An equivalence between the set of graph-knots and the set of homotopy classes of looped graphs

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Abstract

In the present paper we construct a one-to-one correspondence between the set of graph-knots and the set of homotopy classes of looped graphs. Moreover, the graph-knot and the homotopy class constructed from a given knot are related with this correspondence. This correspondence is given by a simple formula.

1 Introduction

The discovery of virtual knot theory by Kauffman [Kau] in mid 1990s was an important step in generalising combinatorial and topological knot theoretical techniques into a larger domain (knots in thickened surfaces), which is an important step towards generalisation of these techniques for knots in arbitrary manifolds.

It turned out that some invariants (Kauffman bracket polynomial) can be generalised for virtual knots immediately [Kau], and some other theories (Khovanov homology theory) need a complete revision of the original construction for a generalisation for the case of virtual knots [Mal]. Virtual knots also sharpened several problems and elicited some phenomena which do not appear in the classical knot theory [FKM], e.g., the existence of a virtual knot with non-trivial

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Jones polynomial and trivial fundamental group emphasises the difficulty of extracting the Jones polynomial information out of the knot group.

In the present paper, we consider two new theories: graph-links introduced in [IM1, IM2] and homotopy classes of looped interlacement graphs introduced by L. Traldi and L. Zulli [TZ]. The two theories are closely connected to both classical and virtual knot theories and, in some sense, are generalisations of virtual knots. Likewise virtual knots appear out of non-realisable Gauss code and thus generalise classical knots (which have realisable Gauss codes), graphs-links and looped graphs come out of intersection graphs: we may consider graphs which realise chord diagrams, and, in turn, virtual knots, and pass to arbitrary graphs which correspond to some mysterious objects generalising knots and virtual knots.

There is a way of coding all virtual knots by Gauss diagrams and there is another way of coding all virtual links by rotating circuits (see, e.g., [Ma2]), see Fig. 1. Looped graphs are a generalisation of intersection graphs of chord diagrams constructed by using the first way of coding and graph-links come out from considering a rotating circuit. In low-dimensional topology both approaches, the Gauss circuit approach and the rotating circuit approach, are very widely used. The Gauss circuit approach is applied in knot theory, namely in the construction of finite-type invariants, Vassiliev invariants [BN, GPV, CDL], and in the planarity problem of immersed curves, [CE1, CE2, RR]. However, for detecting planarity of a framed 4-graph it is more convenient to use the rotating circuit approach,
The criterion of the planarity of an immersed curve, which is the framed 4-graph, is formulated very easy: an immersed curve is planar if and only if the chord diagram obtained from a rotating circuit is a $d$-diagram, i.e. the set of all the chords can be split into two sets and the chords from one set do not intersect each other, see [Ma5]. If we want to generalise the planarity problem to the problem of finding the minimum genus of a closed surface which a given curve can be immersed in, the rotating circuit approach is also more useful. There are criteria giving us the answer to the question what is the minimum genus for a given curve, see [Ma3].

In spite of the fact that graph-links and the homotopy classes of looped graphs are abstract objects the Kauffman bracket polynomial and the Jones polynomial were constructed for them, see [IM1, IM2, TZ]. But the first question which has arisen after the constructions of these two theories was whether or not in each graph-link (each homotopy class of looped graphs) there exists a ‘realisable’ representative, i.e. a graph which is the intersection graph of a chord diagram. Some graphs can not be represented by chord diagrams at all [Bou], see Fig. 2. The answer to this question is negative and this problem was first solved by V. O. Manturov in [Ma6, Ma7] for homotopy classes of looped graphs. The second question is whether there is an equivalence between the two theories. In the present paper we give an explicit formula which gives us an equivalence between the set of graph-knots (graph-links with one component) and the set of homotopy classes of looped graphs. Moreover, under this equivalence ‘realisable’ objects correspond to ‘realisable’ ones. Therefore, by using this equivalence we can give the negative answer to the question of existing a realisable representative for graph-knots.
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2 Basic Definitions and Constructions

2.1 Chord diagrams and Framed 4-Graphs

Throughout the paper all graphs are finite. Let $G$ be a graph with the set of vertices $V(G)$ and the set of edges $E(G)$. We think of an edge as an equivalence class of the two half-edges. We say that a vertex $v \in V(G)$ has degree $k$ if $v$ is incident to $k$ half-edges. A graph whose vertices have the same degree $k$ is called $k$-valent or a $k$-graph. The free loop, i.e. the graph without vertices, is also considered as $k$-graph for any $k$.

Definition 2.1. A 4-graph is framed if for every vertex the four emanating half-edges are split into two pairs of (formally) opposite edges. The edges from one pair are called opposite to each other.

A virtual diagram is a framed 4-graph embedded into $\mathbb{R}^2$ where each crossing is either endowed with a classical crossing structure (with a choice for underpass and overpass specified) or just said to be virtual and marked by a circle. A virtual link is an equivalence class of virtual diagrams modulo generalised Reidemeister moves. The latter consist of usual Reidemeister moves referring to classical crossings and the detour move that replaces one arc containing only virtual intersections and self-intersections by another arc of such sort in any other place of the plane, see Fig. 3. A projection of a virtual diagram is a framed 4-graph obtained from the diagram by considering classical crossings as vertices and virtual crossings are just intersection points of images of different edges. A virtual diagram is connected if its projection is connected. Without loss of generality, all virtual diagrams are assumed to be connected and contain at least one classical crossing [IM1] [IM2].

Definition 2.2. A chord diagram is a cubic graph consisting of a selected cycle (the circle) and several non-oriented edges (chords) connecting points on the circle in such a way that every point on the
A chord diagram is labeled if every chord is endowed with a label \((a, \alpha)\), where \(a \in \{0, 1\}\) is the framing of the chord, and \(\alpha \in \{\pm\}\) is the sign of the chord. If no labels are indicated, we assume the chord diagram has all chords with label \((0, +)\). Two chords of a chord diagram are called linked if the ends of one chord lie in different connected components of the circle with the end-points of the second chord removed.

**Definition 2.3.** By a virtualisation of a classical crossing of a virtual diagram we mean a local transformation shown in Fig. 4.

Having a labeled chord diagram \(D\), one can construct a virtual link diagram \(K(D)\) (up to virtualisation) as follows. Let us immerse this diagram in \(\mathbb{R}^2\) by taking an embedding of the circle and placing some chords inside the circle and the other ones outside the circle. After that we remove neighbourhoods of each of the chord ends and replace them by a pair of lines (connecting four points on the circle which are obtained after removing two neighbourhoods) with a classical crossing if the chord is framed by 0 and a couple of lines with a classical crossing and a virtual crossing if the chord is framed by 1 in the
following way. The choice for underpass and overpass is specified as follows. A crossing can be smoothed in two ways: $A$ and $B$ as in the Kauffman bracket polynomial; we require that the initial piece of the circle corresponds to the $A$-smoothing if the chord is positive and to the $B$-smoothing if it is negative: $A: \xrightarrow{\cdot - \cdot} \xrightarrow{\cdot - \cdot}$, $B: \xrightarrow{\cdot - \cdot} \xrightarrow{\cdot - \cdot}$.

