Higher integrability near the initial boundary for nonhomogeneous parabolic systems of $p$-Laplacian type

Abstract: We establish a sharp higher integrability near the initial boundary for a weak solution to the following $p$-Laplacian type system:

$$\begin{align*}
    u_t - \text{div} A(x, t, \nabla u) &= \text{div} |F|^{p-2} F + f \quad \text{in } \Omega_T, \\
    u &= u_0 \quad \text{on } \Omega \times \{0\},
\end{align*}$$

by proving that, for given $\delta \in (0, 1)$, there exists $\varepsilon > 0$ depending on $\delta$ and the structural data such that

$$|\nabla u_0|^{p+\varepsilon} \in L^1_{\text{loc}}(\Omega) \quad \text{and} \quad |F|^{p+\varepsilon} |f|^{\frac{q(n+2)}{n}} \in L^1(0, T; L^1_{\text{loc}}(\Omega)) \quad \implies \quad |\nabla u|^{p+\varepsilon} \in L^1(0, T; L^1_{\text{loc}}(\Omega)).$$

Our regularity results complement established higher regularity theories near the initial boundary for such a nonhomogeneous problem with $f \not\equiv 0$ and we provide an optimal regularity theory in the literature.

Keywords: Higher integrability, initial boundary data, non-divergence data, parabolic $p$-Laplacian system

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1 Introduction

In this paper, we are interested in finding a sharp higher integrability near the initial boundary to a weak solution to the parabolic system

$$\begin{align*}
    u_t - \text{div} A(x, t, \nabla u) &= \text{div} |F|^{p-2} F + f \quad \text{in } \Omega_T, \\
    u &= u_0 \quad \text{on } \Omega \times \{0\}.
\end{align*}$$

Here, $\frac{2n}{n+2} < p < \infty$, $\Omega$ is a bounded domain in $\mathbb{R}^n$ with $n \geq 2$, $A(x, t, \zeta)$ is modeled after the $p$-Laplacian operator, $u_0 \in W^{1,p}(\Omega, \mathbb{R}^N)$, $F \in L^p(\Omega_T; \mathbb{R}^{Nn})$ and $f \in L^{q^*}(\Omega_T, \mathbb{R}^N)$ for some $N \geq 1$, where $q = p\left(\frac{n+2}{n}\right)$ is the parabolic Sobolev conjugate of $p$ and $q^*$ is Hölder conjugate of $q$.

In the case $f \equiv 0$, interior higher integrability results were proved by Kinnunen and Lewis in [13, 14] by providing a suitable application of DiBenedetto’s intrinsic geometry method from [9] to the setting of Gehring type estimates. Higher integrability results near the initial and lateral boundary were proved by Parviainen in [19, 20]. These regularity results were extended to higher-order systems by Bögelein and Parviainen in [5].
On the other hand, in the case \( f \neq 0 \), a nonlinear relation between the gradient of a weak solution and the non-divergence data \( f \) coming from parabolic embeddings naturally occurs. To be specific, there is an exponent \( \alpha > 1 \) such that \( \| \nabla u \|_{L^p}^p \) is related to \( \| f \|_{L^q}^q \). Indeed, there have been regularity estimates coming from such a nonlinear relation rather than Gehring type estimates. In particular, Kuusi and Mingione in [15] proved gradient \( L^\alpha \) regularity and gradient continuity of a weak solution with a nonlinear exponent \( \alpha \) based on the intrinsic geometry method. The recent paper [2] provides a new representation of the nonlinear relation. It replaces the exponent \( \alpha \) by 1 from the intrinsic geometry method, obtaining interior higher integrability results for a system of \( p(x, t) \)-Laplacian type with scaling invariant estimates and extending gradient continuity results in [15] to the \( p(x, t) \)-Laplacian system. For the elliptic case, there is a divergence representation for the non-divergence data, and Gehring type estimates directly follow from [10, 11, 17].

As already mentioned, the main purpose of this paper is to establish higher integrability results near the initial boundary. In addition, we are considering a class of the nonlinearities whose structures are associated with the divergence data \( F \). Regarding the non-divergence data, \( f \in L^{(\delta m+2)/\delta} \) with \( \delta \in (0, 1) \) is sharp in obtaining the reverse Hölder inequality by the parabolic Sobolev embedding theorem. A noteworthy feature of the present paper is to find the optimality and sharpness of the initial data obtaining the reverse Hölder inequality by the parabolic Sobolev embedding theorem. Due to a suitable application of the Poincaré inequality in a type of the Caccioppoli inequality, it replaces the exponent \( \alpha \) by 1 from the intrinsic geometry method, obtaining interior higher integrability results coming from \( \| \nabla u \|_{L^p}^p \) from the intrinsic geometry method.

Our work can be applicable to different problems including obstacle problem and system of \( p(x, t) \)-Laplacian type problem as in [4, 6] as well as Calderón–Zygmund type estimates as in [1, 3, 7, 8, 16] both when \( f \neq 0 \) and when \( \nabla u_0 \neq 0 \).

The paper is organized as follows. In Section 2, we introduce basic notation and definitions to state our main results. Section 3 is devoted to proving reverse Hölder inequality. Finally, in Section 4, we prove our main results.

## 2 Basic notations and results

### 2.1 Notations

We shall clarify all the notations that will be used in this paper.

(i) We use \( V \) to denote derivatives with respect to the space variable \( x \) and \( \partial_t \) to denote the time derivative.

(ii) In what follows, we always assume the bounds \( \frac{n}{\delta+2} < p < \infty \).

(iii) Let \( z_0 = (x_0, t_0) \in \mathbb{R}^{n+1} \) be a point, \( \rho, s > 0 \) two given parameters, and let \( \lambda \in (1, \infty) \). We use the following notations:

\[
I_s(t_0) := (t_0 - s^2, t_0 + s^2) \subset \mathbb{R}, \quad Q_{\rho, s}(z_0) := B_{\rho(x_0)} \times I_s(t_0) \subset \mathbb{R}^{n+1},
\]

\[
I^1_s(t_0) := (t_0 - \lambda^2 - p s^2, t_0 + \lambda^2 - p s^2) \subset \mathbb{R}, \quad Q^1_{\rho, s}(z_0) := B_{\rho(x_0)} \times I^1_s(t_0) \subset \mathbb{R}^{n+1},
\]

\[
Q^1_{\rho, s}(z_0) := Q^1_{\rho, s}(z_0), \quad Q^1_{\rho, s}(z_0) := Q^1_{\rho, s}(z_0).
\]

(iv) We use \( \int \) to denote the integral with respect to either space variable or time variable and use \( \iint \) to denote the integral with respect to both space and time variables simultaneously.

Analogously, we use \( \int \) and \( \iint \) to denote the integral averages as defined below: for any set \( A \times B \subset \mathbb{R}^n \times \mathbb{R} \), we define

\[
(f)_A := \int_A f(x) \, dx = \frac{1}{|A|} \int_A f(x) \, dx, \quad (f)_{A \times B} := \iint_{A \times B} f(x, t) \, dx \, dt = \frac{1}{|A \times B|} \iint_{A \times B} f(x, t) \, dx \, dt.
\]

(v) We use the notation \( \lesssim_{(a, b, \ldots)} \) to denote an inequality with a constant depending on \( a, b, \ldots \).
Definition 2.1. Let \( \Omega \) be a bounded domain in \( \mathbb{R}^n \) with \( n \geq 2 \) and \( u_0 \in L^2(\Omega, \mathbb{R}^N) \), \( f \in L^\infty(\Omega_t; \mathbb{R}^N) \) for some \( N \geq 1 \). A weak solution \( u \in C(0, T; L^2(\Omega, \mathbb{R}^N)) \cap L^p(0, T; W^{1,p}(\Omega, \mathbb{R}^N)) \) to

\[
\begin{cases}
    u_t - \text{div} \mathcal{A}(x, t, \nabla u) = f & \text{in } \Omega_t, \\
    u = u_0 & \text{on } \Omega \times \{0\}
\end{cases}
\]

is a distributional energy solution in the sense

\[
\iint_{\Omega_t} -u \phi_t + (\mathcal{A}(x, t, \nabla u), \nabla \phi) \, dz = \iint_{\Omega_t} f \phi \, dz \quad \text{for all } \phi \in C_0^\infty(\Omega_T, \mathbb{R}^N),
\]

and

\[
\lim_{h \to 0^+} \frac{1}{h} \iint_0^h |u(x, t) - u_0(x)|^2 \, dx \, dt = 0. \quad (2.2)
\]

2.2 Structures of the operator

We now describe the assumptions on the nonlinear structures in (2.1). Assume \( \mathcal{A}(x, t, \nabla u) \) is a Carathéodory function, i.e., we have that \( (x, t) \mapsto \mathcal{A}(x, t, \zeta) \) is measurable for every \( \zeta \in \mathbb{R}^n \) and \( \zeta \mapsto \mathcal{A}(x, t, \zeta) \) is continuous for almost every \( (x, t) \in \Omega_T \).

We further assume that, for a.e. \( (x, t) \in \Omega_T \) and for any \( \zeta \in \mathbb{R}^n \), there exist two positive constants \( \Lambda_0 \) and \( \Lambda_1 \) such that the following bounds are satisfied by the nonlinear structures:

\[
(\mathcal{A}(x, t, \zeta), \zeta) \geq \Lambda_0 |\zeta|^p - |F(x, t)|^p \quad \text{and} \quad |\mathcal{A}(x, t, \zeta)| \leq \Lambda_1 |\zeta|^{p-1} + |F(x, t)|^{p-1},
\]

where \( F \in L^p(\Omega_T, \mathbb{R}^{nN}) \).

2.3 Main results

Before stating our main theorem, we fix some constants which will be frequently used in this paper.

Definition 2.2. For fixed constants

\[
\max\left\{ \frac{n}{n+2}, \frac{n+1}{(n+2)p} \right\} < \delta < 1 \quad \text{and} \quad \delta := \begin{cases} \frac{p}{2} & \text{if } p \geq 2, \\ \frac{2p}{2p+2} & \text{if } p < 2, \end{cases}
\]

we denote

\[
a := \frac{\delta p(n+2) - 2}{n}, \quad \nu := \alpha'(\min\{2, p\} - \frac{p}{a}) \quad \text{and} \quad \alpha := \left(1 - \frac{d}{p}\nu\right)^{-1}. \]

Remark 2.3. (i) \( 0 < \frac{p}{a} \) holds, and (ii) \( a \) is a parabolic Sobolev conjugate of \( \delta p \).

To see Remark 2.3 (i), we split two cases.

- Case \( p \geq 2 \): There hold \( \frac{p}{a} = \frac{p}{2} - \alpha'(1 - \frac{p}{2}) \) and

\[
\frac{p}{a} - \alpha'(1 - \frac{p}{a}) = 2 - \alpha'(1 - \frac{p}{a}) > 0 \iff \frac{2}{a} - \left(1 - \frac{p}{a}\right) > 0 \iff 2 - \frac{2}{a} > 1 - \frac{p}{a} \iff 1 > \frac{1}{a}(2 - p).
\]

- Case \( p < 2 \): There hold \( \frac{p}{a} = \frac{p}{2} - \alpha'(p - \frac{p}{2}) \) and

\[
0 < \frac{p}{a} = \frac{p(n+2) - 2n}{2} - \alpha'(\frac{p}{2} - \frac{p}{2}) = 0 < \frac{n(p-2)}{a} + p \iff 2n - pn - p < \frac{2n - pn}{a}.
\]

Recall \( a = \frac{\delta p(n+2)}{n} \). Since \( \delta \in (0, 1) \) and

\[
(n+2)p[2n-(n+1)p] \leq n^2(2-p) \iff ((n+2)p - 2n)((n+1)p - n) \geq 0,
\]

the remark follows.
To apply the intrinsic geometry method developed in [13], let us define the following notations.

