On the Cover and Pombra Gaussian Feedback Capacity: Complete Sequential Characterizations via a Sufficient Statistic

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Abstract—The main objective of this paper is to derive a new sequential characterization of the Cover and Pombra [1] characterization of the $n$--finite block or transmission feedback information ($n$-FTFI) capacity, which clarifies several issues of confusion and incorrect interpretation of results in literature.

The optimal channel input processes of the new equivalent sequential characterizations are expressed as functionals of a sufficient statistic and a Gaussian orthogonal innovations process. From the new representations follows that the Cover and Pombra characterization of the $n$--FTFI capacity is expressed as a functional of two generalized matrix difference Riccati equations (DRE) of filtering theory of Gaussian systems. This contradicts results which are redundant in the literature, and equations (DRE) of filtering theory of Gaussian systems. This contradicts results which are redundant in the literature, and illustrates the fundamental complexity of the feedback capacity formula.

I. INTRODUCTION, MOTIVATION, MAIN RESULTS, COMPARISON WITH CURRENT STATE OF KNOWLEDGE

A. The Cover and Pombra Feedback Capacity

Cover and Pombra [1] were concerned with the feedback capacity of the additive Gaussian noise (AGN) channel,

$$ Y_t = X_t + V_t, \quad t = 1, \ldots, n, \quad \frac{1}{n} \mathbb{E} \left\{ \sum_{t=1}^{n} (X_t)^2 \right\} \leq \kappa $$

where $\kappa \in [0, \infty)$ is the total power of the transmitter, $X_t : \Omega \to \mathbb{R}$, $Y_t : \Omega \to \mathbb{R}$, and $V_t : \Omega \to \mathbb{R}$, are the channel input, channel output and noise random variable, respectively, and the distribution of the sequence, $V^n = \{V_1, \ldots, V_n\}$, denoted by $P_{V^n}(\cdot)$, is jointly Gaussian, not necessarily stationary or ergodic. Cover and Pombra considered the set of uniformly distributed messages $W : \Omega \to \mathcal{M}(n) = \{1, 2, \ldots, 2^n R_n\}$, codewords of block length $n$, $X_1 = e_1(W), \ldots, X_n = e_n(W, X^{n-1}, Y^{n-1})$, and decoder functions, $y^n \mapsto d_n(y^n) \in \mathcal{M}(n)$, with average error probability

$$ \mathbb{P}_{\text{error}} = \mathbb{P}\{d_n(y^n) \neq W\} = \frac{1}{2^n R_n} \sum_{w=1}^{2^n R_n} \mathbb{P}\{d_n(y^n) \neq w\}. $$

(1.2)

According to [1, page 39, above Lemma 5], “$X^n$ is causally to $V^n$”, which is equivalent to the following decomposition of the joint probability distribution of $(X^n, Y^n)$:

$$ P_{X^n,Y^n} = P_{V^n|Y^{n-1},X^n} P_{X^n|X^{n-1},Y^{n-1}} \ldots P_{V_1|X^n} P_{X_1|X_1} P_{X_1} $$

$$ = P_{V^n} \prod_{t=1}^{n} P_{X_t|X^{t-1},Y^{t-1}}. $$

That is, $P_{V^n|Y^{n-1},X^n} = P_{V^n|Y^{n-1}}$, or equivalently $X^n \leftrightarrow Y^{n-1} \leftrightarrow V_t$ is a Markov chain, for $t = 1, \ldots, n$. As usual, the messages $W$ are independent of the channel noise $V^n$.

Cover and Pombra [1, Theorem 1], applied the maximum entropy principle of Gaussian RVs, derived direct and converse coding theorem, and characterized feedback capacity, through the $n$--FTFI capacity, given by

$$ C_{\text{fI}}^f(\kappa) \triangleq \sup_{P_{X^n|Y^n}} \frac{1}{2} \log \left( \frac{\mathbb{E} \{ (B^n + I_n) K^n (B^n + I_n)^T + K_{\mathbf{T}} \}^{\frac{n}{2}}}{|K^n|^{\frac{n}{2}}} \right) $$

(1.4)

where $(X^n, Y^n, V^n)$ is jointly Gaussian, provided the supremum exists, and where $H(\cdot)$ denotes differential entropy. Due to the joint Gaussianity of $(X^n, Y^n, V^n)$, and that, any tuple of the RVs $(X^n, Y^n, V^n)$ uniquely specify the third, the $n$--FTFI capacity is given by [1, eqn(10)]

$$ C_{\text{fI}}^f(\kappa) = \max \left\{ \frac{1}{2} \log \left( \frac{\mathbb{E} \{ (B^n + I_n) K^n (B^n + I_n)^T + K_{\mathbf{T}} \}^{\frac{n}{2}}}{|K^n|^{\frac{n}{2}}} \right) \right\} $$

