Convergence for non-commutative rational functions evaluated in random matrices

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Joint work with Tobias Mai, Benoît Collins, Felix Parraud and Sheng Yin
Today’s talk

1. Non-commutative rational functions
   - Evaluation of non-commutative rational functions.
   - Theorem by Mai, Speicher, and Yin.

2. Main results
   - Main result 1: Well-definedness of $r(X^N)$ with $N \times N$ random matrices $X^N$.
   - Main result 2: Convergence in distribution.

3. Strategy of the proof
   - Linearization.
   - Characterization of cumulative distribution functions by projections.
Non-commutative rational function
Non-commutative rational expressions are defined by all possible combinations of $x_1, \ldots, x_d, \mathbb{C}$ with $+, \times, \cdot^{-1}, ()$.

E.g. $x_1 x_2^{-1}, (x_1 + x_2)^{-1}, x_1 + 2x_2^{-1}x_1$

For unital $\mathbb{C}$-algebra $A$ and a nc rational expression $r$, we define

$$\text{dom}_A(r) = \{ X = (X_1, \ldots, X_d) \in A^d : r(X) \in A \}.$$  

For example, $\text{dom}_A((x_1 x_2 - x_2 x_1)^{-1}) = \emptyset$ when $A$ is commutative.
Non-commutative rational functions

- For a nc rational expression \( r \), \( \text{dom}(r) \) is a subset of all square matrices over \( \mathbb{C} \) where evaluation of \( r \) is well-defined.

- Equivalence relation

\[
    r_1 \sim r_2 \iff r_1(a) = r_2(a), \quad \forall a \in \text{dom}(r_1) \cap \text{dom}(r_2) \neq \emptyset.
\]
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  \]

- An equivalence class of nc rational expressions is called a non-commutative rational function.

- A set \( \mathbb{C}\langle x_1, \ldots, x_d \rangle \) of nc rational functions is called the free (skew) field which contains non-commutative polynomials \( \mathbb{C}\langle x_1, \ldots, x_d \rangle \) (Amitur’66, Cohn’94, Kaliuzhnyi-Verbovetskyi and Vinnikov’10).
Example: NC rational expressions and the equivalence relation

\[ r_1 = (x_1(x_1 + x_2)^{-1})x_2, \quad r_2 = x_1((x_1 + x_2)^{-1}x_2), \]
\[ r_3 = (x_1^{-1} + x_2^{-1})^{-1} \] are formally different rational expressions.

\[
\text{dom}(r_1) = \text{dom}(r_2) = \{(X_1, X_2); \det(X_1 + X_2) \neq 0\} \\
\text{dom}(r_3) = \{(X_1, X_2); \det(X_1), \det(X_2), \det(X_1^{-1} + X_2^{-1}) \neq 0\}.
\]

We can see \( \text{dom}(r_3) \subsetneq \text{dom}(r_2) = \text{dom}(r_1) \) and \( r_i \)'s are equivalent since we have the formal calculation,

\[
\{x_1(x_1 + x_2)^{-1}x_2\}^{-1} = x_2^{-1}(x_1 + x_2)x_1^{-1} = x_1^{-1} + x_2^{-1}.
\]

Remark

For any rational expression \( r \) with \( \text{dom}(r) \neq \emptyset \), there exists \( N_0 = N_0(r) \) such that \( \text{dom}(r) \cap M_N(\mathbb{C})^d \neq \emptyset \) for \( N \geq N_0 \).
Evaluation of non-commutative rational functions

- We need to take matrices with large sizes for the evaluation of a nc rational expression.

