K-theory of quiver varieties, q-Fock space and nonsymmetric Macdonald polynomials

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Abstract

We have two constructions of the level-(0, 1) irreducible representation of the quantum toroidal algebra of type $A$. One is due to Nakajima and Varagnolo-Vasserot. They constructed the representation on the direct sum of the equivariant $K$-groups of the quiver varieties of type $A$. The other is due to Saito-Takemura-Uglov and Varagnolo-Vasserot. They constructed the representation on the $q$-deformed Fock space introduced by Kashiwara-Miwa-Stern.

In this paper we give an explicit isomorphism between these two constructions. For this purpose we construct simultaneous eigenvectors on the $q$-Fock space using nonsymmetric Macdonald polynomials. Then the isomorphism is given by corresponding these vectors to the torus fixed points on the quiver varieties.

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1 Introduction

Geometry of quiver varieties, introduced by Nakajima, involves rich mathematical structures.

One of the most notable result is Nakajima’s construction of representations of quantum loop algebras ([13]): the direct sum of torus equivariant K-groups of the quiver varieties is endowed with a structure of a representation of the quantum affinization $U_R(Lg)$ of the corresponding Kac-Moody algebra $g$. The resulting representation is what we call an $l$-highest weight representation, that is to say, a ”highest weight representation” with respect to the triangular decomposition of the quantum loop algebra.

Let us concentrate our attention on quiver varieties of affine type. They appear in gauge theory as framed moduli spaces of instantons on ALE spaces, which originally motivated Nakajima to introduce quiver varieties. They also have interesting connections with some areas in mathematics such as the theory of McKay correspondence and the representation theory of symplectic reflection algebras (see [7] and [6] for example). In this point of view, more careful study about the actions of the quantum toroidal algebras, quantum affinizations of the affine Kac-Moody algebras, on the equivariant K-groups seems to be important.

Schur-Weyl duality is an equivalence between certain categories of representations of $gl_l$ and of $S_n$. Varagnolo-Vasserot show that there exists an analogous duality between the quantum toroidal algebra of type $A$ and a certain double affinization of the Hecke algebra of type $A$, called the toroidal Hecke algebra ([19]). The toroidal Hecke algebra has a remarkable representation called Dunkl-Cherednik representation. Applying Schur-Weyl duality for Dunkl-Cherednik representation, Saito-Takemura-Uglov and Varagnolo-Vasserot construct the representation of the quantum toroidal algebra ([15], [20]). The underlying space is so called the q-Fock space ([8]).

There are much fewer things known about representations of quantum toroidal algebras than of quantum affine algebras. Now, at least, we have two constructions of the representation of the quantum toroidal algebra of type $A$. In this paper we give an explicit isomorphism between these two constructions. We hope it will be helpful for further analyses of the representation, such as study of canonical bases of the representations.

We can describe the representation on the equivariant K-groups in a combinatorial manner using the localization theorem ([22]). In particular, the torus fixed points correspond to simultaneous eigenvectors for the action of a certain subalgebra of the quantum toroidal algebra. Our strategy is to construct simultaneous eigenvectors on the q-Fock space. The isomorphism will be given by corresponding these vectors to the torus fixed points. For the construction of simultaneous eigenvectors, nonsymmetric Macdonald polynomials plays a crucial role, where nonsymmetric Macdonald polynomials are simultaneous eigenvectors for Dunkl-Cherednik operators ([5], [11], [14]).
Takemura-Uglov described the irreducible decomposition of the q-Fock space as the representation of a certain subalgebra of the quantum toroidal algebra, which is isomorphic to the quantum affine algebra \([17]\). They also showed that each irreducible components are isomorphic to tensor products of fundamental representations. For this purpose they introduced specific vectors of the q-Fock space using nonsymmetric Macdonald polynomials (see Remark 6.2.2).

In this paper we introduce new vectors. We also use nonsymmetric Macdonald polynomials, but an additional operation is required (see 6.2.1). They are simultaneous eigenvectors and the main subject of this paper. These simultaneous eigenvectors allow us a combinatorial description of the representation on the q-Fock space and we can see this coincides with the combinatorial description of the representation on the equivariant K-groups.

In \([21]\) and \([16]\), the action of the Hall algebra of the cyclic quiver on the q-Fock space is studied. The Hall algebra of the cyclic quiver is realized using perverse sheaves on the space of representations of the quiver by Lusztig \([9]\). Nakajima’s construction of quiver varieties and representations on their K-groups are, philosophically, parallel to Lusztig’s construction. We could expect this observation gives conceptual interpretation of the isomorphism constructed in this paper. In particular, this isomorphism would help us to study of canonical bases of the K-groups of quiver varieties (see \([10]\) and \([23]\), for quiver varieties of finite type).

In \([33]–[44]\) we are mainly occupied with review of, and arrangement for our use of, the results of \([22]\), \([19]\), \([15]\) and \([20]\). In \([45]\) we construct the simultaneous eigenvectors and in \([47]\) we exhibit the isomorphism.

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2 Preliminaries

2.1 Quantum toroidal algebra

2.1.1

In this paper we usually take \(\mathcal{R} = \mathbb{C}(s^{1/2}, t^{1/2})\) as the coefficient field. We set

\[p = t^l, \quad q = s^{1/2}t^{1/2}, \quad r = s^{-1/2}t^{1/2}.\]

2.1.2

Let us define the quantum toroidal algebra \(U_{\mathcal{R}}^l(sl_{l,tor})\) \((l > 2)\). This is an \(\mathcal{R}\)-algebra generated by \(e_{i,n}, f_{i,n}, K_i^\pm\) and \(h_{i,m}^\pm\) \((i \in I, n \in \mathbb{Z}, m \in \mathbb{Z}_{>0})\). The relations are expressed using the formal series

\[e_i(z) = \sum_{n \in \mathbb{Z}} e_{i,n} z^{-n}, \quad f_i(z) = \sum_{n \in \mathbb{Z}} f_{i,n} z^{-n},\]
as follows:

\[ K_i^+ (z) = K_i^+ \exp \left( \pm (q - q^{-1}) \sum_{m > 0} h_{i,m}^z z^m \right) \]

where \( \delta(Z) = \sum_{n \in \mathbb{Z}} Z^n \).

**Remark.** The quantum toroidal algebra in [22] is "twisted" in their words, which may or may not be isomorphic to ours. See Remark 3.2.2 for the relation between these two algebras.

### 2.1.3

The **horizontal subalgebra** \( U_\mathsc{r}^{(2)} (\mathfrak{sl}_1) \) is the subalgebra of \( U_\mathsc{r}' (\mathfrak{sl}_{l,tor}) \) generated by \( e_{i,0}, f_{i,0} \) and \( K_i^\pm \) (\( i \in I \)). This is isomorphic to \( U_\mathsc{r}' (\mathfrak{sl}_l) \otimes \mathcal{R} \).

The **vertical subalgebra** \( U_\mathsc{r}^{(1)} (\mathfrak{sl}_1) \) is the subalgebra of \( U_\mathsc{r}' (\mathfrak{sl}_{l,tor}) \) generated by \( e_{i,n}, f_{i,n}, K_i^\pm \), and \( h_{i,m}^\pm \) (\( i \neq 0, \ n \in \mathbb{Z}, \ m \in \mathbb{Z}_{\geq 0} \)). Define \( \widehat{e}_{i,n}, \widehat{f}_{i,n}, \widehat{K}_i^\pm \) and \( \widehat{h}_{i,m}^\pm \) by

\[ \widehat{e}_{i,n}(z) = \sum_{m \in \mathbb{Z}} \widehat{e}_{i,n} z^{-m} = e_i(z), \]

\[ \widehat{f}_{i,n}(z) = \sum_{m \in \mathbb{Z}} \widehat{f}_{i,n} z^{-m} = f_i(z), \]

\[ \widehat{K}_i^\pm(z) = K_i^\pm \exp \left( \pm (q - q^{-1}) \sum_{m > 0} h_{i,m}^z z^m \right) = K_i^\pm (r^{-l+i} z). \]
They satisfy the relations in Drinfeld new realization of $U_q'(\mathfrak{sl}_t)$, and so $U_q'^{(1)}(\mathfrak{sl}_t)$ is also isomorphic to $U_q'(\mathfrak{sl}_t) \otimes R$.

## 2.2 Notations for Young diagrams

### 2.2.1

Let $\Pi$ denote the set of all Young diagrams. We identify a Young diagram with a subset of $(\mathbb{Z}_{\geq 0})^2$. A **node** is an element of $(\mathbb{Z}_{\geq 0})^2$.

The **content** of a node $(x, y)$ is the number $x - y$. A node is called an $i$-node if its content equals to $i$ modulo $l$. For $\lambda \in \Pi$ let $d_i(\lambda)$ denote the number of $i$-nodes in $\lambda$ and set $d(\lambda) = (d_i(\lambda))_{i=0, \ldots, l-1} \in \mathbb{Z}^l$. We define the order $\succ$ on the set of nodes according to their contents.

For $\lambda \in \Pi$ a node $(x, y)$ is called **addable** if $(x, y) \notin \lambda$ and $(x - 1, y), (x, y - 1) \in \lambda$. A node $(x, y)$ is called **removable** if $(x, y) \in \lambda$ and $(x + 1, y), (x, y + 1) \notin \lambda$. Let $A_{\lambda, i}$ (resp. $R_{\lambda, i}$) denote the set of all addable (removable) $i$-nodes of $\lambda$.

A **hook** is a pair $((x_h, y_h), (x_t, y_t))$ such that $(x_h, y_h - 1), (x_t, y_t) \in \lambda$ and $(x_h, y_h), (x_t + 1, y_t) \notin \lambda$. The **hook length** of a hook $((x_h, y_h), (x_t, y_t))$ is the number $-x_h + y_h + x_t - y_t$. A hook is called an **$l$-hook** if its length is a multiple of $l$.

### 2.2.2

A **Maya diagram** with charge $c$ is an infinite decreasing sequence of integers $k = (k_1, k_2, \ldots)$ such that $k_a = -a + c$ for sufficiently large $a$. A Maya diagram with charge $c$ can be identified with a Young diagram

$$\lambda = \prod_{a \in \mathbb{Z}_{\geq 0}, 1 \leq b \leq k_a + a - c} (a - 1, b - 1).$$

Let $\Pi_c$ denote the set of all Maya diagrams with charge $c$. Then $\Pi$ and $\Pi_c$ are bijective.

### 2.2.3

We sometimes identify a Maya diagram $k = (k_1, k_2, \ldots)$ with the subset $\{k_1, k_2, \ldots\}$ of $\mathbb{Z}$.

