Introduction

Recently, fractional differential equations have been acquired much attention due to its applications in a number of fields such as physics, mechanics, chemistry, biology, signal and image processing, see for example the books (Baileanu et al., 2012; Kilbas et al., 2006; Lakshimikanatham et al., 2009; Yang et al., 2015).

Some recent works on fractional differential equations involving Riemann Liouville and Caputo-type fractional derivatives are studied using nonlinear analysis methods such as Krasnoselskii fixed-point Theorems (Agarwal and O’Regan, 1998; Ghanmi and Horrigue, 2018; Guo et al., 2007; Guo et al., 2008). Leray-Schauder alternative (Ghanmi and Horrigue, 2019; Qi et al., 2017), sub-solution and super-solution methods (Wang et al., 2019; Mâagli et al., 2015) and iterative techniques (Liu et al., 2013).

Hadamard (1892) introduced an important fractional derivative, which differs from the above-mentioned ones because its definition involves logarithmic function of arbitrary exponent and named as Hadamard derivative. In the last few decades many authors are paying more and more attention to fractional differential equation involving Hadamard derivative, the study of the topic is still in its primary stage. For details and recent developments on Hadamard fractional differential equations, see (Huang and Liu, 2018; Wang et al., 2018; Zhai et al., 2018) and references therein. Recently, some researches have extensively interested in the study of the fractional differential equations with p-Laplacian operators see for examples (Chamekh et al., 2018; Ding et al., 2015).

From the above review of the literature concerning fractional differential equations, most of the authors investigated only the existence of solutions or positive solutions for Hadamard fractional differential equations without considering the pi-Laplacian operator. A very few authors established results along with p-Laplacian operator, see example in (Wang and Wang, 2016), the authors considered the following nonlinear Hadamard fractional functional problem:

\[
\begin{align*}
D^\alpha \left( \phi_p \left( D^\sigma u(t) \right) \right) &= f(t,u(t)), \quad t \in (1,T), \\
u(T) &= \lambda I^\alpha u(\eta), \quad D^\alpha u(1) = 0, \quad u(1) = 0,
\end{align*}
\]

where for an appropriate \( \zeta, D^\alpha \) is the Hadamard fractional derivative of order \( \zeta \), \( 1 < \alpha \leq 2, 0 < \sigma \leq 1, \gamma > 0, \lambda \in \mathbb{R} \) and \( f \in C([1,T] \times \mathbb{R}, \mathbb{R}) \). By using the Schauder fixed point Theorem, the existence of solutions is obtained. In Li and Lin (2013), the authors used the Guo-Krasnosel’skii fixed point Theorem to prove the existence and uniqueness of positive solutions of the following Hadamard fractional boundary value problem with p-Laplacian operator:

\[
\begin{align*}
D^\alpha \left( \phi_p \left( D^\sigma u(t) \right) \right) &= f(t,u(t)), \quad t \in (1,e), \\
u(1) &= u'(1) = u'(e) = 0, \quad D^\alpha u(1) = D^\alpha u(e) = 0,
\end{align*}
\]

where, \( 2 < \alpha \leq 3, 1 < \sigma \leq 2, f \in C([1,e] \times [0,\infty)\times[0,\infty)) \) and the function \( \phi_p(p > 1) \), is called p-Laplacian and is defined in \( \mathbb{R} \) by \( \phi_p(s) = |s|^{p-2}s \). The authors in Zhang et al. (2018) established some existence of positive solutions for the following nonlinear Hadamard fractional differential equations with p-Laplacian operator:

\[
\begin{align*}
D^\alpha \left( \phi_p \left( D^\sigma u(t) \right) \right) &= f(t,u(t)), \quad t \in (1,e), \\
u(1) &= u'(1) = u'(e) = 0, \quad D^\alpha u(1) = 0,
\end{align*}
\]

where \( \phi_p(D^\alpha u(e)) = \mu \int_1^e \phi_p(D^\alpha u(t)) \frac{dt}{t} \).
where, $2 < \alpha \leq 3$, $1 < \sigma \leq 2$, $0 \leq \mu < \sigma$ and $f \in C([1, e] \times \mathbb{R}, \mathbb{R})$.

Motivated by the above mentioned papers, in this study, we study the following fractional Hadamard problem:

$$
[D^\sigma \{ \phi \circ (D^\alpha x) \}](t) = f(t, x),
\quad x(0) = 0
$$

$$
A ^{-\gamma} x(t) + B x(t) = \epsilon_1, \quad 0 < \gamma_1,
A ^{-\gamma} x(t) + B \phi(t) = \epsilon_2, \quad 0 < \gamma_2
$$

(1.1)

where, $\alpha$ and $\sigma$ are in $(1, 2)$, $\eta_1$ and $\eta_2$ are in $(1, e)$, $A_1$, $A_2$, $B_1$, $B_2$, $c_1$ and $c_2$ are fixed real numbers.

For the sake of computational convenience, we set:

$$
\Delta_1 = B_1 + \frac{A_1 \Gamma(\sigma)}{\Gamma(\gamma_1 + \sigma)} (\log \eta_1)^{\gamma_1 - \sigma - 1},
$$

$$
\Delta_2 = B_2 + \frac{A_2 \Gamma(\sigma)}{\Gamma(\gamma_1 + \sigma)} (\log \eta_2)^{\gamma_1 - \sigma - 1}.
$$

(1.2)

And we assume the following conditions.

