PENALIZATION OF NON-SMOOTH DYNAMICAL SYSTEMS WITH NOISE: ERGODICITY AND ASYMPTOTIC FORMULAE FOR THRESHOLD CROSSINGS PROBABILITIES

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Abstract. The purpose of this paper is to prove ergodicity and provide asymptotic formulae for probabilities of threshold crossing related to smooth approximations of three fundamental nonlinear mechanical models: (a) an elasto-plastic oscillator, (b) an oscillator with dry friction, (c) an oscillator constrained by an obstacle (one sided or two sided) and subject to impacts, all three in presence of white or colored noise. Relying on a groundbreaking result on density estimates for degenerate diffusions by Delarue and Menozzi (2010), we identify Lyapunov functions that satisfy appropriate conditions leading to ergodicity (invariant measure and Poisson equation) and a functional central limit theorem. These conditions appear in the very fundamental works of Down, Meyn and Tweedie (1995) and Glynn and Meyn (1996). From an applied mathematics perspective, an important consequence is the access to asymptotic formulae for quantities of interest in engineering and science.

Keywords Moreau-Yosida approximation of variational inequalities, Lyapunov functions, ergodic properties, colored noise, random vibrations.

1. Introduction

Noise and vibration are fundamental features in an extremely wide range of industrial applications. In this context, mechanical structures will accumulate fatigue and then face a risk of failure. This is a major concern which has motivated a considerable effort in the engineering community e.g. see [5–7, 14, 18, 20, 22–25, 27, 28] and reference therein. The challenge is to handle non-smooth stochastic dynamical systems. Strangely enough, it does not attract many experts in probability theory. While this is an important topic for research, mathematical references on the probabilistic analysis and numerical methods associated with such systems are still few in number. In this paper, we focus on smooth approximation of three natural phenomena involving interactions with boundaries, constraints, phase transitions or hysteresis: (a) an elasto-perfectly-plastic oscillator [15], (b) a dry friction one dimensional model [12], (c) an oscillator constrained by an obstacle (one or two sided) and subject to impacts [1], all three in presence of noise.

In section 1.1, we present the type of noise that we consider. Then, in section 1.2, we first write the models shown in Figure 1 in terms of stochastic variational inequalities (SVIs); and then we put the focus on their smooth approximations by penalization.

1.1. White and colored noise. In addition to a white noise forcing in the form \( \dot{W} \) where \( W \) is a real valued Wiener process, we will consider a type of stationary colored noise satisfying

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Figure 1. Rheological models: (a) an elasto-perfectly-plastic oscillator, (b) a dry friction one dimensional model and (c) an oscillator with an obstacle and impacts. A mass (black box) is associated in series with elements which are themselves an association in parallel or in series of elementary rheological models. Each whole system is excited by a time-dependent random forcing $N_t$.

an overdamped Langevin dynamics of the form

\[ \text{d} \eta_t = -v'(\eta_t) \text{d}t + \sigma \text{d}W_t \]

where $v$ is a confining potential satisfying, for some constant $r > 0$, \[ v'(\eta) \eta \geq r |\eta|^2. \]

When $\sigma = \sqrt{2\beta^{-1}}$, $\beta > 0$, it is well known that its invariant probability density function (PDF) on $\mathbb{R}$ is given by

\[ \rho(\eta) = Z_{\beta}^{-1} \exp(-\beta v(\eta)), \quad Z_{\beta} = \int_{\mathbb{R}} \exp(-\beta v(\eta)) \text{d}\eta. \]

Typically, with $v(\eta) = \frac{1}{2} \theta_v \eta^2, \theta_v > 0$, it is clear that the so-called Ornstein-Uhlenbeck process belongs to this class of noise and $\eta_t = \exp(-\theta_v t) \eta_0 + \sigma \int_0^t \exp(-\theta_v (t - s)) \text{d}W_s$. In this case, its corresponding power spectrum is of the form $P(\omega) = \frac{\sigma^2}{2\beta \pi \omega^2 + \theta_v^2 \omega^2}$, $\omega \in \mathbb{R}$. In contrast with a white noise which has constant power spectrum, a colored noise is more interesting for applications since high frequencies are not present. The second class of colored noise belongs to the stochastic Hamiltonian systems of the form

\[ \text{d} \zeta_t = B_1(\eta_t, \zeta_t) \text{d}t + \sigma \text{d}W_t, \quad \text{d} \eta_t = B_2(\eta_t, \zeta_t) \text{d}t, \quad t > 0 \]

where

\[ B_1(\eta, \zeta) \triangleq -\partial_\eta H(\eta, \zeta) - F(\eta, \zeta) \partial_\zeta H(\eta, \zeta), \quad B_2(\eta, \zeta) \triangleq \partial_\zeta H(\eta, \zeta). \]

Here $H(\eta, \zeta)$ is a function called the Hamiltonian and $F$ is a function related to dissipation. Also, the existence and uniqueness of the invariant measure has been shown [26] when the functions $H$ and $F$ satisfy a set of conditions (see Hypothesis 1.1, page 6 in [26]). In particular, it is known that when $F \equiv C > 0$ is a constant, $\sigma = \sqrt{2C \beta^{-1}}$ ($\beta > 0$) and the function $H(\eta, \zeta) \triangleq \frac{1}{2} \zeta^2 + v(\eta)$, then $(\eta_t, \zeta_t)$ has a unique stationary measure state given by

\[ \rho(\eta, \zeta) \triangleq C_{\beta}^{-1} \exp(-\beta H(\eta, \zeta)), \quad C_{\beta} \triangleq \int_{\mathbb{R}^2} \exp(-\beta H(\eta, \zeta)) \text{d}\eta \text{d}\zeta. \]

Remark 1. In earthquake engineering, a realistic type of random forcing to represent seismic excitation is the so-called Kanai-Tajimi model whose power spectral density is of the form $P(\omega) = \sigma \frac{k^2 + \gamma \omega^2}{(k - \omega^2)^2 + \gamma \omega^2}, \omega \in \mathbb{R}$. This type of noise belongs to the class of stochastic Hamiltonian
systems mentioned right above with \( H(\eta, \zeta) = \frac{k}{2}\eta^2 + \frac{1}{2}\zeta^2 \) and \( F \equiv \gamma_0 \). Thus, \( \eta \) of (3) is just the response of a white noise driven linear oscillator (with a stiffness \( k \) and damping \( \gamma_0 \)). Such a model reproduces dynamical properties of the ground [19].

1.2. Stochastic variational inequality formulation and penalization. We first recall a stochastic variational inequality (SVI) formulation for a class of non-smooth dynamical systems with noise. It includes models shown in Figure 1. This type of mathematical structure has been identified as a solid framework in [2–4]. Then, we put the focus on smooth approximation of this type of system via the so-called Moreau-Yosida regularization [21,29]. In Appendix A, for the convenience of the reader, we explicitly write the SVIs and their penalized version for the elasto-plastic, obstacle, and friction problems.

1.2.1. Stochastic variational inequality formulation. Models shown in Figure 1 can be described in terms of SVIs as follows: for a state variable \( Z_t \equiv (Y_t, X_t) \in \mathbb{R}^2 \), where (SVI - 1) 
\[ \forall (y, x) \in \mathbb{R}^2, \forall t > 0, \ (b(Y_t, X_t) + N_t - \dot{Y}_t)(y - Y_t) + (Y_t - \dot{X}_t)(x - X_t) + \varphi(Y_t, X_t) \leq \varphi(y, x), \]
or (SVI - 2) 
\[ \forall x \in \mathbb{R}, \forall t > 0, \ \dot{X}_t = Y_t \text{ and } (b(Y_t, X_t) + N_t - \dot{Y}_t)(x - X_t) + \varphi(X_t) \leq \varphi(x), \]

Here \( b : \mathbb{R}^2 \to \mathbb{R}, b(y, x) \triangleq -U'(x) - C_b y \) where \( U \) is a potential, \( C_b > 0, \varphi \) is a non-smooth convex real valued function on \( \mathbb{R}^d (d = 1, 2) \) and \( N_t \) is a real valued stochastic forcing. See below Table 1 for specifications of the function \( \varphi \) in the different models, with the notation \( K \triangleq [-1, 1], \) and \( \chi_K = 0 \) on \( K \) and \( \infty \) on \( \mathbb{R} - K \). Throughout this paper, the forcing will be of three forms: \( N_t = \dot{W}, N_t = \eta_t \) of (1), or \( N_t = \eta_t \) of (3) and will be used as input in the three mechanical models. Note that the inequalities above must be satisfied for any \((y, x)\) or \( x \) which are variational parameters.

