We establish the asymptotic behavior of change-plane estimators. Two types of estimators of the change-plane parameters are proposed to address the non-uniqueness of the argmins of the objective function. The \( n \) rate of convergence and a full characterization of the limiting distributions of the change-plane estimators are shown through theoretical exposition and numerical simulations. A parametric bootstrap procedure is introduced for inference. We use the model to analyze the ACTG175 AIDS study data for precision medicine.

1. Introduction. Change-plane regression models add flexibility by allowing distinct models on either side of a hyperplane [7, 19, 6]. There is increasing interest in these models especially in applications where domain knowledge suggests the input space is partitioned into (latent) subgroups; e.g., subjects with an enhanced treatment effect [2, 18]. However, the application of these methods is impeded by a lack of rigorous inferential methods. There are two major challenges to establishing theory for change-plane models: non-smoothness of the objective function and non-uniqueness of the estimated parameters. The non-smoothness occurs because the model allows for an abrupt change in parameters at a boundary defined by a hyperplane; consequently, standard M-estimation theory does not apply. The change in parameters is encoded as an indicator function and because of the non-injective binary topology of indicator functions, multiple parameter values form a level set which maps to a single objective value. And, thus, the maximizer of the objective is not unique. This problem occurs not only in finite samples but also asymptotically. Consequently, the argmax continuous mapping theorem is not applicable.

A consequence of the non-smoothness of the objective function in change-plane models is that it is possible to obtain convergence rates faster than \( \sqrt{n} \). In the special case of a change-point model, multiple examples have been given in which estimated parameters converge at \( n \)-rate to the minimizer of a non-Gaussian process. For example, Kosorok and Song [8] showed that a Cox model change-point estimator is \( n \)-consistent and weakly converges to the minimizer of a right-continuous jump process; and Song et al. [16] showed a least-squares estimator in a linear change-point model can attain \( n \)-rate convergence under a correctly specified model but that these rates can be slower under misspecification.

The non-uniqueness issue in change-point estimators can be handled through a modification of the argmax continuous mapping theorem. For example, using the modified theorem, the weak convergence of the smallest and the largest argmax can be obtained in the presence of multiple maximizers [3]. Lan et al. [10] proposed a multistage adaptive procedure to
estimate the change-point in a parametric regression model and investigated the asymptotic properties of the change-point estimator. They proved that the “zoom-in” stage change-point estimator achieves the \( n \)-consistency, and that the joint asymptotic distribution of the smallest and largest minimizers of the zoom-in estimators is the smallest and largest minimizers of a two-sided compound Poisson process. Seijo and Sen [14] proposed a version of the argmax continuous mapping theorem to deal with the multiple maximizers in the change-point regression model defined in the Skorohod topology; the joint weak convergence of the smallest and largest argmax of the stochastic process was obtained when both the stochastic process and the associated pure jump process converged in the Skorohod topology with the smallest and largest argmax.

Among the few studies which have investigated the asymptotic behavior of change-plane models with more than two change-plane parameters, most replace the indicator function with a smooth surrogate to ease the computational and theoretical burden. In this case, the convergence rate depends on the degree of the smoothness and can be slower than \( n \). For example, the change-plane estimator of Seo and Linton [15], which uses an integrated smooth kernel function, has a \( \sqrt{n/\sigma_n} \) rate of convergence, which attains a maximal rate of \( n^{-3/4} \) under the choice of bandwidth \( \sigma_n = \log n/\sqrt{n} \) and converges weakly to a normal distribution when \( \log n/n\sigma_n^2 \to 0 \). Mukherjee et al. [12] showed that their kernel smoothed change-plane estimator can achieve a faster rate, close to \( n^{-1} \), provided \( n\sigma_n \to \infty \).

We establish the large-sample behavior of the prototypical linear change-plane estimator with an arbitrary number of change-plane parameters. Most closely related to our work is the recent paper of Deng et al. [1] in which they study asymptotic properties of the change-plane Cox proportional hazards model. They established \( n \)-rate convergence and suggested the \( m \)-out-of-\( n \) bootstrap for inference. However, the non-uniqueness of the argmax was not fully addressed as the smallest argmax used in their approach is not well-defined in the case of multi-dimensional change-plane parameters. In addition, they required continuity of the change plane covariates and a bounded parameter space.

We propose an M-estimator for the change-plane and establish its asymptotic properties under minimal assumptions; e.g., the change-plane covariate space is allowed to be at least partially discrete. Two types of estimators that uniquely summarize the level set of the argmin are proposed. We establish consistency, rates of convergence, and the weak convergence of the proposed M-estimators using empirical process techniques. The proposed estimators achieve a fast \( n \)-rate of convergence for the change-plane parameter. The most significant contribution of the article is to fully characterize the limiting distribution of the M-estimators, which is a minimizer of a sum of two compound jump processes. Using this limiting distribution, we derive a parametric bootstrap procedure which is more efficient than the \( m \)-out-of-\( n \) bootstrap suggested by Deng et al. [1]. Simulation studies confirm the theoretical \( n \)-rate of convergence as well as the validity of the parametric bootstrap.

The remainder of this article is organized as follows. We describe the change-plane regression model in Section 2 and introduce two types of estimators in Section 3. Consistency, rates of convergence, and the weak convergence of the proposed estimators are established in Sections 4–6. The parametric bootstrap procedure is described in Section 7. Section 8 presents a simulation study of convergence rates and the validity of the parametric bootstrap. We illustrate the application of the proposed methods using data from an AIDS study in Section 9. We conclude the paper with a discussion of future research directions in Section 10.

2. Data and model. We assume that the data consists of \( n \) i.i.d. copies of the triple \((Y, Z, X)\), where \( X \in \mathbb{R}^p \) are the change-plane covariates (to be defined shortly), \( Z \in \mathbb{R}^d \) are the regression covariates, and \( Y \in \mathbb{R} \) is the regression outcome. In addition, we assume that \( Y \) satisfies

\[
Y = \beta_0'ZI\{\omega_0X - \gamma_0 \leq 0\} + \delta_0'ZI\{\omega_0X - \gamma_0 > 0\} + \epsilon, \tag{1}
\]
where: $\epsilon$ is a continuous random variable which is independent of $(X, Z)$ and has mean zero and variance $0 < \sigma^2 < \infty$; $\beta_0, \delta_0 \subset \mathbb{R}^d$ are the classification/regression parameters; $\omega \in S^{p-1}$ and $\gamma_0 \in \mathbb{R}$ are the change-plane parameters, prime denotes transpose, and $S^{p-1}$ is the $p-1$ dimensional unit sphere embedded in $\mathbb{R}^p$ (i.e., $\{\omega \in \mathbb{R}^p : ||\omega|| = 1\}$), for $p \geq 1$. We have used a subscript zero to denote the true parameter values; we omit a subscript when discussing generic parameter values. Let $S^0$ to be the set of points $\{-1, 1\}$, and define the composite parameters $\zeta = (\beta, \delta), \phi = (\omega, \gamma)$, and $\theta = (\zeta, \phi)$. Define $\overline{\omega}_0$ to be the $p \times (p-1)$ matrix consisting of the orthonormal basis vectors for the $p-1$-dimensional subspace in $\mathbb{R}^p$ which is orthogonal to the linear span of $\omega_0$; we denote this subspace by $\mathbb{R}^{p-1}$. We assume the following conditions:

C1. $||X|| \leq k_1 < \infty$ almost surely and its covariance is full rank.

C2. The random variable $U \equiv \omega_0'X - \gamma_0$ is continuous, and, in a neighborhood of zero, it has a continuous density $f$. Let $f_0 \equiv f(0)$, then $0 < f_0 < \infty$. This implies the existence of $0 < c_0, c_1 < \infty$ such that $f(u) \geq c_0$ for all $u \in [-c_1, c_1]$, and hence also implies that $P\{U \leq 0\} > 0$ and $P\{U > 0\} > 0$.

C2’. Condition C2 is strengthened to require that the density $f$ be uniformly bounded on $\mathbb{R}$ and continuous at zero.

C3. Whenever $(\omega, \gamma) \neq (\omega_0, \gamma_0), P\{(\omega'X - \gamma)(\omega_0'X - \gamma_0) < 0\} > 0$. Moreover, the joint distribution of $(Z, X)$ given $U = u$ converges to a probability measure $G$ as $u \to 0$; and, moreover, the covariance of $U = \overline{\omega}_0X$ is full rank under $G$.

C3’. Condition C3 is strengthened to require that $U$ be continuous under $G$.

C4. $||Z|| \leq k_2 < \infty$ almost surely, both $E(ZZ' | \omega_0'X - \gamma_0 \leq 0)$ and $E(ZZ' | \omega_0'X - \gamma_0 > 0)$ are full rank with minimum eigenvalues bounded below by $c_2 > 0$.

C5. $\beta_0 \neq \delta_0$, and $P\{(\beta_0 - \delta_0)'Z = 0\} = 0$ under $G$.

C6. $\epsilon$ has uniformly bounded density $\xi$ on $\mathbb{R}$, where $\xi$ is uniformly equicontinuous, i.e.,

$$\lim_{\eta \downarrow 0} \sup_{s, t \in \mathbb{R} : |s-t| < \eta} |\xi(s) - \xi(t)| = 0.$$  

Condition C1 implies that, without loss of generality, we can assume that $\gamma \in [l_0, u_0]$, where $l_0 = -k_1 - \rho$ and $u_0 = k_1 + \rho$, for any $\rho > 0$, because $1\{\omega'X - \gamma \leq 0\} \geq 1\{\omega'X - l_0 \leq 0\}$ for all $\gamma \leq l_0$ and $1\{\omega'X - \gamma \leq 0\} \leq 1\{\omega'X - u_0 \leq 0\}$ for all $\gamma \geq u_0$, almost surely, for any value of $\omega$. Accordingly, we fix $0 < \rho < \infty, l_0 = -k_1 - \rho$ and $u_0 = k_1 + \rho$ hereafter. Condition C2 is used in establishing the limiting distribution as much of the action in the limit occurs where $U$ is close to zero. This condition is easily satisfied if $X$ is continuous and full rank with density bounded above and below. It is also satisfied if $X$ includes both continuous and categorical variables, provided the weights in the vector $\omega_0$ are positive for at least one of the continuous variables. The strengthening of Condition C2 to C2’ is used for the parametric bootstrap results of Section 7. C3 is needed for model identifiability of $(\omega, \gamma)$ as well as stability in the limit when $U$ is close to zero. The strengthening of Condition C3 to C3’ is used for weak convergence of one of the two estimators which will be introduced in Section 3. Condition C4 is needed for identifiability of $(\beta, \delta)$ as well as stability when $U$ is close to zero. Condition C5 is also crucial for model identifiability of $(\omega, \gamma)$. Condition C6, which is a strengthening of the stated properties of the residual $\epsilon$ above, is used to establish the parametric bootstrap results of Section 7.

### 3. Estimation

Our focus is on estimating $\theta = (\beta, \delta, \omega, \gamma)$. We will use the M-estimator obtained by minimizing

$$\theta \mapsto M_n(\theta) \equiv \mathbb{P}_n \left[ 1\{\omega'X - \gamma \leq 0\}(Y - \delta'Z)^2 + 1\{\omega'X - \gamma > 0\}(Y - \delta'Z)^2 \right],$$

where $\mathbb{P}_n$ is the standard empirical measure. We have established previously that the range of $\theta$ is $K = K_1 \times K_2$, where $K_1 = \mathbb{R}^d \times \mathbb{R}^d$ and $K_2 = S^{p-1} \times [l_0, u_0]$. Let $A_0(\phi) = 1\{\omega'X - \gamma \leq 0\}(Y - \delta'Z)^2 + 1\{\omega'X - \gamma > 0\}(Y - \delta'Z)^2$. 

γ ≤ 0} and \( A_1(\phi) = \{ \omega'X - \gamma > 0 \} \), and define \( D_{0n}(\phi) = \mathbb{P}_n[ZZ' A_0(\phi)] / [\mathbb{P}_n A_0(\phi)] \) and \( D_{1n}(\theta) = \mathbb{P}_n[ZZ' A_1(\phi)] / [\mathbb{P}_n A_1(\phi)] \). Fix a \( c_3 \in (0, c_2) \), where \( c_2 \) comes from assumption C4, and let \( K_{2n} \) be the set of \( \phi \in K_2 \) such that the smallest eigenvalues of \( D_{0n}(\phi) \) and \( D_{1n}(\phi) \) are bounded below by \( c_3 \). Let \( K_{2n}' \cap K_{2n} \) be similarly defined but with the weaker requirement that the minimum eigenvalues of \( D_{0n}(\phi) \) and \( D_{1n}(\phi) \) are both positive. Define \( K_n = K_1 \times K_{2n} \) and \( K_n' = K_1 \times K_{2n}' \), and note that membership in either \( K_{2n} \) or \( K_{2n}' \) does not involve \( \zeta = (\beta, \delta) \). Unfortunately, \( K_n \) (and even \( K_n' \)) may be empty for smaller sample sizes. To address this, we set \( K_n' = K_n \) if \( K_n \) is non-empty, \( K_n'' = K_n' \) if \( K_n \) is empty but \( K_n' \) is non-empty, and \( K_n'' = K \) otherwise. Let \( \theta_n \) be the \( \arg \min \) over \( K_n'' \) of \( M_n(\theta) \). We will, however, need to require that \( K_n \) be non-empty for all \( n \) large enough to ensure identifiability of \( (\beta, \delta) \). The following remark explains why this requirement is feasible.

**Remark 3.1.** Combining Assumption C4 with the strong law of large numbers, it can be seen that \( K_n \) will be non-empty and contain \( \theta_0 \) for all \( n \) large enough, almost surely. Because \( K_n \subset K_n' \), the same is true for \( K_n'' \). Also note, perhaps obviously, that we do not need to know the value of \( \theta_0 \) in advance for these results to be true.

An additional complication with \( M_n(\theta) \) is the presence of certain level sets. To clarify this, fix \( (\omega, \gamma) \) at a point \( (\omega_1, \gamma_1) \in S^{p-1} \times \{0, u_0\} \), and let \( V_i = \{ \omega'_iX_i - \gamma_1 > 0 \} \) and \( V_i = \{ \omega'_iX_i - \gamma_1 \leq 0 \} \), \( 1 \leq i \leq n \). Now let

\[
\Phi_n = \{ (\omega, \gamma) \in S^{p-1} \times \{0, u_0\} : \{ \omega'_iX_i - \gamma > 0 \} - \{ \omega'_iX_i - \gamma \leq 0 \} = V_i, 1 \leq i \leq n \},
\]

and note that \( (\omega, \gamma) \mapsto \Phi_n(\beta, \delta, \omega, \gamma) \) is constant over \( (\omega, \gamma) \in \Phi_n \) and that \( \Phi_n \) is a bounded set by construction. The sets of this form are defined by all possible realizations of \( V \equiv (V_1, \ldots, V_n) \in \{-1, 1\}^n \) that are obtained from partitioning \( X_1, \ldots, X_n \) by hyperplanes in \( \mathbb{R}^p \) into two non-empty groups. Suppose that \( n \) is large enough to ensure that \( K_{2n}' \) is non-empty and that \( (\omega_1, \gamma_1) \in K_{2n}' \). Then \( \Phi_n \subset K_{2n}' \) and there exists \( j, k \in \{1, \ldots, n\} \) such that \( V_j = -1 \) and \( V_k = 1 \). We will now define two types of midpoints of \( \Phi_n \) which we can use to construct unique estimators. We first need to define the maps \( \omega \mapsto C_L(\omega) = \max_{1 \leq i \leq n, V_i = -1} \omega'_iX_i, \omega \mapsto C_U(\omega) = \min_{1 \leq i \leq n, V_i = 1} \omega'_iX_i, \) and \( \omega \mapsto C_R(\omega) = C_L(\omega) - C_U(\omega) \). Note by construction that \( \Phi_n = \{ \phi \in K_2 : C_L(\omega) - \gamma \leq 0 < C_U(\omega) - \gamma \} \), and thus both \( C_L(\omega) > 0 \) and \( C_R(\omega) \leq \gamma < C_U(\omega) \) if and only if \( (\omega, \gamma) \in \Phi_n \).

For the first type of midpoint, we define the “mean-midpoint” of \( \Phi_n \) to be \((\hat{\omega}, \hat{\gamma})\), where

\[
\hat{\omega} = \frac{\int_{R_n} \omega C_R(\omega) d\nu(\omega)}{\int_{R_n} \omega C_R(\omega) d\nu(\omega)},
\]

\( R_n = \{ \omega \in S^{p-1} : C_R(\omega) > 0 \} \), \( \nu \) is the uniform measure on \( S^{p-1} \), and \( \hat{\gamma} = [C_L(\hat{\omega}) + C_U(\hat{\omega})] / 2 \).

In Lemma 3.2 below, we show that the mean-midpoint exists, is well-defined, and is contained in \( \Phi_n \). Note that from a computational perspective, it is not difficult to approximate \((\hat{\omega}, \hat{\gamma})\) to any desired degree of accuracy using rejection sampling; moreover, it is easy to verify that any Monte Carlo estimate based on rejection sampling will be contained in \( \Phi_n \).

**Lemma 3.2.** Assume \( n \) is large enough so that \( K_{2n}' \) is non-empty and \((\omega_1, \gamma_1) \in K_{2n}' \cap \Phi_n \). Then the following are true:

1. The set \( R_n \) has the following properties:
   a) it is open and contains an open ball on \( S^{p-1} \),
   b) it is the intersection of a finite number of open half-spheres \( R_{nj} \), \( j = 1, \ldots, m \), and
   thus if \( \omega \in R_n \) then \(-\omega \notin R_n \) (a property we call “strict positivity”), and
that
2. \( \Phi_n = \{ (\omega, \gamma) \in K_2 : \omega \in R_n, \text{ and } C^0_{\gamma}(\omega) \leq \gamma < C^1_{\gamma}(\omega) \} \); and
3. \((\hat{\omega}, \hat{\gamma}) \in \Phi_n \text{ with } \hat{\gamma} \in [l_0, u_0]\).

\textbf{Proof.}\ Let \( J_R(n) \) be the set of all pairs of indices \( 1 \leq j, k \leq n \) such that for each \((j, k) \in J_R(n), V_j = -1 \) and \( V_k = 1 \). It can be seen that \( J_R(n) \) is finite and also non-empty, as \( n \) is large enough so that \( K_{2n} \) is non-empty and \( (\omega_1, \gamma_1) \in K_{2n}' \). By definition of \( \Phi_n \), we know that \( X_j \neq X_k \).

For \( \omega_1 = (X_k - X_j)/\|X_k - X_j\| \) and the plane in \( \mathbb{R}^p \) orthogonal to \( \omega_1 \) to split \( S^{p-1} \) into two equal parts which exclude the intersection of the plane and the sphere. It is easy to see that each part is an open half-sphere, is geodesically connected, and strictly positive. Take the half-sphere which includes \( \omega_1 \), and label it \( R_{n_1} \). Now do this for all pairs in \( J_R(n) \), and, when there are duplicate open half-spheres, choose only one of them. This leads to the set of all unique open half-spheres, \( R_{nl}, l = 1, \ldots, m \), which are generated in this manner by pairs in \( J_R(n) \). Let \( \tilde{R}_n = \cap_{l=1}^m R_{nl} \). \( \tilde{R}_n \) is by construction the intersection of a finite number of open half-spheres. Moreover, since properties of openness, geodesic connectedness, strict positivity are preserved when taking intersections, \( \tilde{R}_n \) also has all of these properties.

The first part of the lemmas is complete if we show that \( \tilde{R}_n = R_n \). Let \( \omega \in \tilde{R}_n \). We have by construction that for every \((j, k) \in \tilde{R}_n \), \( \omega'(X_k - X_j) > 0 \), and thus \( C^0_R(\omega) > 0 \), which implies that \( \omega \in R_n \), and thus \( \tilde{R}_n \subseteq R_n \). Let \( \omega \in R_n \). Then \( C^0_R(\omega) > 0 \), which implies that \( \max_{1 \leq i \leq n, V_i = -1} \omega'X_i < \min_{1 \leq i \leq n, V_i = 1} \omega'X_i \). Thus \( \omega'(X_k - X_j) > 0 \) for all \((j, k) \in J_R(n) \), and hence \( \omega \in R_n \). Thus \( \tilde{R}_n = R_n \). The fact that \( R_n \) contains an open ball follows automatically because non-empty open sets must contain open balls. Thus Part 1 of the Lemma is proved. Part 2 follows from the definitions involved.

For Part 3, consider the function \( f^R_R(\omega) = C^n_R(\omega) / \int_{\omega \in R_n} C^n_R(\omega) d\nu(\omega) \), and note that this is a well-defined bounded density with respect to \( \nu(\omega) \), and that it has support only on \( R_n \). As the boundary of \( R_n \) is included in the complement of \( R_n \) and because \( C^n_R(\omega) \) is strictly positive for all \( \omega \in R_n \), the projection of \( \int_{R_n} f_R^R(\omega) d\nu(\omega) \) onto \( S^{p-1} \), which is precisely \( \hat{\omega} \), must be \( \in R_n \). The fact that \( \hat{\gamma} \in [l_0, u_0] \) follows from the fact that \( -k_1 \leq C^n_L(\hat{\omega}) \leq C^n_U(\hat{\omega}) \leq k_1 \) by Condition C1.\( \square \)

For the second type of midpoint, we define the “mode-midpoint” of the level set \( \Phi_n \) to be \((\hat{\omega}, \hat{\gamma}^*)\), where \((\hat{\omega}, \hat{\gamma}^*)\) is the \( \text{arg max} \) over \( \Phi_n \) of

\[
(\omega, \gamma) \mapsto \lambda, \ \text{subject to} \n\]

\[
C^0_{\gamma}(\omega) - \gamma \leq 0 \text{ and } C^1_{\gamma}(\omega) - \gamma \geq \lambda. \tag{3}
\]

We assume, as before, that \( n \) is large enough so that \( K_{2n}' \) is non-empty and that \((\omega_1, \gamma_1) \in K_{2n}' \cap \Phi_n \). The following lemma yields that \((\hat{\omega}, \hat{\gamma}^*)\) is unique and contained in \( \Phi_n \). Note that we will utilize a minor modification of this estimator, defined in part 2 of the lemma, which will be our mode-midpoint estimator going forward. The reason for this modification is to ensure that the estimator is in the interior of its associated level set.

\textbf{LEMMA 3.3.}\ Assume \( n \) is large enough so that \( K_{2n}' \) is non-empty and \((\omega_1, \gamma_1) \in K_{2n}' \cap \Phi_n \). Then the following are true:

1. \((\hat{\omega}, \hat{\gamma})\), where \( \hat{\gamma}^* = C^n_L(\hat{\omega}) \in [l_0, u_0] \), is the unique maximizer over \( \Phi_n \) of \((3)\), and

2. both \((\hat{\omega}, \hat{\gamma}^*)\) and \((\hat{\omega}, \hat{\gamma})\) are \( \in \Phi_n \), where \( \hat{\gamma} = [C^n_L(\hat{\omega}) + C^n_U(\hat{\omega})]/2 \).
Proof. The first part of the lemma follows from Lemma 3.4 below by substituting $m = n$, $q = p$, $R = S^{p-1}$, and both $x_j = X_j$ and $v_j = 1\{\omega_j X_j - \gamma > 0\} - 1\{\omega_j X_j - \gamma \leq 0\}$, for $1 \leq j \leq n$. The fact that $\gamma \in [0, u_0]$ follows from the fact that $-k_1 \leq C_k^\ast(\omega) \leq k_1$. The second part of the lemma follows from the straightforward observation that $C_k^\ast(\omega) - \gamma \leq 0$ and $C_k^\ast(\omega) - \gamma > 0$ and also $C_k^\ast(\omega) - \gamma \leq 0$ and $C_k^\ast(\omega) - \gamma > 0$, and thus both pairs of parameter estimators are in $\Phi_n$.

**Lemma 3.4.** Let $(x_j, v_j) \in \mathbb{R}^q \times \{-1\}, 1 \leq j \leq m < \infty$, and let $R \subset S^{q-1}$ be geodesically connected and closed. Assume there exists $(k, l) \in \{1, \ldots, m\}$ such that $v_k = 1$ and $v_l = 1$, and that there exists a $(\omega, u_1) \in R \times \mathbb{R}$ such that

$$v_j = 1\{\omega_j x_j - u_1 > 0\} - 1\{\omega_j x_j - u_1 \leq 0\},$$

for all $1 \leq j \leq m$. Then there exists a unique arg max over $R \times \mathbb{R}$, $(\bar{\omega}, \bar{u})$, of

$$(\omega, u) \mapsto \lambda,$$ subject to

$$\max_{1 \leq j \leq m; v_j = 1} \omega_j x_j - u \leq 0$$ and

$$\min_{1 \leq j \leq m; v_j = 1} \omega_j x_j - u \geq \lambda.$$ Moreover, $\bar{u} = \max_{1 \leq j \leq m; v_j = 1} \omega_j x_j$, and the inequalities in the constraints become equalities when evaluated at this arg max.

Proof. Define the cone $C = \{x \in \mathbb{R}^q : x = rp, \text{ where } p \in R, r \in [0, \infty)\}$, and note that $R \subset C$. We first verify that $C$ is convex. Let $u, v \in C$. If $u = v = 0$, then it is obvious that $\alpha u + (1 - \alpha)v = 0 \in C$ for all $0 < \alpha < 1$. If one is zero but one is not, then it is also easy to verify that $\alpha u + (1 - \alpha)v \in C$ for all $0 < \alpha < 1$. Now suppose both are non-zero. Let $u_1 = u/\|u\|$ and $v_1 = v/\|v\|$. If $u_1 = v_1$, then all points of the form $ru_1$ are in $C$, and thus $\alpha u + (1 - \alpha)v \in C$ for all $0 < \alpha < 1$. If $u_1 = -v_1$, then the inclusion also holds. Now assume that $u_1 \neq v_1$ and $u_1 \neq -v_1$. It is easy to verify that the segment in $\mathbb{R}^q$ joining $u_1$ and $v_1$ and $v_1$ excludes the zero point and has a projection on $S^{q-1}$ which is precisely the shortest geodesic line joining $u_1$ and $v_1$, which geodesic we denote $l(u, v)$. It is now clear that, for any $0 < \alpha < 1$, $\alpha u + (1 - \alpha)v$ projects onto $S^{q-1}$ at some point on $l(u, v)$. Hence, $C$ is convex.

