Abstract. Koopman theory studies dynamical systems in terms of operator theoretic properties of the Perron-Frobenius and Koopman operators \( P \) and \( U \), respectively. This paper derives rates of convergence for estimates of these operators, corresponding to generally nonlinear dynamical systems, under a variety of situations. We also derive convergence rates for some probability measures associated with these operators, as well as for specific data-driven algorithms constructed from them. This paper introduces a suitably general class of priors, which describes the information available for constructing approximations, one that facilitates the development of error estimates in many applications of interest. These priors are defined in terms of the action of \( P \) or \( U \) on certain linear approximation spaces. For cases where it is feasible to obtain eigenfunctions of either the Perron-Frobenius or Koopman operator, priors are defined in terms of the spectral approximation space \( A^r(H) := A^{2r}(T(H)) \subset H \) with \( r > 0 \) the approximation rate, \( T \) a given self-adjoint and compact operator on \( H \), \( \lambda := \lambda(T) \) the eigenvalues of \( T \), and \( H \) a Hilbert space of functions over the domain \( \Omega \subseteq \mathbb{R}^d \). More generally, we utilize priors expressed in terms of the Banach spaces of linear approximation \( A^r(X) := A^{r}(X, \{A_j\}_{j \in \mathbb{N}_0}) \subset X \) with \( X \) a Banach space of functions on the domain \( \Omega \), \( r > 0 \) a measure of the approximation rate (or smoothness), \( \{A_j\}_{j \in \mathbb{N}_0} \) a sequence of approximant spaces, and \( 1 \leq q \leq \infty \). The most common cases studied in the paper choose the Hilbert space \( H \) to be \( U := L^2(\mu) \) with \( \mu \) a measure or \( V \subset U \) a reproducing kernel Hilbert space (RKHS). This paper characterizes the rates of convergence of approximations of \( P \) or \( U \) that are generated by finite dimensional bases of wavelets, multiwavelets, and eigenfunctions, as well as approaches that use samples of the input and output of the system in conjunction with these bases. Since the wavelets and multiwavelets are selected to reproduce piecewise polynomials of a certain order, the results of this analysis also can be used to understand the best attainable approximation rates that are achievable by common spline or finite element approximations in (Petrov-)Galerkin approximations of the Frobenius-Perron or Koopman operators. When the estimates of the operators are generated by samples, it is shown that the error in approximation of the Perron-Frobenius or Koopman operators can be decomposed into two parts, the approximation and sample errors. This result emphasizes that sample-based estimates of Perron-Frobenius and Koopman operators are subject to the well-known trade-off between the bias and variance that contribute to the error, a balance that also features in nonlinear regression and statistical learning theory.

Key words. Dynamical Systems, Koopman Theory, Approximation Spaces, Wavelets, Distributed Learning Theory.

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1. Introduction. Data-driven approaches for the study of dynamical systems have flourished over the past decade. Applications using such methodologies have arisen in several fields of science and engineering, spanning biology [66], neurosciences [6], chemistry [32], solid mechanics [76], and fluid mechanics [24, 42]. The underlying philosophy of building accurate, yet simple, models that predict the output of a dynamical system based on observations from experiments has been a central theme of science and mathematics. While there is an ongoing debate about the fidelity and robustness of such data-driven models as compared to those derived from the first principles of physics, it undeniable that data-driven approaches have been demonstrated to be highly efficient and effective. This success is due in part to advances in computational technologies, especially with respect to data-driven modeling.

The theory underlying many of these data-driven algorithms is referred to as Koopman theory, and has emerged as an important discipline in the study of dynamical systems. The approach studies dynamical systems via an operator theoretic
framework, and it dates back to the papers by Koopman in [36, 37]. It has also found use as a theoretical tool for the study of Markov chains since the 1960’s, see [47] for an overall account of the operators induced by transition probability kernels. The theoretic background of Koopman theory can be found in treatises on operator or ergodic theory such as in [25] or in references on Markov semigroups [45]. Most recently, it has been popularized for data-driven analysis of high dimensional dynamical systems in the seminal work of Mezic in [48, 51, 49, 67].

This paper examines the approximation of the pair of dual or adjoint Perron-Frobenius and Koopman operators $P$ and $U$, respectively, in the context of the study of dynamical systems. A large collection of papers over the past few years have studied the use of Perron-Frobenius and Koopman operators to analyze evolutions in discrete and continuous time, [40], [26], and [13]. Just over the past two years, studies such as [39, 27, 34, 35, 33, 50], and the references cited therein, provide careful analyses that establish the convergence of approximations in many situations. Results are stated for discrete or continuous systems, as well as deterministic or stochastic flows. Both semiflows and flows are studied, and convergence in various norms is examined. As is natural, there is a trade-off between the strength and the generality of the conclusions regarding convergence of approximations in these many references. Still, it is fair to say that while convergence is often studied and proven, the rates of convergence are rarely studied in these references per se.

We limit the study of the Perron-Frobenius and Koopman operators in this paper to those that are easily associated to certain discrete, deterministic flows or stochastic flows generated by Markov chains. While many of the results in this paper can also be used as a foundation for the study of continuous flows, we leave this topic for a future study. We discuss in more detail in Section 3.3 how the form of the operators vary depending on whether they are being used to study the dynamics of deterministic or stochastic flows. For now, it suffices to note that we will focus primarily, but not exclusively, on two particular classes of applications. In the first class of examples, the Perron-Frobenius operator takes the form

\[(1.1) \quad (Pf)(x) := \int p(y, x) f(y) \mu(dy)\]

for $x \in \Omega$, $f \in L^2_\mu(\Omega)$, the kernel $p : \Omega \times \Omega \to \mathbb{R}$, and a measure $\mu$ on the domain $\Omega \subseteq \mathbb{R}^d$. This form of the operator $P$ arises in the study of Markov chains that have a transition probability density $p(y, x)$ so that the transition probability kernel is given by $P(dy|x) := p(y, x) \mu(dy)$. Here the Koopman operator is the adjoint $U = P^*$ and is induced by the adjoint kernel relative to the Hilbert space $L^2_\mu(\Omega)$.

A second class of examples are related to studies of the Koopman operator

\[(1.2) \quad (Uf)(x) = (f \circ w)(x)\]

for $x \in \Omega$ and a mapping $w : \Omega \to \Omega$. This operator arises in connection to discrete, deterministic flows in Equation 1.3 and also for some stochastic flows determined by a Markov chain as in Equation 1.4. In these cases the Perron-Frobenius and Koopman operators are transposes of one another, $P^t = U$, with respect to the dual pairing $\langle \cdot, \cdot \rangle_{X^* \times X}$ for a Banach space $X$. In one setting the transpose is defined in terms of the duality pairing between the continuous functions $C(\Omega)$ and its topological dual space $C^*(\Omega)$, a space of signed measures. This case is quite familiar in the study of Markov chains. [47] This is not the only common choice of $X$. It is also possible to define these operators as dual with respect to the pairing $\langle \cdot, \cdot \rangle_{L^\infty_\mu \times L^1_\mu}$ as in the popular treatise [44], or as adjoints with respect to the inner product on $L^2_\mu(\Omega)$.\]
The limitation of the examples to these two general types enables a cohesive and reasonably general theory of approximations of Perron-Frobenius and Koopman operators. Still, the details of the analysis of these examples remains complex owing the variety of choices for the domain Ω, the density p(·, ·), the mapping w : Ω → Ω, and the measure µ. In addition, the problem of approximating the operators P, U is made all the more difficult because the exact form of the kernel p(·, ·), measure µ, mapping w, or even the domain Ω may be uncertain or unknown. It may be the case that the kernel p is known, but the domain Ω ⊆ \mathbb{R}^d that supports the dynamics is unknown. Or, the domain Ω may be known, but the form of the measure µ over Ω is uncertain. It might be the case that both µ and Ω are unknown. It is clear that there are a large number of approximation problems that can arise depending on what combination of these constituents are known or unknown. We do not systematically consider all combinations of such problems in this paper, but focus on only a few of the key scenarios. These typical cases then can inform the study of problems subject to other types of uncertainty. We briefly discuss some of the specific cases we study in this paper in the next section.

1.1. Types of Approximation Problems. This paper introduces a novel approach for the approximation of a particular class of Perron-Frobenius and Koopman operators whose regularity is characterized by their action on different types of approximation spaces. The most general of these is the Banach space A^{r,q}(X) := A^{r,q}(X, \{A_j\}_{j \in \mathbb{N}_0}) contained in a Banach space X of functions on Ω. Here the parameter r > 0 measures the rate that certain linear approximations converge in this space, \{A_j\}_{j \in \mathbb{N}_0} are a family of approximant subspaces each having dimensions n_j := \#(A_j) from which estimates are constructed, and 1 ≤ q ≤ ∞. These spaces have come to assume a central role in understanding many recent advances in the theoretical foundations of signal and image processing, denoising, compressed sensing, optimal recovery, nonlinear regression, and harmonic analysis. [18] We also construct estimates using the spectral approximation spaces A^{r,q}(H) over a Hilbert space H, which can be shown to be a special case of the A^{r,q}(X) under some circumstances. We show that by combining approximation space theory and probabilistic error estimates based on confidence functions, an overall convergence rate is obtained for approximation of P, U, and some of their associated data-driven algorithms for the study of dynamical systems. We describe below several specific cases treated in this paper.

The case when p, w, Ω, µ are known: In this case we want to define algorithms and create finite dimensional estimates \mathcal{U}_j f or \mathcal{P}_j f that are guaranteed to converge to \mathcal{U} f or \mathcal{P} f, respectively, for all f in a given class of functions. Here the finite dimensional operators \mathcal{U}_j, for instance, are mappings onto some n_j dimensional approximant space A_j contained in the family of functions over which \mathcal{U} is defined. Ideally, we would like to know at what rate r > 0 the estimates \mathcal{P}_j f or \mathcal{U}_j f converge to true values for different classes of functions f. Specifically, we seek conditions that guarantee the error decays such as \| (\mathcal{P} - \mathcal{P}_j) f \|_X \lesssim ao(r, j) for all functions f in a given class. Here ao(r, j) is the approximation order of rate r for estimates constructed from the subspace A_j. Examples of this form with ao(r, j) = 2^{-rj} or ao(r, j) = n_j^{-r} can be found in Theorems 6.2, 6.3, and 7.1. Similar types of convergence rates are also derived in Theorems 5.4 and 5.5 in terms of powers of eigenvalues ao(r, j) = \lambda_j^{-r/2} when the bases used for approximations are eigenfunctions of \mathcal{P} or \mathcal{U}. We also are interested in determining conditions that \mathcal{U}_j, \mathcal{P}_j converge to \mathcal{U}, \mathcal{P} in some suitable operator norm, as stated in Theorem 6.2. In most recent studies of Koopman theory, the approximations are constructed using bases that consist of algebraic polynomials, trigonometric
polynomials, piecewise polynomial spaces such as splines or finite element spaces, as well as eigenfunctions if their calculation is tractable.

This first category of results is important to Koopman theory for a few reasons. As a general rule, we have already observed that it is more common that convergence of approximations is established in many recent studies of Koopman theory [39, 40, 27, 13, 50, 78, 77, 84], but not the rate of convergence. It is also important to note that the case here, which assumes exact knowledge of all the problem data $p, w, \Omega, \mu$, serves as the foundation for treating much more difficult analyses when some problem data are unknown or uncertain. This case is the starting point for the next two cases that feature uncertainty in the problem data.

The case when $p, w, \Omega$ are known, the measure $\mu$ is unknown: Even for the study of deterministic systems, Koopman theory includes aspects of probability theory. The role of a measure, or measures, is central to the approach. It should come as little surprise then that there is an interplay of deterministic and stochastic contributions to the overall error in many of the associated approximation problems. The class of problems when the measure $\mu$ is unknown plays an important theoretical role. It serves to bridge the theoretical gap between the aforementioned case when all problem data $p, w, \Omega, \mu$ are known, and the most complex case studied next when all problem data are unknown or uncertain. It is because the measure is often unknown in applications that algorithms based on samples of the underlying dynamical system are so popular.

Examples of this case are plentiful when we have an accepted model of the discrete evolution, and we seek to understand or characterize the subset or submanifold of the full configuration space over which evolutions are concentrated. One such case arises in developing lower order models from studies of computational fluid dynamics. A fluid flow discrete model arising from a numerical approximation of the Navier-Stokes equations might evolve on a state space having dimension $d \approx O(10^6)$ or $O(10^7)$. So, there does in principle exist an exact model, given by the Navier-Stokes equations. Sufficiently high dimensional grids can be assumed to yield approximations with small, perhaps negligible, error of the large scale flow dynamics. It is not practical to use the direct numerical simulation in many applications that require interactive or near real-time predictions. Such applications can include models of aerodynamic loads for use in air vehicle design or flight control synthesis. In some studies the goal is to gain an understanding of the inherent or underlying mechanics of a particular flow. The determination of regions of recirculation, the identification of coherent structures, or the determination of overall lift and drag trends from the dynamics of shed vortices are but a few of the common examples. Low dimensional proxies are needed for these applications. It is not unusual that low dimensional approximations are generated from data-driven models that evolve on a state space having $d \approx O(10)$ to $O(10^2)$ degrees of freedom. In this application $\mu$ describes the concentration or support of the dynamics on the small subset of $\mathbb{R}^d$, and some low order models can be interpreted in terms of its approximation.

The case when $p, w, \Omega, \mu$ are unknown: In applications it is perhaps of the greatest interest when the only explicit information regarding the system under study is a collection of observations $z := \{(x_i, y_i)\}_{i \leq m} \subset \Omega \times \Omega$ from an experiment that measures the output state $y \in \Omega$ obtained from the input state $x \in \Omega$. We again consider the study of fluid flows for an application. Some fluid flow systems are so complex that it is prohibitively difficult, time-consuming, or just infeasible to construct an accurate numerical simulation of the Navier-Stokes equations without supporting experiments. It is then common that time-indexed experimental measurements of the velocity field
are made using techniques such as particle image velocimetry (PIV), or observations are made of pressure distributions at points on surfaces, say, from pressure sensitive paints. These samples can be the source of discrete approximations of the fluid dynamics. The discrete dynamics between consecutive experimentally observed velocity fields, for instance, can have a dimension that is still large, although perhaps not has large as that in a full resolution simulation of the Navier-Stokes equations. In effect both the measure \( \mu \) describing the concentration of trajectories in the configuration space and the underlying dynamic model may be largely unknown in these cases.

One goal in studying these systems is the generation of data-driven models of the Perron-Frobenius and Koopman operators from the observations \( z = \{(x_i, y_i)\}_{i \leq m} \) of the input-output pairs \((x_i, y_i)\) of the system.

In this situation we want to derive practical estimates \( P_{j,z}f \) or \( U_{j,z}f \) that depend on the samples \( z \) and the approximant subspace \( A_j \) and to prove in what sense these estimates converge to \( Pf \) and \( Uf \). These types of estimators have been proposed and investigated in many of the recent studies [84, 77, 43, 78] and are embodied in popular algorithms such as the Dynamic Mode Decomposition (DMD) method and the Extended Dynamic Mode Decomposition (EDMD). These studies are important in that they make clear the structural relationship between the EDMD algorithm and methods to approximate the Koopman or Frobenius Perron operators, and they establish cases in which convergence is proven. This paper will explore in what sense rates of convergence depend on the number \( m \) of samples and dimension \( n_j \) of the approximant spaces \( A_j \).

1.2. Approach and Philosophy. The primary aim of this paper is to introduce a common theoretical framework for Koopman theory and associated data-driven algorithms that are used in the study of dynamical systems. We focus on determination of the rates of convergence of approximations of the Koopman and Frobenius operators, as well as some classes of measures that arise in the study of associated data-driven algorithms. These results facilitate the study of the rates of convergence of practical data-driven algorithms.

An essential feature of the theory in this paper is the introduction and use of a class of priors that determine the rates of convergence. The set of priors define what information is available about the function, measure, or operator to be approximated [17]. (We are not referring to the notion of priors that arises in Bayesian estimation or stochastic filtering.) In the most general analysis, this paper defines the priors in terms of the linear approximation spaces \( A^{r,q}(X) \) of order \( r > 0 \) with \( 1 \leq q \leq \infty \) that are contained in a Banach space \( X \) of functions defined on a domain \( \Omega \).

We also study the linear spectral approximation spaces \( A^{r,2}(U) \) that are relevant to many approaches in Koopman theory, which can be understood as a special case of the spaces \( A^{r,q}(X) \) in some situations. While we have noted the broad spectrum of advances that have been facilitated by this theory, as far as the authors can tell, the systematic use of such priors has not been pursued in the study of Koopman theory. The intended audience of this paper is primarily the researchers, engineers, scientists, and academics that want to understand or use Koopman theory for the study of dynamical systems. We have worked carefully to try and balance the generality of the approach in this paper, the practicality of the theory for the study of data-driven algorithms, and the intuitive understanding of what membership in an approximation space means, pragmatically speaking. Those who are familiar with the underlying theory of approximation spaces will be well-aware that the methodology introduced here can be generalized substantially.
When they are defined axiomatically, the approximation spaces \( A^{r,q}(X) \) for a (quasi-)Banach space \( X \) can seem quite abstract. A discussion of the theory can be found in the brief paper [58], and a full account of the theory is given in [19]. Reference [18] is a particularly good introduction to the theory and gives a readable, interesting motivation for the approach. Also, Dahmen’s work in [10] is particularly relevant to most of the theory as it is applied in this paper. The approach there utilizes biorthogonal families of bases for the construction of approximation spaces, and the approach here based on orthonormal wavelets or multiwavelets to construct \( A^{r,2}(U) \) is but a specific case of the biorthogonal construction.

Intuition regarding the approximation spaces is perhaps most easily developed considering \( A^{r,2}(H) \) when \( H \) is a Hilbert space of functions over a domain \( \Omega \), the functions \( \{\psi_i\}_{i \in \mathbb{N}} \) are an \( H \)-orthonormal basis for \( H \), and \( A_j \) is the approximant space from which approximations are built. This space can be understood in terms of the generalized Fourier coefficients \( \{(f,\psi_i)_H\}_{i \in \mathbb{N}} \) of \( f \in H \) in the expansion

\[
 f := \sum_{i \in \mathbb{N}} (f,\psi_i)_H \psi_i.
\]

While the precise definition of the approximation space \( A^{r,2}(H) \) in Sections 2, 6, or \( E \) might seem a bit lengthy, it is important to keep in mind a one simple fact. If \( f \in A^{r,2}(H) \), then the approximation \( \Pi_j f \) that is obtained by truncating the orthonormal decomposition \( \Pi_j f = \sum_{i < j} (f,\psi_i)_H \psi_i \), is an example of a linear approximation method. These linear approximation strategies are explained in the context of the more general theory of nonlinear approximation spaces in Appendix E.1 or in references [18, 58]. Linear strategies have an error for \( f \in A^{r,2}(H) \) that decays like

\[
 \|(I - \Pi_j)f\|_H \lesssim n_j^{-r} |f|_{A^{r,2}(H)}.
\]

with \( |\cdot|_{A^{r,2}(H)} \) the seminorm on \( A^{r,2}(H) \), which is defined in Equation 6.3, and \( n_j = \#(A_j) \). In fact a function \( f \in A^{r,2}(H) \) if and only if the sequence \( \{(f,\psi_i)_H\}_{i \in \mathbb{N}} \) is square summable, that is, it is contained in \( \ell^2(\mathbb{N}) \). It follows that the greater the approximation rate \( r > 0 \), the faster the generalized Fourier coefficients must converge to zero for a function \( f \in A^{r,2}(H) \).

We emphasize again that approaches we study in this paper are referred to as linear methods of approximation in the references on approximation theory [18]. This should not be confused with the types of dynamical systems that are to be studied using these techniques. The discrete flow, whether deterministic or stochastic, will generally be nonlinear. Our canonical example of a deterministic system evolves according to the recursion

\[
 x_{n+1} = w(x_n),
\]

where the function \( w : \Omega \to \Omega \) is generally nonlinear. This evolution is discussed in more detail following Equation 3.1. We use linear approximation methods to estimate the Perron-Frobenius or Koopman operators generated by this discrete, nonlinear, deterministic dynamics. An analogous observation is true for our examples of discrete

\[ ^{1}\text{In this particular exemplary case } n_j := \#(A_j) = O(j), \text{ but in many examples } n_j \text{ is some other function of } j. \text{ If we are approximating functions } f : \Omega \subseteq \mathbb{R}^d \to \mathbb{R}^d \text{ using a tensor product wavelet or dyadic spline basis for each coordinate direction, one often obtains an expression such as } n_j \approx O(d \cdot 2^d), \text{ for instance.} \]
stochastic dynamics. Example 18 considers an iterated function system, or IFS, \cite{1, 44}. It is the special case of the discrete, stochastic evolution

\begin{equation}
   x_{n+1} = w(x_n, \lambda_n)
\end{equation}

where \( w : \Omega \times \Lambda \to \Omega \) is generally a nonlinear function and \( \{\lambda_n\}_{n \in \mathbb{N}} \) is a sequence of random values in a symbol space \( \Lambda \). We describes rates of convergence of linear methods of approximation of the Koopman and Perron-Frobenius operators generated by the above discrete, nonlinear, stochastic dynamics.

Intuition about approximation spaces is also improved by noting that they are in many instances equivalent to other function spaces that may be much more familiar. We only use a few of the simplest of the numerous equivalent characterizations of these spaces \cite{19} in this paper. Again, in this motivating introduction, we mostly restrict consideration to \( A^{r,2}(H) \) for a Hilbert space \( H \). For analysts that study evolutionary partial differential equations, it is frequently the case that studies are carried out in terms of the Sobolev space \( W^r(L^2(\Omega)) \). The Sobolev space \( W^r(L^2(\Omega)) \) contains all functions in the Lebesgue space \( L^2(\Omega) \) that have weak derivatives in \( L^2(\Omega) \) of order less than or equal to \( r \). The Banach space \( C^r(\Omega) \), the set of functions having classical derivatives through order \( r \), is a subset of the corresponding Sobolev space, \( C^r(\Omega) \subset W^r(L^2(\Omega)) \). It will be important in many of our examples that linear approximation spaces are often equivalent to Sobolev spaces, \( A^{r,2}(L^2(\Omega)), \{A_j\}_{j \in \mathbb{N}_0} \) \( \approx \) \( W^r(L^2(\Omega)) \) for a range of \( r \) that depends on the smoothness of the basis for \( A_j \). This is made precise via an argument summarized in Appendix E and in the book \cite{19}. This equivalence is the reason why the index \( r \) is understood heurstically as a measure of smoothness and rate of approximation. A practical implication of this fact is that a bound in the error in terms of the approximation space norm (which may seem rather abstract) implies that the bound holds for the more common or conventional space of \( r \)–time continuously differentiable functions.

Analysts who study evolutionary partial or ordinary differential equations encounter Lipschitz conditions in many theorems that guarantee the existence of unique solutions of initial value problems. The Lipschitz spaces play an important role in this paper, and particularly in the examples, in that they are also often equivalent to approximation spaces. For connecting the approximation spaces to certain Lipschitz spaces, we rely on the generalized Lipschitz space \( \text{Lip}^*(r, L^p(\Omega)) \). This space is not usually encountered in theorems that guarantee solutions to initial value problems, but features prominently in approximation theory. As summarized in Section 2.1, the generalized Lipschitz spaces include the spaces \( \text{Lip}(r, C(\Omega)) \) and \( \text{Lip}(r, L^p(\Omega)) \) as special cases for some ranges of smoothness \( r \). In the former, functions satisfy the pointwise Lipschitz inequality that is so common in existence theorems for ODEs. In the latter the Lipschitz condition in defined in terms of an integral in \( L^p(\Omega) \). When \( 0 < r < 1 \), for instance, we have \( \text{Lip}^*(r, L^p(\Omega)) \approx \text{Lip}(r, L^p(\Omega)) \) and \( \text{Lip}(r, L^\infty(\Omega)) \approx \text{Lip}(r, C(\Omega)) \). We have found that the two special cases of Lipschitz spaces make for simpler computations such as in the Example 15, while the generalized Lipschitz spaces are particularly amenable to determine their relationship to linear approximation spaces over a wider range of smoothness \( r \) via Theorem E.1 in Appendix E.2.

One potential limitation of the approach in this paper might be that our choice to use orthonormal bases for the construction of the approximations is too restrictive. In fact, in this paper we mostly limit our analysis to approximations associated with projection onto families of orthonormal wavelets, multiwavelets, and eigenfunctions. By far, most studies of convergence of approximations of Koopman and Perron-Frobenius...
operators use finite dimensional spaces of polynomials, spaces of piecewise polynomials, or eigenfunctions. While the orthogonal eigenfunctions of compact self-adjoint operators do fit nicely within the theory outlined in this paper, the choice of higher order piecewise polynomial approximations in the references are seldom constructed from orthonormal projections. It is much more common, for example, that piecewise polynomial approximations are based on quadratures, an interpolation formula, or on a (Petrov-)Galerkin approximation. There are at least two points we want to make regarding this issue.

First, the general theory of approximation spaces is not cast in terms of orthogonal projections onto finite dimensional subspaces spanned by orthonormal bases. Instead the general theory is presented in many equivalent forms in the literature. A pragmatic general theory can be expressed in terms of quasi-interpolant projections on spline spaces [19] or biorthogonal bases of splines or wavelets [10]. Not too surprisingly, the general theory is more complex to describe and less intuitive compared to the case in this paper. In [10, 12, 11] it requires an introduction of a primal collection of finite dimensional approximation spaces, as well as an associated family of dual approximation spaces. It is precisely this complexity we seek to avoid in this overview of how these methods can be applied to Koopman theory. In short, we have elected to limit the discussion to such orthonormal bases primarily for pedagogic reasons. We have chosen to stick to the theory that is simpler to state, to understand it in an intuitive sense, and to note where the more general case could be brought to bear when the result is fairly direct. Such is the case when we discuss approximation of measures in Section 7.4. In this section we include a discussion of approximations of some measures that are understood in terms of duality to the more general class of linear approximation spaces $A^{r,q}(X)$ for a Banach space $X$. Pragmatic implementations are certainly possible in a more general context. The multiscale analysis defined on a variety of manifolds such as in [12, 11] could play an important role in Koopman theory, for those who are not faint of heart.

Second, we have elected to employ orthonormal wavelets and multiwavelets that exactly reproduce some common families of piecewise polynomials, finite element spaces, or splines in all of our examples. Such choices do generate practical algorithms, of arbitrarily high approximation order, depending on the smoothness of the basis. It should be observed that the approximation rates in spaces such as $A^{r,2}(L^2_\mu(\Omega))$ can be used to obtain the best rates achievable by linear approximation methods based on piecewise polynomial, finite element, or spline functions that are contained in the span of the selected wavelets or multiwavelets. In fact, when the measure $\mu$ is just Lebesgue measure, there are Daubechies orthonormal wavelets, “Coiflets”, and orthonormal multiwavelets that span a large selection of polynomial, piecewise polynomial, and spline spaces. The approach in this paper therefore gives the best possible convergence rates for linear approximations built from all of these more traditional, and perhaps more familiar, approximant spaces that do not enforce any type of orthonormality among the basis functions. This interpretation of the best possible approximation rates for piecewise polynomial spaces is discussed in more detail in Examples 21 and 22.

2. Overview of Primary Results. We have categorized a number of problems that arise naturally in the study of approximations of the Perron-Frobenius and Koopman operators in Section 1.1. Here we specifically summarize the contributions of this paper to each of the categories when 1) the problem data $p,w,\mu,\Omega$ are known, 2) the measure $\mu$ is unknown but $p,w,\Omega$ are known, and 3) the problem data $p,w,\mu,\Omega$
are unknown or our knowledge of them is uncertain, but we are given a collection of observations of the input-output behavior of the dynamical system. Roughly speaking, the presentation in this paper proceeds from rather general results to those that are most specific to applications of Koopman theory. In this sense the paper progresses from a well-known framework to investigate its implications for Koopman theory.

The simplest case is presented first, when the problem data is known. This situation must be studied to address more difficult problems of interest to dynamical systems and Koopman theory. We follow this discussion with a summary of the last two categories, where some of the problem data is uncertain or unknown.

**The case when \( p, w, \mu, \Omega \) are known.** When the problem data \( p, w, \mu, \Omega \) are known, this paper strengthens several existing results that study the convergence of approximations of the Koopman or Perron-Frobenius operators. For the most part these results are direct applications of the theory of linear approximation spaces to Koopman theory. Perhaps the most unique or novel insight in this section are the results that tie the approximation of the dual (or adjoint) Perron-Frobenius and Koopman operators to linear approximation spaces defined in terms of warped wavelets.

This section begins with introduction of the spectral approximation spaces \( A_{\lambda}^{r,2}(U) \) in Equation 5.1 with \( U := L^2_\mu(\Omega) \). The spaces \( A_{\lambda}^{r,2}(U) \) depend on a self-adjoint compact operator \( T \) that has eigenpairs \( \{ (\lambda_i, u_i) \}_{i \in \mathbb{N}} \) with \( \lambda_i \) the eigenvalue corresponding to the eigenvector \( u_i \). From spectral theory we know that the eigenvalues are nonincreasing, and they can only accumulate at zero. The Koopman or Perron-Frobenius operators are assumed to be contained in a family of operators \( A_{\lambda}^{r,2}(U) \) acting on the Hilbert space \( U \) that are analogous to the spaces \( A_{\lambda}^{r,2}(U) \). The principal results in this context are Theorems 5.4 and 5.5 that construct approximations \( \mathcal{P}_j \) of \( \mathcal{P} \) or through duality \( \mathcal{U}_j \) of \( \mathcal{U} \), from the approximant spaces \( \mathcal{A}_j = \text{span}_{i<j} \{ u_i \} \) that converge at a rate \( O(\lambda_{j/2}) \) whenever \( \mathcal{P} f \in A_{\lambda}^{r,2}(U) \). Essentially, the membership of a function in the space \( A_{\lambda}^{r,2}(U) \) guarantees that its projection error decays like \( O(\lambda_{j/2}) \). This analysis can be used to obtain rates of convergence in numerous approaches that currently only guarantee convergence.

The final result for the spectral approximation spaces rewrites the error bounds by grouping the basis functions into blocks of length \( n_j \). This grouping is important when \( \{ n_j \}_{j \in \mathbb{N}_0} \) is a quasigeometric sequence of integers defined in Section 2.1. If the eigenvalues \( \{ \lambda_{2i} \}_{j \in \mathbb{N}} \) are quasigeometric in the sense that they satisfy

\[
\lambda_{n_j}^{1/2} \approx n_j^{-1}
\]

for such a sequence of integers \( \{ n_j \}_{j \in \mathbb{N}_0} \), then we have the error estimate

\[
|\mathcal{P} - \mathcal{P}_{n_j}|_{A_{\lambda}^{r,2}(U)} \lesssim n_j^{-(s-r)}
\]

for all \( s > r > 0 \). Perhaps the most common choice of the sequence selects \( n_j := 2^{dj} \) for \( U := L^2_\mu(\Omega) \) and \( \Omega \subset \mathbb{R}^d \). This result for the spectral approximation spaces is written this way to emphasize its resemblance to the more general error estimates derived in Section 6. Specifically, we have \( n_j \approx 2^j \) when \( d = 1 \) in many applications using wavelets, multiwavelets, or dyadic splines to define the approximant spaces \( A_j \).

For the most part, this paper estimates the Perron-Frobenius and Koopman operators in a more general framework using the approximation spaces \( A_{r,q}^{r,q}(U) \) for \( r > 0 \) and \( 1 \leq q \leq \infty \) that are contained in the Hilbert space \( U \). These are introduced in Section 6. Approximations are constructed from families of orthonormal bases that
determine the approximant space $A_j$ that has dimension $n_j = \#(A_j)$ for $j \in \mathbb{N}$. Theorem 6.4 shows that the spectral approximation spaces $A_j^{r,q}(U)$ can be viewed as special cases of the approximation spaces $A^{r,q}(U)$ in the important case that the eigenvalues are quasigeometric as in Equation 2.1. The approximations in $A^{r,q}(U)$ are cast in this paper in terms of compactly supported wavelets and multiwavelets that reproduce families of piecewise polynomials and splines. It is important to see that these basis functions are readily available: they have been constructed from first principles in a host of references. Perhaps the most well-known are the Daubechies compactly supported orthonormal wavelets described in [14, 15]. Another well-known family of compactly supported $L^2(\mathbb{R})$–orthonormal wavelets are the “Coiflets” described in [16, 53]. We outline the use of compactly supported orthonormal multiwavelets of [21, 20]. Unlike the Daubechies wavelets these functions are piecewise polynomials. The wavelets and multiwavelets above are not constructed from eigenfunctions, and therefore they avoid the challenges associated with computation of the bases from non-trivial Koopman or Perron-Frobenius operators. Approximations $P_j$ are constructed, and when $Pf \in A^{r,q}(U)$, their error is bounded by $\| (P - P_j)f \|_U \approx O(2^{-jr})$ with $j$ being the resolution level of a uniform dyadic grid over $\Omega \subset \mathbb{R}^d$ composed of cells having side length $2^{-j}$. If $d = 1$, we have $n_j := \#(A_j) \approx 2^j$ when using wavelets or multiwavelet families, which again yields the approximation rate $O(n_j^{-r})$ analogous to that for the spectral spaces with quasigeometric eigenvalues.

An additional theoretical result of this approach follows, since the wavelet and multiwavelet bases we use reproduce certain spline spaces. The study of rates of convergence of approximants constructed from the wavelet and multiwavelets determine upper bounds for the the best possible (linear) convergence rates of some common approaches that are expressed in terms of algebraic polynomials, piecewise algebraic polynomials, finite element spaces, or splines. Some of the (generalized) Galerkin projection methods fall into this category, which are discussed in [34], for instance.

Usually in this paper we employ approximation spaces that are contained in a Hilbert space $U$. When the problem data $\mu, p, \Omega$ and $w$ are known, some results for the construction of approximations of signed measures are possible using the Hilbert spaces $A^{r,q}(U)$ contained in the Hilbert space $U$. A bit more generality can be advantageous when we study approximation of signed and probability measures in Section 7.4. Approximation methods that are more widely applicable are obtained with introduction of the spaces $A^{r,q}(X)$ for a Banach space $X$. These priors enable the derivation of rates of convergence for approximations of signed measures or probability measures in a weak* sense. This is the one section of the paper in which we do not restrict attention to approximations in $A^{r,2}(U)$ with $U$ a Hilbert space. Approximation of signed measures is achieved using a uniformly bounded family of dual operators $\Pi_j$ of the linear projection operators $\hat{\Pi}_j : X \to A_j$ that are onto the approximant spaces $A_j$. When approximation of signed measures is carried out in $A^{r,2}(U)$, for instance, it is shown that then we have

$$
\left\| \langle I - \Pi_j \rangle \nu, f \rangle_{C^\bullet(\Omega) \times C(\Omega)} \right\| \approx O(2^{-jr}) \approx O(n_j^{-r})
$$

for $r > 0$, $d = 1$, $f \in A^{r,2}(U) \subset C(\Omega) \subset U$, and $\nu \in U^* \subset C^\bullet(\Omega)$ with $\Pi_j^* : C^\bullet(\Omega) \subset A_j^* \to U^* \subset C^\bullet(\Omega)$ the transpose of the orthogonal projection operator $\hat{\Pi}_j : U \to A_j$ in Theorem 7.1. By construction, this example yields approximations that are signed measures that converge in $C^\bullet(\Omega)$, but the approximations are not guaranteed to be probability measures even if $\nu$ is. An algorithmic approach based on this result that generates approximations that are probability measures is discussed in Section 7.4.2.
Again, to the best of our knowledge, the approach exploiting priors has not been studied in a systematic way for Koopman theory. Several examples in Section 7 illustrate the approximation of signed and probability measures, and likewise demonstrates how they can be used in the study of Perron-Frobenius or Koopman operators. Applications of this strategy are given in Examples 10, 12, and 18. We show in our analysis that it is possible to base approximations $P_j$, and therefore by duality approximations $U_j$, on approximations $\mu_j$ of the measure $\mu$. We show that in some cases this notion of convergence, which relies on the definition of an approximation space, implies convergence in the bounded Lipschitz metric $d_{BL}$ on probability measures.

The case when $p, w, \Omega$ are known but $\mu$ is unknown. The approaches above enable the determination of rates of convergence for several important problems in Koopman theory, but require full knowledge of the data $\mu, p, w, \Omega$ of the dynamic system under study. Also, if the eigenfunctions are to be used to construct approximations, their calculation must be tractable. If these bases are not known or cannot be computed, the theorems and results above do not give a realizable algorithm to approximate the Perron-Frobenius operator $P$ or Koopman operator $U$. Generally, the popularity of Koopman theory over the past few years can be attributed in large degree to its success in deriving approximations from samples when some or much of the problem data is unknown. In this case it is popular to construct approximations of $P$ or $U$ from samples $z_n := \{(x_n, y_n)\} \in \Omega \times \Omega$ of the input state $x_n$ and output state $y_n$ of the dynamic system. In this paper we study two different scenarios for how observations of the system are collected. In the first, we simply make observations of the state process over time for a fixed initial condition $x_0 \in \Omega$, and we have $\{x_n, y_n\}_{n \leq m}$ for time steps $n = 0, 1, \ldots, n$. In this case the observations are along a sample path starting at $x_0$. There is another important case where observations are not collected over a sample path of the system. We consider the case in which a family of initial conditions $\{x_i\}_{i \leq m}$ are determined either deterministically or stochastically, and each $\{z_i\}_{i \leq m} = \{(x_i, y_i)\}_{i \leq m}$ records the single step response of the system for each selected $x_i$. This way of collecting samples can be thought of as a set of input-output responses for a number of test cases. Here, the samples $z_i$ are indexed by the initial conditions or test case, not time step. Of course, there also are hybrids of the two above realizations of experiments: it is possible that initial conditions are selected randomly according to some measure $\mu$ on $\Omega$ and subsequently observations along the sample path of each are made, and so forth. It remains an exciting area of research to explore rates of convergence of approximations for these various cases, with the problem of fusing approximations obtained over different sample paths being one important subproblem.

As discussed in more detail in Section 3 and Appendix B, the assumptions about the statistics of the samples to a large extent determines how the overall rates of convergence can be derived. It is almost always easier to derive approximation rates when the samples are independent and identically distributed (IID). Such is the case if we randomly choose an initial condition $x_i$ independently according to some fixed probability measure $\mu$ on $\Omega$ and then measure the single step response $y_i$ that is generated by this $x_i$. This means that for the deterministic system the states $z_i := (x_i, y_i)$ are IID according to the probability distribution $\nu(dx, dy) := \delta_{w(x)}(dy)\mu(dx)$. More generally, the canonical discrete dynamical systems such as in Equations 1.3 and 1.4 are all examples of a Markov chain. For any Markov chain that has transition probability kernel $P(dy, x)$, as discussed in Section 3.1, the samples of single step input-output response $z_i := (x_i, y_i)$ are IID according to the probability measure $\nu := P(dy, x)\mu(dx)$. 
Here we want to emphasize that the state process is still a Markov chain, as in Equations 1.3 and 1.4, and it generates successive samples \( \{x_n\}_{n \in \mathbb{N}_0} \) over time that are dependent. It is the collection of single step responses \( \{(x_i, y_i)\}_{i \leq m} \) indexed by initial condition or test case that are IID. In this paper, the study of a particular problem is first carried out under the IID assumption: the samples correspond to a collection of test cases or different selections of initial conditions. Then, based on insights gained from the IID scenario, the case when samples are along a sample path is considered. It is important to note that this latter case seems to be the one studied most frequently in the literature on Koopman theory, while the former is closely related to methods of nonlinear regression and statistical learning theory. [28, 31, 17, 9, 79]

Suppose that the family of samples \( \{z_i\}_{i \leq m} \) indexed by initial condition, as discussed above. We define an approximation \( P_{j,z} \) in Section 2 that depends on the \( m \) samples and approximant space \( A_j \), and we show that the error decays like

\[
\|Pf - P_{j,z}f\|_U \lesssim \lambda_j^{r/2} + \epsilon
\]

when the analysis is carried out in a spectral approximation space such as \( A_j^{r/2}(U) \), or

\[
\|Pf - P_{j,z}f\|_U \lesssim 2^{-rj} + \epsilon
\]

where \( j \) is the level of resolution of the grid used in an approximation from \( A_j^{r/2}(U) \). Again, these estimates reflect the same rate for \( d = 1 \) if it so happens that the eigenvalues \( \lambda_j^{1/2} \approx n_j := 2^j \). In both cases \( \epsilon \) is the error due to stochastic contributions from the samples \( z \) to the error. The error bounds derived here therefore depend on the samples and the selected finite dimensional bases. Theorem 8.1 expresses a novel convergence rate for Koopman theory that illustrates the classical bias versus variance tradeoff for probabilistic error estimates. The rate of convergence is cast in terms of an accuracy confidence function \( AC(\epsilon, j) := AC(\epsilon, j; f, P) \) that describes the measure of the set of “bad samples” where the probabilistic error is large. The error \( \epsilon \) is small except for a set of samples that has exponentially low probability. The size of this set of bad samples is measured by the accuracy confidence function. See Theorem 8.1 for a detailed discussion.

This result is generalized and considers observations collected along the sample path of certain types of Markov chain in Theorem 8.3. In this case, the Markov chains are assumed to be exponentially strongly mixing, an assumption closely related to the ergodicity assumptions of many studies in Koopman theory.

The case when \( p, w, \mu, \Omega \) are unknown. Finally, this paper also derives new estimates of the rate of convergence that can be achieved when the problem data \( p, w, \mu, \Omega \) is uncertain or unknown. Specifically, we use a combination of the above error analyses to derive rates of convergence for one version of the data-driven Extended Dynamic Mode Decomposition (EDMD) algorithm that is used in the study of some discrete dynamic systems. Again, this novel error rate is derived based on the tradeoff between contributions of deterministic errors from estimates in the approximation space \( A_j^{r/2}(U) \) and the probabilistic errors that arise from dependence on the samples \( z \). It assumes that the approximant space \( A_j \) is the span of the characteristic functions \( 1_{\square, j,k} \) of dyadic cubes

\[
\square_{j,k} := \{ x \in \mathbb{R}^d \mid k2^{-j} \leq x_i < (k + 1)2^{-j}, \ i = 1, \ldots, d \}
\]

that define a partition of \( \Omega \). We show in this case that the EDMD algorithm can be understood in terms of certain estimates developed via techniques of empirical risk
minimization (ERM) in distribution-free learning theory. We denote by $\mathcal{U}_{j,z}^{edmd}f$ and $\mathcal{U}_{j,z}^{erm}f$ the estimates generated by the EDMD and ERM approaches when the estimates are constructed from the approximant space $A_j$ and $z$ denotes the dependence on a family of observations $z$. When $A_j$ is the span of piecewise constant functions over a dyadic partition of $\Omega \subset \mathbb{R}$, we have

$$\mathcal{U}_{j,z}^{edmd} \equiv \mathcal{U}_{j,z}^{erm} \Pi_j$$

with $\Pi_j$ the orthonormal projection onto $A_j$. In fact it follows that if $f \in A^{s,2}(U) \subset C(\Omega)$ with $U := L^p_2(\Omega)$ we have

$$\|\mathcal{U}_{j,z}^{edmd}f - \mathcal{U}_{j,z}^{erm}f\|_U \lesssim 2^{-(s-r)j}\|f\|_{A^{s,2}(U)}$$

for $s > r > 0$. Also, if the samples $\{z_i\}_{i \leq m} = \{(x_i,y_i)\}_{i \leq m}$ are a collection of observations along the sample path of a Markov chain, and the function $f \in A_j$, we also show that the expected value over $m$ samples of the $L_2^2(\Omega)$-error satisfies

$$\mathbb{E}_m^{\text{erm}} \left( \|Uf - \mathcal{U}_{j,z}^{edmd}f\|_{L_2^2(\Omega)} \right) \lesssim \left( \frac{\log(e(m))}{e(m)} \right)^{2r/(2r+1)}$$

for certain exponentially strongly mixing Markov chains with $e(m)$ the effective number of samples. In the above equation $\mathbb{P}_m^{\text{erm}}(\xi)$ denotes the probability distribution of the first $m$ steps of the Markov chain as summarized in Section 3.1 or in reference [47]. This new result is expressed when $\Omega \subset \mathbb{R}^d$, $d = 1$ in Theorem 8.4.

