PLANAR HARMONIC POLYNOMIALS OF TYPE B

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Abstract. The hyperoctahedral group acting on $\mathbb{R}^N$ is the Weyl group of type $B$ and is associated with a two-parameter family of differential-difference operators $\{T_i : 1 \leq i \leq N\}$. These operators are analogous to partial derivative operators. This paper finds all the polynomials $h$ on $\mathbb{R}^N$ which are harmonic, $\Delta_B h = 0$ and annihilated by $T_i$ for $i > 2$, where the Laplacian $\Delta_B = \sum_{i=1}^N T_i^2$. They are given explicitly in terms of a novel basis of polynomials, defined by generating functions. The harmonic polynomials can be used to find wave functions for the quantum many-body spin Calogero model.

1. Introduction

For each finite reflection group there are families of invariant inner products on the space of polynomials, defined by an algebraic expression, and by integration with respect to invariant weight functions on the sphere or on all of Euclidean space. These inner products essentially coincide on the polynomials harmonic with respect to the associated Laplacian. In this paper we study certain specific explicit harmonic polynomials associated with the hyperoctahedral group on $\mathbb{R}^N$. For the reflection groups on $\mathbb{R}^2$ all the harmonic polynomials are known as expressions in Jacobi polynomials. Also orthogonal bases whose elements are of generalized Hermite (or Laguerre) type, for the Gaussian weight function have been determined by means of a construction using nonsymmetric Jack polynomials. However an orthogonal basis for the weight functions on the sphere (and the ball or the simplex) has not yet been explicitly found. Here we consider the analogue of ordinary harmonic polynomials in two variables, but harmonic for the $N$-variable Laplacian $\Delta_B$; that is, polynomials annihilated by $T_i$ for $i > 2$, where $\{T_i : 1 \leq i \leq N\}$ is the set of differential-difference operators of type $B$ and $\Delta_B = \sum_{i=1}^N T_i^2$. In previous work the author introduced a family of polynomials (the “$p$-basis”) for which it is easy to write down polynomials annihilated by any desired subset of $\{T_i : 1 \leq i \leq N\}$. In this study it is important to select a set of polynomials for which the harmonic polynomials have “nice” coefficients. For example, coefficients of hypergeometric type (Pochhammer symbols) are “nice”. We introduce a set of polynomials which are in the $\mathbb{Q}$-span of the $p$-basis (coefficients independent of the parameters) and which allow nice expressions. The definition is by means of generating functions.

For the harmonic polynomials we will find the values at a special point, $(1, 1, \ldots, 1)$, the leading coefficients, and the $L^2$-norms; the first two are in closed form using $2F_1$ and $3F_2$ summations respectively, the last is a balanced $4F_3$ sum. Finally

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there is a discussion of the important applications of the polynomials, especially as
wave functions for spin Calogero quantum-many-body models.

2. Overview of Results

The finite group of orthogonal transformations which is generated by sign-changes
and permutation of coordinates on \( \mathbb{R}^N \) is called the Weyl group of type \( B \), and will
be denoted by \( W_N \). There is a family of measures associated with \( W_N \) in a natural
way: for positive parameters \( k, k_1 \) let

\[
d\mu(x; k, k_1) = \prod_{i=1}^{N} |x_i|^{2k_1} \prod_{1 \leq i < j \leq N} |x_i^2 - x_j^2|^{2k} \exp\left(-\frac{|x|^2}{2}\right) dx
\]

Analysis for functions related to these measures depends on the differential-difference
operators constructed by the author \[2\]. Note that the polynomial terms in
\( d\mu \) correspond to linear functions vanishing on the reflecting hyperplanes of the Coxeter
group \( W_N \). The reflections in \( W_N \) consist of \{ \( \sigma_i : 1 \leq i \leq N \} \) and \{ \( \sigma_{ij}, \tau_{ij} : 1 \leq i < j \leq N \} \) defined by

\[
\sigma_i = (x_1, \ldots, -\frac{i}{x_i}, \ldots, x_N), \\
\sigma_{ij} = (x_1, \ldots, \frac{i}{x_j}, \ldots, \frac{j}{x_i}, \ldots, x_N), \\
\tau_{ij} = (x_1, \ldots, -\frac{i}{x_j}, \ldots, -\frac{j}{x_i}, \ldots, x_N).
\]

For notational convenience \( \sigma_{ji} = \sigma_{ij} \) and \( \tau_{ji} = \tau_{ij} \). We use the same symbols to
indicate the action on functions, for example, \( \sigma_{ij} f(x) := f(x\sigma_{ij}) \). The differential-difference ("Dunkl") operators of type \( B \) (associated with \( W_N \) are)

\[
T_i := \frac{\partial}{\partial x_i} + k_1 \frac{1 - \sigma_i}{x_i} + k \sum_{j \neq i} \left\{ \frac{1 - \sigma_{ij}}{x_i - x_j} + \frac{1 - \tau_{ij}}{x_i + x_j} \right\}, 1 \leq i \leq N.
\]

They are homogeneous of degree \(-1\) on polynomials and commute, \( T_i T_j = T_j T_i \).
The Laplacian operator is \( \Delta_B := \sum_{i=1}^{N} T_i^2 \).

In this paper we will determine all polynomials \( h \) on \( \mathbb{R}^N \) which satisfy \( \Delta_B h = 0 \)
and \( T_i h = 0 \) for all \( i > 2 \). In the standard case \( k = 0 = k_1 \) this implies that \( h \)
depends only on \( x_1, x_2 \); this explains the name "planar".

Definition 1. The set of polynomials \( \{ \phi_{n,j}, \psi_{n,j} : 0 \leq j \leq n = 0, 1, 2, \ldots \} \) is
defined by

\[
\sum_{n=0}^{\infty} \sum_{j=0}^{n} \phi_{n,j}(x) s^j t^n = F_0 + F_1,
\]

\[
\sum_{n=0}^{\infty} \sum_{j=0}^{n} \psi_{n,j}(x) s^j t^n = x_1(F_0 + sF_1) - x_1 F_1
\]
in terms of the generating functions \((x \in \mathbb{R}^N, \text{ and absolute convergence holds for } |s| < 4/3 \text{ and } |t| < \min(1/x_i^2 : 1 \leq i \leq N)/3):\)

\[
F_0(x; s, t) = \frac{1 - st(x_1^2 + x_2^2) + t^2 x_1^2 x_2^2}{(1 - 2stx_1^2 + t^2x_1^4)(1 - 2stx_2^2 + t^2x_2^4)} \prod_{i=1}^{N} (1 - 2stx_i^2 + t^2x_i^4)^{-k},
\]

\[
F_1(x; s, t) = \frac{t(x_1^2 - x_2^2)}{(1 - 2stx_1^2 + t^2x_1^4)(1 - 2stx_2^2 + t^2x_2^4)} \prod_{i=1}^{N} (1 - 2stx_i^2 + t^2x_i^4)^{-k}.
\]

In each case, the first and second terms on the right hand side produce the basis functions with \(n + j = 0 \text{ mod } 2\) and \(n + j = 1 \text{ mod } 2\), respectively. The polynomials \(\phi_{n,j}\) and \(\psi_{n,j}\) are of degrees \(2n\) and \(2n+1\) in \(x\), respectively, for \(0 \leq j \leq n\). Because of the invariance of \(\Delta_B\) under the subgroup \(W_2\) acting on the first two variables, there is a basis of planar harmonic polynomials in which all the monomials have the same parities of the exponents of \(x_1, x_2\); of course they are even in each of the remaining variables.

