SPECTRA OF PT–SYMMETRIC OPERATORS AND
PERTURBATION THEORY

Emanuela Caliceti\textsuperscript{1}, Sandro Graffi\textsuperscript{2}, Johannes Sjöstrand\textsuperscript{3},

Abstract

Criteria are formulated both for the existence and for the non-existence of complex eigenvalues for a class of non self-adjoint operators in Hilbert space invaraint under a particular discrete symmetry. Applications to the PT-symmetric Schrödinger operators are discussed.

1 Introduction and statement of the results

The Schrödinger operators invariant under the combined application of a reflection symmetry operator $P$ and of the (antilinear) complex conjugation operation $T$ are called PT-symmetric. A standard class of such operators has the form $H = H_0 + iW$ where:

1. $H_0$ is a self-adjoint realization of $-\Delta + V$ on some Hilbert space $L^2(\Omega); \Omega \subset \mathbb{R}^n$, $n \geq 1$; $V$ and $W$ are real multiplication operators.

2. $V$ are even and odd with respect to $P$, respectively: $PV = V$, $PW = -W$. $P$ is the parity operation

$$(P\psi)(x) = \psi((-1)^{j_1}x_1, \ldots, (-1)^{j_n}x_n), \quad \psi \in L^2$$

where $j_i = 0, 1$; $j_i = 1$ for at least one $1 \leq i \leq n$;

If $T$ is the involution defined by complex conjugation: $(T\psi)(x) = \overline{\psi}(x)$, one immediately checks that $(PT)H = H(PT)$.

\textsuperscript{1}Dipartimento di Matematica, Università di Bologna, I- 40127 Bologna (Italy) (caliceti@dm.unibo.it)

\textsuperscript{2}Dipartimento di Matematica, Università di Bologna, I- 40127 Bologna (Italy) (graffi@dm.unibo.it)

\textsuperscript{3}Centre de Mathématiques, École Polytechnique, F-91190 Palaiseau Cedex (France) (johannes@math.polytechnique.fr)
PT-symmetric quantum mechanics (see e.g. [1],[2],[3],[4],[5],[6],[7],[8],[9]) requires
the reality of the spectrum of PT-symmetric operators, recently proved, for in-
stance, for the one dimensional odd anharmonic oscillators [13], [12]. Imposing
boundary conditions along complex directions, however, examples of PT-sym-
metric operators with complex eigenvalues have been constructed [14]. It is therefore
an important issue in this context to determine whether or not the s pectrum of
PT-symmetric Schrödinger operators with standard $L^2$ boundary conditions at infinity
is real. We deal with this problem only in perturbation theory, but we will obtain
criteria both for existence of complex eigenvalues (Theorem 1.1) and for the reality
of the spectrum (Theorem 1.2), in even greater generality than the PT symmetry.

Let $H$ be a Hilbert space with scalar product denoted $(x|y)$, and $H_0 : \mathcal{H} \to \mathcal{H}$
be a closed operator with dense domain $\mathcal{D} \subset \mathcal{H}$. Let $H_1$ be an operator in $\mathcal{H}$ with
$\mathcal{D}(H_1) \supset \mathcal{D}$. This entails that $H_1$ is bounded relative to $H_0$, i.e. there exist $b > 0,$
$a > 0$ such that $\|H_1\psi\| \leq b\|H_0\psi\| + a\|\psi\| \forall \psi \in \mathcal{D}$. We can therefore define on $\mathcal{D}$
the operator family $H_\epsilon := H_\epsilon = H_0 + \epsilon H_1$, $\forall \epsilon \in \mathcal{C}$.

We assume the following symmetry properties: there exists a unitary involution
$J : \mathcal{H} \to \mathcal{H}$ mapping $\mathcal{D}$ to $\mathcal{D}$, such that

$$J H_0 = H_0^* J, \quad J H_1 = H_1^* J$$

(1.1)

In other words, $J$ intertwines $H_0$ and $H_1$ with the corresponding adjoint operators.
Note that:

1. The properties $J^2 = 1$ (involution) and $J^* = J^{-1}$ (unitarity) entail $J^* = J$,
i.e. self-adjointness of $J$;

2. The properties (1.1) entail, if $\epsilon \in \mathcal{R}$, $J H_\epsilon = H_\epsilon^* J$; therefore the spectrum
$\sigma(H_\epsilon)$ of $H_\epsilon$ is symmetric with respect to the real axis if $\epsilon \in \mathcal{R}$.

