Yamabe metrics on cylindrical manifolds

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Abstract

We study a particular class of open manifolds. In the category of Riemannian manifolds these are complete manifolds with cylindrical ends. We give a natural setting for the conformal geometry on such manifolds including an appropriate notion of the cylindrical Yamabe constant/invariant. This leads to a corresponding version of the Yamabe problem on cylindrical manifolds. We affirmatively solve this Yamabe problem: we prove the existence of minimizing metrics and analyze their singularities near infinity. These singularities turn out to be of very particular type: either almost conical or almost cusp singularities. We describe the supremum case, i.e. when the cylindrical Yamabe constant is equal to the Yamabe invariant of the sphere. We prove that in this case such a cylindrical manifold coincides conformally with the standard sphere punctured at a finite number of points. In the course of studying the supremum case, we establish a Positive Mass Theorem for specific asymptotically flat manifolds with two almost conical singularities. As a by-product, we revisit known results on surgery and the Yamabe invariant. Key words: manifolds with cylindrical ends, Yamabe constant/invariant, Yamabe problem, conical metric singularities, cusp metric singularities, Positive Mass Theorem, surgery and Yamabe invariant.

1 Introduction

The goal of this paper is to formulate and solve a natural version of the Yamabe problem for complete manifolds with cylindrical ends. Before describing our main results in detail, we recall the classical situation.

1.1. Classical Yamabe problem for compact manifolds. Let $M$ be a smooth closed manifold (i.e. a smooth compact manifold without boundary) of dim $M = n \geq 3$ and $\text{Riem}(M)$ the space of all Riemannian metrics on $M$.

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We denote by \( R_g \) the scalar curvature and by \( d\sigma_g \) the volume form for each Riemannian metric \( g \in \mathcal{R}_{\text{iem}}(M) \). Then the (normalized) Einstein-Hilbert functional \( I_M : \mathcal{R}_{\text{iem}}(M) \to \mathbb{R} \) is defined as

\[
I_M : g \mapsto \frac{\int_M R_g d\sigma_g}{\text{Vol}_g(M)^{\frac{n}{n-2}}}.
\]  

(1)

The classical Yamabe problem is to find a metric \( \tilde{g} \) in a given conformal class \( C \) such that the Einstein-Hilbert functional attains its minimum on \( C \):

\[
I_M(\tilde{g}) = \inf_{g \in C} I_M(g) =: Y_C(M).
\]

This minimizing metric \( \tilde{g} \) is called a Yamabe metric, and the conformal invariant \( Y_C(M) \) the Yamabe constant.

It is a celebrated result in conformal geometry that the Yamabe problem has an affirmative solution for closed manifolds. This was proven in a series of papers starting with the work of Yamabe \[43\]. Although Yamabe’s proof overlooked the fundamental analytic difficulty concerning Sobolev inequalities with a critical exponent, the general strategy in \[43\] was correct. This difficulty is a profound one, and the proof was ultimately only corrected in stages, first by Trudinger \[42\], then by Aubin \[9\], and was finally completed by Schoen \[34\]. Text-book style proofs are now available in \[40\] and \[27\].

The second case when the Yamabe problem has a positive solution is when a manifold \( M \) has a non-empty boundary. In this case, the Einstein-Hilbert functional has to be restricted to metrics (and corresponding conformal subclasses) with a minimal boundary condition (see \[4\]). The resulting Yamabe problem was solved by Cherrier \[18\] and Escobar \[19\] under some mild restrictions.

1.2. Yamabe problem for open manifolds. In the case of open manifolds, it is important to clarify what is a suitable version of the Yamabe problem which would capture the geometry of their ends. One reasonable version of the Yamabe problem is to find a complete metric of constant scalar curvature in a given conformal class. In this case, however, the concept of a minimizing Yamabe metric does not make sense, and there are serious difficulties in this area. Indeed, there are simple noncompact Riemannian manifolds \( (N, \tilde{g}) \) for which there does not exist any complete metric (conformal to \( \tilde{g} \)) of constant scalar curvature, (cf. \[29\]). Hence one needs to place some geometric restriction (e.g. curvature or injectivity radius bounds) near each end of a noncompact manifold in order to establish the existence of a metric of constant scalar curvature.

A more specific version of the Yamabe problem for open manifolds is the singular Yamabe problem, i.e. when an open manifold \( N \) is a complement
Then the singular Yamabe problem is to find a complete metric \( \tilde{g} \) that is conformal to \( g \) on \( M \setminus \Sigma \) and has constant scalar curvature, see \cite{11,35} for earlier results and also \cite{29} for a survey of results in this area. Here again the concept of a minimizing Yamabe metric is not well-defined.

3. Cylindrical manifolds. In this paper, we consider a particular class of open manifolds, which we call cylindrical manifolds. These are open manifolds with tame ends equipped with a cylindrical metric on each end. Cylindrical manifolds are well-known objects in geometry and topology. First of all, these manifolds are well-suited for the Dirac operator on complete manifolds, as was shown first by Gromov and Lawson in \cite{22}. Secondly, these objects are well known in gauge theory, where cylindrical manifolds have been thoroughly studied, see for example the books \cite{20,30,31,41}. We remark that \( \mathbb{Z}/k \)-manifolds (as geometric objects) can be thought of as cylindrical manifolds with \( k \) identical cylindrical ends (see \cite{16} for results on the existence of positive scalar curvature on \( \mathbb{Z}/k \)-manifolds).

There are several other constructions in geometry where cylindrical manifolds show up as natural limit manifolds. The first example comes from the work describing the process of bubbling out of Einstein manifolds, see \cite{17,13} and \cite{12}. Here it is known that for a sequence of Einstein manifolds \( (X_i, g_i) \) (with some natural restrictions on their geometry) there exists a subsequence \( (X_{i_q}, g_{i_q}) \) which converges (in the Gromov-Hausdorff topology) to a compact Einstein orbifold \( (X_{\infty}, g_{\infty}) \) with a finite set of singular points. Let \( U_p \) be an open neighborhood of a singular point \( p \in X_{\infty} \). Then the tangent cone of \( U_p \setminus \{p\} \) at \( p \) is conformally cylindrical, although \( U_p \setminus \{p\} \) itself need not be. Thus a cylindrical manifold (with an appropriate conformal metric) can be thought of as a linear approximation of the Einstein orbifold \( (X_{\infty}, g_{\infty}) \) (see \cite{4} for the relation of this approximation to the Yamabe invariant of orbifolds).

The second example is related to the Yamabe invariant \( Y(M) = \sup_C Y_C(M) \) of a compact manifold \( M \), see \cite{20,30}. It was shown that \( Y(S^{n-1} \times S^1) = Y(S^n) \), i.e. there exists a sequence of conformal classes \( C_i \) and Yamabe metrics \( \tilde{g}_i \in C_i \) such that the limit \( \lim_i Y_{C_i}(S^{n-1} \times S^1) = Y(S^n) \). The sequence of Riemannian manifolds \( (S^{n-1} \times S^1, \tilde{g}_i) \) converges to the standard sphere \( S^n(1) \) identified with two antipodal points, see \cite{25,30}. After deleting the singular point, the punctured sphere is conformally equivalent to the canonical cylindrical manifold \( S^{n-1} \times \mathbb{R} \). In the general case of a closed manifold \( M \) with positive Yamabe invariant, there is also a compactness result. In \cite{1,2}, the first author proved that, under some restrictions, for a sequence of Yamabe metrics \( \tilde{g}_i \) on \( M \) satisfying \( \lim_i Y_{[\tilde{g}_i]}(M) = Y(M) \)
there exists a subsequence \( \tilde{g}_n \) of \( \tilde{g}_i \) such that \((M, \tilde{g}_n)\) converges (in the Gromov-Hausdorff topology) to a compact metric space \((M_\infty, g_\infty)\), which is a smooth Riemannian manifold away from a finite number of singular points. Again, a cylindrical manifold (with an appropriate conformal metric) serves here as a linear approximation of the limit singular space \((M_\infty, g_\infty)\).

For a given closed manifold \( M \), it is a challenging problem to find a nice sequence of Yamabe metrics \( \hat{g}_i \) satisfying \( \lim_i Y_{\hat{g}_i}(M) = Y(M) \) such that \((M, \hat{g}_i)\) converges to a singular Riemannian space \((M_\infty, g_\infty)\), and to understand the limit singular space \((M_\infty, g_\infty)\) with bubbling out spaces. In our view, cylindrical manifolds may provide a typical conformal model of such a limit singular space \((M_\infty, g_\infty)\).

14. Yamabe problem for cylindrical manifolds. In the smooth category, a cylindrical manifold \( X \) is an open manifold with a relatively compact open submanifold \( W \subset X \) with \( \partial W = Z \) such that \( X \setminus W \cong Z \times [0, \infty) \), where \( Z = \bigsqcup_{j=1}^m Z_j \) and each \( Z_j \) is a connected closed \((n-1)\)-manifold. Throughout this paper, we always assume that \( X \) is connected.

To formulate an appropriate notion of conformal class on cylindrical manifolds requires some care. As a reference metric, we start with a cylindrical Riemannian metric \( \bar{g} \) on \( X \), i.e. \( \bar{g}(x, t) = h(x) + dt^2 \) on \( Z \times [1, \infty) \), for some metric \( h \) on \( Z \). We denote \( \partial_\infty \bar{g} = h \). Let \([\bar{g}]\) denote the conformal class containing \( \bar{g} \). Clearly the Einstein-Hilbert functional \( I_X \) (given by (11)) restricted to the conformal class \([\bar{g}]\) is not well-defined since the scalar curvature may not be integrable and the volume may be infinite. Consider the Dirichlet form

\[
Q_{(X, \bar{g})}(u) = \frac{\int_X \left[ \frac{4(n-1)}{n-2} |du|^2 + R_{\bar{g}}u^2 \right] d\sigma_{\bar{g}}}{\left( \int_X |u|^{\frac{2n}{n-2}} d\sigma_{\bar{g}} \right)^{\frac{n-2}{n}}}
\]

on the space \( C^\infty_\sigma(X) \) (of smooth functions on \( X \) with compact support) associated with the conformal Laplacian of \((X, \bar{g})\). The functional \( Q_{(X, \bar{g})} \) suggests the following natural setting for the Yamabe problem on cylindrical manifolds.

First, we define the \( L^{k, 2}_{\bar{g}} \)-conformal class \([\bar{g}]_{L^{k, 2}_{\bar{g}}} \) consisting of all metrics \( u^{\frac{4}{n-2}} \cdot \bar{g} \), where \( u \) is a smooth positive function on \( X \) and \( u \in L^{k, 2}_{\bar{g}}(X) \), \( k = 1, 2 \). Here \( L^{k, 2}_{\bar{g}}(X) \) denotes the Sobolev space of square-integrable functions on \( X \) (with respect to \( \bar{g} \)) up to their \( k \)-th weak derivatives. Then the functional \( Q_{(X, \bar{g})} \) is well-defined on the space \( C^\infty_\sigma(X) \cap L^{1, 2}_{\bar{g}}(X) \), where \( C^\infty_\sigma(X) = \{ u \in C^\infty(X) \mid u > 0 \} \). On the other hand, the Einstein-Hilbert functional \( I_X \) on the conformal class \([\bar{g}]_{L^{2, 2}_{\bar{g}}(X)} \) is also well-defined. It turns
out that the infima of both functionals coincide:

\[
Y_{[\bar{g}]}^{cyl}(X) := \inf_{\bar{g} \in [\bar{g}], L_{1,2}^2(X)} \int_X (\bar{g}) = \inf_{u \in C^\infty_0(X) \cap L_{1,2}^2(X)} Q(X, \bar{g})(u).
\]

We call the constant \( Y_{[\bar{g}]}^{cyl}(X) \) the cylindrical Yamabe constant of \((X, [\bar{g}])\) and show that it does not depend on the choice of a reference cylindrical metric in the same conformal class as \( \bar{g} \). Now we are ready to state the Yamabe problem on cylindrical manifolds.

**Yamabe Problem.** Given a cylindrical metric \( \bar{g} \) on \( X \), does there exist a metric \( \tilde{\bar{g}} = u^{4/n-2} \cdot \bar{g} \in [\bar{g}], L_{1,2}^2(X) \) such that \( Q(X, \bar{g})(u) = Y_{[\bar{g}]}^{cyl}(X) \)?

We call such a metric \( \tilde{\bar{g}} \) (if it exists) a Yamabe metric and such a function \( u \) a Yamabe minimizer. We affirmatively solve the Yamabe problem for generic cylindrical conformal classes in this setting.

**1.5. The invariant \( \lambda(L_h) \).** First of all, we show that the constant \( Y_{[\bar{g}]}^{cyl}(X) \) satisfies the bounds \(-\infty \leq Y_{[\bar{g}]}^{cyl}(X) \leq Y(S^n)\), where \( Y(S^n) \) is the Yamabe invariant of the \( n \)-sphere. In particular, it is possible that \( Y_{[\bar{g}]}^{cyl}(X) = -\infty \).

To determine when the constant \( Y_{[\bar{g}]}^{cyl}(X) \) is finite, we introduce a new invariant \( \lambda(L_h) \). We introduce the operator

\[
L_h := -\frac{4(n-1)}{n-2} \Delta_h + R_h \quad \text{on} \quad (Z, h).
\]

Notice that the operator \( L_h \) is different from the conformal Laplacian \( \Delta_h = \frac{4(n-2)}{n-3} \Delta_h + R_h \) of \((Z, h)\). Let \( \lambda(L_h) \) be the first eigenvalue of the operator \( L_h \). The invariant \( \lambda(L_h) \) determines the finiteness of the cylindrical Yamabe constant as follows:

- If \( \lambda(L_h) < 0 \), then \( Y_{C}^{cyl}(X) = -\infty \). In particular, if \( R_h < 0 \) on \( Z \), then \( Y_{C}^{cyl}(X) = -\infty \).

- If \( \lambda(L_h) \geq 0 \), then \( Y_{C}^{cyl}(X) > -\infty \). In particular, if \( R_h \geq 0 \) on \( Z \), then \( Y_{C}^{cyl}(X) > -\infty \).

- If \( \lambda(L_h) = 0 \), then \( Y_{C}^{cyl}(X) \leq 0 \). In particular, if \( R_h \equiv 0 \) on \( Z \), then \( Y_{C}^{cyl}(X) \leq 0 \).

Here we remark that if \( Z \) is not connected (i.e. \( Z = \bigcup_{i=1}^m Z_j \) for \( m \geq 2 \)), then \( \lambda(L_h) = \min_{1 \leq j \leq m} \lambda(L_{h|Z_j}) \). Clearly the Yamabe problem does not make
sense if $\lambda(L_h) < 0$. Hence the case we study here is when $\lambda(L_h) \geq 0$. We also observe that, in general, if $\lambda(L_h) = 0$ then there is no solution of the Yamabe problem.

**16. Solution of the Yamabe problem in a “generic case”**. Our first result repeats, in some sense, the classical approach to the Yamabe problem. Recall that in the “generic case” there, i.e. when the Yamabe constant $Y_C(M) < Y(S^n)$, there is a well-known technique giving a solution, see [10]. Here, instead of the standard sphere, we have a canonical cylindrical manifold $(\mathbb{Z} \times \mathbb{R}, h + dt^2)$ (for some metric $h$ on $\mathbb{Z}$) which plays a similar role. In fact, for any cylindrical manifold $(X, \bar{g})$ with the cylindrical end $(\mathbb{Z} \times [1, \infty), h + dt^2)$, we show that the inequality $Y_{cyl}^{\bar{g}}(X) \leq Y_{cyl}^{h + dt^2}(\mathbb{Z} \times \mathbb{R})$ always holds. Here we also remark that if $\mathbb{Z}$ is not connected (i.e. $\mathbb{Z} = \bigsqcup_{j=1}^{m} Z_j$ for $m \geq 2$), then

$$Y_{cyl}^{h + dt^2}(\mathbb{Z} \times \mathbb{R}) = \min_{1 \leq j \leq m} Y_{cyl}^{h + dt^2}(Z_j \times \mathbb{R}).$$

First we study the case when $Y_C^{cyl}(X) < Y_C^{h + dt^2}(\mathbb{Z} \times \mathbb{R})$. We emphasize that the situations when the invariant $\lambda(L_h)$ is positive or zero are very different geometrically. In fact, $Y_{cyl}^{h + dt^2}(\mathbb{Z} \times \mathbb{R}) > 0$ (resp. $Y_{cyl}^{h + dt^2}(\mathbb{Z} \times \mathbb{R}) = 0$) if $\lambda(L_h) > 0$ (resp. $\lambda(L_h) = 0$), and see Theorem B below. However the existence results are similar.

**Theorem A**. Let $X$ be an open manifold of dim $X \geq 3$ with tame ends $\mathbb{Z} \times [0, \infty)$, and $h \in \text{Riem}(Z)$. Let $\bar{C}$ be a conformal class on $X$ containing a cylindrical metric $\bar{g} \in \bar{C}$ with $\partial_\infty \bar{g} = h$. Assume that either

(a) $\lambda(L_h) > 0$ and $Y_C^{cyl}(X) < Y_C^{h + dt^2}(\mathbb{Z} \times \mathbb{R})$, or

(b) $\lambda(L_h) = 0$ and $Y_C^{cyl}(X) < 0$.

Then the Yamabe problem has a solution, i.e. there exists a Yamabe minimizer $u \in C^\infty_+(X) \cap L^2_g(X)$ with $\int_X u^{n+2} d\bar{g} = 1$ such that $Q_{(\bar{g}, \bar{g})}(u) = Y_C^{cyl}(X)$. In particular, the minimizer $u$ satisfies the Yamabe equation $\mathcal{L}_{\bar{g}} u = Y_C^{cyl}(X) u^{\frac{n+2}{n-2}}$.

Next we study the behavior near infinity of the Yamabe metrics $\hat{g} = u^{\frac{n-2}{2}} \cdot \bar{g}$ given by Theorem A. As we mentioned, the cases $\lambda(L_h) > 0$ and $\lambda(L_h) = 0$ lead to completely different geometric situations; in particular, the asymptotics of the Yamabe metrics turn out to be qualitatively distinct.
The first case when \( \lambda(L_h) > 0 \) leads to almost conical metrics. A canonical geometric model here is an open cone over \( Z \), i.e.

\[
\text{Cone}(Z) \cong (Z \times (0, \infty), e^{-2t}(h + dt^2)).
\]

Then, a metric \( g \) on \( X \) is almost conical on a connected end \( Z_j \times [1, \infty) \) if on the end \( Z_j \times [1, \infty) \) it is given as \( g(x, t) = \varphi(x, t)(h(x) + dt^2) \), where \( \varphi(x, t) \) is asymptotically bounded by \( C_1 \cdot e^{-\beta t} \leq \varphi(x, t) \leq C_2 \cdot e^{-\alpha t} \) for some constants \( 0 < \alpha \leq \beta, 0 < C_1 \leq C_2 \).

The case when \( \lambda(L_h) = 0 \) leads to almost cusp metrics. A canonical geometric model here is given by the cusp end of a hyperbolic \( n \)-manifold

\[
\left( (R^{n-1}/\Gamma) \times [1, \infty), \frac{1}{t^2} (h_0 + dt^2) \right)
\]

of curvature \(-1\). Here \((R^{n-1}/\Gamma, h_0)\) is a closed Riemannian manifold uniformized by a flat torus \( T^{n-1} \). Then, a metric \( g \) on \( X \) is almost cusp on a connected end \( Z_j \times [1, \infty) \) if on the end \( Z_j \times [1, \infty) \) it is given as \( g(x, t) = \varphi(x, t)(h(x) + dt^2) \), where the function \( \varphi(x, t) \) is asymptotically bounded by \( C_1 \cdot t^{-2} \leq \varphi(x, t) \leq C_2 \cdot t^{-2} \) for some constants \( 0 < C_1 \leq C_2 \).

**Theorem B.** Let \( X \) be an open manifold of dim \( X \geq 3 \) with tame ends \( Z \times [0, \infty) \), and \( h \in \text{Riem}(Z) \). Let \( C \) be a conformal class on \( X \) containing a cylindrical metric \( \bar{g} \in C \) with \( \partial_{\infty} \bar{g} = h \).

(a) If \( \lambda(L_h) > 0 \) and \( L_C^cyl(X) < L_{[h + dt^2]}^cyl(Z \times R) \), then the Yamabe metric \( \bar{g} = u \frac{\partial}{\partial t} \cdot \bar{g} \) given by Theorem A is almost conical on each connected end.

(b) If \( \lambda(L_h) = 0 \) and \( L_C^cyl(X) < 0 \), then the Yamabe metric \( \bar{g} = u \frac{\partial}{\partial t} \cdot \bar{g} \) given by Theorem A is almost cusp on each connected end \( Z_j \times [1, \infty) \) with \( \lambda(L_{h|z_j}) = 0 \) and almost conical on each connected end \( Z_k \times [1, \infty) \) with \( \lambda(L_{h|z_k}) > 0 \).

7. **Solution of the Yamabe problem for canonical cylindrical manifolds.** As we mentioned, the canonical cylindrical manifolds \((Z \times R, h = h + dt^2)\) play a crucial role with respect to the Yamabe problem, and these manifolds require a special treatment. To state the result, we need the dominant canonical cylindrical manifold \((S^{n-1} \times R, h_+ + dt^2)\), where \( h_+ \) is the standard metric of constant scalar curvature 1 on the sphere \( S^{n-1} \). We remark here that the cylindrical Yamabe constant \( L_{[h_+ + dt^2]}^cyl(S^{n-1} \times R) \) coincides with the Yamabe invariant \( Y(S^n) \).
Theorem C. Let \((Z \times \mathbb{R}, \bar{h} = h + dt^2)\) be a connected canonical cylindrical manifold of \(\dim(Z \times \mathbb{R}) = n \geq 3\) with \(\lambda(L_h) > 0\). Assume that \(Y_{[\bar{h}]}^{cyf}(Z \times \mathbb{R}) < Y_{[0]}^{cyf}(S^{n-1} \times \mathbb{R})\). Then the Yamabe problem has a solution, i.e. there exists a Yamabe minimizer \(u \in C^\infty(\overline{Z \times \mathbb{R}}) \cap L^{1,2}_h(\overline{Z \times \mathbb{R}})\) with \(\int_{Z \times \mathbb{R}} u^\frac{2}{n-1} \, d\bar{h} = 1\) such that \(Q_{(\overline{Z \times \mathbb{R}}, \bar{h})}(u) = Y_{[\bar{h}]}^{cyf}(Z \times \mathbb{R})\). Furthermore, the Yamabe metric \(\bar{g} = u^{-\frac{2}{n-1}} \cdot \bar{g}\) is an almost conical metric.

If \(\lambda(L_h) = 0\), then the Yamabe problem does not have a solution for any canonical cylindrical manifold \((Z \times \mathbb{R}, \bar{h} = h + dt^2)\), see Proposition 5.1.

18. Characterization of the supremum case. First we analyze the supremum case for a connected canonical cylindrical manifold, i.e. when the cylindrical Yamabe constant \(Y_{[\bar{h}]}^{cyf}(Z \times \mathbb{R}) = Y_{[\bar{h}]}^{cyf}(S^{n-1} \times \mathbb{R}) = Y(S^n)\).

Theorem D. Let \((Z \times \mathbb{R}, \bar{h} = h + dt^2)\) be a connected canonical cylindrical manifold of \(\dim(Z \times \mathbb{R}) = n \geq 3\). Assume that \(Y_{[\bar{h}]}^{cyf}(Z \times \mathbb{R}) = Y(S^n)\). Then \((Z, \bar{h})\) is homothetic to the standard sphere \(S^{n-1}(1) = (S^{n-1}, h_+)\).

The proof of Theorem D is somewhat involved. First, as in the compact manifolds case, we prove Theorem D under the condition that the manifold \((Z \times \mathbb{R}, \bar{h})\) is locally conformally flat. The remaining part of the proof is to show that \(Y_{[\bar{h}]}^{cyf}(Z \times \mathbb{R}) < Y(S^n)\) provided that \((Z \times \mathbb{R}, \bar{h})\) is not locally conformally flat. This result splits into two different cases: \(n \geq 6\) and \(n = 3, 4, 5\). In the case \(n \geq 6\), we use the conformal normal coordinates technique and the family of instantons \(u_\epsilon(x) = \left(\frac{\epsilon}{\epsilon + |x|}\right)^{\frac{4}{n-2}}\) on \(\mathbb{R}^n\) to construct nice test functions and to prove the desired statement.

The case when \(n = 3, 4, 5\) is a truly difficult part. We first use a technique similar to the compact manifolds case to construct a minimal positive Green function \(G_p \in C^\infty((\overline{Z \times \mathbb{R}}) \setminus \{p\})\) of the conformal Laplacian \(\Delta_\bar{h}\) on \(Z \times \mathbb{R}\). Then \(\bar{G}_p\) has the expansion \(\bar{G}_p = r^{2-n} + A + O'(r)\) in the conformal normal coordinates, where \(A\) is a constant related to the mass. To complete the proof of Theorem D, we analyze the following situation.

Let \((Z \times \mathbb{R}, h + dt^2)\) be a connected canonical cylindrical manifold with \(\lambda(L_h) > 0\), and \(p \in Z \times \mathbb{R}\). Then for a point \(p \in Z \times \mathbb{R}\), we consider the manifold \(\hat{X} = (Z \times \mathbb{R}) \setminus \{p\}\) with the metric \(\hat{h} = \frac{1}{\bar{G}_p} \cdot \bar{h}\), where \(\bar{G}_p\) is the minimal positive Green function. We obtain that \((\hat{X}, \hat{h})\) is a scalar-flat, asymptotically flat manifold. We emphasize that \((\hat{X}, \hat{h})\) has two almost conical singularities. Then we note that in this case the mass \(m(\hat{h})\) is well-
defined in the same way as in the classical case, see [14]. The following
Theorem is a generalization of the classical positive mass theorem [40, 37, 38]
(cf. [14] and [27]) to the case of specific asymptotically flat manifolds with
two almost conical singularities.

**Theorem E.** (Positive mass theorem) Let \((Z \times \mathbb{R}, \bar{h} = h + dt^2)\) be a connected canonical cylindrical manifold of dim\((Z \times \mathbb{R}) = n\) \((n = 3, 4, 5)\) with
\(\lambda(\mathcal{L}_h) > 0\), and \(p \in Z \times \mathbb{R}\). Let \((\bar{X}, \bar{h}) = ((Z \times \mathbb{R}) \setminus \{p\}, \bar{G}_p^{\mathbb{R}^2} \cdot \bar{h})\) be the scalar-flat, asymptotically flat manifold with two almost conical singularities as above. Then the mass \(m(\bar{h})\) is non-negative. Furthermore, if \(m(\bar{h}) = 0\), then \((\bar{X}, \bar{h})\) is isometric to \(\mathbb{R}^n \setminus \{2 \text{ points}\}\) with the Euclidean metric.

We use Theorem D to analyze the supremum case for general cylindrical manifolds, i.e. when \(Y_{[g]}^{\text{cy}}(X) = Y(S^n)\).

**Theorem F.** Let \((X, \bar{g})\) be a cylindrical manifold of dim\(X \geq 3\) with tame ends \(Z \times [0, \infty)\) and \(\partial_{\infty} \bar{g} = h\). Set \(Z = \bigsqcup_{i=1}^m Z_i\), where each \(Z_i\) is connected. Assume that \(Y_{[\bar{g}]}^{\text{cy}}(X) = Y(S^n)\). Then there exist \(k\) points \(\{p_1, \ldots, p_m\}\) in \(S^n\) such that

(i) each manifold \((Z_j, h_j)\) is homothetic to \((S^{n-1}, h_+)\),

(ii) the manifold 
\((X, \bar{g})\) is conformally equivalent to the punctured sphere 
\((S^n \setminus \{p_1, \ldots, p_m\}, C_{\text{can}})\).

Here \(h_j = h|_{Z_j}\) and \(C_{\text{can}}\) denotes the canonical conformal class on \(S^n\).

119. Cylindrical Yamabe invariant and gluing constructions. The
Yamabe problem on cylindrical manifolds and the cylindrical Yamabe con-
stant naturally leads us to the Yamabe invariant for cylindrical manifolds. For each metric \(h\) on the slice \(Z\), we define the \(h\)-cylindrical Yamabe invariant \(Y^{h-cy}(X)\) and the cylindrical Yamabe invariant \(Y^{\text{cy}}(X)\) as follows:

\[
Y^{h-cy}(X) := \sup_{\bar{g} \in \mathcal{R}_{\text{iem}}^{\text{cy}}(X)} Y_{[\bar{g}]}^{\text{cy}}(X), \quad Y^{\text{cy}}(X) := \sup_{h \in \mathcal{R}_{\text{iem}}(Z)} Y^{h-cy}(X).
\]

We show that the \(h\)-cylindrical Yamabe invariant \(Y^{h-cy}(X)\) is a homothetic invariant, i.e. \(Y^{h-cy}(X) = Y^{k-h-cy}(X)\) for any constant \(k > 0\), but it is not a conformal invariant. This invariant is a natural generalization of the Yamabe invariant for closed manifolds. Indeed, let \(M\) be a closed manifold of dim\(M = n \geq 3\) and \(p_\infty\) a point in \(M\). Then the manifold \(X = M \setminus \{p_\infty\}\) is an open manifold with the tame end \(Z \times \{0, \infty\} = S^{n-1} \times \{0, \infty\}\). Put
Then we show that
\[ Y^{cy}(M \setminus \{p_\infty\}) \geq Y^{h_+ - cy}(M \setminus \{p_\infty\}) = Y(M). \]

We analyze the dependence of these invariants under gluing in the following situation. Let \( W_1, W_2 \) be two compact connected manifolds with boundaries \( \partial W_1 = \partial W_2 = Z \neq \emptyset \). Then we have the open manifolds
\[ X_1 := W_1 \cup Z \times [0, \infty) \quad \text{and} \quad X_2 := W_2 \cup Z \times [0, \infty) \]
with tame ends. We prove the following Kobayashi-type inequality (see \[26\]).

**Theorem G.** Let \( W_1, W_2 \) be compact manifolds of \( \dim W_i \geq 3 \) with \( \partial W_1 = \partial W_2 = Z \). Then, for any metric \( h \in R^{\text{iem}}(Z) \),
\[
Y(W_1 \cup Z \cup W_2) \geq \begin{cases} 
(\frac{1}{2} Y^{h_+ - cy}(X_1) + \frac{1}{2} Y^{h_+ - cy}(X_2))^{\frac{1}{2}} & \text{if } Y^{h_+ - cy}(X_i) \leq 0, \ i = 1, 2, \\
\min \{ Y^{h_+ - cy}(X_1), Y^{h_+ - cy}(X_2) \} & \text{otherwise}
\end{cases}
\]

Finally, we revisit the surgery construction for the Yamabe invariant from \[32\]. We prove the following result.

**Theorem H.** Let \( M \) be a closed manifold of \( \dim M = n \geq 3 \), and \( N \) an embedded closed submanifold of \( M \) with trivial normal bundle. Let \( g_N \in R^{\text{iem}}(N) \) be a given metric on \( N \) and \( h_+ \) the standard metric on \( S^q \). Assume that \( q = n - p \geq 3 \).

1. If \( Y(M) \leq 0 \), then for any \( \varepsilon > 0 \) there exists \( \delta = \delta(\varepsilon, g_N, |Y(M)|) > 0 \) such that
\[
Y((\kappa^2 h_+ + g_N) - cy)(M \setminus N) \geq Y(M) - \varepsilon
\]
for any \( 0 < \kappa \leq \delta \). In particular, \( Y^{cy}(M \setminus N) \geq Y(M) \).

2. If \( Y(M) > 0 \), then there exists \( \delta = \delta(g_N, Y(M)) > 0 \) such that
\[
Y((\kappa^2 h_+ + g_N) - cy)(M \setminus N) > 0
\]
for any \( 0 < \kappa \leq \delta \). In particular, \( Y^{cy}(M \setminus N) > 0 \).

Theorem H combined with Theorem G gives a refinement of the result on surgery and the Yamabe invariant due to Pepean and Yun, see \[32\].

