MINIMAL SUPERSYMMETRIC SCENARIOS FOR SPONTANEOUS CP VIOLATION

M. Masip

Departamento de Física Teórica y del Cosmos
Universidad de Granada
18071 Granada, Spain

A. Rašin

High Energy Section
ICTP
34014 Trieste, Italy

ABSTRACT

We study the possibility of spontaneous $CP$ violation (SCPV) at the tree level in models with an extended Higgs sector. We show that the minimum equations for the complex phases of the vacuum expectation values (VEVs) have always a geometrical interpretation in terms of triangles. To illustrate our method we analyze the minimal supersymmetric (SUSY) model with R-parity violating couplings and sneutrino VEVs, where there is no SCPV. Then we study SUSY models with extra Higgs doublets and/or gauge singlets, and find that the simplest scenario with SCPV must include at least two singlet fields.
1 Introduction

Although the observed $CP$ violation can easily be accommodated in the standard model, allowing for arbitrary $CP$ violating terms leads to phenomenological difficulties in models with an enlarged Higgs sector. For example, in general two-Higgs doublet models with acceptable flavor changing interactions the prediction for $\epsilon_K$ would still be too large [1]. Another example is the minimal SUSY extension of the standard model (MSSM), where arbitrary complex phases in gaugino masses and scalar trilinears would produce too large electric dipole moments [2].

In order to bring the predicted $CP$ violation in such extensions to acceptable levels one can consider imposing additional symmetries, like a flavor symmetry or a discrete symmetry in the Higgs sector. One obvious possibility is to impose $CP$ invariance itself. If $CP$ violation is spontaneous [3], in the sense that it appears via VEVs of scalar fields, then some phases could be naturally suppressed by the ratio of two mass scales present in the model, like the top and bottom quark masses [4, 5] or the electroweak and the SUSY scales. This hierarchy could be used, for example, to accommodate small complex phases in gaugino masses and scalar trilinears together with the large CKM phase required to explain the kaon system.

Another interesting motivation for SCPV arises from the strong $CP$ problem. As far as $CP$ is a good symmetry the QCD phase $\theta$ is zero. The spontaneous breaking of $CP$ could respect this initial value of $\theta$ while creating the observed weak $CP$ violation [4, 6]. Such scenarios find a natural framework in left-right models, where it was recently shown that a minimal model of this type must be supersymmetric and with a low scale of $SU(2)_R$ symmetry breaking [8].

In consequence, SCPV appears as a well motivated possibility in SUSY models. A priori, these models contain enough ingredients and arbitrarity to introduce SCPV: a neutral scalar sector with at least two Higgs fields plus three sneutrinos, and a large number of arbitrary SUSY-breaking terms. However, it is well known that the MSSM has only real minima. In fact, complex VEVs are possible once radiative corrections are included [6], but then the model contains a too light Higgs boson which is experimentally excluded [10]. Therefore, it is necessary to introduce other fields and couplings. Two cases have been considered before: an extra gauge singlet field (2D1S model) and an extra pair of Higgs doublets (4D model). The presence of a singlet is appealing because it defines a scenario where the SUSY mass term of the Higgs is substituted by a trilinear term in the superpotential combined with a singlet VEV, avoiding the $\mu$ problem and relaxing the upper bound on the mass of the lightest neutral Higgs [11]. The singlet model (known as the next–to–MSSM) has
been analyzed in this context by Romao [12], with the result that no $CP$ violating minima exist at tree level\[1\]. The inclusion of an extra pair of Higgs doublets is also an obvious generalization of the MSSM. But the result there is negative as well, the minimum in the 4D model is always real [14]. In Section 4 we review the arguments used to prove these results.

Contrary to non-SUSY models, where the Higgs quartic interactions are arbitrary, SCPV in SUSY scenarios seems to require an increasing number of species and fields. This fact quickly will make the analysis of minimum equations cumbersome. In Section 2 we describe a method to find complex minima that interprets the equations for the phases in terms of a geometrical object combining triangles. The method is general in the sense that it can be applied to any potential (SUSY or not). We illustrate it analyzing a SUSY model with minimal matter content but R-parity breaking couplings and sneutrino VEVs. Our objective in this paper is to find the minimal SUSY scenario giving SCPV in the Higgs sector at the tree level. In Section 3, we extend the Higgs sector of the MSSM with singlets and/or extra doublets (the addition of $SU(2)_L$ triplets is phenomenologically disfavored), and we show that the minimal scenario consists of at least two singlets, regardless of the number of Higgs doublets. In Section 4 we give our final remarks and conclusions.

2 The geometrical method

In this section we describe a geometrical method of analyzing the minima equations for the phases that was recently used in the context of a SUSY model with four Higgs doublets. The method is actually a generalization of the procedure used in the simplest case, where the object representing the equations is just a triangle [12, 15]. Let us start studying the non-SUSY model proposed in [16] to review this simplest case, and then we will describe in some detail a more general scenario.

