ON A RATIONALITY PROBLEM FOR FIELDS OF CROSS-RATIOS

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ABSTRACT. Let $k$ be a field, $n \geq 5$ be an integer, $x_1, \ldots, x_n$ be independent variables and $L_n = k(x_1, \ldots, x_n)$. The symmetric group $\Sigma_n$ acts on $L_n$ by permuting the variables, and the projective linear group $\text{PGL}_2$ acts by applying (the same) fractional linear transformation to each variable. The fixed field $K_n = L_n^{\text{PGL}_2}$ is called “the field of cross-ratios”. Let $S \subset \Sigma_n$ be a subgroup. The Noether Problem asks whether the field extension $L_n^S/k$ is rational, and the Noether Problem for cross-ratios asks whether $K_n^S/k$ is rational. In an effort to relate these two problems, H. Tsunogai posed the following question: Is $L_n^S$ rational over $K_n^S$? He answered this question in several situations, in particular, in the case where $S = \Sigma_n$. In this paper we extend his results by recasting the problem in terms of Galois cohomology. Our main theorem asserts that the following conditions on a subgroup $S \subset \Sigma_n$ are equivalent: (a) $L_n^S$ is rational over $K_n^S$, (b) $L_n^S$ is unirational over $K_n^S$, (c) $S$ has an orbit of odd order in $\{1, \ldots, n\}$.

1. Introduction

Let $k$ be a base field, $n \geq 5$ be an integer, $x_1, \ldots, x_n$ be independent variables, and

$$L_n = k(x_1, \ldots, x_n).$$

The group $\text{PGL}_2$ acts on $L_n$ via

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot x_i \mapsto \frac{ax_i + b}{cx_i + d}$$

for $i = 1, \ldots, n$. The field of invariants $K_n = L_n^{\text{PGL}_2}$ is generated over $k$ by the $n-3$ cross-ratios

$$[x_1, x_2, x_3, x_i] = \frac{(x_i - x_1)(x_3 - x_2)}{(x_i - x_2)(x_3 - x_1)}, \quad i = 4, \ldots, n.$$ 

For this reason we will refer to $K_n$ as the field of cross-ratios. The natural action of the symmetric group $\Sigma_n$ on $L_n$ induced by permuting the variables descends to a faithful action on $K_n$. Suppose $S$ is a subgroup of $\Sigma_n$.

The Noether problem asks whether the fixed field $L_n^S$ is rational (respectively, stably rational or retract rational) over $k$. The Noether Problem for cross-ratios is whether or not $K_n^S$ is rational (respectively, stably rational or retract rational) over $k$. In an effort to relate these two problems, H. Tsunogai [Tsu17] posed the following question:

**Question 1.** Is $L_n^S$ rational over $K_n^S$?

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He answered this question in several situations (see [1su17, Theorems 1, 2, 3]) in particular, in the case, where $S = \Sigma_n$. Our main theorem generalizes his results as follows.

**Theorem 2.** Let $S$ be a subgroup of the symmetric group $\Sigma_n$, where $n \geq 5$. Then the following conditions are equivalent:

(a) $L_n^S$ is rational over $K_n^S$,
(b) $L_n^S$ is unirational over $K_n^S$,
(c) $S$ has an orbit of odd order in $\{1, \ldots, n\}$.

The remainder of this note will be devoted to proving Theorem 2.

2. Recasting the problem in the language of Galois cohomology

Let $G$ be the subgroup of $(\text{GL}_2)^n = \text{GL}_2 \times \cdots \times \text{GL}_2$ consisting of $n$-tuples $(g_1, \ldots, g_n)$ such that $g_1 = \cdots = g_n$. Here $\overline{g}$ denotes the image of $g \in \text{GL}_2$ in $\text{PGL}_2$. In other words, $(g_1, \ldots, g_n) \in (\text{GL}_2)^n$ lies in $G$ if and only if $g_1, \ldots, g_n$ are scalar multiples of each other. The symmetric group $\Sigma_n$ acts naturally on $(\text{GL}_2)^n$ by permuting the entries; $G$ is invariant under this action. For any subgroup $S \subset \Sigma_n$, we will denote the semidirect product $G \rtimes S$ by $G_S$.

