EIGENVALUE INTERACTION FOR A CLASS OF NON-SELFADJOINT OPERATORS UNDER RANDOM PERTURBATIONS

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ABSTRACT. We consider a non-selfadjoint $h$-differential model operator $P_h$ in the semiclassical limit ($h \to 0$) subject to random perturbations with a small coupling constant $\delta$. Assume that $e^{-1/C_h} \ll \delta \ll h^\kappa$ for constants $C, \kappa > 0$ suitably large. Let $\Sigma$ be the closure of the range of the principal symbol.

We study the 2-point intensity measure of the random point process of eigenvalues of the randomly perturbed operator $P_{\delta}h$ and prove an $h$-asymptotic formula for the average 2-point density of eigenvalues. With this we show that two eigenvalues of $P_{\delta}h$ in the interior of $\Sigma$ exhibit close range repulsion and long range decoupling.

RéSUMÉ. Nous considérons un opérateur différentiel non-autoadjoint $P_h$ dans la limite semiclassique ($h \to 0$) soumis à de petites perturbations aléatoires. De plus, nous imposons que la constant de couplage $\delta$ vérifie $e^{-1/C_h} < \delta \ll h^\kappa$ pour certaines constantes $C, \kappa > 0$ choisies assez grandes. Soit $\Sigma$ l’adhérence de l’image du symbole principal de $P_h$.

Dans cet article, nous donnons une formule $h$-asymptotique pour la 2-points densité des valeurs propres en étudiant la mesure de comptage aléatoire des valeurs propres à l’intérieur de $\Sigma$. En étudiant cette densité, nous prouvons que deux valeurs propres sont répulsives à distance courte et indépendantes à long distance.

1. Introduction

In recent times there has been a mounting interest in the spectral theory of non-selfadjoint operators. As opposed to self-adjoint operators, the norm of the resolvent can be very large even far away from the spectrum. As a consequence, the spectrum of such an operator can change a lot, even under small perturbations. A major tool to quantify these zones of spectral instability is given by the $\varepsilon$-pseudospectrum. We follow the work of L.N. Trefethen and M. Embree [25] and define the $\varepsilon$-pseudospectrum of a closed linear operator $A$ on a Banach space $X$ by

$$
\sigma_\varepsilon(A) := \left\{ z \in \mathbb{C} \setminus \sigma(A); \| (z - A)^{-1} \| > \frac{1}{\varepsilon} \right\} \cup \sigma(A),
$$

where $\sigma(A)$ denotes the spectrum of $A$. Equivalently,

$$
\sigma_\varepsilon(A) = \bigcup_{B \in B(X), \| B \| < \varepsilon} \sigma(A + B). \tag{1.1}
$$

Emphasized by the works of L.N. Trefethen, M. Embree, E.B. Davies, M. Zworski, J. Sjöstrand, cf. [25, 26, 5, 4, 6, 8, 20, 28], and many others, spectral instability of non-selfadjoint operators has become a prominent and important subject of research.

In view of (1.1) is natural to study the spectrum of such operators under small random perturbations. One line of recent research interest has focused on the case
of elliptic (pseudo-)differential operators subject to small random perturbations, see for example [10, 1, 9, 2, 11, 20, 21, 19, 27] and [28, 7, 22], as we did ourselves in the present paper.

1.1. **Hager’s model operator.** The central object of this work is the following semiclassical operator: let \( 0 < h \ll 1 \) and consider \( P_h : L^2(S^1) \to L^2(S^1) \), given by

\[
P_h := hD_x + g(x), \quad D_x := \frac{1}{i} \frac{d}{dx}, \quad S^1 = \mathbb{R}/2\pi \mathbb{Z}
\]

where we assume that

(H.1) \( g \in C^\infty(S^1; \mathbb{C}) \) is such that Im \( g \) has exactly two critical points, one minimum and one maximum, say at \( a, b \in S^1 \), with \( \text{Im} \, g(a) \leq \text{Im} \, g(x) \leq \text{Im} \, g(b) \) for all \( x \in S^1 \).

The operator \( P_h \) is a model for a non-selfadjoint semiclassical differential operator and it has first been introduced by M. Hager in [10] and has already been subject to further study in [1, 20]. As the natural domain of \( P_h \) we take the semiclassical Sobolev space

\[
H^1_{sc}(S^1) := \left\{ u \in L^2(S^1) : \left( \| u \|_2^2 + \| hD_x u \|_2^2 \right)^{1/2} < \infty \right\},
\]

where \( \| \cdot \| \) denotes the \( L^2 \)-norm on \( S^1 \) if nothing else is specified. We will use the standard scalar products on \( L^2(S^1) \) and \( \mathbb{C}^N \) defined by

\[
(f|g) := \int_{S^1} f(x)\overline{g(x)} \, dx, \quad f, g \in L^2(S^1),
\]

and

\[
(X|Y) := \sum_{i=1}^N X_i \overline{Y}_i, \quad X, Y \in \mathbb{C}^N.
\]

We denote the semi-classical principal symbol of \( P_h \) by

\[
p(x, \xi) = \xi + g(x), \quad (x, \xi) \in T^* S^1.
\]

The Poisson bracket of \( p \) and \( p \) is given by

\[
\{p, p\} = p_x' \cdot \overline{p}_x - p_x \cdot \overline{p}_x',
\]

**Definition 1.1.** Let \( p(x, \xi) \) be the semiclassical principal symbol of the operator \( P_h \) as in (1.3). Then, we define

\[
\Sigma := \overline{p(T^* S^1)} \subset \mathbb{C}.
\]

In the case of (1.2) and (1.3) \( p(T^* S^1) \) is already closed due to the ellipticity of \( P_h \). The spectrum of \( P_h \) is discrete with simple eigenvalues, given by

\[
\sigma(P_h) = \{ z \in \mathbb{C} : \ z = \langle g \rangle + kh, \ k \in \mathbb{Z} \},
\]

where \( \langle g \rangle := (2\pi)^{-1} \int_0^{2\pi} g(y) \, dy \). Next, for \( z \in \hat{\Sigma} \), consider the equation \( z = p(x, \xi) \). It has precisely two solutions \( \rho_\pm := (x_\pm, \xi_\pm) \) where \( x_\pm \) are given by

\[
\text{Im} \, g(x_\pm) = \text{Im} \, z, \quad \text{with} \quad \pm \text{Im} \, g'(x_\pm) < 0
\]

and \( \xi_\pm = \text{Re} \, z - \text{Re} \, g(x_\pm) \). It has been shown in [27] that for all \( \Omega \Subset \hat{\Sigma} \) and all \( z \in \Omega \)

\[
\| (P_h - z)^{-1} \| \geq C_1 e^{C_2 h},
\]
with $C_1, C_2 > 0$ constants that only depend on $\Omega$. This implies that such an $\Omega$ is inside the $h^\infty$-pseudospectrum of $P_h$. Alternatively, this fact follows as well from the work of N. Dencker, J. Sjöstrand and M. Zworski [8]: since we have that $\{\text{Re } p, \text{Im } p\}(\rho_+(z)) < 0$ for all $z \in \Omega \subseteq \Sigma$ it follows from [8] that we can construct $h^\infty$-quasimodes $u \in L^2(S^1)$ of $P_h$ with the semiclassical wave front set $\text{WF}_h(u) = \{\rho_+(z)\}$. We recall that for $v = v(h)$, $\|v\|_{L^2(S^1)} = \mathcal{O}(h^{-N})$, for some fixed $N$, the semiclassical wave front set of $v$ is defined by

$$\text{WF}_h(v) := \mathcal{C}\{ (x, \xi) \in T^*S^1 : \exists a \in \mathcal{S}(T^*S^1), a(x, \xi) = 1, \|a^w v\|_{L^2(S^1)} = \mathcal{O}(h^\infty) \}$$

where $a^w$ denotes the Weyl quantization of $a$.

Next, by the natural projection $\Pi : \mathbb{R} \to S^1 = \mathbb{R}/2\pi \mathbb{Z}$ and a slight abuse of notation we identify the points $x_\pm, a, b \in S^1$ with points $x_+, a, b \in \mathbb{R}$ such that $x_- - 2\pi < x_+ < x_- \pi$ and $b - 2\pi < a < b$. Furthermore, we will identify $S^1$ with the interval $[b - 2\pi, b]$.

1.2. Adding a random perturbation. Let us consider a random perturbation of $P_h$ given by

$$P_h^\delta := P_h + \delta Q_\omega := hD_x + g(x) + \delta Q_\omega,$$

where $\delta > 0$ and $Q_\omega$ is an integral operator from $L^2(S^1)$ to $L^2(S^1)$ defined as follows: let $[x] := \max\{n \in \mathbb{N} : x \geq n\}$, for $x \in \mathbb{R}$, and define

$$Q_\omega u(x) := \sum_{|j|, |k| \leq \lfloor \frac{C_1}{\delta} \rfloor} \alpha_{j,k}(u)e^k(x),$$

with $C_1 > 0$ big enough and $e^k(x) := (2\pi)^{-1/2}e^{ikx}$, $k \in \mathbb{Z}$. Moreover, $\alpha_{j,k}$ are complex valued independent random variables with complex Gaussian distribution law $\mathcal{N}_C(0, 1)$.

It has been observed by W. Bordeaux-Montrieux in [1] that the Hilbert-Schmidt norm of $Q_\omega$ is bounded with probability close to 1, indeed

$$\|Q_\omega\|_{HS} \leq \frac{C}{h}, \quad \text{with probability} \quad 1 - e^{-\frac{1}{C}}.$$

Hence, we restrict our probability space to a open ball $B(0, R) \subseteq \mathcal{C}^N$, with $N := (2 \lfloor \frac{C_1}{\delta} \rfloor + 1)^2$, of radius $R = C/h$ and centered at 0. There, $Q_\omega$ is a compact operator, so the spectrum of the perturbed operator $P_h^\delta$ is discrete.

In the present work we are interested in the eigenvalues of $P_h^\delta$ in the interior of the pseudospectrum, indeed in the sequel we will always assume that there exists a $C > 1$ such that

(H.2) $\Omega \subseteq \Sigma$ open, convex and simply connected with $\text{dist } (\Omega, \partial \Sigma) > \frac{1}{C}$.

It will be very useful to give bounds on the coupling constant $\delta$ in terms of the imaginary part of the action between $\rho_+(z)$ and $\rho_-(z)$, defined by:

**Definition 1.2.** Let $\Omega$ be as in (H.2), let $p$ denote the semiclassical principal symbol of $P_h$ in (1.3) and let $\rho_\pm(z) = (x_\pm(z), \xi_\pm(z))$ be as above. Define

$$S := \min \left( \text{Im } \int_{x_-}^{x_+} (z - g(y))dy, \text{Im } \int_{x_+}^{x_- - 2\pi} (z - g(y))dy \right).$$
We suppose that the coupling constant $\delta > 0$ in (1.5) satisfies

\[ (H.3) \quad \delta := \delta(h) := \sqrt{h} e^{-\frac{\alpha(h)}{h}} \]

with $(\kappa - \frac{1}{2}) h \ln(h^{-1}) + Ch \leq \epsilon_0(h) < \min_{z \in \Omega} S(z)/C$ for some $\kappa > 7/2$ and $C > 0$ large and where the last inequality is uniform in $h > 0$. Thus, $\delta$ satisfies the inequality

\[ \sqrt{h} \exp \left\{ -\frac{\min_{z \in \Omega} S(z)}{Ch} \right\} < \delta \ll h^\kappa. \]

In [10] M. Hager showed that, with a probability close to 1, the eigenvalues of the perturbed operator $P^h_\delta$ contained in $\Omega$ follow a Weyl law, i.e.

\[ \#(\sigma(P^h_\delta) \cap \Omega) \sim \frac{1}{2\pi h} \text{vol} (\{ \rho \in T^*S^1; p(\rho) \in \Omega \}). \]

This result was extended to a much more general class of semiclassical pseudo-differential operators by M. Hager and J. Sjöstrand in [11] and W. Bordeaux-Montrieux gave in [1] an extension in the 1-dimensional case to sets $\Omega$ which satisfy $\text{dist}(\Omega, \partial \Sigma) \gg (-h \ln[\delta h])^{2/3}$. Furthermore, a similar result has been obtained by T. Christiansen and M. Zworski for randomly perturbed Toeplitz operators in [28].

1.3. Point Process and moments of linear statistics. To the best of the author’s knowledge there has never been a study of the interaction between eigenvalues of the class of non-selfadjoint operators in (1.5) which is crucial for the understanding of the spectral behavior of such operators.

The principal aim of this work is to study the correlation between two points in the spectrum of the perturbed operator $P^h_\delta$. Therefore, following the approach used in [27], we study the moments of linear statistics of the random point process of eigenvalues of the operator $P^h_\delta$, defined as following:

**Definition 1.3.** Let $P^h_\delta$ be as in (1.5), then we define the point process

\[ \Xi := \sum_{z \in \sigma(P^h_\delta)} \delta_z, \quad (1.7) \]

where the eigenvalues are counted according to their multiplicities and $\delta_z$ denotes the Dirac-measure at $z$.

Such an approach is more classical in the study of zeros of random polynomials and Gaussian analytic functions; we refer the reader to the works of B. Shiffman and S. Zelditch [17, 18, 16, 15], M. Sodin [24] an the book [12] by J. Hough, M. Krishnapur, Y. Peres and B. Virág.

In [27], we studied the first moment of the point process $\Xi$ with the random variables $\alpha$ restricted to a ball $B(0, R) \subset \mathbb{C}^N$ with $R = C/h$, i.e. the measure $\mu_1$ defined by

\[ T_1(\varphi) := \mathbb{E}[\Xi(\varphi)1_{B(0,R)}] = \int_C \varphi(z) d\mu_1(z) \]

for all $\varphi \in \mathcal{C}_0(\Omega)$ with $\Omega \subset \Sigma$ as in (H.2). In particular, showed that $\mu_1$ is absolutely continuous with respect to the Lebesgue measure and we gave a precise $h$-asymptotic formula for the Lebesgue density of $\mu_1$.

**Remark 1.4.** The above described approached gives a detailed description of the average density of eigenvalues and in comparison to the works of M. Hager in [10] and of W. Bordeaux-Montrieux in [1],
• we can allow for a smaller coupling constant, i.e. \( \delta > \exp\left\{-\frac{S(\text{Im} \varphi)}{h}\right\} \);
• we can study the distribution of eigenvalues of \( P_h^2 \) in all of \( \Sigma \).

For more details, we refer to [27].

In this work, we are interested in the second moment of the point process \( \Xi \). We recall some facts about second moments of point processes from [12, 3], using the example of \( \Xi \). The second moment of \( \Xi \) is defined by the positive linear functional on \( C_0(\Sigma^2) \), \( T_2 \), defined by

\[
T_2(\varphi) := E \left[ \sum_{z,w \in \sigma(P_h^2)} \varphi(z,w) \mathbb{I}_{B(0,R)} \right] = \int_{C^2} \varphi(z,w) d\mu_2(z,w)
\]

for all \( \varphi \in C_0^\infty(\Sigma^2) \). Note that we have the splitting

\[
T_2(\varphi) = E \left[ \sum_{z \in \sigma(P_h^2)} \varphi(z,z) \mathbb{I}_{B(0,R)} \right] + E \left[ \sum_{z,w \in \sigma(P_h^2) \atop z \neq w} \varphi(z,w) \mathbb{I}_{B(0,R)} \right]
\]

\[
= \int_{C^2} \varphi(z,z) d\tilde{\mu}_2(z,z) + \int_{C^2} \varphi(z,w) d\nu(z,w).
\]

Both terms are positive linear functionals on \( C_0(\Sigma^2) \), and thus the above representation by the two measures \( \tilde{\mu}_2 \) and \( \nu \) is well-defined. The measure \( \tilde{\mu}_2 \) is supported on the diagonal \( D := \{(z,z); z \in \Omega\} \) and is given by the push-forward of \( \mu_1 \) under the diagonal map \( f : \Omega \to D : x \mapsto (x,x) \), i.e. \( \tilde{\mu}_2 = f_* \mu_1 \). The second measure, \( \nu \), is called the two-point intensity measure of \( \Xi \) and it is supported on \( \Omega^2 \setminus D \). Their sum naturally yields \( \mu_2 \), i.e. \( \mu_2 = \tilde{\mu}_2 + \nu \). Hence, \( \mu_2 \) has atoms on the diagonal \( D \) and we see that \( \mu_2 \) is not absolutely continuous with respect to the Lebesgue measure on \( \Sigma^2 \). However, this may be the case for the measure \( \nu \).

Since \( \mu_1 \) has already been treated in [27], the aim of this work is to study the two-point intensity measure \( \nu \), given by

\[
E \left[ \sum_{z,w \in \sigma(P_h^2) \atop z \neq w} \varphi(z,w) \mathbb{I}_{B(0,R)} \right] = \int_{C^2} \varphi(z,w) d\nu(z,w), \tag{1.8}
\]

and to give an asymptotic formula for its Lebesgue density which will give us the correlated behavior of two points of the spectrum of \( P_h^3 \).

**Remark 1.5.** Throughout this work we shall denote the Lebesgue measure on \( \mathbb{C} \) by \( L(dz) \); denote \( d(z) := \text{dist}(z, \partial \Sigma) \); work with the convention that when we write \( O(h)^{-1} \) then we mean implicitly that \( 0 < O(h) \leq Ch \); denote by \( f(x) \asymp g(x) \) that there exists a constant \( C > 0 \) such that \( C^{-1}g(x) \leq f(x) \leq Cg(x) \); write \( \chi_1(x) \asymp \chi_2(x) \), with \( \chi_i \in C_0^\infty \), if \( \text{supp} \chi_2 \subset \mathbb{C} \text{supp} (1 - \chi_1) \).

