Limit theorems for critical first-passage percolation on the triangular lattice

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Abstract

Consider (independent) first-passage percolation on the sites of the triangular lattice \( \mathbb{T} \) embedded in \( \mathbb{C} \). Denote the passage time of the site \( v \) in \( \mathbb{T} \) by \( t(v) \), and assume that \( P(t(v) = 0) = P(t(v) = 1) = 1/2 \). Denote by \( b_{0,n} \) the passage time from 0 to the halfplane \( \{ v \in \mathbb{T} : \text{Re}(v) \geq n \} \), and by \( T(0, nu) \) the passage time from 0 to the nearest site to \( nu \), where \( |u| = 1 \).

We prove that as \( n \to \infty \), \( b_{0,n}/\log n \to 1/(2\sqrt{3}\pi) \) a.s., \( E[b_{0,n}]/\log n \to 1/(2\sqrt{3}\pi) \) and \( \text{Var}[b_{0,n}]/\log n \to 2/(3\sqrt{3}\pi) - 1/(2\pi^2) \); \( T(0, nu)/\log n \to 1/(\sqrt{3}\pi) \) in probability but not a.s., \( E[T(0, nu)]/\log n \to 1/(\sqrt{3}\pi) \) and \( \text{Var}[T(0, nu)]/\log n \to 4/(3\sqrt{3}\pi) - 1/\pi^2 \). This answers a question of Kesten and Zhang (1997) and improves our previous work (2014). From this result, we derive an explicit form of the central limit theorem for \( b_{0,n} \) and \( T(0, nu) \). A key ingredient for the proof is the moment generating function of the conformal radii for conformal loop ensemble CLE_6, given by Schramm, Sheffield and Wilson (2009).

Keywords: critical percolation; first-passage percolation; scaling limit; conformal loop ensemble; law of large numbers; central limit theorem

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1 Introduction

First-passage percolation (FPP) was introduced by Hammersley and Welsh in 1965 as a model of fluid flow through a random medium. We refer the reader to the recent surveys [2, 9]. In this paper, we continue our study of critical FPP on the triangular lattice \( \mathbb{T} \), initiated in [25]. We focus on this particular lattice because our proof relies on the existence of the scaling limit of critical site percolation on \( \mathbb{T} \) (see [4, 19]), and this result has not been proved for other planar percolation processes. For recent progress on general planar critical FPP, see [7].

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Let $T = (V, E)$ denote the triangular lattice, where $V := \{ x + ye^{\pi i/3} \in \mathbb{C} : x, y \in \mathbb{Z} \}$ is the set of sites, and $E$ is the set of bonds, connecting adjacent sites. Let $\{t(v) : v \in V\}$ be an i.i.d. family of Bernoulli random variables:

\[ P[t(v) = 0] = P[t(v) = 1] = \frac{1}{2}. \]

We call this model Bernoulli critical FPP on $T$, and denote by $P$ its probability measure. Note that we can view this model as critical site percolation on $T$ (see e.g. [15, 23] for background on two-dimensional critical percolation). We usually represent it as a random coloring of the faces of the dual hexagonal lattice $H$, each face centered at $v \in V$ being blue ($t(v) = 0$) or yellow ($t(v) = 1$) with probability $1/2$ independently of the others. Sometimes we view the site $v$ as the hexagon in $H$ centered at $v$.

A path is a sequence $v_0, \ldots, v_n$ of distinct sites of $T$ such that $v_{k-1}$ and $v_k$ are neighbors for all $k = 1, \ldots, n$. For a path $\gamma$, we define its passage time as $T(\gamma) := \sum_{v \in \gamma} t(v)$. The first-passage time between two site sets $A, B$ is defined as

\[ T(A, B) := \inf \{ T(\gamma) : \gamma \text{ is a path from a site in } A \text{ to a site in } B \}. \]

For any $u \in \mathbb{C}$ with $|u| = 1$, denote by $T(0, nu)$ the first-passage time from 0 to the nearest site in $V$ to $nu$ (if there are more than one such sites, we choose a unique one by some deterministic method). Denote by $b_{0,n}$ the first-passage time from 0 to the halfplane $\{ v \in V : \text{Re}(v) \geq n \}$.

Our main theorem below answers a question proposed by Kesten and Zhang (see (1.10) and (1.11) in [11]):

**Theorem 1.1.**

\[ \lim_{n \to \infty} \frac{b_{0,n}}{\log n} = \frac{1}{2\sqrt{3}\pi} \text{ a.s.,} \]  
\[ \lim_{n \to \infty} \frac{E[b_{0,n}]}{\log n} = \frac{1}{2\sqrt{3}\pi}, \]  
\[ \lim_{n \to \infty} \frac{\text{Var}[b_{0,n}]}{\log n} = \frac{2}{3\sqrt{3}\pi} - \frac{1}{2\pi^2}. \]

For each $u \in \mathbb{C}$ with $|u| = 1$,

\[ \lim_{n \to \infty} \frac{T(0, nu)}{\log n} = \frac{1}{\sqrt{3}\pi} \text{ in probability but not a.s.,} \]  
\[ \lim_{n \to \infty} \frac{E[T(0, nu)]}{\log n} = \frac{1}{\sqrt{3}\pi}, \]  
\[ \lim_{n \to \infty} \frac{\text{Var}[T(0, nu)]}{\log n} = \frac{4}{3\sqrt{3}\pi} - \frac{1}{\pi^2}. \]

**Remark.** In [23], we proved the law of large numbers for $b_{0,n}$ and $T(0, nu)$, but cannot give exact values of the limits. The proof in that paper relies on the subadditive ergodic theorem, which is a nice tool to show the existence of the limit but gives no insight for the exact value of the limit.
Kesten and Zhang [11] proved a central limit theorem for $b_{0,n}$ and $T(0,nu)$. Combining their result and Theorem 1.1 we obtain the explicit form of the CLT:

**Corollary 1.2.**

\[
\frac{b_{0,n} - \log n}{\sqrt{\left(\frac{2}{3\sqrt{\pi}} - \frac{1}{2\pi}\right) \log n}} \frac{d}{N(0,1)} \text{ as } n \to \infty.
\]

For each $u \in \mathbb{C}$ with $|u| = 1$,

\[
\frac{T(0,nu) - \log n}{\sqrt{\left(\frac{4}{3\sqrt{\pi}} - \frac{1}{2\pi}\right) \log n}} \frac{d}{N(0,1)} \text{ as } n \to \infty.
\]

**Idea of the Proof.** Instead of dealing with $b_{0,n}$ and $T(0,nu)$ directly, we consider $c_n$ which is the first-passage time from 0 to a circle of radius $n$ centered at 0. We shall show analogous limit theorem for $c_n$, then Theorem 1.1 follows from this. Using a color switching trick we obtain that annulus time has the same distribution as the number of cluster boundary loops surrounding 0 in the annulus (under a monochromatic boundary condition). Camia and Newman’s full scaling limit [4] and moment generating function of the conformal radii for CLE$_6$ [16] allow us to derive limit theorem for the scaling limit of annulus times. Then from this we get the limit result for $c_n$ and $E[c_n]$ easily. In order to prove the limit result for $\text{Var}[c_n]$, we will use a martingale approach from [11].

