Conventional Magnetic Superconductors: Coexistence of Singlet Superconductivity and Magnetic Order

Miodrag L. Kulić

1Institute for Theoretical Physics, J. W. Goethe University, Frankfurt/Main, P.O.Box 111932 Frankfurt/Main, Germany
2CPMOH, Université Bordeaux and CNRS, 33405 Talence Cedex, France

Abstract

The basic physics of bulk magnetic superconductors (MS) related to the problem of the coexistence of singlet superconductivity (SC) and magnetic order is reviewed. The interplay between exchange (EX) and electromagnetic (EM) interaction is discussed and argued that the singlet SC and uniform ferromagnetic (F) order practically never coexist. In case of their mutual coexistence the F order is modified into a domain-like or spiral structure depending on magnetic anisotropy. It turns out that this situation is realized in several superconductors such as $ErRh_4B_4$, $HoMo_6S_8$, $HoMo_6Se_8$ with electronic and in $AuIn_2$ with nuclear magnetic order. The later problem is also discussed here.

The coexistence of SC and antiferromagnetism is more favorable than with the modified F order. Very interesting physics is realized in systems with SC and weak-ferromagnetism which results in an very reach phase diagram.

The properties of magnetic superconductors in magnetic field are very peculiar, especially near the (ferro)magnetic transition temperature where the upper critical field becomes smaller than the thermodynamical critical field.

The extremely interesting physics of Josephson junctions based on MS with spiral magnetic order is also discussed. The existence of the triplet pairing amplitude $F_{↑↑}$ ($F_{↑↓}$) in MS with rotating magnetization (the effect recently rediscovered in SFS junctions) gives rise to the so called $\pi$-contact. Furthermore, the interplay of the superconducting and magnetic phase in such a contact renders possibilities for a new type of coupled Josephson-qubits.

Keywords: Superconductivity; Coexistence; Magnetic Order; Triplet amplitude, $\pi$-Josephson contact, Qubits

PACS: 74.70.Tx; 74.20.Mn
1 Introduction

The physics of magnetic superconductors is a very interesting subject due to the pronounced competition of magnetic order and singlet superconductivity in bulk materials. The question of their coexistence was first raised theoretically in the pioneering work by V. L. Ginzburg [1] in 1956, where only the electromagnetic (EM) interaction between magnetic moments and superconductivity was considered. However, the breakthrough in the physics of MS came with experiments after the discovery of ternary rare earth (RE) compounds such as borides (RE)T₃B₄ with transition elements T = Rh, Ir, chalcogenides (RE)Mo₆X₈ (X = S, Se), silicides (RE)₂T₂Si₅ and stannides (RE)TxSn₅ [2]. In most of them type-II superconductivity is realized and in all of them are the localized RE magnetic moments regularly distributed in the crystal lattice. The basic crystallographic structure, for instance in RERh₄B₄, contains localized moments (LMs) which are rather far away from the Rh and B blocks which deliver conduction electrons. Due to this spatial separation the conduction electrons jump rarely on magnetic ions making the direct exchange interaction (EX) \( J_{sf} (\ll 10^3 K) \) much smaller than in transition metallic magnets. In these systems the 4f rare-earth shells are responsible for localized moments in which the f-level lies much below the Fermi energy, \( E_f \ll E_F \). A number of compounds belonging to the above families have shown coexistence of superconductivity with the antiferromagnetic (AF) order - antiferromagnetic superconductors (AFS), such as (RE)Rh₄B₄ (RE=Dy, Sm,...), and in most of them the Neel (AF) transition temperature \( T_N \) is smaller than the superconducting one \( T_c \), i.e. \( T_N \ll T_c \).

However, a lot of research, both experimental and theoretical, starting from the late seventies were devoted to MS systems in which ferromagnetic (F) order and singlet SC compete due to their strong antagonistic characters - we call these systems ferromagnetic superconductors (FS) and they are the main subject of this review. It turned out that the modified F and SC, order can under certain conditions coexist since the F order is transformed (in the presence of superconductivity) into a spiral or domain-like structure - depending on the type and strength of magnetic anisotropy in the system [3], [4]. In the RE ternary compounds this competition is rather strong and therefore these two orderings coexist in a limited temperature interval \( T_{c2} < T < T_m \) (the reentrant behavior), for instance in ErRh₄B₄ and HoMo₆S₈. The coexistence region in ErRh₄B₄ is narrow with \( T_m \approx 0.8 K, T_{c2} \approx 0.7 K \) and \( T_c = 8.7 K \), while for HoMo₆S₈ it is even narrower with \( T_m \approx 0.74 K, T_{c2} \approx 0.7 K \) and \( T_c = 1.8 K \) - see Refs. [2], [3], [4]. In HoMo₆Se₈ where \( T_c = 5.5 K, T_m \approx 0.8 \) the exchange interaction is weaker and the coexistence persists down to \( T = 0 K \).

A new and very interesting research field in the physics of ferromagnetic su-
perconductors was opened in 1997 by Pobell’s group in Bayreuth [5], which
discovered the coexistence of superconductivity and nuclear magnetic order
in the type-I superconductor AuIn$_2$ with $T_c = 0.207$ K and $T_m = 35 \mu K$.
Although there is a tendency to the nuclear ferromagnetic order, supercon-
ducting electrons enforce the appearance of a spiral or domain-like nuclear
magnetic ordering in the SC state below $T_m$ [6].

It turns out that not only bulk properties of FS are exotic, but also Josephson
junctions made of bulk MS with spiral ordering show potentially fascinat-
ing properties, such as $\pi$-contact [7], combination of a magnetic analog of the
Josephson effect for spin current with the ordinary Josephson effect for charge
current [8].

In the following we shall discuss mainly the microscopic and macroscopic the-
ory of ferromagnetic and antiferromagnetic superconductors which takes into
account the most relevant interactions between localized moments and con-
duction electrons - the exchange (EX) and electromagnetic (EM) interaction.
This theory was elaborated by Buzdin, Bulaevskii, Panyukov and present au-
thor - see [3] and references therein, and successfully applied to a number of
systems. Due to the lack of space we shall discuss effects of magnetic field
on magnetic superconductors briefly - for this subject we refer the reader to
Ref.[3].

We would like to point out here that in the last several years there was a huge
activity in studying of hybrid heterogeneous magnetic superconductors such as
S-F multilayers and S-F-S Josephson junctions. This field is not only of impor-
tance for the fundamental solid state physics but it is of enormous interest for
applications in spintronics and quantum computing, especially after the exper-
imental confirmation of the remarkable prediction of the $\pi$-Josephson contact
by Alexander Buzdin and coworkers [9], [10]. This exciting field will be covered
elsewhere in this issue, as well as the physics of other magnetic superconduc-
tors - heavy fermions, borocarbides (RE)$_2$Ni$_2$B$_2$C, cuprates RuSr$_2$GdCu$_2$O$_8$,
ferromagnets with triplet SC such as UGe$_2$.

2 Competition between SC and F order in FS

Here we shall be limited to those magnetic superconductors where the mag-
netic ordering of the localized 4f moments (LM) is due to the indirect exchange
interaction (RKKY) going via the conduction electrons. The characteristic ex-
change energy is $\theta_{ex} \approx N(0)\hbar^2$, where $N(0)$ is the density of states at the Fermi
level (per LM) and $\hbar(= (g-1)nJ_{sf}(0)\langle J_z \rangle)$ is the maximal exchange field act-
ing on conduction electrons. Here, $g$ is the Lande factor, $n$ is the density of lo-
calized magnetic moments (LMs), $J_{sf}(0)$ is the direct exchange energy between
condensation electrons and LMs, $\langle \hat{J}_z \rangle$ is the averaged total angular moment of the LM. Note that the exchange field acting on electrons is $h_{ex}(r) = h_0 S(r)$, where $S(r)(= \langle \hat{J}_z \rangle/J)$ is the localized spin normalized to one. Let us mention in advance that in the RE ternary compounds the exchange field $h_0$ is still rather large, i.e. $h_0 \sim 10^2$ K and $h_0 \gg \Delta_0 \lesssim 10$ K. We shall see below that in spite of the fact that $h_0$ is larger than the Clogston paramagnetic field $h_p$, i.e. $h_0 \gg h_p \approx 0.7 \Delta_0$, there is a coexistence of SC and modified ferromagnetic order. In the presence of magnetic ordering characterized by the magnetization $\mathbf{M}(r)$ there is electromagnetic interaction between localized moments and (super)conducting electrons, since $\mathbf{M}(r) = n \mu \mathbf{S}(r)$ creates the dipolar magnetic field $\mathbf{B}(r) = \text{rot} \mathbf{A}(r)$ which on the other side induce screening current $\mathbf{j}_s$ of conduction electrons (the Meissner effect). The Fourier transformed $\mathbf{j}_s$ is related to $\mathbf{A}$ by the kernel $\mathbf{K}_s(q)$, i.e. $\mathbf{j}_s(q) = -\mathbf{K}_s(q) \mathbf{A}(q)$. Having in mind magnetic superconductors based on RE ternary compounds we shall discuss the physics in the mean-field approximation for SC and magnetic subsystem - quite appropriate approach, i.e. we put $\mathbf{S}(r) \to S(r) = \langle \mathbf{S}(r) \rangle$.