2. Main Results

From the assumptions \((H.1)\) we see that the direct image \( p_*(d\xi \wedge dx) \) of the symplectic volume form \( d\xi \wedge dx \) on \( T^* S^1 \) is absolutely continuous and let \( \sigma(z) \) denote
its Lebesgue density, i.e.

\[ \sigma(z)L(dz) = p_*(d\xi \wedge dx). \]  \hspace{1cm} (2.1)

We give an \( h \)-asymptotic formula for the Lebesgue density of the two-point intensity measure \( \nu \) valid up to a distance \( \gg h^{3/5} \) to the diagonal. For \( \Omega \) as in (H.2) and \( C_2 > 0 \), we define the set

\[ D_h(\Omega, C_2) := \{(z,w) \in \Omega^2; \ |z-w| \leq C_2 h^{3/5}\} \]  \hspace{1cm} (2.2)

and we prove the following result:

**Theorem 2.1.** Let \( \Omega \subset \Sigma \) be as in (H.2). Let \( \delta > 0 \) be as in (H.3) with \( \kappa > 51/10 \). Let \( \nu \) be the measure defined in (1.8) and let \( \sigma(z) \) be as in (2.1). Then, for \( |z-w| \leq 1/C \) with \( C > 1 \) large enough, there exist smooth functions

\[ \begin{align*}
&\bullet \quad \sigma_h(z, w) = \sigma \left( \frac{z+w}{2} \right) + O(h), \\
&\bullet \quad K(z, w; h) = \sigma_h(z, w) \frac{|z-w|^2}{4h}(1 + O(|z-w| + h^\infty)), \\
&\bullet \quad D^\delta(z, w; h) = \frac{\Lambda(z, w)}{(2\pi h)^{2\kappa(1-e^{-2\kappa})}} \left( 1 + O\left( \delta h^{-\frac{1}{2}} \right) \right) + O\left( e^{-\frac{D}{K^2}} \right),
\end{align*} \]

with

\[ \Lambda(z, w) = \sigma_h(z, z)\sigma_h(w, w) + \sigma_h(z, w)^2(1 + O(|z-w|))e^{-2K} + \frac{\sigma_h(z, w)^2(1 + O(|z-w|))}{e^K \sinh(K)} \left( 2K^2 \coth(K) - 4K \right) + O\left( h^\infty + \delta h^{-\frac{31}{16}} \right) \]

and there exists a constant \( c > 0 \) such that for all \( \varphi \in C_0^\infty(\Omega^2 \setminus D_h(\Omega, c)) \) with

\[ \int_{C^2} \varphi(z, w) d\nu(z, w) = \int_{C^2} \varphi(z, w) D^\delta(z, w; h)L(d(z, w)). \]

We prove this theorem in Section 6.

2.1. **Eigenvalue interaction.** We will prove that two eigenvalues of \( P^\delta_h \) exhibit the following interaction:

**Proposition 2.2.** Under the hypothesis of Theorem 2.1, we have that

\[ \begin{align*}
&\bullet \quad \text{for } h^{\frac{1}{4}} \ll |z-w| \ll h^{\frac{1}{2}} \\
&D^\delta(z, w; h) = \frac{\sigma_h^3(z, w)|z-w|^2}{(4\pi h)^2} \left( 1 + O\left( \frac{|z-w|^2}{h} + \delta h^{-\frac{5}{2}} \right) \right); \\
&\bullet \quad \text{for } |z-w| \gg (h \ln h^{-1})^{\frac{1}{2}} \\
&D^\delta(z, w; h) = \frac{\sigma(z)\sigma(w) + O(h)}{(2\pi)^2} \left( 1 + O\left( \delta h^{-\frac{2}{5}} \right) \right).
\end{align*} \]

Let us give some comments on this result: That the we cannot analyze the eigenvalue interaction completely up to the diagonal is due to some technical difficulties.

In the above proposition, two eigenvalues of the perturbed operator \( P^\delta_h \) show the following types of interaction:

\begin{itemize}
  \item **Short range repulsion:** the two-point density decays quadratically in \( |z-w| \) if two eigenvalues are too close.
\end{itemize}
• **Long range decoupling:** if the distance between two eigenvalues is \( \gg (h \ln h^{-1})^{\frac{3}{2}} \) the two-point density is given by the product of two one-point densities. We have shown in [27] (see as well [10]) that the one-point density of eigenvalues in \( \Omega \) as in (H.2), is given by

\[
\mathbb{E}[\Xi(\varphi) 1_{B(0,R)}] = \int \varphi(z) d(z; h) L(dz) \quad \varphi \in C_0(\Omega),
\]

where

\[
d(z; h) = \frac{1}{2\pi h} \sigma(z) + O(1). \tag{2.3}
\]

2.2. **Conditional density function.** It follows from the assumptions (H.1) that \( d(z; h) > 0 \) for all \( z \in \Omega \) as in (H.2). Hence, under the assumptions of Theorem 2.1, the conditional average density of eigenvalues of \( P_h^\delta \) given that \( w_0 \in \sigma(P_h^\delta) \) is well defined and given by

\[
D_{w_0}^\delta (z; h) := \frac{D^\delta(z, w_0; h)}{d(w_0; h)}.
\]

We have the following asymptotic behavior of conditional average density \( D_{w_0}^\delta (z; h) \):

**Proposition 2.3.** Under the hypothesis of Theorem 2.1, we have that for \( w_0 \in \Omega \)

- for \( h^\frac{3}{2} \ll |z - w_0| \ll h^\frac{1}{2} \)

\[
D_{w_0}^\delta (z; h) = \frac{\sigma_h^2(z)|z - w_0|^2}{8\pi h} \left( 1 + O\left( \frac{|z - w_0|^2}{h} + \delta h^{-\frac{8}{5}} \right) \right);
\]

- for \( |z - w_0| \gg (h \ln h^{-1})^{\frac{3}{2}} \)

\[
D_{w_0}^\delta (z; h) = \frac{\sigma(z) + O(h)}{2h\pi} \left( 1 + O\left( \delta h^{-\frac{8}{5}} \right) \right).
\]

In the above proposition we see that, given an eigenvalue \( w_0 \in \sigma(P_h^\delta) \), the density of finding another eigenvalue in the vicinity of \( w_0 \) shows the following behavior:

- **Short range repulsion:** the density \( D_{w_0}^\delta (z; h) \) decays quadratically in \( \sigma_h(z)|z - w_0| \) if the distance between \( z \) and \( w_0 \) is smaller than a term of order \( h^{\frac{3}{2}} \). It follows from (H.1) that \( \sigma(z) \) grows towards the boundary of \( \Sigma \), hence the short range repulsion is weaker for \( \Omega \) close to the boundary of \( \Sigma \), as we expected from the numerical simulations presented in [27].

- **Long range decoupling:** if the distance between \( z \) and \( w_0 \) is larger than a term of order \( (h \ln h^{-1})^{\frac{3}{2}} \), the density \( D_{w_0}^\delta (z; h) \) is given up to a small error by the 1-point density \( d(z; h) \) (see (2.3)). Hence, we see that at these distances two eigenvalues of \( P_h^\delta \) are up to a small error uncorrelated.

To illustrate Proposition 2.3, Figure 1 shows a plot of the principal terms of the conditional density \( D_{w_0}^\delta \) as a function of \( |z| \), for \( w_0 = 0 \) and \( h = 0.01 \), assuming for simplicity that \( \sigma(z) = \text{const} \). On the left hand side of the graph we see the quadratic decay, whereas on the right hand side the density is given by \( (2\pi h)^{-1} \sigma(z) \).

3. **A Formula for the 2-Point Intensity Measure**

In this section we will give a short reminder of a well-posed Grushin problem for the perturbed operator \( P_h^\delta \) which has already been used in [20, 27]. We will then employ the resulting effective Hamiltonians to derive a formula for the two-point intensity measure defined in (1.8).
3.1. **Grushin Problem.** We begin by giving a short refresher on Grushin problems. They have become an essential tool in Microlocal Analysis and are employed with great success in a vast number of works. As reviewed in [23], the central idea is to set up an auxiliary problem of the form

$$\begin{pmatrix} P(z) & R_- \\ R_+ & 0 \end{pmatrix} : \mathcal{H}_1 \oplus \mathcal{H}_- \rightarrow \mathcal{H}_2 \oplus \mathcal{H}_+,$$

where $P(z)$ is the operator under investigation and $R_\pm$ are suitably chosen. We say that the Grushin problem is well-posed if this matrix of operators is bijective. If $\dim \mathcal{H}_- = \dim \mathcal{H}_+ < \infty$, on typically writes

$$\begin{pmatrix} P(z) & R_- \\ R_+ & 0 \end{pmatrix}^{-1} = \begin{pmatrix} E(z) & E_+(z) \\ E_-(z) & E_{-+}(z) \end{pmatrix}.$$

The key observation goes back to the Shur complement formula or, equivalently, the Lyapunov-Schmidt bifurcation method, i.e. the operator $P(z) : \mathcal{H}_1 \rightarrow \mathcal{H}_2$ is invertible if and only if the finite dimensional matrix $E_{-+}(z)$ is invertible and when $E_{-+}(z)$ is invertible, we have

$$P^{-1}(z) = E(z) - E_+(z)E_{-+}^{-1}(z)E_-(z).$$

$E_{-+}(z)$ is sometimes called effective Hamiltonian.

Next, we give a short reminder of the Grushin Problem used to study $P_h^δ$. First, we introduce the following auxiliary operators which have already been used by M. Hager J. Sjöstrand in [11].

3.1.1. **Two auxiliary operators.** For $z \in \mathbb{C}$ we consider $Q(z)$ and $\tilde{Q}(z)$, two $z$-dependent elliptic self-adjoint operators from $L^2(S^1)$ to $L^2(S^1)$, defined by

$$Q(z) := (P_h - z)^*(P_h - z), \quad \tilde{Q}(z) := (P_h - z)(P_h - z)^* \quad (3.1)$$
with natural domains given by $\mathcal{D}(Q(z)), \mathcal{D}(\tilde{Q}(z)) = H^2_{sc}(S^1)$. Since $S^1$ is compact and these are elliptic, non-negative, self-adjoint operators their spectra are discrete and contained in the interval $[0, \infty]$. Since

$$Q(z)u = 0 \Rightarrow (P_h - z)u = 0$$

it follows that $\mathcal{N}(Q(z)) = \mathcal{N}(P_h - z)^\ast$ and $\mathcal{N}(\tilde{Q}(z)) = \mathcal{N}((P_h - z)^\ast)$. Furthermore, if $\lambda \neq 0$ is an eigenvalue of $Q(z)$ with corresponding eigenvector $e_\lambda$ we see that $f_\lambda := (P_h - z)e_\lambda$ is an eigenvector of $\tilde{Q}(z)$ with the eigenvalue $\lambda$. Similarly, every non-vanishing eigenvalue of $\tilde{Q}(z)$ is an eigenvalue of $Q(z)$ and moreover, since $P_h - z$, $(P_h - z)^\ast$ are Fredholm operators of index 0 we see that $\dim \mathcal{N}(P_h - z) = \dim \mathcal{N}((P_h - z)^\ast)$. Hence the spectra of $Q(z)$ and $\tilde{Q}(z)$ are equal

$$\sigma(Q(z)) = \sigma(\tilde{Q}(z)) = \{t^2_0, t^2_1, \ldots\}, \quad 0 \leq t_j \not\to \infty. \quad (3.2)$$

Now consider the orthonormal basis of $L^2(S^1)$

$$\{e_0, e_1, \ldots\} \quad (3.3)$$

consisting of the eigenfunctions of $Q(z)$. By the previous observations we have

$$(P_h - z)(P_h - z)^\ast(P_h - z)e_j = t^2_j(P_h - z)e_j.$$ 

Thus defining $f_0$ to be the normalized eigenvector of $\tilde{Q}$ corresponding to the eigenvalue $t^2_0$ and the vectors $f_j \in L^2(S^1)$, for $j \in \mathbb{N}^\ast$, as the normalization of $(P_h - z)e_j$ such that

$$(P_h - z)e_j = \alpha_j f_j, \quad (P_h - z)^\ast f_j = \beta_j e_j \quad \text{with} \quad \alpha_j \beta_j = t^2_j, \quad (3.4)$$

yields an orthonormal basis of $L^2(S^1)$

$$\{f_0, f_1, \ldots\} \quad (3.5)$$

consisting of the eigenfunctions of $\tilde{Q}(z)$. Since $\alpha_j = (P_h - z)e_j | f_j) = (e_j | (P_h - z)^\ast f_j) = \tilde{\beta}_j$ we can conclude that $\alpha_j \tilde{\sigma}_j = t^2_j$.

3.1.2. A Grushin Problem for the perturbed operator $P_h^t$. Following Sjöstrand in [20], we us the eigenfunctions of the operators $Q$ and $\tilde{Q}$ to create a well-posed Grushin Problem. The sequel is taken from [27], but it originates in the works of Hager [10], Bordeaux-Montreieux [1] and Sjöstrand [20].

**Proposition 3.1** ([27]). Let $z \in \Omega \Subset \Sigma$ with dist $(\Omega, \partial \Sigma) > 1/C$ and let $\alpha_0, e_0$ and $f_0$ be as in (3.4). Define

$$R_+: H^1(S^1) \to \mathbb{C} : \ u \mapsto (u | e_0)$$

$$R_- : \mathbb{C} \to L^2(S^1) : \ u_- \mapsto u_- f_0. \quad (3.6)$$

Then

$$\mathcal{P}(z) := \begin{pmatrix} P_h - z & R_- \\ R_+ & 0 \end{pmatrix} : \ H^1(S^1) \times \mathbb{C} \to L^2(S^1) \times \mathbb{C}$$

is bijective with the bounded inverse

$$\mathcal{E}(z) = \begin{pmatrix} E(z) & E_+(z) \\ E_-(z) & E_{++}(z) \end{pmatrix}$$
where $E_-(z)v = (v|f_0)$, $E_+(z)v_+ = v_+e_0$ and $E(z) = (P_h - z)^{-1}|_{(f_0)^{+} \to (f_0)^{+}}$ and $E_+(z)v_+ = -\alpha_0v_+$. Furthermore, we have the estimates for $z \in \Omega$

\[
\|E_-(z)\|_{L^2 \to C}, \|E_+(z)\|_{C \to H^1} = O(1),
\]

\[
\|E(z)\|_{L^2 \to H^1} = O(h^{-1/2}),
\]

\[
|E_+(z)| = O(\sqrt{h}e^{-\frac{S}{2}}) = O\left(e^{-\frac{C_1}{h}}\right);
\]

(3.7)

**Definition 3.2.** For $x \in \mathbb{R}$ we denote the integer part of $x$ by $[x]$ as in Section 1. Let $C_1 > 0$ be big enough as above and define $N := (2\lfloor \frac{C_1}{R} \rfloor + 1)^2$. Let $e_0$ and $f_0$ be as in (3.4), let $z \in \Omega \subseteq \Sigma$ and let $\hat{e}_0(z; \cdot)$ and $\hat{f}_0(z; \cdot)$ denote the Fourier coefficients of $e_0$ and $f_0$. We define the vector $X(z) = (X_{j,k}(z))_{|j|,|k| \leq \lfloor \frac{C_1}{h} \rfloor} \in \mathbb{C}^N$ to be given by

\[
X_{j,k}(z) = \hat{e}_0(z; k)\overline{f_0(z; j)}, \quad \text{for } |j|, |k| \leq \left\lfloor \frac{C_1}{h} \right\rfloor.
\]

**Proposition 3.3** ([27]). Let $z \in \Omega \subseteq \Sigma$. Let $N$ be as in Definition 3.2 and let $B(0, R) \subset \mathbb{C}^N$ be the ball of radius $R := C/h$, $C > 0$ large, centered at 0. Let $R_-, R_+$ be as in Proposition 3.1. Then

\[
\mathcal{P}_\delta(z) := \left( \begin{array}{cc}
\frac{P_h^\delta - z}{R_-} & R_-
\end{array} \right): H^1(S^1) \times \mathbb{C} \rightarrow L^2(S^1) \times \mathbb{C}
\]

is bijective with the bounded inverse

\[
\mathcal{E}_\delta(z) = \left( \begin{array}{cc}
E_\delta(z) & E_\delta^+(z)
\end{array} \right)
\]

where

\[
E_\delta(z) = E(z) + O(\delta h^{-2}) = O(h^{-1/2})
\]

\[
E_\delta^-(z) = E_-(z) + O(\delta h^{-3/2}) = O(1)
\]

\[
E_\delta^+(z) = E_+(z) + O(\delta h^{-3/2}) = O(1)
\]

and

\[
E_\delta^+(z) = E_-(z) - \delta X(z) \cdot \alpha + T(z, \alpha),
\]

(3.8)

with $X(z) \cdot \alpha = E_-Q_\omega E_+$, $\alpha \in B(0, R)$, and

\[
T(z, \alpha) := \sum_{n=1}^\infty (-\delta)^{n+1}E_-Q_\omega (E\omega)^nE_+ = O(\delta^2 h^{-5/2}).
\]

(3.9)

Here, the dot-product $X(z) \cdot \alpha$ is the natural bilinear one.

**Remark 3.4.** The effective Hamiltonian $E_\delta^+(z)$ depends smoothly on $z \in \Omega$ and holomorphically on $\alpha \in B(0, R) \subset \mathbb{C}^N$. As in (8.6) and Proposition 4.6 in [27], we have the following estimates: for all $z \in \Omega$, all $\alpha \in B(0, R)$ and all $\beta \in \mathbb{N}^2$

\[
\partial_z^\beta E_\delta^+(z) = O\left( h^{-|\beta|+1/2}e^{-\frac{S}{2}} \right), \quad \text{and}
\]

\[
\partial_z^\beta T(z, \alpha) = O\left( \delta^2 h^{-|\beta|+\frac{5}{2}} \right)
\]

where $S$ is as in Definition 1.2.
Moreover, as remarked in [20] the effective Hamiltonian \( E_\delta^\pm(z) \) satisfies a \( \overline{\partial} \)-equation, i.e. there exists a smooth function \( f_\delta : \Omega \to \mathbb{C} \) such that
\[
\partial_\bar{z}E_\delta^\pm(z) + f_\delta(z)E_\delta^\pm(z) = 0.
\]
This implies that the zeros of \( E_\delta^\pm(z) \) are isolated and countable and we may use the same notion of multiplicity as for holomorphic functions.

3.2. Counting zeros. By of the above well-posed Grushin Problem for the perturbed operator \( P_\delta^k \) we have that \( \sigma(P_\delta^k) = (E_\delta^\pm)^{-1}(0) \). Hence, to study the the two-point intensity measure \( \nu \) defined in (1.8), we investigate the integral
\[
\pi^{-N} \int_{B(0,R)} \left( \sum_{z,w \in (E_\delta^\pm)^{-1}(0)} \varphi(z,w) e^{-\alpha^* \cdot \alpha} L(d\alpha) \right) = \int_{\mathbb{C}^2} \varphi(z_1,z_2) d\nu(z_1,z_2)
\]
with \( \varphi \in C_0^\infty(\Omega \times \Omega) \). Using Remark 3.4, we see that the integral is finite since the number of pairs of zeros of \( E_\delta^\pm(\cdot, \alpha) \) in \( \text{supp} \varphi \) is uniformly bounded for \( \alpha \in B(0,R) \).

