GORENSTEIN HOMOLOGICAL PROPERTIES FROM COHERENT RINGS TO ARBITRARY RINGS

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Abstract. In this note, we mainly extend some Gorenstein homological properties from coherent rings to arbitrary rings by introducing the notions of Gorenstein weak injective and projective modules.

Contents

• Gorenstein weak injective modules and dimension
• Gorenstein weak projective modules and dimension
• Derived functors with respect to Gorenstein weak modules
• Tate derived functors with respect to Gorenstein weak modules

1. Introduction and Preliminaries

Throughout $R$ is an associative ring with identity and all modules are unitary. Unless stated otherwise, an $R$-module will be understood to be a left $R$-module. As usual, $pd_R(M)$ and $id_R(M)$ will denote the projective and injective dimensions of an $R$-module $M$, respectively. We also denote by $\mathcal{I}$ and $\mathcal{P}$ the classes of injective and projective $R$-modules, respectively. For unexplained concepts and notations, we refer the readers to [8, 24].

In 1970, to generalize the homological properties from Noetherian rings to coherent rings, Stenström introduced the notion of FP-injective modules in [26]. In this process, finitely generated modules should in general be replaced by finitely presented modules. Recall that an $R$-module $M$ is called FP-injective if $\text{Ext}^1_R(N, M) = 0$ for any finitely presented $R$-module $N$, and accordingly, the FP-injective dimension of $M$, denoted by $FP-id_R(M)$, is defined to be the smallest non-negative integer $n$ such that $\text{Ext}^{n+1}_R(N, M) = 0$ for any finitely presented $R$-module $N$. Recently, as extending work of Stenström’s viewpoint, Gao and Wang introduced the notion of weak injective (and weak flat) modules ([15]). In this process, finitely presented modules were replaced by super finitely presented modules (see [14] or the seventh paragraph in this section for the definition). It was shown that many results of a homological nature may be generalized from coherent rings to arbitrary rings (see [13, 15] for details).

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In 1965, Eilenberg and Moore first introduced the viewpoint of relative homological algebra in [9]. Since then the relative homological algebra, especially the Gorenstein homological algebra, got a rapid development. The main ideal of Gorenstein homological algebra is to replace projective and injective modules by Gorenstein projective and injective modules. Recall from [8] that an $R$-module $M$ is called Gorenstein injective if there exists an exact sequence of injective $R$-modules

$$\cdots \rightarrow I_1 \rightarrow I_0 \rightarrow I^0 \rightarrow I^1 \rightarrow \cdots$$

such that $M = \text{Coker}(I_1 \rightarrow I_0)$ and the functor $\text{Hom}_R(I, -)$ leaves this sequence exact whenever $I$ is an injective $R$-module. Dually, one may give the definition of Gorenstein projective $R$-module. Nowadays, it has been developed to an advanced level (e.g. [2, 4, 5, 7, 8, 17, 18, 19, 27, 31]). However, in the most results of Gorenstein homological algebra, the condition ‘noetherian’ is essential. In order to make the similar properties of Gorenstein homological algebra hold in a wider environment, Ding and his coauthors introduced the notions of Gorenstein FP-injective and strongly Gorenstein flat modules (see [6, 22] for details). Later on, Gillespie renamed Gorenstein FP-injective modules as Ding injective modules, and strongly Gorenstein flat modules as Ding projective modules ([12]). Recall from [12] that an $R$-module $M$ is called Ding injective (or Gorenstein FP-injective in the sense of [22]) if there exists an exact sequence of injective $R$-modules

$$\cdots \rightarrow I_1 \rightarrow I_0 \rightarrow I^0 \rightarrow I^1 \rightarrow \cdots$$

such that $M = \text{Coker}(I_1 \rightarrow I_0)$ and the functor $\text{Hom}_R(E, -)$ leaves this sequence exact whenever $E$ is an FP-injective $R$-module. It was shown that the Ding homological algebra over coherent rings possess many nice properties analogous to the Gorenstein homological algebra over Noetherian rings (see [6, 12, 22, 30] for details). So, in order to make the properties of relative homological algebra hold over any ring, it seems that we have to replace FP-injective modules by weak injective modules.

The main purpose of this paper is to generalize the Gorenstein homological properties from Noetherian or coherent rings to any rings. In Section 2, we introduced the notion of Gorenstein weak injective modules in terms of weak injective modules, and discuss some of the properties of these modules. We show that every Gorenstein weak injective $R$-module is either injective or has weak injective dimension $\infty$ (see Proposition 2.4); and an $R$-module $M$ is Gorenstein weak injective if and only if $M$ has an exact left $\mathcal{W}_I$-resolution and $\text{Ext}_R^i(W, M) = 0$ for any weak injective $R$-module $W$ and any $i \geq 1$ (see Proposition 2.9). We also show that the class of Gorenstein weak injective modules is injectively resolving (see Proposition 2.13). Then we introduce and study the Gorenstein weak injective dimension of modules. Moreover, the existence of Gorenstein weak injective preenvelope is given (see Proposition 2.17). Finally, we introduce the notion of $\mathcal{GW}_I$-copure exact sequence, and further give characterizations of Gorenstein weak injective modules in terms
of it (see Propositions 3.13 and 3.14). In Section 3, we introduce and investigate du-
ally Gorenstein weak projective modules in terms of weak flat modules. In Section 4, we
mainly investigate a class of relative right derived functors, denoted by \( \text{Ext}^n_{GW}(-, -) \), with
respect to Gorenstein weak modules, and study the balance of this functor (see Theorem
4.3). Moreover, we give some characterizations of Gorenstein weak injective and projec-
tive dimensions in terms of this derived functor (see Proposition 4.8). In Section 5, we
continue to investigate another derived functor, denoted by \( \hat{\text{Ext}}^n_{GW}(-, -) \). We show that
the functor \( \hat{\text{Ext}}^n_{GW}(-, -) \) actually measures the distance between the usual right derived
functor \( \text{Ext}^n_R(-, -) \) and the Gorenstein weak right derived functor \( \text{Ext}^n_{GW}(-, -) \). Finally,
we give the balance of this functor (see Theorem 5.15).

In the following, we recall some terminologies and preliminaries. For more details, we
refer the readers to [8, 13, 15].

**Definition 1.1.** ([8]) Let \( \mathcal{F} \) be a class of \( R \)-modules. By an \( \mathcal{F} \)-(pre)envelope of an
\( R \)-module \( M \), we mean a morphism \( \varphi : M \to F \) where \( F \in \mathcal{F} \) such that for any morphism
\( f : M \to F' \) with \( F' \in \mathcal{F} \), there exists a morphism \( g : F \to F' \) such that \( g\varphi = f \), that is,
there is the following commutative diagram:

\[
\begin{array}{ccc}
M & \xrightarrow{\varphi} & F \\
\downarrow f & & \downarrow g \\
F' & & \\
\end{array}
\]

If furthermore, when \( F' = F \) and \( f = \varphi \), the only such \( g \) are automorphisms of \( F \), then
\( \varphi : M \to F \) is called an \( \mathcal{F} \)-envelope of \( M \).

Dually, one may give the notion of \( \mathcal{F} \)-(pre)cover of an \( R \)-module. Note that \( \mathcal{F} \)-envelopes
and \( \mathcal{F} \)-covers may not exist in general, but if they exist, they are unique up to isomor-
phism.

In the process that some results of a homological nature may be generalized from
coherent rings to arbitrary rings, the notion of super finitely presented modules plays a
crucial role. Recall from [14] that an \( R \)-module \( M \) is called super finitely presented if there
exists an exact sequence \( \cdots \to F_1 \to F_0 \to M \to 0 \), where each \( F_i \) is finitely generated
and projective. Then Gao and Wang gave the definition of weak injective (and weak flat)
modules in terms of super finitely presented modules in [15], which is a generalization of
the notion of FP-injective modules.

**Definition 1.2.** ([15]) An \( R \)-module \( M \) is called weak injective if \( \text{Ext}^1_R(N, M) = 0 \) for
any super finitely presented \( R \)-module \( N \). A right \( R \)-module \( M \) is called weak flat if
\( \text{Tor}^1_R(M, N) = 0 \) for any super finitely presented \( R \)-module \( N \).

Accordingly, the weak injective dimension of an \( R \)-module \( M \), denoted by \( \text{wid}_R(M) \),
is defined to be the smallest non-negative integer \( n \) such that \( \text{Ext}^{n+1}_R(N, M) = 0 \) for any
super finitely presented $R$-module $N$, and the \textit{weak flat dimension} of a right $R$-module $M$, denoted by $\text{wfd}_R(M)$, is defined to be the smallest non-negative integer $n$ such that $\text{Tor}^R_{n+1}(M, N) = 0$ for any super finitely presented $R$-module $N$.

We denote by $\mathcal{WI}$ and $\mathcal{WF}$ the classes of weak injective and weak flat $R$-modules, respectively. By [13, Thm. 3.4], every $R$-module has a weak injective preenvelope. So for any $R$-module $M$, $M$ has a right $\mathcal{WI}$-resolution, that is, there exists a $\text{Hom}_R(-, \mathcal{WI})$-exact complex

$$0\rightarrow M\rightarrow E^0\rightarrow E^1\rightarrow E^2\rightarrow \cdots,$$

where each $E^i$ is weak injective. Moreover, since every injective $R$-module is weak injective, this complex is also exact. On the other hand, every $R$-module has a weak injective cover by [13, Thm. 3.1]. So every $R$-module $M$ has a left $\mathcal{WI}$-resolution, that is, there exists a $\text{Hom}_R(\mathcal{WI}, -)$-exact complex

$$\cdots\rightarrow W_2\rightarrow W_1\rightarrow W_0\rightarrow M\rightarrow 0,$$

where each $W_i$ is weak injective. But this complex is not necessarily exact.

Since every $R$-module has a weak flat cover by [13], every $R$-module has a left $\mathcal{WF}$-resolution. So for any $R$-module $M$, there exists a $\text{Hom}_R(\mathcal{WF}, -)$-exact complex

$$\cdots\rightarrow F_2\rightarrow F_1\rightarrow F_0\rightarrow M\rightarrow 0,$$

where each $F_i$ is weak flat. Moreover, since every projective $R$-module is weak flat, this complex is also exact. On the other hand, every $R$-module has a weak flat preenvelope by [15, Thm. 2.15]. So for any $R$-module $M$, $M$ has a right $\mathcal{WF}$-resolution, that is, there exists a $\text{Hom}_R(-, \mathcal{WF})$-exact complex

$$0\rightarrow M\rightarrow F^0\rightarrow F^1\rightarrow F^2\rightarrow \cdots,$$

where each $F^i$ is weak flat. Similarly, this complex is not necessarily exact.

2. \textbf{Gorenstein weak injective modules and dimension}

In this section, we give the definition of Gorenstein weak injective modules in terms of weak injective modules, and discuss some of the properties of these modules.

\textbf{Definition 2.1.} An $R$-module $M$ is called \textit{Gorenstein weak injective} if there exists an exact sequence of injective $R$-modules

$$I = \cdots\rightarrow I_1\rightarrow I_0\rightarrow I^0\rightarrow I^1\rightarrow \cdots$$

such that $M = \text{Coker}(I_1\rightarrow I_0)$ and the functor $\text{Hom}_R(W, -)$ leaves this sequence exact whenever $W$ is a weak injective $R$-module.

We will denote by $\mathcal{GWI}$ the class of Gorenstein weak injective $R$-modules.
Remark 2.2. (1) Every injective $R$-module is Gorenstein weak injective.

(2) Since every FP-injective $R$-module is weak injective, every Gorenstein weak injective is Ding injective $R$-module (in the sense of [12]). If $R$ is a left coherent ring, then the class of Gorenstein weak injective $R$-modules coincides with the class of Ding injective $R$-modules. Moreover, we have the following implications:

$$\text{Gorenstein weak injective } R\text{-modules } \Rightarrow \text{Ding injective } R\text{-modules } \Rightarrow \text{Gorenstein injective } R\text{-modules}.$$ If $R$ is a left Noetherian ring, then these three kinds of $R$-modules coincide.

(3) The class of Gorenstein weak injective $R$-modules is closed under direct products.

(4) If $I = \cdots \to I_1 \to I_0 \to I^0 \to I^1 \to \cdots$ is an exact sequence of injective $R$-modules such that the functor Hom$_R(W, -)$ leaves this sequence exact whenever $W$ is a weak injective $R$-module, then by symmetry, all the images, the kernels and the cokernels of $I$ are Gorenstein weak injective.

Lemma 2.3. Let $M$ be a Gorenstein weak injective $R$-module. Then Ext$_R^i(W, M) = 0$ for any weak injective $R$-module $W$ and any $i \geq 1$.

Proof. By the definition of Gorenstein weak injective $R$-modules, $M$ admits an injective resolution $0 \to M \to I^0 \to I^1 \to \cdots$ which remains exact after applying the functor Hom$_R(W, -)$ for any weak injective $R$-module $W$. So Ext$_R^i(W, M) = 0$ for any $i \geq 1$. □

Proposition 2.4. A Gorenstein weak injective $R$-module is either injective or has weak injective dimension $\infty$. Consequently, GWI \cap W\overline{I} = \mathcal{I}$, where $W\overline{I}$ denote the class of $R$-modules with finite weak injective dimension.

Proof. Let $M$ be a Gorenstein weak injective $R$-module and assume that wid$_R(M) = n < \infty$, that is, Ext$_R^{n+1}(N, M) = 0$ for any super finitely presented $R$-module $N$. Choose a partial injective resolution of $M$: $0 \to M \to I^0 \to I^1 \to \cdots \to I^{n-1} \to V^n \to 0$. It follows from the isomorphism Ext$_R^1(N, V^n) \cong \text{Ext}_R^{n+1}(N, M)$ that $V^n$ is weak injective. Now $0 \to M \to I^0 \to I^1 \to \cdots \to I^{n-1} \to V^n \to 0$ represents an element of Ext$_R^n(V^n, M)$, but this group equals 0 by Lemma 2.3. Thus this sequence is split exact, and so $M$ is injective as a direct summand of $I^0$. □

Corollary 2.5. An $R$-module is injective if and only if it is weak injective and Gorenstein weak injective.

Corollary 2.6. If the class of Gorenstein weak injective $R$-modules is closed under direct sums, then the ring $R$ is left Noetherian.

