Poisson multi-Bernoulli conjugate prior for multiple extended object estimation

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Abstract—This paper presents a Poisson multi-Bernoulli mixture (PMBM) conjugate prior for multiple extended object estimation. A Poisson point process is used to describe the existence of yet undetected targets, while a multi-Bernoulli mixture describes the distribution of the targets that have been detected. The conjugacy property allows the posterior PMBM density to be computed exactly, meaning that given enough computational power the PMBM filter is correct. However, in practice, the data association problem requires approximations. The update and the prediction of the PMBM density parameters are presented and are given interpretations, and a simple linear Gaussian implementation is presented along with methods to handle the data association problem. A simulation study shows that the extended target PMBM filter outperforms the extended target cardinalized probability hypothesis density (CPHD) filter in scenarios where the expected number of detections per target per time step is low.

I. INTRODUCTION

Multiple target tracking (MTT) is the processing of sets of measurements obtained from multiple sources in order to maintain estimates of targets’ current states. Solving the MTT problem is complicated by the fact that—in addition to noise, missed detections and clutter—the number of targets is unknown and time-varying. In an MTT context, a point target is defined as a target that gives rise to at most one measurement at each time step, and an extended target is defined as a target that potentially gives rise to more than one measurement at each time step, where the set of measurements are spatially distributed around the extended target state.

The focus of this paper is on extended targets. Extended targets occur in scenarios where the resolution of the sensor, the size of the target, or the distance between target and sensor, are such that multiple resolution cells of the sensor are occupied by single targets. Examples of such extended target scenarios include vehicle tracking using automotive radars, tracking of sufficiently close airplanes or ships with ground or marine radar stations, and person tracking using laser range sensors. A common extended target measurement model is the inhomogeneous Poisson Point Process (PPP), proposed in [1]. At each time step, a Poisson distributed random number of measurements are generated, distributed around the target.

Random Finite Sets (RFS) and Finite Set Statistics (FISST) [2], [3] is a theoretically elegant and appealing approach to the MTT problem where the targets and the measurements are modelled as sets of random variables. The multiobject Bayes filter is an RFS-type filter that propagates and updates the probability density function (pdf) of the multi-object state in time, where the multi-object state is a random set. Because of the computational complexity of the data association problem, it is generally considered infeasible to implement and use a multi-object filter without using approximations in the data association problem. Computationally feasible approximate filters include the Probability Hypothesis Density (PHD) filters [4], the Cardinalized PHD (CPHD) filters [5], and the various multi-Bernoulli (MB) filters, see e.g. [6].

For the PPP extended target model of [1], a PHD filter was presented in [7] and a CPHD filter was presented in [8], [9]. Implementations, using Gaussian mixtures or gamma Gaussian inverse Wishart mixtures to approximate the PHD intensity, are given in [9]–[12]. A PHD filter for a hierarchical cluster extended target model is presented in [13], [14]; briefly this models the extended target as a set of reflection points located around the target centroid. A Gaussian Mixture Markov Chain Monte Carlo filter for multiple extended target tracking is presented in [15]. Labelled MB (LMB) filters for multiple extended target tracking are presented in [16], [17]. Comparisons have shown that for multiple extended target tracking the LMB filters outperform the CPHD filter, which in turn outperforms the PHD filter, see [9], [16], [17].

In Bayesian probability theory, the prior and posterior distributions are called conjugate if the posterior is in the same distribution family as the prior. Additionally, the prior is then called a conjugate prior for the likelihood function. The conjugacy is a very attractive property because it allows the posterior density to be computed exactly. For example, in random vector estimation, the Gaussian distribution is the conjugate prior for a Gaussian likelihood function. For extended target MTT the LMB filters [17] are conjugate priors. For point target MTT, at least two types of RFS conjugate priors have been presented: an LMB conjugate prior [18], and a PMBM conjugate prior [19]. The PMBM conjugate prior allows an elegant separation of the set of targets into two disjoint subsets: targets that have been detected, and targets that have not yet been detected; and the PMBM has already given rise to computationally efficient algorithms [19], [20]. Note that while the LMB and PMBM conjugate priors are theoretically exact, in practice approximations of the data association problem are required.

The contribution of this paper is the derivation of a PMBM conjugate prior for multiple extended target tracking, under the assumed standard measurement model [1]. The update and the prediction of the parameters of the PMBM density are derived and analyzed, and in addition to this a simple linear Gaussian implementation is presented. Simple solutions to the rapidly
expanding number of MB components in the mixture are also given. The derived conjugate prior is similar to the point target conjugate prior presented in [19].

The paper is organized as follows. In the next section a problem formulation is given. Section III presents some necessary background about RFS modeling for the MTT problem. The main contribution—a PMBM conjugate prior for extended target MTT—is presented in Section IV. A simple linear Gaussian implementation is given in Section V, and results from a simulation study are presented in Section VI. The paper is concluded in Section VII.

II. PROBLEM FORMULATION

Let \( x^i_k \) denote the state of the \( i \)th target at discrete time step \( k \), and let the target set be denoted

\[
X_k = \{ x^i_k \}_{i=1}^{N^z_k} \tag{1}
\]

The target set cardinality \( |X_k| = N^z_k \) is a time-varying discrete random variable, and each target state \( x^i_k \) is a random variable. The set of measurements obtained at time step \( k \) is denoted

\[
Z_k = \{ z^j_k \}_{j=1}^{N^m_k} \tag{2}
\]

where \( N^z_k = |Z_k| \) is the cardinality of the measurement set at time \( k \). There are two types of measurements: clutter measurements and target originated measurements, and the measurement origin is assumed unknown. Note that the sets above are without order and the set indexing is arbitrary; the particular choices \( i = 1, \ldots, N^z_k \) and \( j = 1, \ldots, N^m_k \) are only used for notational simplicity and convenience.

The objective of the paper is to approximate the multi-object distribution at time step \( k \) given all measurement sets up to and including time step \( k \), denoted \( f_{k|k}(X_k|Z^k) \), where \( Z^k \) denotes all measurement sets \( Z_m \) from \( m = 0 \) up to, and including, \( m = k \). The multitarget Bayes filter propagates in time the target set pdf \( f_{k|k-1}(X_k|Z^{k-1}) \) using the Bayes update

\[
f_{k|k}(X_k|Z^k) = \frac{f_k(Z_k|X_k)f_{k|k-1}(X_k|Z^{k-1})}{\int f_k(Z_k|X_k)f_{k|k-1}(X_k|Z^{k-1}) \delta X_k} \tag{3}
\]

and the Chapman-Kolmogorov prediction

\[
f_{k+1|k}(X_{k+1}|Z^k) = \int f_{k+1|k}(X_{k+1}|X_k)f_k(X_k|Z^k) \delta X_k \tag{4}
\]

where \( f_k(Z_k|X_k) \) is the multitarget measurement set density, and where \( f_{k+1|k}(X_{k+1}|X_k) \) is the multi-object transition density.

In MTT the multi-object density \( f_{k|k}(X_k|Z^k) \) is often approximated by a parametric density, e.g., a PPP or a MB process, or by a particle representation. In this paper the set density \( f_{k|k}(X_k|Z^k) \) is approximated by a PMBM density. The PMBM is a conjugate prior, and thus it is possible to compute the posterior density exactly, given enough computational resources. The problem considered in this paper is to derive the update and the prediction of the PMBM parameters under standard modelling assumptions, and to find suitable approximations such that the PMBM filter can be implemented using limited computational resources.