Recall \( \Xi \) defined in (1.7). Using [27, Lemma 7.1], we get the following regularization of the 2-fold counting measure \( \Xi \otimes \Xi \)
\[
\langle \varphi, \Xi \otimes \Xi \rangle = \lim_{\varepsilon \to 0^+} \int \int \varphi(z_1,z_2) 2 \left[ \prod_{j=1}^2 \varepsilon^{-2} \chi \left( \frac{E_\delta^\pm(z_j)}{\varepsilon} \right) \right] |\partial_{z_j} E_\delta^\pm(z_j)|^2 L(dz_1)L(dz_2),
\]
where \( \chi \in C_0^\infty(\mathbb{C}) \) such that \( \int \chi(w) L(dw) = 1 \). Assuming that \( \varphi \in C_0^\infty(\Omega \times \Omega) \) is such that \( \{(z,z); z \in \Omega\} \not\subset \text{supp} \varphi \), we see by the Lebesgue dominated convergence theorem that the two-point intensity measure of the point process \( \Xi \) is given by
\[
\int_{\mathbb{C}^2} \varphi(z_1,z_2) d\nu(z_1,z_2) = \lim_{\varepsilon \to 0^+} \int \int \varphi(z_1,z_2) K_\varepsilon^\delta(z_1,z_2;h) L(dz_1)L(dz_2) \quad (3.10)
\]
with
\[
K_\varepsilon^\delta(z_1,z_2;h) := \int_{B(0,R)} \left[ \prod_{j=1}^2 \varepsilon^{-2} \chi \left( \frac{E_\delta^\pm(z_j)}{\varepsilon} \right) \right] |\partial_{z_j} E_\delta^\pm(z_j)|^2 e^{-\alpha^* \alpha} L(d\alpha).
\]
Using (3.8), we see that the main object of interest is the random vector
\[
F^\delta(z,w,\alpha;h) = \begin{pmatrix} E_\delta^\pm(z) \\ E_\delta^\pm(w) \\ (\partial_\bar{z} E_\delta^\pm)(z) \\ (\partial_\bar{z} E_\delta^\pm)(w) \end{pmatrix} = \begin{pmatrix} E_+^\pm(z) \\ E_-^\pm(w) \\ (\partial_\bar{z} E_+^\pm)(z) \\ (\partial_\bar{z} E_+^\pm)(w) \end{pmatrix} - \delta \begin{pmatrix} X(z) \cdot \alpha \\ X(w) \cdot \alpha \\ (\partial_x X)(z) \cdot \alpha \\ (\partial_x X)(w) \cdot \alpha \end{pmatrix} + \begin{pmatrix} T(z,\alpha) \\ T(w,\alpha) \\ (\partial_\bar{z} T)(z,\alpha) \\ (\partial_\bar{z} T)(w,\alpha) \end{pmatrix}.
\]
The first order term (with respect to \( \delta \)) the correlation matrix of \( F \) is given by the Gramian matrix \( G \) defined by
\[
G := \begin{pmatrix} A & B \\ B^* & C \end{pmatrix} \in \mathbb{C}^{4 \times 4},
\]
with

\[
A := \begin{pmatrix}
(X(z)|X(z)) & (X(z)|X(w)) \\
(X(w)|X(z)) & (X(w)|X(w))
\end{pmatrix},
\]

\[
B := \begin{pmatrix}
(X(z)|\partial_z X(z)) & (X(z)|\partial_w X(w)) \\
(X(w)|\partial_z X(z)) & (X(w)|\partial_w X(w))
\end{pmatrix},
\]

\[
C := \begin{pmatrix}
(\partial_z X(z)|\partial_z X(z)) & (\partial_z X(z)|\partial_w X(w)) \\
(\partial_w X(w)|\partial_z X(z)) & (\partial_w X(w)|\partial_w X(w))
\end{pmatrix}.
\] (3.13)

Notice that the matrices \(A, B, C\) depend on \(h\); see Definition 3.2. Next, we will state a formula for the Lebesgue density of the two-point intensity measure \(\nu\) in terms of the permanent of the Shur complement of \(G\), i.e \(\Gamma := C - B^*A^{-1}B\). For the definition of the permanent of a matrix we refer the reader to [13]. It is defined as follows:

**Definition 3.5.** Let \((M_{ij})_{ij} = M \in \mathbb{C}^{n \times n}\) be a square matrix and let \(S_n\) denote the symmetric group of order \(n\). The permanent of \(M\) is defined by

\[
\text{perm} M := \sum_{\sigma \in S_n} \prod_{i=1}^{n} M_{i\sigma(i)}.
\]

**Remark 3.6.** Although the definition of the permanent resembles closely that of the determinant, the two objects are quite different. Many properties known to hold true for determinants, fail to be true for permanents. For our purposes it is enough to note that it is multi-linear and symmetric. For more details concerning permanents and their properties we refer the reader to [13].

We will prove the following result:

**Proposition 3.7.** Let \(\Omega \Subset \Sigma\) be as in (H.2). Let \(\delta > 0\) be as in Hypothesis (H.3) and let \(\Gamma = C - B^*A^{-1}B\). Moreover, let \(D(\Omega, C_2)\) be as in (2.2). Then, there exists a smooth function

\[
D^\delta(z, w; h) = \frac{\text{perm} \Gamma(z, w; h) + O\left(e^{-\frac{\delta h}{\pi}} + \delta h^{-\frac{\delta}{2n}}\right)}{\pi^2 \left(\sqrt{\det A(z, w; h)} + O\left(\delta h^{-\frac{3}{2}}\right)\right)} + O\left(e^{-\frac{\delta}{h^2}}\right),
\]

and there exists a constant \(C_2 > 0\) such that for all \(\varphi \in C_0^\infty(\Omega^2 \setminus D_h(\Omega, C_2))\)

\[
\int_{\Omega^2} \varphi(z, w) d\nu(z, w) = \int_{\Omega^2} \varphi(z, w) D(z, w, h, \delta) L(d(z, w)).
\]

**Remark 3.8.** The proof of Proposition 3.7 will take up most of the rest of this paper. Therefore we give a short overview on how we will proceed:

In Section 4, we give a formula for the scalar product \((X(z)|X(w))\) by constructing quasimodes for the operators \((P_h - z)\) and \((P_h - z)^*\) to approximate the eigenfunction \(e_0\) and \(f_0\), and by using the method of stationary phase.

In Section 5, we will use this formula to study the invertibility of the matrices \(G, A\) and \(\Gamma\). Furthermore, we will study the permanent of \(\Gamma\).

In Section 6, we give a proof of Proposition 3.7.
4. Stationary Phase

In this section we are interested in the scalar product \((X(z)|X(w))\). Recall from Definition 3.2 that the vector \(X(z)\), \(z \in \Omega\), is given by \(X_{j,k} = \hat{e}_0(z;k)f_0(z;j)\), where \(e_0\) and \(f_0\) are the eigenfunctions of the operators \(Q(z)\) and \(\bar{Q}(z)\), respectively, associated to their first eigenvalue \(t_0^2\).

The Fourier coefficients \(\hat{e}_0(z;k), f_0(z;j)\) and their \(z\)- and \(\bar{z}\)-derivatives are of order \(O(|k|^{-\infty}), O(|j|^{-\infty})\), for \(|j|, |k| \geq C/h\) with \(C > 0\) large enough (cf. [27, Propositions 5.5 and 5.6]). The Parseval identity implies that for \(z, w \in \Omega\)

\[
(X(z)|X(w)) = (e_0(z)|e_0(w))(f_0(w)|f_0(z)) + O_C(h^{\infty}).
\] (4.1)

The aim of this of this section is to prove the following result:

**Proposition 4.1.** Let \(\Omega \subset \Sigma\) be as in hypothesis (H.2) and let \(x\pm(z)\) be as in (1.4). Furthermore, for \(z \in \Omega\) let \(\sigma(z)\) denote the Lebesgue density of the direct image of the symplectic volume form on \(T^*S^1\) under the principal symbol \(p\), i.e. \(\sigma(z)L(dz) = p_*(d\xi \wedge dx)\).

Then, there exists a constant \(C > 0\) such that for all \((z, w) \in \Delta_\Omega(C) := \{(z, w) \in \Omega^2; \ |z - w| < 1/C\}

\[
(X(z)|X(w)) = e^{-\frac{1}{\pi h}\Phi(z;h) - \frac{1}{\pi h}\Phi(w;h)}e^{\frac{2}{\pi h}\Psi(z, w)} + O_C(h^{\infty})
\]

where:

- \(\Phi(\cdot; h) : \Omega \to \mathbb{R}\) is a family of smooth functions depending only on \(i\text{Im} z\), which satisfy

\[
\Phi(z; h) = \text{Im} \int_{x_+(z)}^{x_0} (z - g(y))dy - \text{Im} \int_{x_-(z)}^{y_0} (z - g(y))dy + \frac{h}{4} \left[ \ln \left( \frac{\pi h}{\text{Im} g'(x_+(z))} \right) + \ln \left( \frac{\pi h}{\text{Im} g'(x_-(z))} \right) \right] + O(h^2).
\]

and

\[
\partial^2_{zz}\Phi(z; h) = \frac{1}{4} \sigma(z) + O(h).
\]

- \(\Psi(\cdot, \cdot; h) : \Delta_\Omega(C) \to \mathbb{C}\) is a family of smooth functions which are almost \(z\)-holomorphic and almost \(w\)-anti-holomorphic extensions from the diagonal \(\Delta := \{(z, z); z \in \Omega\} \subset \Delta_\Omega(C)\) of \(\Phi(z; h)\), i.e.

\[
\Psi(z, z; h) = \Phi \left( \frac{1}{2}(z - z); h \right), \quad \partial_\bar{z}\Psi, \partial_w\Psi = O(|z - w|^{\infty}).
\]

Moreover, we have that \(\Psi(z, z) = \Phi(z)\) and for \(z, w \in \Delta_\Omega(C)\) with \(\ |z - w| \ll 1\),

\[
\Psi(z, w; h) = \sum_{\alpha + \beta \leq 2} \frac{1}{2^{\alpha + \beta} \alpha! \beta!} \partial^\alpha_{\bar{z}} \partial^\beta_w \Phi \left( \frac{z + w}{2}; h \right) (z - w)^\alpha (w - z)^\beta + O(|z - w|^{\beta} + h^{\infty}),
\]

and

\[
2\text{Re} \Psi(z, w; h) - \Phi(z; h) - \Phi(w; h)
\]

\[
= -\partial^2_{zz}\Phi \left( \frac{z + w}{2}; h \right) |z - w|^2 (1 + O(|z - w| + h^{\infty}));
\]

13
the function $\Psi(z, w; h)$ has the following symmetries:

$$\Psi(z, w; h) = \overline{\Psi(w, z; h)} \quad \text{and} \quad (\partial_z \Psi)(z, w; h) = \overline{(\partial_w \Psi)(w, z; h)}.$$ 

Let us give some remarks on the above results: Note that the formula for $\Psi$ stated above is simply a special case of the more general Taylor expansion

$$\Psi(z_0 + \zeta, z_0 + \omega; h) = \sum_{|\alpha + \beta| \leq 2} \frac{1}{2^{2|\alpha + \beta|}\alpha!\beta!} \partial_z^{\alpha} \partial_{\omega}^{\beta} \Phi(z_0; h) \zeta^\alpha \omega^\beta + \mathcal{O}((\zeta, \omega)^3 + h^\infty),$$

with $z_0 \in \Omega$ and $|\zeta|, |\omega| \ll 1$.

Next, we define for $(z, w) \in \Delta_1(C)$, as in Proposition 4.1,

$$-K(z, w) := 2\text{Re} \Psi(z, w; h) - \Phi(z; h) - \Phi(w; h) \quad (4.2)$$

$$= -\left(\sigma \left(\frac{z + w}{2}\right) + O(h)\right) \frac{|z - w|^2}{4}(1 + O(|z - w| + h^\infty)).$$

From the above Proposition we can immediately deduce some growth properties of certain quantities that will be become important in the sequel.

**Corollary 4.2.** Under the assumptions of Proposition 4.1,

- $\|(X(z)|X(w))\| = e^{\frac{K(x, w)}{h}} + O_C(h^\infty)$;
- $\|X(z)\|^2\|X(w)\|^2 \pm \|(X(z)|X(w))\|^2 = \left(1 \pm e^{\frac{-2K(x, w)}{h}}\right) + O_C(h^\infty)$;
- $\|X(z)\|^2\|X(w)\|^2\|(X(z), X(w))\|^2 = e^{\frac{-2K(x, w)}{h}} + O_C(h^\infty)$.

To prove Proposition 4.1, we shall study the scalar products $(e_0(z)|e_0(w))$ and $(f_0(w)|f_0(z))$.

### 4.1. The Scalar Product $(e_0(z)|e_0(w))$.

**Proposition 4.3.** Let $\Omega \subset \Sigma$ be as in hypothesis (H.2) and let $x_+(z)$ be as in (1.4). Then, there exists a constant $C > 0$ such that for all $(z, w) \in \Delta_1(C) := \{(z, w) \in \Omega^2; |z - w| < 1/C\}$

$$(e_0(z)|e_0(w)) = e^{-\frac{i}{h}\Phi_1(z; h)}e^{-\frac{i}{h}\Phi_1(w; h)}e^{\frac{i}{\hbar}\Psi_1(z, w; h)} + O(h^\infty), \quad (4.3)$$

where:

- $\Phi_1(\cdot; h) : \Omega \to \mathbb{R}$ is a family of smooth functions depending only on $i\text{Im} z$, which satisfy
  $$\Phi_1(z; h) = \text{Im} \int_{x_+(\text{Im} z)}^{x_0} (z - g(y))dy + \frac{h}{4} \ln \left(\frac{\pi h}{-\text{Im} g'(x_+)}\right) + O(h^2).$$

- $\Psi_1(\cdot, \cdot; h) : \Delta_1(C) \to \mathbb{C}$ is a family of smooth functions which are almost $z$-holomorphic and almost $w$-anti-holomorphic extensions from the diagonal $\Delta := \{(z, z); z \in \Omega\} \subset \Delta_1(C)$ of $\Phi_1(z; h)$, i.e.
  $$\Psi_1(z, z; h) = \Phi_1\left(\frac{1}{2}(z - \overline{z}); h\right), \quad \partial_z \Psi_1, \partial_w \Psi_1 = O(|z - w|).$$
Moreover, for \( z, w \in \Delta_\Omega(C) \) with \( |z - w| \ll 1 \), one has that
\[
\Psi_1(z, w; h) = \sum_{|\alpha + \beta| \leq 2} \frac{1}{2^{\alpha+\beta} \alpha! \beta!} \partial^\alpha_x \partial^\beta_z \Phi_1 \left( \frac{z + w}{2}; h \right) (z - w)^\alpha (\bar{w} - \bar{z})^\beta + \mathcal{O}(|z - w|^3 + h^\infty),
\]
and that
\[
2\Re \Psi_1(z, w; h) - \Phi_1(z; h) - \Phi_1(w; h) = -\partial_x \partial_z \Phi_1 \left( \frac{z + w}{2}; h \right) |z - w|^2 (1 + \mathcal{O}(|z - w| + h^\infty));
\]
- the function \( \Psi_1(z, w; h) \) has the following symmetries:
\[
\Psi_1(z, w; h) = \overline{\Psi_1(w, z; h)} \quad \text{and} \quad (\partial_x \Psi_1)(z, w; h) = (\partial\Psi_1)(w, z; h).
\]

To prove Proposition 4.3, we begin by constructing an oscillating function to approximate \( e_0(z) \). Let us recall from Section 1 that the points \( a, b \in S^1 \) denote the minimum and the maximum of the Im \( g(x) \) and that for \( z \in \Omega \) the points \( x_\pm(z) \in S^1 \) are the unique solutions to the equation \( \text{Im} g(x) = \text{Im} z \). Furthermore, we will identify frequently \( S^1 \) with the interval \([b - 2\pi, b]\). Moreover, let us recall that by the natural projection \( \Pi : \mathbb{R} \rightarrow S^1 = \mathbb{R}/2\pi\mathbb{Z} \) we identify the points \( x_\pm, a, b \in S^1 \) with points \( x_\pm, a, b \in \mathbb{R} \) such that \( b - 2\pi < x_+ < a < x_- < b \).

Let \( K_+ \subset [b - 2\pi, a] \) be an open interval such that \( x_+(z) \in K_+ \) for all \( z \in \Omega \). Let \( \chi \in \mathcal{C}_0^\infty([b - 2\pi, a]) \) and define for \( x \in \mathbb{R} \)
\[
\tilde{e}_0(x, z) := \chi(x) \exp \left( \frac{i}{h} \psi_+(x, z) \right). \tag{4.4}
\]
where, for a fixed \( x_0 \in K_+ \),
\[
\psi_+(x, z) := \int_{x_0}^x (z - g(y)) \, dy. \tag{4.5}
\]

**Remark 4.4.** Note that the function \( u = \exp(i\psi_+(x, z)/h) \) is solution to \((P_h - z)u = 0\) on \( \text{supp} \chi \), since the phase function \( \psi_+ \) satisfies the eikonal equation
\[
p(x, \partial_x \psi_+) = z.
\]
Furthermore, let us remark that \( \tilde{e}_0(x, z) \) depends holomorphically on \( z \).

Next, we are interested in the \( L^2 \)-norm of \( \tilde{e}_0 \).

**Lemma 4.5.** Let \( \Omega \) be as in (H.2). Then, there exists a family of smooth functions \( \Phi_1(\cdot; h) : \Omega \rightarrow \mathbb{R} \), such that
\[
\Phi_1(z; h) = \Phi_1(i\text{Im} z; h) = \text{Im} \int_{x_+(\text{Im} z)}^{x_0} (z - g(y)) \, dy + \frac{h}{4} \ln \left( \frac{\pi h}{-\text{Im} g'(x_+)} \right) + \mathcal{O}(h^2)
\]
and
\[
\|\tilde{e}_0(z)\|^2 = \exp \left\{ \frac{2}{h} \Phi_1(z; h) \right\}.
\]
Proof. In view of the definition of $e_0(z)$, see (4.4) and (4.5), one gets that
\[
||e_0(z)||^2 = \int \chi(x)e^{-\frac{\pi}{h}|\phi(x,z)|}\,dx = \int \chi(x)e^{-\frac{2\pi}{h}\Im \psi_+(x,z)}\,dx.
\]
The critical point for $\Im \psi_+(x,z)$ is given by the equation
\[
\Im \partial_x \psi_+(x,z) = \Im z - \Im g(x) = 0, \quad x \in \text{supp } \chi.
\]
The critical point is unique and is given by $x_+(\Im z)$ and satisfies $\Im g'(x_+(\Im z)) < 0$, see (1.4). This implies in particular that the critical point is non-degenerate. More precisely,
\[
\Im (\partial^2_{xx} \psi_+)(x_+, z) = -\Im g'(x_+) > 0. \tag{4.6}
\]
The critical value of $\Im \psi_+$ is given by
\[
\Im \psi_+(x_+(\Im z), z) = \Im \int_{x_0}^{x_+(\Im z)} (z - g(y))\,dy \leq 0.
\]
Using the method of stationary phase, one gets
\[
||e_0(z)||^2 = \frac{\pi h}{\Im (\partial^2_{xx} \psi_+)(x_+, z)}(1 + O(h)) \exp \left\{ -\frac{2\Im \psi_+(x_+, z)}{h} \right\}
\]
\[
=: \exp \left\{ \frac{2}{h} \Phi_1(z; h) \right\},
\]
where $\Phi_1$ is smooth in $z$. Using (4.6), one gets that
\[
\Phi_1(z; h) = \Im \int_{x_0}^{x_+(\Im z)} (z - g(y))\,dy + \frac{h}{4} \ln \left( \frac{\pi h}{-\Im g'(x_+)} \right) + O(h^2). \tag*{□}
\]

Recall from (3.3) that the function $e_0$ is an eigenfunction of the operator $Q(z)$ (c.f Section 3.1.1) corresponding to its first eigenvalue $t_0^2$. We set
\[
e_0(z) = \frac{\Pi t_0^2 \left( e^{-\frac{i}{h}\Phi_1(z; h)} e_0(z) \right)}{\left\| \Pi t_0^2 \left( e^{-\frac{i}{h}\Phi_1(z; h)} e_0(z) \right) \right\|},
\]
where $\Pi_0 : L^2(S^1) \to C e_0$ denotes the spectral projection for $Q(z)$ onto the eigenspace associated with $t_0^2$.

