Electromechanical instabilities of suspended carbon nanotubes—multi mode excitations

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Abstract. We investigate theoretically the occurrence of ‘shuttle-like’ electromechanical instabilities in systems where charge transfer processes interact with several mechanical degrees of freedom simultaneously. In particular we consider the excitation of the multiple bending modes of a suspended carbon nanotube (CNT) due to the injection of current from the tip of a scanning tunnelling microscope (STM) into the suspended part of the tube. With respect to the shuttle instability, we show that the different bending modes can be treated independently in the limit of weak electromechanical coupling. As a result one or more modes—depending on the strength of the tunnel coupling between the CNT and the STM-tip—can be unstable. We find that it is possible to selectively excite a specific mode by tuning the distance between the STM-tip and the CNT. Furthermore, we have investigated the stable (limit cycle) regime that develops as a result of the shuttle instability when only one bending mode is unstable.
1. Introduction

Nanoelectromechanical systems (NEMS) whose properties are significantly determined by the strength of the coupling between electronic and mechanical degrees of freedom on the nanoscale level [1] have attracted a great deal of attention in recent years. New fundamental physical phenomena occurring in these systems may give rise to novel nano-devices of interest for applications. One such new phenomenon is the ‘shuttle instability’ and the associated ‘shuttle’ mechanism for charge transfer that is expected to occur in a certain class of NEMS systems. A representative shuttle system consists of a movable conducting grain placed in an elastic medium between two electrodes. When a dc bias voltage applied between the electrodes exceeds a certain threshold value an electromechanical instability occurs, resulting in large-amplitude mechanical vibrations of the grain and a shuttle current, which in the low-dissipation limit is proportional to the vibration frequency. Such an instability was predicted in [2] and its consequences have been thoroughly investigated in numerous publications (for reviews see [3, 4]). However, in previous studies of shuttle instabilities a possibly complex mechanical behaviour was reduced to the excitation of a single vibronic mode characterized by one specific frequency. In this work we investigate the shuttle instability in the more general case when several mechanical degrees of freedom are simultaneously involved in the charge transfer process. In particular, this is the situation for a suspended carbon nanotube (CNT) when current is injected into the suspended part through a scanning tunneling microscope (STM) tip.

CNTs are particularly interesting building blocks of NEMS due to their extraordinary mechanical and electrical properties [5, 6], a combination that gives unique possibilities when designing NEMS. Various NEMS-devices, where a CNT is used as the mechanically active component, have been demonstrated e.g. the nanorelay [7] or the nanotweezer [8]. Several aspects of transport through suspended CNTs have been investigated experimentally. Sazonova et al [9] have, e.g. detected bending mode resonances of CNTs and LeRoy et al [10] and Sapmaz et al [11] have found evidence for phonon assisted tunnelling involving different mechanical
modes of the nanotube. Theoretical work on electron–vibron coupling in suspended CNTs has been published by a number of authors [12]–[15]. An important aspect of the suspended CNT set-up is that it is perhaps the most promising set-up for performing quantum-limited position measurements [16]. This is because the amplitude of the bending mode zero point oscillations of the suspended CNT scales as the square root of the tube length (for an unstrained tube), and can be quite large for long nanotubes. Typical, strongly geometry dependent, resonant frequencies for single-walled CNTs are $\sim 0.1$ GHz [9].

Previously we have shown that the fundamental bending mode of a suspended nanotube becomes unstable with respect to small displacements when probing the free-hanging part using an STM-tip, if dissipation is sufficiently small [17]. This results in large amplitude vibrations which modify electron transport through the system. However, this study was limited to the fundamental bending mode and higher order bending modes were neglected. This is possibly a crude approximation since we know that in order to completely describe the mechanics of a suspended beam like the CNT we need to include all modes. In this paper we use a full mechanical description of the flexural vibrations (bending mode) in one dimension and investigate the electromechanical instabilities of different bending modes\(^4\). We show that a single mode can be unstable or several modes can be simultaneously unstable, depending on tunnelling rates, dissipation rates and the strength of the electromechanical coupling. This implies that a simple description of the mechanics is sufficient in some cases whereas in others a more complete description is necessary.

2. Model

The model system investigated in this paper is shown in figure 1. It is analogous with the experimental set-up used by LeRoy et al [10], where an STM-tip is used to inject current into

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\(^4\) The theory developed in this paper does not only apply to bending mode vibrations, but these are the most important modes in the present context.
the suspended part of the tube. A CNT of length $L$ and diameter $D$ is suspended over a trench. An STM tip held at electrochemical potential $eV/2$ is located above the centre of the nanotube. The STM is connected to the nanotube through a tunnel barrier. One end of the tube is connected through another tunnel junction to an electrode at electrochemical potential $-eV/2$. Also, a gate with the potential $eV_g$, capacitively coupled to the other end of the tube, is introduced. The rate of electron tunnelling from the STM tip to the nanotube depends on mechanical deformations of the tube, whereas tunnelling from the tube to the electrode does not. The model of the mechanical and electrical properties of the tube will be briefly presented before giving the full Hamiltonian of the system. A more detailed description can be found in the appendices.

2.1. Tube mechanics

In this work we consider only flexural (bending) deformations of the nanotube. We neglect other modes such as breathing modes, torsional modes and longitudinal stretching modes. Out of these, it is only the radial breathing mode that modifies the tunnelling rate and gives a nonzero electromechanical coupling parameter. However, typical frequencies of these modes $\sim 10$ THz which is several orders of magnitude larger compared to frequencies of the bending mode. We treat the nanotube as an elastic continuum, an approach used by several authors, see e.g. [18, 19]. The accuracy of this approach has been verified by simulations [20].

We consider only transverse deformations of the nanotube profile described by the field $u(z, t)$ with $u(z) = \partial u(z)/\partial z = 0$ at $z = 0, L$. The shape profile of the nanotube can be expanded in the eigenmodes $u_n(z)$ (see appendix A)

$$u(z, t) = \sum_n X_n(t) u_n(z). \quad (1)$$

In terms of the amplitudes $X_n$ and the conjugate momentum $P_n$, the Hamiltonian of the nanotube describing small deformations is a set of independent oscillators with frequencies

$$\omega_n = \frac{c_n}{L^2} \sqrt{\frac{EI}{\rho S}}, \quad n = 1, 2, 3, \ldots . \quad (2)$$

Here $c_n = 22.4, 61.7, 121.9, \ldots$, $\rho$ is the mass density of the tube, $S$ is the cross-section area, $E$ is the Young’s modulus, and $I$ is the moment of inertia of the cross-section. We emphasize that the frequencies strongly depend on the geometry of the nanotube which experimentally allows a large tunability.

