Fibonacci Optical Lattices for Tunable Quantum Quasicrystals

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We describe a new type of quasiperiodic optical lattice, created by a physical realization of the abstract cut-and-project construction underlying all quasicrystals. The resulting potential is a generalization of the Fibonacci tiling. Calculation of the energies and wavefunctions of ultracold atoms loaded into such a lattice demonstrate a multifractal energy spectrum, a singular continuous momentum-space structure, and the existence of controllable edge states. These results open the door to cold atom quantum simulation experiments in tunable or dynamic quasicrystalline potentials, including topological pumping of edge states and phasonic spectroscopy.

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Quasiperiodicity has a profound impact on electronic structure, playing a role in phenomena ranging from the quantum Hall effect to quasicrystalline ordering. However, the formation, stability, excitation, and electronic structure of quasiperiodically ordered systems remain incompletely understood. Open questions include the nature of electronic conductivity or diffusivity, the spectral statistics, the nature of strongly correlated magnetic states on a quasicrystalline lattice, topological properties of quasicrystals, and even the shape of the electronic wavefunctions [1–9].

The exquisite controllability of cold atoms makes them a natural choice for experimental investigation of the open questions regarding quasiperiodicity. Unique features of such experiments would include precisely variable quasiperiodic parameters, tunable interactions, bosonic or fermionic quantum statistics, and the ability to study dynamical phenomena (in modulated or quenched systems, e.g.). Numerous theoretical proposals have explored the rich physics of quasiperiodically trapped cold atoms [9–23]. However, with the exception of some early experiments on non-degenerate atomic gases in 2D quasiperiodic lattices [24, 25], the dominant application of quasiperiodic or incommensurate potentials in cold atomic physics thus far has been as a convenient proxy for disorder [26, 27, e.g.]. The realization of tunable quasicrystalline potentials for cold atoms would open up a broad range of exciting experiments, complementary to those possible with synthesis and characterization of solid-state or photonic quasicrystals.

In this paper, we describe and elucidate the properties of a “generalized Fibonacci” optical lattice which creates a dynamically tunable family of 1D quasicrystals. This lattice, a generalization of the well-known Fibonacci tiling, physically realizes the abstract cut-and-project construction which underlies all quasicrystals. Every quasiperiodic tiling can be defined as a projection of a cut through a lattice which is fully periodic (i.e., crystalline) but exists in a higher dimensional space [28, 29]. For example, the Fibonacci tiling is a projected 1D cut from a 2D square lattice, and the Penrose tiling can be constructed as a projected 2D cut through a 5D crystalline lattice. In these and all other quasicrystals, the hidden degrees of freedom in the higher-dimensional space give rise to unconventional Bragg diffraction, phasonic excitations, and topologically nontrivial phenomena [5, 30, 31]. Generalized Fibonacci optical lattices will provide a flexible platform for realization of tunable quantum quasicrystals, and should enable direct experimental investigation of questions inaccessible to experiments on static, non-tunable, non-interacting systems. Specific topics of interest include studies of edge states, adiabatic quantum pumping, multifractal energy spectra, phasonic spectroscopy, dynamical signatures of many-body localization, and transport in quasicrystals.
The generalized Fibonacci optical lattice is constructed as a direct real-space realization of the cut-and-project procedure, by intersecting an elongated optical trap at a tunable angle with a large-period square lattice, as diagrammed in Fig. 1. The resulting potential is the sum of two simple trap potentials: the large-period square optical lattice, with potential

\[ V_L(x, y) = -A_L \sin^2 \left( \frac{2\pi x}{\lambda_L} \right) - A_L \sin^2 \left( \frac{2\pi y}{\lambda_L} \right), \]

and the elongated cutting beam at an angle \( \alpha \) to the \( x \) axis, with potential

\[ V_C(x, y) = -A_C \exp \left[ -2 \left( \frac{-x \sin(\alpha) + y \cos(\alpha)}{\omega_0} \right)^2 \right]. \]

