ON THE \((C, \alpha)\)-MEANS WITH RESPECT TO THE WALSH SYSTEM

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Abstract. As main result we prove strong convergence theorems for Cesáro means \((C, \alpha)\) on the Hardy spaces \(H_{1/(1+\alpha)}\), where \(0 < \alpha < 1\).

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1. INTRODUCTION

It is well-known that Walsh-Paley system does not form a basis in the space \(L_1(G)\). Moreover, there is a function in the dyadic Hardy space \(H_{1}(G)\), such that the partial sums of \(F\) are not bounded in \(L_1\)-norm. However, in Simon [17] the following estimation was obtained for all \(F \in H_{1}(G)\):

\[
\frac{1}{\log n} \sum_{k=1}^{n} \frac{\|S_k F\|_1}{k} \leq c \|F\|_{H_1},
\]

where \(S_k F\) denotes the \(k\)-th partial sum of the Walsh-Fourier series of \(F\).

(For the trigonometric analogue see in Smith [19], for the Vilenkin system in Gát [5]). Simon [14] (see also [31]) proved that there is an absolute constant \(c_p\), depending only on \(p\), such that

\[
\frac{1}{\log |p|} \sum_{k=1}^{n} \frac{\|S_k F\|^p}{k^{2-p}} \leq c_p \|F\|^p_{H_p}, \quad (0 < p \leq 1),
\]

for all \(F \in H_p\) and \(n \in \mathbb{N}\), where \([p]\) denotes integer part of \(p\).

The second author [23] proved that sequence \(\{1/k^{2-p}\}_{k=1}^{\infty}\) \((0 < p < 1)\) in \(H_1\) is given exactly.

Weisz [32] considered the norm convergence of Fejér means of Walsh-Fourier series and proved that the following is true:

Theorem W1. Let \(F \in H_p\). Then

\[
\|\sigma_k F\|_{H_p} \leq c_p \|F\|_{H_p}, \quad \left(1/2 < p < \infty\right)
\]
This theorem implies that
\[ \frac{1}{n^{2p-1}} \sum_{k=1}^{n} \|\sigma_k F\|_{H_p}^p \leq c_p \|F\|_{H_p}^p, \quad (1/2 < p < \infty). \]

If Theorem W1 held for \(0 < p \leq 1/2\), then we would have
\[ (3) \quad \frac{1}{\log^{1/2+p}[1/n]} \sum_{k=1}^{n} \|\sigma_k F\|_{H_p}^p \leq c_p \|F\|_{H_p}^p, \quad (0 < p \leq 1/2), \]

but the second author \[20\] proved that the assumption \(p > 1/2\) is essential. In particular, he proved that there exists a martingale \(F \in H_p\) \((0 < p \leq 1/2)\), such that
\[ \sup_n \|\sigma_n F\|_p = +\infty. \]

However, the second author \[24\] prove that (3) holds, though (2) is not true for \(0 < p \leq 1/2\).

The weak \((1,1)\)-type inequality for the maximal operator of Fejér means
\[ \mu(\sigma^* f > \lambda) \leq \frac{c}{\lambda} \|f\|_1, \quad (\lambda > 0) \]
can be found in Schipp \[12\] (see also \[11\]). Fuji \[3\] and Simon \[16\] verified that \(\sigma^*\) is bounded from \(H_1\) to \(L_1\). Weisz \[27\] generalized this result and proved the boundedness of \(\sigma^*\) from the martingale space \(H_p\) to the space \(L_p\) for \(p > 1/2\). Simon \[15\] gave a counterexample, which shows that boundedness does not hold for \(0 < p < 1/2\). The counterexample for \(p = 1/2\) due to Goginava \[7\], (see also \[1\] and \[20\]). Weisz \[28\] proved that \(\sigma^*\) is bounded from the Hardy space \(H_{1/2}\) to the space \(L_{1/2,\infty}\).

The second author \[21, 22\] proved that the following is true:

**Theorem T1.** The maximal operators \(\tilde{\sigma}_p^*\) defined by
\[ (4) \quad \tilde{\sigma}_p^* := \sup_{n \in \mathbb{N}} \frac{|\sigma_n|}{n^{1/p-2} \log^{2(1/2+p)}[1/n]} \], \quad (0 < p \leq 1/2, \ n = 2, 3, ...)

where \([1/2 + p]\) denotes integer part of \(1/2 + p\), is bounded from the Hardy space \(H_p\) to the space \(L_p\). Moreover, there was also shown that sequence \(\left\{ n^{1/p-2} \log^{2(1/2+p)}[1/n] : n = 2, 3, ... \right\}\) in (4) can not be improved.

