Large–$N_c$ QCD and Weak Matrix Elements

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Abstract

I report on recent progress \[1, 2\] in calculating electroweak processes within the framework of QCD in the $1/N_c$ expansion.

1. Introduction

In the Standard Model, the physics of non–leptonic $K$–decays is described by an effective Lagrangian which is the sum of four–quark operators modulated by $c$–number coefficients (Wilson coefficients). This effective Lagrangian results from integrating out the fields in the Standard Model with heavy masses ($Z^0, W^\pm, t, b$ and $c$), in the presence of the strong interactions evaluated in perturbative QCD (pQCD) down to a scale $\mu$ below the mass of the charm quark $M_c$. The scale $\mu$ has to be large enough for the pQCD evaluation of the $c$–number coefficients to be valid and, therefore, it is much larger than the scale at which an effective Lagrangian description in terms of the Nambu–Goldstone degrees of freedom ($K, \pi$ and $\eta$) of the spontaneous $SU(3)_L \times SU(3)_R$ symmetry breaking (S$\chi$SB) is appropriate. Furthermore, the evaluation of the coupling constants of the low–energy effective chiral Lagrangian cannot be made within pQCD because at scales $\mu \lesssim 1$ GeV we enter a regime where S$\chi$SB and confinement take place and the dynamics of QCD is then fully governed by non–perturbative phenomena.

The structure of the low–energy effective Lagrangian, in the absence of virtual electroweak interactions, is well–known \[3\]

$$
\mathcal{L}_{\text{eff}} = \frac{1}{4} f_\pi^2 \text{tr}D_\mu U D^\mu U + \cdots + L_{10} \text{tr}U^\dagger F_R^{\mu\nu} UF_L_{\mu\nu} + \cdots. \quad (1)
$$

Here the unitary matrix $U$ collects the meson fields ($K, \pi$ and $\eta$) and $F_L, (F_R)$ denote field–strength tensors associated to external gauge field sources. The dots indicate other terms with the same chiral power counting $O(p^4)$ as the $L_{10}$ term and higher order terms. The important point that

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I wish to emphasize here is that the coupling constants of this effective Lagrangian, correspond to coefficients of the Taylor expansion in powers of momenta (and quark masses), of specific QCD Green’s functions of colour singlet quark–currents. Let us consider as an example, and in the chiral limit where the light quark masses are set to zero, the two–point function

\( Q^2 = -q^2; \quad L^\mu = \bar{q}_\gamma^\mu \frac{1}{2}(1 - \gamma_5)q; \quad R^\mu = \bar{q}_\gamma^\mu \frac{1}{2}(1 + \gamma_5)q \)

\[
\Pi_{LR}^{\mu \nu}(q) = 2i \int d^4x e^{iq \cdot x} \langle 0 | T\left[ L^\mu(x) R^\nu(0) \right]|0 \rangle = (q^\mu q^\nu - g^{\mu \nu} q^2) \Pi_{LR}(Q^2). \tag{2}
\]

For \( Q^2 \) small, \(-Q^2 \Pi_{LR}(Q^2) = f_\pi^2 + 4L_{10} Q^2 + \mathcal{O}(Q^4)\), clearly showing the correspondence stated above.

In the presence of virtual electroweak interactions there appear new couplings in the low–energy effective Lagrangian, like e.g. the term

\[
e^2 C \text{tr} \left( Q_R U Q_L U^\dagger \right) = -2e^2 C \frac{1}{f_\pi^2} (\pi^+ \pi^- + K^+ K^-) + \cdots , \tag{3}
\]

where \( Q_R = Q_L = \text{diag}[2/3, -1/3, -1/3] \), showing that, in the presence of the electroweak interactions, the charged pion and kaon fields become massive. The basic complication in evaluating coupling constants like \( C \) in Eq. (3), which originate in loops with electroweak gauge fields, is that they correspond to integrals over all values of the euclidean momenta of specific combinations of QCD Green’s functions of colour singlet quark–currents. In our particular example, it can be shown [4] that

\[
C = \frac{-1}{8\pi^2} \frac{3}{4} \int_0^\infty dQ^2 Q^2 \left( 1 - \frac{1}{Q^2 + M_Z^2} \right) \Pi_{LR}(Q^2), \tag{4}
\]

with \( Q \) the euclidean momentum of the virtual gauge field; the first term in the parenthesis is the well known \([4]\) contribution from electromagnetism; the second term is the one induced by the weak neutral current \([4]\). It is clear that the evaluation of coupling constants of this type represents a rather formidable task. As we shall see below, it is possible, however, to proceed further within the framework of the \( 1/N_c \)–expansion in QCD \([5]\).

