On weighted occupation times for refracted spectrally negative Lévy processes

Bo Li and Xiaowen Zhou
School of Mathematics and LPMC, Nankai University, Tianjin, P.R.China
Department of Mathematics and Statistics, Concordia University, Canada
September 28, 2018

Abstract
For refracted spectrally negative Lévy processes, we identify expressions of several quantities related to Laplace transforms on their weighted occupation times until first exit times. Such quantities are expressed in terms of unique solutions to integral equations involving weight functions and scale functions for the associated spectrally negative Lévy processes. Previous results on refracted Lévy processes are recovered.

Keywords: spectrally negative Lévy process, refracted process, weighted occupation time, exit time, resolvent, integral equation.

1 Introduction
In this paper, we are interested in evaluating the $\omega$-weighted occupation time of refracted spectrally negative Lévy process, which is formally defined in [14] as the unique solution to stochastic differential equation

$$dX_t = dY_t - \delta 1_{\{X_t \geq a\}} dt = dZ_t + \delta 1_{\{X_t < a\}} dt,$$

where $Y = (Y_t)_{t \geq 0}$ is a spectrally negative Lévy process (SNLP in short) and $Z_t = Y_t - \delta t, t \geq 0$.

A motivation of studying refracted process stems from its applications in stochastic control. In many insurance risk models, see for example [13 8 4] and references therein, to maximise the amount of discounted dividends paid up to time of ruin, among all admissible control strategies the optimal dividend strategy is either paying nothing or paying dividends as much as possible in the so-called solvency regions. When a constant ceiling $\delta$ is imposed for the dividend rate,
under further conditions the optimal policy reduces to the so-called threshold dividend strategy and the surplus process with dividends becomes the refracted process, in which the insurance company pays nothing when the reserve is below a certain critical level, and pays dividends at the maximal rate $\delta$ when the reserve is above the level.

To our best knowledge, the refracted Lévy process is first introduced in \[10, 11\] for Brownian risk mode, where by making use of HJB functional equation, the threshold strategy is shown to be optimal if the dividend rate is bounded above by some constant. The analogous problem for Crémer-Lundberg risk model with exponential claim size distribution is studied in \[5\]. Some actuarial quantities of risk model with threshold dividend are investigated in \[20, 23\]. It is studied under the framework of spectrally negative Lévy process in \[13\] where a sufficient condition on Lévy measure is found under which the threshold strategy is optimal. \[14\] focuses on the existence and uniqueness of solution to (1) and some fluctuation identities for $X$ are also established. \[15\] investigates the occupation times of half lines, \[22\] identifies the distribution of various functionals related, and \[25, 24\] mainly consider general Lévy process with rational jumps.

During the last several years there have been a series of papers concerning occupation time related problems for SNLP. These problems arise from both theoretical interests and the applications in risk theory and finance; see for example \[6, 16, 21, 18, 19, 17, 15, 22, 25, 24\]. Among them using a perturbation approach, \[16\] studies the occupation times of semi-infinite intervals. For the occupation times spent in a certain interval, using a strong approximation approach \[21\] identifies joint Laplace transforms until first passage times. Laplace transforms involving joint occupation times are investigated in \[18, 19\] with a Poisson approach. Expressions involving occupation time over a finite interval and resolvent measure are found in \[7\]. The results are typically expressed using scale functions for SNLP. By further improving the Poisson approach, fluctuation identities on weighted occupation times for SNLP are obtained in \[17\] which generalize many of the previous results. In \[17\] the results are expressed in terms of unique solutions to integral equations specified using the scale functions and the weight function for occupation time.

Given the previous results on refracted SNLP and on occupation times for SNLP, our goal in this paper is to establish identities concerning the Laplace transform of

$$ L(t) := \int_0^t \omega(X_s)ds $$

for a locally bounded measurable nonnegative function $\omega(\cdot)$ on $\mathbb{R}$, which is called a $\omega$-weighted occupation time in \[17\]. Such identities can also be treated as identities for refracted SNLP killed at an occupation time dependent rate. To prove the main results, we adapt the previous approach of \[17\] by replacing the Poisson approach there with a Feyman-Kac type argument.
Our main results are expressed in terms of functions \((w^{(\omega)}, z^{(\omega)})\), which extends the notations introduced in [22] and depends on the classical scale function \((W, Z)\) as well as the weight function \(\omega\). For some simpler examples of weight function, we could find more explicit expressions of \((w^{(\omega)}, z^{(\omega)})\) in terms of \((W, Z)\) and recover previous results in [14, 15, 22].

The remainder of the paper is organised as follows. After the review of previous work on refracted SNLP and occupation times for SNLP and a summary of the main results, in Section 2 we present preliminary results on SNLP, scale functions and exit problems for refracted SNLP. Section 3 contains the main results whose proofs are deferred to Section 5. More detailed discussions are carried out for examples in Section 4.

2 Preliminaries

We first briefly review the spectrally negative Lévy processes, the associated scale functions and some known results. For further details, we refer the readers to Bertoin [2] and Kyprianou [12]. Throughout the paper, \(Y\) denotes an SNLP, \(Z_t = Y_t - \delta t\) and \(X\) is the unique solution to [11] called refracted SNLP. The law of \(X\) for \(X_0 = x\) is denoted by \(\mathbb{P}_x\) and the corresponding expectation by \(\mathbb{E}_x\). We write \(\mathbb{P}\) and \(\mathbb{E}\) when \(x = 0\).

Let \(Y = (Y_t)_{t \geq 0}\) be a spectrally negative Lévy process, that is a stochastic process with stationary and independent increments and without positive jumps. Its Laplace transform exists and is specified by

\[
\mathbb{E}(\exp(\theta Y_t)) = \exp(\psi(\theta)t), \quad \forall \theta \geq 0.
\]

Function \(\psi(\theta)\), known as the Laplace exponent of \(Y\), is continuous, strictly convex on \(\mathbb{R}^+\) and given by the Lévy-Khintchine formula:

\[
\psi(\theta) = \frac{\sigma^2}{2} \theta^2 + \gamma \theta + \int_{(0,\infty)} \left( e^{-\theta x} - 1 + \theta x 1_{\{x < 1\}} \right) \Pi(dx),
\]

where \(\gamma \in \mathbb{R}, \sigma \geq 0\) and the Lévy measure \(\Pi\) is a \(\sigma\)-finite measure on \((0, \infty)\) such that \(\int_{\mathbb{R}^+} (1 \wedge x^2) \Pi(dx) < \infty\). For \(q \geq 0\), the \(q\)-scale function is defined as a continuous and increasing function such that \(W^{(q)}(x) = 0\) for \(x < 0\) and

\[
\int_0^\infty e^{-sy} W^{(q)}(y) dy = \frac{1}{\psi(s) - q}, \quad \text{for } s > \Phi(q),
\]

where \(\Phi(q) := \sup\{s \geq 0, \psi(s) = q\}\) denotes the right inverse of \(\psi(\cdot)\). With the first scale function \(W^{(q)}(\cdot)\), we can define another scale function by

\[
Z^{(q)}(x) := 1 + q \int_0^x W^{(q)}(y) dy, \quad \text{for } x \in \mathbb{R}.
\]

(3)
We write $W(x) = W(0)(x)$ and $Z(x) = Z(0)(x)$ when $q = 0$. It is known that, for $q > 0$

$$\frac{W(q)(x - a)}{W(q)(x)} \to e^{-\Phi(q)a}, \quad e^{-\Phi(q)x}W'(q)(x) \to \Phi'(q) \quad \text{and} \quad \frac{Z(q)(x)}{W(q)(x)} \to \frac{q}{\Phi(q)},$$

(4) as $x \to \infty$. Since $\log W(x)$ is concave on $\mathbb{R}^+$, it is absolutely continuous with respect to Lesbegue measure and $W'(x)$ is well defined a.e. Similar conclusion can be derived for every $W(q)(x)$ by change of measure. We refer to [11, 9] for more detailed discussions and examples of scale functions.

For process $Z$, we denote by $\psi_Z(\theta)$ its Laplace exponent, $\varphi(q) := \sup\{s \geq 0, \psi_Z(s) = q\}$ the right inverse of $\psi_Z(\cdot)$, and $(\mathbb{W}(q), \mathbb{Z}(q))$ the $q$-scale functions associated with $Z$. It can be checked directly that, $\psi_Z(s) = \psi(s) - \delta s, \varphi(q) > \Phi(q)$ and

$$\mathbb{W}(x) = W(x) + \delta \int_0^x \mathbb{W}(x - z)W(dz),$$

(5) where $W(dz)$ stands for the Stieltjes integral on $\mathbb{R}$ induced by function $W$ and $W(\{0\}) = W(0) = \mathbb{W}(0)(1 - \delta W(0))$.