3. An example of $J$ is the parity operator $P$.

Let $H_0$ admit a real isolated eigenvalue $\lambda_0$ of multiplicity 2 (both algebraic and geo-
metric, i.e. we assume absence of Jordan blocks). Let $e_1, e_2$ be linearly independent
eigenvectors, and $E_{\lambda_0}$ the eigenspace spanned by $e_1, e_2$. Clearly $J E_{\lambda_0} := E_{\lambda_0}^*$ is the
eigenspace of $H_0^*$ corresponding to the eigenvalue $\overline{\lambda}_0 = \lambda_0$, and hence the bilinear
form \((u^*|v), u^* \in E_{\lambda_0}^*, v \in E_{\lambda_0}\) is non degenerate. Therefore we can choose \(e_1, e_2\) in \(E_{\lambda_0}\) in such a way that, writing \(u = u_1 e_1 + u_2 e_2\), the quadratic form \(Q(u, u) = (Ju|u)\) on \(E_{\lambda_0}\) assumes the canonical form

\[
Q(u, u) = \tau_1 u^2_1 + \tau_2 u^2_2, \quad \tau_1 = \pm 1, \tau_2 = \pm 1
\]

(1.2)

Notice that if \(e_1^*, e_2^*\) is the dual basis, then (1.2) means that \(Je_j = \tau_j e_j^*\).

Under these circumstances we want to prove the following

**Theorem 1.1** With the above assumptions and notations, consider the operator family \(H_\epsilon\) for \(\epsilon \in \mathbb{R}\). Denote:

\[
H_{11} = (H_1 e_1|e_1), \quad H_{22} = (H_1 e_2|e_2), \quad H_{12} = (H_1 e_1|e_2)
\]

(1.3)

Then \((e_1|H_1 e_1) \in \mathcal{R}, (e_2|H_1 e_2) \in \mathcal{R}\) and there exists \(\epsilon^* > 0\) such that, for \(|\epsilon| < \epsilon^*\):

(i) If \(\tau_1 \cdot \tau_2 = -1\), and

\[
4|H_{12}|^2 > (H_{11} - H_{22})^2
\]

(1.4)

\(H_\epsilon\) has a pair of non real, complex conjugate eigenvalues near \(\lambda_0\);

(ii) If \(\tau_1 \cdot \tau_2 = 1\) \(H_\epsilon\) has a pair of real eigenvalues near \(\lambda_0\).

**Remarks**

1. The above theorem applies to the \(PT\)-symmetric operator family \(H_\epsilon = H_0 + i\epsilon W\), where \(H_0\) and \(iW = H_1\) are as above. Here \(J = P\), and hence \(PH_0 = H_0 P, P(i\epsilon W) = -(i\epsilon W)P = (i\epsilon W)^* P\) so that \(JH_\epsilon = H_\epsilon^* J\). In that case Assumption (1.4) follows from the weaker assumption \(H_{12} \neq 0\) because the \(P\)-symmetry of \(H_0\) and the \(P\)-antisymmetry of \(W\) entail \(H_{11} = H_{22} = 0\). Indeed, we have \(Pe_j = \tau_j e_j\) and

\[
H_{jj} = (iW e_j|e_j) = (iP W e_j, Pe_j) = -(iW Pe_j|Pe_j) = -(iW e_j|e_j) = -H_{jj}
\]

2. The physical relevance of Theorem 1.1 is best illustrated by an elementary example. Let \(\mathcal{H} = L^2(\mathbb{R}^2)\) and \(H_0 : \mathcal{H} \rightarrow \mathcal{H}\) be the (self-adjoint) two dimensional harmonic oscillator with frequencies \(\omega_1, \omega_2\):

\[
H_0 u = -\frac{1}{2} \Delta u + \frac{1}{2}(\omega_1^2 x_1^2 + \omega_2^2 x_2^2)u
\]

3
We have \( \sigma(H_0) = \{E_{k_1,k_2} := \{k_1\omega_1 + k_2\omega_2 + \frac{\omega_1}{2} + \frac{\omega_2}{2}\} \mid k_i = 0, 1, 2 \ldots, i = 1, 2. \)

Let again \( H_\epsilon = H_0 + i\epsilon W, \epsilon \in \mathbb{R} \), with

\[
W(x) = \frac{x_1^2x_2}{1 + x_1^2 + x_2^2}
\]

Then \( W \) is bounded relative to \( H_0 \), and \( PW = -W \) if \( Pu(x_1, x_2) = u(x_1, -x_2) \) or \( Pu(x_1, x_2) = u(-x_1, -x_2) \). Set \( \omega_1 = 1, \omega_2 = 2, k_1 = 2, k_2 = 0 \); i.e., we consider the eigenvalue \( E_{2,0} = E_{0,1} \). Then for \( |\epsilon| > 0 \) small enough \( H_\epsilon \) has a pair of complex conjugate eigenvalues near \( E_{2,0} \).

