ON QUASICONFORMAL CLOSE-TO-CONVEX HARMONIC MAPPINGS INVOLVING STARLIKE FUNCTIONS

ZHI-GANG WANG, XIN-ZHONG HUANG, ZHI-HONG LIU AND RAHIM KARGAR

Abstract. In the present paper, we discuss several basic properties of a class of quasiconformal close-to-convex harmonic mappings with starlike analytic part, such results as coefficient inequalities, an integral representation, a growth theorem, an area theorem, and radii of close-to-convexity of partial sums of the class, are derived.

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1. Introduction

A planar harmonic mapping $f$ in the open unit disk $D$ can be represented as $f = h + g$, where $h$ and $g$ are analytic functions in $D$. We call $h$ and $g$ the analytic part and co-analytic part of $f$, respectively. Since the Jacobian of $f$ is given by $|h'|^2 - |g'|^2$, by Lewy's theorem (see [23]), it is locally univalent and sense-preserving if and only if $|g'| < |h'|$, or equivalently, if $h'(z) \neq 0$ and the dilatation $\omega = g'/h'$ has the property $|\omega| < 1$ in $D$. Let $H$ denote the class of harmonic functions $f = h + g$ normalized by the conditions $f(0) = f_z(0) = 0$, which have the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k + \sum_{k=1}^{\infty} b_k z^k \quad (z \in D). \quad (1.1)$$

Denote by $S_H$ the class of harmonic functions $f \in H$ that are univalent and sense-preserving in $D$. Also denote by $S_H^0$ the subclass of $S_H$ with the additional condition $f_z(0) = 0$. We observe that Clunie and Sheil-Small [8] have proved several fundamental characteristics for the class $S_H$, but other basic problems such as Riemann mapping theorem for planar harmonic mappings, harmonic analogue of Bieberbach conjecture, sharp coefficient inequalities and radius of covering theorem for the class $S_H^0$ are still open (see [9]). The classical family $S$ of analytic univalent and normalized functions in $D$ is a subclass of $S_H^0$ with $g(z) \equiv 0$.

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If a univalent harmonic mapping \( f = h + g \) satisfies the condition
\[
|\omega(z)| = \left| \frac{g'(z)}{h'(z)} \right| \leq k \quad (0 \leq k < 1; \ z \in \mathbb{D}),
\]
then \( f \) is said to be a \( K \)-quasiconformal harmonic mapping, where
\[
K = \frac{1 + k}{1 - k} \quad (0 \leq k < 1).
\]

A domain \( \Omega \) is said to be close-to-convex if \( \mathbb{C} \setminus \Omega \) can be represented as a union of non-crossing half-lines. Following the result due to Kaplan (see [15]), an analytic function \( f \) is called close-to-convex if there exists a univalent convex function \( \phi \) defined in \( \mathbb{D} \) such that
\[
\text{Re} \left( \frac{f'(z)}{\phi'(z)} \right) > 0 \quad (z \in \mathbb{D}).
\]

Furthermore, a planar harmonic mapping \( f : \mathbb{D} \rightarrow \mathbb{C} \) is close-to-convex if it is injective and \( f(\mathbb{D}) \) is a close-to-convex domain. We denote by \( \mathcal{S}_H^0 \) the class of close-to-convex harmonic mappings.

The theory and applications of planar harmonic mappings are presented in the recent monograph by Duren [9]. Furthermore, Bshouty et al. [2–4], Chen et al. [5], Chuaqui and Hernández [7], Kalaj [12], Mocanu [26, 27], Nagpal and Ravichandran [28, 29], Partyla et al. [31], Ponnusamy and Sairam Kaliraj [34, 35], Sun et al. [40, 41], Wang et al. [42, 44] derived several criteria for univalency, or quasiconformality, involving planar harmonic mappings.

Let \( \mathcal{A} \) denote the class of functions \( h \) of the form
\[
h(z) = z + \sum_{k=2}^{\infty} a_k z^k,
\]
which are analytic in \( \mathbb{D} \). Also let \( \mathcal{G}(\alpha) \) be the subclass of \( \mathcal{A} \) whose members satisfy the inequality
\[
\text{Re} \left( 1 + \frac{zh''(z)}{h'(z)} \right) < \alpha \quad (\alpha > 1; \ z \in \mathbb{D}).
\]

For convenience, we write \( \mathcal{G}(3/2) =: \mathcal{G} \). The class \( \mathcal{G} \) plays an important role in the analytic function theory.

