Sequential conditions of integrability of Roumieu ultradistributions

Svetlana Mincheva-Kamińska

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Institute of Mathematics, Faculty of Mathematics and Natural Sciences, University of Rzeszów, Prof. Pigonia 1, 35-310 Rzeszów, Poland
email: minczewa@ur.edu.pl

Abstract

We consider several general conditions for integrability of two Roumieu ultradistributions on $\mathbb{R}^d$ in the space $\mathcal{D}'(\mathcal{M}_p)$ and prove their equivalence. The discussed sequential conditions are based on two classes $\mathcal{U}(\mathcal{M}_p)$ and $\mathcal{U}'(\mathcal{M}_p)$ of approximate units and allow one to introduce sequential definitions of the convolution in $\mathcal{D}'(\mathcal{M}_p)$, analogous to the known definitions in the space $\mathcal{D}'$ of distributions and in the space $\mathcal{D}'(\mathcal{M}_p)$ of ultradistributions of Beurling type.

Keywords: ultradistributions, integrability of ultradistributions, convolution of ultradistributions, approximate unit.

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1 Introduction

Deep investigations of the convolution of two ultradistributions of Roumieu type (that we call shorter Roumieu ultradistributions) in the non-quasianalytic case were carried out via $\varepsilon$-tensor product by Pilipović and Prangoski in [21] and, with important improvements, by Dimovski, Pilipović, Prangoski and Vindas in [5]. The authors gave general functional definitions and proved fundamental results on convolvability and the convolution of Roumieu ultradistributions in a way analogous to the known general approaches of Chevalley and Schwartz in case of distributions. For other aspects of the theory see e. g. [2, 4, 6, 7, 22, 24, 27, 28]. See also the recent article [23] for results concerning the quasianalytic case.

The aim of this paper is to present a characterization of integrable Roumieu ultradistributions (Theorem 1), which allows us to formulate sequential conditions of convolvability of
two Roumieu ultradistributions analogous to those used in the sequential theories of the convolution of distributions (see [26], [3], [9] and [17]) and of ultradistributions of Beurling type (see [11], [12] and [2]). The sequential conditions of integrability are based on two types of \(\mathcal{R}\)-approximate units (Definition 3 and Definition 4), being counterparts of the approximate units in the sense of Dierolf and Voigt (see [3]).

The approximate units are used in several sequential definitions of the convolvability and of the convolution of Roumieu ultradistributions (Definitions 6 and 7 in section 7). We prove their equivalence in [18]. Moreover, we consider in section 8 the notion of multi-convolution of Roumieu ultradistributions and formulate some results concerning associativity of convolution of Roumieu ultradistributions analogous to those given in [13] in the Beurling case.

2 Preliminaries

We consider complex-valued \(C^\infty\)-functions and Roumieu ultradistributions defined on \(\mathbb{R}^d\) (or on an open subset of \(\mathbb{R}^d\)) using the standard multi-dimensional notation in \(\mathbb{R}^d\).

To mark the dimension of \(\mathbb{R}^d\), which is essential in some situations, we denote the considered spaces of test functions and the corresponding spaces of Roumieu ultradistributions simply by adding the index \(d\) at the end of the respective symbol. Moreover, if necessary, the constant function 1 on \(\mathbb{R}^d\) will be denoted by \(1_d\) and the value of \(T \in \mathcal{D}^{(M_p)}_d\) on \(\varphi \in \mathcal{D}_d^{(M_p)}\) by \(\langle T, \varphi \rangle_d\).

The spaces of test functions and Roumieu ultradistributions are defined by a given sequence \((M_p)_{p \in \mathbb{N}_0}\) of positive numbers. Usually some of the following conditions are imposed on the sequence \((M_p)\):

(M.1) (logarithmic convexity)
\[ M^2_p \leq M_{p-1}M_{p+1}, \quad p \in \mathbb{N}; \]

(M.2) (stability under ultradifferential operator)
\[ M_p \leq AH^pM_qM_{p-q}, \quad p, q \in \mathbb{N}_0, \quad q \leq p; \]

(M.2') (stability under differential operator)
\[ M_p \leq AH^pM_{p-1}, \quad p \in \mathbb{N}; \]

(M.3) (strong non-quasi-analyticity)
\[ \sum_{p=q+1}^{\infty} M_{p-1}M_p^{-1} \leq AqM_qM_{q+1}^{-1}, \quad q \in \mathbb{N}; \]

(M.3') (non-quasi-analyticity)
\[ \sum_{p=1}^{\infty} M_{p-1}M_p^{-1} < \infty, \]

for certain constants \(A > 0\) and \(H > 0\). We can and will assume that \(H \geq 1\).

Clearly, conditions (M.2') and (M.3') are particular cases of conditions (M.2) and (M.3), respectively.

For simplicity, we will assume in the whole paper that the sequence \((M_p)\) satisfies the three conditions (M.1), (M.2) and (M.3), not discussing which of them can be weakened or omitted in the formulations of presented theorems.
It follows by induction from (M.1) that
\[
\frac{M_i}{M_{i-1}} \leq \frac{M_{p+i}}{M_{p+i-1}}, \quad p \in \mathbb{N}_0, \ i \in \mathbb{N},
\]
and thus
\[
\frac{M_q}{M_0} = \prod_{i=1}^{\pi} \frac{M_i}{M_{i-1}} \leq \prod_{i=1}^{\pi} \frac{M_{p+i}}{M_{p+i-1}} = \frac{M_{p+q}}{M_p}, \quad p \in \mathbb{N}_0, \ q \in \mathbb{N}.
\]
Consequently, (M.1) implies \(M_p \cdot M_q \leq M_0 M_{p+q}\) for \(p, q \in \mathbb{N}_0\). For simplicity we assume in the sequel that \(M_0 = 1\). With this assumption, the last inequality gets the form:
(2.1) \(M_p \cdot M_q \leq M_{p+q}, \quad p, q \in \mathbb{N}_0\).

It will be convenient to extend the sequence \((M_p)_{p \in \mathbb{N}_0}\) to \((M_k)_{k \in \mathbb{N}_0^d}\) by means of the formula:
\[
M_k := M_{k_1 + \ldots + k_d}, \quad k = (k_1, \ldots, k_d) \in \mathbb{N}_0^d.
\]
Due to the extended notation we immediately get the extended version of inequality (2.1):
(2.2) \(M_j \cdot M_k \leq M_{j+k}, \quad j, k \in \mathbb{N}_0^d\).

