From quadratic Hamiltonians of polymomenta to abstract geometrical Maxwell-like and Einstein-like equations

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Abstract

The aim of this paper is to create a large geometrical background on the dual 1-jet space $J^1(T, M)$ for a multi-time Hamiltonian approach of the electromagnetic and gravitational physical fields. Our geometric-physical construction is achieved starting only from a given quadratic Hamiltonian function

$$H = h_{ab}(t)g^{ij}(t,x)p^a_i p^b_j + U^a_{(t)}(t,x)p^a_i + F(t,x)$$

which naturally produces a canonical nonlinear connection $N$, a canonical Cartan $N$-linear connection $CT(N)$ and their corresponding local distinguished (d-) torsions and curvatures. In such a context, we construct some geometrical electromagnetic-like and gravitational-like field theories which are characterized by some natural geometrical Maxwell-like and Einstein-like equations. Some abstract and geometrical conservation laws for the multi-time Hamiltonian gravitational physical field are also given.

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1 Distinguished Riemannian geometrization of metrical multi-time Hamilton spaces

Recently, the studies of Atanasiu and Neagu (see the papers [2], [3] and [4]) initiated the new way of distinguished Riemannian geometrization for Hamiltonians depending on polymomenta, which represents in fact a natural "multi-time" extension of the already classical Hamiltonian geometry on cotangent bundles (synthesized in the Miron et al.'s book [13]). In what follows, we expose the main geometrical ideas which characterize the distinguished Riemannian geometrical approach of Hamiltonians depending on polymomenta (see for details the Oană and Neagu’s papers [15], [16]).
Let us consider that 
\[ h = (h_{ab}(t)) \] 
is a semi-Riemannian metric on the "multi-time" (temporal) manifold \( T^m \), where \( m = \dim T \). Let \( g = (g^{ij}(t^c, x^k, p^c_i)) \) be a symmetric \( d \)-tensor on the dual 1-jet space \( E^* = J^1(T, M) \), which has the rank \( n = \dim M \) and a constant signature. At the same time, let us consider a smooth multi-time Hamiltonian function \( E^* \ni (t^a, x^i, p^a_i) \rightarrow H(t^a, x^i, p^a_i) \in \mathbb{R} \), which yields the fundamental vertical metrical \( d \)-tensor
\[ G^{(i)(j)}_{(a)(b)}(t^c, x^k, p^c_k) = \frac{1}{2} \frac{\partial^2 H}{\partial p^a_i \partial p^b_j}, \]
where \( a, b = 1, \ldots, m \) and \( i, j = 1, \ldots, n \).

**Definition 1** A multi-time Hamiltonian function \( H : E^* \rightarrow \mathbb{R} \), having the fundamental vertical metrical \( d \)-tensor of the form
\[ G^{(i)(j)}_{(a)(b)}(t^c, x^k, p^c_k) = \frac{1}{2} \frac{\partial^2 H}{\partial p^a_i \partial p^b_j} = h_{ab}(t^c)g^{ij}(t^c, x^k, p^c_k), \]
is called a **Kronecker \( h \)-regular multi-time Hamiltonian function**.

In this context, we introduce the following important geometrical concept:

**Definition 2** A pair \( MH^m_n = (E^* = J^1(T, M), H) \), where \( m = \dim T \) and \( n = \dim M \), consisting of the dual 1-jet space and a Kronecker \( h \)-regular multi-time Hamiltonian function \( H : E^* \rightarrow \mathbb{R} \), is called a **multi-time Hamilton space**.

**Example 3** Let us consider the Kronecker \( h \)-regular multi-time Hamiltonian function \( H_1 : E^* \rightarrow \mathbb{R} \), given by
\[
H_1 = \frac{1}{4mc} h_{ab}(t)\varphi_{ij}(x)p^a_i p^b_j, \tag{1}
\]
where \( h_{ab}(t) \) (\( \varphi_{ij}(x) \), respectively) is a semi-Riemannian metric on the temporal (spatial, respectively) manifold \( T \) (\( M \), respectively) having the physical meaning of gravitational potentials, and \( m \) and \( c \) are the known constants from Theoretical Physics representing the mass of the test body and the speed of light. Then, the multi-time Hamilton space \( GMH^m_n = (E^*, H_1) \) is called the multi-time Hamilton space of the gravitational field.

**Example 4** If we consider on \( E^* \) a symmetric \( d \)-tensor field \( g^{ij}(t, x) \), having the rank \( n \) and a constant signature, we can define the Kronecker \( h \)-regular multi-time Hamiltonian function \( H_2 : E^* \rightarrow \mathbb{R} \), by setting
\[
H_2 = h_{ab}(t)g^{ij}(t, x)p^a_i p^b_j + U^{(i)}(t, x)p^a_i + F(t, x), \tag{2}
\]
where \( U^{(i)}(t, x) \) is a d-tensor field on \( E^* \), and \( F(t, x) \) is a function on \( E^* \). Then, the multi-time Hamilton space \( \mathcal{EDMH}^m_n = (E^*, H_2) \) is called the non-autonomous multi-time Hamilton space of electrodynamics. The dynamical character of the gravitational potentials \( g_{ij}(t, x) \) (i.e., the dependence on the temporal coordinates \( t^c \)) motivated us to use the word “non-autonomous”.

An important role for the subsequent development of our distinguished Riemannian geometrical theory for multi-time Hamilton spaces is represented by the following result (proved in the paper [2]):

**Theorem 5** If we have \( m = \text{dim} \mathcal{T} \geq 2 \), then the following statements are equivalent:

(i) \( H \) is a Kronecker \( h \)-regular multi-time Hamiltonian function on \( E^* \).