1.2.2. Penalization of SVIs. The Moreau-Yosida regularization is a well-known approach to smooth any non-smooth convex function \( \varphi \) as follows:
\[ \varphi_n(z) \triangleq \inf_{y \in \mathbb{R}^d} \left\{ \frac{n}{2}\|z - y\|^2 + \varphi(y) \right\}, \|y\| \triangleq \sqrt{y_1^2 + \cdots + y_d^2}. \]

Here, for every \( n, \varphi_n(z) \) is differentiable with respect to \( z \). For every \( z \in \mathbb{R}^d, \lim_{n \to \infty} \varphi_n(z) = \varphi(z) \). Thus a regularized version of (SVI - 1) or (SVI - 2) consists in replacing \( \varphi \) by \( \varphi_n \). This procedure is also called penalization. Here, the Moreau-Yosida regularization of the functions \( \chi_K \) and the absolute value \( y \to |y| \) are respectively \( \chi_n \) and \( a_n \) defined as follows:
\[ \chi_n(x) \triangleq \frac{n}{2}|x - \text{proj}_K(x)|^2 \text{ and } a_n(y) \triangleq \begin{cases} \frac{|y| - (2n)^{-1}}{n} & \text{if } n|y| \geq 1 \\ \frac{n}{2} y^2 & \text{if } n|y| < 1 \end{cases}. \]

In all cases (for the three mechanical models introduced above), this amounts to replace the SVI problem by a standard stochastic differential equation (SDE) with a nonlinear term depending on \( n \) in the following sense:
\[ (P_n) \quad \dot{Z}_t^n + f_n(Z_t^n) = B(Z_t^n) + \sigma_2 N_t, \text{ where } \sigma_2 = (1, 0)^T, \ B(y, x) \triangleq (b(y, x), y)^T. \]

Here \( N_t = \dot{W} \) is a white noise or a colored noise as shown in (1) or (3). Let us emphasize that a crucial point, for the mathematical perspective, is that the penalization allows us to
have a unified treatment of the three mechanical models.

Table 1. Specification of the the SVI framework and of the functions \( \varphi, \varphi_n \) and \( f_n \) for the three mechanical models of Figure 1.

| SVI | Elasto-perfectly-plastic | Friction | Obstacle |
|-----|---------------------------|-----------|----------|
| \( (SVI - 1) \) | \( \varphi(y, x) = \chi_K(x) \) | \( \varphi(y, x) = |y| \) | \( \varphi(x) = \chi_K(x) \) |
| \( \varphi_n \) | \( \varphi_n(y, x) = \chi_n(x) \) | \( \varphi_n(y, x) = a_n(y) \) | \( \varphi_n(x) = \chi_n(x) \) |
| \( f_n \) | \( f_n(y, x) = (0, \chi'_n(x))^T \) | \( f_n(y, x) = (a'_n(y), 0)^T \) | \( f_n(y, x) = (\chi'_n(x), 0)^T \) |

State variable extension and Markovian structure. In order to remain in a Markovian framework [16], the state variable \( Z^n = (Y^n, X^n) \) is extended to \( Z^n = (\eta, Y^n, X^n) \) for a noise (1) or \( Z^n = (\zeta, \eta, Y^n, X^n) \) for a noise (3). Therefore, we rewrite \( (P_n) \) in terms of a SDE of the form

\[
dZ^n_t = F_n(Z^n_t)dt + \sigma dW_t
\]

with \( F_n \) and \( \sigma \) given in Table 2.

Table 2. \( F_n : \mathbb{R}^d \to \mathbb{R}^d \) and \( \sigma \in \mathbb{R}^d \) for (5). The functions \( f_{n,x} \) and \( f_{n,y} \) are defined, for each mechanical model, in Table 3.

| Type of noise | \( F_n \) | \( \sigma \) |
|---------------|-----------|-----------|
| white noise \( Z^n_t = (Y^n_t, X^n_t)^T \) | \( F_n(y, x) = \begin{pmatrix} -U'(x) - C_b y - f_{n,y}(y) \\ y - f_{n,x}(x) \end{pmatrix} \) | \( \sigma = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \) |
| colored noise of type 1 \( Z_t = (\eta, Y^n_t, X^n_t)^T \) | \( F_n(\eta, y, x) = \begin{pmatrix} -v'(\eta) \\ \eta - U'(x) - C_b y - f_{n,y}(y) \\ y - f_{n,x}(x) \end{pmatrix} \) | \( \sigma = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \) |
| colored noise of type 2 \( Z^n_t = (\zeta, \eta, Y^n_t, X^n_t)^T \) | \( F_n(\zeta, \eta, y, x) = \begin{pmatrix} B_1(\eta, \zeta) \\ B_2(\eta, \zeta) \\ \eta - U'(x) - C_b y - f_{n,y}(y) \\ y - f_{n,x}(x) \end{pmatrix} \) | \( \sigma = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \) |

Table 3. Explicit form of the functions \( f_{n,y}, f_{n,x} \) and potential for each mechanical model. For the obstacle problem, the trick is to move the penalization function directly into the potential.

| model | \( f_{n,y} \) | \( f_{n,x} \) | potential |
|-------|---------------|---------------|-----------|
| elasto-plastic | \( 0 \) | \( \chi'_n(x) \) | \( U(x) \) |
| friction | \( a'_n(y) \) | \( 0 \) | \( U(x) \) |
| obstacle | \( 0 \) | \( 0 \) | \( U_n(x) \pm U(x) + \chi_n(x) \) |

In view of Table 3, an important observation is that, in the three mechanical models (elasto-plastic, friction and obstacle) the functions \( f_{n,x} \) and \( f_{n,y} \) satisfy the following properties

(HPY) \( |f_{n,y}| \leq 1 \),

(HPX) for all \( x \in \mathbb{R} \), \( 0 \leq \text{sign}(x)f_{n,x}(x) \leq n|x| \).
2. Main Results

2.1. Goal of the paper. Relying on [8], we prove the existence of a class of Lyapunov functions covering all the penalization version of the cases shown in Table 2 and satisfying the so-called Foster-Lyapunov condition in the sense of [9] and [13]. This condition leads to ergodicity (existence and uniqueness of an invariant probability measure, rate of convergence for the semi-group, unique solution to the Poisson equation) and a functional central limit theorem. From an applied mathematics perspective, as an important consequence, we now have access to asymptotic formulae for the probabilities of threshold crossing for quantities of interest in engineering, physics and other fields.

2.2. Standing assumptions and conditions. We make the following assumptions.

- \( (A_U) \) - assumptions on \( U \). The potential \( U : \mathbb{R} \to \mathbb{R} \) satisfies the following assumptions:

  \( (HU1) \) \( U \) is of class \( C^1 \) and \( U' \) is Lipschitz for some constant \( \kappa \), that is,
  \[
  \exists \kappa > 0, \forall x, y \in \mathbb{R}, |U'(x) - U'(y)| \leq \kappa |x - y|.
  \]

  \( (HU2) \) \( \exists \beta_1 > 0, \lambda_1 > 0, U(x) \geq \lambda_1 x^2 - \beta_1 \).

  \( (HU3) \) \( \exists \beta_2 > 0, \lambda_2 > 0, xU'(x) \geq \lambda_2 U(x) - \beta_2 \).

  Without loss of generality (up to an additive constant), we will also assume \( (HU4) \) \( U \geq 0 \).

