Obstructions to Coherence: Natural Noncoherent Associativity

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November 14, 2021

Abstract

We study what happens when coherence fails. Categories with a tensor product and a natural associativity isomorphism that does not necessarily satisfy the pentagon coherence requirements (called associative categories) are considered. Categorical versions of associahedra where naturality squares commute and pentagons do not, are constructed (called Catalan groupoids, $A_n$). These groupoids are used in the construction of the free associative category. They are also used in the construction of the theory of associative categories (given as a 2-sketch). Generators and relations are given for the fundamental group, $\pi(A_n)$, of the Catalan groupoids – thought of as a simplicial complex. These groups are shown to be more than just free groups. Each associative category, $B$, has related fundamental groups $\pi(B_n)$ and homomorphisms $\pi(P_n) : \pi(A_n) \longrightarrow \pi(B_n)$. If the images of the $\pi(P_n)$ are trivial, i.e. there is only one associativity path between any two objects, then the category is coherent. Otherwise the images of $\pi(P_n)$ are obstructions to coherence. Some progress is made in classifying noncoherence of associative categories.

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1 Introduction

The history of coherence theory has its beginnings in homotopy theory. In 1963, J. Stasheff [22] investigated the conditions in which an H-space has a homotopy associative multiplication. At around the same time, D.B.A. Epstein [4] came across some associativity questions while dealing with Steenrod operations. With these papers in mind, S. Mac Lane then wrote his classic paper on coherence [16]. He abstracted the problem to the following categorical question: Given a category $\mathcal{B}$ and a tensor product on it $\otimes : \mathcal{B} \times \mathcal{B} \rightarrow \mathcal{B}$ that is associative up to a natural isomorphism $\beta_{A,B,C} : A \otimes (B \otimes C) \rightarrow (A \otimes B) \otimes C$ when is there a unique canonical map between two specified formal combinations of objects? In other words, what conditions on $\mathcal{B}$, $\otimes$ and $\beta$ insure that all combinations of identities and $\beta$ between any two objects are the same. Mac Lane answered the question by giving the following condition:

$$
\begin{array}{c}
A((BC)D) \\
A(B(CD)) \\
(AB)(CD)
\end{array}
\xrightarrow{\beta_{A,B,C,D}}
\begin{array}{c}
(A(BC))D \\
((AB)C)D \\
((AB)(CD))
\end{array}
$$

If this diagram commutes for every four objects $A, B, C$ and $D$, then there is only one canonical map between any two objects. An associativity isomorphism that satisfies the above condition is called coherent and categories with a coherent associativity isomorphism (and a unitary requirement) were later called monoidal or tensor categories.

Other coherence questions arose. There were cartesian closed categories, monoidal closed categories, distributive categories etc. In recent times, coherence problems have arisen in many areas of mathematics and mathematical physics. Quantum groups, quantum field theories, linear logic, knot theory etc. are just a few of the diverse areas that now deal with coherence problems. (See the last section for more examples and some references to current
coherence problems.) Basically, categories with structure and coherence conditions are seen as a type of higher dimensional algebras and such algebras are ubiquitous in the mathematics done today.

What seems to have been left out is what happens if a coherence condition fails. Is there any structure that can still be recovered? Is there a hierarchy of coherence conditions? The prototypical example of a situation in which coherence fails is when the category is $R - \text{Mod}$, the category of left $R$ modules for an arbitrary ring $R$, the $\otimes$ is the usual tensor product and the $\beta$ is the unusual

$$\beta(a \otimes (b \otimes c)) = -1((a \otimes b) \otimes c).$$

Given this $\beta$ the above pentagon does not commute. Since $\beta$ is an isomorphism, we can express this noncommutativity by saying that starting from $A \otimes (B \otimes (C \otimes D))$ and going clockwise around the pentagon, we do not get the identity map. However there is some structure left, namely, going around the pentagon twice does give the identity. The higher complexes do not commute. However, we may be able to say something about the structure of the higher complexes. The goal of this paper is to explore this structure.

We turn back to homotopy theory to study this higher structure. For every natural number $n$, we construct a category $A_n$ whose objects correspond to associations of $n$ letters and whose morphisms correspond to reassociations. Since the reassociations are isomorphisms, the $A_n$’s are in fact groupoids. We call these groupoids the Catalan groupoids. As with all groupoids, they can be thought of as simplicial complexes: the objects are 0-cells, the morphisms are 1-cells and the commuting parts will be 2-cells. One of the goals of this paper is to calculate the fundamental groups of the $A_n$’s and their quotients. If a quotient of the $A_n$’s is indiscrete (i.e. one morphism between any two objects), then it is called coherent. If there is more then one morphism, then we have an obstruction to coherence.

We begin by defining an associative category i.e. a category with a bifunctor that is associative up to a natural isomorphism that does not necessarily satisfy the pentagon condition. Examples of such categories are given. The Catalan groupoids, ( the $A_n$’s ) are constructed in section 3. Section 4 uses the $A_n$’s to construct the free associative category on one generator, $\bar{A}$. The universal properties of $\bar{A}$ are proved. Section 5 goes on to show that all the $A_n$’s together have the structure of an operad. This operad is used in the construction of $A$, the 2-sketch of the theory of associative categories. A
short discussion of associative categories that have units is given in section 6.

In section 8, the fundamental groups of each of the $A_n$ is calculated. To each $A_n$ we assign a maximal indiscrete subgroupoid $T_n$, the categorical analogue of assigning a maximal tree to a simplicial complex. The fundamental groups are then obtained by collapsing the $T_n$ to a point. A way of describing the morphisms of $A_n$ which are the generators of the group is introduced. The general scheme for the generators and relations are provided and the first seven groups are presented. The seventh group is shown not to be a free group. All higher groups are non-free groups. What follows is a discussion of quotients of the 2-sketch of associative categories. An attempt is made to classify the failure of coherence for both unital and nonunital associative categories.

This paper ends with a section that lists some of the possible applications of this work and ways we can go further in the study of the failure of coherence.

I am grateful to my advisor, Prof. Alex Heller for many helpful ideas and long discussions. I would also like to thank my colleague M. Mannucci for many stimulating conversations.

2 Associative Categories

Definition 1 (Associative Category) An associative category is a category $B$, a bifunctor $\otimes_B : B \times B \to B$ called “tensor”, and a natural isomorphism

$$\beta_{B,\otimes} : \otimes_B \circ (Id_B \times \otimes_B) \to \otimes_B \circ (\otimes_B \times Id_B)$$

i.e. for every $A, B, C$ in $B$ an isomorphism

$$\beta_{B,\otimes,A,B,C} : A \otimes_B (B \otimes_B C) \to (A \otimes_B B) \otimes_B C$$

called the “reassociation”.

We reserve the right to abandon the subscripts when there is no concern for ambiguity. Discussions of a unit of the tensor will be left for section 6. The important point is that we do not make any coherence requirements.
In general, all categories have a composition of morphisms that is associative, however, the name “associative category” is used here because the most interesting feature about our categories, is that their tensors are associative up to a isomorphism.

Examples of associative categories abound. Any monoidal category of $\mathcal{L}$ (also called a tensor category in the literature e.g. $\mathcal{H}$) is automatically an associative category. A noncoherent example of an associative category is $R - \text{Mod}$, the category of $R$ modules for a commutative ring $R$. The tensor product is the usual tensor product of modules and the reassociation

$$\beta_{A,B,C}: A \otimes (B \otimes C) \longrightarrow (A \otimes B) \otimes C$$

is defined as

$$a \otimes (b \otimes c) \longrightarrow \zeta(a \otimes b) \otimes c$$

where $\zeta = -1$ (see $\mathcal{L}$) or $\zeta$ is, for example, the fifth root of unity. We shall come back to this example again at the end of section 10.

The above example can be abstracted and put into the language of quantum groups. Let $A = (A, \Delta, \varepsilon)$ be an algebra with a comultiplication and a counit. Let $\Phi$ be an invertible element in $A \otimes A \otimes A$ such that

$$(Id \otimes \Delta)(\Delta(a)) = \Phi((\Delta \otimes Id)(\Delta(a)))\Phi^{-1},$$

for all $a \in A$. Then the category, $A - \text{Mod}$, of $A$ modules, has the structure of an associative category. The tensor product of two modules is constructed using the comultiplication and the associativity isomorphism is given as

$$\beta_{A,B,C}(a \otimes (b \otimes c)) = \Phi((a \otimes b) \otimes c).$$

If $\Phi$ further satisfies

$$[(Id \otimes Id \otimes \Delta)(\Phi)][(\Delta \otimes Id \otimes Id)(\Phi)] = [\Phi_{234}][(Id \otimes \Delta \otimes Id)(\Phi)][\Phi_{123}]$$

where $\Phi_{123} = \Phi \otimes 1$ and $\Phi_{234} = 1 \otimes \Phi$, then $A - \text{Mod}$ is, in fact, a coherent monoidal category (with proper concern given to units). Such an algebra is called a Drinfeld algebra $\mathcal{P}$ or a quasi-bialgebra $\mathcal{Q}$ (again, care must be given to units.)

We will construct $\bar{A}$, the free associative category on one generator. Roughly speaking, the objects in the category will be associations of $n$ letters for any positive integer $n$ (the elements of the free “anomic” algebra
with one binary operation – sometimes called a “magma” – on one generator. Morphisms are called reassociations. The tensor product, \( \otimes \), will concatenate two associations (i.e. multiplication in the anomic algebra). The tensor product of reassociations will be defined similarly.

3 \( \mathbf{A}_n \), The Catalan Groupoids

For each positive integer \( n \), we will construct the groupoid \( \mathbf{A}_n \) which has as objects associations of \( n \) letters and as morphisms reassociations. The free associative category will then be the disjoint union of all the \( \mathbf{A}_n \), i.e.

\[
\bar{\mathbf{A}} = \bigsqcup_{n \in \mathbb{N}^+} \mathbf{A}_n.
\]

These groupoids will be called the Catalan groupoids. The Catalan numbers

\[
c_n = \binom{2n-2}{n-1}
\]

are the number of associations of \( n \) letters with no ambiguity in the multiplication (see e.g. [7]). These groupoids are the categorical version of what people who study finite complexes call associahedra (e.g. [28]).

The categories \( \mathbf{A}_n \) are built up inductively in a manner not unrelated to the way Stasheff’s complexes \( K_n \) are built up ( [22] ). We let \( \mathbf{A}_1 = 1 \), the trivial category with one object and one identity morphism. Now assume that each of the \( \mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_{n-1} \) is defined. We define \( \mathbf{A}_n \) with the following pushout:

\[
\bigsqcup_{i+j+k=n} \mathbf{A}_i \times \mathbf{A}_j \times \mathbf{A}_k \times \hat{I} \xrightarrow{W_{i,j,k}} \bigsqcup_{a+b=n} \mathbf{A}_a \times \mathbf{A}_b \\
\bigsqcup_{i+j+k=n} \mathbf{A}_i \times \mathbf{A}_j \times \mathbf{A}_k \times I \xrightarrow{V_{i,j,k}} \mathbf{A}_n
\]

where \( i, j, k, a \) and \( b \) range over all positive integers and \( I \) (respectively \( \hat{I} \)) is the indiscrete (resp. discrete) category with two objects, 0 and 1. The left
hand vertical map is the obvious inclusion. The map $W_{i,j,k}$ is defined for each $i,j,$ and $k$ as follows: let $f,g,h$ be objects in $A_i, A_j, A_k$ respectively then

$$W_{i,j,k}(f,g,h,t) = \begin{cases} f, U_{j,k}(g,h) & : t = 0 \\ U_{i,j}(f,g),h & : t = 1 \end{cases}$$

where the $U_{j,k}$ and $U_{i,j}$ are defined from the previous pushouts assumed by the induction hypothesis. $W_{i,j,k}$ is defined for morphisms similarly.

