NOTES ON CONVEX FUNCTIONS OF ORDER $\alpha$

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Abstract. Marx and Strohhäcker showed around in 1933 that $f(z)/z$ is subordinate to $1/(1 - z)$ for a normalized convex function $f$ on the unit disk $|z| < 1$. Brickman, Hallenbeck, MacGregor and Wilken proved in 1973 further that $f(z)/z$ is subordinate to $k_\alpha(z)/z$ if $f$ is convex of order $\alpha$ for $1/2 \leq \alpha < 1$ and conjectured that this is true also for $0 < \alpha < 1/2$. Here, $k_\alpha$ is the standard extremal function in the class of normalized convex functions of order $\alpha$ and $k_0(z) = z/(1 - z)$. We prove the conjecture and study geometric properties of convex functions of order $\alpha$. In particular, we prove that $(f + g)/2$ is starlike whenever $f$ and $g$ both are convex of order $3/5$.

1. Introduction and main result

Let $\mathcal{A}$ denote the set of analytic functions on the open unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. Let $\mathcal{A}_1$ be the subclass of $\mathcal{A}$ consisting of functions $f$ normalized by $f(0) = f'(0) - 1 = 0$. Further let $\mathcal{S}$ be the subset of $\mathcal{A}_1$ consisting of functions $f$ univalent on $\mathbb{D}$. The present paper mainly deals with the subfamily of $\mathcal{S}$, denoted by $\mathcal{K}(\alpha)$, consisting of convex functions of order $\alpha$ introduced by Robertson [8]. Here, for a constant $0 \leq \alpha < 1$, a function $f$ in $\mathcal{A}_1$ is called convex of order $\alpha$ if
\[ \Re \left( 1 + \frac{zf''(z)}{f'(z)} \right) > \alpha \]
for $z \in \mathbb{D}$. Note that the class $\mathcal{K}(0) = \mathcal{K}$ is known to consist of convex functions in $\mathcal{A}_1$. Here, a function $f$ in $\mathcal{A}$ is called convex if $f$ maps $\mathbb{D}$ univalently onto a convex domain. A function $f \in \mathcal{A}$ is called starlike if $f$ maps $\mathbb{D}$ univalently onto a domain starlike with respect to $f(0)$. It is clear that every convex function is starlike. We denote by $\mathcal{S}^*$ the set of starlike functions in $\mathcal{A}_1$. By definition, it is obvious that for $0 \leq \alpha < \beta < 1$,
\[ \mathcal{K}(\beta) \subset \mathcal{K}(\alpha) \subset \mathcal{K} \subset \mathcal{S}^* \subset \mathcal{S}. \]

The Koebe function $z/(1 - z)^2$ is often extremal in $\mathcal{S}^*$ or even in $\mathcal{S}$ and thus plays quite an important role in the theory of univalent functions. It is helpful in many respects to have such an extremal function for the class $\mathcal{K}(\alpha)$. Since the function $(1 + (1 - 2\alpha)z)/(1 - z)$ maps $\mathbb{D}$ univalently onto the half-plane $\Re w > \alpha$, indeed, the function $k_\alpha \in \mathcal{K}(\alpha)$ characterized by the following relations serves as an extremal one:
\[ 1 + \frac{zk_\alpha''(z)}{k_\alpha'(z)} = \frac{1 + (1 - 2\alpha)z}{1 - z}, \quad \text{and} \quad k_\alpha(0) = 0, \ k_\alpha'(0) = 1. \]

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Theorem 1.3. The following more refined result. (Brickman et al.)

We say that $D$ is subordinated to $G$ and write $f \prec G$ or $f(z) \prec G(z)$ for it if there is an analytic function $\omega$ on $\mathbb{D}$ such that $\omega(0) = 0$ and $f(z) = g(\omega(z))$ for $z \in \mathbb{D}$. When $g$ is univalent, $f$ is subordinate to $g$ precisely if $f(0) = g(0)$ and $f(\mathbb{D}) \subset g(\mathbb{D})$.

In 1973, Brickman, Hallenbeck, MacGregor and Wilken proved in [2, Theorem 11] the following result for convex functions of order $\alpha$.

**Theorem A** (Brickman et al.). If $f \in K(\alpha)$ for $1/2 \leq \alpha < 1$, then

$$
\frac{f(z)}{z} \prec \frac{k_\alpha(z)}{z} \quad \text{on } \mathbb{D}.
$$

We note that $k_0(z)/z = 1/(1 - z)$ maps $\mathbb{D}$ univalently onto the half-plane $\text{Re } w > 1/2$. Thus the above relation also holds when $\alpha = 0$ by a theorem of Marx and Strohhäcker (see [2, Theorem 10]). In [2], they conjectured that the assertion of Theorem A would hold for $0 < \alpha < 1/2$ as well. They also observed that the conjecture is confirmed if one could show that the function $k_\alpha(z)/z$ is convex. They prove the last theorem by showing it for $1/2 \leq \alpha < 1$ (cf. [2, Lemma 3]). We will show it for all $\alpha$.

**Theorem 1.1.** The function $h_\alpha(z) = k_\alpha(z)/z$ maps $\mathbb{D}$ univalently onto a convex domain for each $0 \leq \alpha < 1$.

