FOCAL VALUES OF PLANE CUBIC CENTERS

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Abstract. We prove that the vanishing of 11 focal values is not sufficient to ensure that a plane cubic system has a center.

1. Introduction

In 1885 Poincaré asked when the differential equation
\[ y' = -\frac{x + p(x, y)}{y + q(x, y)} = -\frac{P(x, y)}{Q(x, y)} \]
with convergent power series \( p(x, y) \) and \( q(x, y) \) starting with quadratic terms, has stable solutions in the neighborhood of the equilibrium solution \( (x, y) = (0, 0) \). This means that in such a neighborhood the solutions of the equivalent plane autonomous system
\[
\begin{align*}
\dot{x} &= y + q(x, y) = Q(x, y) \\
\dot{y} &= -x - p(x, y) = -P(x, y)
\end{align*}
\]
are closed curves around \((0, 0)\).

Poincaré showed that one can iteratively find a formal power series \( F = x^2 + y^2 + f_3(x, y) + f_4(x, y) + \ldots \) such that
\[
\det \begin{pmatrix} F_x & F_y \\ P_x & Q_x \\ Q_y & P_y \\ Q_y & P_y \\ \vdots & \vdots \end{pmatrix} = \prod_{j=1}^{\infty} s_j(x^{2j+2} + y^{2j+2})
\]
with \( s_j \) polynomials in the coefficients of \( P \) and \( Q \). If all \( s_j \) vanish, and \( F \) is convergent then \( F \) is a constant of motion, i.e. its gradient field satisfies \( Pdx + Qdy = 0 \). Since \( F \) starts with \( x^2 + y^2 \) this shows that close to the origin all integral curves are closed and the system is stable. Therefore the \( s_j \)'s are called the focal values of \( Pdx + Qdy \). Often also the notation \( \eta_2 := s_j \) is used, and the \( \eta_i \) are called Liapunov quantities.

Poincaré also showed, that if an analytic constant of motion exists, the focal values must vanish. Later Frommer [Fro34] proved that the systems above are stable if and only if all focal values vanish even without the assumption of convergence of \( F \). (Frommer’s proof contains a gap which can be closed [vW05].)

Unfortunately it is in general impossible to check this condition for a given differential equation because there are infinitely many focal values. In the
case where $P$ and $Q$ are polynomials of degree at most $d$, the $s_j$ are polynomials in finitely many unknowns. Hilbert’s Basis Theorem then implies that the ideal $I_\infty = (s_1, s_2, \ldots)$ is finitely generated, i.e there exists an integer $m := m(d)$ such that

$$s_1 = s_2 = \cdots = s_{m(d)} = 0 \implies s_j = 0 \quad \forall j.$$ 

This shows that a finite criterion for stability exists, but due to the indirect proof of Hilbert’s Basis Theorem no value for $m(d)$ is obtained. In fact even today only $m(2) = 3$ is known. Žołdek [Zo95] and Christopher [Chr05] showed that $m(3) \geq 11$. Since the number of variables for $d = 2$ is six and $m(2) = 6 - 3$ it has been conjectured that for $d = 3$ with 14 variables one has $m(3) = 14 - 3 = 11$.

It is the purpose of this note to prove $m(3) \geq 12$.

The most naive approach to this problem is to calculate a Gröbner Basis of $I_{11} = (s_1, \ldots, s_{11})$ and prove that $s_{12} \notin I_{11}$ by the usual ideal membership test. Unfortunately this is not feasible, since the $s_j$ are very complicated. They involve 14 variables and are of weighted degree $2j$. For example $s_5$ has already 5348 terms and takes about 1.5 hours on a Powerbook G4 to calculate. The polynomials $s_j$, $j \geq 6$ can not at the moment be determined by computer algebra systems. Žołdek and Christopher therefore deduce their result geometrically. They exhibit a component $Y_{11} \subset X_\infty = V(I_\infty)$ that has codimension 11 in the space of all possible $(P, Q)$ of degree at most three. Finding a component of codimension 12 is not an easy task, and indeed we choose a different approach. We prove that there exist a codimension 11 family plane autonomous system of degree 3 with a focus for which nevertheless the first 11 focal values vanish, but 12th one doesn’t. For this we look at the system

$$\dot{x} = y + 3x^2 + 8xy + 5y^2 + 3x^3 + 25x^2y + 20xy^2 + 18y^3$$
$$\dot{y} = -(x + 27x^2 + 9xy + 22y^2 + 11x^3 + 20x^2y + 4xy^2 + 3y^3)$$

and prove that for this system $s_j = 0 \mod 29$ for $j \leq 11$ while $s_{12} \neq 0 \mod 29$. Checking that furthermore the Jacobian matrix of $s_1, \ldots, s_{11}$ has full rank modulo 29 for this system, we can apply a theorem of Schreyer [Sch96] to show the existence of the desired family of foci over $\mathbb{C}$. From this we deduce that $s_{12} \notin I_{11} = (s_1, \ldots, s_{11})$. If fact we even prove the stronger result $s_{12} \notin \text{rad } I_{11}$.

