RESONANCE VARIETIES VIA
BLOWUPS OF \( \mathbb{P}^2 \) AND SCROLLS

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Abstract. Conjectures of Suciu [36] relate the fundamental group of an ar-
rangement complement \( M = \mathbb{C}^n \setminus \mathcal{A} \) to the first resonance variety of \( H^*(M, \mathbb{Z}) \).
We describe a connection between the first resonance variety and the Orlik-
Terao algebra \( C(\mathcal{A}) \) of the arrangement. In particular, we show that non-local
components of \( R^1(\mathcal{A}) \) give rise to determinantal syzygies of \( C(\mathcal{A}) \). As a result,
\( \text{Proj}(C(\mathcal{A})) \) lies on a scroll, placing geometric constraints on \( R^1(\mathcal{A}) \). The key
observation is that \( C(\mathcal{A}) \) is the homogeneous coordinate ring associated to a
nef but not ample divisor on the blowup of \( \mathbb{P}^2 \) at the singular points of \( \mathcal{A} \).

1. Introduction

The fundamental group of the complement \( M \) of an arrangement of hyperplanes
\( \mathcal{A} = \bigcup_{i=1}^{d} H_i \subseteq \mathbb{C}^n \) is a much studied object. The Lefschetz-type theorem of
Hamm-Le [13] implies that taking a generic two dimensional slice of \( M \) yields an
isomorphism at the level of fundamental groups, so to study \( \pi_1(M) \) we may assume
\( \mathcal{A} \subseteq \mathbb{P}^2 \). Even with this simplifying assumption the situation is nontrivial: in [17]
Hirzebruch writes “The topology of the complement of an arrangement of lines in
\( \mathbb{P}^2 \) is very interesting, the investigation of the fundamental group very difficult”.

Presentations for \( \pi_1(M) \) are given by Randell [25], Salvetti [29], Arvola [2],
and Cohen-Suciu [3]. Perhaps the most compact of these is the braid monodromy
presentation of [3], but even this is quite complicated. Somewhat coarser invariants
of \( \pi_1(M) \) are the LCS ranks and Chen ranks. For a finitely generated group
\( G \), let \( G = G_1 \) and define a sequence of normal subgroups inductively by \( G_k = [G_{k-1}, G] \).
This yields an associated Lie algebra
\[
gr(G) \otimes \mathbb{Q} := \bigoplus_{k=1}^{\infty} G_k/G_{k+1} \otimes \mathbb{Q},
\]
with Lie bracket induced by the commutator. The \( k \)-th LCS rank \( \phi_k = \phi_k(G) \) is
the rank of the \( k \)-th quotient. The Chen ranks of a group are the LCS ranks of the
maximal metabelian quotient \( G/[[G, G], [G, G]] \). Work of Papadima and Suciu
[26] shows that the Chen ranks of \( \pi_1(M) \) are combinatorially determined; but save
for some special classes of arrangements, there are no explicit formulas for either
the Chen or LCS ranks. However, there are a beautiful pair of conjectures due
to Suciu [30], giving formulas for the LCS and Chen ranks in terms of the first
resonance variety \( R^1(\mathcal{A}) \). The variety \( R^1(\mathcal{A}) \) is the tangent cone at the origin to the
characteristic variety; the study of \( R^1(\mathcal{A}) \) was pioneered by Falk in [10].

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In the next section, we review the main subjects of investigation: the Orlik-Solomon algebra $A = H^\ast(M, \mathbb{Z})$, the Orlik-Terao algebra $C(A)$, the first resonance variety $R^1(A)$, and blowups of $\mathbb{P}^2$ using certain divisors. Our main result is a description of $C(A)$ as the homogeneous coordinate ring of the blowup $X$ of $\mathbb{P}^2$ at the singular points of $A$, via a specific (nef but not ample) divisor $D_A$. This allows us to give a geometric interpretation of $R^1(A)$ in terms of certain determinantal syzygies; we prove that if $A$ supports a net, then $\text{Proj}(C(A))$ lies on a scroll.

2. Background

In [23], Orlik and Solomon gave a presentation for the cohomology ring of the complement $M$ of a set of hyperplanes $A \subseteq \mathbb{C}^n$. A consequence of their work is that the Betti numbers of $M$ are determined by the intersection lattice $L(A)$. This lattice is ranked by codimension: $x \in L_i(A)$ corresponds to a linear space of codimension $i$ which is an intersection of hyperplanes of $A$. The lattice element $\hat{0}$ corresponds to $\mathbb{C}^n$, and $y \prec x \leftrightarrow x \subseteq y$. We work with $A$ central, so $A$ defines an arrangement in both $\mathbb{C}^n$ and $\mathbb{P}^{n-1}$. We will depict $A$ projectively, as below:

Example 2.1. The reflecting hyperplanes of the Weyl group of $\text{SL}(4)$ are the six hyperplanes in $\mathbb{C}^4$ defined by $V(x_i - x_j), 1 \leq i < j \leq 4$. Projecting along the common subspace $(t, t, t, t)$ yields the braid arrangement of six planes containing the origin in $\mathbb{C}^3$, or six lines in $\mathbb{P}^2$:

![Figure 1. The braid arrangement $A_3$ and its intersection lattice in $\mathbb{C}^3$](image)

**Definition 2.2.** The Möbius function $\mu : L(A) \to \mathbb{Z}$ is defined by

\[
\mu(\hat{0}) = 1 \\
\mu(t) = -\sum_{s \prec t} \mu(s), \text{ if } \hat{0} \prec t
\]

As noted, the Poincaré polynomial of $M$ is determined by $L(A)$:

\[P(M, t) = \sum_{x \in L(A)} \mu(x) \cdot (-t)^{\text{rank}(x)}.\]

In Example 2.2, $P(M, t) = 1 + 6t + 11t^2 + 6t^3$. For a central arrangement in $\mathbb{C}^n$, $M \simeq \mathbb{C}^* \times (\mathbb{P}^{n-1} \setminus A)$, so by Künneth $P(M, t) = (1 + t)P(\mathbb{P}^{n-1} \setminus A, t)$. For $n = 3$, $b_2(M) = \sum_{p \in L_2(A)} \mu(p)$, where $\mu(p)$ is one less than the number of lines through $p$. 