A discussion of this pushout is in order. Each association of $n$ letters must have an “outermost” multiplication i.e. the last two segments of the word to be associated. There are $a$ letters to the left of this multiplication and $b$ letters to the right of this multiplication. Each of these smaller words are also associated. Hence the category $A_a \times A_b$. This outermost multiplication can occur anywhere within the word, hence the coproduct. All these smaller categories also have reassociations and they are carried over to the to the new $A_n$. There are, however, new reassociations that are handled by the left-hand side of the pushout. Reassociations are concerned with three smaller words, hence the $A_1 \times A_j \times A_k$. There are two ways of associating these three words into one whole word. $W_{i,j,k}$ maps these three words into the two ways of associating them. The pushout connects these two ways with isomorphisms.

Let’s look at the first few $A_n$. The objects of $A_n$ are to be thought of as associations of $n$ letters. We write the letters of the associations with non-boldface letters $A,B,C\ldots$ etc. The letters should not be thought of as objects in a category. Rather they are variables or “place holders”. We write $A \otimes (B \otimes C)$ as a shorthand for the functor $(-) \otimes ((-) \otimes (-)) = \otimes \circ (Id \times \otimes)$ from a category cubed to itself. Reassociations will be written as $\alpha$. We also show how each $A_n$ is “built up” from the lower groupoids. This is done by writing $A_a \times A_b$ for its image under $U_{a,b}$. Similarly for the image of $A_1 \times A_j \times A_k \times I$ under $V_{i,j,k}$.

- $A_1$ was defined to be 1. The reader can think of the category as looking like

$$A_1 = A$$

where $A$ is just a variable that corresponds to the single identity functor.
• $A_2$ is defined from the following pushout:

$$
\begin{array}{c}
\emptyset \\
\downarrow \\
1 \times 1 = 1 \\
\downarrow \\
\emptyset \\
\downarrow \\
A_2
\end{array}
$$

The left side of the pushout is a vacuous coproduct. So

$$A_2 = AB$$

• $A_3$

$$
\begin{array}{c}
A_1 \times A_2 \\
A(BC) \xrightarrow{\alpha} A_2 \times A_1 \\
\end{array}
$$

• $A_4$

$$
\begin{array}{c}
A((BC)D) \\
A(B(CD)) \xrightarrow{\alpha} (A(BC))D \\
\end{array}
$$

Note: This diagram does not commute.

• $A_5$. Some of the names of the edges are left out in order to make it
more readable.

\( A_2 \times A_1 \times A_2 \times I \) in the lower left corner describes the lower left quadrilateral. Similarly for \( A_3 \times A_1 \times A_1 \times I \). Notice the map \( \alpha_{AB,C,DE} \) is shown twice: once going around the top and once going around the bottom. There is only one such map and if the page is bent, one will see that \( A_5 \) is really a sphere.

4 \( \bar{A} \), The free associative category

We now define the free associative category on one generator.

\[
(\bar{A} = \coprod_{n \in \mathbb{N}^+} A_n, \otimes_{\bar{A}}, \bar{\alpha}).
\]

The tensor product, \( \otimes_{\bar{A}} \), is defined as follows: given \( \phi : f \to f' \) in \( A_i \), \( \gamma : g \to g' \) in \( A_j \) and \( \eta : h \to h' \) in \( A_k \) then \( f \otimes_{\bar{A}} g = U_{i,j}(f, g) \) in \( A_{i+j} \). The tensor of morphisms is defined as follows: \( \phi \otimes_{\bar{A}} \gamma = U_{i,j}(\phi, \gamma) \) in \( A_{i+j} \). Since \( U_{i,j} \) is a functor, \( \otimes_{\bar{A}} \) is defined. The reassociation is given by

\[
\bar{\alpha}_{f,g,h} = V_{i,j,k}(Id_{f}, Id_{g}, Id_{h}, I) : f \otimes_{\bar{A}} (g \otimes_{\bar{A}} h) \to (f \otimes_{\bar{A}} g) \otimes_{\bar{A}} h
\]
where \( \iota \) is the unique nontrivial isomorphism in \( I \). Naturality means that the following diagram commutes

\[
\begin{array}{ccc}
f \otimes \overline{A} (g \otimes \overline{A} h) & \xrightarrow{\alpha_{f,g,h}} & (f \otimes \overline{A} g) \otimes \overline{A} h \\
\phi \otimes \overline{A} (\gamma \otimes \overline{A} \eta) & & (\phi \otimes \overline{A} \gamma) \otimes \overline{A} \eta \\
f' \otimes \overline{A} (g' \otimes \overline{A} h') & \xrightarrow{\alpha_{f',g',h'}} & (f' \otimes \overline{A} g') \otimes \overline{A} h'.
\end{array}
\]

The simple observation that \( \phi \otimes \overline{A} (\gamma \otimes \overline{A} \eta) = V_{i,j,k}(\phi, \gamma, \eta, Id_0) \) with a similar identity for the right vertical map and the fact that \( V_{i,j,k} \) is a functor shows that the square indeed commutes.

We emphasize that the reassociation \( \overline{\alpha} \) is not coherent.

There is a well-known categorical principle that in any category \( C \), \( \text{Hom}_C(X, G) \) inherits the structure of the codomain object. If a category \( B \) has some higher-order structure (we remain suitably ambiguous) then, loosely speaking,

\[
\prod_{n \in \mathbb{N}^+} \text{Hom}_{\text{Cat}}(X^n, B)
\]

inherits this structure. In our case we set \( X = B \). We claim that if \( B \) has the structure of an associative category then so does

\[
\overline{B} = \prod_{n \in \mathbb{N}^+} \text{Hom}_{\text{Cat}}(B^n, B).
\]

Since this construction will be used often, we feel obliged to go through the gory details at least once. The tensor and reassociation are defined as follows.

Let \( f : B^q \to B \), \( g : B^r \to B \) and \( h : B^s \to B \). Then

\[
f \otimes_B g = \otimes_B (f \times g) : B^q \times B^r \xrightarrow{\phi \times \phi} B \times B \xrightarrow{\otimes_B} B.
\]

The reassociation

\[
\begin{array}{ccc}
B^{q+r+s} \downarrow \beta_{f,g,h} & \xrightarrow{f \otimes_B (g \otimes_B h)} & B \\
(f \otimes_B g) \otimes_B h
\end{array}
\]
is set to
\[ \bar{\beta}_{f,g,h}(b_1, b_2, \ldots, b_{q+r+s}) = \beta_{f(b_1, \ldots, b_q), g(b_{q+1}, \ldots, b_{q+r}), h(b_{q+r+1}, \ldots, b_{q+r+s})}. \]

We have proven the following.

**Lemma 1.** Given an associative category \((B, \otimes, \beta_B)\), then \((\overline{B}, \otimes_{\overline{B}}, \overline{\beta})\) also has the an associative category structure.

**Definition 2.** Given an associative category \((B, \otimes, \beta_B)\), the category of iterates of \(\otimes_B\), denoted \(\text{It}(B, \otimes_B, \beta_B)\) or \(\text{It}(B)\), is the associative subcategory of \((\overline{B}, \otimes_{\overline{B}}, \overline{\beta})\) generated by \(\text{Id}_B \in \text{Hom}(B, B)\).

\(\text{It}(B)\) can be looked at as a disjoint union \(\bigsqcup_{n \in \mathbb{N}^+} \text{It}(B)_n\) where \(\text{It}(B)_n\) is constructed recursively. \(\text{It}(B)_1 = \text{Id}_B\). \(\text{It}(B)_n\) has as objects \(\otimes_B \circ (f \times g)\) where \(f\) is an object in \(\text{It}(B)_a\), \(g\) is an object in \(\text{It}(B)_b\) and \(a + b = n\).

Morphisms of \(\text{It}(B)_n\) are generated like the objects, however, there are also morphisms inherited from the associative category structure of \(\overline{B}\).

**Definition 3 (Strict Tensor Functor).** Given two associative categories \((B, \otimes, \beta)\) and \((B', \otimes', \beta')\), a strict tensor functor between them is a functor \(F : B \rightarrow B'\) satisfying the following two requirements:

1) \(F(A \otimes B) = F(A) \otimes' F(B)\)
2) \(F(\alpha_{A,B,C}) = \alpha'_{F(A), F(B), F(C)}\)

**Proposition 1 (Universality of \(\overline{A}\)).** For every associative category \((B, \otimes, \beta)\), there is a unique strict tensor functor
\[ P^B : (\overline{A} = \bigsqcup_{n \in \mathbb{N}^+} A_n) \rightarrow (\overline{\text{It}(B)} = \bigsqcup_{n \in \mathbb{N}^+} \text{It}(B)_n) \]

such that

1) \(* \in A_1 \mapsto \text{Id}_B \in \text{It}(B)_1\)
2) \(* \in A_2 \mapsto \otimes \in \text{It}(B)_2\)
3) \(\alpha \ (\text{the isomorphism in } A_3) \mapsto \beta \ (\text{the isomorphism in } \text{It}(B)_3)\)
**Proof.** Any functor $S : \mathbb{A} \rightarrow \text{It}(\mathbb{B})$ has as its source a coproduct and hence can be described by its components $S_n : \mathbb{A}_n \rightarrow \text{It}(\mathbb{B})$. Since we require the first summand $A_1$ to go into the first summand, $\text{It}(\mathbb{B})_1$, and $S$ is a strict tensor functor, a quick induction shows that each $S_n$ actually lands in the $n$-th summand of $\text{It}(\mathbb{B})$ i.e. $\text{It}(\mathbb{B})_n$. By the requirements of the theorem $P_1$, $P_2$ and $P_3$ are forced. We construct and show uniqueness of $P_n$ with the following argument. Assume $P_1, P_2, \ldots, P_{n-1}$ are defined. Then $P_n$ is the unique functor making the following pushout complete:

\[ \begin{array}{ccc}
\prod_{i+j+k=n} A_i \times A_j \times A_k \times I & \xrightarrow{W_{i,j,k}} & \prod_{a+b=n} A_a \times A_b \\
U_{a,b} & & U_{a,b} \\
\prod_{i+j+k=n} A_i \times A_j \times A_k \times I & \xrightarrow{V_{i,j,k}} & A_n \\
\psi_{i,j,k} & & \phi_{i,j,k} \\
\end{array} \]

Where

$\varphi_{a,b}(f, g) = P_a(f) \otimes_B P_b(g) = P_2(\tau)(P_a(f) \times P_b(g))$

$\psi_{i,j,k}(f, g, h, 0) = P_i(f) \otimes_B (P_j(g) \otimes_B P_k(h)) = P_3(0)(P_i(f) \times P_j(g) \times P_k(h))$

$\psi_{i,j,k}(f, g, h, 1) = (P_i(f) \otimes_B P_j(g)) \otimes_B P_k(h) = P_3(1)(P_i(f) \times P_j(g) \times P_k(h))$.

Let $\phi : f \to f'$, $\gamma : g \to g'$ and $\eta : h \to h'$ then

$\psi_{i,j,k}(\phi, \gamma, \eta, \tau) = P_3(Id_1)(P_i(f) \times P_j(g) \times P_k(h)) \circ \beta_{f,g,h}$

$= \beta_{f',g',h'} \circ P_3(Id_0)(P_i(f) \times P_j(g) \times P_k(h))$

A simple diagram trace shows that the outer square commutes. The pushout insures that there is a unique map $P_n : \mathbb{A}_n \rightarrow \text{It}(\mathbb{B})_n$.

