Double Covers of Symplectic Dual Polar Graphs

G. Eric Moorhouse\textsuperscript{a,}\textsuperscript{*}, Jason Williford\textsuperscript{a}

\textsuperscript{a}Department of Mathematics, University of Wyoming, Laramie WY 82071 USA

Abstract

Let $\Gamma = \Gamma(2n, q)$ be the dual polar graph of type $Sp(2n, q)$. Underlying this graph is a $2n$-dimensional vector space $V$ over a field $\mathbb{F}_q$ of odd order $q$, together with a symplectic (i.e. nondegenerate alternating bilinear) form $B : V \times V \to \mathbb{F}_q$. The vertex set of $\Gamma$ is the set $V$ of all $n$-dimensional totally isotropic subspaces of $V$. If $q \equiv 1 \mod 4$, we obtain from $\Gamma$ a nontrivial two-graph $\Delta = \Delta(2n, q)$ on $V$ invariant under $PSp(2n, q)$. This two-graph corresponds to a double cover $\hat{\Gamma} \to \Gamma$ on which is naturally defined a $Q$-polynomial $(2n + 1)$-class association scheme on $2|\hat{V}|$ vertices.

Keywords: association scheme, $Q$-polynomial, symplectic group, two-graph, dual polar graph

1. Introduction

Association schemes [2, 6] were first defined by Bose and Mesner [5] in the context of the design of experiments. Philippe Delsarte used association schemes to unify the study of coding theory and design theory in his thesis [9], where he derived his well-known linear programming bound which has since found many applications in combinatorics. There he identified two types of association schemes which were of particular interest: the so-called $P$-polynomial and $Q$-polynomial schemes. Schemes which are $P$-polynomial are precisely those arising from distance-regular graphs, and are well studied. In particular, much effort has gone into the classification of distance-transitive graphs, the $P$-polynomial schemes which are the orbitals of a permutation group; and it is likely that all such examples are known. Also well-studied are the schemes which are both $Q$-polynomial and $P$-polynomial. A well-known conjecture [2, p.312] of Bannai and Ito is the following: for sufficiently large $d$, a primitive scheme is $P$-polynomial if and only if it is $Q$-polynomial.

Classification efforts for $Q$-polynomial schemes are far less advanced than in the $P$-polynomial case; in particular it is likely that more examples from permutation groups are yet to be found. The $Q$-polynomial property has no known

\textsuperscript{*}Corresponding author

Email addresses: moorhous@uwyo.edu (G. Eric Moorhouse), jwillif1@uwyo.edu (Jason Williford)
combinatorial characterization, making their study more difficult. However, the list of known examples (see [13, 15, 8]) indicates that these objects have interesting structure from the viewpoint of designs, lattices, coding theory and finite geometry.

In this paper, we give a new family of imprimitive \(Q\)-polynomial schemes with an unbounded number of classes. These schemes are formed by the orbitals of a group, giving a double cover of the scheme arising from the symplectic dual polar space graph. We note that only one other family of imprimitive \(Q\)-polynomial schemes with an unbounded number of classes is known that is not \(P\)-polynomial, namely the bipartite doubles of the Hermitian dual polar space graphs, which are \(Q\)-bipartite and \(Q\)-antipodal. The schemes in this paper are \(Q\)-bipartite, and have two \(Q\)-polynomial orderings. Except when the field order \(q\) is a square, the splitting field of these schemes is also irrational. We note that this is the only known family of \(Q\)-polynomial schemes with unbounded number of classes and an irrational splitting field. In the last section we give open parameters for hypothetical primitive \(Q\)-polynomial subschemes of this family.

Our paper is organized as follows: Background material on Gaussian coefficients, two-graphs and double covers of graphs, are covered in Sections 2–3. In Section 4 we recall the standard construction of the symplectic dual polar graph \(\Gamma = \Gamma(2n, q)\). There we also introduce the Maslov index, which we use in Section 5 to construct the double cover \(\hat{\Gamma} \to \Gamma\) when \(q \equiv 1 \mod 4\). In Section 6 we construct a \((2n+1)\)-class association scheme \(S = S_{n,q}\) from \(\hat{\Gamma}\); and in Section 7 we show that \(S\) is \(Q\)-polynomial. The \(P\)-matrix of the scheme is constructed in Section 8. A particularly tantalizing open problem is the question whether \(S\) is in general the extended \(Q\)-bipartite double of a primitive \(Q\)-polynomial scheme; see Section 9.

2. Gaussian coefficients

For all integers \(n, k\) we define the Gaussian coefficient

\[
\left[\begin{array}{c}
n \\
k
\end{array}\right]_q = \left[\begin{array}{c}
n \\
k
\end{array}\right]_q = \frac{(q^n-1)(q^{n-1}-1)\cdots(q^{n-k+1}-1)}{(q-1)(q^{k-1}-1)\cdots(q-1)}, \quad \text{if } k \geq 0;
\]

\[
= 0, \quad \text{if } k < 0.
\]

In particular for \(k = 0\) the empty product gives \([n]_0 = 1\). In later sections, \(q\) will be a fixed prime power; but here we may regard \(q\) as an indeterminate, so that for \(n \geq 0\), after cancelling factors we find \([n]_{k} \in \mathbb{Z}[q]\); and specializing to \(q = 1\) gives the ordinary binomial coefficients \([n]_k = (\binom{n}{k})\). For general \(n \in \mathbb{Z}\) we instead obtain a Laurent polynomial in \(q\) with integer coefficients, i.e. \([n]_k \in \mathbb{Z}[q, q^{-1}]\), as follows from conclusion (ii) of the following.

**Proposition 2.1.** Let \(n, k, \ell \in \mathbb{Z}\). The Gaussian coefficients satisfy

\[i) \quad [n]_k = q^k [n-k]_k + [n-1]_{k-1} + q^{n-k} [n-1]_{k-1};\]

\[ii) \quad \left[\begin{array}{c}
n \\
k
\end{array}\right]_q = \frac{(q^n-1)(q^{n-1}-1)\cdots(q^{n-k+1}-1)}{(q-1)(q^{k-1}-1)\cdots(q-1)}.
\]
(ii) \[ \binom{-n}{k} = (-q^{-n})^k \binom{n+k-1}{k}; \]

(iii) \[ \binom{n}{k} = \binom{n}{n-k}; \]

(iv) \[ \binom{n}{k} = \left[ n \atop n-k \right] \text{ whenever } 0 \leq k \leq n. \quad \square \]

Most of the conclusions of Proposition 2.1 are found in standard references such as [1]. However, our definition of \( \binom{n}{k} \) differs from the standard definition found in most sources, which either leave \( \binom{n}{k} \) undefined for \( n < 0 \), or define it to be zero in that case. Our extension to all \( n \in \mathbb{Z} \) means that the recurrence formulas (i) hold for all integers \( n, k \), unlike the ‘standard definition’ which fails for \( n = k = 0 \). Property (i) plays a role in our later algebraic proofs using generating functions. In further defense of our definition, we observe that it has become standard to extend the definition of binomial coefficients \( \binom{n}{k} \) so that \( \binom{-n}{k} = (-1)^k \binom{n+k-1}{k} \) (see e.g. [1, p.12]); and (ii) naturally generalizes this to Gaussian coefficients. We further note that (iii) holds for all \( n, k \in \mathbb{Z} \) whether one takes the standard definition of \( \binom{n}{k} \) or ours. The one advantage of the standard definition is that it renders superfluous the extra restriction \( 0 \leq k \leq n \) in the symmetry condition (iv). The interpretation of \( \binom{n}{k} \) as the number of \( k \)-subspaces of an \( n \)-space over \( \mathbb{F}_q \) is valid for all \( n \geq 0 \).

In Section 8 we will make use of the well-known generating polynomials

\[ E_m(t) = \prod_{i=0}^{m-1} (1 + q^i t) = \sum_{\ell=0}^{\infty} q^{\binom{\ell}{2}} \binom{m}{\ell} t^\ell \quad \text{for } m = 0, 1, 2, \ldots; \]

note that in the latter sum, the terms for \( \ell > m \) vanish, yielding \( E_m(t) \in \mathbb{Z}[q, t] \) (or after specializing to a fixed prime power \( q \), we obtain \( E_m(t) \in \mathbb{Z}[t] \)). Here we see the usual binomial coefficient \( \binom{\ell}{2} = \frac{1}{2} \ell(\ell - 1) \).

In Section 8 we will make use of the following obvious relations:

**Proposition 2.2.** For all \( m \geq 0 \), the generating function \( E_m(t) \) satisfies

(i) \[ E_m(-qt) = \frac{1 + q \binom{m}{m}}{1 + qt} E_m(-t); \]

(ii) \[ E_m(q^2 t) = \frac{1 + q \binom{m+1}{m+1}}{1 + qt} E_m(q \ell) \quad \text{and} \]

(iii) \[ E_m(r^3 t) = \frac{1 + r \binom{m}{m}}{1 + rt} E_m(rt) \quad \text{where } r = \sqrt[3]{q}. \quad \square \]

3. Two-graphs and double covers of graphs

Here we describe the most basic connections between two-graphs and double covers of graphs; see [14, 16, 6, 18] for more details. Our notation is chosen to conform to that used in subsequent sections.

Let \( \mathcal{V} \) be any set. Denote by \( \binom{\mathcal{V}}{k} \) the collection of all \( k \)-subsets of \( \mathcal{V} \) (i.e. subsets of cardinality \( k \)). A two-graph on \( \mathcal{V} \) is a subset \( \Delta \subseteq \binom{\mathcal{V}}{4} \) such that for every 4-set \( \{x, y, z, w\} \in \binom{\mathcal{V}}{4} \), an even number, i.e. 0, 2 or 4, of the triples

\[ x \Delta y, y \Delta z, z \Delta w, w \Delta x \]

form the collection of all \( k \)-subsets of \( \mathcal{V} \).
A graph on \( V \) is a subset \( \Gamma \subseteq \binom{V}{2} \). Elements of \( \Gamma \) are called edges. The complete graph on \( V \) is the graph \( K_V \) with full edge set \( \binom{V}{2} \). In general the complementary set of pairs \( \Gamma = \{\{x,y\} \in \binom{V}{2} : \{x,y\} \notin \Gamma\} \) is the complementary two-graph on \( V \).

Every graph on \( V \) may be identified with a signing of the edges of the complete graph \( K_V \), i.e. a function \( \sigma : \binom{V}{2} \to \{-1, 1\} \). Under this correspondence, the graph corresponding to \( \sigma \) has as its edge set \( \sigma^{-1}(1) = \{\{x,y\} \in \binom{V}{2} : \sigma(x,y) = 1\} \). (Here we abbreviate \( \sigma(\{x,y\}) = \sigma(x,y) \).)

Given \( \Gamma \) and \( \sigma \) as above (which amounts to two graphs which may be entirely unrelated except for sharing the same vertex set \( V \)), we construct a new graph \( \hat{\Gamma} = \hat{\Gamma}_\sigma \) with vertex set \( \hat{V} = V \times \{-1, 1\} \) and adjacency relation defined by

\[
(x,\varepsilon) \sim (y,\varepsilon') \iff x \sim y \text{ and } \varepsilon\varepsilon' = \sigma(x,y).
\]

(Note that \((x,1) \not\sim (x,-1) \) since \( \Gamma \) has no loops.) The map \((x,\varepsilon) \mapsto x \) is a double covering map \( \theta : \hat{\Gamma} \to \Gamma \), also called a double cover or simply a cover; and the fibers of this map are the pairs \( \theta^{-1}(x) = \{(x,1), (x,-1)\} \) where \( x \in V \).

By definition, a covering map of graphs is a graph homomorphism \( \theta : \hat{\Gamma} \to \Gamma \) such that for any vertex \( x \in \Gamma \), the preimage of the neighborhood graph \( \Gamma_x \) is isomorphic to a disjoint union of copies of \( \Gamma_x \); see e.g. [10]. ‘Double’ refers to the condition that the covering map is 2-to-1.) We also say that the vertices \((x,1) \) and \((x,-1) \) are antipodal with respect to the covering map. (Note that antipodal vertices must be at distance \( \geq 2 \); but we deviate from common custom by not requiring pairs of antipodal vertices to be at maximal distance \( \text{diam}\hat{\Gamma} \).)

We denote by \( \zeta \) the transposition interchanging antipodal vertices: \((x,1) \hat{\leftrightarrow} (x,-1) \). Denote by \( \text{Aut}_\zeta \hat{\Gamma} \subseteq \text{Aut} \hat{\Gamma} \) the subgroup consisting of all automorphisms of the graph \( \hat{\Gamma} \) which preserve the antipodality relation. In general, \( \text{Aut}_\zeta \hat{\Gamma} \) is the centralizer of \( \zeta \) in the full automorphism group \( \text{Aut} \hat{\Gamma} \subseteq \text{Sym} \hat{V} \); but in our case we obtain equality \( \text{Aut}_\zeta \hat{\Gamma} = \text{Aut} \hat{\Gamma} \) (see Lemma 5.4). Similarly, two covers \( \theta_i : \hat{\Gamma}_i \to \Gamma \) of the same graph \( \Gamma \) (for \( i = 1, 2 \)) are equivalent or isomorphic if there is a graph isomorphism \( \rho : \hat{\Gamma}_1 \to \hat{\Gamma}_2 \) which preserves antipodality, i.e. \( \theta_1 \circ \rho = \theta_2 \).