(1.5)

where the channel input process $X^n$ is given by [1, eqn(11)]

$$ X_t = \sum_{j=1}^{t-1} B_{ij} V_j + Z_j, \quad t = 1, \ldots, n, $$

(1.6)

$$ X^n = B^n V^n + \bar{Z}, \quad Y^n = \mathbb{B}^{n} + I_n \times n \times n V^n + \bar{Z}, $$

(1.7)

$$ \bar{Z} \sim N(0, K_{\mathbf{T}}) \quad \text{and independent of} \quad V^n, $$

(1.8)

$$ X^n \triangleq \left[ \begin{array}{c} X_1 \\ X_2 \\ \vdots \\ X_n \end{array} \right]^T, $$

(1.9)

$$ \frac{1}{n} \mathbb{E} \left\{ \sum_{t=1}^{n} (X_t)^2 \right\} = \frac{1}{n} \text{trace} \left( \mathbb{E} \left( X^n (X^n)^T \right) \right) \leq \kappa, $$

(1.10)

and where notation $N(0, K_{\mathbf{T}})$ means the random variable $\bar{Z}$ is jointly Gaussian, with zero mean and covariance matrix $K_{\mathbf{T}} = \mathbb{E} \left( \bar{Z} (\bar{Z})^T \right)$, and $I_n$ denotes the $n \times n$ identity matrix. The feedback capacity, $C_{\text{fI}}^f(\kappa)$, is characterized by the per unit time limit of the $n$--FTFI capacity [1, Theorem 1],

$$ C_{\text{fI}}^f(\kappa) \triangleq \lim_{n \to \infty} \frac{1}{n} C_{\text{fI}}^f(\kappa), $$

(1.11)

provided the supremum and limit exist.

B. Main Results of the Paper

The first main results of this paper is

R1) an equivalent sequential characterization of the Cover and Pombra [1] $n$--FTFI capacity, (1.5)-(1.10), in which $X^n$ is expressed as,
\( X_t = \Gamma_t^1 V^{t-1} + \Gamma_t^2 Y^{t-1} + Z_t, \quad X_1 = Z_1, \quad t = 2, \ldots, n \) (I.12)

where \( Z_t \in N(0, K_Z), t = 1, \ldots, n \) is an independent, zero mean, Gaussian sequence, \( Z_t \in N(0, K_Z) \) is independent of \((V^{t-1}, X^{t-1}, Y^{t-1})^T, \quad t = 1, \ldots, n, Z^n \) independent of \( V^n \), and \((\Gamma_1^1, \Gamma_2^1) \in (\infty, \infty) \times (\infty, \infty) \) are nonrandom, and where \( C_n^{b_1}(\kappa) \) is

\[
C_n^{b_1}(\kappa) = \sup_{\{f_{I_t}\in(\infty,\infty)\}} \sum_{t=1}^n \left[ H(I_t) - H(\hat{I}_t) \right]
\]

where \( I_t, \hat{I}_t \) are innovations processes defined,

\[
I_t = Y_t - E\{Y_t|V^{t-1}\}, \quad \hat{I}_t = V_t - E\{V_t|V^{t-1}\}
\]

and where the supremum is over all \((\Gamma_1^1, \Gamma_2^1, K_Z), t = 1, \ldots, n\). The new sequential characterization did not appear elsewhere in the literature.

The second main result is,

R2) the consideration of the partially observable state space (PO-SS) noise realization of Definition 1.1 (below), and an equivalent sequential characterization of the Cover and Pombra [1] \( n \)-FTFI capacity, (I.5)-(I.10), which is expressed as a functional of two generalized matrix difference Riccati equations (DRE) of filtering theory of Gaussian systems.

The third main result is,

R3) the use of a sufficient statistic to expressed \( C_n^{b_1}(\kappa) \), which allows the identification of necessary for the convergence, \( C^{b_1}(\kappa) \triangleq \lim_{n \to \infty} \frac{1}{n} C_n^{b_1}(\kappa) \), in terms of properties of generalized DRE, and its analysis using sequential methods, such as, dynamic programming.

