Platonic polyhedra tune the three-sphere: II. Harmonic analysis on cubic spherical three-manifolds

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Abstract

From the homotopy groups of two distinct cubic spherical three-manifolds, we construct the isomorphic groups of deck transformations acting on the three-sphere. These groups become the cyclic group of order eight and the quaternion group, respectively. By reduction of representations from the orthogonal group to the identity representation of these subgroups we provide two subgroup-periodic bases for the harmonic analysis on the three-manifolds, which have applications to cosmic topology.

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1. Introduction

We view a spherical topological three-manifold \( M \), see [13], as a prototile on its cover \( M = S^3 \). We study in [8] the isometric actions of \( O(4, R) \) on the three-sphere \( S^3 \) and give its basis as well-known homogeneous polynomials in [6, equation (37)]. From the homotopies of \( M \) we find as isomorphic images, see [11], its deck transformations on \( S^3 \). They form a subgroup \( H \) of \( O(4, R) \). By group/subgroup representation theory with intermediate Coxeter groups we construct on \( S^3 \) a \( H \)-periodic basis for the harmonic analysis on \( M \).

Our approach yields in closed analytic form the onset, the selection rules, the multiplicity, projection operators, orthogonality rules and basis for each manifold. Among the Platonic polyhedra it was applied in [6, 7] to the Poincaré dodecahedron with \( H \) the binary icosahedral group. An algorithm due to Everitt in [4] describes homotopies for spherical three-manifolds from five Platonic polyhedra. Following it we found and applied in [8] for the tetrahedron as \( H \) the cyclic group \( C_3 \).

One field of applications for harmonic analysis is cosmic topology, see [9, 10]. The topology of a three-manifold \( M \) is favored if data from the cosmic microwave background can be expanded in its harmonic basis.

Our work is well suited for comparing on \( S^3 \) a family of topologies and their harmonic analysis. Here, we turn to two distinct cubic spherical three-manifolds, with homotopies described in [4]. We employ an intermediate Coxeter group, construct the groups of deck transformations, and derive and compare their harmonic analysis.

2. The Coxeter group \( G \) and the eight-cell on \( S^3 \)

The Cartesian coordinates \( x = (x_0, x_1, x_2, x_3) \in E^4 \) for \( S^3 \), we combine as in [6, 8] in the matrix form

\[
\begin{pmatrix}
  z_1 \\
  -\bar{z}_2 \\
  \bar{z}_1 \\
  1
\end{pmatrix},
\begin{pmatrix}
  z_1 = x_0 - ix_3, \\
  z_2 = -x_2 - ix_1, \\
  z_1\bar{z}_1 + z_2\bar{z}_2 = 1.
\end{pmatrix}
\]

For the group action we start from the Coxeter group \( G < O(4, R) \) [4, p 254], [5], with the diagram

\[
G \cong \begin{pmatrix} 4 & 0 & 3 & 3 \\
 0 & 0 & 0 & 0
\end{pmatrix}.
\]

For the Coxeter diagram equation (2), we give for the four Weyl reflections \( W_i = W_{a_i}, s = 1, 2, 3, 4 \) the Weyl vectors in table 1 and compute for each Weyl vector \( a_i = (a_{i0}, a_{i1}, a_{i2}, a_{i3}) \) the matrix

\[
v_i := \begin{bmatrix}
  a_{i0} - ia_{i3} & -a_{i2} - ia_{i1} \\
  a_{i2} - ia_{i1} & a_{i0} + ia_{i3}
\end{bmatrix} \in SU(2, C).
\]

The matrices equation (3) will be used to relate, see [8], the Weyl reflections to \( SU(2, C) \times SU(2, C) \) acting by left and right multiplication on the coordinates, equation (1). We include in table 1 the inversion \( J_4 \in G \), and the additional Weyl operator \( W_0 \).
The eight-cell projected to the plane according to \([12, \text{Figure 1.}]\), which is a semidirect product of its normal subgroup \(S\). The group is of order \(4! \times 2 = 24\) permutations of these axes, which is a semidirect product of its normal subgroup \(S\).

\[ G = (C_2)^4 \times S(4). \tag{4} \]

The group is isomorphic to the hyperoctahedral group, see \([1, p \ 90]\), which is a semidirect product of its normal subgroup \(S\). We give the action of the Weyl reflections on the coordinates \(x\), and the product form \(g \epsilon \epsilon' p' = \epsilon' p' \epsilon p\).

\[ \epsilon' = (0, \epsilon_1, \epsilon_2, \epsilon_3), \quad \epsilon_j = \pm 1, \quad \epsilon \epsilon' = \epsilon' \epsilon p, \quad \epsilon'' = \epsilon \epsilon' p^{-1}, \quad p'' = p p', \quad (p p^{-1}) = (\epsilon p \epsilon^{-1}(0), \epsilon p^{-1}(1), \epsilon p^{-1}(2), \epsilon p^{-1}(3)). \tag{5} \]

The hyperoctahedral form of \(G\) allows us to study the irreducible representations. We write the permutation \(p \in S(4)\) in cycle form but use the numbers \((0, 1, 2, 3)\) adapted to the enumeration of the coordinates in equation (1). In table 1, we give the action of the Weyl reflections on the coordinates \(x\) and the factors in \(g = \epsilon p\).

The first three Weyl reflections from \(G\) generate, see \([5]\), the cubic Coxeter subgroup \(O = \{0, -0, -0\}\).\(\tag{6}\) isomorphic to the octahedral group \(O \sim (C_2)^3 \times_4 S(3)\) acting on \(E^3 \in E^2\). This group from 48 simplices generates a spherical cube attached to a single vertex. We choose this cube as the prototile on \(S^3\).

