Multi-graviton theories: yes-go and no-go results

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Abstract:

We report yes-go and no-go results on consistent cross-couplings for a collection of gravitons. Motivated by the search of theories where multiplets of massless spin-two fields cross-interact, we look for all the consistent deformations of a positive sum of Pauli-Fierz actions. We also investigate the problem of deforming a (positive and negative) sum of linearized Weyl gravity actions and show explicitly that there exists multi-Weyl-graviton theories. As the single-graviton Weyl theory, these theories do not have an energy bounded from below.

1 Introduction

In this report based on the works [1, 2, 3] done in collaboration with T. Damour, L. Gualtieri and M. Henneaux, we review some results about classical field theories involving several gravitons in interaction. To be more precise, in the following we use the word “graviton” as referring to a tensor $h_{\mu\nu}$ with two symmetric covariant indices. In the case of Einstein gravity, the term is justified, while in the case of Weyl gravity it is no more exact, and the term “Weyl-graviton” will be of use. The main question is to see whether it is possible to have a spin-two analogous of Yang-Mills theories, in the sense of multiplets of gravitons non-linearly cross-interacting through some vertices in the Lagrangian. In the context of Yang-Mills theories, the different massless spin-one fields belong to the adjoint representation of the semi-simple gauge (Lie) algebra and see each other through the Yang-Mills cubic vertex (at first order in the coupling constant) $\mathcal{L}_{cubic} \propto f_{abc} F^{\mu\nu a} A^b_{\mu} A^c_{\nu}$, where the “cubic coupling constant” $f_{abc}$ are the structure constants of the group. We may wonder if there exists similar theories involving multiplets of gravitons, each of these carrying an extra index running over some algebra-like structure and where the various gravitons of the multiplet could feel each other through some vertices in the Lagrangian.

The idea for the construction of such multi-graviton theories is to start from a free theory given by the sum of $N$ separate, completely decoupled free actions, each of these describing a single free graviton, and then try to perturbatively deform this free theory in

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1 Talk given at the RTN meeting “The Quantum Structure of Spacetime and the Geometric Nature of Fundamental Interactions”, Corfu, 13-20 September 2001.

2 “Chercheur F.R.I.A., Belgium”
order that the various gravitons not only self-interact, but also cross-interact. This is to say, we will look for all the possible interactions vertices to add to the free action, order by order in a deformation parameter, until a complete interacting theory is found. We require that in the limit where the deformation parameter goes to zero, the interacting multi-graviton theory (if any) smoothly goes back to the starting one.

Such an exhaustivity of the process (all the possible vertices!) is made possible by using cohomology-BRST tools [4], in the Batalin-Vilkovisky formalism [5]. On the one hand the Noether method is quite useful to build an interacting gauge theory starting from a free one, but on the other hand it is not systematic enough to allow the exhaustivity of the deformation: unless you are very clever, you may miss some deformation vertices, and moreover you are not sure you found them all. The BRST-based deformation method [4] relies on strong mathematical results, clearly organizes the calculation of the non-trivial consistent couplings in terms of cohomologies already known or easily computed, and thus enables you to find all the possible interactions vertices, in an exhaustive way.

When we say that a deformation is consistent, it means that the deformed interacting action possesses the same number of physical degrees of freedom, does not contain any inconsistency between the equations of motion, is free from ghosts, local, etc. In other words, if the starting theory was defined to be consistent, then so will be the deformed one.

The plan is as follows: in section 2 (based on [1, 2]) we give no-go and yes-go results about the consistent deformations of a positive sum of Pauli-Fierz actions. In section 3 (based on [3]) we give no-go and yes-go results about the consistent deformations of a positive and non-positive sum of linearized Weyl actions and finally in section 4 (based on the forthcoming work [11]) we briefly comment about multi-Weyl-graviton theories as gauge theories of non semi-simple extensions of the (super)conformal group.

