Hunt for 3-Schur polynomials

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Abstract

This paper describes our attempt to understand the recent success of Na Wang in constructing the 3-Schur polynomials, associated with the plane partitions. We provide a rather detailed review and try to figure out the new insights, which allowed to overcome the problems of the previous efforts. In result we provide a very simple definition of time-variables \( P_i \geq j \) and the cut-and-join operator \( \hat{W}_2 \), which generates the set of 3-Schur functions. Some coefficients in \( \hat{W}_2 \) remain undefined and require more effort to be fixed.

1 Introduction

Schur functions play a prominent role in modern theoretical physics. Today we understand the basic reason: they provide the right non-trivial basis, adjusted to exhaustive description of Gaussian integrals \([1–6]\). Somewhat unexpectedly, this basis includes an \( a \) priori hidden structure of Young diagrams (partitions) and a good piece of representation theory behind them. Schur functions have a number of generalizations – to Jack, Q-Schur \([7]\), Macdonald \([8, 9]\) and Shiraishi \([10]\) functions, as well as a more mysterious one to Kerov functions \([11–13]\). Generalization to Jacks is the most straightforward and associated to \( \beta \)-deformed Gaussians. Macdonald functions, while formally associated with the further \( q,t \)-deformation, are long anticipated to be also related to plane partitions, i.e. to the still undeveloped theory of 3-Schur functions \([14]\). The lasting failure to develop their theory by elementary means, suggests the full-fledged use of the underlying representation theory, which in this case is the theory of Yangians \([15–18]\) (and further raised to DIM algebras \([19–21]\)). Recently, Na Wang \([22]\) made a very good progress in this direction, based on the old presentations of \([23, 24]\) and further developed in \([25–27]\). Our goal in this paper is to reproduce and understand these results – and to figure out new insights, which allowed to overcome the problems, reported in \([14, 28–30]\).

2 The case of 2-Schur and Jack polynomials

To outline the suggested strategy, we present the sample calculation for Jack polynomials. The ordinary 2-Schur functions arise as a particular case when parameter \( \beta = 1 \). We do not describe the origins of the method, which is inspired by the above-cited literature. It will be discussed in greater details in a lengthier presentation in future works.

The starting point is the cut-and-join operator

\[
\hat{W}_2 = \frac{1}{2} \sum_{a,b=1}^{\infty} \left( ab p_{a+b} \frac{\partial^2}{\partial p_a \partial p_b} + (a+b) p_a p_b \frac{\partial}{\partial p_{a+b}} \right) - \frac{\eta^2 - 1}{2\eta} \sum_{a=2}^{\infty} (a-1)p_a \frac{\partial}{\partial p_a}
\]

where we rescaled \(^1\) the time-variables \( p_a := \eta p_a \) with \( \eta := \sqrt{\beta} \) to make the formulas more symmetric and transparent. In particular, the \( \beta \)-dependent parameter is the one which defines the central charge in the AGT relations \( c = 1 - \frac{6(\eta^2-1)}{\eta^2} \). This will also provide an unusual, but clever normalization of Jacks.

\(^1\)Comparing with ordinary presentation (4.83) from \([24]\). Note the change of variable \( h = \eta \).
In case of Jack polynomials the cut-and-join operator (1) is known in full generality therefore one can derive
all the Jack polynomials as its eigenfunctions. However, we develop another method of deriving Jacks from
cut-and-join operator that will be useful in lifting the whole construction to the level of 3-Schur functions.

Our method involves operators \( \hat{e}_0 \) and \( \hat{e}_1 \) that are defined as follows:
\[
\hat{e}_0 := p_1 \\
\hat{e}_1 := \left[ \hat{W}_2, \hat{e}_0 \right] = \sum_{a=1}^{\infty} a p_{a+1} \frac{\partial}{\partial p_a}
\]

The action of these operators on the vacuum state \( |\varnothing\rangle := 1 \) is independent of \( \eta \),
\[
\hat{e}_0 \cdot 1 = p_1, \quad \hat{e}_1 \cdot 1 = 0 \\
\hat{e}_0 \hat{e}_0 \cdot 1 = p_1^2, \quad \hat{e}_1 \hat{e}_0 \cdot 1 = p_2 \\
\hat{e}_0 \hat{e}_0 \hat{e}_0 \cdot 1 = p_1^3, \quad \hat{e}_1 \hat{e}_0 \hat{e}_0 \cdot 1 = 2p_2p_1, \quad \hat{e}_0 \hat{e}_1 \hat{e}_0 \cdot 1 = p_2 p_1
\]

This \( \eta \)-independence depends on the factor \((a-1)\) in the last term in \( \hat{W}_2 \), which could actually be \( a-c \) with
arbitrary \( c \) – what corresponds to linear combination of \( \hat{W}_2 \) with \( \hat{W}_1 := \sum_{a=1}^{\infty} a p_a \frac{\partial}{\partial p_a} \). We prefer our choice
with \( \eta \)-independent \( \hat{e}_1 \).

Our method also involves the content-function \( \omega_{i,j} \) for the Young diagrams:
\[
\omega_{i,j} := -\eta(i-1) + \frac{(j-1)}{\eta}
\]

We use the convention that the starting box of the Young diagram has coordinate \((1,1)\), therefore \( \omega_{1,1} = 0 \).

We provide illustrating examples of computation of the first three levels of Jacks polynomials and then explain
the construction in great detail.

- 1 level:
\[
\hat{e}_0 \cdot 1 = j_{\square} = p_1
\]

- 2 level:
\[
\begin{pmatrix}
\hat{e}_0 \hat{e}_0 \cdot 1 \\
\hat{e}_1 \hat{e}_0 \cdot 1 \\
\hat{e}_0 \hat{e}_1 \hat{e}_0 \cdot 1 \\
\hat{e}_1 \hat{e}_1 \hat{e}_0 \cdot 1
\end{pmatrix}
= 
\begin{pmatrix}
\begin{bmatrix}
C_{\text{1,1}} \omega_{1,2} C_{\text{2,1}} \\
\eta^2 + 1
\end{bmatrix} \\
\begin{bmatrix}
C_{\text{1,2}} + C_{\text{2,1}} \\
\omega_{1,2} C_{\text{1,2}} + \omega_{2,1} C_{\text{2,1}}
\end{bmatrix}
\end{pmatrix}
\begin{pmatrix}
\begin{bmatrix}
C_{\text{1,1}} \omega_{1,2} C_{\text{2,1}} \\
\eta^2 + 1
\end{bmatrix} \\
\begin{bmatrix}
C_{\text{1,2}} + C_{\text{2,1}} \\
\omega_{1,2} C_{\text{1,2}} + \omega_{2,1} C_{\text{2,1}}
\end{bmatrix}
\end{pmatrix}
\]

\[
\begin{cases}
J_{\square} = \frac{\eta}{\eta^2 + 1}(\eta p_2^2 + p_2) \\
J_{\square} = \frac{1}{\eta^2 + 1}(p_1^2 - \eta p_2)
\end{cases}
\]

