ON DAVENPORT EXPANSIONS, POPOV’S FORMULA, AND FINE’S QUERY

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Abstract. We establish an explicit connection between a Davenport expansion and the Popov sum. Asymptotic analysis follows as a result of these formulas. New solutions to a query of N.J. Fine are offered, and a proof of Davenport expansions is detailed.

Keywords: Davenport expansions; Riemann zeta function; von Mangoldt function

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1. Introduction and Main Formulas

Let \( \Lambda(n) \) denote the von Mangoldt function, \( \zeta(s) \) the Riemann zeta function, and \( \rho \) the non-trivial (complex) zeros of the Riemann zeta function [5, p. 43]. In a recent paper [10] we established a proof of Popov’s formula [12]:

\[
\sum_{n>x} \frac{\Lambda(n)}{n^2} \left( \left\lfloor \frac{n}{x} \right\rfloor - \left\{ \frac{n}{x} \right\} \right)^2 = 2 \frac{-\log(2\pi)}{x} + \sum_{\rho} \frac{x^{\rho-2}}{\rho(\rho - 1)} + \sum_{k \geq 1} \frac{k + 1 - 2k\zeta(2k + 1)}{2k(k + 1)(2k + 1)} x^{-2k-2},
\]

for \( x > 1 \). Here \( \{x\} \) is the fractional part of \( x \), sometimes written as \( \{x\} = x - \lfloor x \rfloor \), where \( \lfloor x \rfloor \) is the floor function. The proof relied on the Mellin transform formula

\[
\frac{1}{2} \left( \{x\}^2 - \{x\} \right) = \frac{1}{2\pi i} \int_{(b)} \left( \frac{s + 1}{2s(s - 1)} - \frac{\zeta(s)}{s} \right) \frac{x^{s+1}}{s+1} ds,
\]

which is valid for \( x > 0 \), \( -1 < b < 0 \). This can be observed by noting (1.2) is known [15, pg. 14, eq. (2.1.4)] for \( x > 1 \), and by [10, pg.405] for \( 0 < x < 1 \),

\[
\frac{1}{2\pi i} \int_{(b)} \frac{x^s}{s(s - 1)} ds = 0.
\]

Let \( f(n) \) be a suitable arithmetic function such that \( L(s) = \sum_{n \geq 1} f(n)n^{-s} \) analytic for \( \Re(s) > 1 \), and \( S(x) = \sum_{n \leq x} f(n) \). Examining the right hand side of (1.2), it is
not difficult to see that
\[
\frac{1}{2} \sum_{n>x} \frac{f(n)}{n^2} \left( \left\{ \frac{n}{x} \right\} - \left\{ \frac{n}{x} \right\}^2 \right) = \frac{1}{2\pi i} \int_{(b)} \frac{x^{-s-1}}{2s-1} L(1-s)ds - \frac{1}{2\pi i} \int_{(b)} \frac{x^{-s-1} \zeta(s)}{s(s+1)} L(1-s)ds.
\]

If we replace \( s \) by \( 1-s \) in the first integral in (1.3) and apply Fubini’s theorem to interchange the integrals, we see that this integral is equal to
\[
\frac{1}{2\pi} \int_0^x \left( \sum_{n \geq 1} \frac{f(n)}{n} \left\{ \frac{ny}{x} \right\} - \frac{1}{2} \right) dy = \frac{1}{2\pi} \int_0^x \frac{S(y)dy}{y^2},
\]
by the Mellin-Perron formula. If we generalize the formula of Segal [14, eq.(5)], we can see that the second integral on the right side of (1.3) is
\[
\int_0^x \left( \frac{1}{2\pi i} \int_{(1-b)} \frac{y^{s-2}}{s} L(s)ds \right) dy = \int_0^x \frac{S(y)dy}{y^2},
\]

assuming uniform convergence where \( F(n) = \sum_{d \mid n} f(d) \). Collecting (1.3), (1.4), and (1.5), we obtain the following theorem upon noting that Davenport’s proof of uniform convergence is dependent on a special estimate [4].

**Theorem 1.1.** Let \( f(n) \) be chosen such that \( L(s) \) is analytic for \( \Re(s) > 1 \), and that \( \sum_{n \leq N} f(n)e^{2\pi inx} = O(x(\log(x)^{-h}) \), for any fixed \( h \). We have for \( x > 1 \),
\[
\frac{1}{2} \sum_{n>x} \frac{f(n)}{n^2} \left( \left\{ \frac{n}{x} \right\} - \left\{ \frac{n}{x} \right\}^2 \right) = \frac{1}{2\pi} \int_0^x S(y)\frac{dy}{y^2} + \frac{1}{2\pi} \sum_{n \geq 1} \frac{F(n)}{n^2} \left( \cos\left(\frac{2\pi nx}{x}\right) - 1 \right).
\]

We have therefore proven the connection between Popov’s formula and the integral \( \int_0^x \frac{S(y)dy}{y^2} \) alluded to in [11] (see also [4, pg.69]). Perhaps even more fascinating is the connection to Davenport expansions [3, 8] through the sum on the far right hand side of (1.5). This Fourier series is known to be connected to the periodic Bernoulli polynomial \( B_2(x) - B_2 = (\{x\}^2 - \{x\}) \). For relevant material on Davenport expansions connected to Bernoulli polynomials, see [2, 9].

