On the distribution of divisors of monic polynomials in function fields

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Abstract

This paper deals with function field analogues of famous theorems of Laudau and R. Hall. Laudau [1] counted the number of integers which have $t$ prime factors and R. Hall [2] researched the distribution of divisors of integers in residue classes. We extend the Selberg-Delange method to handle the following problems. (a) The number of monic polynomials with degree $n$ have $t$ irreducible factors. (b) The number of monic polynomials with degree $n$ in some residue classes have $t$ irreducible factors. (c) The residue classes distribution of divisors of monic polynomials.

Keywords: Function Fields; Selberg-Delange method; Analytic Number Theory

1. Introduction

In 1909, Laudau [1] counted the number of integers which have $t$ prime factors by using prime number theorem. We have

$$N_k(x) = \left| \{ n \leq x : \Omega(n) = k \} \right| \sim \frac{x}{\log(x)} \frac{(\log \log(x))^{k-1}}{(k-1)!}, x \to +\infty$$  

$$\frac{(k-1)!}{(k-1)!}$$

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Another way to solve this problem, devised by Selberg (1954) by identifying $N_k(x)$ as the coefficient of $z_k$ in the expression

$$
\sum_{n \leq x} z^{\Omega(n)}
$$

and then applying Cauchy’s integral formula. We use notation $a \ll c_1, c_2, \ldots b$ or $a = O(c_1, c_2, \ldots b)$ to mean that $|a| \leq C|b|$ for a suitable positive constant $C$ which depend upon parameters $c_1, c_2, \ldots$.

This method is nowadays known as the Selberg-Delange method, we refer the readers to [3] for an excellent exposition of this theory. Indeed, by this method, we can obtain a good estimate for the sum $\sum_{n \leq x} z^{\Omega(n)}$ (can see [3] 301–305), that is

$$
\sum_{n \leq x} z^{\Omega(n)} = x (\log(x))^{-1} \left( \sum_{0 \leq k \leq N} \frac{\nu_k(z)}{(\log(x))^k} + O(A(R_N(x))) \right) \quad (2)
$$

with

$$
R_N(x) := e^{-c_1 \sqrt{\log(x)}} + \left( \frac{c_2 N + 1}{\log(x)} \right)^{N+1}
$$

and some coefficients $\nu_k(z)$. From this formula, we can get an explicit asymptotic formulae for $N_k(x)$. Let

$$
\nu(z) = \frac{1}{\Gamma(z+1)} \prod_p \left( 1 - \frac{z}{p} \right)^{-1} \left( 1 - \frac{1}{p} \right)^z,
$$

then, we have

$$
N_k(x) = \frac{x}{\log(x)} \frac{(\log(x))^{k-1}}{(k-1)!} \left\{ \nu \left( \frac{k-1}{\log(x)} \right) + O \left( \frac{k}{(\log(x))^2} \right) \right\}
$$

for $k \leq (2 - \delta) \log \log(x)$, $0 < \delta < 1$, and

$$
N_k(x) = \frac{1}{4} \prod_{p > 2} \left( 1 + \frac{1}{p(p-2)} \right) \frac{x \log(x)}{2^k} \left( 1 + O \left( (\log(x))^{2k} \right) \right)
$$

for $(2 + \delta) \log \log(x) \leq k \leq A \log \log(x)$.

Let $F_q$ be a finite field with $q$ elements, where $q = p^f$, $p$ is the characteristic of $F_q$. Let $A = F_q[T]$, the polynomial ring over $F_q$. 

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In this paper, we extend the Selberg-Delange method to handle the following problems.

(I) The number of monic polynomials of degree $n$ which have $t$ irreducible factors, i.e.

$$N_t(n) = |\{ f \in \mathbb{F}_q[T] : f \text{ monic}, \deg(f) = n, \Omega(f) = t \}|.$$

(II) The number of monic polynomials of degree $n$ which have $t$ irreducible factors and belong to some residue class, i.e.

$$N_t(n; h, Q) = |\{ f \in \mathbb{F}_q[T] : f \text{ monic}, \deg(f) = n, \Omega(f) = t, f \equiv h \pmod{Q} \}|,$$

where $Q \in \mathbb{A}$ is a polynomial and $h$ is a polynomial such that $(Q, h) = 1$ and $\deg(h) \leq \deg(Q)$.

Let $A_n = \{ f \in \mathbb{A} | \deg(f) = n, f \text{ monic} \}$, $A_n(h, Q) = \{ f \in A_n | f \equiv h \pmod{Q} \}$, $A_n(t) = \{ f \in A_n | \Omega(f) = t \}$ and $A_n(t; h, Q) = \{ f \in A_n | \Omega(f) = t, f \equiv h \pmod{Q} \}$. Then we have $N_t(n) = |A_t(n)|$ and $N_t(n; h, Q) = |A_t(n; h, Q)|$.

To solve these problems, we need to define the analogue of Riemann zeta function and Dirichlet $L$-function. Let $f \in \mathbb{A}$, if $f \neq 0$, set $|f| = |q^{\deg(f)}|$, if $f = 0$, set $|f| = 0$. The zeta function of $\mathbb{A}$, denoted by $\zeta_A(s)$, is defined by

$$\zeta_A(s) = \sum_{f \text{ monic}} \frac{1}{|f|^s} = \frac{1}{1 - q^{1-s}}.$$ 

for $s \in \mathbb{C}$ with $\Re(s) > 1$, where $\Re(s)$ denote the real part of $s$. $\zeta_A(s)$ can be continued to a meromorphic function on the whole complex plane with simple poles at $s = 1 + \frac{2\pi in}{\log(q)}$ for $n \in \mathbb{Z}$. The Riemann zeta function is also a meromorphic function on the whole complex plane, which is holomorphic everywhere except for a simple pole at $s = 1$ with residue 1. It is difficult to solve problem (I) and (II) by using Perron formula since the distribution of poles of Riemann zeta function and $\zeta_A(s)$ are different.

For problem (I), we use Selberg-Delange method, replacing Perron formula with Cauchy integral formula. Then, we obtain the following theorems.

**Theorem 1.1.** Let $y$ be a complex number such that $|y| \leq \rho, \rho < 1$ and $n > 2$
be an integer, then we have

$$\sum_{f \in A_n} y^{\Omega(f)} = \prod_P \left( 1 - \frac{1}{|P|^y} \right)^y \left( 1 - y \right)^{-1} \frac{\kappa(y) q^n}{\Gamma(y) n^{1-y}} + O_{\delta,\rho} \left( \frac{q^n}{n^{2-R(y)}} \right), \quad (3)$$

where

$$\kappa(y) = \exp \left( -(1-y) \int_0^{+\infty} \frac{B_1(t) dt}{(n-1+y+t)(n+t)} \right).$$

Theorem 1.2. The number of monic polynomials of degree $n$, which have exactly $t$ irreducible factors is

$$N_t(n) = \sum_{f \in A_n(t)} 1 = \frac{q^n \log^{t-1}(n)}{n} \sum_{r=1}^t A_r \log^{1-r}(n) + O_{\delta,\rho} \left( \frac{\rho^{-n} q^n}{n^{2-R(y)}} \right) \quad (4)$$

for any $\rho < 1$ and $n > 2$.

Note that if we set $x = q^n$ in the right-hand side of equations (3) and (4), then they are very similar to the equations (2) and (1). If $n = 1$, then equation (4) becomes

$$N_1(n) = \frac{q^n}{n} + O_{\delta,\rho} \left( \frac{\rho^{-n} q^n}{n^{2-R(y)}} \right),$$

since $A_1 = 1$. This result is agree with the previous formula of $N_1(n)$, i.e. the number of irreducible polynomials of degree $n$,

$$N_1(n) = \frac{1}{n} \sum_{d|n} \mu(d) q^\frac{n}{d} = \frac{q^n}{n} + O \left( \frac{q^\frac{n}{n}}{n} \right).$$

The results associates with problem (II) are

Theorem 1.3. Let $y$ be a complex number such that $|y| \leq \rho$, $\rho < 1$ and $n > 2$ be an integer, then we have

$$\sum_{f \in A_n(h,Q)} y^{\Omega(f)} = \frac{1}{\Phi(Q)} \prod_P \left( 1 - \frac{1}{|P|^y} \right)^y \left( 1 - y \right)^{-1} \prod_{P|Q} \frac{1}{1 - \frac{1}{|P|^y}} \frac{\kappa(y) q^n}{\Gamma(y) n^{1-y}} + O_{\delta,\rho} \left( \frac{q^n}{n^{2-R(y)}} \right),$$

where $\kappa(y)$ is defined as above.
Theorem 1.4. The number of monic polynomials, which satisfy \( f \equiv h \mod(Q) \), \( \deg(f) = n \) and \( \Omega(f) = t \) is

\[
\sum_{f \in A_n(t,h,Q)} 1 = \left( \prod_{P \mid Q} \frac{1}{1 - \frac{1}{|P|}} \right) \frac{q^n \log^{t-1}(n)}{\Phi(Q)n} \sum_{r=1}^{t} A_r \frac{\log^{1-r}(n)}{(t-r)!} + O_{\delta,Q,\rho} \left( \rho^{-n} q^n \right)
\]

for any \( \rho < 1 \) and \( n > 2 \).

