Abstract. We introduce K-theoretic GW-invariants of mixed nature: permutation-equivariant in some of the inputs and ordinary in the others, and prove the ancestor-descendant correspondence formula. In genus 0, combining this with adelic characterization, we derive that the range $L_X$ of the big J-function in permutation-equivariant theory is overruled.

The string and dilaton equations

We return to the introductory setup of Part I, and introduce mixed genus-$g$ descendant potentials of a compact Kähler manifold $X$:

$$F_g(x, t) := \sum_{k \geq 0, n \geq 0, d} \frac{Q_d}{k!} \langle x(L), \ldots, x(L); t(L), \ldots, t(L) \rangle_{g, k+n, d}.$$  

The first $k$ seats are occupied by the input $x = \sum_{r \in \mathbb{Z}} x_r q^r$, which is a Laurent polynomial in $q$ with vector coefficients $x_r \in K^0(X) \otimes \Lambda$. We assume that $\Lambda$ includes Novikov’s variables as well. The last $n$ seats are occupied by similar inputs $t = \sum_{r \in \mathbb{Z}} t_r q^r$, $t_r \in K^0(X) \otimes \Lambda$, and only these inputs are considered permutable by renumberings of the marked points. Most of the time we will assume that $t(q) = t$ is constant in $q$, i.e. that the permutable inputs do not involve the cotangent line bundles $L_i$.

We will first treat these generating functions as objects of the ordinary, i.e. permutation-non-equivariant, quantum K-theory, depending however on the parameter $t$. Our nearest aim is to extend to this family of theories some basic facts from the ordinary GW-theory, starting with the genus-0 string and dilaton equations.

On the moduli space $X_{g,m+1,d}$, along with the line bundles $L_i$ formed by the cotangent lines to the curves at the $i$th marked point, consider...
the line bundles $\tilde{L}_i := ft^*_i(L_i)$, $i \geq 2$, where $ft_1 : X_{g,1+m,d} \to X_{g,m,d}$ is the map defined by forgetting the first marked point. In the genus 0 case, it is clear that

$$\langle 1, x_1(\tilde{L}), \ldots, x_k(\tilde{L}); t, \ldots \rangle_{0,1+k+n,d}^{S_n} = \langle x_1(L), \ldots, x_k(L); t, \ldots \rangle_{0,k+n,d}^{S_n}.$$ 

On the other hand, it is well-known how to compare that $(\bigwedge^r L_i) = 1$, and hence $L_i = L_i - (\sigma_i)_1$. Taking into account that $L_i((\sigma_i)_1) = (\sigma_i)_1$, and that $((-\sigma_i)_1)^r = (-\sigma_i)_1(L_i - 1)^{r-1}$, we find by Taylor’s formula (and omitting the subscript $i$):

$$x(\tilde{L}) - x(L) = \sum_{r>0} \frac{x^{(r)}(L)}{r!} (-\sigma_1)^r = -\sigma(\sum_{r>0} \frac{x^{(r)}(1)}{r!} (L - 1)^{r-1}) = \frac{-\sigma x(L) - x(1)}{L - 1}.$$ 

Note that the divisors $\sigma_i$ for different $i$ are disjoint, and that $\sigma_1^*(L_j) = L_j$ if $j \neq i$. Thus

$$\langle 1, x_1(L), \ldots, x_k(L); t, \ldots \rangle_{0,1+k+n,d}^{S_n} = \langle x_1(L), \ldots, x_k(L); t, \ldots \rangle_{0,k+n,d}^{S_n} + \sum_{i=1}^{k} \langle \ldots, x_i(L), \frac{x_i(L) - x_i(1)}{L - 1}, x_{i+1}(L), \ldots; t, \ldots \rangle_{0,k+n,d}^{S_n}.$$ 

This computation is quite standard, since it does not interfere with the permutable inputs, as long as those don’t contain line bundles $L_i$.

**Proposition 1** (string equation). Let $V$ be the linear vector field on the space of vector-valued Laurent polynomials in $q$ defined by

$$V(y) := \frac{y(q) - y(1)}{1 - q}.$$ 

In the genus-0 descendent potential $F_0(x, t)$, introduce the dilaton shift of the origin: $y(q) = 1 - q + t + x(q)$. Then

$$L_V(F_0(y+q-1-t), t)) = F_0(y+q-1-t), t) + \frac{y(1) - y(1)}{2} = \left( \frac{y(1)}{2}, 1 \right),$$ 

where $(a, b) := \chi(X; ab)$ is the $\Lambda$-valued $K$-theoretic Poincaré pairing, and $\Psi^2$ is the $2$nd Adams operation on $K^0(X) \otimes \Lambda$. 
Proof. The linear vector field $V$ becomes inhomogeneous in the unshifted coordinate system:

$$\frac{y(q) - y(1)}{1 - q} = \frac{x(q) - x(1)}{1 - q} + 1.$$ 

Applying the previous, down-to-earth form of the string equation to $F_0(x) := \sum_{k,n,d} Q^d k! \langle x(L), \ldots, x(L); t, \ldots, t \rangle_{0,1+k+n,d}$, we gather that

$$L_V(F_0(x,t)) = F_0(x) + \text{terms} \langle 1, \ldots \rangle_{0,3,0}^{S_n} \text{ with } d = 0 \text{ and } k + n = 2.$$ 

Since $X_{0,3,0} = X \times \mathcal{M}_{0,3} = X$, and $L = 1$ on $\mathcal{M}_{0,3}$, these terms are

$$\frac{1}{2}(x(1), x(1)) + \frac{1}{2}(x(1), t) + \frac{1}{2}(t, t)/2 - \frac{1}{2}(\Psi^2(t), 1).$$

The last two terms come from

$$\langle 1; t, t \rangle_{0,3,0}^{S_2} = \frac{1}{|S_2|} \sum_{h \in S_2} \text{tr}_h(t^{S_2}).$$

All but the last one add up to $(y(1), y(1))/2$. □

Consider now correlators

$$\langle L - 1, x(L), \ldots, x(L); t(L), \ldots, t(L) \rangle_{0,1+k+n,d}^{S_n}.$$ 

The line bundle $L_1$ over $X_{g,1+k+n,d}$ differs from the dualizing sheaf to the fibers of the forgetting map $\mathcal{F}_{1+}$ by the divisor of the marked points. The spaces $H^0(\Sigma, L)$ are formed by holomorphic differentials on $\Sigma$ with at most 1st order poles at the markings, and with at most 1st order poles at the nodes with zero residue sum at each node. In genus 0, if $k + n > 0$, then $H^1(\Sigma, L - 1) = 0$, while the holomorphic differentials are uniquely determined by the residues at the marked points subject to the constrains that the total sum is 0. The residues per se form trivial bundles, but those at the permutable marked points form the standard Coxeter representation of $S_n$, induced from the trivial representation of $S_{n-1}$. Thus,

$$(\mathcal{F}_{1+})_*(L - 1) = k - 2 + \text{Ind}^{S_n}_{S_{n-1}}(1),$$

and this answer is correct even when $k = n = 0$ (in which case $H^1(\Sigma, L) = H^0(\Sigma, 1)* = 1$). On the other hand, $L - 1$ vanishes on
the sections \( \sigma_i : X_{g,k+l,d} \to X_{g,1+k+n,d} \) defined by the markings, where
the differences between \( L_i - L_i, i > 1 \), are supported. We find that
\[
\langle L - 1, \ldots, x(L); t(L), \ldots, \rangle_{0,1+k+n,d}^{S_n} = (k - 2)\langle \ldots, x(L); t(L), \ldots, \rangle_{0,k+n,d}^{S_n} + \langle x(L), \ldots, x(L), t(L); t(L), \ldots, t(L) \rangle_{0,k+n,d}^{S_n-1}
\]
We use here that for any \( S_n \)-module \( V \),
\[
\left( V \otimes \text{Ind}_{S_{n-1}}^{S_n}(1) \right)^{S_n} = \left( \text{Res}_{S_{n-1}}^{S_n}(V) \right)^{S_{n-1}}.
\]

