Approaching $\epsilon$-closed sets via ideals

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Abstract. Our intention of this paper is to initiate the new notions on $\epsilon_1$-closed set and $\epsilon s$-closed sets. Some properties and relationships to other known closed sets are analyzed. Further we define and discuss $\epsilon_1$-continuous and $\epsilon s$-continuous functions and their basic properties.

1. Introduction and Preliminaries

The space $X_T$ is a TS (Topological space) with a topology $\mathcal{T}$ on $X$ and no axioms are considered unless it is precisely specified. Let a point $x \in X_T$, the collection of open neighborhoods of $x$ is represented by $\mathcal{N}(x) = \{U \in \mathcal{T} : x \in U\}$. For any subset $A \subseteq X_T$, we may designate closure of $A$ as $cl(A)$ and interior of $A$ as $int(A)$, for the given TS $X_T$.

A nonempty system of subsets of a set $X$ is called an ideal [2] on $X$, if the subsequent statements hold good: (i) If $L \in \mathcal{I}$ and $M \subseteq L$, then $M \in \mathcal{I}$; (ii) If $L \in \mathcal{I}$ and $M \in \mathcal{I}$, then $L \cup M \in \mathcal{I}$. An ITS (Ideal topological space) or ideal space $(X_T, I)$ means a TS $X_T$ with an ideal $I$ on $X$.

Let $I$ be an ideal and $\mathcal{T}$ be a topology defined on a space $X$. Then for any subset $A$ of $X$, $A^*(I, \mathcal{T}) = \{x \in X : A \cap U \not\in I, \forall U \in \mathcal{N}(x)\}$ is said to be local function of $A$ for the given ideal $I$ and the topology $\mathcal{T}$ [2]. If there is no obscenity, we may denote $A^*(I)$ or simply $A^*$ for $A^*(I, \mathcal{T})$. Also, $cl^*(A) = A^* \cup A$ denotes as a Kuratowski operator for the topology $\mathcal{T}^*$, which is finer than $\mathcal{T}$.

We initiate a definition for $\epsilon_1$-closed sets that is associated to the collection of closed sets in the ITS. We discuss some properties on $\epsilon_1$-closed sets and $\epsilon s$-closed sets using the operations of closure and interior with respect to ITS. Further we define and study $\epsilon_1$-continuous and $\epsilon s$-continuous functions and relationships with same kind of continuous functions.

2. Some results on $\epsilon$-Closed Sets

Talal Al-Hawary introduced and studied $\epsilon$-closed sets [1]. In this subdivision we discuss some of the results about $\epsilon$-closed sets and we observe that the equivalent conditions given for $\epsilon$-closed set in [1] are incorrect.

2.1 Definition
Let $X_T$ be topological space and $A \subseteq X_T$. The $\epsilon$-interior of $A$ is the collection of all open sets of $X$ whose closures are contained in $cl(A)$, and it is represented by $int_\epsilon(A)$. The set $A$ is said to be $\epsilon$-open if $A = int_\epsilon(A)$ and the complement of a $\epsilon$-open set is said to be $\epsilon$-closed. Equivalently, a subset $A$ of $X$ is $\epsilon$-closed if $A = cl_\epsilon(A)$, here $cl_\epsilon(A) = \{x \in X : cl(U) \cap cl(A) \neq \emptyset, x \in U, U \in \mathcal{T}\}$.[1]
Clearly \( \text{int}(A) \subseteq \text{int}(A) \subseteq \text{cl}(A) \) \& \( A \subseteq \text{cl}(A) \subseteq \text{cl}(A) \) and hence every \( \epsilon \)-closed set is closed, but converse is not valid for both the statements. [1]

2.2 \textbf{Remark}

We see that the next example proves the definition given for \( \epsilon \)-closed sets [1] are not equivalent.

2.3 \textbf{Example}

Let \( Z = \{ \theta_1, \theta_2, \theta_3, \theta_4 \} \) and \( \mathcal{T} = \{ \Phi, \{ \theta_2 \}, \{ \theta_3 \}, \{ \theta_2, \theta_3 \}, \{ \theta_1, \theta_2 \}, \{ \theta_1, \theta_2, \theta_3 \}, Z \} \).