(ii) The multi-time Hamiltonian function \( H \) reduces to a multi-time Hamiltonian function of non-autonomous electrodynamic type. In other words, we have

\[ H = h_{ab}(t)g^{ij}(t, x)p^a_i p^b_j + U^{(i)}(t, x)p^a_i + F(t, x). \]

**Corollary 6** The fundamental vertical metrical d-tensor of a Kronecker \( h \)-regular multi-time Hamiltonian function \( H \) has the form

\[ G^{(i)(j)}(a)(b) = \frac{1}{2} \frac{\partial^2 H}{\partial p^a_i \partial p^b_j} = \begin{cases} h_{11}(t)g^{ij}(t, x^k, p^a_k), & m = \text{dim} \mathcal{T} = 1 \\ h_{ab}(t)c^{ij}(t^c, x^k), & m = \text{dim} \mathcal{T} \geq 2. \end{cases} \]

Following now the Miron’s geometrical ideas from [13], the paper [2] proves that any Kronecker \( h \)-regular multi-time Hamiltonian function \( H \) produces a natural nonlinear connection on the dual 1-jet space \( E^* \), which depends only by the given Hamiltonian function \( H \):

**Theorem 7** The pair of local functions \( N = \left( N_{1}^{(a)}(i), N_{2}^{(a)}(i) \right) \) on \( E^* \), where (\( \chi_{bc}^{a} \) are the Christoffel symbols of the semi-Riemannian temporal metric \( h_{ab} \))

\[ N_{1}^{(a)}(i) = \chi_{bc}^{a} \Gamma_{1}^{c}, \]

\[ N_{2}^{(a)}(i) = \frac{h^{ab}}{4} \left[ \frac{\partial g_{ij}}{\partial x^{k}} \frac{\partial H}{\partial p^{b}_k} - \frac{\partial g_{ij}}{\partial p^{b}_k} \frac{\partial H}{\partial x^{k}} + g_{ik} \frac{\partial^2 H}{\partial x^{j} \partial p^{b}_k} + g_{jk} \frac{\partial^2 H}{\partial x^{i} \partial p^{b}_k} \right], \]

represents a nonlinear connection on \( E^* \). This is called the canonical nonlinear connection of the multi-time Hamilton space \( \mathcal{EDMH}^m_n = (E^*, H) \).

Taking into account the Theorem and using the generalized spatial Christoffel symbols of the d-tensor \( g_{ij} \), which are given by

\[ \Gamma_{ij}^{k} = \frac{g^{kl}}{2} \left( \frac{\partial g_{ji}}{\partial x^{l}} + \frac{\partial g_{ij}}{\partial x^{l}} - \frac{\partial g_{lj}}{\partial x^{l}} \right), \]

we immediately obtain the following geometrical result:
Corollary 8 For \( m = \dim T \geq 2 \), the canonical nonlinear connection \( N \) of a multi-time Hamilton space \( MH^m_n = (E^*, H) \), whose Hamiltonian function is given by \( (3) \), has the components

\[
N^{(a)}_{\frac{1}{2} (i)b} = \chi^{a}_{lc} p_{l}^{i}, \quad N^{(a)}_{\frac{1}{2} (i)j} = -\Gamma^{k}_{ij} p_{k}^{a} + T^{(a)}_{(i)j},
\]

where

\[
T^{(a)}_{(i)j} = \frac{h_{ab}}{4} (U_{ib\bullet j} + U_{jb\bullet i}),
\]

and

\[
U_{ib} = g_{ik} U^{(k)}_{(b)}, \quad U_{kb\bullet r} = \frac{\partial U_{kb}}{\partial x^{r}} - U_{sb} \Gamma^{s}_{kr}.
\]

The canonical nonlinear connection \( N = \left( N^{(a)}_{\frac{1}{2} (i)b}, N^{(a)}_{\frac{1}{2} (i)j} \right) \) on \( E^* \) allows us the construction of the adapted bases

\[
\left\{ \frac{\delta}{\delta t^a}, \frac{\delta}{\delta x^i}, \frac{\delta}{\partial p_j^a} \right\} \subset \chi (E^*), \quad \left\{ dt^a, dx^i, \delta p_j^a \right\} \subset \chi^* (E^*),
\]

where

\[
\frac{\delta}{\delta t^a} = \frac{\partial}{\partial t^a} - N^{(a)}_{\frac{1}{2} (i)b} \frac{\partial}{\partial p_j^b}, \quad \frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N^{(a)}_{\frac{1}{2} (i)j} \frac{\partial}{\partial p_j^b}, \quad \delta p_j^a = \partial p_j^a + N^{(a)}_{\frac{1}{2} (i)b} dt^b + N^{(a)}_{\frac{1}{2} (i)j} dx^j.
\]

The main result of the metrical multi-time Hamilton geometry is the Theorem of existence of the Cartan canonical \( h \)-normal \( N \)-linear connection \( \text{CT}(N) \) (see [16]) which allows the subsequent development of our metrical multi-time Hamilton theory of physical fields.