### TABLE I

**Notation**

- \( |X| \): set cardinality, i.e., number of elements in set \( X \).
- \( Z \subseteq Y \), where \( Y \subseteq Z \), denotes set difference, i.e., \( Z \setminus Y \) contains the elements in \( Z \) that are not in the subset \( Y \subseteq Z \).
- \( I_m \times n \): all-one-matrix of size \( m \times n \).
- \( I_m \): identity matrix of size \( m \times m \).
- \( A \otimes B \): Kronecker product of matrices \( A \) and \( B \).
- \( f(a,b) = \int a(x)b(x)dx \): inner product of \( a(x) \) and \( b(x) \).
- \( h^\alpha = \prod_{x^i_k \in X} h(x) \), where \( h^\alpha = 1 \) by definition.
- \( \Phi_i \) denotes the set of mappings \( \alpha : \{1, \ldots, m\} \rightarrow \{0, \ldots, n\} \).

subject to

\[
\{1, \ldots, n\} \subseteq \alpha \{(1, \ldots, m)\} \tag{5b}
\]

\[
\alpha(i) > 0, i \neq \ell \Rightarrow \alpha(i) \neq \alpha(\ell) \tag{5c}
\]

\( \mathcal{P} \subseteq \mathcal{Z} \) denotes that \( \mathcal{P} \) partitions the set \( \mathcal{Z} \) into non-empty subsets \( \mathcal{C} \) (called cells), such that

\[
\bigcup_{\mathcal{C} \subseteq \mathcal{P}} \mathcal{C} = \mathcal{Z} \tag{6a}
\]

\[
\mathcal{C}_a \cap \mathcal{C}_b = \emptyset, \forall \mathcal{C}_a, \mathcal{C}_b \in \mathcal{P} : \mathcal{C}_a \neq \mathcal{C}_b \tag{6b}
\]

- \( \{V_i\}_{i=1}^J \subseteq \{V_i\}_{i=1}^J \) \( V_i = Z \) denotes a disjoint set of \( I \) possibly empty subsets \( V_i \), in other words

\[
\bigcup_{i=1}^I V_i = \mathcal{Z} \tag{7a}
\]

\[
V_{i_1} \cap V_{i_2} = \emptyset, \forall i_1 \neq i_2 \tag{7b}
\]

### III. BRIEF INTRODUCTION TO RFS MODELING

This section first presents a review of random set theory; specifically the probability generating functional (pgfl), the PPP and the MB process. The standard extended target measurement and motion models are presented, and finally the multitarget Bayes filter is given on pgfl form. The pgfl form of the multitarget Bayes filter will be used to derive the update and the prediction of the PMBM density parameters.

Notation is given in Table I. Note the following important differences between a partition \( \mathcal{P} \) of the set \( \mathcal{Z} \), and a disjoint set of \( I \) subsets \( \{V_i\}_{i=1}^J \) whose union is the set \( \mathcal{Z} \):

- The cells \( \mathcal{C} \) in a partition \( \mathcal{P} \) are always non-empty, whereas a subset \( V_i \) may be empty.
- The number of cells in a partition range from one (all measurements in one cell) to \( |\mathcal{Z}| \) (each cell contains one unique measurement), whereas the number of subsets is determined by \( I \), and we may have \( I > |\mathcal{Z}| \).

#### A. Review of random set modeling

1) Probability generating functional: The probability generating functional (pgfl) is a multitarget integral transform defined as [2, p. 371]

\[
G[h] = \int h(X)f(X)\delta X \tag{8}
\]

where \( h(x) \) is a test-function and \( f(X) \) is the pdf of the RFS \( X \). A property of the pgfl is that if \( X \) is the union of independent RFS \( X^j \) with pgfls \( G^j[h] \), then the pgfl of \( X \) is [2, p. 372]

\[
G[h] = \prod_j G^j[h] \tag{9}
\]
The cardinality is Poisson distributed with Poisson rate $\mu$

$$f(X) = \frac{\delta}{\delta X} G[h] \bigg|_{h=0}$$

$$\frac{\delta}{\delta X} G[h] = \frac{\delta |X|}{\prod_{x \in X} \delta x} G[h]$$

$$\frac{\delta}{\delta X} G[h] \triangleq \lim_{\varepsilon \to 0} \frac{G[h + \varepsilon \delta x] - G[h]}{\varepsilon}$$

2) Poisson point process: A PPP is a type of RFS with pdf

$$f(X) = e^{-\mu} \prod_{x \in X} \mu f(x)$$

The cardinality is Poisson distributed with Poisson rate $\mu$ and each target is independent identically distributed (iid) with spatial distribution $f(x)$. The pgfl of a PPP is expressed as, see e.g. [2] p. 373, eq. 11.172

$$G[h] = \exp(\mu \langle f; h \rangle - \mu)$$

where $D(x) = \mu f(x)$ is the PPP intensity function.

3) Multi Bernoulli process: A Bernoulli RFS $X^i$ is a type of RFS that is empty with probability $1 - r^i$ or, with probability $r^i$, contains a single element with distribution $f^i(x)$. The cardinality is Bernoulli distributed with parameter $r^i$ and the pdf of $X^i$ is

$$f(X^i) = \begin{cases} 1 - r^i & X^i = \emptyset \\ r^i \cdot f^i(x) & X^i = \{x\} \\ 0 & |X^i| \geq 2 \end{cases}$$

A Bernoulli RFS has the following pgfl [3] p. 100, eq. 4.119

$$G^{bi}[h] = 1 - r^i + r^i \langle f^i; h \rangle$$

A typical assumption in MTT is that the targets are independent, see e.g. [21]. An MB RFS $X$ is the union of a fixed number $I$ of independent Bernoulli RFSs $X^i$, $X = \bigcup_{i=1}^I X^i$ and is defined by the set of existence probabilities and distributions $\{r^i, f^i(\cdot)\}_{i=1}^I$. Here $I$ is the maximum number of targets that the MB RFS can represent. The MB pgfl for a set $X$ can be expressed as

$$f(X) = \begin{cases} \sum_{\alpha \in \Phi(X)} \prod_{i=1}^I f^i(X^{\alpha(i)}) & |X| \leq I \\ 0 & |X| > I \end{cases}$$

$$X^{\alpha(i)} = \begin{cases} \emptyset & \text{if } \alpha(i) = 0 \\ \{x\} & \text{if } \alpha(i) > 0 \end{cases}$$

where the mapping $\Phi$, defined in Table II describes all possible ways to assign exactly one Bernoulli estimate onto each target in $X^i$, and (any) remaining Bernoulli estimates onto an empty set $\emptyset$. The MB pgfl

$$G^{mb}[h] = \prod_{i=1}^I (1 - r^i + r^i \langle f^i; h \rangle)$$

is given by the combination of the Bernoulli pgfl [14] and the pgfl for a union of independent sets [9].

A MB mixture (MBM) is an RFS whose pdf is a normalized weighted sum of MB pdfs $f^j(X)$,

$$f^{mbm}(X) = \sum_j \mathcal{W}^j f^j(X), \quad \sum_j \mathcal{W}^j = 1$$

where the weights may correspond to, e.g., different data association sequences. The corresponding pgfl can be expressed as

$$G^{mbm}[h] = \sum_j \mathcal{W}^j \prod_{i=1}^I (1 - r^j,i + r^j,i \langle f^j,i; h \rangle)$$

In other words, the pgfl of an MBM is the weighted sum of the MB pgfls.

B. Standard extended target measurement model

The set of measurements $Z_k$ is the union of a set of clutter measurements and a set of target generated measurements; the sets are assumed independent. The clutter is modelled as a PPP with rate $\lambda$ and spatial distribution $c(x)$. An extended target with state $x$ is detected with state dependent probability of detection $p_D(x)$, and, if it is detected, the target measurements are modelled as a PPP with state dependent Poisson rate $\gamma(x)$ and spatial distribution $\phi(z|x)$. For a non-empty set of measurements $|Z| > 0$ the conditional extended target measurement set likelihood is denoted

$$\ell_Z(x) = p_D(x)p(Z|x) = p_D(x)e^{-\gamma(x)} \prod_{z \in Z} \gamma(x)\phi(z|x)$$

Note that this is the product of the probability of detection and the PPP pdf.

The extended target measurement generating process can be understood as follows. The Poisson rate $\gamma(x)$ models the expected number of detections from a target with state $x$. In a practical scenario the rate $\gamma(x)$ often depends on aspects such as the sensor’s resolution and the distance between the target and the sensor. The probability of detection $p_D(x)$ models the probability that a target with state $x$ is detected. In practice, $p_D(x)$ will depend on aspects such as occlusion and the sensor’s field of view (FOV) – a target that is located behind a larger object, or is outside the FOV, is likely to have a low $p_D(x)$, and vice versa. The effective probability of detection for an extended target with state $x$ is

$$p_D(x)(1 - e^{-\gamma(x)})$$

where $1 - e^{-\gamma(x)}$ is the Poisson probability of generating at least one detection. Accordingly, the effective probability of missed detection, i.e., the probability that the target is not detected, is

$$q_D(x) = 1 - p_D(x) + p_D(x)e^{-\gamma(x)}$$

Note that $q_D(x)$ is the conditional likelihood for an empty set of measurements, i.e., $\ell_\emptyset(x) = q_D(x)$ (cf. [20]).

