Abstract. In this paper, we compute explicitly the \( q \)-dimensions of highest weight crystals modulo \( q^n - 1 \) for a quantum group of arbitrary finite type under certain assumption, and interpret the modulo computations in terms of the cyclic sieving phenomenon. This interpretation gives an affirmative answer to the conjecture by Alexandersson and Amini. As an application, under the assumption that \( \lambda \) is a partition of length \( < m \) and there exists a fixed point in \( \text{SST}_m(\lambda) \) under the action \( c \) arising from the crystal structure, we show that the triple \((\text{SST}_m(\lambda), (c), s_\lambda(1, q, q^2, \ldots, q^{m-1}))\) exhibits the cycle sieving phenomenon if and only if \( \lambda \) is of the form \(((am)^b)\), where either \( b = 1 \) or \( m - 1 \). Moreover, in this case, we give an explicit formula to compute the number of all orbits of size \( d \) for each divisor \( d \) of \( n \).

Introduction

The cyclic sieving phenomenon was introduced by Reiner-Stanton-White in [21]. Let \( X \) be a finite set on which a cyclic group \( C \) of order \( n \) acts and \( f(q) \) a polynomial in \( q \) with non-negative integer coefficients. We say that \((X, C, f(q))\) exhibits the cyclic sieving phenomenon if, for all \( c \in C \), we have

\[
\#X^c = f(\zeta_{o(c)}),
\]

where \( X^c \) is the fixed point set under the action of \( c \), \( o(c) \) is the order of \( c \), and \( \zeta_{o(c)} = e^{2\pi i/o(c)} \). Many instances of the cyclic sieving phenomenon have been observed for various combinatorial objects including words, multisets, permutations, and tableaux (see [21, 24] for details).

Let \( \text{pr} \) be the promotion operator due to Schützenberger [25, 26], and let \( \text{SST}_m(\lambda) \) be the set of semistandard Young tableaux of shape \( \lambda \) with entries in \( \{1, 2, \ldots, m\} \). The cyclic sieving phenomenon about \( \text{SST}_m(\lambda) \) and \( \text{pr} \) has drawn a lot of attention from many researchers (see

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One of the most important results in this direction is due to Rhoades [22], who proved that if \( \lambda \) is of rectangular shape, the triple
\[
\left( \text{SST}_m(\lambda), \langle \text{pr} \rangle, \, q^{-\kappa(\lambda)} s_\lambda(1, q, \ldots, q^{m-1}) \right)
\]
exhibits the cyclic sieving phenomenon, where \( \kappa(\lambda) = \sum_{i \geq 1} (i - 1) \lambda_i \) and \( s_\lambda(1, q, \ldots, q^{m-1}) \) is the principal specialization of the Schur polynomial \( s_\lambda(1, x_2, \ldots, x_m) \). However, this result is no longer valid outside rectangular shape in general. If one wants to keep \( \text{SST}_m(\lambda) \) and the principal specialization of the Schur polynomial in the triple, another appropriate action other than \( \text{pr} \) should be considered. In a previous article of the authors, [20], a new cyclic sieving phenomenon triple
\[
\left( \text{SST}_m(\lambda), \langle \text{c} \rangle, \, q^{-\kappa(\lambda)} s_\lambda(1, q, \ldots, q^{m-1}) \right) \tag{0.1}
\]
was provided under the condition \( \gcd(|\lambda|, m) = 1 \), where the action \( \text{c} \) arises naturally from the \( U_q(\mathfrak{sl}_m) \)-crystal structure of \( \text{SST}_m(\lambda) \). Crystal bases theory is one of the most powerful combinatorial tools for studying representations of quantum groups in the viewpoint of graph theory with natural connections to tableaux and functions invariant under the action of the Weyl group like symmetric functions ([7, 10, 14, 15, 16]). The promotion operator \( \text{pr} \) on \( \text{SST}_m(\lambda) \) with entries \( \leq m \) can be defined by \( \text{pr} := \sigma_1 \sigma_2 \cdots \sigma_{m-1} \), where the \( \sigma_i \) are the Bender-Knuth involutions, certain natural involutions on tableaux that exchange the number of \( i \)'s and \( i+1 \)'s. The operator \( \text{c} \) on \( \text{SST}_m(\lambda) \) can similarly be defined by \( \text{c} := s_1 s_2 \cdots s_{m-1} \), where the \( s_i \) are the generators of the symmetric group \( \mathfrak{S}_m \) action on the crystal \( \text{SST}_m(\lambda) \). Since \( s_1 s_2 \cdots s_{m-1} \) is a Coxeter element of \( \mathfrak{S}_m \), it has order \( m \). Thus the cyclic group \( C \) of order \( m \) acts on \( \text{SST}_m(\lambda) \) via the operator \( \text{c} \); for rectangular shape partitions, the action of \( \text{pr} \) has order \( m \), but for other shapes it does not because the Bender-Knuth involutions do not give an action of the symmetric group. We remark that, before crystal theory was developed, the same symmetric group action on \( \text{SST}_m(\lambda) \) was studied at a purely combinatorial level by Lascoux and Schützenberger in [17].

Without the condition \( \gcd(|\lambda|, m) = 1 \), the new triple (0.1) does not exhibit the cyclic sieving phenomenon in general. Thus, it is an interesting problem to find a necessary and sufficient condition for the cyclic sieving phenomenon of the new triple (0.1). To answer this problem, we follow the method of Alexandersson and Amini [1]. To be precise, we ask what conditions guarantee the existence of an action of a cyclic group \( C \) of order \( n \) on \( \text{SST}_m(\lambda) \), without being able to describe it explicitly, such that the triple \( \left( \text{SST}_m(\lambda), C, q^{-\kappa(\lambda)} s_\lambda(1, q, \ldots, q^{m-1}) \right) \) exhibits the cyclic sieving phenomenon.

In this paper, we compute explicitly the \( q \)-dimensions of highest weight crystals modulo \( q^n - 1 \) for a quantum group of arbitrary finite type under certain assumptions, and interpret
the modulo computations in terms of the cyclic sieving phenomenon. Let \( g \) be a finite-dimensional simple Lie algebra over \( \mathbb{C} \) and \( U_q(g) \) be its quantum group. We write \( \Delta^+ \) for the set of positive roots of \( g \). For a dominant integral weight \( \Lambda \), let \( B(\Lambda) \) be the highest weight \( U_q(g) \)-crystal with highest weight \( \Lambda \). We denote by \( \dim_q B(\Lambda) \) the \( q \)-dimension of \( B(\Lambda) \), which is the polynomial in \( q \) obtained from the character \( \text{ch}B(\Lambda) \) by specializing at \( q^{(\rho,-)} \) (see Section 1 for the definition). When \( U_q(g) \) is of type \( A_{m-1} \), i.e., \( g = \mathfrak{sl}_m \), the crystal \( B(\Lambda) \) can be realized as \( \text{SST}_m(\lambda) \) and the \( q \)-dimension \( \dim_q B(\Lambda) \) is equal to the principal specialization \( q^{-\kappa(\lambda)}s_\lambda(1, q, \ldots, q^{m-1}) \) of the Schur polynomial \( s_\lambda(x_1, x_2, \ldots, x_m) \). Here, \( \Lambda \) and \( \lambda \) are related in (3.2).