The maximal operator \(\sigma^{\alpha,*}\) \((0 < \alpha < 1)\) of the Cesáro means means of Walsh-Paley system was investigated by Weisz \[30\]. In his paper Weisz proved that \(\sigma^{\alpha,*}\) is bounded from the martingale space \(H_p\) to the space \(L_p\) for \(p > 1/(1 + \alpha)\). Goginava \[8\] gave counterexample, which shows that boundedness does not hold for \(0 < p \leq 1/(1 + \alpha)\). Recently, Weisz and Simon \[18\] show that in case \(p = 1/(1 + \alpha)\) the maximal operator \(\sigma^{\alpha,*}\) is bounded from the Hardy space \(H_{1/(1+\alpha)}\) to the space \(L_{1/(1+\alpha),\infty}\).

In \[9\] Goginava investigated the behaviour of Cesáro means of Walsh-Fourier series in detail. For some approximation properties of the two dimensional case see paper of Nagy \[10\].
The main aim of this paper is to generalize Theorem T1 and estimation (3) for Cesáro means, when \( p = 1/(1 + \alpha) \).

2. Definitions and Notations

Let \( \mathbb{N}_+ \) denote the set of the positive integers, \( \mathbb{N} := \mathbb{N}_+ \cup \{0\} \). Denote by \( Z_2 \) the discrete cyclic group of order 2, that is \( Z_2 := \{0, 1\} \), where the group operation is the modulo 2 addition and every subset is open. The Haar measure on \( Z_2 \) is given so that the measure of a singleton is 1/2.

Define the group \( G \) as the complete direct product of the group \( Z_2 \) with the product of the discrete topologies of \( Z_2 \)'s. The elements of \( G \) are represented by sequences \( x := (x_0, x_1, \ldots, x_j, \ldots) \) \( (x_k \in \{0, 1\}) \).

It is easy to give a base for the neighbourhood of \( G \)

\[ I_0(x) := G, \quad I_n(x) := \{y \in G \mid y_0 = x_0, \ldots, y_{n-1} = x_{n-1}\} \quad (x \in G, \ n \in \mathbb{N}). \]

Denote \( I_n := I_n(0) \) for \( n \in \mathbb{N} \) and \( I_n := G \setminus I_n \).

\[ e_n := (0, \ldots, 0, x_n = 1, 0, \ldots) \in G \quad (n \in \mathbb{N}) \]

It is evident

\[ \bigcup_{M=2}^{\infty} \bigcup_{k=0}^{M-1} I_{k+1} \left( e_k + e_l \right) \bigcup \left( \bigcup_{k=0}^{M-1} I_M \left( e_k \right) \right) = \bigcup_{n=0}^{\infty} I_M \left( e_k \right) \]

The norm (or quasi-norm) of the space \( L_p(G) \) is defined by

\[ \|f\|_p := \left( \int_G |f(x)|^p \, d\mu(x) \right)^{1/p}, \quad (0 < p < \infty). \]

The space \( L_{p,\infty}(G) \) consists of all measurable functions \( f \) for which

\[ \|f\|_{L_{p,\infty}(G)} := \sup_{\lambda>0} \lambda \mu(f > \lambda)^{1/p} < +\infty. \]

If \( n \in \mathbb{N} \) then for every \( n \) can be uniquely expressed as \( n = \sum_{j=0}^{\infty} n_j 2^j \) where \( n_j \in Z_2 \) \( (j \in \mathbb{N}) \) and only a finite number of \( n_j \)'s differs from zero.

Let \( |n| := \max \{j \in \mathbb{N}, n_j \neq 0\} \), that is \( 2^{|n|} \leq n \leq 2^{|n|+1} \).