Recall that \( h(x) \sim g(x) \) means that \( \lim_{x \to \infty} \frac{h(x)}{g(x)} = 1 \). Letting \( x \to \infty \) and applying L’Hôpital’s rule to Theorem 1, and the Residue Theorem to (1.3)-(1.4), we have the following.
Corollary 1.1.1. Let \( f(n) \) be chosen such that \( L(s) \) is analytic for \( \Re(s) > 1 \).

Suppose that \( S(x) \sim \Delta(x) \) as \( x \to \infty \). Then

\[
\frac{1}{2} \sum_{n > x} \frac{f(n)}{n^2} \left( \left\{ \frac{n}{x} \right\} - \left\{ \frac{n}{x} \right\}^2 \right) - \frac{1}{2\pi^2} \sum_{n \geq 1} \frac{F(n)}{n^2} (\cos(\frac{2\pi nx}{x}) - 1) \sim \frac{\Delta(x)}{x^2},
\]

as \( x \to \infty \).

Notice that the sum on the left hand side of (1.1) is \( \sim \frac{1}{x} \), which corresponds to the Prime Number theorem \( \sum_{n \leq x} \Lambda(n) \sim x \) when coupled with our corollary.

The integral \( \int_0^x \frac{S(y)dy}{y^2} \) has appeared in many recent works in the analytic theory of numbers. Namely, in the case of the von Mangoldt function \( S(x) = \psi(x) = \sum_{n \leq x} \Lambda(n) \), see [13], where we find a study of the function

\[
\sum_{n \leq x} \frac{\Lambda(n)}{n} - \frac{\sum_{n \leq x} \Lambda(n)}{x} = \int_1^x \frac{\psi(y)dy}{y^2}.
\]

For the case of the Möbius function (i.e. \( S(x) = M(x) \) the Mertens function [15, pg.370]), Inoue [7, Corollary 3, \( k = 2 \)] gave, under the assumption of the weak Mertens Hypothesis,

\[
\int_1^x \frac{M(y)dy}{y^2} = x^{-\frac{1}{2}} \sum_{\rho} \frac{x^{s_\rho}}{\zeta'(\rho)\rho(\rho - 1)} + A(2) + O(x^{-1}).
\]

Here \( A(2) \) is a constant, and \( g(x) = O(h(x)) \) means \( |g(x)| \leq c_1 h(x) \), \( c_1 > 0 \) a constant.

We mention there is another form of the Fourier series on the far right side of Theorem 1.1. Note that [15, pg.14, eq.(2.1.5)]

\[
\{x\} = -\frac{1}{2\pi i} \int_{(c)} \frac{\zeta(s)}{s} x^s ds,
\]

where \( x > 0 \), and \( 0 < c < 1 \). Integrating, we have that

\[
\frac{1}{2} \left( \{x\}^2 + \lfloor x \rfloor \right) = \frac{1}{2\pi i} \int_{(c)} \frac{\zeta(s)}{s(s+1)} x^{s+1} ds.
\]

Dividing by \( x \), computing the residue at the pole \( s = 0 \), and inverting the desired series in (1.7), we have

\[
\sum_{n \geq 1} \frac{f(n)}{n} \left( \frac{1}{x} \right) \left( \{nx\}^2 + [nx] \right) - \frac{1}{2} \right) = -\frac{1}{2\pi i} \int_{(c-1)} \frac{\zeta(s)}{s(s+1)} x^s L(1-s) ds.
\]

Here we used the fact that \( \zeta(0) = -\frac{1}{2} \). Hence, after comparing with our computation (1.5), we have proven the following result.
Theorem 1.2. For $x > 0$,
\[
\sum_{n \geq 1} \frac{f(n)}{n} \left( \frac{1}{x^2} (nx)^2 + |nx| \right) - \frac{1}{2} = \frac{1}{2x\pi^2} \sum_{n \geq 1} \frac{F(n)}{n^2} (\cos(2\pi nx) - 1).
\]