- 3 level:
\[
\begin{pmatrix}
\hat{e}_0 \hat{e}_0 \hat{e}_0 \cdot 1 \\
\hat{e}_0 \hat{e}_1 \hat{e}_0 \cdot 1 \\
\hat{e}_1 \hat{e}_0 \hat{e}_0 \cdot 1 \\
\hat{e}_1 \hat{e}_1 \hat{e}_0 \cdot 1
\end{pmatrix}
= 
\begin{pmatrix}
\begin{bmatrix}
C_{\text{1,1}} \omega_{1,2} \omega_{1,3} C_{\text{2,1}} \\
\eta^2 + 1
\end{bmatrix} \\
\begin{bmatrix}
C_{\text{1,2}} + C_{\text{2,1}} \\
\omega_{1,2} C_{\text{1,2}} + \omega_{2,1} C_{\text{2,1}}
\end{bmatrix}
\end{pmatrix}
\begin{pmatrix}
\begin{bmatrix}
C_{\text{1,1}} \omega_{1,2} \omega_{1,3} C_{\text{2,1}} \\
\eta^2 + 1
\end{bmatrix} \\
\begin{bmatrix}
C_{\text{1,2}} + C_{\text{2,1}} \\
\omega_{1,2} C_{\text{1,2}} + \omega_{2,1} C_{\text{2,1}}
\end{bmatrix}
\end{pmatrix}
\]

\[
\begin{cases}
J_{\square} = \frac{\eta^2}{(\eta^2 + 2)(\eta^2 + 1)}(\eta^2 p_3^3 + 3\eta p_2 p_1 + 2p_3) \\
J_{\square} = \frac{3\eta}{(\eta^2 + 2)(2\eta^2 + 1)}(\eta p_1^3 + (1 - \eta^2)p_2 p_1 - \eta p_3) \\
J_{\square} = \frac{1}{(2\eta^2 + 1)(\eta^2 + 1)}(p_1^3 - 3\eta p_2 p_1 + 2\eta^2 p_3)
\end{cases}
\]
2.1 Explanation of the method

In our method Jack polynomials are solutions of the linear system of equations. This system is constructed as follows:

1. The l.h.s. of the equation is the column of "words" of operators $\hat{e}_0, \hat{e}_1$ acting on $|\emptyset\rangle := 1$. On the level $n$ each "word" consist of $n$ operators. Note that $\hat{e}_1 \cdot 1 = 0$.

2. The r.h.s of the equation on the level $n$ is the matrix $M_n$ multiplied by the column of Jack polynomials.

3. The matrix elements of $M_n$ consist of coefficients $C_{\hat{R}}$, where $\hat{R}$ is a Young tableaux, and combinations of content-function computed at specific points. Coefficients $C_{\hat{R}}$ enter the column of the matrix that corresponds to Jack polynomial $J_{\hat{R}}$ of Young diagram $\hat{R}$.

4. The rule to insert content-function is as follows. We look at the positions $k$ of $\hat{e}_1$ in the "word" – counted from the right. Then we look what is the box $\square_k$ in the diagram labeled by this number and put the corresponding factor $\omega_{\square_k}$ in front of the factor $C_{\hat{R}}$. The procedure depends on $k$, i.e. on the "word" at the l.h.s. and on the Young tableaux $\hat{R}$ at the r.h.s. For example, for $\hat{e}_0 \hat{e}_1 \hat{e}_0 \cdot 1$ we have a single $k = 2$, thus the coefficient in front of $C_{\hat{R}}$ will be $\omega_{1,2}$, while that in front of $C_{\hat{0}}$ will be $\omega_{2,1}$. For the word $\hat{e}_1 \hat{e}_1 \hat{e}_0 \cdot 1$ there will be two $k = 2, 3$, and both $C_{\hat{R}}$ and $C_{\hat{0}}$ acquire the same coefficients $\omega_{2,1}\omega_{1,2}$.

5. The "word" in the first line does not contain $\hat{e}_1$ operators, therefore according to p.4 the matrix element corresponding to $J_\emptyset$ is simply the sum of coefficients $C_{\hat{R}}$ for all possible Young tableau of shape $\hat{R}$. To fix the normalization of Jack polynomials we choose the following condition: the sum of coefficients $C_{\hat{R}}$ is equal to the number of Young tableau of shape $\hat{R}$.

\[ \sum_{\hat{R}} C_{\hat{R}} = \text{number of Young tableau of shape } \hat{R} \]  

(11)

In particular, for one line and one column Young diagram the corresponding coefficient is equal to one, because for these Young diagrams there are only one possible Young tableau. For example, $C_{\hat{\{\emptyset\}}} = 1$.

6. We solve this system with respect to Jack polynomials $J_{\hat{R}}$ and unknown coefficients $C_{\hat{R}}$ expanding both sides of equations in a basis of times.

One can check that the above defined Jack polynomials are eigenfunctions of $\hat{W}_2$ with the eigenvalues which depend only on the diagram, not tableaux:

\[ \hat{W}_2 J_{\hat{R}} = \lambda_{\hat{R}} J_{\hat{R}}, \quad \lambda_{\hat{R}} = \sum_{\square \in \hat{R}} \omega_{\square} \]  

(12)

We do not prove this statement here, however we show on a particular example, that the eigenfunction property is in full agreement with our definition of Jacks. Consider the following expression:

\[ \hat{W}_2 \hat{e}_0^3 \overset{(9)}{=} \hat{W}_2 J_{\hat{\{\emptyset\}}} + 2 \hat{W}_2 J_{\hat{\{\square\}}} + \hat{W}_2 J_{\hat{\{\square\}}} \]  

(13)

that should be equal to

\[ \hat{W}_2 J_{\hat{\{\emptyset\}}} + 2 \hat{W}_2 J_{\hat{\{\square\}}} + \hat{W}_2 J_{\hat{\{\square\}}} \overset{(12)}{=} (\omega_{3,1} + \omega_{2,1}) J_{\hat{\{\emptyset\}}} + 2(\omega_{2,1} + \omega_{1,2}) J_{\hat{\{\square\}}} + (\omega_{1,3} + \omega_{1,2}) J_{\hat{\{\square\}}} \]  

(14)

according to the eigenfunction property. Both sides of this equation are indeed equal: the r.h.s.

\[ (\omega_{3,1} + \omega_{2,1}) J_{\hat{\{\emptyset\}}} + 2(\omega_{2,1} + \omega_{1,2}) J_{\hat{\{\square\}}} + (\omega_{1,3} + \omega_{1,2}) J_{\hat{\{\square\}}} \overset{(9)}{=} (\hat{e}_1 \hat{e}_0 + \hat{e}_0 \hat{e}_1) \hat{e}_0 \cdot 1 \]  

(15)

while the l.h.s.

\[ \hat{W}_2 \hat{e}_0^3 = (\hat{e}_1 \hat{e}_0 + \hat{e}_0 \hat{W}_2 \hat{e}_0) \hat{e}_0 \cdot 1 = (\hat{e}_1 \hat{e}_0 + \hat{e}_0 \hat{e}_1) \hat{e}_0 \cdot 1 \]  

(16)

where we used the commutator $[\hat{W}_2, \hat{e}_0] = \hat{e}_1$. 