The total Hamiltonian of the system is given by

$$\hat{H} = \hat{H}_{BCS} + \hat{H}_{CF} + \hat{H}_{imp} + \sum_i [-\mathbf{B}(r_i) g \mu_B \hat{J}_i + \hat{H}_{CF}(\hat{J}_i)]$$  \hspace{1cm} (1)

$$\hat{H}_{BCS} = \int d^3r \{ \hat{\psi}^\dagger(r) [\hat{\epsilon}(\hat{p} - e \hat{A})] - \mu |\hat{\psi}(r)|^2 + i \hat{V}_{ex}(r) \hat{\psi}(r)$$

$$+ \frac{1}{2} \Delta(r) \hat{\psi}^\dagger(r) i \sigma_y \hat{\psi}^\dagger(r) - \frac{1}{2} \Delta^*(r) \hat{\psi}(r) i \sigma_y \hat{\psi}(r) + \frac{|\Delta(r)|^2}{g_{epi}} \}.$$  \hspace{1cm} (2)

Here, $\hat{\epsilon}(\hat{p} - e \hat{A})$ is the band energy of electrons in magnetic field, $\Delta(r)$ is the singlet superconducting order parameter, $\sigma = \{ \sigma_x, \sigma_y, \sigma_z \}$ are Pauli matrices, while the exchange field acting on electrons is given by

$$\hat{V}_{ex}(r) = \begin{pmatrix} h_{ex}^z(r) & h_{ex}^x(r) \\ h_{ex}^x(r) & -h_{ex}^z(r) \end{pmatrix}.$$  \hspace{1cm} (3)

We go slightly in advance by informing the reader, that in the SC state the ferromagnetic order is modified into a spiral or domain-like structure with the wave vector $\mathbf{Q}$, depending on magnetic anisotropy described by $\hat{H}_{CF}$. If the magnetic anisotropy is small (or easy plane) than the spiral structure is realized with $h^\perp(r) = h e^{iQz}$ and $h^z(r) = 0$, while in the opposite case with an easy axis anisotropy one has $h^\perp(r) = 0$ and $h^z(r) = h^z(r + \mathbf{L})$, $\mathbf{L} = 2\pi/Q$. The effect of nonmagnetic impurities is described by $\hat{H}_{imp}$ whose effect is characterized by the mean-free path $l$ and time $\tau$. 

4
2.1 Sinus magnetic structure due to SC for $T \lesssim T_m$

In the RE ternary magnetic superconductors in which the singlet SC and ferromagnetic order compete, the superconducting critical temperature, $T_c$, is much higher than the magnetic one, i.e. $T_m << T_c$. Before discussing the complete phase diagram we shall study the coexistence problem at temperatures near $T_m$, i.e. $T \approx T_m$, where the magnetic order parameter is small $S << 1$.

In case when the easy-axis magnetic anisotropy $D$ is sufficiently large then the sinus structure $S(r) \approx S(T) \sin(Qr)$ appears below $T_m$ (for small $D$ a spiral order is favored - see 2.3). In that case $h_{ex}(r) = h_0 |S(r)| << h_0$, $\Delta$ and the free-energy can be calculated by the perturbation theory

$$F\{\mathbf{S}(r), \Delta(r), \mathbf{A}(r)\} = F_M\{\mathbf{S}\} + F_S\{\Delta\} + F_{\text{Int}}\{\mathbf{S}, \Delta, \mathbf{A}\},$$

(4)

where $F_M$ and $F_S$ are the magnetic and SC functional without mutual interaction, respectively.

$$F_M\{\mathbf{S}(r)\} = n \sum_q \left\{ \frac{1}{2} [(T - T_{m0}) + \theta a^2 q^2] |\mathbf{S}(q)|^2 - D |S_z(q)|^2 \right\}$$

$$+ \int d^3 r \frac{(\mathbf{B} - 4\pi \mathbf{M})^2}{8\pi},$$

(5)

where $\tilde{S}(\mathbf{q})$ is the Fourier transform of $\tilde{S}(\mathbf{r})$. The last term in Eq. (5) is the magnetic energy for a given magnetization $\mathbf{M}(\mathbf{r}) = n\mu \mathbf{S}(\mathbf{r})$ and the magnetic induction $\mathbf{B} = 4\pi \mathbf{M}$. The characteristic energy for the EM interaction is given by $\theta_{em} = (B^2/8\pi n) = 2\pi n\mu^2$ which is $\sim 1$ K in the RE ternary compounds.

Since $T_m << T_c$ and $h \ll \Delta$ the SC free-energy density ($F_S = \int d^3 r \tilde{F}_S$)

$$\tilde{F}_S(\Delta(\mathbf{r})) = -\frac{1}{2} N(0) \Delta^2 \ln \frac{\Delta^2}{\Delta^2_0}$$

(6)

is minimized for $\Delta \approx \Delta_0$ and we omit it from the analysis near $T_m$. The part $F_{\text{Int}}$ describes the EX and EM interaction between SC and magnetic order (note $\mathbf{j}_s(q) = -K_s(q)\mathbf{A}(q)$)

$$F_{\text{Int}} = \sum_q \left\{ \frac{1}{2} K_s(q) |\mathbf{A}(q)|^2 + \theta_{ex} \frac{\chi_n(q) - \chi_s(q)}{\chi_n(0)} |\mathbf{S}(q)|^2 \right\}.$$
After minimizing \( F\{\mathbf{S}(\mathbf{r}), \Delta(\mathbf{r}), \mathbf{A}(\mathbf{r})\} \) with respect to \( \mathbf{A}(\mathbf{r}) \) one gets \( F\{\mathbf{S}(\mathbf{r}), \Delta(\mathbf{r})\} \) in the following form (see more below and in [3])

\[
F\{\mathbf{S}(\mathbf{r}), \Delta(\mathbf{r})\} = n \sum_q \left( \frac{1}{2} \left[ (T - T_{m0}) + \theta a^2 q^2 \right] \right) |\mathbf{S}(\mathbf{q})|^2 - D_z |S_z(\mathbf{q})|^2
+ \frac{\theta_{ex} \chi_n(\mathbf{q}) - \chi_s(\mathbf{q})}{\chi_n(0)} |\mathbf{S}(\mathbf{q})|^2 + \frac{\theta_{em} 4\pi K_s(\mathbf{q}) |\mathbf{S}(\mathbf{q})|^2 + (\mathbf{qS}(\mathbf{q}))(\mathbf{qS}(-\mathbf{q}))}{q^2 + 4\pi K_s(\mathbf{q})} \right) \tag{8}
\]

Here, \( a \) is of the order of lattice constant (magnetic stiffness) and the bare critical temperature \( T_{m0} \) and \( \theta \) take in a subtle way (note \( T_{m0} \neq \theta \) - see details in [3]) into account the indirect EX and direct dipole-dipole (EM) interaction between LMs - see [3]. \( \chi_n(\mathbf{q}) \) and \( \chi_s(\mathbf{q}) \) are electronic susceptibilities in the normal and SC state, respectively. \( \theta_{em} = 2\pi\mu^2 \) characterizes the EM effects in the dipole-dipole interaction between LMs, while \( K_s(\mathbf{q}) \) is the EM kernel which describes the screening effect of the dipole-dipole interaction by the superconducting electrons. \( D_z(>0) \) is the magnetic anisotropy which orients spins along the z-axis.

Due to the singlet SC pairing \( \chi_s(\mathbf{q}) \) is reduced significantly at small wave vectors \( q < \xi_0^{-1} \) where \( \xi_0 \) is the SC coherence length. In the clean limit \( (l \to \infty) \) and at \( T = 0 \) one has \( \chi_s(\mathbf{0}) = 0 \) which means that the ferromagnetic order can not coexist with singlet superconductivity. In Fig.1 we show \( \chi_{s,n}(\mathbf{q}) \) schematically for the cases when the ferromagnetic (1a) or antiferromagnetic order (1b) is realized in the normal state. It is seen that a singlet superconductor behaves as a normal metal at large momenta, i.e. \( \chi_s(q \sim k_F) \) is weakly affected by SC. Therefore AF competes with SC much less than the ferromagnetic order does.

We stress that at finite temperature \( \chi_s(\mathbf{0}) \neq 0 \) but exponentially small in singlet s-wave SC, while in d-wave SC one has \( \chi_s(\mathbf{0}) \sim \chi_n(\mathbf{0})(T/\Delta_0) \). In the presence of the spin-orbit (SO) scattering \( \chi_s(\mathbf{0}) \) is also finite. The general expression for \( \chi_s(\mathbf{q}) \) is calculated in [11]

\[
\chi_s(\mathbf{q}) = 1 - \pi T \sum_{\omega_n} \frac{1}{(1 + u_\omega^2)(P(\omega, q) - 1/2\tau_1)}, \tag{9}
\]

where \( u_\omega = \omega/\Delta \) and

\[
P(\omega, q) = \frac{1}{2} \frac{q v_F}{\arctan{\frac{qv_F}{2\Delta} \sqrt{1 + u_\omega^2 + 1/2\tau_-}}}. \tag{10}
\]

Here, \( \tau_\perp^{-1} = \tau^-^{-1} - (4/3)\tau_{so}^{-1} \), \( \tau^{-1} = \tau^{-1} + \tau_{so}^{-1} \), \( l_{so} = v_F \tau_{so} \) and \( \omega_n = \pi T(2n+1) \). Later, we shall discuss some effects of the SO interaction on the coexistence
Figure 1. Schematic spin susceptibility in the SC and normal state $\chi_{n,s}(\mathbf{q})$: (a) for the ferromagnetic order in the normal state peak at $Q = 0$; (b) for the antiferromagnetic order - peak at $Q_0$.

phase. The effect of the exchange scattering is similar, i.e. $\chi_s(0)$ is finite for finite $\tau_s$.

Since in the following we study the competition between SC and ferromagnetism at low temperatures it is sufficient to give a general expression for $K_s(\mathbf{q})$ in the clean limit

$$K_s(\mathbf{q}) = \frac{3n_e\Delta}{m e q v_F} \int_0^1 dx \frac{1 - x^2}{x} \arcsinh\left(\frac{e q v_F}{2\Delta}\right) \sqrt{1 + \left(\frac{e q v_F}{2\Delta}\right)^2}. \quad (11)$$

The expression for finite $l$ is cumbersome and is omitted here. Some limiting cases of $K_s(\mathbf{q})$ which are relevant for real magnetic superconductors will be studied below.

By minimizing $F\{\mathbf{S}(\mathbf{r}), \Delta(\mathbf{r})\}$ w.r.t. the wave vector $\mathbf{q}$ one obtains the equilibrium magnetic structure which depends on microscopic parameters $a, \xi_0, \lambda_L$ (the London penetration depth), $\theta_{ex}, \theta_{em}$. From the above equation we conclude that due to the EM interaction the magnetic structure is transverse, $\mathbf{q} \cdot \mathbf{S}(\mathbf{q}) = 0$.

