OPTIMAL REINSURANCE AND INVESTMENT STRATEGIES FOR AN INSURER AND A REINSURER UNDER HESTON'S SV MODEL: HARA UTILITY AND LEGENDRE TRANSFORM

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ABSTRACT. The present paper investigates an optimal reinsurance-investment problem with Hyperbolic Absolute Risk Aversion (HARA) utility. The paper is distinguished from other literature by taking into account the interests of both an insurer and a reinsurer. The insurer is allowed to purchase reinsurance from the reinsurer. Both the insurer and the reinsurer are assumed to invest in one risk-free asset and one risky asset whose price follows Heston’s SV model. Our aim is to seek optimal investment-reinsurance strategies to maximize the expected HARA utility of the insurer’s and the reinsurer’s terminal wealth. In the utility theory, HARA utility consists of power utility, exponential utility and logarithmic utility as special cases. In addition, HARA utility is seldom studied in the optimal investment and reinsurance problem due to its sophisticated expression. In this paper, we choose HARA utility as the risky preference of the insurer. Due to the complexity of the structure of the solution to the original Hamilton-Jacobi-Bellman (HJB) equation, we use Legendre transform to change the original non-linear HJB equation into its linear dual one, whose solution is easy to conjecture in the case of HARA utility. By calculations and deductions, we obtain the closed-form solutions of optimal investment-reinsurance strategies. Moreover, some special cases are also discussed in detail. Finally, some numerical examples are presented to illustrate the impacts of our model parameters (e.g., interest and volatility) on the optimal reinsurance-investment strategies.

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1. **Introduction.** The study of optimal reinsurance and investment has been an active field over the past three decades. These problems have been studied in terms of a variety of optimization objectives, for example, [24] and [23] investigated the objective function that minimizing the ruin probability, [2], [16], [31] and [26] considered the optimal investment and reinsurance problems under mean-variance criterion and time-consistent strategy. Likewise, a number of scholars focus on maximizing the expected utility of terminal wealth and obtain a series of results. See, for example, [10], [21], [20], and reference therein. But these results were almost studied under the assumption of exponential utility.

However, there are two aspects worthy to be further explored. On the one hand, the above mentioned researches generally assumed that the risky assets’ prices are driven by geometric Brownian motions (GBMs) in which the volatilities of risky assets’ prices are deterministic, or constant elasticity of variance (CEV) model being a local volatility model. Note that none of these models contains full-fledged stochastic volatility (SV) assumptions. To study more practical financial market, [13] assumed that the volatility of the risky asset was driven by a Cox-Ingersoll-Ross (CIR) process; this model has some computational and empirical advantages. [19] and [30] introduced the Heston’s SV model into the reinsurance and investment problem. Actually, the Heston’s SV model is classical and very popular for option pricing, and has been recognized as an important feature for asset price models. Meanwhile, the Heston’s SV model can explain many well-known empirical findings, such as the volatility smile, the volatility clustering, and the heavy-tailed nature of return distributions.

On the other hand, the insurers should choose different utility function according to the different degree of risk preference. It is all well-known that exponential utility, power utility and logarithmic utility are all special cases of HARA utility. Therefore, it is very necessary to investigate the optimal investment-reinsurance strategy under HARA utility. However, due to the complicated structure of HARA utility, there are very few results concerning the optimal investment-reinsurance problem with HARA utility. [9] presented martingale method to deal with an investment problem with HARA utility. [11] applied Legendre transform technique to tackle an investment problem with HARA utility. [5] derived the optimal consumption-investment strategy under HARA utility. [33] introduced HARA utility into the reinsurance-investment problem, and investigated an optimal reinsurance-investment problem in relation to thinning dependent risks based on Legendre transform.

To the best of our knowledge, there are no published works addressing the optimal reinsurance-investment problem for an insurer and a reinsurer under HARA utility. Most of the existing literature only consider the optimal reinsurance problem from the insurer’s perspective. But in reality, a reinsurance treaty involves an insurer and a reinsurer, which have conflicting interests. An optimal reinsurance treaty for an insurer may not be optimal for a reinsurer and it might be unacceptable for a reinsurer as pointed out by [3]. Thus, it is necessary to take the interest of the reinsurer into consideration. Recently, some scholars begin to investigate the optimal investment-reinsurance problems reflecting the interests of both the insurer and the reinsurer. For example, [27] assumed that the risky asset whose price process is described by CEV model and investigated the optimal reinsurance-investment problem for maximizing the expected exponential utility of the insurer’s and the reinsurer’s terminal wealth. Under the criterion of maximizing the product...
of the utility of an insurer and a reinsurer, [17] studied the optimal reinsurance-investment problem under CEV model. Similar to [17], [15] took the model uncertainty into account and considered a robust optimal investment and reinsurance problem. For other research, one can refer to [16], [14] and [34]. In contrast to the above-mentioned researches, in this paper, we focus on the HARA utility to study the optimal reinsurance-investment problem for an insurer and a reinsurer. The proportional reinsurance treaty is considered, and moreover, both the insurer and reinsurer are assumed to invest in one risk-free asset and one risky asset whose price follows Heston’s SV model. We aim to maximize the expected HARA utility of the insurer’s and the reinsurer’s terminal wealth. Firstly, applying stochastic dynamic programming principle, we derive the HJB equation for the value function. However, the HJB equation in this paper is a non-linear second order partial differential equation (PDE), it is difficult to simplify or solve this equation directly in general. Motived by [5] and [33], we adopt the Legendre transform-dual technique, instead of ordinary method used in the existing literature, to change the HJB equation into its dual one, the form of whose solution is easy to be conjectured. Applying variable change technique, we achieve the closed-form expression of optimal investment and reinsurance strategies for HARA utility. In addition, some special cases of our model are also provided under the criteria of exponential utility, power utility and logarithmic utility, respectively. Comparing with the existing literature, this paper has the main highlights as follows: (i) a new optimal reinsurance-investment problem reflecting the interests of both the insurer and reinsurer under HARA utility is established; (ii) Legendre transform method is used to deal with the optimal control problem.

This paper is organized as follows. The formulation of our model is presented in section 2. Section 3 discusses the optimal investment and reinsurance strategy for the insurer under HARA utility and arrives at the closed form expressions for the optimal results. Section 4 provides the optimal investment and reinsurance strategy for the reinsurer. In Section 5, numerical examples are carried out to illustrate our results in this paper. Finally, we conclude the paper in Section 6.

2. Model formulation. We start with a filtered complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in [0,T]}, P)$, where $T$ represents the terminal time which is a positive finite constant, $\mathcal{F}_t$ stands for the information of the market available up to time $t$. Assume that all processes introduced below are well-defined and adapted processes in this space. In addition, suppose that trading takes place continuously and involves no taxes or transaction costs, and that all securities are infinitely divisible.

2.1. Wealth process of the insurer. The wealth process of the insurer is described by Cramér-Lundberg model as follows

$$R_1(t) = x_0 + ct - \sum_{i=1}^{N(t)} X_i,$$  \hspace{1cm} (1)

where $x_0$ is the deterministic initial reserve of the insurer and the constant $c$ is the premium rate. $X_i$ is the size of the $i$th claim and $\{X_i, i \geq 1\}$ are assumed to be i.i.d. positive random variables with common distribution function $F_X(\cdot)$ and $E(X_i) = \mu_X > 0$. The claim number process $\{N(t), t \geq 0\}$ is assumed to be a Poisson process with intensity $\lambda > 0$ which represents the number of claims occurring in time interval $[0, t]$. 
Moreover, \(\{X_i, i \geq 1\}\) and \(\{N(t), t \geq 0\}\) are mutually independent. In this paper, the insurer’s premium rate is calculated according to the expected value principle, i.e. \(c = (1 + \theta)\lambda \mu_X\), where \(\theta > 0\) is the safety loading of the insurer for the claims.

