (5.35) are related by

\[ \frac{\sinh(2u) \sinh \left( u + \frac{\eta}{2} - \xi^+ \right) \Delta_1(u)}{\sinh(2(u + \eta)) \sinh \left( u + \frac{3\eta}{2} + \xi^+ \right) \Delta_2(u)} = \left. \frac{\sinh(2u) \sinh(u - \xi^+) \delta_1(u)}{\sinh(2u + \eta) \sinh(u + \eta + \xi^+) \delta_2(u)} \right|_{u \rightarrow u + \frac{\eta}{2}}, \] (5.38)

the solutions of the two sets of Bethe equations are related by \( \frac{\eta}{2} \) shifts.

We claim that the off-shell Tarasov–Bethe vectors (5.24) and (5.29) are related to the off-shell fusion-Bethe vectors (5.6) and (5.7) by

\[ |\Psi_m(u_1, \ldots, u_m)\rangle = \tilde{s}(u_1, \ldots, u_m) |\Phi_m \left( u_1 + \frac{\eta}{2}, \ldots, u_m + \frac{\eta}{2} \right) \rangle, \] (5.39)

and

\[ \langle \Psi_m(u_1, \ldots, u_m) | = \langle \Phi_m \left( u_1 + \frac{\eta}{2}, \ldots, u_m + \frac{\eta}{2} \right) | \tilde{s}(u_1, \ldots, u_m) \cosh^m(\eta), \] (5.40)

where

\[ \tilde{s}(u_p, \ldots, u_m) = -2^{-m+p+1} \prod_{i=p}^m \sinh(2u_i) \delta_1 \left( u_i - \frac{\eta}{2} \right) \prod_{j=k=p, j<k} \sinh(u_j + u_k) \sinh(u_j - u_k - 2\eta) \]
\[ \times \sinh(u_j + u_k + \eta) \sinh(u_j - u_k - \eta). \] (5.41)

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A tale of two Bethe ansätze

We now prove (5.39) by induction; the proof of (5.40) is similar. The cases \( m = 1 \) and \( m = 2 \) can be proved by brute-force computation. We begin by using (5.24) to obtain
\[
|\Psi_{m+1}(u_1, \ldots, u_{m+1})\rangle = B_1(u_1) |\Psi_{m}(u_2, \ldots, u_{m+1})\rangle - B_2(u_1) \sum_{i=2}^{m+1} \Gamma_i^{(m+1)}(u_1, \ldots, u_{m+1}) |\Psi_{m-1}(u_2, \ldots, \hat{u}_i, \ldots, u_{m+1})\rangle. \tag{5.42}
\]

The induction hypothesis (5.39) implies
\[
|\Psi_{m+1}(u_1, \ldots, u_{m+1})\rangle = s(u_2, \ldots, u_{m+1}) B_1(u_1) |\Phi_m(\tilde{u}_2, \ldots, \tilde{u}_{m+1})\rangle - B_2(u_1) \sum_{i=2}^{m+1} \Gamma_i^{(m+1)}(u_1, \ldots, u_{m+1}) s(u_2, \ldots, \hat{u}_i, \ldots, u_{m+1}) |\Phi_{m-1}(\tilde{u}_2, \ldots, \hat{u}_i, \ldots, \tilde{u}_{m+1})\rangle,
\]
where we have introduced
\[
s(u_p, \ldots, \hat{u}_l, \ldots, u_m) = 2^{-m-p} \prod_{i=p,i\neq l}^{m} \frac{\sinh(2u_i)\delta_i(u_i - \frac{\eta}{2})}{\sinh(u_i + \frac{\eta}{2})} \prod_{j=k=p,j\neq l}^{m} \frac{\sinh(u_j + u_k) \sinh(u_j - u_k - 2\eta)}{\sinh(u_j + u_k + \eta) \sinh(u_j - u_k - \eta)}, \tag{5.43}
\]
and we have again employed the notation (4.48). We now use the relations
\[
B_1(u) = \frac{\sqrt{2} \cosh(u + \frac{\eta}{2})}{2} A\left(u + \frac{\eta}{2}\right) B\left(u - \frac{\eta}{2}\right) + \frac{\sqrt{2} \sinh(u + \frac{\eta}{2})}{4 \sinh(u + \frac{\eta}{2})} B\left(u + \frac{\eta}{2}\right) D\left(u - \frac{\eta}{2}\right),
\]
\[
B_2(u) = \frac{\sinh(2u)}{2 \sinh(u + \frac{\eta}{2})} B\left(u + \frac{\eta}{2}\right) B\left(u - \frac{\eta}{2}\right), \tag{5.45}
\]
which follow from (3.15) and the reflection algebra (3.13). In this way, we obtain
\[
|\Psi_{m+1}(u_1, \ldots, u_{m+1})\rangle = \frac{\sinh(2u_1)}{2 \sinh(u_1 + \frac{\eta}{2})} \left\{ \sqrt{2} s(u_2, \ldots, u_{m+1}) B(\tilde{u}_1) A(u_1 - \frac{\eta}{2}) |\Phi_m(\tilde{u}_2, \ldots, \tilde{u}_{m+1})\rangle - \sum_{i=2}^{m+1} \Gamma_i^{(m+1)}(u_1, \ldots, u_{m+1}) s(u_2, \ldots, \hat{u}_i, \ldots, u_{m+1}) B(\tilde{u}_1) B(u_1 - \frac{\eta}{2}) |\Phi_{m-1}(\tilde{u}_2, \ldots, \hat{u}_i, \ldots, \tilde{u}_{m+1})\rangle \right\}. \tag{5.47}
\]

