COMPACT EMBEDDED MINIMAL SURFACES IN $S^2 \times S^1$

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Abstract. We prove that closed surfaces of all topological types, except for the non-orientable odd-genus ones, can be minimally embedded in $S^2 \times S^1(r)$, for arbitrary radius $r$. We illustrate it by obtaining some periodic minimal surfaces in $S^2 \times \mathbb{R}$ via conjugate constructions. The resulting surfaces can be seen as the analogy to the Schwarz P-surface in these homogeneous 3-manifolds.

1. Introduction

The geometry and topology of 3-manifolds with non-negative curvature has been an active field of research during the last century. One of the properties of these 3-manifolds is the fact that they generally admit compact (without boundary) embedded minimal surfaces. Many authors have contributed to the study of compact minimal surfaces in order to understand the geometry of the 3-manifold (see, for instance, [1, 5, 13, 20]). In 1970, Lawson [11] proved that any compact orientable surface can be minimally embedded in the constant sectional curvature 3-sphere $S^3$. Lawson’s result has also been extended to different 3-manifolds such as the Berger spheres (see [21]).

The aim of this paper is to determine which compact surfaces admit minimal embeddings in the homogeneous product 3-manifolds $S^2 \times S^1(r)$, $r > 0$, where $S^2$ denotes the constant curvature one sphere, and $S^1(r)$ is the radius $r$ circle, i.e., $S^1(r)$ has length $2\pi r$ and will be identified with $\mathbb{R}/2\pi r$. These 3-manifolds have a distinguished unit vector field $\xi(p,t) = (0,1)$, which is parallel and generates the fibers of the natural Riemannian submersion $\pi : S^2 \times S^1(r) \to S^2$. Given a vector field $X$ we will say that $X$ is horizontal if it is orthogonal to $\xi$ and vertical if it is parallel to $\xi$. Also, given a differentiable curve $\alpha$ in $S^2 \times S^1(r)$ we will say that it is horizontal (resp. vertical) if its tangent vector is pointwise horizontal (resp. vertical). We can express the curvature tensor in terms of the metric and $\xi$ as

$$R(X,Y)Z = \langle Y, Z \rangle X - \langle X, Z \rangle Y + \langle X, \xi \rangle \langle Z, \xi \rangle Y - \langle Y, \xi \rangle \langle Z, \xi \rangle X + \langle \langle X, Z \rangle (Y, \xi) - \langle Y, Z \rangle (X, \xi) \rangle \xi,$$

and hence $\text{Ric}(X) = \|X\|^2 - \langle X, \xi \rangle^2 \geq 0$ for all vector field $X$.

We will now briefly discuss known examples with low genus: First of all, the only spheres minimally immersed in $S^2 \times S^1(r)$ are the horizontal slices $S^2 \times \{t_0\}$ (it follows from lifting such a sphere to a minimal sphere in $S^2 \times \mathbb{R}$ and applying the maximum principle with respect to a foliation of $S^2 \times \mathbb{R}$ by slices), which are always embedded. This implies that the projective plane cannot be minimally

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imprinted in $S^2 \times S^1(r)$ for any $r > 0$. Moreover, they are the only compact minimal stable surfaces in $S^2 \times S^1(r)$ (cf. [22] Corollary 1).

If the Euler characteristic is zero (orientable and non-orientable) the situation is quite different:

- On the one hand, vertical helicoids in $S^2 \times \mathbb{R}$ form a 1-parameter family of minimal surfaces ruled by horizontal geodesics invariant under a 1-parameter group of ambient screw-motions, see [8]. Each vertical helicoid is determined by its pitch $\ell \in [0, +\infty)$ (i.e., the turning speed of the group of screw-motions). Hence, taking quotients by the vertical translation $(p, t) \mapsto (p, t + 2\pi \ell)$ in $S^2 \times \mathbb{R}$, vertical helicoids with appropriate pitches give rise to embedded minimal tori ($\ell = 2\pi$) and Klein bottles ($\ell = 4\pi$) in $S^2 \times S^1(r)$ for any $r > 0$. If $\ell \to +\infty$, the helicoid converges to the vertical cylinder $\Gamma \times \mathbb{R}$, $\Gamma$ being a geodesic of $S^2$.

- On the other hand, one can consider examples foliated by circles: if the centers of the circles lie in the same vertical axis ($p_0$) $\times \mathbb{R}$, then they are the rotationally invariant examples studied by Pedrosa and Ritoré [14]. They are called unduloids by analogy to the Delaunay surfaces and form a 1-parameter family of minimal surfaces in $S^2 \times \mathbb{R}$. Moreover, they are singly periodic in the $\mathbb{R}$ factor with period $T \in [0, 2\pi]$ (if $T \to 0$ the unduloid converges to a double cover of a slice, whereas if $T \to 2\pi$ the unduloid converges to the vertical cylinder).

If the centers of the circle do not lie in the same vertical axis, we get the Riemann-type examples studied by Hauswirth [6 Theorem 4.1]. They are also singly periodic and produce embedded minimal tori in $S^2 \times S^1(r)$ for all $r > 0$ (see also [19, §7] for a beautiful description). Note that, thanks to [23, Theorem 4.8], a compact minimal surface $\Sigma$ immersed in $S^2 \times S^1(r)$, $r \geq 1$, foliated by circles has index greater than or equal to one (the index is one only for $r = 1$ and $\Sigma = \Gamma \times S^1(1)$, $\Gamma \subset S^2$ being a great circle). A similar estimation of the index for $r < 1$ remains an open question (see [23, Remark 4.9]).

