ON THE A-D-E CLASSIFICATION OF
THE SIMPLE SINGULARITIES OF FUNCTIONS

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Introduction.

One of the most marvellous results of singularity theory is the classification
of the simple isolated singularities of analytic functions by the Weyl groups (and
the irreducible root systems) of the types $A_k, D_k, E_6, E_7, E_8$.

There are various interesting relations between the simple singularities
and the algebraic objects (simple Lie groups, root systems, Coxeter groups)
of the types $A_k, D_k, E_6, E_7, E_8$ (see, e.g., [Br1], [Du], [Gr], [Sl]). All these
relations deeply involve the normal forms of the simple singularities obtained
by V.I.Arnold in his paper [Ar1].

In this paper we consider one of such relations not using the normal forms.
We dwell on the construction that associates to a singularity its monodromy
group (for a suspension with an odd number of variables).

It is well known that this construction associates to each of the simple
singularities of the types $A_k, D_k, E_6, E_7, E_8$ the Weyl group of the same type,
and moreover, the monodromy group of the singularity is a finite group gen-
erated by reflections only if the singularity is simple. The normal forms are
essentially used in the proofs of these facts (see [Ar1], [Ar2], [Ar4], [AGV1],
[AGV2], [Tju1]), and the coincidence of the classifications of the simple sin-
gularities and of the (irreducible) Weyl groups looks just like the mysterious
coincidence of the lists obtained by the independently proved classification the-
orems.
That gave rise to the natural problem: to reduce the classification of the simple singularities directly, not using the normal forms, to the classification of the irreducible Weyl groups. This problem was repeatedly mentioned by V.I.Arnold (see e.g. [Ar1], [Ar2], [Ar6]).

In this paper we show how this problem can be solved. More exactly, we prove (see theorems I and II), not using the normal forms, that

a) the monodromy group of a simple singularity is a Weyl group;
b) if the monodromy group of a singularity is finite, then it is an irreducible Weyl group of one of the types $A_k, D_k, E_6, E_7, E_8$, and the singularity is simple;
c) if two simple singularities have isomorphic monodromy groups, then they are equivalent.

We reduce the proofs of a) and b) to the proof of the assertion that the simple singularities coincide with the elliptic ones, i.e. with ones with definite intersection form on the homologies of the Milnor fiber (for a suspension with an odd number of variables), and we obtain this assertion proving that a singularity is simple (elliptic) if and only if the mixed Hodge structure in its vanishing cohomologies is trivial or, in other words, the length of the spectrum of the singularity is less than one.

In the course of proving of a) and b) we also find that

a simple singularity is stably equivalent to the singularity of a quasihomogeneous function of two variables.

We show (not using the normal forms) that

the monodromy operator of a simple singularity is a Coxeter element of the corresponding Weyl group.

From this, by virtue of the purely algebraic result of P.Deligne obtained in [De], we deduce that
the Dynkin diagram of a simple singularity is the canonical Dynkin diagram of
the corresponding Weyl group for some distinguished basis.

Then one can easily find the normal forms of the simple singularities and
determine the types of the corresponding Weyl groups as it is described in
Appendix 1.

Our approach is that instead of the use of the normal forms we apply many
other general and very powerful results of singularity theory and of the theory
of Weyl groups to the particular case of a simple (an elliptic) singularity. Most
of these results were obtained after V.I.Arnold had found the normal forms of
the simple singularities.

We also use the new result (see theorem 1) that

under the canonical identification of the local algebra of a singularity with the
tangent space to the base of a miniversal deformation of the singularity at zero,
the class of the singularity in its local algebra always belongs to the tangent
cone to the stratum \( \mu=\text{const} \).

This theorem has been proved jointly by J.H.M.Steenbrink and the author. For
the case of the singularity of a function which is nondegenerate with respect
to its Newton diagram the result can be sharpened (see Appendix 2) in the way
that

under the mentioned above identification, the class of the singularity in its local
algebra is the tangent vector to a linear 1-parameter \( \mu=\text{const} \) deformation of
the singularity.

The proof of the assertion that the monodromy operator of a simple sin-
gularity is a Coxeter element of the corresponding Weyl group is based in
particular on the interesting purely algebraic remark (see proposition 2.1) that
the Coxeter elements of a Weyl group (of one of the types $A_k, D_k, E_6, E_7, E_8$) are the only elements of maximal length (i.e. those that can be written as an irreducible product of the maximal number of reflections) with trace equal to $-1$.

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§1. Terminology.

In this paragraph we introduce the notation and recapitulate some constructions of singularity theory and the theory of Weyl groups.

1.1. Some notions and constructions of singularity theory. Basic notions of singularity theory can be found in [AGV1], [AGV2], [AGLV].

In this paper we deal only with isolated singularities of analytic functions of several complex variables.
By $O_n$ we denote the space of germs of analytic functions of $n$ complex variables at the point $0 \in \mathbb{C}^n$. Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be an isolated singularity of an analytic function at the point $0 \in \mathbb{C}^n$. By $I_f$ we denote the gradient ideal of $f$, i.e. the ideal in $O_n$ generated by all partial derivatives of $f$ at zero. By $Q_f$ we denote the local algebra of $f$, i.e. $Q_f = O_n/I_f$. By $[f]$ we denote the class of a germ $f$ in its local algebra $Q_f$.

Now we fix such sufficiently small neighbourhoods of zero $U = \{z | \|z\| < \rho\} \subset \mathbb{C}^n$ and $T = \{t | \|t\| < \delta\} \subset \mathbb{C}$ (see [AGV2]) that

a) the point $0 \in \mathbb{C}^n$ is the only critical point of $f$ in the ball $U$;

b) the hypersurface $f^{-1}(t)$ is nonsingular inside $U$ and intersects transversally the boundary of $U$ for all $t \in T \setminus 0$.

Let $X_t = f^{-1}(t) \cap U$, $t \in T \setminus 0$, denote a nonsingular level of $f$ near the critical point $0$. We also fix a point $t \in T \setminus 0$. The manifold $X_* = X_t$ has the homotopy type of the bouquet of $\mu$ $(n - 1)$-dimensional spheres, where $\mu$ is the Milnor number (or the multiplicity) of the singularity $f$ (see [Mi]). So $H_k(X_*) = 0$ if $k \neq 0, n - 1$ and $H_{n-1}(X_*) \cong \mathbb{Z}^\mu$ (for $n = 1$ here and further one should consider the reduced homology group $\tilde{H}_0(X_*)$).

