THE DERIVED FUNCTORS OF UNRAMIFIED COHOMOLOGY

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Abstract. We study the first “derived functors of unramified cohomology” in the sense of [15], applied to the sheaves $\mathbb{G}_m$ and $\mathcal{K}_2$. We find interesting connections with classical cycle-theoretic invariants of smooth projective varieties, involving notably a version of the Griffiths group and the group of indecomposable $(2, 1)$-cycles.

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Introduction

To a perfect field $F$, Voevodsky associates in [25] a triangulated category of (bounded above) effective motivic complexes $\text{DM}^{\text{eff}}(F) = \text{DM}^{\text{eff}}$. In [15], we rather worked with the unbounded version $\text{DM}^{\text{eff}}$. We introduced a triangulated category of birational motivic complexes

Date: September 15, 2017.

2010 Mathematics Subject Classification. 19E15, 14E99.

The first author acknowledges the support of Agence Nationale de la Recherche (ANR) under reference ANR-12-BL01-0005 and the second author that of NSERC Grant 402071/2011.
and constructed a triple of adjoint functors

\[
\begin{align*}
\mathbf{DM}^{\text{eff}} & \xrightarrow{\nu} \mathbf{DM}^0 \\
\mathbf{DM}^{\text{eff}} & \xleftarrow{i_0} \mathbf{DM}^{\nu \le 0} \\
\mathbf{DM}^{\nu \le 0} & \xrightarrow{\nu} \mathbf{DM}^{\text{eff}}
\end{align*}
\]

with \(i_0\) fully faithful. Via \(i_0\), the homotopy \(t\)-structure of \(\mathbf{DM}^{\text{eff}}\) induces a \(t\)-structure on \(\mathbf{DM}^0\) (also called the homotopy \(t\)-structure), and the functors \(\nu \le 0\), \(i_0\) and \(R_{\text{nr}}\) are respectively right exact, exact and left exact.

The heart of \(\mathbf{DM}^{\text{eff}}\) is the abelian category \(\mathbf{HI}\) of homotopy invariant Nisnevich sheaves with transfers (see [25]). The heart of \(\mathbf{DM}^{\nu \le 0}\) is the thick subcategory \(\mathbf{HI}^{\nu \le 0} \subset \mathbf{HI}\) of birational sheaves: an object \(\mathcal{F} \in \mathbf{HI}\) lies in \(\mathbf{HI}^{\nu \le 0}\) if and only if it is locally constant for the Zariski topology.

In [15] we also started to study the right adjoint \(R_{\text{nr}}\). Let \(R_{\text{nr}}^0 = \mathcal{H}^0 \circ R_{\text{nr}} : \mathbf{HI} \to \mathbf{HI}^{\nu \le 0}\) be the induced functor. We proved that \(R_{\text{nr}}^0\) is given by the formula \(R_{\text{nr}}^0 \mathcal{F} = \mathcal{F}_{\text{nr}}\), where for a homotopy invariant sheaf \(\mathcal{F} \in \mathbf{HI}\), \(\mathcal{F}_{\text{nr}}\) is defined by

\[
(1) \quad \mathcal{F}_{\text{nr}}(X) = \text{Ker} \left( \mathcal{F}(K) \to \prod_v \mathcal{F}_{-1}(F(v)) \right).
\]

Here \(X\) is a smooth connected \(F\)-variety, \(K\) is its function field, \(v\) runs through all divisorial discrete valuations on \(K\) trivial on \(F\), with residue field \(F(v)\), and \(\mathcal{F}_{-1}\) denotes the contraction of \(\mathcal{F}\) (see [24] or [18, Lect. 23]). Thus \(R_{\text{nr}}^0\mathcal{F}\) is the unramified part of \(\mathcal{F}\).

Here is the example which connects the above to the classical situation of unramified cohomology. Let \(i \ge 0\), \(n \in \mathbb{Z}\) and let \(m\) be an integer invertible in \(F\). Then the Nisnevich sheaf \(\mathcal{F} = \mathcal{H}^i_{\text{et}}(\mu_m^\otimes n)\) associated to the presheaf

\[
U \mapsto H^i_{\text{et}}(U, \mu_m^\otimes n)
\]

defines an object of \(\mathbf{HI}\), and \(R_{\text{nr}}^0\mathcal{F}\) is the usual unramified cohomology [6].

But the functor \(R_{\text{nr}}\) contains more information: for a general sheaf \(\mathcal{F} \in \mathbf{HI}\), the birational sheaves

\[
R_{\text{nr}}^q \mathcal{F} = \mathcal{H}^q(R_{\text{nr}}\mathcal{F})
\]

need not be 0 for \(q > 0\). Can we compute them, at least in some cases?

In this paper, we try our hand at the simplest examples: \(\mathcal{F} = \mathbb{G}_m(= \mathcal{K}_1)\) and \(\mathcal{F} = \mathcal{K}_2\). We cannot compute explicitly further than \(q = 2\), except for varieties of dimension \(\le 2\); but this already yields interesting connections with other birational invariants. For simplicity, we restrict to the case where \(F\) is algebraically closed; throughout this paper, the
cohomology we use is Nisnevich cohomology. The main results are the following:

**Theorem 1.** Let $X$ be a connected smooth projective $F$-variety. Then

(i) $R^0_{nr} \mathbb{G}_m(X) = F^*$. 
(ii) $R^1_{nr} \mathbb{G}_m(X) \cong \text{Pic}^\tau(X)$. 
(iii) There is a short exact sequence

$$0 \rightarrow D^1(X) \rightarrow R^2_{nr} \mathbb{G}_m(X) \rightarrow \text{Hom}(\text{Griff}_1(X), \mathbb{Z}) \rightarrow 0.$$ 
(iv) For $q \geq 3$, we have short exact sequences

$$(0.1) \quad 0 \rightarrow \text{Ext}_\mathbb{Z}(\text{NS}_1(X, q-3), \mathbb{Z}) \rightarrow R^q_{nr} \mathbb{G}_m(X) \rightarrow \text{Hom}_\mathbb{Z}(\text{NS}_1(X, q-2), \mathbb{Z}) \rightarrow 0.$$ 

Here the notation is as follows: $\text{Pic}^\tau(X)$ is the group of cycle classes in $\text{Pic}(X) = CH^1(X)$ which are numerically equivalent to 0. We write $\text{Griff}_1(X) = \text{Ker} \left( A^1_{\text{alg}}(X) \rightarrow N_1(X) \right)$, where $A^1_{\text{alg}}(X)$ (resp. $N_1(X)$) denotes the group of 1-cycles on $X$ modulo algebraic (resp. numerical) equivalence, and

$$D^1(X) = \text{Coker} \left( N^1(X) \rightarrow \text{Hom}(N_1(X), \mathbb{Z}) \right)$$

where $N^1(X) = \text{Pic}(X)/\text{Pic}^\tau(X)$ and the map is induced by the intersection pairing. Finally, the groups $\text{NS}_1(X, r)$ are those defined by Ayoub and Barbieri-Viale in [1, 3.25].

Note that $D^1(X)$ is a finite group since $N_1(X)$ is finitely generated.

After Colliot-Thélène complained that there was no unramified Brauer group in sight, we tried to invoke it by considering

$$\mathbb{G}^\text{ét}_m = R\alpha_* \alpha^* \mathbb{G}_m$$

where $\alpha$ is the projection of the étale site on smooth $k$-varieties onto the corresponding Nisnevich site. There is a natural map $\mathbb{G}_m \rightarrow \mathbb{G}^\text{ét}_m$, and

**Theorem 2.** The map $R^q_{nr} \mathbb{G}_m \rightarrow R^q_{nr} \mathbb{G}^\text{ét}_m$ is an isomorphism for $q \leq 1$, and for $q = 2$ there is an exact sequence for any smooth projective $X$:

$$0 \rightarrow R^2_{nr} \mathbb{G}_m(X) \rightarrow R^2_{nr} \mathbb{G}^\text{ét}_m(X) \rightarrow \text{Br}(X).$$

Considering now $\mathcal{F} = \mathcal{K}_2$:

**Theorem 3.** We have an exact sequence

$$0 \rightarrow \text{Pic}^\tau(X) F^* \rightarrow R^1_{nr} \mathcal{K}_2(X) \rightarrow H^1_{\text{ind}}(X, \mathcal{K}_2) \rightarrow \text{Hom}(\text{Griff}_1(X), F^*) \rightarrow R^2_{nr} \mathcal{K}_2(X) \rightarrow CH^2(X).$$
for any smooth projective variety $X$. Here

$$H^1_{\text{ind}}(X, \mathcal{K}_2) = \text{Coker} \left( \text{Pic}(X) \otimes F^* \to H^1(X, \mathcal{K}_2) \right)$$

and

$$\text{Pic}^\tau(X) F^* = \text{Im} \left( \text{Pic}^\tau(X) \otimes F^* \to H^1(X, \mathcal{K}_2) \right).$$

The group $H^1(X, \mathcal{K}_2)$ appears in other guises, as the higher Chow group $\text{CH}^2(X, 1)$ or as the motivic cohomology group $H^3(X, \mathbb{Z}(2))$; its quotient $H^1_{\text{ind}}(X, \mathcal{K}_2)$ has been much studied and is known to be often nonzero. Note that, while it is not clear from the literature whether there exist smooth projective varieties $X$ such that $\text{Hom}(\text{Griff}^1(X), \mathbb{Z}) \neq 0$, no such issue arises for $\text{Hom}(\text{Griff}^1(X), F^*)$ since $F^*$ is divisible.

