Engineering Wake Induction Model For Axisymmetric Multi-Kite Systems

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Abstract. Multi-kite airborne wind energy systems (MAWES) fly two or more kites in a high-altitude orbit that allows a main tether to unroll a generator. An engineering wake induction model could simplify the (axisymmetric) aerodynamics of such an unsteady actuator-annulus with variable thrust-coefficient, flight-radius, and downwind-direction apparent velocity, in comparison to high-fidelity wake models. To the authors’ knowledge, existing engineering models do not predict the influence of variable flight radius. The goal of this paper is to synthesize a MAWES engineering model, a closed-form function approximating the axial induction factor $\tilde{a}$, specifically at the kite mid-span, using linearized wake parameters, as available from a dynamics solver. A vortex tube model is used to construct a plausible approximation of the induction, which is then heavily simplified for computation tractability while retaining important flow physics.

1. Introduction

Multi-kite airborne wind energy systems (MAWES) fly two or more kites in a high-altitude orbit that allows a main tether to unroll a generator (see Figure 1). Under the assumption of axisymmetry, this system resembles an unsteady\textsuperscript{5} actuator-annulus, with variable thrust coefficient, flight radius, and downstream-direction apparent velocity. The initial-design and control of a MAWES might be simplified if an engineering induction model is used to represent the wake, rather than a high-fidelity wake model. An engineering model is a simple - and preferably smooth - algebraic function of the system’s current states, rather than its history. Such an engineering model is useful in the framework of optimal control, because it leads to Hessian matrix structures similar to those found when induction is completely neglected.
Because conventional wind turbines have fixed radius, existing engineering models - such as Pitt and Peters\[9\], Øye\[8\], differential ECN\[11\], and most recently Yu\[13\] - do not include the effect of variable flight radius. The Pitt and Peters\[9\] model derives a first order differential equation relating the normalized rotor thrust and moments to induced-flow and rotor-response matrices, using an expanded momentum theory. The Øye\[8\] model uses two first-order differential equations to filter the induced velocity, using time-constants derived from vortex theory. Next, there is the differential ECN\[11\] model which sums the blade element momentum equation with an induced-velocity time derivative found from a cylindrical wake expression. Finally, the recent Yu\[13\] model finds an indicial representation for a vortex tube shed from a rotor with varying load.

Our goal is to find an engineering model to describe the axial induction as would be measured at the mid-span of the kites in an axisymmetric MAWES. This induced velocity is described in terms of the \textit{induction factor} \(a\), the ratio between the current upstream-direction induced velocity and the current downstream-direction apparent velocity. To start, in Section 2, we use limited state and derivative information to construct a plausible vortex-tube model of the wake geometry. Then, in Section 3, we use this plausible model to synthesize a closed-form engineering model \(\tilde{a}\). Finally, Section 4 summarizes some conclusions.

2. Approximating the wake based on limited information

Given the constraint that only limited information is known about the wake’s history, the first step is to approximate the history of the MAWES behavior so that the geometry of the wake can be constructed. For this, we assume two fixed values: a \textit{free-stream wind-speed} \(u_{\infty}\) along the axis of the MAWES’s rotation, and the \textit{span} \(b\) of the kites. Together, these values define a characteristic time scale \(t^* = b/u_{\infty}\).

To represent the wake, we would need to know the time-histories of three non-dimensional parameters:

- \textit{thrust coefficient} \(C_T\), the ratio between the downstream-direction thrust on all kites of the MAWES and the apparent dynamic pressure over the area swept by the flight-path;
- \textit{reel-out factor} \(f\), the ratio between the speed of the MAWES’s downstream-direction motion and the free-stream wind-speed; and
- \textit{relative radius} \(\varrho\), the ratio between the flight radius at the kites’ mid-span and the kite span.

