NON-COMMUTATIVE PROJECTIVE CALABI–YAU
SCHEMES

ATSUSHI KANAZAWA

ABSTRACT. The objective of the present article is to construct the first examples of (non-trivial) non-commutative projective Calabi–Yau schemes in the sense of Artin and Zhang [1].

1. INTRODUCTION

The present article is concerned with certain non-commutative Calabi–Yau projective schemes. Recently non-commutative Calabi–Yau algebras have attracted considerable attention [7, 6, 15, 12] due to their fruitful connections to superstring theory. However, almost all known non-commutative Calabi–Yau algebras are quiver algebras and thus non-commutative analogues of local Calabi–Yau manifolds. The objective of this article is to construct the first examples of (non-trivial) non-commutative projective Calabi–Yau schemes in the sense of Artin and Zhang [1]. The main theorem of the article is the following:

Theorem 1.1 (Theorem 2.1). Let $k$ be an algebraically closed field of characteristic zero and consider the following graded $k$-algebra

$$A_n := k \langle x_1, \ldots, x_n \rangle / \left( \sum_{k=1}^{n} x_k^n, x_i x_j = q_{ij} x_j x_i \right)_{i,j},$$

where the quantum parameters $q_{ij} \in k^\times$ satisfy $q_{ii} = q_{ij}^n = q_{ij} q_{ji} = 1$. Then the quotient category $\text{Coh}(A_n) := \text{gr}(A_n)/\text{tor}(A_n)$ is a Calabi–Yau $(n-2)$ category if and only if $\prod_{i=1}^{n} q_{ij}$ is independent of $1 \leq j \leq n$.

Moreover, we show that there exist quantum parameters $q_{i,j}$'s such that the graded $k$-algebra $A_n$ is not realized as a twisted coordinate ring of a Calabi–Yau $(n-2)$-fold.

One motivation of our study comes from a virtual counting theory of the stable sheaves on a polarized complex Calabi–Yau threefold [13]. In [12], Szendrői introduced a non-commutative version of the theory for the quiver Calabi–Yau 3 algebras [6]. However, it relies on the existence of the global Chern–Simons function on the moduli space of stables modules and cannot

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be readily generalized to the projective case. In [8], the author developed a virtual counting theory of the stable modules over a non-commutative projective Calabi–Yau scheme based on the work [4]. The above $k$-algebra $A_n$ serves as an important example of the general theory [8].

2. NON-COMMUTATIVE CALABI–YAU PROJECTIVE SCHEMES

We begin with a review of the notion of non-commutative projective geometry introduced by Artin and Zhang [1]. Throughout this article, non-commutative means not necessarily commutative.

2.1. Non-commutative Projective Schemes. Let $k$ be a field and $A = \bigoplus_{i=0}^{\infty} A_i$ be a connected noetherian graded $k$-algebra. We assume that each graded piece is finite dimensional and $A_0 \cong k$. We denote by $\text{Gr}(A)$ the category of graded right $A$-modules with morphisms the $A$-module homomorphisms of degree zero and by $\text{gr}(A)$ the subcategory consisting of finitely generated right $A$-modules. The augmentation ideal of $A$ is defined by $m := \bigoplus_{i=1}^{\infty} A_i$.

Let $M = \bigoplus_{i=1}^{\infty} M_i$ be a graded right $A$-module. Let $\text{Tor}(A)$ denote the subcategory of $\text{Gr}(A)$ of torsion modules and $\text{tor}(A)$ denote the intersection of $\text{Tor}(A)$ and $\text{gr}(A)$. For an integer $n \in \mathbb{Z}$ and graded $A$-module $M$ we define $M(n)$ as the graded $A$-module that is equal to $M$ as an $A$-module, but with grading $M(n)_i := M_{n+i}$. We refer to the functor $s : \text{Gr}(A) \to \text{Gr}(A), M \mapsto M(1)$ as the shift functor and $s^n$ as the $n$-th shift functor.

