Modifying partition functions: a way to solve the sign problem

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A possible method to solve the sign problem is developed by modifying the original theory. Considering several modifications of the partition function, the observable in the original theory is reconstructed from the identity connecting the observables in the original and modified theories. We demonstrate that our method gives the correct results even if the original theory has the severe sign problem by using a simple 1-dimensional integral.

PACS numbers: 11.15.Ha

I. INTRODUCTION

In the many fields of physics, the first-principle calculation plays an important role to analyze the nonperturbative properties of theories. However, when the action $S$ is complex, the first-principle calculation is difficult because of the oscillating behavior of the Boltzmann factor $e^{-S}$. This problem is referred to as the sign problem. In spite of many efforts, the sign problem has not been generally solved yet.

A possible solution for the sign problem is the complex Langevin method, which is based on the stochastic quantization \cite{1} with complex actions \cite{2-5} (For reviews, see e.g. \cite{6,7}). However, the complex Langevin method sometimes fails to reproduce the correct results \cite{8-11}. If the integrand of the partition function involves a complex fermion determinant, one possible cause of the failure is the singular drift problem. When configurations generated by the complex Langevin dynamics are close to a zero of the fermion determinant, the drift term becomes large. Such configurations form a power-law tail in the probability distribution of the drift term. If this is the case, the condition for the correctness of the complex Langevin method discussed in \cite{12,13} is not satisfied. Thus, the shape of the distribution of the drift term can be used for a diagnostic test of the complex Langevin method \cite{14,15}. As other reasons, the complex Langevin fails when the probability distribution of configurations has a tail of slow decay in the complex direction \cite{16}, or the ergodicity of the complex Langevin dynamics is not satisfied \cite{17}. In spite of these difficulties, the complex Langevin dynamics for Quantum Chromodynamics (QCD) at finite density is now investigated extensively \cite{17-21}. For a recent progress of this direction, see Ref. \cite{22} for instance.

The purpose of this paper is to develop a way to obtain the correct results even when the complex Langevin method fails. In our previous paper \cite{23}, we proposed a new idea to avoid the sign problem. In this approach, the expectation value of an observable in a given model in which one suffers from the sign problem is reconstructed by that in a modified model which is free from the sign problem through a simple identity. Here, the modified model is defined by adding an analytic function to the fermionic determinant of the original model such that the modified model has a desirable property from a viewpoint of a computational scheme to be applied. Thus, we refer to this approach as the modification method. However, in the previous work, we implicitly assumed that the reweighting factor involved in the identity can be always computed within appropriate precisions. Obviously, this assumption will not be satisfied when the sign problem is quite severe. Thus, the applicability of the modification method may not be different so much from the reweighting method \cite{24-27}.

In this paper, we improve the modification method and demonstrate that it is applicable without computing the reweighting factor. In Sec. II, we review the modification method proposed in the our previous paper \cite{23} and point out that the reweighting factor appears in the key identity. Then, we improve our method and the actual procedure is explained. In Sec. III, we apply our method to a simple model, the Gaussian model, and demonstrate that our method reproduces the correct results. We also discuss the advantages of our method, namely the difference from the reweighting method. Section IV presents our conclusions.

II. MODIFICATION METHOD

We focus on the class of models whose partition function has a following form:

\begin{equation}
Z_f = \int_{D} dx f(x)e^{-S_f(x)},
\end{equation}

where $f(x)$ is a complex-valued function defined on $x \in \mathbb{R}$, $S_f(x)$ is a real-valued action and $D$ is an integration domain on a real axis, $D \subset \mathbb{R}$. Throughout this section, we consider 0-dimensional field theory (1-dimensional integral) for simplicity.

Typically, this type of the partition function (1) appears by integrating out the fermionic sector from the action. For instance, this class of models includes the
Thirring model, the chiral random matrix theories, and QCD. In these cases, $f(x)$ corresponds to the fermion determinant.

By exponentiating $f(x)$ in Eq. (1), we get

$$Z_f = \int_D dx e^{-(S_q(x) - \log f(x))}.$$  \hfill (2)

Clearly seen from this expression, the effective action $S(x) \equiv S_q(x) - \log f(x)$ is complex unless $f(x)$ always takes real and positive values. When $S(x)$ is complex, it causes the sign problem.

