STRUCTURE CONSTANTS IN THE N=1 SUPEROPERATOR ALGEBRA

L. Alvarez-Gaumé

Theory Division, CERN
CH-1211 Geneva 23, Switzerland

Ph. Zaugg*

Département de Physique Théorique
Université de Genève
CH-1211 Genève 4, Switzerland

ABSTRACT

Using the Coulomb gas formulation of N=1 Superconformal Field Theories, we extend the arguments of Dotsenko and Fateev for the bosonic case to evaluate the structure constants of N=1 minimal Superconformal Algebras in the Neveu-Schwarz sector.

* Partially supported by the Swiss National Science Foundation.
1. INTRODUCTION

In this paper we compute the structure constants of the operator algebra for some Superconformal Field Theories in the Neveu-Schwarz sector (NS). For the minimal Conformally Invariant Theories [1] and for the unitary subseries, [2] the structure constants of the operator algebra were computed in general using the Coulomb gas representation for the models in a set of classic papers by Dotsenko and Fateev [3,4,5]. The \( N = 1 \) Superconformal Field Theories [6,2,7] were found soon after the discovery of the minimal models [1], and some of their properties were analyzed in [8,9,10]. In particular, in [9] the Coulomb gas description in [3] was extended to the superconformal case. This construction was used to determine the fusion rules and some of the general properties of four-point correlators [11]. However, the full determination of the structure constants of the operator algebra for the minimal superconformal theories in analogy with the analysis of [3,4,5] is not available in the literature.

We extend the contour manipulation techniques of the work of Dotsenko and Fateev to the case of supercontours and supercontour integral representations of superconformal blocks. We follow their methodology closely, although the superconformal case presents some peculiarities of its own. There are several ways to test the accuracy of our results. The first one consists of verifying that the structure constants have zeroes exactly where indicated by the fusion rules. We can also check that our results agree with those of the tricritical Ising model, where the structure constants can be read off directly from [4,5] or [10]. The third and less trivial check of the consistency of our method is presented in the appendix. It will be argued in the text that in the computation of the structure constants we need to know two things. First we need to know a set of monodromy matrices expressing the behavior of the conformal blocks under braiding and second we need to know explicitly a number of normalization superintegrals. There are two independent ways of computing the monodromy matrices. The first one, as we will discuss in detail, is to take them from [4] after some appropriate changes are made, and the
second is to evaluate them directly in terms of the normalization integrals. These two methods are independent and give the same result thus providing a good verification of our results and methods. This paper is very technical and for the reader interested only in the results, we have summarized the main results at the end of each section.

The structure of this paper is as follows. In section two we collect a number of useful formulae in the theory of superconformally invariant field theories. We have followed the presentations in [12,13] and in [9] for the Coulomb gas formulation of the Superconformal minimal models. We write the Coulomb gas representation of the chiral blocks which need to be computed and at the end of the section we have a short discussion on the issue of open versus closed supercontours in the representation of superconformal blocks and in the solution of super-differential equations. Here, and perhaps unexpectedly, one finds a phenomenon familiar in Superstring perturbation theory with regard to integration ambiguities in supermanifolds with boundaries (see for details and references) although here the problem is not nearly as severe. We also collect a number of useful formula about $SL(2|1)$ and its invariants which are necessary in writing three- and four-point correlators.

In section three we extend to the superconformal case the general analysis of conformal blocks that was carried out for the conformal case in [1] and [4]. The general structure of the operator product expansion (OPE) of two superfield is presented, we define the monodromy matrices and the monodromy invariants used in building physical correlation functions. We write explicitly the holomorphic superconformal blocks to be computed in later sections, we determine carefully the generalization of the conformal block normalization conditions in [1] and itemize all the possible case and how to determine the general form of the monodromy invariant metric allowing us to put together the holomorphic and anti-holomorphic blocks. We do this for the thermal subalgebras and the general algebra of NS superfields. We determine here the type of integrals we need to compute in order to explicitly evaluate the structure constants of the superoperator algebra. This section therefore outlines the computation to follow.
In section four we compute the normalization integrals for the thermal series. We have to separate two substantially different cases depending on whether the number of screening charges is even or odd. The odd case is more difficult than the even case because two types of integrals are necessary. We find the generalization of the recursion relations and functional relations for them which generalize the work of [4], relate the even to the odd integrals by some specific limiting procedures and we are able finally to write the explicit form of the integrals in terms of rather long products of $\Gamma$-functions. The most difficult part of the computation appears in the determination of a set of integers appearing in the arguments of the $\Gamma$-functions.

In section five we compute the normalization constants in the general case. Again the most difficult part is the determination of some integers in the arguments of the $\Gamma$-functions, however, here we can use the results of the thermal series to determine them. In section six we compute the non-symmetric structure constant of the superoperator algebra. These are the ones we can read off directly from the formulae in section three. At the end of this section we collect a number of useful integrals analogous to those appearing in appendix B of [4]. In section seven we extend the arguments in [5] to compute the symmetric (physical) structure constants. Finally, in the appendix we provide a way to compute the monodromy matrices different from the one used in the text. We believe that the methods presented in this paper can be extended to compute the structure constants involving two Ramond fields although we have not yet tried to do so.

*Note added.* When this work was completed we discovered the paper by Kitazawa et al. [15] where the $N = 1$ superconformal structure constants were computed. Our results agree with those of ref. [15], and provide a non-trivial verification of the whole computation. Our method is a manifestly superconformally invariant generalization of the work of Dotsenko and Fateev [3, 4, 5]. The results are complicated enough that an independent computation of these structure constants is not unreasonable. We also feel that some of the technical problems we tackled in dealing with supercontour integrals are interesting in their own right, and they can provide a basis for extension to other cases like for example $N = 2$.
models.
2. GENERAL PROPERTIES OF SUPERCONFORMAL THEORIES

In this section we collect some of the general properties of Superconformal Field Theories (SCFT) [12]. Since there is abundant literature on the subject we mostly establish our notation and review the basic formulae of the Coulomb gas formulation of $N = 1$ SCFT [9]. We also present some useful properties of $SL(2|1)$-transformations. Further details can be found in the literature.

2.1. SUPERCONFORMAL TRANSFORMATIONS, $SL(2|1)$

We follow mainly D. Friedan’s lectures in [12]. A superpoint in the complex superplane $C^{1|1}$ will be denoted by $Z = (z, \theta)$. The superderivative $D$ is given by

$$D = \frac{\partial}{\partial \theta} + \theta \frac{\partial}{\partial z}$$  \hspace{1cm} (2.1)

A function $f(Z, \overline{Z})$ is superanalytic if it satisfies $\overline{D}f = 0$. In components $f = \phi + \theta \psi + \overline{\theta} \lambda + \theta \overline{\theta} \chi$ and the analyticity condition implies $\lambda = \chi = 0$. Furthermore, $\phi, \psi$ are holomorphic functions of $z$: $f = \phi(z) + \theta \psi(z)$. A superconformal transformation is a change of coordinates

$$Z \rightarrow \tilde{Z}(Z) = (\tilde{z}(Z), \tilde{\theta}(Z))$$  \hspace{1cm} (2.2)

under which $D$ transforms covariantly

$$D = D\tilde{\theta} \tilde{D}$$  \hspace{1cm} (2.3)

This implies

$$D\tilde{z} = \tilde{\theta} D\tilde{\theta}$$  \hspace{1cm} (2.4)

A super-Riemann surface is built by gluing $C^{1|1}$ patches with superconformal transformations. In analogy with Conformal Field Theory (CFT) [1] we can introduce
tensor fields. If $dZ$ denotes the superdifferential transforming according to

$$d\tilde{Z} = D\tilde{\theta} dZ$$  \hspace{1cm} (2.5)

under superconformal transformations, a superconformal tensor of rank $h$ is a function $\phi(Z)$ such that

$$\phi(Z) dZ^{2h}$$  \hspace{1cm} (2.6)

is invariant under superconformal transformations. As usual, we omit the anti-holomorphic dependence whenever possible. In components

$$\phi(Z) = \phi_0(z) + \theta \phi_1(z)$$

The standard conformal dimensions of $\phi_0, \phi_1$ are respectively $h$ and $h + 1/2$. The analogues of abelian differentials are tensors of type $1/2$. For these we can define supercontour integrals. Recalling

$$\oint d\theta \theta = 1 \quad \oint d\theta 1 = 0$$  \hspace{1cm} (2.7)

one defines

$$\oint_C dZ \omega(Z) = \oint_C dz \oint_C d\theta \omega(Z) = \oint_C d\omega_1(z)$$  \hspace{1cm} (2.8)

It is also possible to define line integrals

$$f(Z_1, Z_2) = \int_{Z_2}^{Z_1} dZ \omega(Z)$$  \hspace{1cm} (2.9)

according to

$$f(Z_2, Z_2) = 0 \quad D_1 f(Z_1, Z_2) = \omega(Z_1)$$  \hspace{1cm} (2.10)
For example

\[ \theta_{12} \equiv \theta_1 - \theta_2 = \int_{Z_2}^{Z_1} dZ \]

\[ Z_{12} \equiv z_1 - z_2 - \theta_1 \theta_2 = \int_{Z_2}^{Z_1} dZ \int_{Z_2}^{Z'} dZ' \]  

A superanalytic function can be expanded in power series:

\[ f(Z_1) = \sum_{n=0}^{\infty} \frac{1}{n!} Z_{12}^n \partial_2^n (1 + \theta_{12} D_2) f(Z_2) \]

\[ = f(Z_2) + \theta_{12} D_2 f(Z_2) + z_{12} \partial_2 f(Z_2) + \ldots \]  

The fundamental formulae of superconformal calculus are

\[ \frac{1}{2\pi i} \oint_{C_2} dZ_1 Z_{12}^{-n-1} = 0 \]  

\[ \frac{1}{2\pi i} \oint_{C_2} dZ_1 \theta_{12} Z_{12}^{-n-1} = \delta_{n,0} \]  

yielding the generalization of Cauchy’s formula

\[ \frac{1}{2\pi i} \oint_{C_2} dZ_1 f(Z_1) \theta_{12} Z_{12}^{-n-1} = \frac{1}{n!} \partial_2^n f(Z_2) \]

\[ \frac{1}{2\pi i} \oint_{C_2} dZ_1 f(Z_1) Z_{12}^{-n-1} = \frac{1}{n!} \partial_2^n D_2 f(Z_2) \]  

A special type of transformations is the fractional linear transformations. Writing

\[ \tilde{Z} = g(Z) \quad \tilde{z}(Z) = \frac{az + b + \alpha \theta}{cz + d + \beta \theta} \]

\[ \tilde{\theta}(Z) = \frac{\alpha \tilde{z} + \beta \tilde{\alpha} + A \theta}{cz + d + \beta \theta} \]  

one easily solves (2.4) to obtain

\[ \tilde{\alpha} = \frac{a \beta - c \alpha}{\sqrt{ad - bc}} \quad \tilde{\beta} = \frac{b \beta - d \alpha}{\sqrt{ad - bc}} \quad \tilde{A} = \sqrt{ad - bc - 3 \alpha \beta} \]
Define the superdeterminant of $g$ as

$$s\text{det } g \equiv ad - bc - \alpha\beta$$  \hspace{1cm} (2.17)

One easily verifies the following properties:

$$D\tilde{\theta} = \frac{\sqrt{s\text{det } g}}{cz + d + \beta\theta}$$

$$\tilde{Z}_{12} = s\text{det } g \frac{Z_{12}}{(cz_1 + d + \beta\theta_1)(cz_2 + d + \beta\theta_2)}$$  \hspace{1cm} (2.18)

The supergroup $SL(2|1)$ has dimension $3|2$. Therefore given any four points $Z_i, i = 1, 2, 3, 4$ we can fix for example $z_1 = 0, z_2 = 1, z_3 = \infty; \theta_1 = \theta_2 = 0$. In general an $n$-point function of Neveu-Schwarz (NS) fields will depend on $n-3|n-2$ parameters. The harmonic ratio

$$\frac{Z_{12}Z_{34}}{Z_{13}Z_{24}}$$  \hspace{1cm} (2.19)

is $SL(2|1)$-invariant. Given any three points $Z_1, Z_2, Z_3$ we can construct an odd $SL(2|1)$ invariant quantity

$$\eta = (Z_{12}Z_{13}Z_{23})^{1/2}(\theta_1 Z_{23} + \theta_2 Z_{31} + \theta_3 Z_{12} + \theta_1\theta_2\theta_3)$$  \hspace{1cm} (2.20)

It is convenient to perform the transformation from $Z_1, \ldots, Z_4$ to $(0, 0), (1, 0), (\infty, \theta), (z_4, \theta_4)$ in two steps. First we apply the transformation

$$\tilde{z}(Z) = \frac{Z_{13}Z_{23}}{Z_{3}Z_{21}}$$

$$\tilde{\theta}(Z) = -\frac{1}{Z_3} \sqrt{\frac{Z_{23}}{Z_{12}Z_{31}}} (\theta_1 Z_{3} - \theta_2 Z_{13} + \theta_1\theta_2\theta_3)$$  \hspace{1cm} (2.21)

with $Z_{j} = z - z_j - \theta\theta_j$. Then we apply:

$$\tilde{z}(\tilde{Z}) = \tilde{z}(1 + \tilde{\theta}_2\tilde{\theta})$$

$$\tilde{\theta}(\tilde{Z}) = \tilde{\theta} - \tilde{\theta}_2\tilde{z}_2$$  \hspace{1cm} (2.22)

Choosing $Z_1 = (z_1, \theta_1), Z_2 = (z_2, \theta_2), Z_3 = (z_3 + \epsilon, \theta_3), Z_4 = (z_4, \theta_4)$, the application of the previous two transformations has the desired result as $\epsilon \to 0$. 
Since the NS vacuum is \( SL(2|1) \)-invariant, we can write the general form of the \( n \)-point function for NS-fields:

\[
\langle \prod_{i=1}^{n} \Phi_i(Z_i) \rangle = \prod_{i<j} Z_{ij}^{-\gamma_{ij}} F(z_a, \eta_\alpha) \\
a = 1, \ldots, n - 3 \quad \quad \alpha = 1, \ldots, n - 2 \quad (2.23)
\]

\[
\sum_{j \neq i} \gamma_{ji} = 2h_i \quad \gamma_{ij} = \gamma_{ji} \quad \gamma_{ii} = 0
\]

In particular, the three-point function takes the form:

\[
\langle \Phi_1(Z_1)\Phi_2(Z_2)\Phi_3(Z_3) \rangle = Z_{12}^{-\gamma_{12}} Z_{13}^{-\gamma_{13}} Z_{23}^{-\gamma_{23}} (a + b\eta) \quad (2.24)
\]

with \( \eta \) given in (2.20). Both (2.23), (2.24) are direct consequences of (2.15), (2.18). The coefficients \( a, b \) are the structure constants of the superconformal operator algebra, and their computation is the main object of this paper.

### 2.2. FREE SUPERFIELDS AND BACKGROUND CHARGE

The generator of superconformal transformations is the super-energy-momentum tensor

\[
T(Z) = T_F(z) + \theta T_B(z) \quad (2.25)
\]

\[
\delta_v \Phi(Z_2) = \oint_{C_2} dZ_1 v(Z_1) T(Z_1) \Phi(Z_2) \quad (2.26)
\]

For primary superfields the operator product expansion (OPE) of \( T(Z) \) and \( \phi(Z) \) is

\[
T(Z_1) \Phi(Z_2) = \frac{\theta_{12}}{Z_{12}^2} h \Phi(Z_2) + \frac{1}{2} \frac{D_2 \Phi(Z_2)}{Z_{12}} + \frac{\theta_{12}}{Z_{12}} \partial_2 \Phi(Z_2) + \ldots \quad (2.27)
\]

The OPE defining the super-Virasoro algebra is

\[
T(Z_1)T(Z_2) = \frac{\hat{c}}{4Z_{12}^3} + \frac{3\theta_{12}}{2Z_{12}^2} T(Z_2) + \frac{1}{2} \frac{D_2 T(Z_2)}{Z_{12}} + \frac{\theta_{12}}{Z_{12}} \partial_2 T(Z_2) + \ldots \quad (2.28)
\]
The mode expansions defining \( L_n, G_n \) are

\[
T_B(z) = \sum_n L_n z^{-n-2} \quad T_F(z) = \sum_n \frac{1}{2} G_n z^{-n-3/2}
\] (2.29)

where for \( T_F, n \in \mathbb{Z} + 1/2 \) in the NS sector and \( n \in \mathbb{Z} \) in the Ramond (R) sector.

Using Cauchy’s formula (2.14)

\[
[L_n, \Phi(Z)] = z^n \left( z \frac{\partial}{\partial z} + (n + 1)(h + \frac{1}{2} \frac{\theta}{\partial \theta}) \right) \Phi(Z)
\]

\[
[\epsilon G_{n+1/2}, \Phi(Z)] = \epsilon z^n \left( z \left( \frac{\partial}{\partial \theta} - \theta \frac{\partial}{\partial z} \right) - 2h(n + 1) \theta \right) \Phi(Z)
\] (2.30)

A double check on (2.23) can be obtained by writing the \( SL(2|1) \) generators \( G_{\pm 1/2}, L_{\pm 1}, L_0 \) as super-differential operators according to (2.30) and then showing that (2.23) is annihilated by them.

The standard value of the central charge of the Virasoro algebra is \( c = 3 \hat{c}/2 \).

The simplest realization of the superconformal algebra is provided by a free massless scalar superfield.

\[
T(Z) = -\frac{1}{2} : D\phi \partial \phi : (Z)
\]

\[
\phi(Z_1)\phi(Z_2) \sim \log Z_{12}
\] (2.31)

One easily verifies that \( \hat{c} = 1 \) and that the conformal dimension of a vertex operator is given by

\[
h(e^{i\alpha \hat{\phi}}) = \frac{1}{2} \alpha^2
\] (2.32)

The \( n \)-point correlators of vertex operators vanish unless the charge neutrality condition \( \sum \alpha_i = 0 \) is satisfied. The central charge \( \hat{c} \) can be changed to any value by adding a background charge [9] in analogy with the Virasoro case [3,4,5]. Define
a new energy-momentum tensor

\[
T(Z) = -\frac{1}{2} : D\phi \partial \phi : (Z) + \frac{i}{2} \alpha_0 D\partial \phi(Z)
\]  

now

\[
\hat{c} = 1 - 2\alpha_0^2 \quad c = \frac{3}{2} - 3\alpha_0^2
\]  

The conformal dimension of a vertex operator also changes

\[
h(e^{i\alpha\phi}) = \frac{1}{2}\alpha(\alpha - \alpha_0)
\]  

and notice that both \(\alpha\) and \(\alpha = \alpha_0 - \alpha\) give fields with the same conformal dimension. As in the standard Coulomb gas construction \[3\] there are two screening fields of dimension 1/2 leading after contour integration to the screening charges \(Q_{\pm}\):

\[
Q_{\pm} = \oint dZe^{i\alpha_{\pm}\phi}
\]  

with \(\alpha_{\pm}\) satisfying

\[
\frac{1}{2}\alpha_{\pm}(\alpha_{\pm} - \alpha_0) = \frac{1}{2}
\]  

or

\[
\alpha_+ + \alpha_- = \alpha_0
\]

\[
\alpha_+ \alpha_- = -1
\]

\[
\alpha_{\pm} = \frac{1}{2\sqrt{2}}(\sqrt{1 - \hat{c}} \pm \sqrt{9 - \hat{c}})
\]  

The charge neutrality condition for the correlation function of \(n\) vertex operators is changed to \(\sum_i \alpha_i = \alpha_0\). In the NS sector the bosonic (\(\phi_0\)) and fermionic (\(\phi_1\)) fields combine into a superfield \(\phi(Z)\). They are both periodic around \(z = 0\). In the Ramond sector one has instead \(G(e^{2\pi i z}) = -G(z)\); \(\phi_1\) is antiperiodic, \(\phi_0\) is periodic and they do not combine to form a superfield. Furthermore we have to take into
account the spin fields $\sigma^\pm(z)$ associated to $\phi_1$. The vertex operators in the R sector take the general form $\sigma^\pm(z)e^{i\alpha\phi_0(z)}$ with conformal dimension $\frac{1}{16} + \frac{1}{2}(\alpha - \alpha_0)$. This formula is again invariant under $\alpha \mapsto \alpha_0 - \alpha$.

The singular representations of the super-Virasoro algebra are labelled by two positive integers $m, m' \geq 1$ with highest weights

$$h_{m',m} = \frac{\hat{c} - 1}{16} + \frac{1}{8}(m'\alpha_- + m\alpha_+)^2 + \frac{1}{32}(1 - (-1)^{m' - m})$$

(2.40)

When $m' - m$ is even we have NS field, and for $m' - m$ odd we have R fields. In the Coulomb gas picture the singular modules are generated by the vertex operators.

In the NS sector

$$V_{m',m}(Z) = e^{i\alpha_{m',m}\phi(Z)}$$

$$\alpha_{m',m} = \frac{1 - m'}{2}\alpha_- + \frac{1 - m}{2}\alpha_+$$

(2.41)

and in the R sector

$$V_{m',m}(z) = \sigma(z)e^{i\alpha_{m',m}\phi_0(z)}$$

$$\alpha_{m',m} = \frac{1 - m'}{2}\alpha_- + \frac{1 - m}{2}\alpha_+$$

(2.42)

$$m' - m \equiv 0 \pmod{2}$$

$$m' - m \equiv 1 \pmod{2}$$

The charge screening condition for a $n$-point correlator of vertex operators is as usual $\sum_i \alpha_i = \alpha_0$ independently of whether we have NS or R fields. It is useful to note that

$$\bar{\alpha}_{m',m} = \alpha_0 - \alpha_{m',m} = \alpha_{-m',-m}$$

(2.43)

The minimal superconformal theories are those satisfying

$$p'\alpha_- + p\alpha_+ = 0$$

(2.44)

where $p', p$ are positive integers. They are both supposed to be even or odd, and their greatest common divisor is either 2 or 1. The parity condition $m' - m \equiv$
0 (mod 2) follows from the fact that (2.43),(2.44) taken together imply \( \alpha_{m',m} = \alpha_{p'-m', p-m} \) and if \( \alpha_{m',m} \) is in the NS sector, \( \alpha_{p'-m', p-m} \) should also be a NS field.