For a single extended target with state vector $x$, the pgfl of its measurement model pdf is [2] p. 431, eq. 12.217

$$G_k^E[g|x] = 1 - p_D + p_D \exp(\gamma \langle \phi; g \rangle - \gamma)$$
where the probability of detecting the target as a whole has been taken into account. Note that \( \langle \phi; g \rangle = \int g(z) \phi(z|x) dz \) is a function of \( x \). The extended targets are assumed to generate measurements independent of each other, and it follows that the pgfl for the measurements from all targets at time \( k \) is

\[
G_k^T[g|X_k] = (1 - p_D + p_D \exp (\gamma \langle \phi; g \rangle - \gamma))^{X_k} \tag{24}
\]

The set of clutter detections and the set of target detections are assumed independent, and the clutter PPP pgfl is

\[
G_k^C[g] = \exp (\lambda \langle c; g \rangle - \lambda) = \exp (\langle \kappa; g \rangle - \langle \kappa; 1 \rangle) \tag{25}
\]

where \( \kappa(z) = \lambda c(z) \) is the clutter PPP intensity function. It follows that the pgfl for the measurements is the product of the pgfl \( G_k^T[g|X] \) for the target generated measurements and the pgfl \( G_k^C[g] \) for the clutter measurements \([2]\), p. 372, eq. 11.166.

\[
G_k[g|X] = G_k^T[g|X]G_k^C[g] \tag{26}
\]

C. Standard dynamic model

The existing targets—both the detected and the undetected—survive from time step \( k \) to time step \( k + 1 \) with state dependent probability of survival \( ps(x_k) \). The surviving targets evolve according to a Markov process with transition density \( f_{k+1,k}(x_{k+1}|x_k) \). New targets appear independently of the targets that already exist. The target birth is assumed to be a PPP with Poisson rate \( \mu^b_k \) and spatial density \( f^b_{k+1}(x) \), i.e., the intensity is \( D^b_{k+1}(x) = \mu^b_k f^b_{k+1}(x) \). In this work target spawning is omitted, for work on spawning in an extended target context see \([23]\).

The pgfl for this multitarget Markov density is \([2]\) p. 474, eq. 13.61

\[
G_{k+1,k}[h|X_k] = (1 - ps + ps (f_{k+1,k}; h))^X_k \times \exp (\mu^b_k (f^b_{k+1}; h) - \mu^b_{k+1}) \tag{27a}
\]

\[
\langle f_{k+1,k}; h \rangle = \int h(x_{k+1}) f_{k+1,k}(x_{k+1}|x_k) dx_{k+1} \tag{27b}
\]

D. Multitarget Bayes filter on pgfl form

Let \( G_{k|k-1}[h] \) be the pgfl that corresponds to the pdf \( f_{k|k-1}(X_k|Z^{k-1}) \). The pgfl update that corresponds to \( f_{k|k}(X_k|Z^k) \) is \([2]\) pp. 530–531, s. 14.8.2

\[
G_{k|k}[h] = \left. \frac{\delta F[g,h]}{\delta Z_k} \right|_{g=0} \tag{28a}
\]

\[
F[g,h] \triangleq \int h^{X_k} G_k[g|X_k] f_{k|k-1}(X_k|Z^{k-1}) \delta X_k \tag{28b}
\]

\[
G_k[g|X_k] \triangleq \int g^{Z_k} f_k(Z_k|X_k) \delta Z_k \tag{28c}
\]

For the standard measurement model \([24], [25], [26]\), the joint target and measurement pgfl is

\[
F[g,h] = \int h^{X_k} G_k[g|X_k] f_{k|k-1}(X_k|Z^{k-1}) \delta X_k \tag{29a}
\]

\[
= \int h^{X_k} G^T_k[g|X_k]G^C_k[g|X_k]f(X_k|Z^{k-1}) \delta X_k \tag{29b}
\]

\[
G_k^C[g] \int h^{X_k} G^T_k[g|X_k]f(X_k|Z^{k-1}) \delta X_k \tag{29c}
\]

\[
G_k^C[g]G_k|k-1[h (1 - p_D + p_D \exp (\gamma \langle \phi; g \rangle - \gamma))] \tag{29d}
\]

and the updated pgfl is given by combining \((28a)\) and \((29d)\).

The pgfl of the Chapman-Kolmogorov predicted multi-object density \([3]\) is \([2]\) pp. 528–529, s. 14.8.1]

\[
G_{k+1,k+1}[h|X_k] = \int h^{X_{k+1}} f_{k+1,k}(X_{k+1}|X_k) \delta X_{k+1} \tag{30a}
\]

Inserting the standard dynamics model \([27a]\) into \((30a)\) gives the predicted pgfl \([2]\) p. 529, eq. 14.273

\[
G_{k+1,k}[h] = G_k[h] \left[ 1 - ps + ps (f_{k+1,k}; h) \right] \times \exp \left( \mu^b_k (f^b_{k+1}; h) - \mu^b_{k+1} \right) \tag{31}
\]

IV. A CONJUGATE PRIOR

In this section the main result of the paper is presented: a conjugate prior for the standard extended object measurement and motion models presented in the previous section. The conjugate prior is of PMBM form, i.e., it is a combination of a PPP and a MB mixture.

The PMBM model was developed for point target MTT in \([19]\). The PPP describes the distribution of the targets that are thus far undetected, while the MBM describes the distribution of the targets that have been detected at least once. Thus, the set of targets can be divided into two dis-joint subsets,

\[
X_k = X^u_k \cup X^d_k \tag{32}
\]

corresponding to undetected targets \( X^u_k \) and detected targets \( X^d_k \). The PMBM set density at time \( k \) can be expressed as

\[
f_{k|k}(X_k|Z^k) = \sum_{X^u_k \subseteq X_k} f^u_{k|k}(X_k \setminus f^d_{k|k}(X^d_k|Z^k)) \tag{33}
\]

The PPP set density for undetected targets \( f^u_{k|k}(X_k \setminus f^d_{k|k}(Z^k)) \) has Poisson rate \( \mu^u_{k|k} \) and spatial density \( f^u_{k|k}(x) \), and is defined as in \((11)\). The MBM set density for detected targets is

\[
f^d_{k|k}(X^d_k|Z^k) = \sum_{\alpha \in \alpha^d_{k|k}} \sum_{i=1}^{I^d_{k|k}} \prod_{j=1}^{J^d_{k|k}} j^d_{k|k}(X^d_k) \tag{34}
\]
the probability of the \( j \)th MB component is \( \mathcal{W}_{k|k}^j \). Note that each of the \( J_{k|k} \) MBM components corresponds to a unique global hypothesis for the detected targets, i.e. a particular history of data associations for all detected targets. This will be illustrated later.

Propagating in time the PMBM pdf means to propagate in time the PMBM density parameters, using a recursion that consists of an update and a prediction. This recursion can be found either through the multi-object pdf equations (3) and (4), or the corresponding pgfl equations (28a) and (30a). Through the pgfl transform (8) and the inverse pgfl transform (10a), the two alternatives for finding the recursion are equivalent. In this paper the pgfl form is used.

The posterior pgfl at time \( k \), corresponding to the set density (33), is

\[
G_{k|k}[h] = G_{k|k}^u[h]G_{k|k}^d[h] = \exp \left( \mu_{k|k}^u \left[ f_{k|k}^u(h) - \mu_{k|k}^u \right] \right) \\
\times \sum_{j=1}^{J_{k|k}} \mathcal{W}_{k|k}^j \prod_{i=1}^{I_{k|k}} \left( 1 - r_{j,i,k}^+ + r_{j,i,k}^- f_{k|k}(h) \right)
\]

In the following two subsections the updated and predicted pgfls will be presented; this defines how the PMBM parameters are updated and predicted. The pdfs \( f_{k|k}(X_k|Z^k) \) and \( f_{k+1|k}(X_{k+1}|Z^{k+1}) \) are obtained in a straightforward manner from the corresponding pgfls.

A. Update

Assuming that the predicted pgfl at time \( k \) is a PMBM of the form in (35),

\[
G_{k|k-1}[h] = G_{k|k-1}^u[h]G_{k|k-1}^d[h]
\]

the updated pgfl at time \( k \) is given by the Theorem. The proof of the theorem can be found in Appendix A.

**Theorem 1:** Given a prior PMBM pgfl of the form (35) and the standard measurement model (23), (25), (26), the updated pgfl is PMBM and is given in (37) at the top of the next page. The PDF for undetected targets has Poisson rate and spatial density

\[
\mu_{k|k}^u = \left\langle f_{k|k}^u(h) \right\rangle
\]

\[
f_{k|k}^u(x) = \frac{q_D(x)f_{k|k-1}^u(x)}{\left\langle f_{k|k-1}^u(h) \right\rangle}
\]

where the effective probability of missed detection \( q_D(x) \) was defined in (22). The MB component for targets detected for the first time, corresponding to a set \( Y \) and a partition \( \mathcal{P} \) of the set \( Y \), is described by

\[
G_{k|k}[h] = \prod_{C \in \mathcal{P}} \left( 1 - r_C + r_C f_{C|k}(h) \right)
\]

where the updated existence probability, the updated spatial density, and the predicted likelihood are

\[
r_C = \begin{cases} \frac{1}{g_C} & \text{if } |C| > 1 \\ \frac{1}{r_C + L_C} & \text{if } |C| = 1 \end{cases}
\]

\[
f_C(x) = \frac{\ell_C(x)f_{k|k-1}^u(x)}{f_{k|k-1}^u; \ell_C}
\]

\[
L_C = \left\langle D_{k|k-1}^u; \ell_C \right\rangle
\]
ways other than suggested in (37). The choice to separate \( F[j, h] \) into two parts corresponding to the clutter and the undetected targets (\( Y \)), and the detected targets (\( Z \backslash Y \)), will later simplify the complexity reduction in the implementation of the filter. Specifically, it will facilitate the use of standard gating methods and maximum likelihood association. Details are given in a later section.