Conversely, having a connected virtual diagram $K$, one can get a labeled chord diagram $D_C(K)$, see Fig. 5. Indeed, one takes a circuit $C$ of $K$ which is a map from $S^1$ to the projection of $K$. This map is bijective outside classical and virtual crossings, has exactly two preimages at each classical and virtual crossing, goes transversally at each virtual crossing and turns from an half-edge to an adjacent (non-opposite) half-edge at each classical crossing. Connecting the two preimages of a classical crossing by a chord we get a chord diagram, where the sign of the chord is $+$ if the circuit locally agrees with the $A$-smoothing, and $-$ if it agrees with the $B$-smoothing, and the framing of a chord is $0$ (resp., $1$) if two opposite half-edges have the opposite (resp., the same) orientation. It can be easily checked that this operation is indeed inverse to the operation of constructing a virtual link out of a chord diagram: if we take a chord diagram $D$, and construct a virtual diagram $K(D)$ out of it, then for some circuit $C$ the chord diagram $D_C(K(D))$ will coincide with $D$. The rule for setting classical crossings here agrees with the rule described above. This proves the following

**Theorem 2.1** ([Ma5]). For any connected virtual diagram $L$ there is a certain labeled chord diagram $D$ such that $L = K(D)$. 
The Reidemeister moves on virtual diagrams generate the Reidemeister moves on labeled chord diagrams \([IM1 IM2]\).

### 2.2 Reidemeister Moves for Looped Interlacement Graphs and Graph-links

Now we are describing moves on graphs obtained from virtual diagrams by using rotating circuit \([IM1 IM2]\) and the Gauss circuit \([TZ]\). These moves in both cases will correspond to the “real” Reidemeister moves on diagrams. Then we shall extend these moves to all graphs (not only to realisable ones). As a result we get new objects, a graph-link and a homotopy class of looped interlacement graphs, in a way similar to the generalisation of classical knots to virtual knots: the passage from realisable Gauss diagrams (classical knots) to arbitrary chord diagrams leads to the concept of a virtual knot, and the passage from realisable (by means of chord diagrams) graphs to arbitrary graphs leads to the concept of two new objects: graph-links and homotopy classes of looped interlacement graphs (here ‘looped’ corresponds to the writher number, if the writher number is -1 then the corresponding vertex has a loop). To construct the first object we shall use simple labeled graphs, and for the second one we shall use (unlabeled) graphs without multiple edges, but loops are allowed.

**Definition 2.4.** A graph is **labeled** if every vertex \(v\) of it is endowed with a pair \((a, \alpha)\), where \(a \in \{0, 1\}\) is the framing of \(v\), and \(\alpha \in \{\pm\}\) is the sign of \(v\). Let \(D\) be a labeled chord diagram \(D\). The **labeled intersection graph**, cf. [CDL], \(G(D)\) of \(D\) is the labeled graph: 1) whose vertices are in one-to-one correspondence with chords of \(D\), 2) the label of each vertex corresponding to a chord coincides with that of the chord, and 3) two vertices are connected by an edge if and only if the corresponding chords are linked.

**Definition 2.5.** A simple (labeled) graph \(H\) is called **realisable** if there is a (labeled) chord diagram \(D\) such that \(H = G(D)\).

The following lemma is evident.

**Lemma 2.1.** A simple (labeled) graph is realisable if and only if each its connected component is realisable.
Definition 2.6. Let $G$ be a graph and let $v \in V(G)$. The set of all vertices adjacent to $v$ is called the *neighbourhood of a vertex* $v$ and denoted by $N(v)$ or $N_G(v)$.

Let us define two operations on simple unlabeled graphs.

Definition 2.7. (Local Complementation) Let $G$ be a graph. The *local complementation* of $G$ at $v \in V(G)$ is the operation which toggles adjacencies between $a, b \in N(v)$, $a \neq b$, and doesn’t change the rest of $G$. Denote the graph obtained from $G$ by the local complementation at a vertex $v$ by $LC(G; v)$.

Definition 2.8. (Pivot) Let $G$ be a graph with distinct vertices $u$ and $v$. The *pivoting operation* of a graph $G$ at $u$ and $v$ is the operation which toggles adjacencies between $x, y$ such that $x, y \notin \{u, v\}$, $x \in N(u)$, $y \in N(v)$ and either $x \notin N(v)$ or $y \notin N(u)$, and doesn’t change the rest of $G$. Denote the graph obtained from $G$ by the pivoting operation at vertices $u$ and $v$ by $piv(G; u, v)$.

Example. In Fig. 6 the graphs $G$, $LC(G; u)$ and $piv(G; u, v)$ are depicted.

The following lemma can be easily checked.

Lemma 2.2. If $u$ and $v$ are adjacent then there is an isomorphism

$$piv(G; u, v) \cong LC(LC(LC(G; u); v); u).$$
Let us define graph-moves by considering intersection graphs of chord diagrams constructed by using a rotating circuit, and these moves correspond to the Reidemeister moves on virtual diagrams. As a result we obtain a new object — an equivalence class of labeled graphs under formal moves. These moves were defined in [IM1, IM2].

**Definition 2.9.** \(\Omega_g\)

1. The first Reidemeister graph-move is an addition/removal of an isolated vertex labeled \((0, \alpha)\), \(\alpha \in \{\pm\}\).

2. The second Reidemeister graph-move is an addition/removal of two non-adjacent (resp., adjacent) vertices having \((0, \pm \alpha)\) (resp., \((1, \pm \alpha)\)) and the same adjacencies with other vertices.

3. The third Reidemeister graph-move is defined as follows. Let \(u, v, w\) be three vertices of \(G\) all having label \((0, -)\) so that \(u\) is adjacent only to \(v\) and \(w\) in \(G\). Then we only change the adjacency of \(u\) with the vertices \(t \in N(v) \setminus N(w) \cup N(w) \setminus N(v)\) (for other pairs of vertices we do not change their adjacency). In addition, we switch the signs of \(v\) and \(w\) to +. The inverse operation is also called the third Reidemeister graph-move.

4. The fourth graph-move for \(G\) is defined as follows. We take two adjacent vertices \(u\) and \(v\) labeled \((0, \alpha)\) and \((0, \beta)\) respectively. Replace \(G\) with \(\text{piv}(G; u, v)\) and change signs of \(u\) and \(v\) so that the sign of \(u\) becomes \(-\beta\) and the sign of \(v\) becomes \(-\alpha\).

4'. In this fourth graph-move we take a vertex \(v\) with the label \((1, \alpha)\). Replace \(G\) with \(\text{LC}(G; v)\) and change the sign of \(v\) and the framing for each \(u \in N(v)\).

The comparison of the graph-moves with the Reidemeister moves yields the following theorem.

**Theorem 2.2.** Let \(K_1\) and \(K_2\) be two connected virtual diagrams, and let \(G_1\) and \(G_2\) be two labeled intersection graphs obtained from \(K_1\) and \(K_2\), respectively. If \(K_1\) and \(K_2\) are equivalent in the class of connected diagrams then \(G_1\) and \(G_2\) are obtained from one another by a sequence of \(\Omega_g1 - \Omega_g4\) graph-moves.

**Definition 2.10.** A **graph-link** is an equivalence class of simple labeled graphs modulo \(\Omega_g1 - \Omega_g4\) graph-moves.

Let \(D_G(K)\) be the Gauss diagram of a virtual diagram \(K\). Let us construct the graph obtained from the intersection graph of \(D_G(K)\) by adding loops to vertices corresponding to chords with negative writhe
number [TZ]. We refer to this graph as the *looped interlacement graph* or the *looped graph*. Let us construct the moves on graphs. These moves are similar to the moves for graph-links and also correspond to the Reidemeister moves on virtual diagrams.

**Definition 2.11.** Ω1. The first Reidemeister move for looped interlacement graphs is an addition/removal of an isolated looped or unlooped vertex.