Theorem 9 (the Cartan canonical \( N \)-linear connection) On the metrical multi-time Hamilton space \( MH^m_n = (J^1 \times (T, M), H) \), endowed with the canonical nonlinear connection \( N \), there exists an unique \( h \)-normal \( N \)-linear connection

\[
\text{CT}(N) = \left( \chi^a_{bc}, A^i_{jc}, H^i_{jk}, C^{(k)}_{j(c)} \right),
\]

having the metrical properties:

(i) \( g_{ij} \mid k = 0, \quad g^{ij} \mid (k) = 0, \)

(ii) \( A^i_{jc} = \frac{g^{ir}}{2} \frac{\delta g_{lj}}{\delta t^c}, \quad H^i_{jk} = H^i_{kj}, \quad C^{(k)}_{j(c)} = C^{(k)}_{j(c)}, \)

where \( n_{(a)} \), \( n_{(b)} \) and \( n_{(c)} \) represent the local covariant derivatives of \( \text{CT}(N) \). Moreover, the adapted local coefficients \( H^i_{jk} \) and \( C^{(k)}_{j(c)} \) of the Cartan canonical connection \( \text{CT}(N) \) have the expressions

\[
H^i_{jk} = \frac{g^{ir}}{2} \left( \frac{\delta g_{jr}}{\delta x^k} + \frac{\delta g_{kr}}{\delta x^j} - \frac{\delta g_{jk}}{\delta x^r} \right), \quad C^{(k)}_{j(c)} = -\frac{g^{ir}}{2} \left( \frac{\partial g^{jr}}{\partial p_{k}^i} + \frac{\partial g^{kr}}{\partial p_{j}^i} - \frac{\partial g^{jk}}{\partial p_{r}^i} \right).
\]
Remark 10 (i) The Cartan canonical connection $\Gamma(N)$ of the multi-time Hamilton space $MH^m_n$ verifies also the metrical properties

$$h_{ab/c} = h_{ab|k} = h_{ab|k}^{(k)} = 0, \quad g_{ij/c} = 0.$$  

(ii) In the case $m = \dim T \geq 2$, the adapted coefficients of the Cartan canonical connection $\Gamma(N)$ of the multi-time Hamilton space $MH^m_n$ reduce to

$$A^a_{bc} = \chi^a_{bc}, \quad A^i_{jc} = \frac{g^{il}}{2} \frac{\partial g_{lj}}{\partial t^c}, \quad H^i_{jk} = \Gamma^i_{jk}, \quad C^i_{j(k(c)} = 0.$$(7)

Applying the formulas that determine the local d-torsions and d-curvatures of an $h$-normal $N$-linear connection $D\Gamma(N)$ (see [15]) to the Cartan canonical connection $\Gamma(N)$, we obtain (see [16]):

Theorem 11 The torsion tensor $T$ of the Cartan canonical connection $\Gamma(N)$ of the multi-time Hamilton space $MH^m_n$ is determined by the adapted local $d$-components

$$
\begin{array}{|c|c|c|c|c|c|}
\hline
 & h_T & h_M & v & & \\
\hline
m \geq 1 & m = 1 & m \geq 2 & & & \\
\hline
h_T h_T & 0 & 0 & 0 & 0 & R^{(1)}_{(r)ab} \\
\hline
h_M h_T & 0 & T^r_{kj} & T^r_{aj} & P^{(1)}_{(r)(1)} & P^{(1)}_{(r)(a(6)} \\
\hline
v h_T & 0 & 0 & 0 & P^{(1)}_{(r)(1)} & P^{(1)}_{(r)(a(1))} \\
\hline
h_M h_M & 0 & 0 & 0 & R^{(1)}_{(r)ij} & R^{(1)}_{(r)ij} \\
\hline
v h_M & 0 & P^{(1)}_{(r)(i)} & 0 & P^{(1)}_{(r)(i)(1)} & 0 \\
\hline
vv & 0 & 0 & 0 & 0 & 0 \\
\hline
\end{array}
$$

where

(i) for $m = \dim T = 1$, we have

$$T^r_{ij} = -A^r_{j1}, \quad P^{(1)}_{(r)(i)} = C^{(1)}_{(r)(i)}, \quad P^{(1)}_{(r)(1)} = \frac{\partial N}{\partial p_j} + A^r_{r1} - \delta^r_j \chi_{11},$$

$$P^{(1)}_{(r)(i)(1)} = \frac{\partial N}{\partial p_j} + H^i_{ri}, \quad R^{(1)}_{(r)ij} = \frac{\delta N}{\delta x^j} - \frac{\delta N}{\delta t},$$

(ii) for $m = \dim T \geq 2$, using the equality (5) and the notations

$$\chi^c_{fab} = \frac{\partial \chi^c_{fa}}{\partial t^b} - \frac{\partial \chi^c_{fb}}{\partial t^a} + \chi^d_{fa} \chi^c_{db} - \chi^d_{fb} \chi^c_{da},$$

$$\mathcal{W}^k_{lij} = \frac{\partial \Gamma^k_{lij}}{\partial x^j} - \frac{\partial \Gamma^k_{lij}}{\partial x^i} + \Gamma^p_{kli} \Gamma^p_{lij} - \Gamma^p_{klj} \Gamma^p_{lij},$$

5
we have
\[ T^r_{aj} = -A^r_{ja}, \quad P^{(f)}_{(r)ab} = \delta^r_b A^1_{ia}, \quad R^{(f)}_{(r)ab} = \chi^f_{gab} p^g_r, \]
\[ R^{(f)}_{(r)aj} = -\frac{\partial N^{(f)}_{(r)j}}{\partial x^a} - \chi^f_{ja} T^{(c)}_{\cdot j} - T^{(f)}_{(r)j|a}. \]

