A GRONWALL LEMMA FOR FUNCTIONS OF TWO VARIABLES AND ITS APPLICATION TO PARTIAL DIFFERENTIAL EQUATIONS OF FRACTIONAL ORDER

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Abstract. In the paper, a new Gronwall lemma for functions of two variables with singular integrals is proved. An application to weak relative compactness of the set of solutions to a fractional partial differential equation is given.

In the paper, we study the following fractional partial differential equation

\[ D^{\alpha,\beta}_{a+\gamma,x,c+\delta,y} z(x,y) = f(x,y,z(x,y), D^\alpha_{a+\gamma,x} z(x,y), D^\beta_{c+\delta,y} z(x,y), u(x,y)) \] a.e. on \( P \),

with initial conditions

\[
\begin{cases}
I^{1-\alpha,1-\beta}_{a+\gamma,x,c+\delta,y} z(x,c) = \gamma(x), & x \in [a,b] \\
I^{1-\alpha,1-\beta}_{a+\gamma,x,c+\delta,y} z(a,y) = \delta(y), & y \in [c,d]
\end{cases}
\]

where \( \alpha, \beta \in (0,1) \), \( P = [a,b] \times [c,d] \subset \mathbb{R}^2 \), \( \gamma : [a,b] \to \mathbb{R}^n \), \( \delta : [c,d] \to \mathbb{R}^n \), \( f : P \times (\mathbb{R}^n)^3 \times \mathbb{R}^m \to \mathbb{R}^n \), \( z : P \to \mathbb{R}^n \) is an unknown function, \( u : P \to \mathbb{R}^m \) - a functional parameter (control) and \( D^\alpha_{a+\gamma,x,c+\delta,y} z, D^\beta_{c+\delta,y} z \) stand for fractional (in Riemann-Liouville sense) mixed partial derivative of the order \((\alpha,\beta)\), fractional partial derivative of order \(\alpha\) with respect to \(x\) and fractional partial derivative of order \(\beta\) with respect to \(y\) of a function \(z\), respectively, \( I^{1-\alpha,1-\beta}_{a+\gamma,x,c+\delta,y} \) is the integral of order \((\alpha,\beta)\) of the function \(z\). This system is the fractional version of the following continuous Fornasini-Marcesini control system (also called the Goursat-Darboux system)

\[ \frac{\partial^2 z}{\partial x \partial y} (x,y) = f(x,y,z(x,y), \frac{\partial z}{\partial x} z(x,y), \frac{\partial z}{\partial y} z(x,y), u(x,y)) \] a.e. on \( P \)

\[
\begin{cases}
z(x,c) = \gamma(x), & x \in [a,b] \\
z(a,y) = \delta(y), & y \in [c,d]
\end{cases}
\]

With the aid of (3)-(4) a gas absorption process can be described (cf. [9], [3], [6], [7]). More specifically, assuming that the diffusion process plays no essential role in the motion of the gas, the process of absorption of the gas pressed through the

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filter made in the form of a pipe filled up with a substance absorbing gas can be described by the following equation

\[
\frac{\partial^2 z}{\partial x \partial y}(x, y) = -\zeta \omega \frac{\partial z}{\partial y}(x, y) - \frac{\zeta}{v(x,y)} \frac{\partial z}{\partial x}(x, y) - \frac{\zeta^2 u_0}{v(x,y)} e^{-\zeta \omega x}
\]

where \( z(x,y) \) describes the quantity of the gas in the capacity unit (\( x \) denotes the time and \( y \) - distance from the pipe inlet), \( v(x,y) \) is the speed of the gas, and \( \zeta, \omega \) are some characteristics of the gas, \( u_0 \) describes its initial state. Replacing in the above equation the classical partial derivatives \( \frac{\partial^2 z}{\partial x \partial y}, \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y} \) by fractional ones \( D_{\alpha+} z, D_{\beta+} z, D_{\alpha+x, c+y} z \), respectively, we obtain the special case of (1).

Often, the use of ordinary fractional equations is precisely to replace (in an appropriate equation) the derivative of integer order by a fractional one. In some cases, under suitably chosen order of the fractional derivative, the solutions of the obtained fractional model are closer to the real solutions than solutions of the model with classical derivative. So, it seems to be purposeful to study problem (1)-(2) in the context of the gas absorption process.

In the paper [2], theorems on the existence, uniqueness and continuous dependence of the solution \( z \) on the control \( u \) for problem (1)-(2) have been obtained. The aim of our paper is to derive a new Gronwall lemma for functions of two variables, with singular integrals (Lemma 3.1) and apply it in the proof of a theorem on the relative weak compactness of the set of solutions to (1)-(2) (Theorem 4.2). Weak compactness of solutions to control system (1)-(2) is useful in the study of optimal control problems for (1)-(2).

Gronwall type lemmas are one of the basic tools in the theory of functional (differential, integral, integro-differential) equations. They are used to study existence of solutions to such equations, continuous dependence of solutions on functional parameters and initial data as well as boundedness and the mentioned weak compactness of the set of solutions. Recently, special attention has been paid to equations containing the integrals with singular kernels (see [10], [11], [4] for functions of one variable with applications to continuous dependence of solutions on initial data, existence of solutions, integro-differential equations, respectively). This is due to potential applications in the theory of equations of fractional order. As far as we know, our results are the first that concern functions of two variables and singular integrals.

The organization of the paper is the following. In the second section, we recall some notions and facts concerning the fractional partial derivatives of functions of two variables (for more details one can consult [2]). In the third section, we prove a theorem characterizing relatively weakly compact sets in the space of functions \( z \) possessing the mixed fractional derivative \( D_{\alpha+x, c+y} z \). In the fourth section, we prove a new Gronwall lemma for functions of two variables, with integral having singular kernels. In the fifth section, we obtain the mentioned relatively weak compactness of the set of solutions to problem (1)-(2), corresponding to commonly bounded (by an integrable function) controls. In the last part - Appendix, we prove some basic properties of singular integrals and operators described by them that are used in the paper. To the best of our knowledge presented results have not been studied by other authors.
1. Preliminaries. For a function $\varphi \in L^1(P, \mathbb{R}^n)$ and any $\alpha > 0$, we define the integral $I_{a+,x}^{\alpha} \varphi : P \rightarrow \mathbb{R}^n$ of order $\alpha$ of the function $\varphi$ by the formula

$$I_{a+,x}^{\alpha} \varphi(x, y) = \frac{1}{\Gamma(\alpha)} \int_a^x \frac{\varphi(s, y)}{(x-s)^{1-\alpha}} ds,$$

for $(x, y) \in P$ a.e., with the convention that $I_{a+,x}^{0} \varphi = \varphi$. In the Appendix, we show that if $\varphi \in L^1(P, \mathbb{R}^n)$, then $I_{a+,x}^{\alpha} \varphi \in L^1(P, \mathbb{R}^n)$ (in fact, we prove that, for any $1 \leq p < \infty$, $I_{a+,x}^{\alpha} \varphi \in L^p(P, \mathbb{R}^n)$ provided that $\varphi \in L^p(P, \mathbb{R}^n)$). In an analogous way, one defines the operator $I_{c+,y}^{\alpha}$. The semi-group property of these operators is valid for $\alpha_1, \alpha_2 \geq 0$, as is well-known.

The definition of fractional partial derivative $D_{a+,x}^{\alpha} z$ (in Riemann-Liouville sense) with respect to a single variable $x$ on the set $P$, for $\alpha \in (0, 1]$, has been introduced in [2] (see also [8] for a less detailed approach). The following characterization of functions possessing such a derivative has been proved there.

**Theorem 1.1.** Let $\alpha \in (0, 1]$, $z \in L^1(P, \mathbb{R}^n)$. Then $z$ has the derivative $D_{a+,x}^{\alpha} z$ if and only if there exist functions $\varphi \in L^1(P, \mathbb{R}^n)$ and $\mu \in L^1([c, d], \mathbb{R}^n)$ such that

$$z(x, y) = I_{a+,x}^{\alpha} \varphi(x, y) + \frac{1}{\Gamma(\alpha)} \frac{\mu(y)}{(x-a)^{1-\alpha}}, (x, y) \in P \text{ a.e.}$$

In such a case

$$D_{a+,x}^{\alpha} z(x, y) = \varphi(x, y), (x, y) \in P \text{ a.e.},$$

$$I_{a+,x}^{1-\alpha} \mu(a, y) = \mu(y), y \in [c, d] \text{ a.e.}$$

Analogous definition of a fractional partial derivative $D_{c+,y}^{\alpha} z$ with respect to $y$ can be introduced and analogous properties of the class of functions possessing such a derivative can be proved (see [2]).

