Ricci tensor on $\text{RCD}^*(K, N)$ space

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Abstract

We prove an improved Bochner inequality based on the curvature-dimension condition and give a definition of $N$-dimensional Ricci tensor on metric measure space.

Keywords: curvature-dimension condition, Bakry-Émery theory, Bochner inequality, Ricci tensor, metric measure space.

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1 Introduction

Let $M$ be a Riemannian manifold equipped with the metric tensor $\langle \cdot, \cdot \rangle$, we have the Bochner equality

$$\Gamma_2(f) = \text{Ricci}(\nabla f, \nabla f) + \|H_f\|^2_{HS}$$

(1.1)

for any smooth function $f$, where the operator $\Gamma_2$ is defined as

$$\Gamma_2(f) = \frac{1}{2} L \Gamma(f) - \Gamma(f, Lf), \quad \Gamma(\cdot, \cdot) = \langle \nabla \cdot, \nabla \cdot \rangle, \quad \Gamma(f) = \Gamma(f, f)$$

where $L = \Delta$ is the Laplace-Beltrami operator.

In particular, if $M$ has lower Ricci curvature bound $K$, we have the Bochner inequality

$$\Gamma_2(f) \geq K \Gamma(f).$$

(1.2)
Furthermore, if we assume that the dimension of $M$ is $n$, then for any $N \geq n$ we have an improved Bochner inequality

$$\Gamma_2(f) \geq K\Gamma(f) + \frac{1}{N}(\Delta f)^2. \quad (1.3)$$

Using the inequality (1.2) and (1.3) we can study the smooth metric measure space in a abstract setting, i.e. we define the RCD($K,\infty$) (RCD$^*(K,N)$) spaces equipped with a proper Diriclet form $\Gamma$, satisfying (1.2) (1.3) respectively for any $f$ in a well defined algebra $\mathcal{A}$. This method is proposed by Bakry-Émery in [7] and has many applications. One of the applications is to define the abstract Ricci tensor, based on the Bakry-Émery conditions (1.2)(1.3). More precisely, from the inequality (1.2) we can prove

$$\Gamma_2(f) \geq K\Gamma(f) + \|H_f\|_{HS}^2, \quad (1.4)$$

where $H_f(\nabla g, \nabla h) = \frac{1}{2}(\Gamma(\Gamma(f,g),h) + \Gamma(\Gamma(f,h),g) - \Gamma(\Gamma(h,g),f))$. Hence we have reason to define the Ricci tensor as

$$\text{Ricci}(\nabla f, \nabla f)(x) := \min\{\Gamma_2(g) - \|H_g\|_{HS}^2 : \Gamma(f - g)(x) = 0\}. \quad (1.5)$$

We equip $\mathcal{A}$ with the inner product induced by $\Gamma$ and assume that its local dimension is no more than $N$. From (1.3) we can obtain

$$\Gamma_2(f) \geq K\Gamma(f) + \|H_f\|_{HS}^2 + \frac{1}{N - \dim M}(\text{tr}H_f - \Delta f)^2 \quad (1.6)$$

where $\text{tr}H_f = \Delta f$ if $N = \dim M$.

**Remark 1.1.** Since the equality $\text{tr}H_f = \Delta f$ does not hold in general (for example, weighted Riemannian manifolds), we know that the term $\frac{1}{N - \dim M}(\text{tr}H_f - \Delta f)^2$ is not superfluous.

Then we can define the $N$-Ricci tensor as

$$\text{Ricci}_N(\nabla f, \nabla f)(x) := \min\{\Gamma_2(g) - \|H_g\|_{HS}^2 - \frac{1}{N - n(x)}(\text{tr}H_f - \Delta f)^2 : \Gamma(f - g)(x) = 0\}, \quad (1.7)$$

where $n(x)$ is the local dimension of $M$ with respect to $\Gamma$.

In the paper [13] by Sturm, the above definitions of abstract Ricci tensors are proved to be compatible with the ones in smooth metric measure spaces. Therefore we would like to know if we can obtain the similar results in non-smooth case. In metric measure sense, the measure valued operator $\Gamma_2$ and $H_f$ are defined only on the equivalence class with respect to $\Gamma$, therefore the minimum in the definition (1.5) and (1.7) will not appear in metric measure case. In [10] Gigli proves the inequality (1.4) for RCD($K,\infty$) spaces, and define the Ricci tensor as in (1.5). In this paper, we adopt the vocabularies and notations of [10] and prove the finite dimensional case following the ideas in the paper [13]. We firstly prove that the local dimension of a RCD$^*(K,N)$ space is bounded by $N$, then we will prove (1.6) which makes it reasonable to define the Ricci curvature $\text{Ricci}_N$ as (1.7).