Next, we prove that up to an exponentially small error in $1/h$, $e_0$ is given by the normalization of $e_0$.

Lemma 4.6. Let $\Omega$ be as in (H.2). Then, there exists a constant $C > 0$ such that for all $z \in \Omega$ and all $\alpha \in \mathbb{N}^2$
\[
\left\| \partial_{x_0}^\alpha \left( e_0(z) - e^{-\frac{i}{h}\Phi_1(z; h)} e_0(z) \right) \right\| = O \left( h^{-|\alpha|} e^{-\frac{1}{2\pi h}} \right).
\]
Proof. The proof of the lemma is similar to the proof of [27, Proposition 3.11]. \tag*{□}

This result implies that
\[
(e_0(z)|e_0(w)) = e^{-\frac{i}{h}\Phi_1(z; h) - \frac{i}{h}\Phi_1(w; h)}(\bar{e}_0(z)\bar{e}_0(w)) + O_{c\infty} \left( e^{-\frac{1}{2\pi h}} \right). \tag{4.7}
\]
By Remark 4.4, $(\bar{e}_0(z)|\bar{e}_0(w))$ is holomorphic in $z$ and anti-holomorphic in $w$. We can study this scalar product by the method of stationary phase:
Hence, we can write (4.8) as follows:
\[ \Theta := \text{Im} \Theta, \] where \( \tilde{e}_0(x, z) \) is given in (4.4) and \( \Psi_+ \) is defined by
\[ \Psi_+(x, z, w) := \psi_+(x, z) - \overline{\psi_+(x, w)}, \quad z, w \in \Omega. \] Using (4.5), we see that
\[ \Psi(x, z, w) = \int_{x_0}^x \text{Re} (z - w)dy + 2i \int_{x_0}^x \left[ \text{Im} \left( \frac{z + w}{2} \right) - \text{Im} g(y) \right] dy. \] Since the imaginary part of \( \Psi_+ \) can be negative, we shift the phase function by the

**Minimum of \( \text{Im} \Psi_+ \).** The critical points of the function \( x \mapsto \text{Im} \Psi(x, z, w) \) are given by the equation \( \text{Im} (\frac{z + w}{2}) = \text{Im} g(x) \). Since \( \Omega \) is convex, this equation has, for \( |z - w| \) small enough, on the support of \( \chi \) the unique solution \( x_+ (\frac{z + w}{2}) \in \mathbb{R} \) and it satisfies \( \text{Im} g'(x_+ (\frac{z + w}{2})) < 0 \) (cf. (1.4)). Moreover, it depends smoothly on \( z \) and \( w \) since \( g \) is smooth. Therefore,
\[ (\partial_{xx}^2 \text{Im} \Psi_+) (x_+ (\frac{z + w}{2}), z, w) = -2\text{Im} g'_x \left( x_+ \left( \frac{z + w}{2} \right) \right) > 0, \] which implies that \( x_+ (\frac{z + w}{2}) \) is a minimum point, and that
\[ 2\lambda := 2\lambda(z, w) := \text{Im} \Psi_+ \left( x_+ \left( \frac{z + w}{2} \right), z, w \right) \]
\[ = 2 \int_{x_0}^{x_+ (\frac{z + w}{2})} \left[ \text{Im} \left( \frac{z + w}{2} \right) - \text{Im} g(y) \right] dy \leq 0. \] We define \( \Theta_+(x, z, w) := \Psi_+(x, z, w) - i\lambda \), and notice that \( \text{Im} \Theta_+(x, z, w) \geq 0 \). Hence, we can write (4.8) as follows:
\[ I(z, w) = e^{-2i\lambda} \int \chi(x) \exp \left( \frac{i}{\hbar} \Theta_+(x, z, w) \right) dx. \] To study \( I(z, w) \) by the method of stationary phase, we are interested in the

**Critical points of \( \Theta_+ \).** Clearly they are the same as for \( \Psi_+(x, z, w) \). Note that for \( z = w \) one has that
\[ \Psi_+(x, z, z) = 2i\text{Im} \int_{x_0}^x (z - g(y))dy \] which has, on the support of \( \chi \), the unique critical point \( x_+(z) \) and it satisfies \( \text{Im} g'(x_+) < 0 \) (cf. (1.4)). Therefore,
\[ \text{Im} (\partial_{xx}^2 \Psi_+) (x_+(z), z, z) = -2\text{Im} g'_x (x_+(z)) > 0 \] which implies that \( x_+ \) is a non-degenerate critical point.

In the case where \( z \neq w \) the situation is more complicated. By (4.10) we see that if \( \text{Re} (z - w) = 0 \), for \( |z - w| \) small enough, the critical point is real and given by \( x_+ (\frac{z + w}{2}) \), i.e. the minimum point of \( \text{Im} \Psi_+ \).
However, if Re \((z - w)\) \(\neq 0\), we need to consider an almost \(x\)-analytic extension of \(\Psi_+\), which we shall denote by \(\tilde{\Psi}_+\). As described in [14], the “critical point” of \(\tilde{\Psi}_+\) is then given by
\[
\partial_x \tilde{\Psi}_+(x, z, w) = 0,
\]
and we will see, by the following result, that it “moves” to the complex plane.

**Lemma 4.7.** Let \(\Omega \Subset \Sigma\) be as in (H.2). Let \(\chi\) be as in (4.4) and let \(p\) be the principal symbol of \(P_h\) (cf. Section 1). Let \(x_+(z)\) be as in (1.4). Furthermore, let \(\tilde{\psi}_+\) denote an almost analytic extension of \(\psi_+\) to a small complex neighborhood of the support of \(\chi\), and define \(\tilde{\psi}_+^*(x) := \psi_+(\overline{x})\). Then, there exists a \(C > 0\) such that for \((z, w) \in \Delta_\Omega(C)\) the function
\[
\partial_x \tilde{\Psi}_+(x, z, w) = \partial_x \tilde{\psi}_+(x, z) - (\partial_x \tilde{\psi}_+)^*(x, w)
\]
has exactly one zero, \(x_+^e(z, w)\), and:
- it depends almost holomorphically on \(z\) and almost anti-holomorphically \(w\) at the diagonal \(\Delta\), i.e.
\[
\partial_w x_+^c(z, w), \partial_x x_+^c(z, w) = O(|z - w|^{\infty});
\]
- it is non-degenerate in the sense that
\[
(\partial_{xx} \tilde{\Psi}_+)(x_+^c(z, w), z, w) \neq 0;
\]
- for \(z, w \in \Omega\) with \(|z - w| < 1/C\), \(C > 1\) large enough, one has
\[
x_+^c(z, w) = x_+ \left( \frac{z + w}{2} \right) - \frac{\text{Re} (z - w)}{\{p, \overline{p}\}(\rho_+(\frac{z + w}{2}))} + O(|z - w|^2).
\]

**Remark 4.8.** The proof of Lemma 4.7 will be given the proof of Proposition 4.3.

Let \(\tilde{\Psi}_+\) denote an almost \(x\)-analytic extension of \(\Psi_+\). Using the method of stationary phase for complex-valued phase functions (cf. Theorem 2.3 in [14, p.148]) and Lemma 4.7, one gets that
\[
I(z, w) = \exp \left\{ \frac{2\Psi_1(z, w; h)}{h} \right\} + O(h^{\infty}) e^{-\frac{2\lambda}{h}}. \tag{4.13}
\]

Using that Lemma 4.5 and (4.11) imply \(\lambda(z, w) + \Phi(z; h) + \Phi(w; h) \geq 0\), we obtain (4.3) from the above and (4.7).

In (4.13), \(2\Psi_1(z, w)\) is given by the critical value of \(i\tilde{\Psi}_+\) and by the logarithm of the amplitude \(c(z, w, h)\), given by the stationary phase method, i.e.
\[
2\Psi_1(z, w; h) = i\tilde{\Psi}_+(x_+^c(z, w), z, w) + h \ln c(z, w, h)
\]
and \(c(z, w, h) \sim c_0(z, w) + hc_1(z, w) + \ldots\) which depends smoothly on \(z\) and \(w\) in the sense that all \(z, \bar{z}, z, w-\) and \(\bar{w}\)-derivatives remain bounded as \(h \to 0\). \(\tilde{\Psi}_+(x, z, w)\) is by definition \(z\)-holomorphic, \(w\)-anti-holomorphic and smooth in \(x\). By Lemma 4.7, we know that the critical point \(x_+^e(z, w)\) is almost \(z\)-holomorphic and almost \(w\)-anti-holomorphic in \(\Delta_\Omega(C)\), a small neighborhood of the diagonal \(z = w\). Hence, \(\Psi\) is almost \(z\)-holomorphic and almost \(w\)-anti-holomorphic in \(\Delta_\Omega(C)\).
Equivalently, \( \Psi \) is an almost \( z \)-holomorphic and almost \( w \)-anti-holomorphic extension from the diagonal of \( \Psi_1(z, z; h) \). Since \( \Psi_1(z, z; h) = \Phi_1(z; h) \), we obtain by Taylor expansion up to order 2 of \( \Psi \) at \( (\frac{z+w}{2}, \frac{z+w}{2}) \), that

\[
\Psi_1(z, w; h) = \sum_{|\alpha + \beta| \leq 2} \frac{1}{2^{|\alpha + \beta|}|\alpha|!|\beta|!} \partial_z^\alpha \partial_w^\beta \Phi_1 \left( \frac{z + w}{2}; h \right) (z - w)^\alpha (w - z)^\beta + O(|z - w|^3 + h^\infty),
\]

for \( |z - w| \) small enough. Similarly,

\[
\Phi_1(z; h) = \sum_{|\alpha + \beta| \leq 2} \frac{1}{2^{|\alpha + \beta|}|\alpha|!|\beta|!} \partial_z^\alpha \partial_w^\beta \Phi_1 \left( \frac{z + w}{2}; h \right) (z - w)^\alpha (z - w)^\beta + O(|z - w|^3 + h^\infty),
\]

which implies that

\[
2\text{Re} \, \Psi_1(z, w; h) = \Phi_1(z; h) + \Phi_1(w; h) - \partial_z^\alpha \partial_w^\beta \Phi_1 \left( \frac{z + w}{2}; h \right) |z - w|^2 + O(|z - w|^3 + h^\infty),
\]

concluding the second point of the proposition.

Finally, let us give a proof of the stated symmetries. The fact that \( \Psi_1(z, w; h) = \overline{\Psi_1(w, z; h)} \) follows directly from the fact that \( (e_0(z)|e_0(w)) = (e_0(w)|e_0(z)) \). One then computes that

\[
(\partial_z \Psi_1)(z, w; h) = \partial_z \Psi_1(z, w; h) = \overline{\partial_z \Psi_1}(w, z; h) = (\overline{\partial_z \Psi_1})(w, z; h)
\]

which concludes the proof of the Proposition. \( \square \)

**Proof of Lemma 4.7.** We are interested in the solutions of the following equation:

\[
0 = (\partial_z \overline{\psi})(x, z) - (\partial_x \overline{\psi})(x, w) = z - w - \overline{g}(x) + \overline{g'}(x), \tag{4.14}
\]

where \( \overline{g} \) denotes an almost analytic extension of \( g \). Since \( \operatorname{dist}(\Omega, \partial \Sigma) > 1/C \), it follows from the assumptions on \( g \) that \( \text{Im} \, g'(x) > 0 \) for all \( x \in \overline{x}(\Omega) \subset \mathbb{R} \). Since \( g \) depends smoothly on \( x \), there exists a small complex open neighborhood \( V \subset C \) of \( \overline{x}(\Omega) \) such that \( \overline{x}(\Omega) \subset (V \cap \mathbb{R}) \) and such that for all \( x \in V \)

\[
\overline{g'x}(x) - \overline{g'(\overline{x})} \neq 0, \quad \overline{g'x}(x) - \overline{g'(\overline{x})} = O(|\text{Im} \, x| \infty).
\]

Thus, it follows by the implicit function theorem, that for \( (z, w) \in \Delta_\Omega(C) \), with \( C > 0 \) large enough, there exists a unique solution \( x_+(z, w) \) to (4.14) and it depends smoothly on \( (z, w) \in \Delta_\Omega(C) \). Furthermore, we have that \( x_+(z, z) = x_+(z) \in \mathbb{R} \). Taking the \( z \)- and \( \overline{z} \)-derivative of (4.14) at the critical point \( x_+(z) \) yields that

\[
\partial_z x_+^{(z)}(z, w) = \frac{1}{(\partial_z \overline{g})(x_+(z, w)) - (\partial_x \overline{g})(x_+(z, w))} + O(|\text{Im} \, x_+^{(z)}(z, w)| \infty),
\]

\[
\partial_{\overline{z}} x_+^{(z)}(z, w) = \frac{O(|\text{Im} \, x_+^{(z)}(z, w)| \infty)}{(\partial_z \overline{g})(x_+(z, w)) - (\partial_x \overline{g})(x_+(z, w))}, \tag{4.15}
\]

\]

\[
\partial_z x_+^{(z)}(z, w) = \frac{1}{(\partial_z \overline{g})(x_+(z, w)) - (\partial_x \overline{g})(x_+(z, w))} + O(|\text{Im} \, x_+^{(z)}(z, w)| \infty),
\]

\[
\partial_{\overline{z}} x_+^{(z)}(z, w) = \frac{O(|\text{Im} \, x_+^{(z)}(z, w)| \infty)}{(\partial_z \overline{g})(x_+(z, w)) - (\partial_x \overline{g})(x_+(z, w))}, \tag{4.15}
\]
and similarly that
\[
\begin{align*}
\partial_{\overline{w}}x_+^c(z, w) &= \frac{-1 + \mathcal{O}(|\text{Im} x_+^c(z, w)|^\infty)}{\left(\partial_{\overline{w}}\overline{g}(x_+^c(z, w)) - (\partial_{\overline{w}}g)^*(x_+^c(z, w))\right)}, \\
\partial_w x_+^c(z, w) &= \mathcal{O}(|\text{Im} x_+^c(z, w)|^\infty) / \left(\partial_w g(x_+^c(z, w)) - (\partial_w g)^*(x_+^c(z, w))\right) \tag{4.16},
\end{align*}
\]

Using that \(\text{Im} x_+^c(z, z) = 0\), one calculates that for \(z = w\) we have that
\[
(\partial_{\overline{z}}x_+^c)(z, z) = \partial_z x_+^c(z, z) = -(\partial_{\overline{w}}x_+^c)(z, z),
\]
and
\[
(\partial_{\overline{w}}x_+^c)(z, z) = 0 = (\partial_w x_+^c)(z, z), \tag{4.17}
\]
where
\[
(\partial_z x_+^c) = \frac{1}{2\text{Im} g'(x_+^c(z))}.
\]

Taylor’s theorem implies that
\[
x_+^c(z + \zeta, z + \omega) = x_+^c(z) + \frac{\zeta - \overline{\omega}}{2\text{Im} g'(x_+^c(z))} + \mathcal{O}((\zeta, \omega)^2).
\]

Recall that the principal symbol of the operator \(P_h\) is given by \(p(\rho) = \xi + g(x)\) (cf. Section 1), which implies that \(\{p, \overline{p}\}(\rho_\pm(z)) = -2\text{Im} g'(x_+^c(z))\). To conclude the symmetric form of the Taylor expansion stated in the Lemma, we expand around the point \((\frac{z + w}{2}, \frac{z + w}{2})\), for \(|z - w|\) small enough, with \(\zeta = \frac{z - w}{2}\) and \(\omega = -\frac{z - w}{2}\), which is possible since \(\Omega\) is by (H.2) assumed to be convex.

Finally, by taking the imaginary part of the Taylor expansion of \(x_+^c\), we conclude by (4.15) and (4.16) that
\[
\partial_w x_+^c(z, w), \partial_{\overline{z}}x_+^c(z, w) = \mathcal{O}(|z - w|^\infty). \tag*{□}
\]

4.2. The Scalar Product \((f_0(w)|f_0(z))\). We have, as in Section 4.1,

**Proposition 4.9.** Let \(\Omega \in \Sigma\) be as in hypothesis (H.2) and let \(x_+(z)\) be as in (1.4). Then, there exists a constant \(C > 0\) such that for all \((z, w) \in \Delta_\Omega(C) := \{(z, w) \in \Omega^2; |z - w| < 1/C\}
\[
(f_0(w)|f_0(z)) = e^{-\frac{i}{\pi} \Phi_2(z; h)} e^{-\frac{i}{\pi} \Phi_2(w; h)} e^{\frac{\pi}{\text{Im} g'(x_+^c(z))}} + \mathcal{O}(h^\infty),
\]

where:

- \(\Phi_2(\cdot; h) : \Omega \to \mathbb{R}\) is a family of smooth functions depending only on \(\text{Im} z\), which satisfy
  \[
  \Phi_2(z; h) = -\text{Im} \int_{x_-(z)}^{x_0} (z - g(y))dy + \frac{h}{4} \ln \left(\frac{\pi h}{\text{Im} g'(x_+^c(z))}\right) + \mathcal{O}(h^2).
  \]

- \(\Psi_2(\cdot, \cdot; h) : \Delta_\Omega(C) \to \mathbb{C}\) is a family of smooth functions which are almost \(z\)-holomorphic and almost \(w\)-anti-holomorphic extensions from the diagonal \(\Delta := \{(z, z); z \in \Omega\} \subset \Delta_\Omega(C)\) of \(\Phi_2(z; h)\), i.e.
  \[
  \partial_{\overline{z}}\Psi_2, \partial_w \Psi_2 = \mathcal{O}(|z - w|^\infty), \quad \Psi_2(z, z; h) = \Phi_2\left(\frac{1}{2}(z - \overline{z}); h\right).
  \]
Moreover, for \( z, w \in \Delta_{\Omega}(C) \) with \( |z - w| \ll 1 \), one has that
\[
\Psi_2(z, w; h) = \sum_{|\alpha + \beta| \leq 2} \frac{1}{(2\pi h)^{n} |z - w|^3 + h^\infty},
\]
and that
\[
2 \text{Re} \Psi_2(z, w; h) - \Phi_2(z; h) - \Phi_2(w; h)
= -\partial_z \partial_{\bar{z}} \Phi_2 \left( \frac{z + w}{2}; h \right) |z - w|^2 (1 + \mathcal{O}(|z - w| + h^\infty));
\]
• the function \( \Psi_2(z, w; h) \) has the following symmetries:
\[
\Psi_2(z, w; h) = \overline{\Psi_2(w, z; h)} \quad \text{and} \quad (\partial_z \Psi_2)(z, w; h) = (\partial_{\bar{z}} \Psi_2)(w, z; h).
\]

4.3. Link with the symplectic volume. Before the proof of Proposition 4.1, let us give a short description of the connection between the functions \( \Phi_1(z; h), \Phi_2(z; h) \) in Proposition 4.3, 4.9, and the symplectic volume form on the phase space \( T^*S^1 \).