In a similar way, in [2] (see also [8] for a less detailed approach), the definition of the mixed fractional integral $I_{a+,x;c+,y}^{\alpha,\beta} \varphi$ of a function $\varphi$ of two variables has been introduced. Namely, for $\alpha, \beta > 0$, $\varphi \in L^1(P, \mathbb{R}^n)$ one defines the mixed integral of order $(\alpha, \beta)$ of the function $\varphi$ by

$$I_{a+,x;c+,y}^{\alpha,\beta} \varphi(x, y) = \frac{1}{\Gamma(\alpha) \Gamma(\beta)} \int_{(a,x)\times(c,y)} \frac{\varphi(s, t)}{(x-s)^{1-\alpha}(y-t)^{1-\beta}} d(s, t), (x, y) \in P \text{ a.e.}$$

In the Appendix, we show that if $\varphi \in L^p(P, \mathbb{R}^n)$, then $I_{a+,x;c+,y}^{\alpha,\beta} \varphi \in L^p(P, \mathbb{R}^n)$ (1 $\leq p < \infty$) and

$$I_{a+,x;c+,y}^{\alpha,\beta} \varphi(x, y) = I_{a+,x}^{\alpha} I_{c+,y}^{\beta} \varphi(x, y) = I_{c+,y}^{\beta} I_{a+,x}^{\alpha} \varphi(x, y)$$

for $(x, y) \in P$ a.e. In [2], the definition of the mixed fractional derivative $D_{a+,x;c+,y}^{\alpha,\beta} z$ of a function $z$ of two variables (in Riemann-Liouville sense) on the set $P$, for $\alpha, \beta \in (0, 1]$, has been introduced and the following characterization of functions possessing such a derivative has been derived (below, by $I_{a+,x}^{\alpha} \mu$, $I_{c+,y}^{\beta} \nu$ we denote the integrals of fractional order of functions $\mu$, $\nu$ of one variable, given by $I_{a+,x}^{\alpha} \mu(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \frac{\mu(s)}{(x-s)^{1-\alpha}} ds$, $I_{c+,y}^{\beta} \nu(y) = \frac{1}{\Gamma(\beta)} \int_c^y \frac{\nu(t)}{(y-t)^{1-\beta}} dt$ for $x \in [a, b]$ a.e., $y \in [c, d]$ a.e., respectively).

Theorem 1.2. Let $\alpha, \beta \in (0, 1]$ $z \in L^1(P, \mathbb{R}^n)$. Then $z$ has the mixed fractional derivative $D^{\alpha,\beta}_{a^+,x;c^+,y^+} z$ if and only if there exist functions $\varphi \in L^1(P, \mathbb{R}^n)$, $\mu \in L^1([c, d], \mathbb{R}^n)$, $\nu \in L^1([a, b], \mathbb{R}^n)$ and a constant $e \in \mathbb{R}^n$ such that

$$z(x, y) = I^{\alpha,\beta}_{a^+,x;c^+,y^+} \varphi(x, y) + \frac{1}{\Gamma(\alpha)} \int_a^x \mu(t) \, dt + \frac{1}{\Gamma(\beta)} \int_c^y \nu(t) \, dt$$

$$+ \frac{1}{\Gamma(\alpha) \Gamma(\beta)} \int_a^x \int_c^y e^{x-t} e^{y-s} \, ds \, dt$$

for $(x, y) \in P$ a.e. In such a case

$$D^{\alpha,\beta}_{a^+,x;c^+,y^+} z(x, y) = \varphi(x, y), \quad (x, y) \in P \text{ a.e.},$$

$$D^\alpha_{a^+,x}(I^{1-\beta}_{c^+,y^+} z)(x, c) = \mu(x), \quad x \in [a, b] \text{ a.e.},$$

$$D^\beta_{c^+,y}(I^{1-\alpha}_{a^+,x^+} z)(a, y) = \nu(y), \quad y \in [c, d] \text{ a.e.}$$

Remark 1. One can show that

$$D^{\alpha,\beta}_{a^+,x;c^+,y^+} z(x, y) = \frac{\partial}{\partial x} (I^{1-\alpha,1-\beta}_{a^+,x;c^+,y^+} z)(x, c)$$

for $x \in [a, b]$ a.e. and

$$D^\beta_{c^+,y}(I^{1-\alpha}_{a^+,x^+} z)(a, y) = \frac{\partial}{\partial y} (I^{1-\alpha,1-\beta}_{a^+,x;c^+,y^+} z)(a, y)$$

for $y \in [c, d]$ a.e. Moreover (see [2, Remark 17]), any function $z$ of the form (5) has the fractional derivatives

$$D^\alpha_{a^+,x} z(x, y) = I^\alpha_{c^+,y} \varphi(x, y) + \frac{1}{\Gamma(\beta)} \int_c^y \nu(t) \, dt,$$

$$D^\beta_{c^+,y} z(x, y) = I^\beta_{a^+,x} \varphi(x, y) + \frac{1}{\Gamma(\alpha)} \int_a^x \mu(t) \, dt$$

for $(x, y) \in P$ a.e.

Remark 2. From Remark 1 and Theorem 1.2 it follows that if a function $z$ is of the form (5) and satisfies the conditions

$$\begin{cases}
I^{1-\alpha,1-\beta}_{a^+,x;c^+,y^+} z(x, c) = \gamma(x), \quad x \in [a, b] \\
I^{1-\alpha,1-\beta}_{a^+,x;c^+,y^+} z(a, y) = \delta(y), \quad y \in [c, d]
\end{cases}$$

then

$$\gamma'(x) = \mu(x), \quad x \in [a, b] \text{ a.e.},$$

$$\delta'(y) = \nu(y), \quad y \in [c, d] \text{ a.e.}$$

The set of all functions $z : P \to \mathbb{R}^n$ possessing the fractional derivative $D^{\alpha,\beta}_{a^+,x;c^+,y^+} z$ will be denoted by $AC^{\alpha,\beta}_{a^+,x;c^+,y^+}(P)$; in formulas, we shall write shortly $AC^{\alpha,\beta}_{a^+,x;c^+,y^+}$. In a standard way one can check that the functional $\| \cdot \|_{AC^{\alpha,\beta}_{a^+,x;c^+,y^+}} : AC^{\alpha,\beta}_{a^+,x;c^+,y^+} \to \mathbb{R}^+_0$ given by

$$\| z \|_{AC^{\alpha,\beta}_{a^+,x;c^+,y^+}} = \left( \| D^{\alpha,\beta}_{a^+,x;c^+,y^+} z \|_{L^1(P, \mathbb{R}^n)} + \| D^\alpha_{a^+,x} (I^{1-\beta}_{c^+,y^+} z)(\cdot, c) \|_{L^1([a, b], \mathbb{R}^n)} \
+ \| D^\beta_{c^+,y} (I^{1-\alpha}_{a^+,x^+} z)(a, \cdot) \|_{L^1([c, d], \mathbb{R}^n)} + \| I^{1-\alpha,1-\beta}_{a^+,x;c^+,y^+} z(a, c) \|_{\mathbb{R}^n}\right)$$

is a norm in $AC^{\alpha,\beta}_{a^+,x;c^+,y^+}(P)$ and $AC^{\alpha,\beta}_{a^+,x;c^+,y^+}(P)$ with this norm is complete.
2. Relative weak compactness in \( AC^{\alpha,\beta}_{a+x,c+y}(P) \). Let us start with a characterization of the dual space \((AC^{\alpha,\beta}_{a+x,c+y}(P))^*\).

**Proposition 1.** A functional \( \Lambda : AC^{\alpha,\beta}_{a+x,c+y} \to \mathbb{R} \) is linear and continuous if and only if there exist functions \( h \in L^\infty(P, \mathbb{R}^n) \), \( h_1 \in L^\infty([a, b], \mathbb{R}^n) \), \( h_2 \in L^\infty([c, d], \mathbb{R}^n) \) and a constant \( d \in \mathbb{R}^n \) such that

\[
\Lambda z = \int_{\mathbb{R}^n} h(x, y) D^{\alpha,\beta}_{a+x,c+y} z(x, y) \, dx \, dy + \int_a^b h_1(x, y) \frac{\partial}{\partial x} (I^{1-\alpha,1-\beta}_{a+x,c+y} z)(x, c) \, dx
\]

\[
+ \int_a^b h_2(x, y) \frac{\partial}{\partial y} (I^{1-\alpha,1-\beta}_{a+x,c+y} z)(a, y) \, dy + d I^{1-\alpha,1-\beta}_{a+x,c+y} z(a, c)
\]

for \( z \in AC^{\alpha,\beta}_{a+x,c+y} \)