2.1. Notation and Basic Definitions. In this paper we denote by $\mathbb{Z}$, $\mathbb{N}$, and $\mathbb{N}_0$, the integers, integers greater than zero, and nonnegative integers, respectively. We write $a \approx b$ if there are two positive constants $c_1, c_2$ such that $c_1a \leq b \leq c_2a$. The notation $a \lesssim b$ implies that there is a constant $c > 0$ such that $a \leq cb$, and the relation $\gtrsim$ is defined similarly. We use $\#(S)$ to denote the cardinality of any set $S$. The Banach spaces of $p$-summable sequences $\ell^p$ have the usual norms $\|a\|_{\ell^p}^p := \sum_j |a_j|^p$ for $1 \leq p < \infty$ and $\|a\|_{\ell^\infty} := \sup_j |a_j|$. A sequence of integers $\{n_j\}_{j \in \mathbb{J}}$ is quasigeometric provided there exists two positive constants $c_1, c_2$ such that $1 < c_1 \leq n_j/n_{j+1} \leq c_2$ for all $j \in \mathbb{J} \subset \mathbb{N}$. We say that a sequence of real numbers $\{\lambda_j\}_{j \in \mathbb{J}}$, such as eigenvalues of an operator, is quasigeometric whenever there exists positive constants $c_1, c_2$ with $1 < c_1 \leq \lambda_j/\lambda_{j+1} \leq c_2$ for all $j \in \mathbb{J}$. A domain $\Omega$ in this paper is usually either $\Omega := \mathbb{R}^d$, or a compact set $\Omega \subset \mathbb{R}^d$. Some constructions of bases for instance are carried out over $\Omega = \mathbb{R}^d$, and subsequently the basis set is modified so that their support is a compact set $\Omega \subset \mathbb{R}^d$. We denote by $L^p_\mu(\Omega)$ the Banach space of functions $f : \Omega \to \mathbb{R}$ that are $p$-integrable with respect to a measure $\mu$ on $\Omega \subset \mathbb{R}^d$. We have $\|f\|_{L^p_\mu(\Omega)} := \int_{\Omega} |f(\xi)|^p \mu(d\xi)$ for $1 \leq p < \infty$ and $\|f\|_{L^\infty_\mu(\Omega)} = \sup \{|f(\xi)| : \mu-a.e. \ \xi \in \Omega\}$. We overload our notation and also denote the vector-valued Lebesgue space $(L^p_\mu(\Omega))^d$ as $L^p_\mu(\Omega)$. Correspondingly, we define

$$\|f\|_{L^p_\mu(\Omega)} := \int_{\Omega} \|f(\xi)\|_{L^p_\mu(\xi)} \mu(d\xi),$$

$$\|f\|_{L^\infty_\mu(\Omega)} := \text{ess sup}_{\mu-a.e. \ \xi \in \Omega} \|f(\xi)\|_{L^d},$$

for $1 \leq p < \infty$ and functions $f : \Omega \to \mathbb{R}^d$. We define $C(\Omega)$ to be the Banach space of all continuous functions on $\Omega$ with $\|f\|_{C(\Omega)} := \sup_{x \in \Omega} \|f(x)\|_{L^d}$. By $UC(\Omega)$ we
denote the subspace of $C(\Omega)$ that consists of uniformly continuous functions endowed with the norm it inherits. If $\Omega$ is compact, these two spaces are identical, of course.

We will also have occasion to use certain spaces of Lipschitz functions in this paper. The most general of these are the generalized Lipschitz spaces $\text{Lip}^\ast(\alpha, L^p(\Omega))$ for $\alpha > 0$ and $1 \leq p \leq \infty$. Define the $r^{th}$ order difference operator

$$\Delta^r_h(f, x) := \sum_{k=0}^{r} \binom{r}{k} (-1)^{r-k} f(x + kh)$$

and the $r^{th}$ modulus of smoothness

$$\omega_r(f, t) := \sup_{0 < h \leq t} \|\Delta^r_h(f, \cdot)\|_{L^p(\Omega)}.$$

The $r^{th}$ modulus of smoothness is well-defined for $f \in L^p(\Omega)$ with $1 \leq p < \infty$, for $f \in UC(\Omega)$ if $p = \infty$ and $\Omega$ is not compact. The seminorm on the generalized Lipschitz space is given by

$$|f|_{\text{Lip}^\ast(\alpha, L^p(\Omega))} := \sup_{t > 0} \left(t^{-\alpha} \omega_r(f, t)\right).$$

While this definition of a Lipschitz space might seem rather abstract, we introduce it here since it is quite useful in relating linear approximation spaces to Lipschitz spaces in Theorem E.1 in Appendix E. Fortunately, for a restricted range of $\alpha$, these spaces reduce to some more familiar definitions.

For instance, we say that a function $f \in C(\Omega)$ satisfies an $r-$Lipschitz inequality if there is a constant $L > 0$ such that

$$\|f(x) - f(y)\|_{\mathbb{R}^d} \leq L \|x - y\|_{\mathbb{R}^d}$$

for all $x, y \in \Omega$. We define the seminorm

$$|f|_{\text{Lip}(C(\Omega))} := \sup_{x, y \in \Omega} \frac{\|f(x) - f(y)\|_{\mathbb{R}^d}}{\|x - y\|_{\mathbb{R}^d}}.$$

Some references simply refer to this space as Lip $r$ [19]. With the norm

$$\|f\|_{\text{Lip}(C(\Omega))} := \|f\|_{C(\Omega)} + |f|_{\text{Lip}(C(\Omega))},$$

the set of functions

$$\text{Lip}(r, C(\Omega)) \subset UC(\Omega) \subset C(\Omega)$$

is a Banach space for $0 < r \leq 1$. When $0 < r < 1$, functions in $\text{Lip}(r, C(\Omega))$ coincide with functions $\text{Lip}^\ast(r, L^\infty(\Omega))$.

We will also have occasion to employ the space of Lipschitz functions contained in $L^p(\Omega)$ for $1 \leq p \leq \infty$. We define the family $\text{Lip}(L, r, L^p(\Omega))$ for a fixed constant $L \geq 0$ as those functions for which

$$\|f(\cdot + h) - f\|_{L^p(\tilde{\Omega})} \leq L \|h\|^r$$

for every $\|h\| > 0$, with the integration above over the set $\tilde{\Omega} := \{x \in \Omega \mid x, x + h \in \Omega\}$. For $0 < r \leq 1$ the Banach space $\text{Lip}(r, L^p(\Omega))$ is given by

$$\text{Lip}(r, L^p(\Omega)) := \bigcup_{L > 0} \{f \in \text{Lip}(L, r, L^p(\Omega))\},$$

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and the seminorm $|f|_{\text{Lip}(r,L^p(\Omega))}$ is the smallest constant $L$ for which the integral Lipschitz inequality holds for $f$. When $0 < r < 1$, we have $\text{Lip}(r,L^p(\Omega)) \approx \text{Lip}^*(r,L^p(\Omega))$. We have restricted the range of $0 < r < 1$ for $\text{Lip}(r,C(\Omega))$ and $\text{Lip}(r,L^p(\Omega))$ simply because for this range they are equivalent to the generalized Lipschitz spaces $\text{Lip}^*(r,\Omega)$, the latter of which are defined for $r > 0$. There are other ways to extend the definitions $\text{Lip}(r,C(\Omega))$ and $\text{Lip}(r,L^p(\Omega))$ for greater values of $r$, but then their relationship to the generalized spaces becomes a delicate matter. See the detailed discussion in [19], or the summary in [64], for further nuances. We have included all three in this paper since we feel that the definition of $\text{Lip}(r,C(\Omega))$ and $\text{Lip}(r,L^p(\Omega))$ are more convenient for calculations in examples, while the generalized Lipschitz spaces $\text{Lip}^*(r,L^p(\Omega))$ are more readily to related to approximation spaces. The latter topic can be found in [19], page 358, Theorem 2.4, or as as discussed in the Appendix in Section E.2.

For any Banach space $X$ we denote its topological dual $X^*$, and the duality pairing $\langle x^*, x \rangle_{X \times X^*} := x^*(x)$ for all $x \in X$, $x^* \in X^*$. Let $B_1, B_2$ be Banach spaces. The dual or transpose operator $L^*$ of a bounded linear operator $L : B_1 \to B_2$ is the unique bounded linear operator $L^* : B_2^* \to B_1^*$ that satisfies $\langle g^*, Lf \rangle_{B_2^* \times B_1} = \langle L^*g^*, f \rangle_{B_1^* \times B_1}$ for all $f \in B_1$ and $g^* \in B_2^*$. If $L : H_1 \to H_2$ is a bounded linear operator acting between the Hilbert spaces $H_1, H_2$, the adjoint operator $L^* : H_2 \to H_1$ is the unique bounded linear operator that satisfies $\langle Lf, g \rangle_{H_2} = \langle f, L^*g \rangle_{H_1}$ for all $f \in H_1$ and $g \in H_2$. The Riesz map $R_H : H^* \to H$ associated with the Hilbert space $H$ is the isometric isomorphism $R_H : u^* \mapsto u$ defined by

$$< h^*, g >_{H^* \times H^*} := (R_H h^*, g)_H := (h, g)_H$$

for all $g \in H$.

For a compact set $\Omega \subset \mathbb{R}^d$, the topological dual space $C^*(\Omega)$ is identified with the finite, regular, countably additive set functions $\text{rca}(\Omega)$, which are also known as the regular signed measures. We denote by $\mathbb{M}^+(\Omega)$ the positive measures, and by $\mathbb{M}^{+,1}(\Omega)$ the probability measures, contained in $\text{rca}(\Omega)$. For any measure $\mu$ on $\Omega$, $\mu^m$ denotes the product measure on $\Omega^m$. The symbol $\mathbb{P}^m_{\{0,1\}}$ is the probability distribution of the first $m$ steps of the discrete stochastic process $\{z_i\}_{i \in \mathbb{N}}$. A brief discussion of discrete stochastic processes, and in particular Markov chains, is given in the Appendix in Section 3.1.

3. Markov Chains, Perron-Frobenius, and Koopman Operators [34], [44]. In this section we define the class of dynamical systems, and their associated Perron-Frobenius and Koopman operators, that are studied in the paper. Overall, the definitions of the dynamical systems and of their associated operators differ widely depending on the reference. The articles [40, 50] define the operators for the study of continuous and discrete deterministic flows. Reference [27] defines Koopman operators on $L^2_\mu(\Omega)$ for periodic approximations of discrete deterministic evolutions. Other recent studies define the operators for deterministic or stochastic flows in [34], and [35] define them as transfer operators having probability density kernels. The popular text [44] introduces various definitions in the event flows evolve in continuous (Chapter 7) or discrete (Chapter 3) time, and for deterministic (Chapters 3-5) or stochastic (Chapter 10) systems. In fact reference [44] further subcategorizes the Perron-Frobenius and Koopman operators into subclasses such as the Foias and Barnesley operators. The latter arises in some examples in this paper when we discuss certain approximations of evolution supported on fractals. Reference [25] gives an in
depth account of the analysis of flows and semiflows based on the Koopman operator \( \mathcal{U} : f \mapsto f \circ w \) for some fixed mapping \( w \).

As we will see shortly, the operators associated with all of the discrete flows above can be understood in terms of the action on either measures or functions of the transition probability kernel of a Markov chain. [47, 65] The traditional technique for inducing a deterministic flow on signed or probability measures from a Markov chain [65] has been known for some time, so we follow this convention from the outset in the paper. We begin by reviewing a few of the basic definitions for the processes we study in Section 3.1, then discuss realizations of observations in Section 3.2, and subsequently define the classes of operators in Section 3.3.

### 3.1. The Class of Discrete Dynamical Systems.

In the most straightforward case studied in this paper a discrete evolution is defined on a configuration space \( \Omega \subset \mathbb{R}^d \) by the recursion

\[
x_{n+1} = w(x_n)
\]

with \( w : \Omega \to \Omega \) a \( \mu \)-measurable map. Here and below the mapping \( w \) will generally be nonlinear. Trajectories or sample paths starting at some initial condition \( x_0 \in \Omega \) of the dynamical system are just the sequence of iterates \( \{x_n\}_{n \in \mathbb{N}_0} := \{w^n(x_0)\}_{n \in \mathbb{N}_0} \).

In many problems, the dynamics governed by Equation 3.1 may not seem realistic enough since actual experiments are subject to noise, or the model might be uncertain, etc. Common modifications of the above deterministic equation yield the stochastic recursions such as

\[
x_{n+1} = w(x_n) + \xi_n,
\]

\[
x_{n+1} = w(x_n, \lambda_n),
\]

with the sequences \( \{\xi_n\}_{n \in \mathbb{N}_0} \) and \( \{\lambda_n\}_{n \in \mathbb{N}_0} \) a collection of independent and identically distributed (IID) random variables taking values in \( \Omega \) and the finite symbol space \( \Lambda \), respectively. In Equation 3.3 above the function \( w : \Omega \times \Lambda \to \Omega \). While dynamical systems governed by Equation 3.2 include many prosaic physical systems, these equations also define some quite abstract dynamical systems. Equation 3.3 is the form of governing equation for stochastic dynamical systems on fractals [1], for instance. We include an analysis of the approximation of Perron-Frobenius and Koopman operators for this system in Example 18. Of course, many other forms of these stochastic equations are also possible.

All three of the above examples are examples of Markov chains, which is the family of dynamical systems we study in this paper for the representation of state evolution. A brief account of the theoretical foundations of Markov chains is given in Appendix 3.1. A detailed study of the theory over general state spaces, which we employ in this paper, is given in [47]. Suppose that \( (\Omega, \Sigma(\Omega)) \) is a measurable space with \( \Sigma(\Omega) \) a sigma-algebra of subsets of \( \Omega \). A Markov chain is a stochastic process that is defined in terms of a transition probability kernel \( \mathbb{P} : \Sigma(\Omega) \times \Omega \to [0, 1] \). The quantity \( \mathbb{P}(A, x) \) is the probability of a transition from the current state \( x \) to the measurable set \( A \in \Sigma(\Omega) \) in the next step of the discrete stochastic process. The transition probability kernel for the deterministic flow in Equation 3.1 is given by \( \mathbb{P}(A, x) \equiv \delta_{w(x)}(A) \) with \( \delta_{w(x)} \) the Dirac measure concentrated at \( w(x) \in \Omega \). The transition probability kernel of the chain in Equation 3.2 is given by \( \mathbb{P}(A, x) = \mu(A - w(x)) \). See [44] for a discussion of Equation 3.3, or for other examples that underly different types of Perron-Frobenius or Koopman operators.
3.2. Realizations of Observations. In view of the above summary, the most general class of the dynamical systems studied in this paper are those for which the states $x_n$ evolve according to Markov chains having transition kernels $\mathbb{P}(dy, x)$. Up until this point we have not concerned ourselves about how we model observations of a particular dynamic system. The stochastic process that represents the observations of the state equation can have quite different statistical properties, depending on the definition or construction of an experiment. Because there are several ways to model how measurements of a dynamical system are realized in experiments, we briefly review models of a few common setups.

Deterministic Input-Output Samples. In one possible scenario, we assume that $m$ input-output samples

$$z = \{(x_i, y_i)\}_{i \leq m} \subset \Omega \times \Omega$$

for the simple deterministic system are generated by fixing collection of initial conditions $\{x_i\}_{i \leq m} \subset \Omega$ and measuring the single step output for each initial condition

$$y_i := w(x_i).$$

We choose the index $i$ to denote that the samples here are indexed by the initial condition or test case. The input-output samples in this scenario are exact single step observations of a noise-free system. We can then ask how the rates of convergence of approximations of Koopman or Frobenius-Perron operators depend on the collection of test cases. The goal here might be to determine rates of convergence in terms of number of test cases and coverage of the test cases over $\Omega$. For compact domains $\Omega$ we can construct nested grids of initial conditions and analyze convergence rates as the mesh parameter of the grids approaches zero.

Independent Input-Output Samples. In a slight modification of the deterministic scenario we assume we have a fixed probability distribution $\mu$ on $\Omega$, the initial conditions $\{x_i\}_{i \leq m}$ are drawn independently according to $\mu$, and the single step outputs $y_i$ are generated exactly according to Equation 3.4. The initial conditions are said to be independent and identically distributed (IID) with respect to the measure $\mu$. The sequence of samples $\{z_i\}_{i \leq m}$ that are generated this way are IID with respect to the probability measure $\nu(dx, dy) := \delta_{w(x)}(dy)\mu(dx)$ for $z := (x, y) \in \Omega \times \Omega$ in the deterministic, noise-free case. We can easily modify this case somewhat to allow for noisy observations of the output state. Again, we suppose that the initial states $\{x_n\}_{n \leq m}$ are drawn independently according to the fixed probability distribution $\mu$. The single step output states $y_i$ are assumed to be generated by a Markov chain having transition kernel $\mathbb{P}(dy, x)$. In this case the single step samples $z := \{(x_i, y_i)\}_{i \leq m}$ are IID on $\Omega \times \Omega$ with distribution $\nu(dx, dy) := \mathbb{P}(dy, x)\mu(dx)$ for $z = (x, y) \in \Omega$. It should be noted that this manner of collecting observations underlies many strategies for constructing approximations in publications on nonlinear regression or statistical learning theory. [17, 31, 9, 79, 70, 69, 68, 28] However, the standing assumption in these approaches is that rates of convergence for the approximation of a typical function is desired, not approximations of an operator such as $\mathcal{U}$ or $\mathcal{P}$. This is a subtle distinction between nonlinear regression, learning theory, and Koopman theory. Some approaches for nonlinear regression, statistical learning theory, or empirical process estimation study processes that are not IID. It is safe to say, however, that these techniques are not as widely applicable nor as mature as the results based on IID samples.
Dependent Input-Output Samples. In the application of Koopman theory, the assumption that samples are IID is sometimes made. However, it is also frequently the case that observations are measured over multiple time steps for a single initial condition \( x_0 \in \Omega \), instead of over just one time step. In other words, the input-output responses \( \{z_n\}_{n \in \mathbb{N}_0} \) are collected along the sample path of the Markov chain that starts at \( x_0 \). This case can arise in ergodic approximations in Koopman theory. \[25\] We use the index \( n \), the same time index as in the recursions above, for the measurements in this case to emphasize that observations are indexed in terms of the time step. In this case the samples \( \{z_n\}_{n \in \mathbb{N}_0} \) constitute a dependent stochastic process. In the noise-free case we have \( \{x_n\}_{n \in \mathbb{N}_0} \) := \( \{w^n(x_0)\}_{n \in \mathbb{N}} \), while for the stochastic case the observations are along a sample path of the Markov chain having transition probability \( \mathbb{P}(dy, x) \).

Of course, it is also possible that hybrid collections of measurements are made that combine aspects of the above realizations of observations. We could choose initial conditions randomly according to some fixed probability distribution, and then measure the response over time along each sample path for a certain number of time steps. To the authors’ knowledge error rates for such methods have not figured prominently in the literature on Koopman theory.

3.3. Perron-Frobenius and Koopman Operators. Koopman and Perron-Frobenius operators \( U \) and \( P \) are defined in terms of, or associated to, specific dynamical systems. They have many uses including understanding the stability properties of a flow, studying the convergence and rates of convergence of flows to equilibria or attracting sets, or constructing predictors of observations for flows. In view of the conventions for Markov chains \[47, 65\], we define the Koopman operator \( U \) and Perron-Frobenius operator \( P \), respectively, in terms of the transition probability \( P(dy, x) \) as

\[(3.5) \quad (P \nu)(dy) := \int_{\Omega} \mathbb{P}(dy, x) \nu(dx),\]

\[(3.6) \quad (Uf)(x) := \int_{\Omega} \mathbb{P}(dy, x) f(y),\]

for a measure \( \nu \) on \( \Omega \) and function \( f : \Omega \to \mathbb{R} \). We take these expressions as the most general form of the definitions for \( U \) and \( P \) in this paper. We say that a probability measure \( \nu \) is invariant for the Markov chain having a transition probability kernel \( \mathbb{P}(dy, x) \) whenever

\[(P \nu)(A) := \int_{\Omega} \mathbb{P}(A, x) \nu(dx) = \nu \]

for all measurable subsets \( A \subseteq \Omega \). This definition of invariance of measures of a Markov chain \[47\] takes a familiar form if the chain happens to be the simple deterministic evolution law in Equation 3.1. In that case the transition kernel is \( \mathbb{P}(dy, x) := \delta_{w(x)}(dy) \), and we have

\[\nu(A) = \int_{\Omega} \mathbb{P}(A, x) \nu(dx) = \int_{\Omega} \delta_{w(x)}(A) \nu(dx) = \int_{\Omega} \delta_x(w^{-1}(A)) \nu(dx) = \int_{\Omega} 1_{w^{-1}(A)}(x) \nu(dx) = \nu(w^{-1}(A))\]

for all measurable \( A \subseteq \Omega \). Thus, for the deterministic case we say that the measure \( \nu \) is invariant with respect to the mapping \( w : \Omega \to \Omega \) provided \( \nu(A) \equiv \nu(w^{-1}(A)) \).
for all measurable sets $A \subseteq \Omega$. This is the definition of invariance common in ergodic systems or operator theory. [25]

Several specialized definitions of the Koopman and Perron-Frobenius operators can be constructed from this general form, depending on a duality structure. We summarize some of these below.

3.3.1. The Dual Pairing $C^*(\Omega) \times C(\Omega)$. The development of a theory for approximation of the Koopman or Perron-Frobenius operators in this paper makes assumptions regarding the regularity or smoothness of these operators. In our case these regularity conditions will be expressed in terms of specific duality structures associated with the Perron-Frobenius and Koopman operators. One important case studied in this paper regards $\mathcal{U}$ as a bounded linear operator on the continuous functions $C(\Omega)$ on $\Omega$, and $\mathcal{P} : C^*(\Omega) \to C^*(\Omega)$. It is often the case in our analysis that $\Omega$ is a compact subset of $\mathbb{R}^d$, which simplifies some of the duality arguments. See [47] for a discussion of the operators $\mathcal{P}$ and $\mathcal{U}$ when the domain $\Omega$ is not compact.

When $\Omega$ is compact, the normed dual $C^*(\Omega)$ is just the family of regular countably additive set functions, or regular signed measures, denoted $C^*(\Omega) \equiv rca(\Omega)$. [23, 63]

The Koopman operator $\mathcal{U}$ and Perron-Frobenius operator $\mathcal{P}$ are then related by the duality expression

$$\langle \mathcal{U} \nu, f \rangle_{C^*(\Omega) \times C(\Omega)} = \langle \nu, \mathcal{P} f \rangle_{C^*(\Omega) \times C(\Omega)}$$

for all $f \in C(\Omega)$ and $\nu \in C^*(\Omega) := rca(\Omega)$. This identity means that $\mathcal{U} = \mathcal{P}'$, that is, $\mathcal{U}$ is the topological transpose or dual operator of $\mathcal{P}$ relative to the pairing $\langle \cdot, \cdot \rangle_{C^*(\Omega) \times C(\Omega)}$. When we apply this condition for the discrete dynamical flow, which has the transition probability kernel $\mathcal{P}(dy, x) := \delta_{w(x)}(dy)$, we find that

$$\langle \mathcal{U} f, x \rangle = (f \circ w)(x),$$

$$\langle \mathcal{P} f, x \rangle = \nu(w^{-1}(A)).$$

3.3.2. The Dual Pairing $L^\infty_\mu(\Omega) \times L^1_\mu(\Omega)$. We also study transition kernels $\mathcal{P}(dy, x)$ that are given in terms of a transition probability density function $p : \Omega \times \Omega \to \mathbb{R}$ as in

$$\mathcal{P}(dy, x) := p(y, x) \mu(dy),$$

for some probability measure $\mu$ on $\Omega$. If we further suppose that $\nu(dx) := m(x) \mu(dx)$ for some $m \in L^1_\mu(\Omega)$, we then have

$$\langle \mathcal{P} \nu, x \rangle = \int_{\Omega} \mathcal{P}(dy, x) \nu(dx) = \int_{\Omega} p(y, x)m(x) \mu(dx) \cdot \mu(dy) = (\mathcal{P} m)(y) \mu(dy).$$

With suitable restrictions on the density $p$, this last expression leads to an alternate definition of the Perron-Frobenius operator $\mathcal{P} : L^1_\mu(\Omega) \to L^1_\mu(\Omega)$ with

$$\langle \mathcal{P} m, y \rangle = \int_{\Omega} p(y, x)m(x) \mu(dx).$$

In this setup the Koopman operator $\mathcal{U} : L^\infty_\mu(\Omega) \to L^\infty_\mu(\Omega)$ is defined with respect to the dual pairing $\langle \cdot, \cdot \rangle_{L^\infty_\mu(\Omega) \times L^1_\mu(\Omega)}$, since $(L^1_\mu(\Omega))^* = L^\infty_\mu(\Omega)$. That is, we define the Koopman operator $\mathcal{U}$ from the relation

$$\langle \mathcal{U} g, f \rangle_{L^\infty_\mu(\Omega) \times L^1_\mu(\Omega)} = \langle g, \mathcal{P} f \rangle_{L^\infty_\mu(\Omega) \times L^1_\mu(\Omega)}$$
for \( g \in L^\infty_\mu(\Omega) \) and \( f \in L^1_\mu(\Omega) \). It follows that the Koopman operator \( \hat{U} \) is then induced by the dual kernel in

\[
(\hat{U} g)(x) := \int_\Omega p(y, x) g(y) \mu(dy).
\]

3.3.3. Adjoint Operators \( \mathcal{U}, \mathcal{P} \) on a Hilbert Space. We note one last definition of these operators that is found frequently in the literature. When the measure \( \mu \) is finite, that is it satisfies \( \mu(\Omega) < \infty \), and the set \( \Omega \) is compact, we have the embeddings of the primal spaces

\[
C(\Omega) \subseteq L^\infty_\mu(\Omega) \subseteq \cdots \subseteq L^2_\mu(\Omega) \subseteq L^1_\mu(\Omega),
\]

and of the dual spaces

\[
L^\infty_\mu(\Omega) \equiv (L^1_\mu(\Omega))^* \subseteq (L^2_\mu(\Omega))^* \cdots \subseteq C^*(\Omega) \equiv rca(\Omega).
\]

A familiar duality structure can be extracted from the above by identifying \( L^2_\mu(\Omega) \) with itself via the Riesz mapping,

\[
C(\Omega) \subset L^2_\mu(\Omega) \approx (L^2_\mu(\Omega))^* \subseteq (C(\Omega))^* \equiv rca(\Omega).
\]

This is a specific example of a Gelfand triple, a mathematical structure we discuss in some detail in our analysis of the approximation of measures in Section 7.

Not surprisingly, it is quite common to encounter a definition of \( \tilde{\mathcal{U}} \) and \( \tilde{\mathcal{P}} \) as adjoints written in terms of the inner product

\[
(\tilde{\mathcal{P}} f, g)_{L^2_\mu(\Omega)} = (f, \hat{U} g)_{L^2_\mu(\Omega)},
\]

for all \( f, g \in L^2_\mu(\Omega) \). From the definition of the Riesz map \( R_{L^2_\mu(\Omega)} : (L^2_\mu(\Omega))^* \to L^2_\mu(\Omega) \), this identity means that \( \tilde{\mathcal{P}} = R_{L^2_\mu(\Omega)} \hat{\mathcal{P}} \) since

\[
\left\langle \tilde{\mathcal{P}} f, g \right\rangle_{(L^2_\mu(\Omega))^* \times L^2_\mu(\Omega)} = \left\langle R_{L^2_\mu(\Omega)}^{-1} \hat{\mathcal{P}} f, g \right\rangle_{(L^2_\mu(\Omega))^* \times L^2_\mu(\Omega)} = (\hat{\mathcal{P}} f, g)_{L^2_\mu(\Omega)}.
\]

We consider a slight generalization of this setup in some examples in our paper. Above, the Koopman and Perron-Frobenius operators are defined as adjoint operators in the same Hilbert space. For example, if we consider the deterministic system with \( w : \Omega \to \Omega \) an onto mapping, then this may be a fruitful strategy. In many of our examples we consider operators induced by a mapping \( w : \Omega \to \tilde{\Omega} \subseteq \mathbb{R}^d \) and admit the possibility that \( \Omega \) and \( \tilde{\Omega} := w(\Omega) \) do not coincide. Then it may be advantageous to define for some measure \( \tilde{\mu} \) on \( \tilde{\Omega} \)

\[
\tilde{\mathcal{U}} := L^2_{\tilde{\mu}}(\tilde{\Omega}) \to L^2_{\mu}(\Omega),
\]

\[
\tilde{\mathcal{P}} := L^2_{\mu}(\tilde{\Omega}) \to L^2_{\mu}(\Omega),
\]

with \( \tilde{\mathcal{U}} \) and \( \tilde{\mathcal{P}} \) adjoints as in Equation 3.11.

In the remainder of this paper, we use the common notation \( \mathcal{P} \) and \( \mathcal{U} \) for any of the definitions of \( (\mathcal{P}, \mathcal{U}), (\tilde{\mathcal{P}}, \tilde{\mathcal{U}}) \), or \( (\tilde{\mathcal{P}}, \tilde{\mathcal{U}}) \) given above. Whether \( \mathcal{P} \) acts on measures or functions, for example, will be clear from context in each application or example.
4. Reproducing Kernel Hilbert Spaces. In this section we summarize the theory of reproducing kernel Hilbert spaces (RKHS) \( H \) that will enable the formulation of one family of approximations, and the determination of their rates of convergence. We suppose that the evolution law is such that the discrete state remains in the compact set \( \Omega \subseteq \mathbb{R}^d \). In fact, later in the paper, we assume that we are given field observations \( \{x_1, x_2, \ldots, x_N\} \subseteq \Omega \) that are generated as random samples that are distributed in terms of the probability measure \( \mu \) on \( \Omega \). The measure \( \mu \) describes how the samples are concentrated in \( \Omega \). We then are interested in constructing approximations in Koopman theory that somehow reflect the structure of the measure \( \mu \). This is accomplished in this section by introducing a RKHS \( V \subset C(\Omega) \subset U := L_\mu^2(\Omega) \) that depends on the measure \( \mu \).

The construction begins with a continuous, symmetric, positive definite kernel \( K : \Omega \times \Omega \rightarrow \mathbb{R} \) that is assumed to generate a reproducing kernel Hilbert space \((V, \langle \cdot, \cdot \rangle_V)\) over \( \Omega \). [71] The reproducing property of the kernel guarantees that
\[
(K_x, f)_V = f(x)
\]
for all \( f \in V \) and \( x \in \Omega \) with the function \( K_x(\cdot) := K(x, \cdot) \). Alternatively, it is known that if all the evaluation functionals acting on a Hilbert space \( V \) are bounded, then \( V \) is a RKHS. This means that for each \( x \in \Omega \), there is a constant \( c_x \) such that \( f(x) \leq c_x \| f \|_V \). We further assume that the kernel \( K \) is sufficiently regular to continuously embed \( V \) in \( U := L_\mu^2(\Omega) \). In other words the linear injection \( I_K : V \rightarrow U \)
\[
(I_K f)(x) = f(x)
\]
is bounded, and we have \( \| f \|_U \leq \| I_K \| \| f \|_V \) for all \( f \in V \). This fact can be guaranteed if we know that the kernel \( K \) satisfies \( \sup_{x \in \Omega} K(x, x) < \infty \) as shown by Smale and Zhou in [69, 70]. It then also follows that \( V \) is separable and compactly embedded in \( C(\Omega) \). [62]

The adjoint operator \( I_K^* : U \rightarrow V \) is given by
\[
(I_K^* g)_V = (K_q, I_K^* g)_V = (I_K^* g)(q) = \int_\Omega K_q(y)g(y)\mu(dy).
\]
We define the operator \( T_K : U \rightarrow U \) as \( T_K := I_K I_K^* \), and we see that
\[
T_K g := I_K I_K^* g = I_K \int_\Omega K(\cdot, r)g(r)\mu(dr) = \int_\Omega K(\cdot, r)g(r)\mu(dr).
\]
Analogously, we set \( T_\mu := I_K^* I_K \) so that \( T_\mu : V \rightarrow V \). Since \( I_K^* \) is a linear compact operator, both \( T_\mu \) and \( T_K \) are compact and self-adjoint. The relationship among the operators \( T_K, T_\mu, I_K, \) and \( I_K^* \) is depicted in Figure 4.1.

The operators \( T_K \) and \( T_\mu \) have convenient representations that are a consequence of spectral theory. The spectral theory for compact, self-adjoint operators is reviewed
in Appendix C. More extensive summaries can be found in [58, 82]. The eigenvalues of
the operators $T_K$ and $T_\mu$ are identical and are arranged in an extended enumeration,
including multiplicities, in nonincreasing order

$$\lambda_1 \geq \lambda_2 \geq \ldots \geq 0.$$ 

Each eigenspace corresponding to a nonzero eigenvalue is finite dimensional, and the
only possible accumulation point of this infinite sequence is zero. We denote by
$\{v_i\}_{i \in \mathbb{N}} \subseteq V$ and $\{u_i\}_{i \in \mathbb{N}} \subseteq U$ orthonormal eigenvectors of $T_\mu$ and $T_K$, respectively,
associated with the eigenvalues $\{\lambda_i\}_{i \in \mathbb{N}}$. The spectral theory for compact, self-adjoint
operators guarantees that the following expansions are norm-convergent,

$$T_\mu g = \sum_{i \in \mathbb{N}} \lambda_i(g,v_i) v_i \quad \text{in } V,$$

$$T_K f = \sum_{i \in \mathbb{N}} \lambda_i(f,u_i) u_i \quad \text{in } U,$$

$$I_K^i f = \sum_{i \in \mathbb{N}} \sigma_i(f,u_i) u_i v_i \quad \text{in } V,$$

for each $f \in U$ and $g \in V$. By convention these summations are carried out only over
the nonzero eigenvalues. The families $\{u_i\}_{i \in \mathbb{N}}$ and $\{v_i\}_{i \in \mathbb{N}}$ associated with nonzero
eigenvalues are an orthonormal basis for $N(T_K)^\perp \subseteq U$ and $N(T_\mu)^\perp \subseteq V$, respectively.
[62] In these equations $\sigma_i := \sqrt{\lambda_i}$ is the $i^{\text{th}}$ singular value of the operator $I_K^i$. When
the eigenvalues are non-increasing, the decompositions in Equations 4.3,4.4, and 4.5
are also known as the unique monotonic Schmidt decompositions of the compact
operators $T_\mu, T_K$, and $I_K^i$, respectively. [58]

Note that since each $u_i \in U := L^2_2(\Omega)$, it is not defined for all $x \in \Omega$, but only
for $\mu$-a.e. $x \in \Omega$. On the other hand, $v_i \in V \subseteq C(\Omega)$ is defined for each $x \in \Omega$. It is
always possible to extend each $u_i$ to a continuous function $\tilde{u}_i(x) := (T_K u_i)(x)/\lambda_i$ for
all $x \in \Omega$. That is, the function $\tilde{u}_i$ is a continuous representative of the equivalence
class $u_i$. This is the Nyström extension [62], and it is known that $v_i = \sqrt{\lambda_i} \tilde{u}_i$. In the
following we suppress the extension notation $(\cdot)$, but it must be kept in mind when
expressing $v_i$ in terms of $u_i$.

We will use several probabilistic error bounds later in this paper that are readily
cast in terms of spaces of Hilbert-Schmidt operators, a type of operator of the Schatten
class. The Schatten class of operators $S^p(U)$ on $U$ of order $1 \leq p < \infty$ is the Banach
space

$$S^p(U) := \{ T : U \to U \mid T \text{ is compact and } \| T \|_{S^p} < \infty \}$$

with the norm given in the above definition by

$$\| T \|_{S^p} := \left( \sum_{i \in \mathbb{N}} \sigma_i^p(T) \right)^{1/p}.$$ 

Following convention, we define $S^\infty(U) := \{ T \in \mathcal{L}(U) \mid T \text{ is compact} \}$. We then have

$$\| T \|_{S^p} < \infty \quad \implies \quad \| T \|_{S^{p+1}} < \infty,$$

for $1 \leq p \leq \infty$, and therefore

$$S^1(U) \subset S^2(U) \cdots \subset S^\infty(U) \subset \mathcal{L}(U).$$
The Schatten class $S^p(V)$ is defined similarly. The Hilbert-Schmidt operators are obtained by choosing $p = 2$, while trace class operators correspond to $p = 1$.

Because $V \subseteq U$ is an RKHS, more can be said about the relationship of the series expansions in Equations 4.3 and 4.4 by exploiting properties of the Hilbert-Schmidt operators $S^2(V)$. We know that $g(x) = (K_x, f)_V$ for all $x \in \Omega$ and $f \in V$. In this case it is possible to represent the operator $T$ in terms of the Bochner integral $T_\mu := \int K_x \otimes K_x \mu(dx)$ with the tensor product $(K_x \otimes K_x)g = K_x(K_x, g)_V$. [80] For any $g \in V$ we have

$$
\|g\|_U^2 = \int g^2(x)\mu(dx) = \int (K_x, g)_V^2\mu(dx) = \int (K_x \otimes K_x g, g)_V \mu(dx)
= \left(\int K_x \otimes K_x \mu(dx)g, g\right)_V = (T_\mu g, g)_V.
$$

This sequence of steps can be used to show that $(g, v_j)_V = (g, u_j)_U / \sqrt{\lambda_j}$, from which we conclude

$$
T_\mu g = \sum_{i \in \mathbb{N}} \lambda_i^{1/2} (g, u_i)_U u_i.
$$

We thereby can directly compare the norms in terms of their action on the basis $\{u_i\}_{i \in \mathbb{N}}$,

$$
\|T_K f\|_V^2 = \sum_{i \in \mathbb{N}} \lambda_i^2 |(f, u_i)_U|^2,
\|T_\mu g\|_V^2 = \sum_{i \in \mathbb{N}} \lambda_i |(g, u_i)_U|^2,
$$

for $f \in U$ and $g \in V$. If $T_K$ is infinite dimensional, $\lambda_j^2 \leq \lambda_j$ for all $j$ large enough. It is evident that $D(T_\mu) \subseteq D(T_K)$. This means that the generalized Fourier coefficients of functions $f$ in the domain $T_\mu$ decay faster than those in the domain of $T_K$. This idea can be formulated systematically by introducing the spectral approximation spaces, discussed next.

5. Spectral Approximation Spaces $A^r, 2(\Omega)$. This section introduces spectral approximation spaces that are defined in terms of a fixed compact, self-adjoint operator $T : U \to U$. Specifically, the eigenvalues $\lambda_i := \lambda_i(T)$ and $U$-orthonormalized eigenvectors $\{u_i\}_{i \in \mathbb{N}}$ are used to construct $A^r, 2(\Omega)$. Typically, we choose $T := T_K$ as described in the last section, although other choices are also possible. If we happen to have a RKHS $V \subset U$ that satisfies the assumptions of the last section, we find that $U$ and $V$ are two particular spaces in a scale of spectral approximation spaces $A^r, 2(\Omega)$.

For a self-adjoint and compact operator $T$, with non-zero eigenvalues and associated eigenspaces $\{(\lambda_i, u_i)\}_{i \in \mathbb{N}}$, we define spectral approximation spaces $A^r, 2(\Omega)$ for $r \geq 0$ via the formula

$$
A^r, 2(\Omega) := A^r, 2(\lambda(T)) := \left\{ f \in U \mid |f|_{A^r, 2(\Omega)} := \sum_{i \in \mathbb{N}} (\lambda_i^{-r/2} |(u_i, f)|_U)^2 < \infty \right\}.
$$

Note that $A^0, 2(\Omega) \equiv U$ in this definition. Intuitively, these spaces have a simple interpretation: a function $f \in A^r, 2(\Omega)$ provided that the generalized Fourier coefficients $(u_i, f)_U$ decay at a rate that is controlled by the speed that the inverse $\lambda_i^{-r/2}$
of the eigenvalues grow. The next theorem summarizes some standard properties of the spectral spaces.

**Theorem 5.1.** The spectral approximation spaces $A_{r,2}^{s}(U)$ are nested,

$$A_{r,2}^{s}(U) \subset A_{r,2}^{s}(U)$$

for all $s > r$. Let $\Pi_n$ be the $U$-orthogonal projection onto the finite dimensional space of approximants

$$A_n := \text{span}\{u_i \mid i \in \mathbb{N}, i \leq n - 1\}.$$ 

If $f \in A_{r,2}^{r}(U)$, we have the error estimate

$$\|(I - \Pi_n)f\| \lesssim \lambda_n^{r/2} \|f\|_{A_{r,2}^{r}(U)}.$$ 

**Proof.** Nestedness follows since

$$|f|_{A_{r,2}^{s}(U)}^2 = \sum_{i \in \mathbb{N}} \lambda_i^{-r} |(f, u_i)_U|^2 \leq \sum_{i \in \mathbb{N}} \lambda_i^{s-r} \lambda_i^{-s} |(f, u_i)_U|^2$$

$$\lesssim \sum_{i \in \mathbb{N}} \lambda_i^{-s} |(f, u_i)_U|^2 = |f|_{A_{r,2}^{s}(U)}^2,$$

provided that $s > r$. The error in approximation induced by the $U-$orthonormal projection $\Pi_n$ is shown similarly.

$$\|(I - \Pi_n)f\|_U^2 = \sum_{i \geq n} |(f, u_i)_U|^2 \leq \sum_{i \geq n} \lambda_i^{s-r} |(f, u_i)_U|^2$$

$$\lesssim \sum_{i \geq n} \lambda_i^{-s} |(f, u_i)_U|^2 \leq \lambda_n^{s-r} f|_{A_{r,2}^{s}(U)},$$

It is worth noting that the proof of the error bound above can be easily modified to derive

$$\|(I - \Pi_n)f\|_{A_{r,2}^{s}(U)} \leq \lambda_n^{(s-r)/2} \|f\|_{A_{r,2}^{s}(U)}$$

whenever $s > r > 0$ and $f \in A_{r,2}^{r}(U)$. The bound in the theorem can be understood as the limiting case of the above when $r = 0$.

We next see how the approximation spaces $A_{r,2}^{r}(U)$ and $A_{r,2}(V)$ are related when $V$ is a RKHS, and describe some simple mapping properties of the operators $T_\mu$ and $T_K$ when we choose $T = T_K$ in the definition of $A_{r,2}^{r}(U)$. We assume that the general setup discussed in Section 4 holds.

