CONTROL OF THE LINEARIZED STEFAN PROBLEM IN A PERIODIC BOX

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Abstract. In this paper we consider the linearized one-phase Stefan problem with surface tension, set in the strip $T \times (-1,1)$, thus with periodic boundary conditions respect to the horizontal direction $x_1 \in T$. When the support of the control is not localized in $x_1$, namely, is of the form $\omega = T \times (c,d)$, we prove that the system is null-controllable in any positive time. We rely on a Fourier decomposition with respect to $x_1$, and controllability results which are uniform with respect to the Fourier frequency parameter for the resulting family of one-dimensional systems. The latter results are also novel, as we compute the full spectrum of the underlying operator for the non-zero Fourier modes. The zeroth mode system, on the other hand, is seen as a controllability problem for the linear heat equation with a finite-dimensional constraint. We extend the controllability result to the setting of controls with a support localized in a box: $\omega = (a,b) \times (c,d)$, through an argument inspired by the method of Lebeau and Robbiano, under the assumption that the initial data are of zero mean. Numerical experiments motivate several challenging open problems, foraying even beyond the specific setting we deal with herein.

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1. Introduction and main results

The Stefan problem is the quintessential macroscopic model of phase transitions in liquid-solid systems. The physical setup thereof typically consists in considering a domain \( \Omega \subset \mathbb{R}^d \), which is occupied by water (the liquid phase), a part of whose boundary is some interface \( \Gamma \), describing contact with a deformable solid such as ice (the solid phase). Due to melting or freezing, the regions occupied by water and ice will change over time and, consequently, the interface \( \Gamma \) will also change its position and shape. This leads to a free boundary problem. Albeit classical (see [19, 48] for an overview of the mathematical literature), the Stefan problem continues to be of use in many contemporary applications, such as additive manufacturing of alloys ([32]), ice modeling for video rendering in computer graphics ([31]), and, reaching even beyond its original fluid-mechanical nature, in the context of mathematical biology, for modeling the spread of various infectious diseases ([14, 36]).

1.1. Setup. We shall focus on the strong formulation of the one-phase Stefan problem (i.e., where the temperature of the ice is a known constant), with surface tension effects, following [17, 25, 27, 49]. We shall focus on the problem in two spatial dimensions \( (d = 2) \). To describe the geometrical setup, let \( T := \mathbb{R}/(2\pi \mathbb{Z}) \) denote the one-dimensional flat torus, which we identify with \([0, 2\pi]\) and thus described by the equation \( z_2 = 1 + h(t, z_1) \).

Given a time horizon \( T > 0 \) the strong formulation of the one-phase Stefan problem takes the form\(^1\)

\[
\begin{align*}
\partial_t \varrho - \Delta \varrho &= 0 & \text{in } (0, T) \times \Omega(t), \\
\partial_t h + \sqrt{1 + |\partial_{z_1} h|^2} \nabla \varrho_{|\Gamma(t)} \cdot \mathbf{n} &= 0 & \text{on } (0, T) \times \mathbb{T}, \\
\varrho(t, z_1, -1) &= 0 & \text{on } (0, T) \times \mathbb{T} \\
\varrho(t, z_1, z_2) &= -\sigma \kappa(h(t, z_1)) & \text{on } (0, T) \times \Gamma(t), \\
(\varrho, h)|_{t=0} &= (\varrho^0, h^0) & \text{in } \Omega(0) \times \mathbb{T}.
\end{align*}
\]

\(^1\)We make use of the standard notation \((0, T) \times \Omega(t) := \bigcup_{0 \leq t \leq T} t \times \Omega(t)\) and the analog for \((0, T) \times \Gamma(t)\).
This is a coupled system, where the unknown state is the pair \((\rho, h)\). The initial domain \(\Omega(0)\) is therefore given by
\[
\Omega(0) := \left\{ z = (z_1, z_2) \in \mathbb{T} \times \mathbb{R} \mid -1 < z_2 < 1 + h_0(z_1) \right\}.
\]
The constant \(\sigma > 0\) represents the surface tension coefficient\(^2\), whereas \(\kappa(h(t, z_1))\) denotes the mean curvature of the free boundary \(\Gamma(t)\), defined as
\[
\kappa(h) := \frac{\partial_{z_1}^2 h}{(1 + |\partial_{z_1} h|^2)^{3/2}}.
\]
As seen later on, the assumption \(\sigma > 0\) will be a critical part of our study. Finally, \(n = n(t, z_1)\) denotes the unit normal to \(\Gamma(t)\) outward \(\Omega(t)\) and is given by
\[
n := \frac{1}{\sqrt{1 + |\partial_{z_1} h|^2}} \begin{bmatrix} -\partial_{z_1} h \\ 1 \end{bmatrix}.
\]

1.2. Main results. In view of the applications presented just before, analyzing the controlled evolution of trajectories to (1.2) is rather natural. Our interest is the problem of null-controllability for (1.2): given a time horizon \(T > 0\), we seek to steer the temperature \(\rho(t)\), as well as the interface height \(h(t)\), to the equilibrium position \((0, 0)\), by means of some control \(u(t, z_1, z_2)\) actuating either along the bottom boundary \(\mathbb{T} \times \{z_2 = -1\}\) (replacing the Dirichlet boundary condition in (1.2)), or distributed within the fluid domain \(\Omega(t)\) (as a source term in (1.2)).

\(^2\)The condition involving the surface tension and the mean curvature is referred to as the Gibbs-Thomson correction. The physical reason for introducing the Gibbs-Thomson correction stems from the need to account for possible coarsening or nucleation effects ([43]). When \(\sigma = 0\), we are dealing with the classical Stefan problem. The mesoscopic limit \(\sigma \downarrow 0\) has been addressed in [25] (without control).
Note that, due to the presence of the free boundary parametrized by $h$, such a controllability problem would amount to also controlling the domain $\Omega(t)$ to the reference configuration $\Omega$. This is seen in Figure 1. And as a first but necessary step in solving this problem, in this work, we shall focus solely on the system linearized around the equilibrium $(0,0)$.

The linearized system, obtained after fixing the domain in (1.2) and dropping the nonlinearities from the subsequent system (detailed in Appendix B), with a distributed control, will take the form

\[
\begin{align*}
\partial_t y - \Delta y &= u 1_\omega & \text{in } (0,T) \times \Omega, \\
\partial_t h(t, x_1) - \partial_{x_2} y(t, x_1, 1) &= 0 & \text{on } (0,T) \times \mathbb{T}, \\
y(t, x_1, -1) &= 0 & \text{on } (0,T) \times \mathbb{T}, \\
y(t, x_1, 1) &= \sigma \partial_{x_1}^2 h(t, x_1) & \text{on } (0,T) \times \mathbb{T}, \\
(y, h)_{t=0} &= (y^0, h^0) & \text{in } \Omega \times \mathbb{T}.
\end{align*}
\]

For the time being, $\omega \subset \Omega$ is assumed to be open and non-empty. We shall focus on distributed controls since, as it is well known for parabolic equations, a simple extension-restriction argument allows one to obtain results for boundary controls as well.

The controllability properties of (1.3) have not been addressed in the literature, to the best of our knowledge. In fact, even the well-posedness may appear tricky at a first glance, due to the peculiar coupling of the states $y$ and $h$. Nonetheless, we may note the energy dissipation law (when $u \equiv 0$)

\[
\frac{d}{dt} \left\{ \int_\Omega |y(t)|^2 \, dx + \sigma \int_\mathbb{T} |\partial_{x_1} h(t)|^2 \, dx_1 \right\} = -2 \int_\Omega |\nabla y(t)|^2 \, dx,
\]

which will aid in ensuring that the system is well-posed when considered on the energy space $L^2(\Omega) \times H^1(\mathbb{T})$. As a matter of fact, we show that the governing operator of (1.3) generates an analytic semigroup on this energy space (Proposition 3.2 & Corollary 3.1). Identity (1.4) also clearly illustrates the strength of the coupling between $y$ and $h$, and the impact of $\sigma > 0$.

The main results in our paper regard the null-controllability of (1.3). We begin with the following theorem.

**Theorem 1.1.** Suppose $T > 0$ and $\sigma > 0$ are fixed, and suppose that $\omega = \mathbb{T} \times (c,d)$, where $(c,d) \subset (-1,1)$. Then for any $(y^0, h^0) \in L^2(\Omega) \times H^1(\mathbb{T})$, there exists some control $u \in L^2((0,T) \times \omega)$ such that the unique solution $(y, h) \in C^0([0,T]; L^2(\Omega) \times H^1(\mathbb{T}))$ to (1.3) satisfies $y(T, \cdot) \equiv 0$ in $\Omega$ and $h(T, \cdot) \equiv 0$ in $\mathbb{T}$.

The control $u$ in Theorem 1.1 is supported everywhere in the horizontal (i.e., $x_1 \in \mathbb{T}$) direction. This is of critical importance in the strategy of proof – we touch upon the details and reasons behind our methodological choice in Section 1.3 just below.

By slightly tweaking our strategy, we can further localize the support of the control, under the assumption that the initial data are of zero mean.

**Theorem 1.2.** Suppose $T > 0$ and $\sigma > 0$ are fixed, and suppose that $\omega = (a,b) \times (c,d)$ where $(a,b) \subset \mathbb{T}$ and $(c,d) \subset (-1,1)$. Then for any $(y^0, h^0) \in L^2(\Omega) \times H^1(\mathbb{T})$ such that

\[
\int_\mathbb{T} y^0(x_1, x_2) \, dx_1 = \int_\mathbb{T} h^0(x_1) \, dx_1 = 0
\]
for all \( x_2 \in (-1, 1) \), there exists some control \( u \in L^2((0, T) \times \omega) \) such that the unique solution \( (y, h) \in C^0([0, T]; L^2(\Omega) \times H^1(\Omega)) \) to (1.3) satisfies \( y(T, \cdot) \equiv 0 \) in \( \Omega \) and \( h(T, \cdot) \equiv 0 \) in \( \mathbb{T} \).

We postpone further comments to Section 2.

1.3. **Strategy of proof.** When one looks to prove the controllability of (1.3), the primal instinct would be to first write the adjoint system, which reads as

\[
\begin{aligned}
-\partial_t \zeta - \Delta \zeta &= 0 & \text{in } (0, T) \times \Omega, \\
-\partial_t \ell(t, x_1) - \partial^2_{x_1} \partial_{x_2} \zeta(t, x_1, 1) &= 0 & \text{on } (0, T) \times \mathbb{T}, \\
\zeta(t, x_1, -1) &= 0 & \text{on } (0, T) \times \mathbb{T}, \\
\zeta(t, x_1, 1) &= \ell(t, x_1) & \text{on } (0, T) \times \mathbb{T}, \\
(\zeta, \ell)|_{t=T} &= (\zeta_T, \ell_T) & \text{in } \Omega.
\end{aligned}
\]

(1.5)

Note that, since the natural energy space for (1.3) is \( L^2(\Omega) \times H^1(\mathbb{T}) \), the adjoint problem (1.5) ought to be analyzed in the dual space. Proceeding by the Hilbert Uniqueness Method (HUM, [37]), one would look to show an observability inequality of the form

\[
\| \zeta(0) \|^2_{L^2(\Omega)} + \| \ell(0) \|^2_{H^1(\mathbb{T})'} \lesssim_{T, \omega, \sigma} \int_0^T \int_\omega |\zeta(t, x)|^2 \, dt \, dx,
\]

for all \( (\zeta_T, \ell_T) \in L^2(\Omega) \times (H^1(\mathbb{T}))' \). If \( \omega \subset \Omega \) is open, non-empty, but otherwise arbitrary, the canonical way to proceed in such an endeavor would be through the use of Carleman inequalities. At this stage, we are not aware of existing Carleman inequalities which come even close to being adapted to the specific nature of the adjoint problem (1.5), due to the asymmetric nature of the coupling between both states.

There is however, a simpler way in which the problem can be tackled, as a first step. If one first supposes that \( \omega = \mathbb{T} \times (c, d) \), with \((c, d) \subset (-1, 1)\), we may then further exploit the periodicity of the control, and decompose all functions appearing in (1.3) into Fourier series with respect to \( x_1 \in \mathbb{T} \). Namely, we write

\[
y(t, x_1, x_2) = \frac{1}{\sqrt{2\pi}} \sum_{n \in \mathbb{Z}} y_n(t, x_2) e^{inx_1}
\]

(1.7)

with

\[
y_n(t, x_2) = \frac{1}{\sqrt{2\pi}} \int_\mathbb{T} y(t, \xi, x_2) e^{in\xi} \, d\xi,
\]

(1.8)

with analogous decompositions for \( h \) and \( u \), and the initial data \( (y^0, h^0) \). We then find ourselves with a family of systems for the Fourier coefficients, parametrized by \( n \in \mathbb{Z} \):

\[
\begin{aligned}
\partial_t \hat{y}_n - \partial^2_{x_2} \hat{y}_n + n^2 \hat{y}_n &= \hat{u}_n 1_{(c,d)} & \text{in } (0, T) \times (-1, 1), \\
\hat{h}'_n(t) - \partial_{x_2} \hat{y}_n(t, 1) &= 0 & \text{in } (0, T), \\
\hat{y}_n(t, -1) &= 0 & \text{in } (0, T), \\
\hat{y}_n(t, 1) &= -\sigma n^2 \hat{h}_n(t) & \text{in } (0, T), \\
(\hat{y}_n, \hat{h}_n)|_{t=0} &= (\hat{y}_n^0, \hat{h}_n^0) & \text{in } (-1, 1).
\end{aligned}
\]

(1.9)

Clearly, if the Fourier coefficients \((\hat{y}_n, \hat{h}_n)\) solving (1.9) are controllable to 0 in time \( T > 0 \) for any \( n \in \mathbb{Z} \), then by summing up all these Fourier coefficients as in (1.7),
we will deduce the desired controllability result for (1.3) as well. This is summarized in Proposition 4.1. Such procedures have been used in the literature, typically in the context of hypoelliptic operators ([5, 6, 8, 9]).

Curiously, when \( n \neq 0 \), the linear operator governing (1.9) is self-adjoint, and so the controllability of (1.9) can be ensured by proving an appropriate observability inequality for the (simpler) adjoint system, which can now be done by utilizing spectral techniques. This in turn, requires the computation of the full spectrum of the operator, which we provide in Lemma 4.1. Note that the latter is a nontrivial computation, as the operator in (1.9) is not a linear shift of the Laplacian. On another hand, when \( n = 0 \), we see that (1.9) becomes uncoupled, in the sense that we may control the heat component \( \hat{y}_0 \) independently of \( \hat{h}_0 \). Yet, we can view the constraint \( \hat{h}_0(T) = 0 \) as a one-dimensional constraint on the control \( \hat{u}_0 \) – this can then be covered via a compactness-uniqueness argument, and is a rather classical procedure for one-dimensional free boundary problems ([18, 23, 38]). Chaining all of these arguments together, one is led to the statement of Theorem 1.1.