In order to show that $P$ is a strict tensor, let $f$ and $g$ be objects in $\mathbb{A}_a$
and $\mathbb{A}_b$ respectively. The following square commutes.

\[
\begin{array}{ccc}
\mathbb{A} \times \mathbb{A} & \xrightarrow{\oplus} & \mathbb{A} \\
\downarrow & & \downarrow \\
\mathbb{B} \times \mathbb{B} & \xrightarrow{\otimes} & \text{It}(\mathbb{B})
\end{array}
\]

The lower right-hand equality holds because that is just the upper right-hand triangle in the previous pushout diagram. Similar arguments are needed for morphisms. $P$’s uniqueness follows from the fact that $\varphi$ and $\psi$ are used in the proof of $P$ being strict. □

5 A, The 2-sketch of associative categories

In order to define the 2-sketch, we need to give the $\mathbb{A}_n$ an operad structure. An operad is a way of describing the structure of all the operations that an algebraic object has. There are many different definitions of operads, but we shall use the simplest definition that will meet our needs. We basically follow May’s [19] definition of a topological $A_\infty$ operad. For a general introduction to operads, see the first few papers of [14]. For a very general definition of an operad, see [13]. We will give here the definition of a non-$\Sigma$ non-unital operad in $\text{Cat}$.

**Definition 4** An operad $\mathcal{O}$ in $\text{Cat}$ consists of categories $\mathcal{O}_j, j \geq 1$ and composition functors

\[
Q : \mathcal{O}_k \times \mathcal{O}_{j_1} \times \cdots \times \mathcal{O}_{j_k} \longrightarrow \mathcal{O}_{j_1+j_2+\cdots+j_k}
\]
satisfying the following condition. If $\Sigma j_s = j$, $\Sigma i_t = i$, $g_s = j_1 + j_2 + \cdots + j_s$, and $h_s = i_{g_{s-1} + 1} + \cdots + i_{g_s}$ for $1 \leq s \leq k$, then the following “associativity” diagram commutes:

\[
\begin{array}{ccc}
O_k \times \prod_{s=1}^k O_{j_s} \times \prod_{r=1}^{j_s} O_{i_r} & \xrightarrow{Q \times Id} & O_j \times \prod_{r=1}^{j_s} O_{i_r} \\
\downarrow \text{Shuffle} & & \downarrow Q \\
O_k \times (\prod_{s=1}^k O_{j_s} \times (\prod_{q=1}^{j_s} O_{i_{g_{s-1}+q}})) & \xrightarrow{Id \times \prod_s Q} & O_k \times \prod_{s=1}^k O_{h_s}
\end{array}
\]

(2)

Each $O_j$ is to be thought of as the structure of the $j$-ary operations. $Q(c, d_1, \ldots, d_k)$ is the composition of the $c$ operation with the product of the $d_s$.

An operad in our paper is a way of combining associations. Given an association of $n$ letters and $n$ associations of $m_1, m_2, \ldots, m_n$ letters, an operad makes a new association of $t = \sum m_i$ letters by considering each of the $m_i$ letters to be one unit of the original association.

In order to make the discussion of operad more readable, we conform to the following conventions. When we refer to partitions of a set $X$, we mean an ordered set of disjoint nonempty subsets of $X$. Every (ordered) partition of $t$ objects into $n \leq t$ disjoint subsets can be written as an order-preserving surjection $\pi : t \to n$.

**Proposition 2** $A_1, A_2, \ldots, A_n, \ldots$ has the structure of an operad.

**Proof.** We define a functor

\[Q_{n,t,\pi} : A_n \times A_{m_1} \times A_{m_2} \times \cdots \times A_{m_n} \to A_t\]
for all $n$, for all $t \geq n$ and for all partitions $\pi : t \to n$ where $m_i = |\pi^{-1}(i)|$. We define the $Q_{n,t,\pi}$'s by induction on $n$. For $n = 1$, and for any $t \geq 1$ there is a unique $\pi : t \to 1$. We set

$$Q_{1,t,\pi} = \text{Proj} : A_1 \times A_t \to A_t.$$  

Now assume every $Q_{1}, Q_{2}, \ldots, Q_{n-1}$ is defined for each $t$ and partition $\pi$. We define $Q_{n,t,\pi}$ for a given $t$ and partition $\pi$ on objects and $1)$morphisms.

0) Let $f$ be an object in $A_n$ and $g_1, g_2, \ldots, g_n$ be objects in $A_{m_1}, \ldots, A_{m_n}$ respectively. Since each object $f$ in $A_n$ has the property that $f = U_{a,b}(f_1, f_2)$ $(a + b = n)$ for unique $a, b, f_1$ and $f_2$. We let

$$Q_{n,t,\pi}(f, g_1, g_2, \ldots, g_n) = U_{a,b}(Q_{a,t_1,\pi_1}(f_1, g_1, g_2, \ldots, g_a), Q_{b,t_2,\pi_2}(f_2, g_{a+1}, g_{a+2}, \ldots, g_{a+b})).$$

where

$$t_1 = \sum_{i=1}^{a} m_i = \sum_{i=1}^{a} |\pi^{-1}(i)|, \quad t_2 = \sum_{i=a+1}^{a+b} m_i = \sum_{i=a+1}^{a+b} |\pi^{-1}(i)|$$

and $\pi_1$ and $\pi_2$ are the restrictions of $\pi$ to $a$ and $b$.

1)Let $f$ be an object in $A_n$ and $g_1, g_2, \ldots, g_n$ be objects in $A_{m_1}, \ldots, A_{m_n}$ respectively. Since each object $f$ in $A_n$ has the property that $f = U_{a,b}(f_1, f_2)$ $(a + b = n)$ for unique $a, b, f_1$ and $f_2$. We let

$$Q_{n,t,\pi}(f, g_1, g_2, \ldots, g_n) = U_{a,b}(Q_{a,t_1,\pi_1}(f_1, g_1, g_2, \ldots, g_a), Q_{b,t_2,\pi_2}(f_2, g_{a+1}, g_{a+2}, \ldots, g_{a+b})).$$

An example of the way the operad works is in order. Let $n = 4$ and $t = 13$. Let $f \in A_4$ correspond to the following association: $A[[BC]D]$. Let $g_1, g_2, g_3$ and $g_4$ correspond to the following associations:

$$A(BC), \quad D, \quad (EF)((GH)I), \quad ((JK)L)M.$$
Then $Q(f, g_1, g_2, g_3, g_4)$ corresponds to

$$A(BC)[[D((EF)((GH)I))](JK)L)M].$$

For an example of the way reassociations are handled, let $\phi$ correspond to the reassociation $A[[BC]D] \to [AB][CD]$ and

$$A(BC) \quad D \quad (EF)((GH)I) \quad (JK)L)M$$

Then the operad will produce:

$$A(BC)[[D((EF)((GH)I))](JK)L)M]$$

If the $A_n$ is to be a real operad, the $Q_{n,t_1\ldots t}$ must satisfy the “associativity” diagram of Definition 4. We abandon all unnecessary subscripts for the benefit of the reader. $\vec{f}$ is used to mean a sequence of $f$’s of the “right” length. The lemma for morphisms is left to the reader.

**Lemma 2 (Associativity of Q)** Let $h$ be an object in $A_n$. Let $g_i$ be objects $A_{m_i}$. We have the following equality:

$$Q(h, Q(g_1, \vec{f}), \ldots, Q(g_n, \vec{f})) = Q(Q(h, g_1, \ldots, g_n), \vec{f})$$

**Proof.** By induction on $n$. If $n = 1$ then $h = * \in A_1$ and

$LHS = Q(*, Q(g_1, f, \ldots, f)) = Q(g_1, f, \ldots, f) = Q(Q(*, g_1), f, \ldots, f) = RHS.$
Assume the lemma is true for all $a < n$. Then $h \in A_n$ and $h = U_{a,b}(h_a, h_b)$ for unique $a, b, h_a$ and $h_b$. We have

$$LHS = \begin{cases} 1 & Q(h, Q(g_1, f), \ldots, Q(g_n, f)) \\ 2 & Q(U_{a,b}(h_a, h_b), Q(g_1, f), \ldots, Q(g_n, f)) \\ 3 & U_{a,b}(Q_a(h_a, Q(g_1, f), \ldots, Q(g_a, f)), Q_b(h_b, Q(g_{a+1}, f), \ldots, Q(g_{a+b}, f))) \\ 4 & U_{a,b}(Q(Q_a(h_a, g_1, \ldots, g_a), Q_b(h_b, g_{a+1}, \ldots, g_{a+b})), f) \\ 5 & Q(U(Q_a(h_a, g_1, \ldots, g_a), Q_b(h_b, g_{a+1}, \ldots, g_{a+b})), f) \\ 6 & Q(U_{a,b}(h_a, h_b), (g_1, \ldots, g_n), f) \\ 7 & Q(Q(U_{a,b}(h_a, h_b), g_1, \ldots, g_n), f) \end{cases} = RHS.$$

$=2$ and $=7$ are from the definition of $h$. $=3$, $=5$ and $=6$ are from definition of $Q$. $=4$ is from the induction hypothesis. □

A 2-sketch (called an algebraic 2-sketch in [6]) is a strict tensor 2-category whose underlying category (0-cells and 1-cells) is a sketch i.e. a sketch that is enriched over Cat. An algebra $F$ for a 2-sketch $G$ is a strict tensor 2-functor $F : G \rightarrow \text{Cat}$.

At last, we are ready to define the 2-sketch, $A$, of the theory of associative categories. $A$ is a strict tensor 2-category. The objects are the positive natural numbers. In order for $A$ to be a 2-category, it must be enriched over Cat i.e. every hom set must be a category and composition must be a functor. Given any two positive integers $n$ and $k$, we define the category

$$\text{Hom}_A(n, k) = \prod_{\pi : n \rightarrow k} A_{|\pi^{-1}(1)|} \times A_{|\pi^{-1}(2)|} \times \cdots \times A_{|\pi^{-1}(k)|} = \prod_{\pi : n \rightarrow k} \prod_{i=1}^{k} A_{|\pi^{-1}(i)|}$$

where $\pi$ ranges over all partitions of $n$ into $k$ parts. Notice that

$$\text{Hom}_A(n, k) = \begin{cases} A_n & : k = 1 \\ (A_1)^k = A_1 & : k = n \\ \emptyset & : k > n \end{cases}$$

Each object of $\text{Hom}_A(n, k)$ corresponds to a partial association of $n$ letters into $k$ parts. Each of the $k$ parts is totally associated. For example, a typical object in $\text{Hom}(10, 4)$ looks like this

$$(AB), \ (C(DE))F, \ G, \ H(IJ).$$
This object corresponds to the partition 10 = 2 + 4 + 1 + 3. We write objects of $\text{Hom}_A(n, k)$ as

$$f = (f_1, f_2, \ldots, f_k)$$

where each $f_i$ is an object of $A_{\pi_i-1(i)}$. Morphisms of $\text{Hom}_A(n, k)$ correspond to reassociations of partial associations. Since $\text{Hom}_A(n, k)$ is made up of a disjoint union, there are only reassociations of partial associations of the same partition. Morphisms are written as

$$\phi = (\phi_1, \phi_2, \ldots, \phi_k).$$

Composition: Consider the following situation:

Let $f = (f_1, f_2, \ldots, f_k)$ with its corresponding partition $\pi_f : n \rightarrow k$. Let $g = (g_1, g_2, \ldots, g_l)$ with its partition $\pi_g : k \rightarrow l$. Then horizontal composition, $\circ_H$, is defined as

$$g \circ_H f = h = (h_1, h_2, \ldots, h_l)$$

with corresponding partition $\pi_h = \pi_g \circ \pi_f : n \rightarrow l$, where

$$h_i = Q_{\pi_g^{-1}(i), \pi_f^{-1}(i), \pi_f(i)}(g_i, f_{i_1}, f_{i_2}, \ldots, f_{i_s}).$$

The $i_j$ range over $\pi_h^{-1}(i)$ and $\pi_f$ is a restriction of $\pi_f$ to this subset. An example is called for. Let $n = 10$, $k = 4$ and $l = 2$. Let $f$ correspond to

$$(A(BC)), \quad D, \quad (E(F(GH))), \quad (IJ).$$

Let $g$ correspond to

$$[AB], \quad [CD].$$

Then $g \circ_H f$ corresponds to

$$[(A(BC'))D], \quad [(E(F(GH)))(IJ)].$$

Horizontal composition of 2-cells is also done with the operad $Q$. Associativity of horizontal composition follows from lemma [4]. We leave the
details to the reader, however, it is obvious once you look at what the \( Q \)'s are constructed to do.