In [1], Artin and Zhang introduced the notion of a non-commutative projective scheme as follows. We define $\text{Tails}(A) := \text{Gr}(A)/\text{Tor}(A)$. The canonical exact functor from $\text{Gr}(A)$ to $\text{Tails}(A)$ is denoted by $\pi$. We define $\text{tails}(A) := \text{gr}(A)/\text{tor}(A)$ in a similar manner. If $M \in \text{Gr}(A)$, we use the corresponding script letter $\mathcal{M}$ for $\pi(M)$. For example $\mathcal{A} := \pi(A_A)$ where $A_A$ is $A$ viewed as a right $A$-module. The non-commutative projective scheme of a graded right noetherian $k$-algebra $A$ is defined as the triple

$$\text{proj}(A) := (\text{tails}(A), \mathcal{A}, s).$$

Let $X = \text{proj}(A)$. Since $\text{Tails}(A)$ is an abelian category with enough injectives, we may define the functors $\text{Ext}^i_{\text{Tails}(A)}(\mathcal{M}, \star)$ as the $i$-th right derived functor of $\text{Hom}_{\text{Tails}(A)}(\mathcal{M}, \star)$. In particular the global section functor

$$H^0(X, \star) := \text{Hom}_{\text{Tails}(A)}(\mathcal{A}, \star) : \text{Tails}(A) \to \text{Vect}_k$$

induces the higher cohomologies $H^i(X, \mathcal{M}) := \text{Ext}^i_{\text{Tails}(A)}(\mathcal{A}, \mathcal{M})$. The bifunctor $\text{Ext}^i_{\text{Tails}(A)}(\star, \mathcal{M})$ is defined as restriction of $\text{Ext}^i_{\text{Tails}(A)}(\star, \mathcal{M})$ on $\text{tails}(A)$. We say that a noetherian graded $k$-algebra $A$ satisfies condition $\chi$ if $\dim_k \text{Ext}^i_{\text{Tails}(A)}(k, M) < \infty$ for all $i \geq 0$. 

2.2. Calabi–Yau Condition. Let $k$ be an algebraic closed field of characteristic zero. We denote by $A_n$ the non-commutative graded $k$-algebra

$$A_n := k\langle x_1, \ldots, x_n \rangle / \left( \sum_{k=1}^{n} x_k^{n}, \ x_i x_j = q_{ij} x_j x_i \right)_{i,j},$$

where the quantum parameters $q_{ij}$’s are $n$-th roots of unity with $q_{ii} = q_{ij} q_{ji} = 1$. The graded $k$-algebra $A_n$ is of the form $A_n = B_n / (f_n)$ where

$$B_n := k\langle x_1, \ldots, x_n \rangle / (x_i x_j = q_{ij} x_j x_i)_{i,j}, \quad f_n := \sum_{k=1}^{n} x_k^n.$$

The $k$-algebra $B_n$ is a Koszul Artin–Shelter (AS) regular algebra. We observe that $f_n$ is a normalizing element of degree equal to the global dimension of $B_n$. Thus informally $\text{proj}(A_n)$ is the non-commutative Fermat hypersurface in quantum $\mathbb{P}^{n-1}$. This example was previously studied in physics \cite{3, 5} without much mathematical justification.

**Theorem 2.1.** Let $A_n$ be the $k$-algebra defined above. Then $\text{proj}(A_n)$ is a Calabi–Yau $(n-2)$-projective scheme if and only if $\prod_{i=1}^{n} q_{ij}$ is independent of $1 \leq j \leq n$.

Here we say that $\text{proj}(A)$ is a Calabi–Yau $m$ projective scheme if $\text{gl.dim}(\text{tails}(A)) = m$ and $\text{tails}(A)$ has a functorial perfect paring

$$\text{Ext}^i(\mathcal{M}, \mathcal{N}) \otimes_k \text{Ext}^{m-i}(\mathcal{N}, \mathcal{M}) \rightarrow k$$

for all $\mathcal{M}, \mathcal{N} \in \text{tails}(A)$. By passing $\text{tails}(A_n)$ to its derived category, we get a Calabi–Yau triangulated $(n-2)$ category in the sense of \cite{9}.

