Super Domination: Graph Classes, Products and Enumeration

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Abstract

The dominating set problem (DSP) is one of the most famous problems in combinatorial optimization. It is defined as follows. For a given simple graph $G = (V, E)$, a dominating set of $G$ is a subset $S \subseteq V$ such that every vertex in $V \setminus S$ is adjacent to at least one vertex in $S$. Furthermore, the DSP is the problem of finding a minimum-size dominating set and the corresponding minimum size, the domination number of $G$.

In this work we investigate a variant of the DSP, the super dominating set problem (SDSP), which has attracted much attention during the last years. A dominating set $S$ is called a super dominating set of $G$, if for every vertex $u \in S = V \setminus S$, there exists a $v \in S$ such that $N(u) \cap \overline{S} = \{u\}$. Analogously, the SDSP is to find a minimum-size super dominating set, and the corresponding minimum size, the super domination number of $G$. The decision variants of both the DSP and the SDSP have shown to be $\mathcal{NP}$-hard.

In this paper, we present tight bounds for the super domination number of the neighbourhood corona product, $r$-gluing, and the Hajós sum of two graphs. Additionally, we present infinite families of graphs attaining our bounds. Finally, we give the exact number of minimum size super dominating sets for some graph classes. In particular, the number of super dominating sets for cycles has quite surprising properties as it varies between values of the set $\{4, n, 2n, \frac{5n^2-10n}{8}\}$ based on $n \mod 4$.

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1 Introduction

Notations and definitions. Let $G = (V, E)$ be a graph with vertex set $V$ and edge set $E$. Throughout this paper, we consider finite undirected graphs without loops. For each vertex $v \in V$, the set $N(v) = N_G(v) = \{ u \in V \mid uv \in E \}$ refers to the open neighbourhood of $v$ in $G$ and the set $N[v] = N_G[v] = N_G(v) \cup \{ v \}$ refers to the closed neighbourhood of $v$ in $G$. If the graph $G$ is clear from the context, we will omit the corresponding index $G$. The degree of $v$ is the cardinality of $N(v)$. Throughout this paper, for a set $S \subseteq V(G)$, the expression $\overline{S}$ always stands for $V(G) \setminus S$.

A set $S \subseteq V$ is called a dominating set if every vertex in $\overline{S} = V \setminus S$ is adjacent to at least one vertex in $S$. The domination number $\gamma(G)$ is the cardinality of a minimum size dominating set in $G$. For a detailed treatment of domination theory, we refer the reader to [11].

A dominating set $S$ of $G$ is called a super dominating set of $G$, if for every vertex $u \in \overline{S}$, there exists a $v \in S$ such that $N(v) \cap \overline{S} = \{ u \}$. In this case, we say that $v$ super dominates $u$. The super domination number $G$ is the cardinality of a minimum size super dominating set of $G$, denoted by $\gamma_{sp}(G)$ [14]. We refer the reader to [1, 4, 5, 6, 12, 13, 17] for more details on super dominating sets of a graph. Some applications can be found in [14].

In this work, we often call vertices “black” (abbreviated by “B”) if they are contained in a given super dominating set and “white” (abbreviated “W”) if they are not contained in it.

Graph classes. We will consider the following well-known graph classes: path graph, cycle graph, star graph, complete graph, complete bipartite graph. A friendship graph $F_n$ is a collection of $n$ triangles, where all triangles have one vertex, the central vertex, in common (see Figure 1).

Given two simple graphs $G_1$ and $G_2$, the corona product of $G_1$ and $G_2$, denoted by $G_1 \circ G_2$ is the graph arising from the disjoint union of $G_1$ with $|V(G_1)|$ copies of $G_2$, by adding edges between the $i$-th vertex of $G_1$ and all vertices of the $i$-th copy of $G_2$ [10]. The neighbourhood corona product of $G_1$ and $G_2$, denoted by $G_1 \ast G_2$, is the graph obtained by taking one copy of $G_1$ and $|V(G_1)|$ copies of $G_2$ and joining the neighbours of the $i$-th vertex of $G_1$ to every vertex in the $i$-th copy of $G_2$ [8]. Thus, this graph has $|V(G_1)| \cdot (|V(G_2)| + 1)$ vertices. Figure 2 shows $C_4 \ast K_3$, where $C_n$ is the cycle of order $n$ and $K_n$ is the complete graph of order $n$. For more results on the neighbourhood corona product of two graphs, we refer the reader to [2, 7, 15].

Let $G_1$ and $G_2$ be two graphs and $r \in \mathbb{N}_0 := \mathbb{N} \cup \{ 0 \}$ with $r \leq \min\{ \omega(G_1), \omega(G_2) \}$, where $\omega(G)$ is the clique number of $G$. Choose a clique $K_r$ from each $G_i$, $i = 1, 2,$
and form a new graph $G$ from the union of $G_1$ and $G_2$ by identifying the two chosen $r$-cliques in an arbitrary manner. The graph $G$ is called $r$-gluing of $G_1$ and $G_2$ and denoted by $G_1 \cup_K G_2$. If $r = 0$, then $G_1 \cup_{K_0} G_2$ is just its disjoint union. $G_1 \cup_{K_i} G_2$ for $i = 1, 2$, is called vertex and edge gluing, respectively. Notice that there are sometimes several ways to $r$-glue two graphs together (see Figure 3). We refer the reader for some results on the $r$-gluing of two graphs to [7].

Given graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ with disjoint vertex sets, an edge $x_1y_1 \in E_1$, and an edge $x_2y_2 \in E_2$, the Hajós sum $G_3 = G_1(x_1y_1) + H G_2(x_2y_2)$ is the graph obtained as follows: begin with $G_3 = (V_1 \cup V_2, E_1 \cup E_2)$; then in $G_3$ delete the edges $x_1y_1$ and $x_2y_2$, identify the vertices $x_1$ and $x_2$ as $v_H(x_1x_2)$, and add the edge $y_1y_2$ [9]. Figure 4 shows the Hajós sum of $K_4$ and $C_4$ with respect to $x_1y_1$ and $x_2y_2$.

**Previous work.** Super dominating sets have been studied in multiple papers since the inception of the concept in 2015 [14]. In particular, the following tight lower and upper bounds are known for the super domination number.

**Theorem 1.1** [14] Let $G$ be a graph of order $n$ without isolated vertices. Then,

$$1 \leq \gamma(G) \leq \frac{n}{2} \leq \gamma_{sp}(G) \leq n - 1.$$
Figure 3: Graphs $G$, $H$ and all non-isomorphic graphs $G \cup K_4 H$, respectively.

Figure 4: Hajós construction of $K_4$ and $C_4$.

Besides general bounds, the super domination number is known exactly for many graph classes, some stated in the following theorem.

**Theorem 1.2** Let $n \in \mathbb{N}$.

(a) For the path graph $P_n$ it holds that $\gamma_{sp}(P_n) = \lceil \frac{n}{2} \rceil$.

(b) For the cycle graph $C_n$ it holds that

$$\gamma_{sp}(C_n) = \begin{cases} 
\lceil \frac{n+1}{2} \rceil & \text{if } n \equiv 2 \pmod{4}, \\
\lceil \frac{n}{2} \rceil & \text{otherwise}. 
\end{cases}$$

(c) For the complete graph $K_n$, where $n \geq 2$, it holds that $\gamma_{sp}(K_n) = n - 1$.

(d) For the complete bipartite graph $K_{n,m}$, where $\min\{n,m\} \geq 2$, it holds that $\gamma_{sp}(K_{n,m}) = n + m - 2$.

(e) For the star graph $K_{1,n}$ it holds that $\gamma_{sp}(K_{1,n}) = n$.

**Theorem 1.3** Let $n \in \mathbb{N}$.

For the friendship graph $F_n$ it holds that $\gamma_{sp}(F_n) = n + 1$.

Later we will refer to the following known results from [5, 6].
Figure 5: Chain of $n$ graphs $G_1, G_2, \ldots, G_n$.

**Proposition 1.4** [6] Let $G$ be a disconnected graph with components $G_1$ and $G_2$. Then

$$\gamma_{sp}(G) = \gamma_{sp}(G_1) + \gamma_{sp}(G_2).$$

**Theorem 1.5** [6] Let $G_1, G_2, \ldots, G_n$ be a finite sequence of pairwise disjoint connected graphs and let $x_i, y_i \in V(G_i)$. Let $C(G_1, G_2, \ldots, G_n)$ be the chain of graphs $\{G_i\}_{i=1}^n$ with respect to the vertices $\{x_i, y_i\}_{i=1}^k$ obtained by identifying the vertex $y_i$ with the vertex $x_{i+1}$ for $i = 1, 2, \ldots, n - 1$ (see Figure 5). Then, for $n = 2$, we have:

$$\gamma_{sp}(G_1) + \gamma_{sp}(G_2) - 1 \leq \gamma_{sp}(C(G_1, G_2)) \leq \gamma_{sp}(G_1) + \gamma_{sp}(G_2).$$

Furthermore, these bounds are tight.