If $k_a - 1 \notin k$ ($a \in \mathbb{Z}_{>0}$), then a node $(a - 1, k_a + a - c - 1)$ is a removable node. Its content equals to $c - k_a$. If $k_a + 1 \notin k$ ($a \in \mathbb{Z}_{>0}$), then a node $(a - 1, k_a + a - c)$ is an addable node. Its content equals to $c - k_a - 1$.

Note that $\{(a, b) \mid a \in k, b \notin k, a > b\}$ is a finite set. Such a pair $(a, b)$ corresponds to a hook in term of Young diagram. Its hook length is $a - b$.

## 3 K-theory of quiver varieties

In this section we review the representation of $U_{\hat{R}}^*(\mathfrak{sl}_{l, \text{tor}})$ on the equivariant K-groups of the quiver varieties of type $\hat{A}$.

A quiver variety, introduced by Nakajima, is a certain moduli space of representations of a quiver. He also introduced a certain subvariety of the product of two quiver varieties called the Hecke correspondence. Using the Hecke
correspondence we can construct an action of the quantum affinization of the Kac-Moody algebras on the torus equivariant K-groups of the quiver varieties [13].

By the localization theorem, localized equivariant K-groups have bases indexed by fixed points. The fixed points of the quiver varieties of type $A$ are indexed by Young diagrams. The action of the quantum toroidal algebra can be written in terms of Young diagrams ([22], see Theorem 3.2.3).

Nakajima’s definition of quiver varieties involves parameters $v$ and $w$, where $w$ corresponds to the $l$-highest weight of the representation. In this paper we work on the case $w = (1, 0, \ldots, 0)$ only, in other words, we deal with the level 1 representation only.

We do not take the original definition of quiver varieties but another equivalent one, which works only for the case $w = (1, 0, \ldots, 0)$.

We use $d$ instead of $v$.

3.1 Quiver varieties

3.1.1

Let $(\mathbb{C}^2)^{[n]}$ denote the Hilbert scheme of $n$ points on $\mathbb{C}^2$:

$$(\mathbb{C}^2)^{[n]} = \{ J \subset \mathbb{C}[x, y] \mid \dim \mathbb{C}[x, y]/J = n \},$$

and $\text{Sym}^n \mathbb{C}^2$ denote the $n$-th symmetric product of $\mathbb{C}^2$:

$$\text{Sym}^n \mathbb{C}^2 = \left\{ \sum a_ip_i \mid a_i \in \mathbb{Z}_{>0}, \sum a_i = n, p_i \in \mathbb{C}^2 \right\}.$$

Let $\pi$ denote the Hilbert-Chow morphism:

$$\pi: \quad (\mathbb{C}^2)^{[n]} \longrightarrow \text{Sym}^n \mathbb{C}^2, \quad J \longmapsto \text{supp} \mathbb{C}[x, y]/J.$$

We regard $\mathbb{Z}/l\mathbb{Z}$ as the subgroup of $\text{SU}(2)$.

The action of $\mathbb{Z}/l\mathbb{Z}$ on $\mathbb{C}^2$ induces the action of $\mathbb{Z}/l\mathbb{Z}$ on $\text{Sym}^n \mathbb{C}^2$ and $(\mathbb{C}^2)^{[n]}$ so that $\pi$ is $\mathbb{Z}/l\mathbb{Z}$-equivariant. Let $(\text{Sym}^n \mathbb{C}^2)^{\mathbb{Z}/l\mathbb{Z}}$ and $((\mathbb{C}^2)^{[n]})^{\mathbb{Z}/l\mathbb{Z}}$ denote the sets of the fixed points.

Note that for $J \in ((\mathbb{C}^2)^{[n]})^{\mathbb{Z}/l\mathbb{Z}}$, $\mathbb{C}[x, y]/J$ has a canonical $\mathbb{Z}/l\mathbb{Z}$-module structure. For $d = (d_0, \ldots, d_{l-1}) \in \mathbb{Z}^l$ such that $\sum d_i = n$ we define the quiver variety $\mathcal{M}(d)$ by

$$\mathcal{M}(d) = \left\{ J \in ((\mathbb{C}^2)^{[n]})^{\mathbb{Z}/l\mathbb{Z}} \mid \dim \mathbb{C}[x, y]/J \simeq \bigoplus_i \mathbb{C}^{\kappa di} \right\},$$

where $\mathbb{C}_{(i)}$ is the 1-dimensional $\mathbb{Z}/l\mathbb{Z}$-module with weight $i$.

We set

$$\kappa_1 = \frac{1}{2} dC' d + d_0, \quad \kappa_2 = n - \kappa_1 l,$$

where $C$ is the Cartan matrix of type $\hat{A}_{l-1}$. Then we have $\dim \mathcal{M}(d) = 2\kappa_1$ ([12]). Let $\zeta \in \mathbb{Z}/l\mathbb{Z}$ be a generator of $\mathbb{Z}/l\mathbb{Z}$. We define the closed subvariety

$$\mathcal{M}_0(d) = \left\{ \kappa_2 [0] + \sum_{j=1}^{\kappa_1} ([p_j] + \cdots + [\zeta^{j-1} p_j]) \in (\text{Sym}^n \mathbb{C}^2)^{\mathbb{Z}/l\mathbb{Z}} \mid p_j \in \mathbb{C}^2 \right\}. $$
of \((\text{Sym}^n \mathbb{C}^2)^{\mathbb{Z}/l\mathbb{Z}}\). Then we have \(\pi(\mathcal{M}(d)) \subset \mathcal{M}_0(d)\) (in fact we can check \(\pi(\mathcal{M}(d)) = \mathcal{M}_0(d)\)).

For \(d, d' \in \mathbb{Z}^l\) such that \(d_i \leq d'_i\) for all \(i\), we have the inclusion given by

\[
\mathcal{M}_0(d) \hookrightarrow \mathcal{M}_0(d')
\]

\[
X \mapsto X + \kappa_3 [0],
\]

where \(\kappa_3 = \sum(d'_i - d_i)\). We set

\[
\mathcal{M} = \coprod_{d \in \mathbb{Z}^l} \mathcal{M}(d), \quad \mathcal{M}_0 = \bigcup_{d \in \mathbb{Z}^l} \mathcal{M}_0(d),
\]

and

\[
\mathfrak{Z} = \mathcal{M} \setminus \mathcal{M}_0.
\]

Note that we introduce \(\mathcal{M}_0\) just only for terminological reason. We work on \(\mathcal{M}\) and \(\mathfrak{Z}\), of which connected components are finite dimensional.

### 3.1.2 Representation on K-theory of quiver varieties

The natural \(T = (\mathbb{C}^*)^2\)-action on \(\mathbb{C}^2\) induces a \(T\)-action on \(\mathcal{M}\). The \(T\)-fixed points of \(\mathcal{M}\) are indexed by \(\Pi\). For \(\lambda \in \Pi\) the corresponding ideal \(J_\lambda \in (\mathbb{C}^2)^{[\deg \lambda]}\) is the ideal generated by \(\{x^a y^b \mid (a, b) \notin \lambda\}\). Then \(\{[x^a y^b] \in \mathbb{C}[x, y]/J \mid (a, b) \in \lambda\}\) forms a basis of \(\mathbb{C}[x, y]/J\).

For \(\zeta \in \mathbb{Z}/l\mathbb{Z}\) we have \(\zeta \cdot [x^a y^b] = \zeta^{a-b}[x^a y^b]\). So \(J_\lambda \subset \mathcal{M}(\lambda)\).

For \((s, t) \in T\) we have \((s, t) \cdot [x^a y^b] = s^a t^b [x^a y^b]\). So \(\mathbb{C} \cdot [x^a y^b] \subset R(T) = \mathbb{Z}[s^\pm, t^\pm]\), where \(R(T)\) is the representation ring of \(T\) and we identify the coordinate functions of \(T\) with the generators of \(R(T)\). Thus for a node \(X = (a, b)\), we set \([X] = s^a t^b \in R(T)\).

### 3.2 Representation on K-theory of quiver varieties

#### 3.2.1

Let \(e_i\) denote the \(i\)-th coordinate vector in \(\mathbb{Z}^l\). For \(d \in \mathbb{Z}^l\) we define the subvariety of \(\mathfrak{Z}\) by

\[
\mathcal{B}_i(d) = \{(J_1, J_2) \in \mathfrak{Z} \mid J_1 \in \mathcal{M}(d), \ J_2 \in \mathcal{M}(d + e_i), \ J_1 \supset J_2\}
\]

This is called the **Hecke correspondence**.

Let \(p_\varepsilon\) denote the projection from \(\mathfrak{Z}\) to the \(\varepsilon\)-th factor \((\varepsilon = 1, 2)\) and \(q_\varepsilon\) denote its restriction to \(\mathcal{B}_i(d) \subset \mathfrak{Z}\). We define the tautological bundle \(\mathcal{L}\) on \(\mathcal{B}_i(d)\) by \(q_2^* \mathcal{M} / q_1^* \mathcal{M}\).

#### 3.2.2

For a \(T\)-equivariant vector bundle \(\mathcal{B}\) on \(X\), let \(\text{det} \mathcal{B}\) denote its determinant, \(\wedge \mathcal{B}\) denote its \(i\)-th wedge product, and set \(\wedge_i \mathcal{B} = \sum_{z \geq 0} (-z)^i \wedge^i \mathcal{B}\). These operators can be extended to operators on \(K^T(X)\). For a \(\mathbb{Z}/l\mathbb{Z}\)-module \(M\) we set \(M_i = \text{Hom}_{\mathbb{Z}/l\mathbb{Z}}(\mathcal{C}(i), M)\).

