The birth of the contradictory component in random 2-SAT

Sergey Dovgal*

Institut Galilée, Université Paris 13,
99 Avenue Jean Baptiste Clément 93430,
Villetaneuse, France.
email: dovgal-at-lipn.fr

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Abstract

We prove that, with high probability, the contradictory components of a random 2-SAT formula in the subcritical phase of the phase transition have only 3-regular kernels. This follows from the relation between these kernels and the complex component of a random graph in the subcritical phase. This partly settles the question about the structural similarity between the phase transitions in 2-SAT and random graphs. As a byproduct, we describe the technique that allows to obtain a full asymptotic expansion of the satisfiability in the subcritical phase. We also obtain the distribution of the number of contradictory variables and the structure of the spine in the subcritical phase.

Keywords. 2sat, satisfiability, complex component, cores, kernels, spine, analytic combinatorics, generating functions

1 Introduction

1.1 Phase transitions

Phase transitions in random graphs, directed graphs, hypergraphs, random geometric complexes, percolations, real-world networks and constraint satisfiability problems (CSP) have become rapidly developing areas with a huge spread of interconnected publications.

The term “phase transition” was mostly used by physicists, but can be also applied to many combinatorial situations, when a small change of a certain parameter results in a huge asymptotic change of some other parameter. The original studies of the physical phase transitions, including Ising and Potts model, considered graphs which formed certain regular lattices: from rectangular ones to more complicated including maps on surfaces. Of close relation is the percolation theory which is sometimes called “the simplest model displaying a phase transition”. Friendly introductions into percolation theory and Potts models can be found in a PhD Thesis [Dom13], a survey paper [BEMPS10] and in a lecture course [DC17].

Apart from the existing practical applications in hardware in software engineering (e.g. cuckoo hashing [DM03, DGM10]), phase transitions in graphs and digraphs are studied in their own right. Of the most recent references on random graphs and networks can be mentioned the books [JLR11, VDH16]. Concerning the phase transition in directed graphs, the width of the transitions window has been described in [LS09] and a description of the giant core is given in [PP17]. Recent studies reveal the properties of phase transitions in random hypergraphs [dP15, CKP18] and simplicial complexes [CDGKS18].

One of the most recent surveys on satisfiability is a part of the excellent “Art of Computer Programming, Volume 4” by Knuth [Knu15]. The k-SAT problem has also played its role on the intersection of theoretical

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computer science and theoretical physics. One of the surprising connections is that several NP-complete decision problems can be reformulated in terms of positivity of the ground state energy of an Ising model and that the techniques like belief propagation and cavity method used in statistical physics can be also used to predict the behaviour of random formulae, see [MZ97, BCM02, MPZ02]. At the same time, phase transitions in random CSP seem to be closely related to their algorithmic complexity, see [ACO08]. While the existence of satisfiability threshold and its value for k-SAT with \( k \geq 3 \) remains an unsolved open problem, several advances have been made, including a precise asymptotic location of the transition point for \( k \to \infty \) [COP16], phase transitions of \( k \)-XORSAT [HR11, FS16], and more generally, systems of linear equations over \( F_q \) [ACOGM17], improvements in \( k \)-colorability threshold [COV13].

The 2-SAT problem (see Definition 1) is the easiest case of the more general \( k \)-SAT problem which is NP-complete for \( k \geq 3 \), and admits a linear-time algorithm for \( k = 2 \) [APT82]. However, already the problem of maximising the number of satisfying assignments of the 2-SAT, known as MAX-2-SAT is known to be NP-complete as well. The study of the phase transition in 2-SAT and MAX-2-SAT culminated in the papers [BBC+01] and [CGHS04], where the critical width has been determined. Kim [Kim08] improved the bounds and provided an exact constant for the subcritical case. Since then, it has been questioned whether the structural similarities between the transitions in graphs, digraphs and 2-SAT exist. As an example, it is known that for random graphs \( G_{n,m} \) with \( n \) vertices and \( m = \frac{5}{2} (1 + \mu n^{-1/3}) \) edges,

\[
P(G_{n,m} \text{ contains only trees and unicycles}) = 1 - \frac{5 + o(1)}{24|\mu|^3} \quad \text{as } \mu \to -\infty \text{ with } n,
\]

for a random 2-SAT formula \( F_{n,m} \) with \( n \) variables and \( m = n(1 + \mu n^{-1/3}) \) clauses,

\[
P(F_{n,m} \text{ is satisfiable}) = 1 - \frac{1 + o(1)}{16|\mu|^3} \quad \text{as } \mu \to -\infty \text{ with } n,
\]

and for random directed graphs \( D_{n,m} \) with \( n \) vertices and \( m = n(1 + \mu n^{-1/3}) \) oriented edges,

\[
P(\text{every strong component in } D_{n,m} \text{ is either a vertex or a cycle of length } O(n^{1/3} \mu^{-1})) \to 1 \quad \text{as } \mu \to -\infty \text{ with } n,
\]

where \( o(1) \) goes to zero as \( |\mu| \ll n^{1/3} \) (the formula for random graphs requires \( |\mu| \ll n^{1/12} \) and can be further improved to \( |\mu| \ll n^{1/3} \), see [HR11]). The factor \( 5/24 \) is obtained by adding the inverse numbers of automorphisms \( 1/12 \) and \( 1/8 \) of the two possible cubic (i.e. 3-regular) multigraphs with 3 edges and 2 vertices, which appear as the only two possible cores at the point when a graph doesn’t anymore consist only of trees and unicycles, see Section 3.5 and Remark 5 for an explanation of this phenomenon. A different viewpoint on the giant component and satisfiability has been proposed in [Mol08]. Clearly, the structure of a critical implication digraph might not be similar to that of a critical simple graph or a critical digraph, due to the presence of certain symmetries, which have been emphasised in [Kra07]. Nevertheless, it still seems to be possible to draw certain structural similarities between the models.

We follow the approach of analytic combinatorics which was used to obtain the structure of random graph at the point of the phase transition [FKP89, JKL93, FSS04]. The approach makes use of the generating function technique which gives a very clear structural vision of the corresponding combinatorial objects. In addition to the aesthetic benefits (see e.g. a unifying approach for the upper bounds on satisfiability threshold [Puy04]), it allows to give very precise asymptotic descriptions, often accompanied by complete asymptotic expansions. In particular, inside the transition window, the probability that a random graph doesn’t contain a complex component (i.e. consists only of trees and unicycles) has been expressed in terms of Airy function which has appeared in many other contexts in analytic combinatorics [BFSS01, FSS04]. Another interpretation, using the fact that Airy function is linked to the area under a Brownian motion, was discovered by Aldous [Ald97], see also [ABBG12]. The enumeration of unsatisfiable 2-SAT formulæ is equivalent to forbidding a certain contradictory pattern (namely, the contradictory circuit, see Section 1.2).

Of the closest to our approach here is [CDPG18] where the authors use the analytic approach to study the containment problem for small subgraphs. Applying the analytic methods to the phase transitions of SAT-formulæ and directed graphs remains one of the important challenging problems.
The spine of a 2-CNF (see Definition 3) has been introduced in [BBC+01] as a useful tool for combinatorial analysis of the probability of satisfiability inside the critical window. Originally, the spine is defined as the set of variables that are forced to take the FALSE value in any satisfying assignment. Later, the spine has been shown to impact the complexity of the underlying decision problems in a more general setting for various constraint satisfaction problems including $k$-XORSAT, graph bipartition problem, random 3-coloring [Sol, IBP05]. In our analysis of satisfiability we do not use the properties of the spine, but we find this parameter interesting by itself. We obtain a new structural result about the spine of a random formula.

1.2 Definitions

A simple graph $G = (V, E)$, $E \subset \{x, y\}$ with $x, y \in V$, $x \neq y$ is a graph without loops and multiple edges, so that every edge connects a set of two distinct vertices and for every pair of vertices there is at most one edge connecting them. Contrary to a simple graph, a multigraph may contain an arbitrary number of loops and multiple edges. By $\mathcal{G}(n, m)$ we denote the set of all simple graphs with $n$ vertices and $m$ edges. We say that a graph has size $n$ if it contains $n$ vertices.

A simple digraph is a synonym for a simple directed graph which we define as a pair $D = (V, E)$ of the set of vertices $V$ and the set of edges $E \subset \{(x, y) \mid x, y \in V, x \neq y\}$ such that every directed edge $x \to y$ is represented by a pair of vertices $(x, y)$; a simple digraph contains no loops $x \to x$, and no multiple edges; at the same time, any two distinct vertices $x$ and $y$ can have simultaneously both directed edges $x \to y$ and $y \to x$ between them. An unoriented projection of a simple digraph is a multigraph (possibly containing double edges) which is obtained by dropping the orientations of the edges. We denote by $D(2n, m)$ the set of simple digraphs with $2n$ vertices and $m$ oriented edges.

Below we give the definitions related to the 2-SAT model which contains the formulae in 2-conjunctive normal form (2-CNF).

**Definition 1** (Variables, literals and clauses). Consider $n$ Boolean variables $\{x_1, x_2, \ldots, x_n\}$, so that $x_i \in \{0, 1\}$ for all $i \in \{1, \ldots, n\}$. The logical negation of $x_i$ is denoted by $\overline{x_i}$. Each of $\{x_i, \overline{x_i}\}$ is called a literal, while the two literals $\{x_i, \overline{x_i}\}$ refer to the same variable $x_i$. We say that two literals $\xi$ and $\eta$ are complementary if $\xi = \overline{\eta}$. Two literals $\xi$ and $\eta$ are said to be strictly distinct if their underlying variables are distinct. The 2-clauses are disjunctions of two literals, corresponding to distinct Boolean variables, in other words, each 2-clause is of the form $(\xi_j \lor \eta_j)$ where $\xi_j$ and $\eta_j$ belong to the set of $2n$ possible literals $\{x_1, \ldots, x_n, \overline{x_1}, \ldots, \overline{x_n}\}$. We do not distinguish clauses obtained by a change of the order of variables inside a disjunction, so $(\xi_j \lor \eta_j) \equiv (\eta_j \lor \xi_j)$. A 2-CNF is a conjunction of clauses, i.e. a formula of the form $\wedge_{j=1}^m (\xi_j \lor \eta_j)$ where each of the clauses $(\xi_j \lor \eta_j)$ is distinct for $j \in \{1, \ldots, m\}$. A formula is satisfiable (SAT) if there exists variable assignment yielding the truth value of the formula. By $F(n, m)$ we denote the set of 2-CNF formulae with $n$ Boolean variables and $m$ clauses.

In different models, the distinctness condition for clauses is not required, but we show that it is not essential for the phase transition and for our techniques. More explicitly, we require for technical reasons that in a 2-CNF formula in our model the conditions (C1)–(C3) hold:

(C1) clauses $(x_i \lor x_i)$ are not allowed;
(C2) clauses $(x_i \lor \overline{x_i})$ are not allowed;
(C3) all the clauses are distinct.

Each of the conditions (C1)–(C3) may be violated without changing the main results, see Remark 8.

**Definition 2** (Implication digraphs). Any 2-CNF with $n$ Boolean variables and $m$ clauses can be represented in the form of an implication digraph where every clause $(x \lor y)$ is replaced with two directed edges $(\overline{y} \to x)$ and $(\overline{x} \to y)$.

If there is a directed path from a vertex $x$ to a vertex $y$ in a directed graph $D$, we write $x \to y$. In the case when $D$ is an implication digraph, we also say that a literal $x$ implies literal $y$. 3
Definition 3 (Spine, contradictory variables and circuits). The spine of a formula $F$ (denoted $S(F)$) is the set of literals that imply their complementary literals, i.e.

$$S(F) := \{x \mid x \Rightarrow \neg x \text{ in } F\}.$$ 

A variable $x$ is contradictory if $x \in S(F)$ and $\neg x \in S(F)$. The contradictory component $C(F)$ of a formula $F$ is then formed of all the contradictory variables, i.e.

$$C(F) := \{x \mid x \Rightarrow \neg x \Rightarrow x \text{ in } F\}.$$ 

A contradictory circuit is a distinguished directed path passing through $x$, $\neg x$ and $x$. It can be easily shown that any variable belonging to a contradictory circuit is also contradictory (possibly via a different contradictory circuit). It is well known and has been proven in [APT82], that a formula is satisfiable if and only if it contains a so-called contradictory circuit, that is, there exists a literal $x$ such that $x \Rightarrow \neg x$ and $\neg x \Rightarrow x$. Consequently, a formula is satisfiable if and only if its contradictory component is empty.

Example 1. Figure 1 contains an implication digraph of a formula $(x_1 \vee x_2)(x_1 \vee \neg x_2)(x_3 \vee x_4)(x_3 \vee \neg x_4)$. This formula is unsatisfiable because its implication digraph contains a contradictory circuit $1 \rightarrow 2 \rightarrow 1 \rightarrow 3 \rightarrow 4 \rightarrow 3 \rightarrow 1$ passing through 1 and 1.