Ω2. The second Reidemeister move for looped interlacement graphs is an addition/removal of two vertices having the same adjacencies with other vertices and, moreover, one of which is looped and the other one is unlooped.

Ω3. The third Reidemeister move for looped interlacement graphs is defined as follows. Let $u, v, w$ be three vertices such that $v$ is looped, $w$ is unlooped, $v$ and $w$ are adjacent, $u$ is adjacent to neither $v$ nor $w$, and every vertex $x \notin \{u, v, w\}$ is adjacent to either 0 or precisely two of $u, v, w$. Then we only remove all three edges $uv, uw$ and $vw$. The inverse operation is also called the third Reidemeister move.

**Remark 2.1.** The two third Reidemeister moves do not exhaust all the possibilities for representing the third Reidemeister move on Gauss diagrams, see [TZ]. It can be shown that all the other versions of the third Reidemeister move, which involve toggling the non-loop edges in one of the six pictured configurations in Figure 7 (every vertex outside the picture must have either 0 or precisely 2 neighbours among the three vertices that are pictured), are combinations of the second and third Reidemeister moves, see [¨Ost] for details.

**Definition 2.12.** We call an equivalence class of graphs (without multiple edges, but loops are allowed) modulo the three moves listed in Definition 2.11 a *homotopy class* of looped interlacement graphs.

**Remark 2.2.** Looped interlacement graphs encode only knot diagrams but graph-links can encode virtual diagrams with any number of components. The approach using a rotating circuit has an advantage in this sense. In [Tr] L. Traldi introduced the notion of a marked graph by considering any Euler tour (we have vertices which we go transversally and in which we rotate). In the paper we don’t consider “mixes circuits”.
Figure 7: The Possible Configurations of the 3-rd Reidemeister move
3 Graph-Links and Homotopy Classes of Looped Graphs: Their Equivalence

The main goal of this section is to construct an equivalence between the set of graph-knots, see ahead, and the set of homotopy classes of looped graphs such that the graph-knot and the homotopy class of looped graphs constructed from a given knot are related by the equivalence.

Before constructing an equivalence we need some definitions and statements.

Definition 3.1. Let \( G \) be a labeled graph on vertices from the enumerated set \( V(G) = \{v_1, \ldots, v_n\} \). Define the adjacency matrix \( A(G) = (a_{ij}) \) of \( G \) over \( \mathbb{Z}_2 \) as follows: \( a_{ii} \) is equal to the framing of \( v_i \), \( a_{ij} = 1 \), if and only if \( v_i \) is adjacent to \( v_j \) and \( a_{ij} = 0 \) otherwise.

Remark 3.1. Throughout the paper all the matrices are over \( \mathbb{Z}_2 \). Therefore, corank’s and det’s are calculated over \( \mathbb{Z}_2 \).

Definition 3.2. Let us define the number of components in a graph-link \( F \) as corank\( \mathbb{Z}_2(A(G) + E) \) \( + 1 \), here \( G \) is a representative of \( F \). A graph-link \( F \) with corank\( \mathbb{Z}_2(A(G) + E) = 0 \) for any representative \( G \) of \( F \) is called a graph-knot.

Let \( B_i(G) = A(G) + E + E_{ii} \) (all elements of \( E_{ii} \) except for the one in the \( i \)-th column and \( i \)-th row which is one are 0) for each vertex \( v_i \in V(G) \).

Definition 3.3. The writhe number \( w_i \) of \( G \) (with corank\( \mathbb{Z}_2(A(G) + E) = 0 \)) at \( v_i \) is \( w_i = (-1)^{\text{corank}_2 B_i(G)} \) \( \text{sign} \) \( v_i \) and the writhe number of \( G \) is

\[
    w(G) = \sum_{i=1}^{n} w_i.
\]

If \( G \) is a realisable graph by a chord diagram and, therefore, by a virtual diagram then \( w_i \) is the “real” writhe number of the crossing corresponding to \( v_i \).

By \( \hat{C}_{i,j,\ldots,k} \) denote the matrix obtained from a matrix \( C \) by deleting the \( i, j, \ldots, k \)-th rows and the \( i, j, \ldots, k \)-th columns. We shall write \( \hat{B}_{i,j,\ldots,k}(G) \) instead of \( \hat{B}(G)_{i,j,\ldots,k} \).
Lemma 3.1. \( w_i(G) = (-1)^{\text{corank}_2 \hat{B}_i(G)+1} \text{sign } v_i \).

Proof. Without loss of generality we prove the statement of the lemma for \( w_1 \). Let
\[
A(G) + E = \begin{pmatrix} a & b^\top \\ b & C \end{pmatrix}
\]
and
\[
\text{corank}_2 (A(G) + E) = 0 \iff \det (A(G) + E) = 1,
\]
where bold characters indicate column vectors. We have
\[
\det B_1(G) = \det \begin{pmatrix} a + 1 & b^\top \\ b & C \end{pmatrix} = \det \begin{pmatrix} a & b^\top \\ b & C \end{pmatrix} + \det \begin{pmatrix} 1 & 0^\top \\ b & C \end{pmatrix}
\]
and
\[
\hat{B}_1(G) = C, \quad \det(B_1(G)) = \det A(G) + \det C = 1 + \det \hat{B}_1(G).
\]
The last equality gives us the statement of the lemma. \( \square \)

Lemma 3.2. Let \( G \) be labeled graph with \( \det B(G) = 1 \) and \( B(G)^{-1} = (b^{ij}) \). Then \( b^{ii} = \frac{1 - w_i(G) \text{ sign } v_i}{2} \).

Proof. We have
\[
w_i(G) = (-1)^{\text{corank}_2 \hat{B}_i(G)+1} \text{sign } v_i \iff w_i(G) \text{ sign } v_i = (-1)^{\text{corank}_2 \hat{B}_i(G)+1}
\]
\[
\iff \text{corank}_2 \hat{B}_i(G) = \frac{w_i(G) \text{ sign } v_i + 1}{2} \iff b^{ii} = \det \hat{B}_i(G)
\]
\[
= 1 - \text{corank}_2 \hat{B}_i(G) = \frac{1 - w_i(G) \text{ sign } v_i}{2}.
\]
\( \square \)

Definition 3.4. Define the adjacency matrix \( A(L) = (a_{ij}) \) over \( \mathbb{Z}_2 \) for a looped graph \( L \) with enumerated set of vertices as: \( a_{ii} = 1 \) if and only if the vertex numbered \( i \) is looped and \( a_{ii} = 0 \) otherwise; \( a_{ij} = 1 \), \( i \neq j \), if and only if the vertex with the number \( i \) is adjacent to the vertex with the number \( j \) and \( a_{ij} = 0 \) otherwise.

Definition 3.5. We say that an \( n \times n \) matrix \( A = (a_{ij}) \) coincides with an \( n \times n \) matrix \( B = (b_{kl}) \) up to diagonal elements if \( a_{ij} = b_{ij}, i \neq j \).
Lemma 3.3 ([15]). Let $A$ be a symmetric matrix over $\mathbb{Z}_2$. Then there exists a symmetric matrix $\tilde{A}$ with $\det \tilde{A} = 1$ equal to $A$ up to diagonal elements.

The main theorem of the paper is the following one.

