FORWARD CONTROLLABILITY OF A RANDOM ATTRACTOR FOR THE NON-AUTONOMOUS STOCHASTIC SINE-GORDON EQUATION ON AN UNBOUNDED DOMAIN

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Abstract. A pullback random attractor is called forward controllable if its time-component is semi-continuous to a compact set in the future, and the minimum among all such compact limit-sets is called a forward controller. The existence of a forward controller closely relates to the forward compactness of the attractor, which is further argued by the forward-pullback asymptotic compactness of the system. The abstract results are applied to the non-autonomous stochastic sine-Gordon equation on an unbounded domain. The existence of a forward compact attractor is proved, which leads to the existence of a forward controller. The measurability of the attractor is proved by considering two different universes.

1. Introduction. Let Φ be a non-autonomous random dynamical system (cocycle) over a probability space (Ω, ℱ, P), which is often generated from a non-autonomous SPDE. Suppose Φ has a pullback attractor \( \mathcal{A} = \{ \mathcal{A}_t(\omega) : t \in \mathbb{R}, \omega \in \Omega \} \). This type of attractors seems to be first introduced by Crauel et al. [11] and independently by Wang [39] with developments in [1, 13, 14, 26, 27, 41, 44, 45]. Only the compactness of the time-component \( \mathcal{A}_t \) was discussed in these papers.

In this paper, we concern the forward-uniform compactness of \( \mathcal{A} \).

Definition 1.1. The attractor \( \mathcal{A} \) is forward compact (resp. backward compact) if \( \bigcup_{s \geq t} \mathcal{A}_s(\omega) \) (resp. \( \bigcup_{s \leq t} \mathcal{A}_s(\omega) \)) is compact for each \( t \in \mathbb{R} \) and \( \omega \in \Omega \).

The backward compact attractor had been discussed in the recent papers, see [31, 32, 30, 43] in the deterministic case and [33, 40] in the random case. The backward compactness only provides information in the past.

The forward compactness (or uniform compactness) of the attractor was used to discuss the asymptotically autonomous problem for deterministic equations, see [18, 22, 23, 29].

If the random attractor \( \mathcal{A} \) is forward compact, we will prove that there exists a nonempty compact set \( K \) controlling the whole family \( \{ \mathcal{A}_t \} \) in the future, more
precisely, we have the following upper semi-continuity or semi-convergence:

\[
\lim_{t \to +\infty} \text{dist}(A_t(\omega), K(\omega)) = 0, \quad \forall \omega \in \Omega.
\] (1)

The minimum among all of compact sets satisfying (1) is called a forward controller for the attractor \(A\) (or for the cocycle \(\Phi\)). This controller provides information in the future, although it may not completely describe the forward attraction introduced by Kloeden et al. \([12, 21, 24]\).

The existence of a forward controller is almost equivalent to the forward compactness of the attractor, see Theorem 2.3.

Next, we mainly establish the criteria for the forward compactness in terms of the cocycle. A main condition is the forward-pullback compactness of the cocycle \(\Phi\), which means the usual pullback asymptotic compactness of \(\Phi\) is uniform in the future. Another condition is the existence of a decreasing absorbing set, see Theorem 2.8.

In the application part, we consider the non-autonomous stochastic sine-Gordon equation on \(\mathbb{R}^n\):

\[
u_{tt} + u_t - \Delta u + \lambda u + \beta \sin u = f(x, t) + \epsilon u \circ dW \frac{dt}{dt},
\] (2)

where \(W\) is a two-sided scalar Wiener process and the multiplicative noise is in the sense of Stratonovich integrals.

The random dynamics for the equation defined on a bounded domain was investigated in \([2, 5, 7, 8, 9, 16, 17, 35]\), where the coefficients \(\lambda, \beta\) need not to be restricted.

In order to obtain an attractor for the equation defined on the \emph{unbounded} domain, we must restrict the pair \((\lambda, \beta)\) of coefficients (see Hypothesis \(H1\)). We also assume the density \(\epsilon\) is small enough (see Hypothesis \(H2\)). Both hypotheses \(H1\) and \(H2\) are realizable and different from those conditions in \([19, 38, 42]\).

It is also necessary to assume that the time-dependent force \(f\) is forward tempered, forward tail-small and forward complement-small (see Hypothesis \(F1-F3\)). Under these hypotheses, we can verify that the induced cocycle is forward-pullback asymptotically compact and thus we obtain a forward compact (bi-parametric) attractor. So, the attractor is forward controllable in the sense of (1).

Notice that the absorbing set in this paper is an uncountable union of some random sets and thus its measurability is unknown. The measurability of the attractor seems to be unknown.

To overcome the difficulty of measurability, we take two universes \(\bar{\mathcal{D}}\) and \(\mathcal{D}\), where \(\bar{\mathcal{D}}\) is the usual universe forming from all tempered bi-parametric sets and \(\mathcal{D}\) is the universe forming from all forward tempered sets such that \(\mathcal{D}\) is forward-closed. We then prove an important fact that \(A_D = A_{\bar{D}}\) and thus the measurability of the attractor \(A_D\) follows from the measurability of \(A_{\bar{D}}\).

2. Forward controllability of bi-parametric attractors.

2.1. Bi-parametric attractors and random attractors. Let \((\Omega, \mathcal{F}, P)\) be a probability space with a group \(\{\theta_t\}_{t \in \mathbb{R}}\) such that each \(\theta_t : \Omega \to \Omega\) is measure-preserving, and let \((X, \|\cdot\|_X)\) be a separable Banach space.

A measurable mapping \(\Phi : \mathbb{R}^+ \times \mathbb{R} \times \Omega \times X \to X\) is called a \textbf{cocycle} if \(\Phi(0, \tau, \omega) = I_X\) and

\[\Phi(t + s, \tau, \omega) = \Phi(t, \tau + s, \theta_{s\omega})\Phi(s, \tau, \omega), \quad \forall t, s \geq 0, \tau \in \mathbb{R}, \omega \in \Omega.\]
We assume that $\Phi(t, \tau, \omega)x$ is continuous in $t \geq 0$, $\tau \in \mathbb{R}$ and $x \in X$ respectively.

We use a bi-parametric set $D = \{D_{\tau}(\omega)\}$ to denote a set-valued mapping $D : \mathbb{R} \times \Omega \to 2^X \setminus \{\emptyset\}$. A universe $D$ means the collection of some bi-parametric sets, where the elements in $D$ need not measurable.

**Definition 2.1.** Let $\Phi$ be a cocycle on $X$ with a universe $D$. A member $A = \{A_{\tau}(\omega)\}$ in $D$ is called a $D$-pullback bi-parametric attractor for $\Phi$ if

(i) $A$ is compact, that is, each $A_{\tau}(\omega)$ is compact in $X$;

(ii) $A$ is invariant, that is, for all $(t, \tau, \omega) \in \mathbb{R}^+ \times \mathbb{R} \times \Omega$,

$$\Phi(t, \tau, \omega)A_{\tau}(\omega) = A_{t+\tau}(\theta_{t\omega})$$

(iii) $A$ is $D$-pullback attracting, that is, for each $(\tau, \omega, D) \in \mathbb{R} \times \Omega \times D$.

$$\lim_{t \to +\infty} \text{dist}(\Phi(t, \tau - t, \theta_{-t\omega})D_{\tau}(\omega), A_{\tau}(\omega)) = 0.$$  \hspace{1cm} (3)

A bi-parametric attractor $A$ is further called a $D$-pullback random attractor if it is measurable, that is, the mapping $\omega \to d(x, A_{\tau}(\omega))$ is $(\mathcal{F}, \mathcal{B}(\mathbb{R}^+))$-measurable for each $x \in X$ and $\tau \in \mathbb{R}$.

The existence theorems of a pullback random attractor can be found in [39]. Those existence theorems are still true for a bi-parametric attractor if we transitorily disregard the measurability of the attractor.

2.2. **Forward controller of a bi-parametric attractor.** We say a bi-parametric family $\{D_{\tau}(\omega)\}$ or a parametric family $\{D(\omega)\}$ is compact if every component is compact.

**Definition 2.2.** A parametric family $\{E(\omega)\}$ of sets in $X$ is called a controlling set for a bi-parametric attractor $A$ if

$$\lim_{\tau \to +\infty} \text{dist}(A_{\tau}(\omega), E(\omega)) = 0, \forall \omega \in \Omega.$$ \hspace{1cm} (4)

A nonempty compact controlling set $\{E(\cdot)\}$ is called a forward controller if it is the minimum among all of compact controlling sets.

The concept of a forward controller we use here aim at the attractor rather than the PDE (see [25] for the feedback controller of the Navier-Stokes equations).

The existence of a forward controller relates to the forward compactness in Def.1.1, and also the local compactness (see [13, 40]), which means $\bigcup_{\tau \in I} A_{\tau}(\omega)$ is pre-compact in $X$ for any compact interval $I \subset \mathbb{R}$.

**Theorem 2.3.** Suppose a cocycle $\Phi$ has a bi-parametric attractor $A$.

(i) If $A$ is forward compact, then there is a unique forward controller;

(ii) If $A$ is locally compact and there is a compact controlling set $E(\cdot)$, then $A$ is forward compact.

In either case, the forward controller is given by

$$A_{\infty}(\omega) := \{x \in X : \exists \tau_n \uparrow \infty, \ x_n \in A_{\tau_n}(\omega) \text{ s.t. } x_n \to x\}.$$ \hspace{1cm} (5)

**Proof.** (i) Suppose $A$ is forward compact. Then the family

$$\left\{\bigcup_{\tau \geq \tau} A_{\tau}(\omega)\right\}_{\tau \in \mathbb{R}}$$

is nonempty compact and decreasing.
as $\tau \uparrow +\infty$. By the theorem of nested compact sets, their interjection

$$
\bigcap_{\tau \in \mathbb{R}} \bigcup_{r \geq \tau} A_r(\omega) = \bigcap_{r \geq 0} \bigcup_{r \geq \tau} A_r(\omega) =: B(\omega) \quad (6)
$$

is nonempty compact. It is standard to prove that both sets in (5) and (6) are identical, that is, $A_\infty(\omega) = B(\omega)$, and thus $A_\infty(\omega)$ is nonempty compact.

We then prove that $A_\infty(\cdot)$ is a controlling set:

$$
\lim_{\tau \to +\infty} \text{dist}(A_\tau(\omega), A_\infty(\omega)) = 0. \quad (7)
$$

If (7) is not true, then there exist $\delta > 0$ and a sequence $\tau_k \uparrow +\infty$ such that

$$
\text{dist}(A_{\tau_k}(\omega), A_\infty(\omega)) \geq 2\delta, \forall k \in \mathbb{N}. \quad (8)
$$

By the definition of Hausdorff semi-metric, we can take $x_k \in A_{\tau_k}(\omega)$ such that

$$
\text{dist}(x_k, A_\infty(\omega)) \geq \text{dist}(A_{\tau_k}(\omega), A_\infty(\omega)) - \delta \geq \delta, \forall k \in \mathbb{N}. \quad (8)
$$

By the forward compactness of $A$, the sequence $\{x_k\}$ has a convergent subsequence such that $x_{k^*} \to x \in X$ and $x_{k^*} \in A_{\tau_k}(\omega)$. It follows from the definition (5) that $x \in A_\infty(\omega)$, which contradicts with (8).

Finally, we show the minimality of $A_\infty(\omega)$. Let $K(\omega)$ be a compact controlling set (satisfying (4)) and $x \in A_\infty(\omega)$. By the definition (5), there are $\tau_n \uparrow +\infty$ and $x_n \in A_{\tau_n}(\omega)$ such that $x_n \to x$. Hence,

$$
\text{dist}(x, K(\omega)) \leq d(x, x_n) + \text{dist}(x_n, K(\omega)) \leq d(x, x_n) + \text{dist}(A_{\tau_n}(\omega), K(\omega)) \to 0,
$$

which implies $d(x, K(\omega)) = 0$. Since $K(\omega)$ is compact, $x \in K(\omega)$ and thus $A_\infty(\omega) \subset K(\omega)$ as desired.

(ii) We take a sequence $\{x_n\}$ from the union $\bigcup_{r \geq s} A_r(\omega)$ with fixed $s \in \mathbb{R}$. Then there are $r_n \geq s$ such that $x_n \in A_{r_n}(\omega)$.

