ERGODIC SUBEQUIVALENCE RELATIONS INDUCED
BY A BERNOUlli ACTION

by
IONUT CHIFAN and ADRIAN IOANA

ABSTRACT. Let $\Gamma$ be a countable group and denote by $S$ the equivalence relation induced by the Bernoulli action $\Gamma \actson [0, 1]^\mathbb{F}$, where $[0, 1]^\mathbb{F}$ is endowed with the product Lebesgue measure. We prove that for any subequivalence relation $R$ of $S$, there exists a partition $\{X_i\}_{i \geq 0}$ of $[0, 1]^\mathbb{F}$ with $R$-invariant measurable sets such that $R|_{X_0}$ is hyperfinite and $R|_{X_i}$ is strongly ergodic (hence ergodic), for every $i \geq 1$.

§1. Introduction and statement of results.

During the past decade there have been many interesting new directions arising in the field of measurable group theory. One direction came from the deformation/rigidity theory developed recently by S. Popa in order to study group actions and von Neumann algebras ([P5]). Using this theory, Popa obtained striking rigidity results concerning the equivalence relations and the $\Pi_1$ factors induced by a Bernoulli action ([P1-4]).

To recall these results, let $\Gamma$ be a countable group and let $\Gamma \actson [0, 1]^\mathbb{F}$ be the Bernoulli action, where $[0, 1]^\mathbb{F}$ is endowed with the product Lebesgue measure. Denote by $S$ the induced equivalence relation, i.e. $(x_\gamma) \sim (y_\gamma)$ if there exists $\gamma'$ such that $x_\gamma = y_{\gamma' \gamma}$, for all $\gamma$ in $\Gamma$. Popa then proved that if $\Gamma$ satisfies a strong non-amenability condition (e.g. if $\Gamma$ has property (T) or splits as a non-amenable product of two infinite groups), then $S$ remembers both the group and the action ([P3,4]). Moreover, he showed that for any non-amenable group $\Gamma$, the associated $\Pi_1$ factor $L(S)$ is prime ([P4]).

The main goal of this paper is to present a new rigidity phenomenon displayed by the equivalence relation $S$. More precisely, we show that for any countable group $\Gamma$, we have the following structure result for the subequivalence relations of $S$:

**Theorem 1.** Let $R \subset S$ be a subequivalence relation. Then there exists a measurable partition $\{X_i\}_{i \geq 0}$ of $[0, 1]^\mathbb{F}$ with $R$-invariant sets such that

(a). $R|_{X_0}$ is hyperfinite.
(b). $R|_{X_i}$ is strongly ergodic (therefore ergodic), for all $i \geq 1$.

Moreover, the same holds for any quotient equivalence relation $R'$ of $R$. 

Typeset by AM\$-TEX
In particular, Theorem 1 shows that for the non-hyperfinite subequivalence relations of $S$, the notions of ergodicity and strong ergodicity are equivalent. For a more general statement, see Theorem 7.

To prove Theorem 1, we follow an operator algebra approach using Popa’s deformation/rigidity strategy. In this respect, recall first that to every countable, measure preserving (m.p.), equivalence relation $R$ one can associate a finite von Neumann algebra $L(R)$ ([FM]). Then an inclusion of equivalence relations $R \subset S$ gives an inclusion of von Neumann algebras $L(R) \subset L(S)$. On the other hand, remark that for our particular $S$, we can view $L(S)$ as the wreath product $II\_1$ factor $(\bigotimes_{\Gamma} L\_\infty[0,1]) \rtimes \Gamma$. Altogether, we get that $L(R) \subset (\bigotimes_{\Gamma} L\_\infty[0,1]) \rtimes \Gamma$. Theorem 1 is then a consequence of the following general result on controlling relative commutants in wreath product factors.

**Theorem 2.** Let $(B, \tau)$ be an amenable, finite von Neumann algebra and let $\Gamma$ be a countable group. On the infinite tensor product $\bigotimes_{\Gamma} B$ consider the Bernoulli action of $\Gamma$ and denote by $M$ the crossed product $(\bigotimes_{\Gamma} B) \rtimes \Gamma$. Let $p \in M$ be a projection, let $P \subset pMp$ be a von Neumann subalgebra with no amenable direct summand and denote $Q = P' \cap pMp$.

(i) Then there exists a non-zero partial isometry $v \in M$ such that $v^*v \in Q' \cap pMp$ and $vQv^* \subset L\Gamma$.

(ii) Moreover, if $\Gamma$ is ICC (infinite conjugacy class), then there exists a unitary $u \in M$ such that $u(P \vee Q)u^* \subset L\Gamma$.

Theorem 2 has been proved by Popa (Lemma 5.2. in [P4]) under the additional assumption that no corner of $Q$ embeds into $\bigotimes_{\Gamma} B$ by using the malleable deformations of Bernoulli actions. In fact, our proof of Theorem 2 follows closely the proof of Lemma 5.2. in [P4]. Indeed, just as in [P4], we use a spectral gap argument to show that since $Q$ is the commutant of a non-amenable algebra, then it behaves, in some sense (i.e. with respect to certain deformations of $M$), as a rigid subalgebra of $M$.

The main difference in our approach is that we use the (weakly) malleable deformations of Bernoulli actions considered in section 2 of [I] rather than the malleable ones. The benefit of this approach is that we can apply Theorem 3.3. in [I] to get precise information on the position of the "rigid" subalgebra $Q$ in the wreath product factor $M$.

We point out that in the proof of Theorem 1 we actually only use the following consequence of Theorem 2. We will later see that the next Corollary is in fact equivalent to Theorem 1 (Proposition 6).

**Corollary 3.** Assume that $(B, \tau)$ is amenable. Let $p \in M$ be a projection and let $Q \subset p(\bigotimes_{\Gamma} B)p$ be a diffuse von Neumann algebra.

(i) Then $Q' \cap pMp$ is amenable.
(ii) Moreover, if $N \subset pMp$ is a non-amenable subfactor containing $Q$, then $N' \cap Q^\omega = \mathbb{C}1$.

In the case $\Gamma$ is exact, part (i) of Corollary 3 has been first proved by N. Ozawa using $C^*$-algebraic methods (Theorem 4.7. in [O2]). Ozawa’s result thus provides a different approach to proving Theorem 1, in the exact case.

Aside from the introduction, this paper has three more sections. The second section deals with the technical tools that are needed in the proof of Theorem 2. In Section 3, we first prove Theorem 2 and then deduce Corollary 3 and Theorem 1. Lastly, in Section 4, we note a few applications of our main results.

Acknowledgment. We are grateful to Professor Sorin Popa for suggesting to us the idea from which this project arose. The second author is also grateful to Professor Alekos Kechris for useful discussions.

§2. Technicalities.

In this Section we discuss some of the main ingredients of the proof of Theorem 2. Since the context of this proof is the same as in Section 2 in [I], we begin by recalling some notations and constructions from there. Let $(B, \tau)$ be a finite von Neumann algebra with a normal, faithful trace $\tau$. For a countable set $I$, we denote by $\bigotimes_{i \in I} B$ the tensor product von Neumann algebra $\otimes_{i \in I} (B, \tau)_i$.