**Remark 1.** We remark that, for any locally bounded measurable function $f \geq 0$, its Stieltjes integral with respect to $m(\cdot)$, denoted

$$\int_a^b f(z)m(dz) = \int_{[a,b]} f(z)m(dz) = \int_0^{b-a} f(z + a)m(dz + a)$$

is the integral of $f$ on the interval $(a, b]$ and that $\int_a^b m(dz) = m(b) - m(a)$. If the integral is on $[a, b]$, we write $\int_{[a,b]} f(z)m(dz) = \int_{[a,b]} f(z)m(dz)$ as in [13].

The following hypothesis on $Y$ is introduced in [14].

(H) The constant $0 < \delta < \gamma + \int_{(0,1)} x\Pi(dx)$ if $Y$ has paths of bounded variation.

It is shown in [14, Theorem 1] that under hypothesis (H), equation (1) has a unique strong solution. Actually, for the case of bounded variation, we have $W(0) = \frac{1}{\gamma + \int_{(0,1)} x\Pi(dx)}$. Thus, condition (H) is equivalent to the following assumption:

(H') $1 - \delta W(0) > 0$, i.e. $Z$ is not the negative of a subordinator,

which will be in force throughout the remainder of the paper.

We denote by

$$\kappa_b^+ := \inf\{t > 0, X_t > b\} \quad \text{and} \quad \kappa_c^- := \inf\{t > 0, X_t < c\},$$

4
with convention $\inf \emptyset = \infty$, the first passage times of $X$ and for any $x, y \in \mathbb{R}$ define the following auxiliary function,

$$w(x, y) := W(x - y) + \delta \int_a^x \mathbb{W}(x - z) W(dz - y)$$

(6)

$$= \mathbb{W}(x - y) - \delta \int_{y-}^a \mathbb{W}(x - z) W(dz - y)$$

(7)

in light of (6). For $a > 0$, the equation above reduces to

$$w(x, 0) = W(x) + \delta \int_a^x \mathbb{W}(x - z) W'(z) dz$$

and

$$w(x, y) = \begin{cases} W(x - y) + \delta \int_a^x \mathbb{W}(x - z) W'(z - y) dz & \text{for } y \in (0, a] \\ \mathbb{W}(x - y) & \text{for } y \in (a, \infty) \end{cases}$$

which are functions used in [14] for conclusion on interval $(0, b)$, and the following results follow from Theorems 4 and 6 in [14] after a spatial shifting argument.

**Proposition 1.** For any $x, c, b$ such that $a, x \in (c, b)$, we have

$$P_x (\kappa_b^+ < \kappa_c^-) = \frac{w(x, c)}{w(b, c)},$$

(8)

and the resolvent measure $V$ of $X$ killed at exiting $[c, b]$ is given by

$$V f(x) := \int_0^\infty \mathbb{E}_x [f(X_t); t \leq \kappa_b^+ \land \kappa_c^-] dt$$

$$= \int_c^b f(y) \left( \frac{w(x, c)}{w(b, c)} w(b, y) - w(x, y) \right) dy$$

(9)

for any bounded measurable function $f$ on $[c, b]$.

**Remark 2.** Notice that for $c = 0 < x < b$, a version of the density for measure $V(x, dy)$ on $[0, b]$ determined by (9) is

$$\frac{w(x, 0)}{w(b, 0)} w(b, y) - w(x, y), \text{ for } 0 < y < b,$$

and its value at $y = a$ is

$$\frac{w(x, 0)}{w(b, 0)} w(b, a) - w(x, a) = \frac{w(x, 0)}{w(b, 0)} W(b - a) - W(x - a),$$

which is different from the value at $y = a$ of the corresponding density in [14] Theorem 6] given by

$$\frac{w(x, 0)}{w(b, 0)} \mathbb{W}(b - a) - \mathbb{W}(x - a).$$

Otherwise, the density from [9] agrees with that in [14] Theorem 6], and clearly they are associated with the same absolutely continuous measure on $(0, b)$.
3 Main results

Before stating our main results, we introduce two more generalized scale functions \((w^{(\omega)}, z^{(\omega)})\) which, for the \(w(x,y)\) defined in [3], are solutions to

\[ w^{(\omega)}(x,y) = w(x,y) + \int_y^x w(x,z)\omega(z)w^{(\omega)}(z,y)dz \]  \tag{10}

and

\[ z^{(\omega)}(x,y) = 1 + \int_y^x w(x,z)\omega(z)z^{(\omega)}(z,y)dz, \]  \tag{11}

respectively. The existence and uniqueness of solutions to (10) and (11) are assured by Lemma 1.

**Lemma 1.** Let \(R\) admit a unique locally bounded solution on \(w\) which, for the \(w\) of \(x,y\) \(\omega\) and is thus the unique solution to this equation. In particular, for any \(f\geq 0\) locally bounded, \(w^{(\omega)}f(x,y) := \int_y^x w^{(\omega)}(x,z)f(z)dz\) is the solution to

\[ w^{(\omega)}f(x,y) = \int_y^x w(x,z)f(z)dz + \int_y^x w(x,z)\omega(z)\left(w^{(\omega)}f\right)(z,y)dz. \]

Given \(w^{(\omega)}(x,y)\) and Lemma 1 we have the following remark.

**Remark 3.** Let \(\nu(dy)\) be a Radon measure on \(\mathbb{R}\). By Fubini’s Theorem one can check directly that \((w^{(\omega)}\nu)(x,y) := \int_y^x w^{(\omega)}(x,z)\nu(dz)\) satisfies

\[ (w^{(\omega)}\nu)(x,y) = \int_y^x w(x,z)\nu(dz) + \int_y^x w(x,z)\omega(z)\left(w^{(\omega)}\nu\right)(z,y)dz, \]

and is thus the unique solution to this equation. In particular, for any \(f\geq 0\) locally bounded,

\[ w^{(\omega)}f(x,y) = \int_y^x w(x,z)f(z)dz + \int_y^x w(x,z)\omega(z)\left(w^{(\omega)}f\right)(z,y)dz. \]

We now state the main results.

**Theorem 1.** Given \(c < b\), we have for \(x \in [c,b]\)

\[ \mathbb{E}_x \left[ e^{-L(\kappa^+_b)}; \kappa^+_c \leq \kappa_c \right] = \frac{w^{(\omega)}(x,c)}{w^{(\omega)}(b,c)} \]  \tag{13}

and

\[ \mathbb{E}_x \left[ e^{-L(\kappa^-_c)}; \kappa^-_c \leq \kappa^+_b \right] = z^{(\omega)}(x,c) - \frac{w^{(\omega)}(x,c)}{w^{(\omega)}(b,c)}z^{(\omega)}(b,c). \]

For any \(x,y \in (c,b)\), an expression of the resolvent of \(X\) killed at exiting \([c,b]\) is given by

\[ V^{(\omega)}(x,dy) := \int_0^\infty \mathbb{E}_x \left( e^{-L(t)}; X_t \in dy, t \leq \kappa^+_b \wedge \kappa^-_c \right) dt \]

\[ = \left( \frac{w^{(\omega)}(x,c)}{w^{(\omega)}(b,c)}w^{(\omega)}(b,y) - w^{(\omega)}(x,y) \right) dy. \]
The conclusions above are similar to the corresponding results on \(\omega\)-weighted occupation problem for SNLP in (7), where the auxiliary functions \((W^{(\omega)}, Z^{(\omega)})\) are defined as the unique solution, respectively, to the following equations.

\[
W^{(\omega)}(x, y) = W(x - y) + \int_y^x W(x - z)\omega(z)W^{(\omega)}(z, y)dz, \tag{14}
\]
\[
Z^{(\omega)}(x, y) = 1 + \int_y^x W(x - z)\omega(z)Z^{(\omega)}(z, y)dz. \tag{15}
\]

Therefore, we present the following relation between them, which generalises (6) and (7) since for \(\omega(\cdot) \equiv 0\), \(W^{(\omega)}(x, y) = W(x - y)\) by definition.

