Recovering ΛCDM Model From a Cosmographic Study

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Abstract

Using the mathematical definitions of deceleration and jerk parameters we obtain a general functional form for Hubble parameter. By the aid of this exact Hubble function we can exactly reconstruct any other cosmographic parameters. We also obtained a general function for transition redshift as well as spacetime curvature. We may highlight the role of jerk parameter as any other cosmographic parameter could be written as a function of this parameter. Our derived functions clearly impose a lower limit on the jerk parameter which is \( j_{\text{min}} \geq -0.125 \). Moreover, we found that the jerk parameter indicates the geometry of the spacetime i.e any deviation from \( j = 1 \) imply to a non-flat spacetime. In other word \( j \neq 1 \) reevers to a dynamical, time varying, dark energy. From obtained Hubble function we recover the analogue of ΛCDM model. To constrain cosmographic parameters as well as transition redshift and spacetime curvature of the recovered ΛCDM model, we used Metropolis-Hasting algorithm to perform Monte Carlo Markov Chain analysis by using observational Hubble data obtained from cosmic chronometric (CC) technique, BAO/CMB data, Pantheon compilation of Supernovae type Ia, and their joint combination. The only free parameters are \( H, A(\Omega_m) \) and \( j \). From joint analysis we obtained \( H_0 = 69.9 \pm 1.7 \), \( A(\sim \Omega_m) = 0.279^{+0.014}_{-0.016} \), \( B(\sim \Omega_X) = 0.721^{+0.016}_{-0.014} \), \( j_0 = 1.038^{+0.061}_{-0.035} \) and \( z_t = 0.707 \pm 0.034 \).

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1 Introduction

Since 1998 it has been revealed that the cosmic expansion is speeding up \([1, 2, 3, 4]\). In the context of general theory of relativity (GR) the existence of an exotic fluid called “dark energy (DE)” with negative pressure is considered as the source caused present universe accelerated expansion \([2, 3, 4]\). It is also possible to study this cosmic acceleration in the context of modified gravity which is a generalization of general relativity \([5, 6]\). Moreover one may deal with this problem by considering violation of cosmological principle i.e assuming that the spacetime metric is inhomogeneous. Despite of about two decades effort, we still could not propose a realistic cosmological model in order to describe the present day cosmic accelerated expansion precisely. For example, although ΛCDM model \([10]\) excellently fits all observational data, considering cosmological constant \( \Lambda \) as dark energy encounters to coincidence and fine-tuning problems \([12, 13]\). We know that the modern cosmology is based on the Friedmann equations, nevertheless, it is interesting to study universe through a kinematic approach rather than dynamical one (Einstein equations). This model-independent approach is called “Cosmography” or cosmo-kinetics. It is worth noting that in this purely kinematic approach all the derived quantities are also model-independent. Cosmography was firstly introduced by Weinberg \([14]\) in 2008 and later extended by Visser \([15]\). The cornerstone of cosmography is to expand some observables such as the Hubble parameter (or equivalently the scale factor) into power series, and directly relating cosmological parameters to these observable quantities.

However, in practice, the cosmography study confronts to two serious problems with observational data. First of all, as shown in Ref \([17]\), the Taylor expansion fails to reach convergence at redshift \( z > 1 \). This, of course an important shortcoming as many observational probes such as supernovae type Ia (SNIa) and cosmic microwave background (CMB) compilations can span the redshift region up to \( z \sim 1.3 \) and \( z \sim 1100 \) respectively. This problem could be overcome by definition of an improved redshift parametrization such as \( y = 1/(1 + z) \) \([17, 18, 19]\). The second one is the fact that in cosmography we use a finite Taylor series truncation which represent an approximation of the exact function and hence leads to worse estimations. Note that, taking more terms of Taylor series gives raise to more precise approximation but higher errors. Therefore as mentioned in Ref \([19]\) “cosmography is in the dilemma between accuracy and precision”. For recent cosmography studies in the context for GR and Modified GR Reader is advised to see Refs \([19, 20, 21, 22, 23, 24, 25, 26, 27, 28]\) an \([29, 30]\) respectively. Very recently it has been shown that using Weighted Function Regression method \([31]\) improves the usual cosmographic approach by automatically implementing Occams razor criterion \([32]\).