We remark that, in the context of the hypergeometric function, this follows also from results of Küstner in [5] (see the remark at the end of Section 2 for more details). Anyway, the conjecture has been confirmed:

**Corollary 1.2.** Let $0 \leq \alpha < 1$. Then, for $f \in K(\alpha)$, the following subordination holds:

$$
\frac{f(z)}{z} \prec \frac{k_\alpha(z)}{z} \quad \text{on } \mathbb{D}.
$$

In view of the form, it is easy to see that $k_\alpha$ is bounded on $\mathbb{D}$ if and only if $\alpha > 1/2$. By analyzing the shape of the image of $\mathbb{D}$ under the mapping $h_\alpha(z) = k_\alpha(z)/z$, we obtain the following more refined result.

**Theorem 1.3.** Let $0 \leq \alpha < 1$ and $f \in K(\alpha)$. Then the following hold:

(i) $\frac{k_\alpha(-r)}{r} \leq \text{Re } \frac{f(z)}{z} \leq \frac{k_\alpha(r)}{r}$ for $|z| = r < 1$.

(ii) When $0 < \alpha < 1/2$, the asymptotic lines of the boundary curve of $h_\alpha(\mathbb{D})$ are given by $v = \pm \cot(\pi\alpha)(u - \frac{1}{2\alpha - 1})$. In particular, the values of $f(z)/z$ for $z \in \mathbb{D}$ are contained in the sector $S = \{u + iv : |v| < \cot(\pi\alpha)(u - \frac{1}{2\alpha - 1})\}$.

(iii) When $1/2 \leq \alpha < 1$,

$$
|\text{Im } \frac{f(z)}{z}| < M(\alpha), \quad z \in \mathbb{D},
$$
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where

$$M(\alpha) = \max_{0 < \theta < \pi} \text{Im} \left[ e^{-i\theta} k_\alpha(e^{i\theta}) \right] \leq M(\frac{1}{2}) = \frac{\pi}{2}.$$ 

The estimate is sharp.

We remark that the left-hand inequality in (i) was already proved by Brickman et al. [2, Theorem 10] and the right-hand one follows also from Robertson’s theorem (see Lemma 3.1 below). A much simpler proof of (i) is now available thanks to Corollary 1.2. The proof of this theorem and more information about the constant $M(\alpha)$ will be given in Section 3. We also provide an application of our results to an extremal problem for $K(\alpha)$ in Section 3.

Styer and Wright [10] studied (non-)univalence of a convex combination of two convex functions. Among other things, the following result is most relevant to the present study.

**Theorem B** (Styer and Wright). Let $f, g \in K$ be odd convex functions. If $|\text{Im} \left[ f(z)/z \right]| < \pi/4$ and $|\text{Im} \left[ g(z)/z \right]| < \pi/4$ on $|z| < 1$, then $(f + g)/2 \in S^*$.

Styer and Wright suspected that the assumption $|\text{Im} \left[ f(z)/z \right]| \leq \pi/4$ in the theorem was superfluous. They even stated the belief that

\begin{equation}
\frac{f(z)}{z} \prec H_2(z) := \frac{1}{2z} \log \frac{1+z}{1-z} = \sum_{n=0}^{\infty} \frac{z^{2n}}{2n+1}
\end{equation}

if $f \in K$ is odd; namely, $f(-z) = -f(z)$. Note that $|\text{Im} H_2(z)| \leq \pi/4$ on $|z| < 1$. Indeed, Hallenbeck and Ruscheweyh [4] proved that

\begin{equation}
\frac{f(z)}{z} \prec H_1(z) := \frac{1}{2z} \log \frac{1+\sqrt{z}}{1-\sqrt{z}} = \sum_{n=0}^{\infty} \frac{z^n}{2n+1}
\end{equation}

for a function $f \in K$ with $f''(0) = 0$, which implies that $|\text{Im} \left[ f(z)/z \right]| \leq \pi/4$. In this way, they strengthened the above theorem (see [4, Corollary 2]):

**Theorem C** (Hallenbeck and Ruscheweyh). Let $f, g \in K$ satisfy $f''(0) = g''(0) = 0$. Then $(f + g)/2 \in S^*$.

We give another result of this type.

**Theorem 1.4.** $(f + g)/2 \in S^*$ for $f, g \in K(0.6)$.

The proof will be given in Section 4. Note that the constant $0.6 = 3/5$ is not best possible.

We remark that the claim (1.1) for an odd convex function $f$ is not necessarily true. An example will be given in Section 5.