**Theorem 12**
The curvature tensor \( \mathcal{R} \) of the Cartan connection \( \Gamma \) of the multi-time Hamilton space \( MH^n_m \) is determined by the following adapted local curvature d-tensors:

| \( h_T \) | \( h_M \) | \( v \) |
|--------|--------|--------|
| \( h_T h_T \) | \( h_M h_T \) | \( v h_T \) |
| \( h_M h_M \) | \( v h_M \) | \( v v \) |
| \( R^l_{i1k} \) | \( R^l_{ibk} \) | \( P^l_{i(k)} \) |
| \( 0 \) | \( 0 \) | \( 0 \) |
| \( -R^{(f)}_{(i)11k} = -R^{f}_{(i)1} \) | \( -P^{(f)}_{(i)1} (k) = -P^{f}_{(i)1}(k) \) | \( 0 \) |
| \( S^l_{(j)1(1)} \) | \( S^l_{(j)1(1)} \) | \( 0 \) |
| \( 0 \) | \( 0 \) | \( -S^{(d)}_{(j)1(1)} = -S^{d}_{(j)1(1)} \) |

where, for \( m = \dim T \geq 2 \), we have the relations
\[ -R^{(d)(i)}_{(l)(a)abc} = \delta^i_a \chi_{abc} - \delta^a_d R_{i(abc)} = -\delta^d_a R_{ibk}, \quad -R^{(d)(i)}_{(l)(a)jk} = -\delta^a_d \gamma_{ijk}, \]
and, generally, the following formulas are true:

(i) for \( m = \dim T = 1 \), we have
\[ \chi^1_{111} = 0, \]
\[ R^l_{i1k} = \frac{\delta A^1_{i1}}{\delta x^k} - \frac{\delta H^1_{ik}}{\delta t} + A^1_{i1} H^1_{rk} - H^r_{ik} A^1_{r1} + C^1_{i(1)} R^{(1)}_{(r)1k}, \]
\[ R^l_{ijk} = \frac{\delta H^1_{ij}}{\delta x^k} - \frac{\delta H^1_{ik}}{\delta x^j} + H^r_{ij} H^1_{rk} - H^r_{ik} H^1_{rj} + C^1_{i(1)} R^{(1)}_{(r)jk}, \]
\[ P^l_{i1(1)} = \frac{\delta A^1_{i1}}{\delta p^k} - C^{(k)}_{i(1)} + C^{(r)}_{i(1)} P^{(1)}_{(r)1}, \]
\[ P^l_{ij(1)} = \frac{\delta H^1_{ij}}{\delta p^k} - C^{(k)}_{i(1)j} + C^{(r)}_{i(1)} P^{(1)}_{(r)j}, \]
\[ S^{(d)(j)}_{i(1)(1)} = \frac{\partial C^{(r)}_{i(1)}}{\partial p^k} + C^{(r)}_{i(1)} r^{(1)} - C^{(k)}_{i(1)} C^{(j)}_{r(1)}, \]

\[ \]
(ii) for \( m = \dim T \geq 2 \), we have

\[
\chi^d_{abc} = \frac{\partial \chi^d_{ab}}{\partial t^c} - \frac{\partial \chi^d_{ac}}{\partial t^b} + \chi^d_{ac} \chi^d_{fb} - \chi_{ac} \chi^d_{fb},
\]

\[
R^i_{abc} = \frac{\partial A^i_{ab}}{\partial t^c} - \frac{\partial A^i_{ac}}{\partial t^b} + A^i_{ab} A^i_{rc} - A^i_{ac} A^i_{rb},
\]

\[
R^i_{abc} = \frac{\partial A^i_{ab}}{\partial x^c} - \frac{\partial A^i_{ac}}{\partial x^b} + \Gamma^i_{ab} \Gamma^i_{rc} - \Gamma^i_{ac} \Gamma^i_{rb},
\]

\[
\mathcal{R}^i_{ijk} = \frac{\partial \Gamma^i_{jk}}{\partial x^k} - \frac{\partial \Gamma^i_{jk}}{\partial x^j} + \Gamma^i_{ij} \Gamma^i_{rk} - \Gamma^i_{ik} \Gamma^i_{jr},
\]

In the next Sections, following the physical and geometrical ideas of the already classical Lagrangian geometry of physical fields (see [12], [13] and [14]), we construct a possible multi-time Hamiltonian approach of the electromagnetic and gravitational physical fields, which is characterized by some natural geometrical Maxwell-like and Einstein-like equations. To reach this aim, we consider a multi-time Hamilton space \( MH^m_n = (J^1, (T, M), H) \) endowed with its canonical nonlinear connection

\[
N = \left( N^a \right)_{(i)j}, \quad (N^a)^{2}_{(ij)},
\]

and we also consider the Cartan canonical connection of the space \( MH^m_n \), which is locally expressed by

\[
CT(N) = \left( \chi^a_{bc}, A^i_{jc}, H^i_{jk}, C^a_{(j)c} \right).
\]

2 Multi-time Hamilton electromagnetism. Geometrical Maxwell-like equations

Let us consider the canonical Liouville-Hamilton d-tensor field of polymomenta

\[
\mathcal{C}^* = p^a_i \frac{\partial}{\partial p^a_i},
\]

together with the fundamental vertical metrical d-tensor \( G^{(i)(j)}_{(a)(b)} \) of the multi-time Hamilton spaces \( MH^m_n \). These geometrical objects allow us to construct the metrical deflection d-tensors

\[
\Delta^{(i)}_{(a)b} = G^{(i)(k)}_{(a)(c)} \Delta^{(c)}_{(k)b} = p^{(i)}_{(a)b}, \quad \Delta^{(i)}_{(a)j} = G^{(i)(k)}_{(a)(c)} \Delta^{(c)}_{(k)j} = p^{(i)}_{(a)j},
\]

\[
g^{(i)(j)}_{(a)(b)} = G^{(i)(k)}_{(a)(c)} g^{(c)(j)}_{(k)(b)} = p^{(i)}_{(a)(b)},
\]

where \( p^{(i)}_{(a)} = G^{(i)(k)}_{(a)(c)} p^c_k \) and "\( \cdot / b \)" and "\( \cdot / j \)" are the local covariant derivatives induced by the Cartan connection \( CT(N) \).

