Quotients of Buildings as $W$-Groupoids

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Abstract

We introduce structures which model the quotients of buildings by type-preserving group actions. These structures, which we call $W$-groupoids, generalize Bruhat decompositions, chambers systems of type $M$, and Tits amalgams. We define the fundamental group of a $W$-groupoid, and characterize buildings as connected simply connected $W$-groupoids. We give a brief outline of the covering theory of $W$-groupoids, which produces buildings as the universal covers of $W$-groupoids. The local-to-global theorem of Tits concerning spherical 3-resides allows for the construction of $W$-groupoids by gluing together quotients of generalized polygons. In this way, $W$-groupoids can be used to construct exotic, hyperbolic, and wild buildings.

1. Introduction

Conceptually, a building is a set $\Delta$ equipped with a metric $\delta$ which takes its values in a Coxeter group $W = (W, S)$. The elements of $\Delta$ are called chambers, isometric embeddings $W \rightarrow \Delta$ are called apartments, and sequences of chambers such that consecutive chambers are at distance a generator of $W$ are called galleries. This modern ‘$W$-metric space’ approach to buildings is described in [AB08], together with the equivalent ‘simplicial’ approach taken in the early work of Tits (see [Tit74]). The introduction of $W$-metric spaces was motivated by the discovery of twin buildings (see [Tit92]). Buildings are also naturally CAT(0) cell complexes by a construction of Davis, known as the Davis realization (see [Dav94] and [AB08, Chapter 12]). If $W$ is irreducible affine, then the Davis realization is the simplicial complex of the building (up to duality).

By the quotient of a building, we mean a structure naturally associated to the type-preserving action of a group on a building in the manner of stacks in category theory (e.g. orbifolds, graphs of groups, algebraic stacks). By modelling certain quotients of buildings as chamber systems of type $M$, covering theory of buildings was developed by Tits in [Tit81] (expositions of Tits’ work can be found in [Kan86], [Ron89], and [Ron92]). However this theory ‘stops’ at 2-residues, and so is restricted to groups which act freely on the set of 2-residues of a building. On the other hand, quotient data in the form of a Bruhat decomposition is associated to groups which act transitively on chambers and transitively on ‘$W$-spheres’, so called Weyl transitivity (see [AB08, Chapter 5]). More generally, Tits’ amalgam method, which can be found in [Tit85] and [Tit86], constructs quotients of buildings by groups which act transitively on chambers. Over the course of several articles, we’ll show that $W$-groupoids unify and generalize these existing quotient constructions, and we’ll construct $W$-groupoids which give rise to new examples of lattices acting on buildings.

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Alternatively, if one moves to the Davis realization of a building, then a quotient will naturally be a complex of groups (see [BH99, Chapter III.C] for details on complexes of groups). Constructing buildings by taking the universal cover of a complex of groups is a common technique in geometric group theory (see [Bou97], [GP01], [FT11], [Ess13], [NTV17]). If one restricts to actions of torsion-free groups on buildings, then the corresponding complex of groups has trivial local groups, and so cell complexes are sufficient to model quotients (see [CMSZ93], [Vdo02]). However, we’ll argue that $W$-groupoids are the natural structures to work with when studying buildings, and make quotient constructions easier.

### 2. $W$-Groupoid Properties

From now on, $G = (G_0, G_1)$ is a groupoid with vertices $G_0$ and edges $G_1$, $W = (W, S)$ is a Coxeter group with generating set $S$, and ‘$\leq$’ denotes the Bruhat order of $W$ (see [BB06] for details on the Bruhat order). For an edge $g \in G$, we denote the initial and terminal vertices of $g$ by $\iota(g)$ and $\tau(g)$ respectively. For edges $g, h \in G$ such that $\tau(g) = \iota(h)$, we write $gh$ for their composition in $G$. We make the convention that by ‘a function $\delta : G \to W$’ we mean any function of sets $\delta : G_1 \to W$. We call $\delta(g)$ the $W$-length of the edge $g$, and we call vertices $C \in G_0$ the chambers of $G$. We call $G$ simply connected if all its local groups are trivial. Notice that simply connected groupoids are equivalent to equivalence relations, and if $G$ is a connected simply connected groupoid, then a function $\delta : G \to W$ is equivalently a function $\delta : G_0 \times G_0 \to W$ by identifying the unique edge from $x$ to $y$ with $(x, y) \in G_0 \times G_0$. We now introduce a collection of metric space like properties which an arbitrary function $\delta : G \to W$ may satisfy.