3
2.2 Jacks at the level 4

At the next level we encounter 5 Young diagrams and $1 + 3 + 2 + 3 + 1 = 10$ Young tableaux. Five polynomials $J_R$ and $10 - 5 = 5$ constants $C_R$ are defined from the $2^4 = 8$ decompositions:

$$
\hat{e}_0 \hat{e}_0 \hat{e}_0 \hat{e}_0 \cdot 1 = \sum_{i=1}^{3} \left( C_{\text{1,1}} + C_{\text{1,2}} + C_{\text{1,3}} \right) \mathbf{J}_{\text{1,1}} + \sum_{i=2}^{3} \left( C_{\text{2,1}} + C_{\text{2,2}} \right) \mathbf{J}_{\text{2,1}} + \sum_{i=3}^{3} \left( C_{\text{3,1}} + C_{\text{3,2}} + C_{\text{3,3}} \right) \mathbf{J}_{\text{3,1}}
$$

$$
\hat{e}_1 \hat{e}_0 \hat{e}_0 \hat{e}_0 \cdot 1 = \omega_{1,4} \sum_{i=1}^{3} \left( C_{\text{1,1}} + C_{\text{1,2}} + C_{\text{1,3}} \right) \mathbf{J}_{\text{1,1}} + \left\{ \omega_{2,1} \left( C_{\text{2,1}} + C_{\text{2,2}} \right) \mathbf{J}_{\text{2,1}} + \left\{ \omega_{1,3} \left( C_{\text{1,1}} + C_{\text{1,2}} + C_{\text{1,3}} \right) \mathbf{J}_{\text{1,1}} + \right\} \right\}
$$

We do not write the remaining six equations, they can be easily obtained by the rule, formulated in the section 2.1.

The result is:

$$
\begin{align*}
C_{\text{1,1}} &= \frac{3(\eta^2 + 1)}{2(\eta^2 + 2)}, & C_{\text{1,2}} &= \frac{(2\eta^2 + 1)(\eta^2 + 3)}{2(\eta^2 + 2)(\eta^2 + 1)}, & C_{\text{1,3}} &= \frac{\eta^2 + 3}{2(\eta^2 + 1)}; \\
C_{\text{2,1}} &= \frac{2(2\eta^2 + 1)}{3(\eta^2 + 1)}, & C_{\text{2,2}} &= \frac{2(\eta^2 + 2)}{3(\eta^2 + 1)}, & C_{\text{2,3}} &= \frac{3(\eta^2 + 1)}{2(\eta^2 + 1)}; \\
C_{\text{3,1}} &= \frac{3(\eta^2 + 1)}{2(2\eta^2 + 1)}, & C_{\text{3,2}} &= \frac{(\eta^2 + 2)(3\eta^2 + 1)}{2(2\eta^2 + 1)(\eta^2 + 1)}, & C_{\text{3,3}} &= \frac{3\eta^2 + 1}{2(\eta^2 + 1)}.
\end{align*}
$$

$$
\begin{align*}
\mathbf{J}_{\text{2,1}} &= \eta^3 \left( \eta^3 p_1^4 + 6\eta^2 p_2 p_1^2 + 8\eta p_3 p_1 + 3\eta p_2^2 + 6p_4 \right) \\
\mathbf{J}_{\text{1,2}} &= \frac{2\eta^2}{(\eta^2 + 3)(\eta^2 + 1)^2} \left( \eta^2 p_1^4 + (3\eta - \eta^3) p_2 p_1^2 + 2(1 - \eta^2) p_3 p_1 - \eta^2 p_2^2 - 2\eta p_4 \right) \\
\mathbf{J}_{\text{3,1}} &= \frac{3\eta^2}{(2\eta^2 + 1)(\eta^2 + 2)(\eta^2 + 1)^2} \left( \eta^2 p_1^4 + 2(\eta - \eta^3) p_3 p_1^2 - 4\eta^2 p_3 p_1 + (\eta^4 + \eta^2 + 1)p_2^2 + (\eta^3 - \eta)p_4 \right) \\
\mathbf{J}_{\text{2,2}} &= \frac{2\eta}{(3\eta^2 + 1)(\eta^2 + 1)^2} \left( \eta p_1^4 + (1 - 3\eta^2) p_2 p_1^2 + 2(\eta^3 - \eta) p_3 p_1 - \eta p_2^2 + 2\eta^2 p_4 \right) \\
\mathbf{J}_{\text{3,2}} &= \frac{1}{(3\eta^2 + 1)(2\eta^2 + 1)(\eta^2 + 1)} \left( p_1^4 - 6\eta p_2 p_1^2 + 3\eta^2 p_2^2 + 8\eta^2 p_3 p_1 - 6\eta^3 p_4 \right)
\end{align*}
$$

These polynomials are eigenfunctions of $\hat{W}_2$:

$$
\begin{align*}
\hat{W}_2 \cdot \mathbf{J}_{\text{2,1}} &= (\omega_{1,4} + \omega_{3,1} + \omega_{2,1}) \mathbf{J}_{\text{2,1}} \\
\hat{W}_2 \cdot \mathbf{J}_{\text{1,2}} &= (\omega_{3,1} + \omega_{2,1} + \omega_{1,2}) \mathbf{J}_{\text{1,2}} \\
\hat{W}_2 \cdot \mathbf{J}_{\text{3,1}} &= (\omega_{2,2} + \omega_{2,1} + \omega_{1,2}) \mathbf{J}_{\text{3,1}} \\
\hat{W}_2 \cdot \mathbf{J}_{\text{2,2}} &= (\omega_{1,3} + \omega_{2,1} + \omega_{1,2}) \mathbf{J}_{\text{2,2}} \\
\hat{W}_2 \cdot \mathbf{J}_{\text{3,2}} &= (\omega_{1,4} + \omega_{1,3} + \omega_{1,2}) \mathbf{J}_{\text{3,2}}
\end{align*}
$$
Most of these facts about Jack polynomials are well known – from other approaches, like the one reviewed in [13]. The main result of this section is that our current method to define them continues to work pretty well when the number of Young tableaux exceeds the number of equations (which, in its turn, continues to exceed the number of Young diagrams). This encourages its application in the \textit{terra incognita} – to the study of 3-Schurs, in a way which slightly deviates from Wang’s.

### 3 Towards 3-Schur polynomials

Now we apply the procedure to the case of 3-Schur polynomials, taking into account appropriate modifications: 2D Young diagrams become 3D Young diagrams/plane partitions, cut-and-join operator $\hat{W}_2$ and the content-function $\omega$ are deformed.