Next, we introduce on \( G \) an orthonormal system which is called the Walsh system. At first define the Rademacher functions as

\[ r_k(x) := (-1)^{x_k} \quad (x \in G, \ k \in \mathbb{N}). \]

Now define the Walsh system \( w := (w_n : n \in \mathbb{N}) \) on \( G \) as:

\[ w_n(x) := \prod_{k=0}^{\infty} r_k^n (x) = r_{|n|} (x) (-1)^{\sum_{k=0}^{n_j-1} n_k x_k} \quad (n \in \mathbb{N}). \]

The Walsh system is orthonormal and complete in \( L_2(G) \). (see [13]).
If \( f \in L_1(G) \) we can establish the Fourier coefficients, the partial sums of the Fourier series, the Fejér means, the Dirichlet and Fejér kernels in the usual manner:

\[
\hat{f}(k) := \int_G f w_k d\mu \quad (k \in \mathbb{N}),
\]

\[
S_n f := \sum_{k=0}^{n-1} \hat{f}(k) w_k \quad (n \in \mathbb{N}_+, \ S_0 f := 0),
\]

\[
\sigma_n f := \frac{1}{n} \sum_{k=1}^{n} S_k f \quad (n \in \mathbb{N}_+),
\]

\[
D_n := \sum_{k=0}^{n-1} w_k \quad (n \in \mathbb{N}_+),
\]

\[
K_n := \frac{1}{n} \sum_{k=1}^{n} D_k \quad (n \in \mathbb{N}_+),
\]

respectively.

Recall that

\[
D_{2^n} (x) = \begin{cases} 2^n, & \text{if } x \in I_n, \\ 0, & \text{if } x \notin I_n. \end{cases}
\]

For the \(2^n\)-th Fejér kernel we have the following equality (see [4]):

\[
K_{2^n} (x) = \begin{cases} 2^{t-1}, & \text{if } x \in I_n(e_t), \\ \left(2^A + 1\right)/2, & \text{if } x \in I_n, \\ 0, & \text{otherwise}. \end{cases}
\]

for \(n > t, t, n \in \mathbb{N}, x \in I_t \setminus I_{t+1}\).

The Cesáro means, \(((C, \alpha) \text{ means})\) and it’s kernel with respect to the Walsh-Fourier series are defined as

\[
\sigma_{n\alpha} f := \frac{1}{A_{\alpha}^n} \sum_{k=1}^n A_{\alpha-k}^{\alpha-1} S_k f, \quad K_{\alpha}^n f := \frac{1}{A_{\alpha}^n} \sum_{k=1}^n A_{\alpha-k}^{\alpha-1} D_k f,
\]

respectively, where

\[
A_{\alpha}^0 := 0, \quad A_{\alpha}^n := \frac{(\alpha + 1) \ldots (\alpha + n)}{n!}, \quad \alpha \neq -1, -2, \ldots
\]

It is well known that

\[
A_{\alpha}^n = \sum_{k=0}^n A_{\alpha-k}^{\alpha-1}, \quad A_{\alpha}^n - A_{\alpha-1}^n = A_{\alpha-1}^0, \quad A_{\alpha}^n \sim n^\alpha,
\]

and

\[
\sup_n \int_G |K_{\alpha}^n (x)| \, d\mu (x) \leq c < \infty.
\]
The \( \sigma \)-algebra is generated by the intervals \( \{ I_n(x) : x \in G \} \) will be denoted by \( F_n (n \in \mathbb{N}) \). The conditional expectation operators relative to \( F_n (n \in \mathbb{N}) \) are denoted by \( E_n \).

A sequence \( F = (F_n, n \in \mathbb{N}) \) of functions \( F_n \in L_1(G) \) is said to be a dyadic martingale if (for details see e.g. [26])

(i) \( F_n \) is \( F_n \) measurable for all \( n \in \mathbb{N} \),

(ii) \( E_n F_m = F_n \) for all \( n \leq m \).

The maximal function of a martingale \( F \) is defined by

\[
F^* = \sup_{n \in \mathbb{N}} |F_n|.
\]

In case of \( f \in L_1(G) \), the maximal functions are also be given by

\[
f^*(x) = \sup_{n \in \mathbb{N}} \frac{1}{\mu(I_n(x))} \left| \int_{f_n(x)} f(u) d\mu(u) \right|.
\]

For \( 0 < p < \infty \) the Hardy martingale spaces \( H_p(G) \) consist of all martingales for which

\[
\|F\|_{H_p} := \|F^*\|_p < \infty.
\]

A bounded measurable function \( a \) is a \( p \)-atom, if there exists a dyadic interval \( I \), such that

\[
\int_I a d\mu = 0, \quad \|a\|_{\infty} \leq \mu(I)^{-1/p}, \quad \text{supp}(a) \subset I.
\]