Using the result
\[
A(u) |\Phi_m(u_1, \ldots, u_m)\rangle = \delta_1(u) \frac{Q(u - \eta)}{Q(u)} |\Phi_m(u_1, \ldots, u_m)\rangle + \sum_{j=1}^{m} \left[ \frac{\sinh(\eta) \sinh(2u_j)}{\sinh(u - u_j) \sinh(2u_j + \eta)} \delta_1(u_j) \frac{Q_j(u_j - \eta)}{Q_j(u_j)} - \frac{\sinh(\eta)}{\sinh(u + u_j + \eta)} \delta_2(u_j) \frac{Q_j(u_j + \eta)}{Q_j(u_j)} \right] B(u) |\Phi_{m-1}(u_1, \ldots, \hat{u}_j, \ldots, u_m)\rangle \tag{5.48}
\]
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that can be obtained using the reflection algebra (3.13), we find that
\[
|\Psi_{m+1}(u_1, \ldots, u_{m+1})\rangle = \frac{\sinh(2u_1)}{2 \sinh \left( u_1 + \frac{\eta}{2} \right)} \sqrt{2\tilde{s}}(u_2, \ldots, u_{m+1}) \delta_1(u_1 - \frac{\eta}{2})
\]
\[
\times \prod_{k=2}^{m+1} \frac{\sinh(u_1 - u_k - 2\eta) \sinh(u_1 + u_k)}{\sinh(u_1 - u_k - \eta) \sinh(u_1 + u_k + \eta)} |\Phi_{m+1}(\tilde{u}_1, \ldots, \tilde{u}_{m+1})\rangle
\]
\[
+ \frac{\sinh(2u_1)}{2 \sinh \left( u_1 + \frac{\eta}{2} \right)} \sum_{j=2}^{m+1} \left\{ \alpha_j(u_1, \ldots, u_{m+1}) - \Gamma_j^{(m+1)}(u_1, \ldots, u_{m+1}) \tilde{s}(u_2, \ldots, \tilde{u}_j, \ldots, u_{m+1}) \right\}
\]
\[\mathcal{B}(u_1 - \frac{\eta}{2}) |\Phi_{m}(\tilde{u}_1, \ldots, \tilde{u}_j, \ldots, \tilde{u}_{m+1})\rangle, \tag{5.49}\]
where
\[
\alpha_j(u_1, \ldots, u_{m+1}) = \sqrt{2\tilde{s}}(u_2, \ldots, u_{m+1}) \sinh(\eta)
\]
\[
\times \left[ \frac{\sinh(2\tilde{u}_j) \delta_1(\tilde{u}_j)}{\sinh(u_1 - \tilde{u}_j - \frac{\eta}{2}) \sinh(2\tilde{u}_j + \eta)} \prod_{k=2, k \neq j}^{m+1} \frac{\sinh(u_j - u_k - \eta) \sinh(u_j + u_k + \eta)}{\sinh(u_j - u_k) \sinh(u_j + u_k + 2\eta)} \right] \tag{5.50}\]
\[
\delta_2(\tilde{u}_j) - \frac{\sinh(u_j - u_k + \eta) \sinh(u_j + u_k + 3\eta)}{\sinh(u_j - u_k) \sinh(u_j + u_k + 2\eta)} \prod_{k=2, k \neq j}^{m+1} \frac{\sinh(u_j - u_k - \eta) \sinh(u_j + u_k + \eta)}{-\sinh(u_1 + \tilde{u}_j + \frac{\eta}{2})} \]
Finally, using the identities
\[
\tilde{s}(u_1, \ldots, u_{m+1}) = \frac{\sinh(2u_1)}{2 \sinh \left( u_1 + \frac{\eta}{2} \right)} \sqrt{2\tilde{s}}(u_2, \ldots, u_{m+1}) \delta_1(u_1 - \frac{\eta}{2})
\]
\[
\times \prod_{k=2}^{m+1} \frac{\sinh(u_1 - u_k - 2\eta) \sinh(u_1 + u_k)}{\sinh(u_1 - u_k - \eta) \sinh(u_1 + u_k + \eta)} \tag{5.51}\]
and
\[
\alpha_j(u_1, \ldots, u_{m+1}) = \Gamma_j^{(m+1)}(u_1, \ldots, u_{m+1}) \tilde{s}(u_2, \ldots, \tilde{u}_j, \ldots, u_{m+1}),
\]
we conclude that
\[
|\Psi_{m+1}(u_1, \ldots, u_{m+1})\rangle = \tilde{s}(u_1, \ldots, u_{m+1}) |\Phi_{m+1}(\tilde{u}_1, \ldots, \tilde{u}_{m+1})\rangle, \tag{5.52}\]
which completes the inductive proof of (5.39).