Let the content-function have the form

$$\omega_{i,j,k} := \omega_1 \cdot (i - 1) + \omega_2 \cdot (j - 1) + \omega_3 \cdot (k - 1)$$

(21)

where three parameters $\omega_1, \omega_2, \omega_3$ are bound by one condition $\sigma_1 := \omega_1 + \omega_2 + \omega_3 = 0$. The other two invariant combinations $\sigma_2 := \omega_1 \omega_2 + \omega_1 \omega_3 + \omega_2 \omega_3$, $\sigma_3 := \omega_1 \omega_2 \omega_3$ will enter the deformed cut-and-join operator. The case of Jack polynomials in the above sections corresponds to $\omega_1 = -\eta, \omega_2 = \frac{1}{2}, \omega_3 = \eta - \frac{1}{2}$ and is distinguished by the vanishing of the product $(1 + \omega_2)(1 + \omega_1 \omega_2)(1 + \omega_2 \omega_3)$ which eliminates the ”extra” 3d polynomials.

Let the deformed cut-and-join operator have the following form

$$\hat{W}_2 = \frac{1}{2} \sum_{a,b=1}^{\infty} \sum_{i=1}^{a} \sum_{j=1}^{b} \left( A_{i,j-1} \cdot a \cdot \frac{\partial^2}{\partial P_{a,i} \partial P_{a,j}} + B_{i,j-1} \cdot (a + b) \cdot \frac{\partial}{\partial P_{a,i+j-1}} \right) +$$

$$+ \frac{1}{2} \sum_{a=1}^{\infty} \sum_{i=1}^{a} u_{a,i} (a - 1) \frac{\partial}{\partial P_{a,i}} + \frac{1}{2} \sum_{a=1}^{\infty} \sum_{i=1}^{a} (a - 1) \left( v_{a,i} \frac{\partial}{\partial P_{a,i+1}} + w_{a,i} \frac{\partial}{\partial P_{a,i}} \right)$$

(22)

with the triangular set of times $P_{a,i}$, where $i \leq a$ and time-independent functions $A, B, u, v, w$. There are several comments on the form of the cut-and-join operator. The above ansatz is inspired by the form of 2d cut-and-join operator (1). One can write the following general operators

$$P_{a+b,k} \frac{\partial^2}{\partial P_{a,i} \partial P_{b,j}}, \quad P_{a,i} \frac{\partial}{\partial P_{a+b,k}}, \quad P_{a,i} \frac{\partial}{\partial P_{a,j}}$$

(23)

that generalize 2d cut-and-join operator for the case when times have two indices. For first and second operators in (23) we consider only $k = i + j - 1$ as the most simplest law applicable to 2d case (indeed, for 2d case all the times have the form $P_{a,1}$ and $i = j = k = 1$). For the third operator in (23) we also consider the simplest opportunity $i = j \pm 1$ and $i = j$. More involved form of ansatz for $\hat{W}_2$ will be considered in future works.

To reproduce the Wang’s answers [22] we impose a normalization condition

$$A_1 B_1 = 1$$

(24)

This condition actually breaks the homogeneity and we discuss the other normalization in Section 5. Define

$$\hat{e}_0 := P_1,1$$

$$\hat{e}_1 := [\hat{W}_2, \hat{e}_0] = \sum_{a=1}^{\infty} \sum_{i=1}^{a} A_i \cdot a \cdot \frac{\partial}{\partial P_{a,i}}$$

$$\hat{e}_2 := [\hat{W}_2, \hat{e}_1]$$

(25)

The factor $a - 1$ in the last terms in (22) does not allow this term to contribute to $\hat{e}_1$, making $\hat{e}_1$ a pure differential operator.

Now the slightly modify the method from section 2.1 in order to apply it to the more involved case of 3-Schur functions:
1. We consider one more operator $\hat{e}_2$ to produce "words" at the l.h.s. of the system. It looks natural as we proceed from 2d, where we need 2 operators, to 3d case. The p.4 of 2.1 is modified: We look at the positions $k$ of $\hat{e}_i$ in the "word" – counted from the right. Then we look what is the box $\Box_k$ in the diagram labeled by this number and put the corresponding factor $(\omega_{\Box_k})^i$ in front of the factor $C_R$. In other words, for $\hat{e}_2$ the factor is $(\omega_{\Box_k})^2$. This rule is inspired by the constructions of [24].

2. In 3d case we have additional unknown parameters - the coefficients of the ansatz (22), therefore we use the eigenfunction property as additional equation to solve:

$$\hat{W}_2 S_R = \lambda_R S_R, \quad \lambda_R = \sum_{\Box \in R} \omega_{\Box}$$

(26)

3.1 Level 1

At this level there is only one Young diagram and everything is trivial:

$$S_{\Box} := \hat{e}_0 \cdot 1 = P_{1,1}, \quad \hat{W}_2 \cdot S_{\Box} = 0. \quad (27)$$

3.2 Level 2

At this level there are three Young diagrams and corresponding 3-Schur polynomials:

$$\hat{e}_0 \hat{e}_0 \cdot 1 = P_{1,1}^2 = S_{\Box} + S_{\Box \Box} + S_{\Box \Box \Box}$$

$$\hat{e}_1 \hat{e}_0 \cdot 1 = A_1 P_{2,1} = \omega_{2,1,1} S_{\Box} + \omega_{1,2,1} S_{\Box \Box} + \omega_{1,1,2} S_{\Box \Box \Box}$$

$$\hat{e}_2 \hat{e}_0 \cdot 1 = P_{1,1}^2 + A_1 \sigma_3 P_{2,1} + A_1 w_{2,1} P_{2,2} = \omega_{2,1,1}^2 S_{\Box} + \omega_{1,2,1}^2 S_{\Box \Box} + \omega_{1,1,2}^2 S_{\Box \Box \Box}$$

We omit constants $C_{\Box} = C_{\Box \Box} = C_{\Box \Box \Box} = 1$. Then substitute into

$$\hat{W}_2 \cdot S_{\Box} = \omega_{2,1,1} S_{\Box}, \quad \hat{W}_2 \cdot S_{\Box \Box} = \omega_{1,2,1} S_{\Box \Box}, \quad \hat{W}_2 \cdot S_{\Box \Box \Box} = \omega_{1,1,2} S_{\Box \Box \Box}$$

(28)

The above equations fix three parameters:

$$u_{2,1} = \sigma_3, \quad u_{2,2} = -\sigma_3, \quad v_{2,1} = -\frac{(1 + \omega_1 \omega_2)(1 + \omega_1 \omega_3)(1 + \omega_2 \omega_3)}{w_{2,1}} \quad (29)$$

Then

$$S_{\Box} = \frac{(1 + \omega_2 \omega_3) P_{1,1}^2 + A_1 \omega_1 (1 + \omega_2 \omega_3) P_{2,1} + w_{2,1} P_{2,2}}{(\omega_1 - \omega_2)(\omega_1 - \omega_3)} \quad (30)$$

and the other two polynomials are obtained by permutations of $\omega_i$. In other words, a permutation of $\omega_i$ corresponds to transposition of the Young diagram in the corresponding plane:

$$S_{\Box \Box} = S_{\Box}(\omega_1 \leftrightarrow \omega_2), \quad S_{\Box \Box \Box} = S_{\Box}(\omega_1 \leftrightarrow \omega_3) \quad (31)$$