An interesting byproduct of this strategy is the possibility of further localizing the control domain – as stated in Theorem 1.2 –, all the while avoiding the use of Carleman inequalities. Let us corroborate this claim.

First of all, note that the linear operator generating (1.3) is actually self-adjoint on \( \dot{\mathcal{H}} := \left\{ (y, h) \in L^2(\Omega) \times H^1(\mathbb{T}) \mid \int_\mathbb{T} y(x_1, \cdot) \, dx_1 = \int_\mathbb{T} h(x_1) \, dx_1 = 0 \right\} \), endowed with the norm

\[
\| (y, h) \|^2_{\dot{\mathcal{H}}} := \| y \|^2_{L^2(\Omega)} + \sigma \| \partial_{x_1} h \|^2_{L^2(\mathbb{T})},
\]

and the inferred inner product \( \langle f_1, f_2 \rangle_{\dot{\mathcal{H}}} = \frac{1}{2} \| f_1 \|^2_{\dot{\mathcal{H}}} + \frac{1}{2} \| f_2 \|^2_{\dot{\mathcal{H}}} - \frac{1}{2} \| f_1 - f_2 \|^2_{\dot{\mathcal{H}}} \). The adjoint problem for (1.3) taken on \( \dot{\mathcal{H}} \), for initial data in \( \dot{\mathcal{H}} \), then reads as

\[
\begin{align*}
-\partial_t \zeta - \Delta \zeta &= 0 & \text{in } (0, T) \times \Omega, \\
-\partial_t \ell(t, x_1) - \partial_{x_2} \zeta(t, x_1, 1) &= 0 & \text{on } (0, T) \times \mathbb{T}, \\
\zeta(t, x_1, -1) &= 0 & \text{on } (0, T) \times \mathbb{T}, \\
\zeta(t, x_1, 1) &= \sigma \partial_{x_1}^2 \ell(t, x_1) & \text{on } (0, T) \times \mathbb{T}, \\
(\zeta, \ell)|_{t=T} &= (\zeta_T, \ell_T) & \text{in } \Omega.
\end{align*}
\]

When decomposing the solution to (1.12) in Fourier series – the zeroth-mode being removed due to the definition of \( \dot{\mathcal{H}} \) – we end up precisely with the adjoint of (1.9). But if one uses the resulting observability inequality for the low frequency Fourier modes only (namely, corresponding to \( |n| \leq \mu \), for any \( \mu > 0 \)), we can further leverage a classical inequality for the eigenfunctions of the Laplacian\(^3\) (due to Lebeau and Robbiano, [35]) to obtain an observability inequality for the solutions to (1.12) possessing low-frequencies only. To cover the high-frequencies and complete the proof, following the Lebeau-Robbiano argument, we seek to exploit the exponentially stable character of the semigroup for (1.12). But since \( \lambda = 0 \) is, a priori, an eigenvalue of the governing

\(^3\)Here, we consider the Laplacian on the torus \( \mathbb{T} \), whose eigenfunctions are the complex exponentials which span \( L^2(\mathbb{T}) \).
operator in (1.12), identity (1.4) alone will not yield exponential decay of solutions. However, by taking initial data in $H^\infty$, we mod out the constants, and thus $\lambda = 0$ as well. We may consequently ensure the exponential decay of solutions to (1.12), which will lead us to the desired result (Theorem 1.2).

1.4. Scope. The remainder of the paper is organized as follows.

- In Section 2, we provide an in-depth comparison of our results and techniques with existing works on the control of free boundary problems, as well as a commentary on the potential limitations and extensions of our results to more general settings – the latter is also corroborated by numerical experiments.
- In Section 3, we begin by presenting the basic functional setting, as well as the well-posedness, and the symmetry properties in a mean-zero energy space of the operator governing system (1.3).
- In Section 4, we consider the family of systems, indexed by the Fourier frequency parameter, projected with respect to the horizontal direction $x_1 \in \mathbb{T}$. In particular, we show the null-controllability of these systems with a control cost uniform in the frequency parameter. This is done by computing the full spectrum of the underlying operator.
- In Section 5, we make use of the result presented in Section 4 to immediately derive Theorem 1.1.
- In Section 6, we provide the proof to Theorem 1.2, through the Lebeau-Robbiano inspired argument discussed just above.
- Finally, in Section 7, we conclude with a selection of related open problems.

1.5. Notation. Whenever the dependence on parameters of a constant is not specified, we will make use of Vinogradov notation and write $f \lesssim_S g$ whenever a constant $C > 0$, depending only on the set of parameters $S$, exists such that $f \leq Cg$.

2. Discussion

2.1. Previous work. The null-controllability results (Theorem 1.1, Theorem 1.2) we prove in this work are among the first of their kind for multi-dimensional free-boundary problems in which the free boundary depends on the spatial variable – even in the linearized regime. In this sense, our setup differs from existing works on the controllability of multi-dimensional fluid-structure interaction models with rigid bodies ([11, 30]), and the controllability of one-dimensional free boundary problems ([12, 18, 23, 38, 50], see also [15]), as therein, the free boundary is parametrized by the graph of a time-only dependent function, modeling a rigid body. In particular, the spatial regularity of the height function $h$ plays a crucial role in the analysis (or even well-posedness) results.

A partial controllability result for the two-dimensional classical Stefan problem ($\sigma = 0$) is shown in [13] – only the temperature $\varrho$ is controlled to 0 without any consideration of the height function $h(t, z_1)$ defining the free boundary $\Gamma(t)$. In fact, the geometrical setting is also different, as the free boundary $\Gamma(t)$ manifests as the entire boundary of the fluid domain $\Omega(t)$. Moreover, the Stefan law governing the velocity of the height function is regularized by adding a Laplacian term, which significantly simplifies the analysis.
Albeit for a system of different nature to ours, we also refer to [3] (and [1, 2, 53] for related results) for an exact-controllability result of the velocity and the free surface elevation of the water waves equations in two dimensions, by means of a single control actuating along an open subset of the free surface. In the aforementioned works, the two-dimensional geometrical strip-like setting of the free boundary problem is the same as ours. These results are extended to the three dimensional context in [56].

2.2. The (curious) case of $\sigma = 0$. We were unsuccessful in applying our techniques to cover the case $\sigma = 0$. In this case, the linearized system (1.3) is uncoupled, and proceeding by writing the adjoint system directly might appear as an arid endeavor. We provide more insight into some of the possible obstacles.

- Since (1.3) is uncoupled, one can first control the heat equation for $y$ to 0 through HUM, and then see the null-controllability for $h$ as a linear constraint of the form

$$0 = h(T, x_1) = h^0(x_1) + \int_0^T \partial_{x_1} y(t, x_1, 1) \, dt$$

(2.1)

for all $x_1 \in T$. The difference with respect to the one-dimensional case (see, e.g., [16, 23]) is that (2.1) is not a finite-dimensional constraint anymore, due to the fact that $h$ depends on the spatial variable. Hence, the compactness-uniqueness arguments of these works are not directly applicable.

- In this spirit, one can rather proceed by Fourier series decompositions to derive (1.9) with $\sigma = 0$. Now, for any fixed frequency $n$, the compactness-uniqueness arguments of the above-cited works can be used to derive the null-controllability of the full system, since (2.1) will transform into a one-dimensional constraint for the heat control. The caveat is that, due to the compactness-uniqueness argument used for addressing this finite-dimensional constraint, the controllability cost will depend on $n$, with an explicit dependence on $n$ being uncertain. Consequently, we cannot paste the controls for all $n$ to derive the controllability of (1.3) with $\sigma = 0$.

To motivate future work in this direction, we provide illustrations of numerical experiments for finding the minimal $L^2((0, T) \times \omega)$-norm control for (1.3), in the cases $\sigma > 0$ (specifically, $\sigma = 2$) and $\sigma = 0$. The results are displayed in Figure 2 and Figure 3 respectively. Numerical discretization and computing details may be found in Appendix A. For simplicity of the implementation, we worked on a rescaled domain $\Omega = (0, 2) \times (-1, 1)$, whilst keeping periodic boundary conditions with respect to $x_1$. In both cases, we took $T = 0.1^4$, with $h^0(x_1) = x_1(2 - x_1)$ and $y^0(x_1, x_2) = 70 \sin(\pi x_1) \sin(\pi x_2)$ (up to $x_2 = 1$ when $\sigma = 0$, whereas we impose the compatibility condition $y(0, x_1, 1) = \sigma \partial_{x_1}^2 h^0(x_1)$ when $\sigma = 10$). The numerical experiments depicted in Figure 3 insinuate that null-controllability might also hold when $\sigma = 0$. For the time being, a rigorous analytical proof (or disproof) of such a result remains an open problem.

\footnote{Note that the simulations yield the expected results even when $T \gg 1$. We present experiments with $T \ll 1$ since this is somewhat the "interesting" regime in the context of null-controllability of heat-like equations.}
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Figure 2. Controllability of (1.3) with $\sigma = 10$. (Top) The $L^2(\omega)$-norm, over time, of the minimal $L^2((0,T) \times \omega)$-norm control; (Middle) The temperature $y(t,x_1,x_2)$; (Bottom) the height $h(t,x_1)$. The numerical experiment reflects Theorem 1.1.

2.3. Extensions. Let us conclude this section with a brief discussion on the current limitations and possible extensions of our results.

Remark 1 (Geometric setup). (1) Theorem 1.2 actually implies the controllability from any open and non-empty subset $\omega \subset \Omega$ – given any such $\omega$, one can always find a rectangle $\omega_0$ as in the statement, such that $\omega_0 \subset \omega$, and then apply Theorem 1.2 to $\omega_0$.

(2) We parametrize the horizontal variable $x_1$ of any $x = (x_1, x_2) \in \Omega(t)$ (or in $\Gamma(t)$) over the torus $\mathbb{T}$ for convenience. This is actually standard in the literature on free boundary problems. This choice allows us to avoid rather complicated ensuing arguments regarding the regularity of the moving domains. Moreover, our geometrical setting is also amenable to Fourier analysis, which gives us a natural blueprint, based on Fourier decomposition and spectral analysis for one-dimensional problems, for tackling the control problem.
(3) There is no strict need of working in the two-dimensional setting regarding the linear problem (1.3). We do so mainly for convenience. The one-dimensional torus $\mathbb{T}$ may be replaced by $\mathbb{T}^{d-1}$, $d \geq 3$, throughout, without changing the strategy in the slightest. Indeed, just like when $d = 2$, Fourier decomposition can be performed with respect to the first $d - 1$ variables $(x_1, \ldots, x_{d-1})$, when $d \geq 3$ – only the "vertical" one (namely $x_d$) will remain in the projected, one-dimensional system, and the strategy of proof for Theorem 1.1 and Theorem 1.2 can be maintained.

(4) The higher-dimensional ($d \geq 3$) setting may pose an obstacle in a prospective extension of Theorem 1.1 and Theorem 1.2 to the nonlinear setting, where the
dimension would play a role regarding the spatial regularity of solutions (in particular that of the height function \( h(t, \cdot) \)). More specifically, one might have to ensure that the controls for the linearized system are more regular than simply \( L^2_t L^2_x \). We leave this open for a future study – a possible roadmap would include using the penalized-HUM method, as done in [33].

**Remark 2** (The nonlinear problem). In view of Theorem 1.1 and Theorem 1.2, one could stipulate a local controllability result for the nonlinear problem (1.2) by means of a fixed point and smallness of the initial data. This could be accomplished by first adding source terms through the so-called source term method of [38] (see also [7, 21, 29, 33]), and then performing a Banach fixed point for small enough initial data.

Currently, there are a couple of issues vis-à-vis this strategy.

1. First of all, with regard to the source-term method of [38], we need to ensure that the controllability cost in Theorem 1.1 is of the form \( \sim e^{\frac{T}{2}} \) when \( T \ll 1 \). Due to the compactness-uniqueness argument through which the zeroth Fourier mode system is controlled, the explicit time-dependence of the controllability cost is lost for the full system, and so the source term method is not directly applicable. In other words, further study is needed regarding the controllability cost of the system corresponding to the first Fourier mode.

2. When arguing along the lines of the Lebeau-Robbiano method for Theorem 1.2, we can sharpen the proof to ensure that the control cost is of such exponential form. But now the issue is the functional setup of \( \mathcal{H} \), as there is no guarantee that the quadratic nonlinear terms will be of mean zero, therefore posing an impediment regarding the invariance of the fixed point map.

All in all, the nonlinear adaptation of Theorem 1.1 and/or Theorem 1.2 remains open.

3. The Linear Semigroup

3.1. Functional setting. Recalling that \( \Omega := \mathbb{T} \times (-1, 1) \), let us consider the Hilbert space

\[ \mathcal{H} := L^2(\Omega) \times H^1(\mathbb{T}), \]

which, by Plancherel’s theorem, may be endowed with the norm

\[ \|(f, g)\|_{\mathcal{H}}^2 := \sum_{n \in \mathbb{Z}} \left( \|f_n\|_{L^2(-1,1)}^2 + (1 + \sigma n^2) |g_n|^2 \right), \tag{3.1} \]

where the Fourier coefficients \((y_n, g_n)\) are defined as in (1.8). The inner product on \( \mathcal{H} \) can then be inferred from (3.1).

We look to rewrite (1.3) in a canonical first-order form evolving on state space \( \mathcal{H} \). The well-posedness thereof will follow by studying the governing operator generating the semigroup of (1.3). To write (1.3) as an abstract control system evolving in \( \mathcal{H} \), we introduce the unbounded operator \( A : \mathcal{D}(A) \to \mathcal{H} \), defined by

\[ A(y, h) := (\Delta y, \partial_x y(\cdot, 1)), \]
with domain
\[ \mathcal{D}(A) = \left\{ (y, h) \in H^2(\Omega) \times H^{\frac{3}{2}}(\mathbb{T}) \mid \begin{array}{l} y(x_1, 1) = \sigma \partial_x^2 h(x_1) \quad \text{on } \mathbb{T}, \\ y(x_1, -1) = 0 \quad \text{on } \mathbb{T}, \\ \partial_x^2 y(\cdot, 1) \in H^1(\mathbb{T}) \end{array} \right\}. \] (3.2)

It can readily be seen that the operator \( A : \mathcal{D}(A) \to \mathcal{H} \) is closed and densely defined.

