Jordan and Schoenflies in non-metrical analysis situs

Alexandre Gabard and David Gauld*

October 13, 2010

Abstract. We show that both, the Jordan curve theorem and the Schoenflies theorem extend to non-metric manifolds (at least in the two-dimensional context), and conclude by some dynamical applications à la Poincaré-Bendixson.

Key words. Non-metric manifolds, Jordan curve theorem, Schoenflies theorem.

1 Introduction

The intrinsic importance of manifold theory, regarded as the subdiscipline of topology climaxing the venerable Euclidean geometry, can hardly be overemphasised in view of the seminal achievements obtained over the past two centuries (Gauss, Lobatschevskii, Riemann, Poincaré, …, Perelman, … just to name a few). Especially elaborated is the metric theory where it is postulated that a distance function generates the manifold topology. In contradistinction the non-metric case (studied by Cantor, Hausdorff, Vietoris, Alexandroff, Prüfer/Radó, R. L. Moore, Calabi–Rosenlicht, the Kneser family (Hellmuth and Martin), M. E. Rudin, Zenor, Nyikos, …) remains, comparatively, a somewhat marginal branch, whose (in)significance is the object of recurrent cultural controversies.

Whatever the ultimate verdict should be, it is fair to observe that many classical notions like dynamical systems or foliations are perfectly natural—at least well-defined—fields of investigations even in the non-metric realm. Some prolegomena towards a middlebrow foliation theory developed over non-metric manifolds are to be found in [5]. Concerning the allied theory of dynamical systems (from the viewpoint of flows, i.e. continuous actions of the real line \( \mathbb{R} \)), the authors are preparing a modest paper [14] analysing which among the basic principles of dynamics permit an extension to non-metric manifolds. (Topics

*Supported by the Marsden Fund Council from Government funding, administered by the Royal Society of New Zealand.

1Explicit references are given in [5], or also in [41] especially those cited in [9].

2Compare Massey [23, p. 47]: “Such surfaces are usually regarded as pathological, and ignored; …” Hirsch [20, p. 32], as quoted above. A less severe judgement is Milnor [24, p. 7]: “The main object of this exercise is to imbue the reader with suitable respect for non-paracompact manifolds.”, or Carathéodory [11, p. 707]: “[…] so daß unter Umständen eine Triangulation nicht existiert. Daß letzteres möglich ist, zeigt er an einem besonders lehrreichen Beispiel von Heinz Prüfer.”
include the Poincaré-Bendixson theory, G. D. Birkhoff’s minimal systems, the Whitney-Bebutov theory of cross-sections and flow-boxes, Whitney’s flows\(^3\) to the construction of transitive\(^4\) flows à la Sidorov/Anosov-Katok, Anatole Beck’s technique for slowing down flow lines.) Such dynamical motivations led us to inquire about the availability of the Jordan and Schoenflies theorems without any metrical proviso, which is the chief concern of the present note. (This hopefully justifies our somewhat old fashioned title, winking at Veblen’s 1905 paper \[42\], often regarded as the first rigorous proof of the Jordan curve theorem—abbreviated as (JCT) in the sequel.)

If we are permitted to give a slight refinement of Hirsch’s phraseology above, we believe that non-metric manifold theory takes in reality a two-fold incarnation: there is a “soft-side”, usually concerning compact, or even Lindelöf subobjects and a “hard-side” involving the whole manifold itself, and which typically is not safe from the invasion of set-theoretic independence results\(^5\). The Jordan and Schoenflies problems of this note belong to the former class of easy problems, reducible to metrical knowledge as we shall see. A similar metrical reduction occurs with flows, since letting flow any chart\(^6\) \(V\) one generates a Lindelöf submanifold \(f(\mathbb{R} \times V)\), to which one may apply the (metric) Whitney-Bebutov theory of cross-sections and flow-boxes. This extends the availability of the Poincaré-Bendixson theory, as well as the fact that non-singular flows induce foliations. The Lindelöfness of \(f(\mathbb{R} \times V)\) also shows that non-metric manifolds never support minimal flows. Accordingly, it is sometimes much easier to prove things about non-metric than metric manifolds, as corroborated by the elusive Gottschalk conjecture on the (in)existence of a minimal flow on a “baby” manifold like \(S^3\). In contrast Hirsch’s statement remains perfectly vivid when it comes to the existence of smooth structures on 2-manifolds, where it is still much undecided in which category “soft vs. hard” this problem will ultimately fall. Recall that the similar question for PL structures was recently solved by Siebenmann [39, Surface Triangulation Theorem (STT), p. 18–19].

Perhaps another motivation for a non-metric version of Schoenflies, arises in the context of the Bagpipe Theorem of Nyikos [29, Theorem 5.14, p. 666], a far reaching “generalisation” of the classification of compact surfaces, extended to \(\omega\)-bounded\(^7\) surfaces. To be honest the latter is rather a “structure theorem” as the tentacular long pipes emanating from the compact bag (a compact bordered surface) may exhibit a bewildering variety of topological types. Understanding long pipes is tantamount to describing simply-connected \(\omega\)-bounded surfaces, via the canonical bijection given by “filling the pipe with a disc”, whose inverse operation is “disc excision”. In this context, it may be observed that Nyikos (cf. [29, p. 668, §6]) relies on an ad hoc definition of simple-connectivity which is a consequence of the non-metric Schoenflies theorem (Propositions \(9\) and \(11\) below). Hence our results just bridge a little gap between the conventional definition of simple-connectivity (in terms of the vanishing of the fundamental group \(\pi_1\)) and the one adopted by Nyikos (separation by each embedded circle, with at least one residual component having compact closure). Section 6 of Nyikos [29] shows that even under the stringent assumptions \(\pi_1 = 0\) jointly with \(\omega\)-boundedness (which should be regarded as a non-metric pendant of

\(^3\)i.e., a flow parameterising the leaves of a given orientable one-dimensional foliation.

\(^4\)Following G. D. Birkhoff, a flow is transitive if it has at least one dense orbit, and minimal if all orbits are dense.

\(^5\)Formulation borrowed from Nyikos [28, p. 513]. The chief issue is that the answer to an old question of Alexandroff-Wilder relative to the existence of a non-metric perfectly normal manifold (i.e. each closed subset is the zero-set of a continuous real-valued function) turned out to be independent of the usual axiomatic ZFC (Zermelo-Fraenkel-Choice).

\(^6\)By a chart we shall mean an open subset homeomorphic to \(\mathbb{R}^n\).

\(^7\)A topological space is said to be \(\omega\)-bounded if the closure of any countable subset is compact.
compactness) two-dimensional topology permits a menagerie of specimens.