The updated pgfl (37) has three main components: updated PPP for the undetected targets \( G^{u}_{k|k} \), MB for targets that were detected for the first time at time \( k \); \( G^{P}_{k|k} \), and MB for previously detected targets \( G^{(V, i, j=1)}_{k|k} \). The latter two together form an updated MB for detected targets. Below \( G^{P}_{k|k} \) is referred to as new Bernoulli estimates, and \( G^{(V, i, j=1)}_{k|k} \) is referred to as existing Bernoulli estimates.

The PPP pgfl for undetected targets is updated taking the effective probability of missed detection \( q_D(x) \) into account, see (38). The inner product \( \langle f; q_D \rangle \) is the probability that a target with density \( f(x) \) is not detected. In case \( q_D(x) \) is constant over the entire field of view, the effect of the update of the PPP intensity is that the rate is decreased; \( \mu^{u}_{k|k} = q_{D} \mu^{u}_{k|k-1} \) and \( f^{u}_{k|k}(x) = f^{u}_{k|k-1}(x) \). In case \( q_D(x) \) is non-homogenous, e.g., because of a limited sensor field-of-view (FOV), then the effect of the update is that the PPP intensity is decreased only in areas where \( q_D(x) \) was low (typically inside the FOV). In areas that were not observed, the PPP intensity is largely unaffected due to low \( q_D(x) \). The implication of an area with higher/lower undetected PPP intensity is that more/fewer undetected targets are expected to exist in that area.

The MBM pgfl for detected targets is updated as follows. For each predicted MB component (first summation in (37)), a new MB component is generated for all possible data association events, where a data association event corresponds to the following three events,

1) a separation of the measurement set \( Z \) into one set \( Y \) of detections whose origins were either clutter or previously undetected targets, and one set \( Z \backslash Y \) of detections whose origins were previously detected targets. The second summation in (37) is over all possible subsets \( Y \subset Z \).
2) a partition \( P \) of the set \( Y \) into non-empty subsets that are called cells and are denoted \( C \) in (39), (40). For each cell in the partition, a new Bernoulli estimate is generated with existence probability and spatial density as in (40). The third summation in (37) is over all possible partitions \( P \subset Y \).
3) a set \( \{V_{j, i}\} \) of disjoint subsets of \( Z \backslash Y \), i.e., an association of the detections in the set \( Z \backslash Y \) to the Bernoulli components. For a Bernoulli estimate and an associated subset \( V_{j, i} \), the existence probability and spatial density are updated as in (43). Note that the PPP measurement model allows each Bernoulli estimate to be associated to any number of detections, however each detection can only be associated to a single Bernoulli estimate. The fourth summation in (37) is over all possible ways to associate.

The updated parameters of the new Bernoulli estimates, described in (40), can be understood as follows. Regardless of the cardinality of \( C \), the density of the new Bernoulli component is computed by updating the undetected density \( f^{u}_{k|k-1}(x) \) with the measurements in \( C \), see (40b).

For a partition \( P \), the cells \( C \) are interpreted to contain measurements that are from the same source. The clutter process generates independent point measurements, meaning that two clutter detections should not be in the same cell. Thus, if \( |C| \geq 1 \), then \( r_C = 1 \) for the new Bernoulli estimate, see (40a). However, if \( C = \{ \emptyset \} \), i.e., \( |C| = 1 \), then the probability of existence for the new Bernoulli estimate is computed as in the second alternative in (40a), where \( \kappa_C = \lambda_C(z') \) and \( \mathcal{L}_C = \mathcal{L}(z') = (D^{u}_{k|k-1} \ell(z')) \). This evaluates the clutter PPP intensity \( \kappa \) at the measurement of interest \( z' \).

The likelihood \( \mathcal{L}(z') \) that \( z' \) originated form an undetected target takes into account the extended target measurement PPP intensity \( D^{u}_{k|k-1} \) and the conditional likelihood \( \ell(z') \). The inner product integrates out the target state, and \( \mathcal{L}_C \) can be understood as a “detection-from-target intensity”, evaluated at the measurement of interest \( z' \). Thus, the probability of existence is the normalized likelihood that \( z' \) originated form an undetected target and not from clutter.

The updated parameters of existing Bernoulli estimates are understood as follows. If one or more detections are associated to the estimate, the probability of existence is necessarily one, see (42a), and the density is updated with the detections, see (42b). If no detections are associated to the estimate, the probability of existence and the density are updated according to the effective probability of missed detection, see (43a) and (43b).

Note that if the majority of the probability mass of \( f^{u}_{k|k-1}(x) \) is located where \( q_D(x) \) is high – corresponding to an estimate located outside the FOV– the inner product \( \langle f^{u}_{k|k-1}(x); q_D \rangle \) will be closer to 1 and the updated probability of existence will be almost equal to the predicted probability of existence. However, if the majority of the probability mass of \( f^{u}_{k|k-1}(x) \) is located where \( q_D(x) \) is low – corresponding
to an estimate located inside the FOV—the inner product \( \langle f^{j,i}_{k|k-1}; q_{D} \rangle \) will be closer to 0 and the updated probability of existence will decrease. How much it decreases depends on the effective probability of detection: the higher \( p_{D}(x) \) is, the larger the decrease is.

Lastly, the probability of each updated MBM component (37b) is computed using the predicted likelihoods (40c), (42c), (43c), and (44).

B. Prediction

Let the posterior pgfl at time \( k \) be a PMBM of the form (35). The predicted pgfl at time \( k + 1 \) is given by Theorem 2. The proof of the theorem is omitted due to page length constraints, details can be found in [19].

**Theorem 2:** Given a posterior PMBM pgfl of the form (35), and the standard dynamic models (27a), the predicted pgfl is a PMBM

\[
G_{k+1|k}[h] = G_{k+1|k}^{u}[h]G_{k+1|k}^{d}[h]
\]

The predicted PPP pgfl for the undetected targets is

\[
G_{k+1|k}^{u}[h] = \exp \left( \mu_{k+1}^{u} \left( f^{u}_{k+1|k} - \mu_{k+1}^{u} \right) \right)
\]

with Poisson rate and spatial distribution

\[
\mu_{k+1|k} = \mu_{k+1}^{b} + \left( f^{u}_{k+1|k} - \mu_{k+1}^{u} \right)
\]

\[
f^{u}_{k+1|k}(x_{k+1}) = \frac{\mu_{k+1}^{b}}{\mu_{k+1} + \left( f^{u}_{k+1|k} - \mu_{k+1}^{u} \right)} \cdot f^{u}_{k+1}(x_{k+1}) + \frac{\left( f^{u}_{k+1|k} - \mu_{k+1}^{u} \right)}{\mu_{k+1}^{b} + \left( f^{u}_{k+1|k} - \mu_{k+1}^{u} \right)} \cdot \frac{f^{u}_{k+1|k}}{s_{k+1|k}^{u}}
\]

The predicted MBM pgfl for detected targets is

\[
G_{k+1|k}^{d}[h] = \sum_{j=1}^{J} \sum_{i=1}^{I} \mathcal{W}^{j}_{k+1|k} \prod_{i=1}^{I} \left( 1 - r^{j,i}_{k+1|k} + r^{j,i}_{k+1|k} \cdot \left( f^{j,i}_{k+1|k}; h \right) \right)
\]

where \( \mathcal{W}^{j}_{k+1|k} = \mathcal{W}^{j}_{k|k}, J_{k+1|k} = J_{k|k} \) and \( I_{k+1|k} = I_{k|k} \), and the probabilities of existence and the spatial distributions are

\[
r^{j,i}_{k+1|k} = \frac{\left( f^{j,i}_{k+1|k}; s_{k+1} \right)}{r^{j,i}_{k|k}}
\]

\[
f^{j,i}_{k+1|k}(x_{k+1}) = \frac{\left( f^{j,i}_{k+1|k}; s_{k+1} \right)}{r^{j,i}_{k|k}}
\]

Note that for the inner products in (46c) and (47c) that involve the transition density we have

\[
\frac{\left\langle f_{k|k}; s_{k+1} \right\rangle}{\left\langle f_{k|k}; s_{k+1} \right\rangle} = \int f_{k+1,k}(x_{k+1}) p_{S}(x_{k}) f_{k|k}(x_{k}) d x_{k} \int p_{S}(x_{k}) f_{k|k}(x_{k}) d x_{k}
\]

\[
\int f_{k+1,k}(x_{k+1}) \frac{p_{S}(x_{k}) f_{k|k}(x_{k})}{p_{S}(x_{k})} d x_{k}
\]

The inner product in the denominator serves to normalize \( p_{S}(x_{k}) f_{k|k}(x_{k}) \), i.e. it normalizes the posterior multiplied by the probability of survival, and the result is a predicted spatial density where the probability of survival has been taken into account.

