The nonlinear algorithms proposed recently by Abrams and Lloyd [Report No. quant-ph/980104] are fast but make an explicit use of an arbitrarily fast unphysical transfer of information within a quantum computer. It is shown that there exists a simplification of the second Abrams-Lloyd algorithm which eliminates the unphysical effect but keeps the algorithm fast.

I. INTRODUCTION

Any systematic procedure associating with an arbitrary number \( i_0 \ldots i_{n-1}, i_k = 0 \) or 1, another number \( f(i_0 \ldots i_{n-1}) \) can be termed an algorithm. The idea of quantum computation rests on the observation that a binary number \( i_0 \ldots i_{n-1} \) can be represented by a vector (a qubinary number) \( |i_0 \rangle \ldots |i_{n-1} \rangle \) representing an uncorrelated state of \( n \) distinguishable two-level quantum systems. A quantum algorithm is essentially an algorithm based on a qubinary representation of numbers \( |i_0 \ldots i_{n-1} \rangle \).

Typical quantum algorithms use unitary operations and projections \( |i_0 \ldots i_{n-1} \rangle \rightarrow \sum_{k=0}^{2^n-1} \frac{1}{\sqrt{2^n}} |k \rangle \). This is motivated by the unitarity of the standard Schrödinger dynamics and the so-called projection postulate. The latter postulate is typical of the Copenhagen interpretation of quantum mechanics and is not essential to quantum computation (quantum algorithms would look different in, say, the Many Worlds interpretation of quantum mechanics but their fundamental properties would not change). The Schrödinger (linear and unitary) dynamics is not the only dynamics one encounters in quantum theories. Dynamics in the Heisenberg picture is typically nonlinear (in the sense that operators depend nonlinearly on initial conditions). Typical effective dynamics of quantum optical systems (such as two-level atoms) is irreversible and nonunitary. There are also many situations where states of quantum systems evolve in an effectively nonlinear way (optical solitons, Hartree-type approximations). Finally, there is still no proof that the present-day quantum mechanics is not an approximation to a more exact nonlinear theory, and various versions of such a nonlinear generalization have been proposed. It should be stressed that the popular opinion stating that all nonlinear extensions of quantum mechanics must lead to logical absurdisties does not find support in a detailed analysis of nonlinear “nogo” theorems (for a brief discussion cf. [7]).

The assumption of fundamental quantum linearity cannot be regarded as a consequence of experimental data because it is quite typical that a consistent theory is prior to experiments. This point was clear to Wigner [6] who was simultaneously one of the first to associate fundamentally nonlinear phenomena with a theory of brain functioning [8]. Since there is no doubt that human brain is a physical system, there is almost no doubt that at least some of its aspects have to be described by quantum mechanics. The idea of qubinary mathematics (including differentiation and integration) can be traced back to Ollov’s works on a “wave logic” of consciousness [8]. On the other hand, there are serious arguments of Penrose [9,10] for non-algorithmic (in the classical sense) ingredients in brain activity. Human and animal brains are systems that seem to possess a feedback-type property of self-observation, and feedback effects are typically associated with a nonlinear evolution.

Although the above problems may appear somewhat far from standard quantum physics, they naturally lead to the question of possible consequences of a nonlinear quantum dynamics for the theory of quantum computation. The problem was recently addressed by Abrams and Lloyd [11] who showed that a nonlinear evolution in a Hilbert space of states of a quantum computer leads naturally to polynomial-time solutions of NP and #P problems. The Abrams-Lloyd argument was based on a general property of nonlinear evolutions in Hilbert spaces, namely the non-conservation of scalar products between nonlinearly evolving solutions of a nonlinear Schrödinger equation. This effect (in the literature called a mobility phenomenon) was discussed in great detail by Mielnik [12] in his analysis of nonlinear motion semigroups (compare also [13]).

The mobility effect was used in [11] in two algorithms. In the first of them the Authors chose a nonlinear evolution which has \( |0 \rangle \) as a fixed point, but any superposition of \( |0 \rangle \) and \( |1 \rangle \) transports towards \( |1 \rangle \). In the second algorithm a sequence of nonlinear operations was partly disentangling the state of the quantum computer by transformations of the type

\[
\frac{1}{\sqrt{2}} (|0\rangle|0\rangle + |1\rangle|1\rangle) \rightarrow \frac{1}{\sqrt{2}} (|0\rangle|1\rangle + |1\rangle|0\rangle).
\]

Nonlinear evolutions that lead (with an arbitrary accuracy) to the required modification of entangled states can be obtained in a Weinberg-type nonlinear quantum mechanics [14]. There is a problem, however. Using exactly the same trick [i.e. transformation (1)] it was shown in [15] that (1) is responsible for arbitrarily fast influences between noninteracting systems (this should not be confused with another effect discussed by Gisin [16], which was a result of the projection postulate). Therefore, the algorithm of Abrams and Lloyd makes an explicit use of...
an unphysical and arbitrarily fast process, so there is a
danger that this is the reason why it is fast.