**Theorem 5.2.** If $V \subset U$ is a RKHS and the imbedding $\imath_K : V \to U$ is compact and continuous, it follows that

1) $A_{r,2}^{r+1}(U) \approx A_{r,2}^{r}(V)$ for $r > 0$, and

2) the operators $T_K$ and $T_\mu$ are smoothing in the sense that

$$T_K : A_{r,2}^{r}(U) \to A_{r,2}^{r+2}(U),$$

$$T_\mu : A_{r,2}^{r}(V) \to A_{r,2}^{r+2}(V).$$

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Proof. The proof of (1) follows directly from the calculation

\[ |f|^2_{A^{1,2}_r(V)} = \sum_{i \in N} \lambda_i^{-r}|(f, v_i)v|^2, \]

\[ = \sum_{i \in N} \lambda_i^{-r}|\lambda_i^{-1/2}(f, u_i)u|^2, \]

\[ = \sum_{i \in N} \lambda_i^{-(r+1)}|(f, u_i)u|^2 = |f|^2_{A^{r+1,2}_r(U)}. \]

Conclusion (2) in the above theorem holds because

\[ |T_Kf|^2_{A^{r+2,2}_r(U)} = \sum_{i \in N} \lambda_i^{-(r+2)}|(T_Kf, u_i)u|^2, \]

\[ = \sum_{i \in N} \lambda_i^{-(r+2)}\left(\sum_{m \in N} \lambda_m(f, u_m)u_m, u_i\right)_{U}^2, \]

\[ = \sum_{i \in N} \lambda_i^{-(r+1)}|(f, u_i)u|^2 = |f|^2_{A^{r+2}_r(U)}. \]

The result for \( T_\mu \) in (3) follows similarly using eigenfunction expansions in \( V \) in terms of \( \{v_i\}_{i \in N}. \)

The mapping properties described above for \( T_K \) or \( T_\mu \) can also be understood in terms of the operators \( \sqrt{T_K} \) and \( \sqrt{T_\mu}. \) We have

\[ \sqrt{T_K} : A^{r,2}_r(U) \to A^{r+1,2}_r(U) \]

\[ \sqrt{T_\mu} : A^{r,2}_r(V) \to A^{r+1,2}_r(V) \]

This means that we can interpret the square root operators as the (increasing) shift operator on the scale of spaces \( A^{r,2}_r(U) \) and \( A^{r,2}_r(V). \) [10]

5.1. The Compact, Self-Adjoint Operators in \( \mathbb{A}^{r,2}_r(U). \) We define a family of admissible operators \( \mathbb{A}^{r,2}_r(U) \) that are convenient for studying rates of convergence of approximations in terms of the spectral spaces \( A^{r,2}_r(U). \) We define for \( r > 0 \) the family of self-adjoint, compact operators \( \mathbb{A}^{r,2}_r(U) \) via

\[ \mathbb{A}^{r,2}_r(U) := \left\{ \mathcal{P} = \sum_{i \in N} p_i u_i \otimes u_i \in S^\infty(U) \bigg| |\mathcal{P}|_{A^{r,2}_r(U)}^2 := \sum_{i \in N} \lambda_i^{-r}|p_i|^2 < \infty \right\} \]

where the formula for \( \mathcal{P} \) above is a Schatten expansion of \( \mathcal{P} \) in terms of the eigenvector basis \( \{u_i\}_{i \in N} \) of the operator \( T \) used to define \( A^{r,2}_r(U), \) and \( \{p_k\}_{k \in N} \subset \mathbb{R}. \) Admittedly, the family of operators \( \mathbb{A}^{r,2}_r(U) \) contains operators that are highly structured. Each operator \( \mathcal{P} \in \mathbb{A}^{r,2}_r(U) \) is self-adjoint. We emphasize, however, that the operator \( \mathcal{P} \in \mathbb{A}^{r,2}_r(U) \) is not diagonal with respect to some arbitrary orthonormal basis of \( U; \) it is diagonalized in terms of the basis \( \{u_j\}_{j \in N} \) generated from \( T. \) Suppose that \( \{a_j\}_{j \in N} \) is another orthonormal basis for \( U. \) Since, as we show below in Theorem 5.3 that \( \mathbb{A}^{r,2}_r(U) \subset S^2(U), \) the Hilbert-Schmidt operator \( \mathcal{P} \in \mathbb{A}^{r,2}_r(U) \) is guaranteed to have the representation

\[ \mathcal{P} := \sum_{i,j \in N} p_{ij} a_i \otimes a_j \]
with \( p_{ij} = (Pa_i, a_j)_U \) and \( p_{ij} = p_{ji} \). Of course, the value of the norm in \( S^2(U) \) does not depend on orthonormal basis, so we must have

\[
\|P\|_{S^2(U)}^2 := \sum_{i \in \mathbb{N}} (Pu_i, u_i)_U^2 = \sum_{i \in \mathbb{N}} p_i^2 = \sum_{i,j \in \mathbb{N}} (Pa_i, a_j)_U^2.
\]

We have introduced this definition so that proofs of the rates of convergence of the operators \( P \) and \( U \) are particularly simple and illustrative. We will see that this definition can be generalized easily to certain classes of operators defined in Section 5.2 that can contain operators that are not self-adjoint. In fact, essentially all of the error bounds derived in Theorem 5.4 for the family defined in Equation 5.2 hold for the more general class of admissible operators introduced in Section 5.2 that contains non-self-adjoint operators too.

The family of operators \( \mathcal{A}_\lambda^{r,2}(U) \) can again be understood intuitively like the definition of the spectral spaces \( A^{r,2}_\lambda(U) \). A feasible Perron-Frobenius operator \( P \in \mathcal{A}_\lambda^{r,2}(U) \) has a Schatten class representation whose coefficients decay at a rate that is inversely proportional to the rate at which the eigenvalues \( \lambda_i(T) \) converge to zero for some fixed compact, self-adjoint operator \( T \). In this sense, the fixed operator \( T \), by virtue of its eigenstructure, defines rates of convergence in \( \mathcal{A}_\lambda^{r,2}(U) \). When \( V \subseteq U \) is a RKHS, the operators \( T_K \) or \( T_\mu \) that are induced by a symmetric kernel \( K : \Omega \times \Omega \to \Omega \) are a natural choice for the definition of \( \mathcal{A}_\lambda^{r,2}(U) \) or \( \mathcal{A}_\lambda^{r,2}(V) \), respectively. However, the definition above need not be restricted to this case. We summarize a few of the easy properties of the operators in \( \mathcal{A}_\lambda^{r,2}(U) \).

**Theorem 5.3.** For each \( r > 0 \) we have

\[
\mathcal{A}_\lambda^{r,2}(U) \subset S^2(U).
\]

The family of operators \( \mathcal{A}_\lambda^{r,2}(U) \) are nested,

\[
\mathcal{A}_\lambda^{s,2}(U) \subset \mathcal{A}_\lambda^{r,2}(U),
\]

whenever \( s > r > 0 \).

**Proof.** Each \( \mathcal{A}_\lambda^{r,2}(U) \subset S^2(U) \) since

\[
\|\mathcal{P}\|_{S^2(U)}^2 := \sum_{i \in \mathbb{N}} \sigma_i^2(\mathcal{P}) = \sum_{i \in \mathbb{N}} p_i^2 \lesssim \sum_{i \in \mathbb{N}} (p_i \lambda_i^{-r/2})^2 = \|\mathcal{P}\|_{\mathcal{A}_\lambda^{r,2}(U)}^2.
\]

The second assertion follows from

\[
|\mathcal{P}|_{\mathcal{A}_\lambda^{r,2}(U)}^2 = \sum_{i \in \mathbb{N}} \lambda_i^{-r} |p_i|^2 \leq \sum_{i \in \mathbb{N}} \lambda_i^{-s} |p_i|^2 = |\mathcal{P}|_{\mathcal{A}_\lambda^{s,2}(U)}^2,
\]

as long as \( s > r > 0 \).

Note carefully that the larger the approximation index \( r > 0 \), the smaller the space \( \mathcal{A}_\lambda^{r,2}(U) \). A similar inclusion holds for the operators in \( \mathcal{A}_\lambda^{r,2}(U) \). The norm inequality above implies an imbedding of the scale of operators \( \mathcal{A}_\lambda^{r,2}(U) \) that resembles that for the spectral spaces \( \mathcal{A}_\lambda^{r,2}(U) \) in the sense that

\[
\cdots \subset \mathcal{A}_\lambda^{+1,2}(U) \subset \mathcal{A}_\lambda^{2,2}(U) \subset \mathcal{A}_\lambda^{-1,2}(U) \subset \cdots \subset S^2(U) \subset \cdots \subset S^\infty(U) \subset \mathcal{L}(U).
\]
Having defined the spaces $A_{\varphi}^{r,2}(U)$ and the family of admissible operators $A_{\varphi}^{r,2}(U)$, we begin with a rather straightforward result. Although it is nearly self-evident, it is often a building block for more complex error bounds derived later. Specifically, we derive an approximation rate that holds for the family of operators $A_{\varphi}^{r,2}(U)$ and the approximation spaces $A_{\varphi}^{r,2}(U)$.

**Theorem 5.4.** Suppose that $r > 0$ and $P$ has the monotonic Schmidt decomposition $P := \sum_{i \in \mathbb{N}} p_i u_i \otimes u_i \in S^\infty$ with respect to the $U$-orthonormal basis of eigenfunctions $\{u_i\}_{i \in \mathbb{N}}$ of the compact, self-adjoint operator $T$. Define the associated approximation space $A_{\varphi}^{r,2}(U)$ in terms of the eigenstructure of $T$, and denote by $P_n$ the approximation obtained when the Schatten class representation is truncated to $P := \sum_{1 \leq i \leq n-1} p_i u_i \otimes u_i$. If $P \in A_{\varphi}^{r,2}(U)$, we have the error bound

$$
\|(P - P_n) f\|_U \lesssim \lambda_n^{(s-r)/2} |P f|_{A_{\varphi}^{r,2}(U)};
$$

(5.3)

This bound holds in particular for the two important cases when 1) $P \in L(U)$ and $f \in A_{\varphi}^{r,2}(U)$ or when 2) $P \in A_{\varphi}^{r,2}(U)$ and $f \in U$. Suppose that the eigenvalues are quasigeometric in that there are two constants $c_1, c_2$ with

$$
1 < c_1 \leq \frac{\lambda_{n_j-1}}{\lambda_{n_j}} \leq c_2
$$

for all $j \in \mathbb{N}$ with $\{n_j\}_{j \in \mathbb{N}_0}$ a quasigeometric sequence of integers. In this case for $s > r > 0$ we have

$$
|P - P_{n_j}|_{A_{\varphi}^{r,2}(U)} \lesssim \lambda_{n_j}^{(s-r)/2} |P f|_{A_{\varphi}^{r,2}(U)}.
$$

(5.4)

**Proof.** First, we know we have $f = \sum_{i \in \mathbb{N}} (f, u_i) u_i$ since $A_{\varphi}^{r,2}(U) \subset U$. When $P f \in A_{\varphi}^{r,2}(U)$, we compute the error

$$
\|(P - P_n) f\|_U^2 = \left\| \left( \sum_{i \geq n} p_i u_i \otimes u_i \right) \left( \sum_{k \in \mathbb{N}} (f, u_k) u_k u_k \right) \right\|_U^2,
$$

$$
= \sum_{i \geq n} p_i^2 |(f, u_i) V|^2 \leq \sum_{i \geq n} p_i^2 \lambda_i^{s-r} |(f, u_i) V|^2,
$$

$$
\leq \lambda_n^{s-r} \sum_{i \in \mathbb{N}} \lambda_i^{-r} |(f, u_i) V|^2 = \lambda_n^{s-r} |P f|_{A_{\varphi}^{r,2}(U)}^2.
$$

Thus Equation 5.3 holds. When $P \in L(U)$ and $f \in A_{\varphi}^{r,2}(U)$, we have

$$
\|P f\|_{A_{\varphi}^{r,2}(U)}^2 = \sum_{i \in \mathbb{N}} \lambda_i^{-r} |(f, u_i) V|^2 = \sum_{i \in \mathbb{N}} p_i^2 \lambda_i^{s-r} |(f, u_i) V|^2
$$

$$
\leq \|P\|_{L(U)}^2 \|f\|_{A_{\varphi}^{r,2}(U)}^2 < \infty,
$$

which shows that the bound above holds for case (1). Now, at the other extreme, if we only know that $f \in U$, but $P \in A_{\varphi}^{r,2}(U)$, we see that

$$
\|P f\|_{A_{\varphi}^{r,2}(U)}^2 = \sum_{i \in \mathbb{N}} (\lambda_i^{-r} p_i^2 |(f, u_i) V|^2 \leq |P|_{A_{\varphi}^{r,2}(U)}^2 \|f\|_U^2 < \infty,
$$

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which gives the same rate in case (2). We now show that Equation 5.4 is true. We prove the result in the theorem for \( n_j = 2^j \) since this case resembles many of the rates derived later in the paper. See [60, 57] for the details associated with a general quasigeometric sequence. The proof for a general quasigeometric sequence follows similarly. We can write

\[
\sum_{i \geq n} \lambda_{X}^{-r}|p_i|^2 = \sum_{j \geq n} \sum_{k=2^j}^{2^{j+1}-1} \lambda_{X}^{-r}|p_k|^2 \leq \sum_{j \geq n} \lambda_{X}^{-r} \sum_{k=2^j}^{2^{j+1}-1} |p_k|^2
\]

\[
\leq \sum_{j \geq n} \lambda_{X}^{-r} \lambda_{2(2^j+1)}^{-2^j+1} \sum_{k=2^j}^{2^{j+1}-1} |p_k|^2 \lesssim \lambda_{n} \sum_{j \in \mathbb{N}_0} \lambda_{2^{j+1}}^{-2^j+1} \sum_{k=2^j}^{2^{j+1}-1} |p_k|^2
\]

\[
\lesssim \lambda_{n} \sum_{j \in \mathbb{N}_0} \lambda_{2^{j+1}}^{-2^j+1} \sum_{k=2^j}^{2^{j+1}-1} |p_k|^2 \lesssim \lambda_{n} \sum_{j \in \mathbb{N}_0} \sum_{k=2^j}^{2^{j+1}-1} \lambda_{k}^{-2^j} |p_k|^2
\]

\[
= \lambda_{n}' |P|_{\mathcal{A}_X^{*+r}(U)}^2.
\]

The error bound now follows by defining \( \tilde{s} := r + s \) and rewriting the above as

\[
|P - P_j|_{\mathcal{A}_X^{*+s}(U)}^2 \leq \lambda_{n}' |P|_{\mathcal{A}_X^{*}(U)}^2.
\]

The next example describes an overall process by which the preceding analysis is applied. Initially, the operator \( T : U \to U \) is selected, and its eigenvalues and \( U \)-orthonormal eigenfunctions are used to define the approximation spaces \( \mathcal{A}_X^{*+r}(U) \). Then, the approximation rates of operators \( P_j \) are studied, for example, when \( Pf \in \mathcal{A}_X^{*+r}(U) \) or \( P \in \mathcal{A}_X^{*+r}(V) \). This case considers a linear dynamical system for purposes of illustration, but as is clear from several examples that follow, the same general process is applicable to nonlinear systems.

**Example 1 (Discrete Approximation of the Heat Equation).**

**Defining \( T, \mathcal{A}_X^{*+r}(U), \) and \( \mathcal{A}_X^{*+r}(V) \).** Let \( \mathbb{T}^1 \subset \mathbb{R}^2 \) be the unit circle, \( L^2(\mathbb{T}^1) \) be the periodic square integrable functions over \( \mathbb{T}^1 \), and let \( \bar{T} \) be the second order differential operator \( \bar{T}(\cdot) := -d^2(\cdot)/dx^2 \). It is straightforward to check that the eigenvalue problem that seeks a nontrivial solution of

\[
\bar{T} f = \lambda f
\]

subject to the periodic boundary conditions

\[
f(0) = f(2\pi), \quad \frac{df}{dx}(0) = \frac{df}{dx}(2\pi),
\]

generates the orthonormal eigenpairs

\[
\left\{ \left( \lambda_m, \psi_m \right) \right\}_{m \in \mathbb{Z}} = \left\{ \left( m^2, \frac{1}{\sqrt{2\pi}} e^{imx} \right) \right\}_{m \in \mathbb{Z}}.
\]

Orthornormality is defined with respect to the inner product \( \langle f, g \rangle_{L^2(\mathbb{T}^1)} := \int_{\mathbb{T}^1} f(\xi) \overline{g(\xi)} d\xi \) on the \( L^2(\mathbb{T}^1) \) space of complex functions.
A quick check. We know that \( \frac{d\psi_m}{dx} = \hat{j}m\sqrt{\frac{2}{\pi}} e^{-jmx} \). We then have

\[
(\tilde{L}\psi_m)(x) = -\frac{d^2\psi_m}{dx^2}(x) = - \left( \frac{-m^2}{\sqrt{2\pi}} e^{jmx} \right) = m^2\psi_m(x) = \hat{\lambda}_m\psi_m(x).
\]

Also,

\[
\psi_m(0) = \frac{1}{\sqrt{2\pi}} e^0 = \frac{1}{\sqrt{2\pi}} e^{j2\pi m} = \psi_m(2\pi),
\]
\[
\psi'_m(0) = \frac{\hat{j}m}{\sqrt{2\pi}} e^0 = \frac{\hat{j}m}{\sqrt{2\pi}} e^{j2\pi m} = \psi'_m(2\pi).
\]

The operator \( \hat{T} \) is a differential operator that generates a Sturm-Liouville system. The differential operator \( \hat{T} \) can be used define an associated inverse operator \( T := (\hat{T}|_{N(\hat{L})})^{-1} \) that is in fact a linear, self-adjoint, compact integral operator on \( L^2(T^1) \). \[55\] Any function \( f \in L^2(T^1) \) consequently has the Fourier series representation

\[
(5.5) \quad f = \sum_{k \in \mathbb{Z}} (f, \psi_k)_{L^2(T^1)} \psi_k.
\]

Although our theory in Section 4 above studies real-valued functions, only a slight reindexing is needed to modify the definitions to make sense for the complex functions \( \psi_m \). The spectral approximation space generated by this complex orthonormal basis is given by

\[
A^{r,2}_\lambda(L^2(T^1)) := \left\{ f \in L^2(T^1) \left| \sum_{m \in \mathbb{Z} \setminus \{0\}} \lambda_m^{-r} |(f, \psi_m)_{L^2(T^1)}|^2 \leq \infty \right. \right\}.
\]

In this case we have

\[
|f|^2_{A^{r,2}_\lambda(L^2(T^1))} = \sum_{m \in \mathbb{Z} \setminus \{0\}} \lambda_m^{-r} |(f, \psi_m)_{L^2(T^1)}|^2 \approx \sum_{m \in \mathbb{Z} \setminus \{0\}} m^{2r} |(f, \psi_k)_{L^2(T^1)}|^2
\]

It is known that the rightmost series above is in fact equivalent to the seminorm on the Sobolev space \( W^{r,2}(T^1) \). \[18\] We also see that the RKHS space \( V = A^{1,2}_\lambda(U) \approx W^{1,2}(T^1) \). This choice of the operator \( T \), as a compact self-adjoint integral operator, is consistent with the assumptions of Section 4 when \( d = 1 \).

The Sobolev Embedding Theorem states that if \( m > d/2 \), then \( W^{m,2}(T^1) \hookrightarrow C(T) \), and we have

\[
V \approx W^{1,2}(T^1) \subset C(T^1) \subset L^2(T) = U.
\]

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The eigenfunctions above are elements in the \( L^2(T^1) \) space of complex functions. We can further study the kernels that induce the operator \( L \) in terms of the complex eigenfunctions. However, for the form of the real-valued RKHS spaces presented in Section 4, it is more convenient to cast the analysis in terms of the
$L^2(\Omega)$ space of real functions. The eigenvectors $u_{k,i}$ of $T$, viewed as an operator on the $L^2(\Omega)$ space of real functions, are given by

$$u_{k,i}(x) := \begin{cases} 
\frac{1}{\sqrt{2\pi}} & i = 1, k = 0, \\
\frac{\cos kx}{\sqrt{\pi}} & i = 1, k \geq 1, \\
\frac{\sin kx}{\sqrt{2\pi}} & i = 2, k \geq 1,
\end{cases}$$

and the corresponding eigenvalues are

$$\lambda_{k,i} := \begin{cases} 
0 & i = 1, k = 0, \\
1/k^2 & i = 1, k \geq 1, \\
1/k^2 & i = 2, k \geq 1.
\end{cases}$$

Note that $u_{0,2}$ is not defined in this numbering convention. It is easy to check that the functions $u_{k,i}$ are orthonormal with respect to the real inner product $(f, g)_{L^2(\mathbb{T})} = \int_{\Omega} f(\xi)g(\xi)d\xi$. For any $f$ in the real $L^2(\Omega)$ space, we have

$$f = \sum_{k \geq 1} \sum_{i \in \{1, 2\}} (f, u_{k,i})_{L^2(\mathbb{T})} u_{k,i},$$

which yields the same result as in the complex expansion in Equation 5.5 when the function $f$ is real-valued. We define the real Hilbert space $U := L^2(\mathbb{T})$, and in the notation of Sections 4 and 5, we have

$$T_K f := Tf = \sum_{k \geq 1} \sum_{i \in \{1, 2\}} \lambda_{k,i}(u_{k,i}, f)U u_{k,i},$$

$$A^r_{\lambda}(L^2(\mathbb{T})) := \left\{ f \in L^2(\mathbb{T}) \middle| f[\lambda^{-r}_{\lambda}(L^2(\mathbb{T})) := \sum_{k \geq 1} \sum_{i \in \{1, 2\}} \lambda_{k,i}^{-r}(f, \psi_{k,i})_{L^2(\mathbb{T})}] < \infty \right\},$$

and

$$A^r_{\lambda}(L^2(\mathbb{T})) = \left\{ P = \sum_{k \geq 1, i = 1, 2} p_{k,i} u_{k,i} \otimes u_{k,i} \middle| \sum_{k \geq 1, i = 1, 2} (p_{k,i} \lambda_{k,i}^{-r/2})^2 < \infty \right\}.$$
and to the initial condition
\[ \tau(0, x) = \tau_0(x) \quad x \in [0, 2\pi]. \]

It can be shown that the solution (modulo constant functions) of this evolution equation is given by
\[ \tau(t) := \tau(t, \cdot) := \sum_{k \geq 1, i=1,2} e^{-k^2 t (\tau_0, u_{k,i})_{L^2(\Omega)}} u_{k,i}(\cdot), \]
or \( \tau(t) = S(t) \tau_0 \) with \( S(t) \) a linear \( C^0 \)-semigroup of operators. By sampling, this continuous flow induces a discrete flow \( \mathcal{P} : L^2(\Omega) \to L^2(\Omega) \) with
\[ \tau_{i+1} := \tau(t_{i+1}) = S(t_{i+1} - t_i) \tau(t_i) = \mathcal{P} \tau_i, \]
with \( \{\tau_i\}_{i \in \mathbb{N}_0} \subseteq L^2(\Omega) \). The discrete evolution law is induced by a kernel \( p := p_h(\cdot, \cdot) : [0, 2\pi] \times [0, 2\pi] \to \mathbb{R} \) that depends parametrically on the step size \( h > 0 \), with
\[ \tau_{i+1}(x) := (\mathcal{P} \tau_i)(x) := \int_{\Omega} p(x, y) \tau_i(y) dy \]
where
\[ p(x, y) := \sum_{k \geq 1, i=1,2} e^{-hk^2} u_{k,i}(x) u_{k,i}(y). \]

In other words with \( p_{k,i} := e^{-hk^2} \) we have
\[ \mathcal{P} := \sum_{k \geq 1, i=1,2} p_{k,i} u_{k,i} \otimes u_{k,i}, \]
which is the form of a kernel in \( A_{r,2}(U) \). It turns out that the operator \( \mathcal{P} \) above is very smooth. Now consider the sum
\[ \sum_{k \geq k_0} \sum_{i=1,2} \lambda_{k,i}^{-r} p_{k,i}^2 = \sum_{k \geq 1} \sum_{i=1,2} k^{2r} e^{-2hk^2}. \]
The function \( x^{2r} e^{-2hx^2} \) is monotonically decreasing for all \( x > \sqrt{r/h} \) and approaches zero as \( x \to \infty \). This means that
\[ \sum_{k \geq k_0} \sum_{i=1,2} k^{2r} e^{-2hk^2} \leq \int_0^\infty x^{2r} e^{-2hx^2} dx \]
for some any integer \( k_0 \geq \sqrt{r/h} \). Since the integral on the right is finite for every positive \( r \) and \( h \), we conclude that \( \mathcal{P} \in A_{r,2}(U) \) for every \( r > 0 \), and the results of Theorem 5.4 hold.

**Example 2 (Spaces Adapted to a Specific \( \mathcal{P} \)).** In this example we explore a bit more how the results of Theorem 5.4 can be further refined. From the definition
in Equation 5.2 we known that the the operator
\[ \mathcal{P} := \sum_{i \in \mathbb{N}} p_i v_i \otimes v_i. \]

Suppose the coefficients \( \{p_i\}_{i \in \mathbb{N}} \) are nonincreasing and can only accumulate at zero. It is therefore possible to define the approximation space
\[ A^{r,2}_{\mathcal{P}^r}(V) := \left\{ f \in V \left| \|f\|_{A^{r,2}_{\mathcal{P}^r}(V)}^2 := \sum_{i \in \mathbb{N}} |p_i|^{-r} |(f, v_i)_V|^2 \right. \right\}. \]

It is immediate that
\[ |f|_{A^{r,2}_{\mathcal{P}^r}(V)}^2 := \sum_{i \in \mathbb{N}} |\lambda_i|^{-r} |(f, v_i)_V|^2 \leq \sum_{i \in \mathbb{N}} |p_i|^{-r} |p_i|^{-2} |(f, v_i)_V|^2 \leq |f|_{A^{r,2}_{\mathcal{P}^r}(V)}^2, \]
and therefore
\[ A^{r,2}_{\mathcal{P}^r}(V) \subseteq A^{r,2}_{\lambda^r}(V). \]

The space \( A^{r,2}_{\mathcal{P}^r}(V) \) can be endowed with the inner product
\[ (f, g)_{A^{r,2}_{\mathcal{P}^r}(V)} := \sum_{i \in \mathbb{N}} \left( p_i^{-r/2} f_i, p_i^{-r/2} g_i \right)_V \]
where \( f := \sum f_i v_i \) and \( g = \sum g_i v_i \). It is clear from this definition that \( f \in A^{r,2}_{\mathcal{P}^r}(V) \) if and only if
\[ \|f\|_{A^{r,2}_{\mathcal{P}^r}(V)} := (f, f)_{A^{r,2}_{\mathcal{P}^r}(V)} < \infty. \]

It is also immediate that the family of functions \( \{v_{r,i}^r\}_{i \in \mathbb{N}} := \{p_i^{r/2} v_i\}_{i \in \mathbb{N}} \) is an orthonormal basis for \( A^{r,2}_{\mathcal{P}^r}(V) \).

Now suppose that we have a stronger condition that relates the kernels \( \mathcal{K} \) and \( \mathcal{P} \): we assume the equivalence of the sequences \( \{p_i\}_{i \in \mathbb{N}} \approx \{\lambda_i^{r/2}\}_{i \in \mathbb{N}} \).

That is, there exist two constants \( c_1, c_2 > 0 \) such that
\[ c_1 \lambda_i^{r/2} \leq p_i \leq c_2 \lambda_i^{r/2} \]
for all \( i \in \mathbb{N} \). With this assumption we see that \( A^{r,2}_{\mathcal{P}^r}(V) \approx A^{\lambda^r}(V) \). Since the kernel \( \mathcal{K} \) is positive, the kernel of \( \mathcal{P} \) is positive, symmetric, and continuous, and therefore it is possible to define a RKHS \( V_p \subset L^2_\mu(\Omega) \) in terms of \( p(x, y) \). Then we know that
\[ \mathcal{P} := \int_{\Omega} p(x) \otimes p(y) \mu(dx) = \sum_{i \in \mathbb{N}} p_i v_i \otimes v_i. \]

From Theorem 11 of [3] we can compute the reproducing kernel on \( A^{r,2}_{\mathcal{P}^r}(V) \approx A^{\lambda^r}(V) \) from the kernel \( \mathcal{K} \) on \( V \).
5.2. Compact and Non-Self-Adjoint Operators in $A^r_\lambda(U)$. In the last section we presented a definition of feasible operators $A^r_\lambda(U)$ that facilitated the study of convergence rates relative to the spectral approximation spaces $A^r_\lambda(U)$. In a typical application, Theorem 5.1 is used by first fixing some compact self-adjoint operator $T$ whose eigenvalues and eigenfunctions $\{\lambda_i, u_i\}_{i \in \mathbb{N}}$ are used in the construction of the spectral space $A^r_\lambda(U)$. The set of admissible operators consists of compact, self-adjoint operators that have Schatten decompositions that decay faster that the inverse of the eigenvalues used to build $A^r_\lambda(U)$. If we construct a RKHS $V \subset U$, then the integral operator $T_K$ given by

\[(T_K f)(x) := \int_{\Omega} K(x, y) f(y) \mu(dy)\]

is one logical choice for the operator $T$ that defines $A^r_\lambda(U)$. Here, the kernel $K(x, y)$ is symmetric. It is a requirement that ensures the symmetry of the inner product on the real RKHS $V \subset U$.

However, it is important to observe that there is no compelling requirement to restrict attention to Perron-Frobenius, or Koopman operators, that are induced by a symmetric kernel. It is a simple matter to define deterministic or stochastic discrete evolutions for which the associated operators are non-self-adjoint. We now discuss how the setting of the last section can be extended to allow for some operators that are not self-adjoint.

The theory of integral operators and their mapping properties has been studied extensively over the years, and many approaches exist to study them. Comprehensive accounts can be found in [58, 59]. One choice that is canonical and serves as an exemplar for other approaches is the case when it is assumed that $p(x, y)$ is a kernel in $L^2_{\mu \times \mu}(\Omega \times \Omega)$ that induces an operator $P : L^2_{\mu}(\Omega) \rightarrow L^2_{\mu}(\Omega)$. In fact, an operator $P : L^2_{\mu}(\Omega) \rightarrow L^2_{\mu}(\Omega)$ has the form in Equation 5.6 if and only if it is a Hilbert-Schmidt operator. In this case, for any orthonormal basis $\{\psi_i\}_{i \in \mathbb{N}}$ of $L^2_{\mu}(\Omega)$, such an operator $P$ is induced by the kernel

\[p(x, y) = \sum_{m \in \mathbb{N}} \sum_{n \in \mathbb{N}} (P \psi_m, \psi_n) \mu m(x) \psi_n(y),\]

and we also know that

\[\|P\|_{S^2(U)} = \sum_{i \in \mathbb{N}} \sigma_i^2 := \sum_{i \in \mathbb{N}} \|P \psi_i\|_{L^2_{\mu}(\Omega)}^2 = \sum_{m \in \mathbb{N}} \sum_{n \in \mathbb{N}} \|\langle P \psi_m, \psi_n \rangle\|^2.\]

The study of kernels of this type is facilitated by exploiting the equivalence of these norm expressions to the norms of certain infinite matrix operators that act on $\ell^2(\mathbb{N})$. We define the infinite matrices $[p_{m,n}] := [(P \psi_m, \psi_n)]$ and $[D^s_{\lambda_m}] := \text{diag}(\lambda_m^s)$ for $s \in \mathbb{R}$. The induced matrix operator norm $\|p_{m,n}\|_M$ is defined in the usual way

\[\|p_{m,n}\|_M := \sup_{\{z_n\} \in \ell^2(\mathbb{N}) \setminus \{0\}} \left\| \{z_n\} \|_{\ell^2(\mathbb{N})} \right\|.\]

With this notation we have $\|f\|_U = \|\{(f, \psi_m)\}\|_{\ell^2(\mathbb{N})}$, and it is straightforward to show that

\[\|P\|_{\mathcal{L}(U)} = \|p_{m,n}\|_M.\]
It also follows that \( f \in A^{\lambda^2}_\bullet(U) \) if and only if \( \|[D_{\lambda_n}^{-r/2}]\{f_n\}\|_{L^2(\mathbb{N})} < \infty \). We overload the definition of \( D_{\lambda_n}^\bullet \) and also interpret it as an operator on functions by associating its representation

\[
D_{\lambda_n}^\bullet f \sim |D_{\lambda_m}^\bullet|\{f_m\}
\]

with \( f_m := (f, \psi_m)_U \) for \( m \in \mathbb{N} \).

We then can generalize the definition of the families of feasible spectral operators and write

\[
(5.7) \quad \lambda^{\lambda^2}_\bullet(U) := \left\{ P = \sum_{m,n \in \mathbb{N}} p_{m,n} u_m \otimes u_n \subset S^\infty(U) \middle| |P|_{\lambda^{\lambda^2}_\bullet(U)}^2 := \sum_{m,n \in \mathbb{N}} \left( \sum_{n \in \mathbb{N}} \lambda_{m,n}^{-r/2} \right)^2 < \infty \right\}.
\]

The expression above reduces to that in the Definition in Equation 5.2 when the operator \( P \) is diagonalized by the eigenfunctions of the operator \( T \) used to define \( \lambda^{\lambda^2}_\bullet(U) \). The seminorm \( |P|_{\lambda^{\lambda^2}_\bullet(U)} \) is easily shown to have the alternative representations below:

\[
|P|_{\lambda^{\lambda^2}_\bullet(U)}^2 := \text{trace} \left( (D_{\lambda_n}^{-r/2})^* D_{\lambda_n}^{-r/2} \right) = \left( D_{\lambda_n}^{-r/2} P, D_{\lambda_n}^{-r/2} P \right)_{S^2(U)},
\]

\[
= \sum_{k \in \mathbb{N}} \left( D_{\lambda_n}^{-r/2} P \psi_k, D_{\lambda_n}^{-r/2} P \psi_k \right)_U,
\]

\[
= \text{trace} \left( \left[ p_{k,i} [D_{\lambda_n}^{-r/2}] \right]^T \left[ p_{k,i} [D_{\lambda_n}^{-r/2}] \right] \right)
\]

\[
= \text{trace} \left( \left[ p_{k,i} [D_{\lambda_n}^{-r/2}] \right]^2 \right)
\]

\[
= \text{trace} \left( \left[ p_{k,i} [D_{\lambda_n}^{-r/2}] \right]^2 \right)
\]

\[
= \sum_{k \in \mathbb{N}} \left( \sum_{i} p_{k,i} \lambda_i^{-r/2} \right)^2.
\]

Other equivalent forms of the last line follow from the identities \( \text{trace}(A^* B) = \text{trace}(B^* A) \) for any operators \( A, B \in S^2(U) \).

We have the following approximation bounds in terms of this updated definition of \( \lambda^{\lambda^2}_\bullet(U) \).

**Theorem 5.5.** The results of Theorem 5.4 hold with the definition of \( \lambda^{\lambda^2}_\bullet(U) \) in Equation 5.7.

**Proof.** If we define \( f_n := (f, u_n)_U \) for each \( n \in \mathbb{N} \), we can bound the error by writing

\[
\|(P - P_j)f\|_U^2 = \sum_{m \geq j} \left( \sum_{n \geq j} p_{m,n} f_n \right)^2 = \sum_{m \geq j} \left( \sum_{n \geq j} \lambda_{n}^{-r/2} \lambda_{n}^{-r/2} p_{m,n} f_n \right)^2,
\]

\[
\leq \lambda_j^r \sum_{m \geq j} \left( \sum_{n \geq j} p_{m,n} \lambda_{n}^{-r/2} f_n \right)^2 \leq \lambda_j^r \left\| [p_{m,n}][D_{\lambda_n}^{-r/2}]\{f_n\}\right\|_{L^2(\mathbb{N})}^2,
\]

\[
\leq \lambda_j^r \left\| [p_{m,n}]\right\|_M^2 \left\| [D_{\lambda_n}^{-r/2}]\{f_n\}\right\|_{L^2(\mathbb{N})}^2 \leq \lambda_j^r \left\| [P]\right\|_{L^2(U)}^2 \left\| f\right\|_{\lambda^{\lambda^2}_\bullet(U)}^2,
\]

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when $f \in A^r_{\lambda^2}(U)$. It is clear from above that when $f \in U$ only, we get

$$\|(P - P_j)f\|_U^2 \leq \lambda_j^2 \|D_{\lambda_m}^{-r/2}|p_{m,n}\|f_n\|^2_2$$

$$\leq \lambda_j^2 \|D_{\lambda_m}^{-r/2}|p_{m,n}\|f_n\|_U^2$$

$$\leq \lambda_j^2 |P|_U^2 \|\{f_n\}\|_U^2$$

provided that $P \in A^r_{\lambda^2}(U)$. We also see that

$$\|(P - P_j)f\|_U^2 \leq \lambda_j^2 |P|_U^2 \|\{f_n\}\|_U^2$$

when $Pf \in A^r_{\lambda^2}(U)$.

6. The Approximation Spaces $A^r_{\cdot^q}(U)$. In this section we introduce the more general linear approximation spaces $A^r_{\cdot^q}(U)$ for rates $r > 0$ and $1 \leq q \leq \infty$. These will be important to characterize the regularity of a wider variety of Perron-Frobenius or Koopman operators and to determine the rates of convergence of the approximations.

Spectral approximations feature prominently in many of the recent papers that construct approximations in Koopman theory. There are several reasons for introducing the spaces $A^r_{\cdot^q}(U)$, although they do not seem to have been used extensively, or at all, in the study of Koopman theory for dynamical systems. We have chosen to present spectral approximation spaces $A^r_{\lambda^2}(U)$ first as a means of building insight about the more abstract spaces $A^r_{\cdot^q}(U)$. We will see that in some important cases the spectral approximation spaces are special cases of the spaces $A^r_{\cdot^q}(U)$. Essentially, the equivalences result from making assumptions on the rates of convergence of the eigenvalues $\{\lambda_k\}_{k \in \mathbb{N}}$ to zero in the operator expansions in Equations 4.3, 4.4, and 4.5. So approximation spaces provide a reasonable framework to cast approximations in terms of eigenvalues and eigenfunctions of $T_K$ or $T_\mu$, provided their eigenvalues converge to zero at a compatible rate. However, they are much more general and their construction is not given in terms of the eigenstructure of some fixed self-adjoint, compact operator $T$.

One other important reason for the introduction of the scale $A^r_{\cdot^q}(U)$ is simply pragmatics: approximation spaces can be defined in terms of a specific choice of bases used for realizing approximations, although it is also possible to define them in a coordinate free manner just using projections. [10] While the axiomatic foundations of the theory can be abstract, applications of the theory can therefore be a direct source of realizable algorithms. As we have summarized in the introduction, and discuss more fully below, this approach can be used to deduce rates of convergence for a wide variety of bases. These bases include trigonometric polynomials, algebraic polynomials, piecewise algebraic polynomials, splines, and wavelets. This means that in the event that the calculation of eigenfunctions is infeasible, it is still possible to construct estimates and study rates of convergence of approximations.

There is yet another technical reason for pursuing an approximation framework that is not spectral in nature. We have emphasized that the definition of the spectral spaces are essentially tied to a fixed operator $T$, and the approximation rates are stated in terms of eigenvalues of the kernel of a RKHS. In fact, the spectral spaces $A^r_{\lambda^2}(U)$ defined in this way are an example of what is called a “native space” associated with a RKHS. It is a rule of thumb that error rates are easily derived for functions in the native space generated by a particular kernel, but it is often not easy to describe how such estimates can be applied to other more common spaces. Naturally, approximation theorists seek to understand how these approximation rates
in the native space relate to approximation rates in general, “standard” spaces such as Lipschitz, Sobolev, or Besov spaces. A number of such relationships have been derived, see [83]. Researchers who study variants of approximations from RKHS have also referred to this problem as “escaping the native space.” [54]. As a rule, error estimates are more valuable when they apply to a broad family of spaces, not just to ones that are specifically tied to a problem, or operator $T$ at hand.

One other important feature of the approximation spaces $A_{r,q}(U)$ is that there is a rich and systematic theory that establishes the equivalence approximation spaces to interpolation between more common spaces. We will use a few of the fruits of this analysis in our paper, but the interested reader should consult standard references for the details. A discussion of the theory for quite general approximation spaces can be found in classical references such as [60], [58], [19], [2]. See Section E for a very brief overview of the general theory. We will introduce the spaces $A_{r,q}(U)$ here with $U = L^2_\mu(\Omega)$, but the definitions of the approximation spaces $A_{r,q}(V)$ are entirely analogous.

Let $\{A_j\}_{j \in \mathbb{N}_0}$ be a collection of approximant subspaces of a Banach space $U$ that satisfy the properties summarized in the Appendix E. These assumptions include the fact that the approximant spaces are nested and that their closed linear span is dense in $U$. The approximation error $E_n(f,U)$ of $f \in U$ over $A_n$ is given by

$$E_n(f,U) := \inf_{a \in A_n} \|f - a\|_U,$$

and the approximation space $A_{r,q}(U)$ for $r > 0$ is the Banach space

$$A_{r,q}(U) := \left\{ f \in U \mid \|f\|_{A_{r,q}(U)} < \infty \right\}$$

with

$$\|g\|_{A_{r,q}(U)} := \begin{cases} \left( \sum_{n=1}^{\infty} [n^r E_{n-1}(f,U)]^q \right)^{1/q} \frac{1}{n} & 1 \leq q < \infty \\ \sup_{n \geq 1} [n^r E_{n-1}(f,U)]^q & q = \infty \end{cases}$$

There is another useful way to express the norm on $A_{r,2}(U)$ when $U$ is a Hilbert space. Denote by $\Pi_j : U \rightarrow A_j$ the $U$-orthogonal projection of $U$ onto $A_j$ for each $j \in \mathbb{N}_0$. We let $Q_j := \Pi_j - \Pi_{j-1}$ for $j \in \mathbb{N}_0$ and $\Pi_{-1} = 0$ and set

$$[f]_{A_{r,q}(U)} := \left\{ \left( \sum_{j=0}^{\infty} [2^j \|Q_j f\|_U]^q \right)^{1/q} \sup_{j \geq 0} [2^j E_{2j}(f,U)] \right\}^{1/q} \quad 1 \leq q < \infty,$$

$$[f]_{A_{r,q}(U)} := \left\{ \left( \sum_{j=0}^{\infty} [2^j \|Q_j f\|_U]^q \right)^{1/q} \sup_{j \geq 0} [2^j E_{2j}(f,U)] \right\}^{1/q} \quad q = \infty.$$

In this definition $[f]_{A_{r,q}(U)}$ is only a seminorm: the norm is defined as $\|g\|_{A_{r,q}(U)} := \|g\|_U + [g]_{A_{r,q}(U)}$. For $q = 2$ and $U$ a Hilbert space, the space $A_{r,2}(U)$ is a Hilbert space with the inner product given by

$$(f,g)_{A_{r,2}(U)} := (f,g)_U + \sum_{j \in \mathbb{N}} 2^{2jr}(Q_j f, Q_j g)_U = \sum_{j \in \mathbb{N}_0} 2^{2jr}(Q_j f, Q_j g)_U$$

The following theorem is analogous to Theorem 5.1 for the spectral approximation spaces.

**Theorem 6.1.** The approximation spaces are nested,

$$A_{r,2}(U) \subset A_{r,2}(U)$$
for $s > r > 0$. Let $\Pi_j$ be the $U$-orthogonal projection onto the $A_j$ in the definition of $A^{r,2}(U) := A^{r,2}(U, \{ A_j \}_{j \in \mathbb{N}_0})$. If $f \in A^{r,2}(U)$, we have the error estimate

$$\|(I - \Pi_j)f\|_U \leq 2^{-2rj} |f|_{A^{r,2}(U)}.$$ 

Proof. The proof of this theorem resembles the spirit of the proof in Theorem 5.1 when we identify $\lambda_j^{1/2} \sim 2^{-j}$. Nestedness as stated in this theorem is trivial to establish, and in fact there are other nestedness conditions for the general case as summarized in the full theory. [19]. Since $f \in U$, the telescoping series

$$f = \sum_{j \in \mathbb{N}_0} Q_j f$$

converges in $U$. Each $Q_j f$ is perpendicular to all $Q_j f$ for $j \neq i$ by definition, and we have the orthogonal sum

$$\|f - \Pi_j f\|^2_U := \sum_{i \geq j} \|Q_i f\|^2_U \leq \sum_{i \geq j} 2^{2ri} 2^{-2ri} \|Q_i f\|^2_U \leq 2^{-2rj} \sum_{i \geq j} 2^{2ri} \|Q_i f\|^2_U \leq 2^{-2rj} |f|^2_{A^{r,2}(U)}.$$ 

This bound, just like in our analysis of spectral approximation spaces in Theorem 5.1, can be alternatively written as

$$\|(I - \Pi_j)f\|_{A^{r,2}(U)} \leq 2^{-(s-r)j} |f|_{A^{r,2}(U)}$$

for $s > r > 0$ when $f \in A^{r,2}(U)$.

6.1. The Multiscale Structure, Wavelets, and Multiwavelets. Earlier we reviewed how $L^2_\mu(\Omega)$-orthonormal bases can be constructed from the eigenfunctions of linear, compact, self-adjoint operators. In that construction $U := L^2_\mu(\Omega)$ is separable, and so is the reproducing kernel Hilbert space (RKHS) $V$ over a domain $\Omega \subseteq \mathbb{R}^d$. The bases $\{u_i\}_{i \in \mathbb{N}}$ and $\{v_i\}_{i \in \mathbb{N}}$ are natural choices for the spectral approximation spaces $A^{r,2}(U)$ and $A^{r,2}(V)$, respectively. These bases are generally supported globally on $\Omega$ and have no multiscale structure.

In this section we describe some particular bases that, unlike the bases of eigenfunctions, exhibit a multiscale structure. That is, the bases are defined by introducing a family of nested grids over which the basis functions are defined. Here, the bases are selected to be well-known examples of orthonormal wavelets or multiwavelets for the separable Hilbert space $U$. We begin with a rather general overview of the structure of multilevel decompositions induced by such multilevel bases in Section 6.2. We then discuss examples that are applicable to the spaces $U := L^2(\Omega)$. These bases enable development of the theory of approximation spaces $A^{r,2}(L^2(\Omega))$ in a transparent fashion. Such spaces will be applicable to Koopman theory if we know that the measure of interest has the form $\mu(dx) := m(x)dx$ for an integrable function that satisfies $c_1 \leq m(x) \leq c_2$ almost everywhere in $\Omega$ for two positive constants $c_1, c_2$. We finally discuss approximation spaces defined in terms of warped wavelet and warped multiwavelet bases, which enable the treatment of a wider class of measures $\mu$.