The most naive approach, say to the Schoenflies problem, could be the following: given a Jordan curve \( J \) (i.e. an embedded circle) in a non-metric simply-connected surface, try to engulf \( J \) in a chart to conclude via the classical Schoenflies theorem—henceforth abbreviated as (ST)—that \( J \) bounds a disc. This is somewhat hazardous because all the given data (as well the ambient surface as the embedded circle) are a priori extremely large (about the size of an expanding universe). However a refinement of this idea is successful: cover the range of a null-homotopy by a finite number of charts, the union of which provides a metric subsurface into which \( J \) is contractible to point, hence bounds a disc (by a homotopical version of (ST), cf. Section 2 for the details).

Beside this “geometric approach” there is a more “algebraic” one relying on singular homology\(^8\) whose intervention is prompted by the fact that non-metric manifolds are inherently intriangulable\(^9\). We are still hesitant about deciding which of the two approaches provides more insights, so we decided to include both. A useful reference for the singular homology of manifolds is the paper by Samelson \[38\], of which we shall need the basic vanishing result for the top-dimensional homology of an open (connected, Hausdorff) manifold. The surprising issue is that no “extra-terrestrial” non-metric “geometric topology” is required, just easy algebra and finistic topology (classification of compact surfaces) do the job. [This is a fair judgement, modulo the fact that already for the non-metric Jordan theorem, our proof has a reliance on (ST), so fails to be “pure homology”.] At the end of the note we present a converse to the non-metric (ST) (Proposition 11).

Combining both results (Propositions 9 and 11) we can state our main result as:

**Theorem 1** A (Hausdorff) surface \( M \) is simply-connected if and only if each Jordan curve in \( M \) bounds a 2-disc in \( M \).

**TYPO- AND BIBLIOGRAPHICAL CONVENTIONS.** We shall put in small fonts certain digressions not directly relevant to our main purpose (those optional readings, marked by the symbol \( \star \), can be omitted without loss of continuity). Many classical references related to Jordan and Schoenflies are listed in Siebenmann \[39\]: so any lazy referencing, by us, of the form [Jordan, 1887] means that the item can be located in Siebenmann’s bibliography. On the other hand we try to reserve (hopefully not too caricatural) historical comments to footnotes in order to keep clean the logical structure of the argument. Those historical details are provided as distractions, which in the best cases represent only a first-order approximation toward a sharpened picture provided by the first-hand sources. Perhaps the diagram below provides a snapshot view of some of the historical background relevant to our purpose.

## 2 The geometric approach

Before presenting the homological argument, we discuss two alternative approaches that were suggested to us by a closer look to the existing literature.

(1) We realised that the non-metric Jordan theorem is also stated by R. J. Cannon \[10\] Remark on p.97: “The Jordan curve theorem, which is well known in the case of the plane, is true for noncompact simply connected 2-manifolds in general.” Cannon’s proof is rather succinct in its reliance on a “sweeping” theorem of Borsuk, to which alas no

---

\( ^8 \)Initiated by Lefschetz, 1933 and taking its definitive form with Eilenberg, 1944.

\( ^9 \)The easy argument (going back at least to Weyl \[45\], p.24]) is that when enumerating simplices by adjacency, one sees that a triangulated (connected) manifold consists of at most countably many simplices, so is \( \sigma \)-compact, hence metrisable.
explicit reference is provided. The most appropriate reference we were able to locate is Borsuk [6], where previous works of [Leja, 1927] and [Golab, 1928] are revisited. (Golab is apparently responsible for the terminology balayage=sweeping.) Unfortunately we failed to understand the details of Cannon’s proof, as in all sweeping theorems we are aware of ([6] and the two subreferences cited above), the ambient manifold is \( \mathbb{R}^n \), a condition which seems hard to fulfill when covering by charts the range of a null-homotopy.

(2) Now we come to the geometric approach to the (non-metric) Schoenflies problem, which strangely enough permutes the “logical rôles” of Jordan and Schoenflies. It is based on the following exercise in Hirsch’s book [20, p. 207, Ex. 2], stated as: “Let \( M \) be a surface \( \subset \) and \( C \subset M \) a circle. If \( C \) is contractible to a point in \( M \) then \( C \) bounds a disk in \( M \)”. Call this statement (HST) for homotopical Schoenflies theorem. The pleasant issue is that this statement immediately transcends itself beyond the metric realm:

**Proposition 2** Let \( M \) be a (Hausdorff not necessarily metric) surface and \( C \) be a null-homotopic Jordan curve on \( M \). Then \( C \) bounds a 2-disc in \( M \). (In particular if \( M \) is simply-connected, each Jordan curve bounds a disc.)

**Proof.** By compactness we may cover the image of a contracting homotopy shrinking \( C \) to a point by a finite number of charts (open sets homeomorphic to the plane \( \mathbb{R}^2 \)). So \( C \) is contained in a certain Lindelöf (hence metric) subsurface \( M_* \) into which \( C \) is null-homotopic. By (HST) it follows that \( C \) bounds a disc in \( M_* \), which can of course be regarded as embedded in \( M \).

Covering space theoretic proofs of (HST) are proposed in Epstein [13, Theorem 1.7, p. 85] and in Marden–Richards–Rodin [22,13], following a method that goes back at least to Baer [8, §2, (b), p. 106–107]. The idea is simply to lift the problem to the universal covering. So one needs first knowledge of simply-connected (metric) surfaces:

---

10 Leja’s original argument depends on a parametric version of the Riemann mapping theorem due to [Radó, 1923]=[33], and so will doubtfully satisfy the “topologically inclined” reader.
11 Presumably assumed paracompact, cf. the Convention formulated in [20, §5, p. 32–33].
12 This easy argument is not new and appears in Cannon (loc. cit. [10, p. 97]).
13 These authors assume orientability, which is not required (at least for the first part of their statement).
14 Baer assumes his surface \( \bar{C} \) closed of genus \( g > 2 \), but his argument adapts easily to the general case.
15 For Poincaré [31, p. 114], this seems to be obvious a priori (i.e., prior to uniformisation), as he writes:
Lemma 3. A simply-connected metric surface is homeomorphic either to the plane $\mathbb{R}^2$ or to the sphere $\mathbb{S}^2$.

Proof. Of course the statement is a direct consequence of the uniformisation theorem of Riemann surfaces (Klein-Poincaré-Koebe 1882–1907); after using Radó [34] to triangulate and Heins [19] to introduce a $C^\infty$-analytic structure. As an alternative elementary approach one can appeal to a certain enumeration scheme for the triangles of an open triangulated simply-connected 2-manifold proposed by van der Waerden [44] and Reichardt [35]: one can successively aggregate triangles while never introducing a triangle $\Delta_k$ touching the earlier aggregate $\Delta_1 \cup \ldots \cup \Delta_{k-1}$ along one edge plus its opposite vertex. Such an enumeration allows one to construct a homeomorphism with the plane $\mathbb{R}^2$ by considering a suitable subsequence forming an ascending chain of closed discs. One can also conclude with the monotone union theorem of Morton Brown [9]. [In fact van der Waerden’s motivation (cf. also the enthusiastic paper by Carathéodory [11]) was to provide the “simplest possible” foundations to the uniformisation theorem within the frame of pure complex function theory, while avoiding any potential-theoretic intrusion[16].] Other polyhedral proofs are to be found in Ahlfors-Sario [11, §44D, p. 104] or in Massey [23, p. 200, Ex. 5.7]. (We are not aware of an argument bypassing Radó’s triangulation theorem.)