Let \( n \) be a positive integer. Under the assumption that

\[
(\beta, \Lambda) \text{ is divisible by } n \text{ for any } \beta \in \Delta^+,
\]

we provide an explicit expression for \( \dim_q B(\Lambda) \mod q^n - 1 \) using the Weyl character formula as follows:

\[
\dim_q B(\Lambda) \equiv \sum_{d|n} a_d \frac{q^n - 1}{q^d - 1} \pmod{q^n - 1},
\]

where \( a_d \in \mathbb{Z}_{\geq 0} \) are nonnegative integers given explicitly in (2.2) (see Theorem 2.5). Note that when \( U_q(g) \) is of type \( A_{m-1} \) and \( n = m \), condition (0.2) implies \( |\lambda| \) being divisible by \( m \), i.e., \( \gcd(|\lambda|, m) = m \), which case is not covered by the previous result of the authors, [20, Theorem 4.3]. We can also derive a similar result for the \( q \)-dimension \( \dim_q^\vee B(\Lambda) \) of \( B(\Lambda) \), which is obtained by specializing at \( q^{(\rho^\vee,-)} \) (see Remark 2.6). It should be remarked that there are root of unity evaluations of \( \dim_q^\vee B(\Lambda) \) that have been studied in the literature. For instance, letting \( \varphi_\Lambda(q) := q^{-\kappa(\rho, \Lambda)} \dim_q^\vee B(\Lambda) \), it is known by Kac ([12, Exercise 10.15] or [13]) that if \( \omega \) is a root of unity of order equal to the Coxeter number of the Weyl group, then \( \varphi_\Lambda(\omega) = 0 \) or \( \pm 1 \).

From the viewpoint of the cyclic sieving phenomenon, the above computation modulo \( q^n - 1 \) says that there exists an action of a cyclic group \( C \) of order \( n \) on \( B(\Lambda) \), without being able to describe it explicitly, such that the triple \( (B(\Lambda), C, \dim_q B(\Lambda)) \) exhibits the cyclic sieving phenomenon and the number of all orbits of size \( d \) is equal to \( a_d \) for any positive integer \( d \) with \( d|n \) (see Theorem 3.2). In the case where \( g = \mathfrak{gl}_m \), the situation is more interesting. Let \( \lambda = (\lambda_1, \ldots, \lambda_\ell) \) be a partition and \( \Lambda \) the dominant integral weight given in (3.2). In this case, the condition (0.2) is equivalent to

\[
\lambda_i - \lambda_j \text{ is divisible by } n \text{ for all } 1 \leq i < j \leq m,
\]

which means that \( \lambda \) is a \textit{stretched} Young diagram by \( n \), i.e., \( \lambda = n\tilde{\lambda} \) for some Young diagram \( \tilde{\lambda} \). Hence Theorem 3.2 implies that, for a stretched Young diagram \( \lambda \) by \( n \), there exists an action of a cyclic group \( C \) of order \( n \) on \( \text{SST}_m(\lambda) \) such that the triple

\[
(\text{SST}_m(\lambda), C, q^{-\kappa(\lambda)}s_\lambda(1, q, q^2, \ldots, q^{m-1}))
\]
exhibits the cyclic sieving phenomenon. Consequently, we give an affirmative answer to the conjecture [1, Conjecture 3.4] by Alexandersson and Amini (see Corollary 3.4). In this viewpoint, Theorem 3.2 can be understood as an affirmative answer to a crystal-theoretical generalization of this conjecture.

We next focus on the case where $g$ is of type $A_{m-1}$ and the action $c$ arising from the crystal structure. In the previous article of the authors, [20], the case where $\gcd(m, |\lambda|) = 1$ was studied extensively, where every orbit is free. We now consider the case at least one fixed point exists. Under the assumption that $\lambda$ is a partition of length $< m$ and there exists a fixed point in $\text{SST}_m(\lambda)$ under the action of $c$, we show that the triple $(\text{SST}_m(\lambda), \langle c \rangle, s_\lambda(1, q, q^2, \ldots, q^{m-1}))$ exhibits the cycle sieving phenomenon if and only if $\lambda$ is of the form $((am)^b)$, where either $b = 1$ or $m - 1$ (Theorem 4.4). Moreover, in this case, we give an explicit formula to compute the number of all orbits of size $d$ (Proposition 4.6). When $m$ is a prime $p$, combining this result with [20, Theorem 4.3] enables us to characterize when $(\text{SST}_p(\lambda), \langle c \rangle, s_\lambda(1, q, q^2, \ldots, q^{p-1}))$ exhibits the cyclic sieving phenomenon under the assumption that there exists at least one fixed point.

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1. Highest weight crystals and their $q$-dimensions

Let $I$ be a finite index set and let $A = (a_{ij})_{i,j \in I}$ be a Cartan matrix of finite type. We choose a diagonal matrix $D = \text{diag}(d_i \in \mathbb{Z}_{\geq 0} | i \in I)$ such that $\min\{d_i | i \in I\} = 1$ and $DA$ is symmetric. We then consider a quintuple $(A, P, \Pi, P^\vee, \Pi^\vee)$, called a Cartan datum associated with $A$, such that

1. $P$ is a free abelian group of rank $|I|$, called the weight lattice,
2. $\Pi = \{\alpha_i | i \in I\} \subset P$, called the set of simple roots,
(3) $P^\vee = \text{Hom}_\mathbb{Z}(P, \mathbb{Z})$, called the coweight lattice,

(4) $\Pi^\vee = \{ h_i \in P^\vee \mid i \in I \}$, called the set of simple coroots,

which satisfy the following requirements:

- $\langle h_i, \alpha_j \rangle = a_{ij}$ for $i, j \in I$,
- $\Pi$ is linearly independent over $\mathbb{Q}$, and
- for each $i \in I$, there exists $\varpi_i \in P$, called the fundamental weight, such that $\langle h_j, \varpi_i \rangle = \delta_{j,i}$ for all $j \in I$.

We denote by $P^+ := \{ \lambda \in P \mid \langle h_i, \lambda \rangle \geq 0 \text{ for all } i \in I \}$ the set of dominant integral weights.

There exists a nondegenerate symmetric bilinear form $(\cdot, \cdot)$ on $P$ satisfying

$$(\alpha_i, \alpha_j) = d_{ij} a_{ij} \quad (i, j \in I), \quad \text{and} \quad \langle h_i, \lambda \rangle = \frac{2(\alpha_i, \lambda)}{(\alpha_i, \alpha_i)} \quad (\lambda \in P, \ i \in I).$$

Let $Q := \bigoplus_{i \in I} \mathbb{Z} \alpha_i$ be the root lattice, and let $\Delta \subset Q$ be the set of roots associated with $A$.

We write $\Delta^+$ for the set of positive roots.

Fix an indeterminate $q$. Let $U_q(g)$ be the quantum group associated with $(A, P, P^\vee \Pi, \Pi^\vee)$, which is the associative algebra over $\mathbb{C}(q)$ with 1 generated by $f_i, e_i$ $(i \in I)$ and $q^h$ $(h \in P)$ with certain defining relations (see [10, §3] for details). For a dominant integral weight $\Lambda \in P^+$, we denote by $V_q(\Lambda)$ the irreducible highest weight $U_q(g)$-module with highest weight $\Lambda$, and denote by $B(\Lambda)$ its crystal. We denote by $\tilde{e}_i$ and $\tilde{f}_i$ $(i \in I)$ the crystal operators on $B(\Lambda)$.

We refer the reader to [7, 10, 14, 15, 16] for crystals.