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|c|}
\hline
A & \text{int}(A) & \text{cl}(A) & \text{int}(A) & \text{cl}(A) \\
\hline
\Phi & \Phi & \Phi & \Phi & \Phi \\
\{ \theta_1 \} & \Phi & \{ \theta_1, \theta_4 \} & \Phi & Z \\
\{ \theta_2 \} & \{ \theta_2 \} & \{ \theta_1, \theta_2, \theta_4 \} & \{ \theta_1, \theta_2 \} & Z \\
\{ \theta_3 \} & \{ \theta_3 \} & \{ \theta_3, \theta_4 \} & \{ \theta_3 \} & Z \\
\{ \theta_4 \} & \Phi & \{ \theta_4 \} & \Phi & Z \\
\{ \theta_1, \theta_2 \} & \{ \theta_1, \theta_2 \} & \{ \theta_1, \theta_2, \theta_4 \} & \{ \theta_1, \theta_2 \} & Z \\
\{ \theta_1, \theta_3 \} & \{ \theta_3 \} & \{ \theta_1, \theta_3, \theta_4 \} & \{ \theta_3 \} & Z \\
\{ \theta_1, \theta_4 \} & \Phi & \{ \theta_1, \theta_4 \} & \Phi & Z \\
\{ \theta_2, \theta_3 \} & \{ \theta_2, \theta_3 \} & \{ \theta_2, \theta_3 \} & \{ \theta_2, \theta_3 \} & Z \\
\{ \theta_2, \theta_4 \} & \{ \theta_2 \} & \{ \theta_1, \theta_2, \theta_4 \} & \{ \theta_1, \theta_2 \} & Z \\
\{ \theta_3, \theta_4 \} & \{ \theta_3 \} & \{ \theta_3, \theta_4 \} & \{ \theta_3 \} & Z \\
\{ \theta_1, \theta_2, \theta_3 \} & \{ \theta_1, \theta_2, \theta_3 \} & \{ \theta_1, \theta_2, \theta_3 \} & \{ \theta_1, \theta_2, \theta_3 \} & Z \\
\{ \theta_1, \theta_2, \theta_4 \} & \{ \theta_1, \theta_2 \} & \{ \theta_1, \theta_2, \theta_4 \} & \{ \theta_1, \theta_2 \} & Z \\
\{ \theta_1, \theta_3, \theta_4 \} & \{ \theta_3 \} & \{ \theta_1, \theta_3, \theta_4 \} & \{ \theta_3 \} & Z \\
\{ \theta_2, \theta_3, \theta_4 \} & \{ \theta_2, \theta_3 \} & \{ \theta_2, \theta_3 \} & \{ \theta_2, \theta_3 \} & Z \\
\{ \theta_2, \theta_3 \} & \{ \theta_2 \} & \{ \theta_2 \} & \{ \theta_2 \} & Z \\
\{ \theta_3, \theta_4 \} & \{ \theta_3 \} & \{ \theta_3 \} & \{ \theta_3 \} & Z \\
\end{tabular}
\end{table}

From the table 1, \{ \theta_3 \} is \( \epsilon \)-open set but it is given in [1] as not \( \epsilon \)-open set. Also, the set \{ \theta_1, \theta_3 \} is not \( \epsilon \)-open which is wrongly mentioned as \( \epsilon \)-open set.

In the above example, we notice that \( \epsilon \)-open sets are \{ \Phi, \{ \theta_2 \}, \{ \theta_3 \}, \{ \theta_2, \theta_3 \}, \{ \theta_1, \theta_2 \}, \{ \theta_1, \theta_2, \theta_3 \}, Z \} \) and their complements are \{ \Phi, \{ \theta_1, \theta_2, \theta_3 \}, \{ \theta_1, \theta_2 \}, \{ \theta_1 \}, \{ \theta_2 \}, \{ \theta_3 \}, \{ \theta_4 \} \}. We observe that only \( Z, \Phi \) are satisfying the condition \( A = \text{cl}(A) \).

The sets \{ \theta_1, \theta_2, \theta_4 \} and \{ \theta_3, \theta_4 \} are \( \epsilon \)-closed sets in one sense and not \( \epsilon \)-closed sets in other sense.
2.4 Remark

In [1], author proved that arbitrary union of \( \epsilon \)-open set is \( \epsilon \)-open set. From example 2.3, we observe that the sets \( A = \{ \theta_3 \} \) and \( B = \{ \theta_1, \theta_2 \} \) are \( \epsilon \)-open sets but \( A \cup B = \{ \theta_1, \theta_2, \theta_3 \} \) is not \( \epsilon \)-open set.

3. \( \epsilon \)-closed sets via ideal

In this chapter, we are going to initiate some open and closed sets in ITS.

3.1 Definition

Let \( X_I \) be a TS with topology \( \mathcal{T} \) and \( A \subseteq X_I \). The subset \( A \) is called \( \epsilon s \)-closed set of \( X \), if \( A = cl_{\epsilon s}(A) \), where \( cl_{\epsilon s}(A) = \{ x \in X : cl(U) \cap cl(A) \neq \emptyset, x \in U, U \in \mathcal{T} \} \). The complement of \( \epsilon s \)-closed set is \( \epsilon \)-open set.

3.2 Definition

Let \( (X_I, I) \) be an ITS with ideal \( I \) and the \( \epsilon I \)-interior of \( A \) is defined by \( int_{\epsilon I}(A) = U \cap cl(U) \subseteq cl_\epsilon(A) \). The set \( A \) is said to be an \( \epsilon I \)-open set if \( A = int_{\epsilon I}(A) \) and the complement of \( \epsilon I \)-open set is \( \epsilon I \)-closed set.