In the rational case (2.44):

\[
\hat{c} = 1 - \frac{2(p' - p)^2}{pp'} \quad \alpha_+ = \sqrt{p'/p} \quad \alpha_- = -\sqrt{p/p'}
\]  

(2.45)

and we can choose \( p' > p \). The unitary series occurs when \([7] \ p' = p + 2\),

\[
h_{m',m} = \frac{1}{8pp'}[(mp' - m'p)^2 - (p' - p)^2] + \frac{1}{32}(1 - (-1)^{m'-m})
\]  

(2.46)

The primary fields of the minimal theories can be further restricted to lie in the fundamental region \( 1 \leq m' \leq p' - 1, 1 \leq m \leq p - 1, mp' - m'p \leq 0 \). The charge assignments (2.41) can also be derived by requiring the non-vanishing in the four-point function of the lowest component of \( V_\alpha \): \( \langle V_\alpha V_\alpha V_\alpha V_\alpha \rangle \). The charge can only be screened by the insertion of \( Q_+^N Q_-^N \) for the given values of \( \alpha \). A simple consequence of the Coulomb gas representation is the computation of the fusion rules. We only need to write the three possible ways of screening the three-point function

\[
\langle \Phi_{m'_1, m_1}(Z_1) \Phi_{m'_2, m_2}(Z_2) \Phi_{m'_3, m_3}(Z_3) \rangle
\]  

(2.47)

We can conjugate any of the fields in (2.47). For the rational case (2.45) if we compute the fusion rules by counting screenings of both \( \langle V_{\alpha_1} V_{\alpha_2} V_{\alpha_3} \rangle \), \( \langle \bar{V}_{\alpha_1} \bar{V}_{\alpha_2} \bar{V}_{\alpha_3} \rangle \), and requiring compatibility we obtain in the NS sector

\[
[m'_{1}, m_{1}] \times [m'_{2}, m_{2}] = \sum_{|m'_1-m'_2|+1} \sum_{|m_1-m_2|+1}' [m', m]
\]  

(2.48)

Where the prime in the sum means that \( m', m \) jump in steps of two units.
2.3. CORRELATORS IN THE COULOMB GAS REPRESENTATION

We write down in this section the contour integral representation of chiral correlators for minimal $N = 1$ theories in the NS sector. Using (2.21,22) and the covariance properties of conformal superfields, we can transform the four-point function $\langle \Phi_4(Z_4)\Phi_3(Z_3)\Phi_2(Z_2)\Phi_1(Z_1) \rangle$ into the form

$$\lim_{R \to \infty} R^{2h_4} \langle \Phi_4(R, R\eta)\Phi_3(1, 0)\Phi_2(z, \theta)\Phi_1(0, 0) \rangle$$

(2.49)

Representing the superfields as Coulomb gas vertex operators $e^{i\alpha_i\phi(Z_i)}$, the generic chiral four-point function takes the form

$$\lim_{R \to \infty} R^{2h_4} \langle e^{i\alpha_4\phi(R,R\eta)}e^{i\alpha_3\phi(1,0)}e^{i\alpha_2\phi(\theta,\eta)}e^{i\alpha_1\phi(0)} \rangle \int \prod_{a=1}^{m} dZ_a \prod_{a'=1}^{m'} dZ_{a'} e^{i\alpha_+ \phi(Z_a)} e^{i\alpha_- \phi(Z_{a'})}$$

$$= \lim_{R \to \infty} R^{2h_4} R^{\alpha_1\alpha_4}(1-1)\alpha_3\alpha_4 (R-z-R\eta\theta)^{\alpha_2\alpha_4} \int \prod_{a=1}^{m} dZ_a \prod_{a'=1}^{m'} dZ_{a'}$$

$$\prod_{a=1}^{m} u_a^{\alpha_1\alpha_+} (1-u_a)^{\alpha_3\alpha_+} (z-u_a-\theta \theta a)^{\alpha_2\alpha_+} \prod_{a<b}^{m} Z_{a\alpha_+}^{\alpha_+}$$

$$\prod_{a'=1}^{m'} v_{a'}^{\alpha_1\alpha_-} (1-v_{a'})^{\alpha_3\alpha_-} (z-v_{a'}-\theta \omega_{a'})^{\alpha_2\alpha_-} \prod_{a'<b'}^{m'} Z_{a'\alpha_-}^{\alpha_-}$$

$$\prod_{a=1}^{m} Z_{a\alpha_+}^{-1} \prod_{a'=1}^{m'} (R-u_a-R\eta\theta a)^{\alpha_4\alpha_+} \prod_{a'=1}^{m'} (R-v_{a'}-R\eta\omega_{a'})^{\alpha_4\alpha_-}$$

$$= z^{\alpha_1\alpha_2} (1-z)^{\alpha_2\alpha_3} (1-\eta\theta)^{\alpha_2\alpha_4}$$

$$\int \prod_{a=1}^{m} dZ_a \prod_{a'=1}^{m'} dZ_{a'} (1-\alpha_4\alpha_+\eta \sum_{a=1}^{m} \theta a - \alpha_4\alpha_-\eta \sum_{a'=1}^{m'} \omega_{a'})$$

$$\prod_{a=1}^{m} u_a^{\alpha_1\alpha_+} (1-u_a)^{\alpha_3\alpha_+} (z-u_a-\theta \theta a)^{\alpha_2\alpha_+} \prod_{a<b}^{m} Z_{a\alpha_+}^{\alpha_+}$$

$$\prod_{a'=1}^{m'} v_{a'}^{\alpha_1\alpha_-} (1-v_{a'})^{\alpha_3\alpha_-} (z-v_{a'}-\theta \omega_{a'})^{\alpha_2\alpha_-} \prod_{a'<b'}^{m'} Z_{a'\alpha_-}^{\alpha_-} \prod_{c,d'}^{m,m'} Z_{c\alpha_+}^{-\alpha_+} \prod_{c,d'}^{m,m'} Z_{d\alpha_-}^{-\alpha_-}$$

(2.50)

Here $Z_a \equiv (u_a, \theta a)$, $Z_{a'} \equiv (v_{a'}, \omega_{a'})$ and we have used $\alpha_+\alpha_- = -1$. Using (2.21,22)
one can add to (2.50) the appropriate prefactors giving the four-point function for arbitrary points $Z_i$. In the derivation of (2.50) the charge screening condition was crucial. Collecting all powers of $R$ we obtain a prefactor

$$R^{2h_4 + \alpha_4 (\alpha_1 + \alpha_2 + \alpha_3 + m\alpha_+ + m'\alpha_-)}$$

where $m, m'$ are the number of $+, -$ screening charges respectively. Using $2h_4 = \alpha_4 (\alpha_4 - \alpha_0)$ and the screening condition $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + m\alpha_+ + m'\alpha_- = \alpha_0$ the exponent in (2.51) vanishes and there is no $R$-dependence in the $R \to \infty$ limit.

In the next chapter we analyze how to put together chiral conformal blocks to construct physical conformal blocks.

2.4. OPEN AND CLOSED CONTOURS

We come now to a rather delicate issue in the contour definition of supercorrelators and normalization factors. The typical example in the bosonic case is given by the decoupling for a level two null-vector [1]. This yields a hypergeometric differential equation whose solutions can be represented in terms of contour integrals around the singular points according to standard results in the theory of ordinary differential equations [16]. The regular singular points of the hypergeometric equation can be chosen at $(0, 1, \infty)$ for convenience. The solutions $F_1(z), F_2(z)$ can be expressed as contour integrals around $(0, z)$ and $(1, \infty)$. When the hypergeometric equation is applied to the computation of conformal blocks, the normalization of its solutions is determined by the monodromy invariance conditions. It is therefore convenient to write $F_1(z), F_2(z)$ as open line integrals along the cuts joining the singular points. In analogy with (2.50) $F_{1,2}$ are represented up to constants by

$$\oint_{C_1} t^{\alpha}(1 - t)^{\beta}(z - t)^{\gamma}$$

$$\oint_{C_2} t^{\alpha}(t - 1)^{\beta}(t - z)^{\gamma}$$

(2.52)
opening the contours we obtain

\[
\int_0^z t^\alpha (1 - t)^\beta (z - t)^\gamma \\
\int_{\infty}^1 t^\alpha (t - 1)^\beta (t - z)^\gamma
\]  

(2.53)

The line integral representation (2.53) is simpler to use in the computation of monodromy matrices but it has disadvantages in determining the fusion rules and the internal channels in a given conformal block. For practical purposes we will use almost exclusively the representation (2.53). Notice that taking the parameters \((\alpha, \beta, \gamma)\) in the range where the integral converges, the integrands vanish at the end points of the line integrals or they have integrable divergences.

The procedure briefly outlined in the previous paragraphs can be extended without difficulty to multi-contour integrals in the bosonic case. For super-contours we have to be more careful and potential ambiguities may show up in going from closed to open contours. The possible existence of integration ambiguities in super-integrals is a phenomenon encountered in Superstring Perturbation Theory (for a clear exposition of the problem with references to the relevant literature, see [14]). The case at hand is a milder version of this problem. From the definitions (2.8,9) we can compare open and closed contour integrals. We explicitly compute (2.9) satisfying (2.10):

\[
F(Z) = F_0(z) + \theta F_1(z) \\
f(Z_1, Z_2) = \int_{Z_2}^{Z_1} dZF(Z) = \int_{z_2}^{z_1} dzF_1(z) + \theta_1 F_0(z_1) - \theta_2 F_0(z_2)
\]  

(2.54)

The limits of integration on the right-hand side of (2.54) are determined by the even parts of the points \(Z_{1,2}\). When we open a closed supercontour we should use (2.54). It is often unavoidable to generate a nilpotent contribution at the endpoints
$z_1, z_2$. The projected line integral is defined by ignoring the terms linear in $\theta_{1,2}$ in (2.54). In other words it is defined as a closed contour in $\theta$ but as open in $z$. This prescription is inconsistent in general. However, in our case, the integrals of the type (2.50) and similar integrals to be considered in later sections are such that $\theta F_0(z)$ vanishes (possibly in the sense of analytic continuation) at the end points of the line integrals. Effectively this produces a projected integration prescription which is preserved under split superconformal transformations

$$\tilde{z} = f(z) \quad \tilde{\theta} = \theta \sqrt{\frac{\partial f}{\partial z}} \quad (2.55)$$

One easily checks that under (2.55)

$$\int_{\tilde{Z}_1}^{\tilde{Z}_2} d\tilde{Z} F(\tilde{Z}) = \int_{z_1}^{z_2} dz \frac{\partial f}{\partial z} F_1(f(z)) + \tilde{\theta}_1 F_0(\tilde{z}_1) - \tilde{\theta}_2 F_0(\tilde{z}_2) \quad (2.56)$$

Hence, if $\theta F_0(z)$ vanishes at the end points of the integration interval, the projected prescription will be maintained by split transformations. Notice that, strictly speaking, only their difference has to vanish in order to preserve this prescription. Under non-split transformations this prescription has to be modified. In later sections we will only need to use split transformations and the previous arguments justify the use of the projected prescription. We will remind the reader of these apparently obscure considerations when the case arises.
3. SUPERCONFORMAL BLOCKS

In this section we detail the structure of super-OPE, superconformal blocks, and their integral representation. Then we discuss the necessary steps to compute the structure constants of the operator algebra. For simplicity we start with the thermal subalgebra of fields \((1, m)\) or \((m', 1)\). The presentation is tailored closely after the papers [1],[3, 4] and we only emphasize the intrinsic features of the supersymmetric case.

We learned in the previous chapter that the three-point function depends on two arbitrary constants. Through the relation between three-point functions and OPE we derive that the super-OPE involves two independent sets of structure constants. More precisely, \(SL(2|1)\)-invariance constrains the OPE of two NS primary superfields to be of the form

\[
\Phi_m(Z_1)\Phi_n(Z_2) = \sum_p Z_{12}^{-\gamma_{mnp}} A_{mn}^p[\Phi_p(Z_2)]_{even} + Z_{12}^{-\gamma_{mnp}-1/2} B_{mn}^p[\Phi_p(Z_2)]_{odd}
\]

(3.1)

where \(\gamma_{mnp} = h_m + h_n - h_p\), \(A_{mn}^p\) and \(B_{mn}^p\) are the structure constants of the operator algebra, and \([\Phi_p]\) denotes a superconformal family with all its descendant fields. The first representatives are

\[
[\Phi_p(Z_2)]_{even} = \Phi_p(Z_2) + \frac{h_m - h_n + h_p}{2h_p} (\theta_{12} D_2 \Phi_p(Z_2) + Z_{12} \partial_2 \Phi_p(Z_2)) + \ldots
\]

\[
[\Phi_p(Z_2)]_{odd} = \theta_{12} \Phi_p(Z_2) + \frac{1}{2h_p} Z_{12} D_2 \Phi_p(Z_2) + \frac{h_m - h_n + h_p + 1/2}{2h_p} Z_{12} \theta_{12} \partial_2 \Phi_p(Z_2) + \ldots
\]

(3.2)

The subindex refers to the Grassmann parity of the expansion. These formulae are derived by requiring the OPE to be compatible with \(SL(2|1)\)-invariance. The full OPE is obtained by combining the holomorphic and anti-holomorphic contributions.
3.1. THE STRUCTURE OF THERMAL FOUR-POINT FUNCTIONS

In the Coulomb gas representation the number of screening charges determines the Grassmann parity of the correlator. This is a consequence of the use of vertex operators to represent primary fields and of the integration over the fermionic variable implied in every screening charge super-contour. Using (2.50) we see that to an even (resp. odd) number of screening charges corresponds an even (resp. odd) function of the $SL(2|1)$-invariant variables. A similar argument holds for the three-point function. We shall use these remarks in the decomposition of the four-point function into superconformal blocks.

For convenience we denote the thermal primary superfields in the NS sector by $\Phi_m(Z) = \Phi_{(1,2m+1)}(Z)$ and its conjugate field by $\Phi_{\bar{m}}(Z)$. The out-state is as usual

$$\langle \Phi_m(\infty) \rangle = \lim_{R\to\infty} R^{2h_m} \langle 0 | \Phi_m(R, R\eta) \rangle$$

The fusion rule (2.48) reads

$$\Phi_m \times \Phi_n = \sum_{q=|m-n|}^{\min(p-m-n-2m+n)} \Phi_q$$

and the three-point function is

$$\langle \Phi_{\bar{k}}(\infty)|\Phi_l(1,0)\Phi_q(0,0)\rangle = A_{\bar{k}lq} + B_{\bar{k}lq}\eta \quad (3.3)$$

where the number of screenings is $N = l + q - k$.

The four-point function can be expanded in terms of three-point functions when we use the OPE:

$$G = \langle \Phi_{\bar{k}}(\infty)|\Phi_l(1,0)\Phi_m(z, \theta)\Phi_n(0,0)\rangle$$

$$\quad = \sum_q z^{-\gamma_{mnq}} A_{mnq}^{q}. \langle \Phi_{\bar{k}}(\infty)|\Phi_l(1,0)[\Phi_q(0,0)]_{\text{even}} \rangle$$

$$\quad \quad + z^{-\gamma_{mnq}-1/2} B_{mnq}^{q}. \langle \Phi_{\bar{k}}(\infty)|\Phi_l(1,0)[\Phi_q(0,0)]_{\text{odd}} \rangle \quad (3.4)$$

$$\quad = \sum_q G_q$$
where the number of screening charges is $M = l + m + n - k$. Trading the index $q$ for $r = q - |m - n|$ we have

$$ N = l + |m - n| + r - k \equiv M + r \pmod{2} $$

Using the previous observation on the Grassmann parity of the three- and four-point functions, and the expansion (3.2), we see that depending on the values of $N, M \pmod{2}$ four types of conformal blocks $G_q$ appear.

$M$ even: $G_q$ is an even function of $z, \eta, \theta$.

$N$ even: $G_q \sim A_{\bar{k}klq} A_{mn}^q z^{-\gamma} (1 + \eta \theta(\ldots) + z(\ldots) + \ldots)$

$N$ odd: $G_q \sim B_{\bar{k}klq} B_{mn}^q z^{-\gamma-1/2} (\eta \theta + z(\ldots) + z \eta \theta(\ldots) + \ldots)$ (3.5)

$M$ odd: $G_q$ is an odd function of $z, \eta, \theta$.

$N$ even: $G_q \sim A_{\bar{k}klq} B_{mn}^q z^{-\gamma-1/2} (\theta + z \eta(\ldots) + z \theta(\ldots) + \ldots)$

$N$ odd: $G_q \sim B_{\bar{k}klq} A_{mn}^q z^{-\gamma} (\eta + \theta(\ldots) + z \eta(\ldots) + \ldots)$

the ellipsis denotes constants. The knowledge of this structure is essential in order to normalize correctly the superconformal blocks. In deriving (3.5) we have chosen $\Phi_{\bar{k}}$ as the conjugate field. One readily sees that taking the conjugate to be any other field leads to the same values for $N, M \pmod{2}$ which is all the results depend on.

3.2. THERMAL SUPERCONFORMAL BLOCKS AND MONODROMY INVARIANTS

According to (2.50) the integral representation of a chiral superconformal block
in the thermal subalgebra is

\[ J_k^{(m)}(a, b, c, \rho; Z) = T \int_0^\infty dV_1 \cdots \int_0^{V_{m-k}} dV_{m-k} \int_0^{V_{m-k+1}} \cdots \int_0^{V_m} dV_m \]

\[ (1 + a_1 \eta \theta + a_2 \eta \sum_{i=1}^{m-1} (v_i \prod_{i=1}^{m-k} (v_i - 1))^b (v_i - z - \theta_i \theta)^c \]

\[ \prod_{i=1}^{m-k+1} (1 - v_i)^b (z - v_i - \theta \theta)^c \prod_{i<j} V_{ij} \]

where \( V_i = (v_i, \theta_i) \) and

\[ a = \alpha_1 \alpha_+ \quad a_1 = -\alpha_2 \alpha_4 \]
\[ b = \alpha_3 \alpha_+ \quad a_2 = -\alpha_+ \alpha_4 \]
\[ c = \alpha_2 \alpha_+ \quad \rho = \alpha_+^2 \]

For a detailed discussion of the ordering prescription \( T \) and analytic continuation we refer the reader to section four. The contour integrals in (3.6) are ordered as in fig. 3.1. It is often more convenient to work with the ordered integral

\[ I_k^{(m)}(a, b, c, \rho; Z) = \int_0^\infty dV_1 \int_0^{V_{m-k}} dV_{m-k} \int_0^{V_{m-k+1}} \cdots \int_0^{V_m} dV_m \]

\[ (as \ in \ (3.6)) \]

which is simply related to \( J_k^{(m)} \) by (see sec. 4.1 for details)

\[ J_k^{(m)}(a, b, c, \rho; Z) = \lambda_{k-1}(\rho) \lambda_{m-k}(\rho) I_k^{(m)}(a, b, c, \rho; Z) \]

A physical correlation function is a combination of holomorphic and anti-holomorphic
superconformal blocks

\[ G(Z, \bar{Z}) = \sum_{k, l=1}^{m} X_{kl} J_k^{(m)}(Z) \bar{J}_l^{(m)}(Z) \]  

(3.10)

with the requirement that it should be monodromy invariant when we analytically continue the variable \(Z\) (and obviously \(\bar{Z}\)) along the curves shown in fig. 3.2. The constants \(X_{kl}\) are determined using the monodromy properties of \(J_k^{(m)}\) around the points \((0,0), (1,0)\). The integrals \(J_k^{(m)}\) can be thought of as solutions to the superdifferential equations obtained by decoupling null-vectors in the superconformal modules. The equations have regular singularities at the points \((0,0), (1,0), (\infty, \eta \infty)\). We can compute the monodromy when we analytically continue \(J_k^{(m)}\) along \(C_0\) or \(C_1\) in fig. 3.2:

\[ C_i: \quad J_k^{(m)}(Z) \to (g_i)_{kl} J_l^{(m)}(Z) \]  

(3.11)

where \(g_i\) are \((m-1) \times (m-1)\) matrices. With the choice of contours made, the matrix \(g_0\) is diagonal and unitary. Monodromy invariance under \(g_0\) requires

\[ G(Z, \bar{Z}) = \sum_{k=1}^{m} X_k J_k^{(m)}(Z) \bar{J}_k^{(m)}(Z) \]  

(3.12)

The constants \(X_k\) are fixed if we require invariance under \(g_1\) as well. For this purpose it is convenient to use another basis \(\tilde{J}_k^{(m)}(Z)\) with diagonal monodromy at \((1,0)\) and expand \(J_k^{(m)}\) in this new basis. Explicitly \(\tilde{J}_k^{(m)}(Z)\) are given by the integrals

\[ \tilde{J}_k^{(m)}(a, b, c, \rho; Z) = T \int_{0}^{Z} dV_1 \cdots \int_{0}^{Z} dV_{m-k} \int_{1}^{Z} dV_{m-k+1} \cdots \int_{1}^{Z} dV_{m-1} \]

\[ (1+a_1 \eta \theta + a_2 \eta \theta \sum_{1}^{m-1} \theta_i) \prod_{1}^{m-1} (1 - v_i)^b \prod_{1}^{m-k} (-v_i)^a (z - v_i - \theta \theta_i)^c \]

(3.13)

\[ \prod_{m-k+1}^{m-1} v_i^a (v_i - z - \theta \theta_i)^c \prod_{i>j} \nu_{ij} \]

with the contours ordered as in fig.3.3. Denoting by \(\beta_{kl}\) the coefficients in these
linear expansions we get:

\[ J_k^{(m)}(a, b, c, \rho; Z) = \beta_{kl}(a, b, c, \rho) \tilde{J}_l^{(m)}(a, b, c, \rho; Z) \]

\[ \tilde{J}_k^{(m)}(a, b, c, \rho; Z) = \tilde{\beta}_{kl}(a, b, c, \rho) J_l^{(m)}(a, b, c, \rho; Z) \]  \hspace{1cm} (3.14)

Rewriting \( G(Z, \bar{Z}) \) in terms of \( \tilde{J}_k^{(m)}(Z) \) and requiring invariance under \( g_1 \) yields the conditions

\[ \sum_{k=1}^{m} X_k \beta_{kl} \beta_{kn} = 0 \quad l \neq n \]  \hspace{1cm} (3.15)

As in [4] the solution to these constraints can be chosen as

\[ X_k = \frac{\beta_{mm} \tilde{\beta}_{nk}}{\beta_{mm} \beta_{km}} X_m \]  \hspace{1cm} (3.16)

This expression is invariant under simultaneous rescaling of the coefficients \( X_k \). This amounts to changing the normalization of the four-point function. This freedom will be used to simplify the formulae for the structure constants of the operator algebra. Before doing this however we need to compute the matrices \( \beta_{kl} \) and to properly normalize the chiral blocks \( J_k^{(m)} \).