Vertical composition of 2-cells, \( \circ_V \), is much simpler. Let \( \phi = (\phi_1, \phi_2, \ldots, \phi_k) \) be of a reassociation of a particular partition and let \( \phi' = (\phi'_1, \phi'_2, \ldots, \phi'_k) \) be of the same partition, then

\[
\phi' \circ_V \phi = (\phi'_1 \circ \phi_1, \phi'_2 \circ \phi_2, \ldots, \phi'_k \circ \phi_k).
\]

Associativity of vertical composition is obvious.

In order for \( A \) to be an honest 2-category, horizontal and vertical composition must “commute” i.e.

**Lemma 3**

\[
(\gamma' \circ_H \phi') \circ_V (\gamma \circ_H \phi) = (\gamma' \circ_V \gamma) \circ_H (\phi' \circ_V \phi) : g \circ_H f \rightarrow g' \circ_H f'.
\]

**Proof.** LHS\(= (\lambda_1, \lambda_2, \ldots, \lambda_l) \) where

\[
\lambda_i = Q_{[\pi^{-1}_i(i)], m, \pi^i} (\gamma'_i, \phi'_i, \phi'_2, \ldots, \phi'_m) \circ Q_{[\pi^{-1}_i(i)], m, \pi^i} (\gamma_i, \phi_i, \phi_i, \phi_i, \ldots, \phi_i).
\]

RHS\(= (\rho_1, \rho_2, \ldots, \rho_l) \) where

\[
\rho_i = Q_{[\pi^{-1}_i(i)], m, \pi^i} (\gamma'_i \circ \gamma_i, \phi'_i \circ \phi_i, \phi'_i \circ \phi_i, \phi'_i \circ \phi_i, \ldots, \phi'_i \circ \phi_i).
\]

Since \( \pi_\gamma = \pi_\gamma' \) and \( \pi_\phi = \pi_\phi' \), these two \( Q \)'s are actually the same functor and by the functoriality of \( Q \), \( \lambda_i = \rho_i \) and hence LHS = RHS. \( \Box \)

**Proposition 3** \( A \) has a strict 2-tensor structure, \( \otimes_A \).

**Proof.**

0) 0-cells: \( n \otimes_A m = n + m \).

1) 1-cells: Let \( f = (f_1, f_2, \ldots, f_n) \) and \( g = (g_1, g_2, \ldots, g_m) \) be two 1-cells, then \( f \otimes_A g = (f_1, f_2, \ldots, f_n, g_1, g_2, \ldots, g_m) \).

2) 2-cells: This is done the same way as 1-cells. \( \Box \)

In order for the following definition to make sense, we must remind ourselves that \( \text{Cat} \), the 2-category of categories, functors and natural transformations, has a strict 2-tensor structure: product. (We parenthetically note that product is only coherently associative but we think of it - perhaps in error - as strict because of the usual coherence theories.)
Definition 5  Let $\text{Hom}_{\otimes}(A, \text{Cat})$ be the category (we forget, for now, its higher-order structure) whose objects are the strict tensor 2-functors

$$R : A \rightarrow \text{Cat}$$

i.e. the 2-functors that satisfy

0) $R(n \otimes_A m) = R(n) \times R(m)$,
1) $R(f \otimes_A g) = R(f) \times R(g)$ and
2) $R(\phi \otimes_A \gamma) = R(\phi) \times R(\gamma)$.

Morphisms are strict 2-natural transformations

$$F : R \Rightarrow S$$

i.e. 0) For every 0-cell $n$ in $A$ there is a functor $F(n) : R(n) \rightarrow S(n)$ such that $F(n \otimes_A m) = F(n) \times F(m)$.

1) For every 1-cell $f : n \rightarrow m$ in $A$ the following square commutes “on the nose”

$$\begin{array}{ccc}
R(n) & \xrightarrow{F(n)} & S(n) \\
\downarrow R(f) & & \downarrow S(f) \\
R(m) & \xrightarrow{F(m)} & S(m).
\end{array}$$

2) For every 2-cell $\phi : f \Rightarrow f'$ in $A$, the following square commutes “on
where the left and right 2-cells are \( R(\phi) \) and \( S(\phi) \).

**Definition 6** The category \( \text{Assoc} - \text{Cat}_{st} \) has as objects associative categories and as morphisms, strict tensor functors.

**Proposition 4 (A as 2-sketch of associative categories)** The category \( \text{Hom}_{st} \otimes (A, \text{Cat}) \) is equivalent to \( \text{Assoc} - \text{Cat}_{st} \).

**Sketch of Proof.** The proof calls for only a few minutes of staring at the definitions. We will not go through all the hideous details; but shall point the way. Given a strict tensor 2-functor \( R : A \rightarrow \text{Cat} \) we set the underlying category, \( \mathcal{B} \), to be \( R(1) \). Set \( \otimes_{\mathcal{B}} = R(f) \) where \( f \) is the unique morphism from 2 to 1 in \( A \). \( \beta_{\mathcal{B}} = R(\iota) \) where \( \iota \) is the unique nontrivial isomorphism in \( A_3 \).

To a strict 2-natural transformation \( F : R \Rightarrow S \) we assign a strict tensor functor (also called \( F' \)) \( F : R(1) \rightarrow S(1) \). To every positive natural number \( n = 1 + 1 + \cdots + 1 \), there is a functor

\[
[F(n) = F(1)^n] : [R(n) = R(1)^n] \rightarrow [S(n) = S(1)^n].
\]

If we let \( R(1) = \mathcal{B} \) and \( S(1) = \mathcal{B}' \), then the above line looks like the more familiar

\[ F^n : \mathcal{B}^n \rightarrow \mathcal{B}'^n. \]
The two commuting diagrams in the definition of strict 2-natural transformations correspond to the two requirements for a functor to be a strict tensor functor. □

6 Associative categories with units

Definition 7 An associative category with a unit is an associative category 
\((B, \otimes, \beta)\) with a distinguished object \(I \in B\) and the following two natural isomorphisms:

\[ L_A : I \otimes A \to A \quad R_A : A \otimes I \to A. \]

There are times when the following coherence condition will be important. An associative category with a unit in which the following diagram commutes is said to have unital coherence:

\[
\begin{array}{ccc}
(A \otimes I) \otimes B & \xrightarrow{\beta_{A,I,B}} & A \otimes (I \otimes B) \\
\downarrow R_A \otimes \text{Id}_B & & \downarrow \text{Id}_A \otimes L_B \\
A \otimes B & & A \otimes B
\end{array}
\]

If the \(L_A\) and \(R_A\) are identity natural isomorphisms then we say the associative category is unital strict.

The \(L_A\)’s and \(R_A\)’s are isomorphisms connecting associations of \(n + 1\) letters to associations of \(n\) letters. In the free associative category, they would be isomorphisms from the objects of \(A_{n+1}\) to the objects of \(A_n\). These isomorphisms can be formalized with a new inductive scheme of pushouts. We will only sketch this here. In this new formalism, we generate a new sequence of groupoids \(A'_0, A'_1, \ldots, A'_n, \ldots\) Notice that this time we have a \(A'_0\) whereas there is no \(A_0\). Each \(A'_n\) is actually a subgroupoid of \(A'_{n+1}\). The free associative category with unit will not be the coproduct of the \(A'_n\) rather it will be the colimit i.e.

\[ \tilde{A}' = \text{Colim}_{n \geq 0} A'_n. \]

The details of the structure of \(\tilde{A}'\) and its universal properties are straightforward.

22
The $A'_n$ also have the structure of an operad and this operad is used to construct $A'$, the 2-sketch of associative categories with units. There is one interesting difference that is worth pointing out. For $A$ we had

$$\text{Hom}_A(n,k) = \bigoplus_{\pi:n \to k} \prod_{i=1}^{\pi-1(i)} A_{|\pi^{-1}(i)|}.$$ 

where the partitions $\pi$’s are surjective. Here — when talking about units — we allow nonsurjective $\pi$’s. $|\pi^{-1}(i)|$ can equal 0 and we would get $A'_0$ which corresponds to the unit.

7 Maximal indiscrete subgroupoids

The rest of this paper is dedicated to calculating the fundamental group of each of the $A_n$ and their quotients. If the fundamental group of a quotient of $A_n$ is trivial, then there is only one path between any two vertices (objects) in the complex (groupoid) and it is called coherent.

In order to determine the fundamental group of a simplicial complex, one can use the method of maximal trees (see e.g. [20] or [15].) Given any complex $K$, in order to find the fundamental group of $K$, denoted $\pi(K)$, one associates a maximal tree, $T_K$, to $K$. A tree in $K$ is a connected subcomplex of $K$ which has no circuits. A maximal tree in $K$ is a tree in $K$ contained in no larger tree or, equivalently, a tree that contains all vertices of $K$. $\pi(K)$ is then given by the following presentation

- Generators
  1. All edges $(u,v)$ in $K$.

- Relations
  1. $(u,v) = e$ if $(u,v)$ is in $T_K$.
  2. $(u,v)(v,w) = (u,w)$ if $u,v,w$ lie in the same simplex of $K$.

There is a standard theorem that $\pi(K)$ is (up to conjugation) invariant under a change of the maximal tree.

When dealing with the Catalan groupoids, we employ a method analogous to maximal trees. To each groupoid $A_n$, we will assign $T_n$ a maximal
indiscrete subgroupoid (henceforth MIS) - the categorical analog of a maximal tree. A MIS is a subgroupoid with the same objects and exactly one isomorphism between each ordered pair of objects. It must be stressed that $T_n$ is an MIS and not a tree. There may, in fact, be circuits in our MIS but they correspond to a commuting part of the groupoid. (Both the language of trees and the language of categories will be employed. “vertex” and “object” will be interchanged, as will “edge” and “morphism”.)