Remark 1. A contradictory component forms a set of strongly connected components such that there is no directed path between any two strongly connected components. Indeed, if $C_1$ and $C_2$ are such strongly connected components (where each variable is contradictory), and there is a path from $x \in C_1$ to $y \in C_2$, then, the complementary path $\neg y \Rightarrow \neg x$ is also a part of the implication digraph, while $\neg y \in C_2$, $\neg x \in C_1$, since each of the strongly connected components is contradictory. Therefore, the presence of a directed path from $C_1$ to $C_2$ implies a directed path from $C_2$ to $C_1$ and they form a single strongly connected component.

1.3 Structure of the paper

The paper is structured as follows. In Section 2 we introduce the concept of sum-representation which is used throughout the paper. In this section, the main comparisons between simple graphs and implication digraphs of 2-SAT are drawn, and the excess of a contradictory component is introduced. In Section 3 the classical symbolic method for generating functions of simple graphs is explained, along with its variations for weakly connected directed graphs. In this section, the concept of compensation factor of a contradictory component is introduced.

In Section 4, the actual asymptotic analysis is done. The main results of this paper are formed into Theorems 1 to 3. In Theorem 1 we prove that in the subcritical phase of the 2-SAT, the kernels of the contradictory components are typically cubic and that the asymptotic expansion of the probability of satisfiability is linked to the compensation factors of such cubic components. In Theorem 2 we prove that the number of contradictory variables, scaled by $n^{1/3} \mu^{-1}$ follows a mixture of Gamma laws, with the first nontrivial case being Gamma(3) with a scale parameter $\mu^{-1} n^{1/3}$ corresponding to the contradictory component of minimal possible excess 1. Finally, in Theorem 3 we classify the spine literals according to the multiplicities of their
paths to the complementary variables, and prove that a negligible part of the spine can be removed in such a way that the remaining literals form a set of tree-like structures.

Several naturally arising open problems and conjectures are described in Section 5, along with some remarks on how the results presented in this paper could be potentially extended. In the appendix, in Appendix A we present a “proof of concept” full asymptotic expansion of the saddle point lemma which allows in principle to construct the full asymptotic expansion of the satisfiability.

2 Sum-representation of implication digraphs

For the study of the 2-SAT phase transition, it is convenient to consider digraphs with an even number of vertices $2n$ with a special vertex labelling convention. Instead of the labels $\{n + 1, n + 2, \ldots, 2n\}$ we conventionally use the synonyms $\{1, 2, \ldots, n\}$. Under this re-assignment, the nodes with labels $\{1, 2, \ldots, n\}$ correspond to Boolean literals $\{x_1, x_2, \ldots, x_n\}$, and the nodes with labels $\{\overline{1}, \ldots, \overline{n}\}$ correspond to the negations of these literals $\{\overline{x_1}, \ldots, \overline{x_n}\}$. Since the complementary of $x_i$ is $\overline{x_i}$, the same applies to the labels, $i = i$.

**Definition 4** (Conflicts in digraphs). We define a complementary of the edge $x \rightarrow y$ as $y \rightarrow \overline{x}$. We shall say that a pair of complementary edges in a directed graph form a conflict. We say that a digraph is conflict-free if there are no conflicting pairs of edges inside it, i.e. if no two edges are complementary.

Note that in the model that we consider, a random 2-CNF formula contains neither clauses of type $(x \lor x)$ nor of $(x \lor \overline{x})$. This implies that the underlying implication digraph contains neither loops nor multiple edges.

![Figure 2: Example of a sum representation digraph $G$ and its complementary $G^c$ whose edge-union $G + G^c$ gives an implication digraph.](image)

**Definition 5** (Sum-representation). A digraph $G = (V, E)$ with $2n$ conventionally labelled vertices $V = \{1, \cdots, n, \overline{1}, \cdots, \overline{n}\}$ and $|E| = m$ edges is called a sum-representation digraph if it does not contain loops, multiple edges, edges of type $x \rightarrow \overline{x}$ and is conflict-free. The complementary digraph $G^c$ of a sum-representation digraph is obtained by replacing all the edges by their respective complementaries. We say that $G$ is a sum-representation of an implication graph $G + G^c$ which is a digraph obtained by joining the sets of edges of $G$ and $G^c$ (see Figure 2). Every implication digraph with $2m$ edges has $2^m$ sum-representations, since for each of the $m$ clauses of the corresponding formula, 2 choices of complementary edges are available. We denote the set of all sum-representation digraphs with $2n$ vertices and $m$ oriented edges as $D^c(2n, m)$.

Instead of implication digraphs with $2n$ vertices and $2m$ oriented edges we enumerate conflict-free digraphs with $2n$ vertices and $m$ edges. It is worth noticing that $|F(n, m)| = 2^{-m}|D^c(2n, m)|$. Note also that $D^c(2n, m) \subset D(2n, m)$.

An edge rotation is a process of transformation of an edge $x \rightarrow y$ into its complementary edge $\overline{y} \rightarrow \overline{x}$ (see Figure 3). We say that $\pi_1$ is equivalent to $\pi_2$ if it can be obtained by a sequence of edge rotations. This means that $\pi_1$ and $\pi_2$ are both sum-representation of the same implication digraph.
2.1 The spine

Let $y$ be a literal belonging to the spine of some formula $F \in \mathcal{F}(n, m)$, which means that $y \rightarrow \overline{y}$ in the implication digraph corresponding to $F$. Counting such literals $y$ with the multiplicities of the corresponding paths $y \rightarrow \overline{y}$ gives a larger number than just the cardinality of the spine, however, we are going to show that in the subcritical phase, i.e. when $m = n(1 + \mu n^{-1/3}), \mu \rightarrow -\infty$, it gives asymptotically the same result, see Theorem 3 and Corollary 4. The inherent reason behind this is that the majority of the spine components form certain tree-like structures (see Figure 5).

Let us say that a path $y \rightarrow y$ is strictly distinct if all the vertices of the path, except $y$ and $\overline{y}$, are pairwise strictly distinct. If $y \rightarrow \overline{y}$, it is not always possible to find a strictly distinct directed path from $y$ to $\overline{y}$, but it is possible to split it into a few sections, each strictly distinct.

Figure 4: A directed path from $y$ to $\overline{y}$ is split into three strictly distinct sections.

Lemma 1 (Minimal spinal paths). Let $y \rightarrow \overline{y}$ inside an implication digraph. Then there exists a literal $x$ (not necessarily distinct from $y$) such that $y \rightarrow x \rightarrow \overline{x} \rightarrow x$, the paths $y \rightarrow x$ and $x \rightarrow \overline{x}$ are strictly distinct and do not intersect with each other.

As depicted in Figure 4, each arrow represents a directed path with strictly distinct literals. Since $x \rightarrow \overline{x}$ is strictly distinct, it has no intersection with its complementary path. It is convenient to denote by $x \rightarrow_1 \overline{x}$ and $x \rightarrow_2 \overline{x}$ the two complementary versions of the path $x \rightarrow \overline{x}$. We shall call every such quadruple $(y \rightarrow x, x \rightarrow_1 \overline{x}, x \rightarrow_2 \overline{x}, \overline{x} \rightarrow y)$ a minimal spinal path.

Proof. Without loss of generality assume that the path $y \rightarrow \overline{y}$ does not pass through the same vertex twice. Assume that the path $y \rightarrow \overline{y}$ is formed as a sequence of edges $y = a_0 \rightarrow a_1, a_1 \rightarrow a_2, \ldots, a_{k-1} \rightarrow a_k = \overline{y}$. If the path is strictly distinct, we can set $y = x$ and the lemma is proven. Otherwise we choose $x = a_k$ where $k$ is the minimal index such that both $a_k$ and $\overline{a_k}$ belong to the path $y \rightarrow \overline{y}$, and the path $x \rightarrow \overline{x}$ is strictly distinct. It is always possible to choose such an index: if the original path is not strictly distinct, we keep choosing a smaller directed path connecting some literal $x$ and its complementary, until we obtain a path of length at most 2 for which the statement is obviously true since the implication digraph does not contain loops and multiple edges. Note that under such a choice, the path $y \rightarrow \overline{x}$ is also strictly distinct (except for the vertices $x$ and $\overline{x}$), because otherwise it is possible to choose a smaller $k$. The corresponding part $\overline{x} \rightarrow \overline{y}$ can be constructed as a complementary of the path $y \rightarrow x$.

Let $\sigma$ be the random variable denoting the cardinality of the spine in a random formula $F \in \mathcal{F}(n, m)$. 

Figure 3: Example of two equivalent sum-representations obtained by one edge rotation. An edge $1 \rightarrow 3$ is replaced by its complementary $3 \rightarrow 1$. 

Figure 5: The spine components form certain tree-like structures.
Then, since the number of sum-representations of a single implication digraph with \( 2m \) edges is \( 2^m \), the expected value of \( \sigma \) can be expressed as

\[
E\sigma = \frac{|\{(G, x) \mid G \in \mathcal{D}^s(2n, m), x \leadsto \overline{x} \text{ in } G + \overline{G}\}|}{|\mathcal{D}^s(2n, m)|}.
\]

Since the spine literals can be characterised as the literals for which there exists a path connecting a literal and its complementary, the counting of such literals can be handled though inclusion-exclusion. Accordingly,

\[
E\sigma = \frac{|\{(G, p) \mid G \in \mathcal{D}^s(2n, m); p \text{ is a minimal spinal path } y \leadsto \overline{y} \text{ in } G + \overline{G}\}|}{|\mathcal{D}^s(2n, m)|} - \frac{|\{(G, y, p_1, p_2) \mid G \in \mathcal{D}^s(2n, m); p_1, p_2 \text{ are distinct minimal spinal paths } y \leadsto \overline{y} \text{ in } G + \overline{G}\}|}{|\mathcal{D}^s(2n, m)|} + \cdots.
\]

Consider a pattern \( p \in G, G \in \mathcal{D}^s(2n, m) \) consisting of two paths \( y \leadsto x, x \leadsto \overline{x} \), where all the literals on the two paths are pairwise strictly distinct (except for \( x \) and \( \overline{x} \)). Suppose that the pattern \( p \) has total length \( \ell \). Then, among \( 2^m \) equivalent sum-representations of \( G \), \( 2^{m-\ell} \) sum-representations contain the pattern \( p \) unaltered. Among \( 2^\ell \) possible combinations of edge rotations of one of the \( \ell \) edges of \( p \) only, there are exactly 2 possibilities that result in the same type of pattern, the second one obtained by converting the path \( x \leadsto \overline{x} \) into its complementary.

Therefore, the number of sum-representation digraphs with a distinguished pattern \( p = y \leadsto x \leadsto \overline{x} \) of length \( \ell \) counted with multiplicity \( 2^\ell \) enumerates (with multiplicity 2-to-1) implication digraphs with a distinguished minimal spinal path. This allows us to rewrite the first summand of \( E\sigma \) as

\[
\frac{|\{(G, y, p) \mid G \in \mathcal{D}^s(2n, m); p \text{ is a minimal spinal path } y \leadsto \overline{y} \text{ in } G + \overline{G}\}|}{|\mathcal{D}^s(2n, m)|} = \frac{1}{2} \sum_{\ell=1}^{\ell} 2^\ell \frac{|\{(G, y, x, p) \mid G \in \mathcal{D}^s(2n, m); p \text{ is a distinguished pattern } y \leadsto x \leadsto \overline{x} \text{ of length } \ell \text{ in } G\}|}{|\mathcal{D}^s(2n, m)|}.
\]

All the subsequent summands of \( E\sigma \) can be rewritten in the same manner as well, though it requires considering several cases for the mutual configuration of \( p_1 \) and \( p_2 \) and using possibly different factors instead of \( \frac{1}{2} \) for the multiplicities.

### 2.2 Classifying the contradictory components according to their excesses

Recall that contradictory circuits and variables are defined in Definition 3. If a variable \( x \) is contradictory, it can belong to multiple contradictory circuits at the same time. When the complexity of a contradictory component increases, the number of circuits simultaneously containing a given variable can grow exponentially on its excess. However, we show that the excess grows very slowly in the subcritical phase of the 2-SAT transition, so this technique allows to obtain some structural properties before the number of multiplicities blows up.

Similarly to the excess of a complex component in a simple graph which equals to the number of its edges minus the number of its vertices, we introduce the excess of a contradictory graph. (See Figures 6 to 9 for different contradictory components of excess 1 to 2.)
**Definition 6** (Contradictory component and its excess). We call a digraph $D$ with $2n$ conventionally labelled vertices a *contradictory digraph* or a *contradictory component* if for its every edge $e \in D$ its complementary edge $\overline{e}$ is also in $D$, and every vertex $x \in D$ implies its complementary: $x \Rightarrow \overline{x}$. The *excess* of a contradictory component is defined to be equal to the difference between the number of its edges and vertices divided by 2. The excess of an empty graph is defined to be zero.

![Diagram of a contradictory digraph](image1)

**Figure 6**: First possible contradictory component of excess 1.

![Diagram of a contradictory digraph](image2)

**Figure 7**: Second possible contradictory component of excess 1.

![Diagram of a contradictory digraph](image3)

**Figure 8**: A minimal contradictory component of excess 1.

![Diagram of a contradictory digraph](image4)

**Figure 9**: Contradictory strongly component of excess 2 which is not minimal.

In contrast with the phase transition of simple graphs, where the whole structure of the graph is known with high probability, no such information is available for the phase transition of 2-SAT yet. On a positive side, Kim has obtained the number of variables and clauses in a core of a random formula. In this work, we focus on the contradictory component only.