Now note that $(\omega_1, u_1) \in C \times \mathbb{R}$. Define the functions $C_k^\ast(\omega) = \max_{1 \leq j \leq m; v_j = 1} \omega_j X_j$, $C_k^\ast(\omega) = \min_{1 \leq j \leq m; v_j = 1} \omega_j X_j$, and $C_k^\ast(\omega) = C_k^\ast(\omega) - C_k^\ast(\omega)$. Note that the minimum of a finite set of linear functions is convex and the maximum is concave, hence $C_k^\ast(\omega)$ is also convex. Note that, by assumption, $C_k^\ast(\omega_1) - u_1 \leq 0$ and $C_k^\ast(\omega_1) - u_1 > 0$. Hence, $C_k^\ast(\omega_1) > 0$. Let $\eta = C_k^\ast(\omega_1)$, define $\omega_2 = \omega_1 / \eta$, and note that now $C_k^\ast(\omega_2) \geq 1$. Consider minimizing over $C$ the function $\omega \mapsto \|\omega\|^2$ subject to the constraint that $C_k^\ast(\omega) \geq 1$. Because $\omega \mapsto \|\omega\|^2$ is strictly convex, the constraint is convex, and we have a feasible point $\omega_2$: then there must exist a unique solution $\hat{\omega}_2 \in C$. Note that $C_k^\ast(0) = 0$, and this contradicts the constraint. Thus, $\hat{\omega}_2 \neq 0$. Now suppose $C_k^\ast(\hat{\omega}_2) > 1$. Then there exists $0 < \alpha < 1$ such that $C_k^\ast(\alpha \hat{\omega}_2) = 1$. Because $\alpha < 1$, the previous sentence implies that $\alpha \hat{\omega}_2$ is both feasible and smaller in norm than $\hat{\omega}_2$, but this would contradict the uniqueness of the minimum. Thus, $C_k^\ast(\hat{\omega}_2) = 1$. It follows that the minimization we employed is equivalent to minimizing $\|\omega\|^2$ subject to $C_k^\ast(\omega) \geq 1/\|\omega\|$, where the inequality becomes an equality for the minimizer. This is now equivalent to maximizing over $\omega \in R$ subject to $C_k^\ast(\omega) \leq \lambda$, where the equality is again achieved for the maximizer. What we have now shown is that there exists a unique arg max over $R$, $\hat{\omega}_2 = \hat{\omega}_2 / \|\hat{\omega}_2\|$, of $\omega \mapsto \lambda$ subject to $C_k^\ast(\omega) \geq \lambda$, where the equality is achieved.

Now we will verify that $(\hat{\omega}, \hat{u}) = \{\hat{\omega}_2, C_L(\hat{\omega}_2)\}$ is the unique maximizer of (4) with equalities in the constraints. Let $\lambda$ be the corresponding value of $\lambda$ attained by this maximizer.
Suppose there exists some \((\hat{c}_2, \check{u}_3) \in R \times \mathbb{R}\) which is not equal to \((\hat{c}, \check{u})\) but which satisfies (4) for some \(\lambda_3 \geq \lambda\). We know that \(C_{R}^{*}(\hat{c}) \geq \lambda\), and thus \(\hat{c}_3 = \hat{c}\) by the previously established uniqueness of the maximizer of \(C_{R}^{*}\) over \(R\). Hence, \(\lambda_3 = \lambda\). The form of (4) now forces \(\check{u}_3 = \check{u}\), and thus the desired uniqueness of the maximizers, the form of \(\check{u}\), and the equalities in the constraints are established. \(\Box\)

Now we define our two estimators, \(\hat{\theta}_n\) and \(\hat{\theta}_n\), based on the two midpoint approaches. Let \(\hat{\theta}_n\) be as defined at the beginning of this section, and let \(\Phi_n\) be the level set associated with \(\phi_n\). If \(n\) is small enough so that \(K_n^{*}\) is empty, then let \(\theta_n = \theta_n = \hat{\theta}_n\). Otherwise, define \(\hat{n} = (\hat{\theta}_n, \hat{\delta}_n, \hat{\omega}_n, \hat{\gamma}_n)\), where \((\hat{\omega}_n, \hat{\gamma}_n) = (\hat{\omega}, \hat{\gamma})\) is as defined in Lemma 3.2 for the level set \(\Phi_n\). Similarly define \(\hat{n} = (\hat{\theta}_n, \hat{\delta}_n, \hat{\omega}_n, \hat{\gamma}_n)\), where \((\hat{\omega}_n, \hat{\gamma}_n) = (\hat{\omega}, \hat{\gamma})\) as defined in Lemma 3.3 for the level set \(\Phi_n\).

4. Consistency. We now establish consistency of \(\hat{\theta}_n\) and \(\hat{\theta}_n\): 

**Theorem 4.1.** Let \(\hat{\theta}_n\) be any sequence in \(K_n^{*}\) satisfying \(M_n(\hat{\theta}_n) \leq M_n(\theta_0)\) for all \(n\) large enough almost surely. Under Conditions C1, C2, C3, C4 and C5, \(\theta_n \rightarrow \theta_0\) almost surely.

Note, in particular, that both \(\hat{\theta}_n\) and \(\hat{\theta}_n\) are sequences satisfying the conditions of Theorem 4.1.

Before giving the proof of Theorem 4.1, define 

\[
A_{00}(\phi) = 1\{\omega_0'X - \gamma_0 \leq 0, \omega'X - \gamma \leq 0\}, \\
A_{01}(\phi) = 1\{\omega_0'X - \gamma_0 \leq 0, \omega'X - \gamma > 0\}, \\
A_{10}(\phi) = 1\{\omega_0'X - \gamma_0 > 0, \omega'X - \gamma \leq 0\}, \text{ and} \\
A_{11}(\phi) = 1\{\omega_0'X - \gamma_0 > 0, \omega'X - \gamma > 0\},
\]

and note that \(A_0(\phi) = A_0(\phi) + A_{10}(\phi)\) and \(A_1(\phi) = A_{01}(\phi) + A_{11}(\phi)\), where \(A_j(\phi), j = 0, 1\), was defined in Section 3.

**Proof of Theorem 4.1.** Let \(\hat{n} = (\hat{\theta}_n, \hat{\delta}_n, \hat{\omega}_n, \hat{\gamma}_n)\) be any sequence in \(K_n^{*}\) satisfying the conditions of Theorem 4.1. Note that \(K_n^{*}\) can be compactified by applying the metric 

\[
d_1(x, y) = \left\| \frac{x}{1 + \|x\|} - \frac{y}{1 + \|y\|} \right\|
\]

on \(\mathbb{R}^d\), to both copies of \(\mathbb{R}^d\) in \(K\), applying the usual Euclidean metric to \(S^{p-1}\) and \([l_0, u_0]\), and then utilizing the resulting product metric. Note that \(d_1\) is equivalent to the Euclidean norm on restrictions to compact subsets of \(\mathbb{R}^d\).

Define 

\[
H_{00}^n = A_{00}(\phi_n) \left\{ \left[ e - (\hat{\beta}_n - \beta_0)'Z \right]^2 / (1 + \|\hat{\beta}_n\|^2) \right\}, \\
H_{01}^n = A_{01}(\phi_n) \left\{ \left[ e - (\hat{\delta}_n - \beta_0)'Z \right]^2 / (1 + \|\hat{\delta}_n\|^2) \right\}, \\
H_{10}^n = A_{10}(\phi_n) \left\{ \left[ e - (\hat{\beta}_n - \delta_0)'Z \right]^2 / (1 + \|\hat{\beta}_n\|^2) \right\}, \text{ and} \\
H_{11}^n = A_{11}(\phi_n) \left\{ \left[ e - (\hat{\delta}_n - \delta_0)'Z \right]^2 / (1 + \|\hat{\delta}_n\|^2) \right\},
\]
and let $H^n_{0i} = H^n_{00i} + H^n_{10i}$ and $H^n_{1i} = H^n_{01i} + H^n_{11i}$. Note that standard empirical process methods show that the classes $\{A_{ij}(\phi) : (\omega, \gamma) \in S^{p-1} \times [l, u]\}, 0 \leq i, j \leq 1$, are Donsker and bounded, and hence Glivenko-Cantelli. Standard arguments also reveal that $\{(e - (\beta - \beta_0)^2/2, \beta) \in \mathbb{R}^d : \beta \in \mathbb{R}^d\}$ is Glivenko-Cantelli and bounded, and thus $H^n_{00}$ is contained in a Glivenko-Cantelli class. Similar arguments verify that $H^n_{ij}$, for all $0 \leq i, j \leq 1$, are contained in Glivenko-Cantelli classes. Hence $|H^n_{ij} - h^n_{ij}| \to 0$ and $|H^n_{ij} - h^n_{ij}| \to 0$, almost surely, where $h^n_{ij} = E H^n_{ij}$ and $h^n_{ij} = E H^n_{ij}$, for all $0 \leq i, j \leq 1$. Note that, from Remark 3.1, we have $K^n_{Z} = K^n_{a}$ for all $n$ large enough almost surely. Let $\Omega$ be the subset of the probability space where these convergence results hold, and note that the inner probability measure of $\Omega$ is $1$ by the definition of (outer) almost sure convergence. Fix an event $\alpha \in \Omega$.

Let $a^n_{ij} = E A_{ij}(\phi_n)$ and $a^n_{ij} = E A_{ij}(\phi_n)$, $0 \leq i, j \leq 1$, and let $n$ be a subsequence wherein $a^n_{ij} \to a_{ij}$, $d_1(\beta_n, \beta) \to 0$, $d_1(\delta_n, 0) \to 0$, $\|\omega_n - \omega_n\| \to 0$, and $|\gamma_n - \gamma| \to 0$, for some $0 \leq i, j \leq 1$, $\beta_n, 0 \in \mathbb{R}^d$, $(\omega_n, \gamma) \in S^{p-1} \times [l, u]$, and $0 \leq i, j \leq 1$, where $\mathbb{R}^d$ denotes inclusion of infinite values. Let $\tilde{a} = \tilde{a}_{ij}$, $j = 0, 1$, and $\nu_0(X) = \beta_0\{\omega_0X - \gamma_0 \leq 0\} + \delta_0\{\omega_0X - \gamma_0 > 0\}$. Note by construction that $\tilde{a}_0 + \tilde{a}_1 = 1$. We will now show by contradiction that both $\tilde{a}_0 > 0$ and $\tilde{a}_1 > 0$. First assume that $\tilde{a}_1 = 0$. Accordingly, because $k \in \Omega$ and we are tracing along the subsequence $n$,

$$
\lim_{n \to \infty} M_n(\theta_0) \geq \lim_{n \to \infty} M_n(\tilde{\theta}_n)
$$

$$
\geq \lim_{n \to \infty} \left[ E \left\{c_3 \left| \tilde{\beta}_n - \nu_0(X) \right|^2 + \epsilon^2 \right\} \left( A_{00}(\phi_n) + A_{10}(\phi_n) \right) \right] + o(1 + \|\tilde{\beta}_n\|^2)
$$

$$
\geq \lim_{n \to \infty} \left\{ c_3 \left[ \|\tilde{\beta}_n - \beta_0\|^2 \tilde{a}_{00} + \|\tilde{\beta}_n - \delta_0\|^2 \tilde{a}_{10} \right] + \sigma_0^2 + o(1 + \|\tilde{\beta}_n\|^2) \right\},
$$

which implies that

$$
\lim_{n \to \infty} \left\{ c_3 \left[ \|\tilde{\beta}_n - \beta_0\|^2 \tilde{a}_{00} + \|\tilde{\beta}_n - \delta_0\|^2 \tilde{a}_{10} \right] + \sigma_0^2 + o(1 + \|\tilde{\beta}_n\|^2) \right\} \leq 0,
$$

because $M_n(\theta_0) \to \sigma_0^2$. But this leads to a contradiction as C2 implies that both $\tilde{a}_{00}$ and $\tilde{a}_{10}$ are $> 0$ and C5 ensures that $\beta_0 \neq \delta_0$. Thus $\tilde{a}_1 > 0$. We can also symmetrically argue that $\tilde{a}_0 > 0$. Hence we can conclude that $\tilde{a}_0 > 0$, $\tilde{a}_1 > 0$; and, by reapplying C2, that both $\tilde{a}_{00} + \tilde{a}_{01} > 0$ and $\tilde{a}_{10} + \tilde{a}_{11} > 0$.

By recycling previous arguments, we now have, generally, that

$$
\lim_{n \to \infty} M_n(\theta_0) \geq \lim_{n \to \infty} M_n(\tilde{\theta}_n)
$$

$$
\geq \lim_{n \to \infty} \left\{ c_3 \left[ \|\tilde{\beta}_n - \beta_0\|^2 \tilde{a}_{00} + \|\tilde{\beta}_n - \delta_0\|^2 \tilde{a}_{10} + \|\tilde{\beta}_n - \beta_0\|^2 \tilde{a}_{01} + \|\tilde{\beta}_n - \delta_0\|^2 \tilde{a}_{11} \right] + \sigma_0^2 + o(1 + \|\tilde{\beta}_n\|^2 + \|\tilde{\beta}_n\|^2) \right\},
$$

which implies that

$$
\lim_{n \to \infty} \left\{ \tilde{a}_{00}\|\tilde{\beta}_n - \beta_0\|^2 + \tilde{a}_{10}\|\tilde{\beta}_n - \delta_0\|^2 + \tilde{a}_{01}\|\tilde{\beta}_n - \beta_0\|^2 + \tilde{a}_{11}\|\tilde{\beta}_n - \delta_0\|^2 \right\} \leq 0.
$$

The first thing to note is, by the positivity conclusions of the previous paragraph, both $\|\tilde{\beta}_n\| = O(1)$ and $\|\tilde{\beta}_n\| = O(1)$.

Now assume that $\tilde{a}_{00} > 0$ and $\tilde{a}_{11} > 0$, and note that this already satisfies the positivity conclusions. Next note that this assumption forces both $\tilde{a}_{01} = 0$ and $\tilde{a}_{10} = 0$ since, otherwise, we have a contradiction by assumption C5, since a sequence cannot converge simultaneously.
to two distinct values. Hence \( \tilde{\beta} = \beta_0 \) and \( \tilde{\delta} = \delta_0 \). Moreover, we also now have, from the fact that \( \tilde{a}_{01} = 0, \) that

\[
0 \geq \limsup_{n \to \infty} E \left[ A_{01}(\tilde{\phi}_n) \right] \\
\geq \liminf_{n \to \infty} P \left[ \omega'_n X - \gamma_0 < 0, \tilde{\omega}'_n X - \tilde{\gamma}_n > 0 \right] \\
\geq P \left[ \omega'_n X - \gamma_0 < 0, \tilde{\omega}' X - \tilde{\gamma} > 0 \right],
\]

where, for the last inequality, we used the Portmanteau theorem applied to the (trivial) weak convergence of \( \omega'_n X - \tilde{\gamma}_n \) to \( \tilde{\omega}' X - \tilde{\gamma} \). We can use similar arguments to show that \( \tilde{a}_{10} = 0 \) implies that \( P[\omega'_n X - \gamma_0 > 0, \tilde{\omega}' X - \tilde{\gamma} < 0] = 0 \). Thus, by assumption C3, \( \tilde{\omega} = \omega_0 \) and \( \tilde{\gamma} = \gamma_0 \).

We next assume that \( \tilde{a}_{01} > 0 \) and \( \tilde{a}_{10} > 0 \). Note that this also satisfies the positivity conclusions. Arguing along the same lines as in the previous paragraph, we have that \( \tilde{a}_{00} = 0 \) and \( \tilde{a}_{11} = 0 \), and, moreover, that \( \tilde{\beta} = \delta_0 \) and \( \tilde{\delta} = \beta_0 \). Now redefine \( (\beta_0, \delta_0) \) as \( (\delta_0, \beta_0) \) and redefine \( (\omega_0, \gamma_0) \) as \( (-\omega_0, -\gamma_0) \), and note, with this redefinition and after reapplying the assumptions accordingly, we once again conclude that \( \tilde{\theta} = \theta_0 \).

\textbf{Remark 4.2.} Our model is unable to discriminate between \( (\beta_0, \delta_0, \omega_0, \gamma_0) \) and \( (\delta_0, \beta_0, -\omega_0, -\gamma_0) \). Fortunately, this does not matter, provided the assumptions are applied to at least one of the two versions, since the two models generate identical data. Since these two possibilities are indistinguishable, we will simply refer to both versions as \( \theta_0 = (\beta_0, \delta_0, \omega_0, \gamma_0) \).

By careful inspection, we can now deduce that the only way to satisfy the positivity conclusions and not obtain a contradiction is for one of the following two possibilities to hold: (i) \( \tilde{a}_{00} > 0, \tilde{a}_{11} > 0, \) and \( \tilde{a}_{01} = \tilde{a}_{10} = 0 \); or (ii) \( \tilde{a}_{00} > 0, \tilde{a}_{10} > 0, \) and \( \tilde{a}_{01} = \tilde{a}_{11} = 0 \). Allowing for the two possible versions of \( \theta_0 \), we have now established that \( \tilde{\theta}_n \to \theta_0 \). Because the convergent subsequence was arbitrary, we have that \( \tilde{\theta}_n \to \theta_0 \). As \( \kappa \in \Omega \) was arbitrary, and because the inner probability of \( \Omega \) is 1, we now have that \( \tilde{\theta}_n \to \theta_0 \), almost surely.\[ \square \]

5. Rates of convergence. In this section, we derive rates of convergence for the proposed \( M \)-estimators of the change-plane parameters and regression parameters. The rates of obtained from the limiting behavior of the process \( (M_n - M)(\theta) \); our line of argument follows Corollary 14.5 in [9]. Recall that \( \zeta = (\beta, \delta) \) and \( \phi = (\omega, \gamma) \).

\textbf{Theorem 5.1.} (rates of convergence) Under Conditions C1, C2, C3, C4 and C5, \( \sqrt{n} \| \tilde{\zeta}_n - \zeta_0 \| = O_P(1) \) and \( n \| \tilde{\omega}_n - \omega_0 \| + | \tilde{\gamma}_n - \gamma_0 | = O_P(1) \), for any sequence satisfying \( M_n(\tilde{\theta}_n) = \min_{\theta \in K_n} M_n(\theta) \).

Since the sequences \( \tilde{\theta}_n \) and \( \tilde{\theta}_n \) both satisfy the given conditions, Theorem 5.1 yields that \( \sqrt{n} | \tilde{\zeta}_n - \zeta_0 | = O_P(1) \) and \( n \| \tilde{\phi}_n - \phi_0 \| = O_P(1) \) and also \( \sqrt{n} | \tilde{\zeta}_n - \zeta_0 | = O_P(1) \) and \( n \| \tilde{\phi}_n - \phi_0 \| = O_P(1) \).

The proof of the theorem will involve satisfying the conditions of Corollary 14.5 of [9] for the convergence rate \( r_n = \sqrt{n} \) and for convergence of a minimizer rather than a maximizer. This latter adjustment is straightforward since minimizers of an objective function are also maximizers of the negative of the objective function. Before giving the conditions, recall from expression (2) that \( M_n(\theta) = \mathbb{P}_{m_\theta}(X, Y, Z) \), where

\[
m_\theta(Y, Z, X) = 1\{\omega' X - \gamma \leq 0\} \{Y - \beta' Z\}^2 + 1\{\omega' X - \gamma > 0\} \{Y - \delta' Z\}^2,
\]

and define \( M(\theta) = \mathbb{P}_{m_\theta}(X, Y, Z) \). Also, for any \( \eta > 0 \), define \( \Theta_\eta \equiv \{ \theta \in K : \| \tilde{\theta}(\theta, \theta_0) \| \leq \eta \} \), where \( \tilde{\theta}(\theta, \theta_0) = \| \zeta - \zeta_0 \|^2 + \| \omega - \omega_0 \| + | \gamma - \gamma_0 | \). Here are the needed conditions, after adjusting for minimization rather than maximization:
R1 \( \tilde{\theta}_n \) is consistent in outer probability for \( \theta_0 \) and satisfies \( \mathbb{P}_n m_{\tilde{\theta}_n} \leq \inf_{\theta \in \Theta_n} \mathbb{P}_n m_{\theta} + O_p(n^{-1}) \) for some \( n > 0 \).

R2 \( M(\theta) - M(\theta_0) \geq c d^2(\theta, \theta_0) \) for all \( \theta \in \Theta_n \) and some \( 0 < c, \eta < \infty \).

R3 There exists an increasing function \( \phi : [0, \infty) \to [0, \infty) \) such that the following hold for some \( 0 < c < \infty \):

a) \( \eta \to \phi(\eta)/\eta^\alpha \) is decreasing for some \( \alpha < 2 \),

b) \( n \phi(1/\sqrt{n}) \leq c \sqrt{n} \) for some \( 0 < c < \infty \), and

c) \( E^* \| G_n \|_{\mathcal{M}_\eta} \leq c(\phi(\eta)) \) for all \( 0 \leq \eta \leq \eta_1 \) and some \( 0 < c, \eta_1 < \infty \) and all \( n \) large enough, where

\[
\mathcal{M}_\eta = \{ m_{\theta}(Y, Z, X) - m_{\theta_0}(Y, Z, X) : \theta \in \Theta_n \}.
\]

Provided these conditions are satisfied, the corollary yields that \( \sqrt{n} d(\tilde{\theta}_n, \theta_0) = O_p(1) \), and the conclusions of Theorem 5.1 follow.

We will now verify each of the conditions R1–R3. For R1, Conditions C1, C2, C3, C4 and C5 imply that for \( 0 < \eta \) small enough, \( \Theta_\eta \subset K_n \) for all \( n \) large enough almost surely. Combining this with the fact that \( M_n(\tilde{\theta}_n) \leq M_n(\theta_0) \) for all \( n \) large enough almost surely, we obtain consistency of \( \tilde{\theta}_n \) from Theorem 4.1. Combining these two results yields that Condition R1 holds.

The following proposition verifies that Conditions R2 also holds:

**Proposition 5.2.** Under the given conditions, \( M(\theta) - M(\theta_0) \geq c_5 d^2(\theta, \theta_0) \) for all \( \theta \in \Theta_{n_0} \), for some \( 0 < c_5, \eta_5 < \infty \).

For R3, we will use \( \phi(\eta) = \eta \). Part a) is satisfied for any \( 1 < \alpha < 2 \). Part b) is satisfied since \( n \phi(1/\sqrt{n}) = \sqrt{n} \). Finally, Part c) is satisfied through the following proposition:

**Proposition 5.3.** Under the given conditions, \( E^* \| G_n \|_{\mathcal{M}_\eta} \leq b_6 \eta \) for some \( 0 < b_6 < \infty \) and all \( \eta \) small enough.

Since all of the needed conditions are satisfied, the proof is complete. \( \square \)

We now prove the two propositions in reverse order:

**Proof of Proposition 5.3.** Note first that

\[
m_{\theta}(Y, Z, X) - m_{\theta_0}(Y, Z, X) = A_{100}(\phi) \left[ -2 \epsilon (\beta - \beta_0)' Z + (\beta - \beta_0)' ZZ' (\beta - \beta_0) \right] + A_{101}(\phi) \left[ -2 \epsilon (\delta - \delta_0)' Z + (\delta - \delta_0)' ZZ' (\delta - \delta_0) \right] + A_{110}(\phi) \left[ -2 \epsilon (\beta - \beta_0)' Z + (\beta - \beta_0)' ZZ' (\beta - \beta_0) \right] + A_{111}(\phi) \left[ -2 \epsilon (\beta - \beta_0)' Z + (\beta - \beta_0)' ZZ' (\beta - \beta_0) \right] \\
\equiv B_1(\theta) + B_2(\theta) + B_3(\theta) + B_4(\theta).
\]

Standard arguments can be used to verify that each \( B_j = \{ B_j(\theta) : \tilde{d}(\tilde{\theta}_n, \theta_0) \leq \eta \} \) is a VC class, \( 1 \leq j \leq 4 \). It is also not hard to verify that both \( B_1 \) and \( B_4 \) have envelope \( F_1 = \eta(b_1 | \epsilon| + b_2) \) for some finite constants \( 0 < b_1, b_2 < \infty \). It is also not hard to argue that \( A_{101}(\phi) \leq A_{101}^* \equiv 1 \{ -b_3 (||\omega - \omega_0|| + |\gamma - \gamma_0|) < \omega_0 X - \gamma_0 \leq 0 \} \) for some \( 0 < b_3 < \infty \) and all \( \theta \) such that \( \tilde{d}(\tilde{\theta}_n, \theta_0) \) is small enough. Hence there exist constants \( 0 < b_4, b_5 < \infty \) such that \( B_2 \) has envelope \( F_2 = A_{101}^* (b_4 | \epsilon| + b_5) \), where we note that \( EF_2^2 \leq EA_{101}'^* (2b_2 \epsilon^2 + b_5^2) \) and \( EA_{101}' \leq f_b b_2 (\omega - \omega_0) + |\gamma - \gamma_0|) \). We can similarly argue that \( B_3 \) has envelope \( F_3 = A_{101}^* (b_4 | \epsilon| + b_5) \) for \( A_{101}^* \equiv 1 \{ 0 < b_5 X - \gamma_0 \leq b_4 (||\omega - \omega_0|| + |\gamma - \gamma_0|) \} \), after increasing slightly \( b_3, b_4 \) and \( b_5 \) if needed. Hence \( F = 2F_1 + F_2 + F_3 \) is an envelope for \( \mathcal{M}_\eta \) with \( \sqrt{EF^2} \leq b_6 \eta \), for some \( 0 < b_6 < \infty \). Now Theorem 11.1 of [9] yields the desired result since \( F_\eta \) is a VC class. \( \square \)
**Proof of Proposition 5.2.** This proof requires the following three lemmas, which we prove at the end of this section:

**Lemma 5.4.** Under assumptions C2 and C3, the conditional joint distribution of \((\nu^{-1}U, (X, Z))\) given \(|U| \leq \nu\) converges weakly, as \(\nu \downarrow 0\), to \((U_0, W_0)\), where \(U_0 \perp W_0\), \(U_0\) is uniform on \([-1, 1]\), and \(W_0 \sim G\).