Let $(\hat{B}, \hat{\tau})$ denote the free product von Neumann algebra of $(B, \tau)$ with the group von Neumann algebra $L\mathbb{Z}$. On $\bigotimes_{\Gamma} \hat{B}$ consider the Bernoulli action of $\Gamma$ given by $\hat{\sigma}(\gamma)(\otimes_{\gamma'} x_{\gamma'}) = \otimes_{\gamma'} x_{\gamma^{-1} \gamma'}$. Then $\hat{\sigma}$ leaves the subalgebra $\bigotimes_{\Gamma} B \subset \bigotimes_{\Gamma} \hat{B}$ invariant and the restriction $\sigma = \hat{\sigma}|_{\bigotimes_{\Gamma} B}$ is precisely the Bernoulli action of $\Gamma$ on $\bigotimes_{\Gamma} B$. Denote by $M$ (resp. $\hat{M}$) the crossed product von Neumann algebra $(\bigotimes_{\Gamma} B) \rtimes_{\sigma} \Gamma$ (resp. $(\bigotimes_{\Gamma} \hat{B}) \rtimes_{\hat{\sigma}} \Gamma$).

Also, denote by $\{u_\gamma\}_{\gamma \in \Gamma} \subset \hat{M}$ the canonical unitaries implementing $\hat{\sigma}$.

Next, let $u \in L\mathbb{Z}$ be the generating Haar unitary and let $h = h^* \in L\mathbb{Z}$ such that $u = \exp(ih)$. For every $t \in \mathbb{R}$, define the unitary $u_t = \exp(ith) \in L\mathbb{Z}$. Then $\text{Ad}(u_t) \in \text{Aut}(\hat{B})$, so can define the tensor product automorphism

$$\theta_t = \otimes_{\gamma \in \Gamma} (\text{Ad}(u_t))_\gamma \in \text{Aut}(\bigotimes_{\Gamma} \hat{B}), \forall t.$$  

Since $\theta_t$ commutes with $\hat{\sigma}$, it extends to an automorphism $\theta_t$ of $\hat{M}$, for all $t$. Moreover, since $\lim_{t \to 0} ||u_t - 1||_2 = 0$, it follows that $\theta_t \to \text{id}$, as $t \to 0$, in the pointwise $||\cdot||_2$-topology.

Note that our proof of Theorem 2 parallels the proof of Lemma 5.2. in [P4]. For this reason we need to re-establish two main ingredients used there: the "transversality"
of the deformation $\theta_t$ (Lemma 2.1. in [P4]) and the "spectral gap" property of the inclusion $M \subset \tilde{M}$ provided by Lemma 5.1. in [P4].

Firstly, let $\beta$ be the automorphism of $\bigotimes_\Gamma \tilde{B}$ defined by $\beta|_{\bigotimes_\Gamma \tilde{B}} = \text{id}|_{\bigotimes_\Gamma \tilde{B}}$ and by $\beta((u)\gamma) = (u^*)\gamma$, for every $\gamma \in \Gamma$. Since $\beta$ commutes with $\tilde{\sigma}$, it extends to an automorphism of $\tilde{M}$, still denoted $\beta$, which satisfies $\beta^2 = \text{id}$, $\beta|_M = \text{id}_M$ and $\beta \theta_t \beta = \theta_{-t}$, for all $t \in \mathbb{R}$. As pointed out by Popa (Lemma 2.1 in [P4]), any deformation $\theta_t$ which possesses a symmetry $\beta$ satisfying these relations, automatically verifies the following "transversality" condition:

**Lemma 4 ([P4]).** For all $t$ and for all $x \in M$ we have that

$$||\theta_{2t}(x) - x||_2 \leq 2||\theta_t(x) - E_M(\theta_t(x))||_2.$$ 

Secondly, we show that the statement of Lemma 5.1. in [P4] still holds true in our context. Although our proof goes along the same lines as in [P4], there are a few computational differences, which we address below. Briefly, the difference comes from the fact that in [P4], the algebra $\tilde{B}$ is the tensor product $B \bigotimes \mathbb{R}$, rather than the free product $B \ast L\mathbb{Z}$, as in our case. This implies that as a $B-B$ bimodule, $L^2\tilde{B}$ is isomorphic to $L^2B^{\bigoplus \infty}$, in the context of [P4], and, respectively, to $L^2B \oplus (L^2B \bigotimes L^2B)^{\bigoplus \infty}$, in our context. In turn, this gives different formulas for the $M-M$ bimodule $L^2\tilde{M}$, depending on the context.

**Lemma 5.** Assume that $(B, \tau)$ is amenable and let $P \subset M$ be a von Neumann subalgebra with no amenable direct summand. Then $P' \cap M^\omega \subset M^\omega$.

**Proof.** We first prove that the $M-M$ Hilbert bimodule $L^2\tilde{M} \ominus L^2M$ is weakly contained in the $M-M$ Hilbert bimodule $(L^2M \bigotimes L^2M)^{\bigoplus \infty}$.

For this, let $\mathcal{B} = \{\xi_k\}_{k \in \mathbb{N}} \subset B$ be an orthonormal basis for $L^2B$ with $\xi_0 = 1$. Then $\tilde{\mathcal{B}} = \{u^{n_1} \xi_{i_1} u^{n_2} \xi_{i_2} \ldots \xi_{i_k} | n_1, \ldots, n_k \in \mathbb{Z}, i_1, \ldots, i_k \in \mathbb{N}, k \in \mathbb{N}\}$ is an orthonormal basis for $L^2\tilde{B}$ which contains $\mathcal{B}$. Since $\tilde{\mathcal{B}} \setminus \mathcal{B}$ is infinite, we can enumerate $\tilde{\mathcal{B}} \setminus \mathcal{B} = \{\xi_k\}_{k \in \mathbb{Z} \setminus \mathbb{N}}$.

Let $\tilde{I}$ (resp. $I$) be the set of sequences $i = (i_g)_{g \in \Gamma}$ with $i_g \in \mathbb{Z}$, for all $g \in \Gamma$ (resp. $i_g \in \mathbb{N}$, for all $g \in \Gamma$) such that $\Delta_i := \{g \in \Gamma | i_g \neq 0\}$ is finite. For every $i \in \tilde{I}$, define $\eta_i = \bigotimes g \xi_{i_g}$. Then $\tilde{C} = \{\eta_i | i \in \tilde{I}\}$ and $C = \{\eta_i | i \in I\}$ are orthonormal bases for $L^2(\bigotimes_\Gamma \tilde{B})$ and respectively for $L^2(\bigotimes \Gamma B)$.

Now, let $J$ be the set of $i = (i_g)_{g \in \Gamma} \in \tilde{I}$ having the property that for every $i_g \neq 0$ the element $\xi_{i_g}$ starts and ends with a nonzero power of $u$. Then it is clear that

$$L^2(\tilde{M}) \ominus L^2(M) = \bigoplus_{i \in J} L^2(M \eta_i \tilde{M})$$

as Hilbert $M-M$ bimodules.
Note that the Bernoulli action induces an action of $\Gamma$ on $\bar{I}$ by translation and that for every element $i \in \bar{I}$ its stabilizing group $\Gamma_i \leq \Gamma$ under this action is finite. Since $\Gamma_i$ satisfies $\Gamma_i(\Gamma \setminus \Delta_i) = \Gamma \setminus \Delta_i$ we can consider the following crossed product algebra $K_i := (\bigotimes_{\Gamma \setminus \Delta_i} B) \rtimes \Gamma_i$.

