ON ANDREWS’ INTEGER PARTITIONS WITH EVEN PARTS BELOW ODD PARTS

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Abstract. Recently, Andrews defined a partition function $EO(n)$ which counts the number of partitions of $n$ in which every even part is less than each odd part. He also defined a partition function $EO(n)$ which counts the number of partitions of $n$ enumerated by $EO(n)$ in which only the largest even part appears an odd number of times. Andrews proposed to undertake a more extensive investigation of the properties of $EO(n)$. In this article, we prove infinite families of congruences for $EO(n)$. We next study distribution of $EO(n)$. We prove that there are infinitely many integers $N$ in every arithmetic progression for which $EO(2N)$ is even; and that there are infinitely many integers $M$ in every arithmetic progression for which $EO(2M)$ is odd so long as there is at least one. We further prove that $EO(n)$ is even for almost all $n$. Very recently, Uncu has treated a different subset of the partitions enumerated by $EO(n)$. We prove that Uncu’s partition function is divisible by $2^k$ for almost all $k$. We use arithmetic properties of modular forms and Hecke eigenforms to prove our results.

1. Introduction and statement of results

A partition of a nonnegative integer $n$ is a nonincreasing sequence of positive integers whose sum is $n$. In a recent paper, Andrews \cite{1} studied the partition function $EO(n)$ which counts the number of partitions of $n$ where every even part is less than each odd part. He denoted by $EO(n)$, the number of partitions counted by $EO(n)$ in which only the largest even part appears an odd number of times.

For example, $EO(8) = 12$ with the relevant partitions being $8, 6+2, 7+1, 4+4, 4+2+2, 5+3, 5+1+1+1, 2+2+2, 3+3+3+3+1+1, 3+1+1+1+1+1+1$; and $EO(8) = 5$, with the relevant partitions being $8, 4+2+2, 3+3+2, 3+3+1+1, 1+1+1+1+1+1+1$.

Andrews proved that the partition function $EO(n)$ has the following generating function \cite{1, Eqn. (3.2)}:

\begin{equation}
\sum_{n=0}^{\infty} EO(n)q^n = (q^4; q^4)_{\infty} = (q^4; q^4)_{\infty}^3 \frac{(q^2; q^2)_{\infty}^2}{(q; q^2)_{\infty}^2},
\end{equation}

where $(a; q)_{\infty} := \prod_{n \geq 0} (1 - aq^n)$. In the same paper, he proposed to undertake a more extensive investigation of the properties of $EO(n)$. The objective of this

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paper is to study divisibility properties of $\mathcal{EO}(n)$. To be specific, we use the theory of Hecke eigenforms to establish the following two infinite families of congruences for $\mathcal{EO}(n)$ modulo 2 and 8, respectively.

**Theorem 1.1.** Let $k, n$ be nonnegative integers. For each $i$ with $1 \leq i \leq k + 1$, if $p_i \geq 5$ is a prime such that $p_i \equiv 2 \pmod{3}$, then for any integer $j \neq 0 \pmod{p_{k+1}}$

$$\mathcal{EO}\left(p_1^2 \cdots p_{k+1}^2 n + \frac{p_1^2 \cdots p_{k+1}^2 (3j + p_{k+1}) - 1}{3}\right) \equiv 0 \pmod{2}.$$

Let $p \geq 5$ be a prime such that $p \equiv 2 \pmod{3}$. By taking all the primes $p_1, p_2, \ldots, p_{k+1}$ to be equal to the same prime $p$ in Theorem 1.1, we obtain the following infinite family of congruences for $\mathcal{EO}(n)$:

$$\mathcal{EO}\left(p^{2k+1} n + p^{2k+1} j + \frac{p^{2k+1} - 1}{3}\right) \equiv 0 \pmod{2},$$

where $j \neq 0 \pmod{p}$. In particular, for all $n \geq 0$ and $j \neq 0 \pmod{5}$, we have

$$\mathcal{EO}(25n + 5j + 8) \equiv 0 \pmod{2}.$$

**Theorem 1.2.** Let $k, n$ be nonnegative integers. For each $i$ with $1 \leq i \leq k + 1$, if $p_i \equiv 1 \pmod{3}$ is a prime such that $\mathcal{EO}\left(\frac{19p_i - 1}{3}\right) \equiv 0 \pmod{8}$, then for any integer $j \neq 0 \pmod{p_{k+1}}$

$$\mathcal{EO}\left(8p_1^2 \cdots p_{k+1}^2 n + \frac{p_1^2 \cdots p_{k+1}^2 (24j + 19p_{k+1}) - 1}{3}\right) \equiv 0 \pmod{8}.$$

Let $p$ be a prime such that $p \equiv 1 \pmod{3}$ and $\mathcal{EO}\left(\frac{19p - 1}{3}\right) \equiv 0 \pmod{8}$. By taking all the primes $p_1, p_2, \ldots, p_{k+1}$ to be equal to the same prime $p$ in Theorem 1.2, we obtain the following infinite family of congruences for $\mathcal{EO}(n)$:

$$\mathcal{EO}\left(8p^{2k+1} n + 8p^{2k+1} j + \frac{19p^{2k+1} - 1}{3}\right) \equiv 0 \pmod{8},$$

where $j \neq 0 \pmod{p}$. In particular, if we choose $p = 1009$, then $1009 \equiv 1 \pmod{24}$ and $\frac{19 \times 1009 - 1}{3} = 6390$. Using *Mathematica* we verify that $\mathcal{EO}(6390) \equiv 0 \pmod{8}$. Thus, for all $n \geq 0$ and $j \neq 0 \pmod{1009}$, we have

$$\mathcal{EO}(8144648n + 8072j + 6447846) \equiv 0 \pmod{8}.$$

In [1], Andrews proved that, for all $n \geq 0$

$$\mathcal{EO}(10n + 8) \equiv 0 \pmod{5}.$$

In this article, we prove that the congruence (1.2) is also true modulo 4 if $n \neq 0 \pmod{5}$. To be specific, we prove the following result.