Let us analyze $\chi_s(\mathbf{q})$ and $K_s(\mathbf{q})$ in the interesting range of parameters. In the clean limit and for $q\xi_0 \ll 1$ one has

$$K_s(\mathbf{q}) = \frac{1}{4\pi\lambda_L^2}; \chi_n(\mathbf{q}) - \chi_s(\mathbf{q}) = \chi_n(0)(1 - \frac{\pi^2 q^2 \xi_0^2}{30}), \quad (12)$$
while for $q\xi_0 \gg 1$ it holds

$$K_s(q) = \frac{3}{4\lambda_L^2 q\xi_0}; \chi_n(q) - \chi_s(q) = \chi_n(0) \frac{\pi}{2q\xi_0}. \quad (13)$$

Based on Eqs.(12-13) and the expression for the free-energy $F$ in Eq.(8) we obtain that just below the transition temperature $T_m = T_{m0}(1 - 3(\frac{\theta_{ex}}{\theta_{em}})^2/3)$ a transverse $(Q_s \perp S_z)$ sinus structure $S_z(r) \approx S \sin(Q_s r)$ appears. In case when $\xi_0^2 \ll a\lambda_L$ the wave vector $Q_s$ is determined by the EX interaction and for $\theta_{ex}/\theta_{em} \gg (a/\xi_0)^2$ it is given by [12], [13]

$$Q_s = \left(\frac{\pi \theta_{ex}}{4a^2 \xi_0}\right)^{1/3}. \quad (14)$$

For $\theta_{ex}/\theta_{em} \ll (a/\xi_0)^2$ the EM interaction prevails

$$Q_s = \left(\frac{1}{a\lambda_L}\right)^{1/2}. \quad (15)$$

In the opposite limit $\xi_0^2 \gg a\lambda_L$ the EX interaction dominates for $\theta_{ex}/\theta_{em} \gg (a^2\xi_0^3/\lambda_L^3)^{2/5}$ which gives again [12], [13]

$$Q_s = \left(\frac{\pi \theta_{ex}}{4a^2 \xi_0}\right)^{1/3}. \quad (16)$$

For $\theta_{ex}/\theta_{em} \ll (a^2\xi_0^3/\lambda_L^3)^{2/5}$ the EM interaction dominates which gives

$$Q_s \approx \left(\frac{1}{a^2 \xi_0 \lambda_L^2}\right)^{1/5}. \quad (17)$$

From these expressions it is seen that in realistic cases $Q_s$ is determined by the EX interaction - it does not depend on the EM parameter $\lambda_L$, while the EM interaction (with $\lambda_L$ dependence of Q) is dominant only for extremely small EX interaction ($\theta_{ex} \ll \theta_{em}(a/\xi_0)^2$ or $\theta_{ex} \ll \theta_{em}(a^2\xi_0^3/\lambda_L^3)^{2/5}$), i.e. for $(\theta_{ex}/\theta_{em}) \ll 10^{-4} - 10^{-5}$. However, in typical ferromagnetic superconductors, such as ErRh$_4$B$_4$, HoMo$_6$S$_8$, HoMo$_6$Se$_8$, AuIn$_2$, the EX interaction dominates since $\theta_{ex} > 0.1 \theta_{em}$ and $a \ll \xi_0 \lesssim \lambda_L$.

In reality nonmagnetic impurities are always present and one should know $\chi_s(q, l)$ and $K_s(q, l)$ as a function of the mean-free path $l$. The corresponding calculations show that if $(l^5/a^2\xi_0\lambda_L^2) \ll 1$ one has for $\theta_{ex}/\theta_{em} \gg a^2\xi_0/l^3$ [12], [13]

$$Q_s = \left(\frac{\pi \theta_{ex}}{4a^2 \xi_0}\right)^{1/3}. \quad (18)$$
and

$$Q_s \approx \frac{\theta_{ex}}{\theta} \left( \frac{1}{l a^2 \xi_0} \right)^{1/4}$$

(19)

for \((a^2 \xi_0 / l^3) \gg \theta_{ex}/\theta_{em} \gg (l^2/\lambda_L^2)\), \(a^2 l/\xi_0^3\).

In case when \((\theta_{ex}/\theta_{em}) \ll (a^2 \xi_0 / l^3)\) or \((\theta_{ex}/\theta_{em}) \ll l^2/\lambda_L^2\) the EM interaction dominates

$$Q_s \approx \left( \frac{l}{a^2 \xi_0 \lambda_L^2} \right)^{1/4}.$$  

(20)

Let us stress some interesting properties of ferromagnetic superconductors: (i) the ferromagnetic critical temperature is strongly reduced in the presence of SC due to the formation of Cooper pairs in the SC state, i.e. one has \(T_F = T_{m0}(1 - \theta_{ex}^+ \theta_{em}) \ll T_m\). In fact this result is more general and holds also for the coexistence of SC and itinerant ferromagnetism (F) - singlet SC and ferromagnetism do not coexist. In that sense a number of recent papers which claim that the itinerant F and SC coexist should be completely abandoned [14]. However, in some itinerant ferromagnets such as \(Y_9Co_7\) (with \(T_F = 4.5\ K\)) the microscopic parameters favor spiral or domain magnetic structure in the SC state with \(T_c = 2.5\ K\) as it was proposed in [15]; (ii) in isotropic magnetic systems and near the critical temperature \(T_m\) the inverse scattering time due to magnetic fluctuations can diverge and thus destroy SC. However, this divergence is suppressed in the real RE ternary compounds due to the long-range dipole-dipole interaction. The interaction of SC with magnetic fluctuations is described by the free-energy contribution

$$F_{sc,fl} = \frac{\theta_{ex}}{2} \sum_q \langle S_{z,q} S_{z,-q} \rangle \frac{\chi_n(q) - \chi_s(q)}{\chi_n(0)},$$

(21)

where

$$\langle S_{z,q} S_{z,-q} \rangle \sim \frac{1}{\tau + a^2 q^2 + (\theta_{em}/\theta) q^2_z/q^2},$$

(22)

with \(\tau = (T - T_{m0})/\theta\). Due to the large dipole-dipole temperature with \(\theta_{em} \sim \theta\) these fluctuations looks like four-dimensional, thus giving rather small value for the inverse scattering time \(\tau^{-1} \sim \theta \ll T_c\); (iii) the relative strength of the EX and EM interaction is controlled by the parameter

$$r = \frac{F_{Int}^{(EM)}}{F_{Int}^{(EX)}} = \frac{\theta_{em}}{\theta_{ex}} \frac{1}{Q^2 \lambda_L^2}.$$  

(23)
Figure 2. The striped domain magnetic structure \( \mathbf{S}(x) = S_z(x)\mathbf{e}_z \) with the period \( L_D = 2\pi/Q_{DS} \); \( Q \) is along the x-axis.

In the RE ternary compounds the case \( r \ll 1 \) is always realized, due to the large value of \( Q^2\lambda_0^2 \gg 1 \). Therefore, practically in all RE ternary compounds the EX interaction dominates in the formation of magnetic structure, while the EM interaction makes it transversal - see exception in weak ferromagnets below; (iv) In spite of the fact that in the RE ternary compounds the ferromagnetism is stronger phenomenon than SC - the gain in the ferromagnetic energy (per LM and at \( T=0 \) K) \( E_{\text{m}} \approx N(0)\hbar^2/\Theta \) is larger than the gain in the SC condensation energy \( E_{\text{c}} \approx N(0)\Delta^2/\Theta \) since \( \hbar_0(\sim 10^2 \) K) \( \gg \Delta_0(\lesssim 10 \) K), the ferromagnetic order is more "generous" and varies spatially in the SC state. The reason lies in the fact that the magnetic stiffness \( \sim a \) is much smaller than the superconducting stiffness \( \sim \xi_0 \).

2.2 Domain magnetic structure due to SC

By lowering \( T \) the higher term \( \sim S^4(x) \) makes the change of the modulo of \( \mathbf{S}(r) \) unfavorable and the sinus-structure is transformed, as it will be shown below, into the striped domain structure (DS) - see Fig.2. At the same time since the exchange field grows \( h_{\text{ex}} = h_0S(T) \) but for \( h_{\text{ex}}(T) < \Delta \) the mutual interaction of magnetism and SC can be treated by the perturbation theory. In such a case the free-energy density is completed by the density of the domain wall energy \( QE_W/\pi \), where \( E_W \) is the wall-energy per unit surface. In case of sufficiently large magnetic anisotropy \( D_z > \theta \) rotation of the moments in the wall is unfavorable and the linear domain wall is favored, i.e. \( S_z(x) = Sth(x/l_W) \), \( S_x = S_y = 0 \), where \( l_W = a/\sqrt{\tau} \) is the domain-wall thickness [3]. The domain wall energy per unit surface is given by
\[ E_W = (4\sqrt{2}/3)n\theta S^2 a\tau^{1/2} \equiv n\theta S^2\tilde{a} \quad (24) \]

where \( \tau = (T - T_m)/\theta \) - see [3]. The free-energy density \( \tilde{F}_{DS} \) in the DS phase is

\[
\tilde{F}_{DS} = n\theta\left[\frac{1}{2}\tau S^2 + \frac{b}{4}S^4\right] + \frac{Q}{\pi}E_W + n\theta_{ex}\frac{7\zeta(3)}{2\pi}\frac{S^2}{Q\xi_0}.
\]

Note, that if the anisotropy energy is small, i.e. \( \tau > 2D_z/\theta \) the rotating domain wall is realized with the wall thickness \( l_W \approx a(\theta/D_z)^{1/2} \) and the wall energy \( E_W \sim S^2a(\theta D_z)^{1/2} \). In the case of an extremely small anisotropy \( (D_z/\theta) < (a/\xi_0) \sim 10^{-2} - 10^{-3} \) then the spiral structure is realized. Minimizing \( F_{DS} \) in Eq.(25) with respect to \( Q \) one obtains the equilibrium wave vector of the striped DS phase

\[ Q_{DS} \approx 2\left(\frac{\theta_{ex}}{\theta} a\xi_0\right)^{1/2} \]

We conclude that the period of the DS structure is larger than for the sinus phase. It is worth of mentioning that: (i) the here obtained striped domain structure is due to SC and it is property of the bulk, (ii) the problem of the DS phase in SC is mathematically similar to the problem of domain structure in a normal ferromagnetic plate with the magnetization perpendicular to the plate plane, where the role of \( F_{int} \) is played by the magnetic energy dissipated out of plate. Generally, the domain structure is realized when the wall thickness \( a/\sqrt{\tau} \) is much smaller then the striped domain thickness \( \pi/Q \) implying that \( \tau \gg (a/\xi_0)^{2/3} \sim 10^{-2} \).

