A NEW BOUND FOR THE ERROR TERM IN THE APPROXIMATE FUNCTIONAL EQUATION FOR THE DERIVATIVES OF THE HARDY’S Z-FUNCTION

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Abstract. Lavrik and the author gave uniform bounds of the error term in the approximate functional equation for the derivatives of the Hardy’s Z-function. We obtain a new bound of this error term which is much better for high order derivatives.

1. Introduction and main result

Let \( \zeta \) be the Riemann zeta function, and \( Z \) the Hardy function defined by

\[
Z(t) = e^{i\theta(t)}\zeta \left( \frac{1}{2} + it \right)
\]

where

\[
\theta(t) = \arg \left( \pi^{-\frac{1}{2}} \Gamma \left( \frac{1}{4} + it \right) \right)
\]

and the argument is defined by continuous variation of \( t \) starting with the value 0 at \( t = 0 \). The real zeros of \( Z \) coincide with the zeros of \( \zeta \) located on the line of real part \( \frac{1}{2} \). The function \( \theta \) plays a central role in this paper and it is important to mention \([4]\) that

\[
\theta(t) = \frac{t}{2} \log \frac{t}{2\pi} - \frac{t}{2} - \frac{\pi}{8} + O \left( t^{-1} \right)
\]

and

\[
\theta'(t) = \frac{1}{2} \log \frac{t}{2\pi} + O \left( t^{-2} \right).
\]

A weak form of the celebrated Riemann-Siegel formula \([4]\) asserts that

\[
Z(t) = 2 \sum_{1 \leq n \leq \sqrt{t}} \frac{1}{\sqrt{n}} \cos(\theta(t) - t \log n) + O \left( t^{-\frac{1}{4}} \right)
\]

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and, concerning the derivatives of $Z$, the approximate functional equation reads
\[ Z^{(k)}(t) = 2 \sum_{1 \leq n \leq \sqrt{\frac{t}{n}}} \frac{1}{\sqrt{n}} (\theta'(t) - \log n)^k \cos(\theta(t) - t \log n + k \frac{\pi}{2}) + R_k(t) \]
where $R_k(t)$ is the error term. Of particular interest is the set of integers $k$, which depends on $t$, such that, uniformly in $k$,
\[ R_k(t) = o(\theta'(t)^k) \text{ as } t \to \infty \]
which means that $R_k(t)$ is a true error term.
Lavrik [5] proved that (1.4) holds for $0 \leq k \leq c \theta'(t)$ where $c < \frac{1}{2 \log 3} = 0.4551...$, the author [1] extended this result to $c < 1.7955...$ and numerical experiments suggested that (1.4) is probably true for larger $k$. For example $|R_k(10^k)| \leq 0.05 \theta'(10^k)^k$ for $k = 1, \ldots, 117$.
In this paper, we prove that (1.4) holds for $0 \leq k \leq c \theta'(t)^2$ where $c < 3$, which is a consequence of Theorem 1.1. To simplify its proof, and since the case $0 \leq k < \theta'(t)$ is covered by [1], we restrict our attention to the case $k \geq \theta'(t)$.

**Theorem 1.1.** Let $t$ be large enough and $c > 1$ be a fixed constant. Then, for $\theta'(t) \leq k \leq 3 \theta'(t)^2$, we have, uniformly in $k$,
\[ R_k(t) = O \left( t^{-\frac{k}{2}} e^{\frac{k^2}{2} \theta'(t)^2} + t^{-\frac{k}{2}} e^{\frac{k}{2} \theta'(t)^2} k \theta'(t)^{k-1} \right) . \]

The notations used in this paper are standard: $[x]$ and $\lceil x \rceil$ stand for the usual floor and ceiling functions and $\{x\} := x - [x]$. We denote by $(x)_n$ the Pochhammer symbol defined by $(x)_0 = 1$ and $(x)_n = x(x + 1) \cdots (x + n - 1)$ for $n \in \mathbb{N}^*$.
In the next section, we recall some results of the author used in the proof of Theorem 1 of [1] and we state the lemmas needed in the proof of our main result. Section 3 is devoted to the proofs.