**Theorem 1.3.** Let $t \in \{1, 2, 3, 4\}$. Then for all $n \geq 0$ we have

$$\mathcal{EO}(10(5n + t) + 8) \equiv 0 \pmod{40}.$$

We note that Theorem 1.3 is not true if $t = 0$. For example, $\mathcal{EO}(8)$ is not divisible by 4.

For a nonnegative integer $n$, let $p(n)$ denote the number of partitions of $n$. In [12], Ono proved that there are infinitely many integers $N$ in every arithmetic progression for which $p(N)$ is even; and that there are infinitely many integers $M$ in every arithmetic progression for which $p(M)$ is odd so long as there is at least one. Ono’s result gave an affirmative answer to a well-known conjecture on parity
of $p(n)$ in an arithmetic progression. In the following theorem, we prove the same for the partition function $\mathcal{EO}(n)$. We note that $\mathcal{EO}(2n + 1) = 0$ for all $n \geq 0$.

**Theorem 1.4.** For any arithmetic progression $r \pmod{t}$, there are infinitely many integers $N \equiv r \pmod{t}$ for which $\mathcal{EO}(2N)$ is even. Also, for any arithmetic progression $r \pmod{t}$, there are infinitely many integers $M \equiv r \pmod{t}$ for which $\mathcal{EO}(2M)$ is odd, provided there is one such $M$. Furthermore, if there does exist an $M \equiv r \pmod{t}$ for which $\mathcal{EO}(2M)$ is odd, then the smallest such $M$ is less than

$$\frac{2^{9 + 3^2 t^6}}{d^2} \prod_{p \nmid dt} \left(1 - \frac{1}{p^2}\right) - 2^j,$$

where $d = \gcd(12r - 1, t)$ and $2^j > \frac{t}{12}$.

A well-known conjecture of Parkin and Shanks [13] states that the even and odd values of $p(n)$ are equally distributed, that is,

$$\lim_{X \to \infty} \frac{\# \{0 \leq n \leq X : p(n) \equiv r \pmod{2}\}}{X} = \frac{1}{2},$$

where $r \in \{0, 1\}$. Little is known regarding this conjecture. In the following theorem we prove that $\mathcal{EO}(2n)$ is almost always even.

**Theorem 1.5.** Let $n \geq 0$. Then $\mathcal{EO}(8n + 6)$ is almost always divisible by 8, namely,

$$\lim_{X \to \infty} \frac{\# \{0 \leq n \leq X : \mathcal{EO}(8n + 6) \equiv 0 \pmod{8}\}}{X} = 1.$$

Recently, Uncu [17] has treated a different subset of the partitions enumerated by $\mathcal{EO}(n)$. Also see [1, p. 435]. We denote by $\mathcal{EO}_u(n)$ the partition function defined by Uncu, and the generating function is given by

$$\sum_{n=0}^{\infty} \mathcal{EO}_u(n)q^n = \frac{1}{(q^2; q^4)^2}. \quad (1.3)$$

For any fixed positive integer $k$, Gordon and Ono [3] proved that the number of partitions of $n$ into distinct parts is divisible by $2^k$ for almost all $n$. Similar studies are done for some other partition functions, for example see [2, 4, 9, 16]. In this article, we study divisibility of the partition function $\mathcal{EO}_u(n)$ by $2^k$. To be specific, we prove the following result.

**Theorem 1.6.** Let $k$ be a positive integer. Then $\mathcal{EO}_u(2n)$ is almost always divisible by $2^k$, namely,

$$\lim_{X \to \infty} \frac{\# \{0 \leq n \leq X : \mathcal{EO}_u(2n) \equiv 0 \pmod{2^k}\}}{X} = 1.$$
2. Preliminaries

In this section, we recall some definitions and basic facts on modular forms. For more details, see for example [11, 7]. We first define the matrix groups

\[ \text{SL}_2(\mathbb{Z}) := \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in \mathbb{Z}, ad - bc = 1 \right\}, \]

\[ \Gamma_\infty := \left\{ \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix} : n \in \mathbb{Z} \right\}, \]

\[ \Gamma_0(N) := \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \text{SL}_2(\mathbb{Z}) : c \equiv 0 \pmod{N} \right\}, \]

\[ \Gamma_1(N) := \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma_0(N) : a \equiv d \equiv 1 \pmod{N} \right\}, \]

and

\[ \Gamma(N) := \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \text{SL}_2(\mathbb{Z}) : a \equiv d \equiv 1 \pmod{N}, \text{ and } b \equiv c \equiv 0 \pmod{N} \right\}, \]

where \( N \) is a positive integer. A subgroup \( \Gamma \) of \( \text{SL}_2(\mathbb{Z}) \) is called a congruence subgroup if \( \Gamma(N) \subseteq \Gamma \) for some \( N \). The smallest \( N \) such that \( \Gamma(N) \subseteq \Gamma \) is called the level of \( \Gamma \). For example, \( \Gamma_0(N) \) and \( \Gamma_1(N) \) are congruence subgroups of level \( N \). The index of \( \Gamma_0(N) \) in \( \text{SL}_2(\mathbb{Z}) \) is

\[ [\text{SL}_2(\mathbb{Z}) : \Gamma_0(N)] = N \prod_{p|N} (1 + p^{-1}), \]

where \( p \) denotes a prime.

Let \( \mathbb{H} := \{ z \in \mathbb{C} : \text{Im}(z) > 0 \} \) be the upper half of the complex plane. The group

\[ \text{GL}_2^+(\mathbb{R}) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in \mathbb{R} \text{ and } ad - bc > 0 \right\} \]

acts on \( \mathbb{H} \) by

\[ \begin{bmatrix} a & b \\ c & d \end{bmatrix} z = \frac{az + b}{cz + d} \]

We identify \( \infty \) with \( \frac{1}{0} \) and define \( \begin{bmatrix} a & b \\ c & d \end{bmatrix} \frac{r}{s} = \frac{ar + bs}{cr + ds} \) where \( \frac{r}{s} \in \mathbb{Q} \cup \{ \infty \} \). This gives an action of \( \text{GL}_2^+(\mathbb{R}) \) on the extended upper half-plane \( \mathbb{H}^* = \mathbb{H} \cup \mathbb{Q} \cup \{ \infty \} \). Suppose that \( \Gamma \) is a congruence subgroup of \( \text{SL}_2(\mathbb{Z}) \). A cusp of \( \Gamma \) is an equivalence class in \( \mathbb{P}^1 = \mathbb{Q} \cup \{ \infty \} \) under the action of \( \Gamma \).