In order to access the probability of satisfiability, we use the *inclusion-exclusion method*. We consider *minimal* contradictory components which are contradictory implication digraphs that do not have any proper contradictory subgraphs. Two examples of minimal contradictory components of excess 1 and 2 are given in Figures 7 and 8, the arrows play the role of strictly distinct paths. Figure 9 represents a contradictory strongly component of excess 2 which is not minimal.

**Lemma 2.** Let $\xi$ denote the number of minimal contradictory components in a random implication digraph corresponding to a random formula $F \in \mathcal{F}(n, m)$, counted with multiplicities and with possible overlappings. Then, the probability of satisfiability of a random formula $F \in \mathcal{F}(n, m)$ can be expressed using the principle of inclusion-exclusion:

$$
\mathbb{P}(F \in \mathcal{F}(n, m) \text{ is SAT}) = 1 - \mathbb{E}_{n, m, \xi} + \frac{1}{2!}\mathbb{E}_{n, m, \xi}^2(\xi - 1) - \frac{1}{3!}\mathbb{E}_{n, m, \xi}^3(\xi - 1)(\xi - 2) + \cdots.
$$

(2.1)

**Proof.** A formula is satisfiable if and only if the contradictory component is empty, i.e. the number of minimal contradictory components is equal to zero. Let $N$ denote the total possible number of subgraphs in a digraph with $n$ vertices, and let $E_i$ denote the event that $i$th subgraph forms a minimal contradictory component. Using de Morgan’s rule and the inclusion-exclusion principle, we obtain

$$
\mathbb{P}(\xi = 0) = \mathbb{P}(\overline{E_1} \land \cdots \land \overline{E_N}) = 1 - \mathbb{P}(E_1 \lor \cdots \lor E_N) = 1 - \sum_{i=1}^{N} \mathbb{P}(E_i) + \sum_{i<j} \mathbb{P}(E_i \land E_j) - \cdots.
$$

Finally, we note that

$$
\sum_{i_1 < \cdots < i_k} \mathbb{P}(E_{i_1} \land \cdots \land E_{i_k}) = \binom{N}{k} \mathbb{P}(E_1 \land \cdots \land E_k) = \frac{1}{k!}\mathbb{E}_{n, m, \xi}(\xi - 1) \cdots (\xi - k + 1),
$$

which finishes the proof. \hfill \qed
The term $\mathbb{E}\xi(\xi - 1)$ denotes the expected number of pairs of minimal contradictory components, and requires going through different possible cases of their mutual configuration. Further terms will provide even more complicated combinatorial structures, but on the asymptotic level, they will all appear to be negligible in the subcritical phase of the 2-SAT phase transition.

**Example 2.** Figure 9 representing a contradictory component of excess 2 can be also considered as a pair of contradictory components each of excess 1: the first one being $x \sim w \sim \overline{x} \sim \overline{w} \sim y \sim \overline{y} \sim z \sim x$ with a double sequence $y \sim \overline{y}$ and a mirror path $x \sim \overline{w} \sim \overline{x}$, and the second one being described as $z \sim w \sim \overline{x} \sim \overline{w} \sim y \sim \overline{y} \sim z$ with a double path $y \sim \overline{y}$ and a complementary sequence $z \sim x \sim \overline{w} \sim \overline{x}$. In such a way, the first component of excess 1 is obtained by dropping the pair $x \sim w$ and $\overline{w} \sim \overline{x}$, and the second – by dropping $x \sim w$ and $\overline{w} \sim \overline{x}$.

### 2.3 The case of the simplest minimal contradictory component

It is convenient to represent $\xi$ from Lemma 2 as the sum $\xi = \sum_{r \geq 1} \xi_r$ where $\xi_r$ is the number of distinguished minimal contradictory components of excess $r$ in a random implication digraph.

We focus on the case $r = 1$ first. The expected value of $\xi_1$ is equal to the number of implication digraphs with a distinguished contradictory component of excess 1 (e.g. Figure 7) divided by the total number of implication digraphs. In order to count the implication digraphs with distinguished contradictory components, we are going to count their respective sum-representations and then divide by $2^m$ which is the number of sum-representations of a single implication digraph with $m$ edges. Both numerator and denominator of the total fraction are then divided by the same factor $2^m$ which cancels out. In other words, $\mathbb{E}\xi_1$ can be expressed as

$$
\mathbb{E}\xi_r = \frac{\left\{ (G,p) \mid G \text{ is a sum-representation digraph with } 2n \text{ vertices and } m \text{ edges} \right\}}{\left\{ G \mid G \text{ is a sum-representation digraph with } 2n \text{ vertices and } m \text{ edges} \right\}}.
$$

We further note that distinguishing a contradictory pattern in $G + \overline{G}$ can be replaced with distinguishing a different contradictory pattern in a sum-representation digraph $G$ if a proper multiplicity factor is used.

There are only two possible multigraphs which can result in cancelling of a contradictory digraph of excess 1, depicted, respectively, in Figures 6 and 7. Accordingly, we consider two different cases.

**First case.** Consider a sum-representation digraph with a distinguished pattern of type depicted in Figure 10. Each arrow there represents a sequence of strictly distinct literals, and it is also required that all the literals on both sequences (except $x$ and $\overline{x}$) are pairwise strictly distinct, and that the two distinguished literals marked $x$ and $\overline{x}$ are complementary. Suppose that the pattern has $\ell$ oriented edges. Then, among $2^\ell$ graphs obtained by edge rotations of the pattern, there are exactly 4 which belong to the same pattern: it is possible to take a complementary of the path $x \sim \overline{x}$ which gives 2 possibilities, and the complementary of the path $\overline{x} \sim x$.

**Second case.** Consider a sum-representation digraph with a distinguished pattern of type depicted in Figure 11. Again, each arrow corresponds to a sequence, and all the literals on the three sequences $x \sim \overline{x}$, $\overline{x} \sim y$, $y \sim \overline{y}$ are pairwise strictly distinct, except for $x$ and $\overline{x}$, and $y$ and $\overline{y}$, which are required to be complementary. The literals $x$ and $y$ are also required to be strictly distinct. Again, if the total length of the pattern is $\ell$, there are 8 sum-representations among $2^\ell$ equivalent ones, such that these sum-representations form the same pattern. Indeed, 4 options are given by replacing either of the two paths $x \sim \overline{x}$ and $y \sim \overline{y}$ by its complementary. If the path $\overline{x} \sim y$ is replaced by its complementary, then we obtain a different graph constructed of the three sequences $x \sim \overline{x}$, $\overline{x} \sim x$, $y \sim \overline{y}$ which belongs to the same pattern if we swap the variables $x$ and $y$. No other combinations of edge rotations give the same pattern because it is required that all the literals are pairwise strictly distinct.

Suppose that a pattern (either from Figure 10 or from Figure 11) has $\ell$ edges and is a distinguished subgraph of a sum-representation digraph containing $m$ edges in total. Then, among $2^m$ equivalent, $2^{m-\ell}$ contain the pattern unaltered. Therefore, the number of sum-representation digraphs with a distinguished contradictory pattern of length $\ell$ from Figures 10 and 11 counted with multiplicity $2^\ell$ enumerates (with respective
Figure 10: First sum-representation contradictory pattern corresponding to Figure 6.

Figure 11: Second sum-representation contradictory pattern corresponding to Figure 7.

multiplicities 1-to-4 and 1-to-8) implication digraphs with a distinguished contradictory digraph Figures 6 and 7 of excess 1.

Combining the two cases, we obtain a new expression for $E_{ξ_1}$:

$$E_{ξ_1} = \frac{1}{4} \sum_{ℓ=0}^{∞} 2^ℓ \left\{ (G, p_ℓ) \mid G \text{ is a sum-representation digraph with } 2n \text{ vertices and } m \text{ edges} \right\} + \frac{1}{8} \sum_{ℓ=0}^{∞} 2^ℓ \left\{ (G, p_ℓ) \mid G \text{ is a sum-representation digraph with } 2n \text{ vertices and } m \text{ edges} \right\}$$

As we shall see in Section 4, the contribution of the first summand is negligible. An intuitive way to see why this happens is the following: the shape from Figure 6 can be obtained by contracting a path $y \Rightarrow x$ in Figure 7. Therefore, the first case is a degenerate case when the length of the corresponding path is equal to zero and the literal $x$ coincides with the literal $y$. In general, only the structures with cubic kernels have dominant contribution.

A similar decomposition can be obtained for $E_{ξ_r}$ for any finite $r$. Instead the two presented cases, the sum should be taken over all the possible minimal contradictory components of given excess $r$, divided by a corresponding multiplicity factor, which plays the analog of a compensation factor for simple graphs. A detailed explanation of the nature of such compensation factors is given in Section 3.3.

3 Generating functions and saddle point techniques

The symbolic method and analytic combinatorics [FS09] are powerful methods which allow to (i) construct combinatorial objects (like graphs, or 2-SAT formulae, in our case) by combining them in various different ways using a set of allowed operations; (ii) translate this constructions into the language of generating functions; (iii) analyse the asymptotic number of objects using various available integration tools. This section is devoted to describing the symbolic construction of graphs and directed graphs needed for our purpose, and the asymptotic tool needed for our analysis.

3.1 Symbolic method for graphs

Recall that the size of a graph (or a directed graph) is defined to be equal to the number of its vertices. In this paper, labelled graphs are considered: every vertex has a distinct label from the set $\{1, 2, \ldots, n\}$ where $n$ is the total number of vertices of the graph.

Given a family of graphs (or directed graphs) $A$, we construct its corresponding exponential generating function (EGF) which is

$$A(z) := \sum_{n≥0} a_n \frac{z^n}{n!},$$

where $a_n$ is equal to the number of graphs or digraphs of size $n$ from $A$. The operator $[z^n]$ of taking $n$th coefficient is then defined as $[z^n]A(z) = a_n/n!$. All the indefinite integrals $\int F(z)dz$ in this paper should be interpreted as $\int_0^\tau F(\tau)d\tau$. 

Given two families $A$ and $B$, the sum of their respective generating functions $A(z)$ and $B(z)$ represents the union of the two families $A$ and $B$. The family $C := A \times B$ is defined as the family of labelled graphs (or digraphs) whose vertices are partitioned into two sets such that there are no edges between the parts, the underlying graph of the first part is from $A$, and the graph from the second part is from $B$. We can also say that $A \times B$ is the labelled family of ordered tuples of graphs from $A$ and $B$. If the respective EGFs of these families are $A(z)$ and $B(z)$ then the EFG of $C$ is $A(z) \cdot B(z)$.

Consequently, if $A(z)$ is an EGF for a graph family $A$ not containing an empty graph, then $A^k(z)$ is the EGF for sequences of length $k$ of graphs from $A$. Summing over all $k$, we obtain the EGF for all possible sequences (possibly empty) of graphs from $A$, which is $e^{A(z)}$. If the family $A$ does not contain an empty graph, then the EGF for non-ordered sequences of length $k$ (i.e. sets) of graphs from $A$ is $A(z)^k/k!$. Summing over all $k$, we obtain the EGF of all unordered sequences of graphs from $A$, which is $e^{A(z)}$. Finally, a oriented cyclic composition of graphs from $A$ has EGF $\sum_{k \geq 1} A^k(z)/k = \log \frac{1}{1-A(z)}$. If a cycle is not oriented, then the corresponding EGF is $\frac{1}{2} \log \frac{1}{1-A(z)}$.

If all the graphs in a family $A$ are connected, then $e^{A(z)}$ enumerates labelled graphs whose connected components are from $A$. The product of exponential generating functions for labelled graphs can be given further interpretations: if two graph families $A$ and $B$ do not intersect and the graphs from these families are connected, then the product of their EGFs enumerates graphs with two connected components, one from $A$ and the second from $B$. In the same manner, one can multiply generating functions of not necessarily connected graph families provided that the underlying connected components corresponding to the two families, are always distinct. Then, the product can be interpreted as a new family of graphs whose vertices can be partitioned uniquely into two sets, and the graphs constructed on the respective sets, belong to the first and the second family, respectively.

It is well known that the EGF $T(z)$ for rooted trees, also known as Cayley trees, satisfies

$$T(z) = z e^{T(z)} = \sum_{n \geq 0} n^{n-1} \frac{z^n}{n!}.$$ 

Unicyclic graphs are defined as connected graphs whose number of vertices is equal to their number of edges. Every such graph has exactly one undirected cycle inside it and can be represented as a sequence of trees arranged in a cycle of length at least 3, which results in the following EGF $V(z)$:

$$V(z) = \frac{1}{2} \sum_{k \geq 3} \frac{T(z)^k}{2k} = \frac{1}{2} \left[ \log \frac{1}{1-T(z)} - T(z) - \frac{T^2(z)}{2} \right].$$
The EGF for unrooted trees has the form $U(z) = T(z) - \frac{T(z)^2}{2}$. An elegant proof of this fact is given in [FKP89]. We present a sketch of the proof for completeness. The label of the root of a rooted tree can be equal to 1 or greater than 1. The generating function of rooted trees whose root has label 1 is $U(z)$ because an unrooted tree can be canonically rooted at vertex with label 1. Otherwise, a tree whose root label is greater than 1 can be represented as a pair consisting of the subtree whose parent is the root and which contains the vertex with label 1, and the remaining tree which is formed by removing the aforementioned subtree from the initial tree. The generating function of such (unordered) tuples is $T^2(z)/2$. Adding up the two cases, we obtain an identity $T(z) = U(z) + \frac{T^2(z)}{2}$.

