Exact resolution of the Baxter equation for reggeized gluon interactions

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Abstract

The interaction of reggeized gluons in multi-colour QCD is considered in the Baxter-Sklyanin representation, where the wave function is expressed as a product of Baxter functions $Q(\lambda)$ and a pseudo-vacuum state. We find $n$ solutions of the Baxter equation for a composite state of $n$ gluons with poles of rank $r$ in the upper $\lambda$ semi-plane and of rank $n - 1 - r$ in the lower $\lambda$ semi-plane ($0 \leq r \leq n - 1$). These solutions are related by $n - 2$ linear equations with coefficients depending on $\coth(\pi \lambda)$. The poles cancel in the wave function, bilinear combination of holomorphic and anti-holomorphic Baxter functions, guaranteeing its normalizability. The quantization of the intercepts of the corresponding Regge singularities appears as a result of the physical requirements that the holomorphic energies for all solutions of the Baxter equation are the same and the total energies, calculated around two singularities $\lambda, \lambda^* \rightarrow \pm i$, coincide. It results in simple properties of the zeroes of the Baxter functions. For illustration we calculate the parameters of the reggeon states constructed from three and four gluons. For the Odderon the ground state has conformal spin $|m - \tilde{m}| = 1$ and its intercept equals unity. The ground state of four reggeized gluons possesses conformal spin 2 and its intercept turns out to be higher than that for the BFKL Pomeron. We calculate the anomalous dimensions of the corresponding operators for arbitrary $\alpha_s/\omega$.

1 Introduction

The leading logarithmic asymptotics (LLA) of scattering amplitudes in the Regge limit of high energies $\sqrt{s}$ and fixed momentum transfers $q = \sqrt{-t}$ is obtained by calculating and summing all contributions $(g^2 \ln s)^r$, where $g$ is the QCD coupling constant. In this approximation the BFKL Pomeron is a composite state of two reggeized gluons [1]. The BFKL equation for the Pomeron wave function is closely related to the DGLAP equation for the parton distributions [2]. Next-to-leading corrections to its kernel were calculated in QCD [3] and in supersymmetric gauge theories [4].

The asymptotic behaviour $\propto s^{j_0}$ of scattering amplitudes is governed by the $j$-plane singularities of the $t$-channel partial waves $f_j(t)$

$$A(s, t) = \int_{\sigma - i\infty}^{\sigma + i\infty} \frac{dj}{2\pi i} \xi_j s^j f_j(t) \sim \xi_{1+\omega_0} s^{1+\omega_0}. \quad (1)$$
Here the contour of integration in $j$ is situated to the right of the leading singularity $j_0 = 1 + \omega_0$ of $f_j(t)$ ($j_0 < \sigma$) and the signature factor $\xi_j$ is $\simeq i^{n-1}$ for the $t$-channel exchange of $n$-reggeized gluons. Its intercept $\omega_0$ is proportional to the ground state energy $E_0$ of the Schrödinger-like equation [1, 3]:

$$Hf = Ef, \quad \omega_0 = -\frac{g^2}{8\pi^2} N_c E_0.$$ (2)

The wave function $f$ depends on the two-dimensional impact parameters $\rho_k^\pm$ - positions of the reggeized gluons. It is convenient to introduce the holomorphic ($\rho_k = x_k + iy_k$) and anti-holomorphic ($\rho_k^* = x_k - iy_k$) coordinates and their corresponding momenta $p_k = i\frac{\partial}{\partial \rho_k}$ and $p_k^* = i\frac{\partial}{\partial \rho_k^*}$.

In multicolour QCD $N_c \to \infty$ the colour structure of the BFKL equation in LLA is significantly simplified. As a result, each reggeized gluon interacts only with its two neighbours [6]:

$$H = \frac{1}{2} \sum_{k=1}^{n} H_{k,k+1}.$$ (3)

Note, that for three gluon composite state, describing the Odderon responsible for the high energy behaviour of the differences of total cross-sections for particle-particle and particle-anti-particle interactions, this simplification is valid for arbitrary $N_c$ [3, 4].

The Hamiltonian $H$ has the properties of the Möbius invariance [8] and of the holomorphic separability [8]:

$$H = \frac{1}{2}(h + h^*), \quad [h, h^*] = 0,$$ (4)

where the holomorphic and anti-holomorphic Hamiltonians

$$h = \sum_{k=1}^{n} h_{k,k+1}, \quad h^* = \sum_{k=1}^{n} h_{k,k+1}^*$$ (5)

are expressed in terms of the BFKL operator [9]

$$h_{k,k+1} = \log(p_k) + \log(p_{k+1}) + \frac{1}{p_k} \log(\rho_{k,k+1}) p_k + \frac{1}{p_{k+1}} \log(\rho_{k,k+1}) p_{k+1} + 2 \gamma.$$ (6)

Here $\rho_{k,k+1} = \rho_k - \rho_{k+1}$ and $\gamma = -\psi(1)$ is the Euler-Mascheroni constant.

The wave function $f_{m,\tilde{m}}(\rho_1^\pm, \rho_2^\pm, ..., \rho_n^\pm; \rho_0^\pm)$ of the colourless composite state described by the operator $O_{m,\tilde{m}}(\rho_0^\pm)$ belongs to the principal series of unitary representations of the Möbius group [8]. For these representations the conformal weights

$$m = 1/2 + iv + n/2, \quad \tilde{m} = 1/2 + iv - n/2$$ (7)

are expressed in terms of the anomalous dimension $\gamma = 1/2 + iv$ of $O_{m,\tilde{m}}(\rho_0^\pm)$ and its integer conformal spin $n$. Furthermore, the eigenvalues of two Casimir operators $M^2$ and $M^*2$ of the Möbius group are equal to $m(m-1)$ and $\tilde{m}(\tilde{m}-1)$, respectively.

Owing to the holomorphic separability of $H$, the wave function has the property of holomorphic factorization [8]:

$$f_{m,\tilde{m}}(\rho_1^\pm, \rho_2^\pm, ..., \rho_n^\pm; \rho_0^\pm) = \sum_{r,l} c_{r,l} f_{m}(\rho_1, \rho_2, ..., \rho_n; \rho_0) f_{\tilde{m}}^*(\rho_1^*, \rho_2^*, ..., \rho_n^*; \rho_0^*),$$ (8)
where \( r \) and \( l \) enumerate the different solutions of the Schrödinger equations in the holomorphic and anti-holomorphic sub-spaces:
\[
\epsilon_m f_m = h^* f_{m}^* \quad , \quad \epsilon_m f_m = h f_m \quad , \quad E_{m,m} = \epsilon_m + \epsilon_m^* .
\] Similarly to the case of two-dimensional conformal field theories, the coefficients \( c_{r,l} \) are obtained imposing single-valuedness to \( f_{m,m}(\rho_1^1; \rho_2^2; \ldots; \rho_n^r) \) as a function of the two-dimensional variables \( \rho_i \).

There are two different normalization conditions for the wave function [3]:
\[
\| f \|^2 = \int \prod_{r=1}^{n} d^2 \rho_r \left| \prod_{r=1}^{n} \rho_{r+1}^{-1} f \right|^2 , \quad \| f \|^2 = \int \prod_{r=1}^{n} d^2 \rho_r \left| \prod_{r=1}^{n} p_r f \right|^2
\] compatible with the hermiticity of \( H \). This property is related with the fact [3], that \( h \) commutes with the differential operator
\[
A = \rho_{12} \rho_{23} \ldots \rho_{n1} p_1 p_2 \ldots p_n .
\] Furthermore [10], there is a family \( \{ q_r \} \) of mutually commuting differential operators being the integrals of motion:
\[
[q_r, q_s] = 0 \quad , \quad [q_r, h] = 0 .
\] They can be obtained by the small-\( u \) expansion of the transfer matrix for the XXX model [10]
\[
T(u) = tr \left[ L_1(u)L_2(u) \ldots L_n(u) \right] = \sum_{r=0}^{n} u^{n-r} q_r ,
\] where the \( L \)-operators are
\[
L_k(u) = \begin{pmatrix} u + \rho_{k0} p_k & -p_k \\ \rho_{k0} p_k & u - \rho_{k0} p_k \end{pmatrix} = \begin{pmatrix} u & 0 \\ 0 & u \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ \rho_{k0} & 1 \end{pmatrix} p_k .
\]
In particular \( q_0 \) is equal to \( A \) and \( q_2 \) is proportional to \( M^2 \).

The transfer matrix is the trace of the monodromy matrix \( t(u) \):
\[
T(u) = tr \left[ t(u) \right] , \quad t(u) = L_1(u)L_2(u) \ldots L_n(u) .
\] It can be shown [10], [11], that \( t(u) \) satisfies the Yang-Baxter equation:
\[
t_{s_1}^{s_2}_{r_1} (u) t_{s_2}^{s_1}_{r_2} (v) t_{r_1}^{r_2} (v - u) = t_{r_1}^{r_2} (v - u) t_{s_1}^{s_2} (v) t_{s_2}^{s_1}_{r_2} (u) ,
\] where \( l(w) \) is the \( L \)-operator for the well-known Heisenberg spin chain:
\[
l_{s_1}^{s_2} (w) = w \delta_{s_1}^{s_2} + i \delta_{s_1}^{s_2} \delta_{s_2}^{s_1} .
\]

## 2 Baxter-Sklyanin representation

Thus, the problem of finding solutions of the Schrödinger equation for the reggeon gluon interaction reduces to the search of a representation for the monodromy matrix satisfying the Yang-Baxter bilinear relations [10]. For this purpose the algebraic Bethe Ansatz is appropriate [11]. It is important, that the reggeon Hamiltonian in the multi-colour QCD coincides with the
local Hamiltonian of the integrable Heisenberg model with the spins being the generators of the non-compact Möbius group \(SL(2, C)\) \[12, 13\]. For the case of three gluons the wave function and intercepts of the corresponding Regge singularities were found in ref.\[14\] with the use the integral of motion discovered in ref.\[9\].

The integrals of motion and the hamiltonian for \(n\) reggeized gluons have an additional symmetry under the transformation \(15\)

\[ p_k \rightarrow \rho_{k,k+1} \rightarrow p_{k+1}, \]

combined with the operator transposition. This duality symmetry allows to relate the wave function of a composite state with the Fourier transformed wave function of (generally) another physical state. In particular, it gives a possibility to construct a new Odderon solution having the intercept exactly equal to unity \[16\].

The duality symmetry can be interpreted as a symmetry among the states constructed from the reggeons with positive and negative signatures \[16\]. Indeed, the Regge trajectories for these states with gluon quantum numbers are degenerated in multi-colour QCD.

In the framework of the Bethe Ansatz it is convenient to work in the conjugated space \[13\], where the monodromy matrix is parametrized as follows,

\[ \tilde{t}(u) = \tilde{L}_n(u)...\tilde{L}_1(u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}. \]

(19)

Here \(\tilde{L}_k(u)\) is given by

\[ \tilde{L}_k(u) = \begin{pmatrix} u + p_k\rho_{k0} & -p_k \\ p_k\rho_{k0}^2 & u - p_k\rho_{k0} \end{pmatrix}. \]

(20)

Now the equation for the pseudo-vacuum state

\[ C(u) |0\rangle^t = 0 \]

has the following solution \[13\]

\[ |0\rangle^t = \prod_{k=1}^{n} \frac{1}{\rho_{k0}}. \]

(22)

In the total impact parameter space \(\vec{\rho}\) the pseudo-vacuum wave function,

\[ \Psi^{(0)}(\vec{\rho}_1; \vec{\rho}_2; \ldots; \vec{\rho}_n; \vec{\rho}_0) = \prod_{k=1}^{n} \frac{1}{|\rho_{k0}|^4}. \]

(23)

is an eigenfunction of the transfer matrix,

\[ [A(u) + D(u)] |0\rangle^t = [(u - i)^n + (u + i)^n] |0\rangle^t. \]

(24)

The pseudo-vacuum state does not belong to the principal series of unitary representations because it has the conformal weight \(m = n\).

A powerful approach to construct the physical states in the framework of the Bethe Ansatz is based on the use of the Baxter equation for the Baxter function \(Q(\lambda)\) \[17, 18\]. The Baxter equation for the \(n\)-reggeon composite states can be written as follows (see \[13, 14, 20\])

\[ \Lambda^{(n)}(\lambda; \vec{m}) Q(\lambda; m, \vec{m}) = (\lambda + i)^n Q(\lambda + i; m, \vec{m}) + (\lambda - i)^n Q(\lambda - i; m, \vec{m}), \]

(25)
where $\Lambda^{(n)}(\lambda)$ is the polynomial
\[
\Lambda^{(n)}(\lambda; \vec{\mu}) = \sum_{k=0}^{n} (-i)^k \mu_k \lambda^{n-k}, \quad \mu_0 = 2, \quad \mu_1 = 0, \quad \mu_2 = m(m-1).
\] (26)

Here we assume in accordance with ref. [19], that the quantities $\mu_k = i^k q_k$ are real which is compatible with the single-valuedness condition of the wave functions [q_k stand here for the eigenvalues of the integrals of motion].

The eigenfunctions of the holomorphic Schrödinger equation can be expressed through the Baxter function $Q(\lambda)$ using the Sklyanin Ansatz [18]:
\[
f(\rho_1, \rho_2, ..., \rho_n; \rho_0) = Q(\lambda_1; m, \vec{\mu}) Q(\lambda_2; m, \vec{\mu}) ... Q(\lambda_{n-1}; m, \vec{\mu}) |0\rangle^t.
\] (27)

where $\lambda_r$ are the operator zeroes of the matrix element $B(u)$ of the monodromy matrix:
\[
B(u) = -P \prod_{r=1}^{n-1} (u - \lambda_r), \quad P = \sum_{k=1}^{n} p_k.
\] (28)

These expressions are well defined because the operators $\hat{\lambda}_r$ and $P$ commute [18]
\[
[\hat{\lambda}_r, \hat{\lambda}_s] = [\hat{\lambda}_r, P] = 0.
\] (29)

In ref. [13] it was assumed without any convincing arguments, that $Q(\lambda)$ is an entire function in the complex $\lambda$ plane. One of the purposes of our previous paper [19] was to find the class of functions to which the Baxter functions belong. For this purpose we performed an unitary transformation of the wave function of the composite state of $n$ reggeized gluons from the coordinate representation to the Baxter-Sklyanin representation in which the operators $\hat{\lambda}_r$ are diagonal [19] (see also [20]). The kernel of this transformation was expressed through the eigenfunctions of the operators $B(u)$ and $B^*(u)$. For the cases of the Pomeron and Odderon the unitary transformation was constructed in an explicit form [19]. As a consequence of the single-valuedness condition for the kernel of this transformation one obtains the quantization of the arguments of the Baxter functions $Q(\lambda)$ and $Q(\lambda^*)$ in the holomorphic and anti-holomorphic sub-spaces (see [19, 20]):
\[
\lambda = \sigma + i \frac{N}{2}, \quad \lambda^* = \sigma - i \frac{N}{2},
\] (30)

where $\sigma$ and $N$ are real and integer numbers, respectively.

In ref. [13] we proposed a general method of solving the Baxter equation for the $n$-reggeon composite state. To begin with, the simplest $n$-reggeon solution of this equation is searched in the form of a sum over the poles of orders from 1 up to $n-1$ situated in the upper semi-plane
\[
Q^{(n-1)}(\lambda; m, \vec{\mu}) = \sum_{r=0}^{\infty} \frac{P_{r;m,\vec{\mu}}^{(n-2)}(\lambda)}{(\lambda - i r)^{n-1}},
\] (31)

where the $P_{r;m,\vec{\mu}}^{(n-2)}(\lambda)$ are polynomials in $\lambda$ of degree $n-2$. Inserting this Ansatz in the Baxter equation leads to recurrence relations for the polynomials $P_{r;m,\vec{\mu}}^{(n-2)}(\lambda)$, which allows us to calculate them successively starting from $P_{0;m,\vec{\mu}}^{(n-2)}(\lambda)$ [19].

