We present two novel approaches to establish the local density of states as an order parameter field for the Anderson transition problem. We first demonstrate for 2D quantum Hall systems the validity of conformal scaling relations which are characteristic of order parameter fields. Second we show the equivalence between the critical statistics of eigenvectors of the Hamiltonian and of the transfer matrix, respectively. Based on this equivalence we obtain the order parameter exponent $\alpha_0 \approx 3.4$ for 3D quantum Hall systems.

The absence of diffusion in coherent disordered electron systems is known as Anderson localization [1]. In dimensions $d > 2$ a disorder induced localization-delocalization (LD) transition occurs quite generally at some value of the Fermi energy [2]. In $d = 2$ all states are localized unless a certain amount of spin-orbit scattering [3] or a strong magnetic field is present [4,5]. The LD transitions of independent (spinless) 2D electrons subject to a strong perpendicular magnetic field are located at the Landau energies. These transitions are generally believed to be responsible for the integer quantum Hall effect [5].

In general, LD transitions are characterized by the critical exponent $\nu$ of the localization length [6] and by the multifractal $f(\alpha)$ spectrum of the local amplitudes of critical eigenstates [8]. The $f(\alpha)$ spectrum describes the statistics and scaling behavior of the local density of states (LDOS). Although the average density of states does not reflect the LD transition, the typical value (i.e. geometrically averaged) of the LDOS does: it vanishes with an exponent $\beta_{\text{typ}} = (\alpha_0 - d)\nu$ on approaching the LD transition point, where $\alpha_0 > d$ is the maximum position of $f(\alpha)$. It is thus tempting to interpret the LDOS as an order parameter field of the LD problem [8] (see also [10,11]). Our aim is to support this interpretation by establishing two characteristic features of order parameter fields for the LDOS: First, scaling exponents of the order parameter field are related to the critical exponents of the corresponding spatial correlation functions. These correlation functions show conformal invariance. Second, the scaling exponents are universal in the sense of (one-parameter) scaling theory, i.e. any local quantity containing contributions from the relevant scaling field shows asymptotically the same spectrum of scaling exponents.

In this article we demonstrate that the $f(\alpha)$ spectrum of critical eigenstates is related to correlation functions in different geometries by conformal invariance. We derive the conformal mapping relations appropriate for a multifractal situation and check them numerically in 2D quantum Hall systems (QHS). Furthermore, we have calculated numerically $f(\alpha)$ for the local components of transfer matrix eigenvectors in 2D QHS and show that it coincides with $f(\alpha)$ of the Hamiltonian eigenstates. Thus, these two local quantities share the same spectrum of scaling exponents although their microscopic construction is quite different. Our findings support the identification of the LDOS as an order parameter field for the LD transition. From $f(\alpha)$ calculations for the transfer matrix eigenvectors in a 3D QHS we obtain the characteristic order parameter exponent $\alpha_0 \approx 3.4$.

The multifractal analysis of the statistics of critical eigenstates $\psi$ (see e.g. [8]) usually starts from considering the box-probability, $P(L_b) := \int_{b} \mid \psi \mid^2$, of some box with linear size $L_b$. The box-probability is normalized to the total volume $L^d$, $P(L) = 1$. At the LD transition the corresponding distribution function $\pi(P; L_b/L)$ gives rise to power law scaling for the moments

$$\langle P^q(L_b) \rangle_L \propto (L_b/L)^{d+\tau(q)}$$

where $\tau(q)$ is non-linear. The distribution function can be described in terms of a single-humped, positive, function $f(\alpha)$:

