Accuracy of Range-Based Localization in Random Sensor Networks
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Abstract—Location service is essential for many sensor network applications. This paper theoretically assesses the accuracy of range-based localization schemes with respect to network connectivity and scale in sensor networks where sensors are deployed and connected randomly. We first show that the variance of localization errors are proportional to average geometric dilution of precision (AGDOP). We then prove a novel lower bound of expectation of AGDOP (LB-E-AGDOP) and derives a closed-form expression that relates LB-E-AGDOP to only three parameters that describe network connectivity and scale. Furthermore, the paper conjectures a simple relationship between the expectation of AGDOP (E-AGDOP) and its lower bound, LB-E-AGDOP. The closed-form expressions of LB-E-AGDOP and E-AGDOP are used to analyze how accuracy evolves when the network scales up. Finally, we validate the theoretical results via numerical simulations.

Index Terms—Sensor networks, range-based localization, accuracy, connectivity, scale, dilution of precision (DOP), Laplacian matrix

1 INTRODUCTION

Sensors networks represent a new paradigm of large-scale, flexible, robust, cost-effective data collection and information processing in complex environments. They are expected to enable a large variety of applications such as assisted navigation and surveillance, wildlife habitat monitoring, oceanographic data collection, climate control, disaster management, fraud detection, and automated billing [1], [2], [3]. Many of these applications require accurate information about the physical location of each sensor node. To this end, various network localization schemes have been explored over the past decade. These schemes can be generally classified into four categories: range-based [4], [5], angle-based [6], [7], proximity-based [8], [9], and event-driven [10], [11]. In this paper, we focus on range-based schemes because they can achieve better localization accuracy than most other schemes [12].

Fig. 1 illustrates a snapshot of range-based network localization. Anchor nodes know their locations, and sensor nodes determine their locations by inter-node distance information. Since range-based network localization is essentially a graph realization problem [13], [14], [15], connectivity of the graph exerts significant influence on many performance metrics, such as accuracy, energy efficiency, localizability, robustness, and scalability. Although localizability has been extensively studied with respect to connectivity [13], [14], the general relationship between accuracy and connectivity has not been theoretically treated. This paper aims at a general theory to characterize localization accuracy with respect to network connectivity. Specifically, the following two problems are addressed in this paper.

- What is the quantitative relationship between localization accuracy and network connectivity?
- For a certain level of connectivity, how does accuracy vary with the scale of the network?

In this paper, connectivity is defined as the average...
degree (or valency) of the graph. In fact, localization accuracy depends on not only the average degree, but also inter-node ranging accuracy and node geometry. We first show that, under the principle of maximum likelihood (ML), the variance of localization errors for a node is the product of its geometric dilution of precision (DOP) and the variance of range errors. DOP decouples localization accuracy from range accuracy; the question how connectivity affects accuracy becomes how connectivity affects DOP.

Next, we find a closed-form expression of the expectation of average geometric DOP (E-AGDOP) under the assumption that nodes are randomly distributed, and nodes are randomly connected such that the graph of network can reach a certain average degree. Herein lies a big challenge that it is very difficult to evaluate E-AGDOP analytically, because E-AGDOP is proportional to \( \text{tr}(E[(G^T G)^{-1}]) \), where \( G \) is a random matrix (detailed in Section 3). To overcome this challenge, we first prove that \( E[(G^T G)^{-1}] \) has a lower bound \( E(G^T G)^{-1} \), and derive a closed-form expression for the lower bound of E-AGDOP (LB-E-AGDOP). Furthermore, we show that E-AGDOP can be estimated from LB-E-AGDOP using an empirical formula. The analytical expressions of LB-E-AGDOP and E-AGDOP are used to answer the aforementioned two questions. The theoretical conclusions are finally validated by numerical simulations.

1.1 Related work and our contributions

Numerous network localization schemes have been developed thus far (see, e.g., [4], [5], [6], [7], [8], [9], [10], [11]; for an overview, see [16]). Most of past work is about localization algorithms and accuracy analyses specific to certain algorithms. The accuracy analyses are mostly based on Monte Carlo simulations. Although simulations can reveal the relationship between accuracy and connectivity for some specific scenarios (see, e.g., [17]), there is still a dearth of generalized theories.

For range-based localization schemes, four genres are commonly seen in literature: lateration [4], [5], [18], stochastic optimization [19], multidimensional scaling [20], and semidefinite programming [21]. No matter how these genres differ in the way of estimating node locations, and no matter whether locations are estimated in centralized [20], [21] or distributed [4], [5] manners, the accuracy performance is always limited by the Cramér-Rao (CR) bound, i.e., the inverse of the Fisher information [22]. It is worth noting that some prevailing algorithms, e.g., the lateration algorithm used in [18], do not necessarily achieve the CR bound.

There has been some prior work that theoretically analyzed network localization accuracy using the CR bound [17], [23], [24], [25], [26], [27]. However, most of these analyses ended up with the Fisher information matrix, and did not give an explicit closed-form expression to characterize localization accuracy with respect to network connectivity.

This paper distinguishes itself from the past work by the following contributions.

- The paper does not just analyze localization accuracy using the CR bound, but also provides an iterative algorithm (which is extended from the positioning algorithm used in GPS [28]) that can achieve the CR bound (Section 3).
- For the first time (to the best of our knowledge), the paper introduces a lower bound of E-AGDOP, identifies its relationship to graph Laplacians, and derives a closed-form expression that relates LB-E-AGDOP to only three parameters that describe network connectivity and scale (Section 4).
- For the first time (to the best of our knowledge), the paper conjectures a simple relationship between E-AGDOP and LB-E-AGDOP (Section 5).
- For the first time (to the best of our knowledge), this paper analyzes, for a certain level of connectivity, how accuracy evolves when the network scales up (Section 6).

1.2 Outline of the paper

The rest of this paper is organized as follows. Section 2 formulates the network localization problem and introduces the metric of connectivity used throughout this paper. Section 3 analyzes localization accuracy and its relationship to DOP. Section 4 derives a closed-form expression LB-E-AGDOP with respect to network connectivity and scale. Section 5 shows our empirical formula that estimates E-AGDOP from LB-E-AGDOP. Section 6 studies how location accuracy varies with the network scale. Numerical simulation results are presented in Section 7 to validate the theoretical conclusions. Finally, Section 8 concludes the paper. Proofs of key theorems and equations are provided in Appendices A to C.