Besides being studied in exact graph classes, super domination has also been studied for different graph products. In particular, Dettlaff et al. [4] have studied the super domination number of lexicographic products and joins and also shown that determining the super domination number of a graph is $\mathcal{NP}$-hard. Klein et al. [13] have studied Cartesian products and (usual) corona products.

**Our results.** In this paper, we continue the study of the super domination number of a graph, started in [6, 7]. First in Section 3, we present a key lemma which will be used throughout this paper. In Section 3, we find the exact value of the super domination number of the neighbourhood corona product of two graphs. In Section 4, we present tight lower and upper bounds on the $r$-gluing of two graphs and provide infinite families of graphs attaining these bounds. We study the super domination number of the Hajóš sum of two graphs and find tight upper and lower bounds for it, together with infinite families of examples attaining the bounds, in Section 5. Finally, in Section 6, we count exactly the number of minimum size super dominating sets of some graph classes.

## 2 Key lemma

We introduce a technical key lemma for analysing super dominating sets. It will be needed in most of the following results.

**Lemma 2.1** Let $S$ be a super dominating set in a graph $G$.

(a) Then there is a super dominating set $S'$ with same cardinality with $\overline{S} \subseteq S'$ and $\overline{S'} \subseteq S$.

Furthermore, there is a bijective function $f : \overline{S'} \rightarrow \overline{S}$ so that $f(a) = b$ holds if and only if $a$ super dominates $b$ for the super dominating set $S$ and $b$ super dominates $a$ for the super dominating set $S'$.
(b) Let $D = S \cap S'$. Then $V(G)$ can be partitioned as $V(G) = \overline{S} \cup \overline{S'} \cup D$, where it holds that $S = \overline{S'} \cup D$, $S' = \overline{S} \cup D$.

(c) Let $S$ have cardinality $|V(G)|/2$. Then $\overline{S}$ is a super dominating set with the same cardinality and it holds that $\overline{S} = S'$ and $\overline{S'} = S$.

Furthermore, each vertex in $S$ super dominates exactly one vertex in $\overline{S}$ and vice versa, i.e., the function $f : S \rightarrow S'$ from (a) is uniquely determined.

Proof.

(a) Let $S \subseteq V(G)$ be a super dominating set in $G$. Then for each $b \in \overline{S}$ there exists a vertex $a \in S$ such that $N(a) \cap \overline{S} = \{b\}$. We construct the new super dominating set $S'$ by replacing each $a \in S$ by the corresponding $b \in \overline{S}$. Denote $A \subseteq S$ as the set of all vertices $a$ removed from $S$ during this process. Let $f : A \rightarrow \overline{S}$ be the corresponding function with $f(a) = b$ for each $a \in A$ and $b \in \overline{S}$ super dominated by $a$. By construction, $A = \overline{S'}$, $|S| = |S'|$ and $|\overline{S}| = |\overline{S'}|$ hold. As each vertex $a \in \overline{S'}$ can super dominate only one vertex $b \in \overline{S}$, $f$ is well defined. By construction, for each $b$ we have only one $a$, i.e., $f$ is also injective. Because of $|\overline{S'}| = |S|$, it is also bijective.

Claim: $S'$ is a super dominating set in $G$.

Proof (Claim): Let $f(a) = b$ with $a \in \overline{S'}$, $b \in \overline{S}$. Assume that $b$ does not super dominate $a$. Then $b$ is also adjacent to another $a' \in \overline{S'}$. Thus, there is a $b' \in \overline{S}$ with $f(a') = b'$. This would mean that $a'$ super dominates $b'$, but it is also adjacent to $b$, which is a contradiction. Thus, $b$ super dominates $a$, and the claim follows.

(b) This follows easily from (a).

(c) Here $D = S \cap S' = \emptyset$ holds. From (b) it follows that $S = \overline{S'}$ and $S' = \overline{S}$. Since each vertex can super dominate at most one other vertex, each vertex in $S$ (and each vertex in $S'$) super dominates exactly one vertex. Thus, $f$ is uniquely determined. □

3 Super domination number of neighbourhood corona product of two graphs

In this section, we study the super domination number of the neighbourhood corona product of two graphs. Let $G$ and $H$ be two graphs of orders $n, m \in \mathbb{N}$, respectively. It is clear that

$$\gamma_{sp}(G \ast H) \leq \gamma_{sp}(G) + nm,$$

since if we consider all vertices of all copies of $H$ in our super dominating set, then we only need to find a super dominating set for $G$ and the claim follows by the definition of a super dominating set.
Later (in Corollary 3.3) we will show that, under some conditions, it holds that \( \gamma_{sp}(G \ast H) = n(\gamma_{sp}(H) + 1) \). The following proposition shows that this is an improvement over the trivial upper bound from (1) for \( m \geq 2 \).

**Proposition 3.1** Let \( G \) and \( H \) be two connected graphs of orders \( n \) and \( m \neq 1 \), respectively. Then

\[
 n(\gamma_{sp}(H) + 1) < \gamma_{sp}(G) + nm.
\]

**Proof.** By Theorem 1.1 we know that \( \gamma_{sp}(H) \leq m - 1 \). So we have \( n(\gamma_{sp}(H) + 1) = n\gamma_{sp}(H) + n \leq nm \). Hence, we have \( n(\gamma_{sp}(H) + 1) < nm + \gamma_{sp}(G) \) and therefore, we have the result. \( \square \)

Next we present a tight upper bound for \( \gamma_{sp}(G \ast H) \).

**Theorem 3.2** Let \( G \) and \( H \) be two connected graphs of orders \( n, m \in \mathbb{N} \). Then

\[
 \gamma_{sp}(G \ast H) \leq n(\gamma_{sp}(H) + 1).
\]

**Proof.** Let \( S_H \) be a super dominating set for \( H \). We create a set \( S \) for \( G \ast H \) by placing all vertices of \( G \) in \( S \). Next, for each copy of \( H \), we place all vertices corresponding to \( S_H \) in \( S \). In the following, we show that \( S \) is a super dominating set for \( G \ast H \).

Let \( u \in \overline{S} \). Then \( u \in V(H') \) for some copy of \( H \). Let \( v \in V(H') \) super dominate \( u \) in \( H' \). If \( v \) does not super dominate \( u \) in \( G \ast H \), then \( v \) has another neighbour in \( \overline{S} \cap (V(G \ast H) \setminus V(H')) \). However, this is not possible, since \( V(G) \subseteq S \) and there are no edges between different copies of \( H \). Thus, \( S \) is a super dominating set for \( G \ast H \) with cardinality \( n\gamma_{sp}(H) + n = n(\gamma_{sp}(H) + 1) \) and the assertion follows. \( \square \)

In the following theorem, we show that the upper bound in Theorem 3.2 is actually the super domination number of \( G \ast H \), when \( G \) and \( H \neq K_1 \) are connected graphs and \( \gamma_{sp}(H) < m - 1 \) or it holds that \( H = K_m \).

**Theorem 3.3** Let \( G \) and \( H \) be two connected graphs of orders \( n \) and \( m \neq 1 \), respectively, where it holds that \( \gamma_{sp}(H) < m - 1 \) or it holds that \( H = K_m \). Then

\[
 n(\gamma_{sp}(H) + 1) \leq \gamma_{sp}(G \ast H).
\]

**Proof.** Let \( G = (V(G), E(G)) \) and \( H = (V(H), E(H)) \). Let \( S \) be a super dominating set for \( G \ast H \). Let \( H_w \) be the copy of \( H \) corresponding to \( w \in V(G) \). Then it holds that \( V(G \ast H) = V(G) \cup \bigcup_{w \in V(G)} V(H_w) \).

**Claim 1:** In each copy of \( H \), the set \( S \) has at least \( \gamma_{sp}(H) \) vertices.

**Proof (Claim 1):** For a given \( w \in V(G) \), consider the copy \( H_w \) of \( H \). Assume that \( r := \lvert S \cap V(H_w) \rvert < \gamma_{sp}(H) \). As \( H \) is connected, it follows by Theorem 1.1 that \( r < \gamma_{sp}(H) \leq m - 1 \). Thus, this copy of \( H \) has at least two vertices which are contained in \( \overline{S} \).

As \( S \cap V(H_w) \) is not a super dominating set of \( H_w \), there must exist a vertex \( u \in \overline{S} \cap V(H_w) \), such that there does not exist a vertex \( v \in S \cap V(H_w) \) for which
$N_{H_v}(v) \cap \overline{S} = \{u\}$. On the other hand, for this $u \in \overline{S} \cap V(H_v)$, there exists a vertex $v \in (V(G \star H) \setminus V(H_v)) \cap S$ such that $N_{G \star H}(v) \cap \overline{S} = \{u\}$.

As $v \in V(G \star H) \setminus V(H_v)$ holds, we have two possibilities. Either $v \in V(G)$ or $v \in V(H_x)$ for some $x \in V(G) \setminus \{w\}$.

Firstly, $v \in V(G)$ cannot hold, since then all vertices in $V(H_w) \setminus \{u\}$ would lie in $S$, because $v$ is adjacent to all of them. Thus, $r = m - 1$ holds, a contradiction.

Secondly, $v \in V(H_x)$ cannot hold, as there are no adjacent vertices between different copies of $H$. Thus, we have a contradiction again.

