NON-CROSSING PARTITIONS

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Abstract. Non-crossing partitions have been a staple in combinatorics for quite some time. More recently, they have surfaced (sometimes unexpectedly) in various other contexts from free probability to classifying spaces of braid groups. Also, analogues of the non-crossing partition lattice have been introduced. Here, the classical non-crossing partitions are associated to Coxeter and Artin groups of type $A_n$, which explains the tight connection to the symmetric groups and braid groups. We shall outline those developments.

1. The poset of non-crossing partitions

A partition $p$ of a set $U$ is a decomposition of $U$ into pairwise disjoint subsets $B_i$:

$$U = \biguplus_i B_i$$

The subsets $B_i$ are called the blocks of the partition $p$. Another way to look at this is to consider $p$ as an equivalence relation on $U$. In this perspective, the subsets $B_i$ are the equivalence classes. Let $q$ be another partition of the same set $U$. We say that $q$ is a refinement of $p$ if each block of $q$ is contained in a block of $p$. In terms of equivalence relations, if two elements of $U$ are $q$-equivalent, they are also $p$-equivalent. We also say that $q$ is finer than $p$ or that $p$ is coarser than $q$; and we write $q \preceq p$.

Let $P(U)$ be the set of all partitions on the underlying set $U$. The refinement relation $\preceq$ is a partial order on the set $P(U)$, which is therefore a poset. Moreover, it is a lattice, i.e., every non-empty finite subset $\mathcal{P} \subseteq P(U)$ has a least upper bound and a greatest lower bound. We remark that the partition lattice is complete, i.e., even arbitrary infinite subsets have least upper and greatest lower bounds.

Remark 1.1. It is interesting that the definition of a complete lattice can be weakened by breaking the symmetry between upper and lower bounds. If a poset has upper bounds and greatest lower bounds, it is already a complete lattice (i.e. it also has lowest upper bounds).

Sketch of proof. Let $\mathcal{P}$ be a non-empty subset of the poset. We consider the set $B^+(\mathcal{P})$ of all common upper bounds for the non-empty subset $\mathcal{P}$. Since the poset has upper bounds, $B^+(\mathcal{P})$ is non-empty. Hence it has a greatest lower bound, which turns out to be the lowest upper bound of $\mathcal{P}$. □

Consider the following reflexive and symmetric relations on $U$:

$$x \sim y \quad \Leftrightarrow \quad \exists p \in \mathcal{P} : x \text{ and } y \text{ are } p\text{-equivalent}$$

$$x \approx y \quad \Leftrightarrow \quad \forall p \in \mathcal{P} : x \text{ and } y \text{ are } p\text{-equivalent}$$
It is clear that $\approx$ is itself an equivalence relation. It corresponds to the meet $\bigwedge P$ of the partitions in $P$, i.e., the greatest lower bound of $P$. The transitive closure of $\sim$ is an equivalence relation, which corresponds to the join $\bigvee P$ of the partitions in $P$.

Now, we restrict our consideration to finite sets. For a natural number $m \in \mathbb{N}$, let us denote by $[m]$ the set $\{1, 2, 3, \ldots, m\}$. We fix the natural cyclic ordering on $[m]$ and represent its elements as the vertices $v_1, \ldots, v_m$ of a regular $m$-gon inscribed in the unit circle. Let $p$ be a partition of $[m]$. We say that two blocks $B$ and $B'$ of the partition $p$ cross if their convex hulls intersect. The partition $p$ is called non-crossing if its blocks pairwise do not cross. A non-crossing partition can thus be depicted by colouring the convex hulls of its blocks. For blocks of size one or two, we fatten up the convex hull. It is clear from the visualisation that the complements of the coloured regions also are pairwise disjoint. This gives rise to the Kreweras complement. Here, we put dual vertices $w_1, \ldots, w_m$ within the arcs $v_i - v_{i+1}$. There is no natural numbering, and we choose to place $w_1$ within the arc from $v_1$ to $v_2$. Let $p$ be a non-crossing partition. Two dual vertices lie in the same block of the complement $p^c$ if they lie within the same complementary region of the convex hulls of blocks of $p$. The set $\text{NC}(m)$ of all non-crossing partitions of $[m]$ is partially ordered with respect to refinement. It is thus a subposet of the
set of all partitions of \([m]\). It turns out that \(\text{NC}(m)\) is also a lattice. This is clear from Remark 1.1 since greatest lower bounds are inherited from the partition lattice and upper bounds exist trivially since the trivial partition with a single block is noncrossing.

However, the noncrossing partition lattice is not a sublattice of the whole partition lattice: the join operation in both structures differ, i.e., the finest partition coarser than some given non-crossing partitions does not need to be non-crossing; see Remark 1.3 for a counterexample.

The complement map

\[
\text{NC}(m) \longrightarrow \text{NC}(m)
\]

\[
p \longmapsto p^c
\]

is an anti-automorphism of the lattice \(\text{NC}(m)\): it reverses the refinement relation and interchanges the roles of meet and join. In the picture, taking the Kreweras complement twice seems to get you back to the original partition. This is true; however, the indexing of the vertices shifts by one. Thus, the square of the Kreweras complement is given by cyclically rotating the element of the underlying set \(\{1, \ldots, m\}\).

The bottom (finest) element \(\bot\) of \(\text{NC}(m)\) is the partition with \(m\) blocks, each of size one. The top (coarsest) element \(\top\) of \(\text{NC}(m)\) is the partition with a single block. For each non-crossing partition \(p\), we define its rank \(\text{rk}(p)\) in terms of its number of blocks:

\[
\text{rk}(p) := m - \#\{\text{blocks of } p\}
\]

For any non-crossing partition \(p\), all maximal chains from the bottom element \(\bot\) to \(p\) have the same length, which coincides with the rank \(\text{rk}(p)\). Let us summarise the properties and non-properties of the poset of non-crossing partitions:

**Fact 1.2.** The set \(\text{NC}(m)\) of non-crossing partitions of an \(m\)-element is partially ordered by refinement. This poset is a lattice and self-dual with respect to the Kreweras complement, i.e.,

\[
(p \land q)^c = p^c \lor q^c
\]

\[
(p \lor q)^c = p^c \land q^c
\]

for any two \(p, q \in \text{NC}(m)\).

The automorphism \(p \mapsto (p^c)^c\) has order \(m\).

All maximal chains from bottom to top have length \(m - 1\). For any non-crossing partition \(p\), there is a maximal chain from bottom to top going through \(p\). The non-crossing partition lattice is graded and one has

\[
m - 1 = \text{rk}(p) + \text{rk}(p^c)
\]

for any \(p\).

**Remark 1.3.** For \(m \geq 4\), the non-crossing partition lattice \(\text{NC}(m)\) is not a sublattice of the partition lattice: the join operations do not coincide. A counterexample for \(m = 4\) is \(p = \{\{1, 3\}, \{2\}, \{4\}\}\) and \(q = \{\{1\}, \{2, 4\}, \{3\}\}\). The join of these partitions in the partition lattice is \(\{\{1, 3\}, \{2, 4\}\}\) whereas the join in \(\text{NC}(4)\) is the top element. These two partitions also show that the non-crossing
partition lattice $NC(m)$ is not semi-modular, i.e., the following inequality does not hold for all partitions $p$ and $q$,

$$\text{rk}(p) + \text{rk}(q) \geq \text{rk}(p \lor q) + \text{rk}(p \land q).$$

Enumerative properties of the noncrossing partition lattice are well understood. Kreweras counted the number of non-crossing partitions.

**Fact 1.4** (see [42, Cor. 4.2]). For any $m$, we have

$$|NC(m)| = C_m$$

where $C_m = \frac{1}{m+1} \binom{2m}{m} = \frac{(2m)!}{m!(m+1)!}$ is the $m$th Catalan number.

Kreweras also determined the Möbius function for the lattice of non-crossing partitions. Recall that, for a finite poset $P$, the Möbius function $\mu : \{(u,v) \in P \times P \mid u \leq v\} \rightarrow \mathbb{Z}$ is defined by the following recursion:

$$\mu(u,u) = 1,$$

$$\mu(u,v) = - \sum_{u \leq w < v} \mu(u,w).$$

Note that the value $\mu(u,v)$ is completely determined by the isomorphism type (as a poset) of the interval $[u,v] := \{w \in P \mid u \leq w \leq v\}$.

**Fact 1.5** (see [42, Thm. 6] or [14, Cor. 3.2]). For the non-crossing partition poset $NC(m)$, the Möbius function satisfies

$$\mu(\perp, \top) = (-1)^{m-1} C_{m-1} = (-1)^{m-1} \frac{(2m-2)!}{(m-1)!m!}$$

Let $p$ be a non-crossing partition, and consider a non-crossing partition $q \preceq p$. Let $B$ be a block of $p$. The blocks of $q$ contained in $B$ may be thought of as a non-crossing partition of $B$. Thus, we have the following:

**Observation 1.6.** Let $p \in NC(m)$ be a non-crossing partition, and let $B_1, \ldots, B_k$ be its blocks. Then the order ideal $p_\preceq := \{q \in NC(m) \mid q \preceq p\}$ is isomorphic as a poset to the cartesian product $NC(B_1) \times \cdots \times NC(B_k)$.

Let $B'_1, \ldots, B'_{m-k+1}$ be the blocks of the Kreweras complement $p^c$. Since the complement is an antiautomorphism of the non-crossing partition lattice, the filter $p_\succeq := \{q \in NC(m) \mid q \succeq p\}$ is isomorphic as a poset to the cartesian product $NC(B'_1) \times \cdots \times NC(B'_{m-k+1})$.

For non-crossing partitions $p \preceq q$, the interval $[p,q]$ is the filter for $p$ within the order ideal of $q$. Hence, by combining the previous isomorphisms, we see that $[p,q]$ is isomorphic to the product $\prod_B NC(B)$ where $B$ ranges over the blocks of the “blockwise Kreweras complement” of $p$ in $q$.

Since the Möbius function is multiplicative with respect to cartesian products of posets, Observation 1.6 allows one to derive the values of $\mu(p,q)$ in terms of the blockwise complement of $p$ in $q$ from Kreweras’ formula (1).

**Remark 1.7.** To every poset $(P, \leq)$, one associates the order complex $\Delta(P, \leq)$. This is the simplicial complex $\Delta(P, \leq)$ whose vertices are the elements of $P$ and whose simplices are chains in $P$, i.e., non-empty subsets of $P$ on which $\leq$ is a total
order. By a theorem of P. Hall, one can interpret the Möbius function as the Euler characteristic of order complexes [50, Prop. 3.8.6],

$$\mu(u, v) = \chi(\Delta((u,v))), \quad \text{for } u < v.$$ 

Here \((u, v) := \{ w \in P \mid u < w < v \}\) is the open interval from \(u\) to \(v\).

A significant implication is that the Möbius function is invariant with respect to reversing the order relation: let \(\mu_{\leq}\) be the Möbius function of \((P, \leq)\) and let \(\mu_{\geq}\) be the Möbius function of the reversed poset \((P, \geq)\); then, we have

$$\mu_{\leq}(u, v) = \mu_{\geq}(v, u).$$

2. Non-crossing partitions in free probability

Classical probability spaces \((\Omega, F, P)\) can be reformulated using the commutative 

\(*\)-algebra \(A = L^\infty(\Omega, F, P)\) as follows. Real valued (bounded) random variables correspond to elements of \(A\) and their expectations are given by evaluation of the linear functional \(\varphi(a) := \int_\Omega adP\). The 'distribution' of a random variable \(a\) is the induced distribution \(\mu_a(A) := P(a^{-1}(A))\) and its \(k\)th moment is given by

$$\varphi(a^k) = \int_\Omega a^k dP = \int_\mathbb{R} x^k \mu_a(dx) = \int_\mathbb{R} x \mu_{a^k}(dx).$$

This construction admits the following non-commutative extension. Denote by \((M_d(\mathbb{C}), \text{tr})\) the space of \(d \times d\) complex matrices, together with the normalised trace and the usual matrix conjugation. Consider now the algebra of random matrices \(A := M_d(L^\infty(\Omega, F, P))\) together with the linear functional \(\varphi(a) := \int_\Omega \text{tr}(a)dP\).

This represents a genuine non-commutative \(*\)-probability space \((A, \varphi)\), which is a unital \(*\)-algebra over \(\mathbb{C}\) together with a unital and tracial positive linear functional \(\varphi : A \to \mathbb{C}\), that is

$$\varphi(1) = 1, \quad \varphi(a^*a) \geq 0, \quad \varphi(ab) = \varphi(ba), \quad \text{for all } a, b \in A.$$ 

Furthermore, we shall assume that \(\varphi\) is faithful, that is \(\varphi(a^*a) = 0\) is equivalent to \(a = 0\). See the survey [51].

Many constructions in non commutative probability are parallel to those in classical probability, and this is also reflected in the notation: If \(a\) is a self-adjoint
element in $A$, i.e. $a^* = a$, the value $\varphi(a)$ is sometimes called the expectation of $a$, the values $\varphi(a^k)$, $k \in \mathbb{N}$, are called the moments of $a$, and the compactly supported probability measure $\mu_a$ on $\mathbb{R}$ with $\int x^k \mu_a(dx) = \varphi(a^k)$, $k \in \mathbb{N}$, is also called the distribution of $a$ which always exists for self-adjoint elements in a $C^*$-probability space. If the measure $\mu_a$ admits a density $f_a$, the latter is also called the density of $a$. Similarly, given two self-adjoint elements $a$ and $b$ in $A$, the joint moments of $a$ and $b$ are given by the values $\varphi(w)$, $w$ being a “word” in $a$ and $b$.