2 PRELIMINARIES

2.1 Problem formulation

In this paper, a sensor network is modeled as a simple graph \( G = (V, E) \), where \( V = \{1, 2, \ldots, N\} \) is a set of \( N \) nodes (or vertices), and \( E = \{e_1, e_2, \ldots, e_K\} \subseteq V \times V \) is a set of \( K \) links (or edges) that connect the nodes [4].

All nodes are in a \( d \)-dimensional Euclidean space (\( d \geq 1 \)), with the locations denoted by \( p_n \in \mathbb{R}^d \), \( n = 1, \ldots, N \). The first \( N_S \) nodes, labeled 1 through \( N_S \), are sensor nodes (or mobile nodes), whose locations are unknown; the rest \( N_A = N - N_S \) nodes, labeled \( N_S + 1 \) through \( N \), are anchor nodes (or beacon nodes). Anchors are aware of their exact locations through built-in GPS receivers or manual pre-programming during deployment.

An unordered pair \( e_k = (i_k, j_k) \in E \) if and only if there exists a direct ranging link between nodes \( i_k \) and \( j_k \). The link provides inter-node distance information \( \rho_k = r_k + \epsilon_k \), where \( r_k = \|p_{i_k} - p_{j_k}\| \) is the actual Euclidean distance.

1. A simple graph, also known as a strict graph, is an unweighted, undirected graph containing no self-loops or multiple edges [29], [30].
between nodes $i$ and $j$, and $\epsilon_k$ is the range measurement error.

The range measurements $\rho_k$ can be obtained by a variety of methods, such as one-way time of arrival (ToA), two-way ToA, or received signal strength indication (RSSI) \([31]\). One-way ToA can result in biased range measurements due to unsynchronized clocks \([28]\), while two-way ToA and RSSI do not depend on clocks. In this paper, we assume zero clock biases in range measurements. Our assumption holds for the cases of two-way ToA, RSSI, and one-way ToA with perfect clock synchronization.

The network localization problem is to determine the locations of sensor nodes $p_n$, $n = 1, \ldots, N_S$, given a fixed network graph $G$, known locations of anchors $p_n$, $n = N_S + 1, \ldots, N$, and range measurements $\rho_k$, $k = 1, \ldots, K$.

### 2.2 Metrics of connectivity

For all nodes $n = 1, \ldots, N$, we define the following degrees:

- Anchor degree: $\deg_A(n)$, the number of anchor nodes incident to node $n$;
- Sensor degree: $\deg_S(n)$, the number of sensor nodes incident to node $n$;
- Degree: $\deg(n) = \deg_A(n) + \deg_S(n)$, the number of nodes incident to node $n$.

There are no anchor-to-anchor links, i.e., $\deg_A(n) = 0$ for $n = N_S + 1, \ldots, N$, because anchor-to-anchor links are meaningless when locations of anchors are known.

In graph theory, connectivity is usually described by vertex connectivity or edge connectivity: a graph is $\kappa$-vertex/edge-connected if it remains connected whenever fewer than $\kappa$ vertices/edges are removed \([30]\). Unfortunately, vertex/edge connectivity mainly reflects some “minimum” properties of connectivity, such as $\min_{n \in \{1, \ldots, N\}} \deg(n)$ \([30]\), and does not distinguish between sensor and anchor nodes. This paper uses average degrees to characterize the overall connectivity of the network. Average degrees are defined as

$$
\delta_s = \frac{1}{N_S} \sum_{n=1}^{N_S} \deg_s(n),
$$

where the subscript $*$ can be blank, $A$, or $S$, for the average degree, average anchor degree, or average sensor degree, respectively.

Let $K_S$ and $K_A$ denote the number of sensor-to-sensor and anchor-to-sensor links in the network, respectively. It is easy to verify the equalities $K = K_S + K_A$, $N_S \delta_S = 2K_S$, $N_S \delta_A = K_A$, and $\delta = \delta_S + \delta_A$.

### 2.3 List of notations

- $C$ function of $N_S$, $C_1$, and $C_2$, used in the approximation of E-AGDOP
- $\bar{C}$ limit of LB-E-AGDOP as $N_S \to \infty$, $\bar{C} = \lim_{N_S \to \infty} \text{LB-E-AGDOP}$

### 3 Localization Accuracy

Localization is essentially an optimization problem that finds coordinate vectors $p_n \in \mathbb{R}^d$, $n = 1, \ldots, N_S$, such that for each ranging link $e_k = (i_k, j_k) \in E$, the distance $r_k = \|p_{i_k} - p_{j_k}\|$ is as close to the range measurement $\rho_k$ as possible.

Assume that range errors follow a zero-mean Gaussian distribution:

$$
\epsilon_k = \rho_k - r_k \sim \mathcal{N}(0, \sigma_k^2), \quad \forall k = 1, \ldots, K.
$$

Then, the ML estimation of $\{p_n\}^{N_S}_{n=1}$ is equivalent to the...
weighted least squares (LS) problem

\[
\arg \max_{\{p_n\}_{n=1}^{N_S}} P(\{p_k\}_{k=1}^{K} \mid \{p_n\}_{n=1}^{N_S}) = \arg \max_{\{p_n\}_{n=1}^{N_S}} \prod_{k=1}^{K} \frac{1}{2\sigma_k^2} \exp \left( -\frac{\|p_{ik} - p_{jk}\| - \rho_k)^2}{2\sigma_k^2} \right) \\
= \arg \min_{\{p_n\}_{n=1}^{N_S}} \sum_{k=1}^{K} \frac{\|p_{ik} - p_{jk}\| - \rho_k)^2}{\sigma_k^2}.
\]

(3)

The LS problem cannot be directly solved because the distance \(r_k = \|p_{ik} - p_{jk}\|\) is a nonlinear function of the coordinate vectors \(p_{ik}\) and \(p_{jk}\). Let \(r = (r_1, r_2, \ldots, r_K)^T \in \mathbb{R}^K\) and \(p = \text{column}\{p_1, p_2, \ldots, p_{N_S}\} \in \mathbb{R}^{dN_S}\). The first-order linear approximation of the distance function \(r(p)\) with respect to an initial guess \(p_0\) can be written as

\[
r(p_0 + \Delta p) = r(p_0) + G\Delta p,
\]

where the geometry matrix \(G \in \mathbb{R}^{K \times dN_S}\) is given by

\[
G = \begin{bmatrix}
\frac{\partial r_1}{\partial p_1} & \cdots & \frac{\partial r_1}{\partial p_d} \\
\vdots & \ddots & \vdots \\
\frac{\partial r_K}{\partial p_1} & \cdots & \frac{\partial r_K}{\partial p_d}
\end{bmatrix}
\]