Claim 1 follows.

In the following, for a given $w \in V(G)$, we define a block of vertices $B_w(G \star H) = \{w\} \cup V(H_w)$ (or shortly $B_w$). The blocks $B_w$ clearly partition the vertex set $V(G \star H)$.

For the given super dominating set $S$, we define a block as over-satisfied, if it has more than $\gamma_{sp}(H) + 1$ vertices (note that for $m = 1$, i.e., $H = K_1$, a block can never be over-satisfied), as satisfied, if it has exactly $\gamma_{sp}(H) + 1$ vertices, and as under-satisfied, if it has less than $\gamma_{sp}(H) + 1$ vertices. Note that by Claim 1, an under-satisfied block $B_w$ has always exactly $\gamma_{sp}(H)$ vertices in $S$ and that $w$ lies in $\overline{S}$.

Claim 2: Let $w \in V(G)$ and let $B_w$ be under-satisfied. Let $x \in V(G) \setminus \{w\}$ be chosen so that a vertex in $B_x$ super dominates $w \in V(G)$ (and $\{w, x\} \in E(G)$). Then $B_x$ is an over-satisfied block.

Proof (Claim 2): As we have mentioned above, in an under-satisfied block $B_w$ there are exactly $\gamma_{sp}(H)$ vertices in $S$ and $w \in \overline{S}$. Thus, at least one vertex $y \in V(H_w)$ lies in $\overline{S}$.

First, assume that $x \in \overline{S}$. On the one hand, each vertex $z \in V(H_w)$ adjacent to $y$ is also adjacent to $x$, i.e., $z$ cannot super dominate $y$. On the other hand, each vertex $z \notin V(H_w)$ adjacent to $y$ is also adjacent to $w$, i.e., $z$ cannot super dominate $y$ either. In summary, $y$ cannot be super dominated by another vertex of $S$. Thus, by contradiction $x \in S$ follows.

We continue by dividing the proof into two cases.

Case 1: $\gamma_{sp}(H) < m - 1$.

Assume that there are $b, c \in V(H_x) \cap \overline{S}$. Each vertex $z \in V(H_x)$, adjacent to $b$ or $c$, is also adjacent to $w$, and if $z \notin V(H_x)$ is adjacent to $b$, then it is also adjacent to $c$ and vice versa. So neither $b$ nor $c$ can be super dominated by another vertex. Thus, there are at least $\gamma_{sp}(H) + 1$ vertices in $V(H_x) \cap S$, and $B_x$ is over-satisfied.

Case 2: $H = K_m$.

Assume that there is a $b \in V(H_x) \cap \overline{S}$. By the choice of $x \in V(G)$, one vertex of $B_x$ super dominates $w$. This cannot be $x$, as $x$ is also adjacent to $y$. On the other hand, as $H = K_m$, each other vertex in $B_x$ super dominating $w$ is also adjacent to $b$, a contradiction. Thus, there does not exist such $b$, and $B_x$ is over-satisfied.

Claim 2 follows.
Claim 3: No two under-satisfied blocks $B_w$ and $B_{w'}$ can be assigned to the same over-satisfied block $B_x$.

Proof (Claim 3): Assume that this does not hold. If there is a $d \in B_x$ super dominating $w \in V(G)$, then it super dominates also $w' \in V(G)$ and vice versa. On the other hand, no vertex can super dominate two vertices. Thus, we have a contradiction, and Claim 3 follows.

By Claim 1, for each under-satisfied block $B_w$, where $w \in V(G)$, we have exactly $\gamma_{sp}(H)$ vertices in $S$. By Claims 2 and 3, we have a corresponding over-satisfied block with at least $\gamma_{sp}(H) + 2$ vertices in $S$.

In total, we have at least $\gamma_{sp}(H) + 1$ vertices in $S$ for each block. Thus, each super dominating set has cardinality of at least $n(\gamma_{sp}(H) + 1)$ in $G \ast H$.

By using Theorems 3.2 and 3.3 we have the following result which gives us the exact value of the super domination number of $G \ast H$.

Corollary 3.4 Let $G$ and $H$ be two connected graphs of orders $n$ and $m \neq 1$, respectively, with $\gamma_{sp}(H) < m - 1$ or $H = K_m$. Then

$$\gamma_{sp}(G \ast H) = n(\gamma_{sp}(H) + 1).$$

In the following example, we show that “$\gamma_{sp}(H) < |V(H)| - 1$ or $H = K_m$” is a necessary condition for Theorem 3.3 and Corollary 3.4.

Example 3.5 Consider the graph $G$ from Figure 6. One can easily check that $\gamma_{sp}(G) = 3$ and the set of black vertices in $P_3 \ast G$ is a super dominating set for $G$. We have $\gamma_{sp}(P_3 \ast G) = 11 < 12 = 3 \cdot (\gamma_{sp}(G) + 1)$.

Interestingly, the value of Corollary 3.4 is equal to the value attained for the (usual) corona product of two graphs in [13, Theorem 10].

We end this section by determining the super domination number of the neighbourhood corona product of some specific graphs. These results follow directly from Theorems 1.2 (a)-(d) and 1.3 and Corollary 3.4.
Example 3.6 Let $n, m \in \mathbb{N}$.

(a) $\gamma_{sp}(C_n \ast P_{2m}) = n(m + 1)$.

(b) $\gamma_{sp}(P_{2n} \ast C_{4m}) = 4nm + 2n$.

(c) $\gamma_{sp}(C_{2n} \ast K_m) = 2nm$ for $m \geq 2$.

(d) $\gamma_{sp}(C_n \ast K_{2,3}) = 4n$.

(e) $\gamma_{sp}(P_n \ast F_n) = n^2 + 2n$.

4 Super domination number of $r$-gluing of two graphs

In this section, we give exact upper and lower bounds for the super domination number of $r$-gluing of two graphs.

Since for every two graphs $G_1$ and $G_2$, $G_1 \cup K_0 G_2$ is their disjoint union, by Proposition 1.4, we have the following result:

$$\gamma_{sp}(G_1) + \gamma_{sp}(G_2) - 0 \leq \gamma_{sp}(G_1 \cup K_0 G_2) \leq \gamma_{sp}(G_1) + \gamma_{sp}(G_2).$$

Also $G_1 \cup K_1 G_2$ is same as the chain of two graphs and by Theorem 1.5, we have the following result:

$$\gamma_{sp}(G_1) + \gamma_{sp}(G_2) - 1 \leq \gamma_{sp}(G_1 \cup K_1 G_2) \leq \gamma_{sp}(G_1) + \gamma_{sp}(G_2).$$

In Theorem 4.1, we consider the $r$-gluing of two graphs, and we generalize this result for each $r$-clique.

Theorem 4.1 Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs with clique number at least $r \in \mathbb{N}$. Then,

$$\gamma_{sp}(G_1) + \gamma_{sp}(G_2) - r \leq \gamma_{sp}(G_1 \cup K_r G_2) \leq \gamma_{sp}(G_1) + \gamma_{sp}(G_2).$$

Proof. Let $S_1$ and $S_2$ be two minimum size super dominating sets for $G_1$ and $G_2$, respectively. Let the vertex sets $V' = \{v'_i \mid 1 \leq i \leq r\}$ and $V'' = \{v''_i \mid 1 \leq i \leq r\}$ form $r$-cliques in $G_1$ and $G_2$, respectively. Furthermore, let us create a vertex $w_i$ by
identifying the vertices \(v'_i\) and \(v''_i\) for each \(i \in \{1, 2, \ldots, r\}\) (see Figure 7 for the case \(r = 2\)) and denote \(W = \{w_i \mid 1 \leq i \leq r\}\). Denote by \(G\) the graph which we obtain in this way. When \(r = 1\), the claim follows from Theorem 1.5. Hence, we assume from now on that \(r \geq 2\). We divide the proof into two parts, namely the lower and the upper bound.

**Lower bound** \(\gamma_{sp}(G_1) + \gamma_{sp}(G_2) - r \leq \gamma_{sp}(G_1 \cup_{K_r} G_2)\).

We use Lemma 2.1 to show that we can assume two cases for the minimum size super dominating set \(S\):

- \(W \subseteq S\),
- \(|W \cap S| = r - 1\) and one vertex of \(W\), say \(w_1\), super dominates the missing vertex, say \(w_r\).

To show this, assume that \(|S \cap W| \leq r - 1\) and, without loss of generality, \(w_r \notin S\). By Lemma 2.1, there is another minimum size super dominating set \(S'\), the corresponding set \(D = S \cap S'\) and a bijective function \(f : \overline{S'} \to \overline{S}\) with the mentioned characteristics. If no vertex of \(W \cap S\) super dominates another vertex, then \(W \subseteq D\) follows, and we can replace \(S\) by \(S'\), where \(W \subseteq S'\) holds. On the other hand, if \(W \not\subseteq S\) and one vertex of \(W \cap S\) super dominates another vertex, then \(|W \cap S| = r - 1\) and, without loss of generality, \(w_1\) super dominates the missing vertex \(w_r\). This finishes the proof of this assumption. We split our considerations into three subcases.

