Last zero time or Maximum time of the winding number of Brownian motions

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Abstract

In this paper we consider the winding number, \( \theta(s) \), of planar Brownian motion and study asymptotic behavior of the process of the maximum time, the time when \( \theta(s) \) attains the maximum in the interval \( 0 \leq s \leq t \). We find the limit law of its logarithm with a suitable normalization factor and the upper growth rate of the maximum time process itself. We also show that the process of the last zero time of \( \theta(s) \) in \( [0, t] \) has the same law as the maximum time process.

1 Introduction and Main results

In this paper we seek for an analogue of the arcsine law of the linear Brownian motion for the argument of a complex Brownian motion \( \{W(t) = W_1(t) + iW_2(t) : t \geq 0\} \) started at \( W(0) = (1, 0) \). Skew-product representation tells us that there exist two independent linear Brownian motions \( \{B(t) : t \geq 0\} \) and \( \{\hat{B}(t) : t \geq 0\} \) such that

\[
W(t) = \exp(\hat{B}(H(t)) + iB(H(t))) \quad \text{for all } t \geq 0,
\]

where

\[
H(t) = \int_0^t \frac{ds}{|W(s)|^2} = \inf\{u \geq 0 : \int_0^u \exp(2\hat{B}(s))ds > t\},
\]

which entails that \( B \) is independent of \( |W| \) and hence of \( H \), while \( \log |W| \) is time change of \( \hat{B} \) (cf. e.g., [4], Theorem 7.26).

We let \( \theta(t) = B(H(t)) \) so that \( \theta(t) = \arg W(t) \), which we call the winding number. Without loss of generality we suppose \( \theta(0) = 0 \). The well-known result of Spitzer [7] states the convergence of \( \theta(t)/\log t \) in law:

\[
\lim_{t \to \infty} P\left(\frac{\theta(t)}{\log t} \leq a\right) = \frac{1}{\pi} \int_{-\infty}^a \frac{dx}{1 + x^2}.
\]

It is shown in [1] that for any increasing function \( f : (0, \infty) \to (0, \infty) \)

\[
\limsup_{t \to \infty} \frac{\theta(t)}{f(t)} = 0 \text{ or } \infty \quad \text{a.s.} \tag{2}
\]

according as the integral \( \int_0^\infty \frac{1}{f(t)} dt \) converges or diverges and

\[
\liminf_{t \to \infty} \frac{1}{f(t)} \sup\{\theta(s), 1 \leq s \leq t\} = 0 \text{ or } \infty \quad \text{a.s.}
\]

according as the integral \( \int_0^\infty \frac{f(t)}{t (\log t)^2} dt \) converges or diverges; moreover, it is shown that the square root of random time \( H(t) \) is subjected to the same growth law as of \( \theta \) in (2) and \( \liminf \) behavior of \( H(t) \) is also given. Another proof of (2) is given in [6].
Before advancing our result we recall the two arcsine laws whose analogues are studied in this paper. Let \( \{B(t) : t \geq 0\} \) be a standard linear Brownian motion started at zero and denote by \( Z_t \) the time when the maximum of \( B_s \) in the interval \( 0 \leq s \leq t \) is attained. Then the process \( Z_t \) and the process \( \sup\{s \in [0, t] : B(s) = 0\} \), the last zero of Brownian motion in the time interval \( [0, t] \), are subject to the same law, and according to Lévy’s arcsine law the scaled variable \( Z_t/t \) is subject to the arcsin law. (cf. e.g., [4] Theorem 5.26 and 5.28)

For stating the results of this paper we set
\[
V(a) = \frac{4}{\pi^2} \int_0^{\infty} \int_{0 \leq y \leq ax} \frac{dx}{1 + x^2} \frac{dy}{1 + y^2},
\]
and define a random variable \( M_t \in [0, t] \) by
\[
\theta(M_t) = \max_{s \in [0, t]} \theta(s),
\]
the time when \( \theta(s) \) attains the maximum in the interval \( 0 \leq s \leq t \), and a random variable \( L_t \) by
\[
L_t = \sup\{s \in [0, t] : \theta(s) = 0\},
\]
the last zero of \( \theta(s) \) in \([0, t]\). According to Theorem 2.11 of [4] a linear Brownian motion attains its maximum at a single point on each finite interval with probability one. In view of the representation \( \theta(t) = B(H(t)) \), it therefore follows that the maximiser \( M_t \) is uniquely determined for all \( t \) with probability one.