Fortunately, as we shall see below, this is not the case.
To prove it we shall concentrate on the second algorithm.
We will consider a concrete example of a nonlinear
dynamics and will use a formalism that is known to elim-
inate the unphysical influences [17–21]. A detailed dis-
cussion of both algorithms in this context can be found in
[24].

**II. SECOND ABRAMS-LLOYD ALGORITHM**

**Step 1.** We begin with the state
\[
|\psi(0)\rangle = |01, \ldots, 0_n\rangle (0)
\]
where the first \( n \) qubits correspond to the input and the
last qubit represents the output.

Consider the unitary transformation acting as follows
\[
U|0\rangle = \frac{1}{\sqrt{2}} (|0\rangle + |1\rangle)
\]
\[
U|1\rangle = \frac{1}{\sqrt{2}} (-|0\rangle + |1\rangle)
\]

**Step 2.**
\[
|\psi(1)\rangle = U \otimes \ldots \otimes U \otimes |1\psi(0)\rangle
\]
\[
= \frac{1}{\sqrt{2^n}} \sum_{i_1, \ldots, i_n=0}^1 |i_1, \ldots, i_n\rangle |0\rangle
\]
The input consists now of a uniform superposition of all
the numbers \( 0 \leq n \leq 2^n - 1 \).

**Step 3.**
\[
|\psi(2)\rangle = F|\psi(1)\rangle
\]
\[
= \frac{1}{\sqrt{2^n}} \sum_{i_1, \ldots, i_n=0}^1 |i_1, \ldots, i_n\rangle |f(i_1, \ldots, i_n)\rangle
\]
where \( F \) is some unitary transformation (an oracle) that
transforms the input into an output: \( f(i_1, \ldots, i_n) \) equals
1 or 0.

**Step 4.** We assume that \( f(x) = 1 \) for at most one \( x \).
Denote by \( s, s = 0 \) or 1, the number of \( x \)'s that satisfy
\( f(x) = 1 \). The state (8) can be written as
\[
|\psi(2)\rangle = \frac{1}{\sqrt{2^n}} \sum_{i_2, \ldots, i_n}^1 |0_1, i_2, \ldots, i_n\rangle |f(0_1, i_2, \ldots, i_n)\rangle
\]
\[
+ \frac{1}{\sqrt{2^n}} \sum_{i_2, \ldots, i_n}^1 |1_1, i_2, \ldots, i_n\rangle |f(1_1, i_2, \ldots, i_n)\rangle
\]
Let us note that with very high probability the state is
\[
\frac{1}{\sqrt{2^n}} \sum_{i_2, \ldots, i_n}^1 \left( |0_1, i_2, \ldots, i_n\rangle |0\rangle + |1_1, i_2, \ldots, i_n\rangle |0\rangle \right)
\]
(10)

With much smaller probability it is either
\[
\frac{1}{\sqrt{2^n}} \sum_{i_2, \ldots, i_n}^1 \left( |0_1, i_2, \ldots, i_n\rangle |1\rangle + |1_1, i_2, \ldots, i_n\rangle |0\rangle \right)
\]
(11)
or
\[
\frac{1}{\sqrt{2^n}} \sum_{i_2, \ldots, i_n}^1 \left( |0_1, i_2, \ldots, i_n\rangle |0\rangle + |1_1, i_2, \ldots, i_n\rangle |1\rangle \right)
\]
(12)
and is never in the form
\[
\frac{1}{\sqrt{2^n}} \sum_{i_2, \ldots, i_n}^1 \left( |0_1, i_2, \ldots, i_n\rangle |1\rangle + |1_1, i_2, \ldots, i_n\rangle |1\rangle \right)
\]
(13)
since this would mean there are two different numbers
satisfying \( f(x) = 1 \) which contradicts our assumption.
The idea of the algorithm is to apply to the flag qubit
a nonlinearity that leaves (10) unchanged but (11) and
(12) transforms into
\[
\frac{1}{\sqrt{2^n}} \sum_{i_2, \ldots, i_n}^1 \left( |0_1, i_2, \ldots, i_n\rangle |1\rangle + |1_1, i_2, \ldots, i_n\rangle |1\rangle \right)
\]
(14)

Although such a dynamics can be approximated by a
nonlinear Schrödinger dynamics of a Weinberg type, it is
easy to show that it is unphysical [15].