The associated function of the sequence \((M_p)\) is given by
\[
M(\rho) = \sup_{p \in \mathbb{N}_0} \log_+ \frac{\rho^p}{M_p}, \quad \rho > 0.
\]

For an arbitrary \(k = (k_1, \ldots, k_d) \in \mathbb{N}_0^d\) denote by \(D^k\) the differential operator of the form
\[
D^k = D_1^{k_1} \cdots D_d^{k_d} := \left( \frac{1}{i \partial x_1} \right)^{k_1} \cdots \left( \frac{1}{i \partial x_d} \right)^{k_d}.
\]

An essential role in our consideration plays Komatsu’s lemma proved in [16] (see Lemma 3.4 and Proposition 3.5) in which numerical sequences monotonously increasing to infinity are involved. The class of such sequences \((r_p)_{p \in \mathbb{N}_0}\) (with \(r_0 = 1\)) has been denoted by \(\mathcal{R}\) in [21] and [5] and we preserve this notation in our paper.

For every \((r_p) \in \mathcal{R}\) we call \((R_p)\) the *product sequence* corresponding to \((r_p)\) if its elements are of the form \(R_p := \prod_{i=0}^{p} r_i\) for \(p \in \mathbb{N}_0\) (i.e. \(R_0 = 1\)).

Let us recall Komatsu’s lemma in the following equivalent form which emphasizes the symmetry of two assertions:

**Lemma 1.** Let \((a_k)_{k \in \mathbb{N}_0}\) be a sequence of nonnegative numbers.

(I) The following two conditions are equivalent:

\[
(A_1) \quad \exists h > 0 \quad \sup_{k \in \mathbb{N}_0} \frac{a_k}{h^k} < \infty; \quad \text{and} \quad (B_1) \quad \forall (r_k) \in \mathcal{R} \quad \sup_{k \in \mathbb{N}_0} \frac{a_k}{R_k} < \infty; \quad (I)
\]
the following two conditions are equivalent:

\[(A_2) \quad \forall h > 0 \sup_{k \in \mathbb{N}_0} (h^k a_k) < \infty; \]

\[(B_2) \quad \exists (r_k) \in \mathbb{R} \sup_{k \in \mathbb{N}_0} (R^k a_k) < \infty, \]

where \((R_k)\) is the product sequence corresponding to the sequence \((r_k) \in \mathbb{R}\).

**Remark 1.** The above lemma can be easily extended to the \(d\)-dimensional version concerning sequences \((a_k)_{k \in \mathbb{N}_0}\) of nonnegative numbers.

It is worth noticing that Lemma 1 delivers two simple characterizations (dual to each other):

1° of slowly increasing sequences (i.e. satisfying \((A_1)\)), 2° of rapidly decreasing sequences (i.e. satisfying \((A_2)\)) of nonnegative numbers. They are expressed through respective properties of sequences, described by product sequences corresponding to sequences of the class \(\mathfrak{R}\).

It will be convenient to use for \(\lambda > 0\) and \((r_p) \in \mathbb{R}\) the following notation:

\[(2.3) \quad \lambda (r_p) = (\overline{r}_p), \text{ where } \overline{r}_0 = 1 \text{ and } \overline{r}_p = \lambda r_p \text{ for } p \in \mathbb{N}\]

and

\[(2.4) \quad (r_p)^\lambda = (\overline{r}_p), \text{ where } \overline{r}_0 = 1 \text{ and } \overline{r}_p = r_p^\lambda \text{ for } p \in \mathbb{N}.\]

Clearly, if \((r_p) \in \mathfrak{R}\), then \((r_p)^\lambda \in \mathfrak{R}\) for every \(\lambda > 0\) and \(\lambda (r_p) \in \mathfrak{R}\) for \(\lambda \geq 1\). Moreover, \(\lambda (r_p) \in \mathfrak{R}\) for \(0 < \lambda < 1\), if \((r_p) \in \mathfrak{R}\) and \(r_1 \geq \lambda^{-1}\).

We define the following partial ordering between sequences of the class \(\mathfrak{R}\):

**Definition 1.** If \((r_p) \in \mathfrak{R}\) and \((s_p) \in \mathfrak{R}\), then we write \((s_p) < (r_p)\) if \(s_p \leq r_p\) for all \(p \in \mathbb{N}\) and \(\limsup_{p \to \infty} r_p/s_p = \infty\).

### 3 Ultradifferentiable Functions

For a given complex-valued function \(\varphi\) on \(\mathbb{R}^d\) and a compact set \(K\) in \(\mathbb{R}^d\) denote

\[
\|\varphi\|_\infty := \sup_{x \in \mathbb{R}^d} |\varphi(x)|; \quad \|\varphi\|_K := \sup_{x \in K} |\varphi(x)|.
\]

For a given sequence \((M_p)\), a regular compact set \(K\) in \(\mathbb{R}^d\) and \(h > 0\) the symbol \(C^{(M_p)}_{K,h,d}\) will mean the locally convex space (l.c.s.) of all \(C^\infty\)-functions \(\varphi\) on \(\mathbb{R}^d\) such that

\[(3.1) \quad q_{K,h}(\varphi) := \sup_{k \in \mathbb{N}_0} \frac{\|D^k \varphi\|_K}{h^{|k|} M_k} < \infty, \]

with the topology defined by the semi-norm \(q_{K,h}\) given above, while the symbol \(D^{(M_p)}_{K,h,d}\) will mean the Banach space of all \(C^\infty\)-functions \(\varphi\) satisfying (3.1) and having supports contained in \(K\), with the topology of the norm \(q_{K,h}\) in (3.1).
For a fixed sequence \((M_p)\), we consider the following locally convex spaces of ultradifferentiable functions on \(\mathbb{R}^d\):

\[
D_{K,d}^{\{M_p\}} := \lim_{h \to \infty} D_{K,h,d}^{\{M_p\}}; \quad D_d^{\{M_p\}} := \lim_{K \subset \subset \mathbb{R}^d} D_{K,d}^{\{M_p\}};
\]

\[
E_d^{\{M_p\}} := \lim_{K \subset \subset \mathbb{R}^d} \lim_{h \to \infty} E_{K,h,d}^{\{M_p\}};
\]

where the symbol \(K \subset \subset \mathbb{R}^d\) means that compact sets \(K\) grow up to \(\mathbb{R}^d\).

