ON ∗-REPRESENTATIONS OF A CERTAIN CLASS OF ALGEBRAS RELATED TO A GRAPH

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Abstract. We study families of self-adjoint operators with given spectra whose sum is a scalar operator. Such families are ∗-representations of certain algebras which can be described in terms of graphs and positive functions on them. The main result is that in the cases where the graph is one of the extended Dynkin graphs ˜D₄, ˜E₆, ˜E₇ or ˜E₈, all irreducible ∗-representations of the corresponding algebra are finite-dimensional. To prove this fact, we introduce the notion of invariant functional on a graph and give their description.

INTRODUCTION

Let H be a separable Hilbert space, and let A₁, . . . , Aₙ be a family of self-adjoint operators in H with fixed finite spectra, such that A₁ + · · · + Aₙ = λI for some λ ∈ ℜ. Such family of operators can be treated as a representation of certain ∗-algebra with finite number of generators and polynomial relations. The corresponding algebras were studied in a number of recent papers (see [VMS05] and the bibliography therein). The interest to such algebras is due to their relations with deformed preprojective algebras (see, e.g., [CBH98]), representations of quivers (see [KR05]), Horn problem (see [KPS05]), integral operators (see [Vas98]) etc.

Considering a family of self-adjoint operators with a pre-defined spectra, for which A₁ + · · · + Aₙ = λI we can assume that λ > 0, and

σ(Aᵢ) ⊂ Mᵢ = \{0 = αₖᵢ^{(0)} < α₁ᵢ^{(1)} < · · · < αₖᵢ^{(kᵢ)}\}, l = 1, . . . , n.

To study such properties of operators, it is convenient to consider them as ∗-representations of a certain ∗-algebra. Following [VMS05] (see also [MSZ04] and references therein) consider a simply-laced non-oriented graph Γ consisting of n branches, such that l-th branch has kᵢ + 1 vertices, l = 1, . . . , n, and all branches are connected at a single root vertex. Marking the vertices of l-th branch (excluding the root vertex) by positive numbers (α_j^{(l)})_{j=1}^{kᵢ} increasing to the root, we get a function

χ = (α₁^{(1)}, . . . , α_{k₁}^{(1)}; . . . ; α₁^{(n)}, . . . , α_{kₙ}^{(n)})

on the graph Γ defined in all vertices except the root (below this function will be called a character on Γ). The root vertex will be marked by the number λ (notice that the term character in many papers is used to denote a function (χ, λ) on the
whole graph, including the root vertex. For our needs the given notation is more convenient.

Given a graph \( \Gamma \), a character \( \chi \) on \( \Gamma \) and a positive number \( \lambda \), on can construct the following \(*\)-algebra

\[
\mathcal{A}_{\Gamma, \chi, \lambda} = \mathbb{C} \langle a_l = a_l^*, l = 1, \ldots, n \mid p_l(a_l) = 0, l = 1, \ldots, n; \sum_{l=1}^n a_l = \lambda e \rangle,
\]

where \( p_l(x) = x(x - \alpha_{k_1}^{(l)}) \cdots (x - \alpha_{k_t}^{(l)}) \), \( k = 1, \ldots, n \). Then the family \( A_1, \ldots, A_n \) is an \(*\)-representation of \( \mathcal{A}_{\Gamma, \chi, \lambda} \).

Properties of the algebra \( \mathcal{A}_{\Gamma, \chi, \lambda} \), in particular, the structure of its \(*\)-representations, crucially depend on the type of the graph \( \Gamma \): they are quite different for the cases where \( \Gamma \) is a Dynkin graph, extended Dynkin graph or none of them.

If \( \Gamma \) is a Dynkin graph then the corresponding algebra is finite-dimensional and therefore has only finite number of irreducible \(*\)-representations, and all of them are finite-dimensional. In this case, sets of parameters \((\chi, \lambda)\), for which there exist \(*\)-representations, and the \(*\)-representations themselves are described in [KPS05, SZ05].

