On the maximum number of minimum total dominating sets in forests

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received 19\textsuperscript{th} Aug. 2018, revised 18\textsuperscript{th} Dec. 2018, accepted 11\textsuperscript{th} Jan. 2019.

We propose the conjecture that every tree with order $n$ at least 2 and total domination number $\gamma_t$ has at most \[ \left(\frac{n-\gamma_t}{n}\right)^{\frac{\gamma_t}{2}} \] minimum total dominating sets. As a relaxation of this conjecture, we show that every forest $F$ with order $n$, no isolated vertex, and total domination number $\gamma_t$ has at most
\[ \min \left\{ \left(8\sqrt{\pi}\right)^n \left(\frac{n-\gamma_t}{n}\right)^{\frac{\gamma_t}{2}}, (1+\sqrt{2})^{n-\gamma_t}, 1.4865^n \right\} \]
minimum total dominating sets.

Keywords: Tree, forest, total domination, domination

1 Introduction

A set $D$ of vertices of a graph $G$ is a dominating set of $G$ if every vertex of $G$ that is not in $D$ has a neighbor in $D$, and $D$ is a total dominating set of $G$ if every vertex of $G$ has a neighbor in $D$. The minimum cardinalities of a dominating set of $G$ and a total dominating set of $G$ are the well studied domination number $\gamma(G)$ of $G$ and the total domination number $\gamma_t(G)$ of $G$, respectively. A (total) dominating set is minimal if no proper subset is a (total) dominating set. A dominating set of $G$ of cardinality $\gamma(G)$ is a minimum dominating set of $G$, and a total dominating set of $G$ of cardinality $\gamma_t(G)$ is a minimum total dominating set or $\gamma_t$-set of $G$. For a graph $G$, let $\sharp\gamma_t(G)$ be the number of minimum total dominating sets of $G$.

Providing a negative answer to a question of Fricke et al. [6], Bien [2] showed that trees with domination number $\gamma$ can have more than $2^\gamma$ minimum dominating sets. In fact, Bien’s example allows to construct forests with domination number $\gamma$ that have up to $2.0598^\gamma$ minimum dominating sets. In [1] Alvarado et al. showed that every forest with domination number $\gamma$ has at most $2.4606^\gamma$ minimum dominating sets, and they conjectured that every tree with domination number $\gamma$ has $O\left(\frac{\gamma^2}{\ln \gamma}\right)$ minimum dominating sets.

*Research supported in part by the University of Johannesburg.

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In the present paper we consider analogous problems for total domination, which turns out to behave quite differently. As shown by the star $K_{1,n-1}$ which has total domination number 2 but $n-1$ minimum total dominating sets, the number of minimum total dominating sets of a tree is not bounded in terms of its total domination number alone, but in terms of both the order and the total domination number. In Figure 1 we illustrate what we believe to be the structure of trees $T$ with given order $n$ at least 2 and total domination number $\gamma_t$ that maximize $\sharp\gamma_t(T)$.

![Diagram](image)

**Fig. 1:** For the tree $T_{\text{even}}$ on the left, we have $k = \frac{n}{\gamma_t + 1}$, $1 \leq \ell_1, \ldots, \ell_k$, and $(\ell_1 + 1) + \ldots + (\ell_k + 1) = n - k$, while for the tree $T_{\text{odd}}$ on the right, we have $k = \frac{n}{\gamma_t + 2}$, $1 \leq \ell_1, \ldots, \ell_k$, and $(\ell_1 + 1) + \ldots + (\ell_k + 1) = n - k - 2$.

If $\gamma_t$ is even, say $\gamma_t = 2k$, then the tree $T_{\text{even}}$ in the left of Figure 1 satisfies

$$\sharp\gamma_t(T_{\text{even}}) = \prod_{i=1}^{k} (\ell_i + 1) \leq \left( \frac{n - \frac{\gamma_t}{2}}{\frac{\gamma_t}{2}} \right)^{\gamma_t/2},$$

where we use that the geometric mean is at most the arithmetic mean. Similarly, if $\gamma_t$ is odd, say $\gamma_t = 2k + 1$, then the tree $T_{\text{odd}}$ in the right of Figure 1 satisfies

$$\sharp\gamma_t(T_{\text{odd}}) = \sum_{i=1}^{k} \prod_{j=1}^{i-1} \ell_j \prod_{j=i+1}^{k} (\ell_j + 1) \leq \frac{k}{2} \left( \frac{n - \frac{\gamma_t - 1}{2}}{\frac{\gamma_t - 1}{2}} \right)^{k} \left( \frac{n - \frac{\gamma_t + 7}{2}}{\frac{\gamma_t + 7}{2}} \right)^{\gamma_t/2}.$$

In view of these estimates, we pose the following.