For each \(t \in [0, T]\), suppose that the insurer is allowed to reinsure a fraction of his/her claims with the retention level \(q(t) \in [0, T]\) for \(X_i\). In other words, for a claim \(X_i\), the insurer pays \(q(t)X_i\) while the reinsurer pays the rest at time \(t\), i.e. \((1 - q(t))X_i\). However, the insurer has to pay a premium at the rate of \(\delta(q)\) to the reinsurer due to the reinsurance business. In this article, we assume that the reinsurer’s premium is calculated by the expected value principle, i.e.,

\[
\delta(q) = (1 + \eta)(1 - q(t))\lambda \mu_X.
\]

Here \(\eta > \theta\) is the safety loading of the reinsurer. With reinsurance, the wealth process of the insurer is

\[
dX^q(t) = (\lambda \mu_X(\theta - \eta) + \lambda \mu_X q(t)(1 + \eta))dt - q(t) d\sum_{i=1}^{N(t)} X_i. \tag{2}
\]

According to [8], the wealth process (2) can be approximated by the following diffusion model

\[
dX^q(t) = (\lambda \mu_X(\theta - \eta) + \lambda \eta \mu_X(q(t)))dt + \sqrt{\lambda \sigma_X(q(t))}dW_0(t), \tag{3}
\]

where \(\sigma^2_X = E[X_i^2], \{W_0(t)\}_{t \geq 0}\) is a standard \(\{\mathcal{F}_t\}_{t \geq 0}\)-adapted Brownian motion, independent of \(\{X_i, i \geq 1\}\) and \(\{N(t), t \geq 0\}\). The insurer is allowed to invest in a risk-free asset (bond or bank account) whose price process satisfies

\[
dB(t) = rB(t)dt, \ B(0) = b_0, \]

and a risky asset (stock) whose price process is described by the Heston’s SV model

\[
\begin{align*}
    dS(t) &= S(t)[(r + \alpha L(t))dt + \sqrt{L(t)}dW_1(t)], \ S(0) = s_0, \\
    dL(t) &= \beta(\delta - L(t))dt + \sigma \sqrt{L(t)}dW_1(t), \ L(0) = l_0,
\end{align*} \tag{4}
\]

where positive constant \(r\) is the risk-free interest rate, \(\alpha, \beta, \delta\) and \(\sigma\) are all positive constants, representing the appreciation rate, the mean-reversion rate, the long-run mean and the volatility of the volatility parameter, respectively. \(W_1(t)\) is a standard Brownian motion which is independent of \(W_0(t)\). Moreover, we require \(2 \beta \delta \geq \sigma^2\) to ensure that \(L(t)\) is almost surely nonnegative.

The insurer, starting from an initial capital \(x_0\) at time 0, is allowed to dynamically purchase proportional reinsurance and invest in the financial market described above. A trading strategy is denoted by two-dimensional stochastic process \(\alpha_1 := (q(t), \pi_1(t))\), where \(\pi_1(t)\) is the total amount of money invested in the risky asset at time \(t\). Then the amount of money invested in the risk-free asset at time \(t\) is \(X(t) - \pi_1(t)\), where \(X(t)\) is the wealth process associated with strategy \(\alpha_1\).

We call \(\alpha_1\) an admissible strategy if it is \(\mathcal{F}_t\)-progressively measurable and satisfies

\[
q(t) \in [0, 1], E\left(\int_0^t (q(s))^2ds\right) < \infty, \text{ and } E\left(\int_0^t \pi_1^2(s)ds\right) < \infty
\]

for all \(t \geq 0\). Denote the set of all admissible strategies by \(\Pi_1\). For notational convenience, we write \(q(t)\) and \(\pi_1(t)\) as \(q\) and \(\pi_1\), respectively. Then the wealth process \(X(t)\) is given by

\[
\begin{align*}
    dX(t) &= [rX(t) + \pi_1 \alpha L(t) + \lambda \mu_X(\theta - \eta) + \lambda \eta \mu_X q(t)]dt \\
    &\quad + \pi_1 \sqrt{L(t)}dW_1(t) + \sqrt{\lambda \sigma_X(q(t))}dW_0(t), \\
    X(0) &= x_0.
\end{align*}
\]
Assume now that the insurer is interested in maximizing the expected utility of terminal wealth, say at time $T$. Let $u_1(x)$ be the utility function. The insurer aims to maximize the expected utility of terminal wealth $X(T)$, i.e.,

$$
\sup_{\alpha_1} E[u_1(X(T))| X(t) = x, L(t) = l].
$$

(5)

2.2. Wealth process of the reinsurer. In the presence of the proportional reinsurance contract, the wealth process $R_2(t)$ of the reinsurer satisfies

$$
dR_2(t) = (1 + \eta)(1 - p(t))\lambda X dt - (1 - p(t))d\sum_{i=1}^{N(t)} X_i,
$$

(6)

where $p(t)$ is the reinsurance strategy chosen by the reinsurer. In reality, the reinsurer will accept the optimal retention level chosen by the insurer when the reinsurance strategy of the reinsurer is smaller than that of the insurer. While in the opposite case, in order to prevent large losses, the reinsurer may not accept the optimal retention level chosen by the insurer.

According to [8], the wealth process (6) can be approximated by the following diffusion model

$$
dR_2(t) = \eta(1 - p(t))\lambda X dt + \sqrt{2}\sigma X (1 - p(t))dW_0(t).
$$

(7)

Let $\pi_2(t)$ represent the money amount invested in the risky asset at time $t$ by the reinsurer, then $Y(t) - \pi_2(t)$ is the money amount invested in the risk-free asset, where $Y(t)$ is the wealth process associated with strategy $\alpha_2$. We call $\alpha_2 := (p(t), \pi_2(t))$ an admissible strategy if it is $\mathcal{F}_t$-progressively measurable and satisfies

$$
p(t) \in [0, 1], E\left(\int_0^t (p(s))^2 ds\right) < \infty \text{ and } E\left(\int_0^t \pi_2^2(s) ds\right) < \infty
$$

for all $t \geq 0$. Denote the set of all admissible strategies by $\Pi_2$. For notational convenience, we write $p(t)$ and $\pi_2(t)$ as $p$ and $\pi_2$, respectively. Then the wealth process $Y(t)$ is given by

$$
\begin{cases}
\quad dY(t) = [\eta Y(t) + \pi_2 \lambda L(t) + \lambda \eta X (1 - p(t))] dt \\
\quad + \pi_2 \sqrt{2} \lambda L(t) dW_1(t) + \sqrt{2}\sigma X (1 - p(t)) dW_0(t), \\
\quad Y(0) = y_0.
\end{cases}
$$

Let $u_2(x)$ be the utility function of the reinsurer. The objective of the reinsurer is assumed to maximize the expected utility of terminal wealth $Y(T)$, i.e.,

$$
\sup_{\alpha_2 \in \Pi_2} E[u_2(Y(T))| Y(t) = y, L(t) = l].
$$

(8)

For simplicity, we assume that $u_1(x) = u_2(x) = u(x)$, and we will choose Hyperbolic Absolute Risk Aversion (HARA) utility function for our analysis. The HARA utility function with parameters $\gamma, k, v$ can be written as

$$
u(x) = u(\gamma, k, v, x) = \frac{1 - k}{vk} \left(\frac{v}{1 - k} x + \gamma\right)^k, v > 0, k < 1, k \neq 0.
$$

(9)

In reality, HARA utility function recovers exponential utility, power utility and logarithmic utility as special cases, for example

(i) let $\gamma = 1$ and $k \rightarrow -\infty$, then we get $u(1, k, v, x) = \frac{1}{e^v} e^{-vx}$, which is exponential utility;

(ii) suppose that $\gamma = 0$ and $v = 1 - k$, then we arrive at $u(0, k, 1 - k, x) = \frac{x^k}{k}$, which is power utility;
(iii) assume that $\gamma = 0, k \to 0$ and $v \to 1$, then we have $u(0, k, v, x) = \ln x$, which is logarithmic utility.

3. Optimal strategy of the insurer. For an admissible strategy $\alpha_1 \in \Pi_1$, define the value function of the insurer by

$$J(t, x, l) = \sup_{\alpha_1 \in \Pi_1} E[u(X(T)) | X(t) = x, L(t) = l]. \tag{10}$$

In the section below, we try to derive the optimal strategy $\alpha_1^*$ and the optimal value function $J(t, x, l)$. Let $C^{1,2,2}$ be the space of $\varphi(t, x, l)$ such that $\varphi$ and its partial derivatives $\varphi_t, \varphi_x, \varphi_l, \varphi_{xx}, \varphi_{tl}, \varphi_{xl}$ are continuous on $[0, T] \times R \times R$. According to the dynamic programming approach, the value function $J \in C^{1,2,2}$ satisfies the following the HJB equation,

$$\sup_{\alpha_1 \in \Pi_1} A^1 J(t, x, l) = 0, \tag{11}$$

for all $t < T$ with boundary condition

$$J(T, x, l) = u(x), \tag{12}$$

where $A^1$ denotes the generator of the wealth process $X(t)$ controlled by the control variable $\alpha_1$ and is defined by

$$A^1 J(t, x, l) = J_t + (rx + \pi_1 \alpha_1 l + \lambda \mu X(\theta - \eta) + \lambda \eta \mu X q) J_x + \beta(\delta - l) J_l \tag{13}$$

$$+ \frac{1}{2}(\frac{\pi_1^2}{1} + q^2 \lambda \sigma^2 X) J_{xx} + \frac{1}{2} l(\sigma^2 J_{ll} + \pi_1 \sigma l J_{xl}),$$

where $J_t, J_x, J_l, J_{xx}, J_{xl}$ denote partial derivatives of first and second orders with respect to $t, l$ and $x$.

