Singular disk of matter in the Cooperstock–Tieu galaxy model

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Abstract

Recently a new model of galactic gravitational field, based on ordinary General Relativity, has been proposed by Cooperstock and Tieu in which no exotic dark matter is needed to fit the observed rotation curve to a reasonable ordinary matter distribution. We argue that in this model the gravitational field is generated not only by the galaxy matter, but by a thin, singular disk as well. The model should therefore be considered unphysical.

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I. INTRODUCTION

In [1], the authors derived a set of non-linear equations which describes a gravitationally bound rotating cloud of dust in General Relativity. Basing on their result, the authors propose a model of galaxy to which a flat rotation curve can be fitted without dark, unobservable matter. The result is then applied to several galaxies, including the Milky Way.

We shall argue that

1. the set of equations they derive is essentially correct, though does not possess non-trivial, asymptotically flat solutions,

2. the model of galaxy they propose is questionable because it possesses an additional source of gravitational field in the form of a rotating flat disk at $z = 0$.

II. GRAVITATIONALY BOUND CLOUD OF DUST IN GR

In GR a gravitationally bound cloud of dust is described by the stress-energy tensor $T^{\mu\nu} = \rho \, u^\mu \, u^\nu$ such that $u^\mu$ is the dust four-velocity vector and $\rho$ its density (see for example [2]). The metric satisfies the Einstein equations

$$G^{\mu\nu}[g_{\alpha\beta}] = 8\pi G \, \rho \, u^\mu \, u^\nu \quad (1)$$

where the Einstein tensor field is to be considered a functional of the metric field $g_{\alpha\beta}$ (we set $c = 1$ throughout the paper). Note that the Biachi identities immediately imply that the flow of the dust is geodesic and source-free

$$\rho \, u^\mu \, \nabla_\mu u^\nu = 0 \quad (2)$$

$$\nabla_\mu(\rho u^\mu) = 0. \quad (3)$$

Let $\eta$ denote the flat Minkowski metric. The conventional way to simplify this highly non-linear set of equations is to perform a perturbative expansion of some of the quantities in $G$:  

$$g_{\alpha\beta} = \eta_{\alpha\beta} + G \, g^{(1)}_{\alpha\beta} + G^2 \, g^{(2)}_{\alpha\beta} + O(G^3) \quad (4)$$

$$u^\alpha = u^\alpha_{(0)} + G \, u^\alpha_{(1)} + G^2 \, u^\alpha_{(2)} + O(G^3) \quad (5)$$
and obtain the following equation
\[
G \cdot G^{\mu\nu}[g^{(1)}] + G^2 \cdot \left( G^{\mu\nu}[g^{(2)}] + \frac{1}{2} G_{\mu\nu}[g^{(1)}, g^{(1)}] \right) + O(G^3) =
\]
\[
= 8\pi G \rho (u^\mu_{(0)} + G u^\mu_{(1)})(u^\nu_{(0)} + G u^\nu_{(1)}) + O(G^3)
\]
\[
(G^{\mu\nu}[\cdot] \text{ and } G_{\mu\nu}[\cdot, \cdot] \text{ denotes, respectively, the first and second functional derivative of } G^{\mu\nu}[g_{\alpha\beta}] \text{ for } g_{\alpha\beta} = \eta_{\alpha\beta}).\]

We can now attempt to solve these equations order by order.

This approach, however, fails to reproduce the set of equations from [1]. This is due to the fact that the exact solution families approximated there depend on \(G\) via \(\sqrt{G}\), i.e. their dependence on the coupling constant is non-analytic. Since we want to follow the approach of [1], we shall expand (1) in \(\sqrt{G}\) instead of \(G\):

\[
g_{\alpha\beta} = \eta_{\alpha\beta} + G^{1/2} g_{\alpha\beta}^{(1)} + G g_{\alpha\beta}^{(2)} + O(G^{3/2})
\]
(6)
\[
u^\alpha = u^\alpha_{(0)} + G^{1/2} u^\alpha_{(1)} + G u^\alpha_{(2)} + O(G^{3/2})
\]
(7)

and therefore
\[
G^{1/2} \cdot G^{\mu\nu}[g^{(1)}] + G \left( G^{\mu\nu}[g^{(2)}] + \frac{1}{2} G_{\mu\nu}[g^{(1)}, g^{(1)}] \right) + O(G^{3/2}) =
\]
\[
= 8\pi G \rho u^\mu_{(0)} u^\nu_{(0)} + O(G^{3/2}).
\]

Rewritten order by order this is equivalent to

\[
G^{\mu\nu}[g^{(1)}] = 0
\]
(8)
\[
G^{\mu\nu}[g^{(2)}] + \frac{1}{2} G_{\mu\nu}[g^{(1)}, g^{(1)}] = 8\pi \rho u^{(0)} u^{(0)}.
\]
(9)

The first equation is the linearized vacuum Einstein equation. If we impose additionally the Lorentz–De Donder gauge condition (see [2]), (8) is equivalent to

\[
\Box g^{\mu\nu}_{(1)} = 0
\]
(10)