Given \( \sigma : \binom{V}{2} \to \{-1, 1\} \) as above, for every triple \( \{x,y,z\} \in \binom{V}{3} \) we may define

\[
\sigma(x,y,z) = \sigma(x,y)\sigma(y,z)\sigma(z,x) \in \{\pm 1\}.
\]

A triple \( \{x,y,z\} \in \binom{V}{3} \) is called coherent or non-coherent according as \( \sigma(x,y,z) = 1 \) or \( -1 \). The set of all coherent triples forms a two-graph on \( V \), denote by \( \Delta_\sigma \); and the set of non-coherent triples gives the complementary two-graph \( \overline{\Delta}_\sigma \).

Two sign functions \( \sigma_1, \sigma_2 : \binom{V}{2} \to \{-1, 1\} \) (or the corresponding graphs \( \sigma_1^{-1}(1), \sigma_2^{-1}(1) \) on \( V \)) are switching-equivalent in the sense of Seidel [16] if there exists a map \( f : V \to \{\pm 1\} \) such that \( \sigma_2(x,y) = f(x)f(y)\sigma_1(x,y) \) for all \( \{x,y\} \in \binom{V}{2} \).
We have $\Delta_{\sigma_1} = \Delta_{\sigma_2}$ iff $\sigma_1$ and $\sigma_2$ are switching-equivalent. Assuming this holds, then the corresponding covers $\hat{\Gamma}_{\sigma_1}$ and $\hat{\Gamma}_{\sigma_2}$ are isomorphic via $(x, \varepsilon) \mapsto (x, f(x)\varepsilon)$.

In the special case of the complete graph $\Gamma = K_V$, the following three notions are equivalent (see [6, §1.5]): two-graphs on $V$, switching classes of graphs on $V$, and isomorphism classes of double covers of the complete graph $K_V$. For example given a double cover $\hat{K}_V \rightarrow K_V$, the corresponding two-graph is obtained as follows (see [16, p.488]): Each triple $\{x, y, z\}$ of distinct vertices in $V$ induces a triangle $K_{\{x, y, z\}} \subseteq K_V$; and such a triple is coherent iff its preimage in $\hat{K}_V$ induces a pair of triangles, rather than a 6-cycle, in $\hat{K}_V$.

An automorphism of a two-graph $\Delta$ is a permutation of the underlying point set $V$ which preserves the set of coherent triples. We now relate $\text{Aut} \Delta$ to the group $\hat{\text{Aut}} \hat{\Delta} \leq \text{Aut} \hat{\Delta}$ defined above for the associated double cover $\hat{K} \rightarrow K$, where we abbreviate the complete graph $K_V = K$. The following is easy to verify (or see [18, §2], where this isomorphism is denoted $G/Z \cong G$):

**Proposition 3.1.** The group $\hat{\text{Aut}} \hat{\Delta}$ acts naturally on $\Delta$, inducing the full automorphism group of $\Delta$. The kernel of this action is the central subgroup $\langle \zeta \rangle$ of order 2; thus $(\hat{\text{Aut}} \hat{\Delta})/\langle \zeta \rangle \cong \text{Aut} \Delta$.

### 4. Dual polar graphs of type $Sp(2n, q)$, $q$ odd

Fix a finite field $\mathbb{F}_q$ of odd prime power order $q$; an integer $n \geq 1$; a 2n-dimensional vector space $V$ over $\mathbb{F}_q$; and a symplectic (i.e. nondegenerate alternating) bilinear form $B : V \times V \rightarrow \mathbb{F}_q$. The symplectic group $Sp(2n, q)$ consists of all (linear) isometries of $B$, i.e.

$$Sp(2n, q) = \{g \in GL(V) : B(x^g, y^g) = B(x, y) \text{ for all } x, y \in V\}.$$ 

The group of all (linear) similarities of $B$ is

$$GSp(2n, q) = \{g \in GL(V) : \text{for some nonzero } \mu \in \mathbb{F}_q \text{ we have } B(x^g, y^g) = \mu B(x, y) \text{ for all } x, y \in V\};$$

some other notations for this group are $GSp_n(q)$ in [12] or $CSp_n(q)$ in [4, p.31].

Replacing $GL(V)$ by $\Gamma L(V) \cong GL(V) \rtimes \text{Aut } \mathbb{F}_q$, the group of all semilinear transformations of $V$, we obtain the group $\Sigma Sp(2n, q)$ of all semi-isometries, and the group $\Gamma Sp(2n, q)$ of all semi-similarities of $B$, given by

$$\Sigma Sp(2n, q) = \{g \in \Gamma L(V) : \text{for some } \tau \in \text{Aut } \mathbb{F}_q \text{ we have } B(x^g, y^g) = B(x, y)^\tau \text{ for all } x, y \in V\} \cong Sp(2n, q) \rtimes \text{Aut } \mathbb{F}_q;$$

$$\Gamma Sp(2n, q) = \{g \in \Gamma L(V) : \text{for some nonzero } \mu \in \mathbb{F}_q \text{ and } \tau \in \text{Aut } \mathbb{F}_q \text{ we have } B(x^g, y^g) = \mu B(x, y)^\tau \text{ for all } x, y \in V\} \cong GSp(2n, q) \rtimes \text{Aut } \mathbb{F}_q.$$
The projective versions of these groups are

\[ PSp(2n, q) = Sp(2n, q)/(-I), \]
\[ PGSp(2n, q) = GSp(2n, q)/Z, \]
\[ PΣSp(2n, q) = ΣSp(2n, q)/(-I), \]
\[ PΓSp(2n, q) = ΓSp(2n, q)/Z \]

where the central subgroup \( Z \) of order \( q - 1 \) consists of all scalar transformations \( v \mapsto λv \) for \( 0 \neq λ \in \mathbb{F}_q \). We have

\[ [PΓSp(2n, q) : PΣSp(2n, q)] = [PGSp(2n, q) : PSp(2n, q)] = 2 \]

where the nontrivial coset in both cases is represented by \( h \in GSp(2n, q) \) satisfying \( B(u^h, v^h) = ηB(u, v) \) and \( η \in \mathbb{F}_q \) is a non-square.

Our choice of notation for these groups, while not universal, is intended to conform reasonably with [7, 12]. The group \( PΓSp(2n, q) \), for example, is denoted \( PCΓSp_n(q) \) in [4, p.31]. It arises (see Theorem 4.1) as the full automorphism group of the associated dual polar graph, which we now describe.

Denote by \( V \) be the collection of all maximal totally isotropic subspaces with respect to \( B \), i.e.

\[ V = \{ X \subseteq V : X^⊥ = X \} \]

where by definition \( X^⊥ = \{ v \in V : B(x, v) = 0 \text{ for all } x \in X \} \). Members of \( V \) are often called generators, and every \( X \in V \) has dimension \( n \). Denote by \( Γ = Γ(2n, q) \) the graph on \( V \) where two vertices \( X, Y \in V \) are adjacent iff \( X \cap Y \) has codimension 1 in both \( X \) and \( Y \). More generally, the distance between \( X \) and \( Y \) in \( Γ \) is \( d(X, Y) = k \in \{ 0, 1, 2, \ldots, n \} \) where the subspace \( X \cap Y \) has codimension \( k \) in both \( X \) and \( Y \). Let \( Γ_k \) denote the graph of the distance-\( k \) relation on \( V \); i.e. \( Γ_k \) has vertex set \( V \) and two vertices \( X, Y \in V \) are adjacent in \( Γ_k \) iff \( d(X, Y) = k \). The graph \( Γ_1 = Γ \) is called the dual polar graph of type \( Sp(2n, q) \). It is distance regular: given any two vertices \( X, Y \in Γ \) at distance \( k \in \{ 0, 1, 2, \ldots, n \} \), the vertex \( Y \) has \( q^{(n-k)}\binom{n}{k} \) neighbors \( Z \) in \( Γ \), of which

\[ a_k = q^k - 1 \text{ are at distance } k \text{ from } X, \]
\[ b_k = q^{k+1}\binom{n-k}{1} \text{ are at distance } k+1 \text{ from } X, \]
\[ c_k = \binom{k}{1} \text{ are at distance } k-1 \text{ from } X; \]

see [6, §9.4]. The edges of \( Γ_1, Γ_2, \ldots, Γ_n \) partition the non-identical pairs on \( V \), viewed as the edges of the complete graph \( K_V \); and together with the identity relation \( Γ_0 = \{ (X, X) : X \in V \} \) we obtain an \( n \)-class association scheme on \( V \) (see Section 6). This scheme is \( P \)-polynomial since \( Γ \) is distance regular; see [6].

**Theorem 4.1.** For \( n \geq 2 \), the full automorphism group of \( Γ = Γ(2n, q) \) is the group \( PΓSp(2n, q) \) acting naturally on the projective space of \( V \).

**Proof.** See [6, p.275] (where this group is however denoted \( PΣp(2n, q) \)).
Note that when \( n = 1 \), the dual polar graph \( \Gamma(2, q) \) is simply the complete graph \( K_{q+1} \), whose full automorphism group is the symmetric group of degree \( q + 1 \).

For use in Section 5 we record the following well-known fact. Although it follows easily from the axioms of polar geometry (or of near polygons), in the interest of self-containment we include a proof.

**Lemma 4.2.** The ‘diamond’ graph (as shown) is not an induced subgraph of the dual polar graph \( \Gamma \).

![Diagram of a diamond graph]

**Proof.** If \( X, Y, Z \) are mutually adjacent as shown, then \( X \cap Y \) and \( X \cap Z \) are distinct subspaces of codimension 1 in \( X \), so \( X = (X \cap Y) + (X \cap Z) \), whence \( X \subseteq Y + Z \). Thus \( X = X^\perp \supseteq Y^\perp \cap Z^\perp = Y \cap Z \). Similarly, \( W \supseteq Y \cap Z \). Now \( X \cap W \) contains a subspace of dimension \( n-1 \), contradicting \( d(X, W) \geq 2 \). \( \square \)

Now let \( X \) be any \( n \)-dimensional vector space over \( \mathbb{F}_q \). An \( n \)-linear form \( f : X^n \to \mathbb{F}_q \) (i.e. linear in each argument whenever the other \( n-1 \) arguments are fixed) is alternating if \( f(x_1, x_2, \ldots, x_n) = 0 \) whenever two \( x_i \)'s coincide; equivalently, \( f(x_\tau) = -f(x_1, x_2, \ldots, x_n) \) for every odd permutation \( \tau \) of the indices. The space of all such alternating forms is one-dimensional, and is canonically identified with \( (\wedge^n X)^* \), the dual space of \( \wedge^n X \). A determinant function on \( X \) is any nonzero alternating form \( X^n \to \mathbb{F}_q \). Since \( \dim(\wedge^n X)^* = 1 \), a determinant function is determined up to nonzero scalar multiple.

Fix a choice of determinant function \( \delta_X \) for each \( X \in \mathcal{V} \). Although these choices are not canonical, one may proceed by arbitrarily choosing a basis \( \psi_1, \psi_2, \ldots, \psi_n \) for \( X^* = \text{Hom}(X, \mathbb{F}_q) \); then we obtain a determinant function on \( X \) by defining

\[
\delta_X(x_1, x_2, \ldots, x_n) = \det(\psi_i(x_j) : 1 \leq i, j \leq n).
\]

We need to define \( \sigma(X, Y) \in \{ \pm 1 \} \) for any pair \( X \neq Y \) in \( \mathcal{V} \). Let \( k \in \{1, 2, \ldots, n\} \) be the codimension of \( X \cap Y \) in both \( X \) and \( Y \). Choose bases \( x_1, x_2, \ldots, x_n \) and \( y_1, y_2, \ldots, y_n \) for \( X \) and \( Y \) respectively, such that \( x_i = y_i \) (for \( k < i \leq n \)) is a common basis for \( X \cap Y \). (These bases depend on the choice of pair \( (X, Y) \) and so are unrelated to any bases for \( X \) and \( Y \) used as a crutch for constructing the corresponding determinant functions). Define

\[
\sigma(X, Y) = \chi(\delta_X(x_1, x_2, \ldots, x_n)\delta_Y(y_1, y_2, \ldots, y_n) \det[B(x_i, y_j) : 1 \leq i, j \leq k])
\]

where \( \chi : \mathbb{F}_q^* \to \{ \pm 1 \} \) is the quadratic character: \( \chi(a) = 1 \) or \(-1\) according as \( a \in \mathbb{F}_q^* \) is a square or a nonsquare. This definition is implicit in [19, 11]; and inspired by the literature, we refer to \( \sigma(X, Y) \) (or the ternary function \( \sigma(X, Y, Z) \)

\[
\text{Proof.}
\]
defined below) as the \textit{Maslov index}. Note that $B$ induces a nondegenerate bilinear form on the $2k$-space $(X + Y)/(X \cap Y)$, so that the $k \times k$ matrix $[B(x_i, y_j) : 1 \leq i, j \leq k]$ is nonsingular.