Using only current state and derivative values, the wake might be approximated linearly, as:

\[
C_T(t) \approx C_{T0} - \left(\frac{t}{t^*}\right) \dot{C}_{T0}, \quad f(t) \approx f_0 - \left(\frac{t}{t^*}\right) \dot{f}_0, \quad \varrho(t) \approx \varrho_0 - \left(\frac{t}{t^*}\right) \dot{\varrho}_0. \quad (1)
\]

In these expressions, \(t\) designates the dimensional time towards the past, which can be non-dimensionalized by \(t^*\). (This unusual directionality of \(t\) is chosen to reduce the chance of sign errors during integration, by aligning the orientation of \(t\) with a convection distance \(x\) that will be defined in the following section.) This requires that the ‘dot’ symbol for the differentiation of an arbitrary function \(g(t)\) be defined as \(\dot{g}(t) := t^* \frac{dg}{dt}\). The non-dimensionalization of the parameters in (1) will arguably lead to an engineering model which can be easily scaled to ensure the good conditioning of the Hessian of optimal control problems. Further, (1) is signed such that orientation of the derivatives remains consistent with typical time-forwards dynamics expressions.

In the following paper, we use the abbreviations: \(w = [C_{T0}, \dot{C}_{T0}, f_0, \dot{f}_0, \varrho_0, \dot{\varrho}_0]\), and \(W = [u_{\infty}, b; w]\). We also focus our attention around values that have particular meaning to MAWES: \(C_{T0} = \frac{8}{9}\) for Betz\[7\] optimality; \(f_0 = \frac{1}{4}\) for Loyd\[6\] optimality; \(\varrho_0 = 6\) and \(u_{\infty} = 10\ m/s\) as intermediate values inspired by recent\[5\] optimization work; and \(b = 5.5\ m\) from an existing\[10\] power kite.

If (1) describes all times between \(t \in [0, \infty)\), we can then construct the corresponding wake geometry.
2.1. Linearized wake geometry

Assuming an axisymmetric MAWES (see Figure 1), we can define a coordinate system (\(\hat{x}, \hat{r}\)) that is centered on the current combined center of gravity of all kites, with \(\hat{x}\) along the free-stream wind direction, and \(\hat{r}\) along the radial direction. Then, if we assume a high tip-speed ratio, we can use tangential vortex tubes to construct the wake. We neglect axial-direction vorticity, because the torque on the kites is likely to be small in comparison to the thrust on the kites. (Notice that an axisymmetric MAWES produces power with thrust rather than torque.) We also neglect radial-direction vorticity, whose induction would cancel under the assumption of axisymmetry.

For constant circulation along the kites’ spans, these vortex tubes are shed only at the inner and outer wingtips of the kites, and are (assuming a rigid wake, for the sake of simplicity) convected downstream at the uniform and constant free-stream wind speed. Please note that the rigidity assumption neglects self-induction and approximates the flow on either side of any vortex surface to be force-free. If we track the slice of the vortex tube (a.k.a. ring) shed at time \(t=0\), we see that it has convected downstream to a position \(x(t; W)\) relative to the kites, found by integrating the apparent velocity \(u_a(t; W)\). Without any time to convect, \(x(0; W) = 0\). (We now see that the orientation of \(t\) implies that \(x \rightarrow +\infty\) as \(t \rightarrow +\infty\), for intuitive integration later.)

\[
\begin{align*}
   u_a(t; W) &= u_\infty \left( 1 - f_0 + \frac{\dot{f}_0}{f_0} \frac{t}{T_0} \right) = \frac{\partial x}{\partial t}(t; W); \quad x(t; W) = u_\infty t \left( 1 - f_0 + \frac{1}{2} \frac{\dot{f}_0}{f_0} \frac{t}{T_0} \right). 
\end{align*}
\]

(2)