**Theorem 1.1.** (a) For every \( 0 < a < 1 \)
\[
\lim_{t \to \infty} P\left( \frac{\log M_t}{\log t} \leq a \right) = V\left( \frac{a}{1-a} \right).
\]
(b) It holds that
\[
\{L_t : t \geq 0\} =_d \{M_t : t \geq 0\}.
\]

**Theorem 1.2.** Let \( \alpha(t) \) be a positive function that is non-increasing, tends to zero as \( t \to \infty \) and satisfies
\[
2\alpha(t^e) \geq \alpha(t),
\]
and put
\[
I\{\alpha\} = \int_0^{\infty} \alpha(t) |\log \alpha(t)| dt.
\]

Then, with probability one
\[
\liminf_{t \to \infty} \frac{M_t}{t^{\alpha(t)}} = \infty \ or \ 0
\]
according as the integral \( I\{\alpha\} \) converges or diverges.

It may be worth noting that the distribution function \( V(a/(1-a)) \) \((0 \leq a \leq 1)\) is expressed as
\[
V\left( \frac{a}{1-a} \right) = \int_0^a \frac{1}{2u - 1} \log \frac{u}{1-u} du.
\]
Indeed,
\[
V'(c) = \int_0^{\infty} \frac{xdx}{(1+x^2)(1+c^2x^2)} = \frac{\log c}{c^2 - 1} \quad (c \neq 1),
\]
where
\[
\frac{d}{da} V\left( \frac{a}{1-a} \right) = \frac{1}{(1-a)^2} V'\left( \frac{a}{1-a} \right) \quad (a \neq \frac{1}{2}),
\]
and we find the density asserted above.
2 Proofs

2.1 Proof of Theorem [1,1]

Let \(\{N(t) : t \geq 0\}\) be the maximum process of a winding number \(\{\theta(t) : t \geq 0\}\), i.e., the process defined by

\[
N(t) = \max_{s \in [0,t]} \theta(s).
\]

**Lemma 2.1.** If \(a > 0\), then \(P(N(t) > a) = 2P(\theta(t) > a) = P(|\theta(t)| > a)\).

*Proof.* By reflection principle [4], (Theorem 2.21) it holds that for any \(t > 0\)

\[
\max_{0 \leq l \leq t} B(l) =_d |B(t)|,
\]

By Skew-product representation \(B(t)\) is independent of \(|W(t)|\), hence since \(B(l)\) is independent of \(H(t) = \int_0^t \frac{dm}{|W(m)|^2}\), and hence

\[
\max_{0 \leq l \leq t} B(H(l)) =_d |B(H(t))|,
\]

showing the assertion of the lemma. \(\square\)

**Lemma 2.2.** \(\{N(t) - \theta(t) : t \geq 0\} =_d \{|\theta(t)| : t \geq 0\}\).

*Proof.* According to Lévy’s representation of the reflecting Brownian motion [4], (Theorem 2.34) we have

\[
\{\max_{0 \leq l \leq t} B(l) - B(t) : t \geq 0\} =_d \{|B(t)| : t \geq 0\}.
\]

Hence as in the preceding proof,

\[
\{\max_{0 \leq l \leq t} B(H(l)) - B(H(t)) : t \geq 0\} =_d \{|B(H(t))| : t \geq 0\},
\]

as desired. \(\square\)

*Proof of Theorem [1,1].* Lemma 2.2 shows that the process \(\{M_s : s \geq 0\}\) has the same law as \(\{L_s : s \geq 0\}\), being nothing but the last zero of the process \(\{N(t) - \theta(t) : 0 < t \leq s\}\) for any \(s\). So it remains to prove part (b). Fix \(a \in (0,1)\). Set \(T_c = \inf\{l \geq 0 : |W(l)| = c\}\), for which we sometimes write \(T(c)\) for typographical reason. We first prove the upper bound. By (1) it holds that

\[
P(M_t < t^a) = P\left(\max_{0 \leq u \leq t^a} B(H(u)) > \max_{t^a \leq u \leq t} B(H(u))\right)
\]

\[
= P\left(\max_{0 \leq u \leq t^a} B(H(u)) - B(H(t^a)) > \max_{t^a \leq u \leq t} B(H(u)) - B(H(t^a))\right)
\]