$$\pi(P; L_b/L) dP \propto (L_b/L)^{-f(\alpha)} d\alpha,$$

where $\alpha := \ln P/\ln(L_b/L)$. The multifractal spectrum $f(\alpha)$ is the Legendre transform of $\tau(q)$. Thus, the statistics of critical eigenstates is encoded in $f(\alpha)$ or equivalently in $\tau(q)$. The statistics of critical eigenstates transforms to the statistics of the LDOS $\rho(r) = \Delta^{-1} \mid \psi(r) \mid^2$ at criticality. Here $\Delta$ denotes the mean level spacing at the corresponding energy. It is a non-critical quantity, $\Delta \propto L^{-d}$. Consequently, the typical value, $\rho_{\text{typ}} := \exp(\ln \rho) \propto L^{d-\alpha_0}$ where $\alpha_0$ is the maximum position of $f(\alpha)$. Scaling $L$ with the localization length shows that $\rho_{\text{typ}}$ vanishes on approaching the transition point with characteristic exponent $\beta_{\text{typ}} = (\alpha_0 - d)\nu$. This observation motivates to consider the LDOS as an order parameter field of the LD transition [8].

To make contact between $f(\alpha)$ and correlation exponents we first make the general scaling ansatz for the $q$-dependent spatial correlation in square-like geometries
\[ \langle \rho^q(r) \rho^q(r') \rangle_L \propto a(r/L) \cdot r^{-\tilde{\varepsilon}(q)}, r = |r - r'|. \tag{3} \]

Here \( \tilde{\varepsilon}(q) \) fulfills the scaling relation

\[ \tilde{\varepsilon}(q) = 2(1 - q)d + 2\tau(q). \tag{4} \]

In ordinary critical phenomena the coefficient \( a(r/L) \) will approach a constant for \( r \ll L \to \infty \). Then, standard conformal mapping arguments in 2D [3] say that the corresponding correlator in a long strip of width \( L_T \) with periodic boundary conditions in transverse direction decays like

\[ \exp[-2u/\xi^{[q]}(L_T)], \tag{5} \]

where \( u \) is the distance in longitudinal direction and \( \xi^{[q]}(L_T) \) define generalized \( (q\)-dependent) localization lengths in quasi-1D. The localization lengths are then related to \( \tau(q) \) in the following way [14] (see also [14])

\[ \xi^{[q]}(L_T)/L_T = 2[\pi \tilde{\varepsilon}(q)]^{-1}. \tag{6} \]

Equation (6) establishes a relation between generalized localization lengths and \( f(\alpha) \).

In the following we will show that this relation remains valid, although the coefficient \( a(r/L) \) in Eq. (3) cannot be treated as a constant. In a multifractal situation the coefficient \( a(r/L) \) in Eq. (3) remains sensitive to the actual system size \( L \) even for \( L \to \infty \) (at the transition point). It has been shown that \( a(r/L) \propto (r/L)^y_2(q) \) with \( y_2(q) = d + \tau(2q) - 2qd \). Thus, standard conformal mapping arguments do not apply. This fact has been stressed recently by Zirnauer [13]. We recall that the main idea behind conformal mapping arguments is an extension of a homogeneity law for correlation functions with respect to rescaling. Such a law exists also in our case, since any rescaling of all length scales in the correlator of Eq. (3) by the same scaling factor \( s \) leads to

\[ \langle \rho^q(sr) \rho^q(sr') \rangle_{sL} = s^{-\tilde{\varepsilon}(q)} \langle \rho^q(r) \rho^q(r') \rangle_L. \tag{7} \]

We make now the plausible assumption that this law can be extended to conformal mappings which are local scale transformations: conformal mappings transform the correlator of a geometry \( \Omega \) to the corresponding correlator in geometry \( \tilde{\Omega} \). For large but finite 2D systems we are led to

\[ \frac{\langle \rho^q(w(z_1)) \rho^q(w(z_2)) \rangle_{\tilde{\Omega}}}{\langle \rho^q(z_1) \rho^q(z_2) \rangle_{\Omega}} = |w'(z_1)|^{-\tilde{\varepsilon}(q)} |w'(z_2)|^{-\tilde{\varepsilon}(q)}, \tag{8} \]

where \( w(z) \) is any holomorphic function of complex coordinate \( z \) and \( w'(z) \) denotes the derivative.

By choosing \( w(z) = (L_T/2\pi) \ln z \) which maps the plane onto a strip straightforward calculations (cf. [13]) show that the correlator in the strip is given by expression (3) with an additional prefactor, \( \exp[(2\pi y_2(q)/L_T)(u - L_s)] \), where \( L_s \) is the strip length. Note, that the important relation Eq. (3) between \( \xi^{[q]} \) and \( \tilde{\varepsilon}(q) \) remains valid.