If \( \Gamma \) is an extended Dynkin graph, then the algebra is infinite-dimensional of polynomial growth [VMS05]. In this case, there exists a special character \( \chi_\Gamma \) on the corresponding extended Dynkin graph \( \Gamma \). For such special characters, sets \( \Sigma_{\Gamma, \chi_\Gamma} \) of those \( \lambda \), for which there exist representations of the corresponding algebra and the corresponding \(*\)-representations were studied in [OS99, MRS04, MSZ04] etc. In particular, it follows that for the special characters, all irreducible \(*\)-representations are finite-dimensional.

In Section 3 we prove that all irreducible \(*\)-representations of \( \mathcal{A}_{\Gamma, \chi, \lambda} \) are finite-dimensional for any \( \chi \) and \( \gamma \) provided that \( \Gamma \) is an extended Dynkin graph. As a corollary we get that for \( \lambda \neq \omega(\chi) \) the rigidity index for irreducible representation is equal to 2.

To study the properties of \(*\)-representations of \( \mathcal{A}_{\Gamma, \chi, \lambda} \) in the case of arbitrary characters and more general graphs we introduce the notion of invariant functional on the set of characters. We show that such invariant functional is unique if and only if the graph is an extended Dynkin graph; for Dynkin graph there are no invariant functionals, and for other graphs there are exactly two invariant functionals (Section 2). These invariant functionals are described in terms of solutions of the equation

\[
n - s = \sum_{l=1}^n \frac{1}{1 + (s - 1) + \cdots + (s - 1)^{k_l}}, \quad s \geq 1.
\]

In Section 1 we show that this equation has unique solution \( s = 2 \) if and only if \( \Gamma \) is an extended Dynkin graph. For ordinary Dynkin graph this equation has no solutions \( s \geq 1 \), and for other graphs it has two solutions \( 1 < s_1 < 2 < s_2 < n \).

The main tool to study \(*\)-representations of \( \mathcal{A}_{\Gamma, \chi, \lambda} \) are reflection (Coxeter) functors introduced in [Kru02] (for non-involutive case, see [GP71]). Namely there exist two functors, \( S: \text{Rep} \mathcal{A}_{\Gamma, \chi, \lambda} \to \text{Rep} \mathcal{A}_{\Gamma', \chi', \lambda'} \) and \( T: \text{Rep} \mathcal{A}_{\Gamma, \chi, \lambda} \to \text{Rep} \mathcal{A}_{\Gamma'', \chi'', \lambda} \), where

\[
\chi' = (\alpha_{k_1}^{(1)} - \alpha_{k_1-1}^{(1)}, \ldots, \alpha_{k_l}^{(1)} - \alpha_0^{(1)}, \ldots, \alpha_k^{(n)} - \alpha_{k-1}^{(n)}, \ldots, \alpha_{k_l}^{(n)} - \alpha_0^{(n)}),
\]

\[
\lambda' = \alpha_{k_1}^{(1)} + \cdots + \alpha_{k_n}^{(n)} - \lambda,
\]

\[
\chi'' = (\lambda - \alpha_{k_1}^{(1)}, \ldots, \lambda - \alpha_{k_l}^{(1)}, \ldots, \lambda - \alpha_{k_n}^{(n)}, \ldots, \lambda - \alpha_{k_l}^{(n)}).
\]
The action of these functors on \(*\)-representations gives rise to the action on pairs, \(S\): \((\chi, \lambda) \mapsto (\chi', \lambda')\), \(T\): \((\chi, \lambda) \mapsto (\chi'', \lambda)\). This action is extensively used below.

