Bouncing cosmological solutions and their stability

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Abstract

In the present paper we consider the bouncing braneworld scenario, in which the bulk is given by a five-dimensional charged AdS black hole spacetime with matter field confined in a $D_3$ brane. Then, we study the stability of solutions with respect to homogeneous and isotropic perturbations. Specifically, the AdS black hole with zero ADM mass and charge, and open horizon is an attractor, while the charged AdS black hole with zero ADM mass and flat horizon, is a repeller.
1 Introduction

Motivated by string/M theory, the AdS/CFT correspondence, and the hierarchy problem of particle physics, braneworld models were studied actively in recent years [1]-[4]. In these models, our universe is realized as a boundary of a higher dimensional spacetime. In particular, a well studied example is when the bulk is an AdS space. In the cosmological context, embedding of a four dimensional Friedmann-Robertson-Walker universe was also considered when the bulk is described by AdS or AdS black hole [5, 6]. In the latter case, the mass of the black hole was found to effectively act as an energy density on the brane with the same equation of state of radiation. Representing radiation as conformal matter and exploiting AdS/CFT correspondence, the Cardy-Verlinde formula [7] for the entropy was found for the universe (see [8], for the entropy formula in the case of dS black hole).

In either of the above cases, however, the cosmological evolution on the brane is modified at small scales. In particular, if the bulk space is taken to be an AdS black hole with charge, the universe can ‘bounce’ [9]. That is, the brane makes a smooth transition from a contracting phase to an expanding phase. From a four-dimensional point of view, singularity theorems [10] suggest that such a bounce cannot occur as long as certain energy conditions apply. Hence, a key ingredient in producing the bounce is the fact that the bulk geometry may contribute a negative energy density to the effective stress-energy on the brane [11]. At first sight these bouncing braneworlds are quite remarkable, since they provide a context in which the evolution evades any cosmological singularities while the dynamics is still controlled by a simple (orthodox) effective action. In particular, it seems that one can perform reliable calculations without deliberation on the effects of quantum gravity or the details of the ultimate underlying theory. Hence, several authors [12, 13, 14, 15] have pursued further developments for these bouncing braneworlds. However the authors of [16] have found that generically these cosmologies are in fact singular. In particular, they have shown that a bouncing brane must cross the Cauchy horizon in the bulk space. However, the latter surface is unstable when arbitrarily small excitations are introduced in the bulk spacetime.

In this paper we describe solutions of the bouncing braneworld theory and also determine their stability. To do this, we use a set of convenient phase-space variables similar to those introduced in [18, 19]. The critical points of the system of differential equations in the space of these variables describe interesting non-static solutions. A method for evaluating the eigenvalues of the critical points of the Friedmann and Bianchi models was introduced by Goliath and Ellis [19] and further used in the analysis by Campos and Sopuerta [18] of the Randall Sundrum braneworld theory. These latter authors gave a complete description of stationary points in an appropriately chosen phase space of the cosmological setup and investigated their stability with respect to homogeneous and isotropic perturbations. The authors worked in the frames of the Randall Sundrum braneworld theory without the scalar-curvature term in the action for the brane. Cosmological solutions and their stability with respect to homogeneous and isotropic perturbations in the braneworld model with the scalar-curvature term in the action for the brane have further been studied by Iakubovskyi and Shtanov [20].
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\textbf{D}_3 \text{ Brane with Radiative Matter}

We start by considering a 4–dimensional brane in a space-time described by a 5–dimensional charged AdS black hole. The background metric is then given by \[ds^2_5 = -h(a)dt^2 + \frac{da^2}{h(a)} + a^2 d\Omega^2_{k,3},\] (1)

where

\[h(a) = k - \frac{\omega_4 M}{a^2} + \frac{3 \omega_4^2 Q^2}{16a^4} + \frac{a^2}{L^2},\] (2)

in which \(M\) and \(Q\) are ADM mass and charge, respectively, \(L\) is the curvature radius of space, \(k\) is a discrete parameter which indicates the horizon topology, namely \(k = +1, 0\) and \(-1\) for a spherical, flat and hyperbolic geometry respectively, and

\[\omega_4 = \frac{16\pi G_5}{3V_3}.\] (3)

Here \(G_5\) is the five-dimensional Newton constant, and \(V_3\) is the dimensionless volume element associated with \(d\Omega^2_{k,3}\) (a three-dimensional spacelike hypersurface of constant curvature).