We locate the center of the cubic prototile at \(x = (1, 0, 0, 0) \in S^3\). When we include the action of \(W_4\) along with the cubic Coxeter subgroup equation (6), this cube is mapped into seven companions which tile \(S^3\) and generate the eight-cell tiling described in \([12, pp \ 177-8]\), and shown in a projection in figure 1. The center positions of the eight spherical cubes are located at the eight points

\[ \pm (1, 0, 0, 0), (0, \pm 1, 0, 0), (0, 0, \pm 1, 0), (0, 0, 0, \pm 1). \tag{7} \]

Everitt [4] has shown that \(S^3\) admits two cubic spherical three-manifolds with inequivalent first homotopy. He enumerates the faces and edges and gives a graphical algorithm for their gluing, see figure 2. This gluing determines the generators for the first homotopy group.

In figure 3, the three edges of the shaded triangle mark the intersections of the Weyl reflection (hyper-)planes for \(W_1, W_2, W_3\) with the face \(F_1\) of the cube. Face \(F_1\) itself is part of the Weyl reflection (hyper-)plane for \(W_1\). In the homotopy group, there appears a gluing of opposite faces. Use of the inversion \(J_3\) in the center (black circle), followed by the Weyl reflection \(W_4\), converts this gluing of opposite faces into a standard deck operation \(st(1 \equiv 6)\) equation (12). This operation maps the \(J_3\)-inverted and \(W_4\)-reflected initial cube into a new position with its face \(F_6\) glued to face \(F_1\) of the initial cube.

3. The group isomorphism \(deck(C_2) \sim \pi_1(C_2) \sim C_8\) for the cubic spherical three-manifold \(C(2)\)

We start from the algorithm of \([4, \ p \ 259, \ table \ 3]\) on the first homotopy group \(\pi_1(C_2)\) for the cube and construct the explicit isomorphism to the group of deck transformations \(deck(C_2)\). We denote the faces of the cube from figure 2 as \(F_1, \ldots, F_6\). To the first three faces of \(C_2\), the glue

Table 1. The Weyl vectors \(a_s\), \(s = 1, \ldots, 4\) and \(a_0\) for the Coxeter group \(G\) equation (2), the 2 × 2 unitary matrices \(v_s\) equation (3), the action on \(x\), and the product form \(g = \epsilon p\) with \(p\) in cycle form.

| \(s\) | Weyl vector \(a_s\) | Matrix \(v_s\) | \(g x\) | \(\epsilon\) | \(p\) |
|------|-----------------|----------------|---------|---------|-------|
| 1    | (0, 0, 0, 1)    | \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix} | \((x_0, x_1, x_2, x_3)\) | (+ + + +) | \(e\) |
| 2    | \(0, 0, -\sqrt{\frac{1}{2}}, \sqrt{\frac{1}{2}}\) | \begin{bmatrix} \sqrt{\frac{1}{2}} & -i \\ i & 1 \end{bmatrix} | \((x_0, x_1, x_3, x_2)\) | (+ + + +) | \(p_{\text{inv}}\) |
| 3    | \(0, \sqrt{\frac{1}{2}}, -\sqrt{\frac{1}{2}}, 0\) | \begin{bmatrix} 0 & 1-i \\ 1+i & 0 \end{bmatrix} | \((x_0, x_1, x_3, x_2)\) | (+ + + +) | \(p_{\text{inv}}\) |
| 4    | \(-\sqrt{\frac{1}{2}}, \sqrt{\frac{1}{2}}, 0, 0\) | \begin{bmatrix} 1 & -i \\ i & -1 \end{bmatrix} | \((x_1, x_0, x_3, x_2)\) | (+ + + +) | \(p_{\text{inv}}\) |
| 0    | (1, 0, 0, 0)    | \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} | \((-x_0, x_1, x_3, x_2)\) | (+ + + +) | \(e\) |
| \(J_3\) | \((-x_0, -x_1, x_3, -x_2)\) | \((-x_0, -x_1, x_3, -x_2)\) | (+ + + +) | \(e\) |
for the position of the cube and for the partners are

\[ F3 \cup F1, \quad F4 \cup F2, \quad F5 \cup F3. \tag{9} \]

We represent the four edges for a square face by the corresponding numbers from figure 2. Evaluating the glue algorithm for C2 in [4], the first edge glue generator can be depicted from right to left as the map

\[ g_1 = g_1(1 \leftarrow 3) : \begin{bmatrix} 7 & 2 \\ 4 & 2 \end{bmatrix} \xleftarrow{\text{R}_1(\pi/2)} \begin{bmatrix} 7 & 2 \\ 2 & 2 \end{bmatrix}. \tag{10} \]

We use left arrows \( \leftarrow \) in line with the usual sequence of operator products. Now we wish to map this glue generator \( g_1(1 \leftarrow 3) \) isomorphically into a generator from the group deck(C2) of deck transformation acting on the three-sphere \( S^3 \). We follow a similar method as employed in [6, pp 5322–4] for the Poincaré dodecahedral three-manifold. We refer to figure 3 for the position of the cube and for the three orthogonal directions 1, 2, 3. We factorize \( g_1(1 \leftarrow 3) \) into three actions: a positive rotation \( R_1(\pi/2) \) around the three-axis, followed by a standard glue operator \( st(1 \leftarrow 6) \) equation (12), followed by a positive rotation \( R_1(\pi/2) \) around the one-axis. In equation (11), we depict from right to left, the images of the initial face F3. In the second line we write the corresponding triple product of operators.

\[ g_1 = g_1(1 \leftarrow 3) : \begin{bmatrix} 4 & 7 \\ 6 & 2 \end{bmatrix} \xleftarrow{\text{R}_1(\pi/2)} \begin{bmatrix} 4 & 7 \\ 2 & 2 \end{bmatrix} \]

\[ \text{st}(1 \leftarrow 6) \begin{bmatrix} 6 & 12 \\ 3 & 6 \end{bmatrix} \xleftarrow{\text{R}_1(\pi/2)} \begin{bmatrix} 6 & 12 \\ 3 & 6 \end{bmatrix}. \tag{11} \]