**Conjecture 1.** If a tree $T$ has order $n$ at least 2 and total domination number $\gamma_t$, then

$$\sharp\gamma_t(T) \leq \left( \frac{n - \frac{\gamma_t}{2}}{\frac{\gamma_t}{2}} \right)^{\gamma_t/2}.$$

As our first result, we show that Conjecture 1 holds up to a constant factor for bounded values of $\gamma_t$. More precisely, we show the following.

**Theorem 2.** If a forest $F$ has order $n$, no isolated vertex, and total domination number $\gamma_t$, then

$$\sharp\gamma_t(F) \leq \left( 8\sqrt{e} \right)^{\gamma_t} \left( \frac{n - \frac{\gamma_t}{2}}{\frac{\gamma_t}{2}} \right)^{\gamma_t/2}.$$
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The well known estimate $1 + x \leq e^x$ implies

$$\left(\frac{n - \gamma t}{\gamma t}\right)^{\frac{\gamma t}{\gamma t}} = \left(1 + \frac{n - \gamma t}{\gamma t}\right)^{\frac{\gamma t}{\gamma t}} \leq e^{n - \gamma t}.$$ 

In the following theorem we can show an upper bound that is a little better. But since $1 + x \ll e^x$ for large $x$, the estimate is not good for fixed $\gamma t$ and large values of $n$. In this case Theorems 2 and 4 give better upper bounds.

**Theorem 3.** If a forest $F$ has order $n$, no isolated vertex, and total domination number $\gamma t$, then

$$\#\gamma_t(F) \leq (1 + \sqrt{2})^{n - \gamma t},$$

with equality if and only if every component of $F$ is $K_2$.

Note that Theorem 3 is only tight for $\gamma t = n$, which corresponds to the fact that $1 + x = e^x$ only for $x = 0$. For $n$ divisible by 5, the disjoint union of $\frac{n}{5}$ stars of order 5 yields a forest $F$ with $\#\gamma_t(F) = 4^{\frac{n}{5}} \approx 1.3195^n$. Our third result comes close to that value.

**Theorem 4.** If a forest $F$ has order $n$ and no isolated vertex, then $\#\gamma_t(F) \leq 1.4865^n$.

Before we proceed to the proofs of our results, we mention some related research. Connolly et al. [4] gave bounds on the maximum number of minimum dominating sets for general graphs. The maximum number of minimal dominating sets was studied by Fomin et al. [5], and the maximum number of general dominating sets by Wagner [12] and Skupień [11], and by Bród and Skupień [3] for trees. Krzywkowski and Wagner [9] study the maximum number of total dominating sets for general graphs and trees. For similar research concerning independent sets we refer to [10, 13, 14].

The next section contains the proofs of our results. We use standard graph theoretical terminology and notation. An endvertex is a vertex of degree at most 1, and a support vertex is a vertex that is adjacent to an endvertex.

## 2 Proofs

For the proof of Theorem 2, we need the following lemma.

**Lemma 5.** If $T$ is a tree of order $n$ at least 2, and $B$ is a set of vertices of $T$ such that

(i) $|B \cap \{u, v\}| \leq 1$ for every $uv \in E(T)$, and

(ii) $|B \cap N_T(u)| \leq 1$ for every $u \in V(T),$

then $|B| \leq \frac{n}{2}$.

**Proof:** The proof is by induction on $n$. If $T$ is a star, then (i) and (ii) imply $|B| \leq 1 \leq \frac{n}{2}$. Now, let $T$ be a tree that is not a star; in particular, $n \geq 4$. Let $uw \ldots$ be a longest path in $T$. By (i) and (ii), we have $|B \cap (N_T[v] \setminus \{w\})| \leq 1$. By induction applied to the tree $T' = T - (N_T[v] \setminus \{w\})$ and the set $B' = B \cap V(T')$, we obtain $|B| \leq |B'| + |B \cap (N_T[v] \setminus \{w\})| \leq \frac{n(T')}{2} + 1 \leq \frac{n}{2}$. \(\square\)

We are now in a position to present the proof of Theorem 2.
Proof of Theorem 2: Let $F$ be a forest of order $n$ and total domination number $\gamma_t$ such that $\sharp \gamma_t(F)$ is as large as possible. Let $D$ be a $\gamma_t$-set of $F$. Let $F'$ arise by removing from $F$ all endvertices of $F$ that do not belong to $D$. For every $u \in D$, let $L(u) = N_F(u) \setminus N_{F'}(u)$ and $\ell(u) = |L(u)|$, that is, $L(u)$ is the set of neighbors of $u$ in $D$ that are endvertices of $F$ that do not belong to $D$. We call a vertex $u$ in $D$ big if $\ell(u) \geq 2$, and we assume that – subject to the above conditions – the forest $F$ and the set $D$ are chosen such that the number $k$ of big vertices is as small as possible.

Claim 1. No two big vertices are adjacent.