2.1. The harmonic polynomials: The basis elements are labeled \(h_{n,0}\) and \(h_{n,1}\), where \(x_1^2x_2^2\) is the term in \(h\) with the highest power of \(x_1\) and \(\varepsilon = 0\) or 1. The formulas depend on the residues mod 4.

\[
h_{4n,0} = \sum_{j=0}^{n} \frac{(2(N-1)k + k_1 + \frac{3}{2} + 2n)j(\frac{1}{2})}{((N-1)k + k_1 + \frac{3}{2} + n)j(Nk + n + 1)} \phi_{2n+2j},
\]

\[
h_{4n+2,0} = \sum_{j=0}^{n} \frac{(2(N-1)k + k_1 + \frac{3}{2} + 2n)j(\frac{1}{2})}{((N-1)k + k_1 + \frac{3}{2} + n)j(Nk + n + 1)} \phi_{2n+2j},
\]

\[
h_{4n+1,1} = x_1x_2 \sum_{j=0}^{n} \frac{(2(N-1)k + k_1 + \frac{3}{2} + 2n)j(\frac{1}{2})}{((N-1)k + k_1 + \frac{3}{2} + n)j(Nk + n + 1)} \phi_{2n+2j},
\]

\[
h_{4n+3,1} = x_1x_2 \sum_{j=0}^{n} \frac{(2(N-1)k + k_1 + \frac{3}{2} + 2n)j(\frac{1}{2})}{((N-1)k + k_1 + \frac{3}{2} + n)j(Nk + n + 1)} \phi_{2n+2j}.
\]

The following are of mixed parity:

\[
h_{4n+1,0} = \sum_{j=0}^{n} \frac{(2(N-1)k + k_1 + \frac{3}{2} + 2n)j(\frac{1}{2})}{((N-1)k + k_1 + \frac{3}{2} + n)j(Nk + n + 1)} \psi_{2n+2j} + \sum_{j=1}^{n} \frac{(2(N-1)k + k_1 + \frac{3}{2} + 2n)j(\frac{1}{2})}{((N-1)k + k_1 + \frac{3}{2} + n)j(Nk + n + 1)} \psi_{2n+2j-1}
\]

and

\[
h_{4n+3,0} = \sum_{j=0}^{n} \frac{(2(N-1)k + k_1 + \frac{3}{2} + 2n)j(\frac{1}{2})}{((N-1)k + k_1 + \frac{3}{2} + n)j(Nk + n + 2)} \psi_{2n+2j} + \sum_{j=1}^{n+1} \frac{(2(N-1)k + k_1 + \frac{3}{2} + 2n)j(\frac{1}{2})}{((N-1)k + k_1 + \frac{3}{2} + n)j(Nk + n + 2)} \psi_{2n+2j-1}
\]

There are two other basis elements, defined as \(h_{4n,1} := \sigma_{12}h_{4n+1,0}\) and \(h_{4n+2,1} := \sigma_{12}h_{4n+3,0}\). Furthermore, \(\sigma_{12}h_{4n,0} = h_{4n,0}, \sigma_{12}h_{4n+1,1} = h_{4n+1,1}, \text{ and } \sigma_{12}h_{4n+2,0} = \)
constructed as follows: for $1 \leq i \leq n$, the polynomials $h_{n+1,i}$ are (for $i > 1$, that is, a scalar multiple of another polynomial of this family. Specifically the results

2.2. Action of $T_i$: Since $\Delta_B$ commutes with each $T_i$ the action of $T_1$ or $T_2$ on any of the polynomials $h_{n, \varepsilon}$ produces a polynomial annihilated by $\Delta_B$ and $T_i$, $i > 2$, that is, a scalar multiple of another polynomial of this family. Specifically the results are (for $n = 0, 1, 2, \ldots$):

$$T_1 h_{n,0} = 2((N - 1)k + n)h_{4n-1,0},$$
$$T_1 h_{n+2,0} = -2((N - 2)k + k_1 + n + \frac{1}{2})h_{4n+2,0},$$
$$T_1 h_{n+1,0} = 2((N - 1)k + k_1 + n + \frac{1}{2})h_{4n+1,0},$$
$$T_2 h_{n,0} = 2(Nk + n + 1)h_{4n+1,0},$$
$$T_2 h_{n+1,0} = 2((N - 1)k + n)h_{4n+1,1},$$
$$T_2 h_{n+2,0} = 2((N - 2)k + k_1 + n + \frac{1}{2})h_{4n+1,0},$$
$$T_2 h_{n+3,0} = 2((N - 1)k + n)h_{4n+1,1},$$
$$T_2 h_{n+3,1} = -2((N - 1)k + k_1 + n + \frac{3}{2})h_{4n+3,0}.$$

These formulae (together with the claim $T_i h_{n, \varepsilon} = 0$ for $i > 2$) already imply that $\Delta_B h_{n, \varepsilon} = 0$ because $T_2^2 h_{4n,0} = \sigma_2 T_2 h_{4n,0} = \sigma_2 T_2 h_{4n,0} = -c h_{4n-2,0}$. All the other polynomials $h_{m, \varepsilon}$ can be obtained by applying $T_1$ or $T_2$ often enough to some $h_{4n,0}$ with $4n > m$. In subsequent sections we will derive the values of $||h_{n, \varepsilon}||^2_2$, $h_{n, \varepsilon}(1, 1, \ldots, 1)$, $h_{n, \varepsilon}(1, 0, \ldots, 0)$ and coefficients of leading terms. All but the $L^2$-norms are products of linear factors in the parameters, while the norms are expressed as sums of balanced $4F_3$-series.

3. Symbolic Calculus

The results described above depend on the basis of polynomials introduced in $\mathbb{R}$ for type $A$, $\mathbb{F}$ for type $B$. The idea is to replace the variables in the type-$A$ basis by $x_1^2, x_2^2, \ldots, x_N^2$ and then use the expressions for $T_i$ in terms of corresponding type-$A$ operators. Throughout let $y = (y_1, y_2, \ldots, y_N) = (x_1^2, x_2^2, \ldots, x_N^2)$ for $x \in \mathbb{R}^N$.

The type-$A$ Dunkl operator is defined by

$$\tilde{T}_i = \frac{\partial}{\partial y_i} + k \sum_{j \neq i} \frac{1 - (ij)}{y_i - y_j}, 1 \leq i \leq N,$$

where $(ij)$ denotes the transposition of $y_i$ and $y_j$, the effect of $\sigma_{ij}$ or $\tau_{ij}$ on the squared variables. The polynomials in $x \in \mathbb{R}^N$ are spanned by polynomials of the form $x^\varepsilon g(y)$ where $\varepsilon = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_N)$ and $x^\varepsilon = x_1^{\varepsilon_1} x_2^{\varepsilon_2} \ldots$ with each $\varepsilon_i = 0$ or 1.

**Proposition 1.** Let $f(x) = x^\varepsilon g(y)$ with each $\varepsilon_i = 0$ or 1. For $i = 1, 2, \ldots, N$,

$$T_i f(x) = 2x_i x^\varepsilon \tilde{T}_i g(y), \text{ if } \varepsilon_i = 0,$$

$$T_i f(x) = 2x_i x^\varepsilon \left( (k_1 - \frac{1}{2}) g + \tilde{T}_i (y_i g) - k \sum_j ((ij)g : \varepsilon_j = 1, j \neq i) \right), \text{ if } \varepsilon_i = 1.$$

This is Proposition 2.1 in $\mathbb{F}$. The $p$-basis for the symmetric group action is constructed as follows: for $1 \leq i \leq N$ the polynomials $p_n(y_i; y)$ are given by the
Proposition 2. For \( \mathcal{T} \) replaces \( \{ f \} \)

For a power series \( \sum_{n=0}^{\infty} p_n(y_i; y) r^n = (1 - ry_i)^{-1} \prod_{j=1}^{N} (1 - ry_j)^{-k} \);

then for \( \alpha = (\alpha_1, \ldots, \alpha_N) \in \mathbb{Z}^{\mathbb{N}} \) the collection of compositions, the basis element

\[ p_{\alpha} = \prod_{i=1}^{N} p_{\alpha_i}(y_i; y). \]

The key property is that \( \widehat{T}_i p_{\alpha}(y_i; y) = 0 \) for \( j \neq i \). It was shown in \( \square \) that

\[
\widehat{T}_i p_{\alpha} = (Nk + \alpha_i)p_{\alpha_i - 1}(y_i; y) \prod_{m \neq i} p_{\alpha_m}(y_m; y) + k \sum_{j \neq i} \left\{ \sum_{m=0}^{\alpha_j - 1} (p_{\alpha_i+\alpha_j-1-m}(y_i; y)p_{m}(y_j; y) - p_{m}(y_i; y)p_{\alpha_i+\alpha_j-1-m}(y_j; y)) \prod_{n \neq i,j} p_{\alpha_n}(y_n; y) \right\},
\]

if \( \alpha_j > 0 \), and \( \widehat{T}_i p_{\alpha} = 0 \) if \( \alpha_i = 0 \). On the right hand side of this formula the same
term may appear twice, but this does not matter for the intended use. We will set
up a linear isomorphism between the span of the \( p \)-basis and polynomials in the
formal variables \( p_1, p_2, \ldots, p_N \) which is generally not multiplicative but does allow
a simple formula for \( \widehat{T}_i \).