Theorem 3.1. There is a one-to-one correspondence between the set of all graph-knots and the set of all homotopy classes of looped graphs. Moreover, if $K$ is a knot, and $\mathfrak{F}$ is a graph-knot constructed from $K$ and $\mathfrak{L}$ is the homotopy class of looped graphs constructed from $K$ then $\mathfrak{F}$ and $\mathfrak{L}$ are related by this correspondence.

To prove this theorem we construct a map from the set of all graph-knots to the set of all homotopy classes of looped graphs. We shall show that this map has the inverse map.

Let us construct the map $\chi$ from the set of all graph-knots to the set of all homotopy classes of looped graphs. Let $\mathfrak{F}$ be a graph-knot and let $G$ be its representative. Let us consider the simple graph $H$ having the adjacency matrix coinciding with $(A(G) + E)^{-1}$ up to diagonal elements and construct the graph $L(G)$ from $H$ by just adding loops to any vertex of $H$ corresponding to a vertex of $G$ with the negative writhe number. By definition, put $\chi(\mathfrak{F}) = \mathfrak{L}$, here $\mathfrak{L}$ is the homotopy class of $L(G)$.

Theorem 3.2. The map $\chi$ from the set of all graph-knots to the set of all homotopy classes of looped graphs defined by $\chi(\mathfrak{F}) = \mathfrak{L}$, where $\mathfrak{L}$ is the homotopy class of $L(G)$ and $G$ is a representative of $\mathfrak{F}$, is well-defined.

Proof. Let $G_1, G_2$ be two representatives of $\mathfrak{F}$, and let $B(G_i) = A(G_i) + E$, $B(G_i)^{-1} = (b_{kl}^{G_i})$. We have to show that the homotopy classes of $L(G_1)$ and $L(G_2)$ are the same, i.e. $L_1 = L(G_1)$ and $L_2 = L(G_2)$ are related to each other by the Reidemeister moves on looped graphs.

Let us consider four cases.

1) We know that if $G_1$ and $G_2$ are obtained from each other by $\Omega_g \Omega_g'$ (these moves correspond to changing of a rotating circuit in the case of realisable graphs) then the writhe numbers of corresponding vertices of $G_1$ and $G_2$ are the same, see [15], and the graphs $L_1$ and $L_2$ are isomorphic up to loops, see [15]. Therefore, $L_1$ and $L_2$ are isomorphic.
2) Let $G_1$ and $G_2$ be obtained from each other by $\Omega_1$ (we remove the vertex with the number 1). We have

$$B(G_1) = \begin{pmatrix} 1 & 0 \\ 0 & A(G_2) + E \end{pmatrix}, \quad B(G_2) = A(G_2) + E,$$

$$B(G_1)^{-1} = \begin{pmatrix} 1 & 0^\top \\ 0 & (A(G_2) + E)^{-1} \end{pmatrix}, \quad B(G_2)^{-1} = (A(G_2) + E)^{-1},$$

where $0$ indicates column vector with all entries 0. Therefore, $L_2$ is obtained from $L_1$ by $\Omega_1$.

3) Let $G_1$ and $G_2$ be obtained from each other by $\Omega_2$ (we remove the vertex with the numbers 1 and 2). We have two cases. The first case

$$B(G_1) = \begin{pmatrix} 1 & 0 & a^\top \\ 0 & 1 & a^\top \\ a & a & A(G_2) + E \end{pmatrix}, \quad B(G_2) = A(G_2) + E,$$

and the second case

$$B(G_1) = \begin{pmatrix} 0 & 1 & a^\top \\ 1 & 0 & a^\top \\ a & a & A(G_2) + E \end{pmatrix}, \quad B(G_2) = A(G_2) + E.$$

Let us consider only the first case. We know that $w_1(G_1) = -w_2(G_1)$ in $G_1$ [IM1, IM2]. Moreover, as

$$\det \begin{pmatrix} 1 & 0 & \tilde{a}^\top \\ 0 & 1 & \tilde{a}^\top \\ \tilde{a} & \tilde{a} & C \end{pmatrix} = \det \begin{pmatrix} 1 & 0 & \tilde{a}^\top \\ 1 & 1 & 0^\top \\ \tilde{a} & \tilde{a} & C \end{pmatrix} = \det \begin{pmatrix} 1 & 0 & \tilde{a}^\top \\ 1 & 1 & 0^\top \\ 0 & 0 & C \end{pmatrix} = \det C,$$

then

$$B(G_1)^{-1} = \begin{pmatrix} b & c & d^\top \\ c & b & d^\top \\ d & d & (A(G_2) + E)^{-1} \end{pmatrix}, \quad B(G_2)^{-1} = (A(G_2)+E)^{-1}.$$  

This means that $L_1$ and $L_2$ are obtained from each other by $\Omega_2$.

4) Now assume that $G_1$ and $G_2$ are obtained from each other by $\Omega_3$. The corresponding vertices of $G_1$ and $G_2$ under $\Omega_3$ have the same numbers (in [IM1] we have another enumeration). We shall prove that $L_1$ and $L_2$ are obtained from each other by a sequence of $\Omega_2$, $\Omega_3$ moves.
We have

\[ B(G_1) = \begin{pmatrix} 1 & 1 & 1 & 0^\top \\ 1 & 1 & 0 & a^\top \\ 1 & 0 & 1 & b^\top \\ 0 & a & b & C \end{pmatrix}, \]

\[ 1 = \det B(G_1) = \det \begin{pmatrix} 1 & 1 & 1 & 0^\top \\ 1 & 1 & 0 & a^\top \\ 1 & 0 & 1 & b^\top \\ 0 & a & b & C \end{pmatrix} = \det \begin{pmatrix} 0 & 1 & a^\top \\ 1 & 0 & b^\top \\ a & b & C \end{pmatrix}, \]

\[ B(G_2) = \begin{pmatrix} 1 & 0 & 0 & (a + b)^\top \\ 0 & 1 & 0 & b^\top \\ 0 & 0 & 1 & a^\top \\ a + b & b & a & C \end{pmatrix}, \quad \det B(G_2) = 1 \]

Let us show that we have a structure either for \( v_1, v_2, v_3 \in V(L_1) \) or \( \bar{v}_1, \bar{v}_2, \bar{v}_3 \in V(L_2) \) as in Figure 7.

We have \((\text{IM1})\)

\[ w_i(G_1) = w_i(G_2), \quad i = 1, 2, 3, \]

\[ \det \hat{B}_1(G_1) = \det \begin{pmatrix} 1 & 0 & a^\top \\ 0 & 1 & b^\top \\ a & b & C \end{pmatrix} \]

\[ = \det \begin{pmatrix} 1 & 0 & a^\top \\ 1 & 1 & b^\top \\ 0 & b & C \end{pmatrix} + \det \begin{pmatrix} 0 & 0 & a^\top \\ 1 & 1 & b^\top \\ a & b & C \end{pmatrix} \]

\[ = \det \begin{pmatrix} 1 & 0 & a^\top \\ 1 & 1 & b^\top \\ 0 & b & C \end{pmatrix} + \det \begin{pmatrix} 0 & 1 & a^\top \\ 1 & 1 & b^\top \\ a & 0 & C \end{pmatrix} + \det \begin{pmatrix} 0 & 1 & a^\top \\ 1 & 0 & b^\top \\ a & b & C \end{pmatrix}, \]

\[ \det \hat{B}_2(G_1) = \det \begin{pmatrix} 1 & 1 & 0^\top \\ 1 & 1 & b^\top \\ 0 & b & C \end{pmatrix} = \det \begin{pmatrix} 0 & b^\top \\ b & C \end{pmatrix}, \]