Similarly to trees and unicycles, more complex structures can be defined. An excess of a connected graph is equal to the number of its edges minus the number of its vertices. A connected graph with minimal possible excess is a tree, and its excess is equal to $-1$. Next, a unicycle is a connected graph of excess 0. A connected graph of excess 1 is called a bicycle.

Suppose that a connected graph $G$ has excess $r$. The pruning procedure is described as repeated removal of vertices of degree 1 until no vertices of degree 1 remain. The resulting graph is then called the 2-core or just the core of $G$. Consequently, each vertex of degree 2 of $G$ can be removed, while connecting its former neighbours by an edge; this procedure is called cancelling. The resulting graph obtained by pruning and cancelling is called the 3-core, or the kernel of $G$. After cancelling, a simple graph may gain some multiple edges and loops, i.e. become a multigraph. Note that pruning and cancelling do not change the excess of $G$. It is known (see e.g. [JKLP93]) that after pruning and cancelling, the resulting multigraph belongs to a finite set of multigraphs with given excess $r$, see also Section 3.3.

![Figure 15: Pruning and cancelling.](image)

It turns out that in the subcritical and in the critical phases, the dominant contribution comes only from the cubic (3-regular) kernels. If a kernel is not cubic, it can be obtained as a “limiting case” from some cubic kernel by contracting some of its edges. This corresponds to the case when the corresponding paths in the initial graph have length zero. Heuristically, since the average length of a path on the cubic core is of order $\Theta(n^{1/3})$ in the critical phase (see e.g. [JLR11]), the probability that a path has zero vertices, i.e. that the kernel is non-cubic, is of order $O(n^{-1/3})$.

Turning to the case of digraphs and implication digraphs, pruning is not defined for a contradictory component, because it doesn’t contain any vertices of degree 1. Cancellation is then defined in the following way: if a vertex has in-degree 1 and out-degree 1, it is removed and a directed edge between the in-neighbour and the out-neighbour is added. It is easy to see that a directed multigraph obtained after a cancellation procedure with given excess belongs to a finite sets of contradictory reduced multigraphs of given excess (see also Problem 1). If in an implication multidigraph every vertex has the sum of in- and out-degrees equal 3, it is also called cubic.

### 3.2 Weakly connected directed graphs

A weakly connected digraph is a directed graph whose underlying non-oriented projection is connected. We construct the EGFs of the analogs of the trees and unicycles in the world of directed graphs.
A rooted tree of size $n$ has $n - 1$ edges, and for each of the edges there are two orientation choices. For unrooted trees, we can take the vertex with label 1 as a canonical root, so that all of the $2^{n-1}$ edge orientations can be also distinguished and result in distinct oriented unrooted trees. Therefore, the EGFs for, respectively, oriented rooted and unrooted trees (i.e. directed weakly connected graphs whose non-oriented projections are, respectively, rooted and unrooted trees) are, respectively,

$$T\rightarrow(z) = \frac{1}{2}T(2z), \quad U\rightarrow(z) = \frac{1}{2}U(2z) = \frac{1}{2}T(2z) - \frac{1}{4}T^2(2z).$$

(3.1)

The EGF for simple digraphs whose underlying non-oriented projections are cycles of length at least 3, is obtained by substitution $z \mapsto 2z$ into that of cyclic non-oriented graphs, and equals

$$\frac{1}{2} \log \frac{1}{1 - 2z} - (2z) - \frac{(2z)^2}{2}.$$  

In simple digraphs, it is allowed to have a circuit of length 2 on condition that it has the form $x \rightarrow \pi \rightarrow x$, i.e. connects two vertices in both directions. Adding this case corresponds to the summand $z^2/2$.

By substitution, it follows that the EGF for unicyclic directed graphs (directed graphs whose non-oriented projections are unicyclic graphs) is

$$V\rightarrow(z) = \frac{1}{2} \left[ \log \frac{1}{1 - 2T\rightarrow(z)} - (2T\rightarrow(z)) - \frac{(2T\rightarrow(z))^2}{2} \right] + T\rightarrow(z)^2.$$  

(3.2)

The last summand is taken out intentionally. Essentially, $V\rightarrow(z) = V(2z) + T^2(2z)/8$, where $V(z)$ is the previously obtained EGF for unicyclic simple graphs. EGFs for directed graphs with higher excess can be constructed in the same way.

**Remark 2.** Apart from weakly connected digraphs, which come as a more or less direct generalisation of the corresponding non-oriented versions, it is also possible to specify digraphs with respect to their strongly connected components, see [dPD19] and references therein. This may lead to further development of more precise properties of the 2-SAT phase transition, or that of critical random digraphs.

### 3.3 Compensation factors

The compensation factor of a multigraph $M$ is a certain coefficient depending on $M$ which is required to obtain the EGF of the family of graphs reducing to $M$ under pruning and cancelling. While for the purposes of the current paper we take a relatively low-level classical approach, there also exists an elegant unifying framework [dP16] for dealing with compensation factors by introducing bivariate generating functions which are exponential with respect to two variables.

Consider the three possible kernels of excess 1 in Figure 16. Each of these kernels a certain symmetry group which acts on the set of its half-edges and vertices. Since the objects are labelled, the cardinality of the automorphism group acting on its vertices equals $n!$ divided by the number of distinct labelled graphs. The remaining compensation factor can be determined solely on the base of the incidence matrix.

![Figure 16: Kernels of complex components of excess 1 and their respective compensation factors.](image)

**Definition 7.** Consider a multigraph $M$ with $n$ labelled vertices and with $m_{xy}$ edges between vertices with labels $x$ and $y$. In particular, $m_{xx}$ denotes the number of loops of a vertex $x$, and $m_{xy} = m_{yx}$.

A *compensation factor* $\kappa(M)$ is defined as follows:

$$\frac{1}{\kappa(M)} := \prod_{x=1}^{n} 2^{m_{xx}} \prod_{y=x}^{n} m_{xy}!.$$  

(3.3)
The number of ways to construct a single multigraph $M$ with $n$ vertices and $m$ edges by gluing the half-edges of the labelled vertices can be viewed as $2^m n! \pi(M)$. The EGF of multigraphs $G$ reducing under pruning and cancelling to a given multigraph $M$ is then written as

$$\frac{\pi(M)}{n!} \cdot \frac{T(z)^n}{(1 - T(z))^{m}},$$

which can be obtained by substitution of trees into the corresponding sequences and vertices. For a rigorous proof of this expression, see [JKLP93].

For contradictory components in implication digraphs, we define the compensation factors in a very similar way.

**Definition 8** (Compensation factor of sum-representations and implication digraphs). Consider a sum-representation digraph $D \in D^c(2n, m)$ and a contradictory implication digraph $M$ with $2n$ conventionally labelled vertices and $2m$ directed edges. Suppose that for each literals $x$ and $y$ there are $d_{xy}$ oriented edges in $D$ and $m_{xy}$ oriented edges in $M$ between vertices with labels $x$ and $y$. The compensation factors of $D$ and $M$ are then defined as

$$\frac{1}{\pi(D)} := \prod_{x=1}^{n} \prod_{y=1}^{n} d_{xy}!,$$

$$\frac{1}{\pi(M)} := \prod_{x=1}^{n} \prod_{y=1}^{n} m_{xy}!!,$$

(3.4)

where for even $N$, the double factorial is defined as $N!! = 2 \cdot 4 \cdot \ldots \cdot N$.

For the case of simple graphs, the sum over all possible labelled kernels $M$ with $n$ vertices and given excess $r$, $\sum_M \frac{\pi(M)}{m!}$ expresses the coefficient at $|\mu|^{-3r}$ in the probability that a random graph has a complex component of such excess (c.f. the asymptotic probability $5/24 |\mu|^{-3}$ of having a bicyclic complex component, where $5/24$ equals the sum of the compensation factors $1/4$ and $1/6$ divided by $2!$, see also Remark 5).

As we show in Theorem 1, the compensation factor for contradictory components plays exactly the same role. The compensation factor for contradictory components has the following additional interpretation.

**Lemma 3.** Consider a contradictory component $C$ with $n$ Boolean variables and one of its sum-representations $\pi(C)$. We shall say that a sum-representation digraph $\pi_1$ is isomorphic to a digraph $\pi_2$ if $\pi_1$ can be obtained from $\pi_2$ by a permutation of Boolean variables. Then, the number of sum-representations, both equivalent and isomorphic to $\pi(C)$ is equal to $n! \pi(\pi)/\pi(C)$.

**Proof.** If $C_1, \ldots, C_K$ are $K$ possible isomorphic implication digraphs obtained by label permutations, then each of the digraphs has the same number of $2^m$ sum-representations, and therefore, each isomorphic sum-representation is counted with multiplicity $K$.

Let us compute the number $K$ of possible isomorphic digraphs obtained by label permutations. All the edges of $C$ come in pairs, therefore, within each pair there is a cyclic group of order 2, and there are $(m_{xy}/2)!$ permutations between the $(m_{xy}/2)$ pairs of oriented edges between the vertices $x$ and $y$. Multiplying the orders of these groups, we obtain $(m_{xy}/2)! \cdot 2^{m_{xy}/2} = (m_{xy})!!$. Each loop is directed, so there is no additional factors corresponding to the loops. The number $K$ then equal to $n! \pi(\pi)/\pi(C)$ since this is equal to the cardinality of the automorphism group.

**Example 3.** As an illustration of this concept, let us compute the compensation factor of the contradictory component of excess 1 from Figure 7. There are $n = 2$ Boolean variables and two multiple edges, between, respectively, $x$ and $\overline{x}$, and $y$ and $\overline{y}$. Therefore, the compensation factor is $1/2!!^2 = 1/4$. The number of isomorphic equivalent sum-representations is therefore $2! \cdot 4 = 8$, which explains the factor $1/8$ in the second summand of the expression for $\xi_1$ in Section 2.3. The factor $1/4$ corresponding to Figure 6 is computed as $1! \cdot 2!!^2$ because there is one Boolean variable and two double edges.

It is natural to expect that the concept of compensation factor of a contradictory component will prove helpful not only for the subcritical phase of the phase transition, but will allow to give a complete description of the transition curve. See also Section 5 for discussions and open questions.
3.4 Incorporating relations between labels of vertices

We continue to use the labelling convention for digraphs and implication digraphs introduced in Section 2, so that for digraphs having $2n$ vertices, the labels of the vertices are partitioned into the set of positive literals $\{1, \ldots, n\}$ and negative ones $\{\bar{1}, \ldots, \bar{n}\}$.

**Definition 9** (Literal linking construction). Suppose that in a given family $\mathcal{A}$ of digraphs, each graph $G \in \mathcal{A}$ contains two distinguished empty nodes, i.e. vertices for which we do not assign any labels and which do not contribute to the total size of the graph. We define the family $\Lambda[\mathcal{A}]$ as the family obtained from $\mathcal{A}$ by replacing the two empty nodes by complementary literals.

**Lemma 4.** If $A(z)$ is the EGF of a given family $\mathcal{A}$ of digraphs with even number of nodes, two distinguished empty nodes (of size 0), then the EGF of $\Lambda[\mathcal{A}]$ is given by

$$
\Lambda[A](z) := z \int_0^z A(t) dt. 
$$

*Proof. The number of graphs of size $(2n-2)$ from $\mathcal{A}$ is $(2n-2)!![z^{2n-2}]A(z)$. Then, the number of ways to insert the complementary labels into two distinguished positions is $2n$. The total number of graphs of size $2n$ from the family $\Lambda[\mathcal{A}]$ is therefore $2n \cdot (2n-2)!![z^{2n}]z^2 A(z) = \frac{1}{2n-1}(2n)!![z^{2n-2}]A(z)$. We conclude by noting that $(2n)$-th coefficient of $z \int_0^z A(t) dt$ equals to $\frac{1}{2n-1}[z^{2n-2}]A(z)$ and multiplying by $(2n)!$ gives the total number of graphs of size $2n$ from $\Lambda[\mathcal{A}]$. \qed

**Remark 3.** It is possible to insert more than one pair of complementary literals, in this case more than one application of $\Lambda$ is required. Naturally, in this case, it is required that the number of unlabelled empty slots should be equal to two times the number of inserted complementary literals.

3.5 Saddle point lemma for random graphs

Among many tools introduced in [JKLP93], the central one was the saddle point lemma allowing to study the fine structure of a random graph near the point of its phase transition using the generating functions. We give it in a slightly reformulated form without a proof.

**Lemma 5** ([JKLP93, Lemma 3]). Suppose that $m = \frac{n}{2}(1 + \mu n^{-1/3})$, and $H(t)$ is a function analytic at $t = 1$. Then, as $\mu \to -\infty$, and $|\mu| \leq n^{1/12},$

$$
\frac{n!}{|G(n, m)|}[z^n]U(z)^{n-m} \frac{\nu^V(z)}{(n-m)!} \frac{H(T(z))}{(1-T(z))^y} = H(1) \left( \frac{n^{1/3}}{|\mu|} \right)^y (1 + O(|\mu|^{-3})). \tag{3.6}
$$

In order to apply the above lemma, it is needed to construct a graph from trees, unicycles and a finite number of complex components as from “building bricks”, and then extract the asymptotic proportion of such graphs.