As the source of all above mentioned problems lies in the Taylor expansion of the Hubble (scale factor) or luminosity distance, in this paper, in contrast with usual cosmography, we do not expand any of these parameters and instead we try to find an exact function for Hubble parameter on the bases of kinematic parameters of the universe. To do so, we
mix the definition of deceleration and jerk parameters which in turn gives a second order differential equation for squared Hubble parameter. With out any prior assumption, we obtain an exact Hubble parameter as a function of jerk parameter (see sec 3). Through this Hubble parameter we reconstruct all other cosmographic parameters (CS) exactly. Then we use observational Hubble data (OHD) in the redshift range \(0.07 \leq z \leq 1.965\) [32], Pantheon compilation containing 1048 SNIa apparent magnitude measurements over the redshift range of \(0.01 < z < 2.3\) [34], BAO/CMB dataset [35], and their joint combination data to constrain CS parameters. We compare some of our results by those obtained in Refs [18, 20] and 22. This paper is organized as follows. In Sec 2 we briefly discuss cosmography and introducse CS parameters up to the fifth derivative of the scale factor. In Sec 3 we derive a general deferential equation for squared Hubble parameter an reconstruct all CS parameter from it. Subsec 3.1 deals with the derivation (recovering) ΛCDM model from our almost-general Hubble solution and show that how the spacetime geometry is connected to the jerk parameter. We summary reconstruct all CS parameter from it. Subsec 5.1 deals with the results of our fits to data. In Subsec 5.1 we derive a general redshift and constrain it over data. Finally, we summarize our findings and conclusions in Sec 6.

## 2 Cosmography

In this section we shall briefly describe cosmography which may start from Taylor series of scale factor. Taylor expansion of the scale factor \(a(t)\) around the current time \(t_0\) gives

\[
a(t) = a_0 \left[ 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \frac{d^n a}{dt^n} (t - t_0)^n \right].
\]

Without loss of generality we can assume \(a_0 = 1\), where the constant \(a_0\) is the current value of the scale factor. The most important cosmographic series terms i.e the Hubble, deceleration, jerk, snap, and lerk parameters are [16, 32, 36]

\[
H = \frac{1}{a} \frac{da}{dt},
\]

\[
q = -\frac{1}{aH^2} \frac{d^2 a}{dt^2},
\]

\[
j = \frac{1}{aH^3} \frac{d^3 a}{dt^3},
\]

\[
s = \frac{1}{aH^4} \frac{d^4 a}{dt^4},
\]

\[
l = \frac{1}{aH^5} \frac{d^5 a}{dt^5},
\]

respectively. As noted in Ref 20, the first three CS parameters i.e the Hubble rate \(H\), deceleration parameter \(q\) and its first derivative with respect to the cosmic time (or redshift) \(j\) are sufficient to determine the overall kinematics of the Universe. However, at the current time the deceleration parameter is restricted as \(-1 \leq q_0 < 0\) which in turn impose \(j_0 > 0\). For ΛCDM model \(j = 1\) at any time [37].

Since \(a = 1/(1 + z)\), from eqs (2), we can find the derivatives of the Hubble parameter with respect to \(z\) as [19]

\[
\frac{dH}{dz} = \frac{1 + q}{1 + z} H, \quad \frac{d^2 H}{dz^2} = \frac{j - q^2}{(1 + z)^2} H, \quad \frac{d^3 H}{dz^3} = \frac{H}{(1 + z)^3} (3q^2 + 3q^3 - 4qj - s) \tag{3c}
\]

\[
\frac{d^4 H}{dz^4} = \frac{H}{(1 + z)^4} (-12q^2 - 24q^3 - 15q^4 + 32qj + 25q^2 j + 7qs + 12j - 4j^2 + 8s + l). \tag{3d}
\]

Using eqs (3) in the Taylor expansion of the Hubble parameter around \(z = 0\) one can obtain

\[
H(z) = H_0 + \left. \frac{dH}{dz} \right|_{z=0} z + \frac{1}{2} \left. \frac{d^2 H}{dz^2} \right|_{z=0} z^2 + \frac{1}{6} \left. \frac{d^3 H}{dz^3} \right|_{z=0} z^3 + O(z^4)
\]

\[
= H_0 \left[ 1 + (1 + q_0)z + \frac{1}{2}(j_0 - q_0^2)z^2 + \frac{1}{6}(3q_0^2 + 3q_0^3 - 4q_0j_0 - 3j_0 - s_0)z^3 + O(z^4) \right], \tag{4}
\]
In other hand, the cosmographic version of the luminosity distance can be conveniently expressed as

\[
d_L(z) = \frac{cz}{H_0} \left[ 1 + \frac{1}{2}(1 - q_0 z) - \frac{1}{6}(1 - q_0 - 3q_0^2 + j_0)z^2 
+ \frac{1}{24}(2 - 2q_0 - 15q_0^2 - 15q_0^3 + 5j_0 + 10j_0q_0 + s_0)z^3 + O(z^4) \right]
\]

where the subscript “0” indicates the present values of the cosmographic parameters.

