Algebra of non-local charges in the O(N) WZNW model 
at and beyond criticality

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Abstract

We derive the classical algebra of the non-local conserved charges in the $O(N)$ WZNW model and analyze its dependence on the coupling constant of the Wess-Zumino term. As in the non-linear sigma model, we find cubic deformations of the $O(N)$ affine algebra. The surprising result is that the cubic algebra of the WZNW non-local charges does not obey the Jacobi identity, thus opposing our expectations from the known Yangian symmetry of this model.

1 Introduction

Yangian symmetries are expected to play a major role in our understanding of the integrable structure of conformal field theories and their deformations [1, 2]. Some conformal field theories are known to exhibit a Yangian symmetry for any affine Lie algebra at the critical point, with a level-independent structure [3, 4, 5]. The Yangian generators of that symmetry are understood as quantum extensions of classical non-local charges, such as those found in the non-linear sigma model and current algebra models [6]-[15]. Therefore the study of classical algebras of non-local charges may be regarded as a pre-quantum step toward the comprehension of symmetry and integrability properties of this class of field theories.

In previous works [16, 17], we have studied the algebra of the infinite non-local conserved charges in the $O(N)$ non-linear sigma model. The Wess-Zumino-Novikov-Witten (WZNW) model is also known to display an infinite set of non-local charges [18], and the primary aim of this paper is to unveil the classical algebra generated by them. In particular, we are interested in the dependence of this algebra with respect to the Wess-Zumino coupling

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constant. By understanding this dependence we could approach the algebra simultaneously at and beyond the conformal point, defined by a specific value of that coupling. Therefore, one of the applications – and the main motivation – of our algebraic project is the study of integrable perturbations of conformal theories [19, 20, 21].

To construct the charges and the corresponding Dirac brackets, we follow the algebraically inspired strategy outlined in ref.[16] and the diagrammatic technique introduced in ref.[17]. As a result, we observe, just like in the previously studied theories (namely, the bosonic and supersymmetric non-linear sigma models and the Gross-Neveu model), that the WZNW non-local charges turn out to obey a cubic deformation of the $O(N)$ affine algebra. Those cubic algebras are not isomorphic though. Surprisingly, the algebra of the WZNW charges does not satisfy the Jacobi identity, as opposed to the algebra of the chiral non-linear sigma model. These are the main results to be reported in this letter.

In the next section, we take the $O(N)$ WZNW model, recall its current algebra and related properties, and define some notation. In section 3, we discuss the available methods that can be used to construct the non-local charges. In section 4, we consider the WZNW in the conformal point and derive the corresponding algebra of non-local charges. Then we study the same algebra for the WZNW model beyond the conformal point, in section 5, including a discussion on the zero-coupling limit. In section 6 we analyze the issue of the Jacobi identity for the cubic algebra. Section 7 is used to summarize results and comments.

2 Current algebra in the WZNW model

Our starting point is the well-known WZNW action which contains two parcels,

$$S = S_{ch} + nS_{WZ};$$

(1)

$S_{ch}$ is the action of the ordinary chiral model,

$$S_{ch} = -\frac{1}{2\lambda^2} \int d^2x r^{\mu\nu} tr(g^{-1} \partial_\mu g g^{-1} \partial_\nu g),$$

(2)

where the basic field $g(x)$ takes values in a simple Lie group $G$ (we shall take $G = O(N)$), $n$ is an integer and $S_{WZ}$ is the Wess-Zumino term

$$S_{WZ} = \frac{1}{4\pi} \int_0^1 dr \int d^2x \epsilon^{\mu\nu} tr(\tilde{g}^{-1} \partial_r \tilde{g} \tilde{g}^{-1} \partial_\mu \tilde{g} \tilde{g}^{-1} \partial_\nu \tilde{g}).$$

(3)

(Here, $B$ is an appropriate three-dimensional boundary of $G$ and the extended field $\tilde{g}$ is assumed to be constant outside a tubular neighborhood $\Sigma \times [0,1]$ of the boundary $\Sigma$ of $B$; $r$ is the coordinate normal to the boundary.) This model contains a free coupling constant $\lambda$ such that, for $\lambda^2 \to 0$, we recover the ordinary chiral model and, for $\lambda^2 = 4\pi/n$, the conformally invariant WZNW model.

