A comprehensive survey on prime graphs

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Abstract This article presents a short and concise survey on prime graphs. The field of graph theory plays a vital role in various fields. One of the important areas in graph theory is graph labeling used in many applications like “coding theory, x-ray crystallography, radar, astronomy, circuit design, communication network addressing, and data base management”. An assignment of integers to the vertices or edges or to both of a graph $G(V,E)$ subject to certain constraints is called a graph labeling. The notion of “prime labeling” was originated by Entringer and considered in a paper by Tout, Daboucy, and Howalla. A bijection $f:V(G) \rightarrow \{1,2,3,...,|V|\}$ is called a prime labeling of $G$ if for each edge $e = st$, $\text{GCD}(f(s), f(t)) = 1$, where GCD denotes the greatest common divisor. We call $G$ a prime graph if it admits a prime labeling. This article stands divided into six sections. The first and sixth sections are reserved respectively for introduction and a few important references. Sections 2, 3, and 4, respectively deal with the prime labeling of certain classes of graphs such as path, cycle, complete graph, complete bipartite graph, bipartite graphs, join and product graphs, wheel related graphs etc. wherein some known results of high importance have been recalled. The fifth section deals with the enumeration of conjectures and open problems in respect of prime labeling that still remain unsolved.

1. Introduction

In this article we consider only non-trivial, finite, simple, and undirected graphs. A bijection $f:V(G) \rightarrow \{1,2,3,...,|V|\}$ is called a “prime labeling of $G$” if for each edge $e = st$, $\text{GCD}(f(s), f(t)) = 1$, where GCD denotes the greatest common divisor. We call $G$ a prime graph if it admits a prime labeling. Around 1980, Roger Entringer conjectured that “all trees have a prime labeling” which is not settled till date. In 2011, P. Haxell et al. [5] proved that all large trees admit a prime labeling. Seoud et al. [6] provided the necessary and sufficient condition for a graph to be a prime graph. Many researchers have studied prime labeling of graphs. Originally the researchers working in this direction verified this conjecture for special classes of trees which include caterpillars, banana trees, bistars, spiders, paths, etc. We now give below a few definitions which are important for further exposition of the article.

Definition 1: [4]
A tree is a connected and acyclic graph.

Definition 2: [1]
A tree is a “caterpillar” if and only if all nodes of degree \( \geq 3 \) are surrounded by “at most two” nodes of degree two or greater.

**Definition 3:** [9]

A tree with “one vertex of degree at least 3 and all other vertices with degree at most 2” is called a spider tree.

**Definition 4:** [2]

A tree with \( n \) vertices is called a star, denoted by \( S_n \), if one vertex is of degree \( n - 1 \) and all other vertices of degree 1.

**Definition 5:** [2]

A banana tree is a “family of stars” with a new vertex adjoined to one end vertex of each star.

**Definition 6:** [7]

A wheel graph is obtained by connecting a single universal vertex (a vertex adjacent to all other vertices of the graph) to all vertices of a cycle. A wheel graph of order \( n \) is denoted by \( W_n \).

**Definition 7:** [4]

Let \( G_1(V_1, E_1) \) and \( G_2(V_2, E_2) \) be two graphs such that \( V_1 \cap V_2 = \emptyset \). Then join of \( G_1 \) and \( G_2 \), denoted by \( G_1 + G_2 \), is the graph whose vertex set is \( V_1 \cup V_2 \) and edge set contains all the edges of \( E_1 \), \( E_2 \) and the edges obtained by joining each vertex of \( G_1 \) to each vertex of \( G_2 \).

**Definition 8:** [4]

Let \( G_1(V_1, E_1) \) and \( G_2(V_2, E_2) \) be two graphs then the product \( G_1 \times G_2 \) is defined as the graph with vertex set \( V_1 \times V_2 = \{(u, v); u \in V_1 \text{ and } v \in V_2\} \) and edge set \( E_1 \times E_2 \) defined as follows: If \( u = (u_1, u_2) \) and \( v = (v_1, v_2) \in V_1 \times V_2 \). Then \( uv \in E_1 \times E_2 \) whenever \( u_1v_1 \in E_1 \) and \( u_2 = v_2 \) or \( u_2v_2 \in E_2 \) and \( u_1 = v_1 \).