The following theorem was suggested by James Lewis. For a prime $l \neq \text{char} F$, write $e_l$ for the exponent of the torsion subgroup of the $l$-adic cohomology group $H^3(X, \mathbb{Z}_l)$. Then $e_l = 1$ for almost all $l$: in characteristic 0 this follows from comparison with Betti cohomology, and in characteristic $> 0$ it is a famous theorem of Gabber [9]. Set $e = \text{lcm}(e_l)$: in characteristic 0, $e$ is the exponent of $H^3_B(X, \mathbb{Z})_{\text{tors}}$, where $H^*_B$ denotes Betti cohomology.

**Theorem 4.** Assume that homological equivalence equals numerical equivalence on $\text{CH}^1(X) \otimes \mathbb{Q}$. Then, $e\delta = 0$ in Theorem 3.

0.1. Remarks. 1) This hypothesis holds if $\text{char} F = 0$ by Lieberman [17, Cor. 1]. His argument shows that, in characteristic $p$, it holds for $l$-adic cohomology if and only if the Tate conjecture holds for divisors on $X$ — more correctly, for divisors on a model of $X$ over a finitely generated field. In particular, it holds if $X$ is an abelian variety; in this case, $e = 1$.

2) The prime to the characteristic part of the unramified Brauer group also appears in the exact sequence of Theorem 3 as a Tate twist of the torsion of $H^1_{\text{ind}}(X, \mathcal{K}_2)$ [13, Th. 1].

**Theorem 5.** Suppose $\dim X \leq 2$ in Theorems 1 and 3. Then there exists an integer $t > 0$ such that

(i) $R^q_{\text{nr}} \mathbb{G}_m(X) \cong D^1(X)$, $R^q_{\text{nr}} \mathbb{G}_m(X) \cong \text{Ext} \mathbb{Z}(A^\text{alg}_1(X), \mathbb{Z}) \cong \text{Hom} \mathbb{Z}(\text{NS}(X)_{\text{tors}}, \mathbb{Q}/\mathbb{Z})$ and $tR^q_{\text{nr}} \mathbb{G}_m(X) = 0$ for $q > 3$.

(ii) $tR^q_{\text{nr}} \mathcal{K}_2(X) = 0$ for $q > 3$ and $R^3_{\text{nr}} \mathcal{K}_2(X) = 0$. Moreover, if $\dim X = 2$, the last map of Theorem 3 identifies $R^3_{\text{nr}} \mathcal{K}_2(X)$ with an extension of the Albanese kernel by a finite group.

We have $t = 1$ if $\dim X < 2$, and $t$ only depends on the Picard variety $\text{Pic}^0_{X/F}$ if $\dim X = 2$. 
Let $CH^2(X)_{\text{alg}}$ denote the subgroup of $CH^2(X)$ consisting of cycle classes algebraically equivalent to 0. Recall Murre’s higher Abel-Jacobi map

$$AJ^3 : CH^2(X)_{\text{alg}} \to J^3(X)$$

where $J^3(X)$ is an algebraic intermediate Jacobian of $X$ [20]. Theorem 5 (ii) suggests that in general, $\text{Im}(R^2_{nr}(K_2)(X) \to CH^2(X))$ should be contained in $\ker AJ^3$.

A key ingredient in the proofs of Theorems 1 and 3 is the work of Ayoub and Barbieri-Viale [1], which identifies the “maximal 0-dimensional quotient” of the Nisnevich sheaf (with transfers) associated to the presheaf $U \mapsto CH^n(X \times U)$ with the group $A^n_{\text{alg}}(X)$ of cycles modulo algebraic equivalence (see (5.3)).

The example $F = \mathcal{H}^i_{\text{et}}(\mu_m)$ considered at the beginning relates to the sheaves studied in Theorems 1 and 3 through the Bloch-Kato conjecture: Kummer theory for $K_1$ and the Merkurjev-Suslin theorem for $K_2$. Unfortunately, Theorem 1 barely suffices to compute $R^q_{nr}(\mathbb{G}_m/m)$ for $q \leq 1$ and we have not been able to deduce from Theorem 3 any meaningful information on $R^*_nr(K_2/m)$. We give the result for $R^1_{nr}(\mathbb{G}_m/m)$ without proof; there is an exact sequence, where NS($X$) is the Néron-Severi group of $X$:

$$0 \to \text{NS}(X)_{\text{tors}}/m \to R^1_{nr}(\mathbb{G}_m/m) \to mD^1(X) \to 0$$

and encourage the reader to test his or her insight on this issue.

Let us end this introduction by a comment on the content of the statement “the assignment $X \mapsto F(X)$ makes $F$ an object of $\text{HI}^o$”, which applies to the objects appearing in Theorems 1 and 3. It implies of course that $F(X)$ is a (stable) birational invariant of smooth projective varieties, which was already known in most cases; but it also implies some non-trivial functoriality, due to the additional structure of presheaf with transfers on $F$. For example, it yields a contravariant map $i^* : F(X) \to F(Y)$ for any closed immersion $i : Y \to X$. This does not seem easy to prove $a \text{ priori}$, say for $F(X) = D^1(X) = R^2_{nr}\mathbb{G}_m(X)_{\text{tors}}$ in Theorem 1 (iii).

1. Some results on birational motives

We recall here some results from [15].

1.1. Lemma. For any birational sheaf $F \in \text{HI}^o$ and any smooth variety $X$, $H^q(X, F) = 0$ for $q > 0$.

Proof. See [15, Prop. 1.3.3 b)].
For the next proposition, let us write

\[ \nu^{\geq 1} M := \text{Hom}(Z(1), M)(1) \]

for \( M \in \text{DM}^\text{eff} \), where \( \text{Hom} \) is the internal Hom [25, Prop. 3.2.8].

1.2. Proposition. For \( M \) as above, we have a functorial exact triangle

\[ \nu^{\geq 1} M \rightarrow M \rightarrow i^0 \nu^{\leq 0} M \rightarrow \].

Moreover, \( M \in \text{Im} i^0 \) if and only if \( \text{Hom}(Z(1), M) = 0 \).

Proof. See [15, Prop. 3.6.2 and Lemma 3.5.4]. \( \square \)

1.3. Proposition. For any \( \mathcal{F} \in \text{HI} \), the counit map

\[ i^0 R^0_{nr} \mathcal{F} \rightarrow \mathcal{F} \]

is a monomorphism.

Proof. See [15, Prop. 1.6.3]. \( \square \)

1.4. Proposition. Let \( \mathcal{F} \in \text{HI} \). Then \( \mathcal{F} \in \text{HI}^0 \) if and only if \( \mathcal{F}_{-1} = 0 \), where \( \mathcal{F}_{-1} \) is the contraction of \( \mathcal{F} \) ([24] or [18, Lect. 23]).

Proof. This is [15, Prop. 1.5.2]. \( \square \)

1.5. Proposition. Let \( C \in \text{DM}^\text{eff} \), and let \( D = \text{Hom}(Z(1)[1], C) \). Then

\[ \mathcal{H}^i(D) = \mathcal{H}^i(C)_{-1} \]

for any \( i \in \mathbb{Z} \).

Proof. This is [15, (4.1)]. \( \square \)

2. Computational tools

For \( q \geq 0 \), the \( R^q_{nr} \)'s define a cohomological \( \delta \)-functor from \( \text{HI} \) to \( \text{HI}^0 \). Since \( \text{HI} \) is a Grothendieck category (it has a set of generators and exact filtering direct limits), it has enough injectives, so it makes sense to wonder if \( R^q_{nr} \) is the \( q \)-th derived functor of \( R^0_{nr} \). However, if \( \mathcal{I} \in \text{HI} \) is injective, while \( R^0_{nr} \mathcal{I} \) is clearly injective in \( \text{HI}^0 \), it is not clear whether \( R^q_{nr} \mathcal{I} = 0 \) for \( q > 0 \): the problem is similar to the one raised in [25, Rk. 1 after Prop. 3.1.8]. (In particular, the title of this paper should be taken with a pinch of salt.) Thus one cannot \textit{a priori} compute the higher \( R^q_{nr} \)'s via injective resolutions; we give here another approach.
2.1. **Lemma.** Let $F \in HI$, and let $X$ be a smooth variety. Then the hypercohomology spectral sequence\n\[ E_2^{p,q} = H^p(X, R^q_m F) \Rightarrow H^{p+q}(X, R^m_m F) \]\ndegenerates, yielding isomorphisms\n\[ H^n(X, R^m_m F) \simeq H^0(X, R^m_m F). \]

**Proof.** Indeed, $E_2^{p,q} = 0$ for $p > 0$ by Lemma 1.1. \qed

2.2. **Proposition.** Let $C \in DM^{\text{eff}}$, and let $X$ be a smooth variety. Then we have a long exact sequence\n\[ \cdots \to H^n(X, R^m_m C) \to H^n(X, C) \to DM^{\text{eff}}(\nu \geq 1 M(X), C[n]) \to H^{n+1}(X, R^m_m C) \to \cdots \]

In particular, if $C = F[0]$ for $F \in HI$, we get a long exact sequence\n\[ 0 \to R^m_m F(X) \to F(X) \to DM^{\text{eff}}(\nu \geq 1 M(X), F[0]) \to \cdots \]
\[ \to R^m_m F(X) \to H^n(X, F) \to DM^{\text{eff}}(\nu \geq 1 M(X), F[n]) \to \cdots \]

**Proof.** By iterated adjunction, we have\n\[ H^n(X, R^m_m C) \simeq DM^{\text{eff}}(M(X), i^o R^m_m C[n]) \]
\[ \simeq DM^o(\nu \leq 0 M(X), R^m_m C[n]) \simeq DM^{\text{eff}}(i^o \nu \leq 0 M(X), C[n]). \]

The first exact sequence then follows from Proposition 1.2. The second follows from the first, Lemma 2.1 and Proposition 1.3 b). \qed