**Definition 2.** The linear isomorphism \( \Psi \) between the span of the \( p \)-basis and the
space \( \mathcal{P} \) of polynomials in the formal variables \( p_1, p_2, \ldots, p_N \) is given by \( \Psi : p_{\alpha} \mapsto p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_N^{\alpha_N} \)
(and extended by linearity). Further the linear transformations \( \xi_{i,j} \)
and \( \eta_i \) on \( \mathcal{P} \) are defined by \( \xi_{i,j} : p_1 \mapsto p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_N^{\alpha_N} = p_1^{\alpha_1} \prod_{m \neq i,j} p_m^{\alpha_m} \) for \( i \neq j \), and \( \eta_i \) is evaluation at \( p_i = 0 \), for \( 1 \leq i \leq N \). (That is, \( \xi_{i,j} \) replaces \( p_j \) by \( p_i \), and \( \eta_i \) replaces \( p_i \) by \( 0 \).)

It is clear that \( \Psi \) commutes with the \( S_N \)-action. We use the simplified notation
\( \widehat{T}_i \) for the operator \( \Psi \widehat{T}_i \Psi^{-1} \) on \( \mathcal{P} \).

**Proposition 2.** For \( 1 \leq i \leq N \), the operator \( \widehat{T}_i \) acts on polynomials in \( \mathcal{P} \) by

\[
\widehat{T}_i = \frac{\partial}{\partial p_i} + Nk \frac{1 - \eta_i - \sum_{j \neq i} \xi_{i,j} - 1 - (i, j)}{p_i - p_j}.
\]

**Proof.** It suffices to examine the effect of the formula on monomials \( p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_N^{\alpha_N} \)
and for \( i = 1 \). The first two terms produce \( (\alpha_1 + Nk) \) if \( \alpha_1 > 0 \), else 0. In the sum, the
(typical) term for \( j = 2 \) is \( (p_1^{\alpha_1+\alpha_2} + p_2^{\alpha_1+\alpha_2} - p_1^{\alpha_1+\alpha_2} - p_1^{\alpha_2} p_2^{\alpha_2}) \prod_{m=3}^{N} p_m^{\alpha_m} / (p_1 - p_2) \).

A simple calculation shows this is the image under \( \Psi \) of the corresponding term in
equation \( 3.1 \).

We will use generating functions to determine the effects of \( T_1 \) and \( T_2 \) on the
polynomials defined in the previous sections (now we are considering the type-B
operators). For this purpose we consider the cases \( f(p), x_1 f(p), x_1 x_2 f(p) \) where \( f(p) \) is a formal series in \( p = (p_1, p_2) \) (the validity of term-by-term action comes
from the same argument used to justify term-by-term differentiation of a power
series inside its disk of convergence). In the following, we use the notation:

\[
\delta_{1,2} f(p_1, p_2) = (f(p_1, p_1) + f(p_2, p_2) - f(p_1, p_2) - f(p_2, p_1)) / (p_1 - p_2).
\]

**Proposition 3.** For a power series \( f(p_1, p_2) \) (absolutely convergent in the region
\( \{(p_1, p_2) : |p_1| < 1, |p_2| < 1 \} \) the following hold:
1. \( T_1 f(p) = 2x_1 \left( \frac{\partial f(p)}{\partial p_1} + Nk \frac{f(p_1, p_2) - f(0, p_2)}{p_1} + k\delta_{1,2} f(p) \right), \)

2. \( T_1(x_1 f(p)) = 2 \left( ((N - 1)k + k_1 + \frac{1}{2}) f(p) + p_1 \frac{\partial f(p)}{\partial p_1} + k\delta_{1,2}(p_1 f(p)) \right), \)

3. \( T_2(x_1 x_2 f(p)) = 2x_1 \left( ((N - 1)k + k_1 + \frac{1}{2}) f(p) + p_2 \frac{\partial f(p)}{\partial p_2} - k\delta_{1,2}(p_2 f(p)) - k f(p_2, p_1) \right), \)

4. \( T_2(x_1 f(p)) = 2x_1 x_2 \left( \frac{\partial f(p)}{\partial p_2} + Nk \frac{f(p_1, p_2) - f(p_1, 0)}{p_2} - k\delta_{1,2} f(p) \right). \)

Proof. Formulas (1) and (4) follow immediately from Proposition 1. It was shown in Lemma 2.3 of [6] that \( \hat{T}_i y_i = \hat{T}_i \hat{\rho}_i - k, \) where \( \hat{\rho}_i \) is the conjugate under \( \Psi \) of multiplication by \( p_i \) acting on \( P. \) Together with Proposition 2 this proves formulas (2) and (3). \)

By the fundamental properties of the \( p \)-basis, \( T_i(x_1^{r+1} x_2^s f)(p_1, p_2) = 0 \) for all \( i > 2, \) and \( \varepsilon_1, \varepsilon_2 = 0 \) or 1. This applies to all the polynomials used in the sequel. The images under \( \Psi \) of the generating functions \( F_0, F_1 \) defined in section 2 are in fact simple rational functions in \( p = (p_1, p_2). \) Indeed for indeterminates \( u_1, u_2, \) from the definition of the \( p \)-basis it follows that

\[
\Psi \left( (1 - u_1 y_1)^{-1} (1 - u_2 y_2)^{-1} \prod_{i=1}^{N} \{(1 - u_1 y_i)(1 - u_2 y_i)\}^{-k} \right) =
\Psi \left( \sum_{m,n=0}^{\infty} p_{(m,n)} u_1^m u_2^n \right) = \sum_{m,n=0}^{\infty} p_1^m p_2^n u_1^m u_2^n = (1 - u_1 p_1)^{-1} (1 - u_2 p_2)^{-1}
\]

The desired expressions result from changing variables to \( u_1 = tz, u_2 = t^{-1}, \) and also \( u_1 = tz^{-1}, u_2 = tz, \) and \( s = \frac{1}{3}(z + z^{-1}); \) then the two terms are combined by addition and subtraction (symmetric and skew-symmetric under \( (1, 2), \)) respectively. To ensure convergence some region must be chosen, for example, \( |s| < \frac{1}{3} \) and \( |t| < 1/(3 \max(|p_1|, |p_2|)) \) (and \( s, t \in \mathbb{C}. \) This is valid because \( |r - \frac{1}{t}| \leq 2|s| \leq (r + \frac{1}{t}) \) where \( r = |z| \) and \( z \in \mathbb{C}, \) thus \( |s| < \frac{1}{3} \) implies \( \frac{1}{3} < |z| < 3. \)

The method for computing the effect of \( T_i \) on the polynomials \( \phi_{n,j} \) and \( \psi_{n,j} \) is to apply \( T_i \) to the generating functions and express the result by means of combinations of \( \frac{\partial}{\partial s} \) and \( \frac{\partial}{\partial t} \) and multiplication by \( s, t. \) The two basic functions are:

\[
w_1 = (1 - ztp_1)^{-1}(1 - z^{-1}tp_2)^{-1}
\]
\[
w_2 = (1 - z^{-1}tp_1)^{-1}(1 - ztp_2)^{-1}.
\]

Then let

\[
f_0 = \frac{1}{2} (w_1 + w_2) = \frac{1 - st(p_1 + p_2) + t^2 p_1 p_2}{(1 - 2stp_1 + t^2 p_1^2)(1 - 2stp_2 + t^2 p_2^2)}
\]
\[
f_1 = (z - z^{-1})^{-1}(w_1 - w_2) = \frac{t(p_1 - p_2)}{(1 - 2stp_1 + t^2 p_1^2)(1 - 2stp_2 + t^2 p_2^2)}
\]

The same formulas apply to the images under \( \Psi^{-1}. \)

**Proposition 4.** The generating functions in Definition 1 satisfy \( \Psi F_0 = f_0 \) and \( \Psi F_1 = f_1. \)
Proof. Apply $\Psi^{-1}$ to $w_1$ and $w_2$ using equation 3.2, then both $\Psi^{-1}w_1$ and $\Psi^{-1}w_2$ have the common factor $\prod_{i=1}^{N}(1-u_1y_i)(1-u_2y_i))^{-k} = \prod_{i=1}^{N}(1-2sty_i+t^2y_i^2)^{-k}$. The parts of the calculation involving $1-u_1y_i)^{-1}$ and $(1-u_1y_i)^{-1}, i = 1 \text{ or } 2$ proceed just like those with $w_1$ and $w_2$. \[\]

3.1. Action of $T_1$ on the generating functions: Now we can use the symbolic calculus on $f_0$ and $f_1$. Write $g_0 = f_0 + sf_1$ and $g_1 = -f_1$ for the generating functions for $\{\psi_{n,j}\}$. Then $w_1 = f_0 + \frac{1}{2}(z - z^{-1})f_1 = g_0 + z^{-1}g_1$ and $w_2 = f_0 - \frac{1}{2}(z - z^{-1})f_1 = g_0 + zg_1$. First the effect of $\delta_{1,2}$ on various functions is calculated: $\delta_{1,2}f_0 = tf_1, \delta_{1,2}f_1 = 0, \delta_{1,2}(p_2f_0) = sf_1, \delta_{1,2}(p_2f_1) = f_1, \delta_{1,2}(g_0) = tf_1, \delta_{1,2}(g_1) = 0, \delta_{1,2}(p_1g_0) = 0, \delta_{1,2}(p_1g_1) = f_1$. These simple relations are the reason for the use of this particular set of functions.