**Lemma 5.5.** Let \((W, B) \in \mathbb{R}^q \times \mathbb{R}\) be a jointly distributed pair of random variables with \(P(B > 0) = 1\) and \(E(W) = \mu\) finite, and suppose there exists a constant \(\eta > 0\) such that \(E|u'(W - \mu)| \geq \eta\) for all \(u \in S^{q-1}\). Then, there exists a constant \(c > 0\) such that

\[
E \left[ |a'W - b|(B \wedge 1) \right] \geq c(a\|a\| + |b|)
\]

for all \(a \in \mathbb{R}^q\) and \(b \in \mathbb{R}\).

**Lemma 5.6.** For any \(\omega_1, \omega_2 \in S^{p-1}\), \(|(\omega_2 - \omega_1)\omega_1| = \|\omega_2 - \omega_1\|/2\).

We will utilize these lemmas later in the proof. We first note that that Conditions C1, C2, C3, C4 and C5 imply that for all \(0 < \eta\) small enough, we have for any \(\theta \in \Theta_\eta\) that

\[
M(\theta) - M(\theta_0) \geq (c_2/2)a_{00}(\phi)\|\beta - \beta_0\|^2 + (c_2/2)a_{11}(\phi)\|\delta - \delta_0\|^2 \\
+ (c_2/2)a_{01}(\phi)\|\delta - \beta_0\|^2 + (c_2/2)a_{10}(\phi)\|\beta - \delta_0\|^2,
\]

where \(a_{ij}(\phi) = EA_{ij}(\phi)\), for all \(0 \leq i, j \leq 1\). Moreover, we can also verify that there exists an additional constant \(0 < c_3 \leq c_2/2\) such that \((c_2/2)a_{00}(\phi) \geq c_3\), \((c_2/2)a_{11}(\phi) \geq c_3\), \((c_2/2)\|\delta - \beta_0\|^2 \geq c_3\), and \((c_2/2)\|\beta - \delta_0\|^2 \geq c_3\), for all \(\theta \in \Theta_\eta\), decreasing \(\eta\) slightly as needed to \(\eta_1 > 0\). Thus, for some \(0 < c_3, \eta_1 < 1\),

\[
M(\theta) - M(\theta_0) \geq c_3 (\|\beta - \beta_0\|^2 + \|\delta - \delta_0\|^2 + a_{01}(\phi) + a_{10}(\phi)),
\]

for all \(\theta \in \Theta_{\eta_1}\). The proposition is proved if we can establish that there exists \(0 < c_4, \eta_2 < \infty\) such that

\[
a_{01}(\phi) + a_{10}(\phi) \geq c_4 (\|\omega - \omega_0\| + |\gamma - \gamma_0|)
\]

for all \(\theta \in \Theta_{\eta_2}\).

To this end, define \(\nu = (k_1 + 1)\eta^2\), \((X_0, Z_0) \equiv W_0\) where \(W_0\) is as defined in Lemma 5.4, and let \((a)_+\) and \((a)_-\) be the positive and negative parts of \(a \in \mathbb{R}\) respectively. We now have, for all \(\theta \in \Theta_\eta\) for \(\eta > 0\) sufficiently small, that

\[
a_{01}(\phi) = P[\omega_0'X - \gamma_0 \leq 0 < \omega'X - \gamma]
\]

\[
= P[0 \leq -\omega_0'X + \gamma_0 < (\omega - \omega_0)'X - \gamma + \gamma_0]
\]

\[
= P[-(\omega - \omega_0)'X + \gamma - \gamma_0 < \omega_0'X - \gamma_0 \leq 0]
\]

\[
\geq P[-(\omega - \omega_0)'X + \gamma - \gamma_0 < \omega_0'X - \gamma_0 < 0]
\]

\[
= P \left[ -\nu^{-1}\{(\omega - \omega_0)'X - \gamma + \gamma_0\} \leq \nu^{-1}U < 0 \right] - \nu \leq U \leq \nu \right] \cdot P(|U| \leq \nu]
\]

\[
= \left( P \left[ -\nu^{-1}\{(\omega - \omega_0)'X_0 - \gamma + \gamma_0\} < U_0 < 0 \right] + o(1) \right) \cdot 2f_0\nu(1 + o(1))
\]

\[
= P \left[ -\nu^{-1}\{(\omega - \omega_0)'X_0 - \gamma + \gamma_0\} < U_0 < 0 \right] 2f_0\nu + o(\nu)
\]

\[
= \frac{1}{2} E \left( -\nu^{-1}\{(\omega - \omega_0)'X_0 - \gamma + \gamma_0\}\right)_+ \cdot 2f_0\nu + o(\nu)
\]

\[
= f_0 E \left( -\omega_0'X_0 + \gamma - \gamma_0\right)_+ + o(\eta^2),
\]
where the first three equalities and first inequality follow from definitions and straightforward calculations. The first equality after the inequality follows from the use of \( U = \omega_0 X - \gamma_0 \) as defined in Condition C2 combined with the fact that
\[
|\omega - \omega_0)'X - \gamma + \gamma_0| \leq \nu
\]
as a consequence of Condition C1. The next equality follows from Lemma 5.4, the portmanteau theorem (see, e.g., Theorem 7.6 of [9]), and Condition C2. The second-to-last equality follows from the independence of \( U_0 \) and \( X_0 \) combined with the fact that \( U_0 \) is uniform on \([-1, 1]\) (this also explains the leading 1/2 term). The last equality follows from the definition of \( \nu \).

We can use similar arguments to verify that
\[
a_{10}(\phi) \geq f_0 E (-(\omega - \omega_0)'X_0 + \gamma - \gamma_0) + o(\eta^2).
\]
Thus
\[
a_{01}(\phi) + a_{10}(\phi) \geq f_0 E (|\omega - \omega_0)'X_0 - \gamma + \gamma_0| + o(\eta^2).
\]
Since \( \omega_0 \omega'_0 + \bar{\omega}_0 \bar{\omega}'_0 \) equals the identity by definition of \( \bar{\omega}_0 \), we obtain that
\[
(\omega - \omega_0)'X_0 = (\omega - \omega_0)'\bar{\omega}_0 \bar{\omega}'_0 X_0 + (\omega - \omega_0)'\omega_0 \gamma_0
\]
\[
= (\omega - \omega_0)' \bar{\omega}_0 \bar{\omega}'_0 X_0 + O(\eta^4),
\]
where the second term on the right of the first equality follows from the fact that \( \omega_0 X_0 - \gamma_0 = 0 \) and the second equality follows from Lemma 5.6. Letting \( \tilde{\omega} = \bar{\omega}_0 (\omega - \omega_0) \) and \( \tilde{X}_0 = \bar{\omega}_0 X_0 \), and using the fact that \( \tilde{X}_0 \) is full rank, we obtain by application of Lemma 5.5 that there exists a \( c_4 > 0 \) such that
\[
E|\omega - \omega_0)'X_0 - \gamma + \gamma_0| \geq E|\tilde{\omega}' \tilde{X}_0 - \gamma + \gamma_0| + o(\eta^2)
\]
\[
\geq (c_4/f_0)(||\tilde{\omega}|| + |\gamma - \gamma_0|) + o(\eta^2)
\]
\[
\geq (c_4/f_0)(||\omega - \omega_0|| + |\gamma - \gamma_0|) + o(\eta^2),
\]
where the second inequality follows from the fact that
\[
||\tilde{\omega}||^2 = (\omega - \omega_0)'(\bar{\omega}_0 \bar{\omega}'_0 + \omega_0 \omega_0')(\omega - \omega_0) - O(\eta^4).
\]
Thus (9) follows. Note that in using Lemma 5.5, the required condition of the lemma can be shown to be satisfied for \( W = \tilde{X}_0 \) by contradiction. Suppose there does not exist \( \eta > 0 \) such that \( E|u'(W - \mu)| > \eta \) for all \( u \in S^{d-1} \). Then there exists a \( u \in S^{d-1} \) such that \( E|u'(W - \mu)| = 0 \). However, since \( W - \mu \) is full rank, this is not possible for any \( u \in S^{d-1} \). Thus, the condition of Lemma 5.5 is satisfied.

Combining this with (8), we now conclude that for some \( 0 < c_5, \eta_5 < 1 \),
\[
M(\theta) - M(\theta_0) \geq c_5 \tilde{d}^2(\theta, \theta_0),
\]
for all \( \{ \theta : \tilde{d}(\theta, \theta_0) \leq \eta_5 \} \), which is the desired conclusion. □

**Proof of lemma 5.4.** Let \( W = (X, Z) \) and \( W(u) \) be a random variable with distribution equal to the conditional distribution of \( W \) given \( U = u \). Let \( U_\nu \) be a random variable with distribution equal to the conditional distribution of \( U \) give \( |U| \leq \nu \). Then \( (\nu^{-1} U, W) \) given \( |U| \leq \nu \) has the same distribution as \( (\nu^{-1} U_\nu, W(U_\nu)) \). Note that by Condition C3, \( W(u) \rightarrow W(0) = W_0 \sim G \), as \( u \rightarrow 0 \). It is also easy to verify by Condition C2 that \( \nu^{-1} U_\nu \sim U_0 \) as \( \nu \downarrow 0 \). For a complete metric space \((\mathbb{B}, d)\), let \( BL_1(\mathbb{B}) \) be the space of all Lipschitz continuous functions \( f : \mathbb{B} \rightarrow \mathbb{R} \) such that \( |f| \leq 1 \) and \( \sup_{x,y \in \mathbb{B} : d(x,y) \leq \rho} |f(x) - f(y)| \leq \rho \), for all \( \rho > 0 \).
We now have that
\[
\sup_{f \in BL_1(\mathbb{R}^{d+p+1})} \left| Ef (\nu^{-1}U_\nu, W(U_\nu)) - Ef(U_0, W_0) \right|
\leq \sup_{f \in BL_1(\mathbb{R}^{d+p+1})} \left| Ef (\nu^{-1}U_\nu, W(U_\nu)) - Ef(\nu^{-1}U_\nu, W_0) \right|
+ \sup_{f \in BL_1(\mathbb{R}^{d+p+1})} \left| Ef(\nu^{-1}U_\nu, W_0) - Ef(U_0, W_0) \right|
= A_\nu + B_\nu.
\]

If we can show that $A_\nu \to 0$ and $B_\nu \to 0$, as $\nu \downarrow 0$, the desired conclusion for the lemma will follow from the Portmanteau theorem.

Note that for any $f \in BL_1(\mathbb{R}^{d+p+1})$,
\[
\left| Ef (\nu^{-1}U_\nu, W(U_\nu)) - Ef(\nu^{-1}U_\nu, W_0) \right|
\leq E \left| Ef \left( f (\nu^{-1}U_\nu, W(U_\nu)) - f(\nu^{-1}U_\nu, W_0) \right) \right|
\leq E \left\{ \sup_{g \in BL_1(\mathbb{R}^{d+p})} \left| Ef [g(W(U_\nu)) - g(W(0))] \right| \right\}
\to 0,
\]
as $\nu \downarrow 0$, by the fact that $W(U_\nu) \sim W(0)$ combined with the Portmanteau theorem. Since the right-hand-side of the last inequality does not depend on $f$, we obtain that $A_\nu \to 0$.

Note also that for any $f \in BL_1(\mathbb{R}^{d+p+1})$,
\[
\left| Ef(\nu^{-1}U_\nu, W_0) - Ef(U_0, W_0) \right|
\leq E \left| Ef \left( f(\nu^{-1}U_\nu, W_0) - f(U_0, W_0) \right) \right|
\leq E \left\{ \sup_{g \in BL_1(\mathbb{R})} \left| Ef [g(\nu^{-1}U_\nu) - g(U_0)] \right| \right\}
\to 0,
\]
as $\nu \downarrow 0$, using similar arguments as before combined with the fact that $W_0$ is independent of both $\nu^{-1}U_\nu$ and $U_0$. Since, once again, the right-hand-side of the last inequality does not depend on $f$, we obtain that $B_\nu \to 0$. Hence the desired conclusion of the lemma now follows. □

**Proof of Lemma 5.5.** Suppose $\frac{|a^'\mu + b|}{\|a\|} \geq \frac{\eta}{2}$. Then,
\[
E|a^'W + b| = \|a\| E \left| \frac{a^'(W - \mu)}{\|a\|} + \frac{a^'(\mu + b)}{\|a\|} \right|
\geq \|a\| \left| \frac{E \left( \frac{a^'(W - \mu)}{\|a\|} \right) + \frac{a^'(\mu + b)}{\|a\|}}{\|a\|} \right| \quad \text{(by Jensen’s Inequality)}
\]
\[
= \|a\| \left| \frac{a^'(\mu + b)}{\|a\|} \right|
\geq \frac{\eta}{2} \|a\|.
\]
Now suppose $\frac{|a'\mu + b|}{\|a\|} < \frac{\eta}{2}$. Then, 

$$E|a'W + b| = \|a\|E \left| \frac{a'(W - \mu)}{\|a\|} + \frac{a'(\mu + b)}{\|a\|} \right|$$

$$\geq \|a\| \left( E \left| \frac{a'(W - \mu)}{\|a\|} \right| - \left| \frac{a'(\mu + b)}{\|a\|} \right| \right)$$

$$\geq \|a\| \left( \eta - \frac{\eta}{2} \right)$$

$$\geq \frac{\eta}{2} \|a\|.$$

Therefore, for all $a \in \mathbb{R}^q$ and $b \in \mathbb{R}$,

(11) \quad $E|a'(W + b)| \geq \frac{\eta}{2} \|a\|$.

Now suppose $\mu = 0$. Then

$$E|a'W + b| \geq |E(a'W + b)| = |b| \quad \text{(by Jensen’s Inequality)}.$$

Hence, combining with (11), we have

(12) \quad $E|a'W + b| \geq \left( \frac{\eta}{2} \wedge 1 \right) \max (\|a\|, |b|)$.

Next, suppose $\mu \neq 0$. First, assume that $\|b\| > 2\|\mu\|\|a\|$. Then,

$$E|a'W + b| \geq |b| - |a'\mu|$$

$$\geq |b| - \|a\|\|\mu\| (|a'\mu| \leq \|a\|\|\mu\|)$$

$$\geq \frac{|b|}{2}.$$

Therefore,

(13) \quad $E|a'W + b| \geq \frac{\max (|b|, 2\|\mu\|\|a\|)}{2}$.

Now assume that $\|b\| \leq 2\|\mu\|\|a\|$. Then, from (11), we have

$$E|a'W + b| \geq \frac{\eta}{2} \|a\|$$

$$= \frac{\eta}{2} \max (\|a\|, \frac{|b|}{2\|\mu\|})$$

$$= \frac{\eta}{2} \frac{1}{2\|\mu\|} \max (|b|, 2\|\mu\|\|a\|).$$

Combining with (13), we have

$$E|a'W + b| \geq \left( \frac{1}{2} \wedge \frac{\eta}{4\|\mu\|} \right) \max (|b|, 2\|\mu\|\|a\|)$$

$$\geq \left( \frac{1}{2} \wedge \frac{\eta}{4\|\mu\|} \right) (1 \wedge 2\|\mu\|) \max (\|a\|, |b|).$$

Since $\max (\|a\|, |b|) \geq \frac{1}{2}(\|a\| + |b|)$, we now have, combining everything together, that

(14) \quad $E|a'W + b| \geq c_*(\|a\| + |b|)$,
where
\[ c^* = \begin{cases} \frac{1}{2}(\eta/2 \wedge 1), & \text{if } \mu = 0, \\ \frac{1}{2} \left( \frac{\eta}{4\|\mu\|} \right) (1 \wedge 2\|\mu\|), & \text{if } \mu \neq 0. \end{cases} \]

Now suppose, for some \((a, b) \in \mathbb{R}^q \times \mathbb{R} : \|a\|^2 + b^2 = 1\), that \(E(B|a'W - b|) = 0\). Then \(E(\|a'W - b\| B > 0) = 0\) since \(P[B \leq 0] = 0\). But this implies \(E(\|a'W - b\|) \geq c^*\) for all \((a, b) \in \mathbb{R}^q \times \mathbb{R} : \|a\|^2 + b^2 = 1\), and hence the desired result follows with \(c = c^* \wedge (c^*/2)\).

**Proof of lemma 5.6.** Note that the points 0, \(\omega_1\) and \(\omega_2\) form an isosceles triangle with base corresponding to the line segment from \(\omega_1\) and \(\omega_2\). This triangle divides at the midpoint of this base into two identical right triangles which are mirror images of each other, with the line from 0 to \(\omega_1\) forming the hypotenuse of one of them and the line from 0 to \(\omega_2\) forming the hypotenuse of the other. This is illustrated in Figure 1. The angle \(\theta\) between the base and the hypotenuse of either right triangle is the same as the angle between \(\omega_1\) and \(\omega_2 - \omega_1\) and satisfies \(\cos \theta = \|\omega_2 - \omega_1\|/2\) since both \(\omega_1\) and \(\omega_2\) are in \(S^{p-1}\). Hence
\[
|\omega_2 - \omega_1| \cdot \|\omega_1\| \cdot \cos \theta = \frac{\|\omega_2 - \omega_1\|^2}{2}.
\]

**Fig 1.** Partial of the unit circle for the rate of convergence. \(\theta\) is the angle between \(\omega_1\) and \(\omega_2\) which should be small.

**6. Weak convergence.** We now derive the limiting distribution of \(\sqrt{n}(\hat{\zeta}_n - \zeta_0), n(\hat{\phi}_n - \phi_0)\) and \(\sqrt{n}(\tilde{\zeta}_n - \zeta_0), n(\tilde{\phi}_n - \phi_0)\). The basic approach will be to evaluate the suitably standardized minimized process and verify that it converges to a limiting process on compacts in a manner which ensures that the mean-midpoint and mode-midpoint of the arg mins of the processes also converge on compacts. Then we will use this to derive the limiting distribution. However, there are a number of non-standard convergence issues which we will need to address, including the fact that the uniform norm-based Argmax Theorem (see,
e.g., Theorem 14.1 of [9]) is not directly applicable because of the presence of discontinuous indicator functions in the limiting objective function. We will highlight these concerns as they arise.

To begin, let \( H_n = \mathbb{R}^d \times \mathbb{R}^d \times n(S^{p-1} - \omega_0) \times n([l_0, u_0] - \gamma_0) \), let \( h = (h_1, h_2, h_3, h_4) \in H_n \) be an index, and define \( h \mapsto Q_n(h) = Q_n(h_1, h_2, h_3, h_4) = n \left[ M_n(\beta_0 + h_1/\sqrt{n}, \delta_0 + h_2/\sqrt{n}, \omega_0 + h_3/n, \gamma_0 + h_4/n) - M_n(\theta_0) \right] \).

Let \( \tilde{h}_n \in \arg \min_{h \in H_n} Q_n(h) \), and note, by construction, that

\[
\tilde{h}_n = \left( \sqrt{n}(\beta_n - \beta_0), \sqrt{n}(\delta_n - \delta_0), n(\omega_n - \omega_0), n(\gamma_n - \gamma_0) \right),
\]

where the last two parameter estimates are contained in a level set \( n(\Phi_n - \phi_0) \), which we will denote as \( \Phi_n \). By Theorem 5.1, \( \tilde{h}_n \) and the entirety of \( \Phi_n \) are asymptotically bounded in probability. Also define \( \hat{h}_n \) and \( \tilde{h}_n \) by replacing \( \tilde{h}_n \) with \( \hat{h}_n \) and \( \tilde{h}_n \), respectively.

We now characterize the limiting distribution and work towards stating the main result. Let \( W_1 \) and \( W_2 \) be independent mean zero Gaussian random variables with respective covariances \( \Sigma_1 = \sigma^2 (E [ZZ' \mathbf{1}(U \leq 0)])^{-1} \) and \( \Sigma_2 = \sigma^2 (E [ZZ' \mathbf{1}(U > 0)])^{-1} \). Define a new random process indexed by \( g = (g_1, g_2) \in \mathbb{R}^{p-1} \times \mathbb{R} \):

\[
Q_{02}(g) = \sum_{j=1}^{\infty} \mathbf{1}\{ -g_1 X_j^- + g_2 < -\bar{U}_j^- \leq 0 \} \bar{E}_j^- + \mathbf{1}\{ 0 < \bar{U}_j^+ \leq -g_1 X_j^+ + g_2 \} \bar{E}_j^+,
\]

where

- \( \bar{E}_j^- = \left[ (\beta_0 - \delta_0)' Z_j^- \right]^2 + 2\epsilon_j^- (\beta_0 - \delta_0)' Z_j^- , j \geq 1; \)
- \( (\epsilon_j^-, j \geq 1) \) are i.i.d. realizations of the residual \( \epsilon \) from the model (1);
- \( X_j^- = \omega_0 X_j^- , j \geq 1; \)
- \( (Z_j^-, X_j^-) \in \mathbb{R}^d \times \mathbb{R}^p , j \geq 1, \) are i.i.d. draws from the distribution \( G \) of Condition C3;
- \( \bar{U}_j^- = \sum_{k=1}^{j} M_k , j \geq 1, \) where the \( (M_k, k \geq 1) \) are i.i.d. exponential random variables with mean \( f_0^{-1}; \)
- \( \bar{E}_j^+ = \left[ (\beta_0 - \delta_0)' Z_j^+ \right]^2 + 2\epsilon_j^+ (\beta_0 - \delta_0)' Z_j^+ , j \geq 1; \)
- \( (\epsilon_j^+, j \geq 1) \) is an independent random replication of the sequence \( (\epsilon_j^-, j \geq 1) ; \)
- \( X_j^+ = \omega_0 X_j^+ , j \geq 1; \)
- \( (Z_j^+, X_j^+, j \geq 1) \) is an independent random replication of \( (Z_j^-, X_j^-, j \geq 1) \); and
- \( (\bar{U}_j^+, j \geq 1) \) is an independent random realization of \( (\bar{U}_j^- , j \geq 1) \).

We note that \( \bar{U}_1^- , \bar{U}_2^- , \ldots \), are equivalent to the jump points of a stationary Poisson process \( t \mapsto N^- (t) \) over \( 0 < t < \infty \) with intensity \( f_0 \). In other words, for any \( 0 < t < \infty \), \( L(t) = \sum_{j=1}^{\infty} \mathbf{1}\{ \bar{U}_j^- \leq t \} \) has the same distribution as \( N(t) \); and, conditional on \( N(t) = m \), the collection of jump locations of \( N(t) \) has the same distribution as \( \{ \bar{U}_j^- : 1 \leq j \leq m \} \) conditional on \( L(t) = m \). Moreover, this collection of jump locations has the same distribution as an i.i.d. sample of \( m \) uniform\([0, t]\) random variables. These are all direct consequences of the properties of a stationary Poisson process. These same properties hold true for \( \bar{U}_1^+ , \bar{U}_2^+ , \ldots , \) for another independent Poisson process \( t \mapsto N^+(t) \) having the same distribution as \( N^-(t) \).

Now let \( \hat{\Phi}_0 = \arg \min_{g \in \mathbb{R}^{p-1} \times \mathbb{R}} Q_{02}(g) \), where we note that this will typically not be a single number but a level set. We will use \( \hat{g} \) to denote some element of \( \hat{\Phi}_0 \). For any \( g \in \mathbb{R}^{p-1} \times \mathbb{R} \), define \( V_j^-(g) = \mathbf{1}\{ g_1 X_j^- - g_2 - \bar{U}_j^- > 0 \} - \mathbf{1}\{ g_1 X_j^- - g_2 - \bar{U}_j^- \leq 0 \} \) and
\( V_j^+(g) = 1\{g_j^1 \bar{X}_j^+ - g_2 + \bar{U}_j^+ > 0\} - 1\{g_j^1 \bar{X}_j^+ - g_2 + \bar{U}_j^- \leq 0\} \), for all \( j \geq 1 \). The following theorem provides us with some important properties regarding elements of \( \tilde{\Phi}_0 \), the proof of which will be given later in this section:

**Theorem 6.1.** Under Conditions C1, C2, C3, C4 and C5, the following are true:

1. \( T_0 \equiv \sup_{g \in \tilde{\Phi}_0} \| g_1 \| \vee |g_2| \) is a bounded random variable.
2. \( V_j^-(g) = V_j^-(\tilde{g}) \) and \( V_j^+(g) = V_j^+(\tilde{g}) \) for all \( j \geq 1 \) and for any \( g, \tilde{g} \in \tilde{\Phi}_0 \). Accordingly, fix \( \tilde{g} \in \tilde{\Phi}_0 \), and define \( V_j^- = V_j^-(\tilde{g}) \) and \( V_j^+ = V_j^+(\tilde{g}) \) for all \( j \geq 1 \).
3. For all \( j \geq J_0^- \equiv \min\{j \geq 1 : \bar{U}_j^+ > T_0(k_1 + 1)\} \), \( \bar{V}_j^- = -1 \); and, for all \( j \geq J_0^+ \equiv \min\{j \geq 1 : \bar{U}_j^+ > T_0(k_1 + 1)\} \), \( V_j^+ = 1 \).