**Definition 2.4.** Let \( u \in C(0, T; L^2(\Omega, \mathbb{R}^N)) \cap L^p(0, T; W^{1, p}(\Omega, \mathbb{R}^N)) \) be a weak solution of (2.1) under the assumption of (2.2), (2.3) and \( f \in L^q(\Omega, \mathbb{R}^N) \). For any \((x_0, t_0) = z_0 \in \Omega_T \) and \( B_{2r}(x_0) \subset \Omega \), we define

\[
\lambda_0^\rho := \iint_{Q_r(z_0) \cap \Omega_T} \left( |\nabla u| + |F| + 1 \right)^p \, dz + \iint_{Q_r(z_0) \cap \Omega_T} (2n)^a |f|^a \, dz + \left( \iint_{B_{2r}(x_0)} |\nabla u_0|^p \, dx \right)^{\min\{1, \frac{1}{a} \}}.
\]

Now, we state the main theorem.

**Theorem 2.5.** Let \( u \in C(0, T; L^2(\Omega, \mathbb{R}^N)) \cap L^p(0, T; W^{1, p}(\Omega, \mathbb{R}^N)) \) be a weak solution of (2.1) under the assumption of (2.2), (2.3) and \( f \in L^q(\Omega, \mathbb{R}^N) \). Then there exists \( \epsilon_0(\Omega, N, p, \Lambda_0, \Lambda_1, \delta) \) such that, for any \( \epsilon \in (0, \epsilon_0) \) and any \((x_0, t_0) = z_0 \in \Omega_T \) such that \( B_{2r}(x_0) \subset \Omega \), there holds

\[
\iint_{Q_r(x_0) \cap \Omega_T} |\nabla u|^{p+\epsilon} \, dz \leq (n, N, p, \Lambda_0, \Lambda_1, \delta) \lambda_0^\rho \iint_{Q_r(z_0) \cap \Omega_T} |\nabla u|^p \, dz + \iint_{Q_r(z_0) \cap \Omega_T} (F + 1)^{p+\epsilon} \, dz + \iint_{B_{2r}(x_0)} |\nabla u_0|^{p+\epsilon} \, dx + \lambda_0^\rho (l_1 \frac{1}{a} + \frac{1}{2}) \iint_{Q_r(z_0) \cap \Omega_T} (F(f))^{a+\frac{1}{a} \epsilon} \, dz.
\]

Here, \( \lambda_0 \) is defined in Definition 2.4.

**Remark 2.6.** As a consequence of Theorem 2.5, we can also obtain global estimates of the weak solution to

\[
\begin{aligned}
\begin{cases}
u_t - \text{div} A(x, t, \nabla u) = f & \text{in } \Omega_T, \\
u = \phi & \text{on } \partial \Omega \times (0, T), \\
u = u_0 & \text{on } \Omega \times \{0\},
\end{cases}
\end{aligned}
\]

where \( A, f \) and \( u_0 \) are assumed as in (2.1) and \( \phi \in L^p(0, T; W^{1, p}(\Omega, \mathbb{R}^N)) \) such that \( \phi_\epsilon \in L^\left(\frac{p+1}{2}\right, (\Omega_T, \mathbb{R}^N) \). Especially, \( \nabla \phi \) behaves like divergence data \( F \) does while \( \phi_\epsilon \) behaves exactly in the same way as non-divergence data \( F \) does.

Before ending this section, we provide some important lemmas which will be used later in the proof of the main theorem. Let us state Gagliardo–Nirenberg’s inequality (see [18]).

**Lemma 2.7.** Let \( B_\rho(x_0) \subset \mathbb{R}^n \) with \( 0 < \rho \leq 1, \sigma, q, r \in [1, \infty) \) and \( \vartheta \in (0, 1) \) such that \(-\frac{\rho}{\vartheta} \leq \vartheta (1 - \frac{n}{q}) - (1 - \vartheta) \frac{n}{r} \).

Then, for any \( u \in W^{1, q}(B_\rho(x_0)) \), there holds

\[
\iint_{B_\rho(x_0)} \frac{|u|^q}{\rho^q} \, dx \leq (n, \rho, q, r) \left( \iint_{B_\rho(x_0)} \frac{|u|^q}{\rho^q} + |\nabla u|^q \, dx \right)^{\frac{n}{r}} \left( \iint_{B_\rho(x_0)} \frac{|u|^q}{\rho^q} \, dx \right)^{\frac{1-n}{r}}.
\]

The following iteration lemma can be found in [12, Lemma 6.1].

**Lemma 2.8.** Let \( 0 < r < R < \infty \) be given, and let \( h: [r, R] \rightarrow \mathbb{R} \) be a non-negative and bounded function. Furthermore, let \( \theta \in (0, 1) \) and \( A, B, \gamma_1, \gamma_2 \geq 0 \) be fixed constants, and suppose that

\[
h(\rho_1) \leq \theta h(\rho_2) + \frac{A}{(\rho_2 - \rho_1)\gamma_1} + \frac{B}{(\rho_2 - \rho_1)\gamma_2},
\]

holds for all \( r \leq \rho_1 < \rho_2 \leq R \). Then the following conclusion holds:

\[
h(\rho) \leq \left( \frac{\rho}{R} \right)^{\gamma_1} \frac{A}{(R - r)^\gamma_1} + \frac{B}{(R - r)^\gamma_2}.
\]

### 3 Estimates near the initial boundary

In this section, we assume that \( B_{4\rho}(x_0) \subset \Omega, 0 < t_0 \) and \( \lambda \geq 1 \). Also, note that, in the case \( 0 \in I_{1/\rho}(t_0) \), for any \( \rho \leq \rho_1 \leq \rho_2 \leq 4\rho \), there holds

\[
|I^\lambda_{\rho_1}(t_0)| \leq |I^\lambda_{\rho_2}(t_0)| \leq 4^2 |I^\lambda_{\rho_1}(t_0)| \quad \text{and} \quad \frac{1}{2} |I^\lambda_{\rho_2}(t_0)| \leq |I^\lambda_{\rho_1}(t_0) \cap (0, T)| \leq |I^\lambda_{\rho_1}(t_0)|. \quad (3.1)
\]
We will show a reverse Hölder inequality in intrinsic cylinders under the following assumptions.

**Assumption 3.1.** We assume
\[
\iint_{Q_1^b(x_0)\cap\Omega_T}(|\nabla u| + |F| + 1)^p \, dz + \lambda^{a'(1-\frac{p}{a})} \int_{Q_1^b(x_0)\cap\Omega_T}(4p)^{a'}|f|^{a'} \, dz + \int_{B_0(x_0)} |\nabla u_0|^p \, dx \leq \lambda^p, \tag{3.2}
\]
\[
\iint_{Q_1^b(x_0)\cap\Omega_T}(|\nabla u| + |F| + 1)^p \, dz + \lambda^{a'(1-\frac{p}{a})} \int_{Q_1^b(x_0)\cap\Omega_T}(4p)^{a'}|f|^{a'} \, dz + \int_{B_0(x_0)} |\nabla u_0|^p \, dx \geq \lambda^p. \tag{3.3}
\]

Note that, by a choice of \(\delta \in (0, 1)\) in Definition 2.2, \(0 < a'(1 - \frac{p}{a}) < p\) holds, and this exponent \(a'(1 - \frac{p}{a})\) is chosen to preserve a nonlinear relation between the gradient of a weak solution and the non-divergence data. Indeed, using Young’s inequality, (3.3) becomes
\[
\iint_{Q_1^b(x_0)\cap\Omega_T}(|\nabla u| + |F| + 1)^p \, dz + \left( \iint_{Q_1^b(x_0)\cap\Omega_T}(\rho^{a'}|f|^{a'}) \right)^{(1-\frac{a'}{p}(1-\frac{p}{a}))^{-1}} + \int_{B_0(x_0)} |\nabla u_0|^p \, dx \geq \lambda^p.
\]

### 3.1 Caccioppoli inequality and Poincaré type inequality

**Lemma 3.2.** Let \(u\) be a weak solution of (2.1). Suppose \(B_{4\rho}(x_0) \subset \Omega\) and \(0 < t_0\). Then, for any \(p \leq \rho_a < \rho_b < 4\rho\), there holds
\[
\frac{1}{|I_{\rho_a}^1(t_0)|} \sup_{t \in I_{\rho_a}^1(t_0)} \int_{B_{\rho_a}(x_0)} \left| u - (u_0)_{B_{\rho_a}(x_0)} \right|^2 \, dz + \iint_{Q_1^b(x_0)\cap\Omega_T} |\nabla u|^p \, dz \leq \frac{c}{\rho_b - \rho_a}.
\]

**Proof.** The proof basically follows from [5, Lemma 5.1]. For the sake of completeness, we present the details in our setting.

In the space direction, take a cut-off function \(\eta\) satisfying
\[
0 \leq \eta = \eta(x) \in C^\infty_c(B_{\rho_b}(x_0)), \quad \eta \equiv 1 \text{ on } B_{\rho_b}(x_0), \quad |\nabla \eta| \leq \frac{c}{\rho_b - \rho_a}.
\]

For the time direction, we divide two cases. In the case that \(0 \in I_{\rho_a}^1(t_0)\), consider a cut-off function \(\zeta\) satisfying
\[
0 \leq \zeta = \zeta(t) \in C^\infty_c(I_{\rho_b}^1(t_0)), \quad \zeta \equiv 1 \text{ on } I_{\rho_b}^1(t_0), \quad |\partial_t \zeta| \leq \frac{c}{\lambda^{2-p}(\rho_b - \rho_a)^2}.
\]

On the other hand, in the case that \(0 \in I_{\rho_a}^1(t_0)\), consider a cut-off function \(\zeta\) satisfying
\[
0 \leq \zeta = \zeta(t) \in C^\infty_c(I_{\rho_b}^1(t_0)), \quad \zeta \equiv 1 \text{ on } 0 \leq t \leq t_0 + \lambda^{2-p} \rho_b^2, \quad |\partial_t \zeta| \leq \frac{c}{\lambda^{2-p}(\rho_b - \rho_a)^2}.
\]

Taking \((u - (u_0)_{B_{\rho_b}(x_0)})\eta^p \zeta^2\) as a test function in (2.1), we obtain
\[
I + II := \iint_{Q_1^b(x_0)\cap\Omega_T} \partial_t u (u - (u_0)_{B_{\rho_b}(x_0)}) \eta^p \zeta^2 \, dz + \iint_{Q_1^b(x_0)\cap\Omega_T} (A(z, \nabla u), \nabla u) \eta^p \zeta^2 \, dz \leq \iint_{Q_1^b(x_0)\cap\Omega_T} |A(z, \nabla u)||\nabla \eta||u - (u_0)_{B_{\rho_b}(x_0)}|\eta^{p-1} \zeta^2 \, dz + \iint_{Q_1^b(x_0)\cap\Omega_T} |f||u - (u_0)_{B_{\rho_b}(x_0)}|\eta^p \zeta^2 \, dz =: III + IV.
\]
Estimate of I: We use integration by parts and (2.2) to find that
\[
I \geq \frac{1}{2|I_{\rho_a}(t_0)|} \sup_{t \in I_{\rho_a}(t_0)} \left\{ \frac{1}{B_{\rho_a}(x_0)} \left| u - (u_0)_{B_{\rho_a}(x_0)} \right|^2 \eta^2 \xi^2 \, dx - \frac{1}{2|I_{\rho_a}(t_0)|} \right\} \frac{1}{B_{\rho_a}(x_0)} \left| u_0 - (u_0)_{B_{\rho_a}(x_0)} \right|^2 \eta^2 \xi^2 \, dx 
- \frac{1}{2} \mid_{Q_{\rho_a}(z_0) \cap \Omega} \left| u - (u_0)_{B_{\rho_a}(x_0)} \right|^2 \eta^2 \xi \, dz =: I_1 - I_2 - I_3.
\]