Moreover, for a given \((M_p)\), we define \(D_{L^\infty,d}^{\{M_p\}} := \lim_{h \to \infty} D_{L^\infty,h,d}^{\{M_p\}}\), where \(D_{L^\infty,h,d}^{\{M_p\}}\) is the Banach space of all \(C^\infty\)-functions \(\varphi\) on \(\mathbb{R}^d\) such that

\[
\|\varphi\|_{\infty,h} := \sup \left\{ \|D_k \varphi\|_{\infty} : k \in \mathbb{N}_0^d \right\} < \infty,
\]

with the norm \(\|\cdot\|_{\infty,h}\) defined above.

For a given regular compact set \(K \subset \mathbb{R}^d\) and given sequences \((M_p)\) and \((r_p) \in \mathcal{R}\) we denote by \(D_{K,(r_p),d}^{\{M_p\}}\) the Banach space of all \(C^\infty\)-functions \(\varphi\) on \(\mathbb{R}^d\) having supports contained in \(K\) such that

\[
\|\varphi\|_{K,(r_p)} := \sup_{k \in \mathbb{N}_0^d} \frac{\|D_k \varphi\|_K}{R_{|k|} M_k} < \infty
\]

with the norm \(\|\cdot\|_{K,(r_p)}\) defined above.

The following result is essentially due to Komatsu [16], since it is a consequence of his beautiful Lemma [1] recalled above.

**Proposition 1.** We have the equality

\[
D_{K,d}^{\{M_p\}} = \lim_{(r_p) \in \mathcal{R}} D_{K,(r_p),d}^{\{M_p\}},
\]

where the space \(D_{K,d}^{\{M_p\}}\) is defined in (3.2).

**Proof.** The assertion is an immediate consequence of definitions of \(q_{K,h}\) in (3.1) and \(\|\cdot\|_{K,(r_p)}\) in (3.5) and Part (I) of the \(d\)-dimensional version of Lemma [1] with \(a_k := \|D_k \varphi\|_K / M_k\) \((k \in \mathbb{N}_0^d)\) for a given function \(\varphi\) of the considered space. \(\square\)
For given \((M_p)\) and \((r_p) \in \mathcal{R}\) we denote by \(D_{L^\infty,(r_p),d}^{(M_p)}\) the Banach space of all \(C^\infty\)-functions \(\varphi\) on \(\mathbb{R}^d\) such that

\[(3.6) \quad \|\varphi\|_{(r_p)} := \sup_{k \in \mathbb{N}_0} \frac{\|D^k \varphi\|_\infty}{R_{|k|}M_k} < \infty,
\]

with the norm \(\| \cdot \|_{(r_p)}\) defined in \((3.6)\).

For a given sequence \((M_p)\) the following projective description is shown in [5, 23]:

\[\tilde{D}_{L^\infty,d}^{(M_p)} = \lim_{(r_p) \in \mathcal{R}} D_{L^\infty,(r_p),d}^{(M_p)},\]

where the equality holds as l.c.s. We denote by \(\tilde{B}_d^{(M_p)}\) the completion of \(D_d^{(M_p)}\) in \(D_{L^\infty,d}^{(M_p)}\).

**Remark 2.** Notice that, if necessary, we may assume that a sequence \((r_p) \in \mathcal{R}\) satisfies for a given constant \(c > 0\) the inequality \(r_p > c\) for all \(p \in \mathbb{N}\). In fact, if \(r_p \not\to \infty\) as \(p \not\to \infty\), then there is a \(p_0 \in \mathbb{N}\) such that \(r_p > c\) for \(p > p_0\) and we may replace \((r_p)\) by \((\overline{r}_p)\) in \(\mathcal{R}\) defined by 

\[\overline{r}_0 = 1, \quad \overline{r}_p := r_{p+p_0} \quad \text{for all} \quad p \in \mathbb{N}.\]

Thus \(\|\varphi\|_{(r_p)} < \infty\) implies \(\|\varphi\|_{(\overline{r}_p)} < \infty\) for all \(\varphi \in D_d^{(M_p)}\).

**Proposition 2.** If \(\varphi_1, \varphi_2 \in D_{L^\infty,d}^{(M_p)}\), then \(\varphi_1 \cdot \varphi_2 \in D_{L^\infty,d}^{(M_p)}\). Moreover, for every \((r_p) \in \mathcal{R}\) such that \((r_p)/2 \in \mathcal{R}\), the inequality holds:

\[(3.7) \quad \|\varphi_1 \cdot \varphi_2\|_{(r_p)/2} \leq \|\varphi_1\|_{(r_p)/2} \|\varphi_2\|_{(r_p)/2},\]

where \((r_p)/2\) is meant in the sense of \((2.3)\).

**Proof.** Fix \((r_p) \in \mathcal{R}\) such that \((r_p)/2 = (\overline{r}_p) \in \mathcal{R}\) (see \((2.3)\) for \(\lambda = \frac{1}{2}\)). Hence \(\overline{R}_p = 2^{-p}R_p\) for \(p \in \mathbb{N}_0\) and consequently \(\overline{R}_{|k|} = 2^{-k}R_{|k|}\) for \(k \in \mathbb{N}_0\).

If \(\varphi_1, \varphi_2 \in D_{L^\infty,d}^{(M_p)}\), then \(\varphi_1, \varphi_2 \in D_{L^\infty,(r_p)/2,d}^{(M_p)}\) by the projective description of the spaces \(D_{L^\infty,d}^{(M_p)}\). Now it follows from \((3.6)\) that

\[\|D^i \varphi_i\|_\infty \leq 2^{-j}R_{|j|}M_j\|\varphi_i\|_{(r_p)/2} \quad (i = 1, 2)
\]

for all \(j \in \mathbb{N}_0^d\). Hence, by Leibniz’ formula,

\[\|D^k(\varphi_1 \cdot \varphi_2)\|_\infty \leq \sum_{0 \leq j \leq k} \binom{k}{j}\|D^j \varphi_1\|_\infty \cdot \|D^{k-j} \varphi_2\|_\infty \leq 2^{-k}\|\varphi_1\|_{(r_p)/2} \cdot \|\varphi_2\|_{(r_p)/2} \sum_{0 \leq j \leq k} \binom{k}{j} R_{|j|}M_j R_{|k-j|}M_{k-j}
\]

for each \(k \in \mathbb{N}_0^d\).