2. Preliminary results

The functions $\eta_p(d, s) := \sum_{n=1}^{\infty} n^{-s}(d - \log n)^p$ defined for $d \in \mathbb{R}$, $p \in \mathbb{N}$ and $\Re(s) > 1$ have a meromorphic extension to $\Re(s) > 0$ with a pole at $s = 1$ and, as a consequence of the Faà di Bruno formula [6], we have
\[ Z^{(k)}(t) = e^{i \theta(t)} t^k \eta_k(\theta'(t), \frac{1}{2} + it) + e^{i \theta(t)} \sum_{p=0}^{k-2} q_p(t) i^p \eta_p(\theta'(t), \frac{1}{2} + it) \]
where
\[ q_p(t) = \sum_{\substack{2p_2+3p_3+\cdots+k_p \leq k-p \\| p_2, p_3, \ldots, p_k \geq 0}} \frac{k!}{p! p_2! \cdots p_k!} \left( \frac{i \theta''(t)}{2!} \right)^{p_2} \left( \frac{i \theta'''(t)}{3!} \right)^{p_3} \cdots \left( \frac{i \theta^{(k)}(t)}{k!} \right)^{p_k} . \]

The first step in our proof is to get an approximate functional equation for the functions $\tilde{\eta}_p(\theta'(t), \frac{1}{2} + it)$. In [1], we proved that the functions $\tilde{\eta}_p(d, s) := (-1)^p \eta_p(d, s)$
satisfy, for \( d = \theta'(t) \) and \( s = \frac{1}{2} + it \), the relation

\[
\bar{\eta}_p(d, s) = \sum_{1 \leq n \leq N} \phi_p(n) + \frac{N^{1-s}}{s-1} \frac{1}{(\log N - d)^p} \sum_{l=0}^{p} \frac{((s-1)(\log N - d))^{-l}}{(p-l)!} + O\left((1 + |t|)N^{-\frac{1}{2}} \log^p N\right)
\]

(2.3)

where \( \phi_p(x) := x^{-s}(\log x - d)^p \).

In this paper, we fix a constant \( c > 1 \) and for \( t \) sufficiently large, we set \( N_0 = [ce^d] \), \( N_1 = [ce^{2d}] \) and for \( N > N_1 \) we split the sum

\[
\sum_{1 \leq n \leq N} \phi_p(n) = \sum_{1 \leq n \leq N_0} \phi_p(n) + \sum_{N_0 < n \leq N_1} \phi_p(n) + \sum_{N_1 < n \leq N} \phi_p(n).
\]

We use Lemmas 2.1, 2.2, and 2.4 to transform the second sum in a short sum (Lemma 2.3), and Lemma 2.4 to apply the Euler-MacLaurin formula to the third sum (Lemma 2.5).

**Lemma 2.1.** Let \( a < b \) be integers and let \( \varphi \in C^2[a, b] \) and \( f \in C^5[a, b] \) be real-valued functions possessing the following property: There exist constants \( H > 0 \) and \( \varphi(x) \leq H \), \( \varphi'(x) \leq HU^{-1} \), \( \varphi''(x) \leq HU^{-2} \) for all \( x \in [a, b] \).

Let furthermore \( \Theta \) be the function defined on \( [0, \infty[ \times [0, 1] \) by

\[
\Theta(\lambda, \mu) = \int_{0}^{\infty} \frac{\sinh(2\pi(\mu - \frac{1}{2})x)}{\sinh(\pi x)} e^{-i\lambda x^2} dx
\]

and \( x(\cdot) \) be the unique function defined by \( f'(x(y)) = y \) for all \( y \in [f'(a), f'(b)] \). Then

\[
\sum_{a < n \leq b} \varphi(n) e^{2\pi if(n)} = e^{\frac{1}{2}} \int_{f'(a) < n \leq f'(b)} \frac{\varphi(x(n))}{\sqrt{f''(x(n))}} e^{2\pi i f(x(n)) - nx(n)} + R(b) - R(a) + O(H)
\]

where

\[
R(l) = \varphi(l) e^{2\pi if(l)} \Theta(f''(l), \{ f'(l) \}).
\]