The expectation value of an observable $O(x)$ is defined by

$$\langle O \rangle_f \equiv \frac{1}{Z_f} \int_D dx O(x)f(x)e^{-S_q(x)}.$$  \hfill (3)

For the later convenience, we introduce the special notation for the case of the trivial function $f(x) \equiv 1$:

$$Z \equiv Z_1 = \int_D dx e^{-S_q(x)}, \quad \langle O \rangle \equiv \langle O \rangle_1.$$  \hfill (4)

With these definitions, the observable $O(x)$ obeys

$$\langle f \rangle \langle O \rangle_f = \langle fO \rangle.$$  \hfill (5)

By using the identity, we find the following relation for two arbitrary complex-valued functions $f(x)$ and $g(x)$ and the observable $O(x)$:

$$\langle f \rangle \langle O \rangle_f + \langle g \rangle \langle O \rangle_g = \langle f + g \rangle \langle O \rangle_{f+g}.$$  \hfill (6)

If $\langle f \rangle \neq 0$, we obtain

$$\langle O \rangle_f = \langle O \rangle_{f+g} + \left( \langle O \rangle_{f+g} - \langle O \rangle_g \right) \frac{\langle g \rangle}{\langle f \rangle}.$$  \hfill (7)

This is what we have shown in the previous paper [23]. This identity is useful when the expectation value $\langle O \rangle_f$ is difficult to compute due to the sign problem. If one chooses an appropriate $g(x)$ such that the alternative model $Z_g$ and the modified model $Z_{f+g}$ are free from the sign problem, one obtains $\langle O \rangle_f$ through computing $\langle O \rangle_g$ and $\langle O \rangle_{f+g}$. We refer to this technique as the modification method. It is known that this method is applicable to the one site U(1)-link model [11]. A practical way to find an optimal $g(x)$ is also proposed in the previous work [23].

However, there is a caveat. It is non-trivial whether $\langle O \rangle_f$ can be computed within appropriate precisions due to the existence of the factor $\langle g \rangle / \langle f \rangle$. In fact, this factor is rewritten as

$$\frac{\langle g \rangle}{\langle f \rangle} = \int dx g(x)e^{-S_q(x)} \over \int dx f(x)e^{-S_q(x)}$$

$$= \int dx f(x)e^{-S_q(x)} e^{f(x) \langle f \rangle_g - g(x) \langle g \rangle_g} = \left( \frac{f}{g} \right)_g^{-1},$$  \hfill (9)

and the quantity $\langle f/g \rangle_g$ is nothing but the so-called reweighting factor [28]. If the absolute value of the reweighting factor is small, it indicates that the sign problem is severe. Apparently, one can compute $\langle O \rangle_f$ through Eq. (7) when $g(x)$ is chosen such that the factor $| \langle f/g \rangle_g |$ is sufficiently large. Therefore, it seems that our modification method has the same difficulty as the reweighting method [27].

Nevertheless, we shall point out that our method is different from the reweighting method and the reweighting factor is not necessary to calculate in the modification method. To see this, we rewrite Eq. (7) as

$$y = a_g x + b_g.$$  \hfill (10)

Here, we separate the Eq. (7) into the $g$-independent parts

$$y = \langle O \rangle_f, \quad x = \frac{1}{\langle f \rangle},$$  \hfill (11)

and $g$-dependent parts

$$a_g = \langle O \rangle_{f+g} - \langle O \rangle_g \langle g \rangle, \quad b_g = \langle O \rangle_{f+g}.$$  \hfill (12)

Suppose that the $g$-dependent quantities, namely $a_g$ and $b_g$, can be computed without the sign problem, and there are several candidates of such functions $g(x)$. In this case, $g$-independent quantities $y$ and $x$ are obtained as the intersection point of the set of straight lines $\{ y = a_g x + b_g \}$.

In the following section, we demonstrate the method discussed in this section, which is referred to as the multi-modification method in this paper.

III. APPLICATION TO THE GAUSSIAN MODEL

In this section, we apply the multi-modification method to the Gaussian model. To begin with, the sign problem in the model is discussed. We demonstrate the method gives the correct results even when the reweighting factor is small.
A. Sign problem in the Gaussian model

The Gaussian model is a 0-dimensional field theory, which has the partition function
\[ Z_f = \int_{-\infty}^{\infty} dx f(x) e^{-x^2/2}, \quad f(x) = (x + i\alpha)^2, \tag{14} \]
with a positive real parameter $\alpha > 0$. The effective action $S(x) = x^2/2 - \log f(x)$ is complex when $\alpha \neq 0$. It means that this model has the sign problem.