By using the above definitions, we can rewrite (1.3) (for a general source term \( f \) instead of \( u_1(\omega) \)) as a first order system in the Hilbert state space \( \mathcal{H} \):
\[
\begin{aligned}
\partial_t (y, h) &= A(y, h) + (f, 0) \quad \text{in } (0, T), \\
(y, h)|_{t=0} &= (y^0, h^0).
\end{aligned}
\] (3.3)

3.2. **Well-posedness of (3.3).** To study the well-posedness of (3.3), we shall rely on a Fourier decomposition in the horizontal \((x_1 \in \mathbb{T})\) variable. This will lead to a system akin to (1.9). Specifically, for any \( n \in \mathbb{Z} \) we consider the Hilbert space \( H_n := L^2(-1, 1) \times \mathbb{R} \), which we endow with the inner product
\[ \langle (f_1, g_1), (f_2, g_2) \rangle_{H_n} := \langle f_1, f_2 \rangle_{L^2(-1, 1)} + \sigma n^2 g_1 g_2, \]
when \( n \neq 0 \), and the canonical inner product when \( n = 0 \). We then define, for any \( n \in \mathbb{Z} \), the operator \( A_n : \mathcal{D}(A_n) \to H_n \) by
\[ A_n(y, h) := \left( \partial_x^2 y - n^2 y, \partial_x^2 y(1) \right), \]
with domain
\[ \mathcal{D}(A_n) = \left\{ (y, h) \in H^2(-1, 1) \times \mathbb{R} \mid \begin{array}{l} y(-1) = 0, y(1) = -\sigma n^2 h \end{array} \right\}. \]

We will show that the operators \( A \) and \( A_n \) \((n \in \mathbb{Z})\) generate analytic semigroups on \( \mathcal{H} \), and \( H_n \), respectively. The fact that \( A_n \) generates an analytic semigroup on \( H_n \) actually follows directly from [40, Theorem 1.25]. To show that \( A \) generates an analytic semigroup, we shall express its resolvent in terms of the resolvent of the operators \( A_n \). Thus, we shall need resolvent estimates for \( A_n \) which are uniform with respect to \( n \in \mathbb{Z} \).

For \( \theta \in \left( \frac{\pi}{2}, \pi \right) \) and \( \beta > 0 \), we define the sector
\[ \Sigma_{\theta, \beta} := \left\{ \lambda \in \mathbb{C} \setminus \{0\} \mid |\arg(\lambda)| < \theta, |\lambda| \geq \beta \right\}. \]

The following result then holds.

**Proposition 3.1** (Regarding \( A_n \)). Suppose \( n \in \mathbb{Z} \) and \( \sigma > 0 \).

1. If \( n \neq 0 \), the operator \( A_n : \mathcal{D}(A_n) \to H_n \) is self-adjoint, has compact resolvents, and its spectrum \( \text{spec}(A_n) \) consists only of negative eigenvalues.
(2) There exist \( \theta \in \left( \frac{\pi}{2}, \pi \right) \), \( \beta > 0 \) and \( C_{\theta, \beta} > 0 \), all independent of \( n \), such that

\[
|\lambda| \left\| (\lambda \text{Id} - A_n)^{-1} \right\|_{\mathcal{L}(\mathcal{H}_n)} \leq C_{\theta, \beta}
\]

holds for all \( \lambda \in \Sigma_{\theta, \beta} \).

Consequently, for any \( n \in \mathbb{Z} \), the operator \( A_n \) generates an analytic semigroup \( \{e^{tA_n}\}_{t \geq 0} \) on \( \mathcal{H}_n \).

Proof. The fact both claims imply that \( A_n \) generates an analytic semigroup follows from [10, Theorem 2.11 (p. 112), Proposition 2.11 (p. 122)].

Let us begin by proving the first claim. If \( n \neq 0 \), through standard integration by parts, is readily seen that \( A_n \) is self-adjoint and has compact resolvents. Therefore, its spectrum \( \text{spec}(A_n) \) is a discrete subset of \( \mathbb{R} \). Let us conclude the proof the first claim by showing that \( \text{spec}(A_n) \subset (-\infty, 0) \). We argue by contradiction. Let \( \lambda \in \text{spec}(A_n) \) with \( \lambda \geq 0 \). Thus there exists a vector \( (y, h) \in \mathcal{D}(A_n) \setminus \{0\} \) such that

\[
\begin{align*}
\lambda y - \partial^2_{x_2} y + n^2 y &= 0 \quad \text{in} \ (-1, 1), \\
\lambda h - \partial_{x_2} y(1) &= 0, \\
y(-1) &= 0, \\
y(1) &= -\sigma n^2 h.
\end{align*}
\]

We now multiply the first equation by \( y \) and integrate by parts to obtain

\[
\lambda \int_{-1}^{1} |y|^2 \, dx_2 + \int_{-1}^{1} |\partial_{x_2} y|^2 \, dx_2 - \partial_{x_2} y(1)y(1) + n^2 \int_{-1}^{1} |y|^2 \, dx_2 = 0.
\]

Using the boundary conditions, this identity entails

\[
\lambda \int_{-1}^{1} |y|^2 \, dx_2 + \int_{-1}^{1} |\partial_{x_2} y|^2 \, dx_2 + \lambda \sigma n^2 h^2 + n^2 \int_{-1}^{1} |y|^2 \, dx_2 = 0.
\]

Since \( \lambda \geq 0 \) and \( n \neq 0 \), from this identity, we may readily conclude that \( y \equiv h \equiv 0 \). This is a contradiction, and hence \( \lambda \in (-\infty, 0) \).

We now look to prove the second claim. Suppose that \( n \in \mathbb{Z} \) and \( (f_n, g_n) \in \mathcal{H}_n \) are arbitrary. Let us consider the resolvent problem

\[
\begin{align*}
\lambda y_n - \partial^2_{x_2} y_n + n^2 y_n &= f_n \quad \text{in} \ (-1, 1), \\
\lambda h_n = \partial_{x_2} y_n(1) + g_n \\
y_n(-1) &= 0, \\
y_n(1) &= -\sigma n^2 h_n.
\end{align*}
\]

If \( n = 0 \), the operator \( A_0 \) generates an analytic semigroup (see [40, Theorem 1.25]). Therefore, there exist \( \theta_0 \in \left( \frac{\pi}{2}, \pi \right) \), \( \beta_0 > 0 \), and \( C_{\theta_0, \beta_0} > 0 \) such that

\[
|\lambda| \left\| (y_n, h_0) \right\|_{L^2(-1,1) \times \mathbb{C}} \leq C_{\theta_0, \beta_0} \left\| (f_0, g_0) \right\|_{L^2(-1,1) \times \mathbb{C}},
\]

holds for all \( \lambda \in \Sigma_{\theta, \beta} \). Now let \( n \neq 0 \) be arbitrary and fixed. Since \( \text{spec}(A_n) \subset (-\infty, 0) \), we also have

\[
\Sigma_{\theta_0, \beta_0} \subset \text{res}(A_n),
\]
where \( \text{res}(A_n) \) denotes the resolvent set of \( A_n \). Consequently, for any \( n \neq 0 \), \( \lambda \in \Sigma_{\theta, \beta} \), and \( (f_n, g_n) \in \mathcal{H}_n \), (3.5) admits a unique solution \( (y_n, h_n) \in \mathcal{D}(A_n) \). Let us now fix
\[
\lambda = \beta_0 e^{i\theta_0} \in \Sigma_{\theta_0, \beta_0}.
\]
Multiplying the first equation in (3.5) by \( e^{-i\theta_0/2} \) and taking the inner product with \( y_n \), we obtain
\[
\begin{align*}
\beta_0 e^{i\theta_0/2} & \int_{-1}^{1} |y_n|^2 \, dx_2 + e^{-i\theta_0/2} \int_{-1}^{1} |\partial_{x_2} y_n|^2 \, dx_2 - e^{-i\theta_0/2} \partial_{x_2} y_n(1) y_n(1) \\
& + e^{-i\theta_0/2} n^2 \int_{-1}^{1} |y_n|^2 \, dx_2 = e^{-i\theta_0/2} \int_{-1}^{1} f_n y_n \, dx_2.
\end{align*}
\]
Using the boundary conditions, the above identity can be rewritten as
\[
\begin{align*}
\beta_0 e^{i\theta_0/2} & \int_{-1}^{1} |y_n|^2 \, dx_2 + e^{-i\theta_0/2} \int_{-1}^{1} |\partial_{x_2} y_n|^2 \, dx_2 + \beta_0 e^{i\theta_0/2} \sigma n^2 |h_n|^2 + e^{-i\theta_0/2} n^2 \int_{-1}^{1} |y_n|^2 \, dx_2 \\
& = e^{-i\theta_0/2} \int_{-1}^{1} f_n y_n \, dx_2 + e^{-i\theta_0/2} \sigma n^2 h_n g_n \\
& = e^{-i\theta_0/2} \langle (y_n, h_n), (f_n, g_n) \rangle_{\mathcal{H}_n}.
\end{align*}
\]
By taking the real part on both sides in the above identity, and subsequently using Cauchy-Schwarz, we find
\[
\beta_0 \left( \int_{-1}^{1} |y_n|^2 \, dx_2 + \sigma n^2 |h_n|^2 \right) + \int_{-1}^{1} |\partial_{x_2} y_n|^2 \, dx_2 + n^2 \int_{-1}^{1} |y_n|^2 \, dx_2 \\
\leq \| (y_n, h_n) \|_{\mathcal{H}_n} \| (f_n, g_n) \|_{\mathcal{H}_n}.
\]
Taking into account the fact that \( |\lambda| = \beta_0 \), we deduce that
\[
|\lambda| \| (y_n, h_n) \|_{\mathcal{H}_n} \leq \| (f_n, g_n) \|_{\mathcal{H}_n}.
\]
Since \( n \neq 0 \) and \( \lambda \in \Sigma_{\theta_0, \beta_0} \) were taken arbitrary, the above estimate in junction with (3.6) leads us to (3.4).

The above result then leads us to the following.

**Proposition 3.2** (Regarding \( A \)). Suppose \( \sigma > 0 \). There exist \( \theta \in (\frac{\pi}{2}, \pi) \), \( \beta > 0 \) and \( C_{\theta, \beta} > 0 \) such that
\[
|\lambda| \left\| (\lambda \text{Id} - A)^{-1} \right\|_{L(\mathcal{H})} \leq C_{\theta, \beta}
\]
holds for all \( \lambda \in \Sigma_{\theta, \beta} \). Consequently, the operator \( A : \mathcal{D}(A) \to \mathcal{H} \) generates an analytic semigroup \( \{e^{tA}\}_{t \geq 0} \) on \( \mathcal{H} \).

**Proof of Proposition 3.2.** Let \( \theta \in (\frac{\pi}{2}, \pi) \) and \( \beta > 0 \) be the constants stemming from Proposition 3.1. For \( \lambda \in \Sigma_{\theta, \beta} \) and \( (f, g) \in \mathcal{H} \), we consider the eigenvalue problem
\[
\begin{align*}
\lambda y - \Delta y &= f & \text{on } \Omega, \\
\lambda h - \partial_{x_2} y(\cdot, 1) + g &= 0 & \text{on } \mathbb{T}, \\
y(\cdot, -1) &= 0 & \text{on } \mathbb{T}, \\
y(\cdot, 1) &= \sigma \partial^2_{x_1} h(\cdot) & \text{on } \mathbb{T}.
\end{align*}
\]

(3.8)
We decompose all functions appearing in the above eigenvalue problem in Fourier series with respect to the periodic, $x_1$–variable, as in (1.7) – (1.8). We see that for any $n \in \mathbb{Z}$, the pair $(y_n, h_n) \in \mathcal{H}$ of Fourier coefficients solves (3.5), and moreover, because of the Fourier series expansion of $(y, h)$ as in (1.7),

$$(\lambda \text{Id} - A)^{-1} (f, g) = \frac{1}{\sqrt{2\pi}} \sum_{n \in \mathbb{Z}} (\lambda \text{Id} - A_n)^{-1} (f_n, g_n) e^{in\sigma},$$

also holds. Combining the above relation with (3.4), we immediately obtain (3.7). This completes the proof. \hfill \Box

As a consequence of the above result, we have

**Proposition 3.3** (Resolvent of $A$). Let $\lambda \in \Sigma_{\theta, \beta}$, where $\theta \in (\frac{\pi}{2}, \pi)$ and $\beta > 0$ stem from Proposition 3.2. For any $(f, g) \in \mathcal{H}$, the eigenvalue problem (3.8) admits a unique solution $(y, h) \in \mathcal{D}(A)$. In particular, the resolvent of $A$ is compact in $\mathcal{H}$.

**Proof of Proposition 3.3.** From Proposition 3.2, we know that $(y, h) \in \mathcal{H}$. We consider the elliptic boundary value problem (3.8)$_{1-3}$ satisfied by $y$. Note that

$$\partial_{x_1} y(\cdot, 1) = \lambda h - g \in H^1(\mathbb{T}).$$

Therefore, $y \in H^2(\Omega)$ and $y(\cdot, 1) \in H^{3/2}(\mathbb{T})$. Finally, using (3.8)$_{4}$ we obtain $h \in H^{7/2}(\mathbb{T})$. Whence $(y, h) \in \mathcal{D}(A)$, and the compactness readily follows. \hfill \Box

**Remark 3** (Fourier decomposition of the semigroup). In view of Proposition 3.1 and Proposition 3.2, we have

$$e^{tA} (y^0, h^0) = \frac{1}{\sqrt{2\pi}} \sum_{n \in \mathbb{Z}} e^{itA_n} (y^0_n, h^0_n) e^{in\sigma}, \quad (3.9)$$

where $(y^0_n, h^0_n)$ denote the Fourier coefficients of an initial datum $(y^0, h^0)$, defined as in (1.8).

Taking stock of Proposition 3.2, and using standard results from parabolic equations (see e.g. [10, Thm. 2.12, Sect. 2]), we deduce the well-posedness of the linear system (3.3).

**Corollary 3.1.** Suppose $T > 0$ and $\sigma > 0$. For every $(y^0, h^0) \in L^2(\Omega) \times H^1(\mathbb{T})$ and $f \in L^2(0, T; L^2(\Omega))$, there exists a unique solution $(y, h) \in C^0([0, T]; L^2(\Omega) \times H^1(\mathbb{T}))$ to (3.3).

To prove Theorem 1.2, we also need to ensure the well-posedness of the linear system (3.3) on the energy space consisting of mean zero functions, namely $\mathcal{H}$ defined in (1.10). Due to Plancharel’s theorem, the canonical norm on $\mathcal{H}$, induced from the inner product

$$((f_1, g_1), (f_2, g_2))_{\mathcal{H}} := \langle f_1, f_2 \rangle_{L^2(\Omega)} + \sigma \langle \partial_{x_1} g_1, \partial_{x_1} g_2 \rangle_{L^2(\mathbb{T})},$$

is clearly equivalent to

$$\left( \sum_{n \in \mathbb{Z}} \left\{ \| f_n \|^2_{L^2(-1, 1)} + \sigma n^2 |g_n|^2 \right\} \right)^{1/2}.$$
From (3.9), it is clear that the space $\mathcal{H}$ is invariant under the action of $\{e^{tA}\}_{t \geq 0}$, and the operator $A$ may thus be restricted to $\mathcal{H}$.

**Definition 3.1.** [55, Definition 2.4.1] We define the part of $A$ in $\mathcal{H}$ as the restriction of $A$ to the domain $\mathcal{D}(A) \cap \mathcal{H}$.