In the higher genus case, Rosenberg [18] constructed compact minimal surfaces in $S^2 \times S^1(r)$, for all $r > 0$, by a technique similar to Lawson’s. More specifically, he defined a polygon in $S^2 \times \mathbb{R}$ made out of vertical and horizontal geodesics depending on two parameters $\tilde{\kappa} = \tilde{d}$, $d \in \mathbb{N}$, and $\tilde{h} > 0$ (see Figure 1). Then he solved the Plateau problem with respect to this contour and obtained a minimal disk that can be reflected across its boundary in virtue of Schwarz’s reflection principle. This procedure gives a complete surface that projects to the quotient $S^2 \times S^1(r)$, $r = \tilde{h}/\tilde{d}$, as a compact embedded minimal surface. Unfortunately, it fails to be orientable as stated in [18], being a non-orientable surface of Euler characteristic $\chi = 2(1 - d)$ so its (non-orientable) genus $k = 2 - \chi = 2d$ is always even. One can solve this issue by considering $r = 2\tilde{h}/\tilde{d}$ (i.e. doubling the vertical length, which represents a 2-fold cover of the non-orientable examples), so the resulting minimal surfaces have Euler characteristic $\chi = 4(1 - d)$ and their (orientable) genus is $g = 1 - \frac{d}{2} = 2d - 1$, which turns out to be odd. When $d = 2m$ is even, Rosenberg’s examples induce one-sided compact non-orientable embedded minimal surfaces in the quotient $\mathbb{R}P^2 \times S^1(r)$ of odd genus $1 + 2m$, $m \geq 1$.

Very recently, Hoffman, Traizet and White obtained a class of properly embedded minimal surfaces in $S^2 \times \mathbb{R}$ called periodic genus $g$ helicoids. More explicitly, given a genus $g \geq 1$, a radius $r > 0$, and a helicoid $H \subset S^2 \times \mathbb{R}$ containing the horizontal geodesics $\Gamma \times \{0\}$ and $\Gamma \times \{\pm \pi r\}$ for some great circle $\Gamma \subset S^2$, [7]
Theorem 1] yields the existence of two compact orientable embedded minimal surfaces $M^+$ and $M^−$ with genus $g$ in the quotient of $S^2 × R$ by the translation $(p, t) ↦ (p, t + 2πr)$. The existence of infinitely-many non-congruent helicoids $H$ satisfying the conditions above implies the existence of infinitely-many non-congruent compact orientable embedded minimal surfaces in $S^2 × S^1(r)$ with genus $g$. The even genera examples of Hoffman-Traizet-White produce compact minimal surfaces in the quotient $RP^2 × S^1(r)$ of all orientable and non-orientable topological types, for all $r > 0$.

Finally, it is worth mentioning the work of Coutant [2, Theorem 1.0.2], where Riemann-Wei type minimal surfaces in $S^2 × R$ are constructed. Some of them give rise to compact embedded orientable minimal surfaces of arbitrary genus $g ≥ 4$ in the quotient $S^2 × S^1(r)$ for $r$ small enough.

In Section 2, we will state the main result of the paper proving that there are no compact embedded minimal surfaces in $S^2 × S^1(r)$ of the remaining topological types (i.e. non-orientable odd-genus surfaces) for any $r > 0$:

*Every compact surface but the non-orientable odd-genus ones can be minimally embedded in $S^2 × S^1(r)$ for any $r > 0.*

We will also discuss some interesting topological properties of compact minimal surfaces in $S^2 × R$ that will lead to the non-existence result.

Although we have shown that there exist compact embedded minimal examples of all allowed topological types in $S^2 × S^1(r)$, Section 3 will be devoted to obtain new very symmetric examples in $S^2 × R$ via Lawson’s technique [11], and complementing it by considering the conjugate minimal surface (see [2]) instead of the original solution to the Plateau problem for a geodesic polygon. In this case, it is well known that the boundary lines of the conjugate surface are curves of planar symmetry since the initial surface is bounded by geodesic curves (see [12] §2 for a more detailed discussion). Again, reflecting the conjugate surface across its edges will produce a complete example. This technique, known as *conjugate Plateau construction* has been applied in different situations (e.g., see [12] [15] [16]).

It is worth mentioning that a similar construction leads to classical P-Schwarz minimal surfaces in $R^3$, but our construction can be also implemented in the product space $H^2 × R$ to produce embedded triply-periodic minimal surfaces with the symmetries of a tessellation of $H^2$ by regular polygons together with a 1-parameter group of vertical translations.

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2. Topological non-existence results

We will begin by describing the intersection of two compact minimal surfaces in $S^2 × S^1(r)$. The ideas in the proof are adapted from those of Frankel [3] (see also [11] [5]). In [18] Theorem 4.3], a similar result is proved for properly embedded minimal surfaces in $S^2 × R$ by using a different approach.

**Proposition 1.** Let $Σ_1$ and $Σ_2$ be two compact minimal surfaces immersed in $S^2 × S^1(r)$. If $Σ_1 ∩ Σ_2 = ∅$, then $Σ_1$ and $Σ_2$ are two horizontal slices.

