Stringy geometry and topology of orbifolds

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1 Introduction

In 1985, Dixon, Harvey, Vafa and Witten considered string theory on orbifolds (arising as global quotients $X/G$ by a finite group $G$) [DHVW]. Although an orbifold is a singular space, orbifold string theory is surprisingly a smooth theory. Since then, orbifold string theory has become a rather important part of the landscape of string theory. A search in hep-th yields more than 200 papers whose title contains the word "orbifold". Although orbifold string theory has been around for a while, apparently it was poorly explored in geometry and topology. For last fifteen years, only a small piece of orbifold string theory concerning orbifold Euler-Hodge numbers has been studied in geometry and topology in relation to the McKay correspondence in algebraic geometry (see [Re], [B1]). The highlight of previous works was Batyrev’s proof that orbifold Hodge numbers of a global quotient $X/G$ for a finite subgroup $G \subset SL(n, \mathbb{C})$ are the same as Hodge numbers of its crepant resolution. We shall see later that this is only a special case of many "Orbifold string theory conjectures".

One of the reasons for such a slow development of the mathematical aspect of orbifold string theory is poor communication between mathematicians and physicists. The situation has been vastly improved in the last few years due to the increasing number of mathematicians who have been making a serious effort to understand physics. With our improved understanding of the physical ideas behind orbifold string theory, we are starting to be able to explore the full implications of orbifold string theory. In the last year, a series of work [CR1, CR2], [R], [AR] have been carried out on this direction.

Even with a superficial understanding of orbifold string theory, it is obvious that the mathematics surrounding orbifold string theory must be striking. In fact, it has motivated so much new mathematics unique to orbifolds. We believe that there is an emerging "stringy" topology and geometry of orbifolds. The core of this new geometry and topology is the concept of twisted sectors. Roughly speaking, the consistency of orbifold string theory requires that the string Hilbert space has to contain factors called twisted sectors. Twisted sectors can be viewed as the contribution from singularities. All other quantities such as correlation functions have to contain the contributions from the twisted sectors. In another words, the ordinary topology of orbifolds is a WRONG theory. The correct one must incorporate twisted sectors. Furthermore, orbifold string theory has a certain internal freedom (discrete torsion) [V]. Discrete torsion will allow us to twist orbifold string theory [VW]. These are the most important new conceptual ingredients in orbifold string theory. We will emphasis them in our mathematical construction as well.

Once we overcome these basic conceptual hurdles, we can explore the implications of the inclusion of twisted sectors and discrete torsion in all aspect of topology and geometry. This is what I would like call "Stringy geometry and topology of orbifold". Right now, this new subject is very much in its infancy. Currently, most of our motivation comes from physics. As time goes on, I expect that more mathematical motivations will emerge.

In this article, we will survey the new developments on this subject. For a new subject, it is common that there are more problems and speculations than the mathematics we can actually prove. This is also the case for the present subject. Therefore, we will also spend considerable time in talking about problems and conjectures.

One of the major problems of this subject is the lack of references. In section 2, we will give a self-contained introduction to orbifolds. Furthermore, we will introduce the key technical concept of good map.

In section 3, I will carry out the construction of orbifold cohomology [CR1], [R]. Orbifold string theory has an internal freedom (discrete torsion), which will allow us to twist orbifold cohomology. However, discrete torsion is not enough account for all the known examples. In [R], a more general
The notion of inner local system was introduced. We shall construct our orbifold cohomology in this general setting.

In section 4, we will change our point of view to K-theory and develop orbifold K-theory. Again, we will incorporate discrete torsion in our theory. Here, the mathematical motivation is projective representations. We will give a detailed description of this approach and establish the additive isomorphism between orbifold K-theory and orbifold cohomology. A surprising byproduct of orbifold K-theory is the new multiplicative structure between DIFFERENT twisted orbifold cohomologies, which is impossible to observe from the cohomological point of view. Right now, this new multiplicative structure is very intriguing. It is certainly worth more investigation.

In section 5, we will shift from classical theory to quantum theory—orbifold quantum cohomology \([\text{CR}2]\). We will introduce the notion of orbifold stable map, which is a nontrivial generalization of stable maps. Then, we will study various properties of orbifold stable maps and construct orbifold quantum cohomology.

In section 6, we will focus on some of the main predictions from orbifold string theory. The main idea is that orbifold theory should predict the ordinary theory of its desingularizations. A desingularization \(Y\) of a Gorenstein orbifold \(X\) is obtained by deforming \(X\) and then taking a crepant resolution. Two extreme cases are the ones obtained by either deformation or resolution alone. There is a body of conjectures about their relations depending on the particular setting. We call all of them by the term "K or Q-orbifold string theory conjecture". Here, the letter K or Q is indicating the particular setting we are talking about. Furthermore, it can be combined further with author’s quantum minimal model conjecture to extend the orbifold string theory conjecture and the quantum minimal model conjecture in a natural way. Other problems will also be discussed.

Finally, a historical note is in order. So far, the physical construction of orbifold string theory has only been carried out on global quotients of the form \(X/G\). It is not clear how to do it over general orbifolds. However, most of the important examples such as Calabi-Yau orbifolds are not global quotients. In dimension three every Calabi-Yau orbifold admits a crepant resolution. Hence, we can restrict ourselves to smooth models. In higher dimension, this is no longer true. If we want to extend wonderful theories such as mirror symmetry to higher dimension, we are forced to work over singular manifolds. Therefore, it is necessary to be able to construct orbifold string theory over general orbifolds. It is our hope that a better mathematical understanding of general orbifolds will help the physical construction as well. Throughout the paper, we will put an emphasis on developing the theory over general orbifolds.

Even in the case of global quotients, the best understood part of orbifold string theory is orbifold conformal field theory. It is not clear how to do new geometry and topology except the orbifold Hodge number \([\mathbb{Z}]\). However, there are many papers about McKay correspondences for global quotients. One can find relevant references in \([\text{Re}]\). An equivalent formulation of orbifold stable maps was studied in algebraic geometry independently by D. Abramovich and A. Vistoli \([\text{AV}]\).

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2 Basics

One of the difficult of this subject is that there are few references. A detailed description of basic material on orbifold has been given in the appendix of \([\text{CR}3]\), which includes the crucial new concept of good map. Here we review the basic construction. Furthermore, we take a slightly more general definition of orbifold, which corresponds to a smooth Deligne-Mumford stack. It seems to be a
convenient and natural category even if we work with a more restrictive definition, which we call reduced orbifold.

### 2.1 Orbifolds

Primary examples of orbifold are quotients space of smooth manifolds by a smooth finite group action. Here we imagine that the quotient space is uniformized (or modeled) by the manifold with the finite group action. We do not require the group action to be effective. If it happens to be effective, we call it a reduced orbifold. It is clear that we can canonically associate a reduced orbifold to an orbifold by redefining the group. Furthermore, if an element acts nontrivially, we require that the fixed-point set is of codimension at least two. This is the case, for example, when the action is orientation-preserving. This requirement has the consequence that the non-fixed-point set is locally connected.

Let $U$ be a connected topological space, $V$ be a connected $n$-dimensional smooth manifold and $G$ be a finite group acting on $V$ smoothly. An $n$-dimensional uniformizing system of $U$ is a triple $(V,G,\pi)$, where $\pi : V \to U$ is a continuous map inducing a homeomorphism between $V/G$ and $U$. Two uniformizing systems $(V_i,G_i,\pi_i), \ i = 1, 2$, are isomorphic if there is a diffeomorphism $\phi : V_1 \to V_2$ and an isomorphism $\lambda : G_1 \to G_2$ such that $\phi$ is $\lambda$-equivariant, and $\pi_2 \circ \phi = \pi_1$. It is easily seen that if $(\phi, \lambda)$ is an automorphism of $(V,G,\pi)$, then there is a $g \in G$ such that $\phi(x) = g \cdot x$ and $\lambda(a) = g \cdot a \cdot g^{-1}$ for any $x \in V$ and $a \in G$.

Let $i : U' \to U$ be a connected open subset of $U$, and $(V',G',\pi')$ be a uniformizing system of $U'$. We say that $(V',G',\pi')$ is induced from $(V,G,\pi)$ if there is a monomorphism $\lambda : G' \to G$ and a $\lambda$-equivariant open embedding $\phi : V' \to V$ such that $i \circ \pi' = \pi \circ \phi$. We follow Satake $[S]$ and call $(\phi, \lambda) : (V',G',\pi') \to (V,G,\pi)$ an injection. Two injections $(\phi_i, \lambda_i) : (V'_i,G'_i,\pi'_i) \to (V,G,\pi)$, $i = 1, 2$, are isomorphic if there is an isomorphism $(\psi, \tau)$ between $(V'_1,G'_1,\pi'_1)$ and $(V'_2,G'_2,\pi'_2)$, and an automorphism $(\psi, \tau)$ of $(V,G,\pi)$ such that $(\psi, \tau) \circ (\phi_1, \lambda_1) = (\phi_2, \lambda_2) \circ (\psi, \tau)$.

**Lemma 2.1.1:** Let $(V,G,\pi)$ be a uniformizing system of $U$. For any connected open subset $U'$ of $U$, $(V,G,\pi)$ induces a unique isomorphism class of uniformizing systems of $U'$.

**Proof:**

**Existence:** Consider the preimage $\pi^{-1}(U')$ in $V$. $G$ acts as permutations on the set of connected components of $\pi^{-1}(U')$. Let $V'$ be one of the connected components of $\pi^{-1}(U')$, $G'$ be the subgroup of $G$ which fixes the component $V'$ and $\pi' = \pi|_{V'}$. Then $(V',G',\pi')$ is an induced uniformizing system of $U'$.

**Uniqueness:** First of all, different choices of the connected components of $\pi^{-1}(U')$ induce isomorphic uniformizing systems. Secondly, let $(V'_i,G'_i,\pi'_i)$ be any induced uniformizing system of $U'$ and $(\psi, \tau)$ be the injection into $(V,G,\pi)$. We will show that $(\psi, \tau)$ induces an isomorphism between $(V'_1,G'_1,\pi'_1)$ and an induced uniformizing system given by a connected component of $\pi^{-1}(U')$. Suppose $\psi(V'_i)$ lies in the connected component $V'$. We can show that $\psi(V'_i)$ is closed in $V'$. Let $\psi(x_n) \to y_0$ in $V'$, $x_n \in V'_i$, then there exists a $z_0 \in V'_i$ such that $\pi'_1(z_0) = \pi(y_0)$, and $z_n \in V'_1$ such that $z_n \to z_0, \pi'_1(z_n) = \pi(\psi(x_n)) = \pi'_1(x_n)$. So there exist $a_n \in G'_1$ such that $a_n(z_n) = x_n$. Since $G'_1$ is finite, it follows that for large $n$, $a_n = a$ is a constant. So $x_n \to a(z_0)$ in $V'_i$ and $y_0 = \psi(a(z_0))$, i.e., $\psi(V'_i)$ is also open in $V'$. So $\psi$ induces a diffeomorphism between $V'_i$ and $V'$. From this we can easily see that $(V'_1,G'_1,\pi'_1)$ and $(V',G',\pi')$ are isomorphic. □

Let $U$ be a connected and locally connected topological space. For any point $p \in U$, we can define the germ of uniformizing systems at $p$ in the following sense. Let $(V_1,G_1,\pi_1)$ and $(V_2,G_2,\pi_2)$ be uniformizing systems of neighborhoods $U_1$ and $U_2$ of $p$. We say that $(V_1,G_1,\pi_1)$ and $(V_2,G_2,\pi_2)$
are equivalent at $p$ if they induce isomorphic uniformizing systems for a neighborhood $U_3$ of $p$.

**Definition 2.1.2:** Let $X$ be a Hausdorff, second countable topological space. An $n$-dimensional orbifold structure on $X$ is given by the following data: for any point $p \in X$, there is a neighborhood $U_p$ and an $n$-dimensional uniformizing system $(V_p, G_p, \pi_p)$ of $U_p$ such that for any point $q \in U_p$, $(V_p, G_p, \pi_p)$ and $(V_q, G_q, \pi_q)$ are equivalent at $q$ (i.e., define the same germ at $q$). The germ of orbifold structures on $X$ is defined in the following sense: two orbifold structures $(V_p, G_p, \pi_p) : p \in X$ and $(V'_p, G'_p, \pi'_p) : p \in X$ are equivalent if for any $p \in X$, $(V_p, G_p, \pi_p)$ and $(V'_p, G'_p, \pi'_p)$ are equivalent at $p$. With a given germ of orbifold structures on it, $X$ is called an orbifold. We call each $U_p$ a uniformized neighborhood of $p$, and $(V_p, G_p, \pi_p)$ a chart at $p$. An open subset $U$ of $X$ is called a uniformized open set if it is uniformized by $(V, G, \pi)$ such that for each $p \in U$, $(V_p, G_p, \pi_p)$ defines the same germ as $(V_p, G_p, \pi_p)$ at $p$. A point $p \in X$ is called regular or smooth if $G_p$ is trivial; otherwise, it is called singular. The set of smooth points is denoted by $X_{\text{reg}}$, and the set of singular points is denoted by $X_{\Sigma X}$. An orbifold $X$ is called a reduced orbifold if $G_p$ acts effectively on $V_p$. In this case, we can choose $G_p$ to be a subgroup of $O(n)$.

**Remark 2.1.3:** It is obvious that we can associate canonically a reduced orbifold structure $X_{\text{red}}$ to each orbifold $X$ by redefining the local group $G'_p$ to be the quotient of $G_p$ divided by the subgroup fixing $V_p$.

**Remark 2.1.4:** There is a notion of orbifold with boundary, in which we allow the uniformizing systems to be smooth manifolds with boundary, with a finite group action preserving the boundary. If $X$ is an orbifold with boundary, then it is easily seen that the boundary $\partial X$ inherits an orbifold structure from $X$ and becomes an orbifold.

**Example 2.1.5:** Let’s consider the 2-dimensional sphere $S^2$. Let $D_s$, $D_n$ be open disc neighborhoods of the south pole and the north pole such that $S^2 = D_s \cup D_n$. Let $D_s$ be uniformized by $(\hat{D}_s, Z_2, \pi_s)$, and $D_n$ be uniformized by $(\hat{D}_n, Z_3, \pi_n)$ where $Z_2$, $Z_3$ act on $\hat{D}_s$ and $\hat{D}_n$ by rotations. For any point in $S^2$ other than the south pole and the north pole, we take a chart at it induced by either $(\hat{D}_s, Z_2, \pi_s)$ or $(\hat{D}_n, Z_3, \pi_n)$. It is easily seen that this defines a 2-dimensional orbifold structure on $S^2$. Note that as an open subset of both $D_s$ and $D_n$, $D_s \cap D_n$ has non-isomorphic induced uniformizing systems from $(\hat{D}_s, Z_2, \pi_s)$ and $(\hat{D}_n, Z_3, \pi_n)$, although they define the same germ at each point in $D_s \cap D_n$. This also shows that although both $D_s$ and $D_n$ are uniformized, their union $S^2$ can not be uniformized, therefore is not a global quotient.

The notion of orbifold was first introduced by Satake in [5], where a different name, $V$-manifold, was used. Satake’s $V$-manifold corresponds to reduced orbifold in our case. In [5], an orbifold structure on a topological space $X$ is given by an open cover $\mathcal{U}$ of $X$ satisfying the following conditions:

- (2.1.1a) Each element $U$ in $\mathcal{U}$ is uniformized, say by $(V, G, \pi)$.
- (2.1.1b) If $U' \subset U$, then there is a collection of injections $(V', G', \pi') \to (V, G, \pi)$.
- (2.1.1c) For any point $p \in U_1 \cap U_2$, $U_1, U_2 \in \mathcal{U}$, there is a $U_3 \in \mathcal{U}$ such that $p \in U_3 \subset U_1 \cap U_2$.

One can show that our definition is equivalent to Satake’s.

Next we consider a class of continuous maps between two orbifolds which carry an additional structure of differentiability with respect to the orbifold structures. Let $U$ be uniformized by $(V, G, \pi)$ and $U'$ by $(V', G', \pi')$, and $f : U \to U'$ be a continuous map. A $C^l$ lifting, $0 \leq l \leq \infty$, of $f$ is a $C^l$ map $\bar{f} : V \to V'$ and a homomorphism $\lambda : G \to G'$ such that $\pi' \circ \bar{f} = f \circ \pi$, and $\lambda(g) \cdot \bar{f}(x) = \bar{f}(g \cdot x)$ for any $x \in V$. Two liftings $\bar{f}_i : (V_i, G_i, \pi_i) \to (V'_i, G'_i, \pi'_i)$, $i = 1, 2$, are isomorphic if there
exist isomorphisms \((\phi, \tau) : (V_1, G_1, \pi_1) \to (V_2, G_2, \pi_2)\) and \((\phi', \tau') : (V_1', G_1', \pi_1') \to (V_2', G_2', \pi_2')\) such that \(\phi' \circ \tilde{f}_1 = \tilde{f}_2 \circ \phi\). Let \(p \in U\) for any uniformized neighborhood \(U_{f(p)}\) of \(p\) and uniformized neighborhood \(U_{f(p)}\) of \(f(p)\) such that \(f(U_p) \subset U_{f(p)}\), a lifting \(\tilde{f}\) of \(f\) will induce a lifting \(\tilde{f}_p\) for \(f|_{U_p} : U_p \to U_{f(p)}\) as follows: For any injection \((\phi, \tau) : (V_p, G_p, \pi_p) \to (V, G, \pi)\), consider the map \(\tilde{f} \circ \phi : V_p \to V'\), and observe that \(\pi' \circ \tilde{f} \circ \phi(V_p) \subset U_{f(p)}\) implies \(\tilde{f} \circ \phi(V_p) \subset U_{f(p)}\). Therefore there is an injection \((\phi', \tau') : (V(f_p), G_f(p), \pi_f(p)) \to (V', G', \pi')\) such that \(\tilde{f} \circ \phi(V_p) \subset \phi'(V_f(p))\). We define \(\tilde{f}_p = (\phi')^{-1} \circ \tilde{f} \circ \phi\). In this way we obtain a lifting \(\tilde{f}_p : (V_p, G_p, \pi_p) \to (V(f_p), G_f(p), \pi_f(p))\) for \(f|_{U_p} : U_p \to U_{f(p)}\). We can verify that different choices give isomorphic liftings. We define the germ of liftings as follows: two liftings are equivalent at \(p\) if they induce isomorphic liftings on a smaller neighborhood of \(p\).

Now consider orbifolds \(X\) and \(X'\) and a continuous map \(f : X \to X'\). A lifting of \(f\) consists of the following data: for any point \(p \in X\), there exist charts \((V_p, G_p, \pi_p)\) and \((V_{f(p)}, G_{f(p)}, \pi_{f(p)})\) at \(f(p)\) and a lifting \(\tilde{f}_p\) of \(f|_{U_p} : U_p \to U_{f(p)}\) such that for any \(q \in \pi_p(V_p)\), \(\tilde{f}_p(q)\) induce the same germ of liftings of \(f\) at \(q\). We can define the germ of liftings in the sense that two liftings of \(f\) \(\tilde{f}_p : (V_{f,p}, G_{f,p}, \pi_{f,p}) \to (V_{f(p),i}, G_{f(p),i}, \pi_{f(p),i}) : p \in X\), \(i = 1, 2\), are equivalent if for each \(p \in X\), \(\tilde{f}_{p,i} = 1, 2\) induce the same germ of liftings of \(f\) at \(p\).

**Definition 2.1.6:** A \(C^l\) map \((0 \leq l \leq \infty)\) between orbifolds \(X\) and \(X'\) is a germ of \(C^l\) liftings of a continuous map between \(X\) and \(X'\). We denote by \(\tilde{f}\) a \(C^l\) map which is a germ of liftings of a continuous map \(f\).

A sequence of \(C^l\) maps \(\tilde{f}_n\) is said to converge to a \(C^l\) map \(\tilde{f}_0\) in the \(C^l\) topology if there exists a sequence of liftings \(\tilde{f}_{n,p} : (V_{f,p}, G_{f,p}, \pi_{f,p}) \to (V_{f_n(p), G_{f_n(p)}, \pi_{f_n(p)})\) defining the germs \(\tilde{f}_n\) such that for each \(p \in X\) there exists a chart \((V_{f_0(p)}, G_{f_0(p)}, \pi_{f_0(p)})\) and an integer \(n(p) > 0\) with the following property: for each \(n \geq n(p)\), there is an injection \((\psi_{p,n}, \tau_{p,n}) : (V_{f_n(p), G_{f_n(p)}, \pi_{f_n(p)}) \to (V_{f_0(p), G_{f_0(p)}, \pi_{f_0(p)})\) such that \(\psi_{p,n} \circ \tilde{f}_{p,n}\) converges in \(C^l\) to \(\tilde{f}_{p,0}\) which defines the germ \(\tilde{f}_0\).

**Example 2.1.7a:** The real line \(\mathbb{R}\) as a smooth manifold is trivially an orbifold. A \(C^l\) map from an orbifold \(X\) to \(\mathbb{R}\) is called a \(C^l\) function on \(X\). The set of all \(C^l\) functions on \(X\) is denoted by \(C^l(X)\).

**Example 2.1.7b:** Let \(X = \mathbb{R} \times \mathbb{C}\), and be given an orbifold structure by \((\mathbb{R} \times \mathbb{C}, \mathbb{Z}_4, \pi)\) where \(\mathbb{Z}_4\) acts only on the factor \(\mathbb{C}\) by multiplication by \(\sqrt{-1}^t\). Define \(C^1\) maps \(\tilde{f}_1 : \mathbb{R} \to (\mathbb{R} \times \mathbb{C}, \mathbb{Z}_4, \pi)\) by \(t \to (t, t^2)\) and \(\tilde{f}_2 : \mathbb{R} \to (\mathbb{R} \times \mathbb{C}, \mathbb{Z}_4, \pi)\) by \(t \to (t, t^2)\) for \(t \leq 0\) and \((t, \sqrt{-1}t^2)\) for \(t > 0\). Then \(\tilde{f}_1, \tilde{f}_2\) induce the same continuous map \(f : \mathbb{R} \to X\), but they are not isomorphic as \(C^1\) maps.