(\(\Box\) being the flat metric \(\eta\) d’Alembert operator). In particular, for stationary problems

\[
\Delta g^{\mu\nu}_{(1)} = 0
\]
(11)

Note that for an asymptotically flat metric \(g^{\mu\nu}_{(1)} \to 0\) for \(r \to \infty\), which together with (11) implies that \(g^{(1)} = 0\). In other words, no asymptotically flat solutions are possible with the metric depending on \(\sqrt{G}\) in the lowest order.
III. COOPERSTOCK AND TIEU GALACTIC ROTATION CURVE MODELS

Cooperstock and Tieu assumed the metric to be of the form of
\[ g = -e^\nu (dz^2 + dr^2) - r^2 d\phi^2 + (dt - N d\phi)^2 \]  \hspace{1cm} (12)
with
\[ N = r \frac{\partial \Phi}{\partial r} \]  \hspace{1cm} (13)
\[ \Phi = e^{-k|z|} J_0(kr), \]  \hspace{1cm} (14)
both \( \Phi \) and \( N \) being of the order of \( \sqrt{G} \). It is straightforward to verify that this metric fails to satisfy (8), as \( \Delta g^{(1)} \) is not equal to zero, but has a distributional source proportional to \( \delta(z) \). (12) is not, therefore, a well–defined, global metric with rotating dust as the only source of gravity.

In fact, it does not satisfy the first of the two equations derived by Cooperstock and Tieu:
\[ N_{rr} + N_{zz} - \frac{N_r}{r} = 0 \]  \hspace{1cm} (15)
\[ \frac{N_r^2 + N_z^2}{r^2} = 8\pi G \rho \]  \hspace{1cm} (16)
As before, due to the presence of \( |z| \) in (14) the first equation has a distributional singularity at \( z = 0 \) rather than 0 on the right–hand side. In the following section we shall prove that this results in presence of an additional, distributional term in the stress–energy tensor \( T^{\mu\nu} \).

IV. KOMAR INTEGRAL

The shortest way to prove that the galaxy model in \[1\] has a sheet of matter at \( z = 0 \) is to consider the Komar integral of the timelike Killing vector \( \frac{\partial}{\partial t} \). Define \( \tau = g(\frac{\partial}{\partial t}, \cdot) \). It is easy to prove in general that if \( \nabla_\mu \tau_\nu + \nabla_\nu \tau_\mu = 0 \), the following identity holds:
\[ d \star d\tau = \frac{1}{3} R^{\mu\alpha} \tau_\alpha \epsilon_{\mu\nu\rho\sigma} dx^\nu \wedge dx^\rho \wedge dx^\sigma \]  \hspace{1cm} (17)
(\( \epsilon \) is the volume 4–form; if the metric is vacuum the identity reduces to the Komar current conservation formula). We can integrate it over an arbitrary 3–volume \( V \):
\[ \int_{\partial V} \star d\tau = \int_V d \star d\tau = \frac{4\pi G}{3} \int_V (2T^{\alpha\mu} \tau_\alpha - T^{\tau\mu}) \epsilon_{\mu\nu\rho\sigma} dx^\nu \wedge dx^\rho \wedge dx^\sigma. \]  \hspace{1cm} (18)
Take $V$ to be the cylinder defined by the following conditions: $r \leq R$, $-a \leq z \leq a$, $t = \text{const}$, for given positive constants $a$ and $R$. The boundary of this volume consists of a side 2–surface $r = R$ and two circle–shaped 2–surfaces $z = a$ and $z = -a$. Let us take the limit of $a \to 0$, i.e. the flat cylinder limit. If $T^{\mu\nu}$ had no singularities but consisted solely of rotating dust, this limit would be zero, since the volume of the cylinder goes to zero as we shrink it. It is, however, not the case in the metric (12). Indeed, the side surface integral of (18) does converge to 0, but the top and bottom side integrals $I_t$ and $I_b$ do not:

$$I_t + I_b = \int_0^{2\pi} d\varphi \int_0^R dr \frac{N \partial N}{r \partial z} \bigg|_{z=a} - \int_0^{2\pi} d\varphi \int_0^R dr \frac{N \partial N}{r \partial z} \bigg|_{z=-a}. \quad (19)$$

With $N$ given by (13) and (14) $I_t$ and $I_b$ are in fact equal and therefore

$$I_t + I_b = 4\pi \int_0^R dr \frac{N \partial N}{r \partial z} \bigg|_{z=a} \to -4\pi k^3 \int_0^R dr r J_0'(kr)^2 \neq 0. \quad (20)$$

The reason is, once again, that $\frac{\partial N}{\partial z}$ has a discontinuity at $z = 0$ due to the presence of $|z|$ in its definition.

Note that the same argument can be applied to the axial Killing vector $\frac{\partial}{\partial \varphi}$ as well. This indicates that the disk has both mass and angular momentum, simillarily to the singularity in the exact solution of van Stockum [3].

The solution should therefore be interpreted as possesing an additional, distributional source of gravity at $z = 0$ and the resulting discontinuity of the metric’s first derivative as a physical one rather than a coordinate system artifact, as is claimed in [1].

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