Weak localization coexisting with a magnetic field
in a normal-metal–superconductor microbridge

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Abstract

A random-matrix theory is presented which shows that breaking time-reversal symmetry by itself does not suppress the weak-localization correction to the conductance of a disordered metal wire attached to a superconductor. Suppression of weak localization requires applying a magnetic field as well as raising the voltage, to break both time-reversal symmetry and electron-hole degeneracy. A magnetic-field dependent contact resistance obscured this anomaly in previous numerical simulations.

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Weak localization is a quantum correction of order $e^2/h$ to the classical conductance of a metal. The word “localization” refers to the negative sign of the correction, while the adjective “weak” indicates its smallness. In a wire geometry the weak-localization correction takes on the universal value $\delta G = -\frac{2}{3}e^2/h$ at zero temperature, independent of the wire length $L$ or mean free path $\ell$. The classical (Drude) conductance $G_0 \approx (N\ell/L)e^2/h$ is much greater than $\delta G$ in the metallic regime, where the number of scattering channels $N \gg L/\ell$. Theoretically, the weak-localization correction is the term of order $N^0$ in an expansion of the average conductance $\langle G \rangle = G_0 + \delta G + O(N^{-1})$ in powers of $N$. Experimentally, $\delta G$ is measured by application of a weak magnetic field $B$, which suppresses the weak-localization correction but leaves the classical conductance unaffected. The suppression occurs because weak localization requires time-reversal symmetry ($T$). In the absence of $T$, quantum corrections to $G_0$ are of order $N^{-1}$ and not of order $N^0$. As a consequence, the magnetoconductance has a dip around $B = 0$ of magnitude $\delta G$ and width of order $B_c$ (being the field at which one flux quantum penetrates the conductor).

What happens to weak localization if the normal-metal wire is attached at one end to a superconductor? This problem has been the subject of active research. The term $G_0$ of order $N$ is unaffected by the presence of the superconductor. The $O(N^0)$ correction $\delta G$ is increased but remains universal.

$$\delta G = -(2 - 8\pi^{-2}) e^2/h \approx -1.19 e^2/h. \quad (1)$$

In all previous analytical work zero magnetic field was assumed. It was surmised, either implicitly or explicitly, that $\delta G = 0$ in the absence of $T$ — but this was never actually calculated analytically. We have now succeeded in doing this calculation and would like to report the result, which was entirely unexpected.

We find that a magnetic field by itself is not sufficient to suppress the weak-localization correction, but only reduces $\delta G$ by about a factor of two. To achieve $\delta G = 0$ requires in addition the application of a sufficiently large voltage $V$ to break the degeneracy in energy between the electrons (at energy $eV$ above the Fermi energy $E_F$) and the Andreev-reflected holes (at energy $eV$ below $E_F$). The electron-hole degeneracy ($D$) is effectively broken when $eV$ exceeds the Thouless energy $E_c = \hbar v_F \ell/L^2$ (with $v_F$ the Fermi velocity). Weak localization coexists with a magnetic field as long as $eV \ll E_c$. Our analytical results are summarized in Table 1. These results disagree with the conclusions drawn in Ref. 8 on the basis of numerical simulations. We have found that the numerical data on the weak-localization effect was misinterpreted due to the presence of a magnetic-field dependent contact resistance, which was not understood at that time.

| $-\delta G [e^2/h]$ | $T$ | no $T$ |
|----------------------|-----|--------|
| $D$                  | $2 - 8\pi^{-2}$ | $2/3$ |
| no $D$               | $4/3$ | $0$    |

Table 1: Dependence of the weak-localization correction $\delta G$ of a normal-metal wire attached to a superconductor on the presence or absence of time-reversal symmetry ($T$) and electron-hole degeneracy ($D$). The entry in the upper left corner was computed in Refs. 10, 11.
Starting point of our calculation is the general relation between the differential conductance \( G = dI/dV \) of the normal-metal–superconductor (NS) junction and the transmission and reflection matrices of the normal region:

\[
G = (4e^2/h) \text{tr} m(eV) m^*(eV),
\]

\[
m(\varepsilon) = t'(\varepsilon) [1 - \alpha(\varepsilon)r^*(-\varepsilon)r(\varepsilon)]^{-1} t^*(-\varepsilon),
\]