- (Hall) For any $X_i \in M_2(\mathbb{C})$,
  \[
  [[X_1, X_2]^2, X_3] = 0.
  \]

- (Amitsur-Levitzki) For any $X_i \in M_N(\mathbb{C})$,
  \[
  \sum_{\pi \in S_{2N}} \text{sgn}(\pi)X_{\pi(1)} \ldots X_{\pi(2N)} = 0.
  \]

- For a nc rational function $r$, we would like to define
  \[
  \text{dom}_A(r) = \bigcup_{r';[r'] = r} \text{dom}_A(r'), \ r(X) = r'(X), \ X \in \text{dom}_A(r) \cap \text{dom}_A(r').
  \]
Evaluation in operators

- Evaluation of non-commutative rational functions in elements in a unital algebra $\mathcal{A}$ is not well-defined in general. For example, we have $x_1(x_2x_1)^{-1}x_2 = 1$, but for the unilateral shift $S$

$$S(S^*S)^{-1}S^* = SS^* \neq 1.$$  

- Evaluation of non-commutative rational functions is well-defined if $\mathcal{A}$ is stably finite. i.e. we have for each $m \in \mathbb{N}$

$$A, B \in M_m(\mathcal{A}), AB = I_m \iff BA = I_m.$$  

- Every finite von Neumann algebras $\mathcal{M}$ are stably finite.

- The $\ast$-algebra $\widetilde{\mathcal{M}}$ of affiliated operators with $\mathcal{M}$ is also stably finite.
Evaluation of all non-commutative rational functions

Theorem (T. Mai, R. Speicher and S. Yin ‘19)

Let $X = (X_1, \ldots, X_d)$ be a tuple of freely independent self-adjoint operators in a $W^*$-probability space such that each $X_i$ has no atom. Then \( \exists! \mathcal{E}_V : \mathbb{C} \langle x_1, \ldots, x_d \rangle \rightarrow \tilde{\mathcal{M}} \), a homomorphism which extends $\mathbb{C} \langle x_1, \ldots, x_d \rangle \ni P \rightarrow P(X) \in \mathcal{M}$.

- The condition (free + absence of atom) is generalized to maximality of $\Delta(X_1, \ldots, X_d)(=d)$ defined in Connes-Shlyakhtenko’05

$$\dim_{\mathcal{M} \otimes \mathcal{M}^{\text{op}}} \left\{ (T_1, \ldots, T_d) \in \mathcal{F}(L^2(\mathcal{M})) : \sum_{i=1}^{d} [T_i, JX_i J] = 0 \right\} \text{ HS}$$

- Free Haar unitaries $u_1, \ldots, u_d$ satisfy $\Delta(u_1, \ldots, u_d) = d$. 
Atoms of a nc rational function evaluated in free random variables can be computed algebraically (Mai-Speicher-Yin’19, Arizmendi-Cébron-Speicher-Yin’21).

The weight of atoms of a nc rational function evaluated in free random variables is minimal when each distribution is given (Arizmendi-Cébron-Speicher-Yin'21).

nc rational functions are characterized by finite rank commutators, which is an analogue of Kronecker's theorem (Duchamp-Reutenauer’97, Linnell’00, M’22).
Main results
For independent GUE random matrices $X_1^N, \ldots, X_d^N$, we have almost surely,

$$\lim_{N \to \infty} \text{tr}(X_1^N \cdots X_i^N) = \tau(s_1 \cdots s_i),$$

where $s_1, \ldots, s_d$ are free semicircles with respect to $\tau$.

- Let $P \in \mathbb{C}\langle x_1, \ldots, x_d \rangle$ be a self-adjoint polynomial. Then we have almost surely for $f \in C_c(\mathbb{R})$,

$$\lim_{N \to \infty} \int_{\mathbb{R}} f \ d\mu_P(X^N) = \tau[f(P(s))] = \int_{\mathbb{R}} f \ d\mu_P(s).$$

- We replace nc polynomials by nc rational functions.
Main result 1: Evaluation in random matrices

- We work on rational functions evaluated in self-adjoint matrices and unitary matrices.