Next we describe the notion of orbifold vector bundle which corresponds to the notion of smooth vector bundle. When there is no confusion, we simply call it a vector bundle. We begin with local uniformizing systems for vector bundles. Given a uniformized topological space \(U\) and a topological space \(E\) with a surjective continuous map \(pr : E \to U\), a uniformizing system of rank \(k\) vector bundle for \(E\) over \(U\) consists of the following data:

- A uniformizing system \((V, G, \pi)\) of \(U\).
- A uniformizing system \((V \times \mathbb{R}^k, G, \pi)\) for \(E\). The action of \(G\) on \(V \times \mathbb{R}^k\) is an extension of the action of \(G\) on \(V\) given by \(g(x, v) = (gx, \rho(x, g)v)\), where \(\rho : V \times G \to Aut(\mathbb{R}^k)\) is a smooth map satisfying:
  \[
  \rho(gx, h) \circ \rho(x, g) = \rho(x, h \circ g), \quad g, h \in G, x \in V.
  \]
- The natural projection map \(\tilde{pr} : V \times \mathbb{R}^k \to V\) satisfies \(\pi \circ \tilde{pr} = pr \circ \tilde{\pi}\).
We can similarly define isomorphisms between uniformizing systems of vector bundle for $E$ over $U$. The only additional requirement is that the diffeomorphisms between $V \times \mathbb{R}^k$ are linear on each fiber of $\tilde{pr} : V \times \mathbb{R}^k \to V$. Moreover, for each connected open subset $U'$ of $U$, we can similarly prove that there is a unique isomorphism class of induced uniformizing systems of vector bundle for $E' = pr^{-1}(U')$ over $U'$. The germ of uniformizing systems of vector bundle at a point $p \in U$ can also be similarly defined.

**Definition 2.1.8:** Let $X$ be an orbifold and $E$ be a topological space with a surjective continuous map $pr : E \to X$. A rank $k$ vector bundle structure on $E$ over $X$ consists of the following data: For each point $p \in X$, there is a uniformized neighborhood $U_p$ and a uniformizing system of rank $k$ vector bundle for $pr^{-1}(U_p)$ over $U_p$ such that for any $q \in U_p$, the uniformizing systems of vector bundle over $U_p$ and $U_q$ define the same germ at $q$. The germ of rank $k$ vector bundle structures on $E$ over $X$ can be similarly defined. The topological space $E$ with a given germ of vector bundle structures becomes an orbifold and is called a vector bundle over $X$. Each chart $(V_p \times \mathbb{R}^k, G_p, \tilde{\pi}_p)$ is called a local trivialization of $E$. At each point $p \in X$, the fiber $E_p = pr^{-1}(p)$ is isomorphic to $\mathbb{R}^k/G_p$. It contains a linear subspace $E^p$ of fixed points of $G_p$. Two vector bundles $pr_1 : E_1 \to X$ and $pr_2 : E_2 \to X$ are isomorphic if there is a $C^\infty$ map $\psi : E_1 \to E_2$ given by $\psi : (V_1, p) \times \mathbb{R}^k, G_{1,p}, \tilde{\pi}_{1,p}) \to (V_2, p) \times \mathbb{R}^k, G_{2,p}, \tilde{\pi}_{2,p})$ which induces an isomorphism between $(V_1, G_1, \pi_{1,p})$ and $(V_2, G_2, \pi_{2,p})$, and is a linear isomorphism between the fibers of $\tilde{pr}_1$ and $\tilde{pr}_2$. By replacing $\mathbb{R}^k$ with $\mathbb{C}^k$, we have the definition of complex vector bundle.

**Remark 2.1.10:** There is a notion of vector bundle over an orbifold with boundary. One can easily verify that if $pr : E \to X$ is a vector bundle over an orbifold with boundary $X$, then the restriction to the boundary $\partial X$, $E_{\partial X} = pr^{-1}(\partial X)$, is a vector bundle over $\partial X$.

**Remark 2.1.11:** One can define orbifold fiber bundle with fiber a general space in the same manner.

A $C^l$ map $\tilde{s}$ from $X$ to a vector bundle $pr : E \to X$ is called a $C^l$ section if locally $\tilde{s}$ is given by $\tilde{s}_p : V_p \to V_p \times \mathbb{R}^k$ where $\tilde{s}_p$ is $G_p$-equivariant and $\tilde{pr} \circ \tilde{s}_p = Id$ on $V_p$. We observe that

- For each point $p$, $s(p)$ lies in $E^p$, the linear subspace of fixed points of $G_p$.
- The space of all $C^l$ sections of $E$, denoted by $C^l(E)$, has a structure of vector space over $\mathbb{R}$ (or $\mathbb{C}$) as well as a $C^l(X)$-module structure.
- The $C^l$ sections $\tilde{s}$ are in 1 : 1 correspondence with the underlying continuous maps $s$.

**Remark 2.1.12:** If $E \to X$ is an orbifold vector bundle over $X$ which does not introduce an orbifold bundle over the associated reduced orbifold, then $E$ has no nonzero section.

Orbifold vector bundles are more conveniently described by transition maps (see [S]). More precisely, an orbifold vector bundle over an orbifold $X$ can be constructed from the following data: A compatible cover $\mathcal{U}$ of $X$ such that for any injection $i : (V', G', \pi') \to (V, G, \pi)$, there is a smooth map $g_i : V' \to Aut(\mathbb{R}^k)$ giving an open embedding $V' \times \mathbb{R}^k \to V \times \mathbb{R}^k$ by $(x, v) \to (i(x), g_i(x)v)$, and for any composition of injections $j \circ i$, we have

$$g_{j \circ i}(x) = g_j(i(x)) \circ g_i(x), \forall x \in V.$$

Two collections of maps $g^{(1)}$ and $g^{(2)}$ define isomorphic bundles if there are maps $\delta_V : V \to Aut(\mathbb{R}^k)$ such that for any injection $i : (V', G', \pi') \to (V, G, \pi)$, we have

$$g^{(2)}_i(x) = \delta_V(i(x)) \circ g^{(1)}_i(x) \circ (\delta_{V'}(x))^{-1}, \forall x \in V'.$$
Since (2.1.2) behaves naturally under constructions of vector spaces such as tensor product, exterior product, etc. we can define these constructions for vector bundles.

**Example 2.1.3:** For an orbifold $X$, the tangent bundle $TX$ can be constructed because the differential of any injection satisfies (2.1.2). Likewise, we define the cotangent bundle $T^*X$, and the bundles of exterior power or tensor product. All the differential geometry such as de Rham theory, connections, curvature and characteristic classes extends to orbifold vector bundles. Moreover, the de Rham cohomology of an orbifold is isomorphic to the de Rham cohomology of its associated reduced orbifold, which is isomorphic to the singular cohomology of the underlying topological space. Observe also that if $\omega$ is a differential form on $X'$ and $\tilde{f}: X \to X'$ is a $C^\infty$ map, then there is a pull-back form $\tilde{f}^*\omega$ on $X$.

### 2.2 Pull-back bundles and good maps

Let $pr: E \to Y$ be a vector bundle over a topological space $Y$. Then for any continuous map $f: X \to Y$ from a topological space $X$, the pull-back vector bundle $f^*E$ over $X$ is well-defined. However, this is no longer the case for orbifold vector bundles. Let $pr: E \to X'$ be an orbifold vector bundle over $X'$, and $\tilde{f}: X \to X'$ a $C^\infty$ map. By a pull-back bundle of $E$ over $X$ via $\tilde{f}$ we mean an orbifold vector bundle $\pi: E' \to X$ together with a $C^\infty$ map $\tilde{f}: E' \to E$ such that each local lifting of $\tilde{f}$ is an isomorphism restricted to each fiber, and $\tilde{f}$ covers the $C^\infty$ map $\tilde{f}$ between the bases.

Let $\tilde{f}: X \to X'$ be a $C^\infty$ map between orbifolds $X$ and $X'$ whose underlying continuous map is denoted by $f$. Suppose there is a compatible cover $U$ of $X$, and a collection of open subsets $U'$ of $X'$ satisfying (2.1.1a–c) and the following condition: There is a 1:1 correspondence between elements of $U$ and $U'$, say $U \leftrightarrow U'$, such that $f(U) \subset U'$, and an inclusion $U_2 \subset U_1$ implies an inclusion $U_2' \subset U_1'$. Moreover, there is a collection of local $C^\infty$ liftings $\{\tilde{f}_{UV}\}$ of $f$, where $\tilde{f}_{UV}: (V, G, \pi) \to (V', G', \pi')$ satisfies the following condition: each injection $i: (V_2, G_2, \pi_2) \to (V_1, G_1, \pi_1)$ is assigned an injection $\lambda(i): (V_2^U, G_2^U, \pi_2^U) \to (V_1^U, G_1^U, \pi_1^U)$ such that $\tilde{f}_{UV1}' \circ i = \lambda(i) \circ \tilde{f}_{UV2}'$, and for any composition of injections $j \circ i$, the following compatibility condition holds:

$$
(2.2.1) \quad \lambda(j \circ i) = \lambda(j) \circ \lambda(i).
$$

Observe that when the injection $i: (V, G, \pi) \to (V, G, \pi)$ is an automorphism of $(V, G, \pi)$, the assignment of $\lambda(i): (V', G', \pi') \to (V', G', \pi')$ to $i$ satisfies (2.2.1) is equivalent to a homomorphism $\lambda_{UV}: G \to G'$. We call $\lambda_{UV}: G \to G'$ the group homomorphism of $\{\tilde{f}_{UV}, \lambda\}$ on $U$. Such a collection of maps clearly defines a $C^\infty$ lifting of the continuous map $f$. If it is in the same germ as $\tilde{f}$, we call $\{\tilde{f}_{UV}, \lambda\}$ a compatible system of $\tilde{f}$.

**Definition 2.2.1:** A $C^\infty$ map is called good if it admits a compatible system.

**Example 2.2.2:** There can be essentially different compatible systems of the same $C^\infty$ map, as shown in the following example: Let $X = C \times C/G$ where $G = Z_2 \oplus Z_2$ acting on $C \times C$ in the standard way. For the $C^1$ map $\tilde{f}: (C, Z_2) \to (C \times C, G)$ defined by the inclusion of $C \times \{0\}$, there are two compatible systems $(\tilde{f}, \lambda_1): (C, Z_2) \to (C \times C, G)$, $i = 1, 2$, for $\lambda_1(1) = (1, 0)$ and $\lambda_2(1) = (1, 1)$, which are apparently different.

**Lemma 2.2.3:** Let $pr: E \to X'$ be an orbifold vector bundle over $X'$. For any $C^\infty$ compatible system $\xi = \{\tilde{f}_{UV}, \lambda\}$ of a good $C^\infty$ map $\tilde{f}: X \to X'$, there is a canonically constructed pull-back bundle of $E$ over $\tilde{f}$: a bundle $pr: E'_\xi \to X$ together with a $C^\infty$ map $\tilde{f}_\xi: E'_\xi \to E$ covering $\tilde{f}$. Let $c$ be a universal characteristic class defined by the Chern-Weil construction; then $\tilde{f}_\xi^*(c(E)) = c(E'_\xi)$. 


Remark 2.2.9: A homomorphism of the said isomorphism class of compatible systems at the gacy class of homomorphisms. We call such a conjugacy class of group homomorphisms the group orbifold ˜

Definition 2.2.6: An is isomorphic to the quotient bundle over .

Definition 2.2.4: It is easily seen that for each continuous map ˜

Definition 2.2.5: It is easily seen that for each p ∈ X, a compatible system determines a group homomorphism Gp → Gf(p), and an isomorphism class of compatible systems determines a conjugacy class of homomorphisms. We call such a conjugacy class of group homomorphisms the group homomorphism of the said isomorphism class of compatible systems at p.

Following is a very important class of examples of good maps.

Definition 2.2.6: A C∞ map ˜f : X → X′ between orbifolds is called regular if the underlying continuous map f has the following property: ˜f−1(X′reg) is an open dense and connected subset of X.

The following important lemma is proved by Chen-Ruan [CR3].

Lemma 2.2.7: If ˜f is regular, then ˜f is the unique germ of C∞ liftings of f. Moreover, ˜f is good with a unique isomorphism class of compatible systems.

Remark 2.2.8: Here are some examples of regular C∞ maps. Let E be a bundle over a reduced orbifold X. Then any C∞ section s of E, as a C∞ map X → E, is regular, since s(p) is in SE only if p is in ΣX, which is of codimension at least two. Another example: Let E∗ be a pull-back bundle over X of the tangent bundle of TX′ with the C∞ map ˜f : E∗ → TX′; then ˜f is a regular map, since if ˜f(p, v) is in ΣTX′, then f(p) is in ΣX′ and v is in a subset of (TX′)f(p) of codimension at least two.

Remark 2.2.9: Here is another class of regular C∞ maps. A C∞ map ˜f : X → X′ is called a C∞ embedding if each local lifting ˜fp : (Vp, Gp, πp) → (Vf(p), Gf(p), πf(p)) is a λp-equivariant embedding for some isomorphism λp : Gp → Gf(p). It is easily seen that f−1(ΣX′) = ΣX, so that ˜f is regular. For a C∞ embedding ˜f : X → X′, the normal bundle of f(X) in X′ as a bundle is well-defined, and is isomorphic to the quotient bundle (TX′)∗/TX.

Example 2.2.10: Let P1 = {[z0, z1]} be the 1-dimensional complex projective space. We define a Z2 action on it by x · [z0, z1] = [xz0, z1]. Let X = P1/Z2 be the orbifold as quotient space. Similarly, let P2 = {[z0, z1, z2]} be the 2-dimensional complex projective space, with a Z2 ⊕ Z2 action on it, given by (x, y) · [z0, z1, z2] = [xz0, yz1, z2]. Let X′ = P2/(Z2 ⊕ Z2) be the orbifold as quotient space. We consider two sequences of C∞ maps (actually they are holomorphic) ˜fn, ˜gn : X → X′ defined by ˜fn( [z0, z1] ) = [z0, n−1 z1, z1] and ˜gn( [z0, z1] ) = [z0, n−1 z0, z1]. It is easily seen that both sequences consist of regular maps, so they are good maps. As n → ∞, both sequences converge. Let ˜f = lim ˜fn and ˜g = lim ˜gn. Then both ˜f and ˜g are good maps (as we shall see), and ˜f = ˜g as C∞ maps, but
stands for the fixed-point set of \( \tilde{\mathcal{V}} \) where \( V \).

Lemma 3.1.1: Let \( \tilde{f}, \tilde{g} \) be two good \( C^\infty \) maps; then the composition \( \tilde{g} \circ \tilde{f} \) is also a good \( C^\infty \) map, and any isomorphism class of compatible systems of \( \tilde{f} \) and \( \tilde{g} \) determines a unique isomorphism class of compatible systems for the composition \( \tilde{g} \circ \tilde{f} \).

Now consider a good \( C^\infty \) map \( \tilde{f} : X \to X' \) with an isomorphism class of compatible systems \( \xi \). Then we have an isomorphism class of pull-back bundles \( (TX')^\bullet_\xi \) over \( X \) and the good \( C^\infty \) map \( \tilde{f}_\xi : (TX')^\bullet_\xi \to TX' \). For any \( C^\infty \) section \( s \) of \( (TX')^\bullet_\xi \), we take the composition \( \tilde{f}_{\xi,s} = \text{Exp} \circ \tilde{f}_\xi \circ \tilde{s} \) from \( X \) into \( X' \). Then \( \tilde{f}_{\xi,s} \) is a good \( C^\infty \) map with an isomorphism class of compatible systems determined by \( \xi \). A natural question is: given a good map \( \tilde{g} \) nearby \( \tilde{f} \) with an isomorphism class of compatible systems, is there an isomorphism class of compatible systems \( \xi \) of \( \tilde{f} \), and a \( C^\infty \) section \( \tilde{s} \) of the pull-back bundle \( (TX')^\bullet_\xi \) such that \( \tilde{g} \) is realized as \( \tilde{f}_{\xi,s} \)? If there is, will \( \xi \) and \( \tilde{s} \) be unique? These questions seem to be non-trivial in general, but as we shall see, it can be dealt with in certain special cases, e.g., when \( \tilde{f}, \tilde{g} \) are pseudo-holomorphic maps from a complex orbicurve into an almost complex orbifold. We refer reader to [CR3] for details.

3 Orbifold Cohomology

As I mentioned in the introduction, the ordinary cohomology is a wrong theory for orbifolds. The correct one has to incorporate the twisted sectors. Furthermore, the internal freedom of orbifold string theory allows a twisting as well. Such a theory (orbifold cohomology) without twisting was constructed by Chen-Ruan [CR1]. The twisted orbifold cohomology was constructed by Ruan [R]. In the case of global quotients, orbifold cohomology groups were known to physicists [Z], [VW]. However, even in this case, the orbifold cup product is new. This section is a combination of [CR1], [R].

3.1 Twisted sector and inner local system

Let \( X \) be an orbifold. For any point \( p \in X \), let \( (V_p, G_p, \pi_p) \) be a local chart at \( p \). Let \( \Sigma \tilde{X} \) denote the set of pairs \( (p, (g)) \), where \( (g) \) stands for the conjugacy class of \( g = (g_1, \ldots, g_k) \) by an element of \( G_p \). We call the \( X_k \) multi-sectors.

Lemma 3.1.1: The multi-sector \( \Sigma \tilde{X} \) is naturally an orbifold, and is a finite union of closed orbifolds when \( X \) is closed, with the orbifold structure given by

\[
\{ \pi_{p,g} : (V^g_p, C(g)) \to V^g_p / C(g); (p, (g)) \in \Sigma \tilde{X} \} \tag{3.1.1}
\]

where \( V^g_p = V^g_{p_1} \cap V^g_{p_2} \cap \cdots \cap V^g_{p_k} \), \( C(g) = C(g_1) \cap C(g_2) \cap \cdots \cap C(g_k) \). Here \( g = (g_1, \ldots, g_k) \), \( V^g \) stands for the fixed-point set of \( g \in G_p \) in \( V_p \), and \( C(g) \) for the centralizer of \( g \) in \( G_p \), for some local chart \( (V_p, G_p, \pi_p) \) at \( p \).

Proof: First we identify a point \( (g, (h)) \) in \( \Sigma \tilde{X} \) as a point in \( \bigsqcup_{(p, (g)) \in \Sigma \tilde{X}} V^g_p / C(g) \) if \( q \in U_p \). Pick a representative \( y \in V_p \) such that \( \pi_p(y) = q \). Then this gives rise to a monomorphism \( \lambda_y : G_q \to G_p \). Pick a representative \( h = (h_1, \ldots, h_k) \in G_q \times \cdots \times G_q \) for \( (h) \), and let \( g = \lambda_y(h) \).
Then $y \in V_p^g$. So we have a map $\theta : (q, h) \rightarrow (y, g)$. If we change $h$ by $h' = a^{-1}ha$ for some $a \in G_q$, then $g$ is changed to $\lambda y(a^{-1}ha) = \lambda(a)^{-1}g\lambda(a)$. So we have $\theta : (q, a^{-1}ha) \rightarrow (y, \lambda(a)^{-1}g\lambda(a))$ where $y$ is regarded as an element in $V_p^{\lambda(a)^{-1}g\lambda(a)}$. (Note that $\lambda y$ is determined up to conjugacy by an element in $G_q$.) If we take a different representative $y' \in V_p$ such that $\pi_p(y') = q$, and assume $y' = b \cdot y$ for some $b \in G_p$, then we have a different identification $\lambda y' : G_q \rightarrow G_p$ of $G_q$ as a subgroup of $G_p$ where $\lambda y' = b \cdot \lambda y \cdot b^{-1}$. In this case, we have $\theta : (q, h) \rightarrow (y', bgb^{-1})$ where $y' \in V_p^{bgb^{-1}}$. If $g = bgb^{-1}$, then $b \in C(g)$. Therefore we have shown that $\theta$ induces a map sending $(q, h)$ to a point in $\bigcup_{(p, g) \in \Sigma_k X} V_p^g/C(g)$, which can be similarly shown to be one to one and onto. Hence we have shown that $\Sigma_k X$ is covered by $\bigcup_{(p, g) \in \Sigma_k X} V_p^g/C(g)$.

We define a topology on $\Sigma_k X$ so that each $V_p^g/C(g)$ is an open subset for any $(p, g)$. We also uniformize $V_p^g/C(g)$ by $(V_p^g, C(g))$. It remains to show that these charts fit together to form an orbifold structure on $\Sigma_k X$. Let $x \in V_p^g/C(g)$ and take a representative $\hat{x}$ in $V_p^g$. Let $H_x$ be the isotropy subgroup of $\hat{x} \in C(g)$. Then $(V_p^g, C(g))$ induces a germ of uniformizing system at $x$ as $(B_x, H_x)$ where $B_x$ is a small ball in $V_p^g$ centered at $\hat{x}$. Let $\pi_p(\hat{x}) = q$. We need to write $(B_x, H_x)$ as $(V_q^h, C(h))$ for some $h \in G_q \times \cdots \times G_q$. We let $\lambda x : G_q \rightarrow G_p$ be the induced monomorphism resulting from choosing $\hat{x}$ as the representative of $q$ in $V_p$. We define $h = (\lambda x)^{-1}(g)$ (each $g_i$ is in $\lambda x(G_q)$ since $\hat{x} \in V_p^g$ and $\pi_p(\hat{x}) = q$). Then we can identify $B_x$ as $V_q^h$. We also see that $H_x = \lambda x(C(h))$. Hence we have proved that $\Sigma_k X$ is naturally an orbifold with an orbifold structure described above ($\Sigma_k X$ is Hausdorff and second countable with the given topology for similar reasons). The rest of the lemma is obvious. □

**Remark 3.1.2**: $\Sigma_k X$ was introduced by Kawasaki [Ka] in relation to the index theorem. A connected component of $\Sigma_k X$ is called a sector and will contribute to the orbifold cohomology group. $\Sigma_2 X$ will be used to construct the Poincaré pairing. $\Sigma_3 X$ will be used to define cup product and $\Sigma_4 X$ will be used to prove associativity of the orbifold product. $\Sigma_k X$ corresponds to the higher product for $k \geq 4$.

Next, we consider some natural maps between multi-sectors. There are evaluation maps

$$e_{i_1, \ldots, i_l} : \Sigma_k X \rightarrow \Sigma_l X$$

defined by $e_{i_1, \ldots, i_l}(x, (g)) \rightarrow (x, (g_{i_1}, \ldots, g_{i_l}))$. There is an involution

$$I : \Sigma_k X \rightarrow \Sigma_k X$$

defined by $I(x, (g)) = (x, (g^{-1})$, where $g^{-1} = (g_{1}^{-1}, \ldots, g_{k}^{-1})$.