**Example 2.2.** Let $X = \text{Proj}(C) \subset \mathbb{P}^4$ be the Fermat quintic threefold given by

$$C := k[x_1, x_2, x_3, x_4, x_5] / (\sum_{i=1}^{5} x_i^5).$$

Let $q_i$ be a 5-th root of unity for $1 \leq i \leq 5$. Then the map

$$[x_1 : x_2 : x_3 : x_4 : x_5] \mapsto [q_1 x_1 : q_2 x_2 : q_3 x_3 : q_4 x_4 : q_5 x_5]$$

induces a projective automorphism $\sigma$ of $X$. The twisted homogeneous coordinate ring $C^\sigma$ is then given by

$$C^\sigma := k\langle x_1, x_2, x_3, x_4, x_5 \rangle / \left( \sum_{i=1}^{5} x_i^{5}, \ x_i x_j = q_{ij} x_j x_i \right)_{i,j},$$

where $q_{ij} := q_i q_j^{-1}$. A result of Zhang \cite{16} implies an equivalence of categories $\text{tails}(C) \cong \text{tails}(C^\sigma)$. In particular $\text{tails}(C^\sigma)$ is a Calabi–Yau 3 category. Note that for any $1 \leq j \leq 5$ we have $\prod_{i=1}^{5} q_{ij} = q_1 q_2 q_3 q_4 q_5$, which is compatible with Theorem 2.1.
If the graded \( k \)-algebra \( A_n \) is realized as a twisted coordinate ring of a commutative projective scheme \( X \), then \( \text{tails}(A_n) \cong \text{Coh}(X) \) as above and thus \( \text{tails}(A_n) \) is not really interesting. In Section 3 we will show that there exists a non-commutative Calabi–Yau \((n - 2)\)-fold that is not realized as a twisted coordinate ring of a Calabi–Yau \((n - 2)\)-fold.

In the rest of this section, we shall prove Theorem 2.1, assuming that \( \text{gl.dim}(\text{tails}(A_n)) = n - 2 \), the proof of which will be given in Section 2.3. Henceforth we write \( A = A_n \) and \( B = B_n \) for notational convenience. We begin with a study of the balanced dualizing complex \( R_A \) of \( A \), which plays a role of dualizing sheaf in non-commutative graded algebra [17]. It behaves better than a dualizing complex and corresponds, in the commutative case, to the local duality. Since \( A \) has finite global dimension and is finite over its center \( Z(A) \), \( A \) satisfies the condition \( \chi \). Then there is a formula [17, 14] for the balanced dualizing complex \( R_A \) of \( A \) as a graded ring

\[
R_A = R\Gamma_m(A') \in D^b(\text{tails}(A))
\]

where \( \Gamma_m \) denotes local cohomology of \( A \) with respect to the augmentation ideal \( m \). Local cohomology does not depend on the ring with respect to which it is taken so we may compute it using a \( B \)-bimodule resolution of \( A \)

\[
0 \rightarrow B(-n) \xrightarrow{f} B \rightarrow A \rightarrow 0.
\]

Here we used the fact that \( f \in Z(B) \). The exact sequence induces the following triangle in \( D^b(\text{tails}(A)) \).

\[
\begin{array}{ccc}
R\Gamma_m(B(-n)) & \xrightarrow{f} & R\Gamma_m(B) \\
| &  & | \\
R\Gamma_m(A) & \xrightarrow{[1]} & R\Gamma_m(A)
\end{array}
\]

This triangle relates \( R_A \) with \( R_B \).

We start computing the balanced dualizing complex \( R_B \). Let \( C \) be a two-sided noetherian Koszul AS regular algebra of global dimension \( n \). By a result of Smith [11], its Koszul dual \( C^! \) is a Frobenius algebra i.e. \( (C^!)^* \cong C_{\phi^!} \) for some automorphism \( \phi^! \) of \( C^! \). By functionality, \( \phi^! \) is obtained by dualizing an automorphism \( \phi \) of \( C \).

**Theorem 2.3** (Van den Bergh [14, Theorem 9.2]). Let \( C \) be as above and let \( \epsilon \) the automorphism of \( C \) which is multiplication by \((-1)^m\) on the graded piece \( C_m \). Then the balanced dualizing complex of \( C \) is given by \( C_{\phi\epsilon n+1}[n](-n) \).