Here \( A_L \) is the depth of the square lattice, \( \lambda_L/2 = a \) is the lattice constant, \( A_C \) is the trap depth of the cutting beam, \( \omega_0 \) is the beam waist, and we have assumed a Rayleigh range long compared to the trapping region. The natural energy scale is the recoil energy \( E_R = \hbar^2/2m\lambda_L^2 \). The total potential is then

\[ U(x, y) = V_L(x, y) + V_C(x, y, \alpha). \]

This potential is shown in Fig. 1 for \( \tan(\alpha) = 2/(1 + \sqrt{5}) \), and \( A_C = 10A_L \). In order for the total potential to be a good approximation to a true cut-and-project potential, \( A_C/A_L \) must be sufficiently large that we can spectrally distinguish states along the trapping beam from transversely-extended states. To preserve the 1D character of the potential, the waist of the cutting beam should not be large compared to the lattice constant \( a \); however, the calculations presented below indicate that the potentials retain many of their interesting quasiperiodic properties even if this condition is violated. This trap construction can be generalized in a straightforward way to 2D quasiperiodic traps, via intersection of a light sheet with a 3D large-period lattice. This direct experimental realization of the simplest quasiperiodic lattices is also intrinsically tunable: variation of the intersection angle \( \alpha \) tunes the properties of the resulting potential, generating different members of this family of quasicrystals, and variation of the offset transverse to the cut beam axis drives phasonic degrees of freedom.

We now briefly discuss the practical optics required to realize such a potential. If the trapping beam is produced by focusing a gaussian beam of initial diameter \( D \) with a lens of focal length \( F \), the number of lattice sites in one Rayleigh range of the beam is given approximately by

\[ N_\parallel = 16 \frac{\lambda_C}{\pi} \frac{\lambda_L}{\lambda_C} \left( \frac{F}{D} \right)^2, \]

where \( \lambda_C \) is the wavelength of the trapping beam and \( \lambda_L \) is twice the lattice period. Even if these traps are produced by the same laser, \( \lambda_C/\lambda_L \) can be varied by using an angled-beam lattice configuration. The number of lattice sites spanning the width of the trapping beam is

\[ N_\perp = \frac{8}{\pi} \frac{\lambda_C}{\lambda_L} \left( \frac{F}{D} \right), \]

so the aspect ratio of the full trap is \( N_\parallel/N_\perp = 2F/D \). With typical values of \( F \) and \( D \), one can then realize a range of generalized Fibonacci traps, with widths ranging from less than a lattice constant to many lattice constants. The ends of such a trap can be defined for example by tightly-focused blue-detuned light sheets. The intersection angle \( \alpha \) is most easily tuned by rotating the lattice itself; for angled-beam lattices created using a diffractive optical element, this could be straightforwardly achieved with a single rotation stage. As with ordinary optical lattices, adiabatic loading of cold atoms into a Fibonacci-type lattice would be accomplished starting from the elongated optical trap by a slow turn-on of the lattice potential.

Using this trap geometry, one can construct a continuous family of quasiperiodic tilings of the line by placing the trapping beam at any irrational slope. In particular, if the angle of intersection \( \alpha \) satisfies the relationship \( \tan(\alpha) = 1/\tau \) where \( \tau \) is the golden mean \( (1 + \sqrt{5})/2 \), then the resulting potential will approximate the Fibonacci tiling. This one-dimensional structure tiles the line quasiperiodically, exhibits sharp diffraction peaks, and can also be generated algebraically using the deflation rule \( \tau \rightarrow \tau_1, 1 \rightarrow \tau \), which gives rise to the sequence \( (1, \tau, \tau_1, \tau_1\tau, \tau_1\tau_1\tau_1, \tau_1\tau_1\tau_1\tau_1\tau_1\tau_1\tau_1\tau_1\tau_1\tau_1\tau_1... \). As the bottom panel of Fig. 1 demonstrates, the energy minima of the Fibonacci optical lattice are spaced according to the Fibonacci tiling. Because of the deflation symmetry associated with the Fibonacci tiling, if the width of the cut-out strip is reduced, then the resulting one-dimensional projection will simply be an expanded and displaced version of the Fibonacci tiling. In the generalized Fibonacci optical lattice, as the width of the Gaussian cutting beam is increased, the potential no longer approximates a one-dimensional projection, but remains quasiperiodic. This general optical technique for construction of a family of quasiperiodic lattices and their rational approximants is the first main result of this work.