Estimate of I_1: Using the triangle inequality along with the fact that \( \rho \leq \rho_a \leq \rho_b \leq 4\rho_b \) and (3.1), we obtain
\[
I_1 \leq \frac{1}{2|I_{\rho_a}(t_0)|} \sup_{t \in I_{\rho_a}(t_0)} \left\{ \frac{1}{B_{\rho_a}(x_0)} \left| u - (u_0)_{B_{\rho_a}(x_0)} \right|^2 \, dx \leq 1. \right\}
\]

Estimate of I_2: Applying Poincaré's inequality, we have
\[
I_2 \leq \frac{1}{2|I_{\rho_a}(t_0)|} \sup_{t \in I_{\rho_a}(t_0)} \left\{ \frac{1}{B_{\rho_a}(x_0)} \left| u - (u_0)_{B_{\rho_a}(x_0)} \right|^2 \, dx \leq \lambda^{p-2} \left( \frac{\left\| \nabla u \right\|}{\rho_a} \right)^2. \right\}
\]

Estimate of I_3: We get
\[
I_3 \leq \lambda^{p-2} \mid_{Q_{\rho_a}(z_0) \cap \Omega} \left[ \frac{\left| u - (u_0)_{B_{\rho_a}(x_0)} \right|}{\rho_b - \rho_a} \right]^2 \, dz.
\]

Therefore, we obtain
\[
I \geq \frac{1}{2|I_{\rho_a}(t_0)|} \sup_{t \in I_{\rho_a}(t_0)} \left\{ \frac{1}{B_{\rho_a}(x_0)} \left| u - (u_0)_{B_{\rho_a}(x_0)} \right|^2 \, dz - \lambda^{p-2} \left( \frac{\left\| \nabla u \right\|}{\rho_b} \right)^2 \right\}
- \lambda^{p-2} \mid_{Q_{\rho_a}(z_0) \cap \Omega} \left[ \frac{\left| u - (u_0)_{B_{\rho_a}(x_0)} \right|}{\rho_b - \rho_a} \right]^2 \, dz.
\]

Estimate of II: Applying (2.3), we discover
\[
II \geq \mid_{Q_{\rho_a}(z_0) \cap \Omega} \left| \nabla u \right|^p \eta^2 \xi^2 \, dz - \mid_{Q_{\rho_a}(z_0) \cap \Omega} \left| F \right|^p \, dz.
\]

Estimate of III:
\[
III \overset{(2.3)}{\leq} \mid_{Q_{\rho_a}(z_0) \cap \Omega} \left| \nabla u \right|^{p-1} + \mid_{Q_{\rho_a}(z_0) \cap \Omega} \left| F \right|^{p-1} \left[ \frac{\left| u - (u_0)_{B_{\rho_a}(x_0)} \right|}{\rho_b - \rho_a} \right] \eta^{p-1} \xi^2 \, dz
\]
\[
\overset{(a)}{\leq} \gamma \mid_{Q_{\rho_a}(z_0) \cap \Omega} \left| \nabla u \right|^p \eta^p \xi^2 \, dz + \mid_{Q_{\rho_a}(z_0) \cap \Omega} \left[ \frac{\left| u - (u_0)_{B_{\rho_a}(x_0)} \right|}{\rho_b - \rho_a} \right]^p \left| F \right|^p \, dz.
\]

Here, to obtain (a), we used Young's inequality with \( \gamma \in (0, 1) \).

Estimate of IV: Clearly, there holds
\[
IV \leq \mid_{Q_{\rho_a}(z_0) \cap \Omega} \left| f \right| \mid u - (u_0)_{B_{\rho_a}(x_0)} \mid \, dz.
\]

Combining all the above calculations and taking \( \gamma = \gamma(n, N, p, \Lambda_0, \Lambda_1) \) small enough, the conclusion follows.

In our definition of a weak solution to (2.1), there is no differentiability assumption on \( u \) with respect to time. Therefore, we cannot apply Poincaré's inequality directly. Nevertheless, we shall use (2.1) to estimate continuity of \( u \) with respect to time.
Lemma 3.3. Let $u$ be a weak solution of (2.1). Suppose that $B_r(x_0) \subset \Omega$ and $0 < t_0$. Then, for all $\theta \in [1, p]$, there holds
\[
\iint_{Q_1(z_0) \cap \Omega_T} \left| \frac{u - (u_0)_{B_r(x_0)}}{r} \right|^\theta \, dz \lesssim_{n, N, p, A_1} \iint_{Q_1(z_0) \cap \Omega_T} |\nabla u|^\theta \, dz + \left( \lambda^{2-p} \iint_{Q_1(z_0) \cap \Omega_T} |u|^{p-1} + |F|^{p-1} \, dz \right)^\theta
\]
\[+ \lambda^\theta \left( \lambda^{1-p} r \iint_{Q_1(z_0) \cap \Omega_T} |f| \, dz \right)^\theta + \left( \iint_{B_r(x_0)} |u_0| \, dx \right)^\theta.
\]

Proof. Let $0 \leq \eta \in C_c^\infty(B_r(x_0))$ such that $\int_{B_r(x_0)} \eta \, dz = 1$ and $\|\eta\|_{\infty} + r\|\nabla \eta\|_{\infty} \leq C(r)$. For a.e. $t$, we denote weighted integral averages of $u$ by
\[
(u)_\eta(t) := \int_{B_r(x_0)} u(x, t) \eta(x) \, dx.
\]
The triangle inequality gives
\[
\iint_{Q_1(z_0) \cap \Omega_T} \left| \frac{u - (u_0)_{B_r(x_0)}}{r} \right|^\theta \, dz \lesssim_r \iint_{Q_1(z_0) \cap \Omega_T} \left| (u)_\eta(t) - (u)_{B_r(x_0)}(t) \right|^\theta \, dt + \iint_{Q_1(z_0) \cap \Omega_T} \left| u - (u)_\eta(t) \right|^\theta \, dz
\]
\[+ r^\theta \iint_{Q_1(z_0) \cap \Omega_T} \left| (u)_{B_r(x_0)}(t) - (u_0)_{B_r(x_0)} \right| \, dt =: I + II + III.
\]

Estimate of I: Applying Poincaré's inequality in space direction, we obtain
\[
I \leq r^\theta \iint_{Q_1(z_0) \cap \Omega_T} \left| u - (u)_{B_r(x_0)} \right| \eta \, dx \, dt \leq \iint_{Q_1(z_0) \cap \Omega_T} |\nabla u|^\theta \, dz.
\]

Estimate of II: Similarly, we have
\[
II \leq r^\theta \iint_{Q_1(z_0) \cap \Omega_T} \left| u - (u)_{B_r(x_0)}(t) \right|^\theta \, dz \leq \iint_{Q_1(z_0) \cap \Omega_T} |\nabla u|^\theta \, dz.
\]

Estimate of III: Again, the triangle inequality implies
\[
III \leq r^\theta \iint_{Q_1(z_0) \cap \Omega_T} \left| (u)_{\eta}(t) - (u)_{\eta}(t) \right|^\theta \, dt + r^\theta \iint_{Q_1(z_0) \cap \Omega_T} \left| (u)_{\eta}(t) - (u_0)_{\eta} \right|^\theta \, dt
\]
\[\leq \iint_{Q_1(z_0) \cap \Omega_T} |\nabla u|^\theta + r^\theta \iint_{Q_1(z_0) \cap \Omega_T} (u)_{\eta}(t) - (u_0)_{\eta} \, dt + \left( \iint_{B_r(x_0)} |u_0| \, dx \right)^\theta.
\]

To estimate the second term, we test $\eta$ to (2.1) in $B_r(x_0) \times (t_1, t_2) \subset Q_1(z_0) \cap \Omega_T$. Then (2.3) implies that
\[
|\eta(t_1) - (u)_{\eta}(t_2)| \leq_{(A_1)} \int_{t_1}^{t_2} |\nabla u|^{p-1} |\nabla \eta| + |F|^{p-1} |\nabla \eta| + |f| |\eta| \, dx \, dt
\]
\[\leq \lambda^{2-p} r \iint_{Q_1(z_0) \cap \Omega_T} |u|^{p-1} + |F|^{p-1} \, dz + \lambda^{2-p} r^2 \iint_{Q_1(z_0) \cap \Omega_T} |f| \, dz.
\]

Using (2.2), we obtain
\[
\sup_{t \in [t_1, t_2]} |(u)_{\eta}(t) - (u_0)_{\eta}| \leq \lambda^{2-p} r \iint_{Q_1(z_0) \cap \Omega_T} |u|^{p-1} + |F|^{p-1} \, dz + \lambda^{2-p} r^2 \iint_{Q_1(z_0) \cap \Omega_T} |f| \, dz.
\]
Therefore, we get
\[ III \lesssim (n, N, p, \Lambda_1) \llint_{Q^4_{\rho a}(z_0)\cap \Omega_T} |\nabla u|^\theta dz + \left( \lambda^{2-p} \llint_{Q^4_{\rho a}(z_0)\cap \Omega_T} |\nabla u|^{p-1} + |F|^{p-1} \right)^\theta dz \]
\[ + \lambda^\theta \left( \lambda^{1-p} \llint_{Q^4_{\rho a}(z_0)\cap \Omega_T} |f| dz \right)^\theta + \left( \llint_{B_{\rho a}(x_0)} |\nabla u_0| dx \right)^\theta. \]
This completes the proof.

\[ \square \]

3.2 Some crucial estimates

The purpose of this subsection is to refine the estimate \( u \) in \( C(I_t^1(t_0) \cap (0, T); L^2(B_{\rho_b}(x_0), \mathbb{R}^N)) \). To this end, we first estimate the right-hand side of the inequality in Lemma 3.2 using Lemma 3.3 and assumption (3.2), and then return to the left-hand side of the inequality.

**Lemma 3.4.** Let \( u \) be a weak solution of (2.1). Suppose \( B_{4\rho}(x_0) \subset \Omega \) and \( 0 < t_0 \). Also, assume (3.2) for some \( \lambda \geq 1 \). Then, for all \( \theta \in [1, p] \), there holds
\[ \llint_{Q^4_{4\rho_a}(z_0)\cap \Omega_T} \left[ \frac{|u - (u_0)_{B_{4\rho}(x_0)}|}{4\rho} \right]^\theta dz \lesssim (n, N, p, \Lambda_0, \Lambda_1) \lambda^\theta. \]

**Proof.** Applying Lemma 3.3 and Hölder’s inequality, we find
\[ \llint_{Q^4_{4\rho_a}(z_0)\cap \Omega_T} \left[ \frac{|u - (u_0)_{B_{4\rho}(x_0)}|}{4\rho} \right]^\theta dz \lesssim \left( \llint_{Q^4_{4\rho_a}(z_0)\cap \Omega_T} |\nabla u|^p dz \right)^\frac{\theta}{p} + \left( \lambda^{2-p} \llint_{Q^4_{4\rho_a}(z_0)\cap \Omega_T} |\nabla u|^{p-1} + |F|^{p-1} \right)^\frac{\theta}{p} \]
\[ + \lambda^\theta \left( \lambda^{1-p} \llint_{Q^4_{4\rho_a}(z_0)\cap \Omega_T} |f|^{\frac{p}{1-p}} dz \right)^\theta + \left( \llint_{B_{4\rho a}(x_0)} |\nabla u_0|^p dx \right)^\theta. \]

Note that
\[ \lambda^\theta (1-p) \llint_{Q^4_{4\rho_a}(z_0)\cap \Omega_T} (4\rho)^\theta |f|^{\theta a'} dz = \lambda^\theta (1-\frac{\theta}{p}) \llint_{Q^4_{4\rho_a}(z_0)\cap \Omega_T} (4\rho)^{\theta a'} |f|^{\theta a'} dz \leq 1. \tag{3.4} \]
Therefore, making use of (3.2), this completes the proof.