If $\sup_{n \in \mathbb{N}} r_n = \hat{r} < +\infty$, then $\{r_n\} \subset [s, \hat{r}]$ and thus $\{x_n\} \subset \bigcup_{r \in [s, \hat{r}]} A_r(\omega)$.

Thereby, the local compactness of $A$ implies that $\{x_n\}$ has a convergent subsequence.

If $\sup_{n \in \mathbb{N}} r_n = +\infty$, we can choose a subsequence such that $r_n^* \uparrow +\infty$. By the assumption, there is a compact controlling set $E(\cdot)$ and thus

$$
\text{dist}(x_n^*, E(\omega)) \leq \text{dist}(A_{r_n^*}(\omega), E(\omega)) \to 0.
$$

We further choose a sequence $\{y_n^*\} \subset E(\omega)$ such that

$$
d(x_n^*, y_n^*) \leq \text{dist}(x_n^*, E(\omega)) + \frac{1}{n^*} \to 0.
$$

Since $E(\omega)$ is compact, it follows that the sequence $\{y_n^*\}$ has a convergent subsequence and so is $\{x_n\}$. \hfill \Box

By the same method as Theorem 2.3, we can prove that the minimum among all of compact controlling sets is always available.

**Corollary 1.** A bi-parametric attractor $A$ has a forward controller if there is a nonempty compact controlling set.
2.3. Criteria for a forward compact attractor. We require that the $\mathcal{D}$-pullback limit-set compactness of the cocycle is uniform in the future.

**Definition 2.4.** A cocycle $\Phi$ on $X$ is called forward $\mathcal{D}$-pullback compact if for each $(\tau, \omega, \mathcal{D}) \in \mathbb{R} \times \Omega \times \mathcal{D}$,

$$
\lim_{T \to +\infty} \kappa_X \left( \bigcup_{t \geq T, r \geq \tau} \Phi(t, r - t, \theta_{-t}\omega)\mathcal{D}_{r-t}(\theta_{-t}\omega) \right) = 0,
$$

where, $\kappa(\cdot)$ denotes the Kuratowski measure defined by

$$
\kappa_X(A) := \inf \{d > 0 : A \text{ has a finite cover by sets of diameter lesser than } d \}.
$$

For each bi-parametric set $\mathcal{D}$, we define a **forward-pullback limit-set** by

$$
\Upsilon^D_\tau(\omega) := \bigcap_{T > 0} \bigcup_{t \geq T, r \geq \tau} \Phi(t, r - t, \theta_{-t}\omega)\mathcal{D}_{r-t}(\theta_{-t}\omega)
$$

for all $\tau \in \mathbb{R}$ and $\omega \in \Omega$. Then, $\Upsilon_D := \{\Upsilon^D_\tau(\omega)\}$ defines a new bi-parametric set, and it is a forward-uniform version of the $\mathcal{D}$-pullback $\alpha$-limit set:

$$
\alpha^D_\tau(\omega) := \bigcap_{T > 0} \bigcup_{t \geq T} \Phi(t, r - t, \theta_{-t}\omega)\mathcal{D}_{r-t}(\theta_{-t}\omega).
$$

**Lemma 2.5.** (i) $y \in \Upsilon^D_\tau(\omega)$ if and only if there are $r_n \geq \tau, t_n \uparrow +\infty$ and $x_n \in \mathcal{D}_{r_n-t_n}(\theta_{-t_n}\omega)$ such that

$$
y = \lim_{n \to +\infty} \Phi(t_n, r_n - t_n, \theta_{-t_n}\omega)x_n.
$$

(ii) The set-valued mapping $\tau \mapsto \Upsilon^D_\tau(\omega)$ is decreasing under the inclusion relationship. Moreover,

$$
\bigcup_{r \geq \tau} \alpha^D_\tau(\omega) \subset \Upsilon^D_\tau(\omega) = \bigcup_{r \geq \tau} \Upsilon^D_r(\omega), \ \forall \ \tau \in \mathbb{R}.
$$

**Proof.** (i) If $y \in \Upsilon^D_\tau(\omega)$, then, by (10),

$$
y \in \bigcup_{t \geq n} \bigcup_{r \geq \tau} \Phi(t, r - t, \theta_{-t}\omega)\mathcal{D}_{r-t}(\theta_{-t}\omega), \ \forall n \in \mathbb{N}.
$$

We choose three sequences by $r_n \geq \tau, t_n \geq n$ and $x_n \in \mathcal{D}_{r_n-t_n}(\theta_{-t_n}\omega)$ such that

$$
\|\Phi(t_n, r_n - t_n, \theta_{-t_n}\omega)x_n - y\| \leq \frac{1}{n}, \ \forall n \in \mathbb{N},
$$

which implies (12) as desired. On the contrary, it is similar to prove that (12) implies $y \in \Upsilon^D_\tau(\omega)$.

(ii) If $\tau_1 \leq \tau_2$, then we have

$$
\bigcup_{r \geq \tau_1} \Phi(t, r - t, \theta_{-t}\omega)\mathcal{D}_{r-t}(\theta_{-t}\omega) \supset \bigcup_{r \geq \tau_2} \Phi(t, r - t, \theta_{-t}\omega)\mathcal{D}_{r-t}(\theta_{-t}\omega).
$$

By taking the interjection of the closure of the union over $t \geq T$, we have $\Upsilon^D_{\tau_1}(\omega) \supset \Upsilon^D_{\tau_2}(\omega)$, which proves that $\tau \mapsto \Upsilon^D_\tau(\omega)$ is decreasing.

In particular, $\Upsilon^D_\tau(\omega) \supset \Upsilon^D_r(\omega)$ for all $r \geq \tau$ and thus the equality in (13) holds true. On the other hand, by the definitions of $\Upsilon_D$ and $\alpha_D$, we have $\alpha^D_\tau(\omega) \subset \Upsilon^D_\tau(\omega)$ for all $\tau \in \mathbb{R}$. Thereby,

$$
\bigcup_{r \geq \tau} \alpha^D_\tau(\omega) \subset \bigcup_{r \geq \tau} \Upsilon^D_r(\omega) = \Upsilon^D_\tau(\omega).
$$
Suppose the cocycle $\Phi$ is forward $\mathcal{D}$-pullback compact, then, for each $\mathcal{D} \in \mathcal{D}$, both $\alpha^{\mathcal{D}}$ and $\gamma^{\mathcal{D}}$ are forward compact such that their component sets are nonempty. Moreover, $\gamma^{\mathcal{D}}$ forward-pullback attracts $\mathcal{D}$ in the sense that
\[
\lim_{t \to +\infty} \sup_{r \geq \tau} \text{dist}(\Phi(t, r - t, \theta_{-t}\omega)\mathcal{D}_{r-t}(\theta_{-t}\omega), \gamma^{\mathcal{D}}_{\tau}(\omega)) = 0, \quad \forall \tau \in \mathbb{R}, \omega \in \Omega.
\] (14)

**Proof.** We arbitrarily take three sequences such as
\[
\text{dist}(\Phi(t, t - t, \theta_{-t}\omega)\mathcal{D}_{t-t}(\theta_{-t}\omega), \gamma^{\mathcal{D}}_{\tau}(\omega)) \geq \delta > 0, \quad \forall \tau \in \mathbb{R}, \omega \in \Omega.
\] (17)

If (17) is not true, then there are $\delta > 0$ and $t_n \uparrow +\infty$ such that
\[
\text{dist}(\bigcup_{r \geq \tau} \Phi(t_n, r - t_n, \theta_{-t_n}\omega)\mathcal{D}_{r-t_n}(\theta_{-t_n}\omega), \gamma^{\mathcal{D}}_{\tau}(\omega)) \geq 2\delta, \quad \forall n \in \mathbb{N}.
\] (18)

We further choose $r_n \geq \tau$ and $x_n \in \mathcal{D}_{r_n-t_n}(\theta_{-t_n}\omega)$ such that
\[
\text{dist}(\Phi(t_n, r_n - t_n, \theta_{-t_n}\omega)x_n, \gamma^{\mathcal{D}}_{\tau}(\omega)) \geq \delta, \quad \forall n \in \mathbb{N}.
\] (18)

By the forward $\mathcal{D}$-pullback compactness as given in (9), we have
\[
\lim_{m \to +\infty} \kappa_{\mathcal{X}}\{\Phi(t_n, r_n - t_n, \theta_{-t_n}\omega)x_n : n \geq m\} = 0.
\] (19)

Then, by [28, Lemma 2.7], $\{\Phi(t_n, r_n - t_n, \theta_{-t_n}\omega)x_n\}_{n}$ has a convergent subsequence (not relabeled) such that
\[
\Phi(t_n, r_n - t_n, \theta_{-t_n}\omega)x_n \to x \text{ for some } x \in X.
\] (20)

By Lemma 2.5 (i), we have $x \in \gamma^{\mathcal{D}}_{\tau}(\omega)$ and thus $\gamma^{\mathcal{D}}_{\tau}(\omega)$ is nonempty.

We then prove that both $\alpha^{\mathcal{D}}$ and $\gamma^{\mathcal{D}}$ are forward compact for all $\mathcal{D} \in \mathcal{D}$. By (13), it suffices to prove $\gamma^{\mathcal{D}}_{\tau}(\omega)$ is compact. Indeed, let $\{y_n\} \subset \gamma^{\mathcal{D}}_{\tau}(\omega)$. By Lemma 2.5 (i), for each $n \in \mathbb{N}$, there are $r_n \geq \tau$, $t_n \geq \max\{t_{n-1}, n\}$ and $x_n \in \mathcal{D}_{r_n-t_n}(\theta_{-t_n}\omega)$ such that
\[
d(\Phi(t_n, r_n - t_n, \theta_{-t_n}\omega)x_n, y_n) \leq \frac{1}{n}, \quad \forall n \in \mathbb{N}.
\] (15)

By the forward $\mathcal{D}$-pullback compactness of $\Phi$, there are $x \in X$ and a index subsequence $\{n_k\}$ such that
\[
\Phi(t_{n_k}, r_{n_k} - t_{n_k}, \theta_{-t_{n_k}}\omega)x_{n_k} \to x.
\] (16)

Both (15) and (16) imply $y_{n_k} \to x$ as $k \to \infty$ and thus $\gamma^{\mathcal{D}}_{\tau}(\omega)$ is pre-compact. By (16), $x \in \gamma^{\mathcal{D}}_{\tau}(\omega)$ and so $\gamma^{\mathcal{D}}_{\tau}(\omega)$ is compact as desired.

Finally, we prove the forward-uniformly attraction (14), which is equivalent to the following assertion:
\[
\lim_{t \to +\infty} \text{dist}\left(\bigcup_{r \geq \tau} \Phi(t, r - t, \theta_{-t}\omega)\mathcal{D}_{r-t}(\theta_{-t}\omega), \gamma^{\mathcal{D}}_{\tau}(\omega)\right) = 0.
\] (17)

If (17) is not true, then there are $\delta > 0$ and $t_n \uparrow +\infty$ such that
\[
\text{dist}(\bigcup_{r \geq \tau} \Phi(t_n, r - t_n, \theta_{-t_n}\omega)\mathcal{D}_{r-t_n}(\theta_{-t_n}\omega), \gamma^{\mathcal{D}}_{\tau}(\omega)) \geq 2\delta, \quad \forall n \in \mathbb{N}.
\] (18)

We further choose $r_n \geq \tau$ and $x_n \in \mathcal{D}_{r_n-t_n}(\theta_{-t_n}\omega)$ such that
\[
\text{dist}(\Phi(t_n, r_n - t_n, \theta_{-t_n}\omega)x_n, \gamma^{\mathcal{D}}_{\tau}(\omega)) \geq \delta, \quad \forall n \in \mathbb{N}.
\] (18)

By the forward $\mathcal{D}$-pullback compactness as given in (9), we have
\[
\lim_{m \to +\infty} \kappa_{\mathcal{X}}\{\Phi(t_n, r_n - t_n, \theta_{-t_n}\omega)x_n : n \geq m\} = 0.
\] (19)

Then, by [28, Lemma 2.7], there are $x \in X$ and an index subsequence $\{n^*\}$ such that
\[
\Phi(t_{n^*}, r_{n^*} - t_{n^*}, \theta_{-t_{n^*}}\omega)x_{n^*} \to x.
\] (20)

By Lemma 2.5 (i), $x \in \gamma^{\mathcal{D}}_{\tau}(\omega)$, which contradicts with (18).}

\[
\Box
\]
In order to establish the criteria for a forward compact attractor, we need the universe $\mathcal{D}$ of some bi-parametric sets to be special.