Here we again emphasize that there are two main problems arising in the context of cosmography when using eqs (4) and (5) to constrain CS parameters. In fact, it has already been shown that expanding around \( z \sim 0 \) gives raise to the divergence of Taylor series at \( z \geq 0 \). This problem could be limited by transferring \( z \) to \( z/(1+z) \). The second problem is the truncation of the series one may use in analysis, an approximation of the exact function, which may leads to the possible misleading results. Although, this problem can be alleviated by going to higher terms in the expansion, but adding any new term means introducing a new parameter that must be estimated. This indeed increases the divergences of the analysis. Moreover, since this method is based on the Taylor expansion of the scale factor, cosmography is restricted in the scope of Friedmann-Robertson-Walker metric. Hence it is interesting to find a more common kinematic approach which is applicable in other spacetimes posses inhomogeneous properties. In next section we will derive an almost-general model independent solution from which one can reconstruct any CS parameter without any limitation and problems arising in usual cosmography.

### 3 An Exact Cosmographic Solution

In this section we find an exact analytical expressions between the Hubble and jerk parameters. It is interesting to note that in [38] an almost the same expression has been found by considering special parametrization for the jerk parameter, but this solution seems to be wrong as the authors consider a minus sign in the definition of jerk parameter (see eq (5) of this reference). This mistake affects other results. In what follows, first we derive a general differential equation which could be used for any parametrization of the jerk parameter and then we solve it for \( j(z) = j_0 \).

It is possible to obtain the derivatives of the squared Hubble parameter with respect to \( z \) as follows [39]

\[
\begin{align*}
\frac{dH^2}{dz} &= \frac{2H^2}{1+z}(1+q), \\
\frac{d^2H^2}{dz^2} &= \frac{2H^2}{(1+z)^2}(1+2q+j), \\
\frac{d^3H^2}{dz^3} &= \frac{2H^2}{(1+z)^3}(-qj-s) \\
\frac{d^4H^2}{dz^4} &= \frac{2H^2}{(1+z)^4}(4qj + 3qs + 3q^2j - j^2 + 4s + l).
\end{align*}
\]

Combining eqs (6a) and (6b) we find the following differential equation for squared Hubble parameter

\[
\frac{1}{2} (1+z)^2 \frac{(H_0^2)''}{H_0^2} - (1+z) \frac{(H_0^2)'}{H_0^2} + (1-j) = 0,
\]

which in turn, for constant \( j \)\footnote{One can put eqs (6a) and (6b) in eq (6c) to find a similar differential equation as eq (7) in term of snap parameter. Doing so, probably, recovers radiation term.} gives the following general solution for the Hubble parameter

\[
h(z)^2 = \left[ A(1+z)^{\frac{3+\sqrt{1+8j}}{2}} + B(1+z)^{\frac{3-\sqrt{1+8j}}{2}} \right],
\]

where \( h(z) = H(z)/H_0 \). Requiring the consistency of (8) at \( z = 0 \) gives

\[
A + B = 1.
\]

Obviously eq (8) is a model-independent solution. Generally, this solution corresponds to a cosmological model without radiation component\footnote{Note that one can parameterize jerk parameter as \( j = A + B/(1+z) \) to find a solution for non-constant \( j \) which is in consistent with the \( \Lambda \)CDM scenario at current time.} . Note that, as we have shown in next section there is degeneracy between curvature \( \Omega_k \) and \( j \).
It is worth to mention that for spatially flat cosmological constant dark energy model the jerk parameter is \( j(z) = 1 \). Therefore, jerk parameter has been a traditional tool to test the spatially flat \( \Lambda \)CDM model. From eq (8) it is clear that considering \( j = 1 \) recovers flat \( \Lambda \)CDM model. Hence, in this case, one can consider \( A \) and \( B \) as dark energy density (\( \Omega_X \)).