Lemma 4 (HST) (Baer, 1928, H. I. Levine, 1963, Epstein, 1966, Marden et al. 1966). Let $M$ be a metric surface and $C$ be a null-homotopic Jordan curve on $M$. Then $C$ bounds a 2-disc in $M$.

Proof. We follow Baer [3, §2, (b), p. 106–107]. Lift the Jordan curve $C$ to the universal covering $\tilde{\pi}:\tilde{M} \to M$ to obtain another Jordan curve $\Gamma$ (in view of the null-homotopic assumption). By the topological avatar of the Poincaré-Volterra theorem (see Bourbaki [7, Chap.I, §11.7, Corollary 2, p. 116] or [17, p. 197]), the cover $\tilde{M}$ is second countable, hence metric. Therefore $\tilde{M}$ is either $\mathbb{R}^2$ or $\mathbb{S}^2$ by Lemma 3, and it follows that $\Gamma$ bounds a disc $\Delta$ in $\tilde{M}$ by the classical (ST). It is enough to show that the disc $\Delta$ maps bijectively to $M$ via the covering projection $\pi$, yielding the desired disc $D = \pi(\Delta)$ bounding $C$. A priori, three collapsing possibilities could occur:

1. Two points of the boundary of $\Delta$ are equivalent (under a deck-transformation); [this case is obviously impossible]
2. A point interior to $\Delta$ and one lying on its boundary $\Gamma$ are equivalent;
3. Two interior points $p_1, p_2$ of $\Delta$ are equivalent.

Case (2) implies that in the interior of $\Gamma$ there would be a nested sequence of curves lying over $C$, hence an accumulation point (compactness of $\Delta$); violating the discreteness of the fibers of the (universal) covering map. Baer argues that distinct lifts of $C$ are disjoint, as otherwise projecting down to $M$ one would get “multiple points” on $C$ (contradicting $C$ being a simple curve). [This disjunction property can also be seen by considering the restricted covering $\pi: \pi^{-1}(C) \to C$, whose total space is a disjoint union of circles, each simply covering the base circle.] By (JCT), in the plane a Jordan curve possesses a unique (compact) inside, by which we mean the interior of the Jordan curve.

“La surface de Riemann est [...] simplement connexe et ne diffère pas, au point de vue de la Géométrie de situation, de la surface d’un cercle, d’une calotte sphérique ou d’une nappe d’un hyperboloïde à deux nappes.”

16To appreciate fully this fact, we may refer to Gray’s historical survey [16, §6, p. 78]
17In Epstein’s proof this classification is not taken for granted, but rather deduced from (HST). However this is done at the cost of an Appendix on PL technology.
plus the Jordan curve itself. By (ST) one has an alternative argument to Baer’s, as the deck-transformation \( \gamma \) taking \( p_1 \in \Gamma \) to \( p_2 \in \text{int}(\Delta) \) would map the disc \( \Delta \) into itself, violating the Brouwer fixed-point theorem. In both arguments, it must be remarked that the deck-transformation \( \gamma \) carries \( \Gamma \) to the curve \( \gamma(\Gamma) \) which is interior to \( \Gamma \) (by disjunction plus (JCT)), hence since \( \gamma \) is a global homeomorphism of the plane it must take the inside of \( \Gamma \) to the inside of \( \gamma(\Gamma) \). As a Jordan curve \( J_0 \) contained in the inside of a Jordan curve \( J \) has an inside contained in the inside of \( J \) (cf. Siebenmann [39, JORDAN SUBDOMAIN LEMMA, p. 4]), it follows that \( \gamma(\Delta) \subset \Delta \), contradicting either Brouwer, or yielding the infinite sequence \( p_1, \gamma(p_1), \gamma^2(p_1), \ldots \) in the compactum \( \Delta \supset \gamma(\Delta) \supset \gamma^2(\Delta) \supset \ldots \), corrupting the discreteness of the fibres, as argued by Baer).

Case (3) reduces to Case (2). Indeed choose an arc \( A \) inside \( \text{int}(\Delta) \) joining \( p_1, p_2 \in \text{int}(\Delta) \), then its projection is a closed curve \( A \) (loop) \( K \) on \( M \), which is not null-homotopic (as \( p_1 \neq p_2 \)). Considering successive lifts \( A = A_1, A_2, \ldots \) of \( K \), where \( A_i \) starts from the end-point of its predecessor \( A_{i-1} \), shows that we will eventually leave \( \Delta \) (else there would be again a corruption of the discreteness of the fibre). Of course this holds in the absence of a periodic motion, which might be inferred from the torsion-free property of the \( \pi_1 \) of aspherical manifolds. Alternatively, one may argue that a cyclic pattern leads to a string \( J := A_1 \cup \ldots \cup A_k \) of \( A_i \)’s which is a closed curve. Can we arrange it to be Jordan, i.e. simple? If not, there would be double points when projecting down. As yet we have not ensured that \( K \) is a simple closed curve, however a simple trick is to travel along the arc \( A \) and as soon as its projection down to \( M \) exhibits a self-intersection, we may cut out a subarc \( A_0 \subset A \) projecting to an embedded circle; and redefine \( A \) as \( A_0 \). Then a fixed point for \( \gamma \), the deck-translation induced by the loop \( K \), is created in the inside of the Jordan curve \( J \) by the Brouwer fixed-point theorem. This justifies the absence of “periodicity”, so that the end-point of \( A_n \) is not in \( \Delta \) (for some sufficiently large integer \( n \)). Then by (JCT), the path \( A_1 \cup A_2 \cup \ldots \cup A_n \) will meet \( \Gamma \). Since \( A \) does not meet \( \Gamma \), the intersection point \( p \in A_i \cap \Gamma \) occurs on a later arc \( A_i \) \((i \geq 2)\), and therefore the pair consisting of \( p \) and its deck-translation back to \( A \) satisfies the requirement of Case (2).

We may now move towards Jordan, perhaps first recalling the following fine words of Felix Klein [21, p. 531] in 1882, five years before\textsuperscript{20} the official “proof” of [Jordan, 1887]: “[…] daß die jetzt betrachtete Kurve, gleich einer solchen, die sich in einen Punkt zusammenziehen läßt, die gegebene Fläche in getrennte Gebiete zerlegt”. This contains (more-or-less) the following statement:

**Proposition 5** (Klein, 1882) Let \( C \) be a null-homotopic Jordan curve on a Hausdorff surface \( M \) (metric or not). Then \( C \) divides \( M \) (i.e. \( M \setminus C \) is disconnected).