One can normalize this solution imposing the constraint
\[
\lim_{\lambda \to 0} P_{0;m,\vec{\mu}}^{(n-2)}(\lambda) = 1
\] (32)
Then, the remaining coefficients of the polynomial $P_{0,m,\vec{\mu}}^{(n-2)}(\lambda)$ are calculated from the condition
\[
\lim_{\lambda \to \infty} Q^{(n-1)}(\lambda; m, \vec{\mu}) \sim \lambda^{-n+m}
\] (33)
which is a necessary condition for $Q^{(n-1)}(\lambda; m, \vec{\mu})$ to be a solution of the Baxter equation at $\lambda \to \infty$. It is enough to require
\[
\lim_{\lambda \to \infty} \lambda^{n-2} \sum_{r=0}^{\infty} \frac{P_{r,m,\vec{\mu}}^{(n-2)}(\lambda)}{(\lambda - i r)^{n-1}} = 0 .
\] (34)
This condition gives $n-1$ linear equations allowing to calculate all coefficients of the polynomial $P_{0,m,\vec{\mu}}^{(n-2)}(\lambda)$.

The existence of the second independent solution
\[
Q^{(0)}(\lambda; m, \vec{\mu}) = Q^{(n-1)}(-\lambda; m, \vec{\mu}^*) = \sum_{r=0}^{\infty} \frac{P_{r,m,\vec{\mu}}^{(n-2)}(-\lambda)}{(-\lambda - i r)^{n-1}} ,
\] (35)
where
\[
\mu_k^* = (-1)^k \mu_k ,
\]
is related with the invariance of the Baxter equation under the simultaneous transformations
\[
\lambda \to -\lambda , \ \mu \to \mu^* .
\]
One can verify [19] that
\[
Q^* (-\lambda; m, \vec{\mu}^*) = Q \left( \lambda^*; \vec{m}, \vec{\mu}^* \right) .
\]

It turns out [19], that there is a set of the Baxter functions $Q^{(t)} (t = 0, 1, ..., n-1)$ having poles simultaneously in the upper and lower half-$\lambda$ planes.

\[
Q^{(t)}(\lambda; m, \vec{\mu}) = \sum_{r=0}^{\infty} \left[ \frac{P_{r,m,\vec{\mu}}^{(t-1)}}{(\lambda - i r)^t} + \frac{P_{r,m,\vec{\mu}}^{(n-2-t)}}{(-\lambda - i r)^{n-1-t}} \right] ,
\]
where the polynomials $P_{r}^{(t-1)}$ and $P_{r}^{(n-2-t)}$ are fixed by the recurrence relations following from the Baxter equation and from the condition that the above Baxter functions decrease at infinity more rapidly than $\lambda^{-n+2}$. These solutions $Q^{(t)}$ are linear combinations of $Q^{(n-1)}(\lambda; m, \vec{\mu})$ and $Q^{(n-1)}(-\lambda; m, \vec{\mu}^*)$ with the coefficients depending on $\coth(\pi \lambda)$ [19].

Using all these functions in the holomorphic and anti-holomorphic spaces one can construct the normalizable Baxter function $Q_{m,\vec{m},\vec{\mu}}(\lambda)$ in the space $(\sigma, N)$ without poles at $\sigma = 0$ [19]
\[
Q_{m,\vec{m},\vec{\mu}}(\lambda) = \sum_{t,l} C_{t,l} Q^{(t)}(\lambda; m, \vec{\mu}) Q^{(l)}(\lambda^*; \vec{m}, \vec{\mu}^*)
\] (36)
by adjusting for this purpose the coefficients $C_{t,l}$.

Another problem in the Baxter approach is the calculation of the energy $E_{m,\vec{m}}$, because the expression suggested in ref. [13] leads to an infinite result for meromorphic Baxter functions. This problem was solved in ref. [19] by an unitary transformation of the Hamiltonian to the Baxter-Sklyanin representation.
In the region, where the gluon momenta are strongly ordered in their values,

\[ |p_n| << |p_{n-1}| << ... << |p_1| = 1 \]

the unitary transformation of the wave function \( \Psi_{m,\tilde{m}} \) to the Baxter-Sklyanin representation is significantly simplified \[19\]

\[
\Psi_{m,\tilde{m}}(\vec{p}_1, \ldots, \vec{p}_n) \sim \prod_{k=1}^{n-1} \left( \int_{-\infty}^{+\infty} d\sigma_k \sum_{N_k=-\infty}^{+\infty} \frac{i\lambda_k^*}{p_{k-1}^*} \right) \Psi_{m,\tilde{m}}(\lambda_1, \ldots, \lambda_{n-1}). \tag{37}
\]

On the other hand, in this region

\[
\Psi_{m,\tilde{m}}(\vec{p}_1, \ldots, \vec{p}_n) \sim c_n |p_n|^2 \ln \frac{1}{|p_n|^2}, \tag{38}
\]

where \( c_n \) is a constant \[13\]. Therefore \( \Psi_{m,\tilde{m}}(\lambda_1, \ldots, \lambda_{n-1}) \) has the first order poles at \( \lambda_{n-1} = i \) and \( \lambda_{n-1}^* = i \) for \( N_{n-1} = 0 \) and the pole singularities at \( \lambda_k = \lambda_k^* = 0 \) \( (k = 1, 2, \ldots, n - 2) \). The action of the Hamiltonian \( H \) on the function \( \Psi_{m,\tilde{m}} \) near these singularities after the unitary transformation to the \( \lambda \)-space is drastically simplified \[19\]. Moreover, using the Sklyanin factorized expression for the wave function \( \Psi_{m,\tilde{m}}(\lambda_1, \ldots, \lambda_{n-1}) \) we express the energy in terms of the behaviour of \( Q(\lambda_{n-1}) \) near the pole at \( \lambda_{n-1} = i, \lambda_{n-1}^* = i \) and obtain \[13\]

\[
E = i \lim_{\lambda^* \to i} \frac{\partial}{\partial \lambda} \frac{\partial}{\partial \lambda^*} \ln \left[ (\lambda - i)^{n-1}(\lambda^* - i)^{n-1} |\lambda|^{2n} Q_{m,\tilde{m}}(\lambda) \right]. \tag{39}
\]

It is important, that the consideration of the kinematical region with an opposite ordering of the gluon momenta \( p_k \) leads to an analogous expression for \( E \) in terms of the behaviour of \( Q_{m,\tilde{m}}(\lambda) \) near \( \lambda, \lambda^* = -i \). The coincidence of these energies imposes a constraint on the spectrum of the integrals of motion \( q_k \) \( (k = 3, 4, \ldots, n) \).

Since the function \( Q_{m,\tilde{m}}(\lambda) \) is a bilinear combination of the independent Baxter functions \( Q^{(t)}(\lambda) \) and \( Q^{(l)}(\lambda^*) \) \( (t, l = 1, 2, \ldots, n) \), the holomorphic (anti-holomorphic) energies for all solutions should be the same

\[
\epsilon_m = i \lim_{\lambda^* \to i} \frac{\partial}{\partial \lambda} \ln \left[ \lambda^n P^{(t-1)}_{1;m,\tilde{m}}(\lambda) \right]. \tag{40}
\]

This leads to the quantization of the integrals of motion \( q_k \) \[19\].

Note, that this condition can be obtained also as one of the consequences of the absence of poles in \( Q_{m,\tilde{m}}(\lambda) \) at \( \sigma = 0 \) for \( |N| > 0 \). Because the products of poles \( (\lambda - ir)^{-s} \) and \( (-\lambda^* - ir)^{s'} \) appear in the corresponding bilinear combinations, to cancel all singularities one should expand the functions \( Q^{(t)} \), \( Q^{(l)} \) in Laurent series near these poles and keep all relevant terms. The coefficients of the Laurent series satisfy recurrence relations similar to those for the residues of the poles. However, it turns out that they are not proportional to the residues of the poles. Even a special choice of the values for the integrals of motion \( q_r \) does not lead to such proportionality because these recurrence relations are different at \( k = 2 \) \[19\].

We discuss below the solution of the Baxter equation by the method proposed in ref.\[13\]. Note, that for \( n = 3 \) this method gives results in full agreement with those obtained by other approaches \[14, 15\]. We find in the present paper the wave functions and intercepts for the quartet (four reggeons state).
3 Meromorphic solutions of the Baxter equation

Let us rewrite the Baxter equation in a real form introducing the new variable \( x \equiv -i\lambda \),

\[
\Omega(x, \vec{\mu}) Q(x, \vec{\mu}) = (x + 1)^n Q(x + 1, \vec{\mu}) + (x - 1)^n Q(x - 1, \vec{\mu}) ,
\]

where

\[
\Omega(x, \vec{\mu}) = \sum_{k=0}^{n} (-1)^k \mu_k x^{n-k}
\]

and

\[
\mu_0 = 2 , \quad \mu_1 = 0 , \quad \mu_2 = m(m-1) ,
\]

assuming that the eigenvalues of the integrals of motion \( \mu_k (k > 2) \) are real numbers.

Then, the solution of the Baxter equation possessing poles only at \( x = l = 0, 1, 2, \ldots \) can be written as follows,

\[
Q^{(n-1)}(x, \vec{\mu}) = \sum_{l=0}^{\infty} \left[ \frac{a_l(\vec{\mu})}{(x-l)^{n-1}} + \frac{b_l(\vec{\mu})}{(x-l)^{n-2}} + \cdots + \frac{z_l(\vec{\mu})}{x-l} \right] ,
\]

where the residues \( a_l(\vec{\mu}), b_l(\vec{\mu}), \ldots, z_l(\vec{\mu}) \) satisfy recurrence relations which can be easily obtained from the Baxter equation and its derivatives in the limit \( x \to l \). Using these relations we can express all these residues in terms of \( a_0(\vec{\mu}), b_0(\vec{\mu}), \ldots, z_0(\vec{\mu}) \). In addition, imposing the Baxter equation at \( x \to \infty \),

\[
\lim_{x \to \infty} x^s Q^{(n-1)}(x, \vec{\mu}) = 0 \quad , \quad s = 1, 2, \ldots, n - 2 ,
\]

fixes the parameters \( b_0(\vec{\mu}), \ldots, z_0(\vec{\mu}) \) in terms of \( a_0(\vec{\mu}) \). We then require the normalization condition

\[
a_0(\vec{\mu}) = 1 ,
\]

without losing generality. In this normalization the holomorphic energy is (see [19])

\[
\epsilon_n = \frac{b_1}{a_1} + n = b_0 - \frac{\mu_{n-1}}{\mu_n} .
\]

Thus, the solution \( Q^{(n-1)}(x, \vec{\mu}) \) is uniquely defined and can be explicitly constructed.

Let us further introduce a set of the auxiliary functions for \( r = 1, 2, \ldots, n - 1 \)

\[
f_r(x, \vec{\mu}) = \sum_{l=0}^{\infty} \left[ \frac{\tilde{a}_l(\vec{\mu})}{(x-l)^r} + \frac{\tilde{b}_l(\vec{\mu})}{(x-l)^{r-1}} + \cdots + \frac{\tilde{g}_l(\vec{\mu})}{x-l} \right] ,
\]

where the coefficients \( \tilde{a}_l, \ldots, \tilde{g}_l \) satisfy the same recurrent relations as \( a_l, \ldots, z_l \) but with other initial conditions

\[
\tilde{a}_0 = 1 , \quad \tilde{b}_0 = \ldots = \tilde{g}_0 = 0 .
\]

Note, that all functions \( f_r(x, \vec{\mu}) \) are expressed in terms of a subset of pole residues \( \tilde{a}_l, \tilde{z}_l \) for \( Q^{(n-1)}(x, \vec{\mu}) \).

Now we write the Baxter function \( Q^{(n-1)}(x, \vec{\mu}) \) as a linear combination of \( f_r(x, \vec{\mu}) \)

\[
Q^{(n-1)}(x, \vec{\mu}) = \sum_{r=1}^{n-1} C_r(\vec{\mu}) f_r(x, \vec{\mu}) .
\]
In order to impose the asymptotic condition \( [14] \) on \( Q^{(n-1)}(x, \vec{\mu}) \), we use the binomial series for the terms in eq.(45),

\[
\frac{1}{(x-l)^j} = \sum_{n=0}^{\infty} \frac{l^n}{n!} \frac{(j+n-1)!}{(j-1)!} = \sum_{k=j}^{\infty} \frac{l^{k-j}}{x^k} \frac{(k-1)!}{(k-j)!(j-1)!} .
\]

We find a set of \( n-2 \) linear equations on the coefficients \( C_r(\vec{\mu}) \):

\[
\sum_{r=1}^{n-1} G_{k,r}(\vec{\mu}) C_r(\vec{\mu}) = 0 , \quad k = 1, 2, \ldots, n-2 \quad C_{n-1}(\vec{\mu}) = 1 ,
\]

where

\[
G_{k,r}(\vec{\mu}) = \sum_{l=0}^{\infty} \left[ \tilde{a}_l(\vec{\mu}) \frac{(k-1)!}{(k-r)!} \frac{l^{k-r}}{(r-1)!} + \tilde{b}_l(\vec{\mu}) \frac{(k-1)!}{(k-r+1)!} \frac{l^{k-r+1}}{(r-2)!} + \ldots + \tilde{g}_l(\vec{\mu}) l^{k-1} \right] . \tag{48}
\]

Obviously, the symmetric solution

\[
Q^{(0)}(x, \vec{\mu}) = Q^{(n-1)}(-x, \vec{\mu}^\ast) , \quad \text{where} \quad \mu_r^\ast \equiv (-1)^r \mu_r , \tag{49}
\]

can be constructed in a similar way. It has poles at \( x = -l \) \((l = 0, 1, \ldots)\).

But there are other ‘minimal’ solutions \( Q^{(t)}(x, \vec{\mu}) \) \((t = 1, 2, \ldots, n-2)\) of the Baxter equation having \( t \)-order poles at positive integer \( x \) and \( (n-1-t) \)-order poles at negative integer \( x \) \cite{19}

\[
Q^{(t)}(x, \vec{\mu}) = \sum_{r=1}^{t} C_r^{(t)}(\vec{\mu}) f_r(x, \vec{\mu}) + \beta^{(t)}(\vec{\mu}) \sum_{r=1}^{n-1-t} C_r^{(n-1-t)}(\vec{\mu}^\ast) f_r(-x, \vec{\mu}^\ast) , \tag{50}
\]

where the meromorphic functions \( f_r(x, \vec{\mu}) \) were defined above. Such form of the solution is related by the invariance of the Baxter equation under the substitution \( x \rightarrow -x \), \( \vec{\mu} \rightarrow \vec{\mu}^\ast \).