6.2. An Overview of The Multiscale Structure. This section summarizes the structure and indexing of multiscale bases, especially those that arise from wavelets and multiwavelets. The notation is fairly standard in the study of multigrid methods.
or multiresolution analyses. Our discussion is necessarily brief, see [72, 14, 18, 10] for more examples of this common notation.

Suppose that we are given an orthonormal wavelet basis \( \{ \psi_{j,k} \mid j \in \mathbb{N}, k \in \Gamma_{\psi}^j \} \subset U \) with \( \Gamma_{\psi}^j \) the family of admissible indices for each fixed \( j \). Roughly speaking, the integer \( j \) denotes the mesh resolution level and \( k \in \Gamma_{\psi}^j \) ranges over all the functions that are defined on that mesh level. We mostly only consider tensor product (multi)wavelets for \( \Omega \subseteq \mathbb{R}^d \) when \( d > 1 \) to keep our discussion elementary. Example 4 provides one easy example of a domain that is not a tensor product and is included to demonstrate the numbering and indices for such a multiwavelet basis. Such domains, ones that are tilings of a few master subdomains, are one class over which Haar multiwavelets are often straightforward to construct. This class could be important to approximations in Koopman theory since many examples study dynamics over partitions of some domain. Multiscale structures over more general domains can be quite complicated, see [12, 11] for examples. We take the mesh on level \( j \) to be finite union of some collection of dyadic cubes \( \Box_{j,m} \subseteq \Omega \subseteq \mathbb{R}^d \) that have the form

\[
\Box_{j,m} := \left\{ x \in \mathbb{R}^d \ \bigg| \ 2^{-j}m \leq x_i \leq 2^{-j}(m+1), 1 \leq i \leq d \right\}
\]

for \( m \in \mathbb{Z}^d \). The wavelet spaces consist of all the wavelet functions \( \psi_{j,k} \) that are defined on mesh level \( j \),

\[
W_j := \text{span} \left\{ \psi_{j,k} \mid k \in \Gamma_{\psi}^j \right\}.
\]

As explained in more detail in the Examples 21 and 22, the indexing of the functions \( \psi_{j,k} \) in the spaces \( W_j \) uses an overloaded definition that accounts for a variety of possibilities. This notation inherently is designed to reflect as multiscale structure in the analysis that follows. In most of our examples these bases will initially be defined over \( \mathbb{R}^d \), and subsequently the set of basis functions is modified so that that they are a basis for functions over a compact set \( \Omega \). Adapting a given set of wavelets defined over \( \mathbb{R}^d \) to a general set \( \Omega \) is a delicate, difficult, and lengthy process in general. [8, 12, 11] All of our examples are carried out when \( \Omega \) is \([0,1]^d \) or the \( d \)-dimensional torus \( \mathbb{T}^d \). Modifications of a global basis on \( \mathbb{R}^d \) to life on the torus \( \mathbb{T}^d \) are particularly simple since it is only necessary to periodize a finite set of functions on each grid resolution level. Restriction of Haar bases to \([0,1]^d \) is trivial. The modification of the multiscale orthonormal bases of [21, 20] to \([0,1]^d \) for Neumann or Dirichlet boundary conditions is also relatively easy and discussed in these papers. With these considerations in mind, we suppress the bold font style in the remainder of this section for entries \( k \in \Gamma_{\psi}^j \) and just use \( k \) to denote in the set of admissible functions on mesh level \( j \).

We define the approximant spaces \( A_j := \cup_{0 \leq m \leq j-1} W_m \) in terms of the wavelet spaces \( W_j \), and they satisfy \( A_{j+1} = A_j \oplus W_j \) for each \( j \in \mathbb{N}_0 \). [14, 72] The spaces \( A_j \) are also often referred to as the space of scaling functions on level \( j \) in the literature on multiresolution analysis. We denote by \( \Pi_j \) and \( Q_j \) the orthogonal projections on \( A_j \) and \( W_j \), respectively. Common wavelet bases \( \{ \psi_{j,k} \}_{j \in \mathbb{N}, k \in \Gamma_{\psi}^j} \) are constructed so that any \( f \in U \) can be written in the so-called multiscale expansion given by the \( U \)-convergent series

\[
f = \sum_{j \in \mathbb{N}_0} \sum_{k \in \Gamma_{\psi}^j} (f, \psi_{j,k})_U \psi_{j,k}.
\]
The outer summation over $j$ runs over the grids each having uniform mesh parameter $2^{-j}$, and the inner summation spans all the possible functions on a particular grid. In many cases, the summation above in $j$ is expressed over all $0 \leq j_0 \leq j \in \mathbb{N}_0$, where $j_0$ denotes the resolution level of the coarsest grid in the summation. Note that here we have followed the usual convention, discussed more carefully in Section 6.1, of subsuming the scaling functions in the coarsest wavelets on level $j = 0$ or $j = j_0$ in this multiscale representation. [10, 17].

We next discuss how multiscale bases of this type can be realized.

6.3. $L^2(\Omega)$—Orthonormal Wavelets and Multiwavelets. Among the tens of thousands of wavelet and multiwavelet papers that have been published over the past three decades, we have elected to express the theory in our paper using only orthonormal wavelets and multiwavelets. As mentioned in the introduction, this assumption is hardly necessary, but it keeps the treatment intuitive. There is a well-documented collection of $L^2(\Omega)$—orthonormal wavelet systems that can be used to realize the multilevel setup in the last section [14, 21, 20] when the measure $\mu$ is just Lebesgue measure. Much has been said about orthonormal wavelets and multiwavelets in the literature for this case, as well as their generalizations that yield the biorthogonal wavelet families. To review them specifically and in detail here would distract from our primary goal, the determination of convergence rates of approximations of operators in Koopman theory. For those who have not seen these constructions before, we give a detailed summary of the Daubechies compactly supported orthonormal wavelets and some compactly supported orthonormal multiwavelets in Examples 21 and 22, respectively, in the Appendix in Section A.

Here we discuss just two concrete examples of the low approximation order orthonormal wavelets and multiwavelets. These are of course the Haar wavelets and a generalization of them that results in some simple multiwavelets on a triangular domain. We include these since they give clear definitions of the scaling functions $\phi_{j,k}$, wavelets $\psi_{j,k}$, as well as the indexing sets $\Gamma_{\phi}^j$ and $\Gamma_{\psi}^j$ in specific cases. The multiscale decomposition is easy to understand in these canonical examples.

**Example 3 (Haar Wavelets for $d = 1$).** A discussion of the Haar wavelet system appears in many references on wavelets as a beginning example. Despite the simplicity of this basis choice, it should be of interest to researchers that study Koopman theory since so many examples deal with approximations by piecewise constants over partitions of the domain. In one dimension the Haar scaling function $\phi : \mathbb{R} \to \mathbb{R}$ is a constant function, and the Haar wavelet $\psi : \mathbb{R} \to \mathbb{R}$ is a piecewise constant function, over the domain $\Omega := [0, 1)$. They are written as

$$
\phi(x) := \begin{cases} 
1 & x \in [0, 1) \\
0 & \text{otherwise}
\end{cases},
$$

$$
\psi(x) := \phi(2x) - \phi(2x - 1).
$$

We define a family of nested dyadic grids on $\Omega$ formed from dyadic cubes $\Box_{j,k}$ having sidelength $2^{-j}$ that are given by

$$
\Box_{j,k} := \{ x \in \Omega \mid 2^{-j}k \leq x < 2^{-j}(k + 1) \}
$$

for $k = 0, \ldots, 2^j - 1$. The grid that consists of a union of such cubes is said to have resolution level $j$. 39
Based on this enumeration we define the index sets \( \Gamma^\phi_j := \Gamma^\psi_j := \{ k : 0 \leq k \leq 2^j - 1 \} \), and the admissible scaling functions \( \phi_{j,k} := 2^{j/2} \phi(2^j x - k) \) for \( k \in \Gamma^\phi_j \) and wavelets \( \psi_{j,k} := 2^{j/2} \psi(2^j x - k) \) for \( k \in \Gamma^\psi_j \). It can be shown that \( \{ \phi_{j,k} \}_{k \in \Gamma^\phi_j} \) are \( L^2(\Omega) \)-orthonormal for each fixed grid having resolution level \( j \). Also, the family \( \{ \psi_{j,k} \}_{j \in \mathbb{N}_0, k \in \Gamma^\psi_j} \) are \( L^2(\Omega) \)-orthonormal. In a typical approximation problem in \( \Omega \subset \mathbb{R}^d \), we set

\[
A_j := \text{span}\{ \phi_{j,k} \mid k \in \Gamma^\phi_j \} = \text{span}
\{ 1_{\square_{j,k}} \mid k \in \Gamma^\phi_j \}
= \text{span}\{ \psi_{i,k} \mid i \leq j, \ k \in \Gamma^\psi_i \},
\]

so that \( n_j := \#(A_j) = 2^j \). As noted above, we can alternatively expand the family of wavelets to subsume the scaling functions in the coarsest scale. One way to do this is by setting \( \psi_{-1,0} := \phi \) and redefining the index set for the wavelets as

\[
\Gamma^\psi_j := \begin{cases} 
0, \ldots, 2^j - 1 & j \geq 0 \\
0 & j = -1.
\end{cases}
\]

Then we also can write

\[
A_j := \text{span}\{ \psi_{i,k} \mid i \leq j, \ k \in \Gamma^\psi_i \}.
\]

Of course we still have \( n_j = 2^j \). Any \( f \in L^2(\Omega) \) has the multiscale decomposition

\[
f = \sum_{j \in \mathbb{N}_0} \sum_{k \in \Gamma^\psi_j} (f, \psi_{j,k})_{L^2(\Omega)} \psi_{j,k},
\]

and its orthogonal projection \( f_j \in A_j \) can be written

\[
f_j := \Pi_j f := \sum_{k \in \Gamma^\psi_j} (f, \phi_{j,k})_{L^2(\Omega)} \phi_{j,k} = \sum_{i \leq j} \sum_{k \in \Gamma^\psi_i} (f, \psi_{i,k})_{L^2(\Omega)} \psi_{i,k}.
\]

This construction is based on the assumption that \( d = 1 \), but the extension to \( \Omega = [0,1)^d \) via tensor products of the wavelets is routine. An explicit discussion of the process for defining tensor products is given in Examples 21 and 22 in the Appendix. From Theorem E.1 in the Appendix, we know that the approximation space generated by the Haar wavelets \( A^{r,\infty} (L^p(\Omega); \{ A_j \}_{j \in \mathbb{N}_0}) \) is equivalent to the

generalized Lipschitz space \( \text{Lip}^r(\alpha, L^p(\Omega)) \) for all \( 0 < \alpha < r - 1 + 1/p \). This means that we have
\[
\| (I - \Pi_j)f \|_{L^2(\Omega)} \lesssim 2^{-\alpha j} |f|_{\text{Lip}^r(\alpha, L^2(\Omega))}
\]
for all functions \( f \in \text{Lip}^r(r, L^2(\Omega)) \) over the range \( 0 < \alpha < 1/2 \), for example. When \( p = \infty \) and \( f \in UC(\Omega) \), we also have
\[
\| (I - \Pi_j)f \|_{L^\infty(\Omega)} \lesssim 2^{-\alpha j} |f|_{\text{Lip}(\alpha, UC(\Omega))}
\]
over the same range.

As a final observation, if one wants to generate a partition of the domain \( \Omega := [0,1]^d \), the argument above proceeds the same but the definition of the last scaling function on each dyadic level in each coordinate direction is modified. The wavelets on each dyadic level are correspondingly modified and a multiscale decomposition over \( \Omega := [0,1]^d \) is defined.

**Example 4 (Haar Multiwavelets over Triangles).** In this example we consider a slight generalization of the Haar wavelets to define a system of multiwavelets over a different domain, one that is not a product of compact sets. It is an example of a domain that is a self-similar tiling of itself. This construction can be carried out for many similar self-similar tilings, ones that seem to arise often in Koopman theory. The point of this example is to illustrate the form of the numbering schemes and indices for similar multiwavelet systems. The numbering here differs only slightly from the preceding case, but illustrates the situation when there are two indices \((i,k)\), the multiscaling function or multiwavelet number \( i \) and the translation \( k \).

![Fig. 6.2: Triangular Domain and Nested Grids](image)
We define the triangles $\triangle := \triangle_{0,0}$ and $\triangle := \triangle_{1,1}$ on grid level $j = 0$ as

$$\triangle := \triangle_{0,0} := \{(x,y) \mid y \geq 0, x \geq 0, y + x < 1\},$$

$$\triangle_{1,1} := \{(x,y) \mid y < 1, x < 1, y + x \geq 1\},$$

and set the translations $\triangle_{1,0}$ and $\triangle_{0,1}$ of $\triangle_{0,0}$ as depicted in Figure 6.3. The vertical and horizontal side of each triangle $\triangle_{0,0}, \triangle_{1,0}, \triangle_{0,1}$ on grid level $j = 0$ is 1, while $\triangle_{1,1}$ is equal to 1 over its hypotenuse. By recursion, the grid of level $j$ is obtained by the uniform subdivision illustrated in Figure 6.2 where each cell is a scaled and translated copy of $\triangle_{0,0}$ or $\triangle_{1,1}$. A triangle $\triangle_{j,k}$ on grid level $j$ is defined as

$$\triangle_{j,k} := \{x \in \mathbb{R}^2 \mid 2^j x - k \in \triangle_{0,0} \text{ or } \triangle_{1,1}\}$$

for some $k \in \mathbb{Z}^2$. We define the index set $\Gamma_j$ for grid level $j$ to be the indices of the triangles $\triangle_{j,k}$ that meet $\triangle_{0,0}$. In view of the complimentary boundary conditions of $\triangle_{0,0}$ and $\triangle_{1,1}$, the triangles $\{\triangle_{j,k}\}_{k \in \Gamma_j}$ form a partition of $\triangle_{0,0}$.

We set the multiscaling function

$$\phi(x) := \begin{cases} 
\phi_1(x) \\
\phi_2(x) \\
\phi_3(x) \\
\phi_4(x)
\end{cases} := \begin{cases} 
1_{\triangle_{0,0}}(x) \\
1_{\triangle_{1,0}}(x) \\
1_{\triangle_{0,1}}(x) \\
1_{\triangle_{1,1}}(x)
\end{cases}.$$

The $L^2(\Omega)$–orthonormal scaling functions on grid level $j$ are given by

$$\phi_{j,(i,k)}(x) := \frac{1}{|\triangle_{j,k}|^{1/2}} \phi_i(2^j x - k)$$

for $i \in \{1,2,3,4\}$ and some $k \in \mathbb{Z}^2$. For each triangle $\triangle_{j,\ell} \subset \triangle_{0,0}$ there is a unique function $\phi_{j,(i,k)}$ that is supported on $\triangle_{j,\ell}$. We define the collection of admissible indices $\Gamma_j^\phi$ for the multiscaling functions as

$$\Gamma_j^\phi := \{(i,k) \mid i \in \{1,\ldots,4\}, (i,k) \text{ corresponds to } \triangle_{j,\ell} \text{ for some } \ell \in \Gamma_j\},$$

so that the approximant space on level $j$ is then

$$A_j := \text{span}\left\{ \phi_{j,(i,k)} \mid j \in \mathbb{N}_0, (i,k) \in \Gamma_j^\phi \right\}.$$

We have $n_j := \#(A_j) = 2^{dj}$ with $d = 2$. There are a number of ways to define wavelets for this system. We select the multiwavelet $\psi := \{\psi_1, \psi_2, \psi_3\}$ to be the functions

$$\psi_1(x) := (\phi_1(x) - \phi_4(x))/\sqrt{2},$$

$$\psi_2(x) := (\phi_2(x) - \phi_3(x))/\sqrt{2},$$

$$\psi_3(x) := [(\phi_1(x) + \phi_4(x)) - (\phi_2(x) + \phi_3(x))]/2,$$
and then the scaled and translated wavelets are

\[ \psi_{j,(i,k)} := \frac{1}{|\triangle_{j,k}|^{1/2}} \psi_i(2^j x - k) \]

for some \( k \in \mathbb{Z}^2 \). We define the index set of admissible wavelets on level \( j \) to be \( \Gamma_j^\psi \). The \( L^2(\triangle) \)-orthonormal projection of any \( f \in L^2(\Omega) \) onto \( A_j \) is given by the single scale expansion

\[ \Pi_j f := \sum_{(i,k) \in \Gamma_j^\psi} (f, \psi_{j,(i,k)})_{L^2(\Omega)} \phi_{j,(i,k)}, \]

and it has the multiscale representation

\[ f = (f, \phi_{0,(1,0)})_{L^2(\Omega)} \phi_{0,(1,0)} + \sum_{j \in \mathbb{N}_0} \sum_{(i,k) \in \Gamma_j^\psi} (f, \psi_{j,(i,k)})_{L^2(\Omega)} \psi_{j,(i,k)}. \]

As above, or as in Examples 21 and 22, the above multiscale expression can be simplified somewhat by subsuming the scaling function into the collection of wavelets on the coarsest level. We set \( \psi_{-1,0} := \phi_{0,(1,0)} \) and modify the index set \( \Gamma_j^\psi \) accordingly to get

\[ f = \sum_{j \in \mathbb{N}_0} \sum_{(i,k) \in \Gamma_j^\psi} (f, \psi_{j,(i,k)})_{L^2(\Omega)} \psi_{j,(i,k)}. \]

The approximation of functions \( f \in C(\Omega) \subset L^2(\triangle) \) can be constructed in terms of the family of operators \( \tilde{\Pi}_j : C(\Omega) \to A_j \) with

\[ (\tilde{\Pi}_j f(x) := \sum_{\ell \in \Gamma_j} f(\xi_{j,\ell}) 1_{\triangle_{j,\ell}}(x), \]

with \( \{\xi_{j,\ell}\}_{\ell \in \Gamma_j} \) the centroids of the triangles \( \triangle_{j,\ell} \). Following essentially the same steps as in Example 14, we find that

\[ \| (I - \tilde{\Pi}_j) f \|_{C(\Omega)} \lesssim 2^{-rj} |f|_{L^{\infty}(\Omega)}. \]

This situation is a particular example of the more general bound

\[ \| (I - \tilde{\Pi}_j) f \|_{L^p(\Omega)} \lesssim 2^{-rj} |f|_{L^{p}(\Omega)} \]

in Equation 6.16 of [18].

As in the last section, the analysis above can be carried out over the compact set \( \Omega = \overline{\triangle_{0,0}} \) by a simple modification of all scaling functions and multiwavelets that meet the hypotenuse of \( \triangle_{0,0} \).

Using \( L^2(\Omega) \) Wavelets for Approximations in \( L^2_\mu(\Omega) \). The last two examples show that there are a large family of multiscale orthonormal wavelets and multiwavelets that generate a basis for \( L^2(\Omega) := L^2_\mu(\Omega) \) with \( \mu \) simple Lebesgue mea-
If $\mu$ is not Lebesgue measure, warped wavelets [31] based on modifications $L^2(\Omega)$ orthonormal wavelets can be shown to be $L^2_{\mu}(\Omega)$-orthonormal for a reasonable class of measures $\mu$. A discussion of warped wavelets is presented in Section 6.4. However, the approximation of functions in $L^p_{\mu}(\Omega)$ for certain types of measures can also be constructed directly with the bases on $L^2(\Omega)$. Suppose $\Omega \subset \mathbb{R}^d$ is compact, $\mu(dx) := m(x)dx$, and that there are two constants $c_1, c_2$ such that $0 < c_1 \leq m(x) \leq c_2 < \infty$ for $x$ a.e. in $\Omega$. It is immediate that

$$\|f\|_{L^2(\Omega)} \approx \|f\|_{L^2_{\mu}(\Omega)},$$

that is, the two norms are equivalent. The sets $L^2(\Omega)$ and $L^2_{\mu}(\Omega)$ contain the same collection of functions, and the topology on the two spaces is the same. Any $f \in L^2_{\mu}(\Omega)$ can therefore be approximated in terms of the basis $\{\psi_{j,k} \mid j_0 \leq j \in \mathbb{N}_0, k \in \Gamma^\psi_j\}$ for $L^2(\Omega)$, and the approximation spaces $A^{r,q}(L^2(\Omega))$ can be used to approximate functions in $L^2_{\mu}(\Omega)$. In fact, we have

$$A^{r,2}(L^2(\Omega), \{A_j\}_{j \in \mathbb{N}_0}) \equiv A^{r,2}(L^2_{\mu}(\Omega), \{A_j\}_{j \in \mathbb{N}_0})$$

with $A_j := \text{span}\{\phi_{j,k} \mid k \in \Gamma^\phi_j\}$, and the norms on these spaces are equivalent.

### 6.4. Warped Wavelets and Multiwavelets

In this section we discuss the method of warped wavelets [31] and multiwavelets. This approach will be useful for cases when the Koopman or Perron-Frobenius operators are defined over a domain of interest $\tilde{\Omega}$ that is related to a master domain $\Omega$ by a change of variables. In some cases we can think of $\Omega$ as the original domain over which initial conditions are defined, and $\tilde{\Omega}$ as the image of the initial domain under a change of variables. While we are careful to distinguish the initial and image domains in the theory discussed here, the approach is of course applicable to the common examples when the change of variables maps the initial domain onto itself. In both scenarios the measures $\mu$ and $\tilde{\mu}$ are defined over $\Omega$ and $\tilde{\Omega}$, respectively. Figure 6.4 depicts a domain of interest $\tilde{\Omega} \subset \mathbb{R}^d$ that is the image of the master domain $\Omega := [0,1]^d$ under a suitably smooth change of coordinates $\Omega = \tilde{M}(\tilde{\Omega})$. We write $x := \tilde{M}(\tilde{x})$ for each $x \in \Omega$ and $\tilde{x} \in \tilde{\Omega}$, and the determinant of the Jacobian matrix is given by

$$m(\tilde{x}) := \left| \frac{\partial x}{\partial \tilde{x}} \right| := \left| \frac{\partial \tilde{M}}{\partial \tilde{x}} \right|.$$
Let \( \{ \psi_{j,k} \mid j \in \mathbb{N}_0, k \in \Gamma_j^\psi \} \) be any of the orthonormal bases of \( L^2(\Omega) \) constructed above from wavelets or multiwavelets over the master domain \( \Omega \). The bases discussed in Examples 21 and 22 are just two possibilities that can be employed to construct the warped wavelets here. We define an associated collection of bases over the domain \( \tilde{\Omega} \) from the identity

\[
\tilde{\psi}_{j,k}(\tilde{x}) := \psi_{j,k}(\tilde{M}(\tilde{x})),
\]

for \( j \in \mathbb{N}_0 \) and \( k \in \Gamma_j^\psi \). The family \( \{ \tilde{\psi}_{i,k} \}_{i \in \mathbb{N}_0, k \in \Gamma_i^\psi} \) are the warped wavelets generated by \( \{ \psi_{j,k} \} \). We have the integration formula that results from the change of variables,

\[
\delta_{(j,k),(\ell,m)} = \int_{\tilde{\Omega}} \tilde{\psi}_{j,k}(\tilde{x}) \tilde{\psi}_{\ell,m}(\tilde{x}) d\tilde{x} = \int_{\Omega} \psi_{j,k}(M(\tilde{x})) \psi_{\ell,m}(M(\tilde{x})) \left| \frac{\partial \tilde{x}}{\partial x} \right| d\tilde{x}.
\]

The orthonormality of the basis \( \{ \psi_{j,k} \mid j \in \mathbb{N}_0, k \in \Gamma_j^\psi \} \) in the \( L^2(\Omega) \) inner product on \( \Omega \) implies that the basis \( \{ \tilde{\psi}_{j,k} \mid j \in \mathbb{N}_0, k \in \Gamma_j^\psi \} \) is orthonormal in the usual \( \tilde{\mu} \)-weighted inner product

\[
(\tilde{f}, \tilde{g})_{L^2_{\tilde{\mu}}(\tilde{\Omega})} := \int_{\tilde{\Omega}} \tilde{f}(\tilde{x}) \tilde{g}(\tilde{x}) \tilde{\mu}(d\tilde{x})
\]

over \( \tilde{\Omega} \). We finally define the Hilbert space \( \tilde{U} \) as the completion of the set of orthonormal functions \( \{ \tilde{\psi}_{j,k} \mid j \in \mathbb{N}_0, k \in \Gamma_j^\psi \} \) defined over \( \tilde{\Omega} \) in the above \( \tilde{\mu} \)-weighted inner product. The approximation spaces \( A^{r,q}(\tilde{U}) \) are then defined in the usual manner, according the definitions in Section 6 or Appendix E. When the measure \( \tilde{\mu} \) is known, the approximation spaces can be used to build and measure rates of approximations of the Koopman and Perron-Frobenius operators for functions over \( \tilde{\Omega} \). The error rates are identical to that in Theorem 6.1 with \( U \) replaced by \( \tilde{U} \). Examples in Section 8 illustrate that such warped bases are also important in deriving data-dependent approximations from samples.

Note carefully that the discussion above refers to (at least) three different spaces of square integrable functions. We have the space \( L^2(\Omega) \) that is the the usual space of real-valued, Lebesgue square-integrable functions over \( \Omega \). We also refer to the usual \( \tilde{\mu} \)-weighted space of square-integrable functions \( L^2_{\tilde{\mu}}(\tilde{\Omega}) \) and its inner product over \( \tilde{\Omega} \). Finally, we define \( \tilde{U} \) as the Hilbert space of functions defined over \( \tilde{\Omega} \) as the completion of the finite linear span of warped wavelets in the \( \tilde{\mu} \)-weighted inner product on \( L^2_{\tilde{\mu}}(\tilde{\Omega}) \).

Since each of the wavelets \( \tilde{\psi}_{j,k} \) is contained in \( L^2_{\tilde{\mu}}(\tilde{\Omega}) \), we know that

\[
\tilde{U} \subseteq L^2_{\tilde{\mu}}(\tilde{\Omega}),
\]

In general the study of the properties of weighted approximation spaces can be delicate and the reader should see [31] for a presentation of the theory, or consult [4, 5] to see pragmatic adaptive estimators that rely on these spaces.

We discuss in Examples 8, 9, 15 the use of warped wavelets in the approximation of Koopman and Perron-Frobenius operators.
6.5. Approximation of $\mathcal{P}, \mathcal{U}$ over $A^r,2(U)$. Recall that the approximation space $A^r,2(U)$ of order $r \geq 0$ is defined in terms of the seminorm $| \cdot |_{A^r,2(U)}$. When the basis $\{ \psi_{j,k} \mid j \in \mathbb{N}_0, k \in \Gamma^y_j \}$ is orthonormal, it can be expressed as

$$A^r,2(U) := \left\{ f \in U \left| \| f^2_{A^r,2(U)} \| := \sum_{j \in \mathbb{N}_0} 2^{2rj} \| Q_j f \|_U^2 \right. \right\},$$

$$= \left\{ f \in U \left| \| f^2_{A^r,2(U)} \| := \sum_{j \in \mathbb{N}_0} 2^{2rj} \sum_{k \in \Gamma^y_j} |(f, \psi_{j,k})_U|^2 \right. \right\}. $$

Here $Q_j$ the orthogonal projection operator from $U$ onto $W_j$, and $Q_j f$ is simply the sum of the generalized Fourier coefficients for $k \in \Gamma^y_j$ of the function $f$. We now construct approximations of either Koopman or Perron-Frobenius operators using these spaces. In this section we will state the primary results in terms of the Perron-Frobenius operator $\mathcal{P}$, but approximations of the Koopman operator then follow by duality. The discussion that follows is structured like our presentation of the spectral duality. The discussion that follows is structured like our presentation of the spectral spaces described in Theorem 5.1. Corresponding to the family of self-adjoint operators in $A^r,2(U)$, we define the family of admissible self-adjoint Perron-Frobenius operators $\mathcal{K}^r,2(U)$ to be

$$\mathcal{K}^r,2(U) := \left\{ \mathcal{P} := \sum_{j \in \mathbb{N}_0, k \in \Gamma^y_j} p_{j,k} \psi_{j,k} \otimes \psi_{j,k} \in S^\infty(U) \mid |\mathcal{P}|_{A^r,2(U)} < \infty \right\},$$

with the seminorm in the above expression given by

$$|\mathcal{P}|^2_{A^r,2(U)} := \sum_{j \in \mathbb{N}_0} 2^{2rj} \sum_{k \in \Gamma^y_j} |p_{j,k}|^2.$$ 

These operators $\mathcal{K}^r,2(U)$ have the same intuitive interpretation as the spectral spaces $\mathcal{K}^r,2(U)$. The higher the rate $r > 0$, the faster the coefficients $\{p_{j,k}\}$ must converge to zero. Membership in $\mathcal{K}^r,2(U)$ is equivalent to the requirement that the sequence $\{2^{2r}p_{j,k} : j \in \mathbb{N}_0, k \in \Gamma^y_j\}$ is in $\ell^2$. We follow the same general procedure used in the study of the spectral spaces $\mathcal{K}^r,2(U)$. We initially define $\mathcal{K}^r,2(U)$ to consist only of certain self-adjoint operators. This makes the derivation of the error bounds in Theorem 6.2 proceed easily. Then, in Section 6.6 and Theorem 6.3, we show that the same error bounds that are derived for the self-adjoint case apply when $\mathcal{K}^r,2(U)$ is defined as in Equation 6.5.

The following theorem, that gives the fundamental error estimates for the operators when the priors are expressed in terms of the spaces $A^r,2(U)$ or the self-adjoint operators $\mathcal{K}^r,2(U)$, is the analog of the results for the spectral spaces described in Theorem 5.1.

**Theorem 6.2.** Suppose that $\{ \psi_{j,k} \}_{j \in \mathbb{N}_0, k \in \Gamma_j}$ is any orthonormal basis for $U$ and $\mathcal{P}$ has the representation

$$\mathcal{P} := \sum_{j \in \mathbb{N}_0} \sum_{k \in \Gamma_j} p_{j,k} \psi_{j,k} \otimes \psi_{j,k} \in S^\infty(U).$$

When we define $\mathcal{P}_j := \sum_{0 \leq m < j} \sum_{k \in \Gamma_m^y} p_{m,k} \psi_{m,k} \otimes \psi_{m,k}$, we have

$$\| (\mathcal{P} - \mathcal{P}_j) f \|_U \lesssim 2^{-rj} |\mathcal{P} f|_{A^r,2(U)}$$

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whenever $Pf \in A^{r,2}(U)$. Whenever we have $P \in A^{r,2}(U)$ and $s > r$, we also have

$$|P - P_j|_{A^{r,2}(U)} \lesssim 2^{-j(s-r)} |P|_{A^{r,2}(U)}.$$ 

Proof. The proof of these results follows, with slight modifications, the analysis for the spectral approximation spaces in Theorem 5.1. We express the error as

$$\| (P - P_j) f \|_U^2 = \sum_{\ell \geq j} \sum_{k \in \Gamma_\ell^v} |p_{\ell,k}|^2 |(f, \psi_{\ell,m})_U|^2 \leq \sum_{\ell \geq j} \sum_{k \in \Gamma_\ell^v} |p_{\ell,k}|^2 2^{2\ell \tau - 2\ell \tau} |(f, \psi_{\ell,m})_U|^2 \leq 2^{-2j \tau} \sum_{\ell \in \mathbb{N}_0} \sum_{k \in \Gamma_\ell^v} 2^{2\ell \tau} |p_{\ell,m}(f, \psi_{\ell,m})_U|^2 \leq 2^{-2j \tau} |Pf|_{A^{r,2}(U)}^2.$$ 

We also have

$$|P - P_j|_{A^{r,2}(U)}^2 = \sum_{\ell \geq j} \sum_{k \in \Gamma_\ell^v} 2^{2\ell \tau} |p_{\ell,k}|^2 \leq \sum_{\ell \geq j} \sum_{k \in \Gamma_\ell^v} 2^{2\ell \tau} 2^{-2\ell \tau} 2^{2\ell \tau} |p_{\ell,k}|^2 \leq 2^{-2j(s-r)} \sum_{\ell \in \mathbb{N}_0} \sum_{k \in \Gamma_\ell^v} |p_{\ell,k}|^2 = 2^{-2j(s-r)} |Pf|_{A^{r,2}}^2.$$ 

The error bounds for the spectral approximation spaces in Theorem 5.1 should be carefully compared to those in the above theorem. It is important to note in this comparison that the rates in the spectral approximation spaces are referenced to the eigenvalues of a fixed self-adjoint operator, but those in the above theorem are independent of such operator dependence.

Also, before discussing some applications of this theorem, we discuss the relationship between the mesh resolution level $j$, the projector $\Pi_j : U \to A_j$, and the dimension $n_j := \#A_j$ of the finite dimensional approximation spaces $A_j$. In one spatial dimension, on a compact domain such as $\Omega = [0,1]$ or $\Omega = \mathbb{T}$ for instance, we frequently have

$$n_j := \#A_j \approx 2^j.$$ 

This order of dimension holds for any of the Daubechies wavelets, Coiflets, or orthonormal multiwavelets in one dimension. Now suppose we estimate a function $f : \Omega \to \mathbb{R}^d$ for a compact set $\Omega \subset \mathbb{R}^d$. In our examples that follow that use tensor products we have

$$n_j := \#A_j \approx d \cdot 2^j \approx 2^j,$$

where the rightmost equivalence reflects only the asymptotics in the mesh resolution level $j$. This means that an alternative form of the error bounds can be written as

$$\| (I - \Pi_j) f \|_U \lesssim n_j^{-r/d} |f|_{A^{r,2}},$$

$$\| (P - P_j) f \|_U \lesssim n_j^{-r/d} |Pf|_{A^{r,2}(U)},$$

with the multiplying constant a function of $d$, like $Cd^{-r/d}$ with $C$ independent of $d, j$. 

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Example 5 (Haar Wavelet Approximation of \( P : L^2(\Omega) \to L^2(\Omega) \)). In this first example we study a problem when the Perron-Frobenius operator is induced by a kernel \( p \) as in Equation 1.1. The problem is motivated by examples in Sections 2 and 3 of the publication [27] that has appeared in 2018. This paper studies a class of discrete dynamical systems and approximations of the Koopman operator in terms of permutation operators on measurable partitions. As a fundamental step, the approach employs estimates expressed in terms of finite dimensional spaces of characteristic functions over the partitions. Here we study a related problem to see how rates of convergence can be derived in this setting.

We assume the domain \( \Omega = [0,1]^d \) and that the measure \( \mu(dx) := m(x)dx \) for some function \( m \). We further suppose that there exists a pair of constants \( c_1, c_2 > 0 \) such that \( c_1 \leq m(x) \leq c_2 \) for all \( x \in \Omega \). We know then that \( m \in L^\infty(\Omega) \subset L^2(\Omega) \) since the domain is compact and that \( L^2_m(\Omega) \approx \approx L^2(\Omega) \) in this case. We express the domain \( \Omega \) as the union of dyadic cubes \( [j,k] \subset \mathbb{R}^d \) following the definition described in Example 3 for \( d = 1 \). We extend the definition of the Haar scaling functions for \( d = 1 \) given in Example 3 to \( d > 1 \) using the strategy described in Examples 21 or 22 for tensor products, and define the Haar scaling functions \( \phi : \mathbb{R}^d \to \mathbb{R} \). From these we define the scaled and translated Haar scaling functions \( \phi_{j,k} := 2^{jd/2} \phi(2^j x - k) \) and their associated tensor product wavelets \( \psi_{j,k} := 2^{jd/2} \psi(2^j x - k) \) for \( e \in \{0,1\}^d \). These are used to construct the orthonormal basis for \( A_j \subset U \). Finally we construct the approximation spaces \( A_r^\perp(A^2(L^2(\Omega))) \) in Equation 6.3 in terms of the vector-valued, tensor product basis \( \{ \psi_{j,k} : e \in \{0,1\}^d, \ell \leq j, k \in \Gamma_j^\perp \} \). With a bit of bookkeeping, we find that the construction of the tensor product scaling functions and wavelets yields \( n_j := \#(A_j) \approx O(2^{\ell j}) \). When the Perron-Frobenius operator \( P \in A_r^\perp(A^2(L^2(\Omega))) \), we are guaranteed from Theorem 6.2 that \( \| (P - P_j) f \|_U \approx O(n_j^{-r/d}) \).

In addition, in this example problem, we have \( P^* = P = U \). When a kernel \( p(x,y) \) induces \( P \), then \( P^* \) is induced by the kernel that results when \( x,y \) are interchanged. We have \( p(x,y) = \sum_{j,k} p_{j,k}(x) \psi_{j,k}(y) \) in this problem, which is clearly symmetric in \( x,y \). Therefore,

\[
\| (P - P_j) f \|_U = \| (P - P_j^*) f \|_U = \| (U - U_j) f \|_U \lesssim n_j^{-r/d} \| P \|_{A^r} \| f \|_U
\]

when we observe that \( U_j := P_j^* = P_j \). Later in this paper we consider the case when the kernel \( p : \Omega \times \Omega \to \mathbb{R} \) may not be symmetric, when it has a representation of the form

\[
p(x,y) := \sum_{j,\ell \in \mathbb{N}_0} \sum_{k \in \Gamma_j^\perp} \sum_{s \in \Gamma_\ell^\perp} p_{j,k}(x) \psi_{j,k}(y).
\]

In this highly structured example then, as in later examples, the determination of a convergence rate of approximations of one operator implies a rate of convergence for its dual or adjoint.

At this point we have a description of convergence that relies on the fact that the operator \( U f \) or \( P f \) lies in the abstract approximation space \( A_r^\perp(U) \). It is an important issue to understand or interpret this condition in terms of well-known or standard spaces of functions. Here we use one special case of the much more general analysis summarized in [18]. The error \( E_j(f, U) \) of best approximation
from $A_j$ introduced in Section 6 for $U := L^2(\Omega)$ takes the form

$$E_j(f, L^2(\Omega)) := \inf_{a \in A_j} \| f - a \|_{L^2(\Omega)} := \| \Pi_{j+1} f \|_{L^2(\Omega)}.$$  

It is known that for linear approximation methods over the piecewise constant functions on a uniform grid we have

$$f \in \text{Lip}(r, L^2(\Omega)) \implies E_j(f, L^2(\Omega)) \approx O(2^{-jr}) \approx O(n_j^{-r/d})$$

for the range $0 < r < 1/2$. [19, 18] In fact, we have that $\text{Lip}(r, L^2(\Omega))$ is equivalent to the linear approximation space $A^{r, \infty}(L^2(\Omega))$ for this range of the rate $r$ for nice domains. We also know that the approximation space $A^{r, 2}(L^2(\Omega))$ is equivalent to the Sobolev space $W^r(L^2(\Omega))$ over the same range. Both of these results can be deduced from Theorem E.1 in the Appendix in Section E.2. This restriction in the range of the rate $r$ is tied to our choice to use the Haar basis, which is discontinuous. If some other, smoother wavelet system is used, this range would increase up to a value determined by the smoothness of the basis and the domain boundary. Such a case is presented in Example 6.

This analysis has an important implication for those who may feel that the approximation space $A^{r, 2}(L^2(\Omega))$ is, perhaps, too abstract. Suppose that we develop an alternative algorithm that generates a different solution $\tilde{P}jf \in A_j$ for estimating the function $Pjf$. In other words the solution $\tilde{P}jf$ is still constructed in terms of the span of the characteristic functions over the dyadic cubes, but it is different from our approximation. We know that the best we can ever do by such a linear method for functions $Pjf \in \text{Lip}(r, L^2(\Omega))$ is to achieve $O(2^{-rj})$ when $0 < r < 1/2$.

We can make this observation more clear perhaps by thinking about Galerkin and generalized (or Petrov-)Galerkin methods of approximation. First, it is easy to see that the choice of approximation in Theorem 6.2 above

$$P_j := \sum_{0 \leq i \leq j} \sum_{k \in \Gamma^i} p_{i,k} \psi_{i,k} \otimes \psi_{i,k}$$

is nothing more than the solution $P_jf := a$ of the Galerkin equations where we seek an $a \in A_j$ such that

$$(Pf - a, g)_U = 0$$

for all $g \in A_j$. In the generalized Galerkin approximation, we seek a solution $\tilde{P}jf := a$ of the equations

$$(Pf - a, g)_U = 0$$

for all $g \in \tilde{A}_j$. Here the set of test functions $\tilde{A}_j$ is not necessarily the same as the approximation space $A_j$, and these methods will in general yield different solutions $P_jf \neq \tilde{P}jf$, both contained in $A_j$. The comments above apply then to such solutions obtained by the generalized Galerkin approximations. See [34] Section 3 for a discussion of the generalized Galerkin method.
Example 6 (Example Rate of Convergence of EDMD). The EDMD method is a popular method for solving several approximation problems related to Koopman operators. Theorem 3 in the recent paper by Korda and Mezic [39] shows that
\[
\lim_{n \to \infty} \|(U - \Pi_n U)f\|_{L^2(\Omega)} = 0.
\]
for \(\Pi_n\) the orthogonal projection onto the \(n\)-dimensional space of approximations \(A_n := \text{span}\{e_i \mid i \leq n\}\) with \(\{e_i\}_{i \leq n}\) some collection of basis functions. Following convention in approximation theory, we denote by \(\Pi_j\) the orthogonal projection onto the first \(n_j\) basis functions, \(n_j := \#(A_j)\). With the choice of priors in this paper, Theorem 6.2 gives a rate of convergence of \(\|(U - U_j)\|_{L^2(\Omega)} \approx O(2^{-rj})\) when \(Uf\) belongs to \(A^{r,2}(U)\). For example, if \(n_j \approx 2^d\), as is common when using tensor product bases over a domain \(\Omega \subset \mathbb{R}^d\), then we have the rate of convergence \(O(n_j^{-r/d})\). This rate of convergence will hold for a range of \(r\) that depends on the smoothness of the particular choice of wavelet or multiwavelet basis.

Here we can be specific when we choose the Daubechies orthonormal wavelets discussed in detail in Example 21 in the appendix. Define
\[
A_j := \left\{ \phi_{j,k} \mid k \in \Gamma_j^0 \right\} = \left\{ \psi_{i,k} \mid 0 \leq i \leq j, \ k \in \Gamma_j^0 \right\}
\]
in terms of the Daubechies scaling function and wavelet \(\phi := N\phi\) and \(\psi := N\psi\) of order \(N\). The Daubechies wavelet system is discussed in detail in Example 21. From Theorem 6.2 we conclude that the error in approximation \(\|(U - U_j)f\|_{L^2(\Omega)} \approx O(2^{-rj})\) when
\[
Uf \in A^{r,2}(L^2(\Omega)) := A^{r,2}(L^2(\Omega); \{A_j\}_{j \in \mathbb{N}_0}).
\]
At this point, the convergence rate is achieved in a space \(A^{r,2}(L^2(\Omega))\) that is defined in terms of the Daubechies wavelets, and it may not be satisfying that this rate is not (yet) expressed in terms of common or well-known spaces. But more can be said that connects this rather abstract approximation space to other function spaces. The translates of the Daubechies scaling function \(N\phi\) of order \(N\) reproduce polynomials of order \(N\), that is, degree \(N - 1\). This implies that the dyadic spline spaces \(S_r(\Delta_j)\) of order \(r = N\) having dyadic knots \(\Delta_j \subset \Omega\) are contained in the approximant spaces \(A_j := \text{span}\{\phi_{j,k} \mid k \in \Gamma_j^0\}\) for the Daubechies wavelet system of order \(N\). It is easily shown that the approximation space defined in terms of linear approximation by splines \(S_r(\Delta_j)\) is contained in the approximation space generated by the Daubechies wavelet system of order \(N\),
\[
A^{r,2}(L^2(\Omega); \{S_r(\Delta_j)\}_{j \in \mathbb{N}_0}) \subset A^{r,2}(L^2(\Omega); \{A_j\}_{j \in \mathbb{N}_0}).
\]
We conclude that for any \(Uf \in W^r(L^2(\Omega))\) with \(0 < r < N - 3/2\), we obtain a rate of convergence that is \(O(2^{-r})\) := \(O(2^{-Nj})\) in \(L^2(\Omega)\). Theorem E.1 shows that \(A^{r,2}(L^2(\Omega); \{S_r(\Delta_j)\}_{j \in \mathbb{N}_0})\) is equivalent to the Sobolev spaces \(W^r(L^2(\Omega))\) for \(0 < r < s = N - 3/2\). Since \(C^r(\Omega) \subset W^r(L^2(\Omega))\), we conclude this that this approximation rate holds for common \(r\)-times continuously differentiable functions over this range of \(r\).
Example 7 (Haar Multiwavelet Approximation of $\mathcal{P} : L^2(\Delta) \to L^2(\Delta)$).