The dyadic Hardy martingale spaces \( H_p(G) \) for \( 0 < p \leq 1 \) have an atomic characterization. Namely, the following theorem is true (see [29]):

**Theorem W**: A martingale \( F = (F_n, n \in \mathbb{N}) \) is in \( H_p(0 < p \leq 1) \) if and only if there exists a sequence \( (a_k, k \in \mathbb{N}) \) of \( p \)-atoms and a sequence \( (\mu_k, k \in \mathbb{N}) \) of real numbers such that for every \( n \in \mathbb{N} \)

\[
\sum_{k=0}^{\infty} \mu_k S^{2^k} a_k = F_n
\]

and

\[
\sum_{k=0}^{\infty} |\mu_k|^p < \infty,
\]

Moreover, \( \|F\|_{H_p} \sim \inf (\sum_{k=0}^{\infty} |\mu_k|^p)^{1/p} \), where the infimum is taken over all decompositions of \( F \) of the form \( (10) \).

It is easy to check that for every martingales \( F = (F_n, n \in \mathbb{N}) \) and every \( k \in \mathbb{N} \) the limit
(11) \[ \hat{F}(k) := \lim_{n \to \infty} \int_G F_n(x)w_k(x)\,d\mu(x) \]
exists and it is called the \( k \)-th Walsh-Fourier coefficients of \( F \).

If \( F := (E_n f : n \in \mathbb{N}) \) is a regular martingale generated by \( f \in L_1(G) \), then
\[ \hat{F}(k) = \int_G f(x)w_k(x)\,d\mu(x) =: \hat{f}(k), \quad k \in \mathbb{N}. \]

For \( 0 < \alpha < 1 \) let consider maximal operators
\[ \sigma_{\alpha,*} F := \sup_{n \in \mathbb{N}} |\sigma_n^\alpha F|, \quad \sigma_{\alpha,*}^* F := \sup_{n \in \mathbb{N}} \frac{|\sigma_n^\alpha F|}{\log^{1+\alpha} n}. \]

For the martingale
\[ F = \sum_{n=0}^{\infty} (F_n - F_{n-1}) \]
the conjugate transforms are defined as
\[ \widehat{F(t)} = \sum_{n=0}^{\infty} r_n(t) (F_n - F_{n-1}), \]
where \( t \in G \) is fixed. We note that \( \widehat{F(0)} = F \). As it is well known (see \[26\])
\[ \| \widehat{F(t)} \|_{H_p} = \| F \|_{H_p}, \quad \| F \|_{H_p}^p \sim \int_G \| \widehat{F(t)} \|_p^p \, dt. \]

3. FORMULATION OF MAIN RESULTS

**Theorem 1.** a) Let \( 0 < \alpha < 1 \) and \( f \in H_{1/(1+\alpha)} \). Then there exists absolute constant \( c_\alpha \), depending only on \( \alpha \), such that
\[ \| \sigma_{\alpha,*}^* F \|_{H_{1/(1+\alpha)}} \leq c_\alpha \| F \|_{H_{1/(1+\alpha)}}. \]

b) Let \( 0 < \alpha < 1 \) and \( \varphi : \mathbb{N}_+ \to [1, \infty) \) be a non-decreasing function satisfying the condition
\[ \lim_{n \to \infty} \frac{\log^{1+\alpha} n}{\varphi(n)} = +\infty, \]
then there exists a martingale \( f \in H_{1/(1+\alpha)}(G) \), such that
\[ \sup_{n \in \mathbb{N}} \left\| \sigma_n^\alpha f \right\|_{\varphi(n)^{1/(1+\alpha)}} = \infty. \]

**Theorem 2.** Let \( 0 < \alpha < 1 \) and \( f \in H_{1/(1+\alpha)} \). Then there exists an absolute constant \( c_\alpha \), depending only on \( \alpha \), such that
\[ \frac{1}{\log n} \sum_{m=1}^{n} \left\| \sigma_m^\alpha F \right\|_{H_{1/(1+\alpha)}^{1/(1+\alpha)}}^m \leq c_\alpha \| F \|_{H_{1/(1+\alpha)}^{1/(1+\alpha)}}. \]
4. AUXILIARY PROPOSITIONS