The orientation of edges is counter-clockwise for the two right-hand diagrams but clockwise for the left-hand ones. The third operator maps \( F6 \leftarrow F3 \), the first one rotates \( F1 \) while preserving the center \( x = (1, 0, 0, 0) \) of the initial cube. The second operator is crucial for the isomorphism \( \pi_1(C2) \rightarrow \text{deck}(C2) \). It is constructed as the product

\[ \text{st}(1 \leftarrow 6) = W_4\mathcal{J}_3, \quad \mathcal{J}_3 = W_0\mathcal{J}_4, \tag{12} \]

of the inversion operator \( \mathcal{J}_3 \in H \) w.r.t. \( (x_1, x_2, x_3) \), which can be factorized into the Weyl operator \( W_0 \) and the full coordinate inversion \( \mathcal{J}_4 \in G \), followed by the Weyl reflection \( W_4 \). The operator \( \mathcal{J}_3 \) in equation (12) inverts the cube in its center and so maps any face \( F_i \) of the cube into its opposite face, in the present case \( F1 \leftarrow F6 \). The final Weyl reflection \( W_4 \) following \( \mathcal{J}_3 \) reflects the inverted cube in its face \( F1 \) and produces an image of the initial cube with the new center \( (0, 1, 0, 0) \), glued with its face \( F3 \) to the original face \( F1 \). The product equation (12) contains two Weyl reflections and therefore preserves the orientation.

For later use we list three standard glue operators of the cube in table 2. We construct them from equation (12) by the conjugations

\[ \text{st}(2 \leftarrow 4) = (W_3W_2)\text{st}(1 \leftarrow 6)(W_2W_3), \]

\[ \text{st}(3 \leftarrow 5) = (W_2W_3)\text{st}(1 \leftarrow 6)(W_3W_2) \tag{13} \]

listed in table 3.

**Table 2. Standard glue operators in the Coxeter group G.**

| \( g \) | \( gx \) | \( \epsilon \) | \( p \) |
|---|---|---|---|
| \( \text{st}(1 \leftarrow 6) \) | \( (x_1, x_0, x_2, x_3) \) | \(+ + - -\) | \( 01 \) |
| \( \text{st}(2 \leftarrow 4) \) | \( (x_1, -x_1, x_2, -x_3) \) | \(+ - + -\) | \( 02 \) |
| \( \text{st}(3 \leftarrow 5) \) | \( (x_1, x_1, -x_2, -x_3) \) | \(+ + - -\) | \( 03 \) |

The operations equations (13) are in \( G \) but not necessarily in \( \text{deck}(C2) \). The rotations and the product \( W_iW_j \) appearing in equation (12) are even in \( G \), and so all operations preserve orientation.
Next, we express the generator $g_1$ as a product of pairs of Weyl operators and find

$$R_1(\pi/2) = (W_2 W_1), \quad \text{st}(1 \leftrightarrow 6) = (W_4 W_6) J_4,$$

$$R_2(\pi/2) = (W_2 W_3)(W_2 W_1)(W_3 W_2),$$

$$g_1 = (W_2 W_1)(W_4 W_6) J_4 (W_2 W_3)(W_2 W_1) (W_3 W_2).$$

We use [8, equations (60)] for products of Weyl operators in equation (15) and the matrices $\nu_i$ from table 1 to rewrite the operator $T_{g_1}$ in the form

$$T_{g_1} = T_{(w_1, w_3)},$$

$$w_1 = (v_2 v_1^{-1})(v_4 v_0^{-1})(v_2 v_3^{-1})(v_2 v_1^{-1})(v_3 v_2^{-1})$$

$$= \sqrt{2} \begin{bmatrix} -\alpha & 0 \\ 0 & -a \end{bmatrix},$$

$$w_3 = (-1)(v_1^{-1} v_0)(v_4^{-1} v_0)(v_1^{-1} v_3)(v_1^{-1} v_1)(v_3^{-1} v_2)$$

$$= \sqrt{2} \begin{bmatrix} 0 & -a^3 \\ -a & 0 \end{bmatrix},$$

$$a = \exp(\pi i/4).$$

By $(w_1, w_3)$ we denote the elements of the SU$(2, C) \times$ SU$(2, C)$ action $u \rightarrow u_1^{-1} u w_1$ on $S^1$ in the coordinates equation (1). In this scheme, $J_4$ in equation (12) yields the operator

$$T_{J_4} = T_{(e, -e)}$$

which commutes with all rotation operators. In equation (16), we have absorbed this operator into $w_1$. From equation (16), we easily see that $w_1^8 = w_1^2 = e$ so that $g_1, g_1^8 = e$ generates the cyclic group $C_8$ of order 8.

We can construct from the graphs given in [4] the other glue generators of the homotopy group of C2 and map them isomorphically into generators of the group of deck transformations. It turns out that all of these glue generators become powers of the first glue generator $g_1 \in C_8$ equation and its image equation (16) in deck(C2). We list the actions of the eight powers $g_1^t$ in table 3.

**Theorem 1.** The homotopy group and the group of deck transformations for the spherical cubic three-manifold C2 of Everitt [4, p 259, table 3] are isomorphic to the cyclic group

$$\text{deck}(C2) = C_8 = (g_1^t, t = 1, \ldots, 8, g_1^8 = e)$$

with actions on $S^3$ given in table 3. The deck transformations generate fix-point free the eight-cell on $S^3$. The group/subgroup scheme for the cubic three-manifold C2 is

$$O(4, R) > G > C_8.$$

**3.1. The reduction $O(4, R) > C_8$ and harmonic analysis on (C2)**

The irreducible representations of $O(4, R)$, their polynomial basis in terms of spherical harmonics and the matrix elements of Weyl operators we adopt from [8, section 4.2] Here, we consider directly the reduction of representations for the group/subgroup pair

$$O(4, R) > C_8.$$

The operators for the generator of $g_1 \in C_8$ and its powers from [8, equation (60)] have the form

$$T_{g_1} = T_{(w_1, w_3)}, \quad T_{g_1^t} = T_{(w_1^t, w_3^t)},$$

given in table 3 in terms of the SU$(2, C) \times$ SU$(2, C)$ action.