The computation of the matrix \( \beta \) simplifies considerably once we realize that it relies essentially on the monodromy properties of the integrand in (3.6) and therefore we can read off \( \beta \) from the results in [4] with some minor modifications. The main difference between the ordinary chiral conformal blocks in [4] and here lies in the terms \((v_i - v_j - \theta_i \theta_j)^\rho\) replacing the terms \((v_i - v_j)^2\rho\) in the integrands and the fact that the supercontour integrals are Grassmann odd. Thus, analytically continuing \( V_i \) over \( V_j \) yields a phase \( e^{\pm i\pi(\rho-1)} \) in the superconformal case, to be compared with the phase factor \( e^{\pm i\pi 2\rho} \) in the conformal case (this will be explained in more detail in section four). Hence to compute the matrices \( \beta \) we only need to replace \( \rho \) by \((\rho - 1)/2\) in the matrices \( \alpha \) of Dotsenko and Fateev [4]. As a non-trivial check of our computations we evaluate independently in Appendix A the matrix elements \( \beta_{mk} \). This will indeed be an important check of the normalization
integrals computed in section four. The matrix $\tilde{\beta}$ can be derived from $\beta$ using the relation

$$\tilde{J}_k^{(m)}(a, b, c, \rho; (z, \theta); \eta) = \epsilon_m J_k^{(m)}(b, a, c, \rho; (1-z, i\theta); -i\eta)$$  \hspace{1cm} (3.17)$$

where $\epsilon_m = 1$ ($\epsilon_m = -i$) when $m-1$ is even ($m-1$ is odd). (3.17) is obtained by performing the change of variables $(v_j, \theta_j) \to (1-v_j, \sqrt{-1}\theta_j)$ in (3.13). Consequently we find

$$\tilde{\beta}_{kl}(a, b, c, \rho) = \beta_{kl}(b, a, c, \rho)$$ \hspace{1cm} (3.18)$$

This is not surprising if we recall the relation with the $N = 0$ case in [4].

We next turn to the normalization of $J_k^{(m)}$. These integrals possess a singularity as $z \to 0$. Thus we first extract the singularity and then evaluate the resulting integral which yields an analytic function of $z$ near $z = 0$. If in the $0 \to Z$ integrals in (3.8) we make the change of variables $V_{m-k+i} = (v_{m-k+i}, \theta_{m-k+i}) \to S_i = (s_i, \omega_i) = (v_{m-k+i}/z, \theta_{m-k+i}/z^{1/2})$ we obtain

$$I_k^{(m)}(a, b, c, \rho; Z) = z^{(k-1)(1/2+a+c+(k-2)/2)} \int_0^{1} \prod_{1}^{m-k} dV_i \int_0^{1} \prod_{1}^{k-1} dS_i$$

$$(1 + a_1 \eta \theta + a_2 \eta \sum_{1}^{m-k} \theta_i + a_2 z^{1/2} \eta \sum_{1}^{k-1} \omega_i) \prod_{1}^{m-k} v_i^a (v_i - 1)^b (v_i - z - \theta \theta_i)^c \prod_{i < j} V_{ij}^\rho$$

$$\prod_{1}^{k-1} s_i^a (1 - z s_i)^b (1 - s_i - \frac{\theta \omega_i}{z^{1/2}})^c \prod_{i < j} S_{ij}^\rho \prod_{i, j}^{m-k, k-1} (v_i - z s_j - z^{1/2} \theta \omega_j)^\rho$$  \hspace{1cm} (3.19)$$

Due to the presence of $z^{1/2}$ factors, the integrand in (3.19) is still not regular at $z = 0$. We deal with this last singularity by expanding all the terms containing $\theta$
or $z^{1/2}$ in the integrand. This yields

$$I_k^{(m)}(a, b, c, \rho; Z) = z^{\Delta_k} \int_1^{\infty} dV_i \int_0^1 dS_i (1 + a_1 \eta \theta + a_2 \eta \sum_{i=1}^{m-k} \theta_i + a_2 z^{1/2} \eta \sum_{i=1}^{k-1} \omega_i)$$

$$\left(1 - c \sum_{i=1}^{m-k} \frac{\theta \omega_i}{v_i - z}\right) \left(1 - \frac{c}{z^{1/2}} \sum_{i=1}^{k-1} \frac{\theta \eta \omega_i}{1 - s_i}\right) \left(1 - \rho z^{1/2} \sum_{i=1}^{k-1} \frac{\theta \omega_j}{v_i - z s_j}\right)$$

$$\prod_{i=1}^{m-k} v_i^a (v_i - 1)^b (v_i - z)^c \prod_{i<j} V_{ij}^c \prod_{i=1}^{k-1} s_i^a (1 - z s_i)^b (1 - s_i)^c \prod_{i<j} S_{ij}^\rho \prod_{i,j} (v_i - z s_j)^\rho$$

(3.20)

where $\Delta_k = (k - 1)(a + c + (k - 2) \frac{\rho}{2} + \frac{1}{2})$. Depending on the values of $m, k$ we find by inspection the following behaviours:

(m - 1) even : $I_k^{(m)}$ is an even function of $z, \eta, \theta$.

i) $(k - 1)$ even : $I_k^{(m)} \sim z^{\Delta_k} (N_k^{(m)} + \eta \theta (..) + ..)$

ii) $(k - 1)$ odd : $I_k^{(m)} \sim z^{\Delta_k - 1/2} (\eta \theta N_k^{(m)} + z (..) + ..)$

(m - 1) odd : $I_k^{(m)}$ is an odd function of $z, \eta, \theta$.

iii) $(k - 1)$ even : $I_k^{(m)} \sim z^{\Delta_k} (\eta N_k^{(m)} + \theta (..) + ..)$

iv) $(k - 1)$ odd : $I_k^{(m)} \sim z^{\Delta_k - 1/2} (\theta N_k^{(m)} + z \eta (..) + ..)$

where $(..)$ denote some constants. As expected we recognize here the same expansions as in (3.5). The analysis of (3.5) was useful to indicate the leading term to be normalized in the expansions of the superconformal blocks. Once the last $z^{1/2}$ has been taken into account by (3.21) we can set $z = 0$ in the integrand of (3.20) and the $1 \to \infty; 0 \to 1$ integrals decouple. We get for $N_k^{(m)}(a, b, c, \rho)$:

$$i) N_k^{(m)} = \int_0^\infty \prod_{i=1}^{m-k} dV_i v_i^{a+c+\rho(k-1)} (v_i - 1)^b \prod_{i<j} V_{ij}^c \prod_{i=1}^{k-1} dS_i s_i^{a} (1 - s_i)^b \prod_{i<j} S_{ij}^\rho$$

$$= I(0, m - k)(a + c + \rho(k - 1), b, \rho) I(k - 1, 0)(a, c, \rho)$$

$$= I_{m-k}(-1 - a - b - c - \rho(m - 2), b, \rho) I_{k-1}(a, c, \rho)$$

(3.22i)
ii) \( N_k^{(m)} \eta \theta = -a_2 c \int_1^{\infty} \prod_{i=1}^{m-k} dV_i (\sum_j \eta \theta_j v_i^{a+c+\rho(k-1)}(v_i - 1)b \prod_{i<j} V_{ij}^p \)
\[
\int_0^{1} \prod_{i=1}^{k-1} dS_i (\sum_j \frac{\theta \omega_j}{1 - s_j} s_i^{c}(1 - s_i)^c \prod_{i<j} S_{ij}^p \)
\]
\[
= -a_2 c I^{(\Sigma)}(0, m - k)(a + c + \rho(k - 1), b, \rho; \eta) I^{(\Sigma)}(k - 1, 0)(c, a, \rho; \theta) = -a_2 c I^{(\Sigma)}_{m-k}(-1 - a - b - c - \rho(m - 2), b, \rho; \eta) I^{(\Sigma)}_{k-1}(c, a, \rho; \theta) \]
\[\tag{3.22ii}\]

iii) \( N_k^{(m)} \eta = a_2 \int_1^{\infty} \prod_{i=1}^{m-k} dV_i (\sum_j \eta \theta_j v_i^{a+c+\rho(k-1)}(v_i - 1)b \prod_{i<j} V_{ij}^p \)
\[
\int_0^{1} \prod_{i=1}^{k-1} dS_i s_i^{c}(1 - s_i)^c \prod_{i<j} S_{ij}^p \]
\[
= a_2 I^{(\Sigma)}(0, m - k)(a + c + \rho(k - 1), b, \rho; \eta) I(k - 1, 0)(a, c, \rho) = a_2 I^{(\Sigma)}_{m-k}(-1 - a - b - c - \rho(m - 2), b, \rho; \eta) I_{k-1}(a, c, \rho) \]
\[\tag{3.22iii}\]

iv) \( N_k^{(m)} \theta = -c \int_1^{\infty} \prod_{i=1}^{m-k} dV_i v_i^{a+c+\rho(k-1)}(v_i - 1)b \prod_{i<j} V_{ij}^p \)
\[
\int_0^{1} \prod_{i=1}^{k-1} dS_i (\sum_j \frac{\theta \omega_j}{1 - s_j} s_i^{c}(1 - s_i)^c \prod_{i<j} S_{ij}^p \)
\[
= -c I(0, m - k)(a + c + \rho(k - 1), b, \rho) I^{(\Sigma)}(k - 1, 0)(c, a, \rho; \theta) = -c I_{m-k}(-1 - a - b - c - \rho(m - 2), b, \rho) I^{(\Sigma)}_{k-1}(c, a, \rho; \theta) \]
\[\tag{3.22iv}\]

The symbols \( I, \tilde{I}, \tilde{I}^{(\Sigma)}, I^{(\Sigma)} \) denote the different ordering of integrals and they are analyzed in detail in section four. Let us assume for the time being that these integrals have been evaluated. We only need at this point the following properties (see (4.72))

\[
I_{2n}(\alpha, \beta, \rho) = \tilde{I}_{2n}(\alpha, \beta, \rho) \\
\alpha \tilde{I}^{(\Sigma)}_{2n+1}(\alpha, \beta, \rho; \eta) = \eta \tilde{I}_{2n+1}(\alpha, \beta, \rho) = \eta \tilde{I}_{2n+1}(\beta, \alpha, \rho) \]
\[\tag{3.23}\]

These relations are essential in simplifying (3.22). We use (3.23) in (3.22ii,iv) to
absorb the constant $c$. For $a_2$ we make use of the charge screening condition for a four-point function with $m - 1$ screenings:

$$\sum_{i=1}^{4} \alpha_i + (m - 1)\alpha_+ = \alpha_0 = \alpha_+ + \alpha_-$$  \hspace{1cm} (3.24)$$

then

$$a_2 = -\alpha_4\alpha_+ = -1 - a - b - c - \rho(m - 2)$$  \hspace{1cm} (3.25)$$

Hence (3.22) reduces to a nice form:

$$N^{(m)}_{k}(a, b, c, \rho) = (-1)^{m-1} \hat{I}_{m-k}(-1 - a - b - c - \rho(m - 2), b, \rho) \hat{I}_{k-1}(a, c, \rho)$$  \hspace{1cm} (3.26)$$

valid for all values of $m$ and $k$.

Summarizing, we find that the ordered integral representation of the thermal superconformal blocks can be written as

$$I^{(m)}_{k}(a, b, c, \rho; Z) = N^{(m)}_{k}(a, b, c, \rho)\mathcal{F}^{(m)}_{k}(a, b, c, \rho; Z)$$  \hspace{1cm} (3.27)$$

where $\mathcal{F}^{(m)}_{k}(Z)$ are the normalized thermal superconformal blocks

$$\mathcal{F}^{(m)}_{k}(a, b, c, \rho; Z) = z^{\Delta^{(m)}_{k}} f^{(m)}_{k}(a, b, c, \rho; Z)$$  \hspace{1cm} (3.28)$$

with the regular functions $f^{(m)}_{k}(Z)$ having the expansions:

$$i) \quad f^{(m)}_{k} = 1 + \eta\theta(\cdots) + \ldots$$

$$ii) \quad f^{(m)}_{k} = \eta\theta + z(\cdots) + \ldots$$

$$iii) \quad f^{(m)}_{k} = \eta + \theta(\cdots) + \ldots$$

$$iv) \quad f^{(m)}_{k} = \theta + z\eta(\cdots) + \ldots$$  \hspace{1cm} (3.29)$$

and $\Delta^{(m)}_{k}$ can be read off from (3.21). Finally the correlation function of four
thermal vertex operators (2.50) can be written as

\[ G(Z, \bar{Z}) \sim \sum_{1}^{m} S_{k}^{(m)}(a, b, c, \rho) \left| \mathcal{F}_{k}^{(m)}(a, b, c, \rho; Z) \right|^{2} \]  

(3.30)

where the quantities

\[ S_{k}^{(m)}(a, b, c, \rho) = X_{k}(a, b, c, \rho) \left( N_{k}^{(m)}(a, b, c, \rho) \right)^{2} \]  

(3.31)

contain all the information about the structure constants of the operator algebra. These will be evaluated in section six after the normalization integrals are computed.

3.3. GENERAL SUPERCONFORMAL BLOCKS AND MONODROMY INVARIANTS

We now extend the analysis of four-point functions to the full algebra of \((m', m)\) fields. Both +, − screening charges will be involved although the same program can be carried out to the end without major differences. The structure of the four-point function is identical to the thermal case (3.5) where \(M\) (resp. \(N\)) denotes the total number of + and − screenings in the four-point (resp. three-point) function.

The integral representation for a chiral superconformal block is

\[
J_{lk}^{(nm)}(a, b, c, \rho; Z) = T \int_{1}^{n-l} dU_{1} \int_{0}^{Z} \prod_{i=1}^{n-l+1} dU_{i} \int_{1}^{\infty} dV_{1} \int_{0}^{Z} \prod_{i=1}^{m-k+1} dV_{i} \nonumber \\
(1 + a_{1} \eta \theta + a_{2} \eta \sum_{i=1}^{n-1} \theta_{i} + a_{3} \eta \sum_{i=1}^{m-1} \omega_{i}) \prod_{i,j}^{n-1,m-1} (u_{i} - v_{j} - \theta_{i} \omega_{j})^{-1} \nonumber \\
\prod_{i=1}^{n-1} u_{i}^{\rho} \prod_{i=1}^{n-l} (u_{i} - 1)^{b} (u_{i} - z - \theta_{i} \theta)^{c} \prod_{i=1}^{n-1} (1 - u_{i})^{b} (z - u_{i} - \theta_{i} \theta)^{c} \prod_{i<j}^{n-1,m-1} U_{ij}^{\rho} \nonumber \\
\prod_{i=1}^{m-1} v_{i}^{\rho} \prod_{i=1}^{m-k} (v_{i} - 1)^{b} (v_{i} - z - \omega_{i} \theta)^{c} \prod_{i=1}^{m-1} (1 - v_{i})^{b} (z - v_{i} - \theta_{i} \omega_{i})^{c} \prod_{i<j}^{m-1} V_{ij}^{\rho} \]  

(3.32)
where \( U_i = (u_i, \theta_i) \), \( V_i = (v_i, \omega_i) \) and

\[
\begin{align*}
a &= \alpha_1 \alpha_+ \quad a' = \alpha_1 \alpha_- = -\rho' a \\
b &= \alpha_3 \alpha_+ \\b' &= \alpha_3 \alpha_- = -\rho' b \\
c &= \alpha_2 \alpha_+ \\c' &= \alpha_2 \alpha_- = -\rho' c \\
\rho &= \alpha_+^2 \\\rho' &= \alpha_-^2 = \rho^{-1}
\end{align*}
\] (3.33)

The ordering prescription \( T \) is explained in sections 4.1, 5.1 and the contours in (3.32) are ordered according to fig. 3.4. One can also define the superconformal blocks \( \tilde{J}_{ik}^{(nm)}(a, b, c, \rho; Z) \) by a straightforward extension of (3.13). The relation with the ordered integrals

\[
I_{ik}^{(nm)}(a, b, c, \rho; Z) = \int dU_1 \int dU_2 \ldots \int dU_{n-1} \int dU_{n-1} \int dU_{n-2} \ldots \int dU_{n} \int V_{m-2} \ldots \int dV_{m-1} \{ \text{as in (3.32)} \}
\] (3.34)

is

\[
J_{ik}^{(nm)}(a, b, c, \rho; Z) = \lambda_{l-1}(\rho') \lambda_{n-1}(\rho') \lambda_{k-1}(\rho) \lambda_{m-k}(\rho) I_{ik}^{(nm)}(a, b, c, \rho; Z)
\] (3.35)

Notice that the form of the coupling terms \( (u_i - v_j - \theta_i \omega_j)^{-1} \) and the relations (3.33) imply that we can permute the \( C \) and \( S \) contours without affecting the value of the integral. This also implies that the monodromy properties of \( J_{ik}^{(nm)}(a, b, c, \rho; Z) \) can be derived from those of the thermal case, and indeed, we have for the \( \beta \) matrices:

\[
\beta_{(ik)(rs)}(a, b, c, \rho) = \beta_{lr}(a', b', c', \rho') \beta_{ks}(a, b, c)
\] (3.36)

This in turn implies that the coefficients \( X_{ks} \) in the decomposition of the four-point
function:

\[ G(Z, \bar{Z}) = \sum_{k,s=1}^{n,m} X_{ks} I_{ks}(Z) \overline{I_{ks}(\bar{Z})} \]  \hspace{1cm} (3.37)

also split in product of the thermal results \(3.16\)

\[ X_{ks} = X_k(a', b', c', \rho') X_s(a, b, c, \rho) \]  \hspace{1cm} (3.38)

As in the conformal case, the full algebra results are not just products of thermal quantities because of the normalization factors appearing in the superconformal blocks \(3.34\). The procedure for computing the factors \(N_{lk}^{(nm)}(a, b, c, \rho)\) is identical to the thermal case although a bit more cumbersome. We only state some of the necessary steps. After a simple change of variables for the \(0 \rightarrow Z\) integrals and expanding the terms containing either \(\theta\) or \(z^{1/2}\) we end up with a cumbersome expression to evaluate:

\[
I_{lk}^{(nm)}(Z) = z^{\Delta_{lk}} \int_1^{n-l} \prod u_i \int_1^{l-1} \prod t_i \int_1^{m-k} \prod v_i \int_1^{k-1} \prod s_i \hspace{1cm} (3.39)
\]

\[
(1 + a_1 \eta \theta + a_2 \eta \sum_{i=1}^{n-l} \theta_i + a_2 z^{1/2} \eta \sum_{i=1}^{l-1} \zeta_i + a_3 \eta \sum_{i=1}^{m-k} \omega_i + a_3 z^{1/2} \eta \sum_{i=1}^{k-1} \nu_i) \\
(1 - c' \sum_{u_i-z}(1 - \frac{c'}{z^{1/2}} \sum_{1}^{n-l} \frac{\theta_i \zeta_i}{1-t_i} + \frac{\theta_i \zeta_i}{u_i-z t_j})(1 + z^{1/2} \sum_{u_i-z s_j} \frac{\theta_i v_j}{u_i-z s_j}) \\
(1 - c \sum_{v_i-z}(1 - \frac{c}{z^{1/2}} \sum_{1}^{l-1} \frac{\theta_i \zeta_i}{1-s_i} + \frac{\theta_i \zeta_i}{v_i-z s_j})(1 + z^{1/2} \sum_{v_i-z s_j} \frac{\omega_i \nu_j}{v_i-z s_j}) \\
\prod_{1}^{n-l} u_i^{c'}(u_i-1)^{b'}(u_i-z)^c \prod_{1}^{l-1} U_{ij}^{b'} \prod_{1}^{l-1} t_i^{c'}(1-z t_i)^{b'}(1-t_i)^c \prod_{1}^{l-1} T_{ij}^{b'} \\
\prod_{1}^{m-k} v_i^{c'}(v_i-1)^{b'}(v_i-z)^c \prod_{1}^{k-1} V_{ij}^{b'} \prod_{1}^{k-1} s_i^{c'}(1-z s_i)^{b'}(1-s_i)^c \prod_{1}^{k-1} S_{ij}^{b'} \\
\prod_{1}^{n-l} (u_i-z t_j)^{d'} \prod_{1}^{n-l} (v_i-z s_j)^{d'} \prod_{1}^{n-l} (UV)_{ij}^{d} \prod_{1}^{n-l} (TS)_{ij}^{d} \prod_{1}^{n-l} (u_i-z s_j)^{-1} \prod_{1}^{n-l} (z t_i-v_j)^{-1} 
\]

30
with \( T_i = (t_i, \zeta_i) \), \( S_i = (s_i, \nu_i) \) and

\[
\Delta_{lk} = (l-1)(\frac{1}{2}+a'+c' + \frac{\rho'}{2} (l-2))+(k-1)(\frac{1}{2}+a+c+\frac{\rho}{2} (k-2)) - (l-1)(k-1) \tag{3.40}
\]

Depending on the values of \( n, m, l, k \) we may have to compute integrals of the types given in sections 5.1, 5.2. The normalization constants are

\( (n + m) \) even :

\( i) \) \((l + k)\) even : \( N^{nm}_{lk} = I_{n-l,m-k}(-a - b - c - \rho (m - 2) + n - 2, b, \rho) \)

\[ I_{l-1,k-1}(a, c, \rho) \]