**Lemma 2.2.** For \( d \in \mathbb{R} \) and \( p \in \mathbb{N} \), let \( \varphi_p \) be the function defined by \( \varphi_p(x) = x^{-s}(\log x - d)^p \) for \( x \geq 1 \). Then, for \( 2 < d \leq p \), \( e^d \leq a < ce^{2d} \) and \( x \in [a, (2a)^*] \) where \((2a)^* = \min(2a, ce^{2d})\), we have

\[
\varphi_p(x) \leq H, \varphi'_p(x) \leq Ha^{-1}, \varphi''_p(x) \leq Ha^{-2}
\]

where \( H = p^2\varphi_{p-2}((2a)^*) \).
Lemma 2.3. Let $t$ be large enough and assume that $\theta'(t) \leq p \ll t^{\frac{2}{3}}$. Then
\begin{equation}
\sum_{N_0 < n \leq N_1} \phi_p(n) = e^{-2i\theta(t)} \sum_{n \leq N_0} \frac{(\theta'(t) - \log n)^p}{n^{\frac{1}{4} + it}} + O\left(t^{-\frac{1}{2}} e^{\frac{\pi}{4\theta}} \theta'(t)^{p-2}\right).
\end{equation}

The next lemma prepares the application of the Euler-MacLaurin formula to the third sum of (2.4). In [1], the bound we got for the third sum, which depends on an upper bound for $|\phi_p|$ on $[N_1, N]$, is not optimal for $p \geq d$.

Lemma 2.4. For $d \in \mathbb{R}$, $p \in \mathbb{N}$ and $s = \frac{1}{2} + it$, let $g_p$ and $\phi_p$ be the function defined by $g_p(x) = (\log x - d)^p$ and $\phi_p(x) = x^{-s} g_p(x)$ for $x \geq 1$. Then, for $0 < d \leq p$, $x \in [e^{2d}, \infty[$ and $k \in \mathbb{N}$, we have
\begin{equation}
\left|\phi_p^{(k)}(x)\right| \leq k! \left(\frac{p}{d}\right)^k x^{-k} g_p(x).
\end{equation}
Further, for $t$ large enough, let $d = \theta'(t)$, $K \gg t^{\frac{2}{3}}$, $N_1 = [e^{2d}]$, $N_2 = N_1 + 1$ and let $N > N_2$. Then, for $d \leq p \ll t^{\frac{2}{3}}$ and $0 \leq k \leq 2K$, we have
\begin{equation}
\phi_p^{(k)}(x) = (-1)^k (s)_k x^{-s-k} g_p(x) \left(1 + O\left(d^{-1}\right)\right)
\end{equation}
for $x \in [e^{2d}, \infty[$ and moreover
\begin{equation}
\phi_p^{(k)}(N_2) \ll \left(\frac{2\pi}{c}\right)^k t^{-\frac{1}{2}} e^{\frac{\pi}{4d} d^p},
\end{equation}
\begin{equation}
\phi_p^{(k)}(N) \ll \left(\frac{2\pi}{c}\right)^k N^{-\frac{1}{2}} \log^p N
\end{equation}
and
\begin{equation}
\int_{N_2}^{N} |\phi_p^{(2K)}(u)| du \ll \left(\frac{2\pi}{c}\right)^{2K} e^{\frac{\pi}{4d} d^p} + \left(\frac{t}{N}\right)^{2K-\frac{1}{2}} \log^p N.
\end{equation}

Lemma 2.5. Let $t$ be large enough and assume that $\theta'(t) \leq p \ll t^{\frac{1}{2}}$. Then
\begin{equation}
\sum_{N_1 < n \leq N} \phi_p(n) = -N^{1-s} s^{-1} \log N - d)^p p! \sum_{l=0}^{p} \frac{(s-1)(\log N - d)^{-l}}{(p-l)!} + O\left(t^{-\frac{1}{2}} e^{\frac{\pi}{4d} d^p} + N^{-\frac{1}{2}} \log^p N\right).
\end{equation}

Finally, the next lemmas are needed to make use of relation (2.4).