To calculate the generalized localization lengths \( \xi^{[q]} \) in strip-like (quasi-1D) systems \( (L_s \) being the length and \( L_T \ll L_s \) being the width in the remaining \( d - 1 \) directions) one can start from a Hamiltonian \( H \). The Green’s function \( G(r, r'; E) := \langle r' | (E - H + i0^+)^{-1} | r \rangle \) can be calculated for \( r, r' \) situated at opposite ends of a strip by a recursive method [10]. The resulting Green’s function \( G(L_s; E) \) yields (cf. expression (5))

\[ \xi^{[q]}(L_T) = -(2L_s)^{-1} \ln(\langle G(L_s)^{2q} \rangle) = \lambda^{[q]}. \tag{9} \]

By Eq. (3), \( \lambda^{[q]} \) is proportional to \( \tilde{\varepsilon}(q) \) which becomes negative for \( q > 1 \). This corresponds to those rare events where the Green’s function increases with \( L_s \). To detect a reasonable amount of these rare events one has to restrict the numerical calculation to strip-lengths \( L_s \) of a few typical localization lengths.

Alternatively, one can construct a transfer matrix \( T \) which relates either the values of the wavefunction at opposite ends of the strip or the corresponding scattering states [17, 18]. For a transverse width \( L_T = Ml \), \( l \) being a microscopic scale of the problem, the dimension of the transfer matrix \( T \) is proportional to \( M \). Lyapunov exponents, \( \mu_i \), can be defined from the eigenvalues of \( TT^\dagger \) corresponding to eigenvectors \( u_i \),

\[ TT^\dagger u_i = e^{-\mu_i} u_i. \tag{10} \]

The smallest positive Lyapunov exponent, denoted by \( \mu \), determines the generalized inverse localization lengths \( l^{[q]} \),

\[ \lambda^{[q]} = -(2L_s)^{-1} \ln(e^{-\mu q}). \tag{11} \]

We have calculated \( \lambda^{[q]} \) for two models of QHS. The first model is the random Landau model (RLM) describing the Hamiltonian of 2D disordered electrons restricted to the Hilbert space of one Landau band. The disorder is caused by a white-noise potential. The characteristic microscopic scale is the magnetic length \( l_B \). A detailed description of this model is given e.g. in Ref. [19]. We applied the recursive Green’s function method (RLM/GF) and a transfer matrix method (RLM/TM). The second model is the network model (NWM) of Chalker and Coddington [18] (with a characteristic microscopic scale \( l \)) which is formulated in terms of a transfer matrix.

In Fig. 1, we show the corresponding \( \lambda^{[q]}(L_T) \) curves and its first derivative for the NWM in the regime \( |q| \leq 1 \). The conformal mapping relation, Eq. (4), allows for the calculation of \( f(\alpha) \) and, especially, of the most interesting scaling exponent \( \alpha_0 \). For comparison with related work [20] we also give the values for \( \tau(2) \). We find within the RLM by the GF method \( \alpha_0 = 2.28 \pm 0.03 \) for strip widths of 100\( l_B \) and by the TM method \( \alpha_0 = 2.3 \pm 0.03 \) and \( \tau(2) = 1.66 \pm 0.03 \) for strip widths of 50\( l_B \). For the NWM we obtain \( \alpha_0 = 2.27 \pm 0.01 \) and \( \tau(2) = 1.64 \pm 0.03 \) for strip widths of 64\( l \). The results agree for the different models and agree with the values reported in
the literature. We emphasize that our results confirm the conformal mapping relation, Eq. (3).