We set $W$ to be the Weyl group associated with $A$, which is a subgroup of $\text{Aut}(P)$ generated by $s_i(\lambda) := \lambda - \langle h_i, \lambda \rangle \alpha_i$ for $i \in I$ and $\lambda \in P$. Note that $\Delta$ is invariant under the actions of $W$. The Weyl group $W$ also acts on the crystal $B(\Lambda)$ as follows: for $i \in I$ and $b \in B(\Lambda)$, define

$$s_i(b) := \begin{cases} \tilde{e}_i^{\langle h_i, \text{wt}(b) \rangle} b & \text{if } \langle h_i, \text{wt}(b) \rangle \geq 0, \\ \tilde{e}_i^{-\langle h_i, \text{wt}(b) \rangle} b & \text{if } \langle h_i, \text{wt}(b) \rangle < 0. \end{cases}$$

We set $B(\Lambda)_\xi := \{ b \in B(\Lambda) \mid \text{wt}(b) = \xi \}$ so that $B(\Lambda) = \bigsqcup_{\xi \in P} B(\Lambda)_\xi$, and $\text{wt}(B(\Lambda)) = \{ \mu \in P \mid B(\Lambda)_\mu \neq \emptyset \}$. The character $\text{ch}B(\Lambda)$ of $B(\Lambda)$ is defined by

$$\text{ch}B(\Lambda) := \sum_{\xi \in \text{wt}(B(\Lambda))} |B(\Lambda)_\xi| e^\xi,$$

where $|B(\Lambda)_\xi|$ is the number of elements of $B(\Lambda)_\xi$, and $e^\xi$ are formal basis elements of the group algebra $\mathbb{Q}[P]$ with the multiplication given by $e^\xi e^{\xi'} = e^{\xi+\xi'}$. The Weyl character formula says that

$$\text{ch}B(\Lambda) = \frac{\sum_{w \in W} (-1)^{\ell(w)} e^{\langle w(\Lambda+\rho) - \rho \rangle}}{\prod_{\beta \in \Delta^+} (1 - e^{-\beta})},$$

where $\rho = \frac{1}{2} \sum_{\beta \in \Delta^+} \beta$ (for instance, see [13, Theorem 10.4]). Note that $\rho = \sum_{i \in I} \varpi_i$. 
In this paper, we consider the following polynomials in $q$ arising from the crystal $B(\Lambda)$:

$$\dim_q B(\Lambda) := \sum_{\xi \in \text{wt}(B(\Lambda))} |B(\Lambda)_{\xi}|q^{(\rho,\Lambda-\xi)},$$

$$\dim_q^\vee B(\Lambda) := \sum_{\xi \in \text{wt}(B(\Lambda))} |B(\Lambda)_{\xi}|q^{(\rho^\vee,\Lambda-\xi)},$$

where $\vee : \mathfrak{h}^* := \mathbb{C} \otimes \mathfrak{p} \to \mathfrak{h} := \mathbb{C} \otimes \mathfrak{p}^\vee$ is the isomorphism given in [13, §2.1]. These polynomials can be obtained via specializations of $\text{ch} B(\Lambda)$. Define homomorphisms

$$F_\rho : \mathbb{C}[Q] \to \mathbb{C}[q^{\pm1}], \quad e^\lambda \mapsto q^{-(\rho,\lambda)},$$

$$F_{\rho^\vee} : \mathbb{C}[Q] \to \mathbb{C}[q^{\pm1}], \quad e^\lambda \mapsto q^{-(\rho^\vee,\lambda)}.$$ 

One easily sees that $\dim_q B(\Lambda) = F_\rho(e^{-\Lambda}\text{ch} B(\Lambda))$ and $\dim_q^\vee B(\Lambda) = F_{\rho^\vee}(e^{-\Lambda}\text{ch} B(\Lambda))$. Using the Weyl character formula and properties of $\rho$, one can show that

$$\dim_{\nu} B(\Lambda) = \prod_{\beta \in \Delta^+} \frac{1 - q^{(\beta,\Lambda+\rho)}}{1 - q^{(\beta,\rho)}},$$

(1.2)

$$\dim_{\nu}^\vee B(\Lambda) = \prod_{\beta \in \Delta^+} \frac{1 - q^{(\beta^\vee,\Lambda+\rho)}}{1 - q^{(\beta^\vee,\rho)}},$$

(1.3)

(see [13, §10.10] and [28, 29]). In the literature such as [13, §10.10], the right hand side of (1.3) is called the $q$-dimension of $V(\Lambda)$. Note that $\dim_q B(\Lambda) = \dim_{\nu}^\vee B(\Lambda)$ if the Cartan matrix $A$ is symmetric.

2. Congruences of the $q$-dimension of $B(\Lambda)$

In this section, we provide some noteworthy congruences of the $q$-dimensions of $B(\Lambda)$ which are significant not only in itself but also a cyclic sieving phenomenon on $B(\Lambda)$. We fix a Cartan matrix $A$ of finite type and its Cartan datum $(A, P, \Pi, P^\vee, \Pi^\vee)$. Let $\Lambda \in P^+$ and $n \in \mathbb{Z}_{>0}$. From now on, we impose the following condition:

(2.1) **Condition**: for any $\beta \in \Delta^+$, $(\beta, \Lambda)$ is divisible by $n$.

For any $d|n$, we set

$$\Delta_d^+ := \{ \beta \in \Delta^+ | (\beta, \rho) \text{ is divisible by } d \}.$$ 

Denote by $\Phi_d(q)$ the $d$th cyclotomic polynomial. Note that $\Phi_d(q)$ is irreducible over $\mathbb{Z}$ and

$$q^n - 1 = \prod_{d|n} \Phi_d(q).$$
Lemma 2.1. Assume that the condition (2.1) holds. For $d \mid n$, we have

$$
\dim_q B(\Lambda) \equiv \prod_{\beta \in \Delta_n^+} \left( \frac{(\beta, \Lambda)}{(\beta, \rho)} + 1 \right) \pmod{\Phi_d(q)}.
$$

Here, the right hand side is set to be 1 in the case where $\Delta_n^+ = \emptyset$.

Proof. Note that

(i) $q^{(\beta, \Lambda)} \equiv 1 \pmod{\Phi_d(q)}$ for $\beta \in \Delta^+$, and

(ii) $1 - q^{(\beta, \rho)} \not\equiv 0 \pmod{\Phi_d(q)}$ for $\beta \in \Delta^+ \setminus \Delta_n^+$.

It follows from (1.2) that

$$
\dim_q B(\Lambda) = \left( \prod_{\beta \in \Delta_n^+} \frac{1 - q^{(\beta, \Lambda) + \rho}}{1 - q^{(\beta, \rho)}} \right) \left( \prod_{\beta \in \Delta^+ \setminus \Delta_n^+} \frac{1 - q^{(\beta, \Lambda) + \rho}}{1 - q^{(\beta, \rho)}} \right)
$$

$$
\equiv \prod_{\beta \in \Delta_n^+} \frac{1 - q^{(\beta, \Lambda) + \rho}}{1 - q^{(\beta, \rho)}} \equiv \prod_{\beta \in \Delta_n^+} \frac{(1 - q^d)(1 + q^d + q^{2d} + \cdots + q^{d(\frac{(\beta, \Lambda) + \rho)}{d} - 1})}{(1 - q^d)(1 + q^d + q^{2d} + \cdots + q^{d(\frac{(\beta, \rho)}{d} - 1})}
$$

$$
\equiv \prod_{\beta \in \Delta_n^+} \frac{(\beta, \Lambda + \rho)}{(\beta, \rho)} \pmod{\Phi_d(q)},
$$

which completes the proof. \qed

For $d \mid n$, we set

$$
b_d := \prod_{\beta \in \Delta_n^+} \left( \frac{(\beta, \Lambda)}{(\beta, \rho)} + 1 \right),
$$

(2.2)

$$
a_d := \frac{1}{d} \sum_{e \mid d} \mu \left( \frac{d}{e} \right) b_e,
$$

where $\mu$ is the classical Möbius function. Note that

$$
b_d = \sum_{e \mid d} ea_e.
$$

Example 2.2. From the definition, it is easy to see the following.