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1The exponent \( M' \) stands for the Matlis dual of a graded ring \( M \).
Proposition 2.4. Let $B$ be as above. The balanced dualizing complex $R\Gamma_m(B)'$ is $B_\phi[n](-n)$ as a graded $B$-bimodule, where $\phi$ is the automorphism of $B$ which maps $x_j \mapsto \prod_{i=1}^n q_{ij}^{-1} x_j$ for $1 \leq j \leq n$.

Proof. First, $B$ is a Koszul AS regular algebra of global dimension $n$. The Koszul dual $B!$ of $B$ is given by the twisted exterior algebra $B! = \langle y_1, \ldots, y_n \rangle / (q_{ij}y_iy_j + y_jy_i, y_k^2)_{i,j,k}$, where $y_1, \ldots, y_n$ is the dual basis of $x_1, \ldots, x_n$. $B!$ is a Frobenius algebra and $(B!\ast \cong B_\phi\!$, where $\phi\!$ is uniquely determined by the property of Frobenius pairing $(a,b) = (\phi(b),a)$ for any $a,b \in B!$. We hence obtain $ab = \phi\!(b)a$ for any $a \in B_i$ and $b \in B_{n-i}^\!$. It then follows immediately that $\phi\!(y_j) = \prod_{i=1}^n (-q_{ji})y_j$.

By dualizing $\phi\!$, we obtain the desired map $\phi$. This completes the proof. □

Remark 2.5. Let $C$ be a graded $k$-algebra and $C_\psi$ be a graded twisted $k$-algebra of $C$, where $\psi$ is the automorphism of $C$ which acts by multiplication of $c^n$ on the graded piece $C_m$ for some $c \in k$. Such a special automorphism is invisible when passing to the quotient category tails($C$). In other words tensoring with such a bimodule is the identity functor on tails($C$).

We are now ready to prove Theorem 2.1.

Proof of Theorem 2.1. By Proposition 2.4 we obtain the following triangle in the derived category $D^b(\text{tails}(A))$

$$
\begin{array}{ccc}
R_A & \longrightarrow & B_\phi[n](-n) \\
\downarrow[1] & & \downarrow f \\
B_\phi[n] & \longrightarrow & B_\phi]\!
\end{array}
$$

where the automorphism $\phi$ of $B$ is given in Proposition 2.4. Then it immediately follows that $R_A = A_\phi'[n-1]$, where $\phi'$ is the automorphism of $A$ induced by $\phi$. Since tails($A$) has finite global dimension, the Serre functor of tails($A$) is induced by the dualizing complex $R_A$ of $A$. We note that the functor $F(*) = * \otimes A_\phi'[n-1]$ is in general not $(n-1)$-th shift functor in the category gr($A$). However, Remark 2.5 implies that the Serre functor induced by $R_A$ is the $(n-2)$-th shift functor $[n-2]$ on the quotient category tails($A$) if and only if $\prod_{i=1}^n q_{ij}$ is constant independent of $1 \leq j \leq n$. □
2.3. **Proof of** $\text{gl.dim}(\text{tails}(A_n)) = n - 2$. We shall prove that $\text{tails}(A_n)$ has global dimension $n - 2$. As before $k$ is an algebraic closed field of characteristic zero. We begin with some lemmas. Let $R$ be a finitely generated commutative ring and $C$ an $R$-algebra which is finitely generated as an $R$-module. Assume further that $R \subseteq Z(C)$.

**Lemma 2.6.** The ring $C$ has finite global dimension if the projective dimension of every simple module is bounded by some fixed number $m$. The minimum such $m$ is the global dimension of $C$.

**Proof.** We recall the Jordan–Holder decomposition of a module. The assertion follows from the long exact sequence induced from a short exact sequence. \qed

**Lemma 2.7.** Assume that $C$ is a PI ring. If $S$ is a simple $C$-module, then its annihilator $\text{Ann}(S)$ is some maximal ideal $m$ of $R$. We then have $\text{pdim}_C(S) = \text{pdim}_{C_m}(S_m)$. 