where \( \alpha(\varepsilon) \equiv \exp[-2i\arccos(\varepsilon/\Delta)] \). Eq. (2) holds for subgap voltages \( V \leq \Delta/e \), and requires also \( \Delta \ll E_F \) (\( \Delta \) being the excitation gap in \( S \)). We assume that the length \( L \) of the disordered normal region is much greater than the superconducting coherence length \( \xi = (\hbar v_F \ell/\Delta)^{1/2} \). This implies that the Thouless energy \( E_c \ll \Delta \). In the voltage range \( V \lesssim E_c/e \) we may therefore assume that \( eV \ll \Delta \), hence \( \alpha = -1 \). The \( N \times N \) transmission and reflection matrices \( t, t', r, \) and \( r' \) form the scattering matrix \( S(\varepsilon) \) of the disordered normal region (\( N \) being the number of propagating modes at the Fermi level, which corresponds to \( \varepsilon = 0 \)). It is convenient to use the polar decomposition

\[
\begin{pmatrix}
t' & r' \\
t & r
\end{pmatrix} = \begin{pmatrix} v_1 & 0 \\ 0 & w_1 \end{pmatrix} \begin{pmatrix} i\sqrt{R} & \sqrt{T} \\ \sqrt{T} & i\sqrt{R} \end{pmatrix} \begin{pmatrix} v_2 & 0 \\ 0 & w_2 \end{pmatrix}.
\]

Here \( v_1, v_2, w_1, \) and \( w_2 \) are \( N \times N \) unitary matrices, \( T \) is a diagonal matrix with the \( N \) transmission eigenvalues \( T_i \in [0,1] \) on the diagonal, and \( R = 1 - T \). Using this decomposition, and substituting \( \alpha = -1 \), Eq. (2b) can be replaced by

\[
m(\varepsilon) = \sqrt{T(\varepsilon)} \left[ 1 + u(\varepsilon)\sqrt{R(-\varepsilon)}u^*(-\varepsilon)\sqrt{R(\varepsilon)} \right]^{-1}
\cdot u(\varepsilon)\sqrt{T(-\varepsilon)}, \quad u(\varepsilon) \equiv w_2(\varepsilon)w^*_1(\varepsilon).
\]

We perform our calculations in the general framework of random-matrix theory. The only assumption about the distribution of the scattering matrix that we make, is that it is isotropic, i.e. that it depends only on the transmission eigenvalues, in the presence of \( T \) (for \( B \ll B_c \), \( S = S^T \), hence \( w_1 = w_1^T \). (The superscript \( T \) denotes the transpose of a matrix.) If \( T \) is broken, \( w_1 \) and \( w_2 \) are independent. In the presence of \( D \) (for \( eV \ll E_c \)), the difference between \( S(eV) \) and \( S(-eV) \) may be neglected. If \( D \) is broken, \( S(eV) \) and \( S(-eV) \) are independent. Of the four entries in Table I, the case that both \( T \) and \( D \) are present is the easiest, because then \( u = 1 \) and Eq. (2) simplifies to

\[
G = (4e^2/h)\sum_n T_n^2 (2 - T_n)^{-2}.
\]

The conductance is of the form \( G = \sum_n f(T_n) \), known as a linear statistic on the transmission eigenvalues. General formulas, for the weak-localization correction to the average of a linear statistic lead directly to Eq. (1). The three other entries in Table I, where either \( T \) or \( D \) (or both) are broken, are more difficult because \( G \) is no longer a linear statistic. We consider these three cases in separate paragraphs.

(1) \( D \), no \( T \) — Because of the isotropy assumption, \( w_1 \) and \( w_2 \), and hence \( u \), are uniformly distributed in the unitary group \( U(N) \). We may perform the average \( \langle \cdots \rangle \) over the ensemble of scattering matrices in two steps: \( \langle \cdots \rangle = \langle \langle \cdots \rangle \rangle_T \), where \( \langle \cdots \rangle_u \) and \( \langle \cdots \rangle_T \) are,
where we have defined the moment \( \tau_0 \).