From equation (21) and from [8, equation (45)] we find for the characters of the powers $g_1^t$ in the irreducible

**Table 3.** The elements of the cyclic group $C_8$ of deck transformations of the manifold C2 and their actions on $S^3$.

| $t$ | $(g_1^t)^x$ | $w_1^t$ | $w_3^t$ | $e$ | $p$ |
|-----|-------------|---------|---------|-----|-----|
| 1   | $(x_1, -x_3, x_0, x_3)$ | $\begin{bmatrix} -\pi & 0 \\ 0 & -a \end{bmatrix}$ | $\begin{bmatrix} 0 & -a^3 \\ -a & 0 \end{bmatrix}$ | $(+) (-+)$ | (0132) |
| 2   | $(-x_3, -x_2, x_0, x_0)$ | $\begin{bmatrix} \pi^2 & 0 \\ 0 & a^2 \end{bmatrix}$ | $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ | $(-) (++)$ | (03)(12) |
| 3   | $(-x_2, -x_0, -x_3, x_1)$ | $\begin{bmatrix} -\pi^3 & 0 \\ 0 & -a^3 \end{bmatrix}$ | $\begin{bmatrix} 0 & a^3 \\ a & 0 \end{bmatrix}$ | $(-) (-+)$ | (0231) |
| 4   | $(-x_0, -x_1, -x_2, -x_1)$ | $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ | $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ | $(--) e$ | (22) |
| 5   | $(-x_1, x_3, -x_0, x_2)$ | $\begin{bmatrix} \pi & 0 \\ 0 & a \end{bmatrix}$ | $\begin{bmatrix} 0 & -a^3 \\ -a & 0 \end{bmatrix}$ | $(+-+) e$ | (0132) |
| 6   | $(x_3, x_2, -x_1, -x_0)$ | $\begin{bmatrix} \pi^2 & 0 \\ 0 & a^2 \end{bmatrix}$ | $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ | $(++) (-*)$ | (03)(12) |
| 7   | $(x_2, x_0, x_3, -x_1)$ | $\begin{bmatrix} \pi^2 & 0 \\ 0 & a^3 \end{bmatrix}$ | $\begin{bmatrix} 0 & a^3 \\ a & 0 \end{bmatrix}$ | $(+++)$ | (0231) |
| 8   | $(x_0, x_1, x_2, x_3)$ | $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ | $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ | $(+++) e$ |
representation $D^{(j,j)}$ in terms of characters of SU(2, C) the result

$$\chi^{(j,j)}(g_0^i) = \chi^j(w_0^{i1})\chi^j(w_0^{i1}), \quad t = 1, \ldots, 8. \tag{23}$$

These expressions are easily evaluated with the help of [8, appendix]. The multiplicity of the identity representation $D^0$ of $C_8$ is then given by

$$m(C_8(j, j), 0) = \frac{1}{8} \sum_{t=1}^{8} \chi^j(w_0^{i1})\chi^j(w_0^{i1}). \tag{24}$$

In table 4, we give the values of the characters of SU(2, C), SU(2, C) and the onset of multiplicities of $C_8$-periodic states (24) as functions of the degree 2, 0 $\leq$ $j$ $\leq$ 8. The characters $\chi^j$ in this table can be divided into a set with the property $\chi^{j+1}(w_0^{i1}) = \chi^j(w_0^{i1})$ and a set which increases with $j$. Treating these sets in equation (24) separately allows us to derive the following recursion relation for the multiplicities:

$$(2j) = \text{even} : m(C_8(j + 4, j + 4), 0) = m(C_8(j, j), 0) + 8j + 20 + 2(-1)^j. \tag{25}$$

Table 4. The characters $\chi^j(w_0^{i1}), \chi^j(w_0^{i1}), t = 1, \ldots, 8$ in SU(2, C) for $C_8$ in the irreducible representations $D^j$ of SU(2, C). The characters $\chi^{j+1}(g_0^i)$ are given by equation (23) and the multiplicities $m(C_8(j, j), 0)$ by equation (24).

| $(\phi/2)_j$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|----------|----|----|----|----|----|----|----|----|----|
| $\chi^j(w_0^{i1})$ | $\frac{3\pi}{4}$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| $\chi^j(w_0^{i1})$ | $\frac{\pi}{2}$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| $\chi^j(w_0^{i1})$ | $\frac{\pi}{4}$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| $\chi^j(w_0^{i1})$ | $\pi$ | 1 | 3 | 7 | 9 | 11 | 13 | 15 | 17 |
| $\chi^j(w_0^{i1})$ | $\frac{\pi}{4}$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| $\chi^j(w_0^{i1})$ | $\frac{\pi}{2}$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| $\chi^j(w_0^{i1})$ | 0 | 1 | 3 | 5 | 7 | 9 | 11 | 13 | 15 |
| $\chi^j(w_0^{i1})$ | $\frac{\pi}{2}$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| $\chi^j(w_0^{i1})$ | $\frac{\pi}{4}$ | 1 | 3 | 5 | 7 | 9 | 11 | 13 | 15 |
| $\chi^j(w_0^{i1})$ | $\pi$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |

With the representations of SU(2, C) × SU(2, C) given in [8, equation (85)], we construct from table 3 the representations of $C_8$ of the form

$$(T_{(w_0^{i1}, w_0^{i1})}) D_j^{(m_1, m_2)}(u) = D_{m_1, m_2}^{(j)}(w_0^{i1}) + D_j^{(m_1, m_2)}(w_0^{i1}) \tag{29}$$

whose coefficients are given in table 5.