Using eq (8) in eqs (6) we can reconstruct cosmographic parameters as

\[
q = -1 + \frac{1}{2h^2(z)} \left[ AX(1 + z)^X + BY(1 + z)^Y \right],
\]

\[
j = 1 - \frac{1}{2h^2(z)} \left[ A \prod_{i=0}^{1}(X - i)(1 + z)^X + B \prod_{i=0}^{1}(Y - i)(1 + z)^Y \right],
\]

\[
s = -qj - \frac{1}{2h^2(z)} \left[ A \prod_{i=0}^{2}(X - i)(1 + z)^X + B \prod_{i=0}^{2}(Y - i)(1 + z)^Y \right],
\]

\[
l = - (4qj + 3qs + 3q^2j - j^2 + 4s) - \frac{1}{2h^2(z)} \left[ A \prod_{i=0}^{3}(X - i)(1 + z)^X + B \prod_{i=0}^{3}(Y - i)(1 + z)^Y \right],
\]

where \( X = \frac{3+\sqrt{1+8\Delta}}{2} \), and \( Y = \frac{3-\sqrt{1+8\Delta}}{2} \). In the same manner one can reconstruct any other CS parameters.

Although one can use the following dimensionless Hubble parameter

\[
h^2(z) = - \frac{1}{2(4qj + 3qs + 3q^2j - j^2 + 4s + l)} \left[ A \prod_{i=0}^{3}(X - i)(1 + z)^X + B \prod_{i=0}^{3}(Y - i)(1 + z)^Y \right],
\]

(11)

to put constrain on the cosmographic parameters, but, in view of eqs (8) & (9), the only free parameters are \( H_0, A, \) and \( j \).

### 3.1 Recovering \( \Lambda \)CDM Model

In this section we consider that cosmological constant plays the role of dark energy. Therefore, one can write Friedmann equation as

\[
\dot{a}^2 + k = \frac{1}{3} \rho a^2 + \frac{1}{3} \Lambda a^2.
\]

Taking \( \ddot{a} \) we obtain

\[
j = \frac{\ddot{a}}{aH^2} = \Omega_m a^{-3} + \Omega_\Lambda a
\]

(13)

Since, in general, \( \Omega_m + \Omega_\Lambda + \Omega_k = 1 \), we can rewrite above equation as follows

\[
j = 1 + \Omega_m (a^{-3} - a) - \Omega_k a.
\]

(14)

Therefore, we can evaluate the current value of curvature parameter \( \Omega_k \) only in term of jerk parameter as

\[
\Omega_k = 1 - j_0
\]

(15)

This equation clearly shows that any deviation from \( j = 1 \) is an evidence of non-flat universe. **Therefore we may argue that for all models consider cosmological constant as dark energy, the geometry of spacetime must be flat.** Moreover, when \( j \neq 1 \) a time varying dark energy is responsible for current Universe accelerating expansion.

Substituting eq (15) in eq (8), we can easily recover the analogue of \( \Lambda \)CDM model (without radiation) as

\[
h(z)^2 = \left[ A(1 + z)^{3+\sqrt{1+8\Omega_k \Delta+B}} + B(1 + z)^{3-\sqrt{1+8\Omega_k \Delta+B}} \right],
\]

(16)

which clearly shows degeneracy between \( \Omega_k \) and \( j \). In view of eq (16), we may also consider \( A \) as matter density (\( \Omega_m \)) and \( B \) as dark energy density (\( \Omega_X \)) if our estimations indicate \( \Omega_k \sim 0 \), otherwise we consider \( B = \Omega_X + \Omega_k \).

In the next section we will use observational Hubble data (OHD), BAO/CMB data and Pantheon compilation and their joint combination to constrain cosmographic \( \Lambda \)CDM model with following parameters space

\[
\Theta = \{ H_0, q_0, j_0, s_0, l_0, z_t, \Omega_k, A, B \},
\]

(17)

where \( z_t \) is the deceleration-acceleration transition redshift (see subsec. 5.1 for more details). Note that \( 1 + 8j \) must be greater or equal to zero, this in fact imposes a certain lower limit on jerk parameter as \( j \geq -0.125 \). We have to consider this point in our estimations.


4 Data and Method

In this section we briefly describe the astronomical data and the statistical method we have been used to constrain parameter set [17].