**Proposition 3.4.** The part of $A$ in $\mathcal{H}$ is self-adjoint.

**Proof of Proposition 3.4.** First of all, by integrating by parts and using the Schwarz theorem vis à vis the symmetry of second derivatives, we may readily find

$$\left\langle A(y, h), (z, r) \right\rangle_{\mathcal{H}} = \left\langle (y, h), A(z, r) \right\rangle_{\mathcal{H}}$$

for all $(y, h), (\zeta, \ell) \in (\mathcal{D}(A) \cap \mathcal{H})^2$. This shows that $\mathcal{D}(A) \cap \mathcal{H} \subseteq \mathcal{D}(A^*)$, and that $A$ is symmetric. To conclude, we show that $\mathcal{D}(A) \cap \mathcal{H} = \mathcal{D}(A^*)$. To this end, pick an arbitrary $(\zeta, \ell) \in \mathcal{D}(A^*)$. Then, there exists $(f, g) \in \mathcal{H}$ such that

$$\left\langle A(y, h), (\zeta, \ell) \right\rangle_{\mathcal{H}} = \left\langle (y, h), (f, g) \right\rangle_{\mathcal{H}}$$

holds for all $(y, h) \in \mathcal{D}(A) \cap \mathcal{H}$. Let us now set

$$(\varphi, s) := A^{-1}(f, g) \in \mathcal{D}(A) \cap \mathcal{H}.$$ 

Then from the above identities, we may infer that

$$\left\langle (y, h), (f, g) \right\rangle_{\mathcal{H}} = \left\langle (y, h), A(\varphi, s) \right\rangle_{\mathcal{H}} = \left\langle A(y, h), (\varphi, s) \right\rangle_{\mathcal{H}},$$

holds for all $(y, h) \in \mathcal{D}(A) \cap \mathcal{H}$. Therefore, we may also conclude that

$$\left\langle A(y, h), (\varphi, s) - (\zeta, \ell) \right\rangle_{\mathcal{H}} = 0,$$

for all $(y, h) \in \mathcal{D}(A) \cap \mathcal{H}$, whence $(\zeta, \ell) = (\varphi, s) \in \mathcal{D}(A) \cap \mathcal{H}$. This completes the proof. \hfill \Box

4. A FAMILY OF PROJECTED SYSTEMS

To prove Theorem 1.1, we will make use of the periodicity with respect to the $x_1$ variable of the functions appearing in (1.3). We write the Fourier series expansions of $(y, h, u)$ as in (1.7), with the corresponding Fourier coefficients $(y_n, h_n, u_n)$ being defined as in (1.8). It is readily seen that for any $n \in \mathbb{Z}$, these Fourier coefficients satisfy (1.9).

Our objective in this section is thus to prove the null-controllability of (1.9), with a controllability cost which is uniform in $n \in \mathbb{Z}$. This will allow us to simply sum up all of the coefficients as in (1.7), and deduce Theorem 1.1. We postpone the full conclusion to the next section, and solely focus on the result for the Fourier coefficients here.

**Proposition 4.1.** Suppose $T > 0$ and $\sigma > 0$ are fixed. Suppose $(c, d) \subset (-1, 1)$. For any $n \in \mathbb{Z}$, and for any pair of initial data $(y_n^0, h_n^0) \in L^2((-1, 1) \times \mathbb{R})$, there exists some control $u_n \in L^2((0, T) \times (c, d))$ such that the unique solution $(y_n, h_n) \in C^0([0, T]; L^2((-1, 1) \times \mathbb{R})$ to (1.9) satisfy $y_n(T, \cdot) \equiv 0$ in $(-1, 1)$ and $h_n(T) = 0$. Moreover, there exist a constant $C(T, \sigma) > 0$ independent of $n$ such that

$$\|u_n\|_{L^2((0, T) \times (c, d))} \leq C(T, \sigma) \left\|(y_n^0, h_n^0)\right\|_{\mathcal{H}^n}.$$
The full proof may be found in Section 4.2. More specifically, when \( n \neq 0 \), we use the customary Hilbert Uniqueness Method (HUM), which renders the controllability problem equivalent to a proof of an observability inequality for the adjoint system. The observability inequality will be shown by means of spectral arguments (based on results presented in Section 4.1, which come with some degree of difficulty). For the zeroth mode \( n = 0 \), we shall note that the eigenfunctions of the governing linear operator are not orthogonal (the operator is not self-adjoint), but the system is of cascade type and falls into the setting of [23].

4.1. The spectrum of \( A_n \). Recall that, when \( n \in \mathbb{Z} \setminus \{0\} \), the operator \( A_n \) is self-adjoint due to the specific inner product we endowed to \( \mathcal{H}_n \) (see Proposition 3.1). And since \( A_n \) has compact resolvents, by the Hilbert-Schmidt theorem, it may be diagonalized to find an orthonormal basis of \( \mathcal{H}_n \) consisting of eigenfunctions of \( A_n \), associated to a decreasing sequence of eigenvalues.

To prove the controllability of (1.9) when \( n \neq 0 \) by using spectral arguments, we need to explicitly characterize the spectrum of \( A_n \). This is the goal of the following result.

**Lemma 4.1 (The spectrum of \( A_n \)).** Let \( \sigma > 0 \) and \( n \in \mathbb{Z} \setminus \{0\} \) be fixed. Then, the sequence \( \{\lambda_{n,k}\}_{k=0}^{+\infty} \), with \( \lambda_{n,k} < 0 \), of eigenvalues of \( A_n \), reads as follows:

\[
\{\lambda_{n,k}\}_{k=0}^{+\infty} = \left\{-\left(\frac{(k+1)\pi}{2} + \frac{\pi}{4} - \varepsilon_{k+1}\right)^2 - n^2 \right\}_{k=0}^{+\infty}
\]

if \( \sigma > 2 \), for some \( \varepsilon_k \in (0, \frac{\pi}{4}) \), and

\[
\{\lambda_{n,k}\}_{k=0}^{+\infty} = \left\{-\left(\frac{k\pi}{2} + \frac{\pi}{4} - \varepsilon_k\right)^2 - n^2 \right\}_{k=1}^{+\infty} \cup \left\{\nu_0^2 - n^2 \right\} \quad \text{if } \sigma < 2
\]

otherwise, for some \( \nu_0 \in (0, \frac{1}{2}) \) independent of \( n \) and \( k \). (Here, we imply that the second "set" in the union is indexed as \( k = 0 \).)

Furthermore, the following properties hold.

(1) The sequence \( \{\lambda_{n,k}\}_{k=0}^{+\infty} \) is separated uniformly with respect to \( n \in \mathbb{Z} \setminus \{0\} \), in the sense that

\[
\inf_{k \geq 0} |\lambda_{n,k+1} - \lambda_{n,k}| > \gamma
\]

holds for some \( \gamma > 0 \) independent of \( n \).

(2) Moreover,

\[
-\lambda_{n,k} = rk^2 + n^2 + \mathcal{O}(k) \quad \text{as } k \to +\infty
\]

for some \( r > 0 \) independent of \( n \) and \( \sigma \).

(3) Suppose that \( (c, d) \subset (-1, 1) \) is fixed, with \( c < d \). Then, there exists some constant \( C = C(\sigma, c - d) > 0 \) such that for any \( n \in \mathbb{Z} \setminus \{0\} \) and \( k \geq 0 \), the normalized eigenfunctions

\[
\Phi_{n,k} \equiv \left(\varphi_{n,k}, -\frac{1}{\sigma n^2}\varphi_{n,k}(1)\right)
\]
of $A_n$ are such that
\[ \| \varphi_{n,k} \|_{L^2(c,d)} \geq C \] (4.3)
holds.

Before proceeding with the proof of Lemma 4.1, let us make the following important observation.

**Lemma 4.2** (Spectral gap). Suppose $\sigma > 0$ and $n \in \mathbb{Z} \setminus \{0\}$. Then $A_n$ has a spectral gap – namely,
\[ \lambda \leq -\min \left\{ \frac{\sigma}{2}, 1 \right\} n^2 \]
for all $\lambda \in \text{spec}(A_n)$.

In particular, $\lambda \leq -n^2$ if $\sigma \geq 2$. We shall make use of this property in the proof of Lemma 4.1, in particular distinguishing the cases $\sigma \in (0, 2)$ and $\sigma \geq 2$, for convenience.

**Proof of Lemma 4.2.** Let $\lambda < 0$ be such that
\[
\begin{cases}
\lambda y - \partial_x^2 y + n^2 y = 0 & \text{in } (-1, 1), \\
\lambda h - \partial_x y(1) = 0, \\
y(-1) = 0, \\
y(1) = -\sigma n^2 h,
\end{cases}
\] (4.4)
holds for some $(y, h) \in D(A_n) \setminus \{(0, 0)\}$. This entails the identity
\[
-\lambda \left( \int_{-1}^{1} |y|^2 \, dx + \sigma n^2 h^2 \right) = \int_{-1}^{1} |\partial_x y|^2 \, dx + n^2 \int_{-1}^{1} |y|^2 \, dx. \tag{4.5}
\]
Now note that from the boundary condition $y(1) = -\sigma n^2 h$ and an elementary Sobolev embedding\(^5\) for solutions to (4.4), we derive
\[
\sigma^2 n^4 h^2 = |y(1)|^2 \leq 2 \int_{-1}^{1} |\partial_x y|^2 \, dx. \tag{4.6}
\]
Plugging (4.6) in (4.5), we find
\[
-\lambda \left( \int_{-1}^{1} |y|^2 \, dx + \sigma n^2 h^2 \right) \geq \frac{\sigma}{2} \sigma n^4 h^2 + n^2 \int_{-1}^{1} |y|^2 \, dx \\
\geq \min \left\{ \frac{\sigma}{2}, 1 \right\} n^2 \left( \sigma n^2 h^2 + \int_{-1}^{1} |y|^2 \, dx \right),
\]
as desired. \qed

**Proof of Lemma 4.1.** We recall that $A_n : D(A_n) \to \mathcal{H}$ is self-adjoint, has compact resolvents, and its spectrum consists of a decreasing sequence of negative eigenvalues, namely a sequence $\{\lambda_{n,k}\}_{k=0}^{+\infty}$ with $-\infty < \lambda_{n,0} \leq \lambda_{n,1} \leq \ldots \leq \lambda_{n,k} < \ldots < \lambda_{n,0} < 0$. We shall distinguish three different scenarios for the computation of these eigenvalues.

\(^5\)We simply use $y(x) = y(x) - y(-1) = \int_{-1}^{x} \partial_x y(z) \, dz$ combined with the Cauchy-Schwarz inequality. The factor 2 occurs as $\text{meas}([-1, 1]) = 2$. 

Case 1: \[ \lambda < -n^2. \] Suppose that \( \lambda \in (-\infty, 0) \) is an eigenvalue of \( A_n \) which also satisfies \( \lambda < -n^2 \). So, there must exist a vector \((\varphi, \ell) \in D(A_n) \setminus \{(0, 0)\}\) such that
\[
\begin{align*}
\partial_{x_2}^2 \varphi + (-\lambda - n^2) \varphi &= 0 \quad \text{in } (-1, 1), \\
\partial_{x_2} \varphi(1) &= \lambda \ell \\
\varphi(-1) &= 0 \\
\varphi(1) &= -\sigma n^2 \ell.
\end{align*}
\]

In other words, \( \varphi \) would solve the mixed Dirichlet-Robin problem
\[
\begin{align*}
\partial_{x_2}^2 \varphi + (-\lambda - n^2) \varphi &= 0 \quad \text{in } (-1, 1), \\
\varphi(-1) &= 0 \\
\varphi(1) + \frac{\sigma n^2}{\lambda} \partial_{x_2} \varphi(1) &= 0.
\end{align*}
\]

Since \(-\lambda - n^2 > 0\), one may readily see that the solutions to (4.8) are of the form
\[
\varphi(x_2) = C \sin \left( \nu (1 + x_2) \right),
\]
with \( C > 0 \), where \( \nu := \sqrt{-\lambda - n^2} \) is the positive root of the transcendental equation
\[
\left( \frac{\nu^2}{n^2} + 1 \right) \tan(2\nu) = \sigma \nu.
\]

Locating the positive roots of this equation suggest a study of the fixed points of the function \( f(\nu) = \left( \frac{\nu^2}{n^2} + 1 \right) \tan(2\nu) \), defined and non-decreasing on the union of consecutive intervals of the form
\[
\bigcup_{k=1}^{+\infty} \left( \frac{\pi}{4} + \frac{(k - 1)\pi}{2}, \frac{\pi}{4} + \frac{k\pi}{2} \right).
\]

Figure 4. The function \( f(\nu) = (\nu^2/n^2 + 1) \tan(2\nu) \) (blue), with \( \nu \mapsto \sigma x \) superposed (black), with \( n = 10^4 \) and \( \sigma = 1/2 \) (left) and \( \sigma = 5 \) (right). We see how the fixed points of \( f \) are localized over each subinterval.
Moreover, for $k \geq 1$,
\[
\lim_{\nu \searrow \frac{\pi}{2} + \frac{(k-1)\pi}{2}} f(\nu) = -\infty, \quad f \left( \frac{k\pi}{2} \right) = 0, \quad \lim_{\nu \nearrow \frac{\pi}{2} + \frac{\pi}{2}} f(\nu) = +\infty.
\]
Thus, (4.9) has a sequence of positive roots $\{\nu_k\}_{k=1}^{+\infty}$ of the form
\[
\nu_k = \frac{k\pi}{2} + \frac{\pi}{4} - \epsilon_k
\]
for $k \geq 1$, where $\epsilon_k \in (0, \frac{\pi}{2})$ may a priori depend on $\sigma$ and $n$. Consequently, the eigenvalues $\lambda_{n,k} < 0$ in this case are of the form
\[
-\lambda_{n,k} = \left( \frac{k\pi}{2} + \frac{\pi}{4} - \epsilon_k \right)^2 + n^2
\]
for $k \geq 1$.