**Proposition 2.** For \(x \geq y\), we have

\[
w^{(\omega)}(x, y) = W^{(\omega)}(x, y) + \delta \int_a^x W^{(\omega)}(x, z)W^{(\omega)}(dz, y)
\]
\[
= \Psi^{(\omega)}(x, y) - \delta \int_y^a W^{(\omega)}(x, z)W^{(\omega)}(dz, y) \tag{16}
\]

and

\[
z^{(\omega)}(x, y) = Z^{(\omega)}(x, y) + \delta \int_a^x W^{(\omega)}(x, z)Z^{(\omega)}(dz, y)
\]
\[
= Z^{(\omega)}(x, y) - \delta \int_y^a W^{(\omega)}(x, z)Z^{(\omega)}(dz, y). \tag{17}
\]

From (14) and (15), for every \(y \in \mathbb{R}\), it can be shown that \(Z^{(\omega)}(\cdot, y)\) and \(W^{(\omega)}(\cdot, y)\) are increasing functions. The associated Stieltjes measures satisfy respectively,

\[
\begin{cases}
W^{(\omega)}(dx, y) &= W(dx - y) + \int_\mathbb{R} W(dx - z)\omega(z)W^{(\omega)}(z, y)dz, \\
Z^{(\omega)}(dx, y) &= \int_\mathbb{R} W(dx - z)\omega(z)W^{(\omega)}(z, y)dz,
\end{cases} \tag{18}
\]

with \(W^{(\omega)}(\{y\}, y) = W(0)\) and \(Z^{(\omega)}(\{y\}, y) = 0\), noticing that \(0 = W(u - v) = W^{(\omega)}(u, v)\) for \(u < v\). At the refraction point \(a\), we have \(w^{(\omega)}(a, a) = W(0) = \Psi(0)(1 - \delta W(0))\) and

- for \(a > x\), \(w^{(\omega)}(x, y) = W^{(\omega)}(x, y)\) and \(z^{(\omega)}(x, y) = Z^{(\omega)}(x, y)\),
- for \(y > a\), \(w^{(\omega)}(x, y) = \Psi^{(\omega)}(x, y)\) and \(z^{(\omega)}(x, y) = Z^{(\omega)}(x, y)\),
- for \(x = a > y\), \(w^{(\omega)}(a, y) = W^{(\omega)}(a, y)\) and \(z^{(\omega)}(a, y) = Z^{(\omega)}(a, y)\),
- for \(x > a = y\), \(w^{(\omega)}(x, a) = \Psi^{(\omega)}(x, a)(1 - \delta W(0))\), \(z^{(\omega)}(x, a) = Z^{(\omega)}(x, a)\).

In particular, \(w^{(\omega)}(x, y)\) is continuous at \((a, a)\) if and only if \(W(0) = 0\).

We also have the following scale function identities similar to those associated to \((W^{(q)}, Z^{(q)})\).
Therefore, for \( x \)
\[
\therefore \text{for } x, y \in \mathbb{R}
\]
\[
w^{(\omega_2)}(x, y) - w^{(\omega_1)}(x, y) = \int_y^x w^{(\omega_1)}(x, z) (\omega_2(z) - \omega_1(z)) w^{(\omega_2)}(z, y) dz
\]
and
\[
z^{(\omega_2)}(x, y) - z^{(\omega_1)}(x, y) = \int_y^x w^{(\omega_1)}(x, z) (\omega_2(z) - \omega_1(z)) z^{(\omega_2)}(z, y) dz.
\]

**Corollary 1** (First hitting time). For any \( d \in (c, b) \), let \( \kappa^{(d)} := \inf\{ t > 0, X_t = d \} \) be the first hitting time. We have for \( x \in [c, b] \)
\[
\mathbb{E}_x \left[ e^{-L(\kappa^{(d)})}; \kappa^{(d)} \leq \kappa_b^+ \land \kappa_c^- \right] = \frac{w^{(\omega)}(x, c)}{w^{(\omega)}(d, c)} - \frac{w^{(\omega)}(x, d)}{w^{(\omega)}(b, d)} \frac{w^{(\omega)}(b, c)}{w^{(\omega)}(d, c)}.
\]

**Proof of Corollary**. Observing that due to absence of positive jumps,
\[
\{ \kappa_d^- \leq \kappa_b^+ < \infty \} = \{ \kappa^{(d)} \leq \kappa_b^+ < \infty \} \ \mathbb{P}_x \text{-a.s.}
\]
Therefore, for \( x \in [c, b] \)
\[
\mathbb{E}_x \left[ e^{-L(\kappa_b^+)}; \kappa_b^+ \leq \kappa_c^- \right]
\]
\[
= \mathbb{E}_x \left[ e^{-L(\kappa_b^+)}; \kappa_b^+ \leq \kappa_c^-, \kappa_b^- \leq \kappa_d^- \right] + \mathbb{E}_x \left[ e^{-L(\kappa_b^+)}; \kappa_b^- \leq \kappa_c^-, \kappa_d^- \leq \kappa_b^+ \right]
\]
\[
= \mathbb{E}_x \left[ e^{-L(\kappa_b^+)}; \kappa_b^- \leq \kappa_d^- \right] + \mathbb{E}_x \left[ e^{-L(\kappa_b^+)}; \kappa_d^- \leq \kappa_c^- \right]
\]
\[
= \mathbb{E}_x \left[ e^{-L(\kappa_b^+)}; \kappa_b^+ \leq \kappa_d^- \right] + \mathbb{E}_x \left[ e^{-L(\kappa_b^+)}; \kappa_d^- \leq \kappa_c^- \right] \mathbb{E}_d \left[ e^{-L(\kappa_b^+)}; \kappa_b^+ \leq \kappa_c^- \right].
\]
The desired identity then follows from the Markov property and Theorem [1].

Indeed, given the resolvent measure, appealing to compensation formula from [2, O.5] we have the joint distribution:
\[
\mathbb{E}_x \left( e^{-L(\kappa_c^-)}; \kappa_c^- \leq \kappa_b^+, X(\kappa_c^-) \in dy, X(\kappa_c^-) \in dz \right) = V^{(\omega)}(x, dy)(-\Pi(y - dz))
\]
for \( b > y > c \geq z \). Since \( \{ x, \Pi(\{ x \}) > 0 \} \) is at most countable and \( V(x, dy) \) is a continuous measure, the creeping cannot be caused by a jump. Therefore, similar to the case of a Lévy process, a refracted SNLP creeps downward at a lower level with positive probability only if it has a nontrivial Gaussian part. We restrict ourselves to \( \sigma > 0 \) in the following corollary. \( W(\cdot) \) and \( w^{(\omega)}(x, x - \cdot) \) are thus continuously differentiable on \((0, \infty)\) from [3, Theorem 1], \( W(0) = 0 \) and \( W'(0+) = 2/\sigma^2 \).
Corollary 2 (Creeping). For $d < x < b$ and $d \leq a$ we have

$$
\mathbb{E}_x \left[ e^{-L(\kappa_d^-)} ; X(\kappa_d^-) = d, \kappa_d^- \leq \kappa_b^+ \right] = \frac{\sigma^2}{2} \left( \frac{w^{(\omega)}(x, d)}{w^{(\omega)}(b, d)} \partial_y w^{(\omega)}(b, d) - \partial_y w^{(\omega)}(x, d) \right),
$$

where for $x > d$,

$$
\partial_y w^{(\omega)}(x, d) := \left( \lim_{z \to d} \frac{w^{(\omega)}(x, z) - w^{(\omega)}(x, d)}{z - d} \right)_{y=d}.
$$

Proof of Corollary 2. Similar to the discussion in [14, Theorem 7], the desired result is derived by letting $c \to d^-$ in Corollary 1. From the right continuity of $X$, we have

$$
\mathbb{E}_x \left[ e^{-L(\kappa_d^-)} ; X(\kappa_d^-) = d, \kappa_d^- \leq \kappa_b^+ \right] = \lim_{c \to d^-} \mathbb{E}_x \left[ e^{-L(\kappa_d^-)} ; \kappa_d^- \leq \kappa_b^+ \right]
$$

$$
= \lim_{c \to d^-} \frac{d-c}{w^{(\omega)}(d, c)} \frac{1}{d-c} \left( w^{(\omega)}(x, c) - w^{(\omega)}(x, d) \frac{w^{(\omega)}(b, c)}{w^{(\omega)}(b, d)} \right).
$$

On the other hand, from the discussion after Proposition 2, it follows that for $c < d \leq a$,

$$
w^{(\omega)}(d, c) = W^{(\omega)}(d, c) = W(d-c) + \int_c^d W(d-z) \omega(z) W^{(\omega)}(z, c) dz.
$$

The mean value theorem yields $w^{(\omega)}(d, c) = W'(0^+)(d-c) + o(d-c)$. And

$$
\frac{1}{d-c} \left( w^{(\omega)}(x, c) - w^{(\omega)}(x, d) \frac{w^{(\omega)}(b, c)}{w^{(\omega)}(b, d)} \right)
$$

$$
= \frac{w^{(\omega)}(x, c)w^{(\omega)}(b, d) - w^{(\omega)}(x, d)w^{(\omega)}(b, c)}{(d-c)w^{(\omega)}(b, d)} \to \left( w^{(\omega)}(x, d) \frac{\partial_y w^{(\omega)}(b, d)}{w^{(\omega)}(b, d)} - \partial_y w^{(\omega)}(x, d) \right),
$$

by letting $c \to d^-$. This finishes the proof.