The current radial position \(r\) of the kite mid-span is \(r = \rho_0 b\). We define a variable \(\zeta\) along the span, with \(\zeta = -\frac{1}{2}\) on the inner wingtip and \(\zeta = \frac{1}{2}\) on the outer wingtip. For radially-oriented kite-spans, a shed vortex ring has radius \(R(t; \zeta; W)\), as measured radially from the \(\hat{x}\) axis.

\[
R(t; \zeta; W) = \left( \rho_0 - \rho_0 \frac{t}{T_0} \right) + \zeta b. 
\]

(3)

For MAWES angular velocity \(\Omega\), we can find the total bound circulation \(\Gamma\) over all kites with the Kutta-Joukowski relationship: \(\Gamma = \pi C_T u_\infty^2 / \Omega\). Following [2], we solve an equation determining the vortex sheet strength (per unit area) \(\gamma\) to be \(\gamma = -\Gamma / h\), where \(h\) is the pitch of the discrete vortex helix that was smoothed into the vortex tube by the assumption of a large tip-speed ratio. For \(h = (\pi(2u_a + \gamma) / \Omega)\), we find \(\gamma(t; W)\) to be:

\[
\gamma(t; W) = u_\infty \left( 1 - f_0 + \frac{\dot{f}_0}{f_0} \frac{t}{T_0} \right) \left( -1 + \sqrt{1 - C_{\Gamma 0} + \dot{C}_{\Gamma 0} \frac{t}{T_0}} \right). 
\]

(4)

Then, the induction factor \(a(W)\) due to the vorticity shed between time \(t \in [0, \infty)\) from both the inner and outer wingtips reads as:

\[
a(W) \approx \int_0^\infty \alpha(t; \frac{1}{2}; W) - \alpha(t; -\frac{1}{2}; W) \, dt, 
\]

(5)

where \(\alpha(t; \zeta; W)\) is the induction-factor per unit time due to the slice of the vortex tube at spanwise position \(\zeta\). The term \(\alpha\) is based on the induction from a vortex ring[12], accounting for a change in integration variables from \(dx\) to \(dr\), and normalized by the apparent velocity experienced by the MAWES:

\[
\alpha(t; \zeta; W) = -\frac{u_a(t; W)}{u_a(0; W)} \frac{\dot{\gamma}}{2\pi} \frac{\left( \frac{R^2 - r^2 - x^2}{(R-r)^2 + x^2} \right) E \left( k^2 \right) + K \left( k^2 \right)}{\sqrt{(R+r)^2 + x^2}}, \quad k^2(t; \zeta; W) = \frac{4\pi R}{(r + R)^2 + x^2},
\]

(6)
where $K(s)$ and $E(s)$ are, respectively, the complete elliptic integrals of the first and second kind.

Unfortunately, if we were simply to integrate (5) when the derivatives are non-zero, we would not find a meaningful solution, because the wake-geometry becomes non-physical for large $t$ due to the linearization of the wake parameters.

2.2. Non-physicalities inherent in wake linearization

Recall that we linearized the wake so that the resulting engineering model would only depend on the current MAWES states and derivatives. In reality, the kite system would have roughly periodic behavior, and would not accelerate at a constant rate for all of history. Unfortunately, linearization with non-zero derivatives necessarily means that the parameters have non-physical past values.

We describe the parameters as physical when they are within a bounded region set by the system operation and the vortex model assumptions. The thrust coefficient should remain between $C_{T_{\min}} = 0$ and $C_{T_{\max}} = 0.96$, to avoid both propeller mode and the turbulent wake state[7]. We further observe that previous MAWES optimization studies like [5] suggest that MAWES actuator surfaces are unlikely and $C$ physical when they are within a bounded region set by the system operation and the vortex model assumptions. The thrust coefficient should remain between $C_{T_{\min}} = 0$ and $C_{T_{\max}} = 0.96$, to avoid both propeller mode and the turbulent wake state[7]. We further observe that previous MAWES optimization studies like [5] suggest that MAWES actuator surfaces are unlikely to be heavily loaded. Because an assumption of spanwise constant circulation makes less sense for small relative radii where the apparent velocity varies dramatically over the kite-span, we set $\varrho_{\min} = 2$. Practically speaking, the system should reel-out slower than the wind-speed, setting a maximum reel-out factor of $f_{\text{max}} = 1$. The bounds mentioned above have physical meaning for our model. We also chose two more arbitrary bounds so that the domain of $\mathbf{w}$ can be visualized: $\varrho_{\max} = 10$ and $f_{\min} = -5$, based on previous MAWES optimization studies like [5].