\[
= P\left(\max_{0 \leq u \leq t^a} B(H(u)) - B(H(t^a)) > \max_{t^a \leq u \leq t} \tilde{B}(H(u)) - \tilde{B}(H(t^a))\right),
\]

where \(\tilde{B}\) is a linear Brownian motion started at zero which is independent of \(W\). Corresponding to (1) we can write \(\tilde{W}(0) = (1,0)\), \(\arg\tilde{W}(l) = \tilde{B}(\tilde{H}(l))\), \(\tilde{H}(l) = \int_0^l \frac{dm}{|W(m)|^2}\) with \(\tilde{W}\) independent of \(W\), and put \(\tilde{T}_c = \inf\{l \geq 0 : |\tilde{W}(l)| = c\}\). By Lemma 2.1 and Lemma 2.2 we have \(\max_{0 \leq u \leq t^a} B(H(u)) - B(H(t^a)) =_d \max_{0 \leq u \leq t^a} B(H(u))\), and therefore

\[
P\left(\max_{0 \leq u \leq t^a} B(H(u)) - B(H(t^a)) > \max_{t^a \leq u \leq t} \tilde{B}(H(u)) - \tilde{B}(H(t^a))\right)
\]

\[
= P\left(\max_{0 \leq u \leq t^a} B(H(u)) > \max_{t^a \leq u \leq t} \tilde{B}(H(u)) - \tilde{B}(H(t^a))\right).
\]
Given $\epsilon > 0$, it holds that for all sufficiently large $t$

$$P(t^a \leq T_{t^{\frac{a+1}{2}}, T_{t^{\frac{1-\epsilon}{2}}} \leq t}) \geq 1 - \epsilon.$$ 

Therefore, we get

$$P\left(\max_{0 \leq u \leq t^a} B(H(u)) > \max_{t^a \leq u \leq t} \tilde{B}(H(u)) - \tilde{B}(H(t^a))\right)$$

$$\leq P\left(\max_{0 \leq u \leq T(t^{\frac{a+1}{2}})} B(H(u)) > \max_{T(t^{\frac{1-\epsilon}{2}}) \leq u \leq T(t^{\frac{1+\epsilon}{2}})} \tilde{B}(H(u)) - \tilde{B}(H(T_{t^{\frac{a+1}{2}}})) + \epsilon. \right)$$

(7)

Also, strong Markov property tells us

$$\int_{T_t^{\frac{a+1}{2}}}^{T_t^{\frac{1-\epsilon}{2}}} \frac{dm}{|W(m)|^2} = d$$

and $H(T_{t^{\frac{a+1}{2}}}) - H(T_{t^{\frac{1-\epsilon}{2}}})$ is independent of $H(T_{t^{\frac{a+1}{2}}})$.

So if we set for $a, b < \infty$

$$Q(a, b) = P\left(\max_{0 \leq u \leq T(a)} B(H(u)) > \max_{0 \leq u \leq T(b)} \tilde{B}(H(u))\right),$$

it holds that

$$P\left(\max_{0 \leq u \leq T(t^{\frac{a+1}{2}})} B(H(u)) > \max_{T(t^{\frac{1-\epsilon}{2}}) \leq u \leq T(t^{\frac{1+\epsilon}{2}})} \tilde{B}(H(T_{t^{\frac{a+1}{2}}}))\right) = Q(t^{\frac{a+1}{2}}, t^{\frac{1-\epsilon}{2}}).$$

(8)

Note that by Skew-product representation $B(t)$ (resp. $\tilde{B}(t)$) is independent of $H(T_{t^{\frac{a+1}{2}}})$ (resp. $\tilde{H}(T_{t^{\frac{a+1}{2}}})$). Then, if $\tilde{\theta}(l) = \tilde{B}(\tilde{H}(l))$, by reflection principle we get

$$Q(t^{\frac{a+1}{2}}, t^{\frac{1-\epsilon}{2}}) = P(|B(H(T_{t^{\frac{a+1}{2}}}))| > |\tilde{B}(H(T_{t^{\frac{1-\epsilon}{2}}}))|)$$

$$= P(|\theta(T_{t^{\frac{a+1}{2}}})| > |\tilde{\theta}(T_{t^{\frac{1-\epsilon}{2}}}))|).$$

(9)

Moreover, since $\theta(T_r)$ is subject to the Cauchy distribution with parameter $|\log r|$ (cf. e.g., 5, Section 5, Exercise 2.16), we get

$$Q(t^{\frac{a+1}{2}}, t^{\frac{1-\epsilon}{2}}) = P(|\theta(T_{t^{\frac{a+1}{2}}})| > |\tilde{\theta}(T_{t^{\frac{1-\epsilon}{2}}}))|) = V\left(\frac{a + \epsilon}{1 - a - 2\epsilon}\right).$$

(10)

Therefore, since $\epsilon$ is arbitrary, this gives the desired upper bound.

