Stability of Fourier Solutions of Nonlinear Stochastic Heat Equations in 1D

Hazaimeh MH*
Department of Mathematics, Zayed University, Dubai, UAE

Abstract
The main focus of this article is studying the stability of solutions of nonlinear stochastic heat equation and give conclusions in two cases: stability in probability and almost sure exponential stability. The main tool is the study of related Lyapunov-type functionals. The analysis is carried out by a natural N-dimensional truncation in isometric Hilbert spaces and uniform estimation of moments with respect to N.

Nonlinear stochastic heat equation, additive space-time noise, Lyapunov functional, Fourier solution, finite-dimensional approximations, moments, stability.

Keywords: Nonlinear stochastic heat equation; Additive space-time noise; Lyapunov functional; Fourier solution; Finite-dimensional approximations; Moments; Stability

Introduction
In this article we study the stability of solutions of semi-linear stochastic heat equations

$$u_t = \sigma^2 \Delta u + A(u) - B(u) \, dW$$

with cubic nonlinearities $A(u)$ in one dimensions in terms of all systems parameters, i.e., with non-global Lipschitz continuous nonlinearities. Our study focuses on stability of analytic solution $u=u(x,t)$ under the geometric condition

$$\sigma^2 \pi^2 - a_i = \sigma^2 \pi^2 - a_i \pi^2 =: \gamma > 0,$$

where $0 \leq x \leq 1$ such that $D=[0,1]$.

Many authors have treated stochastic heat equations (e.g. [1,2]), semi-linear stochastic heat equations (e.g. [1-3]) or nonlinear stochastic evolution equations (e.g. [4,5]). Also, some authors study the stability of stochastic heat equations like Fournier and Printems [6] study the stability of the mild solution, Walsh reats the stochastic heat equations in one dimension. Chow [1] studies that the null solution of the stochastic heat equation is stable in probability by using the definition. Recall that:

$$L^2(D) = \{ f : D \to \mathbb{R}^d | f(x)^2 \, d\mu(x) < \infty \},$$

Where $\mu$ is the Lebesgue measure in one dimensions. The paper is organized as follows. Section 2 states that the strong Fourier solution of equation (1) is proved. We write the solution using the finite-dimensional truncated system verifies properties of finite-dimensional Lyapunov functional. Section 3 discusses the stability of the strong solution of equation (1) is stable in probability and almost sure exponential stability. Eventually, Section 4 summarizes the most important conclusions on the well-posedness and behaviour of the original infinite-dimensional system (1).

Truncated Fourier Series Solution and Finite-dimensional Lyapunov Functional
Consider the stochastic nonlinear heat equation with additive noise

$$u_t = \sigma^2 u + (a_i - a_i \| u \|^2) u + b_i \, dW$$

with the initial condition $u(x,0)=f(x)$ with $f \in L^2(D)$ (initial position) and $W(t) = \sum_{n=1}^{\infty} \alpha_n W_n(t) c_n(x)$ and $c_n = \frac{\pi^2}{L^2} \sin \frac{n \pi x}{L} \, dW_n$ driven by i.i.d. standard Wiener processes $W_n$ with $E[W_n(t)^2]=0$, $E[W_n(t)^2]=t$. The solution of equation (1) in terms of Fourier series is proved by Schurz [3] and given by

$$u(x,t) = \sum_{n=1}^{\infty} c_n(t) e_n(x).$$

Theorem 1
Assume that $\sum_{n=1}^{\infty} \alpha_n \pi^2 < \infty$, and $\forall u \in L^2(D)$ and $W(x,t) = \sum_{n=1}^{\infty} \alpha_n W_n(t) e_n(x)$, then for all $t \geq 0$, $x \in \Omega=(0,1)_l$, the Fourier-series solutions (2) have Fourier coefficients $c_n$ satisfying (a.s.)

$$\frac{d}{dt} c_n(t) = \left( -\sigma^2 \frac{\pi^2}{L^2} + a_i - a_i \sum_{n=1}^{\infty} (c_n)^2 \right) c_n + b \alpha_n \frac{dW_n}{dt}.$$ (3)

Proof. See Schurz [3].