Note that, as a consequence of \( (HU2) \) and \( (HU3) \), we also have

\[
(HU5) \quad xU'(x) \geq \lambda_3 \left[ U(x) + x^2 \right] - \beta_3, \text{ where } \lambda_3 = \frac{\lambda_1}{1 + \lambda_1}, \text{ and } \beta_3 = \beta_2 + \frac{\lambda_2 \beta_1}{1 + \lambda_1}.
\]

- \( (AH,F) \) - assumptions on \( H, F \). We assume that \( H \) and \( F \) are taken as in [26], satisfying the same assumptions (see Hypothesis 1.1 of [26]) with \( \nu = 2 \). More precisely, we assume \( F \) and \( H \) are of class \( C^\infty \) and there exist two constants \( \delta > 0, M > 0 \) and a function \( R \) on \( \mathbb{R}^2 \) such that for all \( (\eta, \zeta) \in \mathbb{R}^2 \)

\[
(T117) \quad H(\eta, \zeta) + R(\eta, \zeta) + M \geq \delta (|\eta|^2 + |\zeta|^2)
\]

\[
(T120) \quad \left( \frac{1}{2} \partial_\zeta \zeta + B_1 \partial_\zeta + B_2 \partial_\eta \right) \left( H(\eta, \zeta) + R(\eta, \zeta) \right) \leq -\delta [H(\eta, \zeta) + R(\eta, \zeta)] + M.
\]

We recall that \( B_1 \) and \( B_2 \) are defined by (4). We further assume that

\[
(6) \quad \zeta \mapsto \partial_\zeta B_2(\zeta, \eta) \text{ is } \alpha - \text{Hölder continuous with a constant coefficient } \kappa.
\]

\[
(7) \quad \exists l \in \mathbb{R}, \quad \partial_\zeta B_2(\eta, \zeta) \geq l > 0, \quad \text{or} \quad \partial_\zeta B_2(\zeta, \eta) \leq -l < 0.
\]

Inequalities \( (T117) \) and \( (T120) \) correspond to (1.17) and (1.20) of [26] respectively.
2.3. Statement of the results. From now on, unless otherwise specified \( n \) is a fixed positive integer. Let us consider \( Z^n \) solving (5), with \( \sigma = (1,0,\ldots,0) \in \mathbb{R}^d \) for simplicity. Define the transition semi-group \( P_{n,t} \) of \( Z^n \) as follows

\[
\forall \mathcal{O} \in \mathcal{B}([0,\infty)), \forall z \in \mathbb{R}^d, \forall t > 0, \quad P_{n,t}(z,\mathcal{O}) = \mathbb{P}(Z^n_t \in \mathcal{O}|Z^n_0 = z)
\]

and the infinitesimal generator of \( Z^n \) as

\[
\forall \psi \in C^2(\mathbb{R}^d), \quad A_n \psi(z) = \frac{1}{2} \nabla^2 \varphi(z) + \sum_{i=1}^d F_{n,i}(z) \partial^i \psi(z), \quad z \in \mathbb{R}^d.
\]

The notation \( F_n \) is defined in Table 2 and \( F_{n,i} \) is the \( i \)th component of \( F_n \). The following Lemma shows that there exists a Lyapunov function satisfying two key properties [9].

**Lemma 2** (Unbounded off petite set and Foster-Lyapunov drift conditions). Under the assumptions \( A_U \) and \( A_{H,F} \), there exists a function \( V_n : \mathbb{R}^d \rightarrow [1, \infty) \) satisfying:

(HV1) For all \( r \geq 1 \), the set \( B_{V_n}(r) = \{ z \in \mathbb{R}^d : V_n(z) \leq r \} \) is either empty or satisfies: there exists a probability measure \( a \) on \( \mathcal{B}([0,\infty)) \) and a \( \sigma \)-finite measure \( \nu \) on \( \mathcal{B}(\mathbb{R}^d) \) such that

\[
\nu(\cdot) \leq \int_{0}^{\infty} P_{n,t}(z,\cdot) a(\text{d}t), \quad \forall z \in B_{V_n}(r).
\]

For \( r \) fixed, this condition is referred to as \( B_{V_n}(r) \) is petite and the whole condition is referred to as \( V_n \) is unbounded off petite sets.

(HV2) There exist two constants \( \epsilon > 0 \), \( C > 0 \) such that \( \forall z \in \mathbb{R}^d, (A_n V_n + \epsilon V_n)(z) \leq C \).

**Remark 3.** The condition (HV1) can be formulated in other words: when \( B_{V_n}(r) \) is not empty, there is a measure \( \nu(\cdot) \) on \( \mathbb{R}^d \) and a random time \( T_\mathcal{A} \) independent from \( Z^n \) such that for every \( B \in \mathcal{B}(\mathbb{R}) \), the probability for \( Z^n \) starting from an arbitrary point of \( B_{V_n}(r) \) to be in \( B \) at time \( T_\mathcal{A} \) is larger than \( \nu(B) \).

These two conditions have been shown to entail ergodicity with a certain convergence rate (see Theorem 5.2 in [9]). To be more precise, let us recall the notion of \( V \)-uniform ergodicity. As presented in [9], we say that a transition semi-group \( P_t \) is \( V \)-uniformly ergodic if there exist a probability measure \( \pi \) and two constants \( D > 0, \rho \in (0,1) \) such that

\[
\| P_t(\cdot,.) - \pi \|_V \leq V(z) D\rho^t, \quad \forall t \geq 0, z \in \mathbb{R}^d,
\]

where for every signed measure \( \mu \) on \((\mathbb{R}^d,\mathcal{B}(\mathbb{R}^d))\), the \( V \)-norm of \( \mu \) is

\[
\| \mu \|_V \triangleq \sup_{|g| \leq V} \left| \int \mathbb{R}^d g(z)\mu(\text{d}z) \right|.
\]

Our first theorem states the existence and uniqueness of an invariant probability density for \( Z^n \).

**Theorem 4.** The process \( \{ Z^n_t, t \geq 0 \} \) is \( V_n \)-uniformly ergodic. Therefore, \( Z^n \) has a unique invariant probability measure \( \mu_n \) on \( \mathbb{R}^d \), which admits a density \( m_n \) with respect to Lebesgue measure on \( \mathbb{R}^d \) and converges to it in the sense of (11) with an exponential rate. Moreover, \( m_n \) is the unique solution of

\[
\forall f \in C^2(\mathbb{R}^d), \quad \int_{\mathbb{R}^d} A_n f(z) m_n(z) \text{d}z = 0.
\]

Therefore, \( m_n \) is the unique probability density solution, in the sense of distributions, of

\[
\frac{1}{2} \nabla^2 \varphi m_n = \nabla : [F_n m_n], \quad \text{in} \quad \mathbb{R}^d.
\]

As mentioned above, the proof relies on Theorem 5.2 in [9]. Our second theorem states the existence and uniqueness of a solution to the Poisson equation for \( Z^n \).
Theorem 5. Under (HV1) and (HV2) given in Lemma 2, the following properties hold:

1. \( \{Z^n_t, t \geq 0\} \) is positive Harris recurrent, which in particular implies
   \[
   \forall f \text{ s.t. } \int_{\mathbb{R}^d} |f(z)| m_n(z) dz < \infty, \quad \lim_{t \to \infty} \frac{1}{t} \int_0^t f(Z^n_s) ds = \int_{\mathbb{R}^d} f(z)m_n(z) dz, \text{ a.s.}
   \]

   and
   \[
   \int_{\mathbb{R}^d} V_n(z)m_n(z) dz < \infty.
   \]

2. For some constant \( C > 0 \), the Poisson equation
   \begin{equation}
   -A_n u = \bar{g}
   \end{equation}

admits a unique solution, which is denoted by \( u^n_\bar{g} \) in the sequel, and we have the bounds
   \[
   |u^n_\bar{g}(z)| \leq C(V_n(z) + 1), \text{ for a.e. } z.
   \]

Here
   \[
   \bar{g} = g - \int_{\mathbb{R}^d} g(z)m_n(z) dz, \text{ in } \mathbb{R}^d.
   \]

The proof relies on Theorem 3.2 of [13].