Consider the extension of the standard model with three Higgs doublets [16] where one doublet couples to the up quarks, another one to the down quarks, and the third one does not couple to quarks at all. The scalar potential is given by

\[
V_{3D} = m_i^2 H_i^* H_i + \lambda_i (H_i^* H_i)^2 + \lambda_{ij} (H_i^* H_j)(H_j^* H_i) + \lambda_{12} (H_1^* H_2)(H_1^* H_2) + \lambda_{13} (H_1^* H_3)(H_1^* H_3) + \lambda_{23} (H_2^* H_3)(H_2^* H_3) + \text{h.c.},
\]

(1)

We do not consider here singlet extensions where dimensionful couplings are allowed. In such case it was shown that spontaneous $CP$ violation was possible [13].
with all the parameters real. The neutral components of the doublets will have VEVs

$$ \langle H_i \rangle = \frac{1}{\sqrt{2}} v_i e^{i\delta_i} \quad (i = 1, 2, 3), \quad (2) $$

where an hypercharge transformation is used to set $\delta_1 = 0$ (the minimum will be degenerate due to the $U(1)_Y$ symmetry of the potential). The value of $v_i$ and $\delta_i$ will be given by the solution of the minimum equations. For the complex phases these are $\left( \frac{\partial V}{\partial \delta_i} = 0 \right)$ \[17]\n
$$ \lambda_{12} v_1^2 v_2^2 \sin \delta_2 + \lambda_{23} v_2^2 v_3^2 \sin(\delta_2 + \delta_3) = 0 \quad (3) $$

To solve these equations we draw a triangle \[13]\ with sides $a_1^{-1}$, $a_2^{-1}$, $a_3^{-1}$ and opposite angles $\pi - \delta_2$, $\pi - \delta_3$, $\delta_2 + \delta_3 - \pi$, respectively (this is the lower right triangle in Fig. 1). If we define

$$ a_1 = \lambda_{12} v_1^2 v_2^2 $$
$$ a_2 = \lambda_{13} v_1^2 v_3^2 $$
$$ a_3 = \lambda_{23} v_2^2 v_3^2 \quad (4) $$

then the sine law applied to the triangle implies Eq. (3), i.e., the triangle is the solution to Eq. (3). Using the triangle it is now possible to give the values of the cosines of the phases (which appear in the minimum equations for the moduli) in terms of the sides:

$$ \cos \delta_2 = \frac{1}{2} \left[ (\frac{1}{\lambda_{12} v_1^2 v_2^2})^2 - (\frac{1}{\lambda_{13} v_1^2 v_3^2})^2 - (\frac{1}{\lambda_{23} v_2^2 v_3^2})^2 \right] $$
$$ \cos \delta_3 = \frac{1}{2} \left[ (\frac{1}{\lambda_{32} v_3^2 v_2^2})^2 - (\frac{1}{\lambda_{12} v_1^2 v_2^2})^2 - (\frac{1}{\lambda_{13} v_1^2 v_3^2})^2 \right] $$
$$ \cos(\delta_2 + \delta_3) = \frac{1}{2} \left[ (\frac{1}{\lambda_{12} v_1^2 v_2^2})^2 - (\frac{1}{\lambda_{32} v_3^2 v_2^2})^2 - (\frac{1}{\lambda_{13} v_1^2 v_3^2})^2 \right] \quad (5) $$

Substituting these expressions in the three minimum equations for the moduli we will obtain the equations in terms of the three moduli $v_i$ only (with no phases), and these equations can be solved numerically. The value of the moduli will fix the sides of the triangle and, in consequence, the value of the complex phases. For the particular three doublet model under study, it is easy to find solutions $v_i$ for adequate values of the couplings, and then there is SCPV. Below we analyze a similar example where the phases can also be expressed in terms of the moduli, but then the minimum equations for the moduli are always incompatible.

This method of expressing the phases as functions of the moduli can be generalized to models with more than two complex phases. We will show now how to build a set of triangles,
with their angles related to the phases in the VEVs, that solves the phase minimum equations. The method can be used if the equations involve only sines (and not cosines) of the phases, which is the case for Higgs potentials with all the couplings real. In models with \( CP \) odd fields the \( CP \) invariant potential may include purely imaginary couplings. However, since in SUSY models all the scalar fields are complex, the factors of \( i \) in the couplings can always be rotated away by field redefinitions. In non-SUSY cases there may be exceptions with \( CP \) odd real scalars. The (real) VEVs of such fields would break \( CP \), which would be transmitted to the rest of the Lagrangian through complex couplings either through additional fermions \([18]\) or through explicit couplings to the Higgs doublets. The minimal model of the latter type is a two Higgs doublet model with flavor conserving couplings \([19]\) and with one \( CP \) odd real singlet. In this case the minimum of the potential has terms \( \sin(\delta_2) \) and \( \cos(2\delta_2) \), being \( \delta_2 \) the relative phase of the two doublets. Here, however, we can also do a rotation of \( \delta_2 \) by \( \pi/2 \) to convert both terms into cosines, and the minimum equation will only involve sines of the phase.