**Definition 1.1.** A time-varying PO-SS realization of the Gaussian noise \( V^n \in N(0, K_{V^n}) \) is defined by

\[
S_{t+1} = A_S S_t + B_S W_t, \quad t = 1, \ldots, n - 1
\]

\[
V_t = C_S S_t + N_W W_t, \quad t = 1, \ldots, n
\]

\[
S_t \in N(\mu_S, K_S), \quad K_S \succeq 0 \quad \text{(positive semidefinite)}
\]

\[
W_t \in N(0, K_W), \quad K_W \succeq 0, \quad t = 1, \ldots, n
\]

\[
S_t : \Omega \to \mathbb{R}^n, \quad W_t : \Omega \to \mathbb{R}^n, \quad V_t : \Omega \to \mathbb{R}^n
\]

\[
R_t \triangleq N W_t N_T^T > 0
\]

where \( W_t, t = 1, \ldots, n \) is an independent Gaussian process, \( n_S, n_W \) are arbitrary positive integers, and \((A_S, B_S, C_S, N_W, \mu_S, K_S, K_W)\) are nonrandom for all \( t \), and \( n_S, n_W \) are finite positive integers.

The third main result is,

R4) Contrary to the claims in Kim’s [2], the time-domain characterization of feedback capacity [2, Theorem 6.1], does not correspond to the Cover and Pombra code formulation and assumption, and hence recent literature which makes use of [2], such as, [3]–[6], should be read with caution.

The justification of 4) is easy to verify, by using, for example, the channel input process, considered in [2, Page 76], for the autoregressive moving average noise model, \( V_t = c V_{t-1} + W_t - a W_{t-1}, t = 1, \ldots, n \), expressed in state space form, \( S_t \triangleq c S_t + W_t, t = 1, \ldots, n \), \( V_t = (c-a) S_t + W_t, \quad t = 1, \ldots, n \). According to [2, Page 76], the channel input \( X_t \), is

\[
X_t = Z_1, \quad Z_n \text{ is zero mean, variance } K_Z > 0
\]

\[
X_n = \Lambda \left( S_n - E\{S_n Y^{n-1}\} \right), \quad n = 2, \ldots
\]

However, the above channel input depends on the state \( S_n \), and for this to hold it is necessary that the chain of equalities hold:

\[
P_{X_t|Y^{t-1}, Y^{t-2}} = P_{X_t|Y^{t-1}, Y^{t-2}} \text{ holds by (I.1)}
\]

\[
P_{X_t|Y^{t-1}, Y^{t-2}, S_1} \text{ if } S_1 = s \text{ is known to the code}
\]

\[
P_{X_t|S_t, Y^{t-1}, S_1} \text{ if } (V^{t-1}, S_1) \text{ uniquely defines } S_t
\]

The above shows,

a necessary condition for validity of (I.21), (I.22) is: given the initial state of the noise \( S_1 = s = \frac{c-a}{c} \), which should be known to the encoder, the channel noise \( V^n \triangleq \{X_1, \ldots, V_{n-1}\} \) uniquely defines the state variables \( S^n \).

Clearly, the above necessary condition, never holds for the noise of Definition 1.1 with

\[
B_S W_t = B_S^1 W_t^1 + B_S^2 W_t^2, \quad N_S W_t = N_S^1 W_t^1 + N_S^2 W_t^2,
\]

\( W_1^n \) and \( W_2^n \) independent sequences.

We should emphasize that the above necessary condition is explicitly stated by Yang, Kavcic, and Tatikonda in [2], [7].

Although, due to space limitations, the complete proofs of the results of this paper are omitted, these are found [8].

**C. Relation to Past Literature**

Over the years, considerable efforts have been devoted to compute \( C_n^{b_1}(\kappa) \) and \( C_n^{b_1}(\kappa) \), [2]–[7], often under simplified assumptions on the channel noise. In addition, bounds are described in [9], [10], while numerical methods are developed in [11], mostly for time-invariant AGN channel, driven by stationary noise. We should mention that most papers considered a variant of (I.1), by interchanging the per unit time limit and the maximization operations, under the assumption: the joint process \((X^n, Y^n), n = 1, 2, \ldots\) is either jointly stationary or asymptotically stationary (see [2]–[5]), and the joint distribution of the joint process \((X^n, Y^n), n = 1, 2, \ldots\) is time-invariant. A recent investigation of AGN channels driven by autoregressive unit memory stable and unstable noise with and without feedback is [12], [13], while the connection of ergodic theory and feedback capacity of unstable channels is discussed in [14], [15].
II. Equivalent Sequential Characterizations of the Cover and Pombrn n−FTFI

A. Notation

Throughout the paper, we use the following notation.

\( \mathbb{Z} = \{ \ldots, -1, 0, 1, \ldots \}, \mathbb{Z}_+ = \{ 1, 2, \ldots \}, \mathbb{Z}_n^+ = \{ 1, 2, \ldots, n \} \), where \( n \) is a positive integer.

