Nonlinear Reconstruction of the Velocity Field

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Abstract

We propose a new velocity reconstruction method based on the displacement estimation of recently developed methods. The velocity is first reconstructed by transfer functions in Lagrangian space and then mapped into Eulerian space. High-resolution simulations are used to test the performance. We find that the new reconstruction method outperforms the standard velocity reconstruction in the sense of better cross-correlation coefficient, less velocity misalignment, and smaller amplitude difference. We conclude that this new method has the potential to improve large-scale structure sciences involving velocity reconstruction, such as kinetic Sunyaev–Zel’dovich measurement and supernova cosmology.

Unified Astronomy Thesaurus concepts: Large-scale structure of the universe (902); Cosmology (343)

1. Introduction

Reconstructing the velocity field from density is a non-trivial procedure due to the high nonlinearity in the evolved density field and the non-local relationship with the velocity field. Also, usually we only have discrete and biased tracers such as galaxies in the observation, which suffer from density bias, shot noise, and stochasticity. Early attempts, e.g., Nusser & Davis (1994), Fisher et al. (1995), etc., aimed to estimate the peculiar velocities from the pioneering galaxy surveys. These results improved our knowledge of the Local Universe.

Discovering the accelerating expansion of our universe using supernova observations is a triumph of modern cosmology. This discovery utilizes the luminosity distance and redshift relation to constrain the cosmological models. The Doppler effect by the peculiar velocity is one of the systematics in supernova cosmology. Neglecting correlated peculiar velocities can cause an error in the best-fit value of the dark energy equation of state and also an overestimate of the precision of the measurement (Cooray & Caldwell 2006; Hui & Greene 2006; Davis et al. 2011). The low-redshift cutoff is usually applied in order to avoid this systematics. For future supernova surveys attempting statistical error bars of less than about 2%, it is important to correct the peculiar velocity. The reconstructed velocity field is also helpful for reducing the Hubble constant measurement uncertainty from standard sirens (Mukherjee et al. 2019).

The kinetic Sunyaev–Zel’dovich (kSZ) effect offers a unique opportunity to characterize the cosmic peculiar velocity field in the distant universe, and to search for the “missing baryons.” The kSZ measurement benefits from the velocity estimates for avoiding the cancellation of equally likely positive and negative kSZ signals (DeDeo et al. 2005; Ho et al. 2009; Shao et al. 2011; Li et al. 2014; Smith et al. 2018). The significance level depends on the velocity estimation/reconstruction performance. In recent years, works have detected and measured the kSZ signal. Hand et al. (2012) first reported the detection of the kSZ signal by applying the pairwise kSZ estimator to ACT cosmic microwave background (CMB) data using a galaxy catalog from the Sloan Digital Sky Survey (SDSS) III DR9. This measurement was achieved with higher precision using the Baryon Oscillation Spectroscopic Survey (BOSS) DR11 catalog (De Bernardis et al. 2017). With the Planck CMB map, Planck Collaboration et al. (2016) reported a kSZ detection using the Central Galaxy Catalog extracted from SDSS DR7 and Li et al. (2018) presented the measurement using BOSS data. Most of the velocity reconstruction methods used in the above literature are motivated by the linearized continuity equation. One can solve for the velocity field from the observed density field with some preprocessing such as debiasing, smoothing, redshift-space distortion (RSD) correction, and Gaussianization. The reconstruction is performed in Eulerian space and only the irrotational part is reconstructed by design. For future CMB-S4 surveys and Stage-IV galaxy surveys, remote dipole and quadrupole reconstruction from the kSZ effect will benefit from a precisely reconstructed velocity field (Cayuso et al. 2018; Deutsch et al. 2018; McCarthy & Johnson 2019; Pan & Johnson 2019).

Some physical quantities in Lagrangian space suffer less nonlinear effects and they provide alternative angles to study the behavior of our universe. The evolved density field is highly nonlinear in Eulerian space. Part of the nonlinearity comes from the large-scale bulk motion, which dominates the displacement field in Lagrangian picture. The removal of the bulk motion, i.e., the density reconstruction, recovers linear information and sharpens the baryon acoustic oscillation peaks (e.g., Eisenstein et al. 2007). The velocity field is also dominated by the large-scale bulk flow. Thus, the displacement and the velocity field are expected to correlate well in Lagrangian space. By investigating their relation in Lagrangian space, one can develop new velocity reconstruction methods. Exploring how to reconstruct the velocity in Lagrangian space is worthwhile given that the displacement can be estimated from the nonlinear density field.

Reconstruction of the early state of our universe has a long history. Early achievements were accomplished by finding the least-action solution, fast action method, solving the Monge–Ampère equation (e.g., Peebles 1989; Croft & Gaztanaga 1997; Branchini et al. 2002; Brenier et al. 2003), etc. However, the performance was limited by the calculating ability. Lavaux (2008), Lavaux et al. (2008) converted the reconstructed displacement to the velocity field with the linear relation, \( v = \beta \Psi \), with the linear growth factor \( \beta \approx \Omega_m^{5/3} \).
Recently, new algorithms were proposed to reconstruct the initial condition from the highly nonlinear density map, which improves the signal-to-noise in the measurement of the baryon acoustic oscillation sound horizon scale (Schmittfull et al. 2017; Zhu et al. 2017; Hada & Eisenstein 2018; Shi et al. 2018; Sarpa et al. 2019). The performances of biased tracers such as the simulated halos/HOD galaxy samples are tested in Yu et al. (2017), Birkin et al. (2019), and Hada & Eisenstein (2019). Despite the different theoretical motivations and operational procedures in these backward modeling studies, the key to improvement is the same: a better estimate of the nonlinear displacement. Note that the initial condition/displacement could also be obtained from the forward modeling methods such as Hamiltonian Markov Chain Monte Carlo method (Wang et al. 2014), optimization with a Bayesian approach (Seljak et al. 2017; Modi et al. 2018; Jasche & Lavaux 2019; Schmidt et al. 2019), etc.