(5)

where \(p_{m,m} = 1, \ldots, d_i\) is the \(m\)th element of the coordinate vector \(p_i\). Each element of the geometry matrix \(G\) is given by

\[
G_{k,(n-1)d+m} = \frac{\partial r_k}{\partial p_{n,m}} = \begin{cases} 
\frac{p_{ik,m} - p_{jk,m}}{\|p_{ik} - p_{jk}\|} & \text{if } n = i_k, \\
\frac{p_{ik,m} - p_{jk,m}}{\|p_{ik} - p_{jk}\|} & \text{if } n = j_k, \\
0 & \text{otherwise}.
\end{cases}
\]

Each row of \(G\) represents a link. There are only \(d\) nonzero elements in a row for an anchor-to-sensor link, and there are \(2d\) nonzero elements for an sensor-to-sensor link. Given that each row of \(G\) has \(dN_S\) elements, \(G\) is highly sparse when the network contains many nodes.

When the network is localizable, \(G\) must be a tall matrix (i.e., \(K \geq dN_S\)) with full column rank. Then, the weighted LS problem \(\text{(3)}\) can be solved by the following iterative algorithm, which is based on the Newton–Raphson method \[28\].

\[
p^{(n+1)} = p^{(n)} + (G^T\Sigma^{-1}G)^{-1}G^T\Sigma^{-1}(\rho - r(p^{(n)})),
\]

where \(\rho = (\rho_1, \rho_2, \ldots, \rho_K)^T\), \(\Sigma = \text{Cov}(\epsilon, \epsilon)\) is the covariance of range errors, where \(\epsilon = (\epsilon_1, \ldots, \epsilon_K)^T\).

When the initial guess \(p^{(0)}\) is accurate enough and the iteration converges, the localization errors \(\epsilon\) have the following relationship to the range errors \(\epsilon = (\epsilon_1, \ldots, \epsilon_K)^T\):

\[
\epsilon = p^{(\infty)} - p = (G^T\Sigma^{-1}G)^{-1}G^T\Sigma^{-1}(\rho - r) \\
= (G^T\Sigma^{-1}G)^{-1}G^T\Sigma^{-1}\epsilon.
\]

The covariance of localization errors is thus given by

\[
\text{Cov}(\epsilon, \epsilon) = (G^T\Sigma^{-1}G)^{-1}G^T\Sigma^{-1}\text{Cov}(\epsilon, \epsilon) \\
= (G^T\Sigma^{-1}G)^{-1}.
\]

(9)

This has achieved the CR bound \[17\], \[23\], \[24\], \[25\], \[26\], \[27\]. If measurement errors are independent and identically distributed (iid), i.e., \(\Sigma = \text{diag}(\sigma^2, \ldots, \sigma^2)\), we have

\[
\text{Cov}(\epsilon, \epsilon) = (G^T\Sigma^{-1}G)^{-1} = \sigma^2(G^T G)^{-1}.
\]

(10)

The matrix \(H = (G^T G)^{-1} \in \mathbb{R}^{dN_S \times dN_S}\) is referred to as dilution of precision (DOP) matrix. DOP is a term widely used in satellite navigation specifying the multiplicative effect on positioning accuracy due to satellite geometry \[28\]. For network localization, DOP specifies the multiplicative effect due to not only geometry of the nodes but also connectivity of the network. DOP decouples localization accuracy from range accuracy. The smaller DOP is, the better localization accuracy one can expect.

A diagonal element \(H_{(n-1)d+m, (n-1)d+m}\) is the DOP of coordinate \(m\) for node \(n\). The sum of all the diagonal elements, \(tr(H)\), is the geometric DOP (GDOP) of the whole network. In this paper, we define average GDOP (AGDOP) as GDOP divided by the number of sensor nodes, \(tr(H)/N_S\). AGDOP is a performance indicator of localization accuracy due to network geometry and connectivity.

For a network where nodes are deployed and connected randomly, AGDOP is a random variable. The expectation of AGDOP (E-AGDOP) indicates the expected localization accuracy because the root-mean-square localization error is proportional to \(\sqrt{E\text{-AGDOP}}\). We shall use E-AGDOP and its lower bound to study the relationship between localization accuracy and network connectivity in the rest of the paper.

4 LOWER BOUND OF E-AGDOP

E-AGDOP is derived from \(E H = E[(G^T G)^{-1}]\). Unfortunately, it is very difficult to obtain a closed-form expression of \(E H\) directly for a random network (randomly-deployed nodes and randomly-established links) that achieves a certain level of connectivity. Therefore, we consider \(F = G^T G\) here not only because \(E F\) can be evaluated analytically, but also because \(E F^{-1} = E[(G^T G)^{-1}]\) is proven to be a lower bound of \(E[(G^T G)^{-1}]\), as stated by the following theorem.

Theorem 1 (Lower bound of DOP matrix): For a random network with a non-singular geometry matrix \(G\) defined in \[6\],

\[
E[(G^T G)^{-1}] \succeq [E(G^T G)]^{-1},
\]

(11)

2. The DOP used in satellite navigation is usually defined in the form of \(\sqrt{\text{tr}[(G^T G)^{-1}]}\) \[28\]. In this paper, we define DOP in the form of \(\text{tr}[(G^T G)^{-1}]\) to simplify calculation and analysis.
where the operator $X \succeq Y$ means that $X - Y$ is positive semidefinite. A proof of this theorem is shown in Appendix A.

For instance, let us consider a very simple sensor network comprised of 4 nodes and 3 links in 2 dimensions. Nodes 1 to 3 are sensors; node 4 is an anchor ($N_S = 3$, $N_A = 1$, $K_S = 2$, $K_A = 1$). Eq. (15) shows the matrix $\Xi$ for this network.

