Stability of Stochastic Approximations with ‘Controlled Markov’ Noise and Temporal Difference Learning

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Abstract

In this paper we present a ‘stability theorem’ for stochastic approximation (SA) algorithms with ‘controlled Markov’ noise. Such algorithms were first studied by Borkar in 2006. Specifically, sufficient conditions are presented which guarantee the stability of the iterates. Further, under these conditions the iterates are shown to track a solution to the differential inclusion defined in terms of the ergodic occupation measures associated with the ‘controlled Markov’ process. As an application to our main result we present an improvement to a general form of temporal difference learning algorithms. Specifically, we present sufficient conditions for their stability and convergence using our framework. This paper builds on the works of Borkar and Benveniste, Metivier and Priouret.

1 Introduction

Let us begin by considering the general form of stochastic approximation algorithms:

\[ x_{n+1} = x_n + a(n) (h(x_n) + M_{n+1}), \]

where

(i) \( h : \mathbb{R}^d \to \mathbb{R}^d \) is a Lipschitz continuous function;
(ii) \( \{a(n)\}_{n \geq 0} \) is the given step-size sequence such that \( \sum_{n=0}^{\infty} a(n) = \infty \) and \( \sum_{n=0}^{\infty} a(n)^2 < \infty \);
(iii) \( \{M_n\}_{n \geq 1} \) is the sequence of square integrable martingale difference terms.

In 1996, Benaim [3] showed that the asymptotic behavior of recursion (1) can be determined by studying the asymptotic behavior of the associated o.d.e.

\[ \dot{x}(t) = h(x(t)). \]
This technique is popularly known as the ODE method and was originally developed by Ljung in 1977 [9]. In [3] it is assumed that sup \( \| x_n \| < \infty \) a.s., in other words the iterates are assumed to be stable. In many cases the stability assumption becomes a bottleneck in using the ODE method. This bottleneck was overcome by Borkar and Meyn in 1999 [8]. Specifically, they developed sufficient conditions that guarantee the ‘stability and convergence’ of recursion (1).

In many applications, the noise-process is Markovian in nature. Stochastic approximation algorithms with ‘Markov Noise’ have been extensively studied in Benveniste et. al. [5]. These results have been extended to the case when the noise is ‘controlled Markov’ by Borkar [6]. Specifically, the asymptotics of the iterates are described via a limiting differential inclusion (DI) that is defined in terms of the ergodic occupation measures of the Markov process. As explained in [6], the motivation for such a study stems from the fact that in many cases the noise-process is not Markov, but its lack of Markov property comes through its dependence on a time-varying ‘control’ process. In particular this is the case with many reinforcement learning algorithms. In [6], the iterates are assumed to be stable, which as explained earlier poses a bottleneck, especially in analyzing algorithms from reinforcement learning. The aim of this paper is to overcome this bottleneck. In other words, we present sufficient conditions for the ‘stability and convergence’ of stochastic approximation algorithms with ‘controlled Markov’ noise. Finally, as an application setting, we consider a general form of the temporal difference learning algorithms in reinforcement learning and present weaker sufficient conditions (than those in literature) that guarantee their stability and convergence using our framework.

The organization of this paper is as follows:

In Section 2.1 we present the definitions and notations involved in this paper. In Section 2.2 we discuss the assumptions involved in proving the stability of the iterates given by (3).

In Section 3 we show the stability of the iterates under the assumptions outlined in Section 2.2 (Theorem 1).

In Section 4 we present additional assumptions which coupled with assumptions from Section 2.2 are used to prove the ‘stability and convergence’ of recursion (3) (Theorem 2). Specifically, Theorem 2 states that under the aforementioned sets of assumptions the iterates are stable and converge to an internally chain transitive invariant set associated with \( \dot{x}(t) \in h(x(t)) \). For the definition of \( h \) the reader is referred to Section 4.

In Section 5 we discuss an application of Theorem 2. We present sufficient conditions for the ‘stability and convergence’ of a general form of temporal difference learning algorithms, in reinforcement learning.
2 Preliminaries and Assumptions

2.1 Notations & Definitions

In this section we present the definitions and notations used in this paper for the purpose of easy reference. Note that they can be found in Benaïm et. al. [4], Aubin et. al. [1], [2] and Borkar [7].

Marchaud Map: A set-valued map $h : \mathbb{R}^n \to \{\text{subsets of } \mathbb{R}^m\}$ is called a Marchaud map if it satisfies the following properties:

(i) For each $x \in \mathbb{R}^n$, $h(x)$ is convex and compact.

(ii) (point-wise boundedness) For each $x \in \mathbb{R}^n$, $\sup_{w \in h(x)} \|w\| < K (1 + \|x\|)$ for some $K > 0$.

(iii) $h$ is an upper-semicontinuous map.

We say that $h$ is upper-semicontinuous, if given sequences $\{x_n\}_{n \geq 1}$ (in $\mathbb{R}^n$) and $\{y_n\}_{n \geq 1}$ (in $\mathbb{R}^m$) with $x_n \to x$, $y_n \to y$ and $y_n \in h(x_n)$, $n \geq 1$, $y \in h(x)$. In other words, the graph of $h$, $\{(x, y) : y \in h(x), x \in \mathbb{R}^n\}$, is closed in $\mathbb{R}^n \times \mathbb{R}^m$.

If the set-valued map $H : \mathbb{R}^d \to \{\text{subsets of } \mathbb{R}^d\}$ is Marchaud, then the differential inclusion (DI) given by

$$\dot{x}(t) \in H(x(t))$$

(2)

is guaranteed to have at least one solution that is absolutely continuous. The reader is referred to Aubin & Cellina [1] for more details.

If $x$ is an absolutely continuous map satisfying (2) then we say that $x \in \Sigma$.

A set-valued semiflow $\Phi$ associated with (2) is defined on $[0, +\infty) \times \mathbb{R}^d$ as follows:

$$\Phi_t(x) := \{x(t) : x \in \Sigma, x(0) = x\}.$$  

Let $B \times M \subset [0, +\infty) \times \mathbb{R}^d$, define

$$\Phi_B(M) := \bigcup_{t \in B} \Phi_t(x).$$

Let $M \subset \mathbb{R}^d$, the $\omega$-limit set be defined by $\omega_\Phi(M) := \bigcap_{t \geq 0} \Phi_{[t, +\infty)}(M)$. Similarly the limit set of a solution $x$ is given by $L(x) = \bigcap_{t \geq 0} x([t, +\infty))$.

Invariant Set: $M \subset \mathbb{R}^d$ is invariant if for every $x \in M$ there exists a trajectory, $x$, entirely in $M$ with $x(0) = x$. In other words, $x \in \Sigma$ with $x(t) \in M$, for all $t \geq 0$.

Internally Chain Transitive Set: $M \subset \mathbb{R}^d$ is said to be internally chain transitive if $M$ is compact and for every $x, y \in M$, $\epsilon > 0$ and $T > 0$ we have the following: There exist $\Phi_1, \ldots, \Phi_n$ that are $n$ solutions to the differential inclusion $\dot{x}(t) \in h(x(t))$, a sequence $x_1 (= x), \ldots, x_{n+1} (= y) \subset M$ and $n$ real numbers $t_1, t_2, \ldots, t_n$ greater than $T$ such that: $\Phi_{t_i}(x_i) \in N^\epsilon(x_{i+1})$ and $\Phi_{[t_i, t_{i+1}]}(x_i) \subset M$ for $1 \leq i \leq n$. The sequence $(x_1 (= x), \ldots, x_{n+1} (= y))$ is called an $(\epsilon, T)$ chain in $M$ from $x$ to $y$.

Given $x \in \mathbb{R}^d$ and $A \subset \mathbb{R}^d$, define the distance between $x$ and $A$ by $d(x, A) := \inf \{\|a - y\| : y \in A\}$. We define the $\delta$-open neighborhood of $A$ by $N^\delta(A) :=$
\{x \mid d(x, A) < \delta\}$. The $\delta$-closed neighborhood of $A$ is defined by $\overline{N^\delta(A)} := \{x \mid d(x, A) \leq \delta\}$.