Result 1 of Theorem 6.1 will facilitate verifying compactness of our limiting distributions, while Results 2 and 3 allow us to characterize the level set \( \tilde{\Phi}_0 \) using only a finite portion of the \( V_j^- \) and \( V_j^+ \) sequences. We now characterize the two main limiting distributions. First, we need to define the following maps from \( \mathbb{R}^{p-1} \) to \( \mathbb{R} \):

\[
\begin{align*}
g_1 &\mapsto \tilde{C}_L(g_1) = \left( \max_{1 \leq j \leq \tilde{J}_0^-} g_j^1 \bar{X}_j^- - \bar{U}_j^- \right) \land \left( \max_{1 \leq j \leq \tilde{J}_0^+} g_j^1 \bar{X}_j^+ + \bar{U}_j^+ \right), \\
g_1 &\mapsto \tilde{C}_U(g_1) = \left( \min_{1 \leq j \leq \tilde{J}_0^-} g_j^1 \bar{X}_j^- - \bar{U}_j^- \right) \lor \left( \min_{1 \leq j \leq \tilde{J}_0^+} g_j^1 \bar{X}_j^+ + \bar{U}_j^+ \right), \text{ and} \\
g_1 &\mapsto \tilde{C}_R(g_1) = \tilde{C}_U(g_1) - \tilde{C}_L(g_1),
\end{align*}
\]

where we take the maximum of an empty set of numbers to be \(-\infty\) and the minimum of an empty set to be \(+\infty\). By the definitions of \( \tilde{J}_0^- \) and \( \tilde{J}_0^+ \), we know that \( \left( \min_{1 \leq j \leq \tilde{J}_0^-} V_j^- \right) \land \left( \max_{1 \leq j \leq \tilde{J}_0^+} V_j^+ \right) = -1 \) and \( \left( \max_{1 \leq j \leq \tilde{J}_0^-} V_j^- \right) \lor \left( \min_{1 \leq j \leq \tilde{J}_0^+} V_j^+ \right) = 1 \), and thus the above functions are well-defined. Let \( \tilde{R}_0 \equiv \{g_1 \in \mathbb{R}^{p-1} : \tilde{C}_R(g_1) > 0\} \). The following lemma, analogous to Lemma (3.2), provides several useful properties of these functions and the set \( \tilde{R}_0 \):

**Lemma 6.2.** The following are true:

1. \( \tilde{\Phi}_0 = \left\{ (g_1, g_2) \in \mathbb{R}^{p-1} \times \mathbb{R} : \tilde{C}_R(g_1) > 0 \text{ and } \tilde{C}_L(g_1) \leq g_2 < \tilde{C}_U(g_1) \right\} \),
2. \( \tilde{C}_U, -\tilde{C}_L, \) and \( \tilde{C}_R \) are concave functions almost surely, and
3. \( \tilde{R}_0 \) is convex and bounded almost surely.

**Proof.** By the definitions, we know that \((g_1, g_2) \in \tilde{\Phi}_0\) if and only if \( g_j^1 \bar{X}_j^- - g_2 - \bar{U}_j^- \leq 0 \) for all \( 1 \leq j \leq \tilde{J}_0^- : V_j^- = -1 \); \( g_j^1 \bar{X}_j^+ - g_2 + \bar{U}_j^+ \leq 0 \) for all \( 1 \leq j \leq \tilde{J}_0^+ : V_j^+ = -1 \); \( g_j^1 \bar{X}_j^- - g_2 - \bar{U}_j^- > 0 \) for all \( 1 \leq j \leq \tilde{J}_0^- : V_j^- = 1 \); and \( g_j^1 \bar{X}_j^+ - g_2 + \bar{U}_j^+ > 0 \) for all \( 1 \leq j \leq \tilde{J}_0^+ : V_j^+ = 1 \). Thus \( \tilde{\Phi}_0 \) is precisely the set of \((g_1, g_2) \in \mathbb{R}^{p-1} \times \mathbb{R}\) for which \( \tilde{C}_L(g_1) - g_2 \leq 0 \) and \( \tilde{C}_U(g_1) - g_2 > 0 \). This now establishes Part 1. Recall that linear functions are both concave and convex. Since \( \tilde{C}_U \) is the minimum of a finite number of concave functions, it is also concave; since \( \tilde{C}_L \) is the maximum of a finite number of convex functions, it is also convex, and thus its negative is concave. Since \( \tilde{C}_R \) is the sum of two concave functions, it is also
concave. This establishes Part 2. The boundedness of \( R_0 \) follows from Part 1 of the lemma and also Part 1 of Theorem 6.1. Suppose \( g_{11} \) and \( g_{12} \) are both in \( R_0 \), then \( C_R(g_{11}) > 0 \) and \( C_R(g_{12}) > 0 \). By the concavity of \( C_R \), we have for any \( \alpha \in (0, 1) \), that \( C_R(\alpha g_{11} + (1 - \alpha) g_{12}) \geq \alpha C_R(g_{11}) + (1 - \alpha) C_R(g_{12}) > 0 \), and Part 3 is established. \( \square \)

We are now ready for presenting our main results for weak convergence in the following two theorems:

**Theorem 6.3.** Assume Conditions C1, C2, C3, C4 and C5 hold. Then

\[
\left( \begin{array}{c} \sqrt{n}(\hat{\beta}_n - \beta_0) \\ \sqrt{n}(\hat{\delta}_n - \delta_0) \\ n(\hat{\omega}_n - \omega_0) \\ n(\hat{\gamma}_n - \gamma_0) \end{array} \right) \sim \left( \begin{array}{c} W_1 \\ W_2 \\ \omega_0 \hat{g}_1 \\ \hat{g}_2 \end{array} \right),
\]

where \( W_1, W_2, \hat{g} = (\hat{g}_1, \hat{g}_2) \) are mutually independent, and where

\[
\hat{g}_1 = \frac{\int_{R_0} g_1 \tilde{C}_R(g_1) d\mu(g_1)}{\int_{R_0} \tilde{C}_R(g_1) d\mu(g_1)},
\]

\( \mu \) is Lebesgue measure on \( \mathbb{R}^p-1 \), and \( \hat{g}_2 = \left[ \tilde{C}_L(\hat{g}_1) + \tilde{C}_U(\hat{g}_1) \right] / 2. \)

**Theorem 6.4.** Assume Conditions C1, C2, C3‘, C4 and C5 hold. Then

\[
\left( \begin{array}{c} \sqrt{n}(\hat{\beta}_n' - \beta_0) \\ \sqrt{n}(\hat{\delta}_n' - \delta_0) \\ n(\hat{\omega}_n' - \omega_0) \\ n(\hat{\gamma}_n' - \gamma_0) \end{array} \right) \sim \left( \begin{array}{c} W_1 \\ W_2 \\ \omega_0 \hat{g}_1 \\ \hat{g}_2 \end{array} \right),
\]

where \( W_1, W_2, \hat{g} = (\hat{g}_1, \hat{g}_2) \) are mutually independent, and where \( \hat{g}_1 = \arg \max_{g_1 \in R_0} \tilde{C}_R(g_1) \) and \( \hat{g}_2 = \left[ \tilde{C}_L(\hat{g}_1) + \tilde{C}_U(\hat{g}_1) \right] / 2. \)

As will be seen in the proofs, the reason the stronger C3’ condition is needed in Theorem 6.4 is that without the condition, there is no guarantee that the \( \arg \max \) over \( \tilde{C}_R \) is unique. This non-uniqueness issue, however, is not a concern for Theorem 6.3.

Our next step will to more closely examine \( Q_n \) and establish an appropriate convergence on compacts that can facilitate the desired weak convergence. Letting \( U_i = \omega_i' X_i - \gamma_0 \), we obtain after rearranging the constituent components

\[
Q_n(h) = \sum_{i=1}^{n} 1\{U_i \leq 0\} \left( \frac{(h_1' Z_i)^2}{n} - \frac{2 \epsilon_1 h_1' Z_i}{\sqrt{n}} \right) + \sum_{i=1}^{n} 1\{U_i > 0\} \left( \frac{(h_2' Z_i)^2}{n} - \frac{2 \epsilon_1 h_2' Z_i}{\sqrt{n}} \right) + \sum_{i=1}^{n} 1\{-h_3' X_i + h_4 < n U_i \leq 0\}
\]

\[\times \left[ \left( \beta_0 - \delta_0 - \frac{h_2}{\sqrt{n}} \right) Z_i \right] + 2 \epsilon_1 \left( \beta_0 - \delta_0 + \frac{h_1 - h_2}{\sqrt{n}} \right) Z_i - \frac{(h_1' Z_i)^2}{n} \right] + \sum_{i=1}^{n} 1\{0 < n U_i \leq -h_3' X_i + h_4\} \]
\[
\times \left[ \left( \left( \delta_0 - \beta_0 - \frac{h_1}{\sqrt{n}} \right) Z_i \right)^2 + 2 \epsilon_i \left( \delta_0 - \beta_0 + \frac{h_2 - h_1}{\sqrt{n}} \right) Z_i - \frac{(h_2 Z_i)^2}{n} \right]
\]

\[= Q_{1n}(h_1) + Q_{1n}^+(h_2) + Q_{2n}(h) + Q_{2n}^+(h), \]

where we also define

\[E_{in}(h_1, h_2) = \left\{ \left( \delta_0 - \beta_0 - \frac{h_2}{\sqrt{n}} \right) Z_i \right\}^2 + 2 \epsilon_i \left( \delta_0 - \beta_0 + \frac{h_2 - h_1}{\sqrt{n}} \right) Z_i - \frac{(h_2 Z_i)^2}{n}, \]

for \(i = 1, \ldots, n\), so that \(Q_{2n}(h) = \sum_{i=1}^{n} \mathbb{1} \{ -h_2 X_i + h_4 < n U_i \leq 0 \} E_{in}(h_1, h_2) \) and \(Q_{2n}^+(h) = \sum_{i=1}^{n} \mathbb{1} \{ 0 < n U_i \leq -h_2 X_i + h_4 \} E_{in}^+(h) \). Also define \(Q_{1n}(h_1, h_2) = Q_{1n}^+(h_1) + Q_{1n}^+(h_2) \) and \(Q_{2n}(h) = Q_{2n}^-(h) + Q_{2n}^+(h) \).

Note that \(n(\tilde{\omega}_n - \omega_0)' \omega_0 = o_P(1) \) by Lemma 5.6 combined with Theorem 5.1. Thus \(h_{3n} \) lives in \(\mathbb{R}^{p-1}_{\omega_0} \) in the limit as \(n \to \infty \); and, moreover, \(n(S^{p-1} - \omega_0) \) is a \(p \times 1 \) dimensional manifold for all \(n \geq 1 \). We leverage this structure to create a one-to-one map between \(\mathbb{R}^{p-1} \) and \(n(S^{p-1} - \omega_0) \) whenever \(\|h_3\| \leq \sqrt{2n} \), which, by Theorem 5.1, is guaranteed to be almost surely true for all \(n \) large enough. Accordingly, define the function \(h_{sn} : \mathbb{R}^{p-1} \mapsto n(S^{p-1} - \omega_0) \) as

\[h_{sn}(g_1) = \omega_0 g_1 - n \left\{ 1 - \sqrt{\left( 1 - \frac{\|g_1\|^2}{n^2} \right)} + \omega_0 \right\}.
\]

The following lemma provides some useful properties of \(h_{sn} \):
and thus \(\bar{\omega}h_3 \in A_n\). Note also that for any \(g_1 \in A_n\), \(\bar{\omega}g_3 = g_1\), and thus \(h_{sn}^{-1}\) is the inverse of \(h_{sn}\). Moreover, for any \(h_3 \in B_n\),

\[
 h_{sn}(\bar{\omega}h_3) = \bar{\omega}\bar{\omega}h_3 - n \left(1 - \sqrt{1 - \left(\frac{(\bar{\omega}h_3)^2}{n^2}\right)}\right) \omega_0,
\]

\[
 = C_1h_3 + c_{2n}\omega_0,
\]

where \(C_1h_3\) is the projection of \(h_3\) onto the orthocomplement of \(\omega_0\) and \(c_{2n}\), and after recycling previous arguments, can be shown to be the length and sign of the projection of \(h_3\) onto \(\omega_0\). Thus \(h_{sn}(\bar{\omega}h_3) = h_3\); and \(h_{sn}\) is also the inverse of \(h_{sn}^{-1}\). Combining the previous results, we now have that both \(h_{sn}\) and \(h_{sn}^{-1}\) as given are one to one and onto. \(\square\)

We now need to use this to define some suitable compact index sets for convergence. For any \(k \in (0, \infty)\), define \(H_{2n}(k) \equiv \{(g_1, g_2) \in \mathbb{R}^{p-1} \times \mathbb{R} : \|g_1\| \leq 0.9k^{-1}k, \|g_2\| \leq k\}\) and \(H_{2n}(k) \equiv \{(h_3, h_4) \in (sP^{p-1} - \omega_0) \times (\|l, u\| - \gamma_0) : \|h_3\|^2 \leq (0.9k^{-1}k)^2 + r_n^2(k), \|h_4\| \leq k\}\), where

\[
 r_n(k) = n \left[1 - \sqrt{\left(1 - \left(0.9k^{-1}k\right)^2\right)}\right].
\]

Note that Lemma 6.5 yields that for any \(k \in (0, \infty)\) and for all \(n\) large enough, \((h_{sn}(g_1), g_2) \in H_{2n}(k)\) for any \((g_1, g_2) \in H_{2n}(k)\) and also \((h_{sn}^{-1}(h_3), h_4) \in H_{2n}(k)\) for any \((h_3, h_4) \in H_{2n}(k)\). Now define \(H_0(k) = \{(h_1, h_2, h_3, h_4) \in \mathbb{R}^d \times \mathbb{R}^d \times H_{2n}(k) : \|h_1\| + \|h_2\| \leq k\}\) and \(H_n(k) = \{(h_1, h_2, h_3, h_4) \in \mathbb{R}^d \times \mathbb{R}^d \times H_{2n}(k) : \|h_1\| + \|h_2\| \leq k, \|h_3\| \leq k\}\), and note that \(\lim_{n \to \infty} H_0(k) = H_0 \equiv \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^{p-1} \times \mathbb{R}\).

Now let \(\bar{h}_n(k) = (\bar{h}_1n(k), \bar{h}_2n(k), \bar{h}_3n(k), \bar{h}_4n(k)) \in \arg\min_{h \in H_n(k)} Q_n(h)\), and let \(\Phi_n'(k)\) be the level set containing \((\bar{h}_3n(k), \bar{h}_4n(k))\). Let \((\bar{h}_1n(k), \bar{h}_2n(k)) = (\tilde{h}_1n(k), \tilde{h}_2n(k))\), and define \(m_{2n}(k) = \sum_{i=1}^{n} \{4k \leq nU_i < 2k\}\), \(m_{2n}^+(k) = \sum_{i=1}^{n} \{2k < nU_i \leq 4k\}\), and \(F_n(k) = \{m_{2n}(k) \wedge m_{2n}^+(k) \geq 1\}\). Now let \(V_i(k) = \{h_3^i X_i - \tilde{h}_4n + nU_i > 0\} - \{h_3^i X_i - \tilde{h}_4n + nU_i \leq 0\}\), and note that \(\max_{1 \leq i \leq n} |V_i(k)| = 0\) for \(k\) large enough almost surely by Theorem 5.1, where the \(V_i\)'s are as defined in Section 3. Now, we need to define the following maps from \(n(sP^{p-1} - \omega_0)\) to \(\mathbb{R}\), and an associated set, for the setting where \(F_n(k) = 1\): \(h_3 \mapsto C^R_L(h_3) = \max_{1 \leq i \leq n} |V_i(k)| - h_3^i X_i + nU_i\), \(C^R_U(h_3) = \min_{1 \leq i \leq n} |V_i(k)| h_3^i X_i + nU_i\), \(C^R_L(h_3) = C^R_U(h_3) - C^R_L(h_3)\), and \(R_n(k) = \{h_3 \in n(sP^{p-1} - \omega_0) : C^R_L(h_3) > 0, \|h_3\|^2 \leq (0.9k^{-1}k)^2 + r_n^2(k)\}\). Note that by construction, when \(F_n(k) = 1\), \(h_3n \in R_n(k)\) and thus \(R_n(k)\) is nonempty. Note that, also by construction, any \(h_3 \in n(sP^{p-1} - \omega_0)\) satisfying \(\|h_3\|^2 \leq (0.9k^{-1}k)^2 + r_n^2(k)\) also satisfies \(\|h_3\| \leq k^{-1}k\) for all \(n\) large enough since \(\lim_{n \to \infty} r_n(k) = 0\). Hence, for any \((h_3, h_4) \in H_{2n}, |h_3^i X_i - h_4| \leq 2k\) for all \(n\) large enough. Thus when \(F_n(k) = 1\) and \(n\) is large enough, we know that there is a \(nU_i\) for some \(1 \leq i \leq n\) such that \(-4k \leq nU_i < -2k\) and hence \(h_3^i X_i - h_4 + nU_i \leq 0\) \(\forall (h_3, h_4) \in H_{2n}(k)\) and there also exists another \(nU_i\) for some \(1 \leq i \leq n\) for which \(2k < nU_i \leq 4k\) and \(h_3^i X_i - h_4 + nU_i \geq 0\) \(\forall (h_3, h_4) \in H_{2n}(k)\). This guarantees that the maximums and minimums used in the construction of \(C^R_L\) and \(C^R_U\) are over nonempty sets and are thus well defined.

When \(F_n(k) = 1\) and \(n\) is large enough for the conclusions of the the previous paragraph to hold, define

\[
 \tilde{h}_3n(k) = \frac{\int_{R_n(k)} hC^R_L(h) dv_n(h)}{\int_{R_n(k)} C^R_L(h) dv_n(h)},
\]

\[
 \tilde{h}_4n(k) = \frac{\int_{R_n(k)} hC^R_U(h) dv_n(h)}{\int_{R_n(k)} C^R_U(h) dv_n(h)}.
\]
where \( \nu_n \) is the uniform measure on \( n(S^{p-1} - \omega_0) \), and
\[
\hat{h}_{4n}(k) = \left[ C_{L}^{nk}(\hat{h}_{3n}(k)) + C_{U}^{nk}(\hat{h}_{3n}(k)) \right] / 2.
\]
Note that \( \hat{h}_{3n}(k) \in R_n(k) \) by arguments similar to those used in defining \( \hat{\omega}_n \) in Section 3. Also, by construction, both \( |C_{L}^{nk}(\hat{h}_{3n}(k))| \leq k \) and \( |C_{U}^{nk}(\hat{h}_{3n}(k))| \leq k \), and thus \( \hat{h}_{3n}(k), \hat{h}_{4n}(k) \in H_{20}(k) \). Hence we can now define our mean-midpoint estimator \( \hat{h}_n(k) \equiv (\hat{h}_{1n}(k),\hat{h}_{2n}(k),\hat{h}_{3n}(k),\hat{h}_{4n}(k)) \). Moreover, by recycling previous arguments, it is easy to verify that \( n(\theta_n - \theta_0) \to h_n(k) \) for all \( k \) large enough almost surely.

We now define the relevant restricted limiting estimators. Let \( W_1, W_2 \) and \( Q_{02} \) be mutually independent and as defined above. Let \( H_{20}(k) = \{ (g_1, g_2) \in \mathbb{R}^{p-1} \times \mathbb{R} : ||g_1|| \leq 0.9k_{1}^{-1}k, |g_2| \leq k \} \) and \( H^*_0(k) = \{ (h_1, h_2, g_1, g_2) \in \mathbb{R}^{d} \times \mathbb{R}^{d} \times H_{20}(k) : ||h_1|| \vee ||h_2|| \leq k \}. \)

Also define \( \hat{h}_{j0}(k) = \arg\min_{g \in H_{20}(k)} Q_{02}(g) \); \( \hat{g}(k) = \arg\min_{g \in H_{20}(k)} Q_{02}(g) \); \( \hat{h}_{0}(k) = \hat{h}_{10}(k), \hat{h}_{20}(k) \) and \( \hat{g}(k) = \hat{g}(k), \hat{g}_2(k) \). Let \( \hat{h}_{10}(k), \hat{h}_{20}(k) \) and define \( m_{20}^{-}(k) = \sum_{j=1}^{\infty} 1 \{ 2k < \hat{U}_{j}^{-} \leq 4k \}, m_{20}^{+}(k) = \sum_{j=1}^{\infty} 1 \{ 2k < \hat{U}_{j}^{+} \leq 4k \}, F_{0}(k) = 1 \{ m_{20}^{+}(k) \vee m_{20}^{-}(k) \geq 1 \} \).

Now let \( \hat{V}_{j}^{-}(k) = \hat{V}_{j}^{-}(\hat{g}(k)) \) and \( \hat{V}_{j}^{+}(k) = \hat{V}_{j}^{+}(\hat{g}(k)) \), for all \( j \geq 1 \), and note that \( \max_{j \geq 1} |\hat{V}_{j}^{-}(k) - \hat{V}_{j}^{-}(\hat{g})| = 0 \) and \( \max_{j \geq 1} |\hat{V}_{j}^{+}(k) - \hat{V}_{j}^{+}(\hat{g})| = 0 \) for \( k \) large enough almost surely by Theorem 6.1. Let \( \Phi_0(k) \) be the level set in \( \mathbb{R}^{p-1} \times \mathbb{R} \) which contains \( \hat{g}(k) \), i.e., \( \Phi_0(k) \) is the set of all \( g \in \mathbb{R}^{p-1} \times \mathbb{R} \) such that \( V_{j}^{-}(g) = \hat{V}_{j}^{-}(k) \) and \( V_{j}^{+}(g) = \hat{V}_{j}^{+}(k) \), for all \( j \geq 1 \). Let \( j_{0}^{-}(k) = \min\{ j \geq 1 : \hat{U}_{j}^{-} \geq 4k \} \) and \( j_{0}^{+}(k) = \min\{ j \geq 1 : \hat{U}_{j}^{+} \geq 4k \} \), and note that both \( j_{0}^{-}(k) \) and \( j_{0}^{+}(k) \) are finite and well-defined. We now need to define the following maps from \( \mathbb{R}^{p-1} \to \mathbb{R} \), for the setting where \( F_{0}(k) = 1 \):

\[
g_{1} \mapsto \hat{C}_{L}^{k}(g_{1}) = \max_{1 \leq j \leq j_{0}^{-}(k) : \hat{V}_{j}^{-}(k) = -g_{1}^{T} \hat{X}_{j}^{-} - \hat{U}_{j}^{-}} - g_{1}^{T} \hat{X}_{j}^{+} + \hat{U}_{j}^{+},
\]

\[
g_{1} \mapsto \hat{C}_{k}^{k}(g_{1}) = \max_{1 \leq j \leq j_{0}^{-}(k) : \hat{V}_{j}^{+}(k) = -g_{1}^{T} \hat{X}_{j}^{-} - \hat{U}_{j}^{-}} - g_{1}^{T} \hat{X}_{j}^{+} + \hat{U}_{j}^{+},
\]

and \( g_{1} \mapsto \hat{C}_{k}^{k}(g_{1}) = \hat{C}_{k}^{k}(\hat{g}(k)) - \hat{C}_{k}^{k}(g_{1}) \). Let \( \bar{R}_0(k) = \{ g_{1} \in \mathbb{R}^{d} : \hat{C}_{k}^{k}(g_{1}) > 0, ||g_{1}|| \leq 0.9k_{1}^{-1}k \} \). Note that when \( F_{0}(k) = 1, \hat{g}(k) \in \bar{R}_0(k) \) and thus \( \bar{R}_0(k) \) is nonempty. Note also that for any \((g_{1}, g_{2}) \in H_{20}(k), |g_{1}^{T} \hat{X}_{j}^{-} - g_{2}^{T} \hat{U}_{j}^{-} - g_{2}^{T} \hat{U}_{j}^{-}| \leq 2k \) for all \( 1 \leq j \leq j_{0}^{+}(k) \) and \( |g_{1}^{T} \hat{X}_{j}^{+} - g_{2}^{T} \hat{V}_{j}^{+}| \leq 2k \) for all \( 1 \leq j \leq j_{0}^{-}(k) \) and thus when \( F_{0}(k) = 1 \) and \( n \) is large enough, we know that there is an \( \hat{U}_{j}^{-} \in (2k, 4k) \), for some \( 1 \leq j \leq j_{0}^{+}(k) \), and thus also \( g_{1}^{T} \hat{X}_{j}^{-} - g_{2}^{T} \hat{U}_{j}^{-} \leq 0 \) for all \((g_{1}, g_{2}) \in H_{20}(k) \). We also know that there is a \( \hat{U}_{j}^{+} \in (2k, 4k) \), for some \( 1 \leq j \leq j_{0}^{+}(k) \), and hence also \( g_{1}^{T} \hat{X}_{j}^{+} - g_{2}^{T} \hat{U}_{j}^{+} > 0 \) for all \((g_{2}, g_{2}) \in H_{20}(k) \). This means that the minimums and maximums used in the construction of \( \hat{C}_{L}^{k} \) and \( \hat{C}_{k}^{k} \) are well defined and finite.

When \( F_{0}(k) = 1 \), define
\[
\hat{g}_{1}(k) = \frac{\int_{\bar{R}_0(k)} g \hat{C}_{k}^{k}(g) d\mu(g)}{\int_{\bar{R}_0(k)} \hat{C}_{k}^{k}(g) d\mu(g)},
\]
and \( \hat{g}_{2}(k) = \left[ \hat{C}_{k}^{k}(\hat{g}_{1}(k)) + \hat{C}_{k}^{k}(\hat{g}_{1}(k)) \right] / 2 \). Note that \( \hat{g}_{1}(k) \in \bar{R}_0(k) \) by arguments similar to those used previously. Also, by construction, both \( |\hat{C}_{L}^{k}(\hat{g}_{1}(k))| \leq k \) and \( |\hat{C}_{k}^{k}(\hat{g}_{1}(k))| \leq k \), and thus \( (\hat{g}_{1}(k), \hat{g}_{2}(k)) \in H_{20}(k) \). Hence we can now define our mean-midpoint limiting estimator as \( \hat{h}_{0}(k) \equiv (\hat{h}_{10}(k), h_{20}(k), \otimes_{0} \hat{g}_{1}(k), \hat{g}_{2}(k)) \). Moreover, by recycling previous arguments, it
is easy to verify that $\hat{h}_0(k) = (W_1, W_2, \omega_0 \tilde{g}_1, \tilde{g}_2)$ for all $k$ large enough almost surely. Now we are ready for the following convergence theorem for restrictions over compacts:

**Theorem 6.6.** Assume conditions C1, C2, C3, C4 and C5. Then, for every $k \in (0, \infty)$,

$$
\left( \frac{F_n(k) \hat{h}_n(k)}{F_n(k)} \right) \Rightarrow \left( \frac{F_0(k) \hat{h}_0(k)}{F_0(k)} \right).
$$

The proof of this theorem will be given later in this section, but now we have enough results to prove Theorem 6.3.