To see this, remark that \( E_{2,0} = E_2(\omega_1) + E_0(\omega_2) = E_0(\omega_1) + E_1(\omega_2) \), where \( \lambda_i(\omega_i) = (k + 1/2)\omega_i \) are the eigenvalues of the one-dimensional harmonic oscillators with frequencies \( \omega_i, i = 1, 2 \). \( E_{2,0} \) has multiplicity 2. A basis of eigenfunctions is given by

\[
\psi_1(x_1, x_2) = e_2(x_1)f_0(x_2); \quad \psi_2(x_1, x_2) = e_0(x_1)f_1(x_2)
\]

Here \( e_0, e_2 \) are the eigenfunctions corresponding to \( E_0(1) \) and \( E_2(1) \), respectively; \( f_0, f_1 \) are the eigenfunctions corresponding to \( E_0(2) \) and \( E_1(2) \), respectively; note that \( e_0, e_2 \) and \( f_0 \) are even while \( f_1 \) is odd. To first order perturbation theory, the two eigenvalues \( \Lambda_j(\epsilon) : j = 1, 2 \) of \( H_\epsilon \) near \( E_{2,0} \) are given by

\[
\Lambda_j(\epsilon) = E_{2,0} + i\epsilon \lambda_j
\]

where \( \lambda_j : j = 1, 2 \) are the eigenvalues of the \( 2 \times 2 \) matrix

\[
W_{l,k} = \begin{pmatrix}
(W\psi_1|\psi_1) & (W\psi_1|\psi_2) \\
(W\psi_2|\psi_1) & (W\psi_2|\psi_2)
\end{pmatrix}
\]

Now \( \psi_1 \) is even, \( \psi_2 \) is odd, and \( W \) is odd. Therefore \( \tau_1 \cdot \tau_2 = -1 \). Moreover: \((W\psi_1|\psi_1) = (W\psi_2|\psi_2) = 0, (W\psi_2|\psi_1) = (W\psi_1|\psi_2) := w > 0 \) Therefore \( \lambda_j = \pm w \) and \( \Lambda_j(\epsilon) = E_{2,0} \pm i\epsilon w \). Hence the conditions of Theorem 1.1 (i) are satisfied and for \( \epsilon \) small enough \( H_\epsilon \) has a pair complex conjugate eigenvalues near \( E_{2,0} \).

3. By essentially the same proof, the result of Theorem 1.1 remains true under the following more general conditions: under the above assumptions on \( H_0 \)
and $H_1$ let $H_0$ admit two real, simple eigenvalues $E_1, E_2$. Let $d := E_2 - E_1$ be their relative distance; $D := \text{dist}[(\sigma(H_0) \setminus \{E_2, E_1\}), \{E_2, E_1\}]$ their distance from the rest of the spectrum; $e_1, e_2$ the corresponding eigenvectors, all other notation being the same. Then if $d/D$ is small enough the same conclusion of Theorem 1.1 holds provided $|\epsilon H_{12}| > \frac{d}{2D}$.

4. Example: Odd perturbations of quantum mechanical double wells: existence of complex eigenvalues.

Let $H = L^2(\mathbb{R})$, $H_0(h) = -\hbar^2 \frac{d^2}{dx^2} + x^2(1 + x)^2$, $D(H_0) = H^2(\mathbb{R}) \cap L^2(\mathbb{R})$, $W(x) \in L^\infty_{\text{loc}}(\mathbb{R})$, $|W(x)| \leq Ax^4$, $|x| \to \infty$, $W(1-x) = -W(x)$. Here $L^2(\mathbb{R}) = \{u \in L^2(\mathbb{R}) | x^4u \in L^2(\mathbb{R})\}$. In this case it is known that $W$ is bounded relative to $H_0$; moreover $d = O(e^{-1/\hbar})$, $D = O(\hbar)$, $w = O(1)$ if $E_1, E_2$ are the two lowest eigenvalues, $\psi_1, \psi_2$ the corresponding eigenvectors and $w$ is defined as in Point 2 above. Hence the conditions of Theorem 1 are fulfilled in the semiclassical regime provided $W$ is continuous at zero with $W(0) \neq 0$ and that $|(e_1|We_2)| \geq 1/C$ and thus there exist $A > 0, B > 0, C > 0$ such that $H_\epsilon(h) := H_0 + i\epsilon W$ will have at least a pair of complex conjugate eigenvalues for $Ae^{-B/\hbar^2} < \epsilon \wedge << Ch$. Equivalently, we may consider the double well family $H_0(g) = -\hbar^2 \frac{d^2}{dx^2} + x^2(1 + gx)^2$ defined on the same domain. Here $d = O(e^{-1/\hbar^2})$, $D = O(1)$, $w = O(1)$. The same argument holds for the general case $H_0 = -\hbar^2 \Delta + V(x)$, where $V : \mathbb{R}^n \to \mathbb{R}$ is smooth, has two equal quadratic minima and diverges positively as $|x| \to \infty$; $W(x) \in L^\infty_{\text{loc}}(\mathbb{R}^n)$, $|W(x)| \leq AV(x)$ as $|x| \to \infty$ because the estimate for $d$ is the same as above [15].