Recall that condition \((M.1)\) implies inequality \((2.2)\). On the other hand, for every \((r_p) \in \mathcal{R}\) we have

\[R_p \cdot R_q = \prod_{i=0}^{p} r_i \prod_{i=0}^{q} r_i \leq \prod_{i=0}^{p+q} r_i = R_{p+q}
\]
for \( p, q \in \mathbb{N}_0 \), due to monotonicity of \((r_p)\), and hence
\[
R_{|j|} \cdot R_{|k|} \leq R_{|j+k|}, \quad k, j \in \mathbb{N}_0^d.
\]

Therefore, it follows from (3.8) that
\[
\|D^k(\varphi\phi)\|_\infty \leq \|\varphi\|_{(r_p)/2}\|\phi\|_{(r_p)/2} < \infty
\]
for all \( k \in \mathbb{N}_0^d \), which completes the proof. \( \square \)

**Remark 3.** Notice that the assertion of Proposition 2 is true, in particular, for functions \( \varphi_1, \varphi_2 \in D^{(M_p)}_d \) and semi-norms of the form (3.3).

We will need later the following simple consequence of Proposition 2:

**Lemma 2.** Let \( \theta \) be a function from \( D^{(M_p)}_d \) such that \( \theta(x) = 1 \) for all \( x \) in some neighbourhood of \( 0 \) in \( \mathbb{R}^d \), \( |x| \leq 1 \) and let \( \theta_j(x) = \theta(\frac{x}{j}) \) for \( x \in \mathbb{R}^d \) and \( j \in \mathbb{N} \). Then for each \( (r_p) \in \mathfrak{R} \) with \( r_1 \geq 2 \) there exists a constant \( C(\theta, (r_p)) > 0 \) such that
\[
\|(1 - \theta)\varphi\|_{(r_p)} \leq C(\theta, (r_p))\|\varphi\|_{(r_p)/2}, \quad \varphi \in D^{(M_p)}_d.
\]

**Proof.** It suffices to use Leibniz’ formula and Proposition 2. \( \square \)

## 4 Roumieu Ultradistributions

**Definition 2.** We denote the strong dual of the space \( D^{(M_p)}_d \) by \( D^{(M_p)}_d \) and call it the space of Roumieu ultradistributions.

The strong dual of the space \( \mathcal{B}^{(M_p)}_d \), denoted by \( D^{(M_p)}_{L^1,d} \) is called the space of Roumieu integrable ultradistributions.

**Remark 4.** The space \( D^{(M_p)}_d \) is dense in \( \mathcal{B}^{(M_p)}_d \) and the respective inclusion mapping is continuous. Consequently, the space \( D^{(M_p)}_{L^1,d} \) of Roumieu integrable ultradistributions is a subspace of the space \( D^{(M_p)}_d \) of Roumieu ultradistributions.

In the sequel, we assume that the sequence \((M_p)\) satisfies conditions (M.1), (M.3) and (M.3), so we can use the projective description of the considered locally convex spaces of test functions.

**Definition 3.** By an \( \mathfrak{R} \)-approximate unit we mean a sequence \((\pi_n)\) of ultradifferentiable functions \( \pi_n \in D^{(M_p)}_d \) converging to 1 in \( \mathcal{E}^{(M_p)}_d \) such that the following property holds for every sequence \((r_p) \in \mathfrak{R}:\)
\[
\sup_{n \in \mathbb{N}} \|\pi_n\|_{(r_p)} = \sup_{n \in \mathbb{N}} \sup_{k \in \mathbb{N}_0^d} (R_{|k|}M_k)^{-1}\|D^k\pi_n\|_\infty < \infty,
\]
where \((R_p)\) is the product sequence corresponding to \((r_p)\).
**Definition 4.** By a special $R$-approximate unit we mean an $R$-approximate unit $(\Pi_n)$ such that for every compact set $K \subset \mathbb{R}^d$ there exists an index $n_0 \in \mathbb{N}$ such that $\Pi_n(x) = 1$ for all $n \geq n_0$ and $x \in K$.

We denote the class of all $R$-approximate units on $\mathbb{R}^d$ by $U^{(M_p)}_d$ and the class of all special $R$-approximate units on $\mathbb{R}^d$ by $U^{(M_p)}_d$.

**Remark 5.** By the Denjoy-Carleman theorem, the defined above spaces of ultradifferentiable functions as well as the classes $U^{(M_p)}_d$ and $U^{(M_p)}_d$ of approximate units contain sufficiently many members.

5 Integrability of Roumieu Ultradistributions

We are going to prove a characterization (consisting of several equivalent conditions) of integrable Roumieu ultradistributions in a way which is analogous to the theorem of Dierolf and Voigt concerning integrable distributions (see [3]) and to the theorem of Pilipović concerning ultradistributions of Beurling type (see [20]).

**Theorem 1.** Let $V \in \mathcal{D}'^{(M_p)}_d$. The following conditions are equivalent:

(A) $V$ is continuous on $\mathcal{D}^{(M_p)}_d$ in the topology induced by $\mathcal{B}^{(M_p)}_d$, i.e. there are a sequence $(r_p) \in \mathbb{R}$ and a constant $C > 0$ such that the inequality

$$\langle V, \varphi \rangle \leq C \|\varphi\|(r_p)$$

holds for all $\varphi \in \mathcal{D}^{(M_p)}_d$;

(B) there is a sequence $(r_p) \in \mathbb{R}$ with the property that for every $\varepsilon > 0$ there exists a regular compact set $K \subset \mathbb{R}^d$ such that the inequality

$$\langle V, \varphi \rangle \leq \varepsilon \|\varphi\|(r_p)$$

holds for $\varphi \in \mathcal{D}^{(M_p)}_d$ with $\text{supp} \varphi \cap K = \emptyset$;

(C) for every $(n_n) \in U^{(M_p)}_d$ the sequence $(\langle V, n_n \rangle)$ is Cauchy;

(D) for every $(n_n) \in U^{(M_p)}_d$ the sequence $(\langle V, n_n \rangle)$ is Cauchy;

(E) there are a sequence $(r_p) \in \mathbb{R}$, a constant $C > 0$ and a regular compact $K \subset \mathbb{R}^d$ such that inequality (5.1) holds for $\varphi \in \mathcal{D}^{(M_p)}_d$ with $\text{supp} \varphi \cap K = \emptyset$.

