Modified Einstein equations

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Abstract

Standard general relativity fails to take into account the changes in coordinates induced by the variation of metric in the Hilbert action principle. We propose to include such changes by introducing a fundamental compensating tensor field and modifying the usual variational procedure.

1 Introduction

Mathematical representation of spacetime in general relativity relies heavily on the idea that spacetime events form a differentiable manifold. The manifold topology of spacetime (same as that of Euclidean four-dimensional space $\mathbb{R}^4$) is postulated before introduction of any (sufficiently differentiable in that topology) metric.

Finkelstein had argued that the assumption that topology is prior to metric is operationally suspect because experimentally it is always determined from the exchange of signals, governed by the metric. Since the assignment of coordinates to spacetime events is also based on propagation of signals, the usual point of view that coordinates are independent of the metric is equally unsound: there are no coordinates to distinguish one point from another besides the physical fields themselves. The dependence of spacetime coordinates on the metric would modify the Hilbert action principle and, consequently, Einstein’s field equations. This is in contrast to the usual theory which fails to take into account the changes in the coordinates induced by the variation in the metric $\Pi$.

By the metric of spacetime I mean the collection of all possible proper intervals between all possible spacetime events. The ability of such collections of numbers to represent the shapes of differentiable manifolds led to the invention of Riemannian geometry.

In general relativity we mostly deal with two types of metric variations. Variations due to simple changes in coordinates (coordinate variations) do not affect the "shape" of spacetime and represent going from one reference frame to another. In this case the metric transforms in the usual way as a second rank tensor according to

\[
\delta^c : \quad x(P) \rightarrow x'(P), \quad g_{\mu\nu}(x) \rightarrow g'_{\mu\nu}(x') = \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x^\beta}{\partial x'^\nu} g_{\alpha\beta}(x). \quad (1)
\]
Variations of the second type (let’s call them functional variations),

\[ \delta^f : \quad x(P) = \text{const}, \quad g_{\mu\nu}(x) \rightarrow g'_{\mu\nu}(x) = g_{\mu\nu}(x) + \delta^f g_{\mu\nu}(x), \quad (2) \]

which are used in the action principle, do not change coordinates but modify the functional form of the metric. These can be viewed as changes in the “shape” of spacetime. General relativity actually fails to provide the physical meaning of such variations, whose backreaction on spacetime coordinates could become important and would have to be taken into account.

A question arises whether all these variations change the identity of spacetime events/points, or whether the spacetime with all its points “intact” can still be viewed as a stage, albeit the dynamical one, for the matter (the “relative space vs. absolute space” debate [2]). In quantum theory of spacetime these questions would be of paramount importance, because events and elementary particle processes are fundamentally the same. For now, however, we can view spacetime points – marked in one way or another – as just “being there,” even when the metric is varied (similar to how all the points on a mattress would still be there even when it is deformed), and speak of the variational relationship

\[ \delta^f g_{\mu\nu} \rightarrow \delta^f x^\mu. \quad (3) \]

This relationship is clearly nonlocal.\(^1\)

## 2 Variational principle modified

Let us consider how the Hilbert action principle modifies in empty spacetime.

Picture an empty region. Consider a functional variation of the metric \(\delta^f g^{\mu\nu}\) in that region. Observer stationed outside of the region who performs coordinatization by using the same (say, radar) method as before the variation, will find a slight change in the coordinates of the marker events.

Let us assume that such a change is proportional to the functional variation in the metric via

\[ x^\alpha \rightarrow x^\alpha + \delta^f x^\alpha := x^\alpha - \frac{1}{2} \int_{O}^{P(x)} dx' \Lambda \theta_{\lambda\mu\nu}^\alpha(x') \delta^f g^{\mu\nu}(x'), \quad (4) \]

where \(\theta_{\lambda\mu\nu}^\alpha(x)\) is a fundamental compensating tensor field, the coordinate compensator, symmetric with respect to \(\mu \leftrightarrow \nu\) interchange. Integration is performed along the unique geodesic that connects \(P\) to some fixed reference point \(O\). The rare case of several such geodesics – the gravitational lensing – usually corresponds to some sort of coordinate singularity (for example, in the radar coordinatization procedure less than four radar stations are needed to pinpoint the focusing event).

When the Hilbert action

\[ S = \frac{1}{\kappa^2} \int d^4x \sqrt{-g} R \quad (5) \]

is varied, there appears an extra contribution due to the Jacobian of the coordinate transformation,

\[ \delta^f S = \frac{1}{\kappa^2} \int \left( (\delta^f d^4x) \sqrt{-g} R + d^4x \delta^f (\sqrt{-g} R) \right). \quad (6) \]

We have:

\[ \frac{\partial (x^\alpha + \delta^f x^\alpha)}{\partial x^\nu} = \delta^\alpha_\nu + \partial_\nu (\delta^f x^\alpha), \quad (7) \]

\(^1\)I thank David Finkelstein for drawing my attention to this particular point.
and
\[ J \left( \frac{x + \delta^f x}{x} \right) \equiv \det \left( \frac{\partial (x^\alpha + \delta^f x^\alpha)}{\partial x^\nu} \right) = 1 + \partial_\alpha (\delta^f x^\alpha) = 1 - \frac{1}{2} \theta^\alpha_{\alpha\nu} (x) \delta^f g^{\mu\nu} (x), \] (8)
where we have used Mandelstam’s definition for the derivative of the line integral [3]. Therefore the first term in (6) can be written as
\[ (\delta^f d^4 x) \sqrt{-g} R = \frac{1}{2} d^4 x \sqrt{-g} R \theta_{\mu\nu} \delta^f g^{\mu\nu}, \quad \theta_{\mu\nu} \equiv \theta^\alpha_{\alpha\mu\nu}. \] (9)

Varying the second term in (6) gives
\[ d^4 x \delta^f (\sqrt{-g} R) = d^4 x \sqrt{-g} \left(-R^{\mu\nu} + \frac{1}{2} g^{\mu\nu} R \right) \delta^f g_{\mu\nu}. \] (10)
Combining (10) with (9) produces the field equations
\[ R_{\mu\nu} - \frac{1}{2} (g_{\mu\nu} + \theta_{\mu\nu}) R = 0. \] (11)
Einstein’s theory is trivially reproduced in the limit \( \theta^\alpha_{\alpha\mu\nu} = 0 \), but regardless of compensator’s value, the field equations have a flat spacetime solution \( R_{\mu\nu} = 0 \).

Finally, covariant continuity of the Einstein tensor imposes four additional constraints
\[ \nabla^\mu (\theta_{\mu\nu} R) = 0 \] (12)
on the ten components of \( \theta_{\mu\nu} \).

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References
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[2] R. H. Dicke, The many faces of Mach, in H.-Y. Chiu and W. F. Hoffmann, eds., Gravitation and Relativity, pp. 121–141 (W. A. Benjamin, Inc., New York, 1964)

See also the interesting paper by R. F. Marzke and J. A. Wheeler, Gravitation as geometry – I, pp. 40–64, on setting up purely gravitodynamical coordinates, and on the importance of metric for spacetime geometry.
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