\( ii) \) \((l + k)\) odd : \( N^{nm}_{lk} \eta \theta = -a_2 c' I_{n-l,m-k}(-a - b - c - \rho (m - 2) + n - 2, b, \rho; \eta) \)

\[ \tilde{I}_{l-1,k-1}(c, a, \rho; \theta) \tag{3.41} \]

\( (n + m) \) odd :

\( iii) \) \((l + k)\) even : \( N^{nm}_{lk} \eta = -a_2 c' I_{n-l,m-k}(-a - b - c - \rho (m - 2) + n - 2, b, \rho; \eta) \)

\[ I_{l-1,k-1}(a, c, \rho) \]

\( iv) \) \((l + k)\) odd : \( N^{nm}_{lk} \theta = c' I_{n-l,m-k}(-a - b - c - \rho (m - 2) + n - 2, b, \rho) \)

\[ \tilde{I}_{l-1,k-1}(c, a, \rho; \theta) \]

Using (3.33) the charge screening condition for a four-point function with \((n - 1) - \) screenings and \((m - 1) + \) screenings and the definition (5.39) of \( \tilde{I}(\alpha, \beta, \rho) \) we finally get a unified expression for the normalization constants

\[ N^{nm}_{lk}(a, b, c, \rho) = \hat{I}_{n-l,m-k}(-a - b - c - \rho (m - 2) + n - 2, b, \rho) \hat{I}_{l-1,k-1}(a, c, \rho) \tag{3.42} \]

Thus we can define the normalized superconformal blocks \( \mathcal{F}^{(nm)}_{lk}(Z) \) as

\[ I^{(nm)}_{lk}(a, b, c, \rho; Z) = N^{(nm)}_{lk}(a, b, c, \rho) \mathcal{F}^{(nm)}_{lk}(a, b, c, \rho; Z) \tag{3.43} \]

where

\[ \mathcal{F}^{(nm)}_{lk}(a, b, c, \rho; Z) = z^{\Delta^{(nm)}_{lk}} f^{(nm)}_{lk}(a, b, c, \rho; Z) \tag{3.44} \]

\( f^{(nm)}_{lk} \) has an expansion similar to (3.29) and \( \Delta^{(nm)}_{lk} \) is defined by (3.40) with a \(-1/2\) correction depending on the values of \( l, k, n, m \).
Finally the general correlation function of four vertex operators can be written as

\[ G(Z, \bar{Z}) \sim \sum_{k,l} S_{lk}^{(nm)}(a, b, c, \rho) |\mathcal{F}_{lk}^{(nm)}(a, b, c, \rho; Z)|^2 \]  

(3.45)

where

\[ S_{lk}^{(nm)}(a, b, c, \rho) = X_l(a', b', c', \rho') X_k(a, b, c, \rho) \left( N_{lk}^{(nm)}(a, b, c, \rho) \right)^2 \]  

(3.46)

encompasses the full information on the structure constants of the operator algebra.
4. NORMALIZATION INTEGRALS IN THE THERMAL SERIES

4.1. CONTOUR ORDERING

From the analysis of conformal blocks in the previous section we are left with the problem of evaluating a set of normalization integrals before we can explicitly write down the structure constants of the operator algebra. Since the evaluation of these integrals is involved we present the method first in detail for the correlators in the thermal series. The extension of the method to the general case presents some subtleties which require the explicit result for the thermal series. This extension will be the subject of the next section. As previously mentioned, the type of analysis we pursue here is tailored after the treatment of the conformal case in [3,4,5]. There are specific complications to the superconformal case as the reader familiar with these references will see in this and the next section.

Two types of integrals can be distinguished depending on the number of super-contours. We may have an even or an odd number of contours. In the even case the integrals involved are of the form

\[
J(p, q) = T_0 \int \prod_{i=1}^{\infty} dT_i t_i^\alpha (t_i - 1)^\beta \prod_{i>j} (t_i - t_j - \theta_i \theta_j)^\rho \\
\int \prod_{a=1}^{p} dU_a u_a^\alpha (1 - u_a)^\beta \prod_{a<b} (u_a - u_b - \omega_a \omega_b)^\rho \\
\prod_{i,a} (t_i - u_a - \theta_i \omega_a)^\rho \quad (p + q \text{ even})
\]  

(4.1)

where \( T_i = (t_i, \theta_i), U_a = (u_a, \omega_a) \) and \( T_0 \) is the contour ordering prescription. The contours in (4.1) are chosen with the ordering given in fig.4.1. For odd integrals...
there are two subcases to consider

\[
J^k(n - p, p) = T_0 \int_1^{\infty} \prod_{n-p+1}^n dT_i t_i^\alpha (t_i - 1)^\beta \\
\int_0^{1-n-p} \prod_{1}^{n-p} dT_i t_i^\alpha (1 - t_i)^\beta \eta \theta_k \prod_{i > j} (t_i - t_j - \theta_i \theta_j)^p
\]  

(4.2)

and

\[
\tilde{J}^k(n - p, p) = T_0 \int_1^{\infty} \prod_{n-p+1}^n dT_i t_i^\alpha (t_i - 1)^\beta \\
\int_0^{1-n-p} \prod_{1}^{n-p} dT_i t_i^\alpha 1 - (t_i)^\beta \eta \frac{\theta_k}{t_k} \prod_{i > j} (t_i - t_j - \theta_i \theta_j)^p
\]  

(4.3)

The normalization integrals are only

\[
J^\Sigma(n - p, p) = \sum_{k=1}^{n} J^k(n - p, p)
\]  

(4.4)

\[
\tilde{J}^\Sigma(n - p, p) = \sum_{k=1}^{n} \tilde{J}^k(n - p, p)
\]

The contours in (4.2,3) are ordered as in fig. 4.2. The evaluation of (4.2,3) is harder than (4.4). Fortunately only the latter is needed in our computations. The ordering prescription is best illustrated with fig. 4.3 and the integral

\[
J_m = T_0 \int_0^{X} \prod_{1}^{m} dT_i \prod_{i < j} (t_i - t_j - \theta_i \theta_j)^p
\]  

(4.5)
$T_0$ orders the $d\theta_i$’s as well as the integrand

$$T_0 \left\{ \prod_{i=1}^{m} d\theta_i \frac{m}{i<j} (t_i - t_j - \theta_i \theta_j)^\rho \right\} = d\theta_1 d\theta_2 \ldots d\theta_m \prod_{i<j} (t_i - t_j - \theta_i \theta_j)^\rho$$

for $t_1 > t_2 > \ldots > t_m$

$$= -d\theta_2 d\theta_3 \ldots d\theta_m (t_2 - t_1 - \theta_2 \theta_1)^\rho e^{-i\pi \rho} \ldots$$

for $t_2 > t_1 > t_3 > \ldots > t_m$ \hfill (4.6)

According to this definition no residual phase is encountered in the first case in (4.6) and when we braid (analytically continue) $t_i$ around $t_j$ for $i > j$ along the curves in fig. 4.3 from the region $t_i < t_j$ to the region $t_i > t_j$ we pick up a phase $e^{-i\pi (\rho - 1)}$. When there are two sets of contours $(t_i, \theta_i), i = 1, \ldots, m; (u_a, \omega_a), a = 1, \ldots, n$ with $u_a > t_i$ the odd differentials are ordered as $d\omega_1 \ldots d\omega_n d\theta_1 \ldots d\theta_m$ for $u_1 > u_2 > \ldots > u_n > t_1 > \ldots > t_m$. The Grassmann variable with the greatest real variable comes first on the left. With this choice we can define an ordered integral $I_m$:

$$I_m = \int_0^X \int_0^{T_1} \int_0^{T_2} \ldots \int_0^{T_m-1} \prod_{i<j} (t_i - t_j - \theta_i \theta_j)^\rho \quad (4.7)$$

and a simple argument analogous to the manipulation of time orderings in field theory relates $J_m$ to $I_m$:

$$J_m(\rho) = \lambda_m(\rho) \epsilon_m(\rho)^{-1} I_m(\rho)$$

$$\lambda_m(\rho) = \prod_{1}^{m} \frac{s(i\rho^{-1})}{s(i\rho^{-1})} \quad \epsilon_m(\rho) = \prod_{0}^{m-1} e^{i\pi \frac{\rho + 1}{2} k} \quad s(x) = \sin(\pi x) \quad (4.8)$$

In the general case with both types of screenings we will simply get a factor $\lambda_m(\rho) \lambda_m'(\rho') \epsilon_m(\rho)^{-1} \epsilon_m'(\rho')^{-1}$. To make the formulas as simple as possible, in some places we use another ordering prescription $T$ which differs from $T_0$ by the presence of a residual phase $\epsilon_m$. Integrals with this ordering are related to $I_m$ as in (4.8) but without the $\epsilon_m$’s.
4.2. EVEN INTEGRALS

We begin by deriving a set of relations satisfied by (4.1). In the region of \((\alpha, \beta, \rho)\) making the integral converge we pull the top contour labelled by \(p\) from \(0 \to 1\) through the upper half plane. This leads to a relation between \(J(p, q)\) and \(J(p - 1, q + 1)\) (the part of the deformed contour from 1 to \(\infty\)) and an integral \(J^{(0)}(p - 1, q)\) with one contour running from 0 → \(-\infty\). Similarly we can pull the bottom contour through the lower half plane to obtain another relation between these three integrals. Being careful with phases and ordering prescriptions, the top contour yields

\[
J(p, q) = e^{i\pi \alpha} J^{(0)}(p - 1, q) - e^{-i\pi (\rho - 1)(p - 1)} e^{i\pi \beta} J(p - 1, q + 1) \tag{4.9a}
\]

while pulling the bottom contour yields

\[
J(p, q) = e^{-i\pi (\rho - 1)(p - 1) - i\pi \alpha} J^{(0)}(p - 1, q) - e^{i\pi \beta + i\pi (\rho - 1)q} J(p - 1, q + 1) \tag{4.9b}
\]

Eliminating \(J^{(0)}(p - 1, q)\) implies

\[
J(p, q) = -e^{i\pi \frac{\rho - 1}{2}(q - p + 1)} \frac{s(\alpha + \beta + (\rho - 1)(p - 1 + q/2))}{s(\alpha + \frac{\rho - 1}{2}(p - 1))} J(p - 1, q + 1) \tag{4.10}
\]

Iterating this relation we can pull all contours from the region \(0 \to 1\) into \(1 \to \infty\) contours:

\[
J(n, 0) = (-1)^n \prod_{j=0}^{n-1} e^{i\pi \frac{\rho - 1}{2}(2j - n + 1)} \frac{s(\alpha + \beta + (\rho - 1)(n - 1 - j/2))}{s(\alpha + \frac{\rho - 1}{2}(n - 1 - j))} J(0, n)
\]

\[
= \prod_{j=0}^{n-1} \frac{s(\alpha + \beta + (\rho - 1)(n - 1 - j/2))}{s(\alpha + \frac{\rho - 1}{2}j)} J(0, n) \quad n \text{ even} \tag{4.11}
\]

Since \(\sum_{0}^{n-1}(2j - (n - 1)) = 0\), the phase in (4.11) reduces to \((-1)^n = 1\) because \(n\) is even. We can now change from \(J(0, n)\) to \(J(n, 0)\) by a split superconformal
change of variables

\[ U_i = (u_i, \omega_i) \quad u_i = f(t_i) \quad \omega_i = \theta_i \sqrt{\partial_i f} \quad D_i \omega_i = \sqrt{\partial_i f} \]  

(4.12)

with

\[ u_i = \frac{1}{t_i} \quad \omega_i = \frac{\sqrt{-1}}{t_i} \theta_i \quad D_i \omega_i = \frac{\sqrt{-1}}{t_i} \]  

(4.13)

\[ J(0, n)(\alpha, \beta, \rho) = T_0 \int_1^{\infty} \prod_{1}^{n} dT_i t_i^\alpha (t_i - 1)^\beta \prod_{i>j} (t_i - t_j - \theta_i \theta_j)^\rho \]

\[ = \int_{1}^{\infty} dT_n dT_{n-1} \ldots dT_1 (\ldots) \]

\[ = J(n, 0)(-1 - \alpha - \beta - \rho(n - 1), \beta, \rho) \]

leading to

\[ J(n, 0)(\alpha, \beta, \rho) = \prod_{j=0}^{n-1} \frac{s(\alpha + \beta + (\rho - 1)(n - 1 - j/2))}{s(\alpha + \frac{\rho-1}{2}(n - j - 1))} \frac{J(n, 0)(-1 - \alpha - \beta - \rho(n - 1), \beta, \rho)}{J(n, 0)(\beta, \alpha, \rho)} \]  

(4.15)

Furthermore, using the change of variables in (4.1)

\[ u_i = 1 - t_i \quad \omega_i = \sqrt{-1} \theta_i \]  

(4.16)

we obtain a useful symmetry

\[ J(n, 0)(\alpha, \beta, \rho) = J(n, 0)(\beta, \alpha, \rho) \]  

(4.17)

In deriving these formulas we have to be careful in keeping track of the signs originating in the exchange of the \(d\theta\)'s necessary to bring them to the correct ordering. We should also recall the remarks at the end of section two about open versus closed contour integrals. As in [4] (4.10) and (4.11) can be thought of as
analytic continuations of the function \( J(n, 0)(\alpha, \beta) \) to the complex \( \alpha \)-plane. Since the integrals we are dealing with have milder singularities than those treated in [4] we need not repeat their arguments here. The next step consists in determining the behaviour of \( J(n, 0)(\alpha, \beta) \) as \( \alpha \to \infty \). This is achieved using a split superconformal transformation in (4.1) with

\[
 u_i = e^{-t_i/\alpha} \quad D_i \theta_i = \sqrt{-\alpha} e^{-t_i/2\alpha} \quad (4.18)
\]

Keeping track of the phases carefully we arrive at

\[
 J(n, 0)(\alpha, \beta, \rho) \sim \alpha^{-n/2-n\beta-\rho n(n-1)/2} (C_0 + C_1 \alpha^{-1} + \ldots) \quad n \text{ even} \quad (4.19)
\]

With a little extra work we find the asymptotic behaviour of odd integrals like

\[
 J^k(n, 0) = T_0 \int_0^1 \prod_{i=1}^n dT_i t_i^\alpha (1 - t_i)\beta \theta_k \prod_{i>j}(t_i - t_j - \theta_i \theta_j)^\rho \quad n \text{ odd} \quad (4.20)
\]

and \( \tilde{J}^k(n, 0) \) (replace in (4.20) \( \theta_k \to \theta_k/t_k \)). The answer is

\[
 J^k(n, 0) \sim \tilde{J}^k(n, 0) \sim \alpha^{-(n+1)/2-n\beta-\rho n(n-1)/2} (C'_0 + C'_1 \alpha^{-1} + \ldots) \quad n \text{ odd} \quad (4.21)
\]

Using (4.15),(4.17),(4.19) we can write an Ansatz for \( J_n(\alpha, \beta, \rho) = J(n, 0)(\alpha, \beta, \rho) \):

\[
 J_n(\alpha, \beta, \rho) = \prod_{j=0}^{n-1} \frac{\Gamma(1 + \alpha + \frac{\rho-1}{2} j + M_j) \Gamma(1 + \beta + \frac{\rho-1}{2} j + M_j)}{\Gamma(1 + \alpha + \beta + \frac{\rho-1}{2} (n - 1 + j) + N_j)} \mu_n(\alpha, \beta, \rho) \quad (4.22)
\]

From the reflection condition (4.14) we obtain a relation between the integers \( M \) and \( N \):

\[
 N_{n-1-j} + M_j = n - 1 \quad (4.23)
\]

The function \( \mu_n(\alpha, \beta, \rho) \) is symmetric under the exchange of \( \alpha \) and \( \beta \) and it satisfies \( \mu_n(\alpha, \beta, \rho) = \mu_n(-1 - \alpha - \beta - \rho(n-1), \beta, \rho) \). We can obtain more constraints by
matching the large $\alpha$ behaviour (4.19):

$$\sum_{p=0}^{n-1} M_p = \frac{n(n-2)}{4} \quad n \text{ even}$$  \hspace{1cm} (4.24)

Since $\mu(\alpha, \beta, \rho)$ is analytic in $\alpha$ and behaves as a constant in the large $\alpha$ limit, we conclude that $\mu$ is only a function of $\rho$. We can obtain more information about the $M_j$’s if we require them not to depend on $n$. Since the case $n = 2$ can be computed explicitly, $M_0 = M_1 = 0$, if we subtract (4.24) for $n$ and $n + 2$ we obtain the relation

$$M_{2k} + M_{2k+1} = 2k$$  \hspace{1cm} (4.25)

If $[x]$ stands for the integer part of $x$, one easily checks that $M_j = [j/2]$ solves (4.24) and (4.25). As we shall see later this turns out to be the correct answer. To summarize this subsection we have learned that for even $n$

$$J_n(\alpha, \beta, \rho) = \prod_{j=0}^{n-1} \frac{\Gamma(1 + \alpha + \frac{\rho-1}{2} j + M_j) \Gamma(1 + \beta + \frac{\rho-1}{2} j + M_j)}{\Gamma(1 + \alpha + \beta + \frac{\rho-1}{2}(n-1+j) + n-1 - M_j)} \mu_n(\rho)$$  \hspace{1cm} (4.26)

Next we turn to the odd case.

4.3. ODD NUMBER OF CONTOURS

In deriving relations out of pulling contours in the odd case it will be convenient to write one more entry in the arguments of (4.2,3). We write

$$J^k(0, n-p, p) = J^k(n-p, p)$$

The first entry keeps track of contours running from $0 \to -\infty$. The ordering of contours is shown in fig. 4.2. In this case we have to extend the work of [4] to
reach definite answers. The distinguished contour \( k \) in (4.27) (fig. 4.2) may be any of the \( n \) contours. In pulling the top and bottom contours we have to be careful with the position of \( k \). We can distinguish five cases:

1) \( 1 < k < n - p \)

1a) Open the top contour between \( 0 \rightarrow 1 \)

\[
J^k(-)(0, n - p, p) = e^{i\pi \alpha} J^{k(-)}(1, n - p - 1, p) - e^{-i\pi(n-p-1)-i\pi \beta} J^{k(-)}(0, n - p - 1, p + 1)
\] (4.28a)

1b) Open the bottom contour between \( 0 \rightarrow 1 \)

\[
J^k(-)(0, n - p, p) = e^{-i\pi -i\pi (n-p-1)} J^{k-1(-)}(1, n - p - 1, p) - e^{i\pi \beta+i\pi(n-1)p} J^{k-1(-)}(0, n - p - 1, p + 1)
\] (4.28b)

These formulae look cumbersome, but the meaning is simple. By \( J^k(-) \) we mean that the distinguished variable \( \theta_k \) corresponds to the odd integration variable of the \( k \)-th contour from \( 0 \rightarrow 1 \) in fig. 4.2 counting from the bottom. This is why on the right-hand side of (4.28b ) the label of \( J \) has become \( k - 1(-) \). By pulling the bottom contour we have changed the numbering. The arguments \( (1, n - p - 1, p) \) mean that we have one contour running from \( 0 \rightarrow -\infty \), \( n - p - 1 \) from \( 0 \rightarrow 1 \) and \( p \) from \( 1 \rightarrow \infty \). Similarly, if \( \theta_k \) is related to a contour from \( 1 \rightarrow \infty \) we label the corresponding integral by \( J^{k(+)} \).

Notice that the phases in the contour deformations do not depend on \( k \). The same happens in all subsequent cases.