**Remark 4.** For the subcritical phase $\mu \to -\infty$, the condition $|\mu| \leq n^{1/12}$ can be replaced by $|\mu| \ll n^{1/3}$, as shown in [HR11] by a refined technical analysis of the asymptotics. We do not apply this refinement in the current paper as it would require a certain amount of additional technical details.

In a similar context, the enumeration of connected graphs has been pushed even further in [FSS04] using purely analytic methods. The resulting expansions are very tightly related with the Airy function and complex contour integration in the context of the previous result [JKLP93]. The properties of the Airy function and the phenomenon of coalescing saddles in combinatorics are out of the scope of the current paper, and the interested reader can find additional details e.g. in [Ald97, BFSS01].

**Remark 5.** Let us present an exemplary application of Lemma 5 analogous to presented in [JKLP93]. If a simple graph with $n$ vertices and $m$ edges contains only trees and unicycles, then the number of trees is
equal to \( m - n \), because each tree has one edge less than its number of vertices. The EGF for such graphs is then \( e^{V(z)}U(z)^{n-m}/(n-m)! \). The factor \( e^{V(z)} \) can be then rewritten as

\[
e^{V(z)} = \frac{1}{\sqrt{1 - T(z)}} e^{-T(z)/2 - T^2(z)/4}.
\]

When \( m = \frac{5}{2}(1 + \mu n^{-1/3}) \), \( \mu \to -\infty \) with \( n \), and \( |\mu| \leq n^{1/12} \), the probability that a graph consists only of trees and unicycles, is, according to Lemma 10:

\[
\frac{n!}{|G(n,m)|} [z^n] \frac{U(z)^{n-m}}{(n-m)!} e^{V(z)} = 1 - \frac{5 + o(1)}{24|\mu|^3}.
\]

The interpretation of the factor \( 5/24 \) is at least twofold. Formally, it appears as the second expansion term in the saddle point lemma with \( 5/24 = \frac{3y^2 + 3y - 1}{6} \) \( |y=1/2| \). At the same time, it denotes the sum of the compensation factors of the two possible cubic bicyclic multigraphs (see Figure 16), also expressed as the sum of the inverse cardinalities of their automorphism groups. Furthermore, the probability of having only one complex component which is bicyclic, is \( \frac{5}{24}|\mu|^{-3} + O(|\mu|^{-6}) \), and the probability of having a complex component of excess \( r \) is \( e_r|\mu|^{-3r} \), where \( e_r = \frac{r}{24r^3(3r)!} \) correspond to the sum of compensation factors of cubic multigraphs of excess \( r \), see e.g. [Bol85, Chapter 2]. For complete asymptotic expansions in powers of \( |\mu|^{-3} \) we refer to Appendix A.

We give a variant of Lemma 5 for the case of directed graphs.

**Lemma 6.** As \( \mu \to -\infty \), and \( |\mu| \leq n^{1/12} \),

\[
\frac{(2n)!}{|D(2n,m)|} [z^{2n}] U_\to(z)^{2n-m} e^{V_\to(z)} H(T_\to(z)) \left( \frac{H(T_\to(z))}{(1 - 2T_\to(z))^{y}} \right) = H(1/2) \left( \frac{n^{1/3}}{|\mu|} \right)^y (1 + O(|\mu|^{-3}))
\]

(3.7)

\( H(t) \) is a function analytic at \( t = 1/2 \).

**Proof.** As \( \frac{m}{n} \to 1 \) with \( n \) going to infinity, the number of simple digraphs with \( 2n \) vertices and \( m \) edges can be asymptotically related to the number of simple graphs with the same number of vertices and edges using the Stirling’s formula

\[
\frac{|D(2n,m)|}{2^m |G(2n,m)|} = \frac{\binom{2n}{m}}{2^m \binom{2n}{m}} \to e^{1/8}, \quad \text{as } n \to \infty.
\]

Replacing \( |D(2n,m)| \) by its asymptotic equivalent, \( U_\to(z) \) and \( T_\to(z) \) by \( \frac{1}{2} U(2z) \) and \( \frac{1}{2} T(2z) \), and also \( V_\to(z) \) by \( V(2z) + T(2z)^2/8 \), we obtain the expression

\[
e^{-1/8} \frac{(2n)!^2}{|G(2n,m)|} [z^{2n}] e^{-2n+z+m} U(2z)^{2n-m} \frac{e^{V(2z) + T(2z)^2/8} H(T(2z)/2)}{(1 - T(2z))^y}.
\]

After substituting \( 2z = x \), we get an additional \( 2n^2 \) from the operator of coefficient extraction \([z^{2n}]\), and therefore, all the powers of two cancel out. We proceed by applying Lemma 6 and replacing each \( T(x) \) by 1. Since the lemma is designed for a different regime, namely \( m = \frac{5}{2}(1 + \mu n^{-1/3}) \), we can adapt by replacing \( n \to 2n \), \( \mu \to 2^{1/3} \mu \). Finally, the factor \( e^{-1/8} \) also cancels out, because of the presence of the multiple \( e^{T(x)^2/8} \), and the ratio of \((2n)^{1/3}\) to \( 2^{1/3}|\mu| \) becomes again \( n^{1/3}/|\mu| \).

\[
4 \quad \text{Extracting the asymptotics}
\]

**4.1 Contradictory components in simple digraphs**

Before constructing the sum-representation digraphs with marked directed subgraphs, we start by constructing the subcritical simple digraphs first. We are going to mark the same contradictory patterns, but without
an additional assumption that the paths are strictly distinct and that the digraph is conflict free, and without excluding the edges \(x \rightarrow x\).

As we will see later in Section 4.2, this set has a very similar structure to the set of sum-representation digraphs \(\mathcal{D}^0(2n, m)\): the number of edge conflicts in a random digraph \(D \in \mathcal{D}(2n, m)\) asymptotically follows a Poisson distribution with parameter \(1/8\). Excluding edge conflicts is equivalent to conditioning on the value of this Poisson variable being zero.

We start by demonstrating a result directly related to the probability of satisfiability in the subcritical phase, which is shown by Kim in [Kim08] to be asymptotically

\[
\mathbb{P}(F \in \mathcal{F}(n, m) \text{ is SAT}) = 1 - \frac{1 + o(1)}{16|\mu|^3}
\]

for \(m = n(1 + \mu n^{-1/3})\), as \(\mu \to -\infty\) with \(n\), and \(|\mu| = o(n^{1/3})\). In this section we give a new explanation of the factor \(1/16\) and show how to extend this result to obtain a complete asymptotic distribution in powers of \(|\mu|^{-3}\).

The contradictory pattern that we are going to identify first, takes a form of three sequences \(x \Rightarrow x \Rightarrow y \Rightarrow y\) (Figure 11). We start by considering the case when this pattern is a part of a weakly connected component which is a tree (Figure 17).

**Lemma 7.** Consider simple digraphs \(G \in \mathcal{D}(2n, m)\) with a distinguished contradictory pattern \(x \Rightarrow x \Rightarrow y \Rightarrow y\) such that the weakly connected component containing this pattern is a tree, and there are no weakly connected components whose non-oriented projections have positive excess. The proportion of such digraphs among \(\mathcal{D}(2n, m)\) where each such digraph is taken with weight \(2^\ell\), where \(\ell\) is the length of the pattern, is

\[
\frac{(2n)!}{|\mathcal{D}(2n, m)|} \int \int \frac{U_{\rightarrow}(z) z^{2n-m-1}}{(2n-m-1)!} e^{V_{\rightarrow}(z)} \frac{8T_{\rightarrow}(z)^4 z^{-4}}{1 - 2T_{\rightarrow}(z))}\right] dz^2 = \frac{1}{2|\mu|^3} + O(|\mu|^{-6}).
\]

**Proof.** The weakly connected component containing a distinguished pattern \(x \Rightarrow x \Rightarrow y \Rightarrow y\) can be represented as three sequences of trees, each sequence of length at least one, and an additional tree (Figure 17). The generating function for one such sequence, equipped with a weight \(2^\ell\), where \(\ell\) is the length of this sequence, is

\[
\sum_{\ell \geq 1} T_{\rightarrow}(z)^\ell 2^\ell = \frac{2T_{\rightarrow}(z)}{1 - 2T_{\rightarrow}(z)}.
\]

Taking three such sequences and adding a triple multiple 2 corresponding to linking three directed edges between the sequences, and by adding the last tree, we get an EGF \(\frac{8T_{\rightarrow}(z)^4}{1 - 2T_{\rightarrow}(z))}\). The next step is to divide by \(z^4\) which corresponds to erasing the labels of the four distinguished nodes (which will later become the labels \(x, \overline{x}, y, \text{and } \overline{y}\)).

![Figure 17: The case when the weakly connected component of the contradictory pattern is a tree.](image)

The digraphs required in the current lemma are obtained as product of the family of directed trees and unicyles and the constructed contradictory pattern, through literal linking construction (see Definition 9). By applying the complementary label insertion operation \(\Lambda[A](z) = z \int A(z) dz\) from Lemma 4 twice, we obtain the desired EGF. The (weighted) number of digraphs is obtained by taking the \((2n)\)th coefficient of the EGF.
Since the number of trees in the forest of directed trees (without the distinguished one) is \((2n - m - 1)\), the EGF of digraphs containing this forest and set of directed unicycles is then

\[
\frac{U_\to(z)^{2n-m-1}}{(2n-m-1)!} e^{V_\to(z)},
\]

and multiplication by the EGF of the distinguished tree with removed nodes \(\frac{8T_\to(z)^4 z^{-4}}{(1 - 2T_\to(z))^3}\) and double application of the complementary label insertion operator finishes the construction.

The asymptotic analysis of the expression is done in two steps. Firstly, we get rid of the operator \(\Lambda\) by using the properties

\[
[z^n]F(z) = [z^{n-1}]F(z), \quad [z^n] \int F(z) dz = \frac{1}{n}[z^{n-1}]F(z).
\]

We then obtain

\[
[z^{2n}]\int \left[ U_\to(z)^{2n-m-1} e^{V_\to(z)} \frac{8T_\to(z)^4 z^{-4}}{(1 - 2T_\to(z))^3} \right] dz^2
\]

\[
= \frac{1}{(2n-1)(2n-3)} \left[ U_\to(z)^{2n-m-1} e^{V_\to(z)} \frac{8T_\to(z)^4 z^{-4}}{(1 - 2T_\to(z))^3} \right] dz^2
\]

\[
= \frac{1}{(2n-1)(2n-3)} \left[ U_\to(z)^{2n-m-1} e^{V_\to(z)} \frac{8T_\to(z)^4 z^{-4}}{(1 - 2T_\to(z))^3} \right].
\]

The second step is to apply Lemma 6 taking \(H(t)\) such that \(H(T_\to(z)) = 8T_\to(z)^4 (2n - m) U_\to(z)^{-1}, y = 3\). Since \(U_\to(z) = T_\to(z) - T_\to(z)^2\), the resulting value \(H(1/2)\) will be \(8/16 \cdot (2n - m) \cdot 4 \sim 2n\) when \(m/n \to 1\).

An application of the lemma gives

\[
\frac{(2n)!}{|D(2n, m)|} \frac{1}{(2n-1)(2n-3)} \left[ U_\to(z)^{2n-m} e^{V_\to(z)} \frac{H(T_\to(z))}{(1 - 2T_\to(z))^3} \right]
\]

\[
= \frac{2n(1 + \mu n^{-1/3})}{(2n-1)(2n-3)} \left( \frac{n^{1/3}}{\mu} \right)^3 (1 + O(|\mu|^{-3})) = \frac{1}{2|\mu|^3} + O(|\mu|^{-6}).
\]

The weakly connected component containing the pattern \(x \rightarrow \overrightarrow{P} \rightarrow y \rightarrow \overrightarrow{P}\) may happen to be a unicycle (see Figure 18), or a complex component of excess at least 1 (see Figure 19). The constructions analogous to Lemma 7 counting simple digraphs with a distinguished contradictory pattern weighted as \(2^\ell\), where \(\ell\) is the length of the pattern, yield asymptotics of order \(O(|\mu|^{-6})\) or smaller. This concept can be illustrated by the unicyclic case.

Instead of the three sequences with the generating function \(\frac{1}{(1 - 2T_\to(z))^3}\) we consider 6 sequences (possibly empty). There are several different mutual configurations of the distinguished contradictory pattern and the unicycle which contains this pattern, but all can be described by 6 sequences. For one of these sequences the directions of the edges are not fixed, but its length is not accounted in the weight of the graph. Therefore, the generating function for such a sequence is also proportional to \(\frac{1}{(1 - 2T_\to(z))}\), where a factor 2 in front of \(T_\to(z)\) stands for the choice of orientation of each edge. The number of trees in such a graph remains \((2n - m)\), and so, the contribution obtained from Lemma 6 has order \(\Theta\left(\frac{1}{(2n-1)(2n-3)} \cdot \frac{\mu^2}{|\mu|^3}\right) = \Theta(|\mu|^{-6}).