We take such a small perturbation $f_\varepsilon$ of the singularity $f$ that for all sufficiently small $\varepsilon$ the function $f_\varepsilon$ has exactly $\mu$ Morse critical points inside $U$ with different critical values inside $T$. Let’s fix any such $\varepsilon$. A nonsingular level $f_\varepsilon^{-1}(t) \cap U$ is homeomorphic to the level $X_*$ of the function $f$. We take a noncritical value $t_0$ of $f_\varepsilon$ on the boundary of $T$ and then choose $\mu$ non-self-intersecting and mutually nonintersecting (except at the point $t_0$) paths $\gamma_1, \ldots, \gamma_\mu$ going inside $T$ from the point $t_0$ to the critical values of $f_\varepsilon$. The paths are numbered in the same order in which they emanate from the point $t_0$ counting clockwise. Such a system of paths is called a distinguished system of paths (see [Ga1] or [AGV2]). The cycles $\Delta_1, \ldots, \Delta_\mu$ vanishing along the paths
\(\gamma_1, \ldots, \gamma_\mu\) form a basis (a so-called distinguished basis) of \(H_{n-1}(f_{x}^{-1}(t_0) \cap U) \cong H_{n-1}(X_*)\) (see [Br2], [Ga1], [Lam] or [AGV2]).

We assume that the number of variables \(n \equiv 3 \pmod{4}\). Otherwise one takes a suitable suspension of \(f\) – for all such suspensions the monodromy groups are isomorphic and the quadratic forms are the same (see [AGV2]), so further, whenever we speak about the monodromy group or the quadratic form of a singularity, we mean that the appropriate (as it is mentioned above) suspension is considered.

Then (see [AGV2]) the intersection form in the homology group \(H_{n-1}(X_*, \mathbb{R}) \cong \mathbb{R}^\mu\) is symmetric and it defines a quadratic form called the quadratic form of the singularity \(f\). The index of self-intersection of a vanishing cycle is equal to \(-2\), and the monodromy group \(\Gamma\) of the singularity \(f\) is generated by the Picard-Lefschetz transformations \(h_i\) related to the vanishing cycles \(\Delta_i\) (see [Gu] or [AGV2]):

\[
(1) \quad h_i : \sigma \rightarrow \sigma + (\sigma \circ \Delta_i)\Delta_i,
\]

where \(\sigma \in H_{n-1}(X_*)\), \(i = 1, \ldots, \mu\), and \((\cdot \circ \cdot)\) denotes the intersection form. The monodromy operator of the singularity \(f\) is the product \(h_1 \cdot \ldots \cdot h_\mu\).

**Definition** (see [Ar2]). A singularity is said to be elliptic if its quadratic form is negative definite.

**Remark.** This notion of an elliptic singularity should not be confused with one introduced in the works of K.Saito and E.J.N.Looijenga.

The braid group \(Br(\mu)\) acts transitively on the set of the systems of distinguished paths (considered up to a homotopy). Namely, if \(b_1, \ldots, b_{\mu-1}\) are the standard generators of \(Br(\mu)\) then \(b_i\) transfers the system \(\gamma_1, \ldots, \gamma_\mu\) to the system \(\gamma'_1, \ldots, \gamma'_\mu\): \(\gamma'_j = \gamma_j\), \(j \neq i, i+1\), \(\gamma'_i = \gamma_{i+1}\), \(\gamma'_{i+1}\) is homotopic
to $\gamma_i \cup g_{i+1}^{-1}$, where $g_i$ is the simple loop going from the fixed point $t_0$ along the path $\gamma_i$ to a point nearby the end of $\gamma_i$, then once counterclockwise around the end of $\gamma_i$ and then back along $\gamma_i$ to $t_0$. By formula (1) this action of $Br(\mu)$ provides the action of $Br(\mu)$ on the set of the distinguished bases (see [Lo1], [Gu] or [AGV2]):

$$(2) \quad b_i : (\Delta_1, \ldots, \Delta_\mu) \mapsto (\Delta_1, \ldots, \Delta_{i-1}, \Delta_{i+1}, \Delta_i - (\Delta_i \circ \Delta_{i+1})\Delta_{i+1}, \Delta_{i+2}, \ldots, \Delta_\mu).$$

1.2. Some notions on Weyl groups and finite groups generated by reflections. By a finite group generated by reflections we mean a finite group generated by reflections in a Euclidean vector space. If $\mathcal{R}$ is a (reduced) root system then by $W(\mathcal{R})$ we denote the corresponding Weyl group. Basic notions on groups generated by reflections and Weyl groups can be found in [Bou].

Now let $W(\mathcal{R})$ be a Weyl group (of one of the types $A_k, D_k, E_6, E_7, E_8$) of rank $\mu$ and let $S$ denote the set of all $\mu$-tuples $(s_1, \ldots, s_\mu)$ of reflections in $W(\mathcal{R})$ such that
i) $s_1, \ldots, s_\mu$ generate $W(\mathcal{R})$;
ii) the roots corresponding to $s_1, \ldots, s_\mu$ are linearly independent and span $\mathcal{R}$ over $\mathbb{Z}$.

Then the braid group $Br(\mu)$ acts on $S$ in a way similar to (2) (see [Lo1]) :

$$(3) \quad b_i : (s_1, \ldots, s_{i-1}, s_i, s_{i+1}, s_{i+2}, \ldots, s_\mu) \mapsto (s_1, \ldots, s_{i-1}, s_{i+1}, s_{i+1}s_is_{i+1}, s_{i+2}, \ldots, s_\mu).$$

One can easily notice that this action leaves the product $s_1 \cdots s_\mu$ invariant.
§2. Statement of Results.

We use the notation and the agreements introduced in §1.

The main results of this paper are the following two theorems.

**Theorem I.** For any singularity the following conditions are equivalent:

1) The singularity is simple;
2) The singularity is elliptic;
3) The monodromy group of the singularity is finite;
4) The monodromy group of the singularity is isomorphic to a Weyl group of one of the types $A_k, D_k, E_6, E_7, E_8$;
5) The mixed Hodge structure in the vanishing cohomologies of the singularity is trivial (i.e. both the Hodge and the weight filtrations do not contain any nontrivial subspaces);
6) The length of the spectrum of the singularity is less than one.