**Definition 2.6.** A universe $\mathcal{D}$ is called **forward-union closed** if $\mathcal{D}^u \in \mathcal{D}$ as long as $\mathcal{D} \in \mathcal{D}$, where

$$D^u_r(\omega) = \bigcup_{r \geq \tau} D_r(\omega), \ \forall \tau \in \mathbb{R}, \ \omega \in \Omega. \quad (19)$$

A forward-union closed universe is contrary to the backward-union closed universe as given in [33, 40].

We also recall the neighborhood-closed universe given by Wang [39].

**Definition 2.7.** A universe $\mathcal{D}$ is called **neighborhood closed** if for each $\mathcal{D} \in \mathcal{D}$, there exists an $\eta = \eta(\mathcal{D}) > 0$ such that $\hat{D} \in \mathcal{D}$ whenever $D_r(\omega) \subset N_\eta D_r(\omega)$, where

$$N_\eta D_r(\omega) = \{x \in X : d(x, D_r(\omega)) \leq \eta\}, \ \forall \tau \in \mathbb{R}, \ \omega \in \Omega.$$

Recall that a bi-parametric set $K$ is called $\mathcal{D}$-pullback absorbing if, for each $\mathcal{D} \in \mathcal{D}$, $\tau \in \mathbb{R}$ and $\omega \in \Omega$, there is a $T = T(\mathcal{D}, \tau, \omega) > 0$ such that

$$\Phi(t, \tau - t, \theta_{-t} \omega) D_{\tau - t}(\theta_{-t} \omega) \subset K_r(\omega), \ \forall t \geq T. \quad (20)$$

**Theorem 2.8.** Suppose a cocycle $\Phi$ is forward $\mathcal{D}$-pullback compact, where the universe $\mathcal{D}$ is forward-union closed and neighborhood-closed. Then, the followings are equivalent to each other:

(i) $\Phi$ has a decreasing $\mathcal{D}$-pullback absorbing set $K \in \mathcal{D}$;

(ii) $\Phi$ has a unique bi-parametric attractor $A$ such that $A$ is forward compact, in this case, $A_r(\omega) = \alpha^x_r(\omega)$ for all $\tau \in \mathbb{R}$ and $\omega \in \Omega$.

Moreover, in either case, we have the following assertions:

(iii) $\Phi$ has a unique forward controller in the sense of Def. 2.2, given by

$$A_\infty(\omega) = \bigcap_{T > 0} \bigcup_{t \geq T} \alpha^x_r(\omega), \ \forall \omega \in \Omega, \quad (21)$$

which means the forward controller is the omega-limit set of the family $\{\alpha^x_r(\omega) : \tau \in \mathbb{R}\}$.

(iv) The attractor $A$ is random if the absorbing set $K$ is random.

**Proof.** (ii) $\Rightarrow$ (i). By the definition, we know $A \in \mathcal{D}$. Since $\mathcal{D}$ is forward-union closed, it follows $A \in \mathcal{D}$, where the forward-union is given by $A_r(\omega) = \cup_{r \geq \tau} A_r(\omega)$ as in (19). Since $\mathcal{D}$ is neighborhood closed, there is a $\eta = \eta(A) > 0$ such that the $\eta$-neighborhood $N_\eta A \in \mathcal{D}$. We now define a bi-parametric set $K$ by

$$K_r(\omega) := N_\eta A_r(\omega) = N_\eta \left( \bigcup_{r \geq \tau} A_r(\omega) \right), \ \forall \tau \in \mathbb{R}. \quad (22)$$

Then $K \in \mathcal{D}$. Obviously,

$$\bigcup_{r \geq \tau_1} A_r(\omega) \supset \bigcup_{r \geq \tau_2} A_r(\omega), \ \forall \tau_1 \leq \tau_2,$$

and so $K_{\tau_1}(\omega) \supset K_{\tau_2}(\omega)$, which means that the set-valued mapping $\tau \mapsto K_r(\omega)$ is decreasing.

We then show the $\mathcal{D}$-pullback absorption of $K$. Indeed, for each $\mathcal{D} \in \mathcal{D}$, $\tau \in \mathbb{R}$ and $\omega \in \Omega$, it follows from the $\mathcal{D}$-pullback attraction of $A$ that

$$\lim_{t \to +\infty} \text{dist}(\Phi(t, \tau - t, \theta_{-t} \omega) D_{\tau - t}(\theta_{-t} \omega), A_r(\omega)) = 0.$$
Thus, for the number \( \eta \) as in (22), there exists a \( T = T(\eta, \tau, \omega, D) > 0 \) such that for all \( t \geq T \),

\[
\Phi(t, \tau - t, \theta_{-t}\omega)D_{t-t}(\theta_{-t}\omega) \subset N_\eta(\mathcal{A}_r(\omega)) \subset \mathcal{K}_r(\omega),
\]

which implies that \( \mathcal{K} \) absorbs \( D \) as desired.

(i) \( \Rightarrow \) (ii). By (i), \( \mathcal{K} \) is a \( D \)-pullback absorbing set and thus the closure \( \overline{\mathcal{K}} \) is still a \( D \)-pullback absorbing set, where \( \overline{\mathcal{K}}(\omega) = \mathcal{K}_r(\omega) \) for all \( \tau \in \mathbb{R} \) and \( \omega \in \Omega \). By \( \mathcal{K} \in D \) and \( \mathcal{D} \) is neighborhood closed, we know \( \overline{\mathcal{K}} \in D \). So, there is a closed, \( D \)-pullback absorbing set.

We then show \( \Phi \) is \( D \)-pullback asymptotically compact. Let \( D \in D \), \( \tau \in \mathbb{R} \), \( \omega \in \Omega \) and let \( t_n \uparrow +\infty \), \( x_n \in D_{t-t_n}(\theta_{-t_n}\omega) \). By the forward \( D \)-pullback compactness as given in (9), we have

\[
\lim_{m \to +\infty} \kappa x \{ \Phi(t_n, t - t_n, \theta_{-t_n}\omega)x_n : n \geq m \} = 0.
\]

Then, by [28, Lemma 2.7], \( \{ \Phi(t_n, t - t_n, \theta_{-t_n}\omega)x_n \}_n \) has a convergent subsequence, which means \( \Phi \) is \( D \)-pullback asymptotically compact.

Therefore, by the same method given in [39, Theorem 2.23], we obtain a unique bi-parametric attractor \( \mathcal{A} \), given by \( \mathcal{A}_r(\omega) = \alpha^K_r(\omega) = \alpha^K_r(\omega) \) for all \( \tau \in \mathbb{R} \) and \( \omega \in \Omega \).

Finally, we prove \( \mathcal{A} \) is forward compact. Since \( \mathcal{K} \in D \) and \( \Phi \) is forward \( D \)-compact, it follows from Proposition 1 that \( \mathcal{Y}^K_r(\omega) \) is compact for all \( \tau \in \mathbb{R} \) and \( \omega \in \Omega \). By Lemma 2.5 (ii), the set-valued mapping \( \tau \mapsto \mathcal{Y}^K_r(\omega) \) is decreasing and thus

\[
\bigcup_{r \geq \tau} \mathcal{A}_r(\omega) = \bigcup_{r \geq \tau} \alpha^K_r(\omega) \subset \bigcup_{r \geq \tau} \mathcal{Y}^K_r(\omega) = \mathcal{Y}^K_r(\omega).
\]

So, \( \mathcal{A} \) is forward compact.

The assertion (iii) follows from Theorem 2.3. Indeed, since the bi-parametric attractor \( \mathcal{A} \) is forward compact, there is a forward controller in the sense of Definition 2.2, which given by

\[
\mathcal{A}_\infty(\omega) = \bigcap_{T > 0} \bigcup_{t \geq T} \mathcal{A}_t(\omega) = \bigcap_{T > 0} \bigcup_{t \geq T} \alpha^K_t(\omega),
\]

for all \( \omega \in \Omega \). That is, (21) is true.

The assertion (iv) follows from the known abstract result. Indeed, if the absorbing set \( \mathcal{K} \) is random and the cocycle is \( D \)-pullback asymptotically compact (this follows from initial assumption), then the \( \alpha \)-limit set \( \alpha^K_r(\cdot) \) is random for all \( \tau \in \mathbb{R} \) as proved in [15, 37, 39]. \( \square \)

**Remark 1.** As in (iv) of Theorem 2.8, the usual method to prove measurability of the attractor is to show the measurability of the absorbing set. However, as in (22), the absorbing set may be an uncountable union of a family of random sets and thus the measurability of the absorbing set seems to be unknown. In the application part, we will overcome this difficulty by considering two different universes.

3. **Non-autonomous stochastic sine-Gordon equation.** We consider the non-autonomous stochastic sine-Gordon equation on \( \mathbb{R}^n \):

\[
\begin{align*}
\begin{cases}
\begin{aligned}
u_{tt} + u_t + \Delta u + \lambda u + \beta \sin u &= f(x, t) + \epsilon u \circ dW/dt, \\
u(x, \tau) &= u_0, u_t(x, \tau) &= u_1, x \in \mathbb{R}^n, \tau \in \mathbb{R},
\end{aligned}
\end{cases}
\end{align*}
\]
where $\lambda > 0$, $\beta \in \mathbb{R}$, $f \in L^2_{loc}(\mathbb{R}, L^2(\mathbb{R}^n))$ and $W$ is a two-sided real-valued Wiener process.

### 3.1. Wiener space and translation of variables

As in [6, 3], we consider the corresponding canonical Wiener space $(\Omega, \mathcal{F}, P)$ for $W$. Let $\mathcal{C}(\mathbb{R}, \mathbb{R}) = \{\omega \in C(\mathbb{R}, \mathbb{R}) : \omega(0) = 0\}$ and endow it by the compact-open topology. Then, we denote $P$ by the Wiener measure on the Borel $\sigma$-algebra $\mathcal{F}$ of $\mathcal{C}(\mathbb{R}, \mathbb{R})$. We suppose

$$\Omega = \{\omega \in \mathcal{C}(\mathbb{R}, \mathbb{R}) : \lim_{t \to \pm \infty} \frac{\omega(t)}{t} = 0\}$$

where $\mathcal{F}$ is the restriction of $\mathcal{F}$ to $\Omega$. We also define a shift by $\theta_t : \omega \mapsto \omega(t)$ for $\omega \in \Omega$ and $t \in \mathbb{R}$.

By the above identification, the stochastic differential equation $dz + zdt = dW(t)$ has a solution given by the Ornstein-Uhlenbeck process

$$z(t,\omega) = -\int_0^t e^{t-s}(\theta_t\omega)(s)ds, \quad (\omega, t) \in \Omega \times \mathbb{R}. \quad (25)$$

By [4, 16, 17], we have the following convergence.

$$\lim_{t \to \pm \infty} \frac{z(t)}{t} = \lim_{t \to \pm \infty} \frac{1}{t} \int_{-t}^t z(s)ds = 0, \quad (26)$$

$$\lim_{t \to \pm \infty} \frac{1}{t} \int_{-t}^t |z(s)|^m ds = \frac{\Gamma(1+m/2)}{\sqrt{\pi}}, \forall m > 0, \quad (27)$$

where $\Gamma$ is the Gamma function. Let

$$v(t,\omega) = u_t(t,\omega) + \delta u(t,\omega) - \epsilon z(\theta_t\omega) u(t,\omega), \quad (28)$$

where $\delta$ is a suitable constant such that $\delta \in (0, 1)$ and $\delta_\lambda := \delta^2 - \delta + \lambda > 0$. Then (23) can be rewritten as:

$$\left\{ \begin{array}{l}
u_t + \delta u - \epsilon z(\theta_t\omega)u = v, \\
u_t + (1-\delta)v + \delta \chi u - \Delta u + \beta \sin u = (2\delta - \epsilon\varepsilon)uz - \epsilon\varepsilon v + f(x, t), \\
u(x, \tau) = u_\tau = u_0, v(x, \tau) = v_\tau := u_1 + \delta u_0 - u_0 \epsilon z(\theta_\tau\omega). \end{array} \right. \quad (29)$$

We then take the state space by $X = H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$, which is equipped with the following norm:

$$\|\varphi\|^2 := \delta_\lambda \|u\|^2 + \|\nabla u\|^2 + \|v\|^2, \forall \varphi = (u,v). \quad (30)$$

By the same method as in [10, 20, 36], one can prove the well-posedness of Eq.(29). That is, for each $\varphi_{\tau} = (u_\tau, v_\tau) \in X$, there is a unique solution

$$\varphi(\cdot, \tau, \omega, \varphi_{\tau}) = (u(\cdot, \tau, \omega, u_\tau), v(\cdot, \tau, \omega, v_\tau)) \in C([\tau, \infty), X).$$

By the same method as in [15, Corollary 22], the mapping $\omega \mapsto \varphi(t, \tau, \omega, \varphi_{\tau})$ is $\mathcal{F}$-measurable. Then we can define a cocycle $\Phi : \mathbb{R}^+ \times \mathbb{R} \times \Omega \times X \to X$ by

$$\Phi(t, \tau, \omega) \varphi_{\tau} = \varphi(t + \tau, \tau, \theta_{-\tau}\omega, \varphi_{\tau}), \forall (t, \tau, \omega, \varphi_{\tau}) \in \mathbb{R}^+ \times \mathbb{R} \times \Omega \times X. \quad (31)$$
3.2. **Hypotheses and two universes.** In order to provide the absorption estimates, we need to restrict the pair \( (\lambda, \beta) \) of coefficients.