The differentiations can be done on $w_1$ and $w_2$ separately. It is easy to verify that

\[
\frac{\partial w_1}{\partial p_1} = zt(w_1 + t \frac{\partial w_1}{\partial t}) + \frac{z^2 t \partial w_1}{2 \partial z},
\frac{\partial w_2}{\partial p_1} = \frac{t}{z}(w_2 + t \frac{\partial w_2}{\partial t}) - t \frac{\partial w_2}{2 \partial z}.
\]

Further $p_1 \frac{\partial w_1}{\partial p_1} = \frac{1}{2}(\partial \frac{\partial}{\partial t} - \partial \frac{\partial}{\partial z})w_1, p_2 \frac{\partial w_1}{\partial p_1} = \frac{1}{2}(\partial \frac{\partial}{\partial t} + \partial \frac{\partial}{\partial z})w_2, (1 - \eta_1)w_1/p_1 = zw_1$

and $(1 - \eta_1)w_2/p_1 = (t/z)w_2$. In the expressions for $\frac{\partial f_0}{\partial p_1}$ and $\frac{\partial f_1}{\partial p_1}$ the following equations are used:

\[
z w_1 + z^{-1}w_2 = 2sg_0 + 2g_1, z^2 \frac{\partial w_1}{\partial z} - \frac{\partial w_2}{\partial z} = 2(s^2 - 1) \frac{\partial g_0}{\partial s} - 2g_1,
\]

\[
(z - z^{-1})^{-1}(zw_1 - z^{-1}w_2) = g_0, (z - z^{-1})^{-1}(z^2 \frac{\partial w_1}{\partial z} + \frac{\partial w_2}{\partial z}) = \frac{\partial g_0}{\partial s} + \frac{\partial g_1}{\partial s},
\]

\[
z(\frac{\partial w_1}{\partial z} - \frac{\partial w_2}{\partial z}) = -2sg_1 + (1 - s^2) \frac{\partial g_1}{\partial s},
\]

\[
z(z - z^{-1})^{-1}(\frac{\partial w_1}{\partial z} + \frac{\partial w_2}{\partial z}) = \frac{\partial g_0}{\partial s} + g_1 + s \frac{\partial g_1}{\partial s}.
\]

These, as well as the following equations can be proven by direct verification (express $w_1$ and $w_2$ in terms of $g_0$ and $g_1$, or $f_0$ and $f_1$; of course $\frac{\partial}{\partial s} = \frac{1}{2}(1-z^{-1})\frac{\partial}{\partial z}$). The formulas are grouped by type:

3.1.1. Case $T_1f(y) : \phi \to \psi$.

\[
T_1f_0 = 2x_1t \left( [(Nk + 1)s + st \frac{\partial}{\partial t} + \frac{s^2 - 1}{2} \frac{\partial}{\partial s}]g_0 + \left[(N - 1)k + \frac{1}{2} + t \frac{\partial}{\partial t} \right]g_1 \right)
\]

\[
T_1f_1 = 2x_1t \left( [(Nk + 1) + t \frac{\partial}{\partial t} + s \frac{\partial}{\partial s}]g_0 + \frac{1}{2} \frac{\partial}{\partial s}g_1 \right);
\]
3.1.2. Case $T_2(x_1x_2f(y)) : x_1x_2\phi \to \psi$.

$$T_2(x_1x_2f_0) = 2x_1\left[ (N-2)k + k_1 + \frac{1}{2} + \frac{t}{2} \frac{\partial}{\partial t} \right] g_0 +
\left[ s((N-1)k + k_1 + 1 + \frac{t}{2} \frac{\partial}{\partial t} + \frac{s^2 - 1}{2} \frac{\partial}{\partial s} \right] g_1 \right]$$

$$T_2(x_1x_2f_1) = 2x_1 \left( -\frac{1}{2} \frac{\partial}{\partial s} g_0 - \left[ (N-1)k + k_1 + 1 + \frac{t}{2} \frac{\partial}{\partial t} + \frac{s}{2} \frac{\partial}{\partial s} \right] g_1 \right)$$

3.1.3. Case $T_1(x_1f(y)) : \psi \to \phi$.

$$T_1(x_1g_0) = 2\left[ (N-1)k + k_1 + \frac{1}{2} + \frac{t}{2} \frac{\partial}{\partial t} + \frac{s}{2} \frac{\partial}{\partial s} \right] f_0 +
\left[ s((N-1)k + k_1 + 1 + \frac{t}{2} \frac{\partial}{\partial t} + \frac{s^2 - 1}{2} \frac{\partial}{\partial s} \right] f_1 \right]$$

$$T_1(x_1g_1) = 2 \left( -\frac{1}{2} \frac{\partial}{\partial s} f_0 - \left[ (N-2)k + k_1 + \frac{1}{2} + \frac{t}{2} \frac{\partial}{\partial t} \right] f_1 \right)$$

3.1.4. Case $T_2(x_1f(y)) : \psi \to x_1x_2\phi$.

$$T_2(x_1g_0) = 2x_1x_2\left( -\frac{1}{2} \frac{\partial}{\partial s} f_0 + \left[ (N-1)k + 1 + \frac{t}{2} \frac{\partial}{\partial t} \right] f_1 \right),$$

$$T_2(x_1g_1) = 2x_1x_2\left[ \left(Nk + 1\right) + t \frac{\partial}{\partial t} + \frac{s}{2} \frac{\partial}{\partial s} \right] f_0 -
\left[ s(Nk + 3 \frac{1}{2} + \frac{t}{2} \frac{\partial}{\partial t} + \frac{\partial}{\partial s} + \frac{1}{2} \frac{s^2 - 1}{2} \frac{\partial}{\partial s} \right] f_1 \right).$$

3.2. Action of $T_i$ on basis polynomials: Perusal of these formulas reveals that terms involving $k$ are almost “on the diagonal”, that is, setting $k = 0$ does not noticeably simplify the formulas. In the $\{\phi_{n,j}, \psi_{n,j}\}$ basis, the $k \neq 0$ case is not any more complicated than $k = 0$. This is the advantage of these polynomials over the ordinary $x$-basis. In each of the formulas, the result of expanding the equations in $\{\phi_{n,j}\}$ for $f_i, \{\psi_{n,j}\}$ for $g_i, (i = 0 \text{ or } 1)$ and matching up coefficients of $s^it^n$ on both sides leads to the following (grouped by the parity of $n+j$):

3.2.1. $n + j = 0 \mod 2$.

$$T_1\phi_{n,j} = (2Nk + n + j)\psi_{n-1,j-1} +
(2(N-1)k + n)\psi_{n-1,j} - (j+1)\psi_{n-1,j+1},$$

$$T_1\psi_{n,j} = (2(N-1)k + 2k_1 + n + j + 1)(\phi_{n,j} + \phi_{n,j-1}) - (j+1)\phi_{n,j+1},$$

$$T_2(x_1x_2\phi_{n,j}) = (2(N-2)k + 2k_1 + n + 1)\psi_{n,j} +
(2(N-1)k + 2k_1 + n + j + 1)\psi_{n,j-1} - (j+1)\psi_{n,j+1},$$

$$T_2\psi_{n,j} = x_1x_2 \left( (2(N-1)k + n + 1)\phi_{n-1,j} - (j+1)\phi_{n-1,j+1} \right);$$
3.2.2. $n + j = 1 \mod 2$.

\[ T_1 \phi_{n,j} = (2Nk + n + j + 1)\psi_{n-1,j} + (j + 1)\psi_{n-1,j+1}, \]
\[ T_2 \psi_{n,j} = -(2(N - 2)k + 2k_1 + n + 1)\phi_{n,j} - (j + 1)\phi_{n,j+1}, \]
\[ T_2(x_1 x_2 \phi_{n,j}) = -(2(N - 1)k + 2k_1 + n + j + 2)\psi_{n,j} - (j + 1)\psi_{n,j+1}, \]
\[ T_2 \psi_{n,j} = x_1 x_2 \left( (2Nk + n + j + 1)(\phi_{n-1,j} - \phi_{n,j-1}) + (j + 1)\phi_{n-1,j+1} \right). \]

Notice that each expression has no more than three different polynomials on
the right-hand side. In the next section we use these to determine the harmonic polynomials.