Taking into account the expressions of the local covariant derivatives of the Cartan connection \( CT(N) \) (see the paper [15]), by a direct calculation, we obtain
Proposition 13  The metrical deflection $d$-tensors of the multi-time Hamilton spaces $MH^m_n$ have the expressions:

(i) for $m = \dim \mathcal{T} = 1$, we have

\[\Delta_1^{(i)} = -h_{11}g^{ik}A_{k1}p_r^i, \quad \Delta_1^{(j)} = h_{11}g^{ik}\left[-N^{(1)}_{(k)j} - H_{kj}^r p_r^i\right], \quad \vartheta_1^{(i)(j)}(1) = h_{11}g^{ij}\vartheta^{(j)}(1)(1), \quad \vartheta_1^{(i)(j)}(1) = h_{11}g^{ik}C_{k1}^{(j)}p_r^i; \quad \text{(8)}\]

(ii) for $m = \dim \mathcal{T} \geq 2$, we have

\[\Delta_r^{(i)} = -h_{ac}g^{ik}A_{kb}p_r^e, \quad \Delta_r^{(j)} = -\frac{g_{ik}}{4}(U_{ka} \bullet i + U_{ja} \bullet k), \quad \vartheta_r^{(i)(j)}(1)(1) = h_{ab}g^{ij}. \quad \text{(9)}\]

In order to construct our metrical multi-time Hamiltonian theory of electromagnetism, we introduce the following concept:

Definition 14  The distinguished 2-form on $J_1^* (\mathcal{T}, M), \text{ locally defined by}$

\[\mathbb{F} = F_{(a)}^{(i)} \delta p_t^a \wedge dx^j + f_{(a)(b)}^{(i)(j)} \delta p_t^a \wedge \delta p_j^b, \quad \text{(10)}\]

where

\[F_{(a)(b)}^{(i)} = \frac{1}{2} \left[\Delta_{(a)(b)}^{(i)} - \Delta_{(a)(b)}^{(j)}\right], \quad f_{(a)(b)}^{(i)(j)} = \frac{1}{2} \left[\vartheta_{(a)(b)}^{(i)(j)} - \vartheta_{(a)(b)}^{(j)(i)}\right]. \quad \text{(11)}\]

is called the multi-time electromagnetic field of the metrical multi-time Hamilton space $MH^m_n$.

By a straightforward calculation, the Proposition 13 implies

Proposition 15  The components $F_{(a)(b)}^{(i)}$ and $f_{(a)(b)}^{(i)(j)}$ of the multi-time electromagnetic field $\mathbb{F}$, associated to the multi-time Hamilton space $MH^m_n$, have the following expressions:

(i) in the case $m = \dim \mathcal{T} = 1$, we have

\[F_{(1)}^{(i)} = \frac{h_{11}}{2} \left[g^{jk}N^{(1)}_{(k)i} - g^{jk}N^{(1)}_{(k)j} + (g^{jk}H_{ki}^r - g^{jk}H_{kj}^r) p_r^i\right], \quad f_{(1)(1)}^{(i)(j)} = 0; \quad \text{(i)}\]

(ii) in the case $m = \dim \mathcal{T} \geq 2$, we have

\[F_{(a)}^{(i)} = \frac{1}{8} \left[g^{jk}U_{ka} \bullet i - g^{ik}U_{ka} \bullet j + g^{jk}U_{ia} \bullet k - g^{jk}U_{ja} \bullet k\right], \quad f_{(a)(b)}^{(i)(j)} = 0. \quad \text{(ii)}\]

The main result of our abstract geometrical Hamilton multi-time electromagnetism is given by
Theorem 16 The electromagnetic components $F^{(i)}_{(a)j}$ of the multi-time Hamilton space $MH^n_m$ are governed by the following geometrical Maxwell-like equations:

(i) for $m = \dim T = 1$, we have

$$F_{(1)k/1}^{(i)} = \frac{1}{2} A_{(i,j)} \left\{ \Delta_{(1)1|k} + \Delta_{(1)r} T_{rk} + \varphi_{(1)r} G_{(r)j1k} + R_{(r)j1k} \right\}$$

$$\sum_{(i,j,k)} F_{(1)j|1}^{(i)} = \frac{1}{2} \sum_{(i,j,k)} \left\{ \varphi_{(1)1} G_{(r)rj} + R_{(r)rj1k} \right\}$$

$$F_{(1)j/1}^{(i)} = \frac{1}{2} A_{(i,j)} \left\{ \varphi_{(1)(i)} G_{(j)j1} - p_{(j)} R_{(j)j1} - \Delta_{(1)j} C_{(j)j1} - \varphi_{(1)(j)} P_{(j)j1} \right\} ;$$