Such a system is easily quantized by introducing the operators $\hat{x}_n = X_n/l_n$ and $\hat{p}_n = P_n/l_n/\hbar$. Here $l_n = \sqrt{\hbar/M\omega_n}$ is the natural length scale of the oscillators and $M$ is the nanotube’s mass. One finds that the following Hamiltonian describes the mechanical deformation of the nanotube,

$$\hat{H}_{osc} = \hbar \omega_1 \sum_n \beta_n \left( \frac{\hat{x}_n^2}{2} + \frac{\hat{p}_n^2}{2} \right). \quad (3)$$

Here $\beta_n = \omega_n/\omega_1$ is the ratio between the frequencies of mode $n$ and the fundamental mode frequency.
Figure 2. Schematic illustration of the single level regime. Only one single spin degenerate level of the nanotube lies within the energy window allowed for transport.

Figure 3. Illustration of the continuum spectrum regime. There is a band of available states on the nanotube to which the electrons can tunnel, leading to voltage-dependent tunnelling rates.

2.2. Nanotube spectrum and tunnelling

For metallic CNTs of finite length, quantization along the tube axis leads to a set of single particle levels on the nanotube [21, 22]. The typical level spacing $\Delta \sim \hbar v_f / 2L \approx 1.68 \text{meV}/L(\mu\text{m})$. Hence, depending on the nanotube length we can distinguish between two regimes defined by $eV, k_B T \gg \Delta$ and $\Delta \gg k_B T, eV$. In the former case the energy quantization is not important. This will be referred to as the continuum regime. In the latter case we have a strong level quantization and one can expect that only a single level is involved in transport. We call this the single level regime. Schematic illustrations of the energy levels for the two regimes are shown in figure 2 and figure 3. The intermediate regime is not investigated in this paper.

Now we discuss the electromechanical coupling. A combination of two effects leads to a coupling between mechanical and electronic degrees of freedom. The first effect is the electrostatic interaction between the electrons on the tube and the electrodes. The interaction energy depends on the shape of the nanotube and the excess charge ($Q$) on the tube. The linearized interaction energy $E_{\text{int}}$ can be expressed as a function of the tube position with a kernel $K(z)$ as

$$E_{\text{int}}([u(z, t)), Q] = Q \int dz u(z, t) K(z) = Q \sum_n X_n(t) \varepsilon_n,$$  

(4)
where the expansion coefficient $E_n$ is the effective electric field for mode $n$. Assuming that the shape of the STM is symmetric with respect to the nanotube midpoint ($z = L/2$), the effective electric fields for the odd modes (odd with respect to the nanotube midpoint) are zero. Also, when $n \to \infty$ the rapid oscillation of $u_n$ implies that $\lim_{n \to \infty} E_n = 0$. Hence, the coupling to very high modes is small and these modes can be neglected. In general, the Hamiltonian for the electrons on the tube and the electromechanical coupling can be written (using the same units as before)

$$\hat{H}_{\text{tube}} = \sum_{\sigma q} \epsilon_q(V_g) c_{q \sigma}^{\dagger} c_{q \sigma} - \hbar \omega_1 \sum_n d_n \hat{n} \hat{x}_n + \frac{U}{2} \hat{n}^2,$$  \hspace{1cm} (5)

where $U/2$ is the electrostatic energy of the charged nanotube, $\hat{n} = \sum_{\sigma q} c_{q \sigma}^{\dagger} c_{q \sigma} - N_0$ is the number operator with respect to a neutral tube (below we will limit our considerations to the Coulomb blockade (CB) regime, where adding more than one electron to the nanotube is prevented by the charging energy $U/2$). Then $d_n = e \lambda_n \epsilon_n / \hbar \omega_n$ determines the coupling of mode $n$ to the effective electric field. The influence of the gate voltage results in a linear shift of the electronic levels $\epsilon_q(V_g) = \epsilon_q - \alpha g e V_g$. For simplicity, but without loss of generality, the gate coupling efficiency $\alpha_g$ is taken to be one, $\alpha_g = 1$. The index $\sigma = \uparrow, \downarrow$ labels the spin of the electrons and can be neglected outside the Kondo regime. The effect of including spin is to renormalize the tunnelling rates. In the case of a single spin degenerate level the sum over $q$ should be reduced to a single term.

The second effect that contributes to an electromechanical coupling is the dependence of the tunnel coupling (a certain overlap integral) between the STM and the nanotube on the distance between them. The tunnel coupling is therefore a function of the deformation field $u(z, t)$. As for the electrostatic interaction energy, if the deformation $u(z, t)$ is much smaller than the characteristic tunnel length $\Lambda_1$, we can expand the overlap integral $T_L((u(z, t)))$ in powers of $u(z, t)$ and keep the linear term only. Then using an expansion in the eigenmodes one can express the overlap integral in the form

$$T_L([X_n]) = T_0 \left( 1 - \sum_n \frac{X_n}{\Lambda_n} \right) = T_0 \left( 1 - \sum_n \frac{\hat{x}_n}{\lambda_n} \right),$$  \hspace{1cm} (6)

where again only even modes have nonzero expansion coefficients $1/\Lambda_n$, $\Lambda_n$ can be treated as an effective tunnel length for mode $n$. Tunnel matrix elements for the tube–electrode tunnelling are constant $T_R$. From here on we only consider modes for which the electromechanical coupling parameter $d_n/\lambda_n \neq 0$. We can take $\epsilon_n/\Lambda_n = \epsilon_1/\Lambda_1$ for the first few modes and estimate the relation between coupling parameters to find that $d_n/\lambda_n \approx d_1/(\beta_n^{\lambda_1})$.

### 2.3. Hamiltonian

The total Hamiltonian used in this paper is presented below, including the parts that need no detailed description. Note that spin degrees of freedom are neglected for simplicity. Including spin does not change the results, it simply renormalizes the tunnelling rates. The full Hamiltonian is given by

$$\hat{H} = \hat{H}_{\text{leads}} + \hat{H}_{\text{osc}} + \hat{H}_{\text{tube}} + \hat{\mathcal{O}} + \hat{\mathcal{H}}_y,$$  \hspace{1cm} (7)
where the explicit form of the different terms are

\[ \hat{H}_{\text{leads}} = \sum_{ak} \epsilon_{ak} a_\alpha^\dagger a_\alpha, \]  

(8)

\[ \hat{H}_{\text{osc}} = \hbar \omega_1 \sum_n \beta_n \left( \frac{\hat{x}_n^2}{2} + \frac{\hat{p}_n^2}{2} \right), \]  

(9)

\[ \hat{H}_{\text{tube}} = \sum_q \epsilon(V_g) c_q^\dagger c_q + \frac{U}{2} \hat{n}^2 - \hbar \omega_1 \hat{n} \hat{x} \cdot \mathbf{d}, \]  