We observe that the function class \( \mathcal{G}(\alpha) \) was studied extensively by Kargar et al. [16], Kanas et al. [14], Maharana et al. [25], Obradović et al. [30], Ponnusamy and Sahoo [33] and Ponnusamy et al. [38] for differential purposes. It is known that the functions in \( \mathcal{G}(\alpha) \) are starlike in \( \mathbb{D} \) for \( \alpha \in (1, 3/2) \) (see Ponnusamy and Rajasekaran [32], Singh and Singh [39]), whereas not univalent in \( \mathbb{D} \) for \( \alpha \in (3/2, +\infty) \) (see [30]).

Recently, Mocanu [27] posed the following conjecture.

**Conjecture 1.** Let
\[
\mathcal{M} = \left\{ f = h + g \in \mathcal{H} : g' = zh' \text{ and } \text{Re} \left( 1 + \frac{zh''(z)}{h'(z)} \right) > -\frac{1}{2} \ (z \in \mathbb{D}) \right\}.
\]

Then \( \mathcal{M} \subset \mathcal{S}_H^0 \).

By making use of the classical results of close-to-convexity (see Kaplan [15]) and harmonic close-to-convexity (see Clunie and Sheil-Small [8]), Bshouty and Lyzzaik [8] have proved Conjecture [1] by established the following stronger result.
Theorem A. $M \subset C_H^0$.

For more recent general results on the convexity, starlikeness and close-to-convexity of harmonic mappings, we refer the readers to [1, 2, 10, 11, 13, 17, 18, 24, 27, 36, 43].

Recall the following criterion for harmonic close-to-convexity due to Abu Muhanna and Ponnusamy [1, Corollary 3].

Theorem B. Let $h$ and $g$ be normalized analytic functions in $D$ such that

$$\text{Re} \left( 1 + \frac{zh''(z)}{h'(z)} \right) < \frac{3}{2};$$

and

$$g'(z) = \lambda z^n h'(z) \quad \left( 0 < |\lambda| \leq \frac{1}{n+1}; \ n \in \mathbb{N} := \{1, 2, 3, \ldots\} \right).$$

Then the harmonic mapping $f = h + g$ is univalent and close-to-convex in $D$.

Motivated essentially by Theorem B and the definition of quasiconformal harmonic mappings, we introduce and investigate the following subclass $F(\alpha, \lambda, n)$ of quasiconformal close-to-convex harmonic mappings.

Definition 1. A harmonic mapping $f = h + g \in H$ is said to be in the class $F(\alpha, \lambda, n)$ if $h$ and $g$ satisfy the conditions

$$\text{Re} \left( 1 + \frac{zh''(z)}{h'(z)} \right) < \alpha \quad \left( 1 < \alpha \leq \frac{3}{2} \right),$$

and

$$g'(z) = \lambda z^n h'(z) \quad \left( \lambda \in \mathbb{C} \text{ with } |\lambda| \leq \frac{1}{n+1}; \ n \in \mathbb{N} \right).$$

For simplicity, we denote the class $F(\alpha, \lambda, 1)$ by $F(\alpha, \lambda)$. The image of $D$ under the mapping

$$f(z) = z - \frac{1}{2} z^2 + \frac{1}{4} z^2 - \frac{1}{6} z^3 \in F(3/2, 1/2)$$

is presented as Figure 1.

This paper is organized as follows. In Section 2, we provide a counterexample to illustrate the non-univalency of the class $G(\alpha)$ for $\alpha \in (3/2, 2)$. In Section 3, we prove several basic properties of the class $F(\alpha, \lambda, n)$ of quasiconformal close-to-convex harmonic mappings with starlike analytic part, such results as coefficient inequalities, an integral representation, a growth theorem, an area theorem, and radii of close-to-convexity of partial sums of the class, are derived.

2. Non-univalency of the class $\mathcal{G}(\alpha)$ for $\alpha \in (3/2, +\infty)$

Obradović et al. [30] stated that the class $\mathcal{G}(\alpha)$ is not univalent in $D$ for $\alpha \in (3/2, +\infty)$, but they did not give detailed proof about the non-univalency. We note that Kargar et al. [16] given a counterexample to prove the class $\mathcal{G}(\alpha)$ is not univalent in $D$ for $\alpha \in [2, +\infty)$, in this section, we shall give a counterexample to illuminate the non-univalency of the class $\mathcal{G}(\alpha)$ for $\alpha \in (3/2, 2)$.

Theorem 1. $\mathcal{G}(\alpha) \nsubseteq S$ for $\alpha \in (3/2, +\infty)$.
Figure 1. The image of $\mathbb{D}$ under the mapping $f(z) = z - \frac{1}{2}z^2 + \frac{1}{3}z^2 - \frac{1}{6}z^3$.