**Lemma 3.5.** Under the assumptions and the conclusion in Lemma 3.4, we further have
\[ \sup_{I_t^1(t_0) \cap (0, T)} \left\{ \frac{|u - (u_0)_{B_{4\rho}(x_0)}|}{2\rho} \right\}^2 dx \lesssim (n, N, p, \Lambda_0, \Lambda_1) \lambda^2. \]

**Proof.** Take \( 2p \leq \rho_a < \rho_b \leq 4\rho \) in Lemma 3.2 to get
\[ \lambda^{p-2} \sup_{I_t^1(t_0) \cap (0, T)} \llint_{B_{4\rho}(x_0)} \left[ \frac{|u - (u_0)_{B_{4\rho}(x_0)}|}{\rho a} \right]^2 dx \]
\[ \leq \llint_{Q^4_{\rho b}(z_0)\cap \Omega_T} \left[ \frac{|u - (u_0)_{B_{\rho b}(x_0)}|}{\rho b - \rho a} \right]^p dz + \lambda^{p-2} \llint_{Q^4_{\rho b}(z_0)\cap \Omega_T} \left[ \frac{|u - (u_0)_{B_{\rho b}(x_0)}|}{\rho b - \rho a} \right]^2 dz \]
\[ + \llint_{Q^4_{\rho b}(z_0)\cap \Omega_T} |f| |u - (u_0)_{B_{\rho b}(x_0)}| dz + \lambda^{p-2} \left( \llint_{B_{\rho b}(x_0)} |\nabla u_0|^p dx \right)^\frac{1}{2} + \llint_{Q^4_{\rho b}(z_0)\cap \Omega_T} |F|^p dz \]
\[ =: I + II + III + IV + V. \]
Estimate of I: Note that, under the restriction $2\rho \leq \rho_b < \rho_a \leq 4\rho$, we see that
\[
\left(\frac{\rho_b - \rho_a}{\rho_b}\right)^p \leq \left\{\begin{array}{ll}
\int_{Q_b(2)}^{Q_b(4)} & \left[\frac{|u - (u_0)_{B_{t_b}(x_0)}|}{\rho_b}\right]^p dz + \left[\frac{[(u_0)_{B_{t_b}(x_0)} - (u_0)_{B_{t_b}(x_0)}]}{\rho_b}\right]^p \\
& \left[\frac{|u - (u_0)_{B_{t_b}(x_0)}|}{\rho_b}\right]^p dz + \left[\frac{[(u_0)_{B_{t_b}(x_0)} - (u_0)_{B_{t_b}(x_0)}]}{\rho_b}\right]^p dx \leq \lambda^p.
\end{array}\right.
\] (3.5)

Here, to obtain (a), we used Lemma 3.4 and Poincaré’s inequality with (3.2) for initial data.

Estimate of II: There holds that
\[
II = \lambda^{p-2}\left(\frac{\rho_b}{\rho_b - \rho_a}\right)^2 \int_{t_b(x_0)}^{0} \left( \int_{B_{t_b}(x_0)} \left[\frac{|u - (u_0)_{B_{t_b}(x_0)}|}{\rho_b}\right]^p dx \right)^{\frac{1}{p}} dt
\]
\[
(a) \leq \lambda^{p-2}\left(\frac{\rho_b}{\rho_b - \rho_a}\right)^2 \left( \int_{B_{t_b}(x_0)} \left[\frac{|u - (u_0)_{B_{t_b}(x_0)}|}{\rho_b}\right]^p dx \right)^{\frac{1}{p}}
\]
\[
\times \left( \sup_{t_b(x_0)} \left[\frac{|u - (u_0)_{B_{t_b}(x_0)}|}{\rho_b}\right]^p dx \right)^{\frac{1}{p}},
\]
\[
(b) \leq \lambda^{p-2}\left(\frac{\rho_b}{\rho_b - \rho_a}\right)^2 \left( \int_{B_{t_b}(x_0)} \left[\frac{|u - (u_0)_{B_{t_b}(x_0)}|}{\rho_b}\right]^p dx \right)^{\frac{1}{p}}
\]

where, to obtain (a), we used Sobolev’s inequality with respect to space direction and then Hölder’s inequality with respect to the time integral, and to obtain (b), we used (3.5) and (3.2).

Applying Young’s inequality, for any $y \in (0, 1)$, there holds
\[
II \leq \sqrt{\lambda^{p-2}} \sup_{t_b(x_0)} \int_{B_{t_b}(x_0)} \left[\frac{|u - (u_0)_{B_{t_b}(x_0)}|}{\rho_b}\right]^p dx + C(y) \left(\frac{\rho_b}{\rho_b - \rho_a}\right)^q \lambda^p.
\]

Estimate of III: After applying Hölder’s inequality, we get
\[
III \leq \int_{t_b(x_0)}^{0} \left( \int_{B_{t_b}(x_0)} \left[\frac{|u - (u_0)_{B_{t_b}(x_0)}|}{\rho_b}\right]^a dx \right)^{\frac{1}{a}} dt
\]
\[
(a) \leq \int_{t_b(x_0)}^{0} \left( \int_{B_{t_b}(x_0)} \left[\frac{|u - (u_0)_{B_{t_b}(x_0)}|}{\rho_b}\right]^a dx \right)^{\frac{1}{a}} dt
\]
\[
\times \left( \sup_{t_b(x_0)} \left[\frac{|u - (u_0)_{B_{t_b}(x_0)}|}{\rho_b}\right]^a dx \right)^{\frac{1}{a}}
\]
\[
(b) \leq \left( \int_{Q_b(x_0) \cap \Omega_T} \rho_b^a |f|a' dx \right)^{\frac{1}{a'}} \left( \int_{Q_b(x_0) \cap \Omega_T} \left[\frac{|u - (u_0)_{B_{t_b}(x_0)}|}{\rho_b}\right]^\delta p + |\nabla u|^\delta p dx \right)^{\frac{1}{\delta p}}
\]
\[
\times \left( \sup_{t_b(x_0) \cap \Omega_T} \left[\frac{|u - (u_0)_{B_{t_b}(x_0)}|}{\rho_b}\right]^\delta p dx \right)^{\frac{1}{\delta p}}.
\] (3.6)

Here, to obtain (a), we used Lemma 2.7 with $\sigma = a, q = \delta p, r = 2$ and $\vartheta = \frac{a}{a + r}$. To obtain (b), we used Hölder’s inequality with respect to the time integral.

Also, the restriction $\rho \leq \rho_b \leq 4\rho$ and (3.1) imply
\[
\left( \int_{Q_b(x_0) \cap \Omega_T} \rho_b^a |f|a' dx \right)^{\frac{1}{a'}} \leq \left( \int_{Q_b(x_0) \cap \Omega_T} (4\rho)^a |f|a' dx \right)^{\frac{1}{a'}} \leq \lambda^{p-2}(\frac{1}{\delta} - \frac{1}{a}) = \lambda^{p(\frac{1}{\delta} + \frac{1}{a})-1} = \lambda^{p-1}.
\]
Thus, along with (3.5) and (3.2), (3.6) becomes
\[
III \leq \lambda^{p+1}\pi^{\gamma} \left( \sup_{t \in (0,T)} \int_{B_{2\rho}(x_0)} \left| \frac{u - (u_0)_{B_{2\rho}(x_0)}}{\rho_b} \right|^2 dx \right)^{\frac{1}{n}}
\]
\[
= \lambda^{p+1}\pi^{\gamma} \left( \lambda^{p-2} \sup_{t \in (0,T)} \int_{B_{2\rho}(x_0)} \left| \frac{u - (u_0)_{B_{2\rho}(x_0)}}{\rho_b} \right|^2 dx \right)^{\frac{1}{n}},
\]
and Young's inequality gives that
\[
III \leq \frac{1}{2} \lambda^{p-2} \sup_{t \in (0,T)} \int_{B_{2\rho}(x_0)} \left| \frac{u - (u_0)_{B_{2\rho}(x_0)}}{\rho_b} \right|^2 dx + C(y)\lambda^p,
\]
where \( y \in (0, 1) \).

Estimate of IV and V:
\[
IV + V \overset{(3.2)}{\leq} \lambda^p.
\]
Combining estimates and taking \( y = y(n, N, p, \lambda_0, \Lambda_1) \in (0, 1) \) small enough, there holds
\[
\lambda^{p+1}\pi^{\gamma} \left( \sup_{t \in (0,T)} \int_{B_{2\rho}(x_0)} \left| \frac{u - (u_0)_{B_{2\rho}(x_0)}}{\rho_b} \right|^2 dx \right)^{\frac{1}{n}} + C \lambda^p \leq C(n, N, p, \Lambda_0, \Lambda_1),
\]
where \( C = C(n, N, p, \Lambda_0, \Lambda_1) \). Hence, from Lemma 2.8, the conclusion follows.

We will not use following corollary in further estimates for our purpose, but it is worthwhile observing the estimate of the corollary.

**Corollary 3.6.** **Under the assumptions and the conclusion in Lemma 3.5, we derive**
\[
\int_{Q_t^{(2)}(x_0) \cap \Omega_T} \left| \frac{u - (u_0)_{B_{2\rho}(x_0)}}{2\rho} \right|^2 dx \leq \lambda^{n+2}.
\]

**Proof.** Constants \( \sigma = \frac{p(n+2)}{n} \), \( q = p \), \( r = 2 \) and \( \theta = \frac{n}{n+2} \in (0, 1) \) satisfy the condition in Lemma 2.7. Therefore, we obtain
\[
\int_{Q_t^{(2)}(x_0) \cap \Omega_T} \left| \frac{u - (u_0)_{B_{2\rho}(x_0)}}{2\rho} \right|^2 dx \leq \frac{\lambda^{n+2}}{\lambda^{n+2}}.
\]

\[
\int_{Q_t^{(2)}(x_0) \cap \Omega_T} |\nabla u|^p dx \leq \frac{\lambda^{n+2}}{\lambda^{n+2}}.
\]

Where, to obtain (a), we used Lemma 3.4, (3.2), (3.5) and Lemma 3.5.