Example 5 is motivated by work in [27] that constructs approximations of discrete dynamical systems in terms of permutation operators defined over measurable partitions of $\Omega$. The error analysis in Example 5 is carried out when $\Omega$ is a product domain $[0,1]^d$, the partition is realized by dyadic cubes $\square_{j,k}$, and tensor products of classical Haar wavelets and scaling functions are used as bases for approximations. In this example we suppose that we are interested in approximations of the Koopman or Perron-Frobenius operators for permutation operators on measurable partitions over the triangle $\Delta := \Delta_{0,0}$ introduced in Example 4. We suppose that the measure $\mu(dx_1,dx_2) := m(x_1,x_2)dx_1dx_2$ for some $m : \Delta \subset \mathbb{R}^2 \to \mathbb{R}$, and as in Example 4 there are constants $c_1, c_2 > 0$ with

$$c_1 \leq m(x_1,x_2) \leq c_2$$

for all $(x_1,x_2) \in \Delta$. We then define the multiscaling functions $\phi_{j,(i,k)}$, multiwavelets $\psi_{j,(i,k)}$, and index sets $\Gamma^\phi_j$, $\Gamma^\psi_j$ as described in Example 4. We then apply Theorem 6.2 with the choice $U := L^2(\Delta)$ and find that

$$\|P - P_j\|_{L^2(\Delta)} \lesssim 2^{-(s-r)} \|P\|_{L^2(\Delta)}$$

for $s > r \geq 0$. For this case $n_j = \#(A_j) = O(2^d)$ with $d = 2$, and we can alternatively express this bound in the form

$$|P - P_j|_{L^2(\Delta)} \lesssim n_j^{-d(s-r)/d|P|_{L^2(\Delta)}}.$$

This bound is of the same form as that derived in Example 5 for the domain $\Omega = [0,1]^d$ when we choose $r = 0$ above.

Example 8 (Warped Wavelet Approximation of $\tilde{\mathcal{P}} : L^2(\tilde{\Omega}) \to L^2(\tilde{\Omega})$).

Before we summarize our next category of results, we return to Example 5 and discuss how the approach of the last example can be modified for another class of systems. We assume that we must approximate the Perron-Frobenius operator $\tilde{\mathcal{P}}$ where for each $\tilde{y} \in \tilde{\Omega}$

$$(\tilde{\mathcal{P}}f)(\tilde{y}) := \int_{\tilde{\Omega}} \tilde{p}(\tilde{y},\tilde{x})\tilde{f}(\tilde{x})\tilde{m}(\tilde{x})d\tilde{x}$$

with $\tilde{m} \in L^1(\tilde{\Omega})$, $\tilde{\mu}(d\tilde{x}) = \tilde{m}(\tilde{x})d\tilde{x}$, and the domain $\tilde{\Omega}$ is related to a master domain such as $\Omega := [0,1]^d$ by the change of variables $\Omega = M(\tilde{\Omega})$ and $x = M(\tilde{x})$, $|M'(\tilde{x})| = \tilde{m}(\tilde{x})$ for $\tilde{x} \in \tilde{\Omega}$ and $x \in \Omega$. We need to construct an orthonormal basis $\tilde{\psi}_{j,k}$ over $\tilde{\Omega}$ to carry out an error analysis of $\tilde{\mathcal{P}}$ in the spirit of Theorem 6.2. Let $\tilde{\psi}_{j,k}(\tilde{x})$ be any nice orthonormal wavelet or multiwavelet basis for the Lebesgue space $L^2(\tilde{\Omega}) := L^2([0,1]^d)$. Then $\tilde{\psi}_{j,k}(\tilde{x}) := \tilde{\psi}_{j,k}(M(\tilde{x}))$ is easily seen to be a family of $L^2_{\tilde{\mu}}$-orthonormal wavelets over $\tilde{\Omega}$. We define the space $\tilde{U}$ to be the completion in the norm $L^2_{\tilde{\mu}}(\tilde{\Omega})$ of the warped wavelets. Now we define the approximation spaces $A^\infty(\tilde{U})$ in terms of the warped wavelets $\tilde{\psi}_{j,k}$ following the philosophy of [31]. Note that here there is no guarantee that we have an
equivalence $L^2(\Omega) \approx \tilde{U}$. In fact, we do not even know if the usual $\tilde{\mu}$–Lebesgue space $L^2_{\tilde{\mu}}(\tilde{\Omega})$ over $\tilde{\Omega}$ is the same as $\tilde{U}$, although we do know $\tilde{U} \subseteq L^2_{\tilde{\mu}}(\tilde{\Omega})$. Since in this section we assume that the domain $\tilde{\Omega}$ and measure $\tilde{\mu}$ are known, this approach can be used to construct approximations as

$$\tilde{P}_j := \sum_{i<j} \sum_{k \in \Gamma} \tilde{p}_{i,k} \tilde{\psi}_{i,k} \otimes \tilde{\psi}_{i,k},$$

and error rates will have the form $O(2^{-rj})$ in $A^{r,2}(\tilde{U})$. It must be kept in mind that in this case the relation of the approximation spaces to the conventional Lebesgue space $L^2(\tilde{\Omega})$, Sobolev spaces $W^s(L^2(\tilde{\Omega}))$, or Lipschitz space Lip($r, L^2(\tilde{\Omega})$) may not be easy or even feasible to establish. It does, however, suggest a path to feasible algorithms with demonstrable rates of convergence. This strategy is studied more closely in Example 8, so we leave the details to the reader. The next example gives the details of the use of warped wavelets in a slightly different context.

### Example 9 (Warped Wavelet Approximation $U : \tilde{U} \rightarrow U := L^2(\Omega)$)

The last few examples illustrated cases where the Perron-Frobenius or Koopman operator is induced by a kernel and has a representation in terms of orthonormal basis functions. A basic assumption in these examples is that $P$ is an integral operator induced by the kernel $p : \Omega \times \Omega \rightarrow \mathbb{R}$. However, for some problems a specific form for the operator suggests itself, and it may not have this convenient form. The current example shows the benefit of employing the approximation space framework above even if the form of the operator is not exactly that in Theorem 6.2.

Let $\Omega := [0,1]^d$ and suppose that $w : \Omega \rightarrow \tilde{\Omega}$. We consider the discrete iteration

$$x_{n+1} = w(x_n).$$

As noted in the introduction, this deterministic system can always be interpreted as a Markov chain. It has the transition probability kernel $P(A,x) := \delta_{w(x)}(A)$ for any measurable set $S \subseteq \Omega$, and it follows that

$$(Uf)(x) := f(w(x)).$$

We set the change of variables $\tilde{x} := w(x)$ and denote $\tilde{\Omega} := w(\Omega)$. We define the mapping $\tilde{M}$ as in Example 8 in terms of the transition mapping $\tilde{M}(\tilde{x}) := w^{-1}(\tilde{x})$, define the measure $\tilde{\mu}(d\tilde{x}) := \tilde{m}(\tilde{x}) d\tilde{x}$ with $\tilde{m}(\tilde{x}) := |\partial x/\partial \tilde{x}|$. From the identity

$$(Uf,g)_{L^2(\Omega)} = (f, \Pi g)_{L^2(\tilde{\Omega})},$$

we also find that

$$(\Pi g)(\tilde{x}) = g(w^{-1}(\tilde{x}))$$

for all $\tilde{x} := w(x) \in \tilde{\Omega}$. 

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Let \( \psi_{j,k} \) be an orthonormal basis for \( U := L^2(\Omega) \), and define the warped wavelets \( \tilde{\psi}_{j,k}(\tilde{x}) := \psi_{j,k}(w^{-1}(\tilde{x})) \) on \( \tilde{\Omega} \). We have the integration rule

\[
\int_{\Omega} \psi_{j,k}(x) \psi_{l,m}(x) dx = \int_{\tilde{\Omega}} \tilde{\psi}_{j,k}(\tilde{x}) \tilde{\psi}_{l,m}(\tilde{x}) \left| \frac{\partial x}{\partial \tilde{x}} \right| d\tilde{x}
\]

for \( \tilde{x} \in \tilde{\Omega} := w(\Omega) \). As in the last example we define \( \tilde{U} \) as the closed finite span of the set of warped wavelets in \( L^2(\tilde{\Omega}) \). Suppose that \( f \circ w \in L^2(\Omega) \). Then we have

\[
\mathcal{U}f := f \circ w = \sum_{j \in \mathbb{N}_0} \sum_{k \in \Gamma_j} (f \circ w, \psi_{j,k})_{L^2(\Omega)} \psi_{j,k},
\]

(6.6)

This representation implies that

\[
\mathcal{U}f := f \circ w \in A^r(\mathcal{U})
\]

if and only if \( f \in A^r(\tilde{U}) \). This equivalence depends on the mapping \( w \) that is an intrinsic part of the definition of the approximation space \( A^r(\tilde{U}) \). Select the approximation \( \mathcal{U}_j \) to be

\[
\mathcal{U}_j f := \sum_{i < j} \sum_{k \in \Gamma_i} (f, \tilde{\psi}_{i,k})_{\tilde{U}} \tilde{\psi}_{i,k}.
\]

If \( f \in A^r(\tilde{U}) \), then we have

\[
\| (\mathcal{U} - \mathcal{U}_j)f \|_{\mathcal{U}} \lesssim 2^{-jr} \| f \|_{A^r(\tilde{U})},
\]

which is the same rate of convergence as that in Theorem 6.2. On the other hand, by duality, we have the representation

\[
\mathcal{P}g := g \circ (w^{-1}) = \sum_{j \in \mathbb{N}_0} \sum_{k \in \Gamma_j} (g, \psi_{j,k})_{L^2(\Omega)} \psi_{j,k}.
\]

(6.7)

We define the estimate \( \mathcal{P}_j \) by truncating this expression, just as in the definition of \( \mathcal{U}_j \). In this case we see that \( g \in A^r(\mathcal{U}) \) if and only if \( \mathcal{P}g \in A^r(\tilde{U}) \). Carefully note that dual structure in Equations 6.6 and 6.7. In summary, we have

\[
\mathcal{U} : \tilde{U} \to \mathcal{U},
\]

\[
\mathcal{P} : U \to \tilde{U}.
\]

For \( g \in A^r(\tilde{U}) \), we have

\[
\| (\mathcal{P} - \mathcal{P}_j)g \|_{\tilde{U}} \lesssim 2^{-jr} \| \mathcal{P}g \|_{A^r(\tilde{U})},
\]
and for $f \in A^{r,2}(U)$ it follows that
\[
\|(U - U_j)f\|_U \lesssim 2^{-rj}\|Uf\|_{A^{r,2}(U)}.
\]

We return to this Example in 12 when we discuss the approximation of measures where the role of this dual structure again emerges prominently.

6.6. Compact, Non-Self-Adjoint Operators and $A^{r,2}(U)$. Just as in our analysis of the spectral spaces, our initial definition of $A^{r,2}(U)$ in Equation 6.5 includes only self-adjoint operators. We fully expect that it will be necessary to consider discrete evolutions that are characterized by non-self-adjoint Perron-Frobenius and Koopman operators. Here we assume that $P : L^2_\mu(\Omega) \to L^2_\mu(\Omega)$, and it is induced by the kernel
\[
p(x, y) := \sum_{j \in \mathbb{N}_0} \sum_{k \in \Gamma_j^\nu} \sum_{\ell \in \mathbb{N}_0} \sum_{m \in \Gamma_j^\nu} p(j, k, \ell, m) \psi_{j, k}(x) \psi_{\ell, m}(y)
\]
with $p(j, k, \ell, m) := (P \psi_{j, k}, \psi_{\ell, m})_U$. We therefore have $\|P\|_U^2 = \|p\|_{L^2_{\mu, \mu}(\Omega \times \Omega)}^2 = \sum_{j, k} (\sum_{\ell, m} p(j, k, \ell, m))^2$. We define the seminorm
\[
\|P\|_{A^{r,2}(U)}^2 := \sum_{j \in \mathbb{N}_0} \sum_{k \in \Gamma_j^\nu} \left( \sum_{\ell \in \mathbb{N}_0} \sum_{m \in \Gamma_j^\nu} p(j, k, \ell, m) \right)^2,
\]
and the family of possibly non-self-adjoint feasible operators
(6.8)
\[
A^{r,2}(U) := \left\{ P := \sum_{(j, k), (\ell, m)} p(j, k, \ell, m) \psi_{j, k} \otimes \psi_{\ell, m} \in S^\infty(U) \left| \|P\|_{A^{r,2}(U)} < \infty \right. \right\}.
\]

Note that the seminorm above reduces to that in Equation 6.5. Also, we have several equivalent expressions for the seminorm $\|P\|_{A^{r,2}(U)}$. We define the infinite dimensional matrix $[p(j, k, \ell, m)] := [(P \psi_{j, k}, \psi_{\ell, m})_U]$ and introduce the operator $D^r_{2^{2s}} := \sum_{j \in \mathbb{N}_0} \sum_{k \in \Gamma_j^\nu} 2^{-2s} \psi_{j, k} \otimes \psi_{j, k}$ and its associated diagonal matrix representation $[2^{2s}]$. The signs selected in the definition $D^r_{2^{2s}}$ are chosen so as to be analogous to the definition of the operator $D^r_\lambda$ on the spectral spaces. We now can write
\[
\|P\|_{A^{r,2}(U)}^2 := \left( D^{-r/2}_{2^{2s}} P, D^{-r/2}_{2^{2s}} P \right)_{S^2(U)} = \sum_{j \in \mathbb{N}_0} \sum_{k \in \Gamma_j^\nu} \left( D^{-r/2}_{2^{2s}} P \psi_{j, k}, D^{-r/2}_{2^{2s}} P \psi_{j, k} \right)_U.
\]
Theorem 6.3. Let $\mathcal{P} : L^2_{\mu}(\Omega) \to L^2_{\mu}(\Omega)$ be given by

$$\mathcal{P} := \sum_{i < j} \sum_{k \in \Gamma^o_i} \sum_{\ell < j} \sum_{m \in \Gamma^o_\ell} p(i,k),(\ell,m) \psi_{i,k} \otimes \psi_{\ell,m}$$

with $p(i,k),(\ell,m) := (\mathcal{P} \psi_{i,k}, \psi_{\ell,m})_U$ and define the approximation $\mathcal{P}_j$ as

$$\mathcal{P}_j := \sum_{i < j} \sum_{k \in \Gamma^o_i} \sum_{\ell < j} \sum_{m \in \Gamma^o_\ell} p(i,k),(\ell,m) \psi_{i,k} \otimes \psi_{\ell,m}$$

Then the error bounds of Theorem 6.2 hold with the definition of $\mathcal{A}_{r,2}(U)$ in Equation 6.8.

Proof. We follow a similar pattern to the proof of Theorem 5.1.

$$\|\mathcal{P} - \mathcal{P}_j\|_U^2 \leq \sum_{i < j} \sum_{k \in \Gamma^o_i} \sum_{\ell < j} \sum_{m \in \Gamma^o_\ell} \left| p(i,k),(\ell,m) \mathcal{L}(i,k) \psi_{i,k} \otimes \psi_{\ell,m} \right|^2$$

$$\leq 2^{-2rj} \left| \sum_{i \in \mathbb{N}_0} \sum_{k \in \Gamma^o_i} \sum_{\ell \in \mathbb{N}_0} \sum_{m \in \Gamma^o_\ell} p(i,k),(\ell,m) 2^{\ell \cdot 2} \mathcal{L}(i,k) \psi_{i,k} \otimes \psi_{\ell,m} \right|^2$$

$$\leq 2^{-2rj} \left( \sum_{i \in \mathbb{N}_0} \left| \sum_{k \in \Gamma^o_i} p(i,k),(\ell,m) 2^{\ell \cdot 2} \mathcal{L}(i,k) \right|^2 \right)^{1/2}$$

$$= 2^{-2rj} \left| \left| D_{2^{-2i}}^{-r/2} [p(i,k),(\ell,m)] \mathcal{L}(i,k) \right| \right|_{L^2(U)}^2$$

If we know that $f \in U$ and $\mathcal{P} \in \mathcal{A}_{r,2}(U)$, we then see that

$$\|\mathcal{P} - \mathcal{P}_j\|_U^2 \leq 2^{-2rj} \|D_{2^{-2i}}^{-r/2} [p(i,k),(\ell,m)] \mathcal{L}(i,k) \mathcal{L}(i,k) \|_{L^2(U)}^2 \leq 2^{-2rj} \|D_{2^{-2\cdot 2i}}^{-r/2} \mathcal{P} \|_{L^2(U)}^2 \|f\|_U^2 \leq 2^{-2rj} \|D_{2^{-2\cdot 2i}}^{-r/2} \mathcal{P} \|_{L^2(U)}^2 \|f\|_U^2 \leq 2^{-2rj} \|D_{2^{-2\cdot 2i}}^{-r/2} \mathcal{P} \|_{L^2(U)}^2 \|f\|_U^2 = 2^{-2rj} \|\mathcal{P} \|_{\mathcal{A}_{r,2}(U)}^2 \|f\|_U^2.$$
On the other hand, if \( \mathcal{P} \in S^2(U) \) and \( f \in A^{r,2}(U) \), we have

\[
\| (\mathcal{P} - \mathcal{P}_j) f \|^2 \leq 2^{-2rj} \| [D_{2^j}^{-1/2}] [p_{(i,k),\ell,m}] \{ f, k \} \|^2_f \\
\leq 2^{-2rj} \| [p_{(i,k),\ell,m}] [D_{2^j}^{-1/2}] \{ f, k \} \|^2_f \\
\leq 2^{-2rj} \| [p_{(i,k),\ell,m}] \|_2^2 \| [D_{2^j}^{-1/2}] \{ f, k \} \|^2_f \\
\leq 2^{-2rj} \| \mathcal{P} \|^2_{L^2(U)} \| f \|^2_{A^{r,2}(U)} \leq 2^{-2rj} \| \mathcal{P} \|^2_{S^2(U)} \| f \|^2_{A^{r,2}(U)}. \]

\[\square\]

### 6.7. \( A^{r,2}_\alpha(U) \) as a special case of \( A^{r,2}(U) \)

Before we conclude this section, we relate the spectral approximation spaces of Section 5 with the more general approximation spaces in this section. For any compact, self-adjoint operator \( T \), there is no loss of generality by renumbering the eigenfunctions to follow the conventions of the multiscale framework. For instance, when \( d = 1 \), we set \( \Gamma_j^\psi := \{0, 1, \ldots, 2^j - 1\} \).

We subsequently define \( \psi_{j,k} := u_m \) for \( m = 2^j + k \) with \( k \in \Gamma_j \) for \( j \in \mathbb{N}_0 \). This numbering is easily modified for cases when \( d > 1 \).

**Theorem 6.4 (Equivalence of \( A^{r,2}_\alpha(U) \) and \( A^{r,2}(U) \)).** Suppose that the self-adjoint, compact operator \( T : U \to U \) has the Schatten class representation

\[
T = \sum_{i \in \mathbb{N}} \lambda_i u_i \otimes u_i,
\]

so that \( T \in A^{r,2}_\alpha(U) \). Denote by \( \{ \psi_{j,k} \}_{j \in \mathbb{N}_0, k \in \Gamma_j^\psi} \sim \{ u_i \}_{i \in \mathbb{N}_0} \), consistent with the multilevel structure as discussed above. If the eigenvalues satisfy \( \lambda_{2^j} \approx 2^{-2j} \), then \( A^{r,2}_\alpha(U) \approx A^{r,2}(U) \).

**Proof.** Let \( f \in A^{r,2}_\alpha(U) \). We have

\[
|f|_{A^{r,2}_\alpha(U)}^2 = \sum_{i \in \mathbb{N}} \lambda_i^{-r} \| (f, u_i) \|^2_U = \sum_{j \in \mathbb{N}_0} \sum_{k \in \Gamma_j^\psi} \lambda_j^{-r} \| (f, \psi_{j,k}) \|^2_U \\
\leq \sum_{j \in \mathbb{N}_0} \sum_{k \in \Gamma_j^\psi} \lambda_j^{-r} \| (f, \psi_{j,k}) \|^2_U \\
\leq \sum_{j \in \mathbb{N}_0} \lambda_j^{-r} \| (f, \psi_{j,k}) \|^2_U \lesssim \sum_{j \in \mathbb{N}_0} 2^{2jr} \sum_{k \in \Gamma_j^\psi} \| (f, \psi_{j,k}) \|^2_U.
\]

On the other hand, we can write

\[
|f|_{A^{r,2}(U)}^2 = \sum_{i \in \mathbb{N}} \lambda_i^{-r} \| (f, u_i) \|^2_U = \sum_{j \in \mathbb{N}_0} \sum_{k \in \Gamma_j^\psi} \lambda_j^{-r} \| (f, \psi_{j,k}) \|^2_U \\
\geq \sum_{j \in \mathbb{N}_0} \lambda_j^{-r} \sum_{k \in \Gamma_j^\psi} \| (f, \psi_{j,k}) \|^2_U \geq \sum_{j \in \mathbb{N}_0} 2^{2(j+1)r} \sum_{k \in \Gamma_j^\psi} \| (f, \psi_{j,k}) \|^2_U \\
\geq \sum_{j \in \mathbb{N}_0} 2^{2jr} \sum_{k \in \Gamma_j^\psi} \| (f, \psi_{j,k}) \|^2_U.
\]

This concludes the proof. \[\square\]
7. Approximation of Signed and Probability Measures. In this section we study the approximation of signed and probability measures as they arise in the theory of Koopman or Perron-Frobenius operators. The approximation of measures is an important topic in Koopman theory for several reasons. When we study the Perron-Frobenius operator, for instance, one that is associated with a stochastic flow, it is an operator that maps measures into measures. Approximations \( \mathcal{P}_j \) of such a \( \mathcal{P} \) are naturally expressed in terms of finite dimensional subspaces of measures. In analogy to our study of approximations of functions, we are interested in defining certain classes of priors that contain measures, and the priors make it possible to determine rates of convergence of approximations. These convergence rates are of interest in their own right, but there are additional benefits of this analysis. Since both \( \mathcal{P} \) and \( \mathcal{U} \) are sometimes defined explicitly in terms of a kernel and the measure \( \mu \), we will see that methods for approximating \( \mu \) can be used to construct approximations of these operators.

We begin our presentation with a review of basic definitions in Section 7.1 and the introduction of Gelfand triples in Section 7.2. Section 7.3 introduces methods to estimate signed measures with priors defined by duality relative to \( A^r(U) \), \( U \) a Hilbert space. The primary result of this section is encapsulated in Theorem 7.1. In many applications we seek to construct approximations of probability measures, and such estimates are the topic of Section 7.4.

7.1. Measures, Duality, and Weak* Convergence. We assume in this section that \( \Omega \) is a compact subset of \( \mathbb{R}^d \). In this case it is known that the topological dual \( C^*(\Omega) \) is the family of regular countably additive set functions, or regular signed measures, \( rca(\Omega) \) on the set \( \Omega \). For any signed measure \( \mu \) on \( \Omega \), there exist mutually singular positive measures \( \mu^+ \) and \( \mu^- \) such that \( \mu := \mu^+ - \mu^- \). The total variation norm for signed measures over the set \( \Omega \) is given by

\[
\|\mu\|_{TV} := |\mu|(\Omega) := \mu^+(\Omega) + \mu^-(\Omega),
\]

and with this norm \( rca(\Omega) \) is a Banach space [23]. The total variation norm of the difference of two signed measures also has the convenient representation

\[
\|\mu - \nu\|_{TV} = 2 \sup_{S \subset \Omega} |\mu(S) - \nu(S)|.
\]

The total variation norm often induces a topology that is too fine for applications in the study of dynamical systems. When we seek to study the discrete deterministic dynamics in Equation 1.3, the Koopman operator is expressed in terms of the Dirac measure \( \delta_x \) for \( x \in \Omega \). The Perron-Frobenius operator is also expressed in terms of this Dirac measure. Suppose we have a sequence of points generated by a discrete dynamical system, \( \{x_k\}_{k \in \mathbb{N}} \subset \Omega \), that converges to \( x \), but \( x_k \neq x \) for all \( k \in \mathbb{N} \). We easily calculate that \( \|\delta_x - \delta_{x_k}\|_{TV} = 2 \) for all \( k \in \mathbb{N} \), so we conclude that \( \delta_{x_k} \not\rightarrow \delta_x \) in the total variation norm. Intuitively, in the applications to the study of discrete dynamical systems, we would like to have the convergence in some sense of the probability measures \( \delta_{x_k} \rightarrow \delta_x \) for this trajectory. One such topology for which \( x_k \to x \) implies \( \delta_{x_k} \to \delta_x \) is the weak* topology on signed measures.

Recall that if \( X^* \) is the dual space of a Banach space \( X \), the weak* topology on \( X^* \) is the weak topology that it inherits from \( i_{X,X^*}(X) \subset X^{**} \), with \( i_{X,X^*} : X \rightarrow X^{**} \) the canonical injection of \( X \) into its second dual space \( X^{**} \). For any set \( S \subset X \), we denote by \( \text{weak}^*(X^*,S) \) the weak topology on \( X^* \) induced by \( i_{X,X^*}(S) \subset X^{**} \). When we unwrap this definition for \( X := C(\Omega) \) and \( X^* := rca(\Omega) \), we find that a net of measures \( \{\mu_\gamma\}_{\gamma \in \Gamma} \) with \( \Gamma \) a directed set converges to \( \mu \) in the weak* topology if
and only if
\[ \langle \mu, f \rangle_{X^*, X} \rightarrow \langle \mu, f \rangle_{X^*, X} \]
for all \( f \in C(\Omega) \). For computations, this duality statement is just
\[ \int_{\Omega} f(\xi) \mu(\xi) d\xi \rightarrow \int_{\Omega} f(\xi) \mu(\xi) d\xi \]
for all \( f \in C(\Omega) \). Since here the domain \( \Omega \) is a compact metric space, the set of probability measures \( \mathcal{M}^{+, \text{i}}(\Omega) \subset \text{coa}(\Omega) \) is compact and metrizable. Since we always have \( \Omega \subset \mathbb{R}^d \) in this paper, it also suffices to characterize weak* convergence in terms of sequences in this paper. [56]

7.2. The Gelfand Triple. The approximation of measures in this section and the next will be facilitated by the use of a Gelfand triple. The Gelfand triple is a standard construct used in the study of partial differential equations, [85] and we use it here to relate dual pairings and inner products. Let \( X \) be a Banach space that is dense in the Hilbert space \( U \). We suppose that the injection \( i_{X,U} : X \rightarrow U \), which is simply the linear map \( f \in X \mapsto i_{X,U} f = f \in H \), is continuous. This embedding is represented symbolically as \( X \rightarrow U \) and implies that \( \| f \|_U = \| i_{X,U} f \|_U \lesssim \| f \|_X \).

Using the setup described in Section E, we assume that we are given a dense family of approximant subspaces \( \{ A_j \}_{j \in \mathbb{N}} \subset U \) and associated \( U \)-orthogonal projection operators \( \Pi_j : U \rightarrow A_j \) that are onto \( A_j \). The approximation space \( A^{r,q}(U) := A^{r,q}(U, \{ A_j \}_{j \in \mathbb{N}}) \) is defined in the usual way. Finally, we assume that for each \( j \) the space \( A_j \subset X \). This implies that we have the scale of inclusions
\[
A_j \subset X \hookrightarrow U \approx U^* \hookrightarrow X^* \subset A_j^*,
\]
\[
A^{r,2}(U) \subset X \hookrightarrow U \approx U^* \hookrightarrow X^* \subset (A^{r,2}(U))^*.
\]

By duality every \( u^* \in U^* \approx U \) defines an element of the dual space \( i'_{X,U} u^* \in X^* \). We write \( U \approx U^* \) to denote the isometric isomorphism of the Hilbert space onto its dual space that is given by the Riesz map \( R_U \). In the few instances when we want to be explicit about the role of the Riesz map, we use the definition that \( u^* = R_U u \) provided
\[ \langle u^*, v \rangle_{U^*, U} := \langle R_U u, v \rangle_{U^*, U} = \langle u, v \rangle_U \]
for all \( v \in U \). Since the projection operators \( \Pi_j \) map \( U \) to the finite dimensional space \( A_j \), the dual approximation operators \( \Pi'_j \) map \( A_j^* \) to \( U^* \). Symbolically, we depict this relationship as
\[ A_j \leftarrow_{\Pi_j} U \approx U^* \leftarrow_{\Pi'_j} A_j^*. \]

Our primary reason for using the Gelfand structure is that it gives a clear and useful expression for the relationship between the duality pairing \( \langle \cdot, \cdot \rangle_{X^*, X} \) and the inner product \( \langle \cdot, \cdot \rangle_U \) on \( U \). As derived in [85], the inner product on \( U \) can be extended by continuity to represent the duality pairing on \( X \) in the sense that we have
\[ \langle g^*, f \rangle_{X^*, X} = \langle R_U g, f \rangle_{X^*, X} = \langle g, f \rangle_U \]
for all \( f \in X \), \( g \in U \), and \( g^* := R_U g \in U^* \subset X^* \). As a consequence we see that the dual operators \( \Pi'_j \) satisfy the relationship
\[ \langle \Pi'_j g^*, f \rangle_{X^*, X} = \langle g, \Pi_j f \rangle_U \]
whenever \( g \in U \), \( g^* := R_U g \in U^* \) and \( f \in X \).
7.3. Approximation with $\Pi_j$, Priors Dual to $A^{r,2}(U)$. With the Gelfand triple discussed in Section 7.2, we can introduce priors that describe the regularity of signed measures. The notion of regularity of signed measures is determined by duality to the approximation spaces $A^{r,2}(U)$ in this section. This choice is seen to imply a type of convergence of dual forms with respect to the weak* topology on the measures. The rate of convergence of the approximations of the measures is then established by using the rates of convergence of approximations of functions in the space $A^{r,2}(U)$.

**Theorem 7.1.** Suppose $X$ is a Banach space, $U$ is a Hilbert space, and the pair $X \hookrightarrow U$ forms a Gelfand triple. Let the following conditions hold:
1. $A^{r,2}(U) := A^{r,2}(U, \{A_j\}_{j \in \mathbb{N}_0}) \subseteq X$ is an approximation space for $r > 0$,
2. $u^* = R_U u$ for some $u \in U$ with $R_U : U \rightarrow U^*$ the Riesz map, and
3. $A_j \subset X$ for all $j \in \mathbb{N}_0$, and the operator $\Pi_j$ is the $U$-orthogonal projection onto $A_j$.

Then the sequence $\{\Pi_j u^*\}_{j \in \mathbb{N}}$ converges to $u^* \in U^* \subset X^*$ in the weak* $(X^*, A^{r,2}(U))$ topology with the rate

$$
|\langle (I - \Pi_j)^* u^*, f \rangle_{X^* \times X}| \lesssim 2^{-rj} \|u\|_U \|f\|_{A^{r,2}(U)}
$$

for all $f \in A^{r,2}(U)$.

**Proof.** We directly expand and bound the duality pairing using the property in Equation 7.1 for the Gelfand triple:

$$
\left| \langle (I - \Pi_j)^* u^*, f \rangle_{X^* \times X} \right| = \left| \langle u^*, (I - \Pi_j)f \rangle_{X^* \times X} \right|
= \left| \langle R_U u, (I - \Pi_j)f \rangle_{X^* \times X} \right| = \left| \langle u, (I - \Pi_j)f \rangle_U \right|
\leq \|u\|_U \|(I - \Pi_j)f\|_U \leq 2^{-rj} \|u\|_U \|f\|_{A^{r,2}(U)}.
$$

Note that the last line above follows from the approximation rate of functions in $A^{r,2}(U)$ described in Theorem 6.2.

Example 10 that follows presents the rather straightforward case when the measure $\nu$ to be approximated is given as $\nu(dx) := m(x)dx$ for a function $m \in L^2(\Omega)$ and $dx$ denoting ordinary Lebesgue measure. Here we assume additionally that the domain $\Omega$ is compact, and the approximation space $A^{r,2}(U)$ is densely and continuously contained in $X := C(\Omega)$, which in turn is densely and continuously embedded in $U := L^2(\Omega)$. In summary then, the spaces that are used in this analysis define a duality structure where

$$
A_j \subset C(\Omega) \hookrightarrow L^2(\Omega) \approx (L^2(\Omega))^* \hookrightarrow C^*(\Omega) \hookrightarrow A_j^*,
$$

(7.2) $A^{r,2}(U) \hookrightarrow C(\Omega) \hookrightarrow L^2(\Omega) \approx (L^2(\Omega))^* \hookrightarrow C^*(\Omega) \hookrightarrow (A^{r,2}(U))^*$.

The requirement that $A^{r,2}(U) \subset C(\Omega)$ can be relaxed in this example, but when it holds it makes the analysis particularly straightforward.

---

**Example 10** (Approximation of Signed Measures, Duality to $A^{r,2}(U)$).
In Example 5 approximations of functions are defined in terms of piecewise constants. In this example, we consider a somewhat different situation and construct the finite dimensional approximations of functions from the spaces $A_j := \text{span} \{\psi_{i,k} \mid 0 \leq i \leq j, k \in \Gamma_i^j\}$ with $\psi_{i,k}$ the dilates and translates of any...
By hypothesis we also know that \( A \) and \( \psi \) exist. In fact we just have \( \psi \) basis measures in of \([21]\) that are continuous.\([16, 53]\), or any of the orthonormal, compactly supported multiwavelets described in of [21] that are continuous.

To build our approximation of measures, we now carefully discuss how dual measures \( \psi_{j,k}(dx) \in C^*(\Omega) \) are defined relative to the orthonormal functions \( \psi_{j,k} \in U \). The connection between these bases is clear in this example owing to the orthonormality of the basis functions \( \psi_{j,k} \). We summarize this argument in some detail as it serves as the prototype of arguments for more general approximations of measures.

Since \( A \) is finite dimensional, its dual space \( A_\varepsilon \) is finite dimensional. There exists a unique basis \( \{ \psi_{j,k} \mid 0 \leq i \leq j, k \in \Gamma^\psi_i \} \) for \( A_\varepsilon \subset U^* \) that is dual to the basis \( \{ \psi_{j,k} \mid 0 \leq i \leq j, k \in \Gamma^\psi_i \} \subset U^* \) with respect to the pairing \( \langle \cdot, \cdot \rangle_{U^* \times U} \). In fact we just have \( \psi_{j,k} := R_U \psi_{j,k} \) in this example since the dual basis is unique and

\[
\delta_{(\ell,m),(j,k)} = \langle \psi_{\ell,m}, \psi_{j,k} \rangle_U = \langle R_U \psi_{\ell,m}, \psi_{j,k} \rangle_{U^* \times U} = \langle \psi_{\ell,m}, \psi_{j,k} \rangle_{U^* \times U}.
\]

By hypothesis we also know that \( A^{r,2}(U) \subseteq C(\Omega) \subseteq U := L^2(\Omega) \) in this example. Since \( C(\Omega) \subset L^2(\Omega) \approx (L^2(\Omega))^* \subset rca(\Omega) = C^*(\Omega) \), any function in \( L^2(\Omega) \) can be viewed as a signed measure on \( \Omega \). It is immediate that the biorthogonal basis function \( \psi_{j,k} \in U^* \) defines a measure

\[
\psi_{j,k}(d\xi) := \psi_{j,k}(\xi)d\xi \in C^*(\Omega) := rca(\Omega),
\]

and they satisfy

\[
\langle \psi_{j,k}, \psi_{\ell,m} \rangle_{X^* \times X} = \int_\Omega \psi_{j,k}(\xi)\psi_{\ell,m}(\xi)d\xi = \delta_{(j,k),(\ell,m)}
\]

with \( X := C(\Omega) \). We then define the approximation operators \( \Pi_j \) by duality,

\[
\langle \Pi_j \nu, f \rangle_{X^* \times X} = \langle \nu, \Pi_j f \rangle_{X^* \times X} = \int \sum_{0 \leq i \leq j} \sum_{k \in \Gamma^\nu_i} (f, \psi_{i,k}(y)) U\psi_{i,k}(y)\nu(dy)
\]

\[
= \left\langle \sum_{0 \leq i \leq j} \sum_{k \in \Gamma^\nu_i} \psi_{i,k}(y)\nu(dy), f \right\rangle_{X^* \times X}.
\]

This means that

\[
(\Pi_j \nu)(d\xi) = \sum_{0 \leq i \leq j} \sum_{k \in \Gamma^\nu_i} \int \psi_{i,k}(y)\nu(dy)\psi_{i,k}(\xi)d\xi
\]

\[
= \sum_{0 \leq i \leq j} \sum_{k \in \Gamma^\nu_i} \int \psi_{i,k}(y)\nu(dy)\psi_{i,k}(d\xi)
\]

\[
= \sum_{0 \leq i \leq j} \sum_{k \in \Gamma^\nu_i} \nu_{i,k}\psi_{i,k}(d\xi).
\]
In other words the operators $\Pi_j' : X^* \subset A_j^* \to U^* \subset X^* = rca(\Omega)$. Furthermore, we have from Theorem 7.1 that

$$\left| \left< (I - \Pi_j') \nu, f \right>_{X^* \times X^*} \right| = \left| \left< \nu, (I - \Pi_j) f \right>_{X^* \times X^*} \right| \lesssim 2^{-r_i} \|\nu\|_U \|f\|_{A^{r,2}(U)}$$

for $\nu = R_U v \in U^* \subset X^*$, $v \in U$, and all $f \in A^{r,2}(U)$. This last inequality follows again from Theorem 7.1.

We must emphasize that Theorem 7.1 and Example 10 study the approximation of signed measures. Since we have that the probability measures $\mathbb{M}^{+1}_{\Omega}(\Omega) \subset C^*(\Omega)$, any probability measure can be approximated in this way. However, such an approximation $\nu_j := \Pi_j' \nu$ of a probability measure $\nu$ is not guaranteed a priori to be a probability measure.

**Example 11 (An Approximation that is Not a Probability Measure).** Consider again the situation in Example 1. We define $U := L^2(\mathbb{T}^1)$ and set

$$A_j := \text{span} \left\{ u_{k,i} \mid i = 1, 2, \ k \leq j \right\},$$

$$A_j^* := \text{span} \left\{ u^{k,i}(\xi) d\xi \mid i = 1, 2, \ k \leq j \right\},$$

$$\Pi_j f := \sum_{i=1,2} \sum_{k \leq j} (f, u_{k,i})_U u_{k,i},$$

$$A^{r,2}(U) := A^{r,2}(L^2(\mathbb{T}^1); \{A_j\}_{j \in \mathbb{N}}).$$

Here the eigenfunctions are $u_{k,1}(x) := \cos(kx)/\sqrt{\pi}$, and $u_{k,2}(x) := \sin(kx)/\sqrt{\pi}$. By definition we know that $u^{k,i}(S) = \int_S u_{k,i}(\xi) d\xi$. Choose $\nu(dx) = \frac{1}{\pi} 1_{[\pi,2\pi]}(x)(dx)$ and $S = [0, \pi]$. We then have

$$\left( \Pi_j' \nu \right)(S) = \sum_{k \leq 1} \sum_{i=1,2} \int_{[0,2\pi]} u_{k,i}(d\xi) \nu(d\xi) u^{k,i}(S),$$

$$= \frac{1}{\pi^2} \left\{ \int_{[\pi,2\pi]} \cos x dx \int_{[0,\pi]} \cos x dx + \int_{[\pi,2\pi]} \sin x dx \int_{[0,\pi]} \sin x dx \right\},$$

$$= \frac{1}{\pi^2} \left\{ (\sin x \big|_{[\pi,2\pi]}) (\cos x \big|_{[0,\pi]}) + (\cos x \big|_{[\pi,2\pi]}) (\cos x \big|_{[0,\pi]}) \right\},$$

$$= -\frac{4}{\pi^2}.$$

We conclude that while $\nu$ is a probability measure, $\Pi_j' \nu \notin \mathbb{M}^+$ and is not a probability measure.
approximations that yield probability measures by solving constrained optimization problems as in Reference [38]. We return to this issue in Section 7.4.2 and describe an alternative approach that is amenable to the derivation of associated rates of approximation.

In Example 10 it is assumed that the measure to be approximated has a density so that
\[ \nu(dx) := m(x)dx \]
for some function \( m \in L^2_\mu(\Omega) \) and \( dx \) denoting Lebesgue. In this Gelfand triple structure this corresponds to the statement that \( \nu = R_u m \) for some \( m \in U \). In the next example we consider the case when we only know that \( \nu \in C^\ast(\Omega) \). Specifically, we study the choice
\[ \nu(dx) := \delta_w(x) \]
for a sufficiently smooth function \( w : \Omega \rightarrow \Omega \).

**Example 12 (Approximations of Signed Measures and \( \mathcal{U}_j, \mathcal{P}_j \)).** We now return to Example 9 and consider the approximation of the Perron-Frobenius operator or Koopman operator, and associated approximations of measures for the canonical case when \( (\mathcal{U}f)(x) = f(w(x)) \). We begin by reviewing the relationship between the Perron-Frobenius operator, the Koopman operator, and the probability measure \( \delta_w(x) \) in this case. We then derive rates of convergence for approximations of the measure \( \delta_w(x) \), and subsequently we construct the finite dimensional operator approximations
\[ \mathcal{P}_j := \Pi_j \mathcal{P} \Pi_j^* : X^* \rightarrow X^*, \]
\[ \mathcal{U}_j := \Pi_j \mathcal{U} \Pi_j : X \rightarrow A_j \subset X. \]

We will see that these operators coincide with one that is constructed in terms of the approximations \( \Pi_j \delta_w(x) \) of the Dirac measure \( \delta_w(x) \).

**Duality expressions induced by \( w : \Omega \rightarrow \Omega \):**
First we review expressions for the dual pairing. By duality we know that
\[ \langle \mathcal{P}\nu, f \rangle_{X^* \times X} = \langle \nu, \mathcal{U}f \rangle_{X^* \times X} = \int_{\Omega} (f \circ w)(\xi) \nu(d\xi). \]

The Koopman operator in this problem can be re-written as
\[ (\mathcal{U}f)(x) := f(w(x)) = \int_{\Omega} \delta_w(x)(d\xi)f(\xi), \]
and it follows from
\[ \langle \nu, \mathcal{U}f \rangle_{X^* \times X} = \int_{\Omega} \nu(dx) \int_{\Omega} \delta_w(x)(d\xi)f(\xi) = \int_{\Omega} \left( \int_{\Omega} \delta_w(x)(d\xi) \nu(dx) \right) f(\xi) \]
that
\[ \langle \mathcal{P}\nu, f \rangle_{X^* \times X} = \int_{\Omega} \nu(dx) \delta_w(x)(d\xi). \]
We next construct approximations of the measure \( \delta_w(x) \), which is subsequently used to construct the approximate operators \( \mathcal{U}_j \) and \( \mathcal{P}_j \).