**Hypothesis H1.** There is a \( \delta \in (0, 1) \) such that

\[
\delta_1 := \delta_\lambda - \delta = \delta^2 - 2\delta + \lambda > 0, \quad \delta_2 := 1 - \delta - \frac{\beta^2}{4\delta^2} > 0. \tag{32}
\]

The condition (32) is equivalent to \( \lambda > 2\delta - \delta^2 \) and \( \beta^2 < 4\delta^2(1 - \delta) \), which means \( |\beta| \) is small. One can show that the pair \( (\lambda, \beta) \) satisfies (32) provided \( \lambda > \frac{8}{9} \) and \( \beta^2 < \frac{16}{27} \).

We also need the following assumption of small noise.

**Hypothesis H2.** The density \( \epsilon \) of noise is small, that is,

\[
0 < \epsilon \leq \min\{1, \frac{\kappa_2}{8\kappa_1(\frac{2}{\sqrt{\pi}} + 1)}\} =: \epsilon_0, \quad \text{where}
\]

\[
\kappa_1 := \max\{4(\delta + 1), \frac{4(\delta + 1)}{\delta_\lambda}\}, \quad \kappa_2 := \min\{\frac{\delta_1}{\delta_\lambda}, \delta, \delta_2\}. \tag{34}
\]

We then give some hypotheses on the force.

**Hypothesis F1.** \( f \in L^2_{\text{loc}}(\mathbb{R}, L^2(\mathbb{R}^n)) \) is forward tempered:

\[
\sup_{r \geq \tau} \int_{-\infty}^{0} e^{\frac{\kappa_1}{4}\sigma} \|f(\sigma + r)\|^2 d\sigma < +\infty, \quad \forall \; \tau \in \mathbb{R}. \tag{35}
\]

We need two universes. Let \( \mathcal{D} \) be the usual universe of all tempered bi-parametric sets in \( X \). More precisely, \( D \in \mathcal{D} \) if and only if

\[
\lim_{t \to +\infty} e^{-\frac{\kappa_1}{4}t} \left\| \mathcal{D}_{t-}\theta_{-t}\omega \right\|^2_X = 0, \quad \forall (\tau, \omega) \in \mathbb{R} \times \Omega. \tag{36}
\]

We then focus on another universe \( \mathcal{D} \), which consists of all forward tempered sets, that is, \( D \in \mathcal{D} \) if and only if

\[
\lim_{t \to +\infty} e^{-\frac{\kappa_1}{4}t} \sup_{r \geq \tau} \left\| \mathcal{D}_{r-}\theta_{-r}\omega \right\|^2_X = 0, \quad \forall (\tau, \omega) \in \mathbb{R} \times \Omega. \tag{37}
\]

Obviously, \( \mathcal{D} \) is a sub-universe of \( \mathcal{D} \). It is easy to verify that \( \mathcal{D} \) is forward-union closed in the sense of Definition 2.6, while \( \mathcal{D} \) may not be forward-union closed.

4. **Forward-uniform estimates of solutions.** We need the auxiliary estimate of the random variable:

\[
Z(\omega) := |z(\omega)| + |z(\omega)|^2, \quad \omega \in \Omega. \tag{38}
\]

**Lemma 4.1.** If the hypothesis H2 is true, then, for \( \omega \in \Omega \), there are \( T_0 = T_0(\omega) > 0 \) and \( C_0(\omega) > 0 \) such that

\[
\epsilon \kappa_1 \int_{-t}^{0} Z(\theta_s\omega) ds \leq \frac{1}{8} \kappa_2 t, \quad \forall t \geq T_0, \quad \epsilon \leq \epsilon_0, \tag{39}
\]

\[
\epsilon \kappa_1 \int_{-t}^{0} Z(\theta_s\omega) ds \leq \frac{1}{8} \kappa_2 t + C_0(\omega), \quad \forall t \geq 0, \quad \epsilon \leq \epsilon_0. \tag{40}
\]
Proof. By Hypothesis H2 and (27), there is a $T_0 = T_0(\omega) > 0$ such that for all $t \geq T_0$ and $\epsilon \leq \epsilon_0$,

$$\epsilon \kappa_1 \int_{-t}^{0} Z(\theta_\omega)ds \leq \epsilon_0 \kappa_1 \left( \frac{2\Gamma(1)}{\sqrt{\pi}} + \frac{2\Gamma(\frac{3}{2})}{\sqrt{\pi}} \right) t$$

$$= \frac{\kappa_1 \kappa_2}{8 \kappa_1 (\frac{3}{\sqrt{\pi}} + 1)} \left( \frac{2}{\sqrt{\pi}} + 1 \right) t = \frac{1}{8} \kappa_2 t,$$

which proves (39). While (40) is fulfilled if $C_0(\omega) := \epsilon_0 \kappa_1 \int_{-T_0}^{0} Z(\theta_\omega)ds$.

4.1. Forward-pullback absorption. We establish existence of both $\overline{D}$-pullback and $\mathcal{D}$-pullback absorbing sets such that the $\overline{D}$-pullback absorbing set is random and the $\mathcal{D}$-pullback absorbing set is uniform in the future.

Lemma 4.2. Let the hypotheses H1, H2, F1 be fulfilled and let $\tau \in \mathbb{R}$, $\omega \in \Omega$.

(i) For each $\overline{D} \in \overline{\mathcal{D}}$, there is a $\overline{T} := \overline{T}(\tau, \omega, \overline{D}) > 0$ such that

$$||\varphi(\tau, t, \theta_{\tau-r}, \phi_{\tau-t})||^2_X \leq 1 + \epsilon \overline{R}(\tau, \omega)$$

for all $t \geq \overline{T}$ and $\phi_{\tau-t} \in \overline{D}_{\tau-t}(\theta_{\tau-t})$, where

$$\overline{R}(\tau, \omega) = \int_{-\infty}^{0} e^{\kappa_2 \sigma + \epsilon_0 \kappa_1 \int_{\varphi(\tau)}^{0} Z(\theta_\omega)ds} \parallel f(\sigma + \tau) \parallel^2 d\sigma.$$ 

(ii) For each $\mathcal{D} \in \mathcal{D}$, there is a $T := T(\tau, \omega, \mathcal{D}) > 0$ such that

$$||\varphi(r, t, \theta_{\tau-r}, \phi_{\tau-t})||^2_X \leq 1 + \epsilon R(\tau, \omega),$$

$$\int_{r-t}^{r} e^{f(\tau)(\kappa_2 + \epsilon_0 \kappa_1) Z(\theta_\omega)} ds \parallel \varphi(\sigma, r - t, \theta_{\tau-r}, \phi_{\tau-t}) \parallel^2_X d\sigma \leq \epsilon(1 + R(\tau, \omega),$$

for all $r \geq \tau$, $t \geq T$ and $\phi_{\tau-t} \in \mathcal{D}_{\tau-t}(\theta_{\tau-t})$, where

$$R(\tau, \omega) = \sup_{r \geq \tau} \overline{R}(r, \omega).$$

Proof. Let $\tau \in \mathbb{R}$ be fixed and $r \geq \tau$. Taking the inner product of the second equation in (29) with $v(s) := v(s, r - t, \theta_{\tau-r}, \phi_{\tau-t})$, $s \geq r - t$, we obtain

$$\frac{d}{ds} \parallel v \parallel^2 + 2(1 - \delta) \parallel v \parallel^2 + 2 \delta \chi(u, v) - 2(\Delta u, v) + 2(\delta \sin u, v)$$

$$+ \epsilon(\theta_s, \omega)\epsilon(z(\theta_s, \omega))(u, v) - 2 \epsilon \theta_s(z(\theta_s, \omega)) \parallel v \parallel^2 + 2(f(x, s), v).$$

By the first equation of (29), we see

$$2(u, v) = \frac{d}{ds} \parallel u \parallel^2 + 2(\delta - \epsilon \theta_s(z(\theta_s, \omega))) \parallel u \parallel^2.$$

$$- 2(\Delta u, v) = \frac{d}{ds} \parallel \nabla u \parallel^2 + 2(\delta - \epsilon \theta_s(z(\theta_s, \omega))) \parallel \nabla u \parallel^2.$$

Hence, the equality (46) can be rewritten as

$$\frac{d}{ds} \parallel \varphi \parallel^2_X + I_1 + I_2 + I_3 = I_4 + I_5,$$
where $I_1 := 2(1 - \delta) \|v\|^2$. By $\kappa_1 \geq 4$,
\[ I_2 := 2(\delta - \epsilon z(\theta_{s-r}, \omega))(\|u\|^2 + \|\nabla u\|^2) \]
\[ = 2\delta \|u\|^2 + 2\delta \|\nabla u\|^2 - 2\epsilon z(\theta_{s-r}, \omega)(\|u\|^2 + \|\nabla u\|^2) \]
\[ \geq 2\delta \|u\|^2 + 2\delta \|\nabla u\|^2 - \frac{\kappa_1}{2} \epsilon z(\theta_{s-r}, \omega) \|\varphi\|^2_X. \]

By $|\sin u| \leq |u|$ and the Young inequality,
\[ I_3 := 2(\beta \sin u, v) \geq -2\beta^2 \|u\|^2 - \frac{\beta^2}{2\delta} \|v\|^2. \]

Notice that $\delta_1 = \delta - \delta > 0$ and $\delta_2 := 1 - \delta - \frac{\beta^2}{4\delta} > 0$ as in $\textbf{H1}$, we have
\[ I_1 + I_2 + I_3 \geq 2(\delta \|u\|^2 + \delta \|v\|^2 + \delta \|\nabla u\|^2) - \frac{\kappa_1}{2} \epsilon z(\theta_{s-r}, \omega) \|\varphi\|^2_X \]
\[ \geq 2\kappa_2 \|\varphi\|^2_X - \frac{\kappa_1}{2} \epsilon z(\theta_{s-r}, \omega) \|\varphi\|^2_X. \quad (48) \]

On the other hand, by $0 < \epsilon \leq \epsilon_0 < 1$ and so $\epsilon \leq \epsilon_0$, we can write
\[ I_4 := 2(\delta + \epsilon z(\theta_{s-r}, \omega))z(\theta_{s-r}, \omega)(\|u\|^2 + \|v\|^2) \]
\[ \leq (\delta + \epsilon_0 z(\theta_{s-r}, \omega))z(\theta_{s-r}, \omega)((\|u\|^2 + \|v\|^2) + 2\epsilon z(\theta_{s-r}, \omega) \|v\|^2) \]
\[ \leq 2(\delta + 1) \epsilon(z(\theta_{s-r}, \omega) + |z(\theta_{s-r}, \omega)|)(\|u\|^2 + \|v\|^2) \]
\[ \leq \frac{\kappa_1}{2} \epsilon(z(\theta_{s-r}, \omega) + |z(\theta_{s-r}, \omega)|^2) \|\varphi\|^2_X. \quad (49) \]
\[ I_5 := 2(f(x, s), v) \leq 2 \|f(s)\| \|v\| \leq \frac{\kappa_2}{2} \|\varphi\|^2_X + c \|f(s)\|^2. \quad (50) \]