1. Let $\beta_0 \in \Delta^+$ be the highest root. If $(\beta_0, \rho) < n$, then $\Delta_n^+ = \emptyset$ and therefore $a_1 = b_1 = 1$.

2. Let $n = p^l$ for a prime $p$. For $0 < s \leq l$, we have

$$
a_{p^s} = \frac{1}{p^s} (b_{p^s} - b_{p^{s-1}}).$$
**Lemma 2.3.** Let $S$ be a finite set and let $\xi : S \to \mathbb{Z}_{>0}$ be a function. For $s \in S$, let $x_s$ be an indeterminate. For $d | n$, let

$$B_d = \prod_{s \in S, \frac{n}{d} | \xi(s)} (1 + x_s) \quad \text{and} \quad A_d = \sum_{k | d} \mu(k) B_{\frac{n}{d}}.$$ 

Then we have

$$A_d = \sum_{T \subset S, \frac{n}{d} | \gcd(n, \xi(T))} x_T,$$

where $x_T = \prod_{t \in T} x_t$ and $\xi(T) = \gcd(\{\xi(t) \mid t \in T\}$ for any $T \subset S$. Here we set $\xi(\emptyset) = 0$.

**Proof.** It follows from

$$B_d = \prod_{s \in S, \frac{n}{d} | \xi(s)} (1 + x_s) = \sum_{T \subset S, \frac{n}{d} | \xi(T)} x_T$$

that

$$A_d = \sum_{k | d} \mu(k) \left( \sum_{T \subset S, \frac{n}{d} | \xi(T)} x_T \right) = \sum_{T \subset S} x_T \left( \sum_{k | d, \frac{n}{d} | \xi(T)} \mu(k) \right)$$

$$= \sum_{T \subset S} x_T \left( \sum_{\frac{n}{d} | \gcd(n, \xi(T))} \mu(k) \right).$$

Thus the assertion follows from

$$\sum_{\frac{n}{d} | \gcd(n, \xi(T))} \mu(k) = \begin{cases} 1 & \text{if } \gcd(n, \xi(T)) = \frac{n}{d}, \\ 0 & \text{otherwise}. \end{cases}$$

\[\square\]

**Remark 2.4.** In the first arXiv version of the paper, our original lemma used to prove Theorem 2.5 is different from Lemma 2.3. We are very grateful to the anonymous referee for informing us of the present lemma. We also would like to thank Martin Rubey for letting us know the same idea independently.

We now shall prove the following congruence for $\dim_q B(\Lambda)$.

**Theorem 2.5.** Assume that $(\beta, \Lambda)$ is divisible by $n$ for any $\beta \in \Delta^+$.

1. For any $d | n$, $a_d \in \mathbb{Z}_{\geq 0}$.

2. $\dim_q B(\Lambda) \equiv \sum_{d | n} a_d \frac{q^n - 1}{q^d - 1} \pmod{q^n - 1}.$
Proof. We set
\[ B(q) := \sum_{e \mid n} a_e \frac{q^n - 1}{q^e - 1}. \]

Let \( d \in \mathbb{Z}_{>0} \) with \( d \mid n \). Since \( \frac{q^n - 1}{q^d - 1} \equiv 0 \pmod{\Phi_d(q)} \) for \( d \nmid \frac{n}{e} \) and \( \frac{q^n - 1}{q^d - 1} \equiv e \pmod{\Phi_d(q)} \) for \( d \mid \frac{n}{e} \), by (2.3), we have
\[ B(q) \equiv \sum_{e \mid \frac{n}{d}} e a_e = b_{\frac{n}{d}} \pmod{\Phi_d(q)}. \]

Combining Lemma 2.1 with the Chinese Remainder Theorem, we conclude that
\[ \dim_q B(\Lambda) \equiv B(q) \pmod{q^n - 1}. \]

To complete the proof, it remains to see that \( a_d \in \mathbb{Z}_{\geq 0} \) for all \( d \mid n \). Since \( \langle \beta', \Lambda \rangle \geq 0 \) for all \( \beta \in \Delta^+ \), the non-negativity of \( a_d \) follows by applying Lemma 2.3 to the setting \( S = \Delta^+ \) and \( \xi(\beta) = (\beta, \rho) \). We now see that \( a_d \in \mathbb{Z} \). Note that \( B(q) \) is the remainder of \( \dim_q B(\Lambda) \) when divided by \( q^n - 1 \). Since
\[ \dim_q B(\Lambda) \in \mathbb{Z}[q] \]
and \( q^n - 1 \) is monic, it follows that \( B(q) \in \mathbb{Z}[q] \). Let \( d_1 > d_2 \cdots > d_s \) be all divisors of \( n \) such that \( a_d \neq 0 \). Then the leading coefficient of \( B(q) \) is \( a_{d_1} \), so it is an integer. Next, consider
\[ B(q) - a_{d_1} (1 + q^{\frac{n}{d_1}} + \cdots + (q^{\frac{n}{d_1}})^{d_1-1}) \in \mathbb{Z}[q]. \]
Its leading coefficient is given by \( a_{d_2} - a_{d_1} \) if \( d_2 \mid d_1 \) and \( a_{d_2} \) otherwise, thus \( a_{d_2} \in \mathbb{Z} \). In this way, we can see inductively that \( a_d \in \mathbb{Z} \) for all \( d \mid n \). \( \square \)

Remark 2.6. We can also derive an analogue of Theorem 2.5 for \( \dim^\vee_q B(\Lambda) \) in the same manner. For any \( d \mid n \), we set
\[ (\Delta^+_d)^\vee := \{ \beta \in \Delta^+ \mid \langle \beta', \rho \rangle \text{ is divisible by } d \}, \]
\[ b_d^\vee := \prod_{\beta \in (\Delta^+_n/d)^\vee} \left( \frac{\langle \beta', \Lambda \rangle}{\langle \beta', \rho \rangle} + 1 \right), \]
\[ a_d^\vee := \frac{1}{d} \sum_{e \mid d} \mu \left( \frac{d}{e} \right) b_e^\vee. \]
Assume that \( \langle \beta', \Lambda \rangle \) is divisible by \( n \) for all \( \beta \in \Delta^+ \). In the same manner as above, we can derive that
\begin{enumerate}
\item For any \( d \mid n \), \( a_d^\vee \in \mathbb{Z}_{\geq 0} \).
\item \( \dim^\vee_q B(\Lambda) \equiv \sum_{d \mid n} a_d^\vee \frac{q^n - 1}{q^d - 1} \pmod{q^n - 1} \).
\end{enumerate}
3. Cyclic sieving phenomena and $q$-dimensions

As before, let $(A, P, \Pi, P^\vee, \Pi^\vee)$ be a Cartan datum of finite type. We here interpret Theorem 2.5 from the viewpoint of the cyclic sieving phenomenon. To do this, we need the following lemma.