**Proof.** Let $γ: [a, b] → S^2 × S^1(r)$ be a unit-speed curve satisfying $γ(a) ∈ Σ_1$, $γ(b) ∈ Σ_2$ and minimizing the distance from $Σ_1$ to $Σ_2$. This guarantees that $γ$ is orthogonal to $Σ_1$ at $γ(a)$ and to $Σ_2$ at $γ(b)$. For any parallel vector field $X$
along $\gamma$ and orthogonal to $\gamma'$, we can produce a variation $\gamma_t$ of $\gamma = \gamma_0$ with variational field $X$. Note that $\gamma_t$ can be chosen so that $\gamma_t(a) \in \Sigma_1$ and $\gamma_t(b) \in \Sigma_2$ for all $t$, since $X$ is orthogonal to $\gamma'$. The function $\ell_X(t) = \text{Length}(\gamma_t)$ satisfies
\[ \ell_X''(0) = 0 \quad \text{and} \quad \ell_X''(0) = -\int_{\gamma} \text{Ric}(\gamma'). \]
(2.1)

The minimization property tells us that $\ell_X''(0) \geq 0$. Since $\text{Ric}(Y) \geq 0$ for any vector field $Y$ in $S^2 \times S^1(r)$ and $\text{Ric}(Y) = 0$ if and only if $Y$ is vertical (see (1.1)), we deduce from equation (2.1) that $\gamma'$ is vertical. This means that the distance $d$ from $\Sigma_1$ to $\Sigma_2$ is realized by a vertical geodesic. Hence, translating $\Sigma_1$ vertically by distance $d$, we produce a contact point at which the translated surface is locally at one side of $\Sigma_2$. The maximum principle for minimal surfaces tells us that $\Sigma_2$ coincides with the vertical translated copy of $\Sigma_1$ at distance $d$.

Finally, we will suppose that $\Sigma_1$ is not horizontal at some point $p$ and reach a contradiction. In that case, note that the previous argument guarantees the existence of a vertical segment of distance $d$ joining $p$ with a point in $\Sigma_2$, but, since this segment is not orthogonal to any of the two surfaces, we could shorten it and provide a curve from $\Sigma_1$ to $\Sigma_2$ whose length is strictly less than $d$, which is a contradiction.

As a consequence of this result, any compact embedded minimal surface in $\Sigma \subset S^2 \times S^1(r)$ is either connected or a finite union of horizontal slices. Let us now study the orientability of a compact minimal surface in $S^2 \times S^1(r)$.

**Lemma 1.** Let $\Sigma \subset S^2 \times \mathbb{R}$ be a connected properly embedded minimal surface. Then $(S^2 \times \mathbb{R}) \setminus \Sigma$ has two connected components and $\Sigma$ is orientable.

**Proof.** If $\Sigma$ is a horizontal slice, the result is trivial. Otherwise, given $t \in \mathbb{R}$, let us consider $S_t = S^2 \times \{t\}$. Then the set $\Sigma \cap S_t$ is the (non-empty) intersection of two minimal surfaces, hence it consists of a equiangular system of curves in the 2-sphere $S_t$. We will denote by $C_t$ the set of intersection points of these curves (possibly empty), and it is well-known that an even number of such curves meet at each point of $C_t$. The fact that $\Sigma$ is properly embedded guarantees that $\bigcup_{t \in \mathbb{R}} C_t$ consists of isolated points.

We can decompose $S_t \setminus \Sigma = A^1_t \cup A^2_t$ in such a way that $A^i_t, A^j_t \subset S_t$ are open and, given $i \in \{1, 2\}$, the intersection of the closures of two connected components of $A^i_t$ for $i = 1, 2$, is contained in $C_t$. In other words, we are painting the components of $S_t \setminus \Sigma$ in two colors so that adjacent components have different color. Observe that the sets $\Sigma \cap S_t$ depend continuously on $t \in \mathbb{R}$ so it is clear that $A^1_t$ and $A^2_t$ can be chosen in such a way that $W_i = \bigcup_{t \in \mathbb{R}} A^i_t$ is open for $i \in \{1, 2\}$. As $W_1 \cap W_2 = \emptyset$ and $W_1 \cup W_2 = (S^2 \times \mathbb{R}) \setminus \Sigma$, we get that $W_1$ and $W_2$ are the connected components of $(S^2 \times \mathbb{R}) \setminus \Sigma$. In particular, $\Sigma$ is orientable.

If $\Sigma$ is a connected surface embedded in a 3-manifold $M$ and $M \setminus \Sigma$ has two connected components, then $\Sigma$ is well-known to be orientable (the surface $\Sigma$ is said to separate $M$). The converse is false in $S^2 \times S^1(r)$ as horizontal slices show, but they turn out to be the only counterexamples.

**Proposition 2.** Let $\Sigma \subset S^2 \times S^1(r)$ be a compact embedded minimal surface, different from a finite union of horizontal slices. Then $\Sigma$ separates $S^2 \times S^1(r)$ if and only if $\Sigma$ is orientable.
Proof. We will suppose that $\Sigma$ is orientable and prove that it separates $S^2 \times S^1(r)$. In order to achieve this, we consider the projection $\pi: S^2 \times \mathbb{R} \to S^2 \times S^1(r)$ and the lifted surface $\hat{\Sigma} \subset S^2 \times \mathbb{R}$ such that $\pi(\hat{\Sigma}) = \hat{\Sigma}$. Then $\hat{\Sigma}$ is properly embedded, and connected by the same argument as in Proposition 1. Lemma 1 states that we can decompose it in connected components $(S^2 \times \mathbb{R}) \setminus \hat{\Sigma} = W_1 \cup W_2$.

Let $\phi_r: S^2 \times \mathbb{R} \to S^2 \times \mathbb{R}$ be the vertical translation $\phi_r(p, t) = (p, t + 2\pi r)$. As $\phi_r(\hat{\Sigma}) = \hat{\Sigma}$ there are two possible cases, $\phi_r$ either preserves $W_1$ and $W_2$ or swaps them:

1. If $\phi_r(W_1) = W_1$ and $\phi_r(W_2) = W_2$, then $\pi(W_1) \cap \pi(W_2) = \emptyset$. In this case $\Sigma$ separates $S^2 \times S^1(r)$.
2. If $\phi_r(W_1) = W_2$ and $\phi_r(W_2) = W_1$, then $\pi(W_1) = \pi(W_2) = (S^2 \times S^1(r)) \setminus \Sigma$. In particular, $(S^2 \times S^1(r)) \setminus \Sigma$ is connected and $\Sigma$ does not separate $S^2 \times S^1(r)$. Since $\phi_r$ swaps $W_1$ and $W_2$ the unit normal vector field $\hat{N}$ of $\hat{\Sigma}$ does not induce a unit normal vector field on $\Sigma$. This contradicts the fact that $\Sigma$ is orientable because $\hat{N}$ must be the lift of the unit normal vector field $N$ of $\Sigma$.