**Case 2:** $\lambda = -n^2$. We note that $\lambda = -n^2$ is an eigenvalue if and only if $\sigma = 2$. Indeed, should $\lambda = -n^2$ be an eigenvalue, then $\varphi$ in (4.7) would be harmonic, and thus an affine function: $\varphi(x_2) = C_0 x_2 + C_1$, for some $C_0, C_1 \in \mathbb{R}$. Using the boundary conditions, we moreover find that $C_0 = C_1$, as well as $2C_0 = \lambda \sigma \ell$. Since $\partial_{x_2} \varphi(1) = C_0 = \lambda \ell$, we are led to $\sigma = 2$. Summarizing, we find that when $\lambda = -n^2$,
\[
\varphi(x_2) = C_0 (1 + x_2).
\]

**Case 3:** $\lambda \in (-n^2, 0)$. Following Lemma 4.2 regarding the spectral gap, this case may only occur if $\sigma < 2$. So let us henceforth suppose that $\sigma < 2$, and that $\lambda \in (-n^2, 0)$ is an eigenvalue of $A_n$. Consequently, there must exist a vector $(\varphi, \ell) \in \mathcal{D}(A_n) \setminus \{(0,0)\}$ such that (4.7) holds. Then $\varphi$ would again solve the mixed Dirichlet-Robin problem (4.8). Since now $-\lambda - n^2 < 0$, one may readily see that the solutions to (4.8) are now of the form
\[
\varphi(x_2) = C \left( e^{\nu x_2} - e^{-\nu(2+x_2)} \right),
\]
for some $C > 0$, where $\nu := \sqrt{n^2 + \lambda}$ denotes the positive root(s) of the transcendental equation
\[
e^{\nu} - e^{-3\nu} - \frac{\sigma n^2}{n^2 - \nu^2} \left( \nu e^\nu + \nu e^{-3\nu} \right) = 0
\]
in $(0, |n|)$. We may equivalently rewrite the above equation as
\[
\left( -\nu^2 - \sigma n^2 \nu + n^2 \right) + \left( \nu^2 - \sigma n^2 \nu - n^2 \right) e^{-4\nu} = 0.
\]
We designate (4.12) as $f(\nu) = 0$, and we claim that $f$ has a unique root\footnote{One may try to compute this root by using the Lambert $W$ function and its generalizations ([42]). We omit this from our work as it is not needed for our analysis.} in $(0, |n|)$. Let us henceforth focus on proving this claim. Existence follows from the fact that $f$ is increasing and positive near $0$, and decreasing and negative near $|n|$. To ensure uniqueness, we will look to show that $f$ is strictly concave in $[0, |n|]$ (see Figure 5). We shall designate
\[
f_1(\nu) := -\nu^2 - \sigma n^2 \nu + n^2
\]
and
\[ f_2(\nu) := (\nu^2 - \sigma n^2 \nu - n^2) e^{-4\nu}, \]
so that \( f = f_1 + f_2 \). We see that
\[ f_1''(\nu) = -2. \tag{4.13} \]
On another hand, we also have
\[ f_2''(\nu) = \left( 2 - 16\nu + 8\sigma n^2 + 16\nu^2 - 16\sigma n^2 \nu - 16n^2 \right) e^{-4\nu}. \tag{4.14} \]
As \( f = f_1 + f_2 \), and taking (4.13) into account, we see that it suffices to ensure that \( f_2''(\nu) < 2 \) in \([0, |n]| \). This may be shown without too much difficulty. Indeed, whenever \( \nu \in \left[ 0, \frac{1}{2} \right] \) we see that \( 8\sigma \nu^2 - 16n^2 < 0 \) as well as \( 16\nu^2 - 16\nu < 0 \) in (4.14). Similarly, whenever \( \nu \in \left[ \frac{1}{2}, |n]| \right] \), we find that \( 8\sigma n^2 - 16\sigma n^2 \nu < 0 \) as well as \( 16\nu^2 - 16n^2 < 0 \) in (4.14). This yields \( f_2''(\nu) < 2 \) in \([0, |n]| \), whence \( f \) is strictly concave in \([0, |n]| \). Consequently, it follows that \( f \) has at most 2 roots in \([0, |n]| \). (This elementary property is readily shown by arguing by contradiction and Rolle’s theorem.) And since \( f(0) = 0 \), we conclude that any root of \( f \) in \((0, |n]| \) is unique. On another hand, we may readily see that
\[ f_2(\nu) < 0 \quad \text{for} \quad \nu \in [0, |n]|, \]
as well as
\[ f_1(\nu) \leq 0 \quad \text{for} \quad \nu \in \left[ \frac{1}{2} \left( \sqrt{n^4 \sigma^2 + 4n^2} - n^2 \sigma \right), |n| \right). \]
Consequently we find that
\[ f(\nu) < 0 \quad \text{for} \quad \nu \in \left[ \frac{1}{2} \left( \sqrt{n^4 \sigma^2 + 4n^2} - n^2 \sigma \right), |n| \right), \]
and therefore the unique root \( \nu_0 \) of \( f \) must be located in the complement of the above interval, namely \( \nu_0 \in \left( 0, \frac{1}{2} \left( \sqrt{n^4 \sigma^2 + 4n^2} - n^2 \sigma \right) \right) \). Hence, in this case, the first eigenvalue \( \lambda_{n,0} < 0 \) of \( A_n \) will have the form
\[ \lambda_{n,0} = \nu_0^2 - n^2. \tag{4.15} \]
Note that, moreover,
\[ \nu_0 \leq \frac{1}{2} \left( \sqrt{n^4 \sigma^2 + 4n^2} - n^2 \sigma \right) = \frac{4n^2}{2 \left( \sqrt{n^4 \sigma^2 + 4n^2} + n^2 \sigma \right)} \leq \frac{1}{\sigma}, \]
for all \( n \in \mathbb{Z} \setminus \{0\} \).

Having analyzed all the possible scenarios, we collect the sequence of eigenvalues \( \{ \lambda_{n,k} \}_{k=0}^{\infty} \), which if \( \sigma > 2 \) may actually be indexed as \( \{ \lambda_{n,k} \}_{k=1}^{\infty} \) with \( \lambda_{n,k} \) with \( k \geq 1 \) defined in (4.11); if \( \sigma = 2 \), we also have \( \lambda_{n,0} = -n^2 \), and if \( \sigma \in (0, 2) \), we have \( \lambda_{n,0} < 0 \) defined in (4.15). One thus readily sees that (4.2) holds. On another hand, since
Figure 5. Plot of $f_1$ (blue), $f_2$ (green), and $f = f_1 + f_2$ (red), for $\sigma = \frac{2}{5}$, with $n = 5$ (left) and $n = 10^4$ (right). We see that $f$ has a unique root $\nu_0 \in (0, |n|)$, and this root does not collapse to 0 as $n \to +\infty$.

For $\varepsilon_k \in (0, \frac{\pi}{4})$, we see that for $k \geq 1$,

$$-\lambda_{n,k+1} + \lambda_{n,k} = \left(\frac{(k+1)\pi}{2} + \frac{\pi}{4} - \varepsilon_{k+1}\right)^2 - \left(\frac{k\pi}{2} + \frac{\pi}{4} - \varepsilon_k\right)^2$$

$$= \left(k\pi + \pi - \varepsilon_{k+1} - \varepsilon_k\right)\left(\frac{\pi}{2} - \varepsilon_{k+1} + \varepsilon_k\right)$$

$$\geq \frac{3\pi}{2} \cdot \frac{\pi}{4} = \frac{3\pi^2}{8}.$$

Furthermore, if $\sigma \in (0, 2)$,

$$-\lambda_{n,1} + \lambda_{n,0} = \left(\frac{\pi}{2} + \frac{\pi}{4} - \varepsilon_k\right)^2 + \nu_0^2 \geq \frac{\pi^2}{4},$$

whereas if $\sigma = 2$, we similarly find

$$-\lambda_{n,1} + \lambda_{n,0} = \left(\frac{\pi}{2} + \frac{\pi}{4} - \varepsilon_k\right)^2 \geq \frac{\pi^2}{4}.$$

Hence, the separation condition (4.1) holds as well (actually, also uniformly in $\sigma > 0$).

Let us finally prove (4.3). We recall that the normalized eigenfunctions $\Phi_{n,k}$ have the form

$$\Phi_{n,k} = \left(\varphi_{n,k}, -\frac{1}{\sigma n^2} \varphi_{n,k}(1)\right),$$
where \( \varphi_{n,k} \) is given by

\[
\varphi_{n,k}(x_2) = C_{n,k} \sin \left( \sqrt{-\lambda_{n,k} - n^2} (1 + x_2) \right)
\]

for \( k \geq 1 \) (following the indexing of the eigenvalues depending on where \( \sigma \) is located), while

\[
\varphi_{n,0}(x_2) = C_{n,0} \left( e^{\sqrt{n^2 + \lambda_{n,0}} x_2} - e^{-\sqrt{n^2 + \lambda_{n,0}} (2 + x_2)} \right)
\]

if \( \sigma \in (0, 2) \), and

\[
\varphi_{n,0}(x_2) = C_{n,0} (1 + x_2)
\]

if \( \sigma = 2 \). Let us first suppose \( k \geq 1 \). Reusing the notation \( \nu_k := \sqrt{-\lambda_{n,k} - n^2} > 0 \), we note that in order to ensure that the eigenfunctions \( \Phi_{n,k} \) are of norm 1 in \( H_n \), namely

\[
\| \Phi_{n,k} \|_{H_n} = 1
\]

for all \( k \geq 1 \). On the other hand, using elementary trigonometric identities, we may also find

\[
\| \varphi_{n,k} \|_{L^2(c,d)}^2 = \frac{C_{n,k}^2}{2} \left( (d - c) + \frac{\sin (2\nu_k (c + 1))}{2\nu_k} - \frac{\sin (2\nu_k (d + 1))}{2\nu_k} \right), \quad (4.16)
\]

In view of (4.10), namely the asymptotics of \( \nu_k \) when \( k \to +\infty \), we find that there exists \( \delta = \delta(d - c) > 0 \) independent of \( n \in \mathbb{Z} \setminus \{0\} \) and \( k \) such that

\[
(d - c) + \frac{\sin (2\nu_k (c + 1))}{2\nu_k} - \frac{\sin (2\nu_k (d + 1))}{2\nu_k} \geq \delta \quad (4.17)
\]

holds for all \( k \geq 1 \). Therefore, we see from (4.16) and (4.17) that, in order to obtain (4.3), it suffices to have an appropriate lower bound on \( C_{n,k} \) for all \( k \geq 1 \). To this end, we note that

\[
C_{n,k}^2 = \left( \left( 1 - \frac{\sin (4\nu_k)}{4\nu_k} \right) + \frac{\sin^2 (2\nu_k)}{n^2 \sigma} \right)^{-1}.
\]

By virtue of (4.10), we see that

\[
\frac{\sin (4\nu_k)}{4\nu_k} \xrightarrow[k \to +\infty]{} 0 \quad \text{and} \quad \frac{\sin^2 (2\nu_k)}{n^2 \sigma} \xrightarrow[k \to +\infty]{} 0,
\]

hence

\[
C_{n,k}^2 \geq C_*
\]

for some \( C_* > 0 \) independent of \( n, k \) and \( \sigma \). This concludes the proof of (4.3) when \( k \geq 1 \).

Let us now consider the case \( k = 0 \) and \( \sigma \in (0, 2] \). First suppose that \( \sigma \in (0, 2) \). We see that to ensure orthonormality, \( C_{n,0} > 0 \) needs to satisfy

\[
\frac{C_{n,0}^2 \sinh (2\nu_0) - 4\nu_0}{\nu_0} e^{-2\nu_0} + \frac{C_{n,0}^2}{n^2 \sigma} (e^{\nu_0} - e^{-3\nu_0})^2 = 1,
\]
thus

\[ C_{n,0}^2 = \left( \frac{\sinh(2\nu_0) - 4\nu_0 e^{-2\nu_0}}{\nu_0} + \frac{(e^{\nu_0} - e^{-3\nu_0})^2}{n^2 \sigma} \right)^{-1}. \]  (4.18)

Taking (4.18) into stock, we see that since \( \nu_0 \in (0, \frac{1}{\sigma}) \) and \( \frac{\sinh(2x)}{x} \) is positive and continuous for \( x \in [0, \frac{1}{\sigma}] \),

\[
\sinh(2\nu_0) - 4\nu_0 e^{-2\nu_0} + \frac{(e^{\nu_0} - e^{-3\nu_0})^2}{n^2 \sigma} \leq \sinh(2\nu_0) + \frac{1}{\sigma} e^{\frac{1}{2}} \leq C_1(\sigma) + \frac{1}{\sigma} e^{\frac{1}{2}}
\]

holds for some \( C_1(\sigma) > 0 \) independent of \( n \). Consequently,

\[ C_{n,0}^2 \geq \frac{1}{C_1(\sigma) + \frac{1}{\sigma} e^{\frac{1}{2}}}. \]  (4.19)

We also have

\[
\|\varphi_{n,0}\|_{L^2([c,d])}^2 = C_{n,0}^2 \left( -2\nu_0(d-c) - \sinh(2(c+1)\nu_0) + \sinh(2(d+1)\nu_0) \right) e^{-2\nu_0}.
\]

Similarly, using the continuity and the positivity of the function

\[
x \mapsto \frac{\sinh(2(d+1)x) - \sinh(2(c+1)x) - 2(d-c)x}{x}
\]
on \( (0, \frac{1}{\sigma}) \), and using (4.19), we conclude that there exists \( C(\sigma, d-c) > 0 \) independent of \( n \) such that

\[
\|\varphi_{n,0}\|_{L^2([c,d])}^2 \geq C(\sigma, d-c),
\]
holds for all \( n \in \mathbb{Z} \setminus \{0\} \). This is precisely (4.3).

Finally, in the case \( k = 0 \) and \( \sigma = 2 \), since the (normalized) eigenfunction is an affine function, the proof of (4.3) is straightforward. This concludes the proof. \( \square \)

4.2. Proof of Proposition 4.1. We may conclude the study of the family of projected systems.

**Proof of Proposition 4.1.** We distinguish two separate cases.

**Case 1:** \( n = 0 \). By virtue of the results in [16, 23], we know that there exists some control \( u_0 \) such that (1.9) with \( n = 0 \) is null-controllable in time \( T > 0 \). Moreover, there exists a constant \( C(T, \sigma) > 0 \) such that

\[
\|u_0\|_{L^2((0,T) \times (c,d))}^2 \leq C(T, \sigma) \left\| \left( y_0^0, h_0^0 \right) \right\|_{L^2((-1,1) \times \mathbb{R})}^2.
\]  (4.20)

The proof of this result consists in seeing that since \( n = 0 \), the resulting system (1.9) is of cascade type, with

\[ h_0(t) = h_0^0 + \int_0^t \partial_{x_2} y_0(s,1) \, ds. \]

Controlling \( h_0(t) \) to 0 in time \( T \) can then be seen as a one-dimensional constraint on the heat control found by HUM, and may be achieved by a compactness-uniqueness argument within the global Carleman inequality for the heat operator. We refer to [16, 23] for all the necessary details.
Case 2: $n \neq 0$. We thus focus on the case $n \neq 0$. We recall that, by the customary HUM ([37]), (1.9) is null-controllable in any time $T > 0$ by means of a control $u_n$ of minimal $L^2((0, T) \times \omega)$ norm, satisfying
\[ \| u_n \|_{L^2((0, T) \times (c, d))}^2 \leq M_1 e^{M_2 T} \left\| \left( \frac{y_n^0}{h_n^0} \right) \right\|_{\mathcal{H}_n}^2, \]
for some $M_1 > 0$ and $M_2 > 0$ independent of $n$ and $T$, if and only if the functional
\[ J(\zeta, \ell) := \frac{1}{2} \int_0^T \int_c^d |\zeta|^2 \, dx \, dt - \left( (\zeta(0), \ell(0)), (y_n^0, h_n^0) \right)_{\mathcal{H}_n}, \]
where $(\zeta, \ell)$ is the unique solution to the adjoint system
\[ \begin{aligned}
&-\partial_t (\zeta, \ell) = A_n(\zeta, \ell) \quad \text{in } (0, T) \\
&\left( \zeta, \ell \right)_{|t=T} = (\zeta_T, \ell_T),
\end{aligned} \tag{4.21} \]
has a unique minimizer. (We have dropped the subscripts $n$ to declutter the notation.)