Lemma 1. [26] Suppose that an operator $T$ is $\sigma$-sub-linear and for some $0 < p \leq 1$

$$\int |Ta|^p \, d\mu \leq c_p < \infty,$$

for every $p$-atom $a$, where $I$ denotes the support of the atom. If $T$ is bounded from $L_\infty$ to $L_\infty$, then

$$\|Tf\|_{L_p(G)} \leq c_p \|f\|_{H_p(G)}.$$

Lemma 2. [6] Let $0 < \alpha < 1$. Then

$$|K_n^\alpha| \leq \frac{c_\alpha}{A_{n-1}^\alpha} \left\{ \sum_{j=0}^{\lfloor n \rfloor} 2^{ja} K_{2j} \right\},$$

where $K_n$ and $K_n^\alpha$ are kernels of Fejér and Cesáro means, respectively.

Lemma 3. Let $0 < \alpha < 1$ and $n > 2M$. Then

$$\int_{I_M} |K_n^\alpha (x + t)| \, d\mu (t) \leq \frac{c_\alpha 2^{\alpha l + k}}{n^\alpha 2^M}, \text{ for } x \in I_{l+1} (e_k + e_l),$$

$(k = 0, \ldots, M - 2, \ l = k + 1, \ldots, M - 1)$ and

$$\int_{I_M} |K_n^\alpha (x + t)| \, d\mu (t) \leq \frac{c_\alpha}{2^M}, \text{ for } x \in I_M (e_k), (k = 0, \ldots, M - 1).$$

Proof. Let $x \in I_{l+1} (e_k + e_l)$. Then applying (7) we have

$$K_{2A} (x) = 0, \text{ when } A > l.$$

Suppose that $k < A \leq l$. Using (7) we get

$$|K_{2A} (x)| \leq c2^k.$$

Let $A \leq k < l$. Then

$$|K_{2A} (x)| = |K_{2A} (0)| = \frac{2^A + 1}{2} \leq c2^k.$$

If we apply Lemma 2 we conclude that

$$A_n^\alpha \ |K_n^\alpha (x)| \leq c_\alpha \sum_{A=0}^{l-1} 2^{\alpha A} |K_{2A} (x)| \leq c_\alpha \sum_{A=0}^{l-1} 2^{\alpha A + k} \leq c2^{\alpha l + k}.$$

Let $x \in I_{l+1} (e_k + e_l)$, for some $0 \leq k < l \leq M - 1$. Since $x + t \in I_{l+1} (e_k + e_l)$, for $t \in I_M$ and $n \geq 2M$ from (15) we obtain

$$\int_{I_M} |K_n^\alpha (x + t)| \, d\mu (t) \leq \frac{c_\alpha 2^{\alpha l + k}}{n^\alpha 2^M}.$$
Let $x \in I_M (e_k)$, $k = 0, \ldots, M - 1$, then applying Lemma 2 and (7) we have

\begin{equation}
\int_{I_M} A_n^\alpha |K_n^\alpha (x + t)| d\mu (t) \leq \sum_{A=0}^{\lfloor n \rfloor} 2^{\alpha A} \int_{I_M} |K_{2^A} (x + t)| d\mu (t).
\end{equation}

(17)

Let $x \in I_M (e_k)$, $k = 0, \ldots, M - 1$, $t \in I_M$ and $x_q \neq t_q$, where $M \leq q \leq \lfloor n \rfloor - 1$. Using (7) we get

\begin{equation}
\int_{I_M} A_n^\alpha |K_n^\alpha (x + t)| d\mu (t) \leq c_\alpha \sum_{A=0}^{q-1} 2^{\alpha A} \int_{I_M} 2^k d\mu (t) \leq \frac{c_\alpha 2^{k+\alpha q}}{2^M}.
\end{equation}

(18)

Hence

\begin{equation}
\int_{I_M} |K_n^\alpha (x + t)| d\mu (t) \leq \frac{c_\alpha 2^{k+\alpha q}}{n^\alpha 2^M} \leq c_\alpha 2^{k-M}.
\end{equation}

(19)

Let $x \in I_M (e_k)$, $k = 0, \ldots, M - 1$, $t \in I_M$ and $x_M = t_M, \ldots, x_{\lfloor n \rfloor - 1} = t_{\lfloor n \rfloor - 1}$. Applying (7) we have

\begin{equation}
\int_{I_M} |K_n^\alpha (x + t)| d\mu (t) \leq c_\alpha \sum_{A=0}^{\lfloor n \rfloor - 1} 2^{\alpha A} \int_{I_M} 2^k d\mu (t) \leq c_\alpha 2^{k-M}.
\end{equation}

(19)

Combining (16), (18) and (19) we complete the proof of Lemma 3.