Note that the reconstruction of the displacement inspires many potential applications. The reconstructed displacement is an effective displacement that ensures the correct clustering but ignores some complicated processes like shell-crossing. Understanding the reconstructed displacement could help us develop a fast mock generation method. Given the nonlinear density field with an RSD effect, the reconstructed one also contains RSD information and this may improve the RSD modeling since the RSD is more linear after the reconstruction (Zhu et al. 2018). The reconstructed displacement is also useful to measure the relative velocity of the neutrino to DM, which contains important information on the neutrino mass (Zhu & Castorina 2019). The reconstructed displacement also helps with moving the observable in Eulerian space back to its Lagrangian position where it is more physically positioned (such as the angular momentum of the galaxy, Yu et al. 2019).

This paper is an investigation of velocity reconstruction using recently proposed displacement reconstruction methods. In Section 2, the theoretical bases of the standard velocity reconstruction method and the new proposed velocity reconstruction are introduced, and the algorithm is presented. In Section 4, we present the performance of the new velocity reconstruction. Section 6 summarizes the results and provides a discussion. Additional dimensions of observations, such as the shot noise and stochasticity for biased tracers, survey mask and RSD effect, are out of the scope of this paper and will be addressed in future investigations.

2. Motivation

2.1. Standard Velocity Reconstruction

The standard velocity reconstruction method adopts the linearized continuity equation,

$$\frac{\partial \delta(x)}{\partial t} + \nabla \cdot v(x) = 0,$$

(1)

to convert the density maps into velocity maps. The reconstructed velocity field is obtained by the relation in Fourier space:

$$v_r(k) = \frac{aH}{k^2} \frac{ik \delta_L(k)}{b},$$

(2)

in which Gaussian smoothing is usually adopted to reduce the impact from the highly non-Gaussian region, linearizes the field, and galaxy bias is corrected. The prefactor $aH$ comes from the linear theory. $H$ is the Hubble parameter, $f = d \ln D/d \ln a$, and $D$ is the linear growth rate. Throughout the paper, we denote the reconstructed velocity with a subscript $r$ and the true velocity field is labeled with a subscript $t$.

For the purposes of velocity reconstruction, widely adopted Gaussian smoothing might not be optimal. There exist other schemes trying to achieve better performance. For example, one can linearize the density field by a logarithmic transform, and/or obtain the velocity field using second-order Lagrangian perturbation theory (see Planck Collaboration et al. (2016) for a detailed velocity reconstruction comparison for the purposes of kSZ measurement).

Here we extend the standard velocity reconstruction formalism to use the transfer function to ensure that the process is optimal:

$$v_r(k) = \frac{ik}{k^2} T(k) \delta(k).$$

(3)

The transfer function is defined to minimize the error in the reconstruction, and is calibrated from the simulation. Hereafter, we refer to this process as the standard reconstruction.

2.2. Reconstruction in Lagrangian Space

In the Lagrangian scenario, the motion of the fluid element is labeled by its original position $q$ and specified by the displacement $\Psi(q, t) = x(t) - q$ at time $t$. Lagrangian perturbation theory attempts to model the nonlinear displacement in a perturbative way,

$$\Psi = \Psi^{(1)} + \Psi^{(2)} + \ldots,$$

(4)

in which each term collects the contribution from the same order,

$$\Psi^{(n)}(k) = \frac{iD^n}{n!} \int \frac{d^3k_1}{(2\pi)^3} \ldots \frac{d^3k_n}{(2\pi)^3} \delta_D \left( \sum_{j=1}^n k_j - k \right) \times L^{(n)}(k_1, \ldots, k_n) \delta_L(k_1) \ldots \delta_L(k_n),$$

(5)

with $k = k_1 + \ldots + k_n$. The first-order and second-order kernels are given by

$$L^{(1)}(k_1) = \frac{k_1}{k},$$

(6)

$$L^{(2)}(k_1, k_2) = \frac{3}{7} \frac{k}{k_1 k_2} \left[ 1 - \frac{\left( k_1 \cdot k_2 \right)^2}{k_1 k_2} \right].$$

(7)

The displacement divergence $\delta_\psi = -\nabla \cdot \Psi$ is given by

$$\delta_\psi(k) = \delta^{(1)}(k) + \delta^{(2)}(k) + \ldots,$$

(8)

where the first order is just the linear density field, $\delta^{(1)}(k) = \delta_L(k)$, and

$$\delta^{(2)}(k) = \frac{1}{7} \int \frac{d^3k_1}{(2\pi)^3} \frac{d^3k_2}{(2\pi)^3} \delta_D(k_1 + k_2 - k) \times \left\{ 1 - \frac{3}{2} \left( \frac{k_1 \cdot k_2}{k_1 k_2} \right)^2 - \frac{1}{3} \right\} \delta_L(k_1) \delta_L(k_2),$$

(9)

or equivalently in configuration space,

$$\delta^{(2)}(q) = \frac{1}{7} \delta^2(q) - \frac{1}{7} q^2(q).$$

(10)
The tidal term \( K^2(q) \) is given by the contraction of the tidal tensor,
\[
K^2(q) = \frac{3}{2} K_{ij}(q) K_{ij}(q),
\]
(11)
where
\[
K_{ij}(k) = \left( \frac{k_i k_j}{k^2} - \frac{1}{3} \delta_{ij} \right) \delta_L(k).
\]
(12)

Note that both \( \Psi(q) \) and \( \delta_{ij}(q) \) are in Lagrangian configuration space.