(H1) There exist nonnegative functions $a(t)$, $b(t) \in C([1, e], \mathbb{R})$, such that:

$$
[f(t, x)]^{\gamma_1 - 1} \leq a(t) + b(t)[x]^{\gamma_1 - 1}, \text{ for each } [t, u] \subseteq [1, e] \text{ and } x \in \mathbb{R}.
$$

(1.3)

(H2) There exist positive continuous nondecreasing function $p$ on $[0, \infty)$ and a function $p \in C([1, e], \mathbb{R}^+)$ such that:

$$
[f(t, x)]^{\gamma_1 - 1} \leq p(t) g([x]^{1}) \text{ for each } (t, u) \subseteq [1, e] \times \mathbb{R}.
$$

(1.4)

(H3) There exist positive continuous nondecreasing function $q$ on $[0, \infty)$ and a function $p \in C([1, e], \mathbb{R}^+)$ such that:

$$
[f(t, x)]^{\gamma_1 - 1} \leq p(t) g([x]^{1}) \text{ for each } (t, u) \subseteq [1, e] \times \mathbb{R}.
$$

(1.5)

And:

$$
\Theta_1 = \left( \frac{b}{\Gamma(\sigma + 1)} \right)^{\gamma_1 - 1} \frac{\left( 2^{\gamma_1 - 1} \right) \Gamma(p(q - 1) + 1)}{\Gamma(\alpha + \sigma(q - 1) + 1)}
$$

$$
\left( 1 + \frac{A_1}{\Delta_1} + \left( 2^{\gamma_1 - 1} \right) \frac{A_1}{\Delta_1} + \left( 2^{\gamma_1 - 1} \right) \frac{A_1}{\Delta_1} \right)^{-1}
$$

(1.6)

where $\Delta_1, \Delta_2, A_1, A_2, B_1, B_2, \epsilon_1$ and $\epsilon_2$ are fixed real numbers.

The Hadamard fractional integral of order $q > 1$ for a function $g: [1, \infty) \rightarrow \mathbb{R}$ is defined as:

$$
I^q g(t) = \frac{1}{\Gamma(q)} \int_{1}^{t} \left( \log \frac{t}{s} \right)^{q - 1} g(s) \frac{d}{ds} ds,
$$

(2.1)

where, $\log(.) = \log(\cdot)$.

- The Hadamard derivative of fractional order $\alpha$ for a function $g: [1, \infty) \rightarrow \mathbb{R}$ is given by:
\[ D^\alpha g(t) = \left( t \frac{d}{dt} \right)^\alpha I^{n-\alpha}, \quad (2.3) \]

where, \( n-1 < \alpha < n \), \( n = [\alpha] + 1 \), \([\alpha]\) denotes the integer part of the real number \( \alpha \).

**Lemma 2.1**

[12] For any \( t \in [1,e] \), \( c \in \mathbb{R} \) and any constants \( a, \sigma \) in \([1,2] \), we have:

\[
I^n(c)(t) = cI^n1(t) = c \frac{1}{\Gamma(\alpha)} \quad (2.4)
\]

\[
I^n\left[ \left( \log_s \right)^\alpha \right](t) = \frac{1}{\Gamma(\alpha)} \quad (2.5)
\]

We use also the following property:

If \( q > 2 \) and \( \max(|x|,|y|) \leq R \), then:

\[
\left| \Phi_q(x) - \Phi_q(y) \right| \leq (q - 1)R^\alpha |x - y|. \quad (2.6)
\]

The following elementary relation is usefully:

If \( n > 1 \), then for all positive numbers \( a \) and \( b \), we have:

\[
(a + b)^\alpha \leq (2^\alpha - 1)(a^\alpha + b^\alpha). \quad (2.7)
\]

**Lemma 2.2**

Kilbas et al. (2006) Let \( q > 0 \) and \( x \in C[1,\infty)\cap L^1[1,\infty) \). Then the Hadamard fractional differential equation \( D^\alpha x(t) = 0 \) has the solution:

\[
x(t) = \sum_{i=1}^n c_i (\log t)^{\gamma_i}. \quad (2.8)
\]

and the following formula holds:

\[
I^\delta D^\alpha x(t) = x(t) + \sum_{i=1}^n c_i (\log t)^{\gamma_i} . \quad (2.9)
\]

where \( c_i \in \mathbb{R} \), \( i = 1, 2, \ldots, n \) and \( n - 1 < q < n \).