By (48)-(50), we obtain that
\[ \sum_{i=1}^{3} I_i - \sum_{i=4}^{5} I_i \geq \frac{3}{4} \kappa_2 \|\varphi\|^2_X - \epsilon \kappa_1 Z(\theta_{s-r}, \omega) \|\varphi\|^2_X - c \|f(s)\|^2, \quad (51) \]

where $Z(\omega) = |z(\omega)| + |z(\omega)|^2$. We substitute (51) into (47) to find
\[ \frac{d}{ds} \|\varphi\|^2_X + (\kappa_2 - \epsilon \kappa_1 Z(\theta_{s-r}, \omega)) \|\varphi\|^2_X + \frac{\kappa_2}{2} \|\varphi\|^2_X \leq c \|f(s)\|^2. \quad (52) \]

Applying the Gronwall inequality to (52) over $[r - t, s]$, we have
\[ \|\varphi(s, r - t, \theta_{r-t}, \varphi_{r-t})\|^2_X \]
\[ \leq e^{\frac{\kappa_2}{2} \int_{r-t}^{s} e^{\epsilon \kappa_1 Z(\theta_{s-r}, \omega)} \|\varphi(\sigma, r - t, \theta_{r-t}, \varphi_{r-t})\|^2_X \, d\sigma} \]
\[ \leq e^{-\kappa_2(s-r+t) + \epsilon \kappa_1 \int_{r-t}^{s} Z(\theta_{s-r}, \omega) \, d\sigma} \|\varphi(\sigma, r - t, \theta_{r-t}, \varphi_{r-t})\|^2_X \]
\[ + c \int_{r-t}^{s-r} e^{\kappa_2(\sigma+r-s) + \epsilon \kappa_1 \int_{r-t}^{s} Z(\theta_{s-r}, \omega) \, d\sigma} \|f(\sigma + r)\|^2 \, d\sigma. \quad (53) \]

Let $s = r = \tau$ in (53). Then,
\[ \|\varphi(\tau, \tau - t, \theta_{\tau-t}, \varphi_{\tau-t})\|^2_X \leq e^{-\kappa_2 t + \epsilon \kappa_1 \int_{0}^{t} Z(\theta_{s-r}, \omega) \, d\sigma} \|\varphi_{\tau-t}\|^2_X + c \hat{R}(\tau, \omega), \quad (54) \]

where $\hat{R}(\tau, \omega)$ is given by (42). By (40) in Lemma 4.1, we have
\[ e^{-\kappa_2 t + \epsilon \kappa_1 \int_{0}^{t} Z(\theta_{s-r}, \omega) \, d\sigma} \leq e^{C_0(\omega) e^{-\frac{\kappa_2}{2} t}}, \quad \forall t \geq 0, \epsilon \leq \epsilon_0. \quad (55) \]
Given $\tilde{D} \in \tilde{\mathcal{D}}$, then for all $\varphi_{\tau-t} \in \tilde{D}_{\tau-t}(\theta^{-1}\omega)$,
\[
e^{-\kappa_2t+\kappa_1e} \int_{-\infty}^{t} Z(\theta_{\tau-t})d\sigma \|\varphi_{\tau-t}\|_{X}^2 \\
\leq e^{C_0(\omega)} e^{-\frac{\kappa_2}{2} t} \left\|\tilde{D}(t, \theta_{-t}\omega)\right\|_{X}^2 \to 0, \tag{56}
\]
as $t \to +\infty$. Thereby, both (54) and (56) imply (41).

Let $s = r$ in (53) and take the supremum over $r \in [\tau, +\infty)$, we obtain
\[
\sup_{r \geq \tau} \|\varphi(r, r-t, \theta_{-t}\omega, \varphi_{r-t})\|_{X}^2 \\
+ \frac{\kappa_2}{2} \sup_{r \geq \tau} \int_{r-t}^{r} e^{\int_{\tau-t}^{\sigma} (\kappa_2-\kappa_1e) Z(\theta_{\tau-t})d\sigma} \|\varphi(\sigma, r-t, \theta_{-t}\omega, \varphi_{r-t})\|_{X}^2 d\sigma \\
\leq e^{-\kappa_2t+\kappa_1e} \int_{0}^{t} Z(\theta_{\tau-t})d\sigma \sup_{r \geq \tau} \|\varphi_{r-t}\|_{X}^2 + c \sup_{r \geq \tau} \overline{R}(r, \omega). \tag{57}
\]

Given $\mathcal{D} \in \mathcal{D}$, by (55), if $\varphi_{r-t} \in \mathcal{D}_{r-t}(\theta_{-t}\omega)$ with $r \geq \tau$, then
\[
e^{-\kappa_2t+\kappa_1e} \int_{-\infty}^{t} Z(\theta_{\tau-t})d\sigma \sup_{r \geq \tau} \|\varphi_{r-t}\|_{X}^2 \\
\leq e^{C_0(\omega)} e^{-\frac{\kappa_2}{2} t} \left\|\mathcal{D}(r-t, \theta_{-t}\omega)\right\|_{X}^2 \to 0 \tag{58}
\]
as $t \to +\infty$. From (57) and (58), we obtain the assertion (ii) of the lemma. \qed

**Corollary 2.** Let the hypotheses H1, H2, F1 be satisfied and let $\Phi$ be the cocycle associated with Eq. (29).

(i) $\Phi$ has a $\tilde{\mathcal{D}}$-pullback absorbing set $\tilde{K} \in \tilde{\mathcal{D}}$ such that $\tilde{K}$ is random and defined by
\[
\tilde{K}_{\tau}(\omega) = \{\varphi \in X : \|\varphi\|^2 \leq 1 + c \overline{R}(\tau, \omega), \forall (\tau, \omega) \in \mathbb{R} \times \Omega \}. \tag{59}
\]

(ii) $\Phi$ has a $\mathcal{D}$-pullback absorbing set $\mathcal{K} \in \mathcal{D}$, given by
\[
\mathcal{K}_{\tau}(\omega) = \{\varphi \in X : \|\varphi\|^2 \leq 1 + c \sup_{r \geq \tau} \overline{R}(r, \omega) =: 1 + c \overline{R}(\tau, \omega)\} \tag{60}
\]
for all $(\tau, \omega) \in \mathbb{R} \times \Omega$. Moreover, the absorption is uniform in the future, that is, for each $(\tau, \omega, D) \in \mathbb{R} \times \Omega \times \mathcal{D}$, there is a $T := T(\tau, \omega, D) > 0$ such that
\[
\bigcup_{r \geq \tau} \Phi(t, r-t, \theta_{-t}\omega) \mathcal{D}_{r-t}(\theta_{-t}\omega) \subset \mathcal{K}_{\tau}(\omega), \forall t \geq T. \tag{61}
\]

**Proof.** (i) By (41) in Lemma 4.2, $\tilde{K}$ is $\tilde{\mathcal{D}}$-pullback absorbing. The integral $\overline{R}(\tau, \omega)$ as in (42) is obviously measurable in $\omega$ and so $\tilde{K}$ is random. By (62) below, we have $\tilde{K} \in \tilde{\mathcal{D}}$.

(ii) By (43) in Lemma 4.2, $\mathcal{K}$ is $\mathcal{D}$-pullback absorbing such that the absorption is uniform in the future, i.e. (61) holds true. By Lemma 4.1 and Hypothesis F1,
\[
R(\tau, \omega) = \sup_{r \geq \tau} \int_{-\infty}^{0} e^{\kappa_2 \sigma + \kappa_1 e} \int_{-\infty}^{\sigma} Z(\theta_{\tau-t})d\sigma \|f(\sigma + r)\|^2 d\sigma \\
\leq ce^{C_0(\omega)} \sup_{r \geq \tau} \int_{-\infty}^{0} e^{-\frac{\kappa_2}{2} \sigma} \|f(\sigma + r)\|^2 d\sigma < +\infty.
\]
Finally, we show $K \in \mathcal{D}$. It suffices to prove that $R$ is forward tempered with the growth rate $\frac{24}{\kappa}$. Indeed, by Lemma 4.1 and Hypothesis F1,

$$e^{-\frac{24}{\kappa}t} \sup_{r \geq \tau} R(r - t, \theta_{-t}\omega) = e^{-\frac{24}{\kappa}t} R(\tau - t, \theta_{-t}\omega)$$

$$\leq e^{-\frac{24}{\kappa}t} \sup_{r \geq \tau} \int_{-t}^{0} e^{\omega z(s+t) + c \kappa} L(z(\theta_{-t}\omega)) d\sigma \left\| f(\sigma + r) \right\|^2 d\sigma$$

$$\leq e^{-\frac{24}{\kappa}t} e^{C_{0}(\omega)} \sup_{r \geq \tau} \int_{-\infty}^{-t} e^{\frac{24}{2\kappa}(\sigma+t)-\frac{24}{\kappa}\sigma} \left\| f(\sigma + r) \right\|^2 d\sigma$$

$$\leq e^{C_{0}(\omega)} e^{-\frac{24}{\kappa}t} \sup_{r \geq \tau} \int_{-\infty}^{0} e^{\frac{24}{2\kappa}\sigma} \left\| f(\sigma + r) \right\|^2 d\sigma \to 0$$

(62)

as $t \to +\infty$. The proof is complete. $\square$

We remark here that the coefficient $1/24$ in Hypothesis F1 is used in (62) to ensure $K \in \mathcal{D}$.

4.2. Forward tail-estimates. We need a further assumption for $f$.

**Hypothesis F2.** The force $f$ is forward tail-small:

$$\lim_{k \to +\infty} \sup_{r \geq \tau} \int_{Q_k} \int_{-\infty}^{0} e^{\frac{24}{\kappa}\sigma} \left| f(\sigma + r) \right|^2 d\sigma = 0, \quad \forall \tau \in \mathbb{R}. \quad (63)$$

where $Q_k = \{x \in \mathbb{R}^n : |x| < k\}$ and $Q_k^c$ is the complement set of $Q_k$. If we omit the supremum in (63), then it holds true in view of Hypothesis F1 and Lebesgue controlled convergence theorem.

**Lemma 4.3.** Let the hypotheses H1-H2, F1-F2 be satisfied. Then, for $(\tau, \omega, D) \in \mathbb{R} \times \Omega \times \mathcal{D},$

$$\lim_{t,k \to +\infty} \sup_{r \geq \tau} \sup_{\varphi \in D_{r-t}(\theta_{-r}\omega)} \left\| \varphi(r, r - t, \theta_{-t}\omega, \varphi_{r-t}) \right\|_{X(Q_k^c)}^2 = 0. \quad (64)$$

**Proof.** Let $\rho_k(x) := \rho(\frac{|x|^2}{k})$ for each $k \geq 1$ and $x \in \mathbb{R}^n$, where $\rho : \mathbb{R}^+ \to [0, 1]$ is a smooth function such that

$$\rho(s) = \begin{cases} 0, & \text{if } 0 \leq s \leq 1, \\ 1, & \text{if } s \geq 2. \end{cases} \quad (65)$$

Taking the inner product of the second equation in (29) with $\rho_k v(s, r - t, \theta_{-t}\omega)$ and integrating over $\mathbb{R}^n$, we see

$$\frac{d}{ds} \int_{\mathbb{R}^n} \rho_k |v|^2 \, dx + 2(1 - \delta) \int_{\mathbb{R}^n} \rho_k |v|^2 \, dx$$

$$+ 2\delta \int_{\mathbb{R}^n} \rho_k uv \, dx - 2 \int_{\mathbb{R}^n} \rho_k \Delta uv \, dx$$

$$= -2\beta \int_{\mathbb{R}^n} \rho_k v \sin u \, dx + 2(2\delta - \epsilon z(\theta_{s-r}\omega)) \epsilon z(\theta_{s-r}\omega) \int_{\mathbb{R}^n} \rho_k uv \, dx$$

$$- 2\epsilon z(\theta_{s-r}\omega) \int_{\mathbb{R}^n} \rho_k |v|^2 \, dx + 2 \int_{\mathbb{R}^n} \rho_k f(x, s) v \, dx. \quad (66)$$
It follows from the first equation in (29) that

\[
2 \int_{\mathbb{R}^n} \rho_k u v dx = \frac{d}{ds} \int_{\mathbb{R}^n} \rho_k |u|^2 dx + 2(\delta - \epsilon \varepsilon \theta_{s-\tau},\omega)) \int_{\mathbb{R}^n} \rho_k |u|^2 dx. \tag{67}
\]

\[
-2 \int_{\mathbb{R}^n} \rho_k \Delta u v dx = \frac{d}{ds} \int_{\mathbb{R}^n} \rho_k |\nabla u|^2 dx + 2(\delta - \epsilon \varepsilon \theta_{s-\tau},\omega)) \int_{\mathbb{R}^n} \rho_k |\nabla u|^2 dx
+ 2 \int_{\mathbb{R}^n} \nabla \rho_k \nabla u v dx. \tag{68}
\]