We need to truncate the infinite series (2) for practical computations. So, we have to consider finite-dimensional truncations of the form

$$u_n(x,t) = \sum_{n=1}^{N} c_n e_n,$$

with Fourier coefficients $c_n$ satisfying the naturally truncated system of stochastic differential equations (SDEs)

$$\frac{d}{dt} c_n(t) = \left( -\sigma^2 \frac{\pi^2}{L^2} + a_i - a_i \sum_{n=1}^{\infty} (c_n)^2 \right) c_n + b \alpha_n \frac{dW_n}{dt}.$$ (5)

where $\lambda_n = \left( \frac{\pi^2}{L^2} \right)^{1/2}$. Assume that $\sigma^2 \pi^2 - a_i$. Define the Lyapunov functional $V_n$ as follows

*Corresponding author: Hazaimeh MH, Department of Mathematics, University College, Zayed University, P.O.Box 19282, Dubai, UAE. Tel: +971 4 402 1111; E-mail: haziem67@gmail.com

Received September 22, 2016; Accepted October 05, 2016; Published October 15, 2016

Citation: Hazaimeh MH (2016) Stability of Fourier Solutions of Nonlinear Stochastic Heat Equations in 1D. J Appl Computat Math 5: 323. doi: 10.4172/2168-9679.1000323

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Lemma 2
Consider the Lyapunov functional defined in equation (6), and let
\[ \sigma^2 \pi^2 - a = \frac{\sigma^2 \pi^2 - a^2}{l^2} = \gamma > 0. \]
Then if \( \forall \sigma \in L^2(\Omega) \):
\[ V\sigma(u) > \gamma \| u \|_{L^2(\Omega)}^2. \]
Proof: See [7].

Lemma 3
Assume that \( a \geq 0 \). Then, \( \forall N \in \mathbb{N} \), the functional \( V_N \) is
(a) nonnegative and positive semi-definite if \( \sigma^2 \pi^2 > aF \), or \( a_2 \geq 0 \).
(b) positive-definite if \( \sigma^2 \pi^2 > aF \), and
(c) satisfies the condition of radial unboundedness
\[ \lim_{N \to \infty} V_N(c) = +\infty, \text{if } \left[ \sigma^2 \pi^2 - a, a_2 \right] + a_2 > 0. \]
Proof: See [7].

Stability of Fourier Solutions
Recall equation (5) governed by
\[ \frac{d}{dt} c_N(t) = \left[ -\sigma^2 \pi^2 - a + a_N \sum_{i=1}^{n} (c_i)^2 \right] c_N + \sigma^2 \pi^2 \frac{dW}{dt} \]
\[ = \left[ -\sigma^2 \lambda_N + a - a_2 \| u_N \|^2 \right] c_N + \frac{dW}{dt}. \]
To simplify, let
\[ f(u_N) = -\sigma^2 \lambda_N + a - a_2 \| u_N \|^2 \]
and
\[ g(u_N) = \sigma^2 \pi^2 \frac{dW}{dt}. \]
Definition: The trivial solution of system (8) (in terms of norm \( \| u \|_{L^2(\Omega)} \)) is said to be stochastically stable or stable in probability, if for \( 0 < \varepsilon < 1 \) and \( r > 0 \), \( \exists \delta = \delta(\varepsilon, r) \) such that, \( \forall t \geq \delta \), we have
\[ \mathbb{P}( \| u(t) \|_{L^2(\Omega)} < r ) \geq 1 - \varepsilon. \]

Lemma 4
If \( V \) a positive-definite function \( V \in C^2(\mathbb{R}^+ \times [0, \infty), \mathbb{R}) \) such that \( LV(x,t) \leq 0 \) and \( \forall (x,t) \in \mathbb{R}^+ \times [0, \infty), \) then the trivial solution of the equation.
\[ dX(t) = f(x(t),t)dt + g(x(t),t)dw(t) \]
is stochastically stable.
Proof: See Arnold [8].