**Type Ia Supernovae:** We adopt the Pantheon compilation [34] containing 1048 SNIa apparent magnitude measurements over the redshift range of 0.01 < z < 2.3, which includes 276 SNIa (0.03 < z < 0.65) discovered by the Pan-STARRS1 Medium Deep Survey and SNIa distance estimates from SDSS, SNLS and low-zHST samples. It is also possible to use the JLA dataset [40] which combines the SNLS and SDSS SNe to create an extended sample of 740 SNe to reduce the estimation time, but we found that using the Pantheon data slightly improves the parameter estimations. In this case the chi-square is defined as

\[ \chi^2_{SN} = (\mu(z) - \mu_0)^T C^{-1} (\mu(z) - \mu_0), \]

where \( \mu(z) \) is the predicted distance modulus given by

\[ \mu(z) = 5 \log_{10}[3000y(z)(1 + z)] + 25 - 5 \log_{10}(h), \]

and \( C^{-1} \) is the inverse of the 1048 by 1048 Pantheon compilation covariance matrix (\( C_{ij} = \text{diag}(\sigma^2_i) \)). It is worth mentioning that since the parameter \( h(H_0) \) is only an additive constant, thus, marginalizing over \( h \) does not affect the SNe results.

**Observational Hubble Data:** we use OHD data from Table 2 of Ref [33] which includes 31H(z) datapoints in the redshift range 0.07 ≤ z ≤ 1.965. This measurements are uncorrelated and determined using the cosmic chronometric (CC) technique. It is worth nothing that the OHD data can be categorized into the following two classes, (1) BAO based data and (2) cosmic chronometric (CC) based data. To obtain OHD data from BAO, we usually model the redshift space distortions and assume an acoustic scale in a specific model. Therefore, this class of data is model-dependent and hence cannot be used for constraining a cosmological model. Nonetheless, to determine the CC data we use the most massive and passively evolving galaxies based on the galaxy differential age method. consequently, this class of OHD data is model-independent (see ref [41] for more details). Since this compilation includes uncorrelated data, we have \( C_{ij} = \text{diag}(\sigma^2_i) \) as covariance matrix for this class of data. For OHD data the chi-square is given by

\[ \chi^2_{OHD} = (H(z) - H_0)^T C^{-1} (H(z) - H_0). \]

**BAO/CMB Data:** to obtain the BAO/CMB constraints on the model parameters we adopt 1- for BAO we consider six data points (see Table. 1) obtained from the WiggleZ Survey [42], SDSS DR7 Galaxy sample [43] and 6dF Galaxy Survey [44] datasets, 2- for CMB, our considered measurement is derived from the WMAP7 observations [45] (for more details about methodology of obtaining the BAO/CMB constraints on model parameters see [46]). Recently, Mamon et al [47] have used this data to place constraints on a reconstructed dark energy model. The chi-square of this data is given as

\[ \chi^2_{BAO/CMB} = X^T C^{-1} X. \]

Here

\[ X = \begin{pmatrix}
  d_A(z_{s1}) & -30.95 \\
 D_V(0.106) & -17.55 \\
 d_A(z_{s2}) & -10.11 \\
 D_V(0.2) & -8.44 \\
 d_A(z_{s3}) & -6.69 \\
 D_V(0.35) & -5.45 \\
 d_A(z_{s4}) & D_V(0.44) \\
 d_A(z_{s5}) & D_V(0.6) \\
 d_A(z_{s6}) & D_V(0.73)
\end{pmatrix}, \]

where \( d_A(z) = 1/(H_0(1 + z))y(z) \) is the co-moving angular-diameter distance and \( D_V(z) \) is the dilation scale given by

\[ D_V(z) = \left[ (1 + z)^2d_A(z)^2 \frac{z}{H_0h(z)} \right]^{\frac{1}{2}}. \]
It is worth noting that while the estimated values of \( j_1 \) \( Muthukrishna \& Parkinson \) \( BAO/CMB \) and \( zhang et al \) \( SNIa \) in Table .(4). We have also compared our estimated deceleration parameter to those of \( Aviles et al \) \( columns forth and fifth \) \( Table 1 \) \( H_0 \). Therefore, we evaluated the following total likelihood \( \chi^2_{tot} = \chi^2_{OHD} + \chi^2_{BAO/CMB} + \chi^2_{SNIa} \). Since these three datasets are independent, the total chisquare could be written as \( \chi^2_{tot} = \chi^2_{tot} \). Moreover, to check the degeneracy direction between computed parameters we perform covariance matrix which could be obtained from our MCMC runs. Theoretically, covariance matrix is defined as

\[
C_{\alpha\beta} = \rho_{\alpha\beta} \sigma(\theta_{\alpha}) \sigma(\theta_{\beta}),
\]

where the uncertainties in parameters \( \theta_{\alpha} \) and \( \theta_{\beta} \) are given by \( \sigma(\theta_{\alpha}) \) and \( \sigma(\theta_{\beta}) \) are the 1\( \sigma \) respectively, and \( \rho_{\alpha\beta} \) is the correlation coefficient between \( \theta_{\alpha} \) and \( \theta_{\beta} \).