Table 5. Matrix elements in the representation $D^{(j,j)}$ equation (29) of the elements of $C_8$ for $i = 1, \ldots, 8$ with $l = 1, \ldots, 4$.

| $i$ | $D_j^{(m_1, m_2)}(w_0^{i1})$ | $D_j^{(m_1, m_2)}(w_0^{i1})$ |
|-----|-------------------------|-------------------------|
| 1   | $\frac{1}{2}$ $\delta_{m_1, m_1}$ | $\frac{1}{2}$ $\delta_{m_1, m_1}$ |
| 2   | $\frac{1}{2}$ $\delta_{m_1, m_1}$ | $\frac{1}{2}$ $\delta_{m_1, m_1}$ |
| 3   | $\frac{1}{2}$ $\delta_{m_1, m_1}$ | $\frac{1}{2}$ $\delta_{m_1, m_1}$ |
| 4   | $\frac{1}{2}$ $\delta_{m_1, m_1}$ | $\frac{1}{2}$ $\delta_{m_1, m_1}$ |

The projection operator on the identity representation $D^0$ of $C_8$ is

$$P^0 = \frac{1}{8} \sum_{i} T_{(w_0^{i1}, w_0^{i1})}. \tag{31}$$

Its matrix elements from table 5 for fixed $j$ can be rewritten as

$$(P^0)_{m_1, m_2, m_1, m_2} = \frac{1}{2} \left[ 1 + (-1)^{j+1} \right] \frac{1}{2} \left[ 1 + (-1)^{m_1} \right] \times \frac{1}{2} \left[ \delta_{m_1, m_1} \delta_{m_2, m_2} + i m_1 (-1)^{m_1} m_2 \delta_{m_1, m_2} \delta_{m_2, m_1} \right]. \tag{32}$$

The two prefactors imply integer $j$ and even $m_1$. For this case we get by application of the projector.

Table 6. The $C_8$-periodic basis $\phi_j^{m_1, m_2}$ on $S^1$ for the harmonic analysis on the cubic spherical three-manifold $C2$ in terms of spherical harmonics $D_j^{(u)}$ on $S^3$.

$$(\phi_j)^{m_1, m_2} = \frac{\sqrt{2j} + 1}{\sqrt{8\pi}} D_j^{(u)}(\phi_{m_1, m_2}). \tag{33}$$

Theorem 2. An orthonormal basis for the harmonic analysis on the cubic spherical manifold $C2$ is spanned by the $C_8$-periodic polynomials of degree 2, $j = 0, 1, 2, \ldots$ in table 6.
4. The group isomorphism \( \pi_1(C3) \sim \text{deck}(C3) \sim Q \) for the cubic spherical three-manifold (C3)

The second possible homotopy group of the spherical three-cube is given by Everitt [4, p 259, table 3] by a second graphical algorithm. We denote this cubic three-manifold as C3. The order of the homotopy group and the group of deck transformations is again eight. For the homotopy of C3, opposite faces of the cube are glued,

\[
F6 \cup F1, \quad F5 \cup F3, \quad F4 \cup F2.
\]

From the edge gluing we find that each glue is followed by a left-handed rotation by \( \pi/2 \). We construct three glue generators \( q_1, q_2, q_3 \) from the prescription of [4, p 259, table 3], depict the edge gluings from right to left as before and factorize them with the standard glue operators equation (13)

\[
q_1 = q_1(1 \leftarrow 6) : \begin{bmatrix} 2 & 7 \\ 3 & 6 \end{bmatrix} \quad \begin{bmatrix} 10 & 12 \\ 7 & 9 \end{bmatrix},
\]

\[
q_2 = q_2(3 \leftarrow 5) : \begin{bmatrix} 3 & 1 \\ 6 & 5 \end{bmatrix} \quad \begin{bmatrix} 8 & 7 \\ 12 & 9 \end{bmatrix},
\]

\[
q_3 = q_3(2 \leftarrow 4) : \begin{bmatrix} 3 & 8 \\ 10 & 3 \end{bmatrix} \quad \begin{bmatrix} 4 & 5 \\ 11 & 10 \end{bmatrix}.
\]

The three generators in terms of Weyl reflections become

\[
q_1 = (W_1W_2)(W_3W_4), q_2 = (W_3W_4)q_1(W_3W_4), q_3 = (W_2W_3)q_1(W_2W_3).
\]

As mentioned before, any apparent reversion of the edge orientation is compensated by a Weyl reflection. Expressed in \( G \) the generators from equation (36) are given in table 7. Their multiplication yields the relations

\[
q_1^2 = q_2^2 = q_3^2 = q_1q_2q_3 = J_4.
\]

These are exactly the relations characterizing the quaternion group \( Q \), see [1, 8], with \( J_4 \) playing the role of \((-1)\). The eight elements are

\[
deck(C3) = \{e, q_j, J_4, J_4q_j, j = 1, 2, 3\}.
\]

We therefore have shown

**Theorem 3.** The homotopy group and the group of deck transformations for the spherical cubic three-manifold C3 of Everitt [4, p 259, table 3] are isomorphic to the quaternion group \( Q \). The elements of the group deck(C3) act on \( S^3 \) from the left as given in equations (42) and (43) and generate fix-point free the eight-cell on \( S^3 \). By the construction of the generators of \( Q \) in table 7, we have also shown the group/subgroup relation

\[
O(4, R) > G > \text{deck}(C3) = Q.
\]

4.1. The reduction \( O(4, R) > Q \) and harmonic analysis on (C3)

The harmonic analysis for the three-cube (C3) has as its basis the \( Q \)-periodic spherical harmonics on \( S^3 \) of degree \( 2j \). Selection rules eliminate all contributions with \( 2j = \text{odd} \). The periodic states are degenerate with respect to the subgroup \( SU(2, R) \), but the degenerate states can be labeled by a representation index \( m_2, -j \leq m_2 \leq j \). The explicit construction of the \( Q \)-periodic polynomials can be found by the application of Young operators as described in [8, section 4.7].