Theorem 5
Let \( V(u(t)) = \sigma^2 \| u \|_{L^2(\Omega)}^2 - a \| u \|_{L^2(\Omega)}^2 + \frac{a_2}{2} \| u \|_{L^2(\Omega)}^2 \)
If \( (1 - a_2 \sum_{i=1}^{n} c_i^2) \sum_{i=1}^{n} c_i > 0 \), then the trivial solution of equation (8) is stochastically stable i.e., stable in probability.
Proof: From Lemma 3, we know that \( V\sigma(u(t)) \) is positive-definite if \( \forall n \in \mathbb{N}, \sigma^2 \lambda_n - a_2 > 0 \). Define the linear operator \( L \) as in Schurz [3]
\[ L = \sum_{i=1}^{n} -\sigma^2 \lambda_i n^2 + a_1 n \sum_{i=1}^{n} c_i^2 \frac{\partial}{\partial c_i} + \frac{b_i}{2} \sum_{j=1}^{n} \frac{\partial^2}{\partial c_i \partial c_j} \]
The first and second partial derivative of \( V\sigma(u(t)) \) with respect to \( c_i \) are
\[ \frac{\partial V\sigma}{\partial c_i} = 2 \sum_{j=1}^{n} \left( \sigma^2 \lambda_i - a_1 \right) c_i + a_1 \left( \sum_{j=1}^{n} c_j^2 \right) c_i \]
and
\[ \frac{\partial^2 V\sigma}{\partial c_i^2} = 2 \sum_{j=1}^{n} \left( \sigma^2 \lambda_i - a_1 \right) c_i + 4a_1 \left( \sum_{j=1}^{n} c_j^2 \right) c_i + 2a_2 \sum_{j=1}^{n} c_j^2. \]

Then
\[ LV\sigma(c_i) = \left( \sigma^2 \lambda_i - a_1 \right) c_i + 4a_1 \left( \sum_{j=1}^{n} c_j^2 \right) c_i + 2a_2 \sum_{j=1}^{n} c_j^2. \]
But by our assumption that
\[ (1 - a_2 \sum_{i=1}^{n} c_i^2) \sum_{i=1}^{n} c_i > 0, \]
Then thus
\[ LV\sigma(c_i) \leq 0, \]
So by Lemma 4, applied to truncation of (8), the trivial solution of system (8) is stochastically stable.

Corollary 6
Let \( p \geq 2 \) and let \( V \) be as above. Imposing the same assumptions as in Theorem 5 with \( N \to \infty \), then we have \( \forall 0 \leq t \leq T, \)
\[ \mathbb{E}( \| u(t) \|_{L^2(\Omega)}^p ) \leq \frac{1}{\min(1, T)} \mathbb{E}( V^p(u(0)) ). \]
Proof: We know, from the definition of \( V(u) \), and Lemma 2 that
\[ \| u \|_{L^2(\Omega)}^p \leq V(u(t)) \]
It is easy to show that
\[ \mathbb{E}( \| u(t) \|_{L^2(\Omega)}^p ) \leq \frac{1}{\min(1, T)} \mathbb{E}( V^p(u(0)) ). \]

Corollary 7
For \( p \geq 2 \) and \( \forall 0 \leq t \leq T, \) with \( \sigma^2 \lambda_i - a_2 > 0, \) we have \( \forall 0 \leq t \leq T. \)
1) If \( a_2 \geq 0, \) then
\[ \mathbb{E}( \| u(t) \|_{L^2(\Omega)}^p ) \leq \frac{1}{\min(1, T)} \mathbb{E}( V^p(u(0)) ). \]
2) If \( a_2 > 0, \) then
Theorem 9

Let \( V(t) \) be a nonnegative integrable function such that \[ (1-a_c \sum c_n) \sum c_n > 0 \] then the norm of the trivial solution of N-dimensional system (8) is a.s. exponentially stable with sample top Lyapunov exponent

\[ \theta (u_0) \leq 0. \]