Lemma 2.6. Let $t$ be large enough, $c > 1$ be a fixed constant and assume that $\theta'(t) \leq p \ll t^{\frac{1}{2}}$. Then
\begin{equation}
\eta_p\left(\theta'(t), \frac{1}{2} + it\right) = \sum_{1 \leq n \leq \sqrt{\frac{t}{d}}} \frac{(\theta'(t) - \log n)^p}{n^{\frac{1}{4} + it}} + e^{-2\theta(t)} \sum_{1 \leq n \leq \sqrt{\frac{t}{d}}} \frac{(\log n - \theta'(t))^p}{n^{\frac{1}{4} - it}}.
\end{equation}

\begin{equation}
+ O\left(t^{-\frac{1}{2}} e^{\frac{\pi}{4d} \theta'(t)^p}\right).
\end{equation}
Lemma 2.7. Let \( \theta \) be the function defined by (1.7). Then, for \( \nu \geq 2 \) and \( t > 0 \) we have
\[
|\theta^{(\nu)}(t)| \leq \frac{(\nu - 2)!}{2^{\nu-1}} + \frac{2\nu!}{\nu t^\nu}.
\]
Further, let \( t \) be large enough and assume that \( k \ll t^{\frac{1}{\nu}} \). Then
\[
\sum_{\nu=1}^{k} \frac{|\theta^{(\nu)}(t)| t^\nu}{\nu!} \leq t\theta'(t) + \frac{t}{2} \quad \text{and} \quad \sum_{p=0}^{k-2} |q_p(t)| \theta'(t)^p \ll \frac{k}{t} e^{\nu \theta'(t)} \theta'(t)^{k-1}.
\]
where \( q_p \) are the functions defined by (2.4).

3. Proofs

Proof of Lemma 2.2. By computing \( \varphi'_p \) and \( \varphi''_p \) we see that
\[
\varphi'_p(x) \ll \max(p\varphi_{p-1}(x), x\varphi_p(x)) x^{-1}
\]
and
\[
\varphi''_p(x) \ll \max(p\varphi_{p-1}(x), \varphi_p(x) + p^2 \varphi_{p-2}(x)) x^{-2}
\]
for \( x \geq a \) and we complete the proof by noting that for \( p \geq d \) and \( x \in [e^d, ce^{2d}] \) we have
\[
\varphi_p(x) \ll p \varphi_{p-1}(x) \ll p^2 \varphi_{p-2}(x)
\]
and that
\[
\varphi_{p-2}(x) \leq a^{-\frac{1}{2}}(\log(2a)^* - d)^{p-2} \leq 2^{\frac{p}{2}} \varphi_{p-2}((2a)^*)
\]
for \( x \in [a,(2a)^*] \).

Proof of Lemma 2.3. We have
\[
\sum_{N_0 < n \leq N_1} \phi_p(n) = \sum_{0 \leq r \leq l} \sum_{a_r < n \leq a_{r+1}} \phi_p(n)
\]
where \( l \) is an integer such that \( 2^l N_0 < N_1 \leq 2^{l+1} N_0 \), \( a_r = 2^r N_0 \) for \( r = 0, \ldots, l \) and \( a_{l+1} = N_1 \). We introduce the functions \( \varphi_p(x) = x^{-\frac{1}{2}}(\log x - d)^p \) and \( f(x) = -\frac{1}{2\pi} \log x \) so that \( \phi_p(n) = \varphi_p(n) e^{2\pi if(n)} \). Since \( e^d \leq N_0 < N_1 \leq ce^{2d} \) where \( d = \theta'(t) \) and thanks to Lemma 2.2, the assumptions of Lemma 2.1 are satisfied with \( a = a_r, b = a_{r+1}, H = H_r := p^2 \varphi_{p-2}(a_{r+1}), U = a_r, A = \frac{a^2}{2} \) and relation (2.5) reads
\[
\sum_{a_r < n \leq a_{r+1}} \phi_p(n) = e^{\frac{1}{2}} \sum_{f'(a_r) < n \leq f'(a_{r+1})} \frac{\varphi_p(x(n))}{\sqrt{f'''(x(n))}} e^{2\pi i(f(x(n))-nx(n))} + R(a_{r+1}) - R(a_r) + O(H_r)
\]
where \( x(n) = -\frac{1}{2\pi n} \). By definition \( \Theta(\lambda, \mu) = O(\lambda^{-\frac{1}{2}}) \) for \( \mu \in [0,1] \) and \( \Theta(\lambda, \mu) = O_\delta(1) \) for \( \mu \in [\delta,1-\delta] \) which imply that \( R(a_0) \) is a \( O(H_0) \) and \( R(a_{l+1}) \) is a \( O(H_l) \).
We sum the previous relations, noting that $|f'(a_{l+1})| = -1$ and setting $q = -f'(a_0)$, to get