In [22] the parameters are taken to be $A_1 = 1$ and $w_{2,1} = \sqrt{(1 + \omega_1 \omega_2)(1 + \omega_1 \omega_3)(1 + \omega_2 \omega_3)}$, but in our calculation they are not yet fixed at this level. Still we impose these constraints to simplify the formulas and their reduction to the Jack case.
3.3 Level 3

Now we define the 6 polynomials $S_{1,1}$, $S_{1,2}$, $S_{2,1}$, $S_{2,2}$, $S_{3,1}$, $S_{3,2}$ from a system of 9 equations ($i,j = 0,1,2$):

\[ \hat{e}_i \hat{e}_j \cdot \hat{c}_0 = 1 = (\omega_{2,1,1})^{i}(\omega_{3,1,1})^{j} S_{1,1} + (\omega_{1,2,1})^{i}(\omega_{1,3,1})^{j} S_{1,2} + (\omega_{1,1,2})^{i}(\omega_{1,1,3})^{j} S_{2,1} + (\omega_{2,1,1})^{i}(\omega_{2,2,1})^{j} S_{2,2} + (\omega_{1,1,2})^{i}(\omega_{2,1,1})^{j} C_{1,1} + (\omega_{1,1,1})^{i}(\omega_{1,1,1})^{j} C_{2,1} \]

The rule to construct this equation is just the same as p.5 in section 2.1. As we mentioned above, the addition compared to 2d case is that each $\hat{e}_2$ at position $k$ contributes a square of $\omega_{C_{2k}}$. The system is overdefined, but it is resolvable for appropriate choice of parameters. Namely, we need

\[ C_{1,1} + C_{1,2} = C_{1,2} + C_{2,1} = C_{2,1} + C_{2,2} = 2 \] (33)

Then we can ask if the polynomials $S_{\pi}$, deduced from this resolvable system can be eigenfunctions of $\hat{W}_2$. This is not just automatic, the eigenfunction conditions

\[ \hat{W}_2 \cdot S_{1,1} = (\omega_{3,1,1} + \omega_{2,1,1}) S_{1,1} \]
\[ \hat{W}_2 \cdot S_{1,2} = (\omega_{1,3,1} + \omega_{1,2,1}) S_{1,2} \]
\[ \hat{W}_2 \cdot S_{2,1} = (\omega_{1,1,3} + \omega_{1,1,2}) S_{2,1} \]
\[ \hat{W}_2 \cdot S_{2,2} = (\omega_{2,1,1} + \omega_{1,1,2}) S_{2,2} \]
\[ \hat{W}_2 \cdot S_{3,1} = (\omega_{2,1,1} + \omega_{1,2,1}) S_{3,1} \]
\[ \hat{W}_2 \cdot S_{3,2} = (\omega_{2,1,1} + \omega_{1,2,1}) S_{3,2} \]

are satisfied, provided either

\[ A_2 B_2 = -\frac{1}{3}, \quad u_{3,2} = -\sigma_3, \quad u_{3,3} = 0, \quad v_{3,1} = -\frac{2A_1}{A_2} \frac{(1 + \omega_1 \omega_2)(1 + \omega_1 \omega_3)(1 + \omega_2 \omega_3)}{w_{2,1}}, \quad v_{3,2} = \frac{2}{3 w_{3,2}} \] (37)

or

\[ A_2 B_2 = \frac{2}{3}, \quad u_{3,2} = u_{3,3} = -\frac{\sigma_3}{2}, \quad v_{3,1} = -\frac{2A_1}{3A_2} \frac{(1 + \omega_1 \omega_2)(1 + \omega_1 \omega_3)(1 + \omega_2 \omega_3)}{w_{2,1}}, \quad v_{3,2} = -\frac{(2 + \omega_1 \omega_2)(2 + \omega_1 \omega_3)(2 + \omega_2 \omega_3)}{12 w_{3,2}} \] (38)
In what follows we make the second choice \((38)\) as it leads to Wang’s answer. The nature of the other solution \((37)\) remains unclear. Then

\[
\begin{align*}
\mathbf{S}_\pi &= \frac{1}{(\omega_1 - \omega_2)(\omega_1 - \omega_3)(2\omega_1 - \omega_2)(2\omega_1 - \omega_3)} \left[ (1 + \omega_2\omega_3)(2 + \omega_2\omega_3)\mathbf{P}_{1,1}^1 + \\
&+ 3A_1\omega_1(1 + \omega_2\omega_3)(2 + \omega_2\omega_3)\mathbf{P}_{2,1}\mathbf{P}_{1,1} + 3A_1(2 + \omega_2\omega_3)w_{2,1}\mathbf{P}_{2,2}\mathbf{P}_{1,1} + \\
&+ 2A_1^2\omega_1^2(1 + \omega_2\omega_3)(2 + \omega_2\omega_3)\mathbf{P}_{3,1} + 3A_1A_2w_{2,1}\omega_1(2 + \omega_2\omega_3)\mathbf{P}_{3,2} + 6A_1A_2w_{2,1}w_{3,2}\mathbf{P}_{3,3} \right] \\
\mathbf{S}_\bar{\pi} &= \frac{3}{(\omega_3 - \omega_1)(\omega_3 - \omega_2)(2\omega_2 - \omega_1)(2\omega_1 - \omega_1)} \left[ (1 + \omega_2\omega_3)(1 + \omega_1\omega_3)\mathbf{P}_{1,1}^1 + \\
&- A_1\omega_3(1 + \omega_2\omega_3)(1 + \omega_1\omega_3)\mathbf{P}_{2,1}\mathbf{P}_{1,1} + A_1(3 - \omega_3^2)w_{2,1}\mathbf{P}_{2,2}\mathbf{P}_{1,1} + \\
&+ A_1^2\omega_1\omega_2(1 + \omega_2\omega_3)(1 + \omega_1\omega_3)\mathbf{P}_{3,1} + A_1A_2w_{2,1}(-\omega_3 + 3/2\omega_1\omega_2\omega_3)\mathbf{P}_{3,2} + 3A_1A_2w_{2,1}w_{3,2}\mathbf{P}_{3,3} \right]
\end{align*}
\]

(39)

For an appropriate choice of constants:

\[
\begin{align*}
A_1 &= 1, & w_{2,1} &= \sqrt{(1 + \omega_1\omega_2)(1 + \omega_1\omega_3)(1 + \omega_2\omega_3)}, \\
A_2 &= 1, & w_{3,2} &= \frac{1}{6}\sqrt{(2 + \omega_1\omega_2)(2 + \omega_1\omega_3)(2 + \omega_2\omega_3)}.
\end{align*}
\]

(40)

this reproduces the result of \([22]\).

We do not present the results and implications of the complicated analysis of the level-4 example in this short paper. They will be described in a separate more technical presentation.