**Proposition 2** (dilaton equation). The genus-0 descendent potential \( \mathcal{F}_0 \) in dilaton-shifted coordinates satisfies the following homogeneity condition:
\[
L_E(\mathcal{F}_0(y + q - 1 - t, t)) = 2\mathcal{F}_0(y + q - 1 - t, t) - (\Psi^2(t(1)), 1),
\]
where \( E \) is the Euler vector field \( E(y) = y \) in the linear space of vector-valued Laurent polynomials \( y(q) \).

**Proof.** The exceptional terms
\[
\frac{1}{2}\langle L - 1, x(L), x(L) \rangle_{0,3,0} + \langle L - 1, x(L); t(L), t(L) \rangle_{0,3,0}^{S_1} + \langle L - 1, t(L), t(L) \rangle_{0,3,0}^{S_2}
\]
all vanish except for the trace of the non-trivial element in \( S_2 \), which acts by \(-1\) one the cotangent line \( L \). This makes \( L - 1 \) on \( \mathcal{M}_{0,3} \)
equal to \(-2\) (rather than \(0\)), and results in the constant \(-\langle \Psi^2(t(1)), 1 \rangle \).

Therefore the identity derived above yields:
\[
\sum_{k,n,d} \frac{Q^d}{k!} \langle 1 - L + t(L), x(L), x(L), \ldots, x(L); t(L), \ldots, t(L) \rangle_{0,1+k+n,d}^{S_n}
= 2 \sum_{k,n,d} \frac{Q^d}{k!} \langle x(L), \ldots, x(L); t(L), \ldots, t(L) \rangle_{0,k+n,d}^{S_n} - \langle \Psi^2(t(1)), 1 \rangle,
\]
which after shift \( y(q) := 1 - q + t(q) + x(q) \) becomes what we claimed. \( \square \)

**Remark.** Note that we have proved this allowing the permutable input \( t \), i.e. the *parameter* of \( \mathcal{F}_0 \) to depend on \( q \).

**A WDVV-equation**

Let us introduce the gadget
\[
\langle A_1, \ldots, A_m \rangle_{g,m} := \sum_{l,n,d} \frac{Q^d}{l!} \langle A_1, \ldots, A_m; \tau, \ldots, \tau; t, \ldots, t \rangle_{g,m+l+n,d}^{S_n}
\]
for the generating function of \( \tau, t \in K^0(X) \otimes \Lambda \), and the meaning of the inputs \( A_i \) to be specified.
Along with the Poincaré metric \( g_{\alpha\beta} = (\phi_{\alpha}, \phi_{\beta}) \) on \( K^0(X) \), where \( \{\phi_{\alpha}\} \) is a basis, introduce the non-constant metric
\[
G_{\alpha\beta} := g_{\alpha\beta} + \langle \phi_{\alpha}, \phi_{\beta} \rangle_{0,2}.
\]

Note that the inverse tensor has the form
\[
G^{\alpha\beta} = g^{\alpha\beta} - \langle \phi^{\alpha}, \phi^{\beta} \rangle_{0,2} + \sum_{\mu} \langle \phi^{\alpha}, \phi^{\mu} \rangle_{0,2} \langle \phi^{\mu}, \phi^{\beta} \rangle_{0,2} + \ldots,
\]
where \( \{\phi^{\alpha}\} \) is the basis Poincaré-dual to \( \{\phi_{\alpha}\} \).

**Proposition 3 (WDVV-equation).** For all \( \phi, \psi \in K^0(X) \otimes \Lambda \),
\[
(\phi, \psi) + (1 - xy)\langle \frac{\phi}{1 - xL}, \frac{\psi}{1 - yL} \rangle_{0,2} =
\sum_{\alpha,\beta} \left( (\phi, \phi_{\alpha}) + \langle \frac{\phi}{1 - xL}, \phi_{\alpha} \rangle_{0,2} \right) G^{\alpha\beta} \left( (\phi_{\beta}, \psi) + \langle \phi_{\beta}, \frac{\psi}{1 - yL} \rangle_{0,2} \right).
\]

**Proof.** The standard WDVV-argument consists in mapping moduli spaces of genus-0 stable maps with 4+ marked points to the Deligne-Mumford space \( \overline{M}_{0,4} \), and considering the inverse image of a typical point, i.e., in other words, fixing the cross-ratio of the first 4 marked points. When the cross-ratio degenerates into one of the special values 0, 1, \( \infty \), the curves become reducible, with the 4 marked points split into pairs between the two glued pieces in 3 different ways. The WDVV-equation expresses the equality between the three gluings.

We apply the argument to the inputs of the 4 marked points equal to 1, 1, \( \phi/(1 - xL) \), and \( \phi/(1 - yL) \), and arrive at the following identity (see Figure 1):
\[
\sum_{\alpha,\beta} \langle 1, \frac{\phi}{1 - xL}, \phi_{\alpha} \rangle_{0,3} G^{\alpha\beta} \langle \phi_{\beta}, \frac{\psi}{1 - yL}, 1 \rangle_{0,3} =
\sum_{\alpha,\beta} \langle 1, 1, \phi_{\alpha} \rangle_{0,3} G^{\alpha\beta} \langle \phi_{\beta}, \frac{\phi}{1 - xL}, \frac{\psi}{1 - yL} \rangle_{0,3}.
\]

As it is explained in [6], in K-theory the WDVV-argument encounters the following subtlety. The virtual divisor obtained by fixing the cross-ratio and passing to any of the three limits, has self-intersections, represented by curves with more than 2 components (as shown on Figure 1 in shaded areas). As a result, the structure sheaf of the divisor before the limit is identified with the alternated sum of the structure
sheaves of all the self-intersection strata on a manner of the exclusion-inclusion formula. In the identity, this is taken care of by the pairing which involves the tensor $G^{\alpha\beta}$.