At lower temperatures when \( h_{ex}(T) > \Delta \) the nonperturbative problem is studied in the presence of impurities by the quasiclassical ELO equations (see Appendix 7.2) since the period \( L_D \) of the domain structure is much larger than \( a \), i.e. \( L_D \gg a \). This equations are solved for the domain structure with

\[ S_z(r) = \frac{4S(T)}{\pi} \sum_{k=0}^{\infty} \frac{\sin(2k+1)Qr}{2k+1} \equiv \sum_{\mathbf{q}} S_{z,\mathbf{q}} e^{i\mathbf{q}r}. \quad (27) \]

By assuming that \( \mathbf{Q} \) is along the x-axis the solution for the Green’s function are searched in the form

\[ f(v, x) = f_0(v) + \sum_k f_k(v) e^{ikx}, \quad (28) \]

and analogously for \( g(v, x) \), where \( k = (2m + 1)Q \) and \( |f_k| \ll |f_0|, |g_k| \ll |g_0| \). The calculations were done in [12], [3] and here we present only the final
result for the free-energy in the dirty limit \((l \ll \xi_0)\). It turns out, that in that case the interaction of the magnetic domain structure with SC is similar to the case of magnetic impurities with the inverse scattering time \(\tau_m^{-1}\) and with \(\tau_m \Delta > 1\), i.e. \(\tilde{F}_{DS}\) is given by

\[
\tilde{F}_{DS} = n\theta\left[\frac{1}{2}\tau S^2 + \frac{b}{4}S^4\right] + QE_W \\
- \frac{1}{2}N(0)\Delta^2 \ln\left(\frac{\epsilon \Delta^2}{\Delta_0^2}\right) + N(0)\frac{\pi\Delta}{2\tau_m}(1 - \frac{2}{3\pi\tau_m\Delta}).
\]

(29)

\(\tau_m^{-1}\) is given by \((h_{z,q} = h_0 S_{z,q})\)

\[
\tau_m^{-1} = \sum_q \{\frac{\pi h_{z,q} h_{z,-q}}{v_F q} L_1(ql) + \frac{3B_q \cdot B_{-q}}{16\lambda^2 n N(0) v_F q^3} L_2(ql)\}
\]

(30)

where

\[
L_1(y) = \frac{2y \arctan y}{\pi(y - \arctan y)}
\]

(31)

and

\[
L_2(y) = \frac{2}{\pi}[(1 + \frac{1}{y^2}) \arctan y - \frac{1}{y}].
\]

(32)

The magnetic induction \(B_q\) is given by

\[
B_q = \frac{4\pi n \mu [q^2 S_q - q(qS_q)]}{q^2 + K_s(q)(1 - 4/3\pi\tau_m\Delta)}.
\]

(33)

where the Kernel \(K_s(q)\) in the dirty limit has the form

\[
K_s(q) = \frac{3\pi\Delta}{16v_F \lambda^2 q} L_2(y).
\]

(34)

Based on the free-energy in Eq.(29) we can study the coexistence problem in the whole temperature regions and for various \(Ql\) - see more in [12]. We summarize the main results: 

(i) at \(T = T_m\) the sinusoidal magnetic order appears with the wave vector \(Q_s \sim (1/a^2\xi_0)^{1/3}\); 

(ii) by lowering temperature the domain structure appears with \(Q_{DS} \sim (1/a\xi_0)^{1/2}\) which persists up to the temperature of the first order phase transition \(T_{c2}\) where the DS phase passes into the normal ferromagnetic state. At \(T_{c2}\) one has

\[
F_{DS}\{S_{DS}(T_{c2}), \Delta(T_{c2}), Q_{DS,c2}\} = F_{FN}\{S_F(T_{c2}), 0, 0\},
\]

(35)
where $Q_{DS,c2} \approx 1.8(\tilde{a}(T_{c2})\xi_0)^{-1/2} \sim (a\xi_0)^{-1/2}$, $\Delta(T_{c2}) = 0.85\Delta_0$ and $(S_{c2}^2/Q_{c2}) \approx 0.07(\Delta_0v_F/h_0^2)$; (iii) if $S_{DS}(T_{c2}) > 1$ then the DS is stable up to $T = 0 K$ - this situation is realized in systems with small EX interaction (which still dominates over EM), i.e. for $\theta_{ex} < \theta_{ex}^c \sim (T_0^c/h_0^2)$; (iv) in dirty SC with $(h\tau)^2 \ll 1$ there is a gap in the quasiparticle spectrum for $E < \Delta$ in the whole range of the existence of the domain phase. The calculations in clean SC show [4], [3] that when $h(T) \gg \Delta$ the spectrum is gapless. For instance in the DS phase one has for $E \ll \Delta$

$$\frac{N(E)}{N(0)} = \frac{\pi h}{v_F Q} \frac{E}{\Delta} \ln \frac{4\Delta}{E}, \quad (36)$$

while in the case of the spiral order

$$\frac{N(E)}{N(0)} = \frac{\pi h}{2v_F Q} \frac{E}{\Delta}, \quad (37)$$

(v) The spin-orbit interaction decreases the value of $\chi_n(0) - \chi_s(0)$ (and small $q$) and it is detrimental for the DS phase. However the analysis in [16] shows that the S-O scattering destroys the peak in $\chi_s(q)$ only in very dirty systems when $l \sim a$.

We would like to point out, that there were a lot of studies of ferromagnetic superconductors based on the phenomenological theory which takes into account the EM interaction only [17], [18]. However, this interesting phenomenology is inadequate in describing real materials, such as the above numbered RE ternary compounds where the EX interaction prevails in the formation of the oscillatory structure (with $Q \gg \xi_0^{-1}, \lambda_L^{-1}$) in the SC state. We stress that in the above theory the EM interaction, as well as the EX one, is taken into account on the microscopic level, thus giving much more reliable predictions than the phenomenological approach.

### 2.2.1 Experimental situation

In the most important ferromagnetic superconductors HoMo$_6$S$_8$, ErRh$_4$B$_4$, HoMo$_6$Se$_8$ the range of microscopic parameters allows the coexistence of SC and modified ferromagnetic order. For instance in clean systems one has [3]: in ErRh$_4$B$_4$ - $n \sim 10^{22}$ cm$^{-3}$, $\mu = 5.6 \mu_B$, $\tilde{a} \approx 1$ Å, $\lambda_L(0) \approx 900$ Å, $\xi_0 \approx 200$ Å, $\Delta_0 \approx 15.5$ K, $N^{-1}(0) = 1850 K$ spin RE, $v_F \approx 1.3 \times 10^7$ cm$^{-1}$s, $\theta_{ex} \approx 0.5$ K, $h_0 \approx 40$ K, $\tau_m^{-1} \approx 3$ K and $\theta_{es} \approx 1.8$ K; in HoMo$_6$S$_8$ - $n \sim 4 \times 10^{21}$ cm$^{-3}$, $\mu = 9.1 \mu_B$, $\tilde{a} \approx 2.5$ Å, $\lambda_L(0) \approx 1200$ Å, $\xi_0 \approx 1500$ Å, $\Delta_0 \approx 3.2$ K, $N^{-1}(0) = 3600 K$ spin RE, $v_F \approx 1.8 \times 10^7$ cm$^{-1}$s, $\theta_{ex} \approx 0.2$ K, $h_0 \approx 24$ K, $\tau_m^{-1} \approx 0.9$ K and $\theta_{es} \approx 1.3$ K while in HoMo$_6$Se$_8$ a number of parameters are similar to HoMo$_6$S$_8$. 

13
An oscillatory magnetic structure (either sinus or domain-like) due to SC has been observed at least in three compounds: (1) in HoMo$_6$S$_8$ where $T_c = 8.7$ K, $T_m \approx 0.8$ K, $T_{c2} \approx 0.7$ K and $Q_{DS} \sim 0.03 \, \text{Å}^{-1}$; (2) in ErRh$_4$B$_4$ where $T_c = 8.7$ K, $T_m \approx 0.8$ K, $T_{c2} \approx 0.7$ K and $Q_{DS} \sim 0.06 \, \text{Å}^{-1}$; (3) in HoMo$_6$Se$_8$ where $T_c = 5.5$ K, $T_m = 0.5$ K and $Q_{DS} \sim 0.09 - 0.06 \, \text{Å}^{-1}$ and the coexistence persists up to $T = 0$ K! All these results are in a satisfactory agreement with the above theory. The conclusion is that in most RE ternary compounds the EX interaction is responsible for the formation of the oscillatory magnetic structure in the SC state, while the EM interaction makes the structure transverse $Q \cdot S(r) = 0$.

2.3 Domain magnetic structure in thin SC film

In the above calculations we have assumed that the thickness $L$ of the sample is very large, i.e. $L \gg \xi_0$, so that the dissipated magnetic energy (stray field) can be neglected. In case of thin films with $L \sim \xi_0$ the stray magnetic energy $E_{st}$ existing around the domain walls must be added to the free-energy $F_{DS}$ in Eq.(29) (or its simple version in Eq.(25)), i.e. $F_{tot} = F_{DS} + E_{st}$ is given by [13]

$$
\tilde{F}_{tot}/n = (\tilde{F}_{DS}/n) + E_{st} = (\tilde{F}_{DS}/n) + 0.85\theta_{em} S^2(T)/QL.
$$

(38)

In case when the ratio $r(= F^{(EM)}_{Int}/F^{(EX)}_{Int}) \ll 1$ the minimization of $F_{tot}$ w.r.t. $Q$ gives

$$
Q_{tot}^2 = Q_{DS}^2 + Q_F^2,
$$

(39)

where $Q_{DS}$ is the wave vector of the DS phase without stray magnetic energy and $Q_F \approx 1.6(\theta_{em}/\theta aL)^{1/2}$ is the wave vector of the domain structure in the normal ferromagnetic state. From Eq.(39) it is seen that the period of the DS ($d = 2\pi/Q_{tot}$) in thin SC film is decreased due to the stray field. It comes out from Eq.(38) that the transition temperature $T_{c2}$ (for the first order phase transition $DS \to FN$(domain)) can be pushed to zero when $L \leq L_c = 3\xi_0(\theta_{em}S^1_{c2}(L = \infty))/\theta ex(1 - S^1_{c2}(L = \infty))^2$. The experiments on thin films of HoMo$_6$S$_8$ show such a thickness dependence of $T_{c2}$ where $T_{c2}(L) < T_{c2}(\infty)$.