2.3. **Proposition.** Let $X$ be smooth and proper, and let $n \geq 0$. Then\n\[ \underline{\text{Hom}}(\mathbb{Z}(n)[2n], M(X)) \in (DM^{\text{eff}})^{\leq 0}. \]

Moreover,\n\[ \mathcal{H}^0(\underline{\text{Hom}}(\mathbb{Z}(n)[2n], M(X))) = CH^n(X) \]

with\n\[ CH^n(X)(U) = CH^n(X_{\bar{F}(U)}) \]
for any smooth connected variety $U$. Similarly, we have\n\[ \underline{\text{Hom}}(M(X), \mathbb{Z}(n)[2n]) \in (DM^{\text{eff}})^{\leq 0} \]
and\n\[ \mathcal{H}^0(\underline{\text{Hom}}(M(X), \mathbb{Z}(n)[2n])) = CH^n(X) \]

with\n\[ CH^n(X)(U) = CH^n(X_{\bar{F}(U)}). \]
Proof. The first statement is [11, Th. 2.2]. The second is proven similarly. \hfill \square

2.4. Lemma. Let $\mathcal{F} \in \mathbf{H}^0$. Then $R^q_i \mathcal{F} = 0$ for $q > 0$.
Proof. This is obvious from the adjunction isomorphism (due to the full faithfulness of $i^\circ$) $\mathcal{F}[0] \sim R^q_i \mathcal{F}[0]$. \hfill \square

3. Varieties of dimension $\leq 2$

As in [25, §3.4], let $d_{\leq 0} \mathbf{D}^\text{eff}$ be the localising subcategory of $\mathbf{D}^\text{eff}$ generated by motives of varieties of dimension 0: since $F$ is algebraically closed, this category is equivalent to the derived category $D(\mathbf{Ab})$ of abelian groups [25, Prop. 3.4.1]. In [1, Cor. 2.3.3], Ayoub and Barbieri-Viale show that the inclusion functor $j : d_{\leq 0} \mathbf{D}^\text{eff} \hookrightarrow \mathbf{D}^\text{eff}$ has a left adjoint $L\pi_0$.

3.1. Lemma. a) For any smooth connected variety $X$, the structural map $X \to \text{Spec } F$ induces an isomorphism $L\pi_0 M(X) \sim L\pi_0 \mathbb{Z} = \mathbb{Z}$.

b) We have $L\pi_0 \mathbb{G}_m = 0$.

c) If $C$ is a smooth projective irreducible curve with Jacobian $J$ (viewed as an object of $\mathbf{H}^1$), then $L\pi_0 J = 0$.

d) If $A$ is an abelian variety, viewed as an object of $\mathbf{H}^1$ (cf. [23, Lemma 3.2] or [2, Lemma 1.4.4]), then there exists an integer $t > 0$ such that $tL\pi_0 A = 0$. Moreover, $L_0 \pi_0 (A) := H_0 (L\pi_0 (A)) = 0$.

Proof. a) By adjunction and Yoneda’s lemma, we have to show that for any object $C \in D(\mathbf{Ab})$, the map $H^*_\text{Nis}(F, C) \to H^*_\text{Nis}(X, C)$ is an isomorphism. This is well-known: by a hypercohomology spectral sequence, reduce to $C$ being a single abelian group; then $C$ is flasque (see [21, Lemma 1.40]).

b) follows from a), applied to $X = \mathbb{P}^1$ (note that $M(\mathbb{P}^1) \simeq \mathbb{Z} \oplus \mathbb{G}_m[1]$).

c) Let $M^0(C)$ be the fibre of the map $M(C) \to \mathbb{Z}$. By a), $L\pi_0 M^0(C) = 0$. By [25, Th. 3.4.2], we have an exact triangle

$$
\mathbb{G}_m[1] \to M^0(C) \to J[0] \xrightarrow{+1}
$$

so the claim follows from a) and b).

d) As is well-known, there exists a curve $C$ with Jacobian $J$ and an epimorphism $J \to A$, which is split up to some integer $t$ by complete reducibility. The first claim then follows from c).
Let \( \text{NST} \) be the category of Nisnevich sheaves with transfers [25]. To see that \( L_0\pi_0(A) = 0 \), it is equivalent by adjunction to see that \( \text{Hom}_{\text{NST}}(A, \mathcal{F}) = 0 \) for any constant \( \mathcal{F} \in \text{NST} \). We may identify \( \mathcal{F} \) with its value on any connected \( X \in \text{Sm} \). Let \( f : A \to \mathcal{F} \) be a morphism in \( \text{NST} \). Evaluating it on \( 1_A \in A(A) \), we get an element \( f(1_A) \in \mathcal{F}(A) = \mathcal{F} \). If \( X \in \text{Sm} \) is connected and \( a \in A(X) = \text{Hom}_F(X, A) \), then \( f(a) = a^*f(1_A) = f(1_A) \). So \( f \) is constant, and since it is additive it must send 0 to 0. This proves that \( f = 0 \), and thus \( \text{Hom}_{\text{NST}}(A, \mathcal{F}) = 0 \). ✷

3.2. Proposition. Let \( X/F \) be a smooth projective variety of dimension \( \leq 2 \). Then there exists an integer \( t = t(X) > 0 \) such that \( t \text{NS}_1(X, i) = 0 \) for \( i > 0 \). We have \( t = 1 \) for \( \dim X \leq 1 \), and we may take for \( t \) the integer associated to \( \text{Pic}^0(X) \) in Lemma 3.1 d) for \( \dim X = 2 \).

Proof. Recall that \( \text{NS}_1(X, i) := H_i(L\pi_0 \text{Hom}(Z(1)[2], M(X))) \) [1, Def. 3.2.5]. For simplicity, write \( C_X = \text{Hom}(Z(1)[2], M(X)) \). We go case by case, using Poincaré duality as in [11, Lemma B.1]:

If \( \dim X = 0 \), then \( X = \text{Spec} F \) and hence \( M(X) \simeq \mathbb{Z} \) is a birational motive; therefore \( C_X = 0 \) (Proposition 1.2) and \( L\pi_0 C_X = 0 \).

If \( \dim X = 1 \), then Poincaré duality produces an isomorphism

\[
C_X \simeq \text{Hom}(M(X), \mathbb{Z}) \simeq \mathbb{Z}[0].
\]

Hence \( L\pi_0 C_X = \mathbb{Z}[0] \).

Now suppose that \( X \) is a smooth projective surface. By Poincaré duality, we get an isomorphism

\[
C_X \simeq \text{Hom}(M(X), \mathbb{Z}(1)[2]).
\]

By evaluating the latter complex against a varying smooth variety, one computes its homology sheaves as \( \text{Pic}_{X/F} \) and \( \mathbb{G}_m \) in degrees 0 and 1 respectively and zero elsewhere. Hence we have an exact triangle\(^1\)

\[
\mathbb{G}_m[1] \to C_X \to \text{Pic}_{X/F}[0] \xrightarrow{+1}.
\]

We have \( L\pi_0 \mathbb{G}_m[1] = 0 \) by Lemma 3.1 b). On the other hand, the representability of \( \text{Pic}_{X/F} \) yields an exact sequence

\[
0 \to \text{Pic}^0_{X/F} \to \text{Pic}_{X/F} \to \text{NS}_{X/F} \to 0
\]

where \( \text{Pic}^0_{X/F} \) is the Picard variety of \( X \) and \( \text{NS}_{X/F} \) is the (constant) sheaf of connected components of the group scheme \( \text{Pic}_{X/F} \). Hence an exact triangle

\[
L\pi_0 \text{Pic}^0_{X/F} \to L\pi_0 \text{Pic}_{X/F} \to L\pi_0 \text{NS}_{X/F} \xrightarrow{+1}.
\]

\(^1\)It is split by the choice of a rational point of \( X \), but this is useless for the proof.
where $L_{\pi_0} \text{NS}_{X/F} = \text{NS}(X)$. By Lemma 3.1 d), $L_{\pi_0} \text{Pic}_{X/F}^0$ is torsion, which concludes the proof. (The vanishing of $L_{\pi_0} \text{Pic}_{X/F}^0$ gives back the isomorphism $L_{\pi_0} C_X \iso \text{NS}(X)$ of [1], see (5.3) below.) \hfill \square

4. Birational motives and indecomposable $(2,1)$-cycles

In this section, we only assume $F$ perfect; we give proofs of two results promised in [15, Rks 3.6.4 and 3.4.2]. These results are not used in the rest of the paper.

For the first one, let $X$ be a smooth projective variety, and let $M = \underline{\text{Hom}}(M(X), \mathbb{Z}(2)[4])$. Note that $M \iso M(X)$ if $\dim X = 2$ by Poincaré duality (cf. proof of Proposition 3.2). The functor $\nu_{\leq 0}$ is right $t$-exact as the left adjoint of the $t$-exact functor $i_!$ [15, Th. 3.4.1], so $\nu_{\leq 0} M \in (\text{DM}^\text{eff})_{\leq 0}$ by Proposition 2.3. We want to compute the last two non-zero cohomology sheaves of $\nu_{\leq 0} M$. Here is the result:

4.1. Theorem. With the above notation, we have

$$\mathcal{H}^i(\nu_{\leq 0} M) = \begin{cases} CH^2(X) & \text{for } i = 0 \\ H^1_{\text{ind}}(X, K_2) & \text{for } i = -1 \end{cases}$$

where the sections of $H^1_{\text{ind}}(X, K_2)$ over a smooth connected $F$-variety $U$ with function field $K$ are given by the formula

$$H^1_{\text{ind}}(X, K_2)(U) = \text{Coker} \left( \bigoplus_{[L:K]<\infty} \text{Pic}(X_L) \otimes L^* \rightarrow H^1(X_K, K_2) \right)$$

in which the map is given by products and transfers.