The coefficients \( C_r^{(t)}(\vec{\mu}) \), \( C_r^{(n-1-t)}(\vec{\mu}^\ast) \) and \( \beta^{(t)}(\vec{\mu}) \) are obtained imposing the asymptotic validity of the Baxter equation,

\[
\lim_{x \rightarrow -\infty} x^k Q^{(t)}(x, \vec{\mu}) = 0 , \quad k = 1, 2, \ldots, n-2 .
\]

This leads to a system of \( n-2 \) linear equations

\[
\sum_{r=1}^{t} G_{k,r}(\vec{\mu}) C_r^{(t)}(\vec{\mu}) + (-1)^k \beta^{(t)}(\vec{\mu}) \sum_{r=1}^{n-1-t} G_{k,r}(\vec{\mu}^\ast) C_r^{(n-1-t)}(\vec{\mu}^\ast) = 0 ,
\]

\[
k = 1, 2, \ldots, n-2 , \tag{51}
\]

where the matrix elements \( G_{k,r}(\vec{\mu}) \) are defined by eq.(48) and we normalize \( Q^{(t)}(x, \vec{\mu}) \) by choosing

\[
C_t^{(t)}(\vec{\mu}) = C_{n-1-t}^{(n-1-t)}(\vec{\mu}) = 1 . \tag{52}
\]

Moreover, one can obtain the following relations using the symmetry of the Baxter equation,

\[
Q^{(r)}(x, \vec{\mu}) = \beta^{(r)}(\vec{\mu}) Q^{(n-1-r)}(-x, \vec{\mu}^\ast) , \quad \beta^{(r)}(\vec{\mu}) \beta^{(n-1-r)}(\vec{\mu}^\ast) = 1 , \quad \beta^{(0)}(\vec{\mu}) = 1 . \tag{53}
\]
It is important to notice that three subsequent solutions $Q^{(r)}$ for $r = 1, 2, \ldots, n-2$ are linearly related by

$$\left[ \delta^{(r)}(\vec{\mu}) + \pi \cot(\pi x) \right] Q^{(r)}(x, \vec{\mu}) = Q^{(r+1)}(x, \vec{\mu}) + \alpha^{(r)}(\vec{\mu}) Q^{(r-1)}(x, \vec{\mu}).$$

(54)

Indeed, the left and right-hand sides satisfy the Baxter equation everywhere including $x \to \infty$ and have the same singularities. Due to the uniqueness of the ‘minimal’ solutions the quantity $\cot(\pi x) Q^{(r)}(x, \vec{\mu})$ can be expressed as a linear combination of $Q^{(r-1)}(x, \vec{\mu})$, $Q^{(r)}(x, \vec{\mu})$ and $Q^{(r+1)}(x, \vec{\mu})$. Furthermore, the coefficient in front of $Q^{(r)}(x, \vec{\mu})$ is chosen to be 1 taking into account our normalization of $Q^{(r)}(x, \vec{\mu})$. In this normalization we obtain

$$\alpha^{(r)}(\vec{\mu}) = -\frac{\beta^{(r)}(\vec{\mu})}{\beta^{(r-1)}(\vec{\mu})}, \quad \beta^{(r)}(\vec{\mu}) = (-1)^r \prod_{t=1}^{r} \alpha^{(t)}(\vec{\mu}).$$

(55)

The following relations hold for the coefficients $\alpha^{(r)}(\vec{\mu})$ and $\delta^{(r)}(\vec{\mu})$

$$\delta^{(r)}(\vec{\mu}) = -\delta^{(n-1-r)}(\vec{\mu}^5), \quad \alpha^{(r)}(\vec{\mu}) \alpha^{(n-1-r)}(\vec{\mu}^5) = 1.$$

(56)

According to the above recurrence relations the functions $Q^{(r)}(x, \vec{\mu})$ ($r = 1, 2, \ldots, n-2$) can be expressed as linear combinations of $Q^{(n-1)}(x, \vec{\mu})$ and $Q^{(0)}(x, \vec{\mu})$ with the coefficients being periodic functions of $x$ [19]:

$$D \left[ \delta(\vec{\mu}), \alpha(\vec{\mu}), \pi \cot(\pi x) \right] Q^{(r)}(x, \vec{\mu}) =$$

$$= D_0^{(r)} \left[ \delta(\vec{\mu}), \alpha(\vec{\mu}), \pi \cot(\pi x) \right] Q^{(0)}(x, \vec{\mu}) + D_{n-1}^{(r)} \left[ \delta(\vec{\mu}), \alpha(\vec{\mu}), \pi \cot(\pi x) \right] Q^{(n-1)}(x, \vec{\mu})$$

compatible with the relation (53).

The factor $D \left[ \delta, \alpha, \pi \cot(\pi x) \right]$ can be written as the determinant of the matrix $\Lambda$

$$D \left[ \delta, \alpha, \pi \cot(\pi x) \right] = \left| \Lambda \left[ \delta, \alpha, \pi \cot(\pi x) \right] \right|. \quad (58)$$

The matrix $\Lambda$ takes the form,

$$\Lambda \left[ \delta, \alpha, \pi \cot(\pi x) \right] = \pi \cot(\pi x) I + \Delta \left[ \delta, \alpha \right], \quad I = \begin{pmatrix} 1 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 1 \end{pmatrix} \quad (59)$$

and

$$\Delta \left[ \delta, \alpha \right] = \begin{pmatrix} \delta^{(1)} & 0 & \ldots & 0 \\ -\alpha^{(2)} & \delta^{(2)} & \ldots & 0 \\ 0 & -\alpha^{(3)} & \delta^{(3)} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots \\ 0 & 0 & \ldots & \delta^{(n-2)} \end{pmatrix}. \quad (60)$$

In an analogous way $D_0^{(r)} \left[ \delta, \alpha, \pi \cot(\pi x) \right]$ and $D_{n-1}^{(r)} \left[ \delta, \alpha, \pi \cot(\pi x) \right]$ can be expressed in terms of the determinants of the matrices $\Lambda_0^{(r)}$ and $\Lambda_{n-1}^{(r)}$ of (lower) rank $n - 3$ obtained from $\Lambda$ by removing the column $r$ and the first or the last line, respectively:

$$D_0^{(r)} \left[ \delta, \alpha, \pi \cot(\pi x) \right] = (-1)^r \alpha^{(1)} \left| \Lambda_0^{(r)} \right|, \quad D_{n-1}^{(r)} \left[ \delta, \alpha, \pi \cot(\pi x) \right] = (-1)^r \left| \Lambda_{n-1}^{(r)} \right|$$
and

\[
\Lambda_0^{(r)} = \begin{pmatrix}
-\alpha^{(2)} & \delta^{(2)} + \pi \cot(\pi x) & \cdots & \cdots & 0 \\
0 & -\alpha^{(3)} & \cdots & \cdots & 0 \\
\cdots & \cdots & \delta^{(r+1)} + \pi \cot(\pi x) & \cdots & 0 \\
0 & 0 & \cdots & \cdots & \delta^{(n-2)} + \pi \cot(\pi x)
\end{pmatrix},
\]

\[
\Lambda_{n-1}^{(r)} = \begin{pmatrix}
\delta^{(1)} + \pi \cot(\pi x) & -1 & \cdots & \cdots & 0 \\
-\alpha^{(2)} & \delta^{(2)} + \pi \cot(\pi x) & \cdots & \cdots & 0 \\
\cdots & \cdots & -1 & \cdots & 0 \\
0 & 0 & \cdots & \cdots & -1
\end{pmatrix}.
\]

Let us write these combinations in the form

\[
Q^{(r)}(x, \tilde{\mu}) = a^{(r)}[\tilde{\mu}, \pi \cot(\pi x)] Q^{(n)}[x, \tilde{\mu}] + b^{(r)}[\tilde{\mu}, \pi \cot(\pi x)] Q^{(0)}[x, \tilde{\mu}]
\]

for \(n + 1\) reggeons and

\[
Q^{(r)}(x, \tilde{\mu}) = a^{(r)}[\tilde{\mu}, \pi \cot(\pi x)] Q^{(n+1)}(x, \tilde{\mu}) + b^{(r)}[\tilde{\mu}, \pi \cot(\pi x)] Q^{(0)}(x, \tilde{\mu})
\]

for \(n + 2\) reggeons. Then, the following relations are valid

\[
a^{(r)}[\tilde{\mu}, \pi \cot(\pi x)] = a^{(r)}[\tilde{\mu}, \pi \cot(\pi x)] a^{(n)}[\tilde{\mu}, \pi \cot(\pi x)] ,
\]

\[
b^{(r)}[\tilde{\mu}, \pi \cot(\pi x)] = b^{(r)}[\tilde{\mu}, \pi \cot(\pi x)] + a^{(r)}[\tilde{\mu}, \pi \cot(\pi x)] b^{(n)}[\tilde{\mu}, \pi \cot(\pi x)] ,
\]

where

\[
a^{(n)}[\tilde{\mu}, \pi \cot(\pi x)] = \frac{1}{\delta^{(n)}(\tilde{\mu}) + \pi \cot(\pi x) - \alpha^{(n)}(\tilde{\mu}) a^{(n-1)}(\tilde{\mu})} ,
\]

\[
b^{(n)}[\tilde{\mu}, \pi \cot(\pi x)] = b^{(n-1)}[\tilde{\mu}, \pi \cot(\pi x)] a^{(n)}(\tilde{\mu}) a^{(n)}[\tilde{\mu}, \pi \cot(\pi x)] .
\]

Notice that the linear relations (54) among \(Q^{(r)}(x, \tilde{\mu}), Q^{(r+1)}(x, \tilde{\mu})\) and \(Q^{(r-1)}(x, \tilde{\mu})\) are similar to the recurrence relations for the orthogonal polynomials \(P_r(z)\) if we substitute \(\pi \cot(\pi x)\) by a variable \(z\). The Baxter functions also belong to an orthonormalized set of functions.

We use below the formulae of this section for the numerical calculations of the intercepts of the composite states constructed from reggeized gluons.

### 4 Spectrum of eigenvalues of integrals of motion

To quantize the integrals of motion one should impose all physical constraints on the corresponding eigenfunctions. In the impact parameter representation \(\tilde{\rho}\) the wave function \(\Psi\) should be normalized in a conformally - invariant way [8, 10] and the Hamiltonian \(H\) in the space of these functions should be hermitian. The integrals of motion \(q_k\) (\(q'_k\)) are differential operators in the holomorphic (antiholomorphic) subspaces [10]. The wave function in the \(\tilde{\rho}\)-representation is constructed in a bilinear form from all eigenfunctions of the integrals of motion similar to the case of two-dimensional conformal field theories in such a way that its single-valuedness holds.
The single-valuedness condition is an important constraint on the coefficients of the bilinear form and on the possible values of the integrals of motion. But there are other constraints which are significantly simpler.

In the case of the Pomeron one does not have extra integrals of motion apart from two Casimir operators with their eigenvalues depending on the conformal weights \( m \) and \( \tilde{m} \). For the principal series of the unitary representations of the Möbius group the weights depend on the real parameter \( \nu \) and the integer conformal spin \( n \).

For three particles there is an additional integral of motion with its eigenvalue parametrized as \( q_3 = i\mu \). The problem is to find the region to which the parameter \( \mu \) belongs. For hermitian operators the eigenvalues are real, but the integrals of motion \( \hat{q}_k, \hat{q}_k^* \) in holomorphic and anti-holomorphic subspaces are not hermitian. With the use of conformal invariance the integral of motion \( \hat{q}_3 \) for the Odderon can be written as the third order differential operator \[ \hat{q}_3 = a_{1-m} a_m = z (1 - z) \left[ z(1 - z) p^3 + i(2 - m)(1 - 2 z) p^2 + (2 - m)(1 - m) p \right], \]
acting on the functions depending on the anharmonic ratio \[ z = \frac{\rho_{12} \rho_{30}}{\rho_{10} \rho_{32}}. \]

Here \( p = i \frac{\partial}{\partial z} \) and \( a_m \) is the duality operator \[ a_m = z (1 - z) p^{1+m}, \]
transforming a state with the conformal weight \( m \) into a state with conformal weight \( 1 - m \)
\[ a_m \phi_m = l_m \phi_{1-m}, \quad a_{1-m} \phi_{1-m} = l_{1-m} \phi_m. \]

Here \( l_m \) and \( l_{1-m} \) are the eigenvalues of the corresponding transformations depending on the norms of the functions \( \phi_m \). The product of these numbers does not depend on the normalization and is related with the eigenvalue \( \mu \) by
\[ l_m l_{1-m} = q_3 = i\mu. \]

The Odderon wave function \( \phi_{m,\tilde{m}}(\vec{z}) \) is the bilinear combination of holomorphic and anti-holomorphic functions \[ \phi_{m,\tilde{m}}(\vec{z}) = \sum_{ik} C_{ik} \left[ \phi_{m,q_3}^i(z) \phi_{\tilde{m},q_3^*}^k(z^*) + \phi_{m,-q_3}^i(z) \phi_{\tilde{m},-q_3^*}^k(z^*) \right], \]
where the symmetrization under the transformation \( q_3 \rightarrow -q_3 \) is needed for the Bose symmetry of the wave function. The coefficients \( C_{ik} \) are calculated from the single-valuedness of \( \phi_{m,\tilde{m}}(\vec{z}) \).

The sum is performed over three independent eigenfunctions of the integrals of motion \( \hat{q}_3, \hat{q}_3^* \) in the holomorphic and anti-holomorphic spaces. It was argued in ref. [14] that the eigenvalues \( q_3 \) are purely imaginary as a consequence of single-valuedness. We give below additional arguments supporting this conclusion [at least for real values of \( m(m-1) \)].

To begin with, let us perform the simultaneous interchange of \((z, m)\) and \((z^*, \tilde{m})\) in \( \phi_{m,\tilde{m}}(\vec{z}) \). The wave function should remain the same if there is no accidental degeneracy. But this can be true only if \( q_3 \) satisfies one of the two relations \[ q_3 = \pm q_3^*, \]
which is in agreement with the Janik-Wosiek result $q_3 = -q_3^*$ \cite{14}. An analogous argument for the general $n$-reggeons case will lead to the guess that $\vec{\mu}^* = \vec{\mu}$ or $\vec{\mu}^* = \vec{\mu}^2$.

Furthermore, although the holomorphic hamiltonian $h$ is not a single-valued operator (see for example the discussion of the separability properties of the BFKL hamiltonian in ref.\cite{15}), its ambiguity seems to be simple and related with the possibility to add to it some periodic functions of $\partial/\partial (\ln \rho_k)$ cancelling in the total hamiltonian $H$. For example, in the case of the Odderon one can write the holomorphic hamiltonian in the normal form \cite{12}:

$$h = -\ln(z) + \Psi(1-P) + \Psi(-P) + \Psi(m-P) - 3\Psi(1) + \sum_{k=1}^{\infty} z^k f_k(P),$$

where $z$ is the anharmonic ratio, $P = \partial/\partial (\ln z)$ and the explicit formulae for $f_k(P)$ are given in ref.\cite{12}, where $h$ was defined with an extra factor 2 in comparison with the present paper and with ref.\cite{19}. The above expression for $h$ is equivalent to the expressions obtained by adding to it one of two terms

$$-2\pi \cot(\pi P), \quad \pi [\cot(\pi (m-P)) - \cot(\pi P)],$$

cancelling poles of the $\Psi$-functions. Apart from such ambiguities, $h$ can be considered for real $m(m-1)$ as a symmetric operator $h = h^T$ acting on functions $\phi(z, z^*)$ depending on real variables $z, z^*$ with the norm

$$\int_{-\infty}^{+\infty} \frac{dz^*}{z^*(1-z^*)} \int_{-\infty}^{+\infty} \frac{dz}{z(1-z)} |\phi|^2.$$

The integration over the real values of $z$ and $z^*$ can be obtained after the anti-Wick rotation of the vector $\vec{z}$ to the Minkowski space. Therefore the eigenvalues $\epsilon$ of $h$ should be real (for the case when $m(m-1)$ is real). On the other hand, $\epsilon$ is a real function of the eigenvalue $\mu^2$ (apart from constant imaginary contributions cancelling in the total energy $E$). For example, we have for large $\mu$ the expansion of $\epsilon$ in $\mu^{-2}$ \cite{13},

$$\epsilon = \ln(\mu) + 3\gamma + \left[ \frac{3}{448} + \frac{13}{120} \frac{(m-1/2)^2}{(m-1/2)^2} - \frac{1}{12} \frac{(m-1/2)^4}{(m-1/2)^4} \right] \frac{1}{\mu^2} + O \left( \frac{1}{\mu^4} \right).$$

Therefore $\mu$ is real or pure imaginary for real $m(m-1)$ (at least for large $|\mu|$).

Because the operator $q_3$ is anti-symmetric under the permutation of the coordinates $\rho_k$ and $\rho_l$, its matrix element between the symmetric wave functions is zero, but the matrix element of $q_3^2$ can be written as follows

$$\int_{-\infty}^{+\infty} \frac{dz^*}{z^*(1-z^*)} \int_{-\infty}^{+\infty} \frac{dz}{z(1-z)} \hat{q}_3 \phi \hat{q}_3 \phi^*$$

and therefore the eigenvalues $\mu$ of $-\hat{q}_3\hat{q}_3$ are again real or pure imaginary for real $m(m-1)$.

