Relative Turán Numbers for Hypergraph Cycles

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July 20, 2021

Abstract

For an \(r\)-uniform hypergraph \(H\) and a family of \(r\)-uniform hypergraphs \(\mathcal{F}\), the relative Turán number \(\text{ex}(H, \mathcal{F})\) is the maximum number of edges in an \(\mathcal{F}\)-free subgraph of \(H\). In this paper we give lower bounds on \(\text{ex}(H, \mathcal{F})\) for certain families of hypergraph cycles \(\mathcal{F}\) such as Berge cycles and loose cycles. In particular, if \(C^3_\ell\) denotes the set of all \(3\)-uniform Berge \(\ell\)-cycles and \(H\) is a \(3\)-uniform hypergraph with maximum degree \(\Delta\), we prove

\[
\begin{align*}
\text{ex}(H, C^3_4) & \geq \Delta^{-3/4-o(1)} e(H), \\
\text{ex}(H, C^3_5) & \geq \Delta^{-3/4-o(1)} e(H),
\end{align*}
\]

and these bounds are tight up to the \(o(1)\) term.

1 Introduction

Let \(\mathcal{F}\) be a family of \(r\)-uniform hypergraphs, or \(r\)-graphs for short. The Turán number \(\text{ex}(n, \mathcal{F})\) is defined to be the maximum number of edges in an \(\mathcal{F}\)-free \(n\)-vertex \(r\)-graph. The Turán numbers are a central object of study in extremal graph theory, dating back to Mantel’s Theorem [18] and Turán’s Theorem [29]. A more general problem involves studying the relative Turán number \(\text{ex}(H, \mathcal{F})\), which is the maximum number of edges in an \(\mathcal{F}\)-free subgraph of an \(r\)-graph \(H\), and we will say that \(H\) is a host hypergraph. For example, when \(H\) is \(K^r_n\) (the complete \(r\)-graph on \(n\) vertices), we simply recover the original Turán number.

The study of relative Turán numbers was advanced by Foucaud, Krivelevich, and Perarnau [9] and independently by Briggs and Cox [2]. Many results have been obtained for relative Turán numbers, both when \(H\) is a general host, as well as for random hosts [15, 22, 23, 26, 27].

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In this paper we consider relative Turán numbers for hypergraph cycles, and in particular for Berge cycles. If $F$ and $F'$ are hypergraphs with $V(F) \subseteq V(F')$, we say that $F'$ is a Berge-$F$ if there exists a bijection $\phi : E(F) \to E(F')$ such that $e \subseteq \phi(e)$ for all $e \in E(F)$. We let $C'_r$ denote the set of all $r$-uniform Berge-$F$ where $F$ is an $\ell$-cycle. When $\ell = 2$, $F$ is a double edge, and any $r$-graph which is $C'_2$-free is said to be linear. We let $C^r_{[\ell]} = \bigcup_{\ell'=2}^r C_{\ell'}^r$, and any $r$-graph which is $C^r_{[\ell]}$-free is said to have girth $\ell + 1$. In [26], we conjectured

$$ex(n, C^r_{[\ell]}) \geq n^{1+\frac{1}{(r-1)^2}} - o(1),$$

which is a natural generalization of a conjecture of Erdős and Simonovits [7] for graphs. For $\ell, r \geq 3$, Győri and Lemons [14] proved $ex(n, C^r_{\ell}) = O(n^{1+\frac{1}{(r-1)^2}})$, so up to the $o(1)$ term, (*) would be best possible. This conjecture is known to hold for $\ell = 3$ and all $r$ due to work of Ruzsa and Szemerédi [25] and Erdős, Frankl, and Rödl [5], and for $\ell = 4$ and all $r$ due to Lazebnik and the second author [17] and Timmons and the second author [28]. We emphasize that for $r \geq 3$, it is known that the $o(1)$ term in (*) is necessary in general, see Ruzsa and Szemerédi [25] and Conlon, Fox, Sudakov, and Zhao [3].

Our main result is the following, where we recall that an $r$-graph $S$ is a sunflower if there exists a set $K$ called the kernel such that $e \cap e' = K$ for every pair of distinct edges $e, e' \in E(S)$. Throughout this paper, the maximum degree $\Delta$ of a hypergraph $H$ is the maximum degree of any vertex of $H$, and all of our asymptotic statements are with respect to $\Delta$ tending towards infinity.

**Theorem 1.1.** Let $\widehat{C}^r_{[\ell]}$ consist of all the elements of $C^r_{[\ell]}$ which are not sunflowers. If $\ell, r \geq 3$ are such that (*) holds, then for all $r$-graphs $H$ with maximum degree at most $\Delta$, we have

$$ex(H, \widehat{C}^r_{[\ell]}) \geq \Delta^{-1} \cdot (\frac{1}{(r-1)^2})^{\frac{1}{(r-1)^2}} - o(1) \cdot e(H),$$

and this bound is tight up to the $o(1)$ term in the exponent for $H = K^r_{\Delta^{1/(r-1)}}$.

The $r = 2$ case of Theorem 1.1 was proven by Perarnau and Reed [23]. We note that that the bound of Theorem 1.1 assuming (*) can be shown to be tight up to the $o(1)$ term in the exponent by considering $H = K^r_{\Delta^{1/(r-1)}}$.

In Theorem 1.1, it is necessary to consider Berge cycles without sunflowers. Indeed, if $S \in C^r_{[\ell]}$ is a sunflower with kernel of size $k$, and if $H$ is a sunflower with $\Delta$ edges and kernel of size $k$, then $ex(H, S) = \ell - 1 = O(\Delta^{-1}) \cdot e(H)$.

Using Theorem 1.1, known results for classical Turán numbers, the fact that no member of $C^3_3, C^3_4, C^4_4$ is a sunflower, and the observation that $ex(H, F') \geq ex(H, F)$ whenever $F' \subseteq F$, we deduce the following.

**Corollary 1.2.** If $H$ is a 3-graph with maximum degree at most $\Delta$, then

$$ex(H, C^3_3) \geq \Delta^{-1/2} - o(1) \cdot e(H),$$

$$ex(H, C^3_4) \geq \Delta^{-3/4} - o(1) \cdot e(H).$$

If $H$ is a 4-graph with maximum degree at most $\Delta$, then

$$ex(H, C^4_4) \geq \Delta^{-5/6} - o(1) \cdot e(H).$$
Again all of these bounds are tight up to the $o(1)$ term in the exponent. Using a similar argument, we will show in Section 4 that for any 3-graph $H$ with maximum degree at most $\Delta$,  
\[
ex(H, C_3^3) \geq \Delta^{-3/4-o(1)} \cdot e(H),
\]
which does not follow from Theorem 1.1 since the order of magnitude of $\ex(n, C_3^3)$ is not known, see [3].

We next consider the loose cycle $C_r^\ell$, which is the $r$-graph which has $\ell$ edges $e_1, \ldots, e_\ell$ such that $e_i \cap e_{i+1} = \{v_i\}$ for $1 \leq i \leq \ell$ with all $v_i$ distinct and the indices written cyclically, and such that $e_i \cap e_j = \emptyset$ for any other pair of indices $i \neq j$.

While the classical Turán numbers for loose cycles are known exactly (see [10, 12, 16]), it appears to be difficult to find tight bounds for relative Turán numbers for loose cycles. In large part this seems to be because, unlike in Theorem 1.1, the clique $K_{\Delta/\ell(r-1)}^r$ does not give tight bounds for $\ex(H, C_r^\ell)$ in general. Indeed, by using recent results of Mubayi and Yepreyman [21], we show that certain random hypergraphs give stronger bounds for $\ex(H, C_r^\ell)$ compared to the clique when $\ell$ is even:

**Theorem 1.3.** Let $\ell \geq 3$. If $H$ is a 3-graph with maximum degree at most $\Delta$, then  
\[
ex(H, C_r^\ell) \geq \Delta^{-1+1/\ell-o(1)} \cdot e(H).
\]

Moreover, if $\ell$ is even, then there exists a 3-graph $H$ with maximum degree at most $\Delta$ and  
\[
ex(H, C_r^\ell) \leq \Delta^{-1+1/\ell+o(1)} \cdot e(H).
\]

It is possible to extend our arguments to $r$-graphs, but the gap between the lower and upper bound grows considerably with $r$. When the host $H$ is linear, we improve the bounds of Theorem 1.3 to give tight results for all $r$ when $\ell$ is even.