Proof of Claim 1: Suppose, for a contradiction, that $u$ and $v$ are adjacent big vertices. Let $L'$ be a set of $\ell(u) - 1$ vertices in $L(u)$, and let $F' = F - \{ux : x \in L'\} + \{vx : x \in L'\}$, that is, we shift $\ell(u) - 1$ neighbors of $u$ in $L(u)$ to $v$. Clearly, the vertices $u$ and $v$ both belong to every $\gamma_t$-set of $F$ and also to every $\gamma_t$-set of $F'$. This easily implies that a set of vertices of $F$ is a $\gamma_t$-set of $F$ if and only if it is a $\gamma_t$-set of $F'$. It follows that $D$ is a $\gamma_t$-set of $F'$ and that $\sharp \gamma_t(F) = \sharp \gamma_t(F')$. Since $F'$ and $D$ lead to less than $k$ big vertices, we obtain a contradiction to the choice of $F$ and $D$. □

Claim 2. No two big vertices have a common neighbor in $D$.

Proof of Claim 2: Suppose, for a contradiction, that $u$ and $w$ are big vertices with a common neighbor $v$ in $D$. Let

- $\sharp u$ be the number of $\gamma_t$-sets of $F$ that contain a vertex from $L(u)$,
- $\sharp w$ be the number of $\gamma_t$-sets of $F$ that contain a vertex from $L(w)$, and
- $\sharp u, w$ be the number of $\gamma_t$-sets of $F$ that contain no vertex from $L(u) \cup L(w)$.

In view of $v$, no $\gamma_t$-set of $F$ contains a vertex from both sets $L(u)$ and $L(w)$, which implies

$$\sharp \gamma_t(F) = \sharp u + \sharp w + \sharp u, w.$$  

Note that $\frac{\sharp u}{\ell(u)}$ is the number of subsets of $V(F) \setminus L(u)$ that can be extended to a $\gamma_t$-set of $F$ by adding one vertex from $L(u)$. By symmetry, we may assume that $\frac{\sharp u}{\ell(u)} \leq \frac{\sharp w}{\ell(w)}$. Again, let $L'$ be a set of $\ell(u) - 1$ vertices in $L(u)$, and let $F'' = F - \{ux : x \in L'\} + \{vx : x \in L'\}$. Similarly as before, the vertices $u$ and $w$ both belong to every $\gamma_t$-set of $F$ and also to every $\gamma_t$-set of $F'$. It follows that $D$ is a $\gamma_t$-set of $F''$, and that

$$\sharp \gamma_t(F') = \frac{\sharp u}{\ell(u)} + \frac{\sharp w}{\ell(w)} (\ell(u) + \ell(w) - 1) + \sharp u, w \geq \sharp u + \sharp w + \sharp u, w = \sharp \gamma_t(F).$$

Since $F'$ and $D$ lead to less than $k$ big vertices, this contradicts the choice of $F$ and $D$. □

Claim 3. $k \leq \frac{\gamma_t}{2}$.

Proof of Claim 3: This follows immediately by applying Lemma 5 to each component of $F[D]$, choosing $B$ as the set of big vertices in that component. □
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Let \( n' = n(F') \), let \( V_1' \) be the set of endvertices of \( F' \), let \( n_1' = |V_1'| \), and let \( m \) be the number of edges of \( F' \) between \( D \) and \( V(F') \setminus D \). Since the vertices in \( V_1' \) are either endvertices of \( F \) that belong to \( D \) or are adjacent to an endvertex of \( F' \), we obtain that \( V_1' \subseteq D \). Since \( D \) is a total dominating set, we obtain

\[
n' - \gamma_t = |V(F') \setminus D| \leq m \leq \sum_{u \in D} (d_{F'}(u) - 1). \tag{1}
\]

Since \( F' \) is a forest with, say, \( \kappa \) components,

\[
n_1' = 2\kappa + \sum_{u \in V(F') : d_{F'}(u) \geq 2} (d_{F'}(u) - 2) \geq \sum_{u \in D : d_{F'}(u) \geq 2} (d_{F'}(u) - 2) = \sum_{u \in D : d_{F'}(u) \geq 2} d_{F'}(u) - 2(\gamma_t - n_1'),
\]

which implies

\[
2\gamma_t - n_1' \geq \sum_{u \in D : d_{F'}(u) \geq 2} d_{F'}(u). \tag{2}
\]

Now, we obtain

\[
n' \leq \sum_{u \in D} d_{F'}(u) = \sum_{u \in D : d_{F'}(u) \geq 2} d_{F'}(u) + n_1' \leq 2\gamma_t. \tag{3}
\]

Let \( u_1, \ldots, u_k \) be the big vertices. By (3), the forest \( F'' = F - \bigcup_{i=1}^k L(u_i) \) has order at most \( 3\gamma_t \). Let \( D'' \) be a set of vertices of \( F'' \) that is a subset of some \( \gamma_t \)-set \( D \) of \( F \). For every \( i \in \{1, \ldots, k\} \), if \( u_i \) has a neighbor in \( D'' \), then \( D \) contains no vertex from \( L(u_i) \), otherwise, the set \( D \) contains exactly one vertex from \( L(u_i) \). This implies that each of the \( 2^n(F'') \) subsets of \( V(F'') \) can be extended to a \( \gamma_t \)-set of \( F \) in at most \( \prod_{i=1}^k \ell(u_i) \) many ways.