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2 Preliminaries

Let $M = (X, d, m)$ be a metric measure space where $(X, d)$ is a complete and separable geodesic metric space, $m$ is a Borel measure with respect to $d$.

We recall some notations which can be found in [3,4] and [10]. Let $W^{1,2}(M)$ and $W^{2,2}(M)$ be the Sobolev spaces on $M$ and assume that both of them are Hilbert spaces. The weak gradient of a function $f \in W^{1,2}(M)$ is denoted by $|Df|$. Then we define the Diriclet form $\Gamma(f, g) := \frac{1}{4}(|D(f + g)|^2 - |D(f - g)|^2)$ for $f, g \in W^{1,2}(M)$. In particular, the inner product $\langle Df, Dg \rangle := \Gamma(f, g)$ makes sense, and we obtain a meaningful characterization of the ‘co-tangent field’ which inherits the metric of $W^{1,2}(M)$. More precisely, for any Borel set $A, B$ we can define the inner product $\langle \chi_A Df, \chi_B Dg \rangle$ as $\chi_A \cap B \in (D(\Delta))$ is characteristic function. Then by linearity and density of simple functions in $L^\infty(M)$ we can extend this inner product to all ‘co-tangent field’ with the form $\sum a_i Df_i$ where $\{a_i\} \subset L^\infty(M)$. We then define $L^2(T^*M)$ as the $\langle \cdot, \cdot \rangle$ closure of all the ‘co-tangent’ fields with the form above.

We say that $\{Df_i\}_i$, generated $L^2(T^*M)$ if the closure of $\{\sum_i a_i Df_i : a_i \in L^\infty(M)\}$ is $L^2(T^*M)$. In particular, if $\{\sum_i a_i Df_i : a_i \in L^\infty(M)\} = L^2(T^*M)$ we say $L^2(T^*M)$ is spanned by $\{Df_i\}_i$.

As on the Riemannian manifold, we can define the gradient and Laplacian by duality (integration by part).

**Definition 2.1. (Gradient and measure valued Laplacian, [9,10])** Let $f, g \in W^{1,2}(M)$, we define $Df(\nabla g) := (Df, Dg) \in L^1(M)$ and $\langle \nabla f, \nabla g \rangle := Df(\nabla g)$. The space $D(\Delta) \subset W^{1,2}(M)$ is the space of $f \in W^{1,2}(M)$ such that there is a measure $\mu$ satisfying

$$
\int h \, d\mu = - \int \langle \nabla h, \nabla f \rangle \, dm, \forall h : M \to \mathbb{R}, \text{Lipschitz with bounded support}.
$$

In this case the measure $\mu$ is unique and we shall denote it by $\Delta f$. If $\Delta f \ll m$, we still denote its density by $\Delta f$.

From now on, we will focus on RCD (includes RCD$((K, \infty)$ and RCD$^*(K, N)$) spaces. The detailed discussions of these spaces can be found in [4], [7] and [3].

We define the set $\text{TestF}(M) \subset W^{1,2}(M)$ of test functions as

$$
\text{TestF}(M) := \{f \in D(\Delta) \cap L^\infty : |Df| \in L^\infty \text{ and } \Delta f \in W^{1,2}(M)\}.
$$

It is known in [4] that $\text{TestF}(M)$ is dense in $W^{1,2}(M)$ if $M$ is RCD.

Let $f \in \text{TestF}(M)$, we define the measure $\Gamma_2(f)$ as

$$
\Gamma_2(f) = \frac{1}{2} |Df|^2 - \langle f, \Delta f \rangle \, m.
$$

As in the smooth case, we define the Hessian of $f \in W^{2,2}(M)$ as

$$
H_f(\nabla g, \nabla h) = \frac{1}{2} \left( \langle \nabla \langle \nabla f, \nabla g \rangle, \nabla h \rangle + \langle \nabla \langle \nabla f, \nabla h \rangle, \nabla g \rangle - \langle \nabla \langle \nabla g, \nabla h \rangle, \nabla f \rangle \right),
$$

where $g, h \in \text{TestF}(M)$. It can be seen that this definition can be extended to the general vector fields in $L^2(T^*M)$. In particular, the trace of $H_f$ can be defined in the usual way.