**Proof of Theorem 6.3.** Let $F \subset \mathbb{R}^{2d+p+1}$ be closed. Suppose we can show that for any $\eta > 0$, there exists a $0 < k < \infty$ such that $\lim \inf_{n \to \infty} P(F_n(k) = 1) \geq 1 - \eta$, $P(F_0(k) = 1) \geq 1 - \eta$, and $\lim \inf_{n \to \infty} P(F_n(k) \hat{h}_n(k) = \sqrt{n}(\hat{\theta}_n - \theta_0)) \geq 1 - \eta$. Thus, fixing an $\eta > 0$ and finding a corresponding $k < \infty$ which simultaneously satisfy these criteria, we have

$$
\lim \sup_{n \to \infty} P(\sqrt{n}(\hat{\theta}_n - \theta_0) \in F)
\leq \lim \sup_{n \to \infty} P(\sqrt{n}(\hat{\theta}_n - \theta_0) \in F, F_n(k) = 1, F_n(k) \hat{h}_n(k) = \sqrt{n}(\hat{\theta}_n - \theta_0))
+ \lim \sup_{n \to \infty} P(F_n(k) = 0) + \lim \sup_{n \to \infty} P(F_n(k) \hat{h}_n(k) \neq \sqrt{n}(\hat{\theta}_n - \theta_0))
\leq \lim \sup_{n \to \infty} P(F_n(k) \hat{h}_n(k) \in F) + 2\eta
\leq P(F_0(k) \hat{h}_0(k) \in F) + 2\eta
\leq P(\hat{h}_0 \in F, F_0(k) = 1, F_0(k) \hat{h}_0(k) = \hat{h}_0)
+ P(F_0(k) = 0) + P(F_0(k) \hat{h}_0(k) \neq \hat{h}_0) + 2\eta
\leq P(\hat{h}_0 \in F) + 4\eta,
$$

which implies that

$$
\lim \sup_{n \to \infty} P(\sqrt{n}(\hat{\theta}_n - \theta_0) \in F) \leq P(\hat{h}_0 \in F),
$$

since $\eta > 0$ is arbitrary. Since the closed set $F$ was also arbitrary, the desired conclusion follows from the Portmanteau Theorem for weak convergence (see, e.g., Theorem 7.6 of [9]). We next need to verify the required probability bounds for all $\eta > 0$.

We begin by showing that for any $\eta > 0$, there exists a $k < \infty$ for which $P(F_0(k) = 1) \geq 1 - \eta$. Recall that $F_0(k) = 1\{m_{20}(k) \vee m_{20}^+(k) \geq 1\}$, where $m_{20}^{-1}(k)$ is the number of $\tilde{U}_j^-$ values, among all $j \geq 1$, in the interval $(2k, 4k]$. Since the $\tilde{U}_j^-$ are progressive sums of i.i.d. exponentials with mean $f_0^{-1}$, $m_{20}^-$ is Poisson distributed with parameter $2kf_0$. We can similarly reason that $m_{20}^+$ is an independent copy of the same distribution. Thus

$$
\lim \inf_{k \to \infty} P(F_0(k) = 1) = 1,
$$

and we have the desired result for this step. Since Theorem 6.6 yields that $F_n(k) \Rightarrow F_0(k)$ in probability, as $n \to \infty$, we also have that for any $\eta > 0$, there exists a $k < \infty$ for which $\lim \inf_{n \to \infty} P(F_n(k) = 1) \geq 1 - \eta$.

Now let $\Phi'_n$ be the level set containing $\sqrt{n}(\hat{\phi}_n - \phi_0)$. As argued above, the entirety of $\Phi'_n$ is simultaneously asymptotically bounded in probability as a result of Theorem 5.1. Hence for any $\eta > 0$, there exists a $k < \infty$ such that both $\lim \inf_{n \to \infty} P(F_n(k) = 1) \geq 1 - \eta/2$ and

$$
\lim \inf_{n \to \infty} P(\sqrt{n}(\hat{\theta}_n - \theta_0) \in H_n(k)) \geq 1 - \eta/2,
$$

and thus

$$
\lim \inf_{n \to \infty} P(F_n(k) \hat{h}_n(k) = \sqrt{n}(\hat{\theta}_n - \theta_0)) \geq 1 - \eta.
$$

We can apply similar arguments for $\hat{h}_0$ via Theorem 6.1 to obtain that
for any $\eta > 0$ there exists a $k < \infty$ such that $P(F_0(k)\tilde{h}_0(k) = \hat{h}_0) \geq 1 - \eta$, and our proof is therefore complete. □

We now develop the compact convergence needed for proving Theorem 6.4, beginning with the following lemma:

**Lemma 6.7.** Assume Conditions C1, C2, C3’, C4 and C5 hold. Then the following hold:

1. When $F_n(k) = 1$, $h_3 \mapsto C_R^{nk}(h_3)$ has a unique maximum over $R_n(k)$ which we will denote $\tilde{h}_{3n}(k)$.
2. $(\tilde{h}_{3n}(k), \tilde{h}_{4n}(k))$, where $\tilde{h}_{4n}(k) = \left[\frac{C_L^{nk}(\tilde{h}_{3n}(k)) + C_L^{nk}(\hat{h}_{3n}(k))}{2}\right]$, is contained in $\Phi_n(k)$, for all $n$ large enough.
3. When $F_0(k) = 1$, $g_1 \mapsto \tilde{C}_R^{k}(g_1)$ over $g_1 \in \tilde{R}_0(k)$ is concave and has a unique maximum which we will denote $\tilde{g}_1(k)$.
4. $(\tilde{g}_1(k), \tilde{g}_2(k))$, where $\tilde{g}_2(k) = \left[\frac{\tilde{C}_L^k(\tilde{g}_1(k)) + \tilde{C}_U^k(\tilde{g}_1(k))}{2}\right]$, is contained in $\tilde{\Phi}_0(k)$.

**Proof.** When $F_n(k) = 1$, $C_R^{nk}$ is well-defined, and the set $R_n^*(k) = \{h_3 : C_R^{nk}(h_3) \geq C_R^{nk}(\tilde{h}_3(k))\}$ is nonempty, closed and geodesically connected, using arguments similar to those used in Lemma 3.2. Hence the set $R_n^*(k) = \{h_3 \in R_n^*(k) : \|h_3\|^2 \leq (0.9k^{-1})^2 + r_n^k(k)\}$ is also geodesically connected and closed and must contain the set of all $\arg\max$ of $C_R^{nk}$ over $R_n(k)$. Hence the set $\omega_0 + R_n^*(k)/n \subset S^{p-1}$ is also geodesically closed and connected. When $F_n(k) = 1$, all of the conditions of Lemma 3.4 are satisfied, and thus $\tilde{h}_{3n}(k)$ is well-defined and the unique maximizer. Since $\tilde{h}_{3n}(k)$ is the $\arg\max$ of a function over a closed subset of $R_n(k)$, $\|\tilde{h}_{3n}(k)\| \leq k_1^{-1}k$ for all $n$ large enough, and thus $C_L^{nk}(\tilde{h}_{3n}(k)) \leq k$ and $C_U^{nk}(\tilde{h}_{3n}(k)) \leq k$, and hence also $|\tilde{h}_{4n}(k)| \leq k$, for all $n$ large enough. By recycling arguments used in the proof of Lemma 3.2, we obtain the desired containment in $\Phi_n(k)$ by recycling previous arguments; we can verify that when $F_0(k) = 1$, the minimums and maximums in the definitions of the maps $\tilde{C}_L^k$, $\tilde{C}_U^k$ and $\tilde{C}_R^k$ are over non-null sets and are thus well defined, and we can also readily verify the concavity of $\tilde{C}_R^k$. For $\tilde{C}_R^k$ to not have a unique maximum would require $h_3 \mapsto \tilde{C}_R^k(h_3)$ and $h_3 \mapsto \tilde{C}_R^k(h_3)$ to be parallel for at least two distinct points in $\tilde{R}_0(k)$, but this is impossible since the $\tilde{X}_j^-$ and $\tilde{X}_j^+$ values involved are continuously distributed and thus the probability of this happening is zero. Hence Part (3) follows. Part (4) follows by recycling previous arguments. □

Denote $\tilde{h}_n(k) = (h_{1n}(k), h_{2n}(k), h_{3n}(k), h_{4n}(k))$ and $\tilde{h}_0(k) = (\tilde{h}_{10}(k), \tilde{h}_{20}(k), \tilde{h}_{30}(k), \tilde{h}_{40}(k))$. Now we are ready for the following convergence theorem for restrictions over compacts:

**Theorem 6.8.** Assume conditions C1, C2, C3’, C4 and C5. Then, for every $k \in (0, \infty)$,

$$
\begin{pmatrix}
F_n(k)\tilde{h}_n(k) \\
F_n(k)
\end{pmatrix} \rightsquigarrow \begin{pmatrix}
F_0(k)\tilde{h}_0(k) \\
F_0(k)
\end{pmatrix}.
$$

The proof of this theorem will be given later in this section, but now we have enough results to prove Theorem 6.4.

**Proof of Theorem 6.4.** The proof proceeds along very similar arguments as the proof of Theorem 6.3 and will borrow from that proof, especially since all the results of that theorem hold in this proof since Condition C3’ is stronger than Condition C3. Let $F \subset \mathbb{R}^{2d+p+1}$ be closed. As before, we need to show that for any $\eta > 0$, there exists a $0 < k < \infty$ such that $\lim\inf_{n \to \infty} P(F_n(k) = 1) \geq 1 - \eta$, $P(F_0(k) = 1) \geq 1 - \eta$, $\lim\inf_{n \to \infty} P(F_n(k)\tilde{h}_n(k) = \sqrt{n}(\tilde{\theta}_n - \theta_0)) \geq 1 - \eta$, and $P(F_0(k)\tilde{h}_0(k) = \tilde{h}_0(k) = 1 - \eta$. The first two of these follow since these were already shown under Condition C3.
Let $D_1$ be the level set containing $\sqrt{n}(\hat{\theta}_n - \theta_0)$. As previously, the entirety of $\Phi'$ is simultaneously asymptotically bounded in probability as a result of Theorem 5.1. Hence, as before, for any $\eta > 0$, there exists a $k < \infty$ such that both $\liminf n \to \infty P(F_n(k) = 1) \geq 1 - \eta/2$ and $\liminf n \to \infty P(\sqrt{n}(\hat{\theta}_n - \theta_0)) \in H_n(k)) \geq 1 - \eta/2$, and thus $P(F_n(k)\hat{\eta}_n(k) = \sqrt{n}(\hat{\theta}_n - \theta_0)) \geq 1 - \eta$. We can apply similar arguments for $h_0$ via Theorem 6.1 to obtain that for any $\eta > 0$ there exists a $k < \infty$ such that $P(F_0(0)\hat{\eta}_0(k) = \hat{h}_0) \geq 1 - \eta$, and our required probability bounds for all $\eta > 0$ are satisfied.

Hence, fixing an $\eta > 0$ and finding a corresponding $k < \infty$ which simultaneously satisfy these criteria, we have as before that

\[
\lim_{n \to \infty} \sup_{k} P(\sqrt{n}(\hat{\theta}_n - \theta_0) \in F) \\
\leq \lim_{n \to \infty} \sup_{k} P(\sqrt{n}(\hat{\theta}_n - \theta_0) \in F, F_n(k) = 1, F_n(k)\hat{\eta}_n(k) = \sqrt{n}(\hat{\theta}_n - \theta_0)) \\
+ \lim_{n \to \infty} \sup_{k} P(F_n(k) = 0) + \lim_{n \to \infty} \sup_{k} P(F_n(k)\hat{\eta}_n(k) \neq \sqrt{n}(\hat{\theta}_n - \theta_0)) \\
\leq \lim_{n \to \infty} \sup_{k} P(F_n(k)\hat{\eta}_n(k) \in K) + 2\eta \\
\leq P(F_0(0)\hat{\eta}_0(k) \in K) + 2\eta \\
\leq P(\hat{h}_0 \in K, F_0(k) = 1, F_0(k)\hat{h}_0(k) = \hat{h}_0) \\
+ P(F_0(k) = 0) + P(F_0(0)\hat{h}_0(k) \neq \hat{h}_0) + 2\eta \\
\leq P(\hat{h}_0 \in K) + 4\eta,
\]

which, as before, implies the desired weak convergence because of the arbitrariness of $\eta > 0$ and $K$, followed by the Portmanteau Theorem. □

**Proof of Theorem 6.6.** Define $J_n^-(k) = \{1 \leq i \leq n : -4k \leq nU_i \leq 0\}$, $J_n^+(k) = \{1 \leq i \leq n : 0 < nU_i \leq 4k\}$, $m_{1n}^-(k) = \sum_{i=1}^n 1\{\pm 2k \leq nU_i \leq 0\}$, $m_{1n}^+(k) = \sum_{i=1}^n 1\{0 < nU_i \leq 2k\}$, $m_{2n}^-(k) = m_{1n}^-(k) + m_{2n}^-(k)$ and $m_{2n}^+(k) = m_{1n}^+(k) + m_{2n}^+(k)$, where $m_{2n}^-(k)$ and $m_{2n}^+(k)$ are as defined above. Define also $\hat{\Sigma}_{1n} = \sum_{i=1}^n 1\{U_i > 0\}Z_iZ_i^\top$, $\hat{\Sigma}_{2n} = \sum_{i=1}^n 1\{U_i > 0\}Z_iZ_i^\top$, $W_{1n} = n^{-1/2} \sum_{i=1}^n 1\{U_i \leq 0\}\epsilon_iZ_i$, and $W_{2n} = n^{-1/2} \sum_{i=1}^n 1\{U_i > 0\}\epsilon_iZ_i$. Now we define the ensemble process

\[
P_n(k) = \left\{ m_{\pm n}^-(k), (X_i, Z_i, \epsilon_i, -nU_i) : i \in J_n^-(k), m_{1n}^-(k), m_{2n}^-(k), \hat{\Sigma}_{1n}, W_{1n}; m_{\pm n}^+(k), (X_i, Z_i, \epsilon_i, nU_i) : i \in J_n^+(k), m_{1n}^+(k), m_{2n}^+(k), \hat{\Sigma}_{2n}, W_{2n} \right\},
\]

where if either $J_n^-(k)$ or $J_n^+(k)$ are empty, the corresponding list of constituent variables is taken as a null set. Also, when not null, we order the elements $(X_i, Z_i, \epsilon_i, -nU_i)$ for $i \in J_n^-(k)$ in ascending order of $|nU_i|$; and we likewise order $(X_i, Z_i, \epsilon_i, nU_i)$ for $i \in J_n^+(k)$ in ascending order of $|nU_i|$.

Also define $m_{10}^-(k) = \sum_{j=1}^\infty 1\{0 \leq \hat{U}_j^\top \leq 2k\}$, $m_{10}^+(k) = \sum_{j=1}^\infty 1\{0 < \hat{U}_j^\top \leq 2k\}$, $m_0^-(k) = m_{10}^-(k) + m_{20}^-(k)$, and $m_0^+(k) = m_{10}^+(k) + m_{20}^+(k)$, where $m_{20}^-(k)$ and $m_{20}^+(k)$ are as defined above. We now define the limiting ensemble process

\[
P_0(k) = \left\{ m_0^-(k), (X_j^-, Z_j^-, \epsilon_j^-, \hat{U}_j^-) : 1 \leq j \leq m_0^-(k), m_{10}^-(k), m_{20}^-(k), \hat{\Sigma}_1, W_1; m_0^+(k), (X_j^+, Z_j^+, \epsilon_j^+, \hat{U}_j^+) : 1 \leq j \leq m_0^+(k), m_{10}^+(k), m_{20}^+(k), \hat{\Sigma}_2, W_2 \right\},
\]

where we use null sets as needed if either $m_0^-(k)$ or $m_0^+(k)$ are zero.

Our current goal is to show that $P_n(k) \Rightarrow P_0(k)$ with respect to a suitable uniform metric. Let $\mathbb{D}_{\infty}$ be the space of infinite sequences $x_1, x_2, \ldots$, with $x_j \in \mathbb{R}^d$ for all $j \geq 1$, and let $Z_0^+$
be the set of non-negative integers. Define the space \( \mathbb{E}^q \) consisting of the set of elements 
\((x_0, \{x_j : j \geq 1\}) \in \mathbb{Z}^{0+} \times \mathbb{Q}_+ \) such that \(x_j = 0\) for all \(j > x_0\) (and thus consists of all zeros when \(x_0 = 0\)). For \(x, y \in \mathbb{E}^q\), define the metric 
\[d_* (x, y) = |x_0 - y_0| + \max_{1 \leq j \leq x_0 \wedge y_0} \|x_j - y_j\|,\]
where \(d_* (x, y) = 0\) when \(x_0 = y_0 = 0\). It is easy to verify that \(d_*\) satisfies the triangle inequality and is otherwise a valid metric making \((\mathbb{E}^q, d_*)\) into a complete metric space. The actual space and metric for the processes \(P_n (k)\) and \(P_0 (k)\) is \(\mathbb{E}^q \times \mathbb{R}^{2+d^2+d} \times \mathbb{E}^q \times \mathbb{R}^{2+d^2+d}\), where \(q = p + d + 2\), and the metric consisting of the appropriate concatenation of two copies of \(d_*\) and the other needed uniform metrics, which concatenated metric we will denote as \(d_{**}\) and which will be the default uniform metric on this space. We now have the following lemma:

**Lemma 6.9.** Under assumptions C1, C2, C3, C4 and C5, and for any \(0 < k < \infty\), 
\(P_n (k) \rightsquigarrow P_0 (k)\) uniformly, as \(n \to \infty\).

**Proof.** The fact that \(\tilde{\Sigma}_j \to \Sigma_j\) almost surely, as \(n \to \infty\), for \(j = 1, 2\), follows from the strong law of large numbers. Define 
\[W_{1n}^* = n^{-1/2} \sum_{i=1}^n I\{nU_i < -4k\}e_iZ_i\]
and 
\[W_{2n}^* = n^{-1/2} \sum_{i=1}^n I\{nU_i > 4k\}e_iZ_i.\]
Define also
\[P_n^* (k) = \left\{ (m_{1n}^-(k), m_{1n}^+(k), m_{2n}^-(k), m_{2n}^+(k)) \right\}.
\]
If we can show that \(P_n^* (k) \rightsquigarrow P_0 (k)\) uniformly, as \(n \to \infty\), we will be done since 
\[E\|W_{1n} - W_{1n}^*\|^2 \leq 2P\{0 \leq -nU_i \leq 4k\} \sigma_2^2 k_2^2 \to 0\]
and 
\[E\|W_{2n} - W_{2n}^*\|^2 \leq 2P\{0 < nU_i \leq 4k\} \sigma_2^2 k_2^2 \to 0,\]
and thus \(P_n^* \to P_0 \to 0\) uniformly in probability.

If we condition on \(m_{1n}^-(k)\) and \(m_{1n}^+(k)\), then the following four random variables are mutually independent: 
\(W_{1n}^* = \left\{ (X_i, Z_i, e_i, -nU_i) : i \in J_{1n}^-(k), m_{1n}^-(k), m_{2n}^-(k) \right\}\) (as a group), 
\(W_{2n}^* = \left\{ (X_i, Z_i, e_i, nU_i) : i \in J_{1n}^+(k), m_{1n}^+(k), m_{2n}^+(k) \right\}\) (as a group). We will now establish weak convergence of 
\((m_{1n}^-(k), m_{1n}^+(k))\), and then show that the four random variables, conditional on 
\((m_{1n}^-(k), m_{1n}^+(k))\), converge weakly, and this will yield the desired joint weak convergence. It is easy to verify that 
\((m_{1n}^-(k), m_{1n}^+(k))\) come from a trinomial distribution with sample size \(n\) and probabilities 
\(p_1n = P\{-4k \leq nU_i \leq 0\}, p_2n = P\{0 < nU_i \leq 4k\}\), and 
\(p_{3n} = 1 - p_1n - p_2n\). Now standard arguments combined with Condition C2 yield that 
\((m_{1n}^-(k), m_{1n}^+(k)) \rightsquigarrow (m_{0n}^-(k), m_{0n}^+(k))\). Standard arguments also yield that 
\[\limsup_{n \to \infty} P[m_{1n}^-(k) > n^{1/3}] = 0\] and 
\[\limsup_{n \to \infty} P[m_{1n}^+(k) > n^{1/3}] = 0.\]

Now consider the conditional distribution of \(W_{1n}^*\) given \(m_{1n}^-(k) = m_n\) where \(0 \leq m_n \leq n\) is a sequence. Based on previous arguments, we can assume without loss of generality that 
\(0 \leq m_n \leq n^{1/3}\). Letting \(N = n - m_n\), we have that 
\[W_{1n}^* = d_{**} (N + m_n)^{-1/2} \sum_{j=1}^N \tilde{\epsilon}_j \tilde{Z}_j,\]
where \(\tilde{\epsilon}_j\), for \(1 \leq j \leq N\), are i.i.d. realizations from the distribution of \(\epsilon_i\), and \(\tilde{Z}_j\) are i.i.d. draws from the distribution of \(Z_i\) conditional on \(nU_i < -4k\). It is relatively easy to verify that the conditions of the Lindeberg-Feller Theorem apply to the (clearly mean zero) \(\tilde{\epsilon}_j \tilde{Z}_j\) terms for sample size \(N\). Thus 
\[N^{-1/2} \sum_{j=1}^N \tilde{\epsilon}_j \tilde{Z}_j \rightsquigarrow W_1.\]
Since \(N \to 1\) due to the bounds on \(m_n\), we now have that, conditional on \(m_{1n}^-(k)\), 
\(W_{1n}^* \rightsquigarrow W_1\), where \(W_1\) is independent of \(m_{0n}^-(k)\). We can argue similarly that conditional on \(m_{1n}^-(k)\), 
\(W_{n1} \rightsquigarrow W_1\), where \(W_2\) is independent of \(m_{0n}^+(k)\).
Now we consider weak convergence of \( \{(X_i, Z_i, \epsilon_i, -nU_i) : i \in J_n^-(k)\} \) conditional on the number of observations in \( J_{n}^-(k) \) being equal to \( m_{n}^-(k) \). The terms \( (X_i, Z_i, \epsilon_i, -nU_i) \), for \( i \in J_n^-(k) \), are i.i.d. draws from their joint distribution given \( 0 \leq -nU_i \leq 4k \). Since \( \epsilon_i \) is independent from the other terms, its distribution is unaffected by the conditioning. Since \( m_{n}^-(k) = O_P(1) \), it is easy to apply a minor modification of Lemma 5.4 to obtain that the characteristic function of \( \{(X_i, Z_i, \epsilon_i, -nU_i) : i \in J_n^-(k)\} \) given \( m_{n}^-(k) = m \) converges to

\[
\prod_{j=1}^{m} \left[ E \left( e^{i(t_j X_j + t_j Z_j \epsilon_j)} \right) E \left( e^{i(t_j \epsilon_j)} \right) E \left( e^{i(\epsilon_j U_j)} \right) \right],
\]

where \( i = \sqrt{-1} \) in the above expression, \( (t_{1j}, t_{2j}, t_{3j}, t_{4j}) \in \mathbb{R}^{d+1} \), \( j \geq 1 \), \( (X_j, Z_j) \) \( \sim G \), \( U_j^* \) is a uniform \( [0, 4k] \) random variable, and the product when \( m = 0 \) is 1. The minor modification of the lemma involves changing the range of \( U \) from \([-\nu, \nu]\) to \([-\nu, 0]\), with \( \nu = 4k/n \), and then replacing the interval \([-1, 0]\) with \([-1, 0] \). Thus \( \{(X_i, Z_i, \epsilon_i, -nU_i) : i \in J_n^-(k)\} \) conditional on \( m_{n}^-(k) = m \) converges weakly to \( (X_j, Z_j, \epsilon_j, U_j^*) : 1 \leq j \leq m \). We can similarly show that \( \{(X_i, Z_i, \epsilon_i, nU_i) : i \in J_n^+(k)\} \) conditional on \( m_{n}^+(k) = m \) converges weakly to \( (X_j, Z_j, \epsilon_j, U_j^+ : 1 \leq j \leq m \). Combining these results together, we obtain that

\( \mathcal{P}_n(k) \rightarrow \mathcal{P}_0(k) \), and the proof is complete.\( \square \)

It is easy to verify that the data in \( \mathcal{P}_n(k) \) is sufficient to fully generate the processes \( Q_{1n}(h_1) \) and \( Q_{2n}(h_2) \) for all \( h_j \in \mathbb{R}^d : \|h_j\| \leq k \), \( j = 1, 2 \), and also to generate \( F_n(k) \). Also, when \( (h_3, h_4) \in H_{2n}(k) \), \( \|h_3 X_i + h_4\| \leq 2k \) for all \( n \) large enough and for any \( 1 \leq i \leq n \), and thus \( Q_{2n}(h) \) and \( Q_{2n}^+(h) \) is also fully generated by the data in \( \mathcal{P}_n(k) \) for all \( h \in H_{n}(k) \), since for all observations \( 1 \leq i \leq n \) such that \( i \not\in J_n^-(k) \cup J_n^+(k) \) we have that

\[ \{ -h_3 X_i + h_4 < nU_i \leq 0 \} \vee \{ 0 < nU_i \leq -h_3 X_i + h_4 \} = 0 \]

We can also readily verify that the data in \( \mathcal{P}_0(k) \) is sufficient to fully generate the processes \( h_j \rightarrow h_j Z_j h_j - 2 h_j Z_j W_j, \) for \( h_j \in \mathbb{R}^d : \|h_j\| \leq k \), and \( j = 1, 2 \), and also \( F_0(k) \) and \( Q_{02}(g) \), for all \( g \in H_{20}(k) \).

Accordingly, provided \( F_n(k) = 1 \), the data in \( \mathcal{P}_n(k) \) generates \( h_n(k) \) for all \( n \) large enough almost surely; and, similarly, provided \( F_0(k) = 1 \), the data in \( \mathcal{P}_0(k) \) generates \( h_0(k) \).