This theorem claims that monic polynomials in \( \in A_n(t) \) seem to be equally distributed among the \( \Phi(Q) \) reduced residue classes mod \( Q \).

Another problem associates with the distribution of divisors of monic polynomial over function fields is the residue classes distribution of divisors of monic polynomial. R. HALL [2] researched the distribution of divisors of integers in residue classes. In this paper, we research the function fields version of it.

Let \( \tau(f;h,Q) \) be the number of those divisors of \( f \) which are prime to \( Q \). Let \( \tau(f;Q) \) be the number of those divisors of \( f \) which belong to the residue classes \( h \mod(Q) \). The residue classes distribution of divisors of monic polynomial \( f \) can be described by the variance

\[
\mathbb{V}[\tau(f;\circ,Q)] := \frac{1}{\Phi(Q)} \sum_{h,(h,Q) = 1 \atop \deg(h) \leq \deg(Q)} \left( \tau(f;h,Q) - \frac{\tau(f;Q)}{\Phi(Q)} \right)^2.
\]

In this paper, we evaluate the following sum,

\[
\sum_{f \in A_n} y^{\Omega(f)} \mathbb{V}[\tau(f;\circ,Q)]. \tag{5}
\]

For \( y = 1 \), we can obtain that

Theorem 1.5. For integer \( n \), we have

\[
\sum_{f \text{ monic} \atop \deg(f) = n} \mathbb{V}[\tau(f;\circ,Q)] = \frac{1}{\Phi(Q)} \sum_{f \text{ monic} \atop \deg(f) = n} \sum_{(h,Q) = 1 \atop \deg(h) \leq \deg(Q)} \left( \tau(f;h,Q) - \frac{\tau(f;Q)}{\Phi(Q)} \right)^2
\]

\[= A_Q(n + 1)q^{n+1} + B_Qq^{n+1} + O_{Q,\varepsilon}(q^{n+1} - 1).\]

Where \( A_Q \) and \( B_Q \) are constants associated with polynomial \( Q \).
This Theorem shows that the average variance $\mathbb{V}[\tau(f; \circ, Q)]$ over $A_n$ is

$$A_Q(n+1)q + B_Qq.$$ 

For $|y| \leq \rho, \rho < 1$ and $R(y) \leq \frac{1}{2}$, We have a good estimate for equation (5) by using Selberg-Delange method, i.e.

**Theorem 1.6.** Let $y$ be a complex number such that $|y| \leq \rho$ and $\rho < 1$. If $R(y) < \frac{1}{2}$, then

$$\sum_{\substack{f \text{ monic} \ \deg(f) = n}} g^{\Omega(f)}\mathbb{V}[\tau(f; \circ, Q)] = H_1 \left( \frac{1}{q^y} \right) \frac{\kappa(2y)q^n}{\Gamma(2y)n^{1-2y}} + O_{\delta, Q, \rho} \left( \frac{q^n}{n^{2-2R(y)}} \right)$$

for integer $n > 3$, where

$$H_1(u, y) = \left( \prod_{P|Q} \frac{1}{1 + yu^\deg(P)} \right) \frac{(1 - qu^2)^{nu}N^2(u, y)}{\Phi(Q)^2N(u^2, y)} \sum_{\chi \neq \chi_0} \mathcal{L}(u, \chi, y)\mathcal{L}(u, \bar{\chi}, y).$$

If $y = \frac{1}{2}$, then for integer $n$,

$$\sum_{\substack{f \text{ monic} \ \deg(f) = n}} \frac{1}{2^{R(f)}}\mathbb{V}[\tau(f; \circ, Q)] = \frac{q^{n+1}}{\Phi(Q)^2} \prod_{P|Q} \frac{1}{1 + \frac{1}{2y^2P}} \frac{N^2(1, \frac{1}{2})}{\zeta_A(2, \frac{1}{2})} \sum_{\chi \neq \chi_0} |L(1, \chi, \frac{1}{2})|^2$$

$$+ O_{\delta, Q} \left( q^{\left(\frac{5}{2}+\delta\right)n} \right).$$

Theorem 1.6 enable us to obtain the residue classes distribution of divisors of monic polynomials in set $A_n(t)$.

**Theorem 1.7.** For integer $n > 3$, we have

$$\sum_{f \in A_n(t)} \mathbb{V}[\tau(f; \circ, Q)] = \frac{q^{n}(2\log(n))^{t-1}}{n} \sum_{r=1}^{t} \hat{A}_n(2\log(n))^{1-r} \frac{\Gamma(t-r)!}{(t-r)!} + O_{\delta, Q, \rho} \left( \frac{\rho^{-n}q^n}{n^{2-2R(y)}} \right)$$

for any $\rho < 1$.

2. Preliminary

Let $g \in A$ be a monic polynomial and $\chi: \mathbb{F}_q[T]/g\mathbb{F}_q[T]^* \rightarrow \mathbb{C}^*$ is a group homomorphism from invertible elements of $\mathbb{F}_q[T]/g\mathbb{F}_q[T]$ to the non-zero complex
numbers. The Dirichlet character of modulo \( g \) is defined by

\[
\chi(f) = \begin{cases} 
\chi(f \mod g), & \text{if } \gcd(f, g) = 1, \\
0, & \text{otherwise.}
\end{cases}
\]

The Dirichlet character is multiplicative function of the polynomial ring \( \mathbb{F}_q[T] \).

The Dirichlet \( L \)-function \( L(s, \chi) \) associated to Dirichlet character \( \chi \) is defined to be

\[
L(s, \chi) = \sum_{f \text{ monic}} \frac{\chi(f)}{|f|^s}.
\]

The function \( \zeta_A(s) \) and \( L(s, \chi) \) satisfy the Euler product formula

\[
\zeta_A(s) = \prod_{P \text{ irreducible}} \left( 1 - \frac{1}{|P|^s} \right)^{-1},
\]

and

\[
L(s, \chi) = \prod_{P \text{ irreducible}} \left( 1 - \frac{\chi(P)}{|P|^s} \right)^{-1},
\]

respectively, provide \( R(s) > 1 \). The Dirichlet \( L \)-function \( L(s, \chi) \) associated to principal character \( \chi_0 \) is almost the same as \( \zeta_A(s) \). Indeed,

\[
L(s, \chi_0) = \prod_{P \mid g \text{ irreducible}} \left( 1 - \frac{1}{|P|^s} \right) \zeta_A(s). \tag{6}
\]

Let \( \zeta(u) = \frac{1}{1-qu} \) and

\[
\mathcal{L}(u, \chi) = \sum_{f \text{ monic}} \chi(f)u^{\deg(f)}.
\]

We have \( \zeta_A(s) = \zeta(q^{-s}) \) and \( L(s, \chi) = \mathcal{L}(q^{-s}; \chi) \).

For non-principal character \( \chi \), we know that \( \mathcal{L}(u, \chi) \) is a polynomial of \( u \) of degree at most \( \deg(g) - 1 \) and the Generalized Riemann Hypothesis (GRH) states that the all roots of \( \mathcal{L}(u, \chi) \) have modulus 1 or \( q^{-\frac{1}{2}} \). Hence, we have

\[
\mathcal{L}(u, \chi) = \prod_{i=1}^{m(\chi)} (1 - \alpha_i(\chi)u).
\]

where \( |\alpha_i(\chi)| = 1 \) or \( q^{\frac{1}{2}} \) for \( 1 \leq i \leq m(\chi) < \deg(g) \).
At first, we estimate $\zeta(u)$ and $L(u, \chi)$. We have $|\zeta(u)| = |\frac{1}{1-qu}| \leq \frac{1}{|1-q|u|}$ for $u \neq q^{-1}$ and for non-principal character $\chi$ of modulo $g$, we have

$$|L(u, \chi)| = \prod_{i=1}^{m(\chi)} |1 - \alpha_i(\chi)u| \leq (1 + \sqrt{q}|u|)^m$$

(7)

where $m = \deg(g) - 1$. For any $|y| \leq \rho$, $\rho < 1$, let

$$\zeta_A(s, y) = \prod_p \left(1 - \frac{y}{|P|^s}\right)^{-1} = \sum_{f \text{ monic}} \frac{y^{\Omega(f)}}{|f|^s} \chi_{f(y)},$$

$$L(s, \chi, y) = \prod_p \left(1 - \frac{y\chi(P)}{|P|^s}\right)^{-1} = \sum_{f \text{ monic}} \frac{\chi_f(y^{\Omega(f)})}{|f|^s},$$

where $\Omega(f)$ denotes the number of prime divisors of $f$(Including multiplicities).