$$\sum_{0 \leq r \leq l} \sum_{a_r < n \leq a_{r+1}} \phi_p(n) = \sum_{-q < n \leq -1} \frac{\varphi_p(x(n))}{\sqrt{f''(x(n))}} e^{2\pi i(f(x(n))-nz(n)+\frac{d}{2})} + O(\sum_{0 \leq r \leq l} H_r).$$  

Further, using (1.3) and since (3.2) we get

$$\sum_{1 \leq n < q} \frac{\varphi_p(x(-n))}{\sqrt{f''(x(-n))}} e^{2\pi i(f(x(-n))+nz(-n)+\frac{d}{2})} + O(\sum_{0 \leq r \leq l} H_r).$$

Further, using (3.3) and $d = \theta'(t)$, we have for $1 \leq n \leq q$

$$\varphi_p(x(-n)) = (\log \frac{x}{2\pi m})^p = \left(\frac{\theta(t) - \log n}{n^2}\right) + O(p t^{-2} \theta'(t)^{p-1})$$

and making use of (1.2) we get

$$e^{2\pi i(f(x(-n))+nz(-n)+\frac{d}{2})} = e^{-2i(\frac{d}{2} \log \frac{x}{n} - \frac{d}{2} - \frac{d}{2}) + it \log n} = \frac{e^{-2t\theta'(t)}}{n^{-it}} + O(t^{-1}).$$

Further, for $m \geq d - 2$, the function $\varphi_m$ is increasing on $[N_0, N_1]$ and we have

$$\sum_{r=0}^{l} \varphi_m(a_{r+1}) \leq \sum_{r=1}^{l-1} \varphi_m(2^r N_0) + 2\varphi_m(N_1) \leq \int_{l}^{0} \varphi_m(2^u N_0) du + 2\varphi_m(N_1)$$

and since

$$d \leq \log N_0 \leq \log(e^d + 1) = d + \log(1 + e^{-d}) \leq d + e^{-d}$$

and $e^{-d} = O(t^{-\frac{1}{2}})$, we get for $u \geq 1$ and $m \ll t^{\frac{1}{2}}$

$$(u \log 2 + \log N_0 - d)^m \leq (u \log 2)^m \left(1 + \frac{e^{-d}}{u \log 2}\right)^m \leq (u \log 2)^m e^{\frac{e^{-d}}{u \log 2}} \ll (u \log 2)^m$$

and therefore

$$\int_{l}^{0} \varphi_m(2^u N_0) du \ll e^{-\frac{d}{2}} \int_{0}^{l} (u \log 2)^m du \ll e^{-\frac{d}{2}} 2^m \int_{0}^{\frac{\log 2}{2}} e^{-y} y^m dy.$$  

Moreover $2^l N_0 \leq N_1$ and thus $2^l \leq \frac{N_1}{N_0} \leq ce^d$ and $l \log 2 \leq d + \log c$ and this implies that

$$\int_{0}^{\frac{\log 2}{2}} e^{-y} y^m dy \leq \int_{0}^{\frac{\log 2}{2} + \frac{\log c}{2}} e^{-y} y^m dy = \gamma(m + 1, \frac{d}{2} + \frac{\log c}{2})$$

where $\gamma(m, x) = \int_{0}^{x} e^{-t} t^{m-1} dt$ is the lower incomplete gamma function. Setting $x = \frac{d}{2} + \frac{\log c}{2}$ and using integration by parts one checks that

$$\gamma(m + 1, x) = m! e^{-x} \sum_{n=m+1}^{\infty} \frac{x^n}{n!} = e^{-x} \sum_{n=m+1}^{\infty} \frac{x^n}{m+1} \sum_{k=0}^{\infty} \frac{x^k}{(m+2)_k} \ll e^{-x} \frac{2^{-m} e^{\frac{p}{2} d} d^{m+1}}{m}.$$  

Using relations (3.5), (3.6), (3.7), (3.8) with $m = p - 2$ we get

$$\sum_{0 \leq r \leq l} H_r \ll e^{-d} p c \frac{p}{d} d^{p-1} + p^2 \varphi_{p-2}(N_1) \ll e^{-d} p^2 c \frac{p}{d} d^{p-2}.$$
where $x$ since $(\log x)^p$.