We set

\[
\mathcal{H} = (-1 + s + t - st) \mathcal{M} + \mathcal{M} \in K^T(\mathcal{M}).
\]
We define an action of $U'_R\text{span}_{t, t_{tor}}$ on $K^T_R(\mathfrak{m}) = K^T(\mathfrak{m}) \otimes R$ by
\[
e_{i,n}(x) = e_i^{-1}(d)p_{1,*}(p_2^{-1}x \otimes (\xi)^{n+h_i(d)}) \quad x \in K^T_R(\mathfrak{m}(d + e_i)),
\]
\[
f_{i,n}(x) = e_i^{+1}(d)p_{1,*}(p_2^{-1}(x \otimes \det(s^{-1}t^{-1}f_i) \otimes \xi^n)) \quad x \in K^T_R(\mathfrak{m}(d)),
\]
\[
K^\pm_i(z)(x) = e_i^{-1}(d)(c_i^{+1}(d)(\wedge (s^{-1}t^{-1} - 1)H_i))^{\pm} x \quad x \in K^T_R(\mathfrak{m}(d)),
\]
where the index $+$ (resp. $-$) means the expansion as a formal power series in $z^{-1}$ (resp. $z$) and
\[
c_i^{-1}(d) = (-1)^d s^{(2d_i - d_{i+1} + 1)/2} t^{(-2d_i + 2d_i - d_{i+1} + 1)/2},
\]
\[
c_i^{+1}(d) = (-1)^{d_{i+1} - d_i} s^{d_i - d_{i+1}} t^{d_i - d_{i+1}/2},
\]
\[
h_i(d) = d_i - 2d_i + d_{i+1}.
\]

**Remark.** We slightly modify the actions in [22]. In fact we have
\[
e_{i,n} = (-1)^{d_{i+1}} s^{(d_{i+1} + 1)/2} t^{(-d_{i+1} + 1)/2} \Omega_{i,n}^{-},
\]
\[
f_{i,n} = (-1)^{d_i} s^{d_{i+1}/2} t^{d_{i+1}/2} \Omega_{i,n}^{+},
\]
\[
K_i^\pm(z)(x) = s^{(d_{i+1} + 1)/2} t^{(d_i - d_{i+1} + 1)/2} \Theta_i^\pm(z).
\]
Here the operators on right hand side are defined in 3.3 of [22], where we should replace their symbols $q, t, k, s$ with our symbols $t, s, i, n$.

Substitute this to theorem 2 in [22] and the definition of quantum toroidal algebra in [24], we can verify $e_{i,n}, f_{i,n}$ and $K_i^\pm(z)$ satisfy the relation in [24].

### 3.2.3

Let $i_\lambda$ denote the inclusion \{ $J_\lambda$ \} $\hookrightarrow$ $\mathfrak{m}$ and $1_\lambda$ denote the generator of $K^T(\{ J_\lambda \})$.

We set $b_\lambda = i_\lambda(1_\lambda) \in K^T(\mathfrak{m})$.

By the localization theorem
\[
K^T_R(\mathfrak{m}) \simeq \bigoplus_{\lambda \in \Pi} Rb_\lambda.
\]

**Theorem.** ([22] lemma 8) For $\lambda \in \Pi$ such that $d(\lambda) = d$ we have
\[
e_{i,n}(b_\lambda) = (-s^{1/2}t^{-1/2})^{d_{i+1}} \sum_{X \in R_{\lambda,i}} \left[ X^n \prod_{A \in A_{\lambda,i}} \left((st)^{1/2} A^* - (st)^{1/2} X^*\right)^{-1} \right.
\]
\[
\times \prod_{R \in R_{\lambda \setminus X,i}} \left((st)^{-1/2} R^* - (st)^{1/2} X^*\right) b_{\lambda \setminus X},
\]
\[
f_{i,n}(b_\lambda) = (-s^{1/2}t^{-1/2})^{-d_{i+1}} \sum_{X \in A_{\lambda,i}} \left[ X^n \prod_{A \in A_{\lambda \cup X,i}} \left(st A^* - X^*\right) \right.
\]
\[
\times \prod_{R \in R_{\lambda \setminus X}} \left(R^* - X^*\right)^{-1} b_{\lambda \cup X},
\]
\[
K_i^\pm(z)(b_\lambda) = \left( \prod_{A \in A_{\lambda,i}} \frac{(st)^{1/2} A^* z - (st)^{-1/2} X^*}{A^* z - 1} \right) \prod_{R \in R_{\lambda,i}} \frac{(st)^{-1/2} R^* z - (st)^{1/2} R^* z - 1}{R^* z - 1} \pm b_\lambda,
\]
where $(s^a t^b)^* = s^{-a} t^{-b}$ for $s^a t^b \in R(T)$. 

8
4 Schur-Weyl duality

In this section we review Schur-Weyl duality.

One can construct representations of the quantum affine algebra $U_q'(\hat{\mathfrak{sl}}_l)$ from representations of the affine Hecke algebra $\hat{\mathcal{H}}_N$ (see 4.1.3). In this construction the action is given originally in terms of Chevalley generators. One can rewrite the action in terms of Drinfeld generators ([19], see Theorem 4.1.4).

Further, Schur-Weyl duality in [2] can be extended to get representations of the quantum toroidal algebra $U'_R(\mathfrak{sl}_l,\text{tor})$ from representations of the toroidal Hecke algebra $\mathcal{H}_N$. This is done by extending the action of $U_q'(\hat{\mathfrak{sl}}_l)$ to $U'_R(\mathfrak{sl}_l,\text{tor})$ using the rotation automorphism of the Dynkin diagram of type $\hat{A}$ ([VV1], see Theorem 4.2.2).

4.1 Schur-Weyl duality for affine algebras

4.1.1

The finite Hecke algebra $\mathcal{H}_N$ is the $\mathcal{R}$-algebra generated by $T_a^{\pm 1}$ ($a = 1, \ldots, N-1$) with relations:

\[
T_a T_a^{-1} = T_a^{-1} T_a = 1, \\
(T_a + 1)(T_a - q^2) = 0, \\
T_a T_{a+1} T_a = T_{a+1} T_a T_{a+1}, \\
T_a T_b = T_b T_a \quad (|a - b| > 1).
\]

The affine Hecke algebra $\hat{\mathcal{H}}_N$ is the $\mathcal{R}$-algebra generated by $T_a^{\pm 1}$ ($a = 1, \ldots, N-1$), $X_a^{\pm 1}$ ($a = 1, \ldots, N$) with relations:

\[
T_a T_a^{-1} = T_a^{-1} T_a = 1, \\
(T_a + 1)(T_a - q^2) = 0, \\
T_a T_{a+1} T_a = T_{a+1} T_a T_{a+1}, \\
T_a T_b = T_b T_a \quad (|a - b| > 1), \\
X_a X_b = X_b X_a, \\
T_a X_a T_a = q^2 X_{a+1}, \\
X_b T_a = T_a X_b \quad (b \neq a, a+1).
\]

4.1.2

Let $V = \mathcal{R}^l$ with a basis $\{v_0, \ldots, v_{l-1}\}$. We define $\nabla \in \text{End}(V^\otimes 2)$ by

\[
\nabla(v_{i_1} \otimes v_{i_2}) = \begin{cases} 
q^2 v_{i_1} \otimes v_{i_2} & \text{if } i_1 = i_2, \\
qv_{i_2} \otimes v_{i_1} & \text{if } i_1 < i_2, \\
qv_{i_1} \otimes v_{i_2} + (q^2 - 1)v_{i_1} \otimes v_{i_2} & \text{if } i_1 > i_2.
\end{cases}
\]

Then we have a left action of $\mathcal{H}_N$ on $V^\otimes N$ defined by

\[
T_a \mapsto \nabla(T_a) = 1 \otimes a^{-1} \otimes \nabla \otimes 1^\otimes (N-a-1).
\]
4.1.3

Let $M$ be a right $\hat{H}_N$-module. We define the following operators on $M \otimes \hat{H}_N V^\otimes N$ :

$$e_i(m \otimes v) = \sum_{a=1}^{N} m X_a^{\delta_{i,a}} \otimes (K^i_a)^{-1} \cdots (K^i_{a-1})^{-1} F^{i,i-1}_a v,$$

$$f_i(m \otimes v) = \sum_{a=1}^{N} m X_a^{-\delta_{i,a}} \otimes E^{i-1,i}_a K^i_{a+1} \cdots K^i_N v,$$

$$h_i(m \otimes v) = m \otimes K^i_1 \cdots K^i_N v.$$

Here $E^{i,j}_a = 1^\otimes a^{-1} \otimes E^{i,j} \otimes 1^\otimes N-a$, where $E^{i,j} \in \text{End}(V)$ is the matrix unit with respect to the basis $v_0, \ldots, v_{l-1}$ and $K^i_a = q^{E^{i-1,i-1}_a - E^{i,i}_a}$. These operators give a left $U_q(\mathfrak{sl}_N)$-action on $M \otimes \hat{H}_N V^\otimes N$ \cite{[2]}.

4.1.4

An isomorphism between the algebras defined by Chevalley generators and by Drinfeld new realization is given in \cite{[2]}.

For $\mathbf{j} = (j_1, \ldots, j_N) \in \{0, \ldots, l-1\}^N$ let $v_\mathbf{j}$ denote $v_{j_1} \otimes \cdots \otimes v_{j_N} \in V^\otimes N$.

For $1 \leq a, b \leq N$ we define

$$T_{a,b} = \begin{cases} T_a T_{a+1} \cdots T_{b-1} & a < b, \\ T_a T_{a-1} T_{a-2} \cdots T_b & a = b, \\ T_{a-1} T_{a-2} \cdots T_b & a > b. \end{cases}$$

**Theorem.** \cite{[2]} Theorem 3.3 Assume $\mathbf{j}$ is an non-decreasing sequence. We put $n_i = \sharp \{a \mid j_a = i\}$ and $\bar{n}_i = \sum_{i' = 1}^{i} n_{i'}$. Let us write $\mathbf{j} = [n_0, n_1, \ldots]$.

For $m \otimes v_\mathbf{j} \in M \otimes \hat{H}_N V^\otimes N$ the actions of Drinfeld generators of $U_q(\mathfrak{sl}_N)$ are described as follows :

$$\widetilde{e}_i(z)(m \otimes v_\mathbf{j}) = q^{1-n_i} m \left( \sum_{a=\bar{n}_{i-1}+1}^{\bar{n}_i} T_{a,\bar{n}_{i-1}+1} \delta \left(q^{l-i} Y_{\bar{n}_{i-1}+1} z\right) \otimes v_{j_{\mathbf{i}}} \right),$$

$$\widetilde{f}_i(z)(m \otimes v_\mathbf{j}) = q^{1-n_i-1} m \left( \sum_{a=\bar{n}_{i-2}+1}^{\bar{n}_{i-1}} T_{a,\bar{n}_{i-1}} \delta \left(q^{l-i} Y_{\bar{n}_{i-1}} z\right) \otimes v_{j_{\mathbf{i}}} \right),$$

$$\widetilde{K}^\pm_i(z)(m \otimes v_\mathbf{j}) = q^{m_i} \prod_{j_a = 1}^{\mathbf{j}_-} \theta_{j_a}^{\pm} (q^{l-i+1} Y_a z) \prod_{j_b = 1}^{\mathbf{j}_+} \theta_{j_b}^{\pm} (q^{l-i-1} Y_b z) \otimes v_{j_{\mathbf{i}}},$$

where $\mathbf{j}_- = [\ldots, n_{i-1} + 1, n_{i-1} - 1, \ldots]$, $\mathbf{j}_+ = [\ldots, n_{i-1} - 1, n_{i-1} + 1, \ldots]$ and $\theta_{m}(z) = \frac{q^{m} z - 1}{z - q^{m}}$. 