By Lemma 6.9, we know by the almost sure representation theorem (see, e.g., Theorem 7.26 of [9]) that there exists a new probability space where the marginal distributions of \( \mathcal{P}_n(k) \) and \( \mathcal{P}_0(k) \) are unchanged but that \( \mathcal{P}_n(k) \rightarrow \mathcal{P}_0(k) \) uniformly outer almost surely, as \( n \rightarrow \infty \). Note that for clarity we are not changing notation to reflect the new probability space. If we can show that this outer almost sure convergence implies that

\[
\begin{pmatrix}
F_n(k) h_n(k) \\
F_0(k) h_0(k)
\end{pmatrix} \rightarrow \begin{pmatrix}
F_0(k) h_0(k) \\
F_0(k)
\end{pmatrix}
\]

outer almost surely, then our proof of Theorem 6.6 will be complete. Accordingly, assume going forward that we are in this new probability space where \( \mathcal{P}_n(k) \rightarrow \mathcal{P}_0(k) \) uniformly outer almost surely. It is now easy to verify that \( \|h_{jn}(n) - h_{j0}(0)\| \rightarrow 0 \) as \( n \rightarrow \infty \), for \( j = 1, 2 \). It is also easy to verify that if \( x_n \rightarrow x \) on the space of integers, then \( x_n \rightarrow x \) for all \( n \) large enough. Accordingly, we have that \( (F_n(k), m_{n}^-(k), m_{1n}^-(k), m_{2n}^-(k), m_{n}^+(k), m_{1n}^+(k), m_{2n}^+(k)) = (F_0(k), m_{0}^-(k), m_{10}^-(k), m_{20}^-(k), m_{0}^+(k), m_{10}^+(k), m_{20}^+(k)) \) for all \( n \) large enough almost surely. When \( F_0(k) = 0 \), then the results above verify that \( 0 = F_0(k) h_0(k) = F_n(k) h_n(k) \) for all \( n \) large enough, and thus the proof of the theorem is trivial in this case. Accordingly, assume going forward that \( F_0(k) = 1 \). This then assures us that \( F_n(k) = 1 \) for all \( n \) large enough.

Since we also reordered the indices in \( J_n^-(k) \) and \( J_n^+(k) \) by \( |nU_i| \), and since \( \hat{U}_j^- \) and \( \hat{U}_j^+ \) are continuous, we have that the indices in \( J_n^-(k) \) correspond exactly to \( 1 \leq j \leq m_{0}^-(k) \) and those in \( J_n^+(k) \) correspond exactly to those in \( 1 \leq j \leq m_{0}^+(k) \) for all \( n \).
large enough. We will assume $n$ is large enough going forward. This allows us to reconstitute $\mathcal{P}_n(k)$ as follows: replace $(m^-_n(k), m^-_{1n}(k), m^-_{2n}(k), m^-_1(k), m^-_n(k), m^-_{2n}(k))$ with $(m^-_0(k), m^-_{10}(k), m^-_{20}(k), m^-_0(k), m^-_{10}(k), m^-_{20}(k))$; for each $1 \leq j \leq m^-_0(k)$, set $(X_{j-}, Z_{j-}, \epsilon_{j-}, U_{j-}) = (X_i, Z_i, \epsilon_i, nU_i)$ for the matching observation $i \in J^-_n(k)$; and for each $1 \leq j \leq m^-_0(k)$, set $(X_{j+}, Z_{j+}, \epsilon_{j+}, U_{j+}) = (X_i, Z_i, \epsilon_i, nU_i)$ for the matching observation $i \in J^+_n(k)$.

Thus

$$\mathcal{P}_n(k) = \left\{ m^-_0(k), (X_{j-}, Z_{j-}, \epsilon_{j-}, U_{j-}) : 1 \leq j \leq m^-_0(k), m^-_{10}(k), m^-_{20}(k), \hat{\Sigma}_1n, W_1n; \right\}$$

and

$$m^+_0(k), (X_{j+}, Z_{j+}, \epsilon_{j+}, U_{j+}) : 1 \leq j \leq m^+_0(k), m^+_{10}(k), m^+_{20}(k), \hat{\Sigma}_2n, W_2n \right\},$$

where

$$\lim_{n \to \infty} \max_{1 \leq j \leq m^-_0(k)} \left[ \|X_{j-} - X^-_j\| \vee \|Z_{j-} - Z^-_j\| \vee |\epsilon_{j-} - \epsilon^-_j| \vee |U_{j-} - \hat{U}^-_j| \right] = 0$$

and

$$\lim_{n \to \infty} \max_{1 \leq j \leq m^+_0(k)} \left[ \|X_{j+} - X^+_j\| \vee \|Z_{j+} - Z^+_j\| \vee |\epsilon_{j+} - \epsilon^+_j| \vee |U_{j+} - \hat{U}^+_j| \right] = 0,$$

where we take the maximum over a null set to be zero in this situation. Also, for each $1 \leq j \leq m^-_0(k)$, set $E_{jn-}(h_1, h_2) = E^\infty_{jn-}(h_1, h_2)$ for the matching observation $i \in J^-_n(k)$; and for each $1 \leq j \leq m^+_0(k)$, set $E_{jn+}(h_1, h_2) = E^\infty_{jn+}(h_1, h_2)$ for the matching observation $i \in J^+_n(k)$. We can also verify that

$$\lim_{n \to \infty} \max_{1 \leq j \leq m^-_0(k)} \sup_{(h_1, h_2) \in \mathcal{R}^d \times \mathcal{R}^d; \|h_1\| \vee \|h_2\| \leq k} \left| E_{jn-}(h_1, h_2) - \hat{E}^-_j \right| = 0,$$

and

$$\lim_{n \to \infty} \max_{1 \leq j \leq m^+_0(k)} \sup_{(h_1, h_2) \in \mathcal{R}^d \times \mathcal{R}^d; \|h_1\| \vee \|h_2\| \leq k} \left| E_{jn+}(h_1, h_2) - \hat{E}^+_j \right| = 0.$$
\[
\begin{align*}
\text{where, similar to what was done previously, } & \\
& \mathbf{m}_n(k) \equiv \sum_{j=1}^{m_0(n)} \mathbf{1}\{-h_3 X_{j-} + h_4 < -U_{j-} \leq 0\} E_{j-} \left( \hat{h}_{1n}(k), \hat{h}_{2n}(k) \right) \\
& + \sum_{j=1}^{m_0(n)} \mathbf{1}\{0 < U_{j+} \leq -h_3 X_{j+} + h_4\} E_{j+} \left( \hat{h}_{1n}(k), \hat{h}_{2n}(k) \right) \\
& \equiv \hat{Q}^{-+}_{2n}(h_3, h_4) + \hat{Q}^{-+}_{2n}(h_3, h_4).
\end{align*}
\]

Thus \( (\hat{h}_{3n}(k), \hat{h}_{4n}(k)) \in \arg\min_{(h_3, h_4) \in H_{2n}(k)} \hat{Q}^{-+}_{2n}(h_3, h_4) \) depends only on the data in \( \mathcal{P}_n(k) \). Similarly, the level set \( \Phi_n(k) \) depends only on the data in \( \mathcal{P}_n(k) \); and we can also verify that \( V_i(k) \) as defined above, for \( i \in J^- \) \( \cup \) \( J^+ \), also depends on on the data in \( \mathcal{P}_n(k) \). Linking up indices as before, we can define \( V_{j-}(k) = V_i(k) \) for \( 1 \leq j \leq m_0^{-}(k) \) via linking with the appropriate \( i \in J^- \) \( (k) \), and also \( V_{j+}(k) = V_i(k) \) for \( 1 \leq j \leq m_0^{+}(k) \) via the appropriate linking with \( i \in J^+ \) \( (k) \). Thus \( \Phi_n(k) = \{(h_3, h_4) \in H_{2n}(k) : \mathbf{1}\{h_3 X_{j-} - h_4 - U_{j-} > 0\} - \mathbf{1}\{h_3 X_{j-} - h_4 - U_{j-} \leq 0\} = V_{j-}(k), 1 \leq j \leq m_0^{-}(k), \} \) \( \cup \) \( \{(h_3, h_4) \in H_{2n}(k) : \mathbf{1}\{h_3 X_{j+} - h_4 + U_{j+} > 0\} - \mathbf{1}\{h_3 X_{j+} - h_4 + U_{j+} \leq 0\} = V_{j+}(k), \} \) \( 1 \leq j \leq m_0^{+}(k) \} \). Now we can re-express several previously defined quantities in terms of data in \( \mathcal{P}_n(k) \), specifically,

\[
\begin{align*}
\hat{h}_3 & \mapsto C^nk_L(h_3) = \left( \max_{1 \leq j \leq m_0^{-}(k) : V_{j-}(k) = -1} h_3 X_{j-} - U_{j-} \right) \vee \left( \max_{1 \leq j \leq m_0^{-}(k) : V_{j-}(k) = -1} h_3 X_{j+} + U_{j+} \right), \\
\hat{h}_3 & \mapsto C^nk_U(h_3) = \left( \min_{1 \leq j \leq m_0^{-}(k) : V_{j-}(k) = 1} h_3 X_{j-} - U_{j-} \right) \wedge \left( \min_{1 \leq j \leq m_0^{+}(k) : V_{j+}(k) = 1} h_3 X_{j+} + U_{j+} \right),
\end{align*}
\]

and \( C^nk_R, R_n(k) \) and \( \hat{h}_{3n}(k) \) are defined as before relative to \( C^nk_L \) and \( C^nk_U \) and thus depend only on data in \( \mathcal{P}_n(k) \).

By Lemma 6.5 and surrounding arguments, we have that for all \( n \) large enough, \((\varpi_0, h_3, h_4) \in H_{20}(k)\) for all \((h_3, h_4) \in H_{2n}(k)\) and \((h_{sn}(g_1), g_2) \in H_{2n}(k)\) for all \((g_1, g_2) \in H_{20}(k)\). Thus the process \((h_3, h_4) \mapsto \hat{Q}^{-+}_{2n}(h_3, h_4)\) ranging over \( H_{2n}(k) \) is equivalent to the process \((g_1, g_2) \mapsto \hat{Q}^{-+}_{2n}(h_{sn}(g_1), g_2)\) ranging over \((g_1, g_2) \in H_{20}(k)\), for all \( n \) large enough. We can similarly define \( \hat{C}^nk_L(g_1) = C^nk_L(h_{sn}(g_1)), \hat{C}^nk_U(g_1) = C^nk_U(h_{sn}(g_1)), \hat{C}^nk_R = \hat{C}^nk_U - \hat{C}^nk_L \), and \( \hat{R}_n(k) = \{g_1 \in \mathbb{R}^{p-1} : \hat{C}^nk_R(g_1) > 0, \|g_1\| \leq 0.9 k_1^{-1} k\} \). We can now verify that

\[
\hat{h}_{3n}(k) = \frac{\int_{\hat{R}_n(k)} h_{sn}(g_1) \hat{C}^nk_R(g_1) \left( \frac{d\nu_n(h_{sn}(g_1))}{d\mu(g_1)} \right) d\mu(g_1)}{\int_{\hat{R}_n(k)} \hat{C}^nk_R(g_1) \left( \frac{d\nu_n(h_{sn}(g_1))}{d\mu(g_1)} \right) d\mu(g_1)},
\]

where, similar to what was done previously, \( \mu \) is Lebesgue measure on \( \mathbb{R}^{p-1} \) and \( \nu_n \) is Lebesgue measure on the sphere \( nS^{p-1} \) oriented so that a normal vector \( e_p \) in the \( p \)th dimension orthogonal to the hyperplane \( \mathbb{R}^{p-1} \) passes through the center of the sphere and the sphere touches \( \mathbb{R}^{p-1} \) at the point \( e_p = 0 \). We can make this transformation because the orthonormal rotational transformation matrix from \( \mathbb{R}^p \) to \( \mathbb{R}_{\omega_0} \times \mathbb{R}_{\omega_0}^{p-1} \), where \( \mathbb{R}_{\omega_0} \) is just the linear span of \( \omega_0 \), which is the matrix \( I_0 = (\omega_0, \omega_0) \), has determinant \( 1 \). The quantity \( d\nu_n/d\mu \) is the Radon-Nikodym derivative of \( \nu_n \) relative to \( \mu \) defined via the transformation \( g_1 \mapsto h_{sn}(g_1) \). Because the curvature of the surface of \( S^{p-1} \) is always greater than zero, we know that \( d\nu_n/d\mu \geq 1 \). 

We can also show that, for all \( n \) large enough and \( \| g_1 \| \leq 0.9 k_1^{-1} k \), the infinitesimal curvature \( h_{\ast n}(dg_1)/dg_1 \) is bounded above by

\[
\frac{p-1}{\prod_{i=1}^{p} \sqrt{1 + \frac{g_{1,i}^2}{n^2 - \| g_1 \|^2}} } \to 1,
\]

as \( n \to \infty \), where \( g_1 = (g_{1,1}, \ldots, g_{1,p-1})' \). Moreover, we can also readily verify that

\[
\lim \sup_{n \to \infty} \max_{1 \leq j \leq m_0^-(k)} \sup_{(g_1, g_2) \in H_2^-} \left| h_{\ast n}(g_1) X_j^- - g_2 - (g_1' \tilde{X}_j^- - g_2) \right| = 0,
\]

and

\[
\lim \sup_{n \to \infty} \max_{1 \leq j \leq m_0^+(k)} \sup_{(g_1, g_2) \in H_2^+} \left| h_{\ast n}(g_1) X_j^+ - g_2 - (g_1' \tilde{X}_j^+ - g_2) \right| = 0.
\]

Recall \( \tilde{V}_j^- (k) \) and \( \tilde{V}_j^+ (k) \), defined above for all \( 1 \leq j < \infty \), and note that when \( F_0(k) = 1 \), we have that \( \tilde{V}_j^- (k) = -1 \) for all \( m^-_{10}(k) < j < \infty \) and \( \tilde{V}_j^+ (k) = 1 \) for all \( m^+_{10}(k) < j < \infty \).

Thus \( \tilde{g}(k) \in \arg \min_{g \in H_{20}(k)} Q_{02}(g) = \arg \min_{g \in H_{20}(k)} Q_{02}^+(g) \), where

\[
Q_{02}(g) = \sum_{j=1}^{m^-_{0}(k)} \left\{ -g_1' \tilde{X}_j^- + g_2 < -\tilde{U}_j^- \leq 0 \right\} \tilde{E}_j^- + \sum_{j=1}^{m^+_{0}(k)} \left\{ 0 < \tilde{U}_j^+ \leq -g_1' \tilde{X}_j^+ + g_2 \leq 0 \right\} \tilde{E}_j^+.
\]

Based on arguments up to this point, we can also verify that for all \( n \) large enough \( V_{j^-} = \tilde{V}_j^- (k) = -1 \) for all \( m^-_{10}(k) < j \leq m_0^-(k) \) and \( V_{j^+} = \tilde{V}_j^+ (k) = 1 \) for all \( m^+_{10}(k) < j \leq m_0^+(k) \).

Thus the non-zero entries in both \( \tilde{Q}_{2n}(g) \) and \( Q_{02}^+(g) \), for all \( g \in H_{20}(k) \), can only occur when \( \left\{ -h_{\ast n}(g_1) X_j^- + g_2 < -U_j^- \leq 0 \right\} = 1 \) or \( \left\{ -g_1' \tilde{X}_j^- + g_2 < -\tilde{U}_j^- \leq 0 \right\} = 1 \), for \( 1 \leq j \leq m^-_{10}(k) \); or when \( \left\{ 0 < U_j^+ \leq -h_{\ast n}(g_1) X_j^+ + g_2 \right\} = 1 \) or \( \left\{ 0 < \tilde{U}_j^+ \leq -g_1' \tilde{X}_j^+ + g_2 \right\} = 1 \), for \( 1 \leq j \leq m^+_{10}(k) \).

Our next step will be verify that \( V_{j^-} = \tilde{V}_j^- (k) \) for all \( 0 \leq j \leq m^-_{10}(k) \) and \( V_{j^+} = \tilde{V}_j^+ (k) \) for all \( 0 \leq j \leq m^+_{10}(k) \). This will allow us to link the level sets between the processes \( Q_{02}^+ \) and \( \tilde{Q}_{2n} \).

Toward this end, the maximum number of possible level sets in \( Q_{02}^+ \) is at most 2 to the power of \( m^-_{10}(k) + m^+_{10}(k) \), since the number of unique indicator functions is \( m^-_{10}(k) + m^+_{10}(k) \) and each one can be either 0 or 1 (at most). However, not all possibilities may be feasible for \( g \) ranging over \( H_{20}(k) \), so the actual number will generally be lower. For each of the realized level sets, the set of \( g \in H_{20}(k) \) must include an open ball in \( \mathbb{R}^{p-1} \) almost surely since the \( \tilde{U}_j^- \) and \( \tilde{U}_j^+ \) are continuous random variables. Also, there will almost surely be no ties between level sets defined in terms of the indicator functions since all possible values of \( \tilde{E}_j^- \) and \( \tilde{E}_j^+ \) and their finite sums will be unique due to Condition C5 and the assumed continuity of the residual \( \epsilon \). Because of the previously established uniform convergence of the components in \( P_n(k) \) to those in \( P_0(k) \) combined with the compactness of \( H_{20}(k) \), we obtain that all of the boundaries between the level sets in \( \tilde{Q}_{2n} \) and \( \tilde{Q}_{2n} \) will converge uniformly and the magnitudes of the respective sums will converge. Recall that \( \tilde{g}(k) \) is in the interior of \( \tilde{\Phi}_0(k) \cap H_{20}(k) \), the level set of the arg max of \( Q_{02}^+ \). Thus, for all \( n \) large enough \( \tilde{g}(k) \) will be contained in the interior of \( \tilde{\Phi}_n(k) = \{ g \in H_{2n}(k) : h_{\ast n}(g) \in \Phi_n^+(k) \} \), the level set of the arg max of \( \tilde{Q}_{2n} \). Hence \( V_{j^-} = \tilde{V}_j^- (k) \) for all \( 1 \leq j \leq m^-_{0}(k) \) and \( V_{j^+} = \tilde{V}_j^+ (k) \) for all \( 1 \leq j \leq m^+_0(k) \) almost surely, for all \( n \) large enough.
Let $\tilde{C}_L^k(g_1), \tilde{C}_U^k(g_1), \tilde{C}_R^k(g_1), \tilde{R}_0(k)$, and $\tilde{g}_1(k)$, for $(g_1, g_2) \in H_{20}(k)$, be as defined previously but applied to the fixed data in $P_0(k)$; The foregoing convergence results taken together now yield that for each $k < \infty$,

$$\lim_{n \to \infty} \sup_{g_1 \in \mathbb{R}^{n-1}} \| g_1 - \tilde{C}_L^k(g_1) \| = 0,$$

$$\lim_{n \to \infty} \sup_{g_1 \in \mathbb{R}^{n-1}} \| g_1 - \tilde{C}_U^k(g_1) \| = 0,$$

$$\lim_{n \to \infty} \sup_{g_1 \in \mathbb{R}^{n-1}} \| g_1 - \tilde{C}_R^k(g_1) \| = 0.$$

Hence we also have for every closed subset $L$ of the open (non-empty) set $\tilde{R}_0(k)$, $\lim_{n \to \infty} \sup_{g_1 \in L} \inf_{g_1' \in \tilde{R}_n(k)} \| g_1 - g_1' \| = 0$, and for every $\eta > 0$,

$$\lim_{n \to \infty} \inf_{g_1 \in [\tilde{R}_n(k)]^c} \inf_{g_1' \in \tilde{R}_n(k)} \| g_1 - g_1' \| \geq \eta,$$

where superscript $\eta$ on a set denotes the $\eta$-open enlargement of that set and a superscript $c$ denotes complement. Combining this with previous arguments, we now have that $\tilde{h}_{3n}(k) \to 0$ and $\tilde{h}_{4n}(k) \to 0$, as $n \to \infty$, and the proof is complete.

**Proof of Theorem 6.8.** We can utilize essentially all of the results of the proof of Theorem 6.6 above since Condition C3' is stronger than Condition C3. We will assume that we are in the new probability space defined in that theorem and that $F_0(k) = 1$ and $n$ is large enough (as needed). Lemma 6.7 tells us that $h_3 \mapsto C_{R}^{mk}(h_3)$ has a unique maximum over $h_3 \in R_n(k)$ and hence also $g_1 \mapsto \tilde{C}_{R}^{mk}(g_1)$ has a unique maximum over $g_1 \in \tilde{R}_n(k)$ for all $n$ large enough. This same lemma also reveals that $g_1 \mapsto \tilde{C}_{R}^{k}(g_1)$ has a unique maximum over $g_1 \in \tilde{R}_0(k)$. Since we have previously established uniform convergence of $\tilde{C}_{R}^{mk}$ to $\tilde{C}_{R}^{k}$, we now have—after combining with and recycling previous arguments—that $\tilde{h}_{3n}(k) \to 0$ as $n \to \infty$; and, consequently, also $\tilde{h}_{4n}(k) \to 0$ as $n \to \infty$, and the desired weak convergence follows.

**Proof of Theorem 6.1.** Note that

$$Q_{02}(g) = \int_0^\infty \left[ B(t) \mathbf{1}\{ g_1' \tilde{X}(t) - g_2 > t \} \tilde{E}^-(t) + (1 - B(t)) \mathbf{1}\{ -g_1' \tilde{X}(t) + g_2 \geq t \} \tilde{E}^+(t) \right] dN(t) = \tilde{Q}_{02}(g),$$

where $t \mapsto N(t)$ is a homogeneous Poisson process on $[0, \infty)$ with intensity $2f_0$, $B(t)$ is a white-noise type Bernoulli random variable with success probability $1/2$, and where $\tilde{X}(t)$, $\tilde{E}^-(t)$ and $\tilde{E}^+(t)$ are also white-noise type stochastic processes which we will define shortly. By “white-noise type” we mean that there is a new independent draw for every distinct value of $t$. In the given process, these random draws only need occur at jump times in $N$. Define $\tilde{E}^-(t)$ and $\tilde{E}^+(t)$ as follows:

$$\tilde{E}^-(t) = \left( (\beta_0 - \delta_0)' \tilde{Z}(t) \right)^2 + 2\epsilon(t)(\beta_0 - \delta_0)' \tilde{Z}(t),$$

$$\tilde{E}^+(t) = \left( (\beta_0 - \delta_0)' \tilde{Z}(t) \right)^2 - 2\epsilon(t)(\beta_0 - \delta_0)' \tilde{Z}(t),$$

where $\epsilon(t)$ is a white-noise type process with distribution the same as the residual in model (1) and $(\tilde{Z}(t), \tilde{X}(t))$ is joint white-noise type process with joint distribution equal to $G$. What we
are doing is combining the Poisson process associated with the $\tilde{U}_j^-$ values with the Poisson process associated with the $\tilde{U}_j^+$ values. The combined process will have intensity equal to the sum of the two constituent intensities which, in this case, is $2f_0 = f_0 + f_0$. Because of the independence of these two processes and the equality of their intensities, the identity of the two constituent processes can be recaptured through the Bernoulli white-noise type process $B(t)$.

Part 1 of the theorem will following if we can establish that

$$
\lim_{k \to \infty} P \left\{ \inf_{g \in \mathbb{R}^{p-1} \times \mathbb{R}} \| g_1 \| \| g_2 \| > k \tilde{Q}_{02}(g) \leq 0 \right\} = 0,
$$

since we already know that $\tilde{Q}_{02}(0) = 0$, and thus an element $\tilde{g}$ of the arg min of $g \mapsto \tilde{Q}_{02}(g)$ can have $\| g_1 \| \| g_2 \| > k$ when the event in the given probability statement happens. Once we establish Part 1, Part 3 will following since for all $\tilde{U}_j^- > T_0(k_1 + 1), \mathbf{1}\{\tilde{g}_j' \tilde{X}_j^- - \tilde{g}_2 - \tilde{U}_j^- \leq 0\} = 1$, and also $\mathbf{1}\{\tilde{g}_j' \tilde{X}_j^+ - \tilde{g}_2 + \tilde{U}_j^+ > 0\} = 1$ for all $\tilde{U}_j^+ > T_0(k_1 + 1)$. Part 2 will now follow from the positivity and continuity of the $\tilde{E}_j^-$ and $\tilde{E}_j^+$ random variables, using arguments similar to those used in the proof of Theorem 6.6. Thus the proof will be complete once we establish (15).

To this end, for each $0 < k < \infty$ and each integer $j \geq 1$, define $A_j(k) = \{g \in \mathbb{R}^{p-1} \times \mathbb{R} : k^j < \| g_1 \| \| g_2 \| < k(j + 1)^2\}$, and note that

$$
P \left\{ \inf_{g \in \mathbb{R}^{p-1} \times \mathbb{R}} \| g_1 \| \| g_2 \| > k \tilde{Q}_{02}(g) \leq 0 \right\} \leq \sum_{j=1}^{\infty} P \left\{ \inf_{g \in A_{j}(k)} \tilde{Q}_{02}(g) \leq 0 \right\}.
$$

However, since $\| g_1 \| \| g_2 \| < k(j + 1)^2$ for all $g \in A_j$, the restriction of $g \mapsto \tilde{Q}_{02}(g)$ to $g \in A_j$ involves integrating with respect to $dN(t)$ only up to $t = t_j(k) \equiv (k_1 + 1)k(j + 1)^2$, since the involved indicator functions will all be zero for $t > t_j(k)$. Let $N_j(k) = \int_0^{t_j(k)} dN(t)$, and note that this is, by definition, a Poisson random variable with intensity $2f_0 t_j(k)$. Now, conditional on $N_j(k) = m$, and applying the restriction $g \in A_{j}(k)$, we have that

$$
\tilde{Q}_{02}(g) = \sum_{i=1}^{m} \left[ B_i \mathbf{1}\{g_1' \tilde{X}_i^* - g_2 > U_i^*\} \tilde{E}_i^- + (1 - B_i) \mathbf{1}\{-g_1' \tilde{X}_i^* + g_2 \geq U_i^*\} \tilde{E}_i^+ \right],
$$

where $(B_i, \tilde{E}_i^-, \tilde{E}_i^+)$ are i.i.d. realizations of $(B(t), \tilde{E}^-(t), \tilde{E}^+(t))$ and $(\tilde{Z}_i, \tilde{X}_i^*, \tilde{e}_i)$ are i.i.d. realizations of $(\tilde{Z}(t), \tilde{X}(t), \tilde{e}(t))$. Also, the $U_i^*$ values are independent of the other random variables and have a uniform distribution on the interval $[0, t_j(k)]$.