To describe our method in some detail we will consider an extension of the MSSM with \( R \) parity violating couplings (\( \bar{R} \) model). The model has a minimal matter content with one pair of Higgs doublets. However, sneutrino VEVs are allowed. This fact will introduce, in an effective way, three new doublets of \(-1/2\) hypercharge in the scalar Higgs sector. The relevant part of the superpotential is

\[
W = W_{MSSM} + \mu_i H_2 L_i
\]

We will restrict our analysis to the VEVs of the Higgs \((H_1, H_2)\) and lepton \((L_i)\) fields, assuming that we are in the charge conserving part of the parameter space. We will closely follow the notation used in the multi-Higgs case in \([14]\), with odd (even) indices indicating \(-1/2\) (+1/2) hypercharge. We redefine \((L_1, L_2, L_3) \rightarrow (H_3, H_5, H_7)\) and write the VEVs for the neutral components of these doublets as

\[
\langle H_i \rangle = \frac{1}{\sqrt{2}} v_i e^{i\delta_i} \quad (i = 1, 2, 3, 5, 7),
\]

using a global hypercharge transformation to set \( \delta_1 = 0 \). Then the VEV of the scalar potential is

\[
V_R = \frac{1}{2} m_1^2 v_1^2 + \frac{1}{2} m_2^2 v_2^2 + \frac{1}{2} m_3^2 v_3^2 + \frac{1}{2} m_5^2 v_5^2 + \frac{1}{2} m_7^2 v_7^2 + m_{12}^2 v_1 v_2 \cos \delta_2 \\
+ m_{13}^2 v_1 v_3 \cos \delta_3 + m_{15}^2 v_1 v_5 \cos \delta_5 + m_{17}^2 v_1 v_7 \cos \delta_7 + m_{32}^2 v_3 v_2 \cos(\delta_3 + \delta_2) \\
+ m_{52}^2 v_5 v_2 \cos(\delta_5 + \delta_2) + m_{72}^2 v_7 v_2 \cos(\delta_7 + \delta_2) + m_{35}^2 v_3 v_5 \cos(\delta_3 - \delta_5) \\
+ m_{37}^2 v_3 v_7 \cos(\delta_3 - \delta_7) + m_{57}^2 v_5 v_7 \cos(\delta_5 - \delta_7) + V_D
\]

---

\(2\)We thank Goran Senjanović for pointing out this possibility.
where all the mass parameters are real and $V_D$ is the D-term part of the potential

$$V_D = \frac{1}{32} (g^2 + g'^2)(v^2_1 + v^2_3 + v^2_5 + v^2_7 - v^2_2)^2$$  \hspace{1cm} (9)$$

We are seeking $CP$-violating minima, i.e., minima where some phases are different from 0 or $\pi$. First, we solve the minimum conditions for the phases ($-\frac{\partial V}{\partial \delta_i} = 0$):

$$m_{12}^2 v_1 v_2 \sin \delta_2 + m_{32}^2 v_3 v_2 \sin(\delta_3 + \delta_2) + m_{52}^2 v_5 v_2 \sin(\delta_5 + \delta_2) + m_{72}^2 v_7 v_2 \sin(\delta_7 + \delta_2) = 0$$

$$m_{32}^2 v_3 v_2 \sin(\delta_3 + \delta_2) + m_{13}^2 v_1 v_3 \sin \delta_3 + m_{35}^2 v_3 v_5 \sin(\delta_3 - \delta_5) + m_{37}^2 v_3 v_7 \sin(\delta_3 - \delta_7) = 0$$

$$m_{52}^2 v_5 v_2 \sin(\delta_5 + \delta_2) - m_{35}^2 v_3 v_5 \sin(\delta_3 - \delta_5) + m_{15}^2 v_1 v_5 \sin \delta_5 + m_{57}^2 v_5 v_7 \sin(\delta_5 - \delta_7) = 0$$

$$m_{72}^2 v_7 v_2 \sin(\delta_7 + \delta_2) - m_{37}^2 v_3 v_7 \sin(\delta_3 - \delta_7) - m_{57}^2 v_5 v_7 \sin(\delta_5 - \delta_7) + m_{17}^2 v_1 v_7 \sin \delta_7 = 0$$

$$\hspace{1cm} (10)$$

Now we express the solution of the minimum equations for the phases in terms of a combination of triangles. We observe that there are ten independent quantities $m_{ij} v_i v_j$ in the equations and, in consequence, the space of solutions will be 10-dimensional. The geometrical solution must be given by a set of six triangles whose angles involve the different combinations of four phases in Eq. (10). Such a combination of triangles contains ten independent sides. A choice for the six triangles is shown in Fig. 1. Adding the sine law applied to the triangles it is straightforward to obtain the set of equations in (10), where we identify

$$m_{12}^2 v_1 v_2 = a_1 + b_1 + x_1 \hspace{0.5cm}, \hspace{0.5cm} m_{32}^2 v_3 v_2 = a_3 \hspace{0.5cm}, \hspace{0.5cm} m_{52}^2 v_5 v_2 = -b_3 \hspace{0.5cm},$$

$$m_{13}^2 v_1 v_3 = a_2 + c_1 + z_1 \hspace{0.5cm}, \hspace{0.5cm} m_{72}^2 v_7 v_2 = -x_3 \hspace{0.5cm}, \hspace{0.5cm} m_{35}^2 v_3 v_5 = -c_3 \hspace{0.5cm},$$