It only remains to apply the string equation. Since
\[ \frac{1}{L-1} \left( \frac{1}{1-qL} - \frac{1}{1-q} \right) = \frac{x}{(1-x)(1-qL)}, \]
and since $L = 1$ on $X_{0,3,0} = X \times \overline{\mathcal{M}}_{0,3} = X$, we have
\[ \langle 1, \frac{\phi}{1-qL}, \phi \rangle_{0,3} = (\phi, \phi) + \left( 1 + \frac{q}{1-q} \right) \langle \frac{\phi}{1-qL}, \phi \rangle_{0,2}; \]
\[ \sum_{\alpha,\beta} \langle 1, \phi, \psi \rangle_{0,3} G^{\alpha\beta} \langle \frac{\phi}{1-xL}, \frac{\psi}{1-yL} \rangle_{0,3} = \langle 1, \frac{\phi}{1-xL}, \frac{\psi}{1-yL} \rangle_{0,2} \]
\[ = \frac{(\phi, \psi)}{(1-x)(1-y)} + \left( 1 + \frac{x}{1-x} + \frac{y}{1-y} \right) \langle \frac{\phi}{1-xL}, \frac{\psi}{1-yL} \rangle_{0,2}. \]

The result follows.

\[ \square \]

**THE LOOP SPACE FORMALISM**

Here we interpret the string, dilaton, and WDVV-equations using symplectic linear algebra in the space $K$ of rational functions of $q$ with vector coefficients from $\mathcal{K}^0(X) \otimes \Lambda$. To be more precise, we assume that elements of $K$ are such rational functions modulo any power of Novikov’s variables (or in any other topology that may turn out useful in future). We equip $K$ with symplectic form
\[ \Omega(f, g) := - \text{Res}_{q=0, \infty} (f(q^{-1}), g(q)) \frac{dq}{q}. \]

We identify $K$ with $T^*\mathcal{K}_+$ where $\mathcal{K}_+ \subset K$ is the Lagrangian subspace consisting of vector-valued Laurent polynomials in $q$ (in the aforementioned topological sense) by picking the complementary Lagrangian
subspace $K_-$ consisting of rational functions of $q$ regular at $q = 1$ and vanishing at $q = \infty$. We encode K-theoretic genus-0 GW-invariant of $X$ by the big J-function

$$J(x, t) := 1 - q + t(q) + x(q) + \sum_{\alpha, k, n, d} \phi^\alpha Q^d k! \langle \phi^\alpha, x(L), \ldots, x(L); t(L), \ldots, t(L) \rangle^S_{0,1+k+n,d}.$$  

Proposition 4. The big J-function is the dilaton-shifted graph of the differential of the genus-0 descendent potential $F_0$:

$$J(x, t) = 1 - q + t + x + d_x F_0(x, t).$$

Proof. For every $v \in K_+$, we have:

$$L_v F_0 = \sum_{k, n, d} \frac{Q^d}{k!} \langle v(L), x(L), \ldots, x(L); t(L), \ldots, t(L) \rangle^S_{0,1+k+n,d} =$$

$$\sum_{k, n, d} \frac{Q^d}{k!} \langle \text{Res}_{q=L} \frac{v(q)}{(1-L/q)} \frac{dq}{q}, x(L), \ldots, x(L); t(L), \ldots, t(L) \rangle^S_{0,1+k+n,d}$$

$$= - \text{Res}_{q=0,\infty} (J(q^{-1}), v(q)) \frac{dq}{q} = \Omega(J, v).$$

Corollary 1 (dilaton equation). For a fixed value of the parameter $t$, the range of the J-function $x \mapsto J(x, t)$ is a Lagrangian cone $L_t \subset K$ with the vertex at the origin.

Proof. Differentiating the dilaton equation for $F_0$, we find that 1st derivatives of $F_0$ are homogeneous of degree 1. □

Corollary 2 (string equation). For any $t \in K^0(X) \otimes \Lambda$, the linear vector field $f \mapsto f/(1-q)$ on $K$ is tangent to $L_t$.

Remark. We will see later that this is true for any $L_t$, and not only for $t$ independent of $q$.

Proof. Subtracting from the string equation for $F_0$ derived in the previous section a half of the dilaton equation for $F_0$, we obtain a Hamilton-Jacobi equation $L_{V_{-E/2} F_0} = (y(1), y(1))/2$. It expresses the fact that the quadratic Hamiltonian corresponding to this equation vanishes on $L_t$, and hence the Hamiltonian vector field is tangent to $L_t$. We will show that this Hamiltonian vector field is

$$Wf := \frac{f}{1-q} - \frac{f}{2}.$$
Due to Corollary 1, \( f \mapsto f/2 \) is tangent to \( L_t \), and the result about \( f \mapsto f/(1 - q) \) would follow.

The Hamiltonian of \( W \) is \( H(f) := \Omega(f, Wf)/2 = \Omega(f, f/(1 - q))/2 \).

Using the projections \( f_\pm \) of \( f \in \mathcal{K} \) to \( \mathcal{K}_\pm \), we compute \( 2H(f) \):

\[
\Omega \left( f, \frac{f}{1 - q} \right) = \Omega \left( f_+ + f_-, \frac{f_+(1)}{1 - q} + \frac{f_+ - f_+(1)}{1 - q} + \frac{f_-}{1 - q} \right) = \\
-\Omega \left( \frac{f_+(1)}{1 - q}, f_+ \right) + \Omega \left( f_-, \frac{f_+ - f_+(1)}{1 - q} \right) + \Omega \left( \frac{f_+}{1 - q^{-1}}, f_- \right) = \\
\text{Res}_{q=0, \infty} \left( \frac{f_+(1)}{1 - q}, f_+(q) \right) \frac{dq}{q} + \Omega \left( f_-, \frac{f_+ - 2f_+(1) + gf_+}{1 - q} \right) = \\
\Omega \left( \frac{f_+(1)}{1 - q}, f_+(1) \right) = \Omega \left( f_+(1), f_+(1) \right) + 2 \Omega \left( f_-, \frac{f_+ - f_+(1)}{1 - q} + \frac{f_+}{2} \right) = 0.
\]

The last non-zero term is twice the Hamilton function of the vector field \( y \mapsto \frac{y(q^{-1} - y(1))}{1 - q} \) on \( \mathcal{K}_+ \), i.e. \( V - E/2 \), lifted to the cotangent bundle in the standard way. The first non-zero term is twice \( -(y(1), y(1))/2 \). Thus, the quadratic Hamiltonian \( H \) is exactly as claimed. \( \Box \)

Introduce the operator \( S : K^0(X) \otimes \Lambda \to \mathcal{K}_- \) defined by

\[
S(q) \phi = \sum_{\alpha, \beta} \left( \langle \phi, \phi_\alpha \rangle + \langle \frac{\phi}{1 - L/q}, \phi_\alpha \rangle \otimes \phi_\alpha \right) G^{\alpha \beta} \phi_\beta,
\]