Proof. Note that the class of weak injective $R$-modules is closed under direct sums by [15, Prop. 2.3], and every injective $R$-module is weak injective. So a direct sum of injective $R$-modules is weak injective. Moreover, it is also Gorenstein weak injective by
It follows then from Corollary 2.5 that a direct sum of injective \( R \)-modules must be injective. Thus \( R \) is left Noetherian. □

In general, weak injective modules need not be Gorenstein weak injective as shown by the following proposition.

**Proposition 2.7.** A ring \( R \) is left Noetherian if and only if every weak injective module is Gorenstein weak injective.

**Proof.**

\( \Rightarrow \). It follows from the fact that the class of weak injective \( R \)-modules coincide with the class of injective \( R \)-modules over a Noetherian ring \( R \).

\( \Leftarrow \). Let \( W \) be a weak injective \( R \)-module and \( 0 \to W \to E \to N \to 0 \) an exact sequence with \( E \) injective. It is easy to verify that \( N \) is also weak injective, and hence \( \text{Ext}^1_R(N,W) = 0 \) by hypothesis and Lemma 2.3. That is, the sequence \( 0 \to W \to E \to N \to 0 \) is split. Thus \( W \) is injective, and \( R \) is left Noetherian. □

**Remark 2.8.** In order to make the result analogous to the part "if only" in Proposition 2.7 hold without additional assumptions, we introduced in [32] a class of \( R \)-modules of the form \( \text{Coker}(W_1 \to W_0) \) for some exact sequence of weak injective \( R \)-modules

\[
W = \cdots \to W_1 \to W_0 \to W^0 \to W^1 \to \cdots
\]

such that the complex \( \text{Hom}_R(N, W) \) is exact whenever \( N \) is a super finitely presented \( R \)-module with \( \text{pd}_R(N) < \infty \), which is a generalization of [11] over any ring. Accordingly, we also call this class of \( R \)-modules Gorenstein weak injective \( R \)-modules. It is obvious that every weak injective \( R \)-module is Gorenstein weak injective under this definition. Please see [32] for more details.

As what said in [13, Sec. 4], \( \text{Hom}_R(-,-) \) is left balanced on \( _R\mathcal{M} \times _R\mathcal{M} \) by \( \mathcal{W}_I \times \mathcal{W}_I \), where \( _R\mathcal{M} \) denotes the category of left \( R \)-modules. Denote by \( \text{Ext}^i_R(-,-) \) the \( i \)th left derived functor of \( \text{Hom}_R(-,-) \) with respect to \( \mathcal{W}_I \times \mathcal{W}_I \). For any \( R \)-modules \( M \) and \( N \), \( \text{Ext}^i_R(M,N) \) can be computed by using a right \( \mathcal{W}_I \)-resolution of \( M \) or a left \( \mathcal{W}_I \)-resolution of \( N \). That is, let \( \cdots \to W_2 \to W_1 \to W_0 \to N \to 0 \) be a left \( \mathcal{W}_I \)-resolution of \( N \). Applying the functor \( \text{Hom}_R(M,-) \) to it, we have the deleted complex

\[
\cdots \to \text{Hom}_R(M,W_2) \to \text{Hom}_R(M,W_1) \to \text{Hom}_R(M,W_0) \to 0.
\]

Then \( \text{Ext}^i_R(M,N) \) is exactly the \( i \)th homology of the above complex. Now there exists a canonical homomorphism \( \sigma : \text{Ext}^0_R(M,N) \to \text{Ext}^0_R(M,N) = \text{Hom}_R(M,N) \). Let \( \text{Ker} \sigma \) and \( \text{Coker} \sigma \).

We next give some characterizations of Gorenstein weak injective modules.

**Proposition 2.9.** The following are equivalent for an \( R \)-module \( M \):

1. \( M \) is Gorenstein weak injective;
(2) $\text{Ext}_R^i(W,M) = 0 = \text{Ext}_R^R(W,M)$ for any $i \geq 1$ and $\text{Ext}_R^0(W,M) = 0 = \text{Ext}_R^R(W,M)$ for any projective or weak injective $R$-module $W$;
(3) $\text{Ext}_R^i(\widetilde{W},M) = 0 = \text{Ext}_R^R(\widetilde{W},M)$ for any $i \geq 1$ and $\text{Ext}_R^0(\widetilde{W},M) = 0 = \text{Ext}_R^R(\widetilde{W},M)$ for any $R$-module $\widetilde{W}$ with $\text{pd}_R(\widetilde{W}) < \infty$ or $\text{wid}_R(\widetilde{W}) < \infty$;
(4) $M$ has an exact left $\mathcal{W}$-$\mathcal{I}$-resolution and $\text{Ext}_R^i(W,M) = 0$ for any weak injective $R$-module $W$ and any $i \geq 1$;
(5) $M$ has an exact left $\mathcal{W}$-$\mathcal{I}$-resolution and $\text{Ext}_R^i(\widetilde{W},M) = 0$ for any $R$-module $\widetilde{W}$ with $\text{wid}_R(\widetilde{W}) < \infty$ and any $i \geq 1$.
Moreover, if $R$ satisfies $\text{wid}_R(R) < \infty$ as a left $R$-module, then the above conditions are equivalent to
(6) $\text{Ext}_R^i(W,M) = 0$ for any weak injective $R$-module $W$ and any $i \geq 1$.

Proof. (1) $\Rightarrow$ (3). We use induction on $n = \text{wid}_R(\widetilde{W}) < \infty$. If $n = 0$, then the assertions follow from the definition of the functor $\text{Ext}_R^R(-,-)$ and Lemma 2.3. Suppose that the results hold for the case $n - 1$. Consider an exact sequence $0 \to \widetilde{W} \to W \to V \to 0$ with $W$ weak injective. Then $\text{wid}_R(V) = n - 1$. Assume that the sequence $\mathcal{I}$ is defined as in Definition 2.1. Since each term of $\mathcal{I}$ is injective, we have the following exact sequence of complexes

$$0 \to \text{Hom}_R(V, \mathcal{I}) \to \text{Hom}_R(W, \mathcal{I}) \to \text{Hom}_R(\widetilde{W}, \mathcal{I}) \to 0.$$ 

Note that the complex $\text{Hom}_R(W, \mathcal{I})$ is exact by Definition 2.1 and the complex $\text{Hom}_R(V, \mathcal{I})$ is exact by the induction hypothesis. So the complex $\text{Hom}_R(\widetilde{W}, \mathcal{I})$ is also exact. It is easy to verify that (3) holds. By a similar argument, we may also get that (3) holds for the case $\text{pd}_R(\widetilde{W}) < \infty$.

(3) $\Rightarrow$ (2) is trivial.

(2) $\Rightarrow$ (4). Since every $R$-module has a weak injective cover, one easily get that every $R$-module has a left $\mathcal{W}$-$\mathcal{I}$-resolution. Let $\cdots \to W_2 \to W_1 \to W_0 \to M \to 0$ be a left $\mathcal{W}$-$\mathcal{I}$-resolution of $M$. If we set $W = R$, we can easily get that it is also exact by assumption.

(4) $\Rightarrow$ (5) holds by dimension shifting.

(5) $\Rightarrow$ (1). Let $f : W_0 \to M$ be a weak injective cover of $M$. Consider an exact sequence $(\xi) : 0 \to W_0 \xrightarrow{i} E \to N \to 0$ with $E$ injective. It is easy to verify that $N$ is weak injective. So $\text{Ext}_R^1(N, M) = 0$ by hypothesis, and we have the exact sequence $\text{Hom}_R(E, M) \xrightarrow{i^*} \text{Hom}_R(W_0, M) \to 0$ by applying the functor $\text{Hom}_R(-, M)$ to the sequence $(\xi)$. Thus, for $f : W_0 \to M$, there exists $g : E \to M$ such that $gi = i^*(g) = f$. Moreover, since $f$ is a weak injective cover, there exists $h : E \to W_0$ such that $fh = g$. Hence we have $fhi = f$ and thus $hi$ is an isomorphism. This implies that $W_0$ is injective as a direct summand of $E$. Since $f$ is a weak injective cover, we have an exact sequence $\text{Hom}_R(W, W_0) \to \text{Hom}_R(W, \text{Im}f) \to 0$ for any weak injective $R$-module $W$. Moreover,
from the following exact sequence

\[ \text{Hom}_R(W, W_0) \to \text{Hom}_R(W, \text{Im} f) \to \text{Ext}^1_R(W, \text{Ker} f) \to \text{Ext}^1_R(W, W_0) = 0 \]

we obtain that \( \text{Ext}^1_R(W, \text{Ker} f) = 0 \). We repeat the argument by replacing \( M \) with \( \text{Ker} f \) to get a weak injective cover \( f_1 : W_1 \to \text{Ker} f \) and \( W_1 \) is injective. Continue this process, we may obtain a complex \((\varphi) : \cdots \to W_i \to W_0 \to M \to 0\), where each \( W_i \) is weak injective, such that the functor \( \text{Hom}_R(W, -) \) leaves this sequence exact whenever \( W \) is a weak injective \( R \)-module. Moreover, we have \( \text{Ext}^i_R(R, M) = 0 \) for any \( i \geq 1 \) and \( \text{Ext}^0_R(R, M) \cong M \) by hypothesis. It follows then that the complex \((\varphi)\) is exact. On the other hand, we take an injective resolution of \( M \): \( 0 \to M \to I^0 \to I^1 \to \cdots \). Since \( \text{Ext}^i_R(W, M) = 0 \) for any weak injective \( R \)-module \( W \) and any \( i \geq 1 \), it is \( \text{Hom}_R(WI, -) \)-exact. Assembling this sequence with the sequence \((\varphi)\), we get the desired sequence as in Definition \(2.1\) and hence \( M \) is Gorenstein weak injective.

(1) \( \Rightarrow \) (6) follows from Lemma \(2.3\).

(6) \( \Rightarrow \) (1). As a similar argument to the proof of (5) \( \Rightarrow \) (1), we have a \( \text{Hom}_R(WI, -) \)-exact complex \( I = \cdots \to I_1 \to I_0 \to I^0 \to I^1 \to \cdots \) of injective \( R \)-modules with \( M = \text{Coker}(I_1 \to I_0) \). By induction, it is easy to verify that the complex \( \text{Hom}_R(W, I) \) is exact for any \( R \)-module with \( \text{wid}_R(W) = n < \infty \). In particular, \( \text{Hom}_R(R, I) \) is exact by assumption, and thus \( I \) is exact. Therefore, \( M \) is Gorenstein weak injective.

**Remark 2.10.** Following the proof of (5) \( \Rightarrow \) (1) in Proposition \(2.9\), the kernel of a weak injective cover of any Gorenstein weak injective \( R \)-module is Gorenstein weak injective.

**Corollary 2.11.** An \( R \)-module \( M \) is Gorenstein weak injective if and only if there is an exact sequence \( 0 \to L \to I \to M \to 0 \) with \( I \) injective and \( L \) Gorenstein weak injective.

**Proof.** The necessity is clear. For the sufficiency, consider the exact sequence \((\triangleright) : 0 \to L \to I \to M \to 0\) with \( I \) injective and \( L \) Gorenstein weak injective. Since \( \text{Ext}^i_R(W, L) = 0 \) for any weak injective \( R \)-module \( W \) and any \( i \geq 1 \) by Lemma \(2.3\), it is easy to verify that \( \text{Ext}^i_R(W, M) = 0 \) for any weak injective \( R \)-module \( W \) and the sequence \((\triangleright)\) is \( \text{Hom}_R(WI, -) \)-exact. Since \( L \) is Gorenstein weak injective, by the definition, there is a \( \text{Hom}_R(WI, -) \)-exact exact sequence \( \cdots \to I_2 \to I_1 \to L \to 0 \) with each \( I_i \) injective. Assembling this sequence with the sequence \((\triangleright)\), we have the following commutative diagram:

\[
\begin{array}{ccccccc}
\cdots & \to & I_2 & \to & I_1 & \to & I & \to & M & \to & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
& & 0 & & 0 & & & & L & & \\
\end{array}
\]

which shows that \( M \) is Gorenstein weak injective by Proposition \(2.9\).
Proposition 2.12. Given an exact sequence $0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$. If $L$ and $N$ are Gorenstein weak injective, then so is $M$. That is, the class of Gorenstein weak injective $R$-modules is closed under extensions.

Proof. Since $L$ and $N$ are Gorenstein weak injective, we have that $\text{Ext}^i_R(W, L) = 0 = \text{Ext}^i_R(W, N)$ for any weak injective $R$-module $W$ and any $i \geq 1$ by Lemma 2.3. It is easy to verify that $\text{Ext}^i_R(W, M) = 0$ for any weak injective $R$-module $W$ and any $i \geq 1$. So, to prove that $M$ is Gorenstein weak injective, it suffices to show that $M$ has an exact left $\mathcal{W}_L$-resolution by Proposition 2.9. By hypothesis, there exists the following exact left $\mathcal{W}_L$-resolution of $L$ and $N$:

$$
\begin{align*}
\mathcal{I} &= \cdots \longrightarrow I'_1 \xrightarrow{d'_1} I'_0 \xrightarrow{d'_0} L \longrightarrow 0 \\
\mathcal{I}' &= \cdots \longrightarrow I''_1 \xrightarrow{d''_1} I''_0 \xrightarrow{d''_0} N \longrightarrow 0,
\end{align*}
$$

where all $I'_i$ and $I''_i$ are injective, and all kernels of $\mathcal{I}$ and $\mathcal{I}'$ are Gorenstein weak injective (such sequences exist by the definition of Gorenstein weak injective). Consider the following diagram:

$$
\begin{array}{c}
0 \longrightarrow I'_0 \xrightarrow{(1)} I'_0 \oplus I''_0 \xrightarrow{(0, 1)} I''_0 \longrightarrow 0 \\
\downarrow d'_0 \\
0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0 \\
\downarrow d''_0 \\
0 & 0
\end{array}
$$

Since $L$ is Gorenstein weak injective, $\text{Ext}^1_R(I''_0, L) = 0$, and thus we have the following exact sequence

$$
0 \longrightarrow \text{Hom}_R(I''_0, L) \xrightarrow{f_*} \text{Hom}_R(I'_0, M) \xrightarrow{g_*} \text{Hom}_R(I''_0, N) \longrightarrow 0.
$$

Since $g_*$ is epimorphic, there exists $\alpha : I''_0 \rightarrow M$ such that $d'_0 = g_*(\alpha) = g\alpha$. For any $(e'_0, e''_0) \in I'_0 \oplus I''_0$, we define $d_0 : I'_0 \oplus I''_0 \rightarrow M$ by $d_0(e'_0, e''_0) = f d'_0(e'_0) + \alpha(e''_0)$. Then it is easy to verify that $d_0$ makes the above diagram commute. By Snake Lemma, we have
the following commutative diagram:

\[
\begin{array}{ccc}
0 & \rightarrow & \text{Ker}d'_0 \\
\downarrow & & \downarrow \\
0 & \rightarrow & \text{Ker}d_0 \\
\downarrow & & \downarrow \\
0 & \rightarrow & \text{Ker}d''_0 \\
\end{array}
\]

Since \(\text{Ker}d'_0\) and \(\text{Ker}d''_0\) are Gorenstein weak injective, we have \(\text{Ext}^i_R(W, \text{Ker}d'_0) = 0 = \text{Ext}^i_R(W, \text{Ker}d''_0)\) for any weak injective \(R\)-module \(W\) and \(i \geq 1\) by Lemma \[2,3\]. It is easy to verify that \(\text{Ext}^i_R(W, \text{Ker}d_0) = 0\) for any weak injective \(R\)-module \(W\) and \(i \geq 1\).