2 Deformation of a sum of Pauli-Fierz actions

It was shown by Pauli and Fierz [6] that there is a unique, consistent action describing a pure spin-2 massless field. This action happens to be the linearized Einstein-Hilbert action. Therefore, the action for a collection \( \{ h^a_{\mu\nu} \} \) of \( N \) non-interacting, massless spin-2 fields in spacetime dimension \( n \) \( (a = 1, \cdots, N, \mu, \nu = 0, \cdots, n-1) \) must be (equivalent to) the sum of \( N \) separate Pauli-Fierz actions, namely

\[
S_0[h^a_{\mu\nu}] = \sum_{a=1}^{N} \int d^n x \left[ -\frac{1}{2} (\partial_\mu h^a_{\nu\rho}) (\partial^\mu h^{a\nu\rho}) + (\partial_\mu h^{a\mu\nu}) (\partial_\rho h^{a\rho\nu}) 
- (\partial_\nu h^{a\mu\mu}) (\partial_\rho h^{a\rho\nu}) + \frac{1}{2} (\partial_\mu h^{a\nu\nu}) (\partial^\rho h^{a\rho\rho}) \right], \quad n > 2. \tag{2.1}
\]

Our treatment, which is purely algebraic, extends (at least formally) to the case where the collection \( \{ h^a_{\mu\nu} \} \) is, possibly uncountably, infinite (see [4]).

The action (2.1) is invariant under the following linear gauge transformations,

\[
\delta h^a_{\mu\nu} = \partial_\alpha c^a_{\nu} + \partial_{\nu} c^a_{\alpha}, \tag{2.2}
\]

We use the signature “mostly plus”: \(-+++-\cdots\). Furthermore, spacetime indices are raised and lowered with the flat Minkowskian metric \( \eta_{\mu\nu} \). Finally, we take the spacetime dimension \( n \) to be strictly greater than 2 since otherwise, the Lagrangian is a total derivative. Gravity in two dimensions needs a separate treatment.
where the $\epsilon_{\nu}^a$ are $n \times N$ arbitrary, independent functions. These transformations are abelian and irreducible. We rewrite the free action (2.1) as

$$S_0 = \int d^n x k_{ab} \left[ -\frac{1}{2} \left( \partial_\mu h^a_{\nu \rho} \right) \left( \partial^{\mu} h^{b \rho}_{\nu} \right) + \left( \partial_\mu h^a_{\nu} \right) \left( \partial^{\mu} h^{b}_{\nu} \right) \right],$$

(2.3)

with a quadratic form $k_{ab}$ defined by the kinetic terms. In the way of writing the Pauli-Fierz free limit above, equation (2.1), $k_{ab}$ was simply the Kronecker delta $\delta_{ab}$. What is essential for the physical consistency of the theory (absence of negative-energy excitations, or stability of the Minkowski vacuum) is that $k_{ab}$ defines a positive-definite metric in internal space; it can then be normalized to be $\delta_{ab}$ by a simple linear field redefinition.

### 2.1 No-go result

Here comes the no-go result, with the assumptions clearly stated.

**Theorem 2.1** *Under the assumptions of: locality, Poincaré invariance, Eq. (2.1) as free field limit and at most two derivatives in the Lagrangian, the only consistent deformation of Eq. (2.1) involving a collection of spin-2 fields is (modulo field redefinitions) a sum of independent Einstein-Hilbert (or possibly Pauli-Fierz) actions, one for each fields.*

It means that no cross interactions are possible: no Yang-Mills-like massless spin-2 theory.

### 2.2 Yes-go result

Relaxing the hypothesis on the number of derivatives in the Lagrangian, allowing for PT breaking terms (i.e. explicit presence of the completely antisymmetric Levi-Civita tensor) and restricting to $n = 3$ spacetime, we can evade the previous no-go theorem and the positive sum (2.3) admits a possible consistent deformation. Schematically, the action (2.3) writes $S_{0}^{P.F.}[h_{\mu \nu}^a] \sim \int d^3 x \delta_{ab} \partial h^a \partial h^b$ and is deformed as follows, at first order in the deformation parameter $\lambda$ (for an explicit writing of the deformed theory, see [2]):

$$S_{0}^{P.F.}[h_{\mu \nu}^a] \rightarrow S_{0}^{P.F.}[h_{\mu \nu}^a] + \lambda \int d^3 x a_0 + O(\lambda^2)$$