\( \mathbb{R} = ( -\infty, \infty ) \), and \( \mathbb{R}^m \) is the vector space of tuples of the real numbers for an integer \( n \in \mathbb{Z}_+ \).

\( \mathbb{R}^{n \times m} \) is the set of \( n \) by \( m \) matrices with entries from the set of real numbers for integers \( (n,m) \in \mathbb{Z}_+ \times \mathbb{Z}_+ \).

\( (\Omega, \mathcal{F}, \mathbb{P} ) \) denotes a probability space. Given a random variable \( X : \Omega \to \mathbb{R}^n, n \in \mathbb{Z}_+ \), its induced distribution on \( \mathbb{R}^n \) is denoted by \( \mathbb{P}_X \).

\( \mathbb{P}_X \in \mathcal{N}(\mu_X, K_X), K_X \succeq 0 \) denotes a Gaussian distributed RV \( X \), with mean value \( \mu_X = \mathbb{E}(X) \) and covariance matrix \( K_X = \text{cov}(X,X) \succeq 0, \) defined by

\[
K_X = \text{cov}(X,X) \triangleq \mathbb{E}\left\{ (X - \mathbb{E}(X)) (X - \mathbb{E}(X))^T \right\}.
\]

Given another Gaussian random variables \( Y : \Omega \to \mathbb{R}^n, n \in \mathbb{Z}_+ \), which is jointly Gaussian distributed with \( X \), i.e., the joint distribution is \( \mathbb{P}_{X,Y} \), the conditional covariance of \( X \) given \( Y \), \( K_{X|Y} = \text{cov}(X,X|Y) \), is defined by

\[
K_{X|Y} \triangleq \mathbb{E}\left\{ (X - \mathbb{E}(X|Y)) (X - \mathbb{E}(X|Y))^T \right\} = \mathbb{E}\left\{ (X - \mathbb{E}(X|Y)) (X - \mathbb{E}(X|Y))^T \right\}
\]

where the last equality is due to a property of jointly Gaussian distributed RVs.

B. General Equivalent Characterization of Cover and Pombrn n−FTFI Capacity

First, we show validity of R1).

Theorem II.1. The Cover and Pombrn [1] expressions (L.3 - L.10), are equivalently represented by (L.12), (L.13).

Proof. The complete prove is found is given See [8, Section VI.A]. Below, we provide an outline. Consider (L.6) and define the process

\[
Z_t \overset{\triangle}{=} Z_t - \mathbb{E}\{ Z_t \},
\]

(II.23)

\[
Z_t \overset{\triangle}{=} Z_t - \mathbb{E}\{ Z_t \mid X^{t-1}, V^{t-1}, Y^{t-1} \}, \quad t = 2, \ldots, n,
\]

(II.24)

\[
= Z_t - \mathbb{E}\{ Z_t \mid V^{t-1}, Y^{t-1} \}
\]

(II.25)

where the last equality is due to, \( X^{t-1} \) is uniquely defined by \( (V^{t-1}, Y^{t-1}) \). Then \( Z_t \) is a Gaussian orthogonal innovations process, i.e., \( Z_t \) is independent of \( (X^{t-1}, V^{t-1}, Y^{t-1}) \), for \( t = 2, \ldots, n \), and \( \mathbb{E}\{ Z_t \} = 0, \) for \( t = 1, \ldots, n \). By (L.6),

\[
X_t = \sum_{j=1}^{t-1} B_{t,j} V_j + Z_t, \quad t = 1, \ldots, n,
\]

(II.26)

\[
= \sum_{j=1}^{t-1} B_{t,j} V_j + \mathbb{E}\{ Z_t \mid V^{t-1}, Y^{t-1} \} + Z_t, \quad \text{by (II.25)}
\]

(II.27)

\[
= \sum_{j=1}^{t-1} B_{t,j} V_j + \mathbb{E}\{ Z_t \mid V^{t-1}, Y^{t-1} \} + Z_t, \quad \text{for some } \Gamma_t
\]

(II.28)

\[
= \sum_{j=1}^{t-1} \Gamma_{t,j} V_j + Z_t, \quad \text{for some } \Gamma_{t,1}, \Gamma_{t,2}^2
\]

(II.29)

\[
= \sum_{t-1}^{t-1} \Gamma_{t,1} V^{t-1} + \Gamma_{t,2} V^{t-1} + Z_t, \quad \text{by definition (II.30)}
\]

(II.30)

where (a) is due to the joint Gaussianity of \( (Z^n, X^n, Y^n) \). From (II.30) and the independence of \( Z_t \) and \( (V^{t-1}, V^{t-1}, Y^{t-1}, \) for \( t = 2, \ldots, n \), it then follows (L.12) and the properties. To show (L.13), we notice that \( H(V^n) = \sum_{t=1}^{n-1} H(V_{t-1} | V^{t-1}) = \sum_{t=1}^{n-1} H(I_{t-1} | V^{t-1}) = \sum_{t=1}^{n-1} H(I_t) \) by the orthogonality of the innovations process (L.14). Similarly for \( H(V^n) \).