2. PROOF OF THEOREM 1.4

We now show that the function $h_\alpha(z) = k_\alpha(z)/z$ is convex (univalent) on $\mathbb{D}$ for each $0 \leq \alpha < 1$. To this end, we only need to see that $1 + zh_\alpha''(z)/h_\alpha'(z)$ has positive real
part. Since the case $\alpha = 0$ is trivial, we assume that $\alpha > 0$. Put $\beta = 2 - 2\alpha \in (0, 2)$ for convenience. We assume $\alpha \neq 1/2$ so that $\beta \neq 1$ for a while. A simple calculation yields
\[
h'_\alpha(z) = \frac{(1 - \beta z)(1 - z)^{-\beta} - 1}{(1 - \beta)z^2}
\]
and
\[
1 + \frac{zh''_\alpha(z)}{h'_\alpha(z)} = -1 - \frac{\beta(1 - \beta)z^2}{((1 - z)^{\beta} - 1 + \beta z)(1 - z)}.
\]

With the Pochhammer symbol $(a)_n = a(a + 1) \cdots (a + n - 1)$, we compute
\[
(1 - z)^\beta - 1 + \beta z = \sum_{n=2}^{\infty} \frac{(-\beta)_n z^n}{(1)_n}
\]
\[
= -\beta(1 - \beta)z^2 \sum_{n=2}^{\infty} \frac{(2 - \beta)_{n-2} z^{n-2}}{(3)_{n-2}}
\]
\[
= -\beta(1 - \beta)z^2 \sum_{n=0}^{\infty} \frac{(2 - \beta)_{n} z^{n}}{(3)_{n}}.
\]

Letting $b_n = (2 - \beta)_n/(3)_n$ for $n \geq 0$, we obtain
\[
-\frac{((1 - z)^\beta - 1 + \beta z)(1 - z)}{\beta(1 - \beta)z^2} = \frac{1 - z}{2} \sum_{n=0}^{\infty} b_n z^n
\]
\[
= \frac{1}{2} \left( 1 + \sum_{n=1}^{\infty} (b_n - b_{n-1}) z^n \right)
\]
\[
= \frac{1 + \omega(z)}{2},
\]
where
\[
\omega(z) = \sum_{n=1}^{\infty} (b_n - b_{n-1}) z^n.
\]

Hence, we have the expression
\[
1 + \frac{zh''_\alpha(z)}{h'_\alpha(z)} = -1 + \frac{2}{1 + \omega(z)} = \frac{1 - \omega(z)}{1 + \omega(z)}.
\]

Note that this is valid also for $\alpha = 1/2$ as is confirmed directly or by taking limit as $\alpha \to 1/2$.

In order to show $\text{Re} \left( 1 + \frac{zh''_\alpha(z)}{h'_\alpha(z)} \right) > 0$, it suffices to check $|\omega(z)| < 1$. Since
\[
\frac{b_n}{b_{n-1}} = \frac{n + 1 - \beta}{n + 2} = 1 - \frac{1 + \beta}{n + 2} < 1,
\]
we see that $\{b_n\}$ is a decreasing sequence of positive numbers. Therefore,
\[
|\omega(z)| \leq \sum_{n=1}^{\infty} (b_{n-1} - b_n) |z|^n < \sum_{n=1}^{\infty} (b_{n-1} - b_n) = b_0 - \lim_{n \to \infty} b_n \leq b_0 = 1
\]
for $z \in \mathbb{D}$ as required. (Indeed, we can easily show that $b_n \to 0$ as $n \to \infty$.)
We remark that the function $k_\alpha$ can be expressed in terms of the Gauss hypergeometric function

$$2F_1(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \cdot \frac{z^n}{n!}.$$ 

Indeed, by integrating both sides of

$$k'_\alpha(z) = (1-z)^{-\beta} = \sum_{n=0}^{\infty} (\beta)_n \frac{z^n}{n!}$$

with $\beta = 2 - 2\alpha$, we obtain

$$k_\alpha(z) = \sum_{n=0}^{\infty} \frac{(\beta)_n}{n+1} \cdot \frac{z^{n+1}}{n!} = z \sum_{n=0}^{\infty} \frac{(\beta)_n (1)_n}{(2)_n} \cdot \frac{z^n}{n!},$$

and hence

$$h_\alpha(z) = \frac{k_\alpha(z)}{z} = 2F_1(\beta, 1; 2; z).$$

We extract the following result from Küstner’s theorems in [5] (Theorem 1.1 with $r = 1$ and Remark 2.3, see also Corollary 6 (a) in [6]).

**Lemma 2.1** (Küstner). For non-zero real numbers $a, b, c$ with $-1 < a \leq b < c$, let $F(z) = 2F_1(a, b; c; z)$. Then

$$\inf_{z \in \mathbb{D}} \left( 1 + \frac{zF''(z)}{F'(z)} \right) = 1 + \frac{-F''(1)}{F'(1)} \geq 1 - \frac{(a+1)(b+1)}{b+c+2}$$

Since $2F_1(a, b; c; z) = 2F_1(b, a; c; z)$, we can apply the above lemma to our function $h_\alpha(z) = 2F_1(\beta, 1; 2; z)$ for $0 < \alpha < 1$; equivalently, for $0 < \beta < 2$. Hence, by (2.1), we obtain

$$\inf_{z \in \mathbb{D}} \left( 1 + \frac{zh''(z)}{h'(z)} \right) = 1 - \frac{h''(1)}{h'(1)} = \frac{2\beta + 1 - 2 - \beta - \beta^2}{2(1 + \beta - 2\beta)} \geq \begin{cases} \frac{4\alpha - 1}{5}, & 1/2 \leq \alpha < 1, \\ \frac{\alpha}{3 - \alpha}, & 0 < \alpha \leq 1/2. \end{cases}$$

In this way, we have obtained another proof of convexity of $h_\alpha$.