Substitution into Eq. (8) yields the weak-localization correction \( \delta G \). Respectively, the average over the unitary matrix \( u \) to the case of a disordered wire in the limit Eq. (8) is generally valid for any isotropic distribution of the scattering matrix. We apply \( \langle \cdot \cdot \cdot \rangle \) over \( u \). We compute \( \langle \cdot \cdot \cdot \rangle \) of \( u \) over \( U(N) \), we first expand it into a geometric series and then use the general rules for the integration of polynomials of \( u \). The polynomials we need are

\[
\langle G \rangle_u = \frac{4e^2}{h} \sum_{p,q=0}^{\infty} M_{pq}, \quad (4a)
\]

\[
M_{pq} = \langle \text{tr} (u \sqrt{R} u^* \sqrt{R}^T)^p u T u^T (\sqrt{R} u^T \sqrt{R} u^\dagger)^q \rangle_u. \quad (4b)
\]

Neglecting terms of order \( N^{-1} \), we find

\[
M_{pq} = \begin{cases} 
N \tau_1^2 (1 - \tau_1)^{2p} & \text{if } p = q, \\
\tau_1 (\tau_1^2 + \tau_1 - 2\tau_2)(1 - \tau_1)^{p+q-1} - 2 \min(p, q) & \times \tau_1^2(\tau_1^2 - \tau_2)(1 - \tau_1)^{p+q-2} \text{ if } |p - q| \text{ odd,} \\
0 & \text{else},
\end{cases}
\]

where we have defined the moment \( \tau_k = N^{-1} \sum T_k^k \). The summation over \( p \) and \( q \) leads to

\[
\frac{h}{4e^2} \langle G \rangle_u = \frac{N \tau_1}{2 - \tau_1} - \frac{4\tau_1 - 2\tau_2 + 2\tau_3 - 4\tau_2}{\tau_1(2 - \tau_1)^3}. \quad (5)
\]

It remains to average over the transmission eigenvalues. Since \( \tau_k \) is a linear statistic, we know that its sample-to-sample fluctuations \( \delta \tau_k = \tau_k - \langle \tau_k \rangle \) are an order \( 1/N \) smaller than the average. Hence

\[
\langle f(\tau_k) \rangle_T = f(\langle \tau_k \rangle)[1 + \mathcal{O}(N^{-2})], \quad (6)
\]

which implies that we may replace the average of the rational function (4) of the \( \tau_k \)’s by the rational function of the average \( \langle \tau_k \rangle \). This average has the \( 1/N \) expansion

\[
\langle \tau_k \rangle = \langle \tau_k \rangle_0 + \mathcal{O}(N^{-2}), \quad (7)
\]

where \( \langle \tau_k \rangle_0 = \mathcal{O}(N^0). \) There is no term of order \( N^{-1} \) in the absence of \( T \). From Eqs. (5)–(7) we obtain the \( 1/N \) expansion of the average conductance,

\[
\frac{h}{4e^2} \langle G \rangle = \frac{N \langle \tau_1 \rangle_0}{2 - \langle \tau_1 \rangle_0} - \frac{4\langle \tau_1 \rangle_0 - 2\langle \tau_1 \rangle_0^2 + 2\langle \tau_1 \rangle_0^3 - 4\langle \tau_2 \rangle_0}{\langle \tau_1 \rangle_0(2 - \langle \tau_1 \rangle_0)^3} + \mathcal{O}(N^{-1}). \quad (8)
\]

Eq. (8) is generally valid for any isotropic distribution of the scattering matrix. We apply it to the case of a disordered wire in the limit \( N \rightarrow \infty, \ell/L \rightarrow 0 \) at constant \( N\ell/L \). The moments \( \langle \tau_k \rangle_0 \) are given by

\[
\langle \tau_1 \rangle_0 = \ell/L, \quad \langle \tau_2 \rangle_0 = \frac{2}{3}\ell/L. \quad (9)
\]

Substitution into Eq. (8) yields the weak-localization correction \( \delta G = -\frac{2}{3}e^2/h \), cf. Table 1.