(10)

\[ \hat{\Omega}_1 = \sum_{ak} T_{\alpha} (\hat{x})(a_\alpha^\dagger c_q + c_q^\dagger a_\alpha), \]  

(11)

\[ \hat{H}_\gamma = \hat{H}_{\text{bath}} + \hat{H}_{\text{bath-osc}}. \]  

(12)

Here we have introduced \( \hat{x} = (\hat{x}_1, \ldots, \hat{x}_n)^T \) to denote all position operators and \( \mathbf{d} = (d_1, \ldots, d_n)^T \) and the index \( \alpha \) labels the STM (\( \alpha = L \)) or the electrode (\( \alpha = R \)). We also introduced a Hamiltonian \( \hat{H}_\gamma = \hat{H}_{\text{bath}} + \hat{H}_{\text{bath-osc}} \), which is composed of a bath of harmonic oscillators and a linear coupling between the bending modes and the bath [23]. This term gives rise to the damping in the mechanical subsystem. Below the damping is assumed to be weak and characterized by the frequency-independent dissipation rate coefficient \( \gamma \) (Ohmic damping).

In the following derivation we need to assume that the typical voltage scale is large compared to the oscillator energy scale. This implies that our analysis is restricted to mechanical modes with low enough frequency. When considering the bending mode of the nanotube we introduce a cutoff for the allowed modes, which is justified by the fact that the fast oscillations of the high order modes make the electromechanical coupling parameter \( d_n/\lambda_n \) small for large \( n \). Hence, these high modes are inert and can be neglected. The sum over \( n \) is from here on restricted to \( 1, 3, 5, \ldots, 2K - 1 \), where only modes with a nonzero coupling parameter \( d_n/\lambda_n \) are included and \( K \) is the total number of modes included.

### 3. Generalized master equation (GME)

In this section, we present the GME for the reduced density matrix \( \rho_N(t) = \text{Tr}_e \delta(\hat{n} - N)\sigma(t) \) (here \( \sigma(t) \) is the density matrix of the system and \( \text{Tr}_e \) denotes the trace over all electronic degree of freedoms) that governs the mechanical dynamics of the nanotube. The derivation is given in appendix B. With voltages satisfying \( 0 < eV/2 - eV_g + U/2 < U < eV/2 + eV_g + U/2 < 2U \), the only allowed charge states are \( N = 0, 1 \) and the equations are given by

\[
\frac{d}{dt} \rho_0 = -i[\hat{H}_{\text{osc}}(\hat{x}), \rho_0] + \Gamma_R(\hat{x}, V) \rho_0 - \frac{1}{2} \{ \Gamma_L(\hat{x}, V), \rho_0 \} - \frac{\gamma}{2} \sum_n (i[\hat{x}_n, \{ \hat{p}_n, \rho_0 \}] + [\hat{x}_n, [\hat{x}_n, \rho_0]]),
\]

(13)

\[
\frac{d}{dt} \rho_1 = -i[\hat{H}_{\text{osc}}(\hat{x} - \mathbf{d}), \rho_1] + \sqrt{\Gamma_L(\hat{x}, V)} \rho_0 \sqrt{\Gamma_L(\hat{x}, V)} - \Gamma_R(\hat{x}, V) \rho_1 \\
- \frac{\gamma}{2} \sum_n (i[\hat{x}_n, \{ \hat{p}_n, \rho_1 \}] + [\hat{x}_n, [\hat{x}_n, \rho_1]]),
\]

(14)
where \([\bullet, \bullet], (\bullet, \bullet)\) denotes the commutator (anticommutator) and we have used the fact that \(\Gamma_\mathbf{R}(\mathbf{x}, V)\) is independent of \(\dot{x}\). The first terms in the right-hand sides of the above equations describe the coherent dynamics of bending modes for the neutral and charged nanotube respectively. The next two terms represent perturbation of the mechanical subsystem generated by stochastic charge transfer processes. The last term governs the damping of mechanical vibrations resulting from interaction with environment. These equations are valid in both the single level regime and the continuum regime with only a difference in the form of the tunnelling rates. The form of the tunnelling rates \(\Gamma_L(\mathbf{x}, V)\) and \(\Gamma_R(\mathbf{x}, V)\) are given by

\[
\Gamma_\alpha(x, V) = 2\pi D_\alpha D_T T^2_\alpha(x) \left( \frac{eV}{2} \pm eV_k \mp \frac{U}{2} \right)
\]

in the continuum regime or

\[
\Gamma_\alpha(x, V) = 2\pi D_\alpha T^2_\alpha(x),
\]

in the single level regime. Here \(D_\alpha\) is the electron density of states in lead \(\alpha = L, R\) and \(D_T\) is the density of states in the tube. One can see that the difference between the regimes is only the additional voltage dependence of the tunnelling rates in the continuum regime. In the following sections we will analyse these equations.

4. Instability of the mechanical modes

We start to investigate the electromechanical instabilities by studying the time evolution of the average displacements \(x_{n0} = \text{Tr}(x_n \rho_{01})\), the average momenta \(p_{n0} = \text{Tr}(p_n \rho_{01})\), and the probabilities \(W_{01} = \text{Tr}(\rho_{01})\) to find the nanotube in a neutral (charged) state. To obtain the required equation one should multiply equations (13) and (14) by \(x_n\) and \(p_n\) respectively, and take the trace over all electronic degrees of freedom. All higher order cumulants like \(\text{Tr}(x_n x_m \rho_{01})\) can be neglected in the linear regime. First one can immediately find that

\[
\partial_t(W_1 + W_2) = \partial_t W + \sum_{n=0}^{\infty} x_n p_n = 0,
\]

and by normalization we get \(W_+ + W_- = 1\). In what follows we introduce new values \(x_{n\pm} = x_n \pm x_1\), \(p_{n\pm} = p_n \pm p_1\), and \(W_- = W_0 - W_1\) and suppose that only \(K\) modes are involved in charge transfer processes. Then introducing the vector

\[
y = (x_1^+ p_1^+ x_1^- p_1^- \cdots x_K^+ p_K^+ x_K^- p_K^- W_-)^T,
\]

one can present the equations describing time evolution of the average momenta and displacements in the form

\[
\partial_t y = Ay + b,
\]

where \(b\) is a constant vector. \(A\) is an \(4K + 1\) by \(4K + 1\) matrix given by

\[
A = \begin{pmatrix}
A_1 & 0 & 0 & d_1 \\
0 & A_2 & 0 & d_2 \\
0 & 0 & \ddots & \vdots \\
c_1^T & c_2^T & \cdots & -1
\end{pmatrix},
\]
and