$$m_{15}^2 v_1 v_5 = b_2 - c_2 + y_1 \hspace{0.5cm}, \hspace{0.5cm} m_{37}^2 v_3 v_7 = -z_3 \hspace{0.5cm}, \hspace{0.5cm} m_{57}^2 v_5 v_7 = -y_3 \hspace{0.5cm},$$

$$m_{17}^2 v_1 v_7 = x_2 - z_2 - y_2 \hspace{0.5cm},$$

$$\hspace{1cm} (11)$$

Now we have to express $\cos \delta_i$ in terms of $m_{ij} v_i v_j$ and substitute them in the five minimum equations for the moduli. If there we find a solution fixing $v_i$, then we have SCPV; if the equations are incompatible then any minimum that may exist will be $CP$ conserving. Although this procedure sounds simple, in general it is not easy to obtain analytic solutions. In particular, to express the sides of the triangles in terms of $m_{ij} v_i v_j$ involves solving a quartic equation. The multitriangle is useful to generate numerical solutions, as we do in Section 3 to study other models, but it is not enough to solve analytically complicated cases. In the $\mathcal{R}$

\footnote{Note that the four angles and one side of each triangle also fixes the six triangles. In general, the number of phases in the equations plus the number of triangles in the multitriangle must be equal to the number of independent parameters in the equations.}
model under study here, however, we can find a way to simplify the equations that allows for analytical solutions. We can rotate the original fields \( H_1, H_3, H_5, H_7 \) to a new basis so that

\[
m_{13}^2 = m_{15}^2 = m_{17}^2 = m_{35}^2 = m_{37}^2 = m_{57}^2 = 0 .
\]

This rotation redefines the diagonal terms \( m_{13}^2 \) but does not introduce any complex phase in the Lagrangian. From Eq. (10) this set of zero masses forces

\[
\sin \delta_2 = \sin(\delta_2 + \delta_5) = \sin(\delta_2 + \delta_7) = 0
\]

which, if the five VEVs \( v_i \) are non-zero, forces all the phases to be trivial. Then there is no \( CP \) violation in the general case with all \( v_i \neq 0 \).

If the modulus \( v_7 = 0 \) and all the other VEVs nonzero, then the phase \( \delta_7 \) is irrelevant (from (11) we see that \( x_3 = y_3 = z_3 = 0 \) and the corresponding triangles become infinite). Now the three-phase case \( (\delta_2, \delta_3, \delta_5) \) that results has three triangles left and is completely analogous to the 4D model discussed in [14], with no SCPV.

The case with two VEVs zero, \( v_5 = v_7 = 0 \), appeared in [20], where it is claimed that there are \( CP \)-violating minima. Since we disagree with this result, let us consider it in a bit more detail. Now the minimum equations for the phases reduce to

\[
m_{12}^2 v_1 v_2 \sin \delta_2 + m_{32}^2 v_3 v_2 \sin(\delta_2 + \delta_3) = 0
\]
\[
m_{32}^2 v_3 v_2 \sin(\delta_3 + \delta_2) + m_{13}^2 v_1 v_3 \sin \delta_3 = 0
\]

Again, the lower right triangle in Fig. 1 solves these equations if \( a_1 = m_{12} v_1 v_2, a_2 = m_{13} v_1 v_3 \) and \( a_3 = m_{23} v_2 v_3 \). The cosines of the angles can be obtained from the expressions in (4) just exchanging \( \lambda_{ij} v_i^2 v_i^2 \rightarrow m_{ij}^2 v_i v_j \).

Notice that this does not mean yet that there is SCPV, we still have to solve the three minimum equations for the moduli \( (v_i \frac{\partial V}{\partial v_i} = 0) \):

\[
m_{12}^2 v_1^2 + m_{12}^2 v_1 v_2 \cos \delta_2 + m_{13}^2 v_1 v_3 \cos \delta_3 + v_1^2 g(v) = 0
\]
\[
m_{12}^2 v_2^2 + m_{12}^2 v_1 v_2 \cos \delta_2 + m_{23}^2 v_2 v_3 \cos(\delta_2 + \delta_3) - v_2^2 g(v) = 0
\]
\[
m_{13}^2 v_3^2 + m_{13}^2 v_1 v_3 \cos \delta_3 + m_{23}^2 v_2 v_3 \cos(\delta_2 + \delta_3) + v_3^2 g(v) , = 0
\]

where \( g(v) \equiv \frac{1}{8}(g^2 + g'^2)(v_1^2 + v_2^2 - v_3^2) \). We substitute the expressions for \( \cos \delta_i \) in the equations above and obtain

\[
v_1^2 [m_1^2 - \frac{m_{12}^2 m_{13}^2}{m_{32}^2} + g(v)] = 0
\]
There is only one combination of VEVs, \( g(v) \), to solve the system of three equations. Then, unless the masses are fine tuned in such a way that two of the moduli equations are trivial, the three equations can not be satisfied simultaneously. This implies that the minimum equations for the phases must have the trivial solution (phases equal zero or \( \pi \)). We conclude that there is no tree-level SCPV in the extension of the MSSM with R-parity violating couplings.