**Proposition 1.4.** Let $\ell \geq 4$ be even and $r \geq 3$. If $H$ is a linear $r$-graph with maximum degree at most $\Delta$, then  
\[
ex(H, C_r^\ell) \geq \Delta^{-1+1/r-o(1)} \cdot e(H).
\]

Moreover, there exists a linear $r$-graph $H$ with maximum degree at most $\Delta$ and  
\[
ex(H, C_r^\ell) \leq \Delta^{-1+1/r+o(1)} \cdot e(H).
\]

The last hypergraph cycle we consider is $F$, which is the 3-uniform Berge 4-cycle depicted in Figure 1. The Turán number for the hypergraph $F$ is well studied [4, 11, 19, 20, 24]. By Corollary 1.2, we have $\ex(H, F) \geq \Delta^{-3/4-o(1)} e(H)$ for all hosts $H$ with maximum degree at most $\Delta$. We improve this lower bound as follows.

**Theorem 1.5.** If $H$ is a 3-graph with maximum degree at most $\Delta$, then  
\[
ex(H, F) \geq \Delta^{-3/5-o(1)} \cdot e(H).
\]

We do not have a matching upper bound for $\ex(H, F)$, see the concluding remarks for further discussions.

3
1.1 Organization and Notation

All of our proofs for lower bounding $\text{ex}(H, \mathcal{F})$ follow the same basic strategy which we briefly outline here. By considering a dense subgraph of the host hypergraph, we can always assume that $H$ is $r$-partite. If $H$ has small codegrees, i.e. if no set of vertices of $H$ is contained in many edges, then we apply the method of random homomorphisms developed in Section 2. The idea here is to take a dense $\mathcal{F}$-free hypergraph $J$ and define a random mapping $\chi : V(H) \to V(J)$ and use $\chi$ to determine a large $\mathcal{F}$-free subgraph of $H$. If $H$ has large codegrees then the above approach is not effective, and in this case we need tools which we develop in Section 3. The idea here is that there will be some $k$ such that many sets of size $k$ in $H$ will be contained in many edges. Given this, we try and find a subgraph of $H$ which induces a matching on the first $k$ parts.

After we develop these general techniques, we move on to prove our theorems. We prove our main result for Berge cycles, Theorem 1.1, in Section 4, we prove Theorem 1.5 in Section 5, and our results for loose cycles, Theorem 1.3 and Proposition 1.4, are proven in Section 6. Concluding remarks and further problems are given in Section 7.

We gather some notation and definitions that we use throughout the text. A set of size $k$ will be called a $k$-set. If $H$ is an $r$-graph, the number of edges containing a $k$-set $\{u_1, \ldots, u_k\}$ is called the $k$-degree of the set and is denoted by $d_H(u_1, \ldots, u_k)$. Throughout the text we omit ceilings and floors whenever these are not crucial. We often make use of the following basic fact due to Erdős and Kleitman [6]: every $r$-graph $H$ has an $r$-partite subgraph with at least $r^{-r}e(H)$ edges.

We say that an $r$-graph $F$ is non-linear if two of its edges intersect in more than one vertex, or equivalently if it contains an element of $\mathcal{C}_2^r$ as a subgraph. Given an $r$-graph $F$, we say that $A = \{v_1, \ldots, v_\ell\} \subseteq V(F)$ is a core set with respect to $e_1, \ldots, e_\ell \in E(F)$ if $v_i, v_{i+1} \in e_i$ for all
1 \leq i < \ell \text{ and if } v_1, v_\ell \in e_i, \text{ and we simply say that } A \text{ is a core set if such a choice of edges exist. Observe that the existence of a core set } \{v_1, \ldots, v_\ell \} \text{ in } F \text{ is equivalent to saying that } F \text{ contains an element of } C'_\ell \text{ as a subgraph.}

2 \textbf{Hosts with Small Codegrees}

If } \chi \text{ is a map from vertices of } H \text{ and } e = \{u_1, \ldots, u_r \} \in E(H), \text{ we define the set } \chi(e) = \{\chi(u_1), \ldots, \chi(u_r)\}. \text{ If } F \text{ and } F' \text{ are } r\text{-uniform hypergraphs, then we say that a map } \chi : V(F) \rightarrow V(F') \text{ is a local isomorphism if } \chi \text{ is a homomorphism}^1 \text{ and if } \chi(e) \neq \chi(f) \text{ whenever } e, f \in E(F) \text{ with } e \cap f \neq \emptyset. \text{ We define } \mathcal{H}(F) \text{ to be the set of } F' \text{ for which there exists a local isomorphism } \chi : V(F) \rightarrow V(F'). \text{ If } \mathcal{F} \text{ is a family of } r\text{-graphs we define } \mathcal{H}(\mathcal{F}) = \bigcup_{F \in \mathcal{F}} \mathcal{H}(F). \text{ Note that in general } \mathcal{H}(F) \text{ will be infinite. Our main result for this section is the following general result.}

\textbf{Proposition 2.1. Let } H \text{ be an } r\text{-graph, and for all } k < r \text{ let } \Delta_k \text{ denote the maximum } k\text{-degree of } H. \text{ If } t \geq r^24^r \Delta_k^{1/(r-k)} \text{ for all } k, \text{ then for any family of } r\text{-graphs } \mathcal{F}, \begin{align*}
ex(H, \mathcal{F}) \geq \ex(t, \mathcal{H}(\mathcal{F}))t^{-r} \cdot e(H).\end{align*}

\textbf{Proof. Let } J \text{ be an extremal } \mathcal{H}(\mathcal{F})\text{-free } r\text{-graph on } t \text{ vertices and } \chi : V(H) \rightarrow V(J) \text{ chosen uniformly at random. Let } H' \subseteq H \text{ be the random subgraph which keeps the edge } e \in E(H) \text{ if }

\begin{enumerate}
\item \(\chi(e) \in E(J),\)
\item \(\chi(e) \neq \chi(f) \text{ for any other } f \in E(H) \text{ with } |e \cap f| \neq 0.\)
\end{enumerate}

\text{We claim that } H' \text{ is } \mathcal{F}\text{-free. Indeed, assume } H' \text{ contained a subgraph } F \text{ isomorphic to some element of } \mathcal{F}. \text{ Let } F' \text{ be the subgraph of } J \text{ with } V(F') = \{\chi(u) : u \in V(F)\} \text{ and } E(F') = \{\chi(e) : e \in E(F)\}, \text{ and we note that } F \subseteq H' \text{ implies that each edge of } F \text{ satisfies (1), so every element of } E(F') \text{ is an edge in } J. \text{ By conditions (1) and (2), } \chi \text{ is a local isomorphism from } F \text{ to } F', \text{ so } F' \in \mathcal{H}(\mathcal{F}), \text{ a contradiction to } J \text{ being } \mathcal{H}(\mathcal{F})\text{-free.}

\text{It remains to compute } \mathbb{E}[e(H')]. \text{ Fix } e \in E(H). \text{ Let } A \text{ denote the event that (1) is satisfied, let } E_k = \{f \in E(H) : |e \cap f| = k\}, \text{ and let } B \text{ denote the event that } \chi(f) \not\subset \chi(e) \text{ for all } f \in \bigcup_{k=1}^{r-1} E_k, \text{ which in particular implies (2) for the edge } e. \text{ It is not too difficult to see that } \Pr[A] = r!e(J)t^{-r}, \text{ and that for all } f \in E_k \text{ we have } \Pr[\chi(f) \not\subset \chi(e)|A] = (r/t)^{r-k}. \text{ We note that } |E_k| \leq \binom{r}{k} \Delta_k \leq 2^r \Delta_k \text{ for all } k, \text{ since } e \text{ has at most } \binom{r}{k} \text{ subsets of size } k \text{ and each has } k\text{-degree at most } \Delta_k. \text{ Thus taking a union bound we find }

\begin{align*}
\Pr[B|A] \geq 1 - \sum_k |E_k|(r/t)^{-r-k} \geq 1 - \sum_k 2^r \Delta_k (r/t)^{-r-k} \geq 1 - \sum_k (r^{-1}2^{-r}) \geq \frac{1}{2},
\end{align*}

\text{1That is, if for every } e \in E(F) \text{ we have } \chi(e) \in E(F').\)
As a motivating example for the method of this section, consider the case that $H$ is a complete tripartite 3-graph on $V_1 \cup V_2 \cup V_3$ with $|V_1| = |V_2| = o(|V_3|)$. If one wanted to find some $H' \subseteq H$ avoiding the loose 4-cycle $C_4^3$, then an approach would be to take a perfect matching $M$ on $V_1 \cup V_2$ and then to take $H'$ to be all the 3-sets containing a pair from $M$. It turns out that $H'$ is $C_4^3$-free and has many edges. In fact, it will avoid every element of $C_4^3$ except for the 3-graph $F$ defined in Figure 1. To deal with $F$, instead of taking $H'$ to have all edges containing an

where the second to last inequality used $(r4^r)^{k-r} \geq r^{-1}4^{-r}$ for $k < r$. Thus

$$\Pr[e \in E(H')] = \Pr[A] \cdot \Pr[B|A] \geq r!e(J)t^{-r} \cdot \frac{1}{2} \geq e(J)t^{-r},$$

and linearity of expectation gives $\mathbb{E}[e(H')] \geq e(J)t^{-r} \cdot e(H) = \text{ex}(t, \mathcal{H}(F))t^{-r} \cdot e(H)$. Thus there exists some subgraph $H' \subseteq H$ with this many edges which is $\mathcal{F}$-free, giving the result. 