4.2. The multiplicity \( m(Q(j, j), 0) \) of representations in \( O(4, R) > Q \)

The reduction of representations we analyze in section 5 in the scheme equation (40). Here we consider the reduction \( O(4, R) > Q \).

To compute the multiplicity \( m((j, j), 0) \) for the reduction of the irreducible representations \( D^{(j,j)} \) of \( O(4, R) \) to the identity representation of the subgroup \( Q \), denoted as \( D^{(Q)}(Q) \), we need the characters \( \chi_{(j)}^{(j,j)} \) for the eight elements of \( Q \) equation (39). The elements \( e, J_4 \) have the characters

\[
\chi_{(j)}^{(j,j)}(e) = (2j+1)^2, \quad \chi_{(j)}^{(j,j)}(J_4) = (-1)^{2j}(2j+1)^2.
\]

The second equality arises because the basis of \( D^{(j,j)} \) has the homogeneous degree \( 2j \). The elements \( q_1, q_2, q_3 \) by table 8 are conjugate not in \( Q \) but in \( O(4, R) \) and so have the same character w.r.t. \( D^{(j,j)} \). We choose \( q_1 = W_2W_1W_0W_4 \) as representative. Application of [8, equation (60)] for a product of four Weyl operators gives for \( q_1 \) in terms of the \( SU(2, C) \times SU(2, C) \) action

\[
T_{q_1} = T_{W_2W_1W_0W_4} = T_{(i_3i_2^{-1}i_1i_4^{-1}, i_2^{-1}i_3i_2^{-1}i_4)}.
\]
Evaluation of the two matrix products in equation (42) with $v_0 = e$ gives

\[(v_2v_1^{-1})(v_0v_4^{-1}) = w_{11} = \begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix}, \]

\[(v_2^{-1}v_1)(v_1^{-1}v_4) = w_{11} = e \]

and so from equation (43)

\[T_{q_1} = T_{(w_{11})}, \]

\[\chi^{(j,j)}(q_1) = \chi^j(w_{11})\chi^j(e) = \chi^j(w_{11})(2j + 1). \]

The result, equation (43), shows that $q_1$ operates on $u \in S^3$ exclusively by left action corresponding to the subgroup $SU^2(2, C) < (SU^2(2, C) \times SU^2(2, C))$. The same holds true for the operators $T_{q_2}, T_{q_3}$, since the conjugations in equation (36) applied to $T_{q_1}$ preserve the subgroup $SU^2(2, C)$.

For the characters of the element $w_{11} \in SU^2(2, C)$ one finds

\[\chi^{1/2}(w_{11}) = 2 \cos(\phi/2) = 0, \quad \chi^0(w_{11}) = 1, \quad \chi^{j+2}(w_{11}) = \chi^j(w_{11}) = (-1)^j \]

where the period 2 in the last line arises from $\phi/2 = \pi/2$.

With these expressions we find the multiplicity of the $Q$-periodic states for a given representation $D^{(j,j)}$ as

\[m(Q(j, j), 0) = \frac{1}{8} \sum_{g \in Q} \chi^{(j,j)}(g) \]

\[= \frac{1}{8} (1 + (-1)^{(2j)})(2j + 1) \left[ (2j + 1) + 3(-1)^j \right]. \]

The first prefactor eliminates all the states with $(2j) = \text{odd}$.

The second prefactor $(2j + 1)$ arises from the degeneracy with respect to the group $SU^2(2, C)$. For a $Q$-periodic state of polynomial degree $2j$, we can choose $(2j + 1)$ orthogonal basis states with respect to $SU^2(2, C)$, corresponding to the second label $m_2$ in the spherical harmonics $D_{m_1,m_2}(u)$ on $S^3$.

In table 8 we give the onset of the multiplicity equation (46) for the lowest values of $2j$.

### 4.3. An orthogonal $Q$-periodic basis for the harmonic analysis on $C3$

For the quaternionic group $Q$, the orthogonal basis for the harmonic analysis can be given in closed analytic form. We shall use the irreducible representation matrices $D_{m_1,m_2}(u)$ given in [8, equation (85)]. Since $Q < SU^2(2, C)$ acts from the left only, its representations are given by $D_{m_1,m_2}(g), g \in Q$. These matrix elements are given in table 9 for general $j$. In what follows we need them only for integer $j$.

### Table 9. Representation matrices for $g \in Q$ and general $j$.

| $g$ | $D_{m_1,m_2}(g), -j \leq (m_1, m_2) \leq j$ |
|-----|-----------------------------------------------|
| $e$ | $\delta_{m_1,m_2}$ |
| $q_1$ | $\delta_{m_1,-m_2}(1^{-1/2}) \exp(-i\pi m_2)$ |
| $q_2$ | $\delta_{m_1,-m_2}(1)^{j/m_1}$ |
| $q_3$ | $\delta_{m_1,m_2} \exp(-i\pi m_1)$ |
| $J_i$ | $\delta_{m_1,m_2}(-1)^j$ |

We next construct a projection operator on the identity representation $D^0$ of $Q$:

\[P^0 : \sum_{g \in Q} T_g = \frac{1}{4} \left[ T_e + \sum_{i=1}^{3} T_{q_i} \right]. \]

\[(P^0)^2 = P^0. \]

The representation matrix of this operator for fixed integer $j$ is obtained from table 9 in the simple form

\[ (P^0)^{m_2} = \frac{1}{2} [1 + (-1)^{m_2}] \sum_{m_1} \left[ \delta_{m_1,m_2} + \delta_{m_1,-m_2}(-1)^j \right]. \]

The projection operator equation (51) applied to a spherical harmonic does not affect the column index $m_2$. For the projection it proves convenient to separate the even and odd $j$ case. The linearly independent, orthogonal $Q$-periodic polynomials normalized according to equation (28) are then found by projection with equation (51).