This gives rise to the natural split exact sequence

\[ 1 \longrightarrow G \overset{i}\longrightarrow G_S \overset{\phi}\longrightarrow S \longrightarrow 1 \tag{1} \]

We will also be interested in another exact sequence,

\[ 1 \longrightarrow (\text{GL}_2)^n \overset{\alpha}\longrightarrow G_S \overset{\beta}\longrightarrow \text{PGL}_2 \longrightarrow 1 \tag{2} \]

where map $G \to \text{PGL}_2$ sends $(g_1, \ldots, g_n) \in G$ to $\overline{g_1} = \cdots = \overline{g_n}$.

Consider the natural linear action of $G_S$ on the $2n$-dimensional affine space $V = (\mathbb{A}^2)^n$ defined as follows: $(g_1, \ldots, g_n) \in G$ acts on $(\mathbb{A}^2)^n$ by

\[ (g_1, \ldots, g_n) : (v_1, \ldots, v_n) \mapsto (g_1v_1, \ldots, g_nv_n) \]

and $\sigma \in S \subset \Sigma_n$ by

\[ \sigma : (v_1, \ldots, v_n) \mapsto (v_{\sigma(1)}, \ldots, v_{\sigma(n)}). \]

One readily checks that this action is generically free. (Recall that our standing assumption is that $n \geq 5$.) That is, $V$ has a dense $G$-invariant Zariski open subset $V_0$, such that the stabilizer of $v$ in $G_S$ is trivial for every $v \in V_0$. After passing to a smaller $G_S$-invariant open subset, we may assume that $V_0$ is the total space of a $G_S$-torsor $T_S : V_0 \to Z_S$ for some $k$-variety $Z_S$; see [BF03, Theorem 4.7]. We thus obtain the following diagram:

\[ \begin{array}{ccc}
V_0 & \longrightarrow & Z_S \\
\downarrow & & \downarrow \eta \\
Y_S & \longrightarrow & T_S, \text{ a } G_S\text{-torsor} \\
\downarrow & & \downarrow \\
t_S, \text{ a } \text{PGL}_2\text{-torsor} & & \\
\end{array} \]

where $Y_S = V_0/(\text{GL}_2^n \rtimes S)$. The function fields $k(Z_S)$ and $k(Y_S)$ are naturally isomorphic to $L_n^S$ and $K_n^S$, respectively. When we pass to the generic point $\eta$ of $Z_S$, $T_S$ gives rise to a $G_S$-torsor
(V₀)η → Spec(KⁿΣ) and tₛ to a PGL₂-torsor (Yₛ)η → Spec(KⁿΣ), respectively. By abuse of notation we will continue to denote these torsors by Tₛ and tₛ.

Now let K is an arbitrary field. Recall that Gₛ-torsors over Spec(K) are classified by the Galois cohomology set H¹(K, Gₛ), and PGL₂-torsors are classified by H¹(K, PGL₂); see [Se97, §I.5.2]. We will denote the classes of Tₛ and tₛ by [Tₛ] ∈ H¹(KⁿΣ, Gₛ) and [tₛ] ∈ H¹(KⁿΣ, PGL₂), respectively. The exact sequences (1) and (2) of algebraic groups give rise to exact sequences of Galois cohomology sets

\[
\begin{align*}
H¹(K, G) & \xrightarrow{i₁} H¹(K, Gₛ) \xrightarrow{φ₁} H¹(K, S) \\
H¹(K, Gⁿ₂ × S) & \xrightarrow{α₁} H¹(K, Gₛ) \xrightarrow{β₁} H¹(K, PGL₂)
\end{align*}
\]

and

for any field K. If K = KⁿΣ, then by our construction [tₛ] = β₁([Tₛ]). The following proposition recasts Question[1] in the language of Galois cohomology.

**Proposition 3.** The following conditions on a subgroup S ⊂ Σₙ are equivalent:

(a) Lₙ is rational over KⁿS,
(b) Lₙ is unirational over KⁿS,
(c) [tₛ] is the trivial class in H¹(KⁿΣ, PGL₂),
(d) β₁ : H¹(K, Gₛ) → H¹(K, PGL₂) is the trivial for every field K containing k,
(e) α₁ : H¹(K, Gⁿ₂ × S) → H¹(K, Gₛ) is surjective for every field K containing k,
(f) φ₁ : H¹(K, Gₛ) → H¹(K, S) is bijective for every field K containing k,
(g) H¹(K, τG) = 1 for every τ ∈ H¹(K, S).