### 3. Mapping properties of functions in $K(\alpha)$

The present section is devoted to the proof of Theorem 1.3. Before the proof, we note basic results due to Robertson [8] (see also Pinchuk [7]).

**Lemma 3.1** (Robertson). Let $0 \leq \alpha < 1$ and $f \in K(\alpha)$. Then,

$$-k_\alpha(-r) \leq |f(z)| \leq k_\alpha(r) \quad \text{for } |z| = r < 1.$$ 

In particular, the image domain $f(\mathbb{D})$ contains the disk $|w| < -k_\alpha(-1)$.

We will use also the following simple fact.
Lemma 3.2. Let $\Omega$ be an unbounded convex domain in $\mathbb{C}$ whose boundary is parametrized positively by a Jordan curve $w(t) = u(t) + iv(t)$, $0 < t < 1$, with $w(0^+) = w(1^-) = \infty$. Suppose that $u(0^+) = +\infty$ and that $v(t)$ has a finite limit as $t \to 0^+$. Then $v(t) \leq v(0^+)$ for $0 < t < 1$.

Proof. Let $0 \leq t^* \leq 1$ be the number such that $u(t^*) = \inf_{0 < t < 1} u(t)$ and that $u(t) > u(t^*)$ for $0 < t < t^*$. (We interpret $u(0) = u(0^+)$ or $u(1) = u(1^-)$ when $t^* = 0$ or $1$, respectively.) By the assumption $u(0^+) = +\infty$, we have $t^* > 0$. Note that $u(t)$ is strictly decreasing in $0 < t < t^*$. By convexity and orientation, the part $w((t^*,1))$ of the boundary lies below the part $w((0,t^*))$. Thus, it is enough to show that $v(t)$ is non-increasing in $0 < t < t^*$. Let $0 < t_0 < t_1 < t_2 < t^*$ and set $w(t_j) = u_j + iv_j$ for $j = 0, 1, 2$. By convexity, the part $w((t_0, t_2))$ of the boundary lies above the line which passes through the points $w(t_0)$ and $w(t_2)$; equivalently,

$$v(t) \geq v_2 + \frac{v_0 - v_2}{u_0 - u_2} (u(t) - u_2), \quad t_0 < t < t_2.$$ 

We now put $t = t_1$ and let $t_0 \to 0^+$ to obtain $v_1 = v(t_1) \geq v_2 = v(t_2)$. Thus we have shown that $v(t)$ is non-increasing as required. \qed

We are now ready to prove Theorem 1.3.

Proof of Theorem 1.3. Since $h_\alpha(z) = k_\alpha(z)/z$ is convex and symmetric in $\mathbb{R}$, we easily see that $h_\alpha(-r) \leq \Re h_\alpha(z) \leq h_\alpha(r)$ for $|z| = r < 1$. Therefore, assertion (i) immediately follows from Corollary 1.2.

To prove (ii) and (iii), we study mapping properties of the function $h_\alpha(z)$. We remark that $h_\alpha$ analytically extends to $\partial \mathbb{D} \setminus \{1\}$ by its form. Let us investigate the shape of the boundary of $h_\alpha(\mathbb{D})$. In the rest of this section, it is convenient to put $\gamma = 2\alpha - 1 \in [-1, 1)$. Note that $\gamma < 0$ if and only if $\alpha < 1/2$. We write $h_\alpha(e^{i\theta}) = u_\gamma(\theta) + iv_\gamma(\theta)$ for $0 < \theta < 2\pi$.

We remark that the symmetry $h_\alpha(\bar{z}) = \overline{h_\alpha(z)}$ leads to the relations $u_\gamma(2\pi - \theta) = u_\gamma(\theta)$ and $v_\gamma(2\pi - \theta) = -v_\gamma(\theta)$. Thus, we may restrict our attention to the range $0 < \theta \leq \pi$. It is easy to obtain the following expressions for $\gamma \neq 0$:

$$u_\gamma(\theta) = \frac{-1}{\gamma} \left( 2 \sin \frac{\theta}{2} \right)^\gamma \cos \left( -\theta + \frac{\theta - \pi}{2} \gamma \right) - \cos \theta,$$

$$v_\gamma(\theta) = \frac{-1}{\gamma} \left( 2 \sin \frac{\theta}{2} \right)^\gamma \sin \left( -\theta + \frac{\theta - \pi}{2} \gamma \right) + \sin \theta.$$

Observe that for $-1 < \gamma < 0$, both $u_\gamma(\theta)$ and $v_\gamma(\theta)$ tend to $+\infty$ as $\theta \to 0^+$. A simple calculation yields

$$\lim_{\theta \to 0^+} \frac{v_\gamma(\theta)}{u_\gamma(\theta)} = \tan \frac{-\pi \gamma}{2} = -\tan \frac{\pi \gamma}{2}$$

and

$$\lim_{\theta \to 0^+} \left( v_\gamma(\theta) + u_\gamma(\theta) \tan \frac{\pi \gamma}{2} \right) = \lim_{\theta \to 0^+} \left[ \frac{-(2 \sin \frac{\theta}{2})^\gamma \sin(\frac{\pi}{2} - 1) \theta}{\gamma \cos(\pi \gamma/2)} - \frac{1}{\gamma} \tan \frac{\pi \gamma}{2} \cos \theta \right]$$

$$= -\frac{1}{\gamma} \tan \frac{\pi \gamma}{2}.$$


Therefore,
\[ v = -\tan\frac{\pi\gamma}{2}\left(u - \frac{1}{\gamma}\right) \]

\[ = \cot(\alpha\pi)\left(u - \frac{1}{2\alpha - 1}\right) \]
is an asymptotic line of the boundary curve \(\partial h_\alpha(\mathbb{D})\). Since \(h_\alpha(\mathbb{D})\) is a convex domain symmetric in the real axis, we conclude assertion (ii).