Since

(i) \( n(F'') \leq 3\gamma_t \),

(ii) the geometric mean is less or equal the arithmetic mean,

(iii) \( \sum_{i=1}^k \ell(u_i) = n - n(F'') \leq n - \gamma_t \leq n - \frac{\gamma_t}{2} \),

(iv) \( \left(1 + \frac{\gamma_t}{2k}\right)^k \leq e^\frac{\gamma_t}{2} - k \leq e^\frac{\gamma_t}{2} \), and

(v) \( \frac{\gamma_t}{2} \leq 1 \),
we obtain

\[ \sharp_{\gamma_t}(F) \leq 2^{n(F')} \prod_{i=1}^{k} \ell(u_i) \]

\begin{align*}
\leq & \quad 2^{3\gamma_t} \prod_{i=1}^{k} \ell(u_i) \\
\leq & \quad 2^{3\gamma_t} \left( \frac{1}{k} \sum_{i=1}^{k} \ell(u_i) \right)^k \\
\leq & \quad 2^{3\gamma_t} \left( \frac{n - \gamma_t}{k} \right)^k \\
= & \quad 2^{3\gamma_t} \left( 1 + \frac{\gamma_t - k}{k} \right)^k \left( \frac{n - \gamma_t}{n - \gamma_t^2 n} \right)^{\frac{\gamma_t - k}{2}} \left( \frac{n - \gamma_t}{2} \right)^{\frac{\gamma_t}{2}} \\
\leq & \quad 2^{3\gamma_t} e^{\frac{3\gamma_t}{2}} \left( \frac{n - \gamma_t}{n - \gamma_t^2 n} \right)^{\frac{\gamma_t}{2}} \\
\leq & \quad (8\sqrt{e})^{\gamma_t} \left( \frac{n - \gamma_t}{2} \right)^{\frac{\gamma_t}{2}},
\end{align*}

which completes the proof.

There is clearly some room for lowering \(8\sqrt{e}\) to a smaller constant. Since the dependence on \(\gamma_t\) would still be exponential, we did not exploit this for the sake of simplicity. It would be interesting to see whether the bound can be improved to

\[ \left( 1 + o \left( \frac{n}{\gamma_t} \right) \right) \left( \frac{n - \gamma_t}{2} \right)^{\frac{\gamma_t}{2}}. \]

Note that Theorem 2 implies

\[ \sharp_{\gamma_t}(T) \leq \left( \frac{n - \gamma_t}{2} \right)^{\frac{\gamma_t}{2} + o(\gamma_t)}. \]

We proceed to our next proof.

**Proof of Theorem 3:** We proceed by induction on \(n\). If \(n = 2\), then \(F = K_2, \gamma_t = 2\), and \(\sharp_{\gamma_t}(F) = 1 = (1 + \sqrt{2})^0 = (1 + \sqrt{2})^{n - \gamma_t}\). Now, let \(n \geq 3\).

**Claim 1.** If \(F\) contains a component \(T\) that is a star, then \(\sharp_{\gamma_t}(F) \leq (1 + \sqrt{2})^{n - \gamma_t}\), with strict inequality if \(T\) has order at least 3.

**Proof of Claim 1:** Suppose that \(F\) contains a component \(T\) that is a star. Thus, \(T = K_{1,t}\) for some \(t \geq 1\). The forest \(F' = F - V(T)\) has order \(n' = n - t - 1\), no isolated vertex, and total domination number
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\( \gamma'_t = \gamma_t - 2 \). By induction, we obtain

\[
\sharp \gamma_t(F) = t \cdot \sharp \gamma_t(F') \leq t(1 + \sqrt{2})^{n'-\gamma'_t} = t(1 + \sqrt{2})^{n-t-1-(\gamma_t-2)}
\]

\[
= (1 + \sqrt{2})^{n-\gamma_t} (t(1 + \sqrt{2})^{1-t}) \leq (1 + \sqrt{2})^{n-\gamma_t},
\]

where we use \( t(1 + \sqrt{2})^{1-t} \leq 1 \) for \( t = 1 \) and \( t \geq 2 \). Furthermore, if \( t \geq 2 \), then \( t(1 + \sqrt{2})^{1-t} < 1 \), in which case \( \sharp \gamma_t(F) < (1 + \sqrt{2})^{n-\gamma_t}. \)

\[\square\]

**Claim 2.** If \( F \) contains a component \( T \) of diameter 3, then \( \sharp \gamma_t(F) < (1 + \sqrt{2})^{n-\gamma_t}. \)