The MIS’s are defined inductively. $T_1 = A_1 = 1$. Assume $T_1, T_2, \ldots, T_{n-1}$ are defined, then $T_n$ is constructed in a manner similar to pushout (1). We set $j = 1$ in pushout (1) and get the following pushout.

$$\bigsqcup_{i+1+k=n} T_i \times T_1 \times T_k \times W_{i,1,k} \xrightarrow{\bigcup_{a+b=n} T_a \times T_b} W_{a,b} \bigsqcup_{i+1+k=n} T_i \times T_1 \times T_k \times V_{i,1,k} \xrightarrow{U_{a,b}} T_n$$

where $W_{i,1,k}$ is the restriction of $W_{i,j,k}$ of our original pushout. (This is intuitive but actually too swift because we have not yet shown that $T_n$ is a subcategory of $A_n$. So define $W |$ in a similar manner to the way $W$ was in pushout (1).) $T_n$ contains only one class of morphisms between each component. This corresponds to moving the outermost parentheses one place to the left. We use the fact that one can get from any association of $n$ letters to any other association of $n$ letters by only moving the parentheses one letter at a time.

Now inductively define

$$L_n : T_n \rightarrow A_n.$$

$L_1 = Id_1 : T_1 \rightarrow A_1$. Assuming $L_1, L_2, \ldots, L_{n-1}$ are defined, $L_n$ is then
constructed from the following diagram:

\[
\begin{array}{c}
\prod_{i+j+k=n} A_i \times A_j \times A_k \times I \\
W_{i,j,k} \\
\prod_{a+b=n} A_a \times A_b \\
L_a \times L_b \\
\prod_{i+1+k=n} T_i \times T_1 \times T_k \times I \\
\prod_{a+b=n} T_a \times T_b \\
\prod_{i+1+k=n} T_i \times T_1 \times T_k \times I \\
V_{i,j,k} \\
\prod_{i+j+k=n} A_i \times A_j \times A_k \times I \\
L_n \times L \\
\end{array}
\]

The miters of the diagram are made of (co)products of \( L_a \). The upper and left-hand trapezoid commute because the miters are basically inclusions and the parallel morphisms are defined the same way. A diagram chase shows that \( A_n \) satisfies the inner pushout condition and hence there is a unique map \( L_n : T_n \rightarrow A_n \).

In order for \( T_n \) to be a MIS of \( A_n \), \( L_n \) must be bijective on objects (maximal) and for every \( t, t' \) in \( T_n \), \( Hom_{T_n}(t, t') = * \), the one object set (indiscrete).

\( L_n \) can be shown to be bijective on objects with a short inductive proof. The base case is true by definition. The inductive step follows from the fact that the right hand trapezoid commutes; \( U_{a,b} \) is bijective on objects; and the product of bijective-on-objects functors (the miters) is bijective-on-objects. So going around the right hand trapezoid are only functors that are bijective-on-objects.

An inductive proof is used to show that \( Hom(t, t') = * \). Assume \( T_a \) and \( T_b \) are indiscrete subgroupoids. Then the product of indiscrete subgroupoids are indiscrete subgroupoids and hence \( T_a \times T_b \) is an indiscrete subgroupoid. Since there is only one class of morphisms \( <i, 1, k> \) connecting these indiscrete subgroupoids, the entire \( T_n \) is indiscrete.

A few diagrams of the MIS are called for. We shall display the generators of the groupoid of the first few \( T_n \) and the way they sit in \( A_n \). For each \( A_n \) there are three types of generating morphisms:
1. those not in $T_n$ - denoted

![Diagram](image1)

2. those in $T_n$ within $T_a \times T_b$ for some $a$ and $b$ - denoted

![Diagram](image2)

3. those in $T_n$ of the form $A_1 \times A_1 \times A_k \times I$ - denoted

![Diagram](image3)

The first two maximal trees are simple $T_1 = A_1 = 1$, $T_2 = A_2 = 1$.

- $T_3$

![Diagram](image4)

- $T_4$

![Diagram](image5)
Each vertex is reached by $T_n$. The only circuits are naturality squares. The long top map is the same as the long bottom map. The single-line arrows have more than one letter in the center. The double-line arrows have only one letter in the center, but the map is tensored i.e. old. The triple-line arrows have only one letter in the center and are new i.e. not tensored.

Remark. There is nothing canonical about our MIS. We could have chosen as our MIS morphisms those morphisms of the form $<1,j,k>$, or some other scheme. This would have made a different MIS but we would get –up to conjugation– the same group at the end.

As with maximal trees, we must now collapse all the morphisms in the MIS to a point. This is done with the following pushout in the category of
categories or groupoids:

$$\begin{array}{ccc}
T_n & ! & 1 \\
\downarrow L_n & & \downarrow 1 \\
A_n & \rightarrow & \pi(A_n, T_n) \\
\end{array}$$

$\pi(A_n, T_n)$ can be thought of as the fundamental group of $A_n$ relative to $T_n$. Since we shall not change $T_n$ in this paper, we shorten $\pi(A_n, T_n)$ to $\pi(A_n)$. The fact that $L_n$ is surjective on objects, shows that $\pi(A_n)$ is a one object category. Since all the morphisms in $\pi(A_n)$ come from $A_n$ or $1$ which only have isomorphisms, $\pi(A_n)$ is a group. Every morphism in the MIS is sent to the identity of $\pi(A_n)$. The second type of relation comes out of the pushout and the way $T_n$ “sits in” $A_n$.

8 Presentation of the groups

The generators of the groups are equivalence classes of generating isomorphisms of the $A_n$. Two generating isomorphisms are equivalent if they are in the same image of $V_{i,j,k}$. The image of $A_i \times A_j \times A_k \times I$ under $V_{i,j,k}$, loosely speaking, only contributes new morphisms. Each of these morphisms are natural to one another i.e. they are two parallel sides of a square that commutes under naturality. This can be seen by considering the following situation. Let $\phi : f \rightarrow f'$ in $A_i$, $\gamma : g \rightarrow g'$ in $A_j$ and $\eta : h \rightarrow h'$ in $A_k$. Then the following square commutes out of the functoriality of $V_{i,j,k}$

$$
\begin{array}{c}
U(A_i \times A_{j+k}) & U(A_{i+j} \times A_k) \\
\downarrow & \downarrow \\
(f \otimes (g \otimes h)) & (f \otimes g) \otimes h \\
\downarrow & \downarrow \\
(f' \otimes (g' \otimes h')) & (f' \otimes g') \otimes h'.
\end{array}
$$
The left (respectively right) side of the diagram is in the image of \( A_i \times A_{i+j+k} \) under \( U_{i,j} \) (resp. \( A_{i+j} \times A_k \) under \( U_{i,j,k} \)). The entire diagram is contained in \( A_n \) where \( n = i + j + k \). The point is that the entire image of \( V_{i,j,k} \) is really a set of edges of the \( A_n \) and of its \( T_n \). We shall denote this set of edges as \( \langle i, j, k \rangle \). So \( \langle i, j, k \rangle \) can be thought of as a set of morphisms

\[
(f \otimes (g \otimes h)) \xrightarrow{\langle i, j, k \rangle} (f \otimes g) \otimes h.
\]

one for each \( f, g \) and \( h \). We may denote the above element of \( \langle i, j, k \rangle \) as \( \langle i, j, k \rangle_{f,g,h} \).

The presentation of \( \pi(A_n) \) will be given by a set of generators, \( G_n \), and a set of relations, \( R_n \). We represent the operation of the group as \( * \) and the trivial element as \( e \). The generators of the group will be of the form \( X < i, j, k > \) where \( X \) is an element of the free monoid on two generators \( \lambda \) and \( \rho \) (the monoid operation is represented as concatenation). Since the monoid is free, the length of an element is a well defined concept. Intuitively new generators in \( G_n \) are of the form \( \langle i, j, k \rangle \) where there is no prefix. Some generators are from old components and the monoid is used to describe which old component the generator is from. If the generator is from the right side of \( A_a \times A_b \) then we have \( a \) blanks on the left (\( \lambda^a \)). On the other hand, if the generator is on the left, we have \( b \) blanks to the right (\( \rho^b \)). In the latter case, for example, the generator might have come even come from an earlier component and more \( \lambda \)'s and \( \rho \)'s will be before the \( \langle i, j, k \rangle \). For each generator \( X < i, j, k > \) of \( G_n \), the length of \( X \) added to the sum of \( i, j \) and \( k \) will equal \( n \). In other words, a generator first started in \( A_{i+j+k} \) and we use the free monoid to describe how it sits in \( A_n \). In order to facilitate writing out the generators, we define a function \#. Given \( X \), an element of the free monoid and \( G_a \), a set of generators, we let

\[
X \# G_a = \{ Xy : y \in G_a \}.
\]

The relations are given as a set of elements of the form \( y_1 = y_2 \) by which we mean all elements of the form \( y_1 * y_2^{-1} \) are equal to \( e \). Given \( X \) an element of the free monoid and \( R_a \), a set of relations,

\[
X \# R_a = \{ Xy_1 * Xy_2 * \cdots * Xy_m : y_1 * y_2 * \cdots * y_m \in R_a \}.
\]

We begin by discussing the generators and relations and then give a formal definition. The next section has a few calculations carried out.
• Generators. The generators are equivalence classes of generating isomorphisms (edges) in the $A_n$. They form a set $G_n$. There are two types of such morphisms:

1. Morphisms from the old components. This corresponds to the upper right-hand corner of pushout (1). They are written as $X < i, j, k >$ where $ln(X) + i + j + k = n$.

2. New morphisms between the old components. This corresponds to the lower left-hand corner of pushout (1). They are written as $<i, j, k>$ where $i + j + k = n$.

• Relations. Relations come from:

1. commuting squares;
2. setting new morphisms that are in the MIS to the identity of the group. i.e. $< i, j, k > = e$ if $j = 1$;
3. old relations from the old components;
4. product of groupoids $A_a \times A_b$. (as in topological spaces, generators of product groupoids commute.)

Some more words on the first type of relation are needed. The only commuting parts of $A_n$ arise in the following situation. Let $\phi : f \rightarrow f'$ in $A_i$, $\gamma : g \rightarrow g'$ in $A_j$ $\eta : h \rightarrow h'$ in $A_k$ and

$$
\begin{align*}
  f \otimes (g \otimes h) & \xrightarrow{<i,j,k>\otimes\eta} (f \otimes g) \otimes h \\
  f' \otimes (g' \otimes h') & \xrightarrow{<i,j,k>\otimes\eta} (f' \otimes g') \otimes h'.
\end{align*}
$$
This square commutes if the following diagram commutes:

\[
\begin{array}{ccc}
  f \otimes (g \otimes h) & \xrightarrow{<i,j,k>} & (f \otimes g) \otimes h \\
  \phi \otimes (Id_g \otimes Id_h) & \xrightarrow{<i,j,k>} & (\phi \otimes Id_g) \otimes Id_h \\
  f' \otimes (g \otimes h) & \xrightarrow{<i,j,k>} & (f' \otimes g) \otimes h \\
  Id_{f'} \otimes (\gamma \otimes Id_h) & \xrightarrow{(b)} & (Id_{f'} \otimes \gamma) \otimes Id_h \\
  f' \otimes (g' \otimes h) & \xrightarrow{<i,j,k>} & (f' \otimes g') \otimes h \\
  Id_{f'} \otimes (Id_{g'} \otimes \eta) & \xrightarrow{(Id_{f'} \otimes Id_{g'}) \otimes \eta} & (Id_{f'} \otimes Id_{g'}) \otimes \eta \\
  f' \otimes (g' \otimes h') & \xrightarrow{<i,j,k>} & (f' \otimes g') \otimes h' \\
  \end{array}
\]

Squares of these type generate all commuting squares. Lets look at square (b) in depth. The left vertical map, \(Id_{f'} \otimes (\gamma \otimes Id_h)\), is denoted by \(\lambda^i \rho^k y\) for some \(y \in G_j\). Similarly, the right vertical map is denoted \(\rho^k \lambda^i y\). Since the square commutes – i.e. all four edges are “in the same simplex” – there is the relation:

\[<i,j,k> * \rho^k \lambda^i y = \lambda^i \rho^k y * <i,j,k>\]

The multiplication is written as regular multiplication rather then morphism composition. Since \(<i,j,k>\) denotes an isomorphism, in fact, a whole set of isomorphisms, we can take inverses. Thus the above equation looks like

\[<i,j,k> * \rho^k \lambda^i y * <i,j,k>^{-1} = \lambda^i \rho^k y.\]

This looks like the formula for HNN-extensions. This is made more apparent by looking at the categorical constructions of HNN-extensions. Given a group, \(G\), and two subgroups, \(A, B\) with an isomorphism \(f : A \rightarrow B\) between them, the HNN-extension, \(H\), is given as the following pushout in
the category of groupoids:

\[
\begin{array}{c}
A \times \hat{I} \\
\downarrow \hspace{1cm} \\
A \times I \\
\end{array} \xrightarrow{W} \begin{array}{c}
G \\
\downarrow \hspace{1cm} \\
H \\
\end{array}
\]

where \( W(a,o) = a \) and \( W(a,1) = f(a) \). Our pushout is then just a much more complicated version of this. The \(<i,j,k>\)'s are to be thought of as the new generators that extend all the old groups.