**Theorem II.** If two simple singularities have isomorphic monodromy groups then they are (stably) equivalent.

The normal forms of the simple singularities are not used in the proofs of theorems I, II. In the course of the proof of theorem I we also obtain

**Corollary 1.**

A simple (an elliptic) singularity is stably equivalent to the singularity of a quasihomogeneous function of two variables.

**Remark.** The equivalence 1) ⇔ 6) was conjectured by K.Saito in [Sa3].

To prove theorem I we use the following result obtained in this paper (see §4).

**Theorem 1.** Let $F : \mathbb{C}^n \times \Lambda \to \mathbb{C}$ be a miniversal deformation of $f$, and let $T_0(\Lambda)$ denote the tangent space to $\Lambda$ at zero. Then if $v \in T_0(\Lambda)$ is mapped
onto \([f] \in Q_f\) under the canonical identification of \(T_0(\Lambda)\) with \(Q_f\) then \(v\) lies in the tangent cone to the stratum \(\mu=\text{const}\) in the space \(\Lambda\).

This result fits with the theorem of A.N.Varchenko and S.V.Chmutov on the tangent cone to the stratum \(\mu=\text{const}\) of a singularity (see [VC]). For the proof of theorem 1 see §4. Moreover, the following fact is also true (see Appendix 2).

**Theorem 1'**. Let’s fix a monomial basis of \(Q_f\) over \(\mathbb{C}\) and consider \([f]\) as a linear combination of the basic monomials. If \(f\) is nondegenerate with respect to its Newton diagram (see [Kou] or [AGV2]), then for small \(t\) the linear deformation \(f_t = f + t[f] , t \in \mathbb{C}\), is a deformation with constant multiplicity.

We also prove (not using the normal forms) that

**Theorem 2**. The monodromy operator of a simple (an elliptic) singularity with the monodromy group \(W\) is a Coxeter element of the Weyl group \(W\).

**Corollary 2**. Let \(f\) be a simple (an elliptic) singularity. Let a Weyl group \(W\) be the monodromy group of \(f\). Then there exists such a distinguished basis that the Dynkin diagram of \(f\) with respect to this basis is the canonical Dynkin diagram of the Weyl group \(W\).

To prove theorem 2 we use the following algebraic result.

**Proposition 2.1**. Let \(W(\mathcal{R})\) be an irreducible Weyl group of the rank \(\mu\) where \(\mathcal{R}\) is a (reduced) root system of one of the types \(A_k, D_k, E_6, E_7, E_8\). Let \(s = s_1 \cdots s_\mu\) be a product of \(\mu\) reflections in \(W(\mathcal{R})\) corresponding to some linearly independent roots that span the root system \(\mathcal{R}\) over \(\mathbb{Z}\) (i.e. \((s_1, \ldots, s_\mu) \in S\) following the notation from §1). Then \(s\) is a Coxeter element of \(W(\mathcal{R})\) if and only if its trace is equal to \(-1\).

Theorem 2 and corollary 1 also are necessary to check that the results we refer to while proving theorem II can be in fact proved without the normal forms.
Finally we show how using corollary 1 one can find the normal forms of the simple singularities and then, by virtue of theorem 2, determine the types of the corresponding Weyl groups (see Appendix 1).

The scheme of the paper is as follows. In §3 we prove the equivalence of conditions 2), 3) and 4) of theorem I. In §4 we prove theorem 1. In §5 we prove the equivalence of conditions 1), 2), 5) and 6) of theorem I and corollary 1. In §6 we prove proposition 2.1, theorem 2 and corollary 2. In §7 we prove theorem II.

§3. The Proof of the Equivalence of Conditions 2), 3) and 4) in Theorem I.

We shall prove the implications 2) ⇔ 3) and 3) ⇒ 4). The implication 4) ⇒ 3) is obvious. In the proof we use the following well-known facts.

**Proposition 3.1** (see [Sa2]). The quadratic form of a singularity $f$ is the unique integral symmetric even (i.e. with values in $2\mathbb{Z}$) quadratic form on $\mathbb{Z}^\mu \cong H_{n-1}(X_*)$ invariant under the action of the monodromy group $\Gamma$ and such that $-2$ is a value of it.

**Proposition 3.2** (see [Gu], [AGV2]). The monodromy group of a singularity $f$ acts transitively on the set of the vanishing cycles of $f$, i.e. for any vanishing cycles $\Delta$ and $\Delta'$ there exists an element of the monodromy group taking $\Delta$ into $\pm \Delta'$.

The proof of the implications.

2) ⇒ 3)

The monodromy group $\Gamma$ is a subgroup of the automorphism group of the integral lattice $H_{n-1}(X_*, \mathbb{Z}) \cong \mathbb{Z}^\mu \subset H_{n-1}(X_*, \mathbb{R}) \cong \mathbb{R}^\mu$. The elements of $\Gamma$
also preserve the intersection form. Therefore if the intersection form is definite then \( \Gamma \) is a discrete subgroup of a compact group and hence finite.

3) \( \Rightarrow \) 2)

Let \( j \) be a positive definite inner product on \( H_{n-1}(X_*, \mathbb{R}) \cong \mathbb{R}^\mu \) invariant under the action of \( \Gamma \) (such a product exists because \( \Gamma \) is finite). By proposition 3.2, all vanishing cycles have the same length with respect to \( j \). Therefore normalizing \( j \), if necessary, we can assume that all vanishing cycles have the length 2 with respect to \( j \). Since one can choose a basis of \( H_{n-1}(X_*) \) from the set of vanishing cycles (see §1), the form \( j \) restricted to the lattice \( H_{n-1}(X_*, \mathbb{Z}) \cong \mathbb{Z}^\mu \subset H_{n-1}(X_*, \mathbb{R}) \cong \mathbb{R}^\mu \) is integral, even and taking a value 2. Hence, by proposition 3.1, the quadratic form on \( H_{n-1}(X_*, \mathbb{R}) \) defined by the bilinear form \(-j\) is the quadratic form of the singularity. So the singularity is elliptic.