Here we split our considerations into two subcases.

(i) \(W \subseteq S\).

Let

\[
S_1 = (S \cap V_1) \cup V' \quad \text{and} \quad S_2 = (S \cap V_2) \cup V''.
\]

As \(\overline{S_1} \subseteq \overline{S}\), if \(v \in \overline{S_1}\) is super dominated by some vertex \(u\) in \(S\), then it is super dominated by a vertex corresponding to \(u\) in \(S_i\). Thus, \(S_1\) is a super dominating set for \(G_1\), and \(S_2\) is a super dominating set for \(G_2\).

As we replace \(r\) vertices from \(S\) by \(2r\) other vertices to reach \(S_1 \cup S_2\), it follows that \(|S_1| + |S_2| - r \leq \gamma_{sp}(G_1 \cup_{K_r} G_2)\).

(ii) \(|W \cap S| = r - 1\).

Let

\[
S_1 = (S \cap V_1) \cup (V' \setminus \{v'_r\}) \quad \text{and} \quad S_2 = (S \cap V_2) \cup (V'' \setminus \{v''_r\}).
\]
The vertex $v'_r$ is super dominated by $v'_1$, and $v''_r$ is super dominated by $v''_1$. As $S'_i \setminus \{v'_r, v''_r\} \subseteq \overline{S}$, if any other vertex $v \in S'_i$ is super dominated by some vertex $u$ in $S$, then it is super dominated by a vertex corresponding to $u$ in $S_i$.

As we replace $r - 1$ vertices from $S$ by $2r - 2$ other vertices to reach $S_1 \cup S_2$, it follows that $|S_1| + |S_2| - r \leq \gamma_{sp}(G_1 \cup K_r, G_2)$.

**Upper bound** $\gamma_{sp}(G_1 \cup K_r, G_2) \leq \gamma_{sp}(G_1) + \gamma_{sp}(G_2)$.

As in the proof of the lower bound, by Lemma 2.1 we can assume two cases for the minimum size super dominating set $S_1$:

- $V' \subseteq S_1$,
- $|V' \cap S_1| = r - 1$ and one vertex of $V'$, say $v'_1$, super dominates the missing vertex, say $v''_1$.

(By symmetry, this holds analogously for $S_2$ and $V''$.)

(i) $V' \subseteq S_1$ and $V'' \subseteq S_2$.

If for $i \in \{1, 2, \ldots, r\}$ we have $|N(v'_i) \cap S_1| = 1$, then we denote $\{x_i\} = N(v'_i) \cap \overline{S_1}$. If $|N(v'_i) \cap S_1| \neq 1$ but $|N(v''_i) \cap S_2| = 1$, then we denote $\{x_i\} = N(v''_i) \cap \overline{S_2}$. Denote $X = \{x_i \mid 1 \leq i \leq r, \ x_i \text{ exists}\}$. (Notice that the vertices $x_i$ are included in $X$ only if they exist.)

Let $$S = (S_1 \cup S_2 \cup W \cup X) \setminus (V' \cup V'').$$

Clearly, if a vertex $u \in \overline{S}$ was super dominated by a vertex $v$ in $S_1 \setminus V'$ or in $S_2 \setminus V''$, then it is now super dominated by $v \in S \setminus W$. Let us then assume that the vertex $u \in \overline{S}$ was super dominated by a vertex $v$ in $S_1 \cap V'$ or in $S_2 \cap V''$. Since $u \notin X$, we have $v \in S_2 \cap V''$. We may assume that $v = v''_h$ for some $h$ with $1 \leq h \leq r$. Thus, $x_h \in X \subseteq S$ and $N_G(w_h) \cap \overline{S} = \{v\}$. Thus, $S$ is a super dominating set of $G$.

As we replace $2r$ vertices from $S_1 \cup S_2$ by at most $2r$ other vertices to reach $S$, it follows that $|S| \leq \gamma_{sp}(G_1) + \gamma_{sp}(G_2)$, as claimed.

(ii) Either $|S_1 \cap V'| = r - 1$ or $|S_2 \cap V''| = r - 1$.

Notice that the two cases are essentially identical and we may assume, without loss of generality, that $|S_2 \cap V''| = r - 1$.

As we have mentioned above, we can assume that $v''_1$ super dominates $v''_2$ in $S_2$. Let us have $x_i \in N(v'_i) \cap \overline{S_1}$ for each $2 \leq i \leq r$ (if such vertex exists). Define $X = \{x_i \mid 2 \leq i \leq r, \ x_i \text{ exists}\}$.

Let $$S = (S_1 \cup S_2 \cup W \cup X) \setminus (V' \cup V'').$$
Figure 8: $G_1$, $G_2$ and $G_1 \cup K_r G_2$, respectively.

Again, if a vertex $u \in \overline{S}$ was super dominated by a vertex $v$ in $S_1 \setminus V'$ or in $S_2 \setminus V''$, then it is now super dominated by $v \in S \setminus W$. If a vertex $u \in \overline{S}$ was previously super dominated by a vertex $v$ in $V'$ or in $V''$, then $v = v'_1$ since $u \notin W \cup X$ and vertices in $V''$ could super dominate only the vertex $v'_1$. Since $v''_1$ super dominates $v''_r$, the vertex $w_1$ super dominates $u$. Thus, $S$ is a super dominating set of $G$.

As we replace $2r - 1$ vertices from $S_1 \cup S_2$ by at most $2r - 1$ other vertices to reach $S$, it follows that $|S| \leq \gamma_{sp}(G_1) + \gamma_{sp}(G_2)$.

(iii) $|S_1 \cap V'| = r - 1$ and $|S_2 \cap V''| = r - 1$.

We can assume, without loss of generality, that $v'_1$ super dominates $v'_1$ in $S_1$ and $v''_h$ super dominates $v''_p$ in $S_2$, where $1 \leq h < p \leq r$.

Let

$$S = (S_1 \cup S_2 \cup W) \setminus (V' \cup V'').$$

In this case, no vertex in $V'$ or in $V''$ super dominates any vertex in $\overline{S}_1 \setminus V'$ or $\overline{S}_2 \setminus V''$, respectively. Furthermore, $\overline{S} \subseteq \overline{S}_1 \cup \overline{S}_2$. Thus, if $u \in \overline{S}$, then $u \in \overline{S}_1 \cup \overline{S}_2$ and $u$ is super dominated by a vertex in $S_1$ or in $S_2$. That same vertex is in $S$ and super dominates $u$ in $S$. Thus, $S$ is a super dominating set of $G$.

As we replace $2r - 2$ vertices from $S_1 \cup S_2$ by $r \geq 2$ other vertices to reach $S$, it follows that $|S| \leq \gamma_{sp}(G_1) + \gamma_{sp}(G_2)$.

This finishes the proof. \qed

In the following remark, we show that the lower bound in Theorem 4.1 is tight:

**Remark 4.2** Let $r \geq 2$ and consider an $(r + 1)$-vertex graph $G$ which is formed from the complete graph $K_r$ by attaching a single leaf to one of the vertices in $K_r$. We have $\gamma_{sp}(G) = r$. Let $G'$ be the graph with two leaves attached to different vertices obtained by $r$-gluing $G \cup_{K_r} G$ (see Figure 8). One can easily check that $\gamma_{sp}(G') = r = \gamma_{sp}(G) + \gamma_{sp}(G) - r$. So, the lower bound in Theorem 4.1 is tight.

We finish this section by showing that also the upper bound in Theorem 4.1 is tight:

**Remark 4.3** Suppose that $G_1 = G_2 = G$ where $G$ is formed from $K_r$ by attaching two leaves to each vertex in the $r$-clique (see Figure 9). Then one can easily check that $\gamma_{sp}(G) = 2r$ and $\gamma_{sp}(G_1 \cup K_r G_2) = 4r = \gamma_{sp}(G_1) + \gamma_{sp}(G_2)$. So, the upper bound in Theorem 4.1 is tight.
5 Super domination number of the Hajós sum of graphs

In this section, we consider the Hajós sum of two graphs. We start with a tight lower bound for the super domination number of the Hajós sum of two graphs.

**Theorem 5.1** Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs with disjoint vertex sets, $x_1y_1 \in E_1$ and $x_2y_2 \in E_2$. Then for the Hajós sum

$$G_3 = G_1(x_1y_1) +_H G_2(x_2y_2),$$

it holds:

$$\gamma_{sp}(G_1) + \gamma_{sp}(G_2) - 2 \leq \gamma_{sp}(G_3) \leq \gamma_{sp}(G_1) + \gamma_{sp}(G_2).$$

**Proof.** Suppose that we have formed $G_3$. Let $v := v_H(x_1x_2)$ be the vertex identifying the vertices $x_1$ and $x_2$. Again we divide the proof into two parts, namely the lower and the upper bound.

**Lower bound** $\gamma_{sp}(G_1) + \gamma_{sp}(G_2) - 2 \leq \gamma_{sp}(G_3)$.

Let $S$ be a super dominating set for $G_3$. With Lemma 2.1, we may assume that $v \in S$. Moreover, the cases $v \in S$, $y_1 \in S$, $y_2 \in \overline{S}$, and $v \in S$, $y_1 \in \overline{S}$, $y_2 \in S$ are symmetrical and can be studied together. Hence, we have the following three cases.