To demonstrate the non-physicalities inherent in linearization, consider the reel-out factor $f(t)$ and Figure 2. When $f_0 < 0$, there are two non-negative roots of $x(t; W)$: $t/t^* = 0$ and $t/t^* = T = 2(f_0 - 1)/f_0$. This means that the current position of the vorticity shed at time $T$ is concentric with the MAWES’ current location, as the wake folds over itself. We consider this effect of the wake folding over itself to be problematic, because reality appears to be inconsistent with the simplified model assumptions for both small and large values of $T$. That is, if the wake has folded over itself in the recent past (small $T$ for large-magnitude negative $f_0$ values), the model’s assumed neglect of self-induction would be a poor assumption. If, however, the folding-over happened far into the past (large $T$ for small-magnitude negative $f_0$ values), then the air’s viscosity would have already dissipated the vorticity that is conserved in an inviscid formulation.

As Biot-Savart says that the influence of a vortex is inversely proportional to the square of the distance from the observation point, and this distance is identical at $t/t^* = 0$ and $t/t^* = T$ with negative $f_0$; linearization of the wake parameters implies two spikes in $\alpha$ due to the vortex rings shed at non-dimensional time $t/t^*$. Since the spike at $t/t^* = T$ is a result of the wake-folding-over effect that implies an inconsistency in modeling assumptions, there is a great deal of uncertainty in the actual influence of the vortex rings shed around $t/t^* = T$ on the current kite position.

(We can see these two spikes in $\alpha$ for an arbitrarily-selected $W$ with negative $f_0$, in the base-model (gray) curve of Figure 3. Please notice that the selected value of $W$ has zero values for $\varrho_0$ and $C_{T_0}$, such that the spikes are entirely due to the current position of the vortex rings shed at non-dimensional time $t/t^*$. Then, the particular selection of $W$ in Figure 3 puts these two spikes at $t/t^* = 0$ and $t/t^* = T \approx 67$.)

Similar non-physicalities occur when a small $C_{T_0}$ or large $\varrho_0$ implies, respectively, historical values
of $C_T > 1$ or $\rho < 0$. On the arbitrary bounds of the parameters, we have less serious linearization implications, such as the case when $f_0 > 0$, which implies that the MAWES was moving upstream faster than allowed by $f_{\text{min}}$ when $t/t^* > (f_0 - f_{\text{min}})/f_0$.

To prevent the uncertainty on the behavior of those vortex-rings shed at large $t$ from dominating the entire integral over $t \in [0, \infty)$, we adjust (5). We have considered two adjustment models: the vorticity-decay model and the acceleration-limit model. In the absence of real-world validation data, we use the similarities between these models as a plausible guess of the system’s real induction behavior.

### 2.3. Vorticity-decay model

The first method of limiting the impact of non-physical linearization artifacts is to damp out the impact of all vorticity according to the time when it was shed. This can be done by correcting (6) with an exponential time-decay, motivated according to the time when it was shed. This can be done by discounting, Figure 3 shows a comparison between the induction from very-old vortex rings, where the linearization is less physical. To illustrate this discounting, Figure 3 shows a comparison between $\alpha(t; \frac{1}{2}; W)$ and $\beta(t; \frac{1}{2}; \phi; W)$, for the case of a negative $f_0$.