Proof. We consider the analytic function $h_\beta \in \mathcal{A}$ given by

$$h_\beta(z) = \frac{1}{\beta} \left[ 1 - (1 - z)\beta \right] \quad (2 < \beta < 3; \ z \in \mathbb{D}).$$

It follows that

$$1 + \frac{zh''_\beta(z)}{h'_\beta(z)} = \frac{1 - \beta z}{1 - z},$$

and therefore,

$$\text{Re} \left( 1 + \frac{zh''_\beta(z)}{h'_\beta(z)} \right) < \frac{1 + \beta}{2} \quad \left( \frac{3}{2} < \frac{1 + \beta}{2} < 2 \right),$$

which implies that

$$h_\beta \in \mathcal{G}(\frac{1 + \beta}{2}) = \mathcal{G}(\alpha) \quad \left( \frac{3}{2} < \alpha < 2 \right).$$

In what follows, we shall prove that the function $h_\beta$ is not univalent in $\mathbb{D}$. It easily to verify that $h_\beta$ have real coefficients, and thus, $h_\beta(z) = \overline{h_\beta(\overline{z})}$ for all $z \in \mathbb{D}$. In
particular, we see that
\[
\Re \left( h_\beta \left( r e^{i\theta} \right) \right) = \Re \left( h_\beta \left( r e^{-i\theta} \right) \right)
\]
for some \( r \in (0, 1) \) and \( \theta \in (-\pi, 0) \cup (0, \pi) \).

It is sufficient to show that there exist \( r_0 \in (0, 1) \) and \( \theta_0 \in (-\pi, 0) \cup (0, \pi) \) such that \( \Im \left( h_\beta \left( r_0 e^{i\theta_0} \right) \right) = \Im \left( h_\beta \left( r_0 e^{-i\theta_0} \right) \right) = 0 \).

In view of
\[
\Im \left( h_\beta (z) \right) = \Im \left( \frac{1 - (1 - z)^\beta}{\beta} \right) = -\Im \left( \frac{e^{\beta \log(1-z)}}{\beta} \right),
\]
we see that
\[
\Im \left( h_\beta \left( r e^{i\theta} \right) \right) = -\Im \left( \frac{e^{\beta \log(1-re^{i\theta})}}{\beta} \right) = -\frac{e^{\beta \log|1-re^{i\theta}|}}{\beta} \sin \left[ \beta \arg \left( 1 - re^{i\theta} \right) \right],
\]
and
\[
-\Im \left( h_\beta \left( re^{-i\theta} \right) \right) = \frac{e^{\beta \log|1-re^{-i\theta}|}}{\beta} \sin \left[ \beta \arg \left( 1 - re^{-i\theta} \right) \right] = \Im \left( h_\beta \left( re^{i\theta} \right) \right).
\]

By noting that
\[
\arg \left( 1 - re^{i\theta} \right) \in \left( -\frac{\pi}{2}, 0 \right) \cup \left( 0, \frac{\pi}{2} \right),
\]
we deduce that for each \( \beta \in (2, 3) \), there exist \( r_0 \in (0, 1) \) and \( \theta_0 \in (-\pi, 0) \cup (0, \pi) \) such that
\[
\sin \left[ \beta \arg \left( 1 - r_0 e^{i\theta_0} \right) \right] = 0.
\]

It follows that
\[
\Im \left( h_\beta \left( r_0 e^{i\theta_0} \right) \right) = \Im \left( h_\beta \left( r_0 e^{-i\theta_0} \right) \right) = 0.
\]

Therefore, we see that there exist two distinct points \( z_1 = r_0 e^{i\theta_0} \) and \( z_2 = r_0 e^{-i\theta_0} \) in \( \mathbb{D} \) such that \( h_\beta(z_1) = h_\beta(z_2) \), which shows that the function \( h_\beta(z) \) is not univalent in \( \mathbb{D} \). Thus, we deduce that the class \( G(\alpha) \) always contains a non-univalent function for each \( \alpha \in (3/2, 2) \).

Moreover, by noting that the class \( G(\alpha) \) is not univalent in \( \mathbb{D} \) for \( \alpha \in [2, +\infty) \) (see [16, Example 2.1]), we deduce that the assertion of Theorem 1 holds.

To illustrate our counterexample, we present the image domain of \( \mathbb{D} \) under the function \( h_{5/2}(z) = 2/5 \left[ 1 - (1 - z)^{5/2} \right] \) (see Figure 2).

3. Properties and Characteristics of the Class \( \mathcal{F}(\alpha, \lambda, n) \)

Let us recall the following lemma, due to Obradović et al. [30], in a slightly modified form, which will be required in the proof of Theorem 2.
Figure 2. The image of $\mathbb{D}$ under the function $h_{5/2}(z) = 2/5 \left[1 - (1 - z)^{5/2}\right]$.