**Proof.** "Sufficiency" part of the proposition is obvious. So, let us assume that a functional \( \Lambda : AC^{\alpha,\beta}_{a+x,c+y} \to \mathbb{R} \) is linear and continuous and consider the mapping

\[
j : L^1(P, \mathbb{R}^n) \times L^1([a, b], \mathbb{R}^n) \times L^1([c, d], \mathbb{R}^n) \times \mathbb{R}^n \to AC^{\alpha,\beta}_{a+x,c+y}
\]

given by

\[
j(\varphi, \mu, \nu, e) = I^{\alpha,\beta}_{a+x,c+y} \varphi(x, y)
\]

\[
+ \frac{1}{\Gamma(\alpha)} \frac{1}{(x-a)^{1-\alpha}} I^{\beta}_c \nu(y)
\]

\[
+ \frac{1}{\Gamma(\beta)} \frac{1}{(y-c)^{1-\beta}} I^{\alpha}_c \mu(x)
\]

\[
+ \frac{1}{\Gamma(\alpha)} \frac{1}{\Gamma(\beta)} \frac{1}{(x-a)^{1-\alpha}(y-c)^{1-\beta}} e
\]

for \( (\varphi, \mu, \nu, e) \in L^1(P, \mathbb{R}^n) \times L^1([a, b], \mathbb{R}^n) \times L^1([c, d], \mathbb{R}^n) \times \mathbb{R}^n \). Obviously, it is linear and continuous:

\[
\|j(\varphi, \mu, \nu, e)\|_{AC^{\alpha,\beta}_{a+x,c+y}} = \|\varphi\|_{L^1(P, \mathbb{R}^n)} + \|\mu\|_{L^1([c, d], \mathbb{R}^n)} + \|\nu\|_{L^1([c, d], \mathbb{R}^n)} + |e|_{\mathbb{R}^n}
\]

In consequence, functional \( \Lambda \circ j : L^1(P, \mathbb{R}^n) \times L^1([a, b], \mathbb{R}^n) \times L^1([c, d], \mathbb{R}^n) \times \mathbb{R}^n \to \mathbb{R} \) is linear and continuous. So, there exist functions \( h \in L^\infty(P, \mathbb{R}^n) \), \( h_1 \in L^\infty([a, b], \mathbb{R}^n) \), \( h_2 \in L^\infty([c, d], \mathbb{R}^n) \) and a constant \( d \in \mathbb{R}^n \) such that, for \( z \in AC^{\alpha,\beta}_{a+x,c+y} \) being of
Theorem 2.1. A set of one variable on \[ AC \] measurable set \( R \) on \( \mathbb{R} \).

Corollary 1. A sequence \( (z_k) \subset AC^{\alpha,\beta}_{a+,x;c+y} \) is weakly convergent to a \( z_0 \in AC^{\alpha,\beta}_{a+,x;c+y} \) if and only if

(i) \( D^{\alpha,\beta}_{a+,x;c+y}z_k \rightarrow D^{\alpha,\beta}_{a+,x;c+y}z_0 \) weakly in \( L^1(P,\mathbb{R}^n) \)

(ii) \( \frac{\partial}{\partial x}(I_{a+,x;c+y}^{1-\alpha,1-\beta}z_k)(\cdot, c) \rightarrow \frac{\partial}{\partial x}(I_{a+,x;c+y}^{1-\alpha,1-\beta}z_0)(\cdot, c) \) weakly in \( L^1([a, b], \mathbb{R}^n) \)

(iii) \( \frac{\partial}{\partial y}(I_{a+,x;c+y}^{1-\alpha,1-\beta}z_k)(a, \cdot) \rightarrow \frac{\partial}{\partial y}(I_{a+,x;c+y}^{1-\alpha,1-\beta}z_0)(a, \cdot) \) weakly in \( L^1([c, d], \mathbb{R}^n) \)

(iv) \( I_{a+,x;c+y}^{1-\alpha,1-\beta}z_k(a, c) \rightarrow I_{a+,x;c+y}^{1-\alpha,1-\beta}z_0(a, c) \) in \( \mathbb{R}^n \).

Now, we are in a position to derive a characterization of relatively weakly compact sets in \( AC^{\alpha,\beta}_{a+,x;c+y}(P) \). Before we formulate theorem containing such a characterization we recall the notion of equi-absolute integrability of a family of functions on \( P \). Namely, we say that a family of functions \( \{f_s; s \in S\} \subset L^1(P,\mathbb{R}^n) \) is equi-absolutely integrable on \( P \), if for any \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that for any measurable set \( R \subset P \) with measure \( \mu(R) < \delta \) one has \( \int_R |f_s| \, d\mu < \varepsilon \) for any \( s \in S \).

In an analogous way, one defines equi-absolute integrability of a family of functions of one variable on \( [a, b] \) \( ([c, d]) \).

Theorem 2.1. A set \( B \subset AC^{\alpha,\beta}_{a+,x;c+y} \) is relatively weakly compact if and only if

(i) the set \( B_1 = \{D^{\alpha,\beta}_{a+,x;c+y}z; z \in B\} \) is equi-absolutely integrable on \( P \)

(ii) the set \( B_2 = \{\frac{\partial}{\partial x}(I_{a+,x;c+y}^{1-\alpha,1-\beta}z)(\cdot, c); z \in B\} \) is equi-absolutely integrable on \([a, b] \)

(iii) the set \( B_3 = \{\frac{\partial}{\partial y}(I_{a+,x;c+y}^{1-\alpha,1-\beta}z)(a, \cdot); z \in B\} \) is equi-absolutely integrable on \([c, d] \)

(iv) the set \( B_4 = \{I_{a+,x;c+y}^{1-\alpha,1-\beta}z(a, c); z \in B\} \) is bounded in \( \mathbb{R}^n \).

Proof. Necessity. Let us consider a sequence \( \{D^{\alpha,\beta}_{a+,x;c+y}z_{k}\} \) of elements of the set \( B_1 \). Since the set \( B \) is relatively weakly compact, therefore one can choose a subsequence \( \{z_{k_j}\} \) of the sequence \( \{z_k\} \) weakly convergent in \( AC^{\alpha,\beta}_{a+,x;c+y}(P) \) to a function \( z_0 \). From the above corollary it follows that \( D^{\alpha,\beta}_{a+,x;c+y}z_{k_j} \rightarrow D^{\alpha,\beta}_{a+,x;c+y}z_0 \) weakly in
\[ L^1(P, \mathbb{R}^n) \text{.} \] So, the set \( B_1 \) is relatively weakly compact in \( L^1(P, \mathbb{R}^n) \). The Dunford-Pettis theorem (see [1]) implies equi-absolute integrability of \( B_1 \) on \( P \). In the same way, we argue in cases (ii), (iii). In the case (iv), instead of the Dunford-Pettis theorem we use the fact that relatively compact set in \( \mathbb{R}^n \) is bounded.

Sufficiency. Let us assume that conditions (i)-(iv) hold true. Consider any sequence \( (z_k) \subset B \). From (i)-(iv) it follows that there exists a subsequence \( (z_{k_j}) \) and functions \( \varphi \in L^1(P, \mathbb{R}^n) \), \( \mu \in L^1([a, b], \mathbb{R}^n) \), \( \nu \in L^1([c, d], \mathbb{R}^n) \) and a constant \( c \in \mathbb{R}^n \) such that

\[
D_{\alpha + x, c + y}^\beta z_{k_j} \to \varphi \text{ weakly in } L^1(P, \mathbb{R}^n),
\]

\[
\frac{\partial}{\partial x} (I_{\alpha + x, c + y}^{1-\alpha-\beta} z_{k_j})(\cdot, c) \to \mu \text{ weakly in } L^1([a, b], \mathbb{R}^n),
\]

\[
\frac{\partial}{\partial y} (I_{\alpha + x, c + y}^{1-\alpha-\beta} z_{k_j})(a, \cdot) \to \nu \text{ weakly in } L^1([c, d], \mathbb{R}^n),
\]

\[
I_{\alpha + x, c + y}^{1-\alpha-\beta} z_{k_j}(a, c) \to c \text{ in } \mathbb{R}^n.
\]

Let us define a function \( z_0 \in AC_{\alpha + x, c + y}^\beta \) in the following way

\[
z_0(x, y) = I_{\alpha + x, c + y}^{\beta} \varphi(x, y) + \frac{1}{\Gamma(\alpha)} \int_a^x \frac{1}{(x-a)^{1-\alpha}} I_{\beta}^c \nu(y) + \frac{1}{\Gamma(\beta)} \int_c^y \frac{1}{(y-c)^{1-\beta}} I_{\alpha}^a \mu(x)
\]

\[ + \frac{1}{\Gamma(\alpha) \Gamma(\beta)} \int_a^x \int_c^y \frac{\psi(s, t)}{(x-s)^{1-\alpha}(y-t)^{1-\beta}} ds dt + \int_a^x \frac{\psi(s, y)}{(x-s)^{1-\alpha}} ds \]

\[ + \int_c^y \frac{\psi(x, t)}{(y-t)^{1-\beta}} dt \]

for \((x, y) \in P \text{ a.e.} \) In view of Theorem 1.2, Remark 1 and Corollary 1, the above convergences mean that \( z_{k_j} \to z_0 \) weakly in \( L^1(P, \mathbb{R}^n) \) and the proof is completed.

\[ \square \]

3. Gronwall lemma. Now, we shall prove the following Gronwall lemma for functions of two variables with singular integrals.