Next, we claim that for all $i \in J$ we have that

$$L^2(M_{\eta_i} \chi) \cong L^2((M, K_i), Tr)$$

as Hilbert $M - M$ bimodules via the map $x\eta_i y \rightarrow xe_{K_i}y$, where as usual $(M, K_i)$ denotes the basic construction corresponding to the inclusion $K_i \subset M$ and $Tr$ is the canonical trace on it. To show this it suffices to verify that

$$(3) \quad <(xu_\gamma)\eta_i(yu_{\gamma'}), \eta_i >_\gamma = <(xu_\gamma)e_{K_i}(yu_{\gamma'}), e_{K_i} >_{Tr}, \forall x, y \in \bigotimes_{\Gamma} B, \forall \gamma, \gamma' \in \Gamma$$

Note that every element of $\bigotimes_{\Gamma} B$ can be approximated in $|||\cdot|||$ by finite linear combinations of elements from $C$. Thus, in order to prove identity (3) we can assume that $x$ and $y$ are of the form $x = \otimes x_g, y = \otimes y_g$, where $x_g, y_g \in B$, for all $g \in \Gamma$ and $x_g = y_g = 1$, for all but finitely many $g \in \Gamma$. For such $x$ and $y$ the left side of (3) equals

$$(4) \quad <(xu_\gamma)\eta_i(yu_{\gamma'}), \eta_i >_\gamma = \delta_{\gamma\gamma'}, e\tau(\eta_i^*x\delta(\gamma)(\eta_iy)) = \delta_{\gamma\gamma'}, e\prod_{g \in \Gamma} \tau(\xi_{i_g}^* x_{i_g} \xi_{i_{\gamma'g}} y_{i_{\gamma'g}})$$

Remark that if $\xi_{i_1}, \xi_{i_2} \in \tilde{B}$ are either equal to 1 or start and end with a non-zero power of $u$, then for all $x, y \in B$ we have that $\tau(\xi_{i_1}^* x_{i_2} y) = 0$, unless $i_1 = i_2$. Moreover, if $i_1 = i_2 \neq 0$, then $\tau(\xi_{i_1}^* x_{i_2} y) = \tau(x)\tau(y)$ and if $i_1 = i_2 = 0$, then $\tau(\xi_{i_1}^* x_{i_2} y) = \tau(xy)$. Using this remark and (4), we get that left side of (3) equals

$$(5) \quad 1_{\Gamma_i}(\gamma)\delta_{\gamma\gamma'}, e\prod_{g \in \Delta_i} \tau(x_g)\tau(y_{\gamma-1g}) \prod_{g \in \Gamma \setminus \Delta_i} \tau(x_g y_{\gamma-1g})$$

Now, note that if $x = \otimes x_g$ and $\gamma \in \Gamma$, then $E_{K_i}(xu_\gamma) = 1_{\Gamma_i}(\gamma) \prod_{g \in \Delta_i} \tau(x_g)\otimes_{g \in \Gamma \setminus \Delta_i} x_g$. Using this, it is immediate that (5) equals

$$\tau(E_{K_i}(xu_\gamma)E_{K_i}(yu_{\gamma'})) = <(xu_\gamma)e_{K_i}(yu_{\gamma'}), e_{K_i} >_{Tr}.$$

This proves (3) and consequently (2).

Finally, since $B$ is amenable and $\Delta_i$ is finite, we get that $K_i$ is an amenable von Neumann algebra, for all $i \in J$. This further implies that $L^2((M, K_i), Tr) \prec L^2(M)\overline{\otimes} L^2(M)$ and by combining this fact with (1) and (2), we get the desired weak containment of Hilbert bimodules. From this it follows, by the argument in the proof of Lemma 2.2. in [P6], that $P' \cap M^\omega \subset M^\omega$.

We end this section by recalling Popa’s technique of intertwining subalgebras in a finite von Neumann algebra. For this, let \((M, \tau)\) be a finite von Neumann algebra with a normal, faithful trace \(\tau\) and let \(Q, B \subset M\) be two von Neumann subalgebras. Let \(E_B : M \to B\) denote the \(\tau\)-preserving conditional expectation onto \(B\). Then we say that a corner of \(Q\) embeds into \(B\) if one of the following equivalent conditions (see section 2 in [P1] for the proof of the equivalence) holds true:

(a) There exist non-zero projections \(q \in Q\) and \(p \in B\), a normal \(*\)-homomorphism \(\phi : qQq \to pBp\) and a non-zero partial isometry \(v \in M\) such that \(xv = v\phi(x)\), for all \(x \in qQq\) and \(vv^* \in (qQq)' \cap qMq\), \(v^*v \in (\phi(qQq))' \cap pMp\).

(b) There exist \(\varepsilon > 0\) and \(a_1, ..., a_n \in M\) such that \(\sum_{i,j=1}^n ||E_B(a_iua_j^*)||_2^2 \geq \varepsilon\), for all \(u \in U(Q)\).

Also note that if \(B_1, ..., B_k\) are von Neumann subalgebras such that there exist \(\varepsilon > 0\) and \(b_1, ..., b_k \in M\) such that \(\sum_{i=1}^k ||E_{B_i}(b_iub_i^*)||_2^2 \geq \varepsilon, \forall u \in U(Q)\), then a corner of \(Q\) embeds into \(B_i\), for some \(i \in \{1, ..., k\}\) (see the proof of 4.3. in [IPP]).

§ 3. Proofs of main results.

Proof of Theorem 2. By working with amplifications we can assume \(p = 1\). Also, we can assume that \(Q\) is diffuse.

We first use a spectral gap argument due to Popa to show that \(\theta_t\) converge uniformly to \(\text{id}_\delta\) on \((Q)_1\) ([P4]). To this end, let \(\varepsilon > 0\). Since \(P\) has no-amenable direct summand, by Lemma 5 we have that \(P' \cap \delta \subset M^\omega\). Therefore we can find unitaries \(u_1, u_2, ..., u_n\) and \(\delta > 0\) such that if \(x \in M_1\) satisfies \(||[u_i, x]||_2 \leq \delta, \text{ for all } i \in \{1, ..., n\}, \text{ then } ||x - E_M(x)||_2 \leq \varepsilon. \text{ Let } t_0 > 0 \text{ such that for all } t \in [0, t_0] \text{ and all } i, \text{ we have that } ||\theta_t(u_i) - u_i||_2 \leq \delta/2. \text{ Then we deduce that for all } x \in (Q)_1, ||[u_i, \theta_t(x)]||_2 \leq 2||\theta_t(u_i) - u_i||_2 + ||[\theta_t(u_i), \theta_t(x)]||_2 \leq \delta, \forall i \in \{1, ..., n\}.\)

Thus, by the above we get that \(||\theta_t(x) - E_M(\theta_t(x))||_2 \leq \varepsilon, \text{ for all } x \in (Q)_1. \text{ Lemma 4 then gives that } ||\theta_{2t}(x) - x||_2 \leq 2\varepsilon, \text{ for all } x \in (Q)_1 \text{ and all } t \in [0, t_0], \text{ hence } \theta_t \text{ converges uniformly to } \text{id}_\delta \text{ on the unit ball of } Q. \)

Now, recall that Theorem 3.3. in [I] asserts that if \(Q \subset M\) is a relatively rigid von Neumann subalgebra then either (1) a corner of \(Q\) embeds either into \(L\Gamma\) or (2) a corner of \(Q\) embeds into \(\bigotimes_{F} B\), for some finite set \(F \subset \Gamma\). Since the proof of 3.3. in [I] only uses the fact that \(\theta_t\) converges uniformly \(\text{id}_\delta\) on \((Q)_1\), we deduce that either (1) or (2) above are satisfied in our situation.