**Proposition 4.3.** Let $X, Y \in \mathcal{V}$ at distance $d(X, Y) = k \in \{0, 1, 2, \ldots, n\}$.

(i) The value of $\sigma(X, Y)$ is independent of the choice of bases $x_i$ and $y_j$ as above.

(ii) Its dependence on the choice of determinant functions is expressed as follows: Replacing $\delta_X$ by $c\delta_X$ has the effect of multiplying $\sigma(X, Y)$ by $\chi(c)$.

(iii) $\sigma(Y, X) = \chi(-1)^k \sigma(X, Y) = (-1)^{k(q-1)/2} \sigma(X, Y)$.

(iv) Let $g \in \Gamma Sp(2n, q)$, so that there exists a nonzero scalar $\mu_g \in \mathbb{F}_q$ and $\tau_g \in \text{Aut} \mathbb{F}_q$ satisfying $B(x^g, y^g) = \mu_g B(x, y)^{\tau_g}$ for all $x, y \in Y$. Then there exist nonzero scalars $\lambda_{g, U} \in \mathbb{F}_q$ for $U \in \mathcal{V}$, such that

$$\sigma(X^g, Y^g) = \chi(\mu_g^n \lambda_{g, X} \lambda_{g, Y}) \sigma(X, Y).$$

**Proof.** Consider a change of basis on $X \cap Y$ specified by $x'_i = y'_i = \sum_{k \leq j \leq n} a_{ij} x_j$ where $A = (a_{ij} : k < i, j \leq n)$ is any invertible $(n-k) \times (n-k)$ matrix. Then

$$\delta_X(x_1, \ldots, x_k, x'_{k+1}, \ldots, x'_n) = (\det A) \delta_X(x_1, \ldots, x_n)$$

and $\delta_Y(y_1, \ldots, y_n)$ is multiplied by the same factor, $\det A$. The $(n-k) \times (n-k)$ matrix $[B(x_i, y_j) : k < i, j \leq n]$ is unchanged, so the value of $\sigma(X, Y)$ is multiplied by a net factor of $\chi((\det A)^2) = 1$.

Next consider replacing $x_1, \ldots, x_k$ by $x'_1, \ldots, x'_k$ where

$$x'_i = \sum_{1 \leq j \leq k} a_{ij} x_j \mod (X \cap Y)$$

for $i = 1, 2, \ldots, k$ where $A$ is an invertible $k \times k$ matrix, and we leave the basis of $Y$ unchanged. Then

$$\delta_X(x'_1, \ldots, x'_k, x'_{k+1}, \ldots, x'_n) = (\det A) \delta_X(x_1, \ldots, x_n);$$

$$\det [B(x'_i, y_j) : 1 \leq i, j \leq k] = (\det A) \det [B(x_i, y_j) : 1 \leq i, j \leq k]$$

and the $\delta_Y$ factor is unchanged; so once again, the value of $\sigma(X, Y)$ is multiplied by $\chi((\det A)^2) = 1$. The same argument applies if $y_1, \ldots, y_k$ are replaced by $y'_1, \ldots, y'_k$, and so (i) follows. Conclusion (ii) is clear.

Interchanging $X$ and $Y$ has the effect of interchanging the $\delta_X$ and $\delta_Y$ factors, and replacing

$$[B(x_i, y_j) : 1 \leq i, j \leq k] \Rightarrow [B(y_i, x_j) : 1 \leq i, j \leq k] = -[B(x_i, y_j) : 1 \leq i, j \leq k].$$
The determinant of this matrix accrues a factor of \((-1)^k\), whence (iii) holds.

Let \( g \in \Gamma Sp(2n,q) \). There exists a nonzero \( \mu_g \in \mathbb{F}_q \) and \( \tau_g \in \text{Aut} \mathbb{F}_q \) such that \((au + bv)^g = a^{\tau_g}u^g + b^{\tau_g}v^g\) and \( B(u^g, v^g) = \mu_g B(u, v)^{\tau_g} \) for all \( a, b \in \mathbb{F}_q \) and \( u, v \in V \). Now the map

\[
X^n \rightarrow \mathbb{F}_q, \quad (x_1, x_2, \ldots, x_n) \mapsto \delta_X(x_1^n, x_2^n, \ldots, x_n^n)^{\tau_g^{-1}}
\]

is a determinant function on \( X \), so it is a scalar multiple of \( \delta_X(x_1, x_2, \ldots, x_n) \).

Hence there exists a nonzero scalar \( \lambda_X = \lambda_{g,X} \in \mathbb{F}_q \) such that

\[
\delta_X(x_1^n, x_2^n, \ldots, x_n^n) = \lambda_{g,X} \delta_X(x_1, x_2, \ldots, x_n)^{\tau_g}
\]

for all \( x_1, x_2, \ldots, x_n \in X \).

Now given \( X, Y \in \mathcal{V} \) at distance \( k \), fix bases \( x_i, y_i \) as before; then

\[
\sigma(X^g, Y^g) = \chi(\delta_X(x_1^g, x_2^g, \ldots, x_n^g)\delta_Y(y_1^g, y_2^g, \ldots, y_n^g)) \times \det [B(x_i^g, y_j^g) : 1 \leq i, j \leq k] \]

\[
= \chi(\lambda_{g,X} \delta_X(x_1, x_2, \ldots, x_n)^{\tau_g} \lambda_{g,Y} \delta_Y(y_1, y_2, \ldots, y_n)^{\tau_g}) \times \det [\mu_g B(x_i, y_j)^{\tau_g} : 1 \leq i, j \leq k] \]

\[
= \chi(\mu_g^k \lambda_{g,X} \lambda_{g,Y}) \chi(\delta_X(x_1, x_2, \ldots, x_n)^{\tau_g} \delta_Y(y_1, y_2, \ldots, y_n)^{\tau_g}) \times \det [B(x_i, y_j) : 1 \leq i, j \leq k]^{\tau_g} \]

\[
= \chi(\mu_g^k \lambda_{g,X} \lambda_{g,Y}) \sigma(X, Y)
\]

since \( \chi(a^g) = \chi(a) \). This proves (iv). \( \square \)

For each triple \((X,Y,Z)\) with distinct \( X, Y, Z \in \mathcal{V} \), define

\[
\sigma(X, Y, Z) = \sigma(X, Y) \sigma(Y, Z) \sigma(Z, X) \in \{\pm 1\}.
\]

A triple \((X,Y,Z)\) of distinct elements of \( \mathcal{V} \) is coherent or non-coherent according as \( \sigma(X, Y, Z) = 1 \) or \(-1\).

**Theorem 4.4.** Suppose \( q \equiv 1 \mod 4 \). Then the set of coherent triples forms a two-graph \( \Delta_\sigma \) on \( \mathcal{V} \), invariant under \( \Gamma \Sigma Sp(2n,q) \).

**Proof.** Let \( X, Y, Z, W \in \mathcal{V} \) be distinct. Since \( \chi(-1) = 1 \), \((X, Y, Z)\) is coherent iff any permutation of its members yields a coherent triple; so the set of coherent triples may be regarded as a collection of unordered triples \( \{X, Y, Z\} \). Since

\[
\sigma(X, Y, Z) \sigma(X, Y, W) \sigma(X, Z, W) \sigma(Y, Z, W)
\]

\[
= \sigma(X, Y)^2 \sigma(X, Z)^2 \sigma(Z, W)^2 = 1,
\]

evenly many of the triples in \( \{X, Y, Z, W\} \) are coherent. If \( g \in \Gamma Sp(2n,q) \) with \( B(x^g, y^g) = \mu_g B(x, y)^{\tau_g} \), then

\[
\sigma(X^g, Y^g, Z^g) = \chi(\mu_g^{d(X,Y)} \lambda_{g,X} \lambda_{g,Y} \lambda_{g,Y} \lambda_{g,Y}) \chi(\mu_g^{d(Y,Z)} \lambda_{g,Y} \lambda_{g,Y})
\]

\[
\times \chi(\mu_g^{d(Z,X)} \lambda_{g,Z} \lambda_{g,Z}) \sigma(X, Y, Z)
\]

\[
= \chi(\mu_g^{d(X,Y)+d(Y,Z)+d(Z,X)}) \sigma(X, Y, Z).
\]

In particular when \( g \in \Sigma Sp(2n,q) \), \( \mu_g = 1 \) and \( \sigma(X^g, Y^g, Z^g) = \sigma(X, Y, Z) \). \( \square \)
If $q \equiv 3 \mod 4$, or $g \in P\Sigma Sp(2n, q)$ with $g \notin P\Sigma Sp(2n, q)$, the situation is a little trickier: various subsets of the coherent triples form either two-graphs or skew two-graphs in the sense of [14], invariant under $Sp(2n, q)$. We ignore this case here, and henceforth assume that

$q \equiv 1 \mod 4$.

We next show that in a geodesic path, every triple of vertices is coherent.

**Lemma 4.5.** Suppose $q \equiv 1 \mod 4$. Let $X, Y, Z \in \mathcal{V}$ such that $d(X, Y) = j$, $d(Y, Z) = k - j$ and $d(X, Z) = k$ where $1 \leq j < k \leq n$. Then $\sigma(X, Y, Z) = 1$.

**Proof.** Choose a hyperbolic basis $e_1, e_2, \ldots, e_n, f_1, f_2, \ldots, f_n$ for $V$, so that $B(e_i, e_j) = B(f_i, f_j) = 0$ and $B(e_i, f_j) = \delta_{ij}$. Since $Sp(2n, q)$ is transitive on triples of generators satisfying the given distance constraints, by Theorem 4.4 we may suppose that

$$X = \langle e_1, e_2, \ldots, e_n \rangle, \quad Y = \langle f_1, f_2, \ldots, f_j, e_{j+1}, e_{j+2}, \ldots, e_n \rangle,$$

$$Z = \langle f_1, f_2, \ldots, f_k, e_{k+1}, e_{k+2}, \ldots, e_n \rangle.$$

We choose the determinant function $\delta_X$ on $X$ given by

$$\delta_X(x_1, x_2, \ldots, x_n) = \det [B(x_i, f_j) : 1 \leq i, j \leq n] \quad \text{for} \quad x_1, x_2, \ldots, x_n \in X;$$

this is nothing other than the determinant of the $n \times n$ matrix whose columns are the coordinates of $x_1, \ldots, x_n$ with respect to the basis $e_1, e_2, \ldots, e_n$. The determinant functions $\delta_Y, \delta_Z$ on $Y$ and on $Z$ are defined similarly, using the bases for $Y$ and on $Z$ listed above. The computation of $\sigma(X, Z)$ is simplified by the fact that a basis for $X \cap Z$ is $e_{k+1}, e_{k+2}, \ldots, e_n$. We have

$$\delta_X(e_1, e_2, \ldots, e_n) = \delta_Z(f_1, \ldots, f_k, e_{k+1}, \ldots, e_n) = 1$$

and $[B(e_i, f_j) : 1 \leq i, j \leq k]$ is a $k \times k$ identity matrix, with determinant 1; thus $\sigma(X, Z) = 1$. Exactly the same reasoning gives $\sigma(X, Y) = \sigma(Y, Z) = 1$, so $\sigma(X, Y, Z) = 1$. \hfill $\square$

In the case of triples $X, Y, Z$ not lying on geodesic paths, however, $\sigma$ (or its two-graph $\Delta_\sigma$) yields interesting nontrivial information. In particular, the restriction of $\Delta_\sigma$ to partial spreads (sets of vertices of $\Gamma$ mutually at distance $n$) was investigated in [14, §6]. Here we consider triangles in $\Gamma$:

**Lemma 4.6.** Suppose $q \equiv 1 \mod 4$. Let $X, Y \in \mathcal{V}$ such that $d(X, Y) = 1$, i.e. $X$ and $Y$ are adjacent in $\Gamma$. There are $a_1 = q - 1$ common neighbors $Z$ of $X$ and $Y$ in $\Gamma$, and exactly half of the resulting triples $(X, Y, Z)$ are coherent.