**10. The plan of the paper.** In Section 2 we introduce basic terminology and review necessary results. In Section 3 we study some properties of
the cylindrical Yamabe constant/invariant under the gluing of manifolds along their boundary. In particular, we prove Theorem G (Theorem 3.7). In Section 4, we revisit the surgery construction for the Yamabe invariant. In particular, we prove Theorem H (Theorem 4.9). Sections 5 and 6 form the core of this paper. In Section 5, we first give the setting for the Yamabe problem. Then we study the case $\lambda(\mathcal{L}_h) > 0$ and prove Theorem 5.2 and Theorem 5.13 which imply Theorem A(a) and Theorem B(a). Next we study the case $\lambda(\mathcal{L}_h) = 0$ and prove Theorem 5.14 which implies Theorem A(b) and Theorem B(b). In Section 6, we solve the Yamabe problem for canonical cylindrical manifolds. First we consider the case when $Y^{cyf}_{[\bar{h}]}(Z \times \mathbb{R}) < Y^{cyf}(S^n)$ and prove Theorem 6.1 which implies Theorem C. Next we study the supremum case and prove Theorem 6.2 and Corollary 6.3 which imply Theorem D and Theorem F. Theorem E follows from Theorem 6.13 and Theorem 6.16. In the Appendix, we discuss the bottom of the spectrum of the conformal Laplacian $\mathcal{L}_{\bar{g}}$ on a cylindrical manifold $(X, \bar{g})$ and its relationship to the cylindrical Yamabe constant $Y^{cyf}_{[\bar{h}]}(X)$ and the sign of scalar curvature.

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2 Cylindrical manifolds

21. Definition of cylindrical manifolds. Here we review some basic facts and give definitions we need. Let $X$ be an open smooth manifold of $\dim X = n \geq 3$ without boundary $\partial X = \emptyset$.

Definition 2.1 An open smooth manifold $X$ is called a manifold with tame ends if there is a relatively compact open submanifold $W \subset X$ with $\partial W = Z$ such that

1. $Z = \bigsqcup_{j=1}^{m} Z_j$, where each $Z_j$ is connected,

2. $X \setminus W \cong Z \times [0, 1) = \bigsqcup_{j=1}^{m} (Z_j \times [0, 1))$. 
We study such manifolds $X$ equipped with cylindrical metrics, which are defined as follows.

We start by choosing a Riemannian metric $h \in \text{Riem}(Z)$. We identify the interval $[0, 1]$ with the half-line $\mathbb{R}_{\geq 0} = [0, \infty)$, and hence the product $Z \times [0, 1)$ also can be identified with $Z \times [0, \infty)$.

![Figure 1: A manifold with tame ends](image)

**Definition 2.2** A complete Riemannian metric $\bar{g} \in \text{Riem}(X)$ is called a **cylindrical metric modeled by** $(Z, h)$ if there exists a coordinate system $(x, t)$ on $Z \times [0, \infty)$ such that

$$\bar{g}(x, t) = h(x) + dt^2 \quad \text{on} \quad Z \times [1, \infty) \subset Z \times [0, \infty) \cong X \setminus W.$$ 

Let $\text{Riem}^{cyl}(X) (\subset \text{Riem}(X))$ be the space of cylindrical metrics modeled by $(Z, h)$ for some Riemannian metric $h \in \text{Riem}(Z)$.

We also define the map $\partial_\infty : \text{Riem}^{cyl}(X) \to \text{Riem}(Z)$ by $\partial_\infty \bar{g} = h$ if $\bar{g}$ is a cylindrical metric as above. Clearly the map $\partial_\infty$ is onto since any metric $h \in \text{Riem}(Z)$ can be extended to a cylindrical metric on $X$.

![Figure 2: Cylindrical manifold X](image)

**Remarks.** (1) For simplicity, we will often assume that the slice manifold $Z$ is connected. In the general case, i.e. when $Z = \bigsqcup_{j=1}^m Z_j$, one can easily obtain the corresponding results to those we prove here by using that

$$\lambda(\mathcal{L}_h) = \min_{1 \leq j \leq m} \lambda(\mathcal{L}_h|_{Z_j}) \quad \text{and} \quad Y^{cyl}_{[h+dt^2]}(Z \times \mathbb{R}) = \min_{1 \leq j \leq m} Y^{cyl}_{[h|_{Z_j}+dt^2]}(Z_j \times \mathbb{R}).$$
Later on we will also consider the case when a cylindrical manifold $X$ has a non-empty boundary. In that case the definition is modified by assuming that $\partial X = \partial W$ and $\partial W = \partial X \sqcup Z$ (see Fig. 3).

From the viewpoint of conformal geometry, cylindrical manifolds were considered implicitly in [5].

2.2. Cylindrical conformal classes. Next, we define an appropriate notion of a conformal class on a cylindrical manifold $X$. Let $\bar{C} = [\bar{g}]$ denote the regular conformal class for a metric $\bar{g} \in \mathcal{Riem}(X)$. Let $C(X)$ be the space of conformal classes on $X$. We define the space of cylindrical conformal classes

$$C_{cy}^{\ell}(X) := \{ [\bar{g}] \mid \bar{g} \in \mathcal{Riem}_{cy}^{\ell}(X) \} \subset C(X).$$

For $\bar{C} \in C_{cy}^{\ell}(X)$, we fix a cylindrical metric $\bar{g} \in \bar{C} \cap \mathcal{Riem}_{cy}^{\ell}(X)$ to define the $L_{\bar{g}}^{k,2}$-conformal class

$$\bar{C}_{L_{\bar{g}}^{k,2}} := \left\{ u^{\frac{4}{n-2}} \cdot \bar{g} \mid u \in C^{\infty}_{+}(X) \cap L_{\bar{g}}^{k,2}(X) \right\} \subset \bar{C},$$

where $k = 0, 1, 2$.

Now we show that the conformal classes $\bar{C}_{L_{\bar{g}}^{k,2}}$ do not depend on the choice of reference cylindrical metrics $\bar{g}$ for $k = 0, 1$.

Proposition 2.1 Let $\bar{C}$ be a cylindrical conformal class and $\bar{g}, \tilde{g} \in \bar{C} \cap \mathcal{Riem}_{cy}^{\ell}(X)$ two cylindrical metrics. Then there exists a constant $K \geq 1$ such that $K^{-1} \cdot \tilde{g} \leq \bar{g} \leq K \cdot \tilde{g}$ on $X$. In particular,

$$K^{-\frac{2}{n-2}} \cdot \sigma_{\bar{g}} \leq \sigma_{\tilde{g}} \leq K^{\frac{2}{n-2}} \cdot \sigma_{\bar{g}} \text{ on } X, \text{ and }$$

$$L_{\bar{g}}^{k,2}(X) = L_{\tilde{g}}^{k,2}(X), \quad \bar{C}_{L_{\bar{g}}^{k,2}} = \bar{C}_{L_{\tilde{g}}^{k,2}} \text{ for } k = 0, 1.$$

Proof. We have that $\bar{g}(x, t) = h(x) + dt^2$ on $Z \times [1, \infty)$ and $\tilde{g}(y, s) = \tilde{h}(y) + ds^2$ on $\tilde{Z} \times [1, \infty)$, where $y = y(x, t)$ and $s = s(x, t)$ give a diffeomorphism: $Z \times [1, \infty) \cong \tilde{Z} \times [1, \infty)$. By the assumption, $\tilde{g} = \varphi \cdot \bar{g}$ for some function $\varphi \in C^{\infty}_{+}(X)$. We write $\varphi = \varphi(x, t)$ on $Z \times [1, \infty)$. It is enough to show

$$\inf_{X} \varphi > 0, \quad \sup_{X} \varphi < \infty. \quad (2)$$

Indeed, suppose that $\inf_{X} \varphi = 0$. Then there exists a sequence $\{(x_i, t_i)\} \subset Z \times [1, \infty)$ such that $\varphi(x_i, t_i) \to 0$ as $i \to \infty$. In particular, it implies that the injectivity radius $\text{inj}_X \bar{g} = 0$. On the other hand, since $\tilde{g}$ is a cylindrical metric, the injectivity radius $\text{inj}_X \tilde{g} \geq \delta$ for some $\delta > 0$. This leads to a contradiction. A similar argument also shows that $\sup_{X} \varphi < \infty$. Hence the property (2) holds. \qed
Now we define the following functional on the space $L^{1,2}_g(X)$. Set
\[
Q(X, \bar{g})(u) := \frac{\int_X \left[ \alpha_n |du|^2 + R_g u^2 \right] d\sigma_{\bar{g}}}{\left( \int_X |u|^\frac{2n}{n-2} d\sigma_{\bar{g}} \right)^{\frac{n-2}{n}}}, \quad \text{where} \quad \alpha_n = \frac{4(n-1)}{n-2},
\]
for $u \in L^{1,2}_g(X)$ with $u \not\equiv 0$.

**Lemma 2.2** The functional $Q(X, \bar{g})(u)$ is well-defined on the space $L^{1,2}_g(X)$ with $u \not\equiv 0$.

**Proof.** Let $(N, g)$ be a complete Riemannian manifold. We define
\[
C^\infty_c(N) := \{ f \in C^\infty(N) \mid \text{Supp}(f) \text{ is compact} \}.
\]
The following facts are well known (cf. [10]).

**Fact 2.1**

(i) The $L^{k,2}_g$-completion of the space $C^\infty_c(N)$ coincides with the space $L^{k,2}_g(N)$ for $k = 0, 1$.

Now we assume that the sectional curvature $K_g$ and the injectivity radius $\iota_g$ are bounded, i.e. there exist constants $C > 0$ and $\delta > 0$ such that $|K_g| \leq C$ and $\iota_g \geq \delta$. Under this assumption, we have that

(ii) The $L^{2,2}_g$-completion of the space $C^\infty_c(N)$ coincides with the space $L^{2,2}_g(N)$.

(iii) The Sobolev embedding $L^{1,2}_g(N) \subset L^{2n/(n-2)}_g(N)$ holds.

Now we return to a cylindrical manifold $X$. The Sobolev embedding (iii) implies that
\[
\int_X |u|^\frac{2n}{n-2} d\sigma_{\bar{g}} < \infty \quad \text{for} \quad u \in L^{1,2}_g(X).
\]
We define $X(\ell) := W \cup_{Z} (Z \times [0, \ell])$ for $\ell \geq 1$, see Fig. 3.

For $u \in L^{1,2}_g(X)$, we have the following estimates:
\[
\left| \int_X R_g u^2 d\sigma_{\bar{g}} \right| \leq \int_{X(1)} R_g u^2 d\sigma_{\bar{g}} + \int_{X \setminus X(1)} |R_g| u^2 d\sigma_{\bar{g}}
\]
\[
\leq \int_{X(1)} R_g u^2 d\sigma_{\bar{g}} + (\max_{Z} R_h) \cdot \int_{X \setminus X(1)} u^2 d\sigma_{\bar{g}} < \infty.
\]
This completes the proof of Lemma 2.2. □
2.3. Cylindrical Yamabe constant. For $\bar{g} \in \text{Riem}^{cy}\ell(X)$, we define the functional
\[
I_X(\tilde{g}) := \int_X R_{\tilde{g}}d\tilde{\sigma}_{\tilde{g}} \quad \text{Vol}_{\tilde{g}}(X)\tilde{\sigma}_{\tilde{g}}^n
\]
on the space of $L^{2,2}_{\tilde{g}}$-conformal metrics, i.e. $\tilde{g} \in [\bar{g}]_{L^{2,2}_{\tilde{g}}}$. Recall that $\tilde{g} = u^{4-n} \cdot \bar{g}$, where $u \in C^\infty_+(X) \cap L^{2,2}_{\bar{g}}(X)$.

**Lemma 2.3** The functional $I_X(\tilde{g})$ is well-defined for metrics $\tilde{g} \in [\bar{g}]_{L^{2,2}_{\tilde{g}}}$. Furthermore, $I_X(\tilde{g}) = Q_{(X,\bar{g})}(u)$ if $\tilde{g} = u^{4-n} \cdot \bar{g}$.

**Proof.** For $\tilde{g} = u^{4-n} \cdot \bar{g}$ with $u \in C^\infty_+(X) \cap L^{2,2}_{\bar{g}}(X)$, we first notice
\[
\text{Vol}_{\tilde{g}}(X) = \int_X d\tilde{\sigma}_{\tilde{g}} = \int_X u^{\frac{2n}{n-2}}d\tilde{\sigma}_{\tilde{g}} < \infty.
\]
Then we have:
\[
\int_X R_{\tilde{g}}d\tilde{\sigma}_{\tilde{g}} = \int_X u^{-\frac{n+2}{n-2}} (-\alpha_n \Delta_{\tilde{g}}u + R_{\tilde{g}}u) \cdot u^{\frac{2n}{n-2}}d\tilde{\sigma}_{\tilde{g}}
\]
\[
= \int_X (-\alpha_n u \cdot \Delta_{\tilde{g}}u + R_{\tilde{g}}u^2)d\tilde{\sigma}_{\tilde{g}}.
\]
On the other hand, we have
\[
\int_X R_{\tilde{g}}d\tilde{\sigma}_{\tilde{g}} = \lim_{\ell \to \infty} \int_{X(\ell)} R_{\tilde{g}}d\tilde{\sigma}_{\tilde{g}}
\]
\[
= \lim_{\ell \to \infty} \left[ \int_{X(\ell)} (\alpha_n |du|^2 + R_{\tilde{g}}u^2) d\sigma_{\tilde{g}} - 2\alpha_n \int_{Z \times \{\ell\}} u \cdot \partial_t u \ d\sigma_n \right].
\]
Hence to prove that $I_X(\tilde{g}) = Q_{(X,\tilde{g})}(u)$, it is enough to prove the following.
Claim 2.4 \[ \int_{Z \times \{t\}} u \cdot \partial_t u \, d\sigma_h = o(1) \text{ as } \ell \to \infty. \]

**Proof.** Let \( f(t) \) be the function defined as

\[
 f(t) := \int_{Z \times \{t\}} u^2 \, d\sigma_h, \quad \text{then} \quad f'(t) = 2 \int_{Z \times \{t\}} u \cdot \partial_t u \, d\sigma_h.
\]

Since \( u \in L^2_{\bar{g}}(X) \), there exist \( \varepsilon > 0 \) and \( \ell_\varepsilon > 0 \) such that

\[
 \int_{Z \times [\ell_\varepsilon, \infty)} (u^2 + |\partial_t u|^2 + |\partial_t^2 u|^2) \, d\sigma_h \leq \varepsilon.
\]

Note that the volume form \( d\sigma_{\bar{g}} = d\sigma_h \, dt \) on the cylindrical part \( Z \times [1, \infty) \).

Now for any \( t_2 > t_1 \geq \ell_\varepsilon \) we have

\[
|f(t_2) - f(t_1)| = \left| \int_{t_1}^{t_2} f'(t) \right| = 2 \int_{Z \times [t_1, t_2]} u \cdot \partial_t u \, d\sigma_h \, dt 
\leq 2 \left( \int_{Z \times [t_1, t_2]} u^2 \, d\sigma_{\bar{g}} \right)^{\frac{1}{2}} \cdot \left( \int_{Z \times [t_1, t_2]} |\partial_t u|^2 \, d\sigma_{\bar{g}} \right)^{\frac{1}{2}} \leq 2 \varepsilon. \tag{3}
\]

We claim that \( f(\ell_\varepsilon) \leq 3\varepsilon \) and \( f(t) \leq 5\varepsilon \) for all \( t \geq \ell_\varepsilon \).

Indeed, suppose that \( f(\ell_\varepsilon) > 3\varepsilon \). Then \( f(t) > f(\ell_\varepsilon) - 2\varepsilon \geq \varepsilon \) for \( t \geq \ell_\varepsilon \). Therefore one has

\[
\int_{Z \times [\ell_\varepsilon, \infty)} u^2 \, d\sigma_{\bar{g}} = \int_{\ell_\varepsilon}^{\infty} f(t) \, dt \geq \int_{\ell_\varepsilon}^{\infty} \varepsilon \, dt = \infty,
\]

which contradicts the condition \( u \in L^2_{\bar{g}}(X) \). Hence \( f(\ell_\varepsilon) \leq 3\varepsilon \), and \( 0 < f(t) \leq f(\ell_\varepsilon) + 2\varepsilon \leq 5\varepsilon \) for \( t \geq \ell_\varepsilon \). Now we define

\[
g(t) := \int_{Z \times \{t\}} |\partial_t u|^2 \, d\sigma_h \geq 0 \quad \text{with} \quad g'(t) = 2 \int_{Z \times \{t\}} \partial_t u \cdot \partial_t^2 u \, d\sigma_h.
\]

Similarly, one can show that \( g(t) \leq 5\varepsilon \) for all \( t \geq \ell_\varepsilon \). Now we have

\[
\int_{Z \times \{t\}} |u| \cdot |\partial_t u| \, d\sigma_h \leq \left( \int_{Z \times \{t\}} u^2 \, d\sigma_h \right)^{\frac{1}{2}} \cdot \left( \int_{Z \times \{t\}} |\partial_t u|^2 \, d\sigma_h \right)^{\frac{1}{2}} \leq 5\varepsilon
\]

for all \( t \geq \ell_\varepsilon \). Clearly this implies that

\[
\int_{Z \times \{t\}} |u| \cdot |\partial_t u| \, d\sigma_h = o(1)
\]

as \( t \to \infty. \) \( \square \)
This completes the proof of Lemma 2.3. □

Definition 2.3 For $\bar{g} \in \text{Riem}^{cyl}(X)$, we define the constant

$$Y_{\bar{g}}^{cyl}(X) := \inf_{\tilde{g} \in [\bar{g}]_{L^2}} I_X(\tilde{g}).$$

Lemma 2.5 The following identities hold:

$$Y_{\bar{g}}^{cyl}(X) = \inf_{\tilde{g} \in [\bar{g}]_{L^2}} I_X(\tilde{g}) = \inf_{u \in C^\infty_c(X) \cap L^1_2(\bar{g})} Q_{(X,\tilde{g})}(u) = \inf_{u \geq 0, u \not\equiv 0} Q_{(X,\tilde{g})}(u) = Y_{\bar{g}}^{cyl}(X).$$

Proof. The assertion follows from the following inequalities:

$$Y_{\bar{g}}^{cyl}(X) \geq \inf_{u \in C^\infty_c(X) \cap L^1_2(\bar{g})} Q_{(X,\tilde{g})}(u) \geq \inf_{u \in L^1_2(\bar{g}), u \not\equiv 0} Q_{(X,\tilde{g})}(u) \geq \inf_{u \geq 0, u \not\equiv 0} Q_{(X,\tilde{g})}(u) = Y_{\bar{g}}^{cyl}(X).$$

Here we make use of the fact that the $L^1_2$-completion of the space $C^\infty_c(X)$ coincides with the space $L^k_2(\bar{g})$. We use essentially that $\tilde{g} = h + dt^2$ on $Z \times [1, \infty)$ to obtain the last inequality. □

Now we would like to recall the following important observation due to Schoen and Yau [39]. We state the corresponding result in the above terms.

Fact 2.2 [39, Section 2] For any $g', g'' \in \bar{C}$,

$$\inf_{u \in C^\infty_c(X), u \not\equiv 0} Q_{(X,g')}(u) = \inf_{u \in C^\infty_c(X), u \not\equiv 0} Q_{(X,g'')}(u).$$

In particular, these constants are conformally invariant.

We use Fact 2.2 or Proposition 2.1 to conclude the following.

Corollary 2.6 Let $\tilde{g}, \hat{g} \in \bar{C} \cap \text{Riem}^{cyl}(X)$ be any two cylindrical metrics. Then $Y_{\tilde{g}}^{cyl}(X) = Y_{\hat{g}}^{cyl}(X)$. 
Now we define the **cylindrical Yamabe constant** as follows.

**Definition 2.4** Let \( \bar{C} \in C^{cyl}(X) \) be a cylindrical conformal class. Then the **cylindrical Yamabe constant** \( Y^{cyl}_{\bar{C}}(X) \) of \((X, \bar{C})\) is defined by \( Y^{cyl}_{\bar{C}}(X) := Y^{cyl}_{\bar{g}}(X) \) for any cylindrical metric \( \bar{g} \in \bar{C} \cap \text{Riem}^{cyl}(X) \).

Corollary 2.6 implies that the Yamabe constant \( Y_{\bar{C}}(X) \) is well-defined, and Lemma 2.5 gives the formula:

\[
Y^{cyl}_{\bar{C}}(X) = \inf_{\bar{g} \in \bar{C}} \int_X (\alpha_n |du|^2 + R_h u^2) \, d\sigma_h = \inf_{u \in L^1(X), u \neq 0} Q_{(X, \bar{g})}(u).
\]

**2.4. Finiteness of cylindrical Yamabe constants.** Let \((X, \bar{g})\) be a cylindrical manifold modeled by \((Z, h)\) as above. We would like to give a complete criterion for the finiteness of \( Y^{cyl}_{\bar{C}}(X) \) in terms of the geometry of the slice Riemannian manifold \((Z, h)\). Recall that \( \dim X = n \), and hence \( \dim Z = n - 1 \). We define the operator

\[
\mathcal{L}_h := -\frac{4(n-2)}{n-3} \Delta_h + R_h = -\alpha_n \Delta_h + R_h \quad \text{on} \quad (Z, h).
\]

Recall that the operator \( \mathbb{L}_h = -\frac{4(n-2)}{n-3} \Delta_h + R_h = -\alpha_n \Delta_h + R_h \) (when \( n \geq 4 \)) is the conformal Laplacian of \((Z, h)\). Clearly \( \mathcal{L}_h \) resembles the conformal Laplacian \( \mathbb{L}_h \) of \((Z, h)\), but it is not equal to \( \mathbb{L}_h \). The first eigenvalue \( \lambda(\mathcal{L}_h) \) of \( \mathcal{L}_h \) (given by the formula below) captures a complete information on the finiteness of \( Y^{cyl}_{[\bar{g}]}(X) \). We have:

\[
\lambda(\mathcal{L}_h) := \inf_{u \in L^2(Z), u \neq 0} \frac{\int_Z (\alpha_n |du|^2 + R_h u^2) \, d\sigma_h}{\int_Z u^2 \, d\sigma_h}.
\]

**Lemma 2.7** Assume that \( \lambda(\mathcal{L}_h) < 0 \). Then \( Y^{cyl}_{\bar{C}}(X) = -\infty \) for any cylindrical conformal class \( \bar{C} = [\bar{g}] \in C^{cyl}(X) \) with \( \bar{g} \in \text{Riem}^{cyl}(X) \) and \( \partial_{\infty} \bar{g} = h \). In particular, \( Y^{cyl}_{\bar{C}}(X) = -\infty \) if the scalar curvature \( R_h < 0 \) on \( Z \).

**Proof.** Without loss of generality, we may assume that \( Z \) is connected. We define a Lipschitz function \( f_+(t) \in C^{0,1}_c([0, \infty)) \) whose graph is given at Fig. 4. The standard elliptic theory implies that there exists a function \( \varphi \in C^\infty_+(Z) \) such that

\[
\int_Z (\alpha_n |d\varphi|^2 + R_h \varphi^2) \, d\sigma_h = \lambda(\mathcal{L}_h) \quad \text{and} \quad \int_Z \varphi^2 \, d\sigma_h = 1.
\]
We define a family of functions depending on $\varepsilon$ by $u_{\varepsilon}(x,t) := f_{\varepsilon}(t) \cdot \varphi(x) \in C_{c}^{0,1}(X)$. We estimate $Q_{(X,\bar{g})}(u_{\varepsilon})$ from above:

$$Q_{(X,\bar{g})}(u_{\varepsilon}) = \frac{\lambda(L_h) \cdot \int_{0}^{\infty} f_{\varepsilon}^2 dt + \alpha_n \int_{0}^{\infty} (f'_{\varepsilon})^2 dt}{\left( \int_{Z} \varphi \frac{2n}{n-2} d\sigma_h \right)^{\frac{n-2}{n-2}} \cdot \left( \int_{0}^{\infty} \frac{f_{\varepsilon}^2}{\varepsilon} dt \right)^{\frac{n}{n-2}}},$$

$$\leq \frac{\lambda(L_h) \cdot \varepsilon^2 \cdot \frac{1}{\varepsilon} + 2\alpha_n \varepsilon^2}{C^{-1} \left( \varepsilon \frac{2n}{n-2} \cdot \frac{n}{n-2} \right)} \leq \frac{C' (\lambda(L_h) + 2\alpha_n \varepsilon)}{\varepsilon^{\frac{n+2}{n}} \cdot \varepsilon^{\frac{n}{n}}},$$

Here $C$ and $C'$ are some positive constants. Finally we have that $Q_{(X,\bar{g})}(u_{\varepsilon}) \to -\infty$ as $\varepsilon \to 0$ since $\lambda(L_h) < 0$. Hence $Y_{C}^{cy}(X) = -\infty$. $\square$

To proceed further, we recall the following result, which is essentially due to Aubin [9].

**Fact 2.3** (Aubin’s inequality) The cylindrical Yamabe constant $Y_{C}^{cy}(X)$ is bounded from above: $Y_{C}^{cy}(X) \leq Y(S^n)$, where $Y(S^n)$ is the Yamabe invariant of the $n$-sphere.

We conclude that for any cylindrical conformal class $\bar{C} \in C^{cy}(X)$

$$-\infty \leq Y_{C}^{cy}(X) \leq Y(S^n).$$
**Definition 2.5** Let $X$ be an open manifold of dim $X = n \geq 3$ with tame ends. Let $h$ be a metric on $Z$. We define the $h$-cylindrical Yamabe invariant of $X$ by

$$Y_{h-cyl}^c(X) := \sup_{\bar{g} \in \text{Riem}^c(X)} \sup_{\partial\infty \bar{g} = h} \bar{g}.$$

The cylindrical Yamabe invariant of $X$ is also defined by

$$Y_{cyl}^c(X) := \sup_{h \in \text{Riem}(Z)} Y_{h-cyl}^c(X).$$

Then the following inequalities hold:

$$-\infty \leq Y_{h-cyl}^c(X) \leq Y_{cyl}^c(X) \leq Y(S^n).$$

**Lemma 2.8** The $h$-cylindrical Yamabe invariant $Y_{h-cyl}^c(X)$ is a homothetic invariant:

$$Y_{h-cyl}^c(X) = Y_{kh-cyl}^c(X),$$

where $kh(x) := k \cdot h(x)$ for any constant $k > 0$.

**Proof.** Choose any $\bar{g} \in \text{Riem}^c(X)$ with $\partial\infty \bar{g} = kh$. Then $\bar{g} = kh + dt^2$ on $Z \times [1, \infty)$. Set $s = \frac{1}{\sqrt{k}}t$, and then $ds = \frac{1}{\sqrt{k}}dt$. We have

$$\bar{g} = kh + kds^2 = kh + ds^2 \quad \text{on} \quad Z \times [1/\sqrt{k}, \infty).$$

We obtain:

$$Y_{[\bar{g}]}^c(X) = Y_{[\frac{1}{\sqrt{k}}\bar{g}]}^c(X) \leq Y_{h-cyl}^c(X).$$

Hence $Y_{h-cyl}^c(X) \geq Y_{kh-cyl}^c(X)$. Similarly, $Y_{h-cyl}^c(X) \leq Y_{kh-cyl}^c(X)$. \qed

**Remark.** We emphasize that, in general, the invariant $Y_{h-cyl}^c(X)$ is not a conformal invariant with respect to a metric in the conformal class $[h]$.

**Example (1)** Set $Z = T^{n-1}$ for $n \geq 4$. Let $h_0$ be a flat metric on $T^{n-1}$. Then

$$L_{h_0} = -\alpha_n \Delta_{h_0}, \quad \lambda(L_{h_0}) = 0.$$ 

Hence $0 \geq Y_{h_0-cyl}^c(X) > -\infty$. On the other hand, let $h \in [h_0]$ be a non-flat metric on $T^{n-1}$. Clearly there exists a function $u \in C^\infty_+(T^{n-1})$ such that

$$-\alpha_n \Delta_h u + R_h u = 0.$$
Then we have:
\[
\mathcal{L}_h u = -\alpha_n \Delta_h u + R_h u = -\alpha_{n-1} \Delta_h u + R_h u + \frac{4}{(n-2)(n-3)} \Delta_h u.
\]
In particular,
\[
(\mathcal{L}_h u, u) = -\frac{4}{(n-2)(n-3)} \int_Z |du|^2 d\sigma_h < 0.
\]
This implies that \( \lambda(\mathcal{L}_h) < 0 \), and consequently, \( Y^{h-cyl}(X) = -\infty \). Hence we obtain that \( Y^{h-cyl}(X) \neq Y^{h_0-cyl}(X) \).

**Remark.** For any canonical cylindrical manifold, we notice that
\[
Y^{h-cyl}(Z \times \mathbb{R}) = Y^{cyl}_{[h_+ + dt]}(Z \times \mathbb{R}),
\]
see Proposition 2.11.

**Example (2)** Set \( Z = S^{n-1} \) and \( X = S^{n-1} \times \mathbb{R} \) for \( n \geq 3 \). Then
\[
Y^{h_+ - cyc\ell}(X) = Y^{cyl}_{[h_+ + dt]}(X) = Y(S^n).
\]
On the other hand, let \( h \in [h_+] \) be any metric whose sectional curvature is not identically equal to a positive constant. Then we will prove in Theorem 6.2 in Section 6 that
\[
Y^{h-cyl}(X) < Y(S^n).
\]
Hence \( Y^{h-cyl}(X) \neq Y^{h_+ - cyc\ell}(X) \).

**Lemma 2.9** Let \((X, \tilde{g})\) be a cylindrical manifold with \( \partial_{\infty} \tilde{g} = h \), and set \( \tilde{C} = [\tilde{g}] \).

1. If \( \lambda(\mathcal{L}_h) \geq 0 \), then \( Y^{cyl}(X) < -\infty \), and hence \( Y^{h-cyl}(X) < -\infty \). In particular, if \( R_h \geq 0 \) on \( Z \), then \( Y^{cyl}(X) < -\infty \) and \( Y^{h-cyl}(X) > -\infty \).
2. If \( \lambda(\mathcal{L}_h) = 0 \), then \( Y^{cyl}(X) \leq 0 \), and hence \( Y^{h-cyl}(X) \leq 0 \). In particular, if \( R_h \equiv 0 \) on \( Z \), then \( Y^{cyl}(X) \leq 0 \) and \( Y^{h-cyl}(X) \leq 0 \).

**Proof.** (1) Let \( u \in L^{1,2}_g(X) \) be a function with \( \int_X |u|^{2n} d\sigma_g = 1 \). Then we have:
\[
Q_{(X, \tilde{g})}(u) \geq \int_{X^{(1)}} (\alpha_n |du|^2 + R_g u^2) d\sigma_{\tilde{g}}
\]
\[
+ \lambda(\mathcal{L}_h) \cdot \int_{Z^{[1,\infty)}} u^2 d\sigma_h dt + \alpha_n \cdot \int_{Z^{[1,\infty)}} |\partial_h u|^2 d\sigma_h dt.
\]
Since \(\lambda(\mathcal{L}_h) \geq 0\) we have:

\[
Q_{(X,\bar{g})}(u) \geq \int_{X(1)} \left(\alpha u^2 + R_\mathcal{L} u^2\right) d\sigma_g \geq -\left(\max_{X(1)} |R_g|\right) \int_{X(1)} u^2 \ d\sigma_g
\]

\[
\geq -\left(\max_{X(1)} |R_g|\right) \left(\int_{X(1)} |u|^\frac{2n}{n-2} d\sigma_g\right)^{\frac{n-2}{n}} \cdot \Vol_{\bar{g}}(X(1))^\frac{1}{n}
\]

\[
\geq -\left(\max_{X(1)} |R_g|\right) \cdot \Vol_{\bar{g}}(X(1))^\frac{1}{n}.
\]

This gives that \(Y_{h,cyl}^\ell(X) > -\infty\).

(2) Let \(u_\varepsilon\) be the same function as in the proof of Lemma 2.7. Then

\[
Q_{(X,\bar{g})}(u_\varepsilon) \leq C'' \varepsilon \varepsilon_\frac{n-2}{n} \rightarrow 0
\]

as \(\varepsilon \rightarrow 0\), where \(C'' > 0\) is a constant. Thus \(Y_{Cyl}^{cyl}(X) \leq 0\). \(\square\)

We also notice the following fact. The proof is straightforward.