**Lemma 3.1.** If \( \alpha, \beta > 0, P = [a, b] \times [c, d], k > 0 \), functions \( \psi, \omega \in L^1(P, \mathbb{R}) \) are nonnegative and

\[
\psi(x, y) \leq \omega(x, y) + k \int_a^x \int_c^y \frac{\psi(s, t)}{(x-s)^{1-\alpha}(y-t)^{1-\beta}} ds dt + \int_a^x \frac{\psi(s, y)}{(x-s)^{1-\alpha}} ds \]

\[ + \int_c^y \frac{\psi(x, t)}{(y-t)^{1-\beta}} dt \]

for \((x, y) \in P \text{ a.e.}, \) then

\[
\psi(x, y) \leq D(\omega)(x, y)
\]

where \( D : L^1(P, \mathbb{R}) \to L^1(P, \mathbb{R}) \) is a linear bounded operator depending on \( P, \alpha, \beta, k \).

**Remark.** From the proof of the above lemma it follows that

\[
D(\omega)(x, y)
\]

\[ = \sum_{\nu=0}^{\nu_0} G^\nu (I_{\alpha + x, c + y}^\beta + I_{\alpha + x}^\beta + I_{c + y}^\beta)^\nu \omega(x, y)
\]

\[ + r (I_{\alpha + x, c + y}^{1-\beta} + \sum_{i,j \geq 0, (i+j)\alpha < 1} I_{\alpha + x, c + y}^{(i+j)\alpha} + \sum_{i,j \geq 0, (i+j)\beta < 1} I_{\alpha + x, c + y}^{(i+j)\beta}) \omega(x, y)
\]

where \( G \) and \( \nu_0 \) are given by (6) and (8), respectively.
Proof of Lemma 3.1. Clearly, the main assumption of the lemma can be written in the following way

\[
\psi(x, y) \leq \omega(x, y) + k \Gamma(\alpha) \Gamma(\beta) I^\alpha_{a+, x} I^\beta_{c+, y} \psi(x, y) \\
+ k \Gamma(\alpha) I^\alpha_{a+, x} \psi(x, y) + k \Gamma(\beta) I^\beta_{c+, y} \psi(x, y)
\]

for \((x, y) \in P\) a.e. This inequality implies

\[
\psi(x, y) \leq \omega(x, y) + G(I^\alpha_{a+, x} I^\beta_{c+, y} + I^\alpha_{a+, x} + I^\beta_{c+, y}) \psi(x, y)
\]

for \((x, y) \in P\) a.e., where

\[
G = k \max(\Gamma(\alpha) \Gamma(\beta), \Gamma(\alpha), \Gamma(\beta)). \tag{6}
\]

Consequently, for any \(n \in \mathbb{N}, n \geq 2\),

\[
\psi \leq \omega + \sum_{\nu=1}^{n-1} G^{\nu}(I^\alpha_{a+, x} I^\beta_{c+, y} + I^\alpha_{a+, x} + I^\beta_{c+, y}) \nu \omega + G^{n}(I^\alpha_{a+, x} I^\beta_{c+, y} + I^\alpha_{a+, x} + I^\beta_{c+, y}) \nu \psi \tag{7}
\]

a.e. on \(P\).

Now, let us observe that, for any nonnegative \(\chi \in L^1(P, \mathbb{R})\),

\[
(I^\alpha_{a+, x} I^\beta_{c+, y} + I^\alpha_{a+, x} + I^\beta_{c+, y}) \nu \chi(x, y)
= \sum_{i_1, i_2, i_3 \geq 0, i_1 + i_2 + i_3 = \nu} \nu \binom{\nu}{i_1 i_2 i_3} I^\alpha_{a+, x} I^\beta_{c+, y} I^\alpha_{a+, x} I^\beta_{c+, y} \chi(x, y)
= \sum_{i_1, i_2, i_3 \geq 0, i_1 + i_2 + i_3 = \nu} \nu \binom{\nu}{i_1 i_2 i_3} \frac{1}{\Gamma((i_1 + i_2)\alpha)} \frac{1}{\Gamma((i_1 + i_3)\beta)} \\
\times \int_a^x \int_c^y \frac{\chi(s, t)}{(x - s)^{1-(i_1+i_2)\alpha}(y - t)^{1-(i_1+i_3)\beta}} ds dt.
\]

For sufficiently large \(\nu\), more precisely for \(\nu \geq \nu_0\) where \(\nu_0\) is the smallest positive integer such that

\[
\left\lfloor \frac{\nu_0}{2} \right\rfloor \min\{\alpha, \beta\} > \arg \min\{\Gamma(s); \ s > 0\} \tag{8}
\]
(it is important that arg min \{\Gamma(s); \ s > 0\} > 1), we have
\[
\sum_{i_1, i_2, i_3 \geq 0, \ i_1 + i_2 + i_3 = \nu} \int_a^y \int_c^{x-s} (x-s)^{-1-(i_1+i_2)a}(y-t)^{-1-(i_1+i_3)b} dt dt
\leq \sum_{i_1, i_2, i_3 \geq 0, \ i_1 + i_2 + i_3 = \nu} \left( \frac{\nu}{i_1 i_2 i_3} \right) \frac{(b-a)^{\nu a-1}(d-c)^{\nu b-1}}{\Gamma\left(\frac{\nu}{2}\right) \min\{\alpha, \beta\}} I_{a+\alpha, x}^{1, (i_1+i_3)\alpha} \chi(x, y) + \sum_{i_1, i_2, i_3 \geq 0, \ i_1 + i_2 + i_3 = \nu} \left( \frac{\nu}{i_1 i_2 i_3} \right) \frac{(d-c)^{\nu b-1}}{\Gamma\left(\frac{\nu}{2}\right) \min\{\alpha, \beta\}} I_{a+\beta, x}^{1, (i_1+i_2)\beta} \chi(x, y)
\]
where \(E = \min\{\Gamma(s); \ s > 0\}\). It is easy to see that there are nonnegative integers \(i_1, i_2, i_3\) describing the last sum \(^1\). Let us put
\[
B = I_{a+\alpha, x}^{1, \alpha, \beta} + \sum_{i, j \geq 0, \ (i+j)\alpha < 1} I_{a+\alpha, x}^{1, (i+j)\alpha} + \sum_{i, j \geq 0, \ (i+j)\beta < 1} I_{a+\beta, x}^{1, (i+j)\beta}.
\]
Since the above sum consists of a finite number of terms, the operator \(B : L^1(P, \mathbb{R}) \to L^1(P, \mathbb{R})\) is linear and bounded. In consequence, for \(\nu > \nu_0\),
\[
G^\nu (I_{a+\alpha, x}^{1, \alpha, \beta} + I_{a+\alpha, x}^{1, \alpha, \beta} + I_{a+\alpha, x}^{1, \alpha, \beta})^{\nu} (\chi)(x, y) \leq G^\nu 3^\nu \frac{(b-a)^{\nu a-1}(d-c)^{\nu b-1}}{\Gamma\left(\frac{\nu}{2}\right) \min\{\alpha, \beta\}} E + \frac{(b-a)^{\nu a-1}}{\Gamma\left(\frac{\nu}{2}\right) \min\{\alpha, \beta\}} + \frac{(d-c)^{\nu b-1}}{\Gamma\left(\frac{\nu}{2}\right) \min\{\alpha, \beta\}} B(\chi)(x, y).
\]
So, denoting
\[
b_\nu = \frac{G^\nu 3^\nu}{\Gamma\left(\frac{\nu}{2}\right) \min\{\alpha, \beta\}} + (b-a)^{\nu a-1} + (d-c)^{\nu b-1}
\]
for \(\nu > \nu_0\) we have
\[
G^\nu (I_{a+\alpha, x}^{1, \alpha, \beta} + I_{a+\alpha, x}^{1, \alpha, \beta} + I_{a+\alpha, x}^{1, \alpha, \beta})^{\nu} (\chi)(x, y) \leq b_\nu B(\chi)(x, y)
\]
\(^1\)Indeed, let us suppose that there exist nonnegative integers \(i_1, i_2, i_3\) such that \((i_1 + i_2)\alpha < 1, (i_1 + i_3)\beta < 1, i_1 + i_2 + i_3 = \nu\) where \(\nu \geq \nu_0\). Then
\[
\nu \min\{\alpha, \beta\} = (i_1 + i_2 + i_3) \min\{\alpha, \beta\} \leq (i_1 + i_2)\alpha + (i_1 + i_3)\beta < 2
\]
and, in consequence,
\[
1 < \arg \min\{\Gamma(s); \ s > 0\} < \frac{\nu_0}{2} \min\{\alpha, \beta\} \leq \frac{\nu_0}{2} \min\{\alpha, \beta\} \leq \frac{\nu}{2} \min\{\alpha, \beta\} \leq 1.
\]
for \( \nu > \nu_0 \). Convergence of the series \( \sum_{\nu=\nu_0+1}^{\infty} b_\nu \) implies the pointwise convergence (also in \( L^1(P,\mathbb{R}^n) \)) of the series
\[
\sum_{\nu=1}^{\infty} G'(I^\alpha_{a,x}I^\beta_{c,y} + I^\alpha_{a,x} + I^\beta_{c,y})^\nu(\chi)(x,y).
\]
Thus, (7) gives
\[
\psi(x, y) \leq \sum_{\nu=1}^{\infty} G'(I^\alpha_{a,x}I^\beta_{c,y} + I^\alpha_{a,x} + I^\beta_{c,y})^\nu \omega(x, y)
\]
\[
\leq \sum_{\nu=1}^{\nu_0} G'(I^\alpha_{a,x}I^\beta_{c,y} + I^\alpha_{a,x} + I^\beta_{c,y})^\nu \omega(x, y)
\]
\[
+ r(I^{1,1}_{a,x;c+y} + \sum_{i,j \geq 0, (i+j)\alpha < 1} I^{(i+j)\alpha}_{a,x;w+y} + \sum_{i,j \geq 0, (i+j)\beta < 1} I^{(i+j)\beta}_{a,x;w+y}) \omega(x, y)
\]
\[
= A(\omega)(x, y) + rB(\omega)(x, y)
\]
for \((x, y) \in P\) a.e. where \( r = \sum_{\nu=\nu_0+1}^{\infty} b_\nu \) and the linear bounded operator \( A : L^1(P,\mathbb{R}) \to L^1(P,\mathbb{R}) \) is given by
\[
A = \sum_{\nu=0}^{\nu_0} G'(I^\alpha_{a,x}I^\beta_{c,y} + I^\alpha_{a,x} + I^\beta_{c,y})^\nu.
\]
Putting
\[
D(\omega) = A(\omega) + rB(\omega)
\]
we complete the proof. \( \square \)