Then we have the following results.
Proposition 2.2. (Bakry-Émery condition, [2], [8]) Let $M = (X, d, m)$ be a RCD space. Then it is a RCD$^*(K, N)$ space with $K \in \mathbb{R}$ and $N \in [1, \infty]$ if and only if
\[ \Gamma_2(f) \geq (K|Df|^2 + \frac{1}{N}(|\Delta f|^2))m \]
for any $f \in \text{TestF}(M)$.

Lemma 2.3. ([12]) Let $M = (X, d, m)$ be a RCD$^*(K, N)$ space, $n \in \mathbb{N}$, $f_1, \ldots, f_n \in \text{TestF}(M)$ and $\Phi \in C^\infty(\mathbb{R}^n)$ be with $\Phi(0) = 0$. Put $f = (f_1, \ldots, f_n)$, then $\Phi(f) \in \text{TestF}(M)$. In particular, $\Gamma_2(\Phi(f)) \geq (K|D\Phi(f)|^2 + \frac{1}{N}(|\Delta \Phi(f)|^2))m$.

For RCD$^*(K, N)$ space, we have the following theorem characterizing the structure of (co)tangent module.

Theorem 2.4. Let $M = (X, d, m)$ be a RCD$^*(K, N)$ metric measure space where $N < \infty$, then there exists a Borel partition $\{E_n\}_n$ of $X$ and $v_{n,i} = \sum_{k=1}^{N} a_{n,i,k}Df_{n,k}$, $i = 1, 2, \ldots, N_n \leq D$ where $a_{n,i,k} \in L^\infty(M)$ and $\{f_{n,k}\}_{n,k}$ are Lipschitz functions such that $\{v_{n,i}\}_i$ is a unit orthogonal base with respect to the inner product $\langle \cdot, \cdot \rangle$ on each $E_n$.

Proof. By the results of [8], [5] and [?] we know RCD$^*(K, N)$ space is doubling and it satisfies the 1-2 weak Poincaré inequality. Therefore by the result of metric measure version of Rademacher’s theorem proved in [6] and Corollary 2.5.2 in [10] we know there exists a Borel partition $\{E_n\}_n$ of $X$ and Lipschitz functions $\{f_{n,i}\}_1^{N_n}, N_n \leq N$ such that $\{f_{n,i}\}_1^{N_n}$ generates the cotangent module $L^2(T^*M)$. Hence by Proposition 1.4.6 of [10] we know $\{f_{n,i}\}_1^{N_n}$ spanned $L^2(T^*M)$, then by a Gram-Schmidt orthogonalization process we can find a unit orthogonal base as needed.

Therefore in a RCD$^*(K, N)$ space $M$, we can define the smallest number $D$ as the dimension of $M$ and $T^*M$ (and denote it by $\dim M$) and call $N_n$ as the local dimension on $E_n$.

3 Main result

In this part, we will define the Ricci tensor $\text{Ricci}_N$ on the RCD$^*(K, N)$ metric measure spaces. In Theorem 2.4 we know the (co)tangent module is finite generated and has upper dimension bound. Therefore, we want to know if $N$ is an upper bound, even more we would like to know if $N$ is optimal.

First of all, we have a lemma.

Lemma 3.1. (Lemma 3.3.6, [10]) Let $\mu_i = \rho_i m + \mu_i^s$, $i = 1, 2, 3$ be measures with $\mu_i^s \perp m$. We assume that
\[ \lambda^2 \mu_1 + 2\lambda \mu_2 + \mu_3 \geq 0, \quad \forall \lambda \in \mathbb{R}. \]
Then we have
\[ \mu_i^s \geq 0, \quad \mu_3^s \geq 0 \]
and
\[ |\rho_2|^2 \leq \rho_1 \rho_3, \quad m - \text{a.e.} \]
Then we can use this lemma to prove the following proposition.

**Proposition 3.2.** Let \( M = (X, d, m) \) be a RCD\(^*(K, N) \) metric measure space, then \( \dim M \leq N \). Furthermore, if the local dimension on a Borel set \( E \) is \( N \), we know \( \text{tr}H_f(x) = \Delta f(x) \) \( m \)–a.e. \( x \in E \) for \( f \in \text{TestF} \).