The third theorem is a functional central limit theorem for quantities of the form

\[
\Xi^n_{p,g}(t) = \frac{1}{\sqrt{p}} \int_0^t \bar{g}(Z^n_s) ds.
\]

Theorem 6. For any function \( g \) such that \( g^2 \leq V \), there exists a constant \( 0 \leq \gamma^n_g < \infty \) such that for any initial distribution \( \mu \), \( \Xi^n_{p,g} \to \gamma^n_g W, \) \( \mathbb{P}_\mu \) - weakly in \( D[0,1] \) as \( p \to \infty \), where \( W \) is a standard Wiener process. Moreover, the constant \( (\gamma^n_g)^2 \) is characterized by combining the invariant measure of \( Z^n \) and the solution to the Poisson equation (12) as follows:

\[
(\gamma^n_g)^2 = \int_{\mathbb{R}^d} |\partial_z u^n_\bar{g}(z)|^2 m_n(z) dz.
\]

The proof relies on Theorem 4.4 of [13].

3. PROOF OF LEMMA 2

Let us introduce the functions \( V_{n,d} \), for \( d \in \{2, 3, 4\} \), defined as

\[
\begin{align*}
V_{n,2}(y, x) & \quad \triangleq \delta \left( \frac{y^2}{2} + U(x) \right) + xy + C_V, \\
V_{n,3}(\eta, y, x) & \quad \triangleq \Gamma_1(\eta) + V_{n,2}(y, x), \\
V_{n,4}(\zeta, \eta, y, x) & \quad \triangleq \Gamma_2(\zeta, \eta) + V_{n,2}(y, x),
\end{align*}
\]

where

\[
\Gamma_1(\eta) \triangleq \frac{\xi}{2} \eta^2, \quad \Gamma_2(\zeta, \eta) \triangleq K \left( H(\zeta, \eta) + R(\zeta, \eta) + M \right).
\]

Here \( \delta, C_V, \xi \) and \( K \) are large enough constants. To be precise, we require that they satisfy the following inequalities:

\[
\delta > \max \left( \frac{1}{\sqrt{\lambda_1}}, \frac{2}{\xi_0} \left( 2 + \frac{4(C_V + \eta)^2}{3\lambda_3} \right) \right), \quad C_V \geq 1 + \delta \beta_1, \quad \xi > \frac{8}{7} \left[ \frac{\delta}{\lambda_6} + \frac{2}{3\lambda_3} \right], \quad K > \frac{4}{3\delta} \left[ \frac{\delta}{\lambda_6} + \frac{2}{3\lambda_3} \right].
\]

We split the proof into two parts, the first one for (HV1) and the other one for (HV2).
3.1. Proof of Lemma 2 : (HV1). We first show that, in each case, \( V \) is unbounded off compact sets, and then that petite sets are compact. **First step:** Each function \( V \in \{ V_{n,d} \}_{d=2,3,4} \) is unbounded off compact sets in the sense that for all \( r \geq 1 \), the set \( B_r \) is compact (possibly empty). Indeed, \( V \) satisfies \( V(z) \to \infty \) as \( |z| \to \infty \). This is checked by direct calculations as follows. For \( V_{n,2} \), by assumption (HU2) we have

\[
V_{n,2}(y,x) \geq \frac{y^2}{2} \left( \delta - \frac{1}{\delta \lambda_1} \right) + \frac{\delta \lambda_1}{2} x^2 + 1,
\]

and by (14) the coefficient of \( y^2 \) is strictly positive. Also, we deduce readily that \( V_{n,3} \) is unbounded off compact sets. The same conclusion also holds for \( V_{n,4} \) by using (T117). **Second step:** We then check that compact sets are petite. To do so, we exploit density estimates provided by Theorem 1.1 of Delarue and Menozzi [8]. It states (in a simplified form for our present problem (5)) that

Consider a chain of SDEs of the form,

\[
dz^i_t = f_i(z^i_t, \ldots, z^d_t)dt + dW_t, \quad dz^i_t = f_i(z^{i-1}_t, \ldots, z^d_t)dt, \quad 2 \leq i \leq d,
\]

with the initial condition \( (z^i_0, \ldots, z^d_0) = z \in \mathbb{R}^d \) and the following conditions on \( f_1, \ldots, f_d \):

- \( f_1, \ldots, f_d \) are Lipschitz,
- for each \( 2 \leq i \leq n \),
  1. \( z_{i-1} \mapsto f_i(z_{i-1}, z_i, \ldots, z_d) \) is continuously differentiable,
  2. \( z_{i-1} \mapsto \partial_{z_{i-1}} f_i(z_{i-1}, z_i, \ldots, z_d) \equiv 1 \) except for \( d = 4 \) and \( i = 2 \) where it is \( \alpha \)-Hölder continuous with the coefficient \( \kappa \).

Then at any time \( t > 0 \) the solution of (15) admits a density \( z' \in \mathbb{R}^d \mapsto p(t, z, z') \). Moreover, for any \( T > 0 \), there exists a constant \( C_T \geq 1 \), depending on \( T \) and \( \kappa \) such that for any \( 0 < t \leq T \),

\[
C^{-1}_T t^{-\frac{d}{2}} \exp \left( -C_T t ^{\frac{\alpha}{2}} |\theta_t(z) - z'|^2 \right) \leq p(t, z, z') \leq C_T t^{-\frac{d}{2}} \exp \left( -C^{-1}_T t ^{\frac{\alpha}{2}} |\theta_t(z) - z'|^2 \right).
\]

Here \( \theta_t \) is the solution of the deterministic ordinary differential equation (ODE)

\[
d\theta^i_t = f_i(\theta^i_t, \ldots, \theta^d_t)dt, \quad d\theta^i_t = f_i(\theta^{i-1}_t, \ldots, \theta^d_t)dt, \quad 2 \leq i \leq d
\]

with the initial condition \( \theta_0 = z \).

Let \( \mathcal{K} \) be a compact set of \( \mathbb{R}^d \). Fix an arbitrary \( z_0 \in \mathcal{K} \). From (16) we deduce that for any \( \mathcal{O} \in \mathcal{B}(\mathbb{R}^d) \)

\[
\int_{\mathcal{O}} c_1(T) \exp \left( -c_2(T) \sup_{t \in [0,T]} |\theta_t(z_0) - z'|^2 \right) dz' \leq \int_0^T \int_{\mathcal{O}} p(t, z_0, z')d z' dt
\]

with \( c_1(T) = C^{-1}_T T^{-\frac{d}{2}} \) and \( c_2(T) = C_T T \). Hence (10) is verified with \( a \) and \( \nu \) defined by:

\[
a(dt) = \frac{1_{[0,T]}(t)}{T} dt \quad \text{and} \quad \nu(\mathcal{O}) = \frac{c_1(T)}{T} \int_{\mathcal{O}} \exp \left( -c_2(T) \sup_{z \in \mathcal{K}, t \in [0,T]} |\theta_t(z) - z'|^2 \right) dz'.
\]

Note that \( \nu \) is well-defined since \( (t, z) \to \theta_t(z) \) and \( z' \to \sup_{z \in \mathcal{K}, t \in [0,T]} |\theta_t(z) - z'| \) are continuous functions.
3.2. Proof of Lemma 2 : (HV2). The proof is very similar in the three cases. For the sake of clarity we split the analysis into three parts, where we will use the same constant $\epsilon$, taken such that

$$0 < \epsilon < \min \left( \frac{\lambda_3}{\delta_b}, C_b, r \right).$$

3.2.1. Proof in the white noise case. Let us denote by $A_{n,2}$ the infinitesimal generator of $Z^n_t = (Y^n_t, X^n_t)$ in the white noise case; see (9) for the general expression of $A_n$. Thus,

$$A_{n,2} \equiv \frac{1}{2} \partial_{yy} - (C_b y + U'(x) + f_{n,y}(y)) \partial_y + (y - f_{n,x}(x)) \partial_x.$$

We expand the left-hand side of the inequality in the statement of (HV2) and obtain

$$(A_{n,2} V_{n,2} + \epsilon V_{n,2})(y, x) = \left( \frac{\delta}{2} + \epsilon C_V \right) + y^2 \left( 1 - \delta \left[ C_b - \frac{\epsilon}{2} \right] \right) + S_1(x) + S_2(y, x),$$

with the notations

$$S_1(x) \equiv \delta \epsilon U(x) - U'(x) [x + \delta f_{n,x}(x)], \quad S_2(y, x) \equiv -y [x(C_b - \epsilon) + f_{n,x}(x)] - f_{n,y}(y) [\delta y + x].$$

Bounds for $S_1$ and $S_2$ can be derived as shown in the two Claims below.