Let

$$N(s, y) = \zeta_A(s)^{-y} \prod_p \left(1 - \frac{y}{|P|^s}\right)^{-1} = \prod_p \left(1 - \frac{1}{|P|^s}\right)^y$$

and

$$M(s, \chi, y) = L(s, \chi)^{-y} \prod_p \left(1 - \frac{y\chi(P)}{|P|^s}\right)^{-1} = \prod_p \left(1 - \frac{\chi(P)}{|P|^s}\right)^y.$$

Then, $N(s, y)$ and $M(s, \chi, y)$ are convergent and bounded in the half plane $R(s) \geq \frac{1}{2} + \delta$ for any $\delta > 0$. Indeed, Let

$$\frac{(1 - z)^y}{1 - yz} = 1 + \sum_{v \geq 1} a_v z^v,$$

then $a_1 = 0$ and

$$|a_v| = \left|\frac{1}{2\pi i} \int_{C_r} (1 - z)^y z^{v-1} \frac{dz}{1 - yz}\right| \leq \frac{M(\rho)}{r^v}$$

for $v \geq 2$ and $r = \frac{1}{2\rho}$. Where

$$M(\rho) = \max_{|z|=r, |y| \leq \rho} \left|\frac{(1 - z)^y}{1 - yz}\right| \leq 2e^{\rho(\pi+1)} = 2e^{\rho(\pi+1)+\frac{1}{2}}.$$
Let \( z = \frac{1}{\sqrt{q}} \), then

\[
N(s, y) = \prod_p \left( \frac{1 - \frac{1}{|P|^s}}{1 - y^{\deg(P)}} \right)^y = \prod_p \left( 1 + \sum_{v \geq 2} \frac{a_v}{|P|^v s} \right)^y
\]

\[
\ll \delta, \rho \prod_p \left( 1 + \sum_{v \geq 2} \frac{M(\delta)}{r^v |P|^{v(\frac{1}{2} + \delta)}} \right) \ll \delta, \rho \prod_p \exp \left( \sum_{v \geq 2} \frac{1}{r^v |P|^{v(\frac{1}{2} + \delta)}} \right)
\]

\[
= \exp \left( \sum_{n \geq 1} \frac{q^n}{n^2 q^{n(1 + 2\delta)} - r q^{n(\frac{1}{2} + \delta)}} \right) \ll \delta, \rho 1
\]

We can also get \( M(s, \chi, y) \ll \delta, \rho 1 \).

Putting variable substitution \( u = q^{-s} \) in \( \zeta_A(s, y), L(s, \chi, y), N(s, y) \) and \( M(s, \chi, y) \), we denote these new functions with \( \zeta(u, y), L(u, \chi, y), N(u, y) \) and \( M(u, \chi, y) \) respectively. Indeed, we have

\[
\zeta(u, y) = \prod_p \frac{1}{1 - y^{\deg(P)}} = \sum_{f \text{ monic}} y^{\Omega(f)} u^{\deg(f)},
\]

\[
L(u, \chi, y) = \prod_p \frac{1}{1 - y^{\chi(P)} u^{\deg(P)}} = \sum_{f \text{ monic}} y^{\Omega(f)} \chi(f) u^{\deg(f)},
\]

and

\[
N(u, y) = \prod_p \left( \frac{1 - u^{\deg(P)} y}{1 - y^{\deg(P)}} \right)^y, M(u, \chi, y) = \prod_p \left( \frac{1 - \chi(P) y^{\deg(P)}}{1 - y^{\chi(P)} u^{\deg(P)}} \right)^y
\]

Then \( N(u, y), M(u, \chi, y) \) are convergent and bounded if \( |u| \leq \frac{1}{q^{\frac{1}{2} + \delta}} \). Note that \( L(u, \chi) \) is a holomorphic function and has zeros only on the circle \( |u| = 1 \) or \( q^{-\frac{1}{2}} \) (by GRH), so \( L(u, \chi, y) = L(u, \chi)^y M(u, \chi, y) \) is holomorphic function on the disc \( \{ u : |u| \leq \frac{1}{q^{\frac{1}{2} + \delta}} \} \). \( \zeta(u) \) is a meromorphic function on the whole complex plane, which is holomorphic everywhere except for a simple pole at \( u = \frac{1}{q} \). \( \zeta(u) \) has no zero. Thus \( u = \frac{1}{q} \) may be the pole of function \( \zeta(u, y) = \zeta(u)^y \zeta_A(s, y) \).

We now give the bound of \( \zeta(u, y) \) and \( L(u, \chi, y) \) on the disc \( \{ u : |u| \leq \frac{1}{q^{\frac{1}{2} + \delta}} \} \).
Indeed, we have

\[
\zeta(u, y) = e^{-y(\log |1 - qu| + i \arg(1 - qu))} N(u, y) \ll \delta \ e^{-R(y) \log |1 - qu| + I(y) \arg(1 - qu)}
\]

\[
\ll \delta, |1 - qu|^{-R(y)},
\]

(9)

\[
\mathcal{L}(u, \chi, y) = \prod_{i=1}^{m} (1 - \alpha_i(\chi) z)^y \mathcal{M}(u, \chi, y)
\]

\[
\ll \delta, \rho \prod_{i=1}^{m} e^{R(y) \log |1 - \alpha_i(\chi) u|-I(y) \arg(1 - \alpha_i(\chi) u)}
\]

\[
\ll \delta, \rho \ (1 + \sqrt{|u|})^{R(y)m}.
\]

(10)

3. The number of monic polynomials of degree \( n \) which have \( t \) irreducible factors

Let \( B_n(t) \) denote \( n \)-th Bernoulli number as the 1-periodic function. We begin this section with a estimate of \( \log \Gamma(z) \).

**Lemma 3.1.** (Complex Stirling formula) For \( s \in \mathbb{C} - \mathbb{R}^- \), we have

\[
\log \Gamma(s) = \left(s - \frac{1}{2}\right) \log(s) - s + \frac{1}{2} \log(2\pi) - \int_{0}^{+\infty} \frac{B_1(t)dt}{s + t},
\]

(11)

where the complex logarithm is understood as its principal branch, i.e. \( \log(s) = \exp(\log|s| + i \arg(s)) \) and \(-\pi < \arg(s) \leq \pi\).

**Definition 3.1.** (Beta function) For complex numbers \( x, y \) satisfy \( R(x), R(y) > 0 \), we define

\[
B(x, y) = \int_{0}^{+\infty} t^{x-1}(1-t)^{y-1}dt.
\]

The relationship between gamma function and beta function is

\[
B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x + y)}.
\]

(12)

**Lemma 3.2.** Let \( y \) be a complex number such that \( |y| \leq \rho, \rho < 1 \) and \( n > 1 \) be an integer. We have

\[
\frac{\Gamma(n - 1 + y)}{\Gamma(n)} = \frac{\kappa(y)}{n^{1-y}} + O\left(\frac{1}{n^{2-R(y)}}\right),
\]

where

\[
\kappa(y) = \exp\left(-(1 - y)\int_{0}^{+\infty} \frac{B_1(t)dt}{(n - 1 + y + t)(n + t)}\right).
\]
Proof. From equation (11), we have
\[
\log \frac{\Gamma(n-1+y)}{\Gamma(n)} = \left( n - \alpha - \frac{1}{2} \right) \log(n-\alpha) - (n-\alpha) + \frac{1}{2} \log(2\pi) - \int_0^{+\infty} \frac{B_1(t)}{n-\alpha+t} dt
- \left( n - \frac{1}{2} \right) \log(n) + n - \frac{1}{2} \log(2\pi) + \int_0^{+\infty} \frac{B_1(t)}{n+t} dt
= \left( n - \alpha - \frac{1}{2} \right) \log(n-\alpha) - \left( n - \frac{1}{2} \right) \log(n) + C_y
\]
where \( \alpha = 1 - y \) and \( C_y = (1-y) \left( 1 - \int_0^{+\infty} \frac{B_1(t)}{(n-1+y+t)(n+t)} dt \right) \). Thus
\[
\frac{\Gamma(n-1+y)}{\Gamma(n)} = e^{C_y \left( n - \alpha - \frac{1}{2} \right) \log(n-\alpha)} n^{-\alpha} \left( 1 - \frac{\alpha}{n} \right) \frac{\kappa(y)}{n^{1-y}} + O_{\rho,\delta} \left( \frac{1}{n^{2-R(y)}} \right),
\]
where \( \kappa(y) = \exp \left( -(1-y) \int_0^{+\infty} \frac{B_1(t)}{(n-1+y+t)(n+t)} dt \right) \).

**Theorem 3.3.** Let \( y \) be a complex number such that \( |y| \leq \rho, \rho < 1 \) and \( n > 2 \) be an integer. Then we have
\[
\sum_{f \text{ monic}} y^{\Omega(f)} = \prod_{P} \left( 1 - \frac{1}{|P|} \right) y^{1-\frac{|P|}{|P|}} \kappa(y) \frac{q^n}{\Gamma(y) n^{1-y}} + O_{\rho,\delta} \left( \frac{q^n}{n^{2-R(y)}} \right)
\]
where \( \kappa(y) \) is defined as above.