**Approximation of the measure \( \delta_w(x) \):**
We choose \( U := L^2(\Omega) \) and \( X := \mathcal{C}(\Omega) \). By assuming that \( f \circ w \in \mathcal{C}(\Omega) \subset L^2(\Omega) \), we compute the expansion

\[
\langle P \nu, f \rangle_{X^* \times X} = \int_{\Omega} (f \circ w)(\xi) \nu(dx)
\]

in terms of the warped wavelets \( \tilde{\psi}_{j,k} \) as discussed in Example 9. We obtain

\[
\int_{\Omega} (f \circ w)(\xi) \nu(dx) = \int_{\Omega} \left( \sum_{j \in \mathbb{N}_0} \sum_{k \in \Gamma_j^\nu} (f, \tilde{\psi}_{j,k})_{L^2(\tilde{\Omega})} \tilde{\psi}_{j,k}(\xi) \right) \nu(dx),
\]

\[
= \sum_{j \in \mathbb{N}_0} \sum_{k \in \Gamma_j^\nu} (f, \tilde{\psi}_{j,k})_{L^2(\tilde{\Omega})} \int_{\Omega} \tilde{\psi}_{j,k}(\xi) \nu(dx),
\]

\[
= \sum_{j \in \mathbb{N}_0} \sum_{k \in \Gamma_j^\nu} (f \circ w)_{j,k} \nu_{j,k} = \sum_{j \in \mathbb{N}_0} \sum_{k \in \Gamma_j^\nu} \tilde{f}_{j,k} \nu_{j,k},
\]

with \( \tilde{f}_{j,k} := (f, \tilde{\psi}_{j,k})_{L^2(\tilde{\Omega})} \) and \( \nu_{j,k} := \int \tilde{\psi}_{j,k}(\xi) \nu(dx) \). We can generate an approximation of the measure \( \delta_{w(x)} \) via the operator \( \Pi_j \) and obtain

\[
\langle \Pi_j \delta_{w(x)}(\xi), d\eta \rangle := \sum_{0 \leq i \leq j} \sum_{k \in \Gamma_j^\nu} \int_{\Omega} \psi_{i,k}(y) \delta_{w(x)}(dy) \psi_{i,k}(\eta)d\eta
\]

\[
= \sum_{0 \leq i \leq j} \sum_{k \in \Gamma_j^\nu} \psi_{i,k}(w(x)) \psi_{i,k}(d\eta) = \sum_{0 \leq i \leq j} \sum_{k \in \Gamma_j^\nu} \tilde{d}_{i,k}(x) \psi_{i,k}(d\eta)
\]

with \( \tilde{d}_{j,k}(x) := \psi_{j,k}(w(x)) \). We therefore obtain the dual expansions

\[
\langle \Pi_j \delta_{w(x)}, f \rangle_{X^* \times X} = \langle \delta_{w(x)}, \Pi_j f \rangle_{X^* \times X} = \sum_{0 \leq i \leq j} \sum_{k \in \Gamma_j^\nu} \tilde{d}_{i,k}(x) f_{i,k}.
\]

The weak* convergence rate for the approximation of \( \delta_{w(x)} \) is expressed in terms of the duality pairing as

\[
\left| \langle (I - \Pi_j)^2 \delta_{w(x)}, f \rangle_{X^* \times X} \right| = \left| \langle \delta_{w(x)}, (I - \Pi_j)^2 f \rangle_{X^* \times X} \right|,
\]

\[
\leq \| \delta_{w(x)} \|_{X^*} \| (I - \Pi_j)^2 f \|_{X},
\]

\[
\leq \| (I - \Pi_j)^2 f \|_{A^{s,2}(U)} \leq 2^{-(s-r)}j \| f \|_{A^{s,2}(L^2(\Omega))},
\]

for all \( f \in A^{s,2}(L^2(\Omega)) \) with \( s > r > 0 \). Note the difference between the rate of convergence that depends on \( 2^{-(s-r)} \) in this example and that in Theorem 7.1 which is bounded by \( 2^{-r} \). Theorem 7.1 relies on the fact that \( \nu \in U^* \), which enables the use of the Gelfand triple to derive that the error rate is \( O(2^{-r}) \). In this case, the measure \( \nu := \delta_{w(x)} \notin U^* \). In other words there is no \( L^2(\Omega) \) function \( m \) such that \( \delta_{w(x)} := R_\nu m \). We only know that \( \delta_{w(x)} \in C^*(\Omega) := X \), and as measured by the duality structure, the measures in \( C^*(\Omega) \) are less regular.
than those in \(U^*\). We expect the convergence rate of approximations of measures \(\nu \in C^*(\Omega)\) to be lower than those \(\nu \in U^* \subset C^*(\Omega)\).

The approximate operators \(\mathcal{P}_j\) and \(\mathcal{U}_j\):

With the expression for the approximations of the measure \(\delta_{w(x)}\), we define the approximation \(\mathcal{P}_j\) as

\[
(\mathcal{P}_j \nu)(d\xi) = \int (\Pi_j' f_{w(x)}) (d\xi) (\Pi_j \nu) (dx),
\]

and the operator \(\mathcal{U}_j\) is defined by duality. In fact, we find that \(\mathcal{P}_j := \Pi_j' \Pi'' \nu\) and \(\mathcal{U}_j := \Pi_j \mathcal{U}\). To see why this is so, we can rewrite the approximate Perron-Frobenius operator

\[
\langle \mathcal{P}_j \nu, f \rangle_{X^* \times X} = \int (\Pi_j' \nu) (dx) \left( \int (\Pi_j' f_{w(x)}) (d\xi) f(\xi) \right) = \int (\Pi_j' \nu) (dx) \langle \delta_{w(x)}, \Pi_j f \rangle_{X^* \times X} = \langle \nu, \Pi_j \mathcal{U} f \rangle_{X^* \times X} = \langle \nu, \mathcal{U}_j f \rangle_{X^* \times X}.
\]

The approximate Koopman operator \(\mathcal{U}_j\) and approximate Perron-Frobenius operator \(\mathcal{P}_j\) are dual operators since

\[
\langle \mathcal{P}_j \nu, f \rangle_{X^* \times X} = \langle \Pi_j' \Pi'' \nu, f \rangle_{X^* \times X} = \langle \nu, \Pi_j \mathcal{P} \Pi_j \rangle_{X^* \times X} = \langle \nu, \mathcal{U}_j f \rangle_{X^* \times X}.
\]

We will see that the derivation of a rate of convergence for \(\mathcal{U}_j\) in a strong sense will induce a similar rate of convergence in a weak* sense for \(\mathcal{P}_j\). If \(\mathcal{U} \in L(A^{r,2}(L^2(\Omega)))\), we have the bound

\[
\| (\mathcal{U} - \mathcal{U}_j) f \|_{L^2(\Omega)} \leq \| (I - \Pi_j) \mathcal{U} f \|_{L^2(\Omega)} + \| \Pi_j \mathcal{U} (I - \Pi_j) f \|_{L^2(\Omega)},
\]

\[
\lesssim 2^{-rj} \| \mathcal{U} f \|_{A^{r,2}(L^2(\Omega))} + \| \Pi_j \| \| \mathcal{U} f \|_{L^2(\Omega)},
\]

\[
\lesssim 2^{-rj} (\| \mathcal{U} \|_r + \| \mathcal{U} \|_0) \| f \|_{A^{r,2}(L^2(\Omega))},
\]

for any \(f \in A^{r,2}(L^2(\Omega))\). We obtain the weak* rate of convergence

\[
\left| \langle (\mathcal{P} - \mathcal{P}_j) \nu, f \rangle_{X^* \times X} \right| = \left| \langle \nu, (\mathcal{U} - \mathcal{U}_j) f \rangle_{X^* \times X} \right| \lesssim 2^{-rj} \| \nu \|_{X^*} \| f \|_{A^{r,2}(L^2(\Omega))}
\]

for all \(f \in A^{r,2}(L^2(\Omega))\) and \(\nu \in X^*\).

---

**Example 13** (Approximation of \(\delta_{w_n(x)}\) with \(w_n(x) := x^n\)). Here we return to the problem studied in Example 15, but now approximate the measure \(\delta_{w_n(x)}\).
We choose the sequence of finite dimensional approximation spaces as

\[ A_j := \text{span} \left\{ 1_{\Box_{j,k}} \mid k \in \Gamma_j^\circ \right\} = \text{span} \left\{ \phi_{j,k} \mid k \in \Gamma_j^\circ \right\}, \]

\[ = \text{span} \left\{ \psi_{i,k} \mid 0 \leq i \leq j - 1, k \in \Gamma_j^\psi \right\}, \]

with \( \phi_{j,k} \) and \( \psi_{j,k} \) the Haar scaling functions and wavelets for \( d = 1 \) introduced in Example 3. We then have

\[ (\Pi_j' \delta \alpha^\alpha)(d\xi) = \sum_{0 \leq i < j} \sum_{k \in \Gamma_j^\psi} \int_{\Omega} \psi_{j,k}(\xi) \delta \alpha^\alpha(d\eta) \psi_{j,k}(d\xi), \]

\[ = \sum_{0 \leq i < j} \sum_{k \in \Gamma_j^\psi} \psi_{j,k}(x^\alpha) \psi_{j,k}(\xi) d\xi. \]

The action on any \( f \in A_r(\mathcal{L}^2(\Omega)) \) is given by

\[ \langle \Pi_j' \delta \alpha^\alpha, f \rangle_{C^*(\Omega) \times C(\Omega)} = \langle \delta \alpha^\alpha, \Pi_j f \rangle_{C^*(\Omega) \times C(\Omega)} \]

\[ = \sum_{0 \leq i < j} \sum_{k \in \Gamma_j^\psi} \psi_{j,k}(x^\alpha) \int_{\Omega} \psi_{j,k}(\xi) f(\xi) d\xi \]

\[ = \sum_{0 \leq i < j} \sum_{k \in \Gamma_j^\psi} \psi_{j,k}(x^\alpha) (f, \psi_{j,k})_{\mathcal{L}^2(\Omega)}. \]

We then compute the error bound

\[ \left\langle (I - \Pi_j') \delta \alpha^\alpha, f \right\rangle_{C^*(\Omega) \times C(\Omega)} = \left\langle \delta \alpha^\alpha, (I - \Pi_j) f \right\rangle_{C^*(\Omega) \times C(\Omega)} \]

\[ \leq 2^{-(s-r)j} \| f \|_{A^{r,q}(\mathcal{L}^2(\Omega))}. \]

As a second related example, suppose that \( m(x) = x^\alpha \) and \( \mu(dx) := m(x)dx \). In this case we have

\[ (\Pi_j' \mu)(d\xi) = \sum_{0 \leq i < j} \sum_{k \in \Gamma_j^\psi} (x^\alpha, \psi_{j,k})_{\mathcal{L}^2(\Omega)} \psi_{j,k}(\xi) d\xi, \]

and the action on a function \( f \) is given by

\[ \left\langle (I - \Pi_j') \mu, f \right\rangle_{C^*(\Omega) \times C(\Omega)} = \left\langle \mu, (I - \Pi_j) f \right\rangle_{C^*(\Omega) \times C(\Omega)} \]

\[ \leq n_j^{-r} \| \mu \|_{TV} \| f \|_{A^{r,q}(\mathcal{L}^2(\Omega))} \]

for all \( f \in A^{r,q}(\mathcal{L}^2(\Omega)). \)

7.4. Approximation with Dual Operators \( \Pi_j' \), Priors in \( A^{r,q}(X) \). Theorem 7.1 characterizes rates of convergence of approximations \( \Pi_j \nu \) of a signed measure \( \nu \) in terms of duality statements relative to priors in \( A^{r,q}(U) \). In Theorem 7.2 we generalize this analysis and study dual convergence relative to priors in the space \( A^{r,q}(X) \) with
X a Banach space. Also, Theorem 7.2 can be used to guarantee rates of convergence of a some approximations of probability measures in the bounded Lipschitz metric. For many applications, it is required that approximation of a probability measure in fact generates a probability measure, not just a signed or positive measure. When \( \nu \) is a probability measure, Theorem 7.2 is used to develop a general strategy to define approximations \( \hat{\Pi}_n \nu \) that are themselves a probability measure.

### 7.4.1. Approximations of Signed Measures

There are a large number of expressions for a metric on the probability measures \( \mathcal{M}^{+1}(\Omega) \). These include the Wasserstein \( W^p \), Kolmogorov, Levy, Kantorovich, Ky Fan, and bounded Lipschitz metrics, among others. \([22, 61]\) Some of these metrize the induced weak* topology that \( \mathcal{M}^{+1}(\Omega) \) inherits as a subset of the signed measures \( C^*(\Omega) \). In this section, we show that Theorem 7.1 implies convergence in the bounded Lipschitz metric \( d_{BL} \). This is a popular metric that induces the weak* topology on \( \mathcal{M}^{+1}(\Omega) \). We assume throughout this section that the domain \( \Omega \) is compact. The metric \( d_{BL} \) is defined in terms of the bounded Lipschitz functions by the expression

\[
d_{BL}(\mu, \nu) = \sup_{\|f\|_{BL} \leq 1} |\langle \mu - \nu, f \rangle|_{X^* \times X}
\]

with again \( X := C(\Omega) \). In references that discuss probability measures, the bounded Lipschitz norm is usually designated \( \|f\|_{BL} \), and it is synonymous with our notation \( \|f\|_{BL} \approx \|f\|_{\text{Lip}(1, C(\Omega))} := \|f\|_{C(\Omega)} + \|f\|_{\text{Lip}(1, C(\Omega))} \). We use the notation \( \|f\|_{BL} \) in this section to adhere to the more common notation in measure and probability theory. \([22]\) The generalization of the Theorem 7.1, one that employs duality with respect to an arbitrary approximation space \( A^{r,q}(X) \) for a Banach space \( X \), is obtained by introducing a family of near-best approximation operators \( \{\hat{\Pi}_j\}_{j \in \mathbb{N}} \).

**Theorem 7.2 (Dual Convergence of Measures).** Let \( X \) be a Banach space, suppose that a family of subspaces \( \{A_j\}_{j \in \mathbb{N}} \subset X \) define the approximation space \( A^{r,q}(X) \) with \( r > 0 \) and \( 1 \leq q \leq \infty \) as in Section \( E \), and set \( n_j := \#A_j \). Suppose that \( \{\hat{\Pi}_j\}_{j \in \mathbb{N}} \) be a family of uniformly bounded linear projection operators \( \hat{\Pi}_j : X \to A_j \) that are onto \( A_j \) for each \( j \in \mathbb{N} \). When \( \{n_j\}_{j \in \mathbb{N}} \) is a quasigeometric series, we then have

\[
\| (I - \hat{\Pi}_{n_j}) f \|^X \lesssim n_j^{-r} \| f \|_{A^{r,q}(X)}
\]

and

\[
\left| \langle (I - \hat{\Pi}_{n_j}) \nu, f \rangle \right|_{X^* \times X} \lesssim n_j^{-r} \| \nu \|_{X^*} \| f \|_{A^{r,q}(X)}
\]

for all \( j \in \mathbb{N} \), \( \nu \in X^* \), and \( f \in A^{r,q}(X) \).

**Proof.** The approximation error bound in the Banach space \( X \) in Equation 7.4 above is well known and can be found in many places including \([60, 57, 18]\). For completeness we derive the result here in our proof of Equation 7.5. To begin, we can bound the duality pairing

\[
\left| \langle (I - \hat{\Pi}_j) \nu, f \rangle \right|_{X^* \times X} \leq \| \nu \|_{X^*} \| (I - \hat{\Pi}_j) f \|^X.
\]

When \( f \in A^{r,q}(X) \), we have

\[
\| f - \hat{\Pi}_j f \|^X = \| f - \hat{\Pi}_j a + \hat{\Pi}_j a - \hat{\Pi}_j f \| \leq \| f - a \|^X + \| \hat{\Pi}_j \| \| f - a \|
\]

\[
\leq (1 + C) \| f - a \|^X
\]

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for all $a \in A_j$, and we can conclude that $\|f - \Pi_j f\|_X \lesssim E_j(f, X)$. The representation Lemma on page 336 of [57] states that we have the equivalent norms $A^{-q}(X) \approx \left\| \{n_j^r a_j(\cdot, X)\}_j \right\|_{\ell^q}$ with $a_j(f, X) := E_{j-1}(f, X)$ the $j^{th}$ approximation number of the function $f$ in the Banach space $X$. This means that

$$\sum_{j \in \mathbb{N}_0} (n_j^r a_j(f, X))^q \approx \|f\|_{A^{-q}(X)}^q,$$

and therefore $a_n(f, X) \lesssim n_j^{-r} \|f\|_{A^{-q}(X)}$. Since the approximant spaces are nested, we see that

$$E_{n_j}(f, X) \leq E_{n_{j-1}}(f, X) := a_{n_j}(f, X) \lesssim n_j^{-r} \|f\|_{A^{-q}(X)},$$

and the Theorem holds.

In summary then, Theorem 7.2 establishes that the sequence $\{\Pi_{\nu} \}^n_{\nu \in \mathbb{N}_0}$ converges in weak$^*$$(X^*, A^{-q}(X))$, and the weak$^*$ rate is $O(n_j^r)$.

For the Hilbert space $U$, the norm of the error in the orthogonal projection $\|(I - \Pi_n) f\|_U$ is estimated as in Theorem 6.2, while in a Banach space $X$ the approximation error $\|(I - \Pi_n) f\|_X$ is studied in Theorem 7.2. Before proceeding with our discussion of approximation of measures, we illustrate the efficacy of the approximation error bound for functions given in Equation 7.4 in an example application.

**Example 14 (Near Best Approximations in $A^{-q}(L^p(\Omega))$ and $A^{-q}(C(\Omega))$).** In this example, we examine a case where the first conclusion in Example 7.2 in Equation 7.4 holds for approximant spaces $\{A_j\}^n_{j \in \mathbb{N}}$ in $A^{-q}(L^p(\Omega))$. We define the approximant spaces

$$A_j := \text{span}\left\{1_{\square_{j,k}} \mid k \in \Gamma^0_j\right\},$$

which are the same as those built from classical Haar scaling functions and wavelets in Example 3. When we want to use the approximant spaces $\{A_j\}^n_{j \in \mathbb{N}}$ to construct approximations in $A^{-q}(L^p(\Omega))$, we define the family of approximation operators

$$(\Pi_j f)(x) := \sum_{k \leq n_j} \frac{1}{|\square_{j,k}|} \int_{\Omega} f(\xi) 1_{\square_{j,k}}(\xi) d\xi \cdot 1_{\square_{j,k}}(x),$$

so that $\Pi_j f|_{\square_{j,k}}$ is the average of $f$ over $\square_{j,k}$. Clearly each $\Pi_j$ is a linear projection onto $A_j$. From Equation 3.12 in [18] we have

$$\|(I - \Pi_j) f\|_{L^p(\Omega)} \lesssim n_j^{-r} |f|_{L^{ip}(r, L^p(\Omega))}$$

for $1 \leq p \leq \infty$. This result again follows directly from Theorem 7.2 and Theorem E.1 using the fact that for $0 < r < 1/2$ we have

$$A^{r, \infty}(L^p(\Omega)) = Lip(r, L^p(\Omega)).$$
Before we close this example, we carry out an error analysis using a different family of approximation operators $\tilde{\Pi}_j$. Denote the center of each $\Box_{j,k}$ by $\xi_{j,k}$. We introduce the family of associated projection operators 

$$(\tilde{\Pi}_j f)(x) \sum_{k \leq n_j} f(\xi_{j,k}) \mathbb{1}_{\Box_{j,k}}(x).$$

Each $\tilde{\Pi}_j : C(\Omega) \to A_j$ is onto $A_j$. The family of approximation operators $\{\tilde{\Pi}_j\}_{j \in \mathbb{N}}$ is uniformly bounded as maps from $C(\Omega) \to L^\infty(\Omega)$ since we have 

$$\|\tilde{\Pi}_j f\|_{L^\infty(\Omega)} = \sup_{x \text{ a.e. } \in \Omega} \left| \sum_{k \leq n_j} f(\xi_{j,k}) \mathbb{1}_{\Box_{j,k}}(x) \right| \leq \|f\|_{C(\Omega)}.$$ 

From first principles we can conclude that 

$$\|(I - \tilde{\Pi}_j)f\|_{L^\infty(\Omega)} = \sup_{x \text{ a.e. } \in \Omega} \left| f(x) - \sum_{k \leq n_j} f(\xi_{j,k}) \mathbb{1}_{\Box_{j,k}}(x) \right|$$

$$\leq \max_{k \leq n_j} \sup_{x \text{ a.e. } \in \Box_{j,k}} |f(x) - f(\xi_{j,k})|$$

$$\leq \left( \frac{1}{2}(2^{-j}) \right)^\alpha \|f\|_{Lip(r,C(\Omega))} \lesssim 2^{-jr}\|f\|_{Lip(r,C(\Omega))}. \tag{7.6}$$

This final inequality yields the same rate of approximation $O(2^{-\alpha j}) \approx O(n_j^{-r})$ for a continuous function in $f \in C(\Omega)$.

Example 15 (Example of $(Uf)(x) := (f \circ w_\alpha)(x) = x^\alpha$). In this example we give straightforward, concrete case of Examples 8 and 9. We study the case that we are given a deterministic dynamics on the set $[0,1]$ that is generated by the recursion 

$$x_{n+1} = w_\alpha(x_n) := x_n^\alpha$$

over $[0,1]$ for a fixed $0 < \alpha \leq 1$. This example is selected since it provides a good study of how the rate of approximation of the Koopman operator depends on its action on approximation spaces. We know that the Koopman operator satisfies 

$$(Uf)(x) = f(w_\alpha(x)) = f(x^\alpha) = \int_\Omega \delta_{w_\alpha(x)}(d\xi)f(\xi) = \int_\Omega \delta_{x^\alpha}(d\xi)f(\xi).$$

The family of functions $w_\alpha$ for $0 < \alpha \leq 1$ is shown in Figure 15. Intuitively it seems natural to say that the function $w_\alpha$ exhibits a singularity at $x = 0$. 

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We expect that the approximation of the Koopman operator will depend on the severity of the singularity. We show that the definition of priors in terms of approximation spaces enables a rigorous and specific description of this dependence.

Smoothness in $\text{Lip}(s, C(\Omega))$. Initially, we might choose to model the dynamics by choosing a Lipschitz space $\text{Lip}(r, C(\Omega))$, since this smoothness space is one of the simpler to understand. Even though this choice is not of much practical use for studying this deterministic recursion, it is illustrative of how smoothness of the operator $U$, in terms of its action on approximation spaces, plays a critical role in building approximations. The derivative $dw/dx$ is uniformly bounded on every subset of the form $[\epsilon, 1] := \Omega_{\epsilon}$ for $0 < \epsilon < 1$, but unbounded at $x = 0$. We see that $f \not\in \text{Lip}(1, C(\Omega))$. However, $w_{\alpha} \in \text{Lip}(1, C(\Omega_{\epsilon}))$ for each $\epsilon \in (0, 1)$. Now, for any $f \in \text{Lip}(r, C(\Omega))$ with $0 < r \leq 1$, we have

$$|(\mathcal{U}f)(x) - (\mathcal{U}f)(y)| = |(f \circ w_{\alpha})(x) - (f \circ w)(y)|,$$

$$\leq |f|_{\text{Lip}(s, C(\Omega))} |w_{\alpha}(x) - w_{\alpha}(y)|^r,$$

$$\leq |f|_{\text{Lip}(s, C(\Omega))} (|w_{\alpha}|_{\text{Lip}(1, C(\Omega_{\epsilon}))} |x - y|)^r,$$

$$\lesssim |x - y|^r$$

for all $x, y \in \Omega_{\epsilon}$. This inequality shows that $\mathcal{U}f \in \text{Lip}(r, C(\Omega_{\epsilon}))$ for any $0 < \epsilon < 1$ and $0 < r \leq 1$. The operator $\mathcal{U}$ maps between the spaces $\mathcal{U} : \text{Lip}(r, C(\Omega)) \rightarrow \text{Lip}(r, C(\Omega_{\epsilon}))$.

As discussed in [18], it is known that $\text{Lip}(r, C(\Omega_{\epsilon}))$ for $0 < r < 1$ is the linear interpolation space between the spaces $C(\Omega_{\epsilon})$ and $\text{Lip}(1, C(\Omega_{\epsilon}))$. In Example 14 we have shown from first principles that $f \in \text{Lip}(r, C(\Omega_{\epsilon}))$ implies that the operators $\Pi_j$ onto the piecewise constants defined by point evaluation yield errors like $\| (I - \Pi_j)f \|_{C(\Omega_{\epsilon})} \approx O(2^{-rj})$. Following essentially the same steps as in Example 14 for the difference $(\mathcal{U} - \mathcal{U}_j)f$ yields an error bound $O(2^{-rj})$. Reference [18] shows that in fact a uniformly continuous function is approximated via a linear method from the family of piecewise constants with a rate $n_j^{-r}$ if and only if $f \in \text{Lip}(r, L^{\infty}(\Omega_{\epsilon}))$. Since in this example $\Omega_{\epsilon}$ is compact, the result applies here, of course, with $n_j \approx 2^j$. We also note that the choice of the space $\text{Lip}(r, C(\Omega))$
does prove useful for approximations of some related dynamical systems, including stochastic evolutions induced by Markov chains as discussed in Example 18.

**Smoothness in Lip\(s, L^p(\Omega)\).** The analysis above omits the origin since \(w\) exhibits a singularity there, so it can not be used to analyze discrete evolutions over \([0,1]\). We could further expand our analysis above using the smoothness space \(\text{Lip}(r, C(\Omega_\epsilon))\), but it can be fruitful to demonstrate another metric for smoothness in this case. We now study this case using the Lipschitz space \(\text{Lip}(r, L^p(\Omega))\) defined in terms of the \(L^p\)-integrated Lipschitz inequalities. In particular, from [18] page 66 we know that \(w_\alpha \in \text{Lip}(\alpha, L^\infty(\Omega))\), but for no higher index. In this sense, membership in the space \(\text{Lip}(\alpha, L^\infty(\Omega))\) describes the strength of the singularity at the origin of \(w_\alpha(x) := x^\alpha\).

Now suppose the \(f\) is any function in \(\text{Lip}(s, C(\Omega))\) for \(0 < s \leq 1\). We see that
\[
\|\mathcal{U}f(\cdot + h) - \mathcal{U}f\|_{L^p(\Omega)}^p = \int |(f \circ w)(x + h) - f \circ w(x)|^p \, dx,
\]
\[
\leq |f|_{\text{Lip}(s, C(\Omega))}^p \int |w(x + h) - w(x)|^p \, dx,
\]
\[
\leq |f|_{\text{Lip}(s, C(\Omega))}^p (|w|_{\text{Lip}(\alpha, L^p(\Omega))} \cdot h^\alpha)^{sp},
\]
\[
\leq |f|_{\text{Lip}(s, L^p(\Omega))}^p |w|_{\text{Lip}(\alpha, L^p(\Omega))}^{sp} \cdot h^{sp}.
\]

This series of inequalities demonstrates that
\[
\mathcal{U} : \text{Lip}(s, C(\Omega)) \to \text{Lip}(\alpha s, L^p(\Omega)).
\]

Again we can construct estimates and derive rates of convergence based on the fact that \(\text{Lip}(\alpha s, L^p(\Omega))\) is equivalent to certain linear approximation spaces.

Having chosen a measure of smoothness for the study of the dynamical system and Koopman operator, we must choose a basis. Any of the orthonormal wavelets or multiwavelets, such as those described in Examples 21 or 22 could be used here. But for purposes of illustration we choose, yet again, the Haar scaling functions \(\phi_{j,k}\) and wavelets \(\psi_{j,k}\) for \(j \in \mathbb{N}_0\) and \(k \in \Gamma_\phi \) and \(\Gamma_\psi\), respectively, introduced in Example 3. We define the spaces of approximants to be the collections of wavelets
\[
A_j := \text{span}\{\psi_{j,k}\} \quad \text{and} \quad \tilde{A}_j := \text{span}\{\tilde{\psi}_{j,k}\},
\]
as well as the associated approximation spaces
\[
A^r_p(U) := A^r_p(U; \{A_j\}_{j \in \mathbb{N}_0}),
\]
\[
\tilde{A}^r_p(U) := A^r_p(U; \{\tilde{A}_j\}_{j \in \mathbb{N}_0}),
\]
with \(\tilde{\psi}_{j,k}\) the warped wavelet generated from \(\psi_{j,k}\) in terms of the mapping \(w_\alpha\).

We know that if \(f \in \text{Lip}(s, C(\Omega))\), then \(\mathcal{U}f \in \text{Lip}(\alpha s, L^p(\Omega))\). But this space is equivalent to the approximation space \(A^r(\infty)(L^p(\Omega))\) with \(r = \alpha s\) that is obtained by interpolating between the spaces \(L^p(\Omega)\) and \(\text{Lip}(1, L^p(\Omega))\).

We know from page 131 of [18] or [19] that approximations by piecewise constants converge to the function \(f\) with rate \(O(n_j^{-r})\) for \(0 < r < 1/2\) if and only if \(f \in \text{Lip}(s, L^2(\Omega))\). Alternatively, this specific approximation rate follows
from Theorem E.1 upon recognizing that the span of the Haar scaling functions in $A_j$ coincides with the Schoenberg spline space $S_1(\Delta_j)$ of order $1$, and that $\text{Lip}(r, L^2(\Omega)) \approx \text{Lip}^*(r, L^2(\Omega))$ for $0 < r < 1$.

The following corollary is a simple consequence of Theorem 7.2.

**Corollary 7.3.** Let the hypotheses of Theorem 7.2 hold with a family of approximant spaces $\{A_j\}_{j \in \mathbb{N}_0}$ that satisfy
\[
\|f\|_{A^1, \infty(C(\Omega))} \lesssim \|f\|_{\text{Lip}(1, C(\Omega))}.
\]
Then we have
\[
d_{BL}(\Pi'_j, \nu, \nu) \lesssim n_j^{-1}
\]
for all $j \in \mathbb{N}$ and $\nu \in M^{+, 1}(\Omega)$.

**Proof.** The corollary is a direct consequence of the Theorem choosing $r = 1$, $q = \infty$. We have for $X := C(\Omega)$ the bounds
\[
\begin{align*}
\left\langle (I - \Pi'_j)\nu, f \right\rangle_{X^*, X} & = \left\langle \nu, (I - \Pi'_j)f \right\rangle_{X^*, X} \\
& \leq \|\nu\|_{X^*} \| (I - \Pi'_j)f \|_X \\
& \lesssim n_j^{-1} \|f\|_{A^1, \infty(X)}.
\end{align*}
\]
We now take the supremum of both sides of this inequality over the set of $f$ such that $\|f\|_{\text{Lip}(1, C(\Omega))} = 1$, and the theorem follows.

### 7.4.2. Approximation of Probability Measures

As emphasized above, by definition we know that $\Pi'_j : A^*_j \rightarrow C^*(\Omega)$, but we are not guaranteed that it maps from $M^{+, 1}(\Omega) \subset A^*_j$ into $M^{+, 1}(\Omega)$. In many papers and example problems, it is an important requirement to be able to define approximations of probability measures that are in fact probability measures. This problem has been tackled in a more general sense of duality for (generalized) Young’s measures in \[63, 46\]. A Young’s measure $\nu$ is also known as parameterized probability measure, that is, it is a probability measure $\nu_t$ on $\Omega$ for each $t \in T$ of some index set $T$. We might think of $T$ as a set of possible times, and $\nu_t$ as a probability measure in space at time $t \in T$. The approach in these references differs from the strategy here in two important ways. First, they make no explicit use of approximation spaces, although the approximation error hypotheses they rely on can often be inferred from the definition of an approximation space. Secondly, the analysis of error in these references makes systematic use of a duality of a more general nature: the duality between a (generalized) Young’s measure and certain Cartheodory kernels. In the simplest of the cases treated in \[63\], this duality is given by $(L^1(T, C(\Omega)))^* := L^\infty_w(T, rca(\Omega)))$. Here $L^1(T, C(\Omega))$ is the space of integrable Banach space-valued functions taking values in $C(\Omega)$, and $L^\infty_w(T, rca(S))$ is the collection of weakly measurable functions $t \mapsto \nu_t \in rca(S)$. We will not pursue this line of thought in this paper, which employs a simpler duality structure.

### 7.5. Approximations of Probability Measures, Duality in $L^\infty_w(\Omega) \times L^1_w(\Omega)$

We motivate the approach in this section by considering again the simple deterministic recursion
\[
x_{n+1} = w(x_n).
\]
Example 16. From our earlier discussions, the recursion above defines the sample path of a Markov chain with transition probability kernel $P(dy, x) := \delta_{w(x)}(dy)$, which means that the Koopman operator is just

$$(Ug)(x) := \int_\Omega \delta_{w(x)}(dy)g(y) = g(w(x)).$$

Now we suppose that $\mu$ is a probability measure, $\mu \in \mathbb{M}^+$. We can then define the Perron-Frobenius operator by the duality condition

$$<g, Pf >_{L_\infty(\Omega) \times L_1(\Omega)} = <Ug, f >_{L_\infty(\Omega) \times L_1(\Omega)} = \int_\Omega g(w(x))f(x)\mu(dx)$$

for any $g \in L_\infty^\infty(\Omega)$ and $f \in L_1(\Omega)$. When we choose $g(x) = 1_A(x)$ we obtain an alternate representation in Remark 3.2.2 from [44] with

$$(7.7) \int_A Pf(x)\mu(dx) = \int_{w^{-1}(A)} f(x)\mu(dx).$$

As pointed out in Section 3.2 of [44], a useful alternative representation of $P$ can be derived when the measure $\mu$ is Lebesgue measure on the real line. In this strategy we can use the expression above to derive the equation

$$Pf(x) = f(w^{-1}(x)) \frac{d}{dx}(w^{-1}(x))$$

following [44] on page 43, and subsequently build approximations of this expression. Such approximations could be fashioned, for example, by constructing finite dimensional estimates of $f \circ w^{-1}$ and just multiplying the result by $d(w^{-1})/dx$, which is assumed known. The approximations of $f \circ (w^{-1})$ would follow quite similarly to our approach for approximating $Uf := f \circ w$ in Example 15.

However, if the measure $\mu$ is not Lebesgue measure the process above is of no immediate help. An alternative could be to use the identity in Equation 7.7, substituting an approximation $\mu_j$ for $\mu$. In this context, since $\mu$ is assumed to be a probability measure, it seems vital to derive estimates of $\mu$ that are themselves probability measures. We discuss this problem in the remainder of this section.

Denote the bases for the approximant spaces as

$$A_j := \text{span} \{a_{j,k} \mid k \leq n_j\} \subset L_1^1(\Omega).$$

We have the duality structure

$$A_j \subset L_1^1(\Omega) \quad \text{and} \quad (L_1^1(\Omega))^* = L_\infty^\infty(\Omega) \subset A_j^*$$

with $A_j^*$ the topological dual of $A_j$. That is, $A_j^*$ is the topological dual space of $A_j$ when $A_j$ is endowed with its inherited $L_1^1(\Omega)$ norm. There is a unique dual basis $\{a_{j,k}\}_{k \leq n_j} \subset L_\infty^\infty(\Omega)$ for the dual space $A_j^*$ and we have

$$A_j^* := \text{span} \{a_{j,k} \mid k \leq n_j\}.$$
We have the representations
\[
f = \sum_{i \leq n_j} \langle a^{j,k}, f \rangle_{L^\infty_{\mu}(\Omega) \times L^1_{\mu}(\Omega)} a^{j,k} := \sum_{k \leq n_j} a^{j,k}(f)a_{j,k},
\]
\[
h = \sum_{i \leq n_j} \langle h, a^{j,k} \rangle_{L^\infty_{\mu}(\Omega) \times L^1_{\mu}(\Omega)} a^{j,k} := \sum_{k \leq n_j} a_{j,k}(h)a^{j,k},
\]
for each \(f \in A_j\) and \(h \in A_j^*\). Define the approximation operators \(\tilde{\Pi}_j : L^1_{\mu}(\Omega) \to A_j\) by the expression
\[
(\tilde{\Pi}_j f)(x) = \sum_{k \leq n_j} \frac{a^{j,k}(f)}{\|a^{j,k}\|_{L^1_{\mu}(\Omega)}} a_{j,k}(x)
\]
for any \(f \in A_j\). It can be directly shown [46] that the dual operators \(\tilde{\Pi}_j' : A_j^* \to (L^1(\Omega))^* := L^\infty_{\mu}(\Omega)\) are given by
\[
(\tilde{\Pi}_j' h)(x) := \sum_{k \leq n_j} \frac{a_{j,k}(h)}{\|a^{j,k}\|_{L^1_{\mu}(\Omega)}} a^{j,k}(x).
\]
Now, suppose that \(a_{j,k}, a^{j,k}\) are positive functions. Then it is clear that \(\tilde{\Pi}_j h \geq 0\) for all \(h \geq 0\), and \(\tilde{\Pi}_j'\) is a positive operator. Moreover, we can calculate directly that
\[
\|\tilde{\Pi}_j' h\|_{L^1_{\mu}(\Omega)} = \sum_{k \leq n_j} \frac{a_{j,k}(h)\|a^{j,k}\|_{L^1_{\mu}(\Omega)}}{\|a^{j,k}\|_{L^1_{\mu}(\Omega)}} = \sum_{k \leq n_j} a_{j,k}(h).
\]
The \(L^1_{\mu}\)-norm of \(h\) is computed to be
\[
\|h\|_{L^1_{\mu}(\Omega)} = \int_\Omega \left| \sum_{k \leq n_j} a_{j,k}(h) a^{j,k}(\xi) \right| \mu(d\xi) = \sum_{k \leq n_j} a_{j,k}(h)\|a^{j,k}\|_{L^1_{\mu}(\Omega)}.
\]
Suppose that the dual basis is normalized so that \(\|a^{j,k}\|_{L^1_{\mu}(\Omega)} = 1\). We see then that if \(\|h\|_{L^1_{\mu}} = 1\), so that \(h\) is the density of a probability measure, then \(\|\tilde{\Pi}_j' h\|_{L^1(\Omega)} = \|h\|_{L^1(\Omega)} = 1\). The function \(\tilde{\Pi}_j' h\) is consequently the density of a probability measure.

**Example 17 (Approximation of Probability Measures by Piecewise Constants).** We define the bases \(\{a_{j,k}\}_{j \in \mathbb{N}}\) and approximant spaces
\[
A_j := \text{span} \{a_{j,k}\}_{k \leq n_j} := \text{span} \{1_{\square_{j,k}}\}_{k \in \Gamma_j^\circ}
\]
as in Example 5. The dual basis is readily computed to be
\[
a^{j,k}(x) := \frac{1}{\mu(\square_{j,k})} 1_{\square_{j,k}}.
\]
In fact, we have
\[
\|a^{j,k}\|_{L^1_{\mu}(\Omega)} = 1.
\]
The operator \(\tilde{\Pi}_j'\) defined in terms of these bases maps probability density functions into probability density functions. For purposes of illustration, suppose that \(\mu\) is
normalized Lebesgue $\mu(dx) := dx/2\pi$. In this case the definition of $\tilde\Pi_j$ and $\tilde\Pi'_j$ coincide with their definitions in 14. We can conclude that

$$\|(I - \tilde\Pi'_j)f\|_{L^p(\Omega)} \lesssim n_j^{-r}\|f\|_{A^{r,\infty}(L^p(\Omega))}$$

for $0 < r < 1/2$ and $1 \leq p \leq \infty$.

We can then define the approximations of the Koopman operators

$$\left(\tilde U_j f\right)(x) = \left(\tilde\Pi_j(f \circ w)\right)(x),$$

and calculate

$$\|(U - \tilde U_j)f\|_{L^p(\Omega)} = \|(I - \tilde\Pi_j)(f \circ w)\|_{L^p(\Omega)}$$

$$\lesssim n_j^{-r}\|f \circ w\|_{A^{r,\infty}(L^p(\Omega))}$$

$$\approx n_j^{-r}\|f \circ w\|_{Lip^r(L^p(\Omega))}$$

over this range of $r$ with $1 \leq p \leq \infty$. If we do not know the probability measure $\mu$ is Lebesgue measure, we obtain only

$$\|(U - \tilde U_j)f\|_{L^p_{\mu}(\Omega)} = \|(I - \tilde\Pi_j)(f \circ w)\|_{L^p_{\mu}(\Omega)}$$

$$\lesssim n_j^{-r}\|f \circ w\|_{A^{r,\infty}(L^p_{\mu}(\Omega))},$$

since Theorem E.1 applies only for the case that $\mu$ is Lebesgue measure. It cannot be applied directly to establish the equivalence of the Lipschitz space $\text{Lip}(r, L^p_{\mu}(\Omega))$ to the approximation space $A^{r,\infty}(L^p_{\mu}(\Omega))$, for instance.

In this last example we apply the results of this section to study the rate of approximation of the Koopman operator for a non-trivial Markov chain, one that corresponds to an iterated function system (IFS).

**Example 18.** In this example we describe a well-known semidynamical system that is amenable to the estimation of measures as introduced in this section. We briefly review the notion of iterated function systems, the semidynamical systems associated with them, and the construction of fractals. [44, 1] We let $\Omega := [0,1]^d$, and suppose we are given a finite family of Lipschitz maps $w_\lambda \in \text{Lip}(1, C(\Omega))$ and $|w_\lambda|_{\text{Lip}(1, C(\Omega))} = L_\lambda < 1$. Associated with this family we define the set valued map

$$W(S) := \cup_{\lambda \in \Lambda} w_\lambda(S).$$

It is known that the iteration

$$S_{n+1} = W(S_n)$$

is a semidynamical system on the family $\mathcal{H}$ consisting of compact subsets of $\Omega$ when it is endowed with the Hausdorff metric $d_\mathcal{H}$. In fact, since each mapping $w_\lambda$ is a contraction, it follows that $W : \mathcal{H} \to \mathcal{H}$ is a contraction. Using Banach’s fixed point theorem, we know that there is a unique solution to the fixed point
equation $W(S^*) = S^*$, and it is given by

$$S^* := \lim_{n \to \infty} W^o(S_0) \in \mathcal{H}$$

for any initial compact set $S_0 \subseteq \Omega$. This theorem has been used as a constructive way to define a variety of fractals [1].

There is a second probabilistic interpretation of an iterated function system. Intuitively, it proceeds as follows. Let $n = 0$. Given some initial condition $x_0 \in \Omega$, we roll a dice that has $\#(\Lambda)$ sides that are weighted with probabilities $p_{\lambda}$, $\sum_{\lambda \in \Lambda} p_{\lambda} = 1$. If the result is $\lambda_n \in \Lambda$, we set

$$x_{n+1} = w_{\lambda_n}(x_n)$$

and repeat this process. A typical trajectory then has the form

$$\ldots w_{\lambda_n} \circ w_{\lambda_{n-1}} \circ \cdots \circ w_{\lambda_0}(x_0).$$

The relationship between the attractor of the deterministic attractor in Equation 7.8, the stochastic system in Equation 7.9, and the trajectories in 7.10 has been studied in great detail. It provides a model for understanding the interplay of symbolic dynamics, the support of discrete dynamics, and dynamics of probability measures. [1] We have included this example since its associated Frobenius-Perron and Koopman operators are key to understanding the asymptotic behavior of this dynamical system.

We can describe the stochastic recursion in Equation 7.9 in terms of the transition probability kernel

$$P(A, x) = \sum_{\lambda \in \Lambda} p_{\lambda} \delta_{w_{\lambda}(x)}(A).$$

that gives the probability that the next step of the chain is in $A$ given that the current state is $x$. The Koopman and Perron-Frobenius operators are calculated directly from the definitions in Equations 3.5 and 3.6 to obtain

$$(Uf)(x) = \sum_{\lambda \in \Lambda} p_{\lambda} f(w_{\lambda}(x)),$$

$$(P\nu)(A) = \sum_{\lambda \in \Lambda} p_{\lambda} \nu(w_{\lambda}^{-1}(A))$$

for all measurable $A$ and $x \in \Omega$. It is easy to see that the Perron-Frobenius operator $P$ above maps a probability measure $\nu$ into a probability measure since

$$(P\nu)(\Omega) = \sum_{\lambda \in \Lambda} p_{\lambda} \nu(w_{\lambda}^{-1}(\Omega)) = \sum_{\lambda \in \Lambda} p_{\lambda} \cdot 1 = 1.$$

An intuitive explanation of what the Perron-Frobenius operator represents is useful here. Suppose that we are given a probability measure $\mu_0$ that is interpreted as a distribution of initial conditions over the configuration space for the stochastic evolution, and we want to understand the probability that a set $A \subseteq \Omega$ is visited during the first step of the recursion. This probability is given by a measure $\mu_1$ computed from

$$\mu_1(A) = \int_{\Omega} P(A, x) \mu_0(dx) = P\mu_0.$$
This process can be repeated for any finite number of steps \( n \), and \( \mu_n(A) = (P^n \mu_0)(A) \) describes the probability that the \( n \)-th step of the recursion lands in \( A \) when the initial conditions are distributed according to \( \mu_0 \). We see that if \( P^n \mu_0 \) converges to some probability measure \( \mu^* \), then \( \mu^* \) describes where the iterations eventually concentrate.

This argument can be made precise by endowing the set of probability measures \( \mathbb{M}^{+1}(\Omega) \subset rca(\Omega) \) with a metric. The most popular choice is the bounded Lipschitz metric \( d_{BL} \) introduced in Section 7.4.1. It is known that this metric induces the weak* topology that \( \mathbb{M}^{+1}(\Omega) \) inherits from \( C^*(\Omega) = rca(\Omega) \). [56] With this choice, \( (P, (\mathbb{M}^{+1}(\Omega), d_{BL})) \) is a (weak*−) continuous semigroup in discrete time. Generally, the weak* continuity of the semigroup is equivalent to the Feller property, which holds if \( U : C(\Omega) \to C(\Omega) \). In the present case, since the functions \( w_\lambda \) are Lipschitz continuous, the Feller property is immediate.