Claim 7 (bound for $S_1$). For all $x \in \mathbb{R}$,

$$S_1(x) \leq \Gamma - \bar{\Gamma} x^2$$

with $\bar{\Gamma} = \frac{2 \lambda_3}{\delta}$ and $\Gamma = \beta_3 + \frac{(\delta \beta_4 n)^2}{\lambda_3^2}$, where $\beta_4 = \max (\beta_3, \max_{|z| \leq 1} |U'(z)|)$.

Note that $\beta_4 < \infty$ by (HU1).

Proof. First, let us upper bound $-\delta f_{n,x}(x)U'(x)$. We have

$$\text{sign}(x)U'(x) + \beta_4 \geq 0.$$ \hfill (21)

Indeed, from (HU4) and (HU5),

$$\text{sign}(x)U'(x) + \beta_3 \geq 0, \ |x| \geq 1$$

and from (HU1),

$$\text{sign}(x)U'(x) + \max_{z \in [-1,1]} |U'(z)| \geq 0, \ |x| < 1.$$ \hfill (22)

Using (21) together with (HPX), we obtain

$$-\delta f_{n,x}(x)U'(x) = -\delta \text{sign}(x) f_{n,x}(x) \left[ \text{sign}(x)U'(x) + \beta_4 \right] + \delta \beta_4 \text{sign}(x) f_{n,x}(x) \leq 0$$

Second, let us upper bound $-xU'(x) + \epsilon \delta U(x)$. Using the fact that $\epsilon < \frac{\lambda_3}{\delta}$ and (HU5),

$$-xU'(x) + \epsilon \delta U(x) \leq \beta_3 - \lambda_3 x^2.$$ \hfill (23)

The conclusion holds by (22) and (23). \hfill \square

Claim 8 (bound for $S_2$). For all $x, y \in \mathbb{R}$,

$$S_2(y, x) \leq \frac{(C_b + n)^2}{2 \bar{\Gamma}} y^2 + \frac{\bar{\Gamma}}{2} x^2 + \delta |y| + |x|.$$

$$\left( \frac{\lambda_3}{\delta_b}, C_b, r \right).$$
Proof. Using the fact that \( 0 < \epsilon < C_b \) and (HPX) we obtain
\[
|y [x(C_b - \epsilon) + f_{n,x}(x)]| \leq (n + C_b)|xy| \leq \frac{(C_b + n)^2}{2\Gamma} y^2 + \frac{\tilde{r}}{2} x^2,
\]
and using (HPY) we have
\[
|f_{n,y}(y)(\delta y + x)| \leq \delta|y| + |x|,
\]
hence the result. \( \square \)

We plug (20), and (24) into (19), and we use again the fact that \( \delta > \frac{2}{\epsilon_b} \left( 2 + \frac{4(C_b + n)^2}{3\lambda_3} \right) \) to get

\[
(A_{n,2}V_{n,2} + \epsilon V_{n,2}) (y, x) \leq K_2 - K_{2,y} y^2 - K_{2,x} x^2 + \delta|y| + |x|,
\]
where
\[
K_2 = C_V + \Gamma + \frac{\delta}{2}, \quad K_{2,y} = \frac{C_b \delta}{4}, \quad K_{2,x} = \frac{\tilde{r}}{2}.
\]

Since \( K_{2,y} \) and \( K_{2,x} \) are positive constants, there exists a large enough constant \( C \) such that (HV2) is satisfied.

3.2.2. Proof in the case of colored noise of type (1). The proof follows a similar structure and uses the bound (25) obtained in the white noise case. We denote by \( A_{n,3} \) the infinitesimal generator of \( Z_t^n = (\eta_t, Y_t^n, X_t^n) \) in the colored noise of type (1), that is,
\[
A_{n,3} = \frac{1}{2} \hat{\partial}_{\eta} - \nu'(\eta) \psi_{\eta} + \left( \eta - \left[ C_b y + U'(x) + f_{n,y}(y) \right] \right) \hat{\partial}_y + (y - f_{n,x}(x)) \hat{\partial}_x.
\]

We have
\[
(A_{n,3}V_{n,3} + \epsilon V_{n,3})(\eta, y, x) = S_3(\eta) + \eta[\delta y + x] + (A_{n,2}V_{n,2} + \epsilon V_{n,2})(y, x) - \frac{\delta}{2},
\]
with the notation
\[
S_3(\eta) = \frac{1}{2} \Gamma''(\eta) - \nu'(\eta) \Gamma'(\eta) + \epsilon \Gamma(\eta).
\]

We show the following bound on \( S_3 \).

Claim 9 (bound for \( S_3 \)). For all \( \eta \in \mathbb{R} \)
\[
S_3(\eta) \leq \frac{\xi}{2} - \frac{\xi r}{2} \eta^2.
\]

Proof. \( \frac{1}{2} \Gamma''(\eta) - \nu'(\eta) \Gamma'(\eta) + \epsilon \Gamma(\eta) = \frac{\xi}{2} - \nu'(\eta) \xi \eta + \epsilon \xi^2 \eta^2 \leq \xi \left( 1 - \eta^2 \right) \), where we used assumption (2) on \( v \). The conclusion holds since \( r > \epsilon \) by (18). \( \square \)

Using (27) and (25), we get the following bound for (26):

\[
(A_{n,3}V_{n,3} + \epsilon V_{n,3})(\eta, y, x) \leq K_3 - K_{3,\eta} \eta^2 - K_{3,y} y^2 - K_{3,x} x^2 + \delta|y| + |x|
\]
where
\[
K_3 = C_V + \Gamma + \frac{\xi}{2}, \quad K_{3,\eta} = \frac{\xi r}{4}, \quad K_{3,y} = \frac{K_{2,y}}{2}, \quad K_{3,x} = \frac{K_{2,x}}{2}.
\]

This yields (HV2) for \( d = 3 \) (colored noise of type (1)).
3.2.3. Proof in the case of colored noise of type (3). Let us denote by \( A_{n,4} \) the infinitesimal generator \( Z^n_t = (\zeta, \eta, Y^n_t, X^n_t) \) in the colored noise of type (3), that is,

\[
A_{n,4} = \frac{1}{2} \hat{c}_2 + B_1(\zeta, \eta)\hat{c}_2 + B_2(\zeta, \eta)\hat{c}_\eta + (\eta - \left[ C\eta y + U'(x) + f_{n,y}(y) \right]) \hat{c}_y + (y - f_{n,x}(x)) \hat{c}_x.
\]

We have

\[
(A_{n,4}V_{n,4} + \epsilon V_{n,4})(\zeta, \eta, y, x) = S_4(\zeta, \eta) + \eta [\delta y + x] + (A_{n,2}V_{n,2} + \epsilon V_{n,2})(y, x) - \frac{\delta}{2}.
\]

with the notation

\[
S_4(\zeta, \eta) = \left( \frac{1}{2} \hat{c}_2(\zeta, \eta, \Gamma_2 + B_1(\zeta, \eta) \Gamma_2 + B_2(\zeta, \eta) \Gamma_2 + \epsilon \Gamma_2) \right)(\zeta, \eta).
\]

Using (T117) and (T120), one can check that the following bound on \( S_4 \) holds.