## 2 Notation and preliminaries

We denote the underlying probability space by $(\Omega, \mathcal{F}, P)$, where $\Omega = \{0,1\}^V$ (or $\{\text{blue,yellow}\}^V$), $\mathcal{F}$ is the cylinder $\sigma$-field and $P$ is the joint distribution of $\{t(v) : v \in V\}$. A circuit is a path whose first and last sites are neighbors. For a circuit $\mathcal{C}$, define

\[
\overline{\mathcal{C}} := \mathcal{C} \cup \text{interior sites of } \mathcal{C}.
\]

For $r > 0$, let $D_r$ denote the Euclidean disc of radius $r$ centered at 0 and $\partial D_r$ denote the boundary of $D_r$. Write $\mathbb{D} := D_1$. For $v \in V$, let $B(v, r)$ denote the set of hexagons of $\mathbb{H}$ that are contained in $v + D_r$. We will sometimes see $B(v, r)$ as a union of these closed hexagons. For $B(v, r)$, denote by $\partial B(v, r)$ its (topological) boundary and by $\Delta B(v, r)$ its external site boundary (i.e., the set of hexagons that do not belong to $B(v, r)$ but are adjacent to hexagons in it). Write $B(r) := B(0, r)$.

Curves are equivalence classes of continuous functions from the unit interval to $\mathbb{C}$, modulo monotonic reparametrizations. Let $d(\cdot, \cdot)$ denote the uniform metric on curves:

\[
d(\gamma_1, \gamma_2) := \inf_{t \in [0,1]} \sup_{t \in [0,1]} |\gamma_1(t) - \gamma_2(t)|,
\]
where the infimum is taken over all choices of parametrizations of $\gamma_1$ and $\gamma_2$ from the interval $[0,1]$. The distance between two closed sets of curves is defined by the induced Hausdorff metric as follows:

$$\text{dist}(\mathcal{F}, \mathcal{F}') := \inf\{\epsilon > 0 : \forall \gamma \in \mathcal{F}, \exists \gamma' \in \mathcal{F}' \text{ such that } d(\gamma, \gamma') \leq \epsilon \text{ and vice versa} \}. \quad (7)$$

For critical site percolation on $T$, we orient a cluster boundary loop counterclockwise if it has blue sites on its inner boundary and yellow sites on its outer boundary, otherwise we orient it clockwise. We say $B(R)$ has **monochromatic (blue) boundary condition** if all the sites in $\Delta B(R)$ are blue. In [4], Camia and Newman showed the following well-known result (see also Theorem 2 in [5] for the case of a general Jordan domain):

**Theorem 2.1** ([4]). As $\eta \to 0$, the collection of all cluster boundaries of critical site percolation on $\eta T$ in $D$ with monochromatic boundary conditions converges in distribution, under the topology induced by metric (7), to a probability distribution on collections of continuous nonsimple loops in $D$.

Camia and Newman call the continuum nonsimple loop process in Theorem 2.1 the full scaling limit of critical site percolation, which is just the Conformal Loop Ensemble $\text{CLE}_6$ in $D$. The $\text{CLE}_\kappa$ for $8/3 < \kappa < 8$ is the canonical conformally invariant measure on countably infinite collections of noncrossing loops in a simply connected planar domain, and is conjectured to correspond to the scaling limit of a wide class of discrete lattice-based models, see [17, 18]. In the following, $\text{CLE}_6$ means $\text{CLE}_6$ in $D$. We denote by $\mathbb{P}$ the probability measure of $\text{CLE}_6$ and by $\mathbb{E}$ the expectation with respect to $\mathbb{P}$.

Let $\mathcal{L}_k$ be the $k$th largest $\text{CLE}_6$ loop that surrounds 0. Define $U_0 := D$, and let $U_k$ be the connected component of the open set $D \setminus \mathcal{L}_k$ that contains 0. If $D$ is a simply connected planar domain with $0 \in D$, the **conformal radius** of $D$ viewed from 0 is defined to be $\text{CR}(D) := |g'(0)|^{-1}$, where $g$ is any conformal map from $D$ to $\mathbb{D}$ that sends 0 to 0. For $k \in \mathbb{N}$, define

$$B_k := \log \text{CR}(U_{k-1}) - \log \text{CR}(U_k).$$

Proposition 1 in [16] says that $\{B_k\}_{k \in \mathbb{N}}$ are i.i.d. random variables. Furthermore, Schramm, Sheffield and Wilson (see (3) and the remark following the statement of Theorem 1 in [16]) proved the following result for the moment generating function of $B_k$, which is a key ingredient for the proof of our main theorem.

**Theorem 2.2** ([16]). For $-\infty < \lambda < 5/48$ and $k \in \mathbb{N}$,

$$\mathbb{E}[\exp(\lambda B_k)] = \frac{1}{2 \cos(\pi \sqrt{1/9 + 4\lambda/3})}.$$  

Theorem 2.2 implies the following corollary immediately.
Corollary 2.3. For $k \in \mathbb{N}$,
\begin{align*}
E[B_k] &= 2\sqrt{3}\pi, \quad (8) \\
\text{Var}[B_k] &= 16\pi^2 - 12\sqrt{3}\pi. \quad (9)
\end{align*}

To state next proposition we need more definitions. For $1 \leq r < R$, let $A(r, R) := B(R) \setminus B(r)$. Define

\begin{align*}
\rho(r, R) &:= \text{the maximal number of disjoint yellow circuits surrounding 0 in } A(r, R), \\
N(r, R) &:= \text{the number of cluster boundary loops surrounding 0 in } A(r, R), \\
T'(r, R) &:= \inf \{ T(\gamma) : \gamma \text{ is a path connecting } \partial B(r) \text{ and } \partial B(R) \}. \\
\end{align*}

$T'(r, R)$ satisfies two combinatorial properties as follows, the first one is basically the same as (2.39) in [11], and the second one can be derived from the first one and a “color switching trick”. We note that similar trick has appeared in [1, 19, 20].

Proposition 2.4. Suppose $1 \leq r < R$. Then $T'(r, R)$ satisfies the following properties:

\begin{itemize}
  \item $T'(r, R) = \rho(r, R)$.
  \item Assume that $B(R)$ has monochromatic (blue) boundary condition. Then $T'(r, R)$ has the same distribution as $N(r, R)$.
\end{itemize}

Proof. The proof of the equation is essentially the same as that of (2.39) in [11] for critical FPP on $\mathbb{Z}^2$. For completeness, we give the proof for our setting. If $\rho(r, R) = 0$, then there exists a blue path connecting $\partial B(R)$ and $\partial B(r)$, and we get $T'(r, R) = 0$. We assume $\rho(r, R) > 0$ in the following.

Figure 1: A sketch of the color switching trick used in the proof of Proposition 2.4.
First we show $T'(r, R) \geq \rho(r, R)$. This inequality is trivial, since if there are $ho$ disjoint yellow circuits surrounding 0 in $A(r, R)$, any path connecting the two boundary pieces of $A(r, R)$ must intersect these circuits.