**Theorem 2.3**

Given \( y \in C([1, e], \mathbb{R}) \), the unique solution of the problem:

\[
\begin{align*}
D^\alpha \left( \phi^\tau \left( D^\alpha x(t) \right) \right) &= f(t, x), \\
x(1) &= \phi^\tau \left( D^\alpha x(1) \right) = 0, \\
A_i I^{\alpha+\gamma_i} \left( \eta_1 \right) + B_i x(e) &= c_i, \\
A_i I^{\alpha+\gamma_i} \left( \phi^\tau \left( D^\alpha x \right) \right) \left( \eta_2 \right) + B_2 \phi^\tau \left( D^\alpha x \right) (e) &= c_2,
\end{align*}
\]

is given by:

\[
x(t) = I^\tau \phi^\tau \left( I^\alpha f(t, x) \right) + c_1 (\log(t))^{\gamma_i} \quad (2.10)
\]

where, \( c_3 \) and \( c_4 \) are given by:

\[
c_3 = c_2 - A_i I^{\alpha+\gamma_i} \left( f(\eta_2, x) \right) - B_2 I^{\alpha} \left( f(e, x) \right) \quad (2.11)
\]

\[
\text{and:}
\]

\[
\phi^\tau \left( D^\alpha x(t) \right) = I^\alpha f(t, x) + a (\log t)^{\gamma_i} \quad (2.12)
\]

By applying \( I^{\alpha} \) on both sides of (2.12) for \( t = \eta_2 \) and using the property (2.5), we obtain:

\[
\frac{\Gamma(\sigma)}{\Gamma(\eta_2 + \sigma)} (\log \eta_2)^{\gamma_i+\sigma} \quad (2.13)
\]

Since \( \phi^\tau (D^\alpha x(1)) = 0 \), then \( b = 0 \). So, we obtain:
On the other hand, put \( t = e \) in Equation (2.12), we obtain:

\[
\phi_e(D^s_x(e)) = I^s f(e, x) + a. \tag{2.14}
\]

By combining Equations (2.13) and (2.14) and the second boundary condition in (2.8), we obtain:

\[
c_2 = A I^{\gamma_2 - \alpha} f(\eta, x) + a \left[ A \frac{\Gamma(\sigma)}{\Gamma(\gamma_2 + \sigma)} \left( \log \eta_2 \right)^{(\gamma_2 + \sigma) - 1} + B \right] + B I^s f(e, x).
\]

Therefore:

\[
a = c_2 - A I^{\gamma_2 - \alpha} f(\eta, x) - B I^s f(e, x), \tag{2.15}
\]

where:

\[
\Delta_1 = B_2 + A \frac{\Gamma(\sigma)}{\Gamma(\gamma_2 + \sigma)} \left( \log \eta_2 \right)^{(\gamma_2 + \sigma) - 1}.
\]

Then, the solution can be written as follows:

\[
x(t) = I^s \left[ \phi_e \left( I^s f(t, x) + c_1 (\log t)^{\gamma_1 - 1} \right) \right] + a^s (\log t)^{\gamma_1 - 1} + b^s (\log t)^{\gamma_1 - 2}.
\]

Since \( x(1) = 0 \), then \( b^s = 0 \) and we get:

\[
x(t) = I^s \left[ \phi_e \left( I^s f(t, x) + c_1 (\log t)^{\gamma_1 - 1} \right) \right] + a^s (\log t)^{\gamma_1 - 1}. \tag{2.16}
\]

Now, if we apply \( I^{\gamma_1} \) to (2.16) and we replace \( t \) by \( \eta_1 \), then, using the property (2.5), we obtain:

\[
I^{\gamma_1} \left( x(\eta_1) = I^{\gamma_1 - \alpha} \left[ \phi_e \left( I^{\gamma_1 - \alpha} f(\eta_1, x) + c_1 (\log \eta_1)^{\gamma_1 - 1} \right) \right] \right) + a^s \frac{\Gamma(\alpha)}{\Gamma(\eta_1 + \alpha)} (\log \eta_1)^{\gamma_1 - 1}. \tag{2.17}
\]

On the other hand, Equation (2.16) with \( t = e \), yields to:

\[
x(e) = I^s \left[ \phi_e \left( I^s f(e, x) + c_1 \right) \right] + a^s. \tag{2.18}
\]

Finally, by combining Equations (2.17), (2.18) with the first boundary condition \( c_1 \), we obtain:

\[
c_1 = A I^{\gamma_1 - \alpha} \left[ \phi_e \left( I^{\gamma_1 - \alpha} f(\eta_1, x) + c_1 (\log \eta_1)^{\gamma_1 - 1} \right) \right] + a^s \frac{\Gamma(\alpha)}{\Gamma(\eta_1 + \alpha)} (\log \eta_1)^{\gamma_1 - 1} + B I^s f(e, x) + c_1. \tag{2.19}
\]

It follows that:

\[
\phi_e(D^s_x(e)) = I^s f(e, x) + a^s (\log t)^{\gamma_1 - 1} + b^s (\log t)^{\gamma_1 - 2}.
\]

Substituting the values of \( c_1 \) and \( c_4 \) in (2.16), we obtain (2.9). This completes the proof.

To prove the main results of this study, we recall the following theorems.

**Theorem 2.4**

Smart (1974) Let \( X \) be a Banach space. If the operator \( T: X \to X \) is completely continuous and if the set:

\[
V = \{ u \in X \mid u = \mu T u, 0 < \mu < 1 \},
\]

is bounded. Then \( T \) has a fixed point in \( X \).