**Lemma 3.1.3**: $e_{i_1, \ldots, i_l}$ and $I$ are good maps.

**Proof**: Suppose that $\{(V_p, G_p, \pi_p)\}$ is the orbifold structure of $X$. By Lemma 3.1.3, it induces the orbifold structures $\{V_p^g/C(g)\}$, $\{V_p^{(g_1, \ldots, g_l)}/C(g_{i_1}, \ldots, g_{i_l})\}$ of $\Sigma_k X$, $\Sigma_l X$ simultaneously. Together with obvious embedding, they give a compatible system for $e_{i_1, \ldots, i_l}$. The proof for $I$ is similar. We leave it to the reader. □

Next, we would like to describe the connected components of $\Sigma_k X$. Recall that every point $p$ has a local chart $(V_p, G_p, \pi_p)$ which gives a local uniformized neighborhood $U_p = \pi_p(V_p)$. If $q \in U_p$, up to conjugation, there is an injective homomorphism $G_q \rightarrow G_p$. For $g \in G_q$, the conjugacy class $(g)_{G_q}$ is well-defined. We define an equivalence relation $\{(g)_{G_q} \cong (g')_{G_q}\}$. Let $T_k$ be the set of equivalence classes. With abuse of the notation, we often use $(g)$ to denote the equivalence class which $(g)_{G_q}$ belongs to. Let $T_k^p \subset T_k$ be the equivalence class of $(g)$ such that $g_1 \cdots g_k = 1$. 
It is clear that $\Sigma_k X$ is decomposed as a disjoint union of connected components

\begin{equation}
\Sigma_k X = \bigsqcup_{(g) \in T_k} X_{(g)},
\end{equation}

where

\begin{equation}
X_{(g)} = \{(p, (g')_{G_p}) | g' \in G_p, (g')_{G_p} \in (g)\}.
\end{equation}

**Definition 3.1.6:** $X_{(g)}$ for $g \neq 1$ is called a twisted sector. Furthermore, we call $X_{(1)} = X$ the nontwisted sector.

**Example 3.1.5:** Consider the case that the orbifold $X = Y/G$ is a global quotient. We will show that $\Sigma X$ is given by $\bigsqcup_{(g), g \in G} Y^g/C(g)$ where $Y^g$ is the fixed-point set of element $g \in G$. Let $\pi : \Sigma X \to X$ be the surjective map defined by $(p, (g)) \to p$. Then for any $p \in X$, the preimage $\pi^{-1}(p)$ in $\Sigma X$ has a neighborhood described by $W_p = \bigsqcup_{(g), g \in G_p} V^g_p/C(g)$, which is uniformized by $W_p = \bigsqcup_{(g), g \in G_p} V^g_p$. For each $p \in X$, pick a $y \in Y$ that represents $p$, and an injection $(\phi_p, \lambda_p) : (V_p, G_p) \to (Y, G)$ whose image is centered at $y$. This induces an open embedding $f_p : W_p \to \bigsqcup_{(g), g \in G_p} Y^g_p/C(g)$, which induces a homeomorphism $f_p$ from $W_p$ into $\bigsqcup_{(g), g \in G_p} Y^g/C(g)$ that is independent of the choice of $y$ and $(\phi_p, \lambda_p)$. These maps $\{f_p; p \in X\}$ fit together to define a map $f : \Sigma X \to \bigsqcup_{(g), g \in G_p} Y^g/C(g)$ which we can verify to be a homeomorphism.

Now, we introduce the notion of inner local system for orbifold.

**Definition 3.1.6:** Suppose that $X$ is an orbifold (almost complex or not). An inner local system $L = \{L_{(g)}\}_{g \in T_1}$ is an assignment of a flat complex line orbifold-bundle

$L_{(g)} \to X_{(g)}$

to each sector $X_{(g)}$ satisfying the compatibility condition

1. $L_{(1)} = 1$ is trivial.
2. $I^* L_{(g^{-1})} = L_{(g)}$.
3. Over each $X_{(g)}$ with $(g) \in T_1^0$, $\oplus_i e_i^* L_{(g)} = 1$.

If $X$ is a complex orbifold, we assume that $L_{(g)}$ is holomorphic.

**Lemma 3.1.7:** Suppose that $L$ is an inner local system. For any $X_{(g)}$ with $(g) \in T_k^0$ for $k \geq 4$,

\begin{equation}
\oplus_i e_i^* L_{(g)} = 1.
\end{equation}

**Proof:** Suppose that $g = (g_1, \ldots, g_k)$. We can define a sequence of triple elements

$h_1 = (g_1, g_2, (g_1g_2)^{-1}), h_2 = (g_1g_2, g_3(g_1g_2g_3)^{-1}), \ldots, h_{k-2} = (g_1 \cdots g_{k-2}, g_{k-1}, g_k)$.

By the construction, $(h_i) \in T_k^0$. Moreover, the evaluation map $\prod_i e_i$ factors through the evaluation map to $\prod_i X_{(h_i)}$. Then the lemma follows from (3). $\square$
An important way to produce inner local systems is by discrete torsion.

First, we recall the definition of orbifold fundamental group.

**Definition 3.1.8:** A smooth map \( f : Y \to X \) is an orbifold cover iff (1) each \( p \in Y \) has a neighborhood \( U_p/G_p \) such that the restriction of \( f \) to \( U_p/G_p \) is isomorphic to a map \( U_p/G_p \to U_p/\Gamma \) such that \( G_p \subset \Gamma \) is a subgroup. (2) Each \( q \in X \) has a neighborhood \( U_q/G_q \) for which each component of \( f^{-1}(U_q/G_q) \) is isomorphic to \( U_q/\Gamma' \) such that \( \Gamma' \subset G_q \) is a subgroup. An orbifold universal cover \( f : Y \to X \) of \( X \) has the property: (i) \( Y \) is connected; (ii) if \( f' : Y' \to X \) is an orbifold cover, then there exists an orbifold cover \( h : Y \to Y' \) such that \( f = f' \circ h \). If \( Y \) exists, we call \( Y \) the orbifold universal cover of \( X \) and the group of deck translations the orbifold fundamental group \( \pi_1^{orb}(X) \) of \( X \).

By Thurston [1], an orbifold universal cover exists. It is clear from the definition that the orbifold universal cover is unique. Suppose that \( f : Y \to X \) is an orbifold universal cover. Then

\[
(3.1.7) \quad f : Y - f^{-1}(\Sigma X) \to X - \Sigma X
\]

is an honest cover with \( G = \pi_1^{orb}(X) \) as covering group, where \( \Sigma \) is the singular locus of \( X \). Therefore \( X = Y/G \) and there is a surjective homomorphism

\[
(3.1.8) \quad p_f : \pi_1(X - \Sigma X) \to G.
\]

In general, (3.1.7) is not a universal covering. Hence, \( p_f \) is not an isomorphism.

**Remark 3.1.9:** Suppose that \( X = Z/G \) for an orbifold \( Z \) and \( Y \) is the orbifold universal cover of \( Z \). By the definition, \( Y \) is an orbifold universal cover of \( X \). It is clear that there is a short exact sequence

\[
(3.1.9) \quad 1 \to \pi_1(Z) \to \pi_1^{orb}(X) \to G \to 1.
\]

**Example 3.1.10:** Consider the Kummer surface \( T^4/\tau \) where \( \tau \) is the involution

\[
(3.1.10) \quad \tau(e^{it_1}, e^{it_2}, e^{it_3}, e^{it_4}) = (e^{-it_1}, e^{-it_2}, e^{-it_3}, e^{-it_4}).
\]

The universal cover is \( \mathbb{R}^4 \). The group \( G \) of deck translations is generated by translations \( \lambda_i \) by an integral point and the involution

\[
\tau : (t_1, t_2, t_3, t_4) \mapsto (-t_1, -t_2, -t_3, -t_4).
\]

It is easy to check that

\[
(3.1.11) \quad G = \{\lambda_i(i = 1, 2, 3, 4), \tau | \tau^2 = 1, \tau \lambda_i = \lambda_i^{-1} \tau\},
\]

where \( \lambda_i \) represents translation and \( \tau \) represents involution.

**Example 3.1.11:** Let \( T^6 = \mathbb{R}^6/\Gamma \) where \( \Gamma \) is the lattice of integral points. Consider \( \mathbb{Z}_2^4 \) acting on \( T^6 \) lifted to an action on \( \mathbb{R}^6 \) as

\[
\sigma_1(t_1, t_2, t_3, t_4, t_5, t_6) = (-t_1, -t_2, -t_3, -t_4, t_5, t_6)
\]

\[
\sigma_2(t_1, t_2, t_3, t_4, t_5, t_6) = (-t_1, -t_2, t_3, -t_5, -t_6)
\]
\[ \sigma_3(t_1, t_2, t_3, t_4, t_5, t_6) = (t_1, t_2, -t_3, -t_4, -t_5, -t_6). \]

This example was considered by Vafa-Witten [VW]. The orbifold fundamental group

\[ \pi_1^{orb}(T^6/\mathbb{Z}_2^2) = \{ \pi_i(1 \leq i \leq 6), \sigma_j(1 \leq j \leq 3) \} \]

(3.1.12) \[ \sigma_i^2 = 1, \sigma_1 \tau_i = \tau_i^{-1} \sigma_1 (i \neq 5, 6), \sigma_2 \tau_i = \tau_i^{-1} \sigma_2 (i \neq 3, 4), \sigma_3 \tau_i = \tau_i^{-1} \sigma_3 (i \neq 1, 2) \} \]

The following example was taken from [SC]

**Example 3.1.12:** Consider the orbifold Riemann surface \( \Sigma_g \) of genus \( g \) and \( n \) orbifold points \( z = (x_1, \cdots, x_n) \) with orders \( k_1, \cdots, k_n \). Then,

\[ \pi_1^{orb}(\Sigma_g) = \{ \lambda_i(i \leq 2g), \sigma_i(i \leq n) | \sigma_1 \cdots \sigma_n \prod_{i}[\lambda_{2i-1}, \lambda_{2i}] = 1, \sigma_i^{k_i} = 1 \}, \]

where \( \lambda_i \) are the generators of \( \pi_1(\Sigma_g) \) and \( \sigma_i \) are the generators of \( \Sigma_g - z \) represented by a loop around each orbifold point.

Note that \( \pi_1^{orb}(\Sigma_g) \) is just \( \pi_1(\Sigma_g - z) \) modulo the relation \( \sigma_i^{k_i} = 1 \). This suggests that one can first take the cover of \( \Sigma_g - z \) induced by \( \pi_1^{orb}(\Sigma) \). The relation \( \sigma_i^{k_i} = 1 \) implies that the preimage of the punctured disc around \( x_i \) is a punctured disc. Then we can fill in the center point to obtain the orbifold universal cover.

**Definition 3.1.13:** We call an element \( \alpha \in H^2(\pi_1^{orb}(X), U(1)) \) a discrete torsion of \( X \).

If \( X = Z/G \) for a finite group \( G \), by Remark 3.2, there is a surjective homomorphism

\[ \pi : \pi_1^{orb}(X) \rightarrow G. \]

\( \pi \) induces a homomorphism

(3.1.14) \[ \pi^* : H^2(G, U(1)) \rightarrow H^2(\pi_1^{orb}(X), U(1)). \]

Hence, an element of \( H^2(G, U(1)) \) induces a discrete torsion of \( X \).

There are many ways to define \( H^2(G, U(1)) \). The definition \( H^2(G, U(1)) = H^2(BG, U(1)) \) is a very useful definition for computation since we can use algebro-topological machinery. However, we can also take the original definition in terms of cocycles. A 2-cocycle is a map \( \alpha : G \times G \rightarrow U(1) \) satisfying

(3.1.15) \[ \alpha_{g,1} = \alpha_{1,g} = 1, \alpha_{g,hk} = \alpha_{g,h} \alpha_{h,k}, \]

for any \( g, h, k \in G \). We denote the set of two-cocycles by \( Z^2(G, U(1)) \). For any map \( \rho : G \rightarrow U(1) \) with \( \rho_1 = 1 \), its coboundary is defined by the formula

(3.1.16) \[ (\delta \rho)_{g,h} = \rho_g \rho_h \rho_{gh}^{-1}. \]

Let \( B^2(G, U(1)) \) be the set of coboundaries. Then, \( H^2(G, U(1)) = Z^2(G, U(1))/B^2(G, U(1)) \). \( H^2(G, U(1)) \) naturally appears in many important places in mathematics. For example, it classifies the group extensions of \( G \) by \( U(1) \). If we have a unitary projective representation of \( G \), it naturally induces a class in \( H^2(G, U(1)) \). In many instances, this class completely classifies the projective unitary representation. In fact, it is in this context that discrete torsion arises in orbifold string theory.
Definition 3.1.13: For each 2-cocycle $\alpha$, we define its phase

\begin{equation}
(3.1.17) \quad \gamma(\alpha)_{g,h} = \alpha_{g,h}^{-1} \alpha_{h,g}^{-1}.
\end{equation}

It is clear that $\gamma(\alpha)_{g,g} = 1, \gamma(\alpha)_{g,h} = \gamma(\alpha)_{h,g}^{-1}$.

Lemma 3.1.14: Suppose that $gh = hg, gk = kg$. Then

(1) $\gamma(\delta p)_{g,h} = 1$.

(2) $\gamma(\alpha)_{g,hk} = \gamma(\alpha)_{g,h} \gamma(\alpha)_{g,k}$.

Hence, $L^\alpha_g = \gamma_g : C(g) \rightarrow U(1)$ is a representation of $C(g)$.

Proof: (1) is obvious. For (2),

$$
\gamma(\alpha)_{g,hk} = \alpha_{g,hk} \alpha_{hk,g}^{-1} = \alpha_{g,hk} \alpha_{gh,k} \alpha_{h,g,k} \alpha_{h,k,g}^{-1} = \alpha_{g,hk} \alpha_{hk,k} \alpha_{h,k,g}^{-1} = \gamma(\alpha)_{g,h} \gamma(\alpha)_{g,k}.
$$

Recall the following classical definition.

Definition 3.1.15: $g$ is called $\alpha$-regular iff $L^\alpha_g$ is trivial.

Next, we calculate discrete torsion for some groups. We first consider the case of a finite abelian group $G$. In this case $H^i(G, \mathbb{Q}) = 0$ for $i \neq 0$. The exact sequence

$$
0 \rightarrow \mathbb{Z} \rightarrow C \rightarrow C^* \rightarrow 1
$$

implies that $H^2(G, U(1)) = H^2(G, C^*) = H^3(G, \mathbb{Z})$. By the universal coefficient theorem, $H^3(G, \mathbb{Z}) = H_2(G, \mathbb{Z})$.

Example 3.1.16 $G = \mathbb{Z}/n \times \mathbb{Z}/m$: Note that $H^2(G, U(1)) = H_2(G, \mathbb{Z}) = H_2(\mathbb{Z}/n \otimes \mathbb{Z}/m = \mathbb{Z}_{\gcd(n,m)}$.

In this case, one can write down the phase of discrete torsion explicitly [WV]. Let $\xi$ (resp. $\zeta$) be $n$ (resp. $m$) root of unity. Any element of $\mathbb{Z}/n \times \mathbb{Z}/m$ can be written as $(\xi^n, \zeta^m)$. Let $p = \gcd(n,m)$.

The phase of a discrete torsion can be written as

$$
\gamma(\xi^n, \zeta^m) = \omega_p^{m(ab'-ba')}
$$

with $\omega_p = e^{2\pi i/p}, m = 1, \ldots, p$. There are $p$-phases for $p$-discrete torsions. It is trivial to generalize this construction to an arbitrary finite abelian group.

Suppose that $f : Y \rightarrow X$ is the orbifold universal cover and $G$ is the orbifold fundamental group which acts on $Y$ such that $X = Y/G$. Suppose $X_{(g)}$ is a sector (twisted or nontwisted) of $X$. For any $q \in X$, choose an orbifold chart $U_q/G_q$ satisfying Definition 3.1.8. A component of $f^{-1}(U_q/G_q)$ is of the form $U_q/\Gamma'$ for $\Gamma' \subset G_q$. It is clear that $G_q/\Gamma'$ is a subgroup of the orbifold fundamental group. Therefore, we obtain a group homomorphism

\begin{equation}
(3.1.18) \quad \phi_q : G_q \rightarrow \pi_1^{\text{orb}}(X).
\end{equation}

It is easy to check that a different choice of component of $f^{-1}(U_q/G_q)$ or a different choice of $q \in X_{(g)}$ induces a homomorphism differing by a conjugation. Therefore, there is a unique map from the conjugacy classes of $G_q$ to the conjugacy classes of $\pi_1^{\text{orb}}(X)$.
**Definition 3.1.17:** We call \( X(g) \) a dormant sector if \( \phi_p(g) = 1 \).

If \( X(g) \) is a dormant sector, we define \( L(g) = 1 \). It will not receive any correction from discrete torsion. Non-dormant sectors are of the form \( Y_g/C(g) \), where \( Y_g \neq \emptyset \) is the fixed point locus of \( 1 \neq g \in \pi_1^{orb}(X) \). The torsion vector field \( Y_g \) is a smooth suborbifold of \( Y \). It is clear that \( Y_{h^{-1}gh} \) is diffeomorphic to \( Y_g \) by the action of \( h \). By abusing the notation, we denote the twisted sector \( Y_g/C(g) \) by \( X(g) \), where \( C(g) \) is the centralizer of \( g \).

Let \( \alpha \) be a discrete torsion. By Lemma 3.1.14(2), for each \( g \), the phase

\[
L_\alpha^g : C(g) \to U(1)
\]

is a group homomorphism. We can use this group homomorphism to define a flat complex line-bundle

\[
L_g = Y_g \times L_\alpha^g \mathbb{C}
\]

over \( X(g) \).

**Lemma 3.1.18:**

1. \( L_{tgt^{-1}} \) is isomorphic to \( L_g \) by the map

\[
t \times \text{Id} : Y_g \times \mathbb{C} \to Y_{tgt^{-1}} \times \mathbb{C}.
\]

Hence, we can denote \( L_g \) by \( L(g) \).

2. \( L_{(g)}^{-1} = L_{(g^{-1})} \).

3. When we restrict to \( X_{(g_1, \ldots, g_k)} = Y_{g_1} \cap \cdots \cap Y_{g_k}/C(g_1, \ldots, g_k) \), \( L_{(g_1, \ldots, g_k)} = L(g_1) \cdots L(g_k) \), where \( L_{(g_1, \ldots, g_k)} = Y_{g_1} \cap \cdots \cap Y_{g_k} \times \gamma_{g_1 \cdots g_k} \mathbb{C} \).

**Proof:** Recall that there is an isomorphism

\[
t_\#: C(g) \to C(tgt^{-1})
\]

given by \( t_\#(h) = tht^{-1} \). The map

\[
t : Y_g \to X_{tgt^{-1}}
\]

is \( t_\# \)-equivariant. By Lemma 3.8, \( \gamma_{tgt^{-1}}(tht^{-1}) = \gamma_g(h) \) for \( h \in C(g) \). Then,

\[
(3.1.19) \quad (t \times \text{Id})(hx, \gamma(h)(v)) = (thx, \gamma_g(h)(v)) = (tht^{-1}tx, \gamma_{tgt^{-1}}(tht^{-1})(v)).
\]

Then we take the quotient by \( C(g), C(tgt^{-1}) \) respectively to get an isomorphism between \( L_g, L_{tgt^{-1}} \). (2) and (3) follow from the fact that for any \( h \in C(g_1, \ldots, g_k) \),

\[
(3.1.20) \quad \gamma(\alpha)_{g_1 \cdots g_k, h} = \gamma(\alpha)_{h, g_1 \cdots g_k}^{-1} = \gamma(\alpha)_{h, g_1}^{-1} \cdots \gamma(\alpha)_{h, g_k}^{-1} = \gamma(\alpha)_{g_1, h} \cdots \gamma(\alpha)_{g_k, h}.
\]

**Theorem 3.1.19:** \( L_{\alpha} = \{L(g)\}_{(g) \in T_1} \) is an inner local system of \( X \).

**Proof:** Property (1) is obvious. The property (2) follows from Lemma 3.1.17. Let’s prove property (3). Consider the image \( g' = (g'_1, g'_2, g'_3) \) of \( g \) in \( \pi_1^{orb}(X) \) under the homomorphism (3.1.18). Then we still have \( g'_1 g'_2 g'_3 = 1 \). There are three possibilities: (i) \( g'_1 = g'_2 = g'_3 = 1 \) and there is nothing to prove in this case; (ii) \( g'_3 = 1, g'_2 = (g')_1^{-1} \) is nontrivial; (iii) \( g'_1, g'_2, g'_3 \) are all nontrivial.
For the second case, let \( g = g'_1 \). We have the following factorization

\[
e_1 \times e_2 \times e_3 : X(\underline{g}) \rightarrow X(g_1, g_2) \times X(g_3) \rightarrow X(g_1) \times X(g_2) \times X(g_3).
\]

However, \( X(g_1, g_2) = Y_g \cap Y_{g^{-1}}/C(g, g^{-1}) = Y_g/C(g) \). Moreover, over \( X_{g_1, g_2} \)

\[
e^{*}_1 L(g)\epsilon^{*}_2 L(g^{-1}) = L(g)\Gamma L(g^{-1}) = 1.
\]  

In the third case, \( X(\underline{g}) = Y_{g_1} \cap Y_{g_2} \cap Y_{g_3}/C(g_1, g_2, g_3) \). The proof follows from Lemma 3.1.18 (3). \( \square \)

**Remark 3.1.20:** The current definition of discrete torsion is unsatisfactory because of the existence of dormant sectors (see example 3.5.4 as well). It would be desirable to find a better definition of discrete torsion where all the sectors will receive corrections.

### 3.2 Degree shifting and orbifold cohomology group

For the rest of the paper, we will assume that \( X \) is an almost complex orbifold with an almost complex structure \( J \). Recall that an almost complex structure \( J \) on \( X \) is a smooth section of the orbifold bundle \( \text{End}(TX) \) such that \( J^2 = -1 Id \). In this case, multi-sectors \( \Sigma_k X \) naturally inherit an almost complex structure. Moreover, both the evaluation map \( e_{i_1, \ldots, i_l} \) and \( I \) are naturally pseudo-holomorphic, i.e., its differential commutes with the almost complex structures on \( \Sigma_k \tilde{X} \).