The analytic solutions of the observables in this model can be easily obtained. For example, the expectation values for $x^2$, $x^4$, $x^6$ are obtained as
\[
\langle x^2 \rangle_f = \frac{3 - \alpha^2}{1 - \alpha^2}, \tag{15} \\
\langle x^4 \rangle_f = \frac{15 - 3\alpha^2}{1 - \alpha^2}, \tag{16} \\
\langle x^6 \rangle_f = \frac{105 - 15\alpha^2}{1 - \alpha^2}. \tag{17}
\]

In this study, we adopt the complex Langevin method to numerically calculate these observables. The complex Langevin equation of this model is written as
\[ \frac{dz}{dt} = D(z) + \eta(t), \tag{18} \]
where $z$ is the complexified variable $x \rightarrow z \in \mathbb{C}$ and $t$ is the fictitious time of the complex Langevin dynamics. The first term of the right-hand side in Eq. (18) is the drift term, which is expressed as
\[ D(z) = -\frac{\partial S(z)}{\partial z} = -z + \frac{2}{z + i\alpha}, \tag{19} \]
and the second term is the Gaussian noise term, which satisfies
\[ \langle \eta(t) \rangle_{\eta} = 0, \quad \langle \eta(t_1) \eta(t_2) \rangle_{\eta} = 2\delta_{t_1,t_2}, \tag{20} \]
where $\langle \cdots \rangle_{\eta}$ denotes the noise average.

It was shown that the complex Langevin method in this model gives the correct results when the singular drift problem does not occur [14, 15]. The correctness of the complex Langevin method depends on the parameter $\alpha$. In Fig. 1, the numerical result of the complex Langevin method for the observable $O(z) = \text{Re}(z^2)$ is shown. When the parameter $\alpha$ is sufficiently large, say $\alpha \gtrsim 2.7$, the complex Langevin method reproduces the exact results because of the absence of the singular drift problem. The histogram of the absolute value of the drift term $\rho(|D(z)|)$ is obtained from the complex Langevin dynamics, and it is shown in Fig. 2 for $\alpha = 2.7$. Since the histogram exponentially damps as $|D|$ becomes large, it indicates that there is no singular drift problem. On the other hand, the complex Langevin method fails to give the correct results in the other region, in particular around $\alpha \simeq 1$. In Fig. 3, the distribution of the drift term with $\alpha = 1.5$ is shown. Unlike the case with $\alpha \gtrsim 2.7$, the distribution with $\alpha = 1.5$ does not exponentially drop and has the long tail. Therefore, when $\alpha = 1.5$, the complex Langevin method is invalid due to the singular drift problem.

In the numerical calculation, we use the Euler’s method to solve the complex Langevin equation (18) for the total Langevin time $10^7$ with the step size $dt = 10^{-2}$. We take configurations every 1.0 Langevin time after $10^2$ Langevin time. We note that these results are totally consistent with the previous study [14].

![Fig. 1. The observables $\langle \text{Re}(z^2) \rangle$ plotted against the parameter $\alpha$ in the Gaussian model. The solid line denotes the analytical solution and the points denotes the numerical results of the complex Langevin method.](image1)

![Fig. 2. The histogram for the absolute value of the drift term in the complex Langevin equation of the Gaussian model with $\alpha = 2.7$ in log scale.](image2)
In order to reproduce the exact results even in the small-α region, we shall apply the modification method. However, it is difficult to calculate observables within the sufficient precision if we apply the modification formula (7) directly. In the following, we explain that the small reweighting factor causes this problem.

As a modification function $g(x)$, we consider a function

$$g(x) = (x + i\beta)^2,$$  \hspace{1cm} (21)

with a positive real parameter $\beta$. This function has the same form as $f(x)$ given in Eq. (14), but a different parameter $\beta$. The partition function of the modified Gaussian model is defined by

$$Z_{f+g} = \int_{-\infty}^{\infty} dx \, (f(x) + g(x)) \, e^{-x^2/2}.$$  \hspace{1cm} (22)

In order to apply the modification method, we have to choose appropriate $\beta$ such that the following quantities can be correctly calculated:

1. The observables $\langle O \rangle_g$ and $\langle O \rangle_{f+g}$
2. The factor $\langle g \rangle / \langle f \rangle$