**Lemma 2.10** Assume that \(\lambda(\mathcal{L}_h) = 0\) and \(n \geq 4\). Then either \(R_h \equiv 0\) or there exists a metric \(\bar{h} \in [h]\) such that

\[
\begin{cases}
R_\bar{h} \equiv \text{const.} > 0, \\
\lambda(\mathcal{L}_{\bar{h}}) > 0.
\end{cases}
\]

Consider a canonical cylindrical manifold \((Z \times \mathbb{R}, h + dt^2)\). Then the cylindrical Yamabe constant \(Y_{[h+dt^2]}^{cyl}(Z \times \mathbb{R})\) provides a universal upper bound for all \(h\)-cylindrical Yamabe invariants. In more detail, we prove the following.

**Proposition 2.11** (Generalized Aubin’s inequality) Let \(X\) be an open manifold with tame ends \(Z \times [0, \infty)\) and \(h \in \text{Riem}(Z)\) any metric. Then \(Y_{[h+dt^2]}^{cyl}(Z \times \mathbb{R}) \leq Y_{[\bar{h}+dt^2]}^{cyl}(Z \times \mathbb{R})\).

**Proof.** If \(\lambda(\mathcal{L}_h) < 0\), the assertion is obvious from Lemma 2.7. Now assume that \(\lambda(\mathcal{L}_h) \geq 0\). Then \(Y_{[h+dt^2]}^{cyl}(Z \times \mathbb{R}) > -\infty\) by Lemma 2.9. From Lemma 2.6 there exists a sequence \(\{u_i\}\) of functions \(u_i \in C^\infty_c(Z \times \mathbb{R})\) with \(u_i \neq 0\) such that

\[
\lim_{i \to \infty} Q_{(Z \times \mathbb{R}, h+dt^2)}(u_i) = Y_{[h+dt^2]}^{cyl}(Z \times \mathbb{R}).
\]
We define the functions \( \tilde{u}_i \in C^\infty_c(X) \) as follows:

\[
\tilde{u}_i(p) = \begin{cases} 
0 & \text{if } p \in X(1), \\
u_i(x,t - i - 1) & \text{if } p = (x,t) \in Z \times [1, \infty).
\end{cases}
\]

Then we have

\[
Y^\text{cyl}_{[\tilde{g}]}(X) \leq \lim_{i \to \infty} Q(X, \tilde{g})(\tilde{u}_i) = \lim_{i \to \infty} Q(Z \times R, h + dt^2)(u_i) = Y^\text{cyl}_{[h + dt^2]}(Z \times R).
\]

This completes the proof of Proposition 2.11.

□

Now we would like to give an upper bound for the constant \( Y^\text{cyl}_{[h + dt^2]}(Z \times R) \) in terms of the geometry of \((Z, h)\).

\textbf{Proposition 2.12} Assume that \( Z \) is connected. The following inequality holds:

\[
Y^\text{cyl}_{[h + dt^2]}(Z \times R) \leq Y(S^n) \cdot \min \left\{ 1, \left( \frac{\lambda(\mathcal{L}_h)}{(n-1)(n-2)} \right)^{\frac{2(n-2)}{n-1}} \left( \frac{\text{Vol}_n(Z)}{\text{Vol}_n(S^{n-1})} \right)^{\frac{n-2}{n}} \right\}.
\]

\textbf{Proof}. Lemmas 2.7 and 2.9(2) imply that it is enough to consider the case \( \lambda(\mathcal{L}_h) > 0 \).

\textbf{Claim 2.13} Assume that \( \lambda(\mathcal{L}_h) = (n-1)(n-2) \). Then

\[
Y^\text{cyl}_{[h + dt^2]}(Z \times R) \leq Y(S^n) \cdot \left( \frac{\text{Vol}_n(Z)}{\text{Vol}_n(S^{n-1})} \right)^{\frac{n-2}{n}}.
\]

\textbf{Proof}. Set \((X, \tilde{h}) = (Z \times R, h + dt^2)\). By the assumption, there exists a function \( u \in C^\infty_c(Z) \) such that

\[
\{ \begin{align*} 
\mathcal{L}_h u &= (n-1)(n-2)u, \\
\int_Z u^2 \text{d} \sigma_h &= 1.
\end{align*} \]

We consider the function \( \varphi(x,t) = f(t) \cdot u(x) \in C^\infty_c(Z \times R) \) for any \( f \in C^\infty_c(R) \) with \( f(t) \neq 0 \). Then we have

\[
Y^\text{cyl}_{[\tilde{h}]}(X) \leq \inf_{f \in C^\infty_c(R), f \neq 0} Q(X, \tilde{h})(\varphi) \leq \inf_{f \in C^\infty_c(R), f \neq 0} \frac{\int_X (\alpha_n|f'|^2 u^2 + (n-1)(n-2) f^2 u^2) \text{d} \sigma_h \text{d}t}{\left( \int_X |f \cdot u|^{\frac{2n}{n-2}} \text{d} \sigma_h \text{d}t \right)^{\frac{n-2}{n}}}. \quad (4)
\]
We notice that

\[ 1 = \int_Z u^2 d\sigma_h \leq \left( \int_Z |u|^{\frac{2n}{n-2}} d\sigma_h \right)^{\frac{n-2}{2n}} \cdot \text{Vol}_h(Z)^{\frac{2}{n}}, \quad \text{and hence} \]

\[ \frac{1}{\left( \int_Z |u|^{\frac{2n}{n-2}} d\sigma_h \right)^{\frac{n-2}{2n}}} \leq \text{Vol}_h(Z)^{\frac{2}{n}}. \]

From this inequality combined with (4), we obtain

\[ Y_{[h]}^{cyf}(X) \leq \text{Vol}_h(Z)^{\frac{2}{n}} \cdot \inf_{f \in C_\infty^\infty(\mathbb{R})} \frac{\int_{\mathbb{R}} \left( \alpha_n |f'|^2 + (n-1)(n-2)f^2 \right) dt}{\left( \int_{\mathbb{R}} |f|^{\frac{2n}{n-2}} dt \right)^{\frac{n-2}{n}}} \]

\[ = Y(S^n) \cdot \left( \frac{\text{Vol}_h(Z)}{\text{Vol}_{h^{-1}}(S^{n-1})} \right)^{\frac{2}{n}}. \]

The last equality follows from Lemma 4.5 in Section 4. This completes the proof of Claim 2.13. \( \square \)

We continue with the proof of Proposition 2.12. We have that \( \lambda(\mathcal{L}_{\kappa^2,h}) = \kappa^{-2} \cdot \lambda(\mathcal{L}_h) \). In particular, if

\[ \kappa^2 = \frac{\lambda(\mathcal{L}_h)}{(n-1)(n-2)}, \quad \text{then} \quad \lambda(\mathcal{L}_{\kappa^2,h}) = (n-1)(n-2). \]

Also we have that \( \text{Vol}_{\kappa^2,h}(Z) = \kappa^{n-1} \text{Vol}_h(Z) \). Then Claim 2.13 combined with the proof of Lemma 2.14 gives

\[ Y_{[h]}^{cyf}(Z \times \mathbb{R}) \leq Y(S^n) \cdot \left( \frac{\text{Vol}_{\kappa^2,h}(Z)}{\text{Vol}_{h^{-1}}(S^{n-1})} \right)^{\frac{2}{n}} \]

\[ = Y(S^n) \cdot \left( \frac{\lambda(\mathcal{L}_h)}{(n-1)(n-2)} \right)^{\frac{n-1}{n}} \cdot \left( \frac{\text{Vol}_h(Z)}{\text{Vol}_{h^{-1}}(S^{n-1})} \right)^{\frac{2}{n}}. \]

\( \square \)

Now let \( M \) be a closed manifold of dim \( M = n \geq 3 \) and \( p_\infty \) a point in \( M \). Then the manifold \( M \setminus \{ p_\infty \} \) is an open manifold with the tame end \( Z \times [0, \infty) = S^{n-1} \times [0, \infty) \). Similarly, we start with an open manifold \( X \) with tame ends \( Z \times [0, \infty) \), and choose a finite number of points \( p_1, \ldots, p_k \in X \).
Then $X' = X \setminus \{p_1, \ldots, p_k\}$ is also an open manifold with tame ends $Z' \times [0, \infty)$, where $Z' = Z \sqcup (S^{n-1}_1 \sqcup \cdots \sqcup S^{n-1}_k)$ and $S^{n-1}_1 \sqcup \cdots \sqcup S^{n-1}_k$ denotes $k$ disjoint copies of $S^{n-1}$. We denote by $k \cdot h_+$ the metric $h_+ \sqcup \cdots \sqcup h_+$ on $S^{n-1}_1 \sqcup \cdots \sqcup S^{n-1}_k$. With these understood, we prove the following assertion.

**Lemma 2.14** (1) Let $M$ be a closed manifold, and $p_\infty \in M$. Then

$$Y^{cy}(M \setminus \{p_\infty\}) \geq Y^{h_+ - cy}(M \setminus \{p_\infty\}) = Y(M).$$

(2) Let $X$ be an open manifold with tame ends $Z \times [0, \infty)$, and $p_1, \ldots, p_k \in X$. Then

$$Y^{cy}(X \setminus \{p_1, \ldots, p_k\}) \geq \sup_{h \in Riem(Z)} Y^{(h_{\downarrow}(k \cdot h_+)) - cy}(X \setminus \{p_1, \ldots, p_k\}) = Y^{cy}(X).$$

**Proof.** (1) It is well-known from [26] that for any $\varepsilon > 0$ there exists a conformal class $C_\varepsilon \in C(M)$ such that

$$\begin{cases} Y_{C_\varepsilon}(M) \geq Y(M) - \varepsilon, \\ C_\varepsilon \text{ is locally conformally flat near } p_\infty. \end{cases}$$

This implies that there exists a cylindrical metric $\tilde{g}_\varepsilon \in Riem^{cy}(M \setminus \{p_\infty\})$ with $\partial_{\infty} \tilde{g}_\varepsilon = h_+$ such that

$$Y^{cy}_{[\tilde{g}_\varepsilon]}(M \setminus \{p_\infty\}) = Y_{C_\varepsilon}(M) \geq Y(M) - \varepsilon.$$  

This gives

$$Y^{cy}(M \setminus \{p_\infty\}) \geq Y^{h_+ - cy}(M \setminus \{p_\infty\}) \geq Y(M).$$  

On the other hand,

$$Y^{h_+ - cy}(M \setminus \{p_\infty\}) = \sup_{\tilde{g} \in Riem^{cy}(M \setminus \{p_\infty\})} Y^{cy}_{[\tilde{g}]}(M \setminus \{p_\infty\}) \leq Y(M).$$

Therefore, $Y^{h_+ - cy}(M \setminus \{p_\infty\}) = Y(M)$. The proof of the assertion (2) is similar. Hence we omit it. \hfill \Box

Lemma 2.14 shows that the $h$-cylindrical Yamabe invariant is a natural generalization of the Yamabe invariant for closed manifolds. Clearly the dependence of the invariant $Y^{h - cy}(X)$ on the slice metric $h$ is important. There are the following natural questions.
(1) Is it true that $Y^{cy}(M \setminus \{p_{\infty}\}) = Y(M)$?

(2) Is it true that $Y^{cy}(X \setminus \{p_1, \ldots, p_k\}) = Y^{cy}(X)$?

It is easy to see that if $M = S^n$ and $X = S^n \setminus \{\text{finite number of points}\}$, then (1) and (2) hold. We also note the following.

Claim 2.15 Let $M$ be a closed enlargeable manifold (see [22]) with $Y(M) = 0$, and $X := M \setminus \{\text{finite number of points}\}$. Then

$Y^{cy}(M \setminus \{p_{\infty}\}) = Y(M)$ and $Y^{cy}(X \setminus \{p_1, \ldots, p_k\}) = Y^{cy}(X)$.

3 Kobayashi-type inequalities

3.1 Gluing construction. Let $W_1$, $W_2$ be two compact connected manifolds with $\partial W_1 = \partial W_2 = Z \neq \emptyset$. Then let $W = W_1 \cup_Z W_2$ be the union of manifolds $W_1$ and $W_2$ along their common boundary $Z$, see Fig. 5. We

define the corresponding open manifolds with tame ends (see Fig. 6.2) by

$$X_1 := W_1 \cup_Z Z \times [0, \infty),$$

$$X_2 := W_2 \cup_Z Z \times [0, \infty).$$

We denote by $X_1 \sqcup X_2$ the disjoint union of $X_1$ and $X_2$. The following result is analogous to [26, Lemma 1.10]. The proof is similar to [26].

Lemma 3.1 For $\tilde{C}_i \in C^{cy}(X_i)$, $i = 1, 2$, let $\tilde{C}_1 \sqcup \tilde{C}_2$ denote the disjoint union of the conformal classes $\tilde{C}_1$ and $\tilde{C}_2$ on $X_1 \sqcup X_2$. Then

$Y^{cy}_{\tilde{C}_1 \sqcup \tilde{C}_2}(X_1 \sqcup X_2) = \begin{cases} -\left(Y^{cy}_{\tilde{C}_1}(X_1)\right)^\sharp + |Y^{cy}_{\tilde{C}_2}(X_2)|^\sharp & \text{if } Y^{cy}_{\tilde{C}_i}(X_i) \leq 0, \ i = 1, 2, \\
\min \left\{Y^{cy}_{\tilde{C}_1}(X_1), Y^{cy}_{\tilde{C}_2}(X_2)\right\} & \text{otherwise.} \end{cases}$
Lemma 3.1 implies the following assertions.

**Proposition 3.2** Let $X_1, X_2$ be the open manifolds with tame ends as above and $h \in \mathcal{Riem}(Z)$ any metric. Then

$$ Y^{h-cyl}(X_1 \sqcup X_2) = \begin{cases} -\left( |Y^{h-cyl}(X_1)|^\frac{n}{2} + |Y^{h-cyl}(X_2)|^\frac{n}{2} \right)^\frac{2}{n} & \text{if } Y^{h-cyl}(X_i) \leq 0, \quad i = 1, 2, \\ \min \{ Y^{h-cyl}(X_1), Y^{h-cyl}(X_2) \} & \text{otherwise.} \end{cases} $$

$$ Y^{cyl}(X_1 \sqcup X_2) = \begin{cases} -\left( |Y^{cyl}(X_1)|^\frac{n}{2} + |Y^{cyl}(X_2)|^\frac{n}{2} \right)^\frac{2}{n} & \text{if } Y^{cyl}(X_i) \leq 0, \quad i = 1, 2, \\
\min \{ Y^{cyl}(X_1), Y^{cyl}(X_2) \} & \text{otherwise.} \end{cases} $$

We denote $\mathcal{Riem}^+(Z) = \{ h \in \mathcal{Riem}(Z) \mid \lambda(L_h) > 0 \}$. We notice that the space $\mathcal{Riem}^+(Z)$ of positive scalar curvature metrics on $Z$ is contained in $\mathcal{Riem}^*(Z)$, however $\mathcal{Riem}^+(Z) \subsetneq \mathcal{Riem}^*(Z)$.

**Proposition 3.3** Let $W_1, W_2$ be compact manifolds of $\dim W_i = n \geq 3$ with $\partial W_1 = Z = \partial W_2$ and $X_1, X_2$ the corresponding open manifolds with tame ends as above. Assume that $\mathcal{Riem}^*(Z) \neq \emptyset$. Let $h \in \mathcal{Riem}^*(Z)$ be any metric. Then

$$ Y(W_1 \cup_Z W_2) = \begin{cases} -\left( |Y^{h-cyl}(X_1)|^\frac{n}{2} + |Y^{h-cyl}(X_2)|^\frac{n}{2} \right)^\frac{2}{n} & \text{if } Y^{h-cyl}(X_i) \leq 0, \quad i = 1, 2, \\
\min \{ Y^{h-cyl}(X_1), Y^{h-cyl}(X_2) \} & \text{otherwise.} \end{cases} $$

**Remark.** We notice that if $\lambda(L_h) < 0$, then $Y^{h-cyl}(X_i) = -\infty$, $i = 1, 2$, and hence the inequality (5) holds. We study the case when $\lambda(L_h) = 0$ later.

**Proof.** Let $h \in \mathcal{Riem}^*(Z)$ be any metric. First we recall that Lemma 2.10 gives that $Y^{h-cyl}(X_i) > -\infty$ for $i = 1, 2$. 

Figure 6: Cylindrical manifolds $X_1$ and $X_2$. 

We denote $\mathcal{Riem}^*(Z) = \{ h \in \mathcal{Riem}(Z) \mid \lambda(L_h) > 0 \}$. We notice that the space $\mathcal{Riem}^*(Z)$ of positive scalar curvature metrics on $Z$ is contained in $\mathcal{Riem}^*(Z)$, however $\mathcal{Riem}^+(Z) \subsetneq \mathcal{Riem}^*(Z)$.
For any \( \varepsilon > 0 \) there exists a metric \( \bar{g}_i \in \text{Riem}^{cg\ell}(X_i) \) for \( i = 1, 2 \) such that
\[
\begin{align*}
\partial_\infty \bar{g}_1 &= h = \partial_\infty \bar{g}_2, \\
C_i^{cg\ell}(X_1 \cup X_2) &\geq Y^{h-cg\ell}(X_1 \cup X_2) - \varepsilon.
\end{align*}
\]
Here \( \bar{C}_i = [\bar{g}_i] \in \text{C}^{cg\ell}(X_i), \ i = 1, 2. \)

We notice that the condition \( \lambda(L_{h}) > 0 \) implies that \( \lambda(L_{h}^\delta) > 0 \) for sufficiently small \( \delta > 0 \), where \( L_{h}^\delta \) is the operator defined by
\[
L_{h}^\delta := - (\alpha_n - \delta) \Delta_h + R_h.
\]

We construct a family of metrics \( \bar{g}(\ell) \) as below on the manifold \( W = W_1 \cup Z W_2 \) by identifying it with the manifold
\[
W_1 \cup Z W_2 = X_1(1) \cup Z (Z \times [0, \ell]) \cup Z X_2(1);
\]
\[
\bar{g}(\ell) := \bar{g}_1|_{X_1(1)} \cup (h + dt^2) \cup \bar{g}_1|_{X_1(1)}; \quad \text{(see Fig. 7)}.
\]

By the definition, we have

Figure 7: Decomposition of \( W = W_1 \cup Z W_2 \).

\[
Y_{[\bar{g}(\ell)]}(W_1 \cup Z W_2) = \inf_{u > 0} Q_{(W_1 \cup Z W_2, \bar{g}(\ell))}(u),
\]
and hence that for any \( \ell > 0 \) there exists a function \( u_\ell \in C^\infty_+(W_1 \cup Z W_2) \) such that
\[
\alpha_n \int_{W_1 \cup Z W_2} |du_\ell|^2 d\sigma_{\bar{g}(\ell)} + \int_{W_1 \cup Z W_2} R_{\bar{g}(\ell)} u_\ell^2 d\sigma_{\bar{g}(\ell)} \leq Y_{[\bar{g}(\ell)]}(W_1 \cup Z W_2) + \frac{1}{1 + \ell},
\]
and
\[
\int_{W_1 \cup Z W_2} |u_\ell|^2 d\sigma_{\bar{g}(\ell)} = 1.
\]

For simplicity, set
\[
E_{(W_1 \cup Z W_2, \bar{g}(\ell))}(u_\ell) := \alpha_n \int_{W_1 \cup Z W_2} |du_\ell|^2 d\sigma_{\bar{g}(\ell)} + \int_{W_1 \cup Z W_2} R_{\bar{g}(\ell)} u_\ell^2 d\sigma_{\bar{g}(\ell)},
\]
\[
Y_{[\bar{g}(\ell)]} := Y_{[\bar{g}(\ell)]}(W_1 \cup Z W_2).
\]
Claim 3.4 There exists $t_\ell, 0 \leq t_\ell \leq \ell$ such that
\[
\int_{Z \times \{t_\ell\}} (|du_\ell|^2 + u_\ell^2) d\sigma_h \leq \frac{B}{\ell}
\]
for some positive constant $B$ independent of $\ell$.

Proof. Using the inequality
\[
E(\mathcal{W}_1 \cup \mathcal{W}_2, \bar{g}(\ell))(u_\ell) \leq \frac{Y[\bar{g}(\ell)]}{1 + \ell}
\]
we obtain the following
\[
Y[\bar{g}(\ell)] + \frac{1}{1 + \ell} \geq E(X(1), \bar{g}_1)(u_\ell) + E(X(2), \bar{g}_2)(u_\ell) + \lambda(L^a_h) \int_{Z \times [0, \ell]} u_\ell^2 d\sigma_h dt
\]
\[
+ \delta \int_{Z \times [0, \ell]} |du_\ell|^2 d\sigma_h dt + (\alpha_n - \delta) \int_{Z \times [0, \ell]} |\partial_t u_\ell|^2 d\sigma_h dt.
\]
We use the fact that the last term on the right-hand side is positive, and then
\[
Y[\bar{g}(\ell)] + \frac{1}{1 + \ell} \geq \left( \min_{X_1(1)} R^-_{\bar{g}_1} \right) \cdot \text{Vol}_{\bar{g}_1}(X_1(1))^\frac{2}{n} + \left( \min_{X_2(1)} R^-_{\bar{g}_2} \right) \cdot \text{Vol}_{\bar{g}_2}(X_2(1))^\frac{2}{n}
\]
\[
+ \delta_0 \cdot \int_{Z \times [0, \ell]} (|du_\ell|^2 + u_\ell^2) d\sigma_h dt.
\]
(6)

Here $\delta_0 := \min \{\lambda(L^a_h), \delta\} > 0$ and $R^- := \min \{0, R\}$. Hence (6) implies that
\[
\delta_0 \cdot \int_{Z \times [0, \ell]} (|du_\ell|^2 + u_\ell^2) d\sigma_h dt \leq Y[\bar{g}(\ell)] + \frac{1}{1 + \ell} + A \quad \text{with}
\]
\[
A := - \left[ \left( \min_{X_1(1)} R^-_{\bar{g}_1} \right) \cdot \text{Vol}_{\bar{g}_1}(X_1(1))^\frac{2}{n} + \left( \min_{X_2(1)} R^-_{\bar{g}_2} \right) \cdot \text{Vol}_{\bar{g}_2}(X_2(1))^\frac{2}{n} \right] \geq 0,
\]
and thus
\[
\int_{Z \times [0, \ell]} (|du_\ell|^2 + u_\ell^2) d\sigma_h dt \leq \frac{1}{\delta_0} \left( Y[\bar{g}(\ell)] + \frac{1}{1 + \ell} + A \right).
\]
It then follows that there exists \( t_\ell, 0 \leq t_\ell \leq \ell \) such that
\[
\int_{Z \times \{t_\ell\}} (|du_\ell|^2 + u_\ell^2) d\sigma_h \leq \frac{B}{\ell}
\]
with \( B > 0 \). \( \square \)

Now we continue with the proof of Proposition 3.3. We cut the manifold \( W_1 \cup_Z W_2 \) along the slice \( Z \times \{t_\ell\} \), and then attach two copies of the half-cylinder \( Z \times [0, \infty) \) to the corresponding pieces to obtain the cylindrical manifolds \( X_1 \) and \( X_2 \). In other words, we regard \( X_1 \sqcup X_2 \) as
\[
X_1 \sqcup X_2 = (Z \times [0, \infty)) \cup (W_1 \cup_Z W_2 \setminus (Z \times \{t_\ell\})) \cup Z (Z \times [0, \infty)). \quad (7)
\]

We define a non-negative Lipschitz function
\[
U_\ell \in C_c^0(X_1 \sqcup X_2)
\]
by
\[
U_\ell = u_\ell \quad \text{on} \quad (W_1 \cup_Z W_2) \setminus (Z \times \{t_\ell\}), \quad \text{and}
\]
\[
U_\ell(x,t) = \begin{cases} 
(1-t)u_\ell(x,t_\ell) & \text{on } Z \times [0,1], \\
0 & \text{on } Z \times [1,\infty).
\end{cases} \quad (8)
\]

The conditions (8) imply that
\[
E_{(X_1 \sqcup X_2, \bar{g}_1 \sqcup \bar{g}_2)}(U_\ell) \leq Y_{\bar{g}(\ell)} + \frac{C}{\ell}, \quad \int_{X_1 \sqcup X_2} U_\ell^{2n} d\sigma_{\bar{g}_1 \sqcup \bar{g}_2} > 1.
\]

This gives that
\[
Y(W_1 \cup_Z W_2) + \frac{C}{\ell} \geq Y_{\bar{g}(\ell)} + \frac{C}{\ell} \geq Q_{(X_1 \sqcup X_2, \tilde{g}_1 \sqcup \tilde{g}_2)}(U_\ell) \geq \inf_{u \in C_{C_c}^\infty(X_1 \sqcup X_2)} Q_{(X_1 \sqcup X_2, \tilde{g}_1 \sqcup \tilde{g}_2)}(u) \geq Y^{\text{cy}}_{\tilde{C}_1 \sqcup \tilde{C}_2}(X_1 \sqcup X_2). \quad (9)
\]

Now we take \( \ell \to \infty \) in (9) to obtain
\[
Y(W_1 \cup_Z W_2) \geq Y^{\text{cy}}_{\tilde{C}_1 \sqcup \tilde{C}_2}(X_1 \sqcup X_2) \geq Y^{r\text{-cy}}(X_1 \sqcup X_2) - \varepsilon.
\]
Taking $\varepsilon \to 0$, we conclude that
\[
Y(W_1 \cup_Z W_2) \geq Y^{h-cyl}(X_1 \sqcup X_2) \geq \begin{cases} 
-\left(\frac{Y^{h-cyl}(X_1)}{2} + \frac{Y^{h-cyl}(X_2)}{2}\right)^2 & \text{if } Y^{h-cyl}(X_i) \leq 0, \ i = 1, 2, \\
\min\{Y^{h-cyl}(X_1), Y^{h-cyl}(X_2)\} & \text{otherwise}.
\end{cases}
\]
This completes the proof of Proposition 3.3. □

Now we establish the Kobayashi-type inequality in the case when $\lambda(L_h) = 0$.

**Proposition 3.5** Let $h \in \mathcal{Riem}(Z)$ be a metric with $\lambda(L_h) = 0$. Then
\[
Y(W_1 \cup_Z W_2) \geq -\left(\frac{Y^{h-cyl}(X_1)}{2} + \frac{Y^{h-cyl}(X_2)}{2}\right)^2.
\]

**Proof.** Let $h \in \mathcal{Riem}(Z)$ be any metric with $\lambda(L_h) = 0$. Recall that this condition implies that $0 \geq Y^{h-cyl}(X_i) > -\infty$ for our cylindrical manifold $X_i, \ i = 1, 2$. By the definitions, for each $\varepsilon > 0$ there exists a metric $\bar{g}_i \in \mathcal{Riem}^{cyl}(X_i), \ i = 1, 2$, such that
\[
\{ \partial \bar{g}_1 = h = \partial \bar{g}_2, \ Y^{cyl}_{\bar{C}_1 \sqcup \bar{C}_2}(X_1 \sqcup X_2) \geq Y^{h-cyl}(X_1 \sqcup X_2) - \varepsilon. \}
\]
Here $\bar{C}_i := [\bar{g}_i] \in \mathcal{C}^{cyl}(X_i), \ i = 1, 2$. As before, we use the decomposition
\[
W_1 \cup_Z W_2 \cong X_1(1) \cup_Z (Z \times [0, \ell^2]) \cup_Z X_2(1)
\]
with the metric
\[
\bar{g}(\ell^2) = \bar{g}_1|_{X_1(1)} \cup (h + dt^2) \cup \bar{g}_2|_{X_2(1)}
\]
respectively on $X_1(1), Z \times [0, \ell^2]$ and $X_2(1)$. We set the segments
\[
\begin{cases}
I_j := [(j-1)\ell, j\ell], & j = 1, \ldots, \ell, \\
I_{j,k} := [(j-1)\ell + (k-1), (j-1)\ell + k], & j, k = 1, \ldots, \ell.
\end{cases}
\]
We have
\[
Y_{\bar{g}(\ell^2)} := Y_{\bar{g}(\ell^2)}(W_1 \cup_Z W_2) = \inf_{u > 0} Q_{(W_1 \cup_Z W_2, \bar{g}(\ell^2))}(u).
\]
Then for any $\ell \gg 1$ there exists $u_\ell \in C^\infty_+ (W_1 \cup_Z W_2)$ such that
\[
\begin{cases}
E(W_1 \cup_Z W_2, \bar{g}(\ell^2))(u_\ell) \leq Y[\bar{g}(\ell^2)] + \frac{1}{1 + \ell}, \\
\int_{W_1 \cup_Z W_2} \frac{2n}{\ell^2} u_\ell^2 d\sigma_{\bar{g}(\ell^2)} = 1.
\end{cases}
\]

Claim 3.6 There exists $t_\ell$, $0 \leq t_\ell \leq \ell^2$ such that
\[
\int_{Z \times \{t_\ell\}} (|du_\ell|^2 + u_\ell^2) d\sigma_h \leq \frac{B}{\ell^{1/4}},
\]
where $B$ is independent of $\ell$.