The predicted undetected PPP is the union of two independent PPPs: one for the new birth targets, and one for the undetected targets that survive from earlier time steps. The predicted PPP rate is the sum of the birth PPP rate and the posterior undetected PPP rate, taking the probability of survival into account, see (46b). The inner product \( \langle f; s_{k} \rangle \) is the probability that a target with density \( f(x) \) survives to the next time step. The undetected PPP spatial distribution is a normalized weighted sum of the birth PPP spatial distribution and the prediction of the undetected PPP spatial distribution, see (46c).

For the detected MBM components, the probabilities of existence are decreased taking the probability of survival into account, see (47b), and the spatial distributions are predicted according to the motion model, taking the probability of survival into account, see (48b). Note that the number of MB components in the MBM, and the number of Bernoulli estimates in each MB component, are constant through the prediction.

The prediction in Theorem 2 is equal to the PMBM prediction in [19], which was derived for point targets. As expected, assuming that the standard multi-object dynamics model is used, the type of measurement model does not affect the outcome of the prediction.

C. Complexity analysis and global hypothesis illustration

In this section the growing number of global hypotheses is illustrated using a simple example, and the complexity of the proposed prediction and update is analyzed, with a focus on the number of terms in the MBM.

1) Global hypothesis tree illustration: In Section IV a global hypothesis was defined as a particular history of data associations for all detected targets. Further, it was noted that each MB component in the MBM corresponds to a unique global hypothesis. Here this will be illustrated using a simple example. Consider two consecutive measurement sets \( Z_{1} = \{ z_{1}^{1}, z_{1}^{2} \} \) and \( Z_{2} = \{ z_{2}^{1}, z_{2}^{2} \} \). The MBs following two iterations of prediction and update are illustrated as a tree in Figure 1.

Starting from a PPP prior, i.e., a PMBM with an empty MBM, the first update gives a PMBM with two MB components. These hypotheses correspond to the two possible partitions of \( Z_{1} \), and represent hypotheses that there is one target with probability of existence 1, or two targets with probabilities of existence less than 1. During the second update, the number of hypotheses grows considerably. First, there are four possible sets \( Y \) that represent detections from previously undetected targets or clutter. Second, there are different ways to divide the set \( Z \setminus Y \) into subsets, at least under the hypothesis that there are two detected targets (with existence probabilities less than 1). After the second update the PMBM has 15 MB components.

Note that a global hypothesis may contain Bernoulli estimates with uncertain existence, i.e., \( r < 1 \). In contrast,
actual target existence is binary: the target either exists, or it does not. From each global hypothesis with uncertain target existence, global hypotheses with certain target existence can be found. For example, the uncertain global hypotheses \(|\{f(x|z_1^a, z_1^b, z_2^c), r(z_1^a, z_1^c, z_2^b) = 1, [f(x|z_2^c), r(z_2^c) < 1]\}\) from the example in Figure 1 corresponds to the two certain global hypotheses \(\{f(x|z_1^a, z_1^b, z_2^c), f(x|z_2^c)\}\) and \(\{f(x|z_1^a, z_1^b, z_2^c)\}\). If \(|V|\) is the probability of the uncertain global hypothesis, then the corresponding probabilities of the certain global hypotheses are \(|V|v(z_2^c)\) and \(|V\|v(1-r(z_2^c))\).

With uncertain global hypotheses the example hypothesis tree in Figure 1 has 2 hypotheses after the first time step, and 15 hypotheses after the second time step. With certain global hypotheses, the corresponding numbers are 5 hypotheses after the first time step, and 57 hypotheses after the second time step. It can be concluded that global hypotheses with uncertain target existence are a more compact representation than global hypotheses with certain target existence, thereby offering a possibility of lowering the computational cost of an implementation.

2) Complexity of prediction and update: After the prediction step, the number of MB components in the MBM is the same, see Theorem 2. However, following the update, the number of MB components increases.

For a measurement set \(Z\), the number of ways to define two subsets with cardinality \(|Y|\) and \(|Z\) \(\setminus Y|\) is given by the binomial coefficient \(|Z|! \left(\begin{array}{c} |Y|!(|Z| - |Y|)! \right)^{-1}\). The number of ways to partition the set \(Y\) is given by the Bell number of order \(|Y|\) [23], denoted \(B(|Y|).\). The number of ways to assign \(|Z| \setminus Y|\) measurements to \(I_{k|k-1}^j\) Bernoulli components is \(|I_{k|k-1}^j|\). After the update, the number of components in the MBM will be

\[
J_{k|k} = \sum_{j=1}^{J_{k|k-1}} \sum_{|Y|=0}^{|Z|} \frac{|Z|!}{|Y|!(|Z| - |Y|)!} B(|Y|) \left(I_{k|k-1}^j\right)^{|Z| \setminus Y|}
\]

(49)

It can be shown that if the filter is initialised with \(J_{0|0} = 0\), then the number of MB components at time step \(k\) is given by the Bell number whose order is the sum of the measurement set cardinalities,

\[
J_{k|k} = B\left(\sum_{i=1}^k |Z_i|\right)
\]

(50)

The sequence of Bell numbers is log-convex [24] and \(B(n)\) grows very rapidly. The example in Figure 1 has \(J_{2|2} = B(2 + 2) = 15\) hypotheses. A small increase in the number of detections per time step to four (twice the amount in Figure 1), results in an MBM with \(J_{2|2} = B(4 + 4) = 4140\) hypotheses. Needless to say, approximations are necessary to alleviate the rapidly expanding number of hypotheses.

D. Relationship to other work

The update is derived for the PPP extended target measurement model from [1]. This model has been used before to
TABLE II

| Assumptions |
|-------------|
| • Uniform clutter intensity; the clutter Poisson rate $\lambda$ is known and the spatial distribution is uniform, $f(z) = A^{-1}$, where $A$ is the volume of the surveillance region. |
| • Gaussian measurement and motion models: |
| $f_0(x_k|x_{k-1}) = \mathcal{N}(x_k; H x_{k-1}, R)$ (51a) |
| $f_{k+1,k}(x_{k+1}|x_k) = \mathcal{N}(x_{k+1}; F x_k, Q)$ (51b) |
| The Gaussian measurement model leads to |
| $\prod_{z \in Z} \phi(z_k|x_k) = \prod_{z \in Z} \mathcal{N}(z_k; H x_k, R)$ (52a) |
| $= \mathcal{N}(z_{Z}; H Z x_k, R_Z)$ (52b) |
| where $z_Z$ is vertical vectorial concatenation of the measurements $z$ in the set $Z$, $H_Z = 1_{|Z|} \otimes H$, $R_Z = 1_{|Z|} \otimes R$, and block diagonalization of the matrices $A$ and $B$. |
| • Constant probability of detection, probability of survival, and extended target Poisson rate: $p_D(x_k) = p_D$, $p_S(x_k) = p_S$, and $\gamma(x) = \gamma$. |
| • Known birth PPP with rate $\mu_k^B$ and spatial density |
| $f_k^B(x_k) = \sum_{j=1}^{J_k^B} w_k^{b,j} \mathcal{N}(x_k; m_k^{b,j}, P_k^{b,j})$ (53) |
| • Empty initial PMBM: $p_{0|0} = 0$ and $J_{0|0} = 0$. |

derive PHD filters [7], [14], CPD filters [9], and LMB filters [17]. The PHD filters use a PPP for the detected targets, while the CPD filters use an iid cluster process, and the LMB filters use a labelled MBM.

The functional form used here for both undetected and detected targets—PMBM—is most similar to the point target MTT work presented in [19]. As noted above, the prediction is the same, however, owing to the different measurement models used, the updates are very different. An important difference between the PMBM and LMB filters on the one hand, and the PHD and CPD filters on the other hand, is that the PMBM and LMB representations are conjugate priors, while the PHD and CPD filters are not. The PHD and CPD filters are approximate filters that avoid the data association problem. The PMBM and LMB filters are theoretically exact given enough computational power, however the data association problem arises in both filters, and thus any practical implementation requires approximations of the data association problem.