Tuning the brane cosmological constant to zero, we get the following induced equation on the brane:

\[H^2 = -\frac{k}{a^2} + \frac{\omega_4 M}{a^4} - \frac{3 \omega_4^2 Q^2}{16a^6},\] (4)

where \(H = \frac{\dot{a}}{a}\) is the Hubble parameter and \(\dot{a}\) denotes derivation with respect to the brane time \(\tau\). If we consider the more physically relevant case in which a perfect fluid with equation of state of radiation is present on the brane, then the first Friedmann equation takes the following form

\[H^2 = -\frac{k}{a^2} + \frac{8\pi G_4}{3} \rho - \frac{1}{l^2} + \frac{4\pi}{3M_p^2 \rho_0} (\rho_0 + \rho_{br})^2 - \frac{3 \omega_4^2 Q^2}{16a^6},\] (5)

where \(\rho\) is the energy density defined by \(\rho = E_4/V\) and the four-dimensional energy \(E_4\) is given by

\[E_4 = \frac{-3V_3}{16\pi G_5 a} M = \frac{l}{r} E,\] (6)

where \(E\) is the thermodynamical energy of the black hole. Also in Eq.(5) \(\rho_0\) is the tension of the brane, while \(\rho_{br}\) is the energy density of the radiation. The Hubble equation can be rewritten as

\[H^2 = -\frac{k}{a^2} + \frac{8\pi G_4}{3} \rho + \frac{\Lambda_4}{3} + \frac{8\pi}{3M_p^2} \left(\frac{\rho_{br}}{2\rho_0} + \rho_{br}\right) - \frac{3 \omega_4^2 Q^2}{16a^6},\] (7)

where

\[\Lambda_4 = \frac{4\pi \rho_0}{M_p^2} - \frac{3}{l^2} = \Lambda_{br} - \frac{3}{l^2},\] (8)

is the effective cosmological constant of the brane. By tuning the bulk cosmological constant and the brane tension \(\Lambda_{br}\), the effective four dimensional cosmological constant
$\Lambda_4$ can be set to zero. This critical brane is what we would like to consider in the following. The radiation energy density $\rho_{br}$ on the brane is

$$\rho_{br} = \frac{\rho_r}{a^4},$$

where $\rho_r$ is a constant, hence we can rewrite the cosmological equation (7) as

$$H^2 = \frac{-k}{a^2} + \frac{(\omega_4 M + \frac{8\pi \rho_r}{3M_p^2})}{a^4} - \frac{3\omega_4^2 Q^2}{16a^6} + \frac{4\pi \rho_r^2}{3M_p^2 \rho_0 a^8}.$$  

(10)

By defining

$$A = (\omega_4 M + \frac{8\pi \rho_r}{3M_p^2}),$$

(11)

$$B = \frac{3\omega_4^2 Q^2}{16},$$

(12)

and

$$C = \frac{4\pi \rho_r^2}{3M_p^2 \rho_0},$$

(13)

Eq.(10) takes on the following form in terms of the above parameters

$$H^2 = \frac{-k}{a^2} + \frac{A}{a^4} - \frac{B}{a^6} + \frac{C}{a^8}.$$  

(14)

3 

Stability of the Bouncing Solutions

In this section, we describe solutions of the braneworld theory under investigation and also determine their stability. To do this, we use a set of convenient phase-space variables similar to those introduced in [18, 19]. Now, we introduce the notation similar to those of [18]

$$\Omega_k = \frac{-k}{a^2H^2} = \frac{-k}{a^2}, \quad \Omega_A = \frac{A}{a^4H^2}, \quad \Omega_B = \frac{-B}{a^6H^2}, \quad \Omega_C = \frac{C}{a^8H^2}.$$  