**(WG1)** For all identity edges $1 \in G$, we have $\delta(1) = 1$.

In particular, (WG1) allows a non-identity edge of $G$ to have a $W$-length of 1.

**(WG2)** For all edges $g, h \in G$ such that $gh$ is defined in $G$, we have,

$$\delta(gh) \geq \delta(g)\delta(h).$$

We can draw this as:

\[
\begin{array}{c}
\bullet \\
\delta(gh) \geq \delta(g)\delta(h) \\
\bullet \\
\end{array}
\]

Property (WG2) will also be called the triangle inequality, for obvious reasons, although the direction of the inequality is opposite to what one might expect.

**(WG2’)** For all edges $g, h \in G$ such that $g^{-1}h$ is defined in $G$ and $\delta(g^{-1}h) \in S$, putting $w = \delta(g)$ and $s = \delta(g^{-1}h)$, we have:

(i) if $ws < w$, then $\delta(h) \in \{w, ws\}$
(ii) If \( ws > w \), then \( \delta(h) = ws \).

We can draw this as:

\[
\begin{array}{c}
s \\
\{w, ws\} \\
w \\
\end{array}
\]

Property (WG2') will also be called the the **local triangle inequality**. Roughly speaking, the local triangle inequality is the triangle inequality where one of the sides of the triangle has \( W \)-length a generator \( s \in S \). We will describe the precise relationship between the two in Section 4.

**(WG3)** For all edges \( g \in G \) and for each \( s \in S \) such that \( \delta(g)s < \delta(g) \), there exists an edge \( h \in G \) with \( \iota(h) = \iota(g) \) such that,

\[
\delta(h^{-1}g) = s \quad \text{and} \quad \delta(h) = \delta(g)s.
\]

We can draw this as:

\[
\begin{array}{c}
s \\
\delta(h) = \delta(g)s \\
g \\
\end{array}
\]

Property (WG3) can be viewed as the analogue of being a geodesic metric space in classical metric geometry (we make this more precise in Section 4). Finally, we call a function \( \delta : G \rightarrow W \) **weak** if for all chambers \( C \in G \) and all \( s \in S \), there exists an edge \( g \in G \) with \( \iota(g) = C \) and \( \delta(g) = s \). Property (WG3) together with weakness is analogous to the existence of geodesic rays in metric geometry.

### 3. Some Easy Consequences

In this section, we prove some easy consequences of properties (WG1), (WG2), and (WG2').

**Lemma 3.1.** Let \( \delta : G \rightarrow W \) be a function which satisfies properties (WG1) and (WG2'). For any edge \( g \in G \) with \( \delta(g) \in S \), we have \( \delta(g^{-1}) = \delta(g) \).

**Proof.** Let \( \delta(g) = s \), and \( \delta(g^{-1}) = w \). In the definition of (WG2'), put \( h = g^{-1}g \), and redefine \( g \) to be \( g^{-1} \). Then we get,

\[
1 = \delta(h) \in \{w, ws\}.
\]

But by (WG1), \( w \neq 1 \). Thus, \( 1 = ws \) and \( \delta(g^{-1}) = w = s \).
Thus, $\delta$.

Therefore we must have $\delta = \delta(g)\delta(h)$.

Lemma 3.3. Let $\delta : G \to W$ be a function which satisfies properties (WG1) and (WG2). For all edges $g, h \in G$ such that $g^{-1}h$ is defined in $G$ and $\delta(g^{-1}h) = 1$, we have $\delta(g) = \delta(h)$.