This, as is well-known, is nothing else but an integration by parts argument on one side, combined with the Euler-Lagrange equation for the functional $J$ at its minimizer for the converse. On another hand, by the direct method in the calculus of variations, $J$ has a minimizer if the observability inequality
\[ M_1 e^{M_2 T} \int_0^T \| \zeta(t, \cdot) \|_{L^2(c, d)}^2 \, dt \geq \left\| \left( \zeta(0, \cdot), \ell(0) \right) \right\|_{\mathcal{H}_n}^2 \tag{4.22} \]
holds for some $M_1 > 0$ and $M_2 > 0$ independent of $n$ and $T$, and for all $(\zeta_T, \ell_T) \in \mathcal{H}_n$, where $(\zeta, \ell)$ is the solution to (4.21). As $A_n$ has an orthonormal basis of eigenfunctions $\{\Phi_{n,k}\}_{k=0}^{+\infty}$ and corresponding decreasing sequence of negative eigenvalues $\{-\lambda_{n,k}\}_{k=0}^{+\infty}$, we may write the Fourier series decomposition of $\zeta$ as
\[ \zeta(t, x_2) = \sum_{k=0}^{+\infty} e^{-\lambda_{n,k}(T-t)} \left( (\zeta_T, \ell_T), \Phi_{n,k} \right)_{\mathcal{H}_n} \varphi_{n,k}(x_2). \]

Denoting by $\{\psi_j\}_{j=0}^{+\infty}$ the orthonormal basis of $L^2(c, d)$, and via the shift $T - t \mapsto t$, we obtain
\[ \int_0^T \| \zeta(t, \cdot) \|_{L^2(c, d)}^2 \, dt = \sum_{j=0}^{+\infty} \int_0^T \left\| e^{-\lambda_{n,k}t} \left( (\zeta_T, \ell_T), \Phi_{n,k} \right)_{\mathcal{H}_n} \langle \varphi_{n,k}, \psi_j \rangle_{L^2(c, d)} \right\|^2 \, dt. \tag{4.23} \]

Now, making use of (4.1) and (4.2), we deduce from [54, Cor. 3.6] that there exist $M_1 > 0$ and $M_2 > 0$ depending only on $r > 0$ and $\gamma > 0$ such that
\[ M_1 e^{M_2 T} \int_0^T \left\| \sum_{k=0}^{+\infty} |a_k| e^{-\left( \lambda_{n,k} - n^2 \right)t} \right\|^2 \, dt \geq \sum_{k=0}^{+\infty} |a_k|^2 e^{-2\left( \lambda_{n,k} - n^2 \right)T} \tag{4.24} \]
for any \( \{a_k\}_{k=0}^{+\infty} \in \ell^2(\mathbb{R}) \), and hence

\[
M_1 e^{\frac{M^2}{2}} \int_0^T \left| \sum_{k=0}^{+\infty} a_k e^{-\lambda_{n,k} t} \right|^2 dt \geq M_1 e^{\frac{M^2}{2}} \int_0^T \left| \sum_{k=0}^{+\infty} a_k e^{-(\lambda_{n,k}-n^2) t} \right|^2 dt
\]

\[
\geq e^{-n^2 T} \sum_{k=0}^{+\infty} |a_k|^2 e^{-2(\lambda_{n,k}-n^2) T}
\]

\[
= \sum_{k=0}^{+\infty} |a_k|^2 e^{-2\lambda_{n,k} T}.
\]

The above estimate, combined with (4.23), implies that

\[
M_1 e^{\frac{M^2}{2}} \int_0^T \| \zeta(t,\cdot) \|^2_{L^2(c,d)} dt \geq \sum_{j=0}^{+\infty} \sum_{k=0}^{+\infty} e^{-2\lambda_{n,k} T} \left| \langle \zeta_T, \ell_T \rangle, \Phi_{n,k} \rangle_{\mathcal{H}_n} \right|^2 \left| \langle \varphi_{n,k}, \psi_j \rangle_{L^2(c,d)} \right|^2.
\]

After employing the Fubini theorem, we may apply (4.3) to the above estimate, revert the time shift, and deduce that

\[
M_1 e^{\frac{M^2}{2}} \int_0^T \| \zeta(t,\cdot) \|^2_{L^2(c,d)} dt \geq C \| \zeta(0,\cdot), \ell(0) \|^2_{\mathcal{H}_n},
\]

which holds for all \((\zeta_T, \ell_T) \in \mathcal{H}_n\). Here, the constant \(C = C(\sigma, c - d) > 0\) stems from (4.3) in Lemma 4.1. This concludes the proof of (4.22), and thus the proof of the proposition. \(\square\)

5. Proof of Theorem 1.1

We may now proceed with the proof of Theorem 1.1.

**Proof of Theorem 1.1.** Let us define the control \( u \in L^2((0,T) \times \omega) \) by

\[
u(t, x_1, x_2) = \frac{1}{\sqrt{2\pi}} \sum_{n \in \mathbb{Z}} u_n(t, x_2) e^{inx_1} \quad \text{in} \quad (0,T) \times \omega,
\]

where, for \( n \in \mathbb{Z} \), \( u_n \in L^2((0,T) \times (c,d)) \) are the controls provided by Proposition 4.1, thus such that \( y_n \) and \( h_n \), which solve (1.9), vanish at time \( T > 0 \): \( y_n(T,\cdot) \equiv 0 \) in \((-1,1)\) and \( h_n(T) = 0 \). Defining \( y \) and \( h \) via Fourier series similarly as \( u \) above, we readily see that \((y, h)\) is the unique solution to (1.3). Moreover, since all the Fourier coefficients of \( y \) and \( h \) vanish at time \( T \), then also \((y, h)\) vanish at time \( T \). The estimate on the control follows by summing up the estimate of the Fourier coefficient controls over all \( n \), and using the fact that all the constants intervening in this estimate are independent of \( n \). This concludes the proof. \(\square\)
6. Proof of Theorem 1.2

Let us now consider \( \omega = (a, b) \times (c, d) \), where \((a, b) \subset \mathbb{T}\) and \((c, d) \subset (-1, 1)\). We consider the adjoint system

\[
\begin{cases}
-\partial_t \zeta - \Delta \zeta = 0 & \text{in } (0, T) \times \Omega, \\
-\partial_t \ell(t, x_1) - \partial_{x_1} \zeta(t, x_1, 1) = 0 & \text{on } (0, T) \times T, \\
\zeta(t, x_1, 0) = 0 & \text{on } (0, T) \times T, \\
\ell(t, x_1, 1) = \sigma \partial_{x_1}^2 \zeta(t, x_1) & \text{on } (0, T) \times T, \\
(\zeta, \ell)|_{t=T} = (\zeta_T, \ell_T) & \text{in } \Omega \times T.
\end{cases}
\] (6.1)

We also define the filtered space of low frequencies

\[
E_\mu := \bigoplus_{|n| \leq \mu, n \neq 0} H_n \otimes \left\{ \frac{1}{\sqrt{2\pi}} e^{inx_1} \right\}.
\]

Let \( \Pi_\mu : \mathcal{H} \rightarrow \mathcal{H} \) denote the orthogonal projection from \( \mathcal{H} \) onto \( E_\mu \).

6.1. Control of the low frequencies. We first recall the following version of the Lebeau-Robbiano spectral inequality for the eigenfunctions of the Laplacian on \( \mathbb{T} \) with periodic boundary conditions.

**Lemma 6.1** (Spectral inequality). Let \( a, b \in \mathbb{R} \) be such that \( a < b \). There exists \( C > 0 \) such that for every \( \mu > 0 \) and \( \{a_n\}_{n \in \mathbb{Z}} \subset \mathbb{C} \), the inequality

\[
\sum_{|n| \leq \mu} |a_n|^2 \leq Ce^{C\mu} \int_a^b \left| \sum_{|n| \leq \mu} a_n e^{inx_1} \right|^2 \ dx_1
\]

holds.

**Proof.** See [35], and also [34, Theorem 5.4], [5, Proposition 5]. \( \square \)

Using Lemma 6.1, we may derive the following observability inequality for the low frequencies.

**Proposition 6.1** (Filtered observability). Let \( \sigma > 0 \) be fixed, and suppose \( \omega = (a, b) \times (c, d) \), where \((a, b) \subset \mathbb{T}\) and \((c, d) \subset (-1, 1)\). Then, there exists a constant \( C > 0 \) such that for every \( T > 0 \), for every \( \mu > 0 \), and for every \((\zeta_T, \ell_T) \in E_\mu\), the unique solution \((\zeta, \ell) \in C^0([0, T]; L^2(\Omega) \times H^1(\mathbb{T}))\) to (6.1) satisfies

\[
\int_{\Omega} |\zeta(0)|^2 \ dx + \sigma \int_T \int_{\mathbb{T}} |\partial_{x_1} \ell(0)|^2 \ dx_1 \leq Ce^{C(\frac{1}{2} + \mu)} \int_0^T \int_{\mathbb{T}} |\zeta(t)|^2 \ dx \ dt.
\]

**Proof.** By assumption, we can write

\[
\left( \zeta^0(x_1, x_2), \ell^0(x_1) \right) = \frac{1}{\sqrt{2\pi}} \sum_{|n| \leq \mu, n \neq 0} \left( \zeta^0_n(x_2), \ell^0_n \right) e^{inx_1},
\]
with the Fourier coefficients being defined as in (1.8). The solution \((\zeta, \ell)\) to (6.1) is then given by
\[
\left(\zeta(t, x_1, x_2), \ell(t, x_1)\right) = \frac{1}{\sqrt{2\pi}} \sum_{|n| \leq \mu, \ell_n \neq 0} \left(\zeta_n(t, x_2), \ell_n(t)\right) e^{inx_1},
\]
where for every \(n \neq 0\), \((\zeta_n, \ell_n)\) denotes the unique solution to
\[
\begin{align*}
-\partial_t (\zeta_n, \ell_n) &= A_n (\zeta_n, \ell_n) \quad \text{in } (0, T), \\
(\zeta_n, \ell_n)|_{t=T} &= (\zeta_{T,n}, \ell_{T,n}).
\end{align*}
\]
Combining (4.22) with Lemma 6.1, we find that
\[
\int_\Omega |\zeta(0)|^2 \, dx + \sigma \int_0^T \int_\Omega |\partial_x \ell(0)|^2 \, dx \, dt = \sum_{|n| \leq \mu, n \neq 0} \left\| \left(\zeta_n(0), \ell_n(0)\right) \right\|^2_{\mathcal{H}_n}
\]
\[
\leq M_1 e^{\frac{M_2}{\mu}} \sum_{|n| \leq \mu, n \neq 0} \int_0^T \int_c^d \left| \zeta_n(t, x_2) \right|^2 \, dx_2 \, dt
\]
\[
\leq C e^{C \left(\frac{1}{\mu} + \mu\right)} \int_0^T \int_c^d \int_a^b \left\| \sum_{|n| \leq \mu, n \neq 0} \zeta_n(t, x_2) e^{inx_1} \right\|^2 \, dx_1 \, dx_2 \, dt
\]
holds for some constant \(C > 0\) independent of \(T\) and \(\mu\). This is the desired conclusion.

By customary HUM arguments, we also deduce the following result.

**Proposition 6.2** (Filtered control). Let \(\sigma > 0\) be fixed, and suppose \(\omega = (a, b) \times (c, d)\), where \((a, b) \subset \mathbb{T}\) and \((c, d) \subset (-1, 1)\). Then, there exists a constant \(C > 0\) such that for any \(T > 0\), for every \(\mu > 0\), and for every \((y^0, h^0) \in \mathcal{H}\), there exists \(u_\mu \in L^2((0, T) \times \omega)\) such that the unique solution \((y, h) \in C^0([0, T]; \mathcal{H})\) to (1.3) with control \(u_\mu\) satisfies
\[
\Pi_\mu (y(T), h(T)) = 0,
\]
and
\[
\left\| u_\mu \right\|_{L^2((0, T) \times \omega)} \leq C e^{C \left(\frac{1}{\mu} + \mu\right)} \left\| \left( y^0, h^0 \right) \right\|_{\mathcal{H}}.
\]

**Proof.** We again proceed by using customary HUM arguments, as done in the proof of Proposition 4.1, and also closely following [57]. Let us define the functional
\[
\mathcal{J}(\zeta_T, \ell_T) := \frac{1}{2} \int_0^T \int_\omega |\zeta|^2 \, dx \, dt - \left\langle \left(\zeta(0), \ell(0)\right), \left( y^0, h^0 \right) \right\rangle_{\mathcal{H}}.
\]
for \((\zeta_T, \ell_T) \in E_\mu\). The functional \(\mathcal{J}\) has a unique minimizer \((\zeta^*_T, \ell^*_T) \in E_\mu\) by virtue of the observability inequality of Proposition 6.1. Let \((\zeta^*, \ell^*)\) denote the corresponding
solution to (6.1). By writing down the Euler-Lagrange equation, one quickly finds that
the minimizer \((\zeta^*, \ell^*)\) is such that
\[
\mathcal{L}(y(T), h(T), (\varphi_T, s_T), \jmath) = 0
\]
for all \((\varphi_T, s_T) \in \mathcal{E}_\mu\). This yields (6.2), by definition of \(\Pi_\mu\). Estimate (6.3) follows similarly by virtue of Proposition 6.1.

6.2. Decay of the high frequencies. To go beyond the above theorem, as is common with the Lebeau-Robbiano method, we will need to make use of the following exponential decay result for the high frequency components.

Lemma 6.2 (Exponential decay). Suppose \(\sigma > 0\). The following statements hold.

(1) The part of \(A\) in \(\mathcal{H}\) generates an exponentially stable semigroup on \(\mathcal{H}\). More specifically,
\[
\|e^{tA}(y^0, h^0)\|_{\mathcal{H}} \leq e^{-\min\{(\frac{\sigma}{2}, 1)t\}}\|\(y^0, h^0)\|_{\mathcal{H}}
\]
holds for all \(t \geq 0\) and \((y^0, h^0) \in \mathcal{H}\).

(2) Let \(\mu > 0\) be fixed, and suppose that \((y^0, h^0) \in \mathcal{H}\) is such that \(\Pi_\mu(y^0, h^0) = 0\). Then the solution of (1.3) with \(u \equiv 0\) satisfies
\[
\|y(t), h(t)\|_{\mathcal{H}} \leq e^{-\min\{(\frac{\sigma}{2}, 1)\mu^2t\}}\|\(y^0, h^0)\|_{\mathcal{H}}
\]
for all \(t \geq 0\).

