Mechanical Snell’s Law

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Abstract

We investigate the motion of a massive particle constrained to move along a path consisting of two line segments on a vertical plane under an arbitrary conservative force. By fixing the starting and end points of the track and varying the vertex horizontally, we find the least-time path. We define the angles of incidence and refraction similar to the refraction of a light ray. It is remarkable that the ratio of the sines of these angles is identical to the ratio of the average speeds on the two partial paths as long as the horizontal component of the conservative force vanishes.

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I. INTRODUCTION

Snell’s law, which was experimentally discovered in about 1621, [1] states the relationship between the angles of incidence and refraction for a light ray crossing a flat interface of two isotropic media: The ratio of the sines of these angles is equal to the ratio of the phase velocities of light in the two media. This relation derives from Fermat’s principle of least time that a light ray propagates from a fixed point to another following the path that minimizes the elapsed time. [2, 3] The corresponding derivation is also possible by making use of calculus of variations [4] or an algebraic approach. [5]

Invoked by Fermat’s principle, calculus of variations was developed to resolve problems such as the brachistochrone problem and the shape of a hanging chain. [6, 7] This progress eventually has led to the establishment of the Hamilton’s principle of least action. [8] The brachistochrone problem is to find the path $P(x, y)$ that minimizes the elapsed time for a massive particle’s motion between two fixed points $A$ and $C$ on a vertical plane under a uniform gravitational field. Because the horizontal coordinate $x = f(y)$ is a function of the vertical coordinate $y$, the elapsed time $T$ becomes a function of $f$ that is called a functional $T = T[f]$. The function $f(y)$ can be found by solving the Euler-Lagrange equation for the functional $T = T[f]$. The original derivation of Snell’s law based on Fermat’s principle does not require calculus of variations because the determination of the horizontal coordinate $\alpha$ for the intermediate point $B$ on the interface that minimizes the elapsed time $T(\alpha)$ can be carried out by taking the derivative with respect to that single parameter $\alpha$.

In this paper, we investigate a simplified version of the brachistochrone problem by imposing a restriction that a particle slides down the track consisting of two line segments $\overline{AB}$ and $\overline{BC}$ under a conservative force without friction. We keep $A$ and $C$ fixed and vary the horizontal coordinate $\alpha$ of the vertex $B$. Due to this constraint, the least-time path can be found simply by solving the equation $dT/d\alpha = 0$. Our analysis reveals that, as long as the horizontal component of the conservative force vanishes, the ratio of sines of the angles of incidence and refraction, which are defined similarly to the light-ray refraction, is identical to the ratio of the average speeds of the particle on the two partial paths. We call this relation the mechanical Snell’s law. To our best knowledge, a realization of Snell’s law in a purely mechanical system under acceleration is new.

This paper is organized as follows. In Sec. [1] we describe the mechanical system and
derive the mechanical Snell’s law. A formal strategy to compute the corresponding relative index of refraction under an arbitrary potential is given in Sec. [III]. A summary is given in Sec. [IV].

II. THE MECHANICAL SNELL’S LAW

In this section, we introduce a mechanical system consisting of a particle with mass \( m \) constrained to slide along the path consisting of two line segments \( \overline{AB} \) and \( \overline{BC} \) on a vertical plane under a conservative force without friction. We fix the initial (\( A \)) and final (\( C \)) positions and vary the vertex \( B \) horizontally to find the point \( B = B^* \) at which the total elapsed time \( T \) from \( A \) to \( C \) is minimized. Defining the angles of incidence and refraction similar to the light-ray refraction, we derive the relation between the two angles.

A. Definitions

Since the particle’s track is on a plane, it is convenient to employ the two-dimensional Cartesian coordinates: \( A(x_A, y_A) \), \( B(x_B = \alpha, y_B) \), and \( C(x_C, y_C) \), where \( x_Q \) and \( y_Q \) are the horizontal and vertical coordinates, respectively, of \( Q \) for \( Q = A, B, \) or \( C \) with \( x_A < \alpha < x_C \) and \( y_A > y_B > y_C \). We denote \( v_Q \) by the speed of the particle at point \( Q \). Then the total elapsed time \( T(\alpha) \) from \( A \) to \( C \) is a function of a single parameter \( \alpha \). We define \( \alpha^* \) by the horizontal coordinate of the vertex \( B^* \).

