Generalized reduction formula for Discrete Wigner functions of multiqubit systems

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Density matrices and Discrete Wigner Functions are equally valid representations of multiqubit quantum states. For density matrices, the partial trace operation is used to obtain the quantum state of subsystems, but an analogous prescription is not available for discrete Wigner Functions. Further, the discrete Wigner function corresponding to a density matrix is not unique but depends on the choice of the quantum net used for its reconstruction. In the present work, we derive a reduction formula for discrete Wigner functions of a general multiqubit state which works for arbitrary quantum nets. These results would be useful for the analysis and classification of entangled states and the study of decoherence purely in a discrete phase space setting and also in applications to quantum computing.

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I. INTRODUCTION

Wigner distribution functions are phase space representations of continuous variable (CV) quantum systems. These functions find widespread applications in quantum optics. They are real valued and normalized, but unlike genuine probability distribution functions, Wigner functions can take negative values in some regions of the phase space, and are hence called quasi-probability functions [1, 2]. Classical states of light like coherent states have positive Wigner functions [3] but, this is not the case for quantum states of light such as photon added/subtracted coherent states, entangled states and squeezed states [4, 5]. In fact, negative values of the Wigner function attest to the quantum character of the state [6]. Wigner functions can be experimentally reconstructed through homodyne measurements and quantum interference effects are quite nicely brought-out in the visual presentation of the reconstructed state [7]. Given the usefulness of Wigner functions of CV systems, the construction of their finite dimensional analogs has attracted considerable attention [8–14]. Discrete Wigner Functions (DWFs) are particularly relevant for qubit states used in quantum information and quantum computation studies. DWFs find applications in stabilizer codes, quantum error correction, quantum teleportation, study of decoherence and in the construction of toy models in support of epistemic interpretations of the quantum state [15–19]. For multiqubit states, the DWF construction given by Wootters and Gibbons et al., is particularly elegant and the present work is based on this construction [20, 21]. DWFs can be tomographically reconstructed through repeated measurements using mutually unbiased basis sets (MUBS) and in the case of bipartite system they have been directly reconstructed using Hong-Ou-Mandel interferometers [22]. However, for finite dimensional systems such as optical multiqubits, density matrices and Stokes vectors are the most widely used representations of the quantum state. The wider use of DWFs is inhibited by two important limitations: (i). DWF representations of the quantum state are not unique but, depend on the particular way of assigning the MUBS to “lines” in the discrete phase space, with different assignments leading to different versions of the DWF, known as quantum nets. For a given Hilbert space of dimension $N$, there are $N^{N+1}$ possible quantum nets, that is $N^{N+1}$ possible definitions of DWF. The state of the subsystem cannot be easily obtained as in the case of density matrices and Stokes vectors. The problem of reduction of the the composite state DWF to that of the sub-system has been addressed only for two qubit systems [23], based on the tensor product structure of the phase space point operators. In the previous reference, the reduction formula is given only for specific quantum nets called the Wootters and the Aravind nets. For a $4 \times 4$ phase space corresponding to a two qubit system, there are 1024 possible quantum nets, but phase space point operators have a product structure only for 32 of them and the reduction formula of Holmes et al., is applicable only to these cases [21]. For other powers of prime, the existence of the product structure of the phase space point operator has not been investigated. In any case, a reduction procedure for arbitrary multiqubit systems is not known to the best of our knowledge. In the present work, we derive such a generalized reduction formula, that does not require the existence of such a product structure. Recently, we had addressed the problem of carrying out spin flip operations on multiqubit DWFs [24] and based on this result, we had given a formula for quantifying the $n$-concurrence of the multiqubit systems directly from the DWF. The relationship between the Stokes vector representation and DWF for different choices of the quantum was exploited for this purpose [25]. In the present work, we use some of the results obtained in these references, to provide a general method for the reduction of the DWF to that of its subsystems. This prescription works for all possible quantum nets of the global system as well as those of the

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subsystems. The current work is arranged as follows: In section II we give short introduction to the DWF formalism. In section III we discuss some earlier results which are important to the present work, in section IV partial trace of single qubit DWF from two qubit DWF is presented for an arbitrary quantum net as an illustrative example of the present approach. In section V we derive a general partial trace formula for the multiqubit systems. Sections VI sums up the relevance of the present work as conclusions.