Next we prove the lower bound. For all sufficiently large $t$

$$P(T_{t^{\frac{a-\epsilon}{2}}} \leq t^a, t \leq T_{t^{\frac{1+\epsilon}{2}}}) \geq 1 - \epsilon,$$

(11)

and repeating the argument in (7) and (8) we get

$$P\left(\max_{0 \leq u \leq t^a} B(H(u)) > \max_{t^a \leq u \leq t} \tilde{B}(H(u)) - \tilde{B}(H(t^a))\right)
\geq Q(t^{\frac{a+\epsilon}{2}}, t^{\frac{1-a+2\epsilon}{2}}) - \epsilon.$$

Therefore, repeating the arguments in (5), (6), (9) and (10), we get

$$P(M_t < t^a) = P\left(\max_{0 \leq u \leq t^a} B(H(u)) > \max_{t^a \leq u \leq t} \tilde{B}(H(u)) - \tilde{B}(H(t^a))\right)
\geq Q(t^{\frac{a+\epsilon}{2}}, t^{\frac{1-a+2\epsilon}{2}}) - \epsilon
= V\left(\frac{a - \epsilon}{1 - a + 2\epsilon}\right) - \epsilon,$$

yielding the lower bound. ☐
2.2 Proof of Theorem \[1.2\]

Proof of Theorem \[1.2\]. We first prove \(\liminf_{t \to \infty} M_t/t^{\alpha(t)} = \infty\) if \(I\{\alpha\} < \infty\). We may replace \(\alpha(t)\) by \(\alpha(t) \vee (\log \log t)^{-2}\). Because if we set

\[
\bar{\alpha}(t) = \alpha(t) \{\alpha(t) > (\log \log t)^{-2}\} + (\log \log t)^{-2} \{\alpha(t) \leq (\log \log t)^{-2}\},
\]

\(I\{\bar{\alpha}\} < \infty\). By standard large deviation result (cf. e.g., [3], (11) and (12)) for any \(q < \infty\) there exist \(0 < c_1, c_2 < \infty\) such that

\[
P(qt^{4\alpha(t)} < T(t^{1/2-\alpha(t)}) - c_1 \exp(-c_2 t^{\alpha(t)})) \leq c_2 \exp(-c_2 t^{\alpha(t)}).
\]

(12)

Therefore, by the same arguments as made for [3], (11), (13) and (14) we infer that for any \(q < \infty\)

\[
P(M_t < qt^{4\alpha(t)}) = P(\max_{0 \leq u \leq qt^{4\alpha(t)}} B(H(u)) - B(qt^{4\alpha(t)}) > \max_{qt^{4\alpha(t)} \leq u \leq t} B(H(u)) - B(qt^{4\alpha(t)}))
\]

\[
\leq Q(t^{4\alpha(t)}, t^{1/2-5\alpha(t)}) + c_1 \exp(-c_2 t^{\alpha(t)}).
\]

(12)

We set \(t_n = \exp(e^n)\). Then noting that \(V(\alpha(n)) \asymp \alpha(n) \log \alpha(n)\), we deduce from (12) that for some \(C < \infty\)

\[
P(M_t < t^{4\alpha(t_n)}) \leq C \alpha(t_n) \log \alpha(t_n) + c_1 \exp(-c_2 t_n^{\alpha(t_n)}).
\]

The sum of the right-hand sides over \(n\) is finite since \(\sum_{n=1}^\infty \alpha(t_n) \log \alpha(t_n) < \infty\) if \(I\{\alpha\} < \infty\), and \(\alpha(t) \geq (\log \log t)^{-2}\) according to our assumption. Thus by Borel-Cantelli lemma for any \(q < \infty\), with probability one

\[
\frac{M_t}{t^{4\alpha(t_n)}} > q \quad \text{almost all } n.
\]

(13)

Note that if we pick \(t\) such that \(t_n < t \leq t_{n+1}\), then \(t_n^{4\alpha(t_n)} > t^{\alpha(t)}\) and from (13) it follows that \(M_t > M_{t_n} > qt^{\alpha(t)}\) for all sufficiently large \(n\). Hence

\[
\liminf_{t \to \infty} \frac{M_t}{t^{\alpha(t)}} > q \quad a.s.
\]

Since \(q < \infty\) is arbitrary, this concludes the proof.