In case (1), the conclusion follows from the proof of Lemma 5.2. in [P4]. So, to end the proof we need to argue that case (2) leads to a contradiction. Assume therefore that there exist a finite set \(F \subset \Gamma\), projections \(p \in \bigotimes_{F} B, q \in Q\), a homomorphism
\[ \phi : qQq \to p(\bigotimes_f B)p \] and a non-zero partial isometry \( v \in M \) such that \( vv^* \leq q \) and \( xv = v\phi(x) \), for all \( x \in qQq \).

Then \( \phi(qQq) \) is a diffuse von Neumann subalgebra of \( p(\bigotimes_f B)p \). Hence if we denote \( K = FF^{-1} \), then Lemma 1.5. in [I] implies that
\[ \phi(qQq)' \cap pMp \subset \sum_{\gamma \in K} (\bigotimes_f B)u_\gamma. \]

Since \( P \subset Q' \cap M \), it follows that \( v^* P v \subset \phi(qQq)' \cap pMp \). Thus,
\[ v^* P v \subset \sum_{\gamma \in K} (\bigotimes_f B)u_\gamma, \]
hence a corner of \( P \) embeds into \( \bigotimes_f B \). This is however a contradiction, since \( B \) is amenable, while \( P \) has no amenable direct summand.

**Proof of Corollary 3.** To prove (i), suppose by contradiction that \( P = Q' \cap pMp \) is non-amenable. Then we can find a non-zero projection \( z \in Z(P) \) such that \( Pz \) has no amenable direct summand. Since \( [Pz, Qz] = 0 \), by applying Theorem 2, we deduce that there exists a non-zero partial isometry \( v \in M \) such that \( v^* v \in (Qz)' \cap zMz \) and \( vQv^* \subset LT \). Since \( Q \) is diffuse it contains a sequence of unitaries \( \{u_n\}_{n \geq 0} \) which tends weakly to 0. Then \( ||vu_n v^*||_2 = ||v^*||_2 \), for all \( n \). On the other hand, since \( u_n \in p(\bigotimes_f B)p \), for all \( n \) and since \( u_n \to 0 \) weakly, one easily gets that \( ||E_{LT}(vu_n v^*)||_2 \to 0 \), as \( n \to \infty \). This gives a contradiction, thus \( P \) has to be amenable.

For the proof of (ii), suppose that \( N' \cap Q'' \neq C1 \). Since \( N \) is a non-amenable factor, then a result due to Popa (see Lemma 7 in [O1]) implies that we can find a diffuse von Neumann subalgebra \( Q_0 \) of \( Q \) such that \( P = Q'_0 \cap N \) is non-amenable. Part (i) of this corollary then leads to a contradiction.

In the proof of the next result we will use Feldman-Moore’s construction of the von Neumann algebra associated to an equivalence relation, which we now recall ([FM]). Let \( R \) be a countable, m.p. equivalence relation on a probability space \((X, \mu)\). The **full group** of \( R \) (denoted \([R]\)) is the group of automorphisms \( \phi \in \text{Aut}(X, \mu) \) such that \( \phi(x) \sim_R x \), a.e. \( x \in X \). On \( R \) consider the measure \( \nu \) given by \( \nu(K) = \int_X |K \cap R^x| d\mu(x) \), for every \( K \subset R \), and let \( L^2(R, \nu) \) be the associated Hilbert space. For every \( \phi \in [R] \), define the unitary \( u_\phi \in \mathbb{B}(L^2(R, \nu)) \) by
\[ (u_\phi g)(x, y) = g(\phi(x), y), \forall g \in L^2(R, \nu), (x, y) \in R. \]

Also, represent \( L^\infty(X, \mu) \) on \( L^2(R, \nu) \) by
\[ L_f(g)(x, y) = f(x)g(x, y), \forall f \in L^\infty(X, \mu), g \in L^2(R, \nu), (x, y) \in R. \]
The **von Neumann algebra associated to** \( \mathcal{R} \) is then defined as the von Neumann algebra generated by \( \{ u_\phi | \phi \in [\mathcal{R}] \} \) and \( L^\infty(X, \mu) \) and is denoted \( L(\mathcal{R}) \). Note that \( L^\infty(X, \mu) \) is a **Cartan subalgebra** of \( L(\mathcal{R}) \), i.e. is regular and maximal abelian and that \( L(\mathcal{R}) \) is a factor if and only if \( \mathcal{R} \) is **ergodic**. Moreover, every central projection \( p \) of \( L(\mathcal{R}) \) is of the form \( p = 1_Y \), where \( Y \subset X \) is an \( \mathcal{R} \)-invariant measurable subset. In this case, the Cartan subalgebra inclusion associated to \( \mathcal{R}|_Y \) is isomorphic to \( (L^\infty(X, \mu)p \subset L(\mathcal{R})p) \).

Also, recall that \( \mathcal{R} \) is called **strongly ergodic** if whenever \( \{A_n\}_n \subset X \) is a sequence of measurable sets such that \( \lim_{n \to \infty} \mu(A_n) = 0 \), then \( \lim_{n \to \infty} \mu(A_n)(1 - \mu(A_n)) = 0 \) ([Sc], see also [CW],[JS]). By a result of A. Connes, \( \mathcal{R} \) is strongly ergodic if and only if \( L(\mathcal{R})' \cap [L^\infty(X, \mu)]^\omega = \mathcal{C} \), where \( \omega \) is a free ultrafilter on \( \mathbb{N} \) ([C]).

**Proposition 6.** Let \( \mathcal{S} \) be a countable, m.p. equivalence relation on a probability space \( (X, \mu) \). Then the following are equivalent:

1. \( Q' \cap L(\mathcal{S}) \) is amenable, for any diffuse von Neumann subalgebra \( Q \) of \( L^\infty(X, \mu) \).

2. For any subequivalence relation \( \mathcal{R} \subset \mathcal{S} \), there exists a partition \( \{ X_i \}_{i \geq 0} \) of \( X \) with \( \mathcal{R} \)-invariant sets such that (a) \( \mathcal{R}|_{X_0} \) is hyperfinite and (b) \( \mathcal{R}|_{X_i} \) is ergodic and non-hyperfinite, \( \forall i \geq 1 \).

3. For any subequivalence relation \( \mathcal{R} \subset \mathcal{S} \), there exists a partition \( \{ X_i \}_{i \geq 0} \) of \( X \) with \( \mathcal{R} \)-invariant sets such that (a) \( \mathcal{R}|_{X_0} \) is hyperfinite and (b) \( \mathcal{R}|_{X_i} \) is strongly ergodic, \( \forall i \geq 1 \).

4. (3) holds true for any quotient \( \mathcal{R}' \) of a subequivalence relation \( \mathcal{R} \) of \( \mathcal{S} \).

5. For any non-atomic probability space \( (Y, \nu) \) and for every m.p., onto map \( p : X \to Y \), the equivalence relation \( \mathcal{T} = \{(x, y) \in \mathcal{S} | p(x) = p(y)\} \) is hyperfinite.

6. For any non-atomic probability space \( (Y, \nu) \), for every m.p., onto map \( p : X \to Y \) and for every hyperfinite equivalence relation \( \mathcal{V} \) on \( Y \), the equivalence relation \( \mathcal{T} = \{(x, y) \in \mathcal{S} | p(x) \sim_\mathcal{V} p(y)\} \) is hyperfinite.