**Attracting Set:** $A \subseteq \mathbb{R}^d$ is an attracting set if it is compact and there exists a neighborhood $U$ such that for any $\epsilon > 0$ there exists $T(\epsilon) \geq 0$ such that $\Phi_{T(\epsilon), +\infty}(U) \subseteq N^\epsilon(A)$. Then $U$ is called the fundamental neighborhood of $A$. In addition to being compact if the attracting set is also invariant then it is called an attractor. The basin of attraction of $A$ is given by $B(A) = \{x \mid \omega_\Phi(x) \subset A\}$. The set $A$ is Lyapunov stable if for all $\delta > 0$, $\exists \epsilon > 0$ such that $\Phi_{0, +\infty}(N^\delta(A)) \subseteq N^\epsilon(A)$. We use $T(\epsilon)$ and $T_\epsilon$ interchangeably to denote the dependence of $T$ on $\epsilon$.

The open ball of radius $r$ around $0$ is represented by $B_r(0)$, while the closed ball is represented by $\overline{B_r}(0)$.

**Upper limit of sequences of sets:** Let $\{K_n\}_{n \geq 1}$ be a sequence of sets in $\mathbb{R}^d$. The upper-limit of $\{K_n\}_{n \geq 1}$ is given by $\operatorname{limsup}_{n \to \infty} K_n := \{y \mid \lim_{n \to \infty} d(y, K_n) = 0\}$.

### 2.2 Assumptions

Let us consider a stochastic approximation algorithm with ‘controlled Markov’ noise in $\mathbb{R}^d$.

\[ x_{n+1} = x_n + a(n) [h(x_n, y_n) + M_{n+1}], \quad \text{where} \quad (3) \]

(i) $h : \mathbb{R}^d \times S \to \mathbb{R}^d$ is a jointly continuous map with $S$ a compact metric space. The map $h$ is Lipschitz continuous in the first component, further its constant does not change with the second component. Let the Lipschitz constant be $L$. This is assumption (1) in Section 2 of Borkar [6]. Here we call it (A1).

(ii) The step-size sequence $\{a(n)\}_{n \geq 0}$ is such that $a(n) > 0$ for all $n \geq 0$, $\sum_{n=0}^\infty a(n) = \infty$ and $\sum_{n=0}^\infty a(n)^2 < \infty$. Without loss of generality let $\sup_{n \geq 0} a(n) \leq 1$. This is assumption (3) in Section 2 of Borkar [6]. Here we call it (A3).

(iii) $\{M_n\}_{n \geq 1}$ is a sequence of square integrable martingale difference terms, that also contribute to the noise. They are related to $\{x_n\}_{n \geq 0}$ by

\[ E \left[ \|M_{n+1}\|^2 \mid \mathcal{F}_n \right] \leq K(1 + \|x_n\|^2), \quad \text{where} \quad n \geq 0. \]

This is assumption (2) in Section 2 of Borkar [6]. Here we call it (A2).

(iv) $\{y_n\}_{n \geq 0}$ is the $S$-valued ‘Controlled Markov’ process.

Note that $S$ is assumed to be polish in [6]. As stated in (A1), in this paper we let $S$ be a compact metric space, hence polish. Among the assumptions made in [6], (A1) – (A3) are relevant to prove the stability of the iterates. The remaining assumptions are listed in Section 4 where we present the result on the ‘stability and convergence’ of the iterates given by [9]. See Borkar [6] for more details.

For each $c \geq 1$, we define functions $h_c : \mathbb{R}^d \times S \to \mathbb{R}^d$ by $h_c(x, y) := h(cx, y)/c$. 

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We define the limiting map \( h_\infty : \mathbb{R}^d \times S \to \{ \text{subsets of } \mathbb{R}^d \} \) by \( h_\infty(x, y) := \text{Limsup}_{t \to \infty} \{ h_c(x, y) \} \), where Limsup is the upper-limit of a sequence of sets (see Section 2.1).

For each \( x \in \mathbb{R}^d \) define \( H(x) := \overline{\text{conv}} \left( \bigcup_{y \in S} h_\infty(x, y) \right) \).

We replace the stability assumption \( \left( \sup_{n \geq 0} \| x_n \| < \infty \right) \) in \([6]\) with the following two assumptions.

(S1) If \( c_n \uparrow \infty, y_n \to y \) and \( \lim_{n \to \infty} h_{c_n}(x, y_n) = u \) for some \( u \in \mathbb{R}^d \), then \( u \in h_\infty(x, y) \).

(S2) There exists an attracting set, \( A \), associated with \( \dot{x}(t) \in H(x(t)) \) such that \( \sup_{u \in A} \| u \| < 1 \). Further, \( \overline{\mathcal{B}}_1(0) \) is a subset of some fundamental neighborhood of \( A \).

Assumption (T2), discussed in Section \([5]\) is a sufficient condition for (S2) to be satisfied. One could say that (T2) constitutes the ‘Lyapunov function’ condition for DI. We shall show that \( H \) is a Marchaud map in Lemma \([2]\). As explained in \([1]\), it follows that the DI, \( \dot{x}(t) \in H(x(t)) \), has at least one solution that is absolutely continuous. Hence assumption (S2) is meaningful.

We begin by showing that \( h_c \) satisfies (A1) for all \( c \geq 1 \). Fix \( x_1, x_2 \in \mathbb{R}^d \), \( y \in S \) and \( c \geq 1 \), we have

\[
\| h_c(x_1, y) - h_c(x_2, y) \| = \| h(cx_1, y)/c - h(cx_2, y)/c \|,
\]

\[
\left\| h(cx_1, y)/c - h(cx_2, y)/c \right\| \leq L\| cx_1 - cx_2 \|/c, \text{ hence}
\]

\[
\| h_c(x_1, y) - h_c(x_2, y) \| \leq L\| x_1 - x_2 \|.
\]

We thus have that \( h_c \) is Lipschitz continuous in the first component with Lipschitz constant \( L \). Further, for a fixed \( c \) this constant does not change with \( y \). Since \( c \) was arbitrarily chosen it follows that \( L \) is the Lipschitz constant associated with every \( h_c \). It is trivially true that \( h_c \) is a jointly continuous map.

Fix \( c \geq 1, x \in \mathbb{R}^d \) and \( y \in S \), then

\[
\| h_c(x, y) - h_c(0, y) \| \leq L\| x - 0 \|, \text{ hence}
\]

\[
\| h_c(x, y) \| \leq \| h_c(0, y) \| + L\| x \|.
\]

Since \( h(0, \cdot) \) is a continuous function on \( S \) (a compact set) and \( c \geq 1 \) we have \( \| h_c(0, \cdot) \|_\infty \leq \| h(0, \cdot) \|_\infty \leq M \) for some \( 0 < M < \infty \). Thus

\[
\| h_c(x, y) \| \leq K \left( 1 + \| x \| \right), \text{ where } K = L \lor M.
\]

We may assume without loss of generality that \( K \) is such that \( E \left[ \| M_{n+1} \|^2 \mid \mathcal{F}_n \right] \leq K \left( 1 + \| x_n \|^2 \right) \) also holds for all \( n \geq 0 \) (assumption (A2)). Again \( K \) does not change with \( c \).
Fix $x \in \mathbb{R}^d$ and $y \in S$. As explained in the previous paragraph we have,

$$\sup_{c \geq 1} \|h_c(x, y)\| \leq K(1 + \|x\|).$$

The upper-limit of $\{h_c(x, y)\}_{c \geq 1}$, $\text{Limsup}_{c \to \infty} \{h_c(x, y)\}$, is clearly non-empty. Recall that $h_\infty(x, y) = \text{Limsup}_{c \to \infty} \{h_c(x, y)\}$ and $H(x) = \overline{\text{co}} \left( \bigcup_{y \in S} h_\infty(x, y) \right)$. Hence,

$$\sup_{u \in h_\infty(x, y)} \|u\| \leq K(1 + \|x\|)$$

and

$$\sup_{u \in H(x)} \|u\| \leq K(1 + \|x\|).$$

We need to show that $H$ is a Marchaud map. Before we do that, let us prove an auxiliary result.