The group \( \text{GL}_2^+(\mathbb{R}) \) also acts on functions \( f : \mathbb{H} \to \mathbb{C} \). In particular, suppose that \( \gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \text{GL}_2^+(\mathbb{R}) \). If \( f(z) \) is a meromorphic function on \( \mathbb{H} \) and \( \ell \) is an integer, then define the slash operator \( |_\ell \) by

\[ (f|_\ell)(z) := (\det \gamma)^{\ell/2} (cz + d)^{-\ell} f(\gamma z). \]

**Definition 2.1.** Let \( \Gamma \) be a congruence subgroup of level \( N \). A holomorphic function \( f : \mathbb{H} \to \mathbb{C} \) is called a modular form with integer weight \( \ell \) on \( \Gamma \) if the following hold:

1. We have

\[ f \left( \frac{az + b}{cz + d} \right) = (cz + d)^\ell f(z) \]

for all \( z \in \mathbb{H} \) and all \( \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma \).
(2) If $\gamma \in \text{SL}_2(\mathbb{Z})$, then $(f|_\gamma)(z)$ has a Fourier expansion of the form
\[
(f|_\gamma)(z) = \sum_{n \geq 0} a_\gamma(n)q^n_N,
\]
where $q_N := e^{2\pi iz/N}$. That is, $f$ is holomorphic at all the cusps of $\Gamma$.

For a positive integer $\ell$, the complex vector space of modular forms of weight $\ell$ with respect to a congruence subgroup $\Gamma$ is denoted by $M_\ell(\Gamma)$. A modular form $f \in M_\ell(\Gamma)$ is called a cusp form if $f$ vanishes at all the cusps of $\Gamma$. The subspace of $M_\ell(\Gamma)$ consisting of cusp forms is denoted by $S_\ell(\Gamma)$.

**Definition 2.2.** [11, Definition 1.15] If $\chi$ is a Dirichlet character modulo $N$, then we say that a modular form $f \in M_\ell(\Gamma_1(N))$ has Nebentypus character $\chi$ if
\[
f\left(\frac{az+b}{cz+d}\right) = \chi(d)(cz+d)^\ell f(z)
\]
for all $z \in \mathbb{H}$ and all $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma_0(N)$. The space of such modular forms is denoted by $M_\ell(\Gamma_0(N), \chi)$. The corresponding space of cusp forms is denoted by $S_\ell(\Gamma_0(N), \chi)$. If $\chi$ is the trivial character then we write $M_\ell(\Gamma_0(N))$ and $S_\ell(\Gamma_0(N))$ for short.

Recall that Dedekind’s eta-function $\eta(z)$ is defined by
\[
\eta(z) := q^{1/24}(q; q)_\infty = q^{1/24} \prod_{n=1}^\infty (1 - q^n),
\]
where $q := e^{2\pi iz}$ and $z \in \mathbb{H}$. A function $f(z)$ is called an eta-quotient if it is of the form
\[
f(z) = \prod_{\delta \mid N} \eta(\delta z)^{r_\delta},
\]
where $N$ is a positive integer and $r_\delta$ is an integer.

We now recall two theorems from [11, p. 18] which will be used to prove our result.

**Theorem 2.3.** [11, Theorem 1.64 and Theorem 1.65] If $f(z) = \prod_{\delta \mid N} \eta(\delta z)^{r_\delta}$ is an eta-quotient such that $\ell = \frac{1}{2} \sum_{\delta \mid N} r_\delta \in \mathbb{Z}$,
\[
\sum_{\delta \mid N} \delta r_\delta \equiv 0 \pmod{24}
\]
and
\[
\sum_{\delta \mid N} c^2 r_\delta \equiv 0 \pmod{24},
\]
then $f(z)$ satisfies
\[
f\left(\frac{az+b}{cz+d}\right) = \chi(d)(cz+d)^\ell f(z)
\]
for every $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma_0(N)$. Here the character $\chi$ is defined by $\chi(d) := \left(\frac{-1}{d}\right)^{\sum_{\delta \mid N} \delta^\ell r_\delta}\prod_{d \mid N} \frac{\gcd(d, \delta)^{r_\delta}}{\gcd(d, \delta)^{r_\delta d}}$.

In addition, if $c, d$, and $N$ are positive integers with $d \mid N$ and $\gcd(c, d) = 1$, then the order of vanishing of $f(z)$ at the cusp $\frac{a}{d}$ is $\frac{N}{24} \sum_{\delta \mid N} \frac{\gcd(d, \delta)^{r_\delta}}{\gcd(d, \delta)^{r_\delta d}}$. 
Suppose that $f(z)$ is an eta-quotient satisfying the conditions of Theorem 2.3. If $f(z)$ is holomorphic at all of the cusps of $\Gamma_0(N)$, then $f(z) \in M_\ell(\Gamma_0(N), \chi)$.

**Definition 2.4.** Let $m$ be a positive integer and $f(z) = \sum_{n=0}^{\infty} a(n)q^n \in M_\ell(\Gamma_0(N), \chi)$. Then the action of Hecke operator $T_m$ on $f(z)$ is defined by

$$f(z)|T_m := \sum_{n=0}^{\infty} \left( \sum_{d | \gcd(n,m)} \chi(d) d^{\ell-1} a \left( \frac{nm}{d^2} \right) \right) q^n.$$ 

In particular, if $m = p$ is prime, we have

$$(2.1) f(z)|T_p := \sum_{n=0}^{\infty} \left( a(pm) + \chi(p)p^{\ell-1} a \left( \frac{n}{p} \right) \right) q^n.$$ 

We note that $a(n) = 0$ unless $n$ is a nonnegative integer.