V. LINEAR GAUSSIAN IMPLEMENTATION

In this section we present an implementation of the filter under the assumption of linear Gaussian motion and measurement models. The assumptions are listed in Table II. Note the implication of the measurement model: the target extents are approximated by a circle whose size is determined by the measurement noise covariance $R$.

A. Update

1) Undetected targets: The effective probability of missed detection [22] is constant. The undetected PPP rate and density, cf. (38) in Theorem I are

$$\mu_k^u = q_D \mu_{k|k-1}^u$$ (54a)

$$f_k^u(x_k) = f_k^u(x_k)$$ (54b)

2) Targets detected for the first time: The PPP intensity of the previously undetected targets is

$$D_k^{u|k|k-1}(x) = \mu_k^u \sum_{j=1}^{J_k^{u|k-1}} w_k^{u,j} \mathcal{N}(x_k; m_k^{u,j}, P_k^{u,j}) (55)$$

For a cell $C$ in a partition $P$ of $Y$ the updated Bernoulli density, cf. (40b) in Theorem I is

$$f_C(x) = \sum_{j=1}^{J_k^{u|k}} w_k^{u,j} \mathcal{N}(x; m_k^{u,j}, P_k^{u,j})$$ (56a)

$$m_k^{u,j} = m_k^{u,j} + K_k^{u,j} (z_C - H C m_k^{u,j})$$ (56b)

$$P_k^{u,j} = (I - K_k^{u,j} H C) P_k^{u,j}_{k|k-1}$$ (56c)

$$K_k^{u,j} = H C P_k^{u,j}_{k|k-1} H C + R_C$$ (56d)

$$S_k^{u,j} = H C P_k^{u,j}_{k|k-1} H C + R_C$$ (56e)

$$u_k^{u,j} = \frac{w_k^{u,j} \mathcal{N}(z_C; H C m_k^{u,j}, s_k^{u,j})}{\sum_{j'=1}^{J_k^{u|k-1}} w_k^{u,j'} \mathcal{N}(z_C; H C m_k^{u,j'}, s_k^{u,j'})}$$ (56f)

Note that $f_C(x)$ is a Gaussian mixture, i.e., it is multi-modal. This follows from the fact that the birth intensity is multi-modal. However, under the assumption that the modes of the birth intensity are distinct one of the modes of $f_C(x)$ will (typically) dominate the others. The Gaussian mixture $f_C(x)$ can then be approximated by the Gaussian that corresponds to the largest weight $u_k^{u,j}$. Alternatively, if the other weights are not negligible, one can merge the Gaussian mixture into a single Gaussian density.

The predicted likelihood, cf. (40c) in Theorem I is

$$L_C = \mu_k^u p_D e^{-\gamma_C} \mathcal{C} (\lambda A^{-1}) \mathcal{C}$$ (57)

$$\times \sum_{j=1}^{J_k^{u|k-1}} w_k^{u,j} \mathcal{N}(z_C; H C m_k^{u,j}, s_k^{u,j})$$

3) Previously detected targets: The $i$th target estimate in the $j$th MBM component has Gaussian spatial distribution,

$$f_{j|k}^{i,k|k-1}(x) = \mathcal{N}(x; m_{k|k-1}^{i,j}, P_{k|k-1}^{i,j})$$ (58)

If the set of detections associated to the estimate is non-empty, $V_{j,i} \neq \emptyset$, then the updated probability of existence is $r_{k|k}^{i,j} = 1$, and the updated Gaussian parameters and the predicted likelihood, cf. (42) in Theorem I are

$$m_{k|k}^{i,j} = m_{k|k-1}^{i,j} + K_{k|k}^{i,j} (z_{V_{j,i}} - H_{V_{j,i}} m_{k|k}^{i,j})$$ (59a)

$$P_{k|k}^{i,j} (V_{j,i}) = (I - K_{k|k}^{i,j} H_{V_{j,i}}) P_{k|k-1}^{i,j}$$ (59b)

$$K_{k|k}^{i,j} (V_{j,i}) = P_{k|k-1}^{i,j} H_{V_{j,i}} (S_{k|k}^{i,j} + R_{V_{j,i}})^{-1}$$ (59c)

$$S_{k|k}^{i,j} (V_{j,i}) = H_{V_{j,i}} P_{k|k-1}^{i,j} H_{V_{j,i}} + R_{V_{j,i}}$$ (59d)

2Sufficiently separated in the surveillance space.
\[
\mathcal{L}_{V_{j,i}} = r_{j,k-1}^{i,j} p_{DE}^{-\gamma} \mathcal{N}(z_{V_{j,i}}; H \mathcal{Y}_{j,i}, m_{i,k|k-1}^{j,i}, S_{i,k|k-1}^{j,i})
\] (59e)

If the set of detections associated to the estimate is empty, \(V_{j,i} = \emptyset\), then the updated probability of existence and the predicted likelihood, cf. (43) in Theorem 1 are,

\[
r_{j,k|k}^{i,j} = \frac{r_{j,k-1}^{i,j} q_{DE}}{1 - r_{j,k-1}^{i,j} + r_{j,k-1}^{i,j} q_{DE}} \tag{60a}
\]

\[
\mathcal{L}_{V_{j,i}} = 1 - r_{j,k-1}^{i,j} + r_{j,k-1}^{i,j} q_{DE} \tag{60b}
\]

and the updated Gaussian parameters are equal to the prediction, i.e., \(m_{i,k|k}^{j,i} = m_{i,k|k-1}^{j,i}\) and \(P_{i,k|k}^{j,i} = P_{i,k|k-1}^{j,i}\).

B. Prediction

1) Undetected targets: The rate and the spatial distribution of the PPP modeling undetected targets are predicted as follows,

\[
p_{k+1|k}^{i,j} = \mu_{k+1}^{b} + p_{S} \mu_{k}^{b} \tag{61a}
\]

\[
f_{k+1|k}(x) = \frac{p_{S} \mu_{k+1}^{b} + p_{S} \mu_{k}^{b}}{p_{S} \mu_{k+1}^{b} + p_{S} \mu_{k}^{b}} \sum_{j=1}^{J_{k+1}} u_{k+1}^{j,i} \mathcal{N}\left(x; m_{k+1}^{j,i}, P_{k+1}^{j,i}\right) \tag{61b}
\]

\[
= \sum_{j=1}^{J_{k+1}} u_{k+1}^{j,i} \mathcal{N}\left(x; m_{k+1}^{j,i}, P_{k+1}^{j,i}\right) \tag{61c}
\]

2) Detected targets: The probability of existence and Gaussian parameters of the \(i\)th Bernoulli estimate in the \(j\)th MB component are predicted as follows,

\[
r_{j,k|k}^{i,j} = p_{S} r_{j,k}^{i,j} \tag{62a}
\]

\[
m_{i,k|k}^{j,i} = F m_{i,k}^{j,i} \tag{62b}
\]

\[
P_{i,k|k}^{j,i} = F P_{i,k}^{j,i} + Q \tag{62c}
\]

C. Complexity reduction

The number of components in the MBM increase rapidly, and approximations are necessary. This section outlines three approximations that are based on clustering of the measurements, standard MTT ellipsoidal gating, and maximum likelihood association. Gating is commonly used in MTT to reduce the computational cost, see e.g. [21]. Clustering has previously been used successfully in extended target MTT, see e.g. [11], [12]. Note that other alternatives to reduce the complexity may exist, a topic for future work is to investigate the pros and cons of different methods.

The summation over all subsets \(\mathcal{Y}\) of the measurement set \(Z\) generally has too many terms to be considered exhaustively. The following approximation is made to alleviate this problem:

- \(\mathcal{Y} \subseteq Z\): Standard ellipsoidal gating, see e.g. [21], is used to limit the number of \(\mathcal{Y}\) that are considered.

The measurements that fall inside the gate of at least one Bernoulli estimate with probability of existence \(r_{j,k|k-1}^{i,j} = f_{j,k|k-1}^{i,j} p_{S}\) is always included in \(Z \setminus \mathcal{Y}\). The measurements that do not fall inside any gate are always included in \(\mathcal{Y}\). Remaining measurements are clustered using Distance Partition, see [11], with maximum distance threshold, and for the cells all combinations of inclusion in \(\mathcal{Y}\) are considered.