Let us mention that even in the normal ferromagnetic state, which is realized for $T < T_{c2}$, there is possibility that SC exist in the domain walls as it was shown in [19], [20], [21], [3]. This situation can be realized in some pseudoternary compounds where $h_0 \lesssim \Delta_0$. 

14
2.4 Coexistence of Nuclear Magnetism and Superconductivity

In 1997 the Pobell’s group from Bayreuth made an important discovery [5] by observing that superconductivity and nuclear magnetism coexist in AuIn$_2$ with $T_c = 0.207$ K and $T_m = 35$ $\mu$K. At first glance this is not too surprising having in mind smallness of the hyperfine interaction between conduction electrons and nuclear spins. However Buzdin, Bulaevskii and the present author applied in 1997 [6] the theory of magnetic superconductors [3] and found a surprising result - the effective nuclear “exchange” field (the hyperfine contact interaction) is rather large $h_{hyp} \approx 1$ K while $\Delta_0 \approx 0.6$ K, i.e. $h_{hyp} > \Delta_0$! The hyperfine interaction has the same (mathematical) structure as the exchange interaction between the 4f LMs and conduction electrons

$$\hat{H}_{e-nuc} = \int d^3 r \sum_i A_{hyp} \delta(r - R_i) \hat{\psi}^\dagger(r) \sigma \hat{I}_i \hat{\psi}(r)$$ (40)

Here, $A_{hyp}$ is the hyperfine interaction and the ”exchange field” is given by $h_{hyp} = nA_{hyp} \langle \hat{I}_i \rangle$, where $\hat{I}_i$ is the nuclear spin. So, the nuclear magnetism in AuIn$_2$, which shows strong tendency toward ferromagnetism, competes rather strongly with SC. It was estimated from the experiment [5] that $\theta_{em}(= 2\pi n_n \mu_n^2) \approx 1$ K and $\theta_{ex}(\approx N(0)h_{hyp}^2) \approx 35$ $\mu$K, $\xi_0 \approx 10^5$ Å, $\lambda_L \approx 10^5$ Å, $l \approx 3.6 \times 10^4$ Å ($l < \xi_0$) which means that the ”EX” (contact) interaction is much stronger than the EM (dipole-dipole) one and the theory invented for RE ternary compounds is completely applicable to this case. This theory predict, that if the nuclear magnetic anisotropy (due to the dipole-dipole interaction) is small, i.e. $(D/\theta_{ex}) < 10^{-3}$, the spiral magnetic structure should be realized, while in the opposite case $(D/\theta_{ex}) > 10^{-3}$ the striped domain structure is formed. The experiments in magnetic field [5] give evidence that SC and oscillating magnetic order coexist up to $T = 0$ K, i.e. the case $\theta_{ex} < \theta_{ex}^c$ is realized in the type-I superconductor AuIn$_2$. Unfortunately, until now there were no nuclear scattering measurements on AuIn$_2$ which could resolve the nuclear magnetic structure below $T_m = 35$ $\mu$K.

The study of the coexistence of SC and nuclear magnetic order is of enormous importance for the fundamental physics. These systems give an opportunity to study the coexistence problem in cases when the electronic temperature ($T_e$) is different than the nuclear one ($T_n$), i.e. $T_e \neq T_n$. But probably the most interesting problem is the coexistence of SC and nuclear magnetism in the case of negative nuclear temperatures ($T_n < 0$ K).
3 Antiferromagnetic superconductors (AFS)

3.1 Coexistence of antiferromagnetism and superconductivity

An evident experimental fact in the RE ternary compounds is that superconductivity coexists with the antiferromagnetic (AF) order much easier than with the modified ferromagnetic order. The reason is that the effective exchange field in AFS varies on the lattice constant (the AF wave vector is $Q_{AF} \sim a^{-1}$) and it is averaged to zero over the volume of the Cooper pair $\xi_0^3$. Thinking in terms of the electronic susceptibility one has

$$\frac{\chi_n(Q_{AF}) - \chi_s(Q_{AF})}{\chi_n(0)} \approx \frac{\Delta}{v_F Q_{AF}} \sim \frac{T_c}{E_F} \ll 1,$$

which means that $F_{Int}^{(EX)}$ in AFS is very small. Due to the same reason the EM interaction is small since $\delta K_s(Q_{AF}) \sim a^3/(\lambda_0^2 \xi_0)$, i.e. $F_{Int}^{(EM)}(\ll F_{Int}^{(EX)})$

This result is confirmed in a number of RE ternary compounds in which the Neel temperature $T_N(\approx N(0)h^2)$ is in most cases (much) smaller than $T_c$ [2]. In these systems the magnetic scattering above $T_N$ is not harmful for SC since the inverse life time $\tau^{-1}_m \sim T_N \ll T_c$.

However, there are a number of interesting properties of the AF superconductors (AFS) such as the pair-breaking effect of nonmagnetic impurities characterized by the life-time $\tau$. In case when $T_N \ll T_c$ the nonmagnetic impurities in the AFS state are pair-breaking, like magnetic impurities with the inverse scattering time

$$\tau^{-1}_m = \frac{\pi h^2}{2v_F Q_{AF}\sqrt{1 + (h\tau)^2}}. \quad (42)$$

For $h\tau \ll 1$ one gets $\tau^{-1}_m \sim T_N \ll T_c$ which means that in this case the pair-breaking effect of impurities is rather small [22]. Very interesting situation appears for systems with $T_N \gg T_c$. Even in such a case the exchange field does not suppress $T_c$ significantly since the theory (based on Eqs.(63-64) in Appendix 7.1) predicts that $(\delta T_c/T_c) \sim (h/E_F)(\ln h/E_F) \ll 1$. However, in the presence of nonmagnetic impurities $T_c$ is renormalized appreciably and SC disappears for $l < l_c \approx 10\xi_0(h/v_F Q_{AF}) \sim \xi_0 T_N/h$. In that respect there is one very interesting AFS compound $Tb_2Mo_3S_4$ with $T_N = 19 K$ and $T_c = 0.8 K$ where one expects that SC should disappear due to the strong magnetic scattering. However, it turns out that in this system the magnetic anisotropy, in conjunction with large momentum $J = 9$, strongly suppress this pair-breaking effect giving rise for SC.
3.2 Weak ferromagnetism in AFS

In the case of competition of SC and the ferromagnetic order in the RE ternary compounds the theory predicts that in the presence of an appreciable EX interaction SC can coexist only with spiral and DS (or sinus) order - depending on the magnetic anisotropy, while other phases are excluded. It turns out that in AF superconductors with weak ferromagnetism (WF) - of the Moriya-Dyalozhinski type, the phase diagram can be much richer containing also the Meissner phase \((M \neq 0, B = 0)\) and the spontaneous vortex state \([23]\).

We discuss this problem briefly by studying the AF order with two sublattice where the AF order parameter is given by \(l = S_1 - S_2\). In systems which allow WF there is an additional term in the free-energy \(F_{WF} = D[S_1 \times S_2]\) in the total free-energy. If for instance \(l\) is along the xy-plane and \(D\) is so oriented that it allows the appearance of the weak ferromagnetism \(m = S_1 + S_2\) \((M = n\mu m)\) in the xy-plane then \(F_{WF}\) is given by

\[
\tilde{F}_{WF} = \beta n^2 \theta_{ex} (m_x l_y + m_y l_x). 
\]

Since in most systems \(m \sim 10^{-3} l\) it immediately implies that \(\beta \ll 1\). In that case and when \(T_N < T_c\) the interaction part \(F_{int}\) of the total free-energy \((F = F_m + F_s + F_{int})\) is given by Eq.(7), while the magnetic system is described by \(F_m\)

\[
F_m = \int d^3r n\theta_{ex} \left[ a l^2 + \frac{c}{4} l^2 \right] + \beta n^2 \theta_{ex} (m_x l_y + m_y l_x) + \frac{(B - 4\pi M)^2}{8\pi}.
\]

By minimizing \(F\) w.r.t. \(A, l, m\) and \(q\) we get possible phases in AFS with WF \([23]\). The resulting free-energy is similar to the case of ferromagnetic superconductors with an effective magnetic stiffness \(a_{eff} = (ab/\beta) \gg a\). It turns out that if \(\beta \gg a/\xi_0\) the EX interaction dominates in the formation of the magnetic structure, and the sinus structure \((l \sim \sin Q r\) and \(m \sim \sin Q r\)) appears at \(T_N\), while for \((a/\lambda_L) < \beta \ll a/\xi_0\) the EM interaction prevails in the formation of the sinusoidal structure. If \(\beta < (a/\lambda_L) \sqrt{2\theta_{em}/\theta_{ex}}\) than the nonuniform structure is unfavorable and the so called Meissner state (first proposed by Ginzburg in 1956) appears. It is characterized by \(M = \text{const}\) and \(B = 0\) in the bulk sample due to the screening SC current on the surface \((B = 4\pi M \exp\{-z/\lambda_L\})\). By lowering the temperature \(|S_{1,2}|\) grow and it is necessary to take into account higher order terms in \(F\). As a result one gets that for \(\beta \gg \sqrt{a/\xi_0}\) again the EX dominates and the striped DS phase is realized, while for \(\sqrt{a/\lambda_L} \ll \beta \ll \sqrt{a/\xi_0}\) the striped DS phase is realized.
due to the EM interaction. However, by lowering the temperature the domain wall energy grows and it may happen that a spontaneous vortex state (with $4\pi M > H_{c1}$ - the lower critical field) appears for $\beta < \sqrt{a/\xi_0}$ and for the AF vector $l > l_c \sim (H_{c1}/M(0))(\beta \lambda^2_1/a^2)^{1/3}$, $\bar{a} = a[(T_N - T)/T_N]^{1/2}$. From the known RE ternary compounds the good candidate for such a behavior is the body centered tetragonal (b.c.t.) system $ErRh_4B_4$.

4 Magnetic superconductors in magnetic field

There are a number of interesting effects of the magnetic field $H$ either in the coexistence phase or above the magnetic transition temperature $T_m$ where $S(T) = 0$. We discuss some of them briefly - for more details see [3], [16].