**Proof.** Since $C$ is a PI affine $k$-algebra, every simple $C$-module is finite dimensional [10, Theorem 13.10.3]. We now have a map $f : R \rightarrow \text{End}_C(S)$ and $\text{End}_C(S)$ is both a skew field (by Schur’s lemma) and finite dimensional. Thus we conclude that $\text{End}_C(S) = k$ and the map $f$ is surjective. Therefore, the kernel of the map $f$, which is the annihilator of $S$, is a maximal ideal in $R$. This proves the first half of the Lemma. 

Since the localization functor is exact, we have $\text{pdim}_C(S) \geq \text{pdim}_{C_m}(S_m)$. If $M$ and $N$ are finitely generated $C$-modules, then $\text{Ext}^i_C(M, N)$ is a finitely generated $R$-module. Furthermore if $m$ is a maximal ideal in $R$, then $\text{Ext}^i_{C_m}(M, N) = \text{Ext}^i_{C_m}(M_m, N_m)$. Assume that $\text{Ext}^i_{C_m}(S_m, N_m)$ is zero for all $N$. Since $S_n = 0$ for any maximal ideal $n$ in $R$ which is not the annihilator of $S$, we also have $\text{Ext}^i_{C_n}(S_n, N_n) = 0$ for such $n$ and any $C$-module $N$. This means that $\text{Ext}^i_C(S, N) = 0$ and hence $\text{pdim}_C(S) \leq \text{pdim}_{C_m}(S_m)$. This proves the second half of the Lemma. \qed

**Lemma 2.8** ([10, Theorem 7.3.7]). Let $S$ be a right Noetherian ring and $f$ a regular normal element belonging to the Jacobson radical $J(S)$ of $S$. If $\text{gl.dim}(S/(f)) < \infty$ then $\text{gl.dim}(S) = \text{gl.dim}(S/(f)) + 1$.

**Lemma 2.9** ([10, Theorem 7.3.5]). Let $S$ be a ring and $M$ an $S$-module. Take a normalizing non-zero divisor $f \in \text{Ann}(M)$ and assume that $\text{pdim}_{S/(f)}(M)$ is finite. We then have $\text{pdim}_{S/(f)}(M) + 1 = \text{pdim}_S(M)$. 

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2 The ring $C$ above is a PI ring as it is finite over $R \subseteq Z(C)$.
Let us begin with the proof of \( \text{gl.dim}(\text{tails}(A_n)) = n - 2 \). Recall that our non-commutative ring \( A_n \) is of the form

\[
A_n := k\langle x_1, \ldots, x_n \rangle / \left( \sum_{k=1}^{n} x_k^n, \ x_i x_j = q_{ij} x_j x_i \right)_{i,j}.
\]

We write \( t_i = x_i^n \) and

\[
D := k\langle t_1, \ldots, t_n \rangle / \left( \sum_{k=1}^{n} t_k \right).
\]

Then \( \text{proj}(A_n) \) may be seen as the category of modules over the sheaf of algebras \( B \) associated to \( A_n \) on the commutative scheme \( \text{proj}(D) \). The sheaf \( B \) is obtained by gluing five affine patches given by inverting new variables \( t_1, \ldots, t_n \) respectively.

Let us invert for instance \( t_n \). Put \( T_i := t_i / t_n \) and \( X_i = x_i / x_n \) (right denominators). The affine patch under consideration is given by

\[
C := k\langle X_1, \ldots, X_{n-1} \rangle / \left( \sum_{k=1}^{n} X_k^n + 1, \ X_i X_j = Q_{ij} X_j X_i \right)_{i,j},
\]

where \( Q_{ij} := q_{ij} / (q_{ni} q_{nj}) \). We then must show that \( \text{gl.dim}(C) = n - 2 \). Note that \( C \) is a free \( R \)-module with

\[
R := k\langle T_1, \ldots, T_{n-1} \rangle / \left( \sum_{k=1}^{n} T_k + 1 \right),
\]

which is isomorphic to a polynomial ring in three variables.

Let \( \mathfrak{m} = (T_1 - a_1, \ldots, T_{n-1} - a_{n-1}) \) with \( \sum_{i=1}^{n-1} a_i + 1 = 0 \) be a maximal ideal of \( R \). By Lemmas 2.6 it is sufficient to show that \( \text{gl.dim}(C_{\mathfrak{m}}) = n - 2 \).