Remark 6. The same type of reasoning is used to show that the terms \(\xi \in \mathbb{E}\) with \(r \geq 2\) and \(\mathbb{E}\xi (\xi - 1) \cdots (\xi - k + 1)\) for \(k \geq 2\) do not contribute to the term of order \(\Theta(|\mu|^{-3})\) and start contributing from \(\Theta(|\mu|^{-6})\). Refinement of this technique using the complete asymptotic expansion of the saddle point lemma (see the refinement of Lemma 5 in Appendix A) can be used to obtain the complete asymptotic expansions of the probability of satisfiability in powers of \(|\mu|^{-3}\). The problem of enumeration of the mutual combinations of tuples of minimal contradictory components inside an implication digraph, in order to compute the factorial moments of \(\xi\), seems to be a challenging task.
4.2 From simple digraphs to sum-representations

The two details peculiar to sum-representations that were not treated in the previous section are the following: firstly, all the literals of the paths \(x \hookrightarrow \overline{x}, y \hookrightarrow \overline{y}\) of the distinguished pattern, except \(x\) and \(\overline{x}\), and \(y\) and \(\overline{y}\), should be pairwise strictly distinct; secondly, there should be no edge conflicts, i.e. pairs of complementary edges. Note that a presence of an edge \(x \rightarrow \overline{x}\) automatically induces a conflict, so there should be no such edges as well.

Excluding these cases requires inclusion-exclusion: in order to count the instances not containing conflicts and complementary literal pairs, we count the instances with distinguished conflicts and distinguished complementary literal pairs, and then take the alternating sum over them. The two inclusions-exclusions can be done independently.

Lemma 8 (Excluding complementary literals on the paths). Among the digraphs with a distinguished pattern \(x \hookrightarrow \overline{x} \hookrightarrow y \hookrightarrow \overline{y}\), taken with weight 2 to the power of the length of the pattern, the asymptotic proportion of digraphs in which there exist two literals on the pattern which are not strictly distinct (except the pairs \(x\) and \(\overline{x}\), and \(y\) and \(\overline{y}\)), is only \(O(n^{-1/3} |\mu|^{-2})\).

Proof. Let a random variable \(X\) denote the number of pairs of complementary literals on the distinguished pattern, not counting \(x\) and \(\overline{x}\), and \(y\) and \(\overline{y}\). Using the inclusion-exclusion principle, we can express the probability of the event \(X = 0\) as

\[
P(X = 0) = 1 - \mathbb{E}X + \frac{1}{2!}\mathbb{E}X(X - 1) - \cdots ,
\]

where \(\mathbb{E}X\) corresponds to the proportion of digraphs having a distinguished pattern and a distinguished additional pair of complementary literals \(z\) and \(\overline{z}\). Further terms correspond to distinguishing several pairs of complementary literals, etc.

By marking two complementary literals in the case when a contradictory pattern is a tree (see Lemma 7 and Figure 17), we obtain 5 sequences instead of 3, and we need three applications of the label insertion operator \(\Lambda\) instead of just two. We then obtain a generating function

\[
F(z) = \Lambda^3 \left( \frac{U \rightarrow (z)^{2n-m-1}}{(2n-m-1)!} e^{V \rightarrow (z)} \frac{2^5 T \rightarrow (z) z^{6-6}}{(1 - 2T \rightarrow (z))^5} \right)
\]

and after extracting the asymptotics using Lemma 6, we obtain

\[
\mathbb{E}X = \frac{[z^{2n}] \Lambda^3 \left( \frac{U \rightarrow (z)^{2n-m-1}}{(2n-m-1)!} e^{V \rightarrow (z)} \frac{2^5 T \rightarrow (z) z^{6-6}}{(1 - 2T \rightarrow (z))^5} \right)}{[z^{2n}] \Lambda^2 \left( \frac{U \rightarrow (z)^{2n-m-1}}{(2n-m-1)!} e^{V \rightarrow (z)} \frac{2^4 T \rightarrow (z) z^{4-4}}{(1 - 2T \rightarrow (z))^4} \right)} = \Theta(n^{-1/3} |\mu|^{-2})
\]

The same factor \(n^{-1/3}\) multiplied by a polynomial of \(|\mu|^{-1}\) and \(n^{-1/3}\) appears when the weakly connected component containing the contradictory pattern is not necessarily a tree. Further factorial moments of \(X\)
Lemma 9 (Excluding edge conflicts). With an asymptotic probability $e^{−1/8}$, the digraphs from $D(2n, m)$ with a distinguished pattern $x \rightsquigarrow \overline{x} \rightsquigarrow y \rightsquigarrow \overline{y}$ weighted according to $2$ to the power of its length, are conflict-free.

Proof. Following the same inclusion-exclusion principle, we introduce a random variable $X$ denoting the number of conflicting pairs of edges. The probability of the event $[X = 0]$ can be expressed using the inclusion-exclusion principle.

Let us start by computing $\mathbb{E}X$, i.e. the expected number of the conflicting edges. According to the linearity of the mathematical expectation, $\mathbb{E}X$ equals to the total number of possible complementary edge pairs times the probability that one distinguished edge pair, say $1 \rightarrow 2$, $\overline{2} \rightarrow \overline{1}$ is involved in a conflict.

Using Lemma 4, we construct the EGF for the family of digraphs containing a distinguished conflict $x \rightarrow y$, $\overline{y} \rightarrow \overline{x}$ by inserting two pairs of labels $x, \overline{x}$ and $y, \overline{y}$ at the free slots. This insertion can happen in several different ways: each of the edges can belong to the forest of directed trees, to the set of unicycles, and to the complex component (or to the component containing the distinguished pattern $x \rightsquigarrow \overline{x} \rightsquigarrow y \rightsquigarrow \overline{y}$).}

Figure 20: Digraph tree with a marked edge and two empty slots.

First, consider the case when each of the distinguished edges belongs to a distinct tree in a forest of directed trees. We denote the random variable corresponding to the number of such occurrences as $X_1$ which we will show to be asymptotically equivalent to $X$. A tree containing a distinguished edge $v \rightarrow w$ can be represented as a pair of trees with removed roots (which will be later filled by certain labels), see Figure 20. The corresponding EGF for digraphs with two distinguished trees containing edges $v \rightarrow w$, $\overline{w} \rightarrow \overline{v}$, and containing a distinguished pattern $x \rightsquigarrow \overline{x} \rightsquigarrow y \rightsquigarrow \overline{y}$ weighted according to $2$ to the power of the length of the pattern, is then equal to

$$
\Lambda^4 \left[ \frac{1}{2!} \left( \frac{T_{\rightarrow} (z)}{z} \right)^4 U_{\rightarrow} (z)^{2n - m - 3} \frac{e^{V_{\rightarrow} (z)} 8T_{\rightarrow} (z)^4 z^{-4}}{(1 - 2T_{\rightarrow} (z))^3} \right],
$$

where two of the applications of the operator $A$ stand for the insertion of the label pairs $v$ and $\overline{x}$, and $w$ and $\overline{y}$ for the distinguished conflict edge pair, and another two applications for the insertion of complementary literals $x$ and $\overline{x}$, and $y$ and $\overline{y}$ in a distinguished pattern. The term $1/2 \cdot T_{\rightarrow} (z)^4 z^{-4}$ is an EGF of the set of two trees with a distinguished edge. The cardinality $(2n - m)$ of the forest of trees is then reduced by 3.

The expected value of $X_1$ is then expressed as

$$
\mathbb{E}X_1 = \frac{[z^{2n}] \Lambda^4 \left( \frac{1}{2!} \left( \frac{T_{\rightarrow} (z)}{z} \right)^4 U_{\rightarrow} (z)^{2n - m - 3} \frac{e^{V_{\rightarrow} (z)} 8T_{\rightarrow} (z)^4 z^{-4}}{(1 - 2T_{\rightarrow} (z))^3} \right)}{[z^{2n}] \Lambda^2 \left( \frac{U_{\rightarrow} (z)^{2n - m - 1}}{(2n - m - 1)!} e^{V_{\rightarrow} (z)} \frac{8T_{\rightarrow} (z)^4 z^{-4}}{(1 - 2T_{\rightarrow} (z))^3} \right)}.
$$
This expression can be readily analysed by Lemma 4. Using the property that the exchange of the operator $[z^{2n}]$ with $\Lambda$ is done according to the rule $[z^{2n}]\Lambda F(z) = \frac{1}{(2n-1)!} [z^{2n}] z^2 F(z)$, we conclude that the unique multiples in the numerator of $\mathbb{E}X_1$, if carefully counted, appear as follows: (i) a multiple asymptotically equivalent to $(2n)^{-2}$ from the double application of $\Lambda$ and its exchange with $[z^{2n}]$; (ii) a multiple $1/2$ because the two trees with distinguished conflict form a set; (iii) a multiple $2^{-4}$ appear from $T_\rightarrow(z)^4$; (iv) a multiple $4^2$ appears from $U_\rightarrow(z)^2$; (v) a multiple $n^2$ from the change in the factorial $(2n - m - 1)!$ of the EGF of forest; (vi) all the occurrences of $z$ cancel out. Collecting the powers of two, we finally obtain

$$
\mathbb{E}X_1 \sim 2^{-2-1-4+4} = \frac{1}{8}.
$$

So far, we have considered only one particular case, but most of the work has been actually already done.

Without caring too much for the exact coefficients in the asymptotics, let us count the expected values of the following random variables: (i) $X_2$ for conflicting pairs of edges which are located in the same directed tree; (ii) $X_3$ for conflicting pairs, one located in a tree, another located in a unicycle (or a complex component); (iii) $X_4$ for conflicting pairs, neither is located in the forest. For $\mathbb{E}X_2$, for the first conflicting edge we need to fix a pair of rooted trees, and assuming without loss of generality (but with a loss of the exact multiplicative constant) that the second conflicting edge is inside the first tree, we obtain a path from this edge to the root. Therefore, the EGF for such objects is proportional, up to a constant, to $rac{1}{1-2T_\rightarrow(z)} (T_\rightarrow(z)/z)^4$, which corresponds to four trees without roots and a sequence of trees, each term of the sequence equipped with a weight 2 corresponding to the choice of the orientation of the edge in the sequence. Plugging it into the EGF of counted graphs yields

$$
\mathbb{E}X_2 \sim \frac{C_2}{\Theta(|\mu|^{-3})} [z^{2n}] A^4 \left( \frac{1}{1-2T_\rightarrow(z)} \left( \frac{T_\rightarrow(z)}{z} \right)^4 \frac{U_\rightarrow(z)^{2n-m-2}}{(2n-m-2)!} e^{V_\rightarrow(z)} \frac{8T_\rightarrow(z)^4 z^{-4}}{(1-2T_\rightarrow(z))^8} \right).
$$

(4.4)

We immediately notice that using fewer than two trees costs an asymptotic multiple of $n$ because of the factorial in the denominator for the EGF of forest $U_\rightarrow(z)^{2n-m}/(2n-m)!$, while we gain only a multiple $n^{1/3}$ for an additional sequence constructor. Thus, $\mathbb{E}X_2$ is negligible by a factor $n^{2/3}$. The same story happens again for $X_3$ and $X_4$ since in all these cases fewer than two trees are used, and for each lost multiple $n$ it is possible to obtain at most two sequence constructions, contributing each a $n^{1/3}$.

Again, the same principle is applicable for computing the higher moments: the most probable situation is to have all the conflicting edges separately in distinct trees (otherwise the contribution will be negligible by a factor at least $n^{2/3}$), and in this case,

$$
\frac{1}{k!} \mathbb{E}X(X-1) \cdots (X-k+1) \sim \frac{1}{8^k k!},
$$

and therefore, as $n \to \infty$, $\mathbb{P}(X = 0) \to \sum_{k \geq 0} \frac{(-1)^k}{8^k k!} = e^{-1/8}$.

\textbf{Corollary 1} (Formulation of the counting technique). Note that, by simple asymptotic analysis, $|\mathbb{P}(2n,m)| \to e^{1/8}$. Hence, by following Lemma 8, we obtain that the asymptotic probabilities obtained by counting the (weighted) patterns in \textit{simple digraphs}, without excluding conflicting edges and without taking care of strict distinctness of the paths, are equal to the asymptotic probabilities that sum-representation digraphs contain such (weighted) patterns.

\textbf{Remark 7}. As can be seen from the proof, the same technique can be applied to the situations when instead of the pattern $x \rightarrow y \rightarrow x \rightarrow y$ a different pattern is distinguished. This can be either used for the case when the expectations of $\xi_\rightarrow$ are counted, or when further factorial moments of $\xi_\rightarrow$ are being considered. In all such cases, the edge conflicts will most probably appear in the forest component, and excluding such conflicts will always give a multiple $e^{-1/8}$.

\textbf{Remark 8}. Using the same techniques and the inclusion-exclusion method, it is possible to consider the 2-SAT model where some of the conditions (C1)–(C3) are violated. Accordingly, in such models the loops or
multiple edges may appear. The number of the counted graphs will then be multiplied by a certain constant appearing from inclusion-exclusion or from a different symbolic construction, while the total number of graphs in the denominator will be coincidentally multiplied by the same constant. Therefore, the probability of satisfiability and its asymptotic expansion will remain unchanged under the different models.

