SCATTERING RESULTS FOR DIRAC HARTREE-TYPE EQUATIONS WITH SMALL INITIAL DATA

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Abstract. We consider the Dirac equations with cubic Hartree-type nonlinearity which are derived by uncoupling the Dirac-Klein-Gordon systems. We prove small data global well-posedness and scattering results in the full scaling subcritical regularity regime. The strategy of the proof relies on the localized Strichartz estimates and bilinear estimates in $V^2$ spaces, together with the use of the null structure that the nonlinear term exhibits. This result is shown to be almost optimal in the sense that the iteration method based on Duhamel’s formula fails over the supercritical range.

1. Introduction. We consider the following Hartree-type Dirac equation

$(-i\partial_t + \alpha \cdot D + m\beta)\psi = \lambda (V * \langle \psi, \beta\psi \rangle_{C^1})\beta\psi,$

$\psi(0, \cdot) = \psi_0 \in H^s(\mathbb{R}^3) \text{ (or } \dot{H}^s(\mathbb{R}^3),)$

where $D = -i\nabla$, $\lambda \in \mathbb{R} \setminus \{0\}$ and $\psi : \mathbb{R}^3 \to \mathbb{C}^4$ is the Dirac spinor regarded as a column vector. Here, $\beta$ and $\alpha = (\alpha^1, \alpha^2, \alpha^3)$ are the $4 \times 4$ Dirac matrices given by

$\beta = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}, \quad \alpha^j = \begin{pmatrix} 0 & \sigma^j \\ \sigma^j & 0 \end{pmatrix},$

where for $j = 1, 2, 3$ the Pauli matrices $\sigma^j$ are

$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$

The constant $m \geq 0$ is a physical mass parameter and the symbol $*$ denotes the convolution in $\mathbb{R}^3$. In this paper, we consider generalized potentials $V$ which are defined as follows:

Definition 1.1. For $0 \leq \gamma \leq 2$, the potential $V_\gamma$ satisfies $\hat{V}_\gamma \in C^4(\mathbb{R}^3 \setminus \{0\})$ and the growth condition such that for $k = 0, 1, \cdots, 4$

$|\nabla^k \hat{V}_\gamma(\xi)| \lesssim |\xi|^{-\gamma-k}$ for $|\xi| \leq 1,$ \quad $|\nabla^k \hat{V}_\gamma(\xi)| \lesssim |\xi|^{-2-k}$ for $|\xi| > 1.$ \quad (3)

Here, $\hat{V}_\gamma$ denotes the Fourier transform of $V_\gamma$, i.e., $\hat{V}_\gamma(\xi) = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} e^{-i\xi \cdot x} V_\gamma(x) dx$.

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The Yukawa potential $V(x) = e^{-\mu|x|}/|x|$ for $\mu > 0$ and the Coulomb potential $V(x) = 1/|x|$ are well-known examples of $V_γ$ defined in Definition 1.1 which correspond to $γ = 0$ and $γ = 2$, respectively. The equation (1) with Yukawa or Coulomb potential is derived from uncoupling the following Dirac-Klein-Gordon system

$$\begin{cases}
(-i\partial_t + \alpha \cdot D + m\beta)\psi = \phi \beta \psi \\
(\partial_t^2 - \Delta + M^2)\phi = \langle \psi, \beta \psi \rangle_{C^4}.
\end{cases} \tag{4}$$

Precisely, suppose that a scalar field $\phi$ is a standing wave, i.e., $\phi(t, x) = e^{i\lambda t} f(x)$. Then the Klein-Gordon part of (4) becomes

$$(-\Delta - \lambda^2 + M^2)\phi = \langle \psi, \beta \psi \rangle_{C^4}. \tag{5}$$

One immediately verifies that the solutions of (5) are given by

$$\phi = \begin{cases}
c_1 \frac{1}{4\pi|x|} * \langle \psi, \beta \psi \rangle_{C^4}, & \text{if } \lambda = \pm M, \\
c_2 e^{-\mu|x|\sqrt{M^2 - \lambda^2}} / |x| * \langle \psi, \beta \psi \rangle_{C^4}, & \text{if } M > |\lambda|,
\end{cases}$$
for some constants $c_1$ and $c_2$. Putting this into the Dirac part in (4), a spinor $\psi$ results in (1) with potential $V_γ$ for $γ = 2$ and $γ = 0$, respectively. We generalize these two potentials derived from the concrete physical model into $V_γ$ in Definition 1.1 in mathematical viewpoint.