Define, for $i \geq 1$, the random quantity $\tilde{Y}_i = (B_i, \tilde{Z}_i, \tilde{X}_i^*, \tilde{e}_i, U_i^*)$, where the generic version of these random variables are denoted by omitting the $i$ subscript, and also define the class of functions

$$
\mathcal{F}_j(k) = \left\{ B \mathbf{1}\{g_1' \tilde{X}^* - g_2 > U^*\} e^-(\tilde{Z}, \tilde{e}) + (1 - B) \mathbf{1}\{-g_1' \tilde{X}^* + g_2 \geq U^*\} e^+(\tilde{Z}, \tilde{e}) : g \in A_{j}(k) \right\},
$$

where

$$
e^-(\tilde{Z}, \tilde{e}) = \left[ (\beta_0 - \delta_0)' \tilde{Z} \right]^2 + 2 \tilde{e}(\beta_0 - \delta_0)' \tilde{Z} \quad \text{and}
$$

$$
e^+(\tilde{Z}, \tilde{e}) = \left[ (\beta_0 - \delta_0)' \tilde{Z} \right]^2 - 2 \tilde{e}(\beta_0 - \delta_0)' \tilde{Z}.$$
Let $G_m = m^{1/2}(P_m - P)$ be the empirical process associated with a sample of these random variables of size $m$, where $P$ depends on the context, and where a sum of zero elements is zero and thus $G_0 = 0$ almost surely. Specifically, when we apply the empirical process to the class $F_j(k)$, the distribution is the appropriate one defined by the above random variables with the $U_i^*$ being uniform on $[0, t_j(k)]$. We now have the following lemma:

**Lemma 6.10.** There exist constants $0 < c_{s1}, c_{s2} < \infty$ such that for every $j, m \geq 1$ and $0 < k < \infty$, $\|G_m\|_{F_j(k)}^{*} P, 2 \leq c_{s1}$ and $\inf_{f \in F_j(k)} P f \geq c_{s2}$, where superscript $*$ here denotes outer expectation, subscript $P, 2$ indicates the $L_2(P)$ norm, and where $P$ depends on $j, k$ as specified above.

**Proof.** Define $F_j^1(k) = \{B 1\{g'_1 \tilde{X}^* - g_2 > U^*\} e^-(\tilde{Z}, \tilde{c}) : g \in A_j(k)\}$ and $F_j^2(k) = \{(1 - B) 1\{-g'_1 \tilde{X}^* + g_2 \geq U^*\} e^+(\tilde{Z}, \tilde{c}) : g \in A_j(k)\}$, and note that $\|G_m\|_{F_j(k)}^{*} P, 2 \leq \|G_m\|_{F_j^1(k)} P, 2 + \|G_m\|_{F_j^2(k)}^{*} P, 2$. Note that $\{g'_1 \tilde{X}^* - g_2 - U^* : g \in A_j(k)\}$ is a vector space of dimension $p$, and thus, by Lemma 9.6 of [9], is a VC-subgraph class of functions with VC-index $\leq p + 2$. Hence, by Lemma 9.9 Part (iii) of [9], the class $\{1\{g'_1 \tilde{X}^* - g_2 - U^* > 0\} : g \in A_j(k)\}$ is also VC-subgraph with VC-index $\leq p + 2$. Now applying Part (vi) of the same lemma, we obtain that $F_j^1(k)$ is thus VC-subgraph with VC-index $\leq 2p + 3$. We can also readily verify that

$$F_j^1(k) \equiv B \left\{ (\beta_0 - \delta_0)^{\tilde{Z}} \left[ 2(|\beta_0 - \delta_0| + |\tilde{c}|) \right] \right\} \leq B(b_1 + b_2|\tilde{c}|),$$

for universal constants $0 < b_1, b_2 < \infty$, is a measurable envelope for $F_j^1(k)$, by Condition C4. By universal, we mean that the constants depend only on $p, k_2, \beta_0$, and $\delta_0$, and not on $j$ or $k$. We can also similarly verify that $F_j^2(k)$ is VC-subgraph with VC-index $\leq 2p + 3$ and has measurable envelope $F_j^2(k) \equiv (1 - B)(b_1 + b_2|\tilde{c}|)$.

Now Theorem 9.3 of [9] yields that for $l = 1, 2$,

$$\sup_Q \int_0^1 \sqrt{1 + \log N \left( s \|F_j^l(k)\|_{Q, 2}, F_j^l(k), L_2(Q) \right)} ds \leq b_3,$$

where the supremum is taken over all finitely discrete probability measures $Q$ for which $\|F_j^l(k)\|_{Q, 2} > 0$, and where $0 < b_3 < \infty$ is also universal in that in only depends on $p$ and not on $j$ or $k$. We can now apply Theorem 11.1 of [9] to obtain that

$$\|G_m\|_{F_j^l(k)}^{*} P, 2 \leq b_3 b_4 \|F_j^l(k)\|_{P, 2} \leq b_3 b_4 \frac{\sqrt{(b_1^2 + 2b_1b_2\sigma + b_2^2\sigma^2)/2}}{2k(j_1 + 1)(j + 1)^2} \equiv b_5,$$

for $l = 1, 2$, where $b_4$ is another universal constant depending only on $p$. Thus the first assertion of the lemma follows with $c_{s1} = 2b_5$.

To prove the second assertion, fix $j \geq 1$ and $0 < k < \infty$, and let $g \in A_j(k)$. Now we have

$$E \left[ B 1\{g'_1 \tilde{X}^* - g_2 > U^*\} e^-(\tilde{Z}, \tilde{c}) + (1 - B) 1\{-g'_1 \tilde{X}^* + g_2 \geq U^*\} e^+(\tilde{Z}, \tilde{c}) \right]$$

$$= E \left[ \frac{g'_1 \tilde{X}^* - g_2}{2k(j_1 + 1)(j + 1)^2} + \frac{(-g'_1 \tilde{X}^* + g_2)}{2k(j_1 + 1)(j + 1)^2} \right].$$
where the first equality follows from the fact the \( U^* \) is uniform over \([0, t_j(k)]\); and the first inequality, with a new universal constant \(0 < b_6 < \infty\) not depending on \( j \) or \( k \), follows from Lemma 5.5 combined with the assumption from Condition C5 that \( P\{\beta_0 - \delta_0'Z = 0\} = 0\). The second to last inequality follows from the fact that \( \|g_1\| \geq |g_2| > kj^2 \) for all \( g \in A_j(k) \). The last inequality follows from the fact that \( j/(j+1) \geq 1/2 \) for all \( j \geq 1 \). Since \( g \), \( j \) and \( k \) were all arbitrary, we have now established the second assertion with \( c_{s2} = b_6[8(k+1) \] and the proof is complete. □

Continuing with the proof of Theorem 6.1, fix \( 0 < k < \infty \), and note that for each \( j \geq 1 \),

\[
P\left\{ \inf_{g \in A_j(k)} \hat{Q}_{02}(g) \leq 0 \right\}
\]

\[
\leq P\left\{ \inf_{g \in A_j(k)} \hat{Q}_{02}(g) \leq 0, \; N_j(k) \geq f_0t_j(k) \right\} + P\{N_j(k) < f_0t_j(k)\}
\]

\[
\leq P\left\{ \inf_{f \in F_j(k)} \left( N_j(k)^{1/2} \mathbb{G}_N_j(k)f + N_j(k)Pf \right) \leq 0, \; N_j(k) \geq f_0t_j(k) \right\}
\]

\[
+ P\{-N_j(k) - 2f_0t_j(k) > f_0t_j(k)\}
\]

\[
\leq E\left\{ P\left\{ N_j(k)^{1/2} \mathbb{G}_N_j(k) \geq N_j(k) \inf_{f \in F_j(k)} Pf \right\} \left( N_j(k) \geq f_0t_j(k) \right) \right\} + \frac{2f_0t_j(k)}{f_0^2t_j^2(k)}
\]

\[
\leq E\left\{ \frac{N_j(k)c_{s1}}{N_j^2(k) \inf_{f \in F_j(k)} Pf^2} \left( N_j(k) \geq f_0t_j(k) \right) \right\} + \frac{2}{kf_0(k+1)(j+1)^2}
\]

\[
\leq \frac{c_{s1}}{kf_2f_0(k+1)(j+1)^2} + \frac{2}{kf_0(k+1)(j+1)^2}
\]

\[
= \frac{c_{s1} + 2c_{s2}}{kf_2f_0(k+1)(j+1)^2},
\]

where the Markov inequality and Lemma 6.10 were utilized in the third and the fourth inequalities, respectively. This now gives as that

\[
limit_{k \to \infty} \sum_{j=1}^{\infty} P\left\{ \inf_{g \in A_j(k)} \hat{Q}_{02}(g) \leq 0 \right\} = 0,
\]

since \( \sum_{j=1}^{\infty} (j+1)^{-2} < \infty \), and thus (15) follows, completing the proof. □

7. Inference. In this section, we develop a parametric bootstrap type approach to inference. Define \( \hat{U}_i = \hat{\omega}'X_i - \hat{\gamma}_n \) and \( \hat{e}_i = Y_i - 1\{\hat{U}_i \leq 0\} \hat{\beta}_n'Z_i - 1\{\hat{U}_i > 0\} \hat{\delta}_n'Z_i \), for
\[ i = 1, \ldots, n; \text{ and let } \hat{U}_n = n^{-1} \sum_{i=1}^{n} \hat{U}_i, \hat{\epsilon}_n = n^{-1} \sum_{i=1}^{n} \hat{\epsilon}_i, \hat{\epsilon}_n^2 = n^{-1} \sum_{i=1}^{n} (\hat{U}_i - \hat{U}_n)^2, \]
\[ \text{and } \hat{\sigma}_n^2 = n^{-1} \sum_{i=1}^{n} (\hat{\epsilon}_i - \hat{\epsilon}_n)^2. \text{ Define also } \hat{\Sigma}_{1n} = \hat{\sigma}_n^2 \left[ n^{-1} \sum_{i=1}^{n} 1\{\hat{U}_i \leq 0\} Z_i Z_i^\prime \right]^{-1}, \hat{\Sigma}_{2n} = \hat{\sigma}_n^2 \left[ n^{-1} \sum_{i=1}^{n} 1\{\hat{U}_i > 0\} Z_i Z_i^\prime \right]^{-1}, \text{ and } \hat{M}_n \text{ which is an estimator of } \omega_0 \text{ using Gram-Schmidt orthogonalization (for example) wherein the columns of the } p \times (p - 1) \text{ matrix } \hat{M}_n \text{ are orthonormal to each other and orthogonal to } \hat{\omega}_n. \]

We next need to define several density estimators:
\[
\hat{f}_{n0} = \int_{\hat{\eta}_{n1}}^{\hat{\eta}_{n2}} \frac{1}{\eta_{n1}} \phi \left( \frac{t}{\eta_{n1}} \right) \, d\hat{F}_{n0}(t) \quad \text{and} \quad \hat{\xi}_{n}(u) = \int_{\hat{\eta}_{n1}}^{\hat{\eta}_{n2}} \frac{1}{\eta_{n2}} \phi \left( \frac{u - t}{\eta_{n2}} \right) \, d\hat{F}_{n2}(t),
\]

where \( \hat{\eta}_{n1} = 2\hat{\tau}_n n^{-1/5}, \hat{\eta}_{n2} = 2\hat{\tau}_n n^{-1/5} \), \( \phi \) is the standard normal density, \( \hat{F}_{n1}(t) = n^{-1} \sum_{i=1}^{n} 1\{\hat{U}_i \leq t\} \) and \( \hat{F}_{n2}(t) = n^{-1} \sum_{i=1}^{n} 1\{\hat{\epsilon}_i - \hat{\epsilon}_n \leq t\}. \) Also, let \( r_n \) be a sequence such that \( r_n n^{-1/2} \to \infty \) and \( r_n/n \to 0 \), and define \( \hat{t}_n = \sup \left\{ t > 0 : \sum_{i=1}^{n} 1\{|\hat{U}_i| \leq t\} \leq r_n \right\} \) and \( \hat{G}_n = \{(X_i, Z_i) : |\hat{U}_i| \leq \hat{t}_n\}. \)

We are ready to define our parametric bootstrap. Since \( W_1, W_2, \) and the process \( g \mapsto Q_{02}(g) \) are all independent, we can generate the components independently. Realizations of \( W_j \) can be be approximately generated by drawing from a mean zero Gaussian vector with covariance \( \hat{\Sigma}_{jn} \), for \( j = 1, 2 \). It is easy to verify that such a random variable, conditional on the observed sample, will converge weakly to \( W_j, j = 1, 2 \), provide we verify that \( \hat{\Sigma}_{jn} \) is consistent for the covariance of \( W_j, j = 1, 2 \) (which we will verify soon). The main difficulty is generating random realizations of \( g \mapsto Q_{02}(g) \) and applying the two midpoint estimators and verifying that they have the correct conditional limiting distribution. Accordingly, define the process
\[
\hat{Q}_{02}^\ast(g) = \int_{0}^{\infty} \left[ B(t) 1\{g_1^\prime \hat{X}_s(t) - g_2 > t\} \hat{E}_s^-(t) + (1 - B(t)) 1\{-g_1^\prime \hat{X}_s(t) + g_2 \geq t\} \hat{E}_s^+(t) \right] \, d\hat{N}(t),
\]

where \( t \mapsto \hat{N}(t) \) is a homogeneous Poisson process on \([0, \infty)\) with intensity \( 2\hat{f}_{n0} \), \( B(t) \) is a white-noise type Bernoulli random variable with success probability \( 1/2 \) (as previously defined), and where \( \hat{X}_s(t), \hat{E}_s^-(t) \) and \( \hat{E}_s^+(t) \) are also white-noise type stochastic processes similar to processes used in the proof of Theorem 6.1. In this setting, the relevant random draws only occur at jump times in \( \hat{N} \). We define \( \hat{E}_s^-(t) \) and \( \hat{E}_s^+(t) \) as follows:
\[
\hat{E}_s^-(t) = \left[ (\hat{\beta}_s - \hat{\delta}_s)^\prime \hat{Z}_s(t) \right]^2 + 2\hat{\epsilon}_s(t)(\hat{\beta}_s - \hat{\delta}_s)^\prime \hat{Z}_s(t), \quad \text{and}
\hat{E}_s^+(t) = \left[ (\hat{\beta}_s - \hat{\delta}_s)^\prime \hat{Z}_s(t) \right]^2 - 2\hat{\epsilon}_s(t)(\hat{\beta}_s - \hat{\delta}_s)^\prime \hat{Z}_s(t),
\]
for all \( g \in \mathbb{R}^{p-1} \times \mathbb{R} \), where the pair \( (\hat{X}_s(t), \hat{\dot{Z}}_s(t)) = (\hat{M}_n^t X, Z) \) for \( (X, Z) \) drawn independently and with replacement from \( \hat{G}_n \); and the draws for \( \hat{\epsilon}_s(t) \) are taken independently from the density \( \hat{\xi}_n \). Note that drawing from \( \hat{\xi}_n \) is equivalent to first randomly selecting uniformly an \( i \in \{1, \ldots, n\} \) and then adding a normal random variable with standard deviation \( \hat{\eta}_{n2} \) to \( \hat{\epsilon}_i - \hat{\epsilon}_n \). The reason we subtract off \( \hat{\epsilon}_n \) is to ensure that the generated random draws have mean zero, conditional on the data, for all \( n \) almost surely.

We can now define \( \hat{g}_\ast \in \arg \max_{g \in \mathbb{R}^{p-1} \times \mathbb{R}} \hat{Q}_{02}^\ast(g) \). Note that we could also express \( \hat{Q}_{02}^\ast \) in its dual form as an infinite sum, in precisely the same way as \( Q_{02} \) is the dual of \( Q_{02} \), as described in the proof of Theorem 6.1. We will denote this infinite sum dual form of \( \hat{Q}_{02}^\ast \) as \( \hat{Q}_{02}^\ast \) as \( Q_{02} \). We do not provide the details but simply mention this to verify that both the
mean-midpoint \( \arg \min \tilde{g}_s \) and the mode-midpoint \( \arg \min \hat{g}_s \), both based on \( \tilde{g}_s \), \( \tilde{Q}_0^* \) and its infinite-sum dual \( Q_0^* \), are well defined, using the same approach as was applied to \( Q_0^* \) and \( \tilde{Q}_0^* \) to define \( \tilde{g} \) and \( \hat{g} \). We now have the main result for this section which verifies that the forgoing parametric bootstrap algorithm provides an asymptotically valid approach for conducting inference on \( \theta_n \) and \( \theta_n \):

**Theorem 7.1.** Under assumptions C1, C2’, C3, and C4–C6, we have that \( \hat{\Sigma}_j \to \Sigma_j \) in probability, as \( n \to \infty \), for \( j = 1, 2 \); and that, conditional on the observed sample data, \( (\bar{M}_n \hat{g}_1, \hat{g}_2) \) converges weakly to \( (\bar{\omega}_0 \hat{g}_1, \hat{g}_2) \), with probability going to one as \( n \to \infty \), where \( (\hat{g}_1, \hat{g}_2) \equiv \hat{g} \). If, moreover, C3 is strengthened to C3’, and either \( \hat{\phi}_n \) or \( \tilde{\phi}_n \) is used to define \( \hat{U}_i \), then we also have that, conditional on the observed sample data, \( (\bar{M}_n \hat{g}_1, \hat{g}_2) \) converges weakly to \( (\bar{\omega}_0 \hat{g}_1, \hat{g}_2) \), with probability going to one as \( n \to \infty \).

**Proof.** From previous results, it is easy to verify that \( \max_{1 \leq i \leq n} |\hat{U}_i - U_i| \leq e_n \equiv k_1 |\hat{\omega}_n - \omega_0| + |\hat{\gamma}_n - \gamma_0| = O_P(n^{-1}) \). Since \( |U| \leq k_1 + \gamma_0 \) almost surely, we now have that \( \hat{\gamma}_2 \to \text{var}(U) \equiv \tau_0^2 \) in probability. Note also that for all \( t \in \mathbb{R} \),

\[
P_n \{U \leq t - e_n\} \leq \tilde{F}_{n1}(t) \leq P_n \{U \leq t + e_n\},
\]

and thus, by using standard empirical process results combined with Condition C2’, we obtain that for \( F_0(t) = \int_{-\infty}^t f(s)ds \),

\[
\|\tilde{F}_{n1} - F_0\|_R \leq O_P(n^{-1/2}) + \sup_{t \in \mathbb{R}} P\{t - e_n < U \leq t + e_n\} \leq O_P(n^{-1/2}).
\]

By lemma 7.2 below, the proof of which will be given later in this section, we now have that \( \tilde{f}_{n0} \to f_0 \) in probability, since \( \tilde{\eta}_n = \eta_{n1} \) readily satisfies the conditions of the lemma and since C2’ ensures that \( f \) is continuous at \( u = 0 \) as needed for Part 1 of the lemma. We note that the \( n^{1/5} \) power was selected because of its asymptotic optimality when the second derivative of the density being estimated is uniformly equicontinuous (see, e.g., [17]), but this stronger condition is not required since consistency is all we need and will still be achieved under the weaker assumptions we are using.

**Lemma 7.2.** Let \( \tilde{f}_0 \) be a density on \( \mathbb{R} \) that is uniformly bounded, and let \( \tilde{F}_0(t) \equiv \int_{-\infty}^t \tilde{f}_0(s)ds \). Let \( \tilde{F}_n \) be an estimator of \( \tilde{F}_0 \) such that \( \|\tilde{F}_n - \tilde{F}_0\|_R = O_P(n^{-1/2}) \). Also, for \( u \in \mathbb{R} \), define

\[u \mapsto \tilde{f}_n(u) = \int_{\mathbb{R}} \frac{1}{\tilde{\eta}_n} \phi \left( \frac{u - t}{\tilde{\eta}_n} \right) d\tilde{F}_n(t),\]

where \( \tilde{\eta}_n \) is a (possibly random) sequence satisfying \( \tilde{\eta}_n = o_P(1) \) and \( \tilde{\eta}_n^{-1} = o_P(n^{1/2}) \), and where \( \phi \) is the standard normal density. Then we have the following:

1. Provided \( \tilde{f}_0 \) is continuous at \( u \), then \( \tilde{f}_n(u) \to \tilde{f}_0(u) \) in probability.
2. Provided \( \lim_{n \downarrow 0} \sup_{s,t \in \mathbb{R}} |s - t| \leq \eta |\tilde{f}_0(s) - \tilde{f}_0(t)| = 0 \), then \( \|\tilde{f}_n - \tilde{f}_0\|_R = o_P(1) \).

Continuing with the proof of Theorem 7.1, we now verify consistency of \( \bar{M}_n \). As defined, \( \bar{M}_n \) satisfies \( \bar{M}_n \bar{M}_n' = I - \omega_n \omega_n' \equiv \bar{A}_n \) but is not in general unique since, for example, rearranging the columns of \( \bar{M}_n \) does not change \( \bar{A}_n \). This non-uniqueness will be addressed shortly. However, we first address consistency of the projection \( \bar{A}_n \) as an estimator of \( \bar{A}_0 \equiv I - \omega_0 \omega_0' \). Previous arguments yield that for any \( t \in \mathbb{R}^p \), \( t' \bar{A}_n - \bar{A}_0 t = O_P(n^{-1}) \). This implies that \( \bar{A}_n \) converges to \( \bar{A}_0 \) in probability as a projection. Let \( \bar{R}_p \) be the subset of \( \mathbb{R}^p \) equal to the range of the projection \( \bar{A}_n \). Note that \( \bar{g}_1 X_s(t) = g_1 \bar{M}_n X \), for \((X, Z)\) drawn
from $\tilde{G}_n$. Thus minimizing over $g = (g_1, g_2) \rightarrow Q^*_{\alpha_2}(g)$, and then multiplying the resulting minimizer by $\tilde{M}_n$, is the same as minimizing $Q^*_{\alpha_2}$ over $(h_3, g_2)$, after replacing $g_1' \tilde{X}_s(t)$ with $h_3' X$, for $h_3 \in \mathbb{R}_p$. The point of this is that the non-uniqueness of $\tilde{M}_n$ is not a problem provided $\tilde{M}_n M'_n = I - \tilde{\omega}_n \tilde{\omega}'_n$.

Note that, by Condition C4,

$$|\hat{e}_i - e_i| \leq k_2 (||\hat{\beta}_n|| + ||\tilde{\beta}_n - \beta_0|| + ||\hat{\delta}_n - \delta_0||) \equiv \hat{e}_n,$$

where $\hat{E}_i \equiv \{1 \{U_i \leq 0 < U_i\} + 1 \{U_i < 0 < \hat{U}_i\}, \ i = 1, \ldots, n$. Previous arguments verify that $n^{-1} \sum_{i=1}^n \hat{E}_i = O_P(n^{-1})$, and thus, after some derivation, $\hat{\sigma}^2_n = \sigma^2 + O_P(n^{-1/2}), \hat{e}_n = O_P(n^{-1/2})$, and also $\tilde{\Sigma}_{jn} = \Sigma_{j} + O_P(n^{-1/2})$ for $j = 1, 2$. Now, for all $t \in \mathbb{R}$,

$$\mathbb{P}_n \{\epsilon \leq t + \hat{e}_n - \hat{\epsilon}_n\} \leq \hat{F}_{n2}(t) \leq \mathbb{P}_n \{\epsilon \leq t + \hat{\epsilon}_n + \hat{e}_n\},$$

and thus, by using standard empirical process results combined with Condition C6, we obtain that for $F^*_0(t) = \int_{-\infty}^t \xi(s)ds$,

$$\|\hat{F}_{n2} - F^*_0\|_\mathbb{R} \leq O_P(n^{-1/2}) + \sup_{t \in \mathbb{R}} P\{t - \hat{\epsilon}_n < \epsilon \leq t + \hat{\epsilon}_n\}$$

$$+ \sup_{t \in \mathbb{R}} |P\{\epsilon \leq t + \hat{\epsilon}_n\} - F^*_0(t)|$$

$$= O_P(n^{-1/2}),$$

since, regarding the first supremum above, the supremum of a real function over $t \in \mathbb{R}$ is the same as the supremum over $t + \hat{\epsilon}_n \in \mathbb{R}$. Now Condition C6 combined with Part 2 of Lemma 7.2 yield that $\|\hat{\xi}_n - \xi\|_\mathbb{R} = o_P(1)$.