Next we show the converse inequality, $T'(r, R) \leq \rho(r, R)$. We shall construct a path $\gamma$ connecting $\partial B(R)$ and $\partial B(r)$ such that $T(\gamma) = \rho(r, R)$, which implies $T'(r, R) \leq \rho(r, R)$ immediately. Let $D_0 := B(R)$. Take $C_1$ the outermost yellow circuit surrounding 0 in $D_0$ and let $D_1$ be the component of $D_0 \setminus C_1$ that contains 0, then take $C_2$ the outermost yellow circuit surrounding 0 in $D_1$ and let $D_2$ be the component of $D_1 \setminus C_2$ that contains 0, and so on. The process stops after $\rho = \rho(r, R)$ steps. It is easy to see that $C_k \subset A(r, R)$ for $1 \leq k \leq \rho$ and there exists no yellow circuit surrounding 0 in $D_{\rho} \setminus B(r)$. Thus we can take a blue path $\gamma_{\rho}$ connecting $\Delta B(r)$ and a site $v_{\rho}$ in $C_{\rho}$ (note that $\gamma_{\rho}$ may be empty if $C_{\rho}$ intersects $\partial B(r)$, similar case may occur below), then a blue path $\gamma_{\rho-1}$ connecting $v_{\rho}$ and a site $v_{\rho-1}$ in $C_{\rho-1}$ since $C_{\rho}$ is the outermost yellow circuit in $D_{\rho-1}$, and so on. The process stops after $\rho + 1$ steps, and $\gamma_0$ is a blue path connecting $v_1 \in C_1$ and the inner site boundary of $B(R)$. Let $\gamma = \gamma_0 v_1 \ldots \gamma_{\rho-1} v_{\rho} \gamma_{\rho}$. Clearly $\gamma$ connects $\partial B(R)$ and $\partial B(r)$ and $T(\gamma) = \rho(r, R)$.

We now turn to the proof of the second property. See Fig. 1 for an illustration of the following argument. Since $B(R)$ has monochromatic (blue) boundary condition, it is easy to see that $N(r, R)$ equals the maximal number of disjoint circuits $C_1, C_2, \ldots$ surrounding 0 in $A(r, R)$ with alternating colors (yellow, blue, yellow, blue, ...). For any fixed $n \in \mathbb{N} \cup \{0\}$, we shall construct a bijection between the sets $\{\omega : \rho(r, R) = n\}$ and $\{\omega' : N(r, R) = n\}$. Given a configuration $\omega \in \{\rho(r, R) = n\}$, we construct a sequence of yellow circuits $C_1, \ldots, C_n$ from outside to inside and a sequence of connected domains $D_0, \ldots, D_n$ as in the proof of the first property. If $n$ is odd, we switch the colors of the sites in $D_1 \setminus D_2, \ldots, D_{n-2} \setminus D_{n-1}, D_n$; if $n$ is even, we switch the colors of the sites in $D_1 \setminus D_2, \ldots, D_n \setminus D_{n-1}, D_n$. Denote this transformation by $f_n$. Then $f_n(\omega) \in \{N(r, R) = n\}$. This can be seen as follows: When $n$ is odd, after the transformation $f_n$, the color of $C_1, C_3, \ldots, C_n$ is invariant and the color of $C_2, C_4, \ldots, C_{n-1}$ is switched to blue. Then we can find a cluster boundary loop surrounding 0 in each $D_j \setminus D_{j+1}, 0 \leq j \leq n - 1$. Furthermore, none of these domains has two such loops, otherwise it would produce a yellow circuit surrounding 0 between two successive circuits $C_j$ and $C_{j+1}$ or outside $C_1$ in $D_0$ in the original configuration, which contradicts our construction. There does not exist cluster boundary loop surrounding 0 in $D_n \setminus B(r)$ since $\omega \in \{\rho(r, R) = n\}$. Hence, $f_n(\omega) \in \{N(r, R) = n\}$. The argument is similar when $n$ is even.

Given a configuration $\omega' \in \{N(r, R) = n\}$, similarly as above, we let $D'_0 := B(R)$, take $C'_1$ the outermost yellow circuit surrounding 0 in $D'_0$ and let $D'_1$ be the component of $D'_0 \setminus C'_1$ that contains 0, then take $C'_2$ the outermost blue circuit surrounding 0 in $D'_1$ and let $D'_2$ be the component of $D'_1 \setminus C'_2$ that contains 0, and so on. The process stops after $n$ steps. If $n$ is odd, we switch the colors of the sites in $D'_1 \setminus D'_2, \ldots, D'_{n-2} \setminus D'_{n-1}, D'_n$; if $n$ is even, we switch the colors of the sites in $D'_1 \setminus D'_2, \ldots, D'_{n-1} \setminus D'_n$. Note that $\omega'$ is transformed to $\omega' \in \{\rho(r, R) = n\}$ and this transformation is just $f_n^{-1}$. This can be seen as follows: When $n$ is odd, after the transformation, the color of $C'_1, C'_3, \ldots, C'_n$ is invariant and the
color of $C'_2, C'_4, \ldots, C'_{n-1}$ is switched to yellow, and $C'_j$ is the outermost yellow circuit surrounding 0 in $D'_{j-1}$. Furthermore, $\rho(r, R) = n$, otherwise it would produce more than $n$ cluster boundary loops surrounding 0 in $A(r, R)$ before the transformation. Therefore, this transformation is just $f_n^{-1}$ by the construction and the definition of $f_n$. The argument is similar when $n$ is even.

Then the bijection $f_n$ between $\{\omega: \rho(r, R) = n\}$ and $\{\omega': N(r, R) = n\}$ is constructed for each $n \in \mathbb{N} \cup \{0\}$, which completes the proof since we use the uniform measure.

The following lemma gives upper large deviation bound for $T'(r, R)$:

**Lemma 2.5** (Corollary 2.3 in [25]). There exist constants $C_1, C_2 > 0$ and $K > 1$, such that for all $1 \leq r < R$ and $x \geq K \log_2(R/r)$,

$$P[T'(r, R) \geq x] \leq C_1 \exp(-C_2 x).$$

In this paper, $C, C_1, C_2, \ldots$ denote positive finite constants that may change from line to line according to the context.

## 3 Proofs of the main results

### 3.1 Scaling limits of annulus times

For $0 < \epsilon < 1$, denote by $N(\epsilon)$ the number of CLE$_6$ loops surrounding 0 in $\overline{D}\setminus \overline{D}_\epsilon$. Camia and Newman’s full scaling limit allows us to derive a scaling limit of $T'(\rho r, \rho R)$ as $\rho \to \infty$:

**Proposition 3.1.** Suppose $1 \leq r < R$, $\rho > 0$ and $k \in \mathbb{N}$. Assume that $B(\rho R)$ has monochromatic (blue) boundary condition. As $\rho \to \infty$, we have

$$T'(\rho r, \rho R) \overset{d}{\to} N(r/R),$$

(10)

$$E[(T'(\rho r, \rho R))^k] \to E[(N(r/R))^k].$$

(11)

**Proof.** Let

$$\mathcal{F}_\rho := \text{the collection of cluster boundary loops surrounding 0 in } A(\rho r, \rho R),$$

$$\mathcal{F} := \text{the collection of CLE}_6 \text{ loops surrounding 0 in } \overline{D}\setminus \overline{D}_{1/r}. $$

Define event

$$A_{\epsilon, \rho} := \{ \exists \mathcal{L} \in \mathcal{F}_\rho \text{ such that } \mathcal{L} \cap (\overline{D}(1+\epsilon)r \setminus \overline{D}(1-\epsilon)r) \neq \emptyset \}.$$