The second main result of this work is the calculation of the energy spectra and wavefunctions of non-interacting atoms trapped in this family of tunable quasicrystalline potentials. These calculations demonstrate the utility of generalized Fibonacci optical lattices as a tool for the investigation of quasiperiodic quantum phenomena. To determine the energy spectrum of the physically-realized trap, we solved the two-dimensional single-particle Schrödinger equation on a mesh with spacing much smaller than a lattice constant. This approach avoids simplifications inherent in the tight-binding approximation, and makes closer contact with experimen-
FIG. 2. Energies and wavefunctions in a tunable Fibonacci-type potential. a: A portion of the energy spectrum of the generalized Fibonacci optical lattice as a function of cut angle. Here $\alpha = 1$, $A_L = 5/\pi^2$, $A_C = 5A_L$, $w_0 = 1$. Color of points corresponds to center-of-mass of probability density, to enable identification of edge states. b: Spatial probability density at the indicated point (a typical bulk state). c: Spatial probability density at the indicated point (a typical edge state).

tally realizable traps. We did not use periodic boundary conditions, both for more direct comparison with real experiments and so as to accurately model the existence of edge states at the ends of the quasicrystal. Energy eigenvalues as a function of trapping beam angle are shown in Fig. 2. The calculated spectrum has a complex multifractal appearance [34]. Notable features include a hierarchy of minigaps which disperse as the angle is varied, a non-accidental resemblance to the Hofstadter butterfly, and the existence of isolated states in the gaps. We find that the qualitative structure of the energy spectrum remains the same if the waist of the Gaussian beam is increased to several times the size of the lattice constant. The resemblance to the Hofstadter butterfly is to be expected, given the recent demonstration that the generalized Fibonacci quasicrystal and the Harper model of high-magnetic-field 2D integer quantum Hall states are topologically equivalent [35, 36]. The intersection angle $\alpha$ of the Fibonacci quasicrystal plays a role analogous to that of the modulation period of the Harper lattice, or the effective magnetic field in the quantum Hall system.

The wavefunctions of atoms in generalized Fibonacci optical lattices also possess unique characteristics. As the cut angle $\alpha$ is varied, the Fourier transform of the spatial probability density of the ground state, plotted in Fig. 3, shows for irrational $\tan(\alpha)$ a rich singular continuous structure characteristic of quasiperiodic structures. This property is the 1D analogue of the forbidden Bragg diffraction patterns by which 3D quasicrystals were first discovered [37], and recalls the definition of a quasicrystal as a structure which produces a sharply peaked diffraction pattern but lacks translational symmetry. Rational $\tan(\alpha) = p/q$ produces a crystalline superlattice with a periodicity which depends upon $q$. The real-space structure of a typical squared wavefunction in a Fibonacci optical lattice is shown in Fig. 2b. In addition to extended bulk states, isolated states traversing the band gaps are visible in Fig. 2. Fig. 2 shows the squared wavefunction of a typical gap-traversing state, located inside the lowest energy band gap, and demonstrates that the wavefunction is localized towards one end of the lattice. The extent to which these edge states can be considered to be topologically protected is currently debated [3–7, 9], but in any case these states are interesting candidates for realizing topological pumping.