Proof. We use the exact triangle of Proposition 1.2. From the cancellation theorem ([26], [11, Prop. A.1]), we get an isomorphism

$$\nu^{\geq 1} M \iso \underline{\text{Hom}}(M(X), \mathbb{Z}(1)[4])(1) \iso C_X \otimes \mathbb{G}_m[1].$$

where $C_X = \underline{\text{Hom}}(M(X), \mathbb{Z}(1)[2]).$

By Proposition 2.3, $C_X \in (\text{DM}^\text{eff})_{\leq 0}$. On the other hand, $\otimes$ is right $t$-exact because it is induced by a right $t$-exact $\otimes$-functor on $D(\text{NST})$ via the right $t$-exact functor $LC : D(\text{NST}) \rightarrow \text{DM}^\text{eff}$. Hence $\nu^{\geq 1} M \in (\text{DM}^\text{eff})_{\leq -1}$.

Using Proposition 2.3 again, this shows the assertion in the case $i = 0$ (compare [11, Th. 2.2 and its proof]). For the case $i = -1$, the long exact sequence of cohomology sheaves yields an exact sequence:

$$\cdots \rightarrow \mathcal{H}^0(C_X \otimes \mathbb{G}_m) \rightarrow \mathcal{H}^{-1}(M) \rightarrow \mathcal{H}^{-1}(i^* \nu_{\leq 0} M) \rightarrow 0.$$
Let $\mathcal{F} = \mathcal{H}^0(C_X) = CH^1(X)$; then $\mathcal{H}^0(C_X \otimes \mathbb{G}_m) = \mathcal{F} \otimes_{\mathcal{H}_I} \mathbb{G}_m$ by right $t$-exactness of $\otimes$; here $\otimes_{\mathcal{H}_I}$ is the tensor structure induced by $\otimes$ on $\mathcal{H}_I$. For any function field $K/F$, the map induced by transfers

$$\bigoplus_{[L:K]<\infty} \mathcal{F}(L) \otimes \mathbb{G}_m(L) \rightarrow (\mathcal{F} \otimes_{\mathcal{H}_I} \mathbb{G}_m)(K)$$

is surjective [16, 2.14], which concludes the proof. □

The second result which was promised in [15, Rk. 4.3.2] is:

4.2. Proposition. Let $E$ be an elliptic curve over $F$. Then the sheaf

$$\text{Tor}^{\text{DM}}_1(E, E) := \mathcal{H}^{-1}(E[0] \otimes E[0])$$

is not birational. Here $E$ is viewed as an object of $\text{HI}$ [2, Lemma 1.4.4].

(This contrasts with the fact that the tensor product of two birational sheaves is birational, [15, Th. 4.3.1].)

Proof. Up to extending scalars, we may and do assume that $\text{End}(E) = \text{End}(E_F)$. Consider the surface $X = E \times E$. The choice of the rational point $0 \in E$ yields a Chow-K"unneth decomposition of the Chow motive of $E$, hence by [25, Prop. 2.1.4] an isomorphism

$$M(E) \simeq \mathbb{Z}[0] \oplus E[0] \oplus \mathbb{Z}(1)[2]$$

(compare also [25, Th. 3.4.2]). Therefore

$$M(X) \simeq \mathbb{Z}[0] \oplus 2E[0] \oplus 2\mathbb{Z}(1)[2] \oplus E[0] \otimes E[0] \oplus 2E[1][2] \oplus \mathbb{Z}(2)[4].$$

This allows us to compute $\text{Hom}(E[1][2], E[0] \otimes E[0])$ as a direct summand of $\text{Hom}(E[1][2], M(X)) = C_X$. First we have

$$\text{Hom}(E[1][2], \mathbb{Z}[0]) = \text{Hom}(E[1][2], E[0]) = 0.$$

The first vanishing is [11, Lemma A.2], while the second one follows from the Poincaré duality isomorphism $\text{Hom}(E[1][2], M(E)) \simeq \text{Hom}(M(E), \mathbb{Z}) = \mathbb{Z}$ [12, Lemma 2.1 a)]. Hence, using the cancellation theorem:

$$C_X \simeq 2\mathbb{Z}[0] \oplus \text{Hom}(E[1][2], E[0] \otimes E[0]) \oplus 2E[0] \oplus \mathbb{Z}(1)[2]$$

and

$$\text{Pic}_{X/F} = \mathcal{H}^0(C_X) \simeq 2\mathbb{Z} \oplus \mathcal{H}^0(\text{Hom}(E[1][2], E[0] \otimes E[0])) \oplus 2E.$$

On the other hand, using Weil’s formula for the Picard group of a product, we have a canonical decomposition

$$\text{Pic}_{E \times E/F} \simeq \text{Pic}_{E \times E/F}^0 \oplus \text{NS}(E) \oplus \text{NS}(E) \oplus \text{Hom}(E, E) = 2E \oplus 2\mathbb{Z} \oplus \text{End}(E).$$
One checks that the idempotents involved in the two decompositions of \( \text{Pic}_{X/F} \) match to yield an isomorphism

\[
\mathcal{H}^0(\underline{\text{Hom}}(\mathbb{Z}(1)[2], E[0] \otimes E[0])) \simeq \text{End}(E)
\]

where \( \text{End}(E) \) is viewed as a constant sheaf. By the \( t \)-exactness of Voevodsky’s contraction functor \((-)_1 = \underline{\text{Hom}}(\mathbb{G}_m, -) \) [15, Prop. 4.1.1], this yields an isomorphism \( \text{End}(E) \cong \text{Tor}^\text{DM}_1(E, E)_1 \), which proves that \( \text{Tor}^\text{DM}_1(E, E) \) is not birational (see Proposition 1.4).

\[\blacksquare\]

5. The case of \( \mathbb{G}_m \): proof of Theorems 1, 2 and 5 (i)

5.A. Proof of Theorem 1. We apply Proposition 2.2 to \( F = \mathbb{G}_m \). The Nisnevich cohomology of \( \mathbb{G}_m \) is well-known: we have

\[
H^n(X, \mathbb{G}_m) = \begin{cases} 
F^* & \text{if } n = 0 \\
\text{Pic}(X) & \text{if } n = 1 \\
0 & \text{otherwise}.
\end{cases}
\]

Noting that \( \mathbb{G}_m[0] = \mathbb{Z}(1)[1] \) in \( \text{DM}^\text{eff} \), we get

\[
\text{DM}^\text{eff}(\underline{\text{Hom}}(\mathbb{Z}(1)[2], M(X)), \mathbb{Z}[n]) = D(\text{Ab})(\text{L} \pi_0 \underline{\text{Hom}}(\mathbb{Z}(1)[2], M(X)), \mathbb{Z}[n - 1]) =: F_n(X).
\]

The homology group \( H_s(\text{L} \pi_0 \underline{\text{Hom}}(\mathbb{Z}(1)[2], M(X))) \) is denoted by \( \text{NS}_1(X, s) \) in [1, 3.25]. The universal coefficients theorem then gives an exact sequence

\[
0 \to \text{Ext}_{\text{Ab}}(\text{NS}_1(X, n - 2), \mathbb{Z}) \to F_n(X) \to \text{Ab}(\text{NS}_1(X, n - 1), \mathbb{Z}) \to 0.
\]

By Proposition 2.3, \( \underline{\text{Hom}}(\mathbb{Z}(1)[2], M(X)) \in (\text{DM}^\text{eff})_{\leq 0} \). Since the inclusion functor \( j \) is \( t \)-exact, \( \text{L} \pi_0 \) is right \( t \)-exact by a general result on triangulated categories [3, Prop. 1.3.17], hence \( \text{NS}_1(X, n) = 0 \) for \( n < 0 \). For \( n = 0 \), Ayoub and Barbieri-Viale find

\[
\text{NS}_1(X, 0) = A_1^\text{alg}(X)
\]
in [1, Th. 3.1.4].

Gathering all this, we get (i) (which also follows from (1)), an exact sequence

\[ 0 \to R^1_{\text{nr}} \mathbb{G}_m(X) \to \text{Pic}(X) \xrightarrow{\delta} \text{Hom}(A^\text{alg}_1(X), \mathbb{Z}) \to R^2_{\text{nr}} \mathbb{G}_m(X) \to 0 \]

and isomorphisms

\[ F_n(X) \xrightarrow{\sim} R^{n+1}_{\text{nr}} \mathbb{G}_m(X) \]

for \( n \geq 2 \), which yield (iv) thanks to (5.4).

In Lemma 5.1 below, we shall show that \( \delta \) is induced by the intersection pairing. Granting this for the moment, (ii) is immediate and we get a cross of exact sequences

\[ \begin{array}{c}
0 \\
\downarrow \\
\text{Hom}(N_1(X), \mathbb{Z}) \\
\downarrow \\
0 \to N^1(X) \to \text{Hom}(A^\text{alg}_1(X), \mathbb{Z}) \to R^2_{\text{nr}} \mathbb{G}_m(X) \to 0 \\
\downarrow \\
\text{Hom}(\text{Griff}_1(X), \mathbb{Z}) \\
\downarrow \\
0
\end{array} \]

in which the triangle commutes, and where we used that \( N_1(X) \) is a free finitely generated abelian group. The exact sequence of (iii) then follows from a diagram chase.

5.1. **Lemma.** The map \( \delta \) of (5.4) is induced by the intersection pairing.