5.4. Scalar products for Tarasov–Bethe vectors

Combining the scalar product for the fusion-Bethe vectors (5.16) with the relations between Tarasov–Bethe vectors and fusion-Bethe vectors (5.39) and (5.40), it is now straightforward to compute the scalar product between the off-shell state $\langle \Psi_{m}(u_1, \ldots, u_{m}) \rangle$ (5.29) and the on-shell state $|\Psi_{m}(v_1, \ldots, v_{m})\rangle$ (5.24)
A tale of two Bethe ansätze

\[ \langle \Psi_m(u_1, \ldots, u_m) | \Psi_m(v_1, \ldots, v_m) \rangle = \cosh^m(\eta) \bar{s}(u_1, \ldots, u_m) \bar{s}(v_1, \ldots, v_m) \]

\[ \times \prod_{i=1}^m \frac{\delta_2(u_i + \frac{\eta}{2})}{\sinh\left(2\left(v_i + \frac{3\eta}{2}\right)\right) \sinh\left(u_i + \frac{\eta}{2} - \xi^+\right)} \frac{\prod_{j<i}^m \sinh(u_i + u_j + 3\eta)}{\sinh(u_i + u_j + \eta)} \]

\[ \times \frac{\det_m \left( \frac{\partial}{\partial u_i} \delta\left(v_j + \frac{\eta}{2}, u_1 + \frac{\eta}{2}, \ldots, u_m + \frac{\eta}{2}\right) \right)}{\det_m \left( \mathcal{F}(v_i, u_j) \right)} \]  

(5.53)