In particular, the sequence \(0 \rightarrow \text{Ker}d_0 \rightarrow I'_0 \oplus I''_0 \rightarrow M \rightarrow 0\) is \(\text{Hom}_R(W, I, -)\)-exact. Repeating this process, we may get the following commutative diagram:

\[
\begin{array}{ccc}
0 & \rightarrow & I'_0 \\
\downarrow & \mid & \downarrow \mid \\
0 & \rightarrow & I'_1 \\
\downarrow & \mid & \downarrow \mid \\
0 & \rightarrow & I_1' \\
\end{array}
\]

and a \(\text{Hom}_R(W, I, -)\)-exact exact sequence \(\cdots \rightarrow I'_1 \oplus I''_1 \rightarrow I'_0 \oplus I''_0 \rightarrow M \rightarrow 0\), where each \(I'_i \oplus I''_i\) is injective. Therefore, \(M\) is Gorenstein weak injective.

Let \(C\) be a class of \(R\)-modules. Recall from [17] that \(C\) is injectively resolving if the class \(\mathcal{I}\) of injective \(R\)-modules satisfies \(\mathcal{I} \subseteq C\), and for any exact sequence \(0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0\) with \(L \in C\), \(M \in C\) if and only if \(N \in C\).

We have the following proposition.

**Proposition 2.13.** The class \(GW\mathcal{I}\) is injectively resolving.
Proof. It is obvious that \( I \subseteq GWI \). So, for any exact sequence \( 0 \to L \to M \to N \to 0 \) with \( L \) Gorenstein weak injective, it suffices to show that if \( M \) is Gorenstein weak injective, then so is \( N \) by Proposition 2.12. It is easy to verify that \( \text{Ext}^i_R(W, N) = 0 \) for any weak injective \( R \)-module \( W \) and any \( i \geq 1 \). Since \( M \) is Gorenstein weak injective, we have an exact sequence \( 0 \to G \to I_0 \to M \to 0 \) with \( I_0 \) injective and \( G \) Gorenstein weak injective by Corollary 2.11. Consider the following pull-back diagram:

\[
\begin{array}{ccccccccc}
0 & & 0 & & & & & & & \\
& & & & & & & & \downarrow & & \\
G & & G & & & & & & \downarrow & & \\
& & & & & & & & \downarrow & & \\
0 & & Q & & I_0 & & N & & 0 & \\
& & & & & & & & \downarrow & & \\
0 & & L & & M & & N & & 0 & \\
& & & & & & & & \downarrow & & \\
0 & & 0 & & & & & & & \\
\end{array}
\]

By Proposition 2.12 and the second column in the above diagram, we have that \( Q \) is Gorenstein weak injective. Thus \( \text{Ext}^1_R(W, Q) = 0 \) for any weak injective \( R \)-module \( W \), and so the exact sequence \((\#)\) : \( 0 \to Q \to I_0 \to N \to 0 \) is \( \text{Hom}_R(WI, -) \)-exact. In addition, since \( Q \) is Gorenstein weak injective, by the definition of Gorenstein weak injective \( R \)-modules, we have a \( \text{Hom}_R(WI, -) \)-exact exact sequence \( \cdots \to I_2 \to I_1 \to Q \to 0 \), where each \( I_i \) is injective. Assembling this sequence with the sequence \((\#)\), we get the following commutative diagram

\[
\begin{array}{ccccccccc}
\cdots & \to & I_2 & \to & I_1 & \to & I_0 & \to & N & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
& & Q & & & & & & & \\
\end{array}
\]

which shows that \( N \) is Gorenstein weak injective. \( \square \)

Proposition 2.14. The class \( GWI \) is closed under direct summands.

Proof. It follows from Remark 2.2(3), Proposition 2.13 and [17, Prop. 1.4]. \( \square \)

Definition 2.15. The Gorenstein weak injective dimension of an \( R \)-module \( M \), denoted by \( Gwid_R(M) \), is defined as \( \inf\{n \mid \text{there is an exact sequence } 0 \to M \to G^0 \to G^1 \to \cdots \to G^n \to 0 \text{ with } G^i \text{ Gorenstein weak injective for any } 0 \leq i \leq n\} \). If no such \( n \) exists, set \( Gwid_R(M) = \infty \).
Lemma 2.16. Let $M$ be an $R$-module. Consider the following exact sequences:

$$
0 \rightarrow M \rightarrow G^0 \rightarrow G^1 \rightarrow \cdots \rightarrow G^{n-1} \rightarrow V^n \rightarrow 0
$$

$$
0 \rightarrow M \rightarrow \widetilde{G}^0 \rightarrow \widetilde{G}^1 \rightarrow \cdots \rightarrow \widetilde{G}^{n-1} \rightarrow \nabla^n \rightarrow 0
$$

where all $G^i$ and $\widetilde{G}^i$ are Gorenstein weak injective. Then $V^n$ is Gorenstein weak injective if and only if $\nabla^n$ is Gorenstein weak injective.

Proof. It follows from the dual version of [1, Lem. 3.12]. \qed

The following proposition shows the existence of Gorenstein weak injective preenvelope of modules.

Proposition 2.17. Let $M$ be an $R$-module with finite Gorenstein weak injective dimension $n$. Then $M$ admits an injective Gorenstein weak injective preenvelope $\phi : M \hookrightarrow G$, where $V = \text{Coker} \phi$ satisfies $\text{id}_R(V) = n - 1$ (if $n = 0$, this should be interpreted as $V = 0$). Moreover, if $\text{wid}_R(M) < \infty$, then $G$ is injective.

Proof. Assume that $\text{Gwid}_R(M) = n$. Choose a partial injective resolution of $M$:

$$
0 \rightarrow M \rightarrow I^0 \rightarrow I^1 \rightarrow \cdots \rightarrow I^{n-1} \rightarrow G' \rightarrow 0.
$$

By Lemma 2.16 we have that $G'$ is Gorenstein weak injective, and hence, by the definition of Gorenstein weak injective $R$-modules, there is an exact sequence

$$
\text{G} : \ 0 \rightarrow \hat{G} \rightarrow Q^0 \rightarrow Q^1 \rightarrow \cdots \rightarrow Q^{n-1} \rightarrow G' \rightarrow 0,
$$

where $\hat{G}$ is Gorenstein weak injective and each $Q^i$ is injective, such that $\text{Hom}_R(W, \text{G})$ is exact for any weak injective $R$-module $W$. In particular, $\text{Hom}_R(E, \text{G})$ is exact for any injective $R$-module $E$, and hence, by [7, Sec. 8.1], there exists morphisms $I^i \rightarrow Q^i$ and $M \rightarrow \hat{G}$ such that the following diagram is commutative

$$
\begin{array}{ccc}
0 & \rightarrow & M \\
\downarrow & & \downarrow \\
\hat{G} & \rightarrow & Q^0 \\
\downarrow & & \downarrow \\
0 & \rightarrow & Q^1 \\
\downarrow & & \downarrow \\
\cdots & \rightarrow & \cdots \\
\downarrow & & \downarrow \\
\downarrow & & \downarrow \\
G' & \rightarrow & 0 \\
\downarrow & & \downarrow \\
0 & \rightarrow & 0
\end{array}
$$

This diagram gives a chain map of complexes as follows:

$$
\begin{array}{ccc}
0 & \rightarrow & M \\
\downarrow & & \downarrow \\
\hat{G} & \rightarrow & Q^0 \\
\downarrow & & \downarrow \\
0 & \rightarrow & Q^1 \\
\downarrow & & \downarrow \\
\cdots & \rightarrow & \cdots \\
\downarrow & & \downarrow \\
\downarrow & & \downarrow \\
\downarrow & & \downarrow \\
G' & \rightarrow & 0 \\
\downarrow & & \downarrow \\
0 & \rightarrow & 0
\end{array}
$$

which induces an isomorphism in homology. Its mapping cone

$$
0 \rightarrow M \rightarrow I^0 \oplus \hat{G} \rightarrow I^1 \oplus Q^0 \rightarrow \cdots \rightarrow I^{n-1} \oplus Q^{n-2} \rightarrow Q^{n-1} \rightarrow 0
$$
is exact. It is obvious that all $I^i \oplus Q^{i-1}$ and $Q^{n-1}$ are injective, and $I^0 \oplus \hat{G}$ is Gorenstein weak injective. Let $V = \text{Coker}(M \to I^0 \oplus \hat{G})$. Then $\text{id}_R(V) \leq n - 1$ (In fact, $\text{id}_R(V) = n - 1$. Otherwise, $\text{Gwid}_R(M) < n$, which is a contradiction).

Since $\text{id}_R(V) < \infty$, it is easy to verify that $\text{Ext}^1_R(V, \hat{G}) = 0$ for any Gorenstein weak injective $R$-module $\hat{G}$. Thus the sequence $\text{Hom}_R(I^0 \oplus \hat{G}, \hat{G}) \to \text{Hom}_R(M, \hat{G}) \to 0$ is exact. Let $G = I^0 \oplus \hat{G}$. Then $M \to G$ is a Gorenstein weak injective preenvelope of $M$, as desired.

Since $\text{wid}_R(M) < \infty$, $\text{wid}_R(G) < \infty$. By Proposition 2.14, we have that $G$ is injective.

**Corollary 2.18.** Given an exact sequence $0 \to L \to M \to N \to 0$. If $M$ and $N$ are Gorenstein weak injective, then the following are equivalent:

1. $L$ is Gorenstein weak injective;
2. $L$ is Ding injective;
3. $L$ is Gorenstein injective;
4. $\text{Ext}^1_R(I, L) = 0$ for any injective $R$-module $I$;
5. $\text{Ext}^1_R(E, L) = 0$ for any FP-injective $R$-module $E$;
6. $\text{Ext}^1_R(W, L) = 0$ for any weak injective $R$-module $W$.

**Proof.** (1) $\Rightarrow$ (6) $\Rightarrow$ (5) $\Rightarrow$ (4) are trivial. (2) $\Leftrightarrow$ (5) and (3) $\Leftrightarrow$ (4) follow from the dual versions of [17, Cor. 2.11] and [23, Cor. 2.1].

(4) $\Rightarrow$ (1) By hypothesis, $\text{Gwid}_R(L) \leq 1$, and so there is an exact sequence $0 \to L \to G \to I \to 0$ with $G$ Gorenstein weak injective and $I$ injective. By assumption, $\text{Ext}^1_R(I, L) = 0$, and thus this sequence is split. Hence $L$ is Gorenstein weak injective by Proposition 2.14. □

Now we give a functorial description of Gorenstein weak injective dimension of modules.

**Proposition 2.19.** Let $M$ be an $R$-module with finite Gorenstein weak injective dimension. Then the following are equivalent:

1. $\text{Gwid}_R(M) \leq n$;
2. $\text{Ext}^i_R(W, M) = 0$ for any weak injective $R$-module $W$ and any $i \geq n + 1$;
3. $\text{Ext}^i_R(\hat{W}, M) = 0$ for any $R$-module $\hat{W}$ with finite weak injective dimension and any $i \geq n + 1$;
4. For every exact sequence $0 \longrightarrow M \longrightarrow I^0 \longrightarrow \cdots \longrightarrow I^{n-1} \longrightarrow C^n \longrightarrow 0$, where each $I^i$ is injective, $C^n$ is Gorenstein weak injective.
5. For every exact sequence $0 \longrightarrow M \longrightarrow G^0 \longrightarrow \cdots \longrightarrow G^{n-1} \longrightarrow V^n \longrightarrow 0$, where each $G^i$ is Gorenstein weak injective, $V^n$ is Gorenstein weak injective.

Consequently, the Gorenstein weak injective dimension of $M$ is determined by the formulas:
\[ \text{Gwid}_R(M) = \sup \{ n \mid \text{Ext}_R^n(W, M) \neq 0 \text{ for some weak injective } R\text{-module } W \} \]
\[ = \sup \{ n \mid \text{Ext}_R^n(\widetilde{W}, M) \neq 0 \text{ for some } R\text{-module } \widetilde{W} \text{ with } \text{wid}_R(\widetilde{W}) < \infty \}. \]

**Proof.** (1)\(\Rightarrow\)(2). Since \(\text{Gwid}_R(M) \leq n\), we have an exact sequence
\[ 0 \to M \to G^0 \to G^1 \to \cdots \to G^n \to 0, \]
where each \(G^i\) is Gorenstein weak injective. Let \(V^1 = \text{Coker}(M \to G^0)\), \(V^i = \text{Coker}(G^{i-2} \to G^{i-1})\), \(2 \leq i \leq n\). Then for any weak injective \(R\text{-module } W\), we have
\[ \text{Ext}_R^i(W, M) \cong \text{Ext}_R^i(W, V^1) \cong \cdots \cong \text{Ext}_R^{i-n}(W, G^n) = 0, \ i \geq n + 1. \]

(2)\(\Rightarrow\)(3) holds by dimension shifting.