Remark II.1. By (L.13), \( C^n_{fb}(\kappa) \) is expressed in terms of the entropy of two invatations processes, i.e., independent processes. Consequently, its analysis, such as, the existence of the limit \( \lim_{n \to \infty} \frac{1}{n} C^n_{fb}(\kappa) \) is much easier to address, as well its computation.

C. A Sufficient Statistic Approach to the Characterization of the n−FTFI Capacity of AGN Channels Driven by PO-SS Noise Realizations

Now, we turn our attention to the derivation of statements under R2) and R3).

We note that characterization of the \( n−FTFI \) capacity \( C^n_{fb}(\kappa) \) given (L.13), although compactly represented, is not computationally very practical, because the input process \( X^n \) is not expressed in terms of a sufficient statistic that summarizes the information of the channel input strategy [16].

We wish to identify a sufficient statistic for the input process \( X_n \), given by (L.12), called the state of the input, which summarizes the information contained in \( (V^{t-1}, V^{t-1}, \). It will then become apparent that the characterization of the \( n−FTFI \) capacity can be expressed as a functional of two generalized matrix DREs.

First, since \( C^n_{fb}(\kappa) \) is given by (L.13), we need to compute the (differential) entropy \( H(V^n) \) of \( V^n \). The following lemma is useful in this respect.

Lemma II.1. Consider the PO-SS realization of \( V^n \) of Definition (L.11). Define the conditional covariance \( \Sigma_{t,1} = \text{cov}(S_t, S_t | V^{t-1}), \) \( \Sigma_{t,2} = \text{cov}(S_t, S_t) = K_{S_t} \) and the conditional mean of \( S_t \) given \( V^{t-1}, \) \( S_t \overset{\triangle}{=} \mathbb{E}\{ S_t | V^{t-1} \}, \) \( \mu_{S_t} \). Denote also the conditional mean and covariance of \( V_t \) given \( V^{t-1} \), by \( \mu_{V_t | V^{t-1}}, K_{V_t | V^{t-1}} \).
The following hold.

(i) $\hat{S}_t$ satisfies the generalized Kalman-filter recursion

$$
\hat{S}_{t+1} = A_t \hat{S}_t + M_t(\Sigma_t) \hat{I}_t, \quad \hat{S}_1 = \mu_{S_1},
$$

(ii) The covariance of the error, $E = S_t - \hat{S}_t$ is such that $E \{ E_t E_t^T \} = \Sigma_t$ and satisfies the generalized matrix DRE

$$
\Sigma_{t+1} = A_t \Sigma_t A_t^T + B_t K_t W_t B_t^T - \left( A_t \Sigma_t C_t^T + B_t K_t W_t N_t^T \right) \left( N_t W_t^T N_t + C_t \Sigma_t C_t^T \right)^{-1} \left( A_t \Sigma_t C_t^T + B_t K_t W_t N_t^T \right)^T,
$$

$\hat{I}_t \in N(0, K_t)$, $t = 1, \ldots, n$ is an orthogonal innovations process, i.e., $\hat{I}_t$ is independent of $\hat{I}_b$, for all $t \neq s$, $\hat{I}_t$, for all $t \neq s$, and $\hat{I}_t$ is independent of $\hat{V}_t - C_t \hat{S}_t$.

$K_t \triangleq \text{cov}(\hat{I}_t, \hat{I}_b) = C_t \Sigma_t C_t^T + N_t W_t N_t^T$.