Concretely, the value of $\beta$ is constrained in the following way. The observable $\langle O \rangle_g$ should be correctly calculated by the complex Langevin method. Since the modification function $g(x)$ has the same form as the original function $f(x)$, we already know that $\langle O \rangle_g$ is correctly obtained when $\beta > 2.7$. In addition to the observable $\langle O \rangle_g$, the observable in the modified theory $\langle O \rangle_{f+g}$ should also be correctly calculated by the complex Langevin method. We perform the complex Langevin dynamics and investigate the distribution of the drift term in the modified Gaussian model for each $\alpha$ and $\beta$. For example, in Fig. 4, it is shown that the distribution of the absolute value of the drift term exponentially drops and the singular drift problem does not occur when $\alpha = 1.5$ and $\beta = 3.5$. By the similar analysis on the distribution of the drift term, we find that there is no singular drift problem in the modified Gaussian model for arbitrary $\alpha > 0$ if we choose $g(x)$ with $\beta \geq 3.5$. Thus, the first condition of $\beta$ is $\beta \geq 3.5$.

Next, following the second condition, the factor $\langle g \rangle / \langle f \rangle$ should be correctly calculated. However, this factor is problematic in the modification method. As discussed around Eq. (9) in the section II, this factor is rewritten to the inverse of the reweighting factor $\langle f/g \rangle^{-1}$. When the sign problem is severe, the absolute value of the reweighting factor $|\langle f/g \rangle|$ tends to be small, and it is difficult to calculate the observable $\langle O \rangle_f$ within sufficient precisions. Actually, as shown in Fig. 5, the absolute value of the reweighting factor $|\langle f/g \rangle|$ is smaller than 1 when $0 < \alpha < 3$ and $3.5 \leq \beta$. In particular, in the region around $\alpha \sim 1$, where the sign problem is severe, the reweighting factor is almost 0. This happens because of the oscillatory behavior of the quantity $\langle f \rangle$. Thus, there is no $\beta$ satisfying both first and second conditions above. Therefore, it is quite difficult to calculate the observable $\langle O \rangle_f$ by directly using the modification formula (7). This situation is similar to the ordinary reweighting technique [27].

**B. multi-modification method**

In this subsection, we show a trick to avoid the computation of the reweighting factor $\langle f/g \rangle$. We demonstrate that this trick, the multi-modification method, gives the correct results in the multi-modification method even when the sign problem is severe.
As discussed in Sec. II, we rewrite the modification formula (7) to the linear function as Eq. (10). We again use the modification function \( g(x) \) defined in Eq. (21). Each linear function (10) is determined once \( g(x) \) is fixed. Our approach is to obtain the quantities \( \langle f \rangle \) and \( \langle O \rangle_f \), or \( x \) and \( y \), as the intersection point of a set of lines \( \{ y = a_g x + b_g \} \) by calculating \( a_g \) and \( b_g \) for several \( \beta \).

Ideally, a set of lines \( \{ y = a_g x + b_g \} \) has an unique intersection point as shown in Fig. 6. However, the coefficients \( a_g \) and \( b_g \) have the statistical error because the actual calculation for the observables \( \langle O \rangle_{f+g} \), \( \langle O \rangle_g \) and \( \langle g \rangle \) is performed by the complex Langevin method. Therefore, in the actual calculations, one obtains a band instead of a line for each \( g(x) \). Then, the allowed value of the quantities \( \{ f \} \) and \( \langle O \rangle_f \), or \( x \) and \( y \), is obtained as the overlapping region of the bands.

To apply the multi-modification method, we have to choose appropriate function \( g(x) \), namely the parameter \( \beta \) here, so that the following quantities can be correctly calculated:

1. The observables \( \langle O \rangle_g \) and \( \langle O \rangle_{f+g} \)
2. The average \( \langle g \rangle \)

Here we remark that the second condition shown in the previous subsection is replaced. In principle, the average in the quenched theory \( \langle g \rangle \) is always calculable by the Monte Carlo method because \( S_q(x) \) is supposed to be real. However, this quantity becomes difficult to calculate if the modification function \( g(x) \) has violent oscillation on its phase. Thus \( g(x) \) should be chosen so that the sign of \( \text{Re}(g(x)) \) is not frequently changed in the importance sampling of the Monte Carlo simulation.