4.3 The structure of contradictory components

Let us recall that the contradictory component in the implication digraph, defined as the set of contradictory variables $x \leadsto \pi \leadsto x$, forms a set of strongly connected components, see Remark 1. The following theorem gives a description of the contradictory component in the subcritical phase.

**Theorem 1.** Suppose that $m = n(1 + \mu n^{-1/3})$, and $\mu \to -\infty$ with $n$ while remaining $|\mu| \leq n^{1/12}$. The contradictory component of an implication digraph corresponding to a random formula $F \in \mathcal{F}(n, m)$ has an excess $r$ with probability

$$P(\text{excess of the contradictory component} = r) = C_r |\mu|^{-3r}(1 + O(|\mu|^{-3})).$$

Moreover, for every finite $r$, the kernel of this contradictory component is cubic with probability $1 - O(n^{-1/3})$. The coefficient $C_r$ is equal to the sum of $\sum_M 2^{-r} \varphi(M)/(2r)!$ taken over all possible labelled cubic contradictory components of excess $r$.

**Proof.** Let $\xi_r$ denote the random variable which equals to the number of contradictory components of excess $r$ (each component is not necessarily connected and the different components may possibly overlap). Using a manipulation with formal power series that can be interpreted as a variation of the inclusion-exclusion principle, we can express the probability that $\xi_r$ equals 1.

Let $F(x) := \sum_{k \geq 0} P(\xi_r = k)x^k$ be the probability generating function (PGF) of $\xi_r$. Then, $F^{(k)}(z) = \mathbb{E}\xi_r(\xi_r - 1)\cdots(\xi_r - k + 1)$. Using Taylor series expansion at $x = 1$, provided that $F(x)$ is analytic in a circle of radius greater than 1, we obtain

$$F(x) = \sum_{k \geq 0} \frac{\mathbb{E}\xi_r(\xi_r - 1)\cdots(\xi_r - k + 1)}{k!} (x - 1)^k.$$  (4.6)

Using the fact that $P(\xi_r = 1) = F'(0)$, we obtain the expression

$$P(\xi_r = 1) = \frac{\mathbb{E}\xi_r}{0!} - \frac{\mathbb{E}\xi_r(\xi_r - 1)}{1!} + \frac{\mathbb{E}\xi_r(\xi_r - 1)(\xi_r - 2)}{2!} - \cdots.$$  (4.7)

Here, $\mathbb{E}\xi_r$ can be rewritten as the number of implication digraphs with a distinguished contradictory component of excess $r$, $\mathbb{E}\xi_r(\xi_r - 1)$ is the number of implication digraphs with a distinguished pair of contradictory components, etc. The distinguished pair of contradictory components forms a contradictory component by itself, of excess at least $(r + 1)$, as e.g. in Example 2.

Let us choose a contradictory component $\mathcal{C}$ of excess $r$. We choose a contradictory pattern $\pi$ which is chosen by taking an arbitrary sum-representation of the kernel of $\mathcal{C}$, so that no two complementary edges are chosen. Assume that all the isolated vertices are not included into the pattern, so that there might appear distinguished vertices that do not have their complementaries in $\pi$. Finally, every edge of $\pi$ is replaced by a sequence of directed trees to obtain a directed weakly connected component $\mathcal{P}$. By combining the reasoning of Section 2.3 and using Lemma 3, we conclude that $\mathbb{E}\xi_r$ is asymptotically worth the number of all sum-representation digraphs $D^\pi(2n, m)$ containing $\mathcal{P}$, counted with weight $2^\ell$, where $\ell$ is the number of edges of $\pi$, and divided by the compensation factor of $\mathcal{C}$. Note that such a weakly connected component $\mathcal{P}$ may be a part of a larger weakly connected component in a sum-representation digraph, so it is required to take the sum over all possible weakly connected components containing $\mathcal{P}$.

Let us define the following quantities: (i) $\tau(\pi)$ equal to the number of directed edges of $\pi$; (ii) $\nu(\pi)$ equal to the number of pairs of complementary variables in $\pi$; (iii) $\varphi(\pi)$ equal to the number of literals that do not have their complementaries in $\pi$. Then, the number of Boolean variables in $\mathcal{C}$ equals $\nu(\pi) + \varphi(\pi)$, and
the excess \( r \) then equals \( r = \tau(\pi) - \varphi(\pi) - \nu(\pi) \). At the same time, the excess of the unoriented projection of \( \pi \) (in the sense of simple graphs) is equal to \( \tau(\pi) - \varphi(\pi) - 2\nu(\pi) = r - \nu(\pi) \).

The EGF \( f_P(z) \) for digraphs \( D \in D(2n, m) \) containing \( \mathcal{P} \) as a separate weakly connected components, can be expressed as

\[
f_P(z) = \frac{1}{\mathcal{X}(\pi)} \Lambda^\nu \left( \frac{U_{\nu}(z)^{2n-n+(r-\nu(\pi))}}{(2m-n+r-\nu(\pi))!} e^{V_{\nu}(z)(T_{\nu}(z))^{2r(\pi)}} \right). \tag{4.8}
\]

In the case above, each of the literals of \( \pi \) is coloured into a separate distinguished colour, and only the compensation factor of \( \pi \) is considered.

By analysing the asymptotics of \( \frac{(2n)!}{D(2n, m)} z^{2n} f_P(z) \) similarly to Lemma 7, we obtain

\[
\frac{(2n)!}{D(2n, m)} z^{2n} f_P(z) \sim \frac{1}{\mathcal{X}(\pi)} \frac{1}{(2n)^{\nu(\pi)}} 4^{\nu(\pi)-r} n^{\nu(\pi)-r} 2^{-\varphi(\pi)-2\nu(\pi)\frac{3}{2}r(\pi)/3} |\mu|^{-\tau(\pi)}.
\]

By using the relation \( r = \tau(\pi) - \varphi(\pi) - \nu(\pi) \), we obtain

\[
\frac{(2n)!}{D(2n, m)} z^{2n} f_P(z) \sim n^{\tau(\pi)/3-r} |\mu|^{-\tau(\pi)} \frac{1}{\mathcal{X}(\pi)} 2^{-r}.
\]

If the kernel of the component \( C \) is not a cubic multigraph, then \( r > \tau(\pi)/3 \), and the contributions of such terms are negligible. Otherwise, \( \tau(\pi) = 3r \). In the case when \( \mathcal{P} \) is a part of a larger weakly connected component of higher excess, this results of asymptotic of order \( |\mu|^{-3r-3} \) which is negligible compared to \( |\mu|^{-3r} \).

Finally, let us obtain the asymptotics of \( \mathbb{E} \xi_r \). By Corollary 1, enumeration inside simple digraphs gives the same asymptotics as the probability in sum-representations.

The expected value \( \mathbb{E} \xi_r \) can be obtained by adding up the contributions of all possible contradictory components \( C \) of excess \( r \) whose kernels are cubic. We denote such a contribution by \( \mathbb{E} \xi_r \), where \( \xi_r \) is the corresponding random variable. Then, recalling the reasoning from Section 2.3, and by choosing a corresponding sum-representation \( \pi \) of the kernel of \( C \), by using Lemma 3 and the fact that a cubic multigraph of excess \( r \) contains \( 2r \) vertices, we express \( \mathbb{E} \xi_r \) as

\[
\mathbb{E} \xi_r^C \sim \frac{\mathcal{X}(C)}{(2r)! \mathcal{X}(\pi)} \sum_{\ell=0}^{\infty} 2^{\ell} \left\{ \left| \left\{ G, p_\ell \right| G \in D^\pi(2n, m), p_\ell \subset G \text{ is obtained from } \pi \text{ by inserting sequences of trees of total length } \ell \right\} \right| \left| \left\{ G \mid G \in D^\pi(2n, m) \right\} \right|.
\]

Taking the sum over all such \( C \) we obtain the dominant contribution of \( \mathbb{E} \xi_r \). Further terms of the inclusion-exclusion for \( \mathbb{P}(\xi_r = 1) \) have order at most \( |\mu|^{-3r-3} \) and are, therefore, negligible by a factor \( |\mu|^3 \). Collecting the dominant contributions, we conclude that

\[
\mathbb{P}(\xi_r = 1) \sim \sum_{C \text{ of excess } r \text{ with cubic kernels}} \mathbb{E} \xi_r^C \sim \sum_C \frac{\mathcal{X}(C)}{2^r (2r)!} |\mu|^{-3r}.
\]

We present a different proof of a theorem from [Kim08] using the compensation factors of the contradictory components which comes as a corollary of the above theorem.

**Corollary 2.** For a random formula \( F \in \mathcal{F}(n, m) \), when \( m = n(1 + \mu n^{-1/3}) \) and \( \mu \to -\infty \) with \( n \) while remaining \( |\mu| \leq n^{1/12} \),

\[
\mathbb{P}(F \text{ is satisfiable}) = \left( 1 - \frac{1}{16|\mu|^3} \right) (1 + O(\mu n^{-1/3} + \mu^{-3})). \tag{4.11}
\]
Proof. In the subcritical phase, the compensation factor of the only possible cubic contradictory implication multidigraph of excess 1 (viz. Figure 7) is equal to 1/4. Therefore, the probability of having a contradictory component of excess 1 is \(\frac{2}{27}\), \(\frac{1}{16}|\mu|^{-3} = \frac{1}{16}|\mu|^{-3}\). The probability of having a contradictory component of higher excess is then \(\Theta(|\mu|^{-6})\), and so, is negligible.

4.4 Number of contradictory variables

**Theorem 2.** Let \(m = n(1 + \mu n^{-1/3})\), \(\mu \to -\infty\), \(|\mu| \leq n^{1/2}\). Assuming that the excess of the contradictory component is \(r\), and this component has a cubic kernel, the number of contradictory variables \(V\) in a random formula \(F \in \mathcal{F}(n, m)\) follows asymptotically a Gamma law with shape parameter \(3r\) and scale parameter \(n^{1/3}|\mu|^{-1}\), so that

\[
\lim_{n \to \infty} \mathbb{P}\left( V_n = x n^{1/3}|\mu|^{-1} \mid \text{excess } = r \right) = \frac{x^{3r-1}}{\Gamma(3r)} e^{-x}. \tag{4.12}
\]

**Proof.** Fix a contradictory component \(C\) of excess \(r\) with a cubic kernel. Construct a contradictory pattern \(\pi\) by taking an arbitrary sum-representation of the kernel of \(C\), and replace every oriented edge of \(\pi\) by a sequence of directed trees, thus obtaining a digraph \(P\). Consider an EGF \(F_P(z, u)\) for directed graphs with a distinguished weakly connected component \(P\) counted with weight \(2^k\) where \(k\) denotes the length of the 2-core of \(P\). The variable \(u\) then marks all the contradictory variables on \(C\). Then, using the similar constructions as in Theorem 1, we express \(F_P(z, u)\):

\[
F_P(z, u) = \frac{1}{\mathcal{A}(\pi)} 3^{2r} \frac{U_{\rightarrow}(z)^{2m-n-r} (2m-n-r)!}{(2m-n-r)!} e^{V_{\rightarrow}(z)} \frac{(T_{\rightarrow}(z))^{4r} z^{-4r} 2^{3r}}{(1-2u T_{\rightarrow}(z))^{3r}}. \tag{4.13}
\]

The expected value of the number of contradictory variables conditioned on this pattern \(P\) is then

\[
\mathbb{E}[V_n \mid P] = \frac{\partial_u[z^{2n}] F_P(z, u)|_{u=1}}{[z^{2n}] F_P(z, 1)} \sim 3r \cdot n^{1/3}|\mu|^{-1}, \tag{4.14}
\]

and more generally, \(k\)-th factorial moment can be expressed as

\[
\mathbb{E}[V_n \cdots (V_n - k + 1) \mid P] = \frac{\partial^k_u[z^{2n}] F_P(z, u)|_{u=1}}{[z^{2n}] F_P(z, 1)} \sim \frac{\Gamma(3r + k)}{\Gamma(3r)!} (n^{1/3}|\mu|^{-1})^k. \tag{4.15}
\]

Note that the resulting factorial moments do not depend on the choice of \(C\), therefore, these are also the asymptotic moments of the unconditioned variable \(V_n\).

The sequence of moments of the scaled random variable \(\tilde{V}_n := V_n n^{-1/3}|\mu|\) coincides with the sequence of moments of the Gamma distribution with shape parameter \(3r\): if its density is \(f(x) = x^{3r} e^{-x}/\Gamma(3r)\), then \(k\)-th moment is calculated as

\[
\int_0^{\infty} x^{3r+k-1} \frac{e^{-x}}{\Gamma(3r)} dx = \frac{\Gamma(3r+k)}{\Gamma(3r)!}. \tag{4.16}
\]

By checking Carleman’s condition for Stieltjes moment problem on \((0, +\infty)\), we conclude that this distribution is uniquely defined by its moments, which finishes the proof.