**Lemma 1.** Suppose $x_n \to x$ in $\mathbb{R}^d$, $y_n \to y$ in $S$, $c_n \uparrow \infty$ and $\lim_{c_n \to \infty} h_{c_n}(x_n, y_n) = u$. Then $u \in h_\infty(x, y)$.

**Proof.** Consider the following inequality:

$$\|h_{c_n}(x_n, y_n) - u\| \leq \|h_{c_n}(x_n, y_n) - h_{c_n}(x, y_n)\| + \|h_{c_n}(x_n, y_n) - h_{c_n}(x_n, y_n)\|.$$

Since $\|h_{c_n}(x_n, y_n) - h_{c_n}(x_n, y_n)\| \leq L\|x_n - x\|$ and $\lim_{c_n \to \infty} h_{c_n}(x_n, y_n) = u$, we get

$$\lim_{c_n \to \infty} h_{c_n}(x_n, y_n) = u.$$

It follows from (S1) that $u \in h_\infty(x, y)$.

The following is a direct consequence of Lemma 1: If $x_n \to x$ in $\mathbb{R}^d$, $\{y_n\} \subset S$ and $c_n \to \infty$ then $d(h_{c_n}(x_n, y_n), H(x)) \to 0$. If this is not so, then without loss of generality we have that $d(h_{c_n}(x_n, y_n), H(x)) > \epsilon$ for some $\epsilon > 0$. Since $S$ is compact, there exists $\{m(n)\} \subset \{n\}$ such that $\lim_{m(n) \to \infty} y_{m(n)} = y$ and $h_{c_{m(n)}}(x_{m(n)}, y_{m(n)}) \to u$ for some $y \in S$ and some $u \in \mathbb{R}^d$. We have $x_{m(n)} \to x$, $y_{m(n)} \to y$, $c_{m(n)} \to \infty$ and $h_{c_{m(n)}}(x_{m(n)}, y_{m(n)}) \to u$. It follows from Lemma 1 that $u \in h_\infty(x, y) \subseteq H(x)$. This is a contradiction.

**Lemma 2.** $H$ is a Marchaud map.

**Proof.** Recall that $H(x) = \overline{\text{co}} \left( \bigcup_{y \in S} h_\infty(x, y) \right)$. As explained earlier (cf. Lemma 1),

$$\sup_{u \in H(x)} \|u\| \leq K(1 + \|x\|).$$

Hence $H$ is point-wise bounded. From the definition of $H$ it follows that $H(x)$ is convex and compact for each $x \in \mathbb{R}^d$. 

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It is left to show that $H$ is upper semi-continuous. Let $x_n \to x$, $u_n \to u$ and $u_n \in H(x_n)$, $n \geq 1$. We need to show that $u \in H(x)$. If this is not true, then there exists a linear functional on $\mathbb{R}^d$, say $f$, such that $\sup_{v \in H(x)} f(v) \leq \alpha - \epsilon$ and $f(u_0) \geq \alpha + \epsilon$, for some $\alpha \in \mathbb{R}$ and $\epsilon > 0$. Since $u_n \to u$, there exists $N$ such that for each $n \geq N$ $f(u_n) \geq \alpha + \frac{\epsilon}{2}$, i.e., $H(x_n) \cap \{f \geq \alpha + \frac{\epsilon}{2}\} \neq \emptyset$. Also, let $\hat{c}$

Let us construct the linear interpolated trajectory $\mathfrak{t}(t)$ for $t \in [0, \infty)$ from the sequence $\{x_n\}_{n \geq 0}$. Define $\mathfrak{t}(0) := 0$ and $\mathfrak{t}(n) := \sum_{i=0}^{n-1} a(i)$, $\forall n \geq 1$. Let $\mathfrak{t}(t(n)) := x_n$ and for $t \in (t(n), t(n+1))$ let

$$\mathfrak{t}(t) := \left(\frac{t(n+1) - t}{t(n+1) - t(n)}\right) \mathfrak{t}(t(n)) + \left(\frac{t - t(n)}{t(n+1) - t(n)}\right) \mathfrak{t}(t(n+1)).$$

Define $T_0 := 0$ and $T_n := \min\{t(m) : t(m) \geq T_{n-1} + T\}$ for $n \geq 1$. Observe that there exists a subsequence, $\{m(n)\}$, of $\mathbb{N}$ such that $T_n = t(m(n))$ for all $n \geq 0$.

We use $\mathfrak{t}(\cdot)$ to construct the rescaled trajectory, $\hat{x}(t)$, for $t \geq 0$. Let $t \in [T_n, T_{n+1})$ for some $n \geq 0$ and define $\hat{x}(t) := \frac{\mathfrak{t}(t)}{r(n)}$, where $r(n) = ||\mathfrak{t}(T_n)|| \vee 1$. Also, let $\hat{x}(T_{n+1}) := \lim_{t \uparrow T_{n+1}} \hat{x}(t)$, $t \in [T_n, T_{n+1})$. The rescaled martingale difference terms are given by $\hat{M}_{k+1} := \frac{M_{k+1}}{r(n)}$, $t(k) \in [T_n, T_{n+1})$. 

3 Stability Theorem

Let us construct the linear interpolated trajectory $\mathfrak{t}(t)$ for $t \in [0, \infty)$ from the sequence $\{x_n\}_{n \geq 0}$. Define $\mathfrak{t}(0) := 0$ and $\mathfrak{t}(n) := \sum_{i=0}^{n-1} a(i)$, $\forall n \geq 1$. Let $\mathfrak{t}(t(n)) := x_n$ and for $t \in (t(n), t(n+1))$ let

$$\mathfrak{t}(t) := \left(\frac{t(n+1) - t}{t(n+1) - t(n)}\right) \mathfrak{t}(t(n)) + \left(\frac{t - t(n)}{t(n+1) - t(n)}\right) \mathfrak{t}(t(n+1)).$$

Define $T_0 := 0$ and $T_n := \min\{t(m) : t(m) \geq T_{n-1} + T\}$ for $n \geq 1$. Observe that there exists a subsequence, $\{m(n)\}$, of $\mathbb{N}$ such that $T_n = t(m(n))$ for all $n \geq 0$.

We use $\mathfrak{t}(\cdot)$ to construct the rescaled trajectory, $\hat{x}(t)$, for $t \geq 0$. Let $t \in [T_n, T_{n+1})$ for some $n \geq 0$ and define $\hat{x}(t) := \frac{\mathfrak{t}(t)}{r(n)}$, where $r(n) = ||\mathfrak{t}(T_n)|| \vee 1$. Also, let $\hat{x}(T_{n+1}) := \lim_{t \uparrow T_{n+1}} \hat{x}(t)$, $t \in [T_n, T_{n+1})$. The rescaled martingale difference terms are given by $\hat{M}_{k+1} := \frac{M_{k+1}}{r(n)}$, $t(k) \in [T_n, T_{n+1})$. 

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We define a piece-wise constant trajectory, \(\hat{z}(\cdot)\), using the rescaled trajectory as follows: Let \(t \in [t(m), t(m + 1))\) and \(T_n \leq t(m) < t(m + 1) \leq T_{n+1}\). Define \(\hat{z}(t) := h_{r(n)}(\hat{x}(t(m)), y_n)\). Let us define another piece-wise constant trajectory using \(\{y_n\}_{n \geq 0}\) as follows: Let \(\mathbf{y}(t) := y_n\) for all \(t \in [t(n), t(n+1))\).

Recall that \(A\) is an attracting set associated with \(\hat{x}(t) \in H(x(t))\) (see assumption (S2) in section 2.2). Let \(\delta_1 := \sup_{u \in A} \|u\|\), then \(\delta_1 < 1\). Choose \(\delta_2, \delta_3,\) and \(\delta_4\) such that \(\delta_1 < \delta_2 < \delta_3 < \delta_4 < 1\). Fix \(T := T(\delta_2 - \delta_1)\), where \(T(\cdot)\) is defined in section 2.2. Let \(x(\cdot)\) be a solution to \(\hat{x}(t) \in H(x(t))\) such that \(\|x(0)\| \leq 1\), then \(\|x(t)\| < \delta_2\) for all \(t \geq T(\delta_2 - \delta_1)\).