**Proof.** By Proposition \( \text{2} \) \( C \) bounds a disc \( D \) in \( M \). Its interior \( U = \text{int}(D) \) is contained in \( M \setminus C \), and cannot be enlarged without meeting \( C \) (by the Frontier Crossing Lemma as stated in Siebenmann [39, p. 2]). It follows that \( U \) is a connected component of \( M \setminus C \), but not the unique one as otherwise \( U = M \setminus C \), so that when taking closures \( D = M \), violating the assumption that \( M \) is a genuine surface (without boundary).

\textsuperscript{18}A *Jordanbogen* in Baer [3, p. 107]: this follows either from (ST) or more elementarily by a clopen argument that shows that in any connected manifold (whether metric or not), one can join any two given points by an embedded arc, cf. [15, Prop. 1].

\textsuperscript{19}über einer geschlossenen Kurve \( \mathfrak{K} \); it is not perfectly clear if Baer assumes his curve to be simple?

\textsuperscript{20}Of course this does not discredit Jordan’s priority as Klein is always perfectly clear (if not vindicating) the merely heuristic value of his exposition, primarily intended to be a diffusion of Riemann’s ideas.
We may observe that in this geometric approach, the historical as well as the logical order of Jordan and Schoenflies gets reversed. The sequel of the paper presents an alternative “algebraic”, indeed homological approach, where the natural order is restored.

3 Generalised (non-metric) Jordan theorem

We now start with a general (i.e. not metrically confined) formulation of the Jordan curve/separation theorem.

Proposition 6 (Generalised Jordan curve theorem) Let $M$ be a (connected) Hausdorff simply-connected surface. Then $M$ is dichotomic, i.e. each embedded circle $J$ in $M$ divides the surface into exactly two components. Moreover the (topological) frontier of each component of $M - J$ is $J$.

Proof. Note first that the Hausdorff axiom is essential: without it take $M = B \times \mathbb{R}$ the product of the branching line $B$ (as defined e.g. in [4, Figure 1]) with the usual line, on which it is easy to draw a non-dividing circle.

Our proof relies on the following geometric lemma.

Lemma 7 (Tubular neighbourhoods of circles) Let $J$ be a Jordan curve (= a homeomorph of the circle) in a Hausdorff orientable surface $M$. Then there is an open set $T$ in $M$ containing $J$ with a homeomorphism of pairs $(T, J) \approx (S^1 \times \mathbb{R}, S^1 \times \{0\})$.

Proof. Since $J$ is compact, it can be covered by finitely many charts of $M$, so that we can reduce to the case where the ambient manifold $M$ is Lindelöf (hence metric). Then classical results do the work: indeed by Radó [34] metric surfaces can be triangulated, and then the required tubular neighbourhood might be constructed via combinatorial methods. Finally the trivial product structure of $T$ comes from orientability (as opposed to a twisted $\mathbb{R}$-bundle over $S^1$, i.e. a Möbius band, which would violate it).

In fact the metric subsurface $M^*$ engulfing the Jordan curve $J$ (of the previous paragraph) is indeed triangulable but one must ensure that a triangulation of $M^*$ can be arranged in such a way that $J$ is a subcomplex. (In this situation the required tube $T$ may be constructed via regular neighbourhood theory.) A priori, the existence of such a triangulation looks fragile, especially in view of fractal curves or the construction by [Osgood, 1903] of a Jordan curve in the plane $\mathbb{R}^2$ of positive Lebesgue measure. However it is precisely the content of (ST), to ensure that whatever the complexity of a Jordan curve $J$ in $\mathbb{R}^2$ might be, there is still a global homeomorphism of the plane taking $J$ to the unit circle $S^1$ (or to a triangle). In particular, there exists a triangulation of the plane $\mathbb{R}^2$ such that any given Jordan curve $J$ occurs as a subcomplex. Alternatively, one finds a tube around any $J$, just by pulling back a neat annulus around $S^1$. In our situation nothing ensures that $M^*$ is a homeomorphic of $\mathbb{R}^2$. However the classical (ST) allows one to solve the corresponding global problems (i.e. $\mathbb{R}^2$ replaced by an arbitrary metric surface $M$):

---

21See for the classical case [Jordan, 1887], [Veblen, 1905], etc., and for a panoramic view [39].

22The term schlachtartig is also employed, but in the non-metric context it would be misleading.

23In contrast to the homological proofs of the Jordan curve theorem (say by Brouwer [8], or more fairly [Alexander, 1920], [Alexander, 1922]=2) where the ambient manifold is known, either a Euclidean $\mathbb{R}^n$ or a sphere $S^n$, we need here more geometric control, furnished by the classical (ST), in order to construct a tube around the Jordan curve. Hence (at least in our presentation) the non-metric Jordan theorem does not boil down to pure homology theory.

24Initiated by [J. H. C. Whitehead, 1939], developed by [Zeeman, 1962], etc. cf. also [Rourke-Sanderson, 1972].
Lemma 8 (i) Relative surface triangulation theorem. (Compare Siebenmann [39, Remark (b), p. 19]) Given a pair \((M, \Gamma)\) consisting of a graph \(\Gamma\) (locally finite simplicial complex of dimension 1) embedded as a closed subset of a metric surface \(M\), one can construct a triangulation of \(M\) so that \(\Gamma\) occurs as a subcomplex. [In our setting we just need the case where \(\Gamma\) is a circle, which is treated in Epstein [13, Appendix]]

(ii) Tubular neighbourhoods of graphs. (Cf. again [39, Remark (b), p. 19]) Same data as in (i), one can (directly) construct a tubular neighbourhood of the graph \(\Gamma\). The proof uses the local graph taming theorem (cf. Siebenmann [39, §9, p. 16]) jointly with the collaring techniques of Morton Brown.

This lemma completes the proof of Lemma 7.

★ Variant (à la Edwin Evariste Moise): Alternatively it is certainly possible to establish both these results without relying on (ST), following the techniques employed in Moise [25], who is able to prove the (absolute) surface triangulation theorem, without reference to (ST), by founding everything on the PL approximation theorem.