The velocity is the time derivative of the displacement. Thus, the Lagrangian velocity is the summation of the contributions from all orders,
\[
v(q) = a \Psi(q) = a f H \Psi^{(1)} + 2 a f H \Psi^{(2)} + \cdots,
\]
in which the prefactors in each term come from the linear theory for \( v \) and \( \Psi \). In the general case, the velocity divergence is related to densities by a series of transfer functions at each order,
\[
\theta(k) = T_1(k) \delta_{L}(k) + T_2(k) \delta^{(2)}(k) + \cdots.
\]
(14)

The basic idea is that once the linear density field is estimated by the reconstruction algorithm from the observed density field, one can use Equation (14) to obtain the Lagrangian velocity field and further map it into Eulerian space. Note that the reconstruction of the linear density is not perfect. It contains noise and non-Gaussianity induced by the reconstruction algorithm. When high-order terms in Equation (14) are adopted, we need to further process (described below) the reconstructed linear density field to calculate the high-order terms.

### 3. Implementation

#### 3.1. Simulation Setup

To test and compare the velocity reconstruction methods, we use a high-resolution simulation involving 2048\(^3\) dark matter particles in a box with a side length of 600 \( h^{-1} \) Mpc. It is run by the particle–particle–particle-mesh N-body simulation code CUBEP3M (see Harnois-Déraps et al. 2013). The cosmic velocity field has a large correlation scale, typically \( \sim 150 h^{-1} \) Mpc. This simulation box size is insufficient for robust large-scale velocity statistics measurement. However, the following results are mainly based on the cross-correlation analysis. Due to the cancellation of the sample variance, it is sufficient to obtain reliable results. The reconstruction and analysis are performed on 512\(^3\) grids. We assign particle velocity onto uniform grids by the nearest particle (NP) method. Sampling artifacts in the E-mode power spectrum measurement associated with the NP assignment can be neglected in this configuration since the number density of \( \sim 1(h^{-1} \) Mpc\(^{-3}\)) is sufficiently high (see Zhang et al. 2015; Zheng et al. 2015).

#### 3.2. Reconstruction Algorithm

We use three recently developed nonlinear reconstruction algorithms in this work. They are described in Zhu et al. (2017), Shi et al. (2016), and Schmittfull et al. (2017). We denote them as A1, A2, and A3, respectively. They all provide

the reconstructed density field, which has significantly improved correlation with the linear initial condition. Although the performance in recovering the cross-correlation coefficient is similar, these three independently developed procedures produce different behaviors in the reconstructed density field. This leads to slightly different performances for the velocity reconstruction. In the main result below we only show the result from A2 and present the difference in the Appendix.

Once we obtain the reconstructed density field, based on Equation (14), we propose the direct velocity reconstruction in Lagrangian space only using the first-order term (\( O(1) \) reconstruction),
\[
\theta_r(k) = T_1(k) \delta_r(k).
\]
(15)

Here \( \delta_r \) is the reconstructed linear density field and the transfer function is defined as
\[
T_1(k) = \frac{\langle \delta_{L}(k) \rangle}{\delta_r(k)}.
\]
(16)

It is calibrated by the reconstructed density \( \delta_r(q) \) and the true Lagrangian velocity divergence \( \theta_r(q) \) in simulation.

The reconstructed displacement contains nonlinear information. The high-order terms may contain useful information for reconstructing the nonlinear velocity field. Similar to Schmittfull et al. (2017), we propose the \( O(2) \) reconstruction by taking the second term in Equation (14) into further consideration.

First, for the estimated linear density field \( \delta_r \), we use a Wiener filter to remove the spurious power induced by the reconstruction algorithm:
\[
W_W(k) = \frac{\langle \delta_{L}(k) \delta_r(k) \rangle}{\langle \delta_r(k) \rangle},
\]
\[
\delta^{(1)}_r(k) = \delta_r(k) W_{W}(k).
\]
(17)

Note that the reconstructed density fields from different algorithms have different noise powers, and thus different Wiener filters. For a given algorithm, the corresponding Wiener filter should be calibrated from the mocks. The uncertainties in the mock construction may induce uncertainties in this calibration. However, in Wiener filtering the transit from the signal-dominating area to the noise-dominating area is rapid. We do not expect the uncertainties in the Wiener filtering to significantly influence the performance.