**Lemma 1.** If $h(z) = z + \sum_{k=2}^{\infty} a_k z^k$ satisfies the condition (1.2) with $1 < \alpha \leq 3/2$, then

$$|a_k| \leq \frac{2(\alpha - 1)}{(k - 1)k} \quad (k \geq 2),$$

with the extremal function given by

$$h(z) = \int_0^z \left(1 - t^{k-1}\right)^{2(\alpha-1)/k} \, dt \quad (k \geq 2).$$

**Theorem 2.** Let $f = h + g \in \mathcal{F}(\alpha, \lambda, n)$ be of the form (1.1). Then the coefficients $a_k$ ($k \geq 2$) of $h$ satisfy (3.1), furthermore, the coefficients $b_k$ ($k = n+1, n+2, \ldots; n \in \mathbb{N}$) of $g$ satisfy

$$|b_{n+1}| \leq \frac{\lambda}{n + 1} \quad (n \in \mathbb{N}) \quad \text{and} \quad |b_{k+n}| \leq \frac{2\lambda(\alpha - 1)}{(k - 1)k(k + n)} \quad (k \in \mathbb{N} \setminus \{1\}; n \in \mathbb{N}).$$

(3.2)
The bounds are sharp for the extremal function given by
\[
f(z) = \int_0^z \left(1 - t^{k-1}\right)^{\frac{2(\alpha-1)}{k-1}} \, dt + \int_0^z \lambda t^n \left(1 - t^{k-1}\right)^{\frac{2(\alpha-1)}{k-1}} \, dt \quad (n \in \mathbb{N}).
\]

Proof. Comparing the coefficients of \(z^{k+n-1}\) of both sides in (1.4), we obtain
\[
(k + n)b_{k+n} = \lambda k a_k \quad (k, n \in \mathbb{N}; a_1 = 1).
\]
Combining Lemma 1 with (3.3), we readily get the desired coefficient inequalities (3.2) of Theorem 2.

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The Fekete-Szegö functional for \(|a_3 - \delta a_2^3|\) of the class \(G(\alpha)\) with \(\alpha \in (1, 3/2]\) was discussed by Obradović et al. [30], which will be useful in the proof of the upper bounds for \(|b_3 - \delta b_2^3|\) of functions in the class \(F(\alpha, \lambda)\). We here present its modified form.

**Lemma 2.** Let \(f \in G(\alpha)\) with \(\alpha \in (1, 3/2]\). Then
\[
|a_3 - \delta a_2^3| \leq \begin{cases} 
\frac{\alpha-1}{3} |3 + \delta - (2 + \delta)\alpha| \quad \left(\left|\delta - \frac{3 - 2\alpha}{3(\alpha-1)}\right| \geq \frac{1}{3(\alpha-1)}\right), \\
\quad \frac{\alpha-1}{3} \quad \left(\left|\delta - \frac{3 - 2\alpha}{3(\alpha-1)}\right| < \frac{1}{3(\alpha-1)}\right).
\end{cases}
\]

Equality in the Fekete-Szegö functional is attained in each case.

**Theorem 3.** Let \(f \in F(\alpha, \lambda)\) be of the form (1.1). Then
\[
|b_3 - \delta b_2^3| \leq \frac{2(\alpha - 1)|\lambda|}{3} + \frac{|\delta||\lambda|^2}{4}.
\]

The inequality is sharp.

Proof. By noting that \(g'(z) = \lambda z h'(z)\) for \(f \in F(\alpha, \lambda)\), we have
\[
\sum_{k=2}^{\infty} kb_k z^{k-1} = \lambda \sum_{k=1}^{\infty} ka_k z^k \quad (a_1 = 1).
\]
Clearly, we see that
\[
b_2 = \frac{1}{2}\lambda a_1 = \frac{1}{2}\lambda \quad \text{and} \quad b_3 = \frac{2}{3}\lambda a_2.
\]
Therefore, by virtue of (3.4) and (3.6), we obtain
\[
|b_3 - \delta b_2^3| = \left|\frac{2}{3}\lambda a_2 - \frac{1}{4}\delta \lambda^2\right| \leq \frac{2|\lambda||a_2|}{3} + \frac{|\delta||\lambda|^2}{4} \leq \frac{2(\alpha - 1)|\lambda|}{3} + \frac{|\delta||\lambda|^2}{4}.
\]
The proof of Theorem 3 is thus completed.