3) \( \Rightarrow \) 4)

As it has been proved above, if the monodromy group is finite, then the singularity is elliptic. Hence the Picard-Lefschetz transformations are the reflections in the Euclidean vector space \( H_{n-1}(X_*, \mathbb{R}) \) and the vanishing cycles form a root system, so the monodromy group is a Weyl group. It is irreducible because of proposition 3.2. Also by proposition 3.2, all vanishing cycles have the same length, so the irreducible (reduced) root system they form can be only of one of the types \( A_k, D_k, E_6, E_7, E_8 \). The implication is proved.

§4. The Proof of Theorem 1.

If \([f] = 0\) then the statement is trivial. Let \([f] \neq 0\). Then one may choose \( e_1 = 1, e_2, \ldots, e_\mu = f, (e_i \in O_n, i = 1, \ldots, \mu)\), mapping to a basis of \( Q_f \) over \( \mathbb{C} \). Then \( H : \mathbb{C}^n \times \mathbb{C}^\mu \rightarrow \mathbb{C}, H(z, \alpha) = f(z) + \alpha_1 e_1(z) + \ldots + \alpha_\mu e_\mu(z) \),
$z \in \mathbb{C}^n$, $\alpha = (\alpha_1, \ldots, \alpha_\mu) \in \mathbb{C}^\mu$, is a miniversal deformation of $f$. So there exists a biholomorphism $g : \Lambda \to \mathbb{C}^\mu$ such that the deformation $F$ is equivalent to the one induced from $H$ by $g$. One can easily check that $dg_0(v) = (0, \ldots, 0, 1)$, where $dg_0$ is the differential of $g$ at zero. It is also clear that $g$ maps the stratum $\mu=\text{const}$ in $\Lambda$ to the stratum $\mu=\text{const}$ in $\mathbb{C}^\mu$. So it suffices to show that the vector $(0, \ldots, 0, 1) \in T_0\mathbb{C}^\mu$ lies in the tangent cone to the stratum $\mu=\text{const}$ in $\mathbb{C}^\mu$. This is trivial because the family $(1 + t)f$, $t \neq -1$, is $\mu$-constant. The theorem is proved.

§5. The Proofs of the Equivalence of Conditions

1), 2), 5) and 6) in Theorem I and Corollary 1.

Firstly we recall some known facts of singularity theory that will be used in the proof.

Proposition 5.1 (“The Morse lemma with parameters” – see [Ar1] or [AGV1]).

In a neighborhood of a critical point of corank $k$ a holomorphic function $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ is equivalent to a function $\tilde{f}(z_1, \ldots, z_k) + z_{k+1}^2 + \ldots + z_n^2$, where the second differential of $\tilde{f}$ at zero is equal to zero.

Proposition 5.2. If the second differential of the singularity $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ at zero is equal to zero, then the positive (the negative) index of inertia of the quadratic form of the singularity $f$ is not less than the positive (the negative) one of the quadratic form of the singularity $\Theta_n(z_1, \ldots, z_n) = z_1^3 + \ldots + z_n^3$.

The proof of proposition 5.2. The proposition follows, for instance, from the results of G.N.Tjurina – see [Tju1], §1, proposition 2 and theorem 1.

Proposition 5.3 (see [Ga2]). The modality of the singularity is always one less than the dimension of the stratum $\mu=\text{const}$ in the base of a miniversal
deformation of the singularity.

**Proposition 5.4** (see [V5]). The codimension of the stratum \( \mu = \text{const} \) in the base of a miniversal deformation of a singularity \( f \) is not less than the number of spectral numbers of \( f \) that are less than \( l_1 + 1 \), where \( l_1 \) is the minimal spectral number of \( f \).

**Proposition 5.5** (see [St1], [V4]). Let’s assume that the intersection form of the singularity \( f \) is nondegenerate. Let \((\mu_+, \mu_-)\) denote its signature. Then

a) All spectral numbers of \( f \) are not integer;

b) The index \( \mu_+ \) (\( \mu_- \)) is equal to the number of the spectral numbers of \( f \) with odd (even) integral part.

**Proposition 5.6** (see [V2]). If \( \{l_i\}, \ i = 1, \ldots, \mu, \) is the spectrum of the singularity \( f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0), \) then \( \{l_i+1/2\}, \ i = 1, \ldots, \mu, \) is the spectrum of the singularity \( f + z_{n+1}^2 : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0). \)

**Proposition 5.7** (see [V3]). Let \( \{f\} \) denote the operator of multiplication by \( f \) in the local algebra \( Q_f \). Then if a number \( j \) is greater than the length of the spectrum of \( f \), then \( \{f\}^j = 0. \)

**Proposition 5.8** (see [Sa1]). A singularity \( f \) is equivalent to the singularity of a quasihomogeneous function if and only if the class \( [f] = 0 \) in the local algebra \( Q_f \).

We recall that for the singularity of a quasihomogeneous function \( f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0) \) of degree one a monomial from a monomial basis of \( Q_f \) over \( \mathbb{C} \) is called upper (respectively, diagonal or lower) if its quasihomogeneous degree is greater than one (respectively, equal to one or less than one). The total number of upper (diagonal, lower) basic monomials in a monomial basis does not depend on the choice of such a basis of \( Q_f \) (see [Ar3], [AGV1]).
Proposition 5.9 (see [Ar3],[V5]). The modality of the singularity of a quasihomogeneous function \( f \) is equal to the total number of upper and diagonal basic monomials of the local algebra \( Q_f \).

Proposition 5.10 (see [St2]). Let \( f : (\mathbb{C}^n,0) \to (\mathbb{C},0) \) be the singularity of a quasihomogeneous function of degree 1 with the weights \( \nu = (\nu_1, \ldots, \nu_n) \).

Let \( z^{k_i}, i = 1, \ldots, \mu \), be a monomial basis of the local algebra \( Q_f \) over \( \mathbb{C} \). Then the spectrum of \( f \) is the set \( \{ (k_i + 1, \nu) - 1 \} \), \( i = 1, \ldots, \mu \), where \( 1 = (1, \ldots, 1) \).

The proof of the equivalences 1) \( \Leftrightarrow \) 2) \( \Leftrightarrow \) 5) \( \Leftrightarrow \) 6) in theorem I.