Let us consider now the expression for the total energy of the composite state of $n$-reggeons in terms of the holomorphic and anti-holomorphic energies

$$E_{m,\bar{m}} = \epsilon_m(\mu) + \epsilon_{\bar{m}}(\mu^*)$$

valid for the wave function $\phi_{m,\bar{m}}$, satisfying the Schrödinger equation in the Baxter-Sklyanin representation in the limit $\lambda, \lambda^* \to i$ \cite{12}. We can obtain the analogous expression

$$E_{m,\bar{m}} = \epsilon_m(\mu^*) + \epsilon_{\bar{m}}(\mu^*).$$
by taking instead another limit \( \lambda, \lambda^* \to -i \). These two expressions for energies were derived from the Schrödinger equation with the hermitian hamiltonian \([19]\). Therefore they should coincide for its eigenfunction \( \phi_{m,\overline{m}} \). This is possible only if the following property is fulfilled for the quantized values of \( \overline{\mu} \):

\[
\epsilon_m(\overline{\mu}) + \epsilon_m(\mu^{**}) = \epsilon_m(\mu^{**}) + \epsilon_m(\mu^*) ,
\]

The solution of this equation for real \( m(m-1) \) is

\[
\mu^{**} = \mu ,
\]

or \( \mu^* = \mu^s \), if again there is no accidental degeneracy.

The arguments given in this section support our assumption \([19]\), that the eigenvalues \( \mu \) are real for real \( m(m-1) \). For non-vanishing conformal spins \( n \) the values of \( \epsilon \) and \( \mu \) can be obtained by analytic continuation of the anomalous dimensions of the corresponding high-twist operators with \( n = 0 \) to continuous values of the Lorentz spin \( j \) (see sections 6-8 below).

In our opinion the complex values for \( \mu \) found in \([21]\) do not correspond to eigenfunctions of the Schrödinger equation for the Odderon and they were obtained due to an incorrect quantization procedure.

5 Zeroes of the Baxter function and the quantization of the integrals of motion

The zeroes of the Baxter function are very important, because their position \( \lambda_k \) is fixed by the Bethe equations and their knowledge gives a possibility to write the wave function of the composite states in the framework of the Bethe Ansatz as an (infinite) product of the differential operators \( B(\lambda_k) \) applied to the pseudo-vacuum state. The positions of the zeroes of \( Q(\lambda) \) for the Pomeron wave function was investigated in our previous paper \([19]\). For the case of the composite states constructed from \( n > 2 \) reggeons the number of the 'minimal' solutions \( Q^{(r)} \) of the Baxter equation is \( n \) and their linear combinations have rather complicated sets of zeroes.

However, certain linear combinations of functions \( Q^{(r)}(x, \mu) \) have zeroes which are situated at equidistant points \( x_k = x_0 + k, k \in \mathbb{Z} \). Indeed, according to relations \([54]\) among the Baxter functions \( Q^{(r)}(x, \mu) \), \( Q^{(r+1)}(x, \mu) \) and \( Q^{(r-1)}(x, \mu) \) such situation happens for solutions of the equation

\[
Q^{(r+1)}(x, \mu) + \lambda^{(r)}(\mu) Q^{(r-1)}(x, \mu) = 0 ,
\]

where \( r = 1, 2, \ldots, n-2 \). Some of the solutions of eq.\((69)\) coincide with the zeroes of the function \( Q^{(r)}(x, \mu) \) while other roots are situated at the equidistant points

\[
x^{(r)}_k(\mu) = k - \frac{1}{\pi} \arccot \left[ \frac{\delta^{(r)}(\mu)}{\pi} \right] , \quad k \in \mathbb{Z} .
\]

For the eigenstates of the Hamiltonian the parameters \( \mu \) are quantized in accordance with the physical requirement \([19]\) that the holomorphic energies for different Baxter functions are the same. It is equivalent to the condition, that all parameters \( \delta^{(r)}(\mu) \) vanish

\[
\delta^{(r)}(\mu) = 0 ,
\]
because otherwise, the energies for the solutions \( Q^{(r)}(x, \mu) \) and \( Q^{(r+1)}(x, \mu) \) would not coincide. Eq.\([70]\) then implies that for quantized \( \mu \) the above linear combination of \( Q^{(r+1)}(x, \mu) \) and \( Q^{(r-1)}(x, \mu) \) has a sequence of zeroes at the points
\[
x_k = k + \frac{1}{2}, \quad k \in \mathbb{Z}.
\]

Let us consider now the superpositions of the Baxter functions \( Q^{(n-1)}(x, \mu) \) and \( Q^{(0)}(x, \mu) \) with their poles situated only at positive and negative integer points, respectively. It is obvious from the previous section that there are \( n-2 \) different linear combinations of these functions
\[
\Phi^{(t)}(x, \mu) = Q^{(n-1)}(x, \mu) + c^{(t)}(\mu) Q^{(0)}(x, \mu)
\]
(72)
having equidistant zeroes at the points \( x = x_k \) where (see \((58))\)
\[
D \left[ \hat{\delta}(\mu), \hat{\alpha}(\mu), \pi \cot(\pi x) \right] = 0 .
\]
(73)
The last equation has \( n-2 \) different solutions
\[
z^{(t)}(\mu) = \pi \cot \left[ \pi x^{(t)}(\mu) \right], \quad t = 1, 2, \ldots n-2
\]
(74)
and for each solution there is a linear combination of the Baxter functions \( Q^{(n-1)}(x, \mu) \) and \( Q^{(0)}(x, \mu) \) with the relative coefficient
\[
c^{(t)}(\mu) = \frac{D_0^{(r)} \left[ \hat{\delta}(\mu), \hat{\alpha}(\mu), z^{(t)}(\mu) \right]}{D_{n-1}^{(r)} \left[ \hat{\delta}(\mu), \hat{\alpha}(\mu), z^{(t)}(\mu) \right]}. \quad (75)
\]
Note, that \( c^{(t)}(\mu) \) does not depend on the parameter \( r \), if there is no accidental degeneracy.

In the case, when
\[
\delta^{(r)}(\mu) = 0
\]
(76)
for all \( r = 1, 2, \ldots, n-2 \), which corresponds to the quantization of the integrals of motion \( \mu \), the determinant \( D \left[ \hat{\delta}(\mu), \hat{\alpha}(\mu), \pi \cot(\pi x) \right] \) is an even (odd) function of its argument \( \pi \cot(\pi x) \) for even (odd) \( n \). Therefore, the equidistant zeroes of two different functions \( \Phi^{(t)}(x, \mu) \) (or zeroes of the same function \( \Phi^{(t)}(x, \mu) \)) have opposite signs (modulo an integer number):
\[
x^{(t)}(\mu) = -x^{(n-2-t)}(\mu).
\]
(77)

For odd \( n \) one function has a sequence of zeroes at
\[
x_k = k + \frac{1}{2}.
\]

In a general case of \( n \) reggeized gluons we can calculate the important coefficients \( c^{(r)}(\mu) \) in the recurrent relations and the quantized values of \( \mu \) only by finding \( Q^{(n)}, Q^{(0)} \) and all their linear combinations with constant coefficients which have the equidistant zeroes with above properties. Let us consider several examples:

For \( n = 3 \) we have only one function with equidistant zeroes \( x_k = k + 1/2 \)
\[
Q(x) = Q^{(2)}(x, \mu) + \alpha^{(1)}(\mu) Q^{(0)}(x, \mu).
\]
For \( n = 4 \) there are two functions,
\[
Q_{\pm} = Q(3) \pm \alpha^{(1)} \sqrt{\alpha^{(2)}} Q(0)
\]
with zeroes at \( \pi \cot \pi x = \pm \sqrt{\alpha^{(2)}} \), respectively. For \( n = 5 \) there are also two functions with equidistant zeroes:
\[
Q_1 = Q(4) + \alpha^{(1)} \alpha^{(2)} Q(0), \quad Q_2 = Q(4) - \alpha^{(1)} \alpha^{(3)} Q(0)
\]
with the equidistant zeroes at \( \pi \cot \pi x = \pm \sqrt{\alpha^{(2)}} + \alpha^{(3)} \) and \( \pi \cot \pi x = 0 \), respectively. In all these cases, one can calculate all \( \alpha^{(r)} \) and the quantized values of \( \vec{\mu} \) by finding the corresponding combinations of \( Q^{(n)} \) and \( Q^{(0)} \) with the equidistant zeroes.

### 6 Anomalous dimensions of quasi-partonic operators

The \( Q^2 \)-dependence of the inclusive probabilities \( n_i(x, \ln Q^2) \) to have a parton \( i \) with momentum fraction \( x \) inside a hadron with large momentum \( |\vec{p}| \to \infty \) can be found from the DGLAP evolution equation [2]. The eigenvalues of its integral kernels describing the inclusive parton transitions \( i \to k \) coincide with the matrix elements \( \gamma_{jk}(\alpha) \) of the anomalous dimension matrix for twist-2 operators \( O_j \) with the Lorentz spins \( j = 2, 3, \ldots \). These operators are bilinear in the gluon \((i = g)\) or quark \((i = q)\) fields. For example, the twist-2 gluon operator with the Lorentz spin \( j \) can be written as,
\[
O_{\ldots}^{j} = n^{\mu_1} n^{\mu_2} \ldots n^{\mu_j} \text{tr} \, G_{\mu_1 \mu_2} D_{\mu_3} D_{\mu_4} \ldots D_{\mu_{j-1}} G_{\rho \mu_j},
\]
where \( D_\mu = \partial_\mu + g V_\mu \), \( V_\mu \) and \( G_{\rho \mu} = \frac{1}{g} [D_\rho, D_\mu] \) are the covariant derivative, the gluon field and the field tensor, respectively.

The symmetric traceless tensor \( O_{\mu_1 \mu_2 \ldots \mu_j} \) is multiplied by the light-cone vectors \( n_{\mu_r} \)
\[
n_\mu = q_\mu + x p_\mu, \quad n_\mu^2 = 0, \quad p_\mu^2 \simeq 0, \quad q_\mu^2 = -Q^2, \quad x = \frac{Q^2}{2pq},
\]
where \( p_\mu \) and \( q_\mu \) in the deep-inelastic ep scattering are the momenta of the initial proton and virtual photon, respectively.

The matrix elements of the operators \( O_{\ldots}^{j} \) between the hadron states are renormalized as functions of the growing ultraviolet cut-off \( Q^2 \). For example, in the case of the pure Yang-Mills theory with the gauge group \( SU(N_c) \) we have
\[
\langle p | O_{\ldots}^{j} | p \rangle \sim \exp \left( \int_{Q_0^2}^{Q^2} \gamma_j(\alpha_s(Q^2)) d \ln Q^2 \right), \quad \alpha_s(Q^2) \simeq \frac{4\pi}{\beta_2 \ln \frac{Q^2}{\Lambda^2}}, \quad \beta_2 = \frac{11}{3} N_c - \frac{2}{3} n_f.
\]
where \( \Lambda \simeq 200 \text{ Mev} \) is the QCD parameter and the anomalous dimension \( \gamma_j(\alpha) \) can be calculated perturbatively as
\[
\gamma_j(\alpha) = \sum_{k=1}^{\infty} C_j^{(k)} \left( \frac{\alpha N_c}{\pi} \right)^k.
\]
In particular, to the lowest order
\[
C_j^{(1)} = \Psi(1) - \Psi(j - 1) - \frac{2}{j} + \frac{1}{j+1} - \frac{1}{j+2} + \frac{11}{12}.
\]
The gluon anomalous dimension is singular in the non-physical point \( \omega = j - 1 \rightarrow 0 \). In this limit one can calculate it to all orders of perturbation theory [8] 

\[
\gamma_\omega = \frac{\alpha N_c}{\pi \omega} - \Psi''(1) \left( \frac{\alpha N_c}{\pi \omega} \right)^4 + \ldots
\]  

(80)

from the eigenvalue of the kernel of the BFKL equation in LLA [1] at \( n = 0 \):

\[
\omega_{BFKL} = \frac{\alpha N_c}{\pi} \left[ 2 \Psi(1) - \Psi(\gamma) - \Psi(1 - \gamma) \right].
\]  

(81)

Notice that the coefficients of \( \left( \frac{\alpha N_c}{\pi \omega} \right)^2 \) and \( \left( \frac{\alpha N_c}{\pi \omega} \right)^3 \) exactly vanish in eq.(80). Here \( \Psi''(1) = -2\zeta(3) = -2.4044138 \ldots \)

Indeed, the Green function satisfying the inhomogeneous BFKL equation can be written as follows [8]

\[
< \phi(\vec{p}_1) \phi(\vec{p}_2) \phi(\vec{p}_1') \phi(\vec{p}_2') > = \sum_n \int_{-\infty}^{+\infty} d\nu \ C(\nu, n) \int d^2 \rho_0 \frac{E_{\nu,n}(\vec{p}_1, \vec{p}_2) E_{\nu,n}(\vec{p}_1', \vec{p}_2')}{\omega - \omega^0(\nu, n)} ,
\]  

(82)

where \( \omega^0(\nu, n) \) is the eigenvalue of the BFKL kernel and \( C(\nu, n) \) is fixed by the completeness condition for its eigenfunctions

\[
E_{\nu,n}(\vec{p}_1, \vec{p}_2) = <0|\phi(\vec{p}_1) \phi(\vec{p}_2) O_{\nu,n}(\vec{p}_0)|0>
\]

\[
\left( \frac{\rho_{12}}{\rho_{10} \rho_{20}} \right)^m \left( \frac{\rho_{12}'}{\rho_{10} \rho_{20}} \right)^{\tilde{m}}, \ m = \frac{1}{2} + i\nu + \frac{n}{2}, \ \tilde{m} = \frac{1}{2} + i\nu - \frac{n}{2}.
\]  

(83)

In the limit \( |\rho_{1'2'}| \rightarrow 0 \) one can perform a Wilson expansion for the product of the fields \( \phi(\vec{p}_1') \) and \( \phi(\vec{p}_2') \). In this case the integral over \( \vec{p}_0 \) can be calculated by extracting the factor

\[
E_{\nu,n}(\vec{p}_1, \vec{p}_2) \sim \rho_{12}^m \rho_{1'2'}^{\tilde{m}} \sim \left( \frac{\rho_{1'2'}}{\rho_{12}} \right)^{2 \Gamma} |\rho_{1'2'}|^{2\Gamma} , \ \Gamma = \frac{m + \tilde{m}}{2} = 1 - \gamma.
\]

Moreover at \( |\rho_{1'2'}| \rightarrow 0 \) one can then shift the integration contour into the lower half of the \( \nu \)-plane up to the first pole of \( [\omega - \omega^0(\nu, n)]^{-1} \) having \( Im \nu < 0 \) [8, 23];

\[
\lim_{|\rho_{1'2'}| \rightarrow 0} < \phi(\vec{p}_1) \phi(\vec{p}_2) \phi(\vec{p}_1') \phi(\vec{p}_2') > \sim
\]

\[
\sum_n e^{i\gamma n} \int_{\varepsilon - i\infty}^{\varepsilon + i\infty} \frac{\tilde{C}(\gamma, n) \ d\gamma}{\omega - \omega^0(\nu, n)} \left| \frac{\rho_{12} \rho_{1'2'}}{\rho_{11'} \rho_{22'}} \right|^{2(1-\gamma)} , \ \gamma = \frac{1}{2} - i\nu ,
\]  

(84)

where \( \varepsilon \sim \alpha_s / \omega \) and

\[
e^{i\varphi} = \frac{\sqrt{\rho_{11'} \rho_{22'} \rho_{12} \rho_{1'2'}}}{\rho_{12}^2 \rho_{1'2'}^2}.
\]

Note, that in accordance with the fact that the Green function includes the external gluon propagators, the scattering amplitude behaves in the momentum space as \( |k|^{-4+2\gamma} \) for large gluon virtualities \( |k|^2 \).
In particular, for \( n = 0 \) and \( \gamma \to 0 \) one can obtain the above expansion (80) in powers of \( \frac{\pi \omega}{\rho_0} \) for the position \( \gamma_\omega \) of the pole \( (\gamma - \gamma_\omega)^{-1} \). It is important that for real \( \omega \) the pole is situated on the real axis. Therefore, the description of the corresponding state in the framework of the Möbius group approach requires the exceptional series of unitary representations [25] (contrary to the Regge kinematics, where the principal series is used). For the exceptional series the anomalous dimension

\[
\gamma = 1 - \frac{m + \bar{m}}{2}
\]

is real.