We use Metropolis-Hasting algorithm to generate MCMC chains for all parameters. For each parameter we run 4 parallel chains with 6000 separate iterations to stabilize the estimations. We perform Gelman-Rubin and Geweke tests to confirm the convergence of MCMC chains. We also confirm the convergence of all chains by monitoring the trace plots for good mixing and stationarity of the posterior distributions. In our Baysian estimations, we assume the following uniform priors for free parameters of the model (17):

\[
H_0 \sim U(50 - 100) \quad A \sim U(0 - 5) \quad j \sim U(-0.125 - 10).
\]

### 5 Results

In Table. 2 we have listed our statistical analysis on parameter space (17) using OHD, BAO/CMB, SNIa, and their joint combination dataset at 1\( \sigma \) error. It is worth mentioning that, although, SNIa data by itself is not sensitive to the universe expansion rate \( H_0 \), but in the joint analysis, this data constrains other parameters of the model which in turn affect the computation of \( H_0 \). That is why in Table. 2 we observe a change in the value of \( H_0 \) when fitting model to the joint OHD+BAO/CMB+SNIa data. In Table .(3) we have compared our obtained \( H_0 \) for OHD, BAO/CMB and OHD+BAO/CMB+SNIa data to those obtained by other researchers. Results of this table clearly show that when we use joint dataset our computed \( H_0 \) is in high agreement with those obtained by \( Sievers et al \) \( 70 \pm 2.4 \) [49], \( Chen et al \) \( 68.4^{+2.9}_{-3.3} \) [50], and J. Dunkley et al \( 69.7 \pm 2.5 \) [51]. Th estimated \( H_0 \) obtained from fitting to OHD data is in excellent agreement with those of \( Chen \& Ratra \) \( 68 \pm 2.8 \) [48] and \( Chen et al \) \( 68.4^{+2.9}_{-3.3} \) [49]. When we constrained model over BAO/CMB data alone, the computed value of \( H_0 \) is obtained in good agreement with what reported by \( Riess et al \) [66]. We have also compared our estimated deceleration parameter to those of \( Aviles et al \) \( columns forth and fifth of Table 1 \) [18], \( Muthukrishna \& Parkinson \) \( row third of Table 3 \) [23], and \( zhang et al \) \( row forth \) \( Table 1 \) [20] in Table .(4). It is worth noting that while the estimated values of \( j_0, t_0 \) reported in Ref [24] almost not physical (taking high values), our computed values of these parameters are physical for OHD, BAO/CMB, SNIa and their joint datasets. When we

| \( z_{BAO} \) | \( 0.106 \) | \( 0.2 \) | \( 0.35 \) | \( 0.44 \) | \( 0.6 \) | \( 0.73 \) |
|---|---|---|---|---|---|---|
| \( \frac{d_A(z)}{D_V(z_{BAO})} \) | \( 30.95 \pm 1.46 \) | \( 17.55 \pm 0.604 \) | \( 10.11 \pm 0.37 \) | \( 8.44 \pm 0.67 \) | \( 6.69 \pm 0.33 \) | \( 5.45 \pm 0.31 \) |

Also \( z_1 = 1091 \) is the decoupling time and the inverse of covariance matrix for this data is

\[
C^{-1} = \begin{pmatrix}
0.48435 & -0.101383 & -0.164945 & -0.0305703 & -0.097874 & -0.106738 \\
-0.101383 & 3.2882 & -2.45497 & -0.0787898 & -0.252254 & -0.2751 \\
-0.164945 & -2.45499 & 9.55916 & -0.128187 & -0.410404 & -0.447574 \\
-0.0305703 & -0.0787898 & -0.128187 & 2.78728 & -2.75632 & 1.16437 \\
-0.097874 & -0.252254 & -0.410404 & -2.75632 & 14.9245 & -7.32441 \\
-0.106738 & -0.2751 & -0.447574 & 1.16437 & -7.32441 & 14.5022
\end{pmatrix}.
\]
Table 2: Best fit value and 1σ error bars for each cosmographic parameters. We perform a fit by using \( H(z) \) only (column two), BAO/CMB data only (column three), SNIa data only (column four), and by using the combined \( H(z) \) and SNIa data together (column five).