**Proof.** We first prove the equivalence of the conditions (1)-(4) and then we show that (1), (5) and (6) are equivalent.

(1) \( \implies \) (2). Assume that (1) holds true and let \( \mathcal{R} \subset \mathcal{S} \) be a subequivalence relation. Denote by \( \mathcal{Z} \) the center of \( L(\mathcal{R}) \) and let \( p_0 \in \mathcal{Z} \) be the maximal projection such that \( L(\mathcal{R})p_0 \) is amenable.

We claim that \( \mathcal{Z}(1 - p_0) \) is completely atomic. If not, then we could find a non-zero projection \( q \) of \( \mathcal{Z} \) such that \( q \leq 1 - p_0 \) and \( \mathcal{Z}q \) is diffuse. Thus, \( Q = \mathcal{Z}q \oplus L^\infty(X, \mu)(1 - q) \) is a diffuse subalgebra of \( L^\infty(X, \mu) \), so by (1) its relative commutant \( Q' \cap L(\mathcal{S}) \) is amenable. In particular, it follows that \( L(\mathcal{R})q \) is amenable, a contradiction to the maximality of \( p_0 \). Altogether, we derive that \( \mathcal{Z}(1 - p_0) \) is completely atomic, hence we can write \( \mathcal{Z}(1 - p_0) = \oplus_{i \geq 1} \mathcal{C} \), for some projections \( p_i \in \mathcal{Z} \).
Now, for every $i \geq 0$, let $X_i \subset X$ be a $\mathcal{R}$-invariant measurable set such that $p_i = 1_{X_i}$. Since $L(\mathcal{R}) p_0$ is amenable, Connes-Feldman-Weiss’ theorem ([CFW]) implies that $\mathcal{R}|_{X_0}$ is hyperfinite. Also, since $L(\mathcal{R}) p_i$ is a non-amenable factor, we get that $\mathcal{R}|_{X_i}$ is ergodic and non-hyperfinite, for all $i \geq 1$.

$(1) \implies (3)$. In the above context, it suffices to show that $\mathcal{R}|_{X_i}$ is strongly ergodic, for all $i \geq 1$. Assume by contradiction that $\mathcal{R}|_{X_i}$ is not strongly ergodic, for some $i \geq 1$. Then the induced Cartan subalgebra inclusion satisfies $[L(\mathcal{R}) p_i] \cap [L^\infty(X, \mu) p_i] \omega \neq \mathbb{C}$. Since $L(\mathcal{R}) p_i$ is a non-amenable $\Pi_1$ factor, then, as in the proof of part (ii) of Corollary 3, we can find a diffuse von Neumann subalgebra $Q$ of $L^\infty(X, \mu) p_i$, such that $Q \cap L(\mathcal{R}) p_i$ is non-amenable. This however, violates $(1)$.

$(1) \implies (4)$. Let $\mathcal{R}$ be a subequivalence relation of $S$ and let $\mathcal{R}'$ be a quotient of $\mathcal{R}$. Recall that this means that there exists a measurable, measure preserving, onto map $p : (X, \mu) \to (X', \mu')$ and a set $N \subset X$ with $\mu(X \setminus N) = 0$ such that for all $x \in N$, $p$ is a bijection between the $\mathcal{R}$-orbit of $x$ and the $\mathcal{R}'$-orbit of $p(x)$. Then as noted in [P3] (see 1.4.3.) the von Neumann algebra embedding $\theta : L^\infty(X', \mu') \to L^\infty(X, \mu)$ given by $\theta(f) = f \circ p$, for all $f \in L^\infty(X', \mu')$, extends to an embedding $\theta : L(\mathcal{R}') \to L(\mathcal{R})$. Using this observation, the above proof applies verbatim to show that $\mathcal{R}'$ verifies $(3)$.

Since it is obvious that $(4) \implies (3) \implies (2)$, to complete the proof of the equivalence of conditions $(1)-(4)$ we only need to show that $(2) \implies (1)$. To this end, let $Q \subset L^\infty(X, \mu)$ be a diffuse von Neumann subalgebra and denote $P = Q \cap L(\mathcal{R})$. Since $L^\infty(X, \mu) \subset P$, by a result of H. Dye ([D]), we can find a subequivalence relation $\mathcal{R}$ of $S$ such that $P = L(\mathcal{R})$. Now, $(2)$ implies that if $Z$ denotes the center of $L(\mathcal{R})$ and if $p_0 \in Z$ is the maximal projection such that $L(\mathcal{R}) p_0$ is amenable, then $Z(1 - p_0)$ is completely atomic. On the other hand, since $Q \subset Z$, we get that $Z$ is diffuse. Hence, we must have that $p_0 = 1$, therefore $P$ is amenable.

$(1) \iff (5)$. Note that every diffuse von Neumann subalgebra $Q$ of $L^\infty(X, \mu)$ is of the form $L^\infty(Y, \nu) \approx \{ f \circ p | f \in L^\infty(Y, \nu) \}$, where $(Y, \nu)$ is a non-atomic probability space and $p : X \to Y$ is a measure preserving, onto map. For $Q = L^\infty(Y, \mu)$, we claim that $Q' \cap L(S) = L(T)$, where $T = \{ (x, y) \in S | p(x) = p(y) \}$. Since $L(T)$ is amenable if and only if $T$ is hyperfinite ([CFW]), the claim implies the equivalence of $(1)$ and $(5)$. To prove the claim, let $T' \subset S$ be a subequivalence relation such that $Q' \cap L(S) = L(T')$ ([D]). Thus, if $\phi$ is an automorphism of $(X, \mu)$, then $\phi \in [T']$ if and only if $[u_\phi, Q] = 0$. On the other hand, the fact that $u_\phi$ commutes with $Q = L^\infty(Y, \nu)$ is equivalent to $p(\phi(x)) = p(x)$, a.e. $x \in X$. Since the latter condition is in turn equivalent to $\phi \in [T]$, we altogether get that $[T'] = [T]$, thus $T' = T$.

$(5) \iff (6)$. Since $(6)$ clearly implies $(5)$, we only need to show the converse. For this, assume that $(5)$ holds, let $p : X \to Y$ be a m.p., onto map and let $\mathcal{V} = \cup_n \mathcal{V}_n$ be a hyperfinite equivalence relation, where $\mathcal{V}_n$ are finite equivalence relations. For every $n$, let $\pi_n : Y \to Y_n := Y/\mathcal{V}_n$ be the natural projection. Then $\nu_n = \pi_{n*}(\nu)$ is a non-atomic probability measure. Since the map $\pi_n \circ p : X \to Y_n$ is m.p. and
onto, by (5) we deduce that the equivalence relation $T_n = \{(x, y) \in S | \pi_n(p(x)) = \pi_n(p(y))\} = \{(x, y) \in S | p(x) \sim p(y)\}$ is hyperfinite, for all $n$. Finally, since $T = \{(x, y) \in S | p(x) \sim p(y)\} = \cup_n T_n$, we get that $T$ is hyperfinite.

**Proof of Theorem 1.** Denote $B = L^\infty(X, \mu)$ and on $\hat{\otimes}_I B$ we consider the Bernoulli action induced by the action $\Gamma$. Then the inclusion $(L^\infty([0, 1]^I) \subset L(S))$ is naturally identified with the inclusion $(\otimes_I B \subset (\otimes_I B) \rtimes \Gamma)$. Theorem 1 then follows by combining Corollary 3 (i) and Proposition 6.