**Theorem 2.5**

Granas and Dugundji (2003) Let \( X \) be a Banach space, \( C \) be a closed, convex subset of \( X \), \( U \) be an open subset of \( C \) and \( 0 \in C \). Suppose that the operator \( F: U \to C \) is continuous and compact. Then either

(i) \( F \) has a fixed point in \( U \), or

(ii) There is \( u \in \partial U \) and \( \lambda \in (0,1) \), with \( u = \lambda F(u) \).

**Proof of the Main Results**

This section is devoted to prove existence results for the nonlinear boundary value problem (1.1). Also, we shall prove existence and uniqueness results by using different methods. Let us define the operator \( Q: C([1,\varepsilon], \mathbb{R}) \to C([1,\varepsilon], \mathbb{R}) \) by:

\[
Q_x(t) = I^s \phi_e \left( I^s f(t, x) + c_1 \right) + c_1 (\log t)^{\gamma_1 - 1}, \tag{3.1}
\]
where \(c_3\) and \(c_4\) are given respectively by Equation (2.10) and (2.11). Notice that the existence of fixed points of the operator \(Q\) is equivalent to the existence of solutions for problem (1.1).

**Proof of Theorem 1.1**

In this subsection, by using the well known Banach’s fixed point Theorem, we present the existence and uniqueness result for Problem (1.1).

Let \(B_0 = \{x \in C([1, e], \mathbb{R}) : |x|_\infty \leq R\}\). The proof is divided into two steps.

**Step 1:** In this step, we will prove that the operator \(Q\) is completely continuous. Let \(O\) be an open bounded subset of \(C([1, e], \mathbb{R})\). Since \(f\) is continuous, then the operator \(Q\) is continuous and so, \(Q(O)\) is bounded. Next, we will show that \(Q\) is equicontinuous.

Let \(1 \leq t_1 < t_2 \leq e\), then, from Inequality (2.6) we obtain:

\[
\begin{align*}
|Q(x)(t_2) - Q(x)(t_1)| & \leq |I^\sigma f(x,t_2) - I^\sigma f(x,t_1)| + \left|c_1 \int_{t_1}^{t_2} (s-t)^{-\frac{1}{\sigma}}ds \right| \\
& \leq |I^\sigma f(x,t_2) - I^\sigma f(x,t_1)| + \left|c_1 \int_{t_1}^{t_2} (s-t)^{-\frac{1}{\sigma}}ds \right|
\end{align*}
\]

(3.2)

Therefore using (2.1), (2.4) and the fact that \(f\) is bounded, we get:

\[
|Q(x)(t_2) - Q(x)(t_1)| \leq |I^\sigma f(x,t_2) - I^\sigma f(x,t_1)| + \left|c_1 \int_{t_1}^{t_2} (s-t)^{-\frac{1}{\sigma}}ds \right|
\]

(3.3)

for some \(M > 0\). Hence, (2.5), implies that:

\[
|Q(x)(t_2) - Q(x)(t_1)| \leq \frac{M}{\Gamma(\alpha + \sigma + 1)}
\]

(3.4)

By combining (3.2) and (3.4), we obtain:

\[
\begin{align*}
|Q(x)(t_2) - Q(x)(t_1)| & \leq \frac{M}{\Gamma(\alpha + \sigma + 1)} \left| (\log t_2)^{\sigma + \gamma} - (\log t_1)^{\sigma + \gamma} \right|
\end{align*}
\]

(3.5)

On the other hand, from (2.10) and (3.3), we obtain:

\[
|c_2| \leq |c_1| + |A_1 f(x, \eta_1)| + |B_1 f(x, \eta_1)|
\]

(3.6)

Therefore using (2.6), (2.11) and (3.4), we get:
By combining (3.5), (3.6) and (3.7), one has:

$$\lim_{\tau \to \infty} \|Q(t)(t) - Q(x)(t)\| = 0.$$ 

Finally, the Arzelá-Ascoli Theorem implies that $Q: C([1, e], \mathbb{R}) \to C([1, e], \mathbb{R})$ is completely continuous.

Step 2: In this step, we will prove that the set $D = \{ x \in C([1, e], \mathbb{R}): x = \mu Q(x), \mu \in (0, 1) \}$ is bounded. From hypothesis ($H_1$) and using (2.7), we have:

$$|r(t)| = |\mu Q(x)(t)| \leq \max_{\tau \in [1, e]} |r^\tau(1, x) + c_1 \log t|^1 + c_1 \log t^{-1}|$$

$$\leq \max_{\tau \in [1, e]} |r^\tau(1, x) + c_1 \log t|^1 + c_1 \log t^{-1}| + k_1|\log t|^1 + k_1.$$ 

where, $A = \|k\|_\infty + \|b\|_\infty \|x\|_\infty^{-1}$.