Assume that $A_{\epsilon, \rho}$ holds and $\rho$ is large enough (depending on $\epsilon$). Then we have a polychromatic 3-arm event from a ball of radius $3\epsilon$ centered at a point $z \in \partial \overline{D}(1-\epsilon)r$, to a distance of unit order in $\overline{D}\setminus \overline{D}_{1-\epsilon}$. For a fixed $z \in \partial \overline{D}_{1-\epsilon}$, the corresponding 3-arm event happens with probability at most $O(\epsilon^2)$ (see
e.g. Lemma 6.8 in [22]). From this one easily obtains $P[A_{\epsilon, \rho}] \leq O(\epsilon)$. Then Theorem 2.2 implies that $\mathcal{F}_\rho$ converges in distribution to $\mathcal{F}$ as $\rho \to \infty$. Because of the choice of topology, we can find coupled versions of $\mathcal{F}_\rho$ and $\mathcal{F}$ on the same probability space such that dist($\mathcal{F}_\rho, \mathcal{F}$) $\to 0$ a.s. as $\rho \to \infty$ (see e.g. Theorem 6.7 of [3], similar couplings were used in [4]).

Now let us show that for large $\rho$, with high probability the distance $d(\cdot, \cdot)$ between the loops in $\mathcal{F}_\rho$ is not very small. Define event

$$B_{\epsilon, \rho} := \{ \exists L_1, L_2 \in \mathcal{F}_\rho \text{ such that } d(L_1, L_2) < \epsilon \}.$$ 

Assume that $B_{\epsilon, \rho}$ holds and $\rho$ is large enough (depending on $\epsilon$). Suppose $L_2$ is in the interior of $L_1$. Note that the polychromatic half-plane 3-arm exponent $\beta_3$ is 2 and the polychromatic plane 6-arm exponent $\alpha_6$ is larger than 2 (see e.g. [15]).

Let $0 < \alpha < 1 - 2/\alpha_6$ be a fixed number. If $L_1$ comes $\epsilon^\alpha$-2-close to $\partial \mathbb{D}$, then we have a half-plane 3-arm event from a ball of radius $2\epsilon^\alpha$ centered on a point in $\partial \mathbb{D}$ to a distance of unit order in $\mathbb{E}$; otherwise, we have a polychromatic plane 6-arm event from radius $\epsilon$ to $\epsilon^{1/2}$ in $\mathbb{E}$. So, the event $B_{\epsilon, \rho}$ happens with probability at most $O(\epsilon^{-\alpha/\beta_3} + \epsilon^{-2(1-\alpha)(\alpha_6+\alpha(1))}) = O(\epsilon^{\alpha} + \epsilon^{(1-\alpha)(\alpha_6+\alpha(1)-2)})$. This implies that in the above coupling, the number of loops in $\mathcal{F}_\rho$ converges in probability to the number of loops in $\mathcal{F}$ as $\rho \to \infty$. Then we get $N(\rho r, \rho R) \xrightarrow{d} N(r/R)$ immediately. From Proposition 2.4 we obtain (10). Lemma 2.5 and (10) imply (11).

The key ingredient Theorem 2.2 enables us to obtain the following limit theorem for $N(\epsilon)$:

**Proposition 3.2.**

$$\lim_{\epsilon \to 0} \frac{\mathbb{E}[N(\epsilon)]}{\log(1/\epsilon)} = \frac{1}{2\sqrt{3} \pi}, \quad \lim_{\epsilon \to 0} \frac{N(\epsilon)}{\log(1/\epsilon)} = \frac{1}{2\sqrt{3} \pi} \text{ a.s.}, \quad (12)$$

$$\lim_{\epsilon \to 0} \frac{\text{Var}[N(\epsilon)]}{\log(1/\epsilon)} = \frac{2}{3\sqrt{3}\pi} - \frac{1}{2\pi^2}. \quad (13)$$

**Proof.** From Proposition 1 in [10], we know that $\{B_k\}_{k \in \mathbb{N}}$ are i.i.d. random variables. It is clear that $S_n := \sum_{k=1}^{n} B_k = -\log \text{CR}(U_n)$, $n \geq 1$ is a renewal process. We let $N_i := \inf\{n : S_n > t\}$ denote the number of renewals in $[0, t]$. Schwarz Lemma and the Koebe 1/4 Theorem (see e.g. Lemma 2.1 and Theorem 3.17 in [12]) imply that

$$\frac{\text{CR}(U_n)}{4} \leq \text{dist}(0, L_n) \leq \text{CR}(U_n).$$

Hence, $\text{CR}(U_N(\epsilon)) \geq \text{dist}(0, L_N(\epsilon)) > \epsilon$ and $\text{CR}(U_{N(\epsilon)+1}) \leq 4\text{ dist}(0, L_{N(\epsilon)+1}) \leq 4\epsilon$. Then we have

$$N_{\log(1/(5\epsilon))} - 1 \leq N(\epsilon) < N_{\log(1/\epsilon)}. \quad (14)$$

Applying the elementary renewal theorem and law of large numbers for renewal processes (see e.g. (4.1) and (4.2) in Section 3.4 in [8]), we know

$$\lim_{t \to \infty} \frac{\mathbb{E}[N_t]}{t} \to \frac{1}{\mathbb{E}[B_1]}, \quad \lim_{t \to \infty} \frac{N_t}{t} \to \frac{1}{\mathbb{E}[B_1]} \text{ a.s.} \quad (15)$$
Combining (8), (14) and (15), we obtain (12).

There is a variance analogue of the elementary renewal theorem, see e.g. (1.6) in [21]. From this we know

$$\lim_{t \to \infty} \frac{\text{Var}[N_t]}{t} = \frac{\text{Var}[B_1]}{(\mathbb{E}[B_1])^3}.$$ (16)

The triangle inequality for the norm $\| \cdot \|_2 = \sqrt{\mathbb{E}[| \cdot |^2]}$ and (14) give

$$\left| \sqrt{\text{Var}[N(\epsilon)] - \text{Var}[N_{\log(1/\epsilon)}]} \right| \leq \sqrt{\text{Var}[N(\epsilon) - N_{\log(1/\epsilon)}]}$$

$$\leq \sqrt{\mathbb{E}[(N_{\log(1/\epsilon)} - N_{\log(1/(5\epsilon))} + 1)^2]}.$$ It is easy to see that the number of renewals in each time interval of length $\log 5$ can be uniformly dominated by a positive random variable with an exponential tail. Therefore, there exists a constant $C > 0$ such that for all $0 < \epsilon < 1$,

$$\left| \sqrt{\text{Var}[N(\epsilon)]} - \sqrt{\text{Var}[N_{\log(1/\epsilon)}]} \right| \leq C.$$ Combining this inequality, (16) and Corollary 2.3 we get (13).

Remark. In fact, if one uses second-order approximations to the expectation-time and variance-time curves for renewal processes (see e.g. (1.4) and the equation just above Section 1.5 in [21]) in the above proof, one obtains that there exists a constant $C > 0$ such that for all $0 < \epsilon < 1$,

$$\left| \mathbb{E}[N(\epsilon)] - \frac{\log(1/\epsilon)}{2\sqrt{3\pi}} \right| \leq C, \quad \left| \text{Var}[N(\epsilon)] - \frac{\log(1/\epsilon)}{3\sqrt{3\pi}} \right| \leq C.$$ We note that the first inequality above has been proved in [14] (see (3.8) in [14]).