(i) $v \in S$, $y_1 \in \overline{S}$, $y_2 \in \overline{S}$.

There might exist one vertex in $\overline{S}$ which is super dominated by $v$. If such vertex exists, denote it by $v'$. Then it holds that $\overline{S} \cap N(v) = \{v'\}$. We may assume, without loss of generality, that $v' \in V_1$. Observe that $v' \neq y_1$ holds.

Let

$$S_1 = (S \cup \{x_1, v'\}) \setminus (V_2 \cup \{v\})$$

(or $S_1 = (S \cup \{x_1\}) \setminus (V_2 \cup \{v\})$ if $v'$ does not exist) and

$$S_2 = (S \cup \{x_2\}) \setminus (V_1 \cup \{v\}).$$
We have $S_1 \subseteq \overline{S}$. Let $u \in \overline{S}$. If $u$ is super dominated in $G_3$ by some vertex $w \in S$, then $w \neq u$, since $u \neq v'$, and thus $w \in S_1$. Hence, $S_1$ is a super dominating set for $G_1$ and $S_2$ is a super dominating set for $G_2$. Thus,

$$
\gamma_{sp}(G_1) + \gamma_{sp}(G_2) \leq \gamma_{sp}(G_3) + 2.
$$

(ii) $v \in S, y_1 \in S, y_2 \in \overline{S}$.

Let

$$
S_1 = (S \cup \{x_1\}) \setminus (V_2 \cup \{v\}), \quad S_2 = (S \cup \{x_2, y_2\}) \setminus (V_1 \cup \{v\}).
$$

In comparison to Case (i), $y_2$ is added to $S_2$, as it could occur that $y_1$ super dominates $y_2$ in $S$, but $y_2$ is not super dominated in $S_2$. Observe that a vertex $v'$, which was super dominated by $v$ in $S$, as in Case (i), does not have to be added. First, let $v' \in V_1$. Then because of $y_1 \in S$, $x_1$ super dominates $v'$ in $S_1$. Second, let $v' \in V_2$. Then because of $y_2 \in S_2$, $x_2$ super dominates $v'$ in $S_2$.

So $S_1$ is a super dominating set for $G_1$ and $S_2$ is a super dominating set for $G_2$. Thus again,

$$
\gamma_{sp}(G_1) + \gamma_{sp}(G_2) \leq \gamma_{sp}(G_3) + 2.
$$

(iii) $v \in S, y_1 \in S, y_2 \in S$.

Let

$$
S_1 = (S \cup \{x_1\}) \setminus (V_2 \cup \{v\}), \quad S_2 = (S \cup \{x_2\}) \setminus (V_1 \cup \{v\}).
$$

Observe that a vertex $v'$, which was super dominated by $v$ in $S$, as in Case (i), does not have to be added. Because $y_1 \in S$, $y_2 \in S$, $x_1$ super dominates $v'$ in $S_1$ or $x_2$ super dominates $v'$ in $S_2$.

So $S_1$ is a super dominating set for $G_1$ and $S_2$ is a super dominating set for $G_2$. Thus,

$$
\gamma_{sp}(G_1) + \gamma_{sp}(G_2) \leq \gamma_{sp}(G_3) + 1.
$$

**Upper bound** $\gamma_{sp}(G_3) \leq \gamma_{sp}(G_1) + \gamma_{sp}(G_2)$.

Let $S_1$ and $S_2$ be super dominating sets for $G_1$ and $G_2$, respectively. Let $f_i : \overline{S'_i} \rightarrow \overline{S_i}$ be the bijective function introduced in Lemma 221 where $S'_i$ is a super dominating set in $G_i$ for both $i \in \{1, 2\}$. In particular, we can now assume that $y_1 \in S_1$ and $y_2 \in S_2$. Moreover, we may assume that if $x_1 \in \overline{S'_1}$, then it is super dominated by $y_1$. Indeed, if $x_1$ is not super dominated by $y_1$, then $y_1$ does not super dominate any vertex in $V_1$ and so $y_1 \in D$. Thus, $y_1, x_1 \in S'_1$ and we could consider $S'_1$ instead of $S_1$. This holds analogously for $x_2$.

We have the following three cases:

(i) $x_1 \in S_1, x_2 \in S_2$.

There might be vertices $t_1 \in V_1 \cap \overline{S_1}$ and $t_2 \in V_2 \cap \overline{S_2}$ such that $t_1$ is super dominated by $x_1$ and $t_2$ is super dominated by $x_2$. By the definition of a super dominating set, if $t_1$ exists, then all neighbours of $x_1$ are in $S_1$ except $t_1$, and the
same is true for $x_2$. Without loss of generality, if exactly one exists, we assume it is $t_1$. Let

$$S = (S_1 \cup S_2 \cup \{t_1, v\}) \setminus \{x_1, x_2\}$$

(or $(S_1 \cup S_2 \cup \{v\}) \setminus \{x_1, x_2\}$, if $t_1$ does not exist).

We have $|S| \leq |S_1| + |S_2|$, and $S$ is a super dominating set in $G_3$ since each vertex in $S$ is super dominated by the same vertex in $S_1 \cup S_2$ as before except possibly $t_2$ which is super dominated by the vertex $v$.

(ii) $x_1 \in \overline{S_1}$, $x_2 \in S_2$.

By assumption, $y_1$ now super dominates $x_1$ and thus, $f_1(y_1) = x_1$. Thus, by Lemma 2.1 for the super dominating set $S'_1$ of $G_1$ it holds that $x_1 \in S'_1$, $y_1 \in \overline{S'_1}$ and $N(x_1) \cap \overline{S_1} = \{y_1\}$, i.e., $x_1$ super dominates $y_1$. Let us now consider the set

$$S = (S'_1 \cup S_2 \cup \{y_1, v\}) \setminus \{x_1, x_2\}.$$

We have $|S| \leq |S'_1| + |S_2| = |S_1| + |S_2|$ and $S$ is a super dominating set in $G_3$ since each vertex in $S$ is super dominated by the same vertex in $S'_1 \cup S_2$ as before with the (possible) exception that the vertex which was previously super dominated by $x_2$ is now super dominated by $v$.

(iii) $x_1 \in \overline{S_1}$, $x_2 \in \overline{S_2}$.

As in Case (ii) for $x_1$, by assumption $y_i$ now super dominates $x_i$ for both $i \in \{1, 2\}$. Thus, in $S'_1$, $x_1$ super dominates $y_1$. Let us now consider the set

$$S = (S'_1 \cup S_2 \cup \{v\}) \setminus \{x_1\}.$$

We have $|S| \leq |S_1| + |S_2|$. Moreover, $y_1$ is super dominated by $y_2$, $v$ does not super dominate any vertex and all other vertices in $S$ are super dominated by the same vertices in $S'_1 \cup S_2$ as before.

This finishes the proof. \qed

We finish this section by showing that both the lower and upper bound of Theorem 5.1 is tight.

**Remark 5.2** Consider $G_1 = C_{4p+2}$ and $G_2 = C_3$. Then one can easily check that $G_1(x_1y_1) + H G_2(x_2y_2) = C_{4p+4}$, for any two edges $x_1y_1$ and $x_2y_2$ from $G_1$ and $G_2$, respectively. By Theorem 1.2(b), we have $\gamma_{sp}(G_1(x_1y_1) + H G_2(x_2y_2)) = 2p + 2$, $\gamma_{sp}(G_1) = 2p + 2$ and $\gamma_{sp}(G_2) = 2$. Thus, the lower bound of Theorem 5.1 is tight.

**Remark 5.3** Consider the cycles $G_1 = C_{4p}$ and $G_2 = C_3$. Then, we have $G_1(x_1y_1) + H G_2(x_2y_2) = C_{4p+2}$, for any two edges $x_1y_1$ and $x_2y_2$ from $G_1$ and $G_2$, respectively. By Theorem 1.2(b), we have $\gamma_{sp}(G_1(x_1y_1) + H G_2(x_2y_2)) = 2p + 2$, $\gamma_{sp}(G_1) = 2p$ and $\gamma_{sp}(G_2) = 2$. Thus, the upper bound of Theorem 5.1 is tight.
6 The number of minimum size super dominating sets of some graphs

In this section, we initiate the study of the number of minimum size super dominating sets of a graph. Similar research has been conducted, for example for the domination number in multiple papers, see for example, [3]. Let \( N_{sp}(G) \) be the family of super dominating sets of a graph \( G \) with cardinality \( \gamma_{sp}(G) \) and let \( N_{sp}(G) = |N_{sp}(G)| \). By Theorem 1.1 and Lemma 2.1 for every non-empty graph \( G \) we have \( N_{sp}(G) \geq 2 \). In the following, we consider some special graph classes and compute their \( N_{sp} \) values. Following Theorem 1.2, by an easy argument, we have the following result for \( N_{sp} \) of the complete graph, the complete bipartite graph and the star graph.

**Theorem 6.1**

(a) If \( K_n \) is the complete graph, then \( N_{sp}(K_n) = n \).