\[
A_n = \begin{pmatrix}
0 & \omega_n & 0 & 0 \\
-\omega_n & -\gamma & 0 & 0 \\
\Gamma_- & 0 & -\Gamma & \omega_n \\
0 & \Gamma_- & -\omega_n & -\Gamma - \gamma
\end{pmatrix},
\]  
(20)

\[
d_n = \begin{pmatrix} 0 & -\frac{d_n \omega_n}{2} & 0 & \frac{d_n \omega_n}{2} \end{pmatrix}^T,
\]  
(21)

\[
e_n = \begin{pmatrix} \frac{2 \Gamma_0}{\lambda_n} & 0 & -\frac{2 \Gamma_0}{\lambda_n} & 0 \end{pmatrix}^T,
\]  
(22)

where \( \Gamma_0 = \Gamma_L(0, V) \), \( \Gamma = \Gamma_L(0, V) \), \( \Gamma_- = \Gamma_L(0, V) \), and \( \Gamma_\pm(\bullet, \bullet) = \Gamma_R(\bullet, \bullet) \pm \Gamma_L(\bullet, \bullet) \).

The general solution of equation (13) has the form

\[
y(t) = \sum_{m=1}^{4K+1} C_n y_m e^{\alpha_m t} + A^{-1} b,
\]  
(23)

where \( y_m \) are eigenvectors of the matrix \( A \) while the increments \( \alpha_m \) are defined by the associated eigenvalues. If the real part of some \( \alpha_m \) is positive the amplitude of the appropriate eigenvector increases in time which implies that the stationary solution \( y_{st} = A^{-1} b \) is unstable.

The eigenvectors and eigenvalues of the matrix \( A \) can be treated analytically if \( \Gamma_- = 0 \) and the parameter \( d_1/\lambda_1 \ll 1 \) (weak electromechanical coupling, details are given in appendix C). As a result we have found that

\[
\Delta x_\pm^n(t) = \Delta x_\pm^n(0) e^{\pm i \beta_n \omega_1 t} + O(d_1/\lambda_1),
\]  
(24)

where \( \Delta x_\pm^n \) is deviation from the stationary displacement induced by charge transfer processes and

\[
\delta_n = \frac{d_1}{4 \lambda_1 (\Gamma/\omega_1)^2 + \beta_n^2} - \frac{\gamma}{2} + O(d_1^2/\lambda_1^2).
\]  
(25)

This result is the same as one would get if only a single mode were included and the indirect coupling between the different modes through the charge degree of freedom would give a second order contribution in the electromechanical coupling parameters \( d_n/\lambda_n \). In our model \( \lambda_n > 0 \) and in our geometrical configuration \( d_n > 0 \) which leads to a positive value of \( d_n/\lambda_n \). This implies that the electromechanical feedback leads to a net increase of the mechanical energy. We can in principle imagine other geometrical configurations and tunnelling rates where \( d_n/\lambda_n < 0 \), yielding an enhanced dissipation as was found in [14, 24]. We conclude that with respect to the instability, the different modes can be treated independently in the weak coupling limit. Furthermore, the form of the threshold shows that for each mode, the tunnelling rates should be of the order of the mode frequency for optimal conditions. This implies that depending on the tunnelling rates, electromechanical coupling, and dissipation, a single mode can be unstable or several modes can be simultaneously unstable. These two different cases are illustrated in figure 4 for the first two modes with \( d_n/\lambda_n \neq 0 \) (fundamental and third mode) where the relative
Figure 4. The threshold dissipation for the first ($\gamma_{\text{thr},1}$) and third ($\gamma_{\text{thr},3}$) bending modes in the limit of weak electromechanical coupling. A comparison is made between the numerical results when two modes are included and the analytical expression for an arbitrary number of modes. The good agreement shows that the instability of a specific mode is not affected by other modes. An instability of mode $n$ occurs if $\gamma_{\text{thr},n}(\Gamma) > \gamma$. Depending on the $\gamma$ and $\Gamma$, we find that either only the fundamental mode or both modes can be simultaneously unstable.

Peak heights are $1/\beta_3$, based on our estimate for the coupling parameters. We can show that the threshold dissipations $\gamma_{\text{thr},n}$, defined by the condition $\delta_n(\gamma = \gamma_{\text{thr},n}) = 0$, for the two different modes $n$ and $m$ are such that $\gamma_{\text{thr},n} > \gamma_{\text{thr},m}$ when $n < m$, i.e. if dissipation is small enough to make mode $m$ unstable, all modes $n < m$ will also be unstable.

Experimentally, controlling the tunnelling rates of the STM-tube junction is a simple task since the STM–tube distance can be controlled accurately. Changing the tube–electrode tunnelling rates is not as simple although, in the continuum regime, there is some room for changing the tunnelling rates of both junctions using the gate. We analyse the conditions for instability when only the left junction tunnelling rates are changed. The analytical form for the threshold dissipation (equation (25)) is not valid in this case, instead we find the eigenvalues numerically as a function of the rate of tunnelling from the STM to the nanotube $\Gamma_0$. Also, we relax the weak electromechanical coupling approximation to investigate the effects of mode coupling on the instability. For the numerical solution we focus on a case with only two modes included and again find eigenvalues of the form $\omega_n \approx i\omega_n + \delta_n$ which implies oscillating solutions with an increasing or decreasing amplitude depending on the sign of $\delta_n$. From this we find the threshold dissipations for each mode. In figures 5–7, the thresholds for the first and third modes are shown for different values of the electromechanical coupling parameter, $d_1/\lambda_1 = 0.1, d_1/\lambda_1 = 3$, and $d_1/\lambda_1 = 10$, as a function of $\Gamma_0$. Let us analyse these results. In figure 5 the coupling parameter $d_1/\lambda_1 = 0.1$, i.e. in the weak coupling limit with asymmetric tunnelling rates. As was the case for symmetric tunnelling rates, the threshold dissipation for the two modes again have maxima although the location of the maxima depends on $\Gamma_0$. In this case the two modes can be treated independently and $\gamma_{\text{thr},n} > \gamma_{\text{thr},m}$ for $n < m$. When increasing the electromechanical coupling
Figure 5. The threshold dissipation for the first ($\gamma_{\text{thr},1}$) and third ($\gamma_{\text{thr},3}$) bending modes when $d_1/\lambda_1 = 0.1$ and $\Gamma_R = \beta_3/2$. The different modes can be treated independently and $\gamma_{\text{thr},n} > \gamma_{\text{thr},m}$ for $n < m$. By tuning the tunnelling rate $\Gamma_0$ by changing the STM–tube distance the threshold for excitation of a specific mode can be optimized.