3.2 Limit results for $c_n$ and $E[c_n]$

Recall that $c_n$ is the passage time from 0 to $\partial B_n$. In order to prove our limit results for $c_n$ and $E[c_n]$, we need some lemmas. In [25], we proved the following result.

Lemma 3.3 (Lemma 2.5 in [25]).

$$\lim_{n \to \infty} \frac{c_n}{E[c_n]} = 1 \quad a.s.$$ Lemma 2.5 together with RSW and FKG (see e.g. [23]), gives the following lemma. Note that (15) was first proved in [6].
Lemma 3.4. There exist $C_1, C_2 > 0$ such that for all $1 \leq r \leq R/2$,
\[
C_1 \log(R/r) \leq E[T'(r, R)] \leq C_2 \log(R/r). \tag{17}
\]
In particular, for all $n \geq 2$,
\[
C_1 \log n \leq E[c_n] \leq C_2 \log n. \tag{18}
\]

For $x \geq 0$, denote by $S_x$ the maximum number of disjoint yellow circuits that
surround 0 and intersect with $\partial B(2^x)$. Using RSW, FKG and BK inequality, it
is easy to obtain the following result.

Lemma 3.5. There exists a constant $C > 0$, such that for all $x, t \geq 0$,
\[
P[S_x \geq t] \leq \exp(-Ct). \tag{19}
\]
Hence, there is a constant $C_0 > 0$ (independent of $x$), such that
\[
E[S_x] \leq C_0. \tag{20}
\]

Now we are ready to prove the limit result for $c_n$ and $E[c_n]$:

Proposition 3.6.
\[
\lim_{n \to \infty} \frac{c_n}{\log n} = \frac{1}{2\sqrt{3\pi}} \ a.s., \tag{21}
\]
\[
\lim_{n \to \infty} \frac{E[c_n]}{\log n} = \frac{1}{2\sqrt{3\pi}}. \tag{22}
\]

Proof. Note that Lemma 3.3 and (22) imply (21), so it suffices to show (22).
We write $T_{k,j} := T'(2^k(j-1), 2^k)$. By Proposition 2.4, for $k \in \mathbb{N}$ and $n \geq 2^k$, it
is clear that
\[
-1 + \sum_{j=1}^{[\log_2 n]} T_{k,j} \leq c_n \leq 1 + \rho(1,n) \leq 1 + \sum_{j=1}^{[\log_2 n]+1} (T_{k,j} + S_{kj}). \tag{23}
\]
Combining this with (17) and (20), we obtain that there exists $C_1, C_2 > 0$ such that
\[
-1 + \sum_{j=1}^{[\log_2 n]} E[T_{k,j}] \leq E[c_n] \leq C_1 \log_{2^k} n + C_2 k + \sum_{j=1}^{[\log_2 n]} E[T_{k,j}].
\]
This and (18) imply that for each $0 < \epsilon < 1$, there exists $k_0(\epsilon) > 0$, such that
for each $k \geq k_0$, for $n$ sufficiently large (depending on $k$),
\[
1 - \epsilon \leq \frac{\sum_{j=1}^{[\log_2 n]} E[T_{k,j}]}{E[c_n]} \leq 1 + \epsilon. \tag{24}
\]
By the convergence of the Cesàro mean and (11), we obtain

\[
\lim_{m \to \infty} \frac{\sum_{j=1}^{m} E[T_{k,j}]}{m} = E[N(1/2^k)].
\] (25)

Then (22) follows from (12), (24) and (25):

\[
\lim_{k \to \infty} \frac{E[c_n]}{\log n} = \lim_{n \to \infty} \sum_{j=1}^{\lfloor \log_2 n \rfloor} E[T_{k,j} \log n] = \lim_{k \to \infty} \frac{E[N(1/2^k)]}{\log(2^k)} = \frac{1}{2\sqrt{3\pi}}.
\]

3.3 Limit result for \( \text{Var}[c_n] \)

The proof of the result relating to variance turns out to be more involved than that to expectation. We shall use the martingale approach in [11], with a modification by introducing an intermediate scale. Let us mention that an analogous martingale method was used in [24] to prove a CLT for winding angles of arms in 2D critical percolation. We start with some definitions.

For \( j \in \mathbb{N} \cup \{0\} \), define annulus

\[
A(j) := A(2^j, 2^{j+1}).
\]

Furthermore, define

\[
m(j) := \inf\{k \geq j : A(k) \text{ contains a blue circuit surrounding } 0\},
\]

\[
C_j := \text{the innermost blue circuit surrounding } 0 \text{ in } A(m(j)),
\]

\[
\mathcal{F}_j := \sigma\text{-field generated by } \{t(v) : v \in C_j\}.
\]

For all \( j \geq 1 \), denote by \( C_{-j} \) the origin and by \( \mathcal{F}_{-j} \) the trivial \( \sigma \)-field. For \( k, q \in \mathbb{N} \), write

\[
T(0, C_{kq}) - E[T(0, C_{kq})] = \sum_{j=0}^{q} \left( E[T(0, C_{kq}) | \mathcal{F}_{kj}] - E[T(0, C_{kq}) | \mathcal{F}_{kj-1}] \right) := \sum_{j=0}^{q} \Delta_{k,j}.
\]

Then \( \{\Delta_{k,j}\}_{0 \leq j \leq q} \) is an \( \mathcal{F}_{kj} \)-martingale increment sequence. Hence,

\[
\text{Var}[T(0, C_{kq})] = \sum_{j=0}^{q} E[\Delta_{k,j}^2].
\] (26)

We will use this sum to estimate \( \text{Var}[c_n] \).

Let \( (\Omega', \mathcal{F}', P') \) be a copy of \( (\Omega, \mathcal{F}, P) \). Denote by \( E' \) the expectation with respect to \( P' \), and by \( \omega' \) a sample point in \( \Omega' \). Let \( T(\cdot, \cdot)(\omega), m(j, \omega) \) and \( C_j(\omega) \) denote the quantities defined before, but with explicit dependence on \( \omega \). Define \( l(j, \omega, \omega') := m(m(j, \omega) + 1, \omega') \). We need the following results, which were proved in [11]. Note that (27) follows from RSW and FKG, and the proof of (28) is essentially the same as that of Lemma 2 in [11].
Lemma 3.7 ([11]). (i) There exists $C > 0$, such that for all $j, t \in \mathbb{N}$, we have
\[ P[m(j) \geq j + t] \leq \exp(-Ct). \] (27)
(ii) For $j \geq 0$ and $k \geq 1$, $\Delta_{k,j}$ does not depend on $q$. Furthermore,
\[ \Delta_{k,j}(\omega) = T(C_{k(j)}(\omega), C_{kj}(\omega)) + E'[T(C_{kj}(\omega), C_{kl}(\omega', \omega'))(\omega')] - E'[T(C_{k(j)}(\omega), C_{kl}(\omega', \omega'))(\omega')]. \] (28)

Next we bound the variances of $\Delta_{k,j}$ and $T(0, C_j)$.