Consider the following recursion:
\[
\mathbf{z}(t(k + 1)) = \mathbf{z}(t(k)) + a(k) (h(\mathbf{z}(t(k)), y_k) + M_{k+1}),
\]
such that \(t(k), t(k + 1) \in [T_n, T_{n+1})\). Multiplying both sides by \(1/r(n)\), we get the following rescaled recursion:
\[
\hat{x}(t(k + 1)) = \hat{x}(t(k)) + a(k) \left( h_{r(n)}(\hat{x}(t(k)), y_k) + \hat{M}_{k+1} \right).
\] (5)
Note that \(E \left[ \|\hat{M}_{k+1}\|^2 | F_k \right] \leq K \left( 1 + \|\hat{x}(t(k))\|^2 \right)\).

The following two lemmas can be found in Borkar & Meyn [8] (that however does not consider ‘controlled Markov’ noise). It is shown there that the ‘martingale noise’ sequence converges almost surely. We present the results below using our setting.

**Lemma 3.** \(\sup_{t \geq 0} E \|\hat{x}(t)\|^2 < \infty\).

**Proof.** Recall that \(T_n = t(m(n))\) and \(T_{n+1} = t(m(n + 1))\). It is enough to show that
\[
\sup_{m(n) < k \leq m(n + 1)} E \left[ \|\hat{x}(t(k))\|^2 \right] \leq M,
\]
for some \(M(>0)\) that is independent of \(n\). Let us fix \(n\) and \(k\) such that \(n \geq 0\) and \(m(n) < k < m(n + 1)\). Consider the following rescaled recursion:
\[
\hat{x}(t(k)) = \hat{x}(t(k - 1)) + a(k - 1) \left( \hat{x}(t(k - 1)) + \hat{M}_k \right).
\]
Unfolding the above we get,
\[
\hat{x}(t(k)) = \hat{x}(t(m(n))) + \sum_{l=m(n)}^{k-1} a(l) \left( \hat{x}(t(l)) + \hat{M}_{l+1} \right).
\]
Taking expectation of the square of the norms on both sides we get,
\[
E\|\hat{x}(t(k))\|^2 = E \left\| \hat{x}(t(m(n))) + \sum_{l=m(n)}^{k-1} a(l) \left( \hat{x}(t(l)) + \hat{M}_{l+1} \right) \right\|^2.
\]

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Applying the discrete version of the Minkowski inequality that,

\[ E^{1/2} \| \hat{x}(t(k)) \|^2 \leq E^{1/2} \| \hat{x}(T_n) \|^2 + \sum_{l=m(n)}^{k-1} a(l) \left( E^{1/2} \| \hat{x}(t(l)) \|^2 + E^{1/2} \| \hat{M}_{t+1} \|^2 \right). \]

For each \( l \) such that \( m(n) \leq l \leq k - 1 \), \( \| \hat{x}(t(l)) \| = \| h_{x_{[0]}}(\hat{x}(t(l)), \hat{y}(t(l))) \| \leq K \left( 1 + \| \hat{x}(t(l)) \| \right) \). Further, \( E \left[ \| \hat{M}_{t+1} \|^2 | \mathcal{F}_l \right] \leq K \left( 1 + \| \hat{x}(t(l)) \|^2 \right) \). Observe that \( T_{n+1} - T_n \leq T + 1 \) (since \( \sup_n a(n) \leq 1 \)). Using these observations we get the following:

\[ E^{1/2} \| \hat{x}(t(k)) \|^2 \leq 1 + \sum_{l=m(n)}^{k-1} a(l) \left( KE^{1/2} \left( 1 + \| \hat{x}(t(l)) \| \right)^2 \right) + \sqrt{K} E^{1/2} \left( 1 + \| \hat{x}(t(l)) \|^2 \right), \]

\[ E^{1/2} \| \hat{x}(t(k)) \|^2 \leq 1 + \sum_{l=m(n)}^{k-1} a(l) \left( K \left( 1 + E^{1/2} \| \hat{x}(t(l)) \|^2 \right) \right) + \sqrt{K} \left( 1 + E^{1/2} \| \hat{x}(t(l)) \|^2 \right), \]

\[ E^{1/2} \| \hat{x}(t(k)) \|^2 \leq \left[ 1 + (K + \sqrt{K})(T + 1) \right] + (K + \sqrt{K}) \sum_{l=m(n)}^{k-1} a(l) E^{1/2} \| \hat{x}(t(l)) \|^2. \]

Applying the discrete version of the Gronwall inequality we now get,

\[ E^{1/2} \| \hat{x}(t(k)) \|^2 \leq \left[ 1 + (K + \sqrt{K})(T + 1) \right] e^{(K + \sqrt{K})(T + 1)}. \]

Let us define \( M := \left( 1 + (K + \sqrt{K})(T + 1) \right) e^{(K + \sqrt{K})(T + 1)} \). Clearly \( M \) is independent of \( n \) and the claim follows.

**Lemma 4.** The sequence \( \hat{\zeta}_n, n \geq 0 \), converges almost surely, where \( \hat{\zeta}_n := \sum_{k=0}^{n-1} a(k) \hat{M}_{k+1} \) for all \( n \geq 1 \).

**Proof.** It is enough to prove that

\[ \sum_{k=0}^{\infty} E \left[ \| a(k) \hat{M}_{k+1} \|^2 | \mathcal{F}_k \right] < \infty \text{ a.s.} \]

Instead, we prove that

\[ E \left[ \sum_{k=0}^{\infty} a(k)^2 E \left[ \| \hat{M}_{k+1} \|^2 | \mathcal{F}_k \right] \right] < \infty. \]

From assumption (A2) we get

\[ E \left[ \sum_{k=0}^{\infty} a(k)^2 E \left[ \| \hat{M}_{k+1} \|^2 | \mathcal{F}_k \right] \right] \leq \sum_{k=0}^{\infty} a(k)^2 K \left( 1 + E \| \hat{x}(t(k)) \|^2 \right). \]

The claim now follows from Lemma 4 and (A3).
Thus, let \( x^n(t), \ t \in [0, T], \) be the solution (up to time \( T \)) to \( \dot{x}^n(t) = \dot{z}(T_n + t) \) with initial condition \( x^n(0) = \hat{x}(T_n) \). Clearly,

\[
x^n(t) = \hat{x}(T_n) + \int_0^t \dot{z}(T_n + s) \, ds.
\]

\( \text{Lemma 5.} \quad \lim_{n \to \infty} \sup_{t \in [T_n, T_n + T]} \| x^n(t) - \dot{x}(t) \| = 0 \ a.s. \)

\[
\text{Proof. Let } t \in [t(m(n) + k), t(m(n) + k + 1)) \text{ such that } T_n \leq t(m(n) + k) < t(m(n) + k + 1) \leq T_{n+1}, \text{ where } n \geq 0. \text{ First we prove the lemma when } t(m(n) + k + 1) < T_{n+1}. \text{ Consider the following:}
\]

\[
\dot{z}(t) = \left( \frac{t(m(n) + k + 1) - t}{a(m(n) + k)} \right) \dot{x}(t(m(n) + k)) + \left( \frac{t - t(m(n) + k)}{a(m(n) + k)} \right) \dot{x}(t(m(n) + k + 1)).
\]

Substituting for \( \dot{x}(t(m(n) + k)) \) in the above equation we get:

\[
\dot{z}(t) = \left( \frac{t(m(n) + k + 1) - t}{a(m(n) + k)} \right) \dot{x}(t(m(n) + k)) + \left( \frac{t - t(m(n) + k)}{a(m(n) + k)} \right) \left( \dot{x}(t(m(n) + k)) + a(m(n) + k) \left( h_{r(n)}(\dot{x}(t(m(n) + k)), y_{m(n) + k}) + M_{m(n) + k + 1} \right) \right),
\]

hence,

\[
\dot{z}(t) = \dot{x}(t(m(n) + k)) + (t - t(m(n) + k)) \left( h_{r(n)}(\dot{x}(t(m(n) + k)), y_{m(n) + k}) + M_{m(n) + k + 1} \right).
\]