One can calculate from the BFKL equation also anomalous dimensions of higher twist operators by solving the eigenvalue equation near other singular points \( \gamma = -k \) \((k = 1, 2, \ldots)\) or by including in it a dependence from the conformal spin \( |n| \), which also leads to a shift of the pole position to \( \gamma = -|n|/2 \) (see [4]).

But a more important problem is the calculation of the anomalous dimensions for the so-called quasi-partonic operators [24] constructed from several gluonic fields. Indeed, the contribution of these operators at \( j \to 1 \) is responsible for the unitarization of structure functions at high energies. The simplest operator of such type is the product of the twist-2 gluon operators

\[
O^j = \prod_{r=1}^{p} O^j_{r}, \quad j = \sum_{r=1}^{p} j_r = p + \omega, \quad \omega = \sum_{r=1}^{p} \omega_r. \quad (85)
\]

In the limit \( N_c \to \infty \) this operator is multiplicatively renormalized [26] and its dimension is the sum of dimensions of its factors (including their anomalous dimensions \( \gamma(\omega_r) \))

\[
\Gamma = p - \gamma, \quad \gamma = \sum_{r=1}^{p} \gamma(\omega_r), \quad (86)
\]

where in the expression for the total dimension \( \Gamma \) we neglected the small contribution \( \frac{\omega}{2} \).

To investigate the multi-Pomeron configuration, let us write the hamiltonian describing the corresponding composite state in LLA as a sum of pairwise BFKL hamiltonians

\[
H = \sum_{r=1}^{p} H_{a_r b_r},
\]

neglecting the relative Pomeron interactions. In this case its eigenfunction as a product of the Pomeron wave functions:

\[
\Psi(\rho_{a_1}, \rho_{b_1}, \rho_{0_1}, \ldots; \rho_{a_p}, \rho_{b_p}, \rho_{0_p}) = \prod_{r=1}^{p} \left( \frac{\rho_{a_r b_r}}{\rho_{a_r b_r}} \right)^{m_r} \left( \frac{\rho^*_r \rho^*_r}{\rho^*_r \rho^*_r} \right)^{\bar{m}_r},
\]

where \( \rho_{a_r}, \rho_{b_r} \) and \( \rho_{0_r} \) are the coordinates of gluons \( a_r, b_r \) and Pomerons \( 0_r \), respectively, and the quantities \( m_r \) and \( \bar{m}_r \) are their conformal weights. The wave function \( \Psi \) belongs to a reducible representation of the Möbius group and can be expanded in a sum of irreducible representations with the use of the Clebsch-Gordon coefficients \( C^m_{m_1 \ldots m_r \bar{m}_1 \ldots \bar{m}_r} \) [25]

\[
\prod_{r=1}^{p} O_{m_r \bar{m}_r}(\vec{\rho}_r) = \sum_{m, \bar{m}} \int \frac{d^2 \rho_0}{2 \pi} C^m_{m_1 \ldots m_r \bar{m}_1 \ldots \bar{m}_r} (\vec{\rho}_0, \vec{\rho}_1, \vec{\rho}_2, \ldots; \vec{\rho}_p, \vec{\rho}_0) \Phi^{m_1 \ldots m_r}(\vec{\rho}_0).
\]
The contribution to the scattering amplitude in the coordinate space from each irreducible component can be written as follows

\[
\prod_{r=1}^{p} \left[ \sum_{n_r=0}^{\infty} \int_{-\infty}^{\infty} d\nu_r \int d^2 \rho_{0r} \left( \frac{\rho_{a_r b_r}}{\rho_{a_r 0} \rho_{b_r 0}} \right)^{m_r} \left( \frac{\rho^{a_r b_r}_{0*}}{\rho^{b_r 0*}_{0} \rho^{a_r 0}_{0*}} \right)^{\tilde{m}_r} \right] C_{m_r \tilde{m}_r}^{m_1 \tilde{m}_1; \ldots; m_p \tilde{m}_p} \frac{|Q|^{-2(p-\gamma)} e^{i n \varphi}}{\omega - \sum_{s=1}^{p} \omega(n_s, \nu_s)} .
\]

In the Regge regime, \( m_r \) and \( \tilde{m}_r \) belong to the principal series of the unitary representations of the Möbius group and therefore \( m \) and \( \tilde{m} \) also belong to the same series \([25]\). It means, that the position \( \omega_0 \) of the \( t \)-channel partial wave singularity related to the asymptotics of the cross-section \( \sigma_t \sim s^{\omega_0} \) equals \( \omega_0 = p \omega_{BFKL} \).

In the deep-inelastic regime the essential intervals are small \(|\rho_{0r}| \sim 1/Q \ll |\rho_{a_r b_r}| \). From dimensional considerations we obtain at large \(|Q|\) after integration over the essential region of \( \rho_0 \) a power asymptotics of the scattering amplitude

\[
A^{(m, \tilde{m})} \sim \prod_{r=1}^{p} \left[ \sum_{n_r=0}^{\infty} \int_{-\infty}^{\infty} d\nu_r \left( \frac{\rho_{a_r b_r}}{\rho_{a_r 0} \rho_{b_r 0}} \right)^{m_r} \left( \frac{\rho^{a_r b_r}_{0*}}{\rho^{b_r 0*}_{0} \rho^{a_r 0}_{0*}} \right)^{\tilde{m}_r} \right] C_{m_r \tilde{m}_r}^{m_1 \tilde{m}_1; \ldots; m_p \tilde{m}_p} \frac{|Q|^{-2(p-\gamma)} e^{i n \varphi}}{\omega - \sum_{s=1}^{p} \omega(n_s, \nu_s)} , \tag{87}
\]

where \( e^{i \varphi} = \sqrt{Q/Q^*} \), \( n = \frac{1}{2} \sum_{r=1}^{p} (m_r - \tilde{m}_r) \) and

\[
\gamma = p - \frac{1}{2} \sum_{r=1}^{p} (m_r + \tilde{m}_r) \tag{88}
\]

is the anomalous dimension of a composite operator which turns out to be real and small for small \( g^2/\omega \). This expression for \( \gamma \) is in agreement with the known result \([25]\), that in the product of the exceptional representations with real \( \gamma_r = 1 - \frac{m_r + \tilde{m}_r}{2} > 0 \) there is a continuous spectrum of the unitary representations of the principal series and only one representation from the exceptional series having

\[
\gamma = \sum_{r=1}^{p} \gamma_r . \tag{89}
\]

Since \( \omega(n, \nu) \) has a negative second derivative \( \omega''_{\gamma \gamma}(n, \nu) \) at \( 0 < \gamma < 1 \), we obtain after the integration over \( \nu_r \) with the use of the saddle-point method the following result for \( A \) in the Bjorken regime

\[
A^{(m, \tilde{m})} \sim \left[ \left( \frac{\rho_{a_r b_r}}{\rho_{a_r 0} \rho_{b_r 0}} \right)^{m_r} \left( \frac{\rho^{a_r b_r}_{0*}}{\rho^{b_r 0*}_{0} \rho^{a_r 0}_{0*}} \right)^{\tilde{m}_r} \right] |Q|^{-2(p-\gamma)} , \tag{90}
\]

where

\[
\gamma = p \omega(-1)(\omega/p) . \tag{91}
\]

and \( \omega(-1)(\omega) \) is the inverse function to \( \omega = \omega_{BFKL}(\gamma) \).

In particular, for very large \( Q^2 \) corresponding to \( \gamma \to 0 \) and \( n = 0 \) we have \([26]\)

\[
\gamma = \frac{\alpha_s N_c}{\pi \omega} p^2 . \tag{92}
\]

It is possible to calculate also a correction of the relative order \( N_c^{-2} \) to this expression \([26]\).

In accordance with the topics of this paper let us consider now the high energy asymptotics of the irreducible Feynman diagrams in which each of \( n \) reggeized gluons at \( N_c \to \infty \) interacts
only with two neighbours. In the Born approximation the corresponding Green function is the product of free gluon Green functions $\prod_{\nu=1}^{n} \ln |\rho_{r} - \rho_{\nu}'|^{2}$. In LLA $\frac{\alpha_{s}}{\omega} \sim 1$ it can be written as follows (cf. [8])

$$< \phi(\vec{p}_{1}) \ldots \phi(\vec{p}_{n}) \phi(\vec{\rho}) > = $$

$$\sum_{n} \int_{-\infty}^{+\infty} d\nu \sum_{\mu_{1}} \ldots \sum_{\mu_{n}} C_{m,\tilde{m};\mu_{1}, \ldots, \mu_{n}} \int d^{2}\rho_{0} \Psi_{m,\tilde{m};\mu_{1}, \ldots, \mu_{n}}(\vec{\rho}_{1}, \ldots, \vec{\rho}_{n}; \vec{\rho}_{0}) \Psi_{m,\tilde{m};\mu_{1}, \ldots, \mu_{n}}^{*}(\vec{\rho}'_{1}, \ldots, \vec{\rho}'_{n}; \vec{\rho}'_{0})$$

$$\omega - \omega(m, \tilde{m}; \mu_{3}, \ldots, \mu_{n})$$

(93)

where $\omega(m, \tilde{m}; \mu_{3}, \ldots, \mu_{n}) \sim \alpha_{s}$ are the eigenvalues of $H$ depending on the quantized integrals of motion $\mu_{3}, \ldots, \mu_{n}$ and $\Psi_{m,\tilde{m};\mu_{1}, \ldots, \mu_{n}}(\vec{\rho}_{1}, \ldots, \vec{\rho}_{n}; \vec{\rho}_{0})$ are the corresponding eigenfunctions. From the Möbius invariance of the Schrödinger equation we obtain

$$\Psi_{m,\tilde{m};\mu_{1}, \ldots, \mu_{n}}(\vec{\rho}_{1}, \ldots, \vec{\rho}_{n}; \vec{\rho}_{0}) \sim \left( \frac{\rho_{12} \rho_{23} \ldots \rho_{n1}}{\rho_{10} \rho_{20} \ldots \rho_{n0}} \right)^{\frac{m}{n}} \left( \frac{\rho_{12}^{*} \rho_{23}^{*} \ldots \rho_{n1}^{*}}{\rho_{10}^{*} \rho_{20}^{*} \ldots \rho_{n0}^{*}} \right)^{\frac{m}{n}} f_{m,\tilde{m}}(\vec{x}_{1}, \ldots, \vec{x}_{n-2})$$

(94)

where $\vec{x}_{r}$ ($x_{r}$ and $x_{r}^{*}$) are independent anharmonic ratios of $\rho_{k}$ and $\rho_{k}^{*}$ for $k = 0, 1, \ldots, n$.

In the Bjorken region, where

$$|\rho_{r} - \rho_{s}'| \sim Q^{-1} \ll |\rho_{r} - \rho_{s}|$$

the essential integration domain is $|\rho_{0} - \rho_{r}'| \sim 1/Q$ and therefore from dimensional considerations

$$< \phi(\vec{p}_{1}) \ldots \phi(\vec{p}_{n}) > \sim \sum_{n} \int_{-\infty}^{\infty} d\nu \sum_{\mu_{1}} \ldots \sum_{\mu_{n}} C_{m,\tilde{m};\mu_{1}, \ldots, \mu_{n}} \Psi_{m,\tilde{m};\mu_{1}, \ldots, \mu_{n}}(\vec{\rho}_{1}, \ldots, \vec{\rho}_{n}; \vec{\rho}_{0}) Q^{m} Q^{\tilde{m}}$$

$$\omega - \omega(m, \tilde{m}; \mu_{3}, \ldots, \mu_{n})$$

(95)

It means, that in this limit the contour of integration over $\nu$ should be shifted to a lower half of the complex plane up to the first pole of $[\omega - \omega(m, \tilde{m}; \mu_{3}, \mu_{2}, \ldots, \mu_{n})]^{-1}$.

For small coupling constants $\alpha_{s}$ this singularity for $n$ reggeized gluons is situated near the pole of $\omega(m, \tilde{m}; \mu_{3}, \ldots, \mu_{n})$. The position of the leading pole is

$$\frac{m + \tilde{m}}{2} = \frac{n}{2} - \gamma^{(n)} \quad , \quad \gamma^{(n)} = c^{(n)} \frac{\alpha_{s} N_{c}}{\omega} + O\left( \left[ \frac{\alpha_{s} N_{c}}{\omega} \right]^{2} \right)$$

(96)

This relation is in agreement with the above estimate of the anomalous dimension for the diagrams with the $t$-channel exchange of $p$ BFKL Pomeron because here $n = 2p$ and $c^{(2p)} = p^{2}$. Moreover, it can be obtained from the solution of the equation for matrix elements of quasipartonic operators written with a double-logarithmic accuracy in ref. [27]. Let us derive a similar equation starting from the Schrödinger equation for the composite states of reggeized gluons in the multi-colour QCD.

In the case of two reggeized gluons in the momentum space we obtain for $|p_{1}'| \simeq |p_{2}'| \gg |p_{1}|, |p_{2}|$ the equation

$$\omega \Psi(\vec{\rho}'_{1}, \vec{\rho}'_{2}) = -\frac{\alpha_{s} N_{c}}{\pi |p_{1}'|^{2} |p_{2}'|^{2}} \int_{\max(|p_{1}|^{2}, |p_{2}|^{2})}^{\infty} d|p_{1}'|^{2} \int_{0}^{2\pi} \frac{d\phi_{1}}{2\pi} \Psi(\vec{p}'_{1}, -\vec{p}'_{1})$$

Note, that the Bethe-Salpeter amplitude $\Psi(\vec{p}'_{1}, \vec{p}'_{2})$ contains the gluon propagators $|p_{1}|^{-2}, |p_{2}|^{-2}$. The solution of this equation is
\[ \Psi (\vec{p}_1, \vec{p}_2) \sim \left( \frac{\vec{p}_1 \cdot \vec{p}_2}{|p_1|^2 |p_2|^2} \right)^{-\frac{\gamma}{2}}, \quad \gamma = \frac{\alpha_s N_c}{\pi \omega}. \]

In the case of \( n \) reggeized gluons in the multi-colour QCD apart from the product of the propagators \( \prod_{r=1}^{n} |p_r|^{-2} \) due to the gauge invariance one can extract from the amplitude also the momenta \( p^\mu_r \) for each gluon

\[ \Psi (\vec{p}_1, \vec{p}_2, ..., \vec{p}_n) \sim \prod_{r=1}^{n} \frac{p^\mu_r}{|p_r|^2} f_{\mu_1, \mu_2, ..., \mu_n}. \]  

(97)

The factor \( \prod_{r=1}^{n} p^\mu_r / |p_r|^2 \) leads after the Fourier transformation to a singularity in \( \frac{m+\tilde{m}}{2} \) near \( n/2 \). The function \( f_{\mu_1, \mu_2, ..., \mu_n} \) is a tensor depending on \( \xi_r = \ln p^2_r \). The Schrödinger equation for this function takes the form

\[ \omega f_{\mu_1, \mu_2, ..., \mu_n} = \frac{\alpha_s N_c}{4\pi} \sum_{r=1}^{n} \delta_{\mu_r, \mu_{r+1}} \delta_{\mu_r', \mu_{r+1}'} \int_{\xi_r}^{\infty} d\xi'_{r+1} \int_{\xi_{r+1}}^{\infty} d\xi'_{r} \delta (\xi_r' - \xi_{r+1}') f_{\mu_1, \mu_2, ..., \mu_r, \mu_r', ..., \mu_n}. \]  

(98)

The above equation gives a possibility to calculate the anomalous dimension \( \gamma = \gamma (\omega) = c_n \alpha_s / \omega \). A similar equation is discussed in ref. [27]. In particular, for the Odderon \( (n = 3) \) it turns out that \( c_3 = 0 \) according to an unpublished result of M. Ryskin and A. Shuvaev. We confirm this result below [see eq. (117)] by solving the Baxter equation and finding a pole singularity near of \( \frac{m+\tilde{m}}{2} = 2 \) (instead of \( 3/2 \) as it could be expected from above considerations). For \( n = 4 \) in accordance with the general formula \( \frac{m+\tilde{m}}{2} = \frac{n}{2} - \gamma (n) \) we find a pole singularity near \( \frac{m+\tilde{m}}{2} = 2 \) as shown in eq. (130). Moreover, similar to the case of the BFKL Pomeron the anomalous dimensions \( \gamma_3 \) and \( \gamma_4 \) are calculated for arbitrary \( \alpha / \omega \), which is important for finding multi-reggeon contributions to the deep-inelastic processes at small Bjorken’s variable \( x \). We plot \( \pi \omega / (\alpha N_c) \) as a function of \( m = \tilde{m} \) for the Odderon in fig. 1 and for the four-reggeon state in fig. 2. It is important, that for \( m = \frac{1+k}{2} \) the curves allow to calculate the intercepts of some states with conformal spin \( k \).