Proof. Let \( |y| \leq \rho, \rho < 1 \). Note that
\[
\zeta_A(s, y) = \sum_{f \text{ monic}} \frac{y^{\Omega(f)}}{|f|^s}.
\]
Thus, let \( u = q^{-s} \), for \( s \) with \( R(s) > 1 \), we have
\[
\sum_{n \geq 0} \left( \sum_{f \text{ monic}} y^{\Omega(f)} \right) u^n = \zeta(u, y) = \zeta(u)^y N(u, y) = \frac{N(u, y)}{(1-qu)^y}.
\]
provide \( |u| < \frac{1}{q} \). By the Cauchy integral formula, we have
\[
\sum_{f \text{ monic}} y^{\Omega(f)} = \frac{1}{2\pi i} \int_{C_n} \frac{N(z, y)}{z^{n+1}(1-qz)^y} dz,
\]
Figure 1: Region $\Omega_1$ and contour $\gamma_1$

where $C_R$ is a circle centered at 0 with radius $R$. As shown in Figure 1, we consider the region $\Omega_1$ of integration which surrounded by contour $\gamma_1 = \tilde{C}_{q^{\frac{1}{2}-\delta}} \cup \tilde{C}_\epsilon \cup \Gamma_1 \cup \Gamma_2 \cup C_R$. $\gamma_1$ is defined by the following curves,

- $\tilde{C}_{q^{\frac{1}{2}-\delta}}$: semi circle centered at 0 with radius $q^{\frac{1}{2}-\delta}$ from $q^{-\frac{1}{2}-\delta} e^{i\beta}$ to $q^{-\frac{1}{2}-\delta} e^{i(2\pi-\beta)}$.
- $\tilde{C}_\epsilon$: semi circle centered at $z_0 = \frac{1}{q}$ with radius $\epsilon$ from $z_0 + \epsilon e^{i(2\pi-\alpha)}$ to $z_0 + \epsilon e^{i\alpha}$.
- $\Gamma_1$: the straight line from $z_0 + \epsilon e^{i\alpha}$ to $q^{-\frac{1}{2}-\delta} e^{i\beta}$.
- $\Gamma_2$: the straight line from $q^{-\frac{1}{2}-\delta} e^{i(2\pi-\beta)}$ to $z_0 + \epsilon e^{i(2\pi-\alpha)}$.
- $C_R$: circle centered at 0 with radius $R$.

We shall see that the main contribution arises from the integral over the straight lines $\Gamma_1$ and $\Gamma_2$. From (8), the integral over $\tilde{C}_{q^{\frac{1}{2}-\delta}} \triangleq \tilde{C}$ does not
exceed
\[ \ll_{\delta, \rho} \int_{\tilde{C}} \frac{1}{|z|^{n+1}(q^{\frac{1}{2}} - 1)R(y)} dz = \frac{q^n(y^{\delta} + \rho)}{(q^{\frac{1}{2}} - 1)R(y)}. \]
The integral over \( \tilde{C} \), does not exceed
\[ \frac{1}{2\pi i} \int_{\tilde{C}} \frac{N(z, y)}{z^{n+1}(1-qz)^2} dz = \frac{e^{1-y} - e^{(1-y)i\theta - \pi y} N(\frac{1}{q} + \epsilon e^{i\theta}, y)}{(\frac{1}{q} + \epsilon e^{i\theta})^{n+1}} d\theta \ll_{\delta, \rho} e^{1-R(y)}. \]
Then this integral tends to zero with \( \epsilon \to 0 \) since \( R(y) < 1 \). Letting the straight lines \( \Gamma_1, \Gamma_2 \) onto the real line and \( \epsilon \to 0 \) together, we have
\[ \frac{1}{2\pi i} \int_{\Gamma_R} \frac{N(z, y)}{z^{n+1}(1-qz)^2} dz = \frac{q^{-y}}{2\pi i} \int_{\frac{1}{q}}^{\frac{1}{q^{1+y}}} \frac{N(u, y)}{u^{n+1}} e^{-y(\log(u - \frac{1}{q}) - i\pi)} du \]
\[ - \frac{q^{-y}}{2\pi i} \int_{\frac{1}{q}}^{\frac{1}{q^{1+y}}} \frac{N(u, y)}{u^{n+1}} e^{-y(\log(u + 1) + i\pi)} du \]
\[ + O_{\delta, \rho} \left( \frac{q^n(y^{\delta} + \rho)}{(q^{\frac{1}{2}} - 1)R(y)} \right), \]
and therefore, we have
\[ \sum_{f \text{ monic } deg(f) = n} y^{\Omega(f)} = \frac{\sin(y\pi)}{q^n \pi} \int_{\frac{1}{q}}^{\frac{1}{q^{1+y}}} \frac{N(u, y)}{u^{n+1}} (u - \frac{1}{q})^{-y} du + O_{\delta, \rho} \left( \frac{q^n(y^{\delta} + \rho)}{(q^{\frac{1}{2}} - 1)R(y)} \right) \]
\[ = \frac{\sin(y\pi)}{q^n \pi} \int_0^{\omega_0} \frac{N(\frac{1}{q} + \omega, y)}{(\frac{1}{q} + \omega)^{n+1}} \omega^{-y} d\omega + O_{\delta, \rho} \left( \frac{q^n(y^{\delta} + \rho)}{(q^{\frac{1}{2}} - 1)R(y)} \right). \]
where \( \omega_0 = \frac{\frac{1}{q^{1+y}} - \frac{1}{q}}{2} \). By Cauchy’s theorem. We have
\[ \frac{N(\frac{1}{q} + \omega, y)}{\frac{1}{q} + \omega} - qN \left( \frac{1}{q}, y \right) = \frac{\omega}{2\pi i} \int_D \frac{N(z; y)}{z(z - \frac{1}{q} - \omega)(z - \frac{1}{q})} dz = \omega k(\omega), \]
where \( D \) is circle, centre \( \frac{1}{q} + \frac{\omega_0}{2} \) and radius \( \omega_0 \). Thus
\[ \frac{\sin(y\pi)}{q^n \pi} \int_0^{\omega_0} \frac{N(\frac{1}{q} + \omega, y)}{(\frac{1}{q} + \omega)^n} \omega^{-y} d\omega \]
\[ = qN \left( \frac{1}{q}, y \right) \frac{\sin(y\pi)}{q^n \pi} \int_0^{\omega_0} \frac{\omega^{-y}}{(\frac{1}{q} + \omega)^n} d\omega + \frac{\sin(y\pi)}{q^n \pi} \int_0^{\omega_0} \frac{\omega^{1-y}k(\omega)}{(\frac{1}{q} + \omega)^n+1} d\omega \]
\[ = qN \left( \frac{1}{q}, y \right) \frac{\sin(y\pi)}{q^n \pi} \int_0^{\omega_0} \frac{\omega^{-y}}{(\frac{1}{q} + \omega)^n} d\omega + O_{\delta, \rho} \left( \int_0^{\omega_0} \frac{\omega^{1-y}}{(\frac{1}{q} + \omega)^n} d\omega \right). \]
Note that

\[
\int_0^\infty \frac{\omega^{-y}}{(1 + \omega)^n} d\omega = q^{n-1+y} \int_0^{q\omega_0} \frac{x^{-y}dx}{(1 + x)^n} = q^{n-1+y} \int_0^{+\infty} \frac{x^{-y}dx}{(1 + x)^n} + O \left( \frac{(q\omega_0)^{-y}(1 + q\omega_0)^{1-n}}{n-1} \right)
\]

\[
= q^{n-1+y} \int_0^1 (1 - x)^{-y}x^{n-2+y}dx + O \left( \frac{(q\omega_0)^{-y}(1 + q\omega_0)^{1-n}}{n-1} \right)
\]

\[
= q^{n-1+y} \frac{\Gamma(1-y)\Gamma(n-1+y)}{\Gamma(n)} + O \left( \frac{q^{n-1-R(y)}(\frac{1}{2}\delta - \delta)}{n-1} \right)
\]

\[
= q^{n-1+y} \Gamma(1-y) \frac{\kappa(y)}{n^{1-y}} + O_{\delta,\rho} \left( \frac{q^{n-1+R(y)}}{n^{2-R(y)}} \right).
\]

The error term of (13) does not exceed

\[
O_{\delta,Q} \left( \int_0^{q\omega_0} \frac{\omega^{-y}}{(1 + \omega)^n} d\omega \right) = O_{\delta,\rho}(q^{-2+R(y)}|B(2 - y, n - 2 + y)|)
\]

\[
= O_{\delta,\rho} \left( \frac{q^n}{n^{2-R(y)}} \right).
\]

Thus we can obtain that

\[
\sum_{\text{monic } \deg(f)=n} y^{\Omega(f)} = N \left( \frac{1}{q}, y \right) \frac{\sin(y\pi)}{\pi} q^n \Gamma(1-y) \frac{\kappa(y)}{n^{1-y}} + O_{\delta,\rho} \left( \frac{q^n}{n^{2-R(y)}} \right)
\]

\[
= \prod_P \left( 1 - \frac{1}{|P|} \right)^y \left( 1 - \frac{y}{|P|} \right)^{-1} \frac{\kappa(y)q^n}{\Gamma(y)n^{1-y}} + O_{\delta,\rho} \left( \frac{q^n}{n^{2-R(y)}} \right)
\]

by the Reflection formula.