The operator depends on the parameter \( \tau \in K^0(X) \otimes \Lambda \). For each value of the parameter, it can be considered as an operator-valued rational function of \( q \) (a “loop group” element), and in this capacity extends to a map \( S : \mathcal{K} \to \mathcal{K} \) commuting with multiplications by scalar rational functions of \( q \). The WDVV-identity from the previous section can be written as

\[
(1 - xy) \langle \frac{\phi}{1 - xL}, \frac{\psi}{1 - yL} \rangle = \langle \phi, \psi \rangle + \left( S^*(y^{-1})S(x^{-1}) \phi, \psi \right),
\]

where

\[
S^*(q)\psi = \psi + \sum_{\alpha, \beta} \langle \psi, \frac{\phi_\alpha}{1 - L/q} \rangle G^{\alpha \beta} \phi_\beta
\]

is the operator adjoint to \( S(q) \) with respect to the inner product \( (g_{\alpha \beta}) \) on the domain space, and \( (G_{\alpha \beta}) \) on the target space. It follows that

\[
S^*(q^{-1})S(q) = 1, \quad \text{and hence} \quad S(q)S^*(q^{-1}) = 1.
\]

This means that \( S : (\mathcal{K}, \Omega) \to (\mathcal{K}, \bar{\Omega}) \) provides a symplectic isomorphism between two symplectic structures on the loop space: \( \Omega \), based on the metric tensor \( (g_{\alpha \beta}) \), and \( \bar{\Omega} \), based on the metric tensor \( (G_{\alpha \beta}) \).
(and depending therefore on the parameter $\tau \in K^0(X) \otimes \Lambda$). The inverse isomorphism is given by

$$S^{-1}(q) = S^*(q^{-1}).$$

Furthermore, the quantizations $\hat{S}$ and $\hat{S}^{-1}$ provide isomorphisms between the corresponding Fock spaces, which in their formal version consist of expressions

$$D(y) = e^{\mathcal{F}_0(y)/\hbar + \mathcal{F}_1(y) + \hbar \mathcal{F}_2(y) + \hbar^2 \mathcal{F}_3(y) + \cdots}, \quad y \in \mathcal{K}_+,$$

where $\mathcal{F}_g$ is a sequence of scalar-valued functions on $\mathcal{K}_+$. The operators $S^\pm$ have the form of the composition of the operator $G^\pm$ identifying the metric: $(G\phi_{\mu},\phi_{\nu}) = G_{\mu\nu}$, and the operator $(\mathcal{K},\Omega) \to (\mathcal{K},\Omega)$ which is the identity modulo $\mathcal{K}_-$. This means that the quadratic hamiltonian of $\ln S^{-1}G$ contains only $pq$-terms and $q^2$-terms, but no $p^2$-terms. Therefore $S^{-1} := \exp(\ln S^{-1}G)G^{-1}$ will act by a linear change of variables followed by the multiplication by a quadratic form, both depending on $S$. The answer in the finite form is given by Proposition 5.3 in [7]:

$$\hat{(S^{-1}\mathcal{A})}(y) = e^{W(y,y)/\hbar} \mathcal{A}(S(q)(q)(q)_+)$$

where the symmetric bilinear form $W$ is determined by

$$W(x,y) = (\Omega \otimes \Omega) \left( \frac{S^*(x^{-1})S(y^{-1}) - 1}{1 - xy}, x(x) \otimes y(y) \right).$$

Using the WDVV-equation, we find

$$W(x,y) = \text{Res}_{x=0,\infty} \text{Res}_{y=0,\infty} \left( \frac{x(x)}{1 - L/x}, \frac{y(y)}{1 - L/y} \right)_{0.2} \frac{dx}{x} \frac{dy}{y} = \langle \mathcal{A}(L), S(L) \rangle_{0.2}.$$
We introduce ancestor potentials

\[ \bar{F}_g(x, \tau, t) := \sum_{k \geq 0, d} \frac{1}{k!} \langle x(\bar{L}, \ldots, x(\bar{L})) \rangle_{g,k} = \]

\[ \sum_{k,l,n,d} Q^d \frac{1}{k!l!} \langle x(\bar{L}), \ldots, x(\bar{L}); \tau, \ldots, \tau; t, \ldots, t \rangle_{g,k,l+n,d}, \]

where \( \tau, t \in K^0(X) \otimes \Lambda \), and \( \bar{L} \) in the \( i \)th position of the correlator represents the line bundle \( \bar{L}_i \) over the moduli space \( X_{g,k+l+n,d} \) of stable maps to \( X \), obtained by pulling back the universal cotangent line bundle at the \( i \)th marked point over the Deligne-Mumford space \( \overline{M}_{g,k} \) by the contraction map \( \text{ct} : X_{g,k+l+n} \to \overline{M}_{g,k} \). The latter is defined by forgetting the map to \( X \) and the last \( k + n \) marked points, and contracting those components of the curve which have become unstable.

We follow the exposition in Appendix 2 of [3] to relate descendant and ancestor potentials. The geometry of this relationship goes back to the paper of Kontsevich-Manin [10] and Getzler [4].

Let \( L \) be one of the cotangent line bundles over \( X_{g,k+l+n,d} \) (say, the 1st one), and \( \bar{L} \) its counterpart pulled back from \( \overline{M}_{g,k} \). They are the same outside the locus where the 1st marked point lies on a component to be contracted. This shows that there is a holomorphic section of \( \text{Hom}(\bar{L}, L) \) vanishing on the virtual divisor \( j : D \to X_{g,n,d} \) formed by gluing genus \( g \) stable maps, carrying all but the 1st out of the first \( k \) marked points, with genus 0 stable maps, carrying the 1st one. In fact, like in the case of the of WDVV-equation, the divisor has self-intersections (see Figure 2), and we have to refer once again to [6] for a detailed discussion of the K-theoretic exclusion-inclusion formula

\[ \mathcal{O} - \mathcal{O}(-D) = j_* \mathcal{O}_D - j_* \mathcal{O}_{D(2)} + j_* \mathcal{O}_{D(3)} - \cdots \]

which expresses \( 1 - \bar{L}/L = \mathcal{O} - \mathcal{O}(-D) \) in terms of structure sheaves of the strata \( D_{(m)} \) of \( m \)-tuple self-intersections.

We will use this relationship to rid systematically of \( L \)'s in favor of \( \bar{L} \)'s in the correlators. For this, we will have to consider “mixed” correlators, which allow both \( L \) and \( \bar{L} \) at the same seat. Let us use the notation \( \langle \phi L^a \bar{L}^b \rangle \) in correlator expressions which have the specified inputs (here \( \phi \in K^0(X) \otimes \Lambda \)) in the singled out (first) seat, provided that all other inputs in all terms of the expression are the same. For
a > 0, we have:

$$\langle \phi L^a \bar{L}^b \rangle = \langle \phi L^{a-1} \bar{L}^{b+1} \rangle + \langle \phi L^a \bar{L}^b (1 - \bar{L}/L) \rangle$$

$$= \langle \phi L^{a-1} \bar{L}^{b+1} \rangle + \sum_{\alpha,\beta} \langle \phi L^a, \phi_\alpha \rangle_{0,2} G^{\alpha\beta} \langle \phi_\beta \bar{L}^b \rangle .$$

Note that marked points (not shown on Figure 2) which carry the inputs $\tau$ or permutable inputs $t$ can be distributed in any way between the components of the curve, and the above factorization of correlators under gluing is justified by the permutation-equivariant binomial formula from Part I.