### 4 On the definition/evaluation of \(C_\pi\)

The crucial ingredient of the Yangian approach to generalized Schur polynomials (like Jacks for Young diagrams and 3-Schurs for plane partitions) is the idea that they are associated with tableaux rather than with diagrams, i.e. remember the sequence of box additions which lead to building up the diagram. Thus we get a huge set of polynomials \(\mathbf{S}_\pi\) at the r.h.s. of our defining relations

\[
\text{words}\{\hat{e}_2, \hat{e}_1, \hat{e}_0\} \cdot 1 = \mathcal{G}\{\omega\} \cdot \{\mathbf{S}_\pi\}
\]

(41)

\(\mathcal{G}\{\omega\}\) here is the known matrix, build from the function \(\omega_{i,j,k} = \omega_1 \cdot (i - 1) + \omega_2 \cdot (j - 1) + \omega_3 \cdot (k - 1)\), but it is rectangular at each level \(l\), of the size \(d^{l-1} \times \text{tab}^{(l)}_d\), where \(d = 3\) and \(\text{tab}^{(l)}_d\) is the number of tables. The problem is that while \(d^{l-1}\) exceeds the number of diagrams, it is smaller than the number of tables:

\[
\text{dia}^{(l)}_d \leq d^{l-1} \leq \text{tab}^{(l)}_d
\]

(42)

This makes the matrix rectangular in the wrong way – the system of equations for \(P_\pi\) is underdefined and does not have a unique solution.

There are three ways to handle this problem.

**First**, one can extend the number of words, considering the action of higher operators \(\hat{e}_a\), generated by the same recursion rule \(\hat{e}_a = [\hat{W}_2, \hat{e}_{a-1}]\). This would add more lines to the matrix \(\mathcal{G}\) (made from powers \(\omega_{i,j,k}^a\)), and make the system overdefined. Still it will have solutions – due to conspiracy, implied by Yangian representation theory. This approach is most straightforward in the approach of this paper, but it is computationally very hard, because it deals with a linear problem for very big matrices. The above-mentioned conspiracy allows to drastically simplify the calculation – moreover, at least two additional steps are available, which we describe as the two next options in our list.

**Second**, we can use the fact that the dependence of \(P_\pi\) on the table is minor:

\[
\mathbf{S}_\pi\{\mathbf{P}\} = C_\pi\mathbf{S}_\pi\{\mathbf{P}\}
\]

(43)

where \(C_\pi\) are constants, i.e. do not depend on the time-variables \(\mathbf{P}\). This substitutes the system \((41)\)

\[
\text{words}\{\hat{e}_2, \hat{e}_1, \hat{e}_0\} \cdot 1 = \mathcal{G}\{\omega\} \cdot \{\mathbf{S}_\pi\}
\]

(44)
with a much smaller $d^{l-1} \times \text{dia}_d^{(l)}$ matrix $G\{\omega\} = G\{\omega\} \cdot C_{\pi}$ which is already rectangular in the proper way – the linear system for $S_{\pi}$ is overdefined. Indeed, $\text{dia}_d^{(l)}$ is the number of diagrams, generated by MacMahon function

$$\sum_l \text{dia}_d^{(l)} q^l = \prod_n \frac{1}{(1-q^n)^n} = 1 + q + 3q^2 + 6q^3 + 13q^4 + 24q^5 + 48q^6 + 86q^7 + 160q^8 + 282q^9 + \ldots$$

$$< 1 + \sum_{l=1}^3 3^{l-1} q^l \quad (45)$$

This is the 3d analogue of the more familiar

$$\sum_l \text{dia}_2^{(l)} q^l = \prod_n \frac{1}{(1-q^n)^n} = 1 + q + 2q^2 + 3q^3 + 5q^4 + 7q^5 + 11q^6 + 15q^7 + 22q^8 + 30q^9 + \ldots$$

$$< 1 + \sum_{l=1}^2 2^{l-1} q^l \quad (46)$$

To compare,

$$\sum_l \text{tab}_3^{(l)} q^l = 1 + q + 3q^2 + 9q^3 + 33q^4 + 135q^5 + 633q^6 + 3207q^7 + 17602q^8 + 103041q^9 + \ldots$$

$$> 1 + \sum_{l=1}^3 3^{l-1} q^l \quad (47)$$

and

$$\sum_l \text{tab}_2^{(l)} q^l = 1 + q + 2q^2 + 4q^3 + 10q^4 + 26q^5 + 76q^6 + 232q^7 + 764q^8 + 2620q^9 + \ldots$$

$$> 1 + \sum_{l=1}^2 2^{l-1} q^l \quad (48)$$

Solution to the system (44) now depends on the choice of the constants $C_{\pi}$, but since the system is overdefined, there are additional constraints, which actually allow to fix these constants. We illustrated this method in consideration of Jacks for $d = 2$. Still, at $d = 3$ it is still computationally difficult: although the linear system for $S_{\pi}$ is now much smaller, one needs to keep additional parameters $C_{\pi}$ in its coefficients, and there are a lot.

Third, one can deduce the expressions of $C_{\pi}$ through $\omega_{i,j,k}$ from representation theory, and construct matrix $G\{\omega\}$ in (44) explicitly. Then what remains is just to solve the linear system (44) for $S_{\pi}$. This is the strategy which we actually follow for 3-Schurs in this paper.

Explicit formulas for $C_{\pi}$ are implied by the rule [22–24] to add boxes to the table:

$$\tilde{C}_{\pi + \Box} = C_{\pi} \cdot \sqrt{\frac{\omega_{\Box} + \sigma_3}{\omega_{\Box}} \cdot \text{res}_{\Box \in \pi} \phi(\omega_{\Box} - \omega_{\Box'})}.$$

(49)

where

$$\phi(u) := \frac{(u + \omega_1)(u + \omega_2)(u + \omega_3)}{(u - \omega_1 + \epsilon)(u - \omega_2 + \epsilon)(u - \omega_3 + \epsilon)}.$$

(50)

The product over $\pi$ is actually singular at $\epsilon = 0$ and one needs to take a residue at the pole $\epsilon^{-1}$. To get $\tilde{C}_{\pi}$ one needs to take a product of $|\pi| - 1$ such square roots for addition of every box in the process of building up $\tilde{\pi}$. We put tilde over $\tilde{C}_{\pi}$, because these polynomials are actually normalized differently from ours – to get our normalization we impose conditions like (33) on $C_{\pi}$. To obtain our $C_{\pi}$ we need the ratios of these $\tilde{C}_{\pi}$, which are actually full squares, i.e. the square root will not show up in the answers for $C_{\pi}$.

There are simple and elegant formula for the ratios of $\tilde{C}_{\pi}$ that explains the absence of square roots in the final answer. The formula was suggested to us by the anonymous reviewer:

$$\frac{\tilde{C}_{\pi}}{\tilde{C}_{\pi'}} = \prod_{\text{exchanges of boxes } \Box, \Box'} \phi(\omega_{\Box} - \omega_{\Box'}).$$

(51)
where the product is over all pair exchanges of boxes needed to transform \( \pi \) to \( \pi' \). The result is finite and no residues and square roots need to be taken.