### 3.3 Reverse Hölder inequality in intrinsic cylinders

We are now ready to obtain a reverse Hölder inequality under assumptions (3.2) and (3.3). We shall again estimate the right-hand side of Lemma 3.2 with \( \rho_a = \rho \) and \( \rho_b = 2\rho \) using Lemma 3.5. Let us fix a constant
\[
q := \max \left\{ \frac{2n}{n+2}, p - 1, \frac{np}{n+2}, \delta p \right\}.
\]
Lemma 3.7. Let $u$ be a weak solution of (2.1). Suppose $B_{r}(x_{0}) \subset \Omega$ and $0 < t_{0}$. Also, assume (3.2) for some $\lambda \geq 1$. Then, for any $y \in (0, 1)$, there holds

$$
\lambda^{p-2} \iint_{Q_{r}^{1}(x_{0})} \left[ \frac{|u - (u_{0})_{B_{r}(x_{0})}|}{2p} \right]^{2} dz \leq \gamma \lambda^{p} + C(y) \left( \iint_{Q_{r}^{1}(x_{0}) \cap \Omega} |\nabla u|^{q} dz \right)^{\frac{p}{q}} + C(y) \iint_{Q_{r}^{1}(x_{0}) \cap \Omega} |F|^{p} dz
$$

$$
+ C(\gamma)\lambda^{\sigma(1 - \frac{n}{q})} \iint_{Q_{r}^{1}(x_{0}) \cap \Omega} (2p)^{q} |f|^{q} dz + C(y) \iint_{B_{r}(x_{0})} |u|^{p} dx.
$$

Proof. We apply Lemma 2.7 with $\sigma = 2$, $q$ as defined in (3.7), $\gamma = 2$, and any $y \in (0, 1)$ is admissible since

$$
-\frac{n}{2} \leq y \left( 1 - \frac{n}{q} \right) - (1 - \gamma) \frac{n}{2} \iff \frac{2n}{n + 2} \leq q.
$$

Let us take $y \in (0, 1)$ such that

$$
0 < y < \min \left\{ \frac{q}{2}, \frac{p}{2(p - 1)} \right\}.
$$

Then we get

$$
\iint_{Q_{r}^{1}(x_{0})} \left[ \frac{|u - (u_{0})_{B_{r}(x_{0})}|}{2p} \right]^{2} dz \leq \int_{t_{0}}^{t_{1}} \left( \iint_{B_{r}(x_{0})} \left[ \frac{|u - (u_{0})_{B_{r}(x_{0})}|}{2p} \right]^{q} + |\nabla u|^{q} dx \right)^{\frac{2p}{q}} \left( \iint_{B_{r}(x_{0})} \frac{|u - (u_{0})_{B_{r}(x_{0})}|}{2p} \right)^{2} dx^{1 - y} dt
$$

$$
\leq \lambda^{(1 - \gamma)} \left( \iint_{Q_{r}^{1}(x_{0})} \left[ \frac{|u - (u_{0})_{B_{r}(x_{0})}|}{2p} \right]^{q} + |\nabla u|^{q} dx \right)^{\frac{2p}{q}},
$$

where, to obtain (a), we used Hölder’s inequality along with the fact that $\frac{2p}{q} \leq 1$, and to obtain (b), we used Lemma 3.5.

Thus, applying Lemma 3.3 with $\theta = q$ to estimate the second term, we obtain

$$
\lambda^{p-2} \iint_{Q_{r}^{1}(x_{0})} \left[ \frac{|u - (u_{0})_{B_{r}(x_{0})}|}{2p} \right]^{2} dz \leq \lambda^{p-2\theta} \left( \iint_{Q_{r}^{1}(x_{0}) \cap \Omega} |\nabla u|^{q} dz \right)^{\frac{2p}{q}} + \lambda^{p-2\theta} \left( \iint_{Q_{r}^{1}(x_{0}) \cap \Omega} |\nabla u|^{p-1} + |F|^{p-1} dz \right)^{\frac{2p}{q}}
$$

$$
+ \lambda^{p} \left( \iint_{B_{r}(x_{0})} (2p)|f| dz \right)^{2p} + \lambda^{p-2\theta} \left( \iint_{B_{r}(x_{0})} |u|^{p} dx \right)^{2p} =: I + II + III + IV.
$$

Estimate of I: Applying Young’s inequality along with the fact $\frac{2p}{q} < \frac{2p}{p} \leq 1$, we get

$$
I \leq \gamma \lambda^{p} + C(y) \left( \iint_{Q_{r}^{1}(x_{0}) \cap \Omega} |\nabla u|^{q} dz \right)^{\frac{p}{q}}.
$$

Estimate of II: Similarly, since $\frac{2p(p-1)}{p} < 1$, Hölder’s inequality and Young’s inequality give

$$
II \leq \lambda^{p-2\theta(p-1)} \left( \iint_{Q_{r}^{1}(x_{0}) \cap \Omega} |\nabla u|^{q} + |F|^{q} dz \right)^{\frac{2p(p-1)}{p}}.
$$
Proof. We apply Lemma 2.7 with 

\[ \text{Estimate of III: Since } \frac{2q}{p} < 1, \text{ Young's inequality gives} \]

\[ \text{III } \leq \lambda^p \left( \lambda^{a(1-\theta)} \right) \left( \frac{1}{2p} \int_{Q_{2r}^1(z_0)} |f|^{a^*} \, dz \right)^{\frac{2q}{p}} \leq \lambda^p + C(y) \left( \frac{1}{2p} \int_{Q_{2r}^1(z_0)} |f|^{a^*} \, dz \right)^{\frac{2q}{p}} \]

\[ \text{Estimate of IV: Since } \frac{2q}{p} < 1, \text{ Hölder's inequality and Young's inequality give} \]

\[ \text{IV } \leq \lambda^{p-2q} \left( \frac{1}{B_{2r}(x_0)} \int_{B_{2r}(x_0)} |\nabla u_0|^q \, dx \right)^{\frac{2q}{p}} \leq \lambda^p + C(y) \left( \frac{1}{B_{2r}(x_0)} \int_{B_{2r}(x_0)} |\nabla u_0|^p \, dx \right). \]

Therefore, combining all the estimates, the proof is completed. \( \square \)

**Lemma 3.8.** Under the assumptions and the conclusion in Lemma 3.7, we further have

\[ \left( \int_{Q_{2r}^1(z_0)} \left| u - (u_0)_{B_{2r}(x_0)} \right|^p \, dz \right)^{\frac{1}{p}} \leq \lambda^p + C(y) \left( \frac{1}{Q_{2r}^1(z_0)} \int_{Q_{2r}^1(z_0)} |\nabla u|^{q(r)} \, dz \right)^{\frac{1}{q(r)}} + C(y) \left( \frac{1}{Q_{2r}^1(z_0)} \int_{Q_{2r}^1(z_0)} |f|^{a^*} \, dz \right)^{\frac{2q}{p}}. \]

Proof. We apply Lemma 2.7 with \( \sigma = p, q \) as defined in (3.7), \( r = 2 \) and \( \theta = \frac{q}{p} \). Note that (3.7) implies

\[ \frac{np}{n+2} \leq q \iff -\frac{n}{p} \leq \frac{q}{p} \left( 1 - \frac{n}{q} \right) - \left( 1 - \frac{q}{p} \right) \frac{n}{2}. \]

Therefore, we have

\[ \left( \int_{Q_{2r}^1(z_0)} \left| u - (u_0)_{B_{2r}(x_0)} \right|^p \, dz \right)^{\frac{1}{p}} \leq \int_{I_{n,p}^1(t_0)} \left( \int_{B_{2r}(x_0)} \left| u - (u_0)_{B_{2r}(x_0)} \right|^q \, dx \right)^{\frac{1}{q}} \left( \int_{B_{2r}(x_0)} \left| u - (u_0)_{B_{2r}(x_0)} \right|^2 \, dx \right)^{\frac{p-1}{2}} \, dt \]

\[ \leq \lambda^p \left( \int_{Q_{2r}^1(z_0) \cap (0,T)} \left| u - (u_0)_{B_{2r}(x_0)} \right|^q \, dz \right)^{\frac{1}{q}} \left( \int_{B_{2r}(x_0)} \left| u - (u_0)_{B_{2r}(x_0)} \right|^2 \, dx \right)^{\frac{p-1}{2}} \]

\[ \leq \lambda^p \left( \int_{Q_{2r}^1(z_0) \cap (0,T)} \left| u - (u_0)_{B_{2r}(x_0)} \right|^q \, dz \right)^{\frac{1}{q}} \left( \int_{B_{2r}(x_0)} \left| u - (u_0)_{B_{2r}(x_0)} \right|^2 \, dx \right)^{\frac{p-1}{2}} \]

\[ = \lambda^p \left( \int_{Q_{2r}^1(z_0) \cap (0,T)} \left| u - (u_0)_{B_{2r}(x_0)} \right|^q \, dz \right)^{\frac{1}{q}} + \left| \nabla u \right|^q \, dz. \]
Here, to obtain (a), we used $\frac{p\theta}{q} = 1$, and to obtain (b), we used Lemma 3.5. Now, we apply Lemma 3.3 with $\theta = q$ to the first term on the right-hand side to get

$$
\iint_{Q^I_{2p} (x_0)} \left| u - (u_0)_{B_{2p}(x_0)} \right|^p \, dz \leq \lambda^{p - q} \iint_{Q^I_{2p} (x_0) \cap \Omega_T} |\nabla u|^q \, dz + \lambda^{p - q} \left( \lambda^{2 - p} \iint_{Q^I_{2p} (x_0) \cap \Omega_T} |\nabla u|^{p - 1} + |f|^{p - 1} \, dz \right)^q 
$$

$$
+ \lambda^p \left( \lambda^{1 - p} \iint_{Q^I_{2p} (x_0) \cap \Omega_T} (2p |f|) \, dz \right)^q + \lambda^{p - q} \left( \frac{\lambda}{B_{2p}(x_0)} \right| \nabla u_0 | \, dx \right)^q 
$$

$$
= : I + II + III + IV.
$$

Estimate of I: Applying Young’s inequality, there holds

$$
I \leq \gamma \lambda^p + C(\gamma) \left( \iint_{Q^I_{2p} (x_0) \cap \Omega_T} |\nabla u|^q \, dz \right)^\frac{p}{q}.
$$

Estimate of II: Applying Hölder’s inequality and Young’s inequality, we get

$$
II \leq \lambda^{p + q - p} \left( \iint_{Q^I_{2p} (x_0) \cap \Omega_T} |\nabla u|^q + |F|^q \, dz \right)^\frac{p}{p-q} 
$$

$$
\leq \lambda^{p + q - p} \left( \iint_{Q^I_{2p} (x_0) \cap \Omega_T} |\nabla u|^q + |F|^q \, dz \right)^\frac{p}{q} \left( \iint_{Q^I_{2p} (x_0) \cap \Omega_T} |\nabla u|^p + |F|^p \, dz \right)^\frac{(q-1)(p-1)}{q}
$$

$$
\leq \lambda^{p} \left( \iint_{Q^I_{2p} (x_0) \cap \Omega_T} |\nabla u|^q + |F|^q \, dz \right)^\frac{p}{q} + C(\gamma) \iint_{Q^I_{2p} (x_0) \cap \Omega_T} |F|^p \, dz.
$$

Estimate of III: We observe

$$
III \leq \lambda^p \left( \lambda^{a'(1-p)} \iint_{Q^I_{2p} (x_0) \cap \Omega_T} (2p)^a |f|^a \, dz \right)^\frac{q}{a'} 
$$

$$
= \lambda^p \left( \lambda^{a'(1-p)} \iint_{Q^I_{2p} (x_0) \cap \Omega_T} (2p)^a |f|^a \, dz \right)^\frac{1}{a'} \left( \lambda^{a'(1-p)} \iint_{Q^I_{2p} (x_0) \cap \Omega_T} (2p)^a |f|^a \, dz \right)^\frac{a-1}{a'} 
$$

$$
\leq \lambda^p \left( \lambda^{a'(1-p)} \iint_{Q^I_{2p} (x_0) \cap \Omega_T} (2p)^a |f|^a \, dz \right)^\frac{1}{a'} 
$$

$$
= \lambda^p \left( \lambda^{a'(1-p)} \iint_{Q^I_{2p} (x_0) \cap \Omega_T} (2p)^a |f|^a \, dz \right)^\frac{1}{a'}.
$$

Therefore, Young’s inequality gives

$$
III \leq \gamma \lambda^p + C(\gamma) \lambda^{a'(1-\frac{d}{a'})} \iint_{Q^I_{2p} (x_0) \cap \Omega_T} (2p)^a |f|^a \, dz.
$$

Estimate of IV: Hölder’s inequality and Young’s inequality imply

$$
IV \leq \lambda^{p-q} \left( \int_{B_{2p}(x_0)} |\nabla u_0|^p \, dx \right)^\frac{q}{p} \leq \gamma \lambda^p + C(\gamma) \int_{B_{2p}(x_0)} |\nabla u_0|^p \, dx.
$$