Theorem (Collins-Mai-M-Parraud-Yin’22)

Let $r$ be a nc rational function with $d = d_1 + d_2$ formal variables. Let $(X^N, U^N)$ be a tuple of random matrices in $M_N(\mathbb{C})^{d_1}_{sa} \times U_N(\mathbb{C})^{d_2}$ whose law is absolutely continuous with respect to the product measure of Lebesgue measure on $M_N(\mathbb{C}^d)_{sa}$ and Haar measure on $U_N(\mathbb{C})$. Then $\exists N_0 \in \mathbb{N}$ s.t. we have almost surely

$$(X^N, U^N) \in \text{dom}(r), \quad N \geq N_0$$
Main result 2: Convergence in distribution

- $T \in \tilde{\mathcal{M}}$: self-adjoint.
- $P_T(B)$: spectral projection on $B$ for a Borel set $B$.
- $\mu_T$: spectral measure $\mu_T(B) := \tau[P_T(B)]$
- cumulative distribution function $\mathcal{F}_T(t) = \mu_T(-\infty, t]$, $t \in \mathbb{R}$.

**Theorem (Collins-Mai-M-Parraud-Yin’22)**

Let $r$ be a nc rational function. For each $N \in \mathbb{N}$, let $X^N = (X_1^N, \ldots, X_d^N)$ be a tuple of (possibly unbounded) operators affiliated with a $W^*$-probability space $(\mathcal{M}_N, \tau_N)$. We suppose that $X^N$ converges in $*$-distribution towards bounded operators $X = (X_1, \ldots, X_d)$ in a $W^*$-probability space $(\mathcal{M}, \tau)$. We also assume $X^N, X \in \text{dom}(r)$ and $r(X^N), r(X)$ are self-adjoint. Then we have for any continuous point $t \in \mathbb{R}$ of $\mathcal{F}_r(X)$,

$$\lim_{N \to \infty} \mathcal{F}_r(X^N)(t) = \mathcal{F}_r(X)(t).$$
Corollary of main results

By combining our results and the result in Mai-Speicher-Yin, we have the following.

Corollary

Let \((X^N, U^N) \in M_N(\mathbb{C})^{d_1}_{sa} \times U_N(\mathbb{C})^{d_2}\) be a tuple of independent GUE and Haar unitary matrices and \((X, U) \in M^{d_1}_{sa} \times U(\mathcal{M})^{d_2}\) be a tuple of free semicircles and Haar unitaries. Then for any nc rational function \(r\) with \(d_1 + d_2\) indeterminates such that \(r(X, U)\) self-adjoint, the empirical eigenvalue distribution of \(r(X^N, U^N)\) almost surely converges in distribution towards the spectral measure of \(r(X, U)\).
Strategy of the proof
Strategy of the proof: Linearization

Proposition (Linearization)

For a nc rational expression $r$ we can find $A, u, v$ s.t.

$$ r(X) = {}^t uA(X)^{-1}v, \quad X \in \text{dom}(r) \quad \text{(linearization)}, $$

where

- $A \in M_k(\mathbb{C}\langle x_1, \ldots, x_d \rangle)$: linear, i.e.

  $$ A = A_0 + A_1x_1 + \cdots + A_dx_d, \quad A_i \in M_k(\mathbb{C}). $$

- $u, v \in \mathbb{C}^k$.

- $\text{dom}(r) \subset \{ X \in \widetilde{\mathcal{M}}^d; \exists A(X)^{-1} \}$.

- $\text{dom}(r) \neq \emptyset \Rightarrow A$ is full, i.e. there is no $l < k$ s.t.

  $$ A = BC, \quad B \in M_{k \times l}(\mathbb{C}\langle x_1, \ldots, x_d \rangle), \quad C \in M_{l \times k}(\mathbb{C}\langle x_1, \ldots, x_d \rangle). $$
For $r_1 = t u_1 A_1^{-1} v_1$, $r_2 = t u_2 A_2^{-1} v_2$, 

\[
\begin{align*}
    r_1 + r_2 &= \begin{pmatrix} t u_1 & t u_2 \end{pmatrix} \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}^{-1} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \\
    r_1 r_2 &= \begin{pmatrix} t u_1 & 0 \end{pmatrix} \begin{pmatrix} A_1 & -v_1 t u_2 \\ 0 & A_2 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ v_2 \end{pmatrix} \\
    r_1^{-1} &= \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & t u_1 \\ v_1 & A_1 \end{pmatrix}^{-1} \begin{pmatrix} -1 \\ 0 \end{pmatrix}.
\end{align*}
\]
Algorithm for linearization