Next we assume \(\alpha \geq 1/2\) to show (iii). Since \(f(z)/z < k_\alpha(z)/z < k_{1/2}(z)/z\) for \(f \in K(\alpha)\), the assertion is clear except for \(M(1/2) = \pi/2\). A simple computation gives us the expression
\[ v_{1/2}(\theta) = \frac{\pi - \theta}{2} \cos \theta + \sin \theta \log \left(2 \sin \frac{\theta}{2}\right) \]
for \(0 < \theta < \pi\). We easily get \(v_{1/2}(0^+) = \pi/2\). Thus we conclude that \(M(1/2) = \pi/2\) by Lemma 3.2. We have thus proved assertion (iii).

□

We indicate how to compute the value of \(M(\alpha)\) for \(1/2 < \alpha < 1\). Set \(c = \gamma/2 = \alpha - 1/2 \in (0, 1/2)\). Since \(h_\alpha(\mathbb{D})\) is a bounded convex domain symmetric in \(\mathbb{R}\), it is easy to see that \(v_\gamma(\theta)\) has a unique critical point, say, \(\theta_\alpha\) at which \(v_\gamma\) attains its maximum so that \(M(\alpha) = v_\gamma(\theta_\alpha)\). Here, \(\theta = \theta_\alpha\) is a unique solution of the equation
\[ (3.1) \quad \left[c \cot \frac{\theta}{2} + (1 - c) \cot(c\pi + (1 - c)\theta)\right] \left(2 \sin \frac{\theta}{2}\right)^{2c} \sin(c\pi + (1 - c)\theta) - \cos \theta = 0 \]
in \(0 < \theta < \pi\), where \(c = \alpha - 1/2\). By using this equation, we can express \(M(\alpha)\) in a different way:
\[ (3.2) \quad M(\alpha) = \frac{1}{2c} \left[\frac{\cos \theta_\alpha}{c \cot(\theta_\alpha/2) + (1 - c) \cot(c\pi + (1 - c)\theta_\alpha)} - \sin \theta_\alpha\right]. \]

This expression will be used in the proof of Theorem 1.4.

Assertion (ii) of Theorem 1.3 can be applied to an extremal problem for \(K(\alpha)\). For \(0 \leq \alpha < 1\) and \(t \in \mathbb{R}\), we consider the quantity
\[ Q_\alpha(t) = \inf_{f \in K(\alpha), z \in \mathbb{D}} \Re \left[e^{itf(z)}\right]. \]
The quantity \(M(\alpha)\) in Theorem 1.3 is a particular case of this quantity. Indeed, we have \(Q_\alpha(\pi/2) = -M(\alpha)\) for \(1/2 \leq \alpha < 1\). We have the obvious monotonicity \(Q_\alpha(t) \leq Q_\beta(t)\) for \(0 \leq \alpha < \beta < 1\) and the symmetry \(Q_\alpha(-t) = Q_\alpha(t)\). It is thus enough to consider the case when \(0 \leq t \leq \pi\).

**Theorem 3.3.** For \(0 < \alpha < 1\), the function \(\varphi_\alpha(\theta) = \theta + \arg h_\alpha'(e^{\theta})\) maps the interval \((0, \pi]\) onto \((\pi(1 - \alpha), \pi]\) homeomorphically. Furthermore, the following hold.

(i) Suppose \(\alpha = 0\). Then, \(Q_0(0) = 1/2\) and \(Q_0(t) = -\infty\) for \(0 < t \leq \pi\).
(ii) Suppose $0 < \alpha < 1/2$. Then

\[ Q_\alpha(t) = \begin{cases} 
\text{Re} \left[ e^{i(t-\theta_0)}k_\alpha (e^{i\theta_0}) \right], & 0 \leq t < \alpha \pi, \theta_0 = \varphi_\alpha^{-1}(\pi - t), \\
(2\alpha - 1)^{-1} \cos(\alpha \pi), & t = \alpha \pi, \\
-\infty, & \alpha \pi < t \leq \pi.
\end{cases} \]

(iii) Suppose $\alpha = 1/2$. Then

\[ Q_{1/2}(t) = \begin{cases} 
\text{Re} \left[ e^{i(t-\theta_0)}k_{1/2} (e^{i\theta_0}) \right], & 0 \leq t < \pi/2, \theta_0 = \varphi_{1/2}^{-1}(\pi - t), \\
-\pi/2, & t = \pi/2, \\
-\infty, & \pi/2 < t \leq \pi.
\end{cases} \]