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2.1. Orlik-Solomon algebra and $R^1(A)$. The Orlik and Solomon presentation for the cohomology ring of $M = \mathbb{C}^n \setminus A$ is as follows:

**Definition 2.3.** $A = H^*(M, \mathbb{Z})$ is the quotient of the exterior algebra $E = \wedge(\mathbb{Z}^d)$ on generators $e_1, \ldots, e_d$ in degree 1 by the ideal generated by all elements of the form $\partial e_{i_1,\ldots,i_r} := \sum_{q} (-1)^{q-1} e_{i_1} \cdots \hat{e}_q \cdots e_{i_r}$, for which $\text{codim} H_{i_1} \cap \cdots \cap H_{i_r} < r$.

Since $A$ is a quotient of an exterior algebra, multiplication by an element $a \in A^1$ gives a degree one differential on $A$, yielding a cochain complex $(A, a)$:

$$(A, a) : \quad 0 \longrightarrow A^0 \overset{a}{\longrightarrow} A^1 \overset{a}{\longrightarrow} A^2 \overset{a}{\longrightarrow} \cdots \overset{a}{\longrightarrow} A^{r} \longrightarrow 0 .$$

The complex $(A, a)$ is exact as long as $\sum_{i=1}^{n} a_i \neq 0$; the first resonance variety $R^1(A)$ consists of points $a = \sum_{i=1}^{n} a_i e_i \leftrightarrow (a_1 : \cdots : a_n)$ in $P(A^1) \cong P^{d-1}$ for which $H^1(A, a) \neq 0$. Falk initiated the study of $R^1(A)$ in [10]; among his main innovations was the concept of a neighborly partition: a partition $\Pi$ of $A$ is neighborly if, for any rank two flat $Y \in L_2(A)$ and any block $\pi$ of $\Pi$,

$$\mu(Y) \leq |Y \cap \pi| \Longrightarrow Y \subseteq \pi,$$

Falk showed that all components of $R^3(A)$ arise from such partitions, and conjectured that $R^1(A)$ was a subspace arrangement. This was proved, essentially simultaneously, by Cohen–Suciu [3] and Libgober–Yuzvinsky [21]; we will return to this in §4.

2.2. The Orlik-Terao algebra. In [25], Orlik and Terao introduced a commutative analog of the Orlik-Solomon algebra in order to answer a question of Aomoto.

**Definition 2.4.** Let $A = \cup_{i=1}^{d} V(\alpha_i) \subseteq \mathbb{P}^n$, and put $R = \mathbb{C}[y_1, \ldots, y_d]$. For each linear dependency $\Lambda = \sum_{j=1}^{k} c_i \alpha_{i_j} = 0$, define $f_\Lambda = \sum_{j=1}^{k} c_i (y_{i_1} \cdots y_{i_k})$, and let $I$ be the ideal generated by the $f_\Lambda$. The Orlik-Terao algebra $C(A)$ is the quotient of $\mathbb{C}[y_1, \ldots, y_d]$ by $I$, and the Artinian Orlik-Terao algebra (the main object studied in [25]) is $C(A)/(y_1^2, \ldots, y_d^2)$.

**Example 2.5.** Suppose $A \subseteq \mathbb{P}^2$ is defined by the vanishing of $\alpha_1 = x_1, \alpha_2 = x_2, \alpha_3 = x_3, \alpha_4 = x_1 + x_2 + x_3$. The only relation is $\alpha_1 + \alpha_2 + \alpha_3 - \alpha_4 = 0$, so

$$C(A) = \mathbb{C}[y_1, \ldots, y_4]/(y_2y_3y_4 + y_1y_3y_4 + y_1y_2y_4 - y_1y_2y_3).$$

The homogeneous polynomial $y_2y_3y_4 + y_1y_3y_4 + y_1y_2y_4 - y_1y_2y_3$ is irreducible, hence defines a cubic surface in $\mathbb{P}^3$, and a computation shows that the surface has four singular points. A classical result in algebraic geometry is that the linear system of four cubics through six general points in $\mathbb{P}^2$ defines a map from the blowup of $\mathbb{P}^2$ at those points to $\mathbb{P}^3$ whose image is a smooth cubic surface. As the points move into special position the surface acquires singularities, as in this example.

In [32], properties of the Orlik-Terao algebra were studied in relation to 2–formality. An arrangement is 2–formal if any dependency among the linear forms defining the the arrangement can be obtained as a linear combination of dependencies which involve only three of the forms. Among the classes of 2–formal arrangements are $K(\pi, 1)$ arrangements and free arrangements. However, an example of Yuzvinsky [31] shows that 2–formality is not determined by the intersection lattice $L(A)$. The main result of [32] is that 2–formality is determined by the quadratic component of the Orlik-Terao ideal; the key is a computation on the tangent space of $V(I_2) \cap (\mathbb{C}^*)^{d-1}$. 