Proof. From the above, we have
\[
Y(W_1 \cup_Z W_2) + \frac{1}{1 + \ell} \geq Y[\bar{g}(\ell^2)] + \frac{1}{1 + \ell} \geq E(W_1 \cup_Z W_2, \bar{g}(\ell^2))(u_\ell).
\]
We use the same constant $A$ as in the proof of Claim 3.4 to give the following estimate
\[
E(W_1 \cup_Z W_2, \bar{g}(\ell^2))(u_\ell) \geq -A + \int_{Z \times [0, \ell^2]} (\alpha_n |du_\ell|^2 + R_h u_\ell^2) d\sigma_h dt
\]
\[
+ \alpha_n \int_{Z \times [0, \ell^2]} |\partial_t u_\ell|^2 d\sigma_h dt 
\]
\[
\geq -A + \alpha_n \int_{Z \times [0, \ell^2]} |\partial_t u_\ell|^2 d\sigma_h dt
\]
since $\lambda(L_h) = 0$. Clearly there exists an integer $j$ $(1 \leq j \leq \ell)$ such that
\[
\int_{Z \times I_j} u_\ell^{2n/2} d\sigma_h dt \leq \frac{1}{\ell}.
\]
Then there exists also an integer $k$ $(1 \leq k \leq \ell)$ such that
\[
\int_{Z \times I_{j,k}} (\alpha_n |du_\ell|^2 + R_h u_\ell^2) d\sigma_h dt \leq \frac{A'}{\ell},
\]
where
\[
A' := A + Y(W_1 \cup_Z W_2) + \frac{1}{1 + \ell}.
\]
We have
\[ \int_{Z \times I_{j,k}} u_t^2 d\sigma_h \, dt \leq \text{Vol}_h(Z)^{\frac{2}{n}} \cdot \left( \int_{Z \times I_{j,k}} u_{t t}^{\frac{2n}{n-2}} d\sigma_h \, dt \right)^{\frac{n-2}{n}} \leq \frac{C_1}{\ell^{\frac{n-2}{n}}} \quad (10) \]
for some positive constant $C_1$. We define the function $f(t)$ by
\[ f(t) := \int_{Z \times \{t\}} u_t^2 d\sigma_h, \quad \text{where} \quad f'(t) = 2 \int_{Z \times \{t\}} u_t \cdot \partial_t u_t d\sigma_h. \]
It follows from (10) that there exists $t_0 \in I_{j,k}$ such that
\[ f(t_0) \leq \frac{C_1}{\ell^{\frac{n-2}{n}}} \quad (11) \]
We have the following estimate
\[ |f(t_2) - f(t_1)| \leq 2 \left| \int_{Z \times [t_1, t_2]} u_t \cdot \partial_t u_t d\sigma_h \, dt \right| \]
\[ \leq 2 \left( \int_{Z \times [t_1, t_2]} u_t^2 d\sigma_h \, dt \right)^{\frac{1}{2}} \cdot \left( \int_{Z \times [t_1, t_2]} |\partial_t u_t|^2 d\sigma_h \, dt \right)^{\frac{1}{2}} \]
\[ \leq 2 \cdot \frac{C_1^{\frac{2}{n}}}{\ell^{\frac{n-2}{n}}} \cdot \frac{A''}{\ell^{\frac{n-2}{n}}} = \frac{2A'' C_1^{\frac{2}{n}}}{\ell^{\frac{n-2}{n}}} \]
for any $t_1, t_2 \in I_{j,k}$. Here $C_1 > 0$ is the same constant as above. This combined with (11) gives that
\[ f(t) \leq \frac{2A'' C_1^{\frac{2}{n}}}{\ell^{\frac{n-2}{n}}} + \frac{C_1}{\ell^{\frac{n-2}{n}}} \leq \frac{C_2}{\ell^{\frac{n-2}{n}}} \quad (12) \]
for any $t \in I_{j,k}$. Now we have the estimate
\[ \alpha_n \int_{Z \times I_{j,k}} |du_t|^2 d\sigma_h \, dt \leq \frac{A'}{\ell} + \left( \max_Z |R_h| \right) \cdot \int_{Z \times I_{j,k}} u_t^2 \, d\sigma_h \, dt \]
\[ \leq \frac{A'}{\ell} + \frac{C_3}{\ell^{\frac{n-2}{n}}} . \]
Here we used the estimate (10). Then this gives
\[ \int_{Z \times I_{j,k}} |du_t|^2 d\sigma_h \, dt \leq \frac{C_4}{\ell^{\frac{n-2}{n}}} \]
for some positive constant $C_4$. We conclude that there exists $t_\ell \in I_{j,k}$ such that
\[ \int_{Z \times \{t_\ell\}} |du_\ell|^2 d\sigma_h \leq \frac{C_4}{\ell^{\frac{n}{n-2}}} \]  \tag{13}
Now we use the estimates (12) and (13) to obtain the estimate
\[ \int_{Z \times \{t_\ell\}} (|du_\ell|^2 + u_\ell^2) d\sigma_h \leq \frac{C_2 + C_4}{\ell^{\frac{n}{n-2}}} \]
This completes the proof of Claim 3.6. \hfill \square

We continue now with the proof of Proposition 3.5. As above, we regard $X_1 \sqcup X_2$ as in (7). We define a non-negative Lipschitz function $U_\ell \in C^0_c(X_1 \sqcup X_2)$ by

\[ U_\ell = u_\ell \text{ on } (W_1 \cup Z \cup W_2) \setminus (Z \times \{t_\ell\}), \]  
and

\[ U_\ell(x,t) = \begin{cases} (1-t)u_\ell(x,t_\ell) & \text{on } Z \times [0,1], \\ 0 & \text{on } Z \times [1,\infty). \end{cases} \]

Then we obtain the following estimate
\[ E_{(X_1 \sqcup X_2,\bar{g}_1 \sqcup \bar{g}_2)}(U_\ell) \leq Y(W_1 \cup Z \cup W_2) + \frac{C}{\ell^{\frac{n}{n-2}}}, \]  
with
\[ \int_{X_1 \sqcup X_2} U_\ell^{-\frac{n}{n-2}} d\sigma_{\bar{g}_1 \sqcup \bar{g}_2} > 1. \]

This implies
\[ Y(W_1 \cup Z \cup W_2) + \frac{C}{\ell^{\frac{n}{n-2}}} \geq Q_{(X_1 \sqcup X_2,\bar{g}_1 \sqcup \bar{g}_2)}(U_\ell) \]
\[ \geq \inf_{u \in C^\infty_c(X_1 \sqcup X_2)} Q_{(X_1 \sqcup X_2,\bar{g}_1 \sqcup \bar{g}_2)}(u) \]
\[ = Y_{\bar{C}_1 \sqcup \bar{C}_2}^{cyl}(X_1 \sqcup X_2). \]

We take $\ell \to \infty$ to obtain
\[ Y(W_1 \cup Z \cup W_2) \geq Y_{\bar{C}_1 \sqcup \bar{C}_2}^{cyl}(X_1 \sqcup X_2) \geq Y_{\bar{C}_1 \sqcup \bar{C}_2}^{h-cyl}(X_1 \sqcup X_2) - \varepsilon. \]

Finally, we take $\varepsilon \to 0$ and conclude that
\[ Y(W_1 \cup Z \cup W_2) \geq Y_{\bar{C}_1 \sqcup \bar{C}_2}^{h-cyl}(X_1 \sqcup X_2). \]

This combined with Proposition 3.2 completes the proof of Proposition 3.5. \hfill \square
Now we combine Proposition 3.5 with Proposition 3.3 and Lemma 2.7.

**Theorem 3.7** Let $W_1, W_2$ be compact manifolds of $\dim W_i \geq 3$ with $\partial W_1 = Z = \partial W_2$ and $X_1, X_2$ the corresponding open manifolds with tame ends $Z \times [0, \infty)$ as above. Let $h \in \text{Riem}(Z)$ be any metric. Then

$$Y(W_1 \cup_Z W_2) \geq \begin{cases} -\left( |Y^{h-cyl}(X_1)|^{\frac{2}{n}} + |Y^{h-cyl}(X_2)|^{\frac{2}{n}} \right)^{\frac{n}{2}} & \text{if } Y^{h-cyl}(X_i) \leq 0, \ i = 1, 2, \\ \min \{ Y^{h-cyl}(X_1), Y^{h-cyl}(X_2) \} & \text{otherwise.} \end{cases}$$

**Remark.** We proved a similar formula in [4] in terms of the relative Yamabe invariant.

We notice that Theorem 3.7 and Lemma 2.14 recover the original Kobayashi inequality (see [26, Theorem 2(a)]).

**Corollary 3.8** Let $M_1, M_2$ be closed manifolds of $\dim M_i = n \geq 3$, $i = 1, 2$. Then

$$Y(M_1 \# M_2) \geq \begin{cases} -\left( |Y(M_1)|^{\frac{2}{n}} + |Y(M_2)|^{\frac{2}{n}} \right)^{\frac{n}{2}} & \text{if } Y(M_i) \leq 0, \ i = 1, 2, \\ \min \{ Y(M_1), Y(M_2) \} & \text{otherwise.} \end{cases}$$

**Proof.** Theorem 3.7 combined with Lemma 2.14 gives directly the following

$$Y(M_1 \# M_2) = Y((M_1 \setminus D^n) \cup_{S^{n-1}} (M_2 \setminus D^n))$$

$$\geq \begin{cases} -\left( |Y^{h-cyl}(M_1 \setminus \{pt\})|^{\frac{2}{n}} + |Y^{h-cyl}(M_2 \setminus \{pt\})|^{\frac{2}{n}} \right)^{\frac{n}{2}} & \text{if } Y^{h-cyl}(M_i \setminus \{pt\}) \leq 0, \ i = 1, 2, \\ \min \{ Y^{h-cyl}(M_1 \setminus \{pt\}), Y^{h-cyl}(M_2 \setminus \{pt\}) \} & \text{otherwise.} \end{cases}$$

$$= \begin{cases} -\left( |Y(M_1)|^{\frac{2}{n}} + |Y(M_2)|^{\frac{2}{n}} \right)^{\frac{n}{2}} & \text{if } Y(M_i) \leq 0, \ i = 1, 2, \\ \min \{ Y(M_1), Y(M_2) \} & \text{otherwise.} \end{cases} \quad \square$$

## 4 Surgery and cylindrical Yamabe invariant

**4.1. Relative Yamabe constant for a cylindrical manifold with boundary.** Now we concentrate our attention on the case when a cylindrical manifold has a nonempty boundary. Let $X$ be a noncompact manifold
with $\partial X = M \neq \emptyset$. As before, we assume that $X$ is a noncompact manifold with tame ends. More precisely, there exists a relatively compact open submanifold $W \subset X$ with $\partial W = \partial X = M$ and $\partial \overline{W} = M \sqcup Z$ such that

$$
\begin{cases}
Z = \bigcup_{j=1}^{m} Z_j, \text{ where each } Z_j \text{ is connected}, \\
X \setminus W \cong Z \times [0, 1) = \bigcup_{j=1}^{m} (Z_j \times [0, 1)).
\end{cases}
$$

We should refine the definition of suitable cylindrical metrics to the case of a non-empty boundary.

Let $\bar{g} \in \text{Riem}(X)$. We denote by $H_{\bar{g}}$ the mean curvature of $\bar{g}$ on $M$.

Then we define

$$
\text{Riem}^{cg\ell,0}(X) := \{ \bar{g} \in \text{Riem}^{cg\ell}(X) \mid H_{\bar{g}} \equiv 0 \text{ on } M \}.
$$

Let $\bar{C} \in C^{cg\ell}(X)$ be a cylindrical conformal class, and $\bar{g} \in \bar{C} \cap \text{Riem}^{cg\ell,0}(X)$.

We define the normalized $L^{k,2}_{\bar{g}}$-conformal class for $k = 1, 2$ by

$$
\bar{g}^{0} \downarrow_{L^{k,2}_{\bar{g}}} := \bar{C}^{0} \downarrow_{L^{k,2}_{\bar{g}}} := \left\{ \bar{u}^{k-2} \bar{g} \mid \bar{u} \in C_{+}^{\infty}(X) \cap \text{L}^{k,2}_{\bar{g}}(X), \frac{\partial \bar{u}}{\partial \nu} \bigg|_{M} = 0 \right\}.
$$

Here $\nu$ is the outward unit vector field normal to the boundary $M$. Recall that the functional

$$
Q_{(X, \bar{g})}(u) = \frac{\int_{X} \left[ \alpha_{n} |du|^{2} + R_{\bar{g}} u^{2} \right] d\sigma_{\bar{g}}}{\left( \int_{X} |u|^{\frac{4n}{n-2}} d\sigma_{\bar{g}} \right)^{\frac{n-2}{2}}}
$$

is well-defined on the space $L^{1,2}_{\bar{g}}(X)$ with $u \neq 0$.

![Figure 8: A cylindrical manifold $X$ with boundary.](image)
Now we consider the functional

\[ I_X(\tilde{g}) := \int_X R_{\tilde{g}} d\sigma_{\tilde{g}} \frac{Vol_{\tilde{g}}(X)^{\frac{1}{n}}} {\tilde{g}} \]

on the space of normalized $L^2_{\tilde{g}}$-conformal metrics, i.e. $\tilde{g} \in [\bar{g}]_{L^2_{\tilde{g}}}$ with $\bar{g} \in \text{Riem}^{cyl, 0}(X)$. The following lemma is similar to Lemma 2.3.

**Lemma 4.1** The functional $I_X(\tilde{g})$ is well-defined for metrics $\tilde{g} \in [\bar{g}]_{L^2_{\tilde{g}}}$. Furthermore,

\[ I_X(\tilde{g}) = Q(X, \bar{g})(u) \text{ if } \tilde{g} = u^{4} \cdot \bar{g}. \]

Let $\bar{g} \in \text{Riem}^{cyl, 0}(X)$. Then we define the constant

\[ Y_{cyl}^{\bar{g}}(X, M; [\bar{g}]|_M) := \inf_{\tilde{g} \in [\bar{g}]_{L^2_{\tilde{g}}}} I_X(\tilde{g}). \]

The following result follows from [18].

**Fact 4.1** $Y_{cyl}^{\bar{g}}(X, M; [\bar{g}]|_M) \leq Y(S^n_+, S^{n-1}),$ where $Y(S^n_+, S^{n-1})$ is the relative Yamabe invariant of the round hemisphere $S^n_+$ with the equator $S^{n-1} = \partial S^n_+$.

The following lemma is an analogue of Lemma 2.5. The proof is essentially the same.

**Lemma 4.2** (cf. [3, Lemma 2.1]) The following identities hold:

\[ Y_{cyl}^{\tilde{g}}(X, M; [\tilde{g}]|_M) = \inf_{\tilde{g} \in [\bar{g}]_{L^2_{\tilde{g}}}} I_X(\tilde{g}) = \inf_{u \in C^\infty_c(X) \cap L^1_{\tilde{g}}(X)} Q_X(u) \]

\[ = \inf_{u \in L^1_{\tilde{g}}(X), u \not\equiv 0} Q_X(u) = \inf_{u \in C^\infty_c(X), u \not\equiv 0} Q_X(u). \]

We conclude the following:

**Claim 4.3** Let $\tilde{g}, \hat{g} \in \mathcal{C} \cap \text{Riem}^{cyl, 0}(X)$ be any two metrics. Then

\[ Y_{cyl}^{\tilde{g}}(X, M; [\tilde{g}]|_M) = Y_{\tilde{g}}^{\hat{g}}(X, M; [\hat{g}]|_M). \]

This allows us to define the relative cylindrical Yamabe constant.
Definition 4.1 Let $\tilde{C} \in \mathcal{C}^{cy}(X)$ be a cylindrical conformal class and $\tilde{g} \in \tilde{C} \cap \text{Riem}^{cy,0}(X)$ any cylindrical metric. Then the relative cylindrical Yamabe constant $Y^{cy\ell}_{\tilde{g}}(X; M; \partial \tilde{C})$ is defined as

$$Y^{cy\ell}_{\tilde{g}}(X; M; \partial \tilde{C}) := Y^{cy\ell}\left(X, M; [\tilde{g}|_M] \right) = \inf_{\tilde{g} \in \text{C}^{cy\ell,0}(X)} I_X(\tilde{g})$$

It then follows that $Y^{cy\ell}_{\tilde{g}}(X; M; \partial \tilde{C}) \leq Y(S^n_+, S^{n-1})$. We remark here that in the case of compact manifolds with non-empty boundaries, this definition gives the relative Yamabe constant defined in [4].

4.2. Finiteness of relative cylindrical Yamabe constants. Let $h \in \text{Riem}(Z)$ be a metric on the cylindrical end $Z$, and $C \in \mathcal{C}(M)$ be a conformal class. First, we define the relative $h$-cylindrical Yamabe invariant of the triple $(X; M; C)$ as

$$Y^{h-cy\ell}(X; M; C) := \sup_{\tilde{g} \in \text{Riem}^{cy\ell,0}(X)} \left( Y^{cy\ell}\left(X, M; [\tilde{g}|_M] \right) \right)$$

Second, we define the relative $h$-cylindrical Yamabe invariant of the pair $(X; M)$ as

$$Y^{h-cy\ell}(X, M) := \sup_{C \in \mathcal{C}(M)} Y^{h-cy\ell}(X, M; C).$$

Finally, we define the relative cylindrical Yamabe invariant of $X$ as

$$Y^{cy\ell}(X) := \sup_{h \in \text{Riem}(Z)} Y^{h-cy\ell}(X, M).$$

The definition yields the following inequalities:

$$-\infty \leq Y^{h-cy\ell}(X; M; C) \leq Y^{h-cy\ell}(X, M) \leq Y^{cy\ell}(X, M) \leq Y(S^n_+, S^{n-1}).$$

The following lemma is an analogue of Lemmas [27, 28].

Lemma 4.4

(0) If $\lambda(\mathcal{L}_h) < 0$, then $Y^{h-cy\ell}(X, M; C) = -\infty$.

(1) If $\lambda(\mathcal{L}_h) \geq 0$, then $Y^{h-cy\ell}(X, M; C) > -\infty$. 

(2) If $\lambda(L_h) = 0$, then $0 \geq Y^{h_{\mathrm{cyf}}} (X, M; C) > -\infty$.

Now we define the following two constants which play a technical role.

$$A := \inf_{f \in L^{1,2}(\mathbb{R}), f \not\equiv 0} \frac{\int_{\mathbb{R}} (\alpha_n (f')^2 + (n-1)(n-2)f^2) \, dt}{\left(\int_{\mathbb{R}} |f|^\frac{2n}{n-2} \, dt\right)^\frac{2n}{n-2}}$$

$$A_0 := \inf_{f \in L^{1,2}(\mathbb{R}_+, f \not\equiv 0)} \frac{\int_{\mathbb{R}_+} (\alpha_n (f')^2 + (n-1)(n-2)f^2) \, dt}{\left(\int_{\mathbb{R}_+} |f|^\frac{2n}{n-2} \, dt\right)^\frac{2n}{n-2}}.$$

**Lemma 4.5** Let $A, A_0$ be the above constants. Then

(1) $A = n(n-1) \left(\frac{\text{Vol}(S^n(1))}{\text{Vol}(S^{n-1}(1))}\right)^{\frac{2}{n}} = \frac{Y(S^n)}{Y(S^{n-1})}^{\frac{2}{n}}$.

(2) $A_0 = n(n-1) \left(\frac{\text{Vol}(S^n_+(1))}{\text{Vol}(S^{n-1}(1))}\right)^{\frac{2}{n}} = \frac{Y(S^n_+, S^{n-1})}{Y(S^{n-1}(1))}^{\frac{2}{n}}$.

(3) $A = 2^{\frac{2}{n}} A_0$.

**Proof.** (1) From [25] and [30], the Yamabe constant $Y^{h_{\mathrm{cyf}}} (S^{n-1} \times \mathbb{R})$ ($= Y(S^n)$) is attained by the metric $f \frac{1}{n-2} (h_+ + dt^2)$ with a function $f = f(t)$ depending only on $t \in \mathbb{R}$. This gives that $\text{Vol}(S^{n-1}(1))^{\frac{2}{n}} \cdot A = Y(S^n) = n(n-1) \text{Vol}(S^n(1))^{\frac{2}{n}}$, and hence the equality (1).

(2), (3). Similarly, we have

$$\text{Vol}(S^{n-1}(1))^{\frac{2}{n}} \cdot A_0 = Y(S^n_+, S^{n-1}) = n(n-1) \text{Vol}(S^n_+(1))^{\frac{2}{n}}$$

$$= 2^{\frac{2}{n}} \cdot n(n-1) \text{Vol}(S^n(1))^{\frac{2}{n}}$$

$$= 2^{\frac{2}{n}} \cdot \text{Vol}(S^{n-1}(1))^{\frac{2}{n}} \cdot A. \quad \Box$$

**Proposition 4.6** Let $h \in \mathcal{R}(Z)$ be any metric and $\bar{h} = h + dt^2$ a cylindrical metric on $Z \times \mathbb{R}$.

(1) If $\lambda(L_h) < 0$, then $Y^{\bar{h}_{\mathrm{cyf}}} (Z \times \mathbb{R}) = -\infty$, $Y^{\bar{h}_{\mathrm{cyf}}} (Z \times \mathbb{R}_{\geq 0}, Z \times \{0\}; [\bar{h}]) = -\infty$. 


(2) If \( \lambda(\mathcal{L}_h) \geq 0 \), then \( Y^{cy\ell}_{[\bar{h}]}(Z \times \mathbb{R}) \geq 0 \), \( Y^{cy\ell}_{[\bar{h}]}(Z \times \mathbb{R}_{\geq 0}, Z \times \{0\}; [\bar{h}]) \geq 0 \).

(3) If \( \lambda(\mathcal{L}_h) = 0 \), then \( Y^{cy\ell}_{[\bar{h}]}(Z \times \mathbb{R}) = 0 \), \( Y^{cy\ell}_{[\bar{h}]}(Z \times \mathbb{R}_{\geq 0}, Z \times \{0\}; [\bar{h}]) = 0 \).

(4) If \( \lambda(\mathcal{L}_h) > 0 \), then \( Y^{cy\ell}_{[\bar{h}]}(Z \times \mathbb{R}) > 0 \), \( Y^{cy\ell}_{[\bar{h}]}(Z \times \mathbb{R}_{\geq 0}, Z \times \{0\}; [\bar{h}]) > 0 \).

**Proof.** The assertions (1), (2) and (3) follow from arguments similar to the proof of Lemmas 2.22, 2.29. Concerning (4), we postpone the proof that \( Y^{cy\ell}_{[\bar{h}]}(Z \times \mathbb{R}) > 0 \) to Proposition 5.17. Here we only show that \( Y^{cy\ell}_{[\bar{h}]}(Z \times \mathbb{R}_{\geq 0}, Z \times \{0\}; [\bar{h}]) > 0 \).

Assume that \( \hat{Y} = Y^{cy\ell}_{[\bar{h}]}(Z \times \mathbb{R}_{\geq 0}, Z \times \{0\}; [\bar{h}]) \leq 0 \). Then there exists a sequence of nonnegative functions \( u_i \) with \( u_i \in C^\infty(Z \times \mathbb{R}_{\geq 0}) \cap L^{1,2}_h(\mathbb{R} \times \mathbb{R}_{\geq 0}) \) and \( u_i \not\equiv 0 \) such that
\[
\begin{align*}
Q_{(\mathbb{R} \times \mathbb{R}_{\geq 0}, \bar{h})}(u_i) \to \hat{Y} \leq 0, & \quad \text{as } i \to \infty, \\
\frac{\partial u_i}{\partial t} = 0 & \quad \text{on } Z \times \{0\}.
\end{align*}
\]

We set \( \bar{u}_i \in C^1(Z \times \mathbb{R}) \cap L^{1,2}_h(\mathbb{R} \times \mathbb{R}) \) by
\[
\bar{u}_i(x, t) = \begin{cases} 
  u_i(x, t) & \text{for } (x, t) \in Z \times \mathbb{R}_{\geq 0}, \\
  u_i(x, -t) & \text{for } (x, t) \in Z \times \mathbb{R}_{\leq 0}.
\end{cases}
\]

Then \( \limsup_{i \to \infty} Q_{(\mathbb{R} \times \mathbb{R}_{\geq 0}, \bar{h})}(\bar{u}_i) \leq 0 \), and hence \( Y^{cy\ell}_{[\bar{h}]}(Z \times \mathbb{R}) \leq 0 \). This contradicts that \( Y^{cy\ell}_{[\bar{h}]}(Z \times \mathbb{R}) > 0 \). \( \square \)

Let \( X \) be an open manifold with tame ends \( Z \times [0, \infty) \) (and without boundary). We decompose \( X \) as
\[
X = X(1) \cup Z \quad (Z \times [1, \infty)).
\]

Here the manifold \( Z \times [1, \infty) \) endowed with a cylindrical metric \( h + dt^2 \) is considered as a cylindrical manifold with the boundary
\[
\partial(Z \times [1, \infty)) = Z \times \{1\}.
\]

For \( \bar{g} \in \text{Riem}^{cy\ell}(X) \) with \( \partial_\infty \bar{g} = h \), set \( \bar{C} = [\bar{g}] \in C^{cy\ell}(X) \). Then we denote
\[
Y_1 := Y_{\bar{C} |_{X(1)}}(X(1), Z; [\bar{h}]),
\]
\[
Y_2 := Y_{\bar{C} |_{(Z \times [1, \infty])}}^{cy\ell}(Z \times [1, \infty), Z \times \{1\}; [\bar{h}]).
\]

Here \( Y_{\bar{C} |_{X(1)}}(X(1), Z; [\bar{h}]) \) is the relative Yamabe constant defined in [4].
Theorem 4.7 Under the above assumptions, we have

\[ Y^c_{\text{yl}}(X) \geq \begin{cases} - (|Y_1|^\frac{2}{n} + |Y_2|^\frac{2}{n})^\frac{n}{2} & \text{if } Y_1, Y_2 \leq 0, \\ \min \{Y_1, Y_2\} & \text{otherwise.} \end{cases} \]

Proof. First, we notice that if \( Y_2 = -\infty \), then there is nothing to prove. Hence we assume that \( Y_2 > -\infty \).

For any function \( u \in C^\infty_c(X) \) with \( u \neq 0 \),

\[ Q_{(X, \bar{g})}(u) = \frac{\int_X [\alpha_n |du|^2 + R_{\bar{g}} u^2] \, d\sigma_{\bar{g}}}{(\int_{X(1)} |u|^{\frac{2n}{n-2}} \, d\sigma_{\bar{g}} + \int_{Z \times [1, \infty)} |u|^{\frac{2n}{n-2}} \, d\sigma_{\bar{g}})^{\frac{n-2}{n}}} \]

We denote:

\[ \alpha = \int_{X(1)} |u|^{\frac{2n}{n-2}} \, d\sigma_{\bar{g}}, \quad \beta = \int_{Z \times [1, \infty)} |u|^{\frac{2n}{n-2}} \, d\sigma_{\bar{g}}. \]

It is enough to consider the case \( \alpha, \beta > 0 \) and \( \alpha + \beta = 1 \). Then we have

\[ Q_{(X, \bar{g})}(u) = \frac{1}{\left(1 + \frac{\beta}{\alpha}\right)^{\frac{n-2}{n}}} \frac{\int_{X(1)} [\alpha_n |du|^2 + R_{\bar{g}} u^2] \, d\sigma_{\bar{g}}}{\alpha^{\frac{n-2}{n}}} \]

\[ + \frac{1}{\left(1 + \frac{\alpha}{\beta}\right)^{\frac{n-2}{n}}} \frac{\int_{Z \times [1, \infty)} [\alpha_n |du|^2 + R_{\bar{g}} u^2] \, d\sigma_{\bar{g}}}{\beta^{\frac{n-2}{n}}} \]

\[ = \alpha^{\frac{n-2}{n}} \int_{X(1)} [\alpha_n |du|^2 + R_{\bar{g}} u^2] \, d\sigma_{\bar{g}} \]

\[ + \beta^{\frac{n-2}{n}} \int_{Z \times [1, \infty)} [\alpha_n |du|^2 + R_{\bar{g}} u^2] \, d\sigma_{\bar{g}} \]

\[ \geq \alpha^{\frac{n-2}{n}} Y_1 + (1 - \alpha)^{\frac{n-2}{n}} Y_2 \]

for any \( \alpha \in (0, 1) \). Clearly one has

\[ Y^c_{\text{yl}}(X) \geq \inf_{\alpha \in [0, 1]} \left\{ \alpha^{\frac{n-2}{n}} Y_1 + (1 - \alpha)^{\frac{n-2}{n}} Y_2 \right\} \]

\[ = \begin{cases} - (|Y_1|^\frac{2}{n} + |Y_2|^\frac{2}{n})^\frac{n}{2} & \text{if } Y_1, Y_2 \leq 0, \\ \min \{Y_1, Y_2\} & \text{otherwise.} \end{cases} \]
This proves Theorem 4.7.

Theorem 4.7 and Proposition 4.6 (4) imply the following result.

**Corollary 4.8** Let \( X \) be an open manifold with tame ends \( Z \times [0, \infty) \) and \( \bar{g} \in \text{Riem}^{\text{cy}}(X) \) any cylindrical metric with \( \partial_\infty \bar{g} = h \in \text{Riem}(Z) \). Assume that \( \lambda(L_h) > 0 \) on \( Z \). Then \( Y_{\bar{g}}^{\text{cy}}(X) \geq \min\{Y_1, Y_2\} \).

**4.3. Surgery and the cylindrical Yamabe invariant.** Let \( M^n \) be a closed manifold of \( \dim M = n \geq 3 \) and \( N \subset M^n \) be an embedded closed submanifold of \( \dim N = p \leq n-1 \) with trivial normal bundle \( \nu_N \). We observe that the open manifold \( M \setminus N \) is a manifold with a tame end \( Z \times [0, \infty) \), since a tubular neighborhood \( D^n \times N \setminus N \) itself is diffeomorphic to \( S^{q-1} \times N \times (0, 1) \cong S^{q-1} \times N \times R_{>0} \subset M \setminus N, \quad q = n - p \).

With these understood, we prove the following result.

**Theorem 4.9** Let \( M \) be a closed compact manifold of \( \dim M = n \geq 3 \) and \( N \) an embedded closed submanifold of \( M \) with trivial normal bundle. Let \( g_N \in \text{Riem}(N) \) be a given metric on \( N \) and \( h_+ \) the standard metric on \( S^{q-1} \). Assume that \( q = n - p \geq 3 \).

1. If \( Y(M) \leq 0 \), then for any \( \varepsilon > 0 \) there exists \( \delta = \delta(\varepsilon, g_N, |Y(M)|) > 0 \) such that
   
   \[ Y(\kappa^2 h_+ + g_N)^{\text{cy}}(M \setminus N) \geq Y(M) - \varepsilon \]

   for any \( 0 < \kappa \leq \delta \). In particular, \( Y^{\text{cy}}(M \setminus N) \geq Y(M) \).

2. If \( Y(M) > 0 \), then there exists \( \delta = \delta(g_N, Y(M)) > 0 \) such that
   
   \[ Y(\kappa^2 h_+ + g_N)^{\text{cy}}(M \setminus N) > 0 \]

   for any \( 0 < \kappa \leq \delta \). In particular, \( Y^{\text{cy}}(M \setminus N) > 0 \).

**Proof.** We choose a “reference” metric \( g \) on \( M \) and \( \varepsilon > 0 \). Let \( U_\varepsilon(N) \) be an open tubular \( \varepsilon \)-neighborhood of \( N \). Then we define a manifold \( X \) with a tame end \( Z \times [0, \infty) \) as follows. Let

\[
\begin{align*}
W &:= M \setminus U_\varepsilon(N), \\
Z &:= \partial W \cong S^{q-1} \times N, \\
X &:= W \cup_Z (Z \times R_{\geq 0}) \cong M \setminus N.
\end{align*}
\]

From 32 (cf. 21), we recall the following:
For any Riemannian metric $g \in \mathcal{Riem}(M)$ and any $\varepsilon > 0$, there exist $\delta = \delta(\varepsilon, g_N, |Y(M)|)$, $L = L(\varepsilon, g_N, |Y(M)|) > 0$ and a metric $\hat{g} \in \mathcal{Riem}^{cy\ell}(X)$ such that

\begin{align*}
(a) \quad \hat{g} = g & \text{ on } W, \\
(b) \quad R_{\hat{g}} > R_g - \frac{\varepsilon}{4} & \text{ on } X \cong M \setminus N, \\
(c) \quad \text{Vol}_{\hat{g}}(X(L)) \leq \text{Vol}_g(M) + \frac{\varepsilon}{4(1 + |Y(M)|)}, \\
(d) \quad \hat{g} = (\kappa^2 \cdot h_+ + g_N) + dt^2 & \text{ on } Z \times [L, \infty) \text{ for any } 0 < \kappa \leq \delta.
\end{align*}

In particular, $R_{\hat{g}} > 0$ on the cylinder $Z \times [L, \infty)$.

(1) From the assumption $Y(M) \leq 0$, for any $\varepsilon > 0$ there exists a conformal class $C \in \mathcal{C}(M)$ such that

$$0 \geq Y_C(M) \geq Y(M) - \frac{\varepsilon}{4}.$$ 

Then there exists a Yamabe metric $g \in C$ with $\text{Vol}_g(M) = 1$ and and $R_g \equiv Y_C(M) \leq 0$. The above assertion implies that there exists a metric $\hat{g} \in \mathcal{Riem}^{cy\ell}(X)$ satisfying (a)–(d). Now Proposition 4.6(4) implies that

$$Y_{\hat{g}\mid x \times [L, \infty)}(Z \times [L, \infty), Z; [\kappa^2 \cdot h_+ + g_N]) > 0. \quad (14)$$

Theorem 4.7 now gives

$$Y_{\hat{g}\mid x \times [L, \infty)}(X(L), Z; [\kappa^2 \cdot h_+ + g_N]),$$

$$Y_{\hat{g}\mid x \times [L, \infty)}(Z \times [L, \infty), Z; [\kappa^2 \cdot h_+ + g_N]).$$

We see that if $Y_{\hat{g}\mid x \times [L, \infty)}(M \setminus N) \geq 0$, then

$$Y^{(\kappa^2 \cdot h_+ + g_N)-cy\ell}(M \setminus N) \geq Y_{\hat{g}\mid x \times [L, \infty)}(M \setminus N) \geq 0 \geq Y(M) \geq Y(M) - \varepsilon.$$ 

Furthermore, (14) implies that if $Y_{\hat{g}\mid x \times [L, \infty)}(X) < 0$, then

$$0 > Y_{\hat{g}\mid x \times [L, \infty)}(X) \geq Y_{\hat{g}\mid x \times [L, \infty)}(X(L), Z; [\kappa^2 \cdot h_+ + g_N)) \geq \text{Vol}_{\hat{g}}(X(L)) \frac{\varepsilon}{4} \cdot \min_X \{R_g - \frac{\varepsilon}{4} \}
\geq \left( \text{Vol}_g(M) + \frac{\varepsilon}{4(1 + |Y(M)|)} \right) \frac{\varepsilon}{4} \cdot \left( Y_C(M) - \frac{\varepsilon}{4} \right)
\geq Y_C(M) - \frac{3\varepsilon}{4} \geq Y_C(M) - \varepsilon.$$
These imply that $Y^{\left(\kappa^2, h + g_N\right)}_{\text{Yam}}(M \setminus N) \geq Y^{\text{cyf}}_{\left[g\right]}(M \setminus N) \geq Y(M) - \varepsilon$.