An important feature of orbifold cohomology groups is degree shifting, which we shall explain now. Let \( p \in X \) be a singular point of \( X \). The almost complex structure on \( X \) gives rise to an effective representation \( \rho_p : G_p \rightarrow GL(n, \mathbb{C}) \) (here \( n = \dim \mathbb{C} X \)). For any \( g \in G_p \), we write \( \rho_p(g) \) as a diagonal matrix

\[
diag(e^{2\pi i m_{1,g}/m_g}, \ldots, e^{2\pi i m_{n,g}/m_g}),
\]

where \( m_g \) is the order of \( g \) in \( G_p \), and \( 0 \leq m_{i,g} < m_g \). This matrix depends only on the conjugacy class \((g)_{G_p}\) of \( g \) in \( G_p \). We define a function \( \iota : \tilde{X} \rightarrow \mathbb{Q} \) by

\[
\iota(p, (g)_{G_p}) = \sum_{i=1}^{n} \frac{m_{i,g}}{m_g}
\]

It is straightforward to show the following

**Lemma 3.2.1:** The function \( \iota : \tilde{X} \rightarrow \mathbb{Q} \) is locally constant. We will denote it by \( \iota(\underline{g}) \). The function \( \iota(\underline{g}) \) satisfies the following conditions:

- \( \iota(\underline{g}) \) is integral if and only if \( \rho_p(g) \in \text{SL}(n, \mathbb{C}) \).

\[
(3.2.1) \quad \iota(\underline{g}) + \iota(\underline{g}^{-1}) = \text{rank}(\rho_p(g) - I),
\]

which is the “complex codimension” \( \dim \mathbb{C} X - \dim \mathbb{C} X(\underline{g}) = n - \dim \mathbb{C} X(\underline{g}) \) of \( X(\underline{g}) \) in \( X \). As a consequence, \( \iota(\underline{g}) + \dim \mathbb{C} X(\underline{g}) < n \).

**Definition 3.2.2:** \( \iota(\underline{g}) \) is called the degree shifting number.

In the definition of orbifold cohomology groups, we will shift up the degree of cohomology classes of \( X(\underline{g}) \) by \( 2\iota(\underline{g}) \). The reason for such a degree shifting will become clear after we discuss the dimension of the moduli space of ghost maps (see Proposition 3.4.4).
There are two important classes of orbifolds. $X$ is called an $SL$-orbifold if $\rho_p(g) \in SL(n, \mathbb{C})$. $X$ is called an $Sp$-orbifold if $\rho_p(g) \in Sp(n, \mathbb{C})$. In particular, a Calabi-Yau orbifold is a $SL$-orbifold. An holomorphic symplectic orbifold or hyperkahler orbifold is an $Sp$-orbifold.

By the Lemma 3.2.1, $\iota(g)$ is integral if $X$ is a $SL$-orbifold.

We observe that although the almost complex structure $J$ is involved in the definition of degree shifting numbers $\iota(g)$, they do not depend on $J$ because locally the parameter space of almost complex structures, which is the coset $SO(2n, \mathbb{R})/U(n, \mathbb{C})$, is connected.

**Definition 3.2.3:** Let $L$ be an inner local system. We define the orbifold cohomology groups $H^d_{orb}(X; L)$ and compactly supported orbifold cohomology group $H^d_{orb,c}(X; L)$ of $X$ by

\[
H^d_{orb}(X; L) = \bigoplus_{(g) \in T} H^{d-2\iota(g)}(X_{(g)}; L_{(g)}), \quad H^d_{orb,c}(X; L) = \bigoplus_{(g) \in T} H^{d-2\iota(g)}_c(X_{(g)}; L_{(g)})
\]

and orbifold Betti numbers $b^d_{orb, L} = \sum_{(g)} \dim H^{d-2\iota(g)}(X_{(g)}; L_{(g)})$. If $L = L_\alpha$ for some discrete torsion $\alpha$, we define $H^*_{orb, \alpha}(X, C) = H^*_{orb}(X, L_\alpha), H^*_{orb,c, \alpha}(X, C) = H^*_{orb,c}(X, L_\alpha)$.

Note that, in general, orbifold cohomology groups are rationally graded. Traditionally, $H^d_{orb}(X, L)$ for $L = 1$ is called ordinary orbifold cohomology. Other cases are called twisted orbifold cohomology.

Suppose $X$ is a complex orbifold with an integrable complex structure $J$. Then each twisted sector $X_{(g)}$ is also a complex orbifold with the induced complex structure. We consider the Čech cohomology groups on $X$ and on each $X_{(g)}$ with coefficients in the sheaves of holomorphic forms (in the orbifold sense). These Čech cohomology groups are identified with the Dolbeault cohomology groups of $(p, q)$-forms (in the orbifold sense). When $X$ is closed, the harmonic theory can be applied to show that these groups are finite dimensional, and there is a Kodaira-Serre duality between them. When $X$ is a closed Kähler orbifold (so is each $X_{(g)}$), these groups are then related to the singular cohomology groups of $X$ and $X_{(g)}$ as in the smooth case, and the Hodge decomposition theorem holds for these cohomology groups.

**Definition 3.2.4:** Let $X$ be a closed complex orbifold. We define, for $0 \leq p, q \leq \dim_X X$, orbifold Dolbeault cohomology groups

\[
H^{p, q}_{orb}(X; L) = \bigoplus_{(g)} H^{p-\iota(g), q-\iota(g)}(X_{(g)}; L_{(g)}), \quad H^{p, q}_{orb,c}(X; L) = \bigoplus_{(g)} H^{p-\iota(g), q-\iota(g)}_c(X_{(g)}; L_{(g)})
\]

We define orbifold Hodge numbers by $b^{p, q}_{orb}(X, L) = \dim H^{p, q}_{orb}(X; L_{(g)})$.

**Remark 3.3.3:** In the case of global quotient $X = Y/G$, suppose that $\alpha \in H^2(G, S^1)$. $L^\alpha_g$ induces a twisted action of $C(g)$ on the fixed point set $H^*(Y_g, \mathbb{C})$ by $h \circ \beta = L^\alpha_g(h)h^*\beta$. Let $H^*(Y_g, \mathbb{C})^{C^\alpha(g)}$ be the invariant subspace under the twisted action. It is easy to observe that

$H^d_{orb,c, \alpha}(X, C) = \bigoplus_{(g)} H^{d-2\iota(g)}_c(Y_g, \mathbb{C})^{C^\alpha(g)}$

and

$H^{p, q}_{orb,c, \alpha}(X, C) = \bigoplus_{(g)} H^{p-\iota(g), q-\iota(g)}_c(Y_g, \mathbb{C})^{C^\alpha(g)}$.

### 3.3 Poincaré duality

For simplicity, we assume that the orbifold under consideration is closed.

Recall that there is a natural $C^\infty$ map $I : X_{(g)} \to X_{(g^{-1})}$ defined by $(p, (g)) \to (p, (g^{-1}))$, which is an automorphism of $\bar{X}$ as an orbifold, and $I^2 = Id$ (Remark 3.1.4).
Proposition 3.3.1: (Poincaré duality)

For any $0 \leq d \leq 2n$, the pairing

$$< >_{\text{orb}}: H^d_{\text{orb}}(X; \mathcal{L}) \otimes H^{2n-d}_{\text{orb},c}(X; \mathcal{L}) \to \mathbb{C}$$

defined by the direct sum of

$$< (g) >_{\text{orb}}: H^{d-2t(g)}(X(g); \mathcal{L}(g)) \otimes H^{2n-d-2t(g-1)}_c(X(g-1); \mathcal{L}(g-1)) \to \mathbb{C}$$

where

$$(3.3.3) \quad < \alpha, \beta >_{\text{orb}} = \int_{X(g)} \alpha \wedge I^*(\beta)$$

for $\alpha \in H^{d-2t(g)}(X(g); \mathcal{L}(g)), \beta \in H^{2n-d-2t(g-1)}(X(g-1); \mathcal{L}(g-1))$ is nondegenerate.

By Lemma 3.1.18, $I^*L_{(g-1)} = L_{(g)}^{-1}$. Hence, the definition makes sense. Moreover, $< >_{\text{orb}}$ is just the ordinary Poincaré pairing when restricted to the nontwisted sector.

Proof: By (3.2.1), we have

$$(3.3.4) \quad 2n - d - 2t(g-1) = \dim X(g) - d - 2t(g).$$

Furthermore, $I|_{X(g)}: X(g) \to X(g-1)$ is a homeomorphism and $I^*L_{(g-1)} = L_{(g)}^{-1}$. Under this homeomorphism, $< (g) >_{\text{orb}}$ is isomorphic to the ordinary Poincaré pairing with coefficients on $X(g)$. Hence $< >_{\text{orb}}$ is nondegenerate. $\square$

For the case of orbifold Dolbeault cohomology, the following proposition is straightforward.

Proposition 3.3.2: Let $X$ be an $n$-dimensional complex orbifold. There is a Kodaira-Serre duality pairing

$$< >_{\text{orb}}: H^p,q_{\text{orb}}(X; \mathcal{L}) \times H^{n-p,n-q}_{\text{orb},c}(X; \mathcal{L}) \to \mathbb{C}$$

similarly defined as in the previous proposition. When $X$ is closed and Kähler, the following is true:

- $H^r_{\text{orb}}(X; \mathcal{L}) = \oplus_{r=p+q} H^{p,q}_{\text{orb}}(X; \mathcal{L})$
- $H^{\text{p,q}}_{\text{orb}}(X; \mathcal{L}) = \overline{H^{\text{q,p}}_{\text{orb}}(X; \mathcal{L})}$,

and the two pairings (Poincaré and Kodaira-Serre) coincide.

3.4 Orbifold cup product

Our definition of orbifold cup product is motivated by the construction of quantum product. For this approach, we have to construct a Gromov-Witten type invariant of genus zero, homology class zero with three marked points. It involves an analysis of the moduli space of constant (ghost) good maps from the orbifold sphere with three marked points. The construction is lengthy. However, we can skip over the construction of the moduli space and write down the definition explicitly. If the reader wishes to understand the geometric origin of our definition and key properties such as associativity, we encourage the reader to read [CR1].

A key ingredient is orbifold Riemann surface. Every closed orbifold of dimension 2 is complex, with underlying topological space a closed Riemann surface. More precisely, a closed 2-dimensional
orbifold consists of the following data: a closed Riemann surface $\Sigma$ with complex structure $j$, a finite subset of distinct points $z = (z_1, \cdots, z_k)$ on $\Sigma$, each with multiplicity $m_i \geq 2$ (let $m = (m_1, \cdots, m_k)$), such that the orbifold structure at $z_i$ is given by the ramified covering $z \to z^{m_i}$. We will also call a closed 2-dimensional orbifold a complex orbicurve when the underlying complex analytic structure is emphasized.

We observe the following well-known fact (see e.g. [SC]).

**Proposition 3.4.1:** Let $(\Sigma, z, m)$ be a complex orbicurve, where $z = (z_1, \cdots, z_k)$ and $m = (m_1, \cdots, m_k)$, such that either $g_\Sigma \geq 1$, or $g_\Sigma = 0$ with $k \geq 3$ or $k = 2$ and $m_1 = m_2$. Then there is a closed Riemann surface $\tilde\Sigma$ with a finite group $G$ acting on $\tilde\Sigma$ holomorphically, and a holomorphic mapping $\pi : \tilde\Sigma \to \Sigma$, such that $(\tilde\Sigma, G, \pi)$ is a uniformizing system of $(\Sigma, z, m)$.

The construction of cup product follows the procedure to define quantum product. First, we need to define a 3-point function. In our case, the 3-point function is an integral over $X_{(g)}$ for $(g) \in T_3^\alpha$. In order to write down the form we integrate, we need to construct an obstruction bundle and its Euler form over $X_{(g)}$.

Consider the pull-back tangent bundle $e^*TX$ over $X_{(g)}$. Let $x \in X_{(g)}$ be a generic point and its local group in $X$ is $G$. It obviously contains three elements $g_1, g_2, g_3$ with the relation $g_1 g_2 g_3 = 1, g_3^{k_i} = 1$, where $k_i$ is the order of $g_i$. Let $G$ be the subgroup of $G'$ generated by $g_1, g_2, g_3$. Clearly, $G$ acts on $e^*TX$ while fixing $X_{(g)}$.

Consider an orbifold Riemann sphere with three orbifold points $(S^2, (x_1, x_2, x_3), (k_1, k_2, k_2))$. Without any confusion, we simply denote it by $S^2$. Recall Example 3.1.12

$$\pi_{orb}^1(S^2) = \{\lambda_1, \lambda_2, \lambda_3; \lambda_1^{k_1} = 1, \lambda_1 \lambda_2 \lambda_3 = 1\},$$

where $\lambda_i$ is represented by a loop around the marked point $x_i$. There is an obvious surjective homomorphism

$$\pi : \pi_{orb}^1(S^2) \to G.$$

$kerr\pi$ is a subgroup of finite index. Suppose that $\tilde\Sigma$ is the orbifold universal cover of $S^2$. By Proposition 3.4.1, it is uniformized by a closed Riemann surface $\Sigma'$. Hence, $\tilde\Sigma$ is the universal cover of $\Sigma'$ and is smooth. Let $\Sigma = \tilde\Sigma/kerr\pi$. $\Sigma$ is compact and $S^2 = \Sigma/G$. Since $G$ contains the relation $g_3^{k_i} = 1$, $\Sigma$ is smooth. $G$ acts on $H^1(\Sigma)$. Let $e : X_{(g)} \to X$ be the evaluation map. Therefore, we can assume that $G$ acts on both $H^1(\Sigma)$ and $e^*TX$. We view $H^1(\Sigma)$ as a trivial bundle over $X_{(g)}$. The obstruction bundle $E_{(g)}$ we want is is the invariant part of $H^1(\Sigma) \otimes e^*TX$, i.e., $E_{(g)} = (H^1(\Sigma) \otimes e^*TX)^G$. Since we do not assume that $X$ is compact, $X_{(g)}$ could be a non-compact orbifold in general. The Euler class of $E_{(g)}$ depends on a choice of connection on $E_{(g)}$. Let $e_A(E_{(g)})$ be the Euler form computed from the connection $A$ by Chern-Weil theory. It is clear that $e_A(E_{(g)}), e_{A'}(E_{(g)})$ differ by an exact form if $A'$ is another connection on $E_{(g)}$.

Now, we are ready to define our 3-point function. Suppose that $\alpha \in H_{orb,b}^d(X, C), \beta \in H_{orb}^d(X, C), \gamma \in H_{orb,c}^*(X_{(g_3)}, C)$.

**Definition 3.1.3:** We define the 3-point function

$$<\alpha, \beta, \gamma>_{orb} = \sum_{(g) \in T_3^\alpha} \int_{X_{(g)}} e_1^*\alpha \wedge e_2^*\beta \wedge e_3^*\gamma \wedge e_A(E_{(g)}).$$

Note that $e_3^*\gamma$ is compact supported. Therefore, the integral is finite. Moreover, if we choose a different $A'$, then $e_A(E_{(g)}), e_{A'}(E_{(g)})$ differ by an exact form. Hence, the integral is independent of the choice of $A$. 

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Definition 3.1.4: We define the orbifold cup product by the relation

\[ <\alpha \cup_{\text{orb}} \beta, \gamma >_{\text{orb}} = <\alpha, \beta, \gamma >_{\text{orb}}. \]

If \( \alpha, \beta \) are compacted supported orbifold cohomology classes, we can define \( \alpha \cup_{\text{orb}} \beta \in H^*_{\text{orb}, c}(X, \mathcal{L}) \) in the same fashion. Suppose that \( \alpha \in H^*(X_{(g_1)}, \mathcal{C}), \beta \in H^*(X_{(g_2)}, \mathcal{C}). \) \( \alpha \cup_{\text{orb}} \beta \in H^*_{\text{orb}}(X, \mathcal{C}) = \bigoplus_{(g) \in T_1} H^*(X_{(g)}, \mathcal{C}). \) Therefore, we should be able to decompose \( \alpha \cup_{\text{orb}} \beta \) as a sum of its components in \( H^*(X_{(g)}, \mathcal{C}). \) Such a decomposition would be very useful in computation.

Note that when \( g_1 g_2 g_3 = 1 \) the conjugacy class \((g_1, g_2, g_3)\) is uniquely determined by the conjugacy class of the pair \((g_1, g_2)\). We can use it to obtain the following

Decomposition Lemma 3.1.5:

\[ \alpha \cup_{\text{orb}} \beta = \sum_{(h_1, h_2) \in T_2, h_i \in (g_i)} (\alpha \cup_{\text{orb}} \beta)_{(h_1, h_2)}, \]

where \( (\alpha \cup_{\text{orb}} \beta)_{(h_1, h_2)} \in H^*(X_{(h_1 h_2)}, \mathcal{C}) \) is defined by the relation

\[ <(\alpha \cup_{\text{orb}} \beta)_{(h_1, h_2)}, \gamma >_{\text{orb}} = \int_{X_{(h_1, h_2)}} e^*_1 \alpha \wedge e^*_2 \beta \wedge e^*_3 \gamma \wedge e_A(E(g)). \]

for \( \gamma \in H^*(X_{((h_1 h_2)^{-1})}, \mathcal{C}). \)

Remark: In the case of global quotient \( X = Y/G, \) \( X_{(h_1, h_2)} = Y_{h_1} \cap Y_{h_2}/C(h_1) \cap C(h_2). \)

Theorem 3.1.6: Let \( X \) be an almost complex orbifold with almost complex structure \( J \) and \( \dim_{\mathbb{C}} X = n. \) The cup product defined above preserves the orbifold degree \( \cup_{\text{orb}} : H^p_{\text{orb}}(X; \mathcal{C}) \otimes H^q_{\text{orb}}(X; \mathcal{C}) \to H^{p+q}_{\text{orb}}(X; \mathcal{C}) \) for any \( 0 \leq p, q \leq 2n \) such that \( p + q \leq 2n, \) and has the following properties:

1. The total orbifold cohomology group \( H^*_\text{orb}(X; \mathcal{C}) = \bigoplus_{0 \leq d \leq 2n} H^d_\text{orb}(X; \mathcal{C}) \) is a ring with unit \( e^0_X \in H^0(X; \mathcal{C}) \) under \( \cup_{\text{orb}}, \) where \( e^0_X \) is the Poincaré dual to the fundamental class \([X]\).

2. The cup product \( \cup_{\text{orb}} \) is invariant under deformation of \( J. \)

3. When \( X \) is of integer degree shifting numbers, the total orbifold cohomology group \( H^*_\text{orb}(X; \mathcal{C}) \) is integrally graded, and we have supercommutativity

\[ \alpha_1 \cup_{\text{orb}} \alpha_2 = (-1)^{\deg \alpha_1 \deg \alpha_2} \alpha_2 \cup_{\text{orb}} \alpha_1. \]

4. Restricted to the nontwisted sectors, i.e., the ordinary cohomology \( H^*(X; \mathcal{C}), \) the cup product \( \cup_{\text{orb}} \) equals the ordinary cup product on \( X. \)

5. \( \cup_{\text{orb}} \) is associative.

Now we define the cup product \( \cup_{\text{orb}} \) on the total orbifold Dolbeault cohomology group of \( X \) when \( X \) is a complex orbifold. We observe that in this case all the objects we have been dealing with are holomorphic, i.e., \( \Sigma_k X \) is a complex orbifold, \( pr : E(g) \to X_{(g)} \) is holomorphic orbifold bundle, and the evaluation map are holomorphic.
Definition 4.1.7: For any $\alpha_1 \in H_{orb}^{p,q}(X;C)$, $\alpha_2 \in H_{orb}^{p',q'}(X;C)$, we define the 3-point function and orbifold cup product in the same fashion as Definition 3.1.3, 3.1.4.

Note that since the top Chern class of a holomorphic orbifold bundle can be represented by a closed $(r,r)$-form, where $r$ is the rank, it follows that $\alpha_1 \cup_{orb} \alpha_2$ lies in $H_{orb}^{p+p',q+q'}(X;C)$.

The following theorem can be similarly proved.

Theorem 4.1.8: Let $X$ be an $n$-dimensional closed complex orbifold with complex structure $J$. The orbifold cup product

$$\cup_{orb} : H_{orb}^{p,q}(X;C) \otimes H_{orb}^{p',q'}(X;C) \to H_{orb}^{p+p',q+q'}(X;C)$$

defined above has the following properties:

1. The total orbifold Dolbeault cohomology group is a ring with unit $e_X^0 \in H_{orb}^{0,0}(X;C)$ under $\cup_{orb}$, where $e_X^0$ is the class represented by the equaling-one constant function on $X$.

2. The cup product $\cup_{orb}$ is invariant under deformation of $J$.

3. When $X$ is of integral degree shifting numbers, the total orbifold Dolbeault cohomology group of $X$ is integrally graded, and we have supercommutativity

$$\alpha_1 \cup_{orb} \alpha_2 = (-1)^{\text{deg} \alpha_1 \cdot \text{deg} \alpha_2} \alpha_2 \cup_{orb} \alpha_1.$$

4. Restricted to the nontwisted sectors, i.e., the ordinary Dolbeault cohomology $H^{*,*}(X;C)$, the cup product $\cup_{orb}$ equals the ordinary wedge product on $X$.

5. When $X$ is Kähler and closed, the cup product $\cup_{orb}$ coincides with the orbifold cup product over the orbifold cohomology groups $H^*_{orb}(X;C)$ under the relation

$$H^r_{orb}(X;C) = \oplus_{p+q=r} H^{p,q}_{orb}(X;C).$$

The most difficult part of proof is associativity. We refer the proof to [CR1].

Remark 3.4.9:

In many way, the current definition of orbifold cohomology is less than satisfactory. It is a very interesting question whether one can represent a cohomology class from a twisted sector by a differential form on $X$ with certain singularities along singular strata. Recall that a crepant resolution is a map $F : Y \to X$ such that $F^*K_X = K_Y$, where $X$ is a complex orbifold and $Y$ is a smooth complex manifold. An extremely interesting question is to study the relation between the orbifold cohomology of $X$ and the ordinary cohomology of $Y$ (see section 6). The main difficulty is to pull back the class from the twisted sector. If we can establish a theory to use differential forms on $X$ to represent the class from a twisted sector, it would be very useful to understand its relation to crepant resolution.