Next we prove \(\liminf_{t \to \infty} M_t/t^{\alpha(t)} = 0\) assuming that \(I\{\alpha\} = \infty\). For any \(a < b < \infty\), we set

\[
\theta^*[a, b] = \max\{\theta(t) : T_a \leq t \leq T_b\},
\]

and define \(\overline{M}(a, b)\) via

\[
\theta(\overline{M}(a, b)) = \theta^*[a, b] \quad \text{and } T_a \leq \overline{M}(a, b) \leq T_b.
\]

Recall we set \(t_n = \exp(e^n)\). Denote for \(q > 0\) the event

\[
\overline{M}(qt^{\alpha(t_n)}, t_n) \subset T(qt^{2\alpha(t_n)})
\]

by \(A_n\). Bringing in the set \(D = \{n \in \mathbb{N} : \alpha(t_n) > \frac{1}{(\log \log t_n)^2}\}\), we shall prove \(\sum_{n=1, n \in D} P(A_n) = \infty\) and

\[
\liminf_{n \in D, n \to \infty} \frac{\sum_{j=1, j \in D} \sum_{k=1, k \in D} P(A_j \cap A_k)}{(\sum_{j=1, j \in D} P(A_j))^2} \leq 1,
\]

(14)
which together imply \( P(\limsup_{n \in D, n \to \infty} A_n) = 1 \) according to the Borel-Cantelli lemma (cf. [8], p.319). First we prove \( \sum_{n=1}^{\infty} P(A_n) = \infty \). Note that it holds that for \( 0 < a < b < c \)

\[
P(\theta^*[a,b] > \theta^*[b,c]) = P(\theta^*[1, \frac{b}{a}] > \theta^*[\frac{b}{a}, \frac{c}{a}]).
\]

Thus

\[
P(\theta^*[qt^{\alpha(t)}, qt^{2\alpha(t)}] > \theta^*[qt^{2\alpha(t)}, t]) = P(\theta^*[1, t^{\alpha(t)}] > \theta^*[t^{\alpha(t)}, \frac{1}{q} t^{1-\alpha(t)})].
\]

Therefore, we get by the same argument as employed for (5), (6), (7), (8), (9) and (10)

\[
P(M[\alpha(t), t] < T(qt^{2\alpha(t)})) =\]

\[P(\theta^*[1, t^{\alpha(t)}] > \theta^*[t^{\alpha(t)}, \frac{1}{q} t^{1-\alpha(t)}]) =\]

\[P(\max_{u \leq t(t^{\alpha(t)})} B(H(u)) - B(T(t^{\alpha(t)}))) > \max_{T(t^{\alpha(t)}) \leq u \leq T^{\frac{1}{q} t^{1-\alpha(t)}}} \tilde{B}(H(u)) - \tilde{B}(T(t^{\alpha(t)}))) =\]

\[Q(t^{\alpha(t)}, \frac{1}{q} t^{1-2\alpha(t)}) = V(\frac{\alpha(t)}{1 - 2\alpha(t)} - (\log t \log q)^{-1}). \] (15)

Moreover, using \( V(\alpha(n)) \asymp \alpha(n)|\log \alpha(n)| \) again, we get for some \( C > 0 \)

\[
P(A_n) \geq C \alpha(n)|\log \alpha(n)|.
\]

It holds that \( \sum_{n \in D} \alpha(n)|\log \alpha(n)| = \infty \) if \( I(\alpha) = \infty \), since \( \sum_{n \notin D} \alpha(n)|\log \alpha(n)| < \infty \). So we get \( \sum_{n \in D} P(A_n) = \infty \).