The action (1) has a global invariance under the product group $G_L \times G_R$, which acts on $G$ according to

$$g \rightarrow g_L g g_R^{-1}.$$  

(4)

This invariance leads to conserved Noether currents, taking values in the Lie algebra $G_L \oplus G_R$, which may be decomposed as left and right currents. For the purpose of writing a current
This algebra is invariant under the combined change of variables, 
\[ j^L_\mu = -\frac{1}{\lambda^2} \partial_\mu gg^{-1} , \quad j^R_\mu = +\frac{1}{\lambda^2} g^{-1} \partial_\mu g \, , \] (5)
whose algebra, for a general simple Lie group, was derived in ref.\[19\]. Here we shall limit ourselves to the \( O(N) \) algebra. We shall also introduce the parameter
\[ \alpha = \frac{n\lambda^2}{4\pi} \] (6)
which appears in the definition of the conserved covariant currents
\[
\begin{align*}
J^L_\mu &= (\eta_{\mu\nu} + \alpha \epsilon_{\mu\nu}) j^L_{\nu} = -\frac{1}{\lambda^2} (\eta_{\mu\nu} + \alpha \epsilon_{\mu\nu}) \partial^\nu gg^{-1} , \\
J^R_\mu &= (\eta_{\mu\nu} - \alpha \epsilon_{\mu\nu}) j^R_{\nu} = +\frac{1}{\lambda^2} (\eta_{\mu\nu} - \alpha \epsilon_{\mu\nu}) g^{-1} \partial^\nu g . 
\end{align*}
\] (7)
It is quite convenient to use the \( \circ \)-product notation introduced in \[16\], which is a characteristic structure of the \( O(N) \) algebra,
\[
(A \circ B)_{ij,kl} \equiv A_{ik} B_{jl} - A_{il} B_{jk} + A_{jl} B_{ik} - A_{jk} B_{il} ,
\] (8)
to write down the classical current algebra \[19\] in the following way:
\[
\begin{align*}
\{(J^L_0)_{ij}(x) , (J^L_0)_{kl}(y)\} &= (I \circ J^L_0)_{ij,kl}(x) \delta(x - y) - \alpha (I \circ I)_{ij,kl} \delta'(x - y) , \\
\{(J^L_0)_{ij}(x) , (J^L_1)_{kl}(y)\} &= (I \circ J^L_1)_{ij,kl}(x) \delta(x - y) - \frac{(1 + \alpha^2)}{2} (I \circ I)_{ij,kl} \delta'(x - y) , \\
\{(J^L_1)_{ij}(x) , (J^L_1)_{kl}(y)\} &= 2\alpha (I \circ J^L_1)_{ij,kl}(x) \delta(x - y) - \alpha^2 (I \circ J^L_0)_{ij,kl}(x) \delta(x - y) \\
&- \alpha (I \circ I)_{ij,kl} \delta'(x - y) , \\
\{(J^R_0)_{ij}(x) , (J^R_0)_{kl}(y)\} &= (I \circ J^R_0)_{ij,kl}(x) \delta(x - y) + \alpha (I \circ I)_{ij,kl} \delta'(x - y) , \\
\{(J^R_0)_{ij}(x) , (J^R_1)_{kl}(y)\} &= (I \circ J^R_1)_{ij,kl}(x) \delta(x - y) - \frac{(1 + \alpha^2)}{2} (I \circ I)_{ij,kl} \delta'(x - y) , \\
\{(J^R_1)_{ij}(x) , (J^R_1)_{kl}(y)\} &= -2\alpha (I \circ J^R_1)_{ij,kl}(x) \delta(x - y) - \alpha^2 (I \circ J^R_0)_{ij,kl}(x) \delta(x - y) \\
&+ \alpha (I \circ I)_{ij,kl} \delta'(x - y) , \\
\{(J^L_0)_{ij}(x) , (J^R_0)_{kl}(y)\} &= 0 , \\
\{(J^L_0)_{ij}(x) , (J^R_1)_{kl}(y)\} &= -(1 - \alpha^2)(g \circ g)_{ij,kl}(y) \delta'(x - y) , \\
\{(J^L_1)_{ij}(x) , (J^R_0)_{kl}(y)\} &= -(1 - \alpha^2)(g \circ g)_{ij,kl}(x) \delta'(x - y) , \\
\{(J^L_1)_{ij}(x) , (J^R_1)_{kl}(y)\} &= -\alpha(1 - \alpha^2)(g \circ g)_{ij,kl}(y) \delta'(x - y) \\
&+ \alpha(1 - \alpha^2)(g \circ g)_{ij,kl}(x) \delta'(x - y) .
\end{align*}
\]
This algebra is invariant under the combined change \( L \leftrightarrow R \) and \( \alpha \leftrightarrow -\alpha \). Therefore we may concentrate ourselves in one sector and easily extend the results to the other. Let us notice the Schwinger terms, both in time and space-component brackets, whose present form we believe to be underneath the unexpected properties of the algebras to be shown in this paper.
3 Non-local charges