(b) If \( K_{n,m} \) is the complete bipartite graph, then \( N_{sp}(K_{n,m}) = nm \), where \( \min\{n, m\} \geq 2 \).

(c) If \( K_{1,n} \) is the star graph, then \( N_{sp}(K_{1,n}) = n + 1 \).

**Proof.**

(a) By Theorem 1.2(c), \( \gamma_{sp}(K_n) = n - 1 \) holds. Thus, \( |\mathcal{S}| = 1 \) follows. Clearly, any single vertex of \( K_n \) can be chosen as \( \mathcal{S} \). As \( K_n \) has exactly \( n \) vertices, it follows that \( N_{sp}(K_n) = n \).

(b) Let \( \min\{n, m\} \geq 2 \). By Theorem 1.2(d), \( \gamma_{sp}(K_{n,m}) = n + m - 2 \) holds. Thus, \( |\mathcal{S}| = 2 \) follows. Clearly, these two vertices of \( \mathcal{S} \) have to be chosen from two different sides of the bipartition. It follows that \( N_{sp}(K_{n,m}) = nm \).

(c) By Theorem 1.2(e), \( \gamma_{sp}(K_{1,n}) = n \) holds. Thus, \( |\mathcal{S}| = 1 \) follows. Clearly, any single vertex of \( K_{1,n} \) can be chosen as \( \mathcal{S} \). As \( K_{1,n} \) has exactly \( n + 1 \) vertices, it follows that \( N_{sp}(K_{1,n}) = n + 1 \).

In the following, we compute \( N_{sp} \) of the friendship graph.

**Theorem 6.2** Let \( F_n \) be the friendship graph of order \( n \). Then

\[
N_{sp}(F_n) = 2^n.
\]

**Proof.** By Theorem 1.3 we know that \( \gamma_{sp}(F_n) = n + 1 \). Now consider Figure 1. For any dominating set \( S \) of the friendship graph with cardinality less than \( 2n \), if we do not have \( \{x, u_{2t-1}\} \subseteq S \) or \( \{x, u_{2t}\} \subseteq S \), where \( 1 \leq t \leq n \) and \( x \) is the central vertex, then it is clear that \( S \) is not a super dominating set. So we need \( x \) in our super dominating set. Among \( u_{2t-1} \) and \( u_{2t} \), where \( 1 \leq t \leq n \), we choose one of them. So we have \( 2^n \) super dominating sets of size \( n + 1 \), and we have the result.

Now we consider the path graph and compute \( N_{sp}(P_n) \).
Theorem 6.3 Let $P_n$ be the path graph of order $n \geq 2$. Then

$$N_{sp}(P_n) = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even}, \\ \frac{3}{2}(n-1) & \text{if } n \text{ is odd}. \end{cases}$$

Also we have $N_{sp}(P_1) = 1$.

Proof. By Theorem 1.2(a), we have $\gamma_{sp}(P_n) = \lceil \frac{n}{2} \rceil$. We have two cases based on the parity of $n$:

(a) $n$ even.

Let $n = 2k$ with $k \in \mathbb{N}$. Let $V = \{v_1, v_2, \ldots, v_{2k}\}$ be the vertex set of $P_{2k}$ (see Figure 10), and $S$ be a super dominating set of $P_{2k}$ with $|S| = k$. Since $|S| = n/2$, in the partition of Lemma 2.1(b), we have $D = \emptyset$ and $V(G) = S \cup S'$, where $S$ and $S'$ have the same cardinality $n/2$. By Lemma 2.1(a), each vertex in $S$ is adjacent to exactly one vertex in $S'$ and vice versa. Thus, if we choose $v_1 \in S$, then $v_2 \in S'$, $v_3 \in S'$, $v_4 \in S$ and so on. After we know whether $v_1 \in S$, all other vertices have their set decided. The case with $v_1 \in S'$ is analogous; we can just swap $S$ and $S'$. Thus, we have $N_{sp}(P_{2k}) = 2$.

(b) $n$ odd.

Let $n = 2k + 1$ with $k \in \mathbb{N}_0$. It is easy to see that $N_{sp}(P_1) = 1$, $N_{sp}(P_3) = 3$. Now we consider $n \geq 5$ and thus $k \geq 2$. Let $V = \{v_1, v_2, \ldots, v_{2k+1}\}$ be the vertex set of $P_{2k+1}$.
By Theorem 1.2(a), we have $\gamma_{sp}(P_{2k+1}) = k + 1$. Let $S$ be a super dominating set of cardinality $k + 1$ and $S'$ be another super dominating set of the same cardinality with $\overline{S} \subseteq S'$ and $\overline{S'} \subseteq S$. Moreover, by Lemma 2.1, let $f : \overline{S'} \rightarrow \overline{S}$ be a bijective function for which $f(a) = b$ if and only if $a$ super dominates $b$ for $S$ and $b$ super dominates $a$ for $S'$. Observe that $|\overline{S}| = |\overline{S'}| = k$. Again, by Lemma 2.1(b), $|D| = |S \cap S'| = 1$ holds. Let us denote by $w$ the single vertex in $S \cap S'$. Since $w$ is not necessary for super dominating any other vertices, $S \setminus \{w\}$ is a super dominating set for the induced subgraph $P'$ of $V(P_{2k+1}) \setminus \{w\}$. Notice that $P'$ consists of either two paths or a single even-length path (when $w$ is the start or end vertex of the original path). We have $|S \setminus \{w\}| = k$. Thus, each path in $P'$ has even-length since odd paths have more than half of their vertices in any super dominating set. Hence, $w = v_i$ where $i$ is odd. Thus, we have $k + 1$ possible choices for $w$. Furthermore, each of these choices for $w$ yields a super dominating set of the smallest cardinality.

When $w = v_1$, we have $P' = P_{2k}$. Since $N_{sp}(P_{2k}) = 2$, based on the proof of $n$ even, we may choose in this case $S$ in two different ways based on whether $v_2 \in S$. Moreover, if we choose $S$ in one way, then $S'$ is the other super dominating set of the smallest cardinality in $P'$. Thus, $w = v_1$ contributes two super dominating sets. Furthermore, the choice $w = v_{2k+1}$ is symmetrical.

Let us now consider the case $w = v_{2i+1}$ where $i \in \{1, 2, \ldots, k - 1\}$. Now, $P'$ consists of two even paths of lengths $2i$ and $2k - 2i$. Hence, for each choice of $w$ there are four different smallest super dominating sets (two for both even paths).

In summary, we can classify the super dominating sets in eight classes, where by definition, the vertices $w$ are always black.

**SW(0):** $w = v_1$, $v_2$ is white (i.e., does not lie in $S$),

**SB(0):** $w = v_1$, $v_2$ is black (i.e., lies in $S$),

**EW(1):** $w = v_{2k+1}$, $v_{2k}$ is white,

**EB(1):** $w = v_{2k+1}$, $v_{2k}$ is black,

**MBB(i):** $w = v_{2i+1}$, $v_{2i}$ is black, $v_{2i+2}$ is black, where $i \in \{1, 2, \ldots, k - 1\}$,

**MWW(i):** $w = v_{2i+1}$, $v_{2i}$ is white, $v_{2i+2}$ is white, where $i \in \{1, 2, \ldots, k - 1\}$,

**MBW(i):** $w = v_{2i+1}$, $v_{2i}$ is black, $v_{2i+2}$ is white, where $i \in \{1, 2, \ldots, k - 1\}$,

**MWB(i):** $w = v_{2i+1}$, $v_{2i}$ is white, $v_{2i+2}$ is black, where $i \in \{1, 2, \ldots, k - 1\}$.

This leads to

$$2 \cdot 2 + 4 \cdot (k - 1) = 4k$$

super dominating sets. However, some super dominating sets appear in multiple classes. Below we give a formal explanation on the number these classes overlap.

Recall that the eight cases above are formed by first choosing the set $D = \{w\}$. Thus, a super dominating set $S$ is counted in multiple classes if and only if there are multiple ways to choose the set $D$ (and the super dominating set $S'$) for $S$. 19
Moreover, since $|D| = 1$, we can choose the set $D$ in multiple ways if and only if there is a white vertex which is super dominated by two different black vertices. Since the maximum degree of a vertex is 2, we may choose the set $D$ in these cases in exactly two ways, say $D$ and $D'$, where the corresponding vertices $w$ and $w'$ have distance 2 in the path. Hence, in each such case, a super dominating set is included in two classes with consecutive indices.

Finally, if we look into the eight classes above, we notice that in the classes SB(0), EB(k), MBB(i) and MWW(i), the vertex $w$ does not super dominate any vertex. Thus, we do not have any white vertex, which is dominated twice in those classes. On the other hand, we have a white vertex that is super dominated by two black vertices in the classes SW(0), EW(k), MBW(i), and MWB(i). This twice super dominated white vertex is $v_2$, $v_{2k}$, $v_{2i+2}$ and $v_{2i}$, respectively.