Lemma 3.1. (Alexandersson and Amini [1, Theorem 2.7]) Let $f(q) \in \mathbb{N}[q]$ and suppose $f(\omega_n^j) \in \mathbb{N}$ for each $j = 1, \ldots, n$, where $\omega_n$ denotes a primitive $n$th root of unity. Let $X$ be any set of size $f(1)$. Then there exists an action of a cyclic group $C$ of order $n$ on $X$ such that $(X, C, f(q))$ exhibits the cyclic sieving phenomenon if and only if for each $k | n$,

$$\sum_{j | k} \mu(k/j) f(\omega_n^j) \geq 0.$$

Here, $\mu$ is the Möbius function.

Let $X$ be a finite set with an action of a finite cyclic group $C$. We denote by $\text{Orb}_C^d(X)$ the set of all orbits of size $d$. With this notation, we state the following theorem.

Theorem 3.2. Let $\Lambda \in \mathbb{P}^+$ and $n \in \mathbb{Z}_{>0}$. Assume that $(\beta, \Lambda)$ is divisible by $n$ for any $\beta \in \Delta^+$.

1. There exists an action of a cyclic group $C$ of order $n$ on $B(\Lambda)$ such that the triple $(B(\Lambda), C, \dim_q B(\Lambda))$ exhibits the cyclic sieving phenomenon.

2. Let $C$ be a cyclic group of order $n$ acting on $B(\Lambda)$.
   a. The triple $(B(\Lambda), C, \dim_q B(\Lambda))$ exhibits the cyclic sieving phenomenon if and only if the number of orbits in $\text{Orb}_C^d(B(\Lambda))$ is equal to $a_d$ for all $d | n$, where $a_d$ is given in (2.2).
   b. In particular, if $n$ is prime and $(\beta_0, \rho) < n$, then the triple $(B(\Lambda), C, \dim_q B(\Lambda))$ exhibits the cyclic sieving phenomenon if and only if $C$ acting on $B(\Lambda)$ has exactly one fixed point. Here, $\beta_0$ is the highest root and $\rho = \frac{1}{2} \sum_{\beta \in \Delta^+} \beta$.

Proof. (1) Let

$$f(q) = \sum_{d | n} a_d \frac{q^n - 1}{q^d - 1},$$

where $a_d$’s are given in (2.2). For a divisor $k$ of $n$, one can easily see that

$$\frac{(\omega_n^k)^n - 1}{(\omega_n^k)^d - 1} = \begin{cases} d & \text{if } d | k, \\ 0 & \text{if } d \nmid k. \end{cases}$$
Combining this with the fact $f(\omega^i_n) = f(\omega^{\gcd(n,j)}_n)$, one can check that $f(\omega^i_n) \in \mathbb{N}$ for each $j = 1, \ldots, n$. For a divisor $k$ of $n$, note that

$$f(\omega^k_n) = \sum_{d|k} da_d.$$

Applying the Möbius inversion formula to this equality implies that

$$\sum_{j|k} \mu(k/j) f(\omega^j_n) = ka_k \geq 0.$$  

Therefore, the assertion is obtained by combining Theorem 2.5 with Lemma 3.1.

(2) The assertion (a) follows from Theorem 2.5 and [21, Proposition 2.1] and the assertion (b) follows from Example 2.2. \qed

**Remark 3.3.** Assume that $\langle \beta^\vee, \Lambda \rangle$ is divisible by $n$ for all $\beta \in \Delta^+$. Due to Remark 2.6, we can also derive an analogue of Theorem 3.2 for the triple $(B(\Lambda), C, \dim_q B(\Lambda))$.

Let us consider the case where $U_q(\mathfrak{g}) = U_q(\mathfrak{gl}_m)$. Let $I = \{1, 2, \ldots, m-1\}$ and let $\{\epsilon_1, \ldots, \epsilon_m\}$ be the standard orthonormal basis of the Euclidean space $\mathbb{R}^m$. Then one can realize the weight lattice of $U_q(\mathfrak{gl}_m)$ inside $\mathbb{R}^m$ (see [7, §2.1] and [10, §7.1] for example). In this realization, we have

$$\Delta^+ = \{\epsilon_i - \epsilon_j \mid 1 \leq i < j \leq m\}.$$ 

Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ be a partition with $\ell \leq m$ and set

$$(3.2) \quad \Lambda := \sum_{k=1}^\ell \lambda_k \epsilon_k$$

Note that

$$\dim_q B(\Lambda) = q^{-\kappa(\lambda)} s_\lambda(1, q, q^2, \ldots, q^{m-1}),$$

where $\lambda = (\lambda_1, \lambda_2, \ldots)$ is the Young diagram of length $\leq m$ corresponding to $\Lambda$, $\kappa(\lambda) = \sum_{k\geq 1} (k-1)\lambda_k$, and $s_\lambda(1, q, q^2, \ldots, q^{m-1})$ is the principal specialization of the Schur polynomial $s_\lambda(x_1, x_2, \ldots, x_m)$.

It is straightforward to show that $\langle \beta, \Lambda \rangle$ is divisible by $n$ for all $\beta \in \Delta^+$ if and only if

$$(3.3) \quad \lambda_i - \lambda_j \text{ is divisible by } n \text{ for all } 1 \leq i < j \leq m.$$ 

Moreover, we know that

(a) the set $\text{SST}_m(\lambda)$ of all semistandard tableaux of shape $\lambda$ with entries $\{1, 2, \ldots, m\}$ has a $U_q(\mathfrak{gl}_m)$-crystal structure which is isomorphic to $B(\Lambda)$, and

(b) $\dim_q B(\Lambda) = q^{-\kappa(\lambda)} s_\lambda(1, q, q^2, \ldots, q^{m-1})$.

Applying these facts to Theorem 3.2 yields the following corollary, which gives an affirmative answer for the conjecture in [1, Conjecture 3.4].
Corollary 3.4. Let \( m, n \in \mathbb{Z}_{>0} \) and let \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_t) \) be a partition such that \( \lambda_i - \lambda_j \) is divisible by \( n \) for all \( 1 \leq i < j \leq m \). Then there exists an action of a cyclic group \( C \) of order \( n \) on \( \text{SST}_m(\lambda) \) such that the triple \( (\text{SST}_m(\lambda), C, q^{-\kappa(\lambda)} s_\lambda(1, q, q^2, \ldots, q^{m-1})) \) exhibits the cyclic sieving phenomenon.

Remark 3.5. We do not know of an explicit cyclic group action yielding the cyclic sieving phenomenon predicted by Corollary 3.4 in general. However, in the special case where \( n = m \) and \( \lambda = (a^b) \) with \( m \mid a \), Rhoades’ result [22, Theorem 1.4] says that \( (\text{SST}_m(\lambda), (pr), q^{-\kappa(\lambda)} s_\lambda(1, q, \ldots, q^{m-1})) \) exhibits the cyclic sieving phenomenon, where \( pr \) is the promotion operator.