In three dimensions only a few conformal mappings exist. Especially, for three dimensional strips with periodic boundary conditions we have no appropriate conformal mapping at our disposal. We saw that conformal invariance is sufficient to derive relations between \( f(\alpha) \) and \( \lambda^{[q]} \), however it may not be necessary. Therefore, we found it worthwhile to study if a similar relation as Eq. (2) could be valid for 3D strip-like systems. We thus make the ansatz

\[
L_T \lambda^{[q]}(L_T) = C(\tau(q) - 3(q - 1)),
\]

where \( C \) is some unknown constant. Since \( \tau(q) \) has to vanish at \( q = 1 \) and is smaller than \( d(q - 1) \) for \( q > 1 \) this ansatz raises the question if the generalized localization lengths \( \lambda^{[q]} \) vanish for \( q = 1 \) and become negative for \( q > 1 \), for the 3D problem too. To check for such behavior we took a 3D version of the network model, introduced recently by Chalker and Dohmen. The 3D network model consists of coupled layers of 2D networks and the corresponding transfer matrix is described in detail in Ref. [21]. Our numerical findings at LD transition points of the model are shown in Fig. 2. They indicate that \( \lambda^{[q]} \) is compatible with our ansatz. Note that, so far, we have established a relation between \( f(\alpha) \) and \( \lambda^{[q]} \) only at criticality in 2D. Our 3D results suggest that a similar relation might be valid in 3D at criticality. To us, it seems rather unlikely that similar relations hold true far off criticality.

Our second approach to establish the LDOS as an order parameter for the LD transition is based on the assumption that any local quantity in a given disordered system can be viewed as being decomposed in terms of local scaling fields in the sense of critical phenomena theory (e.g. [22]). If one-parameter scaling holds true then, on approaching criticality, only the relevant scaling field has to be considered giving rise to a universal distribution of local amplitudes. Any contribution of irrelevant scaling fields would lead to deviations between the corresponding \( f(\alpha) \) spectra, but eventually die out in the thermodynamic limit.

The components of the eigenvectors \( u_i \) corresponding to the transfer matrix, Eq. (10), in the network model [18] have local meaning (the corresponding scattering channels are defined in real space) and can be subject to a conventional multifractal analysis. From this one obtains a one-dimensional multifractal spectrum. Multiplication with \( d \) yields the associated \( d \)-dimensional spectrum.

When considering the components of transfer matrix eigenvectors which belong to large Lyapunov exponents (small localization lengths) deviations from a universal scaling behavior can clearly be observed in the 2D network model. For the smaller Lyapunov exponents, however, we obtain almost identical results (see Fig. 3). Concentrating on the eigenvector \( u_i \) corresponding to the smallest Lyapunov exponent, we find \( \alpha_0 = 2.3 \pm 0.02 \) and \( \tau(2) = 1.59 \pm 0.05 \). For the 2D network model we have thus evidence that eigenvector components of the transfer matrix give rise to the same \( f(\alpha) \) spectrum as the eigenstates of the Hamiltonian.

We also calculated \( f(\alpha) \) of transfer matrix eigenvectors for the 3D network model at LD transition points. It is worth mentioning that the 3D network model is anisotropic and for most values of the parameter that fixes the coupling between adjacent layers the typical quasi-1d localization lengths are smaller than the system width \( L_T \) in one of the transverse directions, but the basic condition for the applicability of the boxing method is that the localization length has to be larger than \( L_T \). Restricting to those values of the coupling where this condition was fulfilled we find from the statistics of the eigenvector components corresponding to the smallest Lyapunov exponent: \( \alpha_0 = 3.4 \pm 0.2 \) and \( \tau(2) = 2.3 \pm 0.2 \) [23]. We mention that these values of \( \alpha_0 \) and \( \tau(2) \) fulfill the inequality \( 2/\tau(2) < \nu < (\alpha_0 - d)^{-1} \) proposed by Janssen relying on one-parameter scaling arguments (\( \nu \approx 1.35 \)).