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4.2 Schur-Weyl duality for toroidal algebras

4.2.1

The toroidal Hecke algebra $\hat{H}_N$ is the $\mathcal{R}$-algebra generated by $T_a^{\pm 1}$ ($a = 1, \ldots, N - 1$), $X_a^{\pm 1}$ ($a = 1, \ldots, N$), $Y_a^{\pm 1}$ ($a = 1, \ldots, N$) with relations:

\begin{align*}
T_a T_a^{-1} &= T_a^{-1} T_a = 1, \\
(T_a + 1)(T_a - q^2) &= 0,
\end{align*}

Then we have $T_a T_{a+1} T_a = T_{a+1} T_a T_{a+1}$.

Let $\hat{H}_N^{(1)}$ (resp. $\hat{H}_N^{(2)}$) denote the subalgebra generated by $\{T_a^{\pm 1}\}$ and $\{Y_a^{\pm 1}\}$ (resp. $\{T_a^{\pm 1}\}$ and $\{X_a^{\pm 1}\}$). They are isomorphic to $\hat{H}_N$.

4.2.2

Let $M$ be a right $\hat{H}_N$-module. Regarding $M$ as a right $\hat{H}_N^{(1)}$-module we have the action of $U_q'(\mathfrak{sl}_l)$ on $M \otimes_{H_N} V^\otimes N$ by \[ \rho(m \otimes v_{i_1} \otimes \cdots \otimes v_{i_N}) = m X_{i_1}^{\delta_{l,1}} \cdots X_{i_N}^{\delta_{l,N}} \otimes v_{i_{l+1}} \otimes \cdots \otimes v_{i_{N-l}}. \]

Lemma. \[ (19) \text{ Proposition 3.4} \] We set $\mathcal{X}_l(z) = \mathcal{X}_l((r^l)^{-1}z)$ ($\mathcal{X} = e, f, K^{\pm}$). Then we have $\mathcal{X}_l(z) = \rho^{-1} \circ \mathcal{X}_l((r^l)^{-1}z) \circ \rho$.

Theorem. \[ (19) \text{ Theorem 3.5} \] The action of $U_q'(\mathfrak{sl}_l) \otimes \mathcal{R} \simeq U_q'(\mathfrak{sl}_l)$ on $M \otimes_{H_N} V^\otimes N$ can be extended to an action of $U_q'(\mathfrak{sl}_{l,\text{tor}})$ so that the actions of $\mathcal{X}_0$ ($\mathcal{X} = e, f, K^{\pm}$) are given by $\mathcal{X}_0(z) = \rho^{-1} \circ \mathcal{X}_0((r^l)^{-1}z) \circ \rho$.

5 Representation on the q-Fock space

In this section we review the action of $U_q'(\mathfrak{sl}_{l,\text{tor}})$ on the q-Fock space following \[ 15 \] and \[ 20. \]

As a q-analogue of the permutation representation, $\mathcal{R}[z_1^{\pm 1}, \ldots, z_N^{\pm 1}]$ has a right $H_N$-module structure. We define the q-wedge space by $\mathcal{R}[z_1^{\pm 1}, \ldots, z_N^{\pm 1}] \otimes_{H_N} V^\otimes N$. 

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This is the q-analogue of the classical wedge space $\otimes^N V(z)/\oplus \text{Ker}(\text{id}+\sigma_i)$, where $\sigma_i$ is the generator of $\mathfrak{S}_N$.

We define the q-Fock space taking "limit" of the q-wedge space. In other words the q-Fock space is the q-analogue of the classical semi-infinite wedge space.

It is known the right $\mathcal{H}_N$-module structure on $\mathcal{R}[z_1^\pm, \ldots, z_N^\pm]$ can be extended to a right $\mathcal{H}_N$-module structure called Dunkl-Cherednik representation. By Schur-Weyl duality described in 4.2.2, we have an action of $U'_R(\mathfrak{sl}_l,\text{tor})$ on the q-wedge space. This can be naturally lifted to an action on the q-Fock space.

5.1 The q-Fock space

Here we review the definition of the q-Fock space. The reader can refer to [8] for detail.

5.1.1

For $1 \leq a < b \leq N$ let us define an operator $g_{ab}$ on $\mathcal{R}[z_1^\pm, \ldots, z_N^\pm]$ by

$$g_{ab} = \frac{q^{-1}z_a - qz_b}{z_a - z_b}(\sigma_{ab} - 1) + q,$$

where $\sigma_{ab}$ is the operator defined by the permutation of variables $z_a$ and $z_b$.

Then we have a right action of $\mathcal{H}_N$ on $\mathcal{R}[z_1^\pm, \ldots, z_N^\pm]$ defined by

$$T_a \mapsto T_a = (q^2 - 1) - qg_{a,a+1}.$$

5.1.2

Let $V(z) = \mathcal{R}[z^\pm] \otimes V$. We define

$$\wedge^N V(z) = \mathcal{R}[z_1^\pm, \ldots, z_N^\pm] \otimes_{\mathcal{H}_N} V \otimes^N$$

$$= \otimes^N V(z) / \sum_{a=1}^{N-1} \text{Im} \left( \frac{p}{T_a \otimes 1_V \otimes^N - 1_{\mathcal{R}[z_1^\pm, \ldots, z_N^\pm]} \otimes^N T_a} \right).$$

This is called the $\textbf{q}$-wedge space.

5.1.3

We write $u_k = z^m \otimes v_j$ for $k = j - l(m+1)$. Let $u_{k_1} \wedge \cdots \wedge u_{k_N}$ denote the image of $u_{k_1} \otimes \cdots \otimes u_{k_N}$ for the quotient map. We say $u_{k_1} \wedge \cdots \wedge u_{k_N}$ is normally ordered if $k_a > k_b$ for $a < b$.

For $N = 2$ we can verify that if $k \equiv k'$ then

$$u_k \wedge u_{k'} = -u_{k'} \wedge u_k,$$

and if $k < k'$ and $k - k' \equiv i$ ($1 \leq i \leq l - 1$) then

$$u_k \wedge u_{k'} = -qu_{k'} \wedge u_k + (q^2 - 1)(u_{k'-l} \wedge u_{k+l}) - qu_{k'-l} \wedge u_{k+l} + q^2 u_{k'-l-i} \wedge u_{k+l+i} - \cdots$$

where the summation continues as long as the wedge is normally ordered.

The set of all normally ordered wedges forms a basis of $\wedge^N V(z)$.  


For \( c \in \mathbb{Z} \) and \( 0 < N < N' \) we define
\[
\iota_{N,N'}^{c} : \mathbb{Z}^{N} \rightarrow \mathbb{Z}^{N'}
\]
\[
(k_1, \ldots, k_N) \mapsto (k_1, \ldots, k_N, -N + c - 1, \ldots, -N' + c).
\]

For \( k = (k_1, \ldots, k_N) \in \mathbb{Z}^N \) let us write \( u_k = u_{k_1} \wedge \cdots \wedge u_{k_N} \). We can check the well-definedness of the map
\[
\wedge^N V(z) \mapsto \wedge^{N'} V(z)
\]
\[
u_k \mapsto u_{\iota_{N,N'}^{c}(k)}.
\]

We write \( \iota_{N,N'}^{c} \) for this map as well.

We define
\[
F(c) = \lim_{\iota_{N,N'}^{c}} \wedge^N V(z), \quad F = \bigoplus_{c \in \mathbb{Z}} F(c),
\]
and \( \iota_{c,\infty}^{c} \) by the canonical map from \( \wedge^N V(z) \) to \( F(c) \). \( F \) (resp. \( F(c) \)) is called the q-Fock space (with charge \( c \)). An element of \( F \) (resp. \( F(c) \)) is called a semi-infinite wedge (with charge \( c \)).

Let \( k = (k_1, k_2, \ldots) \) be a Maya diagram with charge \( c \) (we use \( k \) both for an element of \( \mathbb{Z}^N \) and for an infinite sequence of integers by abuse of notations), then \( u_k = u_{k_1} \wedge u_{k_2} \wedge \cdots \) is a semi-infinite wedge with charge \( c \). Note that \( \{ u_k \mid k \in \Pi_c \} \) forms a basis of \( F(c) \).

### 5.2 Representation on the q-Fock space

#### 5.2.1

Let us consider the following operators on \( \mathcal{R}[\pm z_1, \ldots, \pm z_N] \) :
\[
Y_a^{(N)} = g_{a,a+1}^{-1} \sigma_a g_{a+1,a}^{-1} g_{a,a,N} p^{D_a} \sigma_{a,N} \sigma_{a-1,a} g_{a-1,a} \cdot \cdot \cdot \sigma_{a-1,a} g_{a-1,a} \quad (a \in \{1, \ldots, N\})
\]
where \( p^{D_a} \) is the difference operator given by
\[
p^{D_a} f(z_1, \ldots, z_a, \ldots, z_N) = f(z_1, \ldots, p z_a, \ldots, z_N), \quad f \in \mathcal{R}[\pm 1, \ldots, \pm N].
\]

The operator \( Y_a^{(N)} \) is called Dunkl-Cherednik operator. Then the action of \( H_N \) defined in [1.1.2] can be extended to the action of \( H_N \) by
\[
T_a \mapsto T_a, \quad \sigma_i \mapsto z_a, \quad Y_a \mapsto q^{1-N} Y_a^{(N)}.
\]

This is called Dunkl-Cherednik representation ([2], [3], [4]).

By the Schur-Weyl duality explained in [1.2.2] we have an action of \( U_q^f [\mathfrak{s}l_{1,tor}] \) on \( \wedge^N V(z) = \mathcal{R}[\pm 1, \ldots, \pm N] \otimes_{H_N} V^N \).
5.2.2

For $k \in \mathbb{Z}^N$ we define $m \in \mathbb{Z}^N$ and $j \in \{0, \ldots, l - 1\}$ by $k_a = j_a - l(m_a + 1)$.
Note that $z^m \otimes v_j = u_k$. We identify $k \in \mathbb{Z}^N$ with the pair $(m, j)$. Let $m^c = (m_1, \ldots, m_N) \in \mathbb{Z}^N$ denote the sequence obtained from $k^c = (c - a)_{1 \leq a \leq N}$.