Topological pumping is possible because the wavefunctions and energy spectra of the generalized Fibonacci optical lattice depend in a non-trivial way on the offset of the trapping beam with respect to the lattice. This offset is a phasonic degree of freedom, something unique to quasicrystals. Just as phonon modes arise from discretely broken real-space translation symmetry, phason modes arise from broken translation symmetry in the higher-dimensional space from which the quasiperiodic lattice is projected [30, 32, 38]. In a Fibonacci-type optical lattice, this corresponds to symmetry under relative translation of the cut beam and the lattice in a direction transverse to the cut beam. A visualization of the effects of continuous adiabatic phasonic driving in the Fibonacci optical lattice is shown in Fig. 4. As the offset of the cutting beam is varied from 0 to 1 lattice constants, an edge state at the right-hand side of the sample with energy in the minigap decreases in energy, merges with the lower band, and later emerges as a left edge state. These cal-
FIG. 4. Edge state topological pumping by phason driving. **Top:** Energy states near the first minigap versus offset of cutting beam along a lattice vector, at the Fibonacci cut slope. As the offset between the lattice and the cut beam is varied, left and right edge states cross the gap. Coloring of points indicates center of mass, with the same mapping as Fig. 2. **Bottom:** Variation of spatial probability amplitude of an initial edge state as offset is adiabatically varied. Position is 0 at the left edge and 1 at the right edge.

Calculations show that adiabatic ramping of the offset can produce a long-range, quantized, oscillatory mass current in a generalized Fibonacci optical lattice. This could be detected, for example, by preferential loading of the edge states in a large-period lattice and direct imaging. Related effects have recently been observed in photonic waveguide lattices [5, 39], and recent theoretical work indicates that bulk Wannier states can be pumped in a similar way in superlattice potentials [40]. The cold atom context, uniquely, would enable realization of topological pumping in the presence of tunable interactions, and with variable adiabaticity. Such experiments would represent a controllable realization of Thouless pumping [41], and could provide a powerful tool for dynamical topological control of atomic wavefunctions.

The tunability of the generalized Fibonacci optical lattice also enables direct study of the effects of diabatic phason driving. Phasons have important but incompletely understood effects on thermal and electronic transport in real quasicrystals [22]. This is of interest not only for fundamental reasons, but also because of potential technological applications of quasicrystals’ anomalous electrical and thermal transport characteristics. The influence of phasons is not understood in large part because of the experimental difficulty of disentangling the effects of domain walls, crystalline impurities, and disorder from those due to phason modes. A unique aspect of the generalized Fibonacci optical lattice is that it enables direct oscillatory driving of phason modes. Measuring the response of the system to driving such modes at variable frequency would constitute a new kind of lattice modulation spectroscopy, in which the modulation occurs in the higher-dimensional space from which the quasiperiodic lattice is projected. This capability, impossible in other quasiperiodic systems, should allow unprecedentedly specific investigation of phason physics.

In conclusion, we have described a novel type of tunable quasiperiodic optical lattice, presented calculations of the properties of quantum gases in such a trap, and showed that this generalized Fibonacci optical lattice will enable experimental realization of topological pumping and phason spectroscopy. Creation of a fully tunable quantum quasicrystal would open the door to a large range of exciting experiments beyond those discussed here, including extensions of these techniques to higher-dimensional quasiperiodic lattices. The unique tools of atomic physics would enable new types of experiments: Feshbach tuning of the scattering length would allow exploration of the poorly understood role of interactions in quasicrystals, and time-varying potentials would enable dynamical experiments impossible in static lattices, such as phason spectroscopy. Artificial quasicrystals such as those we propose allow the exploration of arbitrary quasiperiodic geometries, unrestricted by the laws of chemistry. Experiments on quasiperiodic optical potentials may ultimately prove complementary to synthesis and characterization of solid and photonic quasicrystals: such experiments could open another conceptual angle of attack on the problem of designing and predicting the properties of these complex materials.

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