(1) DS phase in magnetic field - In the case of bulk samples magnetic field penetrates only on the length $\lambda_L$, thus affecting surface of the sample only. However, in thin films the paramagnetic effect of the field is most important [3]. This problem was studied in the case of a thin (along the y-axis) film the thickness $L_y < \xi_0$ when the magnetic field is parallel to the domains, i.e. $H = He_z$ - see Fig. 2. It is found that the magnetization $S_z(x)$ contains besides the odd harmonics also the zeroth-one as well as the even ones

\[
S_z(x) = S\delta + \sum_{k=1}^{\infty} \frac{2S}{\pi k} \{[1 - (-1)^k \cos(\pi k\delta)] \sin(kQx) \\
+ (-1)^k \sin(\pi k\delta) \cos(kQx)\},
\]

with $\delta = \mu H/(2S\theta_{ex})$. This change of harmonics in $S_z(x)$ can be observed by magnetic neutron diffraction experiments. The result in Eq. (45) means that the domains with $M$ parallel to $H$ increase their thickness $d \to d(1 + \delta)$ while those antiparallel decrease it $d \to d(1 - \delta)$. In case when the zeroth component of the exchange field $\bar{h} = h_0 S\delta > \bar{h}_c = \Delta[1 - (1/\tau_m \Delta)^2]^{2/3}$ the DS phase is destroyed due to the Zeeman effect making $\Delta = 0$. For $\bar{h} < \bar{h}_c$ the parameters of the DS phase are renormalized, i.e. $Q(H) < Q(0)$. In case when $H = He_y$ (i.e. orthogonal to the $z$-axis) then all domains have the same thickness and there is no redistribution of intensities of neutron peaks. However, there is only a decrease of intensities of $(2k+1)Q$ peaks by the factor $(1 - \delta^2)$ where $\delta_\perp = \mu H/S(\theta_{ex} + D_z)$ and $D_z$ is the magnetic anisotropy.

(2) MS in magnetic field at $T > T_m$ - The effect of the exchange field on SC in magnetic field is negligible for $T \lesssim T_c$ since for $T_m \ll T_c$ the magnetic susceptibility $\chi_m$ is very small. However, at $T$ near $T_m$ there is a significant increase of $\chi_m$ and accordingly the increase of the paramagnetic effect.
(i) Thermodynamic critical field $H_c(T)$ - We illustrate this effect by analyzing the change of the thermodynamical field $H_c(T)$ (for the transition $N \rightarrow MS$) in magnetic superconductors. In that case the Gibbs energy density of the paramagnetic normal phase is equal to that of the SC phase, $\tilde{G}_N(H_c) = \tilde{G}_{SC}(H_c)$ where

\begin{equation}
\tilde{G}_{SC}(H_c) = \tilde{F}_n(0) - \frac{H^2_c}{8\pi}
\end{equation}

\begin{equation}
\tilde{G}_N(H_c) = F_n(0) - \frac{\mu H^2_c}{8\pi}
\end{equation}

which gives the critical field

\begin{equation}
H_c(T) = \frac{H_{c0}(T)}{\sqrt{1 + 4\pi\chi_m(T)}}.
\end{equation}

Here, $(H^2_{c0}/8\pi) = N(0)\Delta^2/2$ is the SC condensation energy and the magnetic permeability is $\mu = 1 + 4\pi\chi_m$ (here we neglect the conduction electron susceptibility $\chi_e$ since in MS one has $\chi_e \ll \chi_m$). For $T > T_m$ one has $\chi_m(T) \approx (\theta_{em}T_m/4\pi\theta)/(T - T_m)$ and

\begin{equation}
H_c(T) \sim \sqrt{T - T_m}
\end{equation}

i.e. $H_c(T)$ is drastically reduced near $T_m$.

(ii) Upper critical field $H_{c2}(T)$ - Above $T_m$ in the presence of the external field $H_e$ superconductivity is suppressed by the orbital effect of the field $B = H_i(1 + 4\pi\chi_m)$ and by the paramagnetic effect of the exchange field $h$ (and by the much smaller effect due to $B$). Here, $H_i = H_e + H_D$ where $H_D$ is the demagnetization field. The critical field can be calculated by the same formula as for usual SC - see [24], where $\mu_B$ is replaced by $\mu_B = \mu_B + h_0 M/\eta \mu H_i$ and the electron charge $e$ by $\tilde{e} = e(1 + 4\pi M/H_i)$. In the pure limit and for $T \ll T_c$ one gets the modified Gruenberg-Günther formula [25]

\begin{equation}
H_{c2}(T) = \frac{\sqrt{2}}{1 + 4\pi\chi_m(T)} H^*_{c2}(0) \frac{f(\alpha)}{\alpha}
\end{equation}

where $H^*_{c2}(0)$ is the upper orbital critical field in absence of magnetic moments and $f(\alpha)$ is calculated numerically in [25]. The parameter $\alpha$ describes the relative role of the orbital and paramagnetic effect

\begin{equation}
\alpha = \frac{2H^*_{c2}(0)h_0\chi_m(T)}{(1 + 4\pi\chi_m(T))n\mu\Delta_0}.
\end{equation}
In the RE ternary magnetic superconductors one has $h_0 \gg \Delta_0$ and $n\mu$ is one order of magnitude smaller than $H_{c2}^*(0)$ thus giving $\alpha \gg 1$ in the region where $T \ll T_c$. It is known that for $\alpha > 1.8$ in pure superconductors it is realized the Larkin-Ovchinikov-Fulde-Ferrell (LOFF) phase (due to paramagnetic effects) where the SC order parameter oscillates being also zero at some points.

For $\alpha \gg 1$ one has $f(\alpha) \approx 1$ and

$$H_{c2}(T) \approx 1.5 \frac{\Delta_0 T_{m0}}{h_0 \mu \theta} (T - T_{m0})$$

i.e. $H_{c2}(T)$ depends linearly on $T - T_{m0}$ near $T_{m0}$, which is much faster falloff than of $H_c(T)$. This leads to an very interesting effect that by the first order phase transition at $H_c(T)$ the system goes into the Meissner or vortex state depending on the relation between $H_c$ and $H_{c1}$. Let us mention that the lower critical field $H_{c1}$ is very weakly affected by the exchange field due to the localized moments. The theory [3] predicts the following dependence of $H_{c1}$

$$H_{c1} = \frac{\Phi_0}{4\pi \lambda_L^2} \ln \frac{\lambda_L \sqrt{p}}{\xi},$$

where $p$ takes into account screening effects due to EX and EM interaction

$$p(T) = 1 - \frac{\theta_{em}}{\theta_{em} + \theta_{ex} + \frac{\theta(T-T_{m0})}{T_{m0}}},$$

Note, the theory based on the EM interaction (which assumes $\theta_{ex} = 0$) gives $p(T \to T_{m0}) = 0$, which makes the effective penetration depth zero, i.e. $\lambda_{eff} = \lambda_L \sqrt{p} = 0$, and the Ginzburg-Landau parameter $\kappa = (\lambda_{eff}/\xi) \to 0$. If this would be correct than we would have change from type-II SC to type-I SC near $T_{m0}$. This result is apparently incorrect in the RE ternary compounds, since $\theta_{ex} \sim \theta_{em}$, thus making $p$ finite and $\kappa$ stays practically unchanged. So, the change of the type of transition near $T_{m0}$ is not due to the change of $\kappa$ but it is due to the much faster temperature falloff of $H_{c2}(T)$ than of $H_c(T)$. We shall not discuss further this interesting subject but refer the reader to [3] where various phases in the H-T phase diagram were analyzed. Depending on the demagnetization effects several phases can be realized in the same material, such as Meissner-, vortex-, LOFF- or even intermediate-phase.

5 Josephson effect on magnetic superconductors

After the remarkable theoretical discovery by Buzdin and coworkers of the possibility of $\pi$-Josephson junctions in the hybrid $S - F - S$ structure where
Figure 3. The Josephson junction with the insulating contact. $S_L$ and $S_R$ are superconductors with spiral magnetic order. The exchange fields $h_{L,R}$ at the surface make angles $\theta_{L,R}$ with the $y$-axis. $\hat{Q}_{L,R}$ are along the $z$-axis.

F is a ferromagnet [9], [10] the interest in Josephson junctions with magnetic degrees of freedom has grown dramatically - see this issue. In that sense it is natural challenge to investigate this problem in magnetic superconductors.

5.1 $\pi$-contact due to triplet amplitude $F_{\uparrow\uparrow}$ ($F_{\downarrow\downarrow}$)

In [7] the tunnelling Josephson junction was studied with the left-$L$ and right-$R$ bulk magnetic superconductors in which the spiral magnetic ordering is realized - see Fig.3. The spiral magnetic order is characterized by the wave vector $\hat{Q}_{L,R}$ and the exchange fields $h_{\theta_{L,R}} = h_{L,R} e^{i\theta_{L,R}}$, respectively, while superconductivity in the banks is described by the order parameter $\Delta_{L,R} = |\Delta_L| = |\Delta_R| = \Delta$, $h_L = h_R = h$, $|\hat{Q}_L| = |\hat{Q}_R| = Q$ where $\hat{Q}_{L,R} = \chi_{L,R} \hat{Q}_z$ are orthogonal to the tunnelling barrier with the spiral helicity $\chi_{L(R)} = \pm 1$ for $\hat{Q}_{L,R}$ along (+) or opposite(-) to the $z$-axis. Note, that such a junction is characterized by the superconducting phase $\varphi = \varphi_L - \varphi_R \neq 0$ and magnetic phase $\theta = \theta_L - \theta_R \neq 0$. It turns out that besides the singlet amplitude $F_{\uparrow\downarrow}(F_{\uparrow\downarrow})$ the triplet pairing amplitudes $F_{\uparrow\uparrow}$ ($F_{\downarrow\downarrow}$) play very important role in the Josephson effect [7]. (Note, that $F_{\uparrow\uparrow}$ ($F_{\downarrow\downarrow}$) was first introduced in [4] in the study of the spiral magnetic order in SC and first applied to the Josephson junction in [7]. Later on this effect was rediscovered by the Efetov's group in studying S-F-F-S structures with rotating magnetization [26], where $F_{\uparrow\uparrow}$ ($F_{\downarrow\downarrow}$) give rise for new effects see this issue.) It turns out that the Josephson current contains two parts.
\[ J_J (\varphi, \theta) = (J_c^s - J_t^t \cos \theta) \sin \varphi, \]  