Let $M_{N,l}^c$ denote the set of all $m$ such that

- $m$ is non-decreasing with no more than $l$ elements of any given value, and
- $m_a \geq m^c_a$ for all $a$ and $\sum (m_a - m^c_a) = \gamma$.

For $m \in M_{N,l}^c$ we define

$$J(m) = \{ j \in \{0, \ldots, l - 1\}^N \mid j_a < j_b \text{ for } a < b \text{ such that } m_a = m_b \}.$$

We define

$$V_{N}^{c,\gamma} = \bigoplus_{m \in M_{N,l}^c} \bigoplus_{j \in J(m)} R u_k \subset \wedge^N V(z).$$

We can check this is invariant under the $U_{\mathcal{R}}(\mathfrak{sl}_l)$-action.

We can see that for $\alpha, \beta \in \mathbb{Z}$ such that $\alpha l + c > \gamma l$ and $\beta > \alpha$ the restriction

$$i_{\alpha l + c, \beta l + c}^{c,\gamma} = i_{\alpha l + c, \beta l + c}^{c,\gamma} |_{\wedge^N V(z)} : V_{\alpha l + c}^{c,\gamma} \longrightarrow V_{\beta l + c}^{c,\gamma}$$

is an isomorphism as vector spaces.

**Theorem.** ([] Proposition 6) $i_{\alpha l + c, \beta l + c}^{c,\gamma}$ is an isomorphism as $U_{\mathcal{R}}(\mathfrak{sl}_l)$-modules.

5.2.3

For $k \in \Pi^c$ we set $\deg k = \sum (m_a - m^c_a)$. Note that this is well-defined. We set

$$F(c)^{\gamma} = \bigoplus_{\gamma \in \Pi^c, \deg k = \gamma} R u_k \subset F(c).$$

For $\alpha \in \mathbb{Z}$ such that $\alpha l + c > \gamma l$ the restriction

$$i_{\alpha l + c, \beta l + c}^{c,\gamma} = i_{\alpha l + c, \beta l + c}^{c,\gamma} |_{\wedge^N V(z)} : V_{\alpha l + c}^{c,\gamma} \longrightarrow F(c)^{\gamma}$$

is an isomorphism as vector space. By Theorem 5.2.2 we can extend the $U_{\mathcal{R}}(\mathfrak{sl}_l)$-action to $F(c)^r$, and so to $F(c)$.

5.2.4

We define

$$\rho_N : \mathbb{Z}^N \longrightarrow \mathbb{Z}^N$$

$$(k_1, \ldots, k_N) \longmapsto (k_1 - 1, \ldots, k_N - 1).$$

We write $\rho_N$ as well for the map $\wedge^N V(z) \rightarrow \wedge^N V(z)$ given by $u_k \mapsto u_{\rho_N(k)}$.

We can see this is compatible with the construction of $\rho$ in 4.2.2.

We also define

$$\rho_{\infty} : \Pi^c \longrightarrow \Pi_{c-1}$$

$$(k_1, k_2, \ldots) \longmapsto (k_1 - 1, k_2 - 1, \ldots).$$
and $\rho_\infty: F(c) \to F(c - 1)$.

For $0 < N < N' \leq \infty$ we have

$$\iota_{N,N'}^c \circ \rho_N = \rho_{N'} \circ \iota_{N,N'}^c.$$  

Thus the action of $U_R(sl_{1,tor})$ on $\wedge^N V(z)$ can be extended to $F$ so that

$$X_0(z) = \rho_\infty^{-1} \circ X_1(q^{-1} r^{-1} z) \circ \rho_\infty \ (X = e, f, K^\pm).$$

6 Simultaneous eigenvectors

In this section we construct simultaneous eigenvectors for the actions of $K^\pm_i(z)$’s on the q-Fock space, which are the main subjects of this paper.

In §6.1 we review nonsymmetric Macdonald polynomials ([5], [11], [14]). For $\mathbf{m} \in \mathbb{Z}^N$, nonsymmetric Macdonald polynomial $\Phi^\mathbf{m} \in \mathbb{C}[z_1^\pm, \ldots, z_N^\pm]$ is a simultaneous eigenvector of Dunkl-Cherednik operators. The transition matrix between monomials and nonsymmetric Macdonald polynomials is upper triangular with respect to the Bruhat order on $\mathbb{Z}^N$. The actions of the finite Hecke algebra generators $T_a$ on nonsymmetric Macdonald polynomials can be simply described (see Proposition 6.1.4).

For $\mathbf{k} = (k_1 > \cdots > k_N)$ we define a vector $\Psi^\mathbf{k} = \Phi^{\mathbf{m}} \otimes \mathbf{v}_j$ in the q-wedge space (Definition 6.2.2), where $\mathbf{m}$ and $j$ are given by “renumbering” of $(k_1, \ldots, k_N)$ so that $j$ is non-decreasing (6.2.1). It follows immediately from Theorem 4.1.4 that $\Psi^\mathbf{k}$ is a simultaneous eigenvector for the actions of $K^\pm_i(z)$’s $(i \neq 0)$. We can check

- the eigenvalues are multiplicity free (Proposition 6.2.5), and
- the transition matrix between normally ordered wedges and $\{\Psi^\mathbf{k}\}$ is upper triangular (Proposition 6.2.3), in particular $\{\Psi^\mathbf{k}\}$ forms a basis of the q-wedge space.

So the vector $\Psi^\mathbf{k}$ is characterized in term of the actions of $K^\pm_i(z)$’s $(i \neq 0)$ (Corollary 6.2.5). Further, using them we can verify

- $\Psi^\mathbf{k}$ is also a simultaneous eigenvector for $K^\pm_0(z)$ (Corollary 6.2.6), and
- $\Psi^\mathbf{k}$ can be lifted to the q-Fock space (Definition 6.2.8).

We can see the eigenvalues coincide with the eigenvalues of the torus fixed points in the representation on the equivariant K-groups of the quiver varieties.

6.1 Nonsymmetric Macdonald polynomials

6.1.1

Let us define the Bruhat order. This is the partial order on $\mathbb{Z}^N$ given by the transitive closure of the following two relations:

For $\mathbf{x} = (x_1, \ldots, x_N) \in \mathbb{Z}^N$

- If $1 \leq i < j \leq N$ and $x_i > x_j$ then $\mathbf{x} \succ \sigma_{ij}\mathbf{x}$, and
- $1 \leq i < j \leq N$ and $x_i - x_j > 1$ then $\sigma_{ij}\mathbf{x} \succ \mathbf{x} + \mathbf{e}_i - \mathbf{e}_j$ where $\mathbf{e}_i$ is the $i$-th coordinate vector.
6.1.2

For $x \in \mathbb{Z}^N$ let $\sigma_x$ denote the unique element of $\mathfrak{S}_N$ satisfying the following conditions:

- if $\sigma_x(a) < \sigma_x(a')$ then $x_{\sigma_x(a)} \geq x_{\sigma_x(a')}$, and
- if $a < a'$ and $x_{\sigma_x(a)} = x_{\sigma_x(a')}$ then $\sigma_x(a) < \sigma_x(a')$.

6.1.3

We can see that $\mathcal{R}[z_1^{\pm 1}, \ldots, z_N^{\pm 1}]$ has the basis $\{\Phi^m(z) | m \in \mathbb{Z}^N\}$ such that

- $\Phi^m(z) = z^m + \sum_{n < m} c(m, n) z^n$ $(\exists c(m, n) \in \mathcal{R})$,
- $\Phi^m(z) Y_d(N) = \zeta_a(m) \Phi^m(z)$, where $\zeta_a(m) = p^{m_a} q^{2|\sigma_m(N) - N| - 1}$.

$\Phi^m(z)$ is called nonsymmetric Macdonald polynomial ([5, 11, 13]).

6.1.4

Proposition. (see [78] §1.5)

$$\Phi^m(z) \cdot \hat{T}_a = \begin{cases} 
\frac{(-q^2+1)}{x-1} \Phi^m(z) - \frac{(x-q^2)(q^2x-1)}{(x-1)^2} \Phi^{\sigma_a, m}(z) & (m_a > m_{a+1}), \\
\frac{(-q^2+1)}{x-1} \Phi^m(z) & (m_a = m_{a+1}), \\
\frac{(-q^2+1)}{x-1} \Phi^m(z) - \Phi^{\sigma_a, m}(z) & (m_a < m_{a+1}),
\end{cases}$$

where $\sigma_a m = (\ldots, m_{a+1}, m_a, \ldots)$ and $x = \frac{\zeta_{a+1}(m)}{\zeta_a(m)}$.

6.2 Simultaneous eigenvectors and its properties

6.2.1

For $\sigma \in \mathfrak{S}_N$ we define $\hat{\sigma} \in \mathfrak{S}_N$ by $\hat{\sigma}(a) = N - \sigma(a) + 1$.

For $k \in \mathbb{Z}_+^N = \{k \in \mathbb{Z}^N | k_1 > \cdots > k_N\}$ we define $m = (m_1, \ldots, m_N) \in \mathbb{Z}^N$ and $\mathbf{j} = (j_1, \ldots, j_N) \in \{0, \ldots, l-1\}^N$ by

$$m_a = m_{\sigma(a)}, \quad \mathbf{j}_a = \hat{j}_{\sigma(a)}.$$

Note that

- $m$ is non-decreasing, and if $a < b$, $m_a = m_b$ then $j_a > j_b$,
- $\mathbf{j}$ is non-decreasing, and if $a < b$, $\mathbf{j}_a = \mathbf{j}_b$ then $m_a > m_b$,
- $\hat{j}_{\sigma_a}^{-1} = \hat{m}_a$.