II. DISCRETE WIGNER FUNCTIONS

In this section, we review the DWF construction given by Gibbons et al., [21]. In this approach, finite dimensional quantum systems are represented by a discrete phase space of real elements. A N dimensional system is represented as a $N \times N$ discrete lattice of real numbers, with the points being labelled by ordered pairs $(q,p)$ which, are elements of a finite field $F_n$. Since finite fields exist only for prime or prime power dimensions, this condition imposes restrictions on the Hilbert space dimension associated with the system. However, since our interest is in multiqubit systems, this condition is always met. In the discrete phase space of dimension $N \times N$, a subset of $N$-points satisfying the equation $a q + b p = c$, for given values of $a$, $b$ and $c$ is called a line. When the value of $a$ and $b$ are fixed, the variation of $c$ over the finite field elements $F_n$, generates a set of $N$ parallel lines called a striation. In analogy with Euclidean spaces, a set of lines are defined to be parallel if they do not share a common point. In the $N \times N$ discrete phase space, there are $N + 1$ striations, that is $N + 1$ sets of parallel lines. The point $(0,0)$ is called the origin and any line which contains the origin is called a ray, with each striation, containing exactly one ray. For fixed values of field elements $x$ and $y$, if $s$ varies over the field elements $F_n$, the set of $N$ points $(sa, sb) = s(a,b)$ form the rays of each striation. Striations which are formed by the fixed points $(0,1)$ and $(1,0)$ are called the vertical and horizontal striations respectively. The remaining $N-1$ rays are formed by the fixed points $(1, \omega), (1, \omega^2), \ldots , (1, \omega^{N-2})$. For a given prime number $r$, there exists a finite field called the prime field $F_r = \{0, 1, \ldots , r-1\}$. Finite fields of the prime power dimensions are generated from the solutions of the irreducible polynomial of order $r$, with prime field elements being the coefficients of the polynomial. By defining a basis $B = \{a_1, a_2, \ldots , a_n\}$ for the finite field $F_n$, every element of the finite field $F_n$ can be expressed as $q = \sum_{i=1}^{n} q_i a_i$, where the expansion coefficients $q_i$ are the elements of the prime field $F_r$. For example, let $F_2 = \{0, 1\}$ be the prime field, then the elements of the field $F_4$ are generated from the irreducible polynomial of order 2, i.e. $x^2 + x + 1 = 0$. If $\omega$ be one of the solution of the irreducible polynomial, it induces the other solution $\omega^2 = \omega + 1$. Therefore, the finite field of dimension 4 would be given by $F_4 = \{0, 1, \omega, \omega^2\}$. For a $N$ qubit system, one may define $N^2$ translation operators $T_{(q,p)}$ in discrete phase space, whose action on a line shifts each point in the line by an amount $(q,p)$. Using the basis expansion given above, these unitary operators are then defined as,

$$T_{(q,p)} = X^{q_i} Z^{p_i} \otimes \cdots \otimes X^{q_n} Z^{p_n}$$

(1)

where $X$ and $Z$ are the Pauli’s operators and $q_i, p_i \in F_r$. Every line in the discrete phase space is associated with a pure state, represented by a rank one projector $Q(\lambda)$. Lines in the vertical striations are invariant under the translation operators $T_{s(0,1)}$, where $s$ varies over the field elements $F_n$. Therefore, pure states associated with the lines of the vertical striation can be considered to be the eigenstates of the $N^2 - 1$ translation operators $T_{s(0,1)}$. Similarly for every striation, there exists $N^2 - 1$ translation operators which leave the lines in the striation invariant. The state vectors associated with the lines in these striations are simultaneous eigenstates of these $N^2 - 1$ translation operators. However, the association of each these eigenstates to specific lines of a striation is not unique. Each specific association is referred to as a quantum net and leads to a different version of the DWF. For a Hilbert space of dimension $N$, there are $N^{N+1}$ possible quantum nets. Thus, for the quantum state represented by a density matrix $\rho$, one may associate multiple versions of the DWF. This lack of a one-to-one correspondence between the density matrix and the DWF, makes the derivation of general results, independent of the quantum net, particularly problematic. To understand the relationship between the density matrix and the DWF, let $Q(\lambda)$ be the rank one projector associated with the line $\lambda$. Now, the sum of the DWF elements along this line is equal to the probability $p(\lambda) = Tr [Q(\lambda) \rho] = \sum_{\alpha \in \lambda} W_{\alpha}$. With this association the real value taken by the DWF at each point of the phase space is given by,

$$W_{\alpha} = \frac{1}{N} \left[ \sum_{\lambda \in \alpha} Tr (Q(\lambda) \rho) - 1 \right]$$