2.3. **Blowups of $\mathbb{P}^2$.** Fix points $p_1, \ldots, p_n \in \mathbb{P}^2$, and let

\[ X \xrightarrow{\pi} \mathbb{P}^2 \]

be the blow up of $\mathbb{P}^2$ at these points. Then $Pic(X)$ is generated by the exceptional curves $E_i$ over the points $p_i$, and the proper transform $E_i$ of a line in $\mathbb{P}^2$. A classical geometric problem asks for a relationship between numerical properties of a divisor $D_m = mE_0 - \sum a_i E_i$ on $X$, and the geometry of

\[ X \xrightarrow{\phi} \mathbb{P}(H^0(D_m)^\vee). \]

First, some basics. Let $m$ and $a_i$ be non-negative, let $I_{p_i}$ denote the ideal of a point $p_i$, and define

\[ J = \bigcap_{i=1}^n I_{p_i}^{a_i} \subseteq \mathbb{C}[x, y, z] = S. \]

Then $H^0(D_m)$ is isomorphic to the $m^{th}$ graded piece $J_m$ of $J$ (see [13]). In [5], Davis and Geramita show that if $\gamma(J)$ denotes the smallest degree $t$ such that $J_t$ defines $J$ scheme theoretically, then $D_m$ is very ample if $m > \gamma(J)$, and if $m = \gamma(J)$, then $D_m$ is very ample iff $J$ does not contain $m$ collinear points, counted with multiplicity. Note that $\gamma(J) \leq reg(J)$. Now suppose that $A = \bigcup_{i=1}^d L_i \subseteq \mathbb{P}^2$, and fix defining linear forms $\alpha_i$ so that $L_i = V(\alpha_i)$. Let $X$ denote the blowup of $\mathbb{P}^2$ at $Sing(A) = L_2(A)$. The central object of our investigations is the divisor

\[ D_A = (d - 1)E_0 - \sum_{p_i \in L_2(A)} \mu(p_i)E_i. \]

2.4. **Main results.** For an arrangement $A \subseteq \mathbb{P}^2$, let

\[ X \xrightarrow{\phi_A} \mathbb{P}(H^0(D_A)^\vee). \]

We show that $C(A)$ is the homogeneous coordinate ring of $\phi_A(X)$, and that $\phi_A$ is an isomorphism on $\pi^*(\mathbb{P}^2 \setminus A)$, contracts the lines of $A$ to points, and blows up the singularities of $A$. Combining results of Proudfoot-Speyer [27] and Terao [38], we bound the Castelnuovo-Mumford regularity of $C(A)$. Finally, we interpret the resonance varieties studied in [8], [10], [12], [21], [32], [32] in terms of linear subsystems of $D_A$, and connect these jump loci to linear syzygies on $C(A)$.

3. **Connecting $H^0(D_A)$ to the Orlik-Terao algebra**

Let $\alpha = \prod_{i=1}^d \alpha_i$ and define a map $R = \mathbb{C}[y_1, \ldots, y_d] \to \mathbb{C}[1/\alpha_1, \ldots, 1/\alpha_d] = T$. The kernel of this map is the OT ideal (see [34]), so $C(A) \simeq T$. In [38], Terao proved that the Hilbert series for $T$ is given by

\[ HS(T, t) = P\left(A, \frac{t}{1 - t}\right). \]

In this section, we show that for $n = 2$, $C(A)$ is the homogeneous coordinate ring of the image of $X \xrightarrow{\phi_A} \mathbb{P}(H^0(D_A)^\vee)$, with $X$ as in Equation [11]. For brevity, let $l_i = \alpha_i / \alpha_i$.

**Lemma 3.1.** The ideal $L = (l_1, \ldots, l_d)$ defines

\[ \bigcap_{p_i \in L_2(A)} I_{p_i}^{(p_i)} \text{ scheme-theoretically.} \]
Proof. Localize at $I_p$, where $p \in L_2(A)$. Then in $S_{I_p}$, $\alpha_i$ is a unit if $p \notin V(\alpha_i)$. Without loss of generality, suppose forms $\alpha_1, \ldots, \alpha_m$ vanish on $p$, and the remaining forms do not. Thus,

$$L_{I_p} = \langle \alpha_2 \cdot \alpha_m, \alpha_1 \cdot \alpha_3 \cdot \alpha_m, \ldots, \alpha_1 \cdot \alpha_m \rangle.$$ 

Now note that $I_p^{\mu(p)}$ has $\mu(p) + 1$ generators of degree $\mu(p)$. Since $\mu(p) = m - 1$ and the forms in $L_{I_p}$ are linearly independent, equality follows.

Lemma 3.2. The minimal free resolution of $S/L$ is

$$0 \rightarrow S(-d)^{d-1} \xrightarrow{\psi} S(-d + 1)^d \xrightarrow{[l_1, \ldots, l_d]} S \rightarrow S/L \rightarrow 0,$$

where

$$\psi = \begin{bmatrix}
\alpha_1 & 0 & \cdots & 0 \\
-\alpha_2 & \alpha_2 & 0 & \cdots \\
0 & -\alpha_3 & \cdots & \cdots \\
\vdots & \ddots & \ddots & \ddots \\
0 & \cdots & \cdots & 0 & \alpha_{d-1} \\
0 & \cdots & 0 & -\alpha_d
\end{bmatrix}.$$ 

Proof. The columns of $\psi$ are syzygies on $L$. Since the maximal minors of $\psi$ generate $L$, the result follows from the Hilbert-Burch theorem and Lemma 3.1.

Theorem 3.3. $H^0(D_A) \simeq \text{Span}_C \{l_1, \ldots, l_d\}$ and $H^1(D_A) = 0 = H^2(D_A)$.