2. Solution to The N.J. Fine query

In [1], a positive answer was presented to a query of N.J. Fine, who asked for a continuous function $\varphi(x)$ on $\mathbb{R}$, with period 1, $\varphi(x) \neq -\varphi(-x)$, and
\[
\sum_{N \geq k \geq 1} \varphi\left(\frac{k}{N}\right) = 0.
\]
Namely, they gave the solutions
\[
\sum_{n \geq 1} \frac{f(n)}{n} \cos(2\pi nx),
\]
where $f(n)$ is chosen as the Möbius function $\mu(n)$ and the Liouville function $\lambda(n)$, [14]. Their proof utilizes a Ramanujan sum [15, pg.10]
\[
\sum_{N \geq k \geq 1} \cos\left(\frac{2\pi kn}{N}\right),
\]
which is $N$ if $n \equiv 0 \pmod{N}$ and 0 otherwise. It is also dependent on $\sum_{n \geq 1} f(n)/n = 0$. In fact, it is possible to further generalize their result using these properties, which we offer in the following.

Theorem 2.1. Suppose $f(n)$ is a multiplicative arithmetic function chosen such that $\sum_{n \geq 1} f(n)/n = 0$. Then
\[
\sum_{n \geq 1} \frac{f(n)}{n} \cos^m(\pi nx),
\]
and
\[
\sum_{n \geq 1} \frac{f(n)}{n} \sin^{2m}(\pi nx),
\]
for each positive integer $m$ satisfy the properties in Fine’s query.

Proof. We will use [6, pg.31, section 1.320, no.5 and 7] for (2.4) and (2.5) to obtain our $\varphi(x)$. Namely
\[
\cos^{2m}(x) = \frac{1}{2^{2m}} \left( \sum_{m-1 \geq k \geq 0} 2^m \binom{2m}{k} \cos(2(m-k)x) + \binom{2m}{m} \right).
\]
(2.7) \[ \cos^{2m-1}(x) = \frac{1}{2^{2m-2}} \sum_{m \geq k \geq 0} \binom{2m}{k} \cos((2m - 2k - 1)x). \]

Putting \( x = \frac{2\pi lN}{N} \), and summing over \( N \geq l \geq 1 \) we see that the sum is a linear combination of zeros and \( N \)'s depending on whether \( n(2m - k) \) | \( N \). In the case where \( n(2m - k) \) | \( N \), we are left with a linear combination of terms which are independent of \( n \). The last term is simply \( \frac{N}{2^m} \binom{2m}{m} \).

Therefore, summing over \( n \) gives the result upon invoking \( \sum_{n \geq 1} f(n)/n = 0 \). Since (2.5) follows in the same way using another formula from [6, pg.31, section 1.320, no.1], we leave the details to the reader.

We were interested finding more solutions to Fine’s query by constructing a special arithmetic function with the possible property \( \sum_{n \geq 1} f(n)/n \neq 0 \). Define

(2.8) \[ \chi_{m,l}^{\pm}(n) := \begin{cases} \pm (m^l \mp 1), & \text{if } n \equiv 0 \pmod{m}, \\ 1, & \text{if } n \not\equiv 0 \pmod{m}. \end{cases} \]

If \( f(n) \) is completely multiplicative, this tells us that

\[ \sum_{n \geq 1} \frac{\chi_{m,l}^{\mp}(n)f(n)}{n^s} = L(s) - \frac{m^l}{s} \sum_{n \equiv 0 \pmod{m}} \frac{f(n)}{n^s} = (1 - f(m)m^{l-s})L(s). \]

**Definition:** A function is said to be of the class \( \mathcal{N} \) if: (i) it is continuous on \( \mathbb{R} \), (ii) is 1-periodic (iii) is not odd, and (iv) satisfies (2.1) for each \( N \) coprime to \( m \).

**Theorem 2.2.** Suppose that \( L(s) \) is analytic for \( \Re(s) > 1 \). For natural numbers \( m > 1, l > 1, \) and \( N \) is coprime to \( m \), we have \( D_1(x) \in \mathcal{N} \) where

(2.9) \[ D_1(x) = \sum_{n \geq 1} \frac{\chi_{m,l}^{\pm}(n)f(n)}{n^l} \cos(2\pi nx), \]

for a completely multiplicative function with the property \( f(m) = -1 \), and \( D_2(x) \in \mathcal{N} \) where

(2.10) \[ D_2(x) = \sum_{n \geq 1} \frac{\chi_{m,l}^{\pm}(n)f(n)}{n^l} \cos(2\pi nx), \]

for a completely multiplicative function with the property \( f(m) = 1 \).

**Proof.** First we consider (2.9). Note that because \( \chi_{m,l}^{\pm}(n) \) is not completely multiplicative, we need to restrict \( N \) to be coprime to \( m \), since then \( \chi_{m,l}^{\pm}(Nn) = \chi_{m,l}^{\pm}(n) \).