The primary reason for the choice of the bounded Lipschitz metric in the study of the dynamical system \( (P, \mathbb{M}^{+1}(\Omega)) \) is that simple criteria can be developed that ensure that the Perron-Frobenius operator is a contraction. We have

\[
\langle P(\mu - \nu), f \rangle_{X^* \times X} = \langle \mu - \nu, U(f) \rangle_{X^* \times X},
\]

\[
= \sum_{\lambda \in \Lambda} p_\lambda \int_{\Omega} f(w_\lambda(x))(\mu(dx) - \nu(dx)),
\]

\[
= \sum_{\lambda \in \Lambda} p_\lambda L_\lambda \int_{\Omega} \frac{f(w_\lambda(x))}{L_\lambda}(\mu(dx) - \nu(dx)).
\]

But if \(|f|_{BL} \leq 1\), then the function \( |f(w_\lambda)/L_\lambda|_{BL} < 1 \) since

\[
|f(w_\lambda(x)) - f(w_\lambda(y))| \leq |f|_{BL}|w_\lambda|_{BL} \leq L_\lambda.
\]

When we take the supremum of both sides of the inequality in Equation 7.11 over \( \|f\|_{L^p(1,C(\Omega))} \leq 1 \), we conclude that

\[
d_{BL}(P\mu, P\nu) \leq L_\lambda d_{BL}(\mu, \nu)
\]

with \( L_\lambda := \sum_{\lambda \in \Lambda} p_\lambda L_\lambda \). Whenever \( L_\lambda < 1 \), we have a unique solution \( \mu^* \) of the fixed point Equation

\[
\mu^* = P\mu^*;
\]

that is given by

\[
\mu^* = \lim_{k \to \infty} P^k \mu_0
\]

for any initial probability measure \( \mu_0 \). This result is shown in [44] Proposition 12.8.1. See [73] for generalizations that allow for place dependent probabilities \( p_\lambda = p_\lambda(x) \). In fact, we know that

\[
d_{BL}(P^k \mu_0, \mu^*) \leq L_\lambda^k d_{BL}(\mu_0, \mu^*)
\]

By definition, the measure \( \mu^* \) that solves the above fixed point equation is the invariant measure of the discrete dynamical system \( (P, (\mathbb{M}^{+1}(\Omega), d_{BL})) \).

We consider the specific example depicted in Figure 7.2.
This iterated function system is defined in terms of three Lipschitz mappings $w_\lambda$ for $\lambda = 1, 2, 3$ defined as

\begin{align*}
w_1 \left( \begin{array}{c} x_1 \\ x_2 \end{array} \right) & := \frac{1}{2} \begin{array}{c} x_1 \\ x_2 \end{array}, \\
w_2 \left( \begin{array}{c} x_1 \\ x_2 \end{array} \right) & := \frac{1}{2} \left( \begin{array}{c} x_1 \\ x_2 \end{array} + \begin{array}{c} 1 \\ 0 \end{array} \right), \\
w_3 \left( \begin{array}{c} x_1 \\ x_2 \end{array} \right) & := \frac{1}{2} \left( \begin{array}{c} x_1 \\ x_2 \end{array} + \begin{array}{c} 0 \\ 1 \end{array} \right).
\end{align*}

This is an example of a “just touching” IFS, one known as a Sierpinski gasket. We will use the multiwavelet basis defined in Example 4 for the construction of approximations of the Koopman operator for this IFS. From the definition of the mappings $w_\lambda$ it is evident that

\begin{align*}
w_1 : \triangle & \rightarrow \Delta_{1,(0,0)} := \hat{\Delta}_1, \\
w_2 : \triangle & \rightarrow \Delta_{1,(0,1)} := \hat{\Delta}_2, \\
w_3 : \triangle & \rightarrow \Delta_{1,(1,0)} := \hat{\Delta}_3,
\end{align*}

with $\Delta_{1,(0,0)}, \Delta_{1,(1,0)}, \Delta_{1,(0,1)}$ defined in Example 4. We define

$$
\hat{\Delta} := \bigcup_{i \in \Lambda} \hat{\Delta}_i,
$$

$$
L^2_\mu(\hat{\Delta}) := \bigoplus_{i \in \Lambda} L^2_{\hat{\mu}_i}(\hat{\Delta}_i),
$$

with $\Lambda = \{1, 2, 3\}$, $\tilde{x}_i = w_i(x)$, $\hat{\Delta}_i = \Delta_i(\Delta)$, $\hat{M}_i(\tilde{x}_i) = w_i^{-1}(\tilde{x}_i)$, $\hat{m}(\tilde{x}_i) = |\partial \hat{M}_i / \partial \tilde{x}_i|$, and $\hat{\mu}_i(d\tilde{x}_i) := \hat{m}_i(\tilde{x}_i) d\tilde{x}_i$. For any $f$ such that $f \circ w_\lambda \in L^2(\Delta)$
for all \( \lambda \in \Lambda \), we have the expansion

\[
U f = \sum_{\lambda \in \Lambda} \sum_{i \in \mathbb{N}_0} \sum_{k \in \Gamma} (f \circ w_{\lambda, i, k})_{L^2(\Delta)} \psi_{j, k},
\]

\[
= \sum_{\lambda \in \Lambda} \sum_{i \in \mathbb{N}_0} \sum_{k \in \Gamma} \left( f, \tilde{\psi}_{\lambda, i, k} \right)_{L^2(\tilde{\Delta}_\lambda)} \psi_{i, k},
\]

and we define the approximation \( U_j \) via

\[
U_j f = \sum_{\lambda \in \Lambda} \sum_{i \leq j} \sum_{k \in \Gamma} \left( f, \tilde{\psi}_{\lambda, i, k} \right)_{L^2(\tilde{\Delta}_\lambda)} \psi_{i, k}.
\]

In this equation \( \tilde{\psi}_{\lambda, i, k} \) are the warped wavelets over \( \tilde{\Delta}_\lambda \) induced by the mapping \( w_{\lambda} \). For any \( f \in A^r(\tilde{\Delta}) \), we have the error estimate

\[
\| (U - U_j) f \|_{L^2(\Delta)} \lesssim 2^{-rj} |U f|_{A^r(\Delta)}.
\]

By duality, we define the approximation \( P_j \) of the Perron-Frobenius operator \( P \) as

\[
(P_j \nu)(A) := \sum_{\lambda \in \Lambda} \sum_{i \leq j} \sum_{k \in \Gamma} \int_{\Omega} \psi_{i, k}(x) \nu(dx) \int_A \tilde{\psi}_{\lambda, i, k}(\tilde{x}) \tilde{\mu}(d\tilde{x}).
\]
the continuous functions. We can, for instance, choose \( r = 1 \) since the RKHS space \( A_X^{1/2}(U) \approx V \subset C(\Omega) \) by construction. Any larger \( r \geq 1 \) will likewise work.

The Perron-Frobenius operator \( \mathcal{P} \) and its approximation \( \mathcal{P}_j \), as well as their respective kernels are defined as in Theorem 5.1,

\[
\mathcal{P} := \sum_{i \in \mathbb{N}} p_i u_i \otimes u_i, \quad p(x, y) = \sum_{i \in \mathbb{N}} p_i u_i(x) u_i(y)
\]

\[
\mathcal{P}_j := \sum_{i \leq j} p_i u_i \otimes u_i, \quad p_j(x, y) := \sum_{i \leq j} p_i u_i(x) u_i(y).
\]

We define the sample dependent operators \( \mathcal{P}_z \) and \( \mathcal{P}_{j,z} \) as

\[
(\mathcal{P}_z f)(x) := \frac{|\Omega|}{m} \sum_{i=1}^m p(x, x_i) f(x_i),
\]

\[
(\mathcal{P}_{j,z} f)(x) := \frac{|\Omega|}{m} \sum_{i=1}^m p_j(x, x_i) f(x_i),
\]

respectively, for any \( f \in A_X^{1/2}(U) \). In this last expression

\[
p_j(\cdot, x_i) := \Pi_j(p(\cdot, x_i)) = \sum_{k \leq j} \int_\Omega p(\xi, x_i) u_k(\xi) \mu(\xi) \ u_k.
\]

For these operators to make sense, it must be true that pointwise evaluation of the kernels and function \( f \) at \( x_i \in \Omega \) makes sense. From the requirement that \( A_j \subset A_X^{r,2}(U) \hookrightarrow C(\Omega) \subset L_\mu^2(\Omega) \), all of the required evaluation operators acting on \( f \) make sense.

We further assume that \( \bar{\mu} := \sup_{(x,y) \in \Omega^2} |p(x,y)| < \infty \) so that pointwise evaluations of the kernel \( p \) makes sense.

We next study an approximation problem when \( p \) is known, the value \( \mu(\Omega) \) is known, but the measure \( \mu \) is unknown. (The assumption that \( \mu(\Omega) \) is known is not essential.) Specifically, we suppose that the spectral expansions in Equations 8.1 through 8.4 are known, and we are given a closed form expression for the function \( f \). However, since the measure \( \mu \) is unknown, the Fourier coefficients \( (f, u_k)_U \) are not computable. Therefore, we cannot compute \( \mathcal{P} f \) nor \( \mathcal{P}_j f \) using Equations 8.1 and 8.2. We seek an error analysis of how well the sample generated approximat \( \mathcal{P}_{j,z} f \), which is computable, estimates \( \mathcal{P} f \). We will see that this analysis is a precursor to the results presented in Section 8.3.

**Theorem 8.1.** Suppose that the assumptions described above and the definitions in Equations 8.2 through 8.5 hold. Let \( z := (x_1, \ldots, x_m) \subset \Omega \) be IID random samples drawn according the probability measure \( \mu \) on \( \Omega \). Then we have the error estimate

\[
\|\mathcal{P} f - \mathcal{P}_{j,z} f\|_U \lesssim \left( \lambda_{j}^{r/2} + \epsilon \right) \|f\|_{A_X^{r,2}(U)}
\]

for all samples \( z \notin \mathcal{B}(\epsilon, j) := \mathcal{B}(\epsilon, j; \mathcal{P}) \), and the set of bad samples

\[
\mathcal{B}(\epsilon, j) := \{ z \in \Omega^m \mid \|\mathcal{P}_j - \mathcal{P}_{j,z}\|_{S^2(U)} > \epsilon \}
\]

has probability

\[
Pr(\mathcal{B}(\epsilon, j)) < AC(\epsilon, j) := \begin{cases} 
1 & \text{if } \epsilon \leq \alpha \sqrt{j/m} \\
2e^{-\beta_m \epsilon^2/j} & \text{if } \epsilon > \alpha \sqrt{j/m}
\end{cases}
\]
with the constants \( \alpha := 4\sqrt{\ln 2}p \) and \( \beta := 1/16p^2 \). The function
\[
AC(\epsilon, j) := AC(\epsilon, j; P)
\]
is the accuracy confidence function for the fixed operator \( P \).

It is important to note that the error estimate above in Equation 8.6 depends on the fixed function \( f \) and the operator \( P \). Before proving this theorem, we state a corollary that is perhaps the most succinct representation of the bias-variance tradeoff that can arise in the approximation of Koopman or Frobenius-Perron operators from samples.

**Corollary 8.2.** Under the assumptions of Theorem 8.1, we have the error bound in expectation over the samples \( z = \{x_1, \ldots, x_m\} \) given by
\[
E_p^m \left( \|Pf - P_{j,z}f\|_U^2 \right) \lesssim \left( \lambda_j^2 \|P\|_{A^2_j(U)}^2 + \|P_j - P_{j,z}\|_{S^2(U)} \right) \|f\|_U^2
\]
with \( P := \mu/\mu(\Omega) \). The constant in the upper bound above depends on \( \bar{p} \), and hence on the fixed operator \( P \).

The proof of this corollary follows from integration of the accuracy confidence function introduced in Theorem 8.1. The integration strategy is essentially identical to the integration carried out in Example 19, so we omit the proof of the corollary here.

**Proof.** By Proposition 9 of [62], \( P_{j,z} \) is a Hilbert-Schmidt operator on \( U \). For any \( f \in U \), we know that
\[
\|P - P_{j,z}\|_U \leq \|P - P_j\|_U + \|P_j - P_{j,z}\|_U,
\]
\[
\lesssim \left( \lambda_j^2 \|P\|_{A^2_j(U)} + \|P_j - P_{j,z}\|_{S^2(U)} \right) \|f\|_U,
\]
where the last line follows from Theorem 5.1 and the fact that \( \| \cdot \|_{L(U)} \leq \| \cdot \|_{S^2(U)} \).

Note that we have \( P_{j,z} = \Pi_j P_z = P_z \Pi_j \) since
\[
(P_{j,z}f)(x) := \frac{|\Omega|}{m} \sum_{1 \leq i \leq m} p_j(x, x_i)f(x_i),
\]
\[
= \frac{|\Omega|}{m} \sum_{1 \leq i \leq m} \sum_{1 \leq k < j} p_k u_k(x) u_k(x_i)f(x_i),
\]
\[
= \sum_{1 \leq k < j} \left( \frac{|\Omega|}{m} \sum_{1 \leq i \leq m} p_k u_k(x_i)f(x_i) \right) u_k(x)
\]
\[
= (\Pi_j P_z f)(x) = (P_z \Pi_j f)(x),
\]
for each \( x \in \Omega \). From the expansion \( P = \sum_{k \in \mathbb{N}} p_k u_k \otimes u_k \) we have
\[
\|P_j - P_{j,z}\|_{S^2(U)}^2 = \|\Pi_j (P - P_z)\|_{S^2(U)}^2 \leq 2j \|P - P_z\|_{S^2(U)}^2.
\]

From Proposition 10 of [62] we know that \( \|P - P_z\|_{S^2(U)} \leq 8\bar{p}^2 \delta/m \), and thus,
\[
\|P_j - P_{j,z}\|_{S^2(U)}^2 \leq 16\bar{p}^2 \frac{j \delta}{m}
\]
for all samples \( z \in \Omega^m \) in a set having \( \mu^m \)-probability in excess of \( 1 - 2e^{-\delta} \).
We next show that this probabilistic error estimate can be expressed in terms of the accuracy confidence function $AC(\epsilon, j)$ in the theorem. Define the set of bad samples as in Equation 8.7, and denote its complement $G$, the set of good samples. We have

$$\text{Prob}(B) = 1 - \text{Prob}(G) < 1 - (1 - 2e^{-\delta}) = 2e^{-\delta}.$$  

Using the bound $\text{Prob}(B) \leq \min\{1, 2e^{-\delta}\}$, we have

$$\text{Prob}(B) \leq \begin{cases} 
1 & \delta \leq \delta_{cr} \\
2e^{-\delta} & \delta > \delta_{cr}
\end{cases}$$

with $\delta_{cr} := \ln 2$. Define $\epsilon^2 := 16\bar{p}_m^2 \frac{\lambda}{m} \delta$. A change of variables shows that Equation 8.8 holds.

Note carefully that the accuracy confidence function $A(\epsilon, j)$ here is for a fixed choice of the function $f$ and the operator $P$. In principle, it resembles Theorem A of Cucker and Smale in [9], in contrast to the uniform estimates for $f$ in a compact convex subset of $C(\Omega)$ in Theorem C∗ in [9] and to those finer uniform estimates in [17], [30], [74]. This bound has a complicated appearance, but for the problems in which either the measure $\mu$ is unknown, it is a convenient way to express the error. It is one of the ways that error in probability estimates are expressed in nonlinear regression, statistics, approximation theory, and statistical learning theory. [17, 81, 30, 74]

![Fig. 8.1: Typical plot of accuracy confidence function $AC(\epsilon, f, P)$](image)

To develop an intuitive understanding of this bound, it must be kept in mind that the discrete, sample-based approximation $P_z$ depends on the random samples $z$. If you are have a particularly bad day, it is possible that the specific estimator $P_z$ generated by that sample $z$ has a large error. However, the estimate in Theorem 8.1 guarantees that for a resolution level $j$ the size of set of bad samples $B(\epsilon, j)$ decreases exponentially with an increase in the number of samples $m$. Its size decays like $e^{-\beta m \epsilon^2 / j}$. A plot of a typical accuracy confidence function is shown in Figure 8.1. The bound in Theorem 8.1 is comprised of the approximation space error $\mathcal{O}(\lambda_j^{-1})$ and the sample error $\mathcal{O}(\epsilon)$. These are also referred to as the bias and probabilistic error,
Fig. 8.2: Plot of the form of the accuracy confidence function $AC(\epsilon_f, P)$ respectively. Suppose now that the number of samples $m$ is fixed, we are interested in a sample accuracy $\epsilon = \lambda_j^{r/2}$ that matches the size of the approximation space error, and we vary the resolution level $j$. The accuracy confidence function $AC(\lambda_j^{r/2}, j)$ is $O(\epsilon^{-\beta m \lambda_j^r})$ as $j$ increases. The size of the set of bad samples therefore grows as $j$ increases. The plot of the general form of the error in this situation is shown in Figure 8.2 for a case when the number of samples $m$ is fixed and the dimension of the approximation space varies. The portion of the plot exhibits increasing error as the variance term dominates. The form of this plot is analogous to those that are used to discuss fundamental results from statistical learning theory. [79]

**Example 19.** In this example we study a specific case in which the tradeoff between the approximation error and sample errors is readily established. We end up with a worst case upper bound on performance that has the same structure as in Theorem 8.1. But since the example is highly structured, the error bound can also be derived from first principles. The approach in this example, while specific, also provides clues on how to attack the case when the samples $z = \{x_1, \ldots, x_m\}$ are the dependent observations of the Markov chain having a transition probability kernel $P(dy, x) := p(y, x)\mu(dy)$.

We now return the discrete dynamical system discussed in Example 1. Suppose that the measure $\mu$ is just Lebesgue measure on $\mathbb{T}^1$, but we do not have this information available to us to use in building approximations. The dynamical system in Example 1 gives rise to the operators

$$(U_f)(x) := \int_{\Omega} p(y, x)f(y)\mu(dy) = \sum_{k \in \mathbb{N}} \sum_{i=1,2} p_{k,i}(f, u_{k,i})U u_{k,i},$$

$$(U_j f)(x) := \sum_{k \leq j} \sum_{i=1,2} (f, u_{k,i})_U u_{k,i}(x),$$
\( \mathcal{U}(x) := \int_{\Omega} p(y, x) f(y) \left( \frac{|\Omega|}{m} \sum_{\ell \leq m} \delta_{x_{\ell}}(dy) \right), \)
\[
= \sum_{k \in \mathbb{N}} \sum_{i=1,2} p_{k,i} \left( \frac{|\Omega|}{m} \sum_{\ell \leq m} u_{k,i}(x_{\ell}) f(x_{\ell}) \right) u_{k,i}(x),
\]

\( \mathcal{U}_j(x) := \sum_{k \leq j} \sum_{i=1,2} p_{k,i} \left( \frac{|\Omega|}{m} \sum_{\ell \leq m} u_{k,i}(x_{\ell}) f(x_{\ell}) \right) u_{k,i}(x), \)

with \( u_{k,i} \) the eigenfunctions defined in Example 1. We can bound the error using the triangle inequality
\[
\| \mathcal{U} f - \mathcal{U}_j z f \|_U \leq \| \mathcal{U} f - \mathcal{U}_j f \|_U + \| \mathcal{U}_j f - \mathcal{U}_j z f \|_U \lesssim \lambda_j^{r/2} + \| \mathcal{U}_j f - \mathcal{U}_j z f \|_U.
\]

We focus specifically on the sample error and see that
\[
\| \mathcal{U}_j f - \mathcal{U}_j z f \|_U^2 = \sum_{k \leq j} \sum_{i=1,2} p_{k,i}^2 \left( \frac{|\Omega|}{m} \sum_{\ell \leq m} u_{k,i}(x_{\ell}) f(x_{\ell}) - (f, u_{k,i})_U \right)^2,
\]
\[
\lesssim \sum_{k \leq j} \sum_{i=1,2} \left( \frac{1}{m} \sum_{\ell \leq m} u_{k,i}(x_{\ell}) f(x_{\ell}) - \mathbb{E}_{\mathcal{P}}(u_{k,i} f) \right)^2.
\]

with the probability measure \( \mathbb{P}(dx) := \mu(dx)/|\Omega| = dx/\mu(\Omega) \). Again, we emphasize that the constant in the upper bound in Equation 8.9 depends on \( p_{1,1} := \max_{k,i}(p_{k,i}) \), so this estimate is again for a fixed operator \( \mathcal{P} \).

The next critical step employs a concentration of measure result in the form of Hoeffding’s inequality. Suppose that \( |g(x) - \mathbb{E}_{\mathcal{P}}(g)| < M \) for \( x \) a.e. \( \in \Omega \). Hoeffding’s inequality states that
\[
\mathbb{P}^m \left( \left| \frac{1}{m} \sum_{\ell \leq m} g(x_{\ell}) - \mathbb{E}_{\mathcal{P}}(g) \right| \geq \varepsilon \right) \leq 2e^{-m\varepsilon^2/2M^2}.
\]

We want to use this inequality to bound each terms in the parentheses in Equation 8.9, that is, for \( g := u_{k,i} f \). Since \( f \in A_{r,2}^{2,2}(U) \subset C(\Omega) \) and \( \Omega \) is assumed compact, we know that there is a constant \( M_f \) such that \( |f| \leq M_f \). It follows that we have the uniform bound \( |u_{k,i} f| \leq M := M_f / \sqrt{\pi} \) for all \( k, i \). The dependence of our analysis on the constant \( M := M_f \) is important to note, a topic we discuss on completion of the proof. We conclude that
\[
\mathbb{P}^m \left( \left| \frac{1}{m} \sum_{\ell \leq m} u_{k,i}(x_{\ell}) f(x_{\ell}) - \mathbb{E}_{\mathcal{P}}(u_{k,i} f) \right| \geq \varepsilon \right) \leq \min(1, 2e^{-m\varepsilon^2/2M^2})
\]
for all $k \in \mathbb{N}, i = 1, 2$. As in our discussion of Theorem 8.1, we define the accuracy confidence function

$$AC(\epsilon) := \begin{cases} 
\frac{1}{2e^{-m\epsilon^2/2M^2}} & \epsilon \leq \alpha\sqrt{1/m} \\
\epsilon > \alpha\sqrt{1/m} 
\end{cases}$$

with $\alpha = \sqrt{2ln2M}$.

We will now show the utility of the probabilistic bound in terms of the accuracy confidence function $AC(\epsilon)$ for the construction of upper bound on the expectation of the error over $m$ samples. The expected error is given by

$$E_{\text{p}m}\left(\left\| \frac{1}{m} \sum_{\ell \leq m} g(x_\ell) - E_{\text{p}}(g) \right\|^2 \right) = \int_0^\infty P_m\left(\left\| \frac{1}{m} \sum_{\ell \leq m} g(x_\ell) - E_{\text{p}}(g) \right\| \geq \epsilon \right) d\epsilon,$$

$$= \int_0^\infty P_m\left(\left\| \frac{1}{m} \sum_{\ell \leq m} g(x_\ell) - E_{\text{p}}(g) \right\| \geq \sqrt{\epsilon} \right) d\epsilon,$$

$$\leq \int_0^\infty AC(\sqrt{\epsilon}) d\epsilon.$$

Now let $\epsilon = \eta^2$ so that $d\epsilon = 2\eta d\eta$ and

$$\int_0^\infty AC(\sqrt{\epsilon}) d\epsilon = 2\int_0^\infty \eta AC(\eta) d\eta,$$

$$\leq 2 \left( \int_0^{\alpha\sqrt{1/m}} \eta d\eta + 2 \int_{\alpha\sqrt{1/m}}^\infty \eta e^{-\eta^2/2M^2} d\eta \right) \leq \frac{1}{m}$$

for a constant that depends only on the constant $M := M_1$. Combining this result with Equation 8.9, we finally obtain

$$E_{\text{p}m}\left(\|Uf - U_{j,z}f\|_U^2\right) \lesssim \lambda_j^r + \frac{j}{m}. \quad (8.10)$$

This is the same form as upper bound in the conclusions of Theorem 8.1 and Corollary 8.2.

As emphasized in the proof, the coefficient in the upper bound above depends on the constant $M := M_1$ and $p_{1,1}$, so the error estimate holds for the fixed function $f$ and operator $U$. This should be contrasted to the results derived in [9, 17, 31] that give uniform estimates over $f$ in a compact, convex subset of $C(\Omega)$, a much stronger result. References [4] and [28] construct estimates of functions in nonlinear regression, not estimates of Perron-Frobenius operators, that have a form analogous to that in Equation 8.10. These references allow for bases that need not be orthogonal functions, and the variance term is of the order $O(j \log j/m)$ in the more general case.
8.2. Approximations from Observations along a Sample Path. In this section we answer how the analysis of the last section can be viewed as the foundation for the situation in which the observations \( \{x_\ell\}_{\ell \leq m} \subset \Omega \) are collected along the sample path of a Markov chain. We will use the analysis in Example 19 to guide us. When we study the argument in Example 19 carefully, we see that the only place that the IID assumption is required is in the hypotheses of Hoeffding’s inequality. Since the samples \( \{x_\ell\}_{\ell \leq m} \) along the sample path of the Markov chain are generally dependent, we cannot use Hoeffding’s inequality here. Hoeffding’s inequality states what is known as a concentration of measure formula. It turns out that there is a fairly large collection of stochastic processes, including some dependent ones, that satisfy a similar concentration of measure result.

A discrete stochastic process \( \{x_i\}_{i \in \mathbb{Z}} \) is said to be \( k \)-dependent if for all \( \tau \in \mathbb{Z} \) the joint stochastic variables \( \{x_i\}_{i \leq \tau} \) are independent of the joint stochastic variables \( \{x_i\}_{i \geq \tau+k+1} \). This relation can be illustrated by organizing the blocks as in

\[
\left( \cdots, x_{-1}, x_0, x_1, \cdots, x_\tau \right) \quad \left( x_{\tau+1}, \cdots, x_{\tau+k} \right) \quad \left( x_{\tau+k+1}, \cdots \right).
\]

As is well-known, for a Markov chain the state \( x_{\tau+1} \) is only coupled to \( x_\tau \), and \( x_{\tau+2} \) is only coupled to \( x_{\tau+1} \). A Markov chain is therefore always \( k \)-dependent with \( k = 1 \). It turns out that some \( k \)-dependent stochastic processes satisfy a concentration inequality quite similar in form to the Hoeffding inequality. We say that a \( k \)-dependent stochastic process satisfies an \( e(m) \)-effective concentration inequality for a function \( f \) if it is true that

\[
\mathbb{P}_m \left\{ \left| \frac{1}{m} \sum_{\ell \leq m} f(x_\ell) - \mathbb{E}(f(x_1(\cdot))) \right| > \epsilon \right\} \leq 2e^{-e(m)\epsilon^2/2\sigma^2(f(x_1(\cdot)))}
\]

where \( m \) is again the number of samples, now along the sample path, and the function \( m \mapsto e(m) \) is defined to be the effective number of samples. One class that satisfies a concentration inequality of this form is the collection of strongly mixing Markov chains whose \( \alpha \)-mixing coefficients decay at an exponential rate. [52].

**Theorem 8.3.** Let the hypothesis of Theorem 8.1 and Example 19 hold, with the exception that the samples \( \{x_\ell\}_{\ell \leq m} \) are collected along the sample path of a Markov chain, and \( U = L^2_\mu(\Omega) \) with \( \mu \) the probability distribution of \( x_1 \). Suppose that the Markov chain is strongly mixing in that there are three constants \( a,b,c > 0 \) such that the \( \alpha \)-mixing coefficients \( \alpha(i) \) decay exponentially as

\[
\alpha(i) \leq ae^{-bi/c}
\]

for \( i \geq 1 \). Then the Markov chain satisfies the \( e(m) \)-effective concentration inequality in Equation 8.11 for the effective number of samples

\[
e(m) := m \left\lfloor \frac{8m}{b} \right\rfloor^{1/(c+1)} \right]^{-1},
\]

and we have the error bound

\[
\mathbb{E}_{\mathbb{P}(\cdot)} \left( \|u f - U_{j,z} f\|_U^2 \right) \lesssim \lambda_j^2 + \frac{j}{e(m)}.
\]
Here $P^{(x)}$ is the probability distribution on $\Omega^m$ of the process $\{x_1, \ldots, x_m\}$.

Proof. The proof of this inequality follows exactly the same steps as that in the proof of Example 19, but now Hoeffding’s inequality is replaced with the $c(m)$-effective concentration inequality that is guaranteed to hold as discussed in [52]. \qed

8.3. Case with $\mu$ and $p$ unknown. As discussed in the introduction, the popularity of Koopman theory for the study of dynamical systems is due primarily to its utility for systems that are to some degree uncertain or unknown. In this last section we study the case when the primary evidence about a dynamical system under study is a set of samples of its input-output behavior. We will show how, in some cases, it is possible to relate the theory of data-driven algorithms to other popular techniques that synthesize aspects of the theory presented earlier, statistical learning theory, and empirical risk minimization. We begin our discussion in the next example which presents the Extended Dynamic Mode Decomposition (EDMD) algorithm.

EXAMPLE 20. Of the various techniques to approximate the Koopman operator, the Dynamic Mode Decomposition (DMD) or the Extended Dynamic Mode Decomposition (EDMD) methods have seen widespread use. These methods are a popular approach to building approximations of Koopman, and through duality, Perron-Frobenius, operators. They have been studied extensively in applications to the study of evolution of observables of discrete or continuous flows in [26, 34, 33, 35].

In this example we discuss in some detail the EDMD algorithm for estimating the Koopman operator associated with a prototypical discrete dynamical system that evolves on a compact set of $\mathbb{R}$. Observation functions on the configuration space of the discrete flow are assumed to be scalar-valued. The general form of the EDMD algorithm applies to much more general flows, see references [33, 50] for a discussion. These assumptions are not critical to the analysis that follows, but certainly simplify notation. For instance, we can use the same bases for the representations of function $w : \Omega \to \Omega$ and for the observables $f$ in our analysis below. In the general theory these may be different bases.

To be specific, we are given the discrete flows

$$x_{n+1} = y_n := w(x_n) \in \Omega \subset \mathbb{R}$$

on the compact set $\Omega$. Suppose we have a measurement process that provides (possible noisy) estimates of the input-output response $z := \{(x_i, y_i)\}_{i=1}^m \subset \Omega \times \Omega := Z$. We let $\mathcal{F}$ denote the family of observables, or observable functions, on the configurations of the dynamical system. The family $\mathcal{F}$ consists of real-valued functions of a real variable in this example. For a fixed observable $f \in \mathcal{F}$, we therefore have

$$\tilde{y}_n := f(x_{n+1}) := f(w(x_n)) = (Uf)(x_n) \in \Omega \subset \mathbb{R}.$$

We briefly summarize the EDMD algorithm for the estimation of a Koopman operator in the form it is described in [39]. Given the measurements of system input-output response $\{(x_i, y_i)\}_{i=1}^m$, we form the empirical arrays

$$X := [x_1, \ldots, x_m] \in \mathbb{R}^{d \times m},$$

$$Y := [y_1, \ldots, y_m] \in \mathbb{R}^{d \times m},$$
for $d = 1$. We choose a family of functions $A_n := \{\phi_i\}_{i=1}^n$ that will be used as the basis for constructing approximations, $A_n := \text{span} \{\phi_i\}_{i=1}^n$. Each $\phi_i : \Omega \rightarrow \mathbb{R}$.

We define the data matrices

$$
\Phi(X) := [\phi(x_1), \ldots, \phi(x_m)] \in \mathbb{R}^{n \times m},
\Phi(Y) := [\phi(y_1), \ldots, \phi(y_m)] \in \mathbb{R}^{n \times m}
$$

with $\phi := [\phi_1, \ldots, \phi_n]'$ and construct the matrix

$$
M_{n,m} := \Phi(Y)\Phi(X)^\dagger \in \mathbb{R}^{n \times n}
$$

with $\Phi(X)^\dagger := (\Phi(X)\Phi(X))^{-\dagger} \in \mathbb{R}^{m \times n}$ the right pseudoinverse of $\Phi(X)$. Note carefully the dependence of the matrix $M_{n,m}$ on the dimension $n$ of the approximation space $A_n$ and the number of samples $m$. As summarized in [39, 33], the final approximation $U_{n,m}^\text{edmd} : A_n \rightarrow A_n$ is given by

$$
(U_{n,m}^\text{edmd} f)(x) := \phi'(x)M_{m,n}^\prime c
$$

for any $f = \sum_{i \leq n} \phi_i c_i \in A_n \subset F$ and $c \in \mathbb{R}^n$.

Recent references [39, 33] derive various quite general results regarding the convergence of the finite dimensional operators $U_{n,m}^\text{edmd}$ to the Koopman operator $U$. For example, [39] shows that

$$
\lim_{m \rightarrow \infty} \|U_{n,m}^\text{edmd} - U_n f\|_{A_n} = 0
$$

for any $f \in A_n := F$ and any norm on the finite dimensional space $A_n$. We are interested in relating this analysis to methods of distribution-free learning theory and the derivation of stronger convergence rates in a very specific case. This is the subject of the next few theorems.

We now derive an alternative form for $U_{n,m}^\text{edmd} f$ that holds when the approximant space consists of piecewise constants over a uniform dyadic grid. In the example above the subscript $n$ on $A_n$ is equal to dimension of $A_n$, since this is convention that is prevalent in the Koopman theory literature such as [39, 33]. On the other hand, the convention in approximation space theory is that the subscript $j$ on $A_j$ is the resolution level of the grid used to define $A_j$. In our convention we denote by $n_j := \#(A_j)$ the dimension of $A_j$. When we make comparisons of the two approximants, we define $U_{j,m}^\text{edmd} := U_{n,m}^\text{edmd}$ from Example 20 to use equivalent indices for both approaches.

**Theorem 8.4.** Let $U := L^2(\Omega)$ and the approximation spaces $A_j$ be given as

$$
A_j := \text{span} \left\{ \psi_{i,k} : 0 \leq i \leq j, k \in \Gamma^\phi_i \right\} = \text{span} \left\{ \phi_{j,k} : k \in \Gamma^\phi_j \right\},
$$

(8.12)

the span of piecewise constants on a grid of level $j$. Here $\phi_{j,k}$ and $\psi_{i,k}$ are the Haar scaling functions and wavelets, respectively, defined on resolution level $j$. The basis functions $1_{\square_{j,k}}$ are the characteristic functions of the cells $\square_{j,k}$ that define the dyadic grid. They are defined so that their supports define a partition of the domain $\Omega$. Then
the EDMD approximation \( U_{j,m}^{\text{edmd}} \) of \( U \) can be written

\[
(U_{j,m}^{\text{edmd}} f)(x) := \sum_{k \in \Gamma_j^m} p_{j,k} 1_{\square_{j,k}}(x)
\]

for each \( f := \sum_{i \leq n} c_i 1_{\square_{j,i}} \) with

\[
p_{j,k} := \begin{cases} 
\frac{\sum_{t \leq m} \sum_{s \leq m} 1_{\square_{j,k}}(x_t) 1_{\square_{j,s}}(y_s) c_i}{\sum_{s \leq m} 1_{\square_{j,s}}(x_s)} & \{x_s\}_{s \leq m} \cap \square_{j,k} \neq \emptyset, \\
0 & \{x_s\}_{s \leq m} \cap \square_{j,k} = \emptyset.
\end{cases}
\]

Proof. In terms of the discussion in Example 20, we have

\[
U_{j,m}^{\text{edmd}} f := \phi'(x) (\Phi(X)\Phi'(X))' \Phi(X)\Phi'(Y) c
\]

for \( f = \sum_{i \leq n} c_i 1_{\square_{j,i}} \) and \( c := \{c_1, \ldots, c_n\}' \in \mathbb{R}^n, \)

\[
\Phi(X) := \begin{bmatrix}
1_{\square_{j,1}}(x_1) & \cdots & 1_{\square_{j,m}}(x_m) \\
\vdots & \ddots & \vdots \\
1_{\square_{j,n}}(x_1) & \cdots & 1_{\square_{j,n}}(x_m)
\end{bmatrix},
\]

and

\[
\Phi(Y) := \begin{bmatrix}
1_{\square_{j,1}}(y_1) & \cdots & 1_{\square_{j,m}}(y_m) \\
\vdots & \ddots & \vdots \\
1_{\square_{j,n}}(y_1) & \cdots & 1_{\square_{j,n}}(y_m)
\end{bmatrix}.
\]

Since the supports of the basis functions define a partition of the domain \( \Omega \), each column of \( \Phi(X) \), or for that matter of \( \Phi(Y) \), is zero except for one entry that is equal to one. For the purposes of this proof only, suppose that we permute the columns of \( \Phi(X) \) so that the first \( n_1 \) columns correspond to the samples in \( X \) that fall in \( \square_{j,1} \), the next \( n_2 \) columns correspond to samples in \( \square_{j,2} \), and so forth, with \( m = \sum_{i \leq n} n_i \). We permute the matrix \( Y \) based on the new ordering of \( X \). With this reordering of samples, \( \Phi(X)\Phi'(X)' \) is a diagonal matrix, and each entry \((k, k)\) of the major diagonal of \( \Phi(X)\Phi'(X) \) is equal to the number of samples in that lie in \( \square_{j,k} \). It then holds that

\[
[(\Phi(X)\Phi(X)')']_{s,t} := \begin{cases} 
0 & s \neq t, \\
\sum_{k \leq m} 1_{\square_{j,s}}(x_k)^{-1} & s = t \text{ and } \{x_k\}_{k \leq m} \cap \square_{j,s} = \emptyset, \\
0 & s = t \text{ and } \{x_k\}_{k \leq m} \cap \square_{j,s} \neq \emptyset.
\end{cases}
\]

The form in the above theorem now follows by explicitly expanding

\[
\Phi(X)^\dagger := \Phi(X)' (\Phi(X)\Phi(X)')' \Phi(X).
\]

We note that the above derivation has been provided for completeness, and a slightly more general result of the same nature can be found in [34]. In the next section we show how the rates of convergence can derived from Theorem 8.4.
8.4. Distribution-Free Learning Theory. Again, our interest is in approximating the action of the Koopman operator on any function \( f \in \mathcal{F} \). We give noisy measurements \( \{(x_i, y_i)\}_{i \leq m} \) of the input-output behavior of the system. We do not know the distribution of this process since neither \( w \) nor \( \mu \) is known. We will see that the EDMD approximation of the Koopman operator is closely related to approximations that arise in results of distribution-free learning theory \([28, 79]\).

Note that, since it is assumed here that the observable function \( f \) is known, we can define another measurement process

\[
\tilde{y}_n := f(y_n) := f(x_{n+1}) = (f \circ w)(x_n) := (\mathcal{U}f)(x_n).
\]

Each \( \tilde{y}_n \) can be computed from \((x_n, y_n)\) since \( f \) is known. For the remainder of this section, let us just consider the measurement process \( \{(x_i, \tilde{y}_i)\}_{i \leq m} \), which we will use to estimate the function \( f \circ w \) in \( \tilde{y} = f(w(x)) \). Since the distribution of the measurement process \( \{(x_i, y_i)\}_{i \leq m} \) is unknown, so is the distribution of the measurement process \( \{(x_i, \tilde{y}_i)\}_{i \leq m} \). In studies of the convergence of data-driven algorithms it is common to make assumptions regarding the underlying distribution in order to establish convergence. These assumptions can be cast in terms of ergodicity or mixing of the process, for example \([33]\).

Here we depart from these conventional approaches in Koopman theory and employ a somewhat different strategy that is popular in statistical and distribution-free learning theory. As discussed earlier, we are ultimately most interested in the case we are given observations of states along the sample path of a Markov chain. But again, error estimates are usually simpler to derive assuming that the samples are IID. We start by considering the IID assumption, and then discuss how the analysis can be extended to more general dependent processes such as Markov chains.

In view of the above we first suppose that the measurements \( \{(x_i, \tilde{y}_i)\}_{i \leq m} \) are independent and identically distributed with respect to some fixed but unknown measure \( \tilde{\mu} \) on \( Z := \Omega \times \Omega \). Such a collection can be obtained assuming that a set of initial conditions \( \{x_i\}_{i \leq m} \) are selected randomly according to an unknown fixed (marginal) measure \( \mu \) on \( \Omega \), and then observing the state \( y_i \) after a single step of the dynamics. If we define the unknown function \( g := f \circ w \), approaches in distribution-free learning theory seek to construct estimates \( g^* \in L^2_\mu(\Omega) \) that approximately minimize the cost function

\[
\mathcal{J}(g) := \frac{1}{2} \int_{z := (x, \tilde{y}) \in Z} (\tilde{y} - g(x))^2 \tilde{\mu}(dz),
\]

\[
g^* = \arg\min_{g \in L^2_\mu(\Omega)} \mathcal{J}(g).
\]

Any measure \( \tilde{\mu} \) on \( Z \) can be factored as \( \tilde{\mu}(dz) := \tilde{\mu}(dy, x)\mu(dx) \) where \( z := (x, y) \), \( \mu \) is the marginal probability

\[
\mu(A) = \tilde{\mu}(A \times \Omega)
\]

for all measurable \( A \subset \Omega \), and \( \tilde{\mu}(A, x) \) is the conditional probability distribution of a measurable set \( A \subset \Omega \) given \( x \in \Omega \). It is an elementary exercise that the function \( g^* \) that achieves the minimum of the ideal functional \( \mathcal{J} \) is the regressor function given by

\[
g^*(x) := \int_{\Omega} \tilde{y}\tilde{\mu}(d\tilde{y}, x).
\]

Since \( \tilde{\mu} \) and \( \tilde{\mu}(dy, x) \) are unknown, however, it is not possible to use this expression or the cost function \( \mathcal{J} \) directly for the construction of estimates. \([9]\).
It is one of the fundamental results of distribution-free learning theory that it is possible to construct useful estimates of the minimizer $g^*$ from the principle of empirical risk minimization. That is, we seek a $g_{j,z}^*$ that minimizes the empirical risk

$$J_z(g) := J_z(g; f) := \frac{1}{m} \sum_{i \leq m} (\tilde{y}_i - g(x_i))^2 = \frac{1}{m} \sum_{i \leq m} (f(y_i) - g(x_i))^2,$$

$$g_{j,z}^* := \arg\min_{g \in A_j} J_z(g) = \arg\min_{g \in A_j} J_z(g; f).$$

The notation $J_z(g; f)$ emphasizes that here the empirical risk $J_z(g)$ depends parametrically on the fixed function $f \in F$. There is no such dependence in the classical problems of distribution-free learning theory, and in this respect, the approximation problem solved by the EDMD algorithm is more general.

It is shown in [4] that, under the hypotheses of Theorem 8.4, the solution $g_{j,z}^*$ of the above empirical risk minimization problem is written as

$$(8.14) \quad g_{j,z}^*(x) := \sum_{k \in \Gamma_j^\phi} g_{j,k} 1_{\square_{j,k}}(x)$$

with

$$g_{j,k} := \begin{cases} \frac{\sum_{\ell \leq m} 1_{\square_{j,k}}(x_{\ell}) \tilde{y}_{\ell}}{\sum_{s \leq m} 1_{\square_{j,k}}(x_s)} & \{x_s\}_{s \leq m} \cap \square_{j,k} \neq \emptyset, \\ 0 & \{x_s\}_{s \leq m} \cap \square_{j,k} = \emptyset. \end{cases}$$

The similarity of the solution $g_{j,z}^*$ of the empirical risk minimization problem to the construction of the EDMD approximation $U_{edmd}^{n,m}$ is immediate when we note that, for any $f \in A_j$ having a representation $f := \sum_{i \leq n} c_i 1_{\square_{j,i}}$, we have

$$\tilde{y}_{\ell} = f(y_{\ell}) = \sum_{i \leq n} c_i 1_{\square_{j,i}}(x_{\ell}) c_i.$$

We conclude that if $f \in A_j$, the approximant obtained by the EDMD algorithm coincides with that obtained by empirical risk minimization. Based on empirical risk minimization, we therefore can define

$$(U_{erm}^{j,z} f) := g_{j,z}^*,$$

for the function $f \in F$. The above discussion illustrates that

$$U_{edmd}^{j,z} := U_{erm}^{j,z} \Pi_j$$

when $A_j := \text{span} \{1_{\square_{j,k}} | k \in \Gamma_j^\phi\}$ and $\Pi_j$ is the $L^2(\Omega)$-orthonormal projection onto $A_j$.