### Theorem 4. An orthonormal basis for the harmonic analysis on the cubic spherical manifold $C3$ is spanned by the $Q$-periodic polynomials of degree $2j$, $j = 0, 1, 2, \ldots$ in table 10.

The orthogonality and normalization is obtained by use of equation (28). The multiplicities $m(Q(j, j), 0)$ agree with the values derived before in table 8.
denote the two elements by \( C \) little group and little co-group. For the cyclic group \( G \)
choosing coset generators \( c \), \( H \) the little co-group irreducible representations are
direct products of the chosen \( L \)
that defined [2, p 108] as the maximal subgroup \( K \).
The little co-group \( D \) of \( \phi \), \( c \), \( \gamma \) equation (\( \mu \)) we follow Coleman [2], equation (59), of the identity representation of
\( C_8 \) for all partitions \( f \) of \( S(4) \).
\[
\chi^{(\mu,f)}(g) = \sum \delta(c_j^{-1}pc_j, k \in K) D^{\mu}(c_j^{-1}ec_j) \chi_f(k).
\] (58)
To determine these characters we must specify for each little co-group its coset generators. The multiplicity of the identity representation of
the subgroups \( H = (C_8, Q) \) is then given by
\[
m((\mu, f) \uparrow, 0) = \frac{1}{8} \sum_{g \in H} \delta(g, ep) \chi^{(\mu,f)}(ep). \] (59)
We give the corresponding data and the multiplicities in the next subsection.

5.2. The reductions \( G > C_8, G > Q \)

The factorizations \( g = ep \), equation (5), of the subgroup elements are given in tables 3, 7.

Representations of \( G \) with \( \mu = [++++], [----] \), \( K = S(4) \): for all elements \( g \in C_8 \), \( G = Q : g = ep \) one finds in both representations \( D^{\mu}(e) = 1 \).

\[
\begin{array}{c|cccccc}
\chi^{(\mu,f)}(g), & g \in C_8 & | & | & | & | & | \\
\hline
\hline
\end{array}
\] (60)

\[
\begin{array}{c|cccccc}
\chi^{(\mu,f)}(g), & g \in Q & | & | & | & | & | \\
\hline
\hline
\end{array}
\] (61)

\[
\begin{array}{c|cccccc}
\chi^{(\mu,f)}(g), & g \in Q & | & | & | & | & | \\
\hline
\hline
\end{array}
\] (62)

\[
\begin{array}{c|cccccc}
\chi^{(\mu,f)}(g), & g \in Q & | & | & | & | & | \\
\hline
\hline
\end{array}
\] (63)

Representations with \( \mu = [++++], [----] \), \( K = S(3) \times S(1) \) in table 13 we list the four coset generators \( c_j \) of \( K = S(3) \times S(1) \subset S(4) \) in cycle form and give their action
on \( D^{\mu}(e) \).

The representations of the little co-group \( K = S(3) \times S(1) \) are \( f = [3] \times [1], [111] \times [1], 21 \times [1] \). The characters of the elements of \( H = C_8, Q \) are given by equation (58).

The conjugations \( p \to c_j^{-1}pc_j \) cannot change the class of \( p \) encoded by its cycle expression. Therefore the condition \( c_j^{-1}gc_j = k \in S(3) \times S(1) \) eliminates all contributions except
Table 13. Coset generators $c_j$ and their action on $D^\mu(e)$ for the representations of $G$ with $\mu = [+ + +], [- - -]$.

$$
\begin{array}{cllll}
J & c_j \in S(4)/S(3) \times S(1) & D^{\mu=\pm}(c_j^{-1}e c_j) & D^{\mu=\pm}(c_j e c_j) \\
1 & e & \epsilon_1 & \epsilon_1 e_2 \\
2 & (03) & \epsilon_0 & \epsilon_1 e_2 \\
3 & (13) & \epsilon_1 & \epsilon_1 e_2 \\
4 & (23) & \epsilon_2 & \epsilon_1 e_2 \\
\end{array}
$$

Table 14. Non-vanishing characters $\chi^{(\mu f)^T}(g^j)$, equation (58), for $g^j \in C_k$ and multiplicity $m(C_k) = m((\mu f), 0)$, equation (59), of the identity representation of $C_k$ in the representations $\mu = [+ + +], [- - -]$ for all partitions $f = f_1 \times f_2$ of $S(3) \times S(1)$.

$$
\begin{array}{clllll}
g & (g^j) \in C_k & t & [3] \times [1] & [111] \times [1] & [21] \times [1] \\
4 & 4 & -4 & -4 & -4 \\
8 & 4 & 4 & 4 \\
m(C_k) & 0 & 0 & 0 \\
\end{array}
$$

Table 15. Non-vanishing characters $\chi^{(\mu f)^T}(g)$ for $g \in Q$ and multiplicity $m(Q) = m((\mu f), 0)$, equation (59), of the identity representation of $Q$ in the representations $\mu = [+ + +], [- - -]$.

$$
\begin{array}{clllll}
g & \chi^{(\mu f)^T}(g) & t & [3] \times [1] & [111] \times [1] & [21] \times [1] \\
e & \epsilon & 4 & 4 & 4 \\
J_e & -4 & -4 & -4 \\
m(Q) & 0 & 0 & 0 \\
\end{array}
$$

Table 16. Coset generators $c_j$ and their action on $D^\mu(e)$ for the representations of $G$ with $\mu = [+ + -], [- - +]$, $K = S(2) \times S(2)$.