**Corollary 3.** Since in the subcritical phase a random 2-CNF has a contradictory component of excess 1 with probability \(\frac{1}{16|\mu|^3}\), and components of higher excess with a negligible probability, the distribution of the number of contradictory variables can be approximated by a mixture of a deterministic value 0 with probability \(\mathbb{P}(V = 0) = 1 - \frac{3}{16|\mu|^3}\) and of Gamma distribution with parameter 3 and scale \(n^{1/3}|\mu|^{-1}\) with probability \(\frac{1}{16|\mu|^3}\).
4.5 Structure of the spine

In the paper [BBC+ 01] it is proven that the expected size of the spine in the subcritical phase, i.e. for \( m = n(1 + \mu n^{-1/3}) \) when \( \mu \to -\infty \) with \( n \), is asymptotically \( \frac{1}{2} |\mu|^{-2/3} n^{2/3} \). As proven in Lemma 1, for every literal \( y \) from the spine of a formula \( F \in \mathcal{F}(n, m) \) there exists a minimal spinal path of the form \( y \leadsto x \leadsto y \), such that all the internal nodes of the paths \( y \leadsto x \) and \( x \leadsto y \) are pairwise strictly distinct. We show that for almost all literals \( y \) from the spine such a minimal spinal path is unique.

**Theorem 3.** Consider random formulae \( F \in \mathcal{F}(n, m) \) in the subcritical phase, \( m = n(1 + \mu n^{-1/3}) \), \( \mu \to -\infty \), \( |\mu| \leq n^{1/12} \). The expected number of spine variables \( y \) that have exactly \( k \) unique paths from \( y \) to \( y \) is asymptotically equal to \( C_k n^{2/3} |\mu|^{-2-3k} \), where \( C_k \) is some algorithmically computable constant. In particular, the expected proportion of spine variables \( y \) having a unique path \( y \leadsto y \) is \( 1 - O(|\mu|^{-3}) \).

![Figure 21: One possible configuration of two distinct paths \( y \leadsto y \).](image)

**Proof.** Let a random variable \( P_n \) denote the number of spine literals \( y \) of a random formula \( F \in \mathcal{F}(n, m) \), each variable counted with a multiplicity of the number of paths \( y \leadsto y \). We also define \( P_n^{(2)} \) which counts the number of spine literals \( y \) counted with the multiplicities of the pairs of distinct paths \( y \leadsto y \), and similarly \( P_n^{(\ell)} \) for \( \ell \)-tuples of distinct paths.

The cardinality of the spine \( S(F) \), i.e. the number of literals \( y \) for which there exists at least one such path, can be then counted using the variant of the inclusion-exclusion approach:

\[
\mathbb{E}S(F) = \mathbb{E}P_n - \mathbb{E}P_n^{(2)} + \frac{1}{2!} \mathbb{E}P_n^{(3)} - \frac{1}{3!} \mathbb{E}P_n^{(4)} + \cdots. \tag{4.17}
\]

In order to prove the theorem, we show that \( \mathbb{E}P_n = \Theta(n^{2/3} \mu^{-2}) \), \( \mathbb{E}P_n^{(2)} = \Theta(n^{2/3} \mu^{-5}) \), and, generically, \( \mathbb{E}P_n^{(\ell)} = \Theta(n^{2/3} \mu^{-3k \ell + 1}) \).

The expected value of minimal spinal paths can be again counted using sum-representations. By distinguishing a pattern \( y \leadsto x \leadsto y \) and counting such graphs with weight \( 2^\ell \) where \( \ell \) is the length of the pattern, we are counting the minimal spinal paths with multiplicity \( 2 \), as there are \( 2 \) distinct paths \( x \leadsto y \).

Knowing that exclusion of the conflicting edges gives the same result as counting the proportions of graphs with a distinguished pattern in simple digraphs (see Section 4.2), we pass directly to the counting in simple digraphs. The corresponding EGF for simple digraphs \( D \in \mathcal{D}(2n, m) \) with distinguished pattern which forms a weakly connected component which is a tree, is then (taking into account the compensating factor \( 1/2 \) arising from the multiplicity mentioned above)

\[
\frac{1}{2} \Lambda \left( \frac{U \to (z) 2n - m - 1}{(2n - m - 1)!} e^{V \to (z)} 4T \to (z) 3z^{-2}(1 - 2T \to (z))^2 \right)
\]

and therefore, by applying Lemma 6, we obtain the expected value of the dominant term

\[
\mathbb{E}P_n \sim \frac{1}{2} \cdot \frac{1}{2n} \cdot 4 \cdot n \cdot \frac{1}{8} n^{2/3} \mu^{-2} = \frac{1}{2} n^{2/3} \mu^{-2}.
\]
The above expectation also includes the cases when the excess of the weakly connected component containing the pattern \( y \leadsto x \leadsto \top \) is greater than \(-1\), but such cases give contribution at most \( O(n^{2/3} \mu^{-5}) \) and therefore are negligible.

Similarly, by constructing all possible cases of having two distinct paths between \( y \) and \( \top \) in the implication digraph, we note that the dominant contributions come only from the case when vertices of degree 3 are inserted into the core of the pattern, and therefore the pattern is almost cubic (except for the nodes \( y, x \) and \( \top \), and the pairs of complementary literals, where each such pair has a summary degree 3). It is easy to show that such components contribute a factor \( O(n^{2/3} \mu^{-5}) \) (one example of such configuration is given in Figure 21). Considering spine literals of further complexity gives the next orders of asymptotics. This observation finishes the proof.

**Corollary 4.** By removing on average a \( \Theta(|\mu|^{-3}) \) proportion of the spine literals, the spine breaks down into non-intersecting tree-like components, viz. Figure 5. For each component there exists a distinct literal \( x \) such that every literal \( y \) from this component has a unique path to \( x \), and the path \( x \leadsto \top \) is unique and strictly distinct.

### 5 Conclusions and open problems

While the analysis of random graphs and the curve of their phase transition have been described in details in [JKLP93], there is a certain obstacle which doesn’t immediately allow to go inside the critical phase of 2-SAT phase transition using the inclusion-exclusion approach. In [CDPG+18], a notion of a patchwork has been specifically designed for a very similar reason. However, their approach also doesn’t suggest any explicit way for obtaining such patchworks. These considerations give rise to the following questions.

**Problem 1.** What is the number of cubic strongly connected contradictory multigraph components with given excess? How many minimal contradictory (cubic) multigraphs implication multigraphs are there?

A variation of the above question, directly related to the inclusion-exclusion method, leads to the following question.

**Problem 2.** Using a version of [JKLP93, Lemma 3] when \(|\mu| = \Theta(1)\), is it possible to express the probability of satisfiability as a converging sum of Airy functions \( A(y, \mu) \)?

Some of the simulations suggested that the introduced notion of the excess correctly captures the discrete nature of the phase transition. According to the results, the distribution of the excess is discrete for finite \( \mu \) and doesn’t depend on \( n \) in the limit.

**Conjecture 1.** Let \( \xi \) denote the total excess of the strongly connected contradictory component in a random implication digraph corresponding to a formula \( F \) chosen uniformly at random from \( \mathcal{F}(n, m) \). If \( \mu \) is constant and \( m = n(1 + \mu n^{-1/3}) \), the distribution of \( \xi \) is discrete. Moreover, when \( \mu \) is constant, the number of strongly connected components of the contradictory component digraph follows a discrete limiting law; when \( \mu \to +\infty \), the contradictory component is strongly connected with high probability.

When \( \mu \to +\infty \), the expected value of the excess of the complex component in simple graphs is known and is \( \frac{2}{3} \mu^3 \). Some simulations suggest that exactly the same asymptotics may hold when \( \mu \to \infty \) for the expected excess of the contradictory component.

**Problem 3.** When \( m = n(1 + \mu n^{-1/3}) \) and \( \mu \to +\infty \), what is the expected excess of the contradictory component of a random 2-CNF formula?

One of the motivations to study the phase transition in 2-SAT is its similarity to the phase transition of the appearance of the strongly connected component in directed graphs. The papers [LS09, PP17] seem to come the closest to resolving the question, but is it possible to give the exact description?
Problem 4. It is possible to describe a "giant strong component" of a critical directed graph having \( n \) vertices and \( m = n(1 + \mu n^{-1/3}) \) oriented edges in the terms of cubic components and their excesses? Is it then possible to express the probability of having a strong component of excess \( r \) in terms of Airy function depending on \( r \)?

One of the applications of analysis with the help of generating functions, is the study of 2-SAT with a given set of degree constraints, similarly to [dPR16] and [DR18]. It is clear that the same analysis can be done for the case of formulae with literal set degree constraints, however the present analysis is done only for the subcritical phase.

Conjecture 2. The point \( r \) of the phase transition in the 2-SAT model with literal degree constraints from a set \( \Delta = \{ d_1, d_2, \ldots \} \) (possibly weighted) can be computed from the system of equations

\[
\frac{z \omega''(z)}{\omega'(z)} = 1, \quad z \frac{\omega'(z)}{\omega(z)} = r,
\]

where \( \omega(z) := \sum_{d \in \Delta} \frac{e^d}{d!} \) and \( \Delta \) satisfies the condition \( 1 \in \Delta \).

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A Complete asymptotic expansion for the saddle point lemma

Let us show how a complete asymptotic expansion in powers of $\mu^{-3}$ can be obtained in Lemma 5. In [JKLP93], the following formulation is given.

**Lemma 10** ([JKLP93, Lemma 3]). If $m = \frac{1}{2}(1 + \mu n^{-1/3})$ and $y$ is any real constant, $\mathcal{MG}(n, m)$ is the set of all labelled multigraphs with $n$ vertices and $m$ edges, then, as $\mu \to -\infty$ with $n$ while remaining $|\mu| \leq n^{1/12},$

$$\frac{n!}{|\mathcal{MG}(n, m)|} [z^n] \frac{U(z)^{n-m}}{(n-m)! (1 - T(z))^y} = \sqrt{2\pi} A(y, \mu) n^{y/3 - 1/6} \left(1 + O(\mu^4 n^{-1/3})\right),$$

where

$$A(y, \mu) = \frac{1}{\sqrt{2\pi |\mu|^{y-1/2}}} \left(1 - \frac{3y^2 + 3y - 1}{6|\mu|^3} + O(\mu^-6)\right).$$

**Complete asymptotic expansion.** In the proof, it is mentioned that in principle, it is possible to obtain a complete asymptotic series of $|\mu|^{-3}$. Let us describe the procedure that can be used to compute these coefficients.

Let $\alpha = -\mu$. As shown in the proof of [JKLP93, Lemma 3], the function $A(y, \mu)$ can be represented in the form

$$A(y, \mu) = \frac{1}{2\pi \alpha^{y-1/2}} \int_{-\infty}^{+\infty} \left(1 + \frac{it}{\alpha^{3/2}}\right)^{1-y} e^{-t^2/2 - it^3/(3\alpha^{3/2})} dt.$$}

In order to express $A(y, \mu)$ in the form of a complete asymptotic expansion, we introduce $\beta := i\alpha^{-3/2}$ and obtain:

$$A(y, \mu) = \frac{1}{\sqrt{2\pi |\mu|^{y-1/2}}} \int_{-\infty}^{+\infty} (1 + \beta t)^{1-y} e^{-\beta^3 t^3/3} dt \sim \frac{1}{2\pi |\mu|^{y-1/2}} \sum_{r \geq 0} c_r(y) \beta^r,$$

where $(c_r(y))_{r=0}^{\infty}$ are polynomials in $y$. The coefficient $[\beta^k] (1 + \beta t)^{1-y} e^{-\beta^3 t^3/3}$ can be expressed as the convolution of two generating functions

$$[\beta^k] (1 + \beta t)^{1-y} e^{-\beta^3 t^3/3} = \sum_{r=k}^{\infty} t^r \frac{(-t^3/3)^{r-k}}{(r-k)!} = \sum_{r=k}^{\infty} \frac{(-1/3)^{r-k}}{(r-k)!} \beta^{3r-2k} \binom{1-y}{r}.$$

A formal term by term integration (the series is most likely not convergent, but the expansion could be extended up to $r$th term for any finite $r$) yields for even $r$

$$c_{2r}(y) = \sqrt{2\pi} \sum_{k=0}^{2r} \frac{(-1/3)^{2r-k}}{(2r-k)!} \binom{1-y}{k} \int_{-\infty}^{+\infty} e^{-t^2/2} t^{6r-2k} dt = \sum_{k=0}^{2r} \frac{(-1/3)^{2r-k}}{(2r-k)!} \binom{1-y}{k} (6r - 2k - 1)^{r},$$

where the double factorial notation is used $(2n-1)!! := 1 \cdot 3 \cdots (2n-1)$; for odd $r$ the principal value of the integral equals zero, and so, $c_{2r+1}(y) = 0$.

As an example, the first nontrivial term $c_2(y)$ can be computed as

$$c_2(y) = \frac{1}{2!} \cdot 5!! - \frac{1}{4!} \cdot (1-y) \cdot 3!! + \frac{3y^2 + 3y - 1}{6} \cdot 1!! = 3y^2 + 3y - 1$$

and a factor $-1$ in the terms $c_{4r+2}(y)$ in the expansion of $A(y, \mu)$ appears because a multiple $i^{4r+2} = -1$ should be extracted from $\beta = i\alpha^{-3/2}$. 

29