Since for given a given system one can evaluate the $s_j$ using Frommer’s algorithm [vBC07] without knowing the complete Polynomials, this approach is feasible.

We found the above system by performing a random search. Heuristically each $s_i$ vanishes mod 29 for about one of every 29 differential equations [vBS05]. So we expect to find an example as above after checking $29^{11} \approx 10^{16}$ random examples. By parametrizing $s_1$ and $s_2$ we can improve this to $29^9 \approx 10^{13}$ random examples. Indeed we found the example after about $8 \times 10^{12}$ trials. Using an improved version [vBK09] of the program [vBC05] this took 1246 CPU-days. Since this search is easily parallelizable we could
do this calculation in about one month by distributing the work to several computers.

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2. The Proof

**Notation 2.1.** If \( I \subset \mathbb{Z}[x_1, \ldots, x_n] \) is an ideal and \( X_\mathbb{Z} = V(I) \subset \mathbb{A}^n_\mathbb{Z} \) is the variety over \( \text{spec} \mathbb{Z} \) defined by \( I \), then we denote by \( X_{\mathbb{F}_p} \) the fiber of \( X_\mathbb{Z} \) over \( \mathbb{F}_p \) for any prime \( p \). Furthermore we denote by \( X_\mathbb{C} \) the variety defined by \( I \) over \( \mathbb{C} \).

**Theorem 2.2** (Schreyer). Let \( I = (f_1, \ldots, f_k) \subset \mathbb{Z}[x_1, \ldots, x_n] \) be an ideal and \( X_\mathbb{Z} = V(I) \). If \( x \in X_{\mathbb{F}_p} \) is a point with \( \text{codim} T_{X_{\mathbb{F}_p},x} = k \) then there exists an irreducible component \( Y_\mathbb{Z} \subset X_\mathbb{Z} \) with \( x \in Y_\mathbb{Z} \) and \( Y_\mathbb{Z} \not\subset X_{\mathbb{F}_p} \). In particular \( Y_\mathbb{C} \neq \emptyset \)

**Proof.** This is a special case of a theorem of Schreyer \[\text{[Sch06]}\]. See also \[\text{[vBEL05]}\] for a proof. \(\square\)

![Figure 1. A variety over spec \( \mathbb{Z} \)](image)

**Example 2.3.** Consider \( X_\mathbb{Z} = V(3x) \subset \mathbb{A}^1_\mathbb{Z} \). This variety has two components over \( \mathbb{Z} \) namely \( Y_\mathbb{Z} = V(x) \) and \( Z_\mathbb{Z} = V(3) \). Since \( 3 = 0 \) is true only in \( \mathbb{F}_3 \) we have \( Z_\mathbb{Z} = \mathbb{Z}_{\mathbb{F}_3} \). Furthermore \( Z_\mathbb{C} = \emptyset \). On the other hand \( x = 0 \) is possible over all \( \mathbb{F}_p \) and \( Y_\mathbb{C} \neq \emptyset \). See Figure 1.

Indeed, if we consider the point \( x = 0 \in X_{\mathbb{F}_p}, p \neq 3 \) then we have that the derivative \( (3x)' = 3 \neq 0 \) and the tangent space \( T_{0,X_{\mathbb{F}_p}} \) has codimension 1.
Therefore the Theorem applies and the component \( Y_Z \) containing \( x = 0 \) is not contained in \( X_{\mathbb{F}_p} \).

Since \( 3 \cdot 1 = 0 \in \mathbb{F}_3 \) we can also consider the point \( x = 1 \in X_{\mathbb{F}_3} \). Here we have \((3x)' = 3 = 0\) and the tangent space \( T_{1,X_{\mathbb{F}_3}} \) has codimension 0. Hence the Theorem does not apply, and indeed the component \( Z_Z = Z_{\mathbb{F}_3} \) containing \( x = 1_{\mathbb{F}_3} \) is completely contained in \( X_{\mathbb{F}_3} \).

**Corollary 2.4.** If in the situation of Theorem 2.2 we have a further polynomial \( g \in \mathbb{Z}[x_1, \ldots, x_n] \) satisfying \( g(x) \neq 0 \in \mathbb{F}_p \) then \( g \) does not vanish on \( X_\mathbb{C} \).

**Proof.** Assume to the contrary that \( g \) vanishes on \( X_\mathbb{C} \). By Theorem 2.2 we have a component \( Y_Z \subset X_\mathbb{C} \) with \( x \in Y_Z \) and \( Y_\mathbb{C} \neq \emptyset \). Since \( g \) vanishes on \( X_\mathbb{C} \) and \( Y_\mathbb{C} \neq \emptyset \) is also vanishes on \( Y_\mathbb{C} \) and therefore on \( Y_Z \) and \( Y_{\mathbb{F}_p} \). But this contradicts our assumption \( g(x) \neq 0 \).

**Theorem 2.5.** \( m(3) \geq 12 \).

**Proof.** Use our implementation of Frommers algorithm [vBC07], [vBC05], [vBK09] or REDUCE [Hea04] to check that the example in the introduction satisfies the conditions of Corollary 2.4.

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