4. Properties of the harmonic polynomials

Here we demonstrate the action of $T_1, T_2$ on the harmonic polynomials, which
suffices to show $\Delta_B h_{n,e} = 0$ for each such polynomial, as mentioned before. By
construction the polynomials defined in section 2.1 satisfy $T_i h_{n,e} = 0$ for $i > 2$;
it suffices to establish the formulae of section 2.2. In a sense the proofs depend
on induction. Since the computations of $T_i h_{n,e}$ are somewhat repetitive we will
not give details on each formula. The calculations are direct; the definitions of
$F_0, F_1$ and $h_{n,e}$ were formulated after computer-algebra-aided experimentation. In
addition we determine the values at $(1, 1, \ldots, 1), (1, 0, \ldots, 0)$ and the $L^2$-norms.

4.1. The action of $T_1$ on $h$: There are four main cases, each with two parts.
Because $k_1$ always appears in the same way we introduce an (abbreviation) notation:

\[ k_2 = (N - 1)k + k_1 + \frac{1}{2}. \]

4.1.1. Case: $T_1 : h_{2m,0} \to h_{2m-1,0}$. We begin with the proof of $T_1 h_{4n,0} = 2((N - 1)k + n)h_{4n-1,0}$, for $n \geq 1$. We write $T_1 \sum_{i=0}^{m} c_i \phi_{m,i} = \sum_{i=0}^{m-1} (T_1^* c_i) \psi_{m-1,i}$, then

\[ (T_1^* c_i)_{i} = (2(N - 1)k + m)c_i + ic_{i-1}, \text{ for } m + i = 0 \mod 2, \]
\[ (T_1^* c_i)_{i} = (2Nk + m + i + 1)(c_{i+1} + c_i) - ic_{i-1}, \text{ for } m + i = 1 \mod 2. \]

These follow from the equations in section 3.2. Now let $h_{4n,0} = \sum_{j=0}^{n} a_j \phi_{2n,2j}$
with $a_j = \frac{((N - 1)k + k_2 + 2n)_{(1/2)_{j}}}{(k_2 + n)_{j}(Nk + n + 1)_{j}}$. Use the above equations with $m = 2n$, then

\[ T_1 h_{4n,0} = \sum_{j=0}^{n-1} b_j \psi_{2n-1,2j} + \sum_{j=1}^{n} c_j \psi_{2n-1,2j-1} \]
with $b_j = 2((N - 1)k + n)a_j$ and $c_j = 2(Nk + n + j)a_j - 2(j - \frac{1}{2})a_{j-1}$

\[ = 2((N - 1)k + n) \frac{((N - 1)k + k_2 + 2n)_{j-1}((1/2)_{j}}{(k_2 + n)_{j}(Nk + n + 1)_{j-1}}, \]

which is the claimed result. Similarly one can show $T_1 h_{4n+2,0} = 2(Nk + n + 1)h_{4n+1,0}$. 

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4.1.2. Case : $T_1 : h_{2m+1,0} \to h_{2m,0}$. Next we consider the case $T_1 \sum_{i=0}^{n} c_i \psi_{m,i} = \sum_{i=0}^{n} (T_1^* c)_i \phi_{m,i}$, where (see section 3.2)

$$(T_1^* c)_i = (2k_2 + m + i)c_i - ic_{i-1}, \text{ for } m + i = 0 \mod 2,$$

$$(T_1^* c)_i = (2k_2 + m + i + 1)c_{i+1} - (2k_2 - 2k + m)c_i - ic_{i-1}, \text{ for } m + i = 1 \mod 2.$$

Write

$$h_{4n+3,0} = \sum_{j=0}^{n} b_j \psi_{2n+1,2j} + \sum_{j=1}^{n+1} c_j \psi_{2n+1,2j-1},$$

then

$$T_1 h_{4n+3,0} = \sum_{j=0}^{n} a_j \phi_{2n+1,2j} + \sum_{j=1}^{n+1} d_j \phi_{2n+1,2j-1},$$

with $d_j = 2(k_2 + n + j)c_j - (2j - 1)b_{j-1} = 0$ and

$$a_j = 2(k_2 + n + j + 1)c_{j+1} - (2k_2 - 2k + 2n + 1)b_j - 2jc_j$$

$$= -2((N - 2)k + k_2 + n + 1)\left((N - 1)k + k_2 + 2n + 1\right)\frac{j(\frac{1}{2})_j}{(k_2 + n + 1)(Nk + n + 2)_j},$$

the claimed multiple of $h_{4n+2,0}$. Similarly write

$$h_{4n+1,0} = \sum_{j=0}^{n} b_j \psi_{2n,2j} + \sum_{j=1}^{n} c_j \psi_{2n,2j-1},$$

$$T_1 h_{4n+1,0} = \sum_{j=0}^{n} a_j \phi_{2n,2j} + \sum_{j=1}^{n} d_j \phi_{2n,2j-1},$$

with $a_j = 2(k_2 + n + j)b_j - 2jc_j$ which is $2(k_2 + n)$ times the corresponding coefficient of $h_{4n,0}$, while $d_j = 2(k_2 + n + j)b_j - 2(k_2 - k + n)c_j - (2j - 1)b_{j-1} = 0$.

4.1.3. Case : $T_2 : h_{2m+1,0} \to h_{2m-1,1}$. For $T_2 \sum_{i=0}^{n} c_i \psi_{m,i} = \sum_{i=0}^{n-1} (T_2^* c)_i (x_1 x_2 \phi_{m-1,i})$ one has

$$(T_2^* c)_i = (2(N - 1)k + m + 1)c_i - (2Nk + m + i + 2)c_{i+1} + ic_{i-1},$$

for $m + i = 0 \mod 2$,

$$(T_2^* c)_i = (2Nk + m + i + 1)c_i - ic_{i-1}, \text{ for } m + i = 1 \mod 2.$$

4.1.4. Case : $T_2 : h_{2m+1,1} \to h_{2m+1,0}$. For $T_2 \sum_{i=0}^{n} c_i x_1 x_2 \phi_{m,i} = \sum_{i=0}^{n} (T_2^* c)_i \psi_{m,i}$ one has

$$(T_2^* c)_i = (2k_2 - 2k + m)c_i - ic_{i-1}, \text{ for } m + i = 0 \mod 2,$$

$$(T_2^* c)_i = (2k_2 + m + i + 1)(c_{i+1} - c_i) - ic_{i-1}, \text{ for } m + i = 1 \mod 2.$$
4.2. Values at $(1,1,\ldots,1)$. Substituting $x = 1^N = (1,1,\ldots,1) \in \mathbb{R}^N$ in $F_0$ and $F_1$ produces

$$
\sum_{n=0}^{\infty} \sum_{j=0}^{n} \phi_{n,j}(1^N) s^j t^n = (1 - 2st + t^2)^{-(Nk+1)} = \sum_{m,i} t^m s^{m-2i} t^{m-2i}(1)^{j_i} (Nk + 1)_{m-i} \frac{(Nk + 1)_{m-i}}{i!(m-2i)!},
$$

since only terms with $n + j = 0 \mod 2$ can have non-zero values; note that the second equality is familiar as the generating function for Gegenbauer polynomials. To derive this expansion, write

$$
(1 - 2st + t^2)^{-(Nk+1)} = (1 + t^2)^{-(Nk+1)}(1 - \frac{2st}{1 + t^2})^{-(Nk+1)} = \sum_{j=0}^{\infty} \frac{(Nk + 1)_j}{j!} (2st)^j (1 + t^2)^{-(Nk+1+j)} = \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} \frac{(Nk + 1)_j}{j!i!} (2s)^j (1)^{j+i} t^{2i+j},
$$

now let $j = m - 2i$.