We use the following lemma in order to apply Proposition 2.1 to families of Berge cycles.

**Lemma 2.2.**

1. If $F$ is a non-linear $r$-graph, then $\mathcal{H}(F)$ consists of non-linear $r$-graphs.
2. Every element of $\mathcal{H}(C_3^2)$ contains an element of $C_2^3 \cup C_3^3$ as a subgraph.
3. For $\ell, r \geq 2$, every element of $\mathcal{H}(C_{[\ell]}^r)$ contains an element of $C_{[\ell]}^r$ as a subgraph.

**Proof.** For (1), let $e_1, e_2 \in E(F)$ be edges and $u \neq v$ such that $u, v \in e_1 \cap e_2$. Let $\chi : V(F) \to V(F')$ be a local isomorphism, and because $e_1 \cap e_2 \neq \emptyset$ we have $\chi(e_1) \neq \chi(e_2)$. We have $\chi(u) \neq \chi(v)$, as otherwise $|\chi(e_i)| < r$, contradicting $\chi$ being a homomorphism. Thus the distinct edges $\chi(e_1), \chi(e_2)$ intersect in at least two vertices in $F'$ as desired.

For (2), let $F \in C_3^3$ and let $A = \{v_1, \ldots, v_5\}$ be a core set with respect to the edges $e_1, \ldots, e_5 \in E(F)$. Consider a local isomorphism $\chi : V(F) \to V(F')$. Note that $\chi(e_i) \neq \chi(e_{i+1})$ for any $i$ (where here and throughout this proof we write our indices mod 5) since $e_i \cap e_{i+1} \neq \emptyset$ and $\chi$ is a local isomorphism. Similarly, observe that $\chi(v_i) \neq \chi(v_{i+1})$ for any $i$, otherwise this would imply $|\chi(e_i)| < 3$, contradicting $\chi(e_i)$ being an edge of $F'$.

If $\chi(e_i) = \chi(e_{i+2})$, then the distinct edges $\chi(e_i), \chi(e_{i+1})$ both contain the distinct vertices $\chi(v_{i+1}), \chi(v_{i+2})$, so $F'$ contains an element of $C_3^3$ as a subgraph. Thus we will assume $\chi(e_i) \neq \chi(e_{i+2})$ for all $i$, and this together with $\chi(e_i) \neq \chi(e_{i+1})$ implies that all the edges $\chi(e_i)$ are distinct. Similarly if $\chi(v_i) = \chi(v_{i+2})$, then $\chi(e_i), \chi(e_{i+1})$ show that $F'$ contains an element of $C_3^3$. We conclude that all of the $\chi(v_i)$ vertices are distinct, and thus these form a core set for the distinct edges $\chi(e_1), \ldots, \chi(e_5)$, so $F'$ contains an element of $C_3^3$ as a subgraph.

Case (3) is proven in [26, Lemma 4.2] using an argument similar to the proof of (2).

**3 Hosts with Large Codegrees**

As a motivating example for the method of this section, consider the case that $H$ is a complete tripartite 3-graph on $V_1 \cup V_2 \cup V_3$ with $|V_1| = |V_2| = o(|V_3|)$. If one wanted to find some $H' \subseteq H$ avoiding the loose 4-cycle $C_4^3$, then an approach would be to take a perfect matching $M$ on $V_1 \cup V_2$ and then to take $H'$ to be all the 3-sets containing a pair from $M$. It turns out that $H'$ is $C_4^3$-free and has many edges. In fact, it will avoid every element of $C_4^3$ except for the 3-graph $F$ defined in Figure 1. To deal with $F$, instead of taking $H'$ to have all edges containing an
edge of $M$, we take a subset of these edges such that the graph induced by $V_2 \cup V_3$ in $H'$ is $C_4$-free, and this will turn out to be enough to give the desired result.

### 3.1 Induced $k$-graphs

We recall that a hypergraph is a matching if it is a disjoint union of edges. Given an $r$-partite $r$-graph $F$ on $V_1 \cup \cdots \cup V_r$ and a $k$-set $I$, we define $\partial F [\bigcup_{i \in I} V_i]$ to be the $k$-graph $F'$ with

$$V(F') = \bigcup_{i \in I} V_i, \quad E(F') = \{ e \cap \bigcup_{i \in I} V_i : e \in E(F) \},$$

and we call this the $k$-graph induced by $\bigcup_{i \in I} V_i$. Let $\mathcal{R}(F)$ denote the set of all $r$-partitions of $F$, and define

$$\mathcal{P}_k(F) = \{ \partial F [\bigcup_{i \geq k} V_i] : (V_1, \ldots, V_r) \in \mathcal{R}(F), \partial F [\bigcup_{i \leq k} V_i] \text{ is a matching} \}.$$

That is, $\mathcal{P}_k(F)$ consists of all $(r - k + 1)$-graphs which can be obtained by first taking an $r$-partition of $F$ such that its first $k$ parts induce a matching, and then taking the $(r - k + 1)$-graph induced by $\bigcup_{i \geq k} V_i$. For example, $\mathcal{P}_2(F)$ since one can consider the tripartition $V_1 = \{ u_1, v_3 \}$, $V_2 = \{ u_2, v_2 \}$, $V_3 = \{ u_3, v_1 \}$, and it is not too difficult to show that $\mathcal{P}_2(F) = \{ C_4 \}$. If $\mathcal{F}$ is a family of $r$-graphs we define $\mathcal{P}_k(\mathcal{F}) = \bigcup_{F \in \mathcal{F}} \mathcal{P}_k(F)$. Note that if $F$ is not $r$-partite then $\mathcal{P}_k(F) = \emptyset$.

Our main result for this section is the following, which roughly says that one can prove effective lower bounds on $\text{ex}(H, F)$ if $H$ has many edges containing sets with large $k$-degrees and if one knows effective lower bounds for $\text{ex}(G, \mathcal{P}_k(F))$ with $G$ any $(r - k + 1)$-graph.

**Proposition 3.1.** Let $k < r$, $0 \leq \beta \leq 1$, and let $\mathcal{F}$ be a family of $r$-graphs such that $\text{ex}(G, \mathcal{P}_k(\mathcal{F})) = \Omega(\Delta^{-\beta} e(G))$ for any $(r - k + 1)$-graph $G$ with maximum degree at most $\Delta$. If $H$ is an $r$-partite $r$-graph with maximum degree at most $\Delta$ and at least half of the edges of $H$ contain a $k$-set with $k$-degree at least $D$, then

$$\text{ex}(H, \mathcal{F}) = \Omega \left( \frac{D^{1-\beta}}{\Delta (\log \Delta)^2} \right) \cdot e(H).$$

We note that we make no attempt at optimizing the log factor in Proposition 3.1. To prove this result, we establish a few lemmas.

**Lemma 3.2.** Let $H$ be an $r$-partite $r$-graph on $V_1 \cup \cdots \cup V_r$ and $\mathcal{F}$ a family of $r$-graphs. If $\partial H [\bigcup_{i \leq k} V_i]$ is a matching and $\partial H [\bigcup_{i \geq k} V_i]$ is $\mathcal{P}_k(\mathcal{F})$-free, then $H$ is $\mathcal{F}$-free.

**Proof.** Assume $H$ contains a subgraph $F$ isomorphic to some element of $\mathcal{F}$, and consider the $r$-partition $U_1 \cup \cdots \cup U_r$ of $F$ defined by $U_i = V(F) \cap V_i$. Then $\partial F [\bigcup_{i \in I} U_i] \subseteq \partial H [\bigcup_{i \in I} V_i]$ for all sets $I$. In particular, $\partial F [\bigcup_{i \leq k} U_i] \subseteq \partial H [\bigcup_{i \leq k} V_i]$ is a matching, and hence $\partial F [\bigcup_{i \geq k} U_i] \in \mathcal{P}_k(\mathcal{F})$ by definition. This contradicts $\partial F [\bigcup_{i \geq k} U_i] \subseteq \partial H [\bigcup_{i \geq k} V_i]$ and $\partial H [\bigcup_{i \geq k} V_i]$ being $\mathcal{P}_k(\mathcal{F})$-free. \qed
Lemma 3.3. If $H$ is an $r$-graph with maximum degree at most $\Delta > 0$, then $H$ contains a matching with at least $e(H)/r\Delta$ edges.