(ii) for $m = \dim T \geq 2$, we have

$$F_{(a)k/b}^{(i)} = \frac{1}{2} A_{(i,k)} \left\{ \Delta_{(a)b|c} + \Delta_{(a)r} T_{rbc} + \varphi_{(a)r} R_{(r)bck} + R_{(r)bck} \right\}$$

$$\sum_{(i,j,k)} F_{(a)j|b}^{(i)} = \frac{1}{2} \sum_{(i,j,k)} \left\{ \varphi_{(a)(j)} R_{(j)c} + R_{(j)c} \right\}$$

$$\sum_{(i,j,k)} F_{(a)j/1}^{(i)} = 0,$$

where $A_{(i,j)}$ means an alternate sum, $\sum_{(i,j,k)}$ means a cyclic sum, and we used the notations $p_{(j)}^{(i)} = G_{(i)(j)} p_{(j)}^{(i)}$ and $P_{(a)}^{(i)} = G_{(a)(b)} p_{(a)}^{(i)} p_{(b)}^{(i)}$.

**Proof.** The general Ricci identities (see [15] and [16]) applied to $g^{ij}$ lead us to the equalities:

$$g^{ir} R_{rk}^{(i)} + g^{ij} R_{rbc}^{(i)} = 0, \quad g^{ir} R_{rkl}^{(i)} + g^{ij} R_{rkl}^{(i)} = 0,$$

$$g^{ir} P_{rbc}^{(i)} + g^{ij} P_{rkl}^{(i)} = 0. \quad (12)$$

Let us consider the following non-metrical deflection d-tensor identities ([13]):

$$(d_1) \quad \Delta_{(p)k|b}^{(d)} - \Delta_{(p)b|k}^{(d)} = p^d_{(p)} R_{pbc}^{(d)} - \Delta_{(p)r} T_{rbc}^{(d)} - \varphi_{(p)(f)} R_{(f)bck}^{(d)},$$

$$(d_2) \quad \Delta_{(p)j|k}^{(d)} - \Delta_{(p)k|j}^{(d)} = p^d_{(p)} R_{pjc}^{(d)} - \varphi_{(p)(f)} R_{(f)rjk}^{(d)},$$

$$(d_3) \quad \Delta_{(p)j|c}^{(d)} - \varphi_{(p)(f)} p_{(j)c}^{(d)} = p^d_{(p)} R_{pjc}^{(d)} - \Delta_{(p)r} C_{rj(c)}^{(d)} - \varphi_{(p)(f)} R_{(r)rj(c)}^{(d)},$$

where $\Delta_{(a)k|b}^{(d)} = p^d_{(a)} R_{a|b}^{(d)}$, $\Delta_{(a)j|c}^{(d)} = p^d_{(a)} R_{a|c}^{(d)}$, $\varphi_{(a)(f)} p_{(a)}^{(d)} = p_{(a)}^{(d)}$.

Contracting the above deflection d-tensor identities with the fundamental vertical metrical d-tensor $G_{(a)(d)}^{(i)}$, and using the equalities ([12]), we obtain the following geometrical deflection d-tensor identities:

$$(d_1') \quad \Delta_{(a)b|k}^{(i)} - \Delta_{(a)b|k}^{(i)} = -p_{(a)}^{(r)} R_{rk}^{(i)} - \Delta_{(a)r} T_{rbk}^{(i)} - \varphi_{(a)(f)} R_{(f)bck}^{(i)},$$

$$(d_2') \quad \Delta_{(a)j|k}^{(i)} - \Delta_{(a)j|k}^{(i)} = -p_{(a)}^{(r)} R_{rjk}^{(i)} - \varphi_{(a)(f)} R_{(r)rjk}^{(i)};$$
(d^i_3) \Delta^{(i)}_{(a)j}^{(j)}(c) - \varrho^{(i)(k)}_{(a)(c)i}^{(k)} = - p^{(i)}_{(a)} p^{(k)}_{rj(c)} - \Delta^{(i)}_{(a)r} C^{r(k)}_{j(c)} - \varrho^{(i)(r)}_{(a)(f)} p^{(f)}_{rj(c)}.

To obtain the first (respectively, the third) geometrical Maxwell-like equation, we permute the indices $i$ and $k$ in the identity $(d^i_3)$ (respectively, the indices $i$ and $j$ in the identity $(d^j_3)$), and we subtract this new identity from the initial one. For $m = \dim \mathcal{T} \geq 2$ we find the following new identity:

$$F^{(i)}_{(a)j}^{(k)}(c) = \frac{1}{2} \left[ \varrho^{(i)(k)}_{(a)(c)j} - \varrho^{(j)(k)}_{(a)(c)i} \right].$$

Consequently, doing a cyclic sum by $\{i, j, k\}$ for $m \geq 2$, we obtain what we were looking for.

Doing a cyclic sum after the indices $\{i, j, k\}$ in the identity $(d^i_3)$, it follows the second geometrical Maxwell-like equation.

3 Multi-time Hamilton gravitational field. Geometrical Einstein-like equations

Let us consider that $h = (h_{ab}(t))$ is a fixed semi-Riemannian metric on the temporal manifold $\mathcal{T}$ and let

$$N = \left( N^{(a)}_{1(i)b}, N^{(a)}_{2(i)j} \right)$$

be an "a priori" given nonlinear connection on the dual 1-jet space $J^1(\mathcal{T}, M)$.