Unfolding \( \dot{x}(t(m(n) + k)) \), we get (see (5)),

\[
\dot{z}(t) = \dot{x}(T_n) + \sum_{l=0}^{k-1} a(m(n)+l) \left( h_{r(n)}(\dot{x}(t(m(n) + l)), y_{m(n)+l}) + M_{m(n)+l+1} \right) + (t - t(m(n) + k)) \left( h_{r(n)}(\dot{x}(t(m(n) + k)), y_{m(n)+k}) + M_{m(n)+k+1} \right).
\]

Recall that

\[
x^n(t) = \dot{x}(T_n) + \int_0^t \dot{z}(T_n + s) \, ds.
\]

Splitting the above integral, we get

\[
x^n(t) = \dot{x}(T_n) + \sum_{l=0}^{k-1} \int_{t(m(n)+l)}^{t(m(n)+l+1)} \dot{z}(s) \, ds + \int_{t(m(n)+k)}^t \dot{z}(s) \, ds.
\]

Thus,

\[
x^n(t) = \dot{x}(T_n) + \sum_{l=0}^{k-1} a(m(n) + l) h_{r(n)}(\dot{x}(t(m(n) + l)), y_{m(n)+l}) + (t - t(m(n) + k)) h_{r(n)}(\dot{x}(t(m(n) + k)), y_{m(n)+k}) + M_{m(n)+k+1}.
\]
From (7) and (8), we get the following:

\[ \|x^n(t) - \hat{x}(t)\| \leq \left\| \sum_{l=0}^{k-1} a(m(n) + l) \hat{M}_{m(n) + l + 1} \right\| + \left\| (t - t(m(n) + k)) \hat{M}_{m(n) + k + 1} \right\|, \]

\[ \|x^n(t) - \hat{x}(t)\| \leq \left\| \hat{\zeta}_{m(n) + k} - \hat{\zeta}_{m(n)} \right\| + \left\| \hat{\zeta}_{m(n) + k + 1} - \hat{\zeta}_{m(n)} \right\|. \]

If \( t(m(n) + k + 1) = T_{n+1} \) then in the above set of equations we may replace \( \hat{x}(t(m(n) + k + 1)) \) with \( \hat{x}(T_{n+1}) \). The arguments remain the same. Since \( \hat{\zeta}_n \), \( n \geq 1 \), converges almost surely, the lemma follows.

Recall that \( T = T(\delta_2 - \delta_1) \). Let us view \( \{x^n([0, T]) \mid n \geq 0\} \) and \( \{x^n([T_n, T_n + T]) \mid n \geq 0\} \) as subsets of \( C([0, T], \mathbb{R}^d) \) (endowed with the sup-norm, \( \| \cdot \|_\infty \)). We claim that \( \{x^n([0, T]) \mid n \geq 0\} \) is equicontinuous and point-wise bounded almost surely. Since \( \|x^n(0)\| = \|\hat{x}(T_n)\| \leq 1 \), we can use Gronwall inequality to show that \( \sup_{n \geq 0} \|x^n(\cdot)\|_\infty < \infty \) almost surely. Note that \( \|x^n(\cdot)\| = \sup_{t \geq 0} \|x^n(t)\| \).

Hence we conclude that the aforementioned set is almost surely point-wise bounded.

Now we show that the family of functions is almost surely equicontinuous. Recall that \( \sup_{t \geq 0} E\|\hat{x}(t)\|^2 < \infty \) a.s. and \( \|\hat{\zeta}(t)\| \leq K(1 + \|\hat{x}(t)\|) \), where \( |t| := \max\{t(m) \mid t(m) \leq t\} \). Hence \( \sup_{t \geq 0} \|\hat{\zeta}(t)\| < \infty \) a.s. For \( \delta > 0 \), we have the following:

\[ \|x^n(t + \delta) - x^n(t)\| \leq \int_t^{t + \delta} \|\hat{\zeta}(s)\| \, ds. \]

Since \( \sup_{t \geq 0} \|\hat{\zeta}(t)\| < \infty \) a.s. it follows that

\[ \|x^n(t + \delta) - x^n(t)\| \leq \int_t^{t + \delta} M \, ds = M \delta, \]

where \( M \) is a constant (possibly sample path dependent) such that \( \sup_{t \geq 0} \|\hat{\zeta}(t)\| \leq M \).

Hence we conclude that \( \{x^n([0, T]) \mid n \geq 0\} \) is equicontinuous. It follows from Arzela-Ascoli Theorem that \( \{x^n([0, T]) \mid n \geq 0\} \) is relatively compact in \( C([0, T], \mathbb{R}^d) \). From Lemma 5 it follows that \( \{\hat{x}([T_n, T_n + T]) \mid n \geq 0\} \) is also relatively compact in \( C([0, T], \mathbb{R}^d) \).

Using Gronwall’s inequality we can show that \( \sup_{k \geq 0} \|x_k\| < \infty \) a.s. if and only if \( \sup_{n \geq 0} \|\hat{x}(T_n)\| < \infty \) a.s. To prove the stability of the iterates it is enough to show that \( \sup_{n \geq 0} r(n) < \infty \) a.s. given that the recursion satisfies (A1) – (A3), S1 and S2 (see Section 2). If \( \sup_{n \geq 0} r(n) = \infty \) then there exists \( \{l\} \subseteq \{n\} \) such that \( r(l) \uparrow \infty \). In the lemma that follows we characterize the limit set of \( \{\hat{x}([T_l, T_l + T]) \mid \{l\} \subseteq \{n\} \& r(l) \uparrow \infty \} \) in \( C([0, T], \mathbb{R}^d) \).
Lemma 6. Let \( \{l\} \subseteq \{n\} \) such that \( r(l) \uparrow \infty \). Any limit of \( \{\hat{x}(T_l, T_l + T) \mid \{l\} \subseteq \{n\} \& r(l) \uparrow \infty \} \) in \( C([0, T], \mathbb{R}^d) \) is of the form \( x(t) = x(0) + \int_0^t z(s) \, ds \), where \( x(0) \in \mathcal{F}_1(0) \) and \( z : [0, T] \to \mathbb{R}^d \) is a measurable function such that \( z(t) \in H(x(t)), \, t \in [0, T] \).

Proof. For \( t \geq 0 \) define \( [t] := \max\{t(m) \mid t(m) \leq t\} \). Fix \( t_0 \in [T_n, T_{n+1}) \).

\( \hat{z}(t_0) = h_{r(n)}(\hat{x}(t_0), \mathfrak{P}(t_0)) \).

Since \( \|h_{r(n)}(\hat{x}(t_0), \mathfrak{P}(t_0))\| \leq K(1 + \|\hat{x}(t_0)\|) \), we have \( \|\hat{z}(t_0)\| \leq K(1 + \|\hat{x}(t_0)\|) \). It follows from Lemma 3 that \( \|\hat{z}(t)\| < \infty \) a.s. Recall that \( \{\hat{x}(T_l^+ \cdot) \mid \{l\} \subseteq \{n\}\} \) is relatively compact in \( C([0, T], \mathbb{R}^d) \).

Without loss of generality we may assume that

\[ \hat{x}(T_l^+ \cdot) \to x(\cdot) \text{ in } C([0, T], \mathbb{R}^d), \quad \text{for some } x(\cdot) \in C([0, T], \mathbb{R}^d); \]

\[ \hat{z}(T_l^+ \cdot) \to z(\cdot) \text{ weakly in } L^2([0, T], \mathbb{R}^d), \quad \text{for some } z(\cdot) \in L^2([0, T], \mathbb{R}^d). \]

It follows from Lemma 3 that \( x^l(\cdot) \to x(\cdot) \) in \( C([0, T], \mathbb{R}^d) \). Letting \( r(l) \to \infty \) in the following equation,

\[ x^l(t) = x^l(0) + \int_0^t \hat{z}(T_l + s) \, ds, \quad \text{we get} \]

\[ x(t) = x(0) + \int_0^t z(s) \, ds. \]

Since \( \|x^l(0)\| = \|\hat{x}(T_l)\| \leq 1 \), we have that \( \|x(0)\| \leq 1 \). Further, since \( \hat{z}(T_l^+ \cdot) \to z(\cdot) \) weakly in \( L^2([0, T], \mathbb{R}^d) \) it follows from the Banach-Saks Theorem that

\[ \exists \{k(l)\} \subseteq \{l\} \text{ such that } \frac{1}{N} \sum_{l=1}^N \hat{z}(T_{k(l)}^+ \cdot) \to z(\cdot) \text{ strongly in } L^2([0, T], \mathbb{R}^d). \]

Further,

\[ \exists \{m(N)\} \subseteq \{N\} \text{ such that } \frac{1}{m(N)} \sum_{l=1}^{m(N)} \hat{z}(T_{k(l)}^+ \cdot) \to z(\cdot) \text{ a.e. on } [0, T]. \quad \text{(9)} \]

Fix \( t_0 \in [0, T] \) such that \( \text{(9)} \) holds, i.e.,

\[ \lim_{m(N) \to \infty} \frac{1}{m(N)} \sum_{l=1}^{m(N)} \hat{z}(T_{k(l)} + t_0) = z(t_0). \quad \text{(10)} \]

We know that \( \hat{z}(T_{k(l)} + t_0) = h_{r(k(l))}(\hat{x}([T_{k(l)} + t_0]), \mathfrak{P}([T_{k(l)} + t_0])) \). Note that \( \mathfrak{P}([T_{k(l)} + t_0]) = \mathfrak{P}(T_{k(l)} + t_0) \).