The actual condition for \( \beta \) is investigated as follows. From the first condition, \( \beta \) is constrained to \( 3.5 \leq \beta \) as we have already shown in the previous subsection. In this parameter region, the quantity \( \langle g \rangle \) can be correctly obtained by the Monte Carlo method. In fact, the sign of \( \text{Re}(g(x)) \) is almost always negative in the Monte Carlo calculation if \( 3.5 \leq \beta \). This result reflects the fact that the sign problem of the original Gaussian model is not severe when \( 3.5 \leq \alpha \). Therefore, if \( 3.5 \leq \beta \), the above two conditions are satisfied. In other words, the coefficients \( a_g \) and \( b_g \) in Eq. (10) can be correctly calculated when \( 3.5 \leq \beta \) by using the complex Langevin method and the Monte Carlo method.

In our analysis, six values of \( \beta \) are taken from 3.5 to 8.5. With those \( \beta \), the observables \( \langle O \rangle_{f+g} \) and \( \langle O \rangle_g \) are calculated by the complex Langevin method, and the quantity \( \langle g \rangle \) is calculated by the Monte Carlo method. In the complex Langevin method, we also use the Euler’s method to solve the complex Langevin equation for the total Langevin time \( 10^7 \) with the step size \( dt = 10^{-2} \). We take configurations every 1.0 Langevin time after \( 10^2 \) Langevin time. We consider \( \mathcal{O}(x) = x^2 \) as the observable, and then \( \text{Re}(z^2) \) is calculated by the multi-modification method.

In Fig. 7, we show the allowed regions of \( (x, y) \) for each \( \beta \) with \( \alpha = 1.5 \). The black square at \((-0.8, -0.6)\) denotes the analytic solution of \((1/(f), \langle \text{Re}(z^2) \rangle_f)\). The gray region is the overlap of all the allowed regions for each \( \beta \). One can see that the overlapping region is certainly covers the analytic solution. We have performed the similar analysis for the other values of \( \alpha \).

In Fig. 8, we show the numerical results of \( \langle \text{Re}(z^2) \rangle_f \) as a function of \( \alpha \) calculated by the multi-modification method. In addition to the parameter-region where the sign problem is not severe, the multi-modification method certainly reproduces the correct results even when the sign problem is severe and the original complex Langevin method fails to give the correct results.
FIG. 7. The region of the possible values of \((x, y)\) from Eq. (10) for each \(\beta\) and their overlap (gray colored) when \(O(z) = \text{Re}(z^2)\) and \(\alpha = 1.5\). The black circle at \((-0.8, -0.6)\) is the point of the analytic solution.

FIG. 8. The numerical results of \(O(z) = \text{Re}(z^2)\) with \(0 < \alpha < 3\) by the multi-modification method.

FIG. 9. The numerical results of \(O(z) = \text{Re}(z^4)\) with \(0 < \alpha < 3\) by the multi-modification method.

FIG. 10. The numerical results of \(O(z) = \text{Re}(z^6)\) with \(0 < \alpha < 3\) by the multi-modification method.

C. Discussion

In addition to \(O(z) = \text{Re}(z^2)\), we have also performed similar analysis on other observables with higher power of \(z\), \(O(z) = \text{Re}(z^4)\) and \(O(z) = \text{Re}(z^6)\). In Figs. 9 and 10, the numerical results are shown. Although the results are consistent with the analytical solutions within their errors, the errors are large compared with \(\text{Re}(z^2)\).

From Figs. 8, 9 and 10, one can see that the numerical error becomes larger around \(\alpha = 1\). This is because the coefficients \(a_g\) and \(b_g\) are insensitive to \(\beta\) when \(\alpha \sim 1\). As a result, the overlapping region becomes wider and it leads to the large statistical errors.

The multi-modification method has some advantages over the original modification method. In the multi-modification method, we do not have to directly calculate \(\langle f \rangle\), which is the average of the oscillatory function. It is difficult to calculate the quantity with enough precision when the sign problem is severe. However, in our method, both \(\langle f \rangle\) and \(\langle O \rangle_f\) can be simultaneously obtained without the direct calculation of \(f\) itself. In fact, the numerical results of the average \(\langle f \rangle\) obtained in our method well reproduces the correct results, as shown in Fig. 11.

Moreover, as another advantage of the multi-modification method, we can reduce the error of the observable \(\langle O \rangle_f\) systematically. As we consider more...
modification functions \( g(x) \), the allowed region in the \((x, y)\)-plane becomes narrower (see Fig. 7). This improves the precision of the numerical calculation based on our method.