5. PROOF OF THE THEOREMS

Proof of Theorem 1. By Lemma 1 and (9) the proof of first part of theorem 1 will be complete, if we show that

\begin{equation}
\int_{I_M} \left| \sigma_n^a * F(x) \right|^{1/(1+\alpha)} d\mu (x) < \infty,
\end{equation}

for every $1/(1+\alpha)$-atom $a$. We may assume that $a$ be an arbitrary $1/(1+\alpha)$-atom with support $I$, $\mu (I) = 2^{-M}$ and $I = I_M$. It is easy to see that $\sigma_n^a (a) = 0$, when $n \leq 2^M$. Therefore we can suppose that $n > 2^M$.

Let $x \in I_M$. Since $\sigma_n^a$ is bounded from $L_\infty$ to $L_\infty$ (the boundedness follows from (9)) and $\|a\|_\infty \leq c_2 M/(1+\alpha)$ we obtain

\begin{align*}
|\sigma_n^a a (x)| &\leq \int_{I_M} |a (t)| |K_n^\alpha (x + t)| d\mu (t) \leq \|a (x)\|_\infty \int_{I_M} |K_n^\alpha (x + t)| d\mu (t) \\
&\leq c_\alpha 2^{M(1+\alpha)} \int_{I_M} |K_n^\alpha (x + t)| d\mu (t).
\end{align*}

Let $x \in I_{l+1} (e_k + e_l)$, $0 \leq k < l < M$. From Lemma 3 we get

\begin{equation}
|\sigma_n^a a (x)| \leq \frac{c_\alpha 2^{(a l+k)2\alpha M}}{n^\alpha}.
\end{equation}

(20)
Let \( x \in I_M(e_k), 0 \leq k < M \). From Lemma 3 we have

(21) \[ |\sigma_n^\alpha(x)| \leq c_\alpha 2^{\alpha M + k}. \]

Combining (5) and (20-21) we obtain

\[
\mathbf{I}_M \left[ \left( \sum_{l=k+1}^{M} \int_{I_{l+1}(e_k+e_l)} \sup_{\log^{1/\alpha} n > 2^m} |\sigma_n^\alpha(x)|^{1/(1+\alpha)} \right) \right] d\mu(x)
\]

\[
\leq \frac{1}{M} \sum_{k=0}^{M-1} \sum_{l=k+1}^{M} \int_{I_{l+1}(e_k+e_l)} \sup_{\log^{1/\alpha} n > 2^m} |\sigma_n^\alpha(x)|^{1/(1+\alpha)} d\mu(x)
\]

\[
\leq \frac{c_\alpha}{M} \sum_{k=0}^{M-1} \sum_{l=k+1}^{M} 2^{(ak+l)/(1+\alpha)} 2^{\alpha M/(1+\alpha) 2^{k/(1+\alpha)}} + \frac{c_\alpha}{M} \sum_{k=0}^{M-1} 2^{\alpha M/(1+\alpha) 2^{k/(1+\alpha)}}
\]

\[
\leq \frac{c_\alpha}{M} \sum_{k=0}^{M-1} \sum_{l=k+1}^{M} 2^{(ak+l)/(1+\alpha)} 2^{\alpha M/(1+\alpha) 2^{k/(1+\alpha)}} + \frac{c_\alpha}{M} \sum_{k=0}^{M-1} 2^{\alpha M/(1+\alpha) 2^{k/(1+\alpha)}} \leq c_\alpha < \infty.
\]

Now, we prove second part of Theorem 1. Let \( \{\lambda_k, k \in \mathbb{N}_+\} \) be an increasing sequence of positive integers such that

\[
\lim_{k \to \infty} \frac{\log^{1+\alpha} (\lambda_k)}{\varphi(\lambda_k)} = \infty.
\]

It is easy to show that for every \( \lambda_k \) there exists a positive integer \( \{n_k, k \in \mathbb{N}_+\} \subseteq \{\lambda_k, k \in \mathbb{N}_+\} \) such that

\[
\lim_{k \to \infty} \frac{n_k^{1+\alpha}}{\varphi(2^{2n_k+1})} = \infty.
\]