(iii) $\mu_{S[p]}(\cdot)$ and $K_{[p]}(\cdot)$ are given by

$$
\mu_{S[p]}(\cdot) = \hat{C}_t \hat{S}_t, \quad t = 1, \ldots, n,
$$

$$
K_{[p]}(\cdot) = C_t \hat{S}_t^T + N_t W_t N_t^T, \quad t = 1, \ldots, n.
$$

(iv) The entropy of $V^n$, is given by

$$
H(V^n) = \sum_{t=1}^n H(I_t)
$$

$\left( \frac{1}{2} \sum_{t=1}^n \log \left( 2\pi e \left( C_t \Sigma_t C_t^T + N_t W_t N_t^T \right) \right) \right)
$$

Proof. This follows from generalized kalman-filter equations [17]; or [8, Section II.B, proof of Lemma II.1] \qed

Next, we invoke $C_n^{fb}(\cdot)$ given by (1.13) and Lemma II.1 to show that for each time $t$, $X_t$ is expressed as

$$
X_t = \Lambda_t \left( \hat{S}_t - E \left\{ \hat{S}_t | V^{t-1} \right\} \right) + Z_t, \quad t = 1, \ldots, n,
$$

$$
\hat{S}_t \triangleq E \left\{ \hat{S}_t | V^{t-1} \right\}, \quad \tilde{S}_t \triangleq E \left\{ \hat{S}_t | V^{t-1} \right\}
$$

which means, at each time $t$, the state of the channel input process $X_t$ is $\langle \hat{S}_t, \tilde{S}_t \rangle$; this is the sufficient statistic. We show that $\hat{S}_t$ satisfies another generalized Kalman-filter recursion. 

Now, we prepare to prove (1.41) and the main theorem. We start with preliminary calculations.

$$
P \{ Y_t \in dY \} | V^{t-1}, X^t = P_t(\cdot | X_t, V^{t-1}), \quad t = 2, \ldots, n,
$$

$$
P_t(\cdot | X_t, V^{t-1}) = P_t(\cdot | X_t, \hat{S}_t, \hat{I}_t), \quad \text{by} \quad \hat{S}_t = E \left\{ \hat{S}_t | V^{t-1} \right\}
$$

$$
P_t(\cdot | X_t, \hat{S}_t, \hat{I}_t) = P_t(\cdot | X_t, V^{t-1}, \hat{S}_t), \quad \text{by} \quad V_t = C_t \hat{S}_t + \hat{I}_t \quad \text{and} \quad \hat{I}_t \in N(0, K_t)
$$

$$
\begin{align*}
\mathbb{P} \{ Y_t \in dY | V^{t-1}, X^t \} &= P_t(dY | X_t, V^{t-1}), \\
&= P_t(dY | X_t, V^{t-1}, \hat{S}_t), \quad \text{by} \quad \hat{S}_t = E \left\{ \hat{S}_t | V^{t-1} \right\} \\
&= P_t(dY | X_t, V^{t-1}, \hat{S}_t), \quad \text{i.e.,} \quad V_t = C_t \hat{S}_t + \hat{I}_t \\
&= \mathbb{E} \{ Y_t | \hat{S}_t, \hat{I}_t \}, \quad \text{by} \quad Y_t = X_t + V_t = X_t + C_t \hat{S}_t + \hat{I}_t \quad \text{and} \quad \hat{I}_t \in N(0, K_t)
\end{align*}
$$

At $t = 1$, $\mathbb{P} \{ Y_t \in dY | X_1 \} = P_1(dY | X_1)$. By (2.2), it follows that the conditional distribution of $Y_t$ given $Y^{t-1} = y^{t-1}$ is

$$
P_t(dY_t | y^{t-1}) = \int P_t(dY_t | x_t, \hat{S}_t) P_t(dx_t | \hat{S}_t, y^{t-1}) P_t(d\hat{S}_t | y^{t-1}),
$$

$t = 2, \ldots, n$.

$$
P_1(dY_1) = \int P_1(dY_1 | x_1, \hat{S}_1) P_1(dx_1 | \hat{S}_1) P_1(d\hat{S}_1).
$$

From the above distributions, at each time $t$, the distribution of $X_t$ conditioned on $(V^{t-1}, Y^{t-1})$, induced by (1.12), is also expressed as a linear functional of $(\hat{S}_t, Y^{t-1})$, for $t = 1, \ldots, n$. The next theorem further shows that for each $t$, the dependence of $X_t$ on $Y^{t-1}$ is expressed in terms of $E \{ \hat{S}_t | Y^{t-1} \}$ for $t = 1, \ldots, n$, and this dependence gives rise to an equivalent sequential characterization of the Cover and Pombra $n$–FTFI capacity, $C_n^{fb}(\cdot)$.

**Theorem II.2.** Equivalent characterization of $n$–FTFI Capacity $C_n^{fb}(\cdot)$ for PO-SS Noise realizations Consider also the generalized Kalman-filter of Lemma II.1.