We end this section by noticing a stronger version of Theorem 1.

**Theorem 7.** Let $\Gamma \curvearrowright I$ be an action of countable group $\Gamma$ on a set $I$ such that the stabilizer $\Gamma_i = \{\gamma \in \Gamma | \gamma i = i\}$ is amenable, for every $i \in I$. Let $(X, \mu)$ be a probability space and let $S_0 \subset X \times X$ be a hyperfinite, measure preserving equivalence relation. On $(X, \mu)^I$ consider the equivalence relation $S$ given by: $(x_i) \sim_S (y_i)$ if there exists $\gamma \in \Gamma$ and $F \subset I$ finite such that $x_i = y_{\gamma^{-1}i}$, for all $i \in I \setminus F$ and $x_i \sim_{S_0} y_{\gamma^{-1}i}$, for all $i \in F$.

Then $S$ verifies the conclusion of Theorem 1.

**Proof.** Let $A \subset M$ be the Cartan subalgebra inclusion associated to $S$. If we denote $B = L(S_0)$, then it is easy to see that $M$ can be identified with $(\hat{\otimes}_I B) \rtimes_\sigma \Gamma$, where $\sigma$ is the Bernoulli action induced by the action $\Gamma \curvearrowright I$. Moreover, under this identification $A \subset \hat{\otimes}_I B$. Also, note that since $S_0$ is hyperfinite, $B$ is amenable.

To get the conclusion, by the proof of Proposition 6, it suffices to show that if $Q \subset \hat{\otimes}_I B$ is a diffuse von Neumann subalgebra, then $P = Q' \cap M$ is amenable. To this end, we follow the same the lines as in the proof of Theorem 2, leaving some of the details to the reader. We start by observing that the context and results of Section 2 extend here. Indeed, denote $M = (\otimes_I B) \rtimes \Gamma$ and $\hat{\Phi} = (\hat{\otimes}_I B) \rtimes \Gamma$, where $\hat{\Phi} = B \rtimes L\mathbb{Z}$. Then, since every stabilizer $\Gamma_i$ is amenable, Lemma 5 holds true in this context. Moreover, if $\theta_t : M \to \hat{\Phi}$ is defined analogously (i.e. $\theta_t (\otimes_i \Gamma_1 B) = \otimes_i (Ad(u_t))_i$ and $\theta_t(u_{\gamma}) = u_{\gamma}$, for all $\gamma \in \Gamma_i$), then Lemma 4 also holds true.

Next, assume by contradiction that $P$ is non-amenable and let $z \in P$ be a central projection such that $Pz$ has no amenable direct summand. Using the spectral gap argument in the proof of Theorem 2, we deduce that $\theta_t$ converges uniformly to $id_{\hat{\Phi}}$ on $(Qz_1 \hat{\otimes}_I B$, the proof of Theorem 3.3. in [1] implies that a corner of $Q$ embeds into $\hat{\otimes}_B$, for some finite set $I \subset I$.

Thus, we find projections $p \in \hat{\otimes}_d B$, $q \in Q$, a homomorphism $\phi : qQq \to p(\hat{\otimes}_d B)p$ and a non-zero partial isometry $v \in M$ such that $vv^* \in (qQq)' \cap qMq$ and $vx = v\phi(x)$, for all $x \in qQq$. Since $\phi(qQq) \subset p(\hat{\otimes}_d B)p$ is a diffuse von Neumann subalgebra, Lemma 1.5. in [1] implies that

$$
\phi(qQq)' \cap pMp \subset \sum_{\gamma \in K} u_{\gamma}(\hat{\otimes}_d B)
$$
where $K = \{\gamma \in \Gamma | \exists i, j \in F, \gamma i = j\}$.

Further, since $P \subset Q' \cap M$, it follows that $v^*Pv \subset \phi(qQq)' \cap pMp$. By using this together with (1) we get that

$$v^*Pv \subset \sum_{i \in F, \gamma \in L} u_\gamma([\bigotimes_{\Gamma} B] \rtimes \Gamma_i)$$

where $L \subset \Gamma$ is a finite set such that $K \subset \cup_{i \in F} L \Gamma_i$. For every $i \in F$, denote by $M_i = [\bigotimes_{\Gamma} B] \rtimes \Gamma_i$, by $e_i : L^2(M) \to L^2(M_i)$ the orthogonal projection and by $E_{M_i} = e_i|_M : M \to M_i$ the conditional expectation onto $M_i$. Then, for every $\gamma \in \Gamma$, $u_\gamma e_i u_\gamma^*$ is the orthogonal projection onto $L^2(u_\gamma M_i)$, thus (2) rewrites as

$$v^*xv = \bigvee_{i \in F, \gamma \in L} u_\gamma e_i u_\gamma^*(v^*xv), \forall x \in P$$

Since the projections $u_\gamma e_i u_\gamma^*$ mutually commute, we have that $\bigvee_{i \in F, \gamma \in L} u_\gamma e_i u_\gamma^* \leq \sum_{i \in F, \gamma \in L} u_\gamma e_i u_\gamma^*$. By combining this with (3), we get that

$$\|v^*xv\|^2 \leq \sum_{i \in F, \gamma \in L} \|u_\gamma e_i u_\gamma^*(v^*xv)\|^2 = \sum_{i \in F, \gamma \in L} \|E_{M_i}(u_\gamma^*v^*xv)\|^2, \forall x \in P$$

Now, recall that $v v^* \in (qQq)' \cap qMq = Pq$, so we can write $v v^* = p'q$, for some projection $p' \in P$. Then $p'E_P(q) = E_P(p'q) = E_P(v v^*) \neq 0$, hence we can find a projection $p'' \in p'Pp'$ and a constant $C > 0$ such that $p''E_P(q) \geq Cp''$. A simple computation then shows that for all $x \in U(p''Pp'')$, we have that

$$\|v^*xv\|^2 = \tau(v v^* x^* v v^* x) = \tau(p'q x^* p'q x) = \tau(q x^* x) = \tau(E_P(q)x^* x) \geq C\tau(p'' x^* x) = C\tau(p'') .$$

Putting (4) and (5) together we deduce that

$$\sum_{i \in F, \gamma \in L} \|E_{M_i}(u_\gamma^*v^*xv)\|^2 \geq C\tau(p'') , \forall x \in U(p''Pp'').$$

This in turn implies, that we can find $i \in F$ such that a corner of $P$ embeds into $(\bigotimes_{\Gamma} B) \rtimes \Gamma_i$ (by the beginning of the proof of 4.3. in [IPP]). Since $(\bigotimes_{\Gamma} B) \rtimes \Gamma_i$ is amenable (both $B$ and $\Gamma_i$ are amenable), while $P$ has no amenable direct summand, we get a contradiction. \[\square\]
§4. Applications.

4.1. Solid II$_1$ factors. The first examples of solidity in von Neumann algebras are due to Ozawa who proved that $L\Gamma$ is solid, for every hyperbolic group $\Gamma$ ([O1]). Recall in this respect that a II$_1$ factor $M$ is called solid if for any diffuse von Neumann subalgebra $A$ of $M$, the relative commutant $A' \cap M$ is amenable.

**Corollary 8.** Let $\Gamma$ be an ICC countable group. Then $L\Gamma$ is solid if and only if $(\bigotimes \Gamma R) \rtimes \Gamma$ is solid, where $R$ denotes the hyperfinite II$_1$ factor.