Define $t_n = \sup \{t > 0 : \sum_{i=1}^n 1 \{U_i \leq t\} \leq r_n\}$, and note that, by definition of $e_n$, $t_n - e_n \leq \hat{t}_n \leq t_n + e_n$. Since it can also be shown that $t_n = r_n/(2f_0n) + O_P(n^{-1/2})$, we have that $e_n/t_n = o_P(1)$ as a consequence of the fact that $r_n n^{-1/2} \rightarrow \infty$. If we let $t_{n0} = r_n/(2f_0n)$, we now have that both $\hat{t}_n/t_n = 1 + o_P(1)$ and $\hat{t}_n/t_{n0} = 1 + o_P(1)$. Hence, for $\tilde{J}_n = \{i \in \{1, \ldots, n\} : |\hat{U}_i| \leq \hat{t}_n\}$, we have for any $\eta > 0$ that $P\{\max_{i \in \tilde{J}_n} |U_i| \leq t_n(1 + \eta)\} \rightarrow 1$, as $n \rightarrow \infty$. Let $0 < m < \infty$ be an integer, and let $(X_j^*, Z_j^*)$, $j = 1, \ldots, m$ be an i.i.d. sample taken with replacement from $\tilde{G}_n$. Based on the above discussion, this is identical to drawing with replacement $m$ integers from $\tilde{J}_n$, which integers we denote as $i(1), \ldots, i(m)$, and then letting $(X_j^*, Z_j^*) = (X_{i(j)}, Z_{i(j)})$, for $j = 1, \ldots, m$. Since $r_n \rightarrow \infty$, as $n \rightarrow \infty$, we have that the probability that $i(j) = i(k)$, for any $1 \leq j < k \leq m$, is $m(m-1)/(2r_n) \rightarrow 0$, as $n \rightarrow \infty$. Thus, given $1 \leq m < \infty$, we have, with probability going to one, that there are no ties in the indices drawn from a sample of size $m$, as $n \rightarrow \infty$. We now have, by combining previous results, along with reapplication of Lemma 5.4, that for any $(s_1, u_1), \ldots, (s_m, u_m) \in \mathbb{R}_p \times \mathbb{R}_d$,

$$E\left[\exp \left\{\sum_{j=1}^m s_j X_j^* + u_j Z_j^*\right\} \right] \sim_n E\left[\prod_{j=1}^m \exp \left\{i(s_j X_j^* + u_j Z_j^*)\right\} \right] \sim_n \prod_{j=1}^m \chi(s_j, u_j),$$

in probability, as $n \rightarrow \infty$, where $i$ in the above expression is $\sqrt{-1}$. $\chi_n = \{(X_i, Z_i, Y_i), 1 \leq i \leq n\}$ is the observed data from a sample of size $n$, and $\chi$ is the characteristic function of $G$.

We now can rework the theory from the previous section to verify that the restrictions of the estimators $\hat{g}$, and $\tilde{g}_n$, as arg mins of $Q^*_{\alpha_2}(g)$ over the compact set $H^*_2(k)$, for $0 < k < \infty$, respectively.
converge in distribution, conditional on $X_n$, to $\hat{g}(k)$ and $\hat{g}(k)$, respectively, in probability as $n \to \infty$. For the convergence of $\hat{g}_s$ to $\hat{g}$, the asymptotic impossibility of ties in the indices established in the previous paragraph ensures that, under Conditions C3', there are asymptotically no ties in the values of $X_1, X^*, \ldots$ observed in $Q_{02}(g)$ for $g$ restricted to $H_{20}(k)$, for any fixed $0 < k < \infty$. We can also establish the needed compactness of $\hat{g}_s$, $\hat{g}_s$ and $\hat{g}_s$, using arguments parallel to those used in the proof of Theorem 6.1. By Theorem 6.4, we can, under C3', replace $\hat{\phi}_n$ with $\hat{\phi}_n$ in defining $\hat{U}_1$, without changing the conclusions. The desired conditional weak convergence for all components now follows via arguments similar to those used in the proofs of Theorems 6.3 and 6.4. □

**Proof of Lemma 7.2.** Using Riemann-Stieltjes integration by parts, we have for every $u \in \mathbb{R}$,

$$
\int_{\mathbb{R}} \frac{1}{\eta_n} \phi \left( \frac{u - t}{\eta_n} \right) d\hat{F}_n(t) - \int_{\mathbb{R}} \frac{1}{\eta_n} \phi \left( \frac{u - t}{\eta_n} \right) d\hat{F}_0(t) = - \int_{\mathbb{R}} \left[ \hat{F}_n(t) - \hat{F}_0(t) \right] \frac{(u - t)}{\eta_n^3} \phi \left( \frac{u - t}{\eta_n} \right) dt
$$

$\equiv E_1$, where

$$|E_1| \leq \frac{\|\hat{F}_n - \hat{F}_0\|_{\mathbb{R}}}{\eta_n} \equiv E_2,$$

and where $E_2 = O_P(\eta_n^{-1} n^{-1/2}) = o_P(1)$ does not depend on $u$. Moreover, for every $\Delta > 0$,

$$
\int_{\mathbb{R}} \frac{1}{\eta_n} \phi \left( \frac{u - t}{\eta_n} \right) d\hat{F}_0(t) = \int_{u - \Delta}^{u + \Delta} \frac{1}{\eta_n} \phi \left( \frac{u - t}{\eta_n} \right) \tilde{f}_0(u) dt - \int_{u - \Delta}^{u + \Delta} \frac{1}{\eta_n} \phi \left( \frac{u - t}{\eta_n} \right) \tilde{f}_0(u) dt + \int_{(-\infty,u-\Delta)\cup(u+\Delta,\infty)} \frac{1}{\eta_n} \phi \left( \frac{u - t}{\eta_n} \right) \tilde{f}_0(t) dt
$$

$= \tilde{f}_0(u) E_2(u) + E_3(u) + E_4(u),$

where $E_2(u) = [\Phi(\eta_n^{-1} \Delta) - \Phi(-\eta_n^{-1} \Delta)] \equiv E^*_n \to 1$, in probability, as $n \to \infty$. Note that the dependency of $E_2(u)$ on $u$ vanishes via a change of variables in the integral. Next, it is easy to verify that

$$|E_4(u)| \leq [1 - E^*_n] \times \|\tilde{f}_0\|_{\mathbb{R}} \to 0$$

in probability, as $n \to \infty$, since $\|\tilde{f}_0\|_{\mathbb{R}} < \infty$ by assumption. Finally, we have that

$$|E_3(u)| \leq E^*_n \times \sup_{u - \Delta \leq t \leq u + \Delta} |\tilde{f}_0(t) - \tilde{f}_0(u)| \leq \sup_{u - \Delta \leq t \leq u + \Delta} |\tilde{f}_0(t) - \tilde{f}_0(u)| \equiv E_5(u, \Delta),$$

where, since $\Delta$ is arbitrary, we can allow it to get arbitrarily small. For Part 1 of the proof, we have for a fixed $u$, that continuity of $\tilde{f}_0$ at $u$ yields $\lim_{\Delta \downarrow 0} E_5(u, \Delta) = 0$, and thus $\tilde{f}_n(u) \to \tilde{f}_0(u)$ in probability, as $n \to \infty$. Thus Part 1 follows. For Part 2, the uniform equiconvergence of $\tilde{f}_0$ yields that

$$\limsup_{\Delta \downarrow 0} E_5(u, \Delta) = 0,$$

and hence $\|\tilde{f}_n - \tilde{f}_0\|_{\mathbb{R}} \to 0$ in probability, as $n \to \infty$. Thus Part 2 also follows. □

8. Simulation study.
8.1. Numerical estimation procedure. The optimization of \( M_\theta(\theta) \) in (2) can be reduced to finding \((\omega, \gamma)\) that maximizes the total regression sum of squares without explicitly estimating the regression parameters and then finding the optimal \((\beta, \eta)\) given the estimated \((\omega, \gamma)\). Let \( D_{\omega, \gamma} \) denote a diagonal matrix with elements \( \{1 \{\omega^i X_i - \gamma \leq 0\}\}_{i=1}^n \). Let \( Z_+(\omega, \gamma) = D_{\omega, \gamma} Z_n \) and \( M_+(\omega, \gamma) = Z_+(\omega, \gamma)'(Z_+(\omega, \gamma)Z_+(\omega, \gamma))^{-1}Z_+(\omega, \gamma) \), where \( Z_n \) is the vector with elements \( \{X_i\}_{i=1}^n \) and superscript \( -1 \) denotes the Moore-Penrose matrix inverse. Similarly, define \( Z_-(\omega, \gamma) \) and \( M_-(\omega, \gamma) \) using \((I - D_{\omega, \gamma})\) in place of \( D_{\omega, \gamma} \). Let \( \text{SSR}_n(\omega, \gamma) = Y_n'M_+(\omega, \gamma) + M_-(\omega, \gamma))Y_n \), where \( Y_n \) is the vector with elements \( \{Y_i\}_{i=1}^n \). We want to find \((\omega, \gamma)\) that maximizes \( \text{SSR}_n(\omega, \gamma) \). Since, given an \( \omega \) value, there are \( n - 1 \) unique possible non-zero, non-identity \( D_{\omega, \gamma} \) values, only the mid points of the ordered \( \{\omega_j X_i\}_{i=1}^n \) values are considered for the candidates for \( \gamma \). Thus, we aim at finding \( \omega \) that maximizes \( \text{SSR}_n(\omega, \tilde{\gamma}(\omega)) \), where \( \tilde{\gamma}(\omega) = \arg\max_{\gamma} \frac{1}{2} \|\omega^i X_i - (\omega_j X_i + \gamma)\|_2^2 \).

Optimization of the \( \text{SSR}_n \) function can be done by searching over a sufficiently large set of uniform random values on a \( p - 1 \) dimensional unit hypersphere. This can be done by projecting \( p \)-dimensional independent Gaussian random variables onto the sphere. Once we find the optimal \( \omega \) value that maximizes the objective function, we repeat the random search with a narrower search span until no new best objective value has been achieved over the last \( N_0 \) iterations—e.g., \( N_0 = 20 \). The pair \((\tilde{\omega}, \tilde{\gamma})\) is, hence, obtained and \((\beta, \tilde{\eta})\) is immediately obtained through the least squares estimation for each resulting subgroup.

Although uniform random sampling on a subset of a hypersphere can still be done through importance sampling after the projection, it could be highly inefficient for a narrow subset. To circumvent this problem, uniform random values can be drawn by first making \( m_0 \) uniform angles within a range \( a_N[-\pi/2, \pi/2] \) for each dimension, where \( a_N \) characterizing the range becomes narrower from 1 as the iteration goes large—e.g., \( a_N = 0.8a_{N-1} \)—and by randomly drawing \( M \) out of the \( m_0^p \) values with probability proportional to their inverse density. These randomly drawn angles are then transformed to points in the \( S^{p-1} \) space that has the most updated \( \tilde{\omega} \) as the origin. Since \( \omega \) and \( -\omega \) give the same \( \text{SSR}_n \), the actual support of the random variables made of \( p \) angles, each element of which ranges \([-\pi/2, \pi/2]\), exhausts the whole \( p \)-dimensional hypersphere. The density of a random variable \( \omega \), of which Cartesian coordinate angles \((\varphi_1, ..., \varphi_p)\) follow a uniform density over \([-\pi/2, \pi/2]\), is given as 
\[
d(\omega) = \prod_{j=1}^{p-1} \cos^{p-j}(\varphi_j).
\]

Once \( \tilde{\theta} \) is obtained, the mean- and mode-midarginn estimates are obtained using the same uniform random draw technique, because the importance sampling can be highly inefficient. When the sample size \( n \) is large and the change-plane covariates are continuous, the size of the level set becomes tiny. Let \( \Omega_N = \{\omega_1, \omega_2, ..., \omega_{M_N}\} \) denote a set of \( M_N \) uniform random candidates for \( \omega \) in the level set such that \( \langle \omega, \tilde{\omega} \rangle \leq b_N \) for some sequence of decreasing constants \( b_N \leq 1 \) indexed by the number of iterations \( N \). Then obtain \( \{(C_L^n(\omega_j), C_R^n(\omega_j), R_n(\omega_j)) : j = 1, 2, ..., M_N\} \). This routine is repeated until \( \sum_{j=1}^{M_N} \{R_n(\omega_j) > 0\} \geq R_0 \) is satisfied for some large integer \( R_0 > 0 \). At the final iteration, the estimates are obtained as 
\[
\tilde{\omega} = \frac{\sum_{j=1}^{M_N} \omega_j \{R_n(\omega_j) > 0\}}{\sum_{j=1}^{M_N} \|\omega_j \{R_n(\omega_j) > 0\}\|_2}, \quad \tilde{\gamma} = \arg\max_{\omega_j \in \Omega_N} R_n(\omega_j), \quad \hat{\gamma} = 0.5C_L(\tilde{\omega}) + 0.5C_R(\tilde{\omega}), \quad \text{and} \quad \hat{\gamma} = 0.5C_L(\tilde{\omega}) + 0.5C_R(\tilde{\omega}).
\]

8.2. Simulation setup and results. We investigate the finite sample performance of the proposed estimation method in Monte Carlo simulations. We consider three models representing different dimensions of the change-plane covariates \( X \), and for each model, two sets of parameters are given resulting in six scenarios. See Table 1. In Scenario 1, the regression coefficients \((\beta, \eta)\) have opposite signs between the two subgroups, while, in Scenario 2, the associations share the same direction but have different strengths. The ratio of subgroup sizes
Table 1: Models and parameter specification for the simulations. Model 1 is a change-point model, Models 2 and 3 are change-plane models with discrete variables and only with continuous covariates, respectively. $U_p$ is the $p$-dimensional uniform random variables over the following basis support, Bern is a Bernoulli random variable, $\mathbf{1}_d$ is a vector of $d$ many 1's.

| Model | $X$          | $Z$          | $(\omega, \gamma)$ | Scenario 1 | Scenario 2 |
|-------|--------------|--------------|---------------------|------------|------------|
| M1    | $U_1(-2, 2)$ | (1, Bern(0.5)) | (1, 1)              | $\beta_2 - \beta_2$ | 1.52 0.52  |
| M2    | $(U_1(-3, 3), \text{Bern}(0.5))$ | (1, 1)       | $\frac{1}{\sqrt{2}}(1, -1, 1)$ | $\beta_2 - \beta_2$ | 1.52 0.52  |
| M3    | $U_3(-2, 2)$ | $(1, U_2(-2, 2))$ | $\frac{1}{\sqrt{3}}(1, -1, 1)$ | $\beta_3 - \beta_3$ | 1.53 0.53  |

In Figure 2, rate of convergence results for mean-midargmin estimators. Both standard errors (y-axis) and sample sizes (x-axis) are presented on a log scale. The lines and the annotated numbers are the least squares regression and the corresponding slope estimates representing the exponents of the rate of convergence. M1, Model 1; M2, Model 2; M3, Model 3.

is 2:1 for Models 1 and 3, and 3:1 for Model 2, respectively. Sample sizes of $n = 125$, 250, 500, 1000, and 2000 are used, and each scenario is repeated $n_{\text{rep}} = 3000$ times.

The first part of the simulations is to demonstrate the consistency and the rate of convergence of the numerical values of the point estimates. The second part of the simulations is to examine the weak convergence of $\sqrt{n}(\hat{\xi}_n - \xi_0)$ and $n(\hat{\phi} - \phi_0)$. A sample of size $n_{\text{rep}}$ is drawn from the limiting distribution, where the limiting random variables are numerically obtained using an approach analogous to what is described in Section 3. The empirical distributions were first compared marginally between the point estimates $(r_n(\hat{\theta} - \theta_0))$ and random sample from the limiting distribution. The joint weak convergence is studied using the Cramér-Wold device; the marginal comparison approach was repeated for two random linear combinations of the parameters. The final part of the simulations is to examine the empirical validity of the parametric bootstrap procedure. We investigate the coverage probability of the resulting confidence intervals based on the mean-midargmins.

Figure 2 illustrate that the estimation error of all estimates becoming exponentially smaller as sample size becomes larger. Also, the theoretical rate of convergence—$n^{-1}$ for $\hat{\phi}$ and $\hat{\phi}$ and $n^{-1/2}$ for $\hat{\xi}$ and $\hat{\tau}$—is well observed in the numerical results for both mean- and mode-midargmin estimators in most settings.

In Figure 3, the weak convergence simulation results are presented. Note that, for the change-point model (Model 1), mean- and mode-midargmins are identical, and, for the binary change-plane coavriate model (Model 2), $n\hat{\phi} \not\overset{d}{\to} (\tilde{\omega}_0, \tilde{g}_1, \tilde{g}_2)$, as expected, since Assumption $C3'$ is not satisfied. To see the joint convergence, per the Cramér-Wold device, the CDF
The estimated and limiting CDFs for simulation models 1, 2, 3 (mean-midargmin), and 3 (mode-midargmin). The random coefficients are \( \zeta_1 = (-0.47, -0.26, 0.15, 0.82, -0.60, 0.80)^T \), \( \zeta_2 = (0.89, 0.32, 0.26, -0.88, -0.59, -0.65)^T \) for Model 1, \( \zeta_1 = (-0.63, 0.40, 0.15, -0.66, 0.89, 0.89, -0.74)^T \), \( \zeta_2 = (0.67, -0.06, 0.10, 0.11, -0.52, 0.52, -0.64)^T \) for Model 2, and \( \zeta_1 = (-0.66, 0.62, -0.23, -0.34, 0.20, 0.21, -0.75, -0.41, 0.16, 0.26)^T \), \( \zeta_2 = (0.02, 0.01, 0.07, 0.11, 0.74, 0.66, -0.78, 0.41, 0.79, -0.44)^T \) for Model 3.

of linear combinations of the estimates with random coefficients were compared to the corresponding limiting distribution. The numerical results support the weak convergence theory.

We give the parametric bootstrap simulation results in Table 8.2. For most of the settings and parameters, the coverage probabilities are close to the nominal confidence level, 95%. The estimated confidence intervals for \( \phi \) are sometimes too conservative—e.g., Model 3. This is likely a consequence of the non-exhaustive search algorithm for higher-dimensional parameter spaces. The confidence intervals for the linear combinations (\( \zeta_1 \) and \( \zeta_2 \)) have good coverage probabilities, which implies that the parametric bootstrapping framework works reasonably well for both marginal and joint distributions.

| M | S | n  | \( \hat{\omega}_1 \) | \( \hat{\omega}_2 \) | \( \hat{\omega}_3 \) | \( \hat{\gamma} \) | \( \hat{\beta}_1 \) | \( \hat{\beta}_2 \) | \( \hat{\beta}_3 \) | \( \hat{\eta}_1 \) | \( \hat{\eta}_2 \) | \( \hat{\eta}_3 \) | \( \hat{\zeta}_1 \) | \( \hat{\zeta}_2 \) |
|---|---|----|-----------------|-----------------|-----------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|
| 1 | 1 | 125 | 0.97 | 0.93 | 0.94 | 0.95 | 0.92 | 0.90 | 0.91 | 0.95 | 0.94 | 0.93 | 0.94 |
| 1 | 1 | 500 | 0.95 | 0.93 | 0.95 | 0.93 | 0.94 | 0.93 | 0.94 | 0.95 | 0.94 | 0.93 | 0.94 |
| 1 | 1 | 2000 | 0.95 | 0.95 | 0.94 | 0.94 | 0.94 | 0.95 | 0.94 | 0.95 | 0.96 | 0.94 | 0.94 |
| 1 | 2 | 125 | 0.97 | 0.96 | 0.95 | 0.94 | 0.94 | 0.94 | 0.94 | 0.95 | 0.94 | 0.94 | 0.94 |
| 1 | 2 | 500 | 0.97 | 0.94 | 0.95 | 0.94 | 0.94 | 0.94 | 0.94 | 0.95 | 0.94 | 0.94 | 0.94 |
| 1 | 2 | 2000 | 0.96 | 0.91 | 0.91 | 0.95 | 0.95 | 0.93 | 0.95 | 0.95 | 0.95 | 0.95 | 0.95 |
| 2 | 1 | 125 | 0.97 | 0.99 | 0.96 | 0.93 | 0.93 | 0.92 | 0.91 | 0.93 | 0.93 | 0.93 | 0.93 |
| 2 | 1 | 500 | 0.98 | 0.98 | 0.96 | 0.93 | 0.96 | 0.93 | 0.92 | 0.94 | 0.92 | 0.94 | 0.92 |
| 2 | 1 | 2000 | 0.98 | 0.98 | 0.96 | 0.95 | 0.94 | 0.96 | 0.93 | 0.93 | 0.96 | 0.93 | 0.96 |
| 2 | 2 | 125 | 0.97 | 0.96 | 0.93 | 0.94 | 0.95 | 0.93 | 0.95 | 0.95 | 0.96 | 0.95 | 0.96 |
| 2 | 2 | 500 | 0.98 | 0.99 | 0.97 | 0.92 | 0.95 | 0.93 | 0.94 | 0.92 | 0.94 | 0.92 | 0.94 |
| 2 | 2 | 2000 | 0.96 | 0.96 | 0.96 | 0.94 | 0.97 | 0.94 | 0.95 | 0.94 | 0.95 | 0.94 | 0.95 |
| 3 | 1 | 125 | 1.00 | 1.00 | 1.00 | 0.93 | 0.95 | 0.93 | 0.94 | 0.93 | 0.91 | 0.93 | 0.91 | 0.99 | 0.96 |
| 3 | 1 | 500 | 1.00 | 1.00 | 1.00 | 0.93 | 0.95 | 0.93 | 0.94 | 0.93 | 0.94 | 0.94 | 0.94 | 1.00 | 0.93 |
| 3 | 1 | 2000 | 1.00 | 1.00 | 1.00 | 0.95 | 0.91 | 0.91 | 0.94 | 0.96 | 0.94 | 1.00 | 0.93 | 1.00 | 0.93 |
| 3 | 2 | 125 | 0.99 | 0.99 | 0.99 | 0.91 | 0.90 | 0.91 | 0.93 | 0.94 | 0.93 | 0.94 | 0.93 | 0.97 | 0.94 |
| 3 | 2 | 500 | 1.00 | 1.00 | 1.00 | 0.99 | 0.95 | 0.96 | 0.94 | 0.95 | 0.95 | 0.97 | 1.00 | 0.94 |
| 3 | 2 | 2000 | 1.00 | 1.00 | 1.00 | 0.96 | 0.95 | 0.93 | 0.94 | 0.93 | 1.00 | 0.95 |

**Table 2**
The coverage probability of the change-plane mean-midargmin estimators based on 300 replicates. M, Model; S, Scenario; \( \hat{\zeta}_1 \) and \( \hat{\zeta}_2 \) are the linear combinations of the estimates with the coefficients previously described.
and regression parameters were developed using the parametric bootstrap. More, with the limiting distributions being established, valid inferences on the change-plane of the change-plane parameter estimates was proved and illustrated by simulation. Further-

The mean-midargmin estimates and the 95% confidence intervals of model (16) of the CTG175 AIDS study.

9. Application to the ACTG175 AIDS study. We apply the change-plan regression model to the AIDS Clinical Trials Group 175 Study (ACTG175) data [4] to demonstrate the application of the proposed method to a precision medicine problem. The ACTG175 study is a double-blind controlled, randomized trial that compared the effect of four daily regimes in adults infected with human immunodeficiency virus type 1 (HIV-1) whose CD4 cell counts were from 200 to 500 per cubic millimeter. A lower CD4 cell count is indicative of a weakened immune system. These data have been used to study individualized treatment regimes in adults infected with human immunodeficiency virus type 1 (HIV-1) whose CD4 cell counts were from 200 to 500 per cubic millimeter. A lower CD4 cell count is indicative of a weakened immune system. These data have been used to study individualized treatment recommendations in a few recent studies, including [2]. In our analysis, we include 1,046 adults who have been randomized into either 600 mg of zidovudine + 400 mg of didanosine (T = 1) or 600 mg of zidovudine +2.25 mg of zalcitabine (T = 0) as done in [2]. We aim to separate these patients based on their heterogeneous treatment effects and find the best treatment recommendation that maximizes the expected CD4 cell count at 20 ± 5 weeks.

We model CD4 cell counts at 20 weeks (Y) as a function of treatment (T), age (A), and homosexual activity (H) for two subgroups divided by a hyperplane characterized by the last two covariates:

\[
Y = \mathbf{1}\{\omega'X - \gamma \leq 0\}(\beta_0 + \beta_T T + \beta_A A + \beta_H H) + \\
\mathbf{1}\{\omega'X - \gamma > 0\}(\eta_0 + \eta_T T + \eta_A A + \eta_H H) + \epsilon,
\]

where \(X = (A,H)\). In Table 3, the mean-midargmin estimates and the 95% parametric bootstrap confidence intervals are presented.

In the context of precision medicine, the average treatment effect given \(X = x\) is defined by \(\Delta(x) = E\{Y^*(1) - Y^*(0)|X = x\}\), where \(Y^*(t)\) is the potential outcome for treatment \(T = t\). The estimation of \(\Delta(x)\) requires the stable unit treatment value assumption (SUTVA), no unmeasured confounders assumption (that is, \(A \perp \{Y^*(0), Y^*(1)\}|X\)), and positivity, i.e., \(\inf_t \Pr(T = t|X = x) > 0\) [13, 5], all of which are believed to be satisfied for the ACTG 175 study. The average treatment effect implied by Model (16) is given as \(\Delta(x) = \beta_1 + (\eta_1 - \beta_1)\mathbf{1}\{\omega'x - \gamma > 0\}\), and its estimate is given as

\[
\hat{\Delta}((A,H)) = -6.50 + 1\{0.077A - 0.997H - 1.889 > 0\} \times 69.93.
\]

For 571 (54.6%) patients in the data who satisfy \(0.077A - 0.997H - 1.889 > 0\), \(T = 1\) (ZDV + ddI) is recommended over \(T = 0\) (ZDV + zal) to maximize the CD4 cell count at 20 weeks, with an estimated treatment effect \(\hat{\Delta} = 63.44\). For the other subgroup of 475 patients, \(T = 0\) is recommended, as \(T = 1\) compared to \(T = 0\) has a negative estimated treatment effect, \(\hat{\Delta} = -6.50\). The difference of treatment effects in two identified subgroups, \(\eta_T - \beta_T = 69.94\), is relatively large to a degree that the confidence interval, (37.7, 103.1), does not contain zero (\(p < 0.002\)).

10. Discussion. We studied the asymptotic behavior of change-plane estimators. Although there are quite a few theoretical results developed recently, to the best of our knowledge ours is the first to precisely characterize the limiting distribution of the prototypical change-plane estimator without a surrogate loss or smoothing. The \(n\)-rate of convergence of the change-plane parameter estimates was proved and illustrated by simulation. Furthermore, with the limiting distributions being established, valid inferences on the change-plane and regression parameters were developed using the parametric bootstrap.
There are a number of interesting directions for future work. Primary among them are extensions of the present results to multiple change-plane models [11], shared regression slope models, and kernel change-plane models. From a computational point of view, it would be of value to develop efficient algorithms for enumerating all hyperplanes which split a given data cloud into two non-empty sets. Sample size calculation and statistical tests for the existence of a change plane (in the spirit of Fan et al. [2]) is another important direction in which this work might be taken.

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