We now state our main result, which is inspired by [17, Proposition 1].

**Theorem 1.** Let $\Sigma$ be a compact embedded non-orientable minimal surface in $S^2 \times S^1(r)$. Then $\Sigma$ has an even (non-orientable) genus.

Proof. We can lift $\Sigma$ in a natural way to a compact minimal surface $\Sigma_2 \subset S^2 \times S^1(2r)$ so $\Sigma_2$ is a 2-fold cover of $\Sigma$. We also lift $\Sigma$ to a compact minimal surface $\tilde{\Sigma} \subset S^2 \times \mathbb{R}$ and decompose $(S^2 \times \mathbb{R}) \setminus \tilde{\Sigma} = W_1 \cup W_2$ as in the proof of Proposition 2. The map $\hat{\phi}_r: S^2 \times \mathbb{R} \to S^2 \times \mathbb{R}$ given by $\hat{\phi}_r(p, t) = (p, t + 2\pi r)$ swaps $W_1$ and $W_2$, so $\hat{\phi}_{2r} := \hat{\phi}_r \circ \hat{\phi}_r$ preserves $W_1$ and $W_2$, which means that $\Sigma_2$ is orientable. We will prove that $\Sigma$ has odd orientable genus, from where it follows that $\Sigma$ has even non-orientable genus.

Let us consider the isometric involution $F: S^2 \times S^1(2r) \to S^2 \times S^1(2r)$ given by $F(p, t) = (p, t + 2\pi r)$, and the projection $\pi_2: S^2 \times \mathbb{R} \to S^2 \times S^1(2r)$ such that $\pi_2(\hat{\Sigma}) = \Sigma_2$. It satisfies $F(\Sigma_2) = \Sigma_2$ and swaps $\pi_2(W_1)$ and $\pi_2(W_2)$. Given a horizontal slice $S \subset S^2 \times S^1(2r)$ intersecting $\Sigma_2$ transversally, the surface $\Sigma_2 \setminus (S \cup F(S))$ can be split in two isometric surfaces $\Sigma_2'$ and $F(\Sigma_2')$. Decomposing $\Sigma_2'$ in connected components $P_1, \ldots, P_k$, we can compute $\chi(P_i) = 2 - 2g_i - r_i$, where $g_i$ is the genus and $r_i$ the number of boundary components of $P_i$. Thus

\begin{equation}
\chi(\Sigma_2) = 2\chi(\Sigma_2') = 2\sum_{i=1}^{k} \chi(P_i) = 4\left(k - \sum_{i=1}^{k} g_i\right) - 2\sum_{i=1}^{k} r_i.
\end{equation}

The last term in equation (2.2) is a multiple of 4 since the total number of boundary components $\sum_{i=1}^{k} r_i$ is even (note that each of such components appears twice, once in $S$ and once in $F(S)$), say $4m = \chi(\Sigma_2) = 2(1 - g)$. It follows that $g$, the genus of $\Sigma_2$, is odd and we are done.

3. Construction of examples

In this section we are going to apply the conjugate Plateau technique in order to construct compact embedded minimal surfaces in $S^2 \times S^1(r)$. In all cases we will start with a geodesic polygon in $S^2 \times \mathbb{R}$ and we will ensure that the conjugate of the Plateau solution, after successively reflections over its boundary, produces
periodic (in the $\mathbb{R}$ factor) minimal surfaces that are compact examples in the quotient $S^2 \times S^1(r)$.

3.1. Initial minimal piece and conjugate surface. Consider a geodesic triangle with angles $\tilde{\alpha}$, $\tilde{\beta}$ and $\tilde{\gamma}$ in $S^2 \times \{0\}$, lift the hinge given by the angle $\tilde{\gamma}$ in vertical direction and add two vertical geodesics (each of length $\tilde{h}$) to produce a closed curve $\Gamma$ in $S^2 \times \mathbb{R}$ (see Figure 1). For $\tilde{h} \geq 0$, $\tilde{\gamma} < \pi$ and $\tilde{\alpha}, \tilde{\beta} \leq \frac{\pi}{2}$ the polygonal Jordan curve $\Gamma$ bounds a minimal graph $\tilde{M}$ over $S^2 \times \{0\}$ (the existence follows from Radó’s theorem, see [18] for a particular example). Its conjugate surface $M$ is a minimal surface also in $S^2 \times \mathbb{R}$, bounded by a closed Jordan curve $\Gamma$ (see Figure 2) consisting of three symmetry curves in vertical planes and two horizontal symmetry curves (cf. [12, §2]). Two successive curves in vertical planes enclose an angle $\gamma$, the remaining angles in the vertexes of $\Gamma$ are equal to $\frac{\pi}{2}$. The horizontal curves define the angles enclosed by the vertical symmetry planes, which they meet orthogonally. We call them $a$ and $b$. The mean convex domain $\Omega$ with $\Gamma \subset \partial \Omega$ is well-defined by the angles $a$, $b$, $\gamma$, and the height $h$. Our aim is to define $\Gamma$ such that the conjugate surface has the desired properties, i.e., it continues smoothly without branch points by Schwarz reflection about its boundary producing a compact embedded surface in $S^2 \times \mathbb{R}$ after a finite number of reflections.