As shown in Fig. 1, the angle of incidence (refraction) \( \theta_1 \) (\( \theta_2 \)) is the angle between \( \overline{AB} \) (\( \overline{BC} \)) and the vertical. The angles \( \theta_1 \) and \( \theta_2 \) are functions of \( \alpha \) as

\[
\begin{align*}
\theta_1(\alpha) &= \arctan \frac{\alpha - x_A}{y_A - y_B}, \\
\theta_2(\alpha) &= \arctan \frac{x_C - \alpha}{y_B - y_C}.
\end{align*}
\]

Likewise, we use the subscripts 1 and 2 to identify a physical variable for the paths \( \overline{AB} \) and \( \overline{BC} \), respectively, so that \( L_1(\alpha), T_1(\alpha), \) and \( \bar{v}_1(\alpha) \) denote the length of \( \overline{AB} \), the elapsed time to pass \( \overline{AB} \), and the average speed on \( \overline{AB} \), respectively. Thus \( L_2(\alpha), \bar{v}_2(\alpha), \) and \( T_2(\alpha) \) represent the corresponding values for \( \overline{BC} \).
FIG. 1. A track consisting of two line segments $AB$ and $BC$ on a vertical plane. A particle of mass $m$ departs from the point $A$ with the initial speed $v_A$, slides along the path without friction, and arrives at the end point $C$ with the terminal speed $v_C$. $A(x_A,y_A)$ and $C(x_C,y_C)$ are fixed and the vertex $B(x_B = \alpha, y_B)$ varies on the horizontal line $y = y_B$ with $x_A \leq \alpha \leq x_C$. The angle of incidence (refraction) $\theta_1$ ($\theta_2$) is the angle between $AB$ ($BC$) and the vertical.

In summary,

$$L_1(\alpha) = \sqrt{(\alpha - x_A)^2 + (y_A - y_B)^2},$$

$$L_2(\alpha) = \sqrt{(x_C - \alpha)^2 + (y_B - y_C)^2},$$

(2)

and

$$\cos \theta_1(\alpha) = \frac{y_A - y_B}{L_1(\alpha)}, \quad \sin \theta_1(\alpha) = \frac{\alpha - x_A}{L_1(\alpha)},$$

$$\cos \theta_2(\alpha) = \frac{y_B - y_C}{L_2(\alpha)}, \quad \sin \theta_2(\alpha) = \frac{x_C - \alpha}{L_2(\alpha)}.$$  (3)

It is straightforward to express the average speeds $\bar{v}_1(\alpha)$ and $\bar{v}_2(\alpha)$ as

$$\bar{v}_1(\alpha) = \frac{L_1(\alpha)}{T_1(\alpha)} = \frac{y_A - y_B}{T_1(\alpha) \cos \theta_1(\alpha)},$$

$$\bar{v}_2(\alpha) = \frac{L_2(\alpha)}{T_2(\alpha)} = \frac{y_B - y_C}{T_2(\alpha) \cos \theta_2(\alpha)}.$$  (4)

It is manifest that $\bar{v}_1(\alpha)$ and $\bar{v}_2(\alpha)$ depend on $\alpha$ in general.
B. Least-Time Path

Let us find the value for \( \alpha^* \) which minimizes the total elapsed time,

\[
T(\alpha) = T_1(\alpha) + T_2(\alpha) = \frac{y_A - y_B}{\bar{v}_1(\alpha) \cos \theta_1(\alpha)} + \frac{y_B - y_C}{\bar{v}_2(\alpha) \cos \theta_2(\alpha)}.
\]