**Proof.** (A) $\Rightarrow$ (B) Assume (A) and fix a sequence $(r_p) \in \mathbb{R}$ and a constant $C > 0$ such that (5.1) holds for all $\varphi \in \mathcal{D}^{(M_p)}_d$. At the same time, suppose that (B) is not true, i.e. there is an $\varepsilon_0 > 0$ such that for every regular compact set $K \subset \mathbb{R}^d$ one can find a function $\varphi \in \mathcal{D}^{(M_p)}_d$, $\text{supp} \varphi \cap K = \emptyset$, such that $\langle V, \varphi_K \rangle > \varepsilon_0 \|\varphi_K\|(r_p)$. Hence there exists an increasing sequence
of regular compact sets $K_n \subset \mathbb{R}^d$ and a sequence of functions $\varphi_n \in \mathcal{D}_d^{(M_p)}$ such that $\text{supp } \varphi_n \subset \text{Int} K_{n+1} \setminus K_n$, $\|\varphi_n\|_{(r_p)} = 1$ and $\langle V, \varphi_n \rangle > \varepsilon_0$ for all $n \in \mathbb{N}$. Put

$$
\psi_k := \sum_{n=1}^{k} \varphi_n, \quad k \in \mathbb{N}.
$$

Clearly, $\psi_k \in \mathcal{D}_d^{(M_p)}$ and, since the functions $\varphi_n$ have disjoined supports, we have $\|\psi_k\|_{(r_p)} = 1$ and $|\langle V, \psi_k \rangle| \leq C$ for all $k \in \mathbb{N}$. On the other hand, due to the assumption, we get

$$
\langle V, \psi_k \rangle > k \varepsilon_0 \quad \text{for all } k \in \mathbb{N},
$$

which contradicts the above estimate.

$(B) \Rightarrow (C)$ Fix $(\Pi_n) \in \bigcup_{d}^{(M_p)}$ and assume that $(r_p) \in \mathcal{R}$ is a sequence with the property described in $(B)$.

By Remark 2, we may additionally assume that $r_p > 2$ for all $p \in \mathbb{N}$. Under the additional assumption, we have $(r_p)/2 \in \mathcal{R}$ and Lemma 2 can be applied.

Let $\theta$ and $\theta_l$ ($l \in \mathbb{N}$) be functions as described in Lemma 2 so (3.10) is true for some $C(\theta, (r_p)) > 0$. By (4.1), we have

$$
N := 4 \sup \{\|\Pi_n\|_{(r_p)/2} : n \in \mathbb{N}\} < \infty.
$$

Fix $\varepsilon > 0$ and, due to $(B)$, choose a regular compact set $K \subset \mathbb{R}^d$ such that

$$
|\langle V, \varphi \rangle| \leq \frac{\varepsilon \|\varphi\|_{(r_p)}}{NC(\theta, (r_p))}
$$

for all $\varphi \in \mathcal{D}_d^{(M_p)}$ with $\text{supp } \varphi \cap K = \emptyset$. Now select $l \in \mathbb{N}$ so that $\theta_l(x) = 1$ for all $x$ in a neighbourhood of $K$. Then the functions $\varphi_{l,k,n} := (1 - \theta_l)(\Pi_k - \Pi_n)$ have the property $\text{supp } \varphi_{l,k,n} \cap K = \emptyset$ for all $k, n \in \mathbb{N}$. Consequently, in view of (5.3), (3.10) and (5.2), we obtain

$$
|\langle V, \varphi_{l,k,n} \rangle| \leq \frac{\varepsilon}{NC(\theta, (r_p))} \|\varphi_{l,k,n}\|_{(r_p)}
= \frac{\varepsilon}{N} \|\Pi_k - \Pi_n\|_{(r_p)/2}
\leq \frac{\varepsilon}{N} \left(\|\Pi_k\|_{(r_p)/2} + \|\Pi_n\|_{(r_p)/2}\right)
\leq \frac{\varepsilon}{2}.
$$

for all $k, l, n \in \mathbb{N}$. On the other hand, since $(\Pi_n)_{n \in \mathbb{N}}$ converges to 1 in $\mathcal{E}_d^{(M_p)}$ and $\theta_l V \in \mathcal{E}_d^{(M_p)}$, we have

$$
|\langle \theta_l V, (\Pi_k - \Pi_n) \rangle| \leq \frac{\varepsilon}{2}
$$

for sufficiently large $k$ and $n$. It remains to use (5.4) and (5.5) to get property $(C)$.

$(C) \Rightarrow (D)$ is an obvious implication.

$(D) \Rightarrow (E)$ Assume $(D)$ and suppose that condition $(E)$ does not hold. This means that for an arbitrary threesome: a sequence in $\mathcal{R}$, a regular compact set in $\mathbb{R}^d$ and a positive constant,
one can find a function \( \varphi \in D_d^{(M_p)} \) with a support disjoint with the compact set for which (5.1) does not hold. In particular, for a given \((r_p^1) \in \mathcal{R}\) and the compact ball \(K_1 := \overline{B}(0, 1)\) one can find a function \( \varphi_1 \in D_d^{(M_p)} \) with supp \( \varphi_1 \cap K_1 = \emptyset \) and \( |\langle V, \varphi_1 \rangle| > \| \varphi_1 \|_{(r_p^1)} \). According to Definition 2 choose a sequence \((r_p^2) \in \mathcal{R}\) such that \((r_p^2) < (r_p^1)\). For \((r_p^2)^{1/2} \in \mathcal{R}\), the compact ball \(K_2 := \overline{B}(0, 2)\) and the constant \(C_2 := 4\) there exists a function \( \varphi_2 \in D_d^{(M_p)} \) such that supp \( \varphi_2 \cap K_2 = \emptyset \) and \( |\langle V, \varphi_2 \rangle| > 2^2 \| \varphi_2 \|_{(r_p^2)^{1/2}} \) and so on. In this way, we inductively construct a sequence of sequences \((r_p^m) \in \mathcal{R}\) with \((r_p^{m+1}) < (r_p^m)^{1/m}\) and a sequence of functions \( \varphi_m \in D_d^{(M_p)} \) with supp \( \varphi_m \cap K_m = \emptyset \) such that

\[
|\langle V, \varphi_m \rangle| > m^2 \| \varphi_m \|_{(r_p^m)^{1/m}} > 0
\]

for all \( m \in \mathbb{N} \). Define

\[
\psi_m := \frac{\varphi_m}{m \| \varphi_m \|_{(r_p^m)^{1/m}}} \quad \text{for } m \in \mathbb{N}.
\]

Clearly, \( \psi_m \in D_d^{(M_p)} \) and supp \( \psi_m \cap K_m = \emptyset \). Moreover,

\[
\| \psi_m \|_{(r_p^m)^{1/m}} = \frac{1}{m} \quad \text{and} \quad |\langle V, \psi_m \rangle| > m,
\]

by (5.6) and (5.7).