If (11) attains its maximum, then $q$ and $\pi_1$ must satisfy the following expressions

$$q^*(t) = -\frac{\lambda \eta \mu X J_x}{\sigma^2 X} J_{xx}, \quad \pi_1^*(t) = -\frac{\alpha J_x + \sigma J_{xl}}{J_{xx}}. \tag{14}$$

Plugging (14) into (11) yields

$$J_t + (rx + \lambda \mu X(\theta - \eta)) J_x - \left(\frac{\lambda \eta \sigma^2 X}{2 \sigma^2 X} + \frac{\sigma^2}{2} J_{xx} - \frac{1}{2} \sigma^2 J_{xx} l + \frac{\alpha \sigma l J_{xl}}{J_{xx}}\right) \tag{15}$$

$$+ \beta(\delta - l) J_l + \frac{1}{2} \sigma^2 J_{ll} = 0.$$ 

Here, the stochastic control problem has been transformed into a non-linear second order partial differential equation (PDE), it is difficult to solve (15) directly. Therefore, we transform the problem into a dual one and get a linear PDE by applying Legendre transform and dual theory.

In what follows, we aim to solve the equation (15). Applying Legendre transform-dual theory, we firstly convert (15) into its dual one and then get a linear PDE, which is easy to be solved.

**Definition 3.1.** Let $\varpi : \mathbb{R}^n \to \mathbb{R}$ be a convex function. Legendre transform can be defined as follows:

$$\phi(z) = \max_x \{\varpi(x) - \langle z, x \rangle\}, \tag{16}$$

then the function $\phi(z)$ is called Legendre dual function of $\varpi(x)$ (see [33], [5], [28] and [6]).

If $\varpi(x)$ is strictly convex, the maximum in (16) will be attained at just one point, which we denote by $x_0$ and then

$$\phi(z) = \varpi(x_0) - \langle z, x_0 \rangle. \tag{17}$$
According to [28] and [6], we can define a Legendre transform
\[ \hat{J}(t, z, l) = \sup_{x > 0} \{ J(t, x, l) - zx \}, \] (18)
where \( z > 0 \) denotes the dual variable to \( x \). The value of \( x \) which this optimum is attained at is denoted by \( g(t, z, l) \), then we get
\[ g(t, z, l) = \inf_{x > 0} \{ x | J(t, x, l) \geq zx + \hat{J}(t, z, l) \}. \] (19)

The function \( \hat{J}(t, z, l) \) is related to by \( g(t, z, l) \)
\[ g(t, z, l) = - \hat{J}_z(t, z, l). \] (20)
Thus, we shall refer to either one of the two functions \( g(t, z, l) \) and \( \hat{J}(t, z, l) \) as the dual function of \( J(t, x, l) \). In this paper, we choose \( g(t, z, l) \). According to (18) and (19), we get
\[ J_x = z, \quad \hat{J}(t, z, l) = J(t, g(t, l) - zg(t, z, l)) = x. \] (21)

Differentiating (21) with respect to \( t, z, l \), we get the transformation rules as follows
\[ J_t = \hat{J}_t, J_x = z, J_{xx} = -\frac{1}{J_{zz}}, J_l = \hat{J}_l, J_{ll} = \hat{J}_{ll} - \frac{\hat{J}_z}{J_{zz}}, J_{xl} = -\frac{\hat{J}_z}{J_{zz}}. \] (22)

Noting that \( J(T, x, l) = u(x) \), then we can define the following Legendre transform at the terminal time \( T \):
\[ \hat{J}(T, z, l) = \sup_{x > 0} \{ u(x) - zx \}, g(T, z, l) = \inf_{x > 0} \{ x | u(x) \geq zx + \hat{J}(T, z, l) \}. \] (23)
And it leads to \( g(T, z, l) = (u')^{-1}(z) \), where \( (u')^{-1}(z) \) is taken as the inverse of marginal utility.

Plugging (22) into (15) arrives at
\[ \hat{J}_t + (r g + \lambda \mu_X (\theta - \eta)) z + \frac{1}{2} (\lambda \eta^2 \mu_X^2 + \alpha^2) z^2 J_{zz} + \beta (\delta - l) \hat{J}_l + \frac{1}{2} \sigma^2 \hat{J}_{ll} - \alpha \sigma \lambda z J_{zl} = 0. \] (24)

Differentiating both sides of (24) with respect to \( z \) and considering (20), we have
\[ \begin{align*}
g_t - rg + (\lambda \eta^2 \mu_X^2 + \alpha^2) z g_z + \frac{1}{2} (\lambda \eta^2 \mu_X^2 + \alpha^2) z^2 g_{zz} + (\beta (\delta - l) - \alpha \sigma l) g_l + \frac{1}{2} \sigma^2 g_{ll} - \alpha \sigma l z g_{zl} - \lambda \mu_X (\theta - \eta) &= 0, \end{align*} \] (25)
with the boundary condition
\[ g(T, z, l) = \frac{1 - k}{v} (z^{\frac{1}{k-\gamma}} - \gamma). \] (26)

Then we try to conjecture a solution of (25) with the following structure
\[ g(t, z, l) = \frac{1 - k}{v} z^{\frac{1}{k-\gamma}} f(t, l) - \frac{1 - k}{v} \gamma h(t) + I(t), \] (27)
with boundary conditions
\[ f(T, l) = 1, h(T) = 1, I(T) = 0. \] (28)

Substituting all the partial derivatives of \( g(t, z, l) \) and (27) back into (25) and separating the variables, we can derive the following equation
\[ \begin{align*}
\frac{1 - k}{v} z^{\frac{1}{k-\gamma}} [f_t + (\beta (\delta - l) - \frac{\alpha \sigma l}{k-\gamma}) f_1 + (\frac{\lambda \eta^2 \mu_X^2 + \alpha^2}{2(\delta - l)^2} - \frac{k r}{k-\gamma}) f_f + \frac{1}{2} \sigma^2 f_{f1}] - \frac{1 - k}{v} \gamma [h_t - rh] + I_t - r I - \lambda \mu_X (\theta - \eta) &= 0. \end{align*} \] (29)
Eliminating the dependence on \( z \) and \( \gamma \), we get the following three equations:

\[
f_t + (\beta(t) - \frac{k\alpha t}{k - 1})f_t + \left(\frac{k(\lambda^2 \mu^2_X + \alpha^2 t)}{2(k - 1)^2} - \frac{kr}{k - 1}\right)f + \frac{1}{2}\sigma^2 f_{tt} = 0, \tag{30}
\]
\[
h_t - rh = 0, \quad h_t - rI - \lambda \mu_X(\theta - \eta) = 0. \tag{31}
\]

From (31) and (28), we obtain that

\[
h(t) = e^{-r(T-t)}, \quad I(t) = \frac{\lambda \mu_X(\theta - \eta)}{r}(e^{-r(T-t)} - 1). \tag{32}
\]

Inspired by [33], [5] and [22], we summarize the solving process of (30) in Lemma 3.2.