Iterating the procedure, we have:

$$\langle \phi L^a \rangle = \sum_{\alpha,\beta} \langle \phi L^a, \phi_\alpha \rangle_{0,2} G^{\alpha\beta} \langle \phi_\beta \bar{L}^a \rangle$$

$$+ \langle \phi \bar{L}^a \rangle = \langle \phi \bar{L}^a \rangle + \sum_{b=0}^{a-1} \sum_{\alpha,\beta} \langle \phi L^{a-b}, \phi_\alpha \rangle_{0,2} G^{\alpha\beta} \langle \phi_\beta \bar{L}^b \rangle .$$

Similarly, for negative exponents, we have

$$\langle \phi L^{-a} \rangle = \langle \phi L^{-a} \bar{L}^{-1} \rangle - \langle (1 - \bar{L}/L)L^{-1} / \bar{L} \rangle$$

$$= \langle \phi L^{-a} \bar{L}^{-1} \rangle - \sum_{\alpha,\beta} \langle \phi L^{-a}, \phi_\alpha \rangle_{0,2} G^{\alpha\beta} \langle \phi_\beta \bar{L}^{-1} \rangle$$

$$= \langle \phi L^{-a} \bar{L}^{-2} \rangle - \cdots$$

$$= \langle \phi \bar{L}^{-a} \rangle - \sum_{b=0}^a \sum_{\alpha,\beta} \langle \phi L^{-a+b}, \phi_\alpha \rangle_{0,2} G^{\alpha\beta} \langle \phi_\beta \bar{L}^{-b-1} \rangle .$$

In fact the result can be concisely described as

$$\langle x(L) \rangle = \langle [S(L) x(L)]_+ \rangle ,$$
where the operator $S$ is as in the previous section:

$$S(q) \phi = \sum_{\alpha,\beta} \left( (\phi, \phi_{\alpha}) + \langle \frac{\phi}{1 - L/q}, \phi_{\alpha} \rangle_{0,2} \right) G^{\alpha\beta} \phi_{\beta},$$

and $[f(q)]_+$ means extracting from a rational function of $q$ the Laurent polynomial part. The latter procedure, understood as projection along the space of rational functions regular at $q = 0$ and vanishing at $q = \infty$, can be described by Cauchy’s residue formula:

$$[f(q)]_+ = - \text{Res}_{w=0,\infty} \frac{f(w)dw}{w-q}.$$ 

For a Laurent polynomial $x(q)$ we have:

$$- \text{Res}_{w=0,\infty} \frac{x(w)dw}{1 - L/q} = x(L) - x(L') \frac{1 - L/L'}{1 - L'}.$$ 

For $x(L) = L^a$ we get $(L^{a+1} - L')/(L - L') = \sum_{b=0}^a L^{a-b} L^b$, and hence

$$\left[ S(L) \phi L^a \right]_+ = \sum_{\alpha,\beta} \left( (\phi, \phi_{\alpha}) G^{\alpha\beta} \phi L^a \right) + \sum_{b=0}^a \langle \phi L^b, \phi_{\beta} \rangle_{0,2} G^{\alpha\beta} \langle \phi_{\beta} L^{a-b} \rangle,$$

which agrees with what we found earlier, because

$$\sum_{\alpha} (\phi, \phi_{\alpha}) G^{\alpha\beta} = (\phi, \phi_{\beta}) - \sum_{\alpha} \langle \phi, \phi_{\alpha} \rangle_{0,2} G^{\alpha\beta}.$$ 

For $x(L) = L^{-a-1}$, it works out similarly:

$$\frac{L^{-a-1}}{1 - L/L} + \frac{L^{-a-1}}{1 - L'} = \frac{L^{-a} - L^{-a}}{L^{-1} - L^{-1}} = - \sum_{b=1}^a L^{-a+b} L^{-b+1}.$$

The same procedure can be applied to each seat in the correlators. We conclude that for stable values of $(g, m)$,

$$\langle x(L), \ldots, x(L) \rangle_{g,m} = \langle y(L'), \ldots, y(L') \rangle_{g,m},$$

where $y(q) = [S(q)x(q)]_+$ quite analogously to the cohomological results of [10]. Here all the correlators, as well as $S$, depend on the permutable parameters $t$ and non-permutable $\tau$. For a fixed $t$, assembling the correlators into the generating functions $F_g$ and $\tilde{F}_g$, we find:

$$F_g(\tau + x) = \tilde{F}_g^{(r)}([S_{\tau}x]_+) + \delta_{g,1} \langle \rangle_{1,0}^{(r)} + \delta_{g,0} \sum_{m=0}^2 \frac{1}{m!} \langle \ldots, x(L), \ldots \rangle_{0,m}^{(r)},$$

where the decorations by $\tau$ remind on the dependence on the parameter, and the terms on the right represent correlators with unstable values.
of \((g,m) = (1,0), (0,0), (0,1), (0,2)\), present in descendent, but absent in ancestor potentials.

Now we engage the shift of the origin \(x = y + q - 1 - t - \tau\). We have:

\[
\left[\frac{q - 1 - t - \tau}{1 - L/q}\right]_+ = -\text{Res}_{q=0,\infty} \frac{w - 1 - t - \tau}{1 - L/w} \frac{dw}{w - q} = \\
\frac{q - 1 - t - \tau}{1 - L/q} + \frac{L - 1 - t - \tau}{1 - q/L} = L + q - 1 - t - \tau.
\]

Therefore \([S(q)(q - 1 - t - \tau)]_+ =

\[
\sum_{\alpha,\beta} \left( (q - 1 - t - \tau, \phi_\alpha) + \ll \left[\frac{q - 1 - t - \tau}{1 - L/q}\right]_+, \phi_\alpha \rr_{0,2} \right) G^{\alpha\beta} \phi_\beta \\
= q - 1 - t - \tau + \sum_{\alpha,\beta} \ll L, \phi_\alpha \rr_{0,2} G^{\alpha\beta} \phi_\beta = q - 1.
\]

The last equality is due to the string and dilaton equations:

\[
\ll L, \phi_\alpha \rr_{0,2} = \sum_{d,l,n} \frac{Q^d}{l!} \ll L, \phi_\alpha, \tau, \ldots, \tau; t, \ldots, t \rr_{0,2+l+n,d} = \\
\ll L, \phi_\alpha, \tau + t \rr_{0,3,0} + \sum_{d,l,n} \frac{Q^d}{l!} \ll \phi_\alpha, \tau + t, \tau, \ldots, \tau; t, \ldots, t \rr_{0,2+l+n} = \\
(\tau + t, \phi_\alpha) + \ll \tau + t, \phi_\alpha \rr_{0,2},
\]

and hence \(\sum_{\alpha,\beta} \ll L, \phi_\alpha \rr_{0,2} G^{\alpha\beta} \phi_\beta = t + \tau\).