Proof. Return to the analysis of finite N-dimensional equation (5). Recall that

\[ V(u_0(t)) = \int_{\mathbb{R}} \phi E V \left( u_0(t) \right) dt \]

Proof. Note that we have \( (\sigma^2 \lambda - a_i) \| u_0(t) \|_{L^2}^2 \leq E V (u_0(t)) \). Since \( \lambda_n \) is increasing in \( n \),

\[ \int_{\mathbb{R}} (\sigma^2 \lambda - a_i) \| u_0(t) \|_{L^2}^2 \leq E V (u_0(t)). \]

So,

\[ \| u_0(t) \|_{L^2} \leq \frac{V(u_0(t))}{\sigma^2 \lambda - a_i}. \]

Pull over expectation, then

\[ \mathbb{E} \| u(t) \|_{L^2} \leq \frac{V(u(t))}{\sigma^2 \lambda - a_i}. \]

By using Corollary 6, we have

\[ \mathbb{E} \| u(t) \|_{L^2} \leq \frac{V(u(0))}{\sigma^2 \lambda - a_i}. \]

2) From the definition of \( V(u(t)) \), it is clear that \( \frac{d}{dt} \| u(t) \|_{L^2} \leq V(u(t)), \)

so

\[ \frac{d}{dt} \| u(t) \|_{L^2} \leq V(u(t)). \]

Now, take the expectation to both sides, and we get

\[ \frac{d}{dt} \mathbb{E} \| u(t) \|_{L^2} \leq \mathbb{E} V(u(0)), \]

i.e., \( \forall t \leq T \),

\[ \mathbb{E} \| u(t) \|_{L^2} \leq \mathbb{E} \| u(0) \|_{L^2} e^{\int_0^t \mathbb{E} V(u(s)) ds}. \]

Remark: The corollary 7 means that \( \forall t \geq 0 \),

\[ \mathbb{E} \| u(t) \|_{L^2} \leq \min \{ \mathbb{E} V(u(0)), \frac{1}{\sigma^2 \lambda - a_i} \} \mathbb{E} \| u(0) \|_{L^2} e^{\int_0^t \mathbb{E} V(u(s)) ds}. \]

Definition: The trivial solution of system (8) is said to be a.s. exponentially stable if

\[ \theta (u_0) := \limsup_{t \to \infty} \frac{1}{t} \log \mathbb{E} \| u(t) \|_{L^2} < 0 \quad (a.s.) \]

\[ \forall u(0) \in \mathcal{D}. \] The quantity of the left hand side of (12) is called the sample top Lyapunov exponent of \( u_0 \).

Lemma 8

Let \( v(t) \) be a nonnegative integrable function such that \[ (1-a_c \sum c_n) \sum c_n > 0 \] \( \forall t \leq C + A \int_0^t v(s) ds, \]

for some constants \( C, A \). Then \( C \geq 0 \) and

\[ v(t) \leq C \exp (A t), \quad 0 \leq t \leq T, \]

(13)

(14)

Theorem 9

Let \( V(u(t)) \) as in Theorem 5. If \( (1-a_c \sum c_n) \sum c_n > 0 \), then the norm of the v-component of the trivial solution of infinite-dimensional system (1) is a.s. exponentially stable with sample top Lyapunov exponent

\[ \theta (v_0) \leq -k < 0. \]

Proof. Return to the proof of previous Theorem 9 and take the limit \( N \to +\infty \) after the estimation process (16) in the sample Lyapunov exponent \( \theta (v_0) \).

Conclusion

By analyzing appropriate N-dimensional truncations of the original semi-linear heat equations (1), we can verify the asymptotic stability of random Fourier series solutions with strongly unique, Markovian, continuous time Fourier coefficients under the presence of cubic nonlinearities. For this purpose, we introduced and studied an appropriate Lyapunov. The analysis is basically relying on the fact that all estimations of moments of Lyapunov functional are made independent of dimensions \( N \) of their finite-dimensional truncations. Thus, the techniques of our proof are finite-dimensional in character, however...
the conclusions can be drawn to the original infinite-dimensional semilinear equation.

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