**Definition 2.5.** A modular form $f(z) = \sum_{n=0}^{\infty} a(n)q^n \in M_\ell(\Gamma_0(N), \chi)$ is called a Hecke eigenform if for every $m \geq 2$ there exists a complex number $\lambda(m)$ for which

$$f(z)|T_m = \lambda(m)f(z).$$ 

(2.2)

3. **Proof of Theorems 1.1 and 1.2**

We use the theory of Hecke eigenforms to prove Theorems 1.1 and 1.2.

**Proof of Theorem 1.1.** We have

$$\sum_{n=0}^{\infty} \mathcal{EO}(n)q^n = \frac{(q^4; q^4)_\infty^3}{(q^2; q^2)_\infty^2} \equiv (q; q)_{\infty}^8 \quad (\text{mod } 2).$$

This gives

$$\sum_{n=0}^{\infty} \mathcal{EO}(n)q^{3n+1} \equiv \eta^8(3z) \quad (\text{mod } 2).$$

Let $\eta^8(3z) = \sum_{n=1}^{\infty} a(n)q^n$. Then $a(n) = 0$ if $n \not\equiv 1 \pmod{3}$ and for all $n \geq 0$,

$$(3.1) \mathcal{EO}(n) \equiv a(3n+1) \quad (\text{mod } 2).$$

By Theorem 2.3, we have $\eta^8(3z) \in S_4(\Gamma_0(9))$. Since $\eta^8(3z)$ is a Hecke eigenform (see, for example [11]), (2.1) and (2.2) yield

$$\eta^8(3z)|T_p = \sum_{n=1}^{\infty} \left( a(pm) + p^3 a \left( \frac{n}{p} \right) \right) q^n = \lambda(p) \sum_{n=1}^{\infty} a(n)q^n,$$

which implies

$$(3.2) a(pm) + p^3 a \left( \frac{n}{p} \right) = \lambda(p)a(n).$$

Putting $n = 1$ and noting that $a(1) = 1$, we readily obtain $a(p) = \lambda(p)$. Since $a(p) = 0$ for all $p \not\equiv 1 \pmod{3}$, we have $\lambda(p) = 0$. From (3.2), we obtain

$$(3.3) a(pm) + p^3 a \left( \frac{n}{p} \right) = 0.$$

From (3.3), we derive that for all $n \geq 0$ and $p \nmid r$,

$$(3.4) a(p^2n + pr) = 0$$
and
\begin{equation}
(p^2n) = -p^3a(n) \equiv a(n) \pmod{2}.
\end{equation}

Substituting \(n\) by \(3n - pr + 1\) in (3.6) and together with (3.4), we find that
\begin{equation}
\mathcal{E}\mathcal{O}\left(p^2n + \frac{p^2 - 1}{3} + pr\frac{1 - p^2}{3}\right) \equiv 0 \pmod{2}.
\end{equation}

Substituting \(n\) by \(3n + 1\) in (3.3) and using (3.1), we obtain
\begin{equation}
\mathcal{E}\mathcal{O}\left(p^2n + \frac{p^2 - 1}{3}\right) \equiv \mathcal{E}\mathcal{O}(n) \pmod{2}.
\end{equation}

Since \(p \geq 5\) is prime, so \(3 \mid (1 - p^2)\) and \(\gcd\left(\frac{1 - p^2}{3}, p\right) = 1\). Hence when \(r\) runs over a residue system excluding the multiple of \(p\), so does \(\frac{1 - p^2}{3} - r\). Thus (3.6) can be rewritten as
\begin{equation}
\mathcal{E}\mathcal{O}\left(p^2n + \frac{p^2 - 1}{3} + pj\right) \equiv 0 \pmod{2},
\end{equation}
where \(p \nmid j\).

Now, \(p_i \geq 5\) are primes such that \(p_i \equiv 1 \pmod{3}\). Since
\[
p_1^2 \ldots p_k^2 n + \frac{p_1^2 \ldots p_k - 1}{3} = p_1^2 \left(p_2^2 \ldots p_k^2 n + \frac{p_2^2 \ldots p_k - 1}{3}\right) + \frac{p_1^2 - 1}{3},
\]
using (3.7) repeatedly we obtain that
\begin{equation}
\mathcal{E}\mathcal{O}\left(p_1^2 \ldots p_k^2 n + \frac{p_1^2 \ldots p_k - 1}{3}\right) \equiv \mathcal{E}\mathcal{O}(n) \pmod{2}.
\end{equation}

Let \(j \not\equiv 0 \pmod{p_{k+1}}\). Then (3.8) and (3.9) yield
\[
\mathcal{E}\mathcal{O}\left(p_1^2 \ldots p_{k+1}^2 + \frac{p_1^2 \ldots p_{k+1} - 1}{3}(3j + p_{k+1})\right) \equiv 0 \pmod{2}.
\]

This completes the proof of the theorem. \(\square\)

To prove Theorem 1.2 we need that the eta-quotient \(\eta^5(96z)/\eta(24z)\) is an eigenform for the Hecke operators \(T_p\), where \(p \equiv 1 \pmod{24}\). This has been observed to be true by Scott Ahlgren. We now present below the proof given by Ahlgren which was communicated to us through an email. Let \(F_1 = \eta^5(24z)/\eta(96z), \ F_7 = \eta^5(24z)/\eta(96z), \ F_{13} = \eta(24z)/\eta(96z), \ F_{19} = \eta^5(96z)/\eta(24z)\). Then \(F_j\) is supported on exponents congruent to \(j \pmod{24}\). The Hecke operators \(T_p\) for \(p \equiv 5, 11, 17, 23 \pmod{24}\) annihilate each of these forms. The Hecke operators \(T_p\) for \(p \equiv 1, 5, 13, 19 \pmod{24}\) map \(F_j\) to a multiple of \(F'_j\), where \(j' \equiv pj \pmod{24}\). It turns out that a linear combination of the forms \(F_j\) is an eigenform of all of the Hecke operators. In [8, p. 209], equation (13.84) expresses the linear combination as an eigenform. Since the \(F_j\) are supported on distinct classes of coefficients, it follows that \(F_j\) are eigenforms of all the Hecke operators.