Recall that a compactly supported Borel measure $\mu$ on $\mathbb{R}$ (and more generally any $\mu$ with $\int e^{cz} \mu(dx)$ locally analytic around $z = 0$) is uniquely characterised by its moments $\int x^n \mu(dx)$ since then the Fourier transform of $\mu$ is a convergent power series with coefficients given by the moment sequence.

In order to define a corresponding notion of independence for self-adjoint elements (like that for random variables in classical probability theory), recall that two random variables $a, b \in L^\infty(\Omega,\mathcal{F},\mathbb{P})$ endowed with expectation $\varphi$ as above are independent, if $\varphi(a^k b^l) = \varphi(a^k) \varphi(b^l)$ or equivalently

$$\varphi((a^k - \varphi(a^k))(b^l - \varphi(b^l))) = 0$$

for all $k, l \in \mathbb{N}_0$.

Let $A_1$ and $A_2$ denote unital sub-algebras in $A$, for instance generated by elements $a$ and $b$ respectively. They are called ‘free’ if the expectations of all products with factors alternating between elements from $A_1$ and $A_2$ vanish whenever the expectations of all factors vanish. Hence the elements $a, b \in A$ are called free if

$$\varphi((a^{j_1} - \varphi(a^{j_1}))(b^{k_1} - \varphi(b^{k_1})) \cdots (a^{j_m} - \varphi(a^{j_m}))(b^{k_m} - \varphi(b^{k_m}))) = 0$$

for all $m \in \mathbb{N}$ and all $j_1, \ldots, j_m, k_1, \ldots, k_m \in \mathbb{N}$. Hence for $m = 1$ this rule for the evaluation of joint moments coincides with the classical rule $\varphi(ab) = \varphi(a) \varphi(b)$ but is apparently different for $m > 1$. The rules (3) as well as (2) allow to reduce by induction the evaluation of joint moments $\varphi(a^{j_1}b^{k_1} \cdots a^{j_m}b^{k_m})$ of these free or independent elements to the moments $\varphi(a^j)$ and $\varphi(b^k)$, which determine the marginal distribution of $a$ resp. $b$. Thus freeness may be regarded as a (non-commutative) analogue of the notion of independence in classical probability theory, allowing the development of a free probability theory. In particular (3) allows to compute the expectation of $\varphi((a + b)^n)$ for any $n \in \mathbb{N}$, $a \in A_1$ and $b \in A_2$, thus determining the distribution in the sense described above of the ‘free’ sum of $a$ and $b$ via the moments of $a$ and $b$ only. Hence, this assigns to compactly supported measures $\mu, \nu$ (with moments given by those of $a, b$) a free additive convolution $\mu \boxplus \nu$, see the survey [51]. This notion may be considered as an asymptotic limit of a corresponding notion for sequences of random matrices with independent entries of increasing dimension and their limiting spectral measures, [5, Chapter 1].

More generally, a set of unital sub-algebras $A_j \subset A$, $j \in I$, indexed by a set $I$, is called free if for any integer $k$ and $a_j \in A_{i_j}$, $j = 1, \ldots, k, i_j \in I$,

$$\varphi(a_1 \ldots a_k) = 0 \quad \text{provided that} \quad \varphi(a_j) = 0, \quad j = 1, \ldots, k,$$

and $i_1 \neq i_2, i_2 \neq i_3, \ldots, i_{k-1} \neq i_k$,

that is, all adjacent elements in $a_1 \ldots a_k$ belong to different sub-algebras $A_{i_j}$. This notion has similar properties as classical independence. For instance, polynomials $P(a_j)$ of free self-adjoint elements $a_j$ (generating a sub-algebra) are free again.
The density $\psi(x) = \frac{1}{\sqrt{2\pi}} \exp(-x^2/2)$ defines the standard Gaussian distribution. Hence, the classical central limit theorem (CLT) may be stated for independent random elements $a_i, i \in \mathbb{N}$ from a commutative $C^*$-probability space $(A, \varphi)$ with identical distribution such that $\varphi(a_i) = 0, \varphi(a_i^2) = 1$ (such variables are called standardised).

**Theorem 2.1** (Commutative $C^*$-version of CLT). The moments of the normalised sum $S_N := \frac{a_1 + \ldots + a_N}{\sqrt{N}}$ satisfy

$$\lim_{N \to \infty} \varphi(S_N^k) = \int x^k \psi(x) \, dx, \quad k \in \mathbb{N}. \quad (5)$$

Consider free random elements $a_i$ from a (non-commutative) $C^*$-probability space $(A, \varphi)$ , standardized via $\varphi(a_i) = 0, \varphi(a_i^2) = 1$ with identical distribution, that is $\varphi(a_i^j)$ depends on $l$ only. In order to describe a corresponding free `central limit theorem' for this setup we have to determine the asymptotic behaviour of moments of type $\varphi(a_1 \ldots a_k)$ subject to the assumption of freeness (4).

Note that by freeness all mixed moments vanish provided an element $a_j$ occurs only once in the product vanish. (Note that this holds as well for mixed moments of independent random variables). Thus, we only need to consider mixed moments with factors occurring at least twice. For a product $a_1 \ldots a_k$ of $k$ factors, such that $s$ of them, say $b_1, \ldots, b_s$, are different, let $p = \{B_1, \ldots, B_s\}$ denote the corresponding partition of the set $\{1, \ldots, k\}$ into $|p| := s$ nonempty blocks $B_j$ of the positions of $b_j$ in $1 \leq j \leq k$.

One can show by induction that all mixed moments of free or independent elements $\varphi(a_{i_1} a_{i_2} \ldots a_{i_k})$ where $1 \leq i_j \leq N$, can be computed via (4) resp. (2) as above also for $s \geq 2$ in terms of moments $c_l = \varphi(b_j)$ for $j = 1, \ldots, s$ which depend on $l$ only by the assumption of identical distribution. Thus these mixed moments depend on the partition scheme of $i_1, \ldots, i_k$, say $p$, only and will be denoted by $m_p$. The number of such mixed moments in $a_1, \ldots, a_N$ corresponding to a given partition scheme depends on $|p|$ only and is given by $A_{N,p} = N(N-1) \cdots (N - |p| + 1)$.

Thus

$$\varphi(S_N^k) = \sum_p m_p A_{N,p} N^{-k/2}. \quad (6)$$

For a partition $p$ we have $A_{N,p} < N^{|p|}$. If all parts of $p$ satisfy $|B_j| \geq 2$ and one block is of size at least three, the corresponding contribution in (6) is of order $|m_p| A_{N,p} N^{-k/2} \leq |m_p| N^{-1/2}$, that is all these terms are asymptotically negligible as $N$ tends to infinity.

Hence, computing the asymptotic limit of $\varphi(S_N^k)$ reduces to considering all mixed moments of $k$ factors with each random element occurring precisely twice, a consequence being that $\lim_{N \to \infty} \varphi(S_N^k) = 0$ for $k$ odd.

Recall that $\text{NC}(n)$ denoted the lattice of all non-crossing partitions on the set $[n] = \{1, \ldots, n\}$. Furthermore, let $\text{NC}_2(2k)$ denote the subset of non-crossing partitions with blocks of size 2 only, called 'non-crossing pair partitions' on a set of $2k$ elements.

Now consider as an example three free standardised variables $a, b, c$. Then the product $abc^2ab$ corresponds to a pair partition with a crossing, that is $p = \{\{1, 5\}, \{3, 4\}, \{2, 6\}\}$. Hence $\varphi(abc^2ab) = \varphi(abab) \varphi(c^2) = 0$ by freeness, that is (3). Otherwise for a non-crossing pair partition like $ca^2b^2c$ we have $\varphi(ca^2b^2c) =$
\( \varphi(b^2c)\varphi(a^2) = \varphi(cc)\varphi(b^2) = 1 \). These simple observations can be generalised by induction in the following Lemma to determine the values of joint moments 

\[ m_p = \varphi(a_1, a_2, \ldots, a_k) \]

for pair partitions \( p \) of free variables.

**Lemma 2.2.** For any pair partition \( p \),

\[
m_p = \begin{cases} 
0 & \text{if } p \text{ has a crossing} \\
1 & \text{if } p \text{ is non-crossing.}
\end{cases}
\]

Thus, we conclude from (6) and the previous results that

\[
\lim_{N \to \infty} \varphi(S_{Nk}^2) = \lim_{N \to \infty} \sum_{p \in \text{NC}_2(2k)} \frac{A_{N,p}}{N^{k/2}} = |\text{NC}_2(2k)|.
\]

Furthermore, one shows that

\[
C_k := |\text{NC}_2(2k)| = |\text{NC}(k)|,
\]

where \( C_k = \frac{1}{k+1} \binom{2k}{k} \) is the \( k \)th Catalan number. Among its numerous interpretations, it represents as well the \( 2k \)th moment of a compactly supported measure with density \( w(x) := \frac{1}{2\pi} (4 - x^2)^{1/2}, |x| \leq 2 \). This is the so-called Wigner measure or semi-circular distribution. See [45, Rem. 9.5].

Now the free central limit theorem for a sequence of free variables \( a_j, j \in \mathbb{N} \), which are standardised via \( \varphi(a_j) = 0, \varphi(a_j^2) = 1 \), and \( S_N := \frac{a_1 + \cdots + a_N}{\sqrt{N}} \) may be stated as follows.

**Theorem 2.3** (Free Central Limit Theorem). \( S_N \) converges in distribution to \( w \) which serves as the Gaussian distribution in free probability, i.e.

\[
\lim_{N \to \infty} \varphi(S_N^k) = \int x^k w(x) dx, \quad k \in \mathbb{N}.
\]

This means e.g. that the rescaled sum \((a_1 + a_2)/\sqrt{2}\) of two free elements \( a_1, a_2 \) of a non-commutative probability space \((\mathcal{A}, \varphi)\) which both have density \( w(x) \) again has a Wigner distribution. In free probability an element \( s \) of \((\mathcal{A}, \varphi)\) with density \( w(x) \) is called semi-circular and its moments are given by

\[
\varphi(s^n) = \begin{cases} 
\frac{1}{k+1} \binom{2k}{k}, & \text{if } n = 2k, \\
0, & \text{if } n \text{ odd}.
\end{cases}
\]

Recall that \( a \in (\mathcal{A}, \varphi) \) is called positive if there exists an \( c \in (\mathcal{A}, \varphi) \) with \( a = c^*c \). Thus \( a \) is self-adjoint. Define the free multiplicative convolution of two compactly supported measures \( \mu_a, \mu_b \), of positive free elements \( a, b \in (\mathcal{A}, \varphi) \), say \( \mu_a \boxtimes \mu_b \), as follows by specifying its moments. Since in a \( C^* \)-probability space \( \mathcal{A} \) positive square roots \( a^{1/2} \) resp. \( b^{1/2} \) of \( a \) resp. \( b \) as well as the positive element \( p_{a,b} := a^{1/2} b a^{1/2} \) are again in \( \mathcal{A} \), we may define \( \mu_a \boxtimes \mu_b \) by:

\[
\int x^k \, d\mu_a \boxtimes \mu_b(x) := \varphi(p_{a,b}^k), \quad k \in \mathbb{N}.
\]

Since \( \varphi(p_{a,b}^k) = \varphi(p_{b,a}^k), \) \( k \in \mathbb{N} \), because \( \varphi \) is tracial, i.e. \( \varphi(ba) = \varphi(ab) \), we conclude that the free convolution \( \boxtimes \) is commutative. By the same tracial property and the relation of freeness, we show that \( \varphi(p_{a,b}^k) = \varphi((ab)^k) \) and this implies the associativity of \( \boxtimes \). Moreover it follows from this representation that the multiplicative
In order to effectively compute both additive and multiplicative convolution of measures, one needs more properties of the lattice of partitions of \(1, \ldots, n\) into blocks and the subset of non-crossing partitions together with the notion of multi-linear cumulant functionals. As above let \(B_j, j = 1, \ldots s\) denote the blocks of a partition \(p \in \text{NC}(n)\) of \(1, \ldots, n\).

For \(p \in \text{NC}(n)\), the free mixed cumulants are multi-linear functionals \(\kappa_p : \mathcal{A}^n \to \mathbb{C}\) defined in terms of a moment decomposition using the Möbius function \(\mu(q, p)\) of the lattice of non-crossing partitions \(\text{NC}(n)\). We define the general mixed cumulant functionals \(\kappa_p\) as follows:

\[
(11) \quad \kappa_p[a_1, \ldots, a_n] = \sum_{q \in \text{NC}(n), p \preceq q} \varphi_q[a_1, \ldots, a_n] \mu(p, q), \quad \text{where}
\]
and the products \(\prod_{k \in B_j} a_k\) repeat the order of indices within the block \(B_j\). Note that by Hall’s theorem, the coefficient \(\mu(p, q)\) can also be written as \(\mu_{\preceq}(q, p)\) using the relation of reversed refinement (see Remark 1.7).

Then one shows, see [45, Prop. 11.4], that

\[
(12) \quad \varphi(a_1 \cdots a_n) = \sum_{p \in \text{NC}(n)} \kappa_p[a_1, \ldots, a_n].
\]

In the special case \(p = 1_n\) we write \(\kappa_n\) instead of \(\kappa_{1_n}\). The following lemma is proved by induction on \(n\).

