Abstract. For a holomorphic family $(\rho_\lambda)$ of representations $\Gamma \to \mathrm{SL}(d, \mathbb{C})$, where $\Gamma$ is a finitely generated group, we introduce the notion of proximal stability and show that it is equivalent to the pluriharmonicity of Lyapunov exponents of the family (defined using random walks).

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LYAPUNOV EXPO NENTS AND STABILITY

1. Introduction

There exists a deep analogy between the theory of Kleinian groups and one-dimensional complex dynamics. This idea is often referred to as the Sullivan dictionary, because Sullivan explored with great success this connection (see [35], [36], [33]). The dictionary is a conceptual framework whose goal is to “translate” some concepts and methods of one theory into corresponding notions for the other theory. For example, a rational map corresponds to a Kleinian group, and the Julia set of the rational map corresponds to the limit set of the Kleinian group. One of the major achievement of this dictionary has been the proof of the “No wandering domain” theorem by Sullivan [35] (for rational maps) using methods developed by Ahlfors for his proof of the Finiteness theorem [1] (for Kleinian groups). A major common tool is the use of quasiconformal maps and deformations.

In a serie of works, starting by [10], Deroin and Dujardin extended this dictionary. They developped the study of bifurcation currents for Kleinian groups, following the work of DeMarco [8] who introduced bifurcation currents for holomorphic family of rational maps on the complex projective line. Such bifurcation currents live on the parameter space and connect the stability of the family and the potential theory of a certain function, the Lyapunov exponent. The main property of the bifurcation current is that its support is precisely the bifurcation locus (suitably defined for family of Kleinain groups), and this fact gives informations about this locus. For example, they prove that some dynamically defined subvarieties equidistribute toward the bifurcation current. The goal of the present work is to extend the theory of bifurcation currents to higher rank representations, that is the higher dimensional generalization of Kleinian groups, still in parallel with the theory of complex dynamics in higher dimension. First we present some of the existing work which develop the theory of higher rank representations on one side and the theory of higher dimensional complex dynamics on the other side.

The theory of higher rank representations is a body of work aiming at generalizing the theory of Fuchsian and Kleinian groups to subgroups of Lie groups of rank ≥ 2. It is sometime called the Higher Teichmüller Theory. It originated in the study of homogeneous geometric structures (see [13]). The introduction of the Hitchin component [19] as a generalization of the Teichmuller space, which extends the PSL(2, R) theory to PSL(n, R) generated a wealth a subsequent works which explored the properties of this component. In order to understand the Hitchin representations (representations that lie in the Hitchin component), Labourie [20] introduced the notion of Anosov representations, a concept that was greatly studied and extended (see [17], [16], [24], [25], [5] for example). Anosov representations are now believed to be the
“right” higher rank generalization of convex-cocompact representations into rank 1 Lie groups. They have interesting dynamical properties, notably dynamical stability in their limit sets, and form open components of the space of representations.

High dimensional complex dynamics is the study of iteration of rational maps of $\mathbb{P}^d(\mathbb{C})$. A major difficulty in the study of family of rational maps is that the main tools in dimension one, the $\lambda$-lemma and the quasiconformal theory is not available in higher dimensions. This difficulty has been tackled by the use of pluripotential theory, briefly the theory of plurisubharmonic functions and positive currents, see [11] and the survey [13] for an exposition about bifurcation currents and the difficulties of a generalization to higher dimension. A recent work of Berteloot, Bianchi and Dupont [4] studies the stability of family of endomorphisms of $\mathbb{P}^d(\mathbb{C})$ and proves a generalization of the results of DeMarco: there exists a bifurcation current whose support is the bifurcation locus, leading to a richer theory. For polynomial automorphisms of $\mathbb{C}^2$, Dujardin-Lyubich [14] and then Dujardin-Berger [3] introduce another notion of stability/bifurcation and give various dynamical characterization of stability. Part of these works is to define the notions of stability and bifurcation in this context: if in dimension one all notions of stability coincide, this is not the case anymore in higher dimension.

When studying dynamically interesting bifurcation current (as in [8], [10] and [4]), the bifurcation current is defined as the $dd^c$ of the Lyapunov exponent (of the endomorphism or the representation), but it is always a non-trivial task to prove that it correspond to a useful notion of stability, see section 3.5.

Let us now explain the results of the present work. We state the results informally, and refer to the section 2 for the precise definitions. We study holomorphic family of representation $\rho_\lambda : \Gamma \to G$ where $\Gamma$ is a finitely generated group, $G = SL(d, \mathbb{C})$ and $\lambda$ varies in a complex manifold $\Lambda$. As in [10], we define the Lyapunov exponents $(\chi_i(\lambda))_{i=1,\ldots,d}$ of the representation $\rho_\lambda$ using a random walk on $\Gamma$. The bifurcation current is defined as $T_{bif} = dd^c(\chi_1(\lambda) + \chi_d(\lambda))$. Inspired by [3], we define the notion of proximal stability for a representation: a family of representations $(\rho_\lambda)$ is proximally stable if for any $\gamma \in \Gamma$ and $\lambda_0$ such that $\rho_{\lambda_0}(\gamma)$ is proximal, then $\rho_\lambda(\gamma)$ is proximal for every $\lambda$. The stability locus is the set of parameters where this property holds and the bifurcation locus is the complement of the stability locus. Our main result is:

Under some assumptions on the family of representations, the support of the bifurcation current is the bifurcation locus.

This statement is of the same nature as in [8], [10] and [4]. For a precise statement, see [3].
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2. Preliminaries

We collect here the necessary definitions and some useful facts and results about random walks on abstract and linear groups. We also recall some notions of complex analysis. We fix a finitely generated group $\Gamma$, a complex vector space $V$ of dimension $d+1$ and $G = SL(V) = SL(d+1, \mathbb{C})$.

2.1. Properties of linear subgroups. We describes some notions relative to a subgroup (more generally a subsemigroup) of $G$ acting on $V$ and $P V$.

Let $\Lambda \subset G$ a subsemigroup of $G$. We say that $\Lambda$ is irreducible if there does not exist a non-trivial subspace $W$ of $V$ stabilized by all elements of $\Lambda$, i.e. for all $\gamma \in \Lambda$, $gW = W$. We say that is it strongly irreducible if there doesn’t exist a finite number of non-trivial subspaces $W_1, \ldots, W_r$ of $V$ such that the reunion $\bigcup W_i$ is stabilized by all elements of $\Lambda$, i.e. for all $i$ and for $\gamma \in \Lambda$, there exists a $j$ such that $gW_i = W_j$. Remark that $\Lambda$ is strongly irreducible if all of its finite index subsemigroups are irreducible.

The main theme of this work is to understand the dynamical properties of a subsemigroup. These properties are related to the contraction of a transformation on the projective space. Here are the main notions we use.

An element $g \in G$ is proximal if it has a unique eigenvalue (counting multiplicity) of maximal modulus. In this case there exists a fixed point $Fix^+ g \in PV$ and a fixed hyperplane $Fix^- g \in PV^*$ such that the orbit of every point not in $Fix^- g$ converges (exponentially fast) to $Fix^+ g$.

The semigroup $\Lambda$ is proximal if there exists a sequence $g_n \in \Lambda$ and a sequence of scalars $a_n \in \mathbb{C}$ such that $a_ng_n$ converges to an endomorphism of rank 1. If $\Lambda$ contains a proximal element, then it is proximal. If $\Lambda$ is irreducible, the converse is true (see [2, Lem. 4.1]) in this case we think of the action of $\Lambda$ as contracting on $PV$, because the action of a rank 1 endomorphism $\pi$ on $PV$ contracts everything (away from $P(ker \pi)$) to the point $P(Im \pi)$.

When $\Lambda$ is irreducible and proximal, it admits a well-defined limit set $L(\Lambda) \subset PV$. It is the unique minimal $\Lambda$-invariant subset of $PV$. By definition, it is the set of points $x \in PV$ of the form $x = Im(\pi)$ for endomorphisms $\pi$ of rank 1 obtained as limits of sequences $a_ng_n$, where $g_n \in \Lambda$ and $a_n \in \mathbb{C}$ (see [2, Lem. 4.2]). It is the set where the dynamic of $\Lambda$ is concentrated. It is the closure of the set of attracting fixed points of proximal elements of $\Lambda$.  


For a linear representation $\rho : \Gamma \to G$, we say that the representation has the property $P$ if its image $\rho(\Gamma)$ has the property $P$ (where $P$ can be irreducible, proximal, etc.).

### 2.2. Random walks and Lyapunov exponents.

We introduce the notion of random walks on group. We will see that the data of random walk which generates a semigroup $\Lambda$ allows us to quantify all the qualitative properties of $\Lambda$ introduced in the previous paragraph 2.1.

#### 2.2.1. Random walk on the abstract group.

Let us fix a symmetric generator set of $\Gamma$ and the associated word distance $|\cdot|_\Gamma$. Let's fix a probability measure $\mu$ on $\Gamma$ such that the support of $\mu$ generates $\Gamma$ as a semi-group. An example of such a probability is

$$\mu = \frac{1}{\#S} \sum_{s \in S} \delta_s$$

where $S$ is a symmetric generator set for $\Gamma$.

The (left) random walk on $\Gamma$ induced by $\mu$ is the Markov chain with transition probability at $\gamma$ given by $(\gamma^{-1})_\mu$. This means that a trajectory for this process starting at $\gamma_0$ is given by $(\gamma^{-n})$ where $\gamma^{-n} = \gamma_n \cdots \gamma_0$ and $(\gamma_n)$ is a sequence of independent random variables with law $\mu$. The law of the $n$-th step of the random walk is $\mu^n$, the measure $\mu$ convoluted $n$-times with itself. We call $(\gamma_n)$ the sequence of increments and $(\gamma^{(n)})$ the associated trajectory. We denote by $I$ the set $\Gamma^\mathbb{N}$ and call it the space of increments, which we equip with the measure $\bar{\mu} = \mu^{\otimes \mathbb{N}}$. This set parametrizes the trajectories of the random walk.

We will assume that $\mu$ satisfies an exponential moment, that is there exists $\alpha > 0$ such that

$$\int_{\Gamma} \exp(\alpha |\gamma|_\Gamma) d\mu(\gamma) < \infty.$$  

It is always satisfied if the support of $\mu$ is finite. It implies the weaker first moment condition:

$$\int_{\Gamma} |\gamma|_\Gamma d\mu(\gamma) < \infty.$$  

We will use the reversed random walk, defined by: $\bar{\mu}(\gamma) = \mu(\gamma^{-1})$. Remark that it satisfies the same moment conditions as $\mu$.

#### 2.2.2. Linear random walks.

In our setting, linear random walks arise in the following way. Let $\mu$ be a probability on $\Gamma$ as in the previous paragraph and let $\rho : \Gamma \to G$ be a linear representation. It induces a random walk on $G$ given by $\mu_\rho := \rho_* \mu$, supported on $\rho(\Gamma)$.

Because $\mu$ satisfies and exponential moment, $\rho_* \mu$ satisfies a (linear) exponential moment:

$$\int_{\Gamma} \|\rho(\gamma)\|^{\alpha'} d\mu(\gamma) \leq \int_{\Gamma} \exp(\alpha' C_\rho |\gamma|_\Gamma) d\mu(\gamma) < \infty.$$
where $C_\rho = \log \max_s \|\rho(s)\|$ for $s$ in the generating set defining the distance on $\Gamma$, and $\alpha'$ is such that $\alpha' C_\rho \leq \alpha$.

The largest Lyapunov exponent associated to this random walk is by definition

$$
\chi(\rho) := \lim_{n \to \infty} \frac{1}{n} \int_{\Gamma} \log \|\rho(\gamma)\| d\mu^n(\gamma),
$$

where $\|\cdot\|$ is any norm (and the limit does not depend on the norm). The limit is well defined by subadditivity of the sequence appearing and by finiteness of the first moment. We define the Lyapunov spectrum $\chi_1, \ldots, \chi_{d+1}$ inductively by

$$
\chi_1(\rho) + \cdots + \chi_i(\rho) := \chi(\wedge^i \rho),
$$

where $\wedge^i \rho$ is the induced representation on the $i$-th exterior power $\wedge^i V$. We have $\chi_1 \geq \cdots \geq \chi_{d+1}$ and $\chi_1 + \cdots + \chi_{d+1} = 1$, so in particular $\chi_1 \geq 0$. In this article, we will be mainly interested about $\chi_1$, $\chi_2$ and $\chi_{d+1}$.