(2)

This can also be written as,

$$W_{\alpha} = \frac{1}{N} Tr (\rho A_{\alpha})$$

(3)

where the self-adjoint operators $A_{\alpha}$’s are the phase space point operators, defined as,

$$A_{\alpha} = \sum_{\lambda \in \alpha} Q(\lambda) - I$$

(4)

The trace product of the phase space point operators at two different points $\alpha$ and $\beta$ is $Tr(A_{\alpha} A_{\beta}) = N \delta_{\alpha \beta}$. Therefore the set of $N^2$ phase space point operators can be used as a basis for the density matrix as,

$$\rho = \sum_{\alpha} W_{\alpha} A_{\alpha}$$

(5)
The Wigner function elements are therefore the coefficient associated with this basis expansion. One of the crucial questions that arises in the construction of a reduction formula for the DWFs of a composite system is whether the phase space operators have a product structure or not. We shall see from considerations below, that even for the simplest case of \( N = 4 \), this is not always the case.

### A. Equivalence classes of quantum nets

Two quantum nets \( Q \) and \( Q' \) are said to be equivalent if and only if the projection operators \( Q(\lambda) \) and \( Q'(\lambda) \), associated with every line \( \lambda \), are related through a unitary operator \( U \). That is, there exists a unitary operator \( U \) such that for every line \( \lambda \) of the phase space, \( Q'(\lambda) = U Q(\lambda) U^\dagger \). For example, for the \( N = 2 \) case, if lines in the vertical and the horizontal striations are associated with the eigenstates of the Pauli’s \( \sigma_z \) and \( \sigma_x \) operators, the diagonal lines would end-up being the eigenstates of the operator \( \sigma_y \). Now, by assigning the states \( |H\rangle \) and \( |D\rangle = \frac{1}{\sqrt{2}} (|H\rangle + |V\rangle) \) to the rays of the vertical and the horizontal striations, the assignment of the state \( |R\rangle = \frac{1}{\sqrt{2}} (|H\rangle + i|V\rangle) \) or \( |L\rangle = \frac{1}{\sqrt{2}} (|H\rangle - i|V\rangle) \) with the diagonal lines in the vertical and the horizontal striations are associated with the quantum nets. Generalizing this result, a system of dimension \( N \) has \( N^{N-1} \) equivalence classes, where each equivalence class contains \( N^2 \) quantum nets in all. The number of equivalence classes for \( N = 4 \) is 64. Of these, only two of them have the special property that the phase space point operators are tensor products of the \( A_q^\alpha \)'s of the single qubit sub-systems. These operators take the form,

\[
A_\alpha = A_{\alpha_1}^1 \otimes \bar{A}_{\alpha_2}^2
\]  

and

\[
\bar{A}_\alpha = \bar{A}_{\alpha_1}^1 \otimes \bar{A}_{\alpha_2}^2
\]

where \( \alpha = (q, p) \) and \( q, p \in \mathcal{F}_4 \). Finite field elements \( q \) and \( p \) can be expressed as \( q = q_1 e_1 + q_2 e_2 \) and \( p = p_1 f_1 + p_2 f_2 \), where \( \{e_1, e_2\} \) and \( \{f_1, f_2\} \) are the finite field basis for the horizontal and the vertical axes and \( q_1, q_2, p_1, p_2 \in \mathcal{F}_2 \). The phase space points of the individual systems are, \( \alpha_1 = (q_1, p_1) \) and \( \alpha_2 = (q_2, p_2) \). For these two quantum nets, the projectors associated with each line are complex conjugates of each other. Since, each equivalence class contains \( N^2 \) elements, there are only \( 16 \times 2 = 32 \) quantum nets having this special property. With this background, we are now in a position to address the problem of obtaining a reduction formula for the DWF.

### III. Background Related to the Present Work

Before discussing the reduction formula for general multiqubit systems, we now present some results reported in our earlier papers that are relevant to the present work. We shall provide here a summary of results obtained for two qubit systems reported in the literature.