Consider the steady-state case of $C_{T0} = 0 = \dot{\gamma} = f_0 = 0$. When $\phi = 0$, (7) integrates numerically to the momentum-theory expression $\frac{1}{2}(1 - \sqrt{1 - C_{T0}})$. This makes sense, because the assumptions describing the axial induction in this zero-decay, steady-state model are equivalent [3] to the assumptions behind a steady actuator annulus, for neglected wake-rotation. This means that momentum theory provides a good sanity-check for the steady-state case of (7). The smaller $\phi$ is, the closer (see Figure 4) the numerical integral of (7) is to the momentum theory result. For larger $\phi$, the trends in induction factor are simply damped versions of the trends with small $\phi$.

Over the larger domain of $w$, it remains true that increasing $\phi$ smooths the trends found for small $\phi$ (see Figure 5):

- **thrust coefficient** We see that increasing the current thrust coefficient $C_{T0}$ increases $a$, as a greater applied force should

![Figure 3: Influence of vortex rings shed at nondimensional time $t/t^*$, in base model $\alpha(t; \frac{1}{2}; W)$ (light-gray), and vorticity-decay model $\beta(t; \frac{1}{2}; \phi; W)$ (black). Using $\phi = 10^{-8}$, $v = 1.48 \cdot 10^{-5}$ m$^2$/s, and $W = [10m/s, 5.5m/s, 0, \frac{1}{3}, -0.02, 6, 0]$, such that $T \approx 67$.](image-url)

![Figure 4: Steady-state vorticity-decay model (gray, with labeled $\phi$) converges to (black) momentum theory. (Note: $\phi = 10^{-8}$ under black.) Here, $v = 1.48 \cdot 10^{-5}$ m$^2$/s, and $W = [10m/s, 5.5m/s, 0, \frac{1}{3}, 0, 6, 0]$.](image-url)
increase the induced reaction. As the derivative \( \dot{C}_{T0} \) increases, the induction factor decreases. This makes sense because the larger \( \dot{C}_{T0} \) is, the smaller the vortex sheet strength of the wake must have been in the past.

- **relative radius** The induction factor \( a \) is relatively independent of \( \dot{g}_0 \) (for \( \dot{g}_0 \) that are not small), because the induced velocity is described mainly by half of the vorticity strength of the outer tube. The maximum induction occurs when the relative radius is constant in time, \( \dot{\rho}_0 = 0 \). This is because the radial position of the kite mid-span (specifically, the induction measurement position) is only between \( R(t; \frac{1}{2}, W) \) and \( R(t; -\frac{1}{2}, W) \) for \( t \in [0, \infty) \) when \( \dot{\rho}_0 = 0 \). For larger magnitudes of \( \dot{\rho}_0 \), the kite mid-span passes out from between the two tangential vortex tubes (where the induction is strongest) at earlier values of \( t \).

- **reel-out factor** The induction factor peaks when \( f_0 \) approaches one, where the apparent velocity that normalizes \( a \) approaches zero. For positive \( f_0 \), increasing \( \dot{f}_0 \) increases \( a \) because the vortex sheet strength increases towards the past. However, for negative \( f_0 \), decreasing \( \dot{f}_0 \) increases \( a \), as the second occurrence of \( x(t) = 0 \) contributes an additional spike (see Figure 3) to the integral.

Consider, that the steady-state model with \( \phi \geq 10^{-6} \) seem to result (in Figure 4) in neglecting too much of the vorticity’s influence. However, the case of \( \phi = 10^{-8} \) shows a sharp discontinuity in induction factor over \( f_0 = 0 \) (in Figure 5). We might decide, then, that intermediate decay constants around \( \phi = 10^{-7} \) give the most plausible behavior.