**Proof.** Choose a hyperbolic basis $e_i, f_i$ as in the proof of Lemma 4.5. Again without loss of generality,

$$X = \langle e_1, e_2, \ldots, e_n \rangle, \quad Y = \langle f_1, e_2, \ldots, e_n \rangle, \quad Z = \langle e_1 + \alpha f_1, e_2, \ldots, e_n \rangle$$
where \(0 \neq \alpha \in \mathbb{F}_q\). The \(q-1\) choices of \(\alpha\) give exactly the \(q-1\) common neighbors of \(X\) and \(Y\) in \(\Gamma\). These bases of \(X, Y, Z\) give rise to natural choices of determinant functions \(\delta_X, \delta_Y, \delta_Z\) as described in the proof of Lemma 4.5.

When computing \(\sigma(X, Y), \sigma(Y, Z), \sigma(Z, X)\), we use \(e_2, e_3, \ldots, e_n\) as the basis of \(X \cap Y = X \cap Z = Y \cap Z\). Now

\[
\delta_X(e_1, e_2, \ldots, e_n) = \delta_Z(e_1 + \alpha f_1, e_2, \ldots, e_n) = 1
\]

and \(B(e_1, e_1 + \alpha f_1) = \alpha\), so \(\sigma(X, Z) = \chi(\alpha)\). Similarly, \(\sigma(X, Y) = \sigma(Y, Z) = 1\) and

\[
\sigma(X, Y, Z) = \chi(\alpha).
\]

Since exactly half the nonzero elements of \(\mathbb{F}_q\) are squares, the result follows. \(\square\)

**Theorem 4.7.** Suppose \(q \equiv 1 \mod 4\). Let \(X, Y \in \mathcal{V}\) such that \(d(X, Y) = k\). Then \(Y\) has exactly \(a_k = q^k - 1\) neighbors \(Z \in \mathcal{V}\) at distance \(k\) from \(X\) in \(\Gamma\); and exactly half of the resulting triples \((X, Y, Z)\) are coherent.

**Proof.** The result holds for \(k = 1\) by Lemma 4.6, so we may assume \(k \geq 2\). Given \(X, Y \in \mathcal{V}\) with \(d(X, Y) = k\), there are \([k]\) choices of hyperplane \(H < Y\) containing \(X \cap Y\). Each such \(H\) yields an \(\text{Sp}(2, q)\)-space \(H^+ / H\), which contains \(q+1\) subspaces of the form \(Z / H\) with \(Z \in \mathcal{V}\). One such \(Z\) has distance \(k-1\) from \(X\), this being the subspace \(W = (Y \cap Z) + X \cap (Y + Z) = (Y + Z) \cap (X + (Y \cap Z))\). If we exclude \(W\) and \(Y\) itself, this leaves exactly \(q-1\) choices of \(Z\) having the required distances from \(X\) and \(Y\); and this gives \((q-1)[k] = q^k - 1 = a_k\) choices of \(Z\), the full number. But for how many such \(Z\) is the resulting triple \((X, Y, Z)\) coherent? In each case \(\sigma(X, W, Y) = \sigma(X, W, Z) = 1\) by Lemma 4.5; therefore \(\sigma(X, Y, Z) = \sigma(W, Y, Z)\). But by Lemma 4.6, given \(W, Y\) at distance 1, exactly half of the \(q-1\) choices of \(Z\) yield coherent triples \((W, Y, Z)\). Therefore among the \(a_k = (q-1)[k]\) triples \((X, Y, Z)\) with fixed \(X\) and \(Y\), exactly \(\frac{a_k}{2}[k]\) = \((q^k - 1)/2\) such triples are coherent. \(\square\)

5. The Double Cover \(\hat{\Gamma} \to \Gamma\)

The resulting double cover \(\hat{\Gamma} = \hat{\Gamma}(2n, q) \to \Gamma(2n, q)\) has vertex set \(\hat{\mathcal{V}} = \mathcal{V} \times \{\pm 1\}\) and adjacency relation

\[
(X, \varepsilon) \sim (Y, \varepsilon') \iff d(X, Y) = 1 \text{ and } \varepsilon \varepsilon' = \sigma(X, Y).
\]

The covering map is given by \((X, \varepsilon) \mapsto X\).

**Theorem 5.1.** Every geodesic path

\[
X_0 \sim X_1 \sim \cdots \sim X_k
\]

in \(\Gamma\) (meaning that \(d(X_i, X_j) = |j - i|\)) lifts to exactly two paths

\[
(X_0, \varepsilon_0) \sim (X_1, \varepsilon_1) \sim \cdots \sim (X_k, \varepsilon_k)
\]

in \(\hat{\Gamma}\), in which \(\varepsilon_k = \varepsilon_0 \sigma(X_0, X_k)\) for each \(k \geq 1\); thus any one of the \(\varepsilon_i\) determines all the others along this path.
PROOF. We have $\epsilon_1 = \epsilon_0 \sigma(X_0, X_1)$ by definition of adjacency in $\widehat{\Gamma}$. Assuming that $\epsilon_i = \epsilon_0 \sigma(X_0, X_i)$ for some $i \in \{1, 2, \ldots, k-1\}$,

$$\epsilon_{i+1} = \epsilon_i \sigma(X_i, X_{i+1}) = \epsilon_0 \sigma(X_0, X_i) \sigma(X_i, X_{i+1}) = \epsilon_0 \sigma(X_0, X_{i+1})$$

since $\sigma(X_0, X_i, X_{i+1}) = 1$ by Lemma 4.5.

However, not every geodesic path in $\widehat{\Gamma}$ is obtained by lifting a geodesic path in $\Gamma$. For example if $(X, Y, Z)$ is an incoherent triangle in $\Gamma$, say with $\sigma(X, Y) = \epsilon$, $\sigma(Y, Z) = \epsilon'$ and $\sigma(X, Z) = -\epsilon \epsilon'$, then

$$(X, 1) \sim (Y, \epsilon) \sim (Z, \epsilon \epsilon') \sim (X, -1)$$

is a geodesic path of length 3 in $\widehat{\Gamma}$, obtained by lifting a closed path of length 3 (not a geodesic path) in $\Gamma$.

**Lemma 5.2.** Let $X_0 \sim X_1 \sim \cdots \sim X_k$ be a geodesic path of length $k \geq 1$ in $\Gamma$, so that $d(X_i, X_j) = |i - j|$, and let $\epsilon, \epsilon' \in \{\pm 1\}$. Then $(X_0, \epsilon)$ and $(X_k, \epsilon')$ have distance $k$ or $k+1$ in $\widehat{\Gamma}$, according as $\sigma(X_0, X_k) = \epsilon \epsilon'$ or $-\epsilon \epsilon'$. In particular, the diameter of $\widehat{\Gamma}$ is $\max\{n+1, 3\}$.

**Proof.** If $\sigma(X_0, X_k) = \epsilon \epsilon'$, then we have a path

$$(X_0, \epsilon_0) \sim (X_1, \epsilon_1) \sim \cdots \sim (X_k, \epsilon_k)$$

in $\widehat{\Gamma}$ where $\epsilon_i = \epsilon_0 \sigma(X_0, X_i)$ for $i = 1, 2, \ldots, k$; in particular if $\epsilon_0 = \epsilon$ then $\epsilon_k = \epsilon'$. Clearly this path in $\widehat{\Gamma}$ is shortest possible.

Now suppose $\sigma(X_0, X_k) = -\epsilon \epsilon'$. We first obtain a path

$$(X_0, \epsilon) \sim (X_1, \epsilon_1) \sim \cdots \sim (X_{k-1}, \epsilon_{k-1})$$

in $\widehat{\Gamma}$ where $\epsilon_i = \epsilon_0 \sigma(X_0, X_i)$ for $i = 1, 2, \ldots, k-1$. Let $Y \in V$ be adjacent to both $X_{k-1}$ and $X_k$ in $\Gamma$, such that $\sigma(X_{k-1}, Y, X_k) = -1$. (By Lemma 4.6, there are $\frac{q-1}{2} \geq 1$ choices of such $Y \in V$.) Appending

$$(X_{k-1}, \epsilon_{k-1}) \sim (Y, \epsilon'') \sim (X_k, \epsilon'),$$

where $\epsilon'' = \epsilon_{k-1} \sigma(X_{k-1}, Y) = \epsilon' \sigma(Y, X_k)$, we obtain a path of length $k+1$ from $(X_0, \epsilon)$ to $(X_k, \epsilon')$ in $\widehat{\Gamma}$; once again this path is shortest possible.

The fibers of the covering map $\widehat{\Gamma} \rightarrow \Gamma$ are the antipodal pairs $\{(X, 1), (X, -1)\}$ for $X \in V$.

**Lemma 5.3.** Let $(X, \epsilon)$ and $(W, \epsilon')$ be any two vertices of $\widehat{\Gamma}$. Then $(X, \epsilon)$ and $(W, \epsilon')$ are antipodal iff they are at distance $3$ in $\widehat{\Gamma}$ and are joined by exactly $\frac{q}{2}(q^n-1)$ paths of length $3$ in $\widehat{\Gamma}$.
Moreover, let \((X, \varepsilon)\) and \((W, \varepsilon')\) be any two vertices at distance 3 in \(\hat{\Gamma}\). By Lemma 5.2, \(d(X, W) \in \{0, 2, 3\}\) in \(\Gamma\). Consider first the case that \(d(X, W) = 3\); then by Theorem 5.1, every geodesic path from \((X, \varepsilon)\) to \((W, \varepsilon')\) in \(\hat{\Gamma}\) arises from a unique geodesic path \(X \sim Y \sim Z \sim W\) in \(\Gamma\). There are exactly \(c_3 c_2 q_1 = (q^2 + q + 1)(q + 1)\) such geodesic paths from \(X\) to \(W\); and this number clearly cannot equal \(\frac{1}{2} q(q^n - 1)\).

Next suppose \(d(X, W) = 2\) in \(\Gamma\). For every geodesic path

\[(X, \varepsilon) \sim (Y, \varepsilon'') \sim (Z, \varepsilon''') \sim (W, \varepsilon')\]

in \(\hat{\Gamma}\), we have \(X \sim Y \sim Z \sim W\) in \(\Gamma\). Further, the condition \(d(X, W) = 2\) requires either \(X \sim Z\) or \(Y \sim W\) (but not both, by Lemma 4.2). We first count geodesic paths satisfying \(X \sim Z\), noting that the vertex \(W\) has \(c_2 = q + 1\) neighbors \(Z\) in common with \(X\); and in each case \(\sigma(X, Z, W) = 1\) by Lemma 4.5. Moreover \(Z\) has \(a_1 = q - 1\) neighbors \(Y\) in common with \(X\) (all of which satisfy \(\sigma(Y, Z, W) = 1\), again by Lemma 4.5). By the two-graph condition, we have \(\sigma(X, Y, Z) = -1\) if \(\sigma(X, Y, W) = -1\). By Lemma 4.6, for each \(Z\) there are exactly \(\frac{1}{2} (q - 1)\) choices of \(Y\) satisfying the latter condition; and each such pair \((Y, Z)\) yields a unique geodesic path \((X, \varepsilon) \sim (Y, \varepsilon'') \sim (Z, \varepsilon''') \sim (W, \varepsilon')\). We obtain \(q + 1\) \(\frac{1}{2} (q - 1) = \frac{1}{2} (q^2 - 1)\) geodesic paths in this case. There are another \(\frac{1}{2} q(q^n - 1)\) geodesic paths from \((X, \varepsilon)\) to \((W, \varepsilon')\) satisfying \(Y \sim W\), for a total of \(q^2 - 1\) geodesic paths. Once again, this number cannot equal \(\frac{1}{2} q(q^n - 1)\). \(\square\)

**Lemma 5.4.** \(\text{Aut} \, \hat{\Gamma} / \langle \zeta \rangle\) acts naturally on \(\Gamma\), with kernel \(\langle \zeta \rangle\), inducing a proper subgroup \(\text{Aut} \, \hat{\Gamma} / \langle \zeta \rangle < \text{Aut} \, \Gamma\).

**Proof.** By Lemma 5.3, \(\text{Aut} \, \hat{\Gamma}\) permutes fibres of the covering map \(\hat{\Gamma} \to \Gamma\), and so \(\text{Aut} \, \hat{\Gamma}\) acts naturally on \(\Gamma\). It remains to be shown that the induced subgroup \(\text{Aut} \, \hat{\Gamma} / \langle \zeta \rangle < \text{Aut} \, \Gamma\).