4.1 Step 1: randomly-deployed nodes

Recall (6) which describes the elements in $G$. Note that when link $e_{k}$ is incident to node $n$, i.e., $n = \{i_k, j_k\}$,

$$
\sum_{m=1}^{d}(\frac{\partial r_k}{\partial p_{nm}})^2 = \sum_{m=1}^{d}(p_{ik,m} - p_{jk,m})^2 \|p_{ik} - p_{jk}\|^2 = 1. \tag{12}
$$

Assume that the nodes are randomly deployed such that the distribution is the same in all coordinates. Then, $p_{ik,m} - p_{jk,m}$, $m = 1, \ldots , d$ are iid. To satisfy (12), we must have

$$
E(\frac{\partial r_k}{\partial p_{nm}})^2 = \frac{1}{d}, \quad \forall m = 1, \ldots , d. \tag{13}
$$

By (13), the elements of matrix $F = \{F_{ij}\} \in \mathbb{R}^{N_S \times dN_S}$ have the conditional expectation

$$
\Xi_{ij} = E_{\text{nodes}}(F_{ij}) = E \sum_{k=1}^{K} \frac{\partial r_k}{\partial p_{i,m}} \frac{\partial r_k}{\partial p_{j,m}} = \begin{cases} \frac{d}{\delta} \deg (i) & \text{if} \ i = j \text{ and} \ m_1 = m_2, \\ -\frac{1}{\delta} & \text{if} \ (i, j) \in E \text{ and} \ m_1 = m_2, \\ 0 & \text{otherwise}, \end{cases} \tag{14}
$$

where $\delta = (i-1)d + m_1$, $\delta = (j-1)d + m_2$, $1 \leq m_1, m_2 \leq d$.

For instance, let us consider a very simple sensor network shown in Fig. 2. The matrix $\Xi$ for this network is given by

$$
\Xi_{\text{Fig. 2}} = \begin{bmatrix}
\frac{1}{\delta} & 0 & -\frac{1}{\delta} & 0 & 0 & 0 \\
0 & \frac{1}{\delta} & 0 & -\frac{1}{\delta} & 0 & 0 \\
-\frac{1}{\delta} & 0 & 1 & 0 & -\frac{1}{\delta} & 0 \\
0 & -\frac{1}{\delta} & 0 & 1 & 0 & -\frac{1}{\delta} \\
0 & 0 & -\frac{1}{\delta} & 0 & 1 & 0 \\
0 & 0 & 0 & -\frac{1}{\delta} & 0 & 1 \\
\end{bmatrix}. \tag{15}
$$

As shown by the red- and blue-colored elements in (15), $\Xi$ includes $d$ identical submatrices $\tilde{\Xi} = \{\Xi_{ij}\} \in \mathbb{R}^{N_S \times N_S}$, where $\Xi_{ij} = \begin{cases} \frac{d}{\delta} \ deg (i) & \text{if} \ i = j, \\ -\frac{1}{\delta} & \text{if} \ (i, j) \in E, \\ 0 & \text{otherwise}. \end{cases}$

For the sensor network shown by Fig. 2, we can divide the matrix $\Xi$ in (15) into 2 identical submatrices

$$
\tilde{\Xi}_{\text{Fig. 2}} = \begin{bmatrix}
\frac{1}{\delta} & -\frac{1}{\delta} & 0 \\
-\frac{1}{\delta} & \frac{1}{\delta} & -\frac{1}{\delta} \\
0 & -\frac{1}{\delta} & 1 \\
\end{bmatrix}. \tag{17}
$$

The submatrix $\tilde{\Xi}$ (as well as $\Xi$) indicates a relationship between localization accuracy and graph Laplacians [33]. Let $L$ denote the Laplacian matrix of the graph $G$. It can be seen that $d\tilde{\Xi} = [L_{ij} \mid i, j \in \{1, 2, \ldots, N_S\}]$, i.e., $d\tilde{\Xi}$ is the submatrix of $L$ obtained by deleting its last $N_A$ rows and columns that are related to the anchor nodes.

The lower bound of E-AGDOP (LB-E-AGDOP) can be calculated by inverting $F = E\Xi$, or, equivalently, inverting $F = E\tilde{\Xi}$, because $tr([E\Xi]^{-1}) = d tr([E\tilde{\Xi}]^{-1})$.

4.2 Step 2: randomly-established links

Given an average sensor degree $\delta$, the trace of $\tilde{\Xi}$ is given by

$$
tr(\tilde{\Xi}) = \sum_{i=1}^{N_S} \tilde{\Xi}_{ii} = \sum_{i=1}^{N_S} \deg (i)/d = N_S \delta / d. \tag{18}
$$

Given an average sensor degree $\delta_S$, there are $K_S = N_S \delta_S/2$ sensor-to-sensor links in the network, and thus $\tilde{\Xi}$ includes $N_S \delta_S$ off-diagonal elements with a non-zero value of $-1/d$. Assume that the sensor-to-sensor links are chosen uniformly at random from the set $\{(i, j) \mid 1 \leq i < j \leq N_S, i, j \in \mathbb{Z}\}$. Then, each off-diagonal element $\tilde{\Xi}_{ij}$, $i \neq j$ satisfies the Bernoulli distribution

$$
\tilde{\Xi}_{ij} = \begin{cases} -1/d & \text{with probability} \ \frac{\delta_S}{N_S-1}, \\ 0 & \text{with probability} \ 1 - \frac{\delta_S}{N_S-1}. \end{cases} \tag{19}
$$

Then, the expectation of $\tilde{\Xi}$ is given by

$$
E \tilde{\Xi}_{ij} = E_{\text{links}}(\tilde{\Xi}_{ij}) = \begin{cases} \delta_S/d & \text{if} \ i = j, \\ -\delta_S/d(N_S-1) & \text{otherwise}. \end{cases} \tag{20}
$$

Appendix C shows that

$$
tr([E \tilde{\Xi}]^{-1}) = \frac{N_S}{\eta} \left(1 + \frac{\zeta}{1 - N_S \zeta}\right), \tag{21}
$$

where $\eta = d^{-1} [\delta + \delta_S/(N_S - 1)]$ and $\zeta = \delta_S/\delta (N_S - 1) + \delta_S/N_S$. 