Note that \[\prod_P \left( 1 - \frac{1}{|P|} \right)^y \left( 1 - \frac{y}{|P|} \right)^{-1}\] is convergent for \(|y| \leq \rho\), let

\[
\frac{\kappa(y)}{\Gamma(y)} \prod_P \left( 1 - \frac{1}{|P|} \right)^y \left( 1 - \frac{y}{|P|} \right)^{-1} = \sum_{r=1}^{+\infty} A_r y^r,
\]

then we obtain the following theorem of this section.

**Theorem 3.4.** The number of monic polynomials of degree \(n > 2\) which have exactly \(t\) irreducible factors is

\[
\sum_{\text{monic } \deg(f)=n, \Omega(f)=t} 1 = q^n \log^{t-1}(n) \sum_{r=1}^t A_r \log^{1-r}(n) \frac{(t-r)!}{(t-r)!} + O_{\delta,\rho} \left( \frac{\rho^n q^n}{n^{2-R(y)}} \right)
\]

for any \(\rho < 1\).
Proof. By Cauchy’s coefficient formula,
\[
\sum_{\text{monic } f \text{ such that } \deg(f) = n, \Omega(f) = t} \frac{q^n}{n} \sum_{r=1}^{t} \frac{A_r \log(n)^{t-r}}{(t-r)!} + O_{S,\rho} \left( \frac{\rho^{-n}q^n}{n^{2-R(y)}} \right)
\]
for any \( \rho < 1 \). This completes the proof. \( \square \)

4. The number of monic polynomials of degree \( n \) which have \( t \) irreducible factors and belong to some residue classes

Theorem 4.1. Let \( y \) be a complex number such that \( |y| \leq \rho, \rho < 1 \) and \( n > 2 \) be an integer, then we have
\[
\sum_{\text{monic } f \equiv h \mod(Q)} y^\Omega(f) = \frac{1}{\Phi(Q)} \prod_p \left( 1 - \frac{1}{|P|} \right)^y \left( 1 - \frac{y}{|P|} \right)^{-1} \prod_{P|Q} \frac{1}{1 - \frac{1}{|P|} \Gamma(y)n^{1-y}}
\]

\[
+ O_{S,Q,\rho} \left( \frac{q^n}{n^{2-R(y)}} \right).
\]

where \( \kappa(y) \) is defined as above.

Proof. Let \( \chi \) be a Dirichlet character of modulo \( Q \) and \( h \) be a polynomial such that \( (Q,h) = 1 \). From the Orthogonal Relation of Drichlet character and equation (6), we have
\[
\sum_{\text{monic } f \equiv h \mod(Q)} y^\Omega(f) \frac{\prod_{f|f^\omega} \chi(f)\chi(h)}{|f^\omega|} = \frac{1}{\Phi(Q)} \sum_{\chi} \chi(h)L(s,\chi,y)
\]
\[
= \frac{1}{\Phi(Q)} \sum_{\chi} \chi(h)L(s,\chi_0,y) + \sum_{\chi \neq \chi_0} \chi(h)L(s,\chi,y)
\]
\[
= \left( \prod_{P|Q} \frac{1}{1 - \frac{1}{|P|}} \right) \frac{\zeta_A(s,y)}{\Phi(Q)} + \frac{1}{\Phi(Q)} \sum_{\chi \neq \chi_0} \chi(h)L(s,\chi,y).
\]

Thus, let \( u = q^{-s} \), for \( s \) and \( R(s) > 1 \), we have
\[
\sum_{n \geq 0} \left( \sum_{\text{monic } f \equiv h \mod(Q)} y^\Omega(f) \right) u^n = \frac{1}{\Phi(Q)} \sum_{\chi} \chi(h)L(u,\chi,y)
\]
provide $|u| < \frac{1}{q}$. By the Cauchy integral formula, we have

$$
\sum_{\substack{f \text{ monic} \\ \deg(f) = n}} y^{\Omega(f)} = \frac{1}{2\pi i \Phi(Q)} \int_{C_R} \frac{g_Q(z)\zeta(z, y)}{z^{n+1}} dz + \frac{1}{2\pi i \Phi(Q)} \int_{C_R} \sum_{\chi \neq \chi_0} \frac{\chi(h)\mathcal{L}(z, \chi, y)}{z^{n+1}} dz
$$

$$
\triangleq I_1 + I_2,
$$

where $C_R$ is a circle centered at 0 with radius $R$. We consider the same region $\Omega_1$ and contour $\gamma_1$ of the Figure 1. The integral $I_1$ over $\bar{C} = C_{q^{-\frac{1}{2} - \delta}} \triangleq \bar{C}$ does not exceed

$$
\ll_{\delta, Q} g^n(\frac{1}{2} + \epsilon) \left(|q^{\frac{1}{2} - \delta} - 1|^\epsilon R(y)\right).
$$

since $g_Q(z)$ is holomorphic function in the region $\Omega_1$ and we have $g_Q(z) \ll_{\delta, Q} 1$. The integral $I_1$ over $\bar{C}$ does not exceed

$$
\frac{1}{2\pi i \Phi(Q)} \int_{\bar{C}} \frac{g_Q(z)\mathcal{N}(z, y)}{z^{n+1}(1 - qz)^y} dz
$$

$$
= \frac{e^{1-y}}{2\pi i \Phi(Q) q^y} \int_0^{2\pi - \alpha} e^{(1-y)i\theta - \pi y} g_Q(\frac{1}{q} + \epsilon e^{i\theta})\mathcal{N}(\frac{1}{q} + \epsilon e^{i\theta}, y) d\theta
$$

$$
\ll_{\delta, Q, \rho, n} e^{1-R(y)}.
$$

Then this integral tends to zero as $\epsilon \to 0$ since $R(y) < 1$. Letting the $\Gamma_1, \Gamma_2$ onto the real line and $\epsilon \to 0$ together, we have

$$
I_1 = \frac{\sin(y\pi)}{q^y \pi \Phi(Q)} \int_{\frac{1}{q}}^{\frac{1}{q} + \delta} g_Q(u)\mathcal{N}(s, y) \frac{1}{u^{n+1}} (u - \frac{1}{q})^{-y} du + O_{\delta, Q} \left(g^n(\frac{1}{2} + \delta)\right)
$$

$$
= \frac{1}{\Phi(Q)} \prod_P \left(1 - \frac{1}{|P|}\right)^y \left(1 - \frac{y}{|P|}\right)^{-1} \prod_{P \cap Q} \frac{1}{1 - \frac{y}{|P|}} \frac{\kappa(y)q^n}{\Gamma(y)n^{1-y}} + O_{\delta, Q, \rho} \left(\frac{g^n}{n^2 - R(y)}\right).
$$

For integral $I_2$, we consider another region $\Omega_2$ surrounded by contour $\gamma_2 = C_{q^{-\frac{1}{2} - \delta}} \cup C_R$ of integration as shown in the Figure 2,
It follows from (10) that the integral over $C_{q^{-\frac{1}{2}-\delta}}$ satisfies

$$I_2 = \frac{1}{2\pi i \Phi(Q)} \int_{C_{q^{-\frac{1}{2}-\delta}}} \sum_{\chi \neq \chi_0} \frac{\chi(h) \mathcal{L}(z, \chi, y)}{z^{n+1}} \, dz \ll_{\delta, Q} q^n (\frac{1}{2} + \delta) (1 + q^{\frac{1}{2} - \delta}) R(y)^m,$$

(16)

since \( \sum_{\chi \neq \chi_0} \frac{\chi(h) \mathcal{L}(z, \chi, y)}{z^{n+1}} \) is a holomorphic function over region $\Omega_2$. It follows from equations (14), (15) and (16) that

$$\sum_{f \equiv h \mod(Q)} y^{\Omega(f)} = \frac{1}{\Phi(Q)} \prod_P \left( 1 - \frac{1}{|P|} \right)^y \left( 1 - \frac{y}{|P|} \right)^{-1} \prod_{p|Q} \frac{1}{1 - \frac{y}{|P|}} \frac{\kappa(y) q^n}{\Gamma(y) n^{1-y}} + O_{\delta, Q, \rho} \left( \frac{q^n}{n^{2-R(y)}} \right).$$

Applying Cauchy’s integral formula to Theorem 4.1, we have

**Theorem 4.2.** The number of monic polynomials of degree $n$ which satisfy \( f \equiv h \mod(Q) \) and $\Omega(f) = t$ is

$$\sum_{f \in A_n(t; h, Q)} 1 = \left( \prod_{p|Q} \frac{1}{1 - \frac{t}{|P|}} \right) \frac{q^n \log^{t-1}(n)}{\Phi(Q)n} \sum_{r=1}^{t} A_r \frac{\log^{1-r}(n)}{(t-r)!} + O_{\delta, Q, \rho} \left( \frac{\rho^{-n} q^n}{n^{2-R(y)}} \right)$$

for any $\rho < 1$. 

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5. On the residue classes distribution of divisors of monic polynomial

In this section, let $Q \in \mathbb{A}$ be a polynomial. Let $h$ be a polynomial such that $(Q, h) = 1$ and $\text{deg}(h) \leq \text{deg}(Q)$, we research the divisors of monic polynomial $f$ in residue class $h \mod (Q)$.