Choose a hyperbolic basis \(e_1, e_2, \ldots, e_n, f_1, f_2, \ldots, f_n\) for \(V\), so that \(B(e_i, e_j) = B(f_i, f_j) = 0\) and \(B(e_i, f_j) = \delta_{ij}\), and let \(\eta \in F_q\) be a nonsquare. Consider the subspaces \(X, Y, Z, Z' \in V\) defined by \(X = \langle e_1, e_2, \ldots, e_n \rangle\), \(Y = \langle f_1, e_2, \ldots, e_n \rangle\), \(Z = \langle e_1 + f_1, e_2, \ldots, e_n \rangle\) and \(Z' = \langle e_1 + \eta f_1, e_2, \ldots, e_n \rangle\). By straightforward computation, \(\sigma(X, Y, Z) = 1\) and \(\sigma(X, Y, Z') = -1\). Now consider \(g \in GL(V)\) mapping our original ordered basis to the new ordered basis \(e_1, \eta e_2, \ldots, \eta e_n, \eta f_1, f_2, \ldots, f_n\) so that \(B(u^g, v^g) = \eta B(u, v)\) for all \(u, v \in V\).
Proof. Suppose \((\hat{\hat{\sigma}}, \hat{\hat{\varepsilon}})\) is an automorphism of \(\hat{\hat{\Gamma}}(2n,q)\). However, \(\hat{\sigma}\) maps the coherent triple \(\{X,Y,Z\}\) to the non-coherent triple \(\{X,Y,Z'\}\) and so does not preserve \(\Delta_\sigma\). If \(g\) were induced by an automorphism of \(\hat{\hat{\Gamma}}\), this automorphism would map \(\{X,Y,Z\} \times \{\pm 1\}\) to \(\{X,Y,Z'\} \times \{\pm 1\}\). However, the induced subgraphs of \(\hat{\hat{\Gamma}}\) on these two 6-sets of vertices are not isomorphic (a pair of triangles and a 6-cycle, respectively; see Section 3).

The natural action of \(\Sigma \Sp(2n,q)\) on \(\mathcal{V}\) lifts to an action on \(\hat{\hat{\mathcal{V}}}\) as follows: Let \(g \in \Sigma \Sp(2n,q)\) with associated field automorphism \(\tau_g\) in the earlier notation of this section. Given \((U, \varepsilon) \in \hat{\hat{\mathcal{V}}}\), the map
\[
U^n \to \mathbb{F}_q, \quad (u_1, u_2, \ldots, u_n) \mapsto \delta_U(g)(u^g_1, u^g_2, \ldots, u^g_n)^{-1}
\]
is a determinant function; so there exists a nonzero constant \(\lambda_{g,U} \in \mathbb{F}_q\) such that
\[
\delta_U(g)(u^g_1, u^g_2, \ldots, u^g_n) = \lambda_{g,U} \delta_U(g)(u_1, u_2, \ldots, u_n)^{\tau_g}.
\]
Define \((U, \varepsilon)^g = (U^g, \lambda_{g,U,\varepsilon})\). One easily checks that this defines an action of \(\Sigma \Sp(2n,q)\) on \(\hat{\hat{\mathcal{V}}}\). The central element \(-I \in \Sp(2n,q)\) fixes every \(U \in \mathcal{V}\) and since
\[
\delta_U(-u_1, -u_2, \ldots, -u_n) = (-1)^n \delta_U(g)(u_1, u_2, \ldots, u_n)
\]
where \(\chi(-1)^n = 1\), \(-I\) acts trivially on \(\hat{\hat{\mathcal{V}}}\); thus \(\Sigma \Sp(2n,q)\) induces a permutation group \(\Sigma \Sp(2n,q)\) on \(\hat{\hat{\mathcal{V}}}\). The transposition \(\zeta\) which exchanges antipodal vertices via \((U, 1) \leftrightarrow (U, -1)\) is not induced by any element of \(P \Sigma \Sp(2n,q)\) since \(Z(P \Sigma \Sp(2n,q)) = 1\), so we obtain a permutation group \(\langle \zeta \rangle \times P \Sigma \Sp(2n,q)\) acting faithfully on \(\hat{\hat{\mathcal{V}}}\). We show that this permutation group preserves the graph \(\hat{\hat{\Gamma}}\), and in fact its full automorphism group:

**Theorem 5.5.** \(\Aut \hat{\hat{\Gamma}} \cong 2 \times P \Sigma \Sp(2n,q)\) where this group acts as defined above. The full automorphism group of the two-graph associated to \(\sigma\) is \(\Aut \Delta_\sigma \cong P \Sigma \Sp(2n,q)\).

**Proof.** Suppose \((X, \varepsilon) \sim (Y, \varepsilon')\) in \(\hat{\hat{\Gamma}}\), so that \(\sigma(X,Y) = \varepsilon\varepsilon'\); and let \(g \in \Sigma \Sp(2n,q)\) with \(\tau_g \in \Aut \hat{\hat{\Gamma}}\) as above. Then by Proposition 4.3(iv) we have
\[
\sigma(X^g, Y^g) = \chi(\lambda_{g,X} \lambda_{g,Y}) \sigma(X,Y) = (\chi(\lambda_{g,X}) \varepsilon)(\chi(\lambda_{g,Y}) \varepsilon')
\]
so that by definition, \((X, \varepsilon)^g \sim (Y, \varepsilon')^g\). Thus \(P \Sigma \Sp(2n,q)\), acting on \(\hat{\hat{\Gamma}}\) as defined above, preserves the graph \(\hat{\hat{\Gamma}}\). It is clear that the central factor \(\langle \zeta \rangle\) also preserves \(\hat{\hat{\Gamma}}\), so that \(\Aut \hat{\hat{\Gamma}}\) has a subgroup isomorphic to \(\langle \zeta \rangle \times P \Sigma \Sp(2n,q)\). Moreover by Proposition 3.1, \(\Aut \hat{\hat{\Gamma}}/\langle \zeta \rangle \cong \Aut \Delta_\sigma\). (We use the fact that by Lemma 5.4, \(\Aut \hat{\hat{\Gamma}} = \Aut \hat{\hat{\Gamma}}\) in the notation of Proposition 3.1.)

Suppose now that \(n \geq 2\), so that \(\Aut \hat{\hat{\Gamma}} \cong P \Gamma \Sp(2n,q)\) by Theorem 4.1. By Lemma 5.4, \(\Aut \hat{\hat{\Gamma}}\) acts on \(\hat{\hat{\Gamma}}\), inducing a group of automorphisms satisfying
\[
P \Sigma \Sp(2n,q) \leq \Aut \hat{\hat{\Gamma}}/\langle \zeta \rangle < P \Gamma \Sp(2n,q).
\]
This forces $\text{Aut} \hat{\Gamma} \cong \langle \zeta \rangle \times P\Sigma Sp(2n, q)$ and $\text{Aut} \Delta_{\sigma} \cong P\Sigma Sp(2n, q)$.

Finally suppose $n = 1$, so that $\Delta_{\sigma}$ is the Taylor-Paley two-graph on $q + 1$ vertices, with full automorphism group $\text{Aut} \Delta_{\sigma} \cong P\Sigma Sp(2, q) = P\Sigma L(2, q)$ by [18, Theorem 2]; see also [14, §4]. As above, $\text{Aut} \hat{\Gamma}$ has a subgroup isomorphic to $\langle \zeta \rangle \times P\Sigma Sp(2, q)$, and $\text{Aut} \hat{\Gamma}/\langle \zeta \rangle \cong \text{Aut} \Delta_{\sigma} \cong P\Sigma Sp(2, q)$, so we must have equality: $\text{Aut} \hat{\Gamma} \cong 2 \times P\Sigma Sp(2, q) = 2 \times P\Sigma L(2, q)$. $\Box$

6. The Association Scheme

From the double cover $\hat{\Gamma} \to \Gamma$, we now construct association schemes. As we will see in Section 7, this gives a new family of $Q$-polynomial association schemes. We begin with the relevant definitions, following [6, Chapter 2].

Let $\Omega$ be a finite set. A (symmetric) $d$-class association scheme on $\Omega$ is a pair $(\Omega, R)$ such that:

1. $R = \{R_0, \ldots, R_d\}$ is a partition of $\Omega \times \Omega$;
2. $R_0$ is the identity relation on $\Omega$;
3. $R_i = R_i^\top$ for $0 \leq i \leq d$; and
4. there are constants $p_{ij}^k$ such that for any pair $(x, y) \in R_k$, the number of $z \in \Omega$ such that $(x, z) \in R_i$ and $(z, y) \in R_j$ equals $p_{ij}^k$.

For the rest of this paper, all association schemes are symmetric (i.e. the third property above holds). Each relation $R_i$ has adjacency matrix $A_i$ defined by

$$(A_i)_{x,y} = \begin{cases} 1, & \text{if } (x, y) \in R_i; \\ 0, & \text{otherwise.} \end{cases}$$

The axioms above imply that $p_{ij}^k = p_{ij}^k$ and the matrices $A_0, \ldots, A_d$ form an algebra of symmetric matrices satisfying $A_i A_j = \sum_k p_{ij}^k A_k$. This matrix algebra is also closed under Schur (entrywise) multiplication, which we will denote by $\circ$. This algebra is referred to as the Bose-Mesner algebra $\mathfrak{A}$ of the association scheme.

Since $\mathfrak{A}$ is a commutative algebra consisting of symmetric matrices, its elements are simultaneously diagonalizable, and $\mathfrak{A}$ has a second basis consisting of primitive idempotents $E_0, \ldots, E_d$. We define the parameters $Q_{ij}$ by $E_j = \frac{1}{d} \sum_i Q_{ij} A_i$. Similarly we define the parameters $P_{ij}$ by the relation $A_i = \sum_j P_{ij} E_i$. The matrix $P$ of parameters $P_{ij}$ is often referred to as the character table of the scheme. The matrix $Q$ of parameters $Q_{ij}$ satisfies $Q = |\Omega| P^{-1}$.

We say an association scheme is $Q$-polynomial if, after suitably reindexing its idempotents, the idempotent $E_j$ is a degree $j$ polynomial in $E_1$ (where multiplication is done entrywise). This is equivalent to the condition that the $j$th column of the $Q$-matrix is a degree $j$ polynomial of the column 1 of the $Q$-matrix (note that we start indexing the columns at 0).

Permutation groups give many examples of association schemes. Let $G$ be a transitive permutation group acting on a finite set $\Omega$, and suppose the orbits of $G$ on $\Omega \times \Omega$ happen to be symmetric relations; such a group is called generously
transitive). It is not hard to check that the orbits of $G$ on $\Omega \times \Omega$ form an association scheme. We will refer to these schemes as Schurian schemes.

We will now construct a $(2n+1)$-class Schurian association scheme $S = S_{n,q}$ with vertex set $\hat{\mathcal{V}} = \mathcal{V} \times \{\pm 1\}$ of cardinality $|\hat{\mathcal{V}}| = 2|\mathcal{V}| = 2q^n \prod_{i=1}^{n}(q^{2i} - 1)$ using the Maslov index $\sigma$ and the double cover $\hat{\Gamma} \to \Gamma$ (defined as in Section 5).

For $k = 0, 1, 2, \ldots, n$, the $k$th and $(2n+1-k)$th relations are given by

$$R_k = \{((X, \varepsilon), (Y, \varepsilon')) \in \hat{\mathcal{V}} \times \hat{\mathcal{V}} : d(X, Y) = k, \varepsilon \varepsilon' = \sigma(X, Y)\};$$

$$R_{2n+1-k} = \{((X, \varepsilon), (Y, \varepsilon')) \in \hat{\mathcal{V}} \times \hat{\mathcal{V}} : d(X, Y) = k, \varepsilon \varepsilon' = -\sigma(X, Y)\}.$$

These are symmetric relations which clearly partition $\hat{\mathcal{V}} \times \hat{\mathcal{V}}$. In particular, $R_1$ is the adjacency relation of our graph $\hat{\Gamma}$ of Section 5; and the identity and antipodality relations are

$$R_0 = \{((X, \varepsilon), (X, \varepsilon)) : X \in \mathcal{V}, \varepsilon = \pm 1\};$$

$$R_{2n+1} = \{((X, \varepsilon), (X, -\varepsilon)) : X \in \mathcal{V}, \varepsilon = \pm 1\}.$$

We will write

$$(X, \varepsilon) \sim (Y, \varepsilon') \iff ((X, \varepsilon), (Y, \varepsilon')) \in R_i.$$

In the following, the parameters $a_i, b_i, c_i$ are those of the dual polar graph $\Gamma$ as given in Section 4.