Fig. 2. A simple sensor network comprised of 4 nodes and 3 links in 2 dimensions. Nodes 1 to 3 are sensors; node 4 is an anchor ($N_S = 3$, $N_A = 1$, $K_S = 2$, $K_A = 1$). Eq. (15) shows the matrix $\Xi$ for this network.
\[ \delta_S \]. Therefore, LB-E-AGDOP is given by
\[
\text{LB-E-AGDOP} = \frac{\text{tr}[(E F)^{-1}]}{N_S} = \frac{d \text{tr}[(E F)^{-1}]}{N_S} = \frac{d}{\eta} \left( \frac{1}{1 + \frac{1}{\zeta} - N_S} \right) \sim \frac{d^2}{\delta} \left( \frac{N_S - 1}{N_S - 1 + \delta_S/\delta_A} \right).
\]

Thus far, we have obtained a closed-form expression for LB-E-AGDOP. It depends on two parameters of network connectivity, \( \delta_S \) and \( \delta_A \) (note \( \delta = \delta_S + \delta_A \)), and one parameter of network scale, \( N_S \). It can be seen that LB-E-AGDOP is approximately inversely proportional to the average degree, and a low average anchor degree deteriorates accuracy.

5 Estimation of E-AGDOP from LB-E-AGDOP

In this section, we further derive a closed-form expression of E-AGDOP for a deeper insight into the relationship between localization accuracy and network connectivity. As stated in Introduction, it is very difficult to calculate E-AGDOP directly. After studying the relationship between E-AGDOP and LB-E-AGDOP, we discover that E-AGDOP can be estimated from LB-E-AGDOP on the basis of the following theorem.

**Theorem 2 (Approximation of \( E(X^{-1}) \)):** Let \( X \) be a positive random variable \( X \), i.e., \( P(X > 0) = 1 \). Suppose that \( \text{Var}(X) \ll E(X)^2 \) and \( E(\Delta X^n) \ll \text{Var}(X)[E(X)]^{n-2} \) for \( n \geq 3 \), where \( \Delta X = X - E(X) \). Then
\[
\frac{1}{E(X^{-1})} \approx E(X) - \frac{\text{Var}(X)}{E(X)}.
\]

A proof of this theorem can be found in Appendix [3]. Furthermore, Appendix [5] shows that the approximate equal in (23) can be exact for some distributions, such as a chi-squared distribution. An interesting property with a chi-squared random variable \( X \) is that \( \text{Var}(X)/E(X) = 2 \) is a constant.

The diagonal elements of \( F = G^TG \) are all in the form of sum of squares, and hence have a distribution similar to a chi-squared distribution. Inspired by the heuristic in [23], we propose the following approximate estimation of E-AGDOP from LB-E-AGDOP:
\[
\frac{1}{\text{E-AGDOP}} \approx \frac{1}{\text{LB-E-AGDOP}} - C,
\]
where \( C \) represents the term \( \text{Var}(X)/E(X) \) in (23).

As indicated by (20), the calculation of LB-E-AGDOP involves a Laplacian matrix that essentially represents a fully connected graph, in which each sensor-to-senor link is deweighted such that the sum of them is equivalent to a sensor degree of \( \delta_S \). For a network that is not fully connected, the LB-E-AGDOP tends to be more conservative when \( N_S \) is larger. This implies that \( C \) increases with an increasing \( N_S \). This phenomena has been confirmed by the simulations in Section [7]. We find an empirical formula
\[
C \approx C_1 N_S / (C_2 + N_S)
\]
that has a very good fit to the simulation results. Transforming (25) into an affine function of \( N_S^{-1} \),
\[
C^{-1} = (C_2/C_1) N_S^{-1} + 1/C_1,
\]
we can use least squares to determine the parameters \( C_2/C_1 \) and \( 1/C_1 \). Finally, we obtain
\[
C_1 = 0.788 \quad \text{and} \quad C_2 = 6.16
\]
from the simulation results in Section [7].

Substituting (22) and (25) into (24), we finally have the following closed-form approximation of E-AGDOP:
\[
\text{E-AGDOP} \approx \left( \frac{1}{\text{LB-E-AGDOP}} - \frac{C_1 N_S}{C_2 + N_S} \right)^{-1} = \left( \frac{\delta}{N_S - 1 + \delta_S/\delta_A} - \frac{C_1 N_S}{C_2 + N_S} \right)^{-1}.
\]

6 Accuracy for Large-Scale Networks

One of the advantages of sensor network is its flexibility. The coverage and density can be easily improved by deploying more and more sensor nodes. As for range-based network localization, herein lies two question whether localization accuracy is still maintained when more sensor nodes join in the network, and what level of connectivity is required if we want to maintain the localization accuracy. In this section, we address these questions by analyzing how accuracy varies with the network scale for a certain level of connectivity.

6.1 Asymptotic notations of connectivity

In this section, we assume that level of connectivity, or density of links, can be described by two real numbers \( \alpha \) and \( \beta \) such that as \( N_S \to \infty \),
\[
\delta_S \approx N_S^\alpha, \quad \delta_A \approx N_S^\beta.
\]
where \( \approx \) denotes asymptotic equality. Specifically, \( f(n) \approx n^\nu \) means that \( f(n) \) grows at the order of \( n^\nu \). Such a relation is also written as \( f(n) = O(n^\nu) \) [34].

As discussed in [13], [14], [32], \( \delta \geq O(1) \) and \( K_A \geq O(1) \) are necessary for the network to be localizable. For large-scale networks, these necessary conditions for localizability can be expressed as
\[
\max\{\alpha, \beta\} \geq 0, \quad \beta \geq -1
\]

In practice, each sensor node can be connect to at most \( N_S - 1 \) sensor nodes and at most \( N_A \) anchor nodes. Therefore, we have a practicality requirement \( K_S \leq O(N_S) \) and \( K_A \leq O(1) \), which is equivalent to
\[
\alpha \leq 1, \quad \beta \leq 0.
\]
As shown in Fig. [3], the greenish color covers the localizable domain, in which the shaded area indicates the practical domain.
6.2 LB-E-AGDOP for large-scale networks

With the above notations, we can derive asymptotic bounds of LB-E-AGDOP as $N_S \rightarrow \infty$. Theorem 3 states the main results, which are illustrated in Fig. 3 as well.