Lemma 3.8. There exist $C_1, C_2 > 0$ such that for all $j, k \geq 1$,
\[ C_1 k \leq E[\Delta_{k,j}^2] \leq C_2 k, \] (29)
\[ C_1 j \leq \text{Var}[T(0, C_j)] \leq C_2 j. \] (30)

Proof. From (26), we know that (30) is just (2.32) in [11]. The proof of (2.32) in [11] implies that there exist universal $C_1, C_2 > 0$ such that for all $n \in \mathbb{N}$,
\[ C_1 \leq E[\Delta_{1,n}^2] \leq C_2. \] Since
\[ E[\Delta_{k,j}^2] = \sum_{n=kj-k+1}^{kj} E[\Delta_{i,n}^2], \]
then (29) follows.

The next lemma implies that for fixed $k$ and large $q$, if $2^k q \leq n \leq 2^{k(q+1)}$, the variance of $c_n$ is well approximated by that of $T(0, C_{kj})$.

Lemma 3.9. There exists $C > 0$, such that for all $q \geq k \geq 1$ and $2^k q \leq n \leq 2^{k(q+1)}$,
\[ |\text{Var}[T(0, C_{kj})] - \text{Var}[c_n]| \leq C \sqrt{k^{3/2}}. \] (31)
Furthermore, there exist $C_1, C_2 > 0$ such that for all $n \geq 2$,
\[ C_1 \log n \leq \text{Var}[c_n] \leq C_2 \log n. \] (32)

Proof. Recall that $S_j$ denotes the maximum number of disjoint yellow circuits that surround 0 and intersect with $\partial B(2^j)$. Applying Lemma 2.5 and (27), there exist $C_1, C_2 > 0$, such that for all $x \geq 3Kk$, where $K$ is from Lemma 2.5, we have
\[ P[T(0, C_{kj}) - c_n] \geq x \]
\[ \leq P[m(kj) \geq kq + \lfloor x/(2K) \rfloor] + P[T'(2^{kq}, 2^{kq} + \lfloor x/(2K) \rfloor) \geq x/2] + P[S_{kj} \geq x/2] \leq C_1 \exp(-C_2 x). \]

Then there is a universal $C_3 > 0$, such that
\[ E[T(0, C_{kj}) - c_n] \leq C_3 k^2. \]
Therefore, by the triangle inequality for the norm $|| \cdot ||_2 = \sqrt{E[|| \cdot ||^2]}$, we have

$$\left| \sqrt{\text{Var}[T(0, C_{kq})]} - \sqrt{\text{Var}[c_n]} \right| \leq \sqrt{\text{Var}[T(0, C_{kq}) - c_n]} \leq \sqrt{C_3 k}.$$  

Combining this and (30) gives (31).

$(32)$ follows from (30) and (31) immediately.

Recall that $T_{k,j} := T'(2^k(2^{j-1}), 2^{kj})$. The following is a key estimate for $T_{k,j}$, which says that for large $k$, the variance of $T_{k,j}$ is well approximated by that of $\Delta_{k,j}$ (since $E[\Delta^2_{k,j}] \approx k$, from (29)).

**Lemma 3.10.** There exists a constant $C > 1$, such that for all $k, j \in \mathbb{N}$,

$$| \text{Var}[T_{k,j}] - E[\Delta^2_{k,j}] | \leq C\sqrt{k}.$$  

**Proof.** We claim that there exist universal constants $C_1, C_2 > 0$ such that for all $k, j \in \mathbb{N}$ and $x > 0$,

$$P[|T_{k,j} - E[T_{k,j}] - \Delta_{k,j}| \geq x] \leq C_1 \exp(-C_2 x).$$  

Therefore, there is a $C_3 > 0$ such that

$$E[|T_{k,j} - E[T_{k,j}] - \Delta_{k,j}|^2] \leq C_3.$$  

Then we obtain

$$|\{\text{Var}[T_{k,j}]\}^{1/2} - \{\text{Var}[\Delta_{k,j}]\}^{1/2}| \leq \{E[|T_{k,j} - E[T_{k,j}] - \Delta_{k,j}|^2]\}^{1/2} \leq \sqrt{C_3}.$$  

This and (29) imply the desired result:

$$|\text{Var}[T_{k,j}] - E[\Delta^2_{k,j}]| \leq 2\sqrt{C_3 E[\Delta^2_{k,j}]} + C_3 \leq C\sqrt{k}.$$  

We now show our claim (33). By (28), proving (33) boils down to proving that there exist $C_4, C_5, C_6, C_7 > 0$ such that

$$P[|T_{k,j} - T(C_{k(j-1)}, C_{k})| \geq x] \leq C_4 \exp(-C_5 x),$$  

$$P[|E[T_{k,j}] + E[T'(C_{k(j-1)}(\omega), C_{l(kj,\omega,\omega')} (\omega'))] - E'[T'(C_{k(j-1)}(\omega), C_{l(kj,\omega,\omega')} (\omega'))] | \geq x] $$  

$$\leq C_6 \exp(-C_7 x).$$  

We first prove (34). Let $C_8 = 1/(4K)$, where $K$ is from Lemma 2.5. There exist $C_9, C_{10}, C_{11}, C_{12} > 0$ such that if $0 < x \leq k/C_8$,

$$P[|T_{k,j} - T(C_{k(j-1)}, C_{k})| \geq x]$$

$$\leq P[m(k(j-1)) \geq k(j-1) + [C_8 x]] + P[m(kj - [C_8 x]) \geq k] + P[m(kj) \geq kj + [C_8 x]]$$

$$+ P[T'(2^k (2^{j-1}), 2^{kj-1}) \geq x/2] + P[T'(2^k, 2^k x) \geq x/2]$$

$$\leq 3C_9 \exp(-C_{10} x) + 2C_{11} \exp(-C_{12} x)$$  

by (27) and Lemma 2.5.
if $x \geq k/C_8$, 

$$P[T_{k,j} - T(C_{k(j-1)}, C_{kj})] \geq x$$

$$\leq P[m(kj) \geq kj + \lceil C_8 x \rceil] + P[T'(2^{k(j-1)}, 2^{k+\lceil C_8 x \rceil})] \geq x$$

$$\leq C_9 \exp(-C_{10}x) + C_{11} \exp(-C_{12}x) \quad \text{by (27) and Lemma 2.5}$$

Then (34) follows immediately.

Now let us show (35), which follows from the two inequalities below:

$$E'[T(C_{k,j}(\omega), C_{(kj,\omega')}(\omega'))] \leq C_{13} \ a.s., \quad (36)$$

$$P[E'[T_{kj}(\omega')] - E'[T(C_{k(j-1)}(\omega), C_{(kj,\omega')}(\omega'))] \geq x] \leq C_{14} \exp(-C_{15}x), \quad (37)$$

where $C_{13}, C_{14}, C_{15} > 0$ are universal constants. Using (27) and Lemma 2.5 again, we know there exist $C_{16}, C_{17} > 0$ such that for all $C_{kj}(\omega)$,

$$P[T(C_{k,j}(\omega), C_{(kj,\omega')}(\omega'))] \geq x$$

$$\leq P[l(kj, \omega', \omega') \geq m(kj, \omega) + \lceil C_8 x \rceil] + P[T'(2^{m(kj, \omega)}, 2^{m(kj, \omega)+\lceil C_8 x \rceil}) \geq x]$$

$$\leq C_{16} \exp(-C_{17}x),$$

which implies (36).