$$
\begin{array}{clllll}
j & c_j \in S(4)/S(2) \times S(2) & D^{\mu=\pm}(c_j^{-1}e c_j) & D^{\mu=\pm}(c_j e c_j) \\
1 & e & \epsilon_1 e_3 & \epsilon_0 e_2 \\
2 & (12) & \epsilon_1 e_3 & \epsilon_0 e_2 \\
3 & (321) & \epsilon_1 e_3 & \epsilon_0 e_2 \\
4 & (120) & \epsilon_1 e_3 & \epsilon_0 e_2 \\
5 & (1320) & \epsilon_1 e_3 & \epsilon_0 e_2 \\
6 & (02)(13) & \epsilon_1 e_3 & \epsilon_0 e_2 \\
\end{array}
$$

Table 17. Nonvanishing character contributions for $C_k$ for $\mu = [+ + -], [- - +]$. The irreducible representations $f = f_1 \times f_2$ of $S(2) \times S(2)$ are $D^{\mu=\pm}(f(01)(23)) = (-1)^{\mu_1+\mu_2}$, with $\mu_1 = 0, 1$ for $f_1 \times f_2 = [2], [11]$.

$$
\begin{array}{clllll}
g & \chi^{(\mu f)^T}(g) & c_j & \epsilon_j & \epsilon'_j & \epsilon''_j \\
8^4 & [2] & \epsilon_3 & (01)(23) & -1 & -1 \\
8^4 & [2] & \epsilon_4 & (01)(23) & -1 & -1 \\
8^4 & [2] & e & 1 & 1 & 1 \\
8^4 & [2] & \epsilon_1 e_3 & (01)(23) & -1 & -1 \\
8^4 & [2] & \epsilon_1 e_3 & (01)(23) & -1 & -1 \\
8^4 & [2] & \epsilon_1 e_3 & (01)(23) & -1 & -1 \\
\end{array}
$$

for $g = (e, J_e)$. The reduced set of characters and the multiplicities are given in the following tables.

Representations $D^\mu$ with $\mu = [+ + -], [- - +]$, $K = S(2) \times S(2)$: we list the six coset generator of the little co-group in the next table.

For $g^j \in C_8$ and the coset generators in table 16, we now check the condition $c_j^{-1}g^j c_j \in S(2) \times S(2)$. The cycle structure of $S(2) \times S(2)$ admits only the elements $e$, $(01), (23), (01)(23)$, and this condition applies at most for the elements $g^j_1, g^j_4, g^j_6, g^j_8 = e$. For these elements one finds the non-vanishing contributions given in table 17.

Evaluation of the multiplicity equation (59) from this table gives for both representations

$$
\mu = [+ + -], [- - +] : m(C_8) = m(C_8(\mu f) \uparrow, 0) = \frac{1}{3} \left[ 12 - 4(-1)^{\mu_1+\mu_2} \right].
$$

Table 18. Nonvanishing character contributions for $Q$ for $\mu = [+ + -], [- - +]$ in the notation of table 17.

$$
\begin{array}{clllll}
e & \epsilon & 1 & 1 & 1 & 1 \\
\psi_1, \psi_2 & -1 & -1 & 1 & 1 & 1 \\
\psi_1, \psi_2 & -1 & -1 & 1 & 1 & 1 \\
\psi_1, \psi_2 & -1 & -1 & 1 & 1 & 1 \\
\psi_1, \psi_2 & -1 & -1 & 1 & 1 & 1 \\
\psi_1, \psi_2 & -1 & -1 & 1 & 1 & 1 \\
\end{array}
$$

Evaluation of the multiplicity equation (59) from this table gives for these representations

$$
\mu = [+ + -], [- - +] : m(Q) = m(Q(\mu f) \uparrow, 0) = \frac{1}{3} \left[ 12 - 4(-1)^{\mu_1+\mu_2} \right].
$$

Table 19. Representations $D^{\mu f}$ of $G$, their dimensions, and summary on multiplicities for $C_k$- and $Q$-periodic states.

$$
\begin{array}{clll}
\mu & f & D^{\mu f} & \text{dim}(\mu f) \uparrow \\
[+ + +], [- - -] & \{4\} \times \{[23]\} & \{31\} \times \{21\} & \{22\} \\
\text{dim}(\mu f) \uparrow & 1 & 1 & 3 & 2 \\
m(C_k) & 1 & 0 & 0 & 1 \\
m(Q) & 1 & 1 & 0 & 2 \\
\end{array}
$$

$$
\begin{array}{clll}
\mu & f & D^{\mu f} & \text{dim}(\mu f) \uparrow \\
[+ + +], [- - -] & \{[3]\} \times \{[23]\} & \{[111]\} \times \{[21]\} & \{[22]\} \\
\text{dim}(\mu f) \uparrow & 4 & 4 & 8 \\
m(C_k) & 1 & 0 & 0 \\
m(Q) & 0 & 0 & 0 \\
\end{array}
$$

$$
\begin{array}{clll}
\mu & f & D^{\mu f} & \text{dim}(\mu f) \uparrow \\
[+ + +], [- - -] & \{[3]\} \times \{[23]\} & \{[111]\} \times \{[21]\} & \{[22]\} \\
\text{dim}(\mu f) \uparrow & 6 & 6 & 6 \\
m(C_k) & 1 & 2 & 2 \\
m(Q) & 0 & 3 & 3 \\
\end{array}
$$

Note that $H$-periodic states arising from different representations of $G$ are orthogonal. The defining four-dimensional representation of the Coxeter group $G$ acting on the three-sphere appears in this table as $(\mu f) \uparrow = ((+++))[3] \times [1] \uparrow$. 

9
The selection rules for the cubic spherical manifolds $C_2$, $C_3$ now eliminate full representations of the Coxeter group $G$. The multiplicities $m(C_8) = m(Q) = 0$ for the representations with $\mu = \{++-\}, \{---++\}$ of dimensions $[4, 8]$ are easily understood. In these representations one finds for the element $J_4$ of both subgroups from equation (57)

$$D^{(\mu f)}_{is} (J_4) = (-1)^{i} \delta_{ij} \delta_{st}, \quad (71)$$

which excludes the identity representation of $C_8$ and $Q$.

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