The largest Lyapunov exponent controls the exponential growth rate of typical random matrix product. More precisely, we have the following “Law of Large Numbers”:

**Theorem 1** (Furstenberg-Kesten). For $\bar{\mu}$-almost all trajectories $(\gamma_n) \in I$ we have

$$
\lim_{n \to \infty} \frac{1}{n} \log \|\rho(\gamma_n \cdots \gamma_1)\| = \chi_1(\rho).
$$

Of course, if $\chi_1 = 0$ we don’t get much information. For example, if the random walk generates a semigroup supported on a bounded group, it is always the case. This is essentially the only obstruction:

**Theorem 2** (Furstenberg). If $\rho$ is strongly irreducible and unbounded then

$$
\chi_1(\rho) > 0.
$$

This theorem is emblematic of the philosophy of this topic: it improves a qualitative statement (unboundedness) to a quantitative one: almost every trajectory grows exponentially fast (at a given rate). In the same spirit we have

**Theorem 3** (Furstenberg). If $\rho$ is strongly irreducible, then $\rho$ is proximal if and only if $\chi_1(\rho) > \chi_2(\rho)$.

Remark that $\chi_2 - \chi_1$ is the average exponential rate of contraction of tangent vector in $P V$.

We want to be more precise and to describe the behavior of random sequences $\rho(\gamma_n \cdots \gamma_1)v$ for $v \in V$. For that we will assume from now on that $\rho$ is proximal and strongly irreducible (it is not the weakest assumption, but it will be enough for the applications we have in mind).
A measure $\nu$ on $P \mathbb{V}$ is $\mu_\rho$-stationary if $\mu_\rho \ast \nu = \nu$ where $\mu_\rho \ast \nu = \int_{\Gamma} \rho(\gamma)_* \nu d\mu(\gamma)$. Such a measure always exists and under the proximal and strongly irreducible assumption it is unique (see for example [2, Prop. 4.7]). We denote it by $\nu_\rho$. Its support is precisely the limit set $L(\rho)$ of $\rho(\Gamma)$. It is a proper measure: every linear subspace of $P \mathbb{V}$ which is not $P \mathbb{V}$ is given zero mass. The stationary measure describes the distribution of the sequences $(\rho(\gamma_n \cdots \gamma_1)v)$; more precisely, for $\nu_\rho$-almost every $x \in P \mathbb{V}$, the sequence $(\rho(\gamma_n \cdots \gamma_1)x)$ equidistributes according to $\nu_\rho$.

The stationary measure allows us to define the Furstenberg limit map $\theta_\rho : \mathcal{I} \to P \mathbb{V}$, a measurable map which satisfies the following properties:

- for $\bar{\mu}$-almost all $b = (\gamma_n) \in \mathcal{I}$ we have
  $$\rho(\gamma_1 \cdots \gamma_n)_* \nu_\rho \to_{n \to \infty} \delta_{\theta_\rho(b)}$$

- $(\theta_\rho)_* \bar{\mu} = \nu_\rho$

- for $\bar{\mu}$-almost all $b = (\gamma_n) \in \mathcal{I}$, any endomorphism $\pi$ which is a limit point of a sequence $a_n \rho(\gamma_1 \cdots \gamma_n)$ with $a_n \in \mathbb{C}$ is of rank 1 and satisfies $\text{Im}(\pi) = \theta_\rho(b)$.

The first property is actually a definition of the map ([2, Prop. 4.7]). Notice the reversed order of composition in the properties above.

Dually, define the probability $\mu_\rho^t$ as the image of $\mu_\rho$ by the adjoint map $g \mapsto g^t \in G^* = SL(V^*)$. This defines a random walk on $G^*$ which is also strongly irreducible and proximal, and we call it the adjoint random walk. There exists a unique $\mu_\rho^t$-stationary measure $\nu_\rho^t$ on $P \mathbb{V}^*$ for the adjoint random walk, and a limit map $\theta^t : \mathcal{I} \to P \mathbb{V}^*$ which satisfies: for $\bar{\mu}$ almost every $\xi = (\gamma_n) \in \mathcal{I}$, any endomorphism $\pi$ which is a limit point of a sequence $a_n \rho(\gamma_n \cdots \gamma_1)$ with $a_n \in \mathbb{C}$ is of rank 1 and satisfies $\ker(\pi) = \theta^t(\xi)$ (again, notice the order of composition). The adjoint limit map allows us to describe the growth rate of individual vector:

**Proposition 2.1** ([2], Th. 4.8). For $\bar{\mu}$-almost every $\xi = (\gamma_n) \in \mathcal{I}$, for all $v \in P \mathbb{V}$ such that $v \notin \theta^t(\xi)$ we have

$$\lim_{n \to \infty} \frac{1}{n} \log \|\rho(\gamma_n \cdots \gamma_1)v\| = \chi_1(\rho).$$

It implies that for every $v \in V$ and for $\bar{\mu}$-almost every $\xi \in \mathbb{P}$, the limit above holds.

We see that the maximum growth rate is attained by almost every path of the random process. This result can be used to prove the Furstenberg’s formula: ([2, Th. 4.8]):

$$\int_{P \mathbb{V}} \int_{P \mathbb{V}} \log \|\rho(\gamma)v\|_{\|v\|} d\mu(\gamma) d\nu_\rho(v) = \chi_1(\rho).$$

We will need the following result which follows from the average contraction of the random walk (quantified by $\chi_2 - \chi_1 < 0$). Let $P_\rho$ be
the transition operator defined for a continuous function $f$ on $\mathbb{P}V$ by:

$$P_\rho f(x) = \int f(\rho(\gamma)x)d\mu(\gamma).$$

By results of [30], there exists $C > 0$, $\alpha \in (0, 1)$ and $\beta \in (0, 1)$ such that we have

$$\|P_n^\rho f - \int f d\nu\|_{C^\alpha} \leq C \beta^n \|f\|_{C^\alpha},$$

The norm is the standard Holder $C^\alpha$ norm. This result is called the exponential contraction of the transition operator. It is a deep result in the theory which implies many limit laws for the random walk. It assumes an exponential moment for the random walk.

2.2.3. Dual random walk. A important theme in the present work is the interplay between the random walk and the dual random walk.

Given, as in the previous paragraphs, a probability $\mu$ on $\Gamma$ and $\rho$ a linear representation $\Gamma \to G$ the dual random walk is the random walk on $G^* = SL(V^*)$ induced by the probability $\mu$ and the representation $\rho^*(\gamma) = \rho(\gamma^{-1})^t$ (remark that it is different from the adjoint random walk defined earlier). The representation $\rho^*$ is unbounded/irreducible/strongly irreducible/proximal if $\rho$ is.

The Lyapunov exponent ($\chi_1^*$) of the dual random walk satisfy

$$\chi_i^* = -\chi_{d+2-i}.$$

2.3. Complex analysis. We record here the standard notions of complex analysis needed in this article. A reference for all the facts stated is [7]. We fix a connected complex manifold $\Lambda$ of dimension $N$.

A function $f : \Lambda \to \mathbb{R} \cup \{-\infty\}$ is plurisubharmonic if it is upper semicontinous and if $f \circ L$ is subharmonic for every holomorphic map $L : D \to \Lambda$. It is plurisurharmonic if $-f$ is plurisubharmonic, and pluriharmonic if it is plurisubharmonic and plurisurharmonic.

Recall that a current $T$ of degree $p$ (and dimension $2N - p$) on $\Lambda$ is a continuous linear form on the space of differential forms of degree $N - p$ with compact support (it is locally a differential form of degree $p$ with distribution coefficients). The usual operations on forms extend by duality on currents. A current is of bidegree $(p, q)$ if it acts trivially on forms of bidegree different than $(N-p, N-q)$. A current of bidegree $(p, p)$ is positive if its local coefficients are positive measure. The main examples of positive current are given by integration over an analytic set: by a theorem of Lelong, if $A$ is an analytic set of pure dimension $p$ we can define the current of integration over $A$ by

$$[A] \cdot \alpha = \int_{A_{reg}} \alpha,$$

for $\alpha$ a form of bidegree $(p, p)$ with compact support, where $A_{reg}$ is the regular part of $A$; this current is well-defined and is a positive current.
We denote by \( d \) the exterior derivative on the differential forms on \( \Lambda \). The complex structure induces the decomposition \( d = d' + d'' \) where \( d' \) increases the bidegree by \((1,0)\) and \( d'' \) by \((0,1)\). We will use the real operator \( d^c = \frac{1}{2i\pi}(d' - d'') \). It satisfies \( dd^c = \frac{1}{i\pi}d'd'' \). We have the following characterization of plurisubharmonicitity: an uppersemicontinuous function \( f \) (which is not identically \(-\infty\)) is plurisubharmonic if and only if it is locally \( L^1 \) and \( dd^c f \) is a positive current (it is always of bidegree \((1,1)\)). It implies that a function is pluriharmonic if and only if \( dd^c f = 0 \) in the sense of currents.

An important class of examples of plurisubharmonic functions is given by the Poincaré-Lelong formula: if \( f \) is a holomorphic function which is not identically zero then \( \log |f| \) is plurisubharmonic, pluriharmonic outside \( f^{-1}(0) \) and
\[
 dd^c \log |f| = \sum m_i [Z_i],
\]
where the \( Z_i \) are the irreducible components of \( f^{-1}(0) \) and the \( m_i \) are their multiplicities.

Plurisubharmonic functions are convenient to work with because of their compactness properties. A version of Hartog’s Lemma (see [20]) states that if \((f_n)\) is a sequence of plurisubharmonic functions locally bounded above, then either \((f_n)\) converges uniformly to \(-\infty\) or we can extract a sequence \((f_{n_k})\) converging in \( L^1_{\text{loc}} \) to a plurisubharmonic function \( f \), and we have \( \limsup f_{n_k}(x) = f(x) \) almost everywhere. In particular, if \((f_n)\) is a sequence of plurisubharmonic functions locally bounded above and converging pointwise almost everywhere to a function \( f \), then \( f \) is plurisubharmonic. In the same way, if \((f_n)\) is a sequence of pluriharmonic functions locally bounded and converging pointwise almost everywhere to a function \( f \), then \( f \) is pluriharmonic.

Plurisubharmonic functions are subharmonic and therefore satisfy the maximum principle: if a plurisubharmonic function reaches its maximum on an open set, it is constant on this set.

2.4. Family of representations. We fix a connected complex manifold \( \Lambda \), the parameter space. Let \((\rho_\lambda)_{\lambda \in \Lambda} \) be a holomorphic family of representations \( \Gamma \to G \). It means that for all \( \gamma \), the map \( \lambda \mapsto \rho_\lambda(\gamma) \) is holomorphic.

We consider the Lyapunov exponents as functions on \( \Lambda \):
\[
 \chi_i : \lambda \mapsto \chi_i(\rho(\lambda)).
\]
The functions \( \chi_1 + \cdots + \chi_i \) are plurisubharmonic. Indeed, we can use the norm \( \|g\|_2 = \sqrt{\sum g_{ij}^2} \) to define \( \chi \) and it is clear that
\[
 \frac{1}{n} \int_{\Gamma} \log \|\Lambda^i \rho(\gamma)\|_{2d\mu^\gamma}(\gamma)
\]
is a sequence of plurisubharmonic maps, bounded above by the assumption of first moment, converging pointwise to \( \chi_1 + \cdots + \chi_i \).
If at a parameter \( \lambda \) the representation \( \rho_\lambda \) is strongly irreducible and proximal, we denote by \( \nu_\lambda \) and \( \nu^*_\lambda \) the associated stationary measures and by \( \theta_\lambda \) and \( \theta^*_\lambda \) the associated limit maps. We also denote by \( L(\lambda) \) its limit set.

We introduce a notion of dynamical stability for families of representations, which is the main topic of this paper.

**Definition 1.** The family of representations \( (\rho_\lambda) \) is proximally stable if for any \( \lambda_0 \in \Lambda \) and \( \gamma \in \Gamma \), if \( \rho_{\lambda_0}(\gamma) \) is proximal then \( \rho_\lambda(\gamma) \) is proximal for all \( \lambda \in \Lambda \).

In this case, the fact that \( \rho_\lambda(\gamma) \) is proximal is independent of \( \lambda \), and if it is the case we say that \( \gamma \) is proximal. If \( \gamma \) is proximal it is clear that \( \lambda \mapsto \text{Fix}^+\rho_\lambda(\gamma) \) and \( \lambda \mapsto \text{Fix}^-\rho_\lambda(\gamma) \) are holomorphic, and that the logarithm of the spectral radius of \( \rho_\lambda(\gamma) \) is a pluriharmonic function of \( \lambda \). When the context is clear, we denote these maps (or their graph) by \( \text{Fix}^+\gamma \) and \( \text{Fix}^-\gamma \).

We call stability locus the largest subset of \( \Lambda \) on which \( (\rho_\lambda) \) is proximally stable. Its complement is called the bifurcation locus.

We define currents on \( \Lambda \) by \( T_i = dd^c\chi_i \), for \( i = 1, \ldots, d+1 \). The complement of the support of \( T_i \) is the largest open set on which \( \chi_i \) is pluriharmonic. Finally we set \( T_{bif} = T_1 + T_{d+1} \).