\[ \det \hat{B}_3(G_1) = \det \begin{pmatrix} 1 & 1 & 0^\top \\ 1 & 1 & a^\top \\ 0 & a & C \end{pmatrix} = \det \begin{pmatrix} 0 & a^\top \\ a & C \end{pmatrix}, \]

\[ b_{12}^{12} = \det \begin{pmatrix} 1 & 0 & a^\top \\ 1 & 1 & b^\top \\ 0 & b & C \end{pmatrix} = \det \begin{pmatrix} 1 & a^\top + b^\top \\ b & C \end{pmatrix} \]
\[ b_1^{13} = \text{det} \begin{pmatrix} 1 & 0 & a^T \\ 1 & 1 & b^T \\ 0 & a & C \end{pmatrix} = \text{det} \left( \begin{pmatrix} 1 & a^T + b^T \\ a & C \end{pmatrix} \right), \]

\[ b_1^{23} = \text{det} \begin{pmatrix} 1 & 1 & 0^T \\ 1 & 0 & b^T \\ 0 & a & C \end{pmatrix} = \text{det} \left( \begin{pmatrix} 1 & b^T \\ a & C \end{pmatrix} \right), \]

\[ b_2^{12} = \text{det} \begin{pmatrix} 0 & 0 & b^T \\ 0 & 1 & a^T \\ a + b & a & C \end{pmatrix} = \text{det} \left( \begin{pmatrix} 0 & 0 & b^T \\ 1 & a^T \\ b & a & C \end{pmatrix} \right), \]

\[ b_2^{13} = \text{det} \begin{pmatrix} 0 & 1 & b^T \\ 0 & 0 & a^T \\ a + b & b & C \end{pmatrix} = \text{det} \left( \begin{pmatrix} 0 & 0 & a^T \\ 1 & 1 & a^T \\ a & b & C \end{pmatrix} \right), \]

\[ b_2^{23} = \text{det} \begin{pmatrix} 1 & 0 & (a + b)^T \\ 0 & 0 & a^T \\ a + b & b & C \end{pmatrix} = \text{det} \left( \begin{pmatrix} 1 & 0 & b^T \\ 0 & 0 & a^T \\ a & b & C \end{pmatrix} \right), \]

\[ b_1^{12} = b_1^{23} + \text{det} \bar{B}_2(G_1), \quad b_1^{13} = b_1^{23} + \text{det} \bar{B}_3(G_1), \]

\[ b_1^{12} + b_1^{13} = \text{det} \bar{B}_1(G_1) + 1, \]

\[ b_2^{12} = b_1^{12} + 1, \quad b_2^{13} = b_1^{13} + 1, \quad b_2^{23} = b_1^{23} + 1. \]

It is not difficult to show that the last equalities guarantee us that either \( v_1, v_2, v_3 \in V(L_1) \) or \( \tilde{v}_1, \tilde{v}_2, \tilde{v}_3 \in V(L_2) \) have a structure as in
Figure 7. The structure of the other triple is obtained from the first triple by toggling the non-loop edges.

Denote by $\hat{f}_i$ the column vector obtained from $f$ by deleting the $i$-th element and denote by $\hat{C}_j$ (resp., $\hat{C}_j^j$) the matrix obtained from $C$ by deleting the $j$-th column (resp., the $i$-th row and the $j$-th column).

Other equalities for $i, j > 3$ are as follows:

$$b_{ij}^{kl} = \det \begin{pmatrix} 1 & 1 & 1 & 0^\top \\ 1 & 1 & 0 & (\hat{a}^{j-3})^\top \\ 1 & 0 & 1 & (\hat{b}^{i-3})^\top \\ 0 & \hat{a}^{i-3} & \hat{b}^{i-3} & \hat{C}_j^{-1} \end{pmatrix}$$

$$= \det \begin{pmatrix} 1 & 0 & 0 & (\hat{a} + b)^{j-3})^\top \\ 0 & 1 & 0 & (\hat{b}^{i-3})^\top \\ 0 & 0 & 1 & (\hat{a}^{j-3})^\top \\ (a + b) & b & a & \hat{C}_j^{-1} \end{pmatrix} = b_{ij}^{kl},$$

$$b_{2j}^{1j} = \det \begin{pmatrix} 0 & 1 & 0 & (\hat{b}^{i-3})^\top \\ 0 & 0 & 1 & (\hat{a}^{j-3})^\top \\ (a + b) & b & a & \hat{C}_j^{-1} \end{pmatrix}$$

$$= \det \begin{pmatrix} 1 & 1 & 0 \ (\hat{b}^{i-3})^\top \\ 1 & 0 & 1 \ (\hat{a}^{j-3})^\top \\ 0 & b & a & \hat{C}_j^{-1} \end{pmatrix} = b_{2j}^{1j},$$

$$b_{2j}^{2j} = \det \begin{pmatrix} 1 & 0 & 0 & (\hat{a} + b)^{j-3})^\top \\ 0 & 0 & 1 & (\hat{a}^{j-3})^\top \\ a + b & b & a & \hat{C}_j^{-1} \end{pmatrix}$$

$$= \det \begin{pmatrix} 1 & 0 & 0 \ (\hat{b}^{i-3})^\top \\ 1 & 0 & 1 \ 0^\top \\ 0 & b & a & \hat{C}_j^{-1} \end{pmatrix} = \det \begin{pmatrix} 1 & 0 & 0 \ 0 & b & a \ \hat{C}_j^{-1} \end{pmatrix}$$

$$= b_{2j}^{2j},$$

analogously $b_{1j}^{3j} = b_{2j}^{3j}$.

We have to verify that every vertex $x \notin \{v_1, v_2, v_3\}$ is adjacent to either none of $v_1, v_2, v_3$ or precisely two of them in $L_1$ and analogously for $L_2$. This statement is equivalent to both $b_{1j}^{1j} + b_{2j}^{2j} + b_{1j}^{3j} = 0$ and
\[ b_2^{1j} + b_2^{2j} + b_3^{3j} = 0. \] By using the above equalities it is enough to verify only the first equality.

We have

\[
b_2^{2j} = \det \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & a & b \end{pmatrix} \begin{pmatrix} \mathbf{0}^\top \\ \mathbf{C}_{j-3} \end{pmatrix} = \det \begin{pmatrix} 1 & 0 & (\mathbf{b}_{j-3})^\top \\ a & b & \mathbf{C}_{j-3} \end{pmatrix},
\]

\[
b_3^{3j} = \det \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & a & b \end{pmatrix} \begin{pmatrix} \mathbf{0}^\top \\ \mathbf{C}_{j-3} \end{pmatrix} = \det \begin{pmatrix} 0 & 1 & (\mathbf{a}_{j-3})^\top \\ a & b & \mathbf{C}_{j-3} \end{pmatrix},
\]

\[
b_1^{1j} = \det \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & a & b \end{pmatrix} \begin{pmatrix} \mathbf{a}_{j-3} \end{pmatrix}^\top = \det \begin{pmatrix} 1 & 1 & (\mathbf{a} + \mathbf{b})_{j-3}^\top \\ a & b & \mathbf{C}_{j-3} \end{pmatrix} = b_2^{2j} + b_3^{3j}.
\]

We have proven that our triples of vertices are related with each other as in Remark 2.2, therefore, \( L_1 \) and \( L_2 \) are obtained from each other by a sequence of \( \Omega_2 \) and \( \Omega_3 \) moves.