For \( r_1 = t_{u_1}A_1^{-1}v_1 \), \( r_2 = t_{u_2}A_2^{-1}v_2 \),

\[
\begin{align*}
\begin{pmatrix} r_1 \\ r_2 \end{pmatrix} &= \begin{pmatrix} t_{u_1} & t_{u_2} \end{pmatrix} \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}^{-1} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \\
\begin{pmatrix} r_1r_2 \end{pmatrix} &= \begin{pmatrix} t_{u_1} & 0 \end{pmatrix} \begin{pmatrix} A_1 & -v_1t_{u_2} \\ 0 & A_2 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ v_2 \end{pmatrix} \\
\begin{pmatrix} r_1^{-1} \end{pmatrix} &= \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ v_1 \end{pmatrix}^{-1} t_{u_1} \begin{pmatrix} -1 \\ 0 \end{pmatrix}.
\end{align*}
\]

In the third equality, one can see from the formal calculation,

\[
\begin{pmatrix} 0 \\ v \end{pmatrix} t_u A^{-1} \begin{pmatrix} -r^{-1} \\ A^{-1}vr^{-1} \end{pmatrix}^{-1} = \begin{pmatrix} r^{-1}t_{u}A^{-1} \\ A^{-1} - A^{-1}vr^{-1}t_{u}A^{-1} \end{pmatrix}.
\]
Examples of linearization

\[ x_1 = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & -x_1 \\ 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \]

\[ x_1 x_2 = \begin{pmatrix} 1 & 0 | 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & -x_1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -x_2 \\ 0 & 0 & 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \]

\[ (x_1 x_2)^{-1} = \begin{pmatrix} 1 | 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & -x_1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & -x_2 \\ 1 & 0 & 0 & 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \]
**Theorem (J.W.Helton, T.Mai and R.Speicher ‘18)**

Let $r$ be a rational expression and $A$ be a $\ast$-algebra. If $r(X)$ is self-adjoint for $X \in A^d$, then there exists $A \in M_k(\mathbb{C}\langle x_1, \ldots, x_d \rangle)$ and $u \in \mathbb{C}^k$ s.t.

$$A = \sum_{i=1}^{d} A_i \otimes x_i, \quad A_i^* = A_i.$$

$$r(X) = u^* A^{-1}(X)u.$$ 

For $r = t_u A^{-1} v$, consider

$$\begin{pmatrix} v^* & t_u \end{pmatrix} \begin{pmatrix} 0 & A(X) \\ A^*(X) & 0 \end{pmatrix}^{-1} \begin{pmatrix} v \\ \bar{u} \end{pmatrix},$$

which represents $2r(X)$ since $r(X)$ is self-adjoint.
Remark about main result 1

- For the proof, we use $\forall r, \text{dom}(r) \cap M_N(\mathbb{C})^d \neq \emptyset$ for $N \geq \exists N_0$.
- Consider a linearization $r = t u A^{-1} v$, and check by using complex analysis technique,

$$\text{dom}(A^{-1}) \cap M_N(\mathbb{C})^{d_1} \times U_N(\mathbb{C})^{d_2} = \emptyset \implies \text{dom}(A^{-1}) \cap M_N(\mathbb{C})^{d_1 + d_2} = \emptyset.$$  

- If $X_1, X_2$ are $N \times N$ symmetric matrices, then

$$\det(X_1 X_2 - X_2 X_1) = \det(t(X_1 X_2 - X_2 X_1))$$
$$= (-1)^N \det(X_1 X_2 - X_2 X_1)$$
$$= 0 \text{ (}N\text{ is odd)}$$

- This implies $\text{Sym}_N(\mathbb{C}) \not\subset \text{dom}((x_1 x_2 - x_2 x_1)^{-1})$ for odd $N$. 
Estimation of the cumulative distribution function

- \( \text{rank}(T) := \tau(P_T) \), \( P_T \): orthogonal projection onto \( \text{Im} T \)