5) \( \Leftrightarrow \) 6)

Follows from the definitions and the symmetries of the spectrum (see [St1],[V2]).

1) \( \Rightarrow \) 6)

If the singularity \( f \) is simple, then, by proposition 5.3, the stratum \( \mu = \text{const} \) in the base of a miniversal deformation of \( f \) is 1-dimensional and consists only of the coordinate axis \( \lambda_1 \). Hence, by theorem 1, the class \( [f] = 0 \) and, by proposition 5.8, the singularity \( f \) is equivalent to the singularity of a quasihomogeneous function. So let \( f : (\mathbb{C}^n,0) \to (\mathbb{C},0) \) be a simple quasihomogeneous singularity of degree 1 with the weights \( \nu = (\nu_1, \ldots, \nu_n) \). Then, by virtue of proposition 5.10, it can be easily seen that the length of the spectrum of \( f \) is equal to the maximal quasihomogeneous degree of basic monomials of \( Q_f \). Therefore, by proposition 5.9, one finds that the length of the spectrum of \( f \) is less than 1, i.e. all spectral numbers lie in the interval \( (n/2 - 3/2, n/2 - 1/2) \).

6) \( \Rightarrow \) 1)

Follows from propositions 5.4 and 5.3.
Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be an elliptic singularity. We claim that $f$ is stably equivalent to the singularity of a function of two variables.

Indeed, let $k$ be the corank of $f$. Then, by proposition 5.1, the singularity $f$ is stably equivalent to the singularity of a function $\tilde{f} : (\mathbb{C}^k, 0) \to (\mathbb{C}, 0)$, such that the second differential of $\tilde{f}$ at zero is equal to zero. Using propositions 5.10 and 5.5 one computes the spectrum of the singularity $\Theta_k$ and sees that the singularity $\Theta_k$ is elliptic if and only if $k \leq 2$. Therefore, by proposition 5.2, $f$ is equivalent to the singularity of a function $\tilde{f}$ of two variables.

By the symmetry and by proposition 5.5, we find that the spectrum of the singularity $\tilde{f} + z_3^2$ lies in the interval $(0, 1)$, because the singularity $\tilde{f}$ is elliptic. Therefore, using proposition 5.6 we find that the length of the spectrum of the singularity $f$ is less than 1.

The proof of corollary 1.

It has been proved that the simple singularities are the same as the elliptic ones. It has been also proved that an elliptic singularity $f$ is stably equivalent to the singularity of a function $\tilde{f}$ of two variables which has a spectrum of length less than one (by proposition 5.6, the length of the spectrum is the same for all stably equivalent singularities). Now, by propositions 5.7 and 5.8, the singularity $\tilde{f}$ is equivalent to the singularity of a quasihomogeneous function and the corollary follows.
§6. The Proofs of Proposition 2.1, Theorem 2 and Corollary 2.

The proof of theorem 2. Theorem 2 can be obtained immediately from proposition 2.1 and from the following fact.

Proposition 6.1 (see [AC]). For an isolated singularity of a function of $n$ variables the trace of the monodromy operator is equal to $(-1)^n$. In particular, if $n \equiv 3 \pmod{4}$ then the trace is equal to $-1$.

The proof of corollary 2. In the case of an elliptic singularity the action of the braid group $Br(\mu)$ given by formula (2) in §1 is the action on the set of the tuples of roots in a root system and it gives the action of $Br(\mu)$ on the tuples of reflections in the Weyl group given by formula (3) in §1 if one considers the reflections in the hyperplanes orthogonal to the roots.

By virtue of these two actions of the braid group $Br(\mu)$, the corollary follows immediately from the following algebraic result of P.Deligne.

Proposition 6.2 ([De]). Let $W$ be a Weyl group (of one of the types $A_k, D_k, E_6, E_7, E_8$) of rank $\mu$. Let the tuples $(s_1, \ldots, s_\mu)$ and $(s_1', \ldots, s_\mu')$ be any elements of $S$ (for the definition of $S$ see the end of §1) such that $s_1 \cdot \ldots \cdot s_\mu = s_1' \cdot \ldots \cdot s_\mu' = c$, where $c$ is a Coxeter element of $W$. Then these two tuples lie in the same orbit of the action of $Br(\mu)$.

Remark. This result of P.Deligne has been generalized for quasi-Coxeter elements (i.e. the ones like $s$ in the statement of proposition 2.1) by E.Voigt (see [Voi]).

The proof of proposition 2.1. The proposition can be obtained as a consequence of the algebraic theory of the structure of conjugation classes in a Weyl group (see [Ca]). We shall briefly outline the proof.

Definition (see [Ca], §2). Let $w$ be an element of $W$. Then the length $l(w)$ of $w$ is by definition the smallest number $k$ such that $w = w_1 \cdot \ldots \cdot w_k$ where
Proposition 6.3 (see [Ca], lemma 2). The length $l(w)$ is the number of eigenvalues of $w$ which are not equal to 1.

Proposition 6.4 (see [Ca], lemma 3). If a tuple $(s_1, \ldots, s_{\mu})$ lies in $S$, then the length of the element $s_1 \cdot \ldots \cdot s_{\mu}$ is equal to $\mu$.

So the length of the element $s$ from the hypothesis of proposition 2.1 is equal to $\mu$.

By the results in [Ca] (see [Ca], §3 and the corollary after proposition 38) each element $w \in W$ can be represented as $w = w_1 \cdot w_2$, where $w_1$ and $w_2$ can be expressed as products of reflections corresponding to mutually orthogonal roots. The construction described in [Ca], §3, associates a graph (a so-called admissible diagram) $\Upsilon$ to any such a representation of $w$. The number of nodes in $\Upsilon$ is equal to $l(w)$. The admissible diagram for a Coxeter element can be chosen as the canonical Dynkin diagram of $W$. Conversely, one easily obtains

Proposition 6.5. If an admissible diagram of $w$ is the canonical Dynkin diagram of $W$ then $w$ is a Coxeter element of $W$.

Moreover, the following fact turns out to be true.

Proposition 6.6 (see [Ca], lemma 8). If $l(w)$ is equal to the rank of $W$ and if an admissible diagram $\Upsilon$ of $w$ is a tree then $\Upsilon$ is the canonical Dynkin diagram of $W$.

We also need the following assertion.