Proof. The remark following Equation 2 shows that $H^0(D_A) \simeq J_{d-1}$. Since $K = -3E_0 + \sum E_i$, by Serre duality

$$H^2(D_A) \simeq H^0((-d - 2)E_0 + \sum_{p \in L_2(A)} (\mu(p) + 1)E_i),$$ 

which is clearly zero. Using that $X$ is rational, it follows from Riemann-Roch that

$$h^0(D_A) - h^1(D_A) = \frac{D_A^2 - D_A \cdot K}{2} + 1.$$ 

The intersection pairing on $X$ is given by $E_i^2 = 1$ if $i = 0$, and $-1$ if $i \neq 0$, and $E_i \cdot E_j = 0$ if $i \neq j$. Thus,

$$D_A^2 = (d - 1)^2 - \sum_{p \in L_2(A)} \mu(p)^2$$

(6)

$$-D_A K = 3(d - 1) - \sum_{p \in L_2(A)} \mu(p),$$

yielding

$$h^0(D_A) - h^1(D_A) = \left(\frac{d - 1}{2}\right)^2 - \sum_{p \in L_2(A)} \mu(p)^2 + 3\left(\frac{d - 1}{2}\right) - \frac{\sum \mu(p)}{2} + 1$$

(7)

$$= \left(\frac{d + 1}{2}\right) - \frac{\sum \left(\mu(p) + 1\right)}{2}.$$
Double counting the edges between $L_1(A)$ and $L_2(A)$ yields
\[
\binom{d}{2} = \sum_{p \in L_2(A)} \binom{\mu(p) + 1}{2},
\]

hence $h^0(D_A) - h^1(D_A) = d$. From Lemmas 3.1 and 3.2 the Hilbert function satisfies
\[
d = HF((l_1, \ldots, l_d), d - 1) = HF(\bigcap_{p \in L_2(A)} I_{\mu(p)}, d - 1).
\]
The observation after Equation 2 now implies that $h_0(D_A) = d$. □

It follows from Theorem 3.3 that $C(A)$ is the coordinate ring of $\phi_A(X)$. Note also that by Lemmas 3.1 and 3.2, the constant $\gamma(L) = d - 1$, so $dE_0 - \sum_{p \in L_2(A)} \mu(p_i)E_i$

is very ample, and gives a De Concini-Procesi wonderful model [6]: a compactification $\bar{M}$ of $M$ such that $\bar{M} \setminus M$ is a normal crossing divisor. However, since every line of $A$ contains exactly $d - 1$ points counted with multiplicity, the divisor $D_A$ is not very ample. The description of $C(A)$ makes it obvious that $V(I)$ is an irreducible, nondegenerate rational variety, and by [34] $V(I) \setminus V(y_1 \cdots y_d)$ is smooth. Here is a more explicit description of the map:

**Theorem 3.4.** The map $\phi_A$

1. is an isomorphism on $\pi^*(\mathbb{P}^2 \setminus A)$.
2. contracts the lines of $A$ to points on $X$.
3. takes $E_p$ to a rational normal curve of degree $\mu(p)$.

**Proof.** For the first part, without loss of generality suppose that $\alpha_1 \cdot \alpha_2 \cdot \alpha_3 = xyz$ and write $L = \prod_{i=4}^d \alpha_i$. Then

$$\phi_A = [yzL, xzL, xyL, \ldots].$$

Thus, the first three entries of $\phi_A$ define the Cremona transformation, which is an isomorphism from $\mathbb{P}^2 \setminus V(xyz)$ to itself. Since $\mathbb{P}^2 \setminus A$ is contained in $\mathbb{P}^2 \setminus V(xyz)$, (1) follows. For (2), suppose $p$ is a point of $V(\alpha_i)$. Since $\alpha_i$ divides $l_j$ for all $j \neq i$, this means $l_j(p) = 0$ if $j \neq i$. Hence $\phi_A(V(\alpha_i))$ is the $i$th coordinate point of $\mathbb{P}^{d-1}$. The final part follows from the fact that $D_A|_{E_p}$ is a divisor on $E_p$ of degree $D_A \cdot E_p = \mu(p)$, and $E_p \simeq \mathbb{P}^1$. □

### 3.1. Castelnuovo–Mumford regularity and graded betti numbers.

The Castelnuovo–Mumford regularity of a coherent sheaf $\mathcal{F}$ on $\mathbb{P}^n$ is usually phrased in terms of vanishing of certain cohomology modules. Letting $N = \oplus_n H^0(\mathcal{F}(n))$, we may rephrase the condition as

**Definition 3.5.** For a polynomial ring $R$, a finitely generated, graded $R$–module $N$ has Castelnuovo-Mumford regularity $j$ if $j$ is the smallest number such that Tor$_i^R(N, \mathbb{C})_{i+j+1} = 0$ for all $i$. The graded betti numbers of a graded $R$–module $N$ are indexed by

$$b_{ij} = \dim_{\mathbb{C}} \text{Tor}_i^R(N, \mathbb{C})_j.$$
Example 3.6. We revisit Example 2.1. The four triple points yield four quadratic generators for the Orlik-Terao ideal $I$. These four quadrics generate $I$ (see [34]), and a computation in Macaulay2 yields the graded betti numbers of $C(A)$:

|      | 1  | 4  | 5  | 2  |
|------|----|----|----|----|
| total| 0  | 1  |    |    |
|      | 1  | 4  | 2  |    |
|      | 2  |    | 3  | 2  |

This diagram is read as follows: the entry in position $(i,j)$ is simply $b_{i,j}$, e.g.

$$\dim_k \text{Tor}_2^R(C(A), \mathbb{C})_4 = 3.$$ 

The betti table has a very nice interpretation in terms of Castelnuovo-Mumford regularity: the regularity is the index of the last nonzero row.

Theorem 3.7. For $A \subseteq \mathbb{P}^n$, $C(A)$ is $n$-regular.