Now if \((\tilde{n}_n)\) is a given special \( \mathcal{R}\)-approximate unit, then the sequence \((\tilde{n}_n)\), defined by \( \tilde{n}_n = n + \psi_n \) for \( n \in \mathbb{N} \), is also an \( \mathcal{R}\)-approximate unit, because for every \((r_p) \in \mathcal{R}\) we have \( V_p = \prod_{i=0}^{p} y_i \geq (\prod_{i=0}^{p} r_i^m)^{1/m} \) for some \( m \in \mathbb{N} \) and, consequently,

\[
\| \psi_m \|_{(r_p)} \leq \| \psi_m \|_{(r_p^m)^{1/m}} \leq \frac{1}{m}.
\]

On the other hand, we have

\[
|\langle V, \tilde{n}_n \rangle - \langle V, n \rangle| = |\langle V, \psi_n \rangle| > n,
\]

for \( n \in \mathbb{N} \), which means that at least one of the sequences \((\langle V, \tilde{n}_n \rangle)\) and \((\langle V, n \rangle)\) is not convergent. This contradicts the assumed condition (D).

\((E) \Rightarrow (A)\) Fix a sequence \((r_p) \in \mathcal{R}\), a constant \( C > 0 \) and a regular compact set \( K \subset \mathbb{R}^d \). Using similar reasoning as in the proof of \((B) \Rightarrow (C)\) we may assume that \((r_p)/2 \in \mathcal{R}\). Select a relatively compact neighbourhood \( U \) of \( K \) and a function \( \theta \in D_d^{(M_p)} \) such that supp \( \theta \subset U \) and \( \theta(x) = 1 \) for all \( x \) in a certain open neighbourhood of \( K \), contained in \( U \). By \((E)\), we have

\[
|\langle V, (1 - \theta) \varphi \rangle| \leq C \| (1 - \theta) \varphi \|_{(r_p)} \quad \text{for } \varphi \in D_d^{(M_p)}.
\]

The mapping

\[
D_d^{(M_p)} \ni \varphi \mapsto \langle V, \theta \varphi \rangle \in \mathbb{C}
\]
defines the Roumieu ultadistribution \( \theta V \in \mathcal{E}_d^{\{M_p\}} \) given by \( \langle \theta V, \varphi \rangle := \langle V, \theta \varphi \rangle \) for \( \theta \in \mathcal{D}_d^{\{M_p\}} \).

Therefore there exists a constant \( C_1 > 0 \) such that

\[
|\langle \theta V, \varphi \rangle| \leq C_1 \sup \left\{ \frac{|D^k \varphi|}{R_{|k|} M_k} : x \in K, \ k \in \mathbb{N}_0^d \right\} \leq C_1 \| \varphi \|_{(r_p)}
\]

for all \( \varphi \in \mathcal{D}_d^{\{M_p\}} \). Hence

\[
|\langle V, \varphi \rangle| \leq |\langle V, (1-\theta)\varphi \rangle| + |\langle \theta V, \varphi \rangle| \leq C \| (1-\theta)\varphi \|_{(r_p)} + C_1 \| \varphi \|_{(r_p)}
\]

for a certain \( C_2 > 0 \), by (5.8), (5.9) and (3.10) in Lemma 2.

The above estimate shows that \( V \) is continuous in \( \mathcal{D}_d^{\{M_p\}} \) in the topology induced from \( \hat{\mathcal{B}}_d^{\{M_p\}} \).

6 Convolution of Roumieu Ultradistributions

S. Pilipović and B. Prangoski made in [21] a deep study of the convolution of Roumieu ultradistributions. The study was based on the investigation of the \( \varepsilon \) tensor product of the respective spaces of test functions. Let us recall some results proved and observations made in [21].

The authors use the results on the \( \varepsilon \) tensor product from [16] to prove that

\[
\mathcal{B}_d^{\{M_p\}} \cong \mathcal{B}_d^{\{M_p\}} \hat{\otimes} \mathcal{B}_d^{\{M_p\}}
\]

in the sense of an isomorphism. They consider, analogously to ideas applied in [19] to the convolution of measures, the following semi-norms in the space \( \mathcal{D}_d^{\{M_p\}} \): 

\[
q_{g,(r_p)}(\varphi) := \sup_{k \in \mathbb{N}_0^d} \sup_{x \in \mathbb{R}^d} \frac{|g(x)D^k \varphi(x)|}{R_{|k|} M_k}, \quad \varphi \in \mathcal{D}_d^{\{M_p\}}
\]

Denote by \( \mathcal{D}_d^{\{M_p\}} \) the l.c.s. \( \mathcal{D}_d^{\{M_p\}} \) equipped with the topology defined by the family \( \{q_{g,(r_p)} : g \in \mathcal{C}_0, (r_p) \in \mathcal{R}\} \) of semi-norms and the strong dual of \( \mathcal{D}_d^{\{M_p\}} \) by \( \left( \mathcal{D}_d^{\{M_p\}} \right)'_b \). The connection between the strong dual of \( \mathcal{D}_d^{\{M_p\}} \) and the space of integrable distributions \( \mathcal{D}_d^{\{M_p\}} \) was studied in [21]. Such results have recently improved and it is shown that

\[
\left( \mathcal{D}_d^{\{M_p\}} \right)'_b = \mathcal{D}_d^{\{M_p\}},
\]

as locally convex spaces (cf. [23], Prop 5.3 and Prop 5.4).