In the two simple non-SUSY and SUSY cases above the phase minimum equations have been solved in terms of triangles. It is now possible to see how to extend the procedure to more complicated cases. If we add a Higgs field to any of the previous models, this field will introduce one more phase and will increase the number of independent parameters in the equations. To find the solution now we need to draw new triangles, and the number of independent distances in these triangles must be exactly equal to the number of new parameters (couplings) in the equations. This will be always possible because we can draw an arbitrary number of new triangles with the existing phases, and each new triangle will introduce one more independent distance: the overall scale of the triangle, which is not fixed by the phases. We can always add triangles until matching the number of new couplings in the minimum equations. The procedure will become more obvious in the next Section, where we consider SUSY models with additional Higgs fields.

3 Search for the minimal SUSY model

The method described in the previous section has been already applied to minimal SUSY scenarios, namely, to the 2D1S model and to the 4D model. In both cases the results are negative. In the 2D1S model the minimum equations for the phases are also solved by a single triangle. When this triangle is imposed on the equations for the moduli it was found \cite{12} that the solution is always a saddle point: the Hessian matrix has a negative eigenvalue. The scalar mass matrix is given by the second derivatives of the potential, and this negative eigenvalue is equivalent to the negative mass squared distinctive of false vacua. One may rely on radiative corrections to turn the negative mass into positive \cite{21}, but even then the model gives a field which is too light to have escaped detection. In consequence, there is no SCPV in the singlet model.
In the 4D model the equations for the phases are solved by three triangles. When this solution is imposed on the four minimum equations for the moduli we found [14] that they only depend on two combinations of VEVs and, in consequence, there is no solution. A solution can be obtained if the mass parameters of the scalar potential are fine tuned and two of the four equations become trivial (just like in the model studied in Section 2). But even this fine tuned solution is not phenomenologically acceptable, because it is degenerate and predicts two massless fields. Radiative corrections would relax the required amount of fine tuning and would give mass to all the fields, but still the model has two particles which seem to be too light [5]. Therefore, there is no SCPV in the 4D model neither. The $R$ model discussed in section II is in some way a particular case of the 4D model, with an analogous negative result.

In this section we explore further extensions of the MSSM. We shall consider the models with three pairs of Higgs doublets (6D model), with two pairs of doublets plus one singlet (4D1S model) and with two singlets (2S model). When adding singlets, we will not include in the superpotential any couplings with dimensions of mass: $\mu$ terms for the doublets or linear and bilinear terms for the singlets. When such a term does not appear in the superpotential, we will not include the corresponding soft SUSY breaking term in the scalar potential neither (we assume that the SUSY breaking mechanism respects the discrete symmetries of the superpotential).

The scalar potential for the neutral fields is in each case

\[
V_{6D} = \sum_{i=1}^{6} m_i^2 H_i^\dagger H_i + ( m_{13}^2 H_1^\dagger H_3 + m_{15}^2 H_1^\dagger H_5 + m_{35}^2 H_3^\dagger H_5 + \text{h.c.} ) \\
+ ( m_{24}^2 H_2^\dagger H_4 + m_{26}^2 H_2^\dagger H_6 + m_{36}^2 H_4^\dagger H_6 + \text{h.c.} ) \\
+ ( m_{12}^2 H_1 H_2 + m_{14}^2 H_1 H_4 + m_{16}^2 H_1 H_6 + m_{32}^2 H_3 H_2 + m_{34}^2 H_3 H_4 \\
+ m_{36}^2 H_3 H_6 + m_{52}^2 H_5 H_2 + m_{54}^2 H_5 H_4 + m_{56}^2 H_5 H_6 + \text{h.c.} ) \\
+ \frac{1}{8} (g^2 + g'^2) [H_1^2 + H_3^2 + H_5^2 - H_2^2 - H_4^2 - H_6^2]^2 ,
\]

(17)

\[
V_{4D1S} = \sum_{i=1}^{4} m_i^2 H_i^\dagger H_i + m_5^2 S^\dagger S + ( m_{13}^2 H_1^\dagger H_3 + m_{24}^2 H_2^\dagger H_4 + \text{h.c.} ) \\
+ ( \beta_{12} S H_1 H_2 + \beta_{14} S H_1 H_4 + \beta_{32} S H_3 H_2 + \beta_{34} S H_3 H_4 \\
+ \frac{\beta_5}{2} S^3 + \text{h.c.}) + | \alpha_{12} S H_2 |^2 + | \alpha_{12} S H_1 |^2 + | \alpha_{34} S H_4 |^2 \\
+ | \alpha_{34} S H_5 |^2 + | \alpha_{12} H_1 H_2 + \alpha_{34} H_3 H_4 + \lambda S S |^2 \\
+ \frac{1}{8} (g^2 + g'^2) [H_1^2 + H_3^2 - H_2^2 - H_4^2]^2 ,
\]

(18)
and

\[ V_{2S} = \sum_{i=1}^{4} m_i^2 S_i^4 + \frac{1}{2} (\beta_3 S_1^2 S_2 + \beta_4 S_2^2 S_1 + \beta_1 S_3^2 + \beta_2 S_2^3 + \text{h.c.}) \]
\[ + | \alpha_3 S_1 S_2 + \frac{\alpha_4}{2} S_2 + \alpha_1 S_1 |^2 + | \frac{\alpha_3}{2} S_1^2 + \alpha_4 S_2 S_1 + \alpha_2 S_2^2 |^2. \]

(19)

In the 4D1S model we have rotated the doublets in the superpotential so that \( \alpha_{14} = \alpha_{32} = 0 \).

