ON A HOMOTOPY 4-SPHERE

Dedicated to George Floyd

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Abstract. We give a brief survey of some facts about homotopy 4-spheres [A1], then give a proof that the curious homotopy sphere constructed in [A2] is in fact diffeomorphic to the standard $S^4$, and discuss its relation to infinite order loose corks and anti-corks.

0. General remarks

How to generate interesting examples smooth homotopy 4-spheres? and how to decide if they are diffeomorphic to $S^4$? If you don’t mind having 3-handles, an easy way to generate such examples is to start with a balanced presentation of the trivial group (this means that the number of generators and relators in the presentation are the same).

$$G = \{ x_1, x_2, \ldots, x_n \mid r_1(x_1, \ldots, x_n), \ldots, r_n(x_1, \ldots, x_n) \}$$

Then we attach 1-handles to $B^4$ for the generators, and 2-handles for the relators, giving this presentation, and then double the resulting 4 manifold. Then try to solve the resulting Andrews-Curtis problem, which is usually a difficult algebra problem. But turning this handlebody upside down can sometimes result a different handlebody presentation with easier Andrews-Curtis problem, as suggested in the example of [A4] (this approach has not been pursued further than this example).

Any pair of closed smooth simply connected 4-manifolds $M$ and $M'$, that might be exotic copies of each other, can be decomposed into two equal pieces $M = C \sim_{id} W$ and $M' = C \sim_{f} W$, where $W$ is contractible and $f : \partial W \to \partial W$ is some involution ([A3], [M], [CFHS]). Furthermore, in [AM] it was shown that these pieces can be assumed to be Stein manifolds. For this reason, we named the Stein piece $(W, f)$ a cork, and call such a piece without the Stein property a loose cork.

0.1. On constructing homotopy spheres, and detecting $S^4$. In particular any homotopy 4-sphere $\Sigma$ can be decomposed as a union of two Stein contractible manifolds $\Sigma = W_+ \sim_f -W_-$ glued along their common boundaries by some diffeomorphism: $f : \partial W_+ \to \partial W_-$. 

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This could be the starting point of studying homotopy spheres. Often the contractible pieces $W_\pm$ turn out to be ribbon contractible manifolds, that is they are obtained by blowing down $B^4$ along properly imbedded ribbon disks $D_\pm \subset B^4_\pm$ (e.g. 6.2 of [A1], as in Figure 1 or Figure 2).

By putting the pieces together we see that $\Sigma$ is just the Gluck twisted $S^4$ along the imbedded 2-sphere $S^2 = D_+ \sim_\partial D_- \subset B^4_+ \sim_\partial B^4_- = S^4$. Here the 2-sphere $S^2$ can be used to introduce a cancelling 2/3-handle pair, from which we may cancel the 1-handles of $\Sigma$. Afterwards by turning $\Sigma$ upside down we get a 3-handle free handle presentation of $\Sigma$. This is how the 3-handle free handlebodies of the homotopy spheres of [A5], [A7], [A2] were constructed. In the first two cases, by introducing further cancelling 2/3-handle pairs we are able to identify them with the standard $S^4$.

But the homotopy sphere $\Sigma$ of [A2] turned out to be difficult. Even after clearing all of its 3-handles, we weren’t able to decide if it was the standard $S^4$. One source of difficulty was that its gluing diffeomorphism $f$ is unusual, because it is not adapted to the handlebody of $W$ (i.e. it is not induced from an obvious symmetry of the handles of $W$), it is an internal diffeomorphism $R$ of its boundary 3-manifold $Y^3$, pulled back by a boundary identificaion $F : \partial W \rightarrow Y$ (Figure 2). Here the manifold $W$ is the Stein manifold $W(0,1)$ of [A1] and the involution of $\partial W$ is induced by pulling back 180 rotation of $Y$ by $F$ (i.e. conjugating the 180 degree rotation by $F$). This implies that, the Glucked 2-sphere $S^2 \subset S^4$ is obtained by identifying two identical ribbons $D_\pm \subset B^4_\pm$ along their boundaries with 180 degree rotation (Figure 3).

**Theorem 1.** The 3-handle free handlebody picture of $\Sigma$ which was described in [A8] as Figure 4 is in fact diffeomorphic to $S^4$.

**Proof.** First perform the handle slide of Figures 5 to go to Figure 6, then do the handle slide of Figure 6 to arrive to $S^4$. 

![Figure 1. A ribbon contractible manifold](image-url)
Figure 2. Pulling back involution by $F : \partial W \to Y^3$

Figure 3. $\Sigma$ is described as Gluck twisted $S^4$

Figure 4. Gluck twisting $\Rightarrow$ 3-handle free handlebody of $\Sigma$
Remark 1. Note that in [A10] an anti-cork was created by carving a cork $W$. But we now know that infinite order cork automorphisms can be induced from $\delta$-moves, which in term can be interpreted as being induced by carving $B^4$ by infinitely many different ribbons ([A7] [A9]). So, by reverse constructing infinitely many loose corks can be obtained by infinitely many different anti-corks (the anticorks are exotic copies of each other). For example, ribbon disk $D$ which $K\# - K$ bounds in $B^4$ (for some knot $K$) can be extended by concatenating $D$ with the “swallow-follow” isotopy of $K\# - K$ to itself, along the collar $S^3 \times [0, 1]$ of the boundary (Figure 7), as shown in Figure 8. This isotopy extends to an ambient isotopy $F_t : B^3 \times [0, 1] \rightarrow B^3 \times [0, 1]$, which is fixed on $B^3 \times 0 \sim S^2 \times [0, 1]$. Carving $B^4$ along $D_n$ gives the anti-corks in question.

Figure 5. Sliding handles to simplify $\Sigma$

Figure 6. More handle slides to identify $\Sigma = S^4$
Figure 7. Carving infinitely distinct anti-corks from $W$

Figure 8. Swallow-follow concordance on the boundary

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