We assume that the potential is a well-behaved function so that \( T(\alpha) \) is differentiable. Then the first-order derivative of \( T(\alpha) \) is given by

\[
\frac{dT(\alpha)}{d\alpha} = \sin \theta_1(\alpha) \frac{\bar{v}_1(\alpha)}{\bar{v}_1(\alpha) \cos \theta_1(\alpha)} - \sin \theta_2(\alpha) \frac{\bar{v}_2(\alpha)}{\bar{v}_2(\alpha) \cos \theta_2(\alpha)}
- \frac{y_A - y_B}{\bar{v}_1^2(\alpha) \cos \theta_1(\alpha)} \frac{d\bar{v}_1(\alpha)}{d\alpha}
- \frac{y_B - y_C}{\bar{v}_2(\alpha) \cos \theta_2(\alpha)} \frac{d\bar{v}_2(\alpha)}{d\alpha},
\]

where we have used

\[
\frac{d}{d\alpha} \left[ \frac{1}{\cos \theta_1(\alpha)} \right] = \frac{\frac{\alpha - x_A}{y_A - y_B} \sqrt{(x_A - \alpha)^2 + (y_A - y_B)^2}}{\sin \theta_1(\alpha)},
\]

\[
\frac{d}{d\alpha} \left[ \frac{1}{\cos \theta_2(\alpha)} \right] = \frac{-\frac{\frac{\alpha - x_C}{y_B - y_C} \sqrt{(\alpha - x_C)^2 + (y_B - y_C)^2}}{\sin \theta_2(\alpha)}}{y_B - y_C}.
\]

Because \( T(\alpha^*) \leq T(\alpha) \) for all \( \alpha \), we have the constraint \( \frac{dT(\alpha)}{d\alpha} |_{\alpha = \alpha^*} = 0 \) to find that

\[
\frac{\sin \theta_1(\alpha^*)}{\sin \theta_2(\alpha^*)} = \frac{\bar{v}_1(\alpha^*)}{\bar{v}_2(\alpha^*)} \left[ 1 + \frac{y_B - y_C}{y_A - y_B} \frac{d \log \bar{v}_2^2(\alpha)}{d\alpha} \right]_{\alpha = \alpha^*} \frac{d \log \bar{v}_1^2(\alpha)}{d\alpha} \left[ 1 - \frac{y_A - y_B}{sin 2\theta_1(\alpha)} \frac{d \log \bar{v}_1^2(\alpha)}{d\alpha} \right]_{\alpha = \alpha^*}.
\]

If both \( \bar{v}_1(\alpha) \) and \( \bar{v}_2(\alpha) \) are independent of \( \alpha \), then the relation (8) collapses into

\[
n_{12} \equiv \frac{\bar{v}_1}{\bar{v}_2} = \frac{\sin \theta_1(\alpha^*)}{\sin \theta_2(\alpha^*)}.
\]

The relation in Eq. (9) is identical to Snell’s law except that the ratio of the phase velocities in two media is replaced with that of the average speeds of the accelerating particle on the two partial tracks. We call the relation in Eq. (9) the mechanical Snell’s law and the ratio \( n_{12} \) the relative index of mechanical refraction.
According to Eq. (8), the mechanical Snell’s law in Eq. (9) does not hold if \( \bar{v}_1(\alpha) \) or \( \bar{v}_2(\alpha) \) has an explicit dependence on \( \alpha \). We shall find in Sec. III that the mechanical Snell’s law is valid as long as the potential depends only on the vertical coordinate so that the horizontal force vanishes.

### III. VALIDITY OF THE MECHANICAL SNELL’S LAW

In this section, we investigate the requirements of the conservative force to have the particle’s track satisfy the mechanical Snell’s law in Eq. (9). We first consider an elementary example of the motion under a uniform gravitational field. Next we consider the case involving a general conservative force field whose potential is independent of the horizontal coordinate. Both cases are shown to satisfy the mechanical Snell’s law. As the last case, we show that the mechanical Snell’s law in Eq. (9) does not hold but Eq. (8) is still valid if the conservative force has nonvanishing horizontal force.