1. Equation which distinguishes the type of a graph

Let \(n\) and \(k_l, l = 1, \ldots, n\) be given natural numbers, and let \(\Gamma\) be the corresponding graph. In what follows, we will use solutions of the following equation

\[
(1) \quad n - s = \sum_{l=1}^{n} \frac{1}{1 + (s - 1) + \cdots + (s - 1)^{k_l}}, \quad s \geq 1.
\]

**Theorem 1.** The equation \((1)\) have no solutions on \([1, \infty)\) if and only if the corresponding graph \(\Gamma\) is one of the Dynkin graphs \(A_d, d \geq 1, D_4, d \geq 4, E_6, E_7,\) or \(E_8\). The equation \((1)\) has a unique solution on \([1, \infty)\) if and only if the corresponding graph \(\Gamma\) is one of the extended Dynkin graphs \(\tilde{D}_4, \tilde{E}_6, \tilde{E}_7,\) or \(\tilde{E}_8\), this solution is \(s = 2\). In all other cases (i.e., where \(\Gamma\) is neither a Dynkin graph nor an extended Dynkin graph), the equation \((1)\) has on \([1, \infty)\) two solutions \(1 < s_1 < 2 < s_2 < n\).

**Proof.** Consider auxiliary functions \(\phi_k(x) = (1 + x + \cdots + x^k)^{-1}\), so that \((1)\) takes the form

\[
f_\Gamma(s) = n - s - \sum_{l=1}^{n} \phi_{k_l}(s - 1) = 0.
\]

Calculations give the following formula

\[
\phi''_k(x) = (k+1)x^{k-1}(kx^{k-1} + 2(k-1)x^{k-2} + \cdots + (k-1)2x + k)\phi'^3_k(x),
\]

which implies that \(f''_\Gamma(s) < 0, s > 1\).

For each of the extended Dynkin graphs, a direct calculation shows that \(f_\Gamma(2) = 0\) and \(f''_\Gamma(2) = 0\), which gives the uniqueness of the solution for the case of extended Dynkin graphs.

Each of the Dynkin graphs \(D_4, E_6, E_7, E_8\) is a subgraph of the corresponding extended Dynkin graph, which gives inequalities for the corresponding functions, \(f_{D_4}(s) < f_{\tilde{D}_4}(s), f_{E_6}(s) < f_{\tilde{E}_6}(s), f_{E_7}(s) < f_{\tilde{E}_7}(s), f_{E_8}(s) < f_{\tilde{E}_8}(s)\) for \(s \geq 1\). For the graphs \(A_n, n \geq 1,\) and \(D_n, n > 4,\) one checks directly the inequality \(f_\Gamma(s) < 0, s \geq 0,\) where \(\Gamma\) is one of these graphs.

Let \(\Gamma\) be neither Dynkin graph, nor extended Dynkin graph. Then \(\Gamma\) contains subgraph \(\Gamma_0\) which is an extended Dynkin graph, and therefore, the inequality \(f_\Gamma(s) < f_{\Gamma_0}(s)\) holds for \(s \geq 1,\) in particular, \(f_\Gamma(2) > 0\). To complete the proof notice that \(f_\Gamma(1) = -1\) and \(f_\Gamma(n) < 0,\) and \(f''(s) < 0\) guarantees that there are only two solutions on the interval \([1, \infty)\). \(\square\)

2. Invariant functionals on graphs

Let \(\Gamma\) be a graph formed by \(n\) branches connected in a single root vertex, and let \(k_l\) be the number of vertices (excluding root) in \(l\)-th branch. Let \(\chi\) be a character on the graph \(\Gamma\),

\[
(2) \quad \chi = (\alpha_1^{(1)}, \ldots, \alpha_{k_1}^{(1)}; \ldots; \alpha_1^{(n)}, \ldots, \alpha_{k_n}^{(n)}), \quad 0 < \alpha_1^{(l)} < \cdots < \alpha_{k_l}^{(l)}, \quad l = 1, \ldots, n.
\]

Let \(\omega(\cdot)\) be a linear functional, which takes non-negative values on characters.
Definition 1. We say that \( \omega(\cdot) \) is invariant with respect to the functor \( TS \), if

\[
TS(\chi, \omega(\chi)) = (\tilde{\chi}, \tilde{\omega}(\tilde{\chi}))
\]

for any character \( \chi \) on \( \Gamma \).