**Proof of Theorem 1.2** We first recall the following 2-dissection formula from [8, Entry 25, p. 40]:
\begin{equation}
\frac{1}{(q;q)\infty} = \frac{(q^3; q^3)\infty}{(q^2; q^2)\infty} + 2q^4(q^2; q^2)\infty(q^{16}; q^{16})\infty + 2q(q^4; q^4)\infty(q^{16}; q^{16})\infty.
\end{equation}
From (1.1), we have
\[
\sum_{n=0}^{\infty} EO(2n)q^n = \frac{(q^2;q^2)_\infty}{(q;q)_\infty^2}.
\]
Combining (5.1) and (5.2), and then extracting the terms with odd powers of \(q\), we deduce that
\[
\sum_{n=0}^{\infty} EO(4n+2)q^n = 2\frac{(q^2;q^2)_\infty^2(q^8;q^8)_\infty}{(q;q)_\infty^2(q^4;q^4)_\infty}.
\]
We again combine (5.1) and (5.3), and then extract the terms with odd powers of \(q\) to obtain
\[
\sum_{n=0}^{\infty} EO(8n+6)q^n = 4\frac{(q^4;q^4)_\infty^3(q^8;q^8)_\infty^2}{(q;q)_\infty^4}.
\]
Since \((q;q)_\infty^2 \equiv (q^2;q^2)_\infty^2 \pmod{2}\), we have
\[
\sum_{n=0}^{\infty} EO(8n+6)q^n \equiv 4\frac{(q^4;q^4)_\infty^3}{(q;q)_\infty^4} \pmod{8}.
\]
This gives
\[
\sum_{n=0}^{\infty} EO(8n+6)q^{24n+19} \equiv 4\frac{\eta(96z)^5}{\eta(24z)} \pmod{8}.
\]
Let \(\eta(96z)^5 = \sum_{n=1}^{\infty} a(n)q^n.\) It is clear that \(a(n) = 0\) if \(n \not\equiv 19 \pmod{24}\). Also, for all \(n \geq 0\),
\[
\sum_{n=1}^{\infty} a(n)q^n.\]
By Theorem 2.3, we have \(\eta(96z)^5 \in S_2(\Gamma_0(2304)).\) Since \(\eta(96z)^5\) is a Hecke eigenform for the Hecke operator \(T_p\), where \(p \equiv 1 \pmod{24}\), (2.1) and (2.2) yield
\[
a(pm) + p\left(\frac{2}{p}\right)a\left(\frac{n}{p}\right) = \lambda(p)a(n).
\]
Putting \(n = 19\) in (3.14) and noting that \(p \not\equiv 19 \pmod{24}\), we obtain \(a(19p) = \lambda(p)a(19).\) Also, \(a(19) = 1\), and hence \(a(19p) = \lambda(p).\) Thus (3.14) gives
\[
a(pm) + p\left(\frac{2}{p}\right)a\left(\frac{n}{p}\right) = a(19p)a(n).
\]
From (3.15), we obtain that for all \(n \geq 0\) and \(p \nmid r,\)
\[
a(p^2n) + a(n) \equiv a(19p)a(pm) \pmod{2}
\]
and
\[
a(p^2n + pr) = a(19p)a(pm + r).
\]
Let \(A(n) = a(24n+19).\) Let \(p\) be a prime such that \(p \equiv 1 \pmod{24}\). Now, replacing \(n\) by \(24n - pr + 19\) in (3.17), we obtain
\[
A\left(p^2n + 19\frac{p^2 - 1}{24} + pr\frac{1 - p^2}{24}\right) = A\left(19\frac{p - 1}{24}\right) A\left(pm + 19\frac{p - 1}{24} + r\frac{1 - p^2}{24}\right).
\]
From our hypothesis, we have (3.22)

Replacing $n$ by $24n + 19$ in (3.19), we have, modulo 2

Let $p$ be such that $\Omega (19n-1) \equiv 0 \pmod{8}$. Then, using the relation $\Omega (8n+6) \equiv 4A(n) \pmod{8}$, we have $A (19n-1) \equiv 0 \pmod{2}$. Hence, (3.19) and (3.20) imply

\[
A \left( p^2n + 19p^2 - \frac{1}{24} + pj \right) \equiv 0 \pmod{2}
\]

and

\[
A \left( p^2n + 19p^2 - \frac{1}{24} \right) \equiv A(n) \pmod{2}.
\]

From our hypothesis, we have $p_i \geq 5$ are primes such that $p_i \equiv 1 \pmod{24}$ and $A (19n-1) \equiv 0 \pmod{2}$. Now, using (3.22) we deduce that

\[
A \left( p_1^2 \ldots p_k^2n + 19p_1^2 \ldots p_k^2 - \frac{1}{24} \right) \equiv A(n) \pmod{2}.
\]

Replacing $n$ by $p_{k+1}^2n + 19p_{k+1}^2 - \frac{1}{24} + p_{k+1}j$, and then using (3.21) we obtain

\[
A \left( p_1^2 \ldots p_k^2p_{k+1}^2n + 19p_1^2 \ldots p_k^2p_{k+1} - \frac{1}{24} + p_1^2 \ldots p_k^2p_{k+1}j \right) \equiv 0 \pmod{2}.
\]

We complete the proof by using the fact that $\Omega (8n+6) \equiv 4A(n) \pmod{8}$. \qed

4. PROOF OF THEOREM 1.3

We prove Theorem 1.3 using the approach developed in [14, 15]. To this end, we first recall some definitions and results from [14, 15]. For a positive integer $M$, let $R(M)$ be the set of integer sequences $r = (r_\delta)_{\delta | M}$ indexed by the positive divisors of $M$. If $r \in R(M)$ and $1 = \delta_1 < \delta_2 < \cdots < \delta_k = M$ are the positive divisors of $M$, we write $r = (r_{\delta_1}, \ldots, r_{\delta_k})$. Define $c_r(n)$ by

\[
\sum_{n=0}^{\infty} c_r(n)q^n := \prod_{\delta | M} (q^\delta; q^\delta)_\infty^{r_\delta} = \prod_{\delta | M} \prod_{n=1}^{\infty} (1 - q^{n\delta})^{r_\delta}.
\]

The approach to proving congruences for $c_r(n)$ developed by Radu [14, 15] reduces the number of cases that one must check as compared with the classical method which uses Sturm’s bound alone.