(iv) Suppose $1/2 < \alpha < 1$. Then

\[ Q_\alpha(t) = \begin{cases} 
\text{Re} \left[ e^{i(t-\theta_0)}k_\alpha (e^{i\theta_0}) \right], & 0 \leq t < \alpha \pi, \theta_0 = \varphi_\alpha^{-1}(\pi - t), \\
(2\alpha - 1)^{-1} \cos t, & \alpha \pi \leq t \leq \pi.
\end{cases} \]

Proof. When $t = 0$ or $\pi$, the assertions are clear. Assume therefore that $0 < t < \pi$. Let $D_\alpha = h_\alpha(\mathbb{D})$. By Corollary 1.2, we have

\[ Q_\alpha(t) = \inf_{u+iv \in D_\alpha} \text{Re} \left[ e^{it}(u + iv) \right] = \inf_{u+iv \in D_\alpha} \left[ u \cos t - v \sin t \right]. \]

Then, geometrically, we can say that $-Q_\alpha(t)/\sin t$ is the supremum of $y$-intercepts of those lines $y = x \cot t + C$ which intersect with $D_\alpha$. Since $D_\alpha$ does not intersect the $y$-axis, such a line must intersect with $\partial D_\alpha$. Therefore, in the above characterization of $Q_\alpha(t)$, $D_\alpha$ may be replaced by $\partial D_\alpha$. Hence, noting also the symmetry of $D_\alpha$ in $\mathbb{R}$, we further obtain

\[ Q_\alpha(t) = \inf_{u+iv \in \partial D_\alpha} \left[ u \cos t - v \sin t \right] = \inf_{u+iv \in \partial D_\alpha, v \geq 0} \left[ u \cos t - v \sin t \right] = \inf_{0 < \theta < \pi} F(\theta), \]

where

\[ F(\theta) = u_\gamma(\theta) \cos t - v_\gamma(\theta) \sin t. \]

and $u_\gamma, v_\gamma$ are the functions given by $h_\alpha(e^{i\theta}) = u_\gamma(\theta) + iv_\gamma(\theta)$ with $\gamma = 2\alpha - 1$, as before.

When $\alpha = 0$, the function $h_0(z) = 1/(1 + z)$ maps the unit disk onto the half-plane $\text{Re} w > 1/2$ so that assertion (i) is obvious. We thus assume that $0 < \alpha < 1$ in the rest of the proof.

First we analyze the case when $Q_\alpha(t) = -\infty$. Recall that $u_\gamma(\theta) \rightarrow +\infty$ and $v_\gamma(\theta) = u_\gamma(\theta) \cot(\alpha \pi) + O(1)$ as $\theta \rightarrow 0^+$ for $0 < \alpha < 1/2$ by (ii) of Theorem 1.3. This is valid also for $\alpha = 1/2$. Hence,

\[ \sin(\alpha \pi) \left[ u_\gamma(\theta) \cos t - v_\gamma(\theta) \sin t \right] = u_\gamma(\theta) \sin(\alpha \pi - t) + O(1) \rightarrow -\infty \quad (\theta \rightarrow 0^+), \]

whenever $\sin(\alpha \pi - t) < 0$, which confirms the assertion for $\alpha \pi < t < \pi$ and $0 < \alpha \leq 1/2$.

We now show the first assertion of the theorem. Let $\psi_\alpha(\theta) = \arg [u_\gamma'(\theta) + iv_\gamma'(\theta)] \in (\pi/2, 3\pi/2]$ for $0 < \theta \leq \pi$. The strict convexity of $D_\alpha$ implies that $\psi_\alpha$ is strictly increasing. Note that $\psi_\alpha(\theta) = \arg h_\alpha'(e^{i\theta}) + \theta + \pi/2 = \varphi_\alpha(\theta) + \pi/2$. Then we consider the case $0 \leq
$t < \alpha \pi$. From the proof of assertion (ii) of Theorem \ref{thm1.3} we see that $\psi_\alpha(0^+) = 3\pi/2 - \alpha \pi$ for $0 < \alpha < 1/2$. This is valid also for $1/2 \leq \alpha < 1$. Indeed, it follows from

$$\tan \psi_\alpha(0^+) = \lim_{\theta \to 0^+} \frac{u_\gamma(\theta)}{u_\gamma(0^+)} = -\tan \frac{\pi \gamma}{2} = \cot(\alpha \pi)$$

for $1/2 < \alpha < 1$. We can also see that $\psi_{1/2}(0^+) = \pi$ directly. Hence, we conclude that the range of $\psi_\alpha(\theta)$ on $0 < \theta \leq \pi$ is precisely $(\frac{\pi \gamma}{2} - \alpha \pi, \frac{3\pi}{2} \gamma]$, which proves the required assertion.

We now consider the case when $0 \leq t < \alpha \pi$. Then $F'(\theta)$ vanishes precisely when $\tan \psi_\alpha(\theta) = v'_\gamma(\theta)/u'_\gamma(\theta) = \cot t = \tan(3\pi/2 - t)$, namely, $\varphi_\alpha(\theta) = \pi - t$. Thus we see that $F(\theta)$ takes its minimum at $\theta_0 = \varphi_\alpha^{-1}(\pi - t)$ and the corresponding assertions hold.