**Proof.** This map comes from the composition

\[ (5.6) \quad \text{DM}^\text{eff}(M(X), \mathbb{Z}(1)[2]) \]

\[ \to \text{DM}^\text{eff}(\text{Hom}(\mathbb{Z}(1)[2], M(X))(1)[2], \mathbb{Z}(1)[2]) \]

\[ = \text{DM}^\text{eff}(\text{Hom}(\mathbb{Z}(1)[2], M(X)), \mathbb{Z}) \]

\[ \to \text{Hom}_\mathbb{Z}(\text{Hom}(\mathbb{Z}(1)[2], M(X)), \mathbb{Z}) \]

\[ \text{The hypothesis } F \text{ algebraically closed is sufficient for their proof.} \]
in which the first map is induced by the canonical morphism \( \nu^2 \cdot M(X) \rightarrow M(X) \), the equality follows from the cancellation theorem [26] and the third is by taking global sections at \( \text{Spec} \, k \).

Consider the natural pairing

\[
\text{Hom}(M(X), \mathbb{Z}(1)[2]) \otimes \text{Hom}(\mathbb{Z}(1)[2], M(X)) \rightarrow \text{Hom}(\mathbb{Z}(1)[2], \mathbb{Z}(1)[2]) = \mathbb{Z}[0].
\]

By Proposition 2.3, this pairing factors through a pairing

\[
CH^1(X)[0] \otimes CH^1(X)[0] \rightarrow \mathbb{Z}[0].
\]

Taking global sections, we clearly get the intersection pairing.

From the above, we get a commutative diagram

\[
\begin{array}{ccc}
\text{Hom}(M(X), \mathbb{Z}(1)[2]) & \longrightarrow & \text{Hom}(\text{Hom}(\mathbb{Z}(1)[2], M(X)), \mathbb{Z}[0]) \\
\downarrow & & \uparrow \\
CH^1(X)[0] & \longrightarrow & \text{Hom}(CH^1(X)[0], \mathbb{Z}[0]).
\end{array}
\]

Applying the functor \( DM^\text{eff}(\mathbb{Z}, -) \) to this diagram, we get a commutative diagram of abelian groups

\[
\begin{array}{ccc}
DM^\text{eff}(M(X), \mathbb{Z}(1)[2]) & \longrightarrow & DM^\text{eff}(\text{Hom}(\mathbb{Z}(1)[2], M(X)), \mathbb{Z}[0]) \\
\downarrow & & \uparrow b \\
CH^1(X) & \longrightarrow & DM^\text{eff}(CH^1(X)[0], \mathbb{Z}[0]).
\end{array}
\]

In this diagram, one checks easily that \( a \) corresponds to (5.6) via the cancellation theorem. On the other hand, \( b \) is an isomorphism. Now the evaluation functor at \( \text{Spec} \, F, \mathcal{F} \mapsto \mathcal{F}(F) \), yields a commutative triangle

\[
\begin{array}{ccc}
CH^1(X) & \longrightarrow & DM^\text{eff}(CH^1(X)[0], \mathbb{Z}[0]) \\
\cap & \downarrow ev_F & \\
& \text{Hom}(CH^1(X), \mathbb{Z}).
\end{array}
\]

where \( \cap \) is the intersection pairing (see above). But we saw that \( DM^\text{eff}(CH^1(X)[0], \mathbb{Z}[0]) \simeq \text{Hom}(A^1_{\text{alg}}(X), \mathbb{Z}) \) ((5.1), (5.2) and (5.3)); via this isomorphism, \( ev_F \) is induced by the surjection \( CH^1(X) \rightarrow A^1_{\text{alg}}(X) \), hence is injective. This concludes the proof. \( \square \)
5.B. **Proof of Theorem 2.** We use the following lemma:

5.2. **Lemma.** In $\text{DM}^\text{eff}$, the map $\mathbb{G}_m \to \mathbb{G}_m^{\text{ét}}$ is an isomorphism on $H^0$; moreover, $R^1\alpha_*\alpha^*\mathbb{G}_m = 0$ and $R^2\alpha_*\alpha^*\mathbb{G}_m$ is the Nisnevich sheaf $\text{Br}$ associated to the presheaf $U \mapsto \text{Br}(U)^3$. Here, $\alpha : \text{Sm}_\text{ét} \to \text{Sm}_\text{Nis}$ is the change of topology morphism.

**Proof.** The first statement is obvious, the second one follows from the local vanishing of Pic and the third one is tautological. $\square$

To compute $R^i_{nr}\mathbb{G}_m^{\text{ét}}$, we may use the “hypercohomology” spectral sequence

$$E_2^{p,q} = R^p_{nr} R^q\alpha_*\alpha^*\mathbb{G}_m \Rightarrow R^{p+q}_{nr}\mathbb{G}_m^{\text{ét}}.$$ 

From Lemma 5.2, we find an isomorphism

$$R^1_{nr}\mathbb{G}_m \sim R^1_{nr}\mathbb{G}_m^{\text{ét}}$$

and a five term exact sequence

$$0 \to R^2_{nr}\mathbb{G}_m \to R^2_{nr}\mathbb{G}_m^{\text{ét}} \to R^0_{nr}\text{Br} \to R^3_{nr}\mathbb{G}_m \to R^3_{nr}\mathbb{G}_m^{\text{ét}}$$

which yields (a more precise form of) Theorem 2 in view of the obvious isomorphism $R^0_{nr}\text{Br} = \text{Br}_{nr}$, where $\text{Br}_{nr}$ is the unramified Brauer group.

5.C. **Proof of Theorem 5 (i).** Since $\dim X \leq 2$, $\text{Griff}_1(X)$ is torsion hence $\text{Hom}(\text{Griff}_1(X), \mathbb{Z}) = 0$, which gives the first statement. Then, Theorem 1 (iv) and Proposition 3.2 yield isomorphisms

$$\text{Ext}_\mathbb{Z}(\text{NS}_1(X, q - 3), \mathbb{Z}) \sim R^q_{nr}\mathbb{G}_m(X), \quad q \geq 3.$$ 

For $q > 3$, the left hand group is killed by the integer $t$ of Proposition 3.2. Suppose $q = 3$; then $\text{NS}_1(X, q - 3) = A^\text{alg}_1(X)$, which proves Theorem 5 (i) except for the isomorphism involving $\text{NS}(X)_{\text{tors}}$. For this we distinguish 3 cases:

1. If $\dim X = 0$, $A^\text{alg}_1(X) = \text{NS}(X) = 0$ and the statement is true.
2. If $\dim X = 1$, $A^\text{alg}_1(X) \simeq \mathbb{Z} \simeq \text{NS}(X)$ and the statement is still true.
3. If $\dim X = 2$, $A^\text{alg}_1(X) = \text{NS}(X)$. But for any finitely generated abelian group $A$, there is a string of canonical isomorphisms

$$\text{Ext}_\mathbb{Z}(A, \mathbb{Z}) \sim \text{Ext}_\mathbb{Z}(A_{\text{tors}}, \mathbb{Z}) \sim \text{Hom}_\mathbb{Z}(A_{\text{tors}}, \mathbb{Q}/\mathbb{Z}).$$

This concludes the proof.

---

*This presheaf is in fact already a Nisnevich sheaf.*
6. THE CASE OF $\mathcal{K}_2$: PROOF OF THEOREMS 3 AND 5 (ii)

6.A. Preparations.

6.1. **Lemma.** a) The natural map

\[(6.1) \quad \mathbb{Z}(2)[2] \to \mathcal{K}_2[0] \]

induces an isomorphism

\[
\text{cone} \left( i^o R_{nr} \mathbb{Z}(2)[2] \to \mathbb{Z}(2)[2] \right) \sim \text{cone} \left( i^o R_{nr} \mathcal{K}_2[0] \to \mathcal{K}_2[0] \right).
\]

b) The map

\[\text{DM}^{\text{eff}}(\nu \geq 1, \mathbb{Z}(2)[2]) \to \text{DM}^{\text{eff}}(\nu \geq 1, \mathcal{K}(2)[0])\]

for any $C \in \text{DM}^{\text{eff}}$. (See (1.1) for the definition of $\nu \geq 1$.)

**Proof.** By the cancellation theorem, we have

\[
\text{Hom}(\mathbb{Z}(1)[1], \mathbb{Z}(2)[2]) \simeq \mathbb{Z}(1)[1] \simeq G_m[0]
\]

in $\text{DM}^{\text{eff}}$.

Let $\mathcal{H}^i(C)$ denote the $i$-th cohomology sheaf of an object $C \in \text{DM}^{\text{eff}}$. By Proposition 1.5, the $i$-th cohomology sheaf of the left hand side is $\mathcal{H}^i(\mathbb{Z}(2)[2])_{-1}$. Thus the latter sheaf is 0 for $i \neq 0$. By Proposition 1.4, $\mathcal{H}^i(\mathbb{Z}(2)[2]) \in \text{HI}^o$ for $i \neq 0$, hence $\tau_{<0}(\mathbb{Z}(2)[2]) \in \text{DM}^o$. By adjunction, we deduce

\[
\text{cone} \left( i^o R_{nr} \tau_{<0}(\mathbb{Z}(2)[2]) \to \tau_{<0}(\mathbb{Z}(2)[2]) \right) = 0
\]

which in turn implies a).