We claim the following: For any \( \epsilon > 0 \) there exists \( N \) such that for all \( n \geq N \)

\[ \|\hat{x}(t(m)) - \hat{x}(t(m + 1))\| < \epsilon, \quad \text{where } T_n \leq t(m) < t(m + 1) < T_{n+1}. \]

If \( t(m + 1) = T_n \) then we claim that \( \|\hat{x}(t(m)) - \hat{x}(T_n^-)\| < \epsilon \). We shall prove this later, for now we assume it to be true and proceed.

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Since \( \hat{x}(T_{k(l)} + t_0) \to x(t_0) \) it follows from the above claim that \( \hat{x}([T_{k(l)} + t_0]) \to x(t_0) \). Since \( r(k(l)) \uparrow \infty \) it follows from Lemma 1 that

\[
\lim_{r(k(l)) \uparrow \infty} d \left( h_r(k(l)) \left( \hat{x}([T_{k(l)} + t_0]), \mathcal{P}([T_{k(l)} + t_0]) \right), H(x(t_0)) \right) = 0 \text{ i.e.,}
\]

\[
\lim_{r(k(l)) \uparrow \infty} d \left( \hat{x}([T_{k(l)} + t_0]), H(x(t_0)) \right) = 0
\]

Further, since \( H(x(t_0)) \) is convex and compact, it follows from equation (10) that \( z(t_0) \in H(x(t_0)) \). On the measure zero set of \([0, T]\) where (9) does not hold, the value of \( z(\cdot) \) can be modified to ensure that \( z(t) \in H(x(t)) \) for all \( t \in [0, T] \).

It is left to prove the claim that was made earlier. We first show that given any \( \epsilon > 0 \) there exists \( N \) such that \( n \geq N \) implies that \( \| \hat{x}(t(m)) - \hat{x}(t(m+1)) \| < \epsilon \), where \( T_n \leq t(m) < t(m+1) < T_{n+1} \). We know that

\[
\hat{x}(t(m+1)) = \hat{x}(t(m)) + a(n) \left( h_r(n) \left( \hat{x}(t(m)), \mathcal{P}(t(m)) \right), \tilde{M}_{n+1} \right).
\]

Hence,

\[
\| \hat{x}(t(m)) - \hat{x}(t(m+1)) \| \leq a(n) \| h_r(n) \left( \hat{x}(t(m)), \mathcal{P}(t(m)) \right) \| + \| \zeta_{n+1} - \zeta_n \|.
\]

From (9), the above inequality becomes

\[
\| \hat{x}(t(m)) - \hat{x}(t(m+1)) \| \leq a(n) K(1 + \| \hat{x}(t(m)) \|) + \| \zeta_{n+1} - \zeta_n \|.
\]

It follows from Lemmas 3 & 4 that \( a(n) K(1 + \| \hat{x}(t(m)) \|) \to 0 \) and \( \| \zeta_{n+1} - \zeta_n \| \to 0 \) respectively in the ‘almost sure’ sense. In other words, there exists \( N \) (possibly sample path dependent) such that the claim holds. The second part of the unproven claim considers the situation when \( t(m+1) = T_{n+1} \), the proof of which follows in a similar manner.

**Theorem 1 (The Stability Theorem).** Under assumptions (A1)-(A3), (S1) & (S2), sup \( \| x_n \| \leq \infty \) a.s.

**Proof.** Define \( \mathcal{B} := \{ \sup_{t \geq 0} \hat{x}(t) < \infty \} \cap \{ \zeta_n \text{ converges} \} \). It is enough to show that sup \( \| x_n \| \leq \infty \) on \( \mathcal{B} \). Let us assume the contrary i.e., sup \( \| x_n \| = \infty \) on \( \mathcal{D} \subseteq \mathcal{B} \) such that \( P(\mathcal{D}) > 0 \). Fix \( \omega \in \mathcal{D} \cap \mathcal{B} \) and choose \( N \) such that the following hold:

1. For all \( m(l) \geq N \), \( \sup_{t \in [0, T]} \| \hat{x}(T_l + t) - x^l(t) \| < \delta_3 - \delta_2 \). This is possible since \( \| \hat{x}(T_l + t) - x^l(t) \| \to 0 \) on \( \mathcal{B} \) (Lemma 5). Recall that \( \{ m(n) \} \subseteq N \) is such that \( t(m(n)) = T_n \) for all \( n \geq 0 \).

2. For all \( m(l) \geq N \), \( \| \hat{x}(T_{l+1}) \| < \delta_4 \). This is possible since \( \| \hat{x}(T_l + T) - \hat{x}(T_{l+1}) \| \to 0 \) as \( l \to \infty \) and \( \| \hat{x}(T_l + T) \| < \delta_3 \) for large \( m(l) \).

3. For all \( m(l) \geq N \), \( r(l) > 1 \).

We have,

\[
\frac{\| \mathcal{P}(T_{l+1}) \|}{\| \mathcal{P}(T_l) \|} = \frac{\| \hat{x}(T_{l+1}) \|}{\| \hat{x}(T_l) \|}.
\] (11)
For $m(l) > N$, we have $\|\hat{x}(T_{l+1})\| < \delta_4$ and $\|\hat{x}(T_l)\| = 1$. Hence it follows from (11) that
\[
\frac{\|\Psi(T_{l+1})\|}{\|\Psi(T_l)\|} < \delta_4 < 1.
\] (12)

Let us closely analyze the implication of (12). Since $\|\hat{x}(T_l)\| > 1$, it follows that $\|\hat{x}(T_{l+1})\| < \delta_4 \|\hat{x}(T_l)\|$. Further if $\|\hat{x}(T_{l+1})\| \geq 1$, we have $\|\hat{x}(T_{l+2})\| < \delta_4 \|\hat{x}(T_{l+1})\| < \delta_4^2 \|\hat{x}(T_l)\|$. We see that the trajectory has a tendency to fall into the unit ball at an exponential rate (from the outside). Let $t_0 < T_l$ be the last time that the trajectory ‘jumps’ from inside the unit ball to the outside. This is because $t_0$ is the last time before $T_l$ when the trajectory ‘jumps’ outside and $\|\hat{x}(T_l)\| > 1$. It follows from the observation that the trajectory falls exponentially into the unit ball that this jump is at least $\|\hat{x}(T_l)\| - 1$. Since $r(l) \uparrow \infty$ the trajectory is forced to make larger and larger ‘last jumps’ from inside the unit ball to the outside such that the lengths of these jumps ($\geq r(l) - 1$) ‘run off’ to infinity. Further, these jumps are made within time $T + 1$. Using Gronwall’s inequality we however get a contradiction.

4 Convergence Theorem

We begin this section by presenting the additional assumptions imposed on recursion (3). These assumptions are coupled with those made in Section 2.2 to prove that the iterates are stable and converge to an internally chain transitive invariant set associated with a DI that is defined in terms of the ergodic occupation measures associated with the ‘Markov process’.

The additional assumptions made are similar to those in Borkar [6]. We list them below.