Theorem 2. If \( \Gamma \) is a Dynkin graph (one of \( D_4, d \geq 4, E_6, E_7 \) or \( E_8 \)), then there are no invariant functionals on the set of its characters.

If \( \Gamma \) is an extended Dynkin Graph (one of \( D_4, E_6, E_7 \) or \( E_8 \)), then there exists unique invariant functional:

- for \( D_4 \), \( \omega(\alpha; \beta; \gamma; \delta) = \frac{1}{2}(\alpha + \beta + \gamma + \delta) \);
- for \( E_6 \), \( \omega(\alpha_1, \alpha_2; \beta_1, \beta_2; \gamma_1, \gamma_2) = \frac{1}{3}(\alpha_1 + \alpha_2 + \beta_1 + \beta_2 + \gamma_1 + \gamma_2) \);
- For \( E_7 \), \( \omega(\alpha_1, \alpha_2, \alpha_3; \beta_1, \beta_2, \beta_3; \gamma) = \frac{1}{4}(\alpha_1 + \alpha_2 + \alpha_3 + \beta_1 + \beta_2 + \beta_3 + 2\gamma) \);
- for \( E_8 \), \( \omega(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5; \beta_1, \beta_2; \gamma) = \frac{1}{6}(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + 2\beta_1 + 2\beta_2 + 3\gamma) \).

In the case where \( \Gamma \) is neither Dynkin graph nor extended Dynkin graph, there exist two \( TS \)-invariant functionals. They are given by

\[
\omega(\chi) = \sum_{l=1}^{n} \sum_{j=1}^{k_l} a_{j}^{(l)} \alpha_{j}^{(l)}, \quad a_{j}^{(l)} \geq 0, \quad j = 1, \ldots, k_l; \quad l = 1, \ldots, n.
\]

with

\[
a_{j}^{(l)} = \frac{(s - 1)^j}{1 + (s - 1) + \cdots + (s - 1)^{k_l}}, \quad j = 1, \ldots, k_l; \quad l = 1, \ldots, n,
\]

where \( s \) is a solution of \( 1 \).

Proof. Let \( \chi \) be given by (2), then linear functional \( \omega(\cdot) \) can be represented as (3). Then \( (TS)(\chi, \omega(\chi)) = (\tilde{\chi}, \lambda) \), where

\[
\tilde{\chi} = \left( \sum_{l=1}^{n} \sum_{j=1}^{k_l} a_{j}^{(l)} \alpha_{j}^{(l)} - \alpha_{1}^{(l)} \right),
\]

\[
\tilde{\lambda} = \sum_{l=1}^{n} \sum_{j=1}^{k_l} a_{j}^{(l)} \alpha_{j}^{(l)} - \alpha_{1}^{(l)}.
\]

Then

\[
\omega(\tilde{\chi}) - \tilde{\lambda} = \sum_{l=1}^{n} \left( a_{1}^{(l)} ((1 - s)a_{1}^{(l)} + a_{2}^{(l)} + \cdots + a_{k_{l-1}}^{(l)}((1 - s)a_{k_{l-1}}^{(l)} + a_{k_{l}}^{(l)})
\]

\[
+ a_{k_{l}}^{(l)}(s - s_l - 1 + (1 - s)a_{k_{l}}^{(l)}) \right)
\]
where \( s = \sum_{i=1}^{n} s_i, \) \( s_i = \sum_{p=1}^{k_i} a_{kp}^{(l)} \). Since the latter should be zero for arbitrary choice of \( \chi \), we get the following conditions

\[
(5) \quad a_{ki}^{(l)} = (s - 1)a_{ki-1}^{(l)}, \quad \ldots, \quad a_{2}^{(l)} = (s - 1)a_{1}^{(l)},
\]

\[
(6) \quad a_{ki}^{(l)} = 1 - \frac{s_i}{s - 1}, \quad l = 1, \ldots, n,
\]