Proof. Let us prove the first claim. We may write
\[
(y^0(x_1, x_2), h^0(x_1)) = \frac{1}{\sqrt{2\pi}} \sum_{n \neq 0} (y^0_n(x_2), h^0_n) e^{inx_1},
\]
for \((x_1, x_2) \in \Omega\), with the Fourier coefficients being defined as in (1.8). The solution \((y, h)\) to (1.3) with \(u \equiv 0\) is then given by
\[
y(t, x_1, x_2), h(t, x_1) = \frac{1}{\sqrt{2\pi}} \sum_{n \neq 0} (y_n(t, x_2), h_n(t)) e^{inx_1},
\]
where \((y_n, h_n)\) is the unique solution to (1.9) with \(u_n \equiv 0\). On another hand, since \(n \neq 0\), and since \(A_n\) then has a decreasing sequence of negative eigenvalues \(\{\lambda_{n,k}\}_{k=0}^\infty\) with corresponding sequence of eigenfunctions \(\{\Phi_{n,k}\}_{k=0}^\infty\) forming an orthonormal basis of \(\mathcal{H}_n\), we may write
\[
y(t, x_2), h_n(t) = \sum_{k=0}^{+\infty} e^{\lambda_{n,k}t} \mathcal{L}(y_n(t, x_1), h_n(t), \Phi_{n,k}) \Phi_{n,k}(x_2).
\]
Using Lemma 4.2 and \(n^2 \geq 1\), it follows that
\[
\|y(t), h_n(t)\|_{\mathcal{H}_n} \leq e^{\lambda_{n,0}t} \|y_n, h_n\|_{\mathcal{H}_n} \leq e^{-\min\{(\frac{\sigma}{2}, 1)t\}n^2t} \|y_n, h_n\|_{\mathcal{H}_n} \leq e^{-\min\{(\frac{\sigma}{2}, 1)t\}t} \|y_n, h_n\|_{\mathcal{H}_n}
\]
for all \( t \geq 0 \). Combining (6.5), (6.6), and the Plancharel theorem to sum up over \( n \neq 0 \), we arrive to the desired conclusion.

The proof of the second claim is almost identical. We provide details nonetheless. Since \( \Pi_\mu (y^0, h^0) = 0 \), we may write
\[
(y^0(x_1, x_2), h^0(x_1)) = \frac{1}{\sqrt{2\pi}} \sum_{|n| > \mu} \left( y^0_n(x_2), h^0_n \right) e^{inx_1},
\]
with the Fourier coefficients being again defined as in (1.8). The solution \((y, h)\) to (1.3) with \( u \equiv 0 \) is then given by
\[
(y(t, x_1, x_2), h(t, x_1)) = \frac{1}{\sqrt{2\pi}} \sum_{|n| > \mu} \left( y_n(t, x_2), h_n(t) \right) e^{inx_1},
\]
where \((y_n, h_n)\) is the unique solution to (1.9) with \( u_n \equiv 0 \). We again make use of (6.6) and Lemma 4.2, this time to find that
\[
\| (y(t), h(t)) \|_{\mathcal{H}} \leq e^{\lambda_0 n t} \left\| \left( y^0_0, h^0_0 \right) \right\|_{\mathcal{H}} \leq e^{-\min \left\{ \frac{\pi}{2}, 1 \right\} n^2 t} \left\| \left( y^0_0, h^0_0 \right) \right\|_{\mathcal{H}} \leq e^{-\min \left\{ \frac{\pi}{2}, 1 \right\} \mu^2 t} \left\| \left( y^0_0, h^0_0 \right) \right\|_{\mathcal{H}}
\]
holds for all \( t \geq 0 \). Summing up over \(|n| > \mu\) yields the desired conclusion. \( \square \)

Combining Proposition 6.2 and Lemma 6.2, we obtain the following result.

**Proposition 6.3.** Let \( \sigma > 0 \) be fixed, and suppose \( \omega = (a, b) \times (c, d) \), where \((a, b) \subset \mathbb{T} \) and \((c, d) \subset (-1, 1) \). Then, there exists \( C > 0 \) such that for any \( T > 0 \), for any \((y^0, h^0) \in \mathcal{H} \), and for any \( \mu > 0 \), there exists a control \( u_\mu \in L^2((0, T) \times \omega) \) such that the unique solution \((y, h) \in C^0([0, T]; \mathcal{H})\) to (1.3) with control \( u_\mu \) satisfies
\[
\| u_\mu \|_{L^2((0, T) \times \omega)} \leq C e^{C \left( \frac{1}{\sigma} + \mu \right)} \left\| (y^0, h^0) \right\|_{\mathcal{H}}, \tag{6.7}
\]
and
\[
\left\| (y(T), h(T)) \right\|_{\mathcal{H}} \leq \left( 1 + C \sqrt{T} \right) e^{C \left( \frac{1}{\sigma} + \mu \right) - \min \left\{ \frac{\pi}{2}, 1 \right\} \mu^2 \frac{T}{\pi}} \left\| (y^0, h^0) \right\|_{\mathcal{H}}. \tag{6.8}
\]

**Proof.** By virtue of Proposition 6.2, there exists a control \( \pi_\mu \in L^2 \left( (0, \frac{T}{2}) \times \omega \right) \) such that the solution \((\overline{y}, \overline{h})\) to (1.3) with control \( \overline{u}_\mu \) satisfies
\[
\Pi_\mu \left( \overline{y} \left( \frac{T}{2} \right), \overline{h} \left( \frac{T}{2} \right) \right) = 0, \tag{6.9}
\]
and
\[
\| \pi_\mu \|_{L^2((0, \frac{T}{2}) \times \omega)} \leq C e^{C \left( \frac{1}{\sigma} + \mu \right)} \left\| (y^0, h^0) \right\|_{\mathcal{H}}. \tag{6.10}
\]

The constant \( C > 0 \), stemming from Proposition 6.2, is independent of both \( T > 0 \) and \( \mu > 0 \). By virtue of the Duhamel formula, the semigroup contractivity per Lemma 6.2,
the Cauchy-Schwarz inequality, and (6.10), we find
\[
\left\| \left( y \left( T_2 \right), h \left( T_2 \right) \right) \right\|_\mathcal{H} \leq \left\| \left( y^0, h^0 \right) \right\|_\mathcal{H} + \sqrt{T} \| \pi \|_{L^2((0, T) \times \omega)}
\leq \left( 1 + C\sqrt{T} \right) e^{C(\frac{1}{T} + \mu)} \| \left( y^0, h^0 \right) \|_\mathcal{H}.
\] (6.11)

Between \( T_2 \) and \( T \), we let the system dissipate according to Lemma 6.2. Namely, let us define
\[
u(t) := \begin{cases} 
\pi(t) & \text{if } t \in (0, \frac{T}{2}) \\
0 & \text{if } t \in \left( \frac{T}{2}, T \right).
\end{cases}
\]
From the above definition, we immediately have (6.7). Finally, using the second claim in Lemma 6.2 combined with (6.9), and subsequently (6.11), we deduce
\[
\left\| \left( y(T), h(T) \right) \right\|_\mathcal{H} \leq e^{-\min \left\{ \frac{\pi}{2}, 1 \right\} \mu \frac{T}{2}} \left\| \left( y \left( T \right), h \left( T \right) \right) \right\|_\mathcal{H}
\leq \left( 1 + C\sqrt{T} \right) e^{C(\frac{1}{T} + \mu)} e^{-\min \left\{ \frac{\pi}{2}, 1 \right\} \mu \frac{T}{2}} \| \left( y^0, h^0 \right) \|_\mathcal{H}.
\]
This concludes the proof. \( \square \)

**Remark 4.** In the constants appearing in the estimates (6.7) and (6.8), the term \( T \) actually indicates the length of the interval, rather than the endpoint. In other words, the interval \( (0, T) \) may be replaced by an arbitrary interval \( (\tau_1, \tau_2) \) in the above result, in which case the \( T \) appearing in the constants of these estimates would be replaced by \( \tau_2 - \tau_1. \)

### 6.3. Proof of Theorem 1.2
We may now complete the proof of our second main result.

**Proof of Theorem 1.2.** Let \( \beta > 0 \) be fixed and to be chosen suitably later on. For \( j \geq 1 \), we consider the dyadic sequences
\[
T_j := 2^{-j}T \quad \text{and} \quad \mu_j := 2^j \beta,
\]
and we define the sequence
\[
\tau_j := \begin{cases} 
0 & \text{for } j = 0, \\
\sum_{k=1}^{j} T_k & \text{for } j \geq 1.
\end{cases}
\]
We note that
\[
(0, T) = \bigcup_{j=0}^{+\infty} (\tau_j, \tau_{j+1}).
\]
For \( j \geq 1 \), on any interval \( (\tau_{j-1}, \tau_j) \), by virtue of Proposition 6.3, with \( \mu = \mu_j \), we may build a control \( u_j \in L^2((\tau_{j-1}, \tau_j) \times \omega) \) with corresponding state \( (y_j, h_j) \in C^0((\tau_{j-1}, \tau_j); \mathcal{H}) \) such that
\[
\left\| u_j \right\|_{L^2((\tau_{j-1}, \tau_j) \times \omega)} \leq Ce^{C \left( \frac{1}{T_j} + \mu_j \right)} \left\| \left( y_j(\tau_{j-1}), h_j(\tau_{j-1}) \right) \right\|_\mathcal{H},
\] (6.12)
as well as
\[
\left\| \left( y_j (\tau_j), h_j (\tau_j) \right) \right\|_{\mathcal{H}} \leq \left( 1 + C \sqrt{T} \right) e^{C \left( \frac{1}{T_j} + \mu_j \right) - \min \left\{ \frac{\sigma}{2}, 1 \right\} \mu_j^2 T_j} \left\| \left( y_j (\tau_{j-1}), h_j (\tau_{j-1}) \right) \right\|_{\mathcal{H}} \tag{6.13}
\]
hold, with the convention \((y_1 (0), h_1 (0)) = (y^0, h^0)\). From (6.12) and (6.13), we gather
\[
\| u_j \|_{L^2 ((\tau_{j-1}, \tau_j) \times \omega)} \leq C \left( 1 + C \sqrt{T} \right)^{j-1} \exp \left( \sum_{k=1}^{j} C \left( \frac{1}{T_k} + \mu_k \right) - \min \left\{ \frac{\sigma}{2}, 1 \right\} \sum_{k=1}^{j-1} \mu_k^2 T_k \right) \left\| \left( y^0, h^0 \right) \right\|_{\mathcal{H}} \tag{6.14}
\]
as well as
\[
\left\| \left( y_j (\tau_j), h_j (\tau_j) \right) \right\|_{\mathcal{H}} \leq \left( 1 + C \sqrt{T} \right)^{j} \exp \left( \sum_{k=1}^{j} C \left( \frac{1}{T_k} + \mu_k \right) - \min \left\{ \frac{\sigma}{2}, 1 \right\} \mu_k^2 T_k \right) \left\| \left( y^0, h^0 \right) \right\|_{\mathcal{H}} \tag{6.15}
\]
Let us now define
\[
\beta_0 := - \left( \frac{2C}{T} + 2\beta C - \min \left\{ \frac{\sigma}{2}, 1 \right\} \beta^2 T \right)
\]
and
\[
\beta_1 := - \left( \frac{2C}{T} + 2\beta C - \min \left\{ \frac{\sigma}{2}, 1 \right\} \beta^2 T \right),
\]
and we choose \(\beta > 0\) sufficiently large so that \(\beta_0 > 0\) and \(\beta_1 > 0\). We may observe that
\[
\sum_{k=1}^{j} \left[ C \left( \frac{1}{T_k} + \mu_k \right) - \min \left\{ \frac{\sigma}{2}, 1 \right\} \mu_k^2 T_k \right] = \frac{2C}{T} (2^j - 1) + 2\beta C (2^j - 1) - \min \left\{ \frac{\sigma}{2}, 1 \right\} T (2^j - 1) = -2^j \beta_0 + \beta_0,
\]
as well as
\[
\sum_{k=1}^{j} C \left( \frac{1}{T_k} + \mu_k \right) - \min \left\{ \frac{\sigma}{2}, 1 \right\} \sum_{k=1}^{j-1} \mu_k^2 T_k \right) = -2^j \beta_1 + \beta_0.
\]
Thus, from (6.14) and (6.15), we infer that
\[
\| u_j \|_{L^2 ((\tau_{j-1}, \tau_j) \times \omega)} \leq \frac{C}{1 + C \sqrt{T}} \left( 1 + C \sqrt{T} \right)^{j} e^{\beta_0 - 2^j \beta_1} \left\| \left( y^0, h^0 \right) \right\|_{\mathcal{H}} \tag{6.16}
\]
and
\[
\left\| \left( y_j (\tau_j), h_j (\tau_j) \right) \right\|_{\mathcal{H}} \leq \left( 1 + C \sqrt{T} \right)^{j} e^{\beta_0 - 2^j \beta_1} \left\| \left( y^0, h^0 \right) \right\|_{\mathcal{H}} \tag{6.17}
\]
We select \(m = m(T) \in \mathbb{N}\) sufficiently large so that
\[
\frac{1}{1 + C \sqrt{T}} < 2^m. \tag{6.18}
\]
Observe that, since $\beta_1 > 0$, from (6.16) we have
\[
\|u_j\|_{L^2((\tau_j-1,\tau_j) \times \omega)} \leq \frac{C e^{\beta_0 \beta_1^m}}{1 + C \sqrt{T}} \left( (1 + C \sqrt{T}) 2^{-m} \right)^j \| (y^0, h^0) \|_{\mathcal{F}^m} \leq C_1 \left( (1 + C \sqrt{T}) 2^{-m} \right)^j \| (y^0, h^0) \|_{\mathcal{F}^m},
\]
(6.19)
where $C_1 = C_1(T, \sigma) > 0$ is independent of $j$. We define the control $u : (0, T) \times \omega \to \mathbb{R}$ by pasting all the $u_j$ as
\[
u = \sum_{j=1}^{\infty} u_j 1_{(\tau_j-1,\tau_j)}.
\]
By virtue of (6.19) and (6.18), we get
\[
\|u\|_{L^2(0,T;L^2(\omega))}^2 = \sum_{j=1}^{\infty} \|u_j\|_{L^2((\tau_j-1,\tau_j) \times \omega)}^2 = C_2 \| (y^0, h^0) \|_{\mathcal{F}^m}^2 \sum_{j=1}^{\infty} \left( (1 + C \sqrt{T}) 2^{-m} \right)^{2j} = C_2 \| (y^0, h^0) \|_{\mathcal{F}^m}^2,
\]
for some $C_2 = C_2(T, \sigma) > 0$, and therefore $u \in L^2((0, T) \times \omega)$. Now consider the unique solution $(y, h) \in C^0([0, T]; L^2(\Omega) \times H^1(\Omega))$ to (1.3), with control $u$. Clearly
\[(y(t), h(t)) = (y_j(t), h_j(t)) \quad \text{for } t \in [\tau_j-1, \tau_j].
\]
In particular,
\[(y(\tau_j), h(\tau_j)) = (y_j(\tau_j), h_j(\tau_j)),
\]
so from (6.17) we deduce that $(y(T, \cdot), h(T, \cdot)) \equiv 0$ in $\Omega \times \mathbb{T}$. This concludes the proof.

7. Epilogue

We have shown that the linearized Stefan problem with surface tension (Gibbs-Thomson correction) is null-controllable, in the sense that both the temperature and the height function are controllable to zero.