Define the conditional covariance and conditional mean of $\hat{S}_t$ given $Y^{t-1}$, by

$$
K_t \triangleq \text{cov} \left( \hat{S}_t, \hat{S}_t | Y^{t-1} \right) = E \left\{ \left( \hat{S}_t - \hat{S}_t \right) \left( \hat{S}_t - \hat{S}_t \right)^T \right\},
$$

$$
\hat{S}_t \triangleq E \left\{ \hat{S}_t | Y^{t-1} \right\}, \quad t = 2, \ldots, n,
$$

$$
\hat{S}_1 \triangleq \mu_{S_1}, \quad K_1 \triangleq 0.
$$

Then the following hold.

(a) An equivalent characterization of the Cover and Pombra $n$–FTFI capacity $C_n^{fb}(\cdot)$, defined by (1.3)–(1.7), is

$$
C_n^{fb}(\cdot) = \sup_{\mathbb{P}_n} \sum_{t=1}^n H(Y_t | Y^{t-1}) - H(V^n),
$$

where $(X^n, Y^n)$ is jointly Gaussian, $H(V^n)$ is the entropy of $V^n$, $\mathbb{P}_n$ is the innovations process of $V^n$, and

$$
Y_t = X_t + V_t, \quad t = 1, \ldots, n,
$$

$$
V_t = C_t \hat{S}_t + \hat{I}_t,
$$

$$
P_1(dY_1) = \int P_1(dY_1 | x_1, \hat{S}_1) P_1(dx_1 | \hat{S}_1) P_1(d\hat{S}_1),
$$

$$
P_t(dY_t | y^{t-1}) \in N(\mu_{S[p]}(\cdot), K_{[p]}(\cdot)),
$$

$$
\mu_{S[p]}(\cdot) \text{ is linear in } Y^{t-1} \text{ and } K_{[p]}(\cdot) \text{ is nonrandom},
$$

$$
P_t(dx_t | \hat{S}_t, y^{t-1}) \in N(\mu_{X|S[p]}(\cdot), K_{X|S[p]}(\cdot), y^{t-1}),
$$

$$
\mu_{X|S[p]}(\cdot) \text{ is linear in } (\hat{S}_t, Y^{t-1}) \text{ and nonrandom,}
$$

$$
\mathbb{P}_n(\cdot) \triangleq \mathbb{P}_n(\cdot) = \left\{ P_t(dx_t | \hat{S}_t, y^{t-1}), t = 1, \ldots, n : \frac{1}{n} \mathbb{E} \left\{ \left( \sum_{t=1}^n (X_t) \right)^2 \right\} \leq \kappa \right\}.
$$

(II.49)

(II.50)

(II.51)

(II.52)

(II.53)

(II.54)
The optimal jointly Gaussian process \((X^n, Y^n)\) of part (a) is represented, as a function of a sufficient statistic, by

\[
X_t = \Lambda_t \left( \hat{S}_t - \hat{S}_t \right) + Z_t, \quad t = 1, \ldots, n, \tag{II.55}
\]

\[
Z_t \in N(0, K_{Z_t}) \quad \text{independent of} \quad (X^{t-1}, Y^{t-1}, \hat{S}_t, \hat{S}_t, \hat{F}, \hat{Y}^{t-1}), \quad t = 1, \ldots, n,
\]

\[
\hat{I}_t \in N(0, K_{\hat{I}_t}) \quad \text{independent of} \quad (X^{t-1}, Y^{t-1}, \hat{S}_t, \hat{Y}^{t-1}), \quad t = 1, \ldots, n.
\]

\[
Y_t = \Lambda_t \left( \hat{S}_t - \hat{S}_t \right) + C_t \hat{S}_t + \hat{I}_t + Z_t,
\]

\[
\frac{1}{n} \mathbb{E} \left( \sum_{i=1}^{n} (X_i)^2 \right) = \frac{1}{n} \sum_{i=1}^{n} \left( \Lambda_t \Lambda_t^T + K_{\Lambda_t} \right).
\]

where \(\Lambda_t\) is nonrandom.

The conditional mean and covariance, \(\hat{S}_t\) and \(K_t\), are given by generalized Kalman-filter equations, as follows.

(i) \(\hat{S}_t\) satisfies the Kalman-filter recursion

\[
\hat{S}_{t+1} = A_t \hat{S}_t + F_t (\Sigma_t, K_t) \hat{I}_t, \quad \hat{S}_1 = \mu_{S_1},
\]

\[
F_t (\Sigma_t, K_t) \triangleq \left( A_t \Lambda_t + C_t \right)^T + M_t (\Sigma_t) K_t
\]

\[
\left( K_{t+1} + K_{Z_t} + (\Lambda_t + C_t) K_t (\Lambda_t + C_t)^T \right)^{-1}
\]

\[
I_t \triangleq Y_t - \mathbb{E} \left( Y_t | Y^{t-1} \right) = Y_t - C_t \hat{S}_t
\]