There is a comment in order to clarify the above formula. The boxes \( \square \) and \( \square' \) should have adjacent numbers in the Young tableaux and the number of \( \square \) should be greater, otherwise the ratio is inversed. For example:

\[
\frac{\tilde{C}}{C} = \phi(\omega_{2,1,1} - \omega_{1,2,1}), \quad \frac{\tilde{C}}{\hat{C}} = \phi(\omega_{1,2,1} - \omega_{2,1,1})
\]  

(52)

Here the boxes with numbers 2, 3 are exchanged and the formula is correct. We provide another example to show that exchanging boxes with non adjacent numbers gives wrong result:

\[
\frac{\tilde{C}}{C} \neq \phi(\omega_{2,1,1} - \omega_{1,2,1})
\]  

(53)

Namely, one can not exchange boxes with numbers 2 and 4 via one transposition. One should do in three steps 4 \( \leftrightarrow \) 3, 3 \( \leftrightarrow \) 2 and then 4 \( \leftrightarrow \) 3:

\[
\frac{\tilde{C}}{C} = \phi(\omega_{2,1,1} - \omega_{1,2,1}) \cdot \phi(\omega_{1,2,1} - \omega_{1,2,1}) \cdot \phi(\omega_{1,2,1} - \omega_{1,2,1})
\]  

(54)

Now we provide a few examples of calculation at levels three and four using normalization condition from p.5 section 2.1.

**Example. Level 3**

Here we have three pairs of interesting diagrams of which we consider just one, relevant for \( S \):

\[
\tilde{C} = \sqrt{\frac{\omega_{2,1,1} + \sigma_3 \text{res}_{\omega} \phi(\omega_{2,1,1}) \phi(\omega_{2,1,1})}{\omega_{2,1,1} \sigma_3}} \cdot \left[ \frac{\omega_{1,2,1} + \sigma_3 \text{res}_{\omega} \phi(\omega_{1,2,1})}{\omega_{1,2,1} \sigma_3} \right]
\]

(55)

These constants \( \tilde{C}, \tilde{C} \) differ from ours \( C, C \) by overall factor:

\[
\tilde{C} + \tilde{C} \neq 2
\]  

(56)

however, the ratio is the same:

\[
\frac{C}{C} = \frac{\tilde{C}}{\hat{C}} = \phi(\omega_{1,2,1} - \omega_{2,1,1}) = \frac{2\omega_1 - \omega_2}{\omega_1 - 2\omega_2}
\]  

(57)

These equations are enough to compute coefficients in our normalization. In this particular case we reproduce the result of previous sections:

\[
C = \frac{2(2\omega_1 - \omega_2)}{3(\omega_1 - \omega_2)}, \quad \hat{C} = \frac{2(2\omega_2 - \omega_1)}{3(\omega_2 - \omega_1)}
\]  

(58)

Substituting \( \omega_1 = -\eta, \omega_2 = \eta^{-1} \) we get the answer for the Jack case (10):

\[
C = \frac{2(2\eta^2 + 1)}{3(\eta^2 + 1)}, \quad \hat{C} = \frac{2(\eta^2 + 2)}{3(\eta^2 + 1)}
\]  

(59)
Example. Level 4

Using elegant formula (51) we obtain:

\[
C_1 = \frac{(3\omega_1 - \omega_2)(\omega_1 - 2\omega_2)}{3(\omega_1 - \omega_2)^2} \cdot C_3
\]

\[
C_2 = \frac{(3\omega_1 - \omega_2)(\omega_1 - 2\omega_2)}{3(\omega_1 - \omega_2)^2} \cdot C_3
\]

(60)

Using normalization condition \(C_1 + C_2 + C_3 = 3\) we compute the constants:

\[
C_1 = \frac{3(\omega_1 - \omega_2)}{2(2\omega_1 - \omega_2)} \quad C_2 = \frac{(3\eta^2 + 1)(\eta^2 + 2)}{2(2\eta^2 + 1)(\eta^2 + 1)} \quad C_3 = \frac{3\eta^2 + 1}{2(\eta^2 + 1)}
\]

(61)

what reproduce the Jack values (18) at \(\omega_1 = -\eta, \omega_2 = \eta^{-1}\):

\[
C_1 = \frac{3(\eta^2 + 1)}{2(2\eta^2 + 1)}, \quad C_2 = \frac{(3\eta^2 + 1)(\eta^2 + 2)}{2(2\eta^2 + 1)(\eta^2 + 1)} \quad C_3 = \frac{3\eta^2 + 1}{2(\eta^2 + 1)}
\]

(62)

Constants \(C_\pi\) for the other tables that lie in different planes are obtained by the permutations of \(\omega_1, \omega_2, \omega_3\) in (61). For example, \(\omega_1 \leftrightarrow \omega_2\):

\[
C_1 = \frac{3(\omega_2 - \omega_1)}{2(2\omega_2 - \omega_1)}, \quad C_2 = \frac{(3\omega_2 - \omega_1)(\omega_2 - 2\omega_1)}{2(2\omega_2 - \omega_1)(\omega_2 - \omega_1)}, \quad C_3 = \frac{3\omega_2 - \omega_1}{2(\omega_2 - \omega_1)}
\]

(63)

Likewise we deduce

\[
C_1 = \frac{\omega_1 - 2\omega_2}{2\omega_1 - \omega_2} \cdot C_3
\]

(64)

and

\[
C_1 = \frac{2(2\omega_2 - \omega_1)}{3(\omega_2 - \omega_1)} \rightarrow \frac{2(\eta^2 + 2)}{3(\eta^2 + 1)}, \quad C_2 = 2 - C_1 = \frac{2(2\omega_1 - \omega_2)}{3(\omega_1 - \omega_2)} \rightarrow \frac{2(2\eta^2 + 1)}{3(\eta^2 + 1)}
\]

(65)

Finally, using formula (51)

\[
\tilde{C}_1 = \frac{\omega_2 - 2\omega_3}{2\omega_2 - \omega_3} \cdot \tilde{C}_1
\]

\[
\tilde{C}_2 = \frac{\omega_1 - 2\omega_2}{2\omega_1 - \omega_2} \cdot \tilde{C}_1
\]

\[
\tilde{C}_3 = \frac{(\omega_1 - 2\omega_3)(\omega_1 - 2\omega_2)}{2(\omega_1 - \omega_3)(2\omega_1 - \omega_2)} \cdot \tilde{C}_1
\]

\[
\tilde{C}_4 = \frac{(\omega_1 - 2\omega_2)(\omega_1 - 2\omega_3)(\omega_2 - 2\omega_3)}{2(\omega_1 - \omega_2)(2\omega_1 - \omega_3)(2\omega_2 - \omega_3)} \cdot \tilde{C}_1
\]

\[
\tilde{C}_5 = \frac{(\omega_1 - 2\omega_3)(\omega_2 - 2\omega_3)}{2(\omega_1 - \omega_3)(2\omega_2 - \omega_3)} \cdot \tilde{C}_1
\]

\[
\tilde{C}_6 = \frac{(\omega_1 - 2\omega_2)(\omega_2 - 2\omega_3)}{2(\omega_1 - \omega_2)(2\omega_2 - \omega_3)} \cdot \tilde{C}_1
\]

(66)