The proof of Lemma 3.3 is a straightforward greedy argument and we omit its proof. Lastly, we use the Chernoff bound [1].

Proposition 3.4 ([1]). Let $X$ denote a binomial random variable with $N$ trials and probability $p$ of success. For any $\lambda > 0$ we have

$$\Pr[X - pN > \lambda pN] \leq e^{-\lambda^2 pN/2}. \tag{2}$$

With this we can prove a version of Proposition 3.1 under the additional hypothesis that every $k$-set of $\bigcup_{i \leq k} V_i$ is contained in either 0 or roughly $D$ edges.

Proposition 3.5. Let $k < r$, $0 \leq \beta \leq 1$, and let $\mathcal{F}$ be a family of $r$-graphs such that $\text{ex}(G, \mathcal{P}_k(\mathcal{F})) = \Omega((\Delta - \beta)d^k(G))$ for any $(r - k + 1)$-graph $G$ with maximum degree at most $\Delta$. Let $H$ be an $r$-partite $r$-graph on $V_1 \cup \cdots \cup V_r$ with maximum degree at most $\Delta$ and let $D$ be such that $D \leq \Delta / \log \Delta$ and such that if $\{u_1, \ldots, u_k\} \in E(H)$ with $u_i \in V_i$ for all $i$, then $D \leq d_H(u_1, \ldots, u_k) < 2D$. In this case we have

$$\text{ex}(H, \mathcal{F}) = \Omega\left(\frac{(D)^{1-\beta}}{\Delta(\log \Delta)^5}\right) \cdot e(H).$$

Before proving this, let us quickly show that this result implies Proposition 3.1.

Proof of Proposition 3.1 assuming Proposition 3.5. Let $H$ be as in Proposition 3.1 with $V_1 \cup \cdots \cup V_r$ an $r$-partition of $H$. Given $I = \{i_1, \ldots, i_k\} \subseteq [r]$, let $E_I \subseteq E(H)$ be the set of edges $\{u_1, \ldots, u_r\}$ such that $d_H(u_{i_1}, \ldots, u_{i_k}) \geq D$. By the hypothesis of Proposition 3.1, we have

$$\frac{1}{2}e(H) \leq |\bigcup E_I| \leq \sum |E_I|.$$

Thus $|E_I| \geq 2^{-r-1}e(H)$ for some $k$-set $I$, and we will assume $|E_{[k]}| \geq 2^{-r-1}e(H)$ without loss of generality.

Let $E_j \subseteq E_{[k]}$ denote the set of edges $\{u_1, \ldots, u_r\}$ with $2^j D \leq d_H(u_1, \ldots, u_k) < 2^{j+1} D$. The $k$-degree of any $k$-set of $H$ is at most $\Delta$, so the $E_j$ sets with $0 \leq j \leq \log \Delta$ partition $E_{[k]}$. By the pigeonhole principle there exists some $j'$ such that

$$|E_{j'}| \geq \frac{1}{1 + \log \Delta} |E_{[k]}| \geq \frac{1}{2^{r+1} \log \Delta} e(H).$$

Let $H' \subseteq H$ consist of the edges $E_{j'}$ and let $D' = 2^j D$. By construction, if $\{u_1, \ldots, u_r\} \in E(H')$, then $D' \leq d_{H'}(u_1, \ldots, u_k) < 2D'$. Lastly, let $\Delta' = 2\Delta \log \Delta$. Observe that $H'$ has maximum
degree at most $\Delta \leq \Delta'$ and that $D' \leq \Delta \leq \Delta'/\log \Delta'$ for $\Delta$ sufficiently large. Thus $H'$ satisfies the conditions of Proposition 3.5, so we have

$$\text{ex}(H, \mathcal{F}) \geq \text{ex}(H', \mathcal{F}) = \Omega \left( \frac{(D')^{1-\beta}}{\Delta'(\log \Delta')^5} \right) \cdot e(H') \geq \Omega \left( \frac{D^{1-\beta}}{\Delta'(\log \Delta')^5} \right) \cdot e(H),$$

where we used $(D')^{1-\beta} \geq D^{1-\beta}$ since $\beta \leq 1$ and $D' \geq D$. \hfill \square

We now prove Proposition 3.5.

**Proof of Proposition 3.5.** Let $H$ be as in the proposition statement. We assume throughout that $\Delta$ is at least as large as some sufficiently large constant, the result being trivial otherwise. Whenever we consider sets $\{u_1, \ldots, u_r\}$ we will always assume $u_i \in V_i$ for all $i$. We define

$$p = \frac{D \log \Delta}{\Delta} \leq 1.$$

Let $G = \partial H \bigcup_{i \leq k} V_i$. For ease of presentation we refer to edges of $H$ as **hyperedges** and edges of $G$ simply as edges, and in this proof $e$ will always refer to edges of $G$. Define $G_p \subseteq G$ to be the $k$-graph obtained by keeping each edge of $G$ independently with probability $p$, and let $H_p \subseteq H$ comprise all the hyperedges which contain edges of $G_p$.

**Claim 3.6.** For all $u \in V(H)$,

$$\Pr[d_{H_p}(u) > 8D(\log \Delta)^3] \leq \frac{2 \log \Delta}{\Delta^2}.$$

**Proof.** Fix some $u \in V(H)$. Given $e \in E(G)$, let $E_e$ be the set of hyperedges $e' \in E(H)$ which contain $e \cup \{u\}$, and let $I_e$ be the indicator variable which is 1 if $e \in E(G_p)$ and 0 otherwise. Thus $d_{H_p}(u) = \sum_e |E_e| \cdot I_e$. Moreover, if we let $E_j$ be the set of $e \in E(G)$ with $2^j \leq |E_e| < 2^{j+1}$ and let $S_j := \sum_{e \in E_j} I_e$, we find

$$d_{H_p}(u) = \sum_e |E_e| \cdot I_e < \sum_j 2^{j+1} \sum_{e \in E_j} I_e = \sum_j 2^{j+1} S_j.$$

Observe that we only have to consider $0 \leq j \leq \log_2 D \leq 2 \log(\Delta) - 1$ since each $e$ is contained in less than $2D$ hyperedges and $D \leq \Delta$ which is sufficiently large. Thus to prove the result it will be enough to show $\Pr[S_j > 2^{1-j}D(\log \Delta)^2] \leq 2\Delta^{-2}$ for all $j$ in this range and then to apply the union bound to each of these $2 \log \Delta$ events.

Fix some $j$ and let $\alpha$ be such that $|E_j| = \alpha 2^{-j} \Delta$, which means

$$\mathbb{E}[S_j] = p \cdot \alpha 2^{-j} = \alpha 2^{-j} D \log \Delta.$$
Note that $\alpha \leq 1$ since $\Delta \geq d_H(u) \geq 2^j|E_j|$. Using this and the Chernoff bound (2) gives

$$\Pr[S_j > 2^{1-j}D(\log \Delta)^2] \leq \Pr[S_j - \alpha 2^{-j}D \log \Delta > 2^{-j}D(\log \Delta)^2]$$

$$= \Pr[S_j - \mathbb{E}[S_j] > \alpha^{-1}(\log \Delta) \cdot \mathbb{E}[S_j]]$$

$$\leq e^{-\alpha^{-1}2^{-j-1}D(\log \Delta)^3} \leq \Delta^{-2},$$

where this last step used $\alpha \leq 1$, $2^j \leq D$, and that $\Delta$ is sufficiently large. $\blacksquare$

We use Claim 3.6 to prove the following.