### 2.4. Acceleration-limit model

The second method of avoiding non-physical linearization artifacts is to separate the behavior of the MAWES around a switching-time \( \tau > 0 \). For \( t \) more-recent than \( \tau \), the MAWES accelerated linearly; for \( t \) older than \( \tau \), the MAWES had a constant \( C_T \), \( \phi \) and \( f \). This gives the parameter time-histories as:

\[
C_T(t) \approx \begin{cases} 
C_{T0} - \dot{C}_{T0} \frac{t}{\tau} & \text{for } t \leq \tau, \\
C_{T0} - \dot{C}_{T0} \frac{t}{\tau^*} & \text{for } t > \tau
\end{cases} \quad f(t) \approx \begin{cases} 
f_0 - \dot{f}_0 \frac{t}{\tau^*} & \text{for } t \leq \tau, \\
f_0 - \dot{f}_0 \frac{t}{\tau^*} & \text{for } t > \tau
\end{cases} \quad \phi(t) \approx \begin{cases} 
\phi_0 - \dot{\phi}_0 \frac{t}{\tau^*} & \text{for } t \leq \tau, \\
\phi_0 - \dot{\phi}_0 \frac{t}{\tau^*} & \text{for } t > \tau^*.
\end{cases}
\tag{8}
\]

It seems reasonable for the switching-time to be related to the system’s characteristic time \( t^* \), using an unknown proportionality factor \( \theta \) that can again be tuned:

\[
\tau = \theta t^*.
\tag{9}
\]

When we consider (5), the portion of the integral for \( t \in [\tau, \infty) \) then represents two right, tangential vortex cylinders. If we call \( \eta(t_0, t_1; \zeta; W) \) the known induction factor due to a finite vortex cylinder convected between times \( t \in [t_0, t_1] \), then we can split the integral of (5) into two parts:

\[
a(W) \approx \int_0^{\tau} \left( a \left( t; \frac{1}{2}; W \right) - a \left( t; -\frac{1}{2}; W \right) \right) dt + \left( \eta \left( \tau, \infty; \frac{1}{2}; W \right) - \eta \left( \tau, \infty; -\frac{1}{2}; W \right) \right) \tag{10}
\]

Here, \( \eta(t_0, t_1; \zeta; W) \) reads[3] (showing only arguments related to time, for brevity’s sake) as:

\[
\eta(t_0, t_1; \zeta; W) = -\frac{1}{\omega(0) 4\pi \sqrt{R(t_0)}} \left[ x(t) \kappa(t; t_0) \left( K \left( \kappa^2(t; t_0) \right) + \frac{R(t_0)}{R(t_0) + r} \Pi \left( \kappa^2(0; t_0) \kappa^2(t; t_0) \right) \right) \right]_{t=t_0}^{t=t_1},
\tag{11}
\]

where \( \Pi(x|q) \) is the complete elliptic integral of the third kind, and \( \kappa^2(t; t_0; \zeta; W) \) is defined with:

\[
\kappa^2(t; t_0; \zeta; W) = 4 \frac{r(W) R(t_0; \zeta; W)}{(r(W) + R(t_0; \zeta; W))^2 + x(t; W)^2}.
\tag{12}
\]
When we integrate (10) numerically, for various values of $\theta$ and slices of $u$, we find Figure 6. First, notice that the steady-state case of $\dot{C}_{T0} = \dot{\theta}_0 = \dot{f}_0 = 0$ generates a geometry of two semi-infinite vortex cylinders, such that the model is equivalent to momentum theory regardless of $\theta$. Considering the larger region of parameters, the main difference in trends between Figures 5 and 6 is that $a$ decreases also for negative $f_0$, since the induction spike at $t = T$ is avoided altogether. But, neglecting this negative $f_0$ exception, there seems to be a great deal of qualitative similarity between the two models for small $\phi$ and large $\theta$. Since we determined above that the $\phi = 10^{-8}$ case - which resembles very closely the $\theta = 10$ case - gives insufficient decay, we might decide that $\phi = 10^{-7}$ corresponds to a $\theta$ larger than, but on the order of, 1.
next step, then, is to translate these trends into an engineering model. The fact that these two models give such similarities, makes it seem likely that we have actually found relevant physical trends. The available parameter information into a prediction of the induction factor. We have now introduced two models that have used different physical interpretations to convert the available parameter information into a prediction of the induction factor. Numerical integration to accuracy of $10^{-10}$ (black isolines), and giving complex results (gray region). Dimensioned for $b = 5.5$ m and $u_\infty = 10$ m/s.