Consider the \(j\)th predicted MB component, and a separation of the measurement set \(Z\) into subsets \(\mathcal{Y}\) and \(\mathcal{Z} \setminus \mathcal{Y}\). We can write

\[
\sum_{\mathcal{P} \subseteq \mathcal{Y}} \sum_{\mathcal{V}_{j,i}} W_{i|k}^{j,p,\mathcal{V}_{j,i}} G_{k|k}^{j,p} G_{k|k}^{\mathcal{V}_{j,i}} |h| \tag{63}
\]

\[
\propto \mathcal{W}_{i|k-1}^{j} \left( \sum_{\mathcal{P} \subseteq \mathcal{Y}} \mathcal{L}_{\mathcal{P}} G_{k|k}^{j,p} |h| \right) \left( \sum_{\mathcal{V}_{j,i}} \mathcal{L}_{\mathcal{V}_{j,i}} G_{k|k}^{\mathcal{V}_{j,i}} |h| \right) \tag{64}
\]

Both the summation over all possible partitions of \(\mathcal{Y}\), and the summation over all possible ways to allocate the detections in \(\mathcal{Z} \setminus \mathcal{Y}\) to the \(I_{k}^{j}|k-1\) MB estimates, are generally too complex to consider exhaustively. Instead approximations are necessary. Specifically, we make the following two approximations:

- \(\mathcal{P} \subseteq \mathcal{Y}\): Distance Partition, see [11], with minimum and maximum distance threshold is used to obtain a subset of the possible measurement partitions, and the corresponding likelihoods \(\mathcal{L}_{\mathcal{P}}\) are computed. Only the MBM that corresponds to the maximum \(\mathcal{L}_{\mathcal{P}}\) is returned. This means that we approximate

\[
\sum_{\mathcal{P} \subseteq \mathcal{Y}} \mathcal{L}_{\mathcal{P}} G_{k|k}^{j,p} |h| \approx \mathcal{L}_{\mathcal{P}} G_{k|k}^{j,p} |h| \tag{65}
\]

where \(\mathcal{P}\) is the partition with largest likelihood. Note that, typically, this means that we only consider a very small subset of all possible partitions. However, previous work has shown that this is sufficient, see e.g. [9], [12], [17].

- \(\{\mathcal{V}_{i}\}_{i}: \forall_{i} \mathcal{V}_{i} = Z \setminus \mathcal{Y}\): a maximum likelihood assignment is computed for each measurement. This means that we approximate

\[
\sum_{\{\mathcal{V}_{j,i}\}_{i}} \mathcal{L}_{\{\mathcal{V}_{j,i}\}} G_{k|k}^{\mathcal{V}_{j,i}} |h| \approx \mathcal{L}_{\{\mathcal{V}_{j,i}\}} G_{k|k}^{\mathcal{V}_{j,i}} |h| \tag{66}
\]

where \(\{\mathcal{V}_{j,i}\}_{i}\) results from the maximum likelihood assignment.

The above results in

\[
\approx \mathcal{W}_{i|k-1}^{j} \mathcal{L}_{\mathcal{P}} G_{k|k}^{j,p} |h| \mathcal{L}_{\mathcal{P}} G_{k|k}^{\mathcal{V}_{j,i}} |h| \tag{66a}
\]

\[
= \mathcal{W}_{i|k-1}^{j} \mathcal{L}_{\mathcal{P}} G_{k|k}^{j,p} |h| \mathcal{L}_{\mathcal{P}} G_{k|k}^{\mathcal{V}_{j,i}} |h| \tag{66b}
\]

and we get the following approximation of the updated MBM pgfl for detected targets

\[
G_{k|k}^{j,p} |h| \approx \sum_{j=1}^{J_{k|k-1}} \sum_{\mathcal{Y} \subseteq Z} \mathcal{W}_{i|k}^{j,p,\mathcal{V}_{j,i}} G_{k|k}^{\mathcal{V}_{j,i}} |h| \tag{67a}
\]

\[
\mathcal{W}_{i|k}^{j,p,\mathcal{V}_{j,i}} = \frac{\mathcal{W}_{i|k}^{j,p,\mathcal{V}_{j,i}}}{\sum_{j=1}^{J_{k|k-1}} \sum_{\mathcal{Y} \subseteq Z} \mathcal{W}_{i|k}^{j,p,\mathcal{V}_{j,i}} \mathcal{L}_{\mathcal{P}} G_{k|k}^{\mathcal{V}_{j,i}} |h|} \tag{67b}
\]
Lastly, any MB component with weight smaller than a threshold is pruned from the mixture.

VI. SIMULATION STUDY

In this section the results from a Monte Carlo simulation study are presented. A scenario with 7 targets was randomly generated and simulated for different parameter settings, and the PMBM results were evaluated using standard performance measures.

The scenario has 100 time steps, the target state consist of Cartesian position and Cartesian velocity, and the targets appear in, and disappear from, the surveillance area at different time steps. The measurement model is \( H = [I \ 0] \) and the measurement noise covariance is \( R = I \). Constant velocity motion was simulated, with motion model \( F = [1 \ T; 0 \ 1] \otimes I \) and process noise covariance \( Q = \sigma_a^2 GG^T \) with \( \sigma_a = 1 \), \( G^T = [0.5T^2 I_2 \ I_3] \), and sampling time \( T = 1 \). The probability of detection is set to \( p_D = 0.99 \) and the probability of survival is set to \( p_S = 0.99 \). The birth spatial density consists of four Gaussians, with positions in \([-\pm 7.5, \pm 7.5]^T\).

Extended target Poisson rates \( \gamma \in \{2, 5\} \) and clutter Poisson rates \( \lambda \in \{1, 20\} \) were simulated. This gives four combinations in total; each one was simulated 100 times.

An estimate of the set of targets is obtained by taking the mean vector of all Bernoulli estimates with existence combinations in total; each one was simulated 100 times. \( \text{OSPA} \) the optimal sub-pattern assignment (OSPA) metric \([24]\) with cutoff \( c = 10 \) and degree \( p = 1 \). The performance of the PMBM filter is compared to the performance of an extended target CPHD filter \([9]\).

The results are shown in Figures 2 and 3. For \( \gamma = 5 \) the CPHD has larger OSPA when targets appear and disappear. When \( \gamma = 2 \) the PMBM clearly outperforms the CPHD, especially for higher clutter levels (higher \( \lambda \)). It can be seen that for \( \gamma = 2 \) the PMBM performance deteriorates when \( \lambda \) increases, however for \( \gamma = 5 \) such a trend cannot easily be seen. As expected, performance is better for higher \( \gamma \). The reason is simple: for low \( \gamma \) the clusters of detections generated by the targets contain fewer measurements, and therefore they stand out from the clutter to a much smaller degree, compared to higher \( \gamma \). For \( \gamma = 5 \) cardinality performance is equivalent for PMBM and CPHD, however, for the OSPA the increase in error when targets appear/disappear is larger for the CPHD filter, especially for \( \lambda = 20 \). Note that \( \gamma > 5 \) was also simulated, however the PMBM and CPHD results are indistinguishable, and those results are therefore omitted.

VII. CONCLUSIONS AND FUTURE WORK

This paper has presented a Poisson multi-Bernoulli mixture prior for tracking of multiple extended targets. Even though it is a conjugate prior, it is still prohibitive to implement the filtering recursions exactly and approximations are necessary. The presented linear and Gaussian implementation shows how simple clustering methods and standard ellipsoidal gating can be used to alleviate the problem. A simulation study shows improved performance compared to the extended target CPHD filter.

There are many potential directions for future work. One is to implement the filter for non-linear models, e.g., using extended or unscented Kalman filter approximations in the prediction and the update, or using particle filters. Another important direction for the future is estimation of target trajectories.