#### A. Uniform Gravitational Field

We consider the case in which a uniform gravitational field \( g = -g\hat{e}_y \) is applied, where \( g \) is the gravitational acceleration and \( \hat{e}_y \) is the unit vector along the vertical direction. In this case, conservation of the total mechanical energy completely determines the speed of the particle at \( y \) as

\[
v(y) = \sqrt{v_A^2 + 2g(y_A - y)}. \tag{10}
\]

Hence, \( \bar{v}_1(\alpha) \) and \( \bar{v}_2(\alpha) \) are independent of \( \alpha \):

\[
\bar{v}_1 = \frac{1}{2}(v_A + v_B) = \frac{1}{2}\left[v_A + \sqrt{v_A^2 + 2g(y_A - y_B)}\right], \tag{11}
\]

\[
\bar{v}_2 = \frac{1}{2}(v_B + v_C) = \frac{1}{2}\left[\sqrt{v_A^2 + 2g(y_A - y_B)} + \sqrt{v_A^2 + 2g(y_A - y_C)}\right]. \tag{12}
\]

Therefore, the mechanical Snell’s law in Eq. (9) holds.
### B. Arbitrary Conservative Force along the Vertical Direction

If the conservative force field is parallel to the vertical direction, then we can define the potential energy by $V(y)$ which is independent of the horizontal coordinate $x$ and the corresponding force is given by

$$ F = -\nabla V(y) = -\hat{e}_y \frac{dV(y)}{dy}, \quad (13) $$

where $F_y = -dV(y)/dy < 0$ and we have

$$ v(y) = \sqrt{v_A^2 + \frac{2}{m}[V(y_A) - V(y)]}. \quad (14) $$

By making use of the first equality of each line in Eq. (4), we can show that $\bar{v}_1(\alpha)$ and $\bar{v}_2(\alpha)$ are independent of $\alpha$:

$$ \bar{v}_1 = \frac{\int_{T_1(\alpha)}^{T(\alpha)} dt \sqrt{v_A^2 + \frac{2}{m}[V(y_A) - V(y)]}}{\int_{T_1(\alpha)}^{T(\alpha)} dt} = \frac{\int_{y_B}^{y_A} dy \{v_A^2 + \frac{2}{m}[V(y_A) - V(y)]\}^{-1/2}}{y_A - y_B}, \quad (15) $$

$$ \bar{v}_2 = \frac{\int_{T_1(\alpha)}^{T(\alpha)} dt \sqrt{v_A^2 + \frac{2}{m}[V(y_A) - V(y)]}}{\int_{T_1(\alpha)}^{T(\alpha)} dt} = \frac{\int_{y_C}^{y_B} dy \{v_A^2 + \frac{2}{m}[V(y_A) - V(y)]\}^{-1/2}}{y_B - y_C}, $$

where we have changed the integration variable from $t$ to $y$ as

$$ dt = -\frac{dy \sqrt{1 + (dx/dy)^2}}{\sqrt{v_A^2 + \frac{2}{m}[V(y_A) - V(y)]}} = -\frac{dy}{\cos \theta_i \sqrt{v_A^2 + \frac{2}{m}[V(y_A) - V(y)]}}, \quad (16) $$

and used the identity $1 + (dx/dy)^2 = 1/\cos^2 \theta_i$ with $i = 1$ for $AB$ and 2 for $BC$. The elapsed time $t(y)$ for the particle to reach the vertical coordinate $y$ can be expressed as

$$ t(y) = \begin{cases} 
\frac{1}{\cos \theta_1} \int_{y}^{y_A} \frac{dy}{\sqrt{v_A^2 + \frac{2}{m}[V(y_A) - V(y)]}}, & y_A \geq y \geq y_B, \\
(t(y_B) + \frac{1}{\cos \theta_2} \int_{y}^{y_B} \frac{dy}{\sqrt{v_A^2 + \frac{2}{m}[V(y_A) - V(y)]}}), & y_B \geq y \geq y_C,
\end{cases} \quad (17) $$

where $t(y_A) = 0$, $t(y_B) = T_1(\alpha)$, and $t(y_C) = T(\alpha) = T_1(\alpha) + T_2(\alpha)$. Because the average speeds in Eq. (15) are independent of $\alpha$, the mechanical Snell's law in Eq. (9) holds.
C. Arbitrary Conservative Force