### 3. Results

From now on we will assume the standing assumptions:

> for all \( \lambda \in \Lambda \), the representation \( \rho_\lambda \) is strongly irreducible and proximal

Our main result is:

**Theorem 4.** Under the standing assumption, the Lyapunov exponents \( \chi_1 \) and \( \chi_{d+1} \) are pluriharmonic on \( \Lambda \) if and only if \( (\rho_\lambda) \) is proximally stable.

When the probability \( \mu \) is symmetric (that is \( \mu(\gamma) = \mu(\gamma^{-1}) \) for all \( \gamma \in \Gamma \)) the Lyapunov spectrum is symmetric: \( \chi_i = -\chi_{d+2-i} \). In this case \( \chi_1 \) is pluriharmonic if and only if \( \chi_{d+1} \) is, so we can state:

**Corollary 1.** If \( \mu \) is symmetric, under the standing assumption, \( \chi_1 \) is pluriharmonic if and only if \( (\rho_\lambda) \) is proximally stable.

**Remark.** We can state this theorem in term of the bifurcation current \( T_{bif} \):

> Under the standing assumptions, \( T_{bif} \) vanishes on \( \Lambda \) if and only if \( (\rho_\lambda) \) is proximally stable,

or in term of the bifurcation locus

> Under the standing assumptions, the support of \( T_{bif} \) is precisely the bifurcation locus.
We then show that for proximally stable families we can propagate some interesting properties which are true at one parameter, to every parameter.

**Theorem 5.** Under the standing assumption, suppose that \((\rho_\lambda)\) is proximally stable. Let \(P_\lambda\) be one of the following property:

1. the representation \(\rho_\lambda\) is faithful,
2. the representation \(\rho_\lambda\) is discrete,
3. the representation \(\rho_\lambda\) is projectively Anosov.

If \(P_{\lambda_0}\) holds for some \(\lambda_0 \in \Lambda\) then \(P_\lambda\) for every \(\lambda \in \Lambda\).

See section 6.3 for a definition of the Anosov property.

3.1. **Hypothesis of the theorem.** Let us explain the hypothesis of theorem 4. We assume that the representations of the family are proximal, otherwise the property of proximal stability is trivially satisfied. The assumption of strong irreducibility is used to apply results from the theory of random walks. In the section 5.1 we explain how we can weaken these assumptions, to prove the ”if” part of the main theorem. More precisely we can reduce the standing assumptions of the direct implication of theorem 4 to:

there exists one parameter \(\lambda_0\) such that \(\rho_{\lambda_0}\) is strongly irreducible and proximal.

3.2. **Generalization.** Theorem 4 is a result about proximal actions of subgroups of \(G = SL(V)\) acting on \(PV\). It is probable that it can be extented to subgroups of a general semisimple complex Lie group acting proximally on a flag variety.

In this work, the random walk is used as an accessory tool to define the Lyapunov exponents and to see the group \(\Gamma\) as a dynamical system. It is probable that we could use a different dynamical system and obtain the same result. For example, when \(\Gamma\) is the fundamental group of a compact surface, we could use the geodesic flow or the Brownian motion of an hyperbolic structure on this surface as in [9]. A more challenging problem would be to replace the random walk by a subshift of finite type (the random walk corresponding to the full shift), see [26] where this direction is explored for Anosov representations.

3.3. **Equidistribution.** The theorem 4 states that the bifurcation locus coincides with the support of the bifurcation current \(T_{bif}\). The interest of this result is that we can now use tool from potential theory to describe the bifurcation locus. The general machinery developed in [10] applies in our setting (with identical proofs) and we obtain equidistribution results. More precisely:

Let \((\rho_\lambda)_{\lambda \in \Lambda}\) be a family of representation. Fix \(t \in C\) and let \(Z(\gamma, t)\) be the subset of \(\Lambda\) composed of \(\lambda\) such that \(\text{tr}(\rho_\lambda(\gamma)) = t\).
Proposition 3.1. Let $(\rho_\lambda)$ be a family of representations satisfying the standing assumptions. Suppose that $\mu$ is symmetric. Then for $\mu$-almost every $(\gamma_n) \in \mathcal{I}$, the sequence of currents of integration

$$\frac{1}{n} \left[ Z(\gamma_n \cdots \gamma_1, t) \right]$$

converges to $T_{bif}$.

3.4. Outline of the proof. The easy implication (proximally stable implies pluriharmonicity of the Lyapunov exponent) is proved in section 4.

For the reverse implication, the first step is to prove the existence of an holomorphic variation of the Poisson boundary, an object which binds in a family of holomorphic map all the boundary map of the family of representation (section 5.3). To construct this variation, we need a result which allows us to control the volume growth of the trajectories of the random walk applied to certain graphs, in term of the bifurcation current. (section 5.2). Then we show that the holomorphic variation of the Poisson boundary is compact, meaning that the associated family of map is relatively compact in the uniform topology (section 5.5.2). To prove the compactness, we show a non-intersection property between the graphs of the variation of the Poisson boundary and the graphs of the dual variation of the Poisson boundary (section 5.4). Using the compactness, we can conclude and prove that if a matrix is proximal at some parameter, it stays proximal for every parameters (section 5.6).

In the last section 6 we prove some consequences of the proximal stability.

3.5. Comparison of the different notions of stability. In this section we elaborate on the comparison of the different notions of stability we hinted to in the introduction, and compare them with the one we introduce in this article.

For endomorphisms of $\mathbb{P}^1(\mathbb{C})$ Mañé, Sad and Sullivan [32] and independently Lyubich [31] introduced a notion of stability for holomorphic family. They showed that the stability of a family $(f_\lambda)_{\lambda \in \Lambda}$ can be defined by the following equivalent properties, among others (for connected $\Lambda$):

1. the periodic points of $(f_\lambda)$ do not change type.
2. the Julia sets $(J_\lambda)$ vary continuously in the Hausdorff topology.
3. all the $(f_\lambda)$ are quasiconformally-conjugated on their Julia set.

The decomposition stability-bifurcation of the parameter space is analogous to the decomposition Julia-Fatou for a single endomorphism.

For representations with value in $\text{SL}(2, \mathbb{C})$, Sullivan [36] proved that very different notions of stability are in fact equivalent. Let $\rho : \Gamma \to \text{SL}(2, \mathbb{C})$ be a family of representations satisfying the standing assumptions. Suppose that $\mu$ is symmetric. Then for $\mu$-almost every $(\gamma_n) \in \mathcal{I}$, the sequence of currents of integration

$$\frac{1}{n} \left[ Z(\gamma_n \cdots \gamma_1, t) \right]$$

converges to $T_{bif}$.
Let $\rho \colon \Gamma \to \text{SL}(2, \mathbb{C})$ be a faithful, unbounded, non-rigid, torsion-free representation with $\Gamma$ a finitely generated group. Then the following are equivalent:

1. $\rho$ is convex-cocompact.
2. $\rho$ is algebraically stable.
3. $\rho$ is structurally stable.
4. The type of the elements of $\rho$ is stable.
5. $\rho$ satisfies an hyperbolicity condition on its limit set.

Convex-cocompact is a geometric property. Algebraically stable means that nearby representations stay faithful; it is an algebraic property. Structurally stable means that the action on $\mathbb{P}^1$ of nearby representations is (smoothly) conjugated to the one of $\rho$ on their limit set: it is a dynamical property. The stability of the type of elements of $\rho$ means that the type (hyperbolic, parabolic or elliptic) of a matrix $\rho(\gamma)$ doesn’t change for nearby deformation of $\rho$. The hyperbolicity condition is a property of expansion, analogous to the one found in hyperbolic dynamics. It is a remarkable fact, specific to the rank 1, that such different properties are equivalent. Observe that for representations as well as for endomorphisms, the $\lambda$-lemma is a crucial tool to construct conjugacies for stable families.

When generalizing to higher rank groups, the equivalences break down, a priori (for instance, in the case of representations of $\mathbb{Z}$, one easily sees that algebraic stability is not equivalent to structural stability). As said before, the right generalization of convex-cocompact in higher rank is the notion of Anosov representation. It is known that they enjoy all of the previously defined properties of stability. The situation is less clear for the reciprocal. In [23], Kapovich, Kim and Lee explore the generalization of the hyperbolicity property and show that being Anosov is equivalent to a kind of hyperbolicity property (see [23, th. 6.3]).

In this article we study the notion of proximal stability (for family of representations) (see the definition in section 3). It is analogous to the notion of stability of repelling $J$-cycles in [4] for families of endomorphisms of $\mathbb{P}^d(\mathbb{C})$ (which requires that repelling cycles move holomorphically) and the notion of weak $J^*$-stability of [14] for polynomial automorphism of $\mathbb{C}^2$ (which requires that saddle periodic points move holomorphically). We think of it as the generalization of the "stability of type of elements" property for $\text{SL}(2, \mathbb{C})$. We say that a representation is proximally stable if the family composed of all nearby deformations is proximally stable. It is clear that Anosov representations are proximally stable. It would be an interesting question to study if proximally stable representations are also Anosov. For now, we can prove that proximally stable representations satisfy some kinds of stability: they are algebraically stable, stably discrete, and stably Anosov, see section 6. The propagation of the Anosov property for proximally
stable families of representations is an analogue of the propagation of uniform hyperbolicity for weakly stable families of polynomial automorphisms in [3].

4. PROXIMALLY STABLE IMPLIES PLURIHARMONICITY OF THE LYAPUNOV EXPONENT

In this section we prove the “easy” direction of the main theorem[4]. Let \((\rho_\lambda)\) be a family of representations satisfying the standing assumptions and which is proximally stable. We show that \(\chi_1\) and \(\chi_{d+1}\) are pluriharmonic.

By the proposition 4.2 in [10], for almost every \((\gamma_n) \in I\) we have:

\[
\frac{1}{n} \log \|\rho_\lambda(\gamma^{(n)})\| \to \chi_1(\lambda),
\]

where the convergence holds in \(L^1_{loc}\). Remark here that the \((\gamma_n)\) can be chosen independently of \(\lambda\). We now show that it holds also when we replace the norm \(\|\cdot\|\) by the spectral radius \(r\). For that we use the following lemma, which is a variation of the Hartog’s Lemma:

**Lemma 6** ([10], Lemma 4.3). Let \((u_n)\) be a sequence of plurisubharmonic functions converging in \(L^1_{loc}\) to a continuous plurisubharmonic function \(u\) and \((v_n)\) another sequence of plurisubharmonic functions such that

- \(v_n \leq u_n\) pointwise,
- \(v_n(x) \to u_n(x)\) for \(x\) in a dense subset,

then \(v_n \to u\) in \(L^1_{loc}\).

Let \((\lambda_k)\) be a dense sequence of parameters in \(\Lambda\). For almost all \((\gamma_n)\) we have that:

\[
\frac{1}{n} \log r(\rho_\lambda(\gamma^{(n)})) \to \chi_1(\lambda_k),
\]
\[
\frac{1}{n} \log \|\rho_\lambda(\gamma^{(n)})\| \to \chi_1(\lambda),
\]

where the second convergence is in \(L^1_{loc}\) and that \(\rho(\gamma_n)\) is proximal for \(n\) large enough (see [2, Th. 4.12]). Fix a sequence \((\gamma_n)\) such that[4] and these hold.

We define:

\[
u_n : \lambda \mapsto \frac{1}{n} \log r(\rho_\lambda(\gamma^{(n)})).\]

All of these functions are plurisubharmonic and we have \(v_n \leq u_n\), \(u_n \to \chi_1\) and \(v_n(\lambda_k) \to \chi_1(\lambda_k)\). By the Lemma [5] we have that \(u_n\) converges to \(\chi_1\) in \(L^1_{loc}\).
Moreover, each \( v_n \) is pluriharmonic thanks to the proximal stability. This sequence is bounded below by 0 and bounded above by the moment assumption. It follows, by Hartog’s lemma, that the limit \( \chi_1 \) is also pluriharmonic.

We now deduce the pluriharmonicity of \( \chi_{d+1} \) from the pluriharmonicity of \( \chi_1 \):

The top Lyapunov exponent \( \chi_1^* \) of the dual representation is pluriharmonic by the argument above. We have that \( \chi_1^* = -\chi_{d+1} \), so \( \chi_{d+1} \) is also pluriharmonic.

5. Pluriharmonic Lyapunov exponent implies proximal stability

5.1. A series of reductions. In this section we justify the claim made in the discussion of the hypothesis 3.1, that we can weaken the standing assumption to

there exists one parameter \( \lambda_0 \) such that \( \rho_{\lambda_0} \) is strongly irreducible and proximal.

We also explain why we can assume in the proof of the theorem that the dimension of \( \Lambda \) is 1.