**Claim 10 (bound for \( S_4 \)).** For all \( \zeta, \eta \in \mathbb{R} \)

\[
S_4(\zeta, \eta) \leq M(1 + K \delta) - K \delta^2 \zeta^2 - K \delta^2 \eta^2.
\]

Using (30) and (25), we get the following bound for (29):

\[
(A_{n,4}V_{n,4} + \epsilon V_{n,4})(\zeta, \eta, y, x) \leq K_4 - K_{4,\zeta} \zeta^2 - K_{4,\eta} \eta^2 - K_{4,y} y^2 - K_{4,x} x^2 + \delta |y| + |x|.
\]

where

\[
K_4 = C_V + \Gamma + M(1 + K \delta), \quad K_{4,\zeta} = K \delta^2, \quad K_{4,\eta} = \frac{K_2 \delta^2}{2}, \quad K_{4,y} = \frac{K_2 \delta^2}{2}, \quad K_{4,x} = \frac{K_2 \delta^2}{2}.
\]

This yields (HV2) for \( d = 4 \) (colored noise of type (3)).

### 4. Proof of the Theorems

**Proof of Theorem 4.** The existence of the unique invariant measure and the exponential convergence in the sense of \( V_n \)-uniform ergodicity of its semi-group are obtained by application of Theorem 5.1 (page 1679) of [9]. Indeed, our Lemma 2 provides a Lyapunov function \( V_n \) satisfying (HV1)–(HV2), which implies (since \( V_n \) is unbounded off petite sets) that there exist two constant \( b > 0 \) and \( c > 0 \), and a Borel petite set \( \mathcal{K} \), such that

\[
(A_{n} - c)\eta_n(z) = -c V_n(z) + b 1\mathcal{K}(z), \quad \forall z \in \mathbb{R}^d.
\]

To complete the proof of Theorem 4, it remains to show that the limiting distribution has a density with respect to Lebesgue measure on \( \mathbb{R}^d \), denoted here \( \lambda \). Let \( B \) be a Borel subset of \( \mathbb{R}^d \) such that \( \lambda(B) = 0 \). Since the process is \( V_n \)-uniformly ergodic, there exist a probability measure \( \mu_n \) and two constants \( D > 0, \rho \in (0, 1) \) such that, for all \( t \geq 0 \) and \( z \in \mathbb{R}^d \),

\[
V_n(z)D^{\rho t} \geq \| P_{n,t}(\cdot, \cdot) - \mu_n \| V_n = \sup_{|g| \leq V_n} \left| \int (P_{n,t}(z, \cdot) - \mu_n)(dz')g(z') \right| \geq |P_{n,t}(z, B) - \mu_n(B)| \geq |\lambda(B) - \mu_n(B)| - |P_{n,t}(z, B) - \lambda(B)| \geq |\mu_n(B)|.
\]

For the second inequality, we take \( g = 1_B \), which is smaller than \( V \) since \( V \geq 1 \). For the last inequality we use the fact that \( P_{n,t}(\cdot, \cdot) \) is absolutely continuous with respect to \( \lambda \). Letting \( t \to \infty \), we obtain \( \mu_n(B) = 0 \). Hence \( \mu_n \) too is absolutely continuous with respect to \( \lambda \). \( \Box \)
Proof of Theorem 5. The results are readily obtained by application of Theorem 3.2 (page 924) of [13]. Indeed, again our Lemma 2 provides a Lyapunov function $V_n$ satisfying (HV1)-(HV2), which implies the Foster-Lyapunov drift condition of [9], that is, there exist a function $f : \mathbb{R}^d \to [1, +\infty)$, a Borel petite set $\mathcal{K}$, and a constant $b < +\infty$ such that

\[(\mathcal{D}') \quad A_n V_n(z) \leq -f(z) + b 1_{\mathcal{K}}(z), \quad \forall z \in \mathbb{R}^d. \]

\[\square\]

Proof of Theorem 6. The results are readily obtained by application of Theorem 4.4 (page 928) of [13]. Indeed, again our Lemma 2 provides a Lyapunov function $V_n$ satisfying (HV1)-(HV2), which implies that $V_n$ satisfies condition (20) of [13], namely: there exist two constants $b > 0$ and $c > 0$, and a Borel petite set $\mathcal{K}$, such that $(\mathcal{D})$ holds. \[\square\]

5. Application to the penalization of the friction problem with white noise

Our results can be used to study, for a broad class of functions $g$, quantities of the form

\[(32) \quad W^n_g(b, T) = \mathbb{P}\left( \max_{0 \leq t \leq T} |\Delta^n_g(t)| \geq b \right), \quad \text{where } \Delta^n_g(t) = \int_0^t g(Z^n_s) ds. \]

Indeed, we have

\[(33) \quad \lim_{p \to \infty} W^n_p(\sqrt{p} b, pT) = W^* \left( \frac{b}{\gamma^n_g}, T \right) \]

where $W^*(b, T)$ denotes the probability that the absolute value of a standard Wiener process crosses the threshold $b > 0$ before time $T > 0$, which is known to be given by

\[(34) \quad W^*(b, T) = 1 - \frac{4}{\pi} \sum_{k=0}^{+\infty} \frac{(-1)^k}{2k+1} \exp \left( -\frac{(2k+1)^2 \pi^2 T}{8b^2} \right), \]

and the coefficient $\gamma^n_g$ is obtained by combining the invariant probability measure of $Z^n$ with the solution to its Poisson equation, as shown in Theorem 6. Regarding the non-smooth models, so far, such an asymptotic formula was available only in the elasto-perfectly-plastic setting with white noise [10]. For the sake of illustration, we consider the example of the penalization of the friction problem shown in (FP) with $n$ large. For simplicity we consider the white noise case, and we take $C_b = 1$ and $U = 0$ so the problem becomes one-dimensional (see [4] for more details.)

5.1. Parabolic equation for $\gamma^n_g$. From the Feynman-Kac formula, the quantity

\[w^n_g(y, \tau) = \mathbb{E}_y \left[ \int_0^\tau g(Y^n_s) ds \right]\]

is known to satisfy the linear parabolic equation

\[(35) \quad \partial_\tau w^n_g(y, \tau) - \frac{1}{2} \partial_{yy} w^n_g(y, \tau) + (y + a'_n(y)) \partial_y w^n_g(y, \tau) = g(y, \tau), \quad \forall (y, \tau) \in \mathbb{R} \times [0, \infty), \]

subject to the initial condition $w^n_g(y, 0) = 0$. Similarly, the quantity

\[(36) \quad v^n_g(y, \tau) = \mathbb{E}_y \left[ \left( \int_0^\tau [g(Y^n_s) - \mathbb{E}[g(Y^n_s)]] ds \right)^2 \right]\]

satisfies a similar equation but with $|\partial_y w^n_g|^2$ as a different right-hand side, namely,

\[(37) \quad \partial_\tau v^n_g(y, \tau) - \frac{1}{2} \partial_{yy} v^n_g(y, \tau) + (y + a'_n(y)) \partial_y v^n_g(y, \tau) = |\partial_y w^n_g(y, \tau)|^2, \quad \forall (y, \tau) \in \mathbb{R} \times [0, \infty), \]

subject to the initial condition $v^n_g(y, 0) = 0$. \[\square\]
Then, from the ergodic property,
\begin{equation}
\forall y \in \mathbb{R}, \quad u^n_y(y, \tau) \sim \left( \int_{\mathbb{R}} g(y) m_n(y) \, dy \right), \quad \text{as} \quad \tau \to \infty
\end{equation}
and
\begin{equation}
\forall y \in \mathbb{R}, \quad v^n_y(y, \tau) \sim (\gamma^n_g)^2 \tau, \quad \text{as} \quad \tau \to \infty.
\end{equation}

The key point for applications, is that we can use (33) to approximate probabilities of threshold crossings such as (32) by using the explicit formula (34) where an estimate of $\gamma^n_g$ is obtained via a solution to the PDE (37).