For a polynomial $f \in \mathbb{A}$ and Dirichlet character $\chi$ of modulo $Q$, we define

$$\sigma_a(f, \chi) = \sum_{d \mid f, d \text{ monic}} \chi(d) |d|^a.$$  

Let $\sigma_a(f; Q)$ be the sum of the $a$-th “powers” of those divisors of $f$ which are prime to $Q$, i.e.,

$$\sigma_a(f; Q) = \sum_{d \mid f, d \text{ monic}, (d, Q) = 1} |d|^a.$$  

Let $\sigma_a(f; h, Q)$ be the sum of the $a$-th “powers” of those divisors of $f$ which are belong to the residue class $h \mod (Q)$, i.e.,

$$\sigma_a(f; h, Q) = \sum_{d \mid f, d \text{ monic}, d \equiv h \mod (Q)} |d|^a.$$  

Then

$$\sigma_a(f; Q) = \sum_{h, \text{deg}(h) \leq \text{deg}(Q), (h, Q) = 1} \sigma_a(f; h, Q).$$  

The distribution of divisors of monic polynomial in residue classes can be described by the variance

$$\mathbb{V}[\sigma_a(f; \cdot, Q)] := \frac{1}{\Phi(Q)} \sum_{h, \text{deg}(h) \leq \text{deg}(Q), (h, Q) = 1} \left( \sigma_a(f; h, Q) - \frac{\sigma_a(f; Q)}{\Phi(Q)} \right)^2.$$  

**Lemma 5.1.** We have

$$\sum_{h, \text{deg}(h) \leq \text{deg}(Q), (h, Q) = 1} \left( \sigma_a(f; h, Q) - \frac{\sigma_a(f; Q)}{\Phi(Q)} \right)^2 = \frac{1}{\Phi(Q)} \sum_{\chi \neq \chi_0} \sigma_a(f, \chi) \sigma_a(f, \overline{\chi}).$$  

(17)
Proof. We calculate the right side of equation (17). We can obtain that
\[
\frac{1}{\Phi(Q)} \sum_{\chi \neq \chi_0} \sigma_a(f, \chi) \sigma_a(f, \overline{\chi}) \\
= \frac{1}{\Phi(Q)} \sum_{\chi \neq \chi_0} \sum_{d_1, d_2 \mid f \atop d_1, d_2 \text{ monic}} \chi(d_1) \overline{\chi}(d_2) |d_1|^a |d_2|^a \\
= \sum_{d_1, d_2 \mid f \atop d_1, d_2 \text{ monic}} \left( \frac{1}{\Phi(Q)} \sum_{\chi \neq \chi_0} \chi(d_1) \overline{\chi}(d_2) \right) |d_1|^a |d_2|^a \\
= \sum_{d_1, d_2 \mid f \atop d_1 \equiv d_2 \text{ mod}(Q)} |d_1|^a |d_2|^a - \sigma_a(f; Q)^2 \\
= \sum_{h, \deg(h) \leq \deg(Q) \atop (h, Q) = 1} \sigma_a(f; h, Q)^2 - \sigma_a(f; Q)^2.
\]

Note that
\[
\sum_{h, \deg(h) \leq \deg(Q) \atop (h, Q) = 1} \left( \sigma_a(f; h, Q) - \frac{\sigma_a(f; Q)}{\Phi(Q)} \right)^2 \\
= \sum_{h, \deg(h) \leq \deg(Q) \atop (h, Q) = 1} \sigma_a(f; h, Q)^2 + \frac{1}{\Phi(Q)^2} \sigma_a(f; Q)^2 - 2 \frac{\sigma_a(f; Q)}{\Phi(Q)} \sigma_a(f; Q) \\
= \sum_{h, \deg(h) \leq \deg(Q) \atop (h, Q) = 1} \sigma_a(f; h, Q)^2 - \sigma_a(f; Q)^2.
\]

This completes the proof. \(\square\)

A simple calculation shows the following Lemma and we omit the proof.

**Lemma 5.2.** Let \(u, p, q\) be any complex numbers such that the left side of equation (18) is convergent, then we have
\[
\sum_{v \geq 0} u^v \left( \sum_{0 \leq j \leq v} p^j \right) \left( \sum_{0 \leq j \leq v} q^j \right) = \frac{1 - p q u^2}{(1 - u p q)(1 - u p)(1 - u q)(1 - u)}.
\]

**Theorem 5.3.** For integer \(n\) and \(\min\{R(s), R(s - a), R(s - b), R(s - a - \)
b), \(R(2s - a - b)\) \(> 1\), we have

\[
\sum_{f \text{ monic}} \frac{y^{\Omega(f)}}{|f|^s} \mathcal{Y}[\sigma_a(f; \cdot, Q)] = \frac{1}{\Phi(Q)^2} \sum_{\chi \neq \chi_0} \frac{L(s - a, \chi, y)}{L(2s - 2a, \chi_0, y^2)} \sum_{\chi \neq \chi_0} L(s - a, \chi, y)L(s - a, \chi, y).
\]

Proof. From Lemma 5.2, we can get

\[
\sum_{f \text{ monic}} \frac{y^{\Omega(f)}}{|f|^s} \sigma_a(f, \chi_1)\sigma_b(f, \chi_2)
\]

\[
= \prod_P \left( \sum_{i \geq 0} \frac{y^{\Omega(P^i)}}{|P|^{i s}} \sum_{0 \leq j \leq v} \chi_1(P^i)\chi_2(P^j)|P|^{|\alpha_i|}|P|^{|\beta_j|} \right)
\]

\[
= \prod_P \left( 1 - \frac{\chi_1(P)\chi_2(P)y}{|P|^{2s-a-b}} \right) \left( 1 - \frac{y}{|P|^s} \right)^{-1}
\]

\[
= \zeta_A(s, y) L(s - a, \chi_1, y)L(s - b, \chi_2, y)L(s - a - b, \chi_1\chi_2, y)
\]

\[
= \frac{L(2s - a - b, \chi_1\chi_2, y^2)}{L(2s - a - b, \chi_1\chi_2, y^2)},
\]

provide

\[
\min\{R(s), R(s - a), R(s - b), R(s - a - b), R(2s - a - b)\} > 1.
\]

From Lemma 5.1, we have

\[
\sum_{f \text{ monic}} \frac{y^{\Omega(f)}}{|f|^s} \mathcal{Y}[\sigma_a(f; \cdot, Q)] = \frac{1}{\Phi(Q)} \sum_{f \text{ monic}} \frac{y^{\Omega(f)}}{|f|^s} \sum_{h, \deg(h) \leq \deg(Q)} \left( \sigma_a(f; h, Q) - \frac{\sigma_a(f; Q)}{\Phi(Q)} \right)^2,
\]

\[
= \frac{1}{\Phi(Q)^2} \sum_{f \text{ monic}} \frac{y^{\Omega(f)}}{|f|^s} \sum_{\chi \neq \chi_0} \sigma_a(f, \chi)\sigma_a(f, \chi).
\]

Then the result follows. \(\square\)

Let \(a = 0\), then \(\sigma_a(f; Q)\) be the numbers of those divisors of \(f\) which are prime to \(Q\) and \(\sigma_a(f; Q)\) be the numbers of those divisors of \(f\) which belong to the residue class \(h \mod (Q)\). Let \(\tau(f; h, Q) = \sigma_0(f; h, Q)\) and \(\tau(f; Q) = \sigma_0(f; Q)\). We have following theorem.
\[ \sum_{f \text{ monic}, \deg(f) = n} \mathbb{V}[\tau(f; \alpha, Q)] = \frac{1}{\Phi(Q)^2} \sum_{f \text{ monic}, \deg(f) = n} \sum_{h, \deg(h) \leq \deg(Q)} \left( \tau(f; h, Q) - \frac{\tau(f; Q)}{\Phi(Q)} \right)^2 \]

\[ = A_Q(n + 1)q^{n+1} + B_Qq^{n+1} + O_{Q, \varepsilon}(q^{\varepsilon(n-\varepsilon) - 1}) , \]

where

\[ A_Q := \frac{q - 1}{q^2 \Phi(Q)^2} \prod_{P \mid Q} \frac{1}{1 + \frac{1}{|P|}} \sum_{\chi \neq \chi_0} |L(1, \chi)|^2 , \]

\[ B_Q = \frac{\prod_{P \mid Q} (1 + \frac{1}{|P|})^{-1}}{q^2 \Phi(Q)^2} \left( \frac{q - 1}{\log q} \left( 2 \sum_{\chi \neq \chi_0} |L(1, \chi)|^2 R \left( \frac{L'(1, \chi)}{L(1, \chi)} \right) + \sum_{P \mid Q} \frac{\log |P|}{|P| + 1} \right) + 2 \right) . \]

**Proof.** From Theorem 5.3 with \( y = 1 \) and equation (6), the associated Drichlet series is

\[ \frac{1}{\Phi(Q)^2} \sum_{\chi \neq \chi_0} L(s, \chi)L(s, \conjugate{\chi}) \]

\[ = \frac{1}{\Phi(Q)^2} \left( \prod_{P \mid Q} \frac{1}{1 + \frac{1}{|P|}} \right) \frac{\zeta_A(s)}{\zeta_A(2s)} \sum_{\chi \neq \chi_0} L(s, \chi)L(s, \conjugate{\chi}) . \]