To see this assume that the flag qubit does not interact
with the \( n \) input ones. To simplify the discussion take
\( n = 1 \) and consider the transformation (9). We assume that
the nonlinear evolution is applied locally to the flag
system. (11) implies the following transformation of the
reduced density matrix of the first qubit
\[
\frac{1}{2} \left( |0_1\rangle \langle 0_1| + |1_1\rangle \langle 1_1| \right) \rightarrow \frac{1}{2} \left( |0_1\rangle + |1_1\rangle \right) \left( |0_1| + |1_1\rangle \right)
\]
and therefore a fully mixed state evolves into a pure one.
It can be shown [24] that Weinberg’s description of sepa-
rated systems leads to 2-particle Schrödinger equations
that induce this kind of behavior at a distance (“faster-
than-light telegraph”).

Before performing a more detailed analysis let us il-
lustrate the crucial element of the algorithm on a simple
element. Take \( n = 3 \) and \( f(110) = 1 \). The oracle pro-
duces
\[
8^{-1/2} [ |000\rangle \langle 0| + |001\rangle \langle 0| + |010\rangle \langle 0| + |011\rangle \langle 0| + |100\rangle \langle 0| + |101\rangle \langle 0| + |110\rangle \langle 1| + |111\rangle \langle 0| ]
\]
(15)
The nonlinearity now “looks” at the second and the third input slots and sees the above kets as the following pairs

\[
8^{-1/2}[[000]\langle 000| + [100]\langle 100| + \\
[001]\langle 001| + [101]\langle 101| + \\
[010]\langle 010| + [110]\langle 110| + \\
[011]\langle 011| + [111]\langle 111|]]
\] (16)

Now it scans each of the rows and does not do anything when two flag 0’s occur, but when it “notices” one 0 and one 1 it changes 0 to 1. So after this step we get

\[
8^{-1/2}[[000]\langle 000| + [100]\langle 100| + \\
[001]\langle 001| + [101]\langle 101| + \\
[010]\langle 010| + [110]\langle 110| + \\
[011]\langle 011| + [111]\langle 111|]]
\] (17)

Now the nonlinearity looks at the first and the third slots and sees the kets as the following pairs

\[
8^{-1/2}[[000]\langle 000| + [010]\langle 010| + \\
[001]\langle 001| + [011]\langle 011| + \\
[100]\langle 100| + [110]\langle 110| + \\
[101]\langle 101| + [111]\langle 111|]]
\] (18)

It again behaves as before and what we get after this step looks as follows

\[
8^{-1/2}[[000]\langle 000| + [010]\langle 010| + \\
[001]\langle 001| + [011]\langle 011| + \\
[100]\langle 100| + [110]\langle 110| + \\
[101]\langle 101| + [111]\langle 111|]]
\] (19)

Finally our nonlinearity looks at the first and the second slots and the state regroups in the following way

\[
8^{-1/2}[[000]\langle 000| + [001]\langle 001| + \\
[010]\langle 010| + [011]\langle 011| + \\
[100]\langle 100| + [101]\langle 101| + \\
[110]\langle 110| + [111]\langle 111|]]
\] (20)

Now each row contains one 1 and in the final move all flag 0’s are switched to 1’s and the state partly disentangles:

\[
8^{-1/2}[[000] + [001] + \\
[010] + [011] + \\
[100] + [101] + \\
[110] + [111]]
\] (21)

Of course, in case \( s = 0 \) the entire state does not change during the operation and a measurement on the flag qubit gives 0 with certainty. Such an algorithm is fast and allows to distinguish between \( s = 0 \) and \( s \neq 0 \) in a linear time. The number of operations is of the order of \( n \) as compared to \( 2^n \) typical of a slow data-search algorithm.

It follows that we have an algorithm that is fast but simultaneously makes an explicit use of an arbitrarily fast, physically unaccountable process. Now I will show that the faster-than-light effect can be eliminated without any loss in the efficiency of the algorithm.