It remains to show (37). Similarly to the proof of (34), one can show that there exist $C_{18}, C_{19} > 0$, such that

$$P[[T_{m(k(j-1), \omega), m(kj, \omega)+1}(\omega') - T(C_{k(j-1)}(\omega), C_{(kj,\omega')}(\omega'))] \geq x] \leq C_{18} \exp(-C_{19}x),$$

which implies that there is a universal $C_{20} > 0$, such that

$$|E'[T_{m(k(j-1), \omega), m(kj, \omega)+1}(\omega')] - E'[T(C_{k(j-1)}(\omega), C_{(kj,\omega')}(\omega'))]| \leq C_{20}.$$ 

From this proving (37) boils down to proving that there are $C_{21}, C_{22} > 0$ such that

$$P[E'[T_{kj}(\omega')] - E'[T_{m(k(j-1), \omega), m(kj, \omega)+1}(\omega')] \geq x] \leq C_{21} \exp(-C_{22}x). \quad (38)$$

Combining (17) and (20) gives that there exists $C_{24} > 0$, such that for all $x \geq 1$ and $j \geq 0$,

$$E[T'(2^j, 2^j x+1) + S_j + S_{j+x+1}] \leq x/(2C_{24}). \quad (39)$$

Then, using (39) and (27), we get that there exist $C_{25}, C_{26} > 0$ such that if $1/C_{24} \leq x \leq k/C_{24}$,

$$P[|E'[T_{kj}(\omega')] - E'[T_{m(k(j-1), \omega), m(kj, \omega)+1}(\omega')]| \geq x]$$

$$\leq P[m(k(j-1)) \geq k(j-1) + \lceil C_4 x \rceil] + P[m(kj) \geq kj + \lceil C_4 x \rceil]$$

$$+ P[E'[T'(2^{k(j-1)}, 2^{k+\lceil C_4 x \rceil}) + S_{k(j-1)+\lceil C_4 x \rceil}] \geq x]$$

$$+ P[E'[T'(2^{kj}, 2^{kj+\lceil C_4 x \rceil}) + S_{kj}] \geq x]$$

$$\leq C_{25} \exp(-C_{26}x) \quad \text{note that the last two terms above equal 0};$$
if \( x \geq k/C_{24} \),

\[
P[|E'[T_{k,j}(\omega')] - E'[T_{m(k(j-1),\omega),m(kj,\omega)+1}(\omega')]| \geq x] \\ \leq P[m(kj) \geq kj + |C_{24}x|] + P[E'[T'(2^{k(j-1)},2^{kj}+[C_{24}x] + 1)] \geq x] \leq C_{25} \exp(-C_{26}x).
\]

Then (38) follows immediately, which completes the proof.

We are now ready to prove the limit result for \( \text{Var}[c_n] \):

**Proposition 3.11.**

\[
\lim_{n \to \infty} \frac{\text{Var}[c_n]}{\log n} = \frac{2}{3\sqrt{3\pi}} - \frac{1}{2\pi^2}.
\]

**Proof.** Applying Lemma 3.10 we have

\[
\left| \text{Var}[T(0,C_{kj})] - \sum_{j=1}^{q} \text{Var}[T_{k,j}] \right| \leq Cq\sqrt{k} + 1,
\]

where \( C \) is from Lemma 3.10. By the convergence of the Cesàro mean and (11), we obtain

\[
\lim_{q \to \infty} \frac{\sum_{j=1}^{q} \text{Var}[T_{k,j}]}{q} = \text{Var}[N(1/2^k)].
\]

Recall the choice \( q = \lfloor \log_2 n \rfloor \). Combining (13), (31), (32), (40) and (41) gives the desired result:

\[
\lim_{n \to \infty} \frac{\text{Var}[c_n]}{\log n} = \lim_{k \to \infty} \lim_{n \to \infty} \frac{\sum_{j=1}^{\lfloor \log_2 k \rfloor} \text{Var}[T_{k,j}]}{\log n} = \lim_{k \to \infty} \frac{\text{Var}[N(1/2^k)]}{\log(2^k)} = \frac{2}{3\sqrt{3\pi}} - \frac{1}{2\pi^2}.
\]

\[\square\]

### 3.4 Proofs of Theorem 1.1 and Corollary 1.2

**Proof of Theorem 1.1.** The proof of Theorem 1.1 in [25], together with Proposition 3.6, implies (1) and (3) immediately.

By Lemma 2.5 (19) and (27), there exist \( C_1, C_2 > 0 \), such that for all \( 2^q \leq n < 2^{q+1}, \ x \geq 4K \), where \( K \) is from Lemma 2.5, we have

\[
P[b_{0,n} - c_n \geq x] \\ \leq P[T(0,C_{q+1}) - c_{2^q} \geq x] \\ \leq P[m(q+1) \geq q + \lfloor x/(2K) \rfloor] + P[T'(2^q,2^{q+1}\lfloor x/(2K) \rfloor)] \geq x/2] + P[S_q \geq x/2] \\ \leq C_1 \exp(-C_2x).
\]

Then there is a universal \( C_3 > 0 \), such that

\[
E[|b_{0,n} - c_n|] \leq C_3, \quad E[|b_{0,n} - c_n|^2] \leq C_3.
\]

(42)
Then (2) follows from (22) and (42).

By (32) and (42), one easily obtains that there exists \( C_4 > 0 \) such that

\[
| \text{Var}[b_{0,n}] - \text{Var}[c_n] | \leq C_4 \sqrt{\log n}.
\]

Then (3) follows from this and Proposition 3.11.

For \( z \in A(2^q, 2^{q+1}) \) and \( q \geq 2 \), define

\[
Y(z) := T(0, \partial B(2^q - 1)) + T(z, \partial B(z, 2^q - 1)).
\]

It is obvious that \( T(0, z) \geq Y(z) \) and

\[
T(0, z) - Y(z) \leq |T(0, C_{q+2}) - T(0, \partial B(2^q - 1))| + |T(z, C_{q+2}) - T(z, \partial B(z, 2^q - 1))|.
\]

Using this inequality, similarly to the above argument, one can show that there is a universal \( C_5 > 0 \) such that

\[
E[|T(0, z) - Y(z)|] \leq C_5, \quad E[|T(0, z) - Y(z)|^2] \leq C_5,
\]

(43)

see the proofs of Corollary 5.11 and 5.12 in [7]. Combining (22) and (43) gives (5) immediately. By (32) and (43) and using the triangle inequality for the norm \( \| \cdot \|_2 \), there exist universal \( C_6 > 0 \) such that

\[
| \text{Var}[T(0, z)] - 2 \text{Var}[c_{2^{q-1}}] | \leq C_6 \sqrt{q}.
\]

Then we derive (6) from this and Proposition 3.11, finishing the proof.