The second-order term \( \delta^{(2)}_r \) is calculated as Equation (10) by replacing \( \delta_L \) with \( \delta^{(1)}_r \). However, due to the residual non-Gaussianity in \( \delta^{(1)}_r \), \( \langle \delta^{(1)}_r \delta^{(2)}_r \rangle \) is non-zero. Thus, we cannot directly perform second-order reconstruction based on \( \delta^{(2)}_r \); otherwise the reconstructed \( \theta^{(1)}_r \) and \( \theta^{(2)}_r \) are not independent. We use orthogonization technique (Schmittfull et al. 2019) to remove the correlated part in \( \delta^{(2)}_r \) and construct \( \hat{\delta}^{(2)}_r \) which has no correlation with \( \delta^{(1)}_r \),
\[
W_L(k) = \frac{\langle \delta^{(2)}_r \delta^{(1)}_r \rangle}{\langle \delta^{(1)}_r \delta^{(1)}_r \rangle},
\]
\[
\hat{\delta}^{(2)}_r(k) = \delta^{(2)}_r(k) - \delta^{(1)}_r(k) W_L(k).
\]
(18)
leads to an almost perfect correlation at large scales. At small scales, both the velocity and the displacement suffer from nonlinear effects. We expect the influence to be more severe in the velocity field. These nonlinear effects change the small-scale velocity substantially and cause loss of correlation with displacement.

We also plot the cross-correlation between the second-order term $\delta^{(2)}(q)$ and the residual velocity divergence $\theta_m(q) = \theta_r(q) - \theta^{(1)}(q)$ as a blue dotted line. We also find a significant correlation, $r \approx 0.8$ at 0.02 $h$ Mpc$^{-1} < k < 0.2$ Mpc$^{-1}$, implying that $O(2)$ reconstruction could help. However, how much this second-order term improves the result also depends on the power relative to the first-order term. The result for the $O(2)$ reconstruction is shown as a black solid line. A slight improvement is seen at $k < 0.15$ h Mpc$^{-1}$ and $k > 0.4$ h Mpc$^{-1}$.

In the bottom panel we plot the stochasticity, defined as $S = \sqrt{2(1-r)}$. This statistic amplifies the difference when $r \approx 1$. The $O(1)$ reconstruction result is plotted as a red dashed line and the $O(2)$ reconstruction is a black solid line. From this panel we see that $O(2)$ reconstruction has a lower stochasticity than the standard method by a factor of $\sim 3$ at 0.02 h Mpc$^{-1} < k < 0.1$ h Mpc$^{-1}$. This suppression of stochasticity at large scales is important for measuring large-scale effects such as the primordial non-Gaussianity by the sample variance cancellation technique (Münchmeyer et al. 2019).

4.2. Eulerian Velocity

After mapping the reconstructed Lagrangian velocity to Eulerian space by the displacement, we use the NP velocity assignment scheme to obtain the reconstructed velocity field. Note that the true velocity field is obtained by the same NP assignment. The cross-correlation coefficients between the two are presented in Figure 2. The red dashed line, black solid line, and blue dotted line represent the results for $O(1)$ reconstruction, $O(2)$ reconstruction, and the standard reconstruction, respectively. We find that the proposed $O(1)$ reconstruction method performs better than the standard one at scales $k > 0.1$ h Mpc$^{-1}$, and the $O(2)$ reconstruction slightly improves the cross-correlation coefficient at scales $0.1$ h Mpc$^{-1} < k < 1$ h Mpc$^{-1}$.

In the bottom panel of Figure 2 we show the stochasticity for the above reconstruction methods. Adding $O(2)$ reconstruction suppresses the stochasticity at scales $k < 1$ h Mpc$^{-1}$ relative to the $O(1)$ reconstruction, and it performs better than the standard reconstruction method at scales $0.02$ h Mpc$^{-1} < k < 1$ h Mpc$^{-1}$.

One obvious feature is that the cross-correlation coefficient decreases toward small scales for $k < 1$ h Mpc$^{-1}$ but increases at $k > 1$ h Mpc$^{-1}$. However, the result at $k > 1$ h Mpc$^{-1}$ is suspicious due to the fact that this scale is close to the Nyquist frequency of the analysis and this good correlation between the reconstructed and true velocity may partially come from the same systematics by the same velocity assignment.

We compare the reconstructed velocity with the true one at each grid point in Figure 3, which shows the two-dimensional histogram, with the horizontal axis being the reconstructed velocity while the vertical axis is the true velocity. The color indicates the relative counts normalized to unity. The upper panel shows the result from $O(2)$ reconstruction and the lower panel shows the result from the standard method. Compared to the standard reconstruction, we observe an obvious slimmer contour for the $O(2)$ reconstruction.
To quantify the performance, we check the direction and the amplitude of the reconstructed velocity. We define the cosine angle between it and the true one as

$$\mu = \frac{v_r \cdot v_t}{|v_r||v_t|}$$

We plot this cosine angle for the $O(2)$ reconstruction and the standard one of one slice in the middle and bottom panels of Figure 4, respectively. Also plotted is the dark matter density of the same slice in the top panel. Both reconstruction methods perform worse in the high-density region, i.e., the highly nonlinear region. We find that for the $O(2)$ reconstruction, the region with $\mu < 0.95$ (green to blue color) occupies far less volume than the standard reconstruction result, indicating that the reconstruction performs well down to the nonlinear region.

The mean of $\mu$ for the A2 $O(2)$ reconstruction is 0.977, while the standard method has the mean $\langle \mu \rangle = 0.958$. This corresponds to the mean misalignment angle of 12°31 and 16°66 for the A2 $O(2)$ reconstruction and the standard one, respectively. We also plot the histogram for the cosine angle $\mu$ in Figure 5. The 0(2) reconstruction (blue histogram) has much more pixels with very good direction reconstruction ($\mu > 0.995$) than the standard method (red histogram).