Figure 6: Acceleration-limit induction model $a$, with various value of proportionality factor $\theta$ (in rows). Numerical integration to accuracy of $10^{-10}$ (black isolines), and giving complex results (gray region). Dimensioned for $b = 5.5$ m and $u_\infty = 10$ m/s.

We have now introduced two models that have used different physical interpretations to convert the available parameter information into a prediction of the induction factor. The fact that these two models give such similarities, makes it seem likely that we have actually found relevant physical trends. The next step, then, is to translate these trends into an engineering model.

3. Constructing an engineering model to approximate the induction factor

We construct the engineering model in three steps: discretization, approximation, and linearization.
First, we discretize the acceleration-limit integral (10) with a Riemann sum:

\[
a(W) \approx \sum_{m=1}^{M} \left( \eta \left( \frac{m-1}{M} \tau, \frac{m}{M}; \frac{1}{2} \right) - \eta \left( \frac{m-1}{M} \tau, \frac{m}{M}; -\frac{1}{2} \right) \right) + \eta \left( \tau, \infty; \frac{1}{2} \right) - \eta \left( \tau, \infty; -\frac{1}{2} \right),
\]

where we again choose \( \tau = \theta \dot{r} \), and the arguments related to \( W \) are implied but not printed, for brevity.

Second, we approximate the most non-linear components of (11): the complete elliptic integrals \( K(s) \) and \( \Pi(s|q) \). In order to simplify the following linearization step, we might choose to approximate these functions by simpler functions \( \tilde{K}(s) \) and \( \tilde{\Pi}(s|q) \). We construct these approximations so that \( \tilde{K}(s) \) and \( \tilde{\Pi}(s|q) \) have poles corresponding to singularities at \( s = 1 \) and \( q = 1 \), and correct values\(^1\) at \( s = 0 \) and \( q = 0 \). Further, \( \int_0^1 K(s) ds = \int_0^1 \tilde{K}(s) ds \). Consequently, these approximations read as:

\[
\tilde{K}(s) = \frac{4 - \pi}{2\sqrt{1-s}} + \pi - 2, \quad \tilde{\Pi}(s|q) = \tilde{K}(q) + \frac{\pi}{2\sqrt{1-q}} - \frac{\pi}{2}.
\]

We substitute \( \tilde{K}(s) \) and \( \tilde{\Pi}(s|q) \) in (11), to give a function \( \tilde{\eta}(t_0, t_1; \zeta; W) \); and substitute \( \tilde{\eta} \) for \( \eta \) in (13).

Next, we expand \( a \) about \( \dot{C}_{T_0} = \dot{\varrho}_0 = \dot{f}_0 = 0 \), using the abbreviation \( a_0 = (1 - \sqrt{1 - \dot{C}_{T_0}})/2 \):

\[
\tilde{a}(w) = a_0 + \dot{f}_0 c_1 + |\dot{\varrho}_0| c_2 + |\dot{\varrho}_0| f_0 c_3 + \dot{C}_{T_0} c_4 + \dot{C}_{T_0} f_0 c_5 + \dot{C}_{T_0} |\dot{\varrho}_0| c_6 + \dot{C}_{T_0} |\dot{\varrho}_0| f_0 c_7,
\]

enforcing symmetry about \( \dot{\varrho}_0 = 0 \), as seen in both the vorticity-decay and acceleration-limit models. (A smooth norm can be substituted, in applications where smoothness is necessary.) The above parameters \( c_i \) are defined as \( c_i := h_i(C_{T_0}, 0, f_0, 0, \varrho_0, 0) \), where \( h_i \) read as:

\[
h_1 = \frac{\partial a}{\partial f_0}, \quad h_2 = \frac{\partial a}{\partial \varrho_0}, \quad h_3 = \frac{\partial^2 a}{\partial \varrho_0^2}, \quad h_4 = \frac{\partial a}{\partial C_{T_0}}, \quad h_5 = \frac{\partial^2 a}{\partial C_{T_0}^2}, \quad h_6 = \frac{\partial^2 a}{\partial C_{T_0} \partial f_0}, \quad h_7 = \frac{\partial^2 a}{\partial C_{T_0} \partial \varrho_0}.
\]

Notice that \( c_i \) are independent of the fixed values \( b \) or \( u_m \), and converge relatively quickly as \( M \) increases (see Figure 7). Further, the coefficients \( c_i \) can be pre-computed using average values of \( C_{T_0}, \varrho_0 \) and \( f_0 \), to allow for much faster computation times than the full numerical integration: the computational-time for the acceleration-limit numerical-integral (10) is on the order of \( 10^4 \) microseconds in Mathematica, where the computational-time for \( \tilde{a}(w) \) with pre-computed coefficients is on the order of 1 microsecond.

Slices of the engineering model \( \tilde{a} \) and its error with respect to the acceleration limit model can be seen in Figure 8. Notice that \( c_i \) are independent of the fixed values \( b \) or \( u_m \), and converge relatively quickly as \( M \) increases (see Figure 7). Further, the coefficients \( c_i \) can be pre-computed using average values of \( C_{T_0}, \varrho_0 \) and \( f_0 \), to allow for much faster computation times than the full numerical integration: the computational-time for the acceleration-limit numerical-integral (10) is on the order of \( 10^4 \) microseconds in Mathematica, where the computational-time for \( \tilde{a}(w) \) with pre-computed coefficients is on the order of 1 microsecond.

![Figure 7: Convergence in \( c_i \) vs. \( M \), found for \( \theta = 1, C_{T_0} = 8/3, \varrho_0 = 6 \) and \( f_0 = 1/3 \).](image-url)
that the regions corresponding to less than 10 percent error are of moderate size around small-magnitude derivatives. As we might expect, this region is thinner the closer the parameters are to the physically-meaningful bounds of \( C_{T_{min}}, C_{T_{max}}, \) and \( f_{max}. \) The magnitude of the error and the computational time should both be taken into consideration during model selection, especially during iterative tasks like numerical optimization.

4. Conclusion and outlook

The goal of this paper was to find a computationally-inexpensive engineering model that can be used to approximate the induction of an axisymmetric MAWES, given the current values and derivatives of the system’s thrust coefficient, relative radius, and reel-out factor. We compared the engineering model to a reasonable estimate of induction, as described by an acceleration-limit model, and saw a qualitative similarity in landscape. Particularly for small derivative values where the current states are far from the physically-meaningful parameter limits, an accuracy of 10 percent is possible with a speed-up factor of approximately \( 10^4. \)

Future work should attempt to answer three questions. First, how well can the wake of a periodic-trajectory MAWES be represented by a corrected (using, for example, the vorticity-decay model or the acceleration-limit model) linear wake geometry? Second, can the proposed engineering model be easily applied to axisymmetric optimal control problems? And, third, can this methodology be extended to construct an engineering model for the asymmetric case where the reeling out-and-in of a tilted MAWES occurs at a non-zero elevation angle?
Acknowledgments

Thanks to Christoph Sieg and Hans Leuthold for helpful discussions. This research was supported by the EU via H2020-ITN-AWESCO (642 682), by the Federal Ministry for Economic Affairs and Energy (BMWi) via eco4wind (0324125B) and DyConPV (0324166B), and by DFG via Research Unit FOR 2401.

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