4. Fractional Foranisi-Marchesini model. Now, let us consider system (1)-(2) with \( \alpha, \beta \in (0, 1] \) and \( \gamma : [a, b] \to \mathbb{R}^n \), \( \delta : [c, d] \to \mathbb{R}^n \) being absolutely continuous functions such that \( \gamma(a) = \delta(c) \). By a solution of problem (1)-(2) we mean a function \( z \) possessing the fractional derivative \( D^\alpha_{a,x,c+y}z \), satisfying system (1) a.e. on \( P \) and initial conditions (2).

In [2], the following theorem has been derived.

**Theorem 4.1.** If \( f : P \times (\mathbb{R}^n)^3 \times \mathbb{R}^m \to \mathbb{R}^n \) satisfies the conditions
\( I_f \) is measurable with respect to \((x, y) \in P\), continuous with respect to \( u \in \mathbb{R}^m \) and lipschitzian with respect to \((z, z_1, z_2) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n\), i.e. there exists a constant \( L > 0 \) such that
\[
|f(x, y, z, z^1, z^2, u) - f(x, y, w, w^1, w^2, u)| \leq L(|z - w| + |z^1 - w^1| + |z^2 - w^2|)
\]
for \((x, y) \in P\) a.e. and any \( z, z^1, z^2, w, w^1, w^2 \in \mathbb{R}^n \), \( u \in \mathbb{R}^m \),
\( II_f \) for any \( u \in L^1(P,\mathbb{R}^m) \), \( f(\cdot, 0, 0, 0, u(\cdot, \cdot)) \in L^1(P,\mathbb{R}^n) \),
then, for any control \( u \in L^1(P,\mathbb{R}^m) \), there exists a unique solution \( z_u \in AC^\alpha_{a,x,c+y} \) to (1)-(2), corresponding to \( u \).

We shall prove the following
Theorem 4.2. Let \( d \in L^1(P, \mathbb{R}) \) be a fixed function and

\[
\mathcal{U}_d = \{ u \in L^1(P, \mathbb{R}^m); \ |f(x,y,0,0,0,u(x,y))| \leq d(x,y) \text{ for } (x,y) \in P \text{ a.e.} \}.
\]

Then the set of solutions to (1)-(2), corresponding to controls \( u \in \mathcal{U}_d \) is relatively weakly compact in the space \( AC^{\alpha,\beta}_{\alpha,\gamma} (P, \mathbb{R}^n) \).

Proof. It is easy to see (cf. [2]) that, for any fixed \( u \in L^1(P, \mathbb{R}^m) \), problem (1)-(2) has a unique solution \( z_u \) if and only if the following integral equation

\[
\varphi(x,y) = f(x,y,\Gamma_{\alpha,\gamma}^\alpha\varphi(x,y) + \frac{1}{\Gamma(\alpha)} \frac{1}{(x-a)^{1-\alpha}} J_{c+}^\beta (\delta')(y)
+ \frac{1}{\Gamma(\beta)} \frac{1}{(y-c)^{1-\beta}} \Gamma_{\alpha}^\alpha (\gamma')(x) + \frac{1}{\Gamma(\beta)} \frac{1}{(x-a)^{1-\alpha}} (y-c)^{1-\beta} \gamma'(x), I_{\alpha}^\alpha \varphi(x,y) + \frac{1}{\Gamma(\alpha)} \frac{1}{(x-a)^{1-\alpha}} \delta'(y), u(x,y))
\]

for \((x,y) \in P \text{ a.e.}, \) where \( e = \gamma(a) = \delta(c), \) has a unique solution \( \varphi_u \) in the space \( L^1(P, \mathbb{R}^m). \) In such a case, the function

\[
z_u(x,y) = I_{\alpha,\gamma}^\alpha \varphi_u(x,y) + \frac{1}{\Gamma(\alpha)} \frac{1}{(x-a)^{1-\alpha}} J_{c+}^\beta (\delta')(y)
+ \frac{1}{\Gamma(\beta)} \frac{1}{(y-c)^{1-\beta}} \Gamma_{\alpha}^\alpha (\gamma')(x)
+ \frac{1}{\Gamma(\alpha)} \frac{1}{\Gamma(\beta)} \frac{1}{(x-a)^{1-\alpha}} (y-c)^{1-\beta}, \quad (x,y) \in P \text{ a.e.}
\]

is the solution to (1)-(2).

Let us observe that

\[
|\varphi_u(x,y)| = |f(x,y,\Gamma_{\alpha,\gamma}^\alpha\varphi_u(x,y) + \frac{1}{\Gamma(\alpha)} \frac{1}{(x-a)^{1-\alpha}} J_{c+}^\beta (\delta')(y)
+ \frac{1}{\Gamma(\beta)} \frac{1}{(y-c)^{1-\beta}} \Gamma_{\alpha}^\alpha (\gamma')(x) + \frac{1}{\Gamma(\beta)} \frac{1}{(x-a)^{1-\alpha}} (y-c)^{1-\beta} \gamma'(x), I_{\alpha}^\alpha \varphi_u(x,y) + \frac{1}{\Gamma(\alpha)} \frac{1}{(x-a)^{1-\alpha}} \delta'(y), u(x,y)) |
\]

\[
\leq L(\Gamma_{\alpha,\gamma}^\alpha \varphi_u |(x,y), I_{\alpha,\gamma}^\alpha \varphi_u |(x,y) + \frac{1}{\Gamma(\alpha)} \frac{1}{(x-a)^{1-\alpha}} J_{c+}^\beta |(\delta'), \Gamma_{\alpha}^\alpha |(\gamma'), I_{\alpha}^\alpha \varphi_u |(x,y) + \frac{1}{\Gamma(\beta)} \frac{1}{(y-c)^{1-\beta}} \gamma'(x), I_{\alpha}^\alpha \varphi_u |(x,y) + \frac{1}{\Gamma(\alpha)} \frac{1}{\Gamma(\beta)} \frac{1}{(x-a)^{1-\alpha}} |\delta'(y)|, u(x,y)) |
\]

\[
\leq k \int_a^x \int_c^y |\varphi_u(s,t)| \frac{1}{(x-s)^{1-\alpha}(y-t)^{1-\beta}} dsdt + \int_a^x \int_c^y |\varphi_u(s,y)| \frac{1}{(x-s)^{1-\alpha}(y-t)^{1-\beta}} dsdt
+ \int_c^y \varphi_u(x,t) \frac{1}{(y-t)^{1-\beta}} dt + \omega(x,y)
\]
where
\[
\omega(x, y) = L \frac{1}{\Gamma(\alpha)} (I_{x}^{\beta} |\beta'| (y) + |\beta'(y)|) + \frac{L}{\Gamma(\beta)} (I_{y}^{\alpha} |\gamma'| (x) + |\gamma'(x)|)
\]
\[
+ \frac{L}{\Gamma(\alpha)} \frac{1}{\Gamma(\beta)} (x-a)^{1-\alpha} (y-c)^{1-\beta} + d(x, y),
\]
\[
k = L \max\{ \frac{1}{\Gamma(\alpha)} \frac{1}{\Gamma(\beta)} \frac{1}{\Gamma(\alpha)} \frac{1}{\Gamma(\beta)} \}.
\]
Applying the Gronwall lemma with the function \(\psi = |\varphi_u|\) we state that
\[
|\varphi_u(x, y)| \leq D(\omega)(x, y)
\]
for \((x, y) \in P\) a.e. So, the functions \(\varphi_u\) for \(u \in U_d\) are commonly pointwise bounded by the integrable function \(D(\omega)\). Since
\[
D_{\alpha+,x,c+,y}^{\beta+} = \varphi_u,
\]
\[
\frac{\partial}{\partial x} (I_{x}^{\beta} z_u (\cdot, c)) = \gamma',
\]
\[
\frac{\partial}{\partial y} (I_{y}^{\alpha} z_u (a, \cdot)) = \delta',
\]
\[
I_{x}^{\beta} z_u (a, c) = e,
\]
therefore the sets \(B_1, B_2, B_3\) and \(B_4\) from Theorem 2.1 with \(B = \{ z_u \in AC_{\alpha+,x,c+,y}^{\beta+}; u \in U_d \}\) are equi-absolutely integrable and bounded, respectively. This means that the set \(B\) is relatively weakly compact in \(AC_{\alpha+,x,c+,y}^{\beta+}\). The proof is completed. \(\square\)

5. Appendix. In the Riemann-Liouville fractional calculus for functions of two variables the integrals
\[
\int_{a}^{x} \frac{\varphi(s, y)}{(x-s)^{1-\alpha}} ds,
\]
\[
\iint_{(a,x) \times (c,y)} \frac{\varphi(s, t)}{(x-s)^{1-\alpha} (y-t)^{1-\beta}} d(s, t),
\]
where \(\varphi \in L^1(P, \mathbb{R}^n)\), play the fundamental role. Below, we derive some basic properties of them.