As usual, the ratios are nice functions without square roots. Then

\[
\left(1 + \frac{\omega_2 - 2\omega_3}{2\omega_2 - \omega_3} + \frac{\omega_1 - 2\omega_2}{2\omega_1 - \omega_2} + \frac{(\omega_1 - 2\omega_3)(\omega_1 - 2\omega_2)}{2(\omega_1 - \omega_3)(2\omega_1 - \omega_2)} + \frac{(\omega_1 - 2\omega_2)(\omega_1 - 2\omega_3)(\omega_2 - 2\omega_3)}{2(\omega_1 - \omega_2)(2\omega_1 - \omega_3)(2\omega_2 - \omega_3)} + \frac{(\omega_1 - 2\omega_3)(\omega_2 - 2\omega_3)}{2(\omega_1 - \omega_3)(2\omega_2 - \omega_3)} + \frac{(\omega_1 - 2\omega_2)(\omega_2 - 2\omega_3)}{2(\omega_1 - \omega_2)(2\omega_2 - \omega_3)} \right) \cdot C_1 = 6
\]

(67)

and

\[
C_1 = \frac{2(2\omega_1 - \omega_2)(2\omega_1 - \omega_3)}{7(\omega_1 - \omega_2)(\omega_1 - \omega_3)(\omega_2 - 2\omega_3)}
\]

\[
C_2 = \frac{2(2\omega_1 - \omega_2)(2\omega_1 - \omega_3)}{7(\omega_1 - \omega_2)(\omega_1 - \omega_3)(\omega_2 - 2\omega_3)}
\]

\[
C_3 = \frac{2(2\omega_1 - \omega_2)(2\omega_1 - \omega_3)(\omega_2 - 2\omega_3)}{7(\omega_1 - \omega_2)(\omega_1 - \omega_3)(\omega_2 - 2\omega_3)}
\]

\[
C_4 = \frac{2(2\omega_1 - \omega_2)(2\omega_1 - \omega_3)}{7(\omega_1 - \omega_2)(\omega_1 - \omega_3)(\omega_2 - 2\omega_3)}
\]

\[
C_5 = \frac{2(2\omega_1 - \omega_2)(2\omega_1 - \omega_3)(\omega_2 - 2\omega_3)}{7(\omega_1 - \omega_2)(\omega_1 - \omega_3)(\omega_2 - 2\omega_3)}
\]

(68)
As one can see all these coefficients correctly transform into each other under permutations of \( \omega_1, \omega_2, \omega_3 \) that corresponds to transpositions. These constants remain non-vanishing in the Jack case – what should vanish in this case, is the polynomial \( S \).

### 5 Restoration of homogeneity

To summarize, we investigated the possibility to postulate the cut-and-join operator in a rather simple form and impose a normalization condition

\[
A_1 B_1 = \rho \tag{69}
\]

After that we ask:

\[
\text{word}\{\hat{e}_i\} \cdot 1 = G \cdot S = G \cdot S
\]

\[
\hat{W}_2 \cdot S_\pi = \sum_{\square \in \pi} \omega_{\square} \cdot S_\pi \tag{70}
\]

The compatibility conditions for the linear system fix the form of unknown functions \( A, B, u, v, w \) in our ansatz for cut-and-join operator \( \hat{W}_2 \).

In the main text we just put \( \rho = 1 \) to make the formulas looking more like those in [22]. However, this breaks a natural grading in powers of \( \omega_i \) and times \( P_{a,i} \). A better choice implies that \( \rho \) has degree two, and things simplify a lot for particular choice:

\[
\begin{bmatrix}
A_1 = -\sigma_2 \\
B_1 = 1
\end{bmatrix} \tag{71}
\]

For this choice at least the "classical" part of cut-and-join operator has degree 1, provided that the degree of times and \( \omega_i \) is equal to one:

\[
\text{deg} [P_{a,i}] = \text{deg} [\omega_i] = 1 \tag{72}
\]

To illustrate this idea we provide an example on 2 level. The restoration of grading can be seen in the case of cut-and-join operator \( \hat{W}_2 \)

\[
\begin{array}{cccc}
u_{2,1} = -\frac{\sigma_3}{\sigma_2} & u_{2,2} = \frac{\sigma_3}{\sigma_2} & v_{2,1} = -\frac{\sigma_3}{\sigma_2} & w_{2,1} = \frac{\sigma_3}{\sigma_2}
\end{array} \tag{73}
\]

and 3-Schur polynomials:

\[
\frac{1}{(\omega_1 - \omega_2)(\omega_1 - \omega_3)} \left( \omega_1^2 P_{1,1}^2 + \omega_2^2 P_{2,1} - \sigma_3 P_{2,2} \right) \tag{74}
\]

These formulas are direct analogue of (29) and (30) for different choice of normalization of \( A_1 B_1 \). Despite the homogeneous formulas look simpler, the rules for reduction to the Jack case remains unclear. This approach could be continued to higher level, but we do not provide it here.

The main advantage of homogeneous approach is that one can restore the dependence on parameters \( \omega_i \) of the unknown functions \( A, B, u, v, w \) of the cut-and-join operator using degree arguments.

### 6 Conclusion

In this paper we provide a very clear and simple prescription to generate the 3-Schur functions. Namely, we explicitly define the cut-and-join operator \( \hat{W}_2 = \hat{\psi}_3 \) through time variables in (22). Together with the Pieri-rule generators \( \hat{e}_0 = P_{1,1} \) and \( \hat{f}_0 = \frac{\partial}{\partial P_{1,1}} \) this is enough to recursively generate all other \( \hat{e}_{k+1} \) and the 3-Schur functions, which are the common eigenfunctions of all operators \( \hat{\psi}_k \) of the Yangian. In practice these 3-Schur functions are associated with all plane partitions tableaux and for a given plane partition they differ by the easily controlled time-independent factors.

What remains to be fixed are the values of coefficients \( A, B, u, v, w \) in (22), which are just trivial in the case of 2-Schur and Jack functions. We demonstrated that for 3-Schurs they are non-trivial, but more calculations or alternative insights are needed to fix them and understand whether one needs additional terms in (22) or
not. Just like in the previous attempts in [14, 28, 29] the problems seem to mount up at the first essentially 3-dimensional level four, where the formulas of [26] also do not look simple and transparent. Deeper analysis and the relation to Yangian representation theory and free-field representation program [31, 32] will be presented elsewhere.

The four obvious possible reasons for the partial failure of our approach at level 4 are

- computational mistake
- omission of some important contributions to the cut-and-join operator $W_2$ (like changing $i$ by two)
- an overoptimistic/naive assumption that $W_2$ looks natural in variables $p_{a,i}$
- erroneous choice of the constants $C_{\bar{n}}$.

We do not dwell upon the first three options, but provided an explicit list of $C_{\bar{n}}$ in Section 4, which we used – so that an interested reader can check the last option.

We hope that our brief but detailed exposition of the problem will help to attract more researchers to the subject of 3-Schur polynomials, which remains one of the core issues for the theory of DIM algebras and numerous string models with Yangian and DIM symmetries.

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