Noting that the right hand side of the formula is exactly the $y$-derivative of the density of $V^{(\omega)}(x, dy)$ at $(x, y)$, which coincides with conclusion of SNLP without refraction.

4 Examples

The refracted SNLP has been well-defined in [14], but there are not many results yet on its occupation times as far as we know. In this section, we apply our results to some simpler examples of $\omega$ to reproduce the existing results in [14] [15] [22] for which we essentially need to find expressions for $(w^{(\omega)}, z^{(\omega)})$.

Example 1. Firstly, for the case $\omega(z) \equiv q \geq 0$, we want to find $(w^{(\omega)}, z^{(\omega)})$ such that

$$
w^{(\omega)}(x, y) = w(x, y) + q \int_y^x w(x, z) w^{(\omega)}(z, y) dz
$$
and
\[ z^{(\omega)}(x, y) = 1 + q \int_y^x w(x, z)z^{(\omega)}(z, y)dz. \]

Actually, \( W^{(\omega)}(x, y) = W^{(q)}(x - y) \) and \( Z^{(\omega)}(x, y) = Z^{(q)}(x - y) \), see [17] and we have
\[
w^{(\omega)}(x, y) = w^{(q)}(x, y) := W^{(q)}(x - y) + \delta \int_a^x W^{(q)}(x - z)W^{(q)}(dz - y),
\]
and
\[
z^{(\omega)}(x, y) = z^{(q)}(x, y) := Z^{(q)}(x - y) + \delta q \int_a^x W^{(q)}(x - z)W^{(q)}(z - y)dz,
\]
by Proposition [2] where \( w^{(q)} \) and \( z^{(q)} \) coincide with the functions of order \( q \) used in [14, Theorem 4 and 6] and introduced in [22].

With the \( (w^{(q)}, z^{(q)}) \) from above, applying Proposition [3] we have more identities as follows.

\[
\begin{align*}
  w^{(\omega)}(x, y) &= w^{(q)}(x, y) + \int_y^x w^{(q)}(x, z)(\omega(z) - q)w^{(\omega)}(z, y)dz \\
  &= w^{(q)}(x, y) + \int_y^x w^{(\omega)}(x, z)(\omega(z) - q)w^{(q)}(z, y)dz \\
  z^{(\omega)}(x, y) &= z^{(q)}(x, y) + \int_y^x w^{(q)}(x, z)(\omega(z) - q)z^{(\omega)}(z, y)dz \\
  &= z^{(q)}(x, y) + \int_y^x w^{(\omega)}(x, z)(\omega(z) - q)z^{(q)}(z, y)dz
\end{align*}
\]

for \( x, y \in \mathbb{R} \), (23)

by taking \((\omega_2, \omega_1) = (\omega, q)\) and \((\omega_2, \omega_1) = (q, \omega)\) in the equations respectively.

In particular, for the case \( \omega(x) = p + (q - p)1_{\{x < a\}} \), which is the function considered in [15] [22], we have for \( x, y \in \mathbb{R} \)

\[
\begin{align*}
  w^{(\omega)}(x, y) &= w^{(q)}(x, y) - (q - p) \int_a^x w^{(\omega)}(x, z)w^{(q)}(z, y)dz \\
  &= w^{(q)}(x, y) - (q - p) \int_a^x W^{(p)}(x - z)w^{(q)}(z, y)dz, \tag{24} \\
  z^{(\omega)}(x, c) &= z^{(q)}(x, c) - (q - p) \int_a^x W^{(p)}(x - z)z^{(q)}(z, c)dz,
\end{align*}
\]

where the facts \( q - \omega(z) = (q - p)1_{\{z \geq a\}}, \omega(z) = p \) for \( z \geq a \) and \( w^{(\omega)}(x, z) = W^{(\omega)}(x, z) = W^{(p)}(x - z) \) for \( z \geq a \) after Proposition [2] are used, and which gives [22, Theorem 3].

By letting the boundaries go to infinity in Theorem [1] one can solve the one-sided exit problems in principle. However, it is not easy to carry it out when the behaviors of \( \omega(\cdot) \) near infinity are arbitrary. Here, for a simpler case that \( \omega(x) \) is a constant for \(|x| > M_0\) for some \( M_0 > 0 \), we could work out the following Proposition.
Proposition 4. If \( \omega(z) = p \) for all \( z > M_1 \) for some \( p > 0 \) and \( M_1 \in \mathbb{R} \), we have as \( x \to \infty \)
\[
\frac{e^{\varphi(p)(y-z)}w^{(\omega)}(x,y)}{\varphi'(p)} \to 1 + \int_y^\infty (\omega(u) - p)w^{(\omega)}(u,y)e^{\varphi(p)(y-u)}du - \delta \int_y^a e^{\varphi(p)(y-u)}W^{(\omega)}(du,y),
\]
and
\[
\frac{e^{\varphi(p)(y-z)}z^{(\omega)}(x,y)}{\varphi'(p)} \to \frac{p}{\varphi(p)} + \int_y^\infty (\omega(u) - p)z^{(\omega)}(u,y)e^{\varphi(p)(y-u)}du - \delta \int_y^a e^{\varphi(p)(y-u)}Z^{(\omega)}(du,y).
\]
Similarly, if \( \omega(z) = q \) for all \( z < M_2 \) for some \( q > 0 \) and \( M_2 \in \mathbb{R} \), we have as \( y \to -\infty \)
\[
\lim_{y \to -\infty} \frac{e^{\Phi(q)(y-z)}w^{(\omega)}(x,y)}{\Phi'(q)} = \lim_{y \to -\infty} \frac{e^{\Phi(q)(y-z)}z^{(\omega)}(x,y)}{\Phi'(q)} q
= 1 + \int_R w^{(\omega)}(x,u)(\omega(u) - q)e^{\Phi(q)(u-x)}du + \delta \Phi(q) \int_a^x e^{\Phi(q)(u-x)}\mathbb{W}^{(\omega)}(x,u)du.
\]

For the one-sided first passage times, which are studied in [14] [15] [22], we have the following more general results from Proposition 4 and Theorem 1.

Corollary 3 (One-sided first passage times).

1. If \( \omega(z) = p > 0 \) for all \( z > M_1 \) for some \( M_1 \in \mathbb{R} \), we have
\[
\mathbb{E}_x \left( e^{-L(\omega_\cdot)} ; \kappa_c^- < \infty \right) = z^{(\omega)}(x,c) - w^{(\omega)}(x,c)
\times \frac{p}{\varphi(p)} + \int_c^\infty (\omega(u) - p)z^{(\omega)}(u,c)e^{\varphi(p)(c-u)}du - \delta \int_c^a e^{\varphi(p)(c-u)}Z^{(\omega)}(du,c)
\times \frac{1 + \int_c^\infty (\omega(u) - p)w^{(\omega)}(u,c)e^{\varphi(p)(c-u)}du - \delta \int_c^a e^{\varphi(p)(c-u)}W^{(\omega)}(du,c)}{1 + \int_c^\infty w^{(\omega)}(u,c)e^{\varphi(p)(c-u)}du - \delta \int_c^a e^{\varphi(p)(c-u)}W^{(\omega)}(du,c)}.
\]

2. If \( \omega(z) = q > 0 \) for all \( z < M_2 \) for some \( M_2 \in \mathbb{R} \), we have
\[
\mathbb{E}_x \left( e^{-L(\omega_\cdot)} ; \kappa_c^+ < \infty \right) = e^{\Phi(q)(x-b)} \frac{1 + \int_x^\infty w^{(\omega)}(x,u)(\omega(u) - q)e^{\Phi(q)(u-x)}du + \delta \Phi(q) \int_x^a e^{\Phi(q)(u-x)}\mathbb{W}^{(\omega)}(x,u)du}{1 + \int_x^\infty w^{(\omega)}(b,u)(\omega(u) - q)e^{\Phi(q)(u-b)}du + \delta \Phi(q) \int_a^b e^{\Phi(q)(u-b)}\mathbb{W}^{(\omega)}(b,u)du}.
\]

With \((24)\) for \( \omega(x) = p + (q-p)1_{\{x < a\}} \), we here only compare the ratios in Corollary 3 with existing results from [14] Theorem 5 and 6] and [22] Corollary 1 and 2] for \( c = 0 \).