(2) From the assumption $Y(M) > 0$, there exists $C \in C(M)$ such that $Y_C(M) \geq \frac{1}{\varepsilon} Y(M) > 0$. Then there exists a Yamabe metric $g \in C$ with $\text{Vol}_g(M) = 1$ and $R_g \equiv Y_C(M) > 0$. From the above (a), (b) and (d), there exist $\delta = \delta(g_N, Y(M)) > 0$ and a metric $\hat{g} \in \mathcal{R}^\text{Yam}(X)$ such that

\[
\left\{\begin{array}{ll}
R_{\hat{g}} \geq \frac{1}{\varepsilon} R_g > 0 & \text{on } X, \\
\hat{g} = (\kappa^2 \cdot h + g_N) + dt^2 & \text{on } \mathbb{R} \times [L, \infty) \text{ for } 0 < \kappa \leq \delta, \\
R_{\hat{g}} = R_{\kappa^2, h + g_N} \geq \delta_0 > 0 & \text{on } \mathbb{R} \times [L, \infty) \text{ for some } \delta_0 > 0.
\end{array}\right.
\]

Therefore Corollary 3.8 and Proposition 4.10(4) imply that there exists a constant $\delta_1 > 0$ such that $Y^{\left(\kappa^2, h + g_N\right)}_{\text{Yam}}(X) \geq Y^{\text{cyf}}_{\left[g\right]}(X) \geq \delta_1$ for some $\delta_1 > 0$.

**Corollary 4.10** (cf. [32, Theorem 1.1]) Let $M_1, M_2$ be closed manifolds of $\dim M_i = n \geq 3$ and $N \subset M_i$ an embedded closed submanifold of $\dim N = p$ with trivial normal bundle $(i = 1, 2)$. Assume that $g = n - p \geq 3$. Let $M_{1,2}$ be the manifold obtained by gluing $M_1$ and $M_2$ along $N$. Then

(1) If $Y(M_1), Y(M_2) \leq 0$, then $Y(M_{1,2}) \geq - (|Y(M_1)|^{\frac{2}{p}} + |Y(M_2)|^{\frac{2}{p}})^{\frac{p}{2}}$.

(2) If $Y(M_1) \leq 0$ and $Y(M_2) > 0$, then $Y(M_{1,2}) \geq Y(M_1)$.

**Proof.** Let $g_N \in \mathcal{R}^\text{Riem}(N)$ be a metric and $\varepsilon > 0$ a small constant.

(1) Theorem 4.1(1) gives that there exists $\kappa_\varepsilon > 0$ ($\kappa_\varepsilon \to 0$ as $\varepsilon \to 0$) such that

$Y^{\left(\kappa^2, h + g_N\right)}_{\text{Yam}}(M_i \setminus N) \geq Y(M_i) - \varepsilon, \quad i = 1, 2.$

Set $U = N \times D^q$. Then $M_{1,2} = (M_1 \setminus U) \cup_{\partial U} (M_2 \setminus U)$. From Theorem 4.1 we have

$Y(M_{1,2}) = Y((M_1 \setminus U) \cup_{\partial U} (M_2 \setminus U))$

$\geq - \left(|Y^{\left(\kappa^2, h + g_N\right)}_{\text{Yam}}(M_1 \setminus U)|^{\frac{2}{p}} + |Y^{\left(\kappa^2, h + g_N\right)}_{\text{Yam}}(M_2 \setminus U)|^{\frac{2}{p}}\right)^{\frac{p}{2}}$

$\geq - \left(|Y(M_1)|^{\frac{2}{p}} + |Y(M_2)|^{\frac{2}{p}}\right)^{\frac{p}{2}} - C\varepsilon.$

Hence $Y(M_{1,2}) \geq - \left(|Y(M_1)|^{\frac{2}{p}} + |Y(M_2)|^{\frac{2}{p}}\right)^{\frac{p}{2}}$. 

**Corollary 4.10** (cf. [32, Theorem 1.1]) Let $M_1, M_2$ be closed manifolds of $\dim M_i = n \geq 3$ and $N \subset M_i$ an embedded closed submanifold of $\dim N = p$ with trivial normal bundle $(i = 1, 2)$. Assume that $g = n - p \geq 3$. Let $M_{1,2}$ be the manifold obtained by gluing $M_1$ and $M_2$ along $N$. Then
(2) From Theorem 4.9 there exists \( \kappa > 0 \) such that
\[
Y(\kappa^2 h + g_N)_{\text{cy}}(M_1 \setminus N) \geq Y(M_1) - \varepsilon, \quad Y(\kappa^2 h + g_N)_{\text{cy}}(M_2 \setminus N) > 0.
\]
Then Theorem 5.7 implies that
\[
Y(M_{1,2}) \geq Y(\kappa^2 h + g_N)_{\text{cy}}(M_1 \setminus N) \geq Y(M_1) - \varepsilon.
\]
Hence \( Y(M_{1,2}) \geq Y(M_1) \). \( \square \)

5 The Yamabe problem for cylindrical manifolds

5.1 Yamabe problem for cylindrical manifolds. Let \( X \) be an open manifold of \( \dim X = n \geq 3 \) with tame ends \( Z \times [0, \infty) \) (and without boundary). Let \( \bar{g} \) be a cylindrical metric on \( X \) with \( \partial\infty \bar{g} = h \) for \( h \in \text{Riem}(Z) \), and \( \bar{C} = \bar{g} \in C^\text{cy}(X) \). For simplicity, throughout this section we assume that \( Z \) is connected. However, the corresponding results in this section hold even when \( Z \) is not connected.

Recall the following on the Yamabe constant \( Y_{\bar{C}}(X) \):
\[
Y_{\bar{C}}(X) = \inf_{u \in L^1_2(X), \ u \neq 0} Q_{(X, \bar{g})}(u), \quad \text{where}
\]
\[
Q_{(X, \bar{g})}(u) = \frac{E_{(X, \bar{g})}}{\left( \int_X |u|^{\frac{2n}{n-2}} \ d\bar{g} \right)^{\frac{n-2}{n}}} , \quad E_{(X, \bar{g})}(u) = \int_X \left[ a_n |du|^2 + R_{\bar{g}} u^2 \right] \ d\bar{g}.
\]

The following is the Yamabe problem on cylindrical manifolds.

Yamabe Problem. Given a cylindrical metric \( \bar{g} \) on \( X \), does there exist a metric \( \bar{g} = u^{\frac{4}{n-2}} \cdot \bar{g} \in \bar{g} \cdot L^2 \) such that \( Q_{(X, \bar{g})}(u) = Y_{\bar{C}}(X) \)?

If such a function \( u \) exists, we shall call \( u \) a Yamabe minimizer with respect to the metric \( \bar{g} \) and the metric \( \bar{g} = u^{\frac{4}{n-2}} \cdot \bar{g} \) a Yamabe metric in the cylindrical conformal class \( \bar{C} = [\bar{g}] \). First we prove the following.

Proposition 5.1. Let \( (X, \bar{h}) = (Z \times \mathbb{R}, h + dt^2) \) be a canonical cylindrical manifold with \( \lambda(\mathcal{L}_h) = 0 \). Then there does not exist a Yamabe minimizer \( u \in C^\infty_+(X) \cap L^{1,2}_h(X) \) with respect to the metric \( \bar{h} \).
Theorem 5.2

Let $Y$ be a Yamabe minimizer with $\lambda(L_h) = 0$. Then $Y_cy^f(Z \times R) = 0$. The condition $\lambda(L_h) = 0$ also implies that there exists $\varphi \in C^\infty_+(Z)$ such that $L_h \varphi = 0$. We set $\tilde{h} := \varphi^{-1} \cdot \bar{h}$. Clearly we have $R_{\tilde{h}} \equiv 0$, and $d\sigma_{\tilde{h}} = \varphi^{2-n/2} d\sigma_h$. Suppose that there exists a Yamabe minimizer $u \in C^\infty_+(X) \cap L^{1,2}_h(X)$ with $\int_X u^{2-n/2} d\sigma_h = 1$, and hence $E_{(X, \tilde{h})}(u) = 0$. Then from Fact 2.1(i) and Fact 7.1 (in Appendix), $E_{(X, \tilde{h})}(\varphi^{-1} u) = 0$. Set $v = \varphi^{-1} u \in C^\infty_+(X) \cap L^{1,2}_h(X)$. Then

$$
\int_X v^{2-n/2} d\sigma_{\tilde{h}} < \infty \quad \text{and} \quad E_{(X, \tilde{h})}(v) = 0, \tag{15}
$$

and hence

$$
0 = E_{(X, \tilde{h})}(v) = \int_X (\alpha_n |dv|^2 + R_{\tilde{h}} v^2) d\sigma_{\tilde{h}} = \alpha_n \int_X |dv|^2 d\sigma_h.
$$

This combined with (15) gives that $v \equiv 0$. This contradicts that $v > 0$. □

2. Solution of the Yamabe problem. Recall that $Y_cy^f(C) \leq Y_cy^f(Z \times R)$. First, we consider the case when $\lambda(L_h) > 0$ and $Y_cy^f(C) < Y_cy^f(Z \times R)$.

Theorem 5.2 Let $X$ be an open manifold with a connected tame end $Z \times [0, \infty)$ and $h \in \text{Riem}(Z)$ a metric with $\lambda(L_h) > 0$. Let $\bar{g}$ be a cylindrical metric on $X$ with $\partial_\infty \bar{g} = \bar{h}$, and $C = [\bar{g}]$. Assume that

$$
Y_cy^f(C) < Y_cy^f(Z \times R).
$$

Then there exists a Yamabe minimizer $u \in C^\infty_+(X) \cap L^{1,2}_{\bar{g}}(X)$ with $\int_X u^{2-n/2} d\bar{g} = 1$ such that $Q_{(X, \bar{g})}(u) = Y_cy^f(C)$. In particular, the minimizer $u$ satisfies the Yamabe equation:

$$
L_{\bar{g}} u = -\alpha_n \Delta_{\bar{g}} u + R_{\bar{g}} u = Y_cy^f(C) u^{n+2}/n-2.
$$

For the proof of Theorem 5.2, we first prove several lemmas and propositions.

Recall that $X(L) = X \setminus (Z \times (L, \infty))$ for each $L > 0$. From the condition $Y_cy^f(C) < Y_cy^f(Z \times R)$, there exists $L_0 > 1$ such that

$$
Q_L := \inf_{f \in L^{1,2}_{\bar{g}}(X), \ f \neq 0} \frac{Q_{(X, \bar{g})}(f)}{L^{1,2}_{\bar{g}}(X, f \neq 0 \text{ on } Z \times (L, \infty))} < Y_cy^f(Z \times R)
$$

for any $L \geq L_0$. 46
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Then the standard argument combined with the inequality $Y^\text{cyl}_{[h+dt]}(Z \times \mathbb{R}) \leq Y(S^n)$ implies the following fact (cf. [40, Chapter 5]).

Lemma 5.3 Let $(X, \bar{g})$ be a cylindrical manifold as above. Then there exists a nonnegative function $u_L \in C^0(X) \cap C^\infty(\text{Int}(X(L)))$ for each $L \geq L_0$ such that

$$Q_{(X, \bar{g})}(u_L) = Q_L,$$

$$u_L > 0 \text{ on } \text{Int}(X(L)), \quad u_L \equiv 0 \text{ on } Z \times [L, \infty),$$

$$\int_X u_L^{\frac{2n}{n-2}} d\bar{g} = 1.$$

In particular, the function $u_L$ satisfies the equation

$$-\alpha_n \Delta_{\bar{g}} u_L + R_{\bar{g}} u_L = Q_L u_L^{\frac{n+2}{n-2}} \text{ on } \text{Int}(X(L)).$$

Moreover, the following properties of the constant $Q_L$ hold:

- $Q_{L_1} \geq Q_{L_2}$ if $L_1 \leq L_2$.
- $Q_L \to Y^\text{cyl}_C(X)$ as $L \to \infty$.

Lemma 5.4 There exists a positive constant $C_0$ such that $u_L \leq C_0$ on $X(2)$ for any $L \geq L_0$.

Proof. Suppose that there exist $L_i > 0$ and a point $p_i \in X(2)$ for $i = 1, 2, \ldots$ such that

$$L_i \to \infty, \quad \text{and } u_{L_i}(p_i) = \max_{X(2)} u_{L_i} =: m_i \to \infty.$$

Since $X(2)$ is compact, we may assume that $p_i \to p_0 \in X(2)$.

Let $\{U, x = (x^1, \ldots, x^n)\}$ be a normal coordinate system centered at $p_0$. We may also assume that $\{|x| < 1\} \subset U$. We define the functions

$$v_i(x) := m_i^{-1} \cdot u_{L_i} \left( m_i^{-\frac{2}{n-2}} \cdot x + x(p_i) \right) \quad \text{for } x \in \left\{ |x| < m_i^{\frac{2}{n-2}} (1 - |x(p_i)|) \right\}.$$

Similarly to the proof of Theorem 2.1 [40, Chapter 5], there exists a positive function $v \in C^\infty_0(\mathbb{R}^n)$ such that $v_i$ converges to $v$ in the $C^2$-topology on each relatively compact domain in $\mathbb{R}^n$. Let $\Delta_0$ be the Laplacian on $\mathbb{R}^n$ with respect to the Euclidean metric. Then the function $v$ satisfies the following:

$$\begin{cases}
-\alpha_n \Delta_0 v = Y^\text{cyl}_C(X) \cdot v^{\frac{2n}{n-2}} \text{ on } \mathbb{R}^n, \\
\int_{\mathbb{R}^n} v^{\frac{2n}{n-2}} dx \leq \lim \inf_{i \to \infty} \int_{X(2)} u_{L_i}^{\frac{2n}{n-2}} \leq 1.
\end{cases}$$
Hence $Y_{c^k}(X) \geq Y(S^n)$. This contradicts that $Y_{c^k}(X) < Y_{[h+dt^2]}(Z \times R) \leq Y(S^n)$.

We consider $u_L$ on the end $Z \times [1, \infty)$ with the metric $\bar{g} = h + dt^2$. Note that the metric $\bar{g}$ is invariant under parallel translations along the $t$-coordinate. By using this fact, one can obtain (similar to the proof of Lemma 5.4) that

$$u_L \leq K \text{ on } Z \times [2, \infty)$$

(16)

for any $L \geq L_0$, where $K > 0$ is a constant independent of $L$. Set $K_0 := \max \{C_0, K\} > 0$. Then Lemma 5.4 and (16) imply the following.

**Lemma 5.5** There exists a constant $K_0 > 0$ such that $u_L \leq K_0$ on $X$ for any $L \geq L_0$.

**Convention 5.6** Let $(X, \bar{g})$ be a cylindrical manifold with $\bar{g} = h + dt^2$ on $Z \times [1, \infty)$, where $h$ is a metric on $Z$ with $\lambda(L_h) > 0$ (resp. $\lambda(L_h) = 0$). Throughout this and the next sections, we will use the following convention.

By the reason below (replacing the metric $\bar{g}$ by a suitable pointwise conformal metric if necessary), we may assume that the cylindrical metric $\bar{g}$ satisfies the following

$$R_{\bar{g}} = R_h \geq R_{\min} := \min_Z R_h > 0 \text{ (resp. } R_{\bar{g}} = R_h = 0) \text{ on } Z \times [1, \infty).$$

With this understood, we use some constants depending on $R_{\min}$ in this and the next sections. One can easily replace those constants by other ones which depend on the invariant $\lambda(L_h)$ and $\delta_h$ defined below.

The condition $\lambda(L_h) > 0$ (resp. $\lambda(L_h) = 0$) gives that there exists a function $\varphi_h \in C^\infty_+(Z)$ such that

$$\left\{ \begin{array}{l} L_h \varphi_h = \lambda(L_h) \cdot \varphi_h \text{ on } Z, \\ \max_Z \varphi_h = 1. \end{array} \right.$$ 

Let $\varphi \in C^\infty_+(X)$ be a positive smooth function with $\varphi \equiv 1$ on $X(0)$ and $\varphi \equiv \varphi_h$ on $Z \times [1, \infty)$. Set $\tilde{g} = \varphi^{-\frac{4}{n-2}} \cdot \bar{g}$ and $\delta_h := \min_Z \varphi > 0$. Then we have

$$\left\{ \begin{array}{l} \delta_h^{-\frac{4}{n-2}} \cdot \tilde{g} \leq \bar{g} \leq \delta_h^{\frac{4}{n-2}} \cdot \delta_h \geq \min_Z d\sigma_{\bar{g}} \leq d\sigma_{\bar{g}} \text{ on } Z \times [1, \infty). \end{array} \right.$$ 

Clearly $\Delta_{\tilde{g}} f = \varphi^{-\frac{4}{n-2}} \cdot f''$ for any function $f = f(t) \in C^\infty(Z \times [1, \infty))$ which depends only on $t \in [1, \infty)$. We can use this property to construct comparison functions on $Z \times [1, \infty)$. Now we have

$$R_{\tilde{g}} = \lambda(L_h) \cdot \varphi^{-\frac{4}{n-2}} \geq \lambda(L_h) > 0 \text{ (resp. } R_{\tilde{g}} = \lambda(L_h) \cdot \varphi^{-\frac{4}{n-2}} \equiv 0)$$
on \( Z \times [1, \infty) \). We emphasize that the metric \( \tilde{g} = \varphi^{\frac{4}{n-2}} \cdot \bar{g} \) is no longer a product metric on \( Z \times [1, \infty) \), but \( \varphi = \varphi_h \) is a positive smooth function depending only on \( x \in Z \). One can use this property of the metric \( \tilde{g} = \varphi^{\frac{4}{n-2}} \cdot \bar{g} \) to obtain higher derivative estimates on Yamabe minimizers similarly to the cylindrical metric case.

Consider the canonical cylindrical manifold \((X, \bar{h}) = (Z \times R, h + dt^2)\) associated to \((X, \bar{g})\).

**Proposition 5.7** Let \((X, \bar{h}) = (Z \times R, h + dt^2)\) be a canonical cylindrical manifold with \( \lambda(\mathcal{L}_h) > 0 \). Then \( Y_{\text{cyl}}^{\ell}([\bar{h}](X)) > 0 \).

**Proof.** As above, we may assume that \( R_{\min} = \min_Z R_h > 0 \). Then \( \text{Ric}_h \geq (n-1)\kappa \) on \( Z \) since \( Z \) is compact, where \( \kappa \) is a constant (not necessarily positive). Denote by \( B_r(x) \) a geodesic ball of radius \( r \) centered at \( x \in X \). Let \( \Delta_h - \partial / \partial \tau \) be the heat operator on \( X \) and \( p = p(x, y, \tau) \) the heat kernel of \( \Delta_h - \partial / \partial \tau \). Then the following estimate on \( p(x, x, \tau) \) was proved by Li-Yau (see [23]):

\[
p(x, x, \tau) \leq \frac{c(n, \delta)}{\text{Vol}_h(B_{\sqrt{\tau}}(x))} \exp(-c(n)\delta\kappa \tau)
\]

for all \( \delta > 0, \tau > 0 \) and \( x \in X \). Here \( c(n, \delta) \) (resp. \( c(n) \)) is a positive constant depending only on \( n \) and \( \delta > 0 \) (resp. \( n \)). We also notice that

\[
\text{Vol}_h(B_{\sqrt{\tau}}(x)) \geq C^{-1} \cdot \tau^{\frac{n}{2}}
\]

for \( 0 < \tau \leq \sqrt{\frac{1}{2} \text{diam}(Z, h)} \), where \( C > 0 \) is a constant. Set

\[
\tau_0 := \max \left\{ \frac{\alpha_n}{R_{\min}}, \frac{1}{2} \text{diam}(Z, h) \right\} > 0
\]

and \( \delta = 1 \). Using (18) in (17), we then obtain the following estimate

\[
p(x, x, \tau) \leq \frac{C \cdot c(n, \delta)}{\tau^{n/2}} \exp(c(n)|\kappa|\tau_0)
\]

for \( 0 < \tau \leq \tau_0 \). Set \( C_4 := C \cdot c(n, \delta) \exp(c(n)|\kappa|\tau_0) \). Now we use [33 Theorem 2.2] and [10] to obtain

\[
\left( \int_X |f|^{\frac{2n}{n-2}} d\sigma_h \right)^{\frac{n-2}{n}} \leq C' \cdot C_4^{\frac{2}{n}} \alpha_n \int_X \left| df \right|^2 + \frac{1}{\tau_0} f^2 \right) d\sigma_h
\]

\[
\leq C' \cdot C_4^{\frac{2}{n}} \alpha_n \int_X \left( |df|^2 + \frac{R_h}{\alpha_n} f^2 \right) d\sigma_h
\]
for any $f \in C^\infty_c(X)$, where $C' > 0$ is a constant independent of $f$. This implies that

$$Y_{[h]}^{\text{cy}}(X) = Y_{[h+dt]}^{\text{cy}}(Z \times \mathbb{R}) \geq \frac{1}{C' \cdot C_4^2} > 0.$$ 

This completes the proof of Proposition 5.7. \qed

Now we return to consider the cylindrical manifold $(X, \tilde{g})$ in Theorem 5.2. If $Y_{\bar{C}}^{\text{cy}}(X) > 0$, we set

$$Y := Y_{[h+dt]}^{\text{cy}}(Z \times \mathbb{R}) > 0, \quad \delta := \frac{Y_{\bar{C}}^{\text{cy}}(X)}{Y} > 0.$$ 

Clearly $\delta < 1$ by the assumption in Theorem 5.2. In this case, we may assume that

$$0 < \frac{Q_L}{Y} \leq \frac{3\delta + 1}{4} < 1$$

for any $L \geq L_0$. If $Y_{\bar{C}}^{\text{cy}}(X) \leq 0$, we set $\delta = 0$. Using the Moser iteration technique, we then show the following decay estimate of $u_L$ on the cylindrical end $Z \times [1, \infty)$.

**Proposition 5.8** Let $u_L$ be the minimizer obtained in Lemma 5.3 for each $L \geq L_0$. Then there exists

$$L_1 = L_1 \left( \frac{1}{(1 - \delta)^2 R_{\text{min}}}, n \right) \geq 2$$

such that

$$\sup_{Z \times [t_0 - \frac{1}{2}r, t_0 + \frac{1}{2}r]} u_L \leq \frac{C_n}{Y^{\frac{n}{4}}} \cdot \frac{1}{r^{\frac{n}{2}}}$$

for any $t_0, r > 0$ satisfying $t_0 - r \geq L_1$. Here $C_n > 0$ is a constant depending only on $n$.

**Proof.** We give the proof only for the case $Y_{\bar{C}}^{\text{cy}}(X) > 0$. The case $Y_{\bar{C}}^{\text{cy}}(X) \leq 0$ is much easier, hence we omit it. For any $f \in C^\infty_c(X)$ with $\text{Supp}(f) \subset Z \times [1, \infty)$, we have

$$\int_X |f|^{\frac{2n}{n-2}}d\sigma_{\bar{g}} \leq \frac{\alpha_n}{Y} \int_X |df|^2d\sigma_{\bar{g}} + \frac{1}{Y} \int_X R_{\bar{g}} f^2d\sigma_{\bar{g}}. \quad (21)$$

The minimizer $u_L$ satisfies the equation

$$-\Delta_{\bar{g}} u_L + \frac{R_{\bar{g}}}{\alpha_n} u_L = \frac{Q_L}{\alpha_n} u_L^{\frac{n+2}{2}}. \quad (22)$$
with \( u_L \equiv 0 \) on \( X \setminus X(L) \) and \( u_L > 0 \) on \( \text{Int}(X(L)) \). Then \( u_L \) satisfies the following differential inequality

\[
- \Delta \bar{g} u_L + \frac{R_{\bar{g}}}{\alpha_n} u_L \leq \frac{Q_L}{\alpha_n} \frac{\alpha+2}{\alpha} \quad \text{on } X
\]

in the distributional sense.

Let \( \eta \) be a cut-off function with \( \text{Supp}(\eta) \subset Z \times [1, \infty) \) and \( 0 \leq \eta \leq 1 \). Recall that \( \bar{g} = h + dt^2 \) and \( R_{\bar{g}} = R_h \) on the cylindrical end \( Z \times [1, \infty) \). We use (21) and (23) to prove the following assertion.

**Claim 5.9** For any \( \alpha \geq 1 \) and any small \( \varepsilon > 0 \), the following estimate holds:

\[
\left( \int_X (\eta^2 u_L^{\alpha+1}) \frac{n^2}{n-2} \, d\sigma_{\bar{g}} \right)^{\frac{n^2}{n-2}} \leq \left( \frac{(\alpha+1)(\alpha+1+\varepsilon)}{4(\alpha-\frac{\varepsilon}{2})} \right) \frac{Q_L}{Y} \int_X \eta^2 u_L^{\alpha+\frac{n+2}{n-2}} \, d\sigma_{\bar{g}}
\]

\[
+ \frac{C_{\alpha \varepsilon^{-1}}}{Y} \int_X |\eta|^{2} u_L^{\alpha+1} \, d\sigma_{\bar{g}},
\]

where \( C_n > 0 \) is a constant depending only on \( n \).

**Proof of Claim 5.9** We multiply both sides of (23) by \( \eta^2 u_L^\alpha \) for any \( \alpha > 0 \). Then integrating by parts, we obtain

\[
\frac{4\alpha}{(\alpha+1)^2} \int_X \eta^2 \left| du_L^{\alpha+1} \right|^2 \, d\sigma_{\bar{g}} - 2 \int_X \eta |d\eta| u_L^\alpha |du_L| d\sigma_{\bar{g}}
\]

\[
\leq \frac{1}{\alpha_n} \int_X R_h \eta^2 u_L^{\alpha+1} \, d\sigma_{\bar{g}} + \frac{Q_L}{\alpha_n} \int_X \eta^2 u_L^{\alpha+\frac{n+2}{n-2}} \, d\sigma_{\bar{g}}.
\]

The Young inequality implies

\[
|d\eta| u_L^\alpha |du_L| \leq \varepsilon^{-1} \cdot |d\eta|^2 u_L^{\alpha+1} + \varepsilon \cdot \frac{\eta^2}{(\alpha+1)^2} \left| du_L^{\alpha+1} \right|^2
\]

for any \( \varepsilon > 0 \). We use (24) and (25) to obtain the estimate

\[
\int_X \eta^2 \left| du_L^{\alpha+1} \right|^2 \, d\sigma_{\bar{g}}
\]

\[
\leq \frac{(\alpha+1)^2}{4(\alpha-\frac{\varepsilon}{2})} \left[ \int_X (2\varepsilon^{-1} |d\eta|^2 - \frac{1}{\alpha_n} R_h \eta^2) u_L^{\alpha+1} \, d\sigma_{\bar{g}} + \frac{Q_L}{\alpha_n} \int_X \eta^2 u_L^{\alpha+\frac{n+2}{n-2}} \, d\sigma_{\bar{g}} \right]
\]

\[(27)\]
for $0 < \varepsilon < 2\alpha$. It then follows from (21), (26) and (27) that
\[
\left( \int_X \left( \eta \frac{u}{L} \right)^{2n/2n} \, d\bar{g} \right)^{\frac{n-2}{n}} \leq \frac{\alpha_n}{Y} \int_X \left| d \left( \eta \frac{u}{L} \right) \right|^2 \, d\bar{g} + \frac{1}{Y} \int_X R_h \left( \eta \frac{u}{L} \right)^2 \, d\bar{g}
\]
\[
\leq \frac{(\alpha + 1)(\alpha + 1 + \varepsilon)}{4(\alpha - \frac{\varepsilon}{2})} \cdot \frac{QL}{Y} \int_X \eta^2 \frac{u}{L}^{\alpha + \frac{2\alpha}{\varepsilon^2}} \, d\bar{g}
\]
\[
+ \frac{\alpha_n}{Y} \left\{ \frac{(\alpha + 1)(\alpha + 1 + \varepsilon)\varepsilon^{-1}}{2(\alpha - \frac{\varepsilon}{2})} + 1 + (\alpha + 1)\varepsilon^{-1} \right\} \int_X |d\eta|^2 u^{\alpha+1} \, d\bar{g}
\]
\[
- \frac{1}{Y} \left\{ \frac{(\alpha + 1)(\alpha + 1 + \varepsilon)}{4(\alpha - \frac{\varepsilon}{2})} - 1 \right\} \int_X R_h \eta^2 u^{\alpha+1} \, d\bar{g}.
\]
We notice that
\[
\frac{(\alpha + 1)(\alpha + 1 + \varepsilon)}{4(\alpha - \frac{\varepsilon}{2})} - 1 > 0
\]
for $0 < \varepsilon < 2\alpha$. From $R_h > 0$, we have
\[
\int_X R_h \eta^2 u^{\alpha+1} \, d\bar{g} > 0.
\]
Combining these observations with (28), we obtain the inequality (24) for $\alpha \geq 1$.

Now we continue with the proof of Proposition 5.8. Set $\alpha = 1$ in (24), then
\[
\left( \int_X \eta \frac{u}{L}^{\frac{2\alpha}{\varepsilon}} \, d\bar{g} \right)^{\frac{n-2}{n}} \leq \frac{2 + \varepsilon}{2 - \varepsilon} \cdot \frac{QL}{Y} \int_X \eta^2 \frac{u}{L}^{\frac{2\alpha}{\varepsilon^2}} \, d\bar{g}
\]
\[
+ \frac{C_n}{Y} \varepsilon^{-1} \int_X |d\eta|^2 |u|^2 \, d\bar{g}.
\]
Then we use
\[
\int_X \eta \frac{u}{L}^{\frac{2\alpha}{\varepsilon}} \, d\bar{g} \leq \int_X u^{\frac{2\alpha}{\varepsilon}} \, d\bar{g} = 1
\]
to obtain the estimate
\[
\int_X \eta \frac{u}{L}^{\frac{2\alpha}{\varepsilon}} \, d\bar{g} \leq \left( \int_X \frac{u}{L}^{\frac{2\alpha}{\varepsilon}} \, d\bar{g} \right)^{\frac{n-2}{n}}.
\]
From (29) and (30), we also obtain
\[
\left(1 - \frac{2 + \varepsilon}{2 - \varepsilon} \cdot \frac{Q_L}{Y}\right) \int_X \eta^\frac{2n}{n-2} u_L^\frac{2m}{n-2} \, d\sigma_{\bar{g}} \leq \frac{\tilde{C}_n}{Y} \int_X |d\eta|^2 u_L^2 \, d\sigma_{\bar{g}}.
\] (31)

Take \(\varepsilon > 0\) in (31) as \(\varepsilon = \frac{2(1-\delta)}{3 + 5\delta}\). From
\[
0 < \frac{Q_L}{Y} \leq \frac{3\delta + 1}{4} < 1,
\]
we notice that
\[
1 - \frac{2 + \varepsilon}{2 - \varepsilon} \cdot \frac{Q_L}{Y} \geq \frac{1 - \delta}{2}.
\] (32)

It then follows from (31), (32) that
\[
\int_X \eta^\frac{2n}{n-2} u_L^\frac{2m}{n-2} \, d\sigma_{\bar{g}} \leq \frac{C'_n}{(1 - \delta)^2 Y} \int_X |d\eta|^2 u_L^2 \, d\sigma_{\bar{g}},
\] (33)
where \(C'_n > 0\) is a constant depending only on \(n\).

With these understood, we also show the following.

Claim 5.10
\[
\int_{Z \times [r+1, \infty)} u_L^\frac{2m}{n-2} \, d\sigma_{\bar{g}} \leq \frac{4 \cdot C'_n}{(1 - \delta)^2 R_{\min}} \cdot \frac{1}{r^2}
\] (34)
for any \(r \geq 1\).