Proposition 6.7 (see [Ca], proposition 22, and also proposition 6.3 above).

Let $w$ be an element of $W$ with length $l(w) = \mu$ and let $\Upsilon$ be an admissible diagram of $w$. Then the trace of $w$ is given by:

$$\text{tr} w = \text{number of bonds in } \Upsilon - \text{number of nodes in } \Upsilon.$$
Now we notice that if \( l(w) = \mu \) then an admissible diagram corresponding to \( w \) is always connected – otherwise the group \( W \) would not be irreducible. It is well known that the number of nodes in a connected graph is one greater than the number of bonds if and only if the graph is a tree. Hence, by virtue of propositions 6.5, 6.6, 6.7, proposition 2.1 follows.

§7. The Proof of Theorem II.

We shall give two proofs of theorem II. Firstly we recall some notions and results we need for both proofs.

By virtue of corollary 1, we can assume that a simple (an elliptic) singularity is quasihomogeneous.

In what follows we shall always assume that \( f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0) \) is the singularity of a quasihomogeneous function of degree 1 with weights \( \nu_1, \ldots, \nu_n \).

**Definition** (see [AGV2]). Let \( F(z, \lambda) \), \( \lambda \in \Lambda = \mathbb{C}^\mu \), be a miniversal deformation of \( f \). Then the set \( \Sigma_f = \{ \lambda \in \Lambda \mid \text{zero is a critical value of } F(\cdot, \lambda) : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0) \} \), \( \Sigma \subset \Lambda \), is called the bifurcation diagram of \( f \).

Now let \( W \) be a Weyl group (of one of the types \( A_k, D_k, E_6, E_7, E_8 \)) of rank \( \mu \). There is a natural action of \( W \) on the complexification \( \mathbb{C}^\mu \) of the Euclidean vector space \( \mathbb{R}^\mu \) on which \( W \) originally acted by reflections. The functions in \( \mathbb{C}[x] \) invariant under the action of \( W \) form an algebra which is free with \( \mu \) generators; these generators (which are also called the basis invariants of \( W \)) can be chosen as homogeneous polynomials of degrees \( m_i + 1 \), \( i = 1, \ldots, \mu \), where \( m_1, \ldots, m_\mu \) are the exponents of the group \( W \) (see [Ch],[Bou]). Therefore the space of orbits \( B = \mathbb{C}^\mu / W \) is isomorphic to \( \mathbb{C}^\mu \). The set of all singular orbits (i.e. of all ramification points of the ramified
covering $\pi : \mathbb{C}^\mu \to \mathbb{C}^\mu /W$ is a hypersurface $S(W) \subset B$ called the swallowtail of the group $W$.

**Proposition 7.1** (see [Lo2], theorem 4.3). Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be a singularity with multiplicity $\mu$ and monodromy group isomorphic to a Weyl group $W$ (of one of the types $A_k, D_k, E_6, E_7, E_8$) of rank $\mu$. Then there exists a biholomorphic map ping of pairs: $(\Lambda, \Sigma_f) \to (B, S(W))$. Such a biholomorphism maps the fixed origin of coordinates in $B$ into the fixed origin of coordinates in $\Lambda$.

**Remark.** The normal forms of the simple (elliptic) singularities are not actually used in the proof of proposition 7.1 (see [Lo2]). It can be checked that the proof involves only the following information on the singularity $f$ (in addition to the information that the monodromy group of the singularity is an irreducible Weyl group):

a) $f$ is (stably) equivalent to the singularity of a quasihomogeneous function (of three variables);

b) the quasidegrees of the basic monomials of the local algebra $Q_f$ coincide with the numbers $\frac{m_i}{|h|} - \tilde{\nu} + 1$, $i = 1, \ldots, \mu$, where $m_1, \ldots, m_\mu$ and $|h|$ are the exponents and the Coxeter number of the group $W$ respectively, $\tilde{\nu} = \nu_1 + \ldots + \nu_n$.

We have proved both facts not using the normal forms: the first one follows from theorem I and corollary 1 and the second one follows from theorem 2 and proposition 5.10.

So simple singularities with isomorphic monodromy groups have isomorphic bifurcation diagrams. Then there are two ways to complete the proof of theorem II. The first one is to use directly the result of K.Wirthmüller ([Wi]) about singularities determined by their discriminants. The second one is to
find a way to reconstruct the local algebra $Q_f$ from the space of orbits $B$ and then to use the theorem of A.N. Shoshitaishvili (see proposition 7.3) which says that the singularity of a quasihomogeneous function is uniquely determined by its local algebra.

The first proof.

We shall recall some notions we are going to use. Given a singularity $g : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ let $X_g$ denote the germ of the hypersurface $g^{-1}(0)$ at zero. One can consider a miniversal deformation of the hypersurface germ $X_g$ (see [Tju2], [KS]). It is given by the projection $\mathbb{C}^n \times \mathbb{C}^l \to \mathbb{C}^l : (z, \lambda) \mapsto \lambda$, restricted on the hypersurface $G(z, \lambda) = 0$, $G(z, \lambda) = g(z) + \lambda_1 e_1(z) + \ldots + \lambda_l e_l(z)$, where $e_1(z), \ldots, e_l(z)$ determine a $\mathbb{C}$-basis of the vector space $O_n/\langle g, \partial g/\partial z_1, \ldots, \partial g/\partial z_n \rangle$, $z \in \mathbb{C}^n$, $\lambda = (\lambda_1, \ldots, \lambda_l) \in \mathbb{C}^l$. The number $l$ is called the Tjurina number of $g$. The set of critical values of the projection (i.e. the set of $\lambda \in \mathbb{C}^l$ such that the variety $G(\cdot, \lambda) = 0$ is singular) forms a hypersurface in $\mathbb{C}^l$ (more precisely, one should consider a germ of the hypersurface at $0 \in \mathbb{C}^l$) called the discriminant of the deformation.

Proposition 7.2 (the part of the result in [Wi]). Let $X_1$ and $X_2$ be analytic germs of hypersurfaces with isolated singularities at $0 \in \mathbb{C}^n$. If the discriminants of some miniversal deformations of $X_1$ and $X_2$ are isomorphic then $X_1$ and $X_2$ are isomorphic.