**Lemma 3.2.** Given that \( f(t, l) = e^{A_1(l) + A_2(l)l} \) is the solution to (30), with boundary conditions \( A_1(T) = 0 \) and \( A_2(T) = 0 \), then we arrive at

\[
A_2(t) = \begin{cases} 
\frac{v_1 v_2 (1 - e^{-1/2\sigma^2(v_1 - v_2)(T-t)})}{2 + v_1 \sigma^2 (T-t)}, & k < \frac{\beta^2}{(\beta + \sigma \alpha)^2}, k \neq 0, \\
\frac{2 + v_1 \sigma^2 (T-t)}{2 + v_1 \sigma^2 (T-t)}, & k = \frac{\beta^2}{(\beta + \sigma \alpha)^2}, k \neq 0, \\
v_3 - \frac{\sqrt{\Delta}}{\sigma} \tan(\arctan \frac{v_3 \sigma^2}{\sqrt{\Delta}} - \frac{\sqrt{\Delta}}{2} (T-t)), & k > \frac{\beta^2}{(\beta + \sigma \alpha)^2}, k < 1,
\end{cases}
\]

and if \( k < \frac{\beta^2}{(\beta + \sigma \alpha)^2} \) and \( k \neq 0 \), \( A_1(t) \) is given by (36); if \( k = \frac{\beta^2}{(\beta + \sigma \alpha)^2} \) and \( k \neq 0 \), \( A_1(t) \) is given by (38); if \( k > \frac{\beta^2}{(\beta + \sigma \alpha)^2} \) and \( k < 1 \), \( A_1(t) \) is given by (40).

**Proof.** Inserting \( f(t, l) = e^{A_1(l) + A_2(l)l} \) into (30), we get

\[
t \left[ A_2'(t) + \frac{1}{2}\sigma^2 A_2(t) - (\beta + \frac{\sigma \alpha k}{k - 1})A_2(t) + \frac{k \sigma^2}{2(k - 1)^2} \right] + A_1'(t) + \beta \delta A_2(t) + \frac{k \lambda \eta^2 \mu^2_X}{2(k - 1)^2} - \frac{kr}{k - 1} = 0.
\]

Matching the coefficient on both sides, we arrive at

\[
A_2'(t) + \frac{1}{2}\sigma^2 A_2(t) - (\beta + \frac{\sigma \alpha k}{k - 1})A_2(t) + \frac{k \sigma^2}{2(k - 1)^2} = 0, \tag{33}
\]
\[
A_1'(t) + \beta \delta A_2(t) + \frac{k \lambda \eta^2 \mu^2_X}{2(k - 1)^2} - \frac{kr}{k - 1} = 0. \tag{34}
\]

From the equation (33), we find that the solution to (33) depends on the number of the root of the quadratic equation

\[
-\frac{1}{2}\sigma^2 A_2(t) + (\beta + \frac{\sigma \alpha k}{k - 1})A_2(t) - \frac{\alpha^2 k}{2(k - 1)^2} = 0.
\]

We discuss and solve (33) and (34) as follows.

(i) If \( k < \frac{\beta^2}{(\beta + \sigma \alpha)^2} \) and \( k \neq 0 \), considering terminal conditions \( A_1(T) = 0 \) and \( A_2(T) = 0 \) by the method of [33], we arrive at

\[
A_2(t) = \frac{v_1 v_2 (1 - e^{-1/2\sigma^2(v_1 - v_2)(T-t)})}{v_1 - v_2 e^{-1/2\sigma^2(v_1 - v_2)(T-t)}}, \tag{35}
\]
\[
A_1(t) = \frac{2\beta \delta}{\sigma^2} \ln \frac{v_1 - v_2}{v_1 - v_2 e^{-1/2\sigma^2(v_1 - v_2)(T-t)}} + \left(\frac{k \lambda \eta^2 \mu^2_X}{2(k - 1)^2} + \frac{kr}{1 - k}\right)(T-t), \tag{36}
\]

where

\[
v_{1,2} = \frac{k(\beta + \sigma \alpha) - \beta}{(k - 1)\sigma^2} \pm \frac{\sqrt{\Delta}}{\sigma^2}, \quad \Delta = \frac{k(\beta + \sigma \alpha)^2 - \beta^2}{k - 1}.
\]
(ii) If \( k = \frac{\beta^2}{(\delta + \sigma)^2} \) and \( k \neq 0 \), we have

\[
A_2(t) = \frac{v_3^2 \sigma^2 (T - t)}{2 + v_3 \sigma^2 (T - t)},
\]

\[
A_1(t) = \left( \beta \delta + \frac{kr}{1 - k} + \frac{k \lambda \eta^2 \mu_X^2}{2(1 - k)^2} \right)(T - t) + \frac{2 \beta \delta}{\sigma^2} \ln \frac{2}{2 + v_3 \sigma^2 (T - t)},
\]

where \( v_3 = \frac{k(\beta + \sigma)}{(1 - k)^2} \) and \( k < 1 \), we have

\[
A_2(t) = v_3 - \frac{\sqrt{-\Delta}}{\sigma^2} \tan[\arctan \frac{v_3 \sigma^2}{\sqrt{-\Delta}} - \frac{\sqrt{-\Delta}}{2} (T - t)],
\]

\[
A_1(t) = \left( \beta \delta + \frac{kr}{1 - k} + \frac{k \lambda \eta^2 \mu_X^2}{2(1 - k)^2} \right)(T - t) + \frac{2 \beta \delta}{\sigma^2} \ln\cos[\arctan \frac{v_3 \sigma^2}{\sqrt{-\Delta}}]
\]

\[\] \[\]

(iii) If \( k > \frac{\beta^2}{(\delta + \sigma)^2} \) and \( k < 1 \), we have

\[
J(t) = \alpha \left( \frac{\sigma}{1 - k} (x - I(t) + \gamma h(t)) \right)^{k-1} f^{1-k}(t, l).
\]

Taking \( J_x = z \) into account and by integration, we obtain the optimal value function under HARA utility as follows

\[
J(t, x, l) = \frac{-1}{kv} \left( \frac{v}{1 - k} (x - I(t) + \gamma h(t)) \right)^k f^{1-k}(t, l).
\]

According to (20), (22) and (27) yields

\[
\left\{ \begin{array}{l}
\frac{\partial J}{\partial x} = -z J_{xz} = zg = -z \frac{1}{v} z \frac{1}{v} f = -\frac{1}{v} (x + \frac{k - 1}{v} \gamma h(t) - I(t)), \\
\frac{\partial J}{\partial l} = J_{zl} = -g_l = -\frac{1}{v} z \frac{1}{v} f_l = -A_2(t) [x + \frac{1 - k}{v} \gamma h(t) - I(t)].
\end{array} \right.
\]

Then we can derive the optimal investment strategy in (10), which is expressed in the following theorem.

**Theorem 3.3.** For HARA utility (9), the optimal investment strategy of the problem (5) for the insurer is given by

\[
\pi^{HARA}_{1}(t) = \left( \frac{\alpha}{1 - k} + \sigma A_2(t) (x + \frac{1 - k}{v} \gamma h(t) - I(t)),
\right)
\]

and the corresponding optimal value function is given by (44).

According to (44), the optimal reinsurace strategy in (14) can be rewritten as

\[
q^{HARA}_{1}(t) = \frac{\eta X}{\sigma X (1 - k)} \left( x + \frac{\lambda \mu_X (\theta - \eta)}{r} + \frac{\gamma (1 - k)}{v} \frac{\lambda \mu_X (\theta - \eta)}{r} e^{-r(T-t)} \right).
\]

As an optimal reinsurace strategy, the proportion of reinsurace should lie between 0 and 1, i.e., \( 0 \leq q^{HARA}_{1}(t) \leq 1 \). We shall identify in what regions of \( (x, t) \) the reinsurace strategy \( q^{HARA}_{1}(t) \) being of the form (45) or equal to 1 (no reinsurace) or 0 (full reinsurace) respectively.

Denote

\[
x_1^r = \frac{\sigma_X^2 (1 - k)}{\eta \mu_X} + \frac{\lambda \mu_X (\eta - \theta)}{r} (1 - e^{-rT}) - \frac{\gamma (1 - k)}{v} e^{-rT},
\]

\[
x_2^r = \frac{\alpha}{1 - k} + \sigma A_2(t) (x + \frac{1 - k}{v} \gamma h(t) - I(t)),
\]

\[
x_3^r = \frac{\beta^2}{(\delta + \sigma)^2}.
\]
The optimal reinsurance strategy is given by

\[ x^*_2 = \frac{\lambda \mu X (\eta - \theta)}{r} (1 - e^{-rT}) - \frac{\gamma (1 - k)}{v} e^{-rT}, \]

\[ t^*_1 = T + \frac{1}{r} \ln \left( \frac{\sigma^2 (1 - k) \eta \mu X}{\lambda \mu X (\eta - \theta)} - x \right), \]

\[ t^*_2 = T + \frac{1}{r} \ln \left( \frac{\lambda \mu X (\eta - \theta) - x}{\gamma (1 - k) \eta \mu X + \lambda \mu X (\eta - \theta)} \right). \]

If \( x \geq x^*_1 \), the value of (45) is less than 0, thus \( q^*_H A R A(t) = 1 \).