2) \( n - p + 1 < k \leq n \)

2a) Open the top contour between \( 0 \rightarrow 1 \)

\[
J^{k(+)}(0, n - p, p) = e^{i\pi \alpha} J^{k-1(+)}(1, n - p - 1, p) - e^{-i\pi(n-p-1)-i\pi \beta} J^{k+}(0, n - p - 1, p + 1)
\] (4.29a)
2b) Open the bottom contour between 0 \rightarrow 1
\[ J^{k(+)}(0, n - p, p) = e^{-i\pi\alpha - i\pi(\rho - 1)(n - p - 1)}J^{k-1(+)}(1, n - p - 1, p) \]
\[ - e^{i\pi\beta + i\pi(\rho - 1)p}J^{k-1(+)}(0, n - p - 1, p + 1) \] (4.29b)

3) \( k = 1 \)

3a) Open the top contour
\[ J^{1(-)}(0, n - p, p) = e^{i\pi\alpha}J^{1(-)}(1, n - p - 1, p) \]
\[ - e^{-i\pi(\rho - 1)(n - p - 1) - i\pi\beta}J^{1(-)}(0, n - p - 1, p + 1) \] (4.30a)

3b) Open the bottom contour
\[ J^{k(-)}(0, n - p, p) = e^{-i\pi\alpha - i\pi(\rho - 1)(n - p - 1)}J^{(-1)}(1, n - p - 1, p) \]
\[ - e^{i\pi\beta + i\pi(\rho - 1)p}J^{n(+)}(0, n - p - 1, p + 1) \] (4.30b)

4) \( k = n - p \)

4a) Open the top contour
\[ J^{n-p(-)}(0, n - p, p) = e^{i\pi\alpha}J^{(-1)}(1, n - p - 1, p) \]
\[ - e^{-i\pi(\rho - 1)(n - p - 1) - i\pi\beta}J^{n-p(+)}(0, n - p - 1, p + 1) \] (4.31a)

4b) Open the bottom contour
\[ J^{n-p(-)}(0, n - p, p) = e^{-i\pi\alpha - i\pi(\rho - 1)(n - p - 1)}J^{n-p(-)}(1, n - p - 1, p) \]
\[ - e^{i\pi\beta + i\pi(\rho - 1)p}J^{n-p(-)}(0, n - p - 1, p + 1) \] (4.31b)

5) \( k = n - p + 1 \)

5a) Open the top contour
\[ J^{n-p+1(+)}(0, n - p, p) = e^{i\pi\alpha}J^{n-p(+)}(1, n - p - 1, p) \]
\[ - e^{-i\pi(\rho - 1)(n - p - 1) - i\pi\beta}J^{n-p+1(+)}(0, n - p - 1, p + 1) \] (4.32a)

5b) Open the bottom contour
\[ J^{n-p-1(+)}(0, n - p, p) = e^{-i\pi\alpha - i\pi(\rho - 1)(n - p - 1)}J^{n-p(+)}(1, n - p - 1, p) \]
\[ - e^{i\pi\beta + i\pi(\rho - 1)p}J^{n-p(+)}(0, n - p - 1, p + 1) \] (4.32b)
Multiplying all the bottom contours relations by $e^{i\pi\alpha+i\pi(\rho-1)(n-p-1)}$ and all the top contour by $e^{-i\pi\alpha}$ and then adding the top and subtracting the bottom contours we arrive at

$$J^\Sigma(0, n-p, p) = -e^{-i\pi\frac{\rho-1}{2}(n-1-2p)} \frac{s(\alpha + \beta + (\rho - 1)(n - 1 - p/2))}{s(\alpha + \frac{\rho-1}{2}(n - 1 - p))} J^\Sigma(0, n-p-1, p+1)$$

(4.33)

Iterating (4.33) we obtain

$$J^\Sigma(0, n, 0) = (-1)^{p-1} \prod_{j=0}^{n-1} \frac{s(\alpha + \beta + (\rho - 1)(n - 1 - j/2))}{s(\alpha + \frac{\rho-1}{2}j)} \ J^\Sigma(0, 0, n)$$

(4.34)

The same argument can be carried out with $\tilde{J}^\Sigma$. The only differences are sign changes in the $\tilde{J}^{(-1)}$ term in pulling the top contour in 4a) and in pulling the bottom contour in 3b). However, in the sum the same cancellation takes place and we end up with

$$\tilde{J}^\Sigma(0, n, 0) = (-1)^{p-1} \prod_{j=0}^{n-1} \frac{s(\alpha + \beta + (\rho - 1)(n - 1 - j/2))}{s(\alpha + \frac{\rho-1}{2}j)} \ \tilde{J}^\Sigma(0, 0, n)$$

(4.35)

If we next attempt to relate $J^\Sigma(0,0,n)$ with $\tilde{J}^\Sigma(0,0,n)$ by performing the superconformal change (4.13) we find a surprise:

$$J^\Sigma(0,0,n)(\alpha,\beta,\rho) = \tilde{J}^\Sigma(0,0,n)(-1 - \alpha - \beta - \rho(n - 1),\beta,\rho)$$

$$\tilde{J}^\Sigma(0,0,n)(\alpha,\beta,\rho) = J^\Sigma(0,0,n)(-1 - \alpha - \beta - \rho(n - 1),\beta,\rho)$$

(4.36)

Hence we need to do some extra work before we can write an Ansatz for the odd integrals.
A second important difference arises in the symmetry with respect to the exchange of $\alpha$ and $\beta$. For $J^\Sigma$ the change of variables (4.16) produces

$$J^\Sigma(0, n, 0)(\alpha, \beta, \rho) = J^\Sigma(0, n, 0)(\beta, \alpha, \rho)$$  \hspace{1cm} (4.37)

while for $\widetilde{J}^\Sigma$ the correct symmetry is

$$\alpha \widetilde{J}^\Sigma(0, n, 0)(\alpha, \beta, \rho) = \beta \widetilde{J}^\Sigma(0, n, 0)(\beta, \alpha, \rho)$$  \hspace{1cm} (4.38)

To prove the last equation, write

$$\widetilde{J}^k(\alpha, \beta) = T_0 \int_0^1 \prod_i dU_i u_i^\alpha(1 - u_i)^\beta \eta \omega_k \prod_{i<j} U^\rho_{ij}$$  \hspace{1cm} (4.39)

$$\widetilde{J}^k(\beta, \alpha) = T_0 \int_0^1 \prod_i dU_i u_i^\beta(1 - u_i)^\alpha \eta \omega_k \prod_{i<j} U^\rho_{ij}$$  \hspace{1cm} (4.40)

Implementing in the second integral the change of variables $u_i = 1 - t_i, \omega_i = \sqrt{-1} \theta_i$ and keeping track of integration variables ordering, we obtain

$$\alpha \widetilde{J}^k(\alpha, \beta) - \beta \widetilde{J}^k(\beta, \alpha) =$$

$$= T_0 \int_0^1 \prod_i dU_i u_i^\alpha(1 - u_i)^\beta \eta \theta_k \left( \frac{\alpha}{u_k} - \frac{\beta}{1 - u_k} \right) \prod_{i \neq j} u_i^\alpha(1 - u_i)^\beta \prod_{i<j} U^\rho_{ij}$$

$$= -T_0 \int_0^1 \prod_i dU_i u_i^\alpha(1 - u_i)^\beta \eta \theta_k \left( \frac{\partial}{\partial u_k} \right) \prod_{i<j} U^\rho_{ij}$$  \hspace{1cm} (4.41)

In the second step we have integrated by parts. The boundary terms vanish at 0 and 1. To see that (4.41) vanishes note that $Q_k = \theta_k \frac{\partial}{\partial u_k} - \frac{\partial}{\partial \theta_k}$ is the supersymmetry generator acting on $(u_k, \theta_k)$. After summing over $k$, $\sum Q_k$ annihilates $\prod_{i<j} U^\rho_{ij}$ because this function is invariant under global supersymmetry. Hence
(4.41) reduces to an integral of the form

$$-T_0 \int_0^1 \prod_i dU_i \sum_k \frac{\partial}{\partial q_k} \left( \prod_i u_i^\alpha (1 - u_i)^\beta \eta \prod_{i<j} U_{ij}^\rho \right)$$

(4.42)

which vanishes like all total derivatives in the odd directions. Finally, the large $\alpha$ behavior was already computed in (4.21).

The next step in our construction is to relate $J^\Sigma(n,0,0)(\alpha,\beta)$ to $J^\Sigma(0,0,n)(\alpha,\beta)$. This is achieved by considering the contour configuration shown in fig. 4.4. Start with $J^k(p,0,n-p)$. Pulling the top and bottom contours on the right and eliminating the term $J(p,1,n-p-1)$ we obtain

$$J^\Sigma(p,0,n-p) = -e^{-i\pi \frac{p-1}{2}(n-1-2p)} \frac{s(\alpha + \frac{p-1}{2})}{s(\beta + \frac{p-1}{2}(n-1-p))} J^\Sigma(p+1,0,n-p-1)$$

(4.43)

Iterating

$$J^\Sigma(0,0,n) = (-1)^n \prod_{j=0}^{n-1} \frac{s(\alpha + \frac{p-1}{2})}{s(\beta + \frac{p-1}{2})} J^\Sigma(n,0,0)$$

(4.44)

Now by the $SL(2|1)$ transformation $t_i \rightarrow 1 - t_i, \theta_i \rightarrow \sqrt{-1} \theta_i$ we can transform one integral into the other:

$$J^k(n,0,0)(\alpha,\beta) = -J^k(0,0,n)(\beta,\alpha)$$

(4.45)

which together with (4.44) yields

$$J^\Sigma(0,0,n)(\alpha,\beta) = \prod_{j=0}^{n-1} \frac{s(\alpha + \frac{p-1}{2})}{s(\beta + \frac{p-1}{2})} J^\Sigma(0,0,n)(\beta,\alpha)$$

(4.46)

Using (4.36):

$$\tilde{J}^\Sigma(0,n,0)(-1 - \alpha - \beta - \rho(n-1),\beta) = \prod_{j=0}^{n-1} \frac{s(\alpha + \frac{p-1}{2})}{s(\beta + \frac{p-1}{2})} \tilde{J}^\Sigma(0,n,0)(-1 - \alpha - \beta - \rho(n-1),\alpha)$$

(4.47)
Defining

\[ \gamma = -1 - \alpha - \beta - \rho(n - 1) \quad (4.48) \]

we obtain

\[ \tilde{J}_\Sigma(0, n, 0)(\gamma, \beta, \rho) = \prod_{j=0}^{n-1} \frac{s(\gamma + \beta + (\rho - 1)(n - 1 - j/2))}{s(\beta + \rho - 1/2 j)} \quad (4.49) \]

\[ \tilde{J}_\Sigma(0, n, 0)(\gamma, -1 - \gamma - \beta - \rho(n - 1), \rho) \]

This relation can be solved as in the even case

\[ \tilde{J}_n(\alpha, \beta, \rho) = \eta f_n(\alpha) \prod_{j=0}^{n-1} \frac{\Gamma(1 + \beta + \rho - 1/2 j + N_j)}{\Gamma(1 + \alpha + \beta + \rho - 1/2 (n - 1 + j) + n - 1 - N_j)} \mu_n(\rho) \quad (4.50) \]

Since \( \tilde{J}_1 \) is explicitly calculable, we know that \( f_1(\alpha) = \Gamma(\alpha) \). Imposing the symmetry relation \( \alpha \tilde{J}_n(\alpha, \beta) = \beta \tilde{J}_n(\beta, \alpha) \) leads to

\[ f_n(\alpha) = \prod_{j=0}^{n-1} \Gamma(1 + \alpha + \rho - 1/2 j + \tilde{N}_j) \quad (4.51) \]

with

\[ \tilde{N}_0 = -1, \quad N_0 = 0, \quad \tilde{N}_p = N_p, \quad p > 0 \quad (4.52) \]

Summarizing:

\[ \tilde{J}_n(\alpha, \beta, \rho) = \eta \prod_{j=0}^{n-1} \Gamma(1 + \alpha + \beta + \rho - 1/2 (n - 1 + j) + n - 1 - N_j) \mu_n(\rho) \quad (4.53) \]

The large \( \alpha \) behaviour yields

\[ \sum_{p=0}^{n-1} N_p = \frac{(n - 1)^2}{4} \quad n \text{ odd} \quad (4.54) \]
Assuming the $N_p$’s to be independent of $n$ and using $N_0 = 0$ would imply

$$N_{2k+1} + N_{2k+2} = 2k + 1 \quad (4.55)$$

Using (4.36) we finally obtain

$$J_n^\Sigma(\alpha, \beta, \rho) = \eta \prod_{j=0}^{n-1} \frac{\Gamma(1 + \alpha + \frac{\rho-1}{2}j + N_j) \Gamma(1 + \beta + \frac{\rho-1}{2}j + N_j)}{\Gamma(1 + \alpha + \beta + \frac{\rho-1}{2}(n-1+j) + n-1 - N_j)} \mu_n(\rho)$$

$$\tilde{J}_n^\Sigma(\alpha, \beta, \rho) = \eta \prod_{j=0}^{n-1} \frac{\Gamma(1 + \alpha + \frac{\rho-1}{2}j + \tilde{N}_j) \Gamma(1 + \beta + \frac{\rho-1}{2}j + N_j)}{\Gamma(1 + \alpha + \beta + \frac{\rho-1}{2}(n-1+j) + n-1 - N_j)} \mu_n(\rho) \quad (4.56)$$

$$N_0 = 0 \quad \tilde{N}_0 = -1 \quad \sum_{p=0}^{n-1} N_p = \frac{(n-1)^2}{4} \quad n \text{ odd}$$

To complete the computation we need to determine the integers $M_p, N_p$ and the functions $\mu_n(\rho)$ for both even and odd $n$. This we do by relating $J_{2m}$ to $J_{2m-1}^\Sigma$ and $J_{2m+1}^\Sigma$ to $J_{2m}$.

### 4.4. FROM $J_{2m}$ TO $J_{2m-1}^\Sigma$

Start with $J(0, 2m)$ and pull one contour into the $(0, 1)$ region as shown in fig. 4.5. Pulling the top and bottom contours yields

$$J(0, 2m) = -e^{-i\pi \frac{\rho-1}{2}(2m-1)} \frac{s(\alpha)}{s(\alpha + \beta + \frac{\rho-1}{2}(2m-1))} J(1, 2m - 1) \quad (4.57)$$

In the limit $\alpha = -1 + \epsilon$ as $\epsilon \to 0$, $s(\alpha)$ develops a zero. Since the left-hand side

46
does not vanish, the integral on the right-hand side should develop a pole

\[ J(1, 2m - 1) = T_0 \int_0^\infty \prod_{i=1}^{2m} dT_i \frac{t_i^\rho (t_i - 1)^\beta}{\prod_{i > j=2}^2 T_{ij}^{\rho}} \]

\( \int_0^1 dt_1 dt_1 t_1^\alpha (1 - t_1)^\beta \prod_{i=1}^{2m} (t_i - t_1 - \theta_i \theta_1)^\rho \prod_{i > j=2}^2 T_{ij}^{\rho} \)

\[ = T_0 \int_0^\infty \prod_{i=1}^{2m} dT_i \frac{t_i^\rho (t_i - 1)^\beta}{\prod_{i > j=2}^2 T_{ij}^{\rho}} \]

The leading divergence is obtained by expanding the integrand in powers of \( t_1/t_i \).

Multiplying by \( \eta \) and taking the limit as \( \epsilon \to 0 \) leads to

\[ \eta J(0, 2m)(-1, \beta, \rho) = \pi \rho e^{-i \pi \frac{\rho - 1}{2}(2m - 1)} \tilde{J}_\Sigma(0, 2m - 1)(-1 + \rho, \beta, \rho) \]

From the reflection formulae (4.14) and (4.36) we arrive at

\[ \eta J(2m, 0)(-\beta - \rho(2m - 1), \beta, \rho) = \pi \rho e^{-i \pi \frac{\rho - 1}{2}(2m - 1)} \]

\[ J_\Sigma(2m - 1, 0)(-\beta - \rho(2m - 2), \beta, \rho) \]

Substituting the product representations in (4.60), collecting all \( \beta \) dependence on one side, and dropping the \( \eta \) factor:

\[ \prod_{0}^{2m - 1} \frac{\Gamma(1 - \beta - \rho(2m - 1) + \frac{\rho - 1}{2} p + M_0) \Gamma(1 + \beta + \frac{\rho - 1}{2} p + M_0)}{\Gamma(1 - \beta - \rho(2m - 1) + \frac{\rho - 1}{2} p + N_0) \Gamma(1 + \beta + \frac{\rho - 1}{2} p + N_0)} \]

\[ \times \frac{s(\beta + \frac{\rho - 1}{2}(2m - 1))}{\pi} = \rho e^{-i \pi \frac{\rho - 1}{2}(2m - 1)} \frac{\mu_{2m - 1}}{\mu_{2m}} \times \]

\[ \prod_{0}^{2m - 1} \frac{\Gamma(1 - \rho(2m - 1) + (\rho - 1)(2m - 1 - p/2) + 2m - 1 - M_0)}{\Gamma(1 - \rho(2m - 1) + (\rho - 1)(2m - 2 - p/2) + 2m - 2 - N_0)} \]

To cancel \( s(\beta + \frac{\rho - 1}{2}(2m - 1)) \) on the left-hand side we take the \( (2m - 1) \)-th term

47
in the numerator and find that only for $M_{2m-1} = m - 1$ is there a cancellation. Notice that the right-hand side of (4.61) is independent of $\beta$, and therefore all $\beta$-dependence should disappear. By inspection one finds that

$$M_{2m-1} = m - 1, \quad M_p = N_p$$  \hspace{1cm} (4.62)

is the only way to make (4.61) $\beta$-independent. Once the $\beta$-dependence is cancelled, we obtain a recursion relation between $\mu_{2m}$ and $\mu_{2m-1}$:

$$\mu_{2m}(\rho) = (-1)^m e^{-i\pi \frac{\rho-1}{2m-1}} \frac{\prod_{0}^{2m-1} \Gamma(1 - \frac{\rho-1}{2} p - M_p)}{\prod_{0}^{2m-2} \Gamma(1 - \rho - \frac{\rho-1}{2} p - N_p)} \mu_{2m-1}(\rho)$$ \hspace{1cm} (4.63)

4.5. RELATING $J_{2m+1}$ TO $J_{2m}$

As in the previous section we start with $\tilde{J}^{\Sigma}(0,2m+1)(\alpha,\beta,\rho)$ and pull one contour into the $0 \to 1$ region. We obtain

$$\tilde{J}^{\Sigma}(0,2m+1) = -e^{-i\pi(\rho-1)m} \frac{s(\alpha)}{s(\alpha + \beta + (\rho-1)m)} \tilde{J}^{\Sigma}(1,2m)$$ \hspace{1cm} (4.64)

Looking at the $\tilde{J}^k$ component in $\tilde{J}^{\Sigma}$ we easily learn that the leading $\alpha \to 0$ singularity in $\tilde{J}^{\Sigma}$ comes from $\tilde{J}^1$. Taking $\alpha = \epsilon$ and letting $\epsilon \to 0$ we arrive at

$$\tilde{J}^{\Sigma}(0,2m+1)(0,\beta,\rho) = \pi \eta \frac{e^{-i\pi(\rho-1)m}}{s(\beta + (\rho-1)m)} J(0,2m)(\rho,\beta)$$ \hspace{1cm} (4.65)

This yields a relation between $\tilde{J}^{\Sigma}(2m+1,0)$ and $J(2m,0)$. Following the steps of the previous section leads to $M_{2m} = m$. Now we can determine $M_p$ to be

$$M_p = \left\lfloor \frac{p}{2} \right\rfloor$$ \hspace{1cm} (4.66)
and the recursion relations become

\[ \mu_{2m}(\rho) = (-1)^m e^{-i\pi \frac{m-1}{2} \rho} \frac{\Gamma(\frac{1-\rho}{2})}{\Gamma(1-m\rho) \mu_{2m-1}(\rho)} \]

\[ \mu_{2m+1}(\rho) = (-1)^{m+1} e^{-i\pi (\rho-1)m} \frac{\Gamma(\frac{1-\rho}{2})}{\Gamma(\frac{1}{2} - \rho^{2m+1})} \mu_{2m}(\rho) \]

Together, they completely determine \( \mu_n(\rho) \)

\[ \mu_{2m}(\rho) = e^{-i\pi (\rho-1)m(m-1/2)} \left( \frac{\rho}{2} \right)^m \frac{\Gamma(\frac{1-\rho}{2})^{2m}}{\prod_1^m \Gamma(1-pp) \Gamma(\frac{1}{2} - \rho(p-\frac{1}{2}))} \] (4.68a)

\[ \mu_{2m+1}(\rho) = -e^{-i\pi (\rho+1)m(m+1/2)} \left( \frac{\rho}{2} \right)^m \frac{\Gamma(\frac{1-\rho}{2})^{2m}}{\prod_1^m \Gamma(1-pp) \Gamma(\frac{1}{2} - \rho(p+\frac{1}{2}))} \] (4.68b)

These two formulae can be combined into a single one

\[ \mu_n(\rho) = (-1)^n e^{i\pi \frac{M_n}{2}} e^{-i\pi \rho^{\frac{n(n-1)}{4}}} \left( \frac{\rho}{2} \right)^{\frac{M_n}{2}} \frac{\Gamma(\frac{1-\rho}{2})^n}{\prod_1^n \Gamma(1 - \rho^{\frac{n+1}{2}} + M_p)} \] (4.69)

Finally we collect the formulae derived in this section for easy reference

\[ J_n(\alpha, \beta, \rho) = \prod_{p=0}^{n-1} \frac{\Gamma(1 + \alpha + \frac{\rho-1}{2} p + M_p) \Gamma(1 + \beta + \frac{\rho-1}{2} p + M_p)}{\Gamma(1 + \alpha + \beta + \frac{\rho-1}{2} (n-1 + p) + n - 1 - M_p)} \mu_n(\rho) \]

\[ J_n^\omega(\alpha, \beta, \rho) = \eta \prod_{p=0}^{n-1} \frac{\Gamma(1 + \alpha + \frac{\rho-1}{2} p + M_p) \Gamma(1 + \beta + \frac{\rho-1}{2} p + M_p)}{\Gamma(1 + \alpha + \beta + \frac{\rho-1}{2} (n-1 + p) + n - 1 - M_p)} \mu_n(\rho) \] (4.70)

\[ \tilde{J}_n(\alpha, \beta, \rho) = \eta \prod_{p=0}^{n-1} \frac{\Gamma(1 + \alpha + \frac{\rho-1}{2} p + \tilde{M}_p) \Gamma(1 + \beta + \frac{\rho-1}{2} p + \tilde{M}_p)}{\Gamma(1 + \alpha + \beta + \frac{\rho-1}{2} (n-1 + p) + n - 1 - \tilde{M}_p)} \mu_n(\rho) \]

\[ \mu_n(\rho) = (-1)^n e^{i\pi \frac{M_n}{2}} e^{-i\pi \frac{n(n-1)}{4}} \left( \frac{\rho}{2} \right)^{M_n} \frac{\Gamma(\frac{1-\rho}{2})^n}{\prod_1^n \Gamma(1 - \rho^{\frac{n+1}{2}} + M_p)} \]

\[ \tilde{M}_0 = -1, \quad M_p = \left[ \frac{p}{2} \right], \quad \tilde{M}_{p>0} = M_{p>0} \]
It is useful to introduce a new function $\hat{J}_n$ defined by

$$\hat{J}_n(\alpha, \beta, \rho) = \prod_{p=0}^{n-1} \frac{\Gamma(1 + \alpha + \frac{p-1}{2} + M_p) \Gamma(1 + \beta + \frac{p-1}{2} + M_p)}{\Gamma(1 + \alpha + \beta + \frac{p-1}{2} + M_p) \Gamma(1 + \alpha + \beta + \frac{p}{2} + (n-1) + M_p)} \mu_n(\rho)$$  (4.71)

Then

$$J_n(\alpha, \beta, \rho) = \hat{J}_n(\alpha, \beta, \rho) \quad n \text{ even}$$

$$\alpha \hat{J}^\Sigma_n(\alpha, \beta, \rho) = \eta \hat{J}_n(\alpha, \beta, \rho) \quad n \text{ odd}$$  (4.72)

If one works instead with path ordered integrals, the relation between $I$- and $J$-integrals is given in (4.8). We may as well introduce the functions $\hat{I}_n(\alpha, \beta, \rho)$ by

$$\hat{I}_n(\alpha, \beta, \rho) = \lambda_n(\rho) \epsilon_n(\rho)^{-1} \hat{I}_n(\alpha, \beta, \rho)$$

$$\lambda_n(\rho) = \prod_{1}^{n} \frac{s(\frac{\rho-1}{2})}{s(\frac{\rho}{2})}, \quad \epsilon_n(\rho) = \prod_{0}^{n-1} e^{i\pi \frac{\rho-1}{2}}$$  (4.73)
5. NORMALIZATION INTEGRALS: THE GENERAL CASE

We now extend the arguments of the previous section to the case when we have both +, − screening charges. There are some important differences in the determination of the integers \( M, N \) appearing in the arguments of the \( \Gamma \)-functions, but many of the arguments can be translated directly from the thermal case. We therefore present less details than in the previous section. We begin once again with the case of even integrals.