In summary, we have proposed a conformal scaling relation, Eq. (3), between the \( f(\alpha) \) spectrum of the local density of states at criticality and the corresponding generalized Lyapunov exponents in strip-like geometries. We have checked its validity for 2D quantum Hall systems by numerical calculations. We have presented numerical results which support the expectation that similar relations are valid in 3D systems at criticality. Furthermore, our numerical calculations show that eigenvector components of the transfer matrix give rise to the same \( f(\alpha) \) as that of the Hamiltonian eigenstates. We like to stress the following aspects of our results: First, they show that the local density of states has several features in common with order parameter fields in ordinary critical phenomena. Second, the very existence of relations between the critical statistics of eigenstates (or LDOS) of the Hamiltonian, the generalized Lyapunov exponents, and the critical statistics of eigenvectors of the transfer matrix is new and far from obvious. Third, such relations provide alternative ways of computing the \( f(\alpha) \) spectrum of critical eigenstates. As an application of this we calculated the critical order parameter exponent for 3D quantum Hall systems.

We thank John Chalker for pointing out to us that a conformal mapping relation between Lyapunov exponents and multifractal spectra should exist and Bodo Huckestein for useful discussions. This work has been supported by the Sonderforschungsbereich 341 and by MINERVA.
[1] P. W. Anderson, Phys. Rev. 109, 1492 (1958).
[2] B. Kramer, A. MacKinnon, Rep. Prog. Phys. 56, 1496 (1993).
[3] U. Fastenrath, Physica A 189, 27 (1992); L. Schweitzer, J. Phys. Cond. Matt. 7, L281 (1995)
[4] B. Huckestein, Rev. Mod. Phys. 67, 357 (1995).
[5] M. Janssen, O. Viehweger, U. Fastenrath and J. Hajdu, Introduction to the Theory of the Integer Quantum Hall Effect, VCH, Weinheim (1994).
[6] R.E. Prange, S. Girvin (Eds.), The Quantum Hall Effect, Springer, New York (1990).
[7] E. Abrahams et al., Phys. Rev. Lett. 42, 673 (1979).
[8] M. Janssen, Int. J. Mod. Phys.B 8, 943 (1994).
[9] H. Grassbach, M. Schreiber, Phys. Rev. B 51, 663 (1995).
[10] F. Wegner, Z. Phys. B 36, 209 (1980).
[11] A.D. Mirlin, Y.V. Fyodorov, Phys. Rev. Lett. 72, 526 (1991).
[12] K. Pracz, M. Janssen, P. Freche, unpublished numerical calculations.
[13] J.L. Cardy, J. Phys.A 17, L385 (1984).
[14] A.W.W. Ludwig, Nucl. Phys.B 330, 639 (1990).
[15] M.R. Zirnbauer, Ann. Physik. 3, 513 (1994).
[16] A. MacKinnon, B. Kramer, Phys. Rev. Lett. 47, 1546 (1981).
[17] J.L. Pichard, G. Sarma, J. Phys.C 14, L127 (1981).
[18] J.T. Chalker, P.D. Coddington, J. Phys.C 21, 2665 (1988).
[19] A. Crisanti, G. Paladin, A. Vulpiani, Products of Random Matrices, Springer Series in Solid-State Sciences 104, Springer, Berlin (1993).
[20] W. Pook, M. Janssen, Z. Phys.B 82, 295 (1991); B. Huckestein et al., Surface Sciences 263, 125 (1992); R. Klesse, M. Metzler, Europhys. Lett. 32, 229 (1995)
[21] J.T. Chalker, A. Dohmen, Phys. Rev. Lett. 75, 4496 (1995).
[22] C. Domb, M.S. Green (Eds.), Phase Transitions and Critical Phenomena Vol. 6, Academic, London (1976).
[23] The present uncertainty of $\alpha_0 - 3$ in 3D does not allow for a reliable determination of the constant $C$ in Eq. (12).

Figure 1: Generalized inverse localization lengths $\lambda^{[q]}_L L_T$ (A) and first derivative with respect to $q$ (B) for different strip widths $L_T = Ml$ in the 2D network model ($l$ is the corresponding microscopic length scale).

Figure 2: Generalized inverse localization lengths $\lambda^{[q]}_L L_T$ (A) and first derivative with respect to $q$ (B) for different cross-sections $(L_T)^2 = (Ml)^2$ in the 3D network model ($l$ is the corresponding microscopic length scale).

Figure 3: The exponent $\alpha_0$ describing the statistics of eigenvector components corresponding to the $i$-th Lyapunov exponent in the 2D network model.