Example 3.6. Let \( g = gl_3 \) and \( n = 4 \). Then we have \( \Delta^+ = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2\} \) and
\[
\Delta^+_1 = \Delta^+, \quad \Delta^+_2 = \{\alpha_1 + \alpha_2\}, \quad \Delta^+_4 = \emptyset.
\]
Let \( \lambda = (4) \). It is obvious that \( \lambda \) satisfies the condition (3.3). Then we have \( \Lambda = 4e_1 \) by (3.2) and
\[
s_\lambda(x_1, x_2, x_3) = x_1^4 + x_1^3x_2 + x_1^2x_2^2 + x_1x_2^3 + x_2^4 + x_1^3x_3 + x_1^2x_2x_3 + x_1x_2^2x_3 + x_2^3x_3 + x_2^2x_3^2 + x_1^3x_2x_3 + x_1x_2^2x_3 + x_2^3x_3 + x_2^2x_3^2 + x_3^4,
\]
which gives the principal specialization
\[
q^{-\kappa(\lambda)} s_\lambda(1, q, q^2) = q^8 + q^7 + 2q^6 + 2q^5 + 3q^4 + 2q^3 + 2q^2 + q + 1.
\]
On the other hand, it follows from (2.2) and Example 2.2 that
\[
b_1 = 1, \quad b_2 = \frac{1}{2} + 1 = 3, \quad b_4 = \left(\frac{4}{1} + 1\right) \left(\frac{0}{1} + 1\right) \left(\frac{4}{2} + 1\right) = 15,
\]
\[
a_1 = 1, \quad a_2 = \frac{1}{2} (b_2 - b_1) = 1, \quad a_4 = \frac{1}{4} (b_4 - b_2) = 3.
\]
By Theorem 2.5, we have
\[
q^{-\kappa(\lambda)} s_\lambda(1, q, q^2) = \dim_q B(\Lambda) \equiv 1 + (1 + q^2) + 3(1 + q + q^2 + q^3) \pmod{q^4 - 1}.
\]
Thus, Corollary 3.4 tells us that there exists an action of a cyclic group \( C \) of order 4 on \( \text{SST}_3(\lambda) \) such that the triple \( (\text{SST}_3(\lambda), C, q^{-\kappa(\lambda)} s_\lambda(1, q, q^2)) \) exhibits the cyclic sieving phenomenon.

Example 3.7. Let \( g \) be a simple Lie algebra of type \( B_2 \). Then we have
\[
A = \begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix}
\]
and \( \Delta^+ = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2, \alpha_1 + 2\alpha_2\} \). Note that \( \alpha_1 \) and \( \alpha_1 + 2\alpha_2 \) are long roots and \( \alpha_2 \) and \( \alpha_1 + \alpha_2 \) are short roots. We set \( n := 2 \) and \( \Lambda := 2\varpi_1 \). Then \( (\beta, \Lambda) \) and \( \langle \beta^\vee, \Lambda \rangle \) are divisible by \( n \) for all \( \beta \in \Delta^+ \) clearly.
(1) As \((\alpha_1, \varpi_j) = 2\delta_{1,j}\) and \((\alpha_2, \varpi_j) = \delta_{2,j}\), it follows from (1.2) that
\[
\dim_q B(\Lambda) = \left(\frac{1 - q^6}{1 - q^2}\right) \left(\frac{1 - q}{1 - q}\right) \left(\frac{1 - q^7}{1 - q^2}\right) \left(\frac{1 - q^8}{1 - q}\right)
= 1 + q^2 + q^3 + 2q^4 + q^5 + 2q^6 + q^7 + 2q^8 + q^9 + q^{10} + q^{12}
\equiv 10 + 4q \pmod{q^2 - 1}.
\]

Since \(\Delta_1^+ = \Delta^+\) and \(\Delta_2^+ = \{\alpha_1, \alpha_1 + 2\alpha_2\}\), it follows from (2.2) and Example 2.2 that
\[
b_1 = \left(\frac{4}{2} + 1\right) \left(\frac{4}{4} + 1\right) = 6, \quad b_2 = \left(\frac{4}{2} + 1\right) \left(\frac{0}{1} + 1\right) \left(\frac{4}{3} + 1\right) \left(\frac{4}{4} + 1\right) = 14,
\]
\[
a_1 = b_1 = 6, \quad a_2 = \frac{1}{2}(b_2 - b_1) = 4.
\]

By Theorem 2.5, we have
\[
\dim_q B(\Lambda) \equiv 6 + 4(1 + q) \pmod{q^2 - 1}.
\]

(2) As \(\{\beta^\vee \mid \beta \in \Delta^+_\}\} = \{h_1, h_2, 2h_1 + h_2, h_1 + h_2\}\), it follows from (1.3) that
\[
\dim^\vee_q B(\Lambda) = \left(\frac{1 - q^3}{1 - q}\right) \left(\frac{1 - q}{1 - q}\right) \left(\frac{1 - q^2}{1 - q^3}\right) \left(\frac{1 - q^4}{1 - q^2}\right)
= 1 + q + 2q^2 + 2q^3 + 2q^4 + 2q^5 + 2q^6 + q^7 + q^8
\equiv 8 + 6q \pmod{q^2 - 1}.
\]

By Remark 2.6, we have
\[
b^\vee_1 = \left(\frac{2}{2} + 1\right) = 2, \quad b^\vee_2 = \left(\frac{2}{1} + 1\right) \left(\frac{0}{1} + 1\right) \left(\frac{4}{3} + 1\right) \left(\frac{2}{2} + 1\right) = 14,
\]
\[
a^\vee_1 = b^\vee_1 = 2, \quad a^\vee_2 = \frac{1}{2}(b_2 - b_1) = 6,
\]
which implies that
\[
\dim^\vee_q B(\Lambda) \equiv 2 + 6(1 + q) \pmod{q^2 - 1}.
\]

4. APPLICATION TO THE CRYSTAL OPERATOR \(c\) ON \(\text{SST}_m(\lambda)\)

Let \(m\) be a positive integer \(\geq 2\). Recall that \(\text{SST}_m(\lambda)\) has a \(U_q(\mathfrak{gl}_m)\)-crystal structure, thus it is equipped with an action of the Weyl group. Let us consider the operator \(c := s_1s_2 \cdots s_{m-1}\) on \(\text{SST}_m(\lambda)\), where \(s_i\) is the action on the crystal \(\text{SST}_m(\lambda)\) given by the simple reflection \(s_i = (i, i + 1)\) in the Weyl group. Note that the order of \(c\) is \(m\). The cyclic action given by this operator was extensively studied in [20] in the case where \(\ell(\lambda) < m\) and \(\gcd(m, |\lambda|) = 1\). Under this constraint, it was shown that the triple \((\text{SST}_m(\lambda), (c), q^{-\kappa(\lambda)}s_\lambda(1, q, \ldots, q^{m-1}))\)
exhibits the cyclic sieving phenomenon and every orbit is free. We here focus on the case where $\ell(\lambda) < m$ and $|\lambda|$ is divisible by $m$.

Let us collect lemmas which are necessary to develop our arguments. Given $T \in \text{SST}_m(\lambda)$, let $\text{cont}(T) := (c_1, c_2, \ldots, c_m)$, where $c_i$ is the number of $i$’s occurring in $T$.

**Lemma 4.1.** ([9, §2.2. Exercise 2]) Suppose that $\lambda$ and $\mu$ are partitions of $m$. Then $K_{\lambda\mu} > 0$ if and only if $\lambda \geq \mu$, where $K_{\lambda,\mu}$ is the Kostka number.

**Lemma 4.2.** (cf. [7, 10]) Let $\lambda$ be a partition. The set of fixed points of $\text{SST}_m(\lambda)$ under the action of $c$ is nonempty if and only if $|\lambda|$ is divisible by $m$, in which case it is given by

$$\{ T \in \text{SST}_m(\lambda) : \text{cont}(T) = \left( \frac{|\lambda|}{m}, \frac{|\lambda|}{m}, \ldots, \frac{|\lambda|}{m} \right) \}.$$  

**Proof.** If $|\lambda|$ is divisible by $m$, from the definition (1.1) of the action of $s_i$ on $\text{SST}_m(\lambda)$ one can infer that the set of fixed points of $\text{SST}_m(\lambda)$ under the action of $c$ is given by (4.1), which is nonempty by Lemma 4.1. Conversely, if $|\lambda|$ is not divisible by $m$, then there are no fixed points since $\text{cont}(T) \neq \text{cont}(c \cdot T)$ for all $T \in \text{SST}_m(\lambda)$. \hfill $\square$

For any two partitions $\lambda, \mu$, we shall write $\lambda \sim^m \mu$ if they have the same $m$-core. For more information on $m$-cores, see [11, §2.7] or [19, Section I.3. Examples 8]. Assume that $\ell(\lambda) \leq m$. It is easy to see that if $\lambda \sim^m 0$, then there exists a unique permutation $w_\lambda \in S_m$ such that $\lambda + \delta_m \equiv w_\lambda \delta_m \pmod{m}$, where $\delta_m = (m - 1, m - 2, \ldots, 1, 0)$. The following lemma follows from Examples 17 in [19, Section I.3].