**Claim 3.7.** There exists a subgraph $H' \subseteq H$ such that $H'$ has maximum degree at most $8D(\log \Delta)^3$, every $k$-set $S \subseteq \bigcup_{i \leq k} V_i$ has $d_{H'}(S) = 0$ or $d_{H'}(S) \geq D$, and

$$e(H') \geq \frac{D \log \Delta}{2\Delta} \cdot e(H).$$

**Proof.** Let $U$ denote the set of vertices $u$ with $d_{H_p}(u) > 8D(\log \Delta)^3$ and let $\mathbb{1}_u$ be the indicator variable which is 1 if $u \in U$ and 0 otherwise. Let $Y$ denote the set of hyperedges of $H_p$ which contain at least one vertex in $U$, and we think of $Y$ as a set of “bad” hyperedges. Observe that

$$|Y| \leq \sum_{u \in U} d_{H_p}(u) = \sum_u \mathbb{1}_u d_{H_p}(u).$$

Using this, Claim 3.6 and $d_{H_p}(u) \leq d_{H}(u)$ for all $u$, we conclude that

$$\mathbb{E}[|Y|] \leq \frac{2\log \Delta}{\Delta^2} \sum u d_{H}(u) = \frac{2r \log \Delta}{\Delta^2} \cdot e(H).$$

(4)

Define

$$Y' = \{e \cap \bigcup_{i \leq k} V_i : e \in Y\}, \quad Z = \{e \in E(H) : e \cap \bigcup_{i \leq k} U \in Y'\}.$$

That is, $Y'$ consists of edges of $G$ that are contained in the “bad” hyperedges $Y$, and $Z$ consists of the hyperedges of $H$ which contain edges of $Y'$. Observe that $Y \subseteq Z$ and

$$|Z| \leq 2D|Y'| \leq 2D|Y|,$$

(5)

since each $k$-set of $H_p \subseteq H$ is contained at most $2D$ hyperedges. Let $H'_p \subseteq H_p$ be the subgraph obtained by deleting every edge in $Z$. In particular note that because $Y' \subseteq Z$, $H'_p$ has maximum degree at most $8D(\log \Delta)^3$. Also by definition of $Z$, every $k$-set $S \subseteq \bigcup_{i \leq k} V_i$ has $d_{H'_p}(S) = 0$ or $d_{H'_p}(S) \geq D$, depending on whether $S \in Y'$ or not. It thus suffices to show that in expectation $H'_p$ has at least as many edges as in (3). And indeed, each hyperedge of $H$ is in $H_p$ with probability $p$, so we have by (4) and (5)

$$\mathbb{E}[e(H'_p)] = \mathbb{E}[e(H_p) - |Z|] \geq \frac{D \log \Delta}{\Delta} \cdot e(H) - \frac{4r D \log \Delta}{\Delta^2} \cdot e(H)$$

$$\geq \frac{D \log \Delta}{2\Delta} \cdot e(H),$$
where this last step used $\Delta$ being sufficiently large.

Let $H'$ be as in Claim 3.7 and let $G' = \partial H'[\bigcup_{i \leq k} V_i]$. The hypergraph $H'$ has maximum degree at most $8D(\log \Delta)^3$ and every $k$-set in $\bigcup_{i \leq k} V_i$ has $k$-degree 0 or at least $D$, so $G'$ has maximum degree at most $8(\log \Delta)^3$. Similarly $e(G') \geq \frac{1}{2} D^{-1} e(H')$ since each edge of $G' \subseteq G$ is contained in at most $2D$ hyperedges of $H' \subseteq H$. By Lemma 3.3 there exists a matching $M \subseteq G'$ with

$$|M| \geq \frac{e(H')}{{16kD(\log \Delta)^3}} = \Omega \left( \frac{1}{\Delta(\log \Delta)^2} \right) \cdot e(H),$$

where this last step used (3).

Let $G_M$ be the $(r-k+1)$-graph on $M \cup \bigcup_{i > k} V_i$ which contains the edge $\{e, u_{k+1}, \ldots, u_r\}$ if and only if $e \cup \{u_{k+1}, \ldots, u_r\} \in E(H')$. By (6) and the fact that each $k$-set of $\bigcup_{i \leq k} V_i$ has $k$-degree 0 or at least $D$ in $H'$, we have

$$e(G_M) \geq D|M| = \Omega \left( \frac{D}{\Delta(\log \Delta)^2} \right) \cdot e(H).$$

Each $M$ vertex of $G_M$ has degree at most $2D$ since each of these $k$-sets are contained in at most $2D$ hyperedges of $H$, and each $u \in \bigcup_{i > k} V_i$ has degree at most $8D(\log \Delta)^3$ by construction of $H'$. By the hypothesis of the proposition, there exists a $\mathcal{P}_k(\mathcal{F})$-free subgraph $G'' \subseteq G_M$ with

$$e(G'') = \text{ex}(G_M, \mathcal{P}_k(\mathcal{F})) = \Omega(D^{-\beta}(\log \Delta)^{-3\beta}) \cdot \Omega \left( \frac{D}{\Delta(\log \Delta)^2} \right) \cdot e(H)$$

$$= \Omega \left( \frac{D^{1-\beta}}{\Delta(\log \Delta)^5} \right) \cdot e(H),$$

where this last step used $\beta \leq 1$.

Finally, let $H'' \subseteq H'$ be the subgraph which contains the hyperedge $e \cup \{u_{k+1}, \ldots, u_r\}$ if and only if $\{e, u_{k+1}, \ldots, u_r\}$ is an edge of $G''$. Note that $e(H'') = e(G'')$, so by (7) we will be done if we can show that $H''$ is $\mathcal{F}$-free. Indeed, $\partial H'' [\bigcup_{i \leq k} V_i] \subseteq M$ is a matching and $\partial H'' [\bigcup_{i > k} V_i] \cong G''$ is $\mathcal{P}_k(\mathcal{F})$-free, so the result follows from Lemma 3.2.

\section{3.2 Examples of $\mathcal{P}_k(\mathcal{F})$}

To apply Proposition 3.1, we need to understand $\mathcal{P}_k(\mathcal{F})$ for families of cycles $\mathcal{F}$. The simplest case is the loose cycle.

**Lemma 3.8.** For all $\ell \geq 3$, we have $\mathcal{P}_2(C_\ell^3) = \emptyset$.

**Proof.** Let $V_1 \cup V_2 \cup V_3$ be a tripartition of $C_\ell^3$ such that $\partial C_\ell^3 [V_1 \cup V_2]$ is a matching. If $\{u_1, u_2, u_3\} \in E(C_\ell^3)$ with $u_i \in V_i$ for all $i$, then this must be the only edge containing $u_1$ since any other edge must also contain $u_2$, contradicting $C_\ell^3$ being linear. Similarly it is the only edge containing $u_2$, so $\{u_1, u_2, u_3\}$ contains two vertices of degree 1 which does not hold for any edge of $C_\ell^3$, a contradiction. \qed
We define \( S \) to be the set of \( r \)-uniform sunflowers on \( \ell \) edges. We let \( \widehat{S}^r_{\ell} \) denote the set of \( r \)-uniform sunflowers on \( \ell \) edges. We let \( \widehat{C}^r_{\ell} := \widehat{C}^r_{\ell} \setminus S^r_{\ell} \) and \( \widehat{C}^r_{\ell}[r] = \bigcup_{e=2}^r \widehat{C}^r_{\ell} \), which agrees with our definition in Theorem 1.1.

We define \( \tilde{S}^r_{\ell} \) to be the set of \( r \)-graphs \( \tilde{S} \) that can be obtained by adding an edge \( e \) to some \( S \in S^r_{\ell} \) such that \( K \cap e \neq \emptyset \) where \( K \) is the kernel of \( S \), and such that \( S \cup e \) is a sunflower. Note that we do not require \( V(\tilde{S}) = V(S) \), that is, \( e \) might include vertices which are not in the original sunflower \( S \). For any \( \tilde{S} \in \tilde{S}^r_{\ell} \), there exists an edge \( e \) such that \( \tilde{S} – e \) is a sunflower, and we call \( e \) an extra edge of \( \tilde{S} \) (which may not be unique), and we define the kernel of \( \tilde{S} \) (with respect to the extra edge \( e \)) to be the kernel of the sunflower \( \tilde{S} – e \). Finally, we define \( \tilde{S}^r_{\ell}[r] = \bigcup_{e=2}^r \tilde{S}^r_{\ell} \).

**Lemma 3.9.** Every element of \( \mathcal{P}_r(\widehat{C}^r_{\ell} \cup \tilde{S}^r_{\ell}) \) contains an element of \( \widehat{S}^{r-k+1}_{\ell} \cup \tilde{S}^{r-k+1}_{\ell} \) as a subgraph.

**Proof.** We prove the result for any fixed \( r \geq 2 \) by induction on \( k \). We start with \( k = 2 \), and for ease of presentation we define \( F_0 := \partial F[\bigcup_{i=2}^r V_i] \) whenever \( F \) has an \( r \)-partition \( V_1 \cup \cdots \cup V_r \) that is clear from context.

**Claim 3.10.** If \( F \in \tilde{S}^r_{\ell} \), then \( \mathcal{P}_2(F) \subseteq \tilde{S}^{r-1}_{\ell} \).

**Proof.** Let \( e \) be an extra edge of \( F \) and \( K \) the kernel of \( F' := F – e \). Let \( V_1 \cup \cdots \cup V_r \) be an \( r \)-partition of \( F' \) such that \( V_1 \cup V_2 \) induce a matching. Observe that \( F_0 \) is a sunflower with kernel \( K \setminus V_1 \), so \( F_0 \) is this sunflower together with the edge \( e_0 := e \setminus V_1 \). We claim that \( e_0 \) intersects \( K \) \setminus V_1 \). Indeed, by definition of \( \tilde{S}^r_{\ell} \) we have \( u \in K \cap e \) for some vertex \( u \), and if \( u \notin V_1 \) then the subclaim holds. If \( u \in V_1 \), then \( V_1 \cup V_2 \) inducing a matching implies that there is some \( u' \in V_2 \) such that every edge containing \( u \) also contains \( u' \). In particular, \( u' \in e_0 \cap (K \setminus V_1) \), proving the subclaim.