Let us define the map \( \psi \) from the set of all homotopy classes of looped graphs to the set of all graph-knots. Let \( \mathcal{L} \) be the homotopy class of \( L \). By using Lemma 3.3 we can construct a symmetric matrix \( \mathbf{A}(L) = (a_{ij}) \) over \( \mathbb{Z}_2 \) coinciding with the adjacency matrix of \( L \) up to diagonal elements and \( \det \mathbf{A}(L) = 1 \). Let \( G(L) \) be the labeled simple graph having the matrix \( \mathbf{A}(L)^{-1} + E \) as its adjacency matrix (therefore, the first component of the label is equal to the corresponding diagonal element of \( \mathbf{A}(L)^{-1} + E \)), the second component of the label of the vertex with the number \( i \) is \( w_i(1 - 2a_{ii}) \), here \( w_i = 1 \) if the vertex of \( L \) with the number \( i \) does not have a loop, and \( w_i = -1 \) otherwise. Set \( \psi(\mathcal{L}) = \mathfrak{F} \), here \( G(L) \) is a representative of \( \mathfrak{F} \).

**Theorem 3.3.** The map \( \psi \) from the set of all homotopy classes of looped graphs to the set of all graph-knots defined above is well-defined.

**Proof.** Let \( L_1, L_2 \) be two representatives of \( \mathcal{L} \). We have to show that the graph-knots having representatives \( G_1 = G(L_1) \) and \( G_2 = G(L_2) \) respectively are the same. We shall show that a graph-link does not depend on the choice of diagonal elements, and \( G_1 \) and \( G_2 \) are related to each other by Reidemeister graph-moves.
1) The independence (up to the second component of the label of a vertex) on the choice of diagonal elements follows from the results of [15]. Namely, graph-knots obtained from matrices having different diagonal elements are related by $\Omega_g4$ and/or $\Omega_g4'$ graph-moves (up to the second component of the label). We have defined the sign of a vertex in such a way that looped vertices correspond to vertices with the negative writhe number and unlooped vertices correspond to vertices with the positive writhe number (Lemma 3.2). The writhe number doesn’t change under $\Omega_g4$, $\Omega_g4'$ graph-moves, and both the writhe number and the framing allow one to determine the sign of a vertex. Therefore, graph-knots obtained from matrices having different diagonal elements are related by $\Omega_g4$ and/or $\Omega_g4'$ graph-moves.

Now we pass to moves on looped graphs.

2) Let $L_1$ and $L_2$ be obtained from each other by $\Omega1$ (we remove the vertex with the number 1). We have

$$A(L_1) = \begin{pmatrix} a & 0^\top \\ 0 & A(L_2) \end{pmatrix}, \quad a \in \{0, 1\}.$$ 

Assume $\det \tilde{A}(L_2) = 1$, where $\tilde{A}(L_2)$ coincides with $A(L_2)$ up to diagonal elements, and

$$\tilde{A}(L_1) = \begin{pmatrix} 1 & 0^\top \\ 0 & \tilde{A}(L_2) \end{pmatrix}.$$ 

Then

$$\tilde{A}(L_1)^{-1} = \begin{pmatrix} 1 & 0^\top \\ 0 & \tilde{A}(L_2)^{-1} \end{pmatrix},$$

therefore, $G_1$ and $G_2$ are related by a sequence of $\Omega_g1$, $\Omega_g4$, $\Omega_g4'$ graph-moves.

3) Let $L_1$ and $L_2$ be obtained from each other by $\Omega2$ (we remove the vertex with the numbers 1 and 2). We have

$$A(L_1) = \begin{pmatrix} 0 & b & a^\top \\ b & 1 & a^\top \\ a & a & A(L_2) \end{pmatrix}, \quad b \in \{0, 1\}.$$ 

Assume $\det \tilde{A}(L_2) = 1$, where $\tilde{A}(L_2)$ coincides with $A(L_2)$ up to diagonal elements, and

$$\tilde{A}(L_1) = \begin{pmatrix} 1 + b & b & a^\top \\ b & 1 + b & a^\top \\ a & a & \tilde{A}(L_2) \end{pmatrix}, \quad \det \tilde{A}(L_1) = 1.$$
As
\[
\det \begin{pmatrix}
1 + b & b & \bar{a}^T \\
b & 1 + b & \bar{a}^T \\
\bar{a} & \bar{a} & C
\end{pmatrix} = \det \begin{pmatrix}
1 + b & b & \bar{a}^T \\
1 & 1 & 0^T \\
\bar{a} & \bar{a} & C
\end{pmatrix}
\]
\[
= \det \begin{pmatrix}
1 + b & b & \bar{a}^T \\
1 & 1 & 0^T \\
0 & 0 & C
\end{pmatrix} = \det C,
\]
\[
\det \begin{pmatrix}
1 + b & a^T \\
a & \bar{A}(L_2)
\end{pmatrix} = \det \begin{pmatrix}
b & a^T \\
a & \bar{A}(L_2)
\end{pmatrix} + \det \bar{A}(L_2),
\]
then
\[
\bar{A}(L_1)^{-1} = \begin{pmatrix}
f & f + 1 & d^T \\
f + 1 & f & d^T \\
d & d & \bar{A}(L_2)^{-1}
\end{pmatrix}.
\]

From the structure of the matrix \(\bar{A}(L_1)^{-1}\) it follows that the two vertices (which don’t belong to \(G_2\)) have the same framing and the necessary adjacencies, and from the structure of the matrix \(A(L_1)\) and the coincidence of vertices’ framings it follows that the two vertices have the different signs. This means that \(G_1\) and \(G_2\) are obtained from each other by a sequence of \(\Omega_g 2, \Omega_g 4, \Omega_g 4'\) graph-moves.

4) Now assume that \(L_1\) and \(L_2\) are obtained from each other by \(\Omega_3\). Let us enumerate all the vertices of \(V(L_i) = \{v_1^i, \ldots, v_n^i\}\) in such a way that corresponding vertices of \(L_1\) and \(L_2\) under \(\Omega_3\) move have the same number, and without loss of generality we assume that \(v_1^1\) and \(v_3^1\) are looped, \(v_2^1\) is unlooped, \(v_1^2\) is adjacent to \(v_2^1\); \(v_3^1\) is adjacent to neither \(v_1^1\) nor \(v_2^1\) in \(L_1\). The case when \(v_3^1\) is unlooped is obtained from the first case by applying second Reidemeister and one of the third Reidemeister moves.

We have
\[
A(L_1) = \begin{pmatrix}
1 & 1 & 0 & a^T \\
1 & 0 & 0 & b^T \\
0 & 0 & 1 & c^T \\
a & b & c & D
\end{pmatrix}, \quad A(L_2) = \begin{pmatrix}
1 & 0 & 1 & a^T \\
0 & 0 & 1 & b^T \\
1 & 1 & 1 & c^T \\
a & b & c & D
\end{pmatrix},
\]
\[
a + b + c = 0.
\]

Without loss of generality (if necessary we apply second Reid-
meister moves) we may assume that $c \neq 0$ and (Lemma 3.3)

$$
\tilde{A}(L_1) = \begin{pmatrix}
0 & 1 & 0 \\
1 & 1 & 0 \\
0 & 0 & 0 \\
\mathbf{a} & \mathbf{b} & \mathbf{c} & \tilde{D}
\end{pmatrix}
$$

and $\det \tilde{A}(L_1) = 1$.