5.1. EVEN INTEGRALS

We want to evaluate

\[
J_{nm}(\alpha, \beta, \rho) = T_0 \int_{C_i} \prod_{i=1}^n dt_i \int_{S_j} \prod_{j=1}^m dS_j \prod_{i=1}^n t_i^{\alpha'} (1 - t_i)^{\beta'} \prod_{i<j}^n T_{ij}^{\rho'} \\
\prod_{i=1}^m s_i^\alpha (1 - s_i)^\beta \prod_{i<j}^m S_{ij}^{\rho} \prod_{i,j}^{n,m} (t_i - s_j - \theta_i \omega_j)^{-1} \quad (n + m \text{ even})
\]

(5.1)

with \( T_i = (t_i, \theta_i), S_j = (s_j, \omega_j) \) and the integration contours appear in fig. 5.1.

With the notation of section two,

\[
\rho' = \frac{1}{\rho} = \frac{1}{\alpha + 2} \quad \alpha' = -\rho' \alpha \quad \beta' = -\rho' \beta
\]

(5.2)

The ordering prescription is as in the previous chapter, and we should notice that the coupling terms \( (t_i - s_j - \theta_i \omega_j)^{-1} \) do not contribute to the monodromy if we include the signs coming from the exchange of \( dT_i \) and \( dS_j \). One can check as in [4] that exchanging the \( C \) and \( S \) contours does not change the answer. This implies that the monodromies of the conformal blocks will be given as a product of the monodromy matrices for the thermal integrals obtained by ignoring the coupling terms. In all the contour pulling manipulations the \( C_i \) and \( S_j \) contours do not feel each other. To compute (5.1) we define first the integrals \( J\left(\begin{pmatrix} \rho' \\ p' \\ q' \end{pmatrix}\right)(\alpha, \beta, \rho) \) as shown in fig. 5.2.
By opening the top $C_p$ and the bottom $C_1$ contours we can decrease $p'$ by one unit and increase $q'$ by one unit. In this way we can move all the $0 \to 1$ contours $p'$ to $1 \to \infty$ contours. After we are finished with the $C$-type contours we apply the same procedure to the $S$-type contours. The final result is

$$J\left(\begin{pmatrix} n \\ m \end{pmatrix}, \alpha, \beta, \rho \right) = (-1)^{n+m} \prod_{0}^{n-1} \frac{s(\alpha' + \beta' + (\rho' - 1)(n - 1 - i/2))}{s(\alpha' + \rho' - 1/2)} \prod_{0}^{m-1} \frac{s(\alpha + \beta + (\rho - 1)(m - 1 - i/2))}{s(\alpha + \rho - 1/2)} J\left(\begin{pmatrix} 0 \\ m \end{pmatrix}, \alpha, \beta, \rho \right)$$

(5.3)

The matrix label $\succ a'b'ab$ means that there are $a'$ (resp. $a$) $Q_-$ (resp. $Q_+$) contours from $0 \to 1$ and $b'$ (resp. $b$) $Q_-$ (resp. $Q_+$) contours from $1 \to \infty$. Next we use the split superconformal transformation $t_i \to 1/t_i$, $s_i \to 1/s_i$, and keeping in mind the remarks in section 2.4 we obtain

$$J_{nm}(\alpha, \beta, \rho) = \prod_{0}^{n-1} \frac{s(\alpha' + \beta' + (\rho' - 1)(n - 1 - i/2))}{s(\alpha' + \rho' - 1/2)} \prod_{0}^{m-1} \frac{s(\alpha + \beta + (\rho - 1)(m - 1 - i/2))}{s(\alpha + \rho - 1/2)} J_{nm}(-1 - \alpha - \beta - \rho(m - 1) + n, \beta, \rho)$$

(5.4)

This reflection property suggests the Ansatz

$$J_{nm}(\alpha, \beta, \rho) = \prod_{0}^{n-1} \frac{\Gamma(1 + \alpha' + \rho' - 1/2 - j + M_j') \Gamma(1 + \beta' + \rho' - 1/2 - j + M'_j)}{\Gamma(1 + \alpha' + \beta' + \rho'(n - 1) - m - \rho' - 1/2 - j - M_j') \Gamma(1 + \alpha + \beta + \rho(m - 1) - n - \rho - 1/2 - j - M_j')} \mu_{nm}(\rho)$$

(5.5)

With foresight we write $\mu_{nm}(\rho)$ without any dependence on $\alpha, \beta$. The Ansatz (5.5) is symmetrical under the exchange of $\alpha$ and $\beta$ because the original integral had this symmetry. Matching the large $\alpha$ behaviour of (5.1) and (5.5) leads to a constraint
on the integers $M_p', M_p$

\[ 2 \sum_{p=0}^{m-1} M_p + 2 \sum_{p=0}^{n-1} M_p' = \frac{m(m-2)}{2} + \frac{n(n-2)}{2} - nm \quad (5.6) \]

Before analyzing the case $(n + m)$ odd, we can relate the even to the odd case as we did in the thermal case. Starting with $J^{0 \ n}_{0 \ m}$ and pulling one $S$-contour into the $(0, 1)$ region we obtain

\[ J^{0 \ n}_{0 \ m} = \frac{\pi e^{-i\pi \rho'/2} (m-1)}{s(\alpha + \beta + \rho/2)} J^{0 \ n}_{1 \ m-1} \quad (5.7) \]

Explicitly

\[ J^{0 \ n}_{1 \ m-1} = T_0 \int_1^\infty \prod_1^n dT_i t_i^{\alpha'} (t_i - 1)^{\beta'} \prod_{i<j} T_{ij}^{\rho'} \]

\[ \int_1^\infty \prod_{1}^m dS_i s_i^{\alpha} (s_i - 1)^{\beta} \int_0^1 dS_1 s_1^\alpha (1 - s_1)^{\beta} \prod_{i<j} S_{ij}^{\rho} \prod_i (t_i - s_j - \theta \omega_j)^{-1} \]

\[ \quad (5.8) \]

As a consequence of the $T_0$-ordering prescription, $s_1$ is always smaller than all the other variables. Since we are interested in the limit $\alpha = -1 + \epsilon$ as $\epsilon \to 0$, we can expand in powers of $s_1$ and pick up the pole term in $\epsilon$ which cancels the zero from $s(\alpha)$ in (5.7). Multiplying by the odd $\eta$ variable we obtain after some simple manipulations

\[ \eta J^{0 \ n}_{0 \ m} (-1, \beta, \rho) = \pi \frac{e^{-i\pi \alpha'/2} (m-1)}{s(\beta + \rho/2)} \tilde{J}^{\Sigma}(0 \ n \ m-1) (-1 + \rho, \beta, \rho) \quad (5.9) \]

(for the definition of the odd integral $\tilde{J}^{\Sigma}$ see below). The same argument works for the $C$-contours. Now we set $\alpha' = -1 + \epsilon$, take the small $\epsilon$ limit and obtain

\[ \eta J^{0 \ n}_{0 \ m} (\rho, \beta, \rho) = -\pi \rho' \frac{e^{-i\pi \alpha'/2} (n-1)}{s(\beta' + \rho'/2)} \tilde{J}^{\Sigma}(0 \ n-1 \ m) (-1 + \rho, \beta, \rho) \quad (5.10) \]

Using (5.3) and a similar formula for $\tilde{J}^{\Sigma}$ to be derived in the next subsection we
obtain

\[ \eta J_{nm}(-\beta - \rho (m-1) + n, \beta, \rho) = \pi \frac{e^{-i\pi \rho - 1 - \rho}}{s(\beta + \rho - 1/2(m-1))} \tilde{J}_{n,m-1}^\Sigma(-\beta - \rho (m-1) + n, \beta, \rho) \]  

(5.11)

\[ \eta J_{nm}(-1 - \beta - \rho m + n, \beta, \rho) = -\pi \rho' \frac{e^{-i\pi \rho' - 1 - \rho}}{s(\beta' + \rho' - 1/2(n-1))} \tilde{J}_{n-1,m}^\Sigma(-1 - \beta - \rho m + n, \beta, \rho) \]  

(5.12)

Equations (5.11,12) will allow us to obtain recursion relations for \( \mu_{n,m}(\rho) \).

5.2. ODD INTEGRALS

We consider next the \((n + m)\) odd case. We define two types of integrals \( J_{nm}^\Sigma \), \( \tilde{J}_{nm}^\Sigma \).

\[ J_{nm}^\Sigma(\alpha, \beta, \rho) = T_0 \int \prod_{i=1}^n dT_i \int \prod_{j=1}^m dS_j \left( \rho \eta \sum_{1}^{m} \omega_k - \eta \sum_{1}^{n} \theta_k \right) \prod_{1}^{n} t_i^{\alpha} (1 - t_i)^{\beta} \prod_{i<j}^{n} T_{ij}^{\rho} \prod_{1}^{m} s_i^{\alpha} (1 - s_i)^{\beta} \prod_{i<j}^{m} S_{ij}^{\rho} \prod_{i,j}^{n,m} (t_i - s_j - \theta_i \omega_j)^{-1} \]  

\((n + m)\) odd

(5.13)

The \( \rho \) factor appearing in the sums can be understood from the normalization of the conformal blocks. The contours \( C_i, S_i \) are as shown in fig. 5.1. Similarly we define

\[ \tilde{J}_{nm}^\Sigma(\alpha, \beta, \rho) = T_0 \int \prod_{i=1}^n dT_i \int \prod_{j=1}^m dS_j \left( \rho \eta \sum_{1}^{m} \omega_k - \eta \sum_{1}^{n} \theta_k \right) \{\text{same as in (5.13)}\} \]  

(5.14)

For contour manipulations it is convenient to define \( J^k \left( \begin{array}{c} 0 \\ n-p \\ p \end{array} \right) \) and \( J^k \left( \begin{array}{c} 0 \\ n-p \\ m-q \\ q \end{array} \right) \) as in fig. 5.3. The superindex \( k' \) indicates that the factor \( \theta_k \) \((\theta_k/t_k)\) belongs to the \( C \)-contours, and \( k \) that it belongs to the \( S \) contours. The
matrix of labels \((a' b' c')\) counts contours. The first column indicates the contours from \(0 \to -\infty\). The second column counts contours from \(0 \to 1\) and the last column from \(1 \to \infty\). The first row refers to \(Q_-\)-contours and the second to \(Q_+\)-contours. The arguments leading from (4.28) to (4.35) can be repeated here for both \(J^\Sigma\) and \(\tilde{J}^\Sigma\). Using the definitions of fig. 5.3, we can write \(J^\Sigma_{nm}\) as

\[
J^\Sigma_{nm} = \rho \sum_{k=1}^{m} J^k \left( \begin{array}{ccc} 0 & n & 0 \\ 0 & m & 0 \end{array} \right) - \sum_{k'=1}^{n} J^{k'} \left( \begin{array}{ccc} 0 & n & 0 \\ 0 & m & 0 \end{array} \right)
\]

(5.15)

Repeating (4.28)–(4.35) in the present context is more cumbersome and leads to

\[
J^\Sigma \left( \begin{array}{ccc} 0 & n & 0 \\ 0 & m & 0 \end{array} \right) (\alpha, \beta, \rho) = (-1)^{n+m} \prod_{j=0}^{n-1} \frac{s(\alpha' + \beta' + (\rho' - 1)(n - 1 - j/2))}{s(\alpha' + \rho' - 1/2)} \prod_{j=0}^{m-1} \frac{s(\alpha + \beta + (\rho - 1)(m - 1 - j/2))}{s(\alpha + \beta + \rho - 1/2)} J^\Sigma \left( \begin{array}{ccc} 0 & n & 0 \\ 0 & m & 0 \end{array} \right) (\alpha, \beta, \rho)
\]

(5.16)

The same relation holds for \(\tilde{J}^\Sigma\). The change \(t_i \to 1/t_i; s_i \to 1/s_i\) mixes \(J^\Sigma\) and \(\tilde{J}^\Sigma\):

\[
J^\Sigma \left( \begin{array}{ccc} 0 & 0 & n \\ 0 & 0 & m \end{array} \right) (\alpha, \beta, \rho) = \tilde{J}^\Sigma \left( \begin{array}{ccc} 0 & 0 & n \\ 0 & 0 & m \end{array} \right) (-1 - \alpha - \beta - \rho(m - 1) + n, \beta, \rho)
\]

\(\tilde{J}^\Sigma \left( \begin{array}{ccc} 0 & 0 & n \\ 0 & 0 & m \end{array} \right) (\alpha, \beta, \rho) = J^\Sigma \left( \begin{array}{ccc} 0 & n & 0 \\ 0 & 0 & m \end{array} \right) (-1 - \alpha - \beta - \rho(m - 1) + n, \beta, \rho)
\]

(5.17)

As in the thermal case we can pull the \(1 \to \infty\) to \(0 \to -\infty\) contours

\[
J^\Sigma \left( \begin{array}{ccc} 0 & 0 & n \\ 0 & 0 & m \end{array} \right) (\alpha, \beta, \rho) = (-1) \prod_{j=0}^{n-1} \frac{s(\alpha' + \rho - 1/2)}{s(\beta' + \rho - 1/2)} \prod_{j=0}^{m-1} \frac{s(\alpha + \beta) s(\alpha + \beta + \rho - 1/2)}{s(\beta + \rho - 1/2)} J^\Sigma \left( \begin{array}{ccc} n & 0 & 0 \\ m & 0 & 0 \end{array} \right) (\alpha, \beta, \rho)
\]

(5.18)

Now changing the variables \(t_i \to 1 - t_i; s_i \to 1 - s_i\) yields

\[
J^\Sigma \left( \begin{array}{ccc} n & 0 & 0 \\ m & 0 & 0 \end{array} \right) (\alpha, \beta, \rho) = -J^\Sigma \left( \begin{array}{ccc} 0 & 0 & n \\ 0 & 0 & m \end{array} \right) (\beta, \alpha, \rho)
\]

(5.19)
This identity together with (5.17,5.18) implies
\[
\tilde{J}_{nm}(\gamma, -1 - \alpha - \gamma - \rho(m - 1) + n, \rho) = \prod_{j=0}^{n-1} \frac{s(\alpha' + \frac{\rho-1}{2}j)}{s(\alpha + \gamma + \rho' + \rho(\rho-1) - m - \frac{\rho'-1}{2}j)} \prod_{j=0}^{m-1} \frac{s(\alpha + \frac{\rho-1}{2}j)}{s(\alpha + \gamma + \rho(m - 1) - n - \frac{\rho-1}{2}j)} \tilde{J}_{nm}(\gamma, \alpha, \rho)
\]
(5.20)

As in the thermal case we can show that
\[
\alpha \tilde{J}_{nm}(\alpha, \beta, \rho) = \beta \tilde{J}_{nm}(\beta, \alpha, \rho)
\]
(5.21)

This together with (5.20) allows us to write an Ansatz for \(\tilde{J}_{nm}^{\Sigma}\) and \(J_{nm}^{\Sigma}\). The case \(n + m = 1\) can be computed explicitly. Introducing the integers \(N_i, N'_i, \tilde{N}_i, \tilde{N}'_i\) we obtain the Ansätze
\[
\tilde{J}_{nm}^{\Sigma}(\alpha, \beta, \rho) = n \prod_{i=0}^{n-1} \frac{\Gamma(1 + \alpha' + \frac{\rho-1}{2}j + \tilde{N}'_i) \Gamma(1 + \beta' + \frac{\rho-1}{2}j + N'_i) \Gamma(1 + \alpha + \beta + \rho(m - 1) - n - \frac{\rho-1}{2}j)}{\Gamma(1 + \alpha + \beta + \rho(m - 1) - n - \frac{\rho-1}{2}j) \Gamma(1 + \alpha + \beta + \rho' + \rho(n - 1) - m - \frac{\rho'-1}{2}j - N'_i)} \mu_{nm}(\rho)
\]
(5.22)

and
\[
J_{nm}^{\Sigma}(\alpha, \beta, \rho) = \eta \prod_{i=0}^{n-1} \frac{\Gamma(1 + \alpha' + \frac{\rho-1}{2}j + N'_i) \Gamma(1 + \beta' + \frac{\rho-1}{2}j + N'_i) \Gamma(1 + \alpha + \beta + \rho(m - 1) - n - \frac{\rho-1}{2}j - \tilde{N}'_i)}{\Gamma(1 + \alpha + \beta + \rho(m - 1) - n - \frac{\rho-1}{2}j - \tilde{N}'_i) \Gamma(1 + \alpha' + \beta' + \rho'(n - 1) - m - \frac{\rho'-1}{2}j - \tilde{N}'_i)} \mu_{nm}(\rho)
\]
(5.23)

Matching the large \(\alpha\) behaviour we obtain a relation for the integers \(N_p, N'_p, \tilde{N}_p, \tilde{N}'_p\):
\[
\sum_{0}^{n-1} \tilde{N}'_p + N'_p + \sum_{0}^{m-1} \tilde{N}_p + N_p = -\frac{1}{2} + \frac{n(n - 2)}{2} + \frac{m(m - 2)}{2} - nm
\]
(5.24)

Finally we reduce to an even number of contours by taking \(\alpha = \epsilon\) or \(\alpha' = \epsilon\), \(\epsilon \to 0\).
as in the previous subsection. Omitting the details, the results are

\[
\tilde{J}_\Sigma(0_n^m)(0, \beta, \rho) = \eta \pi \rho \frac{e^{-i \pi \frac{n-1}{2}(m-1)}}{s(\beta + \frac{n-1}{2}(m-1))} J\left(0_n^m\right)(\rho, \beta, \rho)
\]

\[
\tilde{J}_\Sigma(0_n^m)(0, \beta, \rho) = -\eta \pi \frac{e^{-i \pi \frac{n-1}{2}(n-1)}}{s(\beta' + \frac{n-1}{2}(n-1))} J\left(0_n^m\right)(-1, \beta, \rho)
\]

(5.25)

Equivalently, using (5.17),

\[
J_{nm}(-1 - \beta - \rho(m-1) + n, \beta, \rho) = \eta \pi \rho \frac{e^{-i \pi \frac{n-1}{2}(m-1)}}{s(\beta + \frac{n-1}{2}(m-1))} J_{n,m-1}(-1 - \beta - \rho(m-1) + n, \beta, \rho)
\]

\[
J_{nm}(-1 - \beta - \rho(m-1) + n, \beta, \rho) = -\eta \pi \frac{e^{-i \pi \frac{n-1}{2}(n-1)}}{s(\beta' + \frac{n-1}{2}(n-1))} J_{n-1,m}(-1 - \beta - \rho(m-1) + n, \beta, \rho)
\]

(5.26)

5.3. COMPUTATION OF \( \mu_{n,m}(\rho) \)

To complete the computation we have to determine the integers \( M_p, N_p, \) etc. This can be done by using the recursion relations established in the two previous subsections. There is, however, a simpler method of obtaining the same answer. Consider first the even case

\[
J_{nm} = \prod_{0}^{n-1} \frac{\Gamma(1 + \alpha' + \frac{p-1}{2} + M_p') \Gamma(1 + \beta' + \frac{p-1}{2} + M_p')}{\Gamma(1 + \alpha' + \beta' + \rho'(n-1) - m - \frac{p-1}{2} + M_p')} \prod_{0}^{m-1} \frac{\Gamma(1 + \alpha + \frac{p-1}{2} + M_p) \Gamma(1 + \beta + \frac{p-1}{2} + M_p)}{\Gamma(1 + \alpha + \beta + \rho(m-1) - n - \frac{p-1}{2} + M_p)} \mu_{nm}(\rho)
\]

The integers \( N_p, M_p, N_p', M_p' \) are independent of \( \rho \). Hence if we take the limit \( \rho \to -1 \) (still within the domain of definition of the integrals if \( \alpha, \beta > 0 \), the
original integral (5.1) becomes

\[ J_{nm} = T_0 \int_0^1 \prod_{i=1}^{n+m} dT_i u_i^\alpha (1 - u_i)^2 \prod_{i<j} (u_i - u_j - \theta_i \theta_j)^{-1} \]  

(5.27)

and it is identical to a thermal integral

\[ J_{n+m}(\alpha, \beta, -1) = \prod_0^{n+m-1} \frac{\Gamma(1 + \alpha - p + C_p)}{\Gamma(1 + \alpha + \beta - (n + m - 1) + p - C_p)} \mu_{n+m}(-1) \]

\[ C_p = \left\lfloor \frac{p}{2} \right\rfloor \]  

(5.28)

Writing the Ansatz (5.5) for \( \rho = -1 \) we can identify the integers \( M, M' \) as

\[ M'_p = C_p \quad p = 0, 1, \ldots, n - 1 \]

\[ M_p = C_{n+p} - n \quad p = 0, 1, \ldots, m - 1 \]  

(5.29)

There is a certain arbitrariness in this choice. We could have taken instead \( M'_p = C_{m+p} - m, \ M_p = C_p. \) This however does not affect the final result. An argument similar to the one employed in the thermal case leads to the same answer. Notice the dependence of \( M \) on \( n. \) The same analysis can be carried out for \( J_{\Sigma}^{nm} \) and it leads to

\[ N'_p = C_p \quad N_p = -n + C_{n+p} \]

\[ \tilde{N}'_p = \tilde{C}_p \quad \tilde{N}_p = -n + \tilde{C}_{n+p} \]  

(5.30)

Then, when \( n \neq 0, \ \tilde{N}_0' = -1 \) and when \( n = 0 \) (i.e. there is no \( \prod_0^{n-1} \tilde{N}_0 = -1 \). With this choice the symmetries \( \alpha \tilde{J}^\Sigma(\alpha, \beta) = \beta \tilde{J}^\Sigma(\beta, \alpha) \) and \( J^\Sigma(\alpha, \beta) = J^\Sigma(\beta, \alpha) \) are automatically satisfied. In the reduction from \( J_{n,m} \) to \( J_{n-1,m} \) or \( J_{n,m-1} \) we have to be careful in taking into account the \( n \)-dependence in \( M, N. \) The recursion relations obtained for \( \mu_{nm} \) are

\[ \mu_{nm}(\rho) = (-1)^{1-M_m} e^{-i\pi \frac{1}{2} (m-1)} \rho^n \frac{\Gamma\left(\frac{1-\rho}{2}\right)}{2 \Gamma(1 + n - \frac{\rho+1}{2} m + M_m)} \mu_{n,m-1}(\rho) \]