By setting \(\delta = 1\) in Lemma 2 respectively Theorem 3 we get the Zalcman type coefficient inequalities of the class \(F(\alpha, \lambda)\) for the case \(k = 2\). For recent developments on this topic (see Li and Ponnusamy [22] and the references therein).

**Corollary 1.** Let \(f \in F(\alpha, \lambda)\) be of the form (1.1). Then
\[
|a_3 - a_2^3| \leq \begin{cases} 
\frac{\alpha-1}{3} |4 - 3\alpha| \quad \left(\left|\delta - \frac{3 - 2\alpha}{3(\alpha-1)}\right| \geq \frac{1}{3(\alpha-1)}\right), \\
\quad \frac{\alpha-1}{3} \quad \left(\left|\delta - \frac{3 - 2\alpha}{3(\alpha-1)}\right| < \frac{1}{3(\alpha-1)}\right).
\end{cases}
\]
and
\[ |b_3 - b_2^2| \leq \frac{2(\alpha - 1)|\lambda|}{3} + \frac{|\lambda|^2}{4} \leq \frac{11}{48}. \]

The inequalities are sharp.

Now, we give an integral representation of the mapping \( f \in \mathcal{F}(\alpha, \lambda, n) \).

**Theorem 4.** Let \( f \in \mathcal{F}(\alpha, \lambda, n) \). Then
\[
f(z) = \int_0^z \exp \left( 2(1 - \alpha) \int_0^\zeta \frac{\varpi(t)}{t(1 - \varpi(t))} \, dt \right) \, d\zeta
+ \lambda \int_0^z \zeta^n \cdot \exp \left( 2(1 - \alpha) \int_0^\zeta \frac{\varpi(t)}{t(1 - \varpi(t))} \, dt \right) \, d\zeta,
\]
where \( \varpi \) is the Schwarz function with \( \varpi(0) = 0 \) and \( |\varpi(z)| < 1 \) \((z \in \mathbb{D})\).

**Proof.** Suppose that \( f \in \mathcal{F}(\alpha, \lambda, n) \). It follows from (1.3) that
\[
1 + \frac{zh''(z)}{h'(z)} \prec \frac{1 - (2\alpha - 1)z}{1 - z} \quad (z \in \mathbb{D}), \tag{3.7}
\]
where “\( \prec \)” denotes the familiar subordination of analytic functions. By virtue of (3.7), we see that
\[
1 + \frac{zh''(z)}{h'(z)} = \frac{1 - (2\alpha - 1)\varpi(z)}{1 - \varpi(z)} \quad (z \in \mathbb{D}), \tag{3.8}
\]
where \( \varpi \) is the Schwarz function with \( \varpi(0) = 0 \) and \( |\varpi(z)| < 1 \) \((z \in \mathbb{D})\). From (3.8), we have
\[
\frac{(zh'(z))'}{zh'(z)} - \frac{1}{z} = \frac{2(1 - \alpha)\varpi(z)}{z(1 - \varpi(z))},
\]
which, upon integration, yields
\[
\log(h'(z)) = 2(1 - \alpha) \int_0^z \frac{\varpi(t)}{t(1 - \varpi(t))} \, dt. \tag{3.9}
\]
We thus find from (3.9) that
\[
h(z) = \int_0^z \exp \left( 2(1 - \alpha) \int_0^\zeta \frac{\varpi(t)}{t(1 - \varpi(t))} \, dt \right) \, d\zeta. \tag{3.10}
\]
Combining (1.4) with (3.10), we obtain
\[
g(z) = \lambda \int_0^z \zeta^n \cdot \exp \left( 2(1 - \alpha) \int_0^\zeta \frac{\varpi(t)}{t(1 - \varpi(t))} \, dt \right) \, d\zeta. \tag{3.11}
\]
Thus, the assertion of Theorem 4 follows from (3.10) and (3.11). \( \square \)

**Remark 1.** Theorem 4 provides a direct integration method for constructing quasi-conformal close-to-convex harmonic mappings by choosing suitable Schwarz functions \( \varpi \).

The following lemma due to Maharana et al. \[25\] will play a crucial role in the proof of our last three results.