Let $m$ be a positive integer. For any integer $s$, let $[s]_m$ denote the residue class of $s$ in $\mathbb{Z}_m := \mathbb{Z}/m\mathbb{Z}$. Let $\mathbb{Z}_m^*$ be the set of all invertible elements in $\mathbb{Z}_m$. Let $S_m \subseteq \mathbb{Z}_m$
be the set of all squares in $\mathbb{Z}_m^\ast$. For $t \in \{0,1,\ldots,m-1\}$ and $r \in R(M)$, we define a subset $P_{m,r}(t) \subseteq \{0,1,\ldots,m-1\}$ by

$$P_{m,r}(t) := \left\{ t' : \exists [s]_{24m} \in \mathbb{S}_{24m} \text{ such that } t' \equiv ts + \frac{s-1}{24} \sum_{\delta | M} \delta r_\delta \pmod{m} \right\}.$$ 

**Definition 4.1.** Suppose $m, M$ and $N$ are positive integers, $r = (r_\delta) \in R(M)$ and $t \in \{0,1,\ldots,m-1\}$. Let $k = k(m) := \gcd(m^2 - 1, 24)$ and write

$$\prod_{\delta | M} \delta^{\nu} = 2^s \cdot j,$$

where $s$ and $j$ are nonnegative integers with $j$ odd. The set $\Delta^\ast$ consists of all tuples $(m, M, N, (r_\delta), t)$ satisfying these conditions and all of the following.

1. Each prime divisor of $m$ is also a divisor of $N$.
2. $\delta \mid M$ implies $\delta \mid mN$ for every $\delta \geq 1$ such that $r_\delta \neq 0$.
3. $kN \sum_{\delta | M} r_\delta mN/\delta \equiv 0 \pmod{24}$.
4. $kN \sum_{\delta | M} r_\delta \equiv 0 \pmod{8}$.
5. $\gcd(-24k - 2\sum_{\delta | M} \delta r_\delta, 24m)$ divides $N$.
6. If $2 \mid m$, then either $4 \mid kN$ and $8 \mid sN$ or $2 \mid s$ and $8 \mid (1-j)N$.

Throughout this section we take $\Gamma = \text{SL}_2(\mathbb{Z})$. Let $m, M, N$ be positive integers.

For $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma$, $r \in R(M)$ and $r' \in R(N)$, set

$$p_{m,r}(\gamma) := \min_{\lambda \in \{0,1,\ldots,m-1\}} \frac{1}{24} \sum_{\delta | M} r_\delta \gcd(\delta a + \delta k\lambda c, mc) \frac{\delta m}{\delta}$$

and

$$p_{r'}(\gamma) := \frac{1}{24} \sum_{\delta | N} r'_\delta \gcd(\delta, c) \frac{\delta}{\delta}.$$ 

**Lemma 4.2.** [14 Lemma 4.5] Let $u$ be a positive integer, $(m, M, N, r = (r_\delta), t) \in \Delta^\ast$ and $r' = (r'_\delta) \in R(N)$. Let $\{\gamma_1, \gamma_2, \ldots, \gamma_n\} \subseteq \Gamma$ be a complete set of representatives of the double cosets of $\Gamma_0(N) \backslash \Gamma / \Gamma_\infty$. Assume that $p_{m,r}(\gamma_i) + p_{r'}(\gamma_i) \geq 0$ for all $1 \leq i \leq n$. Let $t_{\min} = \min_{\nu \in \nu_{m,r}(t)} t'$ and

$$\nu := \frac{1}{24} \left\{ \left( \sum_{\delta | M} r_\delta + \sum_{\delta | N} r'_\delta \right) \left[ \Gamma : \Gamma_0(N) \right] - \sum_{\delta | N} \delta r'_\delta \right\} - \frac{1}{24m} \sum_{\delta | M} \delta r_\delta - \frac{t_{\min}}{m}.$$ 

If the congruence $c_r(\nu + t') \equiv 0 \pmod{u}$ holds for all $t' \in P_{m,r}(t)$ and $0 \leq n \leq \lfloor \nu \rfloor$, then it holds for all $t' \in P_{m,r}(t)$ and $n \geq 0$.

To apply Lemma 4.2 we utilize the following result, which gives a complete set of representatives of the double cosets in $\Gamma_0(N) \backslash \Gamma / \Gamma_\infty$.

**Lemma 4.3.** [18 Lemma 4.3] If $N$ or $\frac{1}{2} N$ is a square-free integer, then

$$\bigcup_{\delta | N} \Gamma_0(N) \left[ \begin{array}{cc} 1 & 0 \\ \delta & 1 \end{array} \right] \Gamma_\infty = \Gamma.$$
Proof of Theorem 1.3. Due to (1.2) we need to prove our congruences modulo 4 only. We have

\[ \sum_{n=0}^{\infty} \overline{EO}(n)q^n = \frac{(q^4; q^4)^3_\infty}{(q^2; q^2)^3_\infty} = \frac{(q^2; q^2)_\infty(q^4; q^4)^3_\infty}{(q^2; q^2)^3_\infty} \equiv \frac{(q^2; q^2)_\infty(q^4; q^4)^3_\infty}{(q^4; q^4)_\infty} \pmod{4} \]

\[ \equiv (q^2; q^2)^3_\infty(q^4; q^4)_\infty \pmod{4}. \]

Let \((m, M, N, r, t) = (50, 8, 10, (0, 2, 1, 0), 18)\). It is easy to verify that \((m, M, N, r, t) \in \Delta^*\) and \(P_{m,r}(t) = \{18, 28, 38, 48\}\). From Lemma 1.3 we know that \(\left\{ \begin{bmatrix} 1 & 0 \\ \delta & 1 \end{bmatrix} : \delta \mid 10 \right\}\) forms a complete set of double coset representatives of \(\Gamma_0(N) \ract \Gamma_\infty\). Let \(r' = (0, 0, 0, 0, 0) \in R(10)\). We have used Sage to verify that \(p_{m,r}(\gamma_0) + p_{r'}(\gamma_0) \geq 0\) for each \(\delta \mid N\), where \(\gamma_0 = \begin{bmatrix} 1 & 0 \\ \delta & 1 \end{bmatrix}\). We compute that the upper bound in Lemma 1.2 is \(|\nu| = 1\). Using Mathematica we verify that \(\overline{EO}(50n + t') \equiv 0 \pmod{4}\) for \(n \leq 1\) and \(t' \in P_{m,r}(t)\). Thus, by Lemma 1.2 we conclude that \(\overline{EO}(50n + t') \equiv 0 \pmod{4}\) for any \(n \geq 0\), where \(t' \in \{18, 28, 38, 48\}\). This completes the proof of the theorem.