Proposition 4.10. Let \( z \in \Omega \subseteq \Sigma \) be as in (H.2) and let \( \Phi_1 \) and \( \Phi_2 \) be as in Propositions 4.3 and 4.9. Furthermore, let \( p \) be the principal symbol of \( P_h \), let \( \rho_\pm \in T^*S^1 \) be the two solutions to \( p(\rho) = z \), see Section 1. Then,
\[
\sigma_h(z) = \left[ (\partial_{\bar{z}} \Phi_1)(z; h) + (\partial_{\bar{z}} \Phi_2)(z; h) \right]
= \frac{1}{4} \left( \frac{1}{\frac{1}{\overline{p}} \{p, \overline{p}\} (\rho_- (z))} + \frac{1}{\frac{1}{p} \{p, \overline{p}\} (\rho_+ (z))} \right) + \mathcal{O}(h)
\]
is, up to an error of order \( h \), the Lebesgue density of the direct image, under the principal symbol \( p \), of the symplectic volume form \( d\xi \wedge dx \) on \( T^*S^1 \), i.e.
\[
\sigma(z) L(dz) = \frac{1}{4} p_\ast (d\xi \wedge dx) + \mathcal{O}(h) L(dz)
\]

Proof. Using that \( x_\pm(t) \), with \( t = \text{Im } z \), is the solution to \( \text{Im } g(x_\pm(t)) = t \) with \( \mp \text{Im } g_\prime(x_\pm(t)) < 0 \) (cf. Section 1), we get that
\[
x_\prime(t) = \pm \frac{1}{\text{Im } g_\prime(x_\pm(t))} < 0.
\]
Using Propositions 4.3 and 4.9, one then computes that
\[
(\partial_{\bar{z}} \Phi_1)(z; h) + (\partial_{\bar{z}} \Phi_2)(z; h) = \frac{1}{4} \left( \frac{1}{\text{Im } g_\prime(x_-(\text{Im } z))} - \frac{1}{\text{Im } g_\prime(x_+(\text{Im } z))} \right) + \mathcal{O}(h).
\]
Since \( -\frac{1}{2} \{p, \overline{p}\}(p_\pm) = \text{Im } g_\prime(x_\pm) \), we conclude by the result of [27, Proposition 6.2] that
\[
[\partial_{\bar{z}} \Phi_1](z; h) + (\partial_{\bar{z}} \Phi_2)(z; h)] L(dz) = \frac{1}{4} p_\ast (d\xi \wedge dx) + \mathcal{O}(h) L(dz).
\]

Proof of Proposition 4.1. The results follow immediately from (4.1) and the Propositions 4.3, 4.9 and 4.10.
5. Gramian matrix

The aim of this section is to study the Gramian matrix \( G \) defined in (3.12) by

\[
G := \begin{pmatrix}
A & B \\
B^* & C
\end{pmatrix} \in \mathbb{C}^{4 \times 4},
\]

where

\[
A := \begin{pmatrix}
(X(z)|X(z)) & (X(z)|X(w)) \\
(X(w)|X(z)) & (X(w)|X(w))
\end{pmatrix},
\]

\[
B := \begin{pmatrix}
(X(z)|\partial_z X(z)) & (X(z)|\partial_w X(w)) \\
(X(w)|\partial_z X(z)) & (X(w)|\partial_w X(w))
\end{pmatrix},
\]

\[
C := \begin{pmatrix}
(\partial_z X(z)|\partial_z X(z)) & (\partial_z X(z)|\partial_w X(w)) \\
(\partial_w X(w)|\partial_z X(z)) & (\partial_w X(w)|\partial_w X(w))
\end{pmatrix}.
\]

The invertibility of the matrix \( G \) will be essential to the proof of Proposition 3.7. Indeed, we prove the following

**Proposition 5.1.** Let \( \Omega \Subset \Sigma \) be as in hypothesis (H.2) and let \( z, w \in \Omega \). Then,

\[
\det G(z, w) > 0 \quad \text{for} \quad h^3 \ll |z - w| \ll 1.
\]

To prove Proposition 5.1 we will first study the matrices \( A \) and, if \( A^{-1} \) exists, the matrix \( \Gamma \) given by the Shur complement formula applied to \( G \), i.e.

\[\Gamma = C - B^* A^{-1} B.\]

5.1. The matrix \( A \). We begin by studying the determinant of \( A \). It is non-zero if and only if the vectors \( X(z) \) and \( X(w) \) are not co-linear. In particular we are interested in a lower bound of this determinant for \( z \) and \( w \) close.

**Proposition 5.2.** Let \( \Omega \Subset \Sigma \) be as in hypothesis (H.2). For \( |z - w| \leq 1/C \), with \( C > 1 \) large enough (cf. Proposition 4.1), we have

\[
\det A(z, w) = 1 - e^{-2K(z,w)/h} + O_C(h^\infty),
\]

where \( K(z, w) \) is as in (4.2). Moreover,

- for \( |z - w| \gg \sqrt{h \ln h^{-1}} \)

\[
\det A(z, w) = 1 + O(h^C), \quad C \gg 1;
\]

- for \( |z - w| \geq \frac{1}{O(1)} \sqrt{h} \)

\[
\det A \geq \frac{1}{O(1)};
\]

- let \( N > 1 \) and let \( C > 1 \) be large enough, then for \( \frac{1}{C} h^N \leq |z - w| \leq \frac{1}{C} \sqrt{h} \),

\[
\det A(z, w) = \frac{|z - w|^2}{2h} \left( \sigma \left( \frac{z + w}{2} \right) + O(h) + O(|z - w|) + O \left( \frac{|z - w|^2}{h} \right) \right)
\]

\[+ O_C(h^\infty)
\geq \frac{h^{2N-1}}{O(1)}.\]
Since the matrix $A$ is self-adjoint, we have a lower bound on the matrix norm of $A$ by its smallest eigenvalue. Using Proposition 4.1 we see that $\text{tr } A = 2 + O(h^{\infty})$ and on calculates that for a fixed $N > 1$ and for $|z - w| \geq \frac{h^N}{O(1)}$ the two eigenvalues of $A$ are given by
\[
\lambda_{1,2}(z, w; h) = 1 \pm e^{-\frac{K(z, w)}{h}} + O(h^{\infty})
\]
and by Taylor expansion we conclude the following result:

**Corollary 5.3.** Under the assumptions of Proposition 5.2, we have that for $N \geq 1$ and $|z - w| \geq h^N$
\[
\min_{\lambda \in \sigma(A)} \lambda \geq \frac{h^{N-\frac{1}{2}}}{O(1)}.
\]

**Proof of Proposition 5.2.** By Corollary 4.2, one has that
\[
\det A(z, w) = 1 - e^{-\frac{2K(z, w)}{h}} + O_C(h^{\infty}),
\]
with
\[
K(z, w) = \left( \sigma \left( \frac{z + w}{2} \right) + O(h) \right) \frac{|z - w|^2}{4} (1 + O(|z - w| + h^{\infty})).
\]
The first two estimates are then an immediate consequence of the above formula. In the case where $|z - w| \leq \frac{1}{C}\sqrt{h}$, one computes, using Taylor’s formula, that
\[
e^{-\frac{2K(z, w)}{h}} = 1 - \frac{|z - w|^2}{2h} \left( \sigma \left( \frac{z + w}{2} \right) + O(h) + O(|z - w|) + O \left( \frac{|z - w|^2}{h} \right) \right),
\]
which implies that
\[
\det A(z, w) = \frac{|z - w|^2}{2h} \left( \sigma \left( \frac{z + w}{2} \right) + O(h) + O(|z - w|) + O \left( \frac{|z - w|^2}{h} \right) \right)
+ O_C(h^{\infty})
\geq \frac{h^{2N-1}}{O(1)}. \quad \square
\]

### 5.2. The matrix $\Gamma$.
We prove the following result.

**Proposition 5.4.** Let $\Omega \subset \Sigma$ be as in (H.2), and let $D_{\Omega}(C)$ and $\Psi(z, w; h)$ for $(z, w) \in D_{\Omega}(C)$ be as in Proposition 4.1. For $(z, w) \in D_{\Omega}(C)$ let $K(z, w)$ be as in (4.2) and define
\[
a_1 := a_1(z, w; h) := (\partial_z \Psi)(z, z; h) - (\partial_z \Psi)(z, w; h),
a_2 := a_2(z, w; h) := -a_1(w, z; h).
\]
Then, for $N > 1$ and $\frac{1}{C} h^N \leq |z - w|$, with $C > 1$ large enough, we have that
\[
\Gamma = \frac{-4}{h^2 \left( 1 - e^{-\frac{2K(z, w)}{h}} \right)} \begin{pmatrix}
a_1 \bar{a}_1 e^{\frac{4}{h}K(z, w)} & a_1 \bar{a}_2 e^{\frac{4}{h}(2\text{Im } \Psi(z, w) - K(z, w))} \\
a_2 \bar{a}_1 e^{\frac{4}{h}(2\text{Im } \Psi(z, w) - K(z, w))} & a_2 \bar{a}_2 e^{\frac{4}{h}K(z, w)}
\end{pmatrix}
+ \frac{2}{h} \begin{pmatrix}
\Psi''(z, z; h) e^{\frac{2}{h}(2\text{Im } \Psi(z, w) - K(z, w))} \\
\Psi''(z, w; h) e^{\frac{2}{h}(2\text{Im } \Psi(z, w) - K(z, w))}
\end{pmatrix}
+ O(h^{\infty}).
\]
We will give a proof of this result further below. First, we state formulae for the trace, the determinant and the permanent of $\Gamma$.

**Corollary 5.5.** Under the assumptions of Proposition 5.4, we have that

\[
\text{tr} \Gamma = \frac{2}{h^2 \left( e^{2\pi K(z,w)} - 1 \right)} \left[ \left( \Psi'_{\pm}(z, z; h) + \Psi'_{\pm}(w, w; h) + O(h^\infty) \right) \left( e^{\frac{2}{\pi} K(z,w)} - 1 \right) \right.
\]

\[\left. - 2h^{-1}(|a_1|^2 + |a_2|^2) \right],
\]

\[
\det \Gamma = - \frac{16}{h^4 \left( 1 - e^{-2\pi K(z,w)} \right)} e^{-\frac{2}{\pi} K(z,w)} \left[ |a_1 a_2|^2 + \frac{h}{2} |a_1|^2 (\partial_{z,w}^2 \Psi)(w, w; h) \right.
\]

\[\left. - 2\text{Re} \left\{ (\partial_{z,w}^2 \Psi)(w, z; h) a_1 \bar{a}_2 \right\} + |a_2|^2 (\partial_{z,w}^2 \Psi)(z, z; h) \right]
\]

\[+ \frac{4}{h^2} \left( (\partial_{z,w}^2 \Psi)(z, z; h)(\partial_{z,w}^2 \Psi)(w, w; h) - (\partial_{z,w}^2 \Psi)(z, w; h)(\partial_{z,w}^2 \Psi)(w, z; h) e^{-\frac{2}{\pi} K(z,w)} \right)
\]

\[+ O(h^\infty) \]

and that

\[
\text{perm} \Gamma = \frac{16}{h^4 \left( 1 - e^{-2\pi K(z,w)} \right)} e^{-\frac{2}{\pi} K(z,w)} |a_1 a_2|^2 \left( 1 + e^{-\frac{2}{\pi} K(z,w)} \right)
\]

\[+ \frac{8}{h^3 \left( 1 - e^{-2\pi K(z,w)} \right)} e^{-\frac{2}{\pi} K(z,w)} \left[ |a_1|^2 (\partial_{z,w}^2 \Psi)(w, w; h) \right.
\]

\[\left. + 2\text{Re} \left\{ (\partial_{z,w}^2 \Psi)(w, z; h) a_1 \bar{a}_2 \right\} + |a_2|^2 (\partial_{z,w}^2 \Psi)(z, z; h) \right]
\]

\[+ \frac{4}{h^2} \left( (\partial_{z,w}^2 \Psi)(z, z; h)(\partial_{z,w}^2 \Psi)(w, w; h) + (\partial_{z,w}^2 \Psi)(z, w; h)(\partial_{z,w}^2 \Psi)(w, z; h) e^{-\frac{2}{\pi} K(z,w)} \right)
\]

\[+ O(h^\infty) \].

**Proof.** The result follows from a direct computation using Proposition 5.4; for the definition of the permanent of a matrix see Definition 3.5. \qed

We have the following bound on the trace of $\Gamma$:

**Proposition 5.6.** Under the assumptions of Proposition 5.4, we have that for $|z - w| \gg h$

\[0 < \text{tr} \Gamma \leq \frac{h^{-1}}{O(1)}.\]

Let us turn to the proofs of the above propositions. We begin by considering a very helpful congruency transformation. In view of Proposition 4.1, we prove

**Lemma 5.7.** Let $\Omega \in \Sigma$ be as in (H.2), and let $D_{\Omega}(C)$, $\Phi(z; h)$ and $\Psi(z, w; h)$ be as in Proposition 4.1, for $(z, w) \in D_{\Omega}(C)$. Define the matrices

\[
\tilde{A} := \begin{pmatrix} e^{\frac{2}{\pi} \Psi(z,z;h)} & e^{\frac{2}{\pi} \Psi(z,w;h)} \\ e^{\frac{2}{\pi} \Psi(w,z;h)} & e^{\frac{2}{\pi} \Psi(w,w;h)} \end{pmatrix} \quad \text{and} \quad \Lambda := \begin{pmatrix} e^{-\frac{1}{\pi} \Phi(z;h)} & 0 \\ 0 & e^{-\frac{2}{\pi} \Phi(w;h)} \end{pmatrix},
\]

\[
\tilde{B} := 2h^{-1} \begin{pmatrix} \Psi'_{\pm}(z, z; h)e^{\frac{2}{\pi} \Psi(z,z;h)} & \Psi'_{\pm}(z, w; h)e^{\frac{2}{\pi} \Psi(z,w;h)} \\ \Psi'_{\pm}(w, z; h)e^{\frac{2}{\pi} \Psi(w,z;h)} & \Psi'_{\pm}(w, w; h)e^{\frac{2}{\pi} \Psi(w,w;h)} \end{pmatrix}.
\]
and 
\[ \tilde{C} := h^{-2} \begin{pmatrix} c(z, z; h) e^{\frac{2}{h} \Psi(z, z; h)} & c(z, w; h) e^{\frac{2}{h} \Psi(z, w; h)} \\ c(w, z; h) e^{\frac{2}{h} \Psi(w, z; h)} & c(w, w; h) e^{\frac{2}{h} \Psi(w, w; h)} \end{pmatrix} \]

with \( c(z, w; h) := 4\Psi_z'(z, w; h)\Psi_{\overline{w}}(z, w; h) + 2h\Psi''_z(z, w; h) \). Then, for \( |z - w| \geq h^N/O(1) \), we have that 
\[ \Gamma = \Lambda(\tilde{C} - \tilde{B}^* \tilde{A}^{-1} \tilde{B}) + \mathcal{O}(h^\infty). \]

**Proof.** To abbreviate the notation, we define for \((z, w) \in D\Omega(C)\) the following function 
\[ F(z, w) := e^{-\frac{1}{h} \Phi(z; h)} e^{-\frac{1}{h} \Phi(w; h)} e^{\frac{2}{h} \Psi(z; w; h)}. \]

By Proposition 4.1, we see that \( F \) is bounded by 1 and that all its derivatives are bounded polynomially in \( h^{-1} \). Furthermore, the matrices \( A, B \) and \( C \) are given by 
\[
A(z, w) = A_0(z, w) + \mathcal{O}(h^\infty),
\]
\[
B(z, w) = B_0(z, w) + \mathcal{O}(h^\infty),
\]
\[
C(z, w) = C_0(z, w) + \mathcal{O}(h^\infty),
\]
where \((z, w) \in D\Omega(C)\) and 
\[
A_0(z, w) = \begin{pmatrix} F(z, z) & F'(z, w) \\ F(w, z) & F(w, w) \end{pmatrix},
\]
and 
\[
B_0(z, w) = \begin{pmatrix} (\partial_{\overline{w}} F)(z, z) & (\partial_{\overline{w}} F)(z, w) \\ (\partial_{\overline{w}} F)(w, z) & (\partial_{\overline{w}} F)(w, w) \end{pmatrix},
\]
and 
\[
C_0(z, w) = \begin{pmatrix} (\partial_{\overline{w}}^2 F)(z, z) & (\partial_{\overline{w}}^2 F)(z, w) \\ (\partial_{\overline{w}}^2 F)(w, z) & (\partial_{\overline{w}}^2 F)(w, w) \end{pmatrix}.
\]

One computes that 
\[
(\partial_{\overline{w}} F)(z, w) = \frac{1}{h} \left[ 2(\partial_{\overline{w}} \Psi)(z, w; h) - (\partial_{\overline{w}} \Phi)(w; h) \right] e^{-\frac{1}{h} \Phi(z; h) - \frac{1}{h} \Phi(w; h)} e^{\frac{2}{h} \Psi(z, w)} + \mathcal{O}(h^\infty),
\]
and that 
\[
(\partial_{\overline{w}}^2 F)(z, w) = \frac{1}{h^2} \left[ 2(\partial_{\overline{w}} \Psi)(z, w; h) - (\partial_{\overline{w}} \Phi)(z; h) \right] \left[ 2(\partial_{\overline{w}} \Psi)(z, w; h) - (\partial_{\overline{w}} \Phi)(w; h) \right] + 2h(\partial_{\overline{w}}^2 \Psi)(z, w; h) e^{-\frac{1}{h} \Phi(z; h) - \frac{1}{h} \Phi(z; h)} e^{\frac{2}{h} \Psi(z, 2z)} + \mathcal{O}(h^{\infty}).
\]

Using that \( \det A_0 = \det A + \mathcal{O}(h^\infty) \) and that \( \det A \geq h^{2N-1}/\mathcal{O}(1) \) for \( |z - w| \geq h^N/\mathcal{O}(1) \) (cf. Proposition 5.2), we see that 
\[ \Gamma = C_0 - B_0^* A_0^{-1} B_0 + \mathcal{O}(h^\infty). \]