Now we go back to the proof of theorem II. Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ and $g : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be simple quasihomogeneous singularities with isomorphic monodromy groups. Then, by proposition 7.1, $f$ and $g$ have isomorphic bifurcation diagrams. One can easily see that since $f$ and $g$ are quasihomogeneous their Tjurina numbers coincide with their multiplicities respectively. So miniversal deformations of $f$ and $g$ provide miniversal deformations of
Theorem II is the immediate consequence of the following facts.

**Proposition 7.3** (see [Sh] or – for more general results – [Ma-Y],[Be]). Let \( f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0) \), \( g : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0) \) be singularities of quasihomogeneous functions. Then the singularities \( f \) and \( g \) are equivalent if and only if their local algebras \( Q_f \) and \( Q_g \) are isomorphic (as algebras).

**Lemma.** For a Weyl group \( W \) (of one of the types \( A_k, D_k, E_6, E_7, E_8 \)) such an algebra \( A(W) \) can be constructed that for any simple (elliptic) singularity \( f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0) \) with the monodromy group isomorphic to \( W \), the local algebra \( Q_f \) is isomorphic (as an algebra) to \( A(W) \).

The proof of the lemma is based on proposition 7.1 and on the V.I.Arnold - A.B.Givental results on the convolution of invariants of finite groups generated by reflections. We shall recapitulate these results (for more details see [Ar5], [Gi]).
Let \( Q_f^* \) denote the dual space to the algebra \( Q_f \). The product of elements \( p, q \) in \( Q_f \) will be denoted \( p \cdot q \).

**Definition** (see [Ar5], [Gi]). A linear functional \( \alpha \in Q_f^* \) is said to be *admissible* if \( \alpha \) is not identically equal to zero on the annihilator of the maximal ideal of \( Q_f \) (i.e. on the 1-dimensional ideal generated by the class of the Hessian of \( f \) at zero – see [AGV1], p.5.11).

A general element of \( Q_f^* \) is admissible. If \( \alpha \in Q_f^* \) is admissible, then the bilinear form \( (p, q) \rightarrow \alpha(p \cdot q) \) on \( Q_f \) is nondegenerate. Let \( N_\alpha : Q_f \rightarrow Q_f^* \) denote the operator of this form. Let also \( D = \nu_1z_1\frac{\partial}{\partial z_1} + \ldots + \nu_nz_n\frac{\partial}{\partial z_n} \) be the Euler derivation in the graded local algebra \( Q_f \) and \( R = E - D \) be the \( \mathbb{C} \)-linear operator on \( Q_f \), where \( E \) is the identity operator.

By the upper star at an operator we shall denote the adjoint operator.

We define a bilinear operation \( P_\alpha : Q_f^* \times Q_f^* \rightarrow Q_f^* \) by the formula \( (a, b) \rightarrow R^* N_\alpha(N_\alpha^{-1}a \cdot N_\alpha^{-1}b) \).

A function on the space of orbits \( B \) is called an *invariant* of the group \( W \). We shall denote the tangent space to \( B \) at \( 0 \in \mathbb{C}^\mu \) by \( T \) and the corresponding dual space – by \( T^* \).

An inner product on \( \mathbb{C}^\mu \) invariant under the action of \( W \) provides the isomorphism \( i : T^* \mathbb{C}^\mu \rightarrow T_* \mathbb{C}^\mu \).

There is a symmetric bilinear operation \( \Phi \) on the set of invariants of \( W \) that associates to each pair \( \phi, \psi \) of invariants the inner product of their Euclidean gradients: \( \Phi(\phi, \psi) = \pi_*(i\pi^*d\phi, i\pi^*d\psi) \).

The invariant \( \Phi(\phi, \psi) \) is called the *convolution* of invariants \( \phi \) and \( \psi \). The operation \( \Phi \) defines a symmetric bilinear operation \( \Phi_0 : T^* \times T^* \rightarrow T^* \) by the formula: \( \Phi_0(d\phi, d\psi) = d\Phi(\phi, \psi) \). The operation \( \Phi_0 \) is called the *linearized convolution of invariants*.

By virtue of proposition 7.1 the space \( T^* \) can be identified with the local
Proposition 7.4 (see [Gi]). Under any biholomorphism \((\Lambda, \Sigma_f) \to (B, S(W))\) the operation \(\Phi_0 : T^* \times T^* \to T^*\) goes over into the operation \(P_\alpha : Q^*_f \times Q^*_f \to Q^*_f\) for some admissible \(\alpha \in Q^*_f\). Moreover, for any admissible \(\alpha \in Q^*_f\) the operation \(P_\alpha\) can be obtained from the operation \(\Phi_0\) by means of such a biholomorphism.

Remark.

1) The normal forms of the simple singularities are not actually used in the proof of proposition 7.4 (see [Gi] and the remark after proposition 7.1).

2) Proposition 7.4 was first proved (by means of the normal forms) for the simple singularities of the types \(A\) and \(D\) (and also for the boundary singularities of the types \(B\) and \(C\)) in the paper [Ar5].

The proof of the lemma (see [Ar5], §9, Remark 7).

Firstly we shall construct such an algebra \(A(W)\). To do it we apply the constructions and assertions mentioned above to the quasihomogeneous simple (elliptic) singularity \(f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)\) and its monodromy group \(W\), which is an irreducible Weyl group of one of the types \(A_k, D_k, E_6, E_7, E_8\).

Let’s define the linear operator \(w_\beta : T^* \to T^*\), \(\beta \in T^*\), by the formula \(w_\beta(\cdot) = \Phi_0(\beta, \cdot)\).

As it was said in the beginning of the paragraph, the basis invariants of \(W\) can be chosen to be homogeneous polynomials. Let’s take \(\beta_0 \in T^*\), \(\beta_0 = d\phi_2\), where \(\pi^*\phi_2(z_1, \ldots, z_n) = z_1^2 + \ldots + z_n^2\) is the homogeneous basis invariant of \(W\) of degree 2. Then, as one easily checks, the operator \(w_{\beta_0}\) is invertible.