**Lemma 2.4** ([45, Thm 11.20]). The elements \(a_1, \ldots, a_n \in \mathcal{A}\) are free if and only if all mixed cumulants satisfy

\[
\kappa_n[a_{j_1}, \ldots, a_{j_k}] = 0,
\]
whenever \(a_{j_1}, \ldots, a_{j_k}, 1 \leq j_i \leq n, 1 \leq k \leq n\) contains at least two different elements.

In contrast to (4), this characterisation of freeness holds even if the \(\varphi(a_j)\) are non-zero.

For a partition \(p \in \text{NC}(n)\), recall that \(p^c\) denotes its Kreweras complement in \(\text{NC}(n)\). Then, one shows that for free elements \(a, b\) the following recursion involving the Kreweras complement holds:

\[
(13) \quad \kappa_n[ab, \ldots, ab] = \sum_{p \in \text{NC}(n)} \kappa_p[a, \ldots, a] \kappa_{p^c}[b, \ldots, b].
\]

See [45, Rem. 14.5]. This entails that the cumulants of \(ab\) and thus by (12) the moments of \(ab\) are indeed determined by multi-linear functionals of \(a\) and \(b\) alone which again by virtue of (11) are determined by the moments of \(a\) together with the moments of \(b\).

The recursive equation (13) and the definition (11) of cumulants may be conveniently encoded as algebraic relations between the following formal generating series. For \(a \in \mathcal{A}\) let \(M_a(z) = \sum_{n=1}^{\infty} \varphi(a^n) z^n\) denote the moment generating series and with \(\kappa_n(a) := \kappa_n[a, \ldots, a]\) let \(R_a(z) := \sum_{n=1}^{\infty} \kappa_n(a) z^n\) and \(R_n(z) := z^{-1} R_a(z)\) denote cumulant generating series. In particular, for free self-adjoint \(a, b \in \mathcal{A}\) we get
by binomial expansion of $\kappa_n(a+b)$ and Lemma 2.4 that $\kappa_n(a+b) = \kappa_n(a) + \kappa_n(b)$ and furthermore, as shown in [45, Lect. 12].

**Lemma 2.5.** One has the following identities:

\[
R_{a+b}(z) = R_a(z) + R_b(z),
\]

(15)

\[
R_a(z M_a(z) + z) = M_a(z), \quad G_a \left( \frac{1 + R_a(z)}{z} \right) = z,
\]

where

\[
G_a(z) := \frac{1}{z} + \sum_{n=1}^{\infty} \frac{\varphi(a^n)}{z^{n+1}} = \frac{1}{z} \left( 1 + M_a \left( \frac{1}{z} \right) \right),
\]

\[
S_a(z) := \frac{1}{z} R_a^{(-1)}(z) = \frac{1 + z}{z} M_a^{(-1)}(z),
\]

\[
S_{ab}(z) = S_a(z)S_b(z)
\]

Since $S_a$ is determined by the spectral measure of $a$, this means with $S_{\mu_a} := S_a$ for measures $\mu = \mu_a$, $\nu = \mu_b$ we have $S_{\mu \ast \nu}(z) = S_{\mu}(z)S_{\nu}(z)$, which uniquely determines the multiplicative free convolution $\mu \boxplus \nu$ in terms of the measures $\mu$ and $\nu$ on the positive reals via the characterising property of the $S$-transform.

Note that by (9), Let $s$ be a semi-circular element as in (9). Then the moment generating functions of $s$ and $s^2$ are given by $M_s = f(z^2)$ and $M_{s^2}(z) = f(z)$ respectively, where $f(z) = \left( 1 - \sqrt{1 - 4z} \right) / (2z) - 1$. The corresponding distribution of $s^2$ is called Marchenko-Pastur or free Poisson law; it is given by the density $p(x) := \frac{1}{2\pi} \sqrt{4/\sqrt{x} - 1}$ on the interval $[0, 4]$. Via the inverse function $f^{(-1)}(z) = z(1 + z)^{-1}$ of $f$ we obtain in view of (16),

\[
S_a(z) = f^{(-1)}(z) \frac{1 + z}{z} = \frac{1}{1 + z}
\]

and hence in view of (16) again $R_a^{(-1)}(z) = \frac{1}{1 + z}$ or $R_{s^2}(z) = \frac{z}{1 + z}$, whereas from (15) we deduce with $g(z) := z(1 + M_a(z))$ and $g^{(-1)}(z) = \frac{z}{1 + z}$, and hence $R_a(z) = \frac{z}{g^{(-1)}(z)} - 1 = z^2$. 

Hence the so-called $R$-transform $R$ of a spectral measure $\mu_a$, introduced by Voiculescu in [51], is determined analytically by the inverse function of the Cauchy transform of $\mu_a$ on the complex plane which is the starting point of the complex analytic theory of the asymptotic approximations of free additive convolution as developed in [23, 21, 22, 24]. Assuming that $\kappa_1 = m_1 \neq 0$, $R_{\mu_a}(z) := R_a(z)$ admits a formal inverse power series $R_a^{(-1)}(z)$. This may be defined via the inverse function of the Cauchy transform of $\mu_a$, which is well defined in a certain region in $\mathbb{C}$.

The so-called $S$-transform

\[
S_a(z) := \frac{1}{z} R_a^{(-1)}(z) = \frac{1 + z}{z} M_a^{(-1)}(z),
\]

of Voiculescu is a multiplicative homomorphism for free multiplicative convolution. That is, see [45, Lect. 18], one has the following result.

**Lemma 2.6.** For two free self-adjoint positive elements $a, b \in \mathcal{A}$, one has

(17)

\[
S_{ab}(z) = S_a(z)S_b(z)
\]
From here, we obtain for free variables $t_1, \ldots, t_l$ with identical distribution given by $s^2$, the so-called Marchenko–Pastur distribution, in view of (18)

\begin{equation}
S_{t_1, \ldots, t_l}(z) = S_{t_1}(z)^l = \frac{1}{(1 + z)^l},
\end{equation}

which determines the so-called free Bessel distributions, $\mu_t$ with support in $[0, K_t]$, $K_t = (l+1)^{l+1}/l!$. Their moments are given by the so-called Fuss–Catalan numbers, that is, if an element $a \in A$ has $S$-transform $S_a(z) = \frac{1}{(1+z)^l}$ we have

\begin{equation}
\varphi(a^k) = \frac{1}{lk + 1} \binom{lk + 1}{k} = C_{k,l}, \quad \text{for all } k \geq 1.
\end{equation}

The proof is based on combinatorial properties of non crossing partitions, see [6].

**Proposition 2.7.** For a sequence of $N \times N$ independent non-Hermitian random matrices, $G_1, \ldots, G_l$, with independent Gaussian centered entries with variance $1/N$, let $W := G_1 \cdots G_l$. Consider the normalised moments of $WW^*$. As $N \to \infty$ they converge as follows:

\begin{equation}
\lim_{N \to \infty} \frac{1}{N} \int_{\Omega} \text{tr}(WW^*)^l d\mathcal{P} = \int_0^{K_l} x^k d\mu_l = C_{k,l}
\end{equation}

This can be shown by induction, using

\begin{equation}
\text{tr}(WW^*)^k = \text{tr}(G_1(G_2 \cdots G_l G_1^* \cdots G_1^*)^{l-1} G_2 \cdots G_l G_1^* \cdots G_1^*),
\end{equation}

which by moving $G_1$ to the right yields

\[
\text{tr}((G_2 \cdots G_l G_1^* \cdots G_1^* G_1)^{l-1} G_2 \cdots G_l G_1^* \cdots G_1^* G_1) = \text{tr}((G_2 \cdots G_l G_1^* \cdots G_1^* G_1)^l) = \text{tr}((G_2 \cdots G_l G_1^* \cdots G_1^* G_1)^l).
\]

Since $(G_2 \cdots G_l G_1^* \cdots G_1^*)$ and $G_1^* G_1$ are asymptotically free of this volume, we get by induction for the asymptotic distribution of $\pi_l$ the recursion $\pi_l = \pi_{l-1} \boxtimes \pi_1$, where $\pi_1$ can be identified with the limiting Marchenko–Pastur distribution of $G_1 G_1^*$. For arbitrary $N \times N$ independent Wigner matrices (which are Hermitian matrices with entries which are independent random variables unless restricted by symmetry) the relation (21) has been shown by combinatorial techniques after an appropriate regularization in [2]. For more details on the asymptotic spectral distribution of products of so-called Girko–Ginibre matrices (having independent and identically distributed random entries) and their inverses using the free probability calculus, see [31]. Strictly speaking one needs to extend the non-commutative $C^*$-probability spaces to spaces of unbounded operators to include distributions with non-compact support like those of Gaussian matrices see e.g. [23].

Remarkably, the same results hold for powers instead of products. Since $G_1^{-1}(G_1^{-1})^*$ and $G_1^* G_1$ are also asymptotically free, a similar argument as above shows that the asymptotic distribution of $(G_1^*)^l(G_1^*)^*$ is also given by $\pi_l$. Similarly as above, these results also extend to powers of non-Gaussian random matrices.

The calculus of $S$-transforms may even be used to describe the asymptotic spectral measure of $WW^*$ when some of the factors in $W = G_1 \cdots G_l$ are inverted, after appropriate regularisation of the inverse matrices [31]. For instance, for
$W = G_1 G_2^{-1}$, the limiting distribution of $WW^*$ is given by the square of a Cauchy distribution.

Moreover, the calculus of $R$-transforms makes it possible, at least in principle, to deal with the case where $W$ is a sum of independent products as above [41]. For instance, for $W = G_1 G_2^{-1} + G_3 G_4^{-1}$, the limiting distribution of $WW^*$ is also given by the square of a Cauchy distribution. This is related to the Cauchy distribution being “stable” under free additive convolution.

3. Braid groups

Let $\mathbb{D}$ be the unit disk. The braid group $B_n$ on $n$ strands can be defined as the fundamental group of the configuration space

$$X_n := \{ \{ z_1, \ldots, z_n \} \subset \mathbb{D} \mid z_i \neq z_j \text{ for } i \neq j \}$$

of unordered $n$-point-subsets in $\mathbb{D}$. One can visualize a path in $X_n$ as a collection of $n$ distinct points moving continuously in $\mathbb{D}$ subject only to the restriction that points are not allowed to collide. Since $X_n$ is connected, the braid group (up to isomorphism) does not depend on the choice of a base point.

We find it convenient to choose as the base point a set $S = \{ v_1, \ldots, v_n \}$ of $n$ points on the boundary circle $\partial \mathbb{D}$ numbered in counter-clockwise order.

Then, we regard $NC(n)$ as the poset of non-crossing partitions of the set $S$, i.e., for any two distinct blocks of the partition, their convex hulls do not intersect. A non-crossing partition $p \in NC(n)$ can be interpreted as a braid on $n$ strands as follows: for each block $B = \{ v_{\alpha_1}, \ldots, v_{\alpha_k} \}$, consider the counter-clockwise rotation of the block by one step:

$$\varrho_B : v_{\alpha_1} \mapsto v_{\alpha_2} \mapsto \cdots \mapsto v_{\alpha_k} \mapsto v_{\alpha_1}$$

The product

$$\sigma_p := \prod_{B : \text{block of } p} \varrho_B$$

describes a loop in the configuration space $X_n$, which does not depend (up to homotopy relative to the basepoint) on the order of factors. We identify it with the corresponding element of the fundamental group $B_n$.

**Figure 4.** The path (braid) that is associated to the partition $\{ \{ 1 \}, \{ 2, 6, 7, 8 \}, \{ 3, 5 \}, \{ 4 \} \}$. 
Fact 3.1. The braid group $B_n$ is generated by the braids $\sigma_i$ corresponding to the counter-clockwise rotations $v_i \mapsto v_{i+1} \mapsto v_i$ for $i = 1, \ldots, n - 1$.

In terms of these generators, the braid group $B_n$ admits the following presentation:

$$B_n = \langle \sigma_1, \ldots, \sigma_{n-1} \mid \sigma_i \sigma_j = \sigma_j \sigma_i \text{ for } |i - j| \geq 2, \sigma_i \sigma_j \sigma_i = \sigma_j \sigma_i \sigma_j \text{ for } |i - j| = 1 \rangle$$

There is an obvious homomorphism

$$\pi : B_n \rightarrow S_n$$

from the braid group on $n$ strands to the symmetric group on $n$ letters. A braid corresponds to a motion of the $n$ points $v_1, \ldots, v_n$, and at the end of this motion, the dots may have changed positions. This way, each braid induces a permutation.

Fact 3.2. The homomorphism $\pi : B_n \rightarrow S_n$ is onto. On the level of presentations, it amounts to making the generators $\sigma_i$ involutions. Formally: the symmetric group has the presentation

$$S_n = \langle s_1, \ldots, s_{n-1} \mid s_i s_j = s_j s_i \text{ for } |i - j| \geq 2, s_i s_j s_i = s_j s_i s_j \text{ for } |i - j| = 1 \rangle$$

and the homomorphism $\pi$ is sending $\sigma_i$ to $s_i$.

Strand diagrams are another frequently used visual representation of braids. Recall that a braid is given by a path in configuration space, i.e. the simultaneous motion of $n$ points in the disk $\mathbb{D}$. Parametrizing time by a real number in $[0, 1]$, each of those moving points traces out a “strand” in $\mathbb{D} \times [0, 1]$. The diagrams we have used so far can be regarded as a “top view” onto the cylinder $\mathbb{D} \times [0, 1]$. A strand diagram is a view from the front. Here, it is useful to put the initial configuration $U$ with the hemicircle fully visible from the front. Figure 5 shows the two representations of the generator $\sigma_2$ in $B_5$. Here, the generator $\sigma_i$ corresponds to a crossing of the $i^{th}$ and the $(i+1)^{th}$ strands. The left strand runs over the right strand. We call such a crossing positive. The inverses of the generators correspond to negative crossings.