**Proof of Claim 2:** Suppose that \( F \) contains a component \( T \) of diameter 3. Note that \( T \) has a unique minimum total dominating set. The forest \( F' = F - V(T) \) has order \( n' \leq n - 4 \), no isolated vertex, and total domination number \( \gamma'_t = \gamma_t - 2 \). By induction, we obtain

\[
\sharp \gamma_t(F) = \sharp \gamma_t(F') \leq (1 + \sqrt{2})^{n'-\gamma'_t} \leq (1 + \sqrt{2})^{n-\gamma_t-2} < (1 + \sqrt{2})^{n-\gamma_t}.
\]

\[\square\]

By Claim 1 and Claim 2, we may assume that there is a component of \( F \) that has diameter at least 4, for otherwise the desired result follows. Let \( T \) be such a component of \( F \). Let \( uvwxxy \ldots r \) be a longest path in \( T \), and consider \( T \) as rooted in \( r \). For a vertex \( z \) of \( T \), let \( V_z \) be the set that contains \( z \) and all its descendants.

**Claim 3.** If \( d_F(w) \geq 3 \), then \( \sharp \gamma_t(F) < (1 + \sqrt{2})^{n-\gamma_t}. \)

**Proof of Claim 3:** Suppose that \( d_F(w) \geq 3 \), which implies that \( w \) belongs to every \( \gamma_t \)-set of \( F \), because either \( w \) is a support vertex or \( w \) is the only neighbor of two support vertices, that is no leaf. Let \( v' \) be a child of \( w \) distinct from \( v \). Let \( F' = F - V_{v'}. \) If \( v' \) is an endvertex, then \( F' \) has order \( n' = n - 1 \), no isolated vertex, and total domination number \( \gamma'_t = \gamma_t - 1 \). By induction, we obtain

\[
\sharp \gamma_t(F) \leq \sharp \gamma_t(F') \leq (1 + \sqrt{2})^{n'-\gamma'_t} = (1 + \sqrt{2})^{n-\gamma_t-1} < (1 + \sqrt{2})^{n-\gamma_t}.
\]

If \( v' \) is not an endvertex, then \( F' \) has order \( n' \leq n - 2 \), no isolated vertex, and total domination number \( \gamma'_t = \gamma_t - 1 \). Note that if \( T \) is a minimum total dominating set of \( F \), \( T - \{v\} \) is a total dominating set of \( F' \), since \( v' \) is a support vertex and \( v \) and \( w \) are part of every minimum total dominating set of \( F \). By induction, we obtain

\[
\sharp \gamma_t(F) \leq \sharp \gamma_t(F') \leq (1 + \sqrt{2})^{n'-\gamma'_t} \leq (1 + \sqrt{2})^{n-\gamma_t-1} < (1 + \sqrt{2})^{n-\gamma_t}.
\]

In both cases, \( \sharp \gamma_t(F) < (1 + \sqrt{2})^{n-\gamma_t}. \)

\[\square\]

By Claim 3, we may assume that \( d_F(w) = 2 \), for otherwise the desired result holds.

**Claim 4.** If \( d_F(v) \geq 3 \), then \( \sharp \gamma_t(F) < (1 + \sqrt{2})^{n-\gamma_t}. \)

**Proof of Claim 4:** Suppose that \( \ell = d_F(v) - 1 \geq 2 \). Let \( F' = F - V_w \), \( F'' = F - (N_F(v) \setminus \{w\}) \), and \( F''' = F - (V_w \cup \{x\}). \) See Figure 2 for an illustration.
The forest $F$  

The forest $F'$  

The forest $F''$  

The forest $F'''$

Fig. 2: The important details of the forests $F$, $F'$, $F''$, and $F'''$. 

- There are at most $\ell \cdot \#\gamma_t(F')$ many $\gamma_t$-sets of $F$ that contain $v$ and a child of $v$ but do not contain $w$. Furthermore, if such a $\gamma_t$-set exists, then $F'$ has order $n' = n - \ell - 2$, no isolated vertex, and total domination number $\gamma'_t = \gamma_t - 2$.

- There are at most $\#\gamma_t(F'')$ many $\gamma_t$-sets of $F$ that contain $v$, $w$, and $x$. Furthermore, if such a $\gamma_t$-set exists, then $F''$ has order $n'' = n - \ell$, no isolated vertex, and total domination number $\gamma''_t = \gamma_t - 1$.

- There are at most $\#\gamma_t(F''')$ many $\gamma_t$-sets of $F$ that contain both $v$ and $w$ but do not contain $x$. Furthermore, if such a $\gamma_t$-set exists, then $F'''$ has order $n''' = n - \ell - 3$, no isolated vertex, and total domination number $\gamma'''_t = \gamma_t - 2$.