Thus, let \( u = q^{-s} \), for \( |u| < \frac{1}{q} \), we have

\[ \sum_{n \geq 0} \left( \sum_{f \text{ monic}, \deg(f) = n} \mathbb{V}[\tau(f; \alpha, Q)] \right) u^n = \frac{g_Q(u)}{\Phi(Q)^2} \frac{\zeta^2(u)}{\zeta(u^2)} \sum_{\chi \neq \chi_0} \mathcal{L}(u, \chi)\mathcal{L}(u, \conjugate{\chi}) , \]

where \( g_Q(u) = \prod_{P \mid Q} \frac{1}{1 + u^{\deg(P)}} \). Let \( 0 < R < \frac{1}{q} \) be arbitrary, by the Cauchy integral formula, we have

\[ \sum_{f \text{ monic}, \deg(f) = n} \mathbb{V}[\tau(f; \alpha, Q)] = \frac{1}{2\pi i} \frac{g_Q(z)}{\Phi(Q)^2} \int_{C_R} \frac{1 - qz^2}{z^{n+1} (1 - qz)^2} \sum_{\chi \neq \chi_0} \mathcal{L}(z, \chi)\mathcal{L}(z, \conjugate{\chi}) \, dz . \]

Let \( \rho < 1 \) be a constant, consider another region \( \Omega_3 \) surrounded by contour \( \gamma_3 = C_\rho \cup C_R \). By Cauchy’s residue theorem, we can obtain that

\[ \frac{1}{2\pi i} \int_{C_\rho} + \int_{C_R} F(z) \, dz = \text{Res} F(z) , \]

where \( F(z) = \frac{g_Q(z)}{z^{n+1} \Phi(Q)^2} \frac{1 - qz^2}{(1 - qz)^2} \sum_{\chi \neq \chi_0} \mathcal{L}(z, \chi)\mathcal{L}(z, \conjugate{\chi}) \). Next we turn to calculate the integral along each of the curves.
We calculate the integral over $C_\rho$ at first. It follows from (7) that
\[
\frac{1}{2\pi i \Phi(Q)^2} \int_{C_\rho} \frac{g_Q(z)}{z^{n+\frac{1}{2}}} \frac{1 - qz^2}{(1 - qz)^2} \sum_{\chi \neq \chi_0} \mathcal{L}(z, \chi) \mathcal{L}(z, \overline{\chi}) dz \\
\leq \frac{1}{2\pi i \Phi(Q)\rho^{n+1}} \int_{C_\rho} |g_Q(z)| \frac{1 + q\rho^2}{(q\rho - 1)z} (1 + \sqrt{q}\rho)^{2m} dz \\
= O_{Q, \rho} \left( \frac{1 + q\rho^2}{\rho^n(q\rho - 1)^2} (1 + \sqrt{q}\rho)^{2m} \right).
\]
Note that $z = \frac{1}{q}$ is a double pole of $F(z)$, so
\[
q^2 \Phi(Q)^2 \text{Res}_{z = \frac{1}{q}} F(z) = \lim_{z \to \frac{1}{q}} \frac{dg_Q(z)}{dz} \left( \frac{g_Q(z)(1 - qz^2)}{z^{n+1}} \sum_{\chi \neq \chi_0} \mathcal{L}(z, \chi) \mathcal{L}(z, \overline{\chi}) \right) \\
= \left( \left( \frac{q - 1}{q} g_Q \frac{1}{q} - 2g_Q \frac{1}{q} \right) q^{n+1} - g_Q \frac{1}{q} (n+1)(q-1)q^{n+2} \right) \sum_{\chi \neq \chi_0} |L(1, \chi)|^2 \\
+ \frac{q - 1}{q} g_Q \frac{1}{q} q^{n+1} \sum_{\chi \neq \chi_0} \left( \mathcal{L}(z, \chi) \mathcal{L}(z, \overline{\chi}) \right)' \bigg|_{z = \frac{1}{q}} \\
= \left( \left( \frac{q - 1}{q} g_Q \frac{1}{q} - 2g_Q \frac{1}{q} \right) q^{n+1} - g_Q \frac{1}{q} (n+1)(q-1)q^{n+1} \right) \sum_{\chi \neq \chi_0} |L(1, \chi)|^2 \\
- 2g_Q \frac{1}{q} \left( \frac{q - 1}{\log q} q^{n+1} \sum_{\chi \neq \chi_0} |L(1, \chi)|^2 \mathcal{R} \left( \frac{L'(1, \chi)}{L(1, \chi)} \right) \right) \\
= \prod_{p|Q} \frac{1}{1 + \frac{1}{|P|}} \left( \left( \frac{q - 1}{\log q} \sum_{p|Q} \frac{|P|}{|P| + 1} - 2 \right) q^{n+1} - (q-1)(n+1)q^{n+1} \right) \sum_{\chi \neq \chi_0} |L(1, \chi)|^2 \\
- 2 \prod_{p|Q} \frac{1}{1 + \frac{1}{|P|}} \frac{q - 1}{\log q} q^{n+1} \sum_{\chi \neq \chi_0} |L(1, \chi)|^2 \mathcal{R} \left( \frac{L'(1, \chi)}{L(1, \chi)} \right).
\]
Hence, we can get
\[
\text{Res}_{z = \frac{1}{q}} F(z) = - (A_Q(n+1)q^{n+1} + B_Qq^{n+1}).
\]
Thus
\[
\sum_{\text{monic } f \text{ degree } Q = n} \forall \tau(f; \circ, Q) = A_Q(n+1)q^{n+1} + B_Qq^{n+1} + O_{Q, \rho} \left( \frac{1 + q\rho^2}{\rho^n(q\rho - 1)^2} (1 + \sqrt{q}\rho)^{2m} \right).
\]
Next, let $\rho = \frac{1}{q^2}(0 < \varepsilon < 1)$, we have
\[
O_{Q, \rho} \left( \frac{1 + q\rho^2}{\rho^n(q\rho - 1)^2} (1 + \sqrt{q}\rho)^{2m} \right) = O_{Q, \varepsilon} (q^{n\varepsilon}(q^{1-\varepsilon} - 1)^{-2}).
\]
This completes the proof of Theorem 5.4. \qed
We now evaluate (5) by Selberg-Delange method.

**Theorem 5.5.** Let $y$ be a complex number such that $|y| \leq \rho$ and $\rho < 1$. If $R(y) < \frac{1}{2}$, then

$$
\sum_{f \text{ monic} \atop \text{deg}(f) = n} y^{\Omega(f)} \psi[\tau(f; \circ, Q)] = H_1 \left( \frac{1}{q}, y \right) \frac{\kappa(2y)q^n}{\Gamma(2y)n^{1-2y}} + O_{\delta, Q, \rho} \left( \frac{q^n}{n^{2-2R(y)}} \right)
$$

for integer $n > 3$, where

$$H_1(u, y) = \left( \prod_{P|Q} \frac{1}{1 + yu^{\deg(P)}} \right) \frac{(1 - au^2)yN^2(u, y)}{\Phi(Q)^2 N(u^2, y)} \sum_{\chi \neq \chi_0} \mathcal{L}(u, \chi, y) \mathcal{L}(u, \overline{\chi}, y).
$$

If $y = \frac{1}{2}$, then for integer $n$,

$$
\sum_{f \text{ monic} \atop \text{deg}(f) = n} \frac{1}{2^{nR(y)}} y^{\Omega(f)} \psi[\tau(f; \circ, Q)] = \frac{q^{n+1}}{\Phi(Q)^2} \prod_{P|Q} \frac{1}{1 + \frac{1}{2^{nR(y)}}} \frac{N^2(1, \frac{1}{2})}{\zeta_A(2, \frac{1}{2})} \sum_{\chi \neq \chi_0} |L(1, \chi, \frac{1}{2})|^2
$$

$$
+ O_{\delta, Q} \left( q^{\left( \frac{1}{2} + \delta \right)n} \right).
$$

**Proof.** From Theorem 5.3 the Drichlet series associates to it is

$$
\frac{1}{\Phi(Q)^2} \zeta_A(s, y)L(s, \chi_0, y) L(2s, \chi_0, y^2) \sum_{\chi \neq \chi_0} L(s, \chi, y)L(s, \overline{\chi}, y)
$$

$$
= \frac{1}{\Phi(Q)^2} \left( \prod_{P|Q} \frac{1}{1 + \frac{1}{2^{nR(y)}}} \right) \frac{\zeta_A^2(s, y)}{\zeta_A(2s, y)} \sum_{\chi \neq \chi_0} L(s, \chi, y)L(s, \overline{\chi}, y).
$$