![Figure 1](image1.png)  
![Figure 2](image2.png)

Figure 1. The boundary curve $\Gamma$ of the Plateau solution $\tilde{M}$.

Figure 2. The boundary curve $\Gamma$ of the conjugate minimal surface $M$ inside the prism $\Omega$ with data $(a, b, \gamma, h)$.

Depending on the surface we want to construct we have to ensure different data $(a, b, \gamma, h)$. Since this data comes from $\Gamma$, we are going to analyze in the following result the relation between $(a, b, \gamma, h)$ and $(\tilde{a}, \tilde{b}, \tilde{\gamma}, \tilde{h})$ and the properties of the conjugate piece $M$.

**Lemma 2.** In the above situation:

(i) $a > \tilde{a}$, $\beta > \tilde{\beta}$ and $\gamma = \tilde{\gamma}$.

(ii) If $\tilde{h}$ is small enough (see inequality (3.1)), then $M$ is an embedded minimal surface with boundary $\Gamma$ and $M \subset \Omega$.
(iii) If \( m \) copies of \( M \) are needed to produce a compact orientable minimal surface \( \Sigma \) by Schwarz reflection, then the genus \( g \) of \( \Sigma \) is \( g = 1 + \frac{m}{\pi \gamma} (\pi - \gamma) \).

Proof. (i) Let \( \tilde{h} > 0 \) so the angle \( \alpha \) is always defined and apply the Gauß-Bonnet-theorem to the domain \( \Omega \) bounded by \( \tilde{T} \) and the edges of \( \Omega \) as in Figure 2. We get \( \alpha = \text{area}(V_a) + \tilde{a} > \tilde{a} \). Likewise, \( \tilde{\beta} > \tilde{\beta} \) by using the other domain \( V_b \). Finally, \( \gamma = \tilde{\gamma} \) since this angle is intrinsic to the surface.

(ii) In order to show the embeddedness of the conjugate minimal surface \( M \) we analyze the conjugate curve of \( \tilde{\Gamma} \). Let us choose the upward-pointing normal \( \tilde{N} \) and parametrize \( \tilde{\Gamma} \) by a curve \( \tilde{c} \) such that \( \langle \tilde{c}', \tilde{n}, \tilde{N} \rangle \) is positively oriented, where \( \tilde{n} \) denotes the co-normal to the side of the surface. Let \( c \) parametrize the boundary curve of \( M \) and let \( N \) and \( \eta \) denote the corresponding normal and co-normal. On the one hand, by the first order description of conjugate surfaces, see [3], we know that the vertical component of \( c' \) coincides with the vertical component of \( \tilde{n} \), i.e. \( \langle c', \xi \rangle = \langle \tilde{n}, \xi \rangle \), where \( \xi \) denotes the vertical Killing field. Moreover we have \( \langle \tilde{c}', \xi \rangle = -\langle \eta, \xi \rangle \). Hence by the boundary maximum principle the symmetry curves in vertical planes which correspond to horizontal geodesics are graphs.

On the other hand, the two horizontal symmetry curves are embedded if \( \tilde{h} \) is small enough, more precisely if

\[
\tilde{h} < \max \left\{ \sqrt{4\pi^2 - (\pi + \tilde{a})^2}, \sqrt{4\pi^2 - (\pi + \tilde{\beta})^2} \right\}.
\]

Let us suppose that the curve \( \tilde{T} \) (cf. Figure 2) is not embedded. Then the curve contains a loop enclosing a domain \( D \). By the Gauß-Bonnet theorem,

\[
\text{area}(D) \geq \pi - \int_{\text{loop}} \kappa \geq \pi - \tilde{a},
\]

where \( \kappa \) is the geodesic curvature, since the total curvature of \( \tilde{T} \) is exactly \( \tilde{a} \). Hence, by the isoperimetric inequality, \( \tilde{h} = \text{length}(\tilde{T}) \geq \text{length(\text{loop})} \geq \sqrt{4\pi^2 - (\pi + \tilde{a})^2} \) which is a contradiction with (3.1). Following a similar reasoning with the curve \( \tilde{M} \) (cf. Figure 2) we get that both horizontal curves are embedded if (3.1) is satisfied. Finally, since the boundary \( \Gamma \) of \( M \) projects injectively to \( S^2 \) the surface \( M \) is embedded due to a classical application of the maximum principle for minimal surfaces.

(iii) The genus of \( \Sigma \) can be worked out by the Gauß-Bonnet theorem: The total curvature of \( M \) is \( \gamma - \pi \) and in order to close the surface we need by assumption \( m \) copies of \( M \). Hence the total curvature of \( \Sigma \) is \( m(\gamma - \pi) \) and so the genus is \( g = 1 + \frac{m}{4\pi}(\pi - \gamma) \). ☐

Remark 1. One could consider the same boundary curve \( \tilde{\Gamma} \) consisting of horizontal and vertical geodesics in \( \mathbb{R}^3 \). This leads to the well-known Schwarz P-surface. The P-surface and its normal is invariant under a primitive cubic lattice \( \Lambda \), which motivates its name. Moreover, there exists another lattice under which the surface is preserved but its orientation is not; \( \Lambda \) is a subset of it. The quotient \( P \) of the surface under the lattice \( \Lambda \) has genus 3 and consists of 16 copies of \( M \) with \( a = \beta = \frac{4}{a} \) and \( \gamma = \frac{4}{a} \). If we set \( a = b = h \) (cp. Figure 3), the quotient \( P \) is contained in a cube of edge length \( 2a \), and these cubes tessellate \( \mathbb{R}^3 \).