If \( \omega(z) \equiv q = p \), then
\[
W^{(\omega)}(x,y) = W^{(q)}(x-y), \quad Z^{(\omega)}(x,y) = Z^{(q)}(x-y), \quad \varphi(q) > \Phi(q),
\]
and the following Laplace transforms satisfy
\[
\tilde{W}^{(q)}(\varphi(q)) = \frac{1}{\delta \varphi(q)} \quad \text{and} \quad d\tilde{W}^{(q)}(\varphi(q)) := \int_0^\infty e^{-\varphi(q)x}W^{(q)}(dx) = \frac{1}{\delta}. \quad (25)
\]
We thus have
\[
\frac{p}{\varphi(p)} + \int_c^\infty (\omega(u) - p) z^{(\omega)}(u, c) e^{\varphi(p)(c-u)} du - \delta \int_c^a e^{\varphi(p)(c-u)} Z^{(\omega)}(du, c)
\]
\[
= \frac{q}{\varphi(q)} - \delta q \int_0^a e^{-\varphi(q)u} W^{(q)}(u) du = q \delta \int_a^\infty e^{-\varphi(q)u} W^{(q)}(u) du.
\]
Applying the Laplace transforms in (25) one gives
\[
1 + \int_c^\infty (\omega(u) - p) w^{(\omega)}(u, c) e^{\varphi(p)(c-u)} du - \delta \int_c^a e^{\varphi(p)(c-u)} W^{(\omega)}(du, c)
\]
\[
= 1 - \delta \int_0^a e^{-\varphi(q)u} W^{(q)}(du) = \delta \int_a^\infty e^{-\varphi(q)u} W^{(q)}(du)
\]
and
\[
1 + \int_{-\infty}^x w^{(\omega)}(x, u)(\omega(u) - q)e^{\Phi(q)(u-x)} du + \delta \Phi(q) \int_a^x e^{\Phi(q)(u-x)} \bar{W}^{(\omega)}(x, u) du
\]
\[
= 1 + \delta \Phi(q) \int_a^x e^{\Phi(q)(u-x)} \bar{W}^{(q)}(x - u) du
\]
which coincide with the formulas of the Laplace transforms in [14, Theorem 5 and 6].

If \( \omega(z) = q 1_{\{z < a\}}, \ p = 0 \) and \( \mathbb{E}[Y_1] > \delta \), we have
\[
\varphi(0) = 0, \quad \frac{p}{\varphi(p)} = \psi'_Y(0) = \mathbb{E}[Y_1] - \delta
\]
and
\[
\omega(z) - p = q 1_{\{z < a\}}, \ w^{(\omega)}(z, c) = W^{(q)}(z), \ z^{(\omega)}(z, c) = Z^{(q)}(z) \text{ for } z < a.
\]
Therefore,
\[
1 + \int_c^\infty (\omega(u) - p) w^{(\omega)}(u, c) e^{\varphi(p)(c-u)} du - \delta \int_c^a e^{\varphi(p)(c-u)} W^{(\omega)}(du, c)
\]
\[
= 1 + q \int_0^a W^{(q)}(z) dz - \delta \int_0^a W^{(q)}(du) = Z^{(q)}(a) - \delta W^{(q)}(a)
\]
and
\[
\frac{p}{\varphi(p)} + \int_c^\infty (\omega(u) - p) z^{(\omega)}(u, c) e^{\varphi(p)(c-u)} du - \delta \int_c^a e^{\varphi(p)(c-u)} Z^{(\omega)}(du, c)
\]
\[
= \psi'_Z(0) + q \int_0^a Z^{(q)}(z) dz - \delta q \int_0^a W^{(q)}(u) du
\]
\[
= \mathbb{E}[Y_1] - \delta + q \int_0^a (Z^{(q)}(y) - \delta Z^{(q)}(a - y) W^{(q)}(y)) dy
\]
where the last equation comes from the integral over \([0, a]\) of functions

\[
Z^{(q)}(x) - Z^{(q)}(x) = \delta q \int_{0}^{x} \mathbb{W}^{(q)}(x - z)W^{(q)}(z)\,dz,
\]

by taking \(\omega(z) \equiv q, y = 0\) in Proposition 2 and they coincide with [22, Corollary 1(i)]. Similarly, \(\omega(z) - q = -q1_{\{z > a\}}\), then

\[
 w^{(\omega)}(x, z) = \mathbb{W}^{(\omega)}(x, z) = \mathbb{W}(x - z) \quad \text{for} \quad z > a.
\]

We have

\[
1 + \int_{-\infty}^{x} w^{(\omega)}(x, u)(\omega(u) - q)e^{\Phi(q)(u-x)}\,du + \delta\Phi(q) \int_{a}^{x} e^{\Phi(q)(u-x)}\mathbb{W}(\omega)(x, u)\,du
\]

\[
= 1 - q \int_{a}^{x} e^{\Phi(q)(u-x)}\mathbb{W}(x - u)\,du + \delta\Phi(q) \int_{a}^{x} e^{\Phi(q)(u-x)}\mathbb{W}(x - u)\,du
\]

\[
= 1 - (q - \delta\Phi(q)) \int_{0}^{x-a} \mathbb{W}(y)e^{-\Phi(q)y}\,dy,
\]

which coincides with [22, Corollary 2(i)].

For the end of this section, we consider the case of \(\omega(\cdot)\) being an \(n\)-step function. Besides the relation given in Proportion 3, an inductive way is provided to define the function, similar to [17],

**Example 2.** Let \(\lambda_j \geq 0\) and \(a_n > a_{n-1} > \cdots > a_1\) be constants. Let

\[
\omega_n(x) = \lambda_0 + \sum_{j=1}^{n} (\lambda_j - \lambda_{j-1})1_{\{x \geq a_j\}}
\]

be a step function. Then \(w^{(\omega_n)}(x, y)\) and \(z^{(\omega_n)}(x, y)\) can be defined inductively as follow. For \(k \geq 1\),

\[
w^{(\omega_k)}(x, y) = w^{(\omega_{k-1})}(x, y) + (\lambda_k - \lambda_{k-1}) \int_{a_k}^{x} w^{(\lambda_k)}(x, z)w^{(\omega_{k-1})}(z, y)\,dz
\]

and for \(a_1 > y\),

\[
z^{(\omega_k)}(x, y) = z^{(\omega_{k-1})}(x, y) + (\lambda_k - \lambda_{k-1}) \int_{a_k}^{x} w^{(\lambda_k)}(x, z)z^{(\omega_{k-1})}(z, y)\,dz
\]

with \(w^{(\omega_0)}(x, y) = w^{(\lambda_0)}(x, y)\) and \(z^{(\omega_0)}(x, y) = z^{(\lambda_0)}(x, y)\) defined in Example 1.

**Proof of Example 2** Observing that \(\omega_k(z) - \omega_{k-1}(z) = 0\) for \(z < a_k\), we have

\[
w^{(\omega_k)}(x, y) = w^{(\omega_{k-1})}(x, y) \quad \text{for} \quad x < a_k.
\]

Since \(\omega_k(z) - \lambda_k = 0\) for \(z \geq a_k\) and \(w^{(\omega_k)}(z, y) = 0\) for \(z > y\), we have from (23)

\[
w^{(\omega_k)}(x, y) = w^{(\lambda_k)}(x, y) + \int_{y}^{x} w^{(\lambda_k)}(x, z)(\omega_k(z) - \lambda_k)w^{(\omega_k)}(z, y)\,dz
\]
\[ w(\lambda_k)(x, y) + \int_y^{a_k} w(\lambda_k)(x, z)(\omega_k(z) - \lambda_k)w(\omega_k)(z, y)dz = w(\lambda_k)(x, y) + \int_y^{a_k} w(\lambda_k)(x, z)(\omega_{k-1}(z) - \lambda_k)w(\omega_{k-1})(z, y)dz. \]

Meanwhile, applying (23) to \( w(\omega_{k-1})(x, y) \) with \( q = \lambda_k \) we have

\[ w(\omega_{k-1})(x, y) = w(\lambda_k)(x, y) + \int_x^y w(\lambda_k)(x, z)(\omega_{k-1}(z) - \lambda_k)w(\omega_{k-1})(z, y)dz. \]

Inductive formula for \( w(\omega_k)(x, y) \) is thus proved by comparing the two identities above. Similar discussion could be applied to derive formulas for \( z(\omega_k)(x, y) \) but under the condition \( a_1 \geq y \).

5 Proofs

Before proving our main results, we comment on the structure of this section. Lemma 1 is slightly postponed and the proof of Theorem 1 is split into two parts. More specifically, instead of showing the existence of solution to (12) directly, using a quantity associated to the event \( \{\kappa_b^+ \leq \kappa_c^-\} \) we first define in (28) an auxiliary function \( w(\omega)(x, y) \), which solves equation (10) and is used in the expression for (13). We then establish the uniqueness of solution for Lemma 1 and find a solution to (12) using function \( w(\omega)(\cdot, \cdot) \). After finishing the proof for Lemma 1 we continue with proofs for the rest of results in Theorem 1.