We can rewrite (5.53) like (5.18)

\[ \langle \Psi_m(u_1, \ldots, u_m) | \Psi_m(v_1, \ldots, v_m) \rangle \]

\[ = \prod_{j<k}^m \frac{\sinh(u_j + u_k) \sinh(u_j - u_k - 2\eta)}{\sinh(u_j + u_k + \eta) \sinh(u_j - u_k - \eta)} \frac{\sinh(v_j + v_k) \sinh(v_j - v_k - 2\eta)}{\sinh(v_j + v_k + \eta) \sinh(v_j - v_k - \eta)} \]

\[ \times \prod_{i=1}^m \frac{\sinh(2u_i + 2\eta) \Delta_2(u_i)}{\sinh(2v_i + 2\eta) \sinh(u_i + \frac{\eta}{2} - \xi^+)} \frac{\prod_{j<i}^m \sinh(u_i + u_j + 3\eta)}{\sinh(u_i + u_j + \eta)} \]

\[ \times \frac{\det_m \left( \sinh(2\eta) \mathcal{F}(u_i, v_j) \mathcal{F}(v_j, u_i) \xi(v_j) Q_i(v_j) \right)}{\det_m \left( \mathcal{F}(v_i, u_j) \right)} \]  

(5.54)

where

\[ \mathcal{F}(u, v) = \frac{1}{\sinh(u - v) \sinh(u + v + 2\eta)}, \]  

(5.55)

i.e. in terms of the Bethe ansatz (5.35) and the Baxter polynomial (5.26).

6. Conclusion

We have considered the Bethe vectors of the ZF model from two different perspectives, namely, the fusion technique and Tarasov’s construction. For the single-row transfer matrix, the relations between the two sets of Bethe vectors are given by (4.40) and (4.41). For the double-row transfer matrix, the associated formulas are given by (5.39) and (5.40). By means of these simple relations, we have computed the Slavnov scalar products for the Bethe vectors within Tarasov’s construction, see (4.58) for the single-row case and (5.54) for the double-row case. The Slavnov formulas have been written in terms of Bethe ansatz and Baxter polynomials, which seems to have more universal structure.

We hope that our results can be useful in the study of the Slavnov scalar products for other 19-vertex models, since they have similar commutation relations in the 3-dimensional auxiliary space representation. One first step would be to derive the formulas (4.58) and (5.54) for the ZF model without resorting to the
A tale of two Bethe ansätze

fusion-Bethe vectors. A particularly important 19-vertex model is the IK model. We hope that scalar product formulas exist for IK analogous to (4.58) or (5.54), at least for the quantum-group-invariant [31] or root of unity cases [32], even though IK (unlike ZF) cannot be obtained from some more elementary model by fusion.

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Appendix. Auxiliary formulas

The auxiliary functions entering the off-shell equations (5.32) and (5.33) are given by

\[ G_1(u, u_j) = \frac{\cosh \left( u + \frac{\eta}{2} \right) \sinh(2\eta) \sinh(2(u + \eta))}{4 \sinh(u - u_j) \sinh(u - \eta - u_j) \sinh(2(\eta + u_j)) \sinh(u + \eta + u_j) \sinh(u + 2\eta + u_j)} \times \left( \sinh\left( u - \frac{\eta}{2} - 2u_j - \xi^+ \right) + \sinh\left( u + \frac{\eta}{2} + \xi^+ \right) - \sinh\left( 3u + \frac{3\eta}{2} - \xi^+ \right) + \sinh\left( u + \frac{5\eta}{2} + \xi^+ \right) + \sinh\left( u + \frac{7\eta}{2} + 2u_j - \xi^+ \right) - \sinh\left( u + \frac{9\eta}{2} + \xi^+ \right) \right), \]

\[ G_2(u, u_j) = \frac{2 \cosh \left( u + \frac{\eta}{2} \right) \sinh(\eta) \sinh(2(u + \eta)) \sinh(u + \frac{3\eta}{2} + \xi^+)}{\sinh(u - \eta - u_j) \sinh(2(\eta + u_j)) \sinh(u + \eta + u_j)}, \]

\[ \beta_1(u, u_j, u_k) = \sinh^2(\eta) \cosh(\eta) \left( -\cosh\left( u - \frac{\eta}{2} \right) - \cosh\left( u + \frac{3\eta}{2} \right) + \cosh\left( 3u + \frac{9\eta}{2} \right) + \cosh\left( 5u + \frac{11\eta}{2} \right) \right) W_1(u, u_j, u_k) \]

\[ W_2(u, u_j, u_k) \]

where

\[ \frac{W_1(u, u_j, u_k)}{W_2(u, u_j, u_k)} \]