5.1. A single integral. Let \(\alpha > 0\) and \(\varphi \in L^1(P, \mathbb{R}^n_+)\). The function
\[
Q_{\Delta} \ni (x, s, y) \mapsto \frac{\varphi(s, y)}{(x-s)^{1-\alpha}} \in \mathbb{R}^+_0,
\]
where \(Q_{\Delta} = P_{\Delta} \times (c, d)\) with \(P_{\Delta} = \{ (x, s) \in \mathbb{R}^2; x \in (a, b), s \in (a, x) \}\), is measurable as a product of two measurable functions:
\[
Q_{\Delta} \ni (x, s, y) \mapsto \varphi(s, y) \in \mathbb{R}^+_0
\]
and
\[
Q_{\Delta} \ni (x, s, y) \mapsto \frac{1}{(x-s)^{1-\alpha}} \in \mathbb{R}^+_0.
\]
Since it is nonnegative, therefore it is integrable (i.e., the Lebesgue integral of it exists). In fact, it is summable. Indeed, from Tonelli theorem it follows that, for \((x, y) \in P\) a.e., the function

\[
(a, x) \ni s \mapsto \frac{\varphi(s, y)}{(x - s)^{1-\alpha}} \in \mathbb{R}_0^+
\]

is integrable, the function

\[
P \ni (x, y) \mapsto \int_a^x \frac{\varphi(s, y)}{(x - s)^{1-\alpha}} ds \in \mathbb{R}_0^+ \cup \{\pm \infty\}
\]

(defined a.e. on \(P\)) is integrable and

\[
\iint_{Q_\triangle} \frac{\varphi(s, y)}{(x - s)^{1-\alpha}} d(s, x, y) = \int_p \int_a^x \frac{\varphi(s, y)}{(x - s)^{1-\alpha}} ds dx \quad d(s, y).
\]

(10)

**Remark 4.** In the same way, using Tonelli theorem, we assert that, for \((s, y) \in P\) a.e., the function

\[
(s, b) \ni x \mapsto \frac{\varphi(s, y)}{(x - s)^{1-\alpha}} \in \mathbb{R}_0^+
\]

is integrable, the function

\[
P \ni (s, y) \mapsto \int_s^b \frac{\varphi(s, y)}{(x - s)^{1-\alpha}} dx \in \mathbb{R}_0^+ \cup \{\pm \infty\}
\]

(defined a.e. on \(P\)) is integrable and

\[
\iint_{Q_\triangle} \frac{\varphi(s, y)}{(x - s)^{1-\alpha}} d(s, x, y) = \int_p \int_s^b \frac{\varphi(s, y)}{(x - s)^{1-\alpha}} dx dy \quad d(s, y).
\]

(11)

Similarly, for \((x, s) \in P_\triangle\) a.e., the function

\[
(c, d) \ni y \mapsto \frac{\varphi(s, y)}{(x - s)^{1-\alpha}} \in \mathbb{R}_0^+
\]

is integrable, the function

\[
P_\triangle \ni (x, s) \mapsto \int_c^d \frac{\varphi(s, y)}{(x - s)^{1-\alpha}} dy \in \mathbb{R}_0^+ \cup \{\pm \infty\}
\]

(defined a.e. on \(P_\triangle\)) is integrable and

\[
\iint_{Q_\triangle} \frac{\varphi(s, y)}{(x - s)^{1-\alpha}} d(s, x, y) = \int_p \int_c^d \frac{\varphi(s, y)}{(x - s)^{1-\alpha}} dy dx \quad d(s, x).
\]

(12)

Thus,

\[
\int_p \int_a^x \frac{\varphi(s, y)}{(x - s)^{1-\alpha}} ds dx \quad d(s, y) = \int_p \int_a^x \frac{\varphi(s, y)}{(x - s)^{1-\alpha}} ds dx 
\]

\[
= \int_p \int_s^b \frac{\varphi(s, y)}{(x - s)^{1-\alpha}} dx \quad d(s, y) = \int_p \varphi(s, y) \int_s^b \frac{dx}{(x - s)^{1-\alpha}} 
\]

\[
= \int_p \varphi(s, y) \frac{1}{\alpha} (b - s)^\alpha d(s, y) \leq \frac{1}{\alpha} (b - a) \int_p \varphi(s, y) d(s, y) < +\infty.
\]

It means that if \(\varphi \in L^1(P, \mathbb{R}_0^+)\), then function (9) is summable on \(P\). Especially, it is finite a.e. on \(P\).
Now, let us assume that $\varphi \in L^1(P, \mathbb{R})$. In the same way as in the case of a nonnegative function, we assert that the function

$$Q_\triangle \ni (x, s, y) \mapsto \frac{\varphi(s, y)}{(x - s)^{1 - \alpha}} \in \mathbb{R}$$

(13)

is measurable. Consequently, the function

$$Q_\triangle \ni (x, s, y) \mapsto \frac{|\varphi(s, y)|}{(x - s)^{1 - \alpha}} \in \mathbb{R}^+_0$$

is measurable. From the previous case it follows that it is summable. Since

$$\left| \frac{\varphi(s, y)}{(x - s)^{1 - \alpha}} \right| = \frac{|\varphi(s, y)|}{(x - s)^{1 - \alpha}},$$

therefore function (13) is summable, too. Fubini theorem implies that the function

$$P \ni (x, y) \mapsto \int_a^x \frac{\varphi(s, y)}{(x - s)^{1 - \alpha}} ds \in \mathbb{R}$$

is defined a.e. on $P$, summable and formula (10) holds true. Of course, formulas (11), (12) remain true.

Thus, we have

**Proposition 2.** If $\alpha > 0$, $\varphi \in L^1(P, \mathbb{R}^n)$, then the function

$$P \ni (x, y) \mapsto \int_a^x \frac{\varphi(s, y)}{(x - s)^{1 - \alpha}} ds \in \mathbb{R}^n$$

belongs to $L^1(P, \mathbb{R}^n)$.

Although the next proposition is used in the previous section in the case of $p = 1$, due to potential applications, we prove it in the general case of $1 \leq p < \infty$. The idea of this proof comes from [5] where theorem of such a type has been derived for the functions of one variable.

**Proposition 3.** Let $\alpha > 0$, $1 \leq p < \infty$. If $\varphi \in L^p(P, \mathbb{R}^n)$, then

$$|I_{a^+}^\alpha \varphi(x, y)|^p \leq \frac{(b - a)^{\alpha p - 1}}{\Gamma(\alpha + 1)p - 1} I_{a^+}^\alpha |\varphi|^p(x, y)$$

(14)

for $(x, y) \in P$ a.e. and the operator $I_{a^+}^\alpha : L^p(P, \mathbb{R}^n) \to L^p(P, \mathbb{R}^n)$ is bounded, i.e.

$$\|I_{a^+}^\alpha \varphi\|_{L^p(P, \mathbb{R}^n)} \leq \frac{(b - a)^\alpha}{\Gamma(\alpha + 1)} \|\varphi\|_{L^p(P, \mathbb{R}^n)}$$

for $\varphi \in L^p(P, \mathbb{R}^n)$. 