Let

$$\delta p^a_i = dp^a_i + N^{(a)}_{1(i)b} dt^b + N^{(a)}_{2(i)j} dx^j$$

be the vertical distinguished 1-forms adapted to the nonlinear connection $N$.

An essential element in the development of our abstract geometrical multi-time Hamilton gravitational theory is given by the following definition:

**Definition 17** From an abstract physical point of view, an adapted metrical $d$-tensor $G$ on the dual 1-jet space $E^* = J^1(\mathcal{T}, M)$, locally expressed by

$$G = h_{ab} dt^a \otimes dt^b + g_{ij} dx^i \otimes dx^j + h_{ab} g^{ij} \delta p^a_i \otimes \delta p^b_j,$$

where $g_{ij} = g_{ij}(t^c, x^k, x^l)$ is a symmetric $d$-tensor field of rank $n = \dim M$ having a constant signature on $E^* = J^1(\mathcal{T}, M)$, is called a multi-time gravitational $h$-potential on $E^*$.

Now, taking a multi-time Hamilton space $M H^m = (E^*, H)$, via its fundamental vertical metrical $d$-tensor $G^{(i)}_{(a)(j)}$ (which is given by (3)) and its canonical nonlinear connection $N$, we naturally construct a corresponding multi-time gravitational $h$-potential on $E^*$, setting

$$G = h_{ab} dt^a \otimes dt^b + g_{ij} dx^i \otimes dx^j + h_{ab} g^{ij} \delta p^a_i \otimes \delta p^b_j.$$
At the same time, let us consider that
\[ C^\Gamma(N) = (\chi^c_{ab}, A^j_k, H^i_{jk}, C_{(c)}^{(k)}) \]
is the Cartan canonical connection of the multi-time Hamilton space \( MH^n_m \).

**Postulate.** We postulate that the geometrical Einstein-like equations, which govern the multi-time gravitational \( h \)-potential \( G \) of the multi-time Hamilton space \( MH^n_m \), are the abstract geometrical Einstein equations attached to the Cartan canonical connection \( C^\Gamma(N) \) and to the adapted metric \( G \) on \( E^* \), namely
\[
\text{Ric}(C^\Gamma) - \frac{\text{Sc}(C^\Gamma)}{2} G = K_T, \quad (13)
\]
where \( \text{Ric}(C^\Gamma) \) represents the Ricci tensor of the Cartan connection, \( \text{Sc}(C^\Gamma) \) is the scalar curvature, \( K \) is the Einstein constant and \( T \) is an intrinsic \( d \)-tensor of matter, which is called the stress-energy \( d \)-tensor of polymomenta.

In the adapted basis of vector fields
\[
(X_A) = \left( \frac{\delta}{\delta t^a}, \frac{\delta}{\delta x^i}, \frac{\partial}{\partial p_i^a} \right),
\]
which is produced by the canonical nonlinear connection \( N \) of the multi-time Hamilton space \( MH^n_m \), the curvature tensor \( \mathcal{R} \) of the Cartan canonical connection \( C^\Gamma(N) \) is locally expressed by
\[
\mathcal{R}(X_C, X_B)X_A = R^D_{ABC}X_D.
\]
It follows that we have
\[
R_{AB} = \text{Ric}(X_A, X_B) = R^D_{ABC}, \quad \text{Sc}(C^\Gamma) = G^{AB}R_{AB},
\]
where
\[
G^{AB} = \begin{cases} 
 h_{ab}, & \text{for } A = a, B = b \\
 g^{ij}, & \text{for } A = i, B = j \\
 h_{ab}g_{ij}, & \text{for } A = (a), B = (i) \\
 0, & \text{otherwise.}
\end{cases} \quad (14)
\]

Taking into account, on the one hand, the form of the metrical \( d \)-tensor \( G = (G_{AB}) \) of the multi-time Hamilton space \( MH^n_m \), and, on the other hand, taking into account the expressions of the local curvature \( d \)-tensors attached to the Cartan canonical connection \( C^\Gamma(N) \), by direct computations, we find

**Proposition 18** The Ricci tensor \( \text{Ric}(C^\Gamma) \) of the Cartan canonical connection \( C^\Gamma(N) \) of the multi-time Hamilton space \( MH^n_m \) is determined by the following adapted components:
(i) for \( m = \dim \mathcal{T} = 1 \), we have

\[ R_{11} := \chi_{11} = 0, \quad R_{11} = R_{i11} = 0, \]
\[ R_{1(1)} := -P_{1(1)}^{1(1)} = 0, \quad R_{i1} = R_{i1r}, \quad R_{ij} = R_{ijr}, \]
\[ R_{(1)(1)} := -S_{(1)(1)} = -S^{(1)(1)}, \quad R_{(1)(1)(1)} := -P_{(1)(1)}^{(1)} = -P^{(1)(1)}, \]

(ii) for \( m = \dim \mathcal{T} \geq 2 \), we have

\[ R_{ab} := \chi_{ab} = \chi_{abf} \]
\[ R_{af(b)} := -P_{af(b)}^{(1)} = 0, \quad R_{fa(b)} = R_{far}, \quad R_{ij} = R_{ijr}, \]
\[ R_{(a)(b)} := -S_{(a)(b)} = -S^{(a)(b)} = 0, \quad R_{(a)(b)} := -P_{(a)(b)}^{(1)} = -P^{(a)(b)} = 0, \]
\[ R_{(a)b} := -P_{(a)b}^{(1)} = -P^{(1)b} = 0, \quad R_{(a)j} := -P_{(a)j}^{(1)} = -P^{(a)j} = 0. \]

Using the notations \( \chi = h^{ab} \chi_{ab} \), \( R = g^{ij} R_{ij} \) and \( S = h_{ij} g_{ij} S^{ij}_{(1)(1)} \), we obtain

**Corollary 19** The scalar curvature \( \text{Sc}(\mathcal{T}) \) of the Cartan canonical connection \( \mathcal{T} \) (\( N \)) of the multi-time Hamilton space \( MH_m^n \) is given by the formulas:

(i) for \( m = \dim \mathcal{T} = 1 \), we have \( \text{Sc}(\mathcal{T}) = R - S \);

(ii) for \( m = \dim \mathcal{T} \geq 2 \), we have \( \text{Sc}(\mathcal{T}) = \chi + R \).