5.1.1. Reduction to the one dimensional case. Assume the theorem holds if \( \Lambda \) is a complex manifold of dimension 1. Let \( (\rho_\lambda)_{\lambda \in \Lambda} \) be a family of representations satisfying the standing assumptions and that \( \chi_1 \) and \( \chi_{d+1} \) are pluriharmonic. Let \( \lambda_0, \lambda_1 \in \Lambda \) and suppose that \( \rho_{\lambda_0}(\gamma) \) is proximal for some \( \gamma \in \Gamma \). We need to show that \( \gamma \) is proximal at \( \lambda_1 \).

It is enough to show that \( \lambda_0 \) and \( \lambda_1 \) can be connected by a sequence of intersecting one dimensional complex manifolds. By compactness of a path joining \( \lambda_0 \) to \( \lambda_1 \) we can find a sequence \( B_0, \ldots, B_N \) of open sets biholomorphic to a standard ball such that \( \lambda_0 \in B_0, \lambda_1 \in B_N \) and \( B_i \cap B_{i+1} \neq \emptyset \). For each \( i \) we can find \( D_i \subset B_i \) such that \( D_i \) is a submanifold biholomorphic to a one-dimensional complex disk, \( \lambda_0 \in D_0, \lambda_1 \in D_N \) and \( D_i \cap D_{i+1} \neq \emptyset \). By applying successively the one-dimensional case theorem to the \( \rho_{|D_i} \), if follows that \( \gamma \) is proximal on each \( D_i \) and then at \( \lambda_1 \).

Now that we assume \( \dim \Lambda = 1 \), the notions of pluriharmonicity, plurisubharmonicity, etc. are equivalent to harmonicity, subharmonicity, etc., so we will drop the prefix ”pluri” from now on.

5.1.2. Proximality at every parameters. We show that if \( (\rho_\lambda) \) is a family which is strongly irreducible at every parameters and proximal at some parameter then it is proximal at every parameter, when \( \chi_1 \) is harmonic.

By [2, Cor. 10.5], the representation \( \rho_\lambda \) is proximal if and only if \( \chi_1(\lambda) > \chi_2(\lambda) \). By assumption this inequality is true at some parameter. The function \( \chi_1 + \chi_2 \) is subharmonic and \( \chi_1 \) is harmonic, so \( \chi_2 \)
is subharmonic. The function $\chi_1 - \chi_2$ is then superharmonic and non-negative. By the minimum principle, if it vanishes at some point it’s constantly equal to 0. It is not the case, so the function is positive. We conclude that for all $\lambda$, $\chi_1(\lambda) > \chi_2(\lambda)$ and that $\rho_\lambda$ is proximal.

5.1.3. Irreducibility and strong irreducibility. We show that if $(\rho_\lambda)$ is irreducible at every parameters and strongly irreducible and proximal at at least one parameter, then it is strongly irreducible at every parameters if $\chi_1$ is harmonic.

By the reasoning of the previous section 5.1.2, we have $\chi_1 > \chi_2$ everywhere. It implies that the family is proximal at every parameter, by irreducibility. It is enough to show the following (classical) lemma for a fixed representation:

**Lemma 7.** If $\rho : \Gamma \to G$ is an irreducible representation which is proximal then $\rho$ is strongly irreducible.

**Proof.** First, remark that there exists a proximal element in the image of $\rho$, because $\rho$ is proximal and irreducible.

Now we assert that an irreducible representation with a proximal element is strongly irreducible. Consider a virtually invariant, strongly irreducible subspace $W$: it’s a subspace such that $\rho(\Gamma)W$ is finite, and no proper subspace is virtually invariant. Assume that $W$ is not all of $\mathbb{C}^{d+1}$, that is $\rho$ is not strongly irreducible. Let $L^+$ and $H^-$ be the attractive line and repulsive hyperplane of $g$. Observe that the orbit $\rho(\Gamma)W$ spans a vector space which is stable by $\rho$: by irreducibility it must be the whole space. In particular, the orbit $\rho(\Gamma)W$ is not contained in $H^-$. Remark that for distincts $W_1, W_2 \in \rho(\Gamma)W$, the intersection $W_1 \cap W_2$ is trivial. Otherwise, it would also be a strongly irreducible virtually invariant subspace and a proper subspace of $W_1$.

The line $L^+$ is in some $W_1 \in \rho(\Gamma)W$ because $\bigcup \rho(\Gamma)W$ is closed, stable by $g$ and if $z \in \bigcup \rho(\Gamma)W \setminus H^-$ then $g^n z$ converges to $L^+$. Then $W_1$ is stable by $g$, because $gW_1 \cap W_1$ contains $L^+$. Now, considering the action of $g$ on the finite set $\rho(\Gamma)W$, we see that the reunion $\bigcup \rho(\Gamma)W \setminus W_1$ is stable by $g$. But for any $z \in \bigcup \rho(\Gamma)W \setminus W_1$, the iterates $g^n z$ converge to $L^+$, which is absurd because $L^+$ is at a positive (projective) distance from $\bigcup \rho(\Gamma)W \setminus W_1$. $\square$

5.1.4. Irreducibility. We first show that if $(\rho_\lambda)_{\lambda \in \Lambda}$ is a family which is strongly irreducible and proximal at one parameter $\lambda_0$, then it is strongly irreducible and proximal on a Zariski dense set $\Lambda_0 \subset \Lambda$, if $\chi_1$ is harmonic. By the results of sections 5.1.2 and 5.1.3, it is enough to show that the family is irreducible on a Zariski dense set. This is true, because being irreducible is an Zariski open condition (see for example [34, prop. 27]).
Now, if we assume that the family \((\rho_\lambda)\) is strongly irreducible and proximal at one parameter \(\lambda_0\), then it is strongly irreducible and proximal on a Zariski dense set \(\Lambda_0 \subset \Lambda\), and we can apply the theorem \([4]\) to the family \((\rho_\lambda)_{\lambda \in \Lambda_0}\). To get the stability of proximal elements at the discrete set of points \(\Lambda \setminus \Lambda_0\), we apply the following lemma \([8]\) at these points.

**Lemma 8.** Let \(M(\lambda)\) be a holomorphic family of matrices indexed by \(\mathbb{D}\) such that for \(\lambda \neq 0\) the matrix \(M(\lambda)\) is proximal. Then \(M(0)\) is proximal.

**Proof.** Let \(\mu_1(\lambda)\) and \(\mu_2(\lambda)\) be the eigenvalues of greatest and second greatest modulus of \(M(\lambda)\), for \(\lambda \neq 0\). Let \(l_1 = \log |\mu_1|\) and \(l_2 = \log |\mu_2|\). The function \(l_1\) is harmonic, because \(\mu_1\) is holomorphic. The function \(l_2\) is subharmonic and continuous because it is the logarithm of the spectral radius of the restriction of \(M(\lambda)\) to its repelling hyperplane. Let \(f = l_1 - l_2\). It is a continuous, superharmonic, positive function on \(\mathbb{D}^*\). It extends to a continuous function at 0 (see \([27, \text{p. II.2}]\)). We show that such a function can’t vanish at 0. Up to restricting and rescaling, we can suppose that \(f\) is defined and continuous on \(\partial \mathbb{D}\).

We pullback \(f\) to the universal covering of \(\mathbb{D}^*\): let \(F = f \circ \pi\) where \(\pi: \mathbb{H} \to \mathbb{D}^*\) is given by \(\pi(z) = \exp(iz)\). The function \(F\) satisfies the same properties as \(f\): it is superharmonic, continuous, and positive. We show that it doesn’t vanish at infinity.

Define, for \(x + iy \in \mathbb{H}\),

\[
P(x + iy) = \frac{y}{x^2 + y^2},
\]

the Poisson kernel for \(\mathbb{H}\). As \(F\) is superharmonic we have for all \(z \in \mathbb{H}\):

\[
F(z) \geq \int_{-\infty}^{+\infty} P(z + t) F(t) dt.
\]

The map \(F\) is continuous, positive and periodic on the real line \(\mathbb{R}\), so it is bounded below by some \(m > 0\). We have for \(y > 0\):

\[
F(iy) \geq m \int_{-\infty}^{+\infty} P(iy + t) dt
\]

\[
\geq m \int_{y}^{+\infty} P(iy + t) dt
\]

\[
= m \int_{y}^{+\infty} \frac{y}{y^2 + t^2} dt
\]

\[
\geq m \int_{y}^{+\infty} \frac{y}{2t^2} dt
\]

\[
= \frac{m}{2} > 0,
\]

so \(F\) doesn’t vanish at infinity, and \(f\) doesn’t vanish at 0. It follows that \(M\) stays proximal at 0. \(\square\)
5.2. The volume of graphs is controlled by the bifurcation current. In this section we show the Proposition 5.1 that asserts that the volume of iterates of graphs under a path of the random walk is controlled, in average, by the bifurcation current. First we introduce some notations and show some technical lemmas.

The results of this section are local in nature, so we will assume that $\Lambda$ is the unit disk. A family $(\rho_o)_o$ of representations naturally gives an action of $\Gamma$ on objects fibered over $\Lambda$. For example, it acts on a function $f: \Lambda \to \mathbb{P}V$ by:
$$\gamma \cdot f := \rho(\gamma) f = \lambda \mapsto \rho_\lambda(\gamma) \cdot f(\lambda),$$
and on function $g: \Lambda \to \mathbb{P}V^*$ by:
$$\gamma \cdot g := \rho^*(\gamma) f = \lambda \mapsto \rho_\lambda(\gamma^{-1})^t \cdot f(\lambda).$$

For a holomorphic map $\sigma$ from $\Lambda$ to $\mathbb{P}V$ we denote $\Gamma(\sigma)$ its graph, which is a complex submanifold of $\Lambda \times \mathbb{P}V$ of dimension 1.

We recall that for an analytic set $A$ of codimension $k$, we denote by $[A]$ the current of integration over $A$ which is a positive $(k, k)$-current.

We denote by $\pi_1$ and $\pi_2$ the projections $\Lambda \times \mathbb{P}V$ over $\Lambda$ and $\mathbb{P}V$. We can suppose $\Lambda$ is equipped with its standard Kähler form, $\omega_1$ and we equip $\mathbb{P}V$ with the Fubini-Study metric $\omega_2$. We equip $\Lambda \times \mathbb{P}V$ with the Kähler form $\omega = \pi_1^* \omega_1 + \pi_2^* \omega_2$.

For a holomorphic map $f: U \to \mathbb{P}V$, we define $dd^c |f|$ as the following. Let $F$ be a local holomorphic lift of $f$ to $V$. Then $dd^c \log |F|$ is (locally) well-defined and another choice of $F$ gives the same result. We denote by $dd^c \log |f|$ the (global) current on $U$ it defines. In fact we have $dd^c \log |f| = f^* \omega_2$ (see [12, Ex. 6.2.14]).

If $T$ is a positive $(1, 1)$-current on $\Lambda$, we denote by $\|T\|_K$ its mass over the (relatively) compact subset $K$ defined by
$$\|T\|_K := \int_K T.$$

The functions $\chi_1$ and $\chi_{d+1}$ are plurisubharmonic on $\Lambda$. We define the positive $(1, 1)$-currents $T_1$, $T_{d+1}$ and $T_{bif}$ on $\Lambda$ by
$$T_1 = dd^c \chi_1, \quad T_{d+1} = dd^c \chi_{d+1}, \quad T_{bif} = T_1 + T_{d+1}.$$

The following lemma gives a formula for the volume of a graph. Let $U$ be a relatively compact open set in $\Lambda$.

**Lemma 9.** Let $f$ be a holomorphic map from $U$ to $\mathbb{P}V$. Then:
$$\text{Vol}(\Gamma(f)) = \text{Vol}(U) + \|(dd^c \log |f|)\|_U.$$

**Proof.** We have
$$\text{Vol}(\Gamma(f)) = \int_{\Gamma_f} \omega = \int_{\Gamma_f} \pi_1^* \omega_1 + \pi_2^* \omega_2.$$
The restriction of the projection $\pi_1 : \Gamma_f \to U$ induces a biholomorphism (that we denote by the same symbol). It gives, for the first term of the sum: $f_1^* \omega_1 = f_1^* \omega_1 = \text{Vol}(U)$.

For the second term of the sum, remark that $\pi_2^{-1} \pi_1$ is just $f$. We then have:

$$\int_{\Gamma_f} \pi_2^* \omega_2 = \int_{\Gamma_f} \pi_1^* \pi_2^{-1} \pi_2^* \omega_2 = \int_U (\pi_2^{-1} \pi_1^*) \omega_2 = \int_U f^* \omega_2.$$ 

Recall that $f^* \omega_2 = \frac{\partial f}{\partial z} \log |f|$, so finally:

$$\int_{\Gamma_f} \pi_2^* \omega_2 = \int_U \frac{\partial f}{\partial z} \log |f| = \|\frac{\partial f}{\partial z} \log |f||_U.$$ 

□

To use the Lemma 9, with $f$ of the form $\lambda \mapsto \gamma_0^{(n)}(\lambda)$ we need estimates on the convergence of potentials to the Lyapunov exponent.