(3)\(\Rightarrow\)(5). For every exact sequence \(0 \to M \to G^0 \to \cdots \to G^{n-1} \to V^n \to 0\), where each \(G^i\) is Gorenstein weak injective. Let \(V^0 = M\), \(V^1 = \text{Coker}(M \to G^0)\) and \(V^j = \text{Coker}(G^{j-2} \to G^{j-1})\), \(2 \leq j \leq n\). Then the sequence \(0 \to V^j \to E^j \to V^{j+1} \to 0\) is exact for any \(0 \leq j \leq n - 1\). Let \(W\) be any weak injective \(R\text{-module}\). Then we have the following exact sequence
\[ \text{Ext}_R^i(W, G^j) \to \text{Ext}_R^i(W, V^{j+1}) \to \text{Ext}_R^{i+1}(W, V^j) \to \text{Ext}_R^{i+1}(W, G^j), \]
where \(\text{Ext}_R^i(W, G^j) = 0 = \text{Ext}_R^{i+1}(W, G^j)\) by Lemma 2.11. So we have
\[ \text{Ext}_R^i(W, V^n) \cong \text{Ext}_R^{i+1}(W, V^{n-1}) \cong \cdots \cong \text{Ext}_R^{i+n}(W, M) = 0, \ i \geq 1. \]

Moreover, since \(\text{Gwid}_R(M) < \infty\), \(\text{Gwid}_R(V^n) < \infty\), and hence we have the following exact sequence for some non-negative integer \(m\)
\[ 0 \to V^n \to \overline{G}^0 \to \overline{G}^1 \to \cdots \to \overline{G}^m \to 0, \]
where each \(\overline{G}^i\) is Gorenstein weak injective. Let \(\overline{V}^0 = V^n\), \(\overline{V}^i = \text{Coker}(V^n \to \overline{G}^0)\) and \(\overline{V}^i = \text{Coker}(\overline{G}^{i-2} \to \overline{G}^{i-1})\), \(2 \leq i \leq m\). Then we have
\[ \text{Ext}_R^i(W, \overline{V}^{m-1}) \cong \text{Ext}_R^2(W, \overline{V}^{m-2}) \cong \cdots \cong \text{Ext}_R^m(W, V^n) = 0. \]

Since \(\overline{V}^{m-1} = \text{Coker}(\overline{G}^{m-3} \to \overline{G}^{m-2})\), we have an exact sequence \(0 \to \overline{V}^{m-1} \to \overline{G}^{m-1} \to \overline{G}^m \to 0\), that is, \(\text{Gwid}_R(\overline{V}^{m-1}) \leq 1\). By Proposition 2.17, there exists an exact sequence \(0 \to \overline{V}^{m-1} \to G \to I \to 0\) such that \(G\) is Gorenstein weak injective and \(I\) is injective. In addition, this sequence is split since \(\text{Ext}_R^i(I, \overline{V}^{m-1}) = 0\). Therefore, \(\overline{V}^{m-1}\) is Gorenstein weak injective. By a similar argument, we have that \(\overline{V}^{m-2}, \ldots, \overline{V}^0\) are Gorenstein weak injective. In particular, \(V^n\) is Gorenstein weak injective.

(5) \(\Rightarrow\)(4) and (4) \(\Rightarrow\)(1) are trivial. \(\square\)

**Proposition 2.20.** **Given an exact sequence** \(0 \to L \to M \to N \to 0\). **If any two of** \(R\text{-modules } L, M, \text{ or } N\) **have finite Gorenstein weak injective dimension, then so has the third.**
It is natural to investigate how much the usual injective dimension differs from the Gorenstein weak injective one. In what follows, \( \text{Gid}_R(M) \) and \( \text{Did}_R(M) \) will denote respectively the Gorenstein injective and Ding injective dimension of an \( R \)-module \( M \) (see [17, Def. 2.8] and [30, Def. 2.3] for details).

**Proposition 2.21.** Let \( M \) be an \( R \)-module. Then

1. \( \text{Gwid}_R(M) \leq \text{id}_R(M) \) with equality, if \( \text{wid}_R(M) < \infty \);
2. \( \text{Gid}_R(M) \leq \text{Did}_R(M) \leq \text{Gwid}_R(M) \) with equalities, if \( \text{Gwid}_R(M) < \infty \).

**Proof.** (1) Clearly, \( \text{Gwid}_R(M) \leq \text{id}_R(M) \). Let \( \text{wid}_R(M) < \infty \). It suffices to prove \( \text{id}_R(M) \leq \text{Gwid}_R(M) \). Without loss of generality, we assume that \( \text{Gwid}_R(M) = n < \infty \) for some non-negative integer \( n \). If \( n = 0 \), that is, \( M \) is Gorenstein weak injective, then there is an exact sequence \( 0 \to L \to I \to M \to 0 \) with \( I \) injective and \( L \) Gorenstein weak injective. Note that \( \text{Ext}^1_R(M, L) = 0 \) since \( \text{wid}_R(M) < \infty \). Thus this sequence is split, and so \( M \) is injective. Now let \( n \geq 1 \). By Proposition 2.17 there is an exact sequence \( 0 \to M \to G \to V \to 0 \) with \( G \) Gorenstein weak injective and \( \text{id}_R(V) = n - 1 \). Since \( G \) is Gorenstein weak injective, there is an exact sequence \( 0 \to G' \to I \to G \to 0 \) with \( I \) injective and \( G' \) Gorenstein weak injective. Consider the following pull-back diagram:

\[
\begin{array}{ccccccccc}
0 & & 0 \\
 & & \\
G' & \rightarrow & G' \\
 & \downarrow & \\
0 & \rightarrow & N & \rightarrow & I & \rightarrow & V & \rightarrow & 0 \\
 & & \downarrow & & \\
0 & \rightarrow & M & \rightarrow & G & \rightarrow & V & \rightarrow & 0 \\
 & & \downarrow & & \\
0 & & 0 \\
\end{array}
\]

Since \( I \) is injective and \( \text{id}_R(V) = n - 1 \), it follows from the middle row in the above diagram that \( \text{id}_R(N) \leq n \). Moreover, \( \text{Ext}^1_R(M, G') = 0 \) since \( \text{wid}_R(M) < \infty \), which shows that the second column in the above diagram is split. Thus \( \text{id}_R(M) \leq \text{id}_R(N) \leq n = \text{Gwid}_R(M) \), as desired.

(2) Since every Gorenstein weak injective \( R \)-module is Ding injective, and every Ding injective \( R \)-module is Gorenstein injective, it is obvious that \( \text{Gid}_R(M) \leq \text{Did}_R(M) \leq \text{Gwid}_R(M) \) for any \( R \)-module \( M \). Now let \( \text{Gwid}_R(M) = n < \infty \). In order to show \( \text{Gid}_R(M) \geq n \), it suffices to find an injective \( R \)-module \( I \) such that \( \text{Ext}^n_R(I, M) \neq 0 \) by [17, Thm. 2.22]. Since \( \text{Gwid}_R(M) = n \), there is some weak injective \( R \)-module \( W \) such that \( \text{Ext}^n_R(W, M) \neq 0 \). Consider an exact sequence \( 0 \to W \to I \to W' \to 0 \) with \( I \) injective. It is easy to verify that \( W' \) is weak injective. Consider the following exact
sequence
\[ \cdots \to \text{Ext}^n_R(I, M) \to \text{Ext}^n_R(W, M) \to \text{Ext}^{n+1}_R(W', M) = 0. \]

It follows then that \( \text{Ext}^n_R(I, M) \neq 0 \), as desired. \( \square \)

**Corollary 2.22.** If an \( R \)-module \( M \) is Gorenstein injective or Ding injective, then either \( M \) is Gorenstein weak injective or \( \text{Gwid}_R(M) = \infty \).

Accordingly, we define the left global Gorenstein weak injective dimension, \( \ell.GwiD(R) \), of a ring \( R \) as follows:

\[ \ell.GwiD(R) = \sup \{ \text{Gwid}_R(M) \mid M \text{ is any } R\text{-module} \}. \]

**Corollary 2.23.** If \( \ell.GwiD(R) < \infty \), then the following are equivalent:

1. \( \ell.GwiD(R) \leq n \);
2. \( \text{pd}_R(\widetilde{W}) \leq n \) for any \( R \)-module \( \widetilde{W} \) with finite weak injective dimension;
3. \( \text{pd}_R(W) \leq n \) for any weak injective \( R \)-module \( W \).

**Proof.** (1) \( \Rightarrow \) (2). Let \( \widetilde{W} \) be an \( R \)-module with finite weak injective dimension. For any \( R \)-module \( M \), \( \text{Gwid}_R(M) \leq n \) by hypothesis, and hence we have \( \text{Ext}^i_R(\widetilde{W}, M) = 0 \) for any \( i \geq n + 1 \). So \( \text{pd}_R(\widetilde{W}) \leq n \).

(2) \( \Rightarrow \) (3) is trivial.

(3) \( \Rightarrow \) (1). Let \( M \) be any \( R \)-module. It follows from Proposition 2.19 that \( \text{Gwid}_R(M) \leq n \). Thus \( \ell.GwiD(R) \leq n \). \( \square \)

It is well-known that pure injective modules play an important role in homological algebra, and the relative version also have been investigated by many authors (e.g. [3, 10, 20, 25, 28, 29]). Inspired by this, we give the following definition.

**Definition 2.24.** An exact sequence \( 0 \to L \to M \to N \to 0 \) is called \( \mathcal{GWI} \)-copure exact if for any \( X \in \mathcal{GWI} \), the induced sequence \( 0 \to \text{Hom}_R(N, X) \to \text{Hom}_R(M, X) \to \text{Hom}_R(L, X) \to 0 \) is exact.

An \( R \)-module \( M \) is called \( \mathcal{GWI} \)-copure projective (resp. \( \mathcal{GWI} \)-copure injective) if \( \text{Hom}_R(M, -) \) (resp. \( \text{Hom}_R(-, M) \)) leaves any \( \mathcal{GWI} \)-copure exact sequence exact.

If \( 0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0 \) is a \( \mathcal{GWI} \)-copure exact sequence, then \( f \) is called a \( \mathcal{GWI} \)-copure injection, and \( g \) is called a \( \mathcal{GWI} \)-copure surjection.

The following proposition shows the necessity of studying \( \mathcal{GWI} \)-copure exact sequences.

**Proposition 2.25.** Let \( M \) be an \( R \)-module with finite Gorenstein weak injective dimension. Then the following are equivalent:

1. \( M \) is Gorenstein weak injective;
(2) For any \( \mathcal{GWI} - \) copure injection \( i : X \to Y \) and any \( h : X \to M \), there exists \( g : Y \to M \) such that the following diagram commute:

\[
\begin{array}{ccc}
X & \xrightarrow{i} & Y \\
\downarrow{h} & & \downarrow{g} \\
M & & \\
\end{array}
\]

(3) The functor \( \text{Hom}_R(-, M) \) is exact with respect to any \( \mathcal{GWI} \) - copure exact sequence;

(4) Every \( \mathcal{GWI} \) - copure exact sequence \( 0 \to M \xrightarrow{f} N \xrightarrow{g} L \to 0 \) is split.

Proof. (1) \(\Rightarrow\) (2). Since \( i : X \to Y \) is \( \mathcal{GWI} \) - copure injective, we have a \( \mathcal{GWI} \) - copure exact sequence \( 0 \to X \xrightarrow{i} Y \xrightarrow{j} Z \to 0 \). Applying the functor \( \text{Hom}_R(-, M) \) to it, since \( M \) is Gorenstein weak injective, we have the following exact sequence

\[
0 \to \text{Hom}_R(Z, M) \xrightarrow{j^*} \text{Hom}_R(Y, M) \xrightarrow{i^*} \text{Hom}(X, M) \to 0 .
\]

Thus for any \( h : X \to M \), there exists \( g : Y \to M \) such that \( i^*(g) = gi = h \), as desired.

(2) \(\Rightarrow\) (3). Given a \( \mathcal{GWI} \) - copure exact sequence \( 0 \to X \xrightarrow{i} Y \xrightarrow{j} Z \to 0 \). Since the functor \( \text{Hom}(-, M) \) is left exact, it suffices to show that \( i^* \) is surjective. For a map \( h : X \to M \), by assumption, there exists \( g : Y \to M \) such that \( gi = h \). Thus, if \( h \in \text{Hom}_R(X, M) \), then \( h = gi = i^*(g) \in \text{im}i^* \), and so \( i^* \) is surjective. Hence the functor \( \text{Hom}_R(-, M) \) is exact with respect to any \( \mathcal{GWI} \) - copure exact sequence.

(3) \(\Rightarrow\) (4). Since \( 0 \to M \xrightarrow{f} N \xrightarrow{g} L \to 0 \) is a \( \mathcal{GWI} \) - copure exact sequence, we have the following exact sequence

\[
0 \to \text{Hom}_R(L, M) \xrightarrow{g^*} \text{Hom}_R(N, M) \xrightarrow{f^*} \text{Hom}_R(M, M) \to 0 ,
\]

that is, there exists a map \( f' : N \to M \) such that \( f^*(f') = f'f = \text{Id}_M \). It follows then that the sequence \( 0 \to M \xrightarrow{f} N \xrightarrow{g} L \to 0 \) is split.

(4) \(\Rightarrow\) (1). Since \( M \) has finite Gorenstein weak injective dimension, by Proposition 2.19 there exists an exact sequence \( 0 \to M \xrightarrow{f} G \xrightarrow{g} V \to 0 \) such that \( f \) is a Gorenstein weak injective preenvelope and \( id_R(V) = \text{Gwid}_R(M) - 1 \). Thus this sequence is \( \mathcal{GWI} \) - copure exact sequence. By (4), it is split, that is, \( G \cong M \oplus V \). Hence \( M \) is Gorenstein weak injective.

\[\square\]

Proposition 2.26.

\[\mathcal{GWI} = \{ \mathcal{GWI} - \text{copure injective } R\text{-modules}\} \cap \widehat{\mathcal{GWI}},\]

where \( \widehat{\mathcal{GWI}} \) denote the class of \( R\) - modules with finite Gorenstein weak injective dimension.
Proof. Clearly, every Gorenstein weak injective $R$-module is $GWl$-copure injective. Let $M$ be $GWl$-copure injective and have finite Gorenstein weak injective dimension. Then there exists an exact sequence $0 \to M \xrightarrow{f} G \xrightarrow{g} V \to 0$ such that $f$ is a Gorenstein weak injective preenvelope and $id_R(V) = Gwid_R(M) - 1 < \infty$. It follows then that $\text{Ext}_R^1(V, G') = 0$ for any Gorenstein weak injective $R$-module $G'$. So this sequence is in fact $\text{Hom}_R(-, GWl)$-exact, that is, it is $GWl$-copure exact. Moreover, since $M$ is $GWl$-copure injective, we have the following exact sequence

$$ 0 \to \text{Hom}_R(V, M) \xrightarrow{g^*} \text{Hom}_R(G, M) \xrightarrow{f^*} \text{Hom}_R(M, M) \to 0. $$

Thus, $0 \to M \xrightarrow{f} G \xrightarrow{g} V \to 0$ is split, and hence $M$ is Gorenstein weak injective as a direct summand of $G$. \hfill \square

3. Gorenstein weak projective modules and dimension

In this section, we give the definition of Gorenstein weak projective modules in terms of weak flat modules, and discuss some of the properties of these modules. The results and their proofs in this section are completely dual to that in Section 2, so we only list the results without proofs.