Since all $\gamma_t$-sets of $F$ are of one of the three considered types, we obtain, by induction,

$$
\#\gamma_t(F) \leq \ell \cdot \#\gamma_t(F') + \#\gamma_t(F'') + \#\gamma_t(F''') \\
\leq \ell(1 + \sqrt{2})^{n-\ell-2-(\gamma_t-2)} + (1 + \sqrt{2})^{n-\ell-(\gamma_t-1)} + (1 + \sqrt{2})^{n-\ell-3-(\gamma_t-2)} \\
= (1 + \sqrt{2})^{n-\gamma_t}(1 + \sqrt{2})^{-\ell-1}(\ell(1 + \sqrt{2}) + (1 + \sqrt{2})^2 + 1) \\
< (1 + \sqrt{2})^{n-\gamma_t},
$$

where we use $\ell(1 + \sqrt{2}) + (1 + \sqrt{2})^2 + 1 < (1 + \sqrt{2})^{\ell+1}$ for all $\ell \geq 2$. 

By Claim 4, we may assume that $d_F(v) = 2$, for otherwise the desired result holds.

**Claim 5.** If $x$ is a support vertex, then $\#\gamma_t(F) < (1 + \sqrt{2})^{n-\gamma_t}$.

**Proof of Claim 5:** Suppose that $x$ is a support vertex, which implies that $v$ and $x$ belong to every $\gamma_t$-set of $F$. Let $F' = F - V_w$ and $F'' = F - (N_F[v] \cup N_F[x])$. 

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- There are at most $\sharp \gamma_t(F')$ many $\gamma_t$-sets of $F$ that contain $w$ but do not contain $u$. Furthermore, if such a $\gamma_t$-set exists, then $F'$ has order $n' = n - 3$, no isolated vertex, and total domination number $\gamma_t' = \gamma_t - 2$.

- There are at most $\sharp \gamma_t(F')$ many $\gamma_t$-sets of $F$ that contain $w$ and at least one other neighbour of $x$. Furthermore, if such a $\gamma_t$-set exists, then $F'$ has order $n' = n - 3$, no isolated vertex, and total domination number $\gamma_t' = \gamma_t - 2$.

- There are at most $\sharp \gamma_t(F'')$ many $\gamma_t$-sets of $F$ that contain $w$ and no other neighbour of $x$. Furthermore, if such a $\gamma_t$-set exists, then $F''$ has order $n'' = n - 5$, no isolated vertex, and total domination number $\gamma_t'' = \gamma_t - 3$.

Since all $\gamma_t$-sets of $F$ are of one of the three considered types, we obtain, by induction,

$$\sharp \gamma_t(F) \leq \sharp \gamma_t(F') + \sharp \gamma_t(F'') < 2(1 + \sqrt{2})^{n-3-(\gamma_t-2)} + (1 + \sqrt{2})^{n-5-(\gamma_t-3)}$$

$$= (1 + \sqrt{2})^{n-\gamma_t}(1 + \sqrt{2})^{-2}(2(1 + \sqrt{2}) + 1) = (1 + \sqrt{2})^{n-\gamma_t},$$

where we use $2(1 + \sqrt{2}) + 1 = (1 + \sqrt{2})^2$. Note that in $F'$ there is a component that contains a path of length two, in particular not every component of $F'$ is a $K_2$.

By Claim 5, we may assume that $x$ is not a support vertex, for otherwise the desired result holds.

**Claim 6.** If $x$ has a child that is a support vertex, then $\sharp \gamma_t(F) < (1 + \sqrt{2})^{n-\gamma_t}$.

**Proof of Claim 6:** Suppose that $x$ has a child $w'$ that is a support vertex. Clearly, the vertex $w'$ is distinct from $w$ and belongs to every $\gamma_t$-set of $F$. The forest $F' = F - V_w$ has order $n' = n - 3$, no isolated vertex, and total domination number $\gamma_t' = \gamma_t - 2$. By induction, we obtain

$$\sharp \gamma_t(F) = 2\sharp \gamma_t(F') \leq 2(1 + \sqrt{2})^{n-3-(\gamma_t-2)} = (1 + \sqrt{2})^{n-\gamma_t}2(1 + \sqrt{2})^{-1} < (1 + \sqrt{2})^{n-\gamma_t},$$

where we use $2 < (1 + \sqrt{2})$.

By Claim 6, we may assume that no child of $x$ is a support vertex, for otherwise the desired result holds. Together with Claims 3 and 4, we may assume that the subforest of $F$ induced by $V_x$ arises from a star $K_{1,q}$ for some $q \geq 1$ by subdividing every edge twice. Let $F' = F - V_x$, $F'' = F - (V_x \cup \{y\})$, and $F''' = F - (V_x \cup N_F[y])$.

- There are at most $2\sharp \gamma_t(F')$ many $\gamma_t$-sets of $F$ that do not contain $x$. Furthermore, if such a $\gamma_t$-set exists, then $F'$ has order $n' = n - 3q - 1$, no isolated vertex, and total domination number $\gamma_t' = \gamma_t - 2q$.