5. Proof of Theorems 1.4, 1.5 and 1.6

We prove Theorem 1.4 by using the approach developed in [12]. Recently, Jameson and Wieczorek [6] have done a similar study for the generalized Frobenius partitions. To make this paper self-contained, we recall two results from [6]. Also see [12]. Let \(M_k^1(\Gamma_0(N_0), \chi)\) denote the space of weakly holomorphic modular forms.

**Theorem 5.1.** [6] Theorem 5] Let \(N_0, \alpha, \beta, t\) be integers with \(N_0, \alpha, t\) positive, and let

\[ \sum_{n=0}^{\infty} c(n)q^{\alpha n + \beta} \in M_k^1(\Gamma_0(N_0), \chi), \]

where \(c(n)\) are algebraic integers in some number field. For any arithmetic progression \(r \pmod{t}\), there are infinitely many integers \(N \equiv r \pmod{t}\) for which \(c(N)\) is even.

**Theorem 5.2.** [6] Theorem 6] Let \(N_0, \alpha, \beta, t\) be integers with \(N_0, \alpha\) positive, and \(t > 1\), and let

\[ \sum_{n=0}^{\infty} c(n)q^{\alpha n + \beta} \in M_k^1(\Gamma_0(N_0), \chi), \]

where \(c(n)\) are algebraic integers in some number field. For any arithmetic progression \(r \pmod{t}\), there are infinitely many integers \(M \equiv r \pmod{t}\) for which \(c(M)\) is odd, provided there is one such \(M\).

Furthermore, if there does exist an \(M \equiv r \pmod{t}\) for which \(c(M)\) is odd, then the smallest such \(M\) is less than \(C_{r,t}\) for

\[ C_{r,t} := \frac{2^j \cdot 12 + k}{12\alpha} \left( \frac{N\alpha^2f^2}{d} \right) \prod_{p \mid N \not\mid t} \left( 1 - \frac{1}{p^2} \right) - 2^j, \]
where \( N := \text{lcm}(at, N_0) \), \( d := \text{gcd}(ar + \beta, t) \), and \( j \) is a sufficiently large integer.

Proof of Theorem 1.4. We have

\[
\sum_{n=0}^{\infty} \frac{E\sigma(2n)}{q^n} = \frac{(q^2; q^2)_\infty^3}{(q; q)_\infty^2}.
\]

We rewrite the above identity in terms of \( \eta \)-quotients, and then use the binomial theorem to obtain

\[
\sum_{n=0}^{\infty} \frac{E\sigma(2n)}{q^{6n+1}} = \frac{\eta(12z)^3}{\eta(6z)^2} \equiv \frac{\eta(12z)^4}{\eta(6z)^3} \pmod{2}.
\]

By Theorem 2.3 we have

\[
\eta(12z)^4 \eta(6z)^4 \in M_0^1(\Gamma_0(72)).
\]

Let \( f_t(z) := \frac{\eta(12z)^4}{\eta(6z)^4} \Delta^{2j}(6tz) \), where \( \Delta(z) := \eta^{24}(z) \). The cusps of \( \Gamma_0(72t) \) are represented by fractions \( \frac{c}{d} \) where \( d | 72t \) and \( \gcd(c, d) = 1 \). Now, \( f_t(z) \) vanishes at the cusp \( \frac{c}{d} \) if and only if

\[
4 \gcd(d, 12)^2 - 4 \gcd(d, 6)^2 + 24 \cdot 2^{j} \gcd(d, 6t)^2 > 0.
\]

We have

\[
4 \frac{\gcd(d, 12)^2}{12} - 4 \frac{\gcd(d, 6)^2}{6} + 24 \cdot 2^j \frac{\gcd(d, 6t)^2}{6t} \geq 2^j \frac{6}{t} - \frac{1}{2}.
\]

Hence, if \( j \) is an integer such that \( 2^j > \frac{1}{12} \), then \( f_t(z) \in S_{12 \cdot 2^j}(\Gamma_0(72t)) \). Finally, our desired result follows immediately by applying Theorems 5.1 and 5.2 to \( \sum_{n=0}^{\infty} \frac{E\sigma(2n)}{q^{6n+1}} \).

Proof of Theorem 1.5. We first recall the following 2-dissection formula from [3, Entry 25, p. 40]:

\[
1 \frac{(q^8; q^8)^5}{(q^4; q^4)^5 (q^{16}; q^{16})^2} + 2q \frac{(q^4; q^8)_\infty^2}{(q^2; q^2)_\infty^5 (q^{16}; q^{16})^2} \frac{(q^{16}; q^{16})^2}{(q^4; q^8)_\infty^5 (q^8; q^8)_\infty}.
\]

From (1.1), we have

\[
\sum_{n=0}^{\infty} \frac{E\sigma(2n)}{q^n} = \frac{(q^2; q^2)_\infty^3}{(q; q)_\infty^2}.
\]

Combining (5.1) and (5.2), and then extracting the terms with odd powers of \( q \), we deduce that

\[
\sum_{n=0}^{\infty} \frac{E\sigma(4n + 2)}{q^n} = 2 \frac{(q^2; q^2)_\infty^2 (q^8; q^8)_\infty^2}{(q; q)_\infty^2 (q^4; q^4)_\infty}.
\]