**Lemma 4.3.** Let $\lambda$ be a partition of length $\leq m$ and let $\zeta_m = e^{2\pi i/m}$. Then we have

$$s_\lambda(1, q, q^2, \ldots, q^{m-1})|_{q=\zeta_m} = \begin{cases} 0 & \text{if } \lambda \sim^m 0, \\ 1 & \text{if } \lambda \sim^m 0 \text{ and } \epsilon(w_\lambda) = 1, \\ -1 & \text{if } \lambda \sim^m 0 \text{ and } \epsilon(w_\lambda) = -1, \end{cases}$$

where $\epsilon(w)$ is the sign of $w$.

When $|\lambda|$ is divisible by $m$, we define $T^0_\lambda$ to be the semistandard tableau in $\text{SST}_m(\lambda)$ of content $\mu = \left( \frac{|\lambda|}{m}, \frac{|\lambda|}{m}, \ldots, \frac{|\lambda|}{m} \right)$ obtained by filling the Young diagram of shape $\lambda$ with entries in the increasing order from left to right and from top to bottom.

**Theorem 4.4.** Let $\lambda$ be a partition of length $< m$. Assume that there exists a fixed point in $\text{SST}_m(\lambda)$ under the action of $c$, that is, $|\lambda|$ is divisible by $m$. Then the following are equivalent.

(a) The triple $(\text{SST}_m(\lambda), (\zeta), s_\lambda(1, q, q^2, \ldots, q^{m-1}))$ exhibits the cyclic sieving phenomenon.
(b) \( \lambda = (am) \) or \(((am)^{m-1})\) for some positive integer \(a\).

**Proof.** First, we assume that (a) holds. In view of (4.2), one sees that there exists only one fixed point, which means that 

\[ T_\lambda^0 \text{ is the unique fixed point in } \text{SST}_m(\lambda). \]

Let \( \lambda = (\lambda_1, \ldots, \lambda_\ell) \) with \( \lambda_\ell > 0 \) and let \( i_k \) be the first entry of the \( k \)th row of \( T_\lambda^0 \) for \( k = 1, \ldots, \ell \).

Suppose that there is a \( k \) such that \( |i_{k+1} - i_k| > 1 \). Let \( b \) be the box \((k + 1, 1) \in \lambda \) and \( b' \) be the rightmost box in the \( k \)th row of \( T_\lambda^0 \) whose entry is less than \( i_{k+1} \). Since \( |i_{k+1} - i_k| > 1 \), the \( k \)th row of \( T_\lambda^0 \) contains all \( i_k + 1 \)'s. Thus we have \( \lambda_k > |\lambda|/m \), which says that the entry of the box just below \( b' \) is larger than \( i_{k+1} \) if it exists. Setting \( T \) to be the tableau obtained from \( T_\lambda^0 \) by swapping the entries of \( b \) and \( b' \), \( T \) is a valid semistandard tableau, which tells us that \( T \) is also a fixed point. This is a contradiction. Thus we conclude that

\[ i_k = k \]

for \( k = 1, \ldots, \ell \).

If \( \ell = 1 \), then \( \lambda = (am) \) for some \(a\). Let us first show that \( 1 < \ell < m - 1 \) is impossible. In this case, since \( i_\ell = \ell < m - 1 \), the \( \ell \)-th row of \( T_\lambda^0 \) should contain all entries equal to \( m - 1 \) and \( m \). Thus \( \lambda_\ell > 2|\lambda|/m \) and therefore also \( \lambda_1 > 2|\lambda|/m \), contradicting \( i_2 = 2 \).

Suppose that \( \ell = m - 1 \). Since the \( \ell \)-th row of \( T_\lambda^0 \) contains all \( m \)'s, it follows that \( \lambda_\ell > |\lambda|/m \) and therefore \( \lambda_k > |\lambda|/m \) for any \( k \). Combining this inequality with (4.3), we see that each \( k \)th row of \( T_\lambda^0 \) has both \( k \) and \( k + 1 \). We assume that there exists an index \( k \) such that \( \lambda_{k+1} < \lambda_k \) and \( \lambda_j = \lambda_k \) for all \( j \leq k \). Let \( b \) be the box \((k, \lambda_k) \in \lambda \) and \( b' \) be the leftmost box \((k + 1, t)\) of the \((k + 1)\)st row whose entry is \( k + 2 \). Note that the entry of \( b \) is \( k + 1 \). Since \( \lambda_k > |\lambda|/m \), the entry of the box \((k, t)\) is \( k \). Hence the tableaux \( T \) obtained from \( T_\lambda^0 \) by swapping the entries of \( b \) and \( b' \) is a valid semistandard tableau. This tells us that \( T \) is also a fixed point, which is a contradiction. Therefore, \( \lambda \) should be of rectangular shape.

We now assume that (b) holds. By [22, Theorem 1.4], it suffices to show that our crystal operator \( c \) coincides with \( \text{pr} \). This is straightforward in the case where \( \lambda = (am) \). So, we assume that \( \lambda = ((am)^{m-1}) \). Pick up any \( T \in \text{SST}_m(\lambda) \).

**Case 1.** Assume that \( 1 \leq i < m \) does not appear in the first column. Then, for all \( i \leq j \leq m - 1 \), the \( j \)th row is filled with only \((j + 1)\)'s. Hence, for all \( i + 1 \leq j \leq m - 1 \), both \( \sigma_j \) and \( s_j \) act on \( T \) as the identity, where \( \sigma_j \) is the \( j \)th Bender-Knuth involution. In case of \( 1 \leq j \leq i \), ignore all entries not equal to \( j \) or \( j + 1 \) and all columns that contain both \( j \) and \( j + 1 \). What remains, which is a sequence of \( j \)'s immediately followed by \((j + 1)\), appears only within one row. This tells us that both \( \sigma_j \) and \( s_j \) act identically for all \( 1 \leq j \leq i \).
Case 2. Assume that $m$ does not appear in the first column. In the same manner as above, one sees that both $\sigma_j$ and $s_j$ act identically for all $1 \leq j \leq m - 1$. 

Remark 4.5. (1) Let $\lambda$ be of rectangular shape. Then, as permutations on $\text{SST}_m(\lambda)$, $c$ and $pr$ have the same order, but they are not conjugate in general. It would be nice to characterize $\lambda$'s such that $c$ and $pr$ are conjugate, equivalently, $\lambda$'s such that $(\text{SST}_m(\lambda), \langle c \rangle, s_{\lambda}(1, q, q^2, \ldots, q^{m-1}))$ exhibits the cyclic sieving phenomenon.

(2) Let $\lambda = (a)$ or $(am^{-1})$, where $|\lambda|$ is not necessarily divisible by $m$. Following the proof of the second part in Theorem 4.4, one can also see that $c$ coincides with $pr$ as operators on $\text{SST}_m(\lambda)$.