Proof. In [27], Proudfoot and Speyer show that the Orlik-Terao algebra is Cohen-Macaulay (for $n = 2$ this also follows from Theorem 3.3). Thus, there exists a regular sequence on $C(A)$ of dim$(V(I)) + 1 = n + 1$ linear forms; quotienting by this sequence yields an Artinian ring whose Hilbert series is the numerator of the Hilbert series of $C(A)$. The regularity of an Artinian module is equal to the length of the module, so the result follows from Equation 5. $\square$

It follows easily from Theorem 3.7 and Terao’s work in [38] that

Proposition 3.8. For $A \subseteq \mathbb{P}^n$ with $|A| = d$, if $I = I_2$, then

$$\dim_c \text{Tor}_2^R(C(A), \mathbb{C})_3 = 2 \left( \binom{d}{3} - 1 \right) - \left( d - 3 \right) \left( \sum_{p_i \in L_2(A)} \mu(p_i) + 1 \right).$$

Example 3.9. The $A_3$ arrangement is supersolvable, so by [34] $I = I_2$, and Proposition 3.8 shows there are two linear first syzygies on $I$. This explains the top row of the betti table in Example 3.6.

4. Nets, syzygies, and scrolls

In [21], Libgober-Yuzvinsky found a surprising connection between nets and the first resonance variety. The approach was further developed by Yuzvinsky in [42], with a beautiful complete picture emerging in Falk and Yuzvinsky’s paper on multinets [12]. In this section, we connect nets to the linear syzygies of $C(A)$, and hence to $R^1(A)$. This allows us to give an interpretation of the first resonance variety in terms of the geometry of $X_A$.

Suppose $Z$ is a subset of the intersection points of $A$, and let $J$ denote the $|Z| \times d$ incidence matrix of points and lines and $E$ denote a $d \times d$ matrix with every entry one. If $\hat{Z}$ is the blowup of $\mathbb{P}^2$ at the points of $Z$, then [21] shows that

$$J^t J - E = Q(\hat{Z})$$

is the intersection form on $\hat{Z}$, and is a generalized Cartan matrix. Using the Vinberg classification of such matrices [19], they show that any component of $R^1(A)$ corresponds to a choice of points $Z$ such that $Q(\hat{Z})$ consists of at least three affine blocks, with no finite or indefinite blocks, and the block sum decomposition of $Q(\hat{Z})$ yields a neighborly partition. Before going into the details of the connection between multinets, divisors and syzygies, we give a pair of motivating examples.
Example 4.1. The matroid \((9_3)_2\) of Hilbert and Cohn-Vossen is realized below by 
\(\mathcal{A} = V(xyz(x + y)(y + z)(x + 2y + z)(x + 2y + 3z)(2x + 3y + 3z)).\) It has nine triple points and nine double points, thus 
\(P(\mathcal{A}, t) = (1 + t)(1 + 8t + 19t^2).\)

![Figure 2. An arrangement realizing \((9_3)_2\)]

The graded betti numbers for \(C((9_3)_2)\) are:

| total | 1  | 11 | 75 | 156 | 145 | 66 | 12 |
|-------|----|----|----|-----|-----|----|----|
| 0     | 1  |    |    |     |     |    |    |
| 1     |    | 9  |    |     |     |    |    |
| 2     |    | 2  | 75 | 156 | 145 | 66 | 12 |

Example 4.2. The \((9_3)_1\) matroid of Hilbert and Cohn-Vossen is realized below by 
\(\mathcal{A} = V(xyz(x - y)(y - z)(x - y - z)(2x + y + z)(2x + y - z)(2x - 5y + z)).\) It has nine triple points and nine double points, so 
\(P((9_3)_1, t) = P((9_3)_2, t).\)

![Figure 3. An arrangement realizing \((9_3)_1\)]

However, the graded betti numbers for \(C((9_3)_1)\) are:

| total | 1  | 13 | 77 | 156 | 145 | 66 | 12 |
|-------|----|----|----|-----|-----|----|----|
| 0     | 1  |    |    |     |     |    |    |
| 1     |    | 9  | 2  |     |     |    |    |
| 2     |    | 4  | 75 | 156 | 145 | 66 | 12 |

The arrangement \((9_3)_1\) possesses a pair of linear first syzygies, while \((9_3)_2\) has no linear first syzygies. An easy check shows that \((9_3)_1\) admits a neighborly partition \([169][258][347]\) and has a corresponding non-local component (see below) in \(R^1(\mathcal{A})\), whereas \((9_3)_2\) does not. To better understand the connection between \(R^1(\mathcal{A})\) and syzygies, we now review two constructions.
4.1. Nets and multinets. It is easy to see that any $p \in L_2(A)$ with $\mu(p) \geq 2$ yields a component $\mathbb{P}_{\mu(p)}^{-1} \subset R^1(A)$. Such components are called local components. Components which are not of this type are called essential. In [42], Yuzvinsky used nets to analyze the essential components of $R^1(A)$.

**Definition 4.3.** Let $3 \leq k \in \mathbb{Z}$. A $k$-net in $\mathbb{P}^2$ is a pair $(A, Z)$ where $A$ is a finite set of distinct lines partitioned into $k$ subsets $A = \bigcup_{i=1}^{k} A_i$ and $Z$ is a finite set of points, such that:

1. for every $i \neq j$ and every $L \in A_i$, $L' \in A_j$, $L \cap L' \in Z$.
2. for every $p \in Z$ and every $i \in \{1, \ldots, k\}$, $\exists$ a unique $L \in A_i$ containing $Z$.

Thus, for a $k$-net, $|A_i| = |L \cap Z|$ for any block $A_i$ and line $L \in A$; denote this number by $m$. Following Yuzvinsky, we call $m$ the order of the net, and refer to a $k$-net of order $m$ as a $(k,m)$-net; note that $|Z| = m^2$. Yuzvinsky shows in [42] that a net must have $k \in \{3, 4, 5\}$, and improves this in [43] to $k \in \{3, 4\}$.