**Example 1:**

![Figure 1. Product of Two Graphs](image)

**Definition 9:** [3]

The helm graph denoted by \( H_n \) is obtained from a wheel \( W_n \) by “attaching a pendant vertex” to each rim vertex.

**Definition 10:** [3]
The flower graph $F_{n}$ is obtained from a helm $H_{n}$ by “joining each pendant vertex to the apex” of the helm.

**Definition 11:** [7]

The gear graph $G_{n}$ is obtained by inserting a vertex between adjacent vertices on the perimeter of a wheel $W_{n}$.

### 2. Prime Labeling of Trees and Special Classes of Banana Trees

In this section, we recall the “sufficient condition” for a tree to admit a prime labeling and some special classes of banana trees admitting prime labeling.

**Theorem 2.1:** [6]

Let $T$ be a tree with vertex set $V$ having $n$ vertices, $n \geq 3$. Let $A = \{v \in V | \deg(v) \neq 1\}$ and $\{u \in A | u$ is adjacent to pendant vertices$\}$. If $|A| \leq |P(1, n)| + 1$ and $|B| \leq \left\lceil P\left(\frac{n}{2}, \frac{n}{2}\right)\right\rceil + 1$, then $T$ admits a prime labeling, where $P(x, y) = \{z | z$ is a prime and $x < z \leq y\}$.

**Example 2:**

Consider a tree $T$ as shown in the Figure 2. Here $n = 31$, $A = \{v_{1}, v_{2}, v_{3}, v_{4}, v_{5}, v_{6}, v_{7}\}$, $B = \{v_{3}, v_{4}, v_{5}, v_{6}, v_{7}\}$, $|A| = 7$, and $|B| = 5$. Also $|P(1, 31)| = 11$, $|P(15, 31)| = 5$. Thus all the conditions of Theorem 2.1 are satisfied and therefore $T$ must admit a prime labeling. One of the prime labeling is given in Figure 2.

**Theorem 2.2:** [2]

Let $T$ be a banana tree corresponding to the family of stars $\{K_{1,n_{1}}, K_{1,n_{2}}, \ldots, K_{1,n_{m}}\}$, $m \geq 2$. If $n_{j} \leq 15$ for each $i = 1, 2, \ldots, m$, then $G$ is prime.

**Example 3:**

Consider a banana tree “corresponding to the family of stars” $\{K_{1,1}, K_{1,5}, K_{1,11}\}$ as shown in the Figure 3 in which one of the prime labeling is also given.
Theorem 2.3: \([2]\)

The banana tree corresponding to the family of stars \(\{K_{1,n_1}, K_{1,n_2}, \ldots, K_{1,n_m}\}\) admits a prime labeling if

\[
1 + m + \sum_{i=1}^{m} n_i
\]

is a prime number.

Example 4:

Consider a banana tree \(T\) “corresponding to the family of stars” \(\{K_{1,3}, K_{1,5}\}\) as shown in the Figure 4. Here

\[
1 + m + \sum_{i=1}^{m} n_i = 1 + 2 + (3 + 5) = 11
\]

which is a prime number. Therefore \(T\) admits a prime labeling.

Theorem 2.4: \([2]\)

Every \(k\) - star is prime for \(k \geq 1\).

Theorem 2.5: \([2]\)

Let \(T\) be a banana tree corresponding to the family of stars \(\{K_{1,n_1}, K_{1,n_2}, \ldots, K_{1,n_m}\}\) such that

\[
n_1 \geq 1, \quad n_2 = n_1 + 3 \quad \text{and} \quad n_j = 2n_{j-1} + 1, j = 3, 4, \ldots, m
\]

then \(T\) admits a prime labeling.
Example 5:

Consider a banana tree $T$ “corresponding to the family of stars” $\{K_{1,2}, K_{1,5}, K_{1,11}\}$ as shown in the Figure 5. Here all the conditions of Theorem 2.5 are satisfied and so $T$ admits a prime labeling.