★ Historical digression on the triangulation of surfaces (Radó, Prüfer, 1922–1925). (For much sharper reports compare [38] especially §6, §9 and [41] End of §8 and §9.) On pages 110–111 of his paper, Radó [34] recalls that the triangulation theorem was in special cases treated by [Weyl, 1913]=[45, p. 21, p. 32] (the case of analytische Gebilde=concrete Riemann surfaces arising via analytic continuation of a holomorphic function-germ), and respectively [H. Kneser, 1924] (triangulability in presence of a Kurvenschar (=foliation) on a compact surface). It is interesting to observe that Radó’s 1925 proof (cf. [34, Hilfssatz 2, p. 111–114]) does not seem to use (ST). In Remmert-Schneider [36] p. 188, §9 (where the early interactions between Prüfer and Radó are beautifully commented on), it is asserted that Radó’s 1925 proof “benutzt den Riemannschen Abbildungssatz”, i.e. relies on the Riemann mapping theorem. [Here we are not sure to agree completely with Remmert–Schneider’s assertion, and we believe instead that Radó gives himself much pain to work at a purely topological level.] As early as 1923, Radó [32, p. 35–36] presents a proof of the triangulation theorem (at least for Riemann surfaces based on the Grenzkreistheorem of Klein-Poincaré). For the general case he provides only a sketch [32, p. 37]. Unfortunately, it seems that Radó condensed his 1923 exposition influenced by the erroneous suggestion of Prüfer that triangulations could exist without any countability proviso. Retrospectively this early mistake of Prüfer looks astonishing in view of the long (line) manifolds of [Cantor, 1883, Hausdorff, 1915, Victoris, 1921, Tietze 1924 and Alexandroff, 1924, 27] but turned out to be extremely fruitful, by leading to a new generation of non-metric manifolds, the so-called Prüfer manifold(s). (The latter was first described in print by Radó [34, but already mentioned in [32, p. 35, footnote 9].) From a strict logical viewpoint, it looks intriguing to question the rigour of Radó’s 1925 proof (a naive minded objection being that while scanning through Radó’s argument one does not encounter any citation to Schoenflies nor to Osgood, but perhaps such a use is implicit somewhere in Radó’s proof). Such a moderate criticism of Radó seems also implicit in Remmert–Schneider’s formulation [36, p. 187]: “Eine heutigen Maßtäben gerecht werdende Behandlung des Triangulierungsproblems . . .”, where of course they refer to the proof presented by Ahlfors-Sario [1, Chap. 1, §8, p. 105–110] (this is perhaps the first place where a reliance on (ST) for triangulability is made explicit 28). It should, however, be emphasised.

25Of course this is not so much of a surprise if one recalls from Siebenmann [39, §4, Historical Notes] that the “Schoenflies theorem” appellation seems to have been coined only in [Wilder, 1949].
26To whom Hausdorff communicated his 1915 construction.
27Accurate references located in [36, 41] [9] where this “long” string of (re)discoverers of the long ray is carefully documented (including the “recent” discovery by E. Brieskorn and W. Purkert in the Univ. Bibl. at Bonn of an unpublished Nachlaß of F. Hausdorff, dated in 1915.) Meanwhile this Nachlaß has been published in [18].
28Moreover it seems that the Ahlfors-Sario proof benefited from some corrections pointed out by G. Thomas (compare page vi of the preface of the 1965 Second Printing of [1]).
sised that the Ahlfors-Sario proof stays very close to the 1925 proof of Radó. Other proofs (usually restricted to the compact case) are given in [Doyle-Moran, 1968]=[12] and [Thomassen, 1992]. In the latter reference there is (on page 116) a (too?) severe criticism that the previous proofs (of triangulability) relied on geometric intuition. In sum, available proofs of the triangulability of metric surfaces (non-compact case included) include the following list (in chronological order): [Radó, 1925]= [33] (with a sketchy precursor in [32]), [Ahlfors-Sario, 1960]= [11], [Moise, 1977]= [25, p. 60] and [Siebenmann, 2005]= [39]. It is to be noted that the proof in [Moise, 1977] does not depend on (ST); and in [25, p. 62] one even finds a serious “3D” justification: “Ordinarily, the triangulation theorem for 2-manifolds is deduced from the Schönflies theorem. This method may be simpler, once the Schönflies theorem is known, but it is in a way misleading. In dimension 3, the Schönflies theorem fails, but the triangulation theorem still holds. Thus we should avoid creating the impression that the latter depends on the former.” In the same vein, it can be observed that in our non-metric two-dimensional context the “reverse situation” occurs: the Schönflies theorem holds, but the triangulation theorem fails (dramatically).

★ A long standing question of Spivak and Nyikos. It is a natural problem to wonder if any surface admits a smooth structure (cf. Spivak [40] page A-18): “I do not know whether every 2-manifold has a $C^\infty$ structure.” and Nyikos [30] p.108: “Are there 2-manifolds and 3-manifolds that do not admit smoothings?”). In the metric (two-dimensional) case the answer is positive either by using methods of Riemann surface theory [29] or by “softer” DIFF methods, albeit an explicit reference seems difficult to locate (as deplored by Remmert-Schneider [36] p. 190): “Erstaunlicherweise scheint hierfür kein direkter Beweis in der Literatur zu existieren.” Does Siebenmann’s existence of a PL structure for non-metric surfaces (cf. e.g. [43, Thm 2.11, p. 47]) bring us closer to a positive answer to the Spivak-Nyikos existence question?

Finishing the Proof of Proposition [6]. Once a tube $T$ around $J$ is available, the proof reduces to homological routines (exact sequence of a pair plus excision). The sequence of the pair $(M, M − J)$ reads (coefficients are taken in $\mathbb{Z}$ and subscripts are the ranks of the homology groups, whose finiteness will be soon evident):

$$H_1(M) \to H_1(M, M − J)_s \to H_0(M − J)_r \to H_0(M)_1 \to H_0(M, M − J).$$

Both groups at the extremities vanish (recall that the first integral homology is the abelianisation of the fundamental group). Therefore $r = s + 1$ (additivity of the rank). We shall use excision to compute $s$ (cf. e.g. [33] Thm 2.11, p.47). By excising the complement of the tube $T$ from the pair $(M, M − J)$, we get an isomorphism $H_1(T, T − J) \approx H_1(M, M − J)$. In turn we may interpret $(T, T − J)$ as the result of excising the two poles of a 2-sphere ($S^2, J$), where $J$ is standardly embedded as the equator; yielding an isomorphism $H_1(T, T − J) \approx H_1(S^2, S^2 − J)$. Writing the sequence of the pair $(S^2, S^2 − J)$ as: $0 = H_1(S^2) \to H_1(S^2, S^2 − J)_s \to H_0(S^2 − J)_2 \to H_0(S^2)_1 \to H_0(S^2, S^2 − J) = 0$, we see that $s = 1$. Hence $r = 2$, completing the proof that $M$ is dichotomic.

Alternatively using reduced homology, as $H_0(M) = 0 \approx H_0(S^2)$, we obtain isomorphisms

$$\tilde{H}_0(M − J) \approx H_1(M, M − J) \approx H_1(T, T − J) \approx H_1(S^2, S^2 − J) \approx \tilde{H}_0(S^2 − J) \approx \mathbb{Z}.$$

The last clause follows easily from the existence of the tube $T$.