Proof of Claim 5.10 For any \(T > 0\), choose a cut-off function \(\eta\) in (33) satisfying
\[
\begin{cases}
\eta = 1 & \text{on } Z \times [r + 1, T + r + 1], \\
\eta = 0 & \text{on } X \setminus (Z \times [1, T + 2r + 1]), \\
|d\eta| \leq \frac{2}{r} & \text{on } Z \times ([1, 1+r] \sqcup [T + r + 1, T + 2r + 1]).
\end{cases}
\]

From (29) and \(\int_X u_L^\frac{2m}{n-2} \, d\sigma_{\bar{g}} = 1\), we notice that
\[
\int_X u_L^2 \, d\sigma_{\bar{g}} \leq \frac{1}{R_{\min}} \int_X (\alpha_n |d\eta|^2 + R_h u_L^2) \, d\sigma_{\bar{g}} \\
\leq \frac{Q_L}{R_{\min}} \int_X u_L^\frac{2m}{n-2} \, d\sigma_{\bar{g}} \leq \frac{Q_L}{R_{\min}}.
\]
This combined with (33) implies
\[
\int_{Z \times [r+1, T+r+1]} u_L^{\frac{2n}{n-2}} d\bar{g} \leq \frac{4C'_n}{(1-\delta)^2 R_{\min}} \cdot \frac{Q_L}{Y} \cdot \frac{1}{r^2},
\]
and hence from \( \frac{Q_L}{Y} \leq 1 \),
\[
\int_{Z \times [r+1, T+r+1]} u_L^{\frac{2n}{n-2}} d\bar{g} \leq \frac{4C'_n}{(1-\delta)^2 R_{\min}} \cdot \frac{1}{r^2}.
\]
Letting \( T \to \infty \), we then obtain the estimate (34).

We return to the proof of Proposition 5.8. From Hölder’s inequality, we notice
\[
\int_X \eta^2 u_L^{\frac{2(n+2)}{n-2}} \ d\bar{g} = \int_X \frac{4}{n-2} \cdot \eta^2 u_L^{\frac{2n}{n-2}} d\bar{g}
\leq \left( \int_{\text{supp}(\eta)} u_L^{\frac{2n}{n-2}} d\bar{g} \right)^{\frac{2}{n-2}} \cdot \left( \int_X \left( \eta^2 u_L^{\frac{2n}{n-2}} \right)^{\frac{n-2}{n-2}} \ d\bar{g} \right)^{\frac{n-2}{n}}.
\]
Set \( \alpha = \frac{n+2}{n-2} \) and \( \varepsilon = 1 \) in (24). Then this combined with the above inequality implies
\[
\left\{ 1 - \frac{2n}{n-2} \left( \frac{2n}{n-2} + 1 \right) \left( \int_{\text{supp}(\eta)} u_L^{\frac{2n}{n-2}} d\bar{g} \right)^{\frac{2}{n-2}} \cdot \left( \int_X \left( \eta^2 u_L^{\frac{2n}{n-2}} \right)^{\frac{n-2}{n-2}} \ d\bar{g} \right)^{\frac{n-2}{n}} \right\} \leq \frac{C_n}{Y} \cdot \frac{n+2}{n-2} \int_X |\eta|^2 u_L^{\frac{2n}{n-2}} d\bar{g}.
\]
From (34), there exists \( L_1 = L_1 \left( \frac{1}{(1-\delta)^2 R_{\min}}, n \right) \geq 2 \) such that
\[
\frac{2n}{n-2} \left( \frac{2n}{n-2} + 1 \right) \left( \int_{\text{supp}(\eta)} u_L^{\frac{2n}{n-2}} d\bar{g} \right)^{\frac{2}{n-2}} \leq \frac{1}{2}
\]
for any \( \eta \) satisfying \( \text{supp}(\eta) \subset Z \times [L_1, \infty) \). It then follows from (34) and (36) that
\[
\left( \int_X \left( \eta^2 u_L^{\frac{2n}{n-2}} \right)^{\frac{n-2}{n}} \ d\bar{g} \right)^{\frac{n-2}{n}} \leq \frac{10 \cdot C'_n}{Y} \int_X |\eta|^2 u_L^{\frac{2n}{n-2}} d\bar{g}.
\]
We obtain the proof of Claim 5.11. By Hölder’s inequality, (37) and the Young inequality, we have
\[
\begin{align*}
\|\chi \cdot u_L\|_{L^2_q(\mathbb{R}^{n-2})} & \leq \left( \frac{250 \cdot C_n}{Y} \cdot \frac{1}{r^2} \right)^{\frac{n-2}{2}}. \\
\end{align*}
\]

Now we specify the cut-off function as follows. Let \( \lambda^- \), \( \lambda^+ \) be positive constants satisfying \( \frac{1}{2} \leq \lambda^- < \lambda^+ \leq \frac{1}{2} \). We set \( \eta \) as
\[
\begin{align*}
\eta & = 1 \quad \text{on } Z \times [t_0 - \frac{3}{4}r, t_0 + \frac{3}{4}r], \\
\eta & = 0 \quad \text{on } X \setminus (Z \times [t_0 - r, t_0 + r]), \\
|d\eta| & \leq \frac{r}{\lambda^-} \quad \text{on } Z \times ([t_0 - r, t_0 - \frac{3}{4}r] \cup [t_0 + \frac{3}{4}r, t_0 + r]).
\end{align*}
\]
Let \( \Phi(q,r) := \left( \int_{Z \times [t_0 - r, t_0 + r]} u_q^\gamma d\sigma_g \right)^{\frac{1}{\gamma}} \) for \( q \geq \frac{2n}{n-2} \). Set \( \gamma = \frac{n}{n-2} > 1 \). We prove the following assertion.

Claim 5.11 There exists a constant \( \hat{C}_n > 0 \) depending only on \( n \) such that the following inequality holds for any \( q \geq \frac{2n}{n-2} \):
\[
\Phi(\gamma q, \lambda^- r) \leq \left( \frac{\hat{C}_n}{Y} \cdot \frac{1}{r^2} \right)^{\frac{n}{2}} \cdot \Phi(q, \lambda^+ r). \tag{38}
\]

Proof of Claim 5.11 By Hölder’s inequality, (37) and the Young inequality, we obtain
\[
\int_X \eta^\alpha u_L^\beta d\sigma_g = \int_X \frac{1}{L_q^{n-2}} \cdot \eta^\alpha u_L^{\alpha+1} d\sigma_g
\leq \|\chi \cdot u_L\|_{L^2_q(\mathbb{R}^{n-2})} \cdot \|\eta \cdot u_L^{\alpha+1}\|_{L^2_q(\mathbb{R}^{n-2})}^{\frac{2n-2}{n-2}} = \left( \frac{250 \cdot C_n}{Y} \cdot \frac{1}{r^2} \right)^{\frac{n}{2}} \cdot \left( \varepsilon' \cdot \|\eta u_L^{\alpha+1}\|_{L^2_q(\mathbb{R}^{n-2})} + \varepsilon' \cdot \|\eta u_L^{\alpha+1}\|_{L^2_q(\mathbb{R}^{n-2})} \right)^{\frac{n-2}{2}}. \tag{39}
\]

\[
\leq \left( \frac{250 \cdot C_n}{Y} \cdot \frac{1}{r^2} \right)^{\frac{n}{2}} \cdot \left( \varepsilon' \cdot \|\eta u_L^{\alpha+1}\|_{L^2_q(\mathbb{R}^{n-2})} + \varepsilon' \cdot \|\eta u_L^{\alpha+1}\|_{L^2_q(\mathbb{R}^{n-2})} \right)^{\frac{n-2}{2}}. \tag{39}
\]
for any $\varepsilon' > 0$. Set $\varepsilon = 1$ in (24). It then follows from (24) and (39) that for any $\alpha > \frac{n+2}{n-2}$

$$
\left( \int_X \left( \eta u^L_{\alpha+1} \right)^{\frac{2n}{n-2}} d\bar{g} \right)^{\frac{n-2}{n}} \leq \frac{C''}{Y} \left[ \frac{Y^{\frac{n-2}{n}}}{r^{\frac{2n}{n-2}}} \left\{ (\varepsilon')^2 \cdot \| \eta u^L_{\alpha+1} \|^{\frac{2n}{n-2}}_{L^\infty(X)} + (\varepsilon')^{-(n-2)} \cdot \| \eta u^L_{\alpha+1} \|^{\frac{2n}{n-2}}_{L^\infty(X)} \right\} \right.

$$

$$
\left. + \int_X |d\eta|^{\frac{2}{\alpha+1}} u^{\alpha+1} \cdot d\bar{g} \right].
$$

Set $(\varepsilon')^2 = \frac{Y^{\frac{n-2}{n}}}{2 C'' n} \alpha$ in (40). Then $(\varepsilon')^{-(n-2)} = \left( \frac{2 C''}{Y} \frac{n-2}{r^{\frac{2n}{n-2}}} \right)$, and hence

$$
\| \eta u^L_{\alpha+1} \|^{\frac{2n}{n-2}}_{L^\infty(X)} \leq \frac{C''}{Y} \frac{n-2}{\alpha} \| \eta u^L_{\alpha+1} \|^{\frac{2n}{n-2}}_{L^\infty(X)} + \frac{C''}{Y} \| |d\eta| \cdot u_{\alpha+1} \|^{\frac{2n}{n-2}}_{L^\infty(X)}.
$$

Finally we set $q = \alpha + 1$, and then this implies the estimate (38). $\square$

Now we complete the proof of Proposition 5.8. The estimate (38) can be iterated to yield the desired result. Indeed, we set

$$
\begin{align*}
q_m &= \frac{\gamma^m 2n}{n-2}, \\
\lambda^-_m &= \frac{1}{2} + 2^{-(m+3)}, \\
\lambda^+_m &= \frac{1}{2} + 2^{-(m+2)} = \lambda^-_{m-1}
\end{align*}
$$

for $m = 0, 1, 2, \ldots$. Then we use (38) to obtain

$$
\Phi \left( \frac{\gamma^m 2n}{n-2}, \frac{1}{2} r \right) \leq \left( \frac{2C_n}{Y} \right) \frac{\gamma^m 2n}{n-2} \cdot \gamma^{-i} \cdot \Phi \left( \frac{2n}{n-2}, r \right)
$$

$$
\leq \frac{C_n}{Y^{\frac{n-2}{n}}} \cdot \frac{1}{r^{\frac{2n}{n-2}}}. \tag{41}
$$

Letting $m \to \infty$ in (41), we obtain the estimate (20). This completes the proof of Proposition 5.8. $\square$

Now we show a more precise decay estimate of the minimizers $u_L$ on the cylindrical end $\mathbb{Z} \times [1, \infty)$. 

Proposition 5.12  Let $u_L$ be the minimizer obtained in Lemma 5.3 for each $L \geq L_0$. For any $a > 0$ satisfying $a < \sqrt{\frac{R_{\min}}{\alpha_n}}$, there exist constants $\bar{K} = \bar{K}(a) > 0$ and $\ell = \ell(a) > 0$ such that

$$u_L \leq \bar{K} \cdot e^{-at} \quad \text{on} \quad Z \times [\ell, \infty) \quad (42)$$

for any $L > \ell$.

Proof. From (20), there exists $\ell = \ell(a) > 0$ such that

$$\sup_{Z \times [\ell, \infty)} u_L \leq \left( \frac{R_{\min} - \alpha_n a^2}{Y(S^n)} \right)^{\frac{n-2}{4}} \quad (43)$$

for any $L > \ell$. We set

$$\bar{K} = e^{a\ell} \left( \frac{R_{\min} - \alpha_n a^2}{Y(S^n)} \right)^{\frac{n-2}{4}} > 0.$$

Consider the function $w_L = u_L - \bar{K} \cdot e^{-at}$ on $Z \times [\ell, L]$. It then follows from (22) and (43) that

$$\Delta_g w_L = \frac{R_h}{\alpha_n} u_L - \frac{Q_L}{\alpha_n} u_L^{\frac{n+2}{n}} - a^2 \bar{K} \cdot e^{-at}$$

$$\geq \left\{ \left( \frac{R_{\min}}{\alpha_n} - a^2 \right) - \frac{Y(S^n)}{\alpha_n} u_L^{-\frac{2}{n-2}} \right\} u_L + a^2 \cdot w_L$$

$$\geq a^2 \cdot w_L \quad \text{on} \quad Z \times (\ell, L),$$

and $w_L|_{Z \times \{\ell\}} \leq 0$, $w_L|_{Z \times \{L\}} < 0$. By the maximum principle, we obtain that

$$w_L = u_L - \bar{K} \cdot e^{-at} \leq 0 \quad \text{on} \quad Z \times [\ell, L].$$

This completes the proof of Proposition 5.12. $\square$

Proof of Theorem 5.2. The $L^p$ and Schauder interior estimates combined with Lemma 5.5 imply that

$$\|u_L\|_{C^{2,\alpha}(X(i))} \leq C(i)$$

for any $L \geq L_0$ and $i = 1, 2, \ldots$, where each constant $C(i) > 0$ is independent of $L$. Then by the argument of diagonal subsequence, there exist
a subsequence \( u_{L_j} \) and a nonnegative function \( u \in C^2(X) \cap L^{1,2}_g(X) \) such that \( u_{L_j} \to u \) with respect to the \( C^2 \)-topology on each \( X(i) \). Furthermore, the function \( u \) satisfies

\[
-\alpha_n \Delta_g u + R_g u = Y^{cy}_C(X) \cdot u^{\frac{n+2}{n-2}} \quad \text{on} \quad X. \tag{44}
\]

Combining Lemma 5.5 with [10, Proposition 3.75], we notice that \( u > 0 \) on \( X \) or \( u \equiv 0 \) on \( X \). Hence from (44), \( u \in C^\infty(X) \).

On the other hand, it follows from (42) that for any small \( \varepsilon > 0 \) there exists \( L(\varepsilon) > 0 \) such that

\[
\int_{X(L(\varepsilon))} u^{\frac{2n}{n-2}} \, d\sigma_g \geq 1 - \varepsilon \quad \text{for any} \quad L \geq L_0.
\]

Then the \( C^2 \)-convergence \( u_{L_j} \to u \) on \( X(L(\varepsilon)) \) implies

\[
\int_{X(L(\varepsilon))} u^{\frac{2n}{n-2}} \, d\sigma_g \geq 1 - \varepsilon.
\]

This implies that \( u > 0 \) on \( X \) and

\[
\int_X u^{\frac{2n}{n-2}} \, d\sigma_g = 1.
\]

This completes the proof of Theorem 5.2.

\[ \square \]

5.3. Singularities of the Yamabe metric. Let \( \hat{g} = u^{\frac{4}{n-2}} \hat{g} \) be the Yamabe metric obtained in Theorem 5.2. We also prove the existence of a Yamabe metric for the case \( \lambda(L_h) = 0 \) (Theorem 5.14). Moreover, we study the singularities of the metric \( \hat{g} \) near infinity of the tame end \( Z \times [0, \infty) \). There are two very different cases here: \( \lambda(L_h) > 0 \) and \( \lambda(L_h) = 0 \).

We start with the case \( \lambda(L_h) > 0 \). A canonical model here is provided by the canonical open cone

\[
\text{Cone}(Z) = (Z \times (0,1), r^2 \cdot h + dt^2)
\]

over \( (Z,h) \). To obtain the cylindrical coordinates, we set \( r = e^{-t} \). Then we identify the cone \( \text{Cone}(Z) \) with

\[
\text{Cone}(Z) \cong (Z \times (0, \infty), e^{-2t}(h + dt^2)).
\]

We use the cone \( \text{Cone}(Z) \) as a canonical model to introduce the following definition.
Definition 5.1 Let $N$ be an open manifold with a connected tame end $Z \times [0, \infty)$. A metric $g \in \mathcal{Riem}(N)$ is called an almost conical metric if there exist a coordinate system $(x,t)$ on $Z \times [0, \infty)$, a metric $h \in \mathcal{Riem}(Z)$, constants $0 < \alpha \leq \beta$, $0 < C_1 \leq C_2$ and a positive function $\varphi \in C_\infty^\infty(Z \times [1, \infty))$ such that

(i) $g(x,t) = \varphi(x,t)(h(x) + dt^2)$ on $Z \times [1, \infty)$,

(ii) $C_1 \cdot e^{-\beta t} \leq \varphi(x,t) \leq C_2 \cdot e^{-\alpha t}$ on $Z \times [1, \infty)$.

Below we state the result on singularities without using Convention 5.6.

Theorem 5.13 Under the same assumptions as in Theorem 5.2, let $u \in C_\infty^\infty(X) \cap L^1_{\bar{g}}(X)$ be the Yamabe minimizer obtained in Theorem 5.2. Then for any constant $a > 0$ satisfying $0 < a < \left( \min_Z \varphi_h \right)^\frac{2}{n-2} \cdot \sqrt{\frac{\lambda(L_h)}{\alpha_n}}$, there exist constants $\overline{C}, \underline{C} > 0$ and $\ell > 0$ such that

$$\overline{C} \cdot e^{-a t} \leq u(x,t) \leq \underline{C} \cdot e^{-a t} \text{ for } (x,t) \in Z \times [\ell, \infty), \text{ where } a_0 = \sqrt{\frac{\lambda(L_h)}{\alpha_n}}.$$

Here $\overline{C} = C(a, n, \lambda(L_h), \min_Z \varphi_h)$, $\underline{C} = C(n, \lambda(L_h), \min_Z \varphi_h, \min_Z u)$, $\ell = \ell(a, n, \lambda(L_h), \min_Z \varphi_h)$, and these constants also depend on $Y_{\{h+dt^2\}}^c(Z \times \mathbb{R})$, $Y_{\bar{C}}(X)$. In particular, the Yamabe metric $\tilde{g} = u^\frac{4}{n-2} \cdot \bar{g}$ on $X$ is an almost conical metric.

Proof. In the proof of Proposition 5.12, we replace the cylindrical metric $\bar{g}$ by another metric $\tilde{g} \in [\bar{g}]$ satisfying $\tilde{g} = \varphi_h^\frac{4}{n-2} \cdot \bar{g}$ on $Z \times [1, \infty)$, where $\varphi_h$ is the positive smooth function given in Convention 5.6.
Let \( \tilde{u} \in C^\infty_+ (X) \cap L^{1,2}_g (X) \) be the Yamabe minimizer with respect to \( \tilde{g} \) with 
\[
\tilde{u} \cdot \tilde{g} = u \cdot g \cdot \tilde{g}.
\]
Recall that \( R_{\tilde{g}} = \lambda (L_h) \cdot \varphi_h^{-1} \), \( \max \varphi_h = 1 \) and 
\[
\Delta \tilde{g} f = \varphi_h^{-1} \cdot f'' \quad \text{for} \quad f = f(t) \in C^\infty (Z \times [1, \infty)).
\]
By these properties, for any \( 0 < a < \left( \min \varphi_h \right) \frac{1}{n} \cdot \frac{\sqrt{\lambda (L_h)}}{\alpha_n} \) there exist \( \tilde{K} = \tilde{K} (a, n, \lambda (L_h), \min \varphi_h) > 0 \) and \( \ell = \ell (a, n, \lambda (L_h), \min \varphi_h) > 0 \) such that 
\[
\tilde{u} \leq \tilde{K} \cdot e^{-at} \quad \text{on} \quad Z \times (\ell, \infty).
\]
A similar argument to the above and Proposition 5.12 combined with the condition \( Y^\text{cusp}_C (X) > 0 \), implies that there exists a constant \( K = K (a, n, \lambda (L_h), \min \varphi_h) > 0 \) such that 
\[
u \geq K \cdot (\min \varphi_h) \cdot e^{-a_{0} t} \quad \text{on} \quad Z \times [1, \infty).
\]
This completes the proof. \( \square \)

Next, we study the case when \( \lambda (L_h) = 0 \). Here a canonical model comes from hyperbolic geometry, namely, it is given by cusp ends of hyperbolic manifolds, see \[15, \text{Chapter D}\].

A cusp end of a hyperbolic \( n \)-manifold of curvature \(-1\) is given as
\[
\left( (R^{n-1} / \Gamma) \times [1, \infty), \frac{1}{r^2} (h_{\text{flat}} + dt^2) \right).
\]

Here \((R^{n-1} / \Gamma, h_{\text{flat}})\) is a closed Riemannian manifold uniformized by a flat torus \( T^{n-1} \). With this understood, we introduce the following definition.

**Definition 5.2** Let \( N \) be an open manifold with a connected tame end \( Z \times [0, \infty) \). A metric \( g \in \text{Riem}(N) \) is called an almost cusp metric if there exist a coordinate system \((x, t)\) on the cylinder \( Z \times [0, \infty) \), a metric \( h \in \text{Riem}(Z) \), constants \( 0 < C_1 \leq C_2 \) and a positive function \( \varphi \in C^\infty_+ (Z \times [1, \infty)) \) such that

(i) \( g(x, t) = \varphi(x, t) (h(x) + dt^2) \) on \( Z \times [1, \infty) \),

(ii) \( C_1 \cdot t^{-2} \leq \varphi(x, t) \leq C_2 \cdot t^{-2} \) on \( Z \times [1, \infty) \).

Recall that \( \lambda (L_h) = 0 \) implies that \( Y^\text{cusp}_C (Z \times \mathbb{R}) = 0 \). Hence \( Y^\text{cusp}_C (X) \leq 0 \).
Theorem 5.14 Let $X$ be an open manifold with a connected tame end $Z \times [0, \infty)$ and $h \in \text{Riem}(Z)$ a metric with $\lambda(L_h) = 0$. Let $\bar{g}$ be a cylindrical metric on $X$ with $\partial_\infty \bar{g} = h$, and $\bar{C} = [\bar{g}]$. Assume that $Y^\text{cy}^\ell_C(X) < 0$. Then there exists a Yamabe minimizer $u \in C^\infty(X) \cap L^1, 2(\bar{g}(X))$ such that $Q_L(u) = Y^\text{cy}^\ell_C(X)$. In particular, the minimizer $u$ satisfies the Yamabe equation:

$$L_{\bar{g}} u = -\frac{4(n-1)}{n-2} \Delta_{\bar{g}} u + R_{\bar{g}} u = Y^\text{cy}^\ell_C(X) u^\frac{n+2}{n-2}. $$

Moreover, there exist constants $0 < C \leq \overline{C}$ such that

$$C \cdot t^{-\frac{n-2}{2}} \leq u(x, t) \leq \overline{C} \cdot t^{-\frac{n-2}{2}} \quad \text{on} \quad Z \times [1, \infty)$$

and

$$\overline{C} = \overline{C}(n, Y^\text{cy}^\ell_C(X), \min_Z \varphi_h, \max_Z u), \quad C = C(n, Y^\text{cy}^\ell_C(X), \min_Z \varphi_h, \min_Z u).$$

In particular, the Yamabe metric $\bar{g} = u^\frac{4}{n-2} \cdot \bar{g}$ on $X$ is an almost cusp metric.

Proof. As in Convention 5.6 we may assume that $R_h \equiv 0$ on $Z$. For any $L \geq 1$, there exists a nonnegative function $u_L \in C^0(X) \cap C^\infty_{+}(\text{Int}(X(L))) \cap L^1, 2(\bar{g}(X))$ satisfying the same properties as in Lemma 5.3. Similarly to the proofs of Lemmas 5.4, 5.5, there exist constants $L_0 >> 1$ and $K_0 > 0$ such that

$$u_L \leq K_0 \quad \text{on} \quad X$$

for any $L \geq L_0$. Here we may assume that

$$Q_L \leq \frac{1}{2} Y^\text{cy}^\ell_C(X) < 0 \quad \text{for} \quad L \geq L_0.$$
We set the constant $\overline{C}$ by
\[
\overline{C} = \max \left\{ \left( \frac{n(n-2)\alpha_n}{2|Y_{cyf}^g(X)|} \right)^{\frac{n-2}{2}} , K_0 \right\} > 0.
\]

Consider the function $\overline{w}_L = u_L - \overline{C} \cdot t^{-\frac{n-2}{2}}$ on $Z \times [1,L]$. It then follows from (22) and (46) that
\[
\Delta_{\overline{g}} \overline{w}_L = \frac{|Q_L|}{\alpha_n} u_L^{\frac{n+2}{2}} - \frac{n(n-2)}{4\overline{C}^{n-2}} \left( \overline{C} \cdot t^{-\frac{n-2}{2}} \right)^{\frac{n+2}{n-2}} 
\geq \frac{|Y_{cyf}^g(X)|}{\alpha_n} \left\{ \left( \overline{C} \cdot t^{-\frac{n-2}{2}} \right)^{\frac{n+2}{n-2}} - u \right\} \quad \text{on } Z \times [1,L],
\]
and $\overline{w}_L|_{Z \times \{1\}} \leq 0$, $\overline{w}_L|_{Z \times \{L\}} < 0$. By the maximum principle, we obtain that
\[
\overline{w}_L = u_L - \overline{C} \cdot t^{-\frac{n-2}{2}} \leq 0 \quad \text{on } Z \times [1,L]. \tag{47}
\]

From (44), for any small $\varepsilon > 0$ there exists $L(\varepsilon) > 0$ such that
\[
\int_{X(L(\varepsilon))} u_L^{\frac{2n}{n-2}} d\overline{g} \geq 1 - \overline{C}^{\frac{n}{n-2}} \cdot \text{Vol}_h(Z) \cdot \int_{L(\varepsilon)}^\infty \frac{1}{t^n} dt \geq 1 - \varepsilon \tag{48}
\]
for any $L \geq L_0$. The estimates (44) and (47) combined with (48) imply that $u_L$ converges to a Yamabe minimizer $u \in C^\infty(X) \cap L^1(\overline{g})$ with
\[
\int_X u^{\frac{2n}{n-2}} d\overline{g} = 1 \quad \text{in the } C^2\text{-topology on each } X(i) \text{ for } i = 1,2,\ldots.
\]

Moreover, $u$ satisfies
\[
u \leq \overline{C} \cdot t^{-\frac{n-2}{2}} \quad \text{on } Z \times [1,\infty), \tag{49}
\]
\[-\alpha_n \Delta_{\overline{g}} u = Y_{cyf}^g(X)u^{\frac{n+2}{2}} \quad \text{on } X.
\]

We set $C = \min \left\{ \left( \frac{n(n-2)\alpha_n}{4|Y_{cyf}^g(X)|} \right)^{\frac{n-2}{2}} , \min_{Z \times \{1\}} u \right\} > 0$. Consider the function
\[
w = C \cdot t^{-\frac{n-2}{2}} - u
\]
on $Z \times [1,\infty)$. It then follows from (49) that
\[
\Delta_{\overline{g}} w &= \frac{n(n-2)}{4C^{n-2}} \left( C \cdot t^{-\frac{n-2}{2}} \right)^{\frac{n+2}{n-2}} - \frac{|Y_{cyf}^g(X)|}{\alpha_n} u^{\frac{n+2}{n-2}} 
\geq \frac{|Y_{cyf}^g(X)|}{\alpha_n} \left\{ \left( C \cdot t^{-\frac{n-2}{2}} \right)^{\frac{n+2}{n-2}} - u \right\} \quad \text{on } Z \times [1,\infty),
\]
and \( w|_{Z \times \{t\}} \leq 0 \), \( \lim_{t \to \infty} w(x, t) = 0 \). The maximum principle now implies that
\[
\bar{w} = C \cdot t^{\frac{n-2}{2}} - u \leq 0 \quad \text{on} \quad Z \times [1, \infty).
\]
This completes the proof. \( \square \)

**Remark.** Let \((X, \bar{g})\) be a cylindrical manifold with tame ends \( Z \times [0, \infty) \) and \( h = \partial_\infty \bar{g} \in \text{Riem}(Z) \) a metric with \( \lambda(L_h) \geq 0 \). We emphasize that the "strict" generalized Aubin’s inequality \( Y_{[\bar{g}]}^{\text{cy}}(X) < Y_{[h+dt^2]}^{\text{cy}}(Z \times \mathbb{R}) \) is a crucial condition for the above solutions of the Yamabe problem. In the case when \( Y_{[\bar{g}]}^{\text{cy}}(X) = Y_{[h+dt^2]}^{\text{cy}}(Z \times \mathbb{R}) \), Proposition 5.1 implies that the Yamabe problem can not be solved in general. Hence, it is a natural problem to characterize conformally cylindrical manifolds \((X, \bar{g})\) satisfying (50). In the next section, we solve this problem in the supremum case, i.e. when \( Y_{[\bar{g}]}^{\text{cy}}(X) = Y_{[h+dt^2]}^{\text{cy}}(Z \times \mathbb{R}) = Y(S^n) \).

**6 Canonical cylindrical manifolds**

Let \((Z, h)\) be a closed Riemannian manifold of \( \dim Z = n-1 \geq 2 \) with \( \lambda(L_h) > 0 \). Throughout in this section, we also assume that \( Z \) is connected (unless we specify otherwise). First, we study the Yamabe problem on the canonical cylindrical manifold \((Z \times \mathbb{R}, \bar{h} = h + dt^2)\). Clearly, canonical cylindrical manifolds do not satisfy the strict generalized Aubin’s inequality in Theorem 5.2. Instead, these Riemannian manifolds are invariant under parallel translations along the \( t \)-coordinate. Using this symmetry for the renormalization technique, we solve the Yamabe problem under the conditions \( \lambda(L_h) > 0 \) and \( Y_{[\bar{h}]}^{\text{cy}}(Z \times \mathbb{R}) < Y(S^n) \). Second, we characterize canonical cylindrical manifolds which satisfy the supremum condition
\[
Y_{[\bar{h}]}^{\text{cy}}(Z \times \mathbb{R}) = Y(S^n)
\]
Furthermore, we characterize general cylindrical manifolds \((X, \bar{g})\) which satisfy
\[
Y_{[\bar{g}]}^{\text{cy}}(X) = Y_{[h+dt^2]}^{\text{cy}}(Z \times \mathbb{R}) = Y(S^n).
\]
Recall that there is no Yamabe minimizer for any canonical cylindrical manifold \((Z \times \mathbb{R}, \bar{h} = h + dt^2)\) with \( \lambda(L_h) = 0 \). Hence we consider only the case \( \lambda(L_h) > 0 \).
Theorem 6.1 Let \((Z \times \mathbb{R}, \bar{h} = h + dt^2)\) be a canonical cylindrical manifold of \(\dim(Z \times \mathbb{R}) = n \geq 3\) with \(\lambda(L_h) > 0\). Assume that
\[
Y_{\bar{h}}^{cy}(Z \times \mathbb{R}) < Y(S^n) = Y_{[h + dt^2]}(S^n - 1 \times \mathbb{R}).
\]
Then there exists a Yamabe minimizer \(u \in C^\infty_+(Z \times \mathbb{R}) \cap L^{1,2}_{\bar{h}}(Z \times \mathbb{R})\) with
\[
\int_{Z \times \mathbb{R}} u^{n-2}_\bar{h} \, d\sigma_\bar{h} = 1
\]
such that
\[
Q_{(Z \times \mathbb{R}, \bar{h})}(u) = Y_{[h]}^{cy}(Z \times \mathbb{R}).
\]
Furthermore, for any constant \(a > 0\) satisfying \(0 < a < (\min_Z \varphi_h)^{-\frac{2}{n-2}} \cdot \sqrt{\frac{\lambda(L_h)}{\alpha_n}}\), there exist constants \(C, C > 0\) and \(\ell > 0\) such that
\[
C \cdot e^{-a_0|t|} \leq u(x, t) \leq C \cdot e^{-a|t|}
\]
for \((x, |t|) \in Z \times [\ell, \infty)\), where \(a_0 = \sqrt{\frac{\lambda(L_h)}{\alpha_n}}\). Here \(C = C(a, n, \lambda(L_h), \min \varphi_h)\), \(C = C(n, \lambda(L_h), \min \varphi_h, \min u, \ell = \ell(a, n, \lambda(L_h), \min \varphi_h, u)\), and the constants \(C, \ell\) also depend on \(Y_{[h + dt^2]}^{cy}(Z \times \mathbb{R})\). In particular, the Yamabe metric \(\tilde{g} = u^{\frac{4}{n-2}} \cdot \bar{g}\) on \(Z \times \mathbb{R}\) is an almost conical metric.