Note that orbifold cohomology is a cohomology theory only. It would be desirable to build a homology theory based on suborbifolds and their intersection theory. Hence, we can realize orbifold Poincaré duality and orbifold cup product geometrically. It seems to the author that good maps should play a critical role in such a homology theory.
3.5 Examples

So far, only a few examples of global quotients have been computed by physicists [VW] [D]. However, orbifold cohomology is very much calculable, as we will demonstrate in examples. Here we compute several examples. The first two are local examples, where the reader should have some general ideas about orbifold cohomology. The third and fourth have nontrivial discrete torsion. One is a global quotient and another one is a non-global quotient. The fourth example has the phenomenon that most of the twisted sectors are dormant sectors. The last one is Joyce’s [JO] example, where there is no nontrivial discrete torsion. However, there are nontrivial inner local systems. We will compute the twisted orbifold cohomology given by nontrivial inner local systems to match Joyce’s desingularizations.

Example 3.5.1: The easiest example is probably a point with a trivial group action of \( G \). In this case, the orbifold cohomology is generated by conjugacy classes of elements of \( G \). All the degree shifting numbers are zero. Only the Poincaré paring and cup products are interesting. We observe that \( X_{(g_1, g_2 (g_1 g_2)^{-1})} \) is a point with a trivial group action of \( C(g_1) \cap C(g_2) \). Hence, the GW-invariants

\[
\int_{X_{(g_1, g_2 (g_1 g_2)^{-1})}} 1 = \frac{1}{|C(g_1) \cap C(g_2)|}.
\]

Let \( x(g) \) be the generators of orbifold cohomology group corresponding to one on sector \( X(g) \). Using decomposition lemma 4.5.1, the cup product

\[
x(g_1) \cup x(g_2) = \sum_{(h_1, h_2) \in (g_1), h_2 \in (g_2)} d_{(h_1, h_2)} x(h_1, h_2),
\]

where \( (h_1, h_2) \) is the conjugacy class of pair \( h_1, h_2 \). We can compute coefficient \( d_{(h_1, h_2)} \) by the relation

\[
\frac{1}{|C(g_1) \cap C(g_2)|} = \int_{X_{(h_1, h_2 (h_1 h_2)^{-1})}} 1 = d_{(h_1, h_2)} \int_{X(h_1, h_2)} 1 = \frac{d_{(h_1, h_2)}}{|C(h_1 h_2)|}.
\]

Hence, \( d_{(h_1, h_2)} = \frac{|C(h_1 h_2)|}{|C(h_1) \cap C(h_2)|} \). Recall that the center \( Z(C[G]) \) of group algebra \( C[G] \) is generated by \( \sum_{h \in (g)} h \) and hence can be identified with conjugacy classes itself. In this sense, we say the center is generated by the set of conjugacy classes. \( Z(C[G]) \) has a natural ring structure inherited from \( C[G] \). Note that if we enumerate all elements in conjugacy class of pair \( (h_1, h_2) \). Its product run through the conjugacy class \( (h_1 h_2) \) precisely \( \frac{|C(h_1 h_2)|}{|C(h_1) \cap C(h_2)|} \) times. Therefore, the orbifold cup product is the same as product of \( Z(C[G]) \).

Suppose that \( \alpha \in H^2(G, U(1)) \) is a discrete torsion. It is clear that twisted orbifold cohomology is generated by conjugacy classes of \( \alpha \)-regular elements. On the another hand, the center of the twisted group algebra \( C_\alpha[G] \) is also generated by conjugacy classes of \( \alpha \)-regular elements. Indeed, they have the same ring structures.

Example 3.5.2: Suppose that \( G \subset GL(n, C) \) is a finite subgroup. Then \( C^n/G \) is an orbifold. Suppose that \( \alpha \in H^2(G, U(1)) \) is a discrete torsion. For any \( g \in G \), the fixed point set \( X(g) \) is a vector subspace and \( X(g) = X(g)/C(g) \). By definition, \( L(g) = X(g) \times L_g \subset C \). Therefore, \( H^*(X(g), L(g)) \) is the subspace of \( H^*(X(g), C) \) invariant under the twisted action of \( C(g) \)

\[
(3.5.2) \quad h \circ \beta = L_g^\alpha(h) h^* \beta
\]

for any \( h \in C(g), \beta \in H^*(X(g), C) \). However, \( H^i(X(g), C) = 0 \) for \( i \geq 1 \). Moreover, if \( L_g^\alpha \) is nontrivial, \( H^0(X(g), L(g)) = 0 \). Therefore, \( H^p_{orb} = 0 \) for \( p \neq q \) and \( H^p_{orb} \) is a vector space generated by conjugacy
classes of $\alpha$-regular elements $g$ with $\iota(g) = p$. Therefore, we have a natural decomposition

\begin{equation}
Z(C_\alpha[G]) = \sum_p H_p,
\end{equation}

where $H_p$ is generated by conjugacy classes of $\alpha$-regular elements $g$ with $\iota(g) = p$. The ring structure is also easy to describe. Let $x(g)$ be the generator corresponding to the zero cohomology class of the twisted sector $X(g)$ such that $g$ is $\alpha$-regular. We would like to get a formula for $x(g_1) \cup x(g_2)$. As we showed before, the multiplication of conjugacy classes can be described in terms of the center of the twisted group algebra $Z(C_\alpha[G])$. But we have further restrictions in this case. Let’s first describe the moduli space $X(h_1,h_2,(h_1,h_2)^{-1})$ and its corresponding GW-invariants. It is clear that

$$X(h_1,h_2,(h_1,h_2)^{-1}) = X_{h_1} \cap X_{h_2}/C(h_1) \cap C(h_2).$$

To have the nonzero invariant, we require that

\begin{equation}
\iota(h_1h_2) = \iota(h_1) + \iota(h_2).
\end{equation}

Then we need to compute

\begin{equation}
\int_{X_{h_1} \cap X_{h_2}/C(h_1,h_2)} e_3^*(vol_c(X_{h_1h_2}/C(h_1,h_2))) \wedge e(E),
\end{equation}

where $vol_c(X_{h_1h_2}/C(h_1,h_2))$ is the compactly supported $C(h_1,h_2)$-invariant top form with volume one over $X_{h_1h_2}$. However,

$$X_{h_1} \cap X_{h_2} \subset X_{h_1h_2}$$

is a submanifold. Therefore, the integral (3.5.5) is zero unless

\begin{equation}
X_{h_1} \cap X_{h_2} = X_{h_1h_2}.
\end{equation}

In this case, we call $(h_1, h_2)$ transverse. In this case, it is clear that the obstruction bundle is trivial. Suppose that the integral is $d_{h_1,h_2}$. Let

\begin{equation}
I_{g_1,g_2} = \{ (h_1, h_2); h_i \in (g_i), \iota(h_1) + \iota(h_2) = \iota(h_1h_2), (h_1, h_2) - \text{transverse}\}.
\end{equation}

Then,

\begin{equation}
x(g_1) \cup x(g_2) = \sum_{(h_1, h_2) \in I_{g_1,g_2}} d_{h_1,h_2} x(h_1h_2).
\end{equation}

A similar computation as previous example yields $d_{h_1,h_2} = \frac{|C(h_1,h_2)|}{|C(h_1)| \cdot |C(h_2)|}$.

Next, we specialize to the symmetric group $S_n$. Recall that any element of the symmetric group can be decomposed into cycles $\gamma_1, \cdots, \gamma_m$ and its conjugacy class is uniquely determined by the cycle classes. Following tradition, for each cycle $\gamma$ of $k$ letters, we define its degree $deg(\gamma) = k - 1$. In particular, the identity element has degree zero. Such a definition of degree is also invariant under the inclusion $S_n \subset S_{n+1}$. Then we define the degree of a conjugacy class as the sum of the degree of its cycles. It is easy to compute that the degree shifting number is the degree. Moreover, we observe that $\iota(h_1h_2) \leq \iota(g_1) + \iota(g_2)$ for $h_i \in (g_i)$. When equality holds, the pair $(h_1, h_2)$ is automatically transverse. Hence,

\begin{equation}
x(g_1) \cup x(g_2) = P^{\iota(g_1) + \iota(g_2)}(x(g_1)x(g_2)),
\end{equation}

24
where $P_p : Z(C_{a}G) \to H_p$ is the projection. The formula is precisely the formula appeared in Lehn-Sorger’s calculation of the cohomology ring of the Hilbert scheme of points of $\mathbb{C}^2$. Therefore, by combining with their calculation, we have

**Corollary 3.5.2a:** $H^*(\text{Hi}lb_k(\mathbb{C}^2), \mathbb{C})$ and $H^*_{\text{orb}}(\text{Sym}_k(\mathbb{C}^2), \mathbb{C})$ are isomorphic as rings.

We would like to emphasize that our calculation works over any finite quotient of affine space and is much more general than the symmetric group.

It is also easy to compute ring structure for the following examples. We leave it to readers.

**Example 3.5.3** $T^4/\mathbb{Z}_2 \times \mathbb{Z}_2$: Here, $T^4 = \mathbb{C}^2/\Lambda$, where $\Lambda$ is the lattice of integral points. Suppose that $g, h$ are generators of the first and the second factor of $\mathbb{Z}_2 \times \mathbb{Z}_2$. The action of $\mathbb{Z}_2 \times \mathbb{Z}_2$ on $T^4$ is defined as

$$g(z_1, z_2) = (-z_1, z_2), h(z_1, z_2) = (z_1, -z_2).$$

The fixed point locus of $g$ is 4 copies of $T^2$. When we divide it by the remaining action generated by $h$, we obtain twisted sectors consisting of 4 copies of $S^2$. The degree shifting number for these twisted sectors is $\frac{1}{2}$. For the same reason, the fixed point locus of $h$ gives twisted sectors consisting of 4 copies of $S^2$ with degree shifting number $\frac{1}{2}$. The fixed point locus of $gh$ is 16 points, which are fixed by the whole group. The degree shifting number of the 16 points is 1. An easy calculation shows that the non-twisted sector contributes one generator to degree 0, 4 orbifold cohomology and two generators to degree 2 orbifold cohomology and no other. Using this information, we can compute the ordinary orbifold cohomology group

$$b^0_{\text{orb,} \alpha} = b^1_{\text{orb,} \alpha} = 1, b^1_{\text{orb,} \alpha} = b^3_{\text{orb,} \alpha} = 8, b^2_{\text{orb}} = 18.$$

By example 2.10, $H^2(\mathbb{Z}_2 \times \mathbb{Z}_2, U(1)) = \mathbb{Z}_2$. By Remark 2.2, the nontrivial generator of $H^2(\mathbb{Z}_2 \times \mathbb{Z}_2, U(1))$ induces a discrete torsion $\alpha$. Next, we compute the twisted orbifold cohomology $H^*_{\text{orb,} \alpha}(T^4/\mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{C})$. Note that $\gamma(\alpha)_{gh,g} = \gamma(\alpha)_{gh,h} = -1$. Hence, the flat orbifold line bundles over the twisted sectors given by the 16 fixed points of $gh$ are nontrivial. Therefore, they contribute nothing to twisted orbifold cohomology. For two-dimensional twisted sectors, let’s consider a component of the fixed point locus of $g$. By the previous description, it is $T^2$. $h$ acts on $T^2$. Then the twisted sector $S^2 = T^2 \{h\}$. We observe that the flat orbifold line bundle over $S^2$ is constructed as $L = T^2 \times \gamma(\alpha)_{g} \mathbb{C}$. Hence $H^*(S^2, L)$ is isomorphic to the space of invariant cohomology of $T^2$ under the action of $h$ twisted by $\gamma(\alpha)_{g} h$. By example 2.10, $\gamma(\alpha)_{g,h} = -1$. The invariant cohomology is $H^1(T^2, \mathbb{C})$. Using the degree shifting number to shift up its degree, we obtain the twisted orbifold cohomology

$$b^0_{\text{orb,} \alpha} = b^1_{\text{orb,} \alpha} = 1, b^1_{\text{orb,} \alpha} = b^3_{\text{orb,} \alpha} = 0, b^2_{\text{orb}} = 18.$$

**Example 3.5.4** $WP(2,2d_1) \times WP(2,2d_2)$ ($d_1, d_2 > 1, (d_1, d_2) = 1$): Here, $WP(2,2d)$ is the weight projective space of weighted $(2,2d)$. $WP(2,2d_1) \times WP(2,2d_2)$ is not a global quotient unless $d_1 = d_2 = 1$. In fact, its orbifold universal cover is $WP(1,d_1) \times WP(1,d_2)$ and $WP(2,2d_1) \times WP(2,2d_2) = WP(1,d_1) \times WP(1,d_2)/\mathbb{Z}_2 \times \mathbb{Z}_2$. Hence, the orbifold fundamental group is $\mathbb{Z}_2 \times \mathbb{Z}_2$. Therefore, there is a nontrivial discrete torsion $\alpha \in H^2(\mathbb{Z}_2 \times \mathbb{Z}_2, U(1))$.

Next, we describe the twisted sectors. Suppose that $p = [0,1], q = [1,0] \in WP(1,d_1)$. We also use $p, q$ to denote its image in $WP(2,2d_1)$. We use $p', q'$ to denote the corresponding points in $WP(1,d_2), WP(2,2d_2)$. $\{p\} \times WP(2,2d_2), \{p'\} \times WP(2,2d_1)$ give rise to two twisted sectors with degree shifting number $\frac{1}{2}$. $\{q\} \times WP(2,2d_2), \{q'\} \times WP(2,2d_1)$ give rise to $2d_1 - 1, 2d_2 - 1$ twisted
sectors with degree shifting numbers \( \frac{i}{2d_1}, \frac{j}{2d_2} \) for \( 1 \leq i \leq 2d_1 - 1, 1 \leq j \leq 2d_2 - 1 \). \( \{p\} \times \{p'\} \) gives rise to a twisted sector with degree shifting number 1. \( \{p\} \times \{q'\} \) gives rise to \( 2d_2 - 1 \) twisted sectors with degree shifting numbers \( \frac{1}{2} + \frac{i}{2d_2} \) for \( 1 \leq i \leq 2d_2 - 1 \). \( \{q\} \times \{p'\} \) gives rise to \( 2d_1 - 1 \) twisted sectors with degree shifting numbers \( \frac{1}{2} + \frac{j}{2d_1} \) for \( 1 \leq i \leq 2d_1 - 1 \). \( \{q\} \times \{q'\} \) gives rise to \( 4d_1d_2 - 1 \) twisted sectors with degree shifting numbers \( \frac{i}{2d_1} + \frac{j}{2d_2} \) for all \( i, j \) except \((i, j) = (0, 0)\). Using this information, we can write down the ordinary orbifold cohomology

\[
\begin{align*}
b^0_{\text{orb}} &= b^4_{\text{orb}} = 1, b^1_{\text{orb}} = b^3_{\text{orb}} = 6, b^2_{\text{orb}} = 6 \\
b^1_{\text{orb}} &= b^5_{\text{orb}} = 1, b^{14}_{\text{orb}} = b^{15}_{\text{orb}} = 3, b^2_{\text{orb}} = b^3_{\text{orb}} = 2, 1 \leq i \leq d_1 - 1, 1 \leq j \leq d_2 - 1
\end{align*}
\]

(3.5.9) \[ b^i_{\text{orb,}\alpha} + d_j = 1, 0 \leq i \leq 2d_1 - 1, 0 \leq j \leq 2d_2, (i, j) \neq (0, 0), (d_1, d_2). \]

Next, we compute \( H^{*,*}_{\text{orb,}\alpha} \). In this example, the most of the twisted sectors are dormant sectors. To find nondormant sectors, recall that \( WP(2, 2d_1) \times WP(2, 2d_2) = WP(1, d_1) \times WP(1, d_2)/\mathbb{Z}_2 \times \mathbb{Z}_2 \). Let \( g \) be the generator of the first factor and \( h \) be the generator of the second factor. The fixed points of \( g \) are \( \{p, q\} \times WP(1, d_2) \). We have two nondormant sectors obtained by dividing by the remaining action generated by \( h \). However, \( \gamma(\alpha)_{g,h} = -1 \). There is no invariant cohomology of \( WP(1, d_2) \) under the action of \( h \) twisted by \( \gamma(\alpha)_{g} \). Hence, these two nondormant twisted sectors give no contribution to twisted orbifold cohomology. Their degree shifting numbers are 1. For the same reason, \( WP(1, d_1) \times \{p', q'\}/g \) gives no contribution to twisted orbifold cohomology. The fixed point locus of \( gh \) consists of 4 points which give 4 nondormant sectors. Again, their degree shifting numbers are 1. As we saw in the last example, their flat orbifold bundles are nontrivial. Hence, they give no contribution to twisted orbifold cohomology. Therefore, the twisted orbifold cohomology is

\[
\begin{align*}
b^0_{\text{orb,}\alpha} &= b^4_{\text{orb,}\alpha} = 1, b^1_{\text{orb,}\alpha} = b^3_{\text{orb,}\alpha} = 2, b^2_{\text{orb,}\alpha} = 2 \\
b^1_{\text{orb,}\alpha} &= b^5_{\text{orb,}\alpha} = 1, b^{14}_{\text{orb,}\alpha} = b^{15}_{\text{orb,}\alpha} = 3, b^2_{\text{orb,}\alpha} = b^3_{\text{orb,}\alpha} = 2, 1 \leq i \leq d_1 - 1, 1 \leq j \leq d_2 - 1
\end{align*}
\]

(3.5.10) \[ b^i_{\text{orb,}\alpha} + d_j = 1, 0 \leq i \leq 2d_1 - 1, 0 \leq j \leq 2d_2, (i, j) \neq (0, 0), (d_1, d_2). \]

**Example 3.5.5** \( T^6/\mathbb{Z}_4 \): Here, \( T^6 = \mathbb{C}^3/\Lambda \), where \( \Lambda \) is the lattice of integral points. The generator of \( \mathbb{Z}_4 \) acts on \( T^6 \) as

(3.5.11) \[ \kappa : (z_1, z_2, z_3) \rightarrow (-z_1, iz_2, iz_3). \]

This example has been studied by D. Joyce \[JO\], where he constructed five different desingularizations. However, there is no discrete torsion in the case which induces nontrivial orbifold cohomology.

First of all, the nontwisted sector contributes one generator to \( H^{0,0}_{\text{orb}}, H^{3,3}_{\text{orb}} \). 5 generators to \( H^{1,1}_{\text{orb}}, H^{2,2}_{\text{orb}} \) and 2 generators to \( H^{2,1}_{\text{orb}}, H^{1,2}_{\text{orb}} \). The fixed point locus of \( \kappa, \kappa^3 \) consists of 16 points

\[ \{(z_1, z_2, z_3) + \Lambda : z_1 \in \{0, \frac{1}{2}, \frac{1}{2} + \frac{i}{2}\}, z_2, z_3 \in \{0, \frac{1}{2} + \frac{i}{2}\}\}. \]

These points are fixed by \( \mathbb{Z}_4 \). Therefore, they generate 32 twisted sectors in which 16 correspond to the conjugacy class \( (\kappa) \) and 16 correspond to the conjugacy class \( (\kappa^3) \). The sector with conjugacy
class \((\kappa)\) has degree shifting number 1. The sector with conjugacy class \((\kappa^3)\) has degree shifting number 2.

The fixed point locus of \(\kappa^2\) is 16 copies of \(T^2\), given by

\[
\{(z_1, z_2, z_3) + \bigwedge z_1 \in \mathbb{C}, z_2, z_3 \in \{0, \frac{1}{2}, \frac{i}{2}, \frac{1}{2} + \frac{i}{2}\}\}
\]

Twelve of the 16 copies of \(T^2\) fixed by \(\kappa^2\) are identified in pairs by the action of \(\kappa\), and these contribute 6 copies of \(T^2\) to the singular set of \(T^6/\mathbb{Z}_4\). On the remaining 4 copies \(\kappa\) acts as \(-1\), so these contribute 4 copies of \(S^2 = T^2/\{\pm1\}\) to the singular set. The degree shifting number of these 2-dimensional twisted sectors is 1.

Next, we construct local systems. We start with two-dimensional twisted sectors. Since \(\kappa^{-2} = \kappa^2\), the condition (2) of Definition 2.1 tells us that the flat orbifold line bundle \(L\) over two-dimensional sectors has the property \(L^2 = 1\). Now, we assign the trivial line bundle to all \(T^2\)-sectors and \(k(k = 0, 1, 2, 3, 4)\) \(S^2 = T^2/\{\pm1\}\)-sectors. For the remaining \(S^2 = T^2/\{\pm1\}\)-sectors, we assign a flat orbifold line bundle \(T^2 \times \mathbb{C}/\{\pm1\}\). For the zero-dimensional sectors, they are all points of two-dimensional sectors. If we assign a trivial bundle on a two-dimensional sector, we just assign the trivial bundle to its point sectors. For these two-dimensional sectors with nontrivial flat line bundle, we need to be careful to choose the flat orbifold line bundle on its point sectors to ensure the condition (3) of Definition 2.1. Suppose that \(\Sigma\) is one of the 2-dimensional sectors supporting a nontrivial flat orbifold line bundle. It contains 4 singular points which generate the point sectors. Let \(x\) be one of the 4 points. \(x\) generates two sectors given by the conjugacy classes \((\kappa), (\kappa^3)\). For condition (3), we have to consider the conjugacy class of the triple \((g_1, g_2, g_3)\) with \(g_1g_2g_3 = 1\). The only nontrivial choices are \((g) = (\kappa, \kappa, \kappa^2), (\kappa^2, \kappa^3, \kappa)\). The corresponding components of \(X_{(g)}\) are exactly these singular points. Clearly, \(x\) is fixed by the whole group \(\mathbb{Z}_4\). The orbifold line bundle is given by the action of \(\mathbb{Z}_4\) on \(\mathbb{C}\). Consider the component of \(X_{(g)}\) generated by \(x\). The pull-back of the flat orbifold line bundle from the 2-dimensional sector \((\kappa^2)\)-sector is given by the action \(\kappa v = -v\). A moment’s thought tells us that for sectors \((\kappa), (\kappa^3)\), we should assign a flat orbifold line bundle given by the action of \(\mathbb{Z}_4\) on \(\mathbb{C}\) as \(\kappa v = iv\). It is easy to check that for the above choices the condition (3) is satisfied for \(X_{(g)}\). Therefore, the twisted sectors given by \((x, (\kappa)), (x, (\kappa^3))\) give no contribution to twisted orbifold cohomology. Suppose that the resulting local system is \(L_k\). For the sectors with trivial line bundle, they contribute \(6 + k\) generators to \(I^{1,1}_{\text{orb}}, I^{2,2}_{\text{orb}}\) and 6 generators to \(H^{2,1,1}_{\text{orb}}, H^{1,2,1}_{\text{orb}}\). Its point sectors contribute \(4k\) generators to \(H^{2,1}_{\text{orb}}, H^{1,2}_{\text{orb}}\). The remaining sectors contribute \(4 - k\) generators to \(H^{2,1}_{\text{orb}}, H^{1,2}_{\text{orb}}\). Its point sectors give no contribution. Moreover, the nontwisted sector contributes

\[
h^{0,0} = h^{3,3} = 2, h^{1,1} = 5.
\]

In summary, we obtain

\[
\dim H^{0,0}_{\text{orb}}(T^6/\mathbb{Z}_4, \mathcal{L}_k) = \dim H^{3,3}_{\text{orb}}(T^6/\mathbb{Z}_4, \mathcal{L}_k) = 1, \dim H^{1,1}_{\text{orb}}(T^6/\mathbb{Z}_4, \mathcal{L}_k) = \dim H^{2,2}_{\text{orb}}(T^6/\mathbb{Z}_4, \mathcal{L}_k) = 11 + 5k,
\]

(3.5.12)

\[
\dim H^{1,2}_{\text{orb}}(T^6/\mathbb{Z}_4, \mathcal{L}_k) = \dim H^{2,1}_{\text{orb}}(T^6/\mathbb{Z}_4, \mathcal{L}_k) = 12 - k.
\]

Our calculation matches the Betti numbers of Joyce’s desingularizations [JO].