Proof of Corollary 1.2. Using the Theorem in [11], one can derive the following result easily: As \( n \to \infty \),

\[
\frac{b_{0,n} - E[b_{0,n}]}{\sqrt{\text{Var}[b_{0,n}]}} \xrightarrow{d} N(0,1),
\]

(44)

\[
\frac{T(0, nu) - E[T(0, nu)]}{\sqrt{\text{Var}[T(0, nu)]}} \xrightarrow{d} N(0,1).
\]

(45)

Note that (45) is a particular case of Corollary 5.13 in [7], and the proof of (44) is similar to that of (45). Corollary 1.2 follows from (44), (45) and Theorem 1.1 immediately.

4 A discussion on the limit shape

Since there is a “shape theorem” for general non-critical FPP (see e.g. [2]), it is natural to ask if there is an analogous result for critical FPP. For that purpose, we view the sites of \( V \) as hexagons, and define \( W(n) := \{ v \in V : T(0, v) \leq n \} \) for \( n \in \mathbb{N} \). Unlike the non-critical case, there are “large holes” in \( W(n) \), and the geometry of \( W(n) \) is not easy to analyze. Instead, it would be easier to deal with the set \( \overline{W}(n) \) obtained from \( W(n) \) by filling in the holes. That is, \( \overline{W}(n) \) is
the closed set surrounded by the outer boundary of $W(n)$. We shall discuss the limit boundary of $W(n)$ below.

Similarly to the proof of Proposition 2.4, we inductively define the $n$-th innermost disjoint yellow circuit $C_n$ surrounding 0. For simplicity, if the hexagon centered at 0 is yellow, we also call it the first innermost circuit surrounding 0. Let $C_n^* := C_n \cup$ blue clusters which touch $C_n$. Observe that $\partial W(n)$ is equal to the outer boundary of $C_n^*$. By a color switching trick one obtains that $\partial W(n)$ is equal in distribution to the $n$-th innermost cluster boundary loop $L_n$ surrounding 0. Thus, it is expected that as $n \to \infty$, the limit boundary of $W(n)$ under appropriate scaling has the same distribution as a “typical” loop of full-plane $\text{CLE}_6$. Note that $W(n)$ grows approximately exponentially as time passes.

In [10], Kemppainen and Werner proved the invariance of full-plane $\text{CLE}_\kappa$ under the inversion $z \mapsto 1/z$ for $0 < \kappa \leq 4$. Although the analogous result has not been proved for the case $4 < \kappa < 8$, Miller and Sheffield [13] proved the time-reversal symmetry of whole-plane $\text{SLE}_\kappa$ for $\kappa \in (0,8]$. To establish the following statement, for a loop $L$ that surrounds 0, we let $R_{\text{in}}(L) := \text{dist}(0,L)$ denote the inner radius of $L$ and let $R_{\text{out}}(L) := \inf\{r : \overline{D_r} \supset L\}$ denote the outer radius of $L$. By the argument above, it is expected that the following statement holds: As $n \to \infty$, the distributions of $\partial W(n)/R_{\text{in}}(\partial W(n))$ and $\partial W(n)/R_{\text{out}}(\partial W(n))$ converge, and we denote the two limit distributions by $\mu_{\text{in}}$ and $\mu_{\text{out}}$, respectively. Furthermore, $\mu_{\text{out}}$ is the image of $\mu_{\text{in}}$ under $z \mapsto 1/z$.

References

[1] Aizenman, M., Duplantier, B., Aharony, A.: Path crossing exponents and the external perimeter in 2D percolation. Phys. Rev. Let. 83, 1359–1362 (1999)
[2] Auffinger, A., Damron, M., Hanson, J.: 50 years of first passage percolation. arXiv:1511.03262 (2016)
[3] Billingsley, P.: Convergence of probability measures. Second Edition. Wiley Series in Probability and Statistics: Probability and Statistics. John Wiley & Sons Inc., New York.
[4] Camia, F., Newman, C.M.: Critical percolation: the full scaling limit. Commun. Math. Phys. 268, 1–38 (2006)
[5] Camia, F., Newman, C.M.: $\text{SLE}_6$ and $\text{CLE}_6$ from critical percolation. In Pinsky, M. and Birnir, B. (eds) Probability, geometry and integrable systems, MSRI Publications, Volume 55, Cambridge University Press, New York, pp. 103–130 (2008)
[6] Chayes, J.T., Chayes, L., Durrett, R.: Critical behavior of two-dimensional first passage time. J. Stat. Phys. 45, 933–951 (1986)
[7] Damron, M., Lam, W.-K., Wang, W.: Asymptotics for 2D critical first passage percolation. arXiv:1505.07544 (2015) To appear in Annals of Probability.
[8] Durrett, R.: Probability: Theory and Examples. Third Edition, Belmont CA: Duxbury Advanced Series (2004)
[9] Grimmett, G., Kesten, H.: Percolation since Saint-Flour. In Percolation at Saint-Flour. Springer-Verlag, Heidelberg. (2012)
[10] Kemppainen, A., Werner, W.: The nested simple conformal loop ensembles in the Riemann sphere. Probab. Theory Relat. Fields 165, 835–866 (2016)

[11] Kesten, H., Zhang, Y.: A central limit theorem for “critical” first-passage percolation in two-dimensions. Probab. Theory Relat. Fields. 107, 137–160 (1997)

[12] Lawler, G.F.: Conformally invariant processes in the plane. Amer. Math. Soc. (2005)

[13] Miller, J., Sheffield, S.: Imaginary geometry IV: interior rays, whole-plane reversibility, and space-filling trees. arXiv:1302.4738 (2013)

[14] Miller, J., Watson, S.S., Wilson, D.B.: The conformal loop ensemble nesting field. Probab. Theory Relat. Fields 163, 769–801 (2015)

[15] Nolin, P.: Near critical percolation in two-dimensions. Electron. J. Probab. 13, 1562–1623 (2008)

[16] Schramm, O., Sheffield, S., Wilson, D.B.: Conformal radii for conformal loop ensembles. Commun. Math. Phys. 288, 43–53 (2009)

[17] Sheffield, S.: Exploration trees and conformal loop ensembles. Duke Math. J. 147, 79–129 (2009)

[18] Sheffield S., Werner W.: Conformal loop ensembles: the Markovian characterization and the loop-soup construction. Ann. Math. 176, 1827–1917 (2012)

[19] Smirnov, S.: Critical percolation in the plane: conformal invariance, Cardy’s formula, scaling limits. C. R. Acad. Sci. Paris Sér. I Math. 333 no. 3, 239–244 (2001)

[20] Smirnov, S., Werner, W.: Critical exponents for two-dimensional percolation. Math. Res. Lett. 8, 729–744 (2001)

[21] Smith, W.L.: Renewal theory and its ramifications. Journal of the Royal Statistical Society. Series B (Methodological), Vol. 20, No. 2, 243–302 (1958)

[22] Sun, N.: Conformally invariant scaling limits in planar critical percolation. Probab. Surveys 8, 155–209 (2011)

[23] Werner, W.: Lectures on two-dimensional critical percolation. In: Statistical mechanics. IAS/Park City Math. Ser., Vol. 16. Amer. Math. Soc., Providence, RI, 297–360 (2009)

[24] Yao, C.-L.: A CLT for winding angles of the arms for critical planar percolation. Electron. J. Probab. 18, no. 85, 1–20 (2013)

[25] Yao, C.-L.: Law of large numbers for critical first-passage percolation on the triangular lattice. Electron. Commun. Probab. 19, no. 18, 1–14 (2014)