![Figure 5](image_url)
3.1. A classifying space for the braid group. Tom Brady [16] has given a construction of a classifying space for braid groups that is strongly related to non-crossing partitions and has found some interesting applications.

Recall that the Cayley graph $\text{CG}_\Sigma(G)$ of a group $G$ relative to a specified generating set $\Sigma$ is the graph with vertex set $G$ and edges connecting $g$ to $gx$ for any $g \in G$ and $x \in \Sigma \setminus \{1\}$. Note that the requirement $x \neq 1$ rules out loops. Obviously, there is more structure here: the edge is oriented from $g$ to $gx$ and should be regarded as labeled by the generator $x$.

Observation 3.3. Since $\Sigma$ is a generating set for $G$, the Cayley graph $\text{CG}_\Sigma(G)$ is connected: if we can write an element $g$ as a word

$$g = x_1^{e_1} \cdots x_k^{e_k}$$

in the generators and their inverses, then

$$1 \rightarrow x_1^{e_1} \rightarrow x_1^{e_1}x_2^{e_2} \rightarrow x_1^{e_1}x_2^{e_2}x_3^{e_3} \rightarrow \cdots \rightarrow g$$

is an edge path connecting the identity element 1 to $g$. Note that the exponents of the generators tell us whether to traverse edges with or against their orientation. □

There are two generating sets for the braid group (and the symmetric group) of particular interest to us. First, we consider the digon generators $\sigma_{ij}$ corresponding to the counter-clockwise rotation $v_i \mapsto v_j \mapsto v_i$. Let $B_n^*$ be the Birman–Ko–Lee-monoid [13, Section 2], i.e., the monoid generated by all the $\sigma_{ij}$. We remark that $B_n^*$ is strictly larger than the submonoid of positive braids (those that can be drawn using positive crossings only), which is the monoid generated by the $\sigma_i$. We define a partial order on the braid group by:

$$\beta \leq \beta' :\iff \beta^{-1}\beta' \in B_n^*$$

The image $s_{ij} \in S_n$ of $\sigma_{ij}$ in the symmetric group is a transposition. Consider the Cayley graph of the symmetric group $S_n$ with respect to the generating set $T \subseteq S_n$ of all transpositions. We define a partial order, called the absolute order, on $S_n$ as follows: For permutations $\xi, \psi \in S_n$ we declare $\xi \preceq_T \psi$ if there is a geodesic (i.e., shortest possible) path in the Cayley graph connecting the identity 1 to $\psi$ and passing through $\xi$.

Our largest generating set is:

$$\Gamma_n := \{ \sigma_p \mid p \in \text{NC}(n) \} \subseteq B_n$$

which is in 1-1 correspondence to the non-crossing partition lattice. Let $s_p$ denote the image of $\sigma_p$ in the symmetric group $S_n$. It turns out that the subset $\{ s_p \mid p \in \text{NC}(n) \} \subseteq S_n$ is the order ideal of the $n$-cycle $1 \mapsto 2 \mapsto \cdots \mapsto n \mapsto 1$ with respect to the partial order $\preceq_T$ just defined, that is the subset consists of all elements in $S_n$ bounded above by the $n$-cycle. In fact, we have isomorphisms of various posets:

Fact 3.4 (see [12, 16]). Let $p, q \in \text{NC}(n)$. Then the following are equivalent:

1. $p \geq q$ in $\text{NC}(n)$,
2. In $\Gamma_n$, the element $\sigma_p$ is a left-divisor of $\sigma_q$, i.e., there exists $r \in \text{NC}(n)$ such that $\sigma_q = \sigma_p\sigma_r$,
3. In $\Gamma_n$, the element $\sigma_p$ is a right-divisor of $\sigma_q$, i.e., there exists $r \in \text{NC}(n)$ such that $\sigma_q = \sigma_r\sigma_p$. 


In the braid group $B_n$, we have $\sigma_p \leq \sigma_q$.

In the symmetric group $S_n$, we have $s_p \leq_T s_q$.

Thus, on $\Gamma_n$ the three partial orderings given by left-divisibility, right-divisibility, and the partial order $\leq$ from $B_n$ coincide. Moreover, we have isomorphisms

$$\text{NC}(n) \cong \{ \sigma_p : p \in \text{NC}(n) \} \cong \{ s_p : p \in \text{NC}(n) \}$$

of posets.

**Example 3.5.** Consider the non-crossing partitions

$$p := \{ \{1,2,8\}, \{3,5\}, \{4\}, \{6\}, \{7\} \} \quad \text{and} \quad q := \{ \{1,2,6,7,8\}, \{3,5\}, \{4\} \}$$

in $\text{NC}(8)$. Here, $p \leq q$ holds and we expect $\sigma_p$ to be a left- and right-divisor of $q$ within $\Gamma_8$. Figure 6 shows the corresponding factorisations. One can interpret the complementary divisors as the blockwise Kreweras complements. In particular, the Kreweras complement yields factorisations of the maximal element in $\Gamma_n$.

The braid group $B_n$ has a particularly nice presentation over the generating set $\Gamma_n$:

**Fact 3.6** ([16, Thm. 4.8]). *The valid equations*

$$\sigma_1 \sigma_2 = \sigma_3 \text{ for } \sigma_1, \sigma_2, \sigma_3 \in \Gamma_n \setminus \{1\}$$

*are a defining set of triangular relations for the braid group $B_n$ with respect to the generating set $\Gamma_n \setminus \{1\}$.*

Let $\tilde{\Gamma}_n$ be the Cayley graph of the braid group $B_n$ with respect to the generating set $\Gamma_n \setminus \{1\}$. A *clique* in $\Gamma_n$ is a set of vertices that are pairwise connected via an edge. As a directed graph, $\tilde{\Gamma}_n$ does not have oriented cycles and each clique is totally ordered by the orientation of edges. Thus, a clique is of the form

$$\{ \beta, \beta \sigma_{p_1}, \beta \sigma_{p_2}, \ldots, \beta \sigma_{p_k} \}$$
where \( p_1 < p_2 < \cdots < p_k \) is an ascending chain in \( \text{NC}(n) \), and \( \beta \in \mathcal{B}_n \) is some element. We denote by \( \tilde{Y}_n \) the simplicial complex of cliques (also known as the flag complex induced by the graph) in \( \Gamma_n \). In particular, \( \Gamma_n \) is the 1-skeleton of \( \tilde{Y}_n \).

**Observation 3.7.** All maximal chains in \( \text{NC}(n) \) have length \( n \). Hence, all maximal simplices in \( \tilde{Y}_n \) have dimension \( n \).

The most important fact about \( \tilde{Y}_n \) is its contractibility.

**Theorem 3.8** ([16, Thm. 6.9 and Cor. 6.11]). The clique complex \( \tilde{Y}_n \) is contractible, and the braid group \( \mathcal{B}_n \) acts freely on it. Consequently, the orbit space

\[
Y_n := \mathcal{B}_n \setminus \tilde{Y}_n
\]

is a classifying space for the braid group \( \mathcal{B}_n \).

**3.2. Higher generation by subgroups.** For a subset \( I \subseteq \{1, \ldots, n\} \) let \( \mathcal{B}_n^I \) be the subgroup of \( \mathcal{B}_n = \pi_1(X_n) \) given by those paths, where the points in \( \{v_i \mid i \in I\} \) do not move at all. For \( k \in \{1, \ldots, n\} \), we put \( \mathcal{B}_n^k := \mathcal{B}_n^{\{v_k\}} \), i.e., \( \mathcal{B}_n^k \) is the group of braids where the \( k \)th strand is rigid. It is, one might say, a group on \( n - 1 \) strands and one rod. However, since \( v_k \) is a point on the boundary \( \partial D \), braiding with the rod is impossible. Thus, \( \mathcal{B}_n^k \) really is just an isomorphic copy of \( \mathcal{B}_{n-1} \) inside of \( \mathcal{B}_n \). Similarly, \( \mathcal{B}_n^I = \bigcap_{k \in I} \mathcal{B}_n^k \) is isomorphic to \( \mathcal{B}_{n-\#I} \).

Let \( \text{NC}^k(n) \) be the lattice of those non-crossing partitions in \( \text{NC}(n) \) where the singleton \( \{k\} \) is a block. For a subset \( I \subseteq \{1, \ldots, n\} \), put \( \text{NC}^I(n) := \bigcap_{k \in I} \text{NC}^k(n) \). Then, \( \Gamma_n^I := \{ \sigma_p \mid p \in \text{NC}^I(n) \} \) is a generating set for \( \mathcal{B}_n^I \).

Note that the inclusion \( \mathcal{B}_n^I \to \mathcal{B}_n \) induces a bijection \( \Gamma_n - \#I \cong \Gamma_n^I \). Recall that \( \Gamma_n - \#I \) is a poset with respect to divisibility. A priori, there are two poset structures on \( \Gamma_n^I \): one from intrinsic divisibility with quotients again in \( \Gamma_n^I \) and one induced from the ambient poset \( \Gamma_n \), i.e., divisibility where quotients are allowed to be anywhere in \( \Gamma_n \). However, since \( \Gamma_n^I = \Gamma_n \cap \mathcal{B}_n^I \), the two poset structures coincide. Then, \( \Gamma_n - \#I \cong \Gamma_n^I \) is an isomorphism of posets.

Moreover, the order preserving bijection \( \{1, \ldots, n - \#I\} \to \{1, \ldots, n\} \setminus I \) induces an isomorphism \( \text{NC}(n - \#I) \cong \text{NC}^I(n) \). This isomorphism is compatible with the poset isomorphism from Fact 3.4, and we have a commutative square of poset isomorphisms:

\[
\begin{array}{ccc}
\Gamma_n - \#I & \cong & \Gamma_n^I \\
\cong & & \cong \\
\text{NC}(n - \#I) & \cong & \text{NC}^I(n)
\end{array}
\]

The identity \( \Gamma_n^I = \Gamma_n \cap \mathcal{B}_n^I \) has another consequence:

**Observation 3.9.** Let \( \tilde{Y}_n^I \) be the full subcomplex spanned by \( \mathcal{B}_n^I \) as a set of vertices in \( \tilde{Y}_n \). Then, \( \tilde{Y}_n^I \) is isomorphic to \( \tilde{Y}_n - \#I \), whence it is contractible by Theorem 3.8. For any coset \( \beta \mathcal{B}_n^I \), regarded as a set of vertices in \( \tilde{Y}_n \), the full subcomplex spanned by \( \beta \mathcal{B}_n^I \) is the translate \( \beta \tilde{Y}_n^I \) and also contractible.

**Observation 3.10.** Assume that two coset complexes \( \beta \tilde{Y}_n^I \) and \( \beta' \tilde{Y}_n^J \) intersect, say in \( \beta \). Then \( \beta \tilde{Y}_n^I = \bar{\beta} \tilde{Y}_n^I \) and \( \beta' \tilde{Y}_n^J = \bar{\beta} \tilde{Y}_n^J \). In this case, the intersection

\[
\bar{\beta} \tilde{Y}_n^I \cap \bar{\beta} \tilde{Y}_n^J = \bar{\beta} \tilde{Y}_n^{I \cup J}
\]
is contractible.

Let \( \mathcal{U} := (U_\alpha)_{\alpha \in A} \) be a family of sets. For a subset \( \sigma \subseteq A \) let

\[
U_\sigma := \bigcap_{\alpha \in \sigma} U_\alpha
\]
denote the associated intersection. The simplicial complex

\[
N(\mathcal{U}) := \{ \sigma \subseteq A \mid \emptyset \neq U_\sigma \}
\]
of all index sets whose associated intersection is non-empty is called the nerve of the family \( \mathcal{U} \). If \( \mathcal{U} \) is a family of subcomplexes in a CW complex, one has the following:

**Theorem 3.11** (Nerve Theorem, see [36, Cor. 4G.3]). Suppose \( \mathcal{U} = (U_\alpha)_{\alpha \in A} \) is a covering of a simplicial complex \( X \) by a family of contractible subcomplexes. Suppose further that, for each \( \sigma \in N(\mathcal{U}) \), the intersection \( U_\sigma \) is contractible. Then, the nerve \( N(\mathcal{U}) \) is homotopy equivalent to \( X \).

According to Observation 3.10, the Nerve Theorem applies in particular to the union:

\[
\tilde{X}_n := \bigcup_k \bigcup_{\beta \in B_n} \beta \tilde{Y}_n^k
\]

We deduce:

**Proposition 3.12.** The complex \( \tilde{X}_n \) is homotopy equivalent to the nerve \( N \) of the family

\[
\{ \beta \mathcal{B}_n^k \mid \beta \in \mathcal{B}_n, 1 \leq k \leq n \}
\]
of cosets.

This relates to higher generation by subgroups as defined by Abels and Holz.

**Definition 3.13** ([1, 2.1]). Let \( G \) be a group and let \( \mathcal{H} \) be a family of subgroups. We say that \( \mathcal{H} \) is \( m \)-generating for \( G \) if the coset nerve

\[
N_G(\mathcal{H}) := N(\{ gH \mid g \in G, H \in \mathcal{H} \})
\]
is \( (m-1) \)-connected.