Defining, 
\[ \Lambda' := \begin{pmatrix} \partial_{\overline{w}} e^{-\frac{1}{h} \Phi(z; h)} & 0 \\ 0 & \partial_{\overline{w}} e^{-\frac{1}{h} \Phi(w; h)} \end{pmatrix}. \]
we see that
\[ A_0 = \Lambda \tilde{\Lambda}, \]
\[ B_0 = \Lambda (\tilde{B}) \Lambda + \Lambda \tilde{\Lambda} (\Lambda') + O_{C^\infty}(h^\infty), \]
\[ C_0 = \Lambda (\tilde{C}) \Lambda + \Lambda (\tilde{B}^*) (\Lambda') + \Lambda' (\tilde{B}) \Lambda + \Lambda' \tilde{\Lambda} (\Lambda') + O_{C^\infty}(h^\infty). \]

A direct computation then yields that
\[ \Gamma = \Lambda (\tilde{C} - \tilde{B}^* \tilde{A}^{-1} \tilde{B}) \Lambda + O_{C^\infty}((\det A)^{-1} h^\infty). \]

**Proof of Proposition 5.4.** In view of Lemma 5.7, it remains to consider the matrix
\[ \tilde{\Gamma} := \tilde{C} - \tilde{B}^* \tilde{A}^{-1} \tilde{B}. \]

In the sequel we will suppress the \( h \)-dependency of the function \( \Psi \) to abbreviate our notation. Recall the definition of \( \tilde{A} \) from Lemma 5.7 and note that
\[ \det \tilde{A} = e^{\frac{2}{h} \Psi(z, z)} e^{\frac{2}{h} \Psi(w, w)} - e^{\frac{2}{h} \text{Re} \, \Psi(z, w)} \]
\[ = e^{\frac{2}{h} \Psi(z, z)} e^{\frac{2}{h} \Psi(w, w)} \left( 1 - e^{-\frac{2}{h} K(z, z)} \right). \quad (5.1) \]

For \( \frac{1}{C} h^N \leq |z - w| \), Proposition 4.1 implies that \( \det \tilde{A} \) is positive. Hence, the inverse of \( \tilde{A} \) exists and is given by
\[ \tilde{A}^{-1} = \frac{1}{\det \tilde{A}} \begin{pmatrix} e^{\frac{2}{h} \Psi(w, w)} & -e^{\frac{2}{h} \Psi(z, w)} \\ -e^{\frac{2}{h} \Psi(w, z)} & e^{\frac{2}{h} \Psi(z, z)} \end{pmatrix}. \]

To calculate \( \tilde{B}^* \), we use Lemma 5.7 and the symmetries of the function \( \Psi(z, w) \) given in Proposition 4.1. Indeed, one gets that
\[ \tilde{B}^* := 2h^{-1} \begin{pmatrix} \Psi'_z(z, z) e^{\frac{2}{h} \Psi(z, z)} & \Psi'_z(z, w) e^{\frac{2}{h} \Psi(z, w)} \\ \Psi'_z(w, z) e^{\frac{2}{h} \Psi(w, z)} & \Psi'_z(w, w) e^{\frac{2}{h} \Psi(w, w)} \end{pmatrix} \]
and one computes that \( M := h \tilde{B}^* \tilde{A}^{-1} h \tilde{B} \) is given by
\[ M = \frac{4}{\det \tilde{A}} \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \]
with
\[ M_{11} = \Psi'_z(z, z) \Psi'_m(z, z) e^{\frac{1}{h} \left( 4 \Psi(z, z) + 2 \Psi(w, w) \right)} + \left[ \Psi'_z(z, w) \Psi'_m(w, z) - \Psi'_z(z, z) \Psi'_m(w, z) \right] e^{\frac{1}{h} \left( 2 \Psi(z, z) + 4 \text{Re} \, \Psi(z, w) \right)}, \]
\[ M_{12} = - \Psi'_z(z, w) \Psi'_m(z, w) e^{\frac{1}{h} \left( 4 \Psi(z, w) + 2 \Psi(w, z) \right)} + \left[ \Psi'_z(z, z) \Psi'_m(z, w) + \Psi'_z(z, w) \Psi'_m(w, w) \right] e^{\frac{2}{h} \left( \Psi(z, z) + \Psi(z, w) + \Psi(w, w) \right)}, \]
and
\[ M_{22} = \Psi'_z(w, w) \Psi'_m(w, w) e^{\frac{1}{h} \left( 2 \Psi(z, z) + 4 \Psi(w, w) \right)} + \left[ \Psi'_z(w, z) \Psi'_m(z, w) - \Psi'_z(w, w) \Psi'_m(w, w) \right] e^{\frac{1}{h} \left( 2 \Psi(w, w) + 4 \text{Re} \, \Psi(z, w) \right)}. \]
Since the matrix $M$ is clearly self-adjoint, one has that $M_{21} = \overline{M}_{12}$. Comparing the coefficients of $M$ with those of $h^2(\det \tilde{A}/4)\tilde{G}$ (cf. Lemma 5.7) and using the symmetries of $\Psi$ (cf. Proposition 4.1), we see that

$$
\begin{align*}
\frac{h^2}{\det A} \left( a_1 \overline{a}_1 e^{2\Psi(z,z)} &+ a_2 \overline{a}_2 e^{2\Psi(z,w)} \\
\frac{1}{2} &\left( \frac{\partial^2 \Psi}{\partial z \partial w}(w, w) + 2 \frac{\partial^2 \Psi}{\partial z \partial w}(w, z) \right)
\right)
\end{align*}
$$

Similarly as in Proposition 4.1, we develop around the point $a_1 \overline{a}_1 e^{2\Psi(z,z)} + a_2 \overline{a}_2 e^{2\Psi(z,w)}$

with $a_i$ as in the hypothesis of Proposition 5.4. Recall from (4.2) that the function $K(z, w)$ is defined by

$$
-K(z, w) = 2 \text{Re} \Psi(z, w) - \Phi(z) - \Phi(w)
$$

where $\Phi(z) = \Psi(z, z)$. Using (5.1), we find that the first matrix in (5.2) is equal to

$$
\begin{align*}
&-\frac{4}{1 - e^{-\frac{2}{h}K(z, w)}} \left( a_1 \overline{a}_1 e^{\frac{1}{h}(2\Psi(z,z) - 2K(z, w))} \\
&\quad + a_2 \overline{a}_2 e^{\frac{1}{h}(2\Psi(z,w) - 2K(z, w))} \right)
\end{align*}
$$

It follows by Lemma 5.7 that

$$
\Gamma = \overline{\Lambda} \Gamma^* + \mathcal{O}_c(h^\infty).
$$

In the last equality we used that $\det A$ is bounded from below by a power of $h$; see Lemma 5.7. Carrying out the matrix multiplication $\overline{\Lambda} \Gamma^*$ implies the statement of the proposition. \hfill \square

**Proof of Proposition 5.1.** The Schur complement formula yields that the determinant of the Gramian matrix $G$ is given by $\det G = \det A \det \Gamma$. Hence, using Proposition 5.2 and Corollary 5.5, we see that

$$
\begin{align*}
\det G &= -\frac{16}{h^4} \left(1 + \mathcal{O}(h^\infty)\right) e^{-\frac{2}{h}K(z, w)} \left[ |a_1 a_2|^2 + \frac{h}{2} \left( |a_1|^2 (\partial^2_{zw} \Psi)(w, w; h) \right. \right.

&\quad - 2 \text{Re} \left\{ (\partial^2_{zw} \Psi)(w, z; h) a_1 \overline{a}_2 \right\} + |a_2|^2 (\partial^2_{zw} \Psi)(z, z; h) \biggr]\right] \\
&\quad + \frac{4}{h^2} \left( (\partial^2_{zw} \Psi)(z, z; h) (\partial^2_{zw} \Psi)(w, w; h) - (\partial^2_{zw} \Psi)(z, w; h) (\partial^2_{zw} \Psi)(w, z; h) e^{-\frac{2}{h}K(z, w)} \right)

&\quad \cdot \left(1 - e^{-\frac{2}{h}K(z, w)} + \mathcal{O}(h^\infty)\right) + \mathcal{O}(h^\infty). \tag{5.3}
\end{align*}
$$

Next we consider the Taylor expansion of the terms $a_1$ and $a_2$ up to first order. Similarly as in Proposition 4.1, we develop around the point $(\frac{z+w}{2}, \frac{z+w}{2})$ and get that

$$
a_1 = (\partial_z \Psi)(z, z) - (\partial_z \Psi)(z, w)
$$

$$
= (\partial^2_{zw} \Psi) \left(\frac{z + w}{2}, \frac{z + w}{2}\right) (z - w) + \mathcal{O}(|z - w|^2 + h^\infty) \tag{5.4}
$$

and

$$
a_2 = (\partial_z \Psi)(w, z) - (\partial_z \Psi)(w, w)
$$

$$
= (\partial^2_{zw} \Psi) \left(\frac{z + w}{2}, \frac{z + w}{2}\right) (z - w) + \mathcal{O}(|z - w|^2 + h^\infty). \tag{5.5}
$$
Moreover, one has that for $\zeta, \omega \in \{z, w\}$
\[
(\partial^2_{z\omega} \Psi)(\zeta, \omega) = (\partial^2_{z\omega} \Phi) \left(\frac{z + w}{2}, \frac{z + w}{2}\right) + O(\sqrt{|z - w|}) + O(h^\infty). \tag{5.6}
\]

Since we suppose that $|z - w| \gg h^{3/5}$, the above error term is equal to $O(\sqrt{|z - w|})$. Since $\partial^2_{z\omega} \Psi$ is evaluated at a point on the diagonal, it follows by (5.8) and (5.9) that
\[
(\partial^2_{z\omega} \Phi) \left(\frac{z + w}{2}, \frac{z + w}{2}\right) = \frac{1}{4} \sigma \left(\frac{z + w}{2}\right) + O(h) =: \frac{1}{4} \sigma_h(z, w). \tag{5.7}
\]

Plugging the above Taylor expansion into (5.3), one gets that
\[
det G = \frac{\sigma_h(z, w)^2}{4h^2} \left\{ 1 + \mathcal{O}(\sqrt{|z - w|}) - \frac{1}{4} \sigma_h(z, w) + \mathcal{O}(h^\infty) \right\}
\]
\[
- 4e^{-\frac{2}{\pi} K(z, w)} \left(\frac{\sigma_h(z, w)|z - w|^2}{4h}\right)^2 \left(1 + \mathcal{O}(\sqrt{|z - w|})\right) + O\left(\frac{\sigma_h(z, w)|z - w|^2}{4h}\right)
\]
\[
+ O(h^\infty)
\]
\[
= \frac{\sigma_h(z, w)^2}{4h^2} \left\{ 1 - \frac{1}{4} \sigma_h(z, w) - \frac{1}{4} \sigma_h(z, w) + \mathcal{O}(h^\infty) \right\}
\]
\[
- 4e^{-\frac{2}{\pi} K(z, w)} \left(\frac{\sigma_h(z, w)|z - w|^2}{4h}\right)^2 \left(1 + \mathcal{O}(\sqrt{|z - w|})\right) + O\left(\frac{\sigma_h(z, w)|z - w|^2}{4h}\right)
\]
\[
+ O\left(\frac{|z - w|^3}{h^2}\right).
\]

Recall from (4.2) that $K(z, w) \approx |z - w|^2$, wherefore we see that $\det G$ is positive for $|z - w| \gg \sqrt{h}$. Hence, one gets that
\[
det G = \frac{\sigma_h(z, w)^2e^{-\frac{2}{\pi} K(z, w)}}{h^2} \left\{ \sinh^2 \frac{K(z, w)}{h} + \mathcal{O}(\sqrt{|z - w|}) \left(\frac{e^{\frac{2}{\pi} K(z, w)}}{h} - 1\right) + \mathcal{O}(h^\infty) \right\}
\]
\[
- \left(\frac{\sigma_h(z, w)|z - w|^2}{4h}\right)^2 \left(1 + \mathcal{O}(\sqrt{|z - w|})\right) + O\left(\frac{\sigma_h(z, w)|z - w|^2}{h^2}\right).
\]

Using the Taylor expansion of the sinh $x$ and (4.2), one gets that
\[
\sinh^2 \frac{K(z, w)}{h} - \left(\frac{\sigma_h(z, w)|z - w|^2}{4h}\right)^2
\]
\[
\geq \left(\frac{1}{3} \frac{\sigma_h(z, w)|z - w|^2}{4h}\right)^4 \left(1 + \mathcal{O}(\sqrt{|z - w|})\right) + O\left(\frac{\sigma_h(z, w)|z - w|^2}{h^2}\right).
\]

Note that the principal term on the right hand side dominates the error terms. The same holds true for the other error terms in (5.8).

Next let us suppose that $h^{3/5} \ll |z - w| \ll \sqrt{h}$. Since
\[
\mathcal{O}(\sqrt{|z - w|}) \left(\frac{e^{\frac{2}{\pi} K(z, w)}}{h} - 1\right) = \mathcal{O}\left(\frac{|z - w|^3}{h}\right),
\]

it follows by (5.8) and (5.9) that $\det G$ is positive for $|z - w| \gg h^{3/5}$. \qed
Proof of Proposition 5.6. Using (5.4), (5.5) and (5.6), one gets that

\[
\text{tr} \Gamma = \frac{\sigma_h(z, w)}{2h} \left[ \left( e^{\frac{2}{\hbar}K(z, w)} - 1 \right) \left( 1 + \mathcal{O}(|z - w|) \right) - \frac{\sigma_h(z, w)|z - w|^2}{2h} \left( 1 + \mathcal{O}(|z - w|) \right) \right].
\]

(5.10)

Since

\[
e^{\frac{2}{\hbar}K(z, w)} - 1 \geq \frac{\sigma_h(z, w)|z - w|^2}{2h} \left( 1 + \mathcal{O}(|z - w|) \right) + \frac{\sigma_h(z, w)|z - w|^4}{8\hbar^2} \left( 1 + \mathcal{O}(|z - w|) \right),
\]

it follows that for \(|z - w| \gg h\) the trace of \(\Gamma\) is positive. Furthermore, the above inequality applied to (5.10), implies the upper bound stated in the Proposition. □

5.3. The permanent of \(\Gamma\). The permanent of the matrix \(\Gamma\) is vital to the 2-point density of eigenvalues and therefore, we shall give a more detailed description of it than the one given in Corollary 5.5.

Proposition 5.8. Let \(\sigma_h(z, w)\) be as in Theorem 2.1 and let \(K(z, w)\) be as in (4.2). Under the assumptions of Proposition 5.4, we have that for \(N > 1\) and \(\frac{1}{\hbar}h^N \leq |z - w|\),

\[
\text{perm} \Gamma(z, w; h) = \frac{1}{4\hbar^2} \left[ \sigma_h(z, z)\sigma_h(w, w) + \sigma_h(z, w)^2 \left( 1 + \mathcal{O}(|z - w|) \right) e^{-\frac{2K(z, w)}{\hbar}} + \mathcal{O}(h^\infty) \right.
\]

\[
+ \frac{\sigma_h(z, w)^2 \left( 1 + \mathcal{O}(|z - w|) \right)}{e^{\frac{K(z, w)}{\hbar}} \sinh \frac{K(z, w)}{\hbar}} \left( \left( \frac{\sigma_h(z, w)|z - w|^2}{4\hbar} \right)^2 2\coth \frac{K(z, w)}{\hbar} - \frac{\sigma_h(z, w)|z - w|^2}{h} \right) \left. \right].
\]

Proof. Applying (5.4), (5.5) and (5.6) to the formula for \(\text{perm} \Gamma\) given in Proposition 5.6 and using the notation introduced in (5.7), one gets that

\[
\text{perm} \Gamma = \frac{8 \coth \frac{K}{\hbar}}{4\hbar^3} e^{-\frac{K}{\hbar}} |4 - 2\sigma_h(z, w)^2 (z - w)^2 (1 + \mathcal{O}(|z - w|))^2
\]

\[
- \frac{\sigma_h(z, w)^3 |z - w|^2 (1 + \mathcal{O}(|z - w|))}{4\hbar^3 \sinh \frac{K}{\hbar}} \right] e^{-\frac{2K(z, w)}{\hbar}} + \mathcal{O}(h^\infty).
\]

Thus, one computes that

\[
\text{perm} \Gamma = \frac{\sigma_h(z, w)^2 \left( 1 + \mathcal{O}(|z - w|) \right) \left( \left( \frac{\sigma_h(z, w)|z - w|^2}{4\hbar} \right)^2 \left( 1 + \mathcal{O}(h^\infty) \right) \right)}{4\hbar^2 e^{\frac{K}{\hbar}} \sinh \frac{K}{\hbar}}
\]

\[
+ \frac{\sigma_h(z, z)\sigma_h(w, w) + \sigma_h(z, w)^2 \left( 1 + \mathcal{O}(|z - w|) e^{-\frac{K(z, w)}{\hbar}} \right)}{4\hbar^2 \left( 1 + \mathcal{O}(h^\infty) \right)}
\]

\[
\text{and we conclude the statement of the proposition.} \]
6. Proof of the Results

We begin by proving the results of Theorem 2.1, Proposition 2.2 and of Proposition 2.3.