Example. For $k = (5, 3, 1, -6, -7, -8, -9, -10)$ we have

$$\begin{pmatrix} m \\ j \end{pmatrix} = \begin{pmatrix} -2 & -1 & -1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 3 & 1 & 4 & 3 & 2 & 1 & 0 \end{pmatrix},$$

$$\begin{pmatrix} m \\ \mathbf{j} \end{pmatrix} = \begin{pmatrix} 1 & -2 & 1 & -1 & 1 & 1 & -1 & 1 \\ 0 & 0 & 1 & 1 & 2 & 3 & 3 & 4 \end{pmatrix}.$$
• enumerate the boxes from lower rows to upper rows and from right to left in a row, then \((j_a, m_a)\) is the coordinate of the \(a\)-th box, and

• enumerate the boxes from left columns to right columns and from the top to the bottom in a column, then \((j_a, m_a)\) is the coordinate of the \(a\)-th box.

\[
\begin{array}{cccccc}
1 & -10 & -9 & -8 & -7 & -6 \\
0 & -5 & -4 & -3 & -2 & -1 \\
-1 & 0 & 1 & 2 & 3 & 4 \\
-2 & 5 & 6 & 7 & 8 & 9 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\end{array}
\]

\[
\uparrow \\
\downarrow
\]

\(m\)

We define a partial order \(<\) on \(\mathbb{Z}_+^N\) by

\[k' \prec k \iff j' = j \text{ and } m' < m.\]

6.2.2
Definition. For \(k \in \mathbb{Z}_+^N\) we define \(\Psi^k = \Phi_m \otimes v_j \in \wedge^N V(z)\).

Remark. Takemura-Uglov introduced vectors \(\Phi_m \otimes v_j \in \wedge^N V(z)\) in [17], which are different from ours.

Proposition. \(\Psi^k\) is a simultaneous eigenvector for the actions of \(K^\pm_i(z)\)'s \((i \in \{1, \ldots, l-1\})\).

Proof. It follows from Theorem 4.1.4 and the definition of \(\Phi_m\) in [6.1.3].

6.2.3
For \(k \in \mathbb{Z}_+^N\) we define

\[\varepsilon(k) = \# \{ (a, b) \mid a < b, \tilde{\sigma}_m(a) > \tilde{\sigma}_m(b) \}.\]

Proposition.

\[\Psi^k = (-q)^{\varepsilon(k)} u_k + \sum_{k' \prec k} c(k, k') u_{k'} \quad (\exists c(k, k') \in \mathcal{R}).\]

Proof. By the definition of nonsymmetric Macdonald polynomials in [6.1.3]

\[\Phi_m \otimes v_j = z_m \otimes v_j + \sum_{m' < m} c(m, m') z_{m'} \otimes v_j \quad (\exists c(m, m') \in \mathcal{R}).\]

On the other hand by the relation in [5.1.2] we can verify

\[z_{m'} \otimes v_j = (-q)^{c(m', j)} z_{m'} \otimes v_{\tilde{\sigma}_m(m')} + \sum_{m'' < m'} c'(m', m') z_{m''} \otimes v_{\tilde{\sigma}_m(m'')} \quad (\exists c'(m', m'') \in \mathcal{R})\]

and \(c(m, j) = \varepsilon(k)\). Then the statement follows.
We define
\[ Z_{\gamma}^N = \{ k \in \mathbb{Z}_+^N \mid m \in M_{\gamma}^N \}. \]
Note that if \( k \in Z_{\gamma}^N \) and \( k' \preceq k \), then \( k' \in Z_{\gamma}^N \).

**Corollary.** If \( N > \gamma l \), then \( \{ \Psi^k \mid k \in Z_{\gamma}^N \} \) is a basis of \( V_N^\gamma \).

6.2.4

For \( k \in \mathbb{Z}_+^{al+c} (al + c > \gamma l) \), let \( \lambda \) denote the Young diagram corresponding to \( \lambda_{al+c,\infty}(k) \in \Pi_c \).

**Proposition.** For \( i = 1, \ldots, l-1 \) we have
\[ K_i^+(z)(\Psi^k) = \left( \prod_{A \in A_{\lambda,i}} \left( \frac{(st)^{1/2}A \begin{vmatrix} t^{-c-1}z - (st)^{-1/2} \end{vmatrix} \right)}{A \begin{vmatrix} t^{-c-1}z - 1 \end{vmatrix} \right)} \right) \pm \Psi^{\lambda} \]

**Proof.** By Theorem 4.1.4 and the defining relations \( K_i^+(z) = \tilde{K}_i^+(r^{l-i}z) \), it is sufficient to show
\[ \Phi^m \prod_{j_a = i-1} \theta_i^+ \left( q^{l-i+1}r^{l-i}Y_a z \right) \prod_{j_b = i} \theta_i^+ \left( q^{l-i+1}r^{l-i}Y_b z \right) \]
\[ = \left( \prod_{A \in A_{\lambda,i}} \left( \frac{(st)^{1/2}A \begin{vmatrix} t^{-c-1}z - (st)^{-1/2} \end{vmatrix} \right)}{A \begin{vmatrix} t^{-c-1}z - 1 \end{vmatrix} \right)} \right) \pm \Phi^m. \]

First we have
\[ \Phi^m q^{l-i+1}r^{l-i}Y_a = q^{l-i+1}r^{l-i}q^{1-N} \begin{vmatrix} m \end{vmatrix} q^{\sigma_m(a)}(a) - N - 1 \Phi^m \]
\[ = q^{\sigma_m(a)} - N + 1/2 q^{\sigma_m(a) - N} \begin{vmatrix} m \end{vmatrix} q^{1/2} \Phi^m \]
\[ = q^{\rho} \sin^{-1}(a) + 3/2 \rho \rho^{-1}(a) - k_a + 1/2 \Phi^m. \]
\[ \Phi^m q^{l-i+1}r^{l-i}Y_b = q^{\rho} \sin^{-1}(b) - N + l/2 \rho \rho^{-1}(b) - k_b + 1/2 \Phi^m. \]

We classify the elements of \( \{ a \mid j_a = i - 1 \} \cup \{ b \mid j_b = i \} \) into three types:

1. \( a \) and \( b \) such that \( m_a = m_b, j_a = i - 1, j_b = i \),
2. \( a \) such that \( j_a = i - 1 \) and \( (m_a, i) \notin k \), and
3. \( b \) such that \( j_b = i \) and \( (m_b, i) - 1 \notin k \).

In the case of type (1), we have \( \tilde{\sigma}_j^{-1}(a) - 1 = \tilde{\sigma}_j^{-1}(b) - k_a + 1 = k_b. \) Thus
\[ \Phi^m q^{l-1}r^{l-i-1}Y_a = \Phi^m q^{l-i-2}r^{l-i-1}Y_b, \]
and so
\[ \Phi^m \theta_i \left( q^{l-1}Y_a r^{l-i-1}z \right) \theta^{-1} \left( q^{l-i-2}Y_b r^{l-i-1}z \right) = \Phi^m. \]
In the case of type (2), the node \( A = (\hat{\sigma}_j^{-1}(a) - 1, \hat{\sigma}_j^{-1}(a) + k_a - c) \) is an addable \( i \)-node. We have
\[
\phi_{\mathbf{m}} \phi_1 \left( q^{l-i} Y_a r^{l-i-1} \right) = \theta_1 \left( s^{-\hat{\sigma}_i^{-1}(a) + 3/2 t - \hat{\sigma}_i^{-1}(a) - k_a + 1/2 z} \right) \phi_{\mathbf{m}} = \theta_1 \left( s^{1/2 t - c + 1/2 |A|} z \right) \phi_{\mathbf{m}}
\]
\[
= \frac{(st)^{1/2} |A| t^{-c} z - (st)^{-1/2}|A| t^{-c} z - 1}{|A| t^{-c} z - 1} \phi_{\mathbf{m}}.
\]

In the case of type (3), the node \( R = (\hat{\sigma}_j^{-1}(b) - 1, \hat{\sigma}_j^{-1}(b) + k_b - c - 1) \) is a removable \( i \)-node. We have
\[
\phi_{\mathbf{m}} \phi_1 \left( q^{l-i-2} Y_b r^{l-i-1} \right) = \theta_1 \left( s^{-\hat{\sigma}_i^{-1}(b) + 1/2 t - \hat{\sigma}_i^{-1}(b) - k_b + 1/2 z} \right) \phi_{\mathbf{m}} = \theta_1 \left( s^{-1/2 t - c - 1/2 |R|} z \right) \phi_{\mathbf{m}}
\]
\[
= \frac{(st)^{-1/2} |R| t^{-c} z - (st)^{1/2} |R| t^{-c} z - 1}{|R| t^{-c} z - 1} \phi_{\mathbf{m}}.
\]

Thus the claim follows.

\[\square\]

6.2.5

**Proposition.** If \( k, k' \in \mathbb{Z}_{\alpha + c} \) (\( \alpha + c > \gamma l \)) and the eigenvalues of \( K_i^\pm(z) \) for \( \Psi^k \) and \( \Psi^{k'} \) coincide for all \( i \in \{1, \ldots, l - 1\} \), then \( k = k' \).

**Proof.** The coincidence of the eigenvalues of \( K_i^\pm(z) \) implies
\[
\prod_{A \in A_{\lambda,i}} (st)^{1/2} |A| t^{-c} z - (st)^{-1/2} \prod_{R \in R_{\lambda,i}} (st)^{-1/2} |R| t^{-c} z - (st)^{1/2} \times \prod_{A \in A_{\lambda',i}} |A| t^{-c} z - 1 \prod_{R \in R_{\lambda',i}} |R| t^{-c} z - 1
\]
\[
= \prod_{A \in A_{\lambda,i}} |A| t^{-c} z - 1 \prod_{R \in R_{\lambda,i}} |R| t^{-c} z - 1 \times \prod_{A \in A_{\lambda',i}} (st)^{1/2} |A| t^{-c} z - (st)^{-1/2} \prod_{R \in R_{\lambda',i}} (st)^{-1/2} |R| t^{-c} z - (st)^{1/2}.
\]

Since \(|\{(s, t) \mid s - t = n\} \cap (A_{\lambda} \cup R_{\lambda})| < 1 \) for any \( n \in \mathbb{Z} \), we have
\[
\prod_{A \in A_{\lambda,i}} \left( (st)^{1/2} |A| t^{-c} z - (st)^{-1/2} \right) \times \prod_{R \in R_{\lambda,i}} \left( (st)^{-1/2} |R| t^{-c} z - (st)^{1/2} \right) \Bigg|_{z = |X|} \neq 0
\]
for any \( X \in A_{\lambda,i} \cup R_{\lambda,i} \). So we have \( X \in A_{\lambda',i} \cup R_{\lambda',i} \), and it follows that \( A_{\lambda,i} \cup R_{\lambda,i} = A_{\lambda',i} \cup R_{\lambda',i} \).

It is easy to see the set \( \bigcup_{i \neq 0} (A_{\lambda,i} \cup R_{\lambda,i}) \) determines \( \lambda \). So the claim follows.

\[\square\]
Corollary. If $X \in \wedge^N V(z)$ is a simultaneous eigenvector for the actions of $K^\pm_i(z)$’s ($i \in \{1, \ldots, l-1\}$) and

$$X = (-q)^{c(k)}u_k + \sum_{k' < k} c(k')u_{k'} \quad (\exists c(k') \in \mathbb{R}),$$

for $k \in \mathbb{Z}_{\alpha l}^{al+c} (\alpha l + c > \gamma l)$, then $X = \Psi^k$.