On the other hand, we have:

$$k_1 \leq \left| c_1 + A_1 \int_0^{t^\gamma_t} f(x, t, \eta_1) \right| + \left| B_1 \int_0^{t^\gamma_t} \int_0^{t^\gamma_t} f(x, t, e) \right|$$

On the other hand, we have:

$$k_1 \leq \left| c_1 + A_1 \Gamma(\frac{\sigma + \gamma_t}{\Gamma(\sigma + 1)} + \frac{B_1}{\Gamma(\sigma + 1)} \right|$$

Then, using Inequality (2.7), we obtain

$$k_1 \leq \left| c_1 + A_1 \Gamma(\frac{\sigma + \gamma_t}{\Gamma(\sigma + 1)} + \frac{B_1}{\Gamma(\sigma + 1)} \right|.$$ 

and:

$$\left(2^{\gamma_t - 1} - 1\right) \left[ A_1 \Gamma(\frac{\sigma + \gamma_t}{\Gamma(\sigma + 1)}) + \frac{B_1}{\Gamma(\sigma + 1)} \right].$$
Therefore, it follows from (2.7), (3.8), (3.9) and (3.10), that:

\[
|x(t)| = |\mu(Qx)(t)| \\
\leq (2^{r^*} - 1) \left( \frac{A}{\Gamma(\sigma + 1)} \right)^{r^* - 1} + \left( \frac{e_3}{\Delta_i} \right)^{r^* - 1} \left[ \frac{A_i + |B_i|}{\Gamma(\alpha + \sigma(q - 1) + 1)} \right]^{r^*} \\
+ \frac{[e_1 + (2^{r^*} - 1)] A_i}{\Gamma(\alpha + 1) A_i} \left( \frac{A_i}{\Gamma(\sigma + 1)} \right)^{r^* - 1} \left( \frac{A_i + |B_i|}{\Gamma(\alpha + \sigma(q - 1) + 1)} \right)^{r^*} \\
+ \frac{e_2 \Gamma(\sigma + 1)}{\Gamma(\alpha + 1) A_i} \left( \frac{A_i}{\Gamma(\sigma + 1)} \right)^{r^* - 1} \left( \frac{A_i + |B_i|}{\Gamma(\alpha + \sigma(q - 1) + 1)} \right)^{r^*} \\
+ \frac{A}{\Gamma(\sigma + 1) \Gamma(\alpha + 1) A_i} \left[ \frac{A_i + |B_i|}{\Gamma(\alpha + \sigma(q - 1) + 1)} \right]^{r^*}.
\]

Using the inequality (2.7), we have:

\[
A^{r^* - 1} = \left( \|e_1\|_{\infty} + \|e_2\|_{\infty} \right)^{r^* - 1} \\
\leq \left( \|e_1\|_{\infty} + \|e_2\|_{\infty} \right)^{r^* - 1}.
\]

Since \( q \geq 2 \), then, \( p \leq 2 \) and \( \|x\|_{\infty} \leq \|x\|_{\infty} \). Thus, we have:

\[
\|x\|_{1} < (\Theta_1 + \Theta_2) \|x\|_{1} + \Theta_3,
\]

where:

\[
\Theta_1 = \left( \frac{\|e_1\|_{\infty}}{\Gamma(\sigma + 1)} \right)^{r^* - 1} \left( \frac{\Gamma(\alpha + \sigma(q - 1) + 1)}{\Gamma(\sigma + 1)} \right)^{r^* - 1} \\
+ \left( \frac{e_2}{\Delta_i} \right)^{r^* - 1} \left( \frac{A_i}{\Gamma(\alpha + 1) \Gamma(\sigma + 1)} \right)^{r^* - 1} \left( \frac{A_i + |B_i|}{\Gamma(\alpha + \sigma(q - 1) + 1)} \right)^{r^*} \\
+ \left( \frac{A}{\Gamma(\sigma + 1) \Gamma(\alpha + 1) A_i} \right)^{r^* - 1} \left( \frac{A_i + |B_i|}{\Gamma(\alpha + \sigma(q - 1) + 1)} \right)^{r^*}.
\]

\[
\Theta_2 = \frac{\|e_1\|_{\infty}}{\Gamma(\sigma + 1) \Gamma(\alpha + 1) A_i} \left[ \frac{A_i + |B_i|}{\Gamma(\alpha + 1) A_i} \right]^{r^* - 1}.
\]

and:

\[
\Theta_3 = \frac{\|e_2\|_{\infty}}{\Gamma(\sigma + 1) \Gamma(\alpha + 1) A_i} \left[ \frac{A_i + |B_i|}{\Gamma(\alpha + 1) A_i} \right]^{r^* - 1}.
\]

This together with condition (H2), gives \( \|u\|_{\infty} < M_1 \). That is \( D \) is bounded, so the operator \( Q \) has at least one fixed point. Which implies that the problem (1.1) has at least one solution.

**Proof of Theorem 1.2**

In this subsection, by using Leray-Schauder’s nonlinear alternative Theorem, we give the proof of Theorem 1.2. The proof is divided into several steps.