Proof of Theorem 2.1. The result follows directly from Proposition 3.7 with the density $D$ given by Proposition 5.8 and by Proposition 5.2. □

Proof of Proposition 2.2. First, let us treat the case of the long range interaction: we suppose that $|z - w| \gg (h \ln h^{-1})^\frac{2}{3}$. Here, we have that for any power $N > 1$ the term

$$\left(\frac{\sigma_h(z, w)|z - w|^2}{4h}\right)^N e^{-K(z, w)}$$

remains bounded. Using that $\sinh K(z, w) \geq O(h^{-C}) > 0$ with $C \gg 1$ and using that $\sigma_h(z, z) = \sigma(z) + O(h)$, it follows that

$$D^\delta(z, w; h) = \frac{\sigma(z)\sigma(w) + O(h)}{(2h\pi)^2} \left(1 + O\left(\delta h^{-\frac{2}{3}}\right)\right).$$

Next, we consider the case where $h^\frac{2}{7} \ll |z - w| \ll h^\frac{2}{7}$. Recall from Theorem 2.1 that

$$D^\delta(z, w; h) = \frac{\Lambda(z, w)}{(2h\pi)^2 \left(1 - e^{-2K(z, w)}\right)} \left(1 + O\left(\delta h^{-\frac{2}{3}}\right)\right) + O\left(e^{-\frac{D}{h^2}}\right) \tag{6.1}$$

with $\Lambda(z, w; h)$ equal to

$$\sigma_h(z, z)\sigma_h(w, w) + \sigma_h(z, z)^2(1 + O(|z - w|))e^{-2K(z, w)} + O\left(h^\infty + \delta h^{-\frac{31}{30}}\right)$$

$$+ \frac{\sigma_h(z, w)^2(1 + O(|z - w|))}{e^{K(z, w)} \sinh K(z, w)} \left(\left(\frac{\sigma_h(z, w)|z - w|^2}{4h}\right)^2 2\coth K(z, w) - \frac{\sigma_h(z, w)|z - w|^2}{h}\right).$$

Similarly to (5.6), we have that $\sigma_h(z, z) = \sigma_h(z, w)(1 + O(|z - w|))$. We start by considering the first term in (6.1):

$$\frac{\Lambda(z, w)}{(2h\pi)^2 \left(1 - e^{-2K(z, w)}\right)}. \tag{6.2}$$

Set $\sigma_h = \sigma_h(z, w)$. Using the Taylor expansions of the functions $\sinh x$, $\coth x$ and $e^{-x}$, one computes, that (6.2) is equal to

$$\frac{1}{h\pi^2 \sigma_h |z - w|^2 \left(1 + O\left(\frac{|z - w|^2}{h}\right)\right)} \left[\sigma_h^2 \left(1 + O\left(|z - w|\right)\right) - \frac{\sigma_h^2 |z - w|^2}{4h} \left(1 + O\left(|z - w|\right)\right)\right]$$

$$+ \frac{\sigma_h^4 |z - w|^4}{4^2 h^2} \left(1 + O\left(\frac{|z - w|^2}{h}\right)\right) + \left\{\frac{\sigma_h^4 |z - w|^4}{3 \cdot 4^4 h^2} \left(1 + O\left(\frac{|z - w|^4}{h^2}\right)\right) - 1\right\}.$$
which simplifies to
\[ \Lambda(z, w; h) = \frac{\sigma^2_h|z - w|^2}{(4\pi h)^2} \left( 1 + O\left( \frac{|z - w|^2}{h} \right) \right). \]

Hence,
\[ D_h^2(z, w; h) = \frac{\sigma^2_h|z - w|^2}{(4\pi h)^2} \left( 1 + O\left( \frac{|z - w|^2}{h} + \delta h^{-\frac{3}{2}} \right) \right) \]
which concludes the proof. \(\square\)

Proof of Proposition 2.3. Using that \(\sigma_h(z, w_0) = \sigma_h(z, z)(1 + O(|z - w_0|))\) (cf. (5.6) and (5.7)), the result of Proposition 2.3 follows from Proposition 2.2. \(\square\)

It remains to prove Proposition 3.7. However, first, we state a global version of the implicit function theorem.

Lemma 6.1. Let \(0 < R_0 < R\), let \(n, m \in \mathbb{N}\), with \(n > m\), and let \(B(0, R) \subset \mathbb{C}^m = \mathbb{C}^{n-m} \times \mathbb{C}^m\) denote the complex open ball of radius \(R > 0\) centered at 0. For \(z \in B_{\mathbb{C}^{n-m}}(0, R_0)\), define \(R(z) := (R^2 - \|z\|_{\mathbb{C}^{n-m}}^2)^{1/2}\). We consider a holomorphic function
\[ F : B(0, R) \rightarrow \mathbb{C}^m \]
such that

- for all \((z, w) \in B(0, R)\) the Jacobian of \(F\) with respect to \(w\) is given by
  \[ \frac{\partial F(z, w)}{\partial w} = A + G(z, w), \]
  where \(G : B(0, R) \rightarrow \mathbb{C}^{m \times m}\) is a matrix-valued holomorphic function and
- \(A \in \text{GL}_m(\mathbb{C})\) such that
  \[ \|A^{-1}\| \cdot \|G(z, w)\| \leq \theta < 1 \]
for all \((z, w) \in B(0, R)\).

Then, for all \(z \in B_{\mathbb{C}^{n-m}}(0, R_0)\) and for all \(y \in B_{\mathbb{C}^m}(F(z, 0), \frac{1-\theta}{\|A^{-1}\|}r)\), with \(0 < r < R(z)\), the equation
\[ F(z, w) = y \] 
has exactly one solution \(w(z, y) \in B_{\mathbb{C}^m}(0, R(z))\), it satisfies \(w(z, y) \in B_{\mathbb{C}^m}(0, r)\) and it depends holomorphically on \(z\) and on \(y\).

Remark 6.2. Observe that the choice of \(R_0 < R\) yields a uniform lower bound on \(R(z)\) and so we can choose the radius of the ball \(B_{\mathbb{C}^m}(F(z, 0), \frac{1-\theta}{\|A^{-1}\|}r)\) uniformly in \(z\). This will become important in the proof of Proposition 3.7.

Proof. Let \(z \in B_{\mathbb{C}^{n-m}}(0, R_0)\) and set
\[ B_{\mathbb{C}^m}(0, R(z)) \ni w \mapsto \tilde{F}(w) := F(z, w). \]

We begin by observing that \(d\tilde{F}(w)\) is invertible for all \(w \in B_{\mathbb{C}^m}(0, R(z))\) and the norm of the inverse is bounded (uniformly in \(z\)). Indeed, for one has that
\[ \left\| \left( d\tilde{F}(w) \right)^{-1} \right\| \leq \|A^{-1}\| \cdot \|(1 + A^{-1}G(z, w))^{-1}\| \leq \frac{\|A^{-1}\|}{1 - \theta}. \]

Claim #1: \(\tilde{F}\) is injective.
Let \( w_0, w_1 \in B_{C^m}(0, R(z)) \) and define \( y_t := \tilde{F}(w_t) \). Hence, with \( w_t := (1 - t)w_0 + tw_1 \), we have that

\[
\frac{d}{dt} \tilde{F}(w_t) = d\tilde{F}(w_t) \cdot (w_1 - w_0) = (A + G(z, w_t)) \cdot (w_1 - w_0).
\]

Thus,

\[
y_1 - y_0 = (A + H(z, w_1, w_0)) \cdot (w_1 - w_0), \quad H(z, w_1, w_0) = \int_0^1 G(z, w_t)dt,
\]

where \( \|H(z, w_1, w_0)\| \leq \sup_{B(0, R)} \|G(z, w)\| \). Therefore, \( \|A^{-1}\| \cdot \|H(z, w_1, w_0)\| \leq \theta < 1 \), and we see that \( (A + H(z, w_1, w_0)) \) is invertible and the norm of its inverse is \( \leq \frac{\|A^{-1}\|}{1 - \theta} \) (uniformly in \( z \)). Hence,

\[
\|w_1 - w_0\| \leq \frac{\|A^{-1}\|}{1 - \theta} \|y_1 - y_0\|,
\]

and we conclude that \( \tilde{F} \) is injective. In particular, we have proven the uniqueness of the solution to the equation (6.3).

**Claim #2:** Let \( 0 < r < R(z) \). Then, for all \( y \in B_{C^m}(\tilde{F}(0), \frac{1 - \theta}{\|A^{-1}\|}r) \) there exists a \( w \in B_{C^m}(0, r) \) such that \( \tilde{F}(w) = y \).

For \( y = \tilde{F}(0) \), we take \( w = 0 \). Using the fact that \( d\tilde{F} \) is invertible everywhere, the implicit function theorem implies that for all \( y \in B(\tilde{F}(0), \rho) \) there exists a solution \( w \in B_{C^m}(0, r) \), if \( \rho > 0 \) is small enough (cf. (6.4)). Let \( y \in B_{C^m}(\tilde{F}(0), \frac{1 - \theta}{\|A^{-1}\|}r) \), and define \( y_t := (1 - t)\tilde{F}(0) + ty \). Let \( t_0 \in [0, 1] \) be the supremum of \( \tilde{t} \in [0, 1] \) such that there exists a solution to \( \tilde{F}(w_t) = y_t \) for all \( 0 \leq t \leq \tilde{t} \).

We have already proven that \( t_0 > 0 \). As \( t \nearrow t_0 \) we have that \( w_t \in B_{C^m}(0, r) \). Since \( B_{C^m}(0, r) \) is relatively compact in \( B_{C^m}(0, R(z)) \), there exists a sequence \( t_j \not\nearrow t_0 \) such that \( w_{t_j} \to \tilde{w} \) with \( \tilde{w} \in B_{C^m}(0, r) \). Thus,

\[
\tilde{F}(\tilde{w}) = y_{t_0},
\]

and we see by (6.4) that \( \tilde{w} \in B_{C^m}(0, r) \).

If \( t_0 < 1 \), we get by the implicit function theorem, that for all \( y \in B(y_{t_0}, \delta) \), with \( \delta > 0 \) small enough, there exists a solution \( w \in B_{C^m}(0, r) \). Therefore, we can solve \( \tilde{F}(w_t) = y_t \) for all \( 0 < t < t_0 + \delta \), which is a contradiction. Hence, \( t_0 = 1 \), which concludes the proof of the existence of a solution.

Finally, note that for all \((z, w) \in B(0, R)\) the Jacobian \( \partial F(z, w) / \partial w \) is invertible and the norm of its inverse is uniformly bounded, indeed

\[
\left\| \left( \frac{\partial F(z, w)}{\partial w} \right)^{-1} \right\| \leq \|A^{-1}\| \cdot \|1 + A^{-1}G(z, w)\|^{-1} \leq \frac{\|A^{-1}\|}{1 - \theta}.
\]

In particular, we have that the determinant of the Jacobian is never equal to 0, and we conclude by the holomorphic implicit function theorem that the solution \( w(z, y) \) to the equation (6.4) depends holomorphically on \( z \) and \( y \). \( \square \)
Proof of Proposition 3.7. In view of (3.10), it remains to study the integral

\[ I(z_1, z_2, h) = \lim_{\varepsilon \to 0^+} \pi^{-N} \int_{B(0, R)} H^\varepsilon_\delta(z_1, z_2, \alpha; h) e^{-\alpha^* \pi L(d\alpha)}. \]  

(6.5)

with

\[ H^\varepsilon_\delta(z_1, z_2, \alpha; h) := \prod_{k=1}^2 \varepsilon^{-2} \chi \left( \frac{E^\varepsilon_\delta(z_k, \alpha)}{\varepsilon} \right) |\partial_{z_k} E^\varepsilon_\delta(z_k, \alpha)|^2 \]

for \( 1/C \geq |z_1 - z_2| \gg h^{3/5} \). We begin by performing a change of variables in the \( \alpha \)-space.

**Change of variables:** For \( X(z) \in \mathbb{C}^N \) as in Definition 3.2, define the matrix

\[ U := \begin{pmatrix} 1 & 0 \\ B^* A^{-1} & 1 \end{pmatrix}. \]

\( U \) is invertible and thus satisfies that \( (U^{-1})^* = (U^*)^{-1} \). Define the matrix

\[ \tilde{G} := \begin{pmatrix} A & 0 \\ 0 & \Gamma \end{pmatrix} \in \mathbb{C}^{4 \times 4}, \]

and notice that

\[ U \begin{pmatrix} A & 0 \\ 0 & \Gamma \end{pmatrix} U^* = \begin{pmatrix} 1 & 0 \\ B^* A^{-1} & 1 \end{pmatrix} \tilde{G} \begin{pmatrix} 1 & A^{-1} B \\ 0 & 1 \end{pmatrix} = G. \]

We see that \( \tilde{G} = U^{-1} G (U^*)^{-1} \). Next, we define the matrix

\[ \tilde{V}^* := (U^{-1} V)^* \tilde{G}^{-\frac{1}{2}} \in \mathbb{C}^{N \times 4}. \]  

(6.6)

\( \tilde{V}^* \) is an isometry since \( \tilde{V} \tilde{V}^* = 1_{\mathbb{C}^4} \). Thus, its columns form an orthonormal family in \( \mathbb{C}^N \). It follows from (6.6) that the kernel of \( V \) and of \( \tilde{V} \) are equal, i.e. \( \mathcal{N}(V) = \mathcal{N}(\tilde{V}) \). The same holds true for the range of \( \tilde{V} \) and of \( V \), i.e. \( \mathcal{R}(V) = \mathcal{R}(\tilde{V}) \).

Next, we choose an orthonormal basis, \( e_1, \ldots, e_N \in \mathbb{C}^N \), of the space of random variables \( \alpha \) such that \( \tilde{V}_i^* \), \( \ldots, \tilde{V}_4^* \), the column vectors of the matrix \( \tilde{V}^* \), are among them. In particular, let \( e_i = \tilde{V}_i^* \) for \( i = 1, \ldots, 4 \), and let \( e_5, \ldots, e_N \) be in the orthogonal complement of the space spanned by \( e_1, \ldots, e_4 \). Hence, we write for \( \alpha \in \mathbb{C}^N \)

\[ \alpha = \sum_{i=1}^N \tilde{\alpha}_i e_i, \]

where \( \tilde{\alpha} = (\tilde{\alpha}_1, \ldots, \tilde{\alpha}_N) \in \mathbb{C}^N \). Moreover, note that

\[ \alpha^* \cdot \alpha = \tilde{\alpha}^* \cdot \tilde{\alpha}. \]  

(6.7)
Remark 6.3. The fact that we can only guarantee the invertibility of $G$ for $h^{3/2} \ll |z - w| \ll 1$ makes (2.2) necessary. This might be avoided by choosing another set of basis vectors.

Next, we apply this change of variables to the vector $F$ given in (3.11) and we get

$$F(z, \alpha(\bar{\alpha}); \delta, h) = \begin{pmatrix} E_{-+}(z_1) \\ E_{-+}(z_2) \\ (\partial_z E_{-+})(z_1) \\ (\partial_z E_{-+})(z_2) \end{pmatrix} - \delta \begin{pmatrix} X(z_1) \\ X(z_2) \\ (\mathcal{L}(\partial_z X))(z_1) \\ (\mathcal{L}(\partial_z X))(z_2) \end{pmatrix} \cdot \alpha(\bar{\alpha}) + \begin{pmatrix} T(z_1, \alpha(\bar{\alpha})) \\ T(z_2, \alpha(\bar{\alpha})) \\ (\partial_z T)(z_1, \alpha(\bar{\alpha})) \\ (\partial_z T)(z_2, \alpha(\bar{\alpha})) \end{pmatrix}.$$ 

Furthermore, one computes that

$$V \tilde{V} = U \tilde{G}^2 = \begin{pmatrix} A^2 & 0 \\ 0 & \Gamma^2 \end{pmatrix},$$

and we get that

$$F(z, \alpha(\bar{\alpha}); \delta, h) = \begin{pmatrix} E_{-+}(z_1) \\ E_{-+}(z_2) \\ (\partial_z E_{-+})(z_1) \\ (\partial_z E_{-+})(z_2) \end{pmatrix} - \delta \tilde{U} \tilde{G}^2 \cdot \begin{pmatrix} \bar{\alpha}_1 \\ \bar{\alpha}_2 \\ \bar{\alpha}_3 \\ \bar{\alpha}_4 \end{pmatrix} + \begin{pmatrix} T(z_1, \alpha(\bar{\alpha})) \\ T(z_2, \alpha(\bar{\alpha})) \\ (\partial_z T)(z_1, \alpha(\bar{\alpha})) \\ (\partial_z T)(z_2, \alpha(\bar{\alpha})) \end{pmatrix}.$$ 

Next, to simplify our notation, we call the $\bar{\alpha}$ variables again $\alpha$. Also, to abbreviate our notation, define

$$\mu(z, w; h) := \begin{pmatrix} E_{-+}(z_1) \\ E_{-+}(z_2) \end{pmatrix} \quad \text{and} \quad \tau(z, \alpha; h, \delta) := \begin{pmatrix} T(z_1, \alpha) \\ T(z_2, \alpha) \end{pmatrix}.$$ 

and

$$\partial_z \mu(z, w; h) := \begin{pmatrix} (\partial_z E_{-+})(z_1) \\ (\partial_z E_{-+})(z_2) \end{pmatrix} \quad \text{and} \quad \partial_z \tau(z, \alpha; h, \delta) := \begin{pmatrix} (\partial_z T)(z_1, \alpha) \\ (\partial_z T)(z_2, \alpha) \end{pmatrix}.$$ 

Remark 6.4. Recall that $T$ (cf. (3.9)) depends on $h$ and on $\delta$, though not explicit in the above notation.

When we write $\partial_z \mu$ and $\partial_z \tau$ the derivatives are to be understood component wise, each of which only depends either on $z_1$ or $z_2$.

Hence,

$$F^\delta(z, \alpha) := F(z, \alpha; \delta, h) = \begin{pmatrix} \mu(z, \alpha, h, \delta) \\ \partial_z \mu(z, \alpha, h, \delta) \end{pmatrix} - \delta \tilde{U} \tilde{G}^2 \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{pmatrix} + \begin{pmatrix} \tau(z, \alpha, h, \delta) \\ \partial_z \tau(z, \alpha, h, \delta) \end{pmatrix}.$$ 

(6.9)

As noted in Remark 3.4, $\mu$ and $\tau$ are smooth in $z$, and $\tau$ is holomorphic in $\alpha$. Moreover, $\tau$ satisfies the estimates

$$\tau_i = \mathcal{O}(h^{-5/2} \delta^2), \quad i = 1, 2 \quad \text{and} \quad \partial_z \tau_i = \mathcal{O}(h^{-7/2} \delta^2), \quad i = 1, 2;$$

(6.10)
and \( \mu \) satisfies the estimates

\[
\mu_i = \mathcal{O}\left(h^{1/2}e^{-\frac{S}{\pi}}\right), \quad \partial_{z_i} \mu_i = \mathcal{O}\left(h^{-1/2}e^{-\frac{S}{\pi}}\right), \quad i = 1, 2 \tag{6.11}
\]

with \( S \) as in Definition 1.2. Finally, we perform the above described change of variables in the integral (6.5), and, using the fact that we chose an orthonormal basis of the \( \alpha \)-space, we get that

\[
H^\delta_\varepsilon(z_1, z_2, \alpha; h) = \prod_{k=1}^{2} \varepsilon^{-2} \chi \left( \frac{F^\delta_k(z_k, \alpha)}{\varepsilon} \right) |F^\delta_{k+2}(z_k, \alpha)|^2.
\]

Next, let \( \alpha = (\alpha_1, \alpha_2, \alpha') = (\widetilde{\alpha}, \alpha') \) and split the ball \( B(0, R) \), \( R = Ck^{-1} \), into two pieces: pick \( C_0 > 0 \) such that \( 0 < C_1 < C_0 < C < 2C_0 \), and define \( R_0 = C_0 h^{-1} \). Then, we perform the splitting: \( I(z, h) = I_1(z, h) + I_2(z, h) \) with

\[
I_1(z, h) := \lim_{\varepsilon \to 0^+} \pi^{-N} \int_{B(0,R)} H^\delta_\varepsilon(z_1, z_2, \alpha; h)e^{-\alpha^* \alpha L(d\alpha)}.
\]

and

\[
I_2(z, h) := \lim_{\varepsilon \to 0^+} \pi^{-N} \int_{B(0,R)} H^\delta_\varepsilon(z_1, z_2, \alpha; h)e^{-\alpha^* \alpha L(d\alpha)}. \tag{6.12}
\]

The integral \( I_1 \) First, we perform a new change of variables in the \( \alpha \)-space. Let \( \beta_1, \ldots, \beta_N \in \mathbb{C} \) such that

\[
\beta_1 = F^\delta_1(z_1, \alpha), \quad \beta_2 = F^\delta_2(z_2, \alpha) \quad \text{and} \quad \beta_i = \alpha_i, \quad \text{for } i = 3, \ldots, N.
\]