5.2. Numerics. To illustrate how our approach can be used, we present here numerical results for the case where $g(y) = y$. A finite difference method is implemented in MATLAB to solve equations (35) and (37), see Appendix C. Figure 2A shows that the results from the PDE method and the Monte Carlo method are consistent. We used a logarithmic scale for time due to the exponential convergence rate towards the asymptotic value (roughly 1.36 here). For the Monte Carlo simulations we used $10^6$ samples. Moreover, Figure 2B illustrates the convergence (33): as $p$ increases, $W^n_g(\sqrt{\bar{p}} b, p T)$ converges to the value given by the analytical formula $W_\tau$. For the Monte Carlo simulations we used $10^5$ samples. Table 4 shows the numerical the convergence of $v^n_g(0, T)$ as $n$ increases.

![Illustration of (39)](A) Illustration of (39): the solid line represents $v^n_g(0, \tau)$, while the dashed and dotted lines represent $W^n_g(\sqrt{\bar{p}} b, p T)$, for 4 different values of $p$, while the solid line represents $W_\tau(b/\gamma^n_g, T)$. Here $b = 0.6, T = 20$ and we used $10^5$ Monte Carlo samples.

![Illustration of (33)](B) Illustration of (33): the dashed and dotted lines represent $W^n_g(\sqrt{\bar{p}} b, p T)$, for 4 different values of $p$, while the solid line represents $W_\tau(b/\gamma^n_g, T)$. Here $b = 0.6, T = 20$ and we used $10^5$ Monte Carlo samples.

**Figure 2.** Numerical results for $(\mathbf{FP}_n)$ with white noise. Here, $g$ is the identity and $n = 100$.

**Table 4.** Convergence of $v^n_g(0, T)$ as $n$ increases, with $T = 100$.

| $n$   | 2   | 5   | 10  | 50  | 100 | 1000 |
|-------|-----|-----|-----|-----|-----|------|
| $v^n_g(0, T)$ | 0.174466 | 0.143272 | 0.138434 | 0.136834 | 0.136784 | 0.136767 |
6. Conclusion and related questions to be investigated

In this work, we have proposed approximations for three non-smooth dynamical systems subjected to different kinds of noise. We have proved existence of Lyapunov functions, from which we have deduced ergodicity. This yields in particular a method to approximate probabilities of threshold crossing of quantities of interest for e.g. engineering and physics. The main ingredient of this work, Lemma 2, crucially relies on the density estimates result of [8]. From then on, a natural question is whether we can obtain such a result for the non-smooth problems. In fact, the aforementioned density estimates result cannot be employed and thus a corresponding version of Lemma 2 cannot be obtained using a similar approach. An other natural question to investigate is the behavior of our results as \( n \) goes to \( \infty \). We know [17] that for the elasto-plastic problem, the solution of \((EPP_n)\) converges, as \( n \) goes to \( \infty \), towards the solution of \((EPP)\) in the sense of the following norm \( \|X\| = \mathbb{E} \left( \sup_{0 \leq t \leq T} |X(t)|^2 \right) \), as a consequence

\[
\lim_{n \to \infty} W^n(b, T) = W(b, T).
\]

However, it is not clear for us in how to prove (and in what sense) the convergence of the solutions of \((FP_n)\) and \((OP_n)\) towards the solutions of \((FP)\) and \((OP)\) respectively. Yet, Figures 3, 4, and 5 suggest that a result of the type of (40) remains true. These issues remain to be investigated by other techniques and are beyond the scope of the present paper.

Appendix A. \((SVI - 1)\) and \((SVI - 2)\) and \((P_n)\) for Elasto-plastic, Obstacle and Friction Problems

A.1. Model definition of an elasto-plastic problem. The so-called elasto-plastic oscillator [15] can be explained as follows. The total deformation is described by \( X_{\text{Tot}}^t \in \mathbb{R} \) and its velocity by \( Y_t \neq X_{\text{Tot}}^t \in \mathbb{R} \). For the elasto-perfectly-plastic oscillator (EPPO) model, the irreversible (plastic) deformation \( \Delta \) and the reversible (elastic) deformation \( X_t \) at time \( t \) satisfy

\[
\begin{align*}
\text{in elastic phase:} & \quad dX_t = dX_{\text{Tot}}^t, \quad d\Delta_t = 0, \\
\text{in plastic phase:} & \quad d\Delta_t = dX_{\text{Tot}}^t, \quad dX_t = 0,
\end{align*}
\]

while \( X_{\text{Tot}}^t = X_t + \Delta_t \). Typically, \( |X_t| \) remains bounded by a given threshold \( P_Y \) at all time \( t \), plastic phase occurs when \( |X_t| = P_Y \) and elastic phase when \( |X_t| < P_Y \). Here \( P_Y \) is an elasto-plastic bound (known as the "Plastic Yield" in the engineering literature). Also, the permanent (plastic) deformation at time \( t \) can then be written as

\[
\Delta_t = \int_0^t 1_{\{|X_s| = P_Y\}} Y_s ds.
\]

Due to the switching of regimes from an elastic phase to a plastic one, or vice versa, it is a non-smooth dynamical system. We take \( P_Y = 1 \) for simplicity.

SVI framework. It has been shown that the dynamics of a nonlinear oscillator can be described mathematically by means of SVIs [3]. The dynamics is then described by the pair \((X_t, Y_t)\) that satisfies

\[
\begin{cases}
\dot{Y}_t = N_t - (U'(X_t) + C_b Y_t), \\
(\dot{X}_t - Y_t)(\zeta - X_t) \geq 0, \forall |\zeta| \leq 1, \quad |X_t| \leq 1,
\end{cases}
\]

and appropriate initial conditions for \( Y_0 \) and \( X_0 \) must be prescribed. \( U \) is a certain type of confining potential.
Penalized framework. The penalized version of (EPP) is:

\[
\begin{align*}
\text{(EPP)}_n & \quad \begin{cases} 
\dot{Y}^n_t = N_t - (U'(X^n_t) + C_b Y^n_t), \\
\dot{X}^n_t = Y^n_t - n (X^n_t - \text{proj}_{[-1,1]}(X^n_t)).
\end{cases}
\end{align*}
\]

The penalization affects the elastic deformation \(X^n\). See Figure 3. As \(n\) tends to \(\infty\), the penalization, acting on \(\dot{X}\), enforces \(X\) to remain between \(-1\) and \(+1\).

A.2. Model definition of an obstacle problem. It is common in the engineering literature [1] to formulate the dynamics of a stochastic obstacle oscillator in terms of a stochastic process \(X_t\), the oscillator displacement, which evolves constrained by obstacles located at \(|X| = P_O\) (position of the obstacle). For general obstacle problems, the non-smooth behavior in such models comes from the collisions in the sense that if at a time \(t\), the state hits the obstacle with incoming velocity \(\dot{X}_t\), it immediately bounces back with velocity \(-\dot{X}_t\), that is, \(\dot{X}_{t+} = -\dot{X}_{t-}\).

SVI framework. From a mathematical viewpoint, the dynamics of such a nonlinear oscillator can be described in the framework of SVIs [2] as follows

\[
\text{(OP)} \quad \begin{cases} 
\dot{Y}_t = \dot{X}_t, \\
(\dot{Y}_t - N_t + U'(X_t) + C_b Y_t)(\zeta - X_t) \geq 0, \quad \forall |\zeta| \leq P_O, \quad |X_t| \leq P_O,
\end{cases}
\]

supplemented by the impact rule: \(Y_{t+} = -Y_{t-}\).

Penalized framework. We take \(P_O = 1\) to simplify. Then, the penalized version of \(\text{(OP)}\) is:

\[
\text{(OP)}_n \quad \begin{cases} 
\dot{X}^n_t = Y^n_t, \\
\dot{Y}^n_t = N_t - (U'(X^n_t) + C_b Y^n_t) - n (X^n_t - \text{proj}_{[-1,1]}(X^n_t)).
\end{cases}
\]

In contrast with \(\text{(EPP)}\), note that here the penalization affects the velocity \(Y\). See Figure 4. As \(n\) tends to \(\infty\), the penalization, acting on \(\dot{Y}\), enforces \(X\) to remain between \(-1\) and \(+1\).