![Figure 5. Prime Labeling of Tree](image)

3. Prime labeling of Join graphs and Product graphs

In this section we recall the prime labeling of some join graphs and product graphs.

**Theorem 3.1:** [9]

The double cone $C_n + 2K_1$ admits prime labeling iff $n = 3$.

**Example 6:**

One can see the double cone $C_3 + 2K_1$ and its prime labeling as shown in the Figure 6.

![Figure 6. Prime Labeling of Double Cone $C_3 + 2K_1$](image)

**Theorem 3.2:** [9]

The join graph $C_n + mK_1, m \geq n$ admits a prime labeling if and only if $n \leq 1 + \left\lfloor \frac{n + m}{2} \right\rfloor$, where $P(x, y) = \{z: z$ is a prime and $x < z \leq y\}$.

**Example 7:**
One can see the join graph $C_3 + 5K_1$ and a prime labeling of it as shown in the Figure 7. Here $3 \leq 1 + |P\left(\frac{3+5}{2,3} + 5\right)| = 1 + |P(4,8)| = 1 + 2 = 3$. Therefore by Theorem 3.2 this graph admits a prime labeling.

![Figure 7. Prime Labeling of $C_3 + 5K_1$](image)

**Theorem 3.3:** [3]
The fan graph $F_n = P_n + K_1$ admits a prime labeling for every $n \in N$.

**Theorem 3.4:** [3]
The double fan $F_n = P_n + 2K_1, n \geq 2$ has a prime labeling if and only if $n$ is odd.

**Example 8:**
The double fan graph $P_8 + 2K_1$ and its prime labeling are shown in the Figure 8.

![Figure 8. Prime Labeling of $P_8 + 2K_1$](image)

**Theorem 3.5:** [9]
The graph $P_n + mK_1$, where $n > 2, m \geq n - 1$ is prime iff $n \leq 1 + |P\left(\frac{n + m}{2}, n + m\right)|$.

**Theorem 3.6:** [8]
Let $n$ be any prime integer, $m \leq 3$ and $m < n$. Then the product graph $P_m \times P_n$ is prime.

**Example 9:**
A prime labeling of $P_3 \times P_{12}$ is given below in the Figure 9.
Figure 9. Prime Labeling of $P_3 \times P_{13}$

**Theorem 3.7:** [10]
The book graph $B_m = K_{1,m} \times P_2$ is a prime graph.

**Example 10:**
A prime labeling of book graph $B_5$ is shown below in the Figure 10.

4. **Prime Labeling of Wheel Related Graphs**
In this section, we highlight the prime labeling of some wheel related graphs.

**Theorem 4.1:** [3]
The helm graph $H_n$ is a prime graph.

**Example 11:**
A prime labeling of $H_{10}$ is shown below in Figure 11.
Theorem 4.2: [3]
Flowers $Fl_n$ are prime graphs.

Example 12:
A prime labeling of $Fl_7$ is shown below in Figure 12.

Figure 12. Prime Labeling of $Fl_7$

Theorem 4.3: [7]
The gear graph $G_n, n \geq 3$ is a prime graph.

Example 13:
A prime labeling of $G_5$ is shown below in Figure 13.

Figure 13. Prime Labeling of $G_5$
5. Conjectures and Open Problems
In this section, we enumerate a few conjectures on prime labeling of various graphs.

Conjecture 5.1: [1]
The ladder \( L_n = P_2 \times P_n \) has a prime labeling for any integer \( n \geq 2 \).

Conjecture 5.2: [5]
All trees are prime graphs.

In addition to the above mentioned conjectures the following problems are also open.
1. Establishing the prime labeling of other classes of graphs.
2. A complete characterization of prime graphs.
3. Exploring the exclusive applications of prime labeling is also an interesting area of research.

Conclusion
A brief survey on prime labeling is given. One can easily identify that the list of graphs which are
proved to be prime and one can also establish a prime labeling for the graphs which are not yet done. A
complete characterization of prime graphs is still open and this is for future work. One can also
explore some new applications of prime graphs.

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