Our next task is to consider the borderline case $t = \alpha \pi$. When $0 < \alpha < 1/2$, Theorem \ref{thm1.3} (ii) implies that the supremum of the $y$-intercepts of the lines $y = x \cot(\alpha \pi) + k$ intersecting with $D_\alpha$ is $\cot(\alpha \pi)/(1 - 2\alpha)$. This case has been confirmed to be true. When $\alpha = 1/2$, the assertion is contained in Theorem \ref{thm1.3} (iii). When $\alpha > 1/2$, this case can be included in the final case below.

We finally consider the case when $1/2 < \alpha < 1$ and $\alpha \pi \leq t < \pi$. In this case the function $F(\theta)$ has no critical point in $0 < \theta < \pi$. Since $F'(\pi) = -v'_\gamma(\pi) \sin t > 0$, we see that $F(\theta)$ is increasing in $0 < \theta < \pi$ so that $Q_\alpha(t) = F(0^+) = (2\alpha - 1)^{-1} \cos t$. \hfill \Box

4. Proof of Theorem \ref{thm1.4}

We denote by $\mathbb{D}_r$ the disk $|z| < r$. Throughout this section, we define $f_a$ for $f \in \mathcal{A}_1$ and $a \in \mathbb{D}$ by $f_a(z) = f(az)/a$. Here, we set $f_0(z) = \lim_{a \to 0} f_a(z) = z$. We begin with the following simple observation.

**Lemma 4.1.** Let $f \in \mathcal{S}$. Suppose that $f(\mathbb{D})$ contains the disk $\mathbb{D}_\rho$ for some $\rho > 0$. Then $\mathbb{D}_\rho \subset f_a(\mathbb{D})$ for $a \in \mathbb{D}$.

**Proof.** It suffices to show that $\mathbb{D}_{\rho r} \subset f(\mathbb{D}_r)$ for $0 < r < 1$. By assumption, $g(w) = f^{-1}(\rho w)$ is a univalent analytic function on $\mathbb{D}$ with $|g(w)| < 1$ and $g(0) = 0$. Then the Schwarz lemma implies that $g(\mathbb{D}_r) \subset \mathbb{D}_r$, which in turn gives us $\mathbb{D}_{\rho r} \subset f(\mathbb{D}_r)$ as required. \hfill \Box

By making use of the idea due to Styer and Wright \[\text{[10]}, \] the following result can now be shown. For convenience of the reader, we reproduce the proof here in a somewhat simplified form.

**Lemma 4.2.** Let $\rho$ be a positive constant. Suppose that two functions $f$, $g \in \mathcal{K}$ satisfy the following two conditions:

1. $f(\mathbb{D})$ and $g(\mathbb{D})$ both contain the disk $\mathbb{D}_\rho$, and
2. $|\text{Im}[f(z)/z]| < \rho$ and $|\text{Im}[g(z)/z]| < \rho$ on $\mathbb{D}$.

Then $(f + g)/2 \in \mathcal{S}^*$.

**Proof.** Put $h = f + g$. For starlikeness, we need to show that $\text{Re}[zh'(z)/h(z)] > 0$ on $\mathbb{D}$. We will show that $\text{Re}[z f'(z)/h(z)] > 0$. Since we can do the same for $g$, it will finish the proof.
Let \( a \in \mathbb{D} \) with \( a \neq 0 \). Since \( f'_a(1)/h_a(1) = af''(a)/h(a) \), it is enough to show the inequality \( \text{Re} [f'_a(1)/h_a(1)] \geq 0 \). Denote by \( W \) the set \( \{ w : |w| \geq \rho, |\text{Im} w| < \rho \} \). Then \( W \) consists of the two connected components \( W_+ \) and \( W_- \), where \( W_+ = \{ w \in W : w > 0 \} \). By Lemma 4.1 and the relation \( f_a(z)/z = f(az)/(az) \), the assumptions imply \( f_a(1) \in W \). Since the (continuous) curve \( t \mapsto f_{a_t}(1), 0 \leq t \leq 1 \), connects \( f_0(1) = 1 \), we see that \( f_a(1) \in W_+ \). Since we have \( g_a(1) \in W_+ \) in the same way and thus \( -g_a(1) \in W_- \), the segment \( [-g_a(1), f_a(1)] \) intersects the disk \( \mathbb{D}_\rho \). Choose a point \( w_0 \in [-g_a(1), f_a(1)] \cap \mathbb{D}_\rho \). Then the vector \( f_a(1) - w_0 \) is directed at the point \( f_a(1) \) outward from the convex domain \( f_a(\mathbb{D}) \). Since the tangent vector of the curve \( f_a(e^{i\theta}) \) at \( \theta = 0 \) is given by \( if'_a(1) \), we have

\[
\arg [if'_a(1)] - \pi \leq \arg [f_a(1) - w_0] = \arg [f_a(1) + g_a(1)] \leq \arg [if'_a(1)],
\]

which is equivalent to \( |\arg [f'_a(1)/h_a(1)]| \leq \pi/2 \). Thus we have shown the desired inequality \( \text{Re} [f'_a(1)/h_a(1)] \geq 0 \).