For \( s_i, \) \( l = 1, \ldots, n \) from (6) we have

\[
(7) \quad s_i = a_{1}^{(l)} + \cdots + a_{ki}^{(l)} = a_{1}^{(l)} (1 + (s - 1) + \cdots + (s - 1)^{k_i - 1})
\]

Substitute in (6) \( a_{ki}^{(l)} = (s - 1)^{k_i - 1}a_{1}^{(l)}, \) then \( (s - 1)^{k_i}a_{1}^{(l)} = s - 1 - s_i \), or

\[
(8) \quad a_{1}^{(l)} = \frac{s - 1}{1 + (s - 1) + \cdots + (s - 1)^{k_i}}
\]

Compare this with (7) we get

\[
(9) \quad s_i = 1 - \frac{1}{1 + (s - 1) + \cdots + (s - 1)^{k_i}}
\]

And taking into account that \( s = \sum_{i=1}^{n} s_i \) we finally get a conditions (11) for \( s \). Solutions of this equations are described by Theorem 1. For such \( s \), using (5), (8), we get expression (4) for \( a_{j}^{(l)} \). \( \square \)

3. Irreducible representations

Let \( \Gamma \) be an extended Dynkin graph, and let \( \chi \) be a character on it. The main result on representations of the corresponding algebra \( A_{\Gamma, \chi, \lambda} \) is the following.

**Theorem 3.** All irreducible families of operators corresponding to extended Dynkin diagrams are finite-dimensional.

**Proof.** Let \( \pi \) be an irreducible representation of the algebra \( A_{\Gamma, \chi, \gamma} \), where \( \Gamma \) is an extended Dynkin graph. We consider two cases.

1. Let \( \lambda = \omega(\chi) \). It is shown in [Mel03, VMS05] that the corresponding algebra is finite-dimensional over its center, and therefore, is a PI-algebra. This implies that the dimensions of all its irreducible representations are bounded.

2. Let \( \lambda < \omega(\chi) \). We proceed as follows. We apply the \( (ST)^n \) functors to the representation of the algebra corresponding to the pair \( (\chi, \lambda) \) to get representations of the algebras corresponding to other pairs \( (\chi_n, \lambda_n) \) and show that at some step either there cannot exist representation (in this case, there are no representations for \( (\chi, \lambda) \)), or the representation is an obvious extension of representation of a subgraph (such subgraph is a Dynkin diagram, and the corresponding algebra is finite dimensional, therefore it has a finite number of representations all of which are finite-dimensional). Then the initial representation \( \pi \) is obtained from some finitely dimensional representation of \( A_{\Gamma, \chi, \lambda} \) as a result of applying of the \( (TS)^n \) functor, and therefore, is finite dimensional as well.

Notice first that if some of the coefficients of \( \chi \) is greater than \( \lambda \), then the corresponding projection is zero. Indeed, let \( P_j, \) \( j = 1, \ldots, m \) be orthogonal projections such that \( \sum_{j=1}^{m} \alpha_j P_j = \lambda I \), where \( \alpha_j > 0, \) \( j = 1, \ldots, m, \) and \( \alpha_k > \lambda \) for some \( k \). Then \( \sum_{j \neq k} \alpha_j P_j P_k P_j = (\lambda - \alpha_k) P_k \) which is possible only for \( P_k = 0 \) since the left-hand side of the latter equality is non-negative, but the right-hand side is non-positive.
In the case where some of the coefficients of $\chi$ are equal to $\lambda$, the corresponding projection commutes with all other projections and therefore, is either identity (in this case all other projections are zero), or zero.

Thus, in both these cases, the representation is in fact a representation of subalgebra in $A_{\Gamma,\chi,\lambda}$ corresponding to some subgraph.

Now let all coefficients of $\chi_k$, $k \leq n$ be positive, and some coefficient of $\chi_{n+1}$ be negative or zero. Taking into account the way the functors act on characters, we easily see that this means that the corresponding coefficient of $\chi_n$ is greater or equal that $\lambda_n$.