Definition 3.1. An $R$-module $M$ is called Gorenstein weak projective if there exists an exact sequence of projective $R$-modules

$$ P = \cdots \to P_1 \to P_0 \to P^0 \to P^1 \to \cdots $$

such that $M = \text{Coker}(P_1 \to P_0)$ and the functor $\text{Hom}_R(-, W)$ leaves this sequence exact whenever $W$ is a weak flat $R$-module.

Remark 3.2. (1) Every projective $R$-module is Gorenstein weak projective.

(2) Since every flat $R$-module is weak flat, every Gorenstein weak projective is Ding projective (in the sense of [12]). If $R$ is a left coherent ring, then the class of Gorenstein weak projective $R$-modules coincides with the class of Ding projective $R$-modules. Moreover, we have the following implications:

Gorenstein weak injective $R$-modules $\Rightarrow$ Ding projective $R$-modules

$\Rightarrow$ Gorenstein projective $R$-modules.

If $R$ is an $n$-Gorenstein ring (i.e. a left and right Noetherian ring with self injective dimension at most $n$ on both sides for some non-negative integer $n$), then these three kinds of $R$-modules coincide.

(3) The class of Gorenstein weak projective $R$-modules is closed under direct sums.

(4) If $P = \cdots \to P_1 \to P_0 \to P^0 \to P^1 \to \cdots$ is an exact sequence of projective $R$-modules such that the functor $\text{Hom}_R(-, W)$ leaves this sequence exact whenever $W$ is
a weak flat $R$-module, then by symmetry, all the images, the kernels and the cokernels of $P$ are Gorenstein weak flat.

**Proposition 3.3.** A Gorenstein weak projective $R$-module is either projective or has weak flat dimension $\infty$. Consequently, $\mathcal{GWP} \cap \mathcal{WF} = \mathcal{P}$, where $\mathcal{WF}$ denote the class of $R$-modules with finite weak flat dimension.

**Corollary 3.4.** An $R$-module is projective if and only if it is weak flat and Gorenstein weak projective.

Let $C$ be a class of $R$-modules. Recall from [17] that $C$ is projectively resolving if the class $\mathcal{P}$ of projective $R$-modules satisfies $\mathcal{P} \subseteq C$, and for any exact sequence $0 \to L \to M \to N \to 0$ with $N \in C$, $L \in C$ if and only if $M \in C$.

We have the following proposition.

**Proposition 3.5.** The class $\mathcal{GWP}$ is projectively resolving and closed under direct summands.

**Definition 3.6.** The Gorenstein weak projective dimension of an $R$-module $M$, denoted by $Gwpd_R(M)$, is defined as $\inf \{n \mid \text{there is an exact sequence } 0 \to G_n \to \cdots \to G_1 \to G_0 \to M \to 0 \text{ with } G_i \text{ Gorenstein weak projective for any } 0 \leq i \leq n\}$. If no such $n$ exists, set $Gwpd_R(M) = \infty$.

**Proposition 3.7.** Let $M$ be an $R$-module with finite Gorenstein weak projective dimension $n$. Then $M$ admits a surjective Gorenstein weak projective precover $\varphi : G \twoheadrightarrow M$, where $K = \text{Ker} \varphi$ satisfies $\text{pd}_R(K) = n - 1$ (if $n = 0$, this should be interpreted as $K = 0$). Moreover, if $\text{wfd}_R(M) < \infty$, then $G$ is projective.

**Corollary 3.8.** Given an exact sequence $0 \to L \to M \to N \to 0$. If $L$ and $M$ are Gorenstein weak injective, then the following are equivalent:

1. $N$ is Gorenstein weak projective;
2. $N$ is Ding projective (in the sense of [12]);
3. $N$ is Gorenstein projective;
4. $\text{Ext}^1_R(N, P) = 0$ for any projective $R$-module $P$;
5. $\text{Ext}^1_R(N, F) = 0$ for any flat $R$-module $F$;
6. $\text{Ext}^1_R(N, W) = 0$ for any weak flat $R$-module $W$.

**Proposition 3.9.** Let $M$ be an $R$-module with finite Gorenstein weak projective dimension. Then the following are equivalent:

1. $Gwpd_R(M) \leq n$;
2. $\text{Ext}^i_R(M, W) = 0$ for any weak flat $R$-module $W$ and any $i \geq n + 1$;
3. $\text{Ext}^i_R(M, \tilde{W}) = 0$ for any $R$-module $\tilde{W}$ with finite weak flat dimension and any $i \geq n + 1$;
(4) For every exact sequence $0 \rightarrow K_n \rightarrow P_{n-1} \rightarrow \cdots \rightarrow P_0 \rightarrow M \rightarrow 0$, where each $P_i$ is projective, $K_n$ is Gorenstein weak projective.

(5) For every exact sequence $0 \rightarrow K'_n \rightarrow G_{n-1} \rightarrow \cdots \rightarrow G_0 \rightarrow M \rightarrow 0$, where each $G_i$ is Gorenstein weak projective, $K'_n$ is Gorenstein weak projective.

Consequently, the Gorenstein weak projective dimension of $M$ is determined by the formulas:

$$Gwpd_R(M) = \sup\{n \mid \Ext^i_R(M, W) \neq 0 \text{ for some weak flat } R\text{-module } W\}$$

$$= \sup\{n \mid \Ext^i_R(M, \tilde{W}) \neq 0 \text{ for some } R\text{-module } \tilde{W} \text{ with } wfd_R(\tilde{W}) < \infty\}.$$

In what follows, $Gpd_R(M)$ and $Dpd_R(M)$ will denote respectively the Gorenstein projective and Ding projective dimension of an $R$-module $M$ (see [17, Def. 2.8] and [30, Def. 2.3] for details).

**Proposition 3.10.** Let $M$ be an $R$-module. Then

1. $Gwpd_R(M) \leq pd_R(M)$ with equality, if $wfd_R(M) < \infty$;
2. $Gpd_R(M) \leq Dpd_R(M) \leq Gwpd_R(M)$ with equalities, if $Gwpd_R(M) < \infty$.

Accordingly, we define the left global Gorenstein weak projective dimension, $\ell.GwpD(R)$, of a ring $R$ as follows:

$$\ell.GwpD(R) = \sup\{Gwpd_R(M) \mid M \text{ is any } R\text{-module}\}.$$ 

**Corollary 3.11.** If $\ell.GwpD(R) < \infty$, then the following are equivalent:

1. $\ell.GwpD(R) \leq n$;
2. $\text{id}_R(\tilde{W}) \leq n$ for any $R$-module $\tilde{W}$ with finite weak flat dimension;
3. $\text{id}_R(W) \leq n$ for any weak flat $R$-module $W$.

**Definition 3.12.** An exact sequence $0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$ is called $GWP$-pure exact if for any $X \in GWP$, the induced sequence

$$0 \rightarrow \Hom_R(X, L) \rightarrow \Hom_R(X, M) \rightarrow \Hom_R(X, N) \rightarrow 0$$

is exact.

An $R$-module $M$ is called $GWP$-pure projective (resp. $GWP$-pure injective) if $\Hom_R(-, M)$ (resp. $\Hom_R(M, -)$) leaves any $GWP$-pure exact sequence exact.

If $0 \rightarrow L \xrightarrow{f} M \xrightarrow{g} N \rightarrow 0$ is a $GWP$-pure exact sequence, then $f$ is called a $GWP$-pure injection, and $g$ is called a $GWP$-pure surjection.

The following proposition shows the necessity of studying $GWP$-pure exact sequences.

**Proposition 3.13.** Let $M$ be an $R$-module with finite Gorenstein weak projective dimension. Then the following are equivalent:

1. $M$ is Gorenstein weak projective;
(2) For any \(\mathcal{GWP}\)-pure surjection \(j : X \rightarrow Y\) and any \(g : M \rightarrow Y\), there exists \(h : M \rightarrow X\) such that the following diagram commute:

\[
\begin{array}{ccc}
M & \xrightarrow{h} & X \\
\downarrow{g} & & \downarrow{j} \\
Y & & Y
\end{array}
\]

(3) The functor \(\text{Hom}_R(M, -)\) is exact with respect to any \(\mathcal{GWP}\)-pure exact sequence;

(4) Every \(\mathcal{GWP}\)-pure exact sequence
\[
0 \rightarrow L \xrightarrow{f} N \xrightarrow{g} M \rightarrow 0
\]
is split.

**Proposition 3.14.**

\[
\mathcal{GWP} = \{\mathcal{GWP}\text{-pure projective } R\text{-modules}\} \cap \tilde{\mathcal{GWP}},
\]

where \(\tilde{\mathcal{GWP}}\) denote the class of \(R\)-modules with finite Gorenstein weak projective dimension.

**4. Derived functors with respect to Gorenstein weak modules**

In this section, we mainly investigate the homological properties of derived functors with respect to Gorenstein weak modules.

Following [8, Def. 8.1.2] or [17, 1.5], we first give the following definition.

**Definition 4.1.** Let \(M\) be an \(R\)-module. A proper right \(\mathcal{GWTL}\)-resolution of \(M\) is a \(\text{Hom}_R(-, \mathcal{GWTL})\)-exact complex

\[
0 \rightarrow M \rightarrow G^0 \rightarrow G^1 \rightarrow \cdots
\]

with each \(G^i\) Gorenstein weak injective.

Dually, a proper left \(\mathcal{GWP}\)-resolution of \(M\) is a \(\text{Hom}_R(\mathcal{GWP}, -)\)-exact complex

\[
\cdots \rightarrow G_1 \rightarrow G_0 \rightarrow M \rightarrow 0
\]

with each \(G_i\) Gorenstein weak projective.

Note that since injective \(R\)-module is Gorenstein weak injective, every proper right \(\mathcal{GWTL}\)-resolution of an \(R\)-module is exact. Similarly, every proper left \(\mathcal{GWP}\)-resolution of an \(R\)-module is also exact.

Following Proposition 2.17, we have the following lemma which shows the existence of proper right \(\mathcal{GWTL}\)-resolutions (resp. proper left \(\mathcal{GWP}\)-resolutions) of an \(R\)-module.

**Lemma 4.2.** (1) Assume that \(M\) is an \(R\)-module with finite Gorenstein weak injective dimension. Then \(M\) admits a proper right \(\mathcal{GWTL}\)-resolution. In particular, if \(\text{Gwid}_R(M) = n < \infty\), then \(M\) admits a proper right \(\mathcal{GWTL}\)-resolution of length \(n\).

(2) Assume that \(M\) is an \(R\)-module with finite Gorenstein weak projective dimension. Then \(M\) admits a proper left \(\mathcal{GWP}\)-resolution. In particular, if \(\text{Gwpd}_R(M) = n < \infty\), then \(M\) admits a proper left \(\mathcal{GWP}\)-resolution of length \(n\).
We denote by $\text{RightRes}_{M}(GWZ)$ and $\text{LeftRes}_{M}(GWP)$ the full subcategory of $R M$ consisting of those $R$-modules that have a proper right $GWZ$-resolution and a proper left $GWP$-resolution, respectively.

Following [18, 2.4], we define two kinds of right derived functors as follows:

$$\text{Ext}^n_{GWZ}(M, -) = R^n_{GWZ}(M, -), \quad \text{Ext}^n_{GWP}(-, N) = R^n_{GWP}(-, N)$$

for fixed $R$-modules $M$ and $N$, and wish to prove the following theorem:

**Theorem 4.3.** Let $M, N$ be an $R$-modules with $\text{Gwpd}_R(M) < \infty$ and $\text{Gwid}_R(N) < \infty$. Then we have isomorphisms

$$\text{Ext}^n_{GWZ}(M, N) \cong \text{Ext}^n_{GWP}(M, N), n \geq 0$$

which are functorial in $M$ and $N$.

Following [18, Thm. 2.6], it suffices to prove the following lemma.

**Lemma 4.4.** (1) Assume that $N$ is an $R$-module with finite Gorenstein weak injective dimension. Let $G^+ = 0 \to N \to G^0 \to G^1 \to \cdots$ be a proper right $GWZ$-resolution of $N$. Then the sequence

$$0 \to \text{Hom}_R(G, N) \to \text{Hom}_R(G, G^0) \to \text{Hom}_R(G, G^1) \to \cdots$$

is exact for any Gorenstein weak projective $R$-module $G$.

(2) Assume that $M$ is an $R$-module with finite Gorenstein weak projective dimension. Let $G^- = \cdots \to G_1 \to G_0 \to M \to 0$ be a proper left $GWP$-resolution of $M$. Then the sequence

$$0 \to \text{Hom}_R(M, G) \to \text{Hom}_R(G_0, G) \to \text{Hom}_R(G_1, G) \to \cdots$$

is exact for any Gorenstein weak injective $R$-module $G$.

**Proof.** (1) We split the proper resolution $G^+$ into short exact sequences. Hence it suffices to show exactness of $\text{Hom}_R(G, V')$ for any Gorenstein weak projective $R$-module $G$ and any exact sequence $V' = 0 \to N \to G' \to V' \to 0$, where $N \to G'$ is a $GWZ$-preenvelope of some $R$-module $N$ with $\text{Gwid}_R(N) < \infty$. By Proposition 2.17, there is a special exact sequence $V'' = 0 \to N \xrightarrow{\mu} G'' \xrightarrow{v} V'' \to 0$, where $N \to G''$ is a $GWZ$-preenvelope of some $R$-module $N$ with $\text{id}_R(V'') < \infty$. It is easy to verify that the complexes $V'$ and $V''$ are homotopy equivalent, and thus so are the complexes $\text{Hom}_R(G, V')$ and $\text{Hom}_R(G, V'')$ for every Gorenstein weak projective $R$-module $G$. Hence it suffices to show the exactness of $\text{Hom}_R(G, V')$ whenever $G$ is Gorenstein weak projective.