The second result concerns the opposite situation, a criterion ensuring the reality of the spectrum. In this case the natural assumption is the simplicity of the spectrum of $H_0$ in addition to its reality. Therefore for the sake of simplicity we assume $H_0$ self-adjoint.

**Theorem 1.2** Let the self-adjoint operator $H_0$ be bounded below (without loss of generality, positive), and let $H_1$ be continuous. Let $H_0$ have discrete spectrum, $\sigma(H_0) = \{0 \leq \lambda_0 < \lambda_1 \ldots < \lambda_l < \ldots\}$, with the property

$$
\delta := \inf_{j \geq 0} [\lambda_{j+1} - \lambda_j]/2 > 0.
$$
Assume that all eigenvalues are simple. Then \( \sigma(H(\epsilon)) \in \mathcal{R} \) if \( \epsilon \in \mathcal{R}, \ |\epsilon| < \frac{\delta}{\|H_1\|} \).

**Example**

Here again \( \mathcal{H} = L^2(\mathcal{R}); H_0 = -\frac{d^2}{dx^2} + V(x), D(H_0) = H^2(\mathcal{R}) \cap D(V). V(x) = kx^{2m}, \ k > 0, \ m \geq 1; \ W(x) \in L^{\infty}(\mathcal{R}), \ W(-x) = -W(x). \) We have: \( \sigma(H_0) = \{\lambda_n\}, n = 0, 1, \ldots; \)

\[
\lambda_n \sim k \frac{1}{^{2m}} n^{-\frac{2m+1}{m}}, \quad n \to \infty
\]

Each eigenvalue \( \lambda_n \) is simple. Clearly \( \delta \geq 1. \) Denote now \( H_\epsilon := H_0 + i\epsilon W \) the operator family in \( L^2(\mathcal{R}) \) defined by \( H_\epsilon = H_0 + H_1, \ H_1 = i\epsilon W, \ D(H_\epsilon) = D(H_0). \) Then \( H_\epsilon \) has real discrete spectrum for \( |\epsilon| < \|W\|^{-1}\infty. \)

## 2 Proof of the results

**Proof of Theorem 1.1**

The proof is based on perturbation theory and consists in two steps. In the first one we show that the \( 2 \times 2 \) matrix generated by restricting the perturbation \( H_1 \) to \( E_{\lambda_0} \) is antihermitian in case (i) of Theorem 1.1 and Hermitian in case (ii). In the second step we show by the method of the Grushin reduction (see, e.g. [16]) that for \( \epsilon \) suitably small the control of the above \( 2 \times 2 \) matrix is enough to establish the result.

Let \( \{e_1, e_1\} \) be once more a basis in \( E_{\lambda_0} \) such that (1.2) holds, and denote by \( e_1^*, e_2^* \) the dual basis in the dual subspace \( E_{\lambda_0}^* = JE_{\lambda_0}. \) Clearly \( Je_j = \tau_j e_j^*, \ \tau_j = \pm 1. \) We denote \( \Pi_0 \) the spectral projection from \( \mathcal{H} \) to \( E_{\lambda_0}. \) Explicitly:

\[
\Pi_0 u = (u|e_1^*)e_1 + (u|e_2^*)e_2 \quad (2.1)
\]

Consider now the rank 2 operator family \( \Pi_0 H_\epsilon \Pi_0 \) acting on \( E_{\lambda_0}. \) The representing \( 2 \times 2 \) matrix is:

\[
H(\epsilon)_{j,k} = \lambda_0 I + \epsilon H_{j,k}^1, \quad H_{j,k}^1 = (H_1 e_k|e_j^*), \ j, k = 1, 2 \quad (2.2)
\]
Now $JH_0 = H_0^* J$, $J\Pi_0 = \Pi_0^* J$. We also have $JH_1 = H_1^* J$. Therefore:

$$(JH_1 e_k|e_j) = (H_1 e_k|J e_j) = \tau_j (H_1 e_k|e_j^*) = \tau_j H_{j,k}^1$$

and in the same way

$$(JH_1 e_k|e_j) = (H_1^* J e_k|e_j) = (J e_k|H_1 e_j) = \tau_k (e_k^*|H_1^1 e_j) = \tau_k \overline{(H_1 e_k|e_j^*)} = \tau_k \overline{H_{k,j}}$$