Proof. For $p = 1$ the inequality (14) is obvious. So, let us assume that $1 < p < \infty$ and $\varphi \in L^p(P, \mathbb{R}^n)$. We have

$$
|I_{a+x}^\alpha \varphi(x, y)|^p \leq \frac{1}{\Gamma(\alpha)^p} \left( \int_a^x |\varphi(s, y)|^p \frac{ds}{(x-s)^{1-\alpha}} \right)^p = \frac{1}{\Gamma(\alpha)^p} \left( \int_a^x \left| \varphi(s, y) \right|^p \left( \frac{1}{(x-s)^{1-\alpha}} \right)^\frac{1}{p} \frac{1}{(x-s)^{1-\alpha}} \frac{1}{(x-s)^{1-\alpha}} \right)^p \\
\leq \frac{1}{\Gamma(\alpha)^p} \int_a^x |\varphi(s, y)|^p \left( \int_a^x \frac{ds}{(x-s)^{1-\alpha}} \right)^{p-1} \\
= \frac{1}{\Gamma(\alpha)^p} I_{a+x}^\alpha |\varphi|^p (x, y) \left( \frac{1}{\alpha} (x-a)^\alpha \right)^{p-1} \\
\leq \frac{((b-a)^\alpha)^{p-1}}{\Gamma(\alpha+1)^p} I_{a+x}^\alpha |\varphi|^p (x, y)
$$

for $(x, y) \in P$ a.e. Thus, $I_{a+x}^\alpha \varphi \in L^p(P, \mathbb{R}^n)$. Moreover, for $1 \leq p < \infty$, we have

$$
\left\| I_{a+x}^\alpha \varphi \right\|_{L^p(P, \mathbb{R}^n)}^p = \int_a^b \int_c^d |I_{a+x}^\alpha \varphi(x, y)|^p dydx \\
\leq \frac{((b-a)^\alpha)^{p-1}}{\Gamma(\alpha+1)^p} \frac{1}{\Gamma(\alpha)} \int_a^b \int_c^d \int_a^x |\varphi(s, y)|^p \frac{dsdy}{(x-s)^{1-\alpha}}.
$$

Let us observe that

$$
\int_a^b \int_c^d \int_a^x \frac{|\varphi(s, y)|^p}{(x-s)^{1-\alpha}} dsdydx = \int_c^d \int_a^b \int_a^x \frac{|\varphi(s, y)|^p}{(x-s)^{1-\alpha}} dsdx dy \\
= \int_c^d \int_a^b \int_a^x |\varphi(s, y)|^p \frac{dx}{(x-s)^{1-\alpha}} dsdy \\
= \int_a^b \int_c^d \int_a^x |\varphi(s, y)|^p \frac{dx}{(x-s)^{1-\alpha}} dsdy = \frac{1}{\alpha} (b-a)^\alpha \int_c^d \int_a^x |\varphi(s, y)|^p dsdy.
$$

So,

$$
\left\| I_{a+x}^\alpha \varphi \right\|_{L^p(P, \mathbb{R}^n)}^p \leq \frac{((b-a)^\alpha)^{p-1}}{\Gamma(\alpha+1)^p} \frac{1}{\Gamma(\alpha)} \frac{(b-a)^\alpha}{\alpha} \int_c^d \int_a^b |\varphi(s, y)|^p dsdy \\
= \frac{((b-a)^\alpha)^p}{\Gamma(\alpha+1)^p} \left\| \varphi \right\|_{L^p(P, \mathbb{R}^n)}^p.
$$

The proof is completed. \qed

5.2. A double integral. Let $\alpha, \beta > 0$ and $\varphi \in L^1(P, \mathbb{R}^n)$. We shall show that, for $(x, y) \in P$ a.e., there exists the finite double integral

$$
\int \int_{(a,x) \times (c,y)} \frac{\varphi(s, t)}{(x-s)^{1-\alpha}(y-t)^{1-\beta}} dsdt,
$$

and the function

$$
P \ni (x, y) \mapsto \int \int_{(a,x) \times (c,y)} \frac{\varphi(s, t)}{(x-s)^{1-\alpha}(y-t)^{1-\beta}} dsdt \in \mathbb{R}
$$

is summable.
First, let us assume that \( \varphi \in L^1(P, \mathbb{R}^+) \). Then the function

\[
P_{\vartriangle} \ni (x, y, s, t) \mapsto \frac{\varphi(s, t)}{(x - s)^{1-\alpha}(y - t)^{1-\beta}} \in \mathbb{R}^+_0
\]

where \( P_{\vartriangle} = \{(x, y, s, t); (x, y) \in \text{Int} P, s \in (a, x), t \in (c, y)\} \), being nonnegative and measurable, is integrable. Tonelli theorem implies that functions

\[
\psi_1 : P \ni (x, y) \mapsto \int_{(a, x) \times (c, y)} \frac{\varphi(s, t)}{(x - s)^{1-\alpha}(y - t)^{1-\beta}} d(s, t) \in \mathbb{R}^+_0 \cup \{+\infty\},
\]

\[
\psi_2 : P \ni (s, t) \mapsto \int_{(s, b) \times (t, d)} \frac{\varphi(s, t)}{(x - s)^{1-\alpha}(y - t)^{1-\beta}} d(x, y) \in \mathbb{R}^+_0 \cup \{+\infty\}
\]

(defined a.e. on \( P \)) are integrable and

\[
\int_P \psi_1(x, y) d(x, y) = \int_P \psi_2(s, t) d(s, t)
\]

(here, we used the fact that section \((P_{\vartriangle})(s, t)\) of the set \( P_{\vartriangle} \) at a point \((s, t) \in \text{Int} P\) is of the form

\[
(P_{\vartriangle})(s, t) = \{(x, y); (x, y, s, t) \in P_{\vartriangle}\} = (s, b) \times (t, d).
\]

But

\[
\int_P \psi_2(s, t) d(s, t) = \int_P \varphi(s, t) \int_{(s, b) \times (t, d)} \frac{d(x, y)}{(x - s)^{1-\alpha}(y - t)^{1-\beta}} d(s, t)
\]

\[
= \int_P \varphi(s, t) \int_y^b \frac{dy}{(y - t)^{1-\beta}} \int_x^s \frac{dx}{(x - s)^{1-\alpha}} d(s, t)
\]

\[
= \int_P \varphi(s, t) \cdot \frac{1}{\beta} (d - t)^{\beta} \cdot \frac{1}{\alpha} (b - s)^\alpha d(s, t) < +\infty
\]

(to obtain the second equality we used Tonelli theorem).

So, the function \( \psi_1 \) is summable on \( P \). In particular, it means that \( \psi_1 \) takes finite values a.e. on \( P \), i.e.

\[
\psi_1(x, y) = \int_{(a, x) \times (c, y)} \frac{\varphi(s, t)}{(x - s)^{1-\alpha}(y - t)^{1-\beta}} d(s, t) \in \mathbb{R}^+_0
\]

for \((x, y) \in P\) a.e. Of course, Tonelli theorem implies that

\[
\int_{(a, x) \times (c, y)} \frac{\varphi(s, t)}{(x - s)^{1-\alpha}(y - t)^{1-\beta}} d(s, t) = \int_a^x \left( \int_c^y \frac{\varphi(s, t)}{(x - s)^{1-\alpha}(y - t)^{1-\beta}} dt \right) ds
\]

(15)

Now, assume that \( \varphi \in L^1(P, \mathbb{R}) \). From the previous case it follows that the function

\[
P_{\vartriangle} \ni (x, y, s, t) \mapsto \frac{|\varphi(s, t)|}{(x - s)^{1-\alpha}(y - t)^{1-\beta}} \in \mathbb{R}^+_0
\]

is summable. Since

\[
\frac{\varphi(s, t)}{(x - s)^{1-\alpha}(y - t)^{1-\beta}} \leq \frac{|\varphi(s, t)|}{(x - s)^{1-\alpha}(y - t)^{1-\beta}},
\]

...
therefore the function
\[ P_{\triangle} \ni (x, y, s, t) \mapsto \frac{\varphi(s, t)}{(x - s)^{1-\alpha}(y - t)^{1-\beta}} \in \mathbb{R}^+ \]
is summable and from Fubini theorem it follows that, for \((x, y) \in P\) a.e., there exists the finite double integral
\[
\iint_{(a,x) \times (c,y)} \frac{\varphi(s, t)}{(x - s)^{1-\alpha}(y - t)^{1-\beta}} d(s, t),
\]
and the function
\[ P \ni (x, y) \mapsto \iint_{(a,x) \times (c,y)} \frac{\varphi(s, t)}{(x - s)^{1-\alpha}(y - t)^{1-\beta}} d(s, t) \in \mathbb{R} \]
(defined a.e. on \(P\)) is summable. Equalities \((15)\) follow from the Fubini theorem applied to the function
\[(a, x) \times (c, y) \ni (s, t) \mapsto \frac{\varphi(s, t)}{(x - s)^{1-\alpha}(y - t)^{1-\beta}} \in \mathbb{R}.\]

Thus, we have

**Proposition 4.** If \(\varphi \in \mathcal{L}^1(P, \mathbb{R}^n)\), then the function
\[ P \ni (x, y) \mapsto \iint_{(a,x) \times (c,y)} \frac{\varphi(s, t)}{(x - s)^{1-\alpha}(y - t)^{1-\beta}} d(s, t) \in \mathbb{R}^n \]
belongs to \(\mathcal{L}^1(P, \mathbb{R}^n)\) and
\[
\iint_{(a,x) \times (c,y)} \frac{\varphi(s, t)}{(x - s)^{1-\alpha}(y - t)^{1-\beta}} d(s, t) = \int_a^x \left( \frac{1}{(x - s)^{1-\alpha}} \int_c^y \frac{\varphi(s, t)}{(y - t)^{1-\beta}} dt \right) ds
= \int_c^y \left( \frac{1}{(y - t)^{1-\beta}} \int_a^x \frac{\varphi(s, t)}{(x - s)^{1-\alpha}} ds \right) dt
\]
for \((x, y) \in P\) a.e.