**Lemma 6.1.** Let $(X, \varepsilon) \overset{k}{\sim} (Y, \varepsilon')$ where $k \in \{0, 1, 2, \ldots, 2n+1\}$. The number of $(Z, \varepsilon'') \in \hat{\mathcal{V}}$ such that $(X, \varepsilon) \overset{k}{\sim} (Z, \varepsilon'') \overset{l}{\sim} (Y, \varepsilon')$ is

$$p_{k,l} = \begin{cases} c_k = \binom{k}{1}, & \text{if } i = k - 1 \leq n; \\
\frac{1}{2}a_k = \frac{1}{2}(q^k - 1), & \text{if } i = k; \\
b_k = q^{k+1}\binom{n-k}{k}, & \text{if } i = k+1 \leq n+2; \\
b_{2n+1-k} = q^{2n+2-k}\binom{k-n-1}{1}, & \text{if } i = k-1 \geq n-1; \\
\frac{1}{2}a_{2n+1-k} = \frac{1}{2}(q^{2n+1-k} - 1), & \text{if } i = 2n+1-k; \\
c_{2n+1-k} = \binom{2n+1-k}{1}, & \text{if } i = k+1 \geq n+1; \\
0, & \text{otherwise.} \end{cases}$$

**Proof.** (i) First suppose $d(X, Y) = k \leq n$, so $\varepsilon \varepsilon' = \sigma(X, Y)$. Then $(Z, \varepsilon'') \in \hat{\mathcal{V}}$ satisfies $(X, \varepsilon) \overset{k}{\sim} (Z, \varepsilon'') \overset{l}{\sim} (Y, \varepsilon')$ if

$$\begin{cases} \text{case (i.a)} \\
i = d(X, Z) \leq n \\
d(Z, Y) = 1 \\
\varepsilon'' = \varepsilon \sigma(X, Z) = \varepsilon' \sigma(Y, Z) \end{cases} \quad \text{or} \quad \begin{cases} \text{case (i.b)} \\
i = 2n+1-d(X, Z) \geq n+1 \\
d(Z, Y) = 1 \\
\varepsilon'' = -\varepsilon \sigma(X, Z) = \varepsilon' \sigma(Y, Z) \end{cases}$$
Moreover, each such \((Z, \varepsilon'')\) satisfies \(d(X, Z) \in \{k-1, k, k+1\}\) by the triangle inequality.

There are exactly \(c_k = \binom{n}{1}\) choices of \(Z \in \mathcal{V}\) satisfying \(d(X, Z) = k-1\) and \(d(Z, Y) = 1\). Each such \(Z\) yields a coherent triple \((X, Y, Z)\) by Lemma 4.5, so \(\varepsilon\varepsilon' = \sigma(X, Y) = \sigma(X, Z)\sigma(Y, Z)\). This yields \(c_k\) pairs \((Z, \varepsilon'')\), all of which satisfy (i.a).

There are exactly \(b_k = q^{k+1} \binom{n-k}{1}\) choices of \(Z \in \mathcal{V}\) satisfying \(d(X, Z) = k+1\) and \(d(Z, Y) = 1\). Each such \(Z\) yields a coherent triple \((X, Y, Z)\) by Lemma 4.5, once again with \(\varepsilon\varepsilon' = \sigma(X, Y) = \sigma(X, Z)\sigma(Y, Z)\). This yields \(b_k\) pairs \((Z, \varepsilon'')\), all of which satisfy (i.a).

There are exactly \(a_k = q^k - 1\) choices of \(Z \in \mathcal{V}\) satisfying \(d(X, Z) = k\) and \(d(Z, Y) = 1\). By Theorem 4.7, exactly \(a_k/2\) of these \(Z\) yield coherent triples \((X, Y, Z)\), in which case \(\varepsilon\varepsilon' = \sigma(X, Y) = \sigma(X, Z)\sigma(Y, Z)\); this yields \(a_k/2\) pairs \((Z, \varepsilon'')\) satisfying (i.a). The remaining \(a_k/2\) of these \(Z\) yield incoherent triples \((X, Y, Z)\), with \(\varepsilon\varepsilon' = \sigma(X, Y) = -\sigma(X, Z)\sigma(Y, Z)\); and the resulting pairs \((Z, \varepsilon'')\) satisfy (i.b). □

In Section 5 we lifted the action of \(P\Sigma Sp(2n, q)\) on \(\mathcal{V}\), to a transitive permutation action of \((\zeta) \times P\Sigma Sp(2n, q)\) on \(\hat{\mathcal{V}}\) (below Lemma 5.4). Theorem 5.5 shows that this group preserves \(R_1\) (the adjacency relation of the graph \(\hat{\Gamma}\)). We next show that this group preserves each of the relations \(R_i\), and so gives the full automorphism group of the scheme.

**Lemma 6.2.** The diagonal action of \(2 \times PSp(2n, q)\) on \(\hat{\mathcal{V}} \times \hat{\mathcal{V}}\) preserves each of the relations \(R_i\). The same conclusion holds for the subgroup \(2 \times P\Sigma Sp(2n, q)\).

**Proof.** Clearly the central factor \((U, \varepsilon) \xrightarrow{\zeta} (U, -\varepsilon)\) preserves each \(R_i\). Now let \(g \in Sp(2n, q)\), and suppose \(X, Y \in \mathcal{V}\) such that \(d(X, Y) = k \in \{0, 1, 2, \ldots, n\}\). Also let \(\varepsilon, \varepsilon' \in \{\pm 1\}\), so that \((X, \varepsilon, (Y, \varepsilon')) \in \mathcal{R}_0\) or \(\mathcal{R}_{2n+1-k}\) according as \(\varepsilon\varepsilon'\sigma(X, Y) = 1\) or \(-1\). Since \(g\) preserves distances in \(\Gamma\), \(d(X^g, Y^g) = k\). Let \(x_1, x_2, \ldots, x_n\) and \(y_1, y_2, \ldots, y_n\) be bases for \(X\) and \(Y\) respectively, such that a basis for \(X \cap Y\) is formed by \(x_{k+1} = y_{k+1}, x_{k+2} = y_{k+2}, \ldots, x_n = y_n\). Then \((X, \varepsilon^g) = (X^g, \chi(\lambda_g, X)\varepsilon)\) and \((Y, \varepsilon')^g = (Y^g, \chi(\lambda_g, Y)\varepsilon')\) where

\[
\chi(\lambda_g, X)\varepsilon(X^g, Y^g) = \varepsilon\varepsilon'\chi(\lambda_g, X)\chi(x_1, \ldots, x_n)\chi(y_1, \ldots, y_n)
\]

\[
\times \det\left[B(x_i, y_j) : 1 \leq i, j \leq k\right]
\]

\[
= \varepsilon\varepsilon'\chi(x_1, \ldots, x_n)\chi(y_1, \ldots, y_n)
\]

\[
= \varepsilon\varepsilon'\chi(X, Y) .
\]

If this value is 1, then both \((X, \varepsilon) \sim (Y, \varepsilon')\) and \((X, \varepsilon^g) \sim (Y, \varepsilon')^g\); but if the latter value is \(-1\), then \((X, \varepsilon) \neq (Y, \varepsilon')\) and \((X, \varepsilon^g) \neq (Y, \varepsilon')^g\).

Thus \(2 \times PSp(2n, q)\) preserves the relations \(R_i\) as claimed. A similar argument holds for \(2 \times P\Sigma Sp(2n, q)\). □

17
It is easy to see that $\langle \zeta \rangle \times P\Sigma S\!p(2n, q)$ acts transitively on each $R_i$, and similarly for $\langle \zeta \rangle \times PSp(2n, q)$. This yields

**Theorem 6.3.** The diagonal action of the group $2 \times PSp(2n, q)$ on $\hat{V} \times \hat{V}$ has orbits $R_0, R_1, \ldots, R_{2n+1}$; so these form the relations of a $(2n+1)$-class Schurian association scheme. The same conclusion holds for $2 \times P\Sigma S\!p(2n, q)$, which is therefore the full automorphism group of the association scheme $\mathcal{S}$. \hfill $\Box$

### 7. The $Q$-polynomial property

In this section we will use some parameters of the scheme to prove that the association scheme $\mathcal{S}$ is $Q$-polynomial. We will benefit from the action of the $A_i$’s by left-multiplication on the Bose-Mesner algebra, resulting in matrices $L_i$ defined by $(L_i)_{kj} = p^k_{ij}$. In particular, the parameter $p^k_{ij}$ of the scheme from Lemma 6.1, is the $(k,j)$-entry of the matrix

$$L_1 = 
\begin{pmatrix}
0 & b_0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
b_0 & a_2 & b_1 & 0 & 0 & \cdots & 0 & 0 & a_2 & 0 \\
0 & c_2 & a_2 & \cdots & \cdots & 0 & \cdots & 0 & \cdots & 0 \\
0 & 0 & \cdots & \cdots & b_{d-1} & 0 & \frac{a_{d-1}}{2} & 0 & 0 & 0 \\
0 & 0 & c_d & \frac{a_d}{2} & \frac{a_d}{2} & \frac{a_d}{2} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & c_d & \frac{a_d}{2} & \frac{a_d}{2} & \frac{a_d}{2} & c_d & 0 & 0 \\
0 & 0 & 0 & a_{d-1} & 2 & b_{d-1} & \cdots & \cdots & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 & 0 & \frac{a_2}{2} & c_2 & 0 \\
0 & \frac{a_2}{2} & 0 & 0 & 0 & 0 & 0 & b_1 & \frac{a_2}{2} & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & b_0 & 0
\end{pmatrix}$$

As it turns out, this matrix has distinct eigenvalues, which in turn will give us a great deal of information about the scheme. In particular by [6, Proposition 2.2.2], the columns of $Q$ are right eigenvectors of $L_1$. We will use the following generalization of [6, Theorem 8.1.1] to prove that $\mathcal{S}$ is $Q$-polynomial.

**Theorem 7.1.** Suppose $A_i$ is a matrix in a $d$-class association scheme $(\Omega, \mathcal{R})$ with $d + 1$ distinct eigenvalues. Then $(\Omega, \mathcal{R})$ is $Q$-polynomial if and only if there is a sequence of distinct complex numbers $\sigma_0, \sigma_1, \ldots, \sigma_d$ and polynomials $s_0(x), s_1(x), \ldots, s_d(x)$ of degree $0, 1, \ldots, d$, respectively, with

$$\sum_j p^k_{ij} \sigma_j^\ell = s_\ell(\sigma_k)$$

for $0 \leq \ell \leq d$. Furthermore, the leading coefficients of the polynomials $s_0(x), s_1(x), \ldots, s_d(x)$ are precisely the eigenvalues of $A_i$ in a $Q$-polynomial ordering.

**Proof.** Without loss of generality we assume $A_1$ has this property. Let $L_1$ be the corresponding intersection matrix. Let $S, T$ be the $d+1$ by $d+1$ matrices with
whereas for even $S_{jk} = \sigma_j^k$ and $T_{jk}$ equal to the coefficient of $x^j$ in the polynomial $s_k(x)$. Then the above statement is equivalent to $L_1 S = ST$. Then $L_i$ is similar to $T$ and since $T$ is upper triangular, the diagonal entries of $T$ are precisely the eigenvalues of $L_1$. Since $T$ is an upper triangular matrix with distinct diagonal entries, an easy induction shows that it can be diagonalized by an upper triangular matrix. Namely, there is an invertible matrix $U$ and a diagonal matrix $D$ with $D_{jj} = T_{jj}$ such that $U^{-1}TU = D$. Then $L_1(SU) = (SU)D$. This implies that the columns of $SU$ are eigenvectors of $L_1$, hence there is a diagonal matrix $D'$ such that $SUD' = Q$. Since the $j$th column of $SUD'$ is a degree $j$ polynomial of the first column of $T$, which is a linear combination of columns 0 and 1 of $SUD'$, it is clear that the $j$th column of $SUD'$ is a degree $j$ polynomial of the first column of $SUD'$. This implies that the columns of $Q$ are in a given $Q$-polynomial ordering, which in turn implies that the ordering of the eigenvalues in $T$ is a $Q$-polynomial ordering. 