29 Cf. M. Heins [19], who (improving works of Stoilow) shows the existence of a complex-analytic structure on any metric orientable surface; hence via the two-fold orientation covering trick, one gets a DIFF structure on any metric surface (in reality one gets much more, namely a so-called “Klein surface” or “dianalytic structure”, much studied by Alling-Greenleaf, etc.).
4 Generalised (non-metric) Schoenflies theorem

Proposition 9 (Non-metric Schoenflies theorem) Let $M$ be a Hausdorff simply-connected surface. Then $M$ is Schoenflies\footnote{Irreducible would perhaps be a more neutral terminology.}, i.e. each embedded circle $J$ in $M$ bounds a 2-disc in $M$.

Proof. No loss of generality results in assuming $M$ to be connected. If $M$ is metric, then Lemma \ref{lemma} implies that $M$ is either $\mathbb{R}^2$ or $\mathbb{S}^2$, and the conclusion is given by the classical $(ST)^\circ$.

So assume that $M$ is non-metric. By Proposition\ref{proposition} $M - J$ has two components, one of which must be non-metric. [If both components were metric, then $M$ could be expressed as the union of those plus $J$ so would be Lindelöf, hence metric.] Pick a non-metric component of $M - J$, and call it the exterior of $J$ (denoted by $J_{ext}$). Call the other component the interior of $J$ (denote it $J_{int}$). Define $W_{int}$ and $W_{ext}$ by adding $J$ to $J_{int}$ and $J_{ext}$ respectively. It is easy to check that both these $W$’s are surfaces-with-boundary (this is a local question which can be handled via the metric version of (ST), compare Lemma\ref{lemma} (ii)).

The sequel depends on the following homological compactness criterion:

Lemma 10 A connected Hausdorff surface-with-boundary $W$ such that $H_1(W) = 0$ and with boundary $\partial W \approx \mathbb{S}^1$ is compact.

Proof. Notice first that the conclusion is easy to corrupt without Hausdorff: consider a 2-disc with infinitely many origins (which is not quasi-compact). Recall from Samelson \ref{lemma} Lemma D that a connected Hausdorff noncompact $n$-manifold $M^n$ has a vanishing top-dimensional (singular) homology, i.e. $H_n(M) = 0$. (Note that Samelson’s proof does not employ any metric assumption.)

Consider the double $2W =: M = W \cup W_s$, where $W_s$ is a copy of $W$. By the Mayer-Vietoris sequence:

$$\ldots \to H_2(W) \oplus H_2(W_s) \to H_2(M) \to H_1(\partial W = W \cap W_s) \to H_1(W) \oplus H_1(W_s) \to \ldots$$

Since the last groups are zero by assumption, $H_2(M)$ surjects onto the nontrivial $H_1(\partial W)$, so is itself non-zero. By the aforementioned (Samelson’s Lemma D) it follows that $M$ is compact, hence $W$ is also compact (because $W$ is closed in $M$).

Now since $W_{ext}$ is non-compact (else its interior would be metric), Lemma\ref{lemma} implies $H_1(W_{ext}) \neq 0$. Write the Mayer-Vietoris sequence of the decomposition $M = W_{int} \cup W_{ext}$, and set $U = W_{int}$ and $V = W_{ext}$ to simplify notation (one should work with open sets obtained by slight collared enlargements of the two $W$’s):

$$\ldots \to H_2(U \cup V) \to H_1(U \cap V) \to H_1(U) \oplus H_1(V) \to H_1(U \cup V) \to \ldots$$

By Samelson’s Lemma D, we have $H_2(U \cup V) = 0$ since $U \cup V = M$ is non-metric hence non-compact. Moreover $H_1(U \cup V) = 0$ as $M$ is assumed to be simply-connected. So exactness gives an isomorphism $H_1(U \cap V) \approx H_1(U) \oplus H_1(V)$. Now as $U \cap V = J \approx \mathbb{S}^1$, \footnote{Compare [Schoenflies, 1906] “versus” [Osgood, 1903]; a thoroughgoing account is to be found in Siebenmann \ref{historical_notes} §4, HISTORICAL NOTES., where the contributions coming from “pure topology” ([Schoenflies, 1906], [Tietze, 1913, 1914], [Antoine, 1921], [R.L. Moore, 1926], [Keldysh, 1966], ...) are analysed, and compared with those coming from “complex analytic methods” ([Osgood, 1903], [Carathéodory, 1913, 1913, 1913], [Koebe, 1913, 1913, 1915], [Osgood-Taylor, 1913], [Study, 1913], ...).}
the first group is \( \mathbb{Z} \). Recalling that \( H_1(V) \neq 0 \), it follows (from the indecomposability of \( \mathbb{Z} \) as a sum of abelian groups) that \( H_1(U) = 0 \). A second application of Lemma 10 shows that \( U = W_{\text{int}} \) is compact. Summarising \( U \) is a connected compact surface-with-boundary with one boundary component and \( H_1(U) = 0 \) (so \( \chi(U) = 1 - 0 + 0 = 1 \) and which is orientable (being embedded in the simply-connected surface \( M \)). The classification of compact surfaces tell us that \( U \) must be the 2-disc, which completes the proof. ■

5 A converse to the non-metric Schoenflies theorem

The purpose of this section is to provide a converse to Proposition 9, i.e. to show the following.

**Proposition 11** Suppose that \( M \) is a Hausdorff surface. If each embedded circle \( J \) in \( M \) bounds a 2-disc in \( M \) then \( M \) is simply-connected.

**Proof.** It is enough to show that if \( \lambda: [0, 1] \to M \) is a non-constant loop in \( M \) then \( \lambda \) is homotopic modulo \( \{0, 1\} \) to an embedded circle. Suppose given such a loop \( \lambda: [0, 1] \to M \). As \( \lambda([0, 1]) \) is compact it may be covered by finitely many coordinate charts, hence lies in a metrisable surface. Like every metrisable surface, this surface is the geometric realisation of a simplicial complex, say \( K \); see for example [25, p. 60] or [34]. We may assume that \( \lambda(0) = \lambda(1) \) is a vertex of \( K \). By the Simplicial Approximation Theorem, see for example [37, Theorem 1.6.11, p. 31], \( \lambda \) is homotopic modulo the base point to a simplicial approximation \( \mu: [0, 1] \to |K| \) to \( \lambda \) such that \( \mu(0) = \mu(1) = \lambda(1) \). Moreover, by General Position, [37, Theorem 1.6.10] we may assume that \( \mu \) is in general position, so that its singular point set is discrete. Thus there is a partition \( \{0 = t_0 < t_1 < \ldots < t_n = 1\} \) of \( [0, 1] \) consisting solely of the singular points of \( \mu \).