We also check whether the amplitude of the velocity is reconstructed well. Here we define three kinds of velocity amplitude difference. The first one is the difference between the true velocity amplitude and the reconstructed one. The second is the difference between the true velocity and the projection of the reconstructed velocity on the true one, i.e., $v_p = v_r \cdot v_t / |v_t|$. The last one is the difference between the velocity component in the z-direction.

The results are shown in Figure 6, in which the distribution of the velocity amplitude difference is plotted. The top plot is the result from the $O(2)$ reconstruction, while the bottom plot is from the standard one. For the first and second distributions (blue and red histogram), the $O(2)$ reconstruction result has a peak closer to zero than the standard reconstruction, i.e., a smaller reconstruction bias in the amplitude. Furthermore, the $O(2)$ reconstruction also has smaller scatters in these distributions than the standard one. For the last statistics, the velocity difference in one direction, the standard method is expected to produce a mean of zero by design. For the $O(2)$ reconstruction, we also find this statistics has a mean of zero, and the width of the distribution is much narrower than the standard one. All the above statistics show that the $O(2)$ reconstruction recovers the velocity amplitude relatively better than the other reconstruction methods.

**5. Ramifications**

5.1. Using Simulated Displacement

The above results slightly depend on the performance of the detailed reconstruction algorithm. We compare the performance of the three different algorithms in the Appendix. Here we want to know the upper limit of the velocity reconstruction
Thus, in this subsection we present the result under the assumption that the displacement estimation is perfect, i.e., the reconstruction is performed using the simulated displacement instead of the reconstructed one. The cross-correlation coefficient in Lagrangian space is shown in the top panel of Figure 7. The red dashed line is the result from \( \mathcal{O}(1) \) reconstruction and the black solid line is for \( \mathcal{O}(2) \) reconstruction. In this case we see that the improvement from including the second-order term is very mild. The nonlinear displacement already includes almost all the information that can be used to reconstruct the nonlinear velocity field.

In the bottom panel we show the cross-correlation in Eulerian space. The performance is better than the results using reconstructed displacement. The cross-correlation coefficient between the reconstructed velocity and the true one reach 0.7 at \( k \sim 1 \, h\,\text{Mpc}^{-1} \) (black solid line). No obvious difference between \( \mathcal{O}(1) \) and \( \mathcal{O}(2) \) reconstruction is observed and the black solid and black dashed lines overlap with each other. We further decompose both the true and the reconstructed velocity field into curl-free E-mode and divergence-free B-mode and correlate them separately. The cross-correlation for the E-mode is slightly lower than the total velocity at scale \( k \sim 1 \, h\,\text{Mpc}^{-1} \). The B-mode cross-correlation coefficient is \( \sim 0.2 \) at \( k < 0.1 \, h\,\text{Mpc}^{-1} \) and rapidly increases to \( \sim 0.8 \) at \( k \sim 0.7 \, h\,\text{Mpc}^{-1} \). Note that the observed B-mode suffers from severe systematics in this configuration (refer to Figure 12 in Zheng et al. 2013) and the power spectrum changes significantly as the simulation configuration changes. We predict that the observed cross-correlation for the B-mode between the true velocity and the reconstructed one mainly comes from the systematics induced by the velocity assignment method, the finite volume effect, and the aliasing effect.

5.2. Considering the B-mode in Lagrangian Space

In the above subsection, the reconstructed B-mode in Eulerian space comes from the nonlinear mapping and is converted from the E-mode in Lagrangian space. Due to the nonlinear evolution of the universe, both the velocity and the displacement have a B-mode component (see e.g., Chan 2014) both in Eulerian and Lagrangian space. Here we are curious about whether the Lagrangian space B-mode correlation helps in the reconstruction. The cross-correlations between the two

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**Figure 4.** Top: a slice of DM density field \( \log(1 + \delta) \). Middle: a slice of the cosine angle between \( v_t \) and \( v_r \) reconstructed by the \( \mathcal{O}(2) \) reconstruction. Bottom: a slice of the cosine angle between \( v_t \) and \( v_r \) reconstructed by the standard method. Large misalignment appears in the high-density region.

**Figure 5.** Distribution of the cosine angle between \( v_t \) and \( \|v_t\|v_r \). The red histogram is for standard reconstruction, while the blue one is for the \( \mathcal{O}(2) \) reconstruction. The \( \mathcal{O}(2) \) reconstruction has much more pixels with \( \mu > 0.993 \). The average cosine angle is 0.9577 for standard reconstruction, corresponding to misalignment of 16.72. For nonlinear reconstruction it is 0.9790 and 11.77.
from simulations are shown as the red dashed line in the upper left panel of Figure 8. Also plotted is the cross-correlation from the E-mode as a black solid line. At large scales, the B-mode velocity and B-mode displacement also have a large cross-correlation coefficient, \( \sim 0.9 \). This B-mode correlation could help in the velocity reconstruction. This also implies that the Cartesian components of the velocity and displacement field contain extra correlations other than the divergence of the two. In the upper right panel we show the cross-correlation coefficient between the true velocity and the velocity reconstructed using the transfer function measured only from the E-mode, \( \langle \theta \rangle = \langle T^E(k) \delta_i(k) \rangle \): (2) using the transfer function measured from the Cartesian components, \( \langle v_i(k) \rangle = T_i(k) \Psi_i(k) \), and \( i \) runs for 3 Cartesian components: and (3) using the transfer functions for the E- and B-modes separately and summing the two reconstructed velocity fields, \( \langle \theta \rangle = T^E(k) \delta_i(k) \) and \( \langle v_i(k) \rangle = T^B(k) \Psi_i^B(k) \).