From Proposition 3.12, we conclude immediately:

**Corollary 3.14.** The family \( \mathcal{B}_n := \{ \mathcal{B}_n^1, \ldots, \mathcal{B}_n^n \} \) is \( m \)-generating for the braid group \( B_n \) if and only if \( \tilde{X}_n \) is \( (m-1) \)-connected.

Recall that \( B_n \) acts freely on the simplicial complex \( \tilde{Y}_n \). The projection \( \tilde{Y}_n \to Y_n \) is a covering space map. In fact, \( \tilde{Y}_n \) is the universal cover of \( Y_n \) and the braid group \( B_n \) acts as the group of deck transformations. The subcomplex \( X_n \) is \( B_n \)-invariant. Let \( X_n \) be its image in \( Y_n \).

**Proposition 3.15.** The family \( \mathcal{B}_n := \{ \mathcal{B}_n^1, \ldots, \mathcal{B}_n^n \} \) is \( m \)-generating for the braid group \( B_n \) if and only if the pair \( (Y_n, X_n) \) is \( m \)-connected.

**Proof.** First, consider the long exact sequence of homotopy groups for the inclusion \( X_n \subseteq \tilde{Y}_n \):

\[
\cdots \to \pi_1(\tilde{X}_n) \to \pi_1(\tilde{Y}_n) \to \pi_1(\tilde{Y}_n, \tilde{X}_n) \to \pi_0(\tilde{X}_n) \to \pi_0(\tilde{Y}_n)
\]
Since $\tilde{Y}_n$ is contractible, we obtain isomorphisms:

$$\pi_{d+1}(\tilde{Y}_n, \tilde{X}_n) \cong \pi_d(\tilde{X}_n)$$

On the other hand, $\tilde{Y}_n \to Y_n$ is a covering space projection and therefore enjoys the homotopy lifting property. Moreover, $\tilde{X}_n$ is the full preimage of $X_n$. Therefore any map

$$\left( B^{d+1}, S^d, * \right) \to (Y_n, X_n, 1)$$

lifts uniquely to a map

$$\left( B^{d+1}, S^d, * \right) \to \left( \tilde{Y}_n, \tilde{X}_n, 1 \right)$$

inducing a map

$$\pi_{d+1}(Y_n, X_n) \to \pi_{d+1}(\tilde{Y}_n, \tilde{X}_n)$$

which is inverse to the map

$$\pi_{d+1}(\tilde{Y}_n, \tilde{X}_n) \to \pi_{d+1}(Y_n, X_n)$$

coming from the covering space projection. Thus, we have isomorphisms

$$\pi_{d+1}(Y_n, X_n) \cong \pi_{d+1}(\tilde{Y}_n, \tilde{X}_n) \cong \pi_d(\tilde{X}_n)$$

and the claim follows from Corollary 3.14. \hfill \Box

We can detect 1-generating and 2-generating families by hand.

**Remark 3.16.** For $n \geq 3$, the family $\mathcal{B}_n$ is $1$-generating for $B_n$, and for $n \geq 4$, it is $2$-generating.

**Proof.** A family $\mathcal{H}$ is $1$-generating for $G$ if and only if $\bigcup_{H \in \mathcal{H}} H$ generates $G$. It is $2$-generating for $G$ if $G$ is the product of the $H \in \mathcal{H}$ amalgamated along their intersections [1, 2.4].

Note that the braid group $B_n$ is generated by counter-clockwise rotations

$$\beta_{ij} := v_i \mapsto v_j \mapsto v_i$$

around digons. Thus, $\mathcal{B}_n := \{ B_1^n, \ldots, B_n^n \}$ generates as long as $n \geq 3$ since then each digon-generator is contained in some $B^n_k$.

Considering the digon-generators for $B_n$, defining relations are given by braid relations, visible in isomorphic copies of $B_3$ inside $B_n$, and commutator relations, visible in isomorphic copies of $B_4$ inside $B_n$. Hence all necessary defining relations are visible in the amalgamated product of the $B^n_k \cong B_{n-1}$ provided $n \geq 5$.

For $n = 4$, the challenge is to derive the commutator relations:

$$\beta_{12}\beta_{34} = \beta_{34}\beta_{12} \quad \text{and} \quad \beta_{23}\beta_{41} = \beta_{41}\beta_{23}$$

We do the first, the second is done analogously. Calculating with only three strands at a time, we find:

$$\beta_{12}\beta_{34}\beta_{24} = \beta_{12}\beta_{23}\beta_{34} = \beta_{23}\beta_{13}\beta_{34} = \beta_{23}\beta_{34}\beta_{14} = \beta_{34}\beta_{24}\beta_{14} = \beta_{34}\beta_{12}\beta_{24}$$

The desired commutator relation follows. \hfill \Box

**Remark 3.17.** The little computation at the end of the preceding proof shows that the commutator relations are redundant in the braid group presentation given in [16, Lem. 4.2]. Accordingly, they are also redundant in the analogous presentation from [13, Prop. 2.1].
Theorem 3.18. For \( n \geq 4 \), the family \( \mathcal{B}_n \) is \( m \)-generating for \( \mathcal{B}_n \) if and only if the homology groups \( H_d(Y_n, X_n) \) are trivial for \( 1 \leq d \leq m \).

Proof. As \( n \geq 4 \), the pair \( (Y_n, X_n) \) is 1-connected by Propositions 3.15 and 3.16. Thus, it follows from the relative Hurewicz theorem that \( m \)-connectivity of the pair is equivalent to \( m \)-acyclicity. By Proposition 3.15, this translates into higher generation of \( \mathcal{B}_n \) by \( \mathcal{B}_n \).

As the pair \( (Y_n, X_n) \) consists of finite complexes that can be described explicitly, Theorem 3.18 implies that it is a finite problem to determine the higher connectivity properties of \( \mathcal{B}_n \) relative to the family \( \mathcal{B}_n \). In particular, the question whether the bounds derived in [5, Example 15.5.4] for higher generation in braid groups are sharp becomes amenable to empirical investigation.

3.3. Curvature in braid groups.

Definition 3.19. For an \( n \times n \) symmetric matrix \( (m_{ij}) \) with entries in \( \{2, 3, \ldots\} \cup \{\infty\} \) we define the associated Artin group to be

\[
\left\langle s_1, \ldots, s_n \mid s_is_js_i\cdots = s_js_is_j\cdots \right\rangle
\]

Here, \( m_{ij} = \infty \) indicates that there is no defining relation for \( s_i \) and \( s_j \). We will refer to the relations appearing above as braid relations (even though some authors reserve this term for the relation with \( m_{ij} = 3 \)).

If one additionally forces the generators \( s_i \) into being involutions, one obtains the associated Coxeter group. A pair consisting of a Coxeter group together with the generating set \( \{s_1, \ldots, s_n\} \) is called a Coxeter system; its rank is defined to be the cardinality of the generating set. If the Coxeter group is spherical, the Coxeter system is said to be spherical as well.

A Coxeter group is spherical if it is finite; an Artin group is spherical if the corresponding Coxeter group is spherical.

Note that the braid group \( \mathcal{B}_n \) is an Artin group and the symmetric group \( S_n \) is the associated Coxeter group. Here, \( m_{ij} = 3 \) for \( |i - j| = 1 \) and \( m_{ij} = 2 \) otherwise. See Fact 3.1

Artin groups form a rich class of groups of importance in geometric group theory and beyond. From geometric group theory perspective they remain in focus largely due to the following conjecture.

Conjecture 3.20 (Charney). Every Artin group is \( \text{CAT}(0) \), i.e. it acts properly and cocompactly on a \( \text{CAT}(0) \) space.

A \( \text{CAT}(0) \) space is a metric space with curvature bounded from above by 0; for details see the book by Bridson–Haefliger [20]. From the current perspective let us list some properties of \( \text{CAT}(0) \) groups: algorithmically, such groups have quadratic Dehn functions and hence soluble word problem; geometrically, all free-abelian subgroups thereof are undistorted; algebraically, the centralisers of infinite cyclic subgroups thereof split; topologically, the space witnessing \( \text{CAT}(0) \)-ness of a group \( G \) is a finite model for \( EG \) and thus, for example, allows to compute the K-theory of the reduced \( C^* \)-algebra \( C^*_r(G) \) provided the Baum–Connes conjecture is known for \( G \).
Conjecture 3.20 has been verified by Charney–Davis for right-angled Artin groups (RAAGs), that is for Artin groups with each $m_{ij}$ equal to $2$ or $\infty$. Outside of this class, the conjecture is mostly open. In particular, it is open (in general) for the braid groups $B_n$.

To prove that a group $G$ is CAT(0), one has to first construct a space $X$ on which $G$ acts properly and cocompactly, and then prove that the space is indeed CAT(0). We shall use the space $\tilde{Y}_n$ from above, on which $B_n$ acts freely and with compact quotient.

What is missing, however, is a metric structure on $\tilde{Y}_n$. Such a metric can be specified by realising the simplices in euclidean space, i.e., by endowing each simplex in $\tilde{Y}_n$ with the metric of a euclidean polytope. Instead of the standard one, we will follow Brady–McCammond [17].

Definition 3.21. Let $e_1, \ldots, e_m$ denote the standard basis of $\mathbb{R}^m$. The $m$-orthoscheme is the convex hull of $\{0, e_1, e_1 + e_2, \ldots, e_1 + e_2 + \cdots + e_m\}$. The orthoscheme has the structure of an $m$-simplex and the vertices come with a grading: the vertex $e_1 + \cdots + e_k$ is declared to be of rank $k$.

We now endow each maximal simplex in $\tilde{Y}_n$ with the orthoscheme metric. Let $\Sigma = \{\beta, \beta \sigma_1, \ldots, \beta \sigma_n\}$ be a maximal simplex. Here, $\beta$ is a braid in $B_n$ and $1 < \sigma_1 < \sigma_2 < \cdots < \sigma_n$ is a maximal chain in $\Gamma_n \cong NC(n)$, which has length $n$ by Observation 3.7. We endow $\Sigma$ with the metric of the standard $n$-orthoscheme by identifying $\beta \sigma_k$ with the vertex of rank $k$ in the orthoscheme. It is easy to see that if two maximal simplices intersect, they induce identical metric on their common face. Thus we have turned $\tilde{Y}_n$ into a metric simplicial complex.

Note that $\tilde{Y}_n$ is obtained by gluing copies of a single shape, the $n$-orthoscheme, and so $\tilde{Y}_n$ is a geodesic metric space by a result of Bridson (finitely many shapes of cells would suffice). Since the shape is euclidean, we may use Gromov’s link condition and deduce the following:

Lemma 3.22. $\tilde{Y}_n$ is CAT(0) if and only if the link of each vertex in $\tilde{Y}_n$ is CAT(1).

Here CAT(1) means that the curvature of the space is bounded above by that of the unit sphere; again, for details see [20].

The poset $\Gamma_n \cong NC(n)$ has a unique maximal element, which is the braid $\gamma$ corresponding to the full counter-clockwise rotation:

$$v_1 \mapsto v_2 \mapsto \cdots \mapsto v_m \mapsto v_1$$

The $n^{th}$ power $\gamma^n$ is central in the braid group $B_n$. In fact, it generates the infinite cyclic center of $B_n$. Brady–McCammond observed in [17] that this algebraic fact has a geometric counterpart: $\tilde{Y}_n$ splits as a cartesian product of the real line $\mathbb{R}$ and another metric space. The $\mathbb{R}$-factor inside $\tilde{Y}_n$ points in the direction of the edges labelled by $\gamma$.

Because of this, instead of looking at the link of a vertex $u$ in $\tilde{Y}_n$, one can look at the link of a midpoint of the (long) edge $(u, u\gamma)$; every two such links are isometric (since $B_n$ acts transitively on the vertices of $\tilde{Y}_n$), and so let $L$ denote any such link.

To compute the curvature of $L$, it is enough to study the subcomplex of $\tilde{Y}_n$ spanned by all simplices containing the edge $(u, u\gamma)$. Clearly, this is the subcomplex spanned by $L$ and $u\sigma$ with $1 \leq \sigma \leq \gamma$, with simplices defined by the chain condition
as before. Thus, such a link is isomorphic as a simplicial complex to the realisation of \( \text{NC}(n) \); the subcomplex also comes with a metric, and it is clear that this coincides with the realisation of \( \text{NC}(n) \) being endowed with its own orthoscheme metric defined as before by identifying each maximal simplex with the \( n \)-orthoscheme. We will refer to the realisation of \( \text{NC}(n) \) with this metric simply as the orthoscheme complex of \( \text{NC}(n) \).

Note that if the orthoscheme complex of \( \text{NC}(n) \) is CAT(0), then \( L \), isometric to the link of the midpoint of the main diagonal, is CAT(1), which implies that \( \tilde{Y}_n \) and so \( B_n \), is CAT(0).

In view of the above, Brady–McCammond formulate the following conjecture.

**Conjecture 3.23** ([17, Conj. 8.4]). For every \( n \), the orthoscheme complex of \( \text{NC}(n) \) is CAT(0), and so the braid group \( B_n \) is CAT(0).

For \( n \leq 4 \), the conjecture is easily seen to be true.

If we know that the orthoscheme complexes of \( \text{NC}(m) \) are CAT(0) for each \( m < n \), then in fact the orthoscheme complex of \( \text{NC}(n) \) is CAT(0) if and only if the link \( L \) is CAT(1). Thus, for \( n = 5 \), it is enough to study \( L \), which is the realisation of the poset obtained from \( \text{NC}(n) \) by removing the trivial and improper partitions, and endowing the realisation with the spherical orthoscheme metric. Knowing that the conjecture is true for all \( m < 5 \) tells us that \( L \) is locally CAT(1). Thus, using the work of Bowditch [15], it is enough to check whether any loop in \( L \) of length less than \( 2\pi \) can be shrunk, i.e., homotoped to the trivial loop without increasing its length in the process.