**Theorem 3 (LB-E-AGDOP for large-scale networks):** As $N_S \rightarrow \infty$, LB-E-AGDOP decreases to zero if

$$\alpha > 0 \text{ and } \beta > -1$$

or $\alpha \leq 0 < \beta$;

LB-E-AGDOP approaches a positive constant if

$$\alpha > 0 \text{ and } \beta = -1$$

or $0 = \alpha \geq \beta \geq -1$ or $\alpha < \beta = 0$.

**Proof:** Let us first simplify (22) into an asymptotical form:

$$\text{LB-E-AGDOP} \approx \frac{1}{N_S^{\alpha} + N_S^{-\beta}} N_S + N_S^{\alpha - \beta}.$$  \hfill (32)

When $\alpha \geq \beta$, the above equation can be written as

$$\text{LB-E-AGDOP} \approx \frac{1}{N_S^{\alpha}} N_S + N_S^{\alpha - \beta}.$$  \hfill (33)

It can be seen that as $N_S \rightarrow \infty$, LB-E-AGDOP approaches zero if $\alpha > 0$ and $\beta > -1$, and LB-E-AGDOP approaches a positive constant if $\alpha = 0$ or $\beta = -1$.

When $\alpha < \beta$, (32) can be written as

$$\text{LB-E-AGDOP} \approx \frac{1}{N_S^{\beta}} N_S + N_S^{\alpha - \beta} = N_S^{-\beta}.$$  \hfill (34)

It can be seen that as $N_S \rightarrow \infty$, LB-E-AGDOP approaches zero if $\beta > 0$, and LB-E-AGDOP approaches a positive constant if $\beta = 0$.

6.3 E-AGDOP for large-scale networks

Let $\tilde{C} = \lim_{N_S \rightarrow \infty} \text{LB-E-AGDOP}$. From (28) we can see, as $N_S \rightarrow \infty$,

$$\text{E-AGDOP} \rightarrow \begin{cases} 0 & \text{if } \tilde{C} = 0, \\ (\tilde{C}^{-1} - C_1)^{-1} & \text{if } \tilde{C} < C_1^{-1}, \\ \infty & \text{if } \tilde{C} \geq C_1^{-1}. \end{cases}$$  \hfill (35)

In addition to knowing the limit E-AGDOP approaches, we are also interested in how E-AGDOP approaches such a limit. Obviously, when $\tilde{C} = 0$ (the greenish area in Fig. 3), LB-E-AGDOP approaches faster than $C_1 N_S / (C_2 + N_S) = O(1)$ for sufficiently large $N_S$. Therefore, E-AGDOP is decreasing for sufficiently large $N_S$. When $\tilde{C} = \infty$ (the reddish area in Fig. 3), E-AGDOP simply increases to infinity with an increasing $N_S$. Both of the two cases are trivial.

In this subsection, we study the behavior of E-AGDOP as the network scale grows to infinity for three nontrivial cases. All of the three cases satisfy $0 < \tilde{C} < \infty$, that is, they are on the blue polyline in Fig. 3.

6.3.1 Case 1: $\delta_S \asymp N_S^0$ and $\delta_A \asymp N_S^{-1}$

The first case is at the bottom left corner of the practical domain in Fig. 3. The connectivity just meets the necessary condition for the network to be localizable.

Since $\delta_S \asymp N_S^0$ and $\delta_A \asymp N_S^{-1}$ are equivalent to a constant $\delta_S$ and a constant $K_A$, (22) can be written as

$$\text{LB-E-AGDOP} = \frac{d^2}{K_A / N_S + \delta_S / (N_S - 1)}.$$  \hfill (37)

It can be seen that with an increasing $N_S$, LB-E-AGDOP increases monotonically. To ensure that E-AGDOP is finite, we must have

$$\tilde{C} = d^2 \left( \frac{1}{\delta_S} + \frac{1}{K_A} \right) < C_1^{-1}.$$  \hfill (38)

It can be seen that with increasing $N_S$, LB-E-AGDOP is an increasing function of $N_S$, so is E-AGDOP. Therefore, for Case 1, increasing the number of sensor nodes always deteriorates localization accuracy.

6.3.2 Case 2: $\delta_S \asymp N_S^0$ and $\delta_A \asymp N_S^0$

The second case is at the top left corner of the practical domain in Fig. 3. Both $\delta_S$ and $\delta_A$ are constant. It can be calculated from (22) that as $N_S \rightarrow \infty$,

$$\text{LB-E-AGDOP} \rightarrow \tilde{C} = d^2 / \delta.$$  \hfill (39)

To ensure that E-AGDOP is finite, we must have

$$\tilde{C} = d^2 / \delta < C_1^{-1}.$$  \hfill (40)

Fig. 3. Asymptotic bounds of LB-E-AGDOP as the network scale grows to infinity ($N_S \rightarrow \infty$). The connectivity is assumed to grow with the network scale as $\delta_S \asymp N_S^0$ and $\delta_A \asymp N_S^{-1}$. 

Unlocalizable, LB-E-AGDOP $\rightarrow \infty$

Localizable, LB-E-AGDOP $\rightarrow 0$

Localizable, LB-E-AGDOP $\rightarrow \infty$

Localizable, LB-E-AGDOP $\rightarrow \delta^2$
As the derivative of 
LB-E-AGDOP with respect to 
$N_S$ is negative,
\[
\frac{\partial \text{LB-E-AGDOP}}{\partial N_S} = -\frac{d^2 \delta_S^2}{\delta_A (\delta N_S - \delta_A)^2} < 0,
\]
increasing the number of sensor nodes always decreases LB-E-AGDOP. Nevertheless, the derivative of 
E-AGDOP with respect to 
$N_S$ is given by
\[
\frac{\partial \text{E-AGDOP}}{\partial N_S} = \frac{\delta_A \delta_S^2}{d^2 (\delta_S + \delta_A (N_S - 1))^2} - \frac{C_1 C_2}{(N_S + C_2)^2},
\]
which indicates that E-AGDOP is not necessarily a monotonic function of $N_S$. As $N_S \rightarrow \infty$, the derivative of 
E-AGDOP with respect to 
$N_S$ is
\[
N_S^2 \frac{\partial \text{E-AGDOP}^{-1}}{\partial N_S} \rightarrow \frac{\delta_A^2}{d^2} - C_1 C_2.
\]
Therefore, for Case 2, when $N_S$ is sufficiently large, increasing $N_S$ can
- deteriorate localization accuracy if $\frac{\delta_A^2}{d^2} < C_1 C_2$;
- improve localization accuracy if $\frac{\delta_A^2}{d^2} > C_1 C_2$.