Finally, using the dilaton equations

\[
\ll L - 1, A \rr_{0,2} = -\ll A \rr_{0,1} + \ll A, t + \tau \rr_{0,2}, \\
\ll L - 1 \rr_{0,1} = -2\ll \rr_{0,0} + \ll t + \tau \rr_{0,1},
\]

we find that

\[
\ll \rr_{0,0} + \ll y + L - 1 - t - \tau \rr_{0,1} + \frac{1}{2} \ll y + L - 1 - t - \tau, y + L - 1 - t - \tau \rr_{0,2}
\]

transforms into \(\ll y, y \rr_{0,2}/2\). Indeed, the terms linear in \(y\)

\[
\ll y \rr_{0,1} + \ll L - 1 - t - \tau, y \rr_{0,2} = \ll y \rr_{0,1} - \ll y \rr_{0,1} + \ll y, t + \tau \rr_{0,2} - \ll t + \tau, y \rr_{0,2}
\]
cancel out. The y-independent terms

\[
\frac{1}{2} \langle L - 1, L - 1 \rangle_{0,2} - \langle L - 1, t + \tau \rangle_{0,2} + \frac{1}{2} \langle t + \tau, t + \tau \rangle_{0,2} \\
+ \langle L - 1 \rangle_{0,1} - \langle t + \tau \rangle_{0,1} + \langle \rangle_{0,0} = -\frac{1}{2} \langle L - 1, t + \tau \rangle_{0,2} + \\
\frac{1}{2} \langle t + \tau, t + \tau \rangle_{0,2} + \frac{1}{2} \langle L - 1 \rangle_{0,1} - \langle t + \tau \rangle_{0,1} + \langle \rangle_{0,0} = \\
\frac{1}{2} \langle t + \tau, t + \tau \rangle_{0,1} - \langle \rangle_{0,0} - \langle t + \tau \rangle_{0,1} + \langle \rangle_{0,0}.
\]

cancel out too. Thus, we obtain

\[
F_g(y + q - 1 - t) = \bar{F}_g([S_y]_+ + q - 1) + \delta_{g,1} \langle \rangle_{1,0} + \frac{\delta_{g,0}}{2} \langle y(L), y(L) \rangle_{0,2}.
\]

In view of Proposition 5 from the previous section, we have proved the following theorem.

**Theorem 1.** The total descendent potential after the shift by 1 - q + t:

\[
D(1 - q + t + \mathbf{x}) = e^{\sum_{g \geq 0} \frac{\bar{F}_g}{\mathcal{g}}} (\mathbf{x}),
\]

and the \( \tau \)-family of total ancestor potentials after the shift by 1 - q:

\[
A_{\tau}(1 - q + \mathbf{x}) := e^{\sum_{g \geq 0} \frac{\bar{F}_g}{\mathcal{g}}} (\mathbf{x}), \quad \tau \in K^0(X) \otimes \Lambda,
\]

are related by the family of quantized operators

\[
D = e^{F_1(\tau)} S^{g-1}_{\tau} A_{\tau},
\]

where \( F_1(\tau) = \langle \rangle_{1,0} := \sum_{k,n,d} \binom{Q^d}{k} \langle \tau, \ldots, \tau; t, \ldots, t \rangle^\mathbf{S}_n_{1,k+n,d} \) is the generating function for primary GW-invariants of genus 1.

Passing to the quasi-classical limit \( \hbar \to 0 \), one obtains

**Corollary 1.** The graph \( \mathcal{L} \subset K \) of the differential of the genus-0 descendent potential \( F_0(y + q - 1 - t) \) and the \( \tau \)-family \( \mathcal{L}^{(\tau)} \subset K \) of the graphs of the differentials of genus-0 ancestor potentials \( F_0^{(\tau)}(y + q - 1) \) are related by symplectic transformations \( S_{\tau} : (K, \Omega) \to (K, \Omega^{(\tau)}) \):

\[
\mathcal{L} = S^{-1}_{\tau} \mathcal{L}^{(\tau)}.
\]

The genus-0 ancestor correlators \( \langle \mathbf{x}(L_1), \ldots, \mathbf{x}(L_0) \rangle_{0,m} \) have the “zero 2-get” property [5]; namely they have zero 2-jet along the subspace \( \mathbf{x} \in K_+ \), where \( \mathbf{x}(1) = 0 \). This is because \( L_i \) are pull-backs of the line bundles \( L_i \) from the Deligne-Mumford space \( \overline{M}_{0,m} \), which is a manifold of dimension \( m - 3 \), and where therefore any product of \( m - 2 \) factors \( L_i - 1 \) vanishes for dimensional reasons. Since the dilaton shift \( 1 - q \) also vanishes at \( q = 1 \), we conclude that \( \mathcal{L}^{\text{taut}} \) is tangent to \( K_+ \).
along \((1 - q)\mathcal{K}_+\). Consequently, \(T_{\text{tan}} := S^{-1}_\tau \mathcal{K}_+\) is tangent to \(\mathcal{L}\) along \((1 - q)T_\tau \subset \mathcal{L}\). In fact, as \(\tau\) varies, these spaces sweep \(\mathcal{L}\) (which is easy to check modulo Novikov’s variables, and then apply the formal Implicit Function Theorem.)

**Corollary 2.** \(\mathcal{L} \subset (\mathcal{K}, \Omega)\) is an overruled Lagrangian cone, i.e. its tangent spaces \(T := T_J \mathcal{L}\) are tangent to \(\mathcal{L}\) exactly along \((1 - q)T_\tau \subset \mathcal{L}\).

**Example:** \(X = pt\)

In Part I, we found that for \(t \in \Lambda\)

\[
\mathcal{J}(0, t) := 1 - q + t + \sum_{n \geq 2} \left( \frac{1}{1 - q^L} ; t, \ldots, t \right)_{0, 1 + n} = (1 - q) e^{\sum_{k > 0} \Psi^k(t)/k(1 - q^k)}.
\]

It follows from the string equation (see Corollary 2 of Proposition 4) that for \(\tau \in \Lambda\)

\[
\mathcal{J}(\tau, t) = 1 - q + t + \tau + \left( \frac{1}{1 - q^L} \right)_{0, 1} = (1 - q) e^{\tau/(1 - q) + \sum_{k > 0} \Psi^k(t)/k(1 - q^k)}.
\]

Taking \(q = 0\) (and using the string equation twice), we find the variable metric

\[
G(\tau) = G_{11}(\tau) := \left(1, 1, 1\right)_{0, 3} = 1 + \tau + t + \left(1\right)_{0, 1} = e^{\tau + \sum_{k > 0} \Psi^k(t)/k}.
\]

Using the string equation once more, we derive that

\[
S_\tau(q) := \left(1 + \left(\frac{1}{1 - \Psi^1/1, 1}\right)_{0, 2}\right) G^{-1}(\tau) = \frac{\mathcal{J}(1/q)}{1 - 1/q} G^{-1}(\tau)
\]

\[
e^{\tau/(q - 1) - \sum_{k > 0} \Psi^k(t)/k(q^k - 1)},
\]

\[
S^{-1}_\tau(q) = e^{\tau/(1 - q)} + \sum_{k > 0} \Psi^k(t)/k(1 - q^k) = \frac{\mathcal{J}(q)}{1 - q},
\]

and find the range \(\mathcal{L}_t\) of the \(J\)-function \(\mathcal{K}_+ \to \mathcal{K}: x \mapsto \mathcal{J}(x, t)\) to be

\[
\mathcal{L}_t = \bigcup_{\tau \in \Lambda} e^{\tau/(1 - q) + \sum_{k > 0} \Psi^k(t)/k(1 - q^k)}(1 - q)\mathcal{K}_+.
\]