**Lemma 3.** If \( h \in \mathcal{G} \), then for \( |z| = r < 1 \), the following statements are true.
\[ \left| \frac{zh''(z)}{h'(z)} \right| \leq \frac{r}{1 - r}. \]

The inequality is sharp and equality is attained for the function

\[ h(z) = z - \frac{z^2}{2}. \tag{3.12} \]

(2)

\[ 1 - r \leq |h'(z)| \leq 1 + r. \tag{3.13} \]

The inequalities are sharp and equalities are attained for the function given by

(3)

If \( h(z) = S_n(z) + \Sigma_n(z) \), with \( \Sigma_n(z) = \sum_{k=n+1}^{\infty} a_k z^k \), then

\[ \left| \Sigma_n'(z) \right| \leq r^n \phi(r, 1, n) \quad \text{and} \quad \left| z \Sigma_n''(z) \right| \leq \frac{r^n}{1 - r}, \]

where \( \phi(r, 1, n) \) is the unified zeta function which is defined by the series

\[ \phi(z, s, a) = \sum_{k=0}^{\infty} \frac{z^k}{(k + a)^s} \quad (|z| < 1; \Re(s) > 1; a \neq 0, -1, -2, \ldots). \]

We now give the growth theorem for the class \( F(\alpha, \lambda, n) \).

**Theorem 5.** Let \( f \in F(\alpha, \lambda, n) \). Then

\[ r \left[ |\lambda| \left( \frac{r}{n + 2} - \frac{1}{n + 1} \right) r^n - \frac{r}{2} + 1 \right] \leq |f(z)| \leq r \left[ |\lambda| \left( \frac{r}{n + 2} + \frac{1}{n + 1} \right) r^n + \frac{r}{2} + 1 \right]. \tag{3.14} \]

The inequalities are sharp.

**Proof.** Assume that \( f = h + \overline{g} \in F(\alpha, \lambda, n) \). By observing that \( h \in \mathcal{G} \), we know that (3.13) holds. Also, let \( \Gamma \) be the line segment joining 0 and \( z \), then

\[ |f(z)| = \left| \int_{\Gamma} \frac{\partial f}{\partial \xi} d\xi + \frac{\partial f}{\partial \xi} d\xi \right| \]

\[ \leq \int_{\Gamma} \left| |h'(\xi)| + |g'(\xi)| \right| |d\xi| \]

\[ = \int_{\Gamma} \left( 1 + |\lambda| |\xi|^n \right) |h'(\xi)| |d\xi| \]

\[ \leq \int_0^r (1 + \xi)(1 + |\lambda| |\xi|^n) d\xi \]

\[ = \frac{1}{2} r \left[ 2|\lambda| \left( \frac{r}{n + 2} + \frac{1}{n + 1} \right) r^n + r + 2 \right]. \tag{3.15} \]
Moreover, let $\tilde{\Gamma}$ be the preimage under $f$ of the line segment joining 0 and $f(z)$, then we obtain

$$|f(z)| = \int_{\tilde{\Gamma}} \left| \frac{\partial f}{\partial \xi} d\xi + \frac{\partial f}{\partial \bar{\xi}} d\bar{\xi} \right|$$

$$\geq \int_{\tilde{\Gamma}} (|h'(\xi)| - |g'(\xi)|) |d\xi|$$

$$= \int_{\tilde{\Gamma}} (1 - |\lambda||\xi|^n) |h'(\xi)| |d\xi|$$

$$\geq \int_0^r (1 - \xi)(1 - |\lambda|\xi^n)d\xi$$

$$= \frac{1}{2}r \left[ 2|\lambda| \left( \frac{r}{n+2} - \frac{1}{n+1} \right) r^n - r^2 \right].$$

It follows from (3.15) and (3.16) that the assertion (3.14) of Theorem 5 holds. $\square$

Denote by $A(f(D_r))$ the area of $f(D_r)$, where $D_r := rD$ for $0 < r < 1$. We now consider the area theorem of mappings $f$ belong to the class $F(\alpha, \lambda, n)$.

**Theorem 6.** Let $f \in F(\alpha, \lambda, n)$. Then for $0 < r < 1$, we have

$$2\pi \int_0^r (1 - |\lambda|^2\xi^{2n}) (1 - \xi)^2\xi d\xi \leq A(f(D_r)) \leq 2\pi \int_0^r (1 - |\lambda|^2\xi^{2n}) (1 + \xi)^2\xi d\xi. \quad (3.17)$$

**Proof.** Suppose that $f = h + \bar{g} \in F(\alpha, \lambda, n)$. Then for $0 < r < 1$, we get

$$A(f(D_r)) = \iint_{D_r} (|h'(z)|^2 - |g'(z)|^2) \, dx \, dy = \iint_{D_r} (1 - |\lambda|^2|z|^{2n}) |h'(z)|^2 \, dx \, dy. \quad (3.18)$$

In view of (3.13) and (3.18), we obtain the result of Theorem 6 $\square$

Finally, we shall discuss the radius problems of mappings $f \in F(\alpha, \lambda)$. The largest value of $r$ so that the partial sums of $f \in F(\alpha, \lambda)$ are close-to-convex in $|z| < r$ are considered. For recent results on partial sums of univalent harmonic mappings (see, e.g., Chen et al. [6], Ghosh and Vasudevarao [11], Li and Ponnusamy [19–21], Ponnusamy et al. [37], Sun et al. [40]).