Proof. Using (WG2) and Lemma 3.2, we get,

$$1 = \delta(g^{-1}h) \geq \delta(g)\delta(h).$$

So $\delta(g^{-1}h) = 1$, or equivalently $\delta(g) = \delta(h)$. \hfill \Box

Lemma 3.4. Let $\delta : G \to W$ be a function which satisfies properties (WG1) and (WG2). Let $g, h \in G$ be edges such that $gh$ is defined in $G$, and $\delta(g), \delta(h) \in S$. Then we have the following:

(i) if $\delta(g) = \delta(h) = s$, then $\delta(gh) \in \{1, s\}$

(ii) if $\delta(g) \neq \delta(h)$, then $\delta(gh) = \delta(g)\delta(h)$.

Proof. Suppose that $\delta(g) = \delta(h) = s$, and let $k = gh$. Then,

$$s = \delta(k) = \delta(g^{-1}k) \geq \delta(g^{-1})\delta(k) = s\delta(k).$$

Thus, $s\delta(k) \in \{1, s\}$, and so $\delta(gh) = \delta(k) \in \{1, s\}$. Now suppose that $\delta(g) \neq \delta(h)$. Let $s = \delta(g)$ and $t = \delta(h)$. Then,

$$t = \delta(h) = \delta(g^{-1}k) \geq \delta(g^{-1})\delta(k) = s\delta(k).$$

Thus, $s\delta(k) \in \{1, t\}$, and so $\delta(k) \in \{s, st\}$. But,

$$\delta(k) = \delta(gh) \geq st.$$ 

Therefore we must have $\delta(gh) = \delta(k) = st$. \hfill \Box

In the presence of (WG1), the local triangle inequality is weaker than the triangle inequality:

Lemma 3.5. Let $\delta : G \to W$ be a function which satisfies properties (WG1) and (WG2). Then $\delta$ satisfies the local triangle inequality (WG2').

Proof. Let $g, h \in G$ with $\delta(g^{-1}h) = s$. Let $k = g^{-1}h$. Then,

$$s = \delta(k) = \delta(g^{-1}h) \geq \delta(g^{-1})\delta(h).$$

Thus, $\delta(g^{-1})\delta(h) \in \{1, s\}$, and so $\delta(h) \in \{\delta(g), \delta(g)s\}$. Also,

$$\delta(h) = \delta(gk) \geq \delta(g)\delta(k) = \delta(g)s.$$ 

Thus, if $\delta(g)s > \delta(g)$, then $\delta(h) = \delta(g)s$. \hfill \Box
4. Galleries and Geodesics.

In this section, we define galleries and geodesics (galleries of reduced type) for functions \( \delta : \mathcal{G} \to W \). We show that property (WG3) ensures that we have geodesics, and we prove the equivalence of the local and global triangle inequalities in the presence of (WG1) and (WG3).

Let \( \delta : \mathcal{G} \to W \) be a function. A gallery of an edge \( g \in \mathcal{G} \) is a word \( g_1 \ldots g_n \) such that:

(i) \( g_k \in G_1 \) and \( \delta(g_k) \in S \) for all \( k \in \{1, \ldots, n\} \)

(ii) \( g_k g_{k+1} \) is defined in \( G \) for all \( k \in \{1, \ldots, n-1\} \)

(iii) \( g = g_1 \ldots g_n \).

We call the word \( \delta(g_1) \ldots \delta(g_n) \) over \( S \) the type of the gallery \( g_1 \ldots g_n \). A geodesic of \( g \) is a gallery \( g = g_1 \ldots g_n \) whose type is a reduced word. In the presence of (WG2') and (WG3), geodesics behave as their name implies:

**Proposition 4.1.** Let \( \delta : \mathcal{G} \to W \) be a function which satisfies properties (WG2') and (WG3). Let \( g = g_1 \ldots g_n \) be a geodesic of a edge \( g \in \mathcal{G} \). Then,

\[
\delta(g) = \delta(g_1) \ldots \delta(g_n).
\]

**Proof.** We prove by induction on \( n \). The result is trivial for \( n = 1 \). Suppose that the result holds for \( n - 1 \). Let \( h = gg_n^{-1} \). Then, by the induction hypothesis, we have,

\[
\delta(h) = \delta(g_1) \ldots \delta(g_{n-1}).
\]

But \( g = hg_n \) and \( \delta(h)\delta(g_n) > \delta(h) \). Therefore \( \delta(g) = \delta(g_1) \ldots \delta(g_n) \) by the local triangle inequality.