**Proposition 5.** For $n = 0,1,2,\ldots$ the following hold:

$$
\begin{align*}
    h_{4n,0}(1^N) &= \frac{(Nk + 1)_n((N - 1)k + 1)_n}{n!(k_2 + n)_n}, \\
    h_{4n+1,0}(1^N) &= \frac{(Nk + 1)_n((N - 1)k + 1)_n}{n!(k_2 + n + 1)_n}, \\
    h_{4n+2,0}(1^N) &= h_{4n+3,1}(1^N) = 0, \\
    h_{4n+1,1}(1^N) &= \frac{(Nk + 1)_n((N - 1)k + 1)_n}{n!(k_2 + n + 1)_n}, \\
    h_{4n+3,0}(1^N) &= \frac{(Nk + 1)_{n+1}((N - 1)k + 1)_{n+1}}{n!(k_2 + n + 1)_{n+1}}.
\end{align*}
$$

**Proof.** The non-zero cases are all $2F_1$ summations. For the first case, $\phi_{2n,2j}(1^N) = \frac{2^{2j}(Nk + 1)_{n+j}(-1)^{n-j}}{(n-j)!(2j)!} = (1)^{n-j}(-n)_j (Nk + 1)_{n+j}$, thus

$$
\begin{align*}
    h_{4n,0}(1^N) &= \frac{(-1)^n(Nk + 1)_n}{n!} \sum_{j=0}^{n} (-n)_j (k_2 + (N - 1)k + 2n)_j (k_2 + n)_j ! \\
    &= \frac{(-1)^n(Nk + 1)_n((N - 1)k - n)_n}{n!(k_2 + n)_n}.
\end{align*}
$$

This uses the Chu-Vandermonde sum $2F_1 \left( \begin{array}{c} -n, b \\ c 
\end{array} ; 1 \right) = \frac{(c - b)_n}{(c)_n}$; and $(-b - n)_n = (-1)^n(b + 1)_n$ for arbitrary $b, c$ and $n = 0,1,2,\ldots$. The other formulas are proved in the same way. □
4.3. Leading Coefficients. Let $\text{cof}(f, x_1^m x_2^n)$ denote the coefficient of the monomial $x_1^m x_2^n$ in the expansion of the polynomial $f$ in terms of $x_1, x_2, \ldots, x_n$. We will determine the values of $\text{cof}(h_{m,n}, x_1^m x_2^n)$. For the case $\varepsilon = 0$ these values agree with evaluation at $(1,0,\ldots,0)$. Thus evaluate $F_0$ and $F_1$ at this point to obtain $(1 - st)(1 - 2st + t^2)^{-(k+1)} = a_0$ and $t(1 - 2st + t^2)^{-(k+1)} = a_1$, respectively. The term $a_1$ is multiplied by $(-1)^n$ to obtain coefficients of $x_2^n$ in $h_{n,0}$. By expansion methods similar to the previous we obtain:

$$a_0 = \sum_{m,i} t^m s^{m-2i} 2^{m-2i-1} (-1)^i \frac{(k+1)_{m-1-i} (2k + m)}{i!(m-2i)!},$$

$$a_1 = \sum_{m,i} t^{m+1} s^{m-2i} 2^{m-2i-1} (-1)^i \frac{(k+1)_{m-i}}{i!(m-2i)!}.$$

4.3.1. Case $(4n + \varepsilon, \varepsilon)$. The computation for $h_{4n,0}(1,0,\ldots)$ proceeds as follows:

$$\phi_{2n,2j}(1,0,\ldots) = (-1)^n \frac{(-n)_{j} (k+1)_n}{nj!\binom{\frac{1}{2}}{j}}$$

and so

$$h_{4n,0}(1,0,\ldots) = \frac{(-1)^n (k+1)_n}{n!} F_3 \left( -n, k+n, k_2 + (N-1)k + 2n ; k_2 + n, Nk + n + 1 \right)$$

$$= \frac{(-1)^n (k+1)_n (k_2 - k)_n ((N-1)k + 1)_n}{n!(k_2 + n)_n (Nk + n + 1)_n}.$$

Clearly $\text{cof}(h_{4n,0}, x_1^{4n}) = \text{cof}(h_{4n,0}, x_2^{4n}) = h_{4n,0}(1,0,\ldots)$. The sum is an application of the Saalschütz formula $\text{F}_3 \left( -n, a, b ; c, d ; 1 \right) = \frac{(c-a)_n(d-a)_n}{(c)_n(d)_n}$, provided $-n + a + b + 1 = c + d$.

The corresponding formula for $h_{4n+1,1}$ are obtained by merely incrementing $k_1$ (and also $k_2$) by 1. Thus

$$\text{cof}(h_{4n+1,1}, x_1^{4n+1} x_2) = \text{cof}(h_{4n+1,1}, x_1 x_2^{4n+1})$$

$$= \frac{(-1)^n (k+1)_n (k_2 + 1 - k)_n ((N-1)k + 1)_n}{n!(k_2 + n + 1)_n (Nk + n + 1)_n}.$$

4.3.2. Case $(4n + 2 + \varepsilon, \varepsilon)$. For $h_{4n+2,0}(1,0,\ldots)$ we begin with $\phi_{2n+1,2j}(1,0,\ldots) = (-1)^n \frac{(-n)_{j} (k+1)_{n+j}}{nj!\binom{\frac{1}{2}}{j}}$ and

$$h_{4n+2,0}(1,0,\ldots) = \frac{(-1)^n (k+1)_n}{n!} F_3 \left( -n, k + n + 1, k_2 + (N-1)k + 2n + 1 ; k_2 + n + 1, Nk + n + 2 \right)$$

$$= \frac{(-1)^n (k+1)_n (k_2 - k)_n ((N-1)k + 1)_n}{n!(k_2 + n + 1)_n (Nk + n + 2)_n}.$$

Further $\text{cof}(h_{4n+2,0}, x_1^{4n+2}) = -\text{cof}(h_{4n+2,0}, x_2^{4n+2}) = h_{4n+2,0}(1,0,\ldots)$. Replace $k_2$ by $k_2 + 1$ to obtain the value of $\text{cof}(h_{4n+3,1}, x_1^{4n+3} x_2) = -\text{cof}(h_{4n+3,1}, x_1 x_2^{4n+3}).$

4.3.3. Case $(4n + 1, 0)$. For $h_{4n+1,0}$ note that $\psi_{2n,2j} = x_1(\phi_{2n,2j} + \phi_{2n,2j-1})$ and $\psi_{2n,2j-1} = -x_1(\phi_{2n,2j} - \phi_{2n,2j-1})$. To sketch the argument, let $h_{4n+1,0} = \sum_{j=0}^{n} a_j \psi_{2n,2j} + \sum_{j=1}^{n} b_j \psi_{2n,2j-1}$ and let $\phi_{2n,2j}(1,0,\ldots) = c_j$ and $\phi_{2n,2j-1}(1,0,\ldots) = d_j$. Then $\sum_{j=0}^{n} a_j c_j + \varepsilon \sum_{j=1}^{n} (a_j - b_j)d_j$ equals $\text{cof}(h_{4n+1,0}, x_1^{4n+1})$ when $\varepsilon = 1$, and
calculation similar to the previous one the first sum is found to equal
\( \frac{(-1)^n(k + 1)_n(k_2 - k + 1)_n((N - 1)k + 1)_n}{n!(k_2 + n + 1)_n(Nk + n + 1)_n} \). For the second sum,
\[
a_j - b_j = ((N - 1)k + n) \left( \frac{(k_2 + (N - 1)k + 2n + 1)_{j-1}(k_2 + (N - 1)k + 2n + 1)_{j-1}}{(k_2 + n + 1)_{j}(Nk + n + 1)_{j}} \right).
\]
The value of \( d_j \) is calculated similarly to \( \phi_{2n+1,2j} \) and thus the second sum equals
\[
\frac{((N - 1)k + n)(-1)^{n-1}(k + 1)_n}{(n - 1)!(k_2 + n + 1)(Nk + n + 1)} \text{Legendre}(1 - n, k + n + 1, k_2 + (N - 1)k + 2n + 1, 1)_{k_2 + n + 2, Nk + n + 2} = \frac{((N - 1)^n(k + 1)_n(k_2 - k + 1)_{n-1}((N - 1)k + 1)_n}{(n - 1)!(k_2 + n + 1)_{n}(Nk + n + 1)_n}.
\]
Combining the two sums and setting \( \varepsilon = 1 \) and -1 respectively we obtain
\[
\text{cof}(h_{4n+1,0}, x_1 x^{4n+1}) = \frac{(-1)^n(k + 1)_n(k_2 - k)_{n}((N - 1)k + 1)_n}{n!(k_2 + n + 1)_n(Nk + n + 1)_n} \frac{k_2 - k + 2n}{k_2 - k}.
\]
and \( \text{cof}(h_{4n+1,0}, x_1 x^{4n} = \text{cof}(h_{4n+1,0}, x_1 x^{4n+1}) \frac{k_2 - k + 2n}{k_2 - k} \).