One can easily check that the conserved covariant currents \((J_{\mu}^{R,L})\) satisfy a curvature-free condition,

\[
\partial_\mu J_{\nu}^{R,L} - \partial_\nu J_{\mu}^{R,L} + \lambda^2 [J_{\mu}^{R,L}, J_{\nu}^{R,L}] = 0 ,
\]

which implies the existence of an infinite set of non-local conserved charges, in both left and right sectors. One could then use the integro-differential algorithm of Brézin et al. \cite{8} to construct an infinite set of non-local conserved charges \(Q^{(n)}, n = 0, 1, \ldots\). Besides the (local) \(O(N)\) generator

\[
(Q^{(0)})_{ij} = \int dx (J_0)_{ij} ,
\]

we also recall the standard expression of the first non-local charge \cite{8}

\[
(Q^{(1)})_{ij} = \int dx (J_1 - \alpha J_0 + 2J_0 \partial^{-1} J_0)_{ij} .
\]

However, from the algebraic point of view, the set of charges thus generated is not necessarily the most suitable. In ref.\cite{16}, after studying the non-linear sigma model, it was shown that the standard charges from Brézin et al.’s algorithm can be recombined into a new set of improved charges, whose algebra is somewhat simpler – it still is a non-linear algebra but the non-linear terms are simply cubic. Taking this “algebraic simplicity” as a guiding criterion, the remaining improved charges are constructed from the algebra itself, using the charge \(Q^{(1)}\) as a step-like generator. This procedure is made possible due to the property

\[
Q^{(n+1)} \propto \text{linear part of \{ }Q^{(n)}, Q^{(1)}\}\] .

Therefore our task is to apply a similar algebraic procedure to the WZNW model and to find the corresponding set of improved non-local charges, whose algebra is supposedly as simple as possible. The calculations involved in this program can be shortened if we use the diagrammatic method developed in ref.\cite{17}. It consists of a graphic representation of charges and brackets, which incorporates currents, non-localities and contraction rules in a rather handy way. The graphic rules from ref.\cite{17} can be adapted to the WZNW model with no major difficulty. The results are discussed in the next sections.

4 Charges and algebra at the critical point

The critical value of the coupling constant, for which the model is conformally invariant, corresponds to \(\alpha = \pm 1\). In that case, the covariant current components are chirally constrained

\[
J_0^L = J_1^L , \quad J_0^R = -J_1^R
\]

and the non-local covariant charges can be written in terms of the time-component \(J_0\) solely,

\[
Q^{(0)} = \int dx (J_0) ,
\]

\[
Q^{(1)} = \int dx J_0 2\partial^{-1} (J_0) ,
\]
\[ Q^{(n)} = \int dx J_0 2 \partial^{-1}(J_0 2 \partial^{-1}(J_0 2 \partial^{-1}(J_0 2 \partial^{-1} \cdots ))) \]  \( (14) \)