A.3. Stick-slip friction. A noise driven particle subject to friction can be expressed in terms of a stochastic process \(X_t\) for the displacement and \(Y_t = \dot{X}_t\) for its velocity. The non-smooth behavior comes from phase transitions between sticky and slippy phases.

SVI framework. From a mathematical viewpoint, the dynamics of such a friction particle can be described in the framework of SVIs [4] as follows

\[
\text{(FP)} \quad \begin{cases} 
\dot{X}_t = Y_t, \\
(\dot{Y}_t - N_t + U'(X_t) + C_b Y_t)(\varphi - Y_t) \geq P_F(|Y_t| - |\varphi|), \quad \forall \varphi \in \mathbb{R}.
\end{cases}
\]

Here \(P_F\) is the amplitude of the friction force. Again we take \(P_F = 1\) for simplicity.

Penalized framework. The penalized version of \(\text{(FP)}\) is:

\[
\text{(FP)}_n \quad \begin{cases} 
\dot{X}^n_t = Y^n_t, \\
\dot{Y}^n_t = N_t - (U'(X^n_t) + C_b Y^n_t) - a_n'(Y^n_t).
\end{cases}
\]

Here the penalization affects the velocity \(Y^n\). See Figure 5. For each \(n\), the penalization term, \(a_n'(Y^n)\), acts as a term reverting \(Y^n\) to 0 (like a damping force). When is large, \(Y^n\) remains 0 most of the time. As \(n\) tends to \(\infty\), \(a_n'(y)\) tends to sign\((y) \in \{-1, +1\}\).
Remark 11. (1) Note that the white noise driven (OP$_n$) model can be written as
\[ dY^n_t = -(U'(X^n_t) + C_b Y^n_t) dt + dW_t, \quad dX^n_t = Y^n_t dt \]
where $U_n(x) \equiv U(x) + \frac{n}{2} x - \text{proj}_{[-1,1]}(x)^2$. Thus, using the notation $H_n(x,y) \equiv \frac{1}{2} |y|^2 + U_n(x)$, the invariant probability density function of $(X_t, Y_t)$ is
\[ m_n(x,y) = C_n^{-1} \exp(-C_b H_n(x,y)), \quad C_n = \int_{\mathbb{R}^2} \exp(-\gamma H_n(x,y)) \, dx \, dy. \]
Here $H_n$ does not satisfies Hypothesis 1.1 of [26] but the calculations for $m_n(x,y)$ are explicit.

(2) The white noise driven (FP$_n$) model has a Hamiltonian structure in the following sense
\[ dY^n_t = -\big(U'(X^n_t) + F_n(Y^n_t)\big) Y^n_t \, dt + dW_t, \quad dX^n_t = Y^n_t \, dt \]
where
\[ F_n(y) = \begin{cases} C_b + \frac{1}{|y|}, & \text{if } |y| > \frac{1}{n} \\ C_b + n, & \text{if } |y| \leq \frac{1}{n}. \end{cases} \]
In spite of the Hamiltonian structure, [26] cannot be applied because $F_n$ does not satisfy Hypothesis 1.1 of [26] (it is not of class $C^\infty$). As a consequence we have to resort to another technique which was previously employed for the white noise driven (EPP$_n$) model [17] (with $U(x) \equiv k \frac{2x}{x^2}$).

(3) We also observe that (EPP$_n$) model does not even have a Hamiltonian structure.

Appendix B. Numerical simulation of trajectories

Below we provide a numerical illustration of the trajectory of $Z^n = (Y^n, X^n)$ in the white noise case for each problem (see Table 2), for different values of $n$.

![Figure 3: Elasto-plastic problem: empirical convergence with respect to $n$.](image)

Appendix C. Discretization of the partial differential equations (35)

To numerically approximate the solutions of (35), we use a finite difference scheme. We discretize the time with $t_k = k \Delta t$. We truncate the domain $\mathbb{R}$ into an interval $[-L, L]$ where $L$ is chosen sufficiently large that the probability of finding $Y^n_t$ outside $[-L, L]$ is negligible. We apply a homogeneous Neumann boundary condition, at $y = \pm L$. We consider a one-dimensional finite difference grid: \( G \equiv \{ y_i = -L + (i - 1) \Delta y, 1 \leq i \leq N \} \) with $\Delta y = \frac{2L}{N-1}$. The total number of nodes in $G$ is $N$. The numerical approximations of $w^n_i(y_i, t_k)$ and $v^n_i(y_i, t_k)$ are denoted by $W^k$ and $V^k$ and the corresponding vectors are $W$ and $V$ (to alleviate the notations we drop the dependence on $g$). We also define $G_i = g(y_i)$ and denote by $G$ the corresponding vector.
Figure 4. Obstacle problem: empirical convergence with respect to $n$.

Figure 5. Friction problem: empirical convergence with respect to $n$.

C.1. Finite differences. For every $2 \leq i \leq N - 1$, at the point $y_i$ and a time $t_k$, $k \geq 1$, the discrete formulation of the parabolic equations (35) results in

$$\frac{W_i^k - W_i^{k-1}}{\Delta t} - \frac{1}{2} \left[ \frac{W_{i+1}^k - 2W_i^k + W_{i-1}^k}{(\Delta y)^2} \right] + \left( y_i + a_n'(y_i) \right) \frac{W_{i+1}^k - W_{i-1}^k}{2\Delta y} = G_i$$

The Neumann boundary conditions at the points $y_1$ and $y_N$ and the time $t_n$ $n \geq 1$ results in

$$\frac{W_N^k - W_{N-1}^k}{\Delta y} = 0 \quad \text{and} \quad \frac{W_2^k - W_1^k}{\Delta y} = 0.$$

Similarly, the same discrete equation is satisfied for $V_i$ with $G_i$ replaced by $|(D W_i^k)|^2$ where

$$(D W_i^k)_i = \frac{W_{i+1}^k - W_{i-1}^k}{2\Delta y}.$$

This results in the following linear systems to be solved for time $t_k$

$$M_{\Delta t} U^k = G_{\Delta t}^k \quad \text{and} \quad M_{\Delta t} V^k = H_{\Delta t}^k$$

where, $M_{\Delta t}$ is a $N \times N$ matrix containing at most three non zeros entries per row. Precisely,

$$(M_{\Delta t})_{1,1} = \frac{1}{\Delta y}, \quad (M_{\Delta t})_{1,2} = -\frac{1}{\Delta y},$$

$$(M_{\Delta t})_{i,i-1} = -\frac{\Delta t}{2} \left( \frac{1}{(\Delta y)^2} + \frac{b(y_i)}{\Delta y} \right), \quad (M_{\Delta t})_{i,i} = 1 + \frac{\Delta t}{(\Delta y)^2},$$

$$(M_{\Delta t})_{i,i+1} = -\frac{\Delta t}{2} \left( \frac{1}{(\Delta y)^2} - \frac{b(y_i)}{\Delta y} \right),$$
and

\((M_{\Delta t})_{N,N-1} = \frac{1}{\Delta y}, \quad (M_{\Delta t})_{N,N} = -\frac{1}{\Delta y}\)

Also,

\((G_{\Delta t}^k)_1 = (G_{\Delta t}^k)_N = 0 \quad \text{and} \quad (G_{\Delta t}^k)_i = \Delta t G_i + U_i^k, \quad 2 \leq i \leq N - 1.\)

Similarly,

\((H_{\Delta t}^k)_1 = (H_{\Delta t}^k)_N = 0 \quad \text{and} \quad (H_{\Delta t}^k)_i = \Delta t (D U_i^k)^2 + V_i^k, \quad 2 \leq i \leq N - 1.\)

We use a MATLAB implementation (available upon request).

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