In the 2S model we shall consider for simplicity only the singlet sector, since this will be enough to prove that there is SCPV.

In the 6D model we search for a complex minimum of type

\[ \langle H_i \rangle = \frac{1}{\sqrt{2}} v_i e^{i\delta_i} \quad (i = 1, 6), \]

(20)

with \( \delta_1 = 0 \). The minimum conditions for the phases \( \delta_i \) give five equations. In these equations there appear 15 combinations of masses and moduli \( m_{ij}^2 v_i v_j \). Following the procedure described in Section 2 we find that the geometrical solution consists of the ten triangles in Fig. (2)\( ^4 \).

A given value of \( m_{ij}^2 v_i v_j \) fixes the multitrangle solution:

\[
\begin{align*}
  m_{12}^2 v_1 v_2 &= a_1 - b_1 - x_1 + y_1, \\
  m_{13}^2 v_1 v_3 &= a_2 - c_2 - w_1 + e_2, \\
  m_{15}^2 v_1 v_5 &= f_2 - b_2 - d_2 + w_2, \\
  m_{14}^2 v_1 v_4 &= x_2 - c_1 - d_1 + z_1, \\
  m_{16}^2 v_1 v_6 &= f_1 - y_2 - z_2 + e_1, \\
  m_{23}^2 v_2 v_3 &= b_3, \\
  m_{24}^2 v_2 v_4 &= x_3, \\
  m_{25}^2 v_2 v_5 &= c_3, \\
  m_{26}^2 v_2 v_6 &= y_3, \\
  m_{36}^2 v_3 v_6 &= e_3, \\
  m_{35}^2 v_3 v_5 &= w_3, \\
  m_{46}^2 v_4 v_6 &= z_3, \\
  m_{54}^2 v_5 v_4 &= d_3, \\
  m_{56}^2 v_5 v_6 &= f_3.
\end{align*}
\]

(21)

We can now choose a particular 10-triangle and find the mass parameters which correspond to that solution. Once we have these parameters we find the second derivatives of the potential to check that the solution is indeed a minimum. It turns out that the minimum is always degenerate: any 10-triangle solution occurs for a set of mass parameters giving two massless eigenstates (in addition to the goldstone boson of the hypercharge). The situation here is completely analogous to the 4D model. The minimum equations have in general no solution. A solution is obtained only for a fine tuned choice of mass parameters that renders two of the six moduli equations trivial. Then there appear the distinctive two massless eigenstates. We

---

\( ^4 \)In a generic SUSY model with 2n Higgs doublets one can choose (n-1)(2n-1) triangles with all possible pairs of phases. These will have n(2n-1) independent sides, equal to the number of \( m_{ij}^2 \) that appear in the equations.
have generated numerically random 10-triangle solutions and have obtained always the same type of nonacceptable minimum. In consequence, we conclude that there is no SCPV in 4D and 6D models: complex minima will require the presence of singlet fields.

Next we study the 4D1S case. The VEVs can be written
\[ \langle H_i \rangle = \frac{1}{\sqrt{2}} v_i e^{i \delta_i} \quad (i = 1, 4) , \]
\[ \langle S \rangle = v_5 e^{i \delta_5} , \]
with \( \delta_1 = 0 \). The minimum conditions for the phases \( \delta_i \) define three equations. It is convenient to rename \((\delta_2 + \delta_5) \rightarrow \delta_2, (\delta_4 + \delta_5) \rightarrow \delta_4, \) and \(3\delta_5 \rightarrow \delta_5, \) and \( \delta_3 \) remains unchanged. In Fig. 3 we plot the combination of triangles that solves the 5 equations. The sides in these triangles are related to the parameters of the scalar potential:

\[
\begin{align*}
\beta_{12} v_1 v_2 v_5 &= a_1 + b_1 + x_2 , \quad \beta_{32} v_3 v_2 v_5 = a_3 , \quad m_{24}^2 v_2 v_4 = y_3 - b_3 , \\
\beta_{34} v_3 v_4 v_5 &= z_1 - c_3 , \quad \beta_{14} v_1 v_4 v_5 = c_2 - b_2 , \quad m_{13}^2 v_1 v_3 = a_2 - y_1 + c_1 , \\
\alpha_{12} \lambda v_1 v_2 v_5^2 &= -x_3 , \quad \alpha_{34} \lambda v_3 v_4 v_5^2 = z_3 , \\
\frac{1}{2} \alpha_{12} \alpha_{34} v_1 v_2 v_3 v_4 &= -y_2 , \quad \beta_5 v_5^3 = -x_1 - z_2 .
\end{align*}
\]

The six triangles in Fig. 3 depend on the ten independent distances above. We proceed like in the 6D model, choosing numerically a particular multitriangle solution and adjusting the parameters in the potential in order to have that minimum. Then we check if the point is really a minimum and we find its properties. We obtain that, although the point given by this method has zero first derivatives, it is never a minimum. Like in the singlet model analyzed by Romao [12], here the Hessian has the negative eigenvalues that characterize a saddle point. For all the (random) cases that we have produced we find negative eigenvalues, never a minimum. We conclude that the extension of the MSSM with one singlet plus an extra pair of doublets does not offer the possibility of SCPV neither.