At \(\tau = 0\), we have here one of the subspace in \(\mathcal{K}\), depending on \(t\), whose union over \(t \in \Lambda_+\), according to the results of Part III, yields the range \(\mathcal{L}\) of the permutation-equivariant \(J\)-function \(t \mapsto \mathcal{J}(0, t)\)

\[
\mathcal{L} = \bigcup_{t \in \Lambda_+} e^{\sum_{k > 0} \Psi^k(t)/k(1 - q^k)}(1 - q)\mathcal{K}_+.
\]

In fact this picture remains true in general, as we will now show.
Adelic characterization

We return now to the mixed genus-0 descendent potential \( F_0(x, t) \) with the permutable input \( t \in \mathbb{K}_+ \) allowed to involve the cotangent line bundles \( L_i \). In the symplectic loop space \( (\mathbb{K}, \Omega) \), it is represented by the dilaton-shifted graph of its differential. According to Proposition 4 and its Corollary 1, it is the range of the J-function

\[
\mathbb{K}_+ \ni x \mapsto J(x, t) := 1 - q + t(q) + x(q) + \sum_{\alpha, k, n, d} \phi_\alpha Q^d \frac{\phi_\alpha}{1 - qL}, x(L), \ldots, x(L); t(L), \ldots, t(L) \rangle_{S_0 1+k+n,d},
\]

and has the form of a Lagrangian cone \( \mathcal{L}_t \), depending on the parameter \( t \in \mathbb{K}_+ \). According to the results of the previous section \( \mathcal{L}_t \) is an overruled Lagrangian cone whenever \( t \) is constant in \( q \). We combine this information with the adelic characterization of the J-function given in [8, 11, 12] and discussed in Part III, to prove the following theorem.

**Theorem 2.** The range \( \mathcal{L} \) of permutation-equivariant J-function \( t \mapsto J(0, t) \) (with \( t \in \mathbb{K}_+ \), and \( t(1) \in K^0(X) \otimes \Lambda_+ \), where \( \Lambda_+ \) is a certain neighborhood of \( 0 \in \Lambda \)) has the form

\[
\mathcal{L} = \bigcup_{t \in K^0(X) \otimes \Lambda_+} (1 - q) S_0^{-1}(q)_t \mathbb{K}_+,
\]

where the operators \( S_\tau(q) \) evaluated at \( \tau = 0 \) still depend on the parameter \( t \in K^0(X) \otimes \Lambda_+ \):

\[
S_0^{-1}(q)_t \psi := \psi + \sum_{\alpha} \phi_\alpha \sum_{n,d} Q^d \langle \psi, \frac{\phi_\alpha}{1 - qL}; t, \ldots, t \rangle_{S_0 2+n,d}.
\]

**Proof.** According to the adelic characterization results, a rational function \( f \in \mathbb{K} \) lies in \( \mathcal{L}_t \) if and only if its Laurent series expansions \( f_\zeta \) near \( q = 1/\zeta \) satisfy the following three conditions:

(i) \( f(1) \in L_{\text{fake}} \subset \hat{\mathbb{K}} \), the range, in the space \( \hat{\mathbb{K}} \) of Laurent series in \( q - 1 \) with vector coefficients in \( K^0(X) \otimes \Lambda \), of the J-function in the fake quantum K-theory of \( X \);

(ii) when \( \zeta \neq 0, 1, \infty \) is a primitive \( m \)th root of unity, \( f_\zeta(q^{1/m}/\zeta) \in \mathcal{L}_t^{(\zeta)} \), a certain Lagrangian subspace in \( \hat{\mathbb{K}} \) which will be specified below;

(iii) when \( \zeta \neq 0, \infty \) is not a root of unity, \( f_\zeta \) is a power series in \( q - 1/\zeta \), i.e. \( f \) has no pole at \( q = 1/\zeta \).

\(^{1}\)Formally speaking, there only the case \( t = 0 \) is considered, but the results extend without change to the general case, where the moduli orbispaces are \( X_{0,1+k+n,d}/S_n \) rather than \( X_{0,1+k,d} \).
To elucidate the situation, recall that in fake K-theory, the genuine holomorphic Euler characteristics \( \chi(\mathcal{M}; V) \) are replaced with their "fake" values given by the right-hand-side of the Hirzebruch–Riemann–Roch formula:

\[
\chi^{\text{fake}}(\mathcal{M}; V) := \int_{[\mathcal{M}]} \text{ch}(V) \text{td}(T\mathcal{M}).
\]

Fake in this sense GW-invariants were studied, e.g. in [2]. In particular, the range of the fake J-function is known to be an overruled Lagrangian cone \( \mathcal{L}^{\text{fake}} \subset (\hat{\mathcal{K}}, \hat{\Omega}) \), where \( \hat{\Omega}(f, g) = \text{Res}_{q=1}(f(q^{-1}), g(q)) q^{-1}dq \).

The moduli spaces of stable maps behave as virtual orbifolds (rather than manifolds), and the genuine holomorphic Euler characteristics are given by the virtual Kawasaki–Riemann–Roch formula [11], summing up certain fake holomorphic Euler characteristics of the inertia orbifold (of the moduli spaces \( X_{0,1+k+n}/S_n \) in our situation). Figure 3, essentially copied from Part III, is to remind us of the recursive device keeping track of all Kawasaki contributions into the J-function.

\[ \begin{align*}
(1-q) + t(q) + x(q)+ & \\
1 - qL & \\
\sum_{\zeta \neq 1} & \\
\text{roots of } 1 & \\
\text{dilaton shift} & \\
\text{inputs} & \\
\text{horns} & \\
\text{head} & \\
\text{arms} & \\
\text{legs} & \\
\text{markings} & \\
\text{spine} & \\
\text{butt} & \\
\text{tail} & \\
\end{align*} \]

**Figure 3. Adelic characterization**

In particular, it shows that the values of the J-function, when expanded into near \( q = 1 \), lie in \( \mathcal{L}^{\text{fake}} \), and when expanded near a primitive \( m \)th root of 1, they are characterized in terms of certain twisted fake invariants of the orbifold target space \( X \times B\mathbb{Z}_m \). The latter, in their turn, are expressed in terms of the untwisted fake invariants of \( X \).