Adding the B-mode improves the velocity reconstruction in Lagrangian space. Reconstruction from the Cartesian components also improves the reconstruction, but it performs worse than directly adding B-mode reconstruction since it neglects the correlation between different Cartesian components.

However, this improvement in Lagrangian space is mainly at scales \( k > 0.2 \ h \ Mpc^{-1} \) and is not mapped into Eulerian space. In the bottom left panel of Figure 8 we present the cross-correlation coefficient for the above three cases and for the total, E-mode, and B-mode components separately. We find that including the B-mode information in Lagrangian space leads to worse performance in Eulerian space. We argue that the noise in the B-mode displacement (the part not correlated with the velocity) is converted into E-mode velocity in Eulerian space, thus contaminating the velocity reconstruction instead of improving it. The bottom right panel shows the E- and B-mode power spectrum of the reconstructed velocity field and the true one. Including the B-mode correlation in Lagrangian space does not change the E-mode power spectrum in Eulerian space much, but the Eulerian B-mode power spectrum is changed significantly. This also implies that the B-mode is mainly noise.

The A1 and A2 reconstruction algorithm has no B-mode displacement by design, but the A3 reconstruction algorithm does. In the left panel of Figure 9, we find that this estimated B-mode displacement has a weaker correlation \( (r \sim 0.5) \) with the true B-mode velocity compared with the case using the real
simulated displacement. Thus, it suffers from more severe noise than the previous case. It is expected that the reconstruction by adding the B-mode in Lagrangian space performs worse in Eulerian space. This is observed in the right panel of Figure 9.

5.3. Using Linear Displacement

For the purpose of mock construction, it is straightforward to start with a linear density field. Here we test the performance of the velocity reconstruction using the linear density field. Combined with fast density map generation methods such as 1LPT, 2LPT, or other techniques, we can obtain mocks with both good density and velocity field. These synthetic mocks are of great importance for future surveys.

The process is roughly the same as the reconstruction with simulated displacement. We just use the linear density field to replace the simulated displacement divergence in the reconstruction algorithm. Since the linear density is Gaussian, no orthogonization is needed.

The transfer function in this case is measured by

\[
T_1(k) = \frac{\langle \delta_L \theta_{1m} \rangle}{\langle \delta_L \delta_L \rangle},
\]

\[
\theta_{1m}(k) = T_1(k) \delta_L(k).
\]  

\[\text{(23)}\]

We could also perform \(\mathcal{O}(2)\) reconstruction by further measuring the second transfer function from the second-order LPT density field \(\delta^{(2)}\) and the residual velocity field \(\theta_m = \theta_t - \theta_{1m}\):

\[
T_2(k) = \frac{\langle \delta^{(2)} \theta_{2m} \rangle}{\langle \delta^{(2)} \delta^{(2)} \rangle},
\]

\[
\theta_{2m}(k) = T_2(k) \delta^{(2)}(k).
\]  

\[\text{(24)}\]

In this case, the \(\mathcal{O}(2)\) reconstruction (i.e., 2LPT) captures more nonlinear velocity information than the \(\mathcal{O}(1)\) reconstruction (i.e., 1LPT). This is observed in the top panel of Figure 10. \(\mathcal{O}(2)\) reconstruction increases the cross-correlation coefficient at \(k \sim 0.1 \text{ h Mpc}^{-1}\) and \(k \sim 1 \text{ h Mpc}^{-1}\) in Lagrangian space.

In the bottom panel, we compare the velocity cross-correlation coefficient in Eulerian space with the reconstruction using the linear displacement (black lines), using the simulated nonlinear displacement (red lines), and using the reconstructed displacement (blue lines). The \(\mathcal{O}(1)\) reconstruction is presented as a dashed line and the \(\mathcal{O}(2)\) reconstruction is a solid line. From this plot we clearly see that using the nonlinear or reconstructed displacement has produces a better
performance than the case using linear displacement. We also note that the velocity reconstruction performance using the reconstructed displacement catches the upper limit down to scale \( k \sim 0.7 \, h \, \text{Mpc}^{-1} \).

5.4. Improvement by Real Space Transfer Functions

According to the result in Section 5.2, we should only use the E-mode displacement in the reconstruction, and the reconstructed velocity field is irrotational in Lagrangian space by design. However, after mapping to Eulerian space, a part of the E-mode is converted into B-mode. To obtain a better velocity reconstruction in real space, we could measure two more transfer functions in Eulerian space to adjust the reconstructed E- and B-mode component:

\[
T^E(k) = \frac{\langle \dot{v}_E \dot{v}_E \rangle}{\langle \dot{v}_E \dot{v}_E \rangle},
\]

\[
T^B(k) = \frac{\langle \dot{v}_B \dot{v}_B \rangle}{\langle \dot{v}_B \dot{v}_B \rangle},
\]

and \( \dot{v}_r(k) = T^E(k) \dot{v}_E(k) + T^B(k) \dot{v}_B(k) \). The result is presented in Figure 11. A very mild improvement is observed in the cross-correlation coefficient at \( k \sim 0.5 \, h \, \text{Mpc}^{-1} \).
Considering that the improvement is negligible and this process may induce noise instead of improving the performance if the transfer function is not sufficiently accurate, we do not propose applying this final step in the reconstruction.