Summing up:

$$\tau_j H_{j,k}^1 = \tau_k \overline{H_{k,j}}$$

Therefore, if $\tau_1 \tau_2 = 1$ the matrix $H(\epsilon)_{j,k}$ is hermitian for $\epsilon \in \mathcal{R}$ and its eigenvalues are real; if instead $\tau_1 \tau_2 = -1$ the matrix $H(\epsilon)_{j,k}$ has a real diagonal part and an antihermitian off diagonal part for $\epsilon \in \mathcal{R}$ and its eigenvalues are complex conjugate.

This completes the first step.

We want now to construct an approximate inverse of $H_\epsilon - z$ near $\lambda_0$ by solving a Grushin problem. In this context it is equivalent to the Feshbach reduction, and provides a convenient formalism for it. To this end, define the operators $R_+, R_-,$ $\mathcal{P}_0(z)$ in the following way:

$$R_+: \mathcal{H} \to \mathbb{C}^2, \quad R_+ u(j) = (u|e_j^*), \quad j = 1, 2; \quad (2.3)$$

$$R_-: \mathbb{C}^2 \to \mathcal{H}, \quad R_- u_- = \sum_{j=1}^2 u_-(j)e_j, \quad (2.4)$$

$$\mathcal{P}_0(z) = \begin{pmatrix} H_0 - z & R_- \\ R_+ & 0 \end{pmatrix}: \mathcal{D} \times \mathbb{C}^2 \to \mathcal{H} \times \mathbb{C}^2. \quad (2.5)$$

Note that we have identified $E_{\lambda_0}$ with its representative $\mathbb{C}^2$, and that $R_+ R_- = I$, the $2 \times 2$ identity matrix.

The associated Grushin system is

$$\begin{cases} (H_0 - z)u + R_- u_- = f \\ R_+ u = f_+ \end{cases} \quad (2.6)$$

where $u \in \mathcal{D}, f \in \mathcal{H}, u_-, f_+ \in \mathbb{C}^2$. $z \in \mathbb{C}$ belongs to a neighborhood of $\lambda_0$ at a positive distance from $\sigma(H_0) \setminus \{\lambda_0\}$. After determining $u_-$ in such a way that $f - R_- u_- \in (1 - \Pi_0)\mathcal{H}$ the first equation can be solved for $u(z) \in (1 - \Pi_0)\mathcal{H}$ and hence the problem is reduced to the the rank 2 equation $R_+ u(z) = f$. To solve
explicitly, remark that, for every \( z \) in the complex complement of \( \sigma(H_0) \setminus \{\lambda_0\} \), \( P_0(z) \) has the bounded inverse,

\[
E_0(z) = \begin{pmatrix}
E^0_0(z) & E^0_+(z) \\
E^0_-(z) & E^0_{-+}(z)
\end{pmatrix},
\]

with

\[
E^0_0(z) = (H_0 - z)^{-1}(1 - \Pi), \quad E^0_+(z) = R_-, \quad E^0_-(z) = R_+, \quad E^0_{-+}(z) = (z - \lambda_0)I.
\]

where \( I \) is the 2×2 identity matrix. The spectral problem within \( E_{\lambda_0} \) is thus reduced to the inversion of \( E^0_{-+}(z) \), and obviously its solution is represented by \( \lambda_0, e_0, e_1 \).

Now restrict the attention to the set of complex \( z \) with \( \text{dist}(z, \{\lambda_0\}) < 1/2R \), where

\[
R := \|E^0_0(\lambda_0)\| = \|(1 - \Pi_0)(H_0 - \lambda_0)^{-1}\|
\]

so that by the geometrical series expansion

\[
\|E^0_0(z)\| \leq \frac{R}{1 - |z - \lambda_0|R}
\]

Consider the operator from \( \mathcal{D} \times \mathbb{C}^2 \) to \( \mathcal{H} \) defined as

\[
P_\varepsilon(z) = \begin{pmatrix}
H_\varepsilon - z & R_- \\
R_+ & 0
\end{pmatrix}.
\]

associated to the Grushin system

\[
\begin{cases}
(H_\varepsilon - z)u + R_- u_+ = f \\
R_+ u = f_+
\end{cases}
\]

Then

\[
P_\varepsilon(z)E_0(z) = 1 + \begin{pmatrix}
i\varepsilon H_1 E^0_0(z) & i\varepsilon H_1 E^0_+(z) \\
0 & 0
\end{pmatrix} =: 1 + \mathcal{K}.
\]