(55)

where

\[ J_c^s \sim T \sum_{k_L, k_R, \omega_n} |T_{k_L,k_R}|^2 F_{\uparrow \downarrow}^\dagger(k_L, \omega_n) F_{\uparrow \downarrow}^\dagger(k_R, -\omega_n) \]  

(56)

(and \( J_c^s \sim \Delta^2 \)) is due to the singlet amplitude and

\[ J_{-\chi}^t \sim -T \sum_{k_L, k_R, \omega_n} |T_{k_L,k_R}|^2 \{ F_{\uparrow \uparrow}^\dagger(k_L, \omega_n)[F_{\uparrow \uparrow}^\dagger(k_R, -\omega_n)]^* + F_{\downarrow \downarrow}^\dagger(k_L, \omega_n)[F_{\downarrow \downarrow}^\dagger(k_R, -\omega_n)]^* \} \]  

(57)

is due to the triplet amplitude. It turns out that

\[ J_{-\chi}^t \sim \Delta^2 h^2[f_1 + (\chi_L \chi_R)f_2(\Delta, h)], \]  

(58)

where \( f_{1,2}(\Delta, h) \) are given in [7], while \( \chi = \chi_L \chi_R \) is the total helicity (in [26] renamed to chirality) of the junction. It was shown in [7] that in some parameter region the triplet effects dominate, i.e. \( |J_{-\chi}^t| > J_c^s \), thus giving rise to the \( \pi \)-Josephson junction. From Eq.(54) it is clear that by changing the magnetic phase \( \theta \) and chirality \( \chi \) one can tune the system from 0- to \( \pi \)-junction. This new degree of freedom in the junction - the magnetic phase \( \theta \), first proposed in [7], opens a new possibility for switching elements and quantum computing. From the physical point of view the above model is a paradigm for analogous effects in S-F-F-S structures with rotating magnetization, in which case is \( \theta \) the angle between magnetizations in neighbouring layers.

5.2 Combined superconducting and magnetic Josephson effect

In [8] the above model is developed further by including the tunnelling of electronic spins and their effect on the energy of the contact. Namely, in ferromagnetic superconductors with rotating of magnetization (such as spiral) besides the standard Green’s function \( G_{\uparrow \uparrow} (G_{\downarrow \downarrow}) \), \( F_{\uparrow \downarrow} (F_{\downarrow \uparrow}) \) for singlet SC other Green’s functions \( G_{\uparrow \downarrow} (G_{\downarrow \uparrow}) \) and \( F_{\uparrow \uparrow} (F_{\downarrow \downarrow}) \) are important [7], [8], since they can produce a static spin current \( J_{\text{spin}} \) through the junction (in absence of voltage),

\[ J_{\text{spin}} = J_{\text{spin},G} + J_{\text{spin},F} \]  

(59)
where

\[ J_{\text{spin,G}} \sim \sum |T|^2 (G_{\uparrow,L}G_{\downarrow,R} - G_{\downarrow,L}G_{\uparrow,R}) \sim \hbar^2 \sin \theta. \] (60)

\[ J_{\text{spin,F}} \sim \sum |T|^2 [F_{\uparrow\uparrow}(k_L, \omega_n)] [F_{\uparrow\uparrow}(k_R, -\omega_n)]^* \]
\[ - F_{\downarrow\downarrow}(k_L, \omega_n)] [F_{\downarrow\downarrow}(k_R, -\omega_n)]^* \sim \hbar^2 \Delta^2 \tilde{f}_\chi \cos \varphi \sin \theta. \] (61)

The exact expression for \( J_{\text{spin,G}} \) and \( J_{\text{spin,F}} \) will be published elsewhere [8]. The energy of this combined magnetic and superconducting Josephson junction \( E = E_{\text{mJ}}(\theta) + E_J(\varphi, \theta) \) is

\[ E(\theta, \varphi) = -A\hbar^2 \cos \theta - \Delta^2 (B + C_\chi \hbar^2 \cos \theta) \cos \varphi, \] (62)

where the cumbersome expressions for \( A, B, C \) are given in [8]. Note, that both the spin \( J_{\text{spin}}(\varphi, \theta) \sim \partial E/\partial \theta \) and the charge \( J_J(\varphi, \theta) \sim \partial E/\partial \varphi \) Josephson current depend on \( \varphi \) and \( \theta \). Thus, by tuning \( \theta \) and \( \varphi \) one can tune these currents. In case of small contacts with small charge and "spin" capacitance the system is in the quantum regime thus giving possibility for a novel Josephson qubit. In fact the latter consists from two qubits - the superconducting and magnetic one, which is of a potential interest for applications [8].

6 Conclusion

The rare earth ternary compounds are reach physical systems which allow coexistence of superconductivity and various magnetic orders, such as ferromagnetic, antiferromagnetic, weak ferromagnetism. It turns out that in these systems superconductivity and ferromagnetism never coexist and the latter is modified into a spiral or domain structure - depending on magnetic anisotropy. This is realized in rare earths ternary compounds as well as in \( \text{AuIn}_2 \) where the modified nuclear ferromagnetism and superconductivity coexist. Although the antiferromagnetic order and superconductivity coexist much easier, these systems show peculiar behavior in the presence of nonmagnetic impurities which surprisingly act as pair-breakers. In case when the antiferromagnetic order is accompanied by the weak ferromagnetism new coexistence phases appear - the Meissner and spontaneous vortex state. Magnetic superconductors show peculiar behavior in magnetic field. Near the magnetic critical temperature the upper critical field goes to zero faster than the thermodynamical field, thus giving rise to the first order transition. Various phases are possible in the \( H - T \) diagram depending on the purity and demagnetization effects of real samples. The lower critical field is weakly affected by the exchange field due to localized moments.
The Josephson junctions based on bulk ferromagnetic superconductors with spiral order are characterized by the superconducting and magnetic phase, opening possibilities for a new kind of coupled qubits. The triplet pairing amplitude gives rise to the $\pi-$junction which can be tuned by changing the magnetic phase and chirality.

This article is submitted for the Special Issue Comptes de l’Academie des Sciences: Problems of the Coexistence of Magnetism and Superconductivity edited by A. Buzdin.

Acknowledgments - I would like to thank Alexander Buzdin for collaboration, support and hospitality I have enjoyed during my stay, in summer 2005, at the University of Bordeaux when this review was written. I thank Igor Kulić for collaboration and support.

7 Appendix

The problem of the coexistence of SC and magnetic order with a wavevector $Q$ can be studied in principle by using Gorkov equations for any $Q$. However, in systems where $a << Q^{-1}$ is fulfilled the quasiclassical Eilenberger-Larkin-Ovchinikov (ELO) equations are more suitable and efficient.

7.1 Gorkov equations for MS

These equations contain normal and anomalous Green’s functions $(\hat{G})_{\alpha\beta}(x,y) = -<\hat{T}\psi_{\alpha}(x)\psi_{\beta}^\dagger(y)>$ and $(\hat{F})_{\alpha\beta}(x,y) = <\hat{T}\psi_{\alpha}^\dagger(x)\psi_{\beta}^\dagger(y)>$, where $x \equiv (\vec{r},\tau)$ and $\alpha,\beta = \uparrow, \downarrow$. The superconducting order parameter is defined by

$$\Delta^*(\vec{r}_i - \vec{r}_j) = -\frac{1}{2}V(\vec{r}_i - \vec{r}_j)[<\psi_{\uparrow}^\dagger(\vec{r}_i)\psi_{\downarrow}^\dagger(\vec{r}_j)> - <\psi_{\downarrow}^\dagger(\vec{r}_i)\psi_{\uparrow}^\dagger(\vec{r}_j)>] \quad (63)$$

where the pairing interaction $V(\vec{r}_i - \vec{r}_j)$ is responsible for the $s$-wave pairing in absence of magnetic order. Note, that in the presence of any (in)commensurate magnetic order the superconducting order parameter should be nonuniform, i.e. $\Delta(\vec{r}_1, \vec{r}_2) = \Delta(\vec{r}_1 - \vec{r}_2, \vec{r})$, $\vec{r} = (\vec{r}_1 + \vec{r}_2)/2$. However, it has been shown in[4] that when $h < hv_F Q$ the nonuniform part of $\Delta$ is small and of the order $(h/h v_F Q) \Delta$. That is the reason for our assumption $\Delta^*(\vec{r}_i, \vec{r}_j) = \Delta^*(\vec{r}_i - \vec{r}_j)$. The set of equations for the Green’s functions is given by

$$[i\omega_n - \hat{E}_0(\hat{p}) - \hat{V}_{ex}(\vec{r})]\hat{G}_{\omega_n}(\vec{r}, \vec{r}') + \int d^2x \hat{\Delta}(\vec{r} - \vec{x})\hat{F}^\dagger_{\omega_n}(\vec{x}, \vec{r}') = \delta(\vec{r} - \vec{r}'),$$

where $\hat{G}_{\omega_n}(\vec{r}, \vec{r}')$ is the normal Green’s function and $\hat{V}_{ex}(\vec{r})$ is the exchange potential.
The SC order parameter is \( \hat{\Delta}(\vec{r}) \), which defines the quasiclassical Green's function \( \hat{g}(\vec{r}, \vec{r}) \).

By integrating out unimportant degrees of freedom at small distances one can obtain the ELO equations for the quasiclassical Green's functions, which read:

\[
\int d^2x \hat{\Delta}^*(\vec{x} - \vec{r}) \hat{G}_{\omega_n}(\vec{x}, \vec{r}') = 0, \quad (64)
\]

where

\[
\hat{\Delta} = \Delta \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad \hat{V}_{ex}(\vec{r}) = \begin{bmatrix} h^z(\vec{r}) & h_{\perp}^e - i(\vec{Q} \cdot \vec{r} + \theta) \\ h_{\perp}^e i(\vec{Q} \cdot \vec{r} + \theta) & -h^z(\vec{r}) \end{bmatrix}. \quad (65)
\]

In case of a spiral magnetic ordering (with spatial rotation of magnetization) besides the singlet pairing amplitude \( F_{\uparrow \uparrow}^\dagger \sim \Delta \cdot h_\theta^* \) (and \( F_{\downarrow \downarrow}^\dagger \) \cite{4}), \( F_{\uparrow \uparrow}^\dagger \) is responsible for singlet pairing with the order parameter \( \Delta R(x) \), while the triplet amplitude \( F_{\uparrow \uparrow}^\dagger \), which is due to the rotating magnetization which mixes spin up and down, does not give rise to the triplet pairing since it is assumed from the very beginning that \( \Delta_{\uparrow \uparrow} = \Delta_{\downarrow \downarrow} = 0 \). Note that \( F_{\uparrow \uparrow}^\dagger \) contains two terms - the first one is odd in frequency \( \omega_n \) and the second one is odd in momentum \( \vec{k} \).