According to the terminology proposed in ref. [16], we would say the critical charges are “saturated”, which just means that each non-local charge is made out of a single chain of time-components \( J_0 \)'s connected by the non-local operator \( \partial^{-1} \). The algebra therefore depends on \( \{ J_0, J_0 \} \) only and is readily derived using the graphic method. The resulting cubic algebra is briefly presented in terms of generators, as follows:

\[ \{ Q(\xi), Q(\mu) \} = (f(\xi, \mu) \circ Q(\xi) - Q(\mu)) \]  \( (15) \)

where \( \xi \) and \( \mu \) are expansion parameters in the charge-generator matrix defined by

\[ Q(\xi) \equiv \sum_{n=0}^{\infty} \xi^{n+1} Q^{(n)} \]  \( (16) \)

and \( f \) is a two-parameter dependent matrix,

\[ f(\xi, \mu) = \frac{1 - 2\alpha(\xi + \mu)}{\xi^{-1} - \mu^{-1}} (I - Q(\xi)Q(\mu)) \]  \( (17) \)

In this formula, \( I \) is the \( N \times N \) identity matrix which leads to the linear part of the algebra. On the other hand, the quadratic term \( Q(\xi)Q(\mu) \) in \( f \) implies the cubic piece in that same algebra. The linear part was derived in full generality, using the graphic method, while the cubic terms were verified up to the order \( n = 4 \) (thus the quadratic part in (17) should be regarded as an ansatz).

We recall that \( \alpha = \pm 1 \) at criticality and the results above are understood to hold only in those cases. Yet it is worth noticing that, should we simply set \( \alpha = 0 \), the resulting algebra would be isomorphic to the cubic algebra of the non-linear sigma model [16, 17]. However, as we show in section 5, the cases \( \alpha \neq \pm 1 \) display an authentically new algebra.

### 5 Non-local charges beyond the critical point

In the non-linear sigma model, the following coincidence was observed: the algebras of the improved charges and of the (non-conserved) saturated charges are identical [14]. It was conjectured in ref. [13] that the same property might hold for the WZNW model, i.e. that the algebra (15) would be obeyed for any value of the coupling \( \alpha \). However, the following results imply that that conjecture is false: we did find a cubic algebra, and some of its brackets were listed below:

\[
\begin{align*}
\{Q^{(0)}, Q^{(0)}\} &= (I \circ Q^{(0)}) , \\
\{Q^{(0)}, Q^{(1)}\} &= (I \circ Q^{(1)}) - 2\alpha(I \circ Q^{(0)}) , \\
\{Q^{(1)}, Q^{(1)}\} &= (I \circ Q^{(2)}) - 4\alpha(I \circ Q^{(1)}) - (Q^{(0)}Q^{(0)} \circ Q^{(0)}) , \\
\{Q^{(0)}, Q^{(2)}\} &= (I \circ Q^{(2)}) - 2\alpha(I \circ Q^{(1)}) - (1 - \alpha^2)(I \circ Q^{(0)}) ,
\end{align*}
\]

\( 5 \)
\{Q^{(1)}, Q^{(2)}\} = (I \circ Q^{(3)}) - 4\alpha(I \circ Q^{(2)}) - (Q^{(1)}Q^{(0)} \circ Q^{(0)}) - \\
- (Q^{(0)}Q^{(0)} \circ Q^{(1)}) + 2\alpha(Q^{(0)}Q^{(0)} \circ Q^{(0)}), \\
\{Q^{(0)}, Q^{(3)}\} = (I \circ Q^{(3)}) - 2\alpha(I \circ Q^{(2)}) - (1 - \alpha^2)(I \circ Q^{(1)}) - \\
- \alpha(1 - \alpha^2)(I \circ Q^{(0)}), \\
\{Q^{(1)}, Q^{(3)}\} = (I \circ Q^{(4)}) - 4\alpha(I \circ Q^{(3)}) - (Q^{(2)}Q^{(0)} \circ Q^{(0)}) - \\
- (Q^{(1)}Q^{(0)} \circ Q^{(1)}) - (Q^{(0)}Q^{(0)} \circ Q^{(2)}) + \\
+ 2\alpha(Q^{(0)}Q^{(1)} \circ Q^{(0)}) + 2\alpha(Q^{(0)}Q^{(0)} \circ Q^{(1)}) + \\
+ 2(1 - \alpha^2)(Q^{(0)}Q^{(0)} \circ Q^{(0)}), \\
\{Q^{(2)}, Q^{(2)}\} = (I \circ Q^{(4)}) - 4\alpha(I \circ Q^{(3)}) - 3\alpha(1 - \alpha^2)(I \circ Q^{(1)}) - \\
- 3\alpha^2(1 - \alpha^2)(I \circ Q^{(0)}) - (Q^{(0)}Q^{(0)} \circ Q^{(2)}) + \\
+ 2\alpha(Q^{(0)}Q^{(1)} \circ Q^{(0)}) + 2\alpha(Q^{(1)}Q^{(0)} \circ Q^{(0)}) - \\
- (Q^{(0)}Q^{(1)} \circ Q^{(1)}) - (Q^{(1)}Q^{(0)} \circ Q^{(1)}) - \\
- (Q^{(0)}Q^{(1)} \circ Q^{(0)}) + 4\alpha(Q^{(0)}Q^{(0)} \circ Q^{(1)}),