**Lemma (Bercovici-Voiculescu’93)**

For any \( t \in \mathbb{R} \), we have

\[
\mathcal{F}_T(t) = \max\{\tau(p); p \in \mathcal{P}(\mathcal{M}), tp \geq pTp\}.
\]

**Lemma**

For \( X, Y \in \mathcal{M}_{sa} \), we have

\[
\sup_{t \in \mathbb{R}} |\mathcal{F}_{X+Y}(t) - \mathcal{F}_X(t)| \leq \text{rank}(Y).
\]
Let $\epsilon > 0$. Approximate the function $g : x \rightarrow x^{-1}$ by continuous functions $f_\epsilon$.

We take $f_\epsilon$ as a continuous function such that $f_\epsilon = g$ on $\mathbb{R} \setminus [-\epsilon, \epsilon]$.

Let $r = w^*Q^{-1}w$ be a self-adjoint linearization. We put $Q_N = Q(X^N)$, $Q_\infty = Q(X)$. Then

$$|F_{w^*Q^{-1}_Nw}(t) - F_{w^*Q^{-1}_\infty w}(t)| \leq |F_{w^*Q^{-1}_Nw}(t) - F_{w^*f_\epsilon(Q_N)w}(t)|$$

$$+ |F_{w^*f_\epsilon(Q_N)w}(t) - F_{w^*f_\epsilon(Q_\infty)w}(t)|$$

$$+ |F_{w^*f_\epsilon(Q_\infty)w}(t) - F_{w^*Q^{-1}_\infty w}(t)|.$$
Strategy for the main result 2: Rank estimation

- From previous Lemma, we have for $X = Q_N, Q_\infty$ ($k \times k$ operator valued matrices),

$$\left| \mathcal{F}_{w^*}X^{-1}w(t) - \mathcal{F}_{w^*}f_\epsilon(X)w(t) \right| \leq \text{rank}(w^*(X^{-1} - f_\epsilon(X))w)$$

$$\leq k \times \text{rank}(ww^*(X^{-1} - f_\epsilon(X))ww^*)$$

$$\leq k \times \text{rank}(X^{-1} - f_\epsilon(X))$$

$$\leq \text{Tr}_k \otimes \tau(1_{[-\epsilon,\epsilon]}(X)).$$

- $\lim_{\epsilon \to 0} \text{Tr}_k \otimes \tau(1_{[-\epsilon,\epsilon]}(Q_\infty)) = \text{Tr}_k \otimes \tau(1_{\{0\}}(Q_\infty)) = 0$ since $Q_\infty$ is invertible.
Strategy for the main result 2: Norm estimation

- For $|\mathcal{F}_{w} f_{\epsilon}(Q_{N}) w(t) - \mathcal{F}_{w} f_{\epsilon}(Q_{\infty}) w(t)|$, we show the convergence in moments

$$\limsup_{N \to \infty} |\tau_{N}[(w^{*} f_{\epsilon}(Q_{N}) w)^{\prime}] - \tau[(w^{*} f_{\epsilon}(Q_{\infty}) w)^{\prime}]| = 0$$

- We use the assumption $Q_{\infty}$ is bounded, and we approximate $f_{\epsilon}$ by a polynomial $P$ on $[-\|Q_{\infty}\| - 1, \|Q_{\infty}\| + 1]$.

- For $|\tau_{N}[(w^{*} P(Q_{N}) w)^{\prime}] - \tau[(w^{*} P(Q_{\infty}) w)^{\prime}]|$, we can use the assumption of convergence in $*$-joint moments.

- For $|\tau_{N}[(w^{*} f_{\epsilon}(Q_{N}) w)^{\prime}] - \tau_{N}[(w^{*} P(Q_{N}) w)^{\prime}]|$, we need additional estimate.
Future perspective

- Positivity of nc rational functions evaluated in free random variables (Cf. Helton’02)

- Other analytic properties of $\mu_r(X)$ (e.g. absolute continuity)

- The case where some variables are commuting (normal operators, $\epsilon$-free, bi-free).
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Thank you for your attention!