Proof. It follows from Corollary 6.2.3 and the previous proposition. □

6.2.6

Proposition. For $k \in \mathbb{Z}_{\alpha l}^{al+c} (\alpha l + c > \gamma l)$, $\Psi^k$ is a simultaneous eigenvector for the actions of $K^\pm_i(z)$’s ($i \in \{0, \ldots, l-1\}$).

Proof. By definition $K^\pm_i(z)$’s commute with each other. Notice that a matrix which commutes with a diagonal matrix with diagonal elements different from each other is diagonal. It follows from Proposition 6.2.2, Corollary 6.2.3 and Proposition 6.2.5 that $\Psi^k$ is also a simultaneous eigenvector for the action of $K^\pm_0(z)$. □

6.2.7

Proposition. For $k \in \mathbb{Z}_{\alpha l}^{al+c} (\alpha l + c > \gamma l)$, we have

$$\rho(q^{(c)}\Psi^k) = q^{c(\rho(k))}\Psi^\rho(k).$$

Proof. By Lemma 4.2.2 and Theorem 4.2.2, $\rho(\Psi^k)$ is also a simultaneous eigenvector of $K^\pm_i(z)$’s ($i \in \{0, \ldots, l-1\}$). Note that

$$\rho(q^{(c)}\Psi^k) = \rho(u_k) + \sum_{k' < k} c(k, k')\rho(u_{k'})$$

$$= u_{\rho(k)} + \sum_{k' < k} c(k, k')u_{\rho(k')}$$

and $\rho$ preserves the order $\triangleleft$. Then the statement follows from Corollary 6.2.5. □

Corollary. For $k \in \mathbb{Z}_{\alpha l}^{al+c} (\alpha l + c > \gamma l)$, the eigenvalue of $\Psi^k$ for $K^\pm_0(z)$ is given by the same formula as in Proposition 6.2.3.

6.2.8

For $\beta > \alpha$ we write simply $i_{\alpha, \beta}^c$ for $i_{\alpha l+c, \beta l+c}^c$.

Lemma. For $k \in \mathbb{Z}_{\alpha l}^{al+c} (\alpha l + c > \gamma l)$, we have

$$i_{\alpha, \beta}^c(q^{c(\rho(k))}\Psi^k) = q^{c(i_{\alpha, \beta}^c(\rho(k))}\Psi^i_{\alpha, \beta}^c(k)).$$
Proof. By Theorem [5.2.2] $\iota_{\alpha,\beta}(\Psi^k) \in \wedge^{\beta + c} V(z)$ is also a simultaneous eigenvector. Note that

\[
\iota_{\alpha,\beta}(q^{-\varepsilon(k)}\Psi^k) = \iota_{\alpha,\beta}(u_k) + \sum_{k' \prec k} c(k, k') \iota_{\alpha,\beta}(u_{k'}) (\exists c(k, k') \in \mathcal{R})
\]

and $\iota_{\alpha,\beta}$ preserves the order $\prec$. The claim follows from Corollary [6.2.5]. □

This lemma assures us of the well-definedness for the following definition:

**Definition.** For a Young diagram $\lambda$, take $k \in \mathbb{Z}^{\alpha l + c}$ such that $\iota_{\alpha,\beta}^c(\Psi^k) \in F(c)$.

7 Isomorphism

In this section we construct an isomorphism between the equivariant K-groups of the quiver varieties and the q-Fock space.

First we express the actions of $e_{i,n}$'s on $\Psi_k$, which can be done completely in terms of Young diagrams (Theorem [7.1.2]). This is basically due to Proposition [6.1.4] although we need the residue theorem and a little complicated induction.

After suitable renormalizations, we arrive at the isomorphism.

7.1 Formula for the representation on the q-Fock space

7.1.1

We will give a formula for the action of $e_{i,n}$ on $\Psi^k$. For this the following proposition is essential:

**Proposition.** (1) For $m_1 < \cdots < m_N$ and $j_1 < j_2 = \cdots = j_N$ we have

\[
\Phi^m \left( \prod_{a=1}^N T_{a,1} \right) \otimes v_j = \sum_{a=1}^N \left( \prod_{b=a+1}^N \frac{-\zeta_b(m) + q^2 \zeta_a(m)}{\zeta_b(m) - \zeta_a(m)} \right) \Phi^{m(a)} \otimes v_j
\]

where $m(a) = (m_a, m_1, \ldots, m_a, \ldots)$.

(2) For $m_1 < \cdots < m_{N-1}$, $m_N = m_a$ ($1 \leq a < N$) and $j_1 = j_2 = \cdots = j_N$ we have

\[
\Phi^m \otimes v_j = 0
\]

(3) For $k$ such that $m_1 < \cdots < m_{N-1}$, $m_a < m_N < m_{a+1}$ and $j_1 = j_2 = \cdots = j_N$ we have

\[
\Phi^m \otimes v_j = \left( \prod_{b=a+1}^{N-1} \frac{-\zeta_b(n) + q^2 \zeta_a(n)}{\zeta_b(n) - \zeta_a(n)} \right) \Phi^{\overline{m}} \otimes v_j
\]

where $\overline{m} = (\ldots, m_a, m_N, m_{a+1}, \ldots)$. 21
Proof. If \( j_a = j_{a+1} \) and \( m_a > m_{a+1} \) we have
\[
q^2 \Phi^m \otimes v_j = \Phi^m \otimes T_a v_j \quad \text{by (4.1.2)}
\]
\[
= \Phi^m T_a \otimes v_j
\]
\[
= \frac{(-q^2 + 1)}{x - 1} \Phi^m \otimes v_j - \frac{(x - q^2)(q^2 x - 1)}{(x - 1)^2} \Phi^{\sigma_a m} \otimes v_j \quad \text{by (6.1.4)}.
\]
\[
\text{where } \sigma_a m = (\ldots, m_{a+1}, m_a, \ldots) \text{ and } x = \frac{\zeta_{a+1}(m)}{\zeta_a(m)}. \text{ Thus}
\]
\[
\Phi^m \otimes v_j = -\frac{\zeta_{a+1}(m) + q^2 \zeta_a(m)}{\zeta_{a+1}(m) - \zeta_a(m)} \Phi^{\sigma_a m} \otimes v_j.
\]
The statement of (3) follows this.

If \( j_a = j_{a+1} \) and \( m_a = m_{a+1} \) then \( x = \frac{\zeta_{a+1}(m)}{\zeta_a(m)} = q^2 \). Thus we have
\[
q^2 \Phi^m \otimes v_j = \Phi^m \otimes T_a v_j \quad \text{by (4.1.2)}
\]
\[
= \Phi^m T_a \otimes v_j
\]
\[
= \frac{(-q^2 + 1)}{x - 1} \Phi^m \otimes v_j \quad \text{by (6.1.4)}
\]
\[
= -\Phi^m \otimes v_j,
\]
and so \( \Phi^m \otimes v_j = 0 \). This shows (2).

We will prove (1) by induction for \( N' \). Assume the statement is true for \( N' < N \). Then
\[
\Phi^m \left( \sum_{a=1}^N T_{a,1} \right) \otimes v_j = \Phi^m \left( \sum_{a=2}^N T_{a,2} \right) \otimes v_j
\]
\[
= \sum_{a=2}^N \left( \prod_{b=a+1}^N \frac{-\zeta_b(m) + q^2 \zeta_a(m)}{\zeta_b(m) - \zeta_a(m)} \Phi^{\tilde{m}(a)} \right) T_1 \otimes v_j,
\]
where \( \tilde{m}(a) = m(a) = (m_1, m_2, \ldots, m_a, \ldots) \). Here we use the assumption of induction. Although the situations are not exactly same, commutativity of
\[
T_1 \text{ with } T_a (a \geq 3) \text{ allows us a parallel argument.}
\]
Further we have
\[
\Phi^{\tilde{m}(a)} T_1 \otimes v_j = \frac{(-q^2 + 1)}{x - 1} \Phi^{\tilde{m}(a)} \otimes v_j - \Phi^{\sigma_1 \tilde{m}(a)} \otimes v_j
\]
\[
= \frac{(-q^2 + 1) \zeta_1(m)}{\zeta_a(m) - \zeta_1(m)} \prod_{b=2}^{a-1} \frac{-\zeta_b(m) + q^2 \zeta_a(m)}{\zeta_b(m) - \zeta_a(m)} \Phi^m \otimes v_j - \Phi^{m(a)} \otimes v_j.
\]
We can see the coefficients of \( m(a) (a \geq 2) \) coincide with required ones. For the coefficient of \( m \) we need to check
\[
\left( \prod_{b=2}^N \frac{\zeta_b(k) - q^2 \zeta_1(k)}{\zeta_b(k) - \zeta_1(k)} \right) = \sum_{a=2}^N \left( \frac{(-q^2 + 1) \zeta_1(m)}{\zeta_a(m) - \zeta_1(m)} \prod_{b=2}^{a-1} \frac{\zeta_b(k) - q^2 \zeta_a(k)}{\zeta_b(k) - \zeta_a(k)} \right) + 1.
\]
This follows the next lemma. ∎
Lemma.

\[
\sum_{a=1}^{N} \left( -q^2 + 1 \prod_{b \neq a} \frac{x_b - q^2 x_a}{x_b - x_a} \right) = \left( \prod_{a=1}^{N} \frac{x_a - q^2}{x_a - 1} \right) - 1
\]

Proof. Apply the residue theorem for a rational function

\[
f(Z) = \frac{1}{Z(Z - 1)} \prod_{a=1}^{N} \frac{x_a - q^2 Z}{x_a - Z}.
\]

\[
\square
\]

7.1.2

Theorem.

\[
e_{i,n}(\Psi^\lambda_c) = \sum_{X \in R_{\lambda,i}} \left( t^\prime X \right)^n \left( \prod_{A \in A_{X,i}} \frac{-(st)^{-1/2} X^* + (st)^{1/2} A^*}{X^* - A} \right) \prod_{R \in R_{\lambda,i}, R > X} \frac{-(st)^{-1/2} R^* + (st)^{1/2} X^*}{R^* - X} \Psi_c^{\lambda \setminus X},
\]

\[
f_{i,n}(\Psi^\lambda_c) = \sum_{X \in A_{\lambda,i}} \left( t^\prime X \right)^n \left( \prod_{A \in A_{\lambda,i} \cup X} \frac{-(st)^{-1/2} X^* + (st)^{1/2} A^*}{X^* - A} \right) \prod_{R \in R_{\lambda,i}, R < X} \frac{-(st)^{-1/2} R^* + (st)^{1/2} X^*}{R^* - X} \Psi_c^{\lambda \cup X},
\]

\[
K^\pm_i(z)(\Psi^\lambda_c) = \left( \prod_{A \in A_{\lambda,i}} \frac{(st)^{1/2} A^{t-c} z}{A^{t-c} - 1} \right)^{\pm} \Psi_c^\lambda.
\]

Proof. The formulas for \( K^\pm_i(z) \)'s are nothing but Proposition 6.2.4 and Corollary 6.2.7. We will check for \( e_{i,n} \)'s.