It is routine to check that \( P_\varepsilon(z) \) has the inverse

\[
E_\varepsilon(z) = \begin{pmatrix}
E^\varepsilon_0(z) & E^\varepsilon_+(z) \\
E^\varepsilon_-(z) & E^\varepsilon_{-+}(z)
\end{pmatrix},
\]

with

\[
E^\varepsilon_0(z) = \sum_{n=0}^{\infty} \left( \frac{\varepsilon}{i} \right)^n E^0_0(H_1 E^0)^n,
\]
\[ E^e_+(z) = \sum_{n=0}^{\infty} \left( \frac{\epsilon}{r} \right)^n (E^0 H_1)^n E^0_+ \]  
(2.16)

\[ E^e_-(z) = \sum_{n=0}^{\infty} \left( \frac{\epsilon}{r} \right)^n (H_1 E^0)^n, \]  
(2.17)

\[ E^e_{-+}(z) = E^0_{-+} + \sum_{n=1}^{\infty} \left( \frac{\epsilon}{r} \right)^n (H_1 E^0)^n E^0_-. \]  
(2.18)

where all the series will be proved to have a positive convergence radius (convergence means here uniform, or, equivalently, in the norm operator sense). We also recall the well known fact that \( z \) is an eigenvalue of \( H_\epsilon \) precisely when \( \det E^e_{-+}(z) = 0 \).

We next derive the appropriate symmetries for the inverse operators [16]. From \( JH_\epsilon = H^*_\epsilon J \) we get:

\[ JR_- u_- = \sum_{j=1}^{2} u_-(j) J e_j = \sum_{j=1}^{2} (\tau u_-)(j)e^*_j, \quad \tau := \begin{pmatrix} \tau_1 & 0 \\ 0 & \tau_2 \end{pmatrix} \]

\[ R^*_+ u_- = \sum_{j=1}^{2} u_-(j) e^*_j \]

where the second equation follows from

\[ (R_+ u|u_-) = \sum_{j=1}^{2} u_-(j)(u|e^*_j) \]

We thus conclude:

\[ J R_- u_- = R^*_+ \tau u_-, \quad R^*_+ J = \tau R_+ \]

Therefore:

\[ \left( \begin{array}{cc} J & 0 \\ 0 & \tau \end{array} \right) \left( \begin{array}{cc} H_\epsilon - z & R_- \\ R_+ & 0 \end{array} \right) = \left( \begin{array}{cc} J(H_\epsilon - z) & J R_- \\ \tau R_+ & 0 \end{array} \right) \]
\[ = \left( \begin{array}{cc} (H^*_\epsilon - z) J & R^*_+ \tau \\ R^*_- J & 0 \end{array} \right) = \left( \begin{array}{cc} (H^*_\epsilon - z) & R^*_+ \\ R^*_- & 0 \end{array} \right) \left( \begin{array}{cc} J & 0 \\ 0 & \tau \end{array} \right) \]

whence

\[ \left( \begin{array}{cc} J & 0 \\ 0 & \tau \end{array} \right) P_\epsilon(z) = P_\epsilon(z)^* \left( \begin{array}{cc} J & 0 \\ 0 & \tau \end{array} \right) \]  
(2.19)

Since \( \mathcal{E}(z) = \mathcal{P}(z)^{-1} \), taking right and left inverses we get

\[ \mathcal{E}(z)^* \left( \begin{array}{cc} J & 0 \\ 0 & \tau \end{array} \right) = \left( \begin{array}{cc} J & 0 \\ 0 & \tau \end{array} \right) \mathcal{E}(z) \]

that is

\[ \left( \begin{array}{cc} E_+(\mathcal{P})^* & E_-(\mathcal{P})^* \\ E_+(\mathcal{P})^* & E_-(\mathcal{P})^* \end{array} \right) \left( \begin{array}{cc} J & 0 \\ 0 & \tau \end{array} \right) = \left( \begin{array}{cc} J & 0 \\ 0 & \tau \end{array} \right) \left( \begin{array}{cc} E(z) & E_+(z) \\ E_-(z) & E_-(z) \end{array} \right) \]  
(2.20)
In particular:

\[ E_{-+}(z)^* \tau = \tau E_{-+}(z) \]

We can thus conclude that, for \( z \in \mathcal{R} \), if \( \tau_1 \cdot \tau_2 = 1 \) the \( 2 \times 2 \) matrix \( E_{-+}(z) \) is Hermitian, and antihermitian off the diagonal with real diagonal elements if if \( \tau_1 \cdot \tau_2 = -1 \).