(18)

but we found no linear change of basis such that it might turn back into the algebra \[14\] for any \(\alpha\). Moreover, if this algebra could really be written in a form similar to \[14\], then \(f(\xi, \mu)\) would not be given by an expression as simple as \[17\]. So far, we do not have an ansatz for the complete algebra. Nevertheless, if necessary, other charges and brackets can be generated from \[18\], using the graphic algorithm.

The zero-coupling case \((\alpha \to 0)\) was studied separately because, in that limit, we expected to reach an algebra isomorphic to the one found in the non-linear sigma model. We constructed some non-local charges and calculated various brackets, up to \(\{Q^{(4)}, Q^{(2)}\}\), and all brackets thus found fit into the following ansatz:

\[
\{Q(\xi), Q(\mu)\} = (f(\xi, \mu) \circ c(\xi, \mu)Q(\xi) - c(\mu, \xi)Q(\mu)),
\]

(19)

\[
Q(\xi) = \sum_{n=0}^{\infty} \xi^{n+1}Q^{(n)}, \quad f(\xi, \mu) = \frac{1}{\xi^{-1} - \mu^{-1}(I - Q(\xi)Q(\mu))},
\]

(20)

\[
c(\xi, \mu) = 1 - \xi^2 + 2\xi^4 - \xi^2\mu^2 + \cdots
\]

(21)

Further terms in the expansion of the function \(c(\xi, \mu)\) would require higher-order brackets. As is turns out, the above algebra is different from the one of the non-linear sigma model \[16, 17\]. This divergence is directly related to the differences between the Schwinger terms in the respective current algebras.

In the search for a general ansatz, we asked ourselves whether we could use the Jacobi identity to derive a closed form for the \(\alpha\)-dependent cubic algebra. This actually happens in the non-linear sigma model: from the Jacobi identity with low-order charges one can calculate higher-order brackets. This procedure is discussed in the next section.

## 6 On the Jacobi identity

Based on the available algebras, we posed ourselves the following question: assuming a cubic algebra with the structure

\[
\{Q(\xi), Q(\mu)\} = (f(\xi, \mu) \circ Q(\xi) - Q(\mu)),
\]

(22)
\[ f(\xi, \mu) = A(\xi, \mu)I + B(\xi, \mu)Q(\xi)Q(\mu) \quad , \tag{23} \]

what constraints would the Jacobi identity impose on the to-be-determined functions \( A(\xi, \mu) \) and \( B(\xi, \mu) \)? To begin with, we considered \( A \) and \( B \) to be ordinary two-parameter functions, in which case the answer is

\[
A(\xi, \mu) = \frac{1}{g(\xi) - g(\mu)} \quad , \tag{24}
\]

\[
B(\xi, \mu) = \text{constant} \times A(\xi, \mu) \quad , \tag{25}
\]