(2.4)

where $a_0$ is the cubic vertex which looks like $a_0 \sim \epsilon^{\mu \nu \rho} a_{abc} (\partial h^a \partial h^b \partial h^c)_{\mu \nu \rho}$. The constants $a_{abc} \equiv k_{ad} a^d_{bc}$ play the same role as the structure constants in Yang-Mills theory, except that here it doesn’t define a Lie algebra, but a commutative, symmetric algebra $\mathcal{A}$. The fields $h^a_{\mu \nu}$ live in $\mathcal{A}$.

The gauge transformations are deformed, at first order in $\lambda$, to become schematically

$$\delta_{\epsilon} h^a_{\mu \nu} = 2 \partial_\mu \epsilon_{\nu}^a + \lambda [\epsilon^{\mu \nu \rho} (\partial h^a \partial h^c)_{\mu \nu \rho} a_{bc}^a].$$

(2.5)

The point is that this deformation can be continued to all orders in $\lambda$, giving rise to a complete exotic theory of non-linearly interacting multiplets of $h^a_{\mu \nu}$-fields:

$$S_{EX}^{\epsilon}[h_{\mu \nu}^a] = S_{0}^{P.F.}[h_{\mu \nu}^a] + \lambda \int d^3 x a_0 + \lambda^2 ...$$

(2.6)

4 the product between two elements $u, v$ of $\mathcal{A}$ writes $z^a = (u \ast v)^a = a^a_{bc} u^b \ast v^c$, in a basis $\{e_a\}_{\alpha=1}^N$ of $\mathcal{A}$. In this basis, the commutativity and symmetry properties read $a^a_{bc} = a^a_{cb}, a_{abc} = a_{(abc)}$, respectively.
The writing of the complete exotic action is done (see \[2\]) by using the first order formulation of gravity, à la Chern-Simons \[7\].

In the proof of theorem \[2.1\], the algebra was constrained to be also associative\[5\]. The positive-definiteness of the internal metric \(k_{ab}\) together with the commutativity, associativity and symmetry of \(A\) implies that \(A\) is a direct sum of one-dimensional ideals, implying in turn the existence of a basis where \(a^a_{bc} = 0\) whenever two indices are different \[4\]: the cross-interactions between the various gravitons can be removed by redefinitions. This is why theorem \[2.1\] holds. If \(k_{ab}\) is of mixed signature, however, the algebra \(A\) need not be trivial, and one can construct truly interacting multi-gravitons theories \[8\]. Also, keeping \(k_{ab}\) positive-definite but with \(A\) no more associative, which is the case of the exotic theory, the interactions between the various gravitons do exist.

3 Deformation of a sum of linearized Weyl actions

The free theory for one Weyl-graviton is now

\[
S_0[h_{\mu\nu}^a] = \frac{1}{2} \int d^4 x W_{\alpha\beta\gamma\delta} W^{\alpha\beta\gamma\delta},
\]

(3.7)

for \(g_{\mu\nu} = \eta_{\mu\nu} + \lambda h_{\mu\nu}\), \(h_{\mu\nu}\) and \(\lambda\) being dimensionless. Here, \(W_{\alpha\beta\gamma\delta}\) is the linearized Weyl tensor constructed out of \(h_{\mu\nu}\). The free action (3.7) is invariant under both linearized diffeomorphisms and the linearized version of the Weyl rescalings,

\[
\delta_{\eta,\phi} h_{\mu\nu} = \partial_\mu \eta_\nu + \partial_\nu \eta_\mu + 2 \phi \eta_{\mu\nu}.
\]

(3.8)

This theory is the linearization of the conformally invariant Weyl gravity action

\[
S^{\text{W}} = \frac{1}{2\lambda^2} \int d^4 x \sqrt{-g} W_{\alpha\beta\gamma\delta} W^{\alpha\beta\gamma\delta},
\]

(3.9)

where \(W_{\beta\gamma\delta}\) is the conformally invariant Weyl tensor.