Namely in the test (ii) above, the subspace \( \mathcal{L}^{(\zeta)}_t \subset \hat{K} \) is obtained from a certain tangent space \( T^{\text{fake}}_t \) to \( \mathcal{L}^{\text{fake}} \) by the linear transformation:

\[
\mathcal{L}^{(\zeta)}_t = e^{\sum_{k>0} \left( \frac{\Psi^k(T_X)}{(1-\zeta^{-1})q^{km}} - \frac{\Psi^km(T_X)}{k(1-q^{km})} \right) \Psi^m(T^{\text{fake}}_t) \otimes \Psi^m(\Lambda) \Lambda}.
\]
In our present discussion, it is important to figure out what determines the application point of the tangent space $T^\text{fake}_t$. On the diagram, it is determined by legs, which are related by the Adams operation $\Psi^n$ to arms (see Part III, or [8]). Note however, that the markings on the legs (each representing $m$ copies of markings on the arms attached to the $m$-fold cover of the spine curve) are allowed to carry permutable inputs $t$, but not allowed to carry the non-permutable inputs $x$, because their numbering would break the $\mathbb{Z}_m$-symmetry of the covering curve). Consequently, the value of $J^\text{fake} \in \mathcal{L}^\text{fake}$, which determines the application point of the tangent space $T_t$, is obtained by the expansion near $q = 1$ of the $J$-function with the non-permutable input $x = 0$: $T_t = T_{J(0,t)(1)} \mathcal{L}^\text{fake}$. In fact, since $\mathcal{L}^\text{fake}$ is overruled, its tangent spaces to $\mathcal{L}^\text{fake}$ are parameterized by $K^0(X) \otimes \Lambda$. Let us analyze the map $t \mapsto (\text{tangent space to } \mathcal{L}^\text{fake})$.

In degree $d = 0$, the $J$-function of $X$ coincides with the $J$-function of the point target space with coefficients in the $\lambda$-algebra $\Lambda' := K^0(X) \otimes \Lambda$. It was described in section Example. For $t = t \in \Lambda'$ (i.e. $q$-independent), we have

$$J(0,t)(1) = (1 - q) e^{\sum_{k>0} \Psi^k(t)/k^2} \times (\text{power series in } q - 1).$$

In other words, $\sum_{k>0} \Psi^k(t)/k^2$ is the parameter value of the tangent space to $\mathcal{L}^\text{fake}$ associated to the input $t \in \Lambda'$ in this approximation.

The series is not guaranteed to converge. E.g. under the identification of $K^0(X) \otimes \mathbb{Q}$ with $H^{\text{even}}(X, \mathbb{Q})$ by the Chern character, $\Psi^k$ acts on $H^{2r}(X)$ as multiplication by $k^r$, and the series $\sum_{k>0} k^{r-2}$ diverges unless $r = 0$. To handle this difficulty, we assume that the ground ring $\Lambda$ is topologized with a filtration $\Lambda \supset \Lambda' \supset \Lambda'' \supset \ldots$ by ideals such that $\Psi^k$ with $k > 1$ increase the filtration. For instance, when $\Lambda = \mathbb{Q}[[Q]]$ is the Novikov ring, $\Psi^k(Q^d) = Q^{kd}$, the filtration by the powers of the maximal ideal is taken. When $\Lambda = \mathbb{Q}[[N_1, N_2, \ldots]]$ is the ring of symmetric functions, $\Psi^k(N_r) = N_{kr}$, the filtration by degrees of symmetric functions satisfies. Then the map $t \mapsto \sum_{k>0} \Psi^k(t)/k^2$ converges for $t \in K^0(X) \otimes \Lambda'$, and is invertible in this range,\footnote{Even in the entire $H^0(X, \Lambda)$, if $\sum_{k>0} k^{-2} = \pi^2/6$ is adjoined to $\Lambda$.} since $\Psi^1(t) = t$.

Returning to the general input $t$ and degree $d \geq$, we conclude from the formal Implicit Function Theorem, that there is a well-defined map

$$T : \{t \in \mathcal{K}_+ \mid t(1) \in K^0(X) \otimes \Lambda_+\} \to K^0(X) \otimes \Lambda_+,$$

such that $T_{J(0,t)(1)} \mathcal{L}^\text{fake} = T_{J(0,t')(1)} \mathcal{L}^\text{fake}$. For all inputs $t$ with the same value $T(t)$, the adelic characterization tests (i), (ii), (iii) coincide.
By the same token, for each \( t \) there is a well-defined map \( \mathcal{K}_+ \to K^0(X) \otimes \Lambda : \mathbf{x} \mapsto \tau(\mathbf{x}) \), such that \( T_{\mathcal{J}(\mathbf{x}, t)} \mathcal{L}_t = T_{\mathcal{J}(\tau(\mathbf{x}), t)} \mathcal{L}_t \). For all inputs \( \mathbf{x} \) with the same \( \tau(\mathbf{x}) \), the values \( \mathcal{J}(\mathbf{x}, t) \) of the J-function form the ruling space \( (1 - q) S_0(q)_t \mathcal{K}_+ \) of the overruled cone \( \mathcal{L}_t \). For all such points, the localizations \( \mathcal{J}(\mathbf{x}, t)(1) \) lie in the same ruling space of \( \mathcal{L}^{fake} \), and moreover, when \( \tau = 0 \), the last ruling space is the one where \( \mathcal{J}(0, t) \) with \( \mathcal{T}(t) = t \) lie. Thus, for rational functions from the space \( (1 - q) S_0(q)_t \mathcal{K}_+ \) and for the values \( \mathcal{J}(0, t) \) with \( \mathcal{T}(t) = t \), the adelic characterization tests (i), (ii), (iii) coincide, i.e. the localizations in test (i) lie in the same ruling space of \( \mathcal{L}^{fake} \), and the tangent spaces to \( \mathcal{L}^{fake} \) involved into test (ii) are the same. Therefore the two sets of rational functions coincide:

\[
\{ \mathcal{J}(0, t) \mid \mathcal{T}(t) = t \} = (1 - q) S_0(q)_t \mathcal{K}_+.
\]

Taking the union over \( t \in \Lambda_+ \) completes the proof. \( \square \)

**Corollary 1.** \( \mathcal{L}_t = \mathcal{L}_t \), where \( t = \mathcal{T}(t) \).

**Corollary 2.** Each \( \mathcal{L}_t \) is an overruled Lagrangian cone invariant under the string flow \( \mathfrak{f} \mapsto e^{\epsilon(1-q)} \mathfrak{f} \), \( \epsilon \in \Lambda \).

**Remark.** The range \( \mathcal{L} \subset (\mathcal{K}, \Omega) \) of the permutation-equivariant J-function \( t \mapsto \mathcal{J}(0, t) \) is a cone ruled by the family \( t \mapsto R_t := (1-q) S_0(q)_t \mathcal{K}_+ \) of isotropic subspaces (and is in this sense “overruled”) but it is not Lagrangian, nor is it invariant under the string flow, as the example of \( X = pt \) readily illustrates. In particular, the spaces \( R_t/(1-q) \) are not tangent to \( \mathcal{L} \), and do not form semi-infinite variations of Hodge structures in the sense of S. Barannikov [1]. Nevertheless from Proposition 2 (dilaton equation), we have:

**Corollary 3.** The permutation-equivariant genus-0 descendent potential

\[
\mathcal{F}_0(0, t) := \sum_{0,n,d} Q^d(t(L), \ldots, t(L)) S_n^{0,n,d}
\]

is reconstructed from the permutation-equivariant J-function by

\[
\frac{1}{2} \Omega ([\mathcal{J}(0, t)]_-, [\mathcal{J}(0, t)]_+) = \mathcal{F}_0(0, t) + \frac{(\Psi^2(t(1)), 1)}{2}.
\]

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