Proof. As in Convention 5.6, we may assume that \(R_{\min} = \min_Z R_h > 0\). By the assumption that \(Y := Y_{[h]}^{cy}(Z \times \mathbb{R}) < Y(S^n)\), there exists \(L_0 >> 1\) such that
\[
Q_L := \inf_{\substack{f \in L^{1,2}_L(Z \times \mathbb{R}), \\ f \not\equiv 0, \\ f \equiv 0 \text{ on } Z \times ((-\infty, -L]|L, \infty))}} Q_{(Z \times \mathbb{R}, \bar{h})}(f) < Y(S^n),
\]
\[
0 < Y \leq \tilde{Q}_L \leq 2Y
\]
for any \(L \geq L_0\). Then,
\[
\tilde{Q}_L \to Y \quad \text{as} \quad L \to \infty.
\]
$u_L \in C^0(Z \times \mathbb{R}) \cap C^\infty(Z \times (-L, L))$ such that
\[
\begin{cases}
Q_{(Z \times \mathbb{R}, \tilde{h})}(u_L) = \tilde{Q}_L, & \int_{Z \times \mathbb{R}} u_L^\infty d\tilde{\sigma}_h = 1 \\
u_L > 0 & \text{on } Z \times (-L, L), \quad u_L \equiv 0 & \text{on } Z \times ((-\infty, -L] \cup [L, \infty)), \quad \text{(53)}
-\alpha_n \Delta_{\tilde{h}} u_L + R_h u_L = \tilde{Q}_L u_L^{n+2} & \text{on } Z \times (-L, L),
\end{cases}
\]
where $K > 0$ is a constant independent of $L$.

By using a parallel translation along the $t$-coordinate for each $L \geq L_0$, we may assume that

\[
u_L(x_L, 0) = \max_{Z \times \mathbb{R}} u_L \leq K,
\]
where $x_L \in Z$. In this case, the support $\text{Supp}(u_L)$ may be no longer equal to $Z \times [-L, L]$ (as in (53)). However, clearly $\text{Supp}(u_L) \subset Z \times [-L, L]$, and $\text{diam}(\text{Supp}(u)) = 2L + \infty$ as $L \to \infty$.

Set $\text{Supp}(u_L) = Z \times [t_L - 2L, t_L]$, where $t_L > 0$ for any $L \geq L_0$. By taking a subsequence if necessary, there exist constants $T^-, T^+ \in [-\infty, \infty]$ with $T^- < T^+$ such that

\[
T^- = \lim_{L \to \infty} (t_L - 2L), \quad T^+ = \lim_{L \to \infty} t_L,
\]
\[-\infty \leq T^- \leq 0 \leq T^+ \leq \infty.
\]
Clearly if $T^- > -\infty$ (resp. $T^+ < \infty$), then $T^+ = \infty$ (resp. $T^- = -\infty$).

We consider only the case $T^+ = \infty$ since the argument in the case $T^- = -\infty$ is similar. Then the properties (53)–(54) allow us to apply the $L^p$ and Schauder interior estimates to conclude that

\[
\|u_L\|_{C^{2,\alpha}(Z \times [T^-(k), k])} \leq C(k)
\]
for any $L >> 1$ and $k = 1, 2, \ldots$. Here

\[
T^-(k) = \begin{cases}
-k & \text{if } T^- = -\infty, \\
T^- + \frac{1}{k} & \text{if } T^- > -\infty,
\end{cases}
\]
and each constant $C(k)$ is independent of $L$. Then we use the argument of diagonal subsequence to obtain a subsequence $\{u_{L_j}\}$ and a nonnegative function $u \in C^0(Z \times \mathbb{R}) \cap C^\infty(Z \times (T^-, \infty))$ such that $u_{L_j} \to u$ in the $C^2$-topology on each cylinder $Z \times [T^-(k), k]$. Here $u \equiv 0$ on $Z \times (-\infty, T^-]$ if $T^- > -\infty$. Furthermore, the nonnegative function $u$ satisfies the equation

\[
-\alpha_n \Delta_{\tilde{h}} u + R_h u = Y u^{n+2} \quad \text{on } Z \times (T^-, \infty).
\]

Combining (54) and (55) with [10, Proposition 3.75], we have that \( u > 0 \) or \( u \equiv 0 \) on \( Z \times (T^-, \infty) \).

Now we recall that \( Y > 0 \) by Proposition 5.7 and \( Y \leq \tilde{Q}_L \leq 2Y \). Then at the maximum point \((x_L, 0)\) for \( u_L \),
\[
u_L(x_L, 0) \geq \left( \frac{R_h(x_L)}{Q_L} \right)^{\frac{2n}{4n-2}} \geq \left( \frac{R_{\min}}{2Y} \right)^{\frac{2n}{4n-2}} > 0.
\]

This implies that \( u > 0 \) on \( Z \times (T^-, \infty) \), and hence \( u \in C^\infty_+(Z \times (T^-, \infty)) \).

We notice also that
\[
0 < \int_{Z \times R} u^{\frac{2n}{4n-2}} d\sigma_{\tilde{h}} \leq \liminf_{L_i \to \infty} \int_{Z \times R} u^{\frac{2n}{4n-2}} L_{i,j} d\sigma_{\tilde{h}} = 1.
\]

If \( \int_{Z \times R} u^{\frac{2n}{4n-2}} d\sigma_{\tilde{h}} < 1 \), then \( Q_{(Z \times R, \tilde{h})}(u) < Y \) from (55). This contradicts the definition of \( Y \), and hence
\[
\int_{Z \times R} u^{\frac{2n}{4n-2}} d\sigma_{\tilde{h}} = 1. \tag{56}
\]

Then we have
\[
Y = Q_{(Z \times R, \tilde{h})}(u) = \inf_{f \in L^{1,2}_h(Z \times R), f \neq 0, f \equiv 0 \text{ on } Z \times (-\infty, T^-)} Q_{(Z \times R, \tilde{h})}(f).
\]

Applying a similar argument in the proof of Proposition 5.8 directly to the minimizer \( u \) with (55) and (56), we obtain that
\[
u(x, t) = o(1) \text{ as } t \to \infty.
\]

To complete the proof, we have to show that \( T^- = -\infty \). Suppose that \( T^- > -\infty \). By the boundary regularity for the Yamabe equation, we notice that \( u \mid_{Z \times [T^-, \infty)} \in C^\infty(Z \times [T^-, \infty)) \). If \( \frac{\partial u}{\partial t} \equiv 0 \) on \( Z \times \{T^-\} \), then \( u \) satisfies
\[
-\alpha_n \Delta_h u + R_h u = Y u^{\frac{2n}{4n-2}} \text{ on } Z \times R
\]
in the distributional sense. Then the standard elliptic regularity implies that \( u \in C^\infty(Z \times R) \), and thus \( u > 0 \) or \( u \equiv 0 \) on \( Z \times R \), and hence \( T^- = -\infty \).

If \( \frac{\partial u}{\partial t} \neq 0 \) on \( Z \times \{T^-\} \), we define a Lipschitz function \( u_\varepsilon \) by
\[
u_\varepsilon(x, t) = \begin{cases} u(x, t) & \text{if } t \geq T^- + \varepsilon, \\ u(x, \frac{1}{2}(T^- + \varepsilon)) & \text{if } T^- - \varepsilon \leq t \leq T^- + \varepsilon, \\ 0 & \text{if } t \geq T^- - \varepsilon. \end{cases}
\]
Then we have

\[ E(Z \times \mathbb{R}, \bar{h}) (u) = E(Z \times \mathbb{R}, \bar{h}) (u) - \frac{1}{2} \alpha_n \left( \int_{Z \times \{T^-\}} \frac{\partial u}{\partial t}^2 \, d\sigma_h \right) \varepsilon + O(\varepsilon^2), \]

and

\[ \|u_\varepsilon\|_{L^2_{\bar{h}}(Z \times \mathbb{R})}^2 = \|u\|_{L^2_{\bar{h}}(Z \times \mathbb{R})}^2 + O(\varepsilon^{1+\frac{2n}{n-2}}), \]

and hence

\[ Q(Z \times \mathbb{R}, \bar{h}) (u) = Q(Z \times \mathbb{R}, \bar{h}) (u) - \frac{1}{2} \alpha_n \left( \int_{Z \times \{T^-\}} \frac{\partial u}{\partial t}^2 \, d\sigma_h \right) \cdot \varepsilon + O(\varepsilon^2) < Y \]

for \(0 < \varepsilon << 1\). This is a contradiction. Therefore, \(T^- = -\infty\).

The above argument also implies that for each \(i \geq 1\) there exist constants \(L^-_i, L^+_i\) and nonnegative functions

\[ u_i \in C^0(Z \times \mathbb{R}) \cap C^\infty_+(Z \times (L^-_i, L^+_i)) \]

such that \(u_i \rightarrow u\) in the \(C^2\)-topology on each cylinder \(Z \times [-j,j]\) for \(j = 1, 2, \ldots\), and

\[
\begin{aligned}
\lim_{i \to \infty} L^-_i &= -\infty, \quad \lim_{i \to \infty} L^+_i = \infty, \\
Q(Z \times \mathbb{R}, \bar{h}) (u_i) &= \tilde{Q}_i := \inf_{f \in L^{1,2}_{\bar{h}}(Z \times \mathbb{R}), f \neq 0, f \equiv 0 \text{ on } Z \times ((-\infty, L^-_i] \cup [L^+_i, \infty))} Q(Z \times \mathbb{R}, \bar{h}) (f), \\
-\alpha_n \Delta_{\bar{h}} u_i + R_{\bar{h}} u_i &= \tilde{Q}_i u_i^{\frac{n+2}{n-2}} \text{ on } Z \times (L^-_i, L^+_i), \\
\int_{Z \times \mathbb{R}} u_i^{\frac{n+2}{n-2}} \, d\sigma_{\bar{h}} = 1, \quad u_i \leq K \text{ on } Z \times \mathbb{R}.
\end{aligned}
\]

Hence the decay estimate of \(u\) follows from a similar argument to the proof of Theorem 5.13. This completes the proof of Theorem 6.1. \(\square\)

Next, we show the following.
Theorem 6.2 Let \((Z \times \mathbb{R}, \bar{h} = h + dt^2)\) be a canonical cylindrical manifold of \(\dim(Z \times \mathbb{R}) = n \geq 3\). Assume that

\[
Y_{[\bar{h}]}(Z \times \mathbb{R}) = Y(S^n) = Y_{[h + dt^2]}(S^{n-1} \times \mathbb{R})
\]

Then \((Z, h)\) is homothetic to the standard sphere \(S^{n-1}(1) = (S^{n-1}, h_+)\).

As a corollary, we also obtain the following.

Corollary 6.3 Let \((X, \bar{g})\) be a cylindrical manifold of \(\dim X = n \geq 3\) with tame ends \(Z \times [0, \infty)\) and \(\partial_{\infty} \bar{g} = h \in \text{Riem}(Z)\). Let \(Z = \bigcup_{j=1}^{k} Z_j\), where each \(Z_j\) is connected. Assume that

\[
Y_{[\bar{g}]}(X) = Y(S^n) = Y_{cy}(S^{n-1}) = Y_{cy}(S^{n-1} \times \mathbb{R})
\]

Then there exist \(k\) points \(\{p_1, \ldots, p_k\}\) in \(S^n\) such that

(i) each manifold \((Z_j, h_j)\) is homothetic to \((S^{n-1}, h_+)\),

(ii) the manifold \((X, [\bar{g}])\) is conformally equivalent to the punctured sphere \((S^n \setminus \{p_1, \ldots, p_k\}, C_{\text{can}})\).

Here \(h_j = h|_{Z_j}\) and \(C_{\text{can}}\) denotes the canonical conformal class on \(S^n\).

Proof of Corollary 6.3 We first notice that Proposition 2.11 implies

\[
Y_{[\bar{g}]}(X) \leq Y_{[h + dt^2]}(Z \times \mathbb{R}) = \min_{1 \leq j \leq k} Y_{[h_j + dt^2]}(Z_j \times \mathbb{R}),
\]

and hence

\[
Y_{[h_j + dt^2]}(Z_j \times \mathbb{R}) = Y(S^n)
\]

for \(j = 1, \ldots, k\). Then Theorem 6.2 implies that \((Z_j, h_j)\) is homothetic to \((S^{n-1}, h_+)\) for \(j = 1, \ldots, k\). By the definition of cylindrical manifold modeled by \((S^{n-1}, h_+)\), there exists a smooth conformal compactification \((\hat{X}, \hat{C})\), \(\hat{X} = X \cup \{p_1, \ldots, p_k\}\), of the conformal manifold \((X, [\bar{g}])\) such that

\[
Y_\hat{C}(\hat{X}) = Y_{[\bar{g}]}(X).
\]

Here \(Y_\hat{C}(\hat{X})\) stands for the Yamabe constant of the closed conformal manifold \((\hat{X}, \hat{C})\).

By Schoen’s Theorem [34, Theorem 2] (cf. [39] and [10], [27], [40]), the equality \(Y_\hat{C}(\hat{X}) = Y(S^n)\) implies that \((\hat{X}, \hat{C})\) is conformally equivalent to \((S^n, C_{\text{can}})\). Hence \((X, [\bar{g}])\) is conformally equivalent to the punctured sphere \((S^n \setminus \{p_1, \ldots, p_k\}, C_{\text{can}})\).

For the proof of Theorem 6.2, we first prepare several lemmas and propositions. Let \(\Gamma\) be a spherical space form group acting freely on \(S^{n-1}(1)\) by isometries. Let denote by \(S^{n-1}(1)/\Gamma\) the spherical space form, that is, the smooth quotient of \(S^{n-1}(1)\) by \(\Gamma\).
Lemma 6.4 Let \((Z \times \mathbb{R}, \bar{h} = h + dt^2)\) be a canonical cylindrical manifold of \(\dim(Z \times \mathbb{R}) = n \geq 3\) with \(\lambda(\mathcal{L}_h) > 0\). If the manifold \((Z \times \mathbb{R}, [\bar{h}])\) is locally conformally flat, then \((Z, h)\) is homothetic to a smooth quotient \(S^{n-1}(1)/\Gamma\) of \(S^{n-1}(1)\).

**Proof.** First we assume that \(n \geq 4\). Let \(W_{\bar{h}} = (\bar{W}_{\alpha\beta\gamma\delta})\) denote the Weyl curvature tensor of the metric \(\bar{h}\), where \(\alpha, \beta, \gamma, \delta = 0, 1, \ldots, n - 1\). Here the index 0 corresponds to the \(t\)-coordinate, and the indices 1, \ldots, \(n - 1\) correspond to local coordinates \(x = (x^1, \ldots, x^{n-1})\) on \(Z\) respectively. Then we obtain that

\[
0 = \bar{W}_{0i0j} = -\frac{1}{n-2} \left( \bar{R}_{ij} - \frac{R_{ij}}{n-1} \bar{h}_{ij} \right) = -\frac{1}{n-2} \left( R_{ij} - \frac{R_{ij}}{n-1} h_{ij} \right)
\]

(57)

for all \(1 \leq i, j \leq n - 1\). Here \((\bar{R}_{ij})\) and \((R_{ij})\) denote respectively the Ricci curvature tensors of the metrics \(\bar{h}\) and \(h\). Then \(\lambda(\mathcal{L}_h) > 0\) implies that \(h\) is an Einstein metric of positive scalar curvature on \(Z\). We use this to obtain that

\[
0 = \bar{W}_{ijk\ell} = R_{ijk\ell} - \frac{R_{ik}}{(n-1)(n-2)} (h_{ik} h_{j\ell} - h_{i\ell} h_{jk})
\]

for all \(1 \leq i, j, k, \ell \leq n - 1\). The right hand side of the above equation is nothing but the concircular curvature tensor of \(h\). Thus \(h\) is a metric of positive constant curvature, and hence \((Z, h)\) is a homothetic to a smooth quotient \(S^{n-1}(1)/\Gamma\) of \(S^{n-1}(1)\).

Next we consider the case \(n = 3\). Let

\[
B_{\bar{h}} = (B_{\alpha\beta\gamma}) = \left( (\bar{\nabla}_\gamma \bar{R}_{\alpha\beta} - \bar{\nabla}_\beta \bar{R}_{\alpha\gamma} - \frac{1}{4} (\partial_\gamma R_{\alpha\beta} \cdot \bar{h}_{\alpha\beta} - \partial_\beta R_{\bar{h}} \cdot \bar{h}_{\alpha\gamma}) \right)
\]

denote the Bak tensor of \(\bar{h}\). Then

\[
0 = B_{00i} = -\frac{1}{4} \partial_i R_{\bar{h}} = -\frac{1}{4} \partial_i R_{h}
\]

for \(i = 1, 2\). This identity combined with the assumption \(\lambda(\mathcal{L}_h) > 0\) and \(\dim Z = 2\) implies that \(h\) is a metric of positive constant Gaussian curvature on \(Z\). Therefore, \((Z, h)\) is homothetic to either \(S^2(1)\) or the projective space \(\mathbb{RP}^2 = S^2(1)/\mathbb{Z}_2\). \(\square\)

Proposition 6.5 Under the same assumptions as in Theorem 6.4, we also assume that \((Z \times \mathbb{R}, \bar{h} = h + dt^2)\) is locally conformally flat. Then \((Z, h)\) is homothetic to the standard sphere \(S^{n-1}(1)\).
Proof. The condition $Y_{[\bar{h}]}^{cyl}(Z \times \mathbb{R}) = Y(S^n) > 0$ implies that $\lambda(L_{\bar{h}}) > 0$, and hence from Lemma 6.4 $(Z, h)$ is homothetic to a smooth quotient $S^{n-1}(1)/\Gamma$ of $S^{n-1}(1)$.

Let $h_\Gamma$ denote the metric on $S^{n-1}(1)/\Gamma$ induced by the metric $h_+$, i.e. $(S^{n-1}/\Gamma, h_\Gamma) = S^{n-1}(1)/\Gamma$. Then we notice that the first eigenvalues $\lambda(L_{h_+})$ and $\lambda(L_{h_\Gamma})$ are equal.

Now we use the above remark and Proposition 2.12 to get the estimate

$$Y_{[\bar{g}]}^{cyl}(Z \times \mathbb{R}) \leq Y(S^n) \cdot \left(\frac{\text{Vol}(S^{n-1}(1)/\Gamma)}{\text{Vol}(S^{n-1}(1))}\right)^{\frac{2}{n}} = \frac{Y(S^n)}{|\Gamma|^\frac{2}{n}},$$

where $|\Gamma|$ denotes the order of $\Gamma$. If $|\Gamma| \geq 2$, then $Y_{[\bar{h}]}^{cyl}(Z \times \mathbb{R}) < Y(S^n)$, which contradicts the assumption. Therefore $|\Gamma| = 1$, and hence $(Z, h)$ is homothetic to $S^{n-1}(1)$. \Box

Remark. The above argument implies that $Y_{[\bar{h}]}^{cyl}(Z \times \mathbb{R}) \leq Y(S^n)/|\Gamma|^\frac{2}{n}$ for any spherical space form group $\Gamma$. Moreover, combining this inequality with $Y_{[\bar{h}]+dt^2]}^{cyl}(S^{n-1} \times \mathbb{R}) = Y(S^n)$, we then obtain that

$$Y_{[\bar{h}]+dt^2]}^{cyl}(S^{n-1} \times \mathbb{R}) = \frac{Y(S^n)}{|\Gamma|^\frac{2}{n}}.$$}

When $\dim(Z \times \mathbb{R}) = n \geq 6$, the following result combined with Proposition 6.6 completes the proof of Theorem 6.2.

**Proposition 6.6** Let $(Z \times \mathbb{R}, \bar{h} = h + dt^2)$ be a canonical cylindrical manifold of $\dim(Z \times \mathbb{R}) = n \geq 6$ with $\lambda(L_h) > 0$. Assume that the manifold $(Z \times \mathbb{R}, [\bar{h}])$ is not locally conformally flat. Then

$$Y_{[\bar{h}]}^{cyl}(Z \times \mathbb{R}) < Y(S^n) = Y_{[\bar{h}]+dt^2]}^{cyl}(S^{n-1} \times \mathbb{R}).$$

For the proof of Proposition 6.6 we first recall the following useful result (see [27] Theorem 5.1 and [17], [23]).

**Lemma 6.7** (Conformal normal coordinates) Let $(M, \bar{g})$ be a Riemannian manifold and $p$ a point in $M$. Then there exists a conformal metric $g \in [\bar{g}]$ on $M$ such that

$$d\sigma_g = dx := dx^1 \land \cdots \land dx^n,$$

i.e. $\det(g_{ij}(x)) = 1$ in a $g$-normal coordinate neighborhood. Here $x = (x^1, \ldots, x^n)$ denotes $g$-normal coordinates at $p$. 
Proof of Proposition 6.6. Since \((Z \times \mathbb{R}, \tilde{h} = h + dt^2)\) is not locally conformally flat, there exists a point \(p = (x_0, 0) \in Z \times \mathbb{R}\) such that \(|W_{\tilde{h}}(p)| \neq 0\). Let \(g \in [\tilde{h}]\) be a conformal metric on \(Z \times \mathbb{R}\) such that 
\[
d\sigma_g = dx \quad \text{on} \quad B_{\rho_0}(0) = \{|x| < \rho_0\},
\]
where \(x = (x^1, \ldots, x^n)\) are \(g\)-normal coordinates at \(p\). Similarly to Aubin’s argument \cite{27} (cf. \cite[Theorem B]{27}), we construct a family of test functions as follows. Let \(\{u_\varepsilon\}_{\varepsilon > 0}\) be the family of positive functions on \(\mathbb{R}^n\) given by 
\[
u_\varepsilon(x) = \left(\frac{\varepsilon}{\varepsilon^2 + |x|^2}\right) \frac{n-2}{2^n} \quad \text{for} \quad x \in \mathbb{R}^n,
\]
i.e. \(\{u_\varepsilon\}_{\varepsilon > 0}\) are the instantons on \((\mathbb{R}^n \cong S^n \setminus \{\text{point}\}, C_{\text{can}})\) centered at \(x = 0\). Let \(\rho > 0\) be a small constant with \(2\rho < \rho_0\), and \(\eta\) a smooth cut-off function of \(r = |x|\) which satisfies 
\[
\begin{align*}
&\eta(x) = 1 \quad \text{for} \quad |x| \leq \rho, \\
&\eta(x) = 0 \quad \text{for} \quad |x| \geq 2\rho, \\
&|\nabla \eta| \leq \frac{2}{\rho} \quad \text{for} \quad \rho \leq |x| \leq 2\rho.
\end{align*}
\]
Set \(\psi_\varepsilon = \eta \cdot u_\varepsilon\) on \(Z \times \mathbb{R}\) for \(\varepsilon > 0\). Then we have the following estimate (see \cite[Proof of Theorem B]{27} for details):
\[
\begin{align*}
Q_{(Z \times \mathbb{R}, g)}(\psi_\varepsilon) &\leq \\
\begin{cases}
Y(S^6) + \frac{1}{\|\psi_\varepsilon\|^2_L^2(Z \times \mathbb{R})} \left[-c_6 \cdot |W_g(p)|^2 \varepsilon^4 \log(1/\varepsilon) + O(\varepsilon^4)\right] &\text{if} \quad n = 6, \\
Y(S^n) + \frac{1}{\|\psi_\varepsilon\|^2_L(L^\infty_{\tilde{h}}(Z \times \mathbb{R}))} \left[-c_n \cdot |W_g(p)|^2 \varepsilon^4 + o(\varepsilon^4)\right] &\text{if} \quad n \geq 7,
\end{cases}
\end{align*}
\]
and \(0 < \|\psi_\varepsilon\|^2_L(L^\infty_{\tilde{h}}(Z \times \mathbb{R})) \leq K\) for any small \(\varepsilon > 0\). Here \(c_6, c_n\) and \(K\) denote positive constants depending only on \(n\). Recall that \(|W_g(p)| \neq 0\). Choosing \(\varepsilon > 0\) sufficiently small, we obtain that 
\[
Q_{(Z \times \mathbb{R}, g)}(\psi_\varepsilon) < Y(S^n).
\] (58)
Then it follows from (58) and Fact 2.2 that

\[
Y_{\hat{h}}(\mathbb{Z} \times \mathbb{R}) = \inf_{f \in C_\infty^c(\mathbb{Z} \times \mathbb{R})} Q(\mathbb{Z} \times \mathbb{R}, \hat{h})(f)
\]

\[
= \inf_{f \in C_\infty^c(\mathbb{Z} \times \mathbb{R})} Q(\mathbb{Z} \times \mathbb{R}, \bar{g})(f) \leq Q(\mathbb{Z} \times \mathbb{R}, g)(\psi_{\epsilon}) < Y(S^n).
\]

This completes the proof. \( \square \)

Now we consider the remaining case when \( n = 3, 4, 5 \) in Theorem 6.2. We start with the existence and uniqueness properties of minimal Green’s functions of the conformal Laplacians in dimensions \( n \geq 3 \).

Lemma 6.8 Let \( (\mathbb{Z} \times \mathbb{R}, \bar{h} = h + dt^2) \) be a canonical cylindrical manifold of \( \dim(\mathbb{Z} \times \mathbb{R}) = n \geq 3 \) with \( \lambda(L_h) > 0 \). Let \( p \in \mathbb{Z} \times \mathbb{R} \) be an arbitrary point. Then there exists a unique normalized minimal positive Green’s function \( \bar{G}_p \) (with pole at \( p \)) of the conformal Laplacian \( \bar{L}_h \) on \( \mathbb{Z} \times \mathbb{R} \). Namely, for each point \( p \in \mathbb{Z} \times \mathbb{R} \) the Green’s function \( \bar{G}_p \in C^\infty(\mathbb{Z} \times \mathbb{R}) \) satisfies the following properties:

(i) \( \bar{L}_h \bar{G}_p = \beta_n \cdot \delta_p \), where \( \beta_n = 4(n-1)\text{Vol}(S^{n-2}(1)) > 0 \), and \( \delta_p \) is the Dirac \( \delta \)-function at \( p \).

(ii) \( \bar{G}_p > 0 \) on \( \mathbb{Z} \times \mathbb{R} \setminus \{p\} \).

(iii) If \( \bar{G'}_p \in C^\infty(\mathbb{Z} \times \mathbb{R} \setminus \{p\}) \) is a normalized positive Green’s function (i.e. \( \bar{G'}_p \) satisfies the conditions (i) and (ii)), then \( \bar{G'}_p \geq \bar{G}_p \).

Proof. Let \( G_p \) be any normalized positive Green’s function with pole at \( p \) of \( L_h \). We notice that for \( u \in C_\infty^c(\mathbb{Z} \times \mathbb{R}) \) the function \( u(p) \bar{h}^{-\frac{n-2}{2}} \cdot u^{-1} \cdot G_p \) is also such a normalized positive Green’s function of the conformal Laplacian \( L_{\bar{h}} \), with respect to the metric \( \bar{h} := u^{\frac{n-2}{2}} \cdot \hat{h} \). Using this observation combined with \( \lambda(L_{\bar{h}}) > 0 \), we assume that \( R_{\bar{h}} > 0 \) on \( \mathbb{Z} \) (as in Convention 5.1). We may also assume that \( p = (x_0, 0) \in \mathbb{Z} \times \mathbb{R} \). Then, since \( R_{\bar{h}} > 0 \), for each \( i \geq 1 \) there exists a unique positive Green’s function

\[
G_p^{(i)} \in C^\infty_\alpha(((\mathbb{Z} \times (-i, i)) \setminus \{p\}) \cap C^0((\mathbb{Z} \times [-i, i]) \setminus \{p\}) \text{ such that}
\]

\[
\left\{
\begin{array}{ll}
L_{\bar{h}}G_p^{(i)} = \beta_n \cdot \delta_p & \text{on } \mathbb{Z} \times (-i, i), \\
G_p^{(i)} = 0 & \text{on } \mathbb{Z} \times \{\pm i\}.
\end{array}
\right.
\]
For each \( G_p^{(i)} \) we use the normalization condition \( \mathbb{L}_h G_p^{(i)} = \beta_n \cdot \delta_p \) to obtain the following expansion in a fixed normal coordinate system \( x \) centered at the point \( p \):

\[
G_p^{(i)}(x) = |x|^{2-n} (1 + o(1)) \quad \text{as} \quad |x| \to 0.
\]

Then by the maximum principle, for any \( i \geq 1 \),

\[
G_p^{(i)} \leq G_p^{(i+1)} \leq G_p^{(i+2)} \leq \cdots \quad \text{on} \quad Z \times [-i, i].
\]

We set \( S^1 = [-1, 1]/\{-1, 1\} \). Let \( N \) denote the closed manifold \( Z \times S^1 \), and let \( g \) be the metric on \( N \) induced by \( \bar{h} = h + dt^2 \) via the Riemannian submersion \( \Phi : (Z \times \mathbb{R}, \bar{h}) \to (Z \times S^1, g) \). Then the condition \( \lambda(\mathbb{L}_h) > 0 \) implies that the first eigenvalue \( \lambda(\mathbb{L}_g) > 0 \). Hence there exists a unique normalized minimal positive Green’s function \( \bar{G}_p \) of \( \mathbb{L}_g \) on \( N \) with pole at \( \bar{p} = \Phi(p) \). The maximum principle also implies that

\[
G_p^{(i)} \leq \Phi^* \bar{G}_p \quad \text{on} \quad Z \times [-i, i]
\]

for any \( i \geq 1 \). Then by Harnack’s convergence theorem, there exists a normalized positive Green’s function \( \bar{G}_p \) with pole at \( p \) of \( \mathbb{L}_g \) on \( Z \times \mathbb{R} \) such that the sequence \( \{ \bar{G}_p^{(i)} \} \) converges uniformly to \( \bar{G}_p \) on each cylinder

\[
(Z \times [-j, j]) \setminus B_{1/j}(p; \bar{h}), \quad \text{where}
\]

\[
B_{1/j}(p; \bar{h}) = \{ x \in Z \times \mathbb{R} \mid \text{dist}_{\bar{h}}(p, x) < 1/j \}.
\]

Now, by the construction of the Green’s function \( \bar{G}_p \) and the maximum principle, \( \bar{G}_p \) is a normalized minimal positive Green’s function of \( \mathbb{L}_h \) with pole at \( p \). The uniqueness for such \( \bar{G}_p \) follows directly from the minimality condition (iii). \( \square \)

**Remark.** Let \( \iota : Z \times \mathbb{R} \to Z \times \mathbb{R} \) denote the involution defined as \( \iota(x, t) = (x, -t) \). Let \( o = (x_0, 0) \in Z \times \mathbb{R} \) be a point. Since \( \iota^* \bar{h} = \bar{h} \), the uniqueness of a normalized minimal positive Green’s function implies that \( \bar{G}_o \) is \( \iota \)-invariant.

Similarly to Theorem 5.13, we obtain the following.

**Lemma 6.9** There exist positive constants \( a \leq b, C_1 \leq C_2 \) such that

\[
C_1 \cdot e^{-b|t|} \leq \bar{G}_o(x, t) \leq C_2 \cdot e^{-a|t|} \quad \text{on} \quad Z \times ((-\infty, -1] \cup [1, \infty)).
\]
To proceed further, we set \( S^1_\ell = [-2\ell, 2\ell]/\{-2\ell\} \) for \( \ell = 1, 2, \ldots \). Let \( N_\ell \) denote the closed manifold \( Z \times S^1_\ell \), and let \( g_\ell \) be the metric on \( N_\ell \) defined via the Riemannian submersion \( \Phi_\ell : (Z \times \mathbb{R}, \bar{h}) \to (N_\ell, g_\ell) \). Let \( G_\ell \) be the normalized minimal positive Green’s function of \( L_{\bar{h}} \) with pole at \( \bar{o} = \Phi_\ell(o) \in N_\ell \). Then we have the following fact (see [40, Chapter 6, Proposition 2.4] for the proof):

**Lemma 6.10** Let \( \Gamma_\ell \subset \pi_1(N_\ell) \) denote the deck transformation group of the covering \( \Phi_\ell : Z \times \mathbb{R} \to N_\ell \). Then

\[
\Phi_\ell^* G_\ell = \sum_{\gamma \in \Gamma_\ell} \gamma_o \bar{G}_o \quad \text{on} \quad (Z \times \mathbb{R}) \setminus \{o\}.
\] (60)

From now on, we assume that \( \dim(Z \times \mathbb{R}) = n = 3, 4, 5 \). Let \( o = (x_0, 0) \in Z \times \mathbb{R} \) be a fixed point and \( \bar{G} = \bar{G}_o \) the normalized minimal positive Green’s function given in Lemma 6.8. Below we suppress the dependence of \( \bar{G} = \bar{G}_o \) on the point \( o \).