Brady–McCammond use a computer to analyse all loops in \( L \) shorter than \( 2\pi \), and show that they are indeed shrinkable, thus establishing:

**Theorem 3.24** ([17, Thm. B]). For \( n \leq 5 \), the braid group \( B_n \) is CAT(0).

Haettel, Kielak and Schwer go beyond that, proving

**Theorem 3.25** ([33, Cor. 4.18]). For \( n \leq 6 \), the braid group \( B_n \) is CAT(0).

Note that their proof is not computer assisted. The crucial improvement in the work of Haettel–Kielak–Schwer is to use the observation (present already in [17]), that the link \( L \) can be embedded into a spherical building, in the following way.

First observe that the vertices of \( L \) are non-trivial proper partitions; let \( p \) be such a partition with blocks \( B_1, \ldots, B_k \). Let \( F \) be the field of two elements; we associate to \( p \) the subspace of \( F^n = \langle b_1, \ldots, b_n \rangle \) which is the intersections of the kernels of the characters

\[
\sum_{j \in B_i} b_j^* = 0
\]

where \( 1 \leq i \leq k \), and \( b_j^* \) is the \( j \)-th character in the basis dual to the \( b_j \).

It is easy to see that this gives a map sending each vertex of \( L \) to a proper non-trivial subspace of \( V := \ker ( \sum_{j=1}^n b_j^* ) \). But these subspaces are precisely the vertices of the spherical building of \( \text{SL}_{n-1}(F) \), and it turns out that our bijection extends to a map sending each maximal simplex in \( L \) onto a chamber (i.e. maximal simplex) in the building in an isometric way. Thus we may view \( L \) as a subcomplex of the building.

The spherical building is CAT(1), and this information gives the extra leverage used to prove Theorem 3.25.
4. Non-crossing partitions in Coxeter groups

In this section, we introduce the general theory of non-crossing partitions and explain how non-crossing partitions appear in group theory. As already observed in the beginning of Section 3.3, the symmetric group $S_n$ is a Coxeter group and $(S_n, S_{tr})$ is a Coxeter system of rank $n - 1$ where

$$S_{tr} := \{(i, i + 1) \mid 1 \leq i \leq n - 1\}$$

is the set of neighbouring transpositions.

Every Coxeter system $(W, S)$ acts faithfully on a real vector space that is equipped with a symmetric bilinear form $(-, -)$ such that for every $s \in S$ there is a vector $\alpha_s \in V$ so that $s$ acts as the reflection

$$r_{\alpha_s} : v \mapsto v - 2 \frac{(v, \alpha_s)}{(\alpha_s, \alpha_s)} \alpha_s$$

on $V$. Thus every Coxeter group is a reflection group that is a group generated by a set of reflections on a vector space $(V, (-, -))$.

The vectors $\alpha_s$ can be chosen so that the subset $\Phi = \{w(\alpha_s) \mid s \in S, w \in W\}$ of $V$ is a so called root system. For a spherical Coxeter system a root system $\Phi$ is characterised by the following three axioms

(R1) $\Phi$ generates $V$;
(R2) $\Phi \cap R\alpha = \{\pm \alpha\}$ for all $\alpha \in \Phi$;
(R3) $s_\alpha(\beta)$ is in $\Phi$ for all $\alpha, \beta \in \Phi$.

The spherical Coxeter groups $W$ are precisely the finite real reflection groups.

Coxeter classified the finite root systems which then also gives a classification of the spherical Coxeter systems: there are the infinite families of type $A_n$, $B_n$, $C_n$ and $D_n$ and some exceptional groups. For instance $(S_n, S_{tr})$ is of type $A_{n-1}$. Note that the groups of type $B_n$ and $C_n$ are isomorphic; and also that the root systems of type $A_n, B_n, C_n$ and $D_n$ are all crystallographic that is

$$\frac{(\alpha, \beta)}{(\alpha, \alpha)} \in \mathbb{Z} \text{ for all } \alpha, \beta \in \Phi.$$ 

We call $T := \cup_{w \in W} w^{-1}Sw$ the set of reflections of the Coxeter system $(W, S)$. If the system is spherical, then $T$ is indeed the set of all reflections.

For instance in the symmetric group $S_n$ the set $T$ is the conjugacy class of transpositions, see also Section 3. There the so called absolute order $\leq_T$ on $S_n$ has been introduced. Let $[id, (1, 2, \ldots, n)]_{\leq_T}$ be the closed interval in $S_n$ with respect to $\leq_T$. In Fact 1.3.4 it has been stated that $(NC(n), \subseteq)$ and $([id, (1, 2, \ldots, n)]_{\leq_T}, \leq_T)$ are posets that are isomorphic. Therefore $NC(n)$ can be thought of being of type $A_{n-1}$.

Out of combinatorial interest, Reiner generalised the concept of non-crossing partitions to the infinite series of type $B_n$ and $D_n$ geometrically [46]. Independently of his work and of each other Brady and Watt [18] as well as Bessis [11] generalised the concept of non-crossing partitions to all the finite Coxeter systems. Their approach agrees with Reiner’s in type $B_n$ [4].

Brady and Watt as well as Bessis started independently the study of the dual Coxeter system $(W, T)$ instead of $(W, S)$. A dual Coxeter system $(W, T)$ of finite rank $n$ has the property that there is a subset $S$ of $T$ such that $(W, S)$ is a Coxeter system [11]. It then follows that $T$ is the set of reflections in $(W, S)$. This concept is called by Bessis dual approach to Coxeter and Artin groups.
A \textit{(parabolic) standard Coxeter element} in \((W, S)\) is the product of all the elements in \((\text{a subset of})\) \(S\) in some order and a \textit{(parabolic) Coxeter element} in \((W, T)\) is a \textit{(parabolic) standard Coxeter element} in \((W, S)\) for some simple system \(S\) in \(T\) for \(W\).

For instance in type \(A_{n-1}\), so in the symmetric group \(S_n\), the standard Coxeter elements with respect to \(S = S_n\) are precisely those \(n\)-cycles in \(S_n\) that can be written as a first increasing and then decreasing cycle. All the \(n\)-cycles in \(S_n\) are the Coxeter elements in the dual system \((S_n, T)\) where \(T\) is the set of reflections, that is the conjugacy class of transpositions.

The partial order \(\leq_T\) on the symmetric group \(S_n\) presented in Section 3 can be generalized to all the dual Coxeter systems \((W, T)\). We consider the Cayley graph \(\text{CG}_T(W)\) of the group \(W\) with respect to the generating set \(T\). For \(u, v \in W\) we declare \(u \leq_T v\) if there is a geodesic path in the Cayley graph connecting the identity to \(v\) and passing through \(u\). This partial order is also called the \textit{absolute order} on \(W\).

We also introduce a length function \(l_T\) on \(W\): for \(u \in W\) we define \(l_T(u) = k\) if there is a geodesic path from the identity to \(u\) of length \(k\) in the Cayley graph. Notice, if \(l_T(u) = m\) then \(u\) is the product of \(m\) reflections, that is \(u = t_1 \cdots t_m\) with \(t_i \in T\), and there is no shorter factorisation of \(u\) in a product of reflections. In this case we say that \(u = t_1 \cdots t_m\) is a \textit{T-reduced factorisation} of \(u\). In particular, if \(u \leq_T v\), then there are \(k, m \in \mathbb{N}\) with \(k \leq m\) and reflections \(t_1, \ldots, t_m\) in \(T\) such that \(u = t_1 \cdots t_k\) and \(v = t_1 \cdots t_m\). Thus
\[
    u \leq_T v \quad \text{if and only if} \quad l_T(u) + l_T(u^{-1}v) = l_T(v).
\]

\textbf{Definition 4.1.} For a dual Coxeter system \((W, T)\) and a Coxeter element \(c\) in \(W\) the set of \textit{non-crossing partitions} is
\[
    \text{NC}(W, c) = \{u \in W \mid u \leq_T c\}.
\]

This definition is conform with the definition in type \(A_n\), see Fact 3.4.

The length function \(l_T\) yields a grading on \(\text{NC}(W, c)\) and the map
\[
    d : \text{NC}(W, c) \to \text{NC}(W, c), \quad x \mapsto x^{-1}c
\]
a duality on \(\text{NC}(W, c)\) that inverses the order relation.

This implies the following.

\textbf{Fact 4.2.} \(\text{NC}(W, c)\) is a poset that is
\begin{itemize}
    \item graded
    \item selfdual
    \item \cite{19, 11} a lattice if \(W\) is spherical.
\end{itemize}

The number of elements in \(\text{NC}(W, c)\) in a finite dual Coxeter system of type \(X\) is the generalised Catalan number of type \(X\). In types \(B_n\) and \(D_n\) there are also nice geometric models for the posets of non-crossing partitions.

Note that in a spherical Coxeter system always \(T \subseteq \text{NC}(W, c)\).

There is also a presentation of \(W\) with generating set \(T\) \cite{11}. The relations are the so called \textit{dual braid relations} with respect to a Coxeter element \(c \in W\):

for every \(s, t, t' \in T\) set \(st = t's\) whenever

the relation \(st = t's\) holds in \(W\) and \(st \leq_T c\).
The Matsumoto property means if we have for some \( w \in W \) two shortest factorisations as products of elements of \( S \), or equivalently two geodesic paths from \( id \) to \( w \) in the Cayley graph \( CG_S(W) \), then we can transform one factorisation or path into the other one just by applying braid relations; that is \( W \) has a group presentation as given in Definition 3.19.

The dual Matsumoto property for a Coxeter element \( c \in W \) is the statement that if we have two shortest factorisations
\[
c = t_1 \cdots t_m = u_1 \cdots u_m \text{ with } t_i, u_i \in T
\]
as products of elements of \( T \), that is two \( T \)-reduced factorisations of \( c \) in \( W \), then one factorisation can be transformed into the other one just by applying dual braid relations. It follows that the dual Matsumoto property holds for \( c \), since
\[
(T \mid \text{dual braid relations})
\]
is a presentation of \( W \).

We obtain the dual Matsumoto property for an arbitrary element \( w \in W \) by replacing \( c \) by \( w \) in the definition of the dual braid relations and of the dual Matsumoto property above.

For an element \( w \in W \), let
\[
\text{Red}_T(w) = \{(t_1, \ldots, t_m) \mid t_i \in T \text{ and } w = t_1 \cdots t_m \text{ is } T\text{-reduced}\}.
\]

The dual Matsumoto property for \( w \in W \) is equivalent to the transitive Hurwitz action of the braid group \( B_{t_T(w)} \) on the set of \( T \)-reduced factorisations \( \text{Red}_T(w) \) of \( w \). For the braid \( \sigma_i \in B_{t_T(w)} \), see Fact 3.1, the action is given by
\[
\sigma_i(t_1, \ldots, t_n) = (t_1, \ldots, t_{i-1}, t_i^{-1}t_{i+1}t_i, t_i, t_{i+2}, \ldots, t_n).
\]
We will discuss this action in more detail in the next section.

The dual approach can also be applied to Artin groups; given a Coxeter system \((W, S)\), we will denote the corresponding Artin group by \( \mathcal{A}(W, S) \). If in the following the Coxeter system \((W, S)\) is of type \( X \), then we abbreviate \( \mathcal{A}(W, S) \) either by \( \mathcal{A}(W) \) or by \( \mathcal{A}_X \). Further we take a copy \( S_a \) of \( S \) in \( \mathcal{A}(W, S) \) and write
\[
\mathcal{A}(W, S) := \langle S_a \mid (s_1)_a(s_2)_a(s_1)_a \cdots = (s_2)_a(s_1)_a(s_2)_a \cdots \text{ for } s_1, s_2 \in S \rangle
\]
in order to distinguish between \( W \) and \( \mathcal{A}(W) \). We call an Artin group \( \mathcal{A}(W) \) spherical if the Coxeter group is spherical. And in the rest of this section, we always consider spherical Artin groups.

Notice that the Matsumoto property implies that one can lift every \( w \in W \) to an element in \( \mathcal{A}(W) \) just by mapping \( w \) to \((s_1)_a \cdots(s_k)_a \in \mathcal{A}_W \) whenever \( w = s_1 \cdots s_k \) is a reduced factorisation of \( w \) into elements of \( S \). We denote this section of \( W \) in \( \mathcal{A}(W) \) by \( \mathcal{W} \).

The non-crossing partitions are a good tool for the better understanding of the spherical Artin groups; for instance they can be used to construct a finite simplicial classifying space for the spherical Artin groups (see Section 3.1), or to solve the word or the conjugacy problem in them, see [18, 11].

The basic idea of this solution of the word and the conjugacy problem in the spherical Artin group \( \mathcal{A}(W) \) is to give a new presentation of \( \mathcal{A}(W) \) as follows. Let \( \text{NC}(W, c)_a \) be a copy of the set of non-crossing partitions \( \text{NC}(W, c) \) with respect to a standard Coxeter element \( c \), that is there is a bijection
\[
a : \text{NC}(W, c) \to \text{NC}(W, c)_a.
\]
Then the new generating set is \( \text{NC}(W, c)_a \); and the new relations are the expressions \((w_1)_a \cdots (w_r)_a\) whenever \(w_1, w_2, \ldots, w_r\) are the vertices of a circuit in \([id, c] \leq_T \subseteq \text{CG}_{\text{NC}(W, c)}(W)\).