- There are at most $(2q - 1)\sharp \gamma_t(F'')$ many $\gamma_t$-sets of $F$ that contain $x$ but do not contain $y$. Furthermore, if such a $\gamma_t$-set exists, then $F''$ has order $n'' = n - 3q - 2$, no isolated vertex, and total domination number $\gamma_t'' = \gamma_t - 2q - 1$.

- There are at most $2\sharp \gamma_t(F''')$ many $\gamma_t$-sets of $F$ that contain both $x$ and $y$. Furthermore, if such a $\gamma_t$-set exists, then $F'''$ has order $n''' \leq n - 3q - 3$, no isolated vertex, and total domination number $\gamma_t''' = \gamma_t - 2q - 2$. 


Since all $\gamma_t$-sets of $F$ are of one of the three considered types, we obtain, by induction,
\[
\sharp\gamma_t(F) \leq 2^q\sharp\gamma_t(F') + (2^q - 1)\sharp\gamma_t(F'') + 2^q\sharp\gamma_t(F''') \\
\leq 2^q(1 + \sqrt{2})^{n-3q-1-(\gamma_t-2q)} + (2^q - 1)(1 + \sqrt{2})^{n-3q-2-(\gamma_t-2q-1)} + 2^q(1 + \sqrt{2})^{n-3q-3-(\gamma_t-2q-2)} \\
= (1 + \sqrt{2})^{n-\gamma_t} + (1 + \sqrt{2})^{-q-1}(2^q + 2^q - 1 + 2^q) \\
< (1 + \sqrt{2})^{n-\gamma_t},
\]
where we use $3 \cdot 2^q - 1 < (1 + \sqrt{2})^{q+1}$ for all $q \geq 1$. This completes the proof of Theorem 3.

We proceed to the proof of Theorem 4, which uses exactly the same approach as Theorem 3.

**Proof of Theorem 4:** By induction on $n$, we show that $\sharp\gamma_t(F) \leq \beta^n$, where $\beta$ is the unique positive real solution of the equation $2\beta + \beta^3 + 1 = \beta^5$, that is, $\beta \approx 1.4865$. If $n = 2$, then $F = K_2$ and $\sharp\gamma_t(F) = 1 < \beta^2$. Now, let $n \geq 3$.

**Claim 1.** If $F$ contains a component $T$ that is a star, then $\sharp\gamma_t(F) \leq \beta^n$.

**Proof of Claim 1:** Suppose that $F$ contains a component $T$ that is a star. Thus, $T = K_{1,t}$ for some $t \geq 1$. The forest $F' = F - V(T)$ has order $n' = n - t + 1$ and no isolated vertex. By induction, we obtain $\sharp\gamma_t(F') = t \cdot \sharp\gamma_t((F')) \leq t\beta^{n-t-1} \leq \beta^n$, where we use $t \leq \beta^{1+1}$.

**Claim 2.** If $F$ contains a component $T$ of diameter 3, then $\sharp\gamma_t(F) \leq \beta^n$.

**Proof of Claim 2:** Suppose that $F$ contains a component $T$ of diameter 3. The forest $F' = F - V(T)$ has order $n' = n - 4$ and no isolated vertex. By induction, we obtain $\sharp\gamma_t(F') = \sharp\gamma_t((F')) \leq \beta^{n'} < \beta^n$.

By Claim 1 and Claim 2, we may assume that every component of $F$ has diameter at least 4, for otherwise the desired result follows. Let $T$ be an arbitrary component of $F$. Let $uvwxy\ldots r$ be a longest path in $T$, and consider $T$ as rooted in $r$. For a vertex $z$ of $T$, let $V_z$ be the set that contains $z$ and all its descendants.

**Claim 3.** If $d_F(w) \geq 3$, then $\sharp\gamma_t(F) \leq \beta^n$.

**Proof of Claim 3:** Suppose that $d_F(w) \geq 3$, which implies that $w$ belongs to every $\gamma_t$-set of $F$. Let $v'$ be a child of $w$ distinct from $v$. The forest $F' = F - V_v$ has order $n' < n$ and no isolated vertex. Since $\sharp\gamma_t(F') \leq \sharp\gamma_t((F'))$, we obtain, by induction, $\sharp\gamma_t(F') \leq \beta^{n'} < \beta^n$.

By Claim 3, we may assume that $d_F(w) = 2$, for otherwise the desired result holds.

**Claim 4.** If $d_F(w) \geq 3$, then $\sharp\gamma_t(F) \leq \beta^n$.

**Proof of Claim 4:** Suppose that $\ell = d_F(v) - 1 \geq 2$. Arguing exactly as in the proof of Claim 4 in the proof of Theorem 3 using the forests $F'$, $F''$, and $F'''$, we obtain, by induction,
\[
\sharp\gamma_t(F) \leq \ell \cdot \sharp\gamma_t(F') + \sharp\gamma_t(F'') + \sharp\gamma_t(F''') \\
\leq \ell\beta^{n-\ell-2} + \beta^{n-\ell} + \beta^{n-\ell-3} \\
= \beta^n \beta^{-\ell-3}(\ell + \beta^3 + 1) \\
\leq \beta^n,
\]
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where we use \( \ell \beta + \beta^3 + 1 \leq \beta^{\ell + 3} \) for all \( \ell \geq 2 \); in fact, this inequality is the reason for the specific choice of \( \beta \).