### 7.2 ELO equations for MS

In problems related to the presence of the exchange field \( \vec{h}(\vec{r}) = h(\vec{r}) \vec{e}_z \) (oriented along the z-axis) and of nonmagnetic potential and spin-orbit scattering a generalization of the ELO equations is needed \cite{12}, \cite{27}. In that case the Gor'kov equations contain 4×4 Green’s functions \( \hat{G}(x_1, x_2) = -\langle \hat{T} \hat{\Psi}(x_1) \hat{\Psi}^\dagger(x_2) \rangle \) with the four-component spinor \( \hat{\Psi}^\dagger(x) = (\psi^\dagger_1 \psi^\dagger_2 \psi^\dagger_1 \psi^\dagger_2) \). Here \( \hat{\sigma}_i \) are Pauli matrices in the spin-space and \( x \equiv (\vec{r}, \tau) \). (Note, that here the functions \( F \) are defined with minus sign.)

The SC order parameter is \( \Delta(\vec{r}) = i \hat{\tau}_+ \hat{\sigma}_2 \hat{G}(\vec{x}, \vec{x})/2 \).

By integrating out unimportant degrees of freedom at small distances one defines the quasiclassical Green’s function \( \hat{g}_{\omega_n}(\vec{R}, \vec{p}_F) \)

\[
\hat{g}_{\omega_n}(\vec{R}, \vec{p}_F) = \int d\xi_p \int d(\vec{r}_1 - \vec{r}_2) e^{-i\vec{p}((\vec{r}_1 - \vec{r}_2))} \hat{\tau}_3 \hat{G}(\vec{r}_1, \vec{r}_2, \omega_n). \quad (67)
\]

Then the ELO equations for the quasiclassical Green’s functions read

\[
i\vec{v}_F \nabla_{\vec{R}} \hat{g}_{\omega_n} = [i\omega_n \hat{\tau}_3 + \hat{\tau}_3 \hat{\Delta}(\vec{R}) + h(\vec{R}) \hat{\sigma}_3 - \hat{\tau}_3 \Sigma_{\omega_n}^{imp}(\vec{R}, \vec{p}_F), \hat{g}_{\omega_n}], \quad (68)
\]
where the impurity self-energy in the Born approximation is given by

$$\Sigma_{\omega n}^{imp}(R, p_F) = c_i N(0) \langle \hat{U}(p_F - p'_F) \hat{g}_{\omega n}(R, p'_F) \hat{U}(p_F - p'_F) \rangle_{p'_F}. \quad (69)$$

The impurity potential (matrix) contains the non-magnetic and spin-orbit scattering

$$\hat{U}(p_F - p'_F) = U_1 \hat{\tau}_3 + i U_{so} [\hat{p}_F \times \hat{p}'_F] \hat{\alpha}, \quad (70)$$

with the vector matrix $$\hat{\alpha} = [(1 + \hat{\tau}_3)\sigma + (1 - \hat{\tau}_3)\sigma_2\sigma_2]/2$$ and $$\hat{p}_F = p_F/|p_F|$$. The matrix $$\hat{g}_{\omega n}(R, p_F)$$ is given by

$$\hat{g}_{\omega n}(R, p_F) = -i \begin{pmatrix}
g_+ & 0 & 0 & i f_+ \\
0 & g_- & i f_- & 0 \\
0 & -i f_\dag & -g_- & 0 \\
-i f_\dag & 0 & 0 & -g_+
\end{pmatrix}. \quad (71)$$

By including also orbital effects in magnetic field, which is determined by the vector potential $$\mathbf{A}(R)$$, the ELO equations written in components $$g_\pm$$ and $$f_\pm$$ read

$$[\tilde{\omega}_{n,\pm} + ie \mathbf{v}_F \cdot \mathbf{A}(R) \pm i h(R) - \frac{1}{2} \mathbf{v}_F \cdot \nabla_R] f_\pm(p_F, R, \omega_n) = \tilde{\Delta}_\pm(p_F, R, \omega_n) g_\pm(p_F, R, \omega_n), \quad (72)$$

$$[\tilde{\omega}_{n,\pm} + ie \mathbf{v}_F \cdot \mathbf{A}(R) \pm i h(R) + \frac{1}{2} \mathbf{v}_F \cdot \nabla_R] f_\dag(p_F, R, \omega_n) = \Delta^*_\pm(p_F, R, \omega_n) g_\pm(p_F, R, \omega_n), \quad (73)$$

with the normalization condition and the self-consistency equation, respectively

$$g^2_\pm + f^\dag_\pm f_\pm = 1 \quad (74)$$

$$\Delta(p_F, R) = \frac{\pi T}{2} \sum_{\omega_n, p'_F} V(p_F, p'_F) \{ f_+(R, p'_F, \omega_n) + f_-(R, p'_F, \omega_n) \}. \quad (75)$$

$$\tilde{\omega}_{n,\pm}$$ and $$\tilde{\Delta}_\pm(p_F, R, \omega_n)$$ are defined by

$$\tilde{\omega}_{n,\pm} = \omega_n + \frac{1}{2\tau_1} \langle g_\pm(p'_F, R, \omega_n) \rangle_{p'_F}$$
+ \frac{3}{2\tau_{^s_o}} \langle g_{\mp}(p'_{F}, R, \omega_n) \sin^2(\theta - \theta') \rangle_{p'_{F}}^{p_{F}}, \quad (76)

\tilde{\Delta}_{\pm}(p_{F}, R, \omega_n) = \Delta(p_{F}, R) + \frac{\Gamma_1}{2} \langle f_{\pm}(p'_{F}, R, \omega_n) \rangle_{p'_{F}}^{p_{F}}

+ \Gamma_{^s_o} \langle f_{\mp}(p'_{F}, R, \omega_n) \sin^2(\theta - \theta') \rangle_{p'_{F}}^{p_{F}} \quad (77)

where \( \Gamma_1 = c_i N(0) | U_1 |^2 \) and \( \Gamma_{^s_o} = c_i N(0) | U_{^s_o} |^2 \).

The microscopic theory of magnetic superconductors which takes into account spin (exchange) and orbital (electromagnetic) effects of magnetic order, as well as of nonmagnetic impurity scattering, has been developed by using the above generalized ELO equations [3], [12].

References

[1] V. L. Ginzburg, Zh. Eksp. Teor. Fiz. 31, 202 (1956)

[2] M. B. Maple and Ø. Fischer (Eds.), in Superconductivity and Magnetism (Springer-Verlag, Berlin, 1982)

[3] A. I. Buzdin, L. N. Bulaevskii, M. L. Kulić and S. V. Panyukov, Advances in Physics 34, 176 (1985)

[4] L. N. Bulaevskii, M. L. Kulić and A. I. Rusinov, Solid State Comm. 30, 59 (1979); J. Low T. Phys. 39, 255 (1980);

[5] S. Rehmann, T. Herrmannsdörfer and F. Pobell, Phys. Rev. Lett. 78, 1122 (1997)

[6] M. L. Kulić, L. N. Bulaevskii and A. I. Buzdin, Phys. Rev. B 56, R11 415 (1997)

[7] M. L. Kulić, I. M. Kulić, Phys. Rev. B 63, 104503 (2001)

[8] M. L. Kulić, I. M. Kulić, in preparation (2005)

[9] A. I. Buzdin, L. N. Bulaevskii, S. V. Panyukov, J.E.T.P. Lett. 35, 147 (1982)

[10] A. I. Buzdin, Rev. Mod. Phys., (2005)

[11] M. Kaufman, O. Entin-Wohlman, Physica 84, 77 (1976)

[12] L. N. Bulaevskii, A. I. Buzdin, M. L. Kulić, S. V. Panyukov, Phys. Rev. B 28, 1370 (1983)

[13] A. I. Buzdin, Dissertation, MGU Lomonosov, (1987)
[14] K. B. Blagoev et al., Phys. Rev. Lett. 82, 133 (1999); ibid Phil. Mag. 83, 3247 (1998); N. I. Karchev et al., Phys. Rev. Lett. 82, 846 (2001)

[15] K. H. Bennemann, M. L. Kulić, P. Stampfli, Sol. State. Comm. 64, 1359 (1987)

[16] L. N. Bulaevskii, A. I. Buzdin, M. L. Kulić, Phys. Rev. B 34, 4928 (1986)

[17] U. Krey, Int. J. Magn., 3, 65 (1973); ibid 4, 153 (1973)

[18] H. S. Greenside, E. I. Blount, C. M. Varma, Phys. Rev. Lett. 46, 49 (1981); M. Tachiki, Physica B 109-110, 1699 (1982)

[19] Yu. V. Kopaev, Fizika tverd. Tela B 7, 2907 (1965) (in Russian)

[20] M. L. Kulić, Phys. Lett. A 83, 4928 (1981)

[21] A. I. Buzdin, L. N. Bulaevskii, S. V. Panyukov, J.E.T.P. 87, 299 (1984)

[22] A. I. Morozov, Fiz. Tverd. Tela (in Russian), 22, 3372 (1980)

[23] A. I. Buzdin, L. N. Bulaevskii, S. S. Krotov, Sol. State Comm. 48, 719 (1983)

[24] N. R. Werthamer, E. Helfland, P. C. Hohenberg, Phys. Rev. 147, 295 (1966)

[25] L. W. Gruenberg, L. Günther, Phys. Rev. Lett. 16, 996 (1966)

[26] F. S. Bergeret, A. F. Volkov, K. B. Efetov, Phys. Rev. Lett. 86, 4096 (2001)

[27] E. A. Demler, G. B. Arnold, M. R. Beasley, Phys. Rev. B 55, 15174 (1997)