These observations allow the authors to give in [21] the following definitions of convolvability and convolution of two Roumieu ultradistributions, analogous to the known definitions of L. Schwartz for distributions (see [25]):
Definition 5. Let $S, T \in \mathcal{D}_d^{(M_p)}$. If the following condition is satisfied:

$$(s) \quad V_\phi := (S \otimes T) \varphi^\triangle \in \mathcal{D}_d^{(M_p)} \quad \text{for all } \varphi \in \mathcal{D}_d^{(M_p)},$$

where $\varphi^\triangle$ is the function of the class $\mathcal{E}_{2d}^{(M_p)}$ defined by

$$(6.2) \quad \varphi^\triangle(x, y) := \varphi(x + y), \quad x, y \in \mathbb{R}^d,$$

then we say that the Roumieu ultradistributions $S, T$ are *convolvable* in the sense of $(s)$. Then the *convolution* $S \ast T$ of $S$ and $T$ in $\mathcal{D}_d^{(M_p)}$ is defined by

$$(6.3) \quad \langle S \ast T, \varphi \rangle_d := \langle V_\varphi, 1 \rangle_{2d}, \quad \varphi \in \mathcal{D}_d^{(M_p)},$$

where $V_\varphi$ is meant, according to (6.1), as an element of the space $\tilde{\mathcal{D}}_{L^\infty,2d}$ and the constant function $1$ is meant as an element of the space $\tilde{\mathcal{D}}_{L^\infty,2d}$.

The following result on equivalence of convolvability conditions for Roumieu ultradistributions was proved in [21]:

**Theorem 2.** Let $S, T \in \mathcal{D}_d^{(M_p)}$. The following conditions are equivalent to condition $(s)$ of convolvability for $S$ and $T$:

1. $(c'_1)$ $S(T \ast \varphi) \in \tilde{\mathcal{D}}_{L^1,d}^{(M_p)}$ for all $\varphi \in \mathcal{D}_d^{(M_p)}$ and for every compact subset $K$ in $\mathbb{R}^d$, the mapping

$$\mathcal{D}_d^{(M_p)} \times \tilde{\mathcal{B}}_d^{(M_p)} \ni (\varphi, \chi) \mapsto \langle S(T \ast \varphi), \chi \rangle \in \mathbb{C}$$

is a continuous bilinear mapping;

2. $(c'_2)$ $(\varphi \ast S)T \in \tilde{\mathcal{D}}_{L^1,d}^{(M_p)}$ for all $\varphi \in \mathcal{D}_d^{(M_p)}$ and, for every compact subset $K$ in $\mathbb{R}^d$, the mapping

$$\mathcal{D}_d^{(M_p)} \times \tilde{\mathcal{B}}_d^{(M_p)} \ni (\varphi, \chi) \mapsto \langle (\varphi \ast S)T, \chi \rangle \in \mathbb{C}$$

is a continuous bilinear mapping;

3. $(c'_3)$ $(\tilde{S} \ast \varphi)(T \ast \psi) \in L_1^1$ for all $\varphi, \psi \in \mathcal{D}_d^{(M_p)}$.

Dimovski, Pilipović, Prangoski and Vindas modified conditions $(c'_1)$ and $(c'_2)$ and proved in [5], under the assumption that the sequence $(M_p)$ satisfies (M.1), (M.2) and (M.3), that they are equivalent to more transparent versions, as given in the following theorem:

**Theorem 3.** Let $S, T \in \mathcal{D}_d^{(M_p)}$. The following conditions are equivalent to condition $(s)$ of convolvability for $S$ and $T$:

1. $(c_1)$ $S(T \ast \varphi) \in \mathcal{D}_d^{(M_p)}$ for all $\varphi \in \mathcal{D}_d^{(M_p)}$;

2. $(c_2)$ $(\varphi \ast \tilde{S})T \in \mathcal{D}_d^{(M_p)}$ for all $\varphi \in \mathcal{D}_d^{(M_p)}$.

In the next section we formulate certain sequential conditions of convolvability of Roumieu ultradistributions, connected with Theorem 1 on integrability in $\mathcal{D}^{(M_p)}$ from section 5.
7 Sequential Definitions of Convolution in $\mathcal{D}'_{d}^{\{M_{p}\}}$

The notion of $\mathcal{A}$-approximate unit makes us possible to consider several sequential definitions of the convolution of Roumieu ultradistributions based on corresponding sequential conditions of convolvability. The conditions require that respective numerical sequences, corresponding to a given pair of Roumieu ultradistributions via certain approximate units, are Cauchy sequence (Cauchy s. in short) for all approximate units from a given class. The first definition of this kind was given for the convolution of distributions by V. S. Vladimirov in [20] and its equivalent versions were discussed in [3] and [9]. Their counterparts for ultradistributions of Beurling type were discussed in [11] (see also [2]).

We prove in [18] that all the sequential definitions are equivalent to the definition of the general convolution of Roumieu ultradistributions in the sense of S. Pilipović and B. Prangoski [21]. Our proof is based on the integrability result proved in the previous section.

**Definition 6.** Let $S, T \in \mathcal{D}'_{d}^{\{M_{p}\}}$. We say that $S, T$ are convolvable in the sense of (v), (ii), (i), (ii), if the corresponding condition below holds for every $\varphi \in \mathcal{D}_{d}^{\{M_{p}\}}$, respectively:

(v) $(\langle S \otimes T, \pi_{n} \varphi^{\triangle} \rangle_{2d})$ is a Cauchy s. for all $(\pi_{n}) \in U_{2d}^{\{M_{p}\}}$;

(ii) $(\langle (\pi_{n}^{1}S) \otimes (\pi_{n}^{2}T), \varphi^{\triangle} \rangle_{2d})$ is a Cauchy s. for all $(\pi_{n}^{1}, \pi_{n}^{2}) \in U_{d}^{\{M_{p}\}}$;

(i) $(\langle (\pi_{n}S) \otimes T, \varphi^{\triangle} \rangle_{2d})$ is a Cauchy s. for all $(\pi_{n}) \in U_{d}^{\{M_{p}\}}$;

(ii) $(\langle S \otimes (\pi_{n}T), \varphi^{\triangle} \rangle_{2d})$ is a Cauchy s. for all $(\pi_{n}) \in U_{d}^{\{M_{p}\}}$.