3.2. Odd genus Schwarz P-type examples. This section is devoted to investigate the most symmetric case in the construction above, which follows from choosing \( \tilde{a} = \tilde{b} \) in the initial contour. Since the solution of the Plateau problem with
boundary \( \tilde{\Gamma} \) is unique and in this particular case \( \tilde{\Gamma} \) is symmetric about a vertical plane, we get that the initial minimal surface \( \tilde{M} \) is also symmetric about the same vertical plane, intersecting \( \tilde{M} \) orthogonally along a curve \( \delta \) (see Figure 1). From [12, §2], this implies that the conjugate surface \( M \) is symmetric about a horizontal geodesic \( \delta \) lying in the interior of \( M \) (see Figure 2). In particular, \( a = b \) and \( \alpha = \beta \). Note also that \( \tilde{\gamma} = \gamma \).

We are going to produce new compact embedded minimal examples in \( S^2 \times S^1(\rho) \), for sufficiently small radius \( \rho > 0 \), coming from two different tessellation of the sphere by isosceles triangles (see the shaded triangles in Figures 3 and 4).

The first one comes from the regular quadrilateral tiling of the sphere once we decompose each square in four isosceles triangles with angles \( \alpha = \beta = \pi / 3 \) and \( \gamma = \pi / 2 \). The second one, as depicted in Figure 4, follows from decomposing \( S^2 \) in \( 4k \) isosceles triangles with angles \( \alpha = \beta = \pi / 2 \) and \( \gamma = \pi / k \). Let’s assume that we can produce, via conjugate Plateau technique, an embedded minimal surface \( M_j \) that fits inside a prism \( \Omega_j \) with data \((\pi / 3, \pi / 3, \pi / 2, h)\) if \( j = 1 \) and \((\pi / 2, \pi / 2, \pi / k, h)\) if \( j = 2 \) for arbitrary small \( h \) (see Figure 2). Then:

- It is clear that 24 congruent copies of \( \Omega_1 \) tessellate \( S^2 \times [0, h] \). Since the tessellation comes from reflections about vertical planes, it implies a smooth continuation of \( M_1 \), which we call \( M_1' \). The surface \( M_1' \) is connected, topologically it is \( S^2 \setminus \{ p_1, \ldots, p_8 \} \), where each \( p_i \) corresponds to one vertex of the quadrilateral tiling. After reflecting \( M_1' \) about one of the horizontal planes \( S^2 \times \{ 0 \} \) or \( S^2 \times \{ h \} \) we get a surface consisting of 48 copies of \( M_1 \). The vertical translation \( T \) about \( 2h \) yields a simply periodic embedded minimal surface \( S \). It follows from Lemma 2 that the quotient \( S / T \) is a compact minimal surface with genus \( 7 \) in \( S^2 \times S^1(\rho) \).
- We can reflect the surface \( M_2 \) about the vertical planes containing the edges \( a \) and \( b \). Repeating this reflection successively the surface closes up in such a way that \( 2k \) copies of \( M_2 \) build a smooth surface in the product of an hemisphere of \( S^2 \) and an interval \( I \) of length \( h \). The surface meets the vertical plane above the bounding great circle orthogonally and a reflection about this plane yields a minimal surface \( M_2' \) which is topologically \( S^2 \setminus \{ p_1, \ldots, p_{2k} \} \). As before reflecting \( M_2' \) about one of the horizontal planes \( S^2 \times \partial I \) we get a surface \( S \) which is invariant under the vertical translation \( T \) of length \( 2h \). Moreover it is invariant under a rotation about the fiber over the north or south pole by an angle of \( \pi / 4 \). The quotient \( S / T \) is compact and consists of \( 8k \) copies of \( M \), so its genus is given by \( g = 2k - 1 \), i.e. an arbitrary odd positive integer.
Now, we are going to show the existence of the embedded minimal surfaces $M_1$ (resp. $M_2$) associated to $\Omega_1$ (resp. $\Omega_2$). Let $\tilde{\gamma} = \gamma = \frac{\pi}{2}$ (resp. $\tilde{\gamma} = \gamma = \frac{\pi}{2}$) and remember that we have fixed $\tilde{a} = \tilde{b}$ in the initial polygon $\tilde{\Gamma}$ and so $a = b$ and $\alpha = \beta$ in the conjugate one $\Gamma$ (see Figures 1 and 3). It is clear that $\alpha = \arccos(\frac{1}{2} \cos \ell(\delta)) = \arccos(\frac{1}{\sqrt{2}} \cos \ell(\hat{\delta}))$ (resp. $\alpha = \arccos(\sin \frac{\pi}{2} \cos \ell(\hat{\delta}))$ since $\delta$ is a horizontal curve and the length $\ell(\hat{\delta}) = \ell(\delta)$. The length of $\hat{\delta}$ varies continuously in $\hat{a}$ and $\hat{h}$ from 0 (when $\hat{a} = 0$) to $+\infty$ (take $\hat{h} \to +\infty$, see Figure 1), so $\alpha$ varies continuously in $\hat{a}$ and $\hat{h}$ from $\frac{\pi}{4}$ to $\frac{3\pi}{4}$ (resp. from $\frac{\pi}{4}(k-1)$ to $\frac{3\pi}{4}(k+1)$). For each $\hat{a} \in [0, \frac{\pi}{4}]$ there exists $\hat{h}(\hat{a})$ satisfying formula (3.1) such that $\alpha = \frac{\pi}{2}$ (resp. $\alpha = \frac{\pi}{4}$). Since $h \leq \ell(2\tilde{h}) = \ell(2\tilde{h}) = 2\arctan(\sin \tilde{a}/\sqrt{1 + \cos^2 \tilde{a}})$ (resp. $\ell(2\hat{h}) = 2\arctan(\sin \hat{a}/\sqrt{\cot^2 \frac{\pi}{4} + \cos^2 \hat{a}})$), we get arbitrarily small $h$ when $\hat{a} \to 0$.