Figure 6. The threshold dissipation for the first ($\gamma_{\text{thr},1}$) and third ($\gamma_{\text{thr},3}$) bending modes when $d_1/\lambda_1 = 3$ and $\Gamma_R = \beta_3/2$. The modes are not independent and the mode interaction results in a modification of the threshold dissipations so that, for some values of $\Gamma_0$, $\gamma_{\text{thr},n} \ll \gamma_{\text{thr},m}$ for $n < m$. This does not occur in the weak coupling limit.

parameter to $d_1/\lambda_1 = 3$, as plotted in figure 6, the modes can no longer be treated independently, but the effect of the mode coupling is not very strong. Finally, for a very large coupling parameter as in figure 7, where $d_1/\lambda_1 = 10$, the mode coupling leads to a qualitatively different behaviour. In this case the peak heights can be of the same order of magnitude, and by changing $\Gamma_0$, we can selectively excite different modes. We emphasize that this can be accomplished by changing the STM–tube distance.
Figure 7. The threshold dissipation for the first ($\gamma_{\text{thr},1}$) and third ($\gamma_{\text{thr},3}$) bending modes when $d_1/\lambda_1 = 10$ and $\Gamma_R = \beta_3/2$. The threshold dissipations for the modes are strongly modified by the mode interaction and we see that $\gamma_{\text{thr},n}$ may be smaller than $\gamma_{\text{thr},m}$ for $n < m$. This allows for the possibility to selectively excite a specific mode by changing $\Gamma_0$, a scenario not possible in the weak coupling limit.

5. Limit cycle with one mode

In the previous section, we showed that one or several modes become unstable with respect to small displacements if dissipation is sufficiently weak. Also, we showed that different modes can be treated independently in the limit of weak electromechanical coupling. In this section, we will analyse the limit cycle behaviour when only one mode is included. We restrict our considerations to weak electromechanical coupling, and parameters are chosen so that only one mode can be unstable. When investigating the limit cycle behaviour we will no longer use linearized tunnelling matrix elements, instead we use an exponential form for the STM–tube tunnel matrix elements in equations (13) and (14), i.e. $\Gamma_L(x, V) = \Gamma_0(V) \exp(-2x/\lambda)$. Since we are only considering a single mode in the rest of this paper we neglect the mode indices $n$. We investigate the steady state density matrix using the Wigner representation of the density operator defined by [25]

$$W_{\pm}(x, p) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{d} \xi \ e^{-i p \xi} \left( x - \frac{d}{2} + \frac{\xi}{2} \right) \rho_{\pm} \left( x - \frac{d}{2} - \frac{\xi}{2} \right),$$

(26)

where $\rho_{\pm} = \rho_0 \pm \rho_1$. The use of the Wigner representation for investigating the steady state of a shuttle was proposed in [26]. In this paper, we use the method developed in [27]. This method can be modified to be applicable for the model investigated in this paper. Writing equations (13) and (14) using the Wigner representation (equation (26)) we get equations for $W_{\pm}$, details are given in [27, 28]. To find the steady state solution it is convenient to introduce polar coordinates $x = \lambda A \cos \phi$, $p = \lambda A \sin \phi$, where $A$ is the amplitude and $\phi$ is the phase. We seek the steady state distribution of amplitudes defined by

$$\bar{W}_+(A) = \frac{1}{2\pi} \int_{0}^{2\pi} \mathrm{d} \phi \ W_+(A, \phi),$$

(27)
Figure 8. The function $\alpha_0(A)$ for $\Gamma = 0.1$ and the two cases $\Gamma_0 > \Gamma_R$ and $\Gamma_0 < \Gamma_R$. This function determines the leading behaviour in the limit cycle. Three different values of the dissipation rate $\gamma$ are shown as horizontal lines (denoted a, b and c) and stable solutions are marked with a ring. If $\gamma < \alpha_0(0)$ (line a) only one solution with $A \geq 0$ is possible which implies that the non-vibrating ground state is unstable. If $\alpha(A) < \gamma < \max \alpha$ (line b), solutions with $A = 0$ and $A \geq 0$ are possible. Finally if $\gamma \geq \max \alpha$ (line c), the only solution corresponds to the non-vibrating ground state with $A = 0$. If $\alpha_0(A)$ has a maximum for a nonzero $A$ the system is bistable, i.e. there is a region where the non-vibrating and the vibrating state coexist. In the limit of $\Gamma \to 0$ a bistable system occurs if $\Gamma_0 > \Gamma_R$.

and seek a solution assuming that $1/\lambda^2, d/\lambda, \gamma \ll 1$, i.e. weak electromechanical coupling and weak dissipation. To leading order in these small parameters the solution is given by

$$W_+(A) = Z^{-1} \exp \left[ - \int_0^A \frac{f(A')}{D(A')} dA' \right],$$

where $Z$ is a normalization constant determined by $2\pi \int_0^\infty dA W_+ = 1$ and

$$f(A) = \frac{A}{2} \left[ \gamma - \frac{d}{\lambda} \alpha_0(A) \right],$$

$$\alpha_0(A) = -\frac{1}{A} \int_0^{2\pi} d\phi \cos \phi \left[ \partial_\phi - \Gamma(A \cos \phi, V) \right]^{-1} \Gamma_-(A \cos \phi, V),$$

$$0 < D(A) \sim \left( \frac{d}{2\lambda} \right)^2,$$

where $\Gamma(\lambda, A \cos \phi, V) = \Gamma_L(\lambda, A \cos \phi, V) + \Gamma_R(0, V)$ and $\Gamma_- = \Gamma_R(0, V) - \Gamma_L(\lambda A \cos \phi, V)$. The distribution has a maximum for $A_M$ which is defined by $f(A_M) = 0$ and $f'(A_M) > 0$. In the vicinity of these points the distribution function is approximately Gaussian with variance $\sigma^2 = D(A_M)/f'(A_M)$. The points that satisfy these conditions are determined by $\alpha_0(A)$ and $\gamma$. These are plotted for different sets of parameters in figure 8. We find a qualitatively different
Figure 9. Threshold dissipation as a function of the tunnelling rates of the two junctions with $d/\lambda = 0.01$. The optimal value is found when $\Gamma \sim 1$, i.e. when the tunnelling rates are of the order of the mode frequency. In a highly non-symmetric set-up the threshold dissipation is small, making an instability unlikely.

Figure 9.