#### A. Spin flipped DWF of the multiqubit systems

In the present work, we use results from our earlier paper on performing the spin flip operation on multiqubit DWFs [24] and quantifying entanglement in such systems by exploiting the relationship between DWFs and generalized Stokes vectors. The spin flip operation for a multi-qubit density matrix \( \rho \) is defined as \( \rho = \sigma_y \rho \sigma_y^\dagger \), where \( \ast \) denotes the complex conjugation in the computational basis and \( \sigma_y \) the Pauli matrix. Let \( W \) be the DWF of the multiqubit system represented as a column vector, \( W^{(\ast)} \) and \( \bar{W} \) are the DWFs associated with \( \rho^\ast \) and \( \bar{\rho} \) respectively. The spin flip operation can now be performed in two steps. In a first step, the complex conjugation can be performed as \( W^{(\ast)} = F W \), where \( F \) is a Hadamard matrix. We have shown that the Hadamard matrix \( F \) is independent of the quantum net. As a next step, the \( \sigma_y^{\otimes n} \) operators can be considered as unitary translation operations \( T_\beta \), which shift each element of the discrete phase space by an amount \( \beta \). Their action on \( F \) merely interchanges the rows, resulting in another Hadamard matrix \( G \). Therefore, the spin flip operation on a multiqubit DWF can be performed by \( \bar{W} = G W \).

#### B. Relationship between Stokes vector and DWFs of multiqubits

In a recent work, we have given a transformation formula relating Stokes vectors and the DWF of a given multiqubit system. We have shown that the Stokes vectors and the DWFs of the multiqubit systems are related through a Hadamard transformation

\[
S = H W
\]

where \( S \) is the Stokes vector and \( W \) the DWF (arranged as a column vector) of the given multiqubit system and \( H \) is a \( N \times N \) Hadamard matrix which depends on the choice of the quantum net. For a multiqubit systems there are \( N^{N+1} \) possible quantum nets, and for each quantum net there exists unique Hadamard matrix. Let \( \mathcal{H}^n \) be the set of all Hadamard matrices for the \( n \)-qubit system. The inverse of these Hadamard matrices takes the Stokes vector \( S \) to the corresponding DWF,

\[
W = H^{-1} S
\]
Thus, the problem of finding the reduction formula for $W$ reduces to that of extracting the sub-system Stokes vector and thereafter applying the inverse of a appropriate Hadamard matrix to it.

C. Reduction formula for the two qubit DWF when point operators have a product structure

M. Holmes et al., have given a method of performing the partial trace operation for two qubit systems [23]. Their result is based on the product structure of the phase space point operators given in the work by Gibbons et al. As mentioned in section II-A, for a two qubit systems, $A_{\alpha}$’s have a product structure only for 32 quantum nets. Consider the two qubit DWF defined in the quantum net, for which the phase space point operator given by $A_{\alpha} = A_{\alpha}^{A} \otimes A_{\alpha}^{B}$. If this definition is used for the reconstruction of the density matrix given in Eq (5), it is easy to show that the density matrices of the subsystems 1 and 2 are,

$$\rho_A = \sum_{\alpha_1} \sum_{\alpha_2} W_{\alpha_1,\alpha_2} A_{\alpha_1}$$

and

$$\rho_B = \sum_{\alpha_1} \sum_{\alpha_2} W_{\alpha_1,\alpha_2} \tilde{A}_{\alpha_2}$$

respectively. It is clear from Eq (10) and Eq (11) that, the phase space point operators of the subsystems are complex conjugates of each other. That is, the DWF of the subsystems 1 and 2 are defined in different quantum nets. The DWF of the first subsystem can be calculated from Eq (3) and Eq (10) as,

$$W^A_{\beta} = \sum_{\alpha_2} W_{\beta,\alpha_2}$$

Since, the DWF of the subsystem-2 is defined on a quantum net where the projection operators associated with each line have been complex conjugated, it is necessary to perform a spin flip operation along the y direction on the DWF of subsystem $B$ i.e. $W^B_{\beta}$ to obtain the correct state. This can be achieved by applying the Hadamard matrix $F$ defined in Section-III A by $W^B = FW^B$. Alternatively, we shall now show that this result can be achieved in the following manner: the phase space point operator $A_{\beta}$ can be used in the place of $\tilde{A}_{\beta}$ in the equation $W^B_{\beta} = \frac{1}{2} Tr(\rho_B A_{\beta})$ to obtain the proper DWF. This is obvious from the fact $\rho_B = \sum_{\alpha_2} W^B_{\beta,\alpha_2} \tilde{A}_{\alpha_2} = \sum_{\alpha_2} W_{\alpha_2} A_{\alpha_2}$, where $W^B_{\beta,\alpha_2}$ is the DWF associated with $\rho_B$. Hence, the DWF of the subsystem-2 takes the form,