In part (g), τG denotes the twist of G by τ via the natural permutation action of S on G. For generalities on the twisting operation, see [Se97, Section I.5.3] or [B10, Section II.5]. Note in particular that τG is an algebraic group over K; it does not descend to k in general.

**Remark 4.** The Galois cohomology set H¹(K, PGL₂) is in a natural (i.e., functorial in K) bijective correspondence with the set of isomorphism classes of quaternion algebras over K; see [Se03, §I.2 and I.3]. Thus condition (c) amounts to saying that a certain quaternion algebra over KⁿS is split.

We defer the proof of Proposition[5] to Section[4]

3. **Generalities on Galois cohomology**

Suppose i : A → B is a morphism of algebraic groups over k, and K is a field containing k. Following the notational conventions of the previous section, we will denote the induced map Hᵈ(K, A) → Hᵈ(K, B) of cohomology sets by iᵈ. Here d = 0 or 1.

The following lemma will be used in the proof of Proposition[3].

**Lemma 5.** Consider the exact sequence

\[
\begin{array}{c}
1 \rightarrow A \xrightarrow{i} B \xrightarrow{π} C \xrightarrow{σ} 1
\end{array}
\]

\[
(5)
\]
of smooth algebraic groups over a field $k$. Then

(a) The map $\pi_1: H^1(K, B) \to H^1(K, C)$ is surjective for every field $K/k$.
(b) $\pi_1$ is injective if and only if $H^1(K, s_1(\gamma)A) = 1$ for every $\gamma \in H^1(K, C)$.

Proof. (a) is clear, since $s_1: H^1(K, C) \to H^1(K, B)$ is a section for $\pi_1$. To prove (b), twist the exact sequence (5) by $\tau = s_1(\gamma)$ to obtain a new exact sequence

$$1 \rightarrow \tau A \rightarrow \tau B \rightarrow \tau C \rightarrow 1$$

of algebraic groups over $K$ and consider the associated long exact sequence

$$H^0(K, \tau B) \rightarrow H^0(K, \tau C) \rightarrow H^1(K, \tau A) \rightarrow H^1(K, \tau B) \rightarrow H^1(K, \tau C)$$

in cohomology. Note that $\tau C$ is naturally isomorphic to $\gamma C$. By [Se97, Corollary I.5.5.2], the fiber of $\gamma$ under $\pi_1$ is in bijective correspondence with the set of $H^0(K, \tau C)$-orbits in $H^1(K, \tau A)$. We are interested in the case, where $\pi_1$ is injective, i.e., this fiber is trivial for every $\gamma \in H^1(K, C)$.

Since $\tau s_0$ is a section for $\tau \pi_0$, we see that $\tau \pi_0$ is surjective. Thus the connecting map $\delta$ in the long exact sequence (6) sends every element of $H^0(K, \tau C)$ to the trivial element of $H^1(K, \tau A)$. Consequently, $H^0(K, \tau C)$ acts trivially on $H^1(K, \tau A)$. We conclude that the fiber of $\gamma$ under $\pi_1$ is in bijective correspondence with $H^1(K, \tau A)$. In particular, $\pi_1$ is injective if and only if $H^1(K, \tau A) = 1$ for every $\gamma$, as claimed. \qed

Corollary 6. Consider the split exact sequence

$$1 \rightarrow \mathbb{G}_m^n \rightarrow \mathbb{G}_m^n \rtimes S \rightarrow S \rightarrow 1,$$

where $S$ is a subgroup of $\Sigma_n$, and $\mathbb{G}_m^n \rtimes S$ is the semidirect product with respect to the natural (permutation) action of $S$ on $\mathbb{G}_m^n$. Then $\pi$ and $s$ induce mutually inverse bijections $\pi_1: H^1(K, \mathbb{G}_m^n \rtimes S) \rightarrow H^1(K, S)$ and $s_1: H^1(K, S) \rightarrow H^1(K, \mathbb{G}_m^n \rtimes S)$ for every field $K$ containing $k$.