7 Solutions of the Baxter equation for the Odderon

We construct explicit solutions of the Baxter equation for the Odderon \( (n = 3) \) following the general method presented in sec. 3.

The Baxter equation for the Odderon takes the real form (11)

\[ B_3 (x; m, \mu) \equiv \left[ 2x^3 + m(m-1)x + \mu \right] Q (x; m, \mu) \]

\[-(x+1)^3 Q (x+1; m, \mu) - (x-1)^3 Q (x-1; m, \mu) = 0, \]  

(99)

where

\[ q_3 = i\mu, \quad Im (\mu) = 0 \]  

(100)

and \( \mu \) is assumed to be real, which is compatible with the single-valuedness condition for the Odderon wave function in coordinate space [14].
The auxiliary functions $f_r$ ([13]) for the Odderon take the form
\[ f_2(x; m, \mu) = \sum_{l=0}^{\infty} \left[ \frac{a_l(m, \mu)}{(x-l)^2} + \frac{b_l(m, \mu)}{x-l} \right] , \]
\[ f_1(x; m, \mu) = \sum_{l=0}^{\infty} \frac{a_l(m, \mu)}{x-l} . \] (101)

Imposing the Baxter equation $B_3 \equiv 0$ at the poles $x = r$ yields the following recurrence relations (cf. [19])
\[ (r + 1)^3 a_{r+1}(m, \mu) = \left[ 2r^3 + m(m-1)r - \mu \right] a_r(m, \mu) - (r - 1)^3 a_{r-1}(m, \mu) , \]
\[ (r + 1)^3 b_{r+1}(m, \mu) = \left[ 2r^3 + m(m-1)r - \mu \right] b_r(m, \mu) - (r - 1)^3 b_{r-1}(m, \mu) + \left[ 6r^2 + m(m-1) \right] a_r(m, \mu) - 3(r + 1)^2 a_{r+1}(m, \mu) - 3(r - 1)^2 a_{r-1}(m, \mu) . \] (102)

We choose,
\[ a_0(m, \mu) = 1 , \quad b_0 = 0 , \] (103)
and all the coefficients $a_r(m, \mu)$ and $b_r(m, \mu)$ become uniquely determined by eqs. (102). In particular,
\[ a_1(m, \mu) = -\mu , \quad 8a_2(m, \mu) = -\mu [2 + m(m-1) - \mu] , \]
\[ b_1(m, \mu) = m(m-1) + 3\mu , \] (104)
\[ 8b_2(m, \mu) = [2 + m(m-1) - \mu] b_1(m, \mu) - \mu [6 + m(m-1)] - 12a_2(m, \mu) . \]

The coefficients of the leading pole singularities $a_r(m, \mu)$ of $f_1$ and $f_2$ obey identical recurrence relations as a consequence of the Baxter equation.

Thanks to eq. (102), $B_3(x; m, \mu)$ for $Q = f_{1,2}$ is an entire function of $x$. We find from eq. (99) that it has generically the form
\[ B_3(x; m, \mu) = [6 - m(m-1)] \lim_{x \to \infty} [x Q(x; m, \mu)] . \]

The Baxter equation is fulfilled provided the limit in the r. h. s. is zero.

Notice that a solution of the Baxter equation multiplied by a periodic function of $x$ with its period equal to 1 is again a solution of the Baxter equation. An example of such a function is $\pi \cot \pi x$ which is a constant at infinity. Furthermore, if $Q(x; m, \mu)$ is a solution of eq. (99), then $Q(-x; m, -\mu)$ is also a solution of eq. (99).

We can now form linear combinations of the auxiliary functions $f_1(x; m, \mu)$ and $f_2(x; m, \mu)$ in order to obtain true solutions of the Baxter equation ([13]):
\[ Q^{(2)}(x; m, \mu) = f_2(x; m, \mu) + B(m, \mu) f_1(x; m, \mu) , \] (105)
\[ Q^{(1)}(x; m, \mu) = f_1(x; m, \mu) + C(m, \mu) f_1(-x; m, -\mu) , \] (106)
\[ Q^{(0)}(x; m, \mu) = Q^{(2)}(-x; m, -\mu). \]

As noticed above, the Baxter equation \( B_3(x; m, \mu) = 0 \) is fulfilled at infinity provided the coefficients of \( x^{-1} \) in \( Q^{(2)}(x; m, \mu) \) and in \( Q^{(1)}(x; m, \mu) \) vanish for large \( x \),

\[
\sum_{r=1}^{\infty} b_r(m, \mu) + B(m, \mu) \sum_{r=0}^{\infty} a_r(m, \mu) = 0, \\
\sum_{r=0}^{\infty} a_r(m, \mu) - C(m, \mu) \sum_{r=0}^{\infty} a_r(m, -\mu) = 0. \tag{107}
\]

This gives the coefficients the result,

\[
B(m, \mu) = -\frac{\sum_{r=1}^{\infty} b_r(m, \mu)}{\sum_{r=0}^{\infty} a_r(m, \mu)}, \quad C(m, \mu) = \frac{\sum_{r=0}^{\infty} a_r(m, \mu)}{\sum_{r=0}^{\infty} a_r(m, -\mu)}. \tag{108}
\]

Therefore, the solutions \( Q^{(2)}(x; m, \mu) \) and \( Q^{(1)}(x; m, \mu) \) are completely determined.

The functions \( Q^{(2)}(x; m, \mu) \), \( Q^{(1)}(x; m, \mu) \) and \( Q^{(0)}(x; m, \mu) = Q^{(2)}(-x; m, -\mu) \) are related by the linear equation \([19]\)

\[
[\pi \cot \pi x + \delta(m, \mu)] Q^{(1)}(x; m, \mu) = Q^{(2)}(x; m, \mu) - C(m, \mu) Q^{(2)}(-x; m, -\mu) \tag{109}
\]

where the coefficients in front of \( Q^{(2)}(x; m, \mu) \) and \( Q^{(2)}(-x; m, -\mu) \) are calculated from the leading asymptotics at \( x \to 1 \) and \( x \to -1 \), respectively. This is a special case of eq.(54) for \( n = 3 \) and \( r = 1 \).

By finding the residues at single poles at \( x = 1 \) in the both sides of the above equation one can calculate \( \delta(m, \mu) \)

\[
\delta(m, \mu) = B(m, \mu) - \frac{C(m, \mu)}{\mu} \sum_{r=0}^{\infty} a_r(m, -\mu) - \frac{1}{\mu} \sum_{r=2}^{\infty} a_r(m, \mu) - 3 - \frac{m(m-1)-1}{\mu}.
\]

The total energy for the Odderon described by the function \( Q_{m,\tilde{m},\mu} \) being a bilinear combination of holomorphic and anti-holomorphic Baxter functions \( Q^{(0)}, Q^{(1)} \) and \( Q^{(2)} \) is given in ref.\([19]\)

\[
E = i \lim_{\lambda, \lambda^* \to i} \frac{\partial}{\partial \lambda} \frac{\partial}{\partial \lambda^*} \ln \left[ (\lambda - i)^2 (\lambda^* - i)^2 |\lambda|^2 Q_{m,\tilde{m},\mu} \left( \frac{\lambda}{\lambda^*} \right) \right], \tag{110}
\]

Thus, the energy is expressed in terms of the ratio of residues for the single and double poles of the Baxter functions at \( x = 1 \).

The holomorphic energies for the Baxter functions \( Q^{(2)}(x, m, \mu) \) and \( \pi \cot(\pi x) Q^{(1)}(x, m, \mu) \) should be the same, which leads to the quantization of \( \mu \) in accordance with ref.\([19]\)

\[
\delta(m, \mu) = 0 \tag{111}
\]

This equation fixes the possible values of \( \mu \) for given \( m \).

Because in the bilinear combination \( Q_{m,\tilde{m},\mu} \left( \frac{\lambda}{\lambda^*} \right) \) there is a term proportional to the product of \( Q^{(2)}(x; m, \mu) \) and \( Q^{(2)}(-x^*; \tilde{m}, -\mu) \), we obtain for the total energy

\[
E(m, \tilde{m}, \mu) = B(m, \mu) + B(\tilde{m}, -\mu). \tag{112}
\]

We found for \( m = \tilde{m} = 1/2 \) the first roots numerically from the above equations (cf.\([19]\))

\[
\mu_1 = 0.205257506 \ldots \quad \mu_2 = 2.3439211 \ldots \quad \mu_3 = 8.32635 \ldots \quad \mu_4 = 20.080497 \ldots \tag{113}
\]
with the corresponding energies

\[ E_1 = 0.49434 \ldots, \quad E_2 = 5.16930 \ldots, \quad E_3 = 7.70234 \ldots, \quad E_4 = 9.46283 \ldots \tag{114} \]

We have followed the eigenvalues \( E_1, \mu_1 \) as functions of \( m \) for \( 0 < m < \frac{1}{2} \). The result is plotted in fig. 1. Notice that only \( m = 0, 1 \) and \( \frac{1}{2} \) are physical values. For other \( m \) the curve describes the behaviour of the anomalous dimension for the corresponding high-twist operator (see sect. 5).

![Figure 1](image)

Figure 1: The energy and \( \mu \) as functions of \( m \) for the odderon eigenvalue \( E_1, \mu_1 \) in the interval \( 0 < m < 1 \). The picture is symmetric under \( m \leftrightarrow 1 - m \).

The energy vanishes at \( m = 0 \). This could be inferred from the fact that \( E(m = 0, \mu \equiv 0) = 0 \) in the expression for \( E(m, \mu \equiv 0) \) given by eq.(74) of ref.[19].

\[ E(m, \mu \equiv 0) = \frac{\pi}{\sin(\pi m)} + \psi(m) + \psi(1 - m) - 2\psi(1). \]

It should be noticed that \( E(m, \mu \equiv 0) \) describes an eigenvalue providing that the function \( Q^{(1)} \) does not enter in the bilinear combination of the total wave function \( Q_{m, \bar{m}, \mu} \) because in this case \( \delta(m, \mu) \) does not vanish.
For the first eigenvalue we obtain numerically,

\[ E_1(m) \overset{m \rightarrow 0}{=} 2.152 \ldots m - 2.754 \ldots m^2 + \mathcal{O}(m^3), \quad \mu_1(m) \overset{m \rightarrow 0}{=} 0.375 \ldots \sqrt{m} - 0.0228 m + \mathcal{O}(m^{\frac{3}{2}}) \]

(115)

Notice that all quantities are functions of \( m \) through the combination \( m(1-m) \). Therefore, they are invariant under the exchange \( m \leftrightarrow 1-m \). Thus, eqs. (115) yield also the behaviour of \( E_1 \) and \( \mu_1 \) near \( m = 1 \)

\[ E_1(m) \overset{m \rightarrow 1}{=} 2.152 \ldots (1-m) - 2.754 \ldots (1-m)^2 + \mathcal{O}[(m-1)^3], \]
\[ \mu_1(m) \overset{m \rightarrow 1}{=} 0.375 \ldots \sqrt{1-m} - 0.0228 (1-m) + \mathcal{O}[(1-m)^{\frac{3}{2}}]. \]

(116)

The state with \( m = 1 \) and \( \tilde{m} = 0 \) (or vice versa) is therefore the ground state of the Odderon. It has a vanishing energy and is situated below eigenstates (114) with \( m = \tilde{m} = 1/2 \).

One can invert eq. (116) to obtain the anomalous dimension \( \gamma \) [see sec. 5] of the operator with \( m = \frac{3}{2} - \gamma \) for \( \omega \rightarrow 0 \). The anomalous dimension is not small at arbitrary \( \alpha/\omega \). On the other hand, we can not interpret the state with \( m = 3/2 \) as a physical state with \( n = 2, \nu = 0 \) and a negative energy (see fig. 3), because \( \mu \) is pure imaginary for this state according to fig. 4. Note, that next-to-leading corrections to the BFKL pair kernel (cf. [3]) can give contributions to the eigenvalue \( \omega \) of the order of \( \alpha^2/(m - \frac{3}{2}) \), which will lead to the usual perturbative expansion of the anomalous dimension \( \gamma \sim \alpha^2/\omega \) for the corresponding operator.

Furthermore, we consider the states with conformal spin \( n = 1 \) where

\[ m = 1 + i \nu, \quad \tilde{m} = i \nu \]

For small \( \nu \) we compute the energy of such state from eqs. (115) and (116) with the result

\[ E = E_1(m) + E_1(\tilde{m}) = 5.51 \ldots \nu^2 + \mathcal{O}(\nu^4). \]

The zero energy state that we find for \( \nu = 0 \) is the one found in ref. [16] by a different approach.