We illustrate these overlapping super dominating sets in Figure 11 for $P_5$ and $P_7$. For $P_5$, in row IV, SW(0) and vertex $w = v_1$ overlap with MBW(1) and vertex $w = v_3$, and in row V, EW(2) and vertex $w = v_5$ overlap with MBW(1) and vertex $w = v_3$. For $P_7$, in row III, SW(0) and vertex $w = v_1$ overlap with MBW(1) and vertex $w = v_3$, in row V, EW(3) and vertex $w = v_7$ overlap with MBW(2) and vertex $w = v_5$, and in row VI, MBW(1) and vertex $w = v_3$ overlap with MBW(2) and vertex $w = v_5$.

That is, we are counting the super dominating sets twice in $1+1+(k-1)+(k-1) = 2k$ classes. Hence, $N_{sp}(P_{2k+1}) = 4k - (2k/2) = 3k$ as claimed. □

In Theorem 6.5, we determine the value $N_{sp}(C_n)$ for each $n$. Interestingly, the value varies quite a lot based on $n$ (mod 4). We will utilize necklace combinatorics for the proof of Case (c). See [16] for an algorithm to calculate the value $N_n(q_1, q_2, \ldots, q_k)$ in the following definition as well as some connections between necklaces and combinatorics on words.

**Definition 6.4** Let $k, n, q_1, q_2, \ldots, q_k \in \mathbb{N}$ with $n = \sum_{i=1}^{k} q_i$.

(a) A $(q_1, q_2, \ldots, q_k)$-necklace of length $n$ consists of a total of $n$ beads where we have $q_i$ beads of type $i$. The beads are placed into a cycle (necklace).

(b) $N_n(q_1, q_2, \ldots, q_k)$ is defined as the number of different $(q_1, q_2, \ldots, q_k)$-necklaces of length $n$, when the $n$ rotations of a necklace are considered as the same necklace.

The following relation is easy to see.

$$N_{k+1}(k, 1) = 1 \text{ for } k \in \mathbb{N}. \quad (2)$$

Furthermore, we have

$$N_{2k+2}(2k, 2) = k + 1 \text{ for } k \in \mathbb{N} \text{ and } N_{2k+3}(2k + 1, 2) = k + 1 \text{ for } k \in \mathbb{N}_0. \quad (3)$$

Indeed, in Eq. (3), let us say that we have two red beads in the necklace and the other beads are blue. Since the rotations of the necklaces are not counted multiple times, each necklace is uniquely determined by the smallest number of blue beads between the two red beads.
Theorem 6.5 Let $C_n$ be the cycle graph of order $n \geq 3$. Then
\[
N_{sp}(C_n) = \begin{cases} 
4 & \text{if } n \equiv 0 \pmod{4}, \\
2n & \text{if } n \equiv 1 \pmod{4}, \\
\frac{5n^2 - 10n}{8} & \text{if } n \equiv 2 \pmod{4}, \\
n & \text{if } n \equiv 3 \pmod{4}.
\end{cases}
\]

Proof. Let $V = \{v_1, v_2, \ldots, v_n\}$ be the vertex set of $C_n$ and $S$ be a minimum size super dominating set for that. We consider the following cases:

(a) $n \equiv 0 \pmod{4}$.

Let $n = 4k$ with $k \in \mathbb{N}$. By Theorem 1.21(b), we have $\gamma_{sp}(C_{4k}) = 2k$. Let $S$ be a super dominating set of cardinality $2k$ in $C_n$. By Lemma 2.1, there is another super dominating set $\overline{S}$ with same cardinality as $S$, $D = \emptyset$, and there is a bijective function $f : \overline{S} \rightarrow \overline{S}$. Thus, we cannot have 3 consecutive vertices in $S$ and for each $v \in S = \overline{S}$, there exists a unique $u \in \overline{S} = S'$ such that $N(v) \cap \overline{S} = \{u\}$ and for each $v \in S' = \overline{S}$ a unique $u \in \overline{S} = S$ such that $N(v) \cap \overline{S} = \{u\}$. Hence, the following four sets are the only super dominating sets in $C_{4k}$:

\begin{align*}
S_1 &= \{v_1, v_2, v_5, v_6, \ldots, v_{4k-3}, v_{4k-2}\}, \\
S_2 &= \{v_2, v_3, v_6, v_7, \ldots, v_{4k-2}, v_{4k-1}\}, \\
S_3 &= \{v_3, v_4, v_7, v_8, \ldots, v_{4k-1}, v_{4k}\}, \\
S_4 &= \{v_1, v_4, v_5, v_8, \ldots, v_{4k-3}, v_{4k}\}.
\end{align*}

Hence, $N_{sp}(C_{4k}) = 4$.

(b) $n \equiv 1 \pmod{4}$.

Let $n = 4k + 1$ with $k \in \mathbb{N}$. By Theorem 1.21(b), we have $\gamma_{sp}(C_{4k+1}) = 2k + 1$. Let $S$ be a super dominating set of size $2k + 1$ in $C_n$. First, notice that $|S| = |\overline{S}| + 1$. Thus, if we consider Lemma 2.1 and the set $D$, we have $|D| = 1$. Without loss of generality, $v_{4k+1} \in D$ holds. If we now contract the edge $v_{4k}v_{4k+1}$, then we get a cycle $C_{4k}$ in which $S \setminus \{v_{4k+1}\}$ is a super dominating set of cardinality $2k$. If we now recall the structure of a minimum size super dominating set for the cycle $C_{4k}$, then we notice that there are two possibilities for $v_{4k+1} \in D$, namely either $N(v_{4k+1}) \subseteq S$ or $N(v_{4k+1}) \subseteq \overline{S}$. Hence, $v_{4k+1}$ is either one of 3 consecutive vertices in $S$ or a single vertex in $S$ surrounded by vertices in $\overline{S}$. Moreover, in both cases, we may rotate the super dominating set around the cycle in $n$ different ways. Thus, $N_{sp}(C_{4k+1}) = 2n$ as claimed.

(c) $n \equiv 2 \pmod{4}$.

Let $n = 4k + 2 = 8q + 4p + 2$ where $q \in \mathbb{N}_0, k \in \mathbb{N}$ and $p \in \{0, 1\}$. By Theorem 1.21(b), we have $\gamma_{sp}(C_{4k+2}) = 2k + 2$. Since choosing any four vertices from
\( C_6 \) gives us a super dominating set, \( \binom{6}{4} = 15 = (5 \cdot 6^2 - 10 \cdot 6)/8 \) is the number of super dominating sets, and it shows that the formula holds for \( n = 6 \). So, let now \( n \geq 10 \).

In the following, we write about \( k \) consecutive vertices in a minimum size super dominating set \( S \) implying that there is not a subset of \( k + 1 \) consecutive vertices in \( S \). We claim that the following cases cannot occur for vertices in \( S \):

- At least 5 consecutive vertices,
- twice 4 consecutive vertices,
- once 4 consecutive vertices and once 1 consecutive vertex,
- once 4 consecutive vertices and once 3 consecutive vertices,
- three times 3 consecutive vertices,
- twice 3 consecutive vertices and once 1 consecutive vertex,
- once 3 consecutive vertices and twice 1 consecutive vertex,
- three times 1 consecutive vertex.

The non-existence of all these cases can be shown in the same way. By Lemma 2.1, we have \(|D| = 2\). On the other hand, if one of these cases occurred, then we would have at least three black vertices surrounded either by two black vertices or by two white vertices. As these vertices do not super dominate other vertices, it follows that \(|D| \geq 3\) leading to a contradiction.

As we are in the case \( n \equiv 2 \pmod{4} \), it remains to consider the following cases for vertices in \( S \), where we do not list the occurrences of 2 consecutive vertices.

(i) Exactly once 4 consecutive vertices,
(ii) exactly twice 3 consecutive vertices,
(iii) exactly once 3 consecutive vertices and exactly once 1 consecutive vertex,
(iv) exactly twice 1 consecutive vertex,
(v) none of that, i.e., only 2 consecutive vertices.

We have illustrated the minimum size super dominating sets for \( C_{10} \) in Figure 12.

As mentioned, we will utilize necklaces (see Definition 6.4). We transform some sequences of vertices into beads according to the following list. After that, we consider the resulting cycle as a necklace.

- **B-bead** (black bead): B,
- **DB-bead** (double black bead): BB,
- **P-bead** (pair bead): WBBW,
- **T-bead** (triple bead): WBBBBW,
- **Q-bead** (quadruple bead): WBBBBBW,
We distinguish between the five cases (i)–(v) mentioned above.

- **DP-bead** (double pair bead): WBBWBBW.

We distinguish between the five cases (i)–(v) mentioned above.

(i) The 4 consecutive vertices form a Q-bead. If we convert the cycle together with the super dominating set into a necklace consisting of $2q - 1 + p$ P-beads and a single Q-bead, then by Eq. 2, we attain $N_{2q+p}(2q - 1 + p, 1) = 1$ different necklaces. Now, this single necklace provides $n$ different super dominating sets in this case.

(ii) We attain a necklace consisting of $2q - 2 + p$ P-beads and two T-beads. We have $N_{2q+p}(2q - 2 + p, 2) = q$ by Eq. 3. Furthermore, we can again rotate these $n$ times. However, if $p = 0$ then the necklace with $q - 1$ P-beads between the two T-beads yields only $n/2$ different super dominating sets. Thus, in this case we have $qn - (1 - p)n/2$ different super dominating sets.