Thus, let $u = q^{-s}$, for $|u| < \frac{1}{q}$, we have

$$
\sum_{n \geq 0} \left( \sum_{f \text{ monic} \atop \text{deg}(f) = n} y^{\Omega(f)} \psi[\tau(f; \circ, Q)] \right) u^n = \hat{\mathcal{G}}_Q(u) \frac{\zeta^2(u, y)}{\Phi(Q)^2 \zeta(u^2, y)} \sum_{\chi \neq \chi_0} \mathcal{L}(u, \chi, y) \mathcal{L}(u, \overline{\chi}, y),
$$

where $\hat{\mathcal{G}}_Q(u) = \prod_{P|Q} \frac{1}{1 + yu^{\deg(P)}}$. Let $0 < R < \frac{1}{q}$ be arbitrary, by the Cauchy integral formula, we have

$$
\sum_{f \text{ monic} \atop \text{deg}(f) = n} y^{\Omega(f)} \psi[\tau(f; \circ, Q)] = \frac{1}{2\pi i} \int_{C_R} \frac{H(z, y)}{z^{n+1}} dz,
$$

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where

\[ H(z, y) = \frac{\hat{g}_Q(z)}{\Phi(Q)^2} \zeta(z, y) \sum_{\chi \neq \chi_0} \mathcal{L}(z, \chi, y) \mathcal{L}(z, \chi, y) \]

\[ = \frac{\hat{g}_Q(z)}{\Phi(Q)^2} \zeta(z, y) \sum_{\chi \neq \chi_0} \mathcal{L}(z, \chi, y) \mathcal{L}(z, \chi, y) \]

\[ = \frac{\hat{g}_Q(z)}{\Phi(Q)^2} \zeta(z, y) \sum_{\chi \neq \chi_0} \mathcal{L}(z, \chi, y) \mathcal{L}(z, \chi, y). \]  \hspace{1cm} (19)

**Case 1:** \( 0 < R(y) < \frac{1}{2}. \) Considering the contours \( \gamma_1 \) of Figure 1, we can obtain that:

\[ \frac{1}{2\pi i} \int_{C_R} H(z, y) \frac{dz}{z^{n+1}} = 0. \]

We estimate the integrals over \( \bar{C}_{q^{-\frac{1}{2}}} \) at first. From (9) and (10), we have

\[ \frac{1}{2\pi i} \int_{\bar{C}} H(z, y) \frac{dz}{z^{n+1}} \ll_{\delta, Q, \rho} \int_{C} \frac{1}{z^{n+1}} |1 - qz^{2}| R(y) (1 + \sqrt{|z|})^{2R(y)m} dz \]

\[ = q^{(\frac{1}{2} + \delta)n} (1 + q^{-2\delta}) R(y) (1 + q^{-\delta})^{2R(y)m} \ll_{\delta, Q, \rho} q^{(\frac{1}{2} + \delta)n}. \]  \hspace{1cm} (20)

Let \( H_1(z, y) = H(z, y)(1 - qz)^{2y} \), then

\[ \frac{1}{2\pi i} \int_{C_R} H_1(z, y) \frac{dz}{z^{n+1}} = \frac{\epsilon^{-2y}}{2\pi y^{2y}} \int_{\alpha}^{2\pi - \alpha} \epsilon^{-2y \theta - 2\pi y} H_1 \left( \frac{1}{y} + \epsilon e^{i\theta} \right) \frac{d\theta}{(1 + \epsilon e^{i\theta})^{n+1}} \ll_{\delta, Q, \rho, \alpha} \epsilon^{-2R(y)}. \]

If \( 0 < R(y) < \frac{1}{2} \), then this integral tends to zero as \( \epsilon \to 0 \). Letting the \( \Gamma_1, \Gamma_2 \) onto the real line and \( \epsilon \to 0 \) together, we have

\[ \frac{1}{2\pi i} \int_{C_R} H(z, y) \frac{dz}{z^{n+1}} = \frac{q^{-2y}}{2\pi i} \int_{\frac{1}{y}}^{\frac{1}{y} + \pi} H_1 \left( \frac{1}{y} + \frac{2y(\log(u - \frac{1}{2}) - i\pi)}{u^{n+1}} \right) du \]

\[ - \frac{q^{-2y}}{2\pi i} \int_{\frac{1}{y}}^{\frac{1}{y} + i\pi} H_1 \left( \frac{1}{y} + \frac{2y(\log(u - \frac{1}{2}) + i\pi)}{u^{n+1}} \right) du \]

\[ + O_{\delta, Q, \rho} \left( q^{(\frac{1}{2} + \delta)n} \right). \]
Thus,

$$
\sum_{\substack{f \text{ monic} \\ \deg(f) = n}} y^{\Omega(f)} \mathcal{V}[\tau(f; \omega, Q)] = \frac{\sin(2\pi y)}{q^{2\pi}} \int_{q^2}^{q^2} H_1(u, y) \frac{1}{u^{n+1}} (u - \frac{1}{q})^{-2y} du + O_{\delta, Q, \rho} \left( q^\left(\frac{n}{2} + \delta \right) \right)
$$

$$
= \frac{\sin(2\pi y)}{q^{2\pi}} \int_0^{\omega_0} H_1\left( \frac{1}{q} + \omega, y \right) \left( \frac{1}{q} + \omega \right)^{n+1} \omega^{-2y} d\omega + O_{\delta, Q, \rho} \left( q^\left(\frac{n}{2} + \delta \right) \right).
$$

Note that

$$
\frac{H_1\left( \frac{1}{q} + \omega, y \right)}{q - \omega} - q H_1\left( \frac{1}{q}, y \right) = \frac{\omega}{2\pi} \int_D \frac{H_1(z, y)}{(z - \frac{1}{q} - \omega)(z - \frac{1}{q})} dz = \omega k_1(\omega),
$$

where $D$ is circle, centre $\frac{1}{q} + \frac{\omega}{2}$ and radius $\omega_0$. We can also obtain that

$$
\frac{\sin(2\pi y)}{q^{2\pi}} \int_0^{\omega_0} H_1\left( \frac{1}{q} + \omega, y \right) \left( \frac{1}{q} + \omega \right)^{n+1} \omega^{-2y} d\omega
$$

$$
= q H_1\left( \frac{1}{q}, y \right) \frac{\sin(2\pi y)}{q^{2\pi}} \int_0^{\omega_0} \omega^{-2y} d\omega + O_{\delta, Q, \rho} \left( q^\left(\frac{n}{2} + \delta \right) \right)
$$

$$
= H_1\left( \frac{1}{q}, y \right) \frac{\sin(2\pi y)}{\pi} q^n \frac{\Gamma(1-2y)\Gamma(n-1+2y)}{\Gamma(n)} + O_{\delta, Q, \rho} \left( \frac{q^n}{n^2-2R(y)} \right)
$$

$$
= H_1\left( \frac{1}{q}, y \right) \frac{\kappa(2y) q^n}{\Gamma(2y)n^{1-2y}} + O_{\delta, Q, \rho} \left( \frac{q^n}{n^2-2R(y)} \right).
$$

as the proof of Theorem 3.3.

**Case 2:** $y = \frac{1}{2}$. We recall the function $H(z, y)$ of equation (19), it has a simple pole at $z = \frac{1}{q}$ with residue

$$
\frac{\tilde{g}_2(\frac{1}{q}) q^{n+1}}{\Phi(Q)^2} \left( 1 - \frac{1}{q} \right)^{1/2} N^2(1, \frac{1}{2}) \sum_{\chi \neq \chi_0} \mathcal{L}\left( \frac{1}{q}, \chi, \frac{1}{2} \right) \mathcal{L}\left( \frac{1}{q}, \chi, \frac{1}{2} \right)
$$

$$
= \frac{q^{n+1}}{\Phi(Q)^2} \prod_{p \mid Q} \frac{1}{1 + \frac{1}{\zeta_A(2, \frac{1}{2})} \sum_{\chi \neq \chi_0} \left| L(1, \chi, \frac{1}{2}) \right|^2}.
$$

Thus, considering the contours $\gamma_2$ of Figure 2, we have

$$
\sum_{\substack{f \text{ monic} \\ \deg(f) = n}} \frac{1}{2\Omega(f)} \mathcal{V}[\tau(f; \omega, Q)] = \frac{q^{n+1}}{\Phi(Q)^2} \prod_{p \mid Q} \frac{1}{1 + \frac{1}{\zeta_A(2, \frac{1}{2})} \sum_{\chi \neq \chi_0} \left| L(1, \chi, \frac{1}{2}) \right|^2}
$$

$$
+ O_{\delta, Q, \rho} \left( q^\left(\frac{n}{2} + \delta \right) \right).
$$

Then the result follows.
Let
\[ H_1 \left( \frac{1}{q^y} \right) \frac{\kappa(2y)}{\Gamma(2y)} = \sum_{r=1}^{+\infty} \hat{A}_r y^r, \]
then we obtain the following theorem.

**Theorem 5.6.**
\[
\sum_{\text{monic } f \atop \deg(f)=n, \Omega(f)=t} \mathbb{V}[\tau(f; \circ, Q)] = \frac{q^n(2 \log(n))^{t-1}}{n} \sum_{r=1}^{t} \frac{\hat{A}_r (2 \log(n))^{1-r}}{(t-r)!} + O_{\delta, Q, \rho} \left( \frac{\rho^{-n} q^n}{n^{2-R(\rho)}} \right)
\]
for any |\rho| < 1.

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