By the general Leibniz rule we have

\[ (2.7) \quad \text{by noting that} \]

\[ \phi \]

\[ \text{since (3.2), (3.3), (3.4), (3.9) and we observe that the sum over } n \text{ such that } 1 \leq n < q \text{ can be replaced by the sum over } n \text{ such that } 1 \leq n \leq N_0 \text{ without changing the order of the error term.} \quad \square \]

**Proof of Lemma 2.4** One can check by induction that the derivatives of $g_p$ are given by

\[ g_p^{(k)}(x) = x^{-k} \sum_{l=0}^{k} c_{k,l} (p-l+1)! (\log x - d)^{p-l} \]

where the $c_{k,l}$ are integers defined recursively by

\[ \begin{cases} c_{0,0} = 1, & c_{k,0} = c_{0,l} = 0 \quad \text{for } k,l \geq 1 \\ c_{k+1,l} = c_{k,l-1} - kc_{k,l} \quad \text{for } k \geq 0, l \geq 1. \end{cases} \]

This shows that $c_{k,l} = S_k^l$ where the $S_k^l$ are the Stirling numbers of first kind. Hence

\[ |g_p^{(k)}(x)| \leq x^{-k} (\log x - d)^p \sum_{l=0}^{k} |S_k^l| (p-l+1)! (\log x - d)^{-l} \]

\[ \leq x^{-k} (\log x - d)^p \sum_{l=0}^{k} |S_k^l| \left( \frac{p}{d} \right)^l \]

since $(\log x - d)^{-l} \leq d^{-l}$ for $x \geq ce^{2d}$. Setting $y = \frac{x}{t} \geq 1$, we complete the proof of (2.7) by noting that

\[ \sum_{l=0}^{k} |S_k^l| y^l = (y)_k = (1 + \frac{1}{y})(1 + \frac{2}{y}) \ldots (1 + \frac{k-1}{y}) y^k \leq k! y^k. \]

By the general Leibniz rule we have

\[ \phi_p^{(k)}(x) = (x^{-s})^{(k)} g_p(x) + \sum_{l=1}^{k} \binom{k}{l} (x^{-s})^{(k-l)} g_p^{(l)}(x) \]

\[ = (-1)^k (s)_k x^{-s-k} g_p(x) (1 + R) \]

where

\[ |R| \leq \sum_{l=1}^{k} \frac{k!}{l! (s+k-l+1)!} \left( \frac{p}{d} \right)^l \leq \sum_{l=1}^{k} \left( \frac{kp}{ld} \right)^l \approx d^{-1}. \]

Since $N_2 = c e^{2d} (1 + O(t^{-1})) = c \frac{1}{2\pi} (1 + O(t^{-1}))$ and $k \ll t^{\frac{1}{2}}$ we deduce that

\[ \phi_p^{(k)}(N_2) \ll t^k \left( \frac{1}{2t} + i \right) \left( \frac{3}{2t} + i \right) \ldots \left( \frac{2k-1}{2t} + i \right) \left( e^{\frac{t}{2\pi}} \right)^{-\frac{s-k}{2}} \ll c^{\frac{k}{2}} d^p \]

\[ \ll \left( \frac{2\pi}{c} \right)^k t^{\frac{s-k}{2}} d^p \]
and similarly
\[ \phi_p^{(k)}(N) \ll \left(\frac{t}{N}\right)^k N^{-\frac{1}{2}} \log^p N \ll \left(\frac{2\pi}{c}\right)^k N^{-\frac{1}{2}} \log^p N. \]