It remains to be proved the norm convergence of the expansions (2.15, 2.17, 2.18). We have, by the relative boundedness condition \( \|H_1 \psi\| \leq b\|H_0 \psi\| + a\|\psi\| \) and (2.10):

\[
\|H^1 E^0\| = \|H^1(H_0 - z)^{-1}(1 - \Pi_0)\| \leq b\|H_0(H_0 - z)^{-1}(1 - \Pi_0)\| + a\|(H_0 - z)^{-1}(1 - \Pi_0)\|
\]

\[
\leq b\|(H_0 - z)(H_0 - z)^{-1}(1 - \Pi_0)\| + b\|z\|((H_0 - z)^{-1}(1 - \Pi_0)\| + a\|(H_0 - z)^{-1}(1 - \Pi_0)\|
\]

\[
\leq b\|1 - \Pi_0\| + \frac{(b|z| + a)R}{1 - |z - \lambda_0|R} < K
\]

for some \( K(z) > 0 \) because \( |z| < R/2 \). Therefore

\[
\|E^0(H^1 E^0)^n\| \leq K^{n+1}, \quad \|(E^0 H^1)^n E^0_+\| \leq K^{n+1},
\]

\[
\|E^0_-(H^1 E^0)^n\| \leq K^{n+1}, \quad \|E^0_-(H^1 E^0)^{n-1} H^1 E^0_+\| \leq K^{n+1}
\]

Hence the expansions (2.15 2.17 2.18) are norm convergent.

To conclude the proof we have to verify that the first order truncation of the expansion for \( E_+(z) \) yields nonreal eigenvalues, and that the higher order terms can be neglected. To this end, first remark that without loss of generality we may assume \( \lambda_0 = 0 \). Then the expansion (2.18) yields:

\[
-E_{--}^{\epsilon}(z) = \left( \frac{\epsilon H_{11} - z}{-\epsilon H_{11}} \quad \frac{\epsilon H_{12}}{\epsilon H_{22} - z} \right) + O(\epsilon^2)
\]

uniformly with respect to \( z, |z| < 1/2R \). Therefore

\[
\det E_{--}^{\epsilon}(z) = z^2 - (H_{11} + H_{22})\epsilon z + \epsilon^2(|H_{12}|^2 + H_{11}H_{22}) + O(\epsilon^3 + \epsilon^2|z|) =
\]

\[
= [z - \epsilon(H_{11} + H_{22})/2]^2 + \epsilon^2[H_{12}^2 - (H_{11} - H_{22})^2/4] + O(\epsilon^3 + \epsilon^2|z|)
\]

Now \( \det E_{--}^{\epsilon}(z) \), which is real for \( z \in \mathcal{R} \), clearly has no zeros for \( z \in \mathcal{C}, \epsilon << |z| << 1 \). On the other hand, for \( z = O(\epsilon) \), i.e. \( z = \epsilon w, w = O(1) \),

\[
\det E_{--}^{\epsilon}(z) = \epsilon^2\{[w - (H_{11} + H_{22})/2]^2 + |H_{12}|^2 - (H_{11} - H_{22})^2/4\}
\]

\[
+ O(\epsilon^3(1 + O(1))
\]
Therefore if \(4|H_{12}|^2 > (H_{11} - H_{22})^2\) there cannot be real zeros for \(\epsilon\) suitably small. We can thus conclude that \(\det E^\epsilon_{-+}(z)\) is zero for \(z = \Lambda_{\pm}(\epsilon)\),

\[
\Lambda_{\pm}(\epsilon) = \frac{1}{2}[H_{11} + H_{22} \pm i\epsilon \sqrt{4|H_{12}|^2 - (H_{11} - H_{22})^2}] + O(\epsilon^2)
\]

and this concludes the proof of the Theorem.

**Proof of Theorem 1.2**

Let us first recall that under the present assumptions \(H_{\epsilon}\) is a type-A holomorphic family of operators in the sense of Kato (see [17], Chapter VII.2) with compact resolvents \(\forall \epsilon \in \mathbb{C}\). Hence \(\sigma(H_{\epsilon}) = \{\lambda_l(\epsilon)\} : l = 0, 1, \ldots\). In particular:

(i) the eigenvalues \(\lambda_l(\epsilon)\) are locally holomorphic functions of \(\epsilon\) with only algebraic singularities;

(ii) the eigenvalues \(\lambda_l(\epsilon)\) are stable, namely given any eigenvalue \(\lambda(\epsilon_0)\) of \(H_{\epsilon_0}\) there is exactly one eigenvalue \(\lambda(\epsilon)\) of \(H_{\epsilon}\) such that \(\lim_{\epsilon \to \epsilon_0} \lambda(\epsilon) = \lambda(\epsilon_0)\);

(iii) the Rayleigh-Schrödinger perturbation expansion for the eigenprojections and the eigenvalues near any eigenvalue \(\lambda_l\) of \(H_0\) has convergence radius \(\delta_l/\|H_1\|\) where \(\delta_l\) is half the isolation distance of \(\lambda_l\).