### 6.3.3 Case 3: $\delta_S \approx N_S^2$ and $\delta_A \approx N_S^{-1}$
Furthermore, let us consider a very benign case that the sensor nodes form a complete graph [30], i.e., range measurements are available for every pair of distinct sensor nodes. This case lies along the right edge of the practical domain in Fig. 3.

Since $\delta_S = N_S - 1$, (22) can be reduced to
\[
\text{LB-E-AGDOP} = \frac{d^2}{\delta_A + N_S} \left(1 + \frac{1}{\delta_A}\right) \rightarrow \tilde{C} = \frac{d^2}{K_A}
\]
as $N_S \rightarrow \infty$. It can be seen that LB-E-AGDOP approaches $\frac{d^2}{K_A}$, just as if each sensor node is directly connected to all anchor nodes. To ensure that E-AGDOP is finite, we must have
\[
\tilde{C} = \frac{d^2}{K_A} < C_1^{-1}.
\]
The derivative of E-AGDOP with respect to $N_S$ is given by
\[
\frac{\partial \text{E-AGDOP}^{-1}}{\partial N_S} = \frac{N_S(N_S - 2) - K_A}{d^2 N_S^2 (1 + N_S/K_A)^2} - \frac{C_1 C_2}{(N_S + C_2)^2},
\]
which indicates that E-AGDOP is not necessarily a monotonic function of $N_S$. As $N_S \rightarrow \infty$, the derivative of 
E-AGDOP with respect to $N_S$ is
\[
N_S^2 \frac{\partial \text{E-AGDOP}^{-1}}{\partial N_S} \rightarrow \frac{K_A^2}{d^2} - C_1 C_2.
\]
Therefore, for Case 3, when $N_S$ is sufficiently large, increasing $N_S$ can
- deteriorate localization accuracy if $\frac{K_A^2}{d^2} < C_1 C_2$;
- improve localization accuracy if $\frac{K_A^2}{d^2} > C_1 C_2$.

Table 1 summarizes the behaviors of LB-E-AGDOP and E-AGDOP for Cases 1 to 3. In general, range-based localization schemes can guarantee a worst bound of accuracy for large-scale network, even for the marginal case $K_S = O(N_S)$ and $K_A = O(1)$ that just guarantees localizability. Nevertheless, to ensure localization accuracy not to deteriorate with an increasing number of nodes, the network must be more densely connected than the marginal case.

### Table 1 Localized accuracy for large-scale networks.

| Network connectivity level | As $N_S \rightarrow \infty$ |
|---------------------------|--------------------------|
| Case 1: $\delta_S \approx N_S^2$ and $\delta_A \approx N_S^{-1}$ | increases to $d^2(1/\delta_S + 1/K_A)$ | always increases |
| Case 2: $\delta_S \approx N_S^2$ and $\delta_A \approx N_S$ | decreases to $d^2/\delta$ | \{ increases if $\delta_S^2/(d^2 \delta_A) < C_1 C_2$
| | | decrease if $\delta_S^2/(d^2 \delta_A) > C_1 C_2$
| Case 3: $\delta_S \approx N_S^3$ and $\delta_A \approx N_S^{-1}$ | decreases to $d^2/K_A$ | \{ increases if $K_A^2/d^2 < C_1 C_2$
| | | decrease if $K_A^2/d^2 > C_1 C_2$

### 7 Simulation Results
In this section, we conduct numerical simulations to validate the theoretical results obtained from Section 4 to Section 5. All simulation results presented in this section are based on the following settings:
- Two dimensions ($d = 2$);
- Sensor nodes are uniformly distributed in the unit square $[0, 1] \times [0, 1]$;
- Four anchors ($K_A = 4$) located at the corners of the unit square, i.e., $(0, 0), (0, 1), (1, 0), \text{and} (1, 1)$;
- Given $K_S$, sensor-to-sensor links are chosen uniformly at random from the set $\{(i, j) | 1 \leq i < j \leq N_S, i, j \in \mathbb{Z}\}$;
- Given $K_A$, anchor-to-sensor links are chosen uniformly at random from $V_S \times V_A$, where $V_S = \{1, 2, \ldots, N_S\}$ is the set of sensors and $V_A = \{N_S + 1, N_S + 2, \ldots, N\}$ is the set of anchors.

Fig. 4 has shown a snapshot excerpted from the simulation with the parameters $N_S = 10$, $K_S = 20$, and $K_A = 8$.

#### 7.1 E-AGDOP and its theoretical lower bound
Fig. 4 compares LB-E-AGDOP from (22) and E-AGDOP obtained from simulations. The simulations are based on
the parameters $N_S = 8, 16, 24, 32, \delta_S = 5, 5.25, \ldots, 8.75$, and $\delta_A = 1, 1.125, \ldots, 2.5$. Each marker in Fig. 4 represents a network configuration with certain $N_S$, $\delta_S$ and $\delta_A$.

It can be seen that the theoretical lower bound is validated by the simulation results as no markers are below the magenta line $y = x$. For a fixed $N_S$, LB-E-AGDOP is a valid performance indicator of localization accuracy because if two different network configurations result in the same LB-E-AGDOP values, they also lead to very close E-AGDOP values. However, the relationship between LB-E-AGDOP and E-AGDOP varies with different values of $N_S$. Eq. (28) captures such a relationship, and the theoretical and simulated E-AGDOP values are compared in Fig. 5. The theoretical E-AGDOP values match the simulated values very well.

The parameters $C_1 = 0.788$ and $C_2 = 6.16$ mentioned in Section 5 are obtained by curve fitting of the theoretical LB-E-AGDOP values and the simulated E-AGDOP values obtained here. Therefore, Fig. 5 essentially shows the “training error” of our model (28), and Fig. 6 to 8 show the “test error.” It can be seen that both training and test errors are small when LB-E-AGDOP is small. This means that our theoretical result (28) is more accurate when the network is more densely connected.

7.2 Accuracy for large-scale networks

As a validation of the theoretical results obtained in Section 5, Fig. 6 to 8 depict how localization accuracy varies for an increasing network scale.

Fig. 6 is based on Case 1. As discussed in Section 6, both LB-E-AGDOP and E-AGDOP increase monotonically with an increasing $N_S$. Increasing $N_S$ deteriorates localization accuracy.