Next we prove (14). We only need to consider \( \sum_{j=1,j \in D} \sum_{k \in A \cap A_k} P(A_j \cap A_k) \). First we consider \( \sum_{j=1,j \in D} \sum_{k \in A \cap A_k} P(A_j \cap A_k) \). Note that for \( a < b \leq c < d < \infty \)

\[
M[a,b] - T_a \text{ is independent of } M[c,d] - T_c. \] (16)

Then, since \( qt^{\alpha(t_k)} < t_k \leq qt^{\alpha(t_j)} < t_j \) when \( k \) is satisfied with \( qt^{\alpha(t_j)} \geq t_k \), it holds that

\[
P(A_j \cap A_k) = P(A_j) P(A_k). \] (17)

So next we consider the case \( qt^{\alpha(t_j)} < t_k \). We denote the event \( M[qt^{\alpha(t_k)}, qt^{2\alpha(t_k)}] < T(qt^{2\alpha(t_k)}) \) by \( A'_{k,j} \). Note that when \( k \) is satisfied with \( qt^{\alpha(t_j)} < t_k \), we have \( A_k \subset A'_{k,j} \) and by (16) \( P(A_j \cap A'_{k,j}) = P(A_j) P(A'_{k,j}) \). Then, since by the same argument for (15) \( P(A'_{k,j}) = V(\frac{e^{k\alpha(t_k)}}{e^{k\alpha(t_j)} - e^{k\alpha(t_k)}}) \), we get

\[
P(A_j \cap A_k) \leq P(A_j \cap A'_{k,j}) = P(A_j) P(A'_{k,j}) = P(A_j) V(\frac{e^{k\alpha(t_k)}}{e^{k\alpha(t_j)} - e^{k\alpha(t_k)}}). \] (18)

On the other hand, since \( \alpha(t_k) \leq 2\alpha(t_k+1) \) owing to the assumption (14), we get

\[
\sum_{j-|\log \alpha(t)| < k < j, k \in D} P(A'_{k,j}) \leq \sum_{j-|\log \alpha(t)| < k < j, k \in D} V(\frac{e^{k\alpha(t_k)}}{e^{k\alpha(t_j)} - e^{k\alpha(t_k)}}) \leq C' \sum_{k=1}^{\infty} \frac{e^{-k}}{(2^k - 2^2)} \leq C'. \] (19)
So by (18) and (19) we get
\[ \sum_{j=1,j\in D}^{n} \sum_{j-|\log \alpha(t_j)|<k<j,k\in D} P(A_j \cap A_k) \leq C \sum_{j=1,j\in D}^{n} P(A_j). \]
Combined with (17) this shows
\[ \sum_{j=1,j\in D}^{n} P(A_j \cap A_k) \leq \sum_{j=1,j\in D}^{n} P(A_j)P(A_k) + C' \sum_{j=1,j\in D}^{n} P(A_j), \]
completing the proof of (14). Therefore, we can conclude that with probability one
\[ M[qt_n^{\alpha(t_n)}, t_n] < T(qt_n^{2\alpha(t_n)}) \text{ infinitely often } n \in D. \] (20)
On the other hand, by standard large deviation result (cf. e.g., [3], (11) and (12)) there exist
\[ 0 < c_3, c_4 < \infty \text{ such that } P(T(qt_n^{2\alpha(t)}) \leq qt_n^{5\alpha(t)}, t_i^{k} \leq T_i) \geq 1 - c_3 \exp(-c_4 t_i^{\alpha(t)}). \]
Moreover, \( \sum_{n\in D} c_3 \exp(-c_4 t_n^{\alpha(t_n)}) < \infty. \) Then, by Borel-Cantelli lemma it holds that with probability one
\[ T(qt_n^{2\alpha(t_n)}) \leq qt_n^{5\alpha(t_n)}, M[t_n^{\frac{1}{5}}, n \in D] \quad \text{almost all } n \in D. \] (21)
So by (20) and (21) it holds that
\[ \liminf_{t\to\infty} \frac{M_t}{qt_n^{2\alpha(t_n)}} \leq \liminf_{n\in D,n\to\infty} \frac{M_n}{qt_n^{2\alpha(t_n)}} \leq \liminf_{n\in D,n\to\infty} \frac{M[t_n^{\frac{1}{5}}, n]}{qt_n^{5\alpha(t_n)}} \leq \liminf_{n\in D,n\to\infty} \frac{M[qt_n^{\alpha(t_n)}, t_n]}{T(qt_n^{2\alpha(t_n)})} < 1 \quad a.s.. \]
Since \( q > 0 \) is arbitrary and we can replace \( \alpha(t) \) by \( \frac{\alpha(t)}{20} \), this completes the proof. \( \square \)

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