Step 1: In this step, we prove that \( Q \) maps bounded sets into equicontinuous sets of \( C([1, e], \mathbb{R}) \). Let \( t_1, t_2 \in [1, e] \) with \( t_1 < t_2 \) and \( x \in B_r \). Then, we have:

\[
I_{t_1} f(x, t_1) - I_{t_2} f(x, t_2) \\
\leq \frac{g(t)}{\Gamma(\sigma)} \left[ \int_1^{t_1} \left( \log \frac{\frac{A_i}{s}}{\frac{|B_i|}{s}} \right)^{r^* - 1} \frac{ds}{s} - \int_1^{t_2} \left( \log \frac{\frac{A_i}{s}}{\frac{|B_i|}{s}} \right)^{r^* - 1} \frac{ds}{s} \right]
\]

and:

\[
\frac{g(t)}{\Gamma(\sigma)} \left[ \int_1^{t_1} \left( \log \frac{\frac{A_i}{s}}{\frac{|B_i|}{s}} \right)^{r^* - 1} \frac{ds}{s} - \int_1^{t_2} \left( \log \frac{\frac{A_i}{s}}{\frac{|B_i|}{s}} \right)^{r^* - 1} \frac{ds}{s} \right]
\]
So, using Inequality (2.6), we obtain:

\[
\left\| Q(x) (t_i) - (Q_0)(t_i) \right\| \\
\leq I_u \Phi \left[ I_{\sigma} f (x, t_i) + c_i (\log t_i)^{q - 1} \right] - I_u \Phi \left[ I_{\sigma} f (x, t_i) + c_i (\log t_i)^{q - 1} \right] + c_i \left[ (\log t_i)^{q - 1} - (\log t_i)^{q - 1} \right] \\
\leq I_u \Phi \left[ I_{\sigma} f (x, t_i) + c_i (\log t_i)^{q - 1} \right] - \Phi \left[ I_{\sigma} f (x, t_i) + c_i (\log t_i)^{q - 1} \right] + c_i \left[ (\log t_i)^{q - 1} - (\log t_i)^{q - 1} \right] \\
\leq I_u \left\{ (q - 1) \epsilon_{\cdot} \left[ I_{\sigma} f (x, t_i) - I_{\sigma} f (x, t_i) \right] + c_i \left[ (\log t_i)^{q - 1} - (\log t_i)^{q - 1} \right] + c_i \left[ (\log t_i)^{q - 1} - (\log t_i)^{q - 1} \right] \\
\right. \\
\leq I_u \left\{ \frac{\left\{ (q - 1) \epsilon_{\cdot} \right\}}{\Gamma (\sigma + 1)} \left[ \left( \log t_i \right)^{q - 1} - \left( \log t_i \right)^{q - 1} \right] + c_i \left[ (\log t_i)^{q - 1} - (\log t_i)^{q - 1} \right] + c_i \left[ (\log t_i)^{q - 1} - (\log t_i)^{q - 1} \right] \\
\leq \frac{\left\{ (q - 1) \epsilon_{\cdot} \right\}}{\Gamma (\sigma + 1)} \left[ \left( \log t_i \right)^{q - 1} - \left( \log t_i \right)^{q - 1} \right] + c_i \left[ (\log t_i)^{q - 1} - (\log t_i)^{q - 1} \right] + c_i \left[ (\log t_i)^{q - 1} - (\log t_i)^{q - 1} \right].
\]

Obviously the right-hand side of the above inequality tends to zero independently of \( x \in B \); as \( t_1 - t_2 \to 0 \).

Step 2: In this step, we will prove that \( Q \) maps bounded sets (balls) into bounded sets in \( C([1, e], \mathbb{R}) \).

\[
\left[ Q(x)(t) \right] \leq \max_{r \in [1, e]} \left\{ \int_{\gamma} \phi_n \left[ I_{\sigma} f (t, x) + c_i (\log t)^{q - 1} \right] + c_i (\log t)^{q - 1} \right\} \\
\leq \max_{r \in [1, e]} \left\{ \int_{\gamma} \phi_n \left[ \frac{1}{\Gamma (\sigma)} \int_{0}^{1} \frac{1}{s} \frac{1}{1 + s + c_i (\log t)^{q - 1}} \right] + c_i (\log t)^{q - 1} \right\} \\
\leq \max_{r \in [1, e]} \left\{ \int_{\gamma} \phi_n \left[ \frac{1}{\Gamma (\sigma + 1)} \int_{0}^{1} \frac{1}{s} + c_i (\log t)^{q - 1} \right] + c_i (\log t)^{q - 1} \right\} \\
\leq \max_{r \in [1, e]} \left\{ \int_{\gamma} \phi_n \left[ \frac{\left\{ \log t \right\}}{\Gamma (\sigma + 1)} + c_i (\log t)^{q - 1} \right] + c_i (\log t)^{q - 1} \right\} \\
\leq \max_{r \in [1, e]} \left\{ \int_{\gamma} \phi_n \left[ \frac{\left\{ \log t \right\}}{\Gamma (\sigma + 1)} + c_i (\log t)^{q - 1} \right] + c_i (\log t)^{q - 1} \right\} \\
\left( 2^{\sigma - 1} - 1 \right) + c_i (\log t)^{q - 1} \right\} \\
\left( 2^{\sigma - 1} - 1 \right) + c_i (\log t)^{q - 1} \right\} + c_i (\log t)^{q - 1} \right\} \\
\leq \left( 2^{\sigma - 1} - 1 \right) + c_i (\log t)^{q - 1} \right\} + c_i (\log t)^{q - 1} \right\} + c_i (\log t)^{q - 1} \right\} \right\} \\
(3.11)
\]