It remains to show that \( F_0 \) is not a sunflower. By definition of \( F \), there exists an edge \( e' \neq e \) such that \( e \cap e' \neq K \), and we will be done if we can show \( (e \cap e') \setminus V_1 \neq K \setminus V_1 \). Assume for the sake of contradiction that \( (e \cap e') \setminus V_1 = K \setminus V_1 \), and this together with \( e \cap e' \neq K \) implies that there exists \( u_1 \in V_1 \) such that \( u_1 \) is in exactly one of \( e \cap e' \) or \( K \). If \( u_1 \in e \cap e' \), then \( V_1 \cup V_2 \) inducing a matching implies that there is some \( u_2 \in V_2 \) such that every edge either contains both of \( u_1 \) or neither of them. In particular, we have \( u_2 \in e \cap e' \) and \( u_2 \notin K \) since \( u_1 \notin K \), contradicting \( (e \cap e') \setminus V_1 = K \setminus V_1 \). Thus we must have \( u_1 \in K \), which similarly implies the existence of some \( u_2 \in K \cap V_2 \) with \( u_2 \notin K \cap e' \), a contradiction. We conclude that \( F_0 \) is not a sunflower, and hence \( F_0 \in \tilde{S}^{r-1}_{\ell} \) as desired.

Recall that \( A = \{v_1, \ldots, v_{\ell}\} \subseteq V(F) \) is a core set if there exist edges \( e_1, \ldots, e_{\ell} \in E(F) \) with \( v_i, v_{i+1} \in e_i \) for all \( 1 \leq i < \ell \) and \( v_1, v_{\ell} \in e_{\ell} \).

**Claim 3.11.** Let \( \hat{F} \in \widehat{C}^r_{\ell} \) with \( \hat{A} = \{v_1, \ldots, v_{\ell}\} \) a core set of \( \hat{F} \), and let \( V_1 \cup \cdots \cup V_r \) be an \( r \)-partition of \( \hat{F} \) such that \( V_1 \cup V_2 \) induce a matching \( M \). If \( \{v_i, v_j\} \notin M \) for all \( i, j \), then \( \hat{F}_0 \in \widehat{C}^{r-1}_{\ell} \).
Proof. For \( \{x, y\} \in M \), define \( \tilde{x} = y \) and \( \tilde{y} = x \). Let \( w_i = v_i \) if \( v_i \notin V_1 \) and \( w_i = \bar{v}_i \) otherwise. By the hypothesis of the claim, \( \hat{A} := \{w_1, \ldots, w_{\ell'}\} \) consists of \( \ell' \) distinct vertices of \( V(\hat{F}_\partial) = V_2 \cup \cdots \cup V_r \). Since every edge containing \( v_i \) also contains \( \bar{v}_i \), this defines a core set for \( \hat{F}_\partial \), and hence \( \hat{F}_\partial \in C_{r-1}^r \).

It remains to show that \( \hat{F}_\partial \) is not a sunflower. Assume for contradiction that it is a sunflower with kernel \( K \). If there exists \( w_2 \in K \cap V_2 \), then because \( V_1 \cup V_2 \) induce a matching and every edge of \( F_\partial \) contains \( w_2 \), there exists some \( v_1 \in V_1 \) contained in every edge of \( F \), so \( F \) is a sunflower with kernel \( K \cup \{v_1\} \), contradicting that \( \hat{F} \in C_{r}^r \). Thus every vertex in \( V_2 \) is in exactly one edge of the sunflower \( \hat{F}_\partial \), and again \( V_1 \cup V_2 \) inducing a matching implies that each vertex of \( V_1 \) is contained in exactly one edge. We conclude that \( \hat{F} \) is a sunflower with kernel \( K \), a contradiction. \( \blacksquare \)

We now return to the proof of Lemma 3.9. Assume for contradiction that there exists some \( 2 \leq \ell' \leq \ell \) and \( F \in C_{\ell'}^r \) which has an \( r \)-partition \( V_1 \cup \cdots \cup V_r \) such that \( V_1 \cup V_2 \) induce a matching \( M \) and such that \( F_\partial \) does not contain an element of \( \hat{C}_{\ell'}^r \cup \bar{S}_{\ell'}^r \) as a subgraph. We further choose \( \ell' \) to be the smallest integer such that there exists such an \( F \in C_{\ell'}^r \). Observe that \( \hat{C}_2^r \) is empty, so \( \ell' \geq 3 \). Let \( \{v_1, \ldots, v_{\ell'}\} \) be a core set of \( F \) with respect to its edges \( e_1, \ldots, e_{\ell'} \).

If \( \{v_i, v_j\} \notin M \) for all \( i \neq j \), then Claim 3.11 shows that \( F_\partial \in C_{\ell'}^r \), a contradiction. Thus by reindexing we will assume \( \{v_1, v_1\} \in M \) for some \( i > 1 \).

Claim 3.12. Let \( F' \subseteq F \) consist of the edges \( e_1, \ldots, e_{i-1} \) and let \( F'' = F \setminus F' \) consist of the edges \( e_i, \ldots, e_{\ell'} \). Then \( F', F'' \) are sunflowers (possibly with one edge) which have kernels \( K', K'' \) containing \( v_1 \).

Proof. We only prove the result for \( F' \), the proof for \( F'' \) being completely analogous. If \( i = 2 \) then \( F' \) has a single edge which contains \( v_1 \), so we will assume \( i > 2 \). We claim that \( A' = \{v_1, \ldots, v_{i-1}\} \) is a core set of \( F' \) with respect to \( \{e_1, \ldots, e_{i-1}\} \). Indeed, because \( \{v_1, \ldots, v_{\ell'}\} \) is a core set with respect to \( \{e_1, \ldots, e_{\ell'}\} \), we automatically have \( v_j, v_{j+1} \in e_j \) for \( 1 \leq j \leq i \), so \( A' \) will be a core set provided \( v_1 \in e_{i-1} \). This follows since \( v_{i-1} \in e_{i-1} \) and \( \{v_1, v_{i-1}\} \in M \) implies that every edge containing \( v_{i-1} \) also contains \( v_1 \).

Thus \( F \in C_{i-1}^r \). If we assume for contradiction that \( F' \) is not a sunflower, then \( F' \in \hat{C}_{i-1}^r \). By the minimality of \( \ell' > i - 1 \), we have that \( F_\partial' \) contains an element of \( \hat{C}_{\ell'}^r \cup \bar{S}_{\ell'}^r \) as a subgraph. This will also be a subgraph of \( F_\partial' \subseteq F_\partial \), a contradiction. Thus it must be that \( F' \) is a sunflower. Moreover, because \( v_1 \) is in a core set of \( F' \), it must have degree at least two in \( F' \), and hence it must be in its kernel \( K' \). \( \blacksquare \)

By Claim 3.12, we see that \( F \) is the union of two sunflowers \( F', F'' \) intersecting at a common vertex \( v_1 \in K' \cap K'' \), so in particular every edge in \( F \) contains \( v_1 \). Since \( \ell' \geq 3 \) we will assume without loss of generality that \( F' \) has at least two edges, i.e. that \( i \geq 3 \). Let \( F^{(i)} \subseteq F' \) be the subgraph with edge set \( \{e_1, \ldots, e_j\} \). Since \( F^{(i)} = F \) is not a sunflower and \( F^{(i-1)} = F' \) is, there exists a largest \( j \) with \( i - 1 \leq j < \ell' \) such that \( F^{(j)} \) is a sunflower on at least two edges. Observe that \( F^{(j)} \) has kernel \( K \) since \( F' \subseteq F^{(j)} \) has this kernel, and that \( F^{(j+1)} \in \bar{S}_{j+1}^r \subseteq S_{\ell'}^r \) since \( v_1 \in e_{j+1} \cap K \). By Claim 3.10 we have that \( F^{(j+1)}_\partial \subseteq F_\partial \) contains an element of \( \bar{S}_{\ell'}^r \) as a
It is known that the hypothesis of Proposition 4.1 holds for theorem of Perarnau and Reed [23]. This result can also be derived from Proposition 2.1, which is essentially a generalization of a Theorem 4.2.

Proof. Let be a 3-graph with at least 3 edges. Thus every element of the lemma, if , then every belongs to some value . Observe that for all we have if contains a subgraph in , and by the inductive hypothesis every element of contains a subgraph in , which completes the inductive step of the proof, giving the desired result.

We make one more observation about .