Let us finally consider the 2S model in Eq. (19). We search for complex minima of type
\[ \langle S_i \rangle = v_i e^{i \delta_i} \quad (i = 1, 2) . \]
The two minimum equations for the phases are solved by the set of four triangles in Fig. 4, where the six independent distances in the triangles are related to the six independent parameters in the scalar potential:

\[
\begin{align*}
\beta_3 v_1^2 v_2 &= -c_1 , \quad \beta_4 v_1 v_2^2 = -a_2 , \quad 3 \beta_1 v_1^3 = 2c_2 + a_3 + c_2 \frac{x_3}{c_1} - a_3 \frac{b_1}{a_2} ,
\end{align*}
\]
\[3 \beta_2 v_2^3 = b_2 - x_1 + b_2 \frac{c_3}{b_3} + x_1 \frac{2a_1}{x_2},\]
\[v_1 v_2 (\alpha_3 \alpha_4 v_2^2 + 2 \alpha_1 \alpha_3 v_1^2 + \alpha_3 \alpha_4 v_1^2 + 2 \alpha_2 \alpha_4 v_2^2) = b_3,\]
\[2 v_1^2 v_2^2 (\alpha_1 \alpha_4 + \alpha_2 \alpha_3) = -d_2.\]

From the triangles we can construct the solution with \(v_1 = 1, v_2 = 1.5, \theta_1 = \pi/6\) and \(\theta_2 = \pi/12\). This minimum corresponds to \(\beta_1 = 0.61, \beta_2 = 0.05, \beta_3 = -0.33, \beta_4 = -0.13,\)
\(\alpha_1 = 2.8, \alpha_2 = 1.3, \alpha_3 = 1, \alpha_4 = -0.58, m_1^2 = -25\) and \(m_2^2 = -6.1.\)

The spectrum in the scalar sector is found diagonalizing the \(4 \times 4\) matrix

\[
M^2 = \begin{pmatrix}
2.8 & -1.0 & -5.8 & 4.3 \\
-1.0 & 0.77 & 2.6 & -2.0 \\
-5.8 & 2.6 & 91. & 6.7 \\
4.3 & -2.0 & 6.7 & 20.
\end{pmatrix}
\]

The eigenvalues give \(m_1 = 9.6, m_2 = 4.6, m_3 = 1.1\) and \(m_4 = 0.56.\) This particular case proves that there is SCPV in SUSY models containing two singlet fields.

### 4 Conclusions

It is well known that the standard model and its minimal SUSY extension do not allow for SCPV. To know whether in a more complicated Higgs sector \(CP\) can be broken spontaneously requires solving equations that, in general, are difficult to handle. We have shown that one can always build a combination of triangles which is a solution of the minimum equations for the complex phases. In simpler cases (3D, 2D1S, \(R\) and 4D models) the triangles are enough to solve also analytically the minimum equations for the moduli, and in more complicated cases they help to find numerical solutions.

Using this method we have analyzed the possibility of SCPV in SUSY models. Despite the large number of arbitrary parameters present in these models, we find that a Higgs sector with only doublets does not provide SCPV: if all the parameters in the Lagrangian are real, then the Higgs VEVs can not be complex. This result has been proven for the 4D model, the \(R\) model with sneutrino VEVs (analogous to a 5D model), and the 6D model. In consequence, SUSY scenarios for SCPV require singlets. We find that, if the singlets do not introduce dimensional parameters (i.e., no linear or bilinear terms in the superpotential), one singlet is not enough to generate SCPV: the 2D1S and the 4D1S models have always real minima.
The MSSM extended with two gauge singlets would be the minimal SUSY model where \( CP \) violation can be generated spontaneously.

Other possibilities consistent with a soft origin of \( CP \) violation would be constrained by this result, like the 4D model in Ref. [1]. There all the parameters in the Lagrangian are taken real except for the mass terms of the Higgs fields. It is argued that these masses could appear at higher energy scales from large VEVs of singlet fields weakly coupled to the Higgs doublets, and thus they can in principle be complex. Since our analysis is valid also in scenarios with a hierarchy between singlet and doublet VEVs, it follows that the model at the large scale must include at least two singlets. Other models that would require at least two singlets to obtain soft \( CP \) violation within specific supersymmetric models can be found in [8, 22, 23].

We would like to emphasize that our method to find complex minima is not restricted to SUSY models. Any potential with all the parameters real will give minimum equations for the phases of the VEVs that involve only sines (no cosines) of the phases. In this case, using the sine law one can define a combination of triangles that solve the equations. This is true for all supersymmetric potentials regardless of the \( CP \) properties of the fields, as well as for all nonsupersymmetric potentials with \( CP \) even fields and even for the physically interesting minimal extension of the SM with one \( CP \) odd real singlet. Of course, there is always the trivial solution with all phases equal to zero and no SCPV. But the search for complex minima beyond the simplest cases seems almost unworkable unless a method like the one described in Section 2 is used.