6. IV-characteristics with a single mode

Next we present the IV-characteristics in the limit where only one mechanical mode is involved in transport. The current operator in the Heisenberg picture is given by

$$\hat{I}_\alpha = -e \frac{d\hat{N}_\alpha(t)}{dt} = -ie[\hat{N}_\alpha, \hat{H}],$$

which is valid for asymmetric tunnelling rates and reduces to equation (25) for $\Gamma_0 = \Gamma_R = \Gamma/2$. Furthermore, we find that the system is bistable if $\alpha'(A) = 0$ for $A > 0$. If the system is bistable, two solutions with zero amplitude and finite amplitude may coexist for a range of $d/(\lambda \gamma)$. The occurrence of bistability depends on the system parameters and in particular on the ratio of the tunnelling rates. This is a result of the position independent tunnelling matrix elements for tube–electrode tunnelling, making the junction properties fundamentally different. From equation (33), we find that $\gamma_{\text{thr}}$ has a maximum for $\Gamma_0 = \Gamma_R = 1/2$. The threshold dissipation goes to zero in the limit of $\Gamma_\alpha \to 0$ and in the limit of $\Gamma_\alpha \to \infty$. In figure 9, the threshold dissipation as a function of junction tunnelling rates is shown. Also, the condition for instability is symmetric in the junction tunnelling rates and for a symmetric set-up with $\Gamma_0 = \Gamma_R$ we get exactly half the threshold as compared to the ordinary shuttle [27]. This is due to the fact that only one junction contributes to the correlation leading to the instability.
The current is constant and given by the tunnel current $I = e \Gamma_L \Gamma_R / \Gamma$ for small voltages. Above a threshold voltage the current is no longer constant but depends on the vibration amplitude. We observe hysteresis in the IV-characteristics for some sets of parameters but for others.

where $\hat{N}_\alpha = \sum_k a_k^\dagger a_k$. Using consistent approximations with the derivation of the GME we find the average current

$$I(t) = e \langle \Gamma_L (\lambda A \cos \phi, V) \rangle_0.$$  \hspace{1cm} (34)

In the semi-classical regime the distribution function $\bar{W}_+(A)$ is sharply peaked around the maximum $A_M$ and, to leading order in the small parameters, we find

$$I(A_M) = \frac{e}{2} \frac{1}{2\pi} \int_0^{2\pi} d\phi \Gamma_L (A_M \cos \phi, V) \left( 1 + \left[ \partial_\phi - \Gamma(A_M \cos \phi, V) \right]^{-1} \Gamma_-(A_M \cos \phi, V) \right).$$  \hspace{1cm} (35)

Solving this numerically gives the current through the system as a function of the limit cycle vibration amplitude. The vibration amplitude is determined by $d/\lambda \gamma$ and $\alpha_0(A)$. The electromechanical coupling parameter depends on the bias voltage scale. We assume a linear relationship, i.e. $d/\lambda = eVd_0/\lambda_0$. We can now find the current–voltage relationship of the device numerically for different sets of parameters. The IV-characteristics for the single level case is shown in figure 10. The current is given by the tunnelling current value $I = e \Gamma_0(0, V) \Gamma_R(0, V) / \Gamma(0, V)$, which is a constant in this regime, until the instability threshold is reached, where the current is modified due to the finite amplitude vibrations. Depending on the tunnelling rates, the vibration amplitude increases either smoothly (not shown) or in a step-like fashion, which is reflected in the IV-curve. We also find the IV-characteristics in the continuum regime when sweeping the applied voltage, which is plotted in figure 11. The current increases smoothly with increased bias, and is given by the tunnelling current value $I = e \Gamma_L(0, V) \Gamma_R(0, V) / \Gamma(0, V)$, until the point when the instability occurs, where the current is

Figure 10. IV-characteristics of the studied device in the single level regime and with $\Gamma_L(0, V) = 0.1$, $\Gamma_R(0, V) = 0.2$, $eV_g = 50$, $U = 100$, and $d_0/(\lambda_0 \gamma) = 0.1$. The current is constant and given by the tunnel current $I = e \Gamma_L \Gamma_R / \Gamma$ for small voltages. Above a threshold voltage the current is no longer constant but depends on the vibration amplitude. We observe hysteresis in the IV-characteristics for some sets of parameters but for others.
Figure 11. IV-characteristics of the studied device in the continuum spectrum regime and with $\Gamma_L(0, V) = 0.005(V/2 + V_g - U/2)$, $\Gamma_R(0, V) = 0.03(eV/2 + eV_g + U/2)$, $eV_g = 50$, $U = 100$, and $d_0/(\lambda_0\gamma) = 0.1$. The current increases smoothly as the bias voltage is increased and is given by the tunnel current $I = e\Gamma_L(0, V)\Gamma_R(0, V)/\Gamma(0, V)$ until the threshold is reached, which is reflected as a step in the IV-characteristics. We observe a hysteretic behaviour with respect to the direction of the voltage sweep.

modified due to the instability. Again we observe a hysteretic behaviour in the IV-characteristics. In this regime, we emphasize that tunnelling rates explicitly depend on bias voltage.

7. Conclusions

We have theoretically investigated electromechanical instabilities of different bending modes of a suspended CNT when probing the suspended part of the tube with an STM-tip. Using a full mechanical description of the flexural vibrations and considering two regimes for the electronics of the nanotube (single level and continuum spectrum), we derive a general quantum master equation for the tube dynamics. We analyse this to show that instabilities of different modes occur under the right conditions. The conditions for instability depend on the electromechanical coupling parameter and the degree of asymmetry of the junction tunnelling rates. Optimal conditions for an instability of a specific mode are found when tunnelling rates are symmetric and of the order of the mode frequency. With optimal conditions in the weak electromechanical coupling limit, $Q$-factors $1/Q \lesssim d/\lambda$ are needed for an instability to occur. Even a pessimistic estimate for the electromechanical coupling parameter in an experimental set-up, $d/\lambda \sim 0.01$, translates into necessary $Q$-factors of the order of 100. Experimental data have shown $Q \sim 10^2 - 10^3$, which makes the instability experimentally accessible and increasing the coupling parameter reduces the necessary $Q$-factors. However, a highly asymmetric set-up increases the $Q$-factors needed by orders of magnitude, making an instability unlikely. In the limit of weak electromechanical coupling the different modes can be treated independently with respect to the instability. We find that only the fundamental mode is unstable for some parameters,
while decreasing the dissipation allows for simultaneous instabilities of several modes. When considering the strong coupling regime we find that a selective instability of higher modes without instability of lower modes is feasible. Finally, we investigate the limit cycle behaviour of the vibrations when only a single mode is unstable, and show that zero and finite amplitude solutions may or may not coexist. We compare the current–voltage characteristics of the two regimes.