where \( g(\xi) \) is some arbitrary function. This solution is compatible with the non-linear sigma model, where it was found \( A = -B = 1/(\xi^{-1} - \mu^{-1}) \). Actually that solution is rather unique: (i) Firstly, by an overall rescaling of the generator \( Q(\xi) \), we could set the constant = -1 in the solution above, so that \( A = -B \) could be taken as a general relation; (ii) As long as \( g(\xi) \) is invertible and the inverse \( g^{-1} \) admits a series expansion, we might take \( \xi = g^{-1}(\xi') \) and define another generator \( Q'(\xi') = Q(g^{-1}(\xi')) = \xi Q^{(n)} \), where \( Q^{(n)} \) would be linear recombinations of the original non-local charges. In the re-parameterized basis, we would reproduce the cubic algebra of the non-linear sigma model.

However, the critical (\( \alpha = 1 \)) WZNW model has

\[
A = -B = \frac{1 - 2(\xi + \mu)}{\xi^{-1} - \mu^{-1}} \quad , \tag{26}
\]

which cannot be written in the form (24) and therefore the cubic algebra (17) does not satisfy the Jacobi identity, as announced in the Introduction. It is important to note that the breaking of the Jacobi identity is not caused by the cubic terms; in fact, the linear part of the algebra is sufficient to break it. The following test, taken from the \( \alpha = 1 \) model, exemplifies this property:

\[
\{\{Q^{(0)}_{ij}, Q^{(1)}_{kl}\}, Q^{(1)}_{mn}\} + \{\{Q^{(1)}_{kl}, Q^{(1)}_{mn}\}, Q^{(0)}_{ij}\} + \{\{Q^{(1)}_{mn}, Q^{(0)}_{ij}\}, Q^{(1)}_{kl}\} = \\
= 4 \left[ (\delta_{ik}\delta_{lm} - \delta_{il}\delta_{km})Q_{jn}^{(0)} + (\delta_{jl}\delta_{km} - \delta_{jk}\delta_{lm})Q_{in}^{(0)} + \\
+ (\delta_{jm}\delta_{kn} - \delta_{km}\delta_{jn})Q_{il}^{(0)} + (\delta_{in}\delta_{kn} - \delta_{kn}\delta_{in})Q_{jl}^{(0)} + \\
+ (\delta_{il}\delta_{km} - \delta_{ik}\delta_{lm})Q_{jn}^{(0)} + (\delta_{jk}\delta_{ln} - \delta_{jl}\delta_{kn})Q_{in}^{(0)} + \right] \. \tag{27}
\]

Although a general ansatz for the off-critical algebra is missing, we have also made tests with the available brackets, such as this:

\[
\text{linear part of } \{\{Q^{(1)}_{ij}, Q^{(1)}_{kl}\}, Q^{(2)}_{mn}\} + \{\{Q^{(1)}_{kl}, Q^{(2)}_{mn}\}, Q^{(1)}_{ij}\} + \{\{Q^{(2)}_{mn}, Q^{(1)}_{ij}\}, Q^{(1)}_{kl}\} = \\
= (\delta_{ik}\delta_{aj}\delta_{kl} - \delta_{il}\delta_{aj}\delta_{km} + \delta_{jl}\delta_{ai}\delta_{bk} - \delta_{jk}\delta_{ai}\delta_{dl}) \times \\
\times [(I \circ Q^{(4)}) - 8\alpha(I \circ Q^{(3)}) - (3 - 19\alpha^2)(I \circ Q^{(2)}) + 4\alpha(1 - \alpha^2)(I \circ Q^{(1)}) + \\\n+ 8\alpha^2(1 - \alpha^2)(I \circ Q^{(0)}))]_{ab,mn} + \\
+ (\delta_{km}\delta_{al}\delta_{bm} - \delta_{kn}\delta_{al}\delta_{bm} + \delta_{in}\delta_{ak}\delta_{bm} - \delta_{im}\delta_{ak}\delta_{bm}) \times \\
\times [(I \circ Q^{(4)}) - 8\alpha(I \circ Q^{(3)}) - (3 - 19\alpha^2)(I \circ Q^{(2)}) + 8\alpha(1 - \alpha^2)(I \circ Q^{(1)}) +
\]
\begin{align}
&+ 2(1 - \alpha^2)^2 (I \circ Q^{(0)})_{ab,ij} + \\
&+ (\delta_{ni}\delta_{am}\delta_{bi} - \delta_{nj}\delta_{an}\delta_{bi} - \delta_{ni}\delta_{am}\delta_{bj}) \times \\
&\times [(I \circ Q^{(4)}) - 8\alpha (I \circ Q^{(3)}) - (3 - 19\alpha^2)(I \circ Q^{(2)}) + 8\alpha (1 - \alpha^2)(I \circ Q^{(1)}) + \\
&+ 2(1 - \alpha^2)^2 (I \circ Q^{(0)})_{ab,kl}] .
\end{align}