For \( a, b \in \{1, \ldots, N\} \) we put

\[
f(a, b) = \frac{q^{-1} \zeta_b(m) - q \zeta_a(m)}{\zeta_b(m) - \zeta_a(m)}.
\]

Then we have
\[ e_{i,n} \left( \Phi^m \otimes v_2 \right) = q^{1-n_i} \Phi^m \left( \sum_{a=\bar{n}_i+1}^{\bar{n}_i} T_{a,\bar{n}_i+1} \right) \left( q^{l_i-1} Y_{\bar{n}_i+1+1} \right)^{-n} \otimes v_2 \]

\[ = q^{1-n_i} \sum_{a=\bar{n}_i+1}^{\bar{n}_i} \left( \prod_{b=a+1}^{\bar{n}_i} -q f(a, b) \right) \Phi^m(a) \left( q^{l_i-1} Y_{\bar{n}_i+1+1} \right)^{-n} \otimes v_2 \]

\[ = q^{1-n_i} \sum_{a=\bar{n}_i+1}^{\bar{n}_i} \left( \prod_{b=a+1}^{\bar{n}_i} -q f(a, b) \right) \Phi^m(a) \otimes v_2 \]

\[ = q^{1-n_i} \sum_{\bar{n}_i+1 < a \leq \bar{n}_i} \left( \prod_{b=a+1}^{\bar{n}_i} -q f(a, b) \right) \Phi^m(a) \otimes v_2 \]

where

- \( m(a) = (\ldots, m_{\bar{n}_i}, m_a, m_{\bar{n}_i+1}, \ldots, m_{\bar{n}_i+1}, \ldots) \),
- \( \bar{m}(a) = (\ldots, m_{\bar{n}_i}, m_a, m_{\bar{n}_i+1}, \ldots, m_{\bar{n}_i+1}, \ldots) \),
- \( X_a = (\bar{\sigma}^{(a)} - 1, \sigma^{(a)} - 1) + k_a - c \) denote the top node on the \( a \)-th line of \( \lambda \), and
- \( n_i^+ = \bar{n}_i - a', n_i^- = a - \bar{n}_i - 1 \).

Since \( \varepsilon(m, j) - \varepsilon(m(a), j^-) = n_i^+ - n_i^- \) we have

\[ e_{i,n} \left( q^{\varepsilon(m, j)} \Phi^m \otimes v_2 \right) = \]

\[ \sum_{\bar{n}_i+1 < a \leq \bar{n}_i} \left( \prod_{b=a+1}^{\bar{n}_i} -f(a, b) \right) \Phi^m(a) \otimes v_2 \]

As in the proof of Proposition 6.3.1, we can arrange the right hand side of the above equation by classify the element of \( \{a+1, \ldots, \bar{n}_i\} \cup \{a' + 1, \ldots, \bar{n}_i - 1\} \) into three types, and finally we get

\[ e_{i,n}(\Psi^\lambda) = \sum_{X \in R_{\lambda,i}} \left( t' X \right)^n \left( \prod_{A \in A_{\lambda,i}} \frac{-(st)^{-1/2} X^A + (st)^{1/2} A^*}{X^A - A^*} \right) \left( \prod_{R \in R_{\lambda,i}} \frac{-(st)^{-1/2} R^A + (st)^{1/2} X^*}{R^A - X^*} \right) \Psi^\lambda \setminus X. \]
7.2 Normalizations

7.2.1
Definition. For \( \lambda \in \Pi \) we define
\[
N(\lambda) = \prod \left( s^{x_h-x_t} t^{y_h-y_t} - 1 \right),
\]
where the product runs over all l-hooks \(((x_h, y_h), (x_t, y_t))\).

We can easily verify the following lemma:
Lemma. If \( X \) is a removable i-node of \( \lambda \), then we have
\[
N(\lambda)/N(\lambda \setminus X) = \prod_{A \in A_{i,<} \lambda} \left( st_{i/2}^{(A)} X - 1 \right) \prod_{A \in A_{i,>} \lambda} \left( st_{i/2}^{(A)} X^* - 1 \right) \prod_{R \in R_{i,>} \lambda \setminus X} \left( st_{i/2}^{(R)} X^* - 1 \right) \prod_{R \in R_{i,<} \lambda \setminus X} \left( st_{i/2}^{(R)} X - 1 \right).
\]

Remark. From geometrical point of view, \( N(\lambda) \) is derived from the Koszul complex of the unstable manifold, with respect to a specific \( C^* \)-action, on which points converge to the fixed point \( \lambda \).

7.2.2
For \( \mu \in \Pi \) we will define \( M(\mu) \in \mathbb{R} \) inductively. First we set \( M(\emptyset) = 1 \). Let \( Y = (a, b) \) be the most right node of the top row of \( \mu \). Then we set
\[
M(\mu) = M(\mu \setminus Y) \left( s^{-1/2} t^{1/2} \right)^{d_{i-1}(\mu)} \prod_{A \in A_{i,<} \mu} \left( st_{i/2}^{(A)} \right) \prod_{R \in R_{i,>} \mu \setminus Y} \left( st_{i/2}^{(R)} \right)^{d_{i,>} \mu \setminus Y}.
\]

Lemma. If \( X \) is a removable i-node of \( \lambda \in \Pi \), then we have
\[
M(\lambda) = M(\lambda \setminus X) \left( s^{-1/2} t^{1/2} \right)^{d_{i-1}(\lambda)} \prod_{A \in A_{i,<} \lambda \setminus X} \left( st_{i/2}^{(A)} \right) \prod_{A \in A_{i,>} \lambda \setminus X} \left( st_{i/2}^{(A)} \right)^{d_{i,>} \lambda \setminus X} \prod_{R \in R_{i,>} \lambda \setminus X} \left( st_{i/2}^{(R)} \right)^{d_{i,>} \lambda \setminus X} \prod_{R \in R_{i,<} \lambda \setminus X} \left( st_{i/2}^{(R)} \right)^{d_{i,>} \lambda \setminus X}.
\]

Proof. We divide \( \lambda \) into \( \lambda_l \) and \( \lambda_r \) by the vertical line on the right of \( X \).
Let us write $\mu < \lambda$ if we can get $\mu$ from $\lambda$ by successive removing the nodes on the top of the most right line of diagrams. Take $\mu \in \Pi$ such that $\lambda_l \leq \mu < \lambda$.
Let $Y$ be the node on the top of the most right line of $\mu$.

Then we can verify

$$
\frac{M(\mu \setminus X)}{M(\mu Y)} / \frac{M(\mu \setminus (X \cap Y))}{M(\mu \setminus X)}
\begin{align*}
&= \left(s^{-1/2}t^{1/2}\right)^{\delta(j-1 \equiv i)} \left(st^{-1/2}A^\ast\right)^{\delta(j \equiv i)} \left((st)^{-1/2}A\right)^{-\delta(j \equiv i)} \\
&\times \left((st)^{-1/2}A^\ast\right)^{\delta(j \equiv i-1)} \left(st^{-1/2}A\right)^{\delta(j \equiv i+1)} \\
&= \left(s^{-1/2}t^{1/2}\right)^{\delta(j-1 \equiv i)} A^{\delta(j \equiv i-1) - 2\delta(j \equiv i) + \delta(j \equiv i+1)}.
\end{align*}
$$

So we have

$$
\frac{M(\lambda)}{M(\lambda_l \setminus X)} / \frac{M(\lambda_l \setminus X)}{M(\lambda_l \setminus X)}
\begin{align*}
&= \left(s^{-1/2}t^{1/2}\right)^{d_{i-1}(\lambda)} A^{\alpha_{i-1}(\lambda_l) - 2\alpha_i(\lambda_l) + \alpha_{i+1}(\lambda_l)}.
\end{align*}
$$

Let $j$ denote the content of the node on the bottom of the most left line be $j$ of $\lambda_r$. Note that we have $\delta(b \equiv 0) = \delta(j \equiv i)$ and

$$
|A_{\lambda_r,i}|-|R_{\lambda_r,i}| = \alpha_{i-1}(\lambda_r) - 2\alpha_i(\lambda_r) + \alpha_{i+1}(\lambda_r) + \delta(j \equiv i).
$$

Finally we have

$$
\frac{M(\lambda)}{M(\lambda \setminus X)}
\begin{align*}
&= \left(s^{-1/2}t^{1/2}\right)^{d_{i-1}(\lambda)} A^{\alpha_{i-1}(\lambda_r) - 2\alpha_i(\lambda_r) + \alpha_{i+1}(\lambda_r)} \\
&\times \left(s^{-1/2}t^{1/2}\right)^{d_{i-1}(\lambda)} A^{\delta(b \equiv 0)} \prod_{A \leq A_{\lambda,\gamma}} \left((st)^{-1/2}A\right) \prod_{R \in R_{\lambda \setminus X,\gamma}} \left((st)^{-1/2}R^\ast\right) \\
&= \left(s^{-1/2}t^{1/2}\right)^{d_{i-1}(\lambda)} A^{|A_{\lambda_r,i}| - |R_{\lambda_r,i}|} \prod_{A \leq A_{\lambda,\gamma}} \left((st)^{-1/2}A\right) \prod_{R \in R_{\lambda \setminus X,\gamma}} \left((st)^{-1/2}R^\ast\right).
\end{align*}
$$

So the claim follows. \hfill \Box

7.3 Main theorem

Now we arrive at the main theorem:

**Theorem.** $K^T_R(\mathfrak{m})$ and $F(0)$ is isomorphic as representations of $U'_R(\mathfrak{sl}_{1, \text{tor}})$. The isomorphism is given by

$$
N(\lambda)b_{\lambda} \longmapsto M(\lambda)\Psi^\lambda_{\Phi_0}.
$$

**Proof.** This follows from Theorem 7.2.2, Theorem 7.1.2, Lemma 7.2.1 and Lemma 7.2.2. \hfill \Box
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