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Appendix A. Nanotube mechanics

Here we derive the Hamiltonian describing the bending dynamics of the nanotube (equation (3)). For small deformations the appropriate Lagrangian is quadratic in the displacement field $u(z, t)$ and given by

$$L = \int_0^L \, dz \left\{ \frac{\rho S}{2} \dot{u}^2(z, t) - \frac{EI}{2}(u''(z, t))^2 \right\},$$

(A.1)

where $\rho$ is the mass density of the tube, $S$ is the cross-section area, $E$ is the Young’s modulus, $I$ is the moment of inertia of the cross-section and field $u(z, t)$ satisfies boundary conditions $u(0) = u(L) = u'(0) = u'(L) = 0$. One can expand $u(z, t)$ in the orthogonal set of functions $u_n(z)$, so that

$$u(z, t) = \sum_n X_n(t) u_n(z).$$

(A.2)

Here $u_n(z)$ are solutions of the differential equation

$$\frac{\partial^4}{\partial x^4} u_n(z) = \kappa_n^4 u_n(z)$$

(A.3)

with the above boundary conditions. One finds that

$$u_n(z) = A_n ((\sin \kappa_n L - \sinh \kappa_n L) (\cos \kappa_n z - \cosh \kappa_n z)$$

$$- (\cos \kappa_n L - \cosh \kappa_n L) (\sin \kappa_n z - \sinh \kappa_n z)),$$

(A.4)

where $\kappa_n$ are given by the equation

$$\cos \kappa_n L \cosh \kappa_n L = 1.$$

(A.5)

Substituting (A.3) into (A.1) one obtains

$$L(X_n) = \sum_n \left( \frac{M}{2} \dot{X}_n^2 - \frac{MEI \kappa_n^4}{2\rho S} X_n^2 \right).$$

(A.6)
This Lagrangian describes a set of independent oscillators with frequencies
\[ \omega_n = \frac{c_n}{L^2} \sqrt{\frac{EI}{\rho S}}, \quad n = 1, 2, 3, \ldots, \]  
(A.7)
where \( c_n = 22.4, 61.7, 121.9, \ldots \).

**Appendix B. Derivation of the GME**

Here we derive the GME for the tube dynamics (equations (13) and (14)). This derivation is valid for the continuum regime but the same procedure can be used in the single level regime.

The time evolution of the density matrix \( \sigma \) is governed by the Liouville–von Neumann equation
\[ i\partial_t \sigma = [\hat{H}, \sigma]. \]  
(B.1)
We define operators in the interaction representation with respect to the non-interacting Hamiltonian \( \hat{H}_0 = \hat{H} - \Omega \)
\[ \tilde{A}(t) = e^{i\hat{H}_0 t} A e^{-i\hat{H}_0 t}, \]  
(B.2)
which results in
\[ i\partial_t \tilde{\sigma}(t) = [\tilde{\Omega}(t), \tilde{\sigma}(t)]. \]  
(B.3)
This is transformed into an equivalent integro-differential form
\[ \partial_t \tilde{\sigma} = -i[\tilde{\Omega}(t), \tilde{\sigma}(\Delta t)] - \int_{-\infty}^t dt \{[\tilde{\Omega}(t), [\tilde{\Omega}(t), \tilde{\sigma}(t)]]\}. \]  
(B.4)
The first term on the right-hand side is zero with a proper choice of boundary conditions. Next we introduce the probability of the tube being charged by \( N \) electrons
\[ \tilde{\rho}_N(t) = \text{Tr}_e \delta(\hat{n} - N) \tilde{\sigma}(t), \]  
(B.5)
where the trace is taken over all electronic degrees of freedom. Then we multiply both sides of equation (B.4) with \( \delta(\hat{n} - N) \)
\[ \partial_t \tilde{\rho}_N = -\sum_{\alpha} \int_{-\infty}^t dt_1 \int d\epsilon D_{\alpha}[e^{i(t-t_1)(\epsilon-u-iU/2(N+1))}]
+ \left( \tilde{T}_{\alpha}(t) \tilde{T}_{\alpha}(t_1) \tilde{\rho}_N(t_1) - e^{i(t-t_1)} \tilde{T}_{\alpha}(t_1) \tilde{\rho}_N(t_1) + e^{i(t-t_1)} \tilde{\rho}_N(t_1) \tilde{T}_{\alpha}(t_1) \tilde{T}_{\alpha}(t_1) \right) f_{\alpha}^+ f_{\alpha}^- + \text{h.c.} \]  
(B.6)
Here \( f_{\alpha}^+ (T) \) is the Fermi function in lead \( \alpha \) (tube) and \( f_{\alpha}^- (T) = 1 - f_{\alpha}^+ (T) \), \( D_{\alpha}(T) \) is the density of states in lead \( \alpha \) (tube) which is constant in the wide band approximation. The next step is to perform the integrations over energies. The kernel of the integral consists of an oscillating part
associated with the scale set by the voltages and a part associated with the scale of the oscillators \( \hbar \omega_n \). If typical voltage differences are large compared to \( \omega_K \) (the maximum frequency of the mechanical modes included) the latter part is approximately constant and the energy integrals are readily solved. Then we divide the integral into its real and imaginary parts, the imaginary part renormalizes the oscillators and is henceforth neglected. The real part describes tunnelling and after going back to the Schrödinger representation we get

\[
\partial_t \rho_N = -i[\hat{H}_{\text{osc}}, \rho_N] - 2\pi \sum_a D_a D^\dagger a \{ \frac{1}{2} \Delta E^{(N)+}_a \theta(\Delta E^{(N)+}_a) \{ T^2_a(\hat{x}), \rho_N \} \\
- \Delta E^{(N-1)+}_a \theta(\Delta E^{(N-1)+}_a) T_a(\hat{x}) \rho_{N-1} T^\dagger a(\hat{x}) + \frac{1}{2} \Delta E^{(N)-}_a \theta(\Delta E^{(N)-}_a) \{ \rho_N, T^2_a(\hat{x}) \} \\
- \Delta E^{(N+1)-}_a \theta(\Delta E^{(N+1)-}_a) T_a(\hat{x}) \rho_{N+1} T^\dagger a(\hat{x}) \} + L_\gamma \rho_N,
\]

where \( \Delta E^{N\pm}_a = \pm eV_a \pm eV_g \pm UN - U/2 \) is the energy window available for tunnelling. If the energy window is negative the process is forbidden at zero temperature. The last term

\[
L_\gamma \rho_N = -\sum_n \left( \frac{i}{2} \{ \hat{x}_n, \{ \hat{p}_a, \rho_N \} \} + \frac{\gamma}{2} \{ \hat{x}_n, [\hat{x}_n, \rho_N] \} \right),
\]

describes dissipation of the mechanical modes. We define

\[
\Gamma^{N\pm}_a(\hat{x}, V) = 2\pi D_a D^\dagger a T^2_a(\hat{x}) \Delta E^{N\pm}_a \theta(\Delta E^{N\pm}_a),
\]

which are the tunnelling rates for tunnelling to/from lead \( \alpha \) resulting in an increase (+) or a decrease (−) of the number of electrons on the tube when the tube is already occupied by \( N \) electrons. With this notation the equation can be written in the compact form

\[
\partial_t \rho_N = -i[\hat{H}_{\text{osc}}, \rho_N] - \sum_a \left( \frac{1}{2} \{ \Gamma^{N+}_a(\hat{x}, V), \rho_N \} + \frac{1}{2} \{ \Gamma^{N-}_a(\hat{x}, V), \rho_N \} \\
- \sqrt{\Gamma^{(N-1)+}_a(\hat{x}, V)} \rho_{N-1} \sqrt{\Gamma^{(N-1)+}_a(\hat{x}, V)} \\
- \sqrt{\Gamma^{(N+1)-}_a(\hat{x}, V)} \rho_{N+1} \sqrt{\Gamma^{(N+1)-}_a(\hat{x}, V)} \right) + L_\gamma \rho_N.
\]

The first term on the right-hand side of the equation describes free evolution of the oscillators around the equilibrium position with \( N \) electrons on the tube, the second term describes tunnelling, and the last term describes damping of the different mechanical modes.