4 Orbifold K-theory

It was known classically that any reduced orbifold can be expressed as \(P/G\), where \(P\) is a smooth manifold and \(G\) is a compact Lie group acting smoothly on \(P\) such that \(G\) has only finite isotropy
subgroups. Therefore, it is natural to use the equivariant theory of $P$ to capture the theory on $P/G$. The classical case is the equivariant cohomology $H^*_G(P)$. If $G$ is connected, it is known that $H^*_G(P, \mathbb{C}) = H^*(P/G, \mathbb{C})$, i.e., the nontwisted sector. In the case of a global quotient $X/G$ for a finite group $G$ Atiyah-Segal and others showed that equivariant $K$-theory $K_G(X)$ carries more information. In fact, $K_G(X) \otimes \mathbb{C} = H^*_G(X/G, \mathbb{C})$. This section has two purposes. (1) We would like to generalize Atiyah-Segal [AS] and results of others to a general orbifold; (2) more importantly, we want to incorporate discrete torsion into our theory. The latter leads to some unexpected structure unique from the $K$-theory point of view. This section is a joint work with Alejandro Adem [AR]. Some related work has been done in the context of the $K$-theory of algebraic vector bundles in algebraic geometry by Vistoli, B. Toen [T], and algebraic $K$-theory of $C^*$-algebra module by Marcolli-Mathai [MM].

### 4.1 Projective representation

Mathematically, our construction is based on projective representation. This subsection is a review of basic material on projective representations of finite group. Throughout this subsection, we will assume that $G$ is finite. Most of the background results which we list appear in [KA] Chapter III.

**Definition 4.1.1:** Let $V$ denote a finite dimensional complex vector space. A mapping $\rho: G \to \text{GL}(V)$ is called a projective representation of $G$ if there exists a function $\alpha: G \times G \to \mathbb{C}^*$ such that $\rho(x)\rho(y) = \alpha(x, y)\rho(xy)$ for all $x, y \in G$ and $\rho(1) = \text{Id}_V$.

Note that $\alpha$ defines a $\mathbb{C}^*$-valued cocycle on $G$, i.e. $\alpha \in Z^2(G, \mathbb{C}^*)$. Also there is a one-to-one correspondence between projective representations of $G$ as above and homomorphisms from $G$ to $P\text{GL}(V)$. We will be interested in the notion of linear equivalence of projective representations.

**Definition 4.1.2:** Two projective representations $\rho_1: G \to \text{GL}(V_1)$ and $\rho_2: G \to \text{GL}(V_2)$ are said to be linearly equivalent if there exists a vector space isomorphism $f: V_1 \to V_2$ such that $\rho_2(g) = f\rho_1(g)f^{-1}$ for all $g \in G$.

If $\alpha$ is the cocycle attached to $\rho$, we say that $\rho$ is an $\alpha$-representation on the space $V$. We list a couple of basic results

**Lemma 4.1.3:** Let $\rho_i$, $i = 1, 2$ be an $\alpha_i$-representation on the space $V_i$. If $\rho_1$ is linearly equivalent to $\rho_2$, then $\alpha_1$ is equal to $\alpha_2$.

It is easy to see that given a fixed cocycle $\alpha$, we can take the direct sum of any two $\alpha$-representations. Hence we can introduce

**Definition 4.1.4:** We define $M_\alpha(G)$ as the monoid of linear isomorphism classes of $\alpha$-representations of $G$. Its associated Grothendieck group will be denoted $R_\alpha(G)$.

In order to use these constructions we need to introduce the notion of a twisted group algebra. If $\alpha: G \times G \to \mathbb{C}^*$ is a cocycle, we denote by $C^\alpha G$ the vector space over $\mathbb{C}$ with basis $\{ g | g \in G \}$ with product

$$\bar{x} \cdot \bar{y} = \alpha(x, y)\bar{xy}$$

extended distributively.

One can check that $C^\alpha G$ is a $\mathbb{C}$-algebra with $\mathbb{1}$ as the identity element. This algebra is called the $\alpha$-twisted group algebra of $G$ over $\mathbb{C}$. Note that if $\alpha(x, y) = 1$ for all $x, y \in G$, then $C^\alpha G = CG$.

**Definition 4.1.5:** If $\alpha$ and $\beta$ are cocycles, then $C^\alpha G$ and $C^\beta G$ are equivalent if there exists a
\[ \psi : C^\alpha G \rightarrow C^\beta G \]

and a mapping \( t : G \rightarrow C^* \) such that \( \psi(\overline{g}) = t(g)\overline{g} \) for all \( g \in G \), where \( \{\overline{g}\} \) and \( \{\overline{g}\} \) are bases for the two twisted algebras.

We have a basic result which classifies twisted group algebras.

**Theorem 4.1.6:** We have an isomorphism between twisted group algebras, \( C^\alpha G \cong C^\beta G \), if and only if \( \alpha \) is cohomologous to \( \beta \); hence if \( \alpha \) is a coboundary, \( \alpha \mapsto \cdot \) induces a bijective correspondence between \( H^2(\Gamma, C^*) \) and the set of equivalence classes of twisted group algebras of \( G \) over \( C \).

Next we recall how these twisted algebras play a role in determining \( R_\alpha(G) \).

**Theorem 4.1.7:** There is a bijective correspondence between \( \alpha \)-representations of \( G \) and \( C^\alpha G \)-modules. This correspondence preserves sums and bijectively maps linearly equivalent (respectively irreducible, completely reducible) representations into isomorphic (respectively irreducible, completely reducible) modules.

Recall the Definition 3.1.15 that an element \( g \in G \) is said to be \( \alpha \)-regular if \( \alpha(g,x) = \alpha(x,g) \) for all \( x \in C_G(g) \).

Note that the identity element is \( \alpha \)-regular for all \( \alpha \). Also one can see that \( g \) is \( \alpha \)-regular if and only if \( g \cdot x = x \cdot g \) for all \( x \in C_G(g) \).

If an element \( g \in G \) is \( \alpha \)-regular, then any conjugate of \( g \) is also \( \alpha \)-regular, hence we can speak of \( \alpha \)-regular conjugacy classes in \( G \). For technical purposes we also want to introduce the notion of a ‘standard’ cocycle; it will be a cocycle \( \alpha \) with values in \( C^* \) such that (1) \( \alpha(x,x^{-1}) = 1 \) for all \( x \in G \) and (2) \( \alpha(x,g)\alpha(xg,x^{-1}) = 1 \) for all \( \alpha \)-regular \( g \in G \) and all \( x \in G \). Expressed otherwise, this simply means that \( \alpha \) is standard if and only if for all \( x \in G \) and for all \( \alpha \)-regular elements \( g \in G \), we have \( \overline{x^{-1}} = x^{-1} \) and \( \overline{xgx^{-1}} = \overline{xgx^{-1}} \). It can be shown that in fact any cohomology class \( c \in H^2(\Gamma, C^*) \) can be represented by a standard cocycle, hence we will make this assumption from now on.

The next result is basic:

**Theorem 4.1.8:** If \( r_\alpha \) is equal to the number of non-isomorphic irreducible \( C^\alpha G \)-modules, then this number is equal to the number of distinct \( \alpha \)-regular conjugacy classes of \( G \). In particular \( R_\alpha(G) \) is a free abelian group of rank equal to \( r_\alpha \).

### 4.2 Twisted Equivariant K–theory and Decomposition Theorem

In this subsection, we assume that \( G \) is a semi-direct product of a compact Lie group \( H \) and a discrete group \( \Gamma \) where \( H \) is a compact Lie group and \( \Gamma \) is a discrete group. Suppose \( \alpha \in H^2(\Gamma, U(1)) \). We have a group extension

\[ 1 \rightarrow S^1 \rightarrow \tilde{\Gamma} \rightarrow \Gamma \rightarrow 1. \]

Let \( \tilde{G} \) be the semi-direct product of \( H \) and \( \tilde{\Gamma} \).

Suppose that \( G \) acts on a smooth manifold \( X \) such that \( X/G \) is compact and the action has only finite isotropy subgroup. It is well-known that \( Y = X/G \) is an orbifold. We are now ready to define a twisted version of equivariant K-theory.

**Definition 4.2.3:** An \( \alpha \)-twisted \( G \)-vector bundle on \( X \) is a complex vector bundle \( E \rightarrow X \) such that \( S^1 \) acts on the fibers through complex multiplication so that the action extends to an action of
$\tilde{G}_\alpha$ on $E$ which covers the given $G$–action on $X$.

In fact $E \to X$ is a $\tilde{G}_\alpha$–vector bundle, where the action on the base is via the projection onto $G$ and the given $G$–action. Note that if we divide out by the action of $S^1$, we obtain a projective bundle over $X$.

**Definition 4.2.4:** We define the $\alpha$–twisted $G$–equivariant $K$–theory of $X$, denoted by $^\alpha K_G(X)$, as the Grothendieck group of isomorphism classes of $\alpha$–twisted $G$–bundles over $X$.

We begin by considering the case $\alpha = 0$; this corresponds to the split extension $G \times S^1$. Any ordinary $G$–vector bundle can be made into a $G \times S^1$–bundle via scalar multiplication on the fibrers; conversely a $G \times S^1$–bundle restricts to an ordinary $G$–bundle. Hence we readily see that $^0 K_G(X) = K_G(X)$.

Next we consider the case when $X$ is equal to a point. It is easy to verify that we obtain $R_0(G)$. More generally, if we consider an orbit $G/H$, then we have $^\alpha K_G(G/H) = R_{res_\alpha}^G(H)$.

The reader may have noticed that our twisted equivariant $K$–theory does not have a product structure. Moreover it depends on a choice of a particular cohomology class in $H^2(\Gamma, S^1)$. Our next goal is to relate the different twisted versions by using a product structure inherited from the additive structure of group extensions.

Suppose we are given $\alpha, \beta$ in $H^2(\Gamma, S^1)$, represented by central extensions $1 \to S^1 \to \tilde{\Gamma}_1 \to \Gamma \to 1$ and $1 \to S^1 \to \tilde{\Gamma}_2 \to \Gamma \to 1$. These give rise to a central extension of the form

$$1 \to S^1 \times S^1 \to \tilde{\Gamma}_1 \times \tilde{\Gamma}_2 \to \Gamma \times \Gamma \to 1.$$

Now we make use of the diagonal embedding $\Delta : \Gamma \to \Gamma \times \Gamma$ and the product map $\mu : S^1 \times S^1 \to S^1$ to obtain a central extension

$$1 \to \mu(S^1 \times S^1) \to \tilde{\Gamma} \to \Delta(\Gamma) \to 1.$$

This operation corresponds to the sum of cohomology classes, i.e. the above extension represents $\alpha + \beta$. Note that $ker \mu = \{(z, z^{-1})\} \subset S^1 \times S^1$. Furthermore, we can pull back the above construction over $G$.

Now consider an $\alpha$–twisted bundle $E \to X$ and a $\beta$–twisted bundle $F \to X$. Consider the tensor product bundle $E \otimes F \to X$. Clearly it will have a $\tilde{G}_1 \times \tilde{G}_2$ action on it, which we can restrict to the inverse image of $\Delta(G)$. Now note that $ker \mu$ acts trivially on $E \otimes F$, hence we obtain a $\tilde{G}$ action on $E \otimes F$, covering the $G$–action on $X$. This is an $\alpha + \beta$–twisted bundle over $X$. Hence we have defined a product

$$^\alpha K_G(X) \otimes^\beta K_G(X) \to ^{\alpha + \beta} K_G(X)$$

which prompts us to introduce the following definition.

**Definition 4.2.5:** The total twisted equivariant $K$–theory of a $G$ space $X$ is defined as

$$TK_G(X) = \bigoplus_{\alpha \in H^2(\Gamma, S^1)} ^\alpha K_G(X)$$

Using the product above, we deduce that $TK_G(X)$ is a graded algebra, as well as a module over $K_G(X)$.

We obtain a purely algebraic construction from the above when $X$ is a point. Namely we obtain the total twisted representation ring of $G$, defined as

$$TR(G) = \bigoplus_{\alpha \in H^2(\Gamma, S^1)} R_\alpha(G),$$

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endowed with the graded algebra structure defined above. Note that if \( \alpha \) is the cocycle representing a cohomology class, then \( \alpha^{-1} \) will represent \(-\alpha\). Hence we see that \( \rho \mapsto \rho^* \) defines an isomorphism between \( R_\alpha(G) \) and \( R_{-\alpha}(G) \) (indeed, using vector bundles instead we can easily extend this to show that \( \alpha K_G(X) \) is isomorphic to \( -\alpha K_G(X) \)).

Next, we relate our (twisted) equivariant K-theory to twisted orbifold cohomology. Note that \( X/H \) is an orbifold covering of \( X/G \) with covering group \( \Gamma \). Therefore, there is a surjective homo-
morphism \( \pi^{orb}_1(X) \to \Gamma \). \( \alpha \in H^2(\Gamma, U(1)) \) induces a discrete torsion of \( X \) (still denoted by \( \alpha \)). Then, we have following theorem.

**Theorem 4.2.6:** Suppose that \( G \) acts on \( X \) such that (i) \( X/G \) is compact; (ii) the action has only finite isotropy subgroups. For any \( \alpha \in H^2(\Gamma, S^1) \) we have a decomposition

\[
\alpha K_G(X) \otimes \mathbb{C} \cong H^*_{orb, \alpha}(X/G, \mathbb{C}),
\]

Recall that \( H^*_{orb, \alpha}(X/G, \mathbb{C}) \) is a summation over each sector. Therefore, Theorem 4.2.6 can be viewed as a decomposition theorem of twisted equivariant K-theory.

**Proof:** We outline a proof in the case that \( G \) is a finite group. The general case requires a more complicated argument. We refer readers to our paper.

The proof requires constructing an \( \alpha \)--twisted equivariant Chern character. Fix the class \( \alpha \in H^2(G, S^1) \); for any subgroup \( H \subset G \) we let \( \alpha_H = res_H^\alpha \). Given any such \( H \), we can associate

\[
H \mapsto R_{\alpha_h}(H)
\]

Note the special case when \( H = \langle g \rangle \), a cyclic subgroup. As \( H^2(\langle g \rangle, S^1) = 0 \), \( \alpha_{\langle g \rangle}(\langle g \rangle) \) is additively isomorphic to \( R(\langle g \rangle) \).

As mentioned before, we can assume that \( \alpha \) is a standard cocycle. If \( z \in C_G(g) \), then it will define an action in the following manner, where we use the \( \alpha \)--twisted product:

\[
\overline{z} \cdot \overline{g} \cdot \overline{z}^{-1} = \alpha(z, g) \alpha(g, z)^{-1} \overline{g}
\]

Recalling our definition of the character \( L^\alpha_g \) for \( C_G(g) \), we see that it agrees precisely with \( z \mapsto \alpha(z, g) \alpha(g, z)^{-1} \). Hence we infer from this that there is an isomorphism of \( C_G(g) \)--modules

\[
R_{\alpha_{\langle g \rangle}}(\langle g \rangle) \otimes \cong R(\langle g \rangle) \otimes L^\alpha_g
\]

for all \( g \in G \). Note that \( R(\langle g \rangle) \) is a trivial \( C_G(g) \)--module.

Now consider \( E \to X \), an \( \alpha \)--twisted bundle over \( X \); it restricts to an \( \alpha_{\langle g \rangle} \)--twisted bundle over the fixed point set \( X_g \). We have isomorphisms of \( C_G(g) \)--modules:

\[
\alpha K_{\langle g \rangle}(X_g) \otimes \cong K(X_g) \otimes R_{\alpha_{\langle g \rangle}}(\langle g \rangle) \otimes \cong K(X_g) \otimes R(\langle g \rangle) \otimes L^\alpha_g.
\]

Let \( u : R(\langle g \rangle) \to \) denote the map \( \chi \mapsto \chi(g) \). Then the composition

\[
\alpha K_G(X) \to K(X_g) \otimes R(\langle g \rangle) \otimes L^\alpha_g \to K(X_g) \otimes L^\alpha_g
\]

has image lying in the invariants under the \( C_G(g) \)--action. Hence we can put these together to yield a map

\[
(4.2.1) \quad \alpha K_G(X) \otimes \mathbb{C} \cong \bigoplus_{(g)} (K(X_g) \otimes L^\alpha_g)^{C_G(g)} \cong H^*_{orb, \alpha}(X/G, \mathbb{C}).
\]

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One checks that this induces an isomorphism on orbits $G/H$; the desired isomorphism follows from using induction on the number of $G$–cells in $X$ and a Mayer-Vietoris argument (as in \cite{AS}).

4.3 Orbifold K-theory

In last section, we develop the theory in an equivariant setting, where our starting point is a smooth manifold with a smooth action of a Lie group. Hence, the quotient is an orbifold. However, there are many different way to represent an orbifold as such a quotient. In this section, our starting point is the orbifold itself. We will use equivariant theory as an intermediate object.

Recall that for every orbifold $X$ there is an orbifold universal cover $Y \to X$ such that the covering group is $\pi_1^{orb}(X)$. Then a discrete torsion $\alpha$ is an element of $H^2(\pi_1^{orb}(X), S^1)$. Suppose that $1 \to S^1 \to \tilde{G} \to G \to 1$ is the extension determined by $\alpha$. We define

**Definition 4.3.1:** We define $\alpha K^{orb}(X)$ to be the Grothendieck group of isomorphism classes of $\alpha$-twisted $\pi_1^{orb}(X)$-orbifold bundles over $Y$ and total orbifold K-theory

\[
TK^{orb}(X) = \bigoplus_{\alpha \in H^2(\pi_1^{orb}(X), S^1)} K^{orb}(X).
\]

Furthermore, we define the (twisted) orbifold Euler characteristic $\chi_{\alpha}(X) = \chi(\alpha K^{orb}(X))$.

Suppose that $X$ is a reduced orbifold. So is the orbifold universal cover $Y$. Choose a Riemannian metric on $X$. The pull-back metric on $Y$ is $\pi_1^{orb}(X)$-invariant. It is well-known that the frame bundle $P(Y)$ is a smooth manifold such that $O(n)$ acts on $P(Y)$ with finite isotropy subgroups. Since $\pi_1^{orb}(X)$ acts as isometries, its action lifts to the action on $P(Y)$. Moreover, the two actions commute. Then, we take $G = SO(n) \times \pi_1^{orb}(X)$. It acts on $P(Y)$ with finite isotropy subgroups. It is obvious that $X = P(Y)/G$. It is easy to check that

\[
\alpha K^{orb}(X) \cong \alpha K_G(P(Y)).
\]

Using our decomposition theorem,

**Theorem 4.3.2:** Suppose that $X$ is a reduced orbifold. Then, for any discrete torsion $\alpha \in H^2(\pi_1^{orb}(X), S^1)$, there is an additive isomorphism

\[
\alpha K^{orb}(X) \otimes \mathbb{C} \cong H_{orb, \alpha}^*(X, \mathbb{C}).
\]

4.4 Examples

**Example 4.4.1:** Suppose that $X$ is a point with a nontrivial group $G$. It is obvious that $\alpha K_G(X) = R_{\alpha}(G)$. Our theorem 4.3.2 yields that $R_{\alpha}(G) \otimes \mathbb{C}$ has rank equal to the number of conjugacy classes of elements in $G$ such that the associated character $L^\alpha_g$ is trivial. This of course agrees with the number of $\alpha$–regular conjugacy classes, as indeed $\alpha K_G(X) = R_{\alpha}(G)$. Moreover, the twisted orbifold Euler characteristic $\chi_{\alpha}$ equals the number of $\alpha$-regular conjugacy classes.

**Example 4.4.2:** We will now consider the case of a weighted projective space. $\mathbb{C}P(d_1, d_2)$ where $(d_1, d_2) = 1$. Let $S^1$ act on the unit sphere in $\mathbb{C}^2$ via

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\((v, w) \mapsto (z^{d_1}v, z^{d_2}w)\).