We combine all the estimates to complete the proof.
Lemma 3.9. Under the assumptions and the conclusion in Lemma 3.7, we further have
\[
\mathbb{E}(I) = \mathbb{E}(\mathcal{Q}_n) + \mathbb{E}(\mathcal{Q}_n) + \mathbb{E}(\mathcal{Q}_n) + \mathbb{E}(\mathcal{Q}_n).
\]

Proof. Apply Lemma 2.7 as in (3.6) and Lemma 3.5 to get
\[
\mathbb{E}(I) \leq \lambda^{p-\frac{n}{p}} \left( \mathbb{E}(\mathcal{Q}_n) + \mathbb{E}(\mathcal{Q}_n) \right) + \lambda^{p-\frac{n}{p}} \left( \lambda^{p-1} \mathbb{E}(\mathcal{Q}_n) \right)
\]
\[
+ \lambda^{p} \left[ \lambda^{1-p} \left( \mathbb{E}(\mathcal{Q}_n) \right) + \mathbb{E}(\mathcal{Q}_n) \right] \mathbb{E}(\mathcal{Q}_n).
\]

Using Lemma 3.3 with \( \theta = q \) to the first term on the right-hand side, we have
\[
\mathbb{E}(I) \leq \lambda^{p-\frac{n}{p}} \left( \mathbb{E}(\mathcal{Q}_n) + \mathbb{E}(\mathcal{Q}_n) \right) + \lambda^{p-\frac{n}{p}} \left( \lambda^{p-1} \mathbb{E}(\mathcal{Q}_n) \right)
\]
\[
+ \lambda^{p} \left[ \lambda^{1-p} \left( \mathbb{E}(\mathcal{Q}_n) \right) + \mathbb{E}(\mathcal{Q}_n) \right] \mathbb{E}(\mathcal{Q}_n).
\]

Estimate of I: Applying Young’s inequality, we obtain
\[
I \leq \gamma \lambda^p + C(y) \left( \mathbb{E}(\mathcal{Q}_n) + \mathbb{E}(\mathcal{Q}_n) \right)^{\frac{p}{q}}.
\]

Estimate of II: Note that \( \left( \frac{p-1}{n+2} \right) \mathcal{Q}_n < 1. \) Apply Hölder’s inequality and Young’s inequality to get
\[
II \leq \lambda^{p-\frac{n}{p}} \left( \mathbb{E}(\mathcal{Q}_n) + \mathbb{E}(\mathcal{Q}_n) \right) \mathbb{E}(\mathcal{Q}_n)
\]
\[
\leq \gamma \lambda^p + C(y) \left( \mathbb{E}(\mathcal{Q}_n) \right)^{\frac{p}{q}} + C(y) \left( \mathbb{E}(\mathcal{Q}_n) \right)^{\mathbb{E}(\mathcal{Q}_n)}.
\]

Estimate of III: Applying Hölder’s inequality and Young’s inequality, we have
\[
III \leq \lambda^{p-(n+2)} \left( \lambda^{p-\frac{n}{p}} \left( \mathbb{E}(\mathcal{Q}_n) \right) + \mathbb{E}(\mathcal{Q}_n) \right) \mathbb{E}(\mathcal{Q}_n)
\]
\[
\leq \gamma \lambda^p + C(y) \left( \mathbb{E}(\mathcal{Q}_n) \right)^{\frac{p}{q}} + C(y) \left( \mathbb{E}(\mathcal{Q}_n) \right)^{\mathbb{E}(\mathcal{Q}_n)}.
\]

Estimate of IV: Again, Hölder’s inequality and Young’s inequality give
\[
IV \leq \gamma \lambda^p + C(y) \left( \mathbb{E}(\mathcal{Q}_n) \right)^{\mathbb{E}(\mathcal{Q}_n)}.
\]

The proof follows.
Lemma 3.10. Let u be a weak solution of (2.1). Suppose \( B_{\rho_0}(x_0) \subset \Omega \) and \( 0 < t_0 \). Also, assume (3.2) and (3.3) for some \( \lambda \geq 1 \). Then there holds

\[
\iint_{Q_{\lambda}^1(z_0) \cap \Omega_T} |\nabla u|^p \, dz \leq (n,N,p,\Lambda_0,\Lambda_1,\delta) \left( \iint_{Q_{\lambda}^1(z_0) \cap \Omega_T} |\nabla u|^q \, dz \right)^{\frac{p}{q}} + \iint_{Q_{\lambda}^1(z_0) \cap \Omega_T} |F|^p \, dz \\
+ \iint_{B_{\rho_0}(x_0)} |\nabla u_0|^p \, dx + \lambda^{\alpha(1-\frac{q}{p})} \iint_{Q_{\lambda}^1(z_0) \cap \Omega_T} (2p)^a |f|^a \, dz.
\]

Proof. From Lemma 3.2 with \( \rho_a = \rho \) and \( \rho_b = 2\rho \), there holds

\[
\iint_{Q_{\lambda}^1(z_0) \cap \Omega_T} |\nabla u|^p \, dz \leq \iint_{Q_{\lambda}^1(z_0) \cap \Omega_T} \left[ \frac{|u - (u_0)_{B_{\rho_0}(x_0)}|}{2\rho} \right]^p \, dz + \lambda^{p-2} \iint_{Q_{\lambda}^1(z_0) \cap \Omega_T} \left[ \frac{|u - (u_0)_{B_{\rho_0}(x_0)}|}{2\rho} \right]^2 \, dz \\
+ \iint_{Q_{\lambda}^1(z_0) \cap \Omega_T} |f|^p \, dz \\
+ \iint_{Q_{\lambda}^1(z_0) \cap \Omega_T} |F|^p \, dz.
\]

We apply Lemma 3.7, Lemma 3.8 and Lemma 3.9 to I, II and III. Use Young’s inequality to estimate IV. Then there holds

\[
\lambda^p \leq \iint_{Q_{\lambda}^1(z_0) \cap \Omega_T} |\nabla u|^p \, dz + \iint_{Q_{\lambda}^1(z_0) \cap \Omega_T} |F|^p \, dz + \iint_{B_{\rho_0}(x_0)} |\nabla u_0|^p \, dx + \lambda^{\alpha(1-\frac{q}{p})} \iint_{Q_{\lambda}^1(z_0) \cap \Omega_T} (2p)^a |f|^a \, dz
\]

Taking \( y = y(n,N,p,\Lambda_0,\Lambda_1,\delta) \) small enough, we finish the proof. \( \square \)

4 Proof of Theorem 2.5

In this section, we prove the main results. First of all, we shall find intrinsic cylinders such that (3.2) and (3.3) hold.

Definition 4.1. Let \( z_0 \in \Omega_T, r \leq r_1 < r_2 < 2r \) and \( \zeta = (\tau, t) \in Q_{r_1}(z_0) \), and let \( d \) and \( \alpha \) be defined in Definition 2.2.

\[
E_{1} := \{ z \in Q_{r_1}(z_0) \cap \Omega_T : |\nabla u|^p > \lambda^p \},
\]

\[
B := \left[ 2 \left( \frac{20r}{2r - r_1} \right)^{\frac{n+2}{2}} + 2 \left( \frac{20r}{2r - r_1} \right)^{\frac{n+2}{2}} \right] \left( \frac{2r - r_1}{r_2 - r_1} \right) + \left( \frac{20r}{2r - r_1} \right)^{\frac{n+2}{2}} ,
\]

\[
G(Q_{\rho}^1(z)) := \iint_{Q_{\rho}^1(z) \cap \Omega_T} |\nabla u|^p \, dz + \iint_{Q_{\rho}^1(z) \cap \Omega_T} |F|^p \, dz + \iint_{B_{\rho}(z)} |\nabla u_0|^p \, dx + \lambda^{\alpha(1-\frac{q}{p})} \iint_{Q_{\rho}^1(z) \cap \Omega_T} (2p)^a |f|^a \, dz.
\]

Lemma 4.2. Let \( \lambda_0 \) be defined in Definition 2.4. Then, for \( \lambda > 3B\lambda_0 \) and \( \zeta = (\tau, t) \in E_{1} \), there exists \( \rho_3 \in (0, r_2 - r_1) \) such that

\[
G(Q_{\rho_3}^1(z)) = \lambda^p \quad \text{and} \quad G(Q_{\rho_3}^1(z)) < \lambda^p \quad \text{for all } \rho \in (\rho_3, r_2 - r_1). \tag{4.1}
\]
Proof. Due to intrinsic geometry, we split the proof into two cases.

Case $p \geq 2$: For any $\frac{r \rho}{2} < \rho < r_2 - r_1$, there holds

\[
G(Q^4_{\rho}(z)) \leq \frac{|Q_{2\rho}(z) \cap \Omega_T|}{|Q^4_{\rho}(z) \cap \Omega_T|} \iint_{Q_{2\rho}(z) \cap \Omega_T} ((|\nabla u| + |F| + 1)^p \, dz \\
+ \frac{|Q_{2\rho}(z) \cap \Omega_T|}{2 Q^4_{\rho}(z) \cap \Omega_T} \iint_{Q_{2\rho}(z) \cap \Omega_T} (2r)^{n^2} |a^\nu| \, dz + \frac{|B_{2\rho}(x_0)|}{|B_\rho(z)|} \int_{B_{2\rho}(x_0)} |\nabla u_0|^p \, dx.
\]

Note that (3.1) implies

\[
\frac{|Q_{2\rho}(z) \cap \Omega_T|}{|Q^4_{\rho}(z) \cap \Omega_T|} \leq \frac{|Q_{2\rho}(z) \cap \Omega_T|}{2 |Q^4_{\rho}(z) \cap \Omega_T|} \leq 2 \left( \frac{2r}{\rho} \right)^{n^2} \lambda^{p-2} \lambda^{\frac{n}{2}}, \quad |\nabla u_0|^p \leq \lambda_0^p.
\]

(4.2)

From Remark 2.3, we have $\frac{p}{\alpha^2} = 2 - a'(1 - \frac{p}{2}) > 0$, $d = \frac{p}{2}$,

\[
\iint_{Q_{2\rho}(z) \cap \Omega_T} ((|\nabla u| + |F| + 1)^p \, dz \leq |Q_{2\rho}(z) \cap \Omega_T| \leq \lambda_0^p, \quad \int_{B_{2\rho}(x_0)} |\nabla u_0|^p \, dx \leq \lambda_0^p,
\]

It follows that

\[
G(Q^4_{\rho}(z)) \leq 2 \left( \frac{20r}{r_2 - r_1} \right)^{n^2} \lambda^{p-2} \left( \lambda_0^p + \lambda^{a'(1-\frac{p}{2})} \lambda_0^{2-a'(1-\frac{p}{2})} \right) + \left( \frac{20r}{r_2 - r_1} \right)^n \lambda_0^p < \lambda_0^p.
\]

On the other hand, since $j \in E_\lambda$, there exists $r_3 \in (0, \frac{r_2 - r_1}{10})$ such that (4.1) holds.