We investigate the global behavior of solutions to (1), especially scattering problem when the initial data is sufficiently small. The massless ($m = 0$) Dirac equation with the Coulomb potential admits the scaling symmetry: If $\psi$ is a solution to (1), then so is $\psi_a(t, x) = a^2 \psi(at, ax)$, and hence the scale invariant data space is $\psi_0 \in L^2(\mathbb{R}^3)$. Solutions to (1) also satisfy conservation of charge $\|\psi(t)\|_{L^2} = \|\psi(0)\|_{L^2}$. Thus, the equation (1) is referred to being charge critical. Our goal is to show the well-posedness and scattering results in full subcritical range ($s > 0$) and ill-posedness in supercritical range ($s < 0$).

The previous researches have been established on the following Hartree-type Dirac equation

$$(-i\partial_t + \alpha \cdot D + m\beta)\psi = \lambda \frac{e^{-\mu|x|}}{|x|} * |\psi|^2 \psi, \mu \geq 0 \tag{6}$$

where the difference from (1) lies in the nonlinear term, namely, the way to define conjugation in the inner product:

$$\langle \psi, \beta \psi \rangle_{C^4} = |\psi_1|^2 + |\psi_2|^2 + |\psi_3|^2 + |\psi_4|^2,$$

$$\langle \psi, \beta \psi \rangle_{C^4} = |\psi_1|^2 + |\psi_2|^2 - |\psi_3|^2 - |\psi_4|^2. \tag{7}$$

The equation (6) with Coulomb potential ($\mu = 0$) appears when the Maxwell-Dirac system with zero magnetic fields is uncoupled [3]. On the other hand, concerning the Yukawa potential ($\mu > 0$), it is conjectured in the same paper [3] that, as in the Maxwell-Dirac case, the equation might be derived from uncoupling the Dirac-Klein-Gordon system (4). Next, Let us mention the Cauchy problem for (6). The first result was obtained in [8], where the authors showed the existence of weak solutions to (6) with the Yukawa potential when mass is zero. Later, in [12], the authors established the small data scattering result for $\psi_0 \in H^s$, $s > \frac{1}{2}$ when the potential is the Yukawa and mass is nonzero. For the same equation, the space regularity threshold could be lowered to $s > \frac{1}{4}$ in [22] by taking angular regularity
into account. We note that all the above results carry over to the equation (1) under the same potential and mass condition since the proofs of the previous results did not enjoy any advantages coming from the cancellation in the nonlinear term (7).

The Cauchy problem for the Dirac equations with other nonlinear terms, \( (|x|^{-\gamma} * |\psi|^{p-1}) \psi \) for 0 < \( \gamma < 3 \) and \( p \geq 3 \) was intensively studied in [16, 18]. Finally, we refer to [1] where the small data scattering result in scaling critical spaces was shown for the Dirac equation with a cubic nonlinear term \( \langle \psi, \beta \psi \rangle \subset \mathbb{C} \beta \psi \). See also the references therein.

In the paper [12], it is suggested that the regularity threshold in their results \( s > \frac{1}{2} \) can be lowered if the null structure of the nonlinear Dirac equation is taken into account, which has initiated our interest in this research. We employ the method developed in [7, 2] diagonalizing the linear Dirac operator and improve the previous results when the cancellation in the nonlinear term thanks to the method developed in [7, 2] diagonalizing the linear Dirac operator and improve the previous results when the cancellation in the nonlinear term thanks to \( \beta \) is exhibited. Thus, we focus on the equation (1) from now on.