Let \( r > 0 \), \( t \in [1, e] \) and \( x \in B \), then, by using the hypothesis (H3) and the inequality (2.7), we obtain:
In the other hand, we have:

\[
\left| c_1 \right| \leq \left| A_1 \right| F^{\gamma_{\eta_1}} f (x, \eta_1) \leq \frac{\left| A_1 \right|}{\Gamma (\sigma + \gamma_2 + 1)} \left( \log \eta_1 \right)^{\gamma_{\eta_1}} + \left| B_1 \right| \left( \log \eta_1 \right)^{\gamma_1} + \left| B_1 \right| \left( \log \eta_1 \right)^{\gamma_1}
\]

\[
\left| c_1 \right| \leq \frac{\left| A_1 \right|}{\Gamma (\sigma + \gamma_2 + 1)} \left( \log \eta_1 \right)^{\gamma_{\eta_1}} + \left| B_1 \right| \left( \log \eta_1 \right)^{\gamma_1} + \left| B_1 \right| \left( \log \eta_1 \right)^{\gamma_1}
\]

(3.12)

\[
\left| c_1 \right| \leq \frac{\left| A_1 \right|}{\Gamma (\sigma + \gamma_2 + 1)} \left( \log \eta_1 \right)^{\gamma_{\eta_1}} + \left| B_1 \right| \left( \log \eta_1 \right)^{\gamma_1} + \left| B_1 \right| \left( \log \eta_1 \right)^{\gamma_1}
\]

(3.13)

and:

\[
\left| c_1 \right| = \frac{\left| A_1 \right|}{\Delta_1} F^{\gamma_{\eta_1}} \phi_f \left( \log \eta_1 \right)^{\gamma_{\eta_1}} + \left| B_1 \right| \left( \log \eta_1 \right)^{\gamma_1} + \left| B_1 \right| \left( \log \eta_1 \right)^{\gamma_1}
\]

(3.14)
It follows, from Inequalities (3.12), (3.13) and (3.14) that the inequality (3.11) becomes:

$$\|Q(x)\| \leq \frac{\alpha_1 (2^{q+1} - 1)}{\Gamma^{q+1}(\sigma+1)}\|P\|_p g(\|L\|)^{q+1} + \frac{\alpha_2 \|P\|_p g(\|L\|)}{\Delta_1} + \omega_3,$$

where:

$$\alpha_1 = \frac{\Gamma(\sigma(q-1)+1)}{\Gamma(\alpha + \sigma(q-1)+1)} \left[1 + \frac{|A_1|}{\Delta_1}\right] + \frac{(2^{q+1} - 1)|A_1 + |B_1|^q}{\Gamma^{q+1}(\sigma+1)},$$

and:

$$\omega_3 = \frac{\alpha_1 (2^{q+1} - 1)^3|c_1|^{1-q}}{\Gamma^{q+1}(\sigma+1)} \left[1 + \frac{|A_1|}{\Delta_1}\right] + \frac{\alpha_2 |B_1|}{\Delta_1 \Gamma(\alpha + 1) \Gamma(\sigma+1)}.$$

Consequently, as $x \in B_\ast$, we have:

$$\|Q(x)\| \leq \frac{\alpha_1 (2^{q+1} - 1)}{\Gamma^{q+1}(\sigma+1)}\|P\|_p g(\|L\|)^{q+1} + \frac{\alpha_2 \|P\|_p g(\|L\|)}{\Delta_1} + \omega_3.$$

Therefore, the Arzelà-Ascoli Theorem implies that $Q: C([1,e], \mathbb{R}) \rightarrow C([1,e], \mathbb{R})$ is completely continuous. Let $x$ be a solution. Then, for $t \in [1,e]$, as in the first step, we have:

$$\|L\| \leq \frac{\alpha_1 (2^{q+1} - 1)}{\Gamma^{q+1}(\sigma+1)}\|P\|_p g(\|L\|)^{q+1} + \frac{\alpha_2 \|P\|_p g(\|L\|)}{\Delta_1} + \omega_3,$$

which implies that:

$$\frac{\alpha_1 (2^{q+1} - 1)}{\Gamma^{q+1}(\sigma+1)}\|P\|_p g(\|L\|)^{q+1} + \frac{\alpha_2 \|P\|_p g(\|L\|)}{\Delta_1} + \omega_3 \leq 1.$$

In view of $(H_k)$, there exists $M_2$ such that $\|x\| \leq M_2$. Let us set:

$$U = \{ u \in C([1,e], \mathbb{R}) : \|u\| \leq M_2 \}.$$

Note that the operator $Q: U \rightarrow C([1,e], \mathbb{R})$ is continuous and completely continuous. From the choice of $U$, there is no $u \in \mathcal{D} U$ such that $u = \lambda Qu$ for some $\lambda \in (0,1)$. Consequently, by the nonlinear alternative of Leray- Schauder type (Lemma 2.5), we deduce that $Q$ has a fixed point $u \in U$ which is a solution of problem (1.1). This completes the proof.

### Examples

**Example 4.1 Consider the Problem**

$$D^{\gamma/2}\left(\phi_{\beta/2}\left(D^{\gamma/2}x(t)\right)\right) = f(t,x),$$

$$x(1) = \phi_{\alpha/2}\left(D^{\gamma/2}x(1)\right) = 0,$$

$$I^{\gamma/2}x(2) + 2x(e) = 0,$$

with

$$\sigma = \frac{7}{4}, \alpha = \frac{3}{2}, p = \frac{3}{2}, q = 3, A_1 = A_2 = B_1 = 1, B_2 = -1, \gamma_1 = \frac{1}{2}, \gamma_2 = \frac{1}{4}, \eta_1 = 2, \eta_2 = \frac{5}{2}, c_1 = 2.$$ 

$= 0$. Then, we have:

$$\Delta_1 = -0.157$$ and $\Delta_2 = 2.61$.