(B1) $\{y_n\}_{n\geq0}$ is an $S$-valued Markov process with two associated control processes: $\{x_n\}_{n\geq0}$ and another random process $\{z_n\}_{n\geq0}$ taking values in a compact metric space $U$. Thus
\[
P(y_{n+1} \in A \mid y_m, z_m, x_m, m \leq n) = \int_A p(dy|y_n, z_n, x_n), \ n \geq 0,
\]
for $A$ Borel in $S$. The map
\[
(y, z, x) \in S \times U \times \mathbb{R}^d \rightarrow p(dw|y, z, x) \in \mathcal{P}(S)
\]
is continuous, further it is uniformly continuous on compacts in the $x$ variable with respect to the other variables. $\mathcal{P}(S)$ is used to denote the space of probability measures on $S$.

Let $\varphi : S \rightarrow \varphi(S) = \varphi(y, dz) \in \mathcal{P}(U)$ be a measurable map. Suppose the Markov process has a (possibly non-unique) invariant probability measure $\eta_x, \varphi(dy) \in \mathcal{P}(S)$, we can define the corresponding ergodic occupation measure
\[
\Psi_x, \varphi(dy, dz) := \eta_x, \varphi(dy)\varphi(dy, dz) \in \mathcal{P}(S \times U).
\] (13)
Let $D(x)$ be the set of all such ergodic occupation measures for a prescribed $x$. It can be shown that $D(x)$ is closed and convex for each $x \in \mathbb{R}^d$. Further, the map $x \mapsto D(x)$ is upper-semicontinuous. For a proof of the aforementioned results the reader is referred to Chapter 6.2 of [7].

(B2) $D(x)$ is compact.

Let us define a $\mathcal{P}(S \times U)$-valued random process $\mu(t) = \mu(t, dydz)$, $t \geq 0$, by $\mu(t) := \delta_{\eta_n, z_n}$, $t \in [t(n), t(n + 1)]$, for $n \geq 0$. For $t > s \geq 0$, define $\mu_s^t \in \mathcal{P}(S \times U \times [s, t])$ by $\mu_s^t(A \times B) := \frac{1}{t-s} \int_B \mu(y, A) dy$ for $A, B$ Borel in $S \times U$, $[s, t]$ respectively.

(B3) Almost surely, for $t > 0$, the set $\{\mu_s^t, s \geq 0\}$ remains tight.

Define $\tilde{h}(x, \nu) := \int h(x, y)\nu(dy, U)$ for $\nu \in \mathcal{P}(S \times U)$. We use this to define the following DI.

\[ \hat{x}(t) \in h(x(t)), \text{ where } \hat{x}(x) := \{\tilde{h}(x, \nu) \mid \nu \in D(x)\}. \quad (14) \]

**Theorem 2** (Stability & Convergence). Under assumptions (A1) – (A3), (S1), (S2) and (B1) – (B3), almost surely the iterates given by (13) are stable and converge to an internally chain transitive invariant set associated with $\hat{x}(t) \in h(x(t))$.

**Proof.** Under assumptions (A1) – (A3), (S1) and (S2) the stability of the iterates follow from Theorem 1. Now, we invoke Theorem 3.1 of Borkar [6] to conclude that the iterates converge to an internally chain transitive invariant set associated with $\hat{x}(t) \in h(x(t))$. \hfill $\square$

5 Application to temporal difference learning

Temporal difference (TD) learning is an important prediction method which combines ideas from *Monte Carlo* and *dynamic programming*. It has been mostly used to solve problems from reinforcement learning. There are several variants of TD algorithms.

Consider the general form of a TD algorithm with ‘controlled Markov noise’.

\[ x_{n+1} = x_n + a(n) (h(x_n, y_n) + M_{n+1}) \text{, where} \]

(i) $h : \mathbb{R}^d \times S \to \mathbb{R}^d$ is of the form $h(x, y) = A(y)x + b(y)$. Here $A : S \to \mathbb{R}^{d \times d}$ is a matrix valued function and $b : S \to \mathbb{R}^d$ is a vector valued function.

(ii) $\{a(n)\}_{n \geq 0}$ is the given step-size sequence such that $\sum_{n=0}^{\infty} a(n) = \infty$ and $\sum_{n=0}^{\infty} a(n)^2 < \infty$. Recall that this is assumption (A3).

(iii) $\{M_n\}_{n \geq 1}$ is the sequence of square integrable martingale difference terms such that

\[ E \left[ \|M_{n+1}\|^2 \mid \mathcal{F}_n \right] \leq K (1 + \|x_n\|^2), \quad n \geq 0. \]

Recall that this is assumption (A2).

(iv) $\{y_n\}_{n \geq 0}$ is a $S$-valued ‘Controlled Markov Process’. We assume that $S$ is a compact metric space.
For a detailed exposition on TD algorithms the reader is referred to Tsitsiklis and Van Roy [10]. In this section, we impose conditions on $A$ and $b$ that guarantee the ‘stability and convergence’ of the iterates given by (15).

Remark: It is important to note that our TD algorithm (cf. [15]) is more general than the regular TD update with function approximation, as in (say) Tsitsiklis and Van Roy [10]. In particular, the regular TD with function approximation can be written (see [10]) as in (15). Note also that unlike the usual analyses of TD, we do not assume that the Markov process $\{y_n\}$ is (a) finite state and (b) ergodic under the given stationary policy.

We state the first of the two assumptions below.

**(T1)** $A : S \to \mathbb{R}^{d \times d}$ and $b : S \to \mathbb{R}^d$ are continuous maps.

We show that (15) satisfies (A1) – (A3) and (S1) if it satisfies (T1). Since $A$ and $b$ are continuous maps, it follows that $h$ is a jointly continuous map. Since $A$ is continuous, the range of $A$, $A(S) \subset \mathbb{R}^{d \times d}$, is compact. Define $L := \sup_{M \in A(S)} \|M\| (< \infty)$. We have the following:

$$\|h(x_1, y) - h(x_2, y)\| \leq \|A(y)\| \times \|x_1 - x_2\| \leq L \|x_1 - x_2\|.$$  

Hence $h$ is Lipschitz continuous in the first component, with Lipschitz constant $L$, further this constant does not change with the second component ((A1) is satisfied). Assumptions (A2) and (A3) are trivially satisfied.

We have $h_n(x, y) = \{A(y)x + b(y)/c\}$ and $h_\infty(x, y) = \{A(y)x\}$. Now, we show that (S1) is satisfied. Let $c_n \uparrow \infty$, $y_n \to y$ and $\lim_{n \to \infty} h_{cn}(x, y_n) = u$. We need to show that $u = h_\infty(x, y)$. Since $A$ is continuous, $\lim_{n \to \infty} A(y_n)x = A(y)x$. Since $b$ is a bounded function, $\lim_{n \to \infty} b(y_n)/c_n = 0$. Hence, we get $u = \lim_{n \to \infty} h_{cn}(x, y_n) = A(y)x \in h_\infty(x, y)$. Before we state our second assumption we present an auxiliary result.

**Lemma 7.** Let $H : \mathbb{R}^d \to \{\text{subsets of } \mathbb{R}^d\}$ be a Marchaud map. Let $A$ be an associated attracting set that is also Lyapunov stable. Let $B$ be a compact subset of the basin of attraction of $A$. Then for all $\epsilon > 0$ there exists $T(\epsilon)$ such that $\Phi_{t \geq T(\epsilon)}(B) \subseteq N^\epsilon(A)$.