APPENDIX A

PROOF OF UPDATE

In this section we show that the prior pgfl \([36]\) and the measurement model \([29]\) results in the PMBM given in Theorem \([4]\). Inserting the prior \( G_k|k-1|h \) into the joint pgfl \( F[g,h] \) (cf. \([28b]\)) gives

\[
F[g,h] = F^C_{u}[g,h] F^d[g,h] \tag{68a}
\]

\[
F^C_{u}[g,h] = G^C[g] G^k_{u|k-1}[h(1 - p_D + p_D e^{\gamma (\phi : g) - \gamma})] \tag{68b}
\]

\[
F^d[g,h] = G^d_{k|k-1}[h(1 - p_D + p_D e^{\gamma (\phi : g) - \gamma})] \tag{68c}
\]

Following the product rule, differentiating \( F[g,h] \) w.r.t. \( Z \) results in

\[
\frac{\delta F[g,h]}{\delta Z_k} = \sum_{Y_k \subseteq Z_k} \frac{\delta F^C_{u}[g,h]}{\delta Y_k} \frac{\delta F^d[g,h]}{\delta Z_k \setminus Y_k} \tag{69}
\]

The differentiation of \( F^C_{u}[g,h] \) w.r.t. \( Y \) is \([7\text{, Lemma } 1]\)

\[
\frac{\delta F^C_{u}[g,h]}{\delta Y} = \kappa^Y F^C_{u}[g,h] \sum_{P \subseteq Y} \prod_{C \in P} d_C[g,h] \tag{70a}
\]

\[
d_C[g,h] = \delta_{1,|C|} \left\{ D^u_{k|k-1} p_D e^{\gamma (\phi : g) - \gamma \langle \gamma \phi \rangle^C_{\kappa^C} \} h \right\} \tag{70b}
\]

where \( \delta_{a,b} = 0 \) if \( a \neq b \) and \( \delta_{a,b} = 1 \) if \( a = b \). Setting \( g(z) = 0 \) gives

\[
\frac{\delta F^C_{u}[g,h]}{\delta Y} \bigg|_{g=0} = \kappa^Y F^C_{u}[0,h] \sum_{P \subseteq Y} \prod_{C \in P} d_C[0,h] \tag{71a}
\]

\[
= F^C_{u}[0,1] F^C_{u}[0,h] \sum_{P \subseteq Y} \kappa^P \prod_{C \in P} d_C[0,h] \tag{71b}
\]

The ratio

\[
\frac{F^C_{u}[0,h]}{F^C_{u}[0,1]} = G^C[0] G^k_{u|k-1}[h p_D] \tag{72a}
\]

\[
= \exp \left\{ \left\langle D^u_{k|k-1} h \right\rangle - \left\langle D^u_{k|k-1} 1 \right\rangle \right\} \tag{72b}
\]

can be identified as a PPP pgfl \( G^k_{u|k}[h] \) with intensity

\[
D^u_{k|k}(x) = q_D(x) D^u_{k|k-1}(x) \tag{73}
\]

The corresponding Poisson rate and spatial density are

\[
\mu^u_{k|k} = \left\langle f^u_{k|k-1} ; q_D \right\rangle \mu^u_{k|k-1} \tag{74a}
\]

\[
f^u_{k|k}(x) = \frac{q_D(x) f^u_{k|k-1}(x)}{\left\langle f^u_{k|k-1} ; q_D \right\rangle} \tag{74b}
\]
Furthermore, we have
\[
d_{C[0, h]} = \delta_{1, |C|} + \left( \frac{D_C; h}{\kappa_C} \right)_{\kappa_C}
\]
\[
D_C(x) = \ell_C(x)D_{k[k-1]}^u(x)
\]
\[
\mathcal{L}_C = (D_C; 1) = \left( D_{k[k-1]}^u; \ell_C \right)
\]
Define the densities \( f_C(x) = D_C(x) / \mathcal{L}_C \). The summands in (70a) can be rewritten
\[
\kappa_C \prod_{C \in P} d_{C[0, h]} = \kappa_C \left( \prod_{C \in P} \left( 1 + \frac{D_C; h}{\kappa_C} \right) \right) \prod_{C \in P} \left( \kappa_C + \mathcal{L}_C \right)
\]
\[
\times \prod_{C \in P} \left( f_C; h \right) \prod_{C \in P} \left( \kappa_C + \mathcal{L}_C \right)
\]
where
\[
\frac{\kappa_C + \mathcal{L}_C (f_C; h)}{\kappa_C + \mathcal{L}_C} = 1 - \frac{\mathcal{L}_C}{\kappa_C + \mathcal{L}_C} + \frac{\mathcal{L}_C}{\kappa_C + \mathcal{L}_C} (f_C; h)
\]
It follows that
\[
\kappa_C \prod_{C \in P} d_{C[0, h]} = \mathcal{L}_P \prod_{C \in P} (1 - r_C + r_C f_C[h])
\]
\[
= \mathcal{L}_P G^P_{k[k]}[h]
\]
\[
\mathcal{L}_P = \prod_{C \in P} \mathcal{L}_C \left( \kappa_C + \mathcal{L}_C \right)
\]
\[
r_C = \left\{ \frac{1}{\kappa_C + \mathcal{L}_C} \right\}_{|C| > 1} \left\{ 1 \right\}_{|C| = 1}
\]

The pgfl \( G^P_{k[k]}[h] \) can be identified as a MB pgfl. To summarize we have
\[
\frac{\delta F^u_{\kappa_C}[h]}{\delta \mathcal{Y}} = \left. F^u_{\kappa_C}|_{g=0} \right|_{\mathcal{Y}} = \left. F^u_{\kappa_C}|_{g=0} \right|_{\mathcal{Y}} \sum_{P \subset \mathcal{Y}} \mathcal{L}_P G^P_{k[k]}[h]
\]
which is a non-normalized PMBM pgfl.

The differentiation of \( F^d_{[g, h]} \) w.r.t. \( \mathcal{Y} \) is
\[
\frac{\delta F^d_{[g, h]}(Z)}{\delta \mathcal{Y}} = \frac{\delta G^d_{k[k-1]}[h]}{\delta \mathcal{Y}} \left( (1 - p_D) + p_D \epsilon^\gamma \overline{\vec{\gamma}} \right)
\]
where
\[
G^d_{k[k-1]}[h] = \left. \left( (1 - p_D) + p_D \epsilon^\gamma \overline{\vec{\gamma}} \right) \right|_{k[k-1]}
\]
and
\[
G^d_{k[k-1]}[h] = \left. \left( (1 - p_D) + p_D \epsilon^\gamma \overline{\vec{\gamma}} \right) \right|_{k[k-1]}
\]
Consider the differentiation of one of the MB pgfls \( C^d_{k[k-1]}[\cdot] \) (indexing w.r.t. time \( k \) and MB component \( j \) is omitted for brevity)
\[
\frac{\delta \prod_{i} (1 - r^i + r^i f^i[h] (1 - p_D + p_D \epsilon^\gamma \overline{\vec{\gamma}}))}{\delta \mathcal{Y}}
\]
\[
= \sum_{\{V_i \}} \prod_{i} d_{V_i}[g, h]
\]
where \( d^d_{V_i} [g, h] \) is defined as follows:

\[
d^d_{V_i} [g, h] = 1 - r^i + r^i \left< f^i; h(1 - pD + pD e^{\gamma \{c;g\} - \gamma}) \right>
\]

(81b)

if \(|V| = 0\) and

\[
d^d_{V_i} [g, h] = r^i \left< f^i; h pD e^{-\gamma (\gamma)} \phi \right> V_i
\]

(81c)

if \(|V| \neq 0\). Let \( g(x) = 0 \), and consider first \( V_i \neq 0 \):

\[
d^d_{V_i} [0, h] = r^i \left< f^i; h qD + pD \phi \right> V_i = L_{V_i} \left< f^i; h \right>
\]

(82a)

\[
L_{V_i} = 1 - r^i + r^i \left< f^i; qD \right>
\]

(82b)

\[
f^i_{V_i} (x) = \left< f^i; h \right> V_i
\]

(82c)

Next, consider \( V_i = \emptyset \):

\[
d^d_{V_i} [0, h] = 1 - r^i + r^i \left< f^i; qD \right>
\]

(83a)

\[
= L_{V_i} \left< f^i; h \right> (1 - r^i + r^i \left< f^i; qD \right>)
\]

(83b)

\[
L_{V_i} = 1 - r^i + r^i \left< f^i; qD \right>
\]

(83c)

\[
f^i_{V_i} (x) = \left< f^i; h \right> V_i
\]

(83d)

with probabilities of existence \( r^i_{V_i} \), densities \( f^i_{V_i} (x) \) and predicted likelihoods \( L_{V_i} \) as in (42) and (43). The pgfl \( G^{V_i} [1] [h] \) can easily be identified as a MB pgfl. In summary, we have

\[
\frac{\delta F^d_{\{g, h\}}}{\delta (Z|Y)} \big|_{g=0} = \sum_{j=1}^{d_{j=k-1}} \sum_{\{V_{j,i}\} \subseteq \{Z\} \setminus \{V\}} \left< L_{V_{j,i}}, G^{V_{j,i}} [1] \right> [h]
\]

(85)

which is a non-normalized MBM pgfl.

From (69), (79) and (85) we get \( \frac{\delta F^d_{\{g, h\}}}{\delta Z} \big|_{g=0} = 0 \) and

\[
\frac{\delta F^d_{\{g, h\}}}{\delta Z} \big|_{q=0, h=1} = \frac{\delta F^d_{\{g, h\}}}{\delta Z} \big|_{g=0}
\]

The updated hybrid MBM pgfl \( G^{k|1} [h] \) then follows from (82a), and was given in (37).

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