We first consider the case when all \( a_i \)'s are different from zero. Then we see that

\[
C/\mathfrak{m} = k\langle X_1, \ldots, X_{n-1} \rangle / (X_i X_j = Q_{ij} X_j X_i, \ X_k^n - a_k)_{i,j,k}.
\]

is a twisted group algebra and hence semi-simple. This means that we have \( \text{gl.dim}(C/\mathfrak{m}) = 0 \).

The generators \( T_i - a_i \) of \( \mathfrak{m} \) in \( R \) form a regular sequence in \( C_{\mathfrak{m}} \). By the Lemma 2.8 we conclude that \( \text{gl.dim}(C_{\mathfrak{m}}) = n - 2 \) because \( C_{\mathfrak{m}} / \mathfrak{m} \cong C/\mathfrak{m} \) has global dimension zero.

We may therefore assume that for instance \( a_{n-1} = 0 \). Let \( S \) be a simple module annihilated by \( \mathfrak{m} \). Since \( T_{n-1} = X_{n-1}^n \) and \( X_{n-1} \) is a normalizing element, \( x_{n-1} S \) is a submodule of \( S \). We thus conclude that the simple
module $S$ is actually annihilated by $X_{n-1}$. Therefore $S$ may be seen as a $C/(X_{n-1})$-module, where
\[ C/(x_{n-1}) = k\langle X_1, \ldots, X_{n-2} \rangle / (X_i X_j = Q_{ij} X_j X_i, X_1^n + \cdots + X_{n-2}^n + 1)_{i,j,k}. \]
According to Lemma 2.9, our problem reduces to showing
\[ \text{pdim}_{C/(x_{n-1})}(S) = n - 3. \]
The ring $C/(X_{n-1})$ is of the same kind of $C$ and we can repeat the above argument; ultimately it is enough to show that the ring
\[ C/(X_2, \ldots, X_{n-1}) = k\langle X_1 \rangle / (X_1^n + 1) \]
has global dimension zero, which is clearly true. This completes the proof.

3. Hilbert Schemes of Points

In this section, we study the abstract Hilbert schemes of points on non-commutative projective schemes [2]. A way to assign geometric objects to a non-commutative scheme is to consider the moduli problem.

**Definition 3.1.** A graded right $A$-module $M$ is called an $m$-point module if

1. $M$ is generated in degree 0 with Hilbert series $h_M(t) = m t - t^m$.
2. There exists a surjection $A \to M$ of $A$-modules.

The isomorphism classes $\text{Hilb}^m(A)$ of $m$-point modules on $A$ is called the abstract Hilbert scheme.

**Example 3.2.** Let $F_n := k\langle x_1, \ldots, x_n \rangle$ be the free associative algebra in $n$ variables. The abstract Hilbert scheme $\text{Hilb}^1(F_n)$ is the set of $\mathbb{N}$-indexed sequences of points in the projective space $\mathbb{P}^{n-1}$. This can be seen as follows. First fix a graded $k$-vector space $M$ of Hilbert series $1 - t$, $M = \bigoplus_{i=0}^{\infty} km_i$ where $m_i$ is a basis of the degree $i$ piece $M_i$. If $M$ is an $A$-module, we have $m_i x_j = \xi_{i,j} m_{i+1}$ for some $\xi_{i,j} \in k$. It is clear that giving $M$ an $A$-module structure is equivalent to giving a sequence $\xi_{i,j} \in k$. Since a point module is cyclic, we need $\xi_{i,j} \neq 0$ for some $j$ for a fixed $i$. Moreover, two point modules determined by sequences $\{\xi_{i,j}\}$ and $\{\xi'_{i,j}\}$ are isomorphic if and only if the vectors $(\xi_{i,1}, \ldots, \xi_{i,n})$ and $(\xi'_{i,1}, \ldots, \xi'_{i,n})$ are scalar multiples for each $i$. This amounts to considering each vector $(\xi_{i,1}, \ldots, \xi_{i,n})$ as a point in $\mathbb{P}^{n-1}$.