The Duhamel’s principle gives the solution to (1) of the form

\[
\psi(t) = U_m(t, D)\psi_0 + \int_0^t U_m(t - \tau, D)(V * (\psi, \beta\psi))(\tau)\beta\psi(\tau)d\tau, \tag{8}
\]

where the linear propagator is defined on \( L^2(\mathbb{R}^3; \mathbb{C}^4) \) by

\[
U_m(t, D) = I \cos \left(t(m^2 - \Delta)^{\frac{1}{2}}\right) - (\alpha + i\beta)(m^2 - \Delta)^{-\frac{1}{2}} \sin \left(t(m^2 - \Delta)^{\frac{1}{2}}\right). \tag{9}
\]

One can check that the essential part of the linear propagator \( U_m(t) \) is \( e^{\pm it(m^2 - \Delta)^{\frac{1}{2}}} \). From this one infers that the proofs of well-posedness and scattering results for the semirelativistic equation with Coulomb Hartree-type nonlinearity would also be applicable to the Dirac equation with the same potential. We review the previous results for the semirelativistic equations, written by

\[
i\partial_t u = \sqrt{m - \Delta} u + \lambda(|x|^{-1} * |u|^2)u, \tag{10}
\]

for \( u : \mathbb{R}^3 \to \mathbb{C}, m > 0 \) and \( \lambda \in \mathbb{R} \setminus \{0\} \). The global well-posedness of (10) in the energy spaces \( H^s \) was established in [15], and this global result was extended to generalized potentials \( |x|^{-\gamma}, 0 < \gamma < 3 \) in [4]. We remark that similar local well-posedness results are only expected to hold for the Dirac equation since the energy functional for the Dirac operator is not positive definite. In [11] the authors proved local well-posedness with low regularity \( s > \frac{1}{4} \) via the Fourier restriction norm method. Concerning the scattering, a negative result that solutions to (10) never scatter even in \( L^2 \) was proved in [4], and modified scattering in the weighted spaces was obtained in [19].

We use the notation \( H^s_m(\mathbb{R}^3) \) to denote homogeneous (or inhomogeneous) Sobolev spaces \( \dot{H}^s(\mathbb{R}^3) \) (or \( H^s(\mathbb{R}^3) \)) if \( m = 0 \) (or \( m > 0 \)), respectively:

\[
H^s_m(\mathbb{R}^3) := \begin{cases} 
\dot{H}^s(\mathbb{R}^3) = \{ |D|^s f \in L^2(\mathbb{R}^3) \} & \text{if } m = 0, \\
H^s(\mathbb{R}^3) = \{ |D|^s f \in L^2(\mathbb{R}^3) \} & \text{if } m > 0.
\end{cases}
\]

Now let us state the first main theorem.

**Theorem 1.2** (Scattering result). Let \( m \geq 0 \) and \( \gamma \) be such that

\[
\begin{cases} 
0 \leq \gamma < 2, & \text{if } m = 0, \\
0 \leq \gamma < 1, & \text{if } m > 0.
\end{cases}
\]
Then for $s > 0$ there exists a $\delta > 0$ such that for all $\psi_0 \in H^s_m(\mathbb{R}^3)$ satisfying
\[
\|\psi_0\|_{H^s_m(\mathbb{R}^3)} \leq \delta,
\]
there exists a global solution $\psi \in C(\mathbb{R}, H^s_m(\mathbb{R}^3))$ to the Dirac equation (1). Furthermore, the solution scatters in $H^s_m(\mathbb{R}^3)$ in the sense that there exists $\psi_{sc} \in H^s_m(\mathbb{R}^3)$ such that
\[
\|\psi(t) - U_m(t, D)\psi_{sc}\|_{H^s_m(\mathbb{R}^3)} \to 0
\]
as time $t$ goes to infinity.

The proof of the main theorem is based on the perturbation method. We employ the standard contraction mapping argument and show that the effect of the nonlinear effect is negligible if the initial data is sufficiently small. Main tasks are to construct appropriate function spaces and to estimate the nonlinear terms based on Duhamel’s formula in the spaces to perform the contraction argument. We find the function spaces by employing the $U^2$ spaces adapted to the linear propagator combined with the dyadic decomposition. The nonlinear term estimates are shown by using the localized linear and bilinear estimates. The null structure exhibited by nonlinearity plays an important role especially when we find bilinear estimates, in contrast to the previous results, and it enables us to prove the scattering result in full subcritical range.

We note that our main theorem covers the Yukawa potential ($\gamma = 0$) for both massless and massive cases, but on the other hand we fail to attack the Coulomb case ($\gamma = 2$). The Hartree type nonlinearity for $\gamma > 0$ has a singularity arising from the potential near zero in the frequency side, and the larger $\gamma$ becomes, the worse the singularity is. We are able to get over this bad behavior for some restricted range of $\gamma$, especially for small $\gamma$. More precisely, we obtain the scattering results for massless case except for the case $\gamma = 2$. Here the null structure in the nonlinearity plays a key role, but for the massive case, the null structure does not provide any gain in estimates near the singularity, which makes the range of $\gamma$ more limited to $0 \leq \gamma < 1$.