We now show that (WG3) can be viewed as the property that ‘all possible’ geodesics exist.

**Lemma 4.2.** Let \( \delta : \mathcal{G} \to W \) be a function which satisfies property (WG3). Let \( g \in \mathcal{G} \) be a edge and put \( w = \delta(g) \). For every reduced decomposition \( f \) of \( w \), there exists a geodesic of \( g \) whose type is \( f \).

**Proof.** We prove by induction on the length \( n \) of \( f \). The result is trivial for \( n = 1 \), since in this case \( g \) is a geodesic of itself. Suppose that the result holds for \( n - 1 \). Let \( f = s_1 \ldots s_n \). By (WG3), there exists a edge \( h \in \mathcal{G} \) such that \( \delta(h) = ws_n \) and \( \delta(h^{-1}g) = s_n \). Put \( g_n = h^{-1}g \). Using the induction hypothesis, let \( g_1 \ldots g_{n-1} \) be a geodesic of \( h \) with type \( s_1 \ldots s_{n-1} \). Then \( g_1 \ldots g_{n-1}g_n \) is the required geodesic of \( g \).

In fact, for each \( f \), the geodesic is unique. Using geodesics, we can now strength the result of Lemma 3.1 by adding (WG3) to the hypothesis:

**Corollary 4.2.1.** Let \( \delta : \mathcal{G} \to W \) be a function which satisfies properties (WG1), (WG2'), and (WG3). Then for any edge \( g \in \mathcal{G} \), we have \( \delta(g^{-1}) = \delta(g)^{-1} \).

**Proof.** By Lemma 4.2, there exists a geodesic \( g_1 \ldots g_n \) of \( g \). Then \( g_n^{-1} \ldots g_1^{-1} \) is a geodesic of \( g^{-1} \). But \( \delta(g_j^{-1}) = \delta(g_j) \) by Lemma 3.1. The result then follows by Proposition 4.1.
In the presence of (WG1) and (WG3), the local triangle inequality implies the triangle inequality:

**Proposition 4.3.** Let \( \delta : \mathcal{G} \to W \) be a function which satisfies properties (WG1), (WG2'), and (WG3). Then \( \delta \) satisfies property (WG2).

**Proof.** Let \( g, h \in \mathcal{G} \) be edges such that \( gh \) is defined in \( \mathcal{G} \), and let,

\[
gh = g_1 \ldots g_n
\]

be a geodesic of \( gh \) with type \( f \). Then \( \delta(hg_n^{-1}) \in \{ \delta(h), \delta(h)\delta(g_n) \} \) by (WG2'). By proceeding inductively in \( n \), we see that \( \delta(g^{-1}) = \delta(h)f'_n \), where \( f' \) is a substring of \( f \). Then \( \delta(g) = f'\delta(h)^{-1} \). Thus, \( \delta(g)\delta(h) = f' \), and it follows that \( \delta(gh) \geq \delta(g)\delta(h) \) in the Bruhat order.

By combining Proposition 4.3 with Lemma 3.5, we see that in the presence of properties (WG1) and (WG3), the triangle inequality and the local triangle inequality are equivalent.

5. **Definition of \( W \)-Groupoids**

We now define \( W \)-groupoids.

**Definition 5.1.** Let \( W \) be a Coxeter group, and let \( \mathcal{G} \) be a groupoid. A **\( W \)-groupoid** on \( \mathcal{G} \) is a weak function \( \delta : \mathcal{G} \to W \) which satisfies the following three properties:

(WG1) For all identity edges \( 1 \in \mathcal{G} \), we have \( \delta(1) = 1 \)

(WG2) For all edges \( g, h \in \mathcal{G} \) such that \( gh \) is defined in \( \mathcal{G} \), we have

\[
\delta(gh) \geq \delta(g)\delta(h)
\]

(WG3) For all edges \( g \in \mathcal{G} \) and for each \( s \in S \) such that \( \delta(g)s < \delta(g) \), there exists an edge \( h \in \mathcal{G} \) with \( \iota(h) = \iota(g) \) such that,

\[
\delta(h^{-1}g) = s \quad \text{and} \quad \delta(h) = \delta(g)s.
\]