Proof. By Lemma 5 it suffices to show that

$$H^1(K, \gamma(\mathbb{G}_m^n)) = 1$$

for every $\gamma \in H^1(K, S)$.

Here $S$ acts on $\mathbb{G}_m^n$ by permuting the $n$ copies of $\mathbb{G}_m$. Thus the twisted group $\gamma(\mathbb{G}_m^n)$ is a quasi-trivial torus, and (8) follows from the Feddeev-Shapiro Lemma [Se97, Section I.2.5]. \qed

4. Proof of Proposition 3

The implication (a) $\implies$ (b) is obvious.

(b) $\implies$ (c): If $L^n_S$ is unirational over $K^n_S$, then $t_S$ has a rational section, and (c) follows.

(c) $\implies$ (a): If $[t_S] = 1$ is the trivial class in $H^1(K^n_S, \text{PGL}_2)$, then $Y_S$ is birationally isomorphic to $Z_S \times \text{PGL}_2$ over $Z_S$. Since the group variety of $\text{PGL}_2$ is rational over $k$, this tells us that $Y_S$ is rational over $Z_S$. Equivalently, $k(Y_S) = L^n_S$ is rational over $k(Z_S) = K^n_S$. \qed
Let us examine the induced sequence $S$ composition is the identity map $H$ trivial in $\beta$ over a finite field $K$, the case, where $K$ is a finite field. By Wedderburn’s “little theorem” every quaternion algebra over a finite field $K$ is split. In view of Remark[4] this translates to $H^2(K, \text{PGL}_2) = \{1\}$. We conclude that the map $H^1(K, G_S) \rightarrow H^1(K, \text{PGL}_2)$ is trivial for every field $K$ containing $k$.

(d) $\implies$ (c): By [Se03] Example I.5.4, $[T_S]$ is a versal $G_S$-torsor. This implies that if $[t_S] = \beta_1([T_S])$ is trivial in $H^1(K_n^S, \text{PGL}_2)$, then the image of every element of $H^1(K, G_S)$ under $\beta_1$ is trivial in $H^1(K, \text{PGL}_2)$ for every infinite field $K$ containing $k$, as desired. It remains to consider the case, where $K$ is a finite field. By Proposition[3] it suffices to show that $r$ is bijective, as claimed.

(d) $\iff$ (e): Follows from the fact that the sequence (4) is exact.

(e) $\iff$ (f): Consider the group homomorphisms $S \xrightarrow{s} \mathbb{G}_m^n \rtimes S \xrightarrow{\alpha} G_S \xrightarrow{\phi} S$, whose composition is the identity map $S \rightarrow S$. Note that here $\phi$, $\alpha$ and $s$ are the same as in (1), (2), and (7), respectively. Let us examine the induced sequence

$$H^1(K, S) \xrightarrow{\simeq} H^1(K, \mathbb{G}_m^n \rtimes S) \xrightarrow{\alpha_1} H^1(K, G_S) \xrightarrow{\phi_1} H^1(K, S)$$

in cohomology. By Corollary[6] $s_1$ is an isomorphism. Thus $\alpha_1$ is surjective if and only if $\phi_1$ is bijective, as claimed.

(f) $\iff$ (g): Immediate from Lemma[5] applied to the exact sequence (1).

5. REDUCTION TO THE CASE, WHERE $S$ IS A 2-GROUP

**Lemma 7.** Let $P$ be a subgroup of $S$. Assume that the index $d = [S : P]$ is odd. Then $[t_S]$ is trivial in $H^1(K_n^S, \text{PGL}_2)$ if and only if $[t_P]$ is trivial in $H^1(K_n^P, \text{PGL}_2)$.

**Proof.** The diagram

$$
\begin{array}{c}
\text{TP} \\
Y_P \quad \text{deg d} \quad Y_S \\
Z_P \quad \text{deg d} \quad Z_S \\
\end{array}
\begin{array}{c}
\text{TS} \\
V_0 \\
\end{array}
\begin{array}{c}
\text{TP} \\
Y_P \\
Z_P \\
\end{array}
\begin{array}{c}
\text{TS} \\
Y_S \\
Z_S \\
\end{array}
\begin{array}{c}
\text{TP} \\
Y_P \quad \text{deg d} \quad Y_S \\
Z_P \quad \text{deg d} \quad Z_S \\
\end{array}
\begin{array}{c}
\text{TS} \\
V_0 \\
\end{array}
$$

shows that $[t_P]$ is the image of $[t_S]$ under the restriction map $r : H^1(K_n^S, \text{PGL}_2) \rightarrow H^1(K_n^P, \text{PGL}_2)$.