**Proof of Theorem 1(1).** We first focus on \( \{\kappa_b^+ \leq \kappa_c^-\} \). To simplify the notation, we denote by

\[ A(x; b) := \mathbb{E}_x \left( \exp \left( -\int_0^{\kappa_b^+} \omega(X_t)dt \right); \kappa_b^+ \leq \kappa_c^- \right), \quad \text{for } c \leq x \leq b. \]

We have from the absence of positive jumps and the Markov property for \( X \) that

\[ A(x; z) = A(x; y)A(y; z) \quad \text{for any } z > y > x > c. \] (26)

On the other hand, for every \( t > 0 \), it holds that

\[ 1 - e^{-L(t)} = e^{-L(t)} \int_0^t e^{L(s)}\omega(X_s)ds = \int_0^{\infty} 1_{s<t} \omega(X_s)e^{-(L(t)-L(s))}ds. \]

Applying Fubini’s theorem we have

\[ \mathbb{E}_x \left( 1 - e^{-L(\kappa_b^+)}; \kappa_b^+ \leq \kappa_c^- \right) = \int_0^{\infty} \mathbb{E}_x \left( \omega(X_s)e^{-(L(\kappa_b^+)-L(s))}; s < \kappa_b^+ \leq \kappa_c^- \right)ds \]
where $\theta_s$ is the shifting operator of $X$ such that $X_t \circ \theta_s = X_{s+t}$ for any $s, t \geq 0$. Applying the Markov property gives

$$\mathcal{A}(x; b) = \mathbb{P}_x(\kappa_b^+ \leq \kappa_c^-) - \int_c^b V(x, dy)\omega(y)\mathcal{A}(y; b)$$

where the identity (26) is applied in the last identity for every $y \in (c, x)$. At the risk of abusing notation define

$$w^{(\omega)}(x, y) := w(x, y) \left(1 - \int_y^x w(x, z)\omega(z)\mathcal{A}(z; x)dz\right) \quad \text{for } x > y \geq c,$$

then $\mathcal{A}(x; b) = w^{(\omega)}(x, c) / w^{(\omega)}(b, c)$. Substituting this identity into (28) again gives

$$w^{(\omega)}(x, y) = w(x, y) + \int_y^x w(x, z)\omega(z)w^{(\omega)}(z, y)dz,$$

which is equation (10) and can be generalised to $\mathbb{R} \times \mathbb{R}$ naturally.

**Proof of Lemma 7.** Equation (12) is a kind of renewal type equation. Similar to the proof of [17, Lemma 2.1], we only need to focus on an arbitrary and fixed cylinder set $[c, b] \times [c, b]$ for the functions involved. Let $M_3 \geq \sup_{x \in [c, b]} \omega(x)$ and $s_0 > 0$ such that $\mathcal{W}(s_0) \leq \frac{1}{2M_3}$.

**Uniqueness** To prove the uniqueness of solution of (12) we show that, for fixed $y_0 \in [c, b]$, $H^{(\omega)}(x, y_0) = 0$ is the only solution of

$$H^{(\omega)}(x, y_0) = \int_{y_0}^x w(x, z)\omega(z)H^{(\omega)}(z, y_0)dz.$$

Actually, we have from (7) and our assumption that, $w(x, z) \leq \mathcal{W}(x - z)$. Then

$$|e^{-sz}H^{(\omega)}(x, y_0)| \leq \sup_{z \in [y_0, x]} |e^{-sz}H^{(\omega)}(z, y_0)| \left(M_3 \int_{y_0}^x e^{-sz} \mathcal{W}(x - z)dz\right)$$

$$\leq \frac{1}{2} \sup_{z \in [y_0, x]} |e^{-sz}H^{(\omega)}(z, y_0)|, \quad \text{for any } x \in [c, b].$$

Thus $|e^{-sz}H^{(\omega)}(x, y_0)| = 0$ and $H^{(\omega)}(\cdot, y_0) \equiv 0$. 

15
Existence  Now, \( w^\omega(x, y) \) is well defined in (28) and is the unique solution to

\[
w^\omega(x, y) = w(x, y) + \int_y^x w(x, z) \omega(z) w^\omega(z, y) dz,
\]

from our previous discussion. For any \( h(x, y) \), define

\[
H^\omega(x, y) := h(x, y) + \int_y^x w^\omega(x, z) \omega(z) h(z, y) dz.
\] (29)

Then we have by change of variable and (10) that

\[
\int_y^x w(x, z) \omega(z) H^\omega(z, y) dz = \int_y^x w(x, z) \omega(z) h(z, y) dz + \int_{x>z>y} w^\omega(z, u) \omega(u) h(u, y) dz du = \int_y^x w^\omega(x, u) \omega(u) h(u, y) du = H^\omega(x, y) - h(x, y).
\]

And this finishes the proof.  

Remark 4. Identity (27) is essentially the Feynman-Kac formula in the context. It can also be derived following the Poisson observation method in [18, 19, 17].

With Lemma 1 proved, one is free to use the results from Remark 3.

Remark 5. Recalling the definition of \( z^\omega \) in (11), a conclusion from (29) with \( h \equiv 1 \) is that

\[
z^\omega(x, y) = 1 + \int_y^x w^\omega(x, z) \omega(z) dz.
\] (30)

And such defined \( z^\omega(x, y) \) is the unique solution to (11).  

Proof of Theorem 7(2). With \((w^\omega, z^\omega)\), we are ready to prove the rest of main results.

The resolvent \( V^\omega \)  For any bounded and measurable \( f \geq 0 \), it follows from (27) that

\[
V^\omega f(x) - V^\omega f(x) = \int_c^b V(x, dy) \omega(y) V^\omega f(y).
\]

Thus, applying Proposition 1 to the equation above gives

\[
V^\omega f(x) = w(x, c) \times c f - \int_c^x w(x, y) f(y) dy + \int_c^x w(x, y) \omega(y) V^\omega f(y) dy
\]

\[
= w^\omega(x, c) \times c f - \int_c^x w^\omega(x, y) f(y) dy
\]

\[
= \int_c^b \left( \frac{w^\omega(x, c)}{w^\omega(b, c)} w^\omega(b, y) - w^\omega(x, y) \right) f(y) dy,
\]

16
where \( c_f \) is a constant given by

\[
c_f = \int_c^b \frac{w(b, y)}{w(b, c)} \left( f(y) - \omega(y)V(\omega)f(y) \right) dy,
\]

the second identity comes from Lemma 1 and Remark 3, and the last identity comes from the boundary condition that \( V(\omega)f(b) = 0 \).

The quantity for \( \{\kappa_c^- \leq \kappa_b^+\} \). First notice that

\[
V(\omega)(x) = \mathbb{E}_x \left( \int_{\kappa_b^+}^{\kappa_c^-} e^{-L(t)}\omega(X_t)dt \right) = 1 - \mathbb{E}_x \left( \exp \left( - \int_{0}^{\kappa_b^+} \omega(X_s)ds \right) \right).
\]

On the other hand, taking use of expression of the resolvent just obtained, we have for \( x \in [c, b] \)

\[
V(\omega)(x) = \int_c^b \left( \frac{w(\omega)(x, c)}{w(\omega)(b, c)} w(\omega)(b, y) - w(\omega)(x, y) \right) \omega(y)dy
\]

\[
= \frac{w(\omega)(x, c)}{w(\omega)(b, c)} (z(\omega)(b, c) - 1) - (z(\omega)(x, c) - 1),
\]

from Remark 5. Therefore, we have from the previous conclusion for event \( \{\kappa_b^+ \leq \kappa_c^-\} \) that

\[
\mathbb{E}_x \left( e^{-L(\kappa_c^-)}; \kappa_c^- \leq \kappa_b^+ \right) = z(\omega)(x, c) - \frac{w(\omega)(x, c)}{w(\omega)(b, c)} z(\omega)(b, c).
\tag{31}
\]

And this ends all the proof of Theorem 1.