6. Conclusion and Discussion

We propose a new velocity reconstruction method based on the estimated displacement field from the nonlinear density maps by recently developed algorithms. The reconstruction is first performed in Lagrangian space by calibrated transfer functions, then the Eulerian velocity is obtained by the mapping. We found that this new velocity reconstruction has a better performance than the standard reconstruction method based on the linearized continuity equation. It produces a velocity field with a better cross-correlation coefficient, less velocity misalignment, and smaller amplitude difference with the true one. Generally, $O(2)$ reconstruction outperforms $O(1)$ reconstruction by taking use of the velocity information residing in the high-order terms. A summary of the statistics we investigated is presented in Table 1.

We explored several extensions. One extension is to consider the correlation between the B-mode component of the velocity and displacement. The other one adopts two more transfer functions in real space to adjust the reconstructed E-mode to the B-mode components in Eulerian space. However, the performance is not improved or very mild. Thus, it complicates the process and is not paid off.

We also explored the upper limit of this new reconstruction method by assuming the displacement is perfectly reconstructed. We found that in this case the difference between $O(1)$ and $O(2)$ is very small. One surprising point is that the $O(2)$ reconstruction performance from A2 is very close to this upper limit in the sense of the cross-correlation coefficient and the misalignment angle. We also attempted to obtain the velocity field from the linear displacement with the same approach. This demonstrates the limit of only using the 1LPT and 2LPT displacements and the transfer functions with calibration.

The reconstruction performance from a biased tracer depends on the understanding of the bias. Wang & Pen (2019) found that the acoustic peaks are recovered best when the linear bias is correctly removed, and thus it is possible to obtain an estimation of the bias in the process of the reconstruction. For the low-density sample with only massive halos, correction of the bias is important. Otherwise, the overestimation of the displacement amplitude significantly degrades the linear density reconstruction. For the high-density sample with bias less than unity, the bias does not influence the results much (e.g., Birkin et al. 2019). We leave the quantification of the velocity reconstruction performance from the biased tracer to future investigations.

With the reconstructed initial condition of some volume of the universe, one could perform the simulation to obtain the velocity field (e.g., Lavaux et al. 2008). However, the reconstruction induces noise and non-Gaussianity in the linear density field. A comparison between the constrained simulation and the original one has not been performed for the reconstruction algorithms investigated in this work. We also leave comparison of the density and the velocity between the constraint simulation and the true one for the future.

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### Appendix

Comparison of A1, A2, and A3

In this work we adopted the three algorithms A1, A2, and A3. They are described in Zhu et al. (2017), Shi et al. (2018), Schmittfull et al. (2017), respectively. These tests have not been performed in a systematical way before and a brief comparison is presented here. We use the default parameters proposed in the literature since the simulation and analysis configuration is similar. The result of the reconstructed Lagrangian velocity is presented in Figure 12. The solid red, dashed green, and dotted blue lines are for A1, A2, and A3, respectively. The left panel is the cross-correlation coefficient of the reconstructed and true velocities in Lagrangian space, while the right panel is for Eulerian space. We find A2 has the best cross-correlation coefficient in Lagrangian space. The good performance is also mapped into Eulerian space, leading to the best cross-correlation coefficient for Eulerian velocity. However, A1 has a similar but slightly worse performance to A2 in Lagrangian space, and has a similar performance with A3 in Eulerian space. A summary is presented in Table 1.

### Table 1

Summary of the Performance from Different Reconstruction Methods, Including the Standard Reconstruction, the $O(1)$ and $O(2)$ Nonlinear Reconstructions by Three Recently Developed Algorithms, the Reconstruction Using the Simulated Displacement, and with the Linear Density Field

| Method                      | $\mu$ | $\Delta v$ | $\Delta v$ | $v_x - v_p$ | $v_y - v_p$ | $\Delta v_1$ | $\Delta v_2$ |
|-----------------------------|-------|------------|------------|-------------|-------------|-------------|-------------|
|                             |       | Mean rms   | Mean rms   | Mean rms    | Mean rms    | Mean rms    | Mean rms    |
| Standard                    | 0.958 | 22.3 71.5  | 33.6 75.8  | 0.02 72.5   |             |             |             |
| A1 $O(1)$                   | 0.965 | −6.34 71.8 | 3.25 77.9  | −17.6 62.3  |             |             |             |
| A1 $O(2)$                   | 0.968 | 10.2 64.2  | 18.7 69.6  | −16.1 57.8  |             |             |             |
| A2 $O(1)$                   | 0.971 | 5.10 67.3  | 13.4 72.9  | 0.08 61.4   |             |             |             |
| A2 $O(2)$                   | 0.977 | 23.5 57.5  | 29.9 62.7  | 0.07 55.4   |             |             |             |
| A3 $O(1)$                   | 0.970 | 5.71 70.7  | 14.2 78.2  | 9.19 63.7   |             |             |             |
| A3 $O(2)$                   | 0.972 | 19.9 63.9  | 27.7 70.5  | 9.75 60.7   |             |             |             |
| Simulated $O(1)$            | 0.978 | 10.9 62.1  | 17.4 69.3  | 0.04 56.5   |             |             |             |
| Simulated $O(2)$            | 0.979 | 17.2 56.0  | 23.3 63.3  | 0.05 54.0   |             |             |             |
| Linear $O(1)$               | 0.955 | 6.25 88.8  | 18.9 95.9  | 0.02 77.9   |             |             |             |
| Linear $O(2)$               | 0.963 | 20.2 70.9  | 30.2 78.1  | 0.10 67.4   |             |             |             |