For any Gorenstein weak projective $R$-module $G$, consider an exact sequence $0 \to G \xrightarrow{d'} P^0 \to G^1 \to 0$, where $P^0$ is projective and $G^1$ is Gorenstein weak projective. Assume that $\text{id}_R(V'') = n < \infty$. Since $G^1$ is Gorenstein weak projective, there exists an exact sequence $0 \to G^1 \to P^1 \to G^2 \to 0$, where $P^1$ is projective and $G^2$ is Gorenstein weak projective. Hence we have that $\text{Ext}_R^1(G^1, V'') \cong \text{Ext}_R^2(G^2, V'')$. Continue this process, we
may choose some Gorenstein weak projective $R$-module $G^{n+1}$ such that $\text{Ext}^1_R(G^1, V'') \cong \text{Ext}^2_R(G^2, V'') \cong \cdots \cong \text{Ext}^{n+1}_R(G^{n+1}, V'') = 0$, and thus get the following exact sequence

$$0 \longrightarrow \text{Hom}_R(G^1, V'') \longrightarrow \text{Hom}_R(P^0, V'') \longrightarrow \text{Hom}_R(G, V'') \longrightarrow 0$$

Therefore, for any $g : G \rightarrow V''$, there exists $h : P^0 \rightarrow V''$ such that the following diagram commute:

$$\begin{array}{ccc}
0 & \longrightarrow & G \\
\downarrow g & & \downarrow h \\
\downarrow V'' & & \\
0 & \longrightarrow & P^0 \\
\end{array}$$

Moreover, since $P^0$ is projective, there exists $\rho : P^0 \rightarrow G''$ such that the following diagram commute:

$$\begin{array}{ccc}
0 & \longrightarrow & N \\
\downarrow \mu & \longrightarrow & G'' \\
\downarrow \nu & \longrightarrow & V'' \\
0 & \longrightarrow & P^0 \\
\downarrow h & \longrightarrow & \longrightarrow \\
\downarrow g & \longrightarrow & \\
\end{array}$$

This shows that for any $g : G \rightarrow V''$, there exists $\phi = \rho f : G \rightarrow G''$ such that $g = \nu \phi = \nu_s(\phi)$, and hence

$$0 \longrightarrow \text{Hom}_R(G, N) \xrightarrow{\mu_*} \text{Hom}_R(G, G'') \xrightarrow{\nu_*} \text{Hom}_R(G, V'') \longrightarrow 0.$$
\[ \text{Ext}^n_{GW}(M, N) \cong \text{Ext}_R^n(M, N), \quad n \geq 0. \]

**Proof.** (1) If \( id_R(N) < \infty \), then it is easy to verify that each injective resolution \( I_N \) of \( N \) is a proper right \( GW \)-resolution of \( N \), and hence

\[ \text{Ext}^n_{GW}(M, N) = H^n(\text{Hom}_R(M, I_N)) = \text{Ext}_R^n(M, N), \quad n \geq 0. \]

If \( N \in \text{RightRes}_{GW}(GW) \) and \( pd_R(M) = m < \infty \), then for any Gorenstein weak injective \( R \)-module \( G \), there exists an exact sequence \( 0 \to G' \to I_{m-1} \to \cdots \to I_1 \to I_0 \to G \to 0 \), where each \( I_i \) is injective. It follows then that \( \text{Ext}_R^1(M, G) \cong \text{Ext}_{G'}^{n+1}(M, G') = 0 \). By [16, III, Prop. 1.2A], we have \( \text{Ext}^n_{GW}(M, N) \cong \text{Ext}_R^n(M, N), \quad n \geq 0. \)

The proof of (2) is similar to that of (1), so we omit it here. (3) follows immediately from (1) and (2). \( \square \)

Following [8, Thm. 8.2.3 and 8.2.5], we have the following long exact sequences induced from \( GW \)-proper and \( GW \)-pure exact sequences respectively.

**Proposition 4.7.** (1) Assume that \( \ell_GwD(R) < \infty \). If the sequence \( 0 \to N' \to N \to N'' \to 0 \) is \( GW \)-proper exact, then we have the following exact sequence

\[ 0 \to \text{Hom}_R(M, N') \to \text{Hom}_R(M, N) \to \text{Hom}_R(M, N'') \to \text{Ext}_R^1(GW, M, N') \to \cdots \]

\[ \to \text{Ext}^n_{GW}(M, N') \to \text{Ext}^n_{GW}(M, N) \to \text{Ext}^n_{GW}(M, N'') \to \text{Ext}^{n+1}_{GW}(M, N') \to \cdots \]

for any \( R \)-module \( M \).

(2) Assume that \( \ell_GwpD(R) < \infty \). If the sequence \( 0 \to M' \to M \to M'' \to 0 \) is \( GW \)-pure exact, then we have the following exact sequence

\[ 0 \to \text{Hom}_R(M'', N) \to \text{Hom}_R(M, N) \to \text{Hom}_R(M', N) \to \text{Ext}_R^1(GP, M'', N) \to \cdots \]

\[ \to \text{Ext}^n_{GW}(M', N) \to \text{Ext}^n_{GW}(M, N) \to \text{Ext}^n_{GW}(M'', N) \to \text{Ext}^{n+1}_{GW}(M', N) \to \cdots \]

for any \( R \)-module \( N \).

Now we further give characterizations of Gorenstein weak injective dimension of \( R \)-modules in terms of Gorenstein weak right derived functors.

**Proposition 4.8.** Let \( \ell_GwD(R) < \infty \). Then the following are equivalent:

(1) \( GwD(M) \leq n \);

(2) \( \text{Ext}_R^i(W, M) = 0 \) for any weak injective \( R \)-module \( W \) and any \( i \geq n + 1 \);

(3) \( \text{Ext}_R^i(\widetilde{W}, M) = 0 \) for any \( R \)-module \( \widetilde{W} \) with finite weak injective dimension and any \( i \geq n + 1 \);

(4) For every exact sequence \( 0 \to M \to I^0 \to \cdots \to I^{n-1} \to C^n \to 0 \), where each \( I^i \) is injective, \( C^n \) is Gorenstein weak injective.

(5) For every exact sequence \( 0 \to M \to G^0 \to \cdots \to G^{n-1} \to V^n \to 0 \), where each \( G^i \) is Gorenstein weak injective, \( V^n \) is Gorenstein weak injective.
(6) \( \text{Ext}_{\text{GWZ}}^{n+1}(M,N) = 0 \) for any \( R \)-module \( M \);
(7) \( \text{Ext}_{\text{GWZ}}^i(M,N) = 0 \) for any \( R \)-module \( M \) and \( i \geq n+1 \).

Proof. (1) \( \Leftrightarrow \) (2) \( \Leftrightarrow \cdots \Leftrightarrow \) (5) hold by Proposition 2.19.
(1) \( \Rightarrow \) (7) Since \( \text{Gwid}_R(N) \leq n \), there exists a proper right \( \text{GWZ} \)-resolution of \( N \):
\[
0 \longrightarrow N \longrightarrow G^0 \longrightarrow G^1 \longrightarrow \cdots \longrightarrow G^{n-1} \longrightarrow G^n \longrightarrow 0 ,
\]
and hence \( \text{Ext}_{\text{GWZ}}^i(M,N) = 0 \) for any \( R \)-module \( M \) and \( i \geq n+1 \).
(7) \( \Rightarrow \) (6) is trivial.
(6) \( \Rightarrow \) (1) Let
\[
0 \longrightarrow N \longrightarrow G^0 \longrightarrow G^1 \longrightarrow \cdots \longrightarrow G^{n-1} \longrightarrow G^n \longrightarrow \cdots
\]
be a proper right \( \text{GWZ} \)-resolution of \( N \), and let \( V^0 = N, V^1 = \text{Coker}(N \to G^0) \) and \( V^i = \text{Coker}(G^{i-2} \to G^{i-1}) \) for any \( i \geq 2 \). Then the exact sequence \( 0 \to V^{n-1} \to G^{n-1} \to V^n \to 0 \) is \( \text{GWZ} \)-copure exact. It follows then from Proposition 4.7 that \( \text{Ext}_{\text{GWZ}}^1(V^{n+1},V^n) \cong \text{Ext}_{\text{GWZ}}^2(V^{n+1},V^{n-1}) = 0 \). Similarly, we have \( \text{Ext}_{\text{GWZ}}^1(V^n,V^{n+1}) \cong \text{Ext}_{\text{GWZ}}^{n+1}(V^{n+1},N) = 0 \), and hence
\[
0 \longrightarrow \text{Hom}_R(V^{n+1},V^n) \longrightarrow \text{Hom}_R(V^{n+1},D^n) \longrightarrow \text{Hom}_R(V^{n+1},V^{n+1}) \longrightarrow 0
\]
is exact. This shows that \( 0 \to V^n \to G^n \to V^{n+1} \to 0 \) is split. Therefore, \( V^n \) is Gorenstein weak injective, as desired.

Similarly, we have

**Proposition 4.9.** Let \( \ell \text{.GwpD}(R) < \infty \). Then the following are equivalent:

(1) \( \text{Gwpd}_R(M) \leq n \);
(2) \( \text{Ext}_R^i(M,W) = 0 \) for any weak flat \( R \)-module \( W \) and any \( i \geq n+1 \);
(3) \( \text{Ext}_R^i(M,\widehat{W}) = 0 \) for any \( R \)-module \( \widehat{W} \) with finite weak flat dimension and any \( i \geq n+1 \);
(4) For every exact sequence \( 0 \longrightarrow K_n \longrightarrow P_{n-1} \longrightarrow \cdots \longrightarrow P_0 \longrightarrow M \longrightarrow 0 \), where each \( P_i \) is projective, \( K_n \) is Gorenstein weak projective.
(5) For every exact sequence \( 0 \longrightarrow K'_n \longrightarrow G_{n-1} \longrightarrow \cdots \longrightarrow G_0 \longrightarrow M \longrightarrow 0 \), where each \( G_i \) is Gorenstein weak projective, \( K'_n \) is Gorenstein weak projective.
(6) \( \text{Ext}_{\text{GWP}}^{n+1}(M,N) = 0 \) for any \( R \)-module \( N \);
(7) \( \text{Ext}_{\text{GWP}}^i(M,N) = 0 \) for any \( R \)-module \( N \) and \( i \geq n+1 \).

5. **Tate derived functors with respect to Gorenstein weak modules**

In this section we continue to investigate another derived functor, \( \text{Ext}^n_{\text{GW}}(-,-) \), which connects the usual right derived functor \( \text{Ext}^n_R(-,-) \) with the Gorenstein weak right derived functor \( \text{Ext}^n_{\text{GW}}(-,-) \).

We first introduce the following related notions.
Definition 5.1. A $\mathcal{WI}$-pure exact complex of injective $R$-modules is an exact complex of injective $R$-modules
\[
I = \cdots \rightarrow I_1 \xrightarrow{d_1} I_0 \xrightarrow{d_0} I^0 \xrightarrow{d^0} I^1 \xrightarrow{d^1} \cdots
\]
such that the complex $\text{Hom}_R(W, I)$ is exact for any weak injective $R$-module $W$.

Note that an $R$-module $M$ is Gorenstein weak injective if and only if there is a $\mathcal{WI}$-pure exact complex $I$ of injective $R$-modules such that $M \cong \text{Coker}(I_1 \rightarrow I_0)$. Moreover, if there is a $\mathcal{WI}$-pure exact complex $I$ of injective $R$-modules, then each kernel, cokernel and image in $I$ are Gorenstein weak injective.

Definition 5.2. Let $M$ be an $R$-module. A $\mathcal{WI}$-pure Tate injective resolution of $M$ is a diagram $M \rightarrow E \xrightarrow{u} T$, where $E$ is a deleted injective resolution of $M$ and $T$ is a $\mathcal{WI}$-pure exact exact complex of injective $R$-modules and $u$ is a morphism of complexes such that $u^n$ is isomorphic for $n \gg 0$.

For example, if $M$ is an $R$-module with $\text{id}_R(M) < \infty$, then the zero complex is a $\mathcal{WI}$-pure Tate injective resolution of $M$, and if $M$ is a Gorenstein weak injective $R$-module such that there is a $\text{Hom}_R(\mathcal{W}, -)$-exact exact complex $I = \cdots \rightarrow I_1 \rightarrow I_0 \rightarrow I^0 \rightarrow \cdots$ and $M \cong \text{Coker}(I_1 \rightarrow I_0)$, then $I$ is a $\mathcal{WI}$-pure Tate injective resolution of $M$, in this case $n = 0$.

Lemma 5.3. Let $M$ be an $R$-module. Then $Gwid_R(M) < \infty$ if and only if $M$ has a $\mathcal{WI}$-pure Tate injective resolution.

Proof. Assume that $Gwid_R(M) = n < \infty$. Consider an injective resolution of $M$: $0 \rightarrow M \rightarrow I^0 \rightarrow I^1 \rightarrow \cdots$. Let $V^n = \text{Coker}(I^{n-2} \rightarrow I^{n-1})$. Then $V^n$ is Gorenstein weak injective, and hence we have the following commutative diagram:

\[
\begin{array}{ccccccc}
0 & \rightarrow & N & \rightarrow & I^0 & \rightarrow & I^1 & \cdots & I^{n-1} & \rightarrow & I^n & \rightarrow & \cdots \\
& & \downarrow \cong & & \downarrow & & \downarrow & & \downarrow \cong & & \downarrow \cong & & \\
\cdots & \rightarrow & E^{-1} & \rightarrow & E^0 & \rightarrow & E^1 & \cdots & E^{n-1} & \rightarrow & I^n & \rightarrow & \cdots
\end{array}
\]

where the bottom row is a $\text{Hom}_R(\mathcal{W}, -)$-exact exact sequence. Thus this diagram is a $\mathcal{WI}$-pure Tate injective resolution of $M$.