Then this presentation can be used to obtain a new normal form for the elements in \(A(W)\) [11]. Notice that this presentation generalises the presentation of the braid group given by Birman, Ko and Lee [13] to all the spherical Artin groups, see also Fact 3.6 in Section 3.1.

Next, we explain this new presentation. Denote the group given by the presentation above \(A(W, c)\). The strategy to prove that \(A(W, c)\) and \(A(W)\) are isomorphic is to use Garside theory. As a first step the presentation above can be transformed into a presentation with set of generators a copy \(T_a = \{t_a \mid t \in T\}\) of \(T\) and set of relations the dual braid relations with respect to \(c\). The next step is to consider the monoid \(A(W, c)^*\) generated by \(T_a\) and the dual braid relations, and to show that this is a Garside monoid. Then using Garside theory one shows that the group of fractions \(\text{Frac}(A(W, c)^*)\) of \(A(W, c)^*\) equals \(A(W, c)\). The last step is to prove that the group of fractions \(\text{Frac}(A(W, c)^*)\) and the Artin group \(A(W)\) are isomorphic.

**Theorem 4.3** ([11]). Let \(A_W\) be a spherical Artin group. Then,
\[
A_W \cong \langle T_a \mid t_a t'_a = (tt')_a t_a \text{ if } t, t' \in T \text{ and } tt' \leq_T c \rangle.
\]

Note also that a basic ingredient in the proof of Theorem 4.3 is the dual Matsumoto property for \(c\), that is the transitivity of the Hurwitz action of the braid group \(B_{\text{Red}_T(c)}\) on \(\text{Red}_T(c)\).

The isomorphism between \(A(W, c)\) and \(A_W\) given by Bessis is difficult to understand explicitly. So an immediate question is what the elements of \(\text{NC}(W, c)_a\) are expressed in the generating set \(S_a\)?

The rational permutation braids, that is, the elements \(xy^{-1}\) where \(x, y \in W\), are also called Mikado braids as they satisfy in type \(A_{n-1}\) a topological condition and are therefore easy to recognise. This condition on an element in the Artin group \(A(W)\) of type \(A_{n-1}\), that is on a braid in the braid group \(B_n\), is that we can lift and remove continuously one strand after the next of the braid without disturbing the remaining strands until we reach an empty braid [27].

**Theorem 4.4.** If \(A_W\) is spherical Artin group and \(c \in W\) a standard Coxeter element, then the dual generators of \(A(W, c)\), that is the elements of \(\text{NC}(W, c)_a\), are Mikado braids in \(A_W\).

*Proof.* This is [27] for those groups of type different from \(D_n\) and [9] for those of type \(D_n\). \(\square\)

Notice that Licata and Queffelec [44] have a proof of Theorem 4.4 in types A,D,E with a different approach using categorification.

In order to be able to find a topological property that characterises the Mikado braids as in type \(A_{n-1}\) topological models for the series of spherical Artin groups \(A_W\) are needed. There is an embedding of Artin groups of type \(B_n\) into those of type \(A_{2n-1}\). The situation in type \(D_n\) is as follows [9]: The root system of type \(D_n\) embeds into the root system of type \(B_n\), which implies that the Coxeter system of type \(D_n\) is a subsystem of that one of type \(B_n\). But there is not an embedding of the Artin group of type \(D_n\) into that one of type \(B_n\) that satisfies a certain natural
condition. Let \((W, S)\) be a Coxeter system of type \(B_n\). Then there is precisely one element \(s \in S\) that is a reflection corresponding to a short root. Let
\[
\overline{A_{B_n}} := A_{B_n} / \langle s^2 \rangle,
\]
where \(\langle s^2 \rangle\) is the normal closure of \(s^2\) in \(A_{B_n}\). Then the following holds.

**Proposition 4.5** ([9, Lem. 2.5 and Prop. 2.7]). There is a natural embedding of \(A_{D_n}\) onto an index-2 subgroup of \(\overline{A_{B_n}}\). More precisely, there is the following commutative diagram

\[
\begin{array}{ccc}
A_{B_n} & \xrightarrow{\pi} & \overline{A_{B_n}} & \xleftarrow{\langle t_1, \ldots, t_n \rangle} & A_{D_n} \\
\downarrow & & \downarrow{\pi} & & \downarrow{\pi_D} \\
A_{B_n} & \xrightarrow{\pi_S} & W_{B_n} & \xleftarrow{W_{D_n}} & \\
\end{array}
\]

The embedding of \(A_{D_n}\) into \(\overline{A_{B_n}}\) makes it possible to associate braid pictures to the \(A_{D_n}\)-elements and to characterise Mikado braids in type \(D_n\) geometrically.

![Figure 7. A Mikado braid in \(A_{B_8}\) whose image in \(\overline{A_{B_8}}\) is a Mikado braid in \(A_{D_8}\).](image)

A reader familiar with Hecke algebras will find it interesting that the Mikado braids satisfy a positivity property involving the canonical Kazhdan-Lusztig basis \(C := \{C_w \mid w \in W\}\) of the Iwahori–Hecke algebra \(H(W)\) related to the Coxeter system \((W, S)\), see [40, 27]. There is a natural group homomorphism \(a : A_W \rightarrow H(W)^\times\) from \(A_W\) into the multiplicative group \(H(W)^\times\) of \(H(W)\). The image of a Mikado braid, that is of a rational permutation braid, in \(H(W)^\times\) has as coefficients Laurent polynomials with non-negative coefficients when expressed in the canonical basis \(C\) by a result by Dyer and Lehrer (see [29, 27]).

### 5. The Hurwitz Action

**Hurwitz action in Coxeter systems.** Deligne showed the dual Matsumoto property in spherical Coxeter systems, that is he showed that the Hurwitz action of the braid group \(B_T(c)\) on \(\text{Red}_T(c)\) is transitive for every Coxeter element \(c\) in \((W, S)\) [26]; and Igusa and Schiffler proved it for arbitrary Coxeter systems [38]. In [7] a new, more general and first of all constructive proof of this property is given:
Theorem 5.1 ([7, Thm. 1.3]). Let \((W,T)\) be a (finite or infinite) dual Coxeter system of finite rank \(n\) and let \(c = s_1 \cdots s_m\) be a parabolic Coxeter element in \(W\). The Hurwitz action on \(\text{Red}_T(c)\) is transitive.

Theorem 5.1 is also more general than Theorem 1.4 in [38], as in [7] dual Coxeter systems are considered while in [38] Coxeter systems, and in general the set of Coxeter elements is in a dual system larger than that one in a Coxeter system.

The proof of Thereom 5.1 is based on a study of the Cayley graphs \(CG_S(W)\) and \(CG_T(W)\). Using the same methods one can also show that every reflection occurring in a reduced \(T\)-factorisation of an element of a parabolic subgroup \(P\) of \(W\) is already contained in that parabolic subgroup.

Theorem 5.2 ([7, Thm. 1.4]). Let \((W,S)\) be a (finite or infinite) Coxeter system, \(P\) a parabolic subgroup and \(w \in P\). Then \(\text{Red}_T(w) = \text{Red}_{T \cap P}(w)\).

This basic fact was not known before and can be seen as a founding stone towards a general theory for ‘dual’ Coxeter systems.

Hurwitz action in the spherical Coxeter systems and quasi-Coxeter elements. In the rest of the section, \((W,T)\) is a finite dual Coxeter system.

In order to understand the dual Coxeter systems \((W,T)\) one also needs to know for which elements in \(W\) the Hurwitz action is transitive. The answer to that question is as follows [8].

A parabolic quasi-Coxeter element is an element \(w \in W\) that has a reduced factorisation into reflections such that these reflections generate a parabolic subgroup of \(W\).

Note if one reduced \(T\)-factorisation of \(w \in W\) generates a parabolic subgroup \(P\) then every reduced \(T\)-factorisation of \(w\) is in \(P\) by Theorem 5.2. It also follows that every such factorisation generates \(P\) [8, Thm. 1.2].

If a factorisation of \(w\) generates the whole group \(W\), it is a quasi-Coxeter element. Clearly every Coxeter element is a quasi-Coxeter element. In type \(A_n\) and \(B_n\) every quasi-Coxeter element is already a Coxeter element. The smallest Coxeter system containing a proper quasi-Coxeter element is of type \(D_4\).

Now we can answer the question above.

Theorem 5.3 ([8, Thm. 1.1]). Let \((W,S)\) be a spherical Coxeter system and let \(w \in W\). The Hurwitz action is transitive on \(\text{Red}_T(w)\) if and only if \(w\) is a parabolic quasi-Coxeter element.

Recently, Wegener showed that the dual Matsumoto property holds for quasi-Coxeter elements in affine Coxeter systems as well [53]. These two results have the following consequence.

Corollary 5.4. Let \((W,T)\) be a dual Coxeter system, \(w \in W\) and \(w = t_1 \cdots t_m\) a reduced \(T\)-factorisation, then the Hurwitz action is transitive on \(\text{Red}_T(w)\) in the Coxeter group \(W' := \langle t_1, \ldots, t_m \rangle\) whenever \(W'\) is a spherical or an affine Coxeter group.

Proof. According to Theorem 3.3 of [28], \(W' := \langle t_1, \ldots, t_m \rangle\) is a Coxeter group. Theorem 5.3 and the main result in [53] then yield the statement. \(\square\)

The (parabolic) quasi-Coxeter elements are interesting for more reasons; for instance also for the following. Let \(\Phi\) be the root system related to \((W,S)\) and let
$L(\Phi) := \mathbb{Z}\Phi$ and $L(\Phi^\vee) := \mathbb{Z}\Phi^\vee$ where $\alpha^\vee := 2\alpha/\langle\alpha, \alpha\rangle$ be the root and the coroot lattices, respectively. Quasi-Coxeter elements are also intrinsic in the dual Coxeter systems as they generate the root as well as the coroot lattice: Let $w = t_1 \cdots t_n$ be a reduced $T$-factorisation of $w \in W$ and let $\alpha_i \in \Phi$ be the root related to the reflection $t_i$ for $1 \leq i \leq n$.

**Theorem 5.5** ([10, Thm. 1.1]). Let $\Phi$ be a finite crystallographic root system of rank $n$. Then $w$ is a quasi-Coxeter element if and only if

1. $\{\alpha_i \mid 1 \leq i \leq n\}$ is a $\mathbb{Z}$-basis of the root lattice $L(\Phi)$, and
2. $\{\alpha_i^\vee \mid 1 \leq i \leq n\}$ is a $\mathbb{Z}$-basis of the coroot lattice $L(\Phi^\vee)$.

Thus if all the roots in $\Phi$ are of the same length, then $L(\Phi) = L(\Phi^\vee)$ and the quasi-Coxeter elements correspond precisely to the basis of the root lattice.

Quasi-Coxeter elements and Coxeter elements share further important properties beyond Hurwitz transitivity.

**Theorem 5.6** ([8, Cor. 6.11]). An element $x \in W$ is a parabolic quasi-Coxeter element if and only if $x \leq_T w$ for a quasi-Coxeter element $w$.

Finally, Gobet observed that, in a spherical Coxeter system, every parabolic quasi-Coxeter element can be uniquely written as a product of commuting parabolic quasi-Coxeter elements [32]. This factorisation of a quasi-Coxeter element can be thought of as a generalisation of the unique disjoint cycle decomposition of a permutation.

6. **Non-crossing partitions arising in representation theory**

In this section, we explain how non-crossing partitions arise naturally in representation theory. For any finite dimensional algebra $A$ over a field $k$ we consider the category $\mod A$ of finite dimensional (right) $A$-modules and denote by $K_0(A)$ its *Grothendieck group*. This group is free abelian of finite rank, and a representative set of simple $A$-modules $S_1, \ldots, S_n$ provides a basis $e_1, \ldots, e_n$ if one sets $e_i = [S_i]$ for all $i$. As usual, we denote for any $A$-module $X$ by $[X]$ the corresponding class in $K_0(A)$. The Grothendieck group comes equipped with the *Euler form* $K_0(A) \times K_0(A) \to \mathbb{Z}$ given by

$$
\langle[X], [Y]\rangle = \sum_{n \geq 0} (-1)^n \dim_k \text{Ext}_A^n(X, Y)
$$

which is bilinear and non-degenerate (assuming that $A$ is of finite global dimension). The corresponding symmetrised form is given by $\langle x, y \rangle = \langle x, y\rangle + \langle y, x\rangle$. For a class $x = [X]$ given by a module $X$, one defines the reflection

$$
\langle x, y \rangle = \langle x, y \rangle + \langle y, x \rangle,
$$

assuming that $\langle x, x \rangle \neq 0$ divides $\langle e_i, x \rangle$ for all $i$. Let us denote by $W(A)$ the group of automorphisms of $K_0(A)$ that is generated by the set of simple reflections $S(A) = \{s_{e_1}, \ldots, s_{e_n}\}$; it is called the Weyl group of $A$.

From now on, assume that $A$ is *hereditary*, that is, of global dimension at most one. Then, one can show that the Weyl group $W(A)$ is actually a Coxeter group. For example, the path algebra $kQ$ of any quiver $Q$ is hereditary and in that case $kQ$-modules identify with $k$-linear representations of $Q$. 


Proposition 6.1 ([37, Thm.B.2]). A Coxeter system \((W, S)\) is of the form \((W(A), S(A))\) for some finite dimensional hereditary algebra \(A\) if and only if it is crystallographic in the following sense:

1. \(m_{st} \in \{2, 3, 4, 6, \infty\}\) for all \(s \neq t\) in \(S\), and
2. in each circuit of the Coxeter graph not containing the edge label \(\infty\), the number of edges labelled 4 (resp. 6) is even.