Case $p < 2$: Let us denote $\rho_1 := \lambda^{\frac{2-p}{2}} r$. Then, for any $\frac{r \rho_1}{10} < \rho < r_2 - r_1$, there holds

\[
G(Q^4_{\rho_1}(z)) \leq \frac{|Q_{2\rho_1}(z) \cap \Omega_T|}{|Q^4_{\rho_1}(z) \cap \Omega_T|} \iint_{Q_{2\rho_1}(z) \cap \Omega_T} ((|\nabla u| + |F| + 1)^p \, dz \\
+ \frac{|Q_{2\rho_1}(z) \cap \Omega_T|}{2 Q^4_{\rho_1}(z) \cap \Omega_T} \iint_{Q_{2\rho_1}(z) \cap \Omega_T} (2r)^{n^2} |a^\nu| \, dz + \frac{|B_{2\rho_1}(z_0)|}{|B_\rho(z)|} \int_{B_{2\rho_1}(x_0)} |\nabla u_0|^p \, dx.
\]

Here, we used the fact that $\rho_1 \leq \lambda^{\frac{2-p}{2}} r$ for the second term. Note that

\[
\frac{|Q_{2\rho_1}(z) \cap \Omega_T|}{|Q^4_{\rho_1}(z) \cap \Omega_T|} \leq \frac{|Q_{2\rho_1}(z) \cap \Omega_T|}{2 |Q^4_{\rho_1}(z) \cap \Omega_T|} \leq 2 \left( \frac{20r}{r_2 - r_1} \right)^{n^2} \lambda^{\frac{2-p}{2} + \frac{n}{2}}.
\]

Again, from Remark 2.3, we have $\frac{p}{\alpha^2} = \frac{p(n+2)-2a}{2} - a'(\frac{p}{2} - \frac{p}{2}) > 0$, $\frac{p}{2} = \frac{p(n+2)-2a}{2}$,

\[
\int_{Q_{2\rho_1}(z) \cap \Omega_T} ((|\nabla u| + |F| + 1)^p \, dz \leq \lambda_0^p, \quad \int_{B_{2\rho_1}(x_0)} |\nabla u_0|^p \, dx \leq \lambda_0^p
\]

It follows that

\[
G(Q^4_{\rho_1}(z)) \leq 2 \left( \frac{20r}{r_2 - r_1} \right)^{n^2} \lambda^{\frac{2-p}{2} + \frac{n}{2}} \lambda_0^p + \lambda^{a'(\frac{p}{2} - \frac{p}{2})} \lambda_0^{2-a'(\frac{p}{2} - \frac{p}{2})} + \left( \frac{20r}{r_2 - r_1} \right)^n \lambda_0^p < \lambda_0^p.
\]

On the other hand, since $j \in E_\lambda$, there exists $r_3 \in (0, \frac{r_2 - r_1}{10})$ such that (4.1) holds. The lemma follows. \qed

We now define upper level sets.

**Definition 4.3.** Let $\eta > 0$ and $\lambda > 3B\lambda_0$. We define the following:

- $\Phi^{\eta}_{\lambda} := \{ z \in Q_{\rho}(z) \cap \Omega_T : |\nabla u|^p(z) > \eta \lambda^p \}$,
- $\Psi^{\eta}_{\lambda} := \{ z \in Q_{\rho}(z) \cap \Omega_T : H(z) > \eta \lambda^p \}$ where $H(z) := |F(z)| + |\nabla u_0(x)| + 1$,
- $\Sigma^0_{\lambda} := \{ z \in Q_{\rho}(z) \cap \Omega_T : |f_\eta(z) - \eta \lambda^p \}$, where $f_\eta := 2^{a'} \eta^{-a'} \lambda_0^p (2r)^n |F(z)|^{a'}$.

Our covering argument is divided into three steps.
Step 1: Let \( \lambda > 3B\lambda_0 \) and \( \xi \in E_\lambda \). Assumptions (3.2) in \( Q^1_{\lambda p_0}(\zeta) \) and (3.3) in \( Q^1_{\rho_3}(\zeta) \) are satisfied by (4.1). Applying Lemma 3.10, there exists \( q < p \) defined in (3.7) such that

\[
G(Q^1_{\rho_3}(\zeta)) \leq \left( \iiint_{Q^1_{\rho_3}(\zeta) \cap \Omega_T} |\nabla u|^q \, dz \right)^{\frac{q}{q'}} + \iiint_{Q^1_{\rho_3}(\zeta) \cap \Omega_T} (|F| + 1)^p \, dz
\]

\[
+ \iiint_{Q^1_{\rho_3}(\zeta) \cap \Omega_T} |\nabla u_0|^p \, dx + \lambda^\xi(1-\frac{\xi}{2}) \iiint_{Q^1_{\rho_3}(\zeta) \cap \Omega_T} (2p_3)^{\xi} |f| \, dz \Longleftrightarrow I + II + III + IV.
\]

Estimate of I + II + III: Let \( \eta \in (0, 1) \) to be chosen later. There holds

\[
I \leq \eta \lambda^p + \frac{1}{|Q^1_{2\rho_3}(\zeta) \cap \Omega_T|} \iiint_{Q^1_{2\rho_3}(\zeta) \cap \Omega_T} |\nabla u|^q \, dz \leq \eta \lambda^p + \frac{1}{|Q^1_{2\rho_3}(\zeta) \cap \Omega_T|} \iiint_{Q^1_{2\rho_3}(\zeta) \cap \Omega_T} |\nabla u|^q \, dz \left( \iiint_{Q^1_{\rho_3}(\zeta) \cap \Omega_T} |\nabla u|^q \, dz \right)^{\frac{q}{q-1}} \leq \eta \lambda^p + \frac{1}{|Q^1_{2\rho_3}(\zeta) \cap \Omega_T|} \iiint_{Q^1_{2\rho_3}(\zeta) \cap \Omega_T} \lambda^{p-\eta} |\nabla u|^q \, dz.
\]

Here, to obtain (a), we used Hölder’s inequality and (4.1). Therefore, we get

\[
I + II + III \leq \eta \lambda^p + \frac{1}{|Q^1_{2\rho_3}(\zeta) \cap \Omega_T|} \iiint_{Q^1_{2\rho_3}(\zeta) \cap \Omega_T} \lambda^{p-\eta} |\nabla u|^q \, dz + \frac{1}{|Q^1_{2\rho_3}(\zeta) \cap \Omega_T|} \iiint_{Q^1_{2\rho_3}(\zeta) \cap \Omega_T} |H|^p \, dz.
\]

Estimate of IV: Let us consider the alternative

\[
\lambda^{\xi(1-\frac{p}{q})} \iiint_{Q^1_{2\rho_3}(\zeta) \cap \Omega_T} (2p_3)^{\xi} |f| \, dz \leq \eta \lambda^p \quad \text{or} \quad \lambda^{\xi(1-\frac{p}{q})} \iiint_{Q^1_{\rho_3}(\zeta) \cap \Omega_T} (2p_3)^{\xi} |f| \, dz \geq \eta \lambda^p. \quad (4.3)
\]

Leaving the first case of (4.3), suppose the second case holds.

Case \( p \geq 2 \): Applying (4.2), we have

\[
\eta \lambda^p \leq 2 \lambda^{\xi(1-\frac{p}{q})+p-\frac{q}{2}} \left( \frac{r}{\rho_3} \right)^{n+2-a'} \iiint_{Q_{2r}(\zeta_0) \cap \Omega_T} (2r|f|) \, dz \leq 2 \lambda^{\xi(1-\frac{p}{q})+p-\frac{q}{2}} \lambda_0^\xi \left( \frac{r}{\rho_3} \right)^{n+2-a'} \cdot (4.4)
\]

Since Definition 2.2 implies the inequality

\[
n + 2 - a' > 1 \Longleftrightarrow n + 1 > a' \Longleftrightarrow \frac{n+1}{n} = (n+1) < a = \delta \frac{p(n+2)}{n} \Longleftrightarrow \frac{n+1}{p(n+2)} < \delta, \quad (4.5)
\]

we see that (4.4) becomes

\[
\frac{\rho_3}{r} \leq 2\eta^{-1} \lambda^{\xi(1-\frac{p}{q})+p-\frac{q}{2}} \lambda_0^{-\frac{a'}{n+2-a'}} \cdot (4.6)
\]

Therefore, we get

\[
\lambda^{\xi(1-\frac{p}{q})} \iiint_{Q^1_{\rho_3}(\zeta) \cap \Omega_T} (2p_3)^{\xi} |f| \, dz = \lambda^{\xi(1-\frac{p}{q})} \left( \frac{\rho_3}{r} \right)^{a'} \iiint_{Q^1_{\rho_3}(\zeta) \cap \Omega_T} (2r)^{a'} |f| \, dz
\]

\[
\leq 2^{a'} \eta^{-a'} \lambda^{\xi(1-\frac{p}{q})+p-\frac{q}{2}+a'(1-\frac{p}{q})-\frac{q}{2}} \lambda_0^{-\frac{a'}{n+2-a'}} \iiint_{Q^1_{\rho_3}(\zeta) \cap \Omega_T} (2r)^{a'} |f| \, dz
\]

\[
\leq 2^{a'} \eta^{-a'} \lambda_0^{\xi(1-\frac{p}{q})} \iiint_{Q^1_{\rho_3}(\zeta) \cap \Omega_T} (2r)^{a'} |f| \, dz. \quad (a)
\]
Here, to obtain (a), we used $\lambda \geq \lambda_0$ and
\[
\frac{a'}{n + 2 - a'} \left(1 - \frac{1}{8}\right) + \frac{a'}{n + 2 - a'} \frac{p}{d a} = a' \left(1 - \frac{p}{a}\right) + \frac{a'}{n + 2 - a'} \left( a' \left(1 - \frac{p}{a}\right) - \frac{p}{d} \right) + \frac{a'}{n + 2 - a'} \frac{p}{d a} = a' \left(1 - \frac{p}{a}\right).
\]

It follows that both cases of (4.3) give
\[
\lambda^{a' (1 - \frac{1}{8})} \iint_{Q_{2\rho_3}(z_0) \cap \Omega_T} (2\rho_3)^{a'} |f|^{a'} dz \leq \eta \lambda^p + 2\rho_3^{a'} \eta^{-a'} \lambda_0^{(1 - \frac{p}{a})} \iint_{Q_{2\rho_3}(z_0) \cap \Omega_T} (2r)^{a'} |f|^{a'} dz.
\]

Case $p < 2$: Since $\rho_3 = \lambda^{\frac{p}{a' - p}} \check{\rho}_3$ for some $\check{\rho}_3 \in (0, \frac{r_0}{10})$, (4.4) becomes
\[
\eta \lambda^p \leq 2\lambda^{a' (1 - \frac{1}{8}) + \frac{a'}{n + 2 - a'} + a' \left(\frac{p}{a} - \frac{p}{10}\right)} \lambda_0^{\frac{p}{a' - p}} (\frac{r}{\check{\rho}_3})^{n + 2 - a'} = 2\lambda^{a' (\xi - \frac{p}{a}) + \frac{a'}{n + 2 - a'} \eta^{-a'} \lambda_0 \left(\frac{r}{\check{\rho}_3}\right)^{n + 2 - a'},
\]
and thus (4.5) gives
\[
\left(\frac{\check{\rho}_3}{r}\right) \leq 2\eta^{-1} \lambda^{-\frac{a'(1 - \frac{1}{8})}{n + 2 - a'} - \frac{a'}{n + 2 - a'} \eta^{-a'} \lambda_0 \left(\frac{r}{\check{\rho}_3}\right)^{n + 2 - a'}.
\]

Analogously, the second case of (4.3) implies
\[
\lambda^{a' (1 - \frac{1}{8})} \iint_{Q_{2\rho_3}(z_0) \cap \Omega_T} (2\rho_3)^{a'} |f|^{a'} dz = \lambda^{a' (\xi - \frac{p}{a})} \left(\frac{\rho_3}{r}\right)^{a'} \iint_{Q_{2\rho_3}(z_0) \cap \Omega_T} (2r)^{a'} |f|^{a'} dz.
\]