$$W^B_{\beta} = \frac{1}{2} Tr(\rho_B A_{\beta})$$

$$W^B_{\beta} = \frac{1}{2} Tr\left[ \left( \sum_{\alpha_1} \sum_{\alpha_2} W_{\alpha_1,\alpha_2} \tilde{A}_{\alpha_2} \right) A_{\beta} \right]$$

$$W^B_{\beta} = \frac{1}{2} \sum_{\alpha_1} \sum_{\alpha_2} W_{\alpha_1,\alpha_2} Tr(\tilde{A}_{\alpha_2} A_{\beta})$$

where the trace product $Tr(\tilde{A}_{\alpha_2} A_{\beta})$ for two different point $\alpha = (q_{\alpha}, p_{\alpha})$ and $\beta = (q_{\beta}, p_{\beta})$ is given by $Tr(\tilde{A}_{\alpha} A_{\beta}) = (-1)^{(q_{\alpha} \oplus q_{\beta})(p_{\alpha} \oplus p_{\beta})}$, where $\oplus$ is addition modulo-2. Therefore, Eq (15) can be written as,

$$W^B_{\beta} = \frac{1}{2} \sum_{\alpha_1} \sum_{\alpha_2} (-1)^{(q_{\alpha_1} \oplus q_{\beta})(p_{\alpha_2} \oplus p_{\beta})} W_{\alpha_1,\alpha_2}$$

From Eq (12) and Eq (16), we can calculate the DWF of the subsystems for a given two qubit DWF. For the other equivalence class, the complex conjugation operation needs to be performed on the first subsystem rather than the second, with the DWF of the second being defined on the chosen net. In this context, as Gibbons et al., have pointed out that the existence of the tensor product structure is itself not established for other powers of prime. In the present work, we provide a reduction formula for multiqubit DWFs where such a product structure is not required. To the best of our knowledge, such a general result is not available in the literature.

IV. A GENERAL REDUCTION FORMULA FOR THE TWO QUBIT DWF FOR ARBITRARY QUANTUM NETS

As shown in the earlier section the approach by Holmes et al., is restricted to only 32 of the possible 1024 quantum nets. In this section, we derive a general result valid for all quantum nets of the global as well as the subsystems. Let $\rho_{AB}$ be the density matrix of the two qubit system, $\rho_A$ and $\rho_B$ be those of its subsystems. In the density matrix representation, the subsystem can be obtained by taking a partial trace on $\rho_{AB}$ i.e. $\rho_A = Tr_B(\rho_{AB})$ and $\rho_B = Tr_A(\rho_{AB})$. Now, to derive a formula for obtaining the DWF of the single qubit subsystem from that of the two qubit DWF, we need to specify the quantum net of both. Hence, the transformation formula must be general enough to accommodate this requirement.

Let $M$ be the observable acting on the subsystem $A$ of the general system $\rho_{AB}$. This can be mathematically represented as $(M \otimes I)\rho_{AB}$. The expectation value of the observable $M$ only on the subsystem $\rho_A$ and the expectation value of the operator $M \otimes I$ on the global system $\rho_{AB}$ are one and the same, that is,

$$< M >_{\rho_A} = < M \otimes I >_{\rho_{AB}}$$

For general two qubit systems, the Stokes vector is a 16 parameter real valued column vector, $S =$
where the entries in the column vector are the expectation values of the generalized two qubit Pauli matrices,

\[ S_{i_1i_2} = \frac{1}{4} Tr(\sigma_{i_1} \otimes \sigma_{i_2}) \]  \tag{18}

where \( i_1, i_2 \in [0, x, y, z] \). Replace the operator \( M \) in the Eq (17) with the Pauli’s operators \( \sigma_i \),

\[ < \sigma_i >_{\rho_A} = Tr[\sigma_i \otimes I]_{\rho_{AB}} \]

These expectation values are essentially the Stokes vector of the first subsystem \( S^A \), given by

\[ S^A = Tr[\sigma_i \rho_A] = Tr[\sigma_i \otimes I]_{\rho_{AB}} \]