For the proofs of Proposition 2 and 3, the fact \( 0 = W(u - v) = w(u, v) = w(\omega)(u, v) \) for \( u < v \) is frequently used, and we could rewrite (6) and (7) simply as

\[
w(x, y) = W(x - y) + \delta \int_{\mathbb{R}} 1_{\{z > a\}} \mathbb{W}(x - z)W(dz - y)
\]

\[
= \mathbb{W}(x - y) - \delta \int_{\mathbb{R}} 1_{\{z \leq a\}} \mathbb{W}(x - z)W(dz - y).
\tag{32}
\tag{33}
\]

Proof of Proposition 2. To obtain identities between \( w(\omega) \) and \( (W(\omega), \mathbb{W}(\omega)) \), denote by

\[
g^{(\omega)}(x, y) := W(\omega)(x, y) + \delta \int_{\mathbb{R}} \mathbb{W}(x, z)W(\omega)(dz, y).
\]

the right-hand side of (30). Making use of (33) for \( w(x, y) \), we have for \( x > y \)

\[
\int_y^x w(x, z)\omega(z)g^{(\omega)}(z, y)dz
\]

\[
= \int_{\mathbb{R}} \left( \mathbb{W}(x - z) - \delta \int_{\mathbb{R}} 1_{\{u \leq a\}} \mathbb{W}(x - u)W(du - z) \right) \omega(z)
\]

17
On the other hand, applying formula (18), (5) and (14), we have
\[
- \delta \int_{u \leq a} \mathbb{W}(x-u) (W(du-z)\omega(z)W^{(\omega)}(z,y)) \, dz,
\]
where the fourth term after the first equality vanishes since \(1_{\{u \leq a\}} 1_{\{v > a\}} W(du - z)\mathbb{W}^{(\omega)}(z,v) \equiv 0 \forall z \in \mathbb{R}\). Further applying (14) to \(\mathbb{W}^{(\omega)}(x,y)\) and (18) to \(W^{(\omega)}(dx,y)\), the equation above equals to
\[
\int_{\mathbb{R}} \mathbb{W}(x-z)\omega(z)W^{(\omega)}(z,y) \, dz + \delta \int_{v > a} (\mathbb{W}^{(\omega)}(x,v) - \mathbb{W}(x-v)) W^{(\omega)}(dv,y) \\
- \delta \int_{u \leq a} \mathbb{W}(x-u) (W^{(\omega)}(du,y) - W(du-y)) \\
= \int_{\mathbb{R}} \mathbb{W}(x-z)\omega(z)W^{(\omega)}(z,y) \, dz - \delta \int_{\mathbb{R}} \mathbb{W}(x-u)W^{(\omega)}(du,y) \\
+ \delta \left( \int_{v > a} \mathbb{W}^{(\omega)}(x,v)W^{(\omega)}(dv,y) + \int_{u \leq a} \mathbb{W}(x-u)W(du-y) \right).
\]

On the other hand, applying formula (18), (5) and (14), we have
\[
\delta \int_{\mathbb{R}} \mathbb{W}(x-u)W^{(\omega)}(du,y) \\
= \delta \int_{\mathbb{R}} \mathbb{W}(x-u) \left( W(du-y) + \int_{\mathbb{R}} W(du-v)\omega(v)W^{(\omega)}(v,y) \, dv \right) \\
= \mathbb{W}(x-y) - W(x-y) + \int_{\mathbb{R}} (\mathbb{W}(x-v) - W(x-v)) \omega(v)W^{(\omega)}(v,y) \, dv \\
= \mathbb{W}(x-y) - W^{(\omega)}(x,y) + \int_{\mathbb{R}} \mathbb{W}(x-z)\omega(z)W^{(\omega)}(z,y) \, dz.
\]

Putting pieces together gives
\[
\int_{y}^{x} w(x,z)\omega(z)g^{(\omega)}(z,y) \, dz \\
= \left( W^{(\omega)}(x,y) + \delta \int_{a}^{x} \mathbb{W}(x,z)W^{(\omega)}(dz,y) \right) - \left( \mathbb{W}(x-y) - \delta \int_{y-a}^{a} \mathbb{W}(x,z)W(dz-y) \right) \\
= g^{(\omega)}(x,y) - w(x,y)
\]
from the definition of \(g^{(\omega)}(x,y)\) and (7). It can be found that \(w^{(\omega)}(x,y)\) and \(g^{(\omega)}(x,y)\) satisfies the same equation. Thus \(w^{(\omega)}(x,y) = g^{(\omega)}(x,y)\).

The second equation of (16) can be proved following the same idea by making use of (32). (17) is a direct consequence of applying Remark 5 to (16). And this finishes the proof. 

Proof of Proposition \[3\] To prove Proposition \[3\] we first claim that

\[
w^{(\omega)}(x, y) = w(x, y) + \int_y^x w^{(\omega)}(x, z)\omega(z)w(z, y)dz. \tag{34}
\]

Denoting by \(k(x, y)\) the right hand side of \(3\), by change of variable and the order of integration we have

\[
\int_R w(x, z)\omega(z)k(z, y)dz
\]

\[
= \int_R w(x, u)\omega(u) \left( w(u, y) + \int_R w^{(\omega)}(u, v)\omega(v)w(v, y)dv \right) du
\]

\[
= \int_R \left( w(x, v) + \int_R w(x, u)\omega(u)w^{(\omega)}(u, v)du \right) \omega(v)w(v, y)dv
\]

\[
= \int_R w^{(\omega)}(x, v)\omega(v)w(v, y)dv = k(x, y) - w(x, y),
\]

by definition, and \(k(x, y) = w^{(\omega)}(x, y)\) follows from the uniqueness of solution to \(10\).

On the other hand, applying \(34\) to \(w^{(\omega_1)}\) and \(10\) to \(w^{(\omega_2)}\) twice in the following computation, we have for \(x > y\)

\[
\int_R w^{(\omega_1)}(x, z)\omega_2(z)w^{(\omega_2)}(z, y)dz
\]

\[
= \int_R \left( w(x, z) + \int_R w^{(\omega_1)}(x, u)\omega_1(u)w(u, z)du \right) \omega_2(z)w^{(\omega_2)}(z, y)dz
\]

\[
= w^{(\omega_2)}(x, y) - w(x, y) + \int_R w^{(\omega_1)}(x, u)\omega_1(u) \left( w^{(\omega_2)}(u, y) - w(u, y) \right) du
\]

\[
= w^{(\omega_2)}(x, y) - w^{(\omega_1)}(x, y) + \int_R w^{(\omega_1)}(x, u)\omega_1(u)w^{(\omega_2)}(u, y)du,
\]

which gives the desired equation. The identity for \((z^{(\omega_1)}, z^{(\omega_2)})\) can be proved by applying Remark \[5\] to the identity of \((w^{(\omega_1)}, w^{(\omega_2)})\). And this finishes the proof.

\[\square\]

Proof of Proposition \[4\] Firstly, let us consider the case of constant \(\omega(\cdot)\). Applying limit identities \(11\) to Proposition \[2\] with \(\omega(\cdot) = p\), one can check that, as \(x \to \infty\)

\[
e^{\varphi(p)(y-x)}w^{(p)}(x, y) = e^{\varphi(p)(y-x)} \left( W^{(p)}(x - y) - \delta \int_{y-}^x W^{(p)}(x - z)W^{(p)}(dz - y) \right)
\]

\[
\to \varphi'(p) \left( 1 - \delta \int_{y-}^a e^{\varphi(p)(y-z)}W^{(p)}(dz - y) \right)
\]

and

\[
e^{\varphi(p)(y-x)}z^{(p)}(x, y) = e^{\varphi(p)(y-x)} \left( Z^{(p)}(x - y) - \delta p \int_y^a W^{(p)}(x - z)W^{(p)}(z - y)dz \right)
\]

19
\[
\rightarrow \varphi'(p) \left( \frac{p}{\varphi'(p)} - p\delta \int_y^a e^{\varphi'(p)(y-z)} W'(p)(z-y) \, dz \right).
\]

Plugging them into (23), with the fact of \( W'(p)(z - y) = 0 \) for \( z < y \) we have

\[
\frac{e^{\varphi'(p)(y-x)} u^{(\omega)}(x, y)}{\varphi'(p)} = \frac{e^{\varphi'(p)(y-x)}}{\varphi'(p)} \left( w^{(p)}(x, y) + \int_{\mathbb{R}} w^{(p)}(x, z)(\omega(z) - p) u^{(\omega)}(z, y) \, dz \right)
\]

\[
\rightarrow 1 - \delta \int_{\mathbb{R}} \left\{ \begin{array}{ll}
1_{\{u \leq a\}} e^{\varphi'(p)(y-u)} W^{(p)}(du - y) + \int_{\mathbb{R}} (\omega(z) - p) u^{(\omega)}(z, y) e^{\varphi'(p)(y-z)} \, dz \\
- \delta \int_{\mathbb{R}} \int_{\{z \geq y\}} W^{(p)}(du - z) (\omega(z) - p) u^{(\omega)}(z, y) e^{\varphi'(p)(y-u)} \, dz
\end{array} \right.
\]

\[
= 1 + \int_{\mathbb{R}} (\omega(u) - p) u^{(\omega)}(u, y) e^{\varphi'(p)(y-u)} \, du - \delta \int_{y-}^a e^{\varphi'(p)(y-u)} W^{(p)}(du, y),
\]

where for \( u \leq a \) the fact \( w^{(p)}(u, y) = W^{(p)}(u - y), u^{(\omega)}(u, y) = W^{(\omega)}(u, y) \) and the following Stieltjes measure from (23) is applied in the last equation,

\[
w^{(\omega)}(du, y) = w^{(p)}(du, y) + \int_{\mathbb{R}} w^{(p)}(du, z)(\omega(z) - p) u^{(\omega)}(z, y) \, dz.
\]