**Lemma 3.13.** Every element of is non-linear.

*Proof.* Let have extra edge and kernel . If then in particular at least two edges of share a common pair of vertices, so we will assume . We actually must have since . If every has , then is a sunflower, which is a contradiction to how is defined. Thus there exists some edge with , and since every edge of contains the vertex of , we must have that .

4 Proof of Theorem 1.1 and a Lower Bound for \( \text{ex}(H, C_3^3) \)

We prove Theorem 1.1 by induction on . Note that \( C_\ell^2 = \{C_3, C_4, \ldots, C_\ell\} \), and that \( \tilde{S}_\ell^2 = \emptyset \) (this latter case can either be seen directly from the definition or from Lemma 3.13). With this we restate a result of Perarnau and Reed [23] in a form that will be convenient to us.

**Proposition 4.1.** If \((*)\) holds for \( \ell \) and \( r = 2 \), then for all graphs \( G \) with maximum degree at most \( \Delta \),

\[
\text{ex}(G, \tilde{C}_\ell^2 \cup \tilde{S}_\ell^2) \geq \Delta^{-1 + \frac{1}{(\ell/2)^2}} - o(1) \cdot e(G).
\]

This result can also be derived from Proposition 2.1, which is essentially a generalization of a theorem of Perarnau and Reed [23].

It is known that the hypothesis of Proposition 4.1 holds for \( \ell = 4, 5 \), see for example [13]. The \( \ell = 5 \) case allows us to prove the following.

**Theorem 4.2.** If \( H \) is a 3-graph with maximum degree at most \( \Delta \), then

\[
\text{ex}(H, C_3^3) \geq \Delta^{-3/4 - o(1)} \cdot e(H).
\]

*Proof.* Let \( H \) be a 3-graph with maximum degree at most \( \Delta \), and let \( H_2 \subseteq H \) be a 3-partite 3-graph with at least \( 3^{-3} e(H) \) edges. First assume that at least half the edges of \( H_2 \) contain a
2-set with 2-degree at least $\Delta^{1/2}$. Using $C_5^3 \subseteq \tilde{C}_5^3$ (i.e. $C_5^3$ contains no sunflowers), Lemma 3.9, Proposition 4.1, and Proposition 3.1 applied to $H$ whenever $\ast$

Assume that $(\ast)$ holds for $\ell, r, t$ and any 2 $\leq k \leq r - 1$. Indeed, given an extremal $C_{[\ell]}^r$-free $r$-graph $J$, we form a $C_{[\ell]}^k$-free $k$-graph $J'$ by including some $k$-set from each edge of $J$, and each of these $k$-sets will be distinct because $J$ is linear. With this $e(J') = e(J)$, and it is straightforward to see that $J'$ will be $C_{[\ell]}^k$-free if $J$ is $C_{[\ell]}^r$-free, proving the claim. Thus by the inductive hypothesis, if $(\ast)$ holds for $\ell, r$ then for all $2 \leq k \leq r - 1$ we have

$$\text{ex}(H', \tilde{C}_{[\ell]}^{r-k+1} \cup \tilde{S}_{[\ell]}^{r-k+1}) \geq \Delta^{-1 + \frac{1}{r(r+k)/2}} - o(1) \cdot e(H'),$$

whenever $H'$ is an $(r - k + 1)$-graph with maximum degree at most $\Delta$.

Assume that $(\ast)$ holds for $\ell, r$. Let $H$ be an $r$-graph with maximum degree at most $\Delta$. We will prove by induction on $2 \leq k \leq r$ that there exists a subgraph $H_k \subseteq H$ such that $H_k$ is $r$-partite, $e(H_k) \geq r^{-k}2^{-k}e(H)$, and $H_k$ has maximum $p$-degree at most $D_p := \Delta^{(r-p)/(r-1)}$.
we can apply Proposition 3.1 to $H$ taking $\hat{H}$.

Recall that $\exists u_1, u_2, v_1, v_2, v_3$ such that

$$
\exists u_1, u_2, v_1, v_2, v_3, \phi(e_1) = \{u_1, u_2, u_3\}, \phi(e_2) = \{u_1, u_2, v_1\}, \phi(e_3) = \{v_1, v_2, v_3\}, \phi(e_4) = \{u_3, v_2, v_3\}.
$$

Observe that $N_F(H)$ is at most the number of such 4-tuples, so it suffices to bound this quantity.

$$
N_F(H) \leq 9D\Delta \cdot e(H).
$$

Proof. Consider all of the ways of choosing a 4-tuple $(e_1, e_2, e_3, e_4)$ of edges of $H$ such that there exists an isomorphism $\phi$ from these edges to $F$ such that $\phi(e_1) = \{u_1, u_2, u_3\}, \phi(e_2) = \{u_1, u_2, v_1\}, \phi(e_3) = \{v_1, v_2, v_3\}, \phi(e_4) = \{u_3, v_2, v_3\}$. The number of such 4-tuples, so it suffices to bound this quantity.

$$
\beta = 1 - \frac{1}{(r - k) \lceil \ell/2 \rceil} + \varepsilon
$$

for any $\varepsilon > 0$ to conclude

$$
ex(H_k, \tilde{C}_{[\ell]} \cup \tilde{S}_{[\ell]}) \geq D_k^{\frac{1}{(r - k)\lceil \ell/2 \rceil} - \varepsilon} \cdot \Delta^{-1-o(1)} \cdot e(H_2) = \Delta^{-1+\frac{1}{(r - k)\lceil \ell/2 \rceil} - \varepsilon} - \frac{1}{r - k} - o(1) \cdot e(H),
$$

and taking $\varepsilon$ arbitrarily small gives (8) as desired. Thus we may assume such a subgraph $H_{k+1}$ exists.

In total we have found a subgraph $H_r \subseteq H$ with $e(H_r) = \Omega(e(H))$ which has maximum $k$-degree at most $D_k$ for all $k$. By Lemmas 3.13 and 2.2(1), every element of $\mathcal{H}(\tilde{S}_{[\ell]})$ contains an element of $\tilde{C}_{[\ell]}$ as a subgraph. Thus by Lemma 2.2(3), every element of $\mathcal{H}(\tilde{C}_{[\ell]} \cup \tilde{S}_{[\ell]})$ contains an element of $\tilde{C}_{[\ell]}$ a subgraph, so

$$
ex(t, \mathcal{H}(\tilde{C}_{[\ell]} \cup \tilde{S}_{[\ell]})) \geq \ex(t, \mathcal{C}_{[\ell]}^r) \geq t^{1+\frac{1}{(r - 1)\lceil \ell/2 \rceil} - o(1)},
$$

with this last step using the assumption that $(*)$ holds for $\ell, r$. We then apply Proposition 2.1 with $t = \Theta(\Delta^{1/(r-1)})$ and conclude

$$
ex(H_r, \tilde{C}_{[\ell]} \cup \tilde{S}_{[\ell]}) \geq \Delta^{-1+\frac{1}{(r - 1)\lceil \ell/2 \rceil} - o(1)} \cdot e(H)
$$

as desired.

\begin{proof}

5 Proof of Theorem 1.5

Recall that $F \in \mathcal{C}_3^3$ is the hypergraph from Figure 1. To improve upon the trivial bound $\ex(H, F) \geq \ex(H, \mathcal{C}_3^3) \geq \Delta^{-3/4-o(1)} e(H)$ from Corollary 1.2, we use the following counting lemma.

Lemma 5.1. Let $N_F(H)$ denote the number of copies of $F$ in the 3-graph $H$. If $H$ has maximum degree at most $\Delta$ and maximum 2-degree at most $D$, then

$$
N_F(H) \leq 9D\Delta \cdot e(H).
$$

Proof. Consider all of the ways of choosing a 4-tuple $(e_1, e_2, e_3, e_4)$ of edges of $H$ such that there exists an isomorphism $\phi$ from these edges to $F$ such that $\phi(e_1) = \{u_1, u_2, u_3\}, \phi(e_2) = \{u_1, u_2, v_1\}, \phi(e_3) = \{v_1, v_2, v_3\}, \phi(e_4) = \{u_3, v_2, v_3\}$. Observe that $N_F(H)$ is at most the number of such 4-tuples, so it suffices to bound this quantity.

\end{proof}

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Any of the $e(H)$ edges of $H$ can be $e_1$ in such a 4-tuple. Given $e_1$, $e_2$ must be one of the at most $3D$ edges intersecting $e_1$ in a pair, and given this $e_3$ must be one of the at most $3\Delta$ edges intersecting $e_2$ in a vertex. Once these three edges have been chosen, $e_4$ is uniquely determined, so we conclude the desired bound. 