**Proof.** Firstly we define a function \( \Phi(x, y, z_1, \ldots, z_{\dim M}) = \lambda(xy + x) - by + \sum_{i}^{\dim M} (z_i - c_i)^2 \) where \( \lambda, b, c_i \in \mathbb{R} \). By Lemma 2.3 we know for \( f = (f, g, h_i) \) where \( f, g = 1, h_i \in \text{TestF} \) we have

\[
\Gamma_2(\Phi(f)) \geq (K|D\Phi(f)|^2 + \frac{(\Delta \Phi(f))^2}{N}) m.
\]

By the chain rules (see reference in Lemma 3.1.4, 3.3.7 of [10] or Lemma 2.1 of [13]) we can deduce:

\[
\lambda^2 \Gamma_2(f) + (2\lambda \sum_i H_f(\nabla h_i, \nabla h_i) + \sum_{i,j} |\langle \nabla h_i, \nabla h_j \rangle|^2 - K\lambda^2 |Df|^2) m
\]

\[
- (\lambda^2 \frac{(\Delta f)^2}{N} + 2\lambda \frac{\Delta f}{N} |Dh_i|^2 + \frac{(\sum_i |Dh_i|^2)^2}{N}) m + E(\lambda, b, c_i) \geq 0
\]

where \( E(\lambda, b, c_i) \) contains either the factor \( (\lambda f - b) \) or \( (h_i - c_i) \). Then we can assume \( E = 0 \) since we can approximate \( \lambda f \) and \( h_i \) by simple functions.

Let \( \gamma_2(f) m \) be the absolutely continuous part of \( \Gamma_2(f) \), by Lemma 3.1 we have the inequality

\[
\left| \sum_i H_f(\nabla h_i, \nabla h_i) - \frac{\Delta f}{N} |Dh_i|^2 \right|^2
\]

\[
\leq (\gamma_2(f) - K |Df|^2 - \frac{(\Delta f)^2}{N}) (\sum_{i,j} |\langle \nabla h_i, \nabla h_j \rangle|^2 - \frac{(\sum_i |Dh_i|^2)^2}{N}).
\]

In particular, we have

\[
\sum_{i,j} |\langle \nabla h_i, \nabla h_j \rangle|^2 \geq \frac{(\sum_i |Dh_i|^2)^2}{N}, \quad m - a.e..
\]

This inequality remains true if we replace \( \nabla h_i \) by \( v := \sum_k \chi_{A_k} \nabla f_k \) where \( f_k \) are test functions and \( A_k \) are disjoint Borel sets. Therefore we can replace \( \{\nabla h_i\}_{1}^{\dim M} \) by \( \{e_{ij} \}_{1}^{\dim M} \) where \( \{e_{ij} \}_{1}^{n(x)} \) is a unit orthogonal base defined on a Borel set \( E_n \) with \( n(x) \leq \dim M \) as in the Theorem 2.4 and \( e_i \equiv 0 \) for \( i = n(x) + 1, \ldots, \dim M \). Then we deduce

\[
n(x) = \sum_{i,j} |\langle e_i, e_j \rangle|^2 \geq \frac{(\sum_i |e_i|^2)^2}{N} = \frac{(n(x))^2}{N}, \quad m - a.e..
\]

Then we know \( \dim M \leq N \) which is the result we need. Furthermore, if \( n(x) = N \) on the Borel set \( E_n \), we know

\[
\left| \sum_{i=1}^{N} H_f(e_i, e_i) - \frac{\Delta f}{N} |e_i|^2 \right| = 0, \quad m - a.e. \ x \in E_n,
\]

which is equivalent to say \( \text{tr}H_f = \Delta f - a.e. \ x \in E_n. \)

\( \square \)
Remark 3.3. The result above can be written in the language of [10] as the following. For any $X = \sum g_i \nabla f_i$ where $g_i, f_i \in \text{TestF}$, recall the identity $\nabla (g \nabla f) = \nabla g \otimes \nabla f + g \nabla (\nabla f)$ and $\text{div}(g \nabla f) = \nabla g \cdot \nabla f + \text{div}(g \nabla f)$ we know $\text{tr}(\nabla X)^b = \text{div} X$ for any $X \in \text{TestV}$. Picking any $X \in H^1_{C^2}(TM)$, we can find a sequence $\{X_i\} \subset \text{TestV}$ such that $X_i \to X$ in $H^1_{C^2}(TM)$. Therefore $\text{div} X_i = \text{tr}(\nabla X_i)^b$ in $L^2$. Since $X_i \to X$ in $L^2(TM)$ and $\text{div}$ is a closed operator, we know $X \in D(\text{div})$ and $\text{div} X = \text{tr}(\nabla X)^b$.