4.3.4. Case \((4n + 3, 0)\). For \( h_{4n+3,0} \) note that \( \psi_{2n+1,2j} = -x_1 \phi_{2n+1,2j} \) and
\[
\phi_{2n+1,2j-1} = x_1(\phi_{2n+1,2j-1} + \phi_{2n+1,2j-2}).
\]
As before, let \( h_{4n+3,0} = \sum_{j=0}^{n} a_j \psi_{2n+1,2j} + \sum_{j=1}^{n+1} b_j \psi_{2n+1,2j-1} \) (not the same coefficients as above) and let \( \phi_{2n+1,2j-1}(1, 0, ...) = c_j \) and \( \phi_{2n+1,2j}(1, 0, ...) = d_j \). Then \( \varepsilon \sum_{j=0}^{n} (-a_j + b_{j+1})d_j + \sum_{j=1}^{n+1} b_jc_j \) equals \( \text{cof}(h_{4n+3,0}, x_1 x^{4n+3}) \) when \( \varepsilon = 1 \) and \( \text{cof}(h_{4n+3,0}, x_1 x^{4n+2}) \) when \( \varepsilon = -1 \). By a calculation similar to the previous one the first sum is found to equal
\[
(k_2 + n + \frac{1}{2}) ((-1)^{n+1}(k + 1)_n(k_2 - k + 1)_n((N - 1)k + 1)_n}{n!(k_2 + n + 1)_n(Nk + n + 2)_n},
\]
and the second sum is
\[
(k + n + \frac{1}{2}) ((-1)^{n}(k + 1)_n(k_2 - k + 1)_n((N - 1)k + 1)_n}{n!(k_2 + n + 1)_n(Nk + n + 2)_n}
\]
Combining the two sums and setting \( \varepsilon = 1 \) and -1 respectively we obtain
\[
\text{cof}(h_{4n+3,0}, x_1 x^{4n+3}) = \frac{(-1)^{n+1}(k + 1)_n(k_2 - k)_{n+1}((N - 1)k + 1)_n}{n!(k_2 + n + 1)_{n+1}(Nk + n + 2)_n} \frac{k_2 + k + 2n + 1}{k_2 - k},
\]
and \( \text{cof}(h_{4n+3,0}, x_1 x^{4n+2}) = -\text{cof}(h_{4n+3,0}, x_1 x^{4n+3}) \frac{k_2 + k + 2n + 1}{k_2 - k} \).

4.4. Norms: For arbitrary polynomials three different inner products have been defined. However for harmonic polynomials there is really only one. Write
\[
d\mu_S(x; k, k_1) = \prod_{i=1}^{N} |x_i|^{2k_1} \prod_{1 \leq i < j \leq N} |x_i^2 - x_j^2|^{2k} d\omega(x)
\]
for the measure on the unit sphere \( S = \{ x \in \mathbb{R}^N : |x| = 1 \} \), where \( d\omega \) denotes the normalized rotation-invariant surface measure. See formula (2.1) for the definition of \( d\mu(x; k, k_1) \).
Proposition 6. Suppose $f, g$ are harmonic ($\Delta_B f = 0 = \Delta_B g$) homogeneous polynomials of degree $m, n$ respectively. Then

$$f(T_1, T_2, \ldots, g(x)|_{x=0} = c \int_{R^N} f(x)g(x)d\mu(x; k, k_1) = \delta_{mn}2^n(Nk_2)_n c_S \int_S fgd\mu_S.$$ 

This was shown in Theorem 3.8 of [3]; the normalizing constants satisfy $c \int_{R^N} d\mu = 1 = c_S \int_S d\mu_S$, (the well-known Macdonald-Mehta-Selberg integral). Specializing to the harmonic polynomials $g = h_{n, \varepsilon}$, for which $T_2^2g = -T_1^2g$ we see that if $f(x) = x_1^m x_2^n f_0(x_1^2, x_2^2, \ldots, x_N^2)$ with $\varepsilon_1, \varepsilon_2 = 0$ or 1, and $f_0$ is homogeneous of degree $m$ then

$$f(T)(g(x) = T_1^4 T_2^2 f_0(T_1^2, -T_1^4, 0, \ldots, 0)g(x) = f_0(1, -1, 0, \ldots, 0) T_1^{4m+2m} T_2^{2m} g(x)$$

By construction the polynomials $h_{n, \varepsilon}$ are pairwise orthogonal so only $h_{n, \varepsilon}(T) h_{n, \varepsilon}(x)$ need be computed. We begin with the calculation of $T_1^4 T_2^2 h_{n, \varepsilon}$. The answers are best stated using a notation introduced in [4].

Definition 3. For $m, n \in Z_+$ and $m \geq n$ let

$$\Lambda(m, n) = (Nk + 1)\alpha((N - 1)k + 1) \beta(k_2)_{m-n} \delta(k_2 - k),$$

where $\alpha = \lfloor \frac{m}{2} \rfloor$ and $\beta = \lfloor \frac{n}{2} \rfloor$.

This is a special case of the generalized Pochhammer symbol for two-part partitions. From the formulæ in section 2.3 we have

$$T_1^{4n} h_{4n, 0} = 2^{4n}(-1)^n \Lambda(2n, 2n),$$

$$T_1^{4n+1} h_{4n+1, 0} = 2^{4n+1}(-1)^n \Lambda(2n + 1, 2n),$$

$$T_1^{4n+2} h_{4n+2, 0} = 2^{4n+2}(-1)^n \Lambda(2n + 2, 2n),$$

$$T_1^{4n+1} T_2 h_{4n+1, 1} = 2^{4n+2}(-1)^n \Lambda(2n + 1, 2n + 1),$$

$$T_1^{4n+3} h_{4n+3, 0} = 2^{4n+3}(-1)^n \Lambda(2n + 2, 2n + 1),$$

$$T_1^{4n+3} T_2 h_{4n+3, 1} = 2^{4n+4}(-1)^n \Lambda(2n + 3, 2n + 1).$$

We turn to the problem of the evaluations at $(1, -1, 0, \ldots, 0)$. In each case, the value will be expressed in terms of a balanced $4F_3$-series which is obviously positive. This is the result of applying the Whipple transformation:

$$4F_3\left(\begin{array}{c} -n, a, b, c \end{array} 1 \right) = \frac{(1 + a - e - n)_n(1 + a - f - n)_n}{(e_n f_n)_n} 4F_3\left(\begin{array}{c} -n, a, d-b, d-c \end{array} d+1, 1 + a - e - n, 1 + a - f - n ; 1 \right)$$

provided $-n + a + b + c + 1 = d + e + f$ (balanced), and $n \in Z_+$. Setting $x = x_0 = (1, \sqrt{-1}, 0, \ldots)$ in the basis polynomials produces the desired values.