As in the previous section, the starting point is really a sum of the actions (3.7), using a quadratic form \(k_{ab}\) to perform the sum.

3.1 No-go result

We have the

**Theorem 3.1** Under the assumptions of: locality, Lorentz invariance, a positive sum of actions (3.7) as free field limit and at most four derivatives in the Lagrangian, the only consistent deformation of the free action involving a collection of Weyl-gravitons is (modulo field redefinitions) a sum of independent Weyl actions, one for each fields.

In this case also, the constant coefficients \(a^a_{bc}\) appearing in the deformation were constrained to define a commutative, associative, symmetric algebra \(A\), and together with the positive-definiteness of the internal metric \(k_{ab}\), it implies that the interactions between the various Weyl-gravitons can be removed, leaving only the self-interactions.

\[5\text{In terms of the } a^a_{bc},\text{ the associativity property reads } a^a_{blc} a^b_{dlc} = 0\]
3.2 Yes-go result

In the case of Weyl gravity, there does not appear to be any particularly strong reason for taking the free Lagrangian to be a positive sum of free Weyl Lagrangians. Any other choice, corresponding to an internal metric $k_{ab}$ that need not be definite positive, would seem to be equally good since the energy is in any case not bounded from below (or above). If one allows non positive definite metrics in internal space, then, non trivial algebras of the type studied in [8] exist and lead to non trivial cross interactions among the various types of Weyl gravitons. We have explicitly given such a two-Weyl-graviton theory in [3].

4 Non semi-simple extensions of the conformal group

It is well known (see [9, 10]) that the Weyl's theory of gravity (3.9) is the gauging of the conformal algebra $so(4,2)$ in four dimensions. In a forthcoming paper [11] we shall show that the two-Weyl-graviton theory given in [3] can be expressed as the gauging of the tensor product $L' = so(4,2) \otimes A$, where $A$ is the commutative, associative algebra encountered in the previous section, with the non positive-definite internal metric $k_{ab}$ making $A$ symmetric. $A$ is spanned by a unit element $e$ and a nilpotent one, $n$, of order two [5]. Similarly, the gauging of $\tilde{L} = su(2,2|1) \otimes A$ along the lines of [12] gives a superconformal theory with two multiplets in interaction. One can see that the presence of nilpotent elements in $A$ makes the tensor product $L' = L \otimes A$ no more semi-simple, even if $L$ was semi-simple.

The extension to $N$ multiplets is straightforward : we just pick out the commutative, associative algebra $A$ spanned by a nilpotent element of order $N$, and find the non positive-definite internal metric $k_{ab}$ which makes the algebra $A$ symmetric.

Although we concentrated on conformal gravity for the reasons explained above, similar considerations apply to standard gravity, provided one replaces the conformal group by the (anti) de Sitter group or its Poincaré contraction [13]. In that case also, one can construct interacting multi-graviton theories if the metric in internal space is not positive definite [14].

Thanks to this formulation in terms of Lie algebras and their gauging, one observes the following fact : all these $N$-field theories can be recovered from the usual one-field starting theory. Just by doing a Taylor expansion of the fields with a nilpotent expansion parameter (of order $N$ for a $N$-field theory). With respect to this observation, it means that even if these multi-field theories are allowed by consistency, they contain no more information than the one-field theory from which they were built, and thus are in some sense trivial.

5 Conclusions

In this report we have given results about the consistent deformations of sums of Pauli-Fierz and linearized Weyl gravity actions, using BRST-cohomology tools [4]. The aim was to build Yang-Mills-like theories of spin-two fields. Except for an exotic $n = 3$ PT-breaking theory, such multi-graviton theories are excluded when the sum of free actions defining the starting theory is a positive one. Allowing for non positive sums of free actions as starting point, we can build truly interacting multi-graviton theories, but at

\[ 6e^2 = e, \ en = ne, \ n^2 = 0 \]
the light of a group-theoretical analysis, it becomes clear that these \(N\)-field theories are also trivial.

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