**Proof of Theorem 1.5.** Let $H' \subseteq H$ be a 3-partite 3-graph with at least $3^{-3}e(H)$ edges. If more than half of the edges of $H'$ contain a pair of vertices with 2-degree at least $\Delta^{4/5}$, then by using Lemmas 3.9 and Proposition 4.1, we apply Proposition 3.1 with $k = 2$ and $\beta = 1/2 + o(1)$ to find

$$\text{ex}(H, F) \geq \text{ex}(H', F) \geq \Omega \left( \frac{(\Delta^{4/5})^{1-\beta}}{\Delta(\log \Delta)^7} \right) \cdot e(H) = \Delta^{-3/5-o(1)} e(H).$$

If instead at most half of the edges of $H'$ contain a pair with 2-degree at least $\Delta^{4/5}$, then we can let $H'' \subseteq H'$ consist of all of these edges. Let $H_p \subseteq H''$ be the random 3-graph obtained by keeping each edge of $H''$ independently and with probability $p = \frac{1}{9} \Delta^{-3/5}$. Observe that

$$\mathbb{E}[e(H_p)] = p \cdot e(H'') \geq \frac{1}{486} \Delta^{-3/5} \cdot e(H),$$

and

$$\mathbb{E}[N_F(H_p)] = p^4 \cdot N_F(H'') \leq 9^{-4} \Delta^{-12/5} \cdot 9\Delta^{9/5} e(H) = \frac{1}{729} \Delta^{-3/5} \cdot e(H),$$

where this last inequality used Lemma 5.1 and that $H''$ has maximum 2-degree at most $D = \Delta^{4/5}$ and that it has at most $e(H)$ edges. In particular, there exists some specific choice of $H'' \subseteq H''$ with $e(H'') - N_F(H'') = \Omega(\Delta^{-3/5} e(H))$. Deleting an edge from each copy of $F$ in $H''$ gives an $F$-free hypergraph with the desired number of edges. 

**6 Proofs of Theorem 1.3 and Proposition 1.4**

To prove Theorem 1.3 and Proposition 1.4 we require the following lemmas.

**Lemma 6.1.** Let $H$ be an $r$-graph with maximum degree $\Delta$ and maximum 2-degree $D$. If $N_\ell(H)$ denotes the number of copies of $C^r_\ell$ in $H$, then

$$N_\ell(H) \leq r^\ell D^\ell e(H).$$

**Lemma 6.2.** If $H$ is an $r$-graph with maximum 2-degree at most $D$, then there exists a linear subgraph $H' \subseteq H$ with $e(H') \geq e(H)/r^2 D$.

The proof of Lemma 6.1 is a straightforward counting argument as in the proof of Lemma 5.1; and the proof of Lemma 6.2 uses a greedy construction. We omit the details.

Lastly, we require a special case of a theorem of Mubayi and Yepremyan [21] for random hypergraphs. We recall that $H^r_{n,p}$ is the random $r$-graph on $n$ vertices obtained by including each edge of $K^n_r$ independently and with probability $p$, and that a statement depending on $n$ holds asymptotically almost surely, or simply a.a.s., if it holds with probability tending to 1 as $n$ tends to infinity.
Proposition 6.3 ([21]). For all even $\ell \geq 4$ and $r \geq 3$, at $p = n^{-r+2}$ we have a.a.s.

$$\text{ex}(H_{n,p}^r, \mathcal{C}_\ell^r) \leq n^{1 + \frac{1}{1+\ell} + o(1)}.$$

Proof of Theorem 1.3. The proof is very similar to that of Theorem 1.5, so we will omit some of the redundant details. By losing at most a constant fraction of the edges, we can assume $H$ is $r$-partite. If more than half of its edges contain a pair with 2-degree at least $D = \Delta^{1/\ell}$, then using Lemma 3.8 we apply Proposition 3.1 with $\beta = 0$ to find $\text{ex}(H, \mathcal{C}_D^r) \geq \Delta^{-1+1/(\ell-1) - o(1)} e(H)$.

Thus by losing at most half of its edges we will assume $H$ has maximum 2-degree at most $D$. Let $H_p$ be $H$ after keeping each edge with probability

$$p = 3^{-1 - 2/((\ell-1)\Delta - 1 + 1/(\ell-1))} D^{-1/((\ell-1))}.$$ 

By Lemma 6.1, we find

$$\mathbb{E}[\mathcal{N}_\ell(H_p)] \leq 3^\ell \Delta^{\ell-2} D p^\ell \cdot e(H) = 3^\frac{2\ell}{\ell-1} \Delta^{-1 + \frac{1}{\ell}} \cdot e(H) = 3^{-1} \cdot \mathbb{E}[e(H_p)].$$

(10)

Thus if we define $H'_p \subseteq H_p$ by deleting an edge from each copy of $\mathcal{C}_D^r$, then we get a subgraph which is $\mathcal{C}_D^r$-free and which has $\Omega(\Delta^{-1+1/(\ell-1)} e(H))$ edges in expectation, proving the lower bound.

For the construction, take $p = n^{2-r}$ and $H = H'_{n,p}$. A straightforward argument using the Chernoff bound (2) gives, for this choice of $p$, that a.a.s. $e(H_{n,p}^r) = \Theta(n^2)$ and that $H_{n,p}^r$ has maximum degree $\Delta = \Theta(n)$. By Proposition 6.3 we have a.a.s.

$$\text{ex}(H_{n,p}^r, \mathcal{C}_\ell^r) \leq n^{1 + \frac{1}{1+\ell} + o(1)} = e(H_{n,p}^r).$$

In particular, such an $H$ exists for $n$ sufficiently large, giving the desired result. \qed

Proof of Proposition 1.4. The proof is nearly identical to that of Theorem 1.3. For the lower bound, we use that $H$ has maximum 2-degree at most 1, so the exact same computation for the proof of (10) works by taking $p = \Theta(\Delta^{-1+1/(\ell-1)})$. For the construction, it is straightforward to use the Chernoff bound (2) to show that a.a.s. $H_{n,p}^r$ at $p = n^{2-r}$ has maximum 2-degree at most $O(\log n)$, so we can apply Lemma 6.2 to find a large linear subgraph $H \subseteq H_{n,p}^r$ and the rest of the proof works out as before. \qed

7 Concluding Remarks

- Using arguments analogous to the ones used throughout the paper, it is not too difficult to show for any $F \in \mathcal{C}_d^i \setminus \{\mathcal{C}_d^i, \mathcal{F}\}$ that $\text{ex}(H, F) \geq \Delta^{-3/2 - o(1)} e(H)$ whenever $H$ has maximum degree at most $\Delta$, and this is often tight by considering $H$ to be the clique. However, our best bounds for $\mathcal{C}_d^3$ and $\mathcal{F}$ given by Theorems 1.3 and 1.5 still have significant gaps.

- Theorem 1.5 shows that $\text{ex}(H, F) \geq \Delta^{3/5 - o(1)} e(H)$ for any host $H$ with maximum degree at most $\Delta$. It is known that $\text{ex}(n, F) = \Theta(n^2)$, so taking $H = K_{\sqrt{3}}^3$ gives $\text{ex}(H, F) =$
$O(\Delta^{-1/2})e(H)$. It is unclear which of these bounds is closer to the truth, and we leave the following as an open problem.

**Problem 7.1.** Determine whether

$$\text{ex}(H, F) \geq \Delta^{-1/2-o(1)} \cdot e(H)$$

whenever $H$ is a 3-graph with maximum degree $\Delta$.

- While Proposition 1.4 gives essentially tight bounds for loose even cycles in linear hosts, our bounds are far from tight for general hosts. In particular, we do not know tight bounds for $C_4^3$.

**Problem 7.2.** Determine whether there exists a 3-graph $H$ with maximum degree $\Delta$ and

$$\text{ex}(H, C_4^3) \leq \Delta^{-3/4+o(1)} \cdot e(H).$$

A reasonable candidate for such an $H$ is $H_{n,p}$ with $p = n^{-2/3}$. It is conjectured by Mubayi and Yepremyan [21] that $\text{ex}(H_{n,p}^3, C_4^3) \leq n^{4/3+o(1)}$ a.a.s. at $p = n^{-2/3}$, and this can be rephrased as saying $\text{ex}(H_{n,p}^3, C_4^3) \leq \Delta^{-3/4+o(1)} e(H)$ a.a.s. Conversely, if there existed a method which improved upon the known lower bounds for $\text{ex}(H_{n,p}^3, C_4^3)$, then it is possible that this method could also be used to improve lower bounds for general hosts $H$.

- Variants of Propositions 2.1 and 3.1 were used to prove essentially tight bounds on the relative Turán numbers for certain kinds of complete $r$-partite $r$-graphs [27], and we suspect that these propositions will continue to be of use for future investigations into relative Turán numbers of hypergraphs.

**Acknowledgments.** The authors thank the anonymous referees for their careful reading of this paper and their many helpful comments.

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