Finally
\[
\int_{N_2}^{N} |\phi_p^{(2K)}(u)| \, du \ll |(s)_{2K}| \int_{N_2}^{N} u^{-\frac{1}{2}-2K} (\log u - d)^p \, du \\
= |(s)_{2K}| \frac{u^{-\frac{1}{2}-2K}}{2-2K} (\log u - d)^p \sum_{l=0}^{p} \left(\frac{p}{(2K - \frac{1}{2})(\log u - d)}\right)^l \left(\frac{p!}{p^l (p-l)!}\right) \bigg|_{N_2}^{N} \\
\ll \left(\frac{2\pi}{c}\right)^{2K} c^{\frac{p}{2}} d^p + \left(\frac{t}{N}\right)^{2K-\frac{1}{2}} \log^p N.
\]

**Proof of Lemma 2.3** We set \( N_2 = N_1 + 1 \) and we use the Euler-MacLaurin formula with \( K \simeq t^{\frac{1}{2}} \) to get
\[
\sum_{N_1 < n \leq N} \phi_p(n) = \int_{N_2}^{N} \phi_p(u) \, du + \frac{1}{2} (\phi_p(N_2) + \phi_p(N)) \\
+ \sum_{l=1}^{K} \frac{B_{2l}}{2l!} (\phi_p^{(2l-1)}(N) - \phi_p^{(2l-1)}(N_2)) + R_{2K}
\]
where
\[ |R_{2K}| \leq \frac{2\zeta(2K)}{(2\pi)^{2K}} \int_{N_2}^{N} \left|\phi_p^{(2K)}(u)\right| \, du. \]

We have
\[
\int_{N_2}^{N} \phi_p(u) \, du = -\frac{u^{1-s}}{s-1} (\log u - d)^p \left(\frac{(s-1)(\log u - d)^{-t}}{(p-l)!}\right) \bigg|_{N_2}^{N} \\
\text{and we observe that}
\]
\[
\frac{N_2^{1-s}}{s-1} (\log N_2 - d)^p \left(\frac{(s-1)(\log N_2 - d)^{-t}}{(p-l)!}\right) \\
= \frac{N_2^{1-s}}{s-1} (\log N_2 - d)^p \sum_{l=0}^{p} \left(\frac{p}{(s-1)(\log N_2 - d)}\right)^l \left(\frac{p!}{p^l (p-l)!}\right) \\
= \frac{N_2^{1-s}}{s-1} (\log N_2 - d)^p \left(1 + O\left(t^{-\frac{1}{4}}\right)\right) \ll t^{-\frac{1}{4}} c^{\frac{p}{2}} d^p.
\]

Further, \( \phi_p(N_2) \ll t^{-\frac{1}{4}} c^{\frac{p}{2}} d^p, \phi_p(N) \ll N^{-\frac{1}{4}} \log^p N \) and since \( B_{2l} = (-1)^{l-1} \frac{2(2l)!}{(2\pi)^{2l}} \zeta(2l) \) we have thanks to (2.8) and (2.9)
\[
\sum_{l=1}^{K} \frac{B_{2l}}{2l!} (\phi_p^{(2l-1)}(N) - \phi_p^{(2l-1)}(N_2)) \ll \sum_{l=1}^{K} \left(\frac{1}{c^{2l-1}} \left(t^{-\frac{1}{4}} c^{\frac{p}{2}} d^p + N^{-\frac{1}{4}} \log^p N\right)\right) \\
\ll t^{-\frac{1}{4}} c^{\frac{p}{2}} d^p + N^{-\frac{1}{4}} \log^p N.
\]
since $c > 1$ is a fixed constant. Finally, since $K = t^{\frac{3}{4}}$, we have $c^{-2K} \ll t^{-\frac{1}{2}}$ and we deduce from (2.30) that $R_{2K} \ll t^{-\frac{1}{2}} c^{\frac{1}{2}} d^p + N^{-\frac{1}{2}} \log^p N$. □