Similarly as in the case of the operator \(I_{a+, x}^{\alpha, \beta}\), the next proposition is used in the case of \(p = 1\) but due to potential applications, we prove it in the general case of \(1 \leq p < \infty\). Also in this case, the idea of the proof comes from [5].

**Proposition 5.** Let \(1 \leq p < \infty\). If \(\varphi \in \mathcal{L}^p(P, \mathbb{R}^n)\) then
\[
\left| I_{a+, x; c, y}^{\alpha, \beta}\varphi(x, y) \right|^p \leq \frac{(b - a)^{\alpha - 1}}{\Gamma(\alpha + 1)} \frac{(d - c)^{\beta - 1}}{\Gamma(\beta + 1)} \int_a^x \int_c^y \left| \varphi(s, t) \right|^p ds dt \leq \frac{(a - b)^{\alpha}}{\Gamma(\alpha + 1)} \frac{(d - c)^{\beta}}{\Gamma(\beta + 1)} \left\| \varphi \right\|_{\mathcal{L}^p(P, \mathbb{R}^n)} (16)
\]
for \((x, y) \in P\) a.e. and the operator \(I_{a+, x; c, y}^{\alpha, \beta} : \mathcal{L}^p(P, \mathbb{R}^n) \rightarrow \mathcal{L}^p(P, \mathbb{R}^n)\) is bounded, i.e.
\[
\left\| I_{a+, x; c, y}^{\alpha, \beta}\varphi \right\|_{\mathcal{L}^p(P, \mathbb{R}^n)} \leq \frac{(b - a)^{\alpha}}{\Gamma(\alpha + 1)} \frac{(d - c)^{\beta}}{\Gamma(\beta + 1)} \left\| \varphi \right\|_{\mathcal{L}^p(P, \mathbb{R}^n)} \text{ for } \varphi \in \mathcal{L}^p(P).
Proof. For $p = 1$ the inequality (16) is obvious. So, let us assume that $1 < p < \infty$ and $\varphi \in L^p(P, \mathbb{R}^n)$. We have

\[
\left| I_{a+\alpha, y}^{\alpha, \beta} \varphi(x, y) \right|^p \leq \frac{1}{\Gamma(\alpha)^p} \frac{1}{\Gamma(\beta)^p} \left( \int_a^x \int_c^y \frac{|\varphi(s, t)|}{(x-s)^{1-\alpha}(y-t)^{1-\beta}} dt ds \right)^p
\]

\[
= \frac{1}{\Gamma(\alpha)^p} \frac{1}{\Gamma(\beta)^p}
\]

\[
\times \left( \int_a^x \int_c^y \frac{|\varphi(s, t)|^p}{(x-s)^{1-\alpha}(y-t)^{1-\beta}} \right)^\frac{1}{p} \left( \frac{1}{(x-s)^{1-\alpha}(y-t)^{1-\beta}} \right)^{1-\frac{1}{p}} dt ds
\]

\[
\leq \frac{1}{\Gamma(\alpha)^p} \frac{1}{\Gamma(\beta)^p} \int_a^x \int_c^y \frac{|\varphi(s, t)|^p}{(x-s)^{1-\alpha}(y-t)^{1-\beta}} dt ds \left( \int_a^x \int_c^y \frac{dt}{(x-s)^{1-\alpha}} \right)^{p-1} \left( \int_c^y \frac{ds}{(y-t)^{1-\beta}} \right)^{p-1}
\]

\[
= \frac{1}{\Gamma(\alpha)^p} \frac{1}{\Gamma(\beta)^p} \int_a^x \int_c^y \frac{|\varphi(s, t)|^p}{(x-s)^{1-\alpha}(y-t)^{1-\beta}} dt ds \left( \frac{1}{\alpha} (x-a)^{\alpha} (1-y+c)^{\beta} \right)^{p-1}
\]

\[
= \frac{1}{\Gamma(\alpha)^p} \frac{1}{\Gamma(\beta)^p} \frac{1}{(x-a)^{\alpha} (y-c)^{\beta}} \left( \frac{1}{\alpha} (x-a)^{\alpha} (1-y+c)^{\beta} \right)^{p-1}
\]

\[
= \frac{(b-a)^{\alpha} (d-c)^{\beta}}{\Gamma(\alpha+1)^{p-1} \Gamma(\beta+1)^{p-1}} \left( \frac{1}{\alpha} (x-a)^{\alpha} (1-y+c)^{\beta} \right)^{p-1}
\]

\[
\leq \frac{(b-a)^{\alpha} (d-c)^{\beta}}{\Gamma(\alpha+1)^{p-1} \Gamma(\beta+1)^{p-1}} \left( \frac{1}{\alpha} (x-a)^{\alpha} (1-y+c)^{\beta} \right)^{p-1}
\]

for $(x, y) \in P$ a.e. So, for $1 \leq p < \infty$, $I_{a+\alpha, y}^{\alpha, \beta} \varphi \in L^p(P, \mathbb{R}^n)$ and

\[
\left\| I_{a+\alpha, y}^{\alpha, \beta} \varphi \right\|_{L^p(P, \mathbb{R}^n)}^p = \int_a^b \int_c^d \int_e^f \frac{|\varphi(s, t)|^p}{(x-s)^{1-\alpha}(y-t)^{1-\beta}} dt ds dy dx
\]

\[
\leq \frac{(b-a)^{\alpha} (d-c)^{\beta}}{\Gamma(\alpha+1)^{p-1} \Gamma(\beta+1)^{p-1}} \frac{1}{\Gamma(\alpha)^p} \frac{1}{\Gamma(\beta)^p} \int_a^b \int_c^d \int_e^f \int_y^y \frac{|\varphi(s, t)|^p}{(x-s)^{1-\alpha}(y-t)^{1-\beta}} dt ds dy dx.
\]

But

\[
\int_a^b \int_c^d \int_e^f \frac{|\varphi(s, t)|^p}{(x-s)^{1-\alpha}(y-t)^{1-\beta}} dt ds dy dx
\]

\[
= \int_a^b \int_c^d \int_e^f \int_y^y \frac{|\varphi(s, t)|^p}{(x-s)^{1-\alpha}(y-t)^{1-\beta}} dt dy dx
\]

\[
= \int_a^b \int_c^d \int_e^f \int_y^y \frac{|\varphi(s, t)|^p}{(x-s)^{1-\alpha}(y-t)^{1-\beta}} dy dx
\]

\[
= \int_a^b \int_c^d \int_e^f \int_y^y \frac{|\varphi(s, t)|^p}{(x-s)^{1-\alpha}(y-t)^{1-\beta}} dy dt dx
\]

\[
= \int_a^b \int_c^d \int_e^f \int_y^y \frac{|\varphi(s, t)|^p}{(x-s)^{1-\alpha}(y-t)^{1-\beta}} dt dy dx
\]

\[
= \int_a^b \int_c^d \int_y^y \frac{|\varphi(s, t)|^p}{(x-s)^{1-\alpha}(y-t)^{1-\beta}} dt ds dy dx
\]

\[
= \int_a^b \int_c^d \int_y^y \frac{|\varphi(s, t)|^p}{(x-s)^{1-\alpha}(y-t)^{1-\beta}} dt ds dy dx
\]

\[
= \frac{1}{\beta} (d-c)^{\beta} \int_c^d \int_y^y \frac{|\varphi(s, t)|^p}{(x-s)^{1-\alpha}(y-t)^{1-\beta}} dt ds dt
\]
\[
\int_a^d \int_c^b |\varphi(s,t)|^p \frac{1}{(x-s)^{1-\alpha}} \, ds \, dt
= \beta (d-c)^\beta \int_c^d \int_a^b |\varphi(s,t)|^p \frac{1}{\alpha} (b-s) \, ds \, dt
\]
\[
\leq \frac{1}{\alpha \beta} (b-a)^\alpha (d-c)^\beta \int_c^d \int_a^b |\varphi(s,t)|^p \, dt \, ds.
\]
Thus,
\[
\left\| I_{a+;x:+;y}^{\alpha,\beta} \right\|_{L^p(P,R^n)}^p
\leq \frac{(b-a)^\alpha (d-c)^\beta}{\Gamma(\alpha+1) \Gamma(\beta+1)} \frac{1}{\Gamma(\alpha) \Gamma(\beta)} \frac{1}{\alpha \beta} (b-a)^\alpha (d-c)^\beta \left\| \varphi \right\|_{L^p(P,R^n)}^p
= \frac{1}{\alpha \beta} (b-a)^\alpha (d-c)^\beta \left\| \varphi \right\|_{L^p(P,R^n)}^p.
\]
The proof is completed.

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