**Lemma 10.** Under the standing assumptions on the family of representations and $\mu$, for any $x \in PV$ we have

$$\chi^{(n)}_1(\lambda) := \int \log \frac{\|\rho_\lambda(\gamma(n))x\|}{\|x\|} d\mu^n(\gamma) = n\chi_1(\lambda) + O(1),$$

where the $O(1)$ is locally uniform in $\lambda$ in the $L^\infty$ norm, and:

$$dd^c \chi^{(n)}_1 = n dd^c \chi_1 + O(1),$$

where the $O(1)$ is in the mass norm over any compact.

**Proof.** We sketch the argument, which is the same as in [10, Prop. 3.8]. We define $\varphi$ on $PV$ by

$$\varphi(x) = \int \log \frac{\|\rho_\lambda(\gamma)x\|}{\|x\|} d\mu(\gamma),$$

and by Furstenberg’s formula we have $\int \varphi d\nu_\lambda = \chi_1(\lambda)$. Recall that $P_\lambda$ is the transition operator. We compute:

$$\int \log \frac{\|\rho_\lambda(\gamma_n \cdot \cdot \cdot \gamma_1)x\|}{\|x\|} d\mu(\gamma_n) \cdot \cdot \cdot d\mu(\gamma_1)$$

$$= \sum_{k=0}^{n-1} \int \log \frac{\|\rho_\lambda(\gamma_{k+1} \cdot \cdot \cdot \gamma_1)x\|}{\|\rho_\lambda(\gamma_k \cdot \cdot \cdot \gamma_1)x\|} d\mu(\gamma_k) \cdot \cdot \cdot d\mu(\gamma_1)$$

$$= \sum_{k=0}^{n-1} F_\lambda^k \varphi(x),$$
so,
\[
\int \log \frac{\|\rho(\gamma_n \cdots \gamma_1)x\|}{\|x\|} \, d\mu(\gamma_n) \cdots d\mu(\gamma_1) - n\chi_1(\lambda) = \sum_{k=0}^{n-1} (P^k_\lambda \varphi(x) - \int \varphi \, d\nu_\lambda)
\]
and as we have
\[
\|P^n_\lambda \varphi - \int \varphi \, d\nu_\lambda\|_{C^\alpha} \leq C \beta^n \|\varphi\|_{C^\alpha},
\]
the last sum is bounded, independently of \(x\). It implies the estimate at a fixed representation:
\[
\log \frac{\|\rho(\gamma_n \cdots \gamma_1)x\|}{\|x\|} \, d\mu(\gamma_n) \cdots d\mu(\gamma_1) = n\chi_1 + O(1).
\]
The estimate is locally uniform in \(\lambda\) because the constants \(\alpha, C, \beta\) can be chosen locally independently of \(\lambda\), because they depend only of the average contraction
\[
\sup_{x \neq y \in \mathbb{V}} \int \left( \frac{d(\rho_\lambda(\gamma)x, \rho_\lambda(\gamma)y)}{d(x, y)} \right)^{\alpha_0} \, d\mu^{\alpha_0}(\gamma) < 1,
\]
(for some \(\alpha_0, n_0\)) and the exponential moment condition on \(\mu\).

The second estimate is deduced from the first by an application of the Chern-Levine-Nirenberg inequality (see [7, prop. 3.3]).

From these lemmas we can deduce the main result of this section, a first step for the study of the stability of fixed points.

**Proposition 5.1.** Let \(\sigma_0\) be a constant map \(U \to \mathbb{P}V\). Under the standing assumptions, if \(T_1 = 0\) on \(U\), the mean volume of the graphs of maps \(\gamma^{(n)}\sigma_0\) is bounded, that is
\[
\int \text{Vol}(\Gamma(\gamma^{(n)}\sigma_0)) \, d\mu^n(\gamma) = O(1).
\]

**Proof.** By lemma 9 it is enough to bound:
\[
\int \|dd^c \log |\gamma^{(n)}\sigma_0|\|_U \, d\mu^n(\gamma) = \int \|dd^c \log |\gamma^{(n)}\sigma_0|d\mu^n(\gamma)\|_U
\]
A lift of \(\gamma^{(n)}\sigma_0\) to \(V\) is given by
\[
F_n(\lambda) = \rho_\lambda(\gamma^{(n)}) \cdot v_0,
\]
where \(v_0\) is a non-zero vector in the line given by \(\sigma_0\). By lemma 10
\[
\int \|dd^c \log |F_n(\lambda)|\|_U \, d\mu^n(\gamma) = O(1),
\]
in the mass norm, as we have \(dd^c \chi_1 = T_1 = 0\). This means that
\[
\| \int \int dd^c \log |\gamma^{(n)}\sigma_0|d\mu^n(\gamma)\|_U = O(1),
\]
and the claim follows.

We can state this result for the dual random walk:
Proposition 5.2. Let $\varphi_0$ be a constant map $U \to \mathbb{P}V^*$. Under the standing assumptions, if $T_{d+1} = 0$ on $U$, we have

$$\int \text{Vol}(\Gamma(\gamma^{(n)}\varphi_0))d\mu^n(\gamma) = O(1).$$

Indeed, recall that the dual Lyapunov exponent $\chi_1^*$ equals $-\chi_{d+1}$.

5.3. Holomorphic variation of the Poisson boundary. In this section we show that the pluriharmonicity of $\chi_1$ allows us to construct a holomorphic variation of the Poisson boundary.

The Poisson boundary of the random walk $(\Gamma, \mu)$ is a measurable space that governs the asymptotic properties of trajectories of the (right) random walk.

We recall briefly a construction of the Poisson boundary. Let $\mathcal{I} = \Gamma^\mathbb{N}$ the space of increments with the measure $\bar{\mu} = \mu^\mathbb{N}$.

The natural map from the space of increment $\mathcal{I}$ to the space of trajectories $\mathcal{T} = \Gamma^\mathbb{N}$:

$$\tau : \mathcal{I} \to \mathcal{T}$$

$$\gamma_n \mapsto (\gamma^{(n)} = \gamma_1 \cdots \gamma_n)$$

pushes the measure $\bar{\mu}$ to a measure $P = \tau_\ast \bar{\mu}$ and induces an isomorphism of measured spaces. The space of trajectories admits a natural action of $\Gamma$, which is coordinate-wise multiplication on the left. Both $\mathcal{I}$ and $\mathcal{T}$ are equipped with the coordinate shift defined by $\sigma((x_n)) = (x_{n+1})$ and we have

$$\tau(\sigma(\gamma_n)) = \gamma_1^{-1}\sigma\tau(i).$$

We define the following equivalence relation on $\mathcal{T}$:

$$t \sim t' \iff \exists k, k' \in \mathbb{N}, \sigma^k(t) = \sigma^{k'}(t').$$

As the measure $P$ is shift-invariant, we obtain a measure on the quotient, still denoted by $P$. The $\Gamma$-action also goes down to the quotient, and $P$ is a $\mu$-stationary measure for this action. The quotient space with this measure is denoted by $(\mathcal{P}, P)$ and is called the Poisson boundary of the random walk $(\Gamma, \mu)$. The notation “let $\xi = (\gamma_n) \in P$” means that $\xi$ is the element of $\mathcal{P}$ corresponding to $(\gamma_n) \in \mathcal{I}$ via the natural maps, and $\gamma^{(n)}$ is the corresponding element of $\mathcal{T}$.

Remark. This definition of the Poisson boundary is not the usual one but it is enough for what is needed in the following. For a general definition see [21].

Consider a random walk induced by $\rho : \Gamma \to G$. Recall from the Furstenberg limit map $\theta_{\rho} : \mathcal{I} \to \mathbb{P}V$. From its definition, it is clear that it factors through $\mathcal{T}$ and $\mathcal{P}$ and induces a map $\theta_{\rho} : \mathcal{P} \to \mathbb{P}V$ and that the stationary measure is $\nu_{\rho} = (\theta_{\rho})_\ast P$. The limit map is $\rho$-equivariant: $\theta_{\rho}(\gamma\xi) = \rho(\gamma)\theta_{\rho}(\xi)$. 
Given a holomorphic family of representation $(\rho_\lambda)$ for $\lambda \in U$, satisfying the standing assumptions, we define a holomorphic variation of the Poisson boundary as a map

$$\theta : \mathcal{P} \times U \to \mathbb{P}V,$$

such that:

- $\theta$ is measurable and defined almost everywhere with respect to the first variable.
- $\theta$ is holomorphic with respect to the second variable.
- $\theta$ is equivariant with respect to the first variable that is $\theta(\gamma \xi, \lambda) = \rho_\lambda(\gamma) \theta(\xi, \lambda)$, for almost all $\xi$ and all $\gamma, \lambda$.

The goal of this section is to prove that, if the current $T_1$ is null on the open set $U$, we can construct a holomorphic variation of the Poisson boundary. In the following, we make this assumption.

For a given parameter $\lambda \in U$, we define the map $\theta(\cdot, \lambda)$ to be $\theta_{\rho_\lambda}$. For each $\lambda$, this map is defined almost everywhere on $\mathcal{P}$. The difficulty is to “glue” all these maps together holomorphically, and to find a full measure subset on which these maps are defined for all $\lambda$. We follow closely [10, Lemma 3.10].

The idea of the proof is the following: We fix a full measure subset of $\xi \in \mathcal{P}$ such that for a countable dense set of parameters $(\lambda_q)$, $\theta_{\rho_{\lambda_q}}$ is defined at $\xi$. Then using the boundedness in average of the volume of the graphs $\Gamma(\gamma_1 \cdots \gamma_n \sigma_0)$ proved in the previous section, we extend these assignment to holomorphic maps. More precisely:

Fix a $P_\varepsilon$-full measure subset $\Omega_1 \subset \mathcal{P}$ such that for a countable dense set of parameter $(\lambda_q)$, the boundary map $\theta_{\rho_{\lambda_q}}$ is defined on $\Omega_1$. We will extend these maps defined only on the $(\lambda_q)$ to holomorphic maps.

For a $\xi = (\gamma_n) \in \Omega_1$, define $f_{\xi,n}$ to be the map $\gamma_1 \cdots \gamma_n \sigma_0$, with the notation from the previous section. By Theorem 5.1, we know that:

$$\int \text{Vol}(\Gamma(f_{\xi,n}))dP_\varepsilon(\xi) = \int \text{Vol}(\Gamma(\gamma^{(n)} \sigma_0))d\mu^n(\gamma) \leq C,$$

for some constant $C$, that is, the volume $\text{Vol}(\Gamma(f_{\xi,n}))$ are bounded in mean. By the Borel-Cantelli lemma we have

$$\int \lim \inf \text{Vol}(\Gamma(f_{\xi,n}))dP_\varepsilon(\xi) \leq \lim \inf \int \text{Vol}(\Gamma(f_{\xi,n}))dP_\varepsilon(\xi) \leq C,$$

and consequently for a $P_\varepsilon$-full measure subset $\Omega_2$, for all $\xi \in \Omega_2$, $\lim \inf \text{Vol}(\Gamma(f_{\xi,n}))$ is finite, and we can extract a subsequence $(f_{\xi,n_k})$ of graphs of bounded volume. We can apply the following lemma (whose proof is postponed to the end of the section):

**Lemma 11.** If $(f_t)$ is a sequence of holomorphic maps on $U$ with graphs $\gamma_t \subset U \times \mathbb{P}V$ of uniformly bounded volume, then, up to a subsequence,
the sequence of graphs converges (as analytic sets) to the union of the
graph of a holomorphic map \( f \) on \( U \) and finitely many "bubbles", that
is analytic sets contained in a fiber \( z_0 \times \mathbb{P}V \).
Moreover, away from the bubbles, the sequence \((f_i)\) converges uni-
formly on compact subset to \( f \).

This lemma gives a holomorphic map \( f_{\xi} : U \to \mathbb{P}V \) such that \((f_{\xi,n})\)
converges (up to a subsequence) to \( f_{\xi} \) uniformly on compact set of \( U \)
with a finite number of points removed. For every \( \lambda_q \), except for a finite
number, we have \( f_{\xi}(\lambda_q) = \theta(\xi, \lambda_q) \). As \((\lambda_q)\) is dense, it means that \( f_{\xi} \)
is the unique holomorphic continuation of the assignment \( \lambda_q \mapsto \theta(\xi, \lambda_q) \).
In particular \( f_{\xi} \) is the only cluster value of \((f_{\xi,n})\). For every
\( \lambda \in U \) we can then define \( \theta(\xi, \lambda) = f_{\xi}(\lambda) \). It is easily checked that \( \theta \) satisfies
the required properties.