Repeating the same calculations for the single level regime gives an equation of the same form as equation (B.10) with a difference in the tunnelling rates. Instead of tunnelling rates according to equation (B.9) tunnelling rates are now given by

\[
\Gamma^{N\pm}_a(\hat{x}, V) = 2\pi D_a D^\dagger a T^2_a(\hat{x}) \theta(\Delta E^{N\pm}_a),
\]

which does not explicitly depend on voltage if we choose voltages so that \( 0 < e(V/2 - V_g) + U/2 < U < e(V/2 + V_g) + U/2 < 2U \) are satisfied. Then several tunnelling processes are forbidden due to energy considerations. The only charge states allowed are \( N = 0, 1 \). The
equations for $\rho_{N \neq 0,1}$ can be solved by inspection with the result $\rho_{N \neq 0,1} = 0$. Then we end up with equations (13) and (14), after using the facts that $\Gamma_R$ is independent of the mechanical degrees of freedom. Now the indices $N \pm$ are redundant which gives tunnelling rates according to equations (15) and (16) in the two different regimes.

Appendix C. Instability of different modes

Here we derive the increments $\alpha_m$ that determine the threshold dissipation of the different mechanical mode in the weak electromechanical coupling limit. We seek the eigenvalues $\alpha_m$ of the $4K + 1$ by $4k + 1$ matrix

$$A = \begin{pmatrix} A_1 & 0 & 0 & d_1 \\ 0 & A_2 & 0 & d_2 \\ 0 & 0 & \ddots & \vdots \\ c_1^T & c_2^T & \cdots & -\Gamma \end{pmatrix},$$  

(C.1)

where

$$A_n = \begin{pmatrix} 0 & \omega_n & 0 & 0 \\ -\omega_n & -\gamma & 0 & 0 \\ \Gamma_- & 0 & -\Gamma & \omega_n \\ 0 & \Gamma_- & -\omega_n & -\Gamma - \gamma \end{pmatrix},$$  

(C.2)

$$d_n = \begin{pmatrix} 0 & -d_n \omega_n/2 & 0 & d_n \omega_n/2 \end{pmatrix}^T,$$  

(C.3)

$$c_n = \begin{pmatrix} 2\Gamma_0/\lambda_n & 0 & 0 \end{pmatrix}^T,$$  

(C.4)

where $\Gamma_0 = \Gamma_L(0, V)$, $\Gamma = \Gamma_+(0, V)$, $\Gamma_- = \Gamma_-(0, V)$, and $\Gamma_{\pm}(\cdot, \cdot) = \Gamma_R(\cdot, \cdot) \pm \Gamma_L(\cdot, \cdot)$. We are mainly interested in the eigenvalues of the form $\alpha_m = i\omega_n + \delta_n$, for which we seek a perturbative solution in the small parameters $\gamma$ and $d_n/\lambda_n$. The eigenvalues of $A$ are determined by $|A - \alpha I_{4K+1}| = 0$, with $I_x$ being an identity matrix of dimension $x$ by $x$. We divide $A - \alpha I_{4K+1}$ into a two by two block matrix

$$A - \alpha I_{4K+1} = \begin{pmatrix} A_B - \alpha I_{4K} & D \\ C^T & -\Gamma - \alpha \end{pmatrix}.$$  

(C.5)

Here the $4K$ by $4K$ matrix $A_B$ is block diagonal with $A_1 \cdots A_m$ on the diagonal, $D = (d_1^T, \ldots, d_m^T)^T$ and $C = (c_1^T, \ldots, c_m^T)^T$. In general the determinant is given by

$$|A - \alpha I_{4m+1}| = |A_1 - \alpha I_4||A_2 - \alpha I_4| \cdots |A_n - \alpha I_4|(-\Gamma - \alpha + C^T(\tilde{A} - \alpha I_{4m})^{-1}D).$$  

(C.6)

With symmetric tunnelling rates, i.e. $\Gamma_- = 0$, $\Delta_n$ is block diagonal with

$$\tilde{A}_n = \begin{pmatrix} 0 & \omega_n \\ -\omega_n & -\gamma \end{pmatrix}.$$  

(C.7)
and $\tilde{A}_n = \tilde{A}_n - \Gamma I_2$ on the diagonal we get a simple expression for the determinant

$$
|A - \alpha I| = -|\tilde{A}_1 - \alpha I_2||\tilde{A}_1 - \alpha I_2| \cdots |\tilde{A}_m - \alpha I_2||\tilde{A}_m - \alpha I_2|
$$

$$
\times \left( \Gamma + \alpha + \sum_n \frac{\Gamma d_n \omega_n^2}{2\lambda_n} \left( \frac{1}{|\tilde{A}_n - I_2\alpha|} + \frac{1}{|\tilde{A}_n - I_2\alpha|} \right) \right). \quad (C.8)
$$

In the limit of weak electromechanical coupling ($d_1/\lambda_1 \ll 1$) we seek eigenvalues of the form $\alpha_m = i\omega_n + \delta\omega_n$, where $\delta\omega_n$ is a small correction to the unperturbed eigenvalue. Then to first order in the small parameters, $|\tilde{A}_n - \alpha_n I_2| \approx i\omega_n \gamma + 2i\omega_n \delta\omega_n$. We get

$$
(\Gamma + i\omega_n)(i\omega_n \gamma + 2i\omega_n \delta\omega_n) + \frac{\Gamma d_n \omega_n^2}{2\lambda_n} = 0, \quad (C.9)
$$

which we solve to first order in the small parameters to find the correction to the unperturbed eigenvalues

$$
\delta\omega_n \approx -\frac{\gamma}{2} + \frac{d_n}{4\lambda_n \omega_n - i\Gamma}. \quad (C.10)
$$

The real part of this expression gives equation (25).

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