PROOF OF LEMMA 2.6. We use the relations (2.28) and (2.30), (2.4) and (2.5) with $c$ replaced by $c^\frac{1}{2}$, and we let $N$ tend to infinity to get

$$
\bar{\eta}_p \left( \theta'(t), \frac{1}{2} + it \right) = \sum_{1 \leq n \leq N} \frac{(\log n - \theta'(t))^p}{n^{\frac{1}{2} + it}} + e^{-2t\theta(t)} \sum_{1 \leq n \leq N_0} \frac{(\theta'(t) - \log n)^p}{n^{\frac{1}{2} - it}} + O \left( t^{-\frac{1}{2}} p^2 c^\frac{1}{2} \theta'(t)^{-p - 2} \right).
$$

We note that $N_0$ can be replaced by $\sqrt{2\pi}$ without changing the order of the error term and to complete the proof, we use the relation $\eta(d, s) = (-1)\bar{\eta}_p(d, s)$ and the inequality $x^2 e^{-x} \leq C_1 e^{-x}$ which holds for $x \geq 0$ and $C_1 = 16 e^{-2(\log c)}$ to check that $p^2 c^\frac{1}{2} \theta'(t)^{p - 2} \leq C_1 c^\frac{1}{2} \theta'(t)^p$. □

PROOF OF LEMMA 2.7. The proof of this lemma is almost exactly the same as that of Lemma 7 of [1]. The only modification is at the end of the proof and it reads

$$
\sum_{p = 0}^{k - 2} |q_p(t)| \theta'(t)^p \leq \frac{k!}{t^k} \sum_{m = 1}^{k - 1} \frac{1}{m!} \left( \sum_{\nu = 1}^{k} \frac{\theta^{(\nu)}(t) t^\nu}{\nu!} \right)^m \leq \frac{k!}{t^k} \sum_{m = 1}^{k - 1} \frac{1}{m!} \left( t\theta'(t) + \frac{t}{2} \right)^m \leq \frac{k!}{t^k (k - 1)!} (t\theta'(t) + \frac{t}{2})^{k - 1} \left( \sum_{l = 0}^{k - 2} \left( \frac{k}{l} \theta'(t) - \frac{k}{l + 1} \right)^l \right)^{k - 1} \leq \frac{k}{t} e^{-c\sqrt{\pi} \theta'(t)} \theta'(t)^{k - 1}.
$$

□

PROOF OF THEOREM 1.1 Thanks to (2.12), the first term of the right hand side of (2.21) reads

$$
e^{i\theta(t) - k \log n} \sum_{1 \leq n \leq \sqrt{N}} \left( \theta'(t) - \log n \right)^k = e^{i\theta(t) - k \log n + k \frac{\pi}{2}} + e^{-i\theta(t) - k \log n + k \frac{\pi}{2}} + O \left( t^{-\frac{1}{2}} e^{\sqrt{\pi} \theta'(t)} \theta'(t)^k \right) = 2 \sum_{1 \leq n \leq \sqrt{N}} \frac{1}{\sqrt{n}} (\theta'(t) - \log n)^k \cos(\theta(t) - t \log n + k \frac{\pi}{2}) + O \left( t^{-\frac{1}{2}} e^{\sqrt{\pi} \theta'(t)} \theta'(t)^k \right).
$$

For $\theta'(t) \leq p \leq 3\theta'(t)^2$, a trivial estimate of the right hand side of (2.12), with the choice $c = e^\frac{1}{2}$, leads to

$$
\eta_p(\theta'(t), \frac{1}{2} + it) = O \left( t^{\frac{1}{2}} \theta'(t)^p \right) + O \left( t^{-\frac{1}{2}} e^{\sqrt{\pi} \theta'(t)} \theta'(t)^p \right) = O \left( t^{\frac{1}{2}} \theta'(t)^p \right).
$$
since $t^{-\frac{k}{2}}e^{2\theta'(t)} \leq t^{-\frac{k}{2}}e^{2\theta'(t)} \leq (\frac{t}{2})^k$. In [1], we proved that the same bound holds for $0 \leq p \leq \theta'(t)$ and Lemma 2.7 implies that the second term of the right hand side of (2.1) satisfies

$$e^{i\theta(t)} \sum_{p=0}^{k-2} q_p(t) t^p \theta'(t), \frac{1}{2} + it = O \left( t^{-\frac{k}{2}} e^{2\theta'(t)} k \theta'(t)^{k-1} \right).$$

□

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