From Eq (19) it is clear that the Stokes vector of the first subsystem \( S^A \) is part of the two qubit Stokes vector \( S^{AB} \), that is, \( S^A = S^A_{10} \). The Stokes vector of the subsystem can be obtained from that of the Stokes vector of the global system by the construction of the transformation matrix \( T_1 \), such that

\[ S^A = T_1 S \]  \tag{20}

where \( T_1 \) is the \( 4 \times 16 \) matrix given by \( T_1 = 2[I O O O] \). Let \( W \) be the DWF of the the system \( \rho_{AB} \). From Eq (8), two qubit Stokes vector \( S \) can calculated from \( W \) by,

\[ S = H_2 W \]  \tag{21}

where \( H_2 \) is the \( 4^2 \times 4^2 \) dimensional Hadamard matrix, which is an element of the set \( S^H_2 \), that is the set of Hadamard matrix for the two qubit systems. Based on the quantum net of \( W \), we can choose the Hadamard matrix from the set \( S^H_2 \). Therefore, from Eq (20) and Eq (21) one can calculate the Stokes vector of the first subsystem directly from the DWF of the two qubit system by,

\[ S^A = T_1 H_2 W \]  \tag{22}

where the product \( T_1 H_2 \) is the \( 4 \times 16 \)-matrix. The inverse transformation from the Eq (9) takes the Stokes vector to the DWF of the subsystem \( A \), by

\[ W^A = H_1^{-1} S^A \]  \tag{23}

where \( H_1 \) is the \( 4 \times 4 \) Hadamard matrix, contained in the set \( S^H_1 \) of single qubit system. Therefore from the Eq (22) and Eq (23), the DWF of the first subsystem can be given by,

\[ W^A = H_1^{-1} T_1 H_2 W^{AB} \]

that is

\[ W^A = P_1 W^{AB} \]  \tag{24}

where \( P_1 = H_1^{-1} T_1 H_2 \) is a \( 4 \times 16 \) matrix. Using this relation one can calculate the DWF of the first subsystem from the DWF of the two qubit system. That is Eq (24) performs the reduction operation for the two qubit DWF. By the similar transformation one can construct the reduction operation for the second subsystem by suitable construction of the matrix \( T_2 \) as,

\[ W^B = P_2 W^{AB} \]  \tag{25}

where \( P_2 = H_1^{-1} T_2 H_2 \). Hence, the reduction operation for the general two qubit DWFs can be performed using Eq (24) and Eq (25). Here, it is important to note that, the reduction formula is general enough to compute the DWF of the subsystem defined in any arbitrary quantum net from the DWF of the two qubit system defined in an arbitrary quantum net. In our transformation equations, the information about the quantum net of the global system and that of the subsystems are contained in the Hadamard matrices \( H_2 \) and \( H_1 \). So equations (24) and (25) carries out the reduction operation for chosen quantum net. The reduction formula given by Holmes et al., in Eq (12) and Eq (16) are the special cases of this formula. As an aside we note that the reduction formula can be used to calculate quantities of interest like concurrence of a bipartite pure state defined by \( W^{AB} \) as,

\[ C(W^{AB}) = \sqrt{2 \left( 1 - \sum_{\alpha} W_{\alpha}^4 \right)} \]

where \( W_{\alpha}^A \) is the DWF of subsystem \( A \).

V. REDUCTION FORMULA FOR THE GENERAL MULTIQUBIT DWF

In the density matrix formalism, from the given \( n \)-qubit state \( \rho \), the state of an arbitrary \( k \)-qubit subsystem can be calculated by the partial trace operation. In this section, we derive a method of performing the equivalent of a partial trace operation on the general \( n \)-qubit DWF by “tracing out” \( n-k \)-qubits. In the DWF setting this can be done using the following facts: the state of the sub-system can be readily extracted from the Stokes vector of the composite state and the transformation formula between Stokes vector and the DWF given in Eq (8) can be used to obtain the DWF of the sub-system. To see how this may be accomplished, consider a density matrix of the general \( n \)-qubit system \( \rho \) and let \( M_i \) be some observable acting on the \( i \)-th qubit. Given \( \rho \), the expectation value of the observable \( M_i \) on \( i \)-th qubit can be calculated as:

\[ < M_i >_{\rho} = Tr(M_i \rho) \]  \tag{26}

\[ = Tr[(I \otimes I \otimes ... \otimes M_i \otimes ... \otimes I) \rho] \]  \tag{27}