**Proof.** Assume that $L\Gamma$ is solid and let $P \subset M := (\bigotimes \Gamma R) \rtimes \Gamma$ be a diffuse von Neumann subalgebra. If the commutant $Q = P' \cap M$ is non-amenable, then we can find a non-zero projection $z \in Z(Q)$ such that $Qz$ has no amenable direct summand.

Since $[Pz, Qz] = 0$ and $\Gamma$ is ICC, we can apply Theorem 2, to deduce that there exists a unitary $u$ in $M$ such that $u(Pz \vee Qz)u^* \subset L\Gamma$. This, however, contradicts the solidity of $L\Gamma$. Thus $Q$ is amenable, hence $M$ is solid. $\blacksquare$

Recently, J. Peterson showed that the existence of a proper cocycle into a multiple of the left regular representation also implies that $L\Gamma$ is solid, for a countable group $\Gamma$ ([Pe]). When combined with Corollary 9, this gives new examples of solid II$_1$ factors.

4.2. Equivalence relations from percolation theory. We start by recalling how certain percolations on graphs naturally induce subequivalence relations of an equivalence relation arising from a Bernoulli action (see [GL] for a reference). Let $\mathcal{G} = (V, E)$ be a transitive graph and let $\Gamma \subset \text{Aut}(\mathcal{G})$ be a subgroup which acts transitively on $V$. Assume that $\Gamma$ acts freely on $E$ with amenable stabilizers, i.e. $\Gamma_e = \{ \gamma \in \Gamma | \gamma e = e \}$ is amenable, for every edge $e \in E$. An example of a such a graph is given by the right Cayley graph $G$ of a countable, finitely generated group $\Gamma$ with respect to a finite generating set $S \subset \Gamma$. More precisely, $\mathcal{G}$ is the graph with vertex set $V = \Gamma$ and edge set $E = \{(\gamma, \gamma s) | \gamma \in \Gamma, s \in S\}$.

On $[0, 1]^E$ (endowed with the product Lebesque measure $\nu$), consider the Bernoulli action of $\Gamma$, given by the action of $\Gamma$ on $E$. Since the stabilizers $\Gamma_e$ are amenable, we get that the induced equivalence relation $\mathcal{S}$ satisfies the conclusion of Theorem 1.

Next, let $\pi : [0, 1]^E \to \{0, 1\}^E$ be a $\Gamma$-equivariant Borel map. Then $\pi$ gives rise to a subequivalence relation $\mathcal{R}$ of $\mathcal{S}$. For this, identify $\{0, 1\}^E$ with the set of subgraphs of $E$ and fix $\rho \in V$. We then say two points $x, y \in [0, 1]^E$ are $\mathcal{R}$-equivalent if and only if there exists $\gamma \in \Gamma$ such that

1. $\gamma x = y$ and
2. $\gamma^{-1} \rho$ and $\rho$ are in the same connected component of $\pi(x)$ (viewed as a subgraph of $E$).

Now, since $\pi$ is $\Gamma$-equivariant, the push-forward measure $\pi_*\nu$ is a $\Gamma$-invariant probability measure on $\{0, 1\}^E$, i.e. a percolation on $\mathcal{G}$. An interesting question is when does this percolation have indistinguishable infinite clusters. Equivalently (by [GL]),
when is the restriction $\mathcal{R}|_{U_\infty}$ ergodic, where $U_\infty$ is the set of all points which have infinite $\mathcal{R}$-classes.

Specifically, the answer to this question is conjectured to be true for the free minimal spanning forest of $\mathcal{G}$ (denoted $\text{FMSF}(\mathcal{G})$) ([LPS]). The FMSF is the percolation/equivalence relation induced by the map $\pi : [0, 1]^E \to \{0, 1\}^E$ defined as follows: for every $\omega \in [0, 1]^E$ label every edge $e \in E$ with $\omega(e) \in [0, 1]$ and then define $\pi(\omega)$ to be the set of edges $e$ which are not maximal (with respect to the labeling) in any cycle containing them. Note that it is known by now that for every finitely generated, non-amenable group $\Gamma$ there exists a Cayley graph $\mathcal{G}$ such that a.e. class of $\text{FMSF}(\mathcal{G})$ is a tree with infinitely many ends ([LPS], [T]). More generally, then same is true for every transitive, unimodular graph $\mathcal{G}$ which satisfies $p_c(\mathcal{G}) < p_u(\mathcal{G})$. By combining these facts with Theorem 1 and the proof of Proposition 11 in [GL], we get the following:

**Corollary 9.** Let $\Gamma$ be a non-amenable, finitely generated group. Then there exists a Cayley graph $\mathcal{G}$ of $\Gamma$ such that the $\text{FMSF}(\mathcal{G})$ admits an invariant, measurable partition $\{X_i\}_{i \geq 1}$ of $[0, 1]^E$ for which the restriction $\text{FMSF}(\mathcal{G})|_{X_i}$ is a (strongly) ergodic, treeable equivalence relation of (normalized) cost $> 1$, $\forall i \geq 1$.

More generally, Corollary 9 holds true for any transitive, unimodal graph $\mathcal{G}$ such that $p_c(\mathcal{G}) < p_u(\mathcal{G})$ and that there exists a group $\Gamma \subset \text{Aut}(\mathcal{G})$ which acts transitively on $V$ and with amenable stabilizers on $E$.

Recently, D. Gaboriau and R. Lyons used certain equivalence relations coming from percolation to show any non-amenable group $\Gamma$ admits $\mathbb{F}_2$ as a "measurable" subgroup ([GL]). Also, they suggested that the free minimal spanning forest might be used to derive their result. We remark that Corollary 9 together with the proof of Proposition 13 in [GL] shows that this is indeed the case.

**4.3. Strong ergodicity vs. spectral gap.** While Theorem 1 proves automatic strong ergodicity of certain non-hyperfinite equivalence relations, it is typically quite difficult to prove strong ergodicity for a given equivalence relation. In case the equivalence relation is induced by a group action, one usually deduces strong ergodicity by proving spectral gap of the action. Recall that a measure preserving, ergodic action $\Gamma \curvearrowright \sigma (X, \mu)$ is said to have spectral gap if the induced representation of $\Gamma$ on $L^2(X, \mu) \ominus C1$ has spectral gap.

The notions of strong ergodicity and spectral gap are however not equivalent in general ([Sc],[HK]). Nevertheless, it is an interesting problem to find classes of group actions for which these notions coincide. This is the case if $\sigma$ is a generalized Bernoulli action ([KT]) or if $\sigma$ comes from an embedding of $\Gamma$ as a dense subgroup of a compact group $G$ ([AN]). Below, we note that the argument in [AN] (Lemma 6) only uses the fact that $\sigma$ has large commutant, thus rendering the following:
Lemma 10. Let $\Gamma \acts^\sigma (X, \mu)$ be a m.p. action. Assume that the commutant of $\Gamma$ in $\text{Aut}(X, \mu)$ acts ergodically on $(X, \mu)$. Then $\sigma$ has spectral gap if and only if $\sigma$ is strongly ergodic.