**Definition 7.** If $S, T \in \mathcal{D}'_{d}^{\{M_{p}\}}$ are convolvable in the sense of (v), (ii), (i), (ii), respectively, then the convolution of $S$ and $T$ in $\mathcal{D}'_{d}^{\{M_{p}\}}$ in the respective sense is defined by the corresponding formula below:

\[
\langle S^{\vee} T, \varphi \rangle_{d} := \lim_{n \to \infty} \langle S \otimes T, \pi_{n} \varphi^{\triangle} \rangle_{2d}, \quad \varphi \in \mathcal{D}_{d}^{\{M_{p}\}}, \quad (\pi_{n}) \in U_{2d}^{\{M_{p}\}};
\]

\[
\langle S^{\Pi} T, \varphi \rangle_{d} := \lim_{n \to \infty} \langle (\pi_{n}^{1}S) \otimes (\pi_{n}^{2}T), \varphi^{\triangle} \rangle_{2d}, \quad \varphi \in \mathcal{D}_{d}^{\{M_{p}\}}, \quad (\pi_{n}^{1}, \pi_{n}^{2}) \in U_{d}^{\{M_{p}\}};
\]

\[
\langle S^{\ii} T, \varphi \rangle_{d} := \lim_{n \to \infty} \langle (\pi_{n}S) \otimes T, \varphi^{\triangle} \rangle_{2d}, \quad \varphi \in \mathcal{D}_{d}^{\{M_{p}\}}, \quad (\pi_{n}) \in U_{d}^{\{M_{p}\}};
\]

\[
\langle S^{\Pi} T, \varphi \rangle_{d} := \lim_{n \to \infty} \langle S \otimes (\pi_{n}T), \varphi^{\triangle} \rangle_{2d}, \quad \varphi \in \mathcal{D}_{d}^{\{M_{p}\}}, \quad (\pi_{n}) \in U_{d}^{\{M_{p}\}};
\]

respectively.

8 Existence Theorems for the convolution of Roumieu ultradistributions

Finally, we will formulate without proofs two theorems concerning existence of the convolution of Roumieu ultradistributions.
The first one says that if the supports of Roumieu ultradistributions satisfy the known condition of compatibility, known also as condition \((\Sigma)\) (see [3], p. 383 and [II], p. 124), then the convolution of these ultradistributions exists in \(D_{d}^{(M_{p})}\). The proof is based on the known representation theorem for Roumieu ultradistributions (see [14], Theorem 8.7) and on the idea in the proof of Theorem 2 from [12] concerning ultradistributions of Beurling type. It should be noticed that the condition of compatibility is optimal in terms of supports of Roumieu ultradistributions, according to the second theorem (which follows from Theorem 5.1 proved in [10]).

We present these existence results not only for the convolution of two Roumieu ultradistributions but in an extended form to the case of \(k\) Roumieu ultradistributions, where \(k \in \mathbb{N}\setminus\{1\}\). Such a general form is useful in the study of various forms of associativity of the convolution (cf. [13]).

**Proposition 3.** Let \(A_{1}, \ldots, A_{k} \subseteq \mathbb{R}^{d}\) be arbitrary sets for \(k \geq 2\). The following conditions are equivalent:

- \((C)\) the set \((A_{1} \times \ldots \times A_{k}) \cap K^{\Delta}\) is bounded in \(\mathbb{R}^{kd}\) for every \(K\) bounded in \(\mathbb{R}^{d}\), where \(K^{\Delta} := \{(x_{1}, \ldots, x_{k}) \in \mathbb{R}^{kd} : x_{1} + \ldots + x_{k} \in K\};\)

- \((c)\) the following implication holds:

\[
\lim_{n \to \infty} \sum_{i=1}^{k} |x_{i,n}| = \infty \Rightarrow \lim_{n \to \infty} \sum_{i=1}^{k} x_{i,n} = \infty,
\]

whenever \(x_{i,n} \in A_{i}\) for \(i \in \{1, \ldots, k\}\) and \(n \in \mathbb{N}\).

**Definition 7.** Sets \(A_{1}, \ldots, A_{k} \subseteq \mathbb{R}^{d}\) are called compatible if any of the two equivalent conditions \((C), (c)\) in Proposition 3 is satisfied.

**Definition 8.** For given Roumieu ultradistributions \(S_{1}, \ldots, S_{k}\) on \(\mathbb{R}^{d}\) we define the convolution \(S_{1} \ast \ldots \ast S_{k}\) in \(D_{d}^{(M_{p})}\) by

\[
\langle S_{1} \ast \ldots \ast S_{k}, \varphi \rangle := \lim_{n \to \infty} \langle S_{1} \otimes \ldots \otimes S_{k}, n n \varphi^{\Delta} \rangle,
\]

where

\[
\varphi^{\Delta}(x_{1}, \ldots, x_{k}) := \varphi(x_{1} + \ldots + x_{k}), \quad x_{1}, \ldots, x_{k} \in \mathbb{R}^{d},
\]

whenever the above limit exists for all \((n) \in \mathbb{U}^{(M_{p})}\) and for all \(\varphi \in D_{d}^{(M_{p})}\). We say then that the convolution \(S_{1} \ast \ldots \ast S_{k}\) exists in \(D_{d}^{(M_{p})}\).

**Theorem 4.** Let \(S_{1}, \ldots, S_{k} \in D_{d}^{(M_{p})}\) and assume that the supports of \(S_{1}, \ldots, S_{k}\) are contained in compatible sets \(A_{1}, \ldots, A_{k} \subseteq \mathbb{R}^{d}\), respectively. Then the convolution \(S_{1} \ast \ldots \ast S_{k}\) exists in \(D_{d}^{(M_{p})}\) and \(\text{supp } (S_{1} \ast \ldots \ast S_{k}) \subseteq A_{1} + \ldots + A_{k}\).

It is interesting that the following converse result is also true:

**Theorem 5.** Let \(A_{1}, \ldots, A_{k} \subseteq \mathbb{R}^{d}\). Suppose that the convolution \(S_{1} \ast \ldots \ast S_{k}\) exists in \(D_{d}^{(M_{p})}\) for arbitrary ultradistributions \(S_{1}, \ldots, S_{k} \in D_{d}^{(M_{p})}(\mathbb{R}^{d})\) such that \(\text{supp } S_{1} \subseteq A_{1}, \ldots, \text{supp } S_{k} \subseteq A_{k}\), respectively. Then \(A_{1}, \ldots, A_{k}\) are compatible.
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