The space \(\mathbb{C}P(d_1, d_2)\) is the quotient under this action, and it has two singular points. \(x = [1, 0]\) and \(y = [0, 1]\). In this case the Lie group used to present the orbifold is \(SO(2) = S^1\) and the corresponding isotropy subgroups are precisely \(\mathbb{Z}/d_1\) and \(\mathbb{Z}/d_2\). Their fixed point sets are disjoint circles in \(S^3\), hence the formula for the orbifold K-theory yields

\[
K^*_\text{orb}(\mathbb{C}P(d_1, d_2)) \cong_{\mathbb{C}} \bigoplus_{d_1 + d_2 - 2} K^*((\ast)) \oplus K^*(\mathbb{C}P(d_1, d_2)).
\]

This is an additive formula which yields the following orbifold Euler characteristic

\[
\chi_{\text{orb}}(\mathbb{C}P(d_1, d_2)) = d_1 + d_2.
\]

**Example 4.4.3:** Let \(G(\mathbb{R})\) denote a semisimple \(\mathbb{Q}\)-group, and \(K\) a maximal compact subgroup. Let \(\Gamma \subset G(\mathbb{Q})\) denote an arithmetic subgroup. Then \(\Gamma\) acts on \(X = G(\mathbb{R})/K\), a space diffeomorphic to euclidean space. Moreover if \(H\) is any finite subgroup of \(\Gamma\), then \(X^H\) is a totally geodesic submanifold, hence also diffeomorphic to euclidean space. We can make use of the Borel-Serre completion \(\overline{X}\). This is a contractible space with a proper \(\Gamma\)-action such that the \(\overline{X}^H\) are also contractible (we are indebted to A.Borel for outlining a proof of this [BO]) but having a compact orbit space \(\overline{X}/\Gamma\). This of course is a basic geometric restriction on these groups, in particular implying that an arithmetic group has only finitely many conjugacy classes of finite subgroups, all of bounded order. Furthermore, using equivariant triangulation, we can assume that \(\overline{X}\) is a proper, finite \(\Gamma\)-CW complex. From this the decomposition theorem allows us to express the orbifold Euler characteristic of \(X/\Gamma\) in terms of group cohomology:

**Theorem 4.4**

\[
\chi_{\text{orb}}(X/\Gamma) = \sum_{\{\gamma \in \Gamma \mid \gamma \text{ of finite order}\}} \chi(Z_{\Gamma}(\gamma))
\]

We now illustrate this with a well-known example:

Let \(K\) be a totally real number field with ring of integers \(\mathcal{O}\), and let \(\zeta_k\) denote the Dedekind zeta function associated to \(k\). The centralizer of every finite subgroup in \(\Gamma = SL_2(\mathcal{O})\) is finite, except for \(\pm 1\). Let \(n(\Gamma)\) denote the number of distinct conjugacy classes of elements of finite order in \(\Gamma\). Then applying the above corollary we obtain

\[
\chi_{\text{orb}}(X/\Gamma) = n(\Gamma) - 2 + 2\chi(X/\Gamma)
\]

Using a formula due to Brown [B] for the regular Euler characteristic, we obtain the following:

\[
\chi_{\text{orb}}(X/\Gamma) = n(\Gamma) - 2 + 4\zeta_k(-1) + \sum_{(H)} (2 - \frac{4}{|H|})
\]

where \(H\) ranges over all \(\Gamma\)-conjugacy classes of maximal finite subgroups in \(\Gamma\).

### 5 Orbifold Quantum Cohomology

In the last two sections, we discuss the stringy topology of orbifold from both the cohomological and \(K\)-theoretic points of view. I hope that I have demonstrated that the stringy topology of
orbifolds is a rich field where geometry, group theory and physics naturally intertwine. However, one should view it only as a beginning. For example, there is a whole range of theories in differential topology related to transversality theory. These theories are very difficult to generalize to singular manifold. I believe that our stringy topology opens a door for such an orbifold differential topology. K-theoretic point of view of stringy topology suggests that there must be much more interaction between such a classical subject as algebraic topology and string theory. However, it is beyond author’s expertise.

Now, we turn to stringy geometry of orbifolds. The stringy geometry is much less understood than stringy topology. Since we are primarily motivated by physics, a natural question of stringy geometry is whether we can quantize the theory. Namely, can we build a theory of orbifold quantum cohomology such that orbifold cohomology is the classical part of orbifold quantum cohomology. The answer is yes. However, we need to restrict ourselves to symplectic or projective orbifolds. It is still an interesting question whether a projective orbifold is always symplectic. This section is joint work of the author with W. Chen [CR2].

5.1 Orbifold stable map

The most important step of our construction of orbifold quantum cohomology is to have an appropriate definition of stable map. This is a nontrivial step because a naive straightforward generalization is wrong. The key new concept is that of good map which we introduced in section 1. The central theorem of this section is that the moduli space of orbifold stable maps is a compact, Hausdorff, metrizable space. An equivalent formulation of orbifold stable map in algebraic geometry was studied independently by D. Abromivich and Vistoli [AV].

We first recall

Definition 5.1.1: A nodal Riemann surface with $k$ marked points is a pair $(\Sigma, z)$ of a connected topological space $\Sigma = \bigcup \pi_{\Sigma_{\nu}}(\Sigma_{\nu})$, where $\Sigma_{\nu}$ is a smooth complex curve and $\pi_{\nu} : \Sigma_{\nu} \to \Sigma$ is a continuous map, and $z = (z_1, \cdots, z_k)$ are distinct $k$ points in $\Sigma$ with the following properties.

- For each $z \in \Sigma_{\nu}$, there is a neighborhood of it such that the restriction of $\pi_{\nu} : \Sigma_{\nu} \to \Sigma$ to this neighborhood is a homeomorphism onto its image.
- For each $z \in \Sigma$, we have $\sum_{\nu} \#\pi_{\nu}^{-1}(z) \leq 2$.
- $\sum_{\nu} \#\pi_{\nu}^{-1}(z_i) = 1$ for each $z_i \in z$.
- The number of complex curves $\Sigma_{\nu}$ is finite.
- The set of nodal points $\{z | \sum_{\nu} \#\pi_{\nu}^{-1}(z) = 2\}$ is finite.

A point $z \in \Sigma_{\nu}$ is called singular if $\sum_{\omega} \#\pi_{\omega}^{-1}(\pi_{\nu}(z)) = 2$. A point $z \in \Sigma_{\nu}$ is said to be a marked point if $\pi_{\nu}(z) = z_i \in z$. Each $\Sigma_{\nu}$ is called a component of $\Sigma$. Let $k_{\nu}$ be the number of points on $\Sigma_{\nu}$ which are either singular or marked, and $g_{\nu}$ be the genus of $\Sigma_{\nu}$; a nodal curve $(\Sigma, z)$ is called stable if $k_{\nu} + 2g_{\nu} \geq 3$ holds for each component $\Sigma_{\nu}$ of $\Sigma$.

A map $\vartheta : \Sigma \to \Sigma'$ between two nodal curves is called as isomorphism if it is homeomorphism and if it can be lifted to biholomorphisms $\vartheta_{\nu} : \Sigma_{\nu} \to \Sigma'_{\nu}$ for each component $\Sigma_{\nu}$ of $\Sigma$. If $\Sigma, \Sigma'$ have marked points $z = (z_1, \cdots, z_k)$ and $z' = (z'_1, \cdots, z'_k)$ then we require $\vartheta(z_i) = z'_i$ for each $i$. Let $Aut(\Sigma, z)$ be the group of automorphisms of $(\Sigma, z)$. 
Each nodal curve \((\Sigma, z)\) is canonically associated with a graph \(T_\Sigma\) as follows. The vertices of \(T_\Sigma\) correspond to the components of \(\Sigma\) and for each pair of components intersecting each other in \(\Sigma\) there is an edge joining the corresponding two vertices. For each point \(z \in \Sigma\) such that \(\#\pi_\nu^{-1}(z) = 2\), there is an edge joining the same vertex corresponding to \(\Sigma_\nu\). For each marked point, there is a half open edge (tail) attaching to the vertex. The graph \(T_\Sigma\) is connected since \(\Sigma\) is connected. We can smooth out all the nodal points to obtain a smooth surface. Its genus is called arithmetic genus of \(\Sigma\). The arithmetic genus can be computed by the formula

\[
g = \sum_\nu g_\nu + \text{rank} H_1(T; \mathbb{Q}).
\]

**Definition 5.1.2:** A nodal orbicurve is a nodal marked curve \((\Sigma, z)\) with an orbifold structure on each component \(\Sigma_\nu\) satisfying the following conditions.

- The singular set \(z_\nu = \Sigma \Sigma_\nu\) (in the sense of orbifolds) of each component \(\Sigma_\nu\) is contained in the set of marked points and nodal points \(z\).
- If \(z_\nu \in \Sigma_\nu\) and \(z_\omega \in \Sigma_\omega\) satisfy \(\pi_\nu(z_\nu) = \pi_\omega(z_\omega)\), then the germs of uniformizing systems at \(z_\nu\) and \(z_\omega\) are the same. (Here \(\nu\) and \(\omega\) may be identical.)

Let a disc neighborhood of \(z = \pi_\nu^{-1}(z_\nu)\) be uniformized by a branched covering map \(z \to z_{m_\nu}\), and a disc neighborhood of \(z\) such that \(\sum_\omega \pi_\omega^{-1}(\pi_\nu(z)) = 2\) be uniformized by \(z \to z_{n_\nu}\). Here \(m_\nu\) and \(n_\nu\) are allowed to be equal to one, i.e., the corresponding orbifold structure is trivial there. We denote the corresponding nodal orbicurve by \((\Sigma, z, m, n)\), where \(m = (m_1, \ldots, m_k)\) and \(n = (n_j)\).

An isomorphism between two nodal orbicurves \(\vartheta : (\Sigma, z, m, n) \to (\Sigma', z', m', n')\) is a collection of \(C^\infty\) isomorphisms \(\vartheta_{\pi_\omega}\) between orbicurves \(\Sigma_\nu\) and \(\Sigma_\omega'\) which induces an isomorphism \(\vartheta : (\Sigma, z) \to (\Sigma', z')\). The group of automorphisms of a nodal orbicurve \((\Sigma, z, m, n)\) is denoted by \(\text{Aut}(\Sigma, z, m, n)\). It is easily seen that \(\text{Aut}(\Sigma, z, m, n)\) is a subgroup of \(\text{Aut}(\Sigma, z)\) of finite index.

**Definition 5.1.3:** Let \((X, J)\) be an almost complex orbifold. An orbifold stable map is a triple \((f, (\Sigma, z), \xi)\) described as follows:

1. \(f\) is a continuous map from a nodal curve \((\Sigma, z)\) into \(X\) such that each \(f_\nu = f \circ \pi_\nu\) is a pseudo-holomorphic map from \(\Sigma_\nu\) into \(X\).

2. \(\xi = \{\xi_\nu\}\) where each \(\xi_\nu\) is an orbifold structure on \(\Sigma_\nu\) together with a compactible system of the (unique) \(C^\infty\) lifting of \(f_\nu : \Sigma_\nu \to X\). The orbifold structure on \(\Sigma_\nu\) has its set of orbifold points contained in \(z_\nu\) where \(z_\nu\) is the union of marked points and singular points in \(\Sigma_\nu\). Moreover, the homomorphism on the local group is injective.

3. Let \(\{f_{\nu, UU'}, \lambda_\nu\}\) be the compatible system defined by \(\xi_\nu\) for each \(\nu\). Then for any \(z_\nu \in \Sigma_\nu\) and \(z_\omega \in \Sigma_\omega\) satisfy \(\pi_\nu(z_\nu) = \pi_\omega(z_\omega)\) (here \(\nu\) and \(\omega\) may be identical); if we let \(p = f(\pi_\nu(z_\nu))\), and the group homomorphism of \(\lambda_\nu\) at \(z_\nu\) be given by \(e^{2\pi i/n_\nu} \to g_\nu\) and the group homomorphism of \(\lambda_\omega\) at \(z_\omega\) be given by \(e^{2\pi i/n_\omega} \to g_\omega\), then \(n_\nu = n_\omega\) and \(g_\nu = g_\omega^{-1}\) in \(G_p\).

4. Let \(k_\nu\) be the order of the set \(z_\nu\), namely the number of points on \(\Sigma_\nu\) which are singular (i.e. nodal or marked); if \(f_\nu\) is a constant map, then \(k_\nu + 2g_\nu \geq 3\).

We will call \(\xi\) a twisted boundary condition of \(f : (\Sigma, z) \to X\). cf. Remark 2.2.10.

A stable map \((f, (\Sigma, z), \xi)\) determines a unique orbifold structure \((\Sigma, z, m, n)\) on \((\Sigma, z)\), as part of \(\xi = \{\xi_\nu\}\). We introduce an equivalence relation amongst the set of stable maps as follows:
two stable maps \((f, (\Sigma, z), \xi)\) and \((f', (\Sigma', z'), \xi')\) are equivalent if there exists an isomorphism \(\vartheta : (\Sigma, z, m, n) \rightarrow (\Sigma', z', m', n')\) such that \(f' \circ \vartheta = f\), and the compatible systems defined by \(\xi'\) pull back via \(\vartheta\) to compatible systems isomorphic to the ones defined by \(\xi\) (we write this as \(\xi' \circ \vartheta = \xi\)).

The automorphism group of a stable map \((f, (\Sigma, z), \xi)\), denoted by \(\text{Aut}(f, (\Sigma, z), \xi)\), is defined by

\[
\text{Aut}(f, (\Sigma, z), \xi) = \{ \vartheta \in \text{Aut}(\Sigma, z, m, n) | f \circ \vartheta = f, \xi \circ \vartheta = \xi \}.
\]

The proof of the following lemma is routine and is left to the readers.

**Lemma 5.1.4:** The automorphism group of an orbifold stable map is finite.

Given a stable map \((f, (\Sigma, z), \xi)\), there is an associated homology class \(f_*([\Sigma])\) in \(H_2(X; \mathbb{Z})\) defined by \(f_*([\Sigma]) = \sum_{\nu}(f \circ \pi_{\nu})_*[\nu]\). On the other hand, for each marked point \(z\) on \(\Sigma_{\nu}\), say \(\pi_{\nu}(z) = z_1 \in z\), \(\xi_{\nu}\) determines, by the group homomorphism at \(z\), a conjugacy class \((g_i)\) where \(g_i \in G_{f(z_1)}\). We thus have a map \(ev\) sending each (equivalence class of) stable map into \(\Sigma_k X\) by \((f, (\Sigma, z), \xi) \rightarrow ((f(z_1), (g_1)), \cdots, (f(z_k), (g_k)))\). Let \(X = \prod_i X_{(g_i)}\) be a connected component in \(X^k\).

**Definition 5.2.5:** A stable map \((f, (\Sigma, z), \xi)\) is said of type \(x\) if \(ev((f, (\Sigma, z), \xi)) \in x\). Given a homology class \(A \in H_2(X; \mathbb{Z})\), we let \(\overline{M}_{g,k}(X, J, A, x)\) denote the moduli space of equivalence classes of orbifold stable maps of genus \(g\), with \(k\) marked points, and of homology class \(A\) and type \(x\), i.e.,

\[
\overline{M}_{g,k}(X, J, A, x) = \{(f, (\Sigma, z), \xi)| f_*([\Sigma]) = A, ev((f, (\Sigma, z), \xi)) \in x\}.
\]

The rest of this subsection is devoted to giving a topology on \(\overline{M}_{g,k}(X, J, A, x)\) and to proving that the moduli space is compact when \((X, J)\) is a compact symplectic orbifold or a projective orbifold.

The set of all isomorphism classes of stable curves of genus \(g\) with \(k\) marked points, denoted by \(\overline{M}_{g,k}\), is called the Deligne-Mumford compactification of the moduli space \(M_{g,k}\) of Riemann surfaces of genus \(g\) with \(k\) marked points (assuming \(k + 2g \geq 3\)). The following differential geometric description of \(\overline{M}_{g,k}\) is standard.

The moduli space \(\overline{M}_{g,k}\) admits a stratification which is indexed by the combinatorial types of the stable curves. More precisely, we can associate a connected graph to each nodal marked Riemann surface by assigning a vertex with an integer (genus) to each component, an edge connecting two vertices if the corresponding components intersect, and a tail to each marked point.

Let \(g_\nu\) be the genus of the component \(\nu\) and \(k_\nu\) be the number of edges and tails containing \(\nu\) (we count twice the edges both of whose vertices are \(\nu\)). Then the data is required to satisfy

\[
k_\nu + 2g_\nu \geq 3, \quad \text{and} \quad \sum_\nu g_\nu + \text{rank} H_1(T; \mathbb{Q}) = g.
\]

Let \(\text{Comb}(g, k)\) be the set of all such objects \((T, (g_\nu))\). For each element \((\Sigma, z) \in \overline{M}_{g,k}\), there is an associated element of \(\text{Comb}(g, k)\) as follows: we take the graph \(T = T_\Sigma\), let \(g_\nu\) be the genus of \(\Sigma_\nu\). The set of combinatorial types \(\text{Comb}(g, k)\) is known to be of finite order.

There is a partial order \(\succ\) on \(\text{Comb}(g, k)\) defined as follows. Let \((T, (g_\nu)) \in \text{Comb}(g, k)\). We consider \((T_\nu, (g_\nu)) \in \text{Comb}(g_\nu, k_\nu)\) for some of the vertices \(\nu = \nu_1, \cdots, \nu_a\) of \(T\). We replace the vertex \(\nu\) of \(T\) by the graph \(T_\nu\), and join the edge containing \(\nu\) to the vertex \(o_\nu(i)\), where \(i \in \{1, \cdots, k_\nu\}\) is the suffix corresponding to this edge. We then obtain a new graph \(\tilde{T}\). The number \(\tilde{g}_\nu\) is determined from \(g_\nu\) and \(g_\nu o\) in an obvious way. It is easily seen that \((\tilde{T}, (\tilde{g}_\nu), \tilde{o})\) is in \(\text{Comb}(g, k)\). We define \((T, (g_\nu), o) \succ (\tilde{T}, (\tilde{g}_\nu), \tilde{o})\).
We need some information about the structure of the Deligne-Mumford compactification $\overline{M}_{g,k}$.

The following are well-known

**Fact 5.1.6:** Let $\mathcal{M}_{g,k}(T,(g_\nu))$ be the set of all stable curves such that the associated object is $(T,(g_\nu))$. Then

- $\overline{M}_{g,k}$ is a compact complex orbifold which admits a stratification by finitely many strata; each stratum is of the form $\mathcal{M}_{g,k}(T,(g_\nu))$.

- There is a fiber bundle $U_{g,k}(T,(g_\nu)) \to \mathcal{M}_{g,k}(T,(g_\nu))$ which has the following property. For each $x = (\Sigma_x,z_\nu) \in \mathcal{M}_{g,k}(T,(g_\nu))$, there is a neighborhood of $x$ in $\mathcal{M}_{g,k}(T,(g_\nu))$ of the form $U_x = V_x/G_x$, where $G_x = \text{Aut}(\Sigma_x, z_\nu)$, such that the inverse image of $U_x$ in $U_{g,k}(T,(g_\nu))$ is diffeomorphic to $V_x \times C_x/G_x$. There is a complex structure on each fiber such that the fiber of $y = (\Sigma_y,z_y)$ is identified with $(\Sigma_y, z_y)$ itself, together with a Kähler metric $\mu_y$ which is flat in a neighborhood of the singular points and varying smoothly in $y$.

- $\mathcal{M}_{g,k}(T',(g'_\nu))$ is contained in the compactification of $\mathcal{M}_{g,k}(T,(g_\nu))$ in $\overline{M}_{g,k}$ only if $(T,(g_\nu)) \succ (T',(g'_\nu))$.

- Different strata are patched together in a way which is described in the following local model of a neighborhood of a stable curve in $\overline{M}_{g,k}$. A neighborhood of $x = (\Sigma, z)$ in $\overline{M}_{g,k}$ is parametrized by

$$V_x \times B_r(\oplus T_{z_\nu} \Sigma_\nu \otimes T_{z_\omega} \Sigma_\omega)/\text{Aut}(\Sigma,z),$$

where $z = \pi_\nu(z_\nu) = \pi_\omega(z_\omega)$ (here it may happen that $\nu = \omega$) runs over all singular points of $\Sigma$; $B_r(W)$ denotes the ball of radius $r$ of vector space $W$. Each $y \in V_x$ represents a stable curve $(\Sigma_y,z_y)$ homeomorphic to $(\Sigma,z)$, with a Kähler metric $\mu_y$ which is flat in a neighborhood of the singular points. Given $y \in V_x$, for each element $\varsigma = (\sigma_z) \in \oplus T_{z_{\nu}} \Sigma_{\nu} \otimes T_{z_{\omega}} \Sigma_{\omega}$ there is an associated stable curve $(\Sigma_{y,\varsigma},z_{y,\varsigma})$ obtained as follows. Each component $\Sigma_{\nu}$ of $\Sigma_y$ is given a Kähler metric $\mu_y$ which is flat in a neighborhood of singular points. This gives a Hermitian metric on each $T_{z_{\nu}} \Sigma_{\nu}$. For each non-zero $\sigma_z \in T_{z_{\nu}} \Sigma_{\nu} \otimes T_{z_{\omega}} \Sigma_{\omega}$, there is a biholomorphic map $\Psi_{\sigma_z} : T_{z_{\nu}} \Sigma_{\nu} \setminus \{0\} \to T_{z_{\omega}} \Sigma_{\omega} \setminus \{0\}$ defined by $u \otimes \Psi_{\sigma_z}(u) = \sigma_z$. Let $|\sigma_z| = R^2$; then for sufficiently large $R$, the map $\exp^{-1} \circ \Psi_{\sigma_z} \circ \exp_{z_{\nu}}$ is a biholomorphism between $D_{z_{\nu}}(R^{-1/2}) \setminus D_{z_{\nu}}(R^{-3/2})$ and $D_{z_{\omega}}(R^{-1/2}) \setminus D_{z_{\omega}}(R^{-3/2})$, where $D_{z_{\nu}}(R^{-1/2})$ is a disc neighborhood of $z_{\nu}$ in $\Sigma_{\nu}$ of radius $R^{-1/2}$ which is flat assuming $R$ is sufficiently large. We glue $\Sigma_{\nu}$ and $\Sigma_{\omega}$ by this biholomorphism. If $\sigma_z = 0$, we do not make any change. Thus we obtain $(\Sigma_{y,\varsigma},z_{y,\varsigma})$.

Moreover, there is a Kähler metric $\mu_{\gamma,\varsigma}$ on $\Sigma_{y,\varsigma}$ which coincides with the Kähler metric $\mu_y$ on $\Sigma_y$ outside a neighborhood of the singular points, and varies smoothly in $\varsigma$. Each $\gamma \in \text{Aut}(\Sigma,z)$ takes $(\Sigma_y,z_y)$ to $(\Sigma_{\gamma(y)},z_{\gamma(y)})$ isometrically, so it acts on $\oplus T_{z_{\nu}} \Sigma_{\nu} \otimes T_{z_{\omega}} \Sigma_{\omega}$. $\gamma$ induces an isomorphism between $(\Sigma_{y,\varsigma},z_{y,\varsigma})$ and $(\Sigma_{\gamma(y,\varsigma)},z_{\gamma(y,\varsigma)})$, which is also an isometry.