Similarly, as \( x \to \infty \),

\[
\frac{e^{\varphi'(p)(y-x)} z^{(\omega)}(x, y)}{\varphi'(p)} = \frac{e^{\varphi'(p)(y-x)}}{\varphi'(p)} \left( \int_{\mathbb{R}} w^{(p)}(x, z)(\omega(z) - p) z^{(\omega)}(z, y) \, dz \right)
\]

\[
\rightarrow \frac{p}{\varphi'(p)} - p\delta \int_{\mathbb{R}} \{ z^{(\omega)} \} (x, y) + z^{(\omega)}(x, y) e^{\varphi'(p)(y-u)} \, du - \delta \int_{y-}^a e^{\varphi'(p)(y-u)} Z^{(\omega)}(du, y),
\]

with \( z^{(\omega)}(u, y) = Z^{(\omega)}(u, y), z^{(p)}(u, y) = Z^{(p)}(u - y) \) for \( u \leq a \) and \( z^{(\omega)}(du, y) = pW^{(p)}(u - y) \, du + \int_{\mathbb{R}} W^{(p)}(du - z)(\omega(z) - p) z^{(\omega)}(z, y) \, dz \)

is needed for the last equation.

Following the same procedure, as \( y \to -\infty \) we have from (11)

\[
e^{\Phi'(q)(y-x)} W^{(q)}(x, y) = \frac{e^{\Phi'(q)(y-x)}}{\Phi'(q)} \left( W^{(q)}(x - y) + \delta \int_x^y W^{(q)}(x-z) W^{(q)}(dz - y) \right)
\]

\[
\rightarrow \Phi'(q) \left( 1 + \delta \Phi'(q) \int_y^x e^{\Phi'(q)(z-x)} W^{(q)}(x-z) \, dz \right).
\]

Plugging it into (23), we have

\[
\frac{e^{\Phi'(q)(y-x)} u^{(\omega)}(x, y)}{\Phi'(q)} = \frac{e^{\Phi'(q)(y-x)}}{\Phi'(q)} \left( w^{(q)}(x, y) + \int_y^x w^{(\omega)}(x, z)(\omega(z) - q) w^{(q)}(z, y) \, dz \right)
\]

20
\[ 1 + \delta \Phi(q) \int_{u>a} e^{\Phi(q)(z-x)} \nu^{(q)}(x-u) \, du + \int_{\mathbb{R}} w^{(\omega)}(x,z)(\omega(z) - q) e^{\Phi(q)(z-x)} \, dz \]

\[ + \delta \Phi(q) \int_{u>a} \int_{z>a} e^{\Phi(q)(u-z)} \nu^{(\omega)}(x,u) \nu^{(q)}(z-u) e^{\Phi(q)(u-x)} \, du \, dz \]

\[ = 1 + \delta \Phi(q) \int_{u>a} e^{\Phi(q)(u-x)} \nu^{(\omega)}(x,u) \, du + \int_{\mathbb{R}} w^{(\omega)}(x,z)(\omega(z) - q) e^{\Phi(q)(z-x)} \, dz, \]

where (23) and the facts \( w^{(\omega)}(x,u) = \nu^{(\omega)}(x,u) \), \( w^{(q)}(x,u) = \nu^{(q)}(x-u) \) for \( u > a \) are applied in the last identity. One can also find

\[ \lim_{y \to -\infty} \frac{e^{\Phi(q)(y-x)} z^{(\omega)}(x,y) \nu^{(q)}(x,y)}{\Phi'(q)}. \]

This completes the proof. \( \square \)

**Acknowledgement**  Bo Li is supported by National Natural Science Foundation of China (No. 11601243). Bo Li and Xiaowen Zhou are supported by NSERC (RGPIN-2016-06704).

**References**

[1] S. Asmussen and M. Taksar. Controlled diffusion models for optimal dividend pay-out. *Insurance: Mathematics and Economics*, 20(1):1–15, Jun 1997.

[2] J. Bertoin. *Lévy Processes*. Cambridge Tracts in Mathematics, 1996.

[3] T. Chan, A. Kyprianou, and M. Savov. Smoothness of scale functions for spectrally negative Lévy processes. *Probability Theory and Related Fields*, 150(3-4):691–708, 2011.

[4] E. C. Cheung and H. Liu. On the joint analysis of the total discounted payments to policyholders and shareholders: threshold dividend strategy. *Annals of Actuarial Science*, 10(02):236–269, Aug 2016.

[5] H. U. Gerber and E. S. W. Shiu. On optimal dividend strategies in the compound poisson model. *North American Actuarial Journal*, 10(2):76–93, Apr 2006.

[6] H. U. Gerber, E. S. W. Shiu, and H. Yang. The omega model: from bankruptcy to occupation times in the red. *Eur. Actuar. J.*, 2(2):259–272, Jul 2012.

[7] H. Guérin and J.-F. Renaud. Joint distribution of a spectrally negative Lévy process and its occupation time, with step option pricing in view. *Eprint Arxiv*, 06 2014.
[8] D. Hernández-Hernández, J.-L. Pérez, and K. Yamazaki. Optimality of refraction strategies for spectrally negative Lévy processes. SIAM Journal on Control and Optimization, 54(3):11261156, Jan 2016.

[9] F. Hubalek and E. Kyprianou. Old and new examples of scale functions for spectrally negative Lévy processes. In R. Dalang, M. Dozzi, and F. Russo, editors, Seminar on Stochastic Analysis, Random Fields and Applications VI, volume 63 of Progress in Probability, pages 119–145. Springer Basel, 2011.

[10] M. Jeanblanc-Picqué and A. N. Shiryaev. Optimization of the flow of dividends. Russian Mathematical Surveys, 50(2):257, 1995.

[11] A. Kuznetsov, A. E. Kyprianou, and V. Rivero. The theory of scale functions for spectrally negative Lévy processes. Lecture Notes in Mathematics, pages 97–186, 2012.

[12] A. E. Kyprianou. Fluctuations of Lévy Processes with Applications. Springer Berlin Heidelberg, 2014.

[13] A. E. Kyprianou, R. Loeffen, and J.-L. Pérez. Optimal control with absolutely continuous strategies for spectrally negative Lévy processes. Journal of Applied Probability, 49(1):150166, Mar 2012.

[14] A. E. Kyprianou and R. L. Loeffen. Refracted Lévy processes. Ann. Inst. H. Poincaré Probab. Statist., 46(1):24–44, Feb 2010.

[15] A. E. Kyprianou, J. C. Pardo, and J. L. Pérez. Occupation times of refracted Lévy processes. Journal of Theoretical Probability, 27(4):1292–1315, 2014.

[16] D. Landriault, J.-F. Renaud, and X. Zhou. Occupation times of spectrally negative Lévy processes with applications. Stochastic Processes and their Applications, 121(11):2629–2641, 2011.

[17] B. Li and Z. Palmowski. Fluctuations of omega-killed spectrally negative Lévy processes. ArXiv e-prints, Mar. 2016.

[18] Y. Li and X. Zhou. On pre-exit joint occupation times for spectrally negative Lévy processes. Statistics & Probability Letters, 94:48–55, Nov 2014.

[19] Y. Li, X. Zhou, and N. Zhu. Two-sided discounted potential measures for spectrally negative Lévy processes. Statistics & Probability Letters, 100:67–76, May 2015.
[20] X. Lin and K. P. Pavlova. The compound poisson risk model with a threshold dividend strategy. Insurance: Mathematics and Economics, 38(1):57–80, Feb 2006.

[21] R. L. Loeffen, J.-F. Renaud, and X. Zhou. Occupation times of intervals until first passage times for spectrally negative Lévy processes. Stochastic Processes and their Applications, 124(3):1408–1435, Mar 2014.

[22] J.-F. Renaud. On the time spent in the red by a refracted Lévy risk process. Journal of Applied Probability, 51(04):1171–1188, Dec 2014.

[23] N. Wan. Dividend payments with a threshold strategy in the compound poisson risk model perturbed by diffusion. Insurance: Mathematics and Economics, 40(3):509–523, May 2007.

[24] L. Wu, J. Zhou, and S. Yu. Occupation times of general Lévy processes. ArXiv e-prints, Apr. 2016.

[25] J. Zhou and L. Wu. Occupation times of refracted double exponential jump diffusion processes. Statistics & Probability Letters, 106:218227, Nov 2015.