\( (n+m) \) even  

(5.31)
and

\[
\mu_{nm}(\rho) = (-1)^{1-M_m} e^{-i \pi \frac{\rho}{2} (m-1)} \rho^m \frac{\Gamma(\frac{1-\rho}{2})}{\Gamma(1 + n - \rho + 1 + M_m)} \mu_{n,m-1}(\rho)
\]

(n+m) odd

(5.32)

The only difference between these two expressions is the factor of \(\rho/2\). Iterating the recursion relations we end up in the thermal case which has already been solved. After some algebraic manipulations we arrive at (up to some irrelevant sign), for \(n \geq 1\)

\[
\mu_{nm}(\rho) = \rho^{nm} \left( \frac{\rho'}{2} \right)^{M_m'} \left( \frac{\rho}{2} \right)^{M_m + M_{m+1}'} e^{-i \pi \frac{\rho'}{2} (n+1)} e^{-i \pi \frac{\rho}{2} m (m-1)} \frac{\Gamma(\frac{1-\rho'}{2})^m \Gamma(\frac{1-\rho}{2})^m}{\prod_1^n \Gamma(1 - \frac{n}{2} + \rho' + M_p') \prod_1^m \Gamma(1 + n - \rho + 1 + M_p)}
\]

(5.33)

For \(n = 0\) we have the thermal result

\[
J_{\Sigma,0,m}(\alpha, \beta, \rho) = \rho J_{\Sigma,m}(\alpha, \beta, \rho)
\]

\[
\mu_{0,m}(\rho) = \left( \frac{1}{2} \right)^{M_m} \rho^{M_{m+1}} e^{-i \pi \frac{\rho'}{2} m (m-1)} \frac{\Gamma(\frac{1-\rho}{2})^m}{\prod_1^n \Gamma(1 - \frac{n}{2} + \rho' + M_p')}
\]

(5.34)

Finally

\[
J_{nm}(\alpha, \beta, \rho) = \prod_{0}^{n-1} \frac{\Gamma(1 + \alpha + \frac{\rho'}{2} p + M_p') \Gamma(1 + \beta + \frac{\rho}{2} p + M_p')}{\Gamma(1 + \alpha' + \beta' + (n-1) - m + \frac{\rho'}{2} p - M_p')}
\]

\[
\prod_{0}^{m-1} \frac{\Gamma(1 + \alpha + \frac{\rho}{2} p + M_p) \Gamma(1 + \beta + \frac{\rho'}{2} p + M_p)}{\Gamma(1 + \alpha + \beta + \rho (m-1) - n + \frac{\rho'}{2} p - M_p)} \mu_{nm}(\rho)
\]

(n+m) even

(5.35a)
\[ J^{\Sigma_{nm}}(\alpha, \beta, \rho) = \eta \prod_{r=0}^{n-1} \frac{\Gamma(1 + \alpha' + \frac{\rho' - 1}{2}p + M'_p) \Gamma(1 + \beta' + \frac{\rho' - 1}{2}p + M'_p)}{\Gamma(1 + \alpha' + \beta' + \rho'(n-1) - m - \frac{\rho' - 1}{2}p + \tilde{M}'_p)} \]
\[ \cdot \prod_{r=0}^{m-1} \frac{\Gamma(1 + \alpha + \frac{\rho - 1}{2}p + M_p) \Gamma(1 + \beta + \frac{\rho - 1}{2}p + M_p)}{\Gamma(1 + \alpha + \beta + \rho(m-1) - n - \frac{\rho - 1}{2}p + \tilde{M}_p)} \mu_{nm}(\rho) \]

\((n+m)\) odd

(5.35b)

where

\[ M'_p = \lceil \frac{p}{2} \rceil \quad M_p = -n + \left[ \frac{n + p}{2} \right] \]
\[ \tilde{M}'_p = \tilde{C}_p \quad \tilde{M}_p = -n + \tilde{C}_{n+p} \quad \text{with} \quad \tilde{C}_0 = -1, \quad \tilde{C}_{p>0} = \left\lceil \frac{p}{2} \right\rceil \]

(5.36)

These results can be unified in a way useful for the computation of \( N^{(nm)}_{lk} \), the normalization constants in the conformal blocks. Define

\[ \hat{J}_{nm}(\alpha, \beta, \rho) = \{ \text{same as in (5.35a), } (n+m) \text{ even or odd} \} \]

(5.37)

Up to an irrelevant sign one can show that

\[ J_{nm}(\alpha, \beta, \rho) = \hat{J}_{nm}(\alpha, \beta, \rho) \quad (n+m) \text{ even} \]
\[ \alpha' \hat{\Sigma}_{nm}(\alpha, \beta, \rho) = \eta \hat{J}_{nm}(\alpha, \beta, \rho) \quad (n+m) \text{ odd} \]

(5.38)

In terms of ordered integrals,

\[ \hat{J}_{nm}(\alpha, \beta, \rho) = \lambda_n(\rho')\epsilon_n(\rho')^{-1}\lambda_m(\rho)\epsilon_m(\rho)^{-1}\hat{I}_{nm}(\alpha, \beta, \rho) \]

\[ \lambda_m(\rho) = \prod_{i=1}^{m} \frac{s(i\rho - 1)}{s(\rho - 1)} \quad \epsilon_m(\rho) = \prod_{k=0}^{m-1} e^{i\pi \frac{\rho - 1}{2} k} \]

(5.39)

In all our previous results we have systematically ignored some signs because in the quantities of interest only the square of the normalization constants is used.
6. STRUCTURE CONSTANTS OF THE OPERATOR ALGEBRA

We have now established all the necessary formulae needed for the computation of the quantities $S_k^{(m)}$ out of which we shall extract the structure constants. We find it convenient to deal first with the thermal case.

6.1. THERMAL STRUCTURE CONSTANTS

Recall that we are considering the NS thermal four-point function $s$, represented with the help of vertex operators as

$$\langle \Phi_{1,q}(Z_4)\Phi_{1,n}(Z_2)\Phi_{1,s}(Z_1) \rangle = \langle V_{\bar{\alpha}_4}(Z_4)V_{\alpha_3}(Z_3)V_{\alpha_2}(Z_2)V_{\alpha_1}(Z_1)Q_{+}^{m-1} \rangle$$

(6.1)

It was shown in previous sections that the four-point correlator takes the form

$$\langle V_{\bar{\alpha}_4}V_{\alpha_3}V_{\alpha_2}V_{\alpha_1}Q_{+}^{m-1} \rangle \sim \sum_{i} S_k^{(m)} \left| F_k^{(m)}(a,b,c,\rho;Z) \right|^2$$

(6.2)

The quantities $S_k^{(m)}$ were given in (3.31):

$$S_k^{(m)}(a,b,c,\rho) = X_k(a,b,c,\rho) \left( N_k^{(m)}(a,b,c,\rho) \right)^2$$

(6.3)

with

$$N_k^{(m)}(a,b,c,\rho) = (-1)^{m-1} \hat{I}_{m-k}(-1 - a - b - c - \rho(m-2), b, \rho) \hat{I}_{k-1}(a, c, \rho)$$

(6.4)

By writing $X_k^{(m)}$ instead of $X_k$ in (6.3) we explicitly indicate that we make a convenient rescaling of $X_k$ (in other words we choose a particular value for $X_m$ in (3.16)). To compute $X_k^{(m)}$ we need the matrices $\beta_{kl}(a,b,c,\rho)$ which can be derived
from the Dotsenko and Fateev results [4] through the substitution \( \rho \rightarrow (\rho - 1)/2 \).

This yields

\[
\beta_{mk}(a, b, c, \rho) = \prod_{0}^{m-k-1} \frac{s(1 + a + b + c + (\rho - 1)(m - 2) - \frac{\rho-1}{2}i)}{s(b + c + (\rho - 1)(m - 2) - \frac{\rho-1}{2}(m - k - 1 + i))} \\
\times \prod_{0}^{k-2} \frac{s(1 + b + \frac{\rho-1}{2}i)}{s(b + c + \frac{\rho-1}{2}(k - 2 + i))}
\]

\[
\beta_{km}(a, b, c, \rho) = \prod_{1}^{m-1} \frac{s(i \frac{\rho-1}{2})}{s(i \frac{\rho-1}{2})} \prod_{1}^{m-k} \frac{s(i \frac{\rho-1}{2})}{s(b + c + \frac{\rho-1}{2}(m - k - 2 + i))} \\
\times \prod_{0}^{k-2} \frac{s(1 + b + \frac{\rho-1}{2}(m - k + i))}{s(b + c + \frac{\rho-1}{2}(m - 2 + i))}
\]

(6.5)

and after some algebra . . .

\[
X^{(m)}_{k} = \frac{\beta_{mm}(a, b)\beta_{mk}(b, a)}{\beta_{mm}(b, a)\beta_{km}(a, b)} C^{(m)} \prod_{0}^{m-2} \frac{s(1 + a + \frac{\rho-1}{2}i)s(1 + c + \frac{\rho-1}{2}i)}{s(a + c + \frac{\rho-1}{2}(m - 2 + i))} \prod_{1}^{m-1} s(i \frac{\rho-1}{2}) \\
= C^{(m)} \prod_{1}^{k-1} \frac{s(i \frac{\rho-1}{2})}{s(i \frac{\rho-1}{2})} \prod_{0}^{k-2} \frac{s(a + \frac{\rho-1}{2}i)s(1 + c + \frac{\rho-1}{2}i)}{s(a + c + \frac{\rho-1}{2}(k - 2 + i))} \\
\times \prod_{1}^{m-k} \frac{s(i \frac{\rho-1}{2})}{s(i \frac{\rho-1}{2})} \prod_{0}^{m-k-1} \frac{s(1 + b + \frac{\rho-1}{2}i)s(a + b + c + (\rho - 1)(m - 2) - \frac{\rho-1}{2}i)}{s(a + c + (\rho - 1)(m - 2) - \frac{\rho-1}{2}(m - k - 1 + i))}
\]

(6.6)

We have introduced the constants \( C^{(m)} = (-1)^{m-1} \pi^2 2^m \Gamma(\frac{1+\rho}{2})^{2m-2} \) for later convenience. Repeatedly using the identity \( s(x)\Gamma(x) = \pi/\Gamma(1 - x) \) and doing some appropriate shifts in the arguments of the sine functions in (6.6) we obtain a reasonably nice expression for \( S^{(m)}_{k} \):
where we have defined $\Delta(x) = \frac{\Gamma(x)}{\Gamma(1-x)}$ and $M_n = \left[\frac{n}{2}\right]$. This equation is more symmetrical than it appears at first sight. Since we restrict ourselves to the thermal subalgebra, the only way to meet the charge screening requirement is by taking a single conjugate vertex operator, which we take to be $V_{\alpha_1}$. Defining $\bar{d} = \alpha_1 + \frac{1}{2}$ and using the charge screening condition we obtain

$$\bar{d} = -1 - a - b - c - \rho(m - 2) \quad (6.8)$$

Then the second product of $S_k^{(m)}$ in (6.7) becomes

$$(\frac{\rho}{2})^{2M_{m-k}} \prod_{1}^{m-k} \Delta(\frac{\rho + 1}{2} i - M_i) \prod_{0}^{m-k-1} \Delta(-b - \bar{d} - \rho(m - k - 1) + \frac{\rho - 1}{2} i + M_i) \Delta(1 + b + \frac{\rho - 1}{2} i + M_i) \Delta(1 + \bar{d} + \frac{\rho - 1}{2} i + M_i) \quad (6.9)$$

with the same structure as the first product in (6.7). Introducing explicitly the Kac labels for the vertex operators, we get the different parameters:

$$a = \alpha_{1,s} \alpha_+ = \frac{1 - s}{2} \rho \quad \quad c = \alpha_{1,n} \alpha_+ = \frac{1 - n}{2} \rho$$
$$b = \alpha_{1,q} \alpha_+ = \frac{1 - q}{2} \rho \quad \quad \bar{d} = \alpha_{1,t} \alpha_+ = -1 + \frac{1 + \bar{t}}{2} \rho \quad (6.10)$$

where $s, n, q, \bar{t}$ are positive odd integers (NS sector) related to the number of screen-
ing charges $m - 1$ through

$$m = \frac{1}{2}(s + n + q - \bar{t}) \quad (6.11)$$

Then, $S_k^{(m)}$ becomes

$$S_k(sn\bar{t}) = \left(\frac{\rho}{2}\right)^{2M_{k-1}} \prod_{1}^{k-1} \Delta(\frac{\rho + 1}{2}i - M_i) \prod_{0}^{k-2} \Delta((\frac{s + n}{2} - k + 1)\rho + \frac{\rho - 1}{2}i + M_i) \Delta(1 + \frac{1-s}{2}\rho + \frac{\rho - 1}{2}i + M_i) \Delta(1 + \frac{1-n}{2}\rho + \frac{\rho - 1}{2}i + M_i)$$

$$(\frac{\rho}{2})^{2M_{m-k}} \prod_{1}^{m-k} \Delta(\frac{\rho + 1}{2}i - M_i) \prod_{0}^{m-k-1} \Delta(1 + \frac{1-q}{2}\rho + \frac{\rho - 1}{2}i + M_i) \Delta(1 + \frac{1-\bar{t}}{2}\rho + \frac{\rho - 1}{2}i + M_i)$$

$$\Delta(1 + \frac{1-q}{2}\rho + \frac{\rho - 1}{2}i + M_i) \Delta(\frac{1+\bar{t}}{2}\rho + \frac{\rho - 1}{2}i + M_i)$$

(6.12)

From this formula we can read off the asymmetric structure constants. The index $k$ labels the intermediate channels contributing to the four-point function. We can establish also the connection between the number $k$ and the conformal dimension of the field exchanged in the corresponding internal channel. This is achieved by choosing a configuration of superpoints $Z_i$ in (6.1) such that

$$|Z_{12}| \sim |Z_{34}| \sim r \ll R \sim |Z_{13}| \sim |Z_{24}|$$

(6.13)

Evaluating the four-point function using the OPE of the fields at $Z_1, Z_3$ we can write

$$\langle \Phi_{1,\bar{t}}(Z_4)\Phi_{1,q}(Z_3)\Phi_{1,n}(Z_2)\Phi_{1,s}(Z_1) \rangle \sim$$

$$\sim \sum_{p} r^{-2(h_s + h_n + h_q + h_{\bar{t}} - 2h_p)} C_{lq}^p C_{ns}^p \langle [\Phi_{1,p}(Z_3)] [\Phi_{1,p}(Z_1)] \rangle$$

(6.14)

For brevity we collectively denote by $C_{ns}^p$ the two structure constants $A_{ns}^p$ and $B_{ns}^p$ and by $[\Phi_{1,p}(Z_1)]$ the superconformal tower of descendant fields including in it the factors $Z_{21}^{-1/2}$ when necessary. This fine structure of the OPE and of the
superconformal blocks is not necessary in the present discussion. For the choice (6.13) we obtain

$$\mathcal{F}_k^{(m)}(a, b, c, \rho; Z) = z^{(k-1)(\frac{1}{2}a+c+\frac{2}{3}(k-2))} (1 + \ldots)$$ (6.15)

Here again we discard the occasional factors $z^{-1/2}$. Including the factors relating (6.1) to (6.2) and after some algebra we arrive at

$$p = s + n + 1 - 2k$$
$$-\bar{p} = -\bar{t} + q + 1 - 2(m - k + 1)$$ (6.16)

Furthermore we can identify the constants

$$S_k(sn\bar{t}) = C_{iq}^{\bar{p}} C_{ns}^{p}$$ (6.17)

Finally we can read off the asymmetric structure constants

$$C_{ns}^{p} = \left(\frac{\rho}{2}\right)^{2M_k} \prod_{1}^{k-1} \Delta \left(\frac{\rho + 1}{2} i - M_i\right) \prod_{0}^{k-2} \Delta \left(1 + \frac{1-s}{2} \rho + \frac{\rho - 1}{2} i + M_i\right)$$
$$\Delta \left(1 + \frac{1-n}{2} \rho + \frac{\rho - 1}{2} i + M_i\right) \Delta \left(\frac{1+p}{2} \rho + \frac{\rho - 1}{2} i + M_i\right)$$

$$C_{tq}^{\bar{p}} = \left(\frac{\rho}{2}\right)^{2M_{l-1}} \prod_{1}^{l-1} \Delta \left(\frac{\rho + 1}{2} i - M_i\right) \prod_{0}^{l-2} \Delta \left(1 + \frac{1-q}{2} \rho + \frac{\rho - 1}{2} i + M_i\right)$$
$$\Delta \left(\frac{1+\bar{t}}{2} \rho + \frac{\rho - 1}{2} i + M_i\right) \Delta \left(1 + \frac{1-\bar{p}}{2} \rho + \frac{\rho - 1}{2} i + M_i\right)$$ (6.18)

where the integers

$$k = \frac{1}{2}(s + n - p + 1)$$
$$l = \frac{1}{2}(q + \bar{p} - \bar{t} + 1)$$ (6.19)

are now chosen to be functions of the quantum numbers $s, n, p$ and $q, \bar{p}, \bar{t}$. It is very interesting to notice that although we had to distinguish in our analysis in section 3.1 between even and odd structure constants, we end up here with a
common expression for both. This will be the same when we compute the physical structure constants in the next section. Using the analyticity properties of the $\Gamma$ functions, it is easy to see that the structure constants we have found reproduce the correct fusion rules mentioned in section two.

6.2. GENERAL STRUCTURE CONSTANTS

This subsection follows the steps of the previous one except for the fact that the computations are more tedious. We will give few details. Once again we are interested in

$$\langle \Phi_{q',q}(Z_4)\Phi_{n',n}(Z_2)\Phi_{s',s}(Z_1) \rangle = \langle V_{a_4}(Z_4)V_{a_3}(Z_3)V_{a_2}(Z_2)V_{a_1}(Z_1)Q_{-}^{n-1}Q_{+}^{m-1} \rangle$$

$$\sim \sum_{k,l} S_{lk}^{(nm)} \left| \mathcal{F}_{lk}(a,b,c,\rho,Z) \right|^2$$

(6.20)

The quantity $S_{lk}^{(nm)}$ is given in (3.46)

$$S_{lk}^{(nm)}(a,b,c,\rho) = X_l(a',b',c',\rho')X_k(a,b,c,\rho) \left( N_{lk}^{(nm)}(a,b,c,\rho) \right)$$

(6.21)

For $X_{l}^{(n)}$, $X_{k}^{(m)}$ we take the same normalization as in (6.6). From (3.42) we obtain

$$N_{lk}^{(nm)}(a,b,c,\rho) = \hat{I}_{n-l,m-k}(-a-b-c-\rho(m-2)+n-2,b,\rho) \hat{I}_{l-1,k-1}(a,c,\rho)$$

(6.22)

Combining $X_{l}^{(n)}$ with the product of $\Gamma$-functions containing the primed quantities and similarly for $X_{k}^{(m)}$ for the unprimed quantities, we end up after a long
calculation with

\[
S_{lk}^{(nm)} = \rho^{2(l-1)(k-1)} \left( \frac{\rho'}{2} \right)^{2M'_{i-1}} \left( \frac{\rho}{2} \right)^{2M_{k-1}+2M'_i} \\
\prod_{1}^{k-1} \Delta(1 - l + \frac{\rho + 1}{2}i - M_i) \prod_{1}^{l-1} \Delta(\frac{\rho' + 1}{2}i - M'_i) \\
\prod_{0}^{k-2} \Delta(1 + a + \frac{\rho - 1}{2}i + M_i) \Delta(1 + c + \frac{\rho - 1}{2}i + M_i) \\
\Delta(-a - c - \rho(k - 2) + l - 1 + \frac{\rho - 1}{2}i + M_i) \\
\prod_{0}^{l-2} \Delta(1 + a' + \frac{\rho' - 1}{2}i + M'_i) \Delta(1 + c' + \frac{\rho' - 1}{2}i + M'_i) \\
\Delta(-a' - c' - \rho'(l - 2) + k - 1 + \frac{\rho' - 1}{2}i + M'_i) \\
\times \rho^{2(l-1)(m-k)} \left( \frac{\rho'}{2} \right)^{2M'_{n-l}} \left( \frac{\rho}{2} \right)^{2M_{m-k}+2M'_{n-l+1}} \\
\prod_{1}^{m-k-1} \Delta(1 - n + \frac{\rho + 1}{2}i - N_i) \prod_{1}^{n-l} \Delta(\frac{\rho' + 1}{2}i - M'_i) \\
\prod_{0}^{m-k-1} \Delta(1 + b + \frac{\rho - 1}{2}i + N_i) \Delta(a + c + \rho(k - 1) - l + 2 + \frac{\rho - 1}{2}i + N_i) \\
\Delta(1 - a - b - c - \rho(m - 2) + n - 2 + \frac{\rho - 1}{2}i + N_i) \\
\prod_{0}^{n-l-1} \Delta(1 + b' + \frac{\rho' - 1}{2}i + M'_i) \Delta(a' + c' + \rho'(l - 1) - k + 2 + \frac{\rho' - 1}{2}i + M'_i) \\
\Delta(1 - a' - b' - c' - \rho'(n - 2) + m - 2 + \frac{\rho' - 1}{2}i + M'_i) \\
\tag{6.23}
\]

where

\[
M_i = 1 - l + \left[ \frac{l - 1 + i}{2} \right] \quad N_i = l - n + \left[ \frac{n - l + i}{2} \right] \quad M'_i = \left[ \frac{i}{2} \right] \tag{6.24}
\]

Using the charge screening condition arising from (6.20) we define
\[ \bar{d} = \alpha_4 \alpha_+ = -a - b - c - \rho (m - 2) + n - 2 \]
\[ \bar{d}' = \alpha_4 \alpha_- = -a' - b' - c' - \rho' (n - 2) + m - 2 \]  

(6.25)

Which helps make the second set of products in (6.23) more similar to the first one. Repeating the arguments leading to (6.16) we find the Kac labels of the intermediate channels

\[ p = s + n + 1 - 2k \quad \quad p' = s' + n' + 1 - 2l \]  

(6.26)

With the simplifying notation \( C_{NS}^P = C_{(w,n),(s',s)}^{(p',p)} \) we find

\[ S_{lk}^{(nm)} = \bar{C}_{TQ} \bar{C}_{NS} \]  

(6.27)

Finally, introducing the Kac labels for the parameters \( a, b, c \ldots \) we obtain the asymmetric structure constants:

\[
C_{NS}^P = \rho^{2(l-1)(l'-1)} \left( \frac{\rho'}{2} \right)^{2M_{i-1}'} \left( \frac{\rho}{2} \right)^{2M_{i}+2M_{l}'} \\
\prod_{1}^{l-1} \Delta(1 - l' + \frac{\rho + 1}{2} i - M_i) \prod_{1}^{l'-1} \Delta(\frac{\rho' + 1}{2} i - M_i') \\
\prod_{0}^{l-2} \Delta(\frac{1 + s'}{2} + \frac{1 - s}{2} \rho + \frac{\rho - 1}{2} i + M_i) \Delta(\frac{1 + n'}{2} + \frac{1 - n}{2} \rho + \frac{\rho - 1}{2} i + M_i) \\
\Delta(\frac{1 - p'}{2} + \frac{1 + p'}{2} \rho + \frac{\rho - 1}{2} i + M_i') \\
\prod_{0}^{l'-2} \Delta(\frac{1 + s}{2} + \frac{1 - s'}{2} \rho' + \frac{\rho' - 1}{2} i + M_i') \Delta(\frac{1 + n}{2} + \frac{1 - n'}{2} \rho' + \frac{\rho' - 1}{2} i + M_i') \\
\Delta(\frac{1 - p}{2} + \frac{1 + p}{2} \rho' + \frac{\rho - 1}{2} i + M_i') 
\]  

(6.28)

with

\[ l = \frac{1}{2}(s + n - p + 1) \quad \quad M_i = 1 - l' + \left[ \frac{l' - 1 + i}{2} \right] \]

\[ l' = \frac{1}{2}(s' + n' - p' + 1) \quad \quad M_i' = \left[ \frac{i}{2} \right] \]
We do not write explicitly the other structure constants \( \tilde{C}_{PQ} \) since they are simply related to (6.28) by

\[
\tilde{C}_{PQ} = C_{PQ}^T
\]  

(6.29)

One readily checks that these structure constants (non-symmetrical) do not reproduce the correct fusion rules due to some cancellations between zeroes and poles of various \( \Delta \) factors. The physical structure constants do agree however with the correct fusion rules, and they are the subject of the next section.