(2) \( T \), no \( D \) — In this case one has \( u^\dagger(-eV) = u(eV) \) and \( u(eV) \) is uniformly distributed in \( U(N) \). A calculation similar to that in the previous paragraph yields for the average over \( u \):
where we have abbreviated \( \tau_{k \pm} = \tau_k(\pm eV) \). The next step is the average over the transmission eigenvalues. We may still use Eq. (6), and we note that \( \langle \tau_k(\varepsilon) \rangle \equiv \langle \tau_k \rangle \) is independent of \( \varepsilon \). (The energy scale for variations in \( \langle \tau_k(\varepsilon) \rangle \) is \( E_F \), which is much greater than the energy scale of interest \( E_c \).) Instead of Eq. (7) we now have the \( 1/N \) expansion

\[
\langle \tau_k \rangle = \langle \tau_k \rangle_0 + N^{-1} \delta \tau_k + \mathcal{O}(N^{-2}),
\]

which contains also a term of order \( N^{-1} \) because of the presence of \( \mathcal{T} \). The \( 1/N \) expansion of \( \langle G \rangle \) becomes

\[
\frac{\hbar}{4e^2} \langle G \rangle_u = \frac{N\langle \tau_1 \rangle_0}{2 - \langle \tau_1 \rangle_0} + \frac{2\delta \tau_1}{(2 - \langle \tau_1 \rangle_0)^2} + \frac{2\langle \tau_1 \rangle_0^3 - 2\langle \tau_1 \rangle_0^3 - 2\langle \tau_2 \rangle_0 + 2\langle \tau_1 \rangle_0\langle \tau_2 \rangle_0 + \mathcal{O}(N^{-1})}{\langle \tau_1 \rangle_0(2 - \langle \tau_1 \rangle_0)^3} + \mathcal{O}(N^{-1}).
\]

For the application to a disordered wire we use again Eq. (3) for the moments \( \langle \tau_k \rangle_0 \), which do not depend on whether \( \mathcal{T} \) is broken or not. We also need \( \delta \tau_1 \), which in the presence of \( \mathcal{T} \) is given by \( \delta \tau_1 = -\frac{1}{3} \). Substitution into Eq. (12) yields \( \delta G = -\frac{4e^2}{h} \), cf. Table I.

(3) no \( \mathcal{T} \), no \( \mathcal{D} \) — Now \( u(eV) \) and \( u(-eV) \) are independent, each with a uniform distribution in \( \mathcal{U}(N) \). Carrying out the two averages over \( \mathcal{U}(N) \) we find

\[
\frac{\hbar}{4e^2} \langle G \rangle_u = \frac{N\langle \tau_1 \rangle_0}{\tau_1 \pm \tau_1 \mp \tau_1 \pm \tau_1 \pm \tau_1}.
\]

The average over the transmission eigenvalues becomes

\[
\frac{\hbar}{4e^2} \langle G \rangle = \frac{N\langle \tau_1 \rangle_0}{2 - \langle \tau_1 \rangle_0} + \mathcal{O}(N^{-1}),
\]

where we have used that \( \delta \tau_1 = 0 \) because of the absence of \( \mathcal{T} \). We conclude that \( \delta G = 0 \) in this case, as indicated in Table I.

This completes the calculation of the weak-localization correction to the average conductance. Our results, summarized in Table I, imply a universal \( B \) and \( V \)-dependence of the conductance of an NS microbridge. Raising first \( B \) and then \( V \) leads to two subsequent increases of the conductance, while raising first \( V \) and then \( B \) leads first to a decrease and then to an increase.

So far we have only considered the \( \mathcal{O}(N^0) \) correction \( \delta G \) to \( \langle G \rangle = G_0 + \delta G \). What about the \( \mathcal{O}(N) \) term \( G_0 \)? From Eqs. (8), (12), and (14) we see that if either \( \mathcal{T} \) or \( \mathcal{D} \) (or both) are broken,

\[
G_0 = \frac{4e^2}{\hbar} \frac{N\langle \tau_1 \rangle_0}{2 - \langle \tau_1 \rangle_0} = (2e^2/h) N \left( \frac{1}{2} + L/\ell \right)^{-1}.
\]

In the second equality we substituted \( \langle \tau_1 \rangle_0 = (1 + L/\ell)^{-1} \), which in the limit \( \ell/L \to 0 \) reduces to Eq. (5). If both \( \mathcal{T} \) and \( \mathcal{D} \) are unbroken, then we have instead the result

\[
G_0 = \frac{4e^2}{\hbar} \frac{1}{2 - \langle \tau_1 \rangle_0} = 4e^2/h.
\]
\[ G_0 = \left(2e^2/h\right) N \left[1 + L/\ell + \mathcal{O}(\ell/L)\right]^{-1}. \]  