As

$$
\det \begin{pmatrix}
0 & 0 & 1 & \mathbf{a}^\top \\
0 & 0 & 1 & \mathbf{b}^\top \\
1 & 1 & 1 & \mathbf{c}^\top \\
\mathbf{a} & \mathbf{b} & \mathbf{c} & \tilde{D}
\end{pmatrix} = \det \begin{pmatrix}
0 & 0 & 1 & \mathbf{a}^\top \\
0 & 0 & 0 & \mathbf{c}^\top \\
1 & 0 & 1 & \mathbf{c}^\top \\
\mathbf{a} & \mathbf{c} & \mathbf{c} & \tilde{D}
\end{pmatrix} = \det \tilde{A}(L_1) = 1, \quad (1)
$$

we have

$$
\tilde{A}(L_2) = \begin{pmatrix}
0 & 0 & 1 & \mathbf{a}^\top \\
0 & 0 & 1 & \mathbf{b}^\top \\
1 & 1 & 1 & \mathbf{c}^\top \\
\mathbf{a} & \mathbf{b} & \mathbf{c} & \tilde{D}
\end{pmatrix}.
$$

Let $\tilde{A}(L_1)^{-1} = (\tilde{a}^{-ij}_1)$, $\tilde{A}(L_2)^{-1} = (\tilde{a}^{-ij}_2)$, and let $G_i = G(L_i)$, $i = 1, 2$, be the two graph-knots having the adjacency matrices $\tilde{A}(L_i)^{-1} + E$. Let us show that $G_1$ and $G_2$ are obtained from each other by a sequence of $\Omega_g2$, $\Omega_g3$, $\Omega_g4$, $\Omega_g4'$ graph-moves.

Performing the same elementary manipulations as in (1) we have $\tilde{a}^{-ij}_1 = \tilde{a}^{-ij}_2$ for $i, j \geq 3$. Further, we get

$$
1 = \det \tilde{A}(L_1) = \det \begin{pmatrix}
0 & 1 & \mathbf{a}^\top \\
1 & 1 & \mathbf{b}^\top \\
0 & 0 & \mathbf{c}^\top \\
\mathbf{a} & \mathbf{b} & \mathbf{c} & \tilde{D}
\end{pmatrix} = \det \begin{pmatrix}
0 & 1 & \mathbf{a}^\top \\
1 & 1 & \mathbf{b}^\top \\
1 & 0 & \mathbf{0}^\top \\
\mathbf{a} & \mathbf{b} & \mathbf{c} & \tilde{D}
\end{pmatrix} = \tilde{a}^{-11}_1,
$$
\[ \tilde{a}_{1}^{12} = \det \begin{pmatrix} 1 & 0 & b^T \\ 0 & 0 & c^T \\ a & c & \tilde{D} \end{pmatrix} = \det \begin{pmatrix} 1 & 0 & b^T \\ 0 & 0 & c^T \\ b & c & \tilde{D} \end{pmatrix} = 1, \]

\[ \tilde{a}_{1}^{13} = \det \begin{pmatrix} 1 & 1 & b^T \\ 0 & 0 & c^T \\ a & b & \tilde{D} \end{pmatrix} = \det \begin{pmatrix} 0 & 1 & b^T \\ 0 & 0 & c^T \\ c & b & \tilde{D} \end{pmatrix} = 1, \]

\[ \tilde{a}_{1}^{1j} = \det \begin{pmatrix} 1 & 1 & 0 & (\hat{b}^{j-3})^T \\ 0 & 0 & 0 & (\hat{c}^{j-3})^T \\ a & b & c & \hat{\tilde{D}}_{j-3} \end{pmatrix} = 0, \quad j \geq 3 \quad (a + b + c = 0), \]

\[ \tilde{a}_{1}^{2j} = \det \begin{pmatrix} 0 & 1 & 0 & (\hat{a}^{j-3})^T \\ 0 & 0 & 0 & (\hat{c}^{j-3})^T \\ a & b & c & \hat{\tilde{D}}_{j-3} \end{pmatrix} = \det \begin{pmatrix} 0 & 0 & (\hat{a}^{j-3})^T \\ 0 & 0 & (\hat{c}^{j-3})^T \\ a & 0 & c & \hat{\tilde{D}}_{j-3} \end{pmatrix}, \]

\[ \tilde{a}_{1}^{3j} = \det \begin{pmatrix} 0 & 1 & 0 & (\hat{a}^{j-3})^T \\ 1 & 1 & 0 & (\hat{b}^{j-3})^T \\ a & b & c & \hat{\tilde{D}}_{j-3} \end{pmatrix} = \det \begin{pmatrix} 1 & 0 & 0 & (\hat{a}^{j-3})^T \\ 0 & 1 & 0 & (\hat{b}^{j-3})^T \\ a & 0 & c & \hat{\tilde{D}}_{j-3} \end{pmatrix}, \]

If either \( \tilde{a}_{1}^{22} = 0 \) or \( \tilde{a}_{1}^{33} = 0 \) we can apply the same second Reidemeister graph-moves to \( G_1 \) and \( G_2 \) and after that applying the \( \Omega_{g}4 \) graph-move we get that the corresponding vertices have the framing 0. Analogously, if \( \tilde{a}_{1}^{32} = 1 \) we can apply the same second Reidemeister graph-moves to \( G_1 \) and \( G_2 \) and after that applying the \( \Omega_{g}4 \) graph-move we get that \( v_1 \) and \( v_3 \) are not adjacent to each other. Therefore, without loss of generality we may assume that \( \tilde{a}_{1}^{22} = \tilde{a}_{1}^{33} = 1, \tilde{a}_{1}^{32} = 0 \).

By using the last equalities we get

\[ \tilde{a}_{1}^{22} = \det \begin{pmatrix} 0 & 0 & a^T \\ 0 & 0 & c^T \\ a & c & \tilde{D} \end{pmatrix} = \det \begin{pmatrix} 0 & 0 & b^T \\ 0 & 0 & c^T \\ b & c & \tilde{D} \end{pmatrix} 
= 1 + \det \begin{pmatrix} 0 & c^T \\ c & \tilde{D} \end{pmatrix} = 1, \]

\[ \tilde{a}_{1}^{33} = \det \begin{pmatrix} 0 & 1 & a^T \\ 1 & 1 & b^T \\ a & b & \tilde{D} \end{pmatrix} = \det \begin{pmatrix} 1 & 0 & c^T \\ 0 & 1 & b^T \\ c & b & \tilde{D} \end{pmatrix} 
= 1 + \det \begin{pmatrix} 1 & b^T \\ b & \tilde{D} \end{pmatrix} = 1, \]
\[ \tilde{a}_{11}^{23} = \det \begin{pmatrix} 0 & 1 & a^T \\ 0 & 0 & c^T \\ b & b & \widetilde{D} \end{pmatrix} = \det \begin{pmatrix} 1 & 1 & b^T \\ 0 & 0 & c^T \\ c & b & \widetilde{D} \end{pmatrix} \\
= 1 + \det \begin{pmatrix} 0 & c^T \\ b & \widetilde{D} \end{pmatrix} = 0. \]