If \( \frac{\sigma^2 (1 - k) \eta \mu X}{\lambda \mu X (\eta - \theta)} - \frac{\gamma (1 - k)}{v} x < x^*_1 \), when \( t^*_1 \leq t \leq T \), the value of (45) is more than 1, thus we have \( q^*_H A R A(t) = 1 \), for \( 0 \leq t < t^*_1 \), the optimal reinsurance strategy is expressed by (45).

If \( x^*_2 < x < \frac{\sigma^2 (1 - k) \eta \mu X}{\lambda \mu X (\eta - \theta)} - \frac{\gamma (1 - k)}{v} \), we have for \( 0 \leq t \leq T \), (45) is the optimal reinsurance strategy.

If \( x \leq x^*_2 \), when \( t^*_2 < t \leq T \), (45) is the optimal reinsurance strategy, when \( 0 \leq t \leq t^*_2 \), the value of (45) is less than 0, thus \( q^*_H A R A(t) = 0 \).

Summarizing what is mentioned above, we arrive at the following theorem.

**Theorem 3.4.** For the problem (5) with HARA utility (9), the optimal reinsurance strategy \( q^*(t) \) for the insurer is given as follows:

1. If \( x \geq x^*_1 \), then \( q^*(t) = 1, 0 \leq t \leq T \).
2. If \( \frac{\sigma^2 (1 - k) \eta \mu X}{\lambda \mu X (\eta - \theta)} - \frac{\gamma (1 - k)}{v} x < x^*_1 \), for \( t^*_1 \leq t \leq T \), we have \( q^*(t) = 1 \), for \( 0 \leq t < t^*_1 \), the optimal reinsurance strategy \( q^*(t) = q^*_H A R A(t) \), which is given by (45);
3. If \( x^*_2 < x < \frac{\sigma^2 (1 - k) \eta \mu X}{\lambda \mu X (\eta - \theta)} - \frac{\gamma (1 - k)}{v} \), for \( 0 \leq t \leq T \), we have \( q^*(t) = q^*_H A R A(t) \);
4. If \( x \leq x^*_2 \), for \( t^*_2 < t \leq T \), \( q^*_H A R A(t) \) in (45) is the optimal reinsurance strategy, for \( 0 \leq t \leq t^*_2 \), we have \( q^*(t) = 0 \).

In what follows, we will discuss three special cases of problem (5), and the results can easily be derived by similar arguments as above, so we only present them without giving the proofs.

Denote

\[ x_{\text{pow}}^* = \frac{\sigma^2 (1 - k)}{\eta \mu X} + \frac{\lambda \mu X (\eta - \theta)}{r} (1 - e^{-rT}), \]

\[ x_{\text{exp}}^* = T + \frac{1}{r} \ln \left( 1 + \frac{r \sigma^2 (1 - k)}{\lambda \eta \mu X (\eta - \theta)} \right), \]

\[ x_{\text{log}}^* = \frac{\sigma^2}{\eta \mu X} + \frac{\lambda \mu X (\eta - \theta)}{r} (1 - e^{-rT}). \]

**Corollary 1.** If \( \gamma = 0, v = 1 - k \), HARA utility is degenerated to power utility. Therefore, the optimal investment strategy of problem (5) for the insurer under \( U_{\text{pow}}(x) = \frac{x^k}{k} \) is given by

\[ \pi_{1_{\text{pow}}(t)}(t) = \left( \frac{\alpha}{1 - k} + \sigma A_2(t) \right) (x - \frac{\lambda \mu X (\theta - \eta)}{r} (e^{-r(T - t)} - 1)), t \in [0, T], \]

And the optimal reinsurance strategy \( q_{\text{pow}}^*(t) \) is given as follows:

1. If \( x \geq x_{\text{pow}}^* \), then \( q_{\text{pow}}^*(t) = 1, 0 \leq t \leq T \).
and the optimal reinsurance strategy is given by
\[ q_{pow}^*(t) = \frac{\eta \mu_X}{\sigma (1-k)} \left( x + \frac{\lambda \mu_X (\theta - \eta)}{r} (1 - e^{-r(T-t)}) \right) ; \] (46)

(3) if \( x_3 < x < \frac{\sigma^2(1-k)}{\eta^2 X} \), then for \( 0 \leq t \leq T \), the optimal reinsurance strategy is given by (46).

(4) if \( x \leq x_3 \), for \( t_3 < t \leq T \), (46) is the optimal reinsurance strategy, for \( 0 \leq t \leq t_3 \), we have \( q_{pow}^*(t) = 0 \).

Remark 1. Based on Legendre transform, we can obtain the explicit expression for the optimal reinsurance-investment strategy with power utility, which is seldom studied in the reinsurance-investment problems. Besides, our method can be used to solve the optimal problem under Vasicek Model in [25] as well as that under CEV Model in [33].

Corollary 2. If \( \gamma = 1, k \rightarrow -\infty \), HARA utility is reduced to exponential utility. Therefore, the optimal investment of problem (5) for the insurer under \( U_{exp}(x) = -\frac{v}{\sigma^2} \) is given by
\[ \pi_{1,exp}^*(t) = \left( \frac{\alpha}{v} - \frac{\alpha^2(1-e^{-(\beta+\sigma\alpha)(T-t)})}{2v(\beta + \sigma\alpha)} \right) e^{-r(T-t)} , t \in [0,T], \]
and the optimal reinsurance strategy \( q_{exp}^*(t) \) of problem (5) for the insurer is given as follows
(1) if \( \frac{\mu_X}{\sigma^2} > 1 \), when \( 0 \leq t < t_{exp}^* \), we have \( q_{exp}^*(t) = \frac{\mu_X}{\sigma^2} e^{-r(T-t)} \); when \( t_{exp}^* \leq t \leq T \), we have \( q_{exp}^*(t) = 1 \);
(2) if \( \frac{\mu_X}{\sigma^2} < 1 \), we have \( q_{exp}^*(t) = \frac{\mu_X}{\sigma^2} e^{-r(T-t)} \) for \( 0 \leq t \leq T \).

Remark 2. From Corollary 2, we can see that the optimal reinsurance strategy \( q_{exp}^*(t) \) coincides with Theorem 4.1 and Proposition 4.2 in [29], and the optimal investment strategy \( \pi_{1,exp}^*(t) \) is consistent with [30], [1] and [18]. This means that our results, i.e., Theorem 3.3 and Theorem 3.4 extend the existing results to the case of HARA utility.

Remark 3. (1) From Theorem 3.3, Theorem 3.4, Corollary 1 and Corollary 2, we find that the optimal investment strategies all depend on the wealth \( x \) except \( \pi_{1,exp}^*(t) \) under exponential utility. The reason is that exponential utility has a constant absolute risk aversion parameter \( v \). Such a utility function is the only function under which the principle of “zero utility” gives a fair premium that is independent of the wealth \( x \) of an insurance company (see [7]).

(2) Based on Corollary 2, \( \pi_{1,exp}^*(t) \) under the Heston’s SV model has the following structure:
\[ D(t) := e^{-r(T-t)} \pi_{1,exp}^*(t) = \frac{\alpha}{v} - \sigma D_1(t) , D_1(t) = \frac{\alpha^2(1-e^{-(\beta+\sigma\alpha)(T-t)})}{2v(\beta + \sigma\alpha)} , t \in [0,T]. \]
Here, \( e^{-r(T-t)} \) stands for an accumulation factor (see [12]). We can decompose \( D(t) \) into two parts (see [29] and [4]): \( \frac{\alpha}{v} \) (the myopic demand) and \( -\sigma D_1(t) \) (the intertemporal demand). The first part is related to the appreciation rate of the risky asset mainly and coincides with \( \pi_{1,exp}^*(t) \). The second part demonstrates the effect resulted from the stochastic volatility under Heston’s SV model, thus it is often referred to in the literature as a “hedging device”, even though this is not
perfect hedging in the sense of complete markets. For further detail on myopic and intertemporal demands, see [4]. Since \( \sigma > 0 \), \( D_1(t) > 0 \), the insurer’s “hedging device” (or intertemporal demand) advises the insurer to reduce investment in risky asset when \( \sigma \) increases. This is consistent with intuition. As described in (4), the correlation coefficient between the risky asset’s price and volatility is positive, which implies that the uncertainties of the two processes change in the same sense, thus the “hedging device” suggests the insurer to invest less in risky asset to reduce risk.