2. Large–\( N_c \) QCD and the OPE

In the limit where the number of colours \( N_c \) becomes infinite, with \( \alpha_s \times N_c \) fixed, the QCD spectrum reduces to an infinite number of zero–width mesonic resonances, and the leading large–\( N_c \) contribution to an \( n \)–point correlator is given by all the possible tree–level exchanges of these resonances in the various channels. In this limit, the analytical structure of an \( n \)–point function is very simple: the singularities in each channel consist only of a succession of simple poles. For example, in the case of \( \Pi_{LR} \) in Eq. (2),

\[
- Q^2 \Pi_{LR}(Q^2) = f_\pi^2 + \sum_A f_A^2 M_A^2 \frac{Q^2}{M_A^2 + Q^2} - \sum_V f_V^2 M_V^2 \frac{Q^2}{M_V^2 + Q^2}, \tag{5}
\]
where the sums extend over all vector \((V)\) and axial–vector \((A)\) states. Furthermore, in the chiral limit, the operator product expansion (OPE) applied to the correlation function \(\Pi_{LR}(Q^2)\) implies

\[
\lim_{Q^2 \to \infty} Q^2 \Pi_{LR}(Q^2) \to 0, \quad \lim_{Q^2 \to \infty} Q^4 \Pi_{LR}(Q^2) \to 0,
\]

and

\[
\lim_{Q^2 \to \infty} Q^6 \Pi_{LR}(Q^2) = -4\pi^2 \left( \frac{\alpha_s}{\pi} + \mathcal{O}(\alpha_s^2) \right) \langle \bar{\psi}\psi \rangle^2.
\]

The first two relations result in the two Weinberg sum rules

\[
\sum_V f_V^2 M_V^2 - \sum_A f_A^2 M_A^2 = f_\pi^2 \quad \text{and} \quad \sum_V f_V^2 M_V^4 - \sum_A f_A^2 M_A^4 = 0.
\]

There is in fact a new set of constraints that emerge in the large–\(N_c\) limit which relate order parameters of the OPE to couplings and masses of the narrow states. In our example, we have from Eqs. (5) and (7), that

\[
\sum_V f_V^2 M_V^6 - \sum_A f_A^2 M_A^6 = -4\pi^2 \left( \frac{\alpha_s}{\pi} + \mathcal{O}(\alpha_s^2) \right) \langle \bar{\psi}\psi \rangle^2.
\]

On the other hand, the coupling constants of the low–energy Lagrangian in the strong interaction sector are also related to couplings and masses of the narrow states of the large–\(N_c\) QCD spectrum; e.g.,

\[
-4L_{10} = \sum_V f_V^2 - \sum_A f_A^2.
\]

It is to be remarked that the convergence of the integral in Eq. (4) in the large–\(N_c\) limit is guaranteed by the two Weinberg sum rules in Eqs. (5). However, in order to obtain a numerical estimate, and in the absence of an explicit solution of QCD in the large–\(N_c\) limit, one still needs to make further approximations. Partly inspired by the phenomenological successes of “vector meson dominance” in predicting e.g., the low–energy constants of the effective chiral Lagrangian [7], we have recently proposed [8] to consider the approximation to large–\(N_c\) QCD, which restricts the hadronic spectrum to a minimal pattern, compatible with the short–distance properties of the QCD Green’s functions which govern the observable(s) one is interested in. In the channels with \(J^P\) quantum numbers \(1^-\) and \(1^+\) this minimal pattern, in the cases which we have discussed so far, is the one with a spectrum which consists of a hadronic lowest energy narrow state and treats the rest of the narrow states as a large–\(N_c\) pQCD continuum; the onset of the continuum being fixed by consistency constraints from the OPE, like the absence of dimension \(d = 2\) operators. We call this the lowest meson dominance (LMD) approximation to large–\(N_c\) QCD. The basic observation
here is that order parameters of \( S \chi \) in QCD have a smooth behaviour at short distances. For example, in the case of the function \( \Pi_{LR} \), and, therefore, the coupling \( C \), this is reflected by the fact that (in the chiral limit) the pQCD continuum contributions in the \( V \)-sum and the \( A \)-sum in Eq. (5) cancel each other. The evaluation of the constant \( C \) in Eq. (4) in this approximation, corresponds to a mass difference \( \Delta m_\pi = 4.9 \) MeV, remarkably close to the experimental result: \( \Delta m_\pi|_{\text{exp.}} = 4.59 \) MeV.