To pass from a) to b), use the fact that, for $C, D \in \text{DM}^{\text{eff}}$, adjunction transforms the exact sequence

\[
\text{DM}^{\text{eff}}(\nu \leq 0, \mathcal{K}_2[0], \mathcal{K}_2[0]) \to \text{DM}^{\text{eff}}(\mathcal{K}_2[0], \mathcal{K}_2[0]) \to \text{DM}^{\text{eff}}(\nu \geq 1, \mathcal{K}_2[0], \mathcal{K}_2[0])
\]

into the exact sequence

\[
\text{DM}^{\text{eff}}(C, i^o R_{nr} D) \to \text{DM}^{\text{eff}}(C, D) \to \text{DM}^{\text{eff}}(C, \text{cone}(i^o R_{nr} D \to D)).
\]

\[\square\]

Applying the exact sequence of Proposition 2.2 to $C = \mathcal{K}_2[0]$ and using Lemma 6.1 b), we get a long exact sequence

\[
\ldots \to H^n(X, R_{nr} \mathcal{K}_2) \to H^n(X, \mathcal{K}_2)
\]

\[
\to \text{DM}^{\text{eff}}(\nu \geq 1, M(X), \mathbb{Z}(2)[n+2]) \to H^{n+1}(X, R_{nr} \mathcal{K}_2) \to \ldots
\]

Using the cancellation theorem, we get an isomorphism

\[
\text{DM}^{\text{eff}}(\nu \geq 1, M(X), \mathbb{Z}(2)[n+2]) \simeq \text{DM}^{\text{eff}}(\text{Hom}(\mathbb{Z}(1)[2], M(X)), \mathbb{Z}(1)[n]).
\]
Since \( Z(1)[n] = \mathbb{G}_m[n-1] \), using Lemma 2.1 we get an exact sequence
\[
\delta : \text{DM}^{\text{eff}}(\text{Hom}(Z(1)[2], M(X)), \mathbb{G}_m[0]) \to (R^2_{nr} K_2)(X) \to CH^2(X)
\]
and isomorphisms for \( q > 3 \)
\[
\text{DM}^{\text{eff}}(\text{Hom}(Z(1)[2], M(X)), \mathbb{G}_m[q-2]) \simeq (R^q_{nr} K_2)(X)
\]
where we also used that \( H^2(X, K_2) \simeq CH^2(X) \) and \( H^i(X, K_2) = 0 \) for \( i > 2 \).

6.B. **Proof of Theorem 3.** The group \( \text{DM}^{\text{eff}}(\text{Hom}(Z(1)[2], M(X)), \mathbb{G}_m[0]) \) may be computed as follows:

\[
\text{DM}^{\text{eff}}(\text{Hom}(Z(1)[2], M(X)), \mathbb{G}_m[0])
\]
\[
\overset{1}{\simeq} H^1(\mathcal{H}_0(\text{Hom}(Z(1)[2], M(X))), \mathbb{G}_m)
\]
\[
\overset{2}{\simeq} H^1(\text{CH}_1(X), \mathbb{G}_m) \overset{3}{\simeq} H^0(\text{CH}_1(X), R^0_{nr} \mathbb{G}_m)
\]
\[
\overset{4}{\simeq} H^1(\text{CH}_1(X), jF^*) \overset{5}{\simeq} \text{Ab}(L_0\pi_0 \text{CH}_1(X), F^*)
\]
\[
\overset{6}{\simeq} \text{Ab}(A_1^{\text{alg}}(X), F^*)
\]

Here, isomorphism 1 follows from the fact that \( \text{Hom}(Z(1)[2], M(X)) \in (\text{DM}^{\text{eff}})^{\leq 0} \) (Proposition 2.3), 2 comes from the computation of \( \mathcal{H}_0 \) (ibid.), 3 follows from adjunction, knowing that \( \text{CH}_1(X) \) is a birational sheaf (ibid.), 4 follows from Theorem 1 (i), 5 comes from adjunction and 6 follows from (5.3).

Thus the homomorphism \( \delta \) corresponds to a pairing
\[
H^1(X, K_2) \times A_1^{\text{alg}}(X) \to F^*.
\]

Let \( d = \dim X \). An argument analogous to that in the proof of Lemma 5.1 shows that this pairing comes from the “intersection” pairing
\[
H^3(X, \mathbb{Z}(2)) \times H^{2d-2}(X, \mathbb{Z}(d-1)) \xrightarrow{\cap} H^{2d+1}(X, \mathbb{Z}(d+1)) \xrightarrow{\pi} H^1(F, \mathbb{Z}(1)) = F^*
\]
where the last map is induced by the “Gysin” morphism \( ^{t}\pi : \mathbb{Z}(d)[2d] \to M(X) \). Here we used the isomorphisms
\[
H^1(X, K_2) \simeq H^3(X, \mathbb{Z}(2)), \quad CH_1(X) \simeq H^{2d-2}(X, \mathbb{Z}(d-1)).
\]
In particular, (6.5) factors through algebraic equivalence. This was proven by Coombes [8, Cor. 2.14] in the special case of a surface; we shall give a different proof below, which avoids the use of (5.3).

Consider the product map
\[ c : CH^1(X) \otimes F^* = H^1(X, \mathcal{K}_1) \otimes H^0(X, \mathcal{K}_1) \rightarrow H^1(X, \mathcal{K}_2). \]

By functoriality, we have a commutative diagram of pairings
\[
\begin{array}{ccc}
CH^1(X) \otimes F^* \times A_1^{\text{alg}}(X) & \longrightarrow & F^* \\
\downarrow c \times 1 & & \downarrow || \\
H^1(X, \mathcal{K}_2) \times A_1^{\text{alg}}(X) & \longrightarrow & F^*
\end{array}
\]

where the top pairing is the intersection pairing \( CH^1(X) \times A_1^{\text{alg}}(X) \rightarrow \mathbb{Z} \), tensored with \( F^* \). Since the latter is 0 when restricted to \( \text{Griff}_1(X) \), we get an induced pairing
\[ H^1_{\text{ind}}(X, \mathcal{K}_2) \times \text{Griff}_1(X) \rightarrow F^* \]
yielding a commutative diagram

\[
\begin{array}{cccc}
0 & \longrightarrow & \text{Pic}^\tau(X) \otimes F^* & \longrightarrow & \text{Pic}(X) \otimes F^* & \overset{\alpha}{\longrightarrow} & \text{Hom}(A_1^{\text{num}}(X), F^*) \\\n\downarrow & & \downarrow & & \downarrow & & \downarrow \\\n0 & \longrightarrow & (R^1_{\nu} \mathcal{K}_2)(X) & \longrightarrow & H^1(X, \mathcal{K}_2) & \overset{\delta}{\longrightarrow} & \text{Hom}(A_1^{\text{alg}}(X), F^*) \\\n\downarrow & & \downarrow & & \downarrow & & \downarrow \\\nH^1_{\text{ind}}(X, \mathcal{K}_2) & \overset{\delta}{\longrightarrow} & \text{Hom}(\text{Griff}_1(X), F^*) \\\n\downarrow & & \downarrow & & \downarrow & & \downarrow \\\n0 & & 0 & & 0 & & 0
\end{array}
\]

In this diagram, all rows and columns are complexes. The middle row and the two columns are exact; moreover, \( \alpha \) is surjective as one sees by tensoring with \( F^* \) the exact sequence
\[ 0 \rightarrow \text{Pic}^\tau(X) \rightarrow \text{Pic}(X) \rightarrow \text{Hom}(A_1^{\text{num}}(X), \mathbb{Z}) \rightarrow D^1(X) \rightarrow 0. \]

Then a diagram chase yields an exact sequence
\[ \text{Pic}^\tau(X) \otimes F^* \rightarrow (R^1_{\nu} \mathcal{K}_2)(X) \rightarrow H^1_{\text{ind}}(X, \mathcal{K}_2) \overset{\delta}{\rightarrow} \text{Hom}(\text{Griff}_1(X), F^*) \]
and the surjectivity of $\alpha$ implies that the map $\text{Hom}(A_1^{\text{alg}}(X), F^*) \to (R^d_{\text{et}}K_2)(X)$ given by (6.2) and (6.4) factors through $\text{Hom}(\text{Griff}_1(X), F^*)$. This concludes the proof.

6.C. **Direct proof that (6.5) factors through algebraic equivalence.** Consider classes $\alpha \in H^3(X, \mathbb{Z}(2))$ and $\beta \in CH^{d-1}(X)$: assuming that $\beta$ is algebraically equivalent to 0, we must prove that $\pi_* (\alpha \cdot \beta) = 0$, where $\pi$ is the projection $X \to \text{Spec } F$.

By hypothesis, there exists a smooth projective curve $C$, two points $c_0, c_1 \in C$ and a cycle class $\gamma \in CH^{d-1}(X \times C)$ such that $\beta = c_0^* \gamma - c_1^* \gamma$. Let $\pi_X : X \times C \to X$ and $\pi_C : X \times C \to C$ be the two projections.

The Gysin morphism $\pi : \mathbb{Z}(d)[2d] \to M(X)$ used in the definition of (6.5) extends trivially to give morphisms $M(d)[2d] \to M \otimes M(X)$ for any $M \in \text{DM}^{\text{eff}}$, which are clearly natural in $M$: this applies in particular to $M = M(C)$, giving a Gysin morphism $\pi_C : M(C)(d)[2d] \to M(X \times C)$ which induces a map

$$\left(\pi_C\right)_* : H^{2d+1}(X \times C, \mathbb{Z}(d+1)) \to H^1(C, \mathbb{Z}(1)).$$

The naturality of these Gysin morphisms then gives

$$\pi_* (\alpha \cdot \beta) = \pi_* (\alpha \cdot (c_0^* \gamma - c_1^* \gamma))
= \pi_* (c_0^* (\pi_X^* \alpha \cdot \gamma) - c_1^* (\pi_X^* \alpha \cdot \gamma)) = (c_0^* - c_1^*) (\pi^* \cdot \gamma) = (c_0^* - c_1^*)(\pi^* \cdot \gamma).$$

But $c_1^* : H^1(C, \mathbb{Z}(1)) \to H^1(F, \mathbb{Z}(1))$ is left inverse to $\pi^* : H^1(F, \mathbb{Z}(1)) \to H^1(C, \mathbb{Z}(1))$ (where $\pi' : C \to \text{Spec } F$ is the structural map), which is an isomorphism since $C$ is proper. Hence $c_0^* = c_1^*$ on $H^1(C, \mathbb{Z}(1))$, and the proof is complete.