It yields from (66)-(68) that

\[
d \frac{d}{ds} \int_{\mathbb{R}^n} \rho_k \phi dx + J_1 + J_2 = J_3 + J_4 + J_5 + J_6, \tag{69}
\]

where \(\phi\) is defined by

\[\phi := \delta \lambda |u|^2 + |\nabla u|^2 + |v|^2,\]

and \(J_i (i = 1, 2, 3, 4, 5, 6)\) are defined and estimated as follows.

\[J_1 := 2(1 - \delta) \int_{\mathbb{R}^n} \rho_k |v|^2 dx.\]

By \(\kappa_1 \geq 4\), we have

\[J_2 := 2(\delta - \epsilon \varepsilon \theta_{s-\tau},\omega)) \left( \delta \lambda \int_{\mathbb{R}^n} \rho_k |u|^2 dx + \int_{\mathbb{R}^n} \rho_k |\nabla u|^2 dx \right) \geq 2\delta \delta_\lambda \int_{\mathbb{R}^n} \rho_k |u|^2 dx + 2\delta \int_{\mathbb{R}^n} \rho_k |\nabla u|^2 dx - \frac{\kappa_1}{2} \epsilon |z(\theta_{s-\tau},\omega))| \int_{\mathbb{R}^n} \rho_k \phi dx.\]

By \(|\sin u| \leq |u|\) and the Young inequality, we obtain

\[J_3 := -2\beta \int_{\mathbb{R}^n} \rho_k v \sin u dx \leq 2\delta^2 \int_{\mathbb{R}^n} \rho_k |u|^2 dx + \frac{\beta^2}{2\delta^2} \int_{\mathbb{R}^n} \rho_k |v|^2 dx.\]

So, we have

\[J_1 + J_2 - J_3 \geq 2(\delta \delta_1 \int_{\mathbb{R}^n} \rho_k |u|^2 dx + \delta_2 \int_{\mathbb{R}^n} \rho_k |v|^2 dx + \delta \int_{\mathbb{R}^n} \rho_k |\nabla u|^2 dx)
- \frac{\kappa_1}{2} \epsilon |z(\theta_{s-\tau},\omega))| \int_{\mathbb{R}^n} \rho_k \phi dx
\geq 2\kappa_2 \int_{\mathbb{R}^n} \rho_k \phi dx - \frac{\kappa_1}{2} \epsilon |z(\theta_{s-\tau},\omega))| \int_{\mathbb{R}^n} \rho_k \phi dx. \tag{70}\]

By \(0 < \epsilon \leq \varepsilon_0 \leq 1\) and so \(\epsilon^2 \leq \epsilon\), and by the Young inequality, we get

\[J_4 := 2(2\delta - \epsilon \varepsilon \theta_{s-\tau},\omega)) \varepsilon \theta_{s-\tau},\omega)) \int_{\mathbb{R}^n} \rho_k uv dx - 2\epsilon \varepsilon \theta_{s-\tau},\omega)) \int_{\mathbb{R}^n} \rho_k |v|^2 dx
\leq 2(\delta + 1) \epsilon (|z(\theta_{s-\tau},\omega))| + |(\theta_{s-\tau},\omega))|^2) \left( \int_{\mathbb{R}^n} \rho_k |u|^2 dx + \int_{\mathbb{R}^n} \rho_k |v|^2 dx \right)
\leq \frac{\kappa_1}{2} \epsilon Z(\theta_{s-\tau},\omega)) \int_{\mathbb{R}^n} \rho_k \phi dx. \tag{71}\]
By $v = u_s + \delta u - \epsilon zu$ as given in (29), $\|\nabla \rho_k\|_\infty \leq \frac{C}{r}$, the Cauchy-Schwarz inequality and the Young inequality, we obtain

\[
J_5 := -2 \int_{\mathbb{R}^n} \nabla \rho_k \nabla uv \, dx \\
= -2 \int_{\mathbb{R}^n} u_s \nabla \rho_k \nabla u \, dx - 2(\delta - \epsilon z(\theta_{s-r}\omega)) \int_{\mathbb{R}^n} u \nabla \rho_k \nabla u \, dx \\
\leq \frac{C}{k} (\|u_s\| \|\nabla u\| + (1 + |z(\theta_{s-r}\omega)|) \|u\| \|\nabla u\|) \\
\leq \frac{C}{k} ((1 + |z(\theta_{s-r}\omega)|) \|u\| + \|v\|) \|\nabla u\| + (1 + |z(\theta_{s-r}\omega)|) \|u\| \|\nabla u\|) \\
\leq \frac{C}{k} (1 + |z(\theta_{s-r}\omega)|)^4 + \|\varphi\|_X^4) \leq \frac{C}{k} (e^{z(\theta_{s-r}\omega)} + \|\varphi\|_X^4).
\]

\[
J_6 := 2 \int_{\mathbb{R}^n} \rho_k f(x, s) v \, dx \leq \frac{K_2}{2} \int_{\mathbb{R}^n} \rho_k \phi \, dx + c \int_{\mathbb{R}^n} \rho_k |f(s)|^2 \, dx.
\]  
(72)

By (70)-(72), for each $\eta > 0$, we can take $k_\eta \in \mathbb{N}$ such that for all $k \geq k_\eta$,

\[
J_1 + J_2 - \sum_{i=3}^{6} J_i \geq \frac{3}{2} K_2 \int_{\mathbb{R}^n} \rho_k \phi \, dx - \epsilon \kappa_1 Z(\theta_{s-r}\omega) \int_{\mathbb{R}^n} \rho_k \phi \, dx \\
- \eta (e^{z(\theta_{s-r}\omega)} + \|\varphi\|_X^4) - c \int_{\mathbb{R}^n} \rho_k |f(s)|^2 \, dx.
\]  
(73)

Substituting (73) into (69), we find that

\[
\frac{d}{ds} \int_{\mathbb{R}^n} \rho_k \phi \, dx + (\kappa_2 - \epsilon \kappa_1 Z(\theta_{s-r}\omega)) \int_{\mathbb{R}^n} \rho_k \phi \, dx + \frac{K_2}{2} \int_{\mathbb{R}^n} \rho_k \phi \, dx \\
\leq c \int_{\mathbb{R}^n} \rho_k |f(s)|^2 \, dx + \eta (e^{z(\theta_{s-r}\omega)} + \|\varphi\|_X^4).
\]  
(74)

Applying the Gronwall inequality to (74) over $[r - t, r]$ and taking the supremum over $r \in [r, +\infty)$, we have

\[
\sup_{r \geq r_\tau} \int_{r - t}^{r} \rho_k \phi \, dx + \frac{K_2}{2} \sup_{r \geq r_\tau} \int_{r - t}^{r} e^{\int_{r_\tau}^{r} (\kappa_2 - \epsilon \kappa_1 Z(\theta_{s-r}\omega)) \, d\sigma} \int_{r_\tau}^{r} \rho_k \phi \, dx \, d\sigma \\
\leq J_7 + J_8 + \eta(J_9 + J_{10}),
\]  
(75)

where $J_i (i = 7, 8, 9, 10)$ are defined and estimated as follows. By Lemma 4.1,

\[
J_7 := \sup_{r \geq r_\tau} e^{-\frac{1}{2} \kappa_2 t + \epsilon \kappa_1 \int_{r_\tau}^{r} Z(\theta_{s-r}\omega) \, d\sigma} \int_{r_\tau}^{r} \rho_k \phi \, dx \leq e^{C_\lambda(\omega)} e^{-\frac{1}{2} \kappa_2 t} \sup_{r \geq r_\tau} \|\varphi_{r-t}\|_X^2 \\
\leq e^{C_\lambda(\omega)} e^{-\frac{1}{2} \kappa_2 t} (e^{-\frac{1}{2} \kappa_2 t} \sup_{r \geq r_\tau} \|D_{r-t}(\theta_{s-r}\omega)\|_X^2) \rightarrow 0,
\]

as $t \rightarrow +\infty$. By Hypothesis \textbf{F2},

\[
J_8 := c \sup_{r \geq r_\tau} \int_{-\infty}^{0} e^{\kappa_2 \sigma + \epsilon \kappa_1 \int_{r_\tau}^{r} Z(\theta_{s-r}\omega) \, d\sigma} \int_{\mathbb{R}^n} \rho_k |f(\sigma + r)|^2 \, dx \, d\sigma \\
\leq ce^{C_\lambda(\omega)} \sup_{r \geq r_\tau} \int_{-\infty}^{0} e^{\frac{1}{2} \kappa_2 \sigma} \int_{Q_k^r} |f(\sigma + r)|^2 \, dx \, d\sigma \rightarrow 0,
\]
as \( k \to +\infty \). By (26) and (40), we have
\[
J_9 := \sup_{r \geq \tau} \int_{-\infty}^{0} e^{\kappa_2 \sigma + \kappa_1 \int_{s}^{0} Z(\theta_s \omega) d\sigma} e^{\kappa_2 \sigma + \kappa_1 \int_{s}^{0} Z(\theta_s \omega) d\sigma} d\sigma \leq e^{2C_0(\omega)} \sup_{r \geq \tau} \int_{-\infty}^{0} e^{\frac{4}{3} \kappa_2 \sigma} d\sigma = \frac{4}{3\kappa_2} e^{2C_0(\omega)} < +\infty.
\]
Finally, we consider the last term:
\[
J_{10} := \sup_{r \geq \tau} \int_{r-t}^{r} e^{\kappa_2 (\sigma-r) + \kappa_1 \int_{s}^{0} Z(\theta_s \omega) d\sigma} \left\| \varphi(\sigma, r-t, \theta_{-r} \omega, \varphi_0) \right\|_{X}^4 d\sigma,
\]
where the fourth power of the norm is involved. By applying (53), we split it as
\[
J_{10} = c(J_{10,1} + J_{10,2}), \quad \text{where}
\]
\[
J_{10,1} := \sup_{r \geq \tau} \int_{r-t}^{r} e^{\kappa_2 (\sigma-r) + \kappa_1 \int_{s}^{0} Z(\theta_s \omega) d\sigma} \left\| \varphi(\sigma, r-t, \theta_{-r} \omega, \varphi_0) \right\|_{X}^4 d\sigma,
\]
\[
J_{10,2} := e^{3} \sup_{r \geq \tau} \int_{r-t}^{r} e^{\kappa_2 (\sigma-r) + \kappa_1 \int_{s}^{0} Z(\theta_s \omega) d\sigma} \left\| f(s+r) \right\|_{X}^2 d\sigma
\]
\[
\leq 8 e^{3C_0(\omega)} \left( \sup_{r \geq \tau} \int_{r-t}^{r} e^{\kappa_2 (\sigma-r) + \kappa_1 \int_{s}^{0} Z(\theta_s \omega) d\sigma} \left\| f(s+r) \right\|_{X}^2 d\sigma \right)^2 < +\infty.
\]
(76)

which is bounded as \( t \to +\infty \) in view of (37). By Hypothesis F1,
\[
J_{10,2} := e^{3} \sup_{r \geq \tau} \int_{r-t}^{r} e^{\kappa_2 (\sigma-r) + \kappa_1 \int_{s}^{0} Z(\theta_s \omega) d\sigma} \left\| f(s+r) \right\|_{X}^2 d\sigma
\]
\[
\leq e^{3} \sup_{r \geq \tau} \int_{r-t}^{r} e^{\kappa_2 (\sigma-r) + \kappa_1 \int_{s}^{0} Z(\theta_s \omega) d\sigma} \left\| f(s+r) \right\|_{X}^2 d\sigma
\]
\[
\leq 8 e^{3C_0(\omega)} \left( \sup_{r \geq \tau} \int_{r-t}^{r} e^{\kappa_2 (\sigma-r) + \kappa_1 \int_{s}^{0} Z(\theta_s \omega) d\sigma} \left\| f(s+r) \right\|_{X}^2 d\sigma \right)^2 < +\infty.
\]
(77)