3. Electroweak Penguin Operators.
Within the framework discussed above, we have also shown [1] that the \( K \to \pi \pi \) matrix elements of the four–quark operator
\[
Q_7 = 6(s_L \gamma^\mu d_L) \sum_{q=u,d,s} e_q(q_R \gamma_\mu q_R),
\]
generated by the electroweak penguin–like diagrams of the Standard Model, can be calculated to first non–trivial order in the chiral expansion and in the \( 1/N_c \) expansion. What is needed here is the bosonization of the operator \( Q_7 \) to next–to–leading order in the \( 1/N_c \) expansion. The problem turns out to be entirely analogous to the bosonization of the operator \( Q_{LR} \equiv (q_L \gamma^\mu Q_L q_L)(q_R \gamma^\mu Q_R q_R) \) which governs the electroweak \( \pi^+ - \pi^0 \) mass difference discussed above. Because of the \( LR \) structure, the factorized component of \( Q_7 \), which is leading in \( 1/N_c \), cannot contribute to order \( \mathcal{O}(p^0) \) in the low–energy effective Lagrangian. The first \( \mathcal{O}(p^0) \) contribution from this operator is next–to–leading in the \( 1/N_c \) expansion and is given by an integral,
\[
\left( \lambda^{(23)}_L \right)_{ij} = \delta_{i2}\delta_{3j} (i, j = 1, 2, 3),
\]

involving the same two–point function as in Eq. (3). Although the resulting \( B \) factors of \( \Delta I = 1/2 \) and \( \Delta I = 3/2 \) transitions are found to depend only logarithmically on the matching scale \( \mu \), their actual numerical values turn out to be rather sensitive to the precise choice of \( \mu \) in the GeV region. Furthermore, because of the normalization to the vacuum saturation approximation (VSA) inherent to the (rather disgraceful) conventional definition of \( B \)–factors, there appears a spurious dependence on the light quark masses as well. In Fig. 1 we show our prediction for the ratio
\[
\hat{B}_7^{(3/2)} = \frac{\langle \pi^+ | Q_7^{(3/2)} | K^+ \rangle}{\langle \pi^+ | Q_7^{(3/2)} | K^+ \rangle_{\text{VSA}}},
\]
versus the matching scale \( \mu \) defined in the \( \overline{MS} \) scheme. This is the ratio considered in recent lattice QCD calculations [3]. In fact, the lattice...
definition of $\tilde{B}_7^{(3/2)}$ uses a current algebra relation between the $K \to \pi\pi$ and the $K \to \pi$ matrix elements which is only valid at order $\mathcal{O}(p^0)$ in the chiral expansion.] In Eq. (13), the matrix element in the denominator is evaluated in the chiral limit, as indicated by the subscript “0”.

![Graph](image)

Fig. 1: The $\tilde{B}_7^{(3/2)}$ factor in Eq. (13) versus $\mu$ in GeV. Solid lines correspond to $(m_s + m_d)(2\text{GeV}) = 158 \text{ MeV}$; dashed lines to $(m_s + m_d)(2\text{GeV}) = 100 \text{ MeV}$.

4. Decay of Pseudoscalars into Lepton Pairs.

The processes $\pi \to e^+e^-$ and $\eta \to l^+l^-$ ($l = e, \mu$) are dominated by the exchange of two virtual photons. It is then useful to consider the ratios $(P = \pi^0, \eta)$

$$R(P \to \ell^+\ell^-) = \frac{Br(P \to \ell^+\ell^-)}{Br(P \to \gamma\gamma)} = 2 \left(\frac{\alpha m_\ell}{\pi M_P}\right)^2 \beta_\ell(M_P^2) |A(M_P^2)|^2,$$

(14)

with $\beta_\ell(s) = \sqrt{1 - 4m_\ell^2/s}$. To lowest order in the chiral expansion, the unknown dynamics in the amplitude $A(M_P^2)$ depends entirely on a low–energy coupling constant $\chi$. We have recently shown [4] that this constant can be expressed as an integral over the three–point function

$$\int d^4x \int d^4y e^{iq_1\cdot x} e^{iq_2\cdot y} < 0 | T \{ j^m_\mu(x) j^m_\nu(y) P^3(0) \} | 0 >$$

$$= \frac{2}{3} \epsilon_{\mu\nu\alpha\beta} q_1^\alpha q_2^\beta \mathcal{H}(q_1^2, q_2^2, (q_1 + q_2)^2),$$