For each \( i = 1, \ldots, n \) either \( \mu|[t_{i-1}, t_i] \) is an embedding or \( \mu(t_{i-1}) = \mu(t_i) \) and \( \mu|[t_{i-1}, t_i] \) is an embedding. In the latter case \( \mu|[t_{i-1}, t_i] \) is an embedded circle so by hypothesis bounds a 2-disc in \( M \). We may use this 2-disc to find a homotopy fixing the end points from \( \mu \) to a loop which agrees with \( \mu \) on \( [0, t_{i-1}] \cup [t_i, 1] \) and is constant on \( [t_{i-1}, t_i] \), then further homotope modulo the end points to a simplicial map which agrees with \( \mu \) on \( [0, t_{i-1}] \cup [t_{i+1}, 1] \) and embeds \( (t_{i-2}, t_{i+1}) \) onto \( \mu((t_{i-2}, t_{i-1}] \cup [t_i, t_{i+1})) \) (with \( t_{i-2} \) replaced by 0 if \( i = 1 \) and \( t_{i+1} \) replaced by 1 if \( i = n \)). Repeating this procedure eventually we reach a loop \( \nu \) which is homotopic to \( \mu \), hence \( \lambda \), modulo the end points and is such that \( \nu([0, 1]) \) is an embedded circle, as required. ■

6 Dynamical applications of Jordan and Schoenflies

Since non-metric manifolds cannot support minimal flows, it is more reasonable to ask: which manifold admits a transitive resp. a non-singular flow (in short a brushing)? The well-known paradigms to the effect that Jordan separation (dichotomy) obstructs transitivity, while Schoenfliesness (more accurately non-vanishing Euler characteristic) impedes brushability, extend beyond the metric (resp. compact) case. Let us be more precise.

The non-metric Jordan theorem (Proposition 5) supplies food to the following “Bendixson type” result:

**Proposition 12** A dichotomic surface (i.e. divided by any embedded circle) cannot support a transitive flow.
Proof. It is a minor adaptation of the classical Bendixson bag argument. Assume by contradiction that there is a point $x$ in the surface $S$ with a dense orbit under a flow $f$. We may draw a cross-section $\Sigma_x$ through $x$ and consider an associated flow-box $f([-\epsilon, \epsilon] \times \Sigma_x)$. Note that the Whitney–Bebutov theory classically stated under a metric assumption [26, p. 333], holds more universally, since the orbit $f(\mathbb{R} \times V)$ of a chart $V$ is Lindelöf. The point $x$ must eventually return to $\Sigma_x$, and we call $x_1$ its first return to $\Sigma_x$. The piece of trajectory from $x$ to $x_1$ closed up by the arc $A$ of $\Sigma_x$ joining $x$ to $x_1$ defines a Jordan curve $J$ on $S$. It is easy to check that the component of $S - J$ containing the near future of $x_1$ (e.g. $f(\epsilon/2, x_1)$) contains in fact the full future of $x_1$. Conclude by noticing that the “short past” of the arc $A$ namely the set $f([-\epsilon, 0] \times \text{int}A)$ is an open subrectangle which cannot intersect the orbit of $x$.

In view of Proposition any simply-connected surface is dichotomic, hence intransitive. Examples include the (original) Prüfer surface described in Radó [34], the Moore surface, the Maungakiekie surface (which is a plane out of which emanates a long ray). A non simply-connected example is the doubled Prüfer surface $2P$ (of Calabi-Rosenlicht, cf. e.g. [5, Example 4.4]), which is clearly dichotomic, hence intransitive.

Schoenflies also has an obvious dynamical implication in relation with its immediate successor Brouwer. Indeed on a Schoenflies surface as soon as a flow line closes up into a periodic orbit, a fixed point is created somewhere (Brouwer’s fixed-point theorem applied to the bounding disc). Of course Schoenfliesness alone is not enough to ensure the presence of a periodic orbit (consider the plane $\mathbb{R}^2$ or the semi-long plane $\mathbb{R} \times \mathbb{L}$ “brushed” along the first factor). However the same condition of $\omega$-boundedness as the one occurring in Nyikos’ Bagpipe theorem, ensures that one will find in the compact closure of an orbit a minimal set (Zorn’s lemma argument), which must be either a point or a periodic orbit (by the Poincaré-Bendixson argument). So picturesquely the motion spirals towards a cycle limite. Hence we get:

Proposition 13 On an $\omega$-bounded, Schoenflies (equivalently simply-connected) surface any flow exhibits a fixed point.

This may be regarded as a non-metric pendant to the “hairy ball theorem” (the 2-sphere cannot be foliated nor brushed). The proposition applies for instance to the long plane $\mathbb{L}^2$ (in which case an alternative proof may also be deduced from the classification of foliations on $\mathbb{L}^2$ given in [5]). It also applies to any space obtained from a Nyikos long pipe, [29], by capping off the short end by a 2-disc, for example the long glass, i.e. the semi-long cylinder $\mathbb{S}^1 \times$ (closed long ray) capped off by a 2-disc.

A more systematic study of the dynamics of non-metric manifolds should appear in a forthcoming paper [14].

References

[1] L. V. Ahlfors and L. Sario, Riemann Surfaces, Princeton Univ. Press, Princeton, N. J., 1960. (Second Printing, 1965.)
[2] J. W. Alexander, A proof and extension of the Jordan–Brouwer separation theorem, Trans. Amer. Math. Soc. 23 (1922), 333–349.
[3] R. Baer, Isotopie von Kurven auf orientierbaren, geschlossenen Flächen und ihr Zusammenhang mit der topologischen Deformation der Flächen, J. reine angew. Math. 159 (1928), 101–116.

32 Recall the fact that the product of a $\sigma$-compact with a Lindelöf space is Lindelöf.
[4] M. Baillif and A. Gabard, *Manifolds: Hausdorffness versus homogeneity*, Proc. Amer. Math. Soc. 136 (2008), 1105–1111.

[5] M. Baillif, A. Gabard and D. Gauld, *Foliations on non-metrizable manifolds: absorption by a Cantor black hole*, arXiv (2009).

[6] K. Borsuk, *Families of compacta and some theorems on sweeping*, Fund. Math. 42 (1955), 240–258.

[7] N. Bourbaki, *General Topology*, Chapters 1–4, Springer, 1989.

[8] L. E. J. Brouwer, *Beweis des Jordanschen Kurvensatzes*, Math. Ann. 69 (1910), 169–175.

[9] M. Brown, *The monotone union of open n-cells is an open n-cell*, Proc. Amer. Math. Soc. 12 (1961), 812–814.

[10] R. J. Cannon, Jr., *Quasiconformal structures and the metrization of 2-manifolds*, Trans. Amer. Math. Soc. 135 (1969), 95–103.

[11] C. Carathéodory, *Bemerkung über die Definition der Riemannschen Flächen*, Math. Z. 52 (1950), 703–708.