Let

\[
f_{n_k} = D_{2^{2n_k+1}} - D_{2^{2n_k}}.
\]

It is evident

\[
\hat{f}_{n_k}(i) = \begin{cases} 
1, & \text{if } i = 2^{2n_k}, \ldots, 2^{2n_k+1} - 1, \\
0, & \text{otherwise}.
\end{cases}
\]
Then we can write

\begin{equation}
S_i f_{n_k} = \begin{cases} 
D_i - D_{2^{2n_k}}, & \text{if } i = 2^{2n_k}, \ldots, 2^{2n_k+1} - 1, \\
0, & \text{otherwise.}
\end{cases}
\tag{22}
\end{equation}

From (22) we get

\begin{equation}
\|f_{n_k}\|_{H^{1/(1+\alpha)}} = \left\|f_{n_k}^*\right\|_{H^{1/(1+\alpha)}} = \|D_{2^{2n_k+1}} - D_{2^{2n_k}}\|_{1/(1+\alpha)} \leq c 2^{-2\alpha n_k}.
\tag{23}
\end{equation}

Let \( q^s_{n_k} = 2^{2n_k} + 2^{2s} \), \( s = 0, \ldots, n_k - 1 \). By (22) we can write:

\begin{equation}
\frac{\left|\sigma^\alpha q^s_{n_k} f_{n_k}\right|}{\varphi (q^s_{n_k})} = \frac{1}{\varphi (q^s_{n_k})} \left| A_{q^s_{n_k}}^\alpha \sum_{j=2^{2n_k+1}}^{q^s_{n_k}} A_{q^s_{n_k}}^{\alpha-1} (D_j - D_{2^{2n_k}}) \right| = \frac{1}{\varphi (q^s_{n_k})} \left| A_{q^s_{n_k}}^{\alpha-1} \sum_{j=1}^{2^{2s}} (D_{j+2^{2n_k}} - D_{2^{2n_k}}) \right|.
\tag{24}
\end{equation}

Since

\begin{equation}
D_{j+2^{2n_k}} - D_{2^{2n_k}} = w_{2^{2n_k}} D_j, \quad j = 1, 2, \ldots, 2^{2n_k} - 1,
\end{equation}

we obtain

\begin{equation}
\frac{\left|\sigma^\alpha q^s_{n_k} f_{n_k}\right|}{\varphi (q^s_{n_k})} \geq \frac{1}{\varphi (q^s_{n_k})} \left| \sum_{j=0}^{2^{2s}} A_{2^{2s-j}}^{\alpha-1} D_j \right|.
\tag{26}
\end{equation}

Let \( x \in I_{2s} \setminus I_{2s+1} \). It is easy to show that

\begin{equation}
\frac{\left|\sigma^\alpha q^s_{n_k} f_{n_k} (x)\right|}{\varphi (q^s_{n_k})} \geq \frac{A_{2^{2s-1}}^{\alpha-1}}{\varphi (q^s_{n_k})} \sum_{j=0}^{2^{2s}} \frac{c 2^{4s} A_{2^{2s-1}}^{\alpha-1}}{\varphi (q^s_{n_k})} \geq \frac{c 2^{2s(1+\alpha)}}{\varphi (2^{2n_k})}. \tag{27}
\end{equation}

Using (27) we have

\[
\int_{G^s} \left|\sigma^\alpha q^s_{n_k} f_{n_k} (x)\right|^{1/(1+\alpha)} \, d\mu (x) \geq c_{\alpha} \sum_{s=1}^{n_k-1} \int_{I_{2s} \setminus I_{2s+1}} \left| \frac{\sigma^\alpha q^s_{n_k} f_{n_k} (x)}{\varphi (q^s_{n_k})} \right|^{1/(1+\alpha)} \, d\mu (x) \geq c_{\alpha} \frac{2^{2s}}{(2^{2n_k+1})^{1/(1+\alpha)}}.
\]

\[
\geq c_{\alpha} \frac{2^{2s}}{(2^{2n_k+1})^{1/(1+\alpha)}} \geq c_{\alpha} n_k.
\]
From (23) we have
\[
\frac{\left(\int_{G\alpha} |\sigma_{n}^{\alpha} f_{n}^{1/(1+\alpha)} \right)^{1+\alpha}}{\|f_{n}\|_{H_{1/(1+\alpha)}}} \geq \frac{c_{\alpha} n^{1+\alpha}}{\varphi(2^{n_{k}+1})} \rightarrow \infty, \text{ when } k \rightarrow \infty.
\]

Theorem 1 is proved.