In [42], Falk and Yuzvinsky extend the notion of a net to a multinet; in a multinet lines may occur with multiplicity. Write $A_w$ for a multiarrangement, where $w \in \mathbb{N}^d$, and $w(L)$ denotes the multiplicity of a line.

**Definition 4.4.** A weak $(k, m)$-multinet on a multi-arrangement $A_w$ is a pair $(\Pi, Z)$ where $\Pi$ is a partition of $A_w$ into $k \geq 3$ classes $A_1, \ldots, A_k$, and $Z$ is a set of multiple points, such that

1. $\sum_{L \in A_i} w(L) = m$, independent of $i$.
2. For every $L \in A_i$ and $L' \in A_j$, with $i \neq j$, $L \cap L' \in Z$.
3. For each $p \in Z$, $\sum_{L \in A_i, p \in L} w(L)$ is a constant $n_p$, independent of $i$.

A multinet is a weak multinet satisfying the additional property

4. For $i \in \{1, \ldots, k\}$ and $L, L' \in A_i$, $\exists$ a sequence $L = L_0, L_1, \ldots, L_r = L'$ such that $L_{j-1} \cap L_j \notin Z$ for $1 \leq j \leq r$.

**Example 4.5.** The reflection arrangement of type $B_3$ is depicted below (there is also a line at infinity). Falk and Yuzvinsky show that this arrangement supports a multinet which is not a net: assign weight two to lines $(3, 6, 8)$ and weight one to the remaining lines.

![Figure 4. The B₃-arrangement](image)

The following lemma of [12] will be useful:

**Lemma 4.6.** Suppose $(A_w, Z)$ is a weak $(k, m)$-multinet. Then

1. $\sum_{L \in A_w} w(L) = km$.
2. $\sum_{p \in Z} n_p^2 = m^2$.
3. For each $L \in A_w$, $\sum_{p \in Z \cap L} n_p = m$. 
4.2. Determinantal syzygies and factoring divisors. One simple way in which linear syzygies can arise comes from a factorization of divisors. First, a definition

**Definition 4.7.** A matrix of linear forms is 1–generic if it has no zero entry, and cannot be transformed by row and column operations to have a zero entry.

For $Y \subseteq \mathbb{P}^n$ irreducible and linearly normal, if there exist line bundles $\mathcal{L}_1$ and $\mathcal{L}_2$ such that $\mathcal{O}_Y(1) = \mathcal{L}_1 \otimes \mathcal{L}_2$ with $h^0(\mathcal{L}_i) = a_i$, then the $a_1 \times a_2$ matrix $\gamma$ representing the multiplication table

$$H^0(\mathcal{L}_1) \otimes H^0(\mathcal{L}_2) \rightarrow H^0(\mathcal{O}_Y(1))$$

is 1–generic. More explicitly (see [8]), if

$$H^0(\mathcal{L}_1) = \text{Span}_\mathbb{C}\{e_1, \ldots, e_{a_1}\} \quad \text{and} \quad H^0(\mathcal{L}_2) = \text{Span}_\mathbb{C}\{f_1, \ldots, f_{a_2}\},$$

then $\gamma$ has $(i, j)$ entry $e_i \otimes f_j$, corresponding to a linear form on $\mathbb{P}^n$, and elements of the ideal $I_2(\gamma)$ of $2 \times 2$ minors of $\gamma$ vanish on $Y$. The most familiar example occurs when $a_1 = 2$ and $a_2 = k$. In this case, the minimal free resolution of $I_2(\gamma)$ is an Eagon-Northcott complex. This relates to geometry via scrolls: let $\Psi$ be the locus of points where a 1–generic matrix

$$\gamma = \begin{bmatrix} l_1 & \cdots & l_k \\ m_1 & \cdots & m_k \end{bmatrix}$$

has rank one. If

$$L_{[\lambda: \mu]} = \{ p \in \mathbb{P}^n \mid \lambda l_1(p) + \mu m_1(p) = \cdots = \lambda l_k(p) + \mu m_k(p) = 0 \},$$

then (see 9.10 of [16])

$$\Psi = \bigcup_{[\lambda: \mu] \in \mathbb{P}^1} L_{[\lambda: \mu]},$$

where $L_{[\lambda: \mu]} \cong \mathbb{P}^{n-k}$, so $\Psi$ is a union of linear spaces. Geometrically, the zero locus of the $2 \times 2$ minors of $\gamma$ is a scroll which contains $V(I_Y)$.

4.3. Connecting nets and determinantal syzygies. The computation in the proof of Theorem 3.3 and the fact that $h^1(D) \geq 0$ shows that if $D_A = A + B$ with $A = mE_0 - \sum a_iE_i$, then

$$h^0(A) \geq \binom{m+2}{2} - \sum_{p \in L_2(A)} \binom{a_i+1}{2}, \quad h^0(B) \geq \binom{d+1-m}{2} - \sum_{p \in L_2(A)} \binom{\mu(p)-a_i+1}{2}.$$

For an arrangement $A$, if there exists a choice of parameters $m$ and $a_i$ such that $h^0(A) = a \geq 2$ and $h^0(B) = b \geq 3$, then the results of the previous section show that there will exist linear first syzygies on $I$.