![FIG. 11. The numerical results of \( \langle f \rangle \) for each \( \alpha \) by the multi-modification method.](image)

In our method, the observable in the original theory \( Z_f \), where the sign problem is severe is obtained from the observables in the other two theories \( Z_g \) and \( Z_{f+g} \), where the sign problem is not severe. This concept is similar to that of the reweighting method. In fact, as we have already shown, the reweighting factor \( (f/g) \), appears when we directly apply the modification formula (7). Therefore, the modification method seems to have the same difficulty as the reweighting method if the sign problem is severe. However, we shall emphasize that the applicability of the multi-modification method is different from the reweighting method. In particular, the multi-modification method has no overlap problem which appears in the reweighting method [29].

In this study, we adopt the complex Langevin method as a calculating tool. Instead of the complex Langevin method, there are some candidates proposed in Ref. [30–39], which are based on the Cauchy’s integral theorem. If one of them is adopted, the appropriate range of \( \beta \) will be different from the case of the complex Langevin method, namely \( 3.5 \leq \beta \). It is nontrivial which method is superior as the actual calculating tool. The reason why we adopt the complex Langevin method is that the correctness of the complex Langevin method is easy to judge by checking the distribution of the drift term, at least in this model.

We also comment on the choice of the modification function \( g(x) \). Though we restrict ourself so that \( g(x) \) has a same form as \( f(x) \), the functional form is arbitrary in general. One of the most interesting class of \( g(x) \) is a function such that it obeys \( \langle g \rangle = 0 \). If this is the case, the coefficient \( a_g \) defined in (12) is replaced by simpler quantity, \( -\langle g \mathcal{O} \rangle \). It may allow us to control the statistical error on \( a_g \) more easily.

Finally, we mention the applicability of the multi-modification method. Although we consider only the 1-dimensional integral in this paper, one can generalize our formulation to a higher dimensional theory in a straightforward way. In the case of \( N \)-dimensional integral, the domain of the integral \( D \) and the variable of the integral \( x \) in Eq. (1) are changed to \( x_i \in D \subset \mathbb{R}^N \) \((i = 1, \cdots, N)\) from \( x \in D \subset \mathbb{R} \). Nevertheless, the formulae (7)-(13) still hold exactly even for the higher dimensional integral. Thus, in general, the same procedure proposed in this paper can be performed in the higher dimensional theories such as the thirring model, the chiral random matrix theories and QCD at finite density, for instance. In these theories, \( f \) in Eq. (1) corresponds to the fermion determinant, and thus it depends on the fermion mass and the chemical potential, as \( f = f(m, \mu) \). While these theories have the sign problem, there are some parameters, say \((m', \mu')\) where the complex Langevin method gives the correct results. Then, it is natural to choose \( g = f(m', \mu') \) as a modification function. To investigate the singularity of the drift term of the modified model \( Z_{f+g} \) for these finite density systems is worth challenging. Another possible choice of \( g \) is a function such that it obeys \( \langle g \rangle = 0 \). As we mentioned, it will be useful to reduce the statistical error. In a lattice model with many degrees of freedom, this is crucial since \( \langle g \rangle \) is expected to be too noisy. To find such class of \( g(x) \) is work in progress.

IV. SUMMARY

In this paper, we have developed a way named multi-modification method to solve the sign problem by improving our previous method [23]. In our method, instead of calculating the observable in the original theory \( \langle \mathcal{O} \rangle_f \) directly, we calculate the observables \( \langle \mathcal{O} \rangle_g, \langle \mathcal{O} \rangle_{f+g} \) and the average \( \langle g \rangle \) in the quenched theory. If \( g(x) \) has appropriate properties, the observable in the original theory can be reconstructed as the intersection point of the linear functions defined in Eq. (10).

By applying our method to a toy model, the Gaussian model, we have demonstrated how our method works. We have chosen the modification function \( g(x) \) such that it has the same form as \( f(x; \alpha) \), but with a different parameter \( \beta \) as defined in Eq. (21). As a result, the correct results are reproduced in the whole parameter region even when the sign problem is severe and the complex Langevin method results in the wrong convergence.

Since it is not difficult, at least formally, to generalize our method to higher dimensional problems, we would like to apply our method to more complicated models such as the Thirring model, the random matrix theory, and finally QCD.
ACKNOWLEDGMENTS

T.M.D. is supported by the RIKEN Special Postdoctoral Researchers Program. The authors thank H. Fujii for fruitful discussions.

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