For a finitely presented graded algebra $A = F_n/I$, $\text{Hilb}^1(A)$ corresponds to a subset $Z \subseteq \prod_{i=0}^{\infty} \mathbb{P}^{n-1} \cong \text{Hilb}^1(F_n)$ determined by an infinite set of equivalence relations. We can take $Z_k$ to be the projection of $Z$ onto the first $k$ copies of $\mathbb{P}^{n-1}$ and define $\text{Hilb}^1(A) = \lim_k Z_k$.

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3 We simply write $\text{Hilb}^m(A)$ rather than $\text{Hilb}^m(\text{proj}(A))$. 
In the following, we always assume that the quantum parameters $q_{ij}$’s are $n$-th roots of unity with $q_{ii} = q_{ij}q_{ji} = 1$. The following proposition may be standard for the experts, but we include it here for the sake of completeness.

**Proposition 3.3.** For the AS regular algebra

$$B_n = \langle x_1, \ldots, x_n \rangle/\langle x_i x_j = q_{ij} x_j x_i \rangle_{i,j},$$

the abstract Hilbert scheme $\text{Hilb}^1(B_n)$ is isomorphic to either $\mathbb{P}^{n-1}$ or the union of some faces of the fundamental $(n-1)$-simplex $\mathbb{P}^{n-1}$ containing all $\mathbb{P}^1$’s making up the 1-faces. The most generic case corresponds to the 1-skeleton of $\mathbb{P}^{n-1}$ consisting of all $\mathbb{P}^1$’s.

**Proof.** We begin with $n = 2$ case. Let

$$A = k\langle x, y, z \rangle/(xy - pxy, yz = qzy, zx = rxz)$$

be the quantum $\mathbb{P}^2$ with some $p, q, r \neq 0$. By the above analysis a point module correspond to a sequence of points in $\mathbb{P}^2$ such that

$$\xi, 1, 2 \xi = p\xi, 2, 3 \xi = q\xi, 3, 1 \xi = r\xi, 1, 3$$

for all $i \geq 0$. Multiplying the RHSs and LHSs above, we get

$$\xi, 1, 2, 3 \xi + 1, 1, 2 \xi + 1, 3 = pqr\xi, 1, 2, 3 \xi + 1, 1, 3 \xi + 1, 2 \xi + 1, 3.$$ 

There are two cases, $pq = 1$ or $pq \neq 1$.

**Case $pq = 1$.** We easily solve the equation on the first pair of points $[\xi_0, 1 : \xi_1, 2 : \xi_0, 3], [\xi_1, 1 : \xi_2, 2 : \xi_1, 3]$ and obtain a linear automorphism $\phi$ of $\mathbb{P}^2$ sending $[a, b, c] \mapsto [a : pb : pqc]$ such that the set of solutions is the graph of $\phi$: $\{(\xi, \phi(\xi))\} \subset \mathbb{P}^2 \times \mathbb{P}^2$. Since the other equations are just the index shift of the first set, it follows that the complete set of solutions is given by

$$\{(\xi, \phi(\xi), \phi^2(\xi), \ldots)\} \subset \prod_{i=0}^\infty \mathbb{P}^2.$$ 

This shows that the isomorphism classes of point modules are parametrized by $\mathbb{P}^2$.

**Case $pq \neq 1$.** Consider the equation on the first pair of points $[\xi_0, 1 : \xi_0, 2 : \xi_0, 3], [\xi_1, 1 : \xi_1, 2 : \xi_1, 3]$. We can check that one of $\xi_0, 1, \xi_0, 2, \xi_0, 3$ must be zero. We set

$$E = \{[\xi_0, 1 : \xi_0, 2 : \xi_0, 3] \in \mathbb{P}^2 \mid \xi_0, 1, \xi_0, 2, \xi_0, 3 = 0\}.$$ 

The solution is again given by $\{(\xi, \phi(\xi)) \mid \xi \in E\} \subset \mathbb{P}^2 \times \mathbb{P}^2$. Observe that the image of $\phi|_E$ is again $E \subset \mathbb{P}^2$. The full set of solution is

$$\{(\xi, \phi(\xi), \phi^2(\xi), \ldots) \mid \xi \in E\} \subset \prod_{i=0}^\infty \mathbb{P}^2.$$
and the isomorphism classes of point modules are parametrized by 3 lines $E \subset \mathbb{P}^2$.