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References
Birkin, J., Li, B., Cautun, M., & Shi, Y. 2019, MNRAS, 483, 5267
Branchini, E., Eldar, A., & Nusser, A. 2002, MNRAS, 335, 53
Bremer, Y., Frisch, U., Hénon, M., et al. 2003, MNRAS, 346, 501
Cayuso, J. I., Johnson, M. C., & Mertens, J. B. 2018, PhRvD, 98, 063502
Chan, K. C. 2014, PhRvD, 89, 083515
Cooray, A., & Caldwell, R. R. 2006, PhRvD, 73, 103002
Croft, R. A. C., & Gaztanaga, E. 1997, MNRAS, 285, 793
Davis, T. M., Hui, L., Frieman, J. A., et al. 2011, ApJ, 741, 67
De Bernardis, F., Aiola, S., Vavagiakis, E. M., et al. 2017, JCAP, 3, 008
DeDeo, S., Spergel, D. N., & Trac, H. 2005, arXiv:astro-ph/0511060
Deutsch, A.-S., Dimastrogiovanni, E., Johnson, M. C., Münchmeyer, M., & Terrana, A. 2018, PhRvD, 98, 123501
Eisenstein, D. J., Seo, H.-J., Sirko, E., & Spergel, D. N. 2007, ApJ, 664, 675
Fisher, K. B., Lahav, O., Hoffman, Y., Lynden-Bell, D., & Zaroubi, S. 1995, MNRAS, 272, 885
Hada, R., & Eisenstein, D. J. 2018, MNRAS, 478, 1866
Hada, R., & Eisenstein, D. J. 2019, MNRAS, 482, 5685
Hand, N., Addison, G. E., Aubourg, E., et al. 2012, PhRvL, 109, 041101
Harnois-Déraps, J., Pen, U.-L., Iliev, I. T., et al. 2013, MNRAS, 436, 540
Hui, L., & Greene, P. B. 2006, PhRvD, 73, 123526
Jasche, J., & Lavaux, G. 2019, A&A, 625, A64
Lavaux, G. 2008, PhD, 237, 2139
Lavaux, G., Mohayaee, R., Colombi, S., et al. 2008, MNRAS, 383, 1292
Li, M., Angulo, R. E., White, S. D. M., & Jasche, J. 2014, MNRAS, 443, 2311
Li, Y.-C., Ma, Y.-Z., Remazeilles, M., & Moodley, K. 2018, PhRvD, 97, 023514
McCarthy, F., & Johnson, M. C. 2019, arXiv:1907.06678
Modi, C., Feng, Y., & Seljak, U. 2018, JCAP, 2018, 028
Mukherjee, S., Lavaux, G., Bouchet, F. R., et al. 2019, arXiv:1909.08627
Münchmeyer, M., Madhavacheril, M. S., Ferraro, S., Johnson, M. C., & Smith, K. M. 2019, PhRvD, 100, 083508
Nusser, A., & Davis, M. 1994, ApJL, 421, L1
Pan, Z., & Johnson, M. C. 2019, PhRvD, 100, 083522
Peebles, P. J. E. 1989, ApJL, 344, L53
Planck Collaboration, Aghanim, N., et al. 2016, A&A, 596, A140
Sarpa, E., Schindl, C., Branchini, E., & Matarrese, S. 2019, MNRAS, 484, 3818
Schmidt, F., Elsner, F., Jasche, J., Nguyen, N. M., & Lavaux, G. 2019, JCAP, 1, 042
Schmittfull, M., Baudrart, T., & Zaldarriaga, M. 2017, PhRvD, 96, 023505
Schmittfull, M., Simonovic, M., Assassi, V., & Zaldarriaga, M. 2019, PhRvD, 100, 043514
Seljak, U., Aslanyan, G., Feng, Y., & Modi, C. 2017, JCAP, 2017, 009
Shao, J., Zhang, P., Lin, W., Jing, Y., & Pan, J. 2011, MNRAS, 413, 628
Shi, F., Yang, X., Wang, H., et al. 2016, ApJ, 833, 241
Shi, Y., Cautun, M., & Li, B. 2018, PhRvD, 97, 023505
Smith, K. M., Madhavacheril, M. S., Münchmeyer, M., et al. 2018, arXiv:1810.13423
Wang, H., Mo, H. J., Yang, X., Jing, Y. P., & Lin, W. P. 2014, ApJ, 794, 94
Wang, X., & Pen, U.-L. 2019, ApJ, 870, 116
Yu, H.-R., Yu, Y., Motloch, P., et al. 2019, arXiv:1904.01029
Yu, Y., Zhu, H.-M., & Pen, U.-L. 2017, ApJ, 847, 110
Zhang, P., Zheng, Y., & Jing, Y. 2015, PhRvD, 91, 043522
Zheng, Y., Zhang, P., & Jing, Y. 2015, PhRvD, 91, 043523
Zheng, Y., Zhang, P., Jing, Y., Lin, W., & Pan, J. 2013, PhRvD, 88, 103510
Zhu, H.-M., & Castorina, E. 2019, arXiv:1905.00361
Zhu, H.-M., Yu, Y., & Pen, U.-L. 2018, PhRvD, 97, 043502
Zhu, H.-M., Yu, Y., Pen, U.-L., Chen, X., & Yu, H.-R. 2017, PhRvD, 96, 123502