Here, to obtain (a), we used $\lambda \geq \lambda_0$ and
\[
\frac{a'}{n + 2 - a'} \left(1 - \frac{1}{8}\right) + \frac{a'}{n + 2 - a'} \frac{p}{d a} = a' \left(1 - \frac{p}{a}\right) + \frac{a'}{n + 2 - a'} \left( a' \left(1 - \frac{p}{a}\right) - \frac{p}{d} \right) + \frac{a'}{n + 2 - a'} \frac{p}{d a} = a' \left(1 - \frac{p}{a}\right),
\]

Therefore, we have
\[
\lambda^{a' (1 - \frac{1}{8})} \iint_{Q_{2\rho_3}(z_0) \cap \Omega_T} (2\rho_3)^{a'} |f|^{a'} dz \leq \eta \lambda^p + \frac{2\rho_3^{a'} \eta^{-a'} \lambda_0^p}{|Q_{2\rho_3}(z_0) \cap \Omega_T|} \iint_{Q_{2\rho_3}(z_0) \cap \Omega_T} (2r)^{a'} |f|^{a'} dz.
\]

Combining all the estimates, we get
\[
G(Q_{\rho_3}(z_0)) \leq \eta \lambda^p + \frac{1}{|Q_{2\rho_3}(z_0) \cap \Omega_T|} \iint_{Q_{2\rho_3}(z_0) \cap \Omega_T} \lambda^{p - q} |\nabla u|^q dz
\]
\[
+ \frac{1}{|Q_{2\rho_3}(z_0) \cap \Omega_T|} \iint_{Q_{2\rho_3}(z_0) \cap \Omega_T} |H|^p dz
\]
\[
+ \frac{2\rho_3^{a'} \eta^{-a'} \lambda_0^p}{|Q_{2\rho_3}(z_0) \cap \Omega_T|} \iint_{Q_{2\rho_3}(z_0) \cap \Omega_T} (2r)^{a'} |f|^{a'} dz.
\]
Taking \( \eta = \eta(n, N, p, \Lambda_0, \Lambda_1, \delta) \in (0, 1) \) small enough, we obtain

\[
\iint_{Q_{10\rho_i}(z_0) \cap \Omega_T} |\nabla u|^p \, dz \leq \frac{1}{|Q_{10\rho_i}(z_0) \cap \Omega_T|} \iint_{Q_{10\rho_i}(z_0) \cap \Omega_T} \lambda^{p-q} |\nabla u|^q \, dz
\]

\[
+ \frac{1}{|Q_{2\rho_i}(z_0) \cap \Omega_T|} \iint_{Q_{2\rho_i}(z_0) \cap \Omega_T} |H|^p \, dz + \lambda \iint_{Q_{2\rho_i}(z_0) \cap \Omega_T} (2r)^{p} |f|^{p} \, dz.
\]

Therefore, for any \( \lambda > 3 \beta_0 \) and \( i \in E_A \), there holds

\[
\iint_{Q_{10\rho_i}(z_0) \cap \Omega_T} |\nabla u|^p \, dz \leq \sum_{1 \leq i < \infty} \iint_{Q_{10\rho_i}(z_0) \cap \Omega_T} \lambda^{p-q} |\nabla u|^q \, dz + \iint_{\Phi_{\lambda}^{q}} |H|^p \, dz + \iint_{\Sigma_{\lambda}^{q}} |f|^{p} \, dz.
\]

where, to obtain (a), we used (4.6) for each \( i \) and disjointness of \( \{Q_{\rho_i}(z_0)\}_{i \in \mathbb{N}} \) in \( \mathbb{R}^{n+1} \).

Also, since there holds

\[
\iint_{\Phi_{\lambda}^{q}} |\nabla u|^p \, dz \leq \iint_{\Phi_{\lambda}^{q}} \lambda^{p-q} |\nabla u|^q \, dz,
\]

it follows

\[
\iint_{\Phi_{\lambda}^{q}} |\nabla u|^p \, dz \leq \iint_{\Phi_{\lambda}^{q}} \lambda^{p-q} |\nabla u|^q \, dz + \iint_{\Sigma_{\lambda}^{q}} |f|^{p} \, dz.
\]

Letting \( \lambda_1 := 3 \eta \frac{1}{\beta} \beta_0 \), for any \( \lambda > \lambda_1 \), we have

\[
\iint_{\Phi_{\lambda}^{q}} |\nabla u|^p \, dz \leq \iint_{\Phi_{\lambda}^{q}} \lambda^{p-q} |\nabla u|^q \, dz + \iint_{\Sigma_{\lambda}^{q}} |f|^{p} \, dz.
\]

**Step 3:** For \( k > \lambda_1 \), let us define

\[
|\nabla u|_k := \min(|\nabla u|, k) \quad \text{and} \quad \Phi_{\lambda, k} := \{ z \in Q_\rho(z_0) \cap \Omega_T : |\nabla u|^p_k > \lambda^p \}.
\]

We see that if \( \lambda > k \), then \( \Phi_{\lambda, k} = \emptyset \), and if \( \lambda \leq k \), then \( \Phi_{\lambda, k} = \Phi_{\lambda}^{q} \). From (4.7), we deduce

\[
\iint_{\Phi_{\lambda, k}^{q}} |\nabla u|^p \, dz \leq \iint_{\Phi_{\lambda, k}^{q}} \lambda^{p-q} |\nabla u|^q \, dz + \iint_{\Sigma_{\lambda}^{q}} |f|^{p} \, dz.
\]

Let \( \epsilon > 0 \) to be chosen later. Multiply (4.8) by \( \lambda^{\epsilon-1} \) and integrate over \( (\lambda_1, \infty) \) to get

\[
I := \int_{\lambda_1}^{\infty} \lambda^{\epsilon-1} \iint_{\Phi_{\lambda, k}^{q}} |\nabla u|^p \, dz \, d\lambda \leq \int_{\lambda_1}^{\infty} \lambda^{\epsilon-1} \iint_{\Phi_{\lambda, k}^{q}} \lambda^{p-q} |\nabla u|^q \, dz \, d\lambda + \int_{\lambda_1}^{\infty} \lambda^{\epsilon-1} \iint_{\Sigma_{\lambda}^{q}} |f|^{p} \, dz \, d\lambda
\]

\[
+ \int_{\lambda_1}^{\infty} \lambda^{\epsilon-1} \iint_{\Sigma_{\lambda}^{q}} |f|^{p} \, dz \, d\lambda = II + III + IV.
\]
Estimate of I: Applying Fubini’s theorem, we get
\[ I = \iint_{\Phi^{1,1}_{i,k}} |\nabla u|^p \, d\lambda \, dz = \frac{1}{\varepsilon} \iint_{\Phi^{1,1}_{i,k}} |\nabla u|^p \, d\lambda \, dz - \frac{1}{\varepsilon} \int_{\Phi^{1,1}_{i,k}} |\nabla u|^p \, d\lambda \, dz. \]

Estimate of II: Again, using Fubini’s theorem, we obtain
\[ II = \iint_{\Phi^{1,1}_{i,k}} |\nabla u|^q \, d\lambda \, dz \leq \frac{1}{p-q} \iint_{\Phi^{1,1}_{i,k}} |\nabla u|^p \, d\lambda \, dz. \]

Estimate of III: Again, by Fubini’s theorem, we have
\[ III = \iint_{\Psi^{1}_{i}} |H|^p \, d\lambda \, dz \leq \frac{1}{\varepsilon} \iint_{\Psi^{1}_{i}} |H|^{p+\varepsilon} \, d\lambda \, dz. \]

Estimate of IV: Similarly, we have
\[ IV = \iint_{\Psi^{2}_{i}} |\tilde{f}_{\eta}|^\frac{1}{\varepsilon} \, d\lambda \, dz \leq \frac{1}{\varepsilon} \iint_{\Psi^{2}_{i}} |\tilde{f}_{\eta}|^{1+\frac{1}{\varepsilon}} \, d\lambda \, dz. \]

It follows that (4, 9) becomes
\[ \iint_{Q_1(z_0 \cap \Omega_1 \setminus \Phi^{1,1}_{i,k})} |\nabla u|^p \, d\lambda \, dz \leq C \frac{\varepsilon}{p-q} \iint_{Q_1(z_0 \cap \Omega_1 \setminus \Phi^{1,1}_{i,k})} |\nabla u|^p \, d\lambda \, dz + \lambda_1 \iint_{Q_1(z_0 \cap \Omega_1 \setminus \Phi^{1,1}_{i,k})} |\nabla u|^p \, d\lambda \, dz \]
\[ + C \iint_{Q_1(z_0 \cap \Omega_1 \setminus \Phi^{1,1}_{i,k})} |H|^{p+\varepsilon} \, d\lambda \, dz + \iint_{\Psi^{1}_{i}} |\tilde{f}_{\eta}|^{1+\frac{1}{\varepsilon}} \, d\lambda \, dz, \]

where \( C = C(n, N, p, \Lambda_0, \Lambda_1, \delta) \).

Since there holds
\[ \iint_{Q_1(z_0 \cap \Omega_1 \setminus \Phi^{1,1}_{i,k})} |\nabla u|^p \, d\lambda \, dz \leq C \frac{\varepsilon}{p-q} \iint_{Q_1(z_0 \cap \Omega_1 \setminus \Phi^{1,1}_{i,k})} |\nabla u|^p \, d\lambda \, dz + C \lambda_0 \iint_{Q_1(z_0 \cap \Omega_1 \setminus \Phi^{1,1}_{i,k})} |\nabla u|^p \, d\lambda \, dz \]
\[ + C \iint_{Q_1(z_0 \cap \Omega_1 \setminus \Phi^{1,1}_{i,k})} |H|^{p+\varepsilon} \, d\lambda \, dz + \iint_{\Psi^{1}_{i}} |\tilde{f}_{\eta}|^{1+\frac{1}{\varepsilon}} \, d\lambda \, dz, \]

Take \( \varepsilon_0 = \varepsilon_0(n, N, p, \Lambda_0, \Lambda_1, \delta) \) so that \( C \frac{\varepsilon_0}{p-q} = \frac{1}{2} \). Then, for any \( \varepsilon \in (0, \varepsilon_0) \), it follows that
\[ \iint_{Q_1(z_0 \cap \Omega_1 \setminus \Phi^{1,1}_{i,k})} |\nabla u|^p \, d\lambda \, dz \leq \frac{1}{2} \iint_{Q_1(z_0 \cap \Omega_1 \setminus \Phi^{1,1}_{i,k})} |\nabla u|^p \, d\lambda \, dz + C \lambda_0 \iint_{Q_1(z_0 \cap \Omega_1 \setminus \Phi^{1,1}_{i,k})} |\nabla u|^p \, d\lambda \, dz \]
\[ + C \iint_{Q_1(z_0 \cap \Omega_1 \setminus \Phi^{1,1}_{i,k})} |H|^{p+\varepsilon} \, d\lambda \, dz + \iint_{\Psi^{1}_{i}} |\tilde{f}_{\eta}|^{1+\frac{1}{\varepsilon}} \, d\lambda \, dz. \]

Hence, applying Lemma 2.8, we get
\[ \iint_{Q_1(z_0 \cap \Omega_1 \setminus \Phi^{1,1}_{i,k})} |\nabla u|^p \, d\lambda \, dz \leq \lambda_0 \iint_{Q_1(z_0 \cap \Omega_1 \setminus \Phi^{1,1}_{i,k})} |\nabla u|^p \, d\lambda \, dz + \iint_{Q_1(z_0 \cap \Omega_1 \setminus \Phi^{1,1}_{i,k})} |H|^{p+\varepsilon} \, d\lambda \, dz + \iint_{\Psi^{1}_{i}} |\tilde{f}_{\eta}|^{1+\frac{1}{\varepsilon}} \, d\lambda \, dz. \]

Let \( k \to \infty \) to derive the desired estimate. This completes the proof.
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