Consider problem (4.1) with:

$$f(t,x) = \frac{1}{10} e^{-x} \sqrt{\cos(t)} + \arctan(1+t).$$

Clearly:

$$\|f(t,x)\| \leq a(t) + b(t) \|L\|_e,$$

where, $a(t) = \arctan(1+t)$ and $b(t) = \frac{1}{10} e^{-x}$. It follows that:

$$\|L\|_e \leq \frac{\pi}{2}, \|L\|_e = \frac{1}{10} e.$$

With the given values, we find that:

$$\theta_1 \leq 8.58 \times 10^{-3}$$ and $\theta_2 \leq 0.121$.

So, $\theta_1 + \theta_2 \leq 1$. Thus, all hypothesis of Theorem 1.1 hold. Therefore, the Hadamard fractional integral boundary value problem (4.1) has at least one solution.
Consider Problem (4.1) with:

\[ f(t, x) = \frac{\log t}{1 + x^2}. \]

It follows that:

\[ |f(t, x)| \leq p(t)g(\|x\|_\infty), \]

for \( p(t) = \log t \) and \( g(\|x\|_\infty) = 1 \). Then, we have:

\[ \|p\|_\infty = 1, \omega_1 = 488.4, \omega_2 = 6.405 \text{ and } \omega_3 = 0.766. \]

Further, the hypothesis \((H_3)\), it is equivalent to show existence of \(M_2\) such that:

\[ M_2 > 496. \]

Thus, all hypothesis of Theorem 1.2 hold. Therefore, the Hadamard fractional integral boundary value problem (4.1) has at least one solution.

Example 4.2

Consider the problem, for \( \lambda > 1 \):

\[
\begin{align*}
  &D^{\alpha} \left( D^\beta x(t) \right) = \frac{\lambda}{10} \log tx^{-1}, \\
  &x(1) = \phi_0 (D^\beta x(1)) = 0, \\
  &A_1 I^\alpha x(\eta_1) + B_1 x(e) = c_1, \\
  &A_2 I^\gamma \left( D^\beta x \right)(\eta_2) + B_2 \phi^\gamma \left( D^\beta x \right)(e) = c_2.
\end{align*}
\]

(4.2)

1. Let \( \alpha = \sigma = 7/4, p = 5/4, q = 5, A_1 = B_1 = -1/4, A_2 = B_2 = 1/4, c_1 = 1, c_2 = 0, \eta_1 = 2, \eta_2 = 5/2, \gamma_1 = 3/2 \) and \( \gamma_2 = 3/2 \). Then, it follows that:

\[ \|p\|_\infty = 0, \|p\|_\infty = \frac{\lambda}{10}, \]

\[ A_1 = 0.34, A_2 = -0.4, \]

\[ \theta_1 = 1.89 \left( \frac{\lambda}{10} \right) \text{ and } \theta_2 = 1.68 \left( \frac{\lambda}{10} \right) \]

The hypothesis \((H_2)\) is equivalent to give a positive real \(M_1\) such that:

\[ 1.89 \left( \frac{\lambda}{10} \right)^4 + 1.68 \left( \frac{\lambda}{10} \right) \leq 1. \]

So, for \( \lambda < 5 \), there exist \(M_1 > 0\) satisfying the above inequality. Thus, all hypothesis of Theorem 1.1 hold.

Therefore, the Hadamard fractional integral boundary value problem (4.1) has at least one solution:

2. Let \( \alpha = 3/2, \sigma = 7/4, p = 3, q = 3/2, A_1 = -1, B_1 = 2, A_2 = 1, B_2 = 0, c_1 = c_2 = 0, \eta_1 = 5/2, \eta_2 = 3/2, \gamma_1 = 1/2 \) and \( \gamma_2 = 3/2 \).

It follows that:

\[ |f(t, x)| \leq p(t)g(\|x\|_\infty), \]

for \( p(t) = \frac{\lambda}{10} \log t \) and \( g(\|x\|_\infty) = 1 \). Then, we have

\[ \|p\|_\infty = \frac{\lambda}{10}, \omega_1 = 5.5, \omega_2 = 7 \text{ and } \omega_3 = 0. \]

Further, the hypothesis \((H_3)\), it is equivalent to show existence of \(M_2\) such that:

\[ 0.7M_2 + \left( \frac{1.5}{\lambda} - 1 \right)M_2 \geq 0. \]

Then, for every \( \lambda > 1 \), there exist \(M_2 > \frac{1.5}{\lambda} - 1 \).

Thus, all hypothesis of Theorem 1.2 hold. Therefore, the Hadamard fractional integral boundary value problem (4.1) has at least one solution.

Ethics

This article is original and contains unpublished material. The corresponding author confirms that all of the other authors have read and approved the manuscript and no ethical issues involved.

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