**Theorem 7.** Let $f \in F(\alpha, \lambda)$ be of the form (1.1). Then for each $m \geq 1$, $l \geq 2$,

$$S_{m,l}(f)(z) = \sum_{k=1}^m a_k z^k + \sum_{k=2}^l b_k z^k \quad (a_1 = 1)$$

is close-to-convex in $|z| < r_c \approx 0.503$, where $r_c$ is the least positive real root in the interval $(0,1)$ of the equation:

$$2 + 2 \ln(1 - r) + r \ln(1 - r) - r + r^2 = 0. \quad (3.19)$$

The bound $r_c$ is sharp.
Proof. Let \( f = h + \overline{g} \in \mathcal{F}(\alpha, \lambda) \) and \( \phi = h + \varepsilon \overline{g} \) with \( |\varepsilon| = 1 \). We observe that \( \text{Re}(\varphi'(z)) > 0 \) for \( \varphi \in \mathcal{A} \) implies that \( \varphi \) is a close-to-convex analytic function. Therefore, it is sufficient to show that each partial sums

\[
S_{m,l}(\phi)(z) = \sum_{k=1}^{m} a_k z^k + \varepsilon \sum_{k=2}^{l} b_k z^k
\]

satisfies the condition

\[
\text{Re}\left(\Gamma_{m,l}(\phi)(z)\right) > 0
\]

in the disk \(|z| < r_c\) for all \(|\varepsilon| = 1\) and \( m \geq 1, \ l \geq 2\), where

\[
\Gamma_{m,l}(\phi)(z) = \sum_{k=1}^{m} a_k z^k + \varepsilon \sum_{k=2}^{l} b_k z^k.
\]

In order to prove the radii of close-to-convexity for the partial sums \( S_{m,l}(f)(z) \), we split it into four cases to prove.

1. For \( m = 1, 2, \ l = 2 \), we have

\[
\Gamma_{1,2}(\phi)(z) = z + \varepsilon b_2 z^2,
\]

and

\[
\Gamma_{2,2}(\phi)(z) = z + a_2 z^2 + \varepsilon b_2 z^2,
\]

it follows that

\[
\Gamma'_{1,2}(\phi)(z) = 1 + \varepsilon \lambda z,
\]

and

\[
\Gamma'_{2,2}(\phi)(z) = 1 + 2a_2 z + \varepsilon \lambda z.
\]

Clearly, \( \text{Re}(\Gamma'_{1,2}(\phi)(z)) > 0 \) in \(|z| < r_1 = 2/3\). By Lemma \([1]\) we know that \(|a_2| \leq \alpha - 1\), thus,

\[
\text{Re}(\Gamma'_{2,2}(\phi)(z)) \geq 1 - 2|a_2||z| - |\lambda||z| \geq 1 - [2(\alpha - 1) + |\lambda||z|] \geq 1 - \frac{3}{2}|z| > 0 \quad (|z| < r_1).
\]

2. For \( m, l \geq 3 \), we find from \([1,3]\) and \([1,4]\) that

\[
\text{Re}(\Gamma'_{m,l}(\phi)(z))
= \text{Re}\left(\mathcal{S}_{m}(h)(z) + \varepsilon \lambda z \mathcal{S}'_{l-1}(h)(z)\right)
= \text{Re}\left((h'(z) - \Sigma_{m}(h)(z)) + \varepsilon \lambda z (h'(z) - \Sigma'_{l-1}(h)(z))\right)
\geq \text{Re}(h'(z)) - |\Sigma_{m}(h)(z)| - |\lambda||z||h'(z)| - |\lambda||z||\Sigma'_{l-1}(h)(z)|
\geq \text{Re}(h'(z)) - |\Sigma_{m}(h)(z)| - \frac{1}{2}|z||h'(z)| - \frac{1}{2}|z||\Sigma'_{l-1}(h)(z)|.
\]