This leads to our main result:

**Theorem 7.2.** The scheme $S$ is $Q$-polynomial. Furthermore, it has two $Q$-polynomial orderings.

**Proof.** Let $r = \sqrt{n}$ and $d = 2n + 1$. We define the sequence of polynomials

$$s_k(x) = \begin{cases} r^\ell \left[ \binom{n+\ell+1}{1} x^\ell + \frac{r^\ell}{r^{\ell-2}} \left[ \binom{\ell-1}{1} x^{\ell-2} \right] \right], & \text{for } \ell \text{ odd;} \\
( r^\ell \left[ \binom{n+\ell+1}{1} - \frac{1}{r} \right] x^\ell + \frac{1}{r^{\ell-2}} \left[ \binom{\ell-1}{1} + 1 \right] x^{\ell-2} \right) x^\ell, & \text{for } \ell \text{ even}
\end{cases}$$

and constants

$$\sigma_j = \begin{cases} \frac{1}{r}, & \text{for } 0 \leq j \leq n; \\
-\frac{1}{x^j}, & \text{for } n + 1 \leq j \leq 2n + 1.
\end{cases}$$

The polynomials $s_0(x), \ldots, s_{2n+1}(x)$ realize $\sigma_0, \ldots, \sigma_{2n+1}$ as a $Q$-sequence for $S$, as we proceed to show by direct computation. For $k \leq n$ we have $\sum_j p^k_{j\ell} \sigma_j^\ell = c_k \sigma_{k-1}^\ell + \frac{a_k}{2} \sigma_k^\ell + b_k \sigma_{k+1}^\ell + \frac{a_k}{2} \sigma_{2n+1-k}^\ell$. For odd $\ell$ this reduces to

$$c_k \sigma_{k-1}^\ell + b_k \sigma_{k+1}^\ell = \frac{1}{r^{n+\ell+1}} \left[ \binom{n+\ell+1}{1} \right] + q \left( \binom{\ell}{1} - \binom{\ell+1}{1} \right) \frac{1}{r^{n+\ell+1}}$$

$$\frac{1}{r^{n+\ell+1}} \left[ \binom{n+\ell+1}{1} - \binom{\ell+1}{1} \right] + \frac{1}{r^{n+\ell+1}} \left[ \binom{\ell}{1} \right] q \left( \binom{\ell+1}{1} - \binom{\ell+2}{1} \right)$$

$$= \frac{1}{r^{n+\ell+1}} \left( \binom{\ell+1}{1} q + \binom{\ell+1}{1} \right)$$

$$= \frac{1}{r^{n+\ell+1}} \left[ \binom{n+\ell+1}{1} - \binom{\ell+1}{1} \right] + \frac{1}{r^{n+\ell+1}} \left[ \binom{\ell+1}{1} - \binom{\ell+2}{1} \right]$$

$$= \frac{1}{r^{n+\ell+1}} \left[ \binom{n+\ell+1}{1} - \binom{\ell+1}{1} \right] + \frac{1}{r^{n+\ell+1}} \left[ \binom{\ell+1}{1} - \binom{\ell+2}{1} \right]$$

$$= s_{\ell}(\sigma_k),$$

whereas for even $\ell$ we have

$$\sum_j p^k_{j\ell} \sigma_j^\ell = c_k \sigma_{k-1}^\ell + a_k \sigma_k^\ell + b_k \sigma_{k+1}^\ell$$

19
Now we deal with $k \geq n + 1$, noting that

$$\sum_j p_{1j}^k \sigma_j^\ell = b_{2n+1-k} \sigma_{k-1} + \frac{a_{2n+1-k}}{2} \sigma_k + c_{2n+1-k} \sigma_{k+1} + \frac{a_{2n+1-k}}{2} \sigma_{2n+1-k}.$$  

For odd $\ell$ this reduces to

$$b_{2n+1-k} \sigma_{k-1} + c_{2n+1-k} \sigma_{k+1} = -b_{2n+1-k} \sigma_{2n+2-k} - c_{2n+1-k} \sigma_{2n-k} = -s_\ell(\sigma_{2n+1-k}) = s_\ell(-\sigma_{2n+1-k}) = s_\ell(\sigma_k),$$

while for even $\ell$ we obtain

$$b_{2n+1-k} \sigma_{k-1} + a_{2n+1-k} \sigma_{k+1} = b_{2n+1-k} \sigma_{2n+2-k} + a_{2n+1-k} \sigma_{2n+1-k} + c_{2n+1-k} \sigma_{2n-k} = s_\ell(\sigma_{2n+1-k}) = s_\ell(-\sigma_{2n+1-k}) = s_\ell(\sigma_k).$$

For nonsquare $q$ the splitting field of $S$ is irrational, implying that it is a quadratic extension of the rationals, namely $\mathbb{Q}(r)$. The Galois group acts faithfully on the idempotents of the scheme, yielding a second $Q$-polynomial ordering. This second $Q$-polynomial ordering can also be obtained by replacing $r \mapsto -r$ in both the $\sigma_i$ and the polynomials $s_\ell(x)$, showing that this second ordering exists for square $q$ as well.

We note that by a result of Suzuki [17], $Q$-polynomial schemes can have at most two $Q$-polynomial orderings.

8. The $P$-matrix

We now compute the $P$-matrix of the scheme $S$, expressing it in terms of the auxiliary matrices $P$ and $\hat{P}$ whose entries are defined by

$$\hat{P}_{ij} = \sum_{\ell=0}^{j} (-1)^{j+2\ell+(j-\ell)^2+\ell^2} \frac{[n-i]}{[j-\ell]};$$

$$\hat{P}_{ij} = \sum_{\ell=0}^{j} (-1)^{j+2\ell+(j-\ell)^2} \frac{[n-i]}{[j-\ell]}.$$
By [6, Proposition 2.2.2], the $P$-matrix is determined by the left-normalized left eigenvectors of $L_1$. We first show that the rows of $\tilde{P}$ and $\hat{P}$ are left eigenvectors of the matrices defined by

$$
\tilde{M} = \begin{pmatrix}
0 & b_0 \\
1 & a_1 & b_1 \\
& c_2 & a_2 & \ddots \\
& & & \ddots & \ddots \\
& & & & \ddots & b_{d-1} \\
& & & & & c_d & a_d
\end{pmatrix},
\hat{M} = \begin{pmatrix}
0 & b_0 \\
1 & 0 & b_1 \\
& c_2 & 0 & \ddots \\
& & & \ddots & \ddots \\
& & & & \ddots & 0 & b_{d-1} \\
& & & & & c_d
\end{pmatrix}
$$

respectively. We will show that the corresponding diagonal forms are

$$
\tilde{D} = \text{diag}(\tilde{P}_0, \tilde{P}_1, \ldots, \tilde{P}_n), \quad \hat{D} = \text{diag}(\hat{P}_0, \hat{P}_1, \ldots, \hat{P}_n).
$$

The ordering we give to the eigenvectors of $\tilde{M}$ and $\hat{M}$ may seem arbitrary, but will be important later.

**Theorem 8.1.** $\tilde{P}\tilde{M} = \tilde{D}\tilde{P}$ and $\hat{P}\hat{M} = \hat{D}\hat{P}$.

**Proof.** Fix $i$ and let $v_i = (\tilde{P}_0, \tilde{P}_1, \ldots, \tilde{P}_n)$. We must show that $v_i \tilde{M} = \tilde{P}_i v_i$. In particular, we need to show the following recurrence holds for all $j$:

$$
b_{j-1} \sum_{\ell=0}^{j-1} (-1)^\ell r^{j-1-2\ell+(j-1-\ell)^2+\ell^2} [n-i] [j-1-\ell] + a_j \sum_{\ell=0}^{j} (-1)^\ell r^{j-2\ell+(j-\ell)^2+\ell^2} [n-i] [j-\ell]
$$

$$+ c_{j+1} \sum_{\ell=0}^{j+1} (-1)^\ell r^{j+1-2\ell+(j+1-\ell)^2+\ell^2} [i] [n-i] [j+1-\ell]
$$

$$= \tilde{P}_i \sum_{\ell=0}^{j} (-1)^\ell r^{j-2\ell+(j-\ell)^2+\ell^2} [n-i] [j-\ell].
$$

Multiplying both sides by $q-1$ and substituting for $b_{j-1}, a_j$ and $c_{j+1}$, we find this is equivalent to showing that the quantity $z_j$, defined as follows, vanishes for all $j$:

$$
z_j = (q^{n+1} - q^j) \sum_{\ell=0}^{j-1} (-1)^\ell r^{j-2\ell-1+(j-\ell-1)^2+\ell^2} [n-i] [j-\ell-1]
$$

$$+ (q-1)(q^j - 1) \sum_{\ell=0}^{j} (-1)^\ell r^{j-2\ell+(j-\ell)^2+\ell^2} [n-i] [j-\ell]
$$

$$+ (q^{j+1} - 1) \sum_{\ell=0}^{j+1} (-1)^\ell r^{j+(j+1)^2+\ell^2} [n-i] [j-\ell+1]
$$

$$- q^j (q^{n-2i+1} - 1) - q + 1 \sum_{\ell=0}^{j} (-1)^\ell r^{j-2\ell+(j-\ell)^2+\ell^2} [n-i] [j-\ell].
$$

21
The second and last sums combine, simplifying to

\[ z_j = (q^{n+1} - q^j) \sum_{\ell=0}^{j-1} (-1)^{\ell} r^{j-2\ell-1+(j-\ell-1)^2+\ell^2} [\ell] [j_{-\ell-1}] \]

\[ + ((q-1)q^j + q^j - q^{n-i+1}) \sum_{\ell=0}^{j} (-1)^{\ell} r^{j-2\ell+(j-\ell)^2+\ell^2} [\ell] [n-i_{-\ell}] \]

\[ + (q^{j+1} - 1) \sum_{\ell=0}^{j+1} (-1)^{\ell} r^{j-2\ell+1+(j-\ell+1)^2+\ell^2} [\ell] [j_{-\ell+1}] . \]

Now it suffices to show that the generating function \( Z(t) = \sum_{j=0}^{\infty} z_j t^j \) vanishes. We first express \( Z(t) \) in terms of the polynomials \( E_m(t) \) defined in Section 2. Using Proposition 2.2(iii), we are able to rewrite our generating function as

\[ Z(t) = \Sigma_1 + \Sigma_2 + \cdots + \Sigma_6 \]

where

\[ \Sigma_1 = q^{n+1} \sum_{j=0}^{\infty} \sum_{\ell=0}^{j-1} (-1)^{\ell} r^{j-2\ell-1+(j-\ell-1)^2+\ell^2} [\ell] [j_{-\ell-1}] t^j \]

\[ = q^{n+1} E_i(-t) E_{n-i}(qt); \]

\[ \Sigma_2 = - \sum_{j=0}^{\infty} q^j \sum_{\ell=0}^{j} (-1)^{\ell} r^{j-2\ell-1+(j-\ell-1)^2+\ell^2} [\ell] [j_{-\ell-1}] t^j \]

\[ = -q^j E_i(-qt) E_{n-i}(q^2 t); \]

\[ \Sigma_3 = (q-1) \sum_{j=0}^{\infty} \sum_{\ell=0}^{j+1} (-1)^{\ell} r^{j-2\ell+(j-\ell)^2+\ell^2} [\ell] [n-i_{-\ell}] t^j \]

\[ = (q-1) E_i(-qt) E_{n-i}(q^2 t); \]

\[ \Sigma_4 = (q^j - q^{n-i+1}) \sum_{j=0}^{\infty} \sum_{\ell=0}^{j+1} (-1)^{\ell} r^{j-2\ell+1+(j-\ell+1)^2+\ell^2} [\ell] [j_{-\ell+1}] t^j \]

\[ = (q^j - q^{n-i+1}) E_i(-t) E_{n-i}(qt); \]

\[ \Sigma_5 = \sum_{j=0}^{\infty} q^{j+1} \sum_{\ell=0}^{j} (-1)^{\ell} r^{j-2\ell+1+(j-\ell+1)^2+\ell^2} [\ell] [j_{-\ell+1}] t^j \]

\[ = \frac{1}{t} E_i(-qt) E_{n-i}(q^2 t); \]

\[ \Sigma_6 = - \sum_{j=0}^{\infty} \sum_{\ell=0}^{j} (-1)^{\ell} r^{j-2\ell+1+(j-\ell+1)^2+\ell^2} [\ell] [n-i_{-\ell+1}] t^j \]

\[ = -\frac{1}{t} E_i(-t) E_{n-i}(qt). \]
Using Proposition 2.2(i,ii), we find
\[ Z(t) = \Sigma_1 + \Sigma_2 + \Sigma_3 + \Sigma_4 + \Sigma_5 + \Sigma_6 \]
\[ = \left( q^{n+1}t + q - q^{n-i+1} - \frac{1}{t} \right) \]
\[ + \left( \frac{1-q^t}{1-t} \right) \left( \frac{1+q^{n-i+1}}{1+q^t} \right) (qt + q - 1 + \frac{1}{q}) \]
\[ E_i(-t)E_{n-i}(qt) = 0 \]
as required.