Both (76) and (77) imply that \( J_{10} \) is bounded as \( t \to +\infty \). Therefore, we infer from (75) that
\[
\sup_{r \geq \tau} \int_{\mathbb{R}^n} \rho_k \phi(r, r-t, \theta_{-r} \omega) dx \to 0,
\]
which proves (64) as desired. \( \square \)
4.3. Forward uniformness of the estimates inside a ball. Let \( \rho_k \) \((k \geq 1)\) be the cut-off function as in the proof of Lemma 4.3. For the solution \( \phi = (u,v) \) of Eq. (29), we denote by
\[
\hat{\rho}_k(x) := 1 - \rho_k(x), \quad x \in \mathbb{R}^n,
\]
\[
\hat{u} := \hat{\rho}_k u, \quad \hat{v} := \hat{\rho}_k v, \quad \hat{\phi} := (\hat{u}, \hat{v}) = \hat{\rho}_k \phi.
\] (78)
Since \( \hat{\rho}_k(x) = 0 \) for \(|x| \geq \sqrt{2}k\), it follows that \( \hat{\phi} \in X(Q_{\sqrt{2}k}) = H^1_0(Q_{\sqrt{2}k}) \times L^2(Q_{\sqrt{2}k}) \), which is endowed with the norm as in (30):
\[
\|\hat{\phi}\|_{X(Q_{\sqrt{2}k})}^2 := \|\hat{\phi}\|_X^2 = \|\hat{u}\|^2 + \|\nabla \hat{u}\|^2 + \|\hat{v}\|^2.
\]
So, \( \hat{\phi} \) has the unique orthogonal decomposition:
\[
\hat{\phi} = P_i(\hat{u}, \hat{v}) + (I - P_i)(\hat{u}, \hat{v}) = (\hat{u}_i, \hat{v}_i) \oplus (\hat{u}_i^+, \hat{v}_i^+), \quad \forall i \in \mathbb{N},
\]
where \( P_i : L^2(Q_{\sqrt{2}k}) \times L^2(Q_{\sqrt{2}k}) \to X_i \times X_i \) is the canonical projection, \( X_i = \text{span}\{e_1, e_2, ..., e_i\} \subset H^1_0(Q_{\sqrt{2}k}) \) and \( \{e_i\}_{i=1}^\infty \) is the family of eigenfunctions for \(-\Delta \) (on \( L^2(Q_{\sqrt{2}k}) \)) with the corresponding eigenvalues:
\[
0 < \lambda_1 \leq \lambda_2 \leq ... \leq \lambda_i \to +\infty.
\]
A further assumption is needed.

**Hypothesis F3.** The force \( f \) is forward complement-small:
\[
\lim_{i,k \to \infty, r \to \tau} \sup_{\sigma \in [0,1]} \int_{-\infty}^0 (I - P_i)(\hat{\rho}_k f(\sigma + r)) \|d\sigma = 0, \quad \forall \tau \in \mathbb{R}. \] (79)

Now, by multiplying (29) by \( \hat{\rho}_k \), we infer from (78) that \( \hat{\phi} \) satisfies the following equation:
\[
\begin{cases}
\hat{u}_t + \delta \hat{u} - \varepsilon \hat{z}(\theta \omega) \hat{u} = \hat{v}, \\
\hat{v}_t + (1 - \delta) \hat{v} + \delta \lambda \hat{u} - \Delta \hat{u} + \beta \hat{\rho}_k \sin u = \\
(2\delta - \varepsilon \omega) \hat{z} \hat{u} - \varepsilon \hat{z} \hat{v} - u \Delta \hat{\rho}_k - 2\nabla \hat{\rho}_k \nabla u + \hat{\rho}_k f(x,t).
\end{cases}
\] (80)

**Lemma 4.4.** Let the hypotheses H1,H2, F1,F3 be satisfied. Then, for \( \mathcal{D} \in \mathcal{D}, \quad \tau \in \mathbb{R} \) and \( \omega \in \Omega, \)
\[
\lim_{i,k \to \infty, r \to \tau} \sup_{s \geq r - t} \|\hat{\phi}(r, r - t, \theta_{r-t}; \hat{\phi}_{r-t})\|_{X(Q_{\sqrt{2}k})}^2 = 0,
\] (81)
uniformly in \( \hat{\phi}_{r-t} \in \mathcal{D}_{r-t}(\theta_{r-t}\omega) \).

**Proof.** We take the inner product of the second equation in (80) with
\[
\hat{v}_i^+(s) := \hat{v}_i^+(s, r - t, \theta_{r-t}; \hat{v}_{i-r-t}), \quad s \geq r - t,
\]
the result is
\[
\frac{d}{ds} \|\hat{v}_i^+\|^2 + 2(1 - \delta) \|\hat{v}_i^+\|^2 + 2\delta \lambda (\hat{u}_i^+, \hat{v}_i^+) - 2(\Delta \hat{u}_i^+, \hat{v}_i^+) \\
= -2\beta (\hat{\rho}_k \sin u, \hat{v}_i^+) + 2(2\delta - \varepsilon \hat{z}(\theta_{r-t}\omega)) \varepsilon \hat{z}(\theta_{r-t}\omega) (\hat{u}_i^+, \hat{v}_i^+) - 2\varepsilon \hat{z}(\theta_{r-t}\omega) \|\hat{v}_i^+\|^2 \\
-2\hat{u} \Delta \hat{\rho}_k + 2\nabla \hat{\rho}_k \nabla u, \hat{v}_i^+) + 2(\hat{\rho}_k f(s), \hat{v}_i^+).
\] (82)
Applying \( I - P_i \) to the first equation in (80), we obtain
\[
\hat{v}_i^+ = \frac{d\hat{u}_i^+}{ds} + \delta \hat{u}_i^+ - \varepsilon \hat{z}(\theta_{r-t}\omega) \hat{u}_i^+,
\]
We infer from (86) and (87) that inequality implies that

\[ 2(\hat{u}_t^+, \hat{v}_t^+) = \frac{d}{ds} \| \hat{u}_t^+ \|^2 + 2(\delta - \epsilon z(\theta_{s-r\omega})) \| \hat{u}_t^+ \|^2. \]  

(83)

\[ -2(\Delta \hat{u}_t^+, \hat{v}_t^+) = \frac{d}{ds} \| \nabla \hat{u}_t^+ \|^2 + 2(\delta - \epsilon z(\theta_{s-r\omega})) \| \nabla \hat{u}_t^+ \|^2. \]  

(84)

We substitute (83) and (84) into (82) to find

\[ \frac{d}{ds} \| \hat{\varphi}_t^+ \|^2_{X(Q,\mathbb{V}_k)} + L_1 + L_2 = L_3 + L_4 + L_5 + L_6, \]  

(85)

where \( L_j(j = 1, \cdots, 6) \) are defined and estimated as follows:

\[ L_1 := 2(1 - \delta) \| \hat{u}_t^+ \|^2, \]

\[ L_2 := 2(\delta - \epsilon z(\theta_{s-r\omega})) \| \delta \| \hat{u}_t^+ \|^2 + \| \nabla \hat{u}_t^+ \|^2, \]

\[ \geq 2\delta \delta \| \hat{u}_t^+ \|^2 + 2\delta \| \nabla \hat{u}_t^+ \|^2 - \frac{k_2}{2} \epsilon |z(\theta_{s-r\omega})| \| \hat{\varphi}_t^+ \|^2_{X(Q,\mathbb{V}_k)}. \]  

(86)

By \((I - P_t)^2 = I - P_t, | \sin u | \leq | u | \) and the Young inequality,

\[ L_3 := -2\beta(\hat{\rho}_k \sin u, \hat{v}_t^+) = -2\beta(\hat{\rho}_k \sin u, (I - P_t)\hat{v}) \]

\[ = -2\beta((I - P_t)\hat{\rho}_k \sin u, (I - P_t)\hat{v}) = -2\beta((I - P_t)\hat{\rho}_k \sin u, \hat{v}_t^+) \]

\[ \leq 2|\beta| \| (I - P_t)\hat{\rho}_k u \| \| \hat{v}_t^+ \| = 2|\beta| \| \hat{u}_t^+ \| \| \hat{v}_t^+ \| \]

\[ \leq 2\beta^2 \| \hat{u}_t^+ \|^2 + \frac{\beta^2}{2\delta^2} \| \hat{v}_t^+ \|^2. \]  

(87)

We infer from (86) and (87) that

\[ L_1 + L_2 - L_3 \geq 2k_2 \| \hat{\varphi}_t^+ \|^2_{X(Q,\mathbb{V}_k)} - \frac{k_1}{2} \epsilon |z(\theta_{s-r\omega})| \| \hat{\varphi}_t^+ \|^2_{X(Q,\mathbb{V}_k)}. \]  

(88)

On the other hand, by \( 0 < \epsilon \leq \epsilon_0 \leq 1 \) and so \( \epsilon^2 \leq \epsilon \), which along with the Young inequality implies that

\[ L_4 := 2(2\delta - \epsilon z(\theta_{s-r\omega})) \epsilon z(\theta_{s-r\omega})(\hat{u}_t^+, \hat{v}_t^+) - 2\epsilon z(\theta_{s-r\omega}) \| \hat{v}_t^+ \|^2 \]

\[ \leq 2(\delta + 1) \epsilon |z(\theta_{s-r\omega})| + |z(\theta_{s-r\omega})|^2 \| \| \hat{u}_t^+ \|^2 + \| \hat{v}_t^+ \|^2 \]

\[ \leq \frac{k_1}{2} \epsilon Z(\theta_{s-r\omega}) \| \hat{\varphi}_t^+ \|^2_{X(Q,\mathbb{V}_k)}. \]  

(89)

\[ L_5 := 2(\hat{\rho}_k f(s), \hat{v}_t^+) \leq 2 \| (I - P_t)\hat{\rho}_k f(s) \| \| \hat{v}_t^+ \| \]

\[ \leq \frac{k_2}{4} \| \hat{\varphi}_t^+ \|^2_{X(Q,\mathbb{V}_k)} + c \| (I - P_t)\hat{\rho}_k f(s) \|^2. \]  

(90)

By the definition of \( \hat{\rho}_k \), we have \( \| \nabla \hat{\rho}_k \|_{\infty} + \| \Delta \hat{\rho}_k \|_{\infty} \leq \frac{\epsilon}{k} \) for \( k \geq 1 \), and thus

\[ L_6 := -2(u\Delta \hat{\rho}_k + 2\nabla \hat{\rho}_k \nabla u, \hat{v}_t^+) \leq \frac{c}{k} (\| u \| + \| \nabla u \|) \| \hat{v}_t^+ \| \]

\[ \leq \frac{c}{k} \| \varphi \|^2_{\mathbb{V}_k} + \frac{k_2}{4} \| \hat{\varphi}_t^+ \|^2_{X(Q,\mathbb{V}_k)}. \]  

(91)

Now, we infer from (88)-(91) that

\[ L_1 + L_2 - \sum_{i=3}^{6} L_i \geq \frac{3}{2} k_2 - \epsilon k_1 Z(\theta_{s-r\omega}) \| \hat{\varphi}_t^+ \|^2_{X(Q,\mathbb{V}_k)} \]

\[ - \frac{c}{k} \| \varphi \|^2_{\mathbb{V}_k} - c \| (I - P_t)\hat{\rho}_k f(s) \|^2. \]  

(92)
We substitute (92) into (85) to obtain
\[
\frac{d}{ds} \left\| \hat{\phi}_1^+ \right\|_{X(Q, \tau_{\omega})} + (\kappa_2 - \epsilon \kappa_1 Z(\theta_{s-r} \omega)) \left\| \hat{\phi}_1^+ \right\|_{X(Q, \tau_{\omega})} \\
\leq \frac{c}{k} \left\| \phi \right\|_{X}^2 + c \left\| (I - P_I) \hat{\rho}_k f(s) \right\|^2.
\] (93)