To complete the proof, it is now sufficient to show that for any $\lambda < \omega(\chi)$ there exists such $n$, that some coefficient of $\chi_n$ is negative or zero.

Let $\chi_1$ be the special character of the corresponding graph, and let $\omega_1 = \omega(\chi_1)$.

If we norm $\chi$ such that $\omega(\chi) = \omega_1$, then the character can be represented as $\chi = \chi_1 + \tilde{\chi}$, where $\tilde{\chi}$ is (not necessarily positive) character, such that $\omega(\tilde{\chi}) = 0$.

Also represent $\lambda = \omega_1 - \lambda$. That is:

- For $D_4$ $\chi = (1 + a_1; 1 + a_2; 1 + a_3; 1 + a_4)$, $\lambda = 2 - \gamma$, $a_1 + a_2 + a_3 + a_4 = 0$;
- For $E_6$ $\chi = (1 + a_1, 2 + a_2; 1 + b_1, 2 + b_2; 1 + c_1, 2 + c_2)$, $\lambda = 3 - \gamma$, $a_1 + a_2 + b_1 + b_2 + c_1 + c_2 = 0$;
- For $E_7$ $\chi = (1 + a_1, 2 + a_2, 3 + a_3; 1 + b_1, 2 + b_2, 3 + b_3; 2 + c_1)$, $\lambda = 4 - \gamma$, $a_1 + a_2 + a_3 + b_1 + b_2 + b_3 + 2c_1 = 0$;
- For $E_8$ $\chi = (1 + a_1, 2 + a_2, 3 + a_3, 4 + a_4, 5 + a_5; 2 + b_1, 4 + b_2; 3 + c_1)$, $\lambda = 6 - \gamma$, $a_1 + a_2 + a_3 + a_4 + a_5 + 2b_1 + 2b_2 + 3c_1 = 0$.

Using this notation, one can directly check that for any $k = 1, 2, \ldots$,

- For $D_4$ $(ST)^{2k}(\chi, \lambda) = (\chi - 2k\gamma\chi_1, \lambda - 4k\gamma)$;
- For $E_6$ $(ST)^{6k}(\chi, \lambda) = (\chi - 3k\gamma\chi_1, \lambda - 9k\gamma)$;
- For $E_7$ $(ST)^{12k}(\chi, \lambda) = (\chi - 4k\gamma\chi_1, \lambda - 16k\gamma)$;
- For $E_8$ $(ST)^{30k}(\chi, \lambda) = (\chi - 6k\gamma\chi_1, \lambda - 36k\gamma)$.

Here $\chi_1$ is the special character for the corresponding graph. The relations listed above complete the proof in the case $\lambda < \omega(\chi)$.

3. Let $\lambda > \omega(\chi)$. Apply the $S$ functor to the pair $(\chi, \lambda)$, then we get a pair $(\chi', \lambda')$ with $\lambda' < \omega(\chi')$ and the arguments above apply. \hfill $\square$

Recall that the rigidity index $\text{Kat96 SV99}$ of a family of operators $A_j$, $j = 1, \ldots, k$ in $n$-dimensional space is

\[ r = n^2(2-k) + \sum_{j=1}^{k} c(A_j) \]

where $c(A_j)$ is the dimension of a centralizer of $A_j$.

**Corollary 1.** The rigidity index is equal to 2 for all irreducible representations of $A_{\Gamma,\chi,\lambda}$, where $\Gamma$ is extended Dynkin graph and $\lambda \neq \omega(\chi)$.

**Proof.** Indeed, rigidity index is preserved by $T$ and $S$ functors. From the proof above it follows that any irreducible representation can be obtained from one-dimensional ones under the action of the Coxeter functors (in fact, it was shown that any irreducible representation is obtained from representation of some algebra related to ordinary Dynkin graph, but this process can be iterated for the subalgebra until we finish in one-dimensional space). Now the result follows from the directly verified fact that for one-dimensional representations $r = 2$. \hfill $\square$
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