A similar argument works for general $n \geq 2$. More precisely, for any choice of 3 commutation relations of the form $xy = pyx$, we can repeat the above argument. \hfill \Box

We call the quantum parameters are generic if any choice of 3 commutation relations $xy = pyx, yz = qzy, zx = ryz$, the condition $pqr \neq 1$ holds. Note that this notion depends on the expression of the generators of relations.

**Proposition 3.4.** Let $S = \text{proj}(A_4)$ be a non-commutative Fermat quartic K3 surface, where

$$A_4 = \langle x_1, \ldots, x_4 \rangle / (\sum_{k=1}^{4} x_k^4, x_i x_j = q_{ij} x_j x_i)_{i,j}$$

for some $q_{ij} \in \mathbb{C}$. Then $\text{Hilb}^1(A_4)$ is either a quartic K3 surface or 24 distinct points. In particular, the Euler number of $\text{Hilb}^1(A_4)$ is always 24, independent of the value of the quantum parameters $q_{ij}$’s.

**Proof.** On case by case basis, it can be checked that $\text{Hilb}^1(B_4)$ is isomorphic to either $\mathbb{P}^3$ or the 1-skelton of $\mathbb{P}^3$ under the Calabi–Yau constraints on $q_{ij}$’s in Theorem 2.1. In the former case, the equation $\sum_{k=1}^{4} x_k^4 = 0$ cuts out a (not necessarily Fermat) quartic K3 surface in $\mathbb{P}^3$. In the latter case, the equation $\sum_{k=1}^{4} x_k^4 = 0$ cuts out 4 distinct points in each line $\mathbb{P}^1$, so $\text{Hilb}^1(A_4)$ consists of $6 \times 4$ distinct points. \hfill \Box

**Proposition 3.5.** Let $\text{proj}(A_5)$ be a non-commutative projective Calabi–Yau 3 scheme. If the quantum parameters $q_{ij}$’s are generic, then $\prod_{i=1}^{5} q_{ij} = 1$ for any $1 \leq j \leq n$, i.e. the element $\prod_{i=1}^{5} x_i$ is central.

**Proof.** This is shown by the aid of computer (there are precisely 3000 parameters choices). \hfill \Box

**Corollary 3.6.** For a generic choice of the quantum parameters, $\text{proj}(A_5)$ admits a deformation in the direction of $\prod_{i=1}^{5} x_i$ preserving the Calabi–Yau condition. More precisely, the following $A_5^{\phi}$ gives a non-commutative projective Calabi–Yau 3 scheme.

$$A_5^{\phi} := k(x_1, \ldots, x_5) / (\sum_{k=1}^{5} x_k^5 + \phi \prod_{l=1}^{5} x_l, x_i x_j = q_{ij} x_j x_i)_{i,j}$$

with any $\phi \in k$.

**Proof.** The proof is almost identical to that of Theorem 2.1, where the fact that $\sum_{i=1}^{n} x_i^n$ is central is crucial. \hfill \Box
An almost identical argument for the K3 surface case applies to the threefold case. When Hilb$^1(B_5) \cong \mathbb{P}^4$, the abstract Hilbert scheme Hilb$^1(A_5)$ is isomorphic to a smooth quintic threefold. On the other hand, in a generic case, Hilb$^1(B_5)$ consists of 10 lines and the equation $\sum_{k=1}^{5} x_k^5 = 0$ cuts out 5 distinct points in each line $\mathbb{P}^1$ to get 50 points. In the latter case, $A_5$ is never realized as the twisted coordinate ring of a variety as Hilb$^1(A_5)$ is discrete (recall Example 2.2 and [16]). The above argument readily generalizes to an arbitrary dimension.

**Proposition 3.7.** For any $n \in \mathbb{N}$, there exists a non-commutative projective Calabi–Yau $n$ scheme that is not realized as a twisted coordinate ring of a Calabi–Yau $n$-fold.

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DEPARTMENT OF MATHEMATICS
HARVARD UNIVERSITY
1 OXFORD STREET
CAMBRIDGE MA 02138 USA
kanazawa@cmsa.fas.harvard.edu