Let us find the remaining elements of \( \widetilde{A}(L_2)^{-1} \). We have

\[ \tilde{a}_{2}^{11} = \det \begin{pmatrix} 0 & 1 & b^T \\ 1 & 1 & c^T \\ b & c & \widetilde{D} \end{pmatrix} = \det \begin{pmatrix} 0 & 1 & b^T \\ 1 & 0 & a^T \\ b & c & \widetilde{D} \end{pmatrix} = \det \begin{pmatrix} 1 & 1 & b^T \\ 1 & 0 & a^T \\ b & c & \widetilde{D} \end{pmatrix} \\
+ \det \begin{pmatrix} 0 & a^T \\ c & \widetilde{D} \end{pmatrix} = \det \begin{pmatrix} 1 & 1 & b^T \\ 0 & 0 & c^T \\ b & c & \widetilde{D} \end{pmatrix} + \det \begin{pmatrix} 0 & b^T \\ c & \widetilde{D} \end{pmatrix} + \det \begin{pmatrix} 0 & c^T \\ c & \widetilde{D} \end{pmatrix} \\
= \det \begin{pmatrix} 1 & 1 & b^T \\ 0 & 0 & c^T \\ b & c & \widetilde{D} \end{pmatrix} + \det \begin{pmatrix} 1 & b^T \\ b & \widetilde{D} \end{pmatrix} + 1 = 1, \]

\[ \tilde{a}_{2}^{22} = \det \begin{pmatrix} 0 & 1 & a^T \\ 1 & 1 & c^T \\ a & c & \widetilde{D} \end{pmatrix} = \det \begin{pmatrix} 1 & 1 & b^T \\ 1 & 1 & c^T \\ b & c & \widetilde{D} \end{pmatrix} = 1, \]

\[ \tilde{a}_{2}^{33} = \det \begin{pmatrix} 0 & 0 & a^T \\ 0 & 0 & b^T \\ a & b & \widetilde{D} \end{pmatrix} = \det \begin{pmatrix} 0 & 0 & b^T \\ 0 & 0 & c^T \\ a & c & \widetilde{D} \end{pmatrix} = 1, \]

\[ \tilde{a}_{2}^{12} = \det \begin{pmatrix} 0 & 1 & b^T \\ 1 & 1 & c^T \\ a & c & \widetilde{D} \end{pmatrix} = \det \begin{pmatrix} 1 & 1 & b^T \\ 1 & 0 & c^T \\ b & c & \widetilde{D} \end{pmatrix} \\
= \det \begin{pmatrix} 1 & 1 & b^T \\ 0 & 0 & c^T \\ b & c & \widetilde{D} \end{pmatrix} + \det \begin{pmatrix} 1 & b^T \\ b & \widetilde{D} \end{pmatrix} = 0, \]

\[ \tilde{a}_{2}^{13} = \det \begin{pmatrix} 0 & 0 & b^T \\ 1 & 1 & c^T \\ a & b & \widetilde{D} \end{pmatrix} = \det \begin{pmatrix} 0 & 0 & b^T \\ 0 & 1 & c^T \\ c & b & \widetilde{D} \end{pmatrix} \\
= \det \begin{pmatrix} 0 & 0 & b^T \\ 0 & 0 & c^T \\ c & b & \widetilde{D} \end{pmatrix} + \det \begin{pmatrix} 0 & b^T \\ c & \widetilde{D} \end{pmatrix} = 0, \]
\[
\tilde{a}_2^{23} = \det \begin{pmatrix} 0 & 0 & a^T \\ a & b & \tilde{D} \\ 1 & 1 & c^T \end{pmatrix} = \det \begin{pmatrix} 0 & 1 & b^T \\ 0 & 1 & c^T \\ c & b & \tilde{D} \end{pmatrix} = 0,
\]
\[
\tilde{a}_2^{1j} = \det \begin{pmatrix} 0 & 0 & 1 \begin{pmatrix} \hat{e}^j \\ -3 \end{pmatrix}^T \\ 1 & 1 & 1 \begin{pmatrix} \hat{c}^j \\ -3 \end{pmatrix}^T \\ a & b & c \end{pmatrix} \tilde{D}_{j-3} = \det \begin{pmatrix} 0 & 1 & 1 \begin{pmatrix} \hat{a}^j \\ -3 \end{pmatrix}^T \\ 1 & 0 & 0 \begin{pmatrix} \hat{b}^j \\ -3 \end{pmatrix}^T \\ a & b & c \end{pmatrix} \tilde{D}_{j-3} = \tilde{a}_1^{2j} + \tilde{a}_3^{3j},
\]
\[
\tilde{a}_2^{2j} = \det \begin{pmatrix} 0 & 0 & 1 \begin{pmatrix} \hat{a}^j \\ -3 \end{pmatrix}^T \\ 1 & 1 & 1 \begin{pmatrix} \hat{c}^j \\ -3 \end{pmatrix}^T \\ a & b & c \end{pmatrix} \tilde{D}_{j-3} = \det \begin{pmatrix} 0 & 1 & 1 \begin{pmatrix} \hat{a}^j \\ -3 \end{pmatrix}^T \\ 1 & 0 & 0 \begin{pmatrix} \hat{b}^j \\ -3 \end{pmatrix}^T \\ a & b & c \end{pmatrix} \tilde{D}_{j-3} = \tilde{a}_1^{3j},
\]
\[
\tilde{a}_2^{3j} = \det \begin{pmatrix} 0 & 0 & 1 \begin{pmatrix} \hat{a}^j \\ -3 \end{pmatrix}^T \\ 0 & 0 & 0 \begin{pmatrix} \hat{b}^j \\ -3 \end{pmatrix}^T \\ a & b & c \end{pmatrix} \tilde{D}_{j-3} = \det \begin{pmatrix} 0 & 0 & 1 \begin{pmatrix} \hat{a}^j \\ -3 \end{pmatrix}^T \\ 0 & 0 & 0 \begin{pmatrix} \hat{c}^j \\ -3 \end{pmatrix}^T \\ a & b & c \end{pmatrix} \tilde{D}_{j-3} = \tilde{a}_1^{2j}.
\]

We see that
\[
\tilde{A}(L_1)^{-1} + E = \begin{pmatrix} 0 & 1 & 1 & 0^T \\ 1 & 0 & 0 & f^T \\ 1 & 0 & 0 & g^T \\ 0 & f & g & H \end{pmatrix},
\]
\[
\tilde{A}(L_2)^{-1} + E = \begin{pmatrix} 0 & 0 & 0 & f^T + g^T \\ 0 & 0 & 0 & g^T \\ 0 & 0 & 0 & f^T \\ f + g & g & f & H \end{pmatrix}.
\]

It is easy to see that the corresponding vertices have the structure as in Definition 2.9. Therefore, \( G_2 \) is obtained from \( G_1 \) by a sequence of \( \Omega_g \), \( \Omega_g' \), \( \Omega_g' \), graph-moves.

From Theorems 3.2, 3.3 and from the definitions of the map \( \chi \) and \( \psi \) it follows that these maps are mutually inverse. Therefore, we have proven Theorem 3.1.

We conclude the paper with the example of non-realisable graph-knot.

**Definition 3.6.** A graph-link (a homotopy class of looped graphs) is called *non-realisable* if it has no realisable representative.
Corollary 3.1. A graph-link $\mathcal{F}$ is non-realisable if and only if $\chi(\mathcal{F})$ is non-realisable.

Let $G$ be the labeled graph depicted in Fig. 8 with each vertex having the framing 1 and signs are chosen arbitrary.

Corollary 3.2. The graph-knot $\mathcal{F}$ generated by $G$ is non-realisable.

Proof. Let $\mathcal{L} = \chi(\mathcal{F})$. It is not difficult to see that the looped graph isomorphic to $G$ (we have no loop) is a representative of $\mathcal{L}$. Therefore, $\mathcal{L}$ is non-realisable, see [Ma6, Ma7], and, in turn, $\mathcal{F}$ is non-realisable graph-knot. \qed

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