To do so I will use an explicit nonlinear dynamics and apply it locally to the flag qubit. By saying that the dynamics is applied locally it is meant that we are using an appropriate \((n + 1)\)-particle extension of a nonlinear 1-particle dynamics. Assume the \( n + 1 \) subsystems do not interact with one another. The extension is local if it satisfies the following condition: A reduced density matrix of any of the \( n + 1 \) subsystems satisfies a Liouville-von Neumann (nonlinear) equation which contains the reduced density matrix and Hamiltonian of only this particular subsystem. In the case of Weinberg’s nonlinear quantum mechanics of pure states and for finite-dimensional systems the extension of this type was introduced by Polchinski [17]. Its generalization to mixed states was given in [8,14]. The extension to more general theories was discussed in [21,22] and applied to concrete problems in [23,24]. The algebraic origin of the Polchinski-Bóna-Jordan formulation was discussed in detail in [25] and [26]. It should be noted that extensions of this type are nonunique if one starts with a nonlinear dynamics of state vectors (cf. [17,24]). They become unique if the dynamics is from the outset given in terms of density matrices. From the point of view of the algorithm the uniqueness problem is irrelevant so we can stick to the nonlinear Schrödinger equation framework.

### III. Modification of the Fourth Step

The above procedure can be, in principle, implemented in terms of a very complicated Schrödinger-type dynamics of the entire \((n + 1)\)-particle system of the quantum computer. It cannot be achieved by applying the nonlinear evolution locally to the flag qubit without generating the unphysical influences between different parts of the computer.

Below I propose a simpler procedure which is based on a nonlinearity which is applied only to the flag qubit.

Let us begin with a 1-qubit system whose dynamics is described by the nonlinear Schrödinger equation

\[
i|\psi⟩ = \epsilon \tanh (\alpha |\psi⟩ A - \eta 1 |\psi⟩) A |\psi⟩
\] (22)

where \( \epsilon \) is the magnitude of the nonlinearity, \( \alpha \) is a very large real number (say, \( \alpha \approx 2^n \)),

\[
A = \eta \left( |0⟩⟨0| - |1⟩⟨1| \right) + \sqrt{1 - \eta^2} \left( |0⟩⟨1| + |1⟩⟨0| \right),
\] (23)

and \( \eta \) is small but nonzero. For \( |\psi⟩ = |0⟩ \) the expression under tanh vanishes. For a small admixture of \( |1⟩ \) and
sufficiently large $\alpha$ the mobility with a nonzero frequency begins and an arbitrarily small amount of $|1\rangle$ can be sufficiently amplified. The Polchinski-type local extension of the dynamics to the entire quantum computer is \[ \rho_{\text{s}} = \frac{2^n - s}{2^n} \langle 0 | 0 \rangle + \frac{s}{2^n} | 1 \rangle \langle 1 | \]

The $(n+1)$-particle solution is

\[ | \Psi_{n+1} \rangle = \left( (1^{(n+1)} - i((n) \otimes A \sin \omega t) | \Psi_0 \rangle \right) \]

with

\[ \omega = \epsilon \tanh \left( \alpha \text{Tr} \rho (A - \eta 1) \right) \]

\[ = \epsilon \tanh \left( \frac{\alpha \eta s}{2^{n-1}} \right) \]

(25)

where

\[ \rho = \text{Tr}_{1,\ldots,n} | \Psi_0 \rangle \langle \Psi_0 | = \frac{2^n - s}{2^n} | 0 \rangle \langle 0 | + \frac{s}{2^n} | 1 \rangle \langle 1 | \]

is the reduced density matrix of the flag system after the first three steps of the original Abrams-Lloyd algorithm.

The average of $\sigma_3 = | 0 \rangle \langle 0 | - | 1 \rangle \langle 1 |$ at the flag subsystem is

\[ \langle \sigma_3 \rangle = \langle \Psi_{n+1} | (1^{(n)} \otimes \sigma_3) | \Psi_{n+1} \rangle \]

\[ = \frac{2^{n-1} - s}{2^{n-1}} \cos 2\omega t + 2\eta^2 \frac{2^{n-1} - s}{2^{n-1}} \sin^2 \omega t \]

(26)

For $s = 0$ the average is constant in time and equals 1. For $s = 1$, $\eta^2 \approx 0$, and sufficiently large $\alpha$ it oscillates with $\omega \approx \epsilon$. For $t \approx \pi / \epsilon$ the average is $\langle \sigma_3 \rangle \approx -1$, which means that almost all flag 0's in $| \Psi \rangle$ have been changed to 1's. Therefore instead of applying a complicated nonlinear dynamics of the original “Step 4” one can use the fact that in the local description the 1-particle nonlinearity is sensitive to the reduced density matrix of the particle.

This kind of algorithm cannot distinguish between different nonzero values of $s$, but clearly distinguishes between $s = 0$ and $s \neq 0$ in a way that is insensitive to small fluctuations of the parameters. It is interesting that the modified “Step 4” is essentially non-algorithmic and more resembles an effect of puncturing a balloon than performing an algorithm.

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