If in addition \( \delta(g) = 1 \) implies that \( g \) is an identity edge, then \( \delta : \mathcal{G} \to W \) is called a **strict \( W \)-groupoid**. We show in Section 6 that buildings are equivalently connected simply connected strict \( W \)-groupoids. It follows from weakness that \( \delta \) is always surjective. We show in Section 7 that injective \( W \)-groupoids are equivalent to Bruhat decompositions. By our previous results, \( W \)-groupoids satisfy the local triangle inequality, and the property that \( \delta(g^{-1}) = \delta(g)^{-1} \) for all edges \( g \in \mathcal{G} \). For a chamber \( C \in \mathcal{G} \), the fundamental group of \( \mathcal{G} \) at \( C \) is the local group of \( \mathcal{G} \) at \( C \).

For \( J \subseteq S \), we denote by \( \mathcal{G}_J \) the restriction of \( \mathcal{G} \) to those edges whose \( W \)-length is an element of \( W_J = \langle J \rangle \leq W \). Then \( \mathcal{G}_J \) is naturally a \( W_J \)-groupoid. The **Borel subgroupoid** \( B \) of \( \mathcal{G} \) is \( \mathcal{G}_\emptyset \), i.e. it is the subgroupoid of \( \mathcal{G} \) whose set of edges is,

\[
B_1 = \{ g \in \mathcal{G}_1 : \delta(g) = 1 \}.
\]
Thus, a $W$-groupoid is strict if and only if its Borel subgroupoid is a bundle of trivial groups. For $J \subseteq S$, we call a connected component of $\mathcal{G}_J$ a $J$-residue; if $|J| = 2$, we say 2-residue. If $J = \{s\}$ for a generator $s \in S$, then we call $\mathcal{G}_J$ the panel groupoid of type $s$. Notice that to determine the function $\delta$ of a $W$-groupoid $\mathcal{G}$, it is sufficient to know the panel groupoids $\mathcal{G}_s \leq \mathcal{G}$ since geodesics will then tell us the value of $\delta$ on all other edges (in the same way that chamber systems determine buildings).

6. $W$-Groupoids and Buildings

We now demonstrate the connection between $W$-groupoids and buildings. For our definition of a building, we take the symmetrical version of the axioms which appear in [AB08, p. 218], which are equivalent by [AB08, Remark 5.18].

**Definition 6.1.** Let $W = (W, S)$ be a Coxeter group. A building $(\Delta, \delta)$ of type $W$ is a set of chambers $\Delta$ equipped with a function $\delta : \Delta \times \Delta \rightarrow W$ such that for all $C, D \in \Delta$, the following three conditions hold:

(WD1) $C = D$ if and only if $\delta(C, D) = 1$

(WD2) if $\delta(C, D) = w$ and $D' \in \Delta$ satisfies $\delta(D, D') = s \in S$, then:

(i) if $ws < w$, then $\delta(C, D') \in \{ws, w\}$

(ii) if $ws > w$, then $\delta(C, D') = ws$

(WD3) if $\delta(C, D) = w$, then for all $s \in S$ there exists a chamber $D' \in \Delta$ such that,

$$\delta(D, D') = s \quad \text{and} \quad \delta(C, D') = ws.$$  

We now show that a building is equivalently a strict $W$-groupoid on a connected simply connected groupoid. Let $(\Delta, \delta)$ be a building, and let $G$ be the connected simply connected groupoid with $G_0 = \Delta$. Define a function,

$$\delta' : G \rightarrow W, \ g \mapsto \delta(\iota(g), \tau(g)).$$

Then $\delta' : G \rightarrow W$ has property (WG1) and is strict because $(\Delta, \delta)$ has property (WD1). Weakness follows from (WD3), as does property (WG3). Also, $\delta' : G \rightarrow W$ satisfies the local triangle inequality because $(\Delta, \delta)$ satisfies (WD2), and so $\delta' : G \rightarrow W$ satisfies property (WG2) by Proposition 4.3. Therefore $\delta' : G \rightarrow W$ is a strict $W$-groupoid. Conversely, let $\delta : G \rightarrow W$ be a strict $W$-groupoid on a connected simply connected groupoid $G$. Put $\Delta = \mathcal{G}_0$, and define a map,