We mentioned that in [4] they proved the nonexistence of scattering in $L^2$ for the semirelativistic equations (10) where the potential is Coulomb. The authors proved that time decay of the $L^2$ norm of the nonlinearity in (10) computed on a free solution is given by $t^{-1}$. It is a key estimate in the proof that the linear propagator decays with $t^{-\frac{3}{2}}$ rate as time evolves. Since the massive Dirac operator has the same time decay [6, Appendix], we can check by slightly modifying their argument that nonexistence of a scattering state also holds for (1) with $m > 0$ and Coulomb potential $\gamma = 2$. We expect that the similar negative result for the massless and Coulomb Dirac equation also might be established, but some new idea seems to be required due to the weaker time decay $t^{-1}$ of massless Dirac propagator.

For the remaining cases for $m > 0$ and $1 \leq \gamma < 2$ we might overcome the singularity and obtain the scattering result if we choose an initial data in radial symmetry spaces or weighted function spaces. We will consider it in future work.

In the supercritical range $s < 0$, we prove the ill-posedness result by adapting the argument in [17]. Thus our result is almost sharp with respect to regularity. We give an additional assumption on the potential $V$ that $\tilde{V}$ should be positive, which seems to be reasonable since the Coulomb and Yukawa clearly satisfy it. The specific free waves that cause the ill-posedness of (1) for $s < 0$ are supported in high frequency region, so it holds for both massive and massless case.
Theorem 1.3. Let $s < 0$ and $T > 0$. We further assume that $\hat{V}$ is positive. Then the flow map $\psi_0 \mapsto \psi$ from $H^s_m(\mathbb{R}^3)$ to $C([0,T]; H^s_m(\mathbb{R}^3))$ cannot be $C^3$ at the origin.

The critical case $s = 0$ remains open for $V_\gamma$ with $0 \leq \gamma < 2$. In future work, we will attack the critical case with some additional angular regularity assumption on initial data.

1.1. Organization of the paper. In section 2, we introduce the notations and preliminaries, the structure of Dirac equations, and $U^p, V^p$ function spaces. In section 3, we prove the main estimates, namely the localized linear and bilinear estimates. First, estimates for the linear solutions are proved, from which the ones for nonlinear solutions are obtained by the transfer principle. Applying the estimates the proof of well-posedness theorem is established in Section 4 based on the standard contraction mapping argument. In section 5, we show the ill-posedness result for the initial data given in negative Sobolev spaces.

Remark 1. While writing on this paper, the author has learned that similar scattering result for (1) with $\gamma = 0$ was independently proved by Tesfahun [21].

2. Notations and preliminaries.

2.1. Notation. Let $\rho \in C^\infty_0(-2,2)$ be real-valued and even function satisfying $\rho(s) = 1$ for $|s| \leq 1$. For $\varphi(\xi) := \rho(|\xi|) - \rho(2|\xi|)$ define $\varphi_k = \varphi(2^{-k}|\xi|)$. Then, $\sum_{k \in \mathbb{Z}} \varphi_k = 1$ on $\mathbb{R}^3 \setminus \{0\}$ and it is locally finite. We define the (spatial) Fourier localization operator $P_k f = F^{-1}(\varphi_k F f)$. Further, we define $\varphi_{\leq k} = \sum_{k' \in \mathbb{Z}, k' \leq k} \varphi_{k'}$ and $P_{\leq k} f = F^{-1}(\varphi_{\leq k} F f), \quad P_{> k} f = f - P_{\leq k} f$. Let $\widetilde{\varphi}_k = \varphi_{k-1} + \varphi_k + \varphi_{k+1}$ and $\widetilde{P}_k f = F^{-1}(\widetilde{\varphi}_k F f)$. Then $\widetilde{P}_k P_k = P_k \widetilde{P}_k = P_k$.

Next, we define the Fourier localization operator $P^m_k$ depending on the mass $m \geq 0$ because our linear propagators show different behaviors in the low frequency part whether the mass $m$ is zero or not. When we consider the case $m = 0$, $P^m