6.D. **Proof of Theorem 5 (ii).** Note that $\text{Griff}_1(X)$ is finite if $\text{dim } X \leq 2$. In view of (6.2) and (6.3), it therefore suffices to prove

6.2. **Proposition.**

a) If $\text{dim } X \leq 2$, we have

$$t \text{DM}^{\text{eff}}(\text{Hom}(\mathbb{Z}(1)[2], M(X)), \mathbb{G}_m[i]) = 0$$

for $i > 1$, and also for $i = 1$ if $\text{dim } X < 2$.

b) Suppose $\text{dim } X = 2$. Then the map $\varphi$ of (6.2) is the Albanese map from [14, (8.1.1)].

a) is a dévissage similar to the one for Proposition 3.2 (using (3.3) and (3.4) for $\text{dim } X = 2$); we leave details to the reader. As for b), we have a diagram in $\text{DM}^{\text{eff}}$

\[
\begin{array}{ccc}
\text{Hom}(M(X), \mathbb{Z}(2)[4]) & \longrightarrow & \text{Hom}(\text{Hom}(\mathbb{Z}(1)[2], M(X)), \mathbb{Z}(1)[2]) \\
\Delta^\ast \mid \downarrow & & \Delta^\ast \mid \downarrow \\
M(X) & \xrightarrow{\text{res}} & \text{Hom}(\text{Hom}(M(X), \mathbb{Z}(1)[2]), \mathbb{Z}(1)[2])
\end{array}
\]
defined as follows. The top row is obtained by applying $\text{Hom}(-, \mathbb{Z}(2)[4])$ to the map $\nu^* \nu \circ M(X) \to M(X)$ of Proposition 1.2, and using the cancellation theorem. The bottom row is obtained by adjunction from the evaluation morphism $M(X) \otimes \text{Hom}(M(X), \mathbb{Z}(1)) \to \mathbb{Z}(1)$. The Poincaré duality isomorphism $\Delta$ is induced by adjunction by the map $M(X \times X) \simeq M(X) \otimes M(X) \to \mathbb{Z}(2)[4]$ defined by the class of the diagonal $\Delta_X \in CH^2(X \times X) = \text{DM}^{\text{eff}}(M(X \times X), \mathbb{Z}(2)[4])$ (see [2, Prop. 2.5.4]). The isomorphism $\Delta^*$ is induced by the isomorphism $\text{Hom}(\mathbb{Z}(1)[2], M(X)) \simeq \text{Hom}(M(X), \mathbb{Z}(1)[2])$ of (3.2), deduced by adjunction from the composition $\text{Hom}(\mathbb{Z}(1)[2], M(X)) \otimes M(X) \to \text{Hom}(\mathbb{Z}(1)[2], M(X) \otimes M(X))$

\[ \xrightarrow{\Delta_X} \text{Hom}(\mathbb{Z}(1)[2], \mathbb{Z}(2)[4]) \simeq \mathbb{Z}(1)[2] \]

where the last isomorphism follows again from the cancellation theorem.4 A tedious but trivial bookkeeping yields:

6.3. Lemma. The diagram (6.6) commutes. \hfill \square

We are therefore left to identify $\text{DM}^{\text{eff}}(\mathbb{Z}, \varepsilon_X)$ (where $\varepsilon_X$ is as in (6.6)) with the Albanese map. For simplicity, let us write in the sequel $\mathcal{F}$ rather than $\mathcal{F}[0]$ for a sheaf $\mathcal{F} \in \text{HI}$ placed in degree 0 in $\text{DM}^{\text{eff}}$. Let $\mathcal{A}_X$ be the Albanese scheme of $X$ in the sense of Serre-Ramachandran, and let $a_X : M(X) \to \mathcal{A}_X$ be the map defined by [23, (7)]. On the other hand, write $D$ for the (contravariant) endofunctor $M \mapsto \text{Hom}(M, \mathbb{G}_m[1])$ of $\text{DM}^{\text{eff}}$, and $\varepsilon : \text{Id}_{\text{DM}^{\text{eff}}} \Rightarrow D^2$ for the biduality morphism, so that $\varepsilon_X = \varepsilon_{M(X)}$. We get a commutative diagram:

\[ M(X) \xrightarrow{\varepsilon_{M(X)}} D^2 M(X) \]

\[ \xrightarrow{a_X} \]

\[ \xrightarrow{D^2(a_X)} \]

\[ \mathcal{A}_X \xrightarrow{\varepsilon_{\mathcal{A}_X}} D^2 \mathcal{A}_X \]

It is sufficient to show:

6.4. Proposition. After application of $\text{DM}^{\text{eff}}(\mathbb{Z}, -) = H_0^{\text{Nis}}(k, -)$ to (6.7), we get a commutative diagram

\[ CH_0(X) \xrightarrow{a_X(k)} \mathcal{A}_X(k) \]

\[ \xrightarrow{u} \]

\[ \mathcal{A}_X(k) \xrightarrow{v} \mathcal{A}_X(k) \oplus Q \]

Note that evaluation and adjunction yield a tautological morphism $\text{Hom}(A, B) \otimes C \to \text{Hom}(A, B \otimes C)$ for $A, B, C \in \text{DM}^{\text{eff}}$. 
where $a_X(k)$ is the Albanese map, $Q$ is some abelian group and $u, v$ are the canonical injections.

The main lemma is:

6.5. **Lemma.** Let $A$ be an abelian $F$-variety. Then there is a canonical isomorphism

$$DA \cong A^* \oplus \tau_{\geq 2}DA$$

where $A^*$ is the dual abelian variety of $A$.

**Proof.** Note that (3.3) holds for any smooth projective variety $Y$, if we replace $C_Y$ by $D(M(Y))$. We shall take $Y = A$ and $Y = A \times A$. Let $p_1, p_2, m : A \times A \to A$ be respectively the first and second projection and the multiplication map. The composition

$$M(A \times A) \xrightarrow{(p_1)_* + (p_2)_* - m_*} M(A) \xrightarrow{a_A} A_A$$

is 0. One characterisation of $\text{Pic}^0_{A/F} \subset \text{Pic}_{A/F}$ is as the kernel of $(p_1)^* + (p_2)^* - m^*$ (e.g. [19, § before Rk. 9.3]). Therefore, the composition

$$DA \to D\!\!_{\text{A}} A \xrightarrow{D(a_A)} D(M(A)) \xrightarrow{(3.3)} \text{Pic}_{A/F}$$

induces a morphism

$$DA \to \text{Pic}^0_{A/F} = A^*.$$ (6.8)

Here we used the canonical splitting of the extension

$$0 \to A \to A_A \to \mathbb{Z} \to 0$$

given by the choice of the origin $0 \in A$. In view of the exact triangle

$$\tau_{\leq 1}DA \to DA \to \tau_{\geq 2}DA \xrightarrow{+1},$$

to prove the lemma we have to show that (6.8) becomes an isomorphism after applying the truncation $\tau_{\leq 1}$ to its left hand side.

For this, we may evaluate on smooth $k$-varieties, or even on their function fields $K$ by “Gersten’s principle” [2, §2.4]. For such $K$, we have to show that the homomorphism

$$\text{Ext}^{1+i}_{\text{NST}}(A_K, \mathbb{G}_m) \to H^i_{\text{Nis}}(K, A^*)$$

is an isomorphism for $i \leq 1$. This is clear for $i < -1$. For $i = -1, 0, 1$, let $\text{EST}$ be the category of étale sheaves with transfers of [18, Lect. 6], and $\text{ES}$ the category of sheaves of abelian groups on $\text{Sm}_{\text{ét}}$, so that we have exact functors

$$\text{NST} \xrightarrow{a^*} \text{EST} \xrightarrow{\omega} \text{ES}$$
where $\alpha^*$ is étale sheafification and $\omega$ is “forgetting transfers”. If $\alpha_*$ denotes the right adjoint of $\alpha^*$, the hyperext spectral sequence

$$E_2^{p,q} = \text{Ext}^{p+q}_{\text{NST}}(A_K, R^q\alpha_*\alpha^*G_m) \Rightarrow \text{Ext}^{p+q}_{\text{EST}}(\alpha^*A_K, \alpha^*G_m)$$

and the vanishing of $R^1\alpha_*\alpha^*G_m$ (Hilbert 90!) yield isomorphisms

$$\text{Ext}^{1+i}_{\text{NST}}(A_K, G_m) \sim \text{Ext}^{1+i}_{\text{EST}}(\alpha^*A_K, \alpha^*G_m), \quad i \leq 0$$

and an injection

$$\text{Ext}^{2}_{\text{NST}}(A_K, G_m) \hookrightarrow \text{Ext}^{2}_{\text{EST}}(\alpha^*A_K, \alpha^*G_m).$$

Finally, by [2, Th. 3.14.2 a)], we have an isomorphism

$$\text{Ext}^{i}_{\text{EST}}(F_K, G) \sim \text{Ext}^{i}_{\text{ES}}(\omega F_K, \omega G)$$

when $F, G \in \text{EST}$ are “1-motivic”, e.g. $F = \alpha^*A, G = \alpha^*G_m$; moreover, these groups vanish for $i \geq 2$. Lemma 6.5 now follows from the obvious vanishing of $H^1_{\text{Nis}}(K, A^*)$, the vanishing of $\text{Hom}_{\text{ES}}(A_K, G_m)$ and the isomorphism

$$\text{Ext}^{1}_{\text{ES}}(A_K, G_m) \sim A^*(K)$$
deduced from the Weil-Barsotti formula. $\square$