**Example 4.8.** We revisit Example 4.2. Let $A = 3E_0 - \sum_{\mu(p)=2} E_p$. Clearly $A^2 = AK = 0$, so we can only guarantee that $h^0(A) \geq 1$. In fact, $h^0(A) = 2$, hence $h^1(A) = 1$. To see this, note that a direct computation shows that the space of cubics passing through the nine multiple points of $A$ is two dimensional. Since

$$\text{Span}_\mathbb{C}\{L_1L_3L_7, L_2L_5L_8\} \subseteq H^0(A),$$

and any two of these are independent, we see that the sections are given by the net. Next, consider the residual divisor $B = D_A - A$. Since $B^2 = 16 - 18 = -2$ and $-BK = 15 - 9 = 6$, we have that $h^0(B) \geq 3$. In fact, equality holds, so $I$ contains the $2 \times 2$ minors of a $2 \times 3$ matrix of linear forms, explaining the linear syzygies.
Lemma 4.9. A $(k, m)$ multinet gives a divisor $A$ on $X$ such that $h^0(A) = 2$.

Proof. Let

$$A = mE_0 - \sum_{p \in Z} n_p E_p.$$ 

Condition (1) of Definition 4.4 implies that for each block $A_i$ of the multinet, $\prod_{L \in A_i} L^{w(L)}$ is homogeneous of degree $m$, and Condition (3) shows that it vanishes to order exactly $n_p$ on $E_p$. In particular, this shows that $\prod_{L \in A_i} L^{w(L)} \in H^0(A)$. If all the blocks of the partition were independent, then this would imply that $h^0(A) \geq k$, but it turns out that the sections are all fibers of a pencil of plane curves, which follows from Theorem 3.11 of [12].

In [12], Falk and Yuzvinsky show that the following are equivalent:

1. $R^1(A)$ contains a nonlocal component $\simeq \mathbb{P}^{k-2}$.
2. $A$ supports a $(k, m)$ multinet.
3. $\exists$ a pencil of plane curves with connected fibers, with at least three fibers (loci of) products of linear forms, and $A$ is the union of all such fibers.

In general, determining the dimension of $h^1(D)$ for $D \in \text{Pic}(X)$ is not easy. However, in the special case of a net, there is enough information to give a lower bound for the dimension of the sections of the residual divisor which is often exact.

Theorem 4.10. If $A$ is a $(k, m)$ net, then $D_A = A + B$, with

$$h^0(A) = 2 \text{ and } h^0(B) \geq km - \left(\frac{m+1}{2}\right).$$

Proof. For a $(k, m)$ net, all lines occur with multiplicity one. Let

$$A = mE_0 - \sum_{p \in Z} E_p.$$ 

By Lemma 4.9 $h^0(A) = 2$. Since $B = D_A - mE_0 + \sum_{p \in Z} E_p$, $B + \frac{B^2 - BK}{2} + 1$

$$= \frac{(d-m-1)(d-m-1+3)}{2} + \left(\sum_{p \in L_A} \mu(p)E_p + \sum_{p \in Z} E_p\right)\left(\sum_{p \in L_A} \mu(p) - 1\right)E_p + \sum_{p \in Z} E_p$$

$$= \left(\sum_{p \in Z} \mu(p)\right)$$

We now compute that

$$\sum_{p \in Z} \mu(p) + |Z| = \sum_{p \in Z} (\mu(p) + 1) = k \sum_{p \in Z} n_p = km^2.$$

The second line follows since $n_p$ lines from each block $A_i$ pass through $p$, and there are $k$ blocks. The third line follows from Lemma 4.9 and the fact that $n_p = 1$ for a net, hence $n_p^2 = n_p$. Since for a $(k, m)$-net $|Z| = m^2$,

$$\sum_{p \in Z} \mu(p) = (k-1)m^2 = dm - m^2.$$
Combining this with the previous calculation shows that for a \((k, m)\)-net
\[
h^0(B) = h^1(B) + \binom{m}{2} - d(m - 1) + dm - m^2 \geq d - \binom{m + 1}{2}.
\]
Recalling that \(km = d\) concludes the proof. \(\Box\)

**Corollary 4.11.** If \(A\) is a \((k, m)\) net with \(k \geq m\), then \(A\) contains the \(2 \times 2\) minors of a 1-generic \(2 \times \left(km - \binom{m + 1}{2}\right)\) matrix. Thus the resolution of \(I\) contains an Eagon-Northcott complex as a subcomplex.

**Proof.** Since \(k \in \{3, 4\}\), if \((k, m) = (3, 2)\) or \((3, 3)\) then by Theorem 4.10 \(h^0(B) \geq 3\), and if \((k, m) = (4, 3)\) or \((4, 4)\) then \(h^0(B) \geq 6\). Note that the only known example of a 4-net is the \((4, 3)\) net corresponding to the Hessian configuration. \(\Box\)

**Example 4.12.** For the arrangement \(A_3\) appearing in Example 2.1 \(Z\) is the collection of multiple points, and
\[
A = 2E_0 - \sum_{\{p | \mu(p) = 2\}} E_p
\]
and
\[
B = 3E_0 - \sum_{p \in L_2(A)} E_p.
\]
So \(d - \binom{m + 1}{2} = 6 - 3 = 3\) and \(I\) contains the \(2 \times 2\) minors of a \(2 \times 3\) matrix.

5. **Connection to Derivations**

In this section, we show that the generators of the Jacobian ideal of \(A \subseteq \mathbb{P}^2\) are contained in \(H^0(D_A)\), and that the associated projection map \(X \to \mathbb{P}^2\) has degree \(\sum_{p \in L_2(A)} \mu(p) - |A| + 1\). This relates \(X\) to one of the fundamental objects in arrangement theory: the module \(D(A)\) of derivations tangent to \(A\).

**Definition 5.1.** \(D(A) = \{\theta \mid \theta(\alpha_i) \in \langle \alpha_i \rangle \text{ for all } \alpha_i \text{ such that } V(\alpha_i) \in A\}\).