**Corollary 3.** If \( \gamma = 0, k \to 0 \) and \( v \to 1 \), HARA utility is reduced to logarithm utility. Therefore, the optimal investment strategy of problem (5) for the insurer under \( U_{\log}(x) = \ln x \) is given by

\[
\pi^*_x(t) = \alpha(x - \frac{\lambda u_x}{r}(e^{-r(T-t)} - 1)), t \in [0, T],
\]

and the optimal reinsurance strategy \( q^*_x(t) \) of problem (5) for the insurer is given as follows

1. if \( x \geq x^*_x \), then \( q^*_x(t) = 1, t \in [0, T] \);
2. if \( \frac{\sigma^2}{\eta u_x} \leq x < x^*_x \), for \( t^*_x \leq t \leq T \), we have \( q^*_x(t) = 1 \), for \( 0 \leq t \leq t^*_x \), the optimal reinsurance strategy is given by

\[
q^*_x(t) = \frac{\eta u_x}{\sigma^2 X} \left( x + \frac{\lambda u_x}{r}(1 - e^{-r(T-t)}) \right);
\]

3. if \( x^*_x < x < \frac{\sigma^2}{\eta u_x} \), for \( 0 \leq t \leq T \), the optimal reinsurance strategy is given by (47);
4. if \( x \leq x^*_x \), for \( t^*_x < t \leq T \), (47) is the optimal reinsurance strategy, for \( 0 \leq t \leq t^*_x \), \( q^*_x(t) = 0 \).

4. **Optimal strategy of the reinsurer.** In this section, we aim to discuss the optimal strategy for the reinsurer. For an admissible strategy \( \alpha_2 \in \Pi_2 \), define the value function by

\[
H(t, y, l) = \sup_{\alpha_2 \in \Pi_2} E[u(Y(T))|Y(t) = y, L(t) = l],
\]

(48)

In what follows, we try to derive the optimal strategy \( \alpha^*_2 \) and the optimal value function \( H(t, y, l) \). According to the dynamic programming approach, the value function \( H \in C^{1,2,2} \) satisfies the following HJB equation,

\[
\sup_{\alpha_2 \in \Pi_2} \mathcal{A}^2 H(t, y, l) = 0,
\]

(49)

for all \( t < T \) with boundary condition

\[
H(T, y, l) = u(y),
\]

(50)

where \( \mathcal{A}^2 \) denotes the generator of the reinsurer’s wealth process \( Y(t) \) controlled by the control variable \( \alpha_2 \), and is defined by

\[
\mathcal{A}^2 H(t, y, l) = H_t + (ry + \pi_2 \alpha l + \lambda u_x \eta(1-p))H_y + \beta(\delta - l)H_l + \frac{1}{2}(\pi^2 \sigma^2 X + (1 - p^2))H_{yy} + \frac{1}{2} l \sigma^2 H_{ll} + \pi_2 \sigma_2 l H_{yl},
\]

(51)

where \( H_t, H_l, H_y, H_{ll}, H_{yy}, H_{yl} \) denote partial derivatives of first and second orders with respect to \( t, l \) and \( y \).
If (49) attains its maximum, then \( p(t) \) and \( \pi_2(t) \) must satisfy the following expressions

\[
p^* = 1 + \frac{\eta \mu_X}{\sigma_X^2} \frac{H_y}{H_{yy}}, \pi_2^*(t) = -\frac{\alpha H_y + \sigma H_{yt}}{H_{yy}}. \tag{52}
\]

Inserting (52) into (49) arrives at

\[
H_t + ryH_y - \left[ \frac{\lambda \eta^2 \mu_X^2}{2 \sigma_X^2} + \frac{\alpha^2 l}{T} \right] \frac{H_y^2}{H_{yy}} - \frac{1}{2} \sigma^2 l H_{yy} H_{yl} - \alpha \sigma \frac{H_y H_{yl}}{H_{yy}} + \beta (\delta - l) H_t + \frac{1}{2} \sigma^2 l H_{ll} = 0. \tag{53}
\]

By similar analysis above, we apply Legendre transform and dual theory to solve (51). Define a Legendre transform

\[
\tilde{H}(t, z, l) = \sup_{y > 0} \{ H(t, y, l) - zy \}, \tag{54}
\]

where \( z > 0 \) denotes the dual variable to \( y \). The value of \( y \) which this optimum is attained at is denoted by \( \tilde{g}(t, z, l) \), then we get

\[
\tilde{g}(t, z, l) = \inf_{y > 0} \{ y | H(t, y, l) \geq zy + \tilde{H}(t, z, l) \}. \tag{55}
\]

The function \( \tilde{H}(t, z, l) \) is related to \( \tilde{g}(t, z, l) \) by

\[
\tilde{g}(t, z, l) = -\tilde{H}_z(t, z, l). \tag{56}
\]

Thus, we shall refer to either one of the two functions \( \tilde{g}(t, z, l) \) and \( \tilde{H}(t, z, l) \) as the dual function of \( H(t, y, l) \). In this paper, we choose \( \tilde{g}(t, z, l) \). According to (54) and (55), we get

\[
H_y = z, \quad \tilde{H}(t, z, l) = H(t, \tilde{g}, l) - z\tilde{g}, \quad \tilde{g}(t, z, l) = y. \tag{57}
\]

Differentiating (57) with respect to \( t, z, l \), we get the transformation rules as follows

\[
H_t = \tilde{H}_t, H_y = z, H_{yy} = -\frac{1}{H_{zz}}, H_l = \tilde{H}_l, H_{ll} = \tilde{H}_l - \frac{\tilde{H}_l^2}{H_{zz}}, H_{yl} = -\frac{\tilde{H}_z}{H_{zz}}. \tag{58}
\]

Noting that \( H(T, y, l) = u(y) \), then we can define the following Legendre transform at the terminal time \( T \):

\[
\tilde{H}(T, z, l) = \sup_{y > 0} \{ u(y) - zy \}, \tilde{g}(T, z, l) = \inf_{y > 0} \{ y | u(y) \geq zy + \tilde{H}(T, z, l) \}. \tag{59}
\]

And it leads to \( \tilde{g}(T, z, l) = (u')^{-1}(z) \), where \( (u')^{-1}(z) \) is taken as the inverse of marginal utility. Plugging (58) into (53) yields

\[
\tilde{H}_t + r\tilde{g}z + \frac{1}{2}(\lambda \eta^2 \mu_X^2 + \alpha^2 l)z^2 \tilde{H}_{zz} + \beta (\delta - l) \tilde{H}_t + \frac{1}{2} l \sigma^2 \tilde{H}_ll - \alpha \sigma l z \tilde{H}_{zl} = 0. \tag{60}
\]

Differentiating both sides of (60) with respect to \( z \) and considering (56), we have

\[
\tilde{g}_t - r\tilde{g} + (\lambda \eta^2 \mu_X^2 + \alpha^2 l - r)z\tilde{g}_z + \frac{1}{2}(\lambda \eta^2 \mu_X^2 + \alpha^2 l)z^2 \tilde{g}_{zz} + (\beta (\delta - l) - \alpha \sigma l)\tilde{g}_l + \frac{1}{2} \sigma^2 \tilde{g}_{ll} - \alpha \sigma l z \tilde{g}_{zl} = 0, \tag{61}
\]

with the boundary condition

\[
\tilde{g}(T, z, l) = \frac{1 - k}{v} (z \frac{1}{v^{\frac{1}{k}}} - \gamma). \tag{62}
\]

Then we try to conjecture a solution of (61) with the following structure

\[
\tilde{g}(t, z, l) = \frac{1 - k}{v} z^{\frac{1}{k}} \tilde{f}(t, l) - \frac{1 - k}{v} \gamma \tilde{h}(t) + \tilde{l}(t), \tag{63}
\]
with boundary conditions
\[ \hat{f}(T, l) = 1, \hat{h}(T) = 1, \hat{I}(T) = 0. \] (64)

Substituting all the partial derivatives of \( \tilde{y}(t, z, l) \) and (63) back into (61) and separating the variables, we can derive the following equation
\[
\begin{align*}
\frac{1-k}{v} z^{\frac{T}{\kappa-1}} \left[ \frac{f_t}{v} + (\beta(\delta - l) - \frac{k\alpha l}{k-1}) f_l + \frac{k(\lambda y^2 \mu^2 + \alpha^2 l)}{2(k-1)^2} - \frac{kr}{k-1} \right] 
- \frac{1-k}{v} \gamma (\hat{h}_t + r\hat{h}) + I_t - r\hat{I} = 0.
\end{align*}
\] (65)