Remark that since \(\delta_l \geq \delta \ \forall l\), all the series will be convergent for all \(\epsilon \in \Omega_{r_0}\);

\(\Omega_{r_0} := \{\epsilon \in \mathbb{C} : |\epsilon| < r_0\}\), where \(r_0 := \delta/\|H_1\|\) is a uniform lower bound for all convergence radii.

Assume now without loss of generality, to simplify the notation, \(\|H_1\| = 1\). By hypothesis \(|\lambda_l - \lambda_{l+1}| \geq 2\delta > 0 \ \forall l \in \mathcal{N}\). First remark that if \(\epsilon \in \mathcal{R}, |\epsilon| < r_0\) and \(\lambda(\epsilon)\) is an eigenvalue of \(H_{\epsilon}\) then \(|\text{Im} \lambda(\epsilon)| < \delta\), i.e. \(\sigma(H_{\epsilon}) \cap \mathcal{C}_\delta = \emptyset\), \(\mathcal{C}_\delta := \{z \in \mathbb{C} : |\text{Im} \ z| \geq \delta\}\). Set indeed

\[
R_0(z) := [H_0 - z]^{-1}, \quad z \notin \sigma(H_0)
\]

Then \(\forall z \in \mathcal{C}\) such that \(|\text{Im} \ z| \geq \delta\) we have

\[
|\epsilon H_1 R_0(z)| \leq |\epsilon| \cdot \|H_1\| \cdot \|R_0(z)\| \leq \frac{|\epsilon|}{\text{dist}[z, \sigma(H_0)]} \leq \frac{|\epsilon|}{|\text{Im} \ z|} \quad (2.21)
\]
Hence the resolvent

\[ R_\epsilon(z) := [H_\epsilon - z]^{-1} = R_0(z)[1 + \epsilon H_1 R_0(z)]^{-1} \]

exists and is bounded if \(|\text{Im} \, z| \geq \delta\) because (2.21) entails the uniform norm convergence of the Neumann expansion for the resolvent:

\[
\| R_\epsilon(z) \| = \|[H_\epsilon - z]^{-1}\| = \|R_0(z) \sum_{k=0}^{\infty} [-\epsilon H_1 R_0(z)]^k \| \leq \| R_0(z) \| \sum_{k=0}^{\infty} |\epsilon|^k \|H_1 R_0(z)\| \leq \frac{|\epsilon|}{|\text{Im} \, z| - \epsilon}
\]

Now \(\forall l \in \mathcal{N}\) let \(Q_l(\delta)\) denote the open square of side \(2\delta\) centered at \(\lambda_l\). Since \(|\lambda_l - \lambda_{l+1}| \geq 2\delta\), it follows as in (2.21) that \(R_\epsilon(z)\) exists and is bounded for \(z \in \partial Q_l(\delta)\), the boundary of \(Q_l(\delta)\). We can therefore, according to the standard procedure (see e.g. [17], Chapter III.2) define the strong Riemann integrals

\[ P_l(\epsilon) = \frac{1}{2\pi i} \int_{\partial Q_l(\delta)} R_\epsilon(z) \, dz, \quad l = 1, 2, \ldots \]

As is well known, \(P_l\) is the spectral projection onto the part of \(\sigma(H_\epsilon)\) inside \(Q_l\). Since \(H_\epsilon\) is a holomorphic family in \(\epsilon\), by well known results (see e.g. [17], Thm. VII.2.1), the same is true for \(P_l(\epsilon)\) for all \(l \in \mathcal{N}\). In particular this entails the continuity of \(P_l(\epsilon)\) for \(|\epsilon| < r_0\). Now \(P_l(0)\) is a one-dimensional: hence the same is true for \(P_l(\epsilon)\). As a consequence, there is one and only one point of \(\sigma(H_\epsilon)\) inside any \(Q_l\). Now \(\sigma(H_\epsilon)\) is discrete, and thus any such point is an eigenvalue; moreover, any such point is real for \(\epsilon\) real because \(\sigma(H_\epsilon)\) is symmetric with respect to the real axis. Finally, we note that if \(z \in \mathcal{R}, \ z \notin \bigcup_{l=1}^{\infty} \lambda_l - \delta, \lambda_l + \delta\) the Neumann series (2.21) is convergent and the resolvent \(R_\epsilon(z)\) is there continuous. This concludes the proof of Theorem 1.2.

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