We have also followed the first eigenstate with \( n = 0 \) as a function of \( \nu \) for real \( \nu \) and

\[ m = \frac{1}{2} + i \nu \]

The result is plotted in fig. 2. The eigenvalues for small \( \nu \) are

\[ E_1(\nu) = 0.49434 \ldots + 1.8179 \ldots \nu^2 + \mathcal{O}(\nu^4), \quad \mu_1(\nu) = 0.205257 \ldots + 0.48579 \ldots \nu^2 + \mathcal{O}(\nu^4) \]

We continue the first eigenstate with \( n = 0 \) for \( m > 1 \) turning \( \mu \) to purely imaginary. In figs. 3 and 4 the energy and \( \Im m \mu \) as functions of \( m \) are plotted for the first eigenvalue in the interval \( 1 \leq m \leq 2 \). This describes the dependence of the anomalous dimension as a function of \( \alpha_c/\omega \) for the corresponding operator with \( m = 2 - \gamma \) (see the previous section).
Figure 2: The energy and $\mu$ as functions of real $\nu$ for the odderon eigenvalue $E_1$, $\mu_1$ setting $m = \frac{1}{2} + i \nu$. 
Near $m = 2$ the energy diverges while $\mu$ tends to zero. We find its behaviour near $m = 2$ as follows

$$E_1(m) \overset{m \to 2}{=} \frac{2}{m-2} + 1 - m + O[(m-2)^2], \quad i\mu \overset{m \to 2}{=} 2 - m - \frac{3}{2}(m-2)^2 + O[(m-2)^3]$$

(117)

Thanks to the $m \leftrightarrow 1 - m$ symmetry we have for $\tilde{m} \to -1$,

$$E_1(\tilde{m}) \overset{\tilde{m} \to -1}{=} -\frac{2}{\tilde{m}+1} + \tilde{m} + 2 + O[(\tilde{m}+1)^2].$$

Now in order to compute the energy of the state with conformal spin $n = 3$ we use $\nu \to 0$ as a regulator. We have from eqs.(83)

$$m = 2 + i\nu, \quad \tilde{m} = -1 + i\nu.$$ 

Hence,

$$E = E_1(m) + E_1(\tilde{m}) \overset{\nu \to 0}{=} \frac{2}{i\nu} + 1 - i\nu - \frac{2}{i\nu} + 1 + i\nu + O(\nu^2) = 2 + O(\nu^2)$$

Inverting eq.(117) as discussed in sec. 5 yields for the anomalous dimension for $\gamma = 2 - m \to 0$,

$$\gamma \overset{\omega \gg \alpha N_c}{=} \alpha N_c \frac{\alpha}{\pi \omega} - 2 \left( \frac{\alpha}{\pi \omega} \right)^2 - 2 \left( \frac{\alpha}{\pi \omega} \right)^3 + O\left[ \left( \frac{\alpha}{\pi \omega} \right)^4 \right].$$

8 The solutions of the Baxter equation for four reggeons: the quartetton

The Baxter equation for the quartetton (four reggeons state) takes the form

$$B_4(x; m, \mu, q_4) \equiv \left[ 2x^4 + m(m-1)x^2 - \mu x + q_4 \right] Q(x; m, \mu, q_4) - (x+1)^4 Q(x+1; m, \mu, q_4) - (x-1)^4 Q(x-1; m, \mu, q_4) = 0.$$ (118)

where $q_3 = i\mu$, $Im(\mu) = 0$. A new integral of motion $q_4$ appears here. $\mu$ and $q_4$ are assumed to be real, which is compatible with the single-valuedness of the wave function in the coordinate space.

Notice that if $Q(x; m, \mu, q_4)$ is a solution of eq.(118) then $Q(-x; m, -\mu, q_4)$ is also a solution of eq.(118).

Following the general method presented in sec. III and [19] we seek solutions of the Baxter equation for the quartetton as a series of poles. We start by finding recurrence relations for the residues of the poles. Then, we impose the validity of the Baxter equation at infinity which gives further linear constraints on the pole residues.

The auxiliary functions (45) take here the form

$$f_3(x; m, \mu, q_4) = \sum_{l=0}^{\infty} \left[ \frac{a_l(m, \mu, q_4)}{(x-l)^3} + \frac{b_l(m, \mu, q_4)}{(x-l)^2} + \frac{c_l(m, \mu, q_4)}{x-l} \right],$$
Figure 3: The odderon energy as a function of $m$ for the eigenvalue 1 in the interval $1 \leq m \leq 2$ and imaginary $\mu$. 
Figure 4: \( \text{Im}\mu \) as a function of \( m \) for the odderon eigenvalue 1 in the interval \( 1 \leq m \leq 2 \). [Here \( \text{Re}\mu = 0 \).]
Imposing the Baxter equations at the poles $x = l$, $l = 0, 1, 2, 3, \ldots$ yields the recurrence relations:

\[(l + 1)^4 a_{l+1}(m, \mu, q_4) = [2l^4 + m(m - 1)l^2 - \mu l + q_4] a_l(m, \mu, q_4) - (l - 1)^4 a_{l-1}(m, \mu, q_4),\]
\[(l + 1)^4 b_{l+1}(m, \mu, q_4) = [2l^4 + m(m - 1)l^2 - \mu l + q_4] b_l(m, \mu, q_4) - (l - 1)^4 b_{l-1}(m, \mu, q_4) + [8l^3 + 2m(m - 1)l - \mu] a_l(m, \mu, q_4) - 4(l + 1)^3 a_{l+1}(m, \mu, q_4) - 4(l - 1)^3 a_{l-1}(m, \mu, q_4),\]
\[(l + 1)^4 c_{l+1}(m, \mu, q_4) = [2l^4 + m(m - 1)l^2 - \mu l + q_4] c_l(m, \mu, q_4) - (l - 1)^4 c_{l-1}(m, \mu, q_4) + [8l^3 + 2m(m - 1)l - \mu] b_l(m, \mu, q_4) - 4(l + 1)^3 b_{l+1}(m, \mu, q_4) - 4(l - 1)^3 b_{l-1}(m, \mu, q_4) - 6(l + 1)^2 a_{l+1}(m, \mu, q_4) - 6(l - 1)^2 a_{l-1}(m, \mu, q_4) + [12l^2 + m(m - 1)] a_l(m, \mu, q_4).\]

We choose,

\[a_0(m, \mu, q_4) = 1, \quad b_0 = 0, \quad c_0 = 0,\]

and all the coefficients $a_l(m, \mu, q_4)$, $b_l(m, \mu, q_4)$ and $c_l(m, \mu, q_4)$ become uniquely determined by eqs.\{120\}. In particular,

\[a_1(m, \mu, q_4) = q_4, \quad b_1(m, \mu, q_4) = -4q_4 - \mu,\]
\[c_1(m, \mu, q_4) = 10q_4 + 4\mu + m(m - 1).\]

Taking linear combinations of $f_1(\pm x; m, \pm \mu, q_4)$, $f_2(x; m, \mu, q_4)$ and $f_3(x; m, \mu, q_4)$ as in eqs.\{17\} and \{30\} we form three independent solutions $Q^{(r)}$ of the Baxter equation \{118\}: 

\[Q^{(3)}(x; m, \mu, q_4) \equiv f_3(x; m, \mu, q_4) + \alpha_1(m, \mu, q_4) f_2(x; m, \mu, q_4) + \alpha_2(m, \mu, q_4) f_1(x; m, \mu, q_4),\]
\[Q^{(2)}(x; m, \mu, q_4) \equiv f_2(x; m, \mu, q_4) + \gamma_1(m, \mu, q_4) f_1(x; m, \mu, q_4) + \gamma_2(m, \mu, q_4) f_1(-x; m, -\mu, q_4).\]

The solution $Q^{(1)}(x; m, \mu, q_4)$ is proportional to $Q^{(2)}(-x; m, -\mu, q_4)$.

The recurrence relations for the coefficients of the poles guarantee that $B_4(x; m, \mu, q_4)$ is an entire function. More precisely, we find from eqs.\{118\} and \{13\} that $B_4(x; m, \mu, q_4) = k_1 + k_2 x$, where $k_1$ and $k_2$ are some constants. These constants vanish and the Baxter equation is fulfilled provided the coefficients of $x^{-1}$ and $x^{-2}$ vanish for large $x$ in the $Q$'s of eq.\{123\}. Because the coefficients of $x^{-1}$ and $x^{-2}$ according to eqs.\{13\} from Appendix A are nonzero for the auxiliary functions $f_1(x; m, \mu, q_4)$, $f_2(x; m, \mu, q_4)$ and $f_3(x; m, \mu, q_4)$, these functions are not solutions of the Baxter equation.

The coefficients $\alpha_1(m, \mu, q_4)$, $\alpha_2(m, \mu, q_4)$, $\gamma_1(m, \mu, q_4)$ and $\gamma_2(m, \mu, q_4)$ are chosen imposing the Baxter equation at infinity. We present the linear equations on $\alpha_1(m, \mu, q_4)$, $\alpha_2(m, \mu, q_4)$, $\gamma_1(m, \mu, q_4)$ and $\gamma_2(m, \mu, q_4)$ and their explicit solutions in Appendix A.
There is one linear relation (124) among the Baxter solutions $Q^{(i)}$ for $n = 4$
\[
\left[\delta^{(2)}(\mu, m, q_4) + \pi \cot \pi x\right] Q^{(2)}(x; m, \mu, q_4) = Q^{(3)}(x; m, \mu, q_4) + \alpha^{(2)}(\mu, m, q_4) Q^{(2)}(-x; m, -\mu, q_4) .
\]

where $\delta^{(2)}(\mu, m, q_4) = \alpha_1(\mu, m, q_4) - \gamma_1(\mu, m, q_4)$. A proof of this equation is given in Appendix B. Another relation is obtained from above one by the substitution $x \to -x$, $\mu \to -\mu$.

The energy of the four Reggeons state is expressed through the bilinear combination $Q_{m,\tilde{m},\mu, q_4}$ of the Baxter functions [19]
\[
E = i \lim_{\lambda, \lambda^* \to i} \frac{\partial}{\partial \lambda} \frac{\partial}{\partial \lambda^*} \ln \left[ (\lambda - i)^{3}(\lambda^* - i)^{3} |\lambda|^8 Q_{m,\tilde{m},\mu, q_4} \left( \frac{\lambda}{\lambda^*} \right) \right] ,
\]
\hspace{1cm} \text{(124)}

Again, $E$ is related to the behavior of the Baxter functions $Q^{(r)}$ near $x = 1$. That is,
\[
Q^{(3)}(x; m, \mu, q_4) \sim \frac{q_4}{(x-1)^3} \left[ 1 - (x-1) \left( 4 + \frac{\mu}{q_4} + \alpha_1(\mu, m, q_4) \right) + O \left( (x-1)^2 \right) \right] ,
\]
\[
\pi \cot[\pi x] Q^{(2)}(x; m, \mu, q_4) \sim -\frac{q_4}{(x-1)^3} \left[ 1 - (x-1) \left( 4 + \frac{\mu}{q_4} + \gamma_1(\mu, m, q_4) \right) + O \left( (x-1)^2 \right) \right].
\]
\hspace{1cm} \text{(125)}

We obtain for the total energy of the three solutions from eqs. (124) and (125),
\[
E^{(3)}(m, \tilde{m}, \mu, q_4) = -\alpha_1(m, \mu, q_4) - \alpha_1(\tilde{m}, -\mu, q_4) ,
\]
\[
E^{(2)}(m, \tilde{m}, \mu, q_4) = -\gamma_1(m, \mu, q_4) - \gamma_1(\tilde{m}, -\mu, q_4) .
\]
\hspace{1cm} \text{(126)}

The eigenvalue conditions
\[
\delta^{(2)}(\mu, m, q_4) \sim \alpha_1(\mu, m, q_4) - \gamma_1(\mu, m, q_4) = 0 , \quad \delta^{(2)}(-\mu, \tilde{m}, q_4) = 0 \hspace{1cm} \text{(127)}
\]
guarantee that the holomorphic energy is the same for these three independent Baxter solutions,
\[
\epsilon^{(3)}(m, \mu, q_4) = \epsilon^{(2)}(m, \mu, q_4) = \epsilon^{(1)}(m, \mu, q_4) .
\]

Eqs. (127) fix possible values of $\mu$ and $q_4$ for given $m$.

We find from the above equations for $m = \tilde{m} = 1/2$ the first roots numerically as,
\[
\mu = 0 \quad , \quad q_4 = 0.1535892 \ldots , \quad E = -1.34832 \ldots ,
\]
\[
\mu = 0.73833 \ldots \quad , \quad q_4 = -0.3703 \ldots , \quad E = 2.34105 \ldots ,
\]
\[
\mu = 0 \ldots \quad , \quad q_4 = -0.292782 \ldots , \quad E = 2.756624 \ldots ,
\]
\[
\mu = 1.4100 \ldots , \quad q_4 = 0.73852 \ldots , \quad E = 3.3581 \ldots ,
\]
\[
\mu = 0 \ldots , \quad q_4 = 1.79992 \ldots , \quad E = 5.67117 \ldots ,
\]
\[
\mu = 0 \ldots , \quad q_4 = 2.185799 \ldots , \quad E = 6.2819490 \ldots . \hspace{1cm} \text{(128)}
\]

The first eigenstate was reported in ref. [21] where it is incorrectly identified as the ground state for the quartetton [see nota added].
In fig. 7 the eigenvalue equations (127) are plotted in the \( \mu, q_4 \) plane for \( m = \tilde{m} = \frac{1}{2} \). The curves intersect at the eigenstates (128).

We find for the first eigenvalues with \( m = 0, \tilde{m} = 1 \) corresponding to \( n = -1 \) in eq.(88) [\( n = 1 \) gives the same state with \( m \leftrightarrow \tilde{m} \)].

\[
\begin{align*}
\mu &= 0, \quad q_4 = 0.12167 \ldots, \quad E = -2.0799 \ldots, \\
\mu &= 0.51214 \ldots, \quad q_4 = -0.33288 \ldots, \quad E = 2.2007 \ldots, \\
\mu &= 0, \quad q_4 = -0.2905426 \ldots, \quad E = 2.441210 \ldots.
\end{align*}
\] (129)

Therefore, the ground state of the quartet with \( |n| = 1 \) corresponds to \( m = 0, \tilde{m} = 1 \). Its energy, \( E = -2.0799 \ldots \) is below the lowest energy \( E = -1.34832 \ldots \) of the states with \( m = \tilde{m} = \frac{1}{2} \).

We plot in fig. 8 the eigenvalue equations (127) in the \( \mu, q_4 \) plane for \( m = 0, \tilde{m} = 1 \). The curves intersect at the eigenstates (129). It should be noticed that the curves in figs. 7 and 8 turn to be qualitatively similar. The eigenstates with \( m = 0, \tilde{m} = 1 \) follow from those with \( m = \tilde{m} = \frac{1}{2} \) by an analytic continuation in \( m, \tilde{m} \).

We have followed the first eigenvalue as a function of \( m \) for \( 0 < m < \frac{1}{2} \). The result is plotted in fig. 6. Contrary to the Odderon case, the energy eigenvalue does not vanish for \( m = 0 \). The energy decreases with \( m \) for \( 0 < m < \frac{1}{2} \) and takes the value \( E = -2.0799 \ldots \) at \( m = 0 \) while \( q_4 \) takes there the value 0.12167 \ldots\. Further, \( \mu \) vanishes for this eigenvalue for all \( m \) [see fig. 5].

For \( m = 2 \) the lowest energy state of four reggeons goes to minus infinity and \( q_4 \) vanishes \([q_3 \text{ is zero for all } m \text{ in this state}]\). We find the following behaviour,

\[
E_1 \stackrel{m \to 2}{\longrightarrow} \frac{4}{m - 2} + 2 + 2 - m + \mathcal{O} \left[ (m - 2)^2 \right], \quad q_4 \stackrel{m \to 2}{\longrightarrow} \frac{1}{4} (m - 2)^2 + \mathcal{O} \left[ (m - 2)^3 \right].
\] (130)

Thanks to the \( m \leftrightarrow 1 - m \) symmetry we have for \( \tilde{m} \to -1 \),

\[
E_1 \stackrel{\tilde{m} \to -1}{\longrightarrow} -\frac{4}{\tilde{m} + 1} + 3 + \tilde{m} + \mathcal{O} \left[ (\tilde{m} + 1)^2 \right].
\]

Now in order to compute the energy of the state with conformal spin \( n = 3 \) we use \( \nu \to 0 \) as a regulator as in the odderon case. We find in this way,

\[
E = E_1(m) + E_1(\tilde{m}) \stackrel{\nu \to 0}{\longrightarrow} 4 + \mathcal{O}(\nu^2)
\]

Inverting eq.(130) as discussed in sec. 5 yields for the anomalous dimension for \( \gamma = 2 - m \to 0 \),

\[
\gamma \stackrel{\omega > \alpha N_c}{\longrightarrow} \frac{4}{\pi} \frac{\alpha N_c}{\pi \omega} + 8 \left( \frac{\alpha N_c}{\pi \omega} \right)^2 + \mathcal{O} \left[ \left( \frac{\alpha N_c}{\pi \omega} \right)^4 \right].
\]

The state with \( m = 3/2 \) (with \( n = 2, \nu = 0 \)) can be considered as a physical ground state for the quartet because for it the eigenvalue of \( q_4 \) is real according to fig. 5. It has a large negative energy comparable with the energy of the BFKL pomeron constructed from two reggeized gluons. But to prove this conclusion it is needed to construct a bilinear combination of the corresponding Baxter functions to verify the normalizability of the corresponding solution.
Figure 5: The $q_4$ as a function of $m$ for the quartetion eigenvalue 1 in the interval $0 < m < 2$. We have for this eigenvalue $\mu = 0$ for all $m$. 
Figure 6: The energy as a function of $m$ for the quartet no eigenvalue 1 in the interval $0 < m < 2$. We have for this eigenvalue $\mu = 0$ for all $m$. 
Figure 7: The quartetton eigenvalue equations $\alpha_1(\mu, m, q_4) - \gamma_1(\mu, m, q_4) = 0$ and $\alpha_1(-\mu, \tilde{m}, q_4) - \gamma_1(-\mu, \tilde{m}, q_4) = 0$ for $m = \tilde{m} = \frac{1}{2}$ in the $\mu, q_4$-plane.
Figure 8: The quartetion eigenvalue equations \( \alpha_1(\mu, m, q_4) - \gamma_1(\mu, m, q_4) = 0 \) and \( \alpha_1(-\mu, \tilde{m}, q_4) - \gamma_1(-\mu, \tilde{m}, q_4) = 0 \) for \( m = 0, \tilde{m} = 1 \) in the \( \mu, q_4 \)-plane
TABLE 1. Conformal spins, conformal weights and corresponding lowest energy eigenvalues for the pomeron, odderon and quartetron states. The odderon state with imaginary $\mu$ is discarded as non-physical. The pomeron and quartetron states with $n = 3$ have the same energy and are related presumably by the duality symmetry.