**Proof of Proposition 6.4.** Let $A = A^0_{X/F}$ be the Albanese variety of $X$. Lemma 6.5 yields an isomorphism

$$D^2A \simeq A \oplus \tau_{\geq 2}DA^* \oplus D(\tau_{\geq 2}DA)$$

hence a split exact triangle

$$A_X \xrightarrow{\varepsilon_A} D^2A_X \to \tau_{\geq 2}DA^* \oplus D(\tau_{\geq 2}DA) \xrightarrow{+1} .$$

Let now $M^0(X)$ be the reduced motive of $X$, sitting in the (split) exact triangle $M^0(X) \to M(X) \to \mathbb{Z} \xrightarrow{+1}$, as in the proof of Lemma 3.1 c). The map $a_X$ induces a map $a^0_X : M^0(X) \to A$, hence a dual map

$$D(a^0_X) : A^* \oplus \tau_{\geq 2}DA \simeq DA \to DM^0(X) \simeq \text{Pic}_{X/F}$$

where the left (resp. right) hand isomorphism follows from Lemma 6.5 (resp. from (3.3)). By construction, $D(a^0_X)$ restricts to the isomorphism $A^* \sim \text{Pic}^0_{X/F}$. Dualising the resulting exact triangle $A^* \to DM^0(X) \to \text{NS}_X \xrightarrow{+1}$ and reusing Lemma 6.5, we get an exact triangle

$$\text{NS}_X[1] \to D^2M^0(X) \to A \oplus \tau_{\geq 2}DA^* \xrightarrow{+1}$$

where $\text{NS}_X^*$ is the Cartier dual of $\text{NS}_X$. It follows that

$$H^0(k, D^2M^0(X)) = A(k)$$

and therefore that $H^0(k, D^2M(X)) = A_X(k)$, the map induced by $D^2(a_X)$ being the canonical injection. We thus get the requested diagram, with $Q = H^0(k, D(\tau_{\geq 2}DA))$. $\square$
7. Proof of Theorem 4

Instead of Lewis’ idea to use the complex Abel-Jacobi map, we use the \( l \)-adic Abel-Jacobi map in order to cover the case of arbitrary characteristic.

We may find a regular \( \mathbb{Z} \)-algebra \( R \) of finite type, a homomorphism \( R \to F \), and a smooth projective scheme \( p : X \to \text{Spec} R \), such that \( X = X \otimes R F \). By a direct limit argument, it suffices to show the theorem when \( F \) is the algebraic closure of the quotient field of \( R \) and, moreover, to show that the composition

\[
H^1(X, \mathcal{K}_2) \to H^1_{\text{ind}}(X, \mathcal{K}_2) \xrightarrow{\delta} \text{Hom}(\text{Griff}_1(X), F^*)
\]

has image killed by \( e \).

Let \( l \) be a prime number different from \( \text{char} F \). We may assume that \( l \) is invertible in \( R \). We have \( l \)-adic regulator maps

\[
H^1(X, \mathcal{K}_2) \xrightarrow{c} H_\text{ét}^3(X, \mathbb{Z}_l(2)), \quad H^{d-1}(X, \mathcal{K}_{d-1}) \xrightarrow{c'} H_\text{ét}^{2d-2}(X, \mathbb{Z}_l(d-1))
\]

and two compatible pairings

(7.1) \[
H^1(X, \mathcal{K}_2) \times H^{d-1}(X, \mathcal{K}_{d-1}) \to H^d(X, \mathcal{K}_{d+1}) \xrightarrow{p^*} H^0(R, \mathcal{K}_1) = R^*
\]

(7.2) \[
H_\text{ét}^3(X, \mathbb{Z}_l(2)) \times H_\text{ét}^{2d-2}(X, \mathbb{Z}_l(d-1)) \to H_\text{ét}^{2d+1}(X, \mathbb{Z}_l(d+1)) \xrightarrow{p^*} H_\text{ét}^1(R, \mathbb{Z}_l(1)).
\]

The Leray spectral sequence for the projection \( p \) yields a filtration \( F^r H_\text{ét}^*(X, \mathbb{Z}_l(\bullet)) \) on the \( l \)-adic cohomology of \( X \).

Let \( H^{d-1}(X, \mathcal{K}_{d-1})_0 = c^{-1}(F^1 H_\text{ét}^{2d-2}(X, \mathbb{Z}_l(d-1))) \) and \( H^1(X, \mathcal{K}_2)_0 = c^{-1}(F^1 H_\text{ét}^3(X, \mathbb{Z}_l(2))) \).

7.1. Lemma. The restriction of (7.1) to \( H^1(X, \mathcal{K}_2)_0 \times H^{d-1}(X, \mathcal{K}_{d-1})_0 \) has image in \( R^* \{l' \} \), the subgroup of \( R^* \) of torsion prime to \( l \).

Proof. Since \( R \) is a finitely generated \( \mathbb{Z} \)-algebra, its group of units \( R^* \) is a finitely generated \( \mathbb{Z} \)-module, hence the map \( R^* \otimes \mathbb{Z}_l \to H^1_\text{ét}(R, \mathbb{Z}_l(1)) \) from Kummer theory is injective; therefore the induced map \( R^* \to H^1_\text{ét}(R, \mathbb{Z}_l(1)) \) has finite kernel of cardinality prime to \( l \). It therefore suffices to show that the restriction of (7.2) to

\[
F^1 H_\text{ét}^3(X, \mathbb{Z}_l(2)) \times F^1 H_\text{ét}^{2d-2}(X, \mathbb{Z}_l(d-1))
\]

is 0. By multiplicativity of the Leray spectral sequences, it suffices to show that \( p_*(F^2 H_\text{ét}^{2d+1}(X, \mathbb{Z}_l(d+1))) = 0 \).
Since \( \dim X = d \), we have \( H^0_{\text{ét}}(R, H^{2d+1}_{\text{ét}}(X, \mathbb{Z}_l(d+1))) = 0 \) and hence \( H^{2d+1}_{\text{ét}}(\mathcal{X}, \mathbb{Z}_l(d+1)) = F^1 H^{2d+1}_{\text{ét}}(\mathcal{X}, \mathbb{Z}_l(d+1)) \). The edge map
\[
F^1 H^{2d+1}_{\text{ét}}(\mathcal{X}, \mathbb{Z}_l(d+1)) \to H^1_{\text{ét}}(R, H^{2d}_{\text{ét}}(X, \mathbb{Z}_l(d+1)))
\]
coinsides with the map \( p^* \) of (7.2) via the isomorphism
\[
H^{2d}_{\text{ét}}(X, \mathbb{Z}_l(d+1)) \xrightarrow{p_*} H^0_{\text{ét}}(F, \mathbb{Z}_l(1)) = \mathbb{Z}_l(1).
\]
This concludes the proof. \( \square \)

Passing to the \( \lim_{\to} \) in Lemma 7.1, we find that the pairing
\[
H^1(X, K_2) / H^1(X, K_2)_0 \to F^*
\]
has image in \( F^*\{l'\} \).

7.2. Lemma. The group \( H^1(X, K_2) / H^1(X, K_2)_0 \) is finite of exponent dividing \( e_l \).

Proof. It suffices to observe that the regulator map
\[
H^1(X, K_2) \to H^3_{\text{ét}}(X, \mathbb{Z}_l(2))
\]
has finite image [7, Th. 2.2]. \( \square \)

Lemmas 7.1 and 7.2 show that the pairing \( H^1(X, K_2) \times CH^{d-1}(X)_0 \to F^* \) has image in a group of roots of unity whose \( l \)-primary component is finite of exponent \( e_l \) for all primes \( l \neq \text{char } F \). This completes the proof of Theorem 4.

8. Questions and remarks

(1) Does the conclusion of Proposition 3.2 remain true when \( \dim X > 2 \)?

(2) Can one give an \( a \text{ priori} \), concrete, description of the extension in Theorem 1 (iii)?

(3) It is known that \( \text{Griff}_1(X) \otimes \mathbb{Q} \) (resp \( \text{Griff}_1(X)/l \) for some primes \( l \)) may be nonzero for some 3-folds \( X \) [10, 4]; these groups may not even be finite dimensional, \( e.g. \) [5, 22]. Can one find examples for which \( \text{Hom}(\text{Griff}_1(X), \mathbb{Z}) \neq 0 \)?

(4) To put the previous question in a wider context, let \( A \) be a torsion-free abelian group. Replacing \( \mathbb{G}_m \) by \( F = \mathbb{G}_m \otimes A \) in Theorem 1 yields the following computation (with same proofs):

(i) \( R^0_{\text{nr}} \mathcal{F}(X) = F^* \otimes A \).

(ii) \( R^1_{\text{nr}} \mathcal{F}(X) \cong \text{Pic}^r(X) \otimes A \).

(iii) There is a short exact sequence

(8.1) \( 0 \to D^1(X) \otimes A \to R^2_{\text{nr}} \mathcal{F}(X) \to \text{Hom}(\text{Griff}_1(X), A) \to 0. \)
Taking $A = \mathbb{Q}$ we get examples, from the nontriviality of $\text{Griff}_1(X) \otimes \mathbb{Q}$, where $R^2_{nr} F(X)$ is not reduced to $D^1(X) \otimes A$. But, choosing $X$ such that $\text{Griff}_1(X) \otimes \mathbb{Q}$ is not finite dimensional and varying $A$ among $\mathbb{Q}$-vector spaces, (8.1) also shows that the functor $F \mapsto R^2_{nr} F$ does not commute with infinite direct sums. (Therefore $R^0_{nr}$ cannot have a right adjoint.) This is all the more striking as $R^0_{nr}$ does commute with infinite direct sums, which is clear from Formula (1) in the introduction.

We don’t know whether $R^1_{nr}$ commutes with infinite direct sums or not.

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