Eliminating the dependence on \( z \) and \( \gamma \), we get the following three equations
\[
\begin{align*}
\hat{f}_t + (\beta(\delta - l) - \frac{k\alpha l}{k-1}) \hat{f}_l + \frac{k(\lambda y^2 \mu^2 + \alpha^2 l)}{2(k-1)^2} - \frac{kr}{k-1} f_l + \frac{1}{2} \sigma^2 f_{tt} &= 0, \quad \text{(66)}
\hat{h}_t - r\hat{h} = 0, \quad \text{(67)}
I_t - r\hat{I} &= 0.
\end{align*}
\]

By similar arguments as Section 3, from (66), (67) and (64), we obtain that
\[ \tilde{f}(t, l) = f(t, l), \quad \tilde{h}(t) = e^{-r(T-t)}, \quad \tilde{I}(t) = 0, \] (68)

where \( f(t, l) \) is given by (63). According to (63), and by using \( \tilde{g}(t, z, l) \), we obtain
\[ z = \left( \frac{v}{1-k} (y + \gamma \hat{h}(t)) \right)^{k-1} \tilde{f}^{1-k}(t, l). \] (69)

Taking \( H_z = y \) into account and by integration, we obtain the optimal value function for the reinsurer under HARA utility as follows
\[ H(t, y, l) = \frac{1-k}{v} \left( \frac{v}{1-k} (y + \gamma e^{-r(T-t)}) \right)^k \tilde{f}^{1-k}(t, l). \] (70)

According to (66), (68) and (63) yields
\[ \begin{cases} 
\frac{H_z}{H_y} = -z\tilde{h}_z = z\tilde{g}_z = -\frac{1}{v} z^{\frac{T}{\kappa-1}} f_t = -\frac{1}{v} (y + \frac{1-k}{v} \gamma \hat{h}(t)), \\
\frac{H_{zz}}{H_{y^2}} = \tilde{h}_{zz} = -\tilde{g}_t = -\frac{1}{v} z^{\frac{T}{\kappa-1}} f_l = -A_2(t) [y + \frac{1-k}{v} \gamma \hat{h}(t)].
\end{cases} \] (71)

According to (71) and (52), we can have the following theorem.

**Theorem 4.1.** For HARA utility (9), the optimal investment strategy of the problem (8) for the reinsurer is given by
\[ \pi_{HARA}^*(t) = \left( \frac{\alpha}{1-k} + \sigma A_2(t) \right) \left( y + \frac{1-k}{v} \gamma e^{-r(T-t)} \right), \quad 0 \leq t \leq T, \] (72)

where \( A_2(t) \) is given by Lemma 3.2, and the corresponding optimal value function is given by (70).

According to (71), the optimal reinsurance strategy in (52) can be rewritten as
\[ p_{HARA}^*(t) = 1 - \frac{\eta \mu_X}{\sigma_X^2 (1-k)} \left( y + \frac{\gamma (1-k)}{v} e^{-r(T-t)} \right). \] (73)

As an optimal reinsurance strategy, the proportion of reinsurance should lie between 0 and 1, i.e., \( 0 \leq p_{HARA}^*(t) \leq 1 \). We shall identify in what regions of \((y, t)\) the reinsurance strategy \( p_{HARA}^*(t) \) being of the form (73) or equal to 1 (no reinsurance) or 0 (full reinsurance) respectively.

Denote
\[ \hat{t}_1 = T + \frac{1}{r} \ln \left( \frac{\gamma (1-k)}{v} (\frac{\sigma_X^2 (1-k)}{\eta \mu_X} - y) \right), \hat{t}_2 = T + \frac{1}{r} \ln \left( \frac{-yv}{\gamma (1-k)} \right). \]

Summarizing what is mentioned above, we arrive at the following theorem.
Theorem 4.2. For the problem (8) with HARA utility (9), the optimal reinsurance strategy $p^*(t)$ for the reinsurer is given as follows:

1. if $y \geq \frac{\sigma_X^2}{\sigma_X^2 + \lambda} - \frac{\gamma(1-k)}{v}$, for $0 \leq t < t^*_1$, the optimal reinsurance strategy $p^*(t) = p^*_{HARA}(t)$, for $t^*_1 \leq t \leq T$, we have $p^*(t) = 0$;
2. if $-\gamma(1-k)e^{-rT} \leq y < \frac{\sigma_X^2}{\sigma_X^2 + \lambda} - \frac{\gamma(1-k)}{v}$, for $0 \leq t \leq T$, the optimal reinsurance strategy is $p^*(t) = p^*_{HARA}(t)$;
3. if $-\gamma(1-k)e^{-rT} < y < \frac{\sigma_X^2}{\sigma_X^2 + \lambda} - \frac{\gamma(1-k)}{v}$, for $0 \leq t < t^*_2$, the optimal reinsurance strategy $p^*(t) = 1$, for $t^*_2 < t \leq T$, we have $p^*(t) = p^*_{HARA}(t)$;
4. if $y < -\frac{\gamma(1-k)}{v}$, for $0 \leq t \leq T$, the optimal reinsurance strategy is $p^*(t) = 1$.

Proof. The solving process is similar to that of Theorem 3.4.

In particular, similar to Corollary 1 and Corollary 3, it is easy to derive the optimal reinsurance-investment strategy for the reinsurer under power utility and logarithm utility, respectively, so we omit them. Here we only present the optimal strategy under the exponential utility. If $\gamma = 1, k \to -\infty$, HARA utility is reduced to exponential utility $U_{exp}(x) = \frac{-e^{-vx}}{v}$, according to Theorem 4.1 and Theorem 4.2, we obtain the following result.

Corollary 4. If the reinsurer takes $U_{exp}(x) = \frac{-e^{-vx}}{v}$ into account, the optimal investment strategy of problem (8) is given by

$$\pi^*_2(t) = \left(\frac{\alpha}{v} - \frac{\alpha^2 \sigma(1 - e^{-\beta + \sigma \alpha(T - t)})}{2v(\beta + \sigma \alpha)}\right) e^{-r(T - t)}, t \in [0, T],$$

and the optimal reinsurance strategy $p^*_{exp}(t)$ is given as follows

1. if $\frac{\mu_X}{\sigma_X^2} > 1$, when $0 \leq t < t^*_exp$, we obtain that reinsurance strategy is $p^*_{exp}(t) = 1 - \frac{\mu_X}{\sigma_X^2} e^{-r(T - t)}$, when $t^*_exp \leq t \leq T$, we have $p^*_{exp}(t) = 0$;
2. if $\frac{\mu_X}{\sigma_X^2} \leq 1$, we have $p^*_{exp}(t) = 1 - \frac{\mu_X}{\sigma_X^2} e^{-r(T - t)}$ for $0 \leq t \leq T$.

Remark 4. $\pi^*_2(t)$ is the same as $\pi^*_1(t)$, this is because that the utility function chosen by the insurer is the same as that of the reinsurer. If the utility functions chosen by the insurer are different from that of the reinsurer, the two optimal investment strategies will be different. The economic analysis of $\pi^*_2(t)$ is similar to that of Remark 3.

5. Numerical analysis. In this section, we investigate the effects of parameters on our results and present some numerical simulations to illustrate the effects. We assume that the claim size $\xi_i$ is exponentially distributed with parameter $\gamma = 2$. Throughout this section, unless otherwise stated, the basic parameters are given by $k = -0.5, v = 3, t = 0, T = 10, r = 0.05, \gamma = 0.6, \lambda = 3, \mu_X = 0.5, \sigma_X^2 = 0.5, \theta = 0.2, \eta = 0.24, \sigma = 0.16, \beta = 2, \delta = 0.3, \alpha = 1.5, x_0 = 6, y_0 = 5$. We can draw the following conclusions from Figs.1-11.