Proof of Theorem 1.4. Let \( f, g \in \mathcal{K}(3/5) \). We will apply the last lemma to these two functions. Let \( \rho = -k_{3/5}(-1) = 5(2^{1/5} - 1) = 0.743491 \ldots \). By Theorem 1.3, we have only to show that \( M(3/5) \leq \rho \). We denote by \( F(\theta) \) the function in the left-hand side in (3.1) for \( c = \frac{3}{5} - \frac{1}{2} = \frac{1}{10} \). A numerical computation gives us \( F(0.11) = 0.0050 \cdots > 0 \) and \( F(0.114) = -0.0010 \cdots < 0 \). Thus we have \( 0.11 < \theta_{3/5} < 0.114 \). By (3.2), we have the expression \( M(3/5) = 5G(\theta_{3/5}) \), where

\[
G(\theta) = \frac{\cos \theta}{c \cot(\theta/2) + (1 - c) \cot(c\pi + (1 - c)\theta)} - \sin \theta = \frac{\cos \theta}{H(\theta)} - \sin \theta.
\]

We observe that \( H(\theta) \) is positive and decreasing in \( 0 < \theta < \frac{1/2 - c}{1-c} \pi = 4\pi/9 \), because

\[
H'(\theta) = -\frac{c}{2 \sin^2(\theta/2)} - \frac{(1 - c)^2}{\sin^2(c\pi + (1 - c)\theta)} < 0.
\]

Also, we see that \(-H'(\theta)\) is positive and decreasing in \( 0 < \theta < 4\pi/9 \) by its form. Since

\[
G'(\theta) = -\frac{\sin \theta}{H(\theta)} - \frac{H'(\theta) \cos \theta}{H(\theta)^2} - \cos \theta,
\]

letting \( \theta_0 = 0.11 \) and \( \theta_1 = 0.114 \), we estimate on \( \theta_0 \leq \theta \leq \theta_1 \) in the form

\[
G'(\theta) > -\frac{\sin \theta_1}{H(\theta_1)} - \frac{H'(\theta_1) \cos \theta_1}{H(\theta_0)^2} - \cos \theta_0 = 0.326 \cdots > 0.
\]

Hence \( G(\theta) \) is increasing in this interval so that

\[
M(3/5) = 5G(\theta_{3/5}) < 5G(\theta_1) = 0.743487 \cdots < \rho.
\]

The proof is now complete. \( \square \)
5. An example

We conclude the present note by giving an example of an odd convex function \( f \in K \) such that

\[
\frac{f(z)}{z} \not\preccurlyeq H_{2}(z) = \frac{1}{2z} \log \frac{1+z}{1-z} \quad \text{on } \mathbb{D}.
\]

The following result due to Alexander [1] (see also Goodman [3]) is useful for our aim here.

**Lemma 5.1** (Alexander). The function \( f(z) = z + a_{2}z^{2} + a_{3}z^{3} + \ldots \) is convex univalent on \( \mathbb{D} \) if

\[
\sum_{n=1}^{\infty} n^{2} |a_{n}| \leq 1.
\]

We also need the following auxiliary result which is a special case of Theorem 5 of Ruscheweyh [9] with \( n = 1 \).

**Lemma 5.2** (Ruscheweyh). The function \( q_{\gamma}(z) = \sum_{j=1}^{\infty} \gamma + 1 \gamma + j z^{j} \) belongs to \( K \) for \( \text{Re} \gamma \geq 0 \).

In particular, the function \( H_{1} \) given in (1.2) is univalent because \( H_{1} = 1 + q_{1/2}/3 \).

We now consider the function

\[
f(z) = z + \frac{z^{3}}{100} + \frac{z^{5}}{50}.
\]

Then, by Alexander’s lemma, \( f \) is an odd convex function. Secondly, we observe that \( f \) has a non-zero fixed point \( z_{0} \) in \( \mathbb{D} \). Indeed, by solving the algebraic equation \( f(z) = z \), we obtain \( z_{0} = \pm i/\sqrt{2} \).

We now show that \( f(z)/z \) is not subordinate to \( H_{2}(z) \) given in (1.1). Suppose, to the contrary, that

\[
\frac{f(z)}{z} \preccurlyeq H_{2}(z) = \frac{1}{2z} \log \frac{1+z}{1-z} \quad z \in \mathbb{D}.
\]

Then there exists an analytic function \( \omega \) on \( \mathbb{D} \) with \( \omega(0) = 0 \) and \( |\omega| < 1 \) such that

\[
\frac{f(z)}{z} = H_{2}(\omega(z)) = H_{1}(\omega(z)^{2}).
\]

Thus

\[
\frac{zf'(z) - f(z)}{z^{2}} = 2\omega(z)\omega'(z)H_{1}'(\omega(z)^{2}).
\]

Since \( f(z_{0}) = z_{0} \), we have \( H_{1}(\omega(z_{0})^{2}) = 1 = H_{1}(0) \). Univalence of \( H_{1} \) enforces the relation \( \omega(z_{0}) = 0 \) to hold. Hence, \( z_{0}f'(z_{0}) - f(z_{0}) = 0 \) which is equivalent to \( f'(z_{0}) = 1 \).

By solving the equation \( f'(z) = 1 \), we obtain \( z_{0} = \pm i\sqrt{3}/10 \). This is a contradiction. Therefore, \( f(z)/z \) is not subordinate to \( H_{2}(z) \).
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