The strategy for showing \( \hat{PM} = \hat{D}P \) is very similar but the details are sufficiently different that we provide the details here. Fix \( i \) and let \( v_i = (\hat{P}_0, \hat{P}_1, \ldots, \hat{P}_m) \). We must show that \( v_iM = \hat{P}_1v_i \). In particular, we need to show the following recurrence holds for all \( j \):
\[ b_{j-1} \sum_{\ell=0}^{j-1} (-1)^{\ell} r^{(j-1-\ell)^2+\ell^2} \left[ i \right] \left[ j-\ell \right] \]
\[ + c_{j+1} \sum_{\ell=0}^{j+1} (-1)^{\ell} r^{(j+1-\ell)^2+\ell^2} \left[ i \right] \left[ j+1-\ell \right] \]
\[ = \hat{P}_1 \sum_{\ell=0}^{j} (-1)^{\ell} r^{(j-\ell)^2+\ell^2} \left[ i \right] \left[ j-\ell \right] \]

Multiplying both sides by \( q - 1 \) and substituting for \( b_{j-1}, c_{j+1} \), we find this is equivalent to showing that the following is zero for all \( j \):
\[ z_j = (q^{n+1} - q^t) \sum_{\ell=0}^{j-1} (-1)^{\ell} r^{(j-\ell-1)^2+\ell^2} \left[ i \right] \left[ j-\ell-1 \right] \]
\[ + (q^{j+1} - 1) \sum_{\ell=0}^{j+1} (-1)^{\ell} r^{(j+1-\ell+1)^2+\ell^2} \left[ i \right] \left[ j-\ell+1 \right] \]
\[ - q^t (q^{2n-i} - 1) \sum_{\ell=0}^{j} (-1)^{\ell} r^{(j-\ell)^2+\ell^2} \left[ i \right] \left[ j-\ell \right] \]

Again, it suffices to show that the generating function \( Z(t) = \sum_{j=0}^{\infty} z_j t^j \) vanishes. As before, we first rewrite our generating function as \( Z(t) = \Sigma_1 + \Sigma_2 + \cdots + \Sigma_6 \) where
\[ \Sigma_1 = q^{n+1} \sum_{\ell=0}^{j-1} (-1)^{\ell} r^{(j-\ell-1)^2+\ell^2} t^m \left[ i \right] \left[ j-\ell-1 \right] t^m = q^{n+1} t E_i(-rt)E_{n-i}(rt); \]
\[ \Sigma_2 = -q^t \sum_{\ell=0}^{j} (-1)^{\ell} r^{(j-\ell-1)^2+\ell^2} t^m \left[ i \right] \left[ j-\ell-1 \right] t^m = -qt E_i(-r^3 t)E_{n-i}(r^3 t); \]
\[ \Sigma_3 = q^{j+1} \sum_{\ell=0}^{j+1} (-1)^{\ell} r^{(j+1-\ell+1)^2+\ell^2} t^m \left[ i \right] \left[ j-\ell+1 \right] t^m = \frac{1}{r} E_i(-r^3 t)E_{n-i}(r^3 t); \]
\[ \Sigma_4 = - \sum_{\ell=0}^{j+1} (-1)^{\ell} r^{(j+\ell+1)^2+\ell^2} t^m \left[ i \right] \left[ j+\ell+1 \right] t^m = -\frac{1}{r} E_i(-rt)E_{n-i}(rt); \]
\[\Sigma_5 = -rq^{2n} \sum_{\ell=0}^{j} (-1)^{\ell} r^{(j-\ell)^2+\ell^2} t^{[\ell]} \lfloor \frac{n-\ell}{j-\ell} \rfloor t^m = -rq^{n-i} E_i(-rt)E_{n-i}(rt);\]

\[\Sigma_6 = r q^{i} \sum_{\ell=0}^{j} (-1)^{\ell} r^{(j-\ell)^2+\ell^2} t^{[\ell]} \lfloor \frac{n-\ell}{j-\ell} \rfloor t^m = rq^{i} E_i(-rt)E_{n-i}(rt)\]

in terms of the polynomials \(E_m(t)\) defined in Section 2. Using Proposition 2.2(iii), we find

\[Z(t) = \Sigma_1 + \Sigma_2 + \Sigma_3 + \Sigma_4 + \Sigma_5 + \Sigma_6\]

\[= \left( q^{n+1} t - \frac{1}{t} - rq^{n-i} + rq^{i}\right) + \left( \frac{1}{t} - qt \right) \frac{(1-rq^t)(1+rq^{n-i})}{(1-rt)(1+rt)} E_i(-rt)E_{n-i}(rt) = 0\]

as required. \(\square\)

**Corollary 8.2.** The \(P\)-matrix of the \(Q\)-polynomial scheme \(S\) is given by

\[
P_{i,j} = P_{i,2n+1-j} = \hat{P}_{i,j} \quad \text{for } i \text{ even, } 0 \leq j \leq n;\]

\[
P_{i,j} = -P_{i,2n+1-j} = \hat{P}_{i,j} \quad \text{for } i \text{ odd, } 0 \leq j \leq n.\]

**Proof.** If \((v_0, \ldots, v_n)\) is a left eigenvector of \(\tilde{M}\) or \(\hat{M}\), it is easily seen that either \((v_0, \ldots, v_n, v_n, \ldots, v_0)\) or \((v_0, \ldots, v_n, -v_n, \ldots, -v_0)\) is a left eigenvector of \(L_1\), respectively. The fact that this ordering of the eigenvalues of \(L_1\) is a \(Q\)-polynomial ordering follows from Theorem 7.2. \(\square\)

**9. A hypothetical subscheme**

We ask whether \(S\) is the extended \(Q\)-bipartite double (in the sense of [13]) of a primitive \(Q\)-polynomial scheme. We investigated these parameters up to \(n=20\) and found they satisfied the Krein conditions, had integral eigenvalue multiplicities and nonnegative integral \(p_{ij}^k\), and satisfy the handshaking lemma for all square \(q\). This appears to give an infinite family of feasible parameters for primitive \(Q\)-polynomial schemes with an unbounded number of classes. Detailed parameters and a proof of feasibility will be given in a forthcoming paper of Eiichi Bannai and Jianmin Ma.

We give the smallest case below for which existence is unknown:

\[
P = \begin{pmatrix}
1 & r^4 + r^3 + r^2 + r & r^4 - r^3 + r^2 - r & r^6 + r^4 & r^6 - r^4 \\
2 & \frac{1}{r^3 + r^2 + r - 1} & \frac{1}{r^3 + r^2 - r - 1} & \frac{1}{r^2 - 2} & \frac{1}{r^2 - 2} \\
1 & \frac{r^2 - 1}{2} & \frac{r^2 - 1}{2} & -r^2 & 0 \\
1 & \frac{-r^2}{2} & \frac{-r^2}{2} & 0 & r^2 \\
1 & \frac{-r^2 - r^2 + r - 1}{2} & \frac{r^3 - r^2 + r - 1}{2} & \frac{r^4 + r^2}{2} & \frac{-r^4 + r^2}{2}
\end{pmatrix}
\]

24
\[
Q = \begin{pmatrix}
1 & r^4 & r^6 + r^4 + r^2 + 1 & r^6 - r^4 + r^2 - 1 & r^4 + 1 \\
1 & r^2 & r^6 - r^2 - 1 & r^6 + r^2 - 1 & r^2 + 1 \\
1 & r^2 - 2r - 1 & r^2 - 1 & r^2 - 1 & r^2 - 1 \\
1 & r^2 - 2r + 1 & r^2 r - 1 & r^2 r - 1 & r^2 r - 1 \\
1 & r^2 - 2r^2 - 1 & 0 & r^2 - 2r^2 - 1 & 0
\end{pmatrix}
\]

\[
L_0 = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix}
\]

\[
L_1 = \begin{pmatrix}
0 & r^4 + r^3 + r^2 + r & 0 & 0 & 0 \\
1 & r^2 + 2r + 1 & r^2 - 1 & r^2 + 1 & 0 \\
0 & r^2 + 2r + 1 & r^2 - 1 & r^2 + 1 & r^2 + 1 \\
0 & r^2 + 1 & 0 & r^2 + 1 & r^2 + 1 \\
0 & r^2 + 1 & 0 & r^2 + 1 & r^2 + 1
\end{pmatrix}
\]

\[
L_2 = \begin{pmatrix}
0 & 0 & r^4 - r^3 + r^2 - r & 0 & 0 \\
0 & r^4 - r^3 + r^2 - r & 0 & 0 & 0 \\
1 & r^4 - r^3 + r^2 - r & 0 & 0 & 0 \\
0 & r^4 - r^3 + r^2 - r & 0 & 0 & 0 \\
0 & r^4 - r^3 + r^2 - r & 0 & 0 & 0
\end{pmatrix}
\]

\[
L_3 = \begin{pmatrix}
0 & 0 & 0 & r^6 + r^4 & 0 \\
0 & 0 & 0 & r^6 + r^4 & r^6 + r^4 + 2r^3 \\
0 & 0 & 0 & r^6 + r^4 + 2r^3 & r^6 + r^4 + 2r^3 \\
0 & 0 & 0 & r^6 + r^4 + 2r^3 & r^6 + r^4 + 2r^3 \\
1 & r^4 + r^3 + r^2 - r & r^4 + r^3 + r^2 - r & r^4 + r^3 + r^2 - r & r^4 + r^3 + r^2 - r
\end{pmatrix}
\]

\[
L_4 = \begin{pmatrix}
0 & 0 & 0 & 0 & r^6 - r^4 \\
0 & 0 & 0 & 0 & r^6 - r^4 \\
0 & 0 & 0 & 0 & r^6 - r^4 \\
0 & 0 & 0 & 0 & r^6 - r^4 \\
1 & r^4 + r^3 + r^2 + r & r^4 + r^3 + r^2 + r & r^4 + r^3 + r^2 + r & r^4 + r^3 + r^2 + r
\end{pmatrix}
\]

\[
L_0^* = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix}
\]
ware package MAGMA [3] was instrumental in finding the initial example s of jamin/Cummings, Menlo Park, 1984.

\[
L_1 = \begin{pmatrix}
    0 & r^4 & 0 & 0 & 0 & 0 \\
    r^4 & 0 & 0 & 0 & 0 & 0 \\
    0 & r^4 & 0 & 0 & 0 & 0 \\
    0 & 0 & r^4 & 0 & 0 & 0 \\
    0 & 0 & 0 & r^4 & 0 & 0 \\
    0 & 0 & 0 & 0 & r^4 & 0 \\
\end{pmatrix}
\]

\[
L_2 = \begin{pmatrix}
    0 & 0 & r^6 & 0 & 0 & 0 \\
    0 & r^6 & 0 & 0 & 0 & 0 \\
    0 & 0 & r^6 & 0 & 0 & 0 \\
    0 & 0 & 0 & r^6 & 0 & 0 \\
    0 & 0 & 0 & 0 & r^6 & 0 \\
    0 & 0 & 0 & 0 & 0 & r^6 \\
\end{pmatrix}
\]

\[
L_3 = \begin{pmatrix}
    0 & 0 & 0 & 0 & r^6 & 0 \\
    0 & 0 & 0 & 0 & 0 & r^6 \\
    0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\]

\[
L_4 = \begin{pmatrix}
    0 & 0 & 0 & 0 & 0 & r^6 \\
    0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\]

Acknowledgements

This research was supported in part by NSF grant DMS-1400281. The software package MAGMA [3] was instrumental in finding the initial examples of these schemes, as well as in computations related to verifying our proofs.

References

References

[1] M. Aigner, A Course in Enumeration, Springer-Verlag, Berlin, 2007.

[2] E. Bannai and T. Ito, Algebraic combinatorics I. Association schemes, Benjamin/Cummings, Menlo Park, 1984.
[3] W. Bosma, J. Cannon and C. Playoust, The Magma algebra system I. The user language, J. Symbolic Comput. 24 (1997) 235–265.

[4] J.N. Bray, D.F. Holt and C.M. Roney-Dougal, The Maximal Subgroups of the Low-Dimensional Finite Classical Groups, Camb. Univ. Press, Cambridge, 2013.

[5] R.C. Bose and D.M. Mesner, On linear associative algebras corresponding to association schemes of partially balanced designs, Ann. Math. Statist. 30 (1959) 21–38.

[6] A.E. Brouwer, A.M. Cohen and A. Niemaier, Distance-Regular Graphs, Springer-Verlag, Berlin, 1989.

[7] J.H. Conway, R.T. Curtis, S.P. Norton, R.A. Parker and R.A. Wilson, Atlas of Finite Groups: Maximal Subgroups and Ordinary Characters for Simple Groups, Oxford Univ. Press, Oxford, 1986.

[8] E.R. van Dam, W.J. Martin and M. Muzychuk, Uniformity in association schemes and coherent configurations: cometric Q-antipodal schemes and linked systems, J. Combin. Theory Ser. A 120 (2013) 1401–1439.

[9] P. Delsarte, An algebraic approach to the association schemes of coding theory, Philips Res. Rep. Suppl. No. 10 (1973).

[10] C. Godsil and G. Royle, Algebraic Graph Theory, Springer, New York, 2001.

[11] M. Kashiwara and P. Schapira, Sheaves on Manifolds, Springer-Verlag, Berlin, 1990.

[12] P. Kleidman and M. Liebeck, The Subgroup Structure of the Finite Classical Groups, Camb. Univ. Press, Cambridge, 1990.

[13] W.J. Martin, M. Muzychuk and J. Williford, Imprimitive cometric association schemes: constructions and analysis, J. Algebraic Combin. 25 (2007) 399–415.

[14] G. E. Moorhouse, Two-graphs and skew two-graphs in finite geometries, Linear Alg. Appl. 226–22 (1995) 529–551.

[15] T. Penttila and J. Williford, New families of Q-polynomial association schemes, J. Combin. Theory Ser. A 118 (2011) 502–509.

[16] J.J. Seidel, A survey of two-graphs, pp.481–511 in Teorie Combinatorie, Accademia Naz. dei Lincei, 1976.

[17] H. Suzuki, Association schemes with multiple Q-polynomial structures, J. Algebraic Combin. 7 (1998) No. 2, 181–196.
[18] D.E. Taylor, Two-graphs and doubly transitive groups, J. Comb. Theory Ser. A 61 (1992) 113–122.

[19] T. Thomas, The character of the Weil representation, J. London Math. Soc. (2) 77 (2008) 221–239.