That is to say that \( Nn \equiv 0 \pmod{m} \) is solved by \( n \equiv 0 \pmod{m} \) provided that \( N \)
is coprime to $m$. The same argument applies in the case $Nn \not\equiv 0 \pmod{m}$. Using (2.3) and the method applied in [1] we compute that

$$
\sum_{N \geq k \geq 1} \sum_{n \geq 1} \frac{\chi_{m,l}(n)f(n)}{n^l} \cos(\frac{2\pi nk}{N}) = N \sum_{n \equiv 0 \pmod{N}} \frac{\chi_{m,l}(n)f(n)}{n^l} = f(N) \lim_{s \to 1} (1 - m^{-s})L(s) = 0.
$$

The computation for (2.10) is similar, and so we leave the details for the reader. □

An example for $D_2(x)$ is if $f(n) = \lambda(n)$, and $m = 4, l > 1$, since $\lambda(4) = 1$. One for $D_1(x)$ is if $f(n) = \mu(n)$, and $m = 5, l > 1$, since $\mu(5) = -1$.

3. Fourier Analysis of Davenport expansions

In [8, pg.280–281], it is noted that Davenport’s expansion may be obtained from standard Fourier techniques, and evaluating a Fourier integral involving the fractional part function. Here we will give a detailed proof to obtain a further expansion.

**Lemma 3.1.** Let $\phi(x)$ be a 1-periodic function on $[0,1]$. Then $\phi(x)$ admits the expansion

$$
\phi(x) = \sum_{n \geq 1} c_n \sin(\pi nx) \cos(\pi nx),
$$

where

$$
c_n = 8 \int_0^1 \phi(y) \sin(\pi ny) \cos(\pi ny) dy.
$$

**Proof.** A computation gives us

$$
\int_0^1 \sin(\pi ny) \cos(\pi ny) \sin(\pi my) \cos(\pi my) dy = \frac{1}{16\pi} \left( \frac{\sin(2\pi(m-n))}{m-n} - \frac{\sin(2\pi(m+n))}{m+n} \right)
$$

$$
= \begin{cases} 
\frac{1}{8}, & \text{if } n = m, \\
0, & \text{if } n \neq m.
\end{cases}
$$

Hence, provided $\phi(x)$ satisfies the hypothesis of the Lemma, we find the result follows. □

We also require a formula to evaluate integrals involving the fractional function.
Lemma 3.2. ([15, pg.13] ) Suppose $\phi(x)$ has a continuous derivative in $[a, b]$. Then we have,

\begin{equation}
\sum_{a < n \leq b} \phi(n) = \int_a^b \phi(y)dy + \int_a^b (\{y\} - \frac{1}{2})\phi'(y)dy + ((a) - \frac{1}{2})\phi(a) - ((b) - \frac{1}{2})\phi(b).
\end{equation}

Noting that $\sin(2x) = 2 \sin(x) \cos(x)$, it follows that we should have the following Davenport expansion.

**Theorem 3.3.** We have,

\[ \sum_{n \geq 1} \frac{f(n)}{n} (\{nx\} - \frac{1}{2}) = 2 \sum_{n \geq 1} c_n \sin(\pi nx) \cos(\pi nx), \]

where

\[ c_n = -\frac{1}{n\pi} \sum_{d|n} f(d). \]

**Proof.** In Lemma 3.2, we put $a = 0$, $b = N$, and set $\phi(x) = \sin(\frac{2\pi x}{N})^2$. In this case we have

\begin{equation}
\sum_{0 < n \leq N} \sin(\frac{\pi nm}{N})^2 = N^2/2 + 2\pi \int_0^N (\{y\} - \frac{1}{2})\sin(\frac{\pi ym}{N})\cos(\frac{\pi ym}{N})dy
\end{equation}

\[ = \frac{N}{2} + 2\pi \int_0^1 (\{yN\} - \frac{1}{2})\sin(\pi ym)\cos(\pi ym)dy \]

The sum on the left side of (3.2) may be evaluated in the same way as (2.3) to find that

\begin{equation}
\sum_{0 < n \leq N} \sin(\frac{\pi nm}{N})^2 = \begin{cases} 
0, & \text{if } m \equiv 0 \pmod{N}, \\
\frac{N}{4}, & \text{otherwise}.
\end{cases}
\end{equation}

Collecting (3.2) and (3.3), it follows that

\begin{equation}
\int_0^1 (\{yN\} - \frac{1}{2})\sin(\pi ym)\cos(\pi ym)dy = \begin{cases} 
-\frac{N}{4\pi m}, & \text{if } m \equiv 0 \pmod{N}, \\
0, & \text{otherwise}.
\end{cases}
\end{equation}

Summing over the desired series in (3.4) for $m$ and $N$ gives the result. \qed
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