The module \(D(A)\) is a graded \(S = \mathbb{C}[x_0, \ldots, x_n]\) module, and over a field of characteristic zero, \(D(A) \simeq E \oplus D_0(A)\), where \(E\) is the Euler derivation and \(D_0(A)\) corresponds to the module of syzygies on the Jacobian ideal \(J_\alpha\) of the defining polynomial \(\alpha = \prod_{i=1}^d \alpha_i\) of \(A\). An arrangement \(A\) is free if \(D(A) \simeq \oplus S(-a_i)\); the \(a_i\) are called the exponents of \(A\). Terao’s theorem [37] is that if \(D(A) \simeq \oplus S(-a_i)\), then \(P(M, t) = \prod (1 + a_i t)\). Supersolvable arrangements are free, so Example 2.1 is of this type, and
\[
P(A_3, t) = (1 + t)(1 + 2t)(1 + 3t)
\]
On the other hand, the arrangement \(A\) of Example 2.4 is not free, and \(P(A, t)\) does not factor. However, it is shown in [30] that for \(A \subseteq \mathbb{P}^2\) the Poincaré polynomial is \((1 + t) \cdot c_t(D_0^L)\), where \(c_t\) is the Chern polynomial and \(D_0^L\) is the dual of the rank two vector bundle associated to \(D_0\). An easy localization argument [31] shows that in this case, the Jacobian ideal is a local complete intersection with
\[
deg(J_\alpha) = \sum_{p \in L_2(A)} \mu(p)^2.
\]
For \(A \subseteq \mathbb{P}^n\) with \(n \geq 3\), some generalizations are possible, see [22].
Proposition 5.2. For an arrangement $A \subseteq \mathbb{P}^2$, 

$$J_\alpha \subseteq L = \langle l_1, \ldots, l_d \rangle = \langle \frac{\alpha}{\alpha_1}, \ldots, \frac{\alpha}{\alpha_d} \rangle.$$ 

Proof. By Lemma 3.2

$$L = \bigcap_{p \in L_2(A)} I_{\mu(p)} \subseteq \mathbb{C}[x, y, z].$$

The ideal $J_\alpha$ is generated in degree $d-1$. The result of [31] mentioned above implies that at any point $p \in L_2(A)$, the localization $(J_\alpha)_p$ is a local complete intersection: changing coordinates so that $p = (0:0:1)$, and writing $A = V(L_0 L_1)$ with $L_0$ the product of the defining linear forms which vanish at $p$ and $L_1$ the product of the remaining forms, we have

$$(J_\alpha)_p = \langle \partial(L_0)/\partial x, \partial(L_0)/\partial y \rangle.$$ 

In particular, both generators are of degree $\mu(p)$, so in the primary decomposition of $J_\alpha$, the primary component associated to $I(p)$ is contained in $I_{\mu(p)}$. Now,

$$\text{Sing}(A) = \bigcup_{I(p) \in \text{Ass}(\sqrt{J_\alpha})} V(I(p)).$$

If $Q_p$ is the $I(p)$-primary component of $J_\alpha$, then

$$\bigcap_{I(p) \in S} Q_p \subseteq L.$$ 

Since the saturation of $J_\alpha$ with respect to $(x, y, z)$ is the left hand side, and $J_\alpha$ is generated in degree $d-1$, the result follows.

The inclusion $W = J_\alpha \subseteq H^0(D_A)$ corresponds to an induced map

$$\begin{array}{ccc}
\mathbb{P}^2 \setminus V(\alpha) & \xrightarrow{\phi_A} & \mathbb{P}(H^0(D_A)) \\
\psi & \downarrow \pi & \\
\mathbb{P}(W) & \xrightarrow{} & \\
\end{array}$$

Proposition 5.3. The degree of $\pi$ is $\sum_{p \in L_2(A)} \mu(p) - |A| + 1$.

Proof. In [7], Dimca and Papadima show that on a projective hyperplane complement $\mathbb{P}^n \setminus V(\alpha)$, the degree of the gradient map $\psi$

$$\begin{array}{ccc}
\mathbb{P}^n \setminus V(\alpha) & \xrightarrow{[\partial(\alpha)/\partial x_0 : \ldots : \partial(\alpha)/\partial x_n]} & \mathbb{P}^n \\
\end{array}$$

is equal to $b_0(\mathbb{P}^n \setminus V(\alpha))$; for a configuration of $A \subseteq \mathbb{P}^2$ this means the degree of the gradient map is

$$\sum_{p \in L_2(A)} \mu(p) - |A| + 1.$$ 

By Theorem 3.4 $\phi_A$ is an isomorphism on $\mathbb{P}^2 \setminus V(\alpha)$, and the result follows.

Concluding Remarks and Questions
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(1) To study $\text{Tor}_i^R(C(A), C)_{i+1}$, it suffices to restrict to the case of line arrangements. This follows since the quadratic generators of $C(A)$ depend only on $L_2(A)$, hence taking the intersection of $A$ with a generic $\mathbb{P}^2$ leaves these generators (and relations among them) unchanged. The graded betti numbers for $I_2$ are not combinatorial invariants; it would be interesting to understand how the geometry of $A$ governs $b_{ij}$, even for $j = i + 1$.

(2) Can freeness of $D(A)$ be related to the surface $X$ and divisor $D_A$? It seems possible that there is a connection between $D_A$ and multiarrangements, studied recently in [1], [39], [40].

(3) For $n \geq 3$, is $C(A)$ the homogeneous coordinate ring of a blowup of $\mathbb{P}^n$ along a locus related to the arrangement $A$? Since $C(A)$ is Cohen-Macaulay, if this is true, then combining Riemann-Roch with Terao’s result would yield a formula for the global sections of $D_A$.

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