\[
= \left( \Lambda_t + C_t \right) \left( \hat{S}_t - \hat{S}_t \right) + \hat{I}_t + Z_t, \quad t = 1, \ldots, n,
\]

\[
I_t \in N(0, K_{I_t}), \quad t = 1, \ldots, n \quad \text{is an orthogonal innovations process, i.e.,} \quad I_t \text{ is independent of } I_s, \quad \text{for all } t \neq s
\]

\[
\text{and } I_t \text{ is independent of } Y_t^{-1}
\]

\[
K_{Y_t|Y^{t-1}} = K_{K_t} \triangleq \text{cov} (I_t, I_t)
\]

\[
= \left( \Lambda_t + C_t \right) K_t \left( \Lambda_t + C_t \right)^T + K_{\Lambda_t} + K_{Z_t},
\]

\[
K_{I_t} \text{ given by } II.35.
\]

(ii) \(K_t = \mathbb{E} \left( \hat{E}_t, \bar{E}_t \right)\) satisfies the generalized DRE

\[
K_{t+1} = A_t \Lambda_t \Lambda_t^T + M_t (\Sigma_t) K_t \left( M_t (\Sigma_t) \right)^T - \left( A_t \Lambda_t + C_t \right)^T
\]

\[
+ M_t (\Sigma_t) K_t \left( K_{Z_t} + K_{Z_t} + (\Lambda_t + C_t) K_t (\Lambda_t + C_t)^T \right)^{-1}
\]

\[
. \left( A_t \Lambda_t + C_t \right)^T M_t (\Sigma_t) K_t \right), \quad K_t \succeq 0,
\]

\[
t = 1, \ldots, n, \quad K_1 = 0.
\]

(c) An equivalent characterization of the \(n\)-FTFI capacity \(C_{\text{FTFI}}^b(\kappa)\), defined by \([13, 14, 10]\), using the sufficient statistics of part (b), is

\[
C_{\text{FTFI}}^b(\kappa) = \sup_{\Lambda_t, K_{Z_t}, K_t} \left\{ \frac{1}{n} \mathbb{E} \left( \sum_{i=1}^{n} (X_i)^2 \right) \leq \kappa \right\}
\]

\[
= \sup_{\Lambda_t, K_{Z_t}, K_t} \left\{ \frac{1}{2} \sum_{i=1}^{n} \log \left( \frac{\Lambda_t + C_t}{K_{I_t}} \left( \Lambda_t + C_t \right)^T + K_{\Lambda_t} + K_{Z_t} \right) \right\}.
\]

Proof. The derivation, although lengthy, is based on the preliminary calculations prior to the statement of the theorem (see [8, Section VI.D]).

The analysis of the per unit time limit \(C_{\text{FTFI}}^b(\kappa) \triangleq \lim_{n \to \infty} C_{\text{FTFI}}^b(\kappa)\) is carried out in [8].

Remark II.2. The characterization of \(n\)-FTFI capacity \(C_{\text{FTFI}}^b(\kappa)\) given by (II.65), involves the generalized matrix DRE \(K_t\) which is also a functional of the generalized matrix DRE \(\Sigma_t\) of the error covariance of the state \(S_t\) from the noise output \(V^n\). This feature does not appear in [2] and recent literature [2], [4]–[6], [18], because as explained in R4).

Further to the above remark, if the conditions below hold,

\(\Lambda.1\) The feedback code assumes knowledge of the initial state of the noise or the channel, \(S_1 = s\), at the encoder and the decoder, and

\(\Lambda.2\) the noise sequence \(V^{t-1}\) and initial state \(S_1 = s\) uniquely defines the noise state sequence \(S_t\) and vice-versa for \(t = 1, \ldots, n\),

then in Theorem II.2 \(X_t\) is reduced to \(X_t = \Lambda_t \left( S_t - \mathbb{E} \left( S_t | Y^{t-1}, S_1 \right) \right) + Z_t, t = 1, \ldots, n\), and all equations are simplified, precisely as in Yang, Kanvic and Tatikonda [7].

III. CONCLUSION

New equivalent sequential characterizations of the Cover and Pombra [1] “\(n\)-block” feedback capacity formulas are derived using time-domain methods, for additive Gaussian noise (AGN) channels driven by nonstationary Gaussian noise. The new feature of the equivalent characterizations are the representation of the optimal channel input process by a sufficient statistic and Gaussian orthogonal innovations process. The sequential characterizations of the \(n\)-block feedback capacity formula are expressed as a functional of two generalized matrix difference Riccati equations (DRE) of filtering theory of Gaussian systems.

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