**Proof of Theorem 2.** Suppose that
\[
\frac{1}{\log n} \sum_{m=1}^{n} \left\|\sigma_{m}^{\alpha} F_{m}\right\|_{H_{1/(1+\alpha)}}^{1/(1+\alpha)} \leq c_{\alpha} \left\|F\right\|_{H_{1/(1+\alpha)}}^{1/(1+\alpha)}.
\]

Using (12) we have
\[
1 \frac{1}{\log n} \sum_{m=1}^{n} \left\|\sigma_{m}^{\alpha} F_{m}\right\|_{H_{1/(1+\alpha)}}^{1/(1+\alpha)} = \frac{1}{\log n} \sum_{m=1}^{n} \int_{G} \left\|\sigma_{m}^{\alpha} F(t)\right\|_{H_{1/(1+\alpha)}}^{1/(1+\alpha)} dt.
\]

We obtain
\[
= \frac{1}{\log n} \sum_{m=1}^{n} \int_{G} \sigma_{m}^{\alpha} F(t) dt \leq \frac{1}{\log n} \sum_{m=1}^{n} \sigma_{m}^{\alpha} F(t) dt \sim c_{\alpha} \int_{G} F^{1/(1+\alpha)} dt = c_{\alpha} \left\|F\right\|_{H_{1/(1+\alpha)}}.
\]

By Theorem W and (28) the proof of theorem 2 will be complete, if we show that
\[
\frac{1}{\log n} \sum_{m=1}^{n} \left\|\sigma_{m}^{\alpha} a\right\|_{H_{1/(1+\alpha)}}^{1/(1+\alpha)} \leq c_{\alpha} < \infty,
\]
for every 1/ (1 + \alpha)-atom \(a\). We may assume that \(a\) be an arbitrary 1/ (1 + \alpha)-atom with support \(I\), \(\mu(I) = 2^{-M}\) and \(I = I_{M}\). It is easy to see that \(\sigma_{n}(a) = 0\), when \(n \leq 2^{M}\). Therefore we can suppose that \(n > 2^{M}\).

Let \(x \in I_{M}\). Since \(\sigma_{n}\) is bounded from \(L_{\infty}\) to \(L_{\infty}\) (the boundedness follows from (29)) and \(\|a\|_{\infty} \leq c_{2^{M/(1+\alpha)}}\) we obtain
\[
\int_{I_{M}} |\sigma_{m}^{\alpha} a(x)|^{1/(1+\alpha)} d\mu \leq \|a(x)\|_{\infty}^{1/(1+\alpha)} / 2^{M} \leq c_{\alpha} < \infty.
\]

Hence
\[
\frac{1}{\log n} \sum_{m=1}^{n} \int_{I_{M}} |\sigma_{m}^{\alpha} a(x)|^{1/(1+\alpha)} d\mu \leq \frac{c_{\alpha}}{\log n} \sum_{m=1}^{n} \frac{1}{m} \leq c_{\alpha} < \infty.
\]

Combining (5) and (20,21) we obtain
\[ \frac{1}{\log n} \sum_{m=2^M+1}^{n} \int_{I_m} \left| \sigma_m^\alpha a(x) \right|^{1/(1+\alpha)} d\mu(x) \]\n\[ = \frac{1}{\log n} \sum_{m=2^M+1}^{n} \sum_{l=k+1}^{M-1} \int_{I_{l+1}(e_k+e_l)} \left| \sigma_m^\alpha a(x) \right|^{1/(1+\alpha)} d\mu(x) \]
\[ + \frac{1}{\log n} \sum_{m=2^M+1}^{n} \sum_{k=0}^{M-1} \int_{I_{l+1}(e_k)} \left| \sigma_m^\alpha a(x) \right|^{1/(1+\alpha)} d\mu(x) \]
\[ \leq \frac{1}{\log n} \left( \sum_{m=2^M+1}^{n} \frac{c_\alpha 2^{\alpha M/(1+\alpha)}}{m^{\alpha/(1+\alpha)+1}} + \sum_{m=2^M+1}^{n} \frac{c_\alpha}{m} \right) < c_\alpha < \infty. \]
which completes the proof of Theorem 2.

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