This can be generalized for any \( k \)-partite subsystem. When this problem is cast in terms of Stokes vectors, the
observables are the Pauli operators and the expectation values $\langle \sigma_j \rangle_{\rho^i}$'s are the Stokes parameters $S_j^i$ of the subsystem $i$. That is, $S_j^i = \langle \sigma_j \rangle_{\rho^i}$. But from Eq (27), it is clear that, $\langle \sigma_j \rangle_{\rho^i} = \langle I \otimes I \otimes ... \otimes \sigma_j^i \otimes ... \otimes I \rangle_{\rho}$. This implies,

$$S_j^i = S_0...j...0 \quad (28)$$

Therefore, in the Stokes vector representation, the state of the $i$-th subsystem is a part of the multiqubit Stokes vector $S$. Similarly, for any $k$-partite subsystem, its state is contained in the multiqubit Stokes vector from which it can be easily extracted. Consider the case of tracing out the last $(n - 1)$ subsystems from the $n$-qubit state, giving the state of the first system $S^1$. The Stokes vector of the first system can be calculated as,

$$S^1 = T_1 S \quad (29)$$

where $T_1$ is the $4 \times 4^n$ matrix, given by $T_1 = 2^{n-1}I \otimes O ... O$ with $I$ is a $4 \times 4$ identity matrix and $O$ is a $4 \times 4$ matrix with all entries being zero. Similarly any $k$-qubit Stokes vector can be constructed from the Stokes vector of the multiqubit systems with the help of a suitable transformation matrix $T_k$.

$$S^k = T_k S \quad (30)$$

where $T_k$ is a $4^k \times 4^n$ matrix. For a given multiqubit DWF $W$, the corresponding Stokes vector can be calculated using the $4^n \times 4^n$ Hadamard matrix $H_n \in S_n^H$ by,

$$S = H_n W \quad (31)$$

where the Hadamard matrix $H_n \in S_n^H$. Let $S_n^H$ be the set containing $N^{N+1}$ possible Hadamard matrices associated with each quantum net. Therefore, from Eqs (30) and (31),

$$S^k = T_k H_n W \quad (32)$$

where the subscript $n$ of $H_n$ indicate that the Hadamard matrix picked up from the set $S_n^H$ of the $n$-qubit systems. Eq (29) allows us to calculate the Stokes vector of the $k$-qubit subsystem from the multiqubit DWF. Here the knowledge of the quantum net of $W$ is implicitly available in the Hadamard matrix $H_n$. Using the inverse formula given in Eq (9), we can find the DWF of the $k$-qubit system as,

$$W^k = H_k^{-1} T_k H_n W \quad (33)$$

where $H_k \in S_k^H$.

$$W^k = P_k W \quad (34)$$

Therefore, Eq (34) helps us perform the reduction operation for the multiqubit DWF. Thus, we find that the quantum net of the global system and the subsystems are to be obtained from the Hadamard matrices $H_n \in S_n^H$ and $H_k \in S_k^H$ respectively. Hence, the choice of the quantum net of the global system and that of the reduced system can be freely made by an appropriate choice of the corresponding $H_n$ and $H_k$.

VI. CONCLUSIONS

There are many contexts in the fields of quantum computation and quantum information where access to subsystem information is vital. The quantification of entanglement present in a composite bipartite system through Concurrence and the derivation monogamy relationships from tripartite entangled states are typical examples. Similarly, in the case of multiqubit systems, the distribution of entanglement over suitably partitioned subsystems is a problem of interest. Frequently, one also requires to enlarge the Hilbert space by taking a tensor product of the system with that of the environment, subjecting the joint system to a unitary evolution and eventually tracing out either the environment or the system. The theory of POVMs and weak measurements are typical examples of such procedures. Hitherto, such techniques have been uniquely applied to the case where the state of the system is represented in terms of the density matrix and an equivalent approach was not available at least in the case of systems represented by the discrete Wigner function. While the representation of the state of continuous systems by Wigner functions has found widespread use, this not the case for the DWF due to some obvious limitation. An important limitation with DWF as stated earlier has been the absence of a general reduction formula, which problem has been addressed in the present work. While its true that DWF, density matrix and Stokes vector representations are but linear transforms of each other, experimental situations could make one choice or the other more favorable and experimental reconstructions of the the different representation are also different. Going by the experience with continuous system, where the phase space representation of the state provides certain unique insights, further development of its discrete analog is warranted. Motivated by such considerations, the present work is a step in the direction of developing the relevant tools for the Discrete Wigner function of multiqubit systems.

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