We again combine (5.1) and (5.3), and then extract the terms with odd powers of \( q \) to obtain

\[
\sum_{n=0}^{\infty} \frac{E\sigma(8n + 6)}{q^n} = 4 \frac{(q^2; q^2)_\infty (q^4; q^4)_\infty (q^8; q^8)_\infty^2}{(q; q)_\infty^2}.
\]
Since \((q; q)_\infty^n \equiv (q^2; q^2)_\infty \pmod{2}\), we have
\[
\sum_{n=0}^{\infty} \mathcal{EO}(8n + 6)q^n \equiv 4(q^4; q^4)_\infty^5 (q; q)_\infty \pmod{8}.
\]

We rewrite the above equation in terms of \(\eta\)-quotients and obtain
\[
\sum_{n=0}^{\infty} \mathcal{EO}(8n + 6)q^{24n + 19} \equiv 4 \frac{\eta^5(96z)}{\eta(24z)} \pmod{8}.
\]

Let \(A(z) = \frac{\eta^2(24z)}{\eta(48z)}\). Then, \(A^2(z) \equiv 1 \pmod{4}\). Also, let \(B(z) = \frac{\eta^5(96z)\eta^3(24z)}{\eta^2(48z)}\).

Then we have
\[
B(z) = \frac{\eta^5(96z)}{\eta(24z)} A^2(z) \equiv \frac{\eta^5(96z)}{\eta(24z)} \pmod{4}.
\]

The cusps of \(\Gamma_0(2304)\) are represented by fractions \(c/d\) where \(d \mid 2304\) and \(\gcd(c, d) = 1\). By Theorem 2.3, \(B(z)\) is holomorphic at the cusp \(c/d\) if and only if
\[
5 \gcd(d, 96)^2 + 3 \gcd(d, 24)^2 - 2 \gcd(d, 48)^2 / 48 \geq 0.
\]

Now,
\[
5 \frac{\gcd(d, 96)^2}{96} + 3 \frac{\gcd(d, 24)^2}{24} - 2 \frac{\gcd(d, 48)^2}{48}
= \frac{\gcd(d, 48)^2}{24} \left( 5 \frac{\gcd(d, 96)^2}{48} + 3 \frac{\gcd(d, 24)^2}{48} - 1 \right) > 0.
\]

Hence, by Theorem 2.3, \(B(z) \in S_3(\Gamma_0(2304), \left(\frac{-4}{t}\right))\).

Let \(m\) be a positive integer. By a deep theorem of Serre [11, p. 43], if \(f(z) \in M_\ell(\Gamma_0(N), \chi)\) has Fourier expansion
\[
f(z) = \sum_{n=0}^{\infty} c(n)q^n \in \mathbb{Z}[[q]],
\]
then there is a constant \(\alpha > 0\) such that
\[
\# \{ n \leq X : c(n) \not\equiv 0 \pmod{m} \} = \mathcal{O} \left( \frac{X}{(\log X)^\alpha} \right).
\]

Since \(B(z) \in S_3(\Gamma_0(2304), \left(\frac{-4}{t}\right))\), the Fourier coefficients of \(B(z)\) are almost always divisible by \(m\). Hence, using (5.5) and (5.4) we complete the proof of the theorem.

\(\square\)

**Proof of Theorem 1.6.** The generating function of \(EO_u(2n)\) is given by
\[
\sum_{n=0}^{\infty} \mathcal{EO}_u(2n)q^n = \frac{1}{(q; q^2)_\infty} = (q^2; q^2)_\infty / (q; q)_\infty.
\]

We note that \(\eta(24z) = q \prod_{n=1}^{\infty} (1 - q^{24n})\) is a power series of \(q\). As in the proof of Theorem 1.5 let
\[
A(z) = \prod_{n=1}^{\infty} \frac{(1 - q^{24n})^2}{(1 - q^{48n})} = \frac{\eta^2(24z)}{\eta(48z)}.
\]
Then using binomial theorem we have

\[ A_{2k}^k(z) = \frac{\eta^{2k+1}(24z)}{\eta^{2k}(48z)} \equiv 1 \pmod{2^{k+1}}. \]  

Define \( B_k(z) \) by

\[ B_k(z) = \left( \frac{\eta(48z)}{\eta(24z)} \right)^2 A_{2k}^k(z). \]

Modulo \( 2^{k+1} \), we have

\[ B_k(z) \equiv \eta^2(48z) \eta^2(24z) A_{2k}^k(z) \equiv q^2(q^{48}; q^{48})_\infty^2. \]

Combining (5.6) and (5.9), we obtain

\[ B_k(z) \equiv \sum_{n=0}^{\infty} E_0(2n)q^{24n+2} \pmod{2^{k+1}}. \]

The cusps of \( \Gamma_0(576) \) are represented by fractions \( \frac{c}{d} \) where \( d \mid 576 \) and \( \gcd(c, d) = 1 \). By Theorem 2.3, it is easily seen that \( B_k(z) \) is a form of weight \( 2k-1 \) on \( \Gamma_0(576) \). Therefore, \( B_k(z) \in M_{2k-1}(\Gamma_0(576)) \) if and only if \( B_k(z) \) is holomorphic at the cusp \( \frac{c}{d} \). We know that \( B_k(z) \) is holomorphic at a cusp \( \frac{c}{d} \) if and only if

\[ \gcd(d, 24) \left( 2^{k+1} - 2 \right) + \gcd(d, 48) \left( 1 - 2^{k-1} \right) \geq 0. \]

Now,

\[ \gcd(d, 24)^2 \left( 2^{k+1} - 2 \right) + \gcd(d, 48)^2 \left( 1 - 2^{k-1} \right) \]

\[ = \gcd(d, 48)^2 \left( \frac{\gcd(d, 24)^2}{\gcd(d, 48)^2} (2^{k+1} - 2) + (1 - 2^{k-1}) \right) \]

\[ \geq \frac{1}{4} \left( 2^{k+1} - 2 \right) + (1 - 2^{k-1}) \]

\[ > 0. \]

Hence, \( B_k(z) \in M_{2k-1}(\Gamma_0(576)) \). Now, using Serre’s theorem [11, p. 43] as shown in the proof of Theorem 1.5 we arrive at the desired result due to (5.10). \( \square \)

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