5.1. Comparative analyses on $q^*(t)$ and $p^*(t)$. A fair reinsurance contract should consider the interests of both insurer and reinsurer, while they both prefer to choose a reinsurance proportion for their own utility maximization. Assume that the insurer and the reinsurer have complete information on the risk of insurance and investment.
If \( q^*(t) = p^*(t) \), both the insurer and reinsurer will obtain the maximum value of terminal wealth, thus the reinsurer will accept the optimal reinsurance strategy chosen by the insurer. From Theorem 3.4 and Theorem 4.2, we will have \( q_{HARA}^* = p_{HARA}^* \) if the parameters satisfy some conditions given by Theorem 3.4 and Theorem 4.2, i.e.,

\[
\frac{\gamma^2(1-k)}{v^2 - \gamma(1-k)} \leq x \leq x^*, 0 \leq t < \min\{t_1^*, t_2^*\}
\]

or

\[
-\frac{\gamma(1-k)}{v} < y = -\frac{\gamma(1-k)}{v} e^{-rt}, x^* < x < \frac{\sigma_\nu^2(1-k)}{\nu^2} - \frac{\gamma(1-k)}{v}, t_2^* < t \leq T.
\]

In this case, the corresponding optimal investment strategies \( \pi_{HARA}^* \) and \( \pi_{HARA}^* \) are also given by (44) and (72), but same parameters in (44) and (72) will be changed, as a result, the optimal investment strategies \( \pi_{HARA}^* \) and \( \pi_{HARA}^* \) may be different from that case of \( p_{HARA}^* \neq q_{HARA}^* \), which are shown in Figs 6-8 with initial time 0 for simplicity.

If \( q^*(t) > p^*(t) \), the optimal retention level proposed by the insurer is larger than that of the reinsurer, it means that the insurer would like to purchase less reinsurance and undertake more risk himself, in this case, the reinsurer who prefers a higher level of reinsurance will accept the optimal reinsurance strategy proposed by the insurer. But in the opposite case, if the insurer’s optimal retention level is smaller than that of the reinsurer, i.e., \( q^*(t) < p^*(t) \), this means that the insurer prefers to spread more risks to the reinsurer, which may not be accepted by the reinsurer. The insurer may need to find other reinsurers to undertake the rest proportion of reinsurance, i.e., \( p^*(t) - q^*(t) \).

5.2. Sensitivity analysis of reinsurance strategies. In Fig.1, we analyze the influence of the wealth \( x \) on the optimal reinsurance strategy \( q_{HARA}^* \). Without loss of generality, we only consider the optimal reinsurance strategy \( q_{HARA}^* \) at initial time 0 for simplicity. From Fig.1, we can see that the optimal reinsurance strategy equals 0 when the initial wealth \( x \) is smaller, then increases with \( x \) when \( x \) is larger and finally becomes 1 when \( x \) is much larger. That is, when the initial wealth \( x \) is smaller, the insurer’s wealth is less, he/she is unable to take insurance risks and prefers to purchase 100 percent reinsurance to reduce the underlying claims risk, in this case, the retention level of reinsurance \( q_{HARA}^* \) is 0; as \( x \) increases, the insurer’s wealth becomes more, he/she is able to take more insurance risks, so prefers to raise the retention level of reinsurance; when \( x \) is much larger, he/she is able to take more claim risks and doesn’t need to purchase reinsurance, so the retention level of reinsurance is 100 percent.

From Fig.2, we find that \( y \) exerts negative effect on \( p_{HARA}^* \). As \( y \) becomes larger, the reinsurer will obtain more reinsurance premium from the insurer, while he/she has to accept higher reinsurance proportion. This is consistent with intuition.

In addition, we provide some numerical simulations to illustrate the optimal reinsurance strategy under exponential utility. Fig.3 shows \( \exp^*_H \) is an increasing function of time \( t \). As time \( t \) passes, the insurer will obtain much wealth from insurance business and investment. Then he/she is able to take more insurance risks and prefers to raise the retention level of reinsurance. \( \exp^*_H \) is decreasing with \( t \). The reinsurer prefers to take more share of the reinsurance to gain more profits.

Fig.4 demonstrates that \( \exp^*_H \) decreases with respect to \( v \). Note that is the insurer’s risk aversion coefficient, a large value of \( v \) means more risk averse. This implies that if the insurer is more risk averse, he/she will purchase more reinsurance. Thus, with the increase of \( v \), the insurer would like to reduce the retention level of
reinsurance and transfer a larger portion of the underlying risk to a reinsurer. The aversion coefficient $v$ exerts a positive effect on $p_{HARA}^\ast(t)$. As $v$ increases, the reinsurer is more conservative and he/she will accept lower reinsurance proportion.

Fig.5 illustrates that $q_{HARA}^\ast(t)$ decreases with respect to the risk-free interest rate $r$. As $r$ increases, the insurer will earn more money from investment in risk-free asset. Hence, he/she will have enough money to purchase reinsurance and reduce the retention level. $p_{HARA}^\ast(t)$ increases with $r$. As $r$ becomes larger, the reinsurer will gain more profits from investment in risk-free asset, he/she prefers to accept lower reinsurance proportion.

5.3. Sensitivity analysis of investment strategies. From Figs.6-7, we can see that the optimal investment strategies $\pi_{1HARA}^\ast(t)$ and $\pi_{2HARA}^\ast(t)$ increase as the value of the initial wealth $x$ and $y$ increase at time 0, respectively. This displays that both the insurer and the reinsurer want to invest more money in the risky assets.

Fig.8 indicates that the risk aversion coefficient $v$ exerts negative effect on the optimal investment strategies $\pi_{1HARA}^\ast(t)$ and $\pi_{2HARA}^\ast(t)$, which means that both the insurer and the reinsurer will invest less money in the risky assets to avoid risks.
The optimal reinsurance strategy

In particular, the analysis of the optimal investment strategy under power utility and logarithm utility is similar to that of the case under exponential utility, so we only take the optimal investment strategy \( \pi_{1\text{exp}}^* (t) \) (or \( \pi_{2\text{exp}}^* (t) \)) under exponential utility for example and analyze the effects of some parameters on \( \pi_{1\text{exp}}^* (t) \) (or \( \pi_{2\text{exp}}^* (t) \)). From Corollary 2 and Corollary 4, we find that the wealth \( x \) (or \( y \)) has no influence on the optimal investment strategy \( \pi_{1\text{exp}}^* (t) \) (or \( \pi_{2\text{exp}}^* (t) \)) under exponential utility. The reason is that this utility has constant absolute risk aversion parameter \( v \). Such a utility function is the only function under which the principle of “zero utility” gives a fair premium that is independent of the wealth \( x \) of an insurance company (see [7]). Therefore, the optimal amount invested in the risky asset is independent of the wealth \( x \) (or \( y \)).

Since the optimal investment strategy of the insurer is the same as that of the reinsurer, i.e., \( \pi_{1\text{exp}}^* (t) = \pi_{2\text{exp}}^* (t) \), we take \( \pi_{1\text{exp}}^* (t) \) for example, and provide the effects of parameters \( \alpha, \sigma, \beta \) on the optimal investment strategy. As shown in Fig.9, the optimal investment strategy \( \pi_{1\text{exp}}^* (t) \) decreases with respect to \( \sigma \). That is, if \( \sigma \) increases, the volatility of risky asset will fluctuate a little drastically and then the insurer would like to invest less in the risky asset. From Fig.10, we can find that the optimal investment strategy \( \pi_{1\text{exp}}^* (t) \) increases with respect to \( \beta \). A larger
The optimal investment strategy $\pi_{1}^{*} \exp$

Figure 9. Effect of $\beta$ on $\pi_{1}^{*} \exp$

Figure 10. Effect of $\alpha$ on $\pi_{1}^{*} \exp$

Figure 11. Effect of $\sigma$ on $\pi_{1}^{*} \exp$

$\beta$ means a more stable volatility of risky asset, and then the insurer will increase the investment in risky asset. Fig.11 displays that the optimal investment strategy increases with respect to $\alpha$. In other words, the appreciation rate of risky asset increases with respect to $\alpha$, therefore a larger $\alpha$ leads to more investment in the risky asset.

6. Conclusions. This paper takes the interests of both the insure and the reinsurer and investigates the optimal reinsurance and investment problem with HARA utility under Heston’s SV model. Considering the difficulties in conjecturing the form of the value function under HARA utility, we apply Legendre transform method and stochastic control theory to obtain the optimal reinsurance-investment strategy. We also provide some special cases of our model under the criteria of exponential utility, power utility and logarithmic utility, respectively. More importantly, based on Legendre transform, the optimal investment-reinsurance strategy with power utility is derived, which is seldom studied in the optimal investment-reinsurance problem. Finally, we present some numerical simulations to illustrate our results. These results display that the Legendre transform-dual theory is an effective methodology in dealing with our problems with HARA utility.

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