(15)

and work in the 4-dimensional $\Omega$-space ($\Omega_k, \Omega_A, \Omega_B, \Omega_C$). In this space, the $\Omega$ parameters are not independent since the Friedmann equation (4) reads

$$\Omega_k + \Omega_A + \Omega_B + \Omega_C = 1.$$  

(16)

In this case the state space defined by the variables ($\Omega_k, \Omega_A, \Omega_B, \Omega_C$) is no longer compact ( because now $\Omega_B < 0$ ). However, we can introduce another set of variables describing a compact state space. Firstly, instead of using the Hubble function $H$ we will use the following quantity

$$D = \sqrt{H^2 + \frac{B}{a^6}},$$

(17)

and from it, let us define the following dimensionless variables

$$Z = \frac{H}{D},$$

(18)
\[ \tilde{\Omega}_k = -\frac{k}{a^2 D^2}, \]  
\[ \tilde{\Omega}_A = \frac{A}{a^4 D^2}, \]  
\[ \tilde{\Omega}_C = \frac{C}{a^8 D^2}. \]  

From these definitions we see that now the case \( H = 0 \) is included. Moreover, the Friedmann equation takes the following form
\[ \tilde{\Omega}_k + \tilde{\Omega}_A + \tilde{\Omega}_C = 1, \]  
which, together with the fact that \(-1 \leq Z \leq 1 \) [see Equation (17)], implies that the state space defined by the new variables is indeed compact.

Introducing the primed time derivative
\[ ' = \frac{1}{D} \frac{d}{dt}, \]  
one obtains the system of first-order differential equations
\[ D' = -Z D [1 + (q - 2\Omega_B)Z^2], \]  
\[ Z' = -Z^2[q - (q - 2\Omega_B)Z^2], \]  
\[ \tilde{\Omega}_k' = 2\tilde{\Omega}_k(q - 2\Omega_B)Z^3, \]  
\[ \tilde{\Omega}_A' = 2\tilde{\Omega}_A[-1 + (q - 2\Omega_B)Z^2]Z, \]  
\[ \tilde{\Omega}_C' = 2\tilde{\Omega}_C[-3 + (q - 2\Omega_B)Z^2]Z, \]  
where
\[ 1 + (q - 2)Z^2 = 4\tilde{\Omega}_C - \tilde{\Omega}_k. \]  
The evolution equation for \( D \) is not coupled to the rest, so we will not consider it for the dynamical study. Therefore, we just study the dynamical system for the variables \( \tilde{\Omega} = (Z, \tilde{\Omega}_k, \tilde{\Omega}_A, \tilde{\Omega}_C) \), determined by the equations (24).

The behavior of this system of equations in the neighborhood of its stationary point is determined by the corresponding matrix of its linearization. The real parts of its eigenvalues tell us whether the corresponding cosmological solution is stable or unstable with respect to the homogeneous perturbations [20]. To begin with, we have to find the critical points of this dynamical system, which can be written in vector form as follows
\[ \Omega' = f(\Omega), \]  
where \( f \) can be extracted from(24). The critical points, \( \Omega^* \), namely the points at which the system will stay provided it is initially at there, are given by the condition
\[ f(\Omega^*) = 0. \]  
Their dynamical character is determined by the eigenvalues of the matrix
\[ \left. \frac{\partial f}{\partial \Omega} \right|_{\Omega=\Omega^*}. \]
If the real part of the eigenvalues of a critical point is not zero, the point is said to be hyperbolic [18]. In this case, the dynamical character of the critical point is determined by the sign of the real part of the eigenvalues: If all of them are positive, the point is said to be a repeller, because arbitrarily small deviations from this point will move the system away from this state. If all of them are negative the point is called an attractor because if we move the system slightly from this point in an arbitrary way, it will return to it. Otherwise, we say the critical point is a saddle point.