We set \( \hat{X} = (Z \times \mathbb{R}) \setminus \{o\} \) with the metric \( \hat{h} = \bar{G}^{\frac{2}{n-2}} \cdot h \). Then the condition \( L_{\hat{h}} \bar{G} = \beta_o \delta_o \) implies that

\[
R_{\hat{h}} \equiv 0 \quad \text{on} \quad \hat{X}.
\] (61)

Let \( x = (x^1, \ldots, x^n) \) be conformal normal coordinates at \( o = (x_0, 0) \in Z \times \mathbb{R} \) satisfying

\[
d\sigma(x) = dx.
\]

Then we use Lemma 6.4 and Theorem 6.5 in [27] to obtain the following result. First, we need some notations. We denote by \( \nabla \) for the covariant derivative with respect to the metric \( \hat{h} \). Then we write \( f = O(r^k) \) to mean that \( f = O(r^k), \nabla f = O(r^{k-1}) \) and \( \nabla^2 f = O(r^{k-2}) \) (following the notations from [27]).

**Lemma 6.11** In the conformal normal coordinates \( x = (x^i) \), the Green’s function \( \bar{G} \) has the following expansion:

\[
\bar{G}(x) = r^{2-n} + A + O''(r), \quad A \equiv \text{const.}
\] (62)

Furthermore, in the inverted conformal normal coordinates \( y = (y^i = r^{-2} x^i) \), the metric \( \hat{h} \) has the following expansion:

\[
\begin{align*}
\hat{h}_{ij}(y) &= \gamma(y)^{\frac{n-2}{n-4}} \left( \delta_{ij} + O'((\rho)^{-2}) \right), \\
\gamma(y) &= 1 + A \rho^{2-n} + O''(\rho^{1-n}),
\end{align*}
\] (63)

where \( \rho = |y| = r^{-1} \).
A similar argument to the proofs of [34, Theorem 1] and [40, Chapter 5, Theorem 4.1] implies the following.

**Proposition 6.12** If $A > 0$ in (62), then $Y_{\hat{h}}^{cyt}(Z \times \mathbb{R}) < Y(S^n)$.

Now we notice that (61) and (63) imply that $(\hat{X}, \hat{h})$ is a scalar-flat, asymptotically flat manifold of order 1 (if $n = 3$), and of order 2 (if $n = 4, 5$). We emphasize that $(\hat{X}, \hat{h})$ has two almost conical singularities and $\hat{h}$ is $\iota$-invariant, see Fig. 12. We notice that in this case the mass $m = m(\hat{h}, (y^i))$

![Figure 12: The manifold $(\hat{X}, \hat{h})$.](image)

is well-defined in the standard way:

$$m(\hat{h}, (y^i)) = \lim_{R \to \infty} \frac{1}{\text{Vol}(S^{n-1})} \int_{\{|y|=R\}} \sum_{i,j=1}^{n} \left( \partial_i \hat{h}_{ij} - \partial_j \hat{h}_{ii} \right) \partial_j \mu.$$  

We emphasize that the mass $m(\hat{h}) = m(\hat{h}, (y^i))$ depends only on the metric $\hat{h}$, see [14, Theorem 4.2]. However, for any large constant $L > 0$, the region $(X \times [-L, L]) \cap \hat{X}$ is not convex with respect to the metric $\hat{h}$. Hence we cannot apply directly the technique (developed in [37]) for asymptotically flat manifolds with many ends to proving that the mass $m \geq 0$.

We also remark that if $n = 3$ or $\hat{h}$ is locally conformally flat near $o$, then $\hat{h}$ has the following expansion:

$$\hat{h}_{ij}(y) = \left( 1 + \frac{4}{n-2} A \rho^{2-n} \right) \delta_{ij} + O'(\rho^{1-n}).$$

Then the mass $m(\hat{h}) = m(\hat{h}, (y^i))$ is given as

$$m(\hat{h}) = 4(n-1)A.$$  

(64)

Even if $\hat{h}$ is not locally conformally flat on any open set inside of $Z \times \mathbb{R}$, the equality (64) still holds (if $3 \leq n \leq 5$), see [27, Lemma 9.7].
The following result is a version of the positive mass theorem for the asymptotically flat manifold \((\hat{X}, \hat{h})\) with almost conical singularities.

**Theorem 6.13** (First part of the positive mass theorem) Let \((\hat{X}, \hat{h})\) be the asymptotically flat manifold with two almost conical singularities and \(\dim \hat{X} = n = 3, 4, 5\), as above. Then

\[
A \geq 0. \tag{65}
\]

**Proof.** Let \(\hat{N}_\ell = N_\ell \setminus \{\hat{o}\}\), where \(\hat{o} = \Phi_\ell(o)\), with the metric \(\hat{g}_\ell = G_{\ell}^{\frac{1}{n-2}} \cdot g_\ell\) for \(\ell = 1, 2, \ldots\). Since \(\Phi_\ell\) is a local isometry near \(o\) with respect to \(\bar{h}\) and \(g_\ell\), we can use the same conformal normal coordinates \(x = (x^i)\) near \(\hat{o} \in \hat{N}_\ell\). Then each Green’s function \(G_\ell\) has the following expansion:

\[
G_\ell(x) = r^{2-n} + A_\ell + O''(r), \quad A_\ell \equiv \text{const}. \tag{66}
\]

The mass of the asymptotically flat manifold \((\hat{N}_\ell, \hat{g}_\ell)\) is also given by

\[
m(\hat{g}_\ell) = 4(n-1)A_\ell.
\]

It then follows from the positive mass theorem \([40, 37, 38]\) (cf. \([14]\) and \([27]\)) that

\[
A_\ell \geq 0 \quad \text{for any } \ell \geq 1. \tag{67}
\]

Under the identification \(\Phi_\ell|_{Z \times [-2^\ell, 2^\ell]}\) and using that \(\iota^*G_\ell = G_\ell\), we may regard \(G_\ell\) as a function on \((Z \times [0, 2]) \setminus \{o\}\). Then the maximum principle combined with (60) implies that

\[
G_\ell \geq G_{\ell+1} \geq \cdots \geq \bar{G} > 0 \quad \text{on } (Z \times [-2^\ell, 2^\ell]) \setminus \{o\}. \tag{68}
\]

Hence \(\{G_\ell\}\) converges uniformly to \(\bar{G}\) on each cylinder \((Z \times [-i, i]) \setminus B_o(1/i; \bar{h})\). The estimate \([50]\) and the properties \([60], [58]\) give that

\[
\bar{G}(x) \leq G_\ell(x) \leq \bar{G}(x) + \frac{2C_2 \cdot e^{-a \cdot 2^\ell}}{1-e^{-a \cdot 2^\ell}}
\]

for \(\ell \gg 1\). We then obtain that

\[
|\bar{G}(x) - G_\ell(x)| \leq C \cdot e^{-a \cdot 2^\ell} \quad \text{on } U_o \setminus \{o\} \tag{69}
\]

for \(\ell \gg 1\). Here \(U_o\) denotes a coordinate neighborhood centered at \(o\) and \(C > 0\) is a constant independent of \(\ell\) and \(x\). From \([62]\) and \([68]\), the function \(\bar{G}(x) - G_\ell(x)\) is smooth on \(U_o\). Thus by letting \(|x| \to 0\) in \([60]\), we obtain

\[
|A - A_\ell| \leq C \cdot e^{-a \cdot 2^\ell} \quad \text{for } \ell \gg 1,
\]
and hence by letting $\ell \to \infty$, we prove that
\[ A = \lim_{\ell \to \infty} A_\ell. \] (70)
Therefore, it follows from (67) and (70) that $A \geq 0$. \qed

Proposition 6.14 If the metric $\hat{h} = \bar{G}^{-\frac{1}{n-2}} \cdot \bar{h}$ is not Ricci-flat on $\hat{X}$, then $A > 0$, where $A$ is the same constant as in (62).

Proof. For any symmetric 2-tensor $S = (S_{ij})$ with compact support in $\hat{X} = (Z \times \mathbb{R}) \setminus \{o\}$, we define the family of smooth metrics
\[ \bar{h}_s = \bar{h} + s \cdot \bar{G}^{-\frac{1}{n-2}} \cdot S \] on $Z \times \mathbb{R}$ for small $s \geq 0$. Notice that $\bar{h}_s$ is no longer a product metric on $Z \times \mathbb{R}$ for $s > 0$. However, the arguments in Lemmas 6.8, 6.9, 6.10, 6.11 and Theorem 6.13 are still valid since $\text{Supp}(S)$ is compact in $\hat{X}$. Hence there exists a small constant $\delta_0 > 0$ such that the following holds:

For any $s$ with $0 \leq s \leq \delta_0$ there exists a unique normalized minimal positive Green’s function $\bar{G}^s$ with pole at $o$ of $L_{\bar{h}_s}$ on $\hat{X}$ such that $\bar{G}^s$ has the expansion (as in (62)):
\[ \bar{G}^s(x) = |x|^{2-n} + A_s + O''(|x|), \quad A_s \equiv \text{const} \geq 0. \]

Then we can apply the argument [34, Lemma 3] to proving that $A > 0$. \qed

From Propositions 6.12, 6.14 and Theorem 6.13 in order to complete the proof of Theorem 6.13 it is enough to show the following.

Proposition 6.15 Under the same assumptions as in Theorem 6.13 we also assume that $\hat{h}$ is Ricci-flat
\[ \text{Ric}_{\hat{h}} = 0 \quad \text{on} \quad \hat{X} = (Z \times \mathbb{R}) \setminus \{o\}. \] (71)
Moreover, assume that $A = 0$ in (64) when $n = 3$. Then $(\hat{X}, \hat{h})$ is isometric to $\mathbb{R}^n \setminus \{2 \text{ points}\}$ with the Euclidean metric. In particular, $(Z, \bar{h})$ is homothetic to the sphere $S^{n-1}(1)$.

As a complementary assertion to Theorem 6.13 we also obtain the following.

Theorem 6.16 (Second part of the positive mass theorem) Let $(\hat{X}, \hat{h})$ be an asymptotically flat manifold with two almost conical singularities and $\dim \hat{X} = n = 3, 4, 5$, as above. Then if $A = 0$, then $(\hat{X}, \hat{h})$ is isometric to $\mathbb{R}^n \setminus \{2 \text{ points}\}$ with the Euclidean metric.
Proof of Theorem 6.16. The condition $A = 0$ combined with Theorem 6.13 and Proposition 6.14 implies that $\hat{h}$ is Ricci-flat on $\hat{X}$. It then follows from Proposition 6.15 that $(\hat{X}, \hat{h})$ is isometric to $\mathbb{R}^n \setminus \{2 \text{ points}\}$. \qed

Proof of Proposition 6.15. Recall that, in the inverted conformal normal coordinates $y = (y^1, \ldots, y^n)$, the metric $\hat{h}$ has the following expansion ($n = 3$ with $A = 0$ or $n = 4, 5$):

$$\hat{h}_{ij}(y) = \delta_{ij} + O'(\rho^{-2}) \quad \text{as} \quad \rho = |y| \to \infty.$$ 

We choose a sufficiently large $L_0 > 0$ and fix it. Set $\hat{X}_0 = \hat{X} \cap (\mathbb{Z} \times [-L_0, L_0])$.

Let $L^{k,p}_\delta(\hat{X}_0)$ denote the weighted Sobolev space with weight $\delta \in \mathbb{R}$ on $(\hat{X}_0, \hat{h}; (y^i))$, $k = 0, 1, 2, 1 \leq p \leq \infty$, defined in [14]. We need the following result to complete the proof of Proposition 6.15.

Lemma 6.17 (Harmonic coordinates near infinity) Let $\varepsilon (0 < \varepsilon < 1)$, $q > n$ and $\tau (\frac{n}{2} \leq \tau < 2)$ be positive constants satisfying $q > \frac{n}{2} + \frac{\varepsilon}{2}$. Then there exist smooth functions $z^i \in C^\infty(\hat{X}_0)$ for $i = 1, \ldots, n$ such that

$$\begin{cases}
\Delta_{\hat{h}} z^i = 0 & \text{on} \quad \hat{X}_0,
\frac{\partial z^i}{\partial \nu} = 0 & \text{on} \quad \partial \hat{X}_0,
y^i - z^i \in L^{2,q}_\tau(\hat{X}_0),
y^i - z^i = O''(\rho^{-(1-\varepsilon)}) & \text{as} \quad \rho = |y| \to \infty.
\end{cases}$$

(72)

Here $\nu$ is the outward unit vector field normal to the boundary $\partial \hat{X}_0$ with respect to the metric $\hat{h}$. In particular, $z = (z^i)$ are harmonic coordinates near infinity.

Proof of Lemma 6.17. We extend each function $y^i$ to a smooth function on $\hat{X}_0$ satisfying $\frac{\partial y^i}{\partial \nu} = 0$ on $\partial \hat{X}_0$. We notice that

$$\Delta_{\hat{h}} y^i = \hat{h}^{jk} \hat{\Gamma}^i_{jk} = O'(\rho^{-3}) \quad \text{on} \quad \hat{X}_0.$$ 

Here we write $f = O'(\rho^k)$ if $f = O(\rho^k)$ and $\hat{\nabla} f = O(\rho^{k-1})$.

Modifying the $L^{\frac{2n}{n-2}}$-theory in [37] Lemmas 3.1, 3.2] and using the $L^q$-estimates in the linear elliptic theory, we can show the following:

Claim 6.18 There exist unique smooth functions $u^i \in L^q(\hat{X}_0)$ such that

$$\begin{cases}
\Delta_{\hat{h}} u^i = \Delta_{\hat{h}} y^i = O'(\rho^{-3}) & \text{on} \quad \hat{X}_0,
\frac{\partial u^i}{\partial \nu} = 0 & \text{on} \quad \partial \hat{X}_0,
u^i = O''(\rho^{-(1-\varepsilon)}) & \text{as} \quad \rho \to \infty.
\end{cases}$$
Then the weighted Sobolev estimate \[14\] Proposition 1.6] and the scale-broken estimate \[14\] Theorem 1.10] imply that \( u^i \in L^{2,q}_{-\tau}(\hat{X}_0) \). Set \( z^i = y^i - u^i \in C^\infty(\hat{X}_0) \). Then the functions \( z = (z^1, \ldots, z^n) \) satisfy the properties (72). This completes the proof of Lemma 6.17.

We return to the proof of Proposition 6.15. Let \( z = (z^1, \ldots, z^n) \) be the harmonic coordinates near infinity of \( \hat{X}_0 \) obtained in Lemma 6.17. Then the Ricci-flatness of \( \hat{h} \) combined with \[14\] Proposition 3.3] implies that, in the harmonic coordinates \( z = (z^i) \), the metric \( \hat{h}(z) = (\hat{h}^{ij}(z)) \) satisfies

\[
\hat{h}^{ij}(z) - \delta^{ij} \in L^{2,q}_{-\eta}(\hat{X}_0), \quad i, j = 1, \ldots, n \quad (73)
\]

for any \( \eta > n - 2 \). It then follows from \[14\] and \[14\] Theorem 4.3] that the mass \( m(\hat{h}) = m(\hat{h}, (z^i)) = 0 \).

Now we can apply the Bochner technique to complete the proof. The harmonicity of \( (z^i) \) implies that \( \{dz^i\} \) are harmonic 1-forms on \((\hat{X}_0, \hat{h})\). From the Bochner formula for 1-forms \( (z^i) \) and the condition that \( \frac{\partial z^i}{\partial \nu} = 0 \) on \( \partial \hat{X}_0 \), we obtain that

\[
\sum_{i=1}^{n} \int_{\hat{X}_0 \cap \{|z|<R\}} \left[ |\hat{\nabla} dz^i|^2 + \text{Ric}_{\hat{h}}(dz^i, dz^i) \right] d\sigma_{\hat{h}} = \\
\sum_{i,j=1}^{n} \int_{\{|z|=R\}} \hat{h}^{jk}(dz^i, \hat{\nabla}_j dz^i) \partial_k \downarrow d\sigma_{\hat{h}} \quad (74)
\]

for any \( R \gg 1 \). Letting \( R \to \infty \) in (74] and using the Ricci-flatness \( \text{Ric}_{\hat{h}} \equiv 0 \) on \( \hat{X} \), then we have

\[
m(\hat{h}) = \frac{1}{\text{Vol}(S^{n-1}(1))} \sum_{i=1}^{n} \int_{\hat{X}_0} |\hat{\nabla} dz^i|^2 d\sigma_{\hat{h}}, \quad (75)
\]

see \[14\] Theorem 4.4] or \[21\] Proposition 10.2]. Now we use that \( m(\hat{h}) = 0 \) in (75]. Then we obtain that the 1-forms \( \{dz^i\} \) are parallel on \( \hat{X}_0 \) with respect to \( \hat{h} \). Since the coframe \( \{dz^i\} \) is orthonormal at infinity, then \( \{dz^i\} \) is a parallel orthonormal coframe everywhere on \( \hat{X}_0 \). This implies that the map \( z = (z^1, \ldots, z^n) : (\hat{X}_0, \hat{h}) \to \mathbb{R}^n \) is a local isometry, and hence \( \hat{h} \) is locally conformally flat on \( \hat{X} \). From Lemma 6.14, \( (Z, h) \) is homothetic to a smooth quotient \( S^{n-1}(1)/\Gamma \) of \( S^{n-1}(1) \).

**Claim 6.19** The manifold \( (Z, h) \) is homothetic to \( S^{n-1}(1) \).
Proof of Claim 6.19  Let \( \iota : Z \times \mathbb{R} \to Z \times \mathbb{R} \) be the involution given by \( \iota(x,t) = (x,-t) \). Recall that \( G \) is \( \iota \)-invariant, and hence the metric \( \bar{h} = G^{\frac{1}{4}} \cdot \tilde{h} \) is an \( \iota \)-invariant metric. This implies that \( (Z \setminus \{x_0\}) \times \{0\} \) is a totally geodesic submanifold of \( (\tilde{X}, \tilde{h}) \). Using this fact and that \( z \) is a local isometry, we conclude that the restriction \( z \) to \( (Z \setminus \{x_0\}) \times \{0\} \) is a global isometry onto a hyperplane in \( \mathbb{R}^n \). Hence \( (Z,h) \) is homothetic to \( S^{n-1}(1) \).

From Claim 6.19, the manifold \( (\tilde{X}, \tilde{h}), \tilde{h} = G^{\frac{1}{4}} \cdot \bar{h}, \) is conformally equivalent to \( \mathbb{R}^n \setminus \{2 \text{ points}\} \) with the Euclidean metric \( g_0 \). Under the identification \( \tilde{X} \sim = \mathbb{R}^n \setminus \{2 \text{ points}\} \), the metric \( g_0 \) is represented as \( g_0 = G^{\frac{1}{4}} \cdot \bar{h} \) on \( \tilde{X} \). Here \( G \) is a normalized positive Green’s function with pole at \( o \) of \( \mathbb{L}_{\tilde{h}} \), which satisfies \( G(x,t) = o(1) \) as \( |t| \to \infty \) for \( (x,t) \in Z \times \mathbb{R} \). Using the maximum principle combined with the minimality of \( \bar{G} \) and the normalization for \( G \), we obtain that \( \bar{G} = G \) on \( \tilde{X} \). Hence \( \tilde{h} = g_0 \) on \( \tilde{X} \). This completes the proof of Proposition 6.15.

7 Appendix

Let \( (X, \tilde{g}) \) be a cylindrical manifold of \( \dim X = n \geq 3 \) with tame ends \( Z \times [0, \infty) \) and \( \partial_\infty \tilde{g} = h \in \text{Riem}(Z) \). Here we study some properties of the conformal Laplacian \( \mathbb{L}_{\tilde{g}} \), and its relationship to the constant \( Y_{[\tilde{g}]} \) and the sign of the scalar curvature \( R_{\tilde{g}} \) of a conformal metric \( \tilde{g} \in [\tilde{g}] \). In the case of a closed conformal manifold \( (M, C) \), the signs of the first eigenvalues of \( \mathbb{L}_g \) and \( \mathbb{L}_{\tilde{g}} \) for \( g, \tilde{g} \in C \) are identical. Since \( (X, \tilde{g}) \) is not compact, \( \mathbb{L}_{\tilde{g}} \) has no first eigenvalue in general. Instead, we consider the bottom of the spectrum of \( \mathbb{L}_{\tilde{g}} \), defined as

\[
\lambda(\mathbb{L}_{\tilde{g}}) := \inf_{f \in L^1_{\tilde{g}}(X), \ f \neq 0} \frac{E_{(X, \tilde{g})}(f)}{\|f\|_{L^2_{\tilde{g}}}},
\]

where

\[
E_{(X, \tilde{g})}(f) = \int_X \left[ \alpha_n |df|^2 + R_{\tilde{g}} f^2 \right] d\sigma_{\tilde{g}}.
\]

It is easy to show that \( \lambda(\mathbb{L}_{\tilde{g}}) > -\infty \) since \( \tilde{g} = h + dt^2 \) on \( Z \times [1, \infty) \). Indeed, one has the following estimate

\[
\lambda(\mathbb{L}_{\tilde{g}}) \geq \min_X R_{\tilde{g}} = \min \left\{ \min_{\lambda(1)} R_{\tilde{g}}, \min_Z R_h \right\} > -\infty.
\]

Similarly to the case of closed manifolds, the sign of \( \lambda(\mathbb{L}_{\tilde{g}}) \) is uniquely determined by the conformal class \([\tilde{g}]\) provided that \( \tilde{g} \) is a cylindrical metric.
Proof. The proof of the following statement is straightforward.

Fact 7.1 Let $\tilde{g} = \varphi^{\frac{-4}{n-2}} \cdot \check{g}$ (here $\check{g}$ and $\tilde{g}$ are not necessarily cylindrical), where $\varphi \in C^\infty_0(X)$. Then $L_{\tilde{g}} = \varphi^{-\frac{4}{n-2}} \cdot L_{\check{g}} \circ \varphi$, and $E_{(X,\tilde{g})}(\varphi f) = E_{(X,\check{g})}(f)$ for any function $f \in C^\infty_c(X)$.

Clearly we have $d\sigma_{\check{g}} = \varphi^{\frac{n-2}{2}} d\sigma_{\tilde{g}}$. This implies
\[
\left(\inf_X \varphi\right)^{\frac{n-2}{2}} \|\varphi f\|_{L_{\tilde{g}}}^2 \leq \|f\|_{L_{\check{g}}}^2 \leq \left(\sup_X \varphi\right)^{\frac{n-2}{2}} \|\varphi f\|_{L_{\tilde{g}}}^2
\]
for any $f \in C^\infty_c(X)$. We proved in Proposition 2.1 that for cylindrical metrics $\check{g}$ and $\tilde{g}$ with $\tilde{g} = \varphi^{\frac{-4}{n-2}} \cdot \check{g}$ there exists a constant $K \geq 1$ such that $0 < K^{-1} \leq \varphi \leq K < \infty$ on $X$. This implies the estimate
\[
K^{-\frac{n-2}{2}} \cdot \frac{E_{(X,\tilde{g})}(\varphi f)}{\|\varphi f\|_{L_{\tilde{g}}}^2} \leq \frac{E_{(X,\check{g})}(f)}{\|f\|_{L_{\check{g}}}^2} \leq K^{\frac{n-2}{2}} \cdot \frac{E_{(X,\tilde{g})}(\varphi f)}{\|\varphi f\|_{L_{\tilde{g}}}^2}
\]
for any $f \in C^\infty_c(X)$. It then follows from (76) that
\[
K^{-\frac{n-2}{2}} \lambda(L_{\tilde{g}}) \leq \lambda(L_{\check{g}}) \leq K^{\frac{n-2}{2}} \lambda(L_{\tilde{g}}).
\]
This completes the proof. \qed

Recall that $Y^{cyt}_c(X) = -\infty$ when $\lambda(L_0) < 0$. Hence the finiteness of $\lambda(L_{\check{g}})$ does not imply that of the cylindrical Yamabe constant $Y^{cyt}_c(X)$. However, the signs of $\lambda(L_{\tilde{g}})$ and $Y^{cyt}_c(X)$ are still related as follows.

Proposition 7.2 Let $\tilde{g}$ be a cylindrical metric on $X$. Then

(i) $\lambda(L_{\tilde{g}}) \geq 0$ if and only if $Y^{cyt}_c(X) \geq 0$,

(ii) $\lambda(L_{\tilde{g}}) < 0$ if and only if $Y^{cyt}_c(X) < 0$.

Proof. It is enough to prove only (ii). We first note that
\[
Q_{(X,\tilde{g})}(f) = \frac{E_{(X,\tilde{g})}(f)}{\|f\|_{L_{\tilde{g}}}^2} = \frac{E_{(X,\check{g})}(f)}{\|f\|_{L_{\check{g}}}^2} \cdot \frac{\|f\|_{L_{\check{g}}}^2}{\|f\|_{L_{\tilde{g}}}^2} \leq\frac{\|f\|_{L_{\check{g}}}^2}{\|f\|_{L_{\tilde{g}}}^2}
\]
for any $f \in C^\infty_c(X)$ with $f \neq 0$. This implies that $\lambda(L_{\tilde{g}}) < 0$ if and only if $Y^{cyt}_c(X) < 0$. \qed
It follows from (i) that \( \lambda(\mathbb{L}_g) \geq 0 \) implies \( \lambda(\mathcal{L}_h) \geq 0 \) for any cylindrical metric \( \hat{g} \) with \( \partial_\infty \hat{g} = h \).

For a closed Riemannian manifold \((M, g)\), the following result for the first eigenvalue \( \lambda(\mathbb{L}_g) \) of \( \mathbb{L}_g \) is well-known (cf. [S], [24]):

There exists a conformal metric \( \hat{g} \in [g] \) such that the scalar curvature \( R_{\hat{g}} \) satisfies \( R_{\hat{g}} > 0, R_{\hat{g}} < 0, \) or \( R_{\hat{g}} \equiv 0 \) on \( X \). Moreover, the sign of \( R_{\hat{g}} \) is identical with the sign of \( \lambda(\mathbb{L}_g) \).

Under the assumption \( \lambda(\mathcal{L}_h) > 0 \) for \((X, \hat{g})\), a similar relationship holds.

**Proposition 7.3** Let \((X, \hat{g})\) be a cylindrical manifold of \( \dim X = n \geq 3 \) with tame ends \( Z \times [0, \infty) \) and \( h = \partial_\infty \hat{g} \in \text{Riem}(Z) \) a metric with \( \lambda(\mathcal{L}_h) > 0 \). Then there exists a conformal metric \( \hat{g} = v^{\frac{2}{n-2}} \cdot \hat{g} \) with \( v \in C^\infty_+(X) \cap L^1_2(X) \) such that the scalar curvature \( R_{\hat{g}} \) satisfies \( R_{\hat{g}} > 0, R_{\hat{g}} < 0, \) or \( R_{\hat{g}} \equiv 0 \) on \( X \). Moreover, the sign of \( R_{\hat{g}} \) is identical with the sign of \( \lambda(\mathbb{L}_g) \).

**Proof.** By the conformal-rescaling argument below, we may assume that \( \lambda(\mathbb{L}_\hat{g}) < \lambda(\mathcal{L}_h) \). Indeed, if \( \lambda(\mathbb{L}_\hat{g}) \geq \lambda(\mathcal{L}_h) \), it is enough to change \( \hat{g} \) to an appropriate conformal metric \( \hat{g}_\varphi = e^{2\varphi} \cdot \hat{g} \) with

\[
\varphi = \begin{cases} 
  k \equiv \text{const.} > 0 & \text{on } W = X \setminus (Z \times (0, \infty)), \\
  0 & \text{on } Z \times [1, \infty). 
\end{cases}
\]

In particular, \( \mathcal{L}_\hat{g}_\varphi = e^{-2k} \cdot \mathcal{L}_\hat{g} \) on \( W \) and \( \hat{g}_\varphi = \hat{g} \) on \( Z \times [1, \infty) \). By using this combined with \( \lambda(\mathcal{L}_h) > 0 \), the Dirichlet first eigenvalue \( \lambda(\mathbb{L}_\hat{g}_\varphi; W) \) on \( W \) satisfies \( \lambda(\mathbb{L}_\hat{g}_\varphi; W) < \lambda(\mathcal{L}_h) \) for sufficiently large \( k > 0 \). Hence by the domain monotonicity of the Dirichlet eigenvalues, \( \lambda(\mathbb{L}_\hat{g}_\varphi) \leq \lambda(\mathbb{L}_\hat{g}_\varphi; W) < \lambda(\mathcal{L}_h) \). Using this and Proposition 7.3, we may assume that \( \hat{g} \) itself satisfies \( \lambda(\mathbb{L}_\hat{g}) < \lambda(\mathcal{L}_h) \).

Then the standard argument implies the existence of a positive minimizer \( v \in C^\infty_+(X) \cap L^1_2(X) \) with \( \lambda(\mathbb{L}_\hat{g}) = E_{(X, \hat{g})}(v)/|v|^2_{L^2_\hat{g}} \). Hence \( v \) satisfies

\[
-\alpha_n \Delta_{\hat{g}} v + R_{\hat{g}} v = \lambda(\mathbb{L}_\hat{g}) v \quad \text{on } X,
\]

and thus \( R_{\hat{g}} = \lambda(\mathbb{L}_\hat{g}) v^{-\frac{n-2}{n-2}} \) for \( \hat{g} = v^{\frac{2}{n-2}} \cdot \hat{g} \). This completes the proof. \( \square \)

From Theorem 7.2 and Propositions 7.2, 7.3 we obtain the following.

**Corollary 7.4** Under the same assumptions as in Proposition 7.3, then

(i) \( \lambda(\mathbb{L}_\hat{g}) > 0 \) if and only if \( Y^{cy}_\phi(X) > 0 \),
(ii) \( \lambda(\mathcal{L}_{\bar{g}}) = 0 \) if and only if \( Y^{cyle}_{\bar{g}}(X) = 0 \).

(iii) \( \lambda(\mathcal{L}_{\bar{g}}) < 0 \) if and only if \( Y^{cyle}_{\bar{g}}(X) < 0 \).

Proof. From Proposition 7.2, it is enough to prove that \( \lambda(\mathcal{L}_{\bar{g}}) = 0 \) if and only if \( Y^{cyle}_{\bar{g}}(X) = 0 \).

First we assume that \( \lambda(\mathcal{L}_{\bar{g}}) = 0 \). From the argument in the proof of Proposition 7.3, there exists a non-zero function \( v \in C^\infty(X) \cap L^2(\bar{g}) \) such that \( E(X, \bar{g})(v)/||v||^2_{L^2_{\bar{g}}} = \lambda(\mathcal{L}_{\bar{g}}) = 0 \). Using this in (77), we then obtain

\[
Y^{cyle}_{\bar{g}}(X) \leq Q(X, \bar{g})(v) = 0.
\]

If \( Y^{cyle}_{\bar{g}}(X) < 0 \), then \( \lambda(\mathcal{L}_{\bar{g}}) < 0 \) by Proposition 7.2 (ii). This is a contradiction. Hence, \( Y^{cyle}_{\bar{g}}(X) = 0 \).

Next we assume that \( Y^{cyle}_{\bar{g}}(X) = 0 \). Note that \( Y^{cyle}_{\bar{g}}(X) = 0 < \lambda(\mathcal{L}_h) \).

From Theorem 5.2, there exists a Yamabe minimizer \( u \in C^\infty(X) \cap L^2_{\bar{g}}(X) \) such that \( Q(X, \bar{g})(u) = Y^{cyle}_{\bar{g}}(X) = 0 \). Using this also in (77), we then obtain

\[
\lambda(\mathcal{L}_{\bar{g}}) \leq \frac{E(X, \bar{g})(u)}{||u||^2_{L^2_{\bar{g}}}} = 0.
\]

If \( \lambda(\mathcal{L}_{\bar{g}}) < 0 \), then \( Y^{cyle}_{\bar{g}}(X) < 0 \) by Proposition 7.2 (ii). This is also a contradiction. Hence, \( \lambda(\mathcal{L}_{\bar{g}}) = 0 \). This completes the proof. \( \square \)

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