(3) Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra over $\mathbb{C}$. The longest Weyl group element $w_0$ defines an involution on the simple roots by $\alpha_i \mapsto -w_0(\alpha_i)$. Consider the automorphism of $U(\mathfrak{g})$ defined by

$$
\phi(e_i) = f_i, \quad \phi(f_i) = e_i, \quad \phi(h_i) = -h_i.
$$

Let $\Lambda^\vee := -w_0(\Lambda)$, $v_\Lambda$ be the highest weight vector and $v_\Lambda^{\text{low}}$ the lowest weight vector of $B(\Lambda)$. By [6, Proposition 21.1.2] and [18, Proposition 7.1], one has the bijection $\phi_\Lambda : B(\Lambda) \to B(\Lambda^\vee)$ satisfying that $v_\Lambda \mapsto v_\Lambda^{\text{low}}$ and

$$
\phi_\Lambda(\tilde{f}_i u) = \tilde{e}_i \phi_\Lambda(u), \quad \phi_\Lambda(\tilde{e}_i u) = \tilde{f}_i \phi_\Lambda(u), \quad \text{for } u \in B(\Lambda) \text{ and } i \in I.
$$

Using (1.1), it is not difficult to see that $\phi_\Lambda : B(\Lambda) \to B(\Lambda^\vee)$ is an isomorphism as $W$-sets. Hence, in type $A_{m-1}$, we have the isomorphism $\phi_{am\infty} : \text{SST}_m((am)) \cong \text{SST}_m((am)^{m-1})$ as $\langle c \rangle$-sets. This isomorphism $\phi_{am\infty}$ explains why both of $(am)$ and $((am)^{m-1})$ appear in Theorem 4.4. Note that the isomorphism $\phi_{am\infty}$ can be understood as a modification of Schützenberger’s or Luszting’s involution.

For each divisor $d$ of $m$, let $\#\text{Orb}^d_{\langle c \rangle}(\text{SST}_m(\lambda))$ denote the number of orbits of size $d$ in $\text{SST}_m(\lambda)$ under the action of $\langle c \rangle$. If $\lambda = (am)$ or $((am)^{m-1})$ for any positive integer $a$, then it satisfies the condition (2.1). This enables us to use (2.2) in computing $\#\text{Orb}^d_{\langle c \rangle}(\text{SST}_m(\lambda))$.

**Proposition 4.6.** Assume that $\lambda$ is either $(am)$ or $((am)^{m-1})$ for any positive integer $a$. For each divisor $d$ of $m$, we have

$$
\#\text{Orb}^d_{\langle c \rangle}(\text{SST}_m(\lambda)) = \frac{1}{d} \sum_{e \mid d} \mu \left( \frac{d}{e} \right) \prod_{1 \leq k < e} \left( \frac{ae}{k} + 1 \right).
$$

(4.4)
Proof. Let \( n = m \) and \( \Lambda = am\epsilon_1 \). It is not difficult to see that
\[
\Delta^+_d = \{ \alpha \in \Delta^+ : \text{ht}(\alpha) \text{ is divisible by } d \}
= \bigcup_{1 \leq k < m/d} \{ \alpha \in \Delta^+ : \text{ht}(\alpha) = kd \}
= \{ \epsilon_i - \epsilon_{i+kd} : 1 \leq k < m/d \text{ and } 1 \leq i \leq m - kd \}.
\]
Since
\[
\frac{(\epsilon_i - \epsilon_{i+kd}, am\epsilon_1)}{(\epsilon_i - \epsilon_{i+kd}, \rho)} = \frac{am}{kd} \delta_{i1},
\]
we have
\[
(4.5) \quad b_d = \prod_{1 \leq k < d} \left( \frac{ad}{k} + 1 \right).
\]
For the definition of \( b_d \), see (2.2). Here, \( \delta \) denotes the Kronecker delta function and the right hand side of (4.5) is understood as 1 in the case where \( d = m \). Therefore, our assertion follows from (2.2). \( \Box \)

Example 4.7. Note that the right hand side of (4.4) does not depend on the choices of \( m \). Let \( \lambda = (am) \) or \((am)^{m-1}\). For every even positive integer \( m \), we have \#\text{Orb}_{(\epsilon)}(\text{SST}_m(\lambda)) = \frac{a}{2}(a + 1)\).

In the rest of this section, we assume that \( n \) is a prime \( p \geq m \) and \( \lambda \) is a partition of length \( \leq p \). Recall that
\[
(4.6) \quad q^{-\kappa(\lambda)} s_\lambda(1, q, q^2, \ldots, q^{m-1}) = \prod_{1 \leq i < j \leq m} \frac{1 - q^{(\lambda_i - i) - (\lambda_j - j)}}{1 - q^{j-i}}
\]
(for instance, see [27, Theorem 7.21.2]). Let \( \mathcal{A} \) be the set of all partitions \( \lambda \) of length \( \leq p \) satisfying that \( \lambda_i - i \equiv \lambda_j - j \pmod{p} \) for some \( 1 \leq i < j \leq m \).

Proposition 4.8. Let \( p \) be a prime \( \geq m \) and \( \lambda \) a partition of length \( \leq p \).
\begin{enumerate}
\item \( s_\lambda(1, q, q^2, \ldots, q^{m-1}) \equiv 0 \pmod{\Phi_p(q)} \) if and only if \( \lambda \in \mathcal{A} \).
\item If \( \lambda \in \mathcal{A} \), then there exists an action of a cyclic group \( C \) of order \( p \) on \( \text{SST}_m(\lambda) \) such that the triple \((\text{SST}_m(\lambda), C, q^{-\kappa(\lambda)} s_\lambda(1, q, q^2, \ldots, q^{m-1}))\) exhibits the cyclic sieving phenomenon.
\item There exists an action of a cyclic group \( C \) of order \( p \) on \( \text{SST}_p(\lambda) \) such that the triple \((\text{SST}_p(\lambda), C, s_\lambda(1, q, q^2, \ldots, q^{p-1}))\) exhibits the cyclic sieving phenomenon if and only if either \( \lambda \sim_p 0 \) or else \( \lambda \sim_p 0 \) and \( \epsilon(w_\lambda) = 1 \).
\end{enumerate}
Proof. (1) Note that $\mathbb{Q}[q]/(\Phi_p(q))$ is a field and $1 - q^{j-i}$ appearing in the denominator is a unit for all $1 \leq i < j \leq n$. Applying this fact to the right hand side of (4.6), we obtain the desired result.

(2) Let $\lambda \in \mathcal{A}$. By (1), we have that $q^{-\kappa(\lambda)} s_\lambda(1, q, q^2, \ldots, q^{m-1}) \equiv a_p(1 + q + \cdots + q^{p-1}) \mod q^p - 1$ for some positive integer $a_p$. Therefore, our assertion can be proven in the same way as in Theorem 3.2 (1).

(3) Due to Lemma 4.3, the condition “either $\lambda \sim_p 0$ or else $\lambda \sim_p 0$ and $\epsilon(w_\lambda) = 1$” is equivalent to saying that $s_\lambda(1, q, q^2, \ldots, q^{m-1}) \equiv a_1 + a_p(1 + q + \cdots + q^{p-1}) \mod q^p - 1$ for some nonnegative integers $a_1$ and $a_p$. Therefore, our assertion can also be proved in the same way as in Theorem 3.2 (1). □

Remark 4.9. By virtue of the congruence to Kac [13, Exercise 10.15], one can derive an analogue of Proposition 4.8 for highest weight crystals of any finite type.

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