To sum up, we can always choose a polygon $\tilde{\Gamma}$ with data $(\tilde{a}, \tilde{b}, \frac{\pi}{2}, \tilde{h})$ (resp. $(\hat{a}, \hat{b}, \frac{\pi}{2}, \hat{h})$) satisfying the inequality (3.1) such as the conjugate of the Plateau solution of $\tilde{\Gamma}$ is embedded and its boundary is contained in the prism $\Omega_1$ (resp. $\Omega_2$), for sufficiently small height $h$.

3.3. Arbitrary genus Schwarz P-type examples. To construct a compact orientable minimal surface $M_k$ with arbitrary genus, we have to ensure that the existence of a minimal surface $M$ that matches a prism $\Omega$ with data $(\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, h)$ for some $h > 0$. This leads to less symmetric examples since $\alpha \neq \beta$ and we have to use a degree argument as in [110] to guarantee existence. If $M \subset \Omega$ exists, then Schwarz reflection about the vertical mirror planes continues the surface smoothly and $4k$ copies of $M$ build a minimal surface $M'$ in the product $S^2 \times [0, h]$ (see Figure 5). Topologically $M'$ is $S^2 \setminus \{p_1, \ldots, p_{k+2}\}$. As in the symmetric cases above, reflecting $M'$ about the bounding horizontal mirror planes gives a complete surface $M''$, which is invariant under vertical translation about $2h$. The quotient is the desired surface $M_k$. Moreover, $M''$ is also invariant under rotation about those vertical fibers which are intersection of vertical mirror planes by angles $\frac{2\pi}{2k}$ (resp. $\pi$). As in the previous cases one computes the genus and gets $1 + k$, since $8k$ copies of $M$ build the compact surface $M_k$ (see Lemma 2).

Proposition 3. There exists a compact orientable minimal surface $M_k$ with genus $k + 1$, embedded in $S^2 \times S^1(r)$, for any $k \geq 3$ and $0 < r \leq r(k)$.

Proof. We only need to prove the existence of the minimal surface $M$ with data $(\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, h)$, the existence of $M_k$ follows directly by Schwarz reflection.
We start with a polygonal Jordan curve $\tilde{\gamma}$ as in Figure 1 with $\tilde{\gamma} = \frac{\pi}{2}$ and some $\tilde{h} > 0$. We have seen that the minimal surface $M$ is then uniquely defined by $(\tilde{a}, \tilde{b})$, so this defines a map $f : \mathbb{R}^+ \times [0, \frac{\pi}{2}]^2 \rightarrow \mathbb{R}^3$, $(\tilde{h}, \tilde{a}, \tilde{b}) \mapsto (h, a, b)$. The map $f$ is continuous, since a sequence of minimal sections $\tilde{M}_n$ given by a converging sequence $(\tilde{h}_n, \tilde{a}_n, \tilde{b}_n)$ has a converging subsequence with unique limit given by the data $\lim_{n}(\tilde{h}_n, \tilde{a}_n, \tilde{b}_n)$.

In order to prove the existence of a minimal surface $M$, i.e., the existence of a triple $(\tilde{h}_0, \tilde{a}_0, \tilde{b}_0)$ such that $f(\tilde{h}_0, \tilde{a}_0, \tilde{b}_0) = (h_0, a_0, b_0)$ for some $h_0 > 0$, we use a degree argument: We consider the map

$$f_h : (0, \frac{\pi}{2}]^2 \rightarrow \mathbb{R}^2, (\tilde{a}, \tilde{b}) \mapsto (a, b)$$

and show there exists a closed Jordan curve $c : \mathbb{R} \rightarrow (0, \frac{\pi}{2}]^2$ such that the image $f_h \circ c$ is a closed curve around $(\frac{\pi}{2}, \frac{\pi}{2})$. We compose $c$ of four straight lines $c_i$, $i \in \{1, \ldots, 4\}$, namely,

- $c_1(t) = (\frac{\pi}{2}, t)$ with $t \in [1, \frac{\pi}{2}]$,
- $c_2(t) = (t, \frac{\pi}{2})$ with $t \in [\frac{\pi}{2}, \pi]$,
- $c_3(t) = (\frac{\pi}{2}, t)$ with $t \in [\frac{\pi}{2}, \pi]$,
- $c_4(t) = (t, \frac{\pi}{2})$ with $t \in [\frac{\pi}{2}, \pi]$,

and we claim there exists $\tilde{h} > 0$ such that $f_h \circ c$ has the desired property.

Since $\tilde{\gamma} = \tilde{a} = \frac{\pi}{2}$ along $c_1$, we have $\tilde{\beta} > \tilde{\beta} = \frac{\pi}{2}$ along $f_h \circ c_1$ (see Lemma 2). Likewise, $\alpha > \tilde{\beta} > \frac{\pi}{2}$ along $f_h \circ c_2$. Along $c_3$ (resp. $c_4$), elementary spherical trigonometry shows that $\tilde{\beta} < \frac{\pi}{2}$ (resp. $\tilde{\beta} < \frac{\pi}{2}$). Since $\alpha \rightarrow \tilde{\beta} \rightarrow \tilde{\beta}$ for $\tilde{h} \rightarrow 0$, we deduce that there exists $\tilde{h}_0 > 0$ such that $\tilde{\beta} < \frac{\pi}{2}$ along $f_h \circ c_3$ and $\alpha < \frac{\pi}{2}$ along $f_h \circ c_4$ for all $\tilde{h} \leq \tilde{h}_0$. Note that we can choose $\tilde{h}_0$ such that (3.1) is satisfied.