(28)

The conclusion is that the Jacobi identity also breaks beyond the critical point, thus for any value of \( \alpha \). This is one of the main results of this paper.

7 Final remarks

In spite of several similarities between the algebras of non-local charges in the \( O(N) \) non-linear sigma model and the WZNW model, we have found a major difference: the former obeys the Jacobi identity while the latter does not. This result came as a surprise because we expected to find classical Yangian algebras. Indeed, the presence of Yangians in WZNW models is well established [22]-[25] and Yangians do satisfy the Jacobi identity. So we looked for other indications of non-Yangian features: going back to those brackets in (18), we noticed that the generator \( Q^{(1)} \) does not transform in the usual manner under the \( O(N) \) symmetry:

\[ \{Q^{(0)}, Q^{(1)}\} = (I \circ Q^{(1)}) - 2\alpha (I \circ Q^{(0)}) , \]  

(29)

except when \( \alpha = 0 \). This implies that one of the defining relations of a Yangian – the property sometimes referred to as \( Y(2) \) – is not generally obeyed here. Therefore the cubic algebra (18) cannot be a classical Yangian algebra. Neither can the algebra (15) of critical charges, for which we nevertheless found a rather simple cubic ansatz.

As neither the case \( \alpha = 0 \) obeys the Jacobi identity, we have searched for more fundamental reasons of this violation. An important difference between the sigma model and WZNW current algebras is the presence (or absence) of the intertwiner field in the Schwinger terms: in the sigma model, the vanishing behavior of the intertwiner as \( x \to \pm \infty \) eliminates many boundary contributions. In the algebra, such contributions come up as integrals of the type

\[ \int dx dy (j \circ \partial^{-1} j_0)(x) \delta'(x - y) \to 0 \quad \text{because } j(\pm \infty) \to 0 \]  

(30)

where \( j \) is the intertwiner field. We have used the prescription \( \int dx dy F(x)\delta'(x - y) \propto [F(+\infty) - F(-\infty)] \), according to ref.[26]. In the WZNW model, if we consider either the left or right sector separately, there will be no intertwiner, and the corresponding integrals will not vanish, due to contributions of boundary charges:

\[ \int dx dy (I \circ \partial^{-1} J_0)(x) \delta'(x - y) \propto (I \circ (\partial^{-1} J_0)(+\infty) - (\partial^{-1} J_0)(-\infty)) = (I \circ Q^{(0)}) \]  

(31)

For instance, the unusual second term on the rightmost side of eq.(29) is originated in this way. Hence the algebra is bound to become more involved in the WZNW model. We suggest the ref.[27] for a discussion on the potential breaking of the Jacobi identity by c-number Schwinger terms in current algebras, although a specific interpretation based on the WZNW-field dynamics would be most valuable.
As concerns the integrability of the off-critical WZNW model, we have not fully caught on the consequences of this intriguing violation of the Jacobi identity. Moreover, the algebraic study of this paper has not gone beyond the classical level. Extrapolating the examples from the non-linear sigma model, we believe that the commutator algebra of the infinitesimal symmetry transformations – in other words, the action of the Yangian symmetry – generated by the non-local charges through some Lie-Poisson action, should obey the Jacobi identity (see ref. [16] for a discussion of the Lie-Poisson action of generators in the non-linear sigma model). In that case, the Yangian symmetry would remain associative as expected. Further investigations in this direction are in progress.

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