Proof of Lemma 11. By a result of Bishop (see [6, p. 15.5]), a sequence
of analytic sets of pure dimension 1 with uniformly bounded volume
converges (up to a subsequence) to an analytic set of pure dimension
1. Denote by \( A \) such an analytic set for the sequence \((\gamma_i)\). It is of finite
volume.
Pick a fiber \( F = F_z = z \times \mathbb{P}V \). There is 2 cases: either \( A \cap F \) is
finite, or \( A \cap F = A_z \) is a an analytic set of pure dimension 1. In the
second case, \( A_z \) is an algebraic set in \( F = \mathbb{P}V \), so its volume is bounded
below by \( \frac{2^d}{d!} \). In particular, this case can occurs for only finitely many
\( z \). Denote by \( B \) this finite set.
In the first case by the continuity of the intersection index (Prop. 2
of [6, p. 12.2]), because the intersection index of \( \gamma_i \) and \( F \) is 1 for all \( i \),
the intersection index of \( A \) and \( F \) is 1. It means that the intersection of
\( A \) and \( F \) is a single point with multiplicity 1. In particular, this point
is a point of irreducibility for \( A \).
Consider \((A_j)\) the irreducible components of \( A \). Each \( A_z \) for \( z \in B \)
is a union of irreducible components, that we call vertical. Let \( A_j \) be
a non-vertical component. For some \( z_0 \notin B \), \( A_j \cap F_{z_0} \) is not empty,
so it is a singe point. The intersection index of \( A_j \) and \( F_{z_0} \) depends
only on the homology class of \( F_{z_0} \), in particular it doesn’t depend on
\( z_0 \), so when \( A_j \) and \( F_z \) intersect at isolated points, their intersection
is a single points. In fact \( A_j \) always intersects \( F_z \) at isolated points: otherwise if would be completly contained in \( F_z \) and would be a vertical
component.
This imply that there is only one non-vertical irreducible component
that we denote by \( A_0 \). We have seen that it intersects every fiber
transversally at a single point. By [6, 3.3, Prop. 3], \( A_0 \) is the graph
of a holomorphic map \( f \). By the definition of convergence of analytic
set (see [6, p. 15.5]), the sequence \((f_i)\) converges to \( f \), uniformly on
compact set not meeting \( B \). \( \square \)
By using the pluriharmonicity of $\chi_{d+1}$ and doing the same construction as in the previous section we obtain a holomorphic variation of the Poisson boundary $\mathcal{P}^*$ associated to the dual random walk in $\mathbb{P}V^*$ denoted by:

$$\theta^* : \mathcal{P}^* \times U \rightarrow \mathbb{P}V^*,$$

equivariant with respect to the dual representation $\rho^*$.

### 5.4. Intersections of curves and hyperplanes

In this section we prove:

**Proposition 5.3.** There exists a full measure subset $D \subset \mathcal{P} \times \mathcal{P}^*$ such that for $(\xi, \xi^*) \in D$ the graphs $\theta_\xi$ and $\theta^*_\xi^*$ don't intersect.

Fix a relatively compact open subset $D$ inside $U$. For $(\xi, \xi^*) \in \mathcal{P} \times \mathcal{P}^*$ define $i_D(\xi, \xi^*)$ as the number of intersection points of the graphs $\theta_\xi$ and $\theta^*_\xi$, in $D$. Remark that either $\theta_\xi$ intersects $\theta^*_\xi$ in a finite number of points in $D$ or $\theta_\xi$ is entirely contained in $\theta^*_\xi$. Because $\nu_\lambda$ don’t charge any linear subspace, for almost all $(\xi, \xi^*)$ the latter doesn’t happen and $i_D(\xi, \xi^*)$ is finite.

As $\theta$ and $\theta^*$ are equivariant, $i_D$ is invariant by the diagonal action of $\Gamma$. By the double ergodicity theorem of Kaimanovich [22], this function is constant almost surely, equal to a constant we still denote by $i_D$.

The map $D \mapsto i_D$ extends to an integer-valued measure on $U$, which is finite on relatively compact subsets. It follows that it is a sum of Dirac masses with integer coefficients, supported on a discrete set $F$. It means that there exists a full measure subset $D \subset \mathcal{P} \times \mathcal{P}^*$ such that for all $p \in U \setminus F$, if $(\xi, \xi^*) \in D$ then $\theta_\xi$ and $\theta^*_\xi$ don’t intersect.

We show that this set $F$ is empty. Let $p \in F$. For almost all $(\xi, \xi^*) \in \mathcal{P} \times \mathcal{P}^*$ the graphs $\theta_\xi$ and $\theta^*_\xi$ intersect at $p$. Fix a $\xi^*$ and a full measure subset of $\xi \in \Omega \subset \mathcal{P}$ such that this holds. Then the image $\theta(p, \Omega)$ is contained in the kernel of $\theta^*(p, \xi^*)$. Because the stationary measure $\nu_p$ verifies $\nu_p = \theta(p, \cdot)_* P_\xi$, it implies that the support of $\nu_p$ is contained in the kernel of $\theta^*(p, \xi^*)$, which is absurd because the stationary measure doesn’t charge linear subspaces. We have proved the proposition 5.3.

### 5.5. Compactness of the holomorphic variation

The goal of this section is to prove Proposition 5.3 which asserts that the maps $\theta_\xi$ form a normal family when $\xi$ varies in some full measure subset.

We will use a technical lemma which asserts that some space of holomorphic maps avoiding some hyperplanes is compact. It is just a slight variation on a classical result in complex hyperbolic space, but we need to detail some constructions related to hyperplanes in projective space.
5.5.1. Some technical results about hyperplanes. In order to use a result of [28, Th. 3.10.27] we need to study hyperplanes in general position. We first explain what this means, then how to construct such family and finally extend these definition to graphs of hyperplanes.

We say that a family of hyperplanes is in general position if any \((d + 1)\) of them are linearly independent. Let \((H_i)_{i=0,\ldots,d+1}\) be a family of \(d+2\) hyperplanes in general position. We call such a family a system of hyperplanes. There exists a (unique, up to homotheties) basis of \(V\), called the normal basis, such that in the coordinates of this basis, the hyperplanes are defined by the equations:

\[
(H_0) : x_0 = 0 \\
\cdots \\
(H_d) : x_d = 0 \\
(H_{d+1}) : x_0 + \cdots + x_d = 0.
\]

We say that the system is in normal form.

The diagonals associated to this system are the hyperplanes \(\delta_I\), where \(I\) is a subset of \(\{0,\ldots,d\}\) which is not a singleton neither the entire set, defined in the normal basis by:

\[
(\delta_I) : \sum_{i \in I} x_i = 0.
\]

We will also denote \(H_{d+1}\) by \(\Delta\).

If we work in the dual projective space, and consider an hyperplane as a point of the dual projective space, we remark the following relations: for all subset \(I\), \(\delta_I\) is the intersection of the space spanned by the \(((H_j)_{j \not\in I}, \Delta)\) with the space spanned by the \((H_i)_{i \in I}\). Another way to say it is that \(\delta_I\) is the image of \(\Delta\) by the projection to \(\text{span}(H_i)_{i \in I}\) parallel to \(\text{span}(H_k)_{k \not\in I}\).

Now, let’s consider \((S^1,\ldots,S^k)\) a family of systems of hyperplanes. We say that this family is in general position if the family of all associated hyperplanes and all the associated diagonals are in general position. We can construct such family in the following generic way:

**Lemma 12.** Given an integer \(k\) we can construct by induction a family of systems of hyperplanes \((S^1,\ldots,S^k)\), and \(S^j = (H^j_i)_{i=0,\ldots,d+1}\), in general position, in the following way: Choose any \(H^1_i \in \mathbb{P}V^*\). Suppose we have constructed the hyperplanes up to some \(H^i_i\). Then for the next hyperplane \(H^j_{i+1}\) or \(H^{i+1}_1\), we can choose any hyperplane in \(\mathbb{P}V^*\) that is not in an exceptional set, which is a finite union of proper linear subspaces.

**Proof.** We choose any \(H^1_i \in \mathbb{P}V^*\). Suppose we have constructed the hyperplanes up to some \(H^i_i\), with an exceptional set \(E_{ij}\).

If \(i < d\), we can choose for \(H^j_{i+1}\) any hyperplane which is in general position with all the hyperplanes already constructed and the diagonals...
hyperplanes associated to the systems already constructed. So we add to the exceptional set the linear spaces spanned by any \( d \) of theses hyperplanes. The same applies for \( i = d + 1 \) to construct \( H^i_{d+1} \).

If \( i = d \), we have to choose \( H^i_{d+1} = \Delta^i \). Fix an admissible subset of indices \( I \). The choice of \( \Delta^i \) determines the diagonal \( \delta^i_I \), explicitely \( \delta^i_I \) is is image of \( \Delta^i \) by the projection to \( \text{span}(H^i_{i})_{i \in I} \) parallel to \( \text{span}(H^i_{k})_{k \notin I} \). We want \( \delta^i_I \) to be in general position with all the already constructed hyperplane, that is to avoid the spaces spanned by any \( d + 1 \) of them. These spaces intersect \( \text{span}(H^i_{i})_{i \in I} \) transversally , because the previous hyperplanes and diagonals are all in general position. So the inverse images of the spaces by the projection are proper subspaces and we just have to add them to the exceptional set. \( \Box \)

Remark that this lemma imply that being a system, or a family of systems of hyperplanes is an open condition in the appropriate space.

5.5.2. Compactness of maps avoiding varying hyperplanes. Let \( U \) be the unit disk in \( \mathbb{C} \). A graph of hyperplane \( H \) (above \( U \)) is a holomorphic map \( H : U \to \mathbb{P}V^* \) where we confuse \( H(\lambda) \) and the hyperplane it defines in \( \mathbb{P}V \). We define in the same way graph of linear subspaces of any dimension.

We extend all the definitions of the section 5.5.1 about hyperplanes to graphs of hyperplanes in the obvious way.

Given a graph of subspaces \( K \), we say that a sequence \( f_n \) of maps \( U \to \mathbb{P}V \) converges to \( K \) if the image of any compact by \( f_n \) is contained in any neighborhood of \( K \) for \( n \) large enough.

We say that a subspace \( F \) of \( \text{Hol}(U, \mathbb{P}V) \) is relatively compact modulo a graph of subspaces \( K \) if any sequence of elements of \( F \) is either convergent up to a subsequence, or convergent to \( K \).

Lemma 13. Let \((H_k)_{k=0,...,d+1}\), be a graph of systems of hyperplanes above \( U \). Let \( F \) be a set of holomorphic maps \( U \to \mathbb{P}V \). Assume that any \( f \in F \) avoids all the \( H_k \). Then \( F \) is relatively compact in \( \text{Hol}(U, \mathbb{P}V) \) modulo the diagonals hyperplanes of \((H_k)\).

Proof. For each \( \lambda \in U \), because the \( H_k(\lambda) \) are in general position, we can find a projective automorphism \( \Phi_\lambda \), (which vary holomorphically with \( \lambda \)) which put the system \((H_k(\lambda))\) in normal form. By replacing \( F \) by \( \{ \Phi_\lambda f; f \in F \} \) and the \( H_k \) by the \( \Phi_\lambda H_k \), we can consider that the graphs of hyperplanes are constant.

The theorem [28, Th. 3.10.27] states the space of holomorphic maps \( U \to \mathbb{P}V \) avoiding \( d + 2 \) hyperplanes in general position is relatively compact modulo the diagonals. Composing everything with \( \Phi_\lambda^{-1} \), we get the result. \( \Box \)

Now we can prove:
Proposition 5.4. There exists a full measure subset $\Omega$ of the Poisson boundary $\mathcal{P}$ such that the set of holomorphic maps

$$\Theta = \{ \theta(\xi, \cdot); \xi \in \Omega \},$$

is relatively compact in $\text{Hol}(U_0, \mathbb{P}V)$, where $U_0$ a Zariski dense open set of $U$. Moreover, $U_0$ can be chosen to contain any relatively compact subset of $U$.

Proof. Choose a $\lambda_0 \in U$. The stationary measure $\nu^{*,\lambda_0}$ at the parameter $\lambda_0$ for the dual representation doesn’t charge any linear subspace of $\mathbb{P}V^*$.

By using Lemma 12 we can construct a family of systems in general position $(S_1, \ldots, S_{d+1})$ with every hyperplanes of these systems in the support of $\nu^{*,\lambda_0}$, because it doesn’t charge linear subspaces.

Because the image of $\theta^*(\mathcal{P}^*, \lambda_0)$ is dense in the support of $\nu^{*,\lambda_0}$, we can find trajectories such that their traces at $\lambda_0$ are close enough to the hyperplanes of $(S_1, \ldots, S_{d+1})$, and so they also form a family of systems of hyperplanes in general position.

We fix such trajectories, $\xi_i^* \in \mathcal{P}^*$. Let $U_0$ be the subspace such that for all $\lambda \in U_0$ the family of systems $\theta^*(\xi_i^*, \lambda)$ is still in general position. By the result of Lemma 12, the complement of $U_0$ is an analytic set of positive codimension, so $U_0$ is a Zariski dense open set of $U$.