We expect that this result should lead to a local controllability result for the nonlinear problem, pending technical developments. There are, however, several questions and problems which we believe merit further attention and clarity, even in the linearized regime. In addition to addressing the case $\sigma = 0$, discussed in greater depth in Section 2, other problems may include

1. Control on the free boundary. While not exactly perfectly clear to interpret in physical terms, one can consider the problem of controlling through the free boundary, which would mean putting a control $u 1_{\omega}$ in the second equation in (1.3) (namely the evolution equation for $h$). This would be in the spirit of works in control of water waves ([3]), and also the simplified piston problem ([12]).

\footnote{As a matter of fact, assuming $T < 1$, and with suitable choice of $\beta > 0$ above, we can refine this estimate to obtain
\[
\|u\|_{L^2((0,T) \times \omega)}^2 \leq C \| (y^0, h^0) \|_{\mathcal{F}^m}^2,
\]
for some $C = C(\sigma) > 0$ depending on $\sigma$, but independent of $T$.}
expect this problem to be significantly more challenging to address than the one we had considered here.

(2) Spectral optimization. The governing operator $A$ of (1.3) appears somewhat opaque, and a variety of alternative (control) problems can be envisaged and studied regarding (1.3). We believe that further analysis is warranted in the analysis of the spectral properties of $A$, in particular with regard to extension of various control results to general geometries (beyond strips).

As we have noted, $-A$ is self-adjoint when considered on the space $\mathcal{H}$ defined in (1.10). In particular, one could define the first eigenvalue $\lambda_1 > 0$ of $-A$ through the min-max theorem as the Rayleigh quotient

$$\lambda_1 := \inf_{f\in\mathcal{D}(A)\setminus\{0\}} \frac{\langle -Af, f \rangle_{\mathcal{H}}}{\|f\|_{\mathcal{H}}^2}.$$  

But as is typical for the Laplacian, one seeks to use the symmetry and obtain a more tractable representation. In [28, Lemma 4.5] (see also [26, Chapter 3, Section 4, Lemma 4.5]), it also is stated that

$$\lambda_1 = \inf_{f=(f_1,f_2)\in S}\frac{\int_{\Omega} |\nabla f_1|^2 \, dx}{\int_{\Omega} |f_1|^2 \, dx + \sigma \int_{\mathbb{T}} |\partial_{x_1} f_2|^2 \, dx_1}, \quad (7.1)$$

where the space $S$ is defined as

$$S := \left\{ f = (f_1,f_2) \in H^1(\Omega) \times H^{5/2}(\mathbb{T}) \mid \int_{\Omega} f_1(x_1,\cdot) \, dx_1 + \int_{\mathbb{T}} f_2(x_1) \, dx_1 = 0 \right\}.$$  

Taking stock of (7.1), there are a variety of different spectral optimization problems one could then envisage for (1.3), such as characterizing optimal actuator and observer domains $\omega$ in the spirit of [24, 45, 46, 47], and in particular, comparing how these designs differ from that of the classical heat equation, or the limit of these designs as $\sigma \searrow 0$ (should controllability hold for the latter).

(3) The obstacle problem. It is by now well-known that the classical Stefan problem ($\sigma = 0$), without source terms, is related to the parabolic obstacle problem through the so-called Duvaut transform (see [19, 51] and the references therein). For control purposes, one could envisage transferring results from the Stefan problem to the parabolic obstacle problem (which is actually a problem to be studied in its own right, [20, 51]). But this is highly nontrivial due to the fact that the Duvaut transform applies to non-negative solutions of the Stefan problem, and it is not clear if existing techniques on controllability under positivity constraints ([39], or the so-called staircase method [41, 44, 52]) would be applicable here. The bottom line is that the controllability properties of the parabolic obstacle problem remain widely open. (See [22, Section 1.5.1].)

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Appendix A. Numerics

A.1. Discretizing (1.3). As we did not find precisely the same scheme in the literature, for completeness and future reproducibility purposes, let us briefly discuss the numerical discretization we used for computing, and obtaining the simulations presented in Figure 2 and Figure 3.

1. Setup. We shall focus on $\Omega = (0, 2) \times (-1, 1)$ for simplicity. The control domain $\omega$ is a thin neighborhood of a line ranging from $x_1 = \frac{1}{2}$ to $x_1 = \frac{3}{2}$, tilted at an angle of $45^\circ$ (i.e. with slope 1) with respect to the horizontal axis (chosen for the relative simplicity of numerical implementation, and the sparsity of the resulting matrix). We choose the same mesh-size $\Delta x > 0$ for the horizontal and vertical variables, and we set $n_x := \frac{2}{\Delta x} - 1$. Since we are working with periodic boundary conditions in the $x_1$-variable, and Dirichlet boundary conditions in the $x_2$-variables, $y(t, x_1, x_2)$ will be unknown at $(n_x + 1)n_x$ points, and $h(t, x_1)$ at $(n_x + 1)$ points.

2. Finite difference semi-discretization. We define an equi-distributed grid $\left\{x_1^i, x_2^j\right\}_{i \in \{0, \ldots, n_x+1\}, j \in \{0, \ldots, n_x+1\}}$ of $\Omega$ through

$$x_1^i = i \Delta x \quad \text{and} \quad x_2^j = -1 + j \Delta x.$$ 

We discretize the two-dimensional Laplacian $\Delta x_1, x_2$ with the classical 5-point finite-difference stencil, and the Neumann trace $\partial_{x_2} y(t, x_1, 1)$ with a centered difference scheme. Henceforth denoting

$$y_{i,j}(t) := y(t, x_1^i, x_2^j),$$

with analog definitions for $u_{i,j}$ and $h_i$, the finite-difference semi-discretization of (1.3) reads as

$$\begin{align*}
\dot{y}_{i,j} &= \frac{y_{i+1,j} + y_{i-1,j} + y_{i,j+1} + y_{i,j-1} - 4y_{i,j}}{\Delta x^2} u_{i,j} \chi_{\omega_{n_x}} \quad \{1, \ldots, n_x + 1\} \times \{1, \ldots, n_x\}, \\
\dot{h}_i &= \frac{y_{i,n_x+1} - y_{i,n_x-1}}{2 \Delta x} \quad \{1, \ldots, n_x + 1\}, \\
y_{0,j} &= y_{n_x+1,j} \quad \{1, \ldots, n_x\}, \\
y_{i,0} &= 0 \quad \{1, \ldots, n_x + 1\}, \\
y_{i,n_x+1} &= \sigma \frac{h_{i+1} + h_{i-1} - 2h_i}{\Delta x^2} \quad \{1, \ldots, n_x + 1\},
\end{align*}$$

(A.1)
for \( t \in (0, T) \). With analog definitions for \( u_{[j]} \) and \( h \), setting

\[
y_{[j]} := \begin{bmatrix} y_{1,j} \\ y_{2,j} \\ \vdots \\ y_{n_{x}+1,j} \end{bmatrix}
\]

for \( j \in \{1, \ldots, n_{x}\} \), as well as \( z := (y_{[1]}, \ldots, y_{[n_{x}]}, h) \) and then, similarly, setting \( u := (u_{[1]}, \ldots, u_{[n_{x}]}, 0_{\mathbb{R}^{n_{x}+1}}) \), we may rewrite (A.1) as a canonical finite-dimensional linear system \( \dot{z} = A_{n_{x}} z + B_{n_{x}} u \), where

\[
A_{n_{x}} := \begin{bmatrix} A_{0} & A_{1} \\ A_{1} & A_{0} \\ & \ddots & \ddots \\ & & A_{1} & A_{0} \\ & & & A_{1} & A_{0} \\ & & & & A_{1} & A_{0} \\ & & & & & A_{2} \\ & & & & & & A_{3} \\ & & & & & & & A_{4} \\
\end{bmatrix}
\]

where \( A_{1} = \frac{1}{\Delta x^{2}} \text{Id}_{n_{x}+1} \) and \( A_{3} = -\frac{1}{2\Delta x} \text{Id}_{n_{x}+1} \), whereas

\[
A_{0} = \frac{1}{\Delta x^{2}} \begin{bmatrix} -4 & 1 & 1 \\ 1 & -4 & 1 \\ & \ddots & \ddots \\ & & 1 & -4 & 1 \\ & & & 1 & -4 \end{bmatrix}
\]

and \( A_{\ell} = c_{\ell} \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ & \ddots & \ddots \\ & & 1 & -2 & 1 \\ & & & 1 & -2 \end{bmatrix} \),

for \( \ell \in \{2, 4\} \), with \( c_{2} = \frac{\sigma}{\Delta x} \) and \( c_{4} = \frac{2\sigma}{\Delta x^{2}} \).

![Figure 6. The sparsity pattern of the matrix \( A_{n_{x}} \) when \( \sigma = 10 \) (left) and \( \sigma = 0 \) (right).](image-url)
(3) **Time-stepping.** Given nodes \( \{t_k\}_{k \in \{0, \ldots, n_t\}} \) with \( t_k = k \Delta t \) for some \( \Delta t = \frac{T}{n_t} \), we set \( z^k = z(t_k) \). Time-stepping for (1.3) is done with a Crank-Nicolson method

\[
\frac{z^{k+1} - z^k}{\Delta t} = \frac{1}{2} A_{n_x} (z^k + z^{k+1}) + \frac{1}{2} B_{n_x} (u^k + u^{k+1})
\]

for \( k \in \{0, \ldots, n_t\} \), which is unconditionally stable.

A.2. **Computing.** We solve

\[
\min_{u \in L^2((0,T) \times \omega)} \|u\|_{L^2((0,T) \times \omega)}^2 \text{ subject to } (g,h) \text{- solve (1.3)} \quad (g(T),h(T)) = 0
\]

by making use of the above discretization for parametrizing the PDE constraints. This is a convex program, which can be solved using interior point methods. We use the IpOpt solver embedded into Casadi for Matlab ([4]). For the experiments in Figure 2 and Figure 3, we took \( \Delta x = \frac{2}{13} \), and thus \( n_x = 12 \), with \( n_t = 200 \). All codes are openly made available at https://github.com/borjanG/2022-stefan-control.

**APPENDIX B. LINEARIZATION**

B.1. **Fixing the domain.** A simple change of variables which allows us to pass from (1.2), set in the moving domain \( \Omega(t) \), to a nonlinear problem in the time-independent reference domain \( \Omega \), consists in defining the map \( \Psi(t, \cdot) : \overline{\Omega} \rightarrow \overline{\Omega(t)} \) for \( t \geq 0 \) by

\[
\Psi(t, x) := \left( x_1, \left(1 + h(t,x_1)\right)x_2 \right), \quad x = (x_1, x_2) \in \overline{\Omega}.
\]

If \( h(t, \cdot) \in H^s(\mathbb{T}) \), then \( \Psi(t, \cdot) \in H^s(\Omega) \) – namely, the transformation \( \Psi(t, \cdot) \) preserves the spatial regularity of \( h \). This is not desirable when \( d \geq 3 \), since one would have to ensure that \( h \) is rather smooth, and consequently, the control too.

One can exhibit a slightly different transformation which entails a gain in spatial regularity with respect to that of the height function \( h(t, \cdot) \). Given \( h \in C^0([0,T]; H^s(\mathbb{T})) \) for some \( s \geq 0 \), for any \( t \geq 0 \) we consider the solution \( \psi(t, \cdot) \) to the Poisson problem

\[
\Delta \psi(t, \cdot) = 0 \quad \text{in } \Omega
\]

\[
\psi(t, x_1, 0) = 0 \quad \text{on } \mathbb{T}
\]

\[
\psi(t, x_1, 1) = h(t, x_1) \quad \text{on } \mathbb{T},
\]

and we define \( \Psi(t, \cdot) : \overline{\Omega(t)} \rightarrow \overline{\Omega(t)} \) by

\[
\Psi(t, x) := (x_1, x_2 + \psi(t,x)).
\]

In this case, if \( h(t, \cdot) \in H^s(\mathbb{T}) \) then \( \Psi(t, \cdot) \in H^{s+1/2}(\Omega) \). Note that \( \Psi \) is similar to the transformation defined in [25, Eq. (1.6)]. From elliptic estimates, it can be seen that

\[
\|\Psi(t, \cdot) - \text{Id}\|_{H^{s+1/2}(\Omega)} \lesssim \|h(t, \cdot)\|_{H^s(\mathbb{T})}
\]

for all \( t \geq 0 \), so whenever \( h(t, \cdot) \) is sufficiently small, \( \Psi(t, \cdot) \) is a diffeomorphism from \( \overline{\Omega(t)} \) onto \( \overline{\Omega(t)} \) by the inverse function theorem.
B.2. **Linearizing.** In this case, we denote by $X(t, \cdot) = [\Psi(t, \cdot)]^{-1}$ the inverse of $\Psi(t, \cdot)$ for all $t \geq 0$, and consider

$$y(t, x) = \rho(t, \Psi(t, x)) \quad \text{for } (t, x) \in (0, T) \times \Omega.$$ 

In other words,

$$\rho(t, z) = y(t, X(t, z)) \quad \text{for } (t, z) \in (0, T) \times \Omega(t).$$

We also introduce the standard notation

$$B_{\Psi} := \text{Cof}(\nabla \Psi), \quad \text{and} \quad A_{\Psi} := \frac{1}{\det(\nabla \Psi)} B_{\Psi}^\top B_{\Psi},$$

where $\delta_{\Psi} := \det(\nabla \Psi)$ denotes the Jacobian determinant of $\nabla \Psi$, and $\text{Cof}(M)$ denotes the cofactor matrix of $M$, satisfying $M(\text{Cof}(M))^\top = (\text{Cof}(M))^\top M = \det(M) \text{Id}$. System (1.2) can then be equivalently rewritten as

$$\begin{cases}
\partial_t y - \Delta y = N_1(y, h) & \text{in } (0, T) \times \Omega, \\
\partial_t h(t, x, 1) = (\nabla y(t, x, 1, 1) - e_2) + (N_3(y(t, x, 1, 1)) \cdot e_2) & \text{on } (0, T) \times \mathbb{T}, \\
y(t, x_1, -1) = 0 & \text{on } (0, T) \times \mathbb{T}, \\
y(t, x_1, 1) = \sigma \partial^2_{x_1} h(t, x_1) + N_2(h(t, x_1)) & \text{on } (0, T) \times \mathbb{T}, \\
(y, h)|_{t=0} = (y^0, h^0) & \text{in } \Omega \times \mathbb{T},
\end{cases} \quad (B.1)$$

where $y^0(\cdot) := \rho^0(\Psi(0, \cdot), e_2 := (0, 1)^\top$, with the nonlinear terms having the form

$$N_1(y, h) := -(\det(\nabla \Psi) - 1) \partial_t y + \text{div}(N_3), \quad N_3(y) := (A_{\Psi} - \text{Id}) \nabla y.$$ 

and

$$N_2(h) := \sigma (\kappa(h) - \partial^2_{x_1} h).$$

The linearized problem can then be obtained simply by dropping $N_j$ from (B.1).

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