We construct our models as follows:

(1) The model $k$, or $(Z, \tilde{\Omega}_k, \tilde{\Omega}_A, \tilde{\Omega}_C) = (\epsilon, 1, 0, 0)$, where $\epsilon \equiv \text{sgn}(Z)$. We have

$$q = 2 - \frac{2}{Z^2} = 0,$$

and the eigenvalues are

$$(\lambda_Z, \lambda_k, \lambda_A, \lambda_C) = (0, 0, -4Z, -12Z) = -\epsilon(0, 0, 4, 12)$$

(2) The model $A$, or $(Z, \tilde{\Omega}_k, \tilde{\Omega}_A, \tilde{\Omega}_C) = (\epsilon, 0, 1, 0)$. We have

$$q = 2 - \frac{1}{Z^2} = 1,$$

and the eigenvalues are

$$(\lambda_Z, \lambda_k, \lambda_A, \lambda_C) = (2Z, 2Z, 0, -4Z) = \epsilon(2, 2, 0, -4)$$

(3) The model $C$, or $(Z, \tilde{\Omega}_k, \tilde{\Omega}_A, \tilde{\Omega}_C) = (\epsilon, 0, 0, 1)$. We have

$$q = 2 + \frac{3}{Z^2} = 5,$$

and the eigenvalues are

$$(\lambda_Z, \lambda_k, \lambda_A, \lambda_C) = (10Z, 10Z, 8Z, 4Z) = \epsilon(10, 10, 8, 4)$$

(4) The model $O$, or

$$(Z, \tilde{\Omega}_k, \tilde{\Omega}_A, \tilde{\Omega}_C) = (0, \tilde{\Omega}_k^*, \tilde{\Omega}_A^*, \tilde{\Omega}_C^*)$$

where $\tilde{\Omega}_k^*, \tilde{\Omega}_A^*$ and $\tilde{\Omega}_C^*$ are constants satisfying (22) and (27). The eigenvalues of these points can be obtained in a straightforward manner and show a saddle point, which has not been included here. Hence, $O$ represents a set of infinite saddle points whose line element is that of an open universe ($k = -1$) with $H = 0$.

The dynamical system (24) has three hyperbolic critical points as follows:

i) The model $k$ ($k = -1$),

$$A = B = C = 0, \ a(t) = t,$$

with the critical point of an attractor type.

ii) The model $A$,

$$k = B = C = 0, \ a(t) = (A)^{1/4} \sqrt{2t},$$
with the critical point of a saddle point type. 

iii) The model $C$, 

$$k = A = B = 0, \quad a(t) = (4\sqrt{C}t)^{1/4},$$

with the critical point of a replicer type.

iv) The model $O$, 

$$Z = 0, \tilde{\Omega}^*_k, \tilde{\Omega}^*_A, \tilde{\Omega}^*_C,$$

with the critical point of a saddle point type.

The four models have been depicted within the compact state space, in Fig. (1). There are just trajectories on the planes, which are invariant submanifolds of the state space.

4 Conclusion

In this work, we have studied bouncing cosmological solutions and their stability with respect to homogeneous and isotropic perturbations in a braneworld theory in which the bulk is given by a five-dimensional charged AdS black hole spacetime with matter field confined in a D3 brane. The effects of the charge of five-dimensional black hole in the bulk has been considered. By including this quantity in the analysis we have obtained four models with the critical points of an attractor, a couple saddle point and a replicer, respectively, and constructed the complete state space for these cosmological models.

Acknowledgment

This work has been supported by Research Institute for Astronomy and Astrophysics of Maragha.

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Figure Captions

Figure 1. State space for the bouncing braneworld models. The points $k_+, A_+$ and $C_+$ describe the critical points of an attractor, a saddle point and a repeller, respectively. The points $k_-, A_-$ and $C_-$ describe the critical points of a repeller, a saddle point and an attractor, respectively. The points $O$ represent a set of infinite saddle points. Only trajectories on the invariant planes, $(\tilde{\Omega}_k = 0, \tilde{\Omega}_A = 0,$ and $\tilde{\Omega}_C = 0)$ which outline the whole dynamics, are drawn.
This figure "bouncing.jpg" is available in "jpg" format from:

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