By the Proposition 5.3, there exists a full measure subset $\Omega$ of the Poisson boundary $\mathcal{P}$, such that the graphs $\theta(\xi, \cdot)$ doesn’t intersect the graphs of hyperplanes $\theta^*(\xi_i^*, \cdot)$ over $U_0$, for $\xi \in \Omega$.

We can now apply the Lemma 13 to the system of graphs of hyperplanes $(\theta_{\xi}^*)$. We deduce that the family $(\theta_{\xi})_{\xi \in \Omega}$ is relatively compact modulo the diagonal $\Delta^1$.

We show now that it is relatively compact (not only modulo the diagonal). Consider a sequence $\theta_{\xi_k}$ which converges to a diagonal graph $\delta^1$. We apply again Lemma 13 but now to $(\theta_{\xi_k}^*)$. Then this sequence is also relatively compact modulo the diagonal $\Delta^2$ associated to this new system. Now $\theta_{\xi_k}$ is either convergent (up to a subsequence) or convergent to a $\delta^2$, and so convergent to $\delta^1 \cap \delta^2$, which is a graph of linear subspaces of codimension 2 (by the general positions of the diagonal hyperplanes). Iterating the argument at most $d + 1$ times, either the sequences is convergent up to a subsequence or it must converges to an empty set, which is impossible.

We prove that $U_0$ can be chosen to contain any relatively compact subset of $U$. Given a relatively compact subset $K \subset U$, the difference $K \setminus U_0$ consists of finitely many points. Around each of these points $\lambda$, repeat the previous argument with $\lambda$ instead of $\lambda_0$. We obtain a full measure subset $\Omega$ such that $(\theta_{\xi})_{\xi \in \Omega}$ is relatively compact over a neighborhood of each of these points, so over $K$. \qed
All the results of this section can be done dually by exchanging the role of $\theta$ and $\theta^*$. Consequently we can state the dual version of proposition 5.4:

**Proposition 5.5.** There exists a full measure subset $\Omega^*$ of the dual Poisson boundary $P^*$ such that the set of holomorphic maps

$$\tilde{\Theta}^* = \{\theta^*(\xi^*, \cdot); \xi^* \in \Omega^*\},$$

is relatively compact in $\text{Hol}(U_0, \mathbb{P}V^*)$, where $U_0$ a Zariski dense open set of $U$.

We can, and will, fix a set $U_0$ which works for proposition 5.4 and proposition 5.5.

Recall that the proposition 5.3 gives $D \subset P \times P^*$ of full measure such that for $(\xi, \xi^*) \in D$, the graph $\theta^\xi$ and the graph of hyperplanes $\theta^*\xi^*$ don't intersect. We will show that an analogous non-intersection property also holds for every couples of $\tilde{\Theta} \times \tilde{\Theta}^*$, not only almost surely.

We consider the closure $\bar{\Theta}$ of $\Theta$ in $\text{Hol}(U_0, \mathbb{P}V)$. By the proposition 5.4, the closure is compact. Then, pushing the probability $P$ by $\theta$, $\bar{\Theta}$ is equipped with a probability measure. We replace $\bar{\Theta}$ by the support of this measure. The same can be done for $\bar{\Theta}^*$ and we have that for almost all $(f, H) \in \bar{\Theta} \times \bar{\Theta}^*$, $f$ and $H$ don't intersect.

**Lemma 14.** Let $(f, H) \in \bar{\Theta} \times \bar{\Theta}^*$. Then either $f$ and $H$ are disjoints, or $f$ is entirely contained in $H$.

**Proof.** Consider $(f, H) \in \bar{\Theta} \times \bar{\Theta}^*$. Suppose that the graph of $f$ and the hyperplane graph of $H$ admit a non-trivial isolated intersection. By the continuity of intersection of analytic set [6, sec. 12.3], if $f$ and $H$ admit an isolated intersection then there exists a neighborhood $N$ of $(f, H)$ in $\bar{\Theta} \times \bar{\Theta}^*$ of couple which also admit a non-trivial isolated intersection.

But as $(f, H)$ is in the support of the measure, any of its neighborhood has positive measure. This contradicts the fact that the set non-intersecting couples is of full measure.

We deduce that if $f$ and $H$ intersect then the graph of $f$ is entirely included in the graph of $H$. \(\square\)

The compactness of $\bar{\Theta}$ allows many constructions, as in the following lemma.

**Lemma 15.** Let $\lambda_0 \in V$ and $z_0 \in \text{supp}\nu_{\lambda_0}$. Then there exists a $f \in \bar{\Theta}$ such that $f(\lambda_0) = z_0$.

**Proof.** The image of $\theta_{\lambda_0}$ is dense in $\text{supp}\nu_{\lambda_0}$. Choosing $\xi_k$ such that $\theta(\lambda_0, \xi_k) \to z_0$, we can choose $f$ to be any cluster value of the sequence $(\theta_{\xi_k})$ in $\bar{\Theta}$. \(\square\)

An analogous lemma holds dually.

We summarize the results of the previous sections 5.3, 5.4, 5.5.
Proposition 5.6. If \((\rho_\lambda)_{\lambda \in \Lambda}\) satisfies the standing assumption and \(\chi_1\) and \(\chi_{d+1}\) are harmonic on \(\Lambda\) then there exists two family of holomorphic maps
\[
\overline{\Theta} \subset \text{Hol}(\Lambda, \mathbb{P}V),
\]
\[
\overline{\Theta}^* \subset \text{Hol}(\Lambda, \mathbb{P}V^*),
\]
which are normal and which are transverse: for any \(f \in \overline{\Theta}\) and \(H \in \overline{\Theta}^*\), either \(f\) and \(H\) don’t intersect or \(f\) is completely contained in \(H\).

This construction is central to the present work. We can compare these family of graphs with the equilibrium web and equilibrium lamination constructed in [4], which play similar roles in the proof. It seems that the construction is easier in our case, due to the interplay between graphs and dual graphs, which does not have an analog in the theory of endomorphisms. They also defines branched holomorphic motions, in the sense of [3], and play a similar role.

5.6. The attracting fixed points move holomorphically. In this section we first show that a proximal element at some parameter \(\lambda_0\) stays proximal on the whole domain \(U_0\) given by Proposition 5.4, then on all of \(U\).

The proof is in two steps. First we show that if \(\gamma\) is proximal at \(\lambda_0\) then we can find a graph \(f\) in \(\overline{\Theta}\) which is fixed by \(\rho_\lambda(\gamma)\) for all \(\lambda \in U_0\), and which is the attractive fixed point of \(\rho_{\lambda_0}(\gamma)\) at \(\lambda_0\). Then we show that this graph of fixed point is in fact a graph of attractive fixed points, that is there exists a graph of hyperplane \(H\) in \(\overline{\Theta}^*\) such at every \(\lambda\), \(\rho_\lambda(\gamma)\) contracts every point not in \(H(\lambda)\) to \(f(\lambda)\).

Graph of fixed points. Let’s fix a parameter \(\lambda_0 \in U_0\) and a group element \(\gamma\) such that \(\gamma\) is proximal at \(\lambda_0\). For \(\lambda\) in a small neighborhood \(W\) of \(\lambda_0\), the matrix \(\rho_\lambda(\gamma)\) stays proximal. Let \(f : W \to \mathbb{P}V\) be the graph of attractive fixed points and \(H : W \to \mathbb{P}V^*\) be the graph of repulsive fixed hyperplane of \(\rho(\gamma)\) on \(W\). The goal is to extend these graphs to all of \(U_0\).

By lemma [15] there exists a graph \(g \in \overline{\Theta}\) such that \(g(\lambda_0) = f(\lambda_0)\) and a graph of hyperplane \(K \in \overline{\Theta}^*\) such that \(K(\lambda_0) = H(\lambda_0)\), because an attractive fixed point belongs to the support of the stationary measure. We’re going to show that \(f = g\) and \(H = K\) on \(W\).

Suppose, by contradiction, that \(f \neq g\) in \(W\). Shrinking \(W\) if necessary we can assume that \(f\) and \(g\) intersect only at \(\lambda_0\). Let \(g_n = \gamma^{-n}g\). The sequence \((g_n)\) is in \(\overline{\Theta}\), so it has a cluster value \(g_\infty\). We have \(g_0(\lambda_0) = g(\lambda_0) = g_\infty(\lambda_0)\) and \(g_n(\lambda) = \rho_\lambda(\gamma^{-n})g(\lambda)\) which converges to a point in \(H(\lambda)\), so \(g_\infty(\lambda) \in H(\lambda)\), for all \(\lambda \neq \lambda_0\). This is impossible because \(g(\lambda_0) = f(\lambda_0) \notin H(\lambda_0)\) and we have obtained the desired contradiction. We must have \(f = g\) in \(W\). Dually, by the same reasoning we have that \(H = K\) in \(W\).
As \( g \) and \( K \) are defined on all of \( U_0 \), we have extended \( f \) and \( H \) to \( U_0 \). By analytic continuation, \( f \) stays a graph of fixed points and \( H \) stays a graph of fixed hyperplanes.

*Graph of attractive fixed points.* We need to show that \( f \) and \( H \) stays attractive fixed points and repulsive hyperplanes on all of \( U_0 \), that is for all \( \lambda \), for all \( z \in \mathbb{P}V \setminus H(\lambda) \), we have \( \rho_\lambda(\gamma^n)z \to f(\lambda) \). In fact, it is enough to show it for only one generic \( z \in \mathbb{P}V \); more precisely, suppose that \( z \in \mathbb{P}V \) has all of its coordinates non zero in a basis adapted to the direct sum \( f(\lambda) \oplus H(\lambda) \). Then \( \rho_\lambda(\gamma^n)z \to f(\lambda) \) implies that the spectral radius of \( \rho_\lambda(\gamma)_{H(\lambda)} \) is strictly smaller than the maximal eigenvalue of \( \rho_\lambda(\gamma) \), and so that \( \rho_\lambda(\gamma) \) is proximal.

Fix some \( \lambda_1 \in U_0 \). As the stationary measure \( \nu_{\lambda_1} \) doesn’t charge linear subspaces, there is a generic \( z \notin H(\lambda_1) \) in its support. Suppose that we don’t have \( \rho_{\lambda_1}(\gamma^n)z \to f(\lambda_1) \). By the lemma \[13\] there exists a graph \( j \in \Theta \) such that \( j(\lambda_1) = z \). As \( j(\lambda_1) \notin H(\lambda_1) \), the graph \( j \) and \( H \) are disjoints. The sequence \( \gamma^n j \) has a cluster value \( j_\infty \). For all \( \lambda \in W \), we have \( \gamma^n j(\lambda) \to f(\lambda) \) because \( j(\lambda) \notin H(\lambda) \) and \( f \) is a graph of attracting fixed points over \( W \), so \( j_\infty(\lambda) = f(\lambda) \) for all \( \lambda \in W \). By analytic continuation we must have \( j_\infty = f \) on all of \( U_0 \). It implies that \( j_\infty(\lambda_1) = f(\lambda_1) \) but that is impossible because \( j_\infty(\lambda_1) \) is a cluster value of \( \rho_{\lambda_1}(\gamma^n)z \). We conclude that we have the convergence \( \rho_{\lambda_1}(\gamma^n)z \to f(\lambda_1) \) for this generic \( z \), and so for all \( z \notin H(\lambda_1) \).

*Alternative proof.* Let \( \mu_1 \) be the log of the modulus of the eigenvalue of \( \rho_{\lambda_1}(\gamma) \) at \( f(\lambda) \) and \( \mu_2 \) be the log of the spectral radius of \( \rho_{\lambda_1}(\gamma) \) on \( H(\lambda) \). The map \( \mu_1 \) is pluriharmonic and the map \( \mu_2 \) is plurisubharmonic, so \( \mu_1 - \mu_2 \) is a plurisubharmonic non-negative map. If it vanishes somewhere, it is identically null, but it is not zero at \( \lambda_0 \) by assumption. Consequently we have \( \mu_1 > \mu_2 \) everywhere and \( \rho_{\lambda_1}(\gamma) \) stays proximal on \( U_0 \).

6. **Propagation**

In this section we prove the theorem \[5\]. This theorem is about *propagation*: we assume that a property is satisfied at some parameter and we prove that it is satisfied at every parameters. In the following proofs the main tools is the existence of compact and transverse families of graphs \( \Theta \) and \( \Theta^* \), see proposition \[3,6\]. We fix a family \( (\rho_\lambda) \) satisfying the standing assumptions and such that \( \chi_1 \) and \( \chi_{d+1} \) are harmonic, and equivalently, by theorem \[4\] which is proximally stable.

6.1. **Faithfulness.** Suppose that for some parameter \( \lambda_0 \) the representation is faithful. We prove that for every \( \lambda, \rho_\lambda \) is faithful.