By Proposition[3] it suffices to show that $r$ has trivial kernel. By Remark[4] elements of the Galois cohomology set $H^1(K, \text{PGL}_2)$ can be identified with quaternion algebras over $K$ (up to $K$-isomorphism). The map $r$ sends a quaternion algebra $A$ over $K_n^S$ to the quaternion algebra $A \otimes_{K_n^S} K_n^P$ over $K_n^P$. Since $K_n^P / K_n^S$ is a field extension of odd degree, $A \otimes_{K_n^S} K_n^P$ is split if and only if $A$ is split. Thus $r$ has trivial kernel, as claimed.

Combining Lemma[7] with the equivalence of (a), (b), (c) in Proposition[3] we see that for the purpose of proving Theorem[2], $S$ may be replaced its $2$-Sylow subgroup $P$. Note that $S$ has an orbit of odd order in $\{1, 2, \ldots, n\}$ if and only if $P$ has a fixed point, by the equivalence of parts (a), (b) and (g) in Proposition[3] in order to complete the proof of Theorem[2] it suffices to establish the following.
Proposition 8. Let $S$ be a 2-subgroup of $\Sigma_n$. Then the following conditions are equivalent.

(i) $H^1(K, \tau G) = 1$ for every $\tau \in H^1(K, S)$.

(ii) $S$ has a fixed point in $\{1, \ldots, n\}$.

6. Conclusion of the proof of Theorem 2

In this section we will complete the proof of Theorem 2 by establishing Proposition 8.

Denote the orbits of $S$ in $\{1, \ldots, n\}$ by $O_1, \ldots, O_t$ where $O_i \simeq S/S_i$ as a $G$-set. Here $S_i$ is the stabilizer of a point in $O_i$. The groups $S_1, \ldots, S_t$ are uniquely determined by the embedding $S \hookrightarrow \Sigma_n$ up to conjugacy and reordering. Note that $S_1, \ldots, S_t$ may not be distinct.

Recall that elements of $\tau \in H^1(K, S)$ are in a natural bijective correspondence with $S$-Galois algebras $L/K$. Here by an $S$-Galois algebra $L/K$ we mean an étale algebra (i.e., a direct sum of finite separable field extensions of $K$) equipped with a faithful action of $S$ such that $\dim_K(L) = |S|$ and $L^S = K$; see [Se03, Example 2.2]. To an $S$-Galois algebra $L/K$ one can naturally associate the étale $K$-algebra

$$E = L^{S_1} \times \cdots \times L^{S_t}$$

of degree $n$.

Now observe that the group $G$ (defined at the beginning of Section 2) admits the following alternative description. Consider the natural surjective map $f : GL_2 \times \mathbb{G}^m \rightarrow G$ given by

$$(g, t_1, \ldots, t_n) \mapsto (gt_1, gt_2, \ldots, gt_n).$$

The kernel of $f$ is $\Delta = \{(tI_2, t, \ldots, t) | t \in \mathbb{G}^m\} \simeq \mathbb{G}^m$. Thus $f$ induces an isomorphism $G \simeq (GL_2 \times \mathbb{G}^m)/\Delta$ of algebraic groups. Moreover, this isomorphism is $S$-equivariant with respect to the natural actions of $S$ on $G$ (described at the beginning of section 2) and $(GL_2 \times \mathbb{G}^m)/\Delta$ (via permuting the $n$ components of $\mathbb{G}^m$). The twisted forms $\tau G$ of $G$ and the Galois cohomology sets $H^1(K, \tau G)$ are explicitly described in [FR18]. In particular, if $\tau \in H^1(K, S)$ corresponds to the $n$-dimensional étale $K$-algebra $E$ as above, then $\tau G \simeq (GL_2 \times R_{E/K}(\mathbb{G}^m))/\Delta_K$, where $R_{E/K}$ denotes Weil restriction and $\Delta_K \simeq \tau \Delta$ is $\mathbb{G}^m$ (over $K$), diagonally embedded into $GL_2 \times R_{E/K}(\mathbb{G}^m)$; see [FR18, Section 4]. Moreover,

$$H^1(K, \tau G) \simeq \{\text{isom. classes of quaternion $K$-algebras $A$ such that $A$ is split by $E \otimes_K K$}\};$$

see [FR18] Lemma 5.1. Note that $A$ is split by $E \otimes_K K$ if and only if $A$ is split by $L^{S_i} \otimes_K K$ for every $i = 1, \ldots, t$. This explicit description of $H^1(K, \tau G)$ reduces Proposition 8 to the following equivalent form.