In view of \([3.13]\), we obtain

\[
\min_{|z|=r<1} \{\text{Re}(h'(z))\} \geq \min_{|z|=r<1} \{\text{Re}(1 - z)\} \geq 1 - r.
\]
From Lemma [3](3), for \(|z| = r < 1\), we know that
\[
|\Sigma_n'(z)| \leq \sum_{k=0}^{\infty} \frac{r^{k+n}}{k+n} = -\ln(1-r) - \sum_{k=1}^{n-1} \frac{r^k}{k} =: \Delta(n),
\]
and
\[
\Delta(n + 1) - \Delta(n) = -\frac{r^n}{n} < 0 \quad (n \geq 2).
\]
Therefore, \(\Delta(n)\) is a decreasing function of \(n\). For all \(m, l \geq 3\), we see that
\[
\Delta(m) \leq \Delta(3) = -\ln(1-r) - r - \frac{r^2}{2}, \quad (3.22)
\]
and
\[
\Delta(l - 1) \leq \Delta(2) = -\ln(1-r) - r. \quad (3.23)
\]
Moreover, it follows from Lemma [3](2) that
\[
|h'(z)| \leq |(1 + |z|) = r(1 + r) \quad (|z| = r < 1). \quad (3.24)
\]
From the relationships (3.20), (3.21), (3.22), (3.23) and (3.24), it follows that
\[
\text{Re} \left( \Gamma_n'(\phi(z)) \right) \geq 1 + \ln(1-r) - \frac{r}{2} + \frac{1}{2} r \ln(1-r) + \frac{1}{2} r^2 > 0
\]
for all \(l \geq 3\) and \(|z| = r < r_2 \approx 0.503\), where \(r_2\) is the least positive root in the interval \((0, 1)\) of the equation:
\[
2 + 2 \ln(1-r) + r \ln(1-r) - r + r^2 = 0.
\]
(3) For \(m = 1, 2, l \geq 3\), we see that
\[
\text{Re} \left( \Gamma_1'(\phi(z)) \right) = \text{Re} \left( \Sigma'_1(h(z) + \varepsilon \Sigma'_1(g(z)) \right)
\]
\[
= \text{Re} \left( 1 + 2a_2z + \varepsilon \lambda z \Sigma'_{l-1}(h(z)) \right)
\]
\[
\geq 1 - 2|a_2||z| - |\lambda||z||h'(z)| - |\lambda||z||\Sigma'_{l-1}(h(z))|
\]
\[
\geq 1 - \frac{1}{2} |z| - \frac{1}{2} |z||h'(z)| - \frac{1}{2} |z||\Sigma'_{l-1}(h(z))|.
\]
From (3.22) and (3.23), we know that
\[
\text{Re} \left( \Gamma_2'(\phi(z)) \right) \geq 1 - \frac{1}{2} r - \frac{r(1+r)}{2} + \frac{1}{2} r \ln(1-r) + r > 0
\]
for all \(l \geq 3\) and \(|z| = r < r_3 \approx 0.653575\), where \(r_3\) is the least positive root in the interval \((0, 1)\) of the equation:
\[
2 - 2r + r \ln(1-r) = 0.
\]
Similarly, for all \(l \geq 3\) and \(|z| = r < r_3\), we have
\[
\text{Re} \left( \Gamma_1'(\phi(z)) \right) \geq 1 - \frac{1}{2} |z||h'(z)| - \frac{1}{2} |z||\Sigma'_{l-1}(h(z))| \geq 1 - \frac{r}{2} + \frac{r}{2} \ln(1-r) > 0.
\]
(4) For $m \geq 3, l = 2$, we deduce from (3.21) and (3.22) that
\[
\text{Re} \left( \Gamma_{m,2}(\phi)(z) \right) = \text{Re} \left( S_m'(h)(z) + zS_2'(g)(z) \right)
\geq \text{Re} \left( h'(z) \right) - |S_m'(h)(z)| - |\lambda||z|
\geq 1 - \frac{1}{2}r + \ln(1-r) + \frac{r^2}{2} > 0,
\]
where $|z| = r < r_4 \approx 0.584628$, where $r_4$ is the least positive root in the interval $(0,1)$ of the equation:
\[
2 - r + 2 \ln(1-r) + r^2 = 0.
\]
By setting
\[
r_c := \min \{ r_1, r_2, r_3, r_4 \} = r_2,
\]
we see that $\text{Re} \left( \Gamma_{m,l}(\phi)(z) \right) > 0$ for all $|z| < r_c$ and $m \geq 1, l \geq 2$. The proof of Theorem 7 is thus completed. \qed

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Zhi-Gang Wang

School of Mathematics and Computing Science, Hunan First Normal University, Changsha 410205, Hunan, P. R. China.

E-mail address: wangmath@163.com

Xin-Zhong Huang

School of Mathematical Sciences, Huaqiao University, Quanzhou 362021, Fujian, P. R. China.

E-mail address: huangxz@hqu.edu.cn

Zhi-Hong Liu

College of Science, Guilin University of Technology, Guilin 541004, Guangxi, P. R. China.

E-mail address: liuzhihongmath@163.com

Rahim Kargar

Department of Mathematics and Statistics, University of Turku, Turku, Finland.

E-mail address: rakarg@utu.fi