Applying the Gronwall inequality to (93) over \([r - t, r]\) and taking the supremum over \(r \in [\tau, +\infty]\), we have
\[
\sup_{r \geq \tau} \left\| \hat{\phi}_1^+ (r, r - t, \theta_{s-r} \omega, \hat{\phi}_1^+ (r, r - t)) \right\|_{X(Q, \tau_{\omega})} \\
\leq \sup_{r \geq \tau} e^{\int_{r-t}^{r} (\kappa_2 - \epsilon \kappa_1 Z(\theta_{s-r} \omega)) d\sigma} \left\| \hat{\phi}_1^+ \right\|_{X(Q, \tau_{\omega})} \\
+ \frac{c}{k} \sup_{r \geq \tau} \int_{r-t}^{r} e^{\int_{r-t}^{r} (\kappa_2 - \epsilon \kappa_1 Z(\theta_{s-r} \omega)) d\sigma} \left\| \phi \right\|_{X}^2 d\sigma \\
+ c \sup_{r \geq \tau} \int_{r-t}^{r} e^{\int_{r-t}^{r} (\kappa_2 - \epsilon \kappa_1 Z(\theta_{s-r} \omega)) d\sigma} \left\| (I - P_I) \hat{\rho}_k f(\sigma) \right\|^2 d\sigma.
\] (94)

By \(\| I - P_I \| \leq 1\), we have \(\left\| \hat{\phi}_1^+ (r, r - t, \theta_{s-r} \omega, \hat{\phi}_1^+ (r, r - t)) \right\|_{X(Q, \tau_{\omega})} \leq \| \phi_{r-t} \|_{X}^2\). By (40),
\[
\sup_{r \geq \tau} e^{\int_{r-t}^{r} (\kappa_2 - \epsilon \kappa_1 Z(\theta_{s-r} \omega)) d\sigma} \left\| \hat{\phi}_1^+ \right\|_{X(Q, \tau_{\omega})} \\
\leq e^{C_0(\omega)} e^{-\frac{\epsilon}{\kappa_2} t} \left( \sup_{r \geq \tau} \left\| \phi_{r-t} \right\|_{X}^2 \right) \\
\leq e^{C_0(\omega)} e^{-\frac{\epsilon}{\kappa_2} t} \left( \sup_{r \geq \tau} \left\| \mathcal{D}_{r-t} (\theta_{s-r} \omega) \right\|_{X}^2 \right) \to 0
\] (95)
as \(t \to +\infty\). By (44) in Lemma 4.2,
\[
\frac{c}{k} \sup_{r \geq \tau} \int_{r-t}^{r} e^{\int_{r-t}^{r} (\kappa_2 - \epsilon \kappa_1 Z(\theta_{s-r} \omega)) d\sigma} \left\| \phi \right\|_{X}^2 d\sigma \leq \frac{c}{k} (1 + R(\tau, \omega)) \to 0
\] (96)
as \(k \to +\infty\). By Hypothesis F3,
\[
c \sup_{r \geq \tau} \int_{r-t}^{r} e^{\int_{r-t}^{r} (\kappa_2 - \epsilon \kappa_1 Z(\theta_{s-r} \omega)) d\sigma} \left\| (I - P_I) \hat{\rho}_k f(\sigma) \right\|^2 d\sigma \\
\leq ce^{C_0(\omega)} \sup_{r \geq \tau} \int_{-\infty}^{0} e^{\frac{\epsilon}{\kappa_2} \sigma} \left\| (I - P_I) \hat{\rho}_k f(\sigma + r) \right\|^2 d\sigma \to 0
\] (97)
as \(i, k \to +\infty\). We substitute (95)-(97) into (94) to obtain the conclusion. \(\square\)

5. Forward compact attractor and forward controller. This section gives the proof for the application result.

**Theorem 5.1.** Suppose all hypotheses H1-H2, F1-F3 are satisfied. Let \(\Phi\) be the cocycle associated with the non-autonomous stochastic sine-Gordon equation (2) and let \(\mathcal{D}\) (resp. \(\tilde{\mathcal{D}}\)) be the universe of forward tempered sets (resp. tempered sets). Then,

(i) \(\Phi\) has a \(\mathcal{D}\)-pullback bi-parametric attractor \(A \in \mathcal{D}\) such that \(A\) is forward compact in \(X = H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)\);
(ii) \(\Phi\) has a \(\mathcal{D}\)-pullback attractor \(\bar{A} \in \tilde{\mathcal{D}}\) such that \(\bar{A}\) is random;
(iii) \(A_r(\omega) = \bar{A}_r(\omega)\) for all \((\tau, \omega) \in \Omega\).

That is, \(A \in \mathcal{D}\) is a \(\mathcal{D}\)-pullback random attractor with the forward compactness. Moreover,
(iv) $\Phi$ has a unique forward controller in the sense of Def.2.2, given by

$$A_\infty(\omega) = \bigcap_{t_0 > 0} \bigcup_{t \geq t_0} \Phi(t, \tau - t, \theta_{-t} \omega)K_{\tau - t}^{\theta_{-t}}(\theta_{-t} \omega), \ \forall \omega \in \Omega,$$

(98)

where $K \in \mathcal{D}$ is the $\mathcal{D}$-pullback absorbing set as given in (60), more precisely, $A_\infty(\omega)$ is the minimal compact set such that

$$\lim_{\tau \to +\infty} dist_X(A_\tau(\omega), A_\infty(\omega)) = 0, \ \forall \omega \in \Omega.$$

(99)

Proof. (i) We mainly prove that $\Phi$ is forward $\mathcal{D}$-pullback compact. Let $(\tau, \omega, \mathcal{D}) \in \mathbb{R} \times \Omega \times \mathcal{D}$ and denote by

$$B_T := \{\varphi(r, r - t, \theta_{-r} \omega, \varphi_{-r}) : r \geq \tau, t \geq T, \varphi_{-r} \in \mathcal{D}_{r-t}(\theta_{-t} \omega)\}$$

for all $T > 0$. It suffices to prove $\kappa_X B_T \to 0$ as $T \to +\infty$.

Indeed, given $\eta > 0$, by Lemma 4.3, there exist $T_1 > 0$ and $k_1 \geq 1$ such that

$$\|B_{T_1}\|_{X(Q_{k_1})} \leq \eta.$$  

(100)

By Lemma 4.4, there are $i \in \mathbb{N}$, $k \geq k_1$ and $T_2 \geq T_1$ such that

$$\|(I - P_i)(\hat{\rho}_k B_{T_2})\|_{X(Q_{k_2})} \leq \eta.$$  

(101)

In addition, by Lemma 4.2, there exists a $T_3 \geq T_2$ such that

$$\|B_{T_3}\|_{X}^2 \leq 1 + cR(\tau, \omega) < +\infty,$$

which means $B_{T_3}$ is bounded in $X$. Hence, both $\hat{\rho}_k B_{T_3}$ and $P_i(\hat{\rho}_k B_{T_3})$ are bounded in $X(Q_{k_2})$. Since $P_i$ has a finite-dimensional range, it follows that $P_i(\hat{\rho}_k B_{T_3})$ is pre-compact in $X(Q_{k_2})$ and so

$$\kappa_X(Q_{k_2}) P_i(\hat{\rho}_k B_{T_3}) = 0.$$  

(102)

Since $B_{T_2} \supset B_{T_3}$, it follows from (101) and (102) that

$$\kappa_X(Q_{k_2}) (\hat{\rho}_k B_{T_3}) \leq \kappa_X(Q_{k_2}) P_i(\hat{\rho}_k B_{T_3}) + \kappa_X(Q_{k_2}) (I - P_i)(\hat{\rho}_k B_{T_3}) \leq \kappa_X(Q_{k_2}) (I - P_i)(\hat{\rho}_k B_{T_3}) \leq 2\eta.$$

For all $\varphi \in X$, we have $\hat{\rho}_k \varphi = \varphi$ on $Q_k$ and thus

$$\kappa_X(Q_k) B_{T_3} = \kappa_X(Q_k) (\hat{\rho}_k B_{T_3}) \leq \kappa_X(Q_{k_2}) (\hat{\rho}_k B_{T_3}) \leq 2\eta.$$  

(103)

Note that $k \geq k_1$ and $T_3 \geq T_1$, we infer from (100) that

$$\kappa_X(Q_k) B_{T_3} \leq \kappa_X(Q_{k_1}) B_{T_3} \leq 2\eta.$$  

(104)

Now, both (103) and (104) imply that

$$\kappa_X(Q_k) B_{T_3} \leq \kappa_X(Q_{k_1}) B_{T_3} + \kappa_X(Q_{k_2}) B_{T_3} \leq 4\eta,$$

which proves the forward $\mathcal{D}$-pullback compactness of $\Phi$.

By Corollary 2 (ii), the cocycle $\Phi$ has a decreasing, $\mathcal{D}$-pullback absorbing set $\mathcal{K} \in \mathcal{D}$ as given in (60). Note that $\mathcal{D}$ is forward-union closed. We infer from Theorem 2.8 that $\Phi$ has a $\mathcal{D}$-pullback bi-parametric attractor $\mathcal{A} \in \mathcal{D}$ such that $\mathcal{A}$ is forward compact and given by

$$\mathcal{A}_\tau(\omega) = \alpha^\mathcal{K}_\tau(\omega) = \bigcap_{t_0 > 0} \bigcup_{t \geq t_0} \Phi(t, \tau - t, \theta_{-t} \omega)K_{\tau - t}^{\theta_{-t}}(\theta_{-t} \omega),$$  

(105)

where the measurability of $\mathcal{K}$ is unknown. However, we will prove that $\mathcal{A}$ is random in (ii) and (iii).
(ii) By Corollary 2 (i), the cocycle $\Phi$ has a $\widetilde{D}$-pullback absorbing set $\widetilde{K} \subset D$ as in (59) and particularly $\widetilde{K}$ is random. By using the same estimates as in Lemmas 4.3 and 4.4, one can show that $\Phi$ is $\widetilde{D}$-pullback limit-set compact. By the abstract result in [34], $\Phi$ has a $\widetilde{D}$-pullback random attractor $\tilde{A} \subset \widetilde{D}$ given by
\begin{equation}
\tilde{A}_\tau(\omega) = \alpha^*_\tau(\omega) = \bigcap_{t_0 > 0} \bigcup_{\tau \geq t_0} \Phi(t, \tau - t, \theta_{-t}\omega)\tilde{K}_{\tau-t}(\theta_{-t}\omega). \tag{106}
\end{equation}

(iii) We show $A = \tilde{A}$. By (42) and (45), the absorbing radii satisfy
\[ \bar{R}(\tau, \omega) \leq \sup_{t \geq \tau} \bar{R}(t, \omega) =: R(\tau, \omega), \quad \forall \tau \in \mathbb{R} \]
and thus we have the inclusion $\tilde{K}_\tau(\omega) \subset K_\tau(\omega)$. Hence, both (105) and (106) imply
\[ \tilde{A}_\tau(\omega) \subset A_\tau(\omega), \quad \forall \tau \in \mathbb{R}, \quad \omega \in \Omega. \tag{107} \]
On the other hand, since $D$ is a sub-universe of $\widetilde{D}$ and $A \subset D$, we have $A \subset \widetilde{D}$ and so the $\widetilde{D}$-pullback attractor $\tilde{A}$ attracts $A$. By the invariance of $A$, we have
\[ \text{dist}_X(A_\tau(\omega), \tilde{A}_\tau(\omega)) \rightarrow \text{dist}_X(\Phi(t, \tau - t, \theta_{-t}\omega)A_{\tau-t}(\theta_{-t}\omega), \tilde{A}_\tau(\omega)) \rightarrow 0 \]
as $t \rightarrow +\infty$. By the compactness of $\tilde{A}$, we have $A_\tau(\omega) \subset \tilde{A}_\tau(\omega)$, which together with (107) implies that $A_\tau(\omega) = \tilde{A}_\tau(\omega)$ as desired.

(iv) Since the attractor $A$ is forward compact, it follows from Theorem 2.3 that $\Phi$ has a forward controller $A_\infty(\omega)$ as given by (98). \hfill \Box

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