Acknowledgments

We thank Alex Pomarol, Goran Senjanović and Atsushi Yamada for helpful comments and discussions. M.M. thanks ICTP for its hospitality during the course of this work. The work of M.M. was supported by CICYT under contract AEN96-1672 and by the Junta de Andalucía under contract FQM-101. The work of A.R. was supported in part by EEC grant under the TMR contract ERBFMRX-CT960090. Part of this work has been done during the ICTP Extended Workshop on Astroparticle Physics.

References

[1] L. Hall and S. Weinberg, Phys. Rev. D48, 979 (1993), hep-ph/9303241.
[2] W. Buchmüller and D. Wyler, Phys. Lett. **B121**, 321 (1983); J. Polchinski and M.B. Wise, Phys. Lett. **B125**, 393 (1983); A. de Rújula *et al.*, Phys. Lett. **B245**, 640 (1990).

[3] T.D. Lee, Phys. Rev. **D8**, 1226 (1973); Phys. Rep. **96**, 143 (1976).

[4] K.S. Babu and S.M. Barr, Phys. Rev. Lett. **72**, 2831 (1994), [hep-ph/9308217](http://arxiv.org/abs/hep-ph/9308217).

[5] M. Masip and A. Rašin, Nucl. Phys. **B460**, 449 (1996), [hep-ph/9508365](http://arxiv.org/abs/hep-ph/9508365).

[6] H. Georgi, Hadronic J. **1**, 155 (1978); M.A.B. Beg and H.-S. Tsao, Phys. Rev. Lett. **41**, 278 (1978); R.N. Mohapatra and G. Senjanović, Phys. Lett. **79B**, 283 (1978); G. Segre and H.A. Weldon, Phys. Rev. Lett. **42**, 1191 (1979); S. Barr and P. Langacker, Phys. Rev. Lett. **42**, 1654 (1979);

[7] A. Nelson, Phys. Lett. **136B**, 165 (1984); S.M. Barr, Phys. Rev. Lett. **53**, 329 (1984); Phys. Rev. **D30**, 1805 (1984); S.M. Barr and A. Zee, Phys. Rev. Lett. **55**, 2253 (1985); L. Lavoura, Phys. Lett. **B400**, 152 (1997), [hep-ph/9701221](http://arxiv.org/abs/hep-ph/9701221).

[8] R.N. Mohapatra, A. Rašin and G. Senjanović, Phys. Rev. Lett. **79**, 4744 (1997), [hep-ph/9707281](http://arxiv.org/abs/hep-ph/9707281).

[9] N. Maekawa, Phys. Lett. **B282**, 387 (1992).

[10] A. Pomarol, Phys. Lett. **B287**, 331 (1992).

[11] M. Masip, R. Muñoz-Tapia and A. Pomarol, Univ. Granada preprint UG-FT-84/97, [hep-ph/9801437](http://arxiv.org/abs/hep-ph/9801437).

[12] J.C. Romão, Phys. Lett. **B287**, 331 (1986).

[13] A. Pomarol, Phys. Rev. **D47**, 273 (1993), [hep-ph/9208203](http://arxiv.org/abs/hep-ph/9208203).

[14] M. Masip and A. Rašin, Phys. Rev. **D52**, 3768, (1995), [hep-ph/9506471](http://arxiv.org/abs/hep-ph/9506471).

[15] G. Branco, Phys. Rev. Lett. **44**, 504 (1980). Phys. Rev. **D22**, 2901 (1980).

[16] S. Weinberg, Phys. Rev. Lett. **37**, 657 (1976).

[17] N.G. Deshpande and E. Ma, Phys. Rev. **D16**, 1583 (1977).

[18] G. Dvali, A. Melfo and G. Senjanović, Phys. Rev. **D54**, 7857 (1996), [hep-ph/9601370](http://arxiv.org/abs/hep-ph/9601370).

For a model of this type but with a complex singlet, see L. Bento and G. Branco, Phys. Lett. **B245**, 599 (1990).
[19] S.L. Glashow and S. Weinberg, Phys. Rev. D15, 1958 (1977).

[20] A.S. Joshipura and M. Nowakowski, Phys. Rev. D51, 5271 (1995), hep-ph/9403349.

[21] K.S. Babu and S.M. Barr, Phys. Rev. D49, 2156 (1994), hep-ph/9308217.

[22] P.H. Frampton and O.C.W. Kong, Phys. Lett. B402, 297 (1997).

[23] G. Eyal and Y. Nir, hep-ph/9801411.
Figure 1: The six triangles that make up the geometrical solution for the minimum phase equations for the $R$ model. The lower right triangle is the solution for the 3 Higgs doublet extension of the SM. The relations between the sides of the triangles and the mass parameters are given in the text.
Figure 2: 6D model.
Figure 3: 4D1S model.
Figure 4: 2S model.