Lemma 1. For $0 \leq j \leq n$, $\phi_{2n, 2j+1}(x_0) = 0$, $\phi_{2n+1, 2j+1}(x_0) = 0$ and

$$\phi_{2n, 2j}(x_0) = \frac{(2k + 1)_{n+j}(-1)^n(-n)_j(2k + 2n + 1)}{n!j!(k + \frac{1}{2})j(2k + 1)},$$

$$\phi_{2n+1, 2j}(x_0) = \frac{2(2k + 2)_{n+j}(-1)^n(-n)_j}{n!j!(k + \frac{3}{2})j}.$$
Proof. Substituting $x = x_0$ in the generating functions yields $\sum_{n=0}^{\infty} \sum_{j=0}^{n} \phi_{n,j}(x_0) = (1 + 2t - t^2)((1 + t^2)^2 - 4x^2 t^2)^{-(k+1)}$. The latter term expands to

$$\sum_{i,j=0}^{\infty} 2^{2j}(k+1)_j(2k+2+2j)_j, \text{ for } j!t!.$$ 

Now multiply top and bottom by $(2k+2)2j$, replace $i+j$ by $n$, (also a simple calculation to multiply the resulting series by $(1-t^2)$), and in the denominator expand $(2k+2)2j = 2^{2j}(k+1)_j(k+\frac{3}{2})_j$. \(\blacksquare\)

To illustrate the intermediate steps, consider the case $h_{4n,0}(x_0) = (-1)^n \frac{(2k+1)_n(2k+2n+1)}{n!(2k+1)} 4F_3 \left( \begin{array}{c} -n, k_2 + (N-1)k + 2n, \frac{k}{2}, 2k + n + 1 \\ k + \frac{3}{2}, k_2 + n, Nk + n + 1 \end{array} | 1 \right)$

and transform the series, using $a = k_2 + (N-1)k + 2n$ and $d = k + \frac{3}{2}$. The other cases are done similarly (the case $h_{4n+3,0}$ incorporates one additional step, see case 4.3.4 above). The results are:

$$h_{4n,0}(x_0) = \frac{(-1)^n (2k+1)_n(2k+2n+1)((N-1)k+1)_n(k_2-k)_n}{n!(2k+1)(k_2+n)_n(Nk+n+1)_n} 4F_3 \left( \begin{array}{c} -n, k_2 + (N-1)k + 2n, k + 1, -n - k + \frac{1}{2} \\ k + \frac{3}{2}, k_2 - k, (N-1)k + 1 \end{array} | 1 \right),$$

$$h_{4n+2,0}(x_0) = \frac{(-1)^n 2(2k+2)_n((N-1)k+1)_n(k_2-k)_n}{n!(k_2+n+1)_n(Nk+n+2)_n} 4F_3 \left( \begin{array}{c} -n, k_2 + (N-1)k + 2n, k + 1, -n - k + \frac{1}{2} \\ k + \frac{3}{2}, k_2 - k, (N-1)k + 1 \end{array} | 1 \right),$$

$$h_{4n+1,0}(x_0) = \frac{(-1)^n (2k+1)_n(2k+2n+1)((N-1)k+1)_n(k_2-k)_n}{n!(2k+1)(k_2+n+1)_n(Nk+n+1)_n} 4F_3 \left( \begin{array}{c} -n, k_2 + (N-1)k + 2n, k + 1, -n - k + \frac{1}{2} \\ k + \frac{3}{2}, k_2 - k + 1, (N-1)k + 1 \end{array} | 1 \right),$$

$$h_{4n+3,0}(x_0) = \frac{(-1)^n+1 (2k_2+2n+1)(2k+2)_n((N-1)k+1)_n(k_2-k)_n}{n!(k_2+n+1)_{n+1}(Nk+n+2)_n} 4F_3 \left( \begin{array}{c} -n, k_2 + (N-1)k + 2n, 2k + 1, -n - k + \frac{1}{2} \\ k + \frac{3}{2}, k_2 - k + 1, (N-1)k + 1 \end{array} | 1 \right).$$

The values of the even parts of $h_{4n+1,1}(x_0)$ and $h_{4n+3,1}(x_0)$ are obtained by replacing $k_2$ by $k_2 + 1$ in $h_{4n+0,0}(x_0)$ and $h_{4n+2,0}(x_0)$ respectively (“even part” refers to $f_0$ in the expressions $h_{2m+1,1}(x) = x_1x_2f_0(x_1^2, x_2^2, \ldots)$). This completes the calculation of the $L^2$-norms of the harmonic polynomials. The $4F_3$–series allow no further simplification.

5. DISCUSSION

To conclude, we discuss the significance of the results, especially with regard to applications and indications for further research. The problem that was solved here
is, in a sense, the minimal approach to constructing harmonic polynomials of type $B$. It may turn out that a different normalization may be more useful or concise; for example the value $h_{4n,0}(1,0,\ldots)$ can be written as $(-1)^n(k+1)n!\Lambda(4n,0)$ and similar expressions hold for the other formulae in section 4.3. The expression for $\|h_{4n,0}\|^2$ is also somewhat simplified by changing the normalization to $\frac{\Lambda(4n,0)}{\Lambda(2n,2n)}h_{4n,0}$. Of course the $4F3$ part stays.

5.1. Application: There is a quantum many-body exactly-solvable model associated with $\Delta_B$, namely the spin Calogero model of Yamamoto and Tsuchiya [10][11]. This deals with $N$ identical particles on a line with inverse-square mutual repulsion potential and an external harmonic confinement potential. As well, the particles have a two-valued spin which can be exchanged among them. The construction of eigenfunctions in terms of nonsymmetric Jack and generalized Hermite polynomials was discussed in [1]. The Hamiltonian for the system (with $\omega, k, k > 0$) is

$$\mathcal{H} = \sum_{i=1}^N \left\{ -\left(\frac{\partial}{\partial x_i}\right)^2 + \omega^2 x_i^2 + \frac{k_1(k_1-\sigma_i)}{x_i^2} \right\} + 2k \sum_{1 \leq i < j \leq N} \left\{ \frac{k-\sigma_{ij}}{(x_i-x_j)^2} + \frac{k-\tau_{ij}}{(x_i+x_j)^2} \right\}. $$

The ground state for the system is

$$\psi(x) = \prod_{i=1}^N |x_i|^{k_1} \prod_{1 \leq i < j \leq N} |x_i^2 - x_j^2|^k \exp(-\omega|x|^2/2).$$

Then the conjugate $\psi^* \mathcal{H} \psi^{-1} = 2\omega(\sum_{i=1}^N x_i \frac{\partial}{\partial x_i} + Nk_2) - \Delta_B$. Let $f_m(x)$ be a harmonic and homogeneous polynomial of degree $m$, then for $n = 0, 1, 2, \ldots$ the function $L_n^{(c)}(\omega|x|^2)f_m(x)\psi(x)$ is an eigenfunction of $\mathcal{H}$ with eigenvalue $2\omega(m + 2n + Nk_2)$, where $c = m + Nk_2 - 1$. Here

$$L_n^{(c)}(t) = \frac{(c + 1)_n}{n!} \sum_{j=0}^n \frac{(-n)_j}{(c + 1)_j} \frac{t^j}{j!}$$

denotes the Laguerre polynomial of index $c$ and degree $n$. The set of all such functions with $m + 2n = s$ spans all the eigenfunctions with eigenvalue $2\omega(s + Nk_2)$. The set $\{L_n^{(c)}(\omega|x|^2)f_m(x)\}$ as a basis for polynomials was used in the study of inner products [8] and the Hankel transform [1]. Van Diejen [5] considered $W_N$-invariant eigenfunctions of this type for $\mathcal{H}$. The polynomials $h_{2n,0}$ can produce such invariants by summing over translates:

$$(1 + \sum_{j>2} \sigma_{2j} + \sum_{2<i<j\leq N} \sigma_{1; \sigma_{2j}})h_{2n,0}(x).$$

5.2. Further work: It is still an open problem to find an orthogonal basis for the harmonic homogeneous polynomials. Such bases are useful in approximation theory and numerical cubature (see Xu [10][11]). It is not difficult to write down self-adjoint operators on polynomials, for example $(x_iT_j - x_jT_i)^2$ for $1 \leq i < j \leq N$. This is a useful method for abelian reflection groups. However computer algebra calculations reveal that the characteristic polynomials of these operators on polynomials of not large degree do not factor linearly in $Q(k, k_1)$ (for type $B$). Hence one does not
expect tractable eigenfunction decompositions. It seems worthwhile to try to extend
to more variables the generating function construction for $F_0, F_1$ which was the main
device for this paper (that is, consider harmonic polynomials annihilated by $T_i$ for
$i > n_0$; already the case $n_0 = 3$ is interesting). Obviously there will need to be
a more sophisticated way of handling the different cases. The present approach is
just barely tolerable for the different basis functions involved in the representation
theory of $B_2$. It certainly seems that finding orthogonal bases is considerably more
complicated then the construction of nonsymmetric Jack polynomials.

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