3.3 Definition

Let \( A \) be a subset of an ITS \( (X_I, I) \), The \( \epsilon I \)-closed set is defined by \( A = cl_{\epsilon I}(A) \) and \( cl_{\epsilon I}(A) = \{ x \in X : cl(U) \cap cl_\epsilon(A) \neq \emptyset, x \in U, U \in \mathcal{T} \} \). And the complement of \( \epsilon I \)-closed set is \( \epsilon I \)-open set.

3.4 Remark

Let \( \epsilon c(X_I) \) represent the system of all \( \epsilon \)-closed sets in \( X_I \), \( \epsilon s c(X_I) \) represent the system of all \( \epsilon s \)-closed sets in \( X_I \), \( \epsilon c_I(X_I, I) \) represent the system of all \( \epsilon I \)-closed sets in \( X_I \), \( \epsilon s c_I(X_I, I) \) represent the system of all \( \epsilon I \)-closed sets in \( X_I \).

3.5 Proposition

The subsequent statements are true for any ITS

i) \( cl_{\epsilon I}(X_I, I) \subseteq \epsilon c(X_I) \) ii) \( \epsilon s c_I(X_I, I) \subseteq \epsilon s c(X_I) \) but converse need not to be true.

3.6 Proposition

In any ITS, the following statements hold good

i) \( cl_{\epsilon I}(X_I, I) = \epsilon c(X_I) \) ii) \( \epsilon s c_I(X_I, I) = \epsilon s c(X_I) \) if and only if \( I = \{ \emptyset \} \)

3.7 Remark

In the next example we are going to show that in any TS, \( \epsilon s \)-closed sets and \( \epsilon \)-closed sets are independent.

3.8 Example

Let us consider the space \( Z = \{ \theta_1, \theta_2, \theta_3 \} \) with topology \( \mathcal{T} = \{ \emptyset, \{ \theta_2 \}, \{ \theta_1 \}, \{ \theta_1, \theta_2 \}, Z \} \) and their complement is \( \mathcal{T}^c = \{ \emptyset, \{ \theta_1 \}, \{ \theta_2 \}, \{ \theta_3 \}, \{ \theta_1, \theta_2 \}, \{ \theta_1, \theta_3 \}, \{ \theta_2, \theta_3 \}, \{ \theta_1, \theta_2, \theta_3 \}, Z \} \), ideal with respect to topology is \( I = \{ \emptyset, \{ \theta_3 \}, \{ \theta_1 \}, \theta_3 \} \)
4.1 Definition

A function \( F : (X, I) \rightarrow (Y, J) \) is called as i) \( cs \)-continuous function if \( F^{-1}(\mathcal{P}) \) is \( cs \)-closed set in \( (X, I) \) for every closed set \( \mathcal{P} \) in \( (Y, J) \), ii) \( cs_\ell \)-continuous function if \( F^{-1}(\mathcal{P}) \) is \( cs_\ell \)-closed set in \( (X, I) \) for every \( cs_\ell \)-closed set \( \mathcal{P} \) in \( (Y, J) \).

### 4. \( cs \)-continuous function and \( cs_\ell \)-continuous function

#### 4.1 Definition

#### 4.2 Example
4.2 Theorem
Every εs–continuous function is εs–continuous function
Proof: Using the proposition 3.5 every εsI closed set is εs–closed set but converse need not to be true.

4.3 Example
Let us consider the space Z={θ₁, θ₂, θ₃} with Topology $T = \{\Phi, \{\theta_2\}, \{\theta_1, \theta_2\}, Z\}$ and their complement is $T^c = \{\Phi, \{\theta_2\}, \{\theta_1, \theta_2\}, \{\theta_3\}, Z\}$, Ideal with respect to topology is $I = \{\Phi, \{\theta_3\}, \{\theta_1, \theta_3\}\}$. Now we define $F: Z \rightarrow Y$ such that $F(\theta_1) = F(\theta_2) = \theta_1, F(\theta_3) = \theta_3$

This implies that $F^{-1}\{\theta_1\} = F^{-1}\{\theta_2\} = \theta_1, F^{-1}\{\theta_3\} = \theta_1, F^{-1}\{\theta_3\} = \theta_3$.

Now consider the closed sets $F^{-1}\{Z\} = Z, F^{-1}\{\Phi\} = \Phi, F^{-1}\{\theta_2\} = \{\theta_2\}, (\{\theta_3\} = \theta_3$, since $\{\Phi, \{\theta_3\}, Z\}$ are εs-closed sets $\Rightarrow$ F is εs-continuous function. But $\{\theta_1\}$ is not εs–closed sets $\Rightarrow$ F is not εs-continuous function.

εs-continuous function $\Rightarrow$ εs-continuous function, but converse is not valid.

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