(15)

involving the electromagnetic current $j^m_\mu$ and the density current $P^3 = \frac{1}{2}(\bar{u}i\gamma_5 u - \bar{d}i\gamma_5 d)$. More precisely, ($d = \text{space–time dimension}$,)

$$\frac{\chi(\mu)}{32\pi^4} \frac{< \bar{\psi}\psi >}{F_\pi^2} = -\left(1 - \frac{1}{d}\right) \int \frac{d^dq}{(2\pi)^d} \left(\frac{1}{q^2}\right)^2 \times$$

5
\[
\lim_{(p' - p) \to 0} (p' - p)^2 \left[ \mathcal{H}(q^2, q^2, (p' - p)^2) - \mathcal{H}(0, 0, (p' - p)^2) \right].
\]

The evaluation of this coupling in the LMD approximation to large–\(N_c\) QCD which we have discussed above, leads to the result \(\chi_{\text{LMD}}(\mu = M_V) = 2.2 \pm 0.9\). The corresponding branching ratios are shown in Table 1.

| \(R(P \to \ell^+\ell^-)\) | LMD | Experiment |
|--------------------------|-----|------------|
| \(\mathcal{R}(\pi^0 \to e^+e^-) \times 10^8\) | 6.2 \pm 0.3 | 7.13 \pm 0.55 |
| \(\mathcal{R}(\eta \to \mu^+\mu^-) \times 10^8\) | 1.4 \pm 0.2 | 1.48 \pm 0.22 |
| \(\mathcal{R}(\eta \to e^+e^-) \times 10^8\) | 1.15 \pm 0.05 | ? |

It was shown in ref. [12] that, when evaluated within the chiral \(U(3)\) framework and in the \(1/N_c\) expansion, the \(|\Delta S| = 1\) \(K_L^0 \to \ell^+\ell^-\) transitions can also be described by an expression like in Eq. (14) with an effective constant \(\chi_{K_L}^0\) containing an additional piece from the short–distance contributions [13]. The most accurate experimental determination [14] gives: \(\text{Br}(K_L^0 \to \mu^+\mu^-) = (7.18 \pm 0.17) \times 10^{-9}\). In the framework of the \(1/N_c\) expansion and using the experimental branching ratio [11] \(\text{Br}(K_L^0 \to \gamma\gamma) = (5.92 \pm 0.15) \times 10^{-4}\), this leads to a unique solution for an effective \(\chi_{K_L}^0 = 5.17 \pm 1.13\). Furthermore, following Ref. [12], \(\chi_{K_L}^0 = \chi - N^\prime \delta_{\chi SD}\) where \(N = (3.6/g_8 c_{\text{red}})\) normalizes the \(K_L^0 \to \gamma\gamma\) amplitude. The coupling \(g_8\) governs the \(\Delta I = 1/2\) rule, the constant \(c_{\text{red}}\) is defined in Ref. [12] and \(\delta_{\chi SD} = (+1.8 \pm 0.6)\) is the short–distance contribution in the Standard Model [13]. Therefore, a test of the short–distance contribution to this process completely hinges on our understanding of the long–distance constant \(N^\prime\) and therefore of the \(\Delta I = 1/2\) rule in the \(1/N_c\) expansion. Moreover, \(c_{\text{red}}\) is regrettably very unstable in the chiral and large–\(N_c\) limits, a behaviour that surely points towards the need to have higher order corrections under control. The analysis of Ref. [12] uses \(c_{\text{red}} \approx +1\) and \(g_8 \approx 3.6\), where these numbers are obtained phenomenologically by requiring agreement with the two–photon decay of \(K_L^0, \pi^0, \eta\) and \(\eta^\prime\) as well as \(K \to 2\pi, 3\pi\). Should we use these values of \(c_{\text{red}}\) and \(g_8\) with our result \(\chi_{\text{LMD}}(\mu = M_V) = 2.2 \pm 0.9\) we would obtain \(\chi_{K_L}^0 = 0.4 \pm 1.1\), corresponding to a ratio \(R(K_L^0 \to \mu^+\mu^-) = (2.24 \pm 0.41) \times 10^{-5}\) which is 2.5\(\sigma\) above the experimental value \(R(K_L^0 \to \mu^+\mu^-) = (1.21 \pm 0.04) \times 10^{-5}\).

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