**Proof.** Since $A$ is Lyapunov stable, corresponding to $N^\epsilon(A)$ there exists $N^{\delta}(A)$ such that $\Phi_{[0, +\infty)}(N^{\delta}(A)) \subseteq N^\epsilon(A)$. Fix $x_0 \in B$. Since $B$ is contained in the basin of attraction of $A$, $\exists \delta(x_0) > 0$ such that $\Phi_{t(x_0)}(x_0) \subseteq N^{\delta/4}(A)$. Further, from the upper semi-continuity of flow it follows that, for all $x \in N^{\delta(x_0)}(x_0)$, $\Phi_{t(x_0)}(x) \subseteq N^{\delta/4}(\Phi_{t(x_0)}(x_0))$ for some $\delta(x_0) > 0$, see Chapter 2 of Aubin and Cellina [11]. Hence $\Phi_{t(x_0)}(x) \subseteq N^{\delta}(A)$ for all $x \in N^{\delta(x_0)}(x_0)$. Since $A$ is Lyapunov stable, we get $\Phi_{t(x_0), +\infty}(x) \subseteq N^\epsilon(A)$. In this manner for each $x \in B$ we calculate $t(x)$ and $\delta(x)$, the collection $\{N^{\delta(x)}(x) : x \in B\}$ is an open cover for $B$. Let $\{N^{\delta(x)}(x) : 1 \leq i \leq m\}$ be a finite sub-cover. If we define $T(\epsilon) := \max\{t(x_i) : 1 \leq i \leq m\}$ then $\Phi_{[T(\epsilon), +\infty)}(B) \subseteq N^\epsilon(A)$.
We have $H(x) = \overline{\text{co}}\{A(y)x \mid y \in S\}$. It follows from Lemma 2 that $H$ is a Marchaud map. We state our second assumption below.

(T2) Let $\varepsilon > 0$ and $V : \overline{B}_{1+\varepsilon}(0) \to [0, \infty)$. Let $\Lambda$ be a compact subset of $B_1(0)$, clearly $\sup_{u \in \Lambda} ||u|| < 1$. Let the following hold:

(i) For all $t \geq 0$, $\Phi_t(B_{1+\varepsilon}(0)) \subseteq B_{1+\varepsilon}(0)$, where $\Phi_t(\cdot)$ is a solution to the DI $\dot{x}(t) \in H(x(t))$.

(ii) $V^{-1}(0) = \Lambda$.

(iii) $V$ is continuous and for all $x \in \overline{B}_{1+\varepsilon}(0) \setminus \Lambda$. Further, for $y \in \Phi_t(x)$ and $t > 0$ we have $V(y) < V(x)$.

Proposition 3.25 from Benaim et. al. [4] : Under (T2), $\Lambda$ is a Lyapunov stable attracting set, further there exists an attractor, $A$, contained in $\Lambda$ whose basin contains $B_{1+\varepsilon}(0)$.

Since $A \subset \Lambda$ and $B_{1+\varepsilon}(0)$ is contained in the basin of attraction of $A$, it follows that $B_{1+\varepsilon}(0)$ is contained in the basin of attraction of $\Lambda$. We have that $\overline{B}_1(0)$ is contained in some fundamental neighborhood of $\Lambda$ (Lemma 7). Further, $\sup_{z \in \Lambda} ||z|| < 1$. Hence (S2) is satisfied. Note that the attracting set associated with $\dot{x}(t) \in \overline{\text{co}}\{A(y)x(t) \mid y \in S\}$ in (S2) is $\Lambda$.

Theorem 3. Under assumptions (A1) – (A3), (T1), (T2) and (B1) – (B3), almost surely the iterates given by (15) are stable and converge to an internally chain transitive invariant set associated with $\dot{x}(t) \in \hat{h}(x(t))$.

Proof. We have shown that assumptions (A1) – (A3), (S1) & (S2) are satisfied by (15). It follows from Theorem 1 that the iterates are stable. Further, we have assumed that (B1) – (B3) are satisfied by (15). It follows from Theorem 2 that the iterates converge to an internally chain transitive invariant set associated with $\dot{x}(t) \in \hat{h}(x(t))$.

Let us consider the special case when $A$ is a constant map i.e., $A(y) = M$ for all $y \in S$. Thus, we get the following recursion:

$$x_{n+1} = x_n + a(n) [Mx_n + b(y_n) + M_{n+1}], \quad (16)$$

where $M \in \mathbb{R}^{d \times d}$ and $b : \mathbb{R}^d \to \mathbb{R}^d$ is a continuous map. Hence (16) satisfies assumption (T1). As explained before, (16) also satisfies (A1) – (A3) and (S1). It follows from the definition of $\{h_c\}_{c \geq 1}$ and $H$ that

$$h_c(x, y) = Mx + b(y)/c \text{ and } h_\infty(x, y) = Mx;$$

$$H(x) = \overline{\text{co}} \left( \bigcup_{y \in Z} h_\infty(x, y) \right) = Mx.$$ 

Note that the DI $\dot{x}(t) \in H(x(t))$ is really the o.d.e. $\dot{x}(t) = Mx$, here.
Let us assume that all eigenvalues of $M$ have strictly negative real parts. Then, the origin is a globally asymptotic stable equilibrium point (a globally attracting set that is also Lyapunov stable) associated with $\dot{x}(t) = Mx(t)$ (see 11.2.3 of Borkar [7]). Now, we show that recursion (16) satisfies assumption (T2). Solving $\dot{x}(t) = Mx(t)$, we get $\Phi_t(x(0)) = e^{Mt}x(0)$ for $t \geq 0$. Let $t > 0$ and $x(0) \in \mathbb{R}^d \setminus \{0\}$, we have that $\|\Phi_t(x(0))\| < \|x(0)\|$ since all the eigenvalues of $M$ have strictly negative real parts.

Let us define the following:

1. $\epsilon := 1$.
2. $V(x) : \overline{B}_2(0) \to [0, \infty)$ as $V(x) := \|x\|$.
3. $\Lambda := \{0\}$ (origin).

As explained earlier, for $t \geq 0$ we have $\|\Phi_t(x)\| \leq \|x\|$, hence $\Phi_t(B_2(0)) \subseteq B_2(0)$ (\(T2)(i)\) holds).

Recall that $V(x) = \|x\|$ for all $x \in \overline{B}_2(0)$. It follows from the definition of $V$ that $V^{-1}(0) = \Lambda$ (\(T2)(ii)\) holds).

Fix $x_0 \in B_2(0) \setminus \{0\}$ and $t > 0$, we have $\|\Phi_t(x_0)\| < \|x_0\|$, hence $V(\Phi_t(x_0)) < V(x_0)$ (\(T2)(iii)\) holds).

Since the recursion given by (16) satisfies (T1) & (T2), it follows from Theorem [3] that the iterates are ’stable and convergent’.

6 Conclusions

We presented in this paper general sufficient conditions for stability and convergence of stochastic approximation algorithms with ’controlled Markov’ noise. To the best of our knowledge this is the first time that sufficient conditions for stability of stochastic approximations with ’controlled Markov’ noise have been provided. We further studied an application of our results as a temporal difference learning algorithm and showed that the algorithm is stable and asymptotically convergent under weaker requirements than those in the other analyses in the literature. An interesting future direction would be to extend this analysis to the case of multi-timescale stochastic approximations that would encompass actor-critic algorithms - another important class of algorithms in reinforcement learning.

References

[1] J. Aubin and A. Cellina. Differential Inclusions: Set-Valued Maps and Viability Theory. Springer, 1984.
[2] J. Aubin and H. Frankowska. Set-Valued Analysis. Birkhäuser, 1990.
[3] M. Benaïm. A dynamical system approach to stochastic approximations. SIAM J. Control Optim., 34(2):437–472, 1996.
[4] M. Benaïm, J. Hofbauer, and S. Sorin. Stochastic approximations and differential inclusions. SIAM Journal on Control and Optimization, pages 328–348, 2005.
[5] A. Benveniste, M. Metivier, and P. Priouret. *Adaptive Algorithms and Stochastic Approximations*. Springer Publishing Company, Incorporated, 1st edition, 2012.

[6] V. S. Borkar. Stochastic approximation with 'controlled markov' noise. *Systems & Control Letters*, 55(2):139–145, 2006.

[7] V. S. Borkar. *Stochastic Approximation: A Dynamical Systems Viewpoint*. Cambridge University Press, 2008.

[8] V. S. Borkar and S.P. Meyn. The O.D.E. method for convergence of stochastic approximation and reinforcement learning. *SIAM J. Control Optim*, 38:447–469, 1999.

[9] L. Ljung. Analysis of recursive stochastic algorithms. *Automatic Control, IEEE Transactions on*, 22(4):551–575, 1977.

[10] J. N. Tsitsiklis and B. Van Roy. An analysis of temporal-difference learning with function approximation. *IEEE Transactions on Automatic Control*, 1997.