Conversely, suppose that $M$ has a $\mathcal{WI}$-pure Tate injective resolution as in the above diagram. Then $V^n$ is Gorenstein weak injective, and hence $Gwid_R(M) \leq n < \infty$. □

Definition 5.4. If an $R$-module $M$ has a $\mathcal{WI}$-pure Tate injective resolution $M \rightarrow E \rightarrow T$, then we define the relative Tate cohomology of $M$ with coefficient in an $R$-module $N$ as $\widehat{\text{Ext}}^i_{G\mathcal{W}}(N, M) = H^i(\text{Hom}_R(N, T))$. 
We first claim that the above definition doesn’t depend on the choice of \( W \)'-pure Tate injective resolutions of \( M \). Indeed, assume that \( M \rightarrow E \xrightarrow{u} T \) and \( M \rightarrow E' \xrightarrow{u'} T' \) are two \( W \)'-pure Tate injective resolutions of \( M \) such that \( u'' \) is isomorphic for \( n' \gg 0 \) and \( u'' \) is isomorphic for \( n'' \gg 0 \). Let \( n = \max\{n', n''\} \). If \( i > n \), then \( H^i(\text{Hom}_R(N, T)) \cong \text{Ext}^i_R(N, M) \cong H^i(\text{Hom}_R(N, T')) \). If \( i \leq n \), we consider an exact sequence \( 0 \rightarrow N \rightarrow W \rightarrow V^0 \rightarrow 0 \) with \( W \) a weak injective preenvelope of \( N \), then we have the following exact sequence of complexes

\[
0 \rightarrow \text{Hom}_R(V^0, T) \rightarrow \text{Hom}_R(W, T) \rightarrow \text{Hom}_R(N, T) \rightarrow 0,
\]

which induces a long exact sequence of \( R \)-modules

\[
\cdots \rightarrow H^i(\text{Hom}_R(W, T)) \rightarrow H^i(\text{Hom}_R(N, T)) \rightarrow H^{i+1}(\text{Hom}_R(V^0, T)) \rightarrow H^{i+1}(\text{Hom}_R(W, T)) \rightarrow \cdots.
\]

By Definition \ref{def:5.1}, we have \( H^i(\text{Hom}_R(W, T)) = 0 = H^{i+1}(\text{Hom}_R(W, T)) \), and hence \( H^i(\text{Hom}_R(N, T)) \cong H^{i+1}(\text{Hom}_R(V^0, T)) \). Repeating this process, we may find \( V^j \) such that \( H^i(\text{Hom}_R(N, T)) \cong H^{i+j+1}(\text{Hom}_R(V^j, T)) \) and \( i + j + 1 > n \). Hence \( H^i(\text{Hom}_R(N, T)) \cong \text{Ext}^{i+j+1}_R(V^j, M) \). Similarly, we also have \( H^i(\text{Hom}_R(N, T')) \cong \text{Ext}^{i+j+1}_R(V^j, M) \).

**Proposition 5.5.** Let \( M \) be an \( R \)-module with \( \text{Gwid}_R(M) < \infty \). For an exact sequence \( 0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0 \) of \( R \)-modules, we have the following exact sequence

\[
\cdots \rightarrow \widehat{\text{Ext}}^{i-1}_{GW}(A, M) \rightarrow \widehat{\text{Ext}}^i_{GW}(C, M) \rightarrow \widehat{\text{Ext}}^i_{GW}(B, M) \rightarrow \widehat{\text{Ext}}^{i-1}_{GW}(A, M) \rightarrow \widehat{\text{Ext}}^{i+1}_{GW}(C, M) \rightarrow \cdots
\]

for any \( i \in \mathbb{Z} \).

**Proof.** By Lemma \ref{lem:5.3}, \( M \) has a \( W \)'-pure Tate injective resolution \( M \rightarrow E \xrightarrow{u} T \). Since each term of \( T \) is injective, we have the following exact sequence of complexes \( 0 \rightarrow \text{Hom}_R(C, T) \rightarrow \text{Hom}_R(B, T) \rightarrow \text{Hom}_R(A, T) \rightarrow 0 \), which induces a long exact sequence

\[
\cdots \rightarrow H^{i-1}(\text{Hom}_R(A, T)) \rightarrow H^i(\text{Hom}_R(C, T)) \rightarrow H^i(\text{Hom}_R(B, T)) \rightarrow H^i(\text{Hom}_R(A, T)) \rightarrow H^{i+1}(\text{Hom}_R(C, T)) \rightarrow \cdots,
\]

as required. \( \square \)

The following theorem shows the case of vanishing of relative Tate cohomology defined as in Definition \ref{def:5.4}.

**Theorem 5.6.** Let \( M \) be an \( R \)-module with \( \text{Gwid}_R(M) = n < \infty \). The following are equivalent:

1. \( \text{id}_R(M) \leq n \);
(2) $\text{id}_R(M) < \infty$;
(3) $\widehat{\text{Ext}}^i_{GW}(N, M) = 0$ for any $R$-module $N$ and any $i \in \mathbb{Z}$;
(4) $\widehat{\text{Ext}}^i_{GW}(R/I, M) = 0$ for any left ideal $I$ of $R$ and any $i \in \mathbb{Z}$.

Proof. (1)⇒(2) and (3)⇒(4) are trivial.
(2)⇒(3). Since $\text{id}_R(M) < \infty$, we may take a $\mathcal{WI}$-pure Tate injective resolution of $M$ to be the zero complex, and thus $\widehat{\text{Ext}}^i_{GW}(N, M) = 0$ for any $N \in R\text{-Mod}$ and any $i \in \mathbb{Z}$.
(4)⇒(1). We use induction on $n = \text{Gwid}_R(M) < \infty$. If $\text{Gwid}_R(M) = 0$, then $\text{Ext}^1_R(R/I, M) \cong \text{Ext}^{-1}_{GW}(R/I, M) = 0$ for any left ideal $I$ of $R$, which implies that $M$ is injective, i.e. $\text{id}_R(M) = 0$. Now we assume that $\text{Gwid}_R(M) > 0$, and let $M \rightarrow E \xrightarrow{u} T$ be a $\mathcal{WI}$-pure Tate injective resolution of $M$ and $M' = \text{Coker}(M \rightarrow E^0)$. Then we have an exact sequence $0 \rightarrow M \rightarrow E^0 \rightarrow M' \rightarrow 0$ with $E^0$ injective. Moreover, $\text{Gwid}_R(M') \leq n - 1$ and $T[-1]$ is a weak Tate injective resolution of $M'$. This implies that $\widehat{\text{Ext}}^i_{GW}(N, M') \cong \widehat{\text{Ext}}^{i-1}_{GW}(N, M)$ for any $N \in R\text{-Mod}$ and any $i \in \mathbb{Z}$. In particular, $\widehat{\text{Ext}}^i_{GW}(R/I, M') \cong \widehat{\text{Ext}}^{i-1}_{GW}(R/I, M) = 0$ for any left ideal $I$ of $R$ and any $i \in \mathbb{Z}$. This implies $\text{id}_R(M') \leq n - 1$ by the induction hypothesis, and hence $\text{id}_R(M) \leq n$.

We also have the following long exact sequence with respect to the usual cohomology, the Gorenstein weak cohomology and the relative Tate cohomology, which is similar to that in [21, Sec. 4]:

**Lemma 5.7.** Let $M$ be an $R$-module with $\text{Gwid}_R(M) < \infty$. Then we have a long exact sequence

$$0 \rightarrow \text{Ext}^1_{GW}(N, M) \rightarrow \text{Ext}^1_R(N, M) \rightarrow \widehat{\text{Ext}}^1_{GW}(N, M) \rightarrow \text{Ext}^2_{GW}(N, M) \rightarrow \cdots$$

for any $R$-module $N$.

Both this and the following proposition show that the relative Tate cohomology measures the distance between the cohomology and the Gorenstein weak cohomology.

**Proposition 5.8.** Let $M$ and $N$ be $R$-modules with $\text{Gwid}_R(M) = n < \infty$. If $\text{id}_R(M) < \infty$, then the natural transformation $\text{Ext}^i_{GW}(N, M) \rightarrow \text{Ext}^i_R(N, M)$ is a natural isomorphism for any $0 \leq i \leq n$, and $\text{Ext}^i_R(N, M) = 0$ for any $i > n$.

Proof. If $0 < i \leq n$, then it follows from Theorem 5.6 and Lemma 5.7. Moreover, $\text{Ext}^0_{GW}(N, M) \cong \text{Hom}_R(N, M) \cong \text{Ext}^0_R(N, M)$. So the assertion holds for $0 \leq i \leq n$. Furthermore, $\text{Ext}^i_{GW}(N, M) = 0 = \widehat{\text{Ext}}^i_{GW}(N, M)$ whenever $i > n$, which implies that $\text{Ext}^i_R(N, M) = 0$ for all $i > n$ by the exact sequence of Lemma 5.7.

**Lemma 5.9.** Let $M$ and $N$ be $R$-modules with $\text{id}_R(N) < \infty$ or $\text{pd}_R(N) < \infty$. If $M$ admits a $\mathcal{WI}$-pure Tate injective resolution $M \rightarrow E \xrightarrow{u} T$. Then $\widehat{\text{Ext}}^i_{GW}(N, M) = 0$ for any $i \in \mathbb{Z}$. 


Proof. We only prove the case \( id_R(N) < \infty \), the proof of the case \( pd_R(N) < \infty \) is similar.

By Definition 5.4, it suffices to prove that the complex \( \text{Hom}_R(N, T) \) is exact.

We use induction on \( n = id_R(N) < \infty \). If \( n = 0 \), then \( \text{Hom}_R(N, T) \) is exact. Now we assume that \( n > 0 \), and consider an exact sequence \( 0 \rightarrow N \rightarrow E \rightarrow N' \rightarrow 0 \) with \( E \) injective and thus \( id_R(N') = n - 1 \). Then we have the following exact sequence of complexes

\[
0 \rightarrow \text{Hom}_R(N', T) \rightarrow \text{Hom}_R(E, T) \rightarrow \text{Hom}_R(N, T) \rightarrow 0.
\]

Note that the complex \( \text{Hom}_R(E, T) \) is exact and the complex \( \text{Hom}_R(N', T) \) is also exact by the induction hypothesis, which implies that the complex \( \text{Hom}_R(N, T) \) is exact, as desired. \( \square \)

By this lemma, we can refine Proposition 5.8 as follows.

**Proposition 5.10.** Let \( M \) and \( N \) be \( R \)-modules with \( \text{Gwid}_R(M) = n < \infty \). If \( id_R(M) < \infty \) or \( id_R(N) < \infty \), then the natural transformation \( \text{Ext}^i_{\text{Gwid}}(N, M) \rightarrow \text{Ext}^i_R(N, M) \) is a natural isomorphism for any \( 0 \leq i \leq n \), and \( \text{Ext}^i_R(N, M) = 0 \) for any \( i > n \).

Similar to Definitions 5.1, 5.2 and 5.4, we give the following definitions.

**Definition 5.11.** A \( \mathcal{WF} \)-copure exact complex of projective \( R \)-modules is an exact complex of projective \( R \)-modules

\[
P = \cdots \rightarrow P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} P^0 \xrightarrow{d^0} P^1 \xrightarrow{d^1} \cdots
\]

such that the complex \( \text{Hom}_R(P, W) \) is exact for any weak flat \( R \)-module \( W \).

Note that \( M \in R\text{-Mod} \) is Gorenstein weak projective if and only if there is a \( \mathcal{WF} \)-copure exact complex \( P \) of projective \( R \)-modules such that \( M \cong \text{Coker}(P_1 \rightarrow P_0) \). Moreover, if there is a \( \mathcal{WF} \)-copure exact complex \( P \) of projective \( R \)-modules, then each kernel, cokernel and image in \( P \) are Gorenstein weak projective.

**Definition 5.12.** Let \( M \) be an \( R \)-module. A \( \mathcal{WF} \)-copure Tate projective resolution of \( M \) is a diagram \( T \rightarrow P \xrightarrow{u} M \), where \( P \) is a deleted projective resolution of \( M \) and \( T \) is a \( \mathcal{WF} \)-copure exact complex of projective \( R \)-modules and \( u \) is a morphism of complexes such that \( u_n \) is isomorphic for \( n \gg 0 \).

For example, if \( M \in R\text{-Mod} \) with \( pd_R(M) < \infty \), then the zero complex is a \( \mathcal{WF} \)-copure Tate projective resolution of \( M \), and if \( M \in R\text{-Mod} \) is a Gorenstein weak projective module such that there is a \( \text{Hom}_R(-, \mathcal{WF}) \)-exact exact complex \( P = \cdots \rightarrow P_1 \rightarrow P_0 \rightarrow P^0 \rightarrow \cdots \) and \( M \cong \text{Coker}(P_1 \rightarrow P_0) \), then \( P \) is a \( \mathcal{WF} \)-copure Tate projective resolution of \( M \).
Theorem 5.15. Let \( \text{N} \) or a injective resolution of \( \text{Ext} \) as \( \text{WF} \) on the choice of \( \text{M} \) weak projective. Let above resolution is of the form by the definition of \( \text{WF} \). It is obvious that \( \text{H} \) that \( \text{Ext} \) \( i \) \( \in \text{WF} \) \( \in \text{M} \) then \( \text{Ext} \) \( i \) \( \in \text{WF} \) \( \in \text{M} \), then we define the relative Tate cohomology of \( \text{M} \) with coefficient in an \( \text{R} \)-module \( \text{N} \) as
\[
\widehat{\text{Ext}}_{\text{GW}}(\text{M}, \text{N}) = \text{H}^i(\text{Hom}_\text{R}(\text{T}, \text{N})).
\]

As a similar argument in the above, we may show that this definition doesn’t depend on the choice of \( \text{WF} \)-copure Tate projective resolutions of \( \text{M} \).

It is well-known that \( \text{Ext}^n_i(\text{M}, \text{N}) \) can be compute by a projective resolution of \( \text{M} \) or a injective resolution of \( \text{N} \). It is natural to ask that whether \( \widehat{\text{Ext}}_{\text{GW}}^n(\text{M}, \text{N}) = \widehat{\text{Ext}}_{\text{GW}}^n(\text{M}, \text{N}) \) hold or not? The following theorem gives an affirmative answer.

Lemma 5.13. Let \( \text{M} \) and \( \text{N} \) be \( \text{R} \)-modules with \( \text{Gwpd}_\text{R}(\text{M}) < \infty \) and \( \text{GwId}_\text{R}(\text{M}) < \infty \). Then \( \widehat{\text{Ext}}^i_{\text{GW}}(\text{M}, \text{N}) = \widehat{\text{Ext}}^i_{\text{GW}}(\text{M}, \text{N}) \) for any \( i \in \mathbb{Z} \).