$$\delta' : \Delta \times \Delta \rightarrow W, \ (C, D) \mapsto \delta(g)$$

where $g$ is the unique edge travelling from $C$ to $D$. Then $(\Delta, \delta')$ has property (WD1) because $\delta : G \rightarrow W$ is strict and has property (WG1). Also, $(\Delta, \delta')$ satisfies (WD2) because $\delta : G \rightarrow W$ satisfies the local triangle inequality. Property (WD3) follows from weakness in the case where $ws > w$, and (WG3) in the case where $ws < w$. 

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7. **W-Groupoids and Bruhat Decompositions**

We now consider the case where $G$ has only one chamber. Such a $W$-groupoid will be the quotient of a chamber-transitive action of a group on a building. In this case, $G$ is naturally a group, and we put $G = G$ and $B = B$. It follows from Lemma 3.3 and Corollary 4.2.1 that $\delta$ factors through,

$$ G \to B \backslash G / B, \quad g \mapsto BgB $$

to give a function,

$$ \delta_B : B \backslash G / B \to W. $$

We now show that a Bruhat decomposition is equivalently a $W$-groupoid $\delta : G \to W$ with one chamber such that $\delta_B$ is a bijection. Let $\delta : G \to W$ be such a $W$-groupoid, and let,

$$ C : W \to B \backslash G / B $$

be the inverse of $\delta_B$. Then the local triangle inequality may be written as,

(B') For all $w \in W$ and all $s \in S$, we have,

(i) if $ws < w$, then $C(w)C(s) \subseteq C(ws) \cup C(w)$

(ii) if $ws > w$, then $C(w)C(s) \subseteq C(ws)$.

And (WG3) may be written as,

(B'') For all $w \in W$ and $s \in S$ such that $ws < w$, we have,

$$ C(w) \subseteq C(ws)C(s). $$

If we substitute $w' = ws$, we obtain the following alternate form of (B''); for all $w' \in W$ and $s \in S$ such that $w's > w'$, we have,

$$ C(w's) \subseteq C(w')C(s). $$

In fact, this inclusion holds in general:

**Proposition 7.1.** Let $\delta : G \to W$ be a $W$-groupoid with one chamber such that $\delta_B$ is a bijection. Let $w \in W$ and $s \in S$ such that $ws < w$, then,

$$ C(ws) \subseteq C(w)C(s). $$

**Proof.** Put $w' = ws$, then $w's > w'$ and so $C(w')C(s) \subseteq C(w's)$ by (B'). Then $C(ws)C(s) \subseteq C(w)$, and so $C(ws) \subseteq C(w)C(s)$ since $C(s)$ is closed under inverses by Lemma 3.1. \hfill \Box

Then, combining (B'), (B''), and Proposition 7.1, we obtain,

(B) For all $w \in W$ and all $s \in S$, we have,

(i) if $ws < w$, then $C(ws) \subseteq C(w)C(s) \subseteq C(ws) \cup C(w)$

(ii) if $ws > w$, then $C(w)C(s) = C(ws)$. 

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This is the axiom for a bijective function $\delta_B : B\backslash G/B \to W$ to be a Bruhat decomposition which appears in [AB08, Section 6.2.1]. Conversely, let $G$ be a group with subgroup $B \leq G$, and suppose that we have a bijective function $\delta_B : B\backslash G/B \to W$ with property (B). Let $\delta : G \to W$ be the composition of $\delta_B$ with the projection $G \to B\backslash G/B$. Then $\delta$ has properties (B') and (B''), and so has properties (WG2') and (WG3). Finally, $\delta$ is clearly weak, and has property (WG1) since if $C(w) = B$, then for any $s \in S$ we have,

$$C(ws) \subseteq C(s)C(w) = C(s).$$

Therefore $ws = s$ and $w = 1$.

We now discuss some future developments in the theory of $W$-groupoids, in particular their connection with chamber systems of type $M$, their covering theory, and their ‘presentations’.