We use the following notation: \( \beta = (\beta_1, \beta_2, \beta') = (\tilde{\beta}, \alpha') \). It is sufficient to check that we can express \( \tilde{\alpha} = (\alpha_1, \alpha_2) \) as a function of \( (\tilde{\beta}, \alpha') \). Therefore, we apply Lemma 6.1 to the function

\[
F^\delta(z, \alpha) = \left( \begin{array}{c} F^\delta(z_1, \alpha) \\ F^\delta(z_2, \alpha) \end{array} \right),
\]

where \( \alpha \) plays the role of \((z, w)\) in the Lemma. In particular, \(\tilde{\alpha}\) plays the role of \( w \). Let us check that the assumptions of Lemma 6.1 are satisfied: \( F^\delta(z, \alpha) \) is by definition holomorphic in \( \alpha \). Using (6.9) and (6.8), we see that its Jacobian, with respect to the variables \(\tilde{\alpha}\), is given by

\[
\frac{\partial F(z, \alpha)}{\partial \tilde{\alpha}} = \frac{\partial \tau}{\partial \tilde{\alpha}} - \delta A^\frac{1}{2} \tag{6.13}
\]

The Cauchy inequalities and (6.10) imply that

\[
\frac{\partial \tau_i}{\partial \tilde{\alpha}_j} = \mathcal{O}\left(\delta^2 h^{-\frac{3}{2}}\right), \quad i, j = 1, 2.
\]

This estimate is uniform in \( \alpha \in B(0, R) \) and \((z_1, z_2) \in \text{supp } \varphi \). Expansion of the determinant yields that

\[
\det \left( \frac{\partial \tau}{\partial \tilde{\alpha}} - \delta A^\frac{1}{2} \right) = \delta^2 \left( \sqrt{\det A} + \mathcal{O}\left(\delta h^{-\frac{3}{2}}\right) \right). \tag{6.14}
\]
Using that $A$ is self-adjoint, we see by Corollary 5.3 that for $(z_1, z_2) \in \text{supp} \varphi$
\begin{equation}
\|A^{-\frac{1}{2}}\| \leq \frac{1}{\min_{\lambda \in \sigma(A)} \sqrt{\lambda}} \leq O \left( h^{-\frac{3}{2}} \right). \tag{6.15} \end{equation}

By the hypothesis (H.3), we have that $\delta \ll h^{7/2}$. Hence, one gets that for all $\alpha \in B(0, R)$
$$
\delta^{-1}||A^{-\frac{1}{2}}|| \cdot ||\partial_\alpha \tau|| \leq O \left( \delta h^{-\frac{3}{2}} \right) \ll 1.
$$

Hence $\mathcal{F}(z, \alpha)$ satisfies the assumptions of Lemma 6.1. In the integral $I_1$ we restricted $\alpha'$ to the open ball $||\alpha'||_{CN-2} < R_0$. It follows by Lemma 6.1 that for all
$$
\tilde{\beta} \in B_{C^2} \left( \mathcal{F}(z; 0, \alpha'), r \right) \tag{6.16}
$$
with
$$
r := \left( \delta ||A^{-\frac{1}{2}}||^{-1}(1 - \max_{\alpha \in B(0, R)} \delta^{-1}||A^{-\frac{1}{2}}|| \cdot ||\partial_\alpha \tau||) \right) \sqrt{R^2 - R_0^2} \geq \frac{\delta h^{\frac{1}{2}} - 1}{O(1)} > 0,
$$
the equation $\tilde{\beta} = \mathcal{F}(z, \tilde{\alpha}, \alpha')$ has exactly one solution $\tilde{\alpha}(\tilde{\beta}, \alpha'; z)$ in the ball $B \left( 0, \sqrt{R^2 - ||\alpha'||_{CN-2}^2} \right)$. Moreover, the solution satisfies $\tilde{\alpha}(\tilde{\beta}, \alpha'; z) \in B(0, \sqrt{R^2 - R_0^2})$, and it depends holomorphically on $\tilde{\beta}$ and $\alpha'$ and is smooth in $z$. Using (6.9), we see that the solution is implicitly given by
$$
\tilde{\alpha}(\tilde{\beta}, \alpha') = -\delta^{-1}A^{\frac{1}{2}} \left( \tilde{\beta} - \nu(z, \tilde{\alpha}(\tilde{\beta}, \alpha'), \alpha', h, \delta) \right). \tag{6.17}
$$
with
$$
\nu := (\nu_1, \nu_2)^t := \mu(z, h) + \tau(z, \tilde{\alpha}(\tilde{\beta}, \alpha'), \alpha', h, \delta)
$$
where $\tau$ satisfies the estimate (6.10). Since the support of $\chi$ is compact (cf. Section 3.2), we can restrict our attention to $\tilde{\beta}$ and $\mathcal{F}(z; 0, \alpha')$ in a small poly-disc of radius $K \varepsilon > 0$ centered at 0, with $K > 0$ large enough such that $\text{supp} \chi \subset D(0, K)$. By choosing $\varepsilon < \delta h/C$, $C > 0$ large enough, we see that $\tilde{\beta}, \mathcal{F}(z; 0, \alpha') \in D(0, K \varepsilon) \times D(0, K \varepsilon)$ implies (6.16).

From (6.8), (6.9) and (6.17), it follows that
$$
\begin{pmatrix}
F_3^\delta(z, \tilde{\alpha}(\tilde{\beta}, \alpha'), \alpha') \\
F_4^\delta(z, \tilde{\alpha}(\tilde{\beta}, \alpha'), \alpha')
\end{pmatrix} = \partial_\nu \nu + B^*A^{-1}(\tilde{\beta} - \nu) - \delta \Gamma \frac{1}{2} \begin{pmatrix} \alpha_3 \\ \alpha_4 \end{pmatrix}, \tag{6.18}
$$
with
$$
\partial_\nu \nu = (\partial_\nu \nu_1, \partial_\nu \nu_2)^t = (\partial_\nu \mu)(z, h) + (\partial_\nu \tau)(z, \tilde{\alpha}(\tilde{\beta}, \alpha'), \alpha', h, \delta)
$$
where $\partial_\nu \tau$ satisfies the estimate given in (6.10). Furthermore, (6.13) and (6.14) imply that
$$
L(d\tilde{\alpha}) = \delta^{-4} \left( \sqrt{\text{det} A} + O \left( \delta h^{-\frac{3}{2}} \right) \right)^{-2} L(d\tilde{\beta}) =: J(\tilde{\beta}, \alpha') L(d\tilde{\beta}) \tag{6.19}
$$
By performing this change of variables in the integral $I_1$ and by picking $\varepsilon > 0$ small enough as above, we get that $I_1$ is equal to

$$
\lim_{\varepsilon \to 0^+} \pi^{-N} \int \int_{(\tilde{\alpha}(\tilde{\beta}, \alpha'), \alpha') \in B(0, R)} H^\delta(z_1, z_2, \tilde{\alpha}(\tilde{\beta}, \alpha'), \alpha'; h) e^{-\Phi(\tilde{\beta}, \alpha')} J(\tilde{\beta}, \alpha') L(d\alpha') L(d\tilde{\beta}),
$$

where

$$
\Phi(\tilde{\beta}, \alpha') := \tilde{\alpha}(\tilde{\beta}, \alpha')^* \cdot \tilde{\alpha}(\tilde{\beta}, \alpha') + (\alpha')^* \cdot \alpha'.
$$

The integrand of $I_1$ depends continuously on $\tilde{\beta}$. Hence, by performing the limit $\varepsilon \to 0^+$, we get

$$
I_1(z, h) = \pi^{-N} \int_{(\tilde{\alpha}(0, \alpha'), \alpha') \in B(0, R)} H^\delta(z_1, z_2, \tilde{\alpha}(0, \alpha'), \alpha'; h) e^{-\Phi(0, \alpha')} J(0, \alpha') L(d\alpha')
$$

with

$$
H^\delta(z_1, z_2, \tilde{\alpha}(0, \alpha'), \alpha'; h) = |F_3(z, 0, \alpha') F_4(z, 0, \alpha')|^2.
$$

Using (6.17), one computes that

$$
\Phi(0, \alpha') = \frac{1}{\delta^2} \nu^* A^{-1} \nu + (\alpha')^* \cdot \alpha'
$$

and, using (6.18), we get

$$
\left( \frac{F_3^\delta(z, \tilde{\alpha}(0, \alpha'), \alpha')}{F_4^\delta(z, \tilde{\alpha}(0, \alpha'), \alpha')} \right) = \partial_z \nu - B^* A^{-1} \nu - \delta \Gamma \frac{2}{2} (\frac{\alpha_3}{\alpha_4}),
$$

where $\nu = \nu(z, \tilde{\alpha}(0, \alpha'), \alpha', h, \delta)$. Using (6.10), (6.11) and (6.15) one computes that

$$
\|\tilde{\alpha}(0, \alpha')\|^2 = \frac{1}{\delta^2} \nu^* A^{-1} \nu \leq \frac{C}{h^{\frac{4}{10}}} \left[ \mathcal{O}\left(\delta^{-2} e^{-\frac{4}{\pi} \delta} \right) + \mathcal{O}(\delta^2 h^{-5}) \right],
$$

where the constant $C > 0$ comes from the upper bound of $\|A^{-1/2}\|^{-1}$ given in (6.15). By the Hypothesis (H.3), we conclude that

$$
\|\tilde{\alpha}(0, \alpha')\|^2 \ll \frac{1}{h^{\frac{4}{10}}},
$$

which implies that $(\tilde{\alpha}(0, 0, \alpha'), \alpha') \in B(0, R)$ for all $\alpha'$ with $\|\alpha'\|_{C^{N-2}} \leq R_0$. Hence,

$$
I_1(z, h) = \pi^{-N} \int_{\|\alpha'\|_{C^{N-2}} \leq R_0} |F_3(z, 0, \alpha') F_4(z, 0, \alpha')|^2 e^{-\Phi(0, \alpha')} J(0, \alpha') L(d\alpha').
$$

Next, we want to apply a multi-dimensional version of the mean value theorem for integrals to (6.23). Indeed, let $U \subset \mathbb{R}^n$ be open, relatively compact and path-connected, it then holds true that for a continuous function $f : \overline{U} \to \mathbb{R}$ and a positive integrable function $g : \overline{U} \to \mathbb{R}$, there exists a $y \in \overline{U}$ such that

$$
f(y) \int_U g(x) dx = \int_U f(x) g(x) dx.
$$
Hence, the mean value theorem applied to (6.23) yields that

\[ I_1(z, h) = \pi^{-N} J e^{-\frac{c}{h} L} \int_{|\alpha'|_{C^{N-2}} \leq R_0} \left| F_3(z, 0, \alpha') F_4(z, 0, \alpha') \right|^2 e^{-\alpha' \bar{\alpha}} L(d\alpha'). \]

Here, \( J \) denotes the evaluation of the Jacobian \( J(0, \alpha') \) (cf. (6.19)) at the intermediate point for \( \alpha' \) given by mean value theorem. Note that \( J \) depends smoothly on \( z_1 \) and \( z_2 \) because \( \tau \) and \( A \) do.

Similarly, \( \tilde{\nu} \) above denotes the evaluation of the function \( \nu(z, \tilde{\alpha}(0, \alpha'), \alpha', h, \delta) \) at the intermediate point for \( \alpha' \) given by mean value theorem. It depends smoothly on \( z_1 \) and \( z_2 \) because \( \mu \) and \( \tau \) do. Moreover, using (6.10), we see that it satisfies

\[ \tilde{\nu} = \left( E_{-+}(z_1) \right) + \mathcal{O} \left( \delta^2 h^{-\frac{5}{2}} \right). \]

In remains to study the integral

\[ \tilde{I}_1(z, h) := \pi^{-N} \int_{|\alpha'|_{C^{N-2}} \leq R_0} \left| F_3(z, 0, \alpha') F_4(z, 0, \alpha') \right|^2 e^{-\alpha' \bar{\alpha}} L(d\alpha'). \]

Define the linear forms

\[ l_1(\alpha') = [\Gamma^{\frac{1}{2}}]_{11} \alpha_3 + [\Gamma^{\frac{1}{2}}]_{12} \alpha_4, \quad l_2(\alpha') = [\Gamma^{\frac{1}{2}}]_{21} \alpha_3 + [\Gamma^{\frac{1}{2}}]_{22} \alpha_4. \]

Using (6.21), we get that

\[ F_3(z, 0, \alpha') = (\partial_2 \nu - B^* A^{-1} \nu)_1 - \delta l_1(\alpha') = \mathcal{O} \left( h^{-\frac{3}{2}} e^{-\frac{\alpha}{h}} + \delta^2 h^{-\frac{6}{h}} \right) - \delta l_1(\alpha'). \]
\[ F_4(z, 0, \alpha') = (\partial_2 \nu - B^* A^{-1} \nu)_2 - \delta l_2(\alpha') = \mathcal{O} \left( h^{-\frac{3}{2}} e^{-\frac{\alpha}{h}} + \delta^2 h^{-\frac{6}{h}} \right) - \delta l_2(\alpha'). \]

In the last equation we used (6.10), (6.11), and the fact that the Hilbert-Schmidt norm of \( B^* \) is \( \leq \frac{1}{10^{10}} \) which follows from the fact that elements of the matrix \( B^* \) are bounded by a term of order \( h^{-1} \).

By Proposition 5.6, one gets that the Hilbert-Schmidt norm of \( \Gamma^{\frac{1}{2}} \) is bounded, indeed one has that

\[ \|\Gamma^{\frac{1}{2}}\|_{HS} = \sqrt{\text{tr} \Gamma} \leq \mathcal{O}(h^{-\frac{1}{2}}). \]

Since \( \|\alpha'|_{C^{N-2}} \leq R_0 \), one gets

\[ |F_3(z, 0, \alpha') F_4(z, 0, \alpha')|^2 = \delta^4 \left( |l_1(\alpha') l_2(\alpha')|^2 + \mathcal{O}(\delta^{-1} e^{-\frac{\alpha}{h}} + \delta h^{-\frac{5}{2}}) \right), \]

where the error estimate is uniform in \( \alpha' \). Here we used as well that by the hypothesis (H.3), we have that \( \mathcal{O}(\delta^{-1} e^{-\frac{\alpha}{h}}) = \mathcal{O}(e^{-\frac{\alpha}{h}}) \). Hence,

\[ \tilde{I}_1(z, h) = \delta^4 \pi^{-N} \int_{|\alpha'|_{C^{N-2}} \leq R_0} \left| l_1(\alpha') l_2(\alpha') \right|^2 e^{-\alpha' \bar{\alpha}} L(d\alpha') + \mathcal{O} \left( \delta^4 e^{-\frac{\alpha}{h}} + \delta^5 h^{-\frac{5}{2}} \right). \]

Extend the function \( |l_1(\alpha') l_2(\alpha')|^2 \) to the whole of \( C^{N-2} \) by a function that satisfies the same bounds, i.e. bounded by a term of order \( h^{-5} \), and note that

\[ \pi^{2-N} \int_{|\alpha'|_{C^{N-2}} \geq R_0} |l_1(\alpha') l_2(\alpha')|^2 e^{-\alpha' \bar{\alpha}} L(d\alpha') \leq \mathcal{O}(e^{-\frac{\alpha}{h^2}}). \]
Integration by parts yields that
\[ \pi^{2-N} \int_{C^{N-2}} |l_1(\alpha')l_2(\alpha')|^2 e^{-\alpha'\overline{\alpha}} L(d\alpha') \]
\[ = \pi^{-2} \int_{C^{N-2}} e^{-\alpha\overline{\alpha}} \prod_{k=1}^{2} l_k(\overline{\alpha}) \left( \prod_{n=1}^{2} l_n(\alpha) \right) L(d\alpha). \]

Note that for any permutation \( \sigma \in S_n \), where \( S_n \) is the symmetric group, we have that \( (l_i|l_{\sigma(i)}) = \Gamma_{\sigma(i)} \). Thus, in view of Definition 3.5, we have that
\[ \prod_{k=1}^{2} l_k(\overline{\alpha}) \left( \prod_{n=1}^{2} l_n(\alpha) \right) = \sum_{\sigma \in S_2} (l_1|l_{\sigma(1)})(l_2|l_{\sigma(2)}) = \text{perm } \Gamma \]

We conclude that
\[ I_1(z, h) = \frac{\text{perm } \Gamma + O\left(e^{-\frac{4}{C} + \delta h^{-\frac{51}{10}}} \right)}{\pi^2 \left( \sqrt{\text{det } A} + O\left(\delta h^{-\frac{3}{2}} \right) \right)^2} + O\left(e^{-\frac{5}{C}} \right), \]

where we used the fact that \( \text{det } A \geq \frac{h^2}{C^2} \) for \( 1/C \geq |z - w| \gg h^{3/5} \), see Proposition 5.4, to obtain the last equality.

The integral \( I_2 \) In this step we will estimate the second integral of equation (6.12). Therefore, we will increase the space of integration
\[ \pi^{-N} \int_{B(0,R)} \prod_{k=1}^{2} e^{-2\chi \left( \frac{F_k(z, \alpha)}{\varepsilon} \right)} |\partial_{z_k} F_k(z, \alpha)|^2 e^{-\alpha\overline{\alpha}} L(d\alpha) \]
\[ \leq \pi^{-N} \int_{B(0,2\Delta R)} \prod_{k=1}^{2} e^{-2\chi \left( \frac{F_k(z, \alpha)}{\varepsilon} \right)} |\partial_{z_k} F_k(z, \alpha)|^2 e^{-\alpha\overline{\alpha}} L(d\alpha) =: W_\varepsilon. \]

It is easy to see that Lemma 6.1 holds true for the set \( B(0,2R) \cap \{R_0 < ||\alpha'||_{C^{N-2}} < 2R_0 \} \). Therefore, we can proceed as for the integral \( I_1 \): perform the same change of variables and perform the limit of \( \varepsilon \to 0 \).

As for \( I_1 \), the integrand remains bounded by at most a finite power of \( h^{-1} \) which then yields that
\[ \lim_{\varepsilon \to 0} W_\varepsilon = O\left(e^{-\frac{L}{\pi^2}} \right), \]
where the exponential decay comes from the fact that \( R_0 < ||\alpha'||_{C^{N-2}} \). Therefore,
\[ \int_{C^2} \varphi_1(z_1)\varphi_2(z_2)dv(z_1, z_2) = \int_{C^2} \varphi_1(z_1)\varphi_2(z_2)D(z, h)L(dz_1dz_2) \]
with
\[ D(z, h, \delta) = \frac{\text{perm } \Gamma + O\left(e^{-\frac{4}{C} + \delta h^{-\frac{51}{10}}} \right)}{\pi^2 \left( \sqrt{\text{det } A} + O\left(\delta h^{-\frac{3}{2}} \right) \right)^2} + O\left(e^{-\frac{5}{C^2}} \right). \]
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