**8.5. Error Bounds for EDMD.** We see from the above study that when $f \in A_j$, the approximations generated by the EDMD algorithm and that constructed from empirical risk minimization coincide. This result is instructive and of interest in its own right. In addition, it is possible to use this analogy to obtain rates of convergence of approximation that rely on choices of $\tilde{y}$ priors, just as in the last few sections. Recall
again that $U := L^2_2(U)$ in this section. Using the decomposition of the error in terms of approximation space and variance contributions as discussed in Section 2, we have

$$
\|Uf - U_{j,z}^{edmd} f\|_U \leq \left[\|Uf - U_{j} f\|_U + \|U_{j} f - U_{j,z}^{erm} f\|_U + \|U_{j,z}^{erm} f - U_{j,z}^{edmd} f\|_U\right].
$$

The first term will be bounded by a constant $O(2^{-rj/2})$ when $Uf \in A^{r,2}(U)$, and the second term will be bounded by a constant $c$ that holds for all samples outside a bad set of samples. As before the measure of the bad set of samples decays exponentially as a function of the number of samples, but grows as a function of the dimension of the approximant space $A_j$. We have discussed the analysis of these terms and the form of the bounds that arise for them. Only the last term is new in this inequality, so we study it more carefully. We have

$$
\|U_{j,z}^{erm} f - U_{j,z}^{edmd} f\|_U = \|U_{j,z}^{erm} (I - \Pi_j) f\|_U.
$$

But, in this section it is assumed that the function $f \in C(\Omega)$ to ensure that the evaluation operator $f \mapsto f(x_k)$ is well-defined for all samples. It follows that for any fixed $z = \{ (x_i, y_i) \}_{1 \leq m}$ and $j \in \mathbb{N}$ we have

$$
\|U_{j,z}^{erm} f\|_{L^\infty(\Omega)} := \sup_{x \in \Omega} \left| \sum_{k \in \Gamma^r} \sum_{1 \leq m} \frac{1_{[0,1)}(x_k) f(y_k) 1_{[0,1)}(x_k)}{1_{[0,1)}(x_k)} \right| \leq \|f\|_{L^\infty(\Omega)},
$$

and this inequality holds for all $f \in C(\Omega)$. Now, since we have assumed that $A^{r,2}(U) \subset C(\Omega) \subset U$, we have

$$
\|U_{j,z}^{erm} f - U_{j,z}^{edmd} f\|_U = \int \left| (U_{j,z}^{erm} (I - \Pi_j) f)(x) \right|^2 \mu(dx),
$$

$$
\lesssim \mu^2(\Omega) \|U_{j,z}^{erm} (I - \Pi_j) f\|_{L^\infty(\Omega)}^2 \lesssim \|(I - \Pi_j) f\|_{L^\infty(\Omega)}^2.
$$

In this section the approximant space $A_j$ is the span of the Haar scaling functions on a grid having resolution level $j$, and the operators $\Pi_j$ are the $L^2(\Omega)$-orthonormal projections onto $A_j$. Just for this particular basis, the orthonormal projection $\Pi_j f$ is identical to the averaging operator studied in Example 14. From that example, we know that $\|(I - \Pi_j) f\|_{L^\infty(\Omega)} \approx 2^{-rj}$ when $f \in \text{Lip}^*(r, L^\infty(\Omega))$. The overall estimate of the error in the EDMD approximation now results from combining this bound with the triangle inequality in Equation 8.15. One way to derive an error bound based on Equation 8.15 assumes IID samples $\{ (x_k, y_k) \}_{1 \leq m}$, such as those generated from a random collection of initial conditions and measurement of the resulting one step response state. The analysis for this case is essentially the same as in our previous discussion. We really are interested with the samples are along the sample path of a Markov chain. We briefly discuss this case next.

There are a number of precise upper bounds in the literature on the second term in Equation 8.15. A good overview can be found in [4]. Given the tradeoff between the approximation space and bias errors, much effort has been devoted to “equilibrating” the leftmost two terms in Equation 8.15 to achieve a balanced overall rate of convergence. Let us suppose that $f \in A_j$ to keep the exposition simple, so
that the rightmost term in Equation 8.15 is zero. Furthermore, also suppose that \( f \) is in a compact, convex subset of \( C(\Omega) \). The equilibration techniques work by choosing a relationship between the number of samples \( m \) and the dimension of approximant space \( n_j \) as these numbers grow. One example rule in [4] calculates the required mesh resolution \( j \) of the space \( A_j \), and therefore the number of basis functions to be used, in terms of the number of samples \( m \). With that rule, Equation 8.15 can be used following [4] to show that

\[
E_{\nu_m} \left( \| U_f - U_{\text{edmd}}^j, z f \|_U \right) \lesssim \left( \frac{\log m}{m} \right)^{\frac{2r}{(2r+1)}}.
\]

Here \( E_{\nu_m} (\cdot) \) is the expectation over the samples \( z:=\{(x_i, \tilde{y}_i)\}_{i\leq m} \) with respect to the product measure \( \nu^m \) on \( Z^m \). It is known that this estimate is the best possible rate of convergence except for the \( \log m \) term, and therefore it is referred to as a semi-optimal bound. [17] As discussed in more detail in [41], an analogous result follows for some strongly mixing Markov chains. By modifying the rule derived in [4] to employ the effective number of samples \( e(m) \) as in Reference [41], it follows that

\[
E_{\nu_m} (\cdot) \left( \| U_f - U_{\text{edmd}}^j, z f \|_U \right) \lesssim \left( \frac{\log e(m)}{e(m)} \right)^{\frac{2r}{(2r+1)}}.
\]

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Appendix A. Detailed Examples of $L^2(\Omega)$–Orthonormal Wavelets.

**Example 21** (Daubechies’ Orthonormal Wavelets). We initially study the case when $\Omega \subseteq \mathbb{R}$ and $f : \Omega \rightarrow \mathbb{R}$. Once the conventions for the one dimensional domain are understood, we briefly discuss the generalization for certain sets $\Omega \subseteq \mathbb{R}^d$ and for vector-valued functions $f : \Omega \rightarrow \mathbb{R}^d$. In this example we consider bases for the approximation space that are given by Daubechies orthonormal, compactly supported wavelets. This example can be considered a model or prototype for more general multiscale bases discussed later. It introduces the specifics of the conventional numbering, indexing, and definition of bases when the approximation space is defined in terms of a wavelet basis. This case has perhaps the most limited applicability among the examples we discuss: the limitations for the most part arise due to the fact that the class of orthonormal, compactly supported wavelets in [14] do not have a closed form expression. Still, computations can be carried out using quadratures that can derived as described in [10]. Despite these limitations, the approach discussed in this example can be applied directly to Perron-Frobenius and Koopman operators defined over the torus $T^d \subseteq \mathbb{R}^d$. This is an important application in itself.

When $\Omega \subseteq \mathbb{R}$, we choose the orthonormal scaling function to be $\phi := \phi_1$, the Daubechies orthonormal scaling function of order $N \geq 2$. [14] The associated orthonormal wavelet is denoted as $\psi := \psi_N$. The support of the functions $\phi_1$ and $\psi_N$ is $[0, 2N − 1]$. It is well-known that these wavelets can be constructed so that they have arbitrarily high Lipschitz smoothness by choosing the index $N$ large enough. Moreover, the scaling functions reproduce polynomial functions of order $N − 1$ and the wavelets have $N − 1$ vanishing moments. These properties are crucial in establishing the approximation rates for spaces of piecewise polynomials contained in the span of these functions. In fact, we find that approximation by the order $N$ Daubechies scaling functions and wavelets yield convergence that is at least $O(2^{−r})$ for functions in $A^r(\Omega) \approx W^r(\Omega)$ with $N = r$. This topic is discussed in more detail in Sections E.2 and E.2.2.

The scaling functions and wavelets over a dyadic grid of cubes having side length $2^{−j}$ are defined to be $\phi_{j,k}(x) := 2^{jd/2} \phi(2^j x − k)$ and $\psi_{j,k}(x) := 2^{jd/2} \psi(2^j x − k)$ for $k \in \mathbb{Z}^d$ and $j \in \mathbb{N}_0$ with $d = 1$. In the conventional notation of multiresolution analyses, we set the scaling spaces $V_j := \text{span} \left\{ \phi_{j,k} \mid k \in \Gamma_j^\phi \right\}$ and detail spaces $W_j := \text{span} \left\{ \psi_{j,k} \mid k \in \Gamma_j^\psi \right\}$ with $\Gamma_j^\phi$ and $\Gamma_j^\psi$ the set of admissible translates for the scaling functions and wavelets on grid level $j$. In this first example, when we assume $\Omega := \mathbb{R}$, we have $\Gamma_j^\phi = \Gamma_j^\psi = \mathbb{Z}$. But this will change when we consider compact domains $\Omega \subset \mathbb{R}^d$ below or multiscale wavelets in the next example. We have the representation

$$f := \sum_{k \in \Gamma_0^\phi} (f, \phi_{0,k}) U \phi_{0,k} + \sum_{j \in \mathbb{N}_0} \sum_{k \in \Gamma_j^\psi} (f, \psi_{j,k}) U \psi_{j,k} \tag{A.1}$$

for any $f \in U$. It is standard to modify this representation and create the multiscale summation

$$f := \sum_{j \in \mathbb{N}_0} \sum_{k \in \Gamma_j^\psi} (f, \tilde{\psi}_{j,k}) L^2 \tilde{\psi}_{j,k} \tag{A.2}$$
with the modified set of indices

\[ \tilde{\psi}_{j,k} := \psi_{j,k}, \quad k \in \tilde{\Gamma}_j^\psi := \Gamma_j^\psi \]
\[ \tilde{\psi}_{0,k} := \phi_{0,k}, \quad \tilde{k} := (k,0) \in \tilde{\Gamma}_0^\psi := \Gamma_0^\phi \times \Gamma_0^\psi, \quad \text{for } j = 0, \]
\[ \tilde{\psi}_{0,k} := \psi_{0,k}, \quad \tilde{k} := (0,k) \in \tilde{\Gamma}_0^\psi := \Gamma_0^\phi \times \Gamma_0^\psi, \quad \text{for } j = 0. \]

In the discussion that follows, whenever we refer to a multiscale representation, we suppress the notation and assume that the wavelets, scaling functions, and index sets have been modified so as to obtain a form similar to that in Equation A.2, or just as in Equation 6.2.

So far we have selected \( \Omega = \mathbb{R} \). In this case the index sets \( \Gamma_j^\psi, \Gamma_j^\phi := \mathbb{Z} \) are infinite. However, suppose that we instead consider the compact domain \( \Omega \subset \mathbb{R} \). For illustration suppose \( \Omega := [0,1] \). When we identify basis functions modulo periodization over \([0,1]\), we can arrange that the index sets \( \Gamma_j^\psi \) and \( \Gamma_j^\phi \) contain \( O(2^j) \) functions for all \( j \) larger than some fixed level \( j_0 \). For example, for \( \phi := N \phi \) with \( N = 3 \), we obtain \( 2^j \) indices in \( \Gamma_0^\phi \) for \( j \geq 3 \). The approximation spaces are defined to be \( A_{j+1} := V_{j+1} := V_j \oplus W_j \) for each \( j \geq 0 \), and now \( V_j, W_j \) are finite dimensional. In the one dimensional case we have

\[ \#(\Gamma_j^\psi) = 2^j \quad \text{and} \quad \#(\Gamma_j^\phi) = 2^j \]

for all \( j \geq j_0 \) large enough. The orthogonal projections \( \Pi_j := U \rightarrow A_j \) for each \( j \geq j_0 \) are written as

\[ \Pi_j f := \sum_{k \in \Gamma_j^\phi} (\phi_{j,k}, f)_U \phi_{j,k} = \sum_{j_0 \leq m \leq j} \sum_{k \in \Gamma_m^\phi} (\psi_{j,k}, f)_U \tilde{\psi}_{j,k}, \]

where again the latter multiscale decomposition is modified to include both wavelets and scaling functions in the lowest level resolution grid \( j_0 \). We thereby obtain easily computed representations of the norm and inner product on the approximation spaces \( \mathbb{A}^{r,2}(U) \). We have

\[ (f,g)_{\mathbb{A}^{r,2}(U)} = \sum_{j_0 \leq j \in \mathbb{N}_0} 2^{2jr} \sum_{k \in \Gamma_j^\phi} (f, \psi_{j,k})_U (g, \psi_{j,k})_U, \]
\[ \|f\|_{\mathbb{A}^{r,2}(U)}^2 = \sum_{j_0 \leq j \in \mathbb{N}_0} 2^{2jr} \sum_{k \in \Gamma_j^\phi} |(f, \psi_{j,k})_U|^2, \]

for \( f,g \in \mathbb{A}^{r,2}(U) \). No generality is lost in that the summation above starts for a coarsest grid level \( j_0 \) that may be greater than 0.

The modification of the above framework for domains \( \Omega \subset \mathbb{R}^d \) for \( d > 1 \) often employs tensor products of bases. First, consider only the case when \( f : \Omega \rightarrow \mathbb{R} \) and \( \Omega := \mathbb{R}^d \). We set \( \psi^\phi = \phi \) and \( \psi^1 := \psi \). Let \( e := (e_1, e_2, \ldots, e_d) \) be the coordinates of the corners of the unit cube \( [0,1]^d \) in \( \mathbb{R}^d \). We set the tensor product basis \( \psi_e(x) := \psi^{e_1}(x_1) \cdots \psi^{e_d}(x_d) \), and analogously to the setup above we define the dilated and translated wavelets as \( \psi_{j,(e,k)} := 2^{jd/2} \psi_e(2^j x - k) \). We then have an expansion of the form in Equation A.2 with \( \tilde{k} := (e,k) \in \tilde{\Gamma}_j^\phi := \{0,1\}^d \times \Gamma_j^\psi \) for \( j \geq 1 \). Modifications are also made for \( \tilde{\Gamma}_0^\phi \), similar to the above, to include
scaling functions and wavelets on the coarsest level $j_0 \geq 0$. The restriction of the domain $\Omega \subseteq \mathbb{R}^d$ above to generate bases for periodic functions on the torus $\mathbb{T}^d$ is straightforward.

Finally, we note that when the family of functions
\[
\left\{ \psi_{j,e,k} \mid j \in \mathbb{N}_0, e \in \{0,1\}^d, k \in \Gamma_j^\psi \right\}
\]
form an orthonormal basis for $U := L^2(\Omega)$, it also generates a component-wise orthonormal basis for the functions $f := \Omega \rightarrow \mathbb{R}^d$ on $\Omega \subseteq \mathbb{R}^d$. If $\{E_\ell\}_{1 \leq \ell \leq d}$ is the canonical basis for $\mathbb{R}^d$, then
\[
\left\{ \psi_{j,e,k} E_\ell \mid j \in \mathbb{N}_0, e \in \{0,1\}^d, k \in \Gamma_j^\psi, 1 \leq \ell \leq d \right\}
\]
is an orthonormal basis for $U := (L^2(\Omega))^d$ in its usual inner product. This family of vector-valued functions serves as the basis for the approximation spaces $A^{r,2}(U)$ in this case.

**Example 22 (Orthonormal Multiwavelets).** In our discussion of Example 21 it was noted that the orthonormal wavelets, not having a closed form expression, can be more difficult to employ than other wavelet constructions. The definition of wavelets that have many of the desirable properties of orthonormal, compactly supported wavelets has been studied carefully over the past 25 years. In this example we consider compactly supported multiwavelets [29, 21]. These wavelets have closed form expressions, are written in terms of piecewise polynomials, exhibit useful symmetry and antisymmetry properties, and have good approximation properties. The essential difference in this framework is that there is a finite family of multiscaling functions $\{\phi_p\}_{p=1,\ldots,b}$ and an associated family of multiwavelets $\{\psi_q\}_{q=1,\ldots,a}$. The multiscaling functions and multiwavelets are then defined as $\phi_{j,(p,k)} := 2^{jd/2}\phi_p(2^j x - k)$ and $\psi_{j,(q,k)} := 2^{jd/2}\psi_q(2^j x - k)$, respectively. We set the spaces
\[
V_j := \text{span} \left\{ \phi_{j,(p,k)} \mid (p,k) \in \Gamma_j^\phi \right\},
\]
\[
W_j := \text{span} \left\{ \phi_{j,(q,k)} \mid (q,k) \in \Gamma_j^\psi \right\},
\]
for $j \in \mathbb{N}_0$. For any $f \in L^2(\mathbb{R}^d)$, when the level $j = 0$ is redefined to include both scaling functions and wavelets, the multiscale expansion takes the form
\[
f := \sum_{j \in \mathbb{N}_0} \sum_{(q,k) \in \Gamma_j^\psi} (f, \psi_{j,(q,k)}) L^2 \psi_{j,(q,k)},
\]
which again can be recast in the form of Equations 6.2 or A.2 with an appropriate re-definition of the indices and index set.

The multiscale expansion in Equation A.3 contains an infinite number of terms in each $\Gamma_j^\psi$ when $\Omega = \mathbb{R}^d$, and it must be modified for representations of functions over a compact domain $\Omega$. This will result in a finite number of terms in each $\Gamma_j^\psi$. One of the nice features of these multiwavelets is that it is a simple matter to adapt the expansion over compact domains that are the union
of dyadic cubes on some fixed resolution level $j_0$. The index sets $\Gamma^\psi_j$, $\Gamma^\phi_j$ are then modified by retaining only those linearly independent restrictions of basis functions whose support intersects $\Omega$ and satisfy the desired boundary conditions on $\partial\Omega$. This is particularly easy task for domains $\sim [0,1]^d$ using symmetry and antisymmetry of the functions. [21]

Once the modifications of the index sets $\Gamma^\psi_j, \Gamma^\phi_j$ is complete, so that they are finite dimensional, we again define the finite dimensional approximation spaces $A_{j+1} := V_{j+1} := V_j \oplus W_j$. The orthogonal projections $Q_j : L^2(\Omega) \to A_j$ are now written in the form

$$Q_j f := \sum_{(q,k) \in \Gamma^\psi_j} (\psi_{j,(q,k)}, f) \psi_{j,(q,k)}.$$  

When $\Omega \subset \mathbb{R}$, that is $d = 1$, the index set $\Gamma^\psi_j$ contains $O(2^j)$ functions in this case. The norm on the approximations is written as

$$\|f\|^2_{L^2(\Omega)} = \sum_{j \in \mathbb{N}_0} 2^{2jr} \sum_{(q,k) \in \Gamma^\psi_j} |(f, \psi_{j,(q,k)})|^2,$$

which again just gives a weighted sum of the generalized Fourier coefficients of the function $f$ in the orthonormal multiwavelet basis. Modifications for $d > 1$ are constructed similarly to the discussion above.

Appendix B. Discrete Stochastic Processes and Markov Chains. This section summarizes some of the basic definitions encountered in the study of stochastic processes and Markov chains. The reader is referred to [47] for a full treatment of the details. Let $(Z, \Sigma_Z)$ be a measurable space and $(Q, \mu, \Sigma_Q)$ be a measure space. A random variable is a function $f : Q \to Z$ such that $A \in \Sigma_Z$ implies that $f^{-1}(A) \in \Sigma_Q$. The probability distribution $\mu_f$ on $Z$ of the random variable $f$ is given by $\mu_f(\cdot) := \mu(f^{-1}(\cdot))$, or sometimes is expressed in the form

$$\mu_f(A) = \text{Prob} \{ q \in Q \mid f(q) \in A \} = \mu \{ q \in Q \mid f(q) \in A \}$$

for any measurable set $A \in \Sigma_Z$. A discrete stochastic process $z := \{z_i\}_{i \in \mathbb{N}_0}$ indexed by $\mathbb{N}_0$ is a sequence of random variables $z_i : Q \to Z$. If we denote the product space $Z := \mathbb{Z}^{\mathbb{N}_0}$, then $z$ may be understood as a random variable $z : Q \to Z$ from the measure space $(Q, \mu, \Sigma_Q)$ to the measurable space $(Z, \Sigma_Z)$. The probability distribution of the process $z$ is the probability measure $\mu_z := \mu(z^{-1}(\cdot))$ for each measurable $\mathbb{A} \subset \mathbb{Z}$. We denote this probability measure on $\mathbb{A}$ by

$$\mathbb{P}_{\{z\}}(\mathbb{A}) = \text{Prob} \{ \mathbb{A} \subset Z \}$$

and refer to it as the distribution of the process. A cylinder set $C$ contained in $Z^m$ is a set that has the form $C^\ell \subset C_1 \times C_2 \times \cdots \times C_m$ with each $C_i \in \Sigma_Z$ for $1 \leq i \leq m$. We define the law or probability distribution $\mu_{\{i_1, \ldots, i_m\}}$ on $Z^m$ for a finite collection of coordinates indexed by $\{i_1, \ldots, i_m\}$ by setting

$$\mu_{\{i_1, \ldots, i_m\}}(C) := \mu \{ q \in Q \mid z_{i_k}(q) \in C_k, \ 1 \leq km\ell \}$$
for each cylinder set $C^\ell \subset Z^m$. The statistical properties of the process $z$ is determined by the laws of all such finite dimensional subsets of coordinates. When the index set \( \{i_1, \cdots, i_m\} \) is just the first $m$ coordinates, we follow the convention above and define the distribution of the first $m$ states of the process by

\[
\mathbb{P}^m_{\{i\}}(C^m) := \mu^{(1, \ldots, m)}(C^m).
\]

We say that the stochastic process is independent and identically distributed whenever $\mu_{z_i} := \mu$ for some fixed measure $\mu$ on $Z$ and all the $z_i$ are independent of one another. In this case for any finite set of indices $\{i_1, \ldots, i_m\}$ we have

\[
\mu^{m}_{i_1, \ldots, i_m}(d\xi_1 \cdots d\xi_m) = \mu_{z_{i_1}}(d\xi_1) \cdots \mu_{z_{i_m}}(d\xi_m) = \otimes_{k \leq \ell} \mu(d\xi_k)
\]

with $\mu^m(d\xi_1, \ldots, d\xi_m) := \mu(d\xi_k)$ the product measure on $Z^\ell$. In this case we simply write

\[
\mathbb{P}^m(C^m) := \mathbb{P}^m_{\{i\}}(C^m) = \mu^m(C^m),
\]

and the distribution of the first $m$ states of the process is just the product measure $\mu^m$.

We will limit considerations in this paper to stochastic processes that are Markov chains. A Markov chain on a state space $Z$ is a discrete stochastic process whose statistics are entirely determined by a transition probability kernel. The transition probability kernel is a map on $\mathbb{P} : \Sigma_Z \times Z \mapsto [0, 1]$. For each fixed measurable set $A \subset Z$, $\xi \mapsto \mathbb{P}(A, \xi)$ is a measurable function. And, for each fixed $\xi \in Z$ the mapping $A \mapsto \mathbb{P}(A, \xi)$ is a probability measure. Intuitively, the transition probability kernel $\mathbb{P}(A, \xi)$ defines what the probability is that the next single step in the chain lands in the set $A$ given that the current state is $\xi$. As discussed in [47], p. 66, if the probability distribution of the initial state $x_0$ is the probability measure $\mu_0$ on $Z$, the probability distribution of the first $m$ steps of the chain belonging to the cylinder set $C := C_1 \times \cdots \times C_m$ is given by

\[
\mathbb{P}^m_{\{i\}}(C_1 \times \cdots \times C_m) = \int_{\xi_{m-1} \in C_{m-1}} \cdots \int_{\xi_0 \in A_0} \mathbb{P}(C_m, \xi_{m-1}) \mathbb{P}(d\xi_{m-1}, \xi_{m-2}) \cdots \mathbb{P}(d\xi_1, \xi_0) \mu_0(d\xi_0).
\]

We are interested specifically in this paper in integrating expressions such as

\[
\mathbb{E}_m(g)
\]

for a function $g : Z^m \rightarrow \mathbb{R}$. For a general Markov chain this can be a formidable task.

However, there are a few cases in which the general expression simplifies. The first case is an independent and identically distributed, or IID, stochastic process. For this process, each of the coordinates $z_i$ is independent of all other coordinates and has the probability distribution $\mu_{z_i} = \mu$ for some fixed measure $\mu$. In the language of Markov chains the transition kernel is just $\mathbb{P}(dz, \xi) = \mu(dz)$. This means that the transition kernel does not care where it starts. If we want to compute the expectation of a function $g : Z^m \rightarrow \mathbb{R}$ over the sequence of random variables $(z_1, \ldots, z_m)$, we just have

\[
\mathbb{E}_p(g) = \mathbb{E}_\mu(g) = \mathbb{E}_{\mu^m}(g) = \mathbb{E}_{\mu^m}(g) = \int_{Z^m} g(\xi_1, \ldots, \xi_m) \mu(\xi_1) \mu(\xi_2) \cdots \mu(\xi_m)
\]
with $\mu^n$ the product measure on $Z^m$.

In the second case, if the process is just the deterministic system having the transition probability $P(dy, x) := \delta_{w(x)}(dy)$, it can be shown that the probability \( \{z_1, \cdots, z_m\} \in C := C_1 \times \cdots \times C_m \) that \( \{z_i\}_{i \leq m} \) is contained in the cylinder set \( C \) is given by

\[
P_m(\{x\})(C_1 \times \cdots \times C_m) = \delta_{w(x)}(C_1) \cdot \delta_{w^2(x)}(C_2) \cdots \delta_{w^n(x)}(C_n),
\]

and the expectation becomes

\[
E_{\mathbb{P}_m}(g) = \int_{Z^m} g(\xi_1, \ldots, \xi_m) \delta_{w(x_0)}(d\xi_1) \cdot \delta_{w^2(x_0)}(d\xi_2) \cdots \delta_{w^n(x_0)}(d\xi_m)
\]

\[
= g(w^1(x_0), w^2(x_0), \cdots, w^n(x_0)).
\]

**Appendix C. Spectral Decomposition: Compact, Self-Adjoint Operators.** In this section we review the fundamental properties of compact, self-adjoint operators and their spectral decompositions, which are used extensively in Section 4. We first review the construction of the singular value decomposition for matrices, and subsequently discuss the generalization to the Schmidt decomposition of compact operators acting between Hilbert spaces.

For any matrix $T \in \mathbb{R}^{m \times n}$, the symmetric and positive semidefinite matrix $T^*T$ has a collection of real eigenvalues $\lambda_k := \lambda_k(T^*T)$ that can be arranged with multiplicities in nonincreasing order $\lambda_1 \geq \cdots \geq \lambda_n \geq 0$. When the corresponding eigenvectors $v_1, \ldots, v_n \in \mathbb{R}^{n \times n}$ are chosen to be of unit length and mutually orthogonal, we obtain the spectral factorization of $T^*T$ in the form

\[
T^*T = V \Sigma_n^2 V'
\]

where $\Sigma_n \in \mathbb{R}^{n \times n}$ is the diagonal matrix $\Sigma_n = \text{diag}(\sigma_1, \ldots, \sigma_n)$ and $V \in \mathbb{R}^{n \times n}$ is an orthogonal matrix whose columns are the associated eigenvectors $V = [v_1, \ldots, v_n]$. If we carry out the same construction for the matrix $TT^* \in \mathbb{R}^{m \times m}$, we obtain

\[
TT^* = U \Sigma_m^2 U'
\]

where $\Sigma_m \in \mathbb{R}^{m \times m}$ is the diagonal matrix of non-increasing eigenvalues $\sigma_m := \sqrt{\lambda_m} := \sqrt{\lambda_m(TT^*)}$ and $U = [u_1, \ldots, u_m] \in \mathbb{R}^{m \times n}$ is an orthogonal matrix whose columns are the corresponding orthonormal eigenvectors of $TT^*$. It is a quick calculation to show that these decompositions are further related: we always have

\[
u_k = \frac{1}{\sigma_k} T v_k
\]

for $k = 1, \ldots, \min(m, n)$.

The singular value decomposition of the matrix $T$ is then defined as

\[
T = U \Sigma V'
\]

where $\Sigma \in \mathbb{R}^{m \times n}$ is diagonal with entries $\sigma_1, \ldots, \sigma_{\min(m,n)}$ on the diagonal. It is convenient for comparison to the infinite dimensional operators discussed later to express these decompositions in terms of operators acting on arbitrary vectors. We
have

\[ Tx = \sum_{k=1,\ldots,m} \sigma_k(v_k, x) R u_k, \]

\[ TT^* y = \sum_{k=1,\ldots,m} \sigma_k^2(u_k, y) R u_k, \]

\[ T^* Tx = \sum_{k=1,\ldots,m} \sigma_k^2(v_k, x) R v_k, \]

for each \( x \in \mathbb{R}^n \) and \( y \in \mathbb{R}^m \).

Now consider the case when \( U, V \) are Hilbert spaces and \( T : V \to U \) is a linear, compact operator. Since \( T^* T \) is a compact, self-adjoint operator, all of its eigenvalues are real and are contained in the interval \([0, \|T^* T\|]\). The number of eigenvalues greater than any given positive constant is finite, so the only possible accumulation point of the eigenvalues is zero. Each eigenspace corresponding to a nonzero eigenvalue is finite dimensional. The eigenvalues \( \lambda_k(T^* T) \equiv \lambda_k(T T^*) \) are assumed to be arranged in an extended enumeration that includes multiplicities in nonincreasing order

\[ \lambda_1 \geq \lambda_2 \geq \cdots \geq 0. \]

The \( k^{th} \) singular value \( \sigma_k(T) \) of \( T : V \to U \) is defined as

\[ \sigma_k(T) := \lambda_k^{1/2}(T^* T). \]

There exist orthonormal collections of eigenvectors \( \{u_k\}_{k=1}^\infty \) for \( TT^* \) and \( \{v_k\}_{k=1}^\infty \) for \( T^* T \) such that

\[ Tf = \sum_{k=1}^\infty \sigma_k(v_k, f) v_k \quad \text{convergence in } U, \]

\[ TT^* g = \sum_{k=1}^\infty \sigma_k^2(u_k, g) u_k \quad \text{convergence in } U, \]

\[ T^* T f = \sum_{k=1}^\infty \sigma_k^2(v_k, f) v_k \quad \text{convergence in } V, \]

for each \( f \in V \) and \( g \in U \).

The expansions above are vital to the development in Section 4. One further result is also needed to define the square root operator \( \sqrt{T} \). Suppose that \( g : \mathbb{R} \to \mathbb{R} \).

For any self-adjoint compact operator \( T : U \to U \), the function \( g(T) \) can be defined in terms of the spectral expansion

\[ (g(T))(f) := \sum_{k=1}^\infty g(\lambda_k)(u_k, f) u_k, \]

provided that \( g \) is continuous on the spectrum of \( T \).

Appendix D. Schatten Operators. The spectral decomposition summarized in Section C motivates the definition of the Schatten class of operators \( S^p(V, U) \) acting between the Hilbert spaces \( V \) and \( U \). Whenever \( T : V \to U \) is compact, the operator \( T^* T : V \to V \) is compact, nonnegative, and self-adjoint. By the spectral theory
presented in Section C, there is a unique square root operator \((T^\ast T)^{1/2}\). We define \(|T| := (T^\ast T)^{1/2} : V \to V\) given by

\[
|T|f := \sum_{k=1}^{\infty} \sigma_k(v_k, f)v_k, \quad \text{convergence in } V,
\]

for each \(v \in V\). We say the operator \(T \in S^p(V, U)\) for \(0 < p < \infty\) provided that \(T\) is compact and the \(S^p\)-norm defined as

\[
\|T\|_{S^p} := \left(\sum_{k=1}^{\infty} \sigma_k^p(T)\right)^{1/p} := \left\|\{\sigma_j\}_{j=1}^{\infty}\right\|_{\ell^p} < \infty
\]

with \(\sigma_j(T)\) the singular values of \(|T|\). We also define

\[
S^\infty(V, U) := \{T \in \mathcal{L}(V, U) \mid T \text{ is compact}\}.
\]

Perhaps the two most well-known subclasses of the Schatten operators are the Hilbert-Schmidt operators \(S^2(V, U)\) and the trace class operators \(S^1(V, U)\). For these two cases we have

\[
\|T\|_{S^1} := \sum_{j=1}^{\infty} (Tv_j, v_j)_V,
\]

\[
\|T\|_{S^2} := \left\|\{\sigma_j\}_{j=1}^{\infty}\right\|_{\ell^2} = \left(\sum_{j=1}^{\infty} (Tv_j, v_j)_V\right)^{1/2}
\]

Appendix E. Approximation Spaces. Our error estimates will be based on either linear approximation methods as they arise in the construction of approximation spaces. See [60], or more recently [19], for a thorough theoretical discussion of these spaces. An approximation method is a pair \((X, \{A_n\}_{n \in \mathbb{N}_0})\) where \(X\) is a (quasi-)Banach space and \(\{A_n\}_{n \in \mathbb{N}_0}\) is a sequence of subsets of \(X\) that satisfy the following conditions:

(i) \(\{0\} = A_0 \subseteq A_1 \subseteq A_2 \subseteq \cdots \subseteq X\),

(ii) \(aA_n = A_n\) for all \(a \in \mathbb{R}\) and \(n = 1, 2, \ldots\),

(iii) there is a constant \(c\) such that \(A_n + A_n \subseteq A_{cn}\) for \(m, n = 1, \ldots\),

(iv) \(\cup_{n \in \mathbb{N}_0} A_n\) is dense in \(X\),

(v) for each \(f \in X\) there exists a best approximation of \(f\) in \(A_n\).

In general the subsets \(\{A_n\}_{n \in \mathbb{N}_0}\) need not be linear spaces, and the assumptions above define the foundation of nonlinear approximation methods. In this paper we only consider the case when they are linear subspaces and refer to the \(A_n\) as the approximant subspaces. When we define the \(n^{th}\) approximation number as

\[
(E.1) \quad a_n(f, X) := E_{n-1}(f, X) := \inf_{a \in A_{n-1}} \|f - a\|_X
\]

for each \(n > 0\), the approximation space \(A^{r,q} := A^{r,q}(X)\) for \(r > 0, 1 \le q \le \infty\) is defined to be the collection of \(f \in X\) for which the sequence

\[
\left\{n^{r-1/q}a_n(f, X)\right\}_{n \in \mathbb{N}} \in \ell^q.
\]
The approximation space $A^{r,q}$ is a (quasi-)Banach space when we set

$$
\|f\|_{A^{r,q}} := \left\{ \left\| n^{r-1/q} a_n (f, X) \right\|_{\ell^q} \right\} = \left( \sum_{n=1}^{\infty} \left| n^r E_{n-1} (f, X) \right|^q \frac{1}{n} \right)^{1/q}.
$$

Note that since the first term in the sequence above is $\|f\|_X$, so that if $\|f\|_{A^{r,u}} = 0$, we have $f = 0$. Many of the basic properties of approximation spaces are described in [60, 57, 19].

One of the important properties that we will use in this paper is a theorem that guarantees the representation of elements in an approximation space in terms of quasi-geometric sequences. A sequence of integers $\{n_k\}_{k=0}^\infty$ is said to be quasi-geometric if $n_0 = 1$ and there are two positive constants $a, b$ such that

$$
1 < a \leq \frac{n_{k+1}}{n_k} \leq b < \infty
$$

for all $k \geq 0$. The representation theorem [57] is stated below:

**Proposition 1.** Let $\{n_k\}_{k=0}^\infty$ be a quasi-geometric series. A function $f \in X$ belongs to $A^{r,q}$ if and only if there exists a representation of the form

$$
f = \sum_{k=0}^{\infty} x_{n_k}
$$

such that $x_{n_k} \in A_{n_k}$ and

$$
\{n_k \|x_{n_k}\|_X\}_{k} \in \ell^q.
$$

In this case we have

$$
\|f\|_{A^{r,q}} \approx \inf_{f = \sum_{k=0}^{\infty} x_{n_k}} \|\{n_k \|x_{n_k}\|_X\}_{k}\|_{\ell^q}.
$$

This last characterization of the approximation spaces $A^{r,q}$ sometimes are presented in slightly different forms. As noted in [19], we can derive an equivalent expression for the seminorm $|f|_{A^{r,u}}$ choosing the quasigeometric sequence $n_k \equiv 2^k$ for $k = 0, 1, \ldots$.

$$
|f|_{A^{r,u}} = \left( \sum_{n=0}^{\infty} \left| 2^{nr} E_{2^n} (f, X) \right|^q \right)^{1/q}.
$$

Approximation spaces have been studied for a wide range of choices of the underlying space $X$ and sequence of approximating subsets $\{A_n\}_{n \in \mathbb{N}_0}$. In this paper we will restrict consideration to approximation spaces $X$ is in fact a Hilbert space and the subsets $\{A_n\}_{n \in \mathbb{N}_0}$ are defined in terms of an orthonormal basis for $X$. See [30] Section (2) for discussion of this classical choice, where it is compared to the more general cases in which the family is generated from an unconditional or greedy basis for a (quasi-)Banach space $X$.

**E.1. Linear Approximation Methods.** Suppose that $\{x_k\}_{k=1}^\infty$ is an orthonormal basis for the Hilbert space $X$. A linear approximation method chooses the sets $A_n := \text{span} \{x_k\}_{k \leq n}$. In this case we have

$$
a_n = \inf_{g \in A_{n-1}} \|f - g\|_X = \|(I - \Pi_{n-1}) f\|_X = E_{n-1} (f, X)
$$
where $\Pi_n$ is the orthogonal projection onto $A_n$. We always have
\[
f = \sum_{k \in \mathbb{N}_0}^{\infty} Q_k f := \sum_{k} (\Pi_{2^k} - \Pi_{2^{k-1}}) f
\]
when we define $\Pi_{2^{-1}} := 0$. It follows that for $1 \leq q < \infty$ we have a seminorm on $A^{r,q}$
\[
|f|_{A^{r,q}} = \begin{cases} \left\| \{2^{kr}\|Q_k f\|_{X}\}_{k \in \mathbb{N}_0} \right\|_{\ell_q} & 1 \leq q < \infty \\
\sup_{n \in \mathbb{N}_0} |2^{nr} E_{2^n}(f)| & q = \infty
\end{cases}
\]

**E.2. Linear Approximation Spaces for Dyadic Splines.** A various points in the paper, we refer to the connection of the linear approximation spaces
\[
A^{r,q}(X; \{A_j\}_{j \in \mathbb{N}_0})
\]
to other more common spaces such as the generalized Lipschitz spaces $\text{Lip}(r, L^p(\Omega))$ and Sobolev spaces $W^r(L^q(\Omega))$. Here we summarize some of these relationships for approximations constructed from families of dyadic splines on $\Omega = [0, 1]$. A dyadic B-spline approximation over the domain $\Omega = [0, 1]$ on a grid of level $j$ selects the knots of the spline to be the dyadic grid points $\Delta_j := \{k2^{-j} | 1 \leq k \leq 2^j - 1\}$. A spline of order $r$ on grid level $j$ is a piecewise polynomial of order $r - 1$ over each of the dyadic intervals of the mesh. We denote by $S_r(\Delta_j)$ the linear space of the dyadic splines of order $r$ having dyadic knots $\Delta_j$ in $[0, 1]$. The normalized B-splines $N^r_{\mathbb{N}}$ over the interval $[0, r]$ having integer knots $\{0, \ldots, r\}$ are used to construct bases for $S_r(\Delta_j)$. The functions $N^r_{\mathbb{N}}$ can be defined by induction. Let $N^1 := \chi_{[0,1]}$ be the characteristic function of $[0, 1]$. Then $N^r$ is defined by recursion of the convolution
\[
N^r := N^{r-1} \ast \chi_{[0,1]} \quad \text{for } r \in \mathbb{N}.
\]
Given $N^r$, we set $N^r_{j,k} := N^r(2^j x - k)$ for $x \in [0, 1]$ and $0 \leq k \leq n_j - 1$. We have $n_j := 2^j = \#(S_r(\Delta_j))$. Each function $f$ in $S_r(\Delta_j)$ has a spline series representation
\[
f = \sum_{k \leq n_j} c_{j,k} N^r_{j,k}
\]
for a unique set of coefficients $\{c_{j,k}\}_{k \leq n_j}$. The following theorem gives a concise characterization of the approximation spaces in terms of the Besov spaces [19, 64], a class of smoothness spaces that contain many well-known function spaces.

**Theorem E.1** (Theorem 3.3, page 361, [19]). Let $r \in \mathbb{N}, 1 \leq p \leq \infty, s := r - 1 + 1/p$, and
\[
A^{\alpha,q}(L^p(\Omega)) := A^{\alpha,q}(L^p(\Omega); \{S_r(\Delta_j)\}_{j \in \mathbb{N}_0})
\]
be the approximation spaces generated by the approximant spaces $A_j := S_r(\Delta_j)$ of dyadic splines. For all $0 < \alpha < s$ and $0 < q \leq \infty$, the approximation space $A^{\alpha,q}(L^p(\Omega))$ is equivalent to the Besov space $B^{\alpha,q}(L^p(\Omega))$.

The scale of Besov spaces $B^{\alpha,q}(L^p(\Omega))$ contains a wide variety of other common function spaces, and it is useful to us specifically because it contains the Sobolev and (generalized) Lipschitz spaces. From [64], page 14, we see that
\[
\begin{align*}
\text{Lip}^*(\alpha, L^p(\Omega)) &= B^{\alpha,\infty}(L^p(\Omega)) & \alpha > 0, 1 < p \leq \infty, \\
\text{Lip}(\alpha, L^p(\Omega)) &= B^{\alpha,\infty}(L^p(\Omega)) & \text{noninteger } \alpha > 0, 1 < p \leq \infty, \\
W^{\alpha}(L^p(\Omega)) &= B^{\alpha,p}(L^p(\Omega)) & \text{noninteger } \alpha > 0, 1 < p \leq \infty, \\
W^{\alpha}(L^2(\Omega)) &= B^{\alpha,2}(L^2(\Omega)) & \alpha > 0.
\end{align*}
\]
E.2.1. Approximation by Haar Wavelets. The results above improve our understanding of several special cases used in our examples. The first of these cases are the examples that employ Haar scaling functions and wavelets for approximation of the Koopman or Perron-Frobenius operators. The span of scaled and translated Haar scaling functions are equivalent to the span of first order splines over $\Omega = [0, 1]$,

$$A_j := \text{span} \left\{ \phi_{j,k} \mid k \in \Gamma_j^0 \right\} \equiv \text{span} \left\{ N_{j,k}^1 \mid k \in \Gamma_j^0 \right\}$$

From Theorem E.1, it follows that

$$A_\alpha,\infty(L^p(\Omega); \{A_j\}_{j \in \mathbb{N}_0}) = \text{Lip}^*\left(\alpha, L^p(\Omega)\right), \quad \text{and} \quad A_\alpha,2(L^2(\Omega), \{A_j\}_{j \in \mathbb{N}_0}) = W^\alpha(L^2(\Omega)),$$

for $0 < \alpha < 1/2$.

E.2.2. Approximation by Orthonormal Wavelets that Reproduce Polynomials. In this paper we have elected to employ families of $L^2(\Omega)$–orthonormal wavelets that reproduce certain polynomials. This general property is referred to the degree of exactness of a wavelet or multiwavelet system, and it is often described in terms of “zero moment” conditions on wavelets [7, 14, 15]. This property states, roughly speaking, that if $\phi, \psi$ are orthonormal scaling functions and wavelets, then if the wavelets satisfy $N$ moment conditions of the form

$$\int_{\mathbb{R}} x^k \psi(x) dx = 0 \quad 0 \leq k \leq N - 1,$$

then the scaling function $\phi$ reproduces polynomials in the sense that

$$x^k = \sum_{\ell} a_p(\phi(x - \ell)) \quad 0 \leq k \leq N - 1.$$

When the scaling functions $\phi$ are compactly supported, the summation at a fixed $x$ in the reproduction formula involves only a finite number of terms.

Recall that each B-spline $N^r$ of order $r$ is a piecewise polynomial of degree at most $r - 1$ between its knots. This means that each $N^r$ can be expressed in terms of a finite linear combination of the scaling functions $\phi(x - k)$, provided the wavelets satisfy zero moment conditions for $0, \ldots, r - 1$. From this we conclude that the spaces of approximants

$$\tilde{A}_j := S_r(\Delta_j) \subseteq \{\phi_{j,k} \mid k \in \Gamma_j^0\} := A_j$$

are nested. From the definition of the seminorm

$$|f|_{A^{r,s}(X; \{A_j\})} := \sum_{n \in \mathbb{N}_0} \left[ 2^{nr} E_{2^n}(f, X; A_{2^n}) \right]^q,$$

we conclude that

$$|f|_{A^{r,s}(X; \{A_j\})} \leq |f|_{A^{r,s}(X; \{\tilde{A}_j\})}.$$

Among other things, this inclusion implies that linear approximation by the wavelet functions converge at least as fast as approximations by the B-splines that are contained in the span of the wavelets. By Theorem E.1 we conclude that the approximation rates using wavelets is at least $O(2^{-\alpha j})$ for the range of $0 < \alpha < s = r - 1 + 1/p$ with $r$ the order of the splines contained in the range of the wavelets.