We may assume that the simple \(A\)-modules are numbered in such a way that \(\langle e_i, e_j \rangle = 0\) for \(i > j\), and we set \(c = s_{e_1} \cdots s_{e_n}\). Note that \(c = c(A)\) is a Coxeter element which is determined by the formula

\[
\langle x, y \rangle = -\langle y, c(x) \rangle \quad \text{for} \quad x, y \in K_0(A).
\]

We are now in a position to formulate a theorem which provides an explicit bijection between certain subcategories of \(\text{mod} \, A\) and the non-crossing partitions in \(\text{NC}(W(A), c)\). Call a full subcategory \(C \subseteq \text{mod} \, A\) thick if it is closed under direct summands and satisfies the following two-out-of-three property: any exact sequence \(0 \to X \to Y \to Z \to 0\) of \(A\)-modules lies in \(C\) if two of \(\{X, Y, Z\}\) are in \(C\).

A subcategory is coreflective if the inclusion functor admits a right adjoint.

Theorem 6.2. Let \(A\) be a hereditary finite dimensional algebra. Then, there is an order preserving bijection between the set of thick and coreflective subcategories of \(\text{mod} \, A\) (ordered by inclusion) and the partially ordered set of non-crossing partitions \(\text{NC}(W(A), c)\). The map sends a subcategory which is generated by an exceptional sequence \(E = (E_1, \ldots, E_r)\) to the product of reflections \(s_E = s_{E_1} \cdots s_{E_r}\).

The rest of this article is devoted to explaining this result. In particular, the crucial notion of an exceptional sequence will be discussed.

This result goes back to beautiful work of Ingalls and Thomas [39]. It was then established for arbitrary path algebras by Igusa, Schiffler, and Thomas [38], and we refer to [37] for the general case. Observe that path algebras of quivers cover only the Coxeter groups of simply laced type (via the correspondence \(A \mapsto W(A)\)); so there are further hereditary algebras.

We may think of Theorem 6.2 as a categorification of the poset of non-crossing partitions. There is an immediate (and easy) consequence which is not obvious at all from the original definition of non-crossing partitions; the first (combinatorial) proof required a case by case analysis.

Corollary 6.3. For a finite crystallographic Coxeter group, the corresponding poset of non-crossing partitions is a lattice.

Proof. Any finite Coxeter group can be realised as the the Weyl group \(W(A)\) of a hereditary algebra of finite representation type. In that case any thick subcategory is coreflective. On the other hand, it is clear from the definition that the intersection of any collection of thick subcategories is again thick. This yields the join, but also the meet operation; so the poset of thick and coreflective subcategories is actually a lattice; see Remark 1.1

This categorification provides some further insight into the collection of all posets of non-crossing partitions. This is based on the simple observation that any thick and coreflective subcategory \(C \subseteq \text{mod} \, A\) (given by an exceptional sequence \(E = (E_1, \ldots, E_r)\)) is again the module category of a finite dimensional
hereditary algebra, say \( \mathcal{C} = \text{mod} \, B \). Then the inclusion \( \text{mod} \, B \to \text{mod} \, A \) induces not only an inclusion \( K_0(B) \to K_0(A) \), but also an inclusion \( W(B) \to W(A) \) for the corresponding Weyl groups, which identifies \( W(B) \) with the subgroup of \( W(A) \) generated by \( s_{E_1}, \ldots, s_{E_n} \), and identifies the Coxeter element \( c(B) \) with the non-crossing partition \( s_E \) in \( W(A) \). Moreover, the inclusion \( W(B) \to W(A) \) induces an isomorphism

\[
\text{NC}(W(B), c(B)) \cong \{ x \in \text{NC}(W(A), c(A)) \mid x \leq s_E \}.
\]

The following result summarises this discussion; it reflects the fact that there is a category of non-crossing partitions. This means that we consider a poset of non-crossing partitions not as a single object but look instead at the relation with other posets of non-crossing partitions.

**Corollary 6.4** ([37, Cor. 5.8]). Let \( \text{NC}(W, c) \) be the poset of non-crossing partitions given by a crystallographic Coxeter group \( W \). Then, any element \( x \in \text{NC}(W, c) \) is the Coxeter element of a subgroup \( W' \leq W \) that is again a crystallographic Coxeter group. Moreover,

\[
\text{NC}(W', x) = \{ y \in \text{NC}(W, c) \mid y \leq x \}. \quad \square
\]

7. **Generalised Cartan lattices**

Coxeter groups and non-crossing partitions are closely related to root systems. The approach via representation theory provides a natural setting, because the Grothendieck group equipped with the Euler form determines a root system; we call this a generalised Cartan lattice and refer to [37] for a detailed study.

The following definition formalises the properties of the Grothendieck group \( K_0(A) \). A generalised Cartan lattice is a free abelian group \( \Gamma \cong \mathbb{Z}^n \) with an ordered standard basis \( e_1, \ldots, e_n \) and a bilinear form \( \langle -, - \rangle : \Gamma \times \Gamma \to \mathbb{Z} \) satisfying the following conditions.

1. \( \langle e_i, e_i \rangle > 0 \) and \( \langle e_i, e_j \rangle \) divides \( \langle e_i, e_j \rangle \) for all \( i, j \).
2. \( \langle e_i, e_j \rangle = 0 \) for all \( i > j \).
3. \( \langle e_i, e_j \rangle \leq 0 \) for all \( i < j \).

The corresponding symmetrised form is

\[
(x, y) = \langle x, y \rangle + \langle y, x \rangle \quad \text{for} \, x, y \in \Gamma.
\]

The ordering of the basis yields the Coxeter element

\[
\text{cox}(\Gamma) := s_{e_1} \cdots s_{e_n}.
\]

We can define reflections \( s_x \) as in (24) and denote by \( W = W(\Gamma) \) the corresponding Weyl group, which is the subgroup of \( \text{Aut}(\Gamma) \) generated by the simple reflections \( s_{e_1}, \ldots, s_{e_n} \). We write \( \text{NC}(\Gamma) = \text{NC}(W, c) \) with \( c = \text{cox}(\Gamma) \) for the poset of non-crossing partitions, and the set of real roots is

\[
\Phi(\Gamma) := \{ w(e_i) \mid w \in W(\Gamma), 1 \leq i \leq n \} \subseteq \Gamma.
\]

A real exceptional sequence of \( \Gamma \) is a sequence \( (x_1, \ldots, x_r) \) of elements that can be extended to a basis \( x_1, \ldots, x_n \) of \( \Gamma \) consisting of real roots and satisfying \( \langle x_i, x_j \rangle = 0 \) for all \( i > j \). A morphism \( \Gamma' \to \Gamma \) of generalised Cartan lattices is given by an isometry (morphism of abelian groups preserving the bilinear form \( \langle -, - \rangle \)) that maps the standard basis of \( \Gamma' \) to a real exceptional sequence of \( \Gamma \). This yields a category of generalised Cartan lattices.
What is this category good for? One of the basic principles of category theory is Yoneda’s lemma which tells us that we understand an object \( \Gamma \) by looking at the representable functor \( \text{Hom}(-, \Gamma) \) which records all morphisms that are received by \( \Gamma \). In our category all morphisms are monomorphisms, so \( \text{Hom}(-, \Gamma) \) amounts to the poset of subobjects (equivalence classes of monomorphisms \( \Gamma' \to \Gamma \)).

**Theorem 7.1** ([37, Thm 5.6]). The poset of subobjects of a generalised Cartan lattice \( \Gamma \) is isomorphic to the poset of non-crossing partitions \( \text{NC}(\Gamma) \). The isomorphism sends a monomorphism \( \phi: \Gamma' \to \Gamma \) to \( s_{\phi(e_1)} \cdots s_{\phi(e_r)} \) where \( \text{cox}(\Gamma') = s_{e_1} \cdots s_{e_r} \). Moreover, the assignment \( w \mapsto w|_{\Gamma'} \) induces an isomorphism

\[
W(\Gamma) \cong (s_{\phi(e_1)}, \ldots, s_{\phi(e_r)}) \xrightarrow{\sim} W(\Gamma').
\]

3. **Braid group actions on exceptional sequences**

The link between representation theory and non-crossing partitions is based on the notion of an exceptional sequence and the action of the braid group on the collection of complete exceptional sequences. This will be explained in the following section.

There are two sorts of abelian categories that we need to consider. This follows from a theorem of Happel [34, 35] which we now explain. Fix a field \( k \) and consider a connected hereditary abelian category \( \mathcal{A} \) that is \( k \)-linear with finite dimensional \( \text{Hom} \) and \( \text{Ext} \) spaces. Suppose in addition that \( \mathcal{A} \) admits a tilting object. This is by definition an object \( T \) in \( \mathcal{A} \) with \( \text{Ext}_1^A(T, T) = 0 \) such that \( \text{Hom}_\mathcal{A}(T, A) = 0 \) and \( \text{Ext}_1^A(T, A) = 0 \) imply \( A = 0 \). Thus the functor \( \text{Hom}_\mathcal{A}(T, -): \mathcal{A} \to \text{mod} \Lambda \) into the category of modules over the endomorphism algebra \( \Lambda = \text{End}_\mathcal{A}(T) \) induces an equivalence

\[
\mathbf{D}^b(\mathcal{A}) \xrightarrow{\sim} \mathbf{D}^b(\text{mod} \Lambda)
\]

of derived categories [3]. There are two important classes of such hereditary abelian categories admitting a tilting object: module categories over hereditary algebras, and categories of coherent sheaves on weighted projective lines in the sense of Geigle and Lenzing [30]. Happel’s theorem then states that there are no further classes.

**Theorem 8.1** (Happel). A hereditary abelian category with a tilting object is, up to a derived equivalence, either of the form \( \text{mod} A \) for some finite dimensional hereditary algebra \( A \) or of the form \( \text{coh} \mathcal{X} \) for some weighted projective line \( \mathcal{X} \). □

It is interesting to observe that these abelian categories form a category: Any thick and coreflective subcategory is again an abelian category of that type; so the morphisms are given by such inclusion functors.

Now, fix an abelian category \( \mathcal{A} \) which is either of the form \( \mathcal{A} = \text{mod} A \) or \( \mathcal{A} = \text{coh} \mathcal{X} \), as above. Note that in both cases the Grothendieck group \( K_0(\mathcal{A}) \) is free of finite rank and equipped with an Euler form, as explained before. An object \( X \) in \( \mathcal{A} \) is called exceptional if it is indecomposable and \( \text{Ext}_1^\mathcal{A}(X, X) = 0 \). A sequence \( (X_1, \ldots, X_r) \) of objects is called exceptional if each \( X_i \) is exceptional and \( \text{Hom}_\mathcal{A}(X_i, X_j) = 0 = \text{Ext}_1^\mathcal{A}(X_i, X_j) \) for all \( i > j \). Such a sequence is complete if \( r \) equals the rank of the Grothendieck group \( K_0(\mathcal{A}) \). Let \( n \) denote rank of \( K_0(\mathcal{A}) \). Then, the braid group \( B_n \) on \( n \) strands is acting on the collection of isomorphism classes of complete exceptional sequences in \( \mathcal{A} \) via mutations, and it is an important theorem that this action is transitive (due to Crawley-Boevey [25] and Ringel [47] for module categories, and Kussin–Meltzer [43] for coherent sheaves).
Any tilting object $T$ admits a decomposition $T = \bigoplus_{i=1}^{n} T_i$ such that $(T_1,\ldots,T_n)$ is a complete exceptional sequence. We denote by $W(A)$ the group of automorphisms of $K_0(A)$ that is generated by the corresponding reflections $s_{T_1},\ldots,s_{T_n}$; it is the Weyl group with Coxeter element $c = s_{T_1} \cdots s_{T_n}$ and does not depend on the choice of $T$. Thus we can consider the poset of non-crossing partitions and we have the Hurwitz action on factorisations of the Coxeter element as product of reflections. But it is important to note that $W(A)$ is not always a Coxeter group when $A = \text{coh}X$, and it is an open question whether the Hurwitz action is transitive.

The key observation is now the following.

**Proposition 8.2.** The map

$$(E_1,\ldots,E_r) \mapsto s_{E_1} \cdots s_{E_r}$$

which assigns to an exceptional sequence in $A$ the product of reflections in $W(A)$ is equivariant for the action of the braid group $B_r$. \hfill \Box$

The proof is straightforward. But a priori it is not clear that the product $s_{E_1} \cdots s_{E_r}$ is a non-crossing partition. In fact, the proof of Theorem 6.2 hinges on the transitivity of the Hurwitz action on factorisations of the Coxeter element. So the analogue of Theorem 6.2 for categories of type $A = \text{coh}X$ remains open. A proof would provide an interesting extension of the theory of crystallographic Coxeter groups and non-crossing partitions, which seems very natural in view of Happel’s theorem since the Grothendieck group $K_0(A)$ is a derived invariant.

Partial results were obtained recently by Wegener in his thesis [52]. In fact, when a weighted projective line $X$ is of tubular type (that is, the weight sequence is up to permutation of the form $(2,2,2,2)$, $(3,3,3)$, $(2,4,4)$ or $(2,3,6)$), then the Grothendieck group gives rise to a tubular elliptic root system [48, 49]. Wegener showed the transitivity of the Hurwitz action in this case. Thus, one has in particular the analogue of Theorem 6.2 for $\text{coh}X$ in the tubular case.

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