Now we define a topology on the moduli space $\overline{M}_{g,k}(X,J,A,x)$. We put a Hermitian metric $h$ on $(X,J)$ and the distance function on $X$ is assumed to be induced from $h$.

**Definition 5.1.7:** A sequence of equivalence classes of stable maps $x_n$ in $\overline{M}_{g,k}(X,J,A,x)$ is said to converge to $x_0 \in \overline{M}_{g,k}(X,J,A,E)$ if there are representatives $(f_n,(\Sigma_n, z_n),\xi_n)$ of $x_n$ and a representative $(f_0,(\Sigma_0, z_0),\xi_0)$ of $x_0$ the following conditions hold.

- For each $n$ (including $n = 0$), there is a set of distinct regular points $\{z_{n,1}, \ldots, z_{n,n}\}$ (it may happen that this set is empty) on $\Sigma_n$ which is disjoint from the marked point set $z_n$ such that
after adding this set to \( z_n \) we obtain a stable curve in \( \overline{M}_{g,k+a} \), denoted by \((\Sigma_n, z_n)^+\). Let \((f_n^+, (\Sigma_n, z_n)^+, \xi_n^+)\) be the sequence of stable maps naturally obtained.

- The sequence \((\Sigma_n, z_n)^+\) converges to \((\Sigma_0, z_0)^+\) in \( \overline{M}_{g,k+a} \). This means that for sufficiently large \( n \), \((\Sigma_n, z_n)^+\) is identified with \((\Sigma_{y_n, z_n}, z_{y_n, z_n})\) for some \((y_n, z_n)\) in the canonical model of a neighborhood of \((\Sigma_0, z_0)^+\). Let \( z_n \) be given by \((\sigma_{z,n})\) and \(|\sigma_{z,n}| = R_{z,n}^2\) (here \( R_{z,n} \) is allowed to be \( \infty \)). For each \( \mu > \max_z (R_{z,n}^{-1}) \) we put

\[
W_{z,n}(\mu) = (D_{z_0}(\mu) \setminus D_{z_0}(R_{z,n}^{-1})) \cup (D_{z_0}(\mu) \setminus D_{z_0}(R_{z,n}^{-1})), \quad \text{and} \quad W_n(\mu) = \bigcup_z W_{z,n}(\mu).
\]

Then the following holds. First, for each \( \mu > 0 \), when \( n \) is sufficiently large, the restriction of \( f_n^+ \) to \( \Sigma_{y_n, z_n} \setminus W_n(\mu) \) converges to \( f_0^+ \) in the \( C^\infty \) topology as a \( C^\infty \) map with an isomorphism class of compatible systems. Secondly, \( \lim_{n \to \infty} \limsup_{n \to \infty} \text{Diam}(f_n(W_{z,n}(\mu))) = 0 \) for each singular point \( z \) of \( \Sigma_0 \).

**Proposition 5.1.8:** Suppose \( X \) is either a symplectic orbifold with a symplectic form \( \omega \) and an \( \omega \)-compatible almost complex structure \( J \), or a projective orbifold with an integrable almost complex structure \( J \). Then the moduli space \( \overline{M}_{g,k}(X, J, A, x) \) is compact and metrizable.

For the proof, the reader is referred to [CR2].

### 5.2 Orbifold Gromov-Witten Invariants

For any component \( x = (X_{(g_1)}, \ldots, X_{(g_k)}) \), there are \( k \) evaluation maps.

\[
e_i : \overline{M}_{g,k}(X, J, A, x) \to X_{(g_i)}, \quad i = 1, \ldots, k.
\]

For any set of cohomology classes \( \alpha_i \in H^{2g_i}(X_{(g_i)}; \mathbb{Q}) \subset H^\ast_{\text{orb}}(X; \mathbb{Q}), \) \( i = 1, \ldots, k \), the orbifold Gromov-Witten invariant is defined as the virtual integral

\[
\Psi_{(g,k,A,x)}^{X,J}(\alpha_1^{l_1}, \ldots, \alpha_k^{l_k}) = \int_{\overline{M}_{g,k}(X, J, A, x)}^{\text{vir}} \prod_{i=1}^{k} c_1(L_i)^{l_i} e_i^\ast \alpha_i,
\]

where \( L_i \) is the line bundle defined by the cotangent space of the \( i \)-th marked point.

When \( g = 0 \) and \( A = 0 \), the moduli space \( \overline{M}_{g,k}(X, J, A, x) \) admits a very nice and elementary description, based on which we gave an elementary construction of genus zero, degree zero orbifold Gromov-Witten invariants in [CR1]. Even in this case, virtual integration is needed where there is an obstruction bundle. The orbifold cup product (cf. Theorem 2.3) is defined through these orbifold Gromov-Witten invariants. In the general case, we need to use the full scope of the virtual integration machinery developed by [FO], [L1], [R2] and [Si].

Singularities of an orbifold impose additional difficulties in carrying out virtual integration in the orbifold case. Due to the presence of singularities, even on a closed orbifold the function of injective radius of the exponential map does not have a positive lower bound. As a consequence, it is not known that a neighborhood of a (good) \( C^\infty \) map into an orbifold can be completely described by \( C^\infty \) sections of the pull-back tangent bundle via the exponential map. Our approach is a combination of techniques developed in the smooth case and some additional techniques for orbifolds.

The main results of this work are summarized in the following
The first Chern class $C_i$ is non-zero in the twisted sector. Let $D$ be the divisor class. Therefore, there is only one orbifold holomorphic map covering the line. This one is regular and hence good. Therefore, $\Psi_{(A_2,0)}(\alpha, \beta) = 1$.

5.3 Examples

It is generally a difficult problem to compute orbifold quantum cohomology. Much machinery has been developed to compute ordinary quantum cohomology. The most important ones are localization and surgery techniques. They should have their counterparts in orbifold quantum cohomology as well. The problem is that this subject is so young that there has not been enough time to develop all the machinery. Here, we compute some simple examples by direct computations.

Example 5.3.1: Let's consider weighted projective space $WP(1,1,2,2,2)$. This is an important example in mirror symmetry. It has a twisted sector $WP(0,0,2,2,2)$ with local group $\mathbb{Z}_2$. Let $\tau$ be the generator. Its degree shifting number is one. Recall that $WP(0,0,2,2,2)$ is isomorphic to $\mathbb{P}^2$. Therefore,

$$H^{2i}_{orb}(WP(1,1,2,2,2), \mathbb{C}) = \begin{cases} \mathbb{C} & i = 0 \\ \mathbb{C} \otimes \mathbb{C} & 1 \leq i \leq 3 \\ \mathbb{C} & i = 4 \end{cases}$$

All others are zero. Let $D_i$ be the hyperplane divisor where $i$-th homogeneous coordinate is zero. The first Chern class $C = D_1 + D_2 + D_3 + D_4 + D_5$ with $2D_1 = 2D_2 = D_3 = D_4 = D_5$. The weighted projective space is much more complicated than projective space. For example, there are three types of lines in $WP(1,1,2,2,2)$, depending on whether it is in the twisted sector, intersects the twisted sector transversely or is disjoint from the twisted sector. Their examples are $\{[0,0,t,s,0]\}, \{[0,t,s,0,0]\}, \{[t,s,0,0,0]\}$. Let $A_i$ be its fundamental class. It is easy to calculate $C_1(A_1) = 2, C_1(A_2) = 4, C_1(A_3) = 8$. Let's compute the invariant for $A_2$; the other two are more difficult to compute. The second kind of line has an orbifold point. Consider the moduli space of orbifold stable spheres with orbifold points of order $(1,2)$. The complex dimension of the moduli space is 6. Now, we choose a point class $\alpha$ from non-twisted sector and a point class $\beta$ from the twisted sector. We are interested in computing the orbifold GW-invariant $\Psi_{(A_2,0)}(\alpha, \beta)$. We observe that there are only two kinds of holomorphic curves with homology class $A_2$, a line intersecting twisted sector transversely or a conic in twisted sector. A conic in the twisted sector does not pass through a point not in the twisted sector. Therefore, for the purpose of calculating the invariant, we only have to consider the lines intersecting transversely with the twisted sector. We choose our two points as $[0,1,0,0,0], [0,0,1,0,0]$. There is only one such line passing through these two points. Therefore, there is only one orbifold holomorphic map covering the line. This one is regular and hence good. Therefore, $\Psi_{(A_2,0)}(\alpha, \beta) = 1$. 

Theorem 5.2.1: Let $X$ be a closed symplectic or projective orbifold. The orbifold Gromov-Witten invariants defined in (5.2) satisfy the quantum cohomology axioms of Witten-Ruan for ordinary Gromov-Witten invariants (cf. [R3]) except that in the Divisor Axiom, the divisor class is required to be in the nontwisted sector (i.e. in $H^2(X; \mathbb{Q})$). In the formulation of axioms, the ordinary cup product is replaced by the orbifold cup product $\cup_{orb}$ (cf. Theorem 2.3).
6 Orbifold String Theory Conjectures

The physicists believe that orbifold string theory is equivalent to ordinary string theory of its desingularizations. This belief motivated a body of conjectures which we call the Orbifold string theory conjecture. At present, the case without discrete torsion is much better understood than the case with discrete torsion. Classical theory is better understood than quantum theory. Therefore, we shall start from the classical theory without discrete torsion and then discuss the case with discrete torsion. We finish by discussing the case of quantum theory. In the physics literature, physicists concern only with 3-dimensional Calabi-Yau orbifolds. But it is clear that much more is true beyond 3-dimensional Calabi-Yau orbifolds. Here, we restrict ourselves to the case of Gorenstein reduced orbifolds $X$. In this case, all the local groups are subgroup of $SL_2(C)$ (i.e., a $SL$-orbifold) and hence have integral degree shifting number.

6.1 Classical Case Without Discrete Torsion

Recall

Definition 6.1: A deformation of $X$ is a triple $\pi : U \to \Delta$ such that $\Delta$ is a disc around the origin, $\pi$ is holomorphic and $X = \pi^{-1}(0)$. We call $\pi$ a Kähler or projective deformation if $U$ is Kähler or projective. We call $X$ the central fiber and $X_t = \pi^{-1}(t)$ a generic fiber. Suppose that $Z$ is a singular space such that $K_Z$ is Cartier. A crepant resolution $Y$ of $Z$ is a morphism $p : Y \to Z$ such that $Y$ is smooth and $p^*K_Z = K_Y$. A desingularization $Y$ of $X$ is defined as a crepant resolution of a generic fiber $X_t$ of a deformation of $X$.

Two extreme cases are (i) $Y$ is a crepant resolution of $X$; (ii) The generic fiber $X_t$ is smooth. In this case, we call $Y = X_t$ a smoothing of $X$.

K-Orbifold String Theory Conjecture: Suppose that $X$ is a Gorenstein orbifold and $\pi : Y \to X$ is a crepant resolution. Then there is a natural isomorphism between $K_{\text{orb}}(X)$ and $K(Y)$.

Many weaker forms of this conjecture have been studied intensively in literature under the name of the McKay correspondence. For example, we can replace $K$-theory by the Euler number, which we call the $E$-Orbifold string theory conjecture. One can also consider a weaker version of $K$-orbifold string theory conjecture by dropping naturality. Namely, we only consider the corresponding dimensions. We label it as $WK$-orbifold string theory conjecture. The best result in this direction so far is Batyrev’s proof of the $WK$-orbifold string theory conjecture for global quotients [B2]. Batyrev used a number theoretic method called motivic integration invented by Kontsevich [KO]. Actually, Batyrev proved a stronger result of the equivalence of Hodge numbers. However, this method does not yield a natural map. Moreover, the conjecture is completely open for general orbifolds. The hardest part of this conjecture is to get a natural map between $K_{\text{orb}}(X)$ and $K(Y)$. Such a map is necessary for us to compare orbifold quantum cohomology to quantum cohomology of $Y$. One of the difficulties in constructing such a map is that the projection $Y \to X$ does not pull back class from twisted sector. I believe that another formulation of orbifold cohomology is needed here and topological methods may play an important role. If we go beyond the Gorenstein orbifold, orbifold cohomology is rationally graded. Batyrev defined string theoretic Hodge numbers in terms of its resolution (not necessarily crepant). The generating function of his string theoretic Hodge numbers is not necessarily a polynomial, which echoed the rationality of grading of orbifold cohomology. It would be an interesting question to investigate their relation. In the meantime, we have very few examples of non-global quotients which we have calculated. It is also very important
to calculate more examples. Calabi-Yau hypersurface of simplicial toric varieties will be a good place to start. The case of complex dimension three has been calculated recently by M. Poddar [P] which gives further evidence to the $K$-orbifold string theory conjecture.

The example attracting a lot of attention is the symmetric product of the algebraic surface, which is also the best understood example.

**Example 6.2:** Let $S$ be an algebraic surface and $X_k = \text{Sym}^k(S)$ be the $k$-fold symmetric product of $S$. Then, $X_k$ is a $SL$-orbifold and $K_{X_k}$ is Cartier. It is well-known that the Hilbert scheme $\text{Hilb}_k(S)$ of points of length $k$ is a crepant resolution of $X_k$. This case has been extensively studied and occupies a special place in stringy geometry and topology. Many years ago, Göttsche calculated the generating function of the Euler number of $\text{Hilb}_k(S)$ and discovered a modularity property. Motivated by orbifold string theory, Vafa-Witten [VW] calculated the orbifold cohomology group of $X_k$ and showed that $\mathcal{H} = \oplus_k H^\ast_{orb}(X_k, \mathbb{C})$ is a "Fock space" of orbifold conformal field theory. This means that $\mathcal{H}$ is a representation of Heisenberg algebra. Under the framework of conformal field theory, the Euler characteristic is the genus one correlation function and hence modular by definition. The $K$-orbifold string theory conjecture predicts that $\oplus_k \text{Hilb}_k(S)$ is also a representation of Heisenberg algebra. This was verified by Nakajima [N]. Currently, the ring structure of $\text{Hilb}_k(S)$ is still unknown and is a hot topic right now. Recently, Lehn-Sorger calculated ring structure of $\text{Hilb}_k(\mathbb{C}^2)$ [LS]. Combined with example 3.5.2, it implies that $\text{Hilb}_k(\mathbb{C}^2)$ and $\text{Sym}_k(\mathbb{C}^2)$ have the same ring structure.

In the general case, we conjecture

**Conjecture 6.3:** Suppose that $\text{Hilb}_k(S)$ has hyperkähler structure. Then $\text{Hilb}_k(S)$ and $\text{Sym}_k(S)$ have the same ring structure. In general, if $X$ and its crepant resolution $Y$ have hyperKähler structure, $Y$ and $X$ have the same ring structure.

Examples includes $T^4$, $K3$ and many other open manifolds such as $\mathbb{C}^2$. This conjecture is a consequence of the quantum version of the orbifold string theory conjecture.

### 6.2 Classical Case With Discrete Torsion

The geometry of orbifold cohomology with discrete torsion is much less understood. Originally, Vafa-Witten proposed to use it to identify the cohomology of a general desingularization which may not be a straightforward crepant resolution. D. Joyce [JO] showed that this proposal fails badly to count the desingularizations of some simple examples such as $T^6/\mathbb{Z}_4$, where there is no discrete torsion but there are at least five different desingularizations. To overcome this difficulty, I introduced a more general notion of *inner local system* and constructed a twisted orbifold cohomology for any inner local system. Inner local systems were successfully used to count all the examples Joyce constructed. We propose the following conjecture:

**D-Orbifold String Theory Conjecture:** Suppose that $Y$ is a desingularization of $X$. Then there is an inner local system $\mathcal{L}$ such that the ordinary cohomology group of $Y$ is a direct sum of $H^\ast_{orb}(X, \mathcal{L})$ and the cohomologies generated by the exceptional locus corresponding to small resolution. In particular, if $Y$ is a smoothing, its cohomology is the same as twisted orbifold cohomology for an inner local system.

**Remark 6.4:** A generic fiber $X_t$ may not be smooth in general. To obtain a crepant resolution, a small resolution may be needed. Here a small resolution is a resolution where the exceptional locus is of codimension at least two.

The best known example of small resolution is the small resolution of a nodal point in three
dimension, where the exceptional locus is a rational curve of the normal bundle $O(-1) + O(-1)$.

**Remark 6.5:** The inverse of the D-orbifold string theory conjecture is obviously false. We can easily construct an example with nontrivial discrete torsion, where the singularities are of codimension $\geq 3$. In this case, there is no deformation.

It is also an unknown question to identify desingularizations corresponding to discrete torsion. It is also very important to understand more examples. Recall that symmetric group has a non-trivial discrete torsion. It would be an interesting problem to find out if there is a desingularization of $Sym_k(S)$ realizing the twisted orbifold cohomology.

### 6.3 Quantum Case

The ultimate goal of the orbifold string theory conjecture is to compare orbifold quantum cohomology of $X$ with the quantum cohomology of $Y$. I do not know a precise statement to which I could not find a counterexample. However, it is a very useful general goal to motivate other better formulated conjectures. I do not know how to twist orbifold quantum cohomology using an inner local system or a discrete torsion. Therefore, we focus on the case without discrete torsion.

**Q-Orbifold String Theory conjecture:** Suppose that $Y$ is a crepant resolution of $X$ and $\pi : K_{orb}(X) \to K(Y)$ is the natural additive isomorphism given by the $K$-orbifold string theory conjecture. Then $\pi$ induces an isomorphism of orbifold quantum cohomology up to a mirror transformation.

The mirror transformation here is similar to the one appearing in mirror symmetry [K]. An interesting case is when $Y$ is a hyperkähler manifold. In this case, there are no quantum corrections and the quantum cohomology of $Y$ is the same as ordinary cohomology. Our $Q$-orbifold string theory conjecture becomes a statement for the orbifold cohomology of $X$. This is the origin of conjecture 6.3.

The physical prediction in the quantum case is very imprecise. Basically, there is a family of superconformal field theories containing both orbifold theory and the theory of the resolution. Physics predicts that there is an analytic continuation from one theory to the other. This tells very little about the precise relation between them. I believe that orbifold quantum cohomology is different from quantum cohomology of crepant resolutions in general and a mirror transformation is needed. Actually, orbifold cohomology should be naturally related to relative quantum cohomology. Suppose that $Z$ is the exceptional divisor of the projection $Y \to X$. We want to identify orbifold GW-invariants of $X$ with the relative GW-invariants of the pair $(Y, Z)$ introduced by Li-Ruan [LR]. Then we can relate relative GW-invariants of $(Y, Z)$ with ordinary GW-invariants of $X$. In fact, a generalization of Li-Ruan’s surgery technique to the orbifold category should be very useful for this purpose.

### 6.4 Generalization of Orbifold String Theory Conjecture

Note that resolution is a special class of birational maps. It is natural to recast orbifold string theory conjectures in the context of birational geometry. Several years ago, Batyrev [33] and Wang [W] proved that smooth birational minimal models have the same Betti numbers. In fact, their results are slightly more general.

**Definition 6.6:** $X,Y$ are called $K$-equivalent if there is a common resolution $\phi : Z \to X, \psi : Z \to Y$ such that $\phi^* K_X = \psi^* K_Y$. 

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Batyrev and Wang proved that $K$-equivalent smooth projective manifolds have the same Betti numbers. At the same time, An-Min Li and I \[LR\] proved that a smooth flop in three dimensions induces an isomorphism of quantum cohomology. These two results inspired the author to propose \[R1\]

**Quantum minimal model conjecture:** Smooth birational minimal models have isomorphic quantum cohomology.

Here, we observe that if $\phi : Y \to X$ is a crepant resolution, then $Y, X$ are $K$-equivalent. Therefore, we can combine orbifold string theory conjectures with quantum minimal model conjecture to formulate even more general conjectures.

**$K, Q$-Conjectures:** Suppose that $Y, X$ are $K$-equivalent orbifolds. The same statements for $K, Q$-orbifold string theory conjectures are true.

I feel that in the more general context of birational geometry our $K, Q$-conjectures give us a better understanding of relations between orbifold string theory and birational geometry than the orbifold string theory conjectures itself.

### 6.5 Orbifold Mirror Symmetry and Mirror Symmetry in Higher Dimension

One of my first impressions of mirror symmetry is how lucky we are in three dimensions where every Calabi-Yau orbifold has a crepant resolution. Hence, we have the luxury to consider smooth Calabi-Yau 3-folds only. In higher dimensions, it is no longer true that every Calabi-Yau orbifold has a crepant resolution. We are stuck with orbifolds. Therefore, it makes sense to talk about orbifold mirror symmetry, i.e., mirror symmetry among orbifolds. Moreover, if one looks at the physical literature, physicists clearly consider orbifolds as well. For example, the first physical proof of mirror symmetry by Greene and Plesser \[GP\] was based on some orbifold model. It is clear that orbifold theory is central to mirror symmetry. However, during the mathematization of mirror symmetry, orbifold model was replaced by its crepant resolution. I tried some simple examples such as Calabi-Yau hypersurfaces of weighted projective space $WP(1, 1, 2, 2, 2)$. It is not clear how mirror symmetry predicts its orbifold quantum cohomology. I believe that it is important to do some soul-searching on the role of orbifold string theory in mirror symmetry. This is related to another important question of generalizing mirror symmetry to higher dimensions, where crepant resolution no longer exists. However, it still makes perfect sense to talk about mirror symmetry between Calabi-Yau orbifolds.

### 6.6 Other Problems

Orbifold String Theory Conjectures or our general $K, Q$-conjectures are certainly outstanding problems in stringy geometry and topology. There are other very interesting problems in this subject as well. Here are some of my favorite problems.

1. **Orbifold Differential Topology:** As I remarked in (3.4.11), it is a very interesting problem to establish a homology theory reflecting orbifold cohomology. This should have a profound impact on transversality theory of singular spaces. It should also be useful for stack theory in algebraic geometry.

2. **Relation between two product structures:** We have two different products on orbifold cohomology groups coming from cohomology and $K$-theory. Each seems to reflect one aspect
of orbifold theory. The relation between is still mysterious to me. It is certainly worth further investigation.

Finally, we remark that we only talked about the part of stringy geometry and topology motivated by so called type IIA, IIB orbifold string theory. There are other types of physical orbifold theories which also generate interesting mathematics. Unfortunately, the author has very little understanding of other types of physical orbifold theories. I apologize for the omission. However, I would like to mention the orbifold Landau-Ginzburg theory [KU] and the orbifold elliptic genus [BL] (orbifold heterotic string theory). Obviously, there is much more rich mathematics waiting for us to explore!

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