There is a special case of the above situation. When \( j = 1 \) the \(<i,j,k>\) generator is set to \( e \) and the above equation looks like

\[
\rho^k \lambda^i y = \lambda^i \rho^k y.
\]

This is the relation for a free product with amalgamation of groups. This can be thought of as changing the \( I \) in the lower left-hand corner of the above pushout into \( 1 \), the trivial one object groupoid (with the left vertical map as the projection on \( A \)).

We have only looked at the square \((b)\). There are, however, similar equations for the other two types of boxes and they are given in the scheme.

We give the inductive scheme for the generators and relations. Throughout the scheme, \( i, j \) and \( k \) are positive integers. \( G_1 = R_1 = \emptyset \). Assume we have all the \( G_k \) and \( R_k \) for \( k \leq n - 1 \) then \( G_n \) and \( R_n \) is given as:

- **Generators.** \( G_n = \)

1. \([((\lambda^i \# G_{n-i}) \cup (\rho^{n-i} \# G_i)) : i = 1, 2, \ldots, n-1]) \cup \)
2. \(<i,j,k> : i + j + k = n\>.

- **Relations.** \( R_n = \)

1. For each \(<i,j,k>\) such that \( i + j + k = n \) we have the unions of the following relations.

   (a) \(<i,j,k> \ast \rho^k \rho^j x \ast <i,j,k>^{-1} = \rho^{j+k} x : x \in G_i\>\)

   (b) \(<i,j,k> \ast \rho^k \lambda^i y \ast <i,j,k>^{-1} = \lambda^i \rho^k y : y \in G_j\>

32
(c) \{ <i,j,k> * \lambda^i j^k z * <i,j,k>^{-1} = \lambda^i j^k z : z \in G_k \}

2. \{ <i,1,k> = e : i + 1 + k = n \} \cup

3. \{ [(\lambda^i \# R_{n-i}) \cup (\rho^{n-i} \# R_i) : i = 1, 2, \ldots, n - 1] \} \cup

4. \{ \rho^{n-i} x * \lambda^i y = \lambda^i y * \rho^{n-i} x : x \in G_i, y \in G_{n-i} \} \}

9 The first few groups

In order to give the presentations in an clear fashion, we conform to the following conventions about listing the generators and relations:

- The old edges are listed in columns below the names of the components that contributed them. The new edges are listed at the bottom.

- If an old edge was set to \( e \), we do not list it in further groups. For example \( <1,1,1> \) will be listed in \( G_3 \). Since it is set to \( e \), we do not list it or any of its “progeny” (e.g. \( \lambda <1,1,1> \) or \( \rho \lambda^3 \rho <1,1,1> \)).

- We do not list relations about edges that were set to \( e \).

- Since the first nontrivial relation is in \( R_6 \) we do not list a relation of type 3 until \( R_7 \).

- Since the first nontrivial generator is in \( G_4 \), the first time we have a relation of type 4 is in \( R_8 \). Due to the fact that \( G_8 \) and \( R_8 \) will not be listed in this paper, we feel obliged to give this relation of type 4:

\[
\rho^4 <1,2,1> * \lambda^4 <1,2,1> = \lambda^4 <1,2,1> * \rho^4 <1,2,1> .
\]

- If two generators are set to be equal, then, in the future, we only list one of them. We usually choose the shorter name e.g. \( \lambda^2 \rho <1,2,1> \) rather than \( \lambda \lambda \rho <1,2,1> \).

Here are the groups:

\[
\begin{array}{|c|c|}
\hline
G_3 & A_1 \times A_2 & A_2 \times A_1 \\
\hline
<1,1,1> & \hline
\end{array}
\]
So $\pi(A_3)$ is the trivial group.

| $A_1 \times A_3$ | $A_2 \times A_2$ | $A_3 \times A_1$ |
|------------------|------------------|------------------|
| $\lambda < 1,1,1>$ | $\rho < 1,1,1>$ |                  |
|                  | $< 1,2,1>$       | $< 1,1,2>, <2,1,1>$ |

$\pi(A_4)$ is the free group generated by the single generator $< 1,2,1 >$. This can be thought of as the fundamental group of the noncommuting pentagon ($=\pi(S^1)$)

| $A_1 \times A_4$ | $A_2 \times A_3$ | $A_3 \times A_2$ | $A_4 \times A_1$ |
|------------------|------------------|------------------|------------------|
| $\lambda < 1,2,1>$ |                  | $\rho < 1,2,1>$ |                  |
|                  | $< 1,3,1>$       |                  |                  |
|                  | $< 1,2,2>, <2,2,1>$ |                  |                  |
|                  | $< 1,1,3>, <2,1,2>, <3,1,1>$ |                  |                  |
$\pi(A_5)$ is the free group generated by five generators. There are six pentagons in $A_5$ but they are linked up in such a way that there are only five generators. This is similar to the fact that although the cube has six faces (squares) there are only five generators i.e. only five faces must commute in order for the entire cube to commute.

There is one unwritten trivial relation

$$<1,3,1> \ast \rho \lambda <1,1,1> \ast <1,3,1>^{-1} = \lambda \rho <1,1,1>.$$ 

However since $X <1,1,1>$ is set to $e$, the relation is superfluous. This relation corresponds to the center square in $A_5$. 

\begin{verbatim}
R_5

\begin{tabular}{|c|}
\hline
$<1,1,3> = e$

$<2,1,2> = e$

$<3,1,1> = e$

\hline
\end{tabular}
\end{verbatim}
The first HNN-extension is the first non-trivial relation that we have. The second relation is an amalgamation because \( <1,1,4> = e \) and so the relation reduces to

\[ \lambda^2 <1,2,1> = \lambda \lambda <1,2,1> . \]

This essentially shows the associativity of the free monoid on two generators. Similarly for the third relation. So \( \pi(A_6) \) is a group with 22 generators (12 old + 10 new.) Two pairs of old generators are set equal to each other and four new generators are set to \( e \). We are left with 16 generators (10 old + 6 new.) There is one nontrivial relation on these generators. However the group is isomorphic to the free group on 15 generators.
| $A_1 \times A_6$ | $A_2 \times A_5$ | $A_3 \times A_4$ | $A_4 \times A_3$ | $A_5 \times A_2$ | $A_6 \times A_1$ |
|------------------|------------------|------------------|------------------|------------------|------------------|
| $\lambda\lambda < 1,2,1 >$ | $\lambda^2 < 1,2,1 >$ | $\lambda < 1,2,1 >$ | $\lambda^2 < 1,2,1 >$ | $\lambda^2 < 1,2,1 >$ | $\lambda^2 < 1,2,1 >$ |
| $\lambda\lambda < 1,2,1 >$ | $\lambda^2 < 1,2,1 >$ | $\lambda < 1,2,1 >$ | $\lambda^2 < 1,2,1 >$ | $\lambda^2 < 1,2,1 >$ | $\lambda^2 < 1,2,1 >$ |
| $\lambda < 1,3,1 >$ | $\lambda < 1,3,1 >$ | $\lambda < 1,2,2 >$ | $\lambda < 1,2,2 >$ | $\lambda < 1,2,2 >$ | $\lambda < 1,2,2 >$ |
| $\lambda < 2,2,1 >$ | $\lambda^2 < 2,2,1 >$ | $\lambda < 1,2,1 >$ | $\lambda^2 < 1,2,1 >$ | $\lambda^2 < 1,2,1 >$ | $\lambda^2 < 1,2,1 >$ |
| $\lambda < 1,3,1 >$ | $\lambda < 1,3,1 >$ | $\lambda < 1,2,2 >$ | $\lambda < 1,2,2 >$ | $\lambda < 1,2,2 >$ | $\lambda < 1,2,2 >$ |
| $\lambda < 2,2,1 >$ | $\lambda^2 < 2,2,1 >$ | $\lambda < 1,2,1 >$ | $\lambda^2 < 1,2,1 >$ | $\lambda^2 < 1,2,1 >$ | $\lambda^2 < 1,2,1 >$ |
| $\lambda < 1,4,1 >$ | $\lambda < 1,4,1 >$ | $\lambda < 1,3,2 >$ | $\lambda < 1,3,2 >$ | $\lambda < 1,3,2 >$ | $\lambda < 1,3,2 >$ |
| $\lambda < 2,3,1 >$ | $\lambda < 2,3,1 >$ | $\lambda < 1,2,3 >$ | $\lambda < 1,2,3 >$ | $\lambda < 1,2,3 >$ | $\lambda < 1,2,3 >$ |
| $\lambda < 2,3,2 >$ | $\lambda < 2,3,2 >$ | $\lambda < 1,2,2 >$ | $\lambda < 1,2,2 >$ | $\lambda < 1,2,2 >$ | $\lambda < 1,2,2 >$ |
| $\lambda < 3,2,1 >$ | $\lambda < 3,2,1 >$ | $\lambda < 1,2,1 >$ | $\lambda < 1,2,1 >$ | $\lambda < 1,2,1 >$ | $\lambda < 1,2,1 >$ |

$< 1,5,1 >$
$< 1,4,2 >, < 2,4,1 >$
$< 1,3,3 >, < 2,3,2 >, < 3,3,1 >$
$< 1,2,4 >, < 2,2,3 >, < 3,2,2 >, < 4,2,1 >$
$< 1,1,5 >, < 2,1,4 >, < 3,1,3 >, < 4,1,2 >, < 5,1,1 >$
\[ R_7 \]