https://doi.org/10.1088/1742-5468/aab851

25
A tale of two Bethe ansätze

\[ W_1(u, u_j, u_k) = - \cosh (2u + u_j - u_k - 2\xi^+ - 3\eta) + \cosh (3u_j - u_k - 2\xi^+ + \eta) + \cosh (2u - u_j + u_k - 2\xi^+ - 3\eta) + \cosh (u_j + u_k - 2\xi^+ + \eta) + \cosh (4u + u_j + u_k - 2\xi^+ + 3\eta) - \cosh (u_j + u_k - 2\xi^+ + 3\eta) - \cosh (2u + 3u_j + u_k - 2\xi^+ + 3\eta) + \cosh (3u_k - u_j - 2\xi^+ + \eta) - \cosh (2u + u_j + u_k - 2\xi^+ + 3\eta) + \cosh (u_j + u_k - 2\xi^+ + 3\eta) + \cosh (3u_k + u_k - 2\xi^+ + 5\eta) + \cosh (3u_j - u_k + 2\xi^+ + 3\eta) + \cosh (3u_j - u_k + 2\xi^+ + 5\eta) - \cosh (2u - u_j + u_k + 2\xi^+ + 3\eta) - \cosh (2u - u_j + u_k + 2\xi^+ + 5\eta) - \cosh (u_j + u_k + 2\xi^+ + 3\eta) + \cosh (u_j + u_k + 2\xi^+ + 5\eta) - \cosh (-u_j - u_k - 2\xi^+ - \eta) + \cosh (-u_j - u_k + 2\xi^+ + 3\eta) + \cosh (-u_j + u_k + 2\xi^+ + \eta) - \cosh (4u - u_j - u_k + 2\eta) + \cosh (2u - 4\eta - u_j - u_k) + \cosh (2u - u_j - u_k + 2\eta) + \cosh (2u + u_j + u_k + 6\eta) + \cosh (2u + u_j + u_k + 4\eta) + \cosh (2u + u_j - u_k) - \cosh (2u + u_j + u_k - 2\xi^+ - 3\eta) - \cosh (2u - u_j - u_k) - 3 \cosh (u_j - u_k) - \cosh (3 (u_j - u_k)) + \cosh (3 (u_j + u_k + 6\eta)) + \cosh (2u - u_j - u_k) - \cosh (3u_k + u_k + 6\eta) + \cosh (2u - u_j - u_k) - 3 \cosh (u_j - u_k) - \cosh (3 (u_j - u_k)) + \cosh (2u - 3u_j + u_k) - \cosh (2u + u_j + u_k) \right) .
\]

\[ W_2(u, u_j, u_k) = 32 \sinh (u - u_j) \sinh (u - u_k) \sinh (u - u_j - \eta) \sinh (2 (u_j + \eta)) \times \sinh (u + u_j + \eta) \sinh (u + u_j + 2\eta) \sinh (u - u_k - \eta) \sinh (2 (u_k + \eta)) \times \sinh (u + u_k + \eta) \sinh (u + u_k + 2\eta) \sinh (u_j - u_k + \eta) \sinh (u_j - u_k + \eta) \sinh (u_k - u_j + \eta) \sinh (u_j + u_k + \eta) \times \sinh (u_j + u_k + 2\eta) \sinh \left( u_j - \xi^+ + \frac{\eta}{2} \right) \sinh \left( u_k - \xi^+ + \frac{\eta}{2} \right) ,
\]

\[ \beta_2(u, u_j, u_k) = \frac{2 \cosh \left( u + \frac{\eta}{2} \right) \sinh^2(\eta) \sinh(2\eta) \sinh(2(u + \eta)) \sinh(2u + 3\eta)}{\sinh(u - u_j) \sinh(u - u_j - \eta) \sinh(u + u_j + \eta) \sinh(u + u_j + 2\eta) \sinh(2(u_k + \eta))} \times \frac{\sinh(2(u_j + \eta)) \sinh(u_j - u_k) \sinh(u_j - u_k + \eta) \sinh(u_j + u_k + \eta) \sinh(u_j + u_k + 2\eta) \sinh(u_j + u_k + \frac{\eta}{2} - \xi^+)}{\sinh(u_k + \frac{\eta}{2} - \xi^+)} .
\]

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27
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