| Parameter | OHD (CC) | BAO/CMB | SNIa (Pantheon) | CC + ABO/CMB + Pantheon |
|-----------|----------|---------|-----------------|------------------------|
| \( H_0 \) | 67.9 ± 2.9 | 70.1 ± 5.8 | – | 69.9 ± 1.7 |
| \( A \) | 0.328±0.052 | 0.26 ± 0.14 | 0.278±0.014 | 0.279±0.014 |
| \( B \) | 0.672±0.052 | 0.74 ± 0.14 | 0.722±0.014 | 0.721±0.014 |
| \( q_0 \) | −0.508±0.080 | −0.62 ± 0.22 | −0.590±0.024 | −0.587±0.024 |
| \( \Omega_{0k} \) | 0.001 ± 0.058 | 0.001 ± 0.057 | −0.039±0.019 | −0.038±0.014 |
| \( j_0 \) | 0.999 ± 0.057 | 0.999 ± 0.057 | 1.039±0.019 | 1.038±0.015 |
| \( s_0 \) | −0.47±0.28 | −0.23 | −0.299 ± 0.059 | −0.307 ± 0.060 |
| \( l_0 \) | −2.55±0.83 | −3.5±2.7 | −3.35 ± 0.21 | −3.32 ± 0.21 |

The contour plots, at 1σ level, depict the correlation matrix for OHD (5a), BAO/CMB (5b), SNIa (5c) and OHD+BAO/CMB+SNIa (5d).

Figure 5. depicts the correlation matrix for OHD (5a), BAO/CMB (5b), SNIa (5c) and OHD+BAO/CMB+SNIa (5d).
Figure 1: One-dimensional marginalized distribution, and three-dimensional contours with 68% CL, 95% CL, and 99% CL for some parameters from parameter space $\Theta$ using CC+ABO/CMB+Pantheon data. The vertical dashed red line stands for $\Omega_{0k} = 0$.

Table 5: The value of $\Omega_{0k}$ at 1$\sigma$ obtained by different researches.

| Researchers          | 9yearsWMAP  | Planck (2015) | Park & Ratra | This work |
|----------------------|-------------|---------------|--------------|-----------|
| $\Omega_{0k}$        | $-0.0027^{+0.0039}_{-0.0038}$ | $0.0008^{+0.0040}_{-0.0039}$ | $-0.0083 \pm 0.0016$ | $-0.038^{+0.054}_{-0.061}$ |

5.1 Transition Redshift

It is well known that the universe expansion phase has changed from decelerating to accelerating at a specific redshift called “Transition redshift $z_t$” [65] [66]. Mathematically we can find this decelerating-accelerating redshift by imposing
Figure 2: The plot of Hubble rate versus the redshift $z$ at $1\sigma$ and $2\sigma$ confidence level for OHD (green color), BAO/CMB (red color) and OHD+BAO/CMB+SNIa (yellow color). The points with bars indicate the experimental data summarized in Table 2 of Ref [33]. It is clear that using joint datasets gives rise to better fit to the data.

Figure 3: Schematic representation of $H_0$ (at $1\sigma$) for model 16 ($\text{OHD}$ (purple color), $\text{BAO/CMB}$ (black color) and $\text{CC} + \text{BAO/CMB} + \text{SNIa}$ (cyan color)). Constraints from the direct measurement by Riess et al. (2016) (red color) WMAP (green color), Dunkley et al (gold color), and Planck (2015) (blue color) are also shown.

Figure 4: The plots of deceleration parameters at $1\sigma$ and $2\sigma$ confidence level. Green, Blue, Red and orange figures show our fit to OHD, BAO/CMB, SNIa, and OHD+BAO/CMB+SNIa respectively.

Figure 5: Plots of correlation matrix of parameter space $\Theta$ using: (a) Hubble (OHD), (b) BAO(CMB), (c) SNIa, and (d) OHD+BAO(CMB)+SNIa data. The color bars share the same scale.