As displayed in Table 1 the intercept for the quartetron ground state possessing conformal spin $n = 2$ is larger than that for the BFKL Pomeron. This result is not very surprising because four reggeized gluons clustered into two pomerons have an even larger intercept. A large intercept for the state with the conformal spin 2 may lead to such unphysical results as negative cross sections. But it is known that the unitarization of scattering amplitudes is not solved within a framework where the number of reggeons is fixed.

9 Acknowledgements

We thank E. A. Antonov, J. Bartels, A. A. Belavin, A. P. Bukhvostov, R. Kirschner, E. A. Kuraev, G. Marchesini, G. P. Vacca, W. von Schlippe and further participants to the PNPI Winter School for stimulating discussions on the basic results of this paper. Subsequent discussions with V. Fateev, P. Mitter, A. Neveu, F. Smirnov and Al. Zamolodchikov were especially fruitful. One of us (LNL) thanks LPTHE for the hospitality during his visit to the University of Paris VI in March 2002.

Nota Added: After completion of this work we have seen the preprint \[22\] studying similar problems.

It is stated in refs.\[21, 22\] that the ground state for three and four reggeons corresponds to vanishing conformal spin $m - \tilde{m} = 0$. We show here that the lowest energy eigenvalue corresponds in both cases to a non-zero conformal spin $m = 1/2$ and $m = 1$, respectively.

It is claimed in ref.\[22\] that the eigenvalues $\mu$ can be complex. In our opinion the reason for this disagreement with our results is, that the authors of ref.\[22\] did not take into account, that in the Baxter-Sklyanin representation there are two different expressions for the energy in terms of $Q_m(\lambda), Q_{\tilde{m}}(\lambda^*)$ obtained at $\lambda \to \pm i$. These two energy values coincide if and only if $\psi_{m, \tilde{m}}$ is an eigenfunction of the Hamiltonian. It leads to the constraint eq. (68) on the spectrum of the integrals of motion.

We have computed the energies for the odderon states displayed in table 1 of ref.\[22\]. States $(0, 2)$ and $(6, 2)$ coincide with our first two eigenstates 1 and 2 [see eqs. (113) and (114)] within
the three figures precision of table 1 of ref. [22]. States (2, 2), (4, 2), (8, 2) and (10, 2) possess complex \( \mu \) and complex energy. Moreover, they fulfill the real part of our eigenvalue equation (111) up to \( \mathcal{O}(10^{-3}) \) while the imaginary part of \( \delta(m, \mu) \) turns to be of the order one. Our understanding is that these states with complex \( \mu \) are not eigenstates of the hamiltonian.

We furthermore computed the energies for the quarteton states displayed in table 2 of ref. [22]. States (2, 0), (2, 2), (4, 0) and (3, 3) coincide with our quarteton eigenvalues [see eqs. (128)] 1, 3, 5 and 6, respectively. (Again, within the three figures precision of table 2 of ref. [22]). States (3, 1) and (4, 2) possess complex \( q_4 \) and complex energy. The real parts of \( E^{(3)} \) and \( E^{(2)} \) computed from eqs. (126) coincide up to \( \mathcal{O}(10^{-3}) \) while the imaginary parts of \( E^{(3)} \) and \( E^{(2)} \) differ by amounts of the order one. Our understanding is again that these states with complex \( q_4 \) are not eigenstates of the hamiltonian.

### A Asymptotic constraints on the solutions of the Baxter equation. The quartet case.

We impose here the Baxter equation at infinity on the solutions for four reggeons. As derived in sec. VI, the coefficients of \( x^{-1} \) and \( x^{-2} \) in \( Q^{(3)}(x; m, \mu, q_4) \) and \( Q^{(2)}(x; m, \mu, q_4) \) for \( x \to \infty \) must vanish.

For large \( x \) the series (119) yield the following formal expansions,

\[
\begin{align*}
    f_3(x; m, \mu, q_4) & \xrightarrow{|x| \to \infty} \frac{1}{ix} \sum_{r=0}^{\infty} c_r(m, \mu, q_4) + \frac{1}{(ix)^2} \sum_{r=0}^{\infty} [b_r(m, \mu, q_4) - r c_r(m, \mu, q_4)] + \mathcal{O}\left(\frac{1}{x^3}\right), \\
    f_2(x; m, \mu, q_4) & \xrightarrow{|x| \to \infty} \frac{1}{ix} \sum_{r=0}^{\infty} b_r(m, \mu, q_4) + \frac{1}{(ix)^2} \sum_{r=0}^{\infty} [a_r(m, \mu, q_4) - r b_r(m, \mu, q_4)] + \mathcal{O}\left(\frac{1}{x^3}\right), \\
    f_1(x; m, \mu, q_4) & \xrightarrow{|x| \to \infty} \frac{1}{ix} \sum_{r=0}^{\infty} a_r(m, \mu, q_4) - \frac{1}{(ix)^2} \sum_{r=0}^{\infty} r a_r(m, \mu, q_4) + \mathcal{O}\left(\frac{1}{x^3}\right). \quad (131)
\end{align*}
\]

For \( Q^{(3)}(x; m, \mu, q_4) \) we find the conditions,

\[
\begin{align*}
    \alpha_1(m, \mu, q_4) & \sum_{r=0}^{\infty} b_r(m, \mu, q_4) + \alpha_2(m, \mu, q_4) \sum_{r=0}^{\infty} a_r(m, \mu, q_4) = - \sum_{r=0}^{\infty} c_r(m, \mu, q_4), \\
    \alpha_1(m, \mu, q_4) & \sum_{r=0}^{\infty} [a_r(m, \mu, q_4) - r b_r(m, \mu, q_4)] - \alpha_2(m, \mu, q_4) \sum_{r=0}^{\infty} r a_r(m, \mu, q_4) = \\
    & = \sum_{r=0}^{\infty} [r c_r(m, \mu, q_4) - b_r(m, \mu, q_4)]. \quad (132)
\end{align*}
\]

Imposing the same constraint on \( Q^{(2)}(x; m, \mu, q_4) \) yields,

\[
\begin{align*}
    \gamma_1(m, \mu, q_4) & \sum_{r=0}^{\infty} a_r(m, \mu, q_4) - \gamma_2(m, \mu, q_4) \sum_{r=0}^{\infty} a_r(m, \mu, q_4) = - \sum_{r=0}^{\infty} b_r(m, \mu, q_4), \\
    \gamma_1(m, \mu, q_4) & \sum_{r=0}^{\infty} r a_r(m, \mu, q_4) + \gamma_2(m, \mu, q_4) \sum_{r=0}^{\infty} r a_r(m, \mu, q_4) = \\
    & = \sum_{r=0}^{\infty} [a_r(m, \mu, q_4) - r b_r(m, \mu, q_4)]. \quad (133)
\end{align*}
\]
Eqs. (132) can be easily solved yielding,

$$\alpha_1(m, \mu, q_4) = \frac{1}{\Delta_\alpha(m, \mu, q_4)} \left\{ \sum_{r=0}^{\infty} a_r(m, \mu, q_4) \sum_{n=0}^{\infty} c_n(m, \mu, q_4) - \sum_{n=0}^{\infty} a_n(m, \mu, q_4) \sum_{r=0}^{\infty} \left[ r c_r(m, \mu, q_4) - b_r(m, \mu, q_4) \right] \right\} ,$$

$$\alpha_2(m, \mu, q_4) = \frac{1}{\Delta_\alpha(m, \mu, q_4)} \left\{ \sum_{n=0}^{\infty} a_n(m, \mu, q_4) \sum_{r=0}^{\infty} \left[ a_r(m, \mu, q_4) - r b_r(m, \mu, q_4) \right] - \sum_{r=0}^{\infty} b_r(m, \mu, q_4) \sum_{n=0}^{\infty} n a_n(m, \mu, q_4) \right\}$$

where

$$\Delta_\alpha(m, \mu, q_4) \equiv \sum_{n=0}^{\infty} a_n(m, \mu, q_4) \sum_{r=0}^{\infty} \left[ r b_r(m, \mu, q_4) - a_r(m, \mu, q_4) \right] - \sum_{r=0}^{\infty} b_r(m, \mu, q_4) \sum_{n=0}^{\infty} n a_n(m, \mu, q_4) .$$

We analogously obtain from eqs. (133)

$$\gamma_1(m, \mu, q_4) = \frac{1}{\Delta_\gamma(m, \mu, q_4)} \left\{ \sum_{n=0}^{\infty} a_n(m, -\mu, q_4) \sum_{r=0}^{\infty} \left[ r a_r(m, -\mu, q_4) - r b_r(m, \mu, q_4) \right] \right\} ,$$

$$\gamma_2(m, \mu, q_4) = \frac{1}{\Delta_\gamma(m, \mu, q_4)} \left\{ \sum_{n=0}^{\infty} a_n(m, \mu, q_4) \sum_{r=0}^{\infty} \left[ a_r(m, \mu, q_4) - r b_r(m, \mu, q_4) \right] + \sum_{n=0}^{\infty} b_n(m, \mu, q_4) \sum_{r=0}^{\infty} a_r(m, \mu, q_4) \right\}$$

where,

$$\Delta_\gamma(m, \mu, q_4) \equiv \sum_{n=0}^{\infty} a_n(m, \mu, q_4) \sum_{r=0}^{\infty} r a_r(m, -\mu, q_4) + \sum_{n=0}^{\infty} a_n(m, -\mu, q_4) \sum_{r=0}^{\infty} r a_r(m, \mu, q_4) .$$

B Linear Relations among the solutions of the Baxter equation for the quartet case.

We present here a proof of the linear recurrence relations (134) for the quartet case. The Baxter solutions $Q^{(3)}(x; m, \mu, q_4)$ and $Q^{(2)}(x; m, \mu, q_4)$ are related by

$$[\alpha_1(\mu, m, q_4) - \gamma_1(\mu, m, q_4) + \pi \cot \pi x] Q^{(2)}(x; m, \mu, q_4) = Q^{(3)}(x; m, \mu, q_4) - \gamma_2(\mu, m, q_4) .$$

This is eq. (134) for $n = 4, r = 2$ and $\delta^{(2)}(\mu, m, q_4) = \alpha_1(\mu, m, q_4) - \gamma_1(\mu, m, q_4), \alpha^{(2)}(\mu, m, q_4) = -\gamma_2(\mu, m, q_4)$. Notice that $Q^{(1)}(x; m, \mu, q_4)$ is proportional to $Q^{(2)}(-x; m, -\mu, q_4)$ according to eq. (133) and therefore eq. (136) is the only independent three-terms linear relation between Baxter solutions.
In order to prove this relations we compute its triple, double and simple poles at \( x = l \in \mathbb{Z} \). In the course of these calculations we use the fact, that \( \pi \cot \pi x \) as a function of \( x \) has unit residues in all these poles.

Moreover, we have from eqs. (119) and (123) that

\[
Q^{(2)}(x; m, \mu, q_4) \xrightarrow{x \rightarrow r > 0} a_r(m, \mu, q_4) \frac{b_r(m, \mu, q_4) + \gamma_1(\mu; m, q_4) a_r(m, \mu, q_4)}{(x - r)^2} + \tilde{A}_r(m, \mu, q_4) + \mathcal{O}(x - r)
\]

\[
Q^{(2)}(x; m, \mu, q_4) \xrightarrow{x \rightarrow r < 0} -\gamma_2(\mu, m, q_4) \frac{a_r(m, -\mu, q_4)}{x + r} + \tilde{A}_r(m, \mu, q_4) + \mathcal{O}(x + r)
\]

Inserting eqs. (137) into the Baxter equation (118) yields the recurrence relations

\[
(r + 1)^4 A_{r+1}(m, \mu, q_4) = \left[2 r^4 + m(m - 1) r^2 - \mu r + q_4 \right] A_r(m, \mu, q_4) - (r - 1)^4 A_{r-1}(m, \mu, q_4)
\]

\[
+ \left[8 r^3 + 2m(m - 1) r - \mu \right] b_r(m, \mu, q_4) - 4 (r + 1)^3 b_{r+1}(m, \mu, q_4) - 4 (r - 1)^3 b_{r-1}(m, \mu, q_4)
\]

\[-6(r + 1)^2 a_{r+1}(m, \mu, q_4) - 6(r - 1)^2 a_{r-1}(m, \mu, q_4) + [12 r^2 + m(m - 1)] a_r(m, \mu, q_4)
\]

\[+
\gamma_1(\mu; m, q_4) \left\{ \left[8 r^3 + 2m(m - 1) r - \mu \right] a_r(m, \mu, q_4)
\right\}.
\]

\[
(138)
\]

and

\[
(r + 1)^4 \tilde{A}_{r+1}(m, \mu, q_4) = \left[2 r^4 + m(m - 1) r^2 + \mu r + q_4 \right] \tilde{A}_r(m, \mu, q_4) - (r - 1)^4 \tilde{A}_{r-1}(m, \mu, q_4)
\]

\[+
\gamma_2(\mu; m, q_4) \left\{ \left[8 r^3 + 2m(m - 1) r + \mu \right] a_r(m, -\mu, q_4)
\right\}.
\]

\[
-4 (r + 1)^3 a_{r+1}(m, -\mu, q_4) - 4 (r - 1)^3 a_{r-1}(m, -\mu, q_4)
\]

\[
(139)
\]

These recurrence relations can be solved in terms of the coefficients \( a_r \), \( b_r \) and \( c_r \) using eq. (120). One has to take into account that \( A_r(m, \mu, q_4) \) and \( \tilde{A}_r(m, \mu, q_4) \) do not obey the initial conditions (121). We find,

\[
A_r(m, \mu, q_4) = c_r(m, \mu, q_4) + \gamma_1(\mu; m, q_4) b_r(m, \mu, q_4) + A_0(m, \mu, q_4) a_r(m, \mu, q_4)
\]

\[
\tilde{A}_r(m, \mu, q_4) = \gamma_2(\mu; m, q_4) b_r(m, -\mu, q_4) + A_0(m, \mu, q_4) a_r(m, -\mu, q_4)
\]

\[
(140)
\]

Using eqs. (119), (123) and (140) we find that eq. (136) holds at all of its poles. Therefore, the r. h. s. and the l. h. s. of eq. (136) can only differ on an entire function. Taking into account the asymptotic behaviour of the Baxter functions we conclude that this entire function is identically zero.
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