Proof. Denote by $\Lambda$ the commutant of $\Gamma$ in $\text{Aut}(X, \mu)$. We claim that for every measurable sets $A, B \subset X$ and every $\varepsilon > 0$ we can find $\theta \in \Lambda$ such that $\mu(\theta(A) \cap B) \leq (1 + \varepsilon)\mu(A)\mu(B)$. This claim is folklore, but we include a proof for the sake of completeness. Assuming that the claim is false, then

$$\mu(\theta(A) \cap B) > (1 + \varepsilon)\mu(A)\mu(B), \forall \theta \in \Lambda.$$  

Thus, if $K$ denotes the $\|\cdot\|_2$-closure of the convex hull of the set $\{1_{\theta(A)}|\theta \in \Lambda\} \subset L^2(X, \mu)$, then $\int_B f d\mu > \mu(A)\mu(B)$ and $\int_X f d\mu = \mu(A)$, for all $f \in K$. Let $f \in K$ be the element of minimal $\|\cdot\|_2$. Then, since $K \ni g \mapsto \theta(g) \in K$ is a $\|\cdot\|_2$-preserving map, for all $\theta \in \Lambda$, and since $f$ is unique, we get that $f$ is $\Lambda$-invariant. Using the fact that $\Lambda$ acts ergodically on $(X, \mu)$ we get that $f \in C1$. Thus $f = \mu(A)1_X$, which contradicts the inequality $\int_B f d\mu > (1 + \varepsilon)\mu(A)\mu(B)$.

Equivalently, the claim shows that for every $A, B \subset X$ and every $\varepsilon > 0$ there exists $\theta \in \Lambda$ such that $\mu(\theta(A) \cup B) \geq (1 - \varepsilon) - (1 - \mu(A))(1 - \mu(B))$. Let $A \subset X$ be a measurable set. By using the claim we can inductively find (as in [AN]) a sequence $\theta_1 = 1, \theta_2, ..., \theta_k, ... \in \Lambda$ such that $\mu(\bigcup_{i=1}^{k} \theta_i(A)) \geq 1 - (1 - \mu(A))^{k-1}$, for all $k$.

Now, assume that $\sigma$ is strongly ergodic but does not have spectral gap. Thus, we can find measurable sets $\{A_n\} \subset X$ such that $\mu(A_n) > 0$, $\lim_{n \to \infty} \mu(A_n) = 0$ and $\lim_{n \to \infty} \mu(\gamma A_n A_n^\perp) = 0$, for all $\gamma \in \Gamma$. For every $n$, define $k_n = \lceil 1/(2\mu(A_n)) \rceil + 1$. Then by the above, we can find $\theta_{n,1}, ..., \theta_{n,k_n} \in \Lambda$ such that $B_n := \bigcup_{i=1}^{k_n} \theta_i(A_n)$ verifies $\mu(B_n) > 1 - (1 - \mu(A_n))^{k_n-1}$. Then, since $\mu(A_n) \to 0$, it is easy to check that for a large enough $n$ we have that

$$1 - 1/\sqrt{e} \leq \mu(B_n) \leq 2/3$$

Also, using the fact that $\Gamma$ and $\Lambda$ commute, it follows that

$$\mu(\gamma B_n A_n^\perp) \leq \sum_{i=1}^{k_n} \mu(\gamma (\theta_i(A_n)) A_n^\perp) = k_n \mu(\gamma A_n A_n^\perp) \leq \mu(\gamma A_n A_n^\perp)/(2\mu(A_n)), \forall \gamma \in \Gamma.$$  

Finally, (1) and (2) contradict the assumption that $\sigma$ is strongly ergodic. 

Remark that Lemma 11 implies that if the commutant of $\sigma$ acts weakly mixing on $(X, \mu)$ then $\sigma$ has double spectral gap if and only if $\sigma$ has double strongly ergodic (see [P4] for definitions).
References

[AN] M. Abért, N. Nikolov: The rank gradient from a combinatorial point of view, preprint [math.GR/0701925].

[C] A. Connes: Outer conjugacy classes of automorphisms of factors, Ann. Ec. Norm. Sup. 8 (1975), 383-419.

[CFW] A. Connes, J. Feldman, B. Weiss: An amenable equivalence relations is generated by a single transformation, Ergodic Th. Dynam. Sys. 1(1981), 431-450.

[CW] A. Connes, B. Weiss: Property (T) and asymptotically invariant sequences, Israel J. Math. 37(1980), 209-210.

[D] H. Dye: On groups of measure preserving transformations II, Amer. J. Math. 85(1963), 551-576.

[FM] J. Feldman, C.C. Moore: Ergodic equivalence relations, cohomology, and von Neumann algebras, II, Trans. Am. Math. Soc. 234(1977), 325-359.

[GL] D. Gaboriau, R. Lyons: A-Measurable-Group-Theoretic Solution to von Neumann’s Problem, [math.GR/0711164].

[HK] G. Hjorth, A. Kechris: Rigidity theorems for actions of product groups and countable Borel equivalence relations, Memoirs of the Amer. Math. Soc., 177, No. 833, 2005.

[I] A. Ioana: Rigidity results for wreath product II1 factors, Journal of Functional Analysis 252(2007) 763-791.

[IPP] A. Ioana, J. Peterson, S. Popa: Amalgamated Free Products of w-Rigid Factors and Calculation of their Symmetry Groups, [math.OA/0505589], to appear in Acta Mathematica.

[JS] V.F.R. Jones, K. Schmidt: Asymptotically invariant sequences and approximate finiteness, Amer. J. Math. 109(1987), 91–114.

[KT] A. Kechris, T. Tsankov: Amenable actions and almost invariant sets, preprint 2006, to appear in the Proceedings of the AMS.

[LPS] R. Lyons, Y. Peres, O. Schramm: Minimal spanning forests, Ann. Probab., 27(2006), 1665-1692.

[MvN] F. Murray, J. von Neumann: Rings of operators, IV, Ann. Math. 44(1943), 716-808.

[O1] N. Ozawa: Solid von Neumann algebras, Acta Math., 192 (2004), 111-117.

[O2] N. Ozawa: A Kurosh type theorem for type II1 factors, Int. Math. Res. Not. (2006) Vol. 2006, article ID 97560.

[Pe] J. Peterson: L2-rigidity in von Neumann algebras, preprint [math.OA/0605033].

[P1] S. Popa: Strong rigidity of II1 factors arising from malleable actions of w-rigid groups I, Invent. Math. 165(2006), 369–408.

[P2] S. Popa: Strong rigidity of II1 factors arising from malleable actions of w-rigid groups II, Invent. Math. 165(2006), 409–451.

[P3] S. Popa: Cocycle and orbit equivalence superrigidity for Bernoulli actions of Kazhdan groups, Invent. Math, 170(2007), 243-295.
S. Popa: *On the superrigidity of malleable actions with spectral gap*, Journal of the AMS (math.GR/0608429).

S. Popa: *Deformation and rigidity for group actions and von Neumann algebras*, International Congress of Mathematicians. Vol. I, 445–477, Eur. Math. Soc., Zrich, 2007.

S. Popa: *On Ozawa’s property for free group factors*, Int. Math. Res. Not. (2007) Vol. 2007, article ID rnm036.

K. Schmidt: *Amenability, Kazhdan’s property T, strong ergodicity and invariant means for ergodic group-actions*, Ergod. Th. Dynam. Sys. 1 (1981), 223-236.

Á. Timár: * Ends in minimal spanning forests*, Ann. Probab., 34 (2006), 865-869.

Math Dept, UCLA, LA, CA 90095-1555 and IMAR, Bucharest, Romania
E-mail address: ichifan@math.ucla.edu

Math Dept, Caltech, Pasadena, 91125 and IMAR, Bucharest, Romania
E-mail address: aioana@caltech.edu