A direct application of this result is the evaluation of some surface integrals corresponding to correlators where the screening charges are integrated over the whole plane instead of contours. Consider the three-point function

\[
J_{i'i}(2D) = \lim_{R \to \infty} R^{4h_M} (V_{\alpha_M}(R, R\eta)) V_{\alpha_N}(1, 0) V_{\alpha_S}(0, 0) \int \prod_{1}^{l'} d^2Z_i^r V_{\alpha_m}(Z_i^r, Z_i^l) \int \prod_{1}^{l} d^2Z_i V_{\alpha_+}(Z_i, Z_i)
\]

\[
= \int \prod_{1}^{l'} d^2Z_i^r \int \prod_{1}^{l} d^2Z_i \xi \prod_{i=1}^{l'} |z_i'|^{2\alpha} |1 - z_i'|^{2\beta} \prod_{i<j}^{l'} |z_i' - z_j'|^{2\theta} \prod_{i,j}^{l} |z_i - z_j - \theta_i \theta_j'|^{-2}
\]

(6.30)

where

\[
\xi = |1 - \alpha_M \alpha_+ \sum_i \eta \theta_i - \alpha_M \alpha_+ \sum_i \eta \theta_i'|^2
\]

\[
a = \alpha_S \alpha_+ \quad b = \alpha_N \alpha_+ \quad \text{etc.}
\]
Then with the help of (6.28) we can express it as

\[
J_{ll'}^{(2D)}(a, b, \rho) = (-1)^{M_{l'+1}} \pi^{l'+l'} l!' l! \Delta\left(\frac{1-\rho'}{2}\right)\Delta\left(\frac{1-\rho}{2}\right) \rho^{2l'} \left(\frac{\rho'}{2}\right)^{2M_{l'}} \left(\frac{\rho}{2}\right)^{2M_{l}+2M_{l'+1}} \prod_{1}^{l} \Delta(-l' + \frac{\rho + 1}{2} i - M_{i}) \prod_{1}^{l'} \Delta(\frac{\rho' + 1}{2} i - M'_{i})
\]

\[
\prod_{0}^{l-1} \Delta(1 + a + \frac{\rho - 1}{2} i + M_{i}) \Delta(1 + b + \frac{\rho - 1}{2} i + M_{i})
\]

\[
\Delta(-a - b - \rho(l - 1) + l' + \frac{\rho - 1}{2} i + M_{i})
\]

\[
\prod_{0}^{l'-1} \Delta(1 + a' + \frac{\rho' - 1}{2} i + M'_{i}) \Delta(1 + b' + \frac{\rho' - 1}{2} i + M'_{i})
\]

\[
\Delta(-a' - b' - \rho'(l' - 1) + l + \frac{\rho' - 1}{2} i + M'_{i})
\]

(6.31)

with \(M_{i} = -l' + \left[\frac{l'+i}{2}\right]\), \(M'_{i} = \left[\frac{i}{2}\right]\). The proportionality factor in (6.31) is determined by taking the limit \(\rho \to 0\) for the thermal case, in which limit the surface integrals decouple and are easily evaluated. In deriving these results we choosed the convention that \(\int d^2 \theta |\theta|^2 = 1\) and we omitted to write the factor \(|\eta|^2\) that appears in the right-hand side when \((l' + l)\) is odd. Equation (6.31) is indeed the generalization to the super case of equation (B.10) in [4].
7. PHYSICAL STRUCTURE CONSTANTS

All the material necessary for the computation of the physical structure constants has already been collected in previous sections. We follow closely the methodology of [5]. By physical structure constants we mean the constants $D_{SN}^P$ entering the OPE of two NS primary fields

$$\Phi_S(Z_1, \bar{Z}_1)\Phi_N(Z_2, \bar{Z}_2) = \sum_P D_{SN}^P |Z_{12}|^{-2(h_S + h_N - h_P)} [\Phi_P(Z_2, \bar{Z}_2)]_{\text{odd}}$$

(7.1)

For convenience $S = (s', s), \ldots$ and no distinction is made between the odd and even parts of this expansion (this can easily be done by counting screening charges in the three-point function). We shall impose the normalization condition

$$D_{SS}^1 = 1$$

(7.2)

corresponding to a diagonal two-point function

$$\langle \Phi_S(Z_1, \bar{Z}_1)\Phi_N(Z_2, \bar{Z}_2) \rangle = \delta_{S,N} |Z_{12}|^{-4h_S}$$

(7.3)

and to totally symmetric structure constants. Their determination is made with the help of the quantities $S^P(TQSN)$ defined in section 6. Consider a four-point function with the choice of arguments as in (6.13),

$$\langle \Phi_S(Z_4)\Phi_N(Z_3)\Phi_S(Z_2)\Phi_N(Z_1) \rangle = \sum_P \frac{D_{SN}^P D_{SN}^P}{r^4(h_S + h_N - h_P)} \langle [[\Phi_P(Z_3)][\Phi_P(Z_1)] \rangle$$

$$\langle \Phi_S(Z_4)\Phi_S(Z_3)\Phi_N(Z_2)\Phi_N(Z_1) \rangle = \sum_P \frac{D_{SS}^P D_{NN}^P}{r^4(h_S + h_N - h_P)} \langle [[\Phi_P(Z_3)][\Phi_P(Z_1)] \rangle$$

(7.4)

also written in section 6 as

$$\langle SNSN \rangle \sim \sum_P S^P(SNSN) \left| F^P(Z) \right|^2$$

$$\langle SSNN \rangle \sim \sum_P S^P(SSNN) \left| F^P(Z) \right|^2$$

(7.5)

With the present normalization the coefficient at the main singularity corresponding to the identity intermediate channel is equal to unity whereas $S^1(SSNN)$...
generally is not equal to 1. Hence the appropriate definition for the square of the physical structure constants is

\[
\left( D^P_{SN} \right)^2 = \frac{S^P(SNSN)}{S^1(SSNN)}
\]  

(7.6)

Using the asymmetric structure constants computed in section 6 we arrive at

\[
\left( D^P_{SN} \right)^2 = \frac{\tilde{C}^P_{SN}C^P_{SN}}{C^1_{SS}C^1_{NN}} = \frac{C^S_{PN}C^P_{SN}}{C^1_{NN}}
\]  

(7.7)

We shall shortly prove the relation

\[
C_{SNP} \equiv C^{-P}_{SN} = -\frac{\rho}{4} \Delta(\frac{\rho - 1}{2}) \Delta(\frac{\rho - 1}{2}) C^1_{PP} C^P_{SN}
\]  

(7.8)

showing that \( C^1_{PP} \) plays the role of a metric for raising or lowering indices. Substituting in (7.7) we obtain

\[
D^P_{SN} = -4\rho' \Delta(\frac{3 - \rho}{2}) \Delta(\frac{1 - \rho'}{2}) (C^1_{SS}C^1_{NN}C^1_{PP})^{-1/2} C_{SNP}
\]  

(7.9)

As expected we see the symmetry of \( D^P_{SN} \) under interchange of pairs of indices.

The proof of (7.8) is tedious and will be roughly sketched here. One uses the \( \Delta \)-functions properties listed below

\[
\Delta(1 - x) = \Delta(x)^{-1}
\]

\[
\Delta(x + n) = (-1)^n \prod_{0}^{n-1} (i + x)^2 \Delta(x)
\]  

(7.10)

\[
\Delta(x - n) = (-1)^n \prod_{0}^{n-1} (-1 - i + x)^{-2} \Delta(x)
\]

In the definition of \( C^P_{SN} \) the bounds of the products are

\[
l' = \frac{1}{2}(s' + n' - p' + 1) \quad l = \frac{1}{2}(s + n - p + 1)
\]  

(7.11)
whereas in $C_{SNP} = C_{SN}^P$ they are

$$\tilde{l} = \frac{1}{2}(s' + n' + p' + 1) \quad \tilde{\ell} = \frac{1}{2}(s + n + p + 1) \quad (7.12)$$

We first show that in the ratio $C_{SNP}/C_{SNP}^P$ the $(s', s)$ and $(n', n)$ dependence cancel out (up to some power of $\rho$). This is achieved using the simple relations for the products

$$\prod_{0}^{l-2} f(i) = \prod_{0}^{l-2} f(l - 2 - i)$$

$$\prod_{0}^{\tilde{l}-2} f(i) = \prod_{0}^{\tilde{l}-2} f(i) \prod_{0}^{p-1} f(i + l - 1) \quad (7.13)$$

Then the $(p', p)$ dependent part of $C_{SNP}$ cancels against $\mu_{\mu'}$ (and similarly for the $(p',p)$ dependence in $C_{SNP}^P$ against $\tilde{\mu}_{\tilde{\mu}'}$) up to some factors that turn out to be proportional to $C_{PP}^l$. The NS condition is frequently used during these simplifications. The determination of the power of $\rho$ in (7.8) is a rather delicate issue. Actually, $C_{SNP}$ contains poles and zeroes that cancel each other, and the exponent of $\rho$ is a direct consequence of the regularization procedure. We confirmed the consistency of this procedure by comparing the result obtained for $D_{SN}^P$ using either (7.7) or (7.9). More on this ?
APPENDIX

This appendix is devoted to the computation of the matrix elements $\beta_{mk}$ entering in the linear expansion

$$I_m^{(m)}(a, b, c, \rho; Z) = \sum_{k=1}^{m} \beta_{mk} \tilde{I}_k^{(m)}(a, b, c, \rho; Z) \quad (A.1)$$

From the normalization procedure (3.19) for $I_k^{(m)}$ and the relation between $I_k^{(m)}$ and $\tilde{I}_k^{(m)}$ we know that when $z \to 1$, $\tilde{I}_k^{(m)}$ has the singular behavior

$$\tilde{I}_k^{(m)}(Z) \sim (1 - z)^{(k-1)(1/2+b+c+(k-2)\rho/2)} \quad (\text{integral}) \quad (A.2)$$

As in (3.19) the integral is not necessarily a regular function as $z \to 1$. Depending on the values of $k, m$ it may exhibit a $(1 - z)^{-1/2}$ singular behaviour. Nevertheless the power in front of the integral in (A.2) characterizes the block sufficiently.

The procedure to evaluate $\beta_{mk}$ is to find in $I_k^{(m)}$ the same singular behaviour as $z \to 1$ as in (A.2). The coefficient in front of this divergence will only be proportional to $\beta_{mk}$ because we are not taking into account the normalization of the blocks $\tilde{I}_k^{(m)}(Z)$. We start with the integral representation of $I_m^{(m)}$

$$I_m^{(m)} = z^{\Delta_0} \int_0^1 dS_i (1 + a_1 \eta \theta + a_2 z^{1/2} \eta \sum_i \omega_i) \prod_{i=1}^{m-1} s_i^a (1 - z s_i)^b (1 - s_i - \theta \omega_i z^{1/2})^c \prod_{i<j} s_{ij}^{\rho} \quad (A.3)$$

where $\Delta_0 = (m - 1)(1/2 + a + c + \rho/2(m - 2))$. Performing on the first $k - 1$ variables $s_i$ the change of variables

$$t_i = \frac{1 - z}{1 - s_i + 1 - z} \quad \theta_i = \frac{t_i}{(1 - z)^{1/2}} \omega_i \quad (A.4)$$

and letting:

$$\epsilon = t_i(s_i = 0) = \frac{1 - z}{2 - z} \to 0 \quad \text{when} \quad z \to 1$$

$$1_{\epsilon} = s_{k-1}(t_{k-1}) = 1 - (1 - z) \frac{1 - t_{k-1}}{t_{k-1}} \to 1 \quad \text{when} \quad z \to 1 \quad (A.5)$$
we obtain after relabelling the $S_i$ and expanding the terms containing either $\theta$ or $(1-z)^{1/2}$:

\[
I^{(m)}_k = z \Delta_0 (1 - z)^{(k-1)(1/2+b+c+(k-2)\rho)/2} \int_0^{1} \int_0^{1} \int_0^{1} \int_0^{1} dT_1 dT_{k-1} dS_1 dS_{m-k-1} \\
(1 + a_1 \eta \theta + a_2 z^{1/2} (1 - z)^{1/2} \eta \sum_{i=1}^{k-1} \frac{\theta_i}{t_i} + a_2 \eta \sum_{i=1}^{m-k} \omega_i)(1 - \frac{c}{(1-z)^{1/2}} \sum_{i=1}^{m-k} \frac{\theta \theta_i}{1-t_i}) \\
(1 - c \sum_{i=1}^{k-1} \frac{\theta \omega_i}{1-s_i})(1 - \rho (1 - z)^{1/2} \sum_{i=1}^{m-k} \frac{\theta_i \omega_j}{(1-(1-z)^{1/2} \sum_{i=1}^{m-k} \frac{\theta \theta_i}{1-t_i})} \\
\prod_{i=1}^{k-1} \frac{1}{t_i} (1-b-c-\rho(k-2))(1-(1-z)^{1/2} \frac{1-t_i}{t_i})^b (1-t_i)^c \prod_{i<j} T_{ij}^\rho \\
\prod_{i=1}^{m-k} s_i^\rho (1-z s_i)^b (1-s_i)^c \prod_{i<j} S_{ij}^\rho \prod_{i,j} (1-(1-z)^{1/2} \frac{1-t_i}{t_i} - s_j)^\rho
\]

(A.6)

As expected from (3.21) we find four types of expansions for this integral depending on the values of $k$ and $m$. In each case, the coefficients $\beta'_{mk}$ we are looking for are given by the regular part of the integral evaluated at $z = 1$. Then the $T_i$ and $S_i$ integrals decouple and we get

\[
i) \beta'_{mk} = \int_0^{1} \prod_{i=1}^{k-1} dT_i t_i^{1-b-c-\rho(k-2)} (1-t_i)^c \prod_{i<j} T_{ij}^\rho \\
\int_0^{1} \prod_{i=1}^{m-k} dS_i s_i^\rho (1-s_i)^b+c+\rho(k-1) \prod_{i<j} S_{ij}^\rho \\
= I_{k-1}(-1 - b - c - \rho(k-2), c, \rho) I_{m-k}(a, b + c + \rho(k-1), \rho)
\]

(A.7i)
\[ \beta'_{mk} = \epsilon_{1} \hat{I}_{m-k}(a, b + c + \rho(k-1), \rho) \hat{I}_{k-1}(c, -1 - b - c - \rho(k-2), \rho) \]

(A.10)
with \( \epsilon_1 = 1 \) except for case \( \text{iv} \) where \( \epsilon_1 = -1 \).

In order to obtain the coefficients \( \beta_{mk} \) we only need to divide \( \beta_{mk}' \) by the normalization factor of the superconformal block \( \tilde{I}_k^{(m)} \),

\[
\beta_{mk}(a, b, c, \rho) = \frac{\beta_{mk}'(a, b, c, \rho)}{\tilde{N}_k^{(m)}(a, b, c, \rho)} \quad (A.11)
\]

The normalization factor \( \tilde{N}_k^{(m)} \) is derived from \( N_k^{(m)} \) by using (3.17). A careful analysis of the four possible cases leads to

\[
\tilde{N}_k^{(m)}(a, b, c, \rho) = \epsilon_2 N_k^{(m)}(b, a, c, \rho) \quad (A.12)
\]

with \( \epsilon_2 = 1 \) except for case \( \text{iii} \) where \( \epsilon_2 = -1 \). The last relation together with (3.26) enables us to write

\[
\beta_{mk} = (-1)^{m-1} \prod_{0}^{m-k-1} \frac{s(1 + a + b + c + (\rho - 1)(m - 2) - \frac{p-1}{2}i)}{s(b + c + (\rho - 1)(m - 2) - \frac{p-1}{2}(m - k - 1 + i))} \prod_{0}^{k-2} \frac{s(1 + b + \frac{p-1}{2}i)}{s(b + c + \frac{p-1}{2}(k - 2 + i))} \quad (A.13)
\]

This result is exactly the same as the result obtained by Dotsenko and Fateev [4] (3.16) provided we implement in their formulae the substitution \( \rho \to (\rho - 1)/2 \). The sign difference \( (-1)^{m-1} \) arises from the difference between our definition of conformal blocks and that of [4]. This result was expected, as explained in the text, but it provides a rather non-trivial test of our evaluation of the normalization integrals, the main computational difficulty in this paper.
REFERENCES

1. A.A. Belavin, A.M. Polyakov and A.B. Zamolodchikov: Nucl. Phys. B241 (1984) 333.

2. D. Friedan, Z. Qiu and S. Shenker: Phys. Rev. Lett. 51 (1984) 1575.

3. V.S. Dotsenko and V.A. Fateev: Nucl. Phys. B240 (1984) 312.

4. V.S. Dotsenko and V.A. Fateev: Nucl. Phys. B251 (1985) 691.

5. V.S. Dotsenko and V.A. Fateev: Phys. Lett. 154B (1985) 291.

6. D. Friedan, Z. Qiu and S. Shenker: In Vertex Operators in Mathematical Physics. J. Lepowsky ed. Springer Verlag, 1984.

7. D. Friedan, Z. Qiu and S. Shenker: Phys. Lett. 151B (1985) 37.

8. H. Eichenherr: Phys. Lett. 151B (1985) 26.

9. M. Bershadsky, V. Knizhnik and A. Teitelman: Phys. Lett. 151B (1985) 31.

10. Z. Qiu: Nucl. Phys. B270 (1986) 205.

11. G. Mussardo, G. Sotkov and H. Stanishkov: Phys. Lett. 195B (1987) 397; Nucl. Phys. B305 (1988) 69.

12. D. Friedan: Notes on String Theory and Two Dimensional Conformal Field Theory. In Proceedings of the Santa Barbara Workshop. M.B. Green and D.J. Gross eds. World Scientific 1985.

13. D. Friedan, E. Martinec and S. Shenker: Nucl. Phys. B271 (1986) 93.

14. J. Atick, G. Moore and A. Sen: Nucl. Phys. B308 (1988) 1.

15. Y. Kitazawa, N. Ishibashi, A. Kato, K. Kobayashi, Y. Matsuo and S. Odake: Nucl. Phys. B306(1988) 425.

16. E.L. Ince Ordinary Differential Equations. Dover 1927.
Figure captions

Fig.3.1. Contour ordering for $J_k^{(m)}(Z)$.

Fig.3.2. Contours for the analytic continuation of $J_k^{(m)}$.

Fig.3.3. Contour ordering for $\tilde{J}_k^{(m)}(Z)$.

Fig.3.4. Contour ordering for $J_{lk}^{(nm)}(Z)$.

Fig.4.1. Ordering of contours chosen in (4.1).

Fig.4.2. Contour ordering in (4.2,3).

Fig.4.3. Explicit definition of contour ordering.

Fig.4.4. Contours used in the definition of $J^{\Sigma}(p, 0, n - p)$.

Fig.4.5. Pulling one contour in $J(0, 2m)$.

Fig.5.1. Integration contours in (5.1).

Fig.5.2. Contours used for the evaluation of the even integral $J_{p'+q',p+q}^{p'}$.

Fig.5.3. Contours used for the evaluation of the odd integral $J_{nm}^{k'}$. 
Fig 4.1.

Fig 4.2.

Fig 4.3.

Fig 4.4.
Fig 4.5.

Fig 5.1.

\[ J_{p' q'}^{p q} = T_0 \]

Fig 5.2.

\[ J^{k'}_{0 n-p p 0 m-q q} = \]

Fig 5.3.