Theorem 3.4. Let $M = (X, d, m)$ be a $\text{RCD}^*(K, N)$ metric measure space, where $N \geq \text{dim} M$. We denote the local dimension of $M$ by $n(x)$, then

$$\Gamma_2(f) \geq (K|Df|^2 + \|H_f\|^2_{\text{HS}} + \frac{1}{N - n(x)}(\text{tr}H_f - \Delta f)^2) \, dm$$

holds for any $f \in \text{TestF}$.

Proof. First of all, we see that the right hand side of the inequality is well defined by the result of Proposition 3.2. We define the function $\Phi$ as

$$\Phi(x, y_1, ..., y_N) := x - \frac{1}{2} \sum_{i,j} c_{i,j} (y_i - c_i)(y_j - c_j) - c \sum_{i=1}^N (y_i - c_i)^2$$

where $c, c_i, c_{i,j} = c_j, i$ are constants. Then we have

$$\Phi_{x,i} = 0, \quad \Phi_{i,j} = -c_{i,j} - \frac{c}{n} \delta_{ij}$$

$$\Phi_{x} = 1, \quad \Phi_i = -\sum_j c_{i,j} (y_j - c_j) - \frac{c}{n} (y_i - c_i)$$

Therefore by using the chain rules we have

$$|D\Phi(f, h_1, ..., h_n)|^2 = |Df|^2 + \sum_i (h_i - c_i)I_i,$$

$$\Gamma_2(\Phi(f, h_1, ..., h_N)) = \Gamma_2(f) - 2 \sum_{i,j} (c_{i,j} + 2c\delta_{ij})H_i(\nabla h_i, \nabla h_j) \, m$$

$$- \sum_{i,j,k,l} (c_{i,j} + 2c\delta_{ij})(c_{k,l} + 2c\delta_{kl})(\nabla h_i, \nabla h_k, \nabla h_l, \nabla h_j) \, m + \sum_i (h_i - c_i)J_i \, m,$$

$$\Delta \Phi(f, h_1, ..., h_N) = \Delta f - \sum_{i,j} (c_{i,j} + 2c\delta_{ij}) \langle \nabla h_i, \nabla h_j \rangle + \sum_i (h_i - c_i)K_i \, m,$$

where $I_i, J_i, K_i$ are some positive terms.

Then we apply Lemma 2.3 to the function $\Phi(f, h_1, ..., h_N)$, we have the inequality
\[ \Gamma_2(f) - 2 \sum_{i,j} (c_{i,j} + 2c\delta_{ij}) H_f(\nabla h_i, \nabla h_j) m \]
\[ - \sum_{i,j,k,l} (c_{i,j} + 2c\delta_{ij})(c_{k,l} + 2c\delta_{kl}) \langle \nabla h_i, \nabla h_k \rangle \langle \nabla h_l, \nabla h_j \rangle m + \sum_i (h_i - c_i) J_i m \]
\[ \geq K (|Df|^2 + \sum_i (h_i - c_i) J_i) m + \]
\[ \frac{1}{N}(\Delta(f) - \sum_{i,j} (c_{i,j} + 2c\delta_{ij}) \langle \nabla h_i, \nabla h_j \rangle + \sum_i (h_i - c_i) K_i)^2 m. \]

Considering the inequality above is an inequality about measures, by an approximation of \( h_i, c_{i,j} \) by simple functions we can assume \( h_i = c_i \), hence we obtain

\[ \Gamma_2(f) - 2 \sum_{i,j} (c_{i,j} + 2c\delta_{ij}) H_f(\nabla h_i, \nabla h_j) m \]
\[ - \sum_{i,j,k,l} (c_{i,j} + 2c\delta_{ij})(c_{k,l} + 2c\delta_{kl}) \langle \nabla h_i, \nabla h_k \rangle \langle \nabla h_l, \nabla h_j \rangle m \]
\[ \geq K |Df|^2 m + \frac{1}{N}(\Delta(f) - \sum_{i,j} (c_{i,j} + 2c\delta_{ij}) \langle \nabla h_i, \nabla h_j \rangle)^2 m. \]