The difference between Eqs. (15) and (16) is a contact resistance, which equals \( h/4Ne^2 \) in Eq. (15) but is twice as large in Eq. (16). In contrast, in a normal-metal wire the contact resistance is \( h/2Ne^2 \), independent of \( B \) or \( V \). The \( B \) and \( V \)-dependent contact resistance in an NS junction is superimposed on the \( B \) and \( V \)-dependent weak-localization correction. Since the contribution to \( \langle G \rangle \) from the contact resistance is of order \( (e^2/h)N(\ell/L)^2 \), while the weak-localization correction is of order \( e^2/h \), the former can only be ignored if \( N(\ell/L)^2 \ll 1 \). This is an effective restriction to the diffusive metallic regime, where \( \ell/L \ll 1 \) and \( N\ell/L \gg 1 \). To measure the weak-localization effect without contamination from the contact resistance if \( N(\ell/L)^2 \) is not \( \ll 1 \), one has two options: (1) Measure the \( B \)-dependence at fixed \( V \gg E_c/e \); (2) Measure the \( V \)-dependence at fixed \( B \gg B_c \). In both cases we predict an increase of the conductance, by an amount \( 4e^2/h \) and \( 2e^2/h \), respectively. In contrast, in the normal state weak localization leads to a \( B \)-dependence, but not to a \( V \)-dependence.

We performed numerical simulations similar to those of Ref. [8] in order to test the analytical predictions. The disordered normal region was modeled by a tight-binding Hamiltonian on a square lattice (lattice constant \( a \)), with a random impurity potential at each site (uniformly distributed between \( \pm 1/2U_o \)). The Fermi energy was chosen at \( E_F = 1.57U_0 \) from the band bottom \( (U_0 = h^2/(2ma^2)) \). The length \( L \) and width \( W \) of the disordered region are \( L = 167a \), \( W = 35a \), corresponding to \( N = 15 \) propagating modes at \( E_F \). The mean free path is obtained from the conductance \( G = \left(2e^2/h\right)N(1 + L/\ell)^{-1} \) of the normal region in the absence of \( \mathcal{T} \). The scattering matrix of the normal region was computed numerically at \( \epsilon = \pm eV \), and then substituted into Eq. (2) to obtain the differential conductance.

In Fig. 1, we show the \( V \)-dependence of \( G \) (averaged over some \( 10^3 \) impurity configurations), for three values of \( \ell \). The left panel is for \( B = 0 \), the right panel for a flux of \( 6h/e \) through the disordered region. The \( V \)-dependence for \( B = 0 \) is mainly due to the contact resistance effect of order \( N(\ell/L)^2 \), and indeed one sees that the amount by which \( G \) increases depends significantly on \( \ell \). The \( V \)-dependence in a \( \mathcal{T} \)-violating magnetic field is entirely due to the weak-localization effect, which should be insensitive to \( \ell \) (as long as \( \ell/L \ll 1 \ll N\ell/L \)). This is indeed observed in the simulation. Quantitatively, we would expect that application of a voltage increases \( \langle G \rangle \) by an amount \( 2e^2/h \) for the three curves in the right panel, which agrees very well with what is observed. In the absence of a magnetic field the analytical calculation predicts a net increase in \( \langle G \rangle \) by 0.79, 0.46, and 0.25 \( \times e^2/h \) (from top to bottom), which is again in good agreement with the simulation.

In conclusion, we have shown that weak localization can coexist with a time-reversal symmetry breaking magnetic field in a disordered normal-metal–superconductor junction. One needs to apply a magnetic field and to raise the voltage in order to suppress the weak-localization correction to the conductance. This leads to an unusual \( B \) and \( V \)-dependence of the differential conductance, which should be observable in experiments.

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FIGURES

FIG. 1. Numerical simulation of the voltage dependence of the average differential conductance for $B = 0$ (left panel) and for a flux $6 \hbar/e$ through the disordered normal region (right panel). The filled circles are for an NS junction; the open circles represent the $V$-independent conductance in the normal state. The three sets of data points correspond, from top to bottom, to $\ell/L = 0.31$, 0.23, and 0.18, respectively. The arrows indicate the theoretically predicted net increase of $\langle G \rangle$ between $V = 0$ and $V \gg E_c/e$. 