Fig. 5. Comparison between E-AGDOP from (28) and E-AGDOP from simulations with the parameters $N_S = 8, 16, 24, 32, \delta_S = 5, 5.25, \ldots, 8.75$, and $\delta_A = 1, 1.125, \ldots, 2.5$. The magenta line shows $y = x$.

Fig. 6. Accuracy of range-based localization schemes for large-scale network (Case 1: $\delta_S = 6$ and $\delta_A = 16/N_S$).

Fig. 7 is based on Case 2, in which LB-E-AGDOP always decreases with an increasing $N_S$. Because $\frac{\delta_S^2}{2x^3}$ is $9 > C_1C_2$, E-AGDOP is also a decreasing function of $N_S$. Increasing $N_S$ improves localization accuracy.

Fig. 8 shows a very benign case where the sensor nodes form a complete graph, and the number of anchor-to-sensor links is equal to the number of sensors. This case is at the top right corner of the practical domain in Fig. 5. According to the discussion in Section 6, as $N_S \to \infty$, LB-E-AGDOP approaches 0, so does E-AGDOP. The simulation result confirms this conclusion.

8 CONCLUSION

This paper has studied how connectivity and scale affect the accuracy of range-based localization schemes in ran-
random sensor networks. We have shown that the variance of localization errors are proportional to AGDOP. We have proven a novel lower bound of expectation of AGDOP and derived two closed-form formulas (22) and (28) that relate LB-E-AGDOP and E-AGDOP to only three parameters: the average sensor degree, average anchor degree, and number of sensor nodes. We have then used both formulas to study how localization accuracy varies with connectivity and the network scale. The following conclusions are drawn from our theoretical analysis.

- Localization accuracy is approximately inversely proportional to the average degree.
- When network connectivity merely guarantees localizability, increasing sensor nodes deteriorates localization accuracy. When a network is sufficiently densely connected, increasing sensor nodes improves localization accuracy.

The simulation results have validated the theoretical results, and shown that our formulas (22) and (28) can correctly describe the expected accuracy of range-based localization in random sensor networks. The theories and results presented in this paper provide guidelines on the design of range-based localization schemes and the deployment of sensor networks.

**APPENDIX A**

**Proof of Theorem 1**

There are a few approaches to proving Theorem 1. One of the simplest proofs is based on a recent result about the Cauchy–Schwarz inequality for the expectation of random matrices [35], [36].

**Lemma 1 (Cauchy–Schwarz inequality [35], [36]):** Let $A \in \mathbb{R}^{n \times p}$ and $B \in \mathbb{R}^{n \times p}$ be random matrices such that $E\|A\|^2 < \infty$, $E\|B\|^2 < \infty$, and $E(A^T A)$ is non-singular. Then

$$E(B^T B) \succeq E(B^T A)[E(A^T A)]^{-1} E(A^T B).$$  (47)

With the substitutions $A = G$ and $B = G(G^T G)^{-1}$ into the above inequality, we have

$$U = E[(G^T G)^{-1}] \succeq V = [E(G^T G)]^{-1},$$ (48)

which already proves Theorem 1.

Since the diagonal elements of a positive semidefinite matrix must be non-negative, we have

$$U_{ii} \geq V_{ii}, \quad \forall i = 1, \ldots, dN_S,$$  (49)

where $U = [U_{ij}]$ and $V = [V_{ij}]$. In particular, the expectation of GDOP, $\text{tr}(U)$, has a lower bound $\text{tr}(V)$.

**APPENDIX B**

**Proof of Theorem 2**

Under the assumption of Theorem 2, using Taylor series leads to

$$E(X^{-1}) = E\left(\frac{1}{E(X) + \Delta X}\right)$$
$$= \frac{1}{E(X)} E\left(\frac{1}{1 + \Delta X / E(X)}\right)$$
$$\approx \frac{1}{E(X)} E\left(1 - \frac{\Delta X}{E(X)} + \frac{\Delta X^2}{E(X)^2}\right)$$
$$= \frac{1}{E(X)} \left(1 + \frac{\text{Var}(X)}{|E(X)|^2}\right)$$
$$\approx \frac{1}{E(X)(1 - \text{Var}(X)/|E(X)|^2)}$$
$$= \frac{1}{E(X) - \text{Var}(X)/|E(X)|}.$$

It is worth noting that the approximate equals in (50) can be exact for some distributions. For instance, if $X$
is distributed according to the chi-squared distribution with $\kappa$ degrees of freedom, we have
\[
E(X^{-1}) = \frac{1}{\kappa - 2},
\]
(51)
\[
E(X) = \kappa,
\]
(52)
\[
\text{Var}(X) = 2\kappa.
\]
(53)

The above equations directly lead to
\[
\frac{1}{E(X^{-1})} = E(X) - \frac{\text{Var}(X)}{E(X)} = E(X) - 2.
\]
(54)

**APPENDIX C**

**Proof of Eq. (21)**

**Lemma 2** (Sherman–Morrison formula [37]): Suppose $A$ is an invertible square matrix, and $u$ and $v$ are vectors. Suppose furthermore that $1 + v^T A^{-1} u \neq 0$. Then the Sherman–Morrison formula states that
\[
(A + uv^T)^{-1} = A^{-1} - A^{-1}uv^T A^{-1} + \frac{1}{1 + v^T A^{-1} u}.
\]
(55)

With $\eta = d^{-1} [\delta + 6\kappa/(N\kappa S - 1)]$, (20) can be written as
\[
\eta^{-1} E\mathbf{\hat{F}} = I - uu^T,
\]
(56)
where $u = \sqrt{\zeta}(1, 1, \ldots, 1)^T$, and $\zeta = \delta S / [\delta(N\kappa S - 1) + \delta S]$. Letting $u = -v = \sqrt{\zeta}(1, 1, \ldots, 1)^T$, by the Sherman–Morrison formula we have
\[
(I - uu^T)^{-1} = I + uu^T / (1 - u^T u),
\]
(57)
and thus
\[
\eta \text{tr}[(E\mathbf{\hat{F}})^{-1}] = \text{tr}[(I - uu^T)^{-1}]
\]
\[= N\kappa S + N\kappa S \zeta / (1 - N\kappa \zeta).
\]
(58)

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