As the argument before, we can restrict the inequality above on Borel set \( E_n \) where \( n \leq N \) and we can also regard the constant \( c, c_{i,j} \) as functions defined on \( E_n \). Replacing \( \{\nabla h_i\}_{1}^{n} \) by an unit orthogonal base \( \{e_i\}_{1}^{n} \), we obtain

\[ \Gamma_2(f) - 2 \sum_{i,j=1}^{n} (c_{i,j} + 2c\delta_{ij}) H_f(e_i, e_j) m + \sum_{i,j=1}^{n} (c_{i,j} + 2c\delta_{ij})(c_{i,j} + 2c\delta_{ij}) m \]
\[ \geq K |Df|^2 m + \frac{1}{N}(\Delta(f) - \sum_{i=1}^{n} (c_{i,i} + 2c))^2 m. \]

Regarding the above inequality as a positive definite quadratic form of \( c \), then the inequality holds if and only if it is true when \( c \) get the minimal point. By a direct computation, we know this point is

\[ c = \frac{N\text{tr}H_f - n\Delta f - (N - n)\text{tr}C}{2n(N - n)} \]

where \( C = (c_{i,j}) \) is a matrix.

Now we define the functional \( F(H_f, C) \) as:

\[ F : = -2 \sum_{i,j} (c_{i,j} - \frac{\text{tr}C}{n}\delta_{ij}) H_f(e_i, e_j) + \|C\|_{HS}^2 + 4\text{tr}C + 4nc^2 \]

which is the only term contains both \( H_f \) and \( C \). By computation, we have:

\[ F(H_f, C) = \|C - (H_f - \frac{\text{tr}H_f}{n}\text{Id})\|_{HS}^2 + G(H_f), \]
where $G$ has nothing to do with $C$. We can pick $C = H_f - \frac{\text{tr} H_f}{n} \text{Id}_n$ to obtain the minimum of the functional as we need.

Using the vocabularies developed in [10], we can define the Ricci curvature on arbitrary ‘vector field’. Firstly, for any $f \in \text{TestF}(M)$, we define

$$\text{Ricci}_N(\nabla f, \nabla f) := \Gamma_2(f) - (\|H_f\|_{HS}^2 + \frac{1}{N - n(x)}(\text{tr} H_f - \Delta f)^2) \text{m}.$$ 

Secondly, it is rest to verify if $\text{Ricci}_N$ can be (uniquely) extended to $[H^1_H(TM)]^2$. Since $\delta X = \text{div} X$, the operator $\text{div}$ is continuous in $[H^1_H(TM)]^2$. Corollary 3.6.4 of [10] tells us that $[H^1_H(TM)]^2 \subset [H^1_C(TM)]^2$ and $\text{tr}(\nabla X)^b$ is continuous in $[H^1_C(TM)]^2$, hence the map $R : [\text{TestV}(M)]^2 \mapsto \text{Meas}(M)$ with

$$R(X, Y) = (\text{tr}(\nabla X)^b - \text{div} X)(\text{tr}(\nabla Y)^b - \text{div} Y) \text{m}$$

is continuous w.r.t. the $[H^1_H(TM)]^2$-norm. Combining the result of Theorem 3.6.7 of [10], we know $\text{Ricci}_N$ is well defined.

**Definition 3.5. (Ricci tensor)** We define $\text{Ricci}_N$ as a measure valued continuous map on $[H^1_H(TM)]^2$ such that for any $X, Y \in \text{TestV}(M)$ it holds

$$\text{Ricci}_N(X, Y) = \Gamma_2(f) - (\nabla X : \nabla Y) \text{m}$$

$$- \frac{1}{N - n(x)}(\text{tr}(\nabla X)^b - \text{div} X)(\text{tr}(\nabla Y)^b - \text{div} Y) \text{m},$$

where

$$\Gamma_2(X, Y) = \Delta \frac{\langle X, Y \rangle}{2} + \frac{1}{2} \langle X, (\Delta H Y)^b \rangle + \frac{1}{2} \langle Y, (\Delta H X)^b \rangle \text{m}.$$ 

A direct consequence of Theorem 3.4 is:

**Corollary 3.6.** Let $M$ be a $\text{RCD}^*(K, N)$ space, then $\text{Ricci}_N \geq K$, and the inequality

$$\Gamma_2(X, Y) \geq \text{Ricci}_N(X, Y) + (\nabla X : \nabla Y) \text{m}$$

$$+ \frac{1}{N - n(x)}(\text{tr}(\nabla X)^b - \text{div} X)(\text{tr}(\nabla Y)^b - \text{div} Y) \text{m}.$$ 

holds for any $X, Y \in H^1_H(TM)$.

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