$q(z) = \ddot{a} = 0$ in (10a). Doing so, and after some algebra we find the following general transition redshift.

$$z_t = \left( -\frac{B(Y - 2)}{A(X - 2)} \right)^{\frac{1}{X - Y}} - 1,$$

(29)

which can be simplified and rewritten in terms of CS parameters as

$$z_t = \left[ -\frac{B(\sqrt{1 + 8j} + 1)}{A(\sqrt{1 + 8j} - 1)} \right]^{\frac{1}{\sqrt{1 + 8j}}} - 1.$$

(30)

Note that we may set $A \approx \Omega_m$ and $B \approx \Omega_X$ in above equation. According to the previous discussion, for $\Lambda$CDM model, we can rewrite eq 30 and relate transition redshift to the spacetime curvature as follows

$$z_t = \left[ -\frac{B(\sqrt{1 + 8j} + 1)}{A(\sqrt{1 + 8j} - 1)} \right]^{\frac{1}{\sqrt{9 - 8\Omega_k}}} - 1.$$

(31)
Our statistical analysis on transition redshift for both datasets and their joint combination could be seen in Table 6. Figures 6 and 7 depict the robustness of our fits for \( z_t \). Gastri et al [46] have recently have used different SNIa data in combination with BAO/CMB data to constrain transition redshift. For MLCS2k2+BAO/CMB (see [67] for details of MLCS2k2 data) they obtained \( z_t = 0.56^{+0.13}_{-0.10} \) and for SALT2+BAO/CMB (see [68] for details of SALT2 data) they found \( z_t = 0.64^{+0.11}_{-0.07} \). Also Farooq et al [69] used 38 \( H(z) \) data and found \( z_t = 0.72 \pm 0.05(0.84 \pm 0.03) \) for two Hubble constant priors as \( H_0 = 68 \pm 2.8(73.24 \pm 1.74) \) at 1\( \sigma \) error. Moreover, recently Capozziello et al [70] through an effective cosmographic construction, in the framework of \( f(T) \) gravity have obtained \( z_t = 0.43^{+0.034}_{-0.030} \) at 1\( \sigma \) error. Note that in ref [70] authors set \( \Omega_m = 0.315 \) (from Plank (2015)) to obtain above mentioned transition redshift but when they consider this parameter as a free one they obtain \( z_t = 0.24^{+0.345} \) which is not accurate value. Generally, our obtained transition redshifts, except for BAO/CMB data, are consistent with what is expected in the cosmological models with present-epoch energy budget dominated by dark energy as well as standard spatially flat \( \Lambda \)CDM model.

| Parameter | OHD (CC) | BAO/CMB | SNIa | CC+BAO/CMB+Pantheon |
|-----------|----------|---------|------|---------------------|
| \( z_t \) | \( 0.683^{+0.087}_{-0.21} \) | \( 1.181^{+0.095}_{-0.96} \) | \( 0.711^{+0.031}_{-0.35} \) | \( 0.707 \pm 0.034 \) |

Table 6: Our estimated transition redshift \( z_t \) at 1\( \sigma \) for both datasets and their joint combination.

6 Summary

Cosmography is based on the Taylor expansion of the scale factor around \( z = 0 \). This expansion is the source of two main problems arise in this kinematic approach of the study of universe. In stead of expanding scale factor we have combined the mathematical definitions of deceleration and jerk parameters (see eqs 1b,2c) which results in a general second order differential equation for squared Hubble parameter. Although the solution of this equation could lead to a general function for Hubble parameter, but it seems to be much complicated. However, it is possible to find some reasonable solutions by considering jerk parameterization. In this paper we assumed \( j(z) = j_0 \) and found a Hubble function in terms of jerk parameter. Using this function we have reconstructed other cosmographic parameters as well as deceleration-acceleration transition redshift. It is worth mentioning that since our approach is totally model-independent, we can constrain any derived quantity without any doubt on the validity of the estimations. Next we recovered \( \Lambda \)CDM model from the obtained Hubble function. It is found that when cosmological constant is responsible for the current cosmic accelerating expansion the geometry of spacetime is necessarily should be flat. For any other dynamical (time varying) dark energy scenarios the spacetime geometry should be non-flat. Probably this approach could be used for other spacetime posses some inhomogenities. We have constrained cosmographic parameters as well as transition redshift and spacetime curvature over observational Hubble data [33], BAO/CMB [35] and SNIa (Pantheon compilation) [34], and their joint combination. Our results are in agreement with almost all other results such as 9years WMAP and Planck (2015) collaboration.

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