Proof. We use induction on \( n = \text{Dpd}_\text{R}(\text{M}) < \infty \). If \( \text{M} \) is Gorenstein weak projective, then \( \text{M} \) admits a \( \text{WF} \)-copure Tate projective resolution \( \text{T} \xrightarrow{u} \text{P} \xrightarrow{\pi} \text{M} \), where \( T_i = P_i \) and \( u_i = \text{Id}_{P_i} \) for any \( i \geq 0 \). Thus \( \widehat{\text{Ext}}^i_{\text{GW}}(\text{M}, \text{N}) \cong \text{Ext}^i_{\text{R}}(\text{M}, \text{N}), i \geq 1 \). Note that \( \text{Ext}^i_{\text{GW}}(\text{M}, \text{N}) \cong \text{Ext}^i_{\text{GW}}(\text{M}, \text{N}) = 0 \) for any \( i \geq 1 \), and hence \( \widehat{\text{Ext}}_{\text{GW}}^i(\text{M}, \text{N}) \cong \text{Ext}^i_{\text{R}}(\text{M}, \text{N}) \) by Lemma 5.7. Therefore, \( \widehat{\text{Ext}}_{\text{GW}}^i(\text{M}, \text{N}) = \widehat{\text{Ext}}_{\text{GW}}^i(\text{M}, \text{N}) \) for any \( i \geq 1 \). Now we see the case \( i \leq 0 \). Since \( \text{N} \) has finite Gorenstein weak injective dimension, we may take a \( \text{WF} \)-pure Tate injective resolution \( \text{N} \xrightarrow{\text{E}} \text{T}' \). Assume that \( \text{T} \) in the above resolution is of the form
\[
\cdots \xrightarrow{d_1} \text{P}_1 \xrightarrow{d_0} \text{P}_0 \xrightarrow{d_0} \text{P}_{-1} \xrightarrow{d_{-1}} \text{P}_{-2} \xrightarrow{d_{-2}} \cdots .
\]

By the definition of \( \text{WF} \)-copure Tate projective resolution, each \( \text{M}_i := \text{Im}d_i \) is Gorenstein weak projective. Let \( \text{M}_{-1} = \text{Im}d_{-1} \). Then we have an exact sequence
\[
0 \xrightarrow{1} \text{M} \xrightarrow{d_{-1}} \text{M}_{-1} \xrightarrow{d_{-2}} 0.
\]

Since each term of \( \text{T}' \) is injective, we have the following exact sequence of complexes
\[
0 \xrightarrow{1} \text{Hom}_\text{R}(\text{M}_{-1}, \text{T}') \xrightarrow{1} \text{Hom}_\text{R}(\text{P}_{-1}, \text{T}') \xrightarrow{1} \text{Hom}_\text{R}(\text{M}, \text{T}') \xrightarrow{1} 0,
\]
which induced the following exact sequence
\[
\cdots \xrightarrow{1} \text{H}^i(\text{Hom}_\text{R}(\text{P}_{-1}, \text{T}')) \xrightarrow{1} \text{H}^i(\text{Hom}_\text{R}(\text{M}, \text{T}')) \xrightarrow{1} \text{H}^{i+1}(\text{Hom}_\text{R}(\text{M}_{-1}, \text{T}')) \xrightarrow{1} \text{H}^{i+1}(\text{Hom}_\text{R}(\text{P}_{-1}, \text{T}')) \xrightarrow{1} \cdots .
\]

It is obvious that \( \text{H}^i(\text{Hom}_\text{R}(\text{P}_{-1}, \text{T}')) = 0 = \text{H}^{i+1}(\text{Hom}_\text{R}(\text{P}_{-1}, \text{T}')) \). So \( \text{H}^i(\text{Hom}_\text{R}(\text{M}, \text{T}')) \cong \text{H}^{i+1}(\text{Hom}_\text{R}(\text{M}_{-1}, \text{T}')) \), that is, \( \widehat{\text{Ext}}_{\text{GW}}^i(\text{M}, \text{N}) \cong \text{Ext}_{\text{GW}}^{i+1}(\text{M}_{-1}, \text{N}) \). Repeating this process, we may get that \( \widehat{\text{Ext}}_{\text{GW}}^i(\text{M}, \text{N}) \cong \text{Ext}_{\text{GW}}^{i+1}(\text{M}_{i-1}, \text{N}) \). On the other hand, it is
obvious to verify that $\mathbf{T}[−1] \to \mathbf{P}[−1] \to M_{−1}$ is a $\mathcal{WF}$-copure Tate projective resolution of $M_{−1}$, and hence $\widehat{\operatorname{Ext}}^i_{\text{GWP}}(M_{−1}, N) \cong \widehat{\operatorname{Ext}}^{i−1}_{\text{GWP}}(M, N)$.

By a similar argument as in the above, we have $\widehat{\operatorname{Ext}}^i_{\text{GWP}}(M, N) \cong \widehat{\operatorname{Ext}}^i_{\text{GWP}}(M_{i−1}, N)$, and hence

$$\widehat{\operatorname{Ext}}^i_{\text{GWP}}(M, N) \cong \widehat{\operatorname{Ext}}^{i−1}_{\text{GWP}}(M_{i−1}, N) \cong \widehat{\operatorname{Ext}}^i_{\text{GWP}}(M, N), \ i \leq 0.$$ 

Therefore, we have $\widehat{\operatorname{Ext}}^i_{\text{GWP}}(M, N) = \widehat{\operatorname{Ext}}^i_{\text{GWP}}(M, N), i \in \mathbb{Z}$, whenever $M$ is Gorenstein weak projective.

Assume that the assertion holds for the case $n−1$. Let $\text{Gwpd}_R(M) = n$ and consider a $\mathcal{WF}$-copure Tate projective resolution of $M$ as follows:

$$\begin{align*}
&\begin{array}{c}
M \\
\phantom{=} \\
\uparrow \\
\mathbf{P} \\
\phantom{=} \\
\uparrow \\
\mathbf{T}
\end{array}
\quad = \quad \begin{array}{c}
M \\
\phantom{=} \\
\uparrow \\
\cdots \to P_2 \to P_1 \to P_0 \\
\phantom{=} \\
\uparrow \\
\cdots \to T_2 \to T_1 \to T_0 \to T_{−1} \to \cdots
\end{array}
\end{align*}$$

Let $M_1 = \ker(P_0 \to M)$. Then we have an exact sequence $0 \to M_1 \to P_0 \to M \to 0$, and it is easy to verify that the following diagram

$$\begin{align*}
&\begin{array}{c}
M_1 \\
\phantom{=} \\
\uparrow \\
\mathbf{P}[1] \\
\phantom{=} \\
\uparrow \\
\mathbf{T}[1]
\end{array}
\quad = \quad \begin{array}{c}
M_1 \\
\phantom{=} \\
\uparrow \\
\cdots \to P_3 \to P_2 \to P_1 \\
\phantom{=} \\
\uparrow \\
\cdots \to T_3 \to T_2 \to T_1 \to T_0 \to T_{−1} \to \cdots
\end{array}
\end{align*}$$

is a $\mathcal{WF}$-copure Tate projective resolution of $M_1$. Thus $\widehat{\operatorname{Ext}}^i_{\text{GWP}}(M, N) = \widehat{\operatorname{Ext}}^{i−1}_{\text{GWP}}(M_1, N)$ for any $i \in \mathbb{Z}$.

On the other hand, with a similar argument to the first step, we have the following exact sequence of complexes

$$0 \to \operatorname{Hom}_R(M, \mathbf{T}') \to \operatorname{Hom}_R(P_0, \mathbf{T}') \to \operatorname{Hom}_R(M_1, \mathbf{T}') \to 0$$

which induced the following exact sequence

$$\cdots \to H^{i−1}(\operatorname{Hom}_R(P_0, \mathbf{T}')) \to H^{i−1}(\operatorname{Hom}_R(M_1, \mathbf{T}')) \to H^i(\operatorname{Hom}_R(M, \mathbf{T}')) \to H^i(\operatorname{Hom}_R(P_0, \mathbf{T}')) \to \cdots$$

It is obvious that $H^{i−1}(\operatorname{Hom}_R(P_0, \mathbf{T}')) = 0 = H^i(\operatorname{Hom}_R(P_0, \mathbf{T}'))$ for any $i \in \mathbb{Z}$. Thus $H^i(\operatorname{Hom}_R(M_1, \mathbf{T}')) \cong H^{i+1}(\operatorname{Hom}_R(M, \mathbf{T}'))$, that is, $\widehat{\operatorname{Ext}}^{i−1}_{\text{GWP}}(M_1, N) \cong \widehat{\operatorname{Ext}}^i_{\text{GWP}}(M, N)$. Since $Dpd_R(M_1) = n−1$, we have $\widehat{\operatorname{Ext}}^i_{\text{GWP}}(M_1, N) \cong \widehat{\operatorname{Ext}}^{i−1}_{\text{GWP}}(M_1, N)$ by the induction hypothesis. Therefore, $\widehat{\operatorname{Ext}}^i_{\text{GWP}}(M, N) \cong \widehat{\operatorname{Ext}}^i_{\text{GWP}}(M, N)$ for any $i \in \mathbb{Z}$, as desired. □
**Definition 5.16.** Let $M$ and $N$ be $R$-modules with $Gwpd_R(M) < \infty$ and $Gwid_R(N) < \infty$. Then we define

$$\widehat{\text{Ext}}_n^{GW}(M, N) := \widehat{\text{Ext}}_n^{GW_T}(M, N) \cong \widehat{\text{Ext}}_n^{GW_P}(M, N), \ n \geq Z$$

and call it the $n$th Gorenstein weak Tate cohomology.

Following Lemma 5.7, we have the following proposition.

**Proposition 5.17.** Let $M$ and $N$ be $R$-modules with $Gwid_R(M) < \infty$ and $Gwid_R(N) < \infty$. Then we have a long exact sequence

$$0 \to \text{Ext}^1_{GW}(N, M) \to \text{Ext}^1_R(N, M) \to \widehat{\text{Ext}}^1_{GW}(N, M) \to \text{Ext}^2_{GW}(N, M) \to \cdots$$

**References**

[1] M. Auslander, M. Bridger, Stable Module Theory, Am. Math. Soc., 1969
[2] L. L. Avramov, A. Martsinkovsky, Absolute, relative, and Tate cohomology of modules of finite Gorenstein dimension, Proc. Lond. Math. Soc., 2006, 85: 393-440
[3] M. Behboodi, A. Ghorbani, A. Moradzadeh-Dehkordi, S. H. Shojaee, C-pure projective modules, Comm. Algebra, 2013, 41(12): 4559-4575
[4] L. W. Christensen, Gorenstein Dimension, Lecture Notes in Math., vol. 1747, Springer-Verlag, Berlin, 2000
[5] L. W. Christensen, A. Frankild, H. Holm, On Gorenstein projective, injective and flat dimensions—a functorial description with applications, J. Algebra, 2006, 302: 231-279
[6] N. Ding, Y. Li, L. Mao, Strongly Gorenstein flat modules, J. Aust. Math. Soc., 2009, 86: 323-338
[7] E. E. Enochs, O. M. G. Jenda, Gorenstein injective and projective modules, Math. Z. 1995, 220: 611-633
[8] E. E. Enochs, O. M. G. Jenda, Relative Homological Algebra, de Gruyter Exp. Math., vol. 30, Walter de Gruyter, Berlin, New York, 2000
[9] S. Eilenberg, J. C. Moore, Foundations of relative homological algebra, Am. Math. Soc., 1965
[10] S. M. Fakhruddin, Pure-injective modules, Glasgow Math. J., 1973, 14(02): 120-122
[11] Z. Gao, Coherent rings and Gorenstein FP-injective modules, Comm. Algebra, 2012, 40: 1669-1679
[12] J. Gillespie, Model structures on modules over Ding-Chen rings, Homology, Homotopy Appl., 2010, 12(1): 61-73
[13] Z. Gao, Z. Huang, Weak injective covers and dimension of modules, preprint available at: http://math.nju.edu.cn/~huangzy/papers/weakinj.pdf
[14] Z. Gao, F. Wang, All Gorenstein hereditary rings are coherent, J. Algebra Appl., 2014, 13(4): 1350140 (5 pages)
[15] Z. Gao, F. Wang, Weak injective and weak flat modules, Comm. Algebra (to appear)
[16] R. Hartshorne, Algebraic Geometry, Springer, 1977
[17] H. Holm, Gorenstein homological dimensions, J. Pure Appl. Algebra, 2004, 189: 167-193
[18] H. Holm, Gorenstein derived functors, Proc. Am. Math. Soc., 2004, 132: 1913-1923
[19] Z. Huang, Proper resolutions and Gorenstein categories, J. Algebra, 2013, 393(1): 142-169
[20] Z. Huang, Duality of preenvelopes and pure injective modules. Canad. Math. Bull., 2014, 57(2): 318-325
[21] A. Iacob, Generalized Tate cohomology, Tsukuba J. Math., 2005, 29: 389-404
[22] L. Mao, N. Ding, Gorenstein FP-injective and Gorenstein flat modules, J. Algebra Appl., 2008, 7(4): 491-506
[23] N. Mahdou, M. Tamekkante, Strongly Gorenstein flat modules and dimensions, Chin. Ann. Math., 2011, 32B(4): 533-548
[24] J. J. Rotman, An Introduction to Homological Algebra, New York: Springer, 2009
[25] G. Simmons, Cyclic-purity: a generalization of purity for modules, Houston J. Math., 1987, 13(1): 135-150
[26] B. Stenström, Coherent rings and FP-injective modules, J. London Math. Soc., 1970, 2: 323-329
[27] S. Sather-Wagstaff, T. Sharif, D. White, Stability of Gorenstein categories, J. London Math. Soc., 2008, 77: 481-502
[28] R. B. Warfield, Jr., Purity and algebraic compactness for modules, Pacific J. Math., 1969, 28(3): 699-719
[29] R. Wisbauer, Foundations of Module and Ring Theory, CRC Press, 1991.
[30] G. Yang, Homological properties of modules over Ding-Chen rings, J. Korean Math. Soc., 2012, 49(1): 31-47
[31] F. Zareh-Khoshschehreh, M. Asgharzadeh, K. Divaani-Aazar, Gorenstein homology, relative pure homology and virtually Gorenstein rings, J Pure Appl. Algebra (to appear)
[32] T. Zhao, Y. Xu, On Gorenstein weak injective modules, arXiv:1405.0648, 2014

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