Fix a doubly-proximal element, that is a \( \gamma_0 \in \Gamma \), such that \( \rho_{\lambda_0}(\gamma_0) \) and \( \rho_{\lambda_0}(\gamma_0)^{-1} \) are proximal (such an element exist by \[18\]). Fix a \( h \in \Gamma \) which is not the identity. We want to show that we can’t have \( \rho_{\lambda_1}(h) = id \) for some \( \lambda_1 \in U \).
Let $\gamma_1 = h\gamma_0 h^{-1}$. It is proximal at $\lambda_0$, so it stays proximal on all of $U$. We show that the graphs of fixed points of $\gamma_0$ and $\gamma_1$ are disjoint (possibly by changing $\gamma_0$) so the element $h$ can’t be mapped to the identity at any parameter. For a proximal element $\gamma$ (it doesn’t depend on the parameter by assumption) we denote by $\text{Fix}^{+\gamma}$ its graph of attractive fixed points and by $\text{Fix}^{-\gamma}$ its hyperplane graph of repulsive hyperplane.

We have $\text{Fix}^{+\gamma_0} \subset \text{Fix}^{-\gamma_0^{-1}}$, because $\gamma_0$ and $\gamma_0^{-1}$ are proximals. So if $\text{Fix}^{+\gamma_1}$ avoids $\text{Fix}^{-\gamma_0^{-1}}$, we’re done. If not, it means that $\text{Fix}^{+\gamma_1}$ is entirely contained in $\text{Fix}^{-\gamma_0^{-1}}$ by Lemma 14, i.e.

$$h(\text{Fix}^{+\gamma_0}) \subset \text{Fix}^{-\gamma_0^{-1}}.$$ 

We now work at the parameter $\lambda_0$, but we omit it in the notation. Denote $L = \text{Fix}^{+\gamma_0}$ and $H = \text{Fix}^{-\gamma_0^{-1}}$, so we have $h(L) \subset H$. If we replace $\gamma_0$ by a conjugate $g\gamma_0 g^{-1}$, then $L$ and $H$ are replaced by $gL$ and $gH$. We show that for some $g$, the relation $h(gL) \subset gH$ doesn’t hold.

The existence of a doubly-proximal element implies that the group $\rho_{\lambda_0}(\Gamma)$ is proximal on the flag variety $\mathcal{F} = \{(L, H); L \subset H\}$. By results of [2] there is a unique stationary measure $\nu_{\mathcal{F}}$ on $\mathcal{F}$. This measure doesn’t charge any subvariety of $\mathcal{F}$, in particular the subvariety $\{(L, H); h(L) \subset H\}$. As almost every orbit $g_n \cdots g_1(L, H)$ equidistributes towards $\nu_{\mathcal{F}}$, for some $g = g_n \cdots g_1$ a big enough random product, we don’t have $h(gL) \subset gH$.

For such a $g$, replacing $\gamma_0$ by $g\gamma_0 g^{-1}$, we have that $\text{Fix}^{+\gamma_1}$ is not entirely contained in $\text{Fix}^{-\gamma_0^{-1}}$, because it’s false at parameter $\lambda_0$, and so $\text{Fix}^{+\gamma_1}$ and $\text{Fix}^{-\gamma_0^{-1}}$ don’t intersect. Finally, the graphs $\text{Fix}^{+\gamma_0}$ and $\text{Fix}^{+\gamma_1}$ don’t intersect, and the element $h$ can’t be trivial at any parameter.

6.2. Discreteness. Suppose that for some parameter $\lambda_0$ the representation is discrete. We prove that for all parameters $\lambda$ the representation $\rho_\lambda$ is discrete.

Fix a sequence $(h_n)$ of elements of $\Gamma$ going to infinity in $\Gamma$. We’re going to show that $\rho_{\lambda_n}(h_n)$ can’t converge to the identity, for all $\lambda_n$.

As $\rho_{\lambda_1}$ is discrete, the sequence $\rho_{\lambda_n}(h_n)$ goes to infinity. By replacing $(h_n)$ with a subsequence, we can assume that it converges (up to scalar) to an endomorphism $\pi$ with range $R$ and (non-trivial) kernel $K$.

The idea of the proof is the following: let $\gamma_n = h_n^* \gamma_0 h_n^{-1}$ for a carefully chosen doubly-proximal $\gamma_0$. We show that the graphs $\text{Fix}^{+\gamma_0}$ converges to a graph $f$ that is disjoint from $\text{Fix}^{+\gamma_0}$. It readily implies that $\rho_{\lambda_1}(h_n)$ can’t converge to the identity, for all $\lambda_1$, as it would cause an intersection at $\lambda_1$ between $\text{Fix}^{+\gamma_0}$ and $f$. 

First step. Like in the previous section 6.1, for each \(n\), we can choose \(\gamma_n^0\) so that \(\text{Fix}^+\gamma_n\) avoids \(\text{Fix}^-\gamma_n^0\). The first step is to show that we can choose a \(\gamma_0\) that works for all \(n\).

Like before, the graph \(\text{Fix}^+\gamma_n\) is disjoint from \(\text{Fix}^-\gamma_n^0\) if the former in not entirely contained in the latter. We show that at the parameter \(\lambda_0\) they don’t intersect. We fix everything at \(\lambda_0\), but we omit it in the notation. Let \(L = \text{Fix}^+\gamma_0\) and \(H = \text{Fix}^-\gamma_0^0\). As we have \(h_nL = \text{Fix}^+\gamma_n\) we want to show that we can choose \(\gamma_0\) such that \(h_nL \notin H\). With the notation from 6.1, we want that \((L, H) \notin \mathcal{F}_{h_n}\). If we replace \(\gamma_0\) by \(g_{\gamma_0}g^{-1}\), then \((L, H)\) is replaced by \((gL, gH)\). We have to find a \(g\) such that \((gL, gH)\) is not in \(\bigcup \mathcal{F}_{h_n}\).

Let \(V = \bigcup \mathcal{F}_{h_n} \cup \mathcal{F}_\pi\), where \(\mathcal{F}_\pi\) is the subset of \(\mathcal{F}\) composed by the elements \((L, H)\) such that either \(L \in K\) or \(L \notin K\) and \(\pi L \notin H\). Then \(V\) is a closed subset of \(\mathcal{F}\) and is a measure 0 subset for \(\nu_\mathcal{F}\). Now, for almost all large enough random product \(g = g_1 \cdots g_n\), the orbit \(g(L, H)\) is not in \(V\).

Replacing \(\gamma_0\) by \(g_{\gamma_0}g^{-1}\), we have that \(h_nL \notin H\) for all \(n\), and consequently \(\text{Fix}^+\gamma_n\) is disjoint from \(\text{Fix}^-\gamma_n^0\), because they don’t intersect at the parameter \(\lambda_0\).

Second step. By compactness, the sequence of graphs \(\text{Fix}^+\gamma_n\) converges (up to replacing it with a subsequence) to a graph \(f\). We proved that all these graphs avoids \(\text{Fix}^-\gamma_0\). By continuity of the intersection, the limit \(f\) is either disjoint from it or entirely contained in it. We will prove that at the parameter \(\lambda_0\) they don’t intersect, and consequently that they are disjoint.

The value of \(f\) at \(\lambda_0\) is the point \(P\) which is the limit of \(h_n(\text{Fix}^+\gamma_0)\), that is \(\pi(L)\). But because of the way we chose \(\gamma_0\), we have that \((L, H) \notin W\), that is \(L \notin K\) and \(\pi(L) \notin R \cap H\). This means that \(f(\lambda_0) \notin H\), and so \(f\) doesn’t intersect \(\text{Fix}^-\gamma_0^0\) at \(\lambda_0\). This means that \(f\) and \(\text{Fix}^-\gamma_0^0\) are disjoint.

6.3. The Anosov property. Suppose that \(\Gamma\) is hyperbolic. Suppose that the family \((\rho_\lambda)\) is irreducible at every parameter and projectively Anosov at some parameter \(\lambda_0\). We prove that for every parameter \(\lambda\) the representation \(\rho_\lambda\) is projectively Anosov.

By [17], Prop. 4.10, an irreducible representation \(\rho\) is projectively Anosov if and only if there exists two injective, \(\rho\)-equivariant, continuous maps

\[\xi^+ : \partial_\infty \Gamma \to \mathbb{P}V \text{ and } \xi^- : \partial_\infty \Gamma \to \mathbb{P}V^*,\]

which are:

- **transverse**: for \(\eta \neq \eta' \in \partial_\infty \Gamma\), the line \(\xi^+(\eta)\) is not contained in the hyperplane \(\xi^-(\eta')\)
- **dynamic preserving**: for \(\gamma \in \Gamma\) of infinite order, \(\rho(\gamma)\) is proximal and if \(\gamma_+\) is its attractive fixed point on \(\partial_\infty \Gamma\), then \(\xi^+(\gamma_+)\) is
the attractive fixed point of $\rho(\gamma)$ and if $\gamma_-$ is its repulsive fixed point on $\partial_\infty \Gamma$, then $\xi^- (\gamma_-)$ is the repulsive hyperplane of $\rho(\gamma)$. These maps are called the boundary maps of $\rho$.

By hypothesis, $\rho_{\lambda_0}$ is Anosov. As being Anosov is an open property, there exists a neighborhood $U$ of $\lambda_0$ such that for $\lambda \in U$ the representation $\rho_\lambda$ is Anosov. Denote the associated boundary maps by $\xi^+_\lambda$ and $\xi^-_\lambda$. We are going to prove that such maps exists at every parameter, by defining them on the dense set of points of the form $\gamma_t^{+\infty}$ and then extend these maps to $\partial_\infty \Gamma$ of infinite order and then extend these maps to $\partial_\infty \Gamma$.

For every $\gamma \in \Gamma$ of infinite order, because $\rho_{\lambda_0}(\gamma)$ is proximal, $\gamma$ is proximal at every parameter. By the proof of 5.6 we know that the fixed point of $\gamma$ of $\gamma$ the restriction of $H_{\lambda}$ and $\xi$ never intersect by lemma 14. We deduce that $f$ and $\eta$ graphs with all the $\eta_n, \eta'_n$ distincts, such that $f_{\eta_n} \rightarrow f$ and $f_{\eta'_n} \rightarrow f'$. Then $f$ and $f'$ are disjoint or equal. Indeed, we have $f_{\eta_n} \subset H_{\gamma_n}$, $f_{\eta_n}$ and $H_{\eta_n}$ are disjoints and we can assume that $H_{\eta_n}$ converges to some $H'$. By continuity of the intersection, $f$ and $H'$ are disjoint, or $f$ is contained in $H'$. We can assume that $f_{\eta_n}$ converges to $\eta$ and $f_{\eta'_n}$ converges to $\eta'$. By continuity of $\xi^+_\lambda$ for $\lambda \in U$, we have that $f(\lambda)$ is equal to $\xi^+_\lambda (\eta)$, $f'(\lambda)$ is equal to $\xi^+_\lambda (\eta')$ and $H'(\lambda)$ is equal to $\xi^-_\lambda (\eta')$. If $\eta$ is different from $\eta'$ then by transversality of the boundary maps, $f(\lambda_0)$ is not contained in $H'(\lambda_0)$, so $f$ is disjoint from $H'$. If $\eta$ is equal to $\eta'$, then $f$ and $f'$ coincide on $U$. By analytic continuation, they coincide everywhere.

This strong transversality has the following consequence. Let $\mathcal{F}$ be the closure of the set of $f_{\eta}$ for $\eta \in \partial_\infty \Gamma$ an attractive fixed point. Then for each point $z$ of $L(\lambda_0) = \xi^+_\lambda(\partial_\infty \Gamma)$, there exists exactly one $f \in \mathcal{F}$ such that $f(\lambda_0) = z$. Denote by $f_z$ this graph. It is clear that for each $\lambda$:

$$\{f_z(\lambda); z \in L(\lambda_0)\} = L(\lambda),$$

and that the same property holds: for each $x \in L(\lambda)$, there exists exactly one $f \in \mathcal{F}$ such that $f(\lambda) = x$. Fix a $\lambda$ and let:

$$\varphi : L(\lambda_0) \rightarrow L(\lambda)$$

$$z \mapsto f_z(\lambda).$$

We show that $\varphi$ is continuous. Let $z_n \rightarrow z$ be a convergent sequence in $L(z_0)$. Let $l$ be the limit of a converging subsequence $(\varphi(z_{n_k}))$. The sequence of graphs $(f_{z_{n_k}})$ has a cluster value $f$. Evaluating at $\lambda_0$...
we have that \( f(\lambda_0) = z \), so by uniqueness, \( f = f_z \). This implies that \( l = \varphi(z) \). The only cluster value of the sequence \( (\varphi(z_n)) \) is \( \varphi(z) \), so \( \varphi(z_n) \to \varphi(z) \). The map \( \varphi \) is continuous.

Define \( \xi^+ \) to be \( \varphi \circ \xi^0 \). We construct similarly \( \xi^- \). Then \( \xi^\pm \) are boundary maps for \( \rho_\lambda \). It follows that \( \rho_\lambda \) is Anosov.

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