Proposition 9. Let $S \subset \Sigma_n$ be a 2-group. Then the following conditions on $S$ are equivalent.

(a) There exists a field a field extension $K/k$, a quaternion division algebra $A/K$ and $S$-Galois algebra $L/K$ such that $A$ splits over $L^{S_i}$ for every $i = 1, \ldots, t$.

(b) $S$ does not have a fixed point in $\{1, \ldots, n\}$.

To prove (a) $\implies$ (b), assume that $S$ has a fixed point. That is, one of the orbits of $S$ in $\{1, \ldots, n\}$, say $O_1$, consists of a single point. Equivalently, $S_1 = S$. Clearly a quaternion division algebra over $K$, cannot split over $L^{S_1} = L^K = K$.
To prove (b) $\implies$ (a), assume $S$ does not have a fixed point in $\{1, 2, \ldots, n\}$, i.e., $|O_i| = |S/S_i| \geq 2$ for every $i = 1, \ldots, t$. Since $S$ is a 2-group, each $S_i$ is contained in a maximal proper subgroup of $S$. That is, for each $i = 1, \ldots, t$, there exists a subgroup $S_i \subseteq H_i \subsetneq S$ such that $[S : H_i] = 2$.

Note that $H_i$, being a subgroup of index 2, is normal in $S$.

Now let $M/F$ be an $S$-Galois field extension. For example, we can let $S$ act on $M = k(x_1, \ldots, x_n)$ by permuting the variables and set $F = M^S$. Since $H_i$ is normal in $S$, $M^{H_i}/F$ is a Galois extension of degree 2 for each $i = 1, \ldots, t$. By a theorem of M. Van den Bergh and A. Schofield [VdB-S94 Theorem 3.8], there exists a field extension $K/F$ and a quaternion division algebra $A/K$ such that $A$ contains $M^{H_i} \otimes_F K$ as a maximal subfield for each $i = 1, \ldots, t$. Now consider the $S$-Galois algebra $L = M \otimes_F K$ over $K$. For each $i = 1, \ldots, t$, $L^{S_i}$ contains $L^{H_i} = M^{H_i} \otimes_F K$. Hence, each $L^{S_i}$, splits $A$, as desired.

This completes the proof of Proposition 9 and thus of Proposition 8 and Theorem 2.$\square$

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REFERENCES

[B10] G. Berhuy, An introduction to Galois cohomology and its applications, London Mathematical Society Lecture Note Series, 377, Cambridge University Press, Cambridge, 2010. MR2723693

[BF03] G. Berhuy and G. Favi, Essential dimension: a functorial point of view (after A. Merkurjev), Doc. Math. 8 (2003), 279–330. MR2029168

[FR18] M. Florence, Z. Reichstein, The rationality problem for forms of moduli spaces of $\overline{M}_{0,n}$, Bull. London Math. Soc., 50, no. 1 (2018), 148–158.

[Se97] J.-P. Serre, Galois cohomology, Springer-Verlag, Berlin, 1997.

[Se03] J.-P. Serre, Cohomological invariants, Witt invariants, and trace forms, notes by Skip Garibaldi, in Cohomological invariants in Galois cohomology, 1–100, Univ. Lecture Ser., 28, Amer. Math. Soc., Providence, RI, 2003. MR1999384

[Tsu17] H. Tsunogai, Toward Noether’s problem for the fields of cross-ratios, Tokyo J. Math. 39 (2017), no. 3, 901–922. MR3634298

[VdB-S94] M. Van den Bergh and A. Schofield, Division algebra coproducts of index $n$, Trans. Amer. Math. Soc. 341 (1994), no. 2, 505–517. MR1033236

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