8. \textit{W-Groupoids and Chamber Systems of Type $M$.}

A \textit{chamber system} is an indexed collection of equivalence relations on a set, and a \textit{chamber system of type $M$} is a chamber system whose 2-residues are buildings (see [Tit81] for more details). The relationship between chamber systems of type $M$ and $W$-groupoids will be described in [Nor], where we show that strict $W$-groupoids whose 2-residues are simply connected are equivalent to the chamber systems of type $M$ which are covered by buildings. A $W$-groupoid is obtained from a chamber system of type $M$ by taking the groupoid of homotopy classes of galleries. Conversely, the panel groupoids of a strict $W$-groupoid whose 2-residues are simply connected will also be simply connected, and one recovers the associated chamber system of type $M$ by taking for equivalence relations the panel groupoids.

9. \textit{Strict W-Groupoids and Weyl Graphs.}

In [Nor], we introduce ‘presentations’ of strict $W$-groupoids, which we call \textit{Weyl graphs}. Roughly speaking, Weyl graphs are to strict $W$-groupoids what chambers systems are to buildings. A Weyl graph consists of an indexed collection of ‘generating’ panel groupoids together with a collection of ‘relations’ which we call \textit{suites}. Suites are exactly the images of apartments of the 2-residues of the building which covers the Weyl graph.

Weyl graphs generalize chamber systems of type $M$ by allowing 2-residues to be quotients of generalized polygons; they are the quotients of actions which are free on chambers, but not necessarily free on the set of 2-residues. For example, Figure 1 shows the Weyl graph of the Fano plane and of its quotient by a Singer cycle (all of the groupoids involved are equivalence relations). The galleries in the quotient which lift to apartments of the Fano plane are the suites of the quotient. We develop covering theory of strict Weyl groupoids in the language of Weyl graphs, which reduces to Tits’ covering theory of chamber systems of type $M$ if one assumes that coverings are injective on 2-residues.

Weyl graphs provide a framework in which quotients of generalized polygons by groups acting freely on chambers (flags) can be glued together to form quotients of buildings. For example, Essert’s Singer lattices in [Ess13] of type $A_2$ are constructed by gluing together three copies of the quotient of a projective plane by a Singer group. Using Weyl graphs, this construction is easily generalized to type $M$, where $m_{st} \in \{2, 3, \infty\}$ for all $s, t \in S$. 

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Figure 1: The Fano plane and its quotient by a Singer cycle, constructed using the difference set \{0, 1, 3\}.

10. Stacky Covering Theory of \(W\)-Groupoids

We can obtain (stacky) covering theory of \(W\)-groupoids directly from covering theory of groupoids (see [Bro06] for details on groupoids); the key step is to somehow allow the Borel subgroupoid to account for non-trivial isotropy. We outline the main ideas here.

A covering of \(W\)-groupoids \(p : \mathcal{G} \to \mathcal{G}'\) is a surjective groupoid homomorphism which preserves \(W\)-length such that for all chambers \(C \in \mathcal{G}\), the restriction of \(p\) to the edges which issue from \(C\) is a bijection into the edges which issue from \(p(C)\). Naturally associated to the free action of a group \(G\) on a \(W\)-groupoid \(\mathcal{G}\) is the quotient \(\mathcal{G} \to G\backslash \mathcal{G}\), which is a covering map of \(W\)-groupoids. Conversely, the fundamental group of a strict \(W\)-groupoid \(\mathcal{G}\) acts freely on the building which covers \(\mathcal{G}\).

If the action of a group on a building isn’t free, one can move to a free action by replacing each chamber \(C\) by \text{stab}(C)-many chambers, all at distance 1 \(\in W\). One then takes the ordinary quotient, which will have a non-trivial Borel subgroupoid which is a bundle of groups. Conversely, the fundamental group of a (possibly non-strict) \(W\)-groupoid acts freely on its universal cover, which develops non-trivial isotropy when one moves to a building by identifying chambers at distance 1 \(\in W\).

One should be able to define ‘presentations’ of \(W\)-groupoids in general, which might be called Weyl graphs of groups, and develop covering theory in this language. Tits’ amalgams are Weyl graphs of groups with a single chamber since the associated \(W\)-groupoid is generated from a collection of panel groups, whose suites are determined by the amalgamation data.
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