Hence for any $\tilde{h} \leq \tilde{h}_0$ there exists a pair $(\tilde{a}, \tilde{b})$ such that $f_h(\tilde{a}, \tilde{b}) = (\frac{\pi}{2}, \frac{\pi}{2})$. Now the height $h > 0$ of $M$ depends continuously on the triple $(\tilde{h}, \tilde{a}, \tilde{b})$, but for $\tilde{h} \rightarrow 0$ we have $h \rightarrow 0$. For $\tilde{h} = \tilde{h}_0$ we get the maximum height, which defines the maximal radius $r(k) = \frac{1}{\pi}$ max $h$.

**Remark 2.** Note that The Schwarz P-type example of genus $2k - 1$, $k \geq 2$, (see Section 3.4) induces a compact embedded minimal surface in $\mathbb{R}P^2 \times S^1(r)$ if and only if $k = 2m$. In this case, we get a two-sided non-orientable embedded minimal surface of genus $4m$, $m \geq 1$.

The orientable arbitrary genus embedded minimal surfaces $M_{k'}$, $k \geq 3$, constructed in Proposition 3 induces a compact embedded minimal surface in $\mathbb{R}P^2 \times S^1(r)$ if and only if $k = 2m$. In this case, the induced surfaces are two-sided non-orientable embedded minimal with genera $2(1 + m)$, $m \geq 2$.

### 3.4. Final remarks

In this section we are going to point out the reason why it is difficult to obtain a better result using the Plateau construction technique for all $r$. First we state the following general properties:

**Proposition 4.**

(i) Let $\Sigma$ be a properly embedded minimal surface of $S^2 \times \mathbb{R}$. If $\Sigma$ contains a vertical geodesic, then it also contains the antipodal geodesic. More precisely if $\{p\} \times \mathbb{R} \subset \Sigma$, then $\{-p\} \times \mathbb{R} \subset \Sigma$.

(ii) Let $\Sigma$ be an oriented, embedded minimal surface of $S^2 \times S^1(r)$ different from a finite union of horizontal slices. If $\Sigma$ contains a horizontal geodesic, then it also contains another horizontal geodesic at vertical distance $\pi r$. More precisely, if $\Gamma \times \{1\} \subset \Sigma$, $\Gamma$ a great circle of $S^2$, then $\Gamma \times \{\pi r\} \subset \Sigma$. 

Proof. Both assertions follow from two facts: (1) under the hypothesis the surface \( \Sigma \) separates the ambient space in two different connected components; (2) there exists an isometry \( \rho \) of the ambient space which preserves the surface by the Schwarz reflection principle but interchanges the connected components of the complement of \( \Sigma \). Hence, the fixed points of \( \rho \) must be contained in \( \Sigma \). In the first case, \( \rho \) is the reflection around \( \{p\} \times \mathbb{R} \) whereas in the second case \( \rho \) is the reflection around \( \Gamma \times \{0\} \). The first assertion appears in [9] §2.

One important consequence of this proposition is that, generically, the oriented compact embedded minimal surfaces constructed solving and reflecting a Plateau problem have odd genus. More precisely, let \( M \) a minimal disk (different from an open subset of a slice) spanning a polygon \( \Gamma \) made of horizontal and vertical geodesics. Suppose that \( M \) produces a compact surface \( \Sigma \) by Schwarz reflection, and the angle function \( \nu = \langle N, \xi \rangle \), where \( N \) is the unit normal to \( M \), satisfies \( \nu^2|_M < 1 \) and \( \nu^2|_\Gamma = 1 \) only at the points of \( \Gamma \) where two horizontal geodesics meet. Then \( \Sigma \) has odd genus if it is orientable or it has even genus if it is not.

In order to prove this, we consider the vector field \( T = \xi - \nu N \) which is the tangent part of \( \xi \), so \( |T|^2 = 1 - \nu^2 \), i.e., \( T \) only vanishes in the points where \( \nu^2 = 1 \) that turns out to be the points where the Hopf differential associated to \( \Sigma \) vanishes. The zeros of \( T \) appear in multiples of 4 since if \( (p,0) \in \Sigma \) is such a point then it is an intersection point of two horizontal geodesics \( \gamma_1 \times \{0\} \) and \( \gamma_2 \times \{0\} \) contained in the surface \( \Sigma \) and so the point \( (p,0) \in (\gamma_1 \cap \gamma_2) \times \{0\} \) is also in \( \Sigma \) and \( T_{(p,0)} = 0 \). Moreover, by Proposition 4, the horizontal geodesics \( \gamma_j \times \{\pi r\} \), \( j = 1,2 \), are contained in \( \Sigma \) so \( (p,\pi r) \) and \( (p,-\pi r) \) are also zeros of \( T \). Applying the Poincaré-Hopf theorem to the vector field \( T \), the Euler characteristic of \( \Sigma \) must be \( -4k \), that is, the genus of \( \Sigma \) is \( g = 2k + 1 \) if \( \Sigma \) is orientable or \( g = 2(1 + 2k) \) if it is not.

Note that this trick is not useful in general for the conjugate surface of \( M \) (we assume that the conjugate piece can be reflected to obtain a compact surface): Although the angle function is preserved by conjugation and so are the zeros of the vector field \( T \), we cannot guarantee in general that the zeros of \( T \) occur in the intersection of two horizontal geodesics in the conjugate piece.

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