In this context, the main result of the Hamilton geometrical multi-time gravitational theory is offered by

**Theorem 20** The geometrical Einstein-like equations, which govern the multi-time gravitational h-potential \( \mathcal{G} \) of the multi-time Hamilton space \( MH_m^n \), have the following adapted local form:

(i) for \( m = \dim \mathcal{T} = 1 \), we have

\[
\begin{align*}
-\frac{R - S}{2} h_{11} &= \kappa T_{11} \\
R_{ij} - \frac{R - S}{2} g_{ij} &= \kappa T_{ij} \\
-\frac{S^{(j)}}{T_{(1)(1)}} - \frac{R - S}{2} h_{11} g^{j} &= \kappa T^{(j)}_{(1)(1)} \\
0 &= T_{11}, \quad R_{i1} = \kappa T_{i1} \\
0 &= T_{(1)(1)}, \quad P_{i(1)}^{(j)} = \kappa T_{(1)(1)}^{(j)} \quad P_{(1)(1)}^{j} = \kappa T_{(1)(1)}^{j}.
\end{align*}
\]

(15)
(ii) for \( m = \dim T \geq 2 \), we have

\[
\begin{align*}
\chi_{ab} - \frac{\chi + R}{2} h_{ab} &= K T_{ab} \\
R_{ij} - \frac{\chi + R}{2} g_{ij} &= K T_{ij} \\
0 &= T_{ai}, \quad R_{ia} = K T_{ia} \\
0 &= T^{(j)}_{(a) b}, \quad 0 = T^{(j)}_{a (b)} \\
0 &= T^{(j)}_{i (b)}, \quad 0 = T^{(j)}_{(a) j},
\end{align*}
\]

(16)

where \( T_{AB}, A, B \in \{ a, i, (a), (i) \} \) represent the adapted components of the stress-energy d-tensor of matter \( T \).

**Remark 21** In order to have the compatibility of the system of geometrical Einstein-like equations, it is necessary the "a priori" vanishing of certain adapted components of the stress-energy d-tensor of matter \( T \).

From a physical point of view, it is well known that in the classical Riemannian theory of gravity, the stress-energy d-tensor of matter have to verify some conservation laws. By a natural extension of the classical Riemannian conservation laws, in our geometrical Hamiltonian context, we postulate the following **generalized conservation laws** of the stress-energy d-tensor of polymomenta \( T \):

\[
T_{A B}^{B} = 0, \quad \forall A \in \{ a, i, (a), (i) \},
\]

where \( T_{A B}^{B} = G^{BD} T_{DA} \). Consequently, by straightforward computations, we obtain

**Theorem 22** The **generalized conservation laws** of the Einstein-like equations of the multi-time Hamilton space \( MH^{n}_{m} \) are expressed by the following formulas:

(i) for \( m = \dim T = 1 \), we have

\[
\begin{align*}
\left[ \frac{R - S}{2} \right]_{1}^{1} &= R_{1}^{(r)} - P_{1(1)}^{(r)} \\
\left[ R_{j}^{r} - \frac{R - S}{2} \delta_{j}^{r} \right]_{r}^{(r)} &= P_{j(1)}^{(r)} \\
\left[ S_{(1)(1)}^{(1)(j)} + \frac{R - S}{2} \delta_{j}^{r} \right]_{(1)}^{(r)} &= -P_{(j)(1)}^{(r)},
\end{align*}
\]

(17)

where
\[
R^i_1 = g^{iq} R^q_1, \quad P^{(1)}_{(1)i} = h^{11} g^{iq} P^{(q)}_{(1)1}, \quad R^j_i = g^{iq} R^q_j,
\]
\[
P^{(1)}_{(1)ij} = h^{11} g^{iq} P^{(q)}_{(1)j}, \quad P^{(j)}_{(1)i} = g^{iq} P^{(q)}_{j(1)}, \quad S^{(1)(j)}_{(1)(1)} = h^{11} g^{iq} S^{(q)(j)}_{(1)(1)}.
\]

(ii) for \( m = \text{dim} \mathcal{T} \geq 2 \), we have

\[
\begin{align*}
\left[ \chi^f_b - \frac{X + R}{2} \delta^f_b \right] & = -R^r_{bj} \\
\left[ R^r_j - \frac{X + R}{2} \delta^r_j \right] & = 0,
\end{align*}
\]

where

\[
\chi^f_b = h^f_c \chi^c_b, \quad R^i_j = g^{iq} R^q_j, \quad R^i_b = g^{iq} R^q_b.
\]

Open problem. The authors of this paper consider that the finding of a possible real physical meaning of the present multi-time Hamiltonian geometrical theories of gravity and electromagnetism may be an open problem for physicists.

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