(iii) We attain a necklace consisting of $2q - 1 + p$ P-beads, one T-bead and one B-bead. We have $N_{2q+p+1}(2q - 1 + p, 1, 1) = 2q + p$. Indeed, the number of P-beads $h$ which we can have between the B-bead and the T-bead is $0 \leq h \leq 2q - 1 + p$. Moreover, $n$ rotations give $n$ different super dominating sets for each of these necklaces. Thus, we have $n(2q + p)$ different super dominating sets in this case.

(iv) We attain a necklace consisting of $2q + p$ P-beads and two B-beads. We have $N_{2q+p+2}(2q + p, 2) = q + 1$ by Eq. 3. However, we do not consider the case where we have two adjacent B-beads since in that case we would actually have 2 consecutive black vertices and we would be considering Case

Figure 12: All super dominating sets of $C_{10}$.
(v) As in Case (ii), if we have \( p = 0 \), then the necklace which has \( q \) P-beds between the two B-beads give only \( n/2 \) different super dominating sets with rotations. In all other cases, we get \( n \) different super dominating sets. Thus, we have \( qn - (1 - p)n/2 \) different super dominating sets in this Case (iv).

(v) In this case, we have two vertices \( u_1, u_2 \in \mathcal{S} \) such that they are both super dominated by two different vertices. Let us first consider the case where \( u_1 \) and \( u_2 \) have distance 3. We attain a necklace consisting of \( 2q + p \) P-beds and one DB-bead. By Eq. (2), we have \( N_{2q+p}(2q + p, 1) = 1 \). We can rotate this case \( n \) times and thus, we have \( n \) different super dominating sets.

Let us then assume that \( u_1 \) and \( u_2 \) have distance of at least 7. Here the white vertex in the middle corresponds to \( u_1 \) or \( u_2 \). In this case, we attain a necklace consisting of \( 2q - 3 + p \) DB-beads and two DP-beads. We have \( N_{2q-3+p}(2q - 3 + p, 2) = q - 1 + p \) by Eq. (3). Notice that when \( p = 1 \) the necklace in which we have \( q - 1 \) DB-beads between the two DP-beads gives us only \( n/2 \) different super dominating sets. All other necklaces give \( n \) different super dominating sets. Thus, we have \( n(q - 1) + pn/2 \) different super dominating sets in this Case (v).

By summing all these different cases together, we get

\[
N_{sp}(C_n) = n + qn - (1 - p)n/2 + n(2q + p) + qn - (1 - p)n/2 + n + n(q - 1) + pn/2
\]

\[
= 5qn + (5/2)pn
\]

\[
= \frac{5n^2 - 10n}{8}
\]

super dominating sets, as claimed.

(d) \( n \equiv 3 \pmod{4} \).

Let \( n = 4k + 3 \) with \( k \in \mathbb{N}_0 \). It is easy to see that \( N_{sp}(C_3) = 3 \). Now let \( n \geq 7 \).

By Theorem 1.2(b), \( \gamma_{sp}(C_{4k+3}) = 2k + 2 \). So we need to choose \( 2k + 2 \) vertices in a proper way to have a super dominating set. First, we show that it is not possible to have 3 consecutive vertices in a minimum size super dominating set \( S \). Suppose that we have 3 consecutive vertices \( v, v', v'' \) and contract one of the two corresponding edges \((vv' \text{ or } v'v'')\). Then we have a super dominating set of size \( 2k + 1 \) in \( C_{4k+2} \), a contradiction. Second, with the same contraction technique, we notice that it is not possible among 3 consecutive vertices for the middle one to be in \( S \) and for the two others to be in \( \overline{S} \). Hence, we can only have 2 consecutive vertices in \( S \).

Since \( \gamma_{sp}(C_{4k+3}) = 2k + 2 \), we have \( k + 1 \) sets of size two with consecutive vertices in \( S \). Moreover, we have \( |\overline{S}| = 2k + 1 \). Thus, we have one vertex \( v \in \overline{S} \) with \( N(v) \subseteq S \) and \( 2k \) sets of size two with consecutive vertices in \( \overline{S} \). Assume first
that $v = v_1$. Now we can construct $n$ different super dominating sets by rotating them around the cycle. Thus, $N_{sp}(C_{4k+3}) = n$.

Therefore, we have the result. □

7 Conclusions and future work

In this paper, we obtained tight results on the super domination number of graphs, particularly on the neighbourhood corona product, $r$-gluing and Hajós sum of two graphs. For each of these, we presented tight lower and upper bounds together with constructions attaining these upper bounds. Moreover, we gave the exact number of minimum size super dominating sets of some graph classes such as paths and cycles. Finally, we provide the following suggestions for future research.

(a) The formula $\gamma_{sp}(G \ast H) = n(\gamma_{sp}(H) + 1)$ of Corollary 3.4 does not cover the case $H = K_1$. Thus, it would be interesting to compute $\gamma_{sp}(G \ast K_1)$. By Eq. (1), $\gamma_{sp}(G \ast K_1) \leq \gamma_{sp}(G) + n$ is a (trivial) upper bound, but it is tight only for some graphs $G$.

(b) We found a connection for the super domination number between the usual corona product of two graphs [13] and the neighbourhood corona product of two graphs. Does a generalization exist for the corona products that preserve the result of Corollary 3.4?

(c) Give general upper and lower bounds for the number of minimum size super dominating sets with help of $\gamma_{sp}(G)$ and possibly some other graph parameters.

(d) Count the number of (minimum size) super dominating sets in trees.

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References

[1] R. Alfarisi, Dafik, R. Adawiyah, R.M. Prihandini, E. R. Albirri, I.H. Agustin, Super domination number of unicyclic graphs, IOP Conf. Series: Earth and Environmental Science, 243 (2019) 012074. DOI:10.1088/1755-1315/243/1/012074.

[2] S. Barik, G. Sahoo, On the Laplacian spectra of some variants of corona, Linear Algebra and its Applications, 512 (2017) 32–47. DOI:10.1016/j.laa.2016.09.030.
[3] S. Connolly, Z. Gabor, A. Godbole, B. Kay, and T. Kelly, Bounds on the maximum number of minimum dominating sets, *Discrete Mathematics* 339(5) (2016) 1537–1542. DOI:10.1016/j.disc.2015.12.030.

[4] M. Dettlaff, M. Lemańska, J.A. Rodríguez-Velázquez, R. Zuazua, On the super domination number of lexicographic product graphs, *Discrete Applied Mathematics*, 263 (2019) 118–129. DOI:10.1016/j.dam.2018.03.082.

[5] N. Ghanbari, Some results on the super domination number of a graph, submitted, arXiv:2204.10666 (2022).

[6] N. Ghanbari, Some results on the super domination number of a graph II, submitted, arXiv:2205.02634 (2022).

[7] N. Ghanbari, S. Alikhani, More on the total dominator chromatic number of a graph, *Journal of Information and Optimization Sciences*, 40(1) (2019) 157–169. DOI:10.1080/02522667.2018.1453665.

[8] I. Gopalapillai, The spectrum of neighborhood corona of graphs, *Kragujevac Journal of Mathematics*, 35(3) (2011) 493–500.

[9] G. Hajós, Über eine Konstruktion nicht n-färbarer Graphen, *Wiss. Z. Martin-Luther-Univ. Halle-Wittenberg Math.-Natur. Reihe*, 10 (1961) 116–117.

[10] F. Harary, Graph Theory, *Addison-Wesley, Reading, MA*, (1969).

[11] T.W. Haynes, S.T. Hedetniemi, P.J. Slater, Fundamentals of domination in graphs, *Marcel Dekker*, New York, (1998).

[12] B. Krishnakumari, Y.B. Venkatakrishnan, Double domination and super domination in trees, *Discrete Mathematics, Algorithms and Applications*, 08(04) (2016) 1650067. DOI:10.1142/S1793830916500671.

[13] D.J. Klein, J.A. Rodríguez-Velázquez, E. Yi, On the super domination number of graphs, *Communications in Combinatorics and Optimization*, 5(2) (2020) 83–96. DOI:10.22049/CCO.2019.26587.1122.

[14] M. Lemańska, V. Swaminathan, Y. B. Venkatakrishnan, and R. Zuazua, Super dominating sets in graphs, *Proceedings of the National Academy of Sciences, India, Section A: Physical Sciences*, 85(3) (2015) 353–357. DOI:10.1007/s40010-015-0208-2.

[15] X. Liu, S. Zhou, Spectra of the neighbourhood corona of two graphs, *Linear and Multilinear Algebra*, 62(9) (2014) 1205–1219. DOI:10.1080/03081087.2013.816304.

[16] J. Sawada, A fast algorithm to generate necklaces with fixed content, *Theoretical Computer Science* 301(1–3) (2003) 477–489. DOI:10.1016/S0304-3975(03)00049-5.

[17] W. Zhuang, Super domination in trees, *Graphs and Combinatorics*, 38(1) (2022). DOI:10.1007/s00373-021-02409-3.