[12] P. H. Doyle and D. A. Moran, *A short proof that compact 2-manifolds can be triangulated*, Invent. Math. 5 (1968), 160–162.

[13] D. B. A. Epstein, *Curves on 2-manifolds and isotopies*, Acta Math. 115 (1966), 83–107.

[14] A. Gabard and D. Gauld, *Dynamics of manifolds: beyond the metric case*, in preparation.

[15] D. Gauld, *Metrisability of manifolds*, arXiv (2009).

[16] J. Gray, *On the history of the Riemann mapping theorem*, Rend. Circ. Mat. Palermo (2) Suppl. N. 34 (1994), 47–94.

[17] J. Guenot et R. Narasimhan, *Introduction à la théorie des surfaces de Riemann*, Monogr. Ens. Math. 23, 1976.

[18] F. Hausdorff, *Metrische und topologische Räume*. Nachlass Hausdorff: Kapsel 33: Fasz. 223. Greifswald, 25.5.1915. In: *Gesammelte Werke, Band IV, Analysis, Algebra und Zahlentheorie*. Herausgegeben von S. D. Chatterji, R. Remmert und W. Scharlau. Springer-Verlag, 2002.

[19] M. Heins, *Interior mapping of an orientable surface in $S^2$*, Proc. Amer. Math. Soc. 2 (1951), 951–952.

[20] M. W. Hirsch, *Differential Topology*, Graduate Texts in Mathematics 33, Springer, 1976.

[21] F. Klein, *Über Riemanns Theorie der algebraischen Funktionen und ihrer Integrale*. B. G. Teubner, Leipzig, 1882. Also in: *Gesammelte mathematische Abhandlungen, Band III*. Herausgegeben von R. Fricke, H. Vermeil und E. Bessel-Hagen. Springer-Verlag, 1923. (Reprint, 1973).

[22] A. Marden, I. Richards and B. Rodin, *On the regions bounded by homotopic curves*, Pacific J. Math. 16 (1966), 337–339.

[23] W. S. Massey, *Algebraic Topology: An Introduction*, Graduate Texts in Mathematics 56, Springer-Verlag, 1967 (New Printing, 1977).

[24] J. Milnor, *Foliations and foliated vector bundles*, Notes from lectures given at MIT, Fall 1969. File located at http://www.foliations.org/surveys/FoliationLectNotes_Milnor.pdf

[25] E. E. Moise, *Geometric Topology in Dimensions 2 and 3*, Graduate Texts in Mathematics 47, Springer-Verlag, New York, Heidelberg, Berlin, 1977.

[26] V. V. Nemytskii and V. V. Stepanov, *The Qualitative Theory of Differential Equations*, Dover 1989; republication of the Princeton edition, 1960.

[27] P. J. Nyikos, *The topological structure of the tangent and cotangent bundles on the long line*, Topology Proceedings 4 (1979), 271–276.

[28] P. J. Nyikos, *Set-theoretic topology of manifolds*. In: *General Topology and its Relations to Modern Analysis and Algebra V*, Proc. Fifth Topol. Symp. 1981, J. Novak (ed.), 513–526, Heldermann Verlag, Berlin, 1982.
[29] P. J. Nyikos, *The theory of nonmetrizable manifolds*. In: *Handbook of Set-theoretic Topology*, 633–684, North-Holland, Amsterdam, 1984.

[30] P. J. Nyikos, *Mary Ellen Rudin’s contributions to the theory of nonmetrizable manifolds*, Ann. New York Acad. Sci. (1993), 92–113.

[31] H. Poincaré, *Sur un théorème de la Théorie générale des fonctions*, Bull. Soc. Math. France 11 (1883), 112–125.

[32] T. Radó, *Bemerkung zur Arbeit des Herrn Bieberbach: Über die Einordnung des Hauptsatzes der Uniformisierung in der Weierstraßsche Funktionentheorie* (Math. Annalen 78), Math. Ann. 90 (1923), 30–37.

[33] T. Radó, *Sur la représentation conforme des domaines variables*, Acta Univ. Szeged 1 (1923), 180–186.

[34] T. Radó, *Über den Begriff der Riemannschen Fläche*, Acta Univ. Szeged 2 (1925), 101–121.

[35] H. Reichardt, *Lösch der Aufgabe 274*, Jahresb. Deutsch. Math.-Verein. 51 (1941), 23–24.

[36] R. Remmert und M. Schneider, *Analysis Situs und Flächentheorie*. In: Hermann Weyl, *Die Idee der Riemannschen Fläche*, B. G. Teubner Verlagsgesellschaft, Stuttgart, Leipzig, 1997 (annotated re-edition of the first edition of [45]), 183–195.

[37] T. B. Rushing, *Topological Embeddings*, Pure and Applied Math. 52, Academic Press, New York and London, 1973.

[38] H. Samelson, *On Poincaré duality*, J. Anal. Math. 14 (1965), 323–336.

[39] L. Siebenmann, *The Osgood-Schoenflies theorem revisited*, Russian Math. Surveys 60 (2005), 645–672. In fact we refer rather to the online version available in the Hopf archive: [http://hopf.math.purdue.edu/cgi-bin/generate?/Siebenmann/Schoen-02Sept2005](http://hopf.math.purdue.edu/cgi-bin/generate?/Siebenmann/Schoen-02Sept2005) (from which a number of misprints in the printed version have been removed.)

[40] M. Spivak, *A Comprehensive Introduction to Differential Geometry. Vol. One.*, Published by M. Spivak, Brandeis Univ., Waltham, Mass. 1970. (See especially Appendix A.)

[41] P. Ullrich, *The Poincaré-Volterra theorem: from hyperelliptic integrals to manifolds with countable topology*, Arch. Hist. Exact Sci. 54 (2000), 375–402.

[42] O. Veblen, *Theory of plane curves in non-metrical analysis situs*, Trans. Amer. Math. Soc. 6 (1905), 83–98.

[43] J. W. Vick, *Homology Theory: An Introduction to Algebraic Topology*, Second Edition, Graduate Texts in Mathematics 145, Springer-Verlag, 1994.

[44] B. L. van der Waerden, *Aufgabe 274. Eine elementare kombinatorisch-topologische Aufgabe, deren Lösung für eine einfache Begründung der Uniformisierungstheorie von großer Bedeutung ist.* Jahresb. Deutsch. Math.-Verein. 49 (1939), 1–1.

[45] H. Weyl, *Die Idee der Riemannschen Fläche*, B. G. Teubner, Leipzig, 1913.

Alexandre Gabard  
Université de Genève  
Section de Mathématiques  
2-4 rue du Lièvre, CP 64  
CH-1211 Genève 4  
Switzerland  
alexandregabard@hotmail.com

David Gauld  
Department of Mathematics  
The University of Auckland  
Private Bag 92019  
Auckland  
New Zealand  
d.gauld@auckland.ac.nz