By Claim 4, we may assume that \( d_F(v) = 2 \), for otherwise the desired result holds.

**Claim 5.** If \( x \) is a support vertex, then \( \sharp \gamma_t(F) \leq \beta^n \).

**Proof of Claim 5:** Suppose that \( x \) is a support vertex. Arguing exactly as in the proof of Claim 5 in the proof of Theorem 3 using the forests \( F' \) and \( F'' \), we obtain, by induction,

\[
\sharp \gamma_t(F) \leq 2^q \gamma_t(F') + \sharp \gamma_t(F'') \leq 2\beta^{n-3} + \beta^{n-5} = \beta^n \beta^{-5}(2\beta^2 + 1) \leq \beta^n,
\]

where we use \( 2\beta^2 + 1 \leq \beta^5 \).

By Claim 5, we may assume that \( x \) is not a support vertex, for otherwise the desired result holds.

**Claim 6.** If \( x \) has a child that is a support vertex, then \( \sharp \gamma_t(F) \leq \beta^n \).

**Proof of Claim 6:** Suppose that \( x \) has a child \( w' \) that is a support vertex. Arguing exactly as in the proof of Claim 6 in the proof of Theorem 3 using the forest \( F' \), we obtain, by induction,

\[
\sharp \gamma_t(F) = 2^q \gamma_t(F') \leq 2\beta^{n-3} = \beta^n 2\beta^{-3} \leq \beta^n,
\]

where we use \( 2 < \beta^3 \).

Now, arguing exactly as at the end of the proof of Theorem 3 using the forests \( F', F'', \) and \( F''' \), we obtain, by induction,

\[
\sharp \gamma_t(F) \leq 2^q \gamma_t(F') + (2^q - 1)\gamma_t(F'') + 2^q \gamma_t(F''')
\leq 2^q \beta^{n-3q-1} + (2^q - 1)\beta^{n-3q-2} + 2^q \beta^{n-3q-3}
= \beta^n \beta^{-3q-3} (2^q \beta^2 + (2^q - 1)\beta + 2^q)
\leq \beta^n \beta^{-3q-3} 2^q (\beta^2 + \beta + 1)
\leq \beta^n,
\]

where we use \( 2^q (\beta^2 + \beta + 1) \leq \beta^{3q+3} \) for all \( q \geq 1 \).

**References**

[1] J.D. Alvarado, S. Dantas, E. Mohr, and D. Rautenbach, On the maximum number of minimum dominating sets in forests, Discrete Mathematics 342 (2019) 934–942.

[2] A. Bień, Properties of gamma graphs of trees, presentation at the 17th Workshop on Graph Theory Colourings, Independence and Domination (CID 2017), Piechowice, Poland.

[3] D. Bród and Z. Skupień, Trees with extremal numbers of dominating sets, The Australasian Journal of Combinatorics 35 (2006) 273–290.

[4] S. Connolly, Z. Gabor, A. Godbole, B. Kay, and T. Kelly, Bounds on the maximum number of minimum dominating sets, Discrete Mathematics 339 (2016) 1537–1542.
[5] F.V. Fomin, F. Grandoni, A.V. Pyatkin, and A.A. Stepanov, Bounding the number of minimal dominating sets: a measure and conquer approach, Lecture Notes in Computer Science 3827 (2005) 573–582.

[6] G.H. Fricke, S.M. Hedetniemi, S.T. Hedetniemi, and K.R. Hutson, $\gamma$-graphs of graphs, Discussiones Mathematicae Graph Theory 31 (2011) 517–531.

[7] T.W. Haynes, S.T. Hedetniemi, and P. Slater, Fundamentals of domination in graphs, Marcel Dekker, Inc., New York, 1998.

[8] M.A. Henning and A. Yeo, Total Domination in Graphs, Springer 2013.

[9] M. Krzywkowski and S. Wagner, Graphs with few total dominating sets, Discrete Mathematics 341 (2018) 997–1009.

[10] K.M. Koh, C.Y Goh, and F.M. Dong, The maximum number of maximal independent sets in unicyclic connected graphs, Discrete Mathematics 308 (2008) 3761–3769.

[11] Z. Skupień, Majorization and the minimum number of dominating sets, Discrete Applied Mathematics 165 (2014) 295–302.

[12] S. Wagner, A note on the number of dominating sets of a graph, Utilitas Mathematica 92 (2013) 25–31.

[13] I. Włoch, Trees with extremal numbers of maximal independent sets including the set of leaves, Discrete Mathematics 308 (2008) 4768–4772.

[14] J. Zito, The structure and maximum number of maximum independent sets in trees, Journal of Graph Theory 15 (1991) 207–221.