If we consider an arbitrary conservative force field, we can define the potential energy by \( V(x, y) \) and the corresponding force is given by
\[
F = -\nabla V(x, y) = -\hat{e}_x \frac{\partial V(x, y)}{\partial x} - \hat{e}_y \frac{\partial V(x, y)}{\partial y}.
\]
(18)
Here, we have assumed that \( F_x = -\frac{\partial V}{\partial x} < 0 \) and \( F_y = -\frac{\partial V}{\partial y} < 0 \). Thus the speed of the particle at a point \( Q(x, y) \) is expressed as
\[
v(x, y) = \sqrt{v_A^2 + \frac{2}{m}[V(x_A, y_A) - V(x, y)]}.
\]
(19)
Hence, both \( \bar{v}_1(\alpha) \) and \( \bar{v}_2(\alpha) \) depend on \( \alpha \):
\[
\bar{v}_1 = \frac{\alpha - x_A}{\int_{\alpha}^{x_A} dx \left\{ v_A^2 + \frac{2}{m}[V(x_A, y_A) - V(x, y_1)] \right\}^{-1/2}},
\]
\[
\bar{v}_2 = \frac{x_C - \alpha}{\int_{x_C}^{\alpha} dx \left\{ v_A^2 + \frac{2}{m}[V(x_A, y_A) - V(x, y_2)] \right\}^{-1/2}},
\]
(20)
where \( y_1 \) and \( y_2 \) are functions of \( x \):
\[
y_1(x) = y_B + (y_A - y_B) \frac{\alpha - x}{\alpha - x_A},
\]
\[
y_2(x) = y_C + (y_B - y_C) \frac{x_C - x}{x_C - \alpha}.
\]
(21)
In deriving Eq. (20), we have followed the same procedure to obtain Eq. (15) except that we have changed the integration variable from \( t \) to \( x \) as
\[
dt = \frac{dx}{\sqrt{1 + (dy/dx)^2}} \frac{dx}{\sqrt{v_A^2 + \frac{2}{m}[V(x_A, y_A) - V(x, y)]}}
\]
\[
= \frac{dx}{\sin \theta_i \sqrt{v_A^2 + \frac{2}{m}[V(x_A, y_A) - V(x, y_i)]}},
\]
(22)
where we have used the identity \( 1 + (dy/dx)^2 = 1/\sin^2 \theta_i \) with \( i = 1 \) for \( \overline{AB} \) and \( 2 \) for \( \overline{BC} \).

It is manifest that, if the conservative force field has non-vanishing horizontal component, then the mechanical Snell’s law in Eq. (9) does not hold but the most general form in Eq. (8) should be applied. This result is consistent with a previous observation in light-ray refractions: If the medium does not exert forces parallel to the interface on the incident photon, then the tangential momentum of the refracted photon is conserved to satisfy Snell’s law. [9]
IV. SUMMARY

We have investigated the relationship between the angles of incidence and refraction of an accelerating particle sliding down a path consisting of two line segments on a vertical plane without friction in the presence of an arbitrary conservative force, where the angles are defined similarly to the case of the light-ray refraction crossing a flat interface of two isotropic media. Our main result which is given in Eq. (9) states that, if we choose the least-time path, then the ratio of the sines of these angles is equal to the ratio of the average speeds of the particle on the two partial paths as long as the horizontal component of the conservative force vanishes. The mechanical Snell’s law allows us to compute the relative index of mechanical refraction once the explicit form of the potential is known.

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