Now one can consider the family \(A(W)\) of operators \(u_\beta = w_{\beta_0}^{-1}w_\beta\), \(u_\beta : T^* \to T^*\), where \(\beta\) runs over the entire space \(T^*\). The correspondence \(\beta \mapsto u_\beta\) provides \(A(W)\) with the structure of a \(\mu\)-dimensional vector space. We shall show that \(A(W)\) is the algebra isomorphic to the local algebra \(Q_f\).
Indeed, let’s fix an admissible element \( \alpha \in Q_f^* \) and let’s fix such an identification of \( T^* \) and \( Q_f^* \) (see propositions 7.1, 7.4) that the operation \( \Phi_0 \) goes over into the operation \( P_\alpha \) under this identification. Let’s define for any \( q \in Q_f \) the linear operator \( V_q : Q_f \to Q_f \) by the formula \( V_q = M_q R \), where \( M_q \) is the operator of multiplication by \( q \) in \( Q_f \). Using only the definitions and proposition 7.4 one checks (see [Ar5], §9, propositions 3 and 4) that under such an identification \( w_\beta = V_q^* = R^* M_q^* \), where \( \beta = N_\alpha q \). Notice that since \( w_{\beta_0} \) and \( R \) are invertible \( q_0 = N^{-1}_\alpha \beta_0 \) is an invertible element of \( Q_f \).

Now \( u_\beta = w_{\beta_0}^{-1} w_\beta = (M_{q_0}^*)^{-1} (R^*)^{-1} R^* M_q^* = (M_{q_0}^*)^{-1} M_q^* \). If \( \beta \) runs over the entire space \( T^* \), then \( q = N^{-1}_\alpha \beta \) runs over the entire space \( Q_f \). The operators \( M_q^* \) (\( q \) runs over the entire space \( Q_f \)) form (with respect to the operator multiplication) an algebra isomorphic to \( \tilde{Q}_f \), where \( \tilde{Q}_f \) is the algebra obtained by introducing on the vector space \( Q_f \) a new multiplication operation \( \ast : a \ast b = q_0^{-1} \cdot a \cdot b \), \( a, b \in Q_f \). One easily checks that \( \tilde{Q}_f \) is isomorphic to \( Q_f \) as an algebra. Hence \( A(W) \) is also an algebra and it is isomorphic (as an algebra) to \( Q_f \).

**Appendix 1. The Normal Forms.**

Using the information on the simple (elliptic) singularities obtained so far one rather easily finds their normal forms and determines the types of the corresponding Weyl groups.

**Theorem** (cf. [Ar1] and [AGV1], §§11,13).

i) Any simple (elliptic) singularity is stably equivalent to one of the following singularities:

1) \( f(x,y) = x^{k+1} + y^2 \), \( k \geq 1 \);  
2) \( f(x,y) = x^2 y + y^{k-1} \), \( k \geq 4 \);  
3) \( f(x,y) = x^3 + y^4 \);
4) \( f(x, y) = x^3 + xy^3 \);
5) \( f(x, y) = x^3 + y^5 \).

ii) The Weyl groups corresponding to the singularities 1)-5) are of the types \( A_k, D_k, E_6, E_7, E_8 \) respectively. In particular, all singularities 1)-5) are mutually not equivalent.

The proof.

i) As we know from corollary 1, any simple (elliptic) singularity is stably equivalent to the singularity of a quasihomogeneous function of two variables. By virtue of proposition 5.9, the diagonal to which all the exponents of the monomials contained in that quasihomogeneous function belong lies below the point \((2, 2)\) (in the plane of exponents). So to obtain the normal forms one sorts out all such lines and uses the following very easy assertions as well as proposition 5.1 ("The Morse lemma with parameters").

**Lemma 1.** Any singularity of a function of one variable is equivalent to \( f(x) = x^k \) for some \( k \geq 2 \).

**Lemma 2** (see [AGV1], §11.2). A cubic form of two variables can be reduced by a \( \mathbb{C} \)-linear transformation to one of the forms: (1) \( x^2y + y^3 \), (2) \( x^2y \), (3) \( x^3 \), (4) 0.

**Lemma 3** (see [AGV1], §11.2). A polynomial \( Ax^2y + Bxy^{k+1} + Cy^{2k+1} \), \( A \neq 0 \), can be reduced by a linear transformation to the same form with \( B = 0 \).

(ii) We can easily compute the spectra of the singularities 1)-5) and hence, by means of the proposition 5.10, the order and the eigenvalues of the monodromy operator. By theorem 2, this enables us to compute the exponents and the Coxeter numbers of the corresponding Weyl groups. Then we compare these numbers with ones in the tables (see e.g [Bou]). A Weyl group can be unambiguously determined by such data and the theorem follows.
Appendix 2. The proof of theorem 1’.

For the terminology used in this proof see §1 and also [Kou] or [AGV1], [AGV2].

Let $\Delta$ denote the Newton diagram of $f$. It is known (see [Ar3] or [AGV1]) that one can always choose such a monomial basis $e_1, \ldots, e_\mu$ of $Q_f$ over $\mathbb{C}$ (a so called regular basis – see [Ar3]), that for any number $D$ the basic monomials of Newton degree $D$ are linearly independent modulo the sum of the gradient ideal $I_f$ and the space of functions (more precisely, elements of $O_n$) of Newton order greater than $D$. (The Newton degree and the Newton order are defined by means of the diagram $\Delta$).

Let’s take a miniversal deformation $F$ of the singularity $f$ defined by such a choice of $e_1, \ldots, e_\mu$, so $F(z, \lambda) = f(z) + \lambda_1 e_1 + \ldots + \lambda_\mu e_\mu$, $\lambda = (\lambda_1, \ldots, \lambda_\mu) \in \mathbb{C}^\mu$, $z \in \mathbb{C}^n$. Let $[f]$ be now the class of $f$ in $Q_f$ considered as a linear combination of monomials $e_1, \ldots, e_\mu$. One sees that for small $t \in \mathbb{C}$ the Newton diagram of the function $f_t = f + t[f]$ coincides with $\Delta$.

The set $A$ of the principal parts that are nondegenerate with respect to $\Delta$ is open in the space of all principal parts corresponding to the diagram $\Delta$ (see [Kou]). Since for all functions with the Newton diagram $\Delta$ and with principal part belonging to the set $A$ the multiplicity of the critical point $0 \in \mathbb{C}^n$ is the same (see [Kou]), the linear 1-parameter deformation $f_t = f + t[f]$ of the singularity $f$ is a $\mu = \text{const}$ one (for small $t$). The theorem is proved.
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