\(< 1, 5, 1 > * \rho \lambda \lambda < 1, 2, 1 > * < 1, 5, 1 >^{-1} = \lambda \rho \lambda < 1, 2, 1 >
\(< 1, 5, 1 > * \rho \lambda \rho < 1, 2, 1 > * < 1, 5, 1 >^{-1} = \lambda \rho \rho < 1, 2, 1 >
\(< 1, 5, 1 > * \rho \lambda < 1, 3, 1 > * < 1, 5, 1 >^{-1} = \lambda \rho < 1, 3, 1 >
\(< 1, 5, 1 > * \rho \lambda < 1, 2, 2 > * < 1, 5, 1 >^{-1} = \lambda \rho < 1, 2, 2 >
\(< 1, 5, 1 > * \rho \lambda < 2, 2, 1 > * < 1, 5, 1 >^{-1} = \lambda \rho < 2, 2, 1 >
\(< 1, 4, 2 > * \rho^2 \lambda < 1, 2, 1 > * < 1, 4, 2 >^{-1} = \lambda \rho^2 < 1, 2, 1 >
\(< 2, 4, 1 > * \rho \lambda^2 < 1, 2, 1 > * < 2, 4, 1 >^{-1} = \lambda^2 \rho < 1, 2, 1 >
\)

\[ \star < 1, 2, 4 > * \lambda^3 < 1, 2, 1 > * < 1, 2, 4 >^{-1} = \lambda \lambda \lambda < 1, 2, 1 >
\[ \star < 4, 2, 1 > * \rho \rho^2 < 1, 2, 1 > * < 4, 2, 1 >^{-1} = \rho^3 < 1, 2, 1 >
\[ < 1, 1, 5 >:
\lambda^2 \lambda < 1, 2, 1 > = \lambda \lambda^2 < 1, 2, 1 >
\lambda^2 \rho < 1, 2, 1 > = \lambda \lambda \rho < 1, 2, 1 >
\lambda^2 < 1, 3, 1 > = \lambda \lambda < 1, 3, 1 >
\lambda^2 < 1, 2, 2 > = \lambda \lambda < 1, 2, 2 >
\lambda^2 < 2, 2, 1 > = \lambda \lambda < 2, 2, 1 >
\[ < 5, 1, 1 >:
\rho \rho \lambda < 1, 2, 1 > = \rho^2 \lambda < 1, 2, 1 >
\rho^2 < 1, 2, 1 > = \rho^2 \rho < 1, 2, 1 >
\rho \rho < 1, 3, 1 > = \rho^2 < 1, 3, 1 >
\rho \rho < 1, 2, 2 > = \rho^2 < 1, 2, 2 >
\rho \rho < 2, 2, 1 > = \rho^2 < 2, 2, 1 >
\[ < 2, 1, 4 >:
\lambda^3 < 1, 2, 1 > = \lambda \lambda \lambda < 1, 2, 1 >
\[ < 4, 1, 2 >:
\rho^2 \rho < 1, 2, 1 > = \rho^3 < 1, 2, 1 >
\]

\[ \lambda \# R_6 : \]
\lambda < 1, 4, 1 > * \lambda \rho \lambda < 1, 2, 1 > * \lambda < 1, 4, 1 >^{-1} = \lambda \lambda \rho < 1, 2, 1 >
\rho \# R_6 :
\rho < 1, 4, 1 > * \rho \rho \lambda < 1, 2, 1 > * \rho < 1, 4, 1 >^{-1} = \rho \lambda \rho < 1, 2, 1 >

38
There are 57 (42 old + 15 new) generators. After amalgamations and setting some of the new generators to $e$ we have 42 (32 old + 10 new) generators. There are 11 relations on these 42 generators.

If you combine all the relations you get

$$\rho^3 < 1, 2, 1 > = \rho \rho^2 < 1, 2, 1 > = \rho^2 \rho < 1, 2, 1 >$$

i.e. $\rho$ is associative. Similarly for $\lambda$.

This group has examples of the first nontrivial relations. Consider

$$< 4, 2, 1 > * \rho^2 < 1, 2, 1 > * < 4, 2, 1 >^{-1} = \rho^3 < 1, 2, 1 >$$

which reduces to

$$< 4, 2, 1 > * \rho^3 < 1, 2, 1 > * < 4, 2, 1 >^{-1} = \rho^3 < 1, 2, 1 >$$

or

$$< 4, 2, 1 > * \rho^3 < 1, 2, 1 > = \rho^3 < 1, 2, 1 > * < 4, 2, 1 >$$

This relation can actually be seen. Look at the following diagram of associations and reassociations.
The horizontal maps move the round parenthesis and the vertical maps move the square parenthesis. The top row and the bottom row are the same. As are the left side and the right side. Each square commutes out of naturality. The whole thing is a torus. As with every torus, the generators commute. The two generators have been marked off. Going around $\rho^3 < 1, 2, 1 >$ and then going around $< 4, 2, 1 >$ is the same as doing it vice versa. All other maps are part of equivalence classes that have a 1 in the center position of the $< i, j, k >$ bracket and hence are set equal to $e$.

The important point is that the two stared relations show commutativity of generators. These generators do not show up in any other relations and hence the group is not free! The higher groups contain this group and hence they are also not free.

## 10 Fundamental groups of quotients

In order to look at quotients of associative categories, we must define congruences that respect the associative category structure.

**Definition 8** A congruence on an associative category $(\mathcal{B}, \otimes, \beta)$ is an equivalence relation $\sim$ for each Hom set of morphisms of $\mathcal{B}$ such that

1. composition is respected;
2. the $\otimes$ is respected, i.e. if $f \sim f'$ and $g \sim g'$, then $f \otimes g \sim f' \otimes g'$;
3. naturality is respected, i.e. for any $f : A \rightarrow A'$, $g : B \rightarrow B'$ and $h : C \rightarrow C'$, we have

$$((f \otimes g) \otimes h) \circ \beta_{A,B,C} \sim \beta_{A',B',C'} \circ (f \otimes (g \otimes h)).$$

By the customary universal algebra tricks, to every congruence on $(\mathcal{B}, \otimes, \beta)$ there exists a unique associative category $(\tilde{\mathcal{B}}, \tilde{\otimes}, \tilde{\beta})$ and a unique strict tensor functor $\Pi : \mathcal{B} \rightarrow \tilde{\mathcal{B}}$ that satisfies universal properties. $\tilde{\mathcal{B}}$ is isomorphic on objects. Morphisms are equivalence classes of morphisms of $\mathcal{B}$, denoted $[f]$. The tensor product is defined as usual: $[f] \tilde{\otimes} [g] = [f \otimes g]$. The associativity is $\tilde{\beta}_{A,B,C} = [\beta_{A,B,C}]$. Condition 3 insures naturality of the associativity.

The next step is to look at quotients of the 2-sketch $\mathbf{A}$ of the theory of associative categories. Since $\mathbf{A}$ is a strict tensor 2-category, we must define
what congruences of such categories are like. Our only interest is when 2-cells of \(A\) (which correspond to 1-cells of \(\tilde{A}\)) are set equal to each other, not 0-cells or 1-cells. So we have the following definition.

**Definition 9** A 2-congruence on a strict tensor 2-category \((C, \otimes)\) is an equivalence relation \(\sim\) on each set of 2-cells between any two 1-cells of \(C\) such that

1. the \(\otimes\) is respected, i.e. if \(\phi \sim \phi'\) and \(\gamma \sim \gamma'\), then \((\phi \otimes \gamma) \sim (\phi' \otimes \gamma')\);

2. vertical composition is respected, i.e. if \(\phi \sim \phi'\) and \(\gamma \sim \gamma'\), then 
   \[
   (\phi \circ_V \gamma) \sim (\phi' \circ_V \gamma');
   \]

3. horizontal composition is respected (similar to 2).

By generalized universal algebra, to every 2-congruence on a strict tensor 2-category \((C, \otimes)\) there exists a unique strict tensor 2-category \((\tilde{C}, \tilde{\otimes})\) and a unique strict tensor 2-functor \(\Pi : C \rightarrow \tilde{C}\) that satisfies the usual universal properties. \(\tilde{C}\) is isomorphic on 0-cells and 1-cells. 2-cells are equivalence classes of 2-cells of \(C\), denoted \([\phi]\). The tensor product, vertical composition and horizontal composition are defined as usual. The “commutativity” of vertical composition and horizontal composition in \(\tilde{C}\), falls out of the fact that \(C\) has this property. The converse of this statement is also true: given two strict tensor 2-categories and a strict tensor 2-functor between them that is isomorphic on 0-cells and 1-cells, we get such a 2-congruence.

We shall now examine what a 2-congruence for our 2-sketch \(A\) is like. The fact that the 2-congruence preserves tensor product means that if \(\phi = (\phi_1, \ldots, \phi_k) \sim (\phi_1', \ldots, \phi_k')\) and \(\gamma = (\gamma_1, \ldots, \gamma_l) \sim (\gamma_1', \ldots, \gamma_l') = \gamma'\) then

\[
\phi \otimes \gamma = (\phi_1, \ldots, \phi_k, \gamma_1, \ldots, \gamma_l) \sim (\phi_1', \ldots, \phi_k', \gamma_1', \ldots, \gamma_l') = \phi' \otimes \gamma'.
\]

Since we are interested in associative categories in \(\text{Cat}\) i.e. strict tensor 2-functors \(F\) from \((A, \otimes)\) to \((\text{Cat}, \times)\), we have the following

\[
F(\phi) \times F(\gamma) = F(\phi \otimes \gamma) = F(\phi' \otimes \gamma') = F(\phi') \times F(\gamma'),
\]

and hence using usual properties of the product in \(\text{Cat}\) \(F(\phi) = F(\phi')\) and \(F(\gamma) = F(\gamma')\). Such functors determine quotients and hence if \((\phi \otimes \gamma) \sim\)
\((\phi' \otimes \gamma')\) \(\sim \phi'\) \(\text{and} \ \gamma \sim \gamma'\). By a small inductive argument it suffices to discuss only individual components of the 2-cells. Hence we shall talk about \(\phi_i \sim \phi'_i\) where they are both elements of \(A_{n_i}\) for some \(n_i\). Hence we have, that every 2-congruence on \(A\) induces a congruence on each of the \(A_{n_i}\).

Vertical composition of 2-cells in \(A\) is simply composition in each \(A_{n_i}\).

Finally we are left with horizontal composition which is the heart of the matter. We remind the reader that horizontal composition is defined with the use of the operad \(Q\). A review of how the operad is defined and employed would be beneficial at this time. If \(\phi \sim \phi'\) and \(\gamma \sim \gamma'\) then

\[
\Lambda = (\phi \circ_H \gamma) \sim (\phi' \circ_H \gamma') = \Delta.
\]

By the argument above, this means that

\[
\Lambda_i = Q(\phi_i, \gamma_{n_1}, \ldots, \gamma_{n_s}) \sim Q(\phi'_i, \gamma'_{n_1}, \ldots, \gamma'_{n_s}) = \Delta_i
\]

where the subscripts of \(Q\) are abandoned since they are the same \(Q\).

Between every two 1-cells of \(A\) there is a distinct 2-cell, denoted \(\tilde{\gamma}\), that has all its components \(\tilde{\gamma}_i\) in the MIS \(T_{n_i}\). We call such a 2-cells a “branch” (branches make up trees.) Given two branches \(\tilde{\gamma}\) and \(\tilde{\gamma}'\), their tensor product \(\tilde{\gamma} \otimes \tilde{\gamma}'\) is also a branch. Similarly for vertical composition. “Branchness” is also preserved by horizontal composition because it is preserved by the operad \(Q\) i.e.

\[
Q(\tilde{\phi}, \tilde{\gamma}_{n_1}, \ldots, \tilde{\gamma}_{n_s}) = \tilde{\phi}'.
\]

This is shown by an inductive proof on the construction of the \(Q\)’s and from the fact that the \(V_{i,j,k}\) are built up from earlier \(V_{i,j,k}\).

For every congruence, there is a unique equivalence class between every two 1-cells, namely, those 2-cells congruent to the unique branch 2-cell. We denote this equivalence class by \([\tilde{\phi}]\).

Given an associative category \((B, \otimes, \beta)\), there are unique strict tensor functors \(P_n : A_n \rightarrow \text{It}(B)_n\) as was shown in Section 4. We use \(P_n\) in the
Both squares are pushouts. The inner square is familiar from Section 7. The result of the outer pushout is \( \pi(\text{It}(B)_n) \) which is shortened to \( \pi(B)_n \) (this should not seem so strange since the tensor product and the reassociation of \( \text{It}(B)_n \) is basically the same as the tensor product and the reassociation of \( B \).) The induced arrow is denoted by \( \pi(P_n) \). The functorial properties of \( \pi(-) \) will not be discussed here. Since the left hand vertical map is surjective on objects, \( \pi(B)_n \) is a group and we call it the fundamental group of \( B_n \). We must stress that \( P_n \) is not necessarily full (surjective on morphisms). A typical example is when \( B \) is a strict tensor category with \( \beta \) being something other than the identity. Then \( A_3 \) is the indiscrete category with two objects whereas the \( \text{It}(B)_3 \) has one object and - by composition - infinite morphisms. Our interest lies not in \( \pi(B)_n \) but in the image of \( \pi(P_n) \) since there are to be found the morphisms that are of concern to coherence.

Now for a short discussion of the way \( Q \) works. \( Q(\phi, \gamma, \ldots, \gamma) \) takes \( \phi \) and for each letter that \( \phi \) reassociates, puts in other letters. How many other letters? One or more, depending on what partition we have. Now if \( \phi \) is a generating morphism of the form \( \phi = V_{i,j,k}(\phi_i, \phi_2, \phi_3) \) then

\[
Q(\phi, \gamma, \ldots, \gamma) = V_{i',j',k'}(\phi'_i, \phi'_2, \phi'_3)
\]

by the definition of \( Q \) given in Section 1.4. This can be denoted in our group theoretic notation as

\[
Q(<i, j, k>, \gamma, \ldots, \gamma) = <i', j', k'>
\]
where \( i', j' \geq j \) and \( k' \geq k \). Similarly if \( \phi = \phi_a \otimes \phi_b \) then
\[
Q(\phi, \gamma, \ldots, \gamma) = Q(\phi_a, \gamma, \ldots, \gamma) \otimes Q(\phi_b, \gamma, \ldots, \gamma).
\]

In group theoretic notation we have
\[
Q(X < i, j, k >, \gamma, \ldots, \gamma) = X' < i', j', k' >
\]
where \( i', j' \geq j, k' \geq k \) and \( \text{length}(X') \geq \text{length}(X) \).

The important point is that every 2-cell (of \( A \) or 1-cell of \( A_n \)) that is a branch, goes to \( e \), the identity of \( \pi(B_n) \).

Putting this all together we have the following.

**Theorem 1** If \( j \geq 2, i + j + k = n \) and \( \pi(P_n)(< i, j, k >) = e \) (the identity of the \( \text{It}(B_n) \)), then for all \( i', j', k' \) with \( i' + j' + k' = n' \) we have \( \pi(P_{n'})(< i', j', k'>) = e \).

This theorem is a proper generalization of the Mac Lane's coherence theorem and we have:

**Corollary 1 (Mac Lane’s Coherence Theorem)** If \( \pi(P_4)(< 1, 2, 1 >) = e \), then for all \( i', j', k' \) with \( i' + j' + k' = n' \) we have \( \pi(P_{n'})(< i', j', k'>) = e \) and hence \( \pi(P_{n'}) \) is the trivial map for all \( n' \).

**Proof.** Set \( i = 1, j = 2 \) and \( k = 1 \) in the above theorem. The fact that \( \pi(P_{n'}) \) is trivial for all \( n' \) comes from the fact that all the generators of \( \pi(A_n) \) are of the form \( X < i, j, k > \) where \( X \) is an element of the free monoid on \( \lambda, \rho \). If \( < i, j, k > = e \), then \( X < i, j, k > = Xe = e \).

Let us return to a typical category that was briefly discussed in Section 2. Let \( R \) be a ring with a unit and \( R - \text{Mod} \) denote the category of left and right \( R \) modules. \( R - \text{Mod} \) has the usual tensor product. Then we have the set of all nonpathological reassociation natural transformations
\[
\beta_{A,B,C} : A \otimes (B \otimes C) \rightarrow (A \otimes B) \otimes C
\]
of the form
\[
a \otimes (b \otimes c) \rightarrow \phi(a' \otimes b') \otimes c'.
\]
Since \( \beta \) must be natural, \( \phi \) is a central element of the ring. Since \( \beta \) must be an isomorphism, \( \phi \) must have a multiplicative inverse. So \( \phi \in CU(R) \),

45
the central units of $R$. $CU(R)$ forms an abelian multiplicative group. Let $\phi \in CU(R)$, then $O(\phi) \in \mathbb{N} \cup \{\infty\}$ is the number of times that you must go around the pentagon in order to get to the identity. If $O(\phi) = 1$ then $\phi$ induces a coherent reassociator. This order turns out to be nothing more then asking what power do we have to raise $\phi$ to, in order to get the unit i.e. $O(\phi) = n$ iff $\phi^n = 1$. So noncoherence for It($R - \text{Mod}$)$_4$ is classified by $CU(R)$.

The conclusion is that there are many forms of noncoherence but we can classify them as a tree. If going around a certain loop in $A_n$ is set equal the identity, then there are implications for the higher $A_n$. That is, every relation in $A_n$ has ramifications for the $A_k$ for $k \geq n$. However, for non-unital associative categories, there are no ramifications for $A_l$ for $l \leq n$. When we add the unit, we make the classification more interesting.

11 Quotients of unital associative categories

Since we have not given a formal construction of $\bar{A}'$ (see Section 6,) the free associative category with a unit or of $A'$ the theory of associative categories with units, we shall give an intuitive rather than a formal discussion of their fundamental groups. All congruences must also take into account the morphisms $L, R : A_n \rightarrow A_{n-1}$. Hence there are, in fact, fewer congruences if we insist upon units. Here is another way of looking at it. Assume there is a loop in It($B$)$_n$: we can write it as

$$f_1 \xrightarrow{\phi_1} f_2 \xrightarrow{\phi_2} \cdots \xrightarrow{\phi_n} f_1$$

where the $f_i$'s are associations of any $n$ objects in $B$. In particular, one of the objects may be the unit object $I$. Let $f_i^*$ be the association $f_i$ without that unit object. $f_i^*$ is then an association of $n - 1$ objects. Hence we have the following commutative diagram:

$$\begin{array}{c}
\text{It}(B)_n \\
\begin{array}{cccccccc}
& f_1 & \xrightarrow{\phi_1} & f_2 & \xrightarrow{\phi_2} & \cdots & \xrightarrow{\phi_n} & f_1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & \\
\text{It}(B)_{n-1} & f_1^* & \xrightarrow{\phi_1^*} & f_2^* & \xrightarrow{\phi_2^*} & \cdots & \xrightarrow{\phi_n^*} & f_1^* \\
\end{array}
\end{array}$$

46
where the vertical maps are identities tensored with $L$ or $R$. The squares commute only if we assume the identity triangle coherence requirement discussed in Section 6. If the top loop is the identity, then the bottom loop must also be the identity. In other words, when there are no units a relation in $A_n$ causes a relation in $A_{n+t}$. In contrast, when there are units, relations in $A_{n+t}$ go down to the lower $A_n$. Hence there are fewer congruences possible. So, noncoherence for unital associative categories are classified by $\mathbb{Z}$ the integers. To every unital associative category, there is an integer $v$ assigned to it. $v$ is the least number that one must go around a generator of some – hence all – $A_n$ to get the identity.

12 Future Directions

There are several places that the work done here might be applied. Most of the second half of [21] is dedicated to calculating the cohomology of Drinfeld algebras (bialgebras that are coassociative only up to a coherent isomorphism.) It would be interesting to study the relationship of these cohomology groups to the cohomology of bialgebras that are coassociative only up to an isomorphism - without a coherence requirement. Such bialgebras arise when representing (in the sense of, say, [27]) an associative category. We also feel that the groups presented in section 8 will be of importance to the study of pentagonal algebras and homotopy associative (Lie) algebras (see [23].)

The second part of [24] deals with functors between associative categories that do not necessarily satisfy the hexagon coherence condition (see [4].) An induction scheme to calculate the fundamental groups of “mapping funnals” is given. A generalisation of Proposition 4 in Section 5 is proved where the restriction to strict tensor functors is lifted. There is an attempt to classify noncoherent tensor functors. Much work remains to be done in this area. The standard coherence theorem states that any coherent tensor category is equivalent to a strict tensor category via a coherent tensor functor. Given a noncoherent tensor category can we say that it is equivalent to a strict tensor category via a (weaker) noncoherent tensor functor?

Many of the assumptions made in this work can be relaxed for more interesting computations. For example, in this paper, the reassociations are always considered to be isomorphisms. What happens if we relax this requirement and only ask for a morphism? We would then be working with general
categories rather than (Catalan) groupoids. Would we get a fundamental group or a fundamental monoid? Similarly for noncoherent tensor functors (25), we assumed there is an isomorphism $F(A) \otimes' F(B) \rightarrow F(A \otimes B)$. What happens if we loosen this requirement and ask only for a morphism between them? These situations are known to arise “in nature”. For instance, (27) is concerned with braidings that are not necessarily isomorphisms (termed “pre-braidings”). Also, when using the forgetful functor, $U : R - \text{Mod} \rightarrow \text{Ab}$ from the module category of an arbitrary commutative ring $R$ to the category of abelian groups, the morphism $U(A) \otimes U(B) \rightarrow U(A \otimes B)$ is in general not an isomorphism.

The next coherence requirement that is under investigation is commutativity (26). This area is of utmost importance to the study of quantum groups, quasi-triangular Hopf algebras, quantum field theory, etc. (8) has given a three-level hierarchy of coherence requirements for commutativity: symmetric, balanced and braided. Each of these coherence conditions correspond to different algebraic structures. Is this hierarchy complete? Are there intermediate levels? Each one of these coherence rules corresponds to “filling in” part of the permutoassociahedrons (9). We would like to look at the fundamental groups of each of the related polytopes and see if there are any other interesting coherence conditions.

Another area of extreme interest is categorical logic and coherence. We would like to look more carefully at the coherence requirements for cartesian closed categories. It has been shown (e.g. (18) for a general survey, (12)) that these coherence requirements are intimately related to the cut-elimination theorem which is central to proof theory of (intuitionistic) logic. The goal would be to furnish a classification of categories that fail this coherence condition and hence to see what can be learned about logical systems in which the cut-elimination theorem fails. Coherence requirements have also shown up in the area of linear logic. Linear logic deals with – the more general – monoidal closed categories (see e.g. (23)).

There are numerous other coherence problems which we can explore using generalizations of methods used in this paper. For example, we can look at Laplaza’s distributivity categories, Shum’s tortile tensor categories, Crane and Yetter’s Hopf categories etc.

The long term goals are to study n-categories as forms of higher dimensional algebras. Recently, these algebras have become quite popular. Even mathematical physicists (see e.g. (3) or (1)) have shown an interest in such
structures. In 1972, Kelley ([11]) formulated the notion of a club in order to present coherence problems. A club is like a presentation of a universal algebraic theory. However, instead of just having operations and identities (or generators and relations), there are two levels of operations and identities. Loosely speaking, coherence requirements are second dimensional identities. (From this point of view, the Catalan groupoids constructed in here are the 2-dimensional versions of the Cayley graphs of groups.) In the thesis, we study categories with structures that are “free” or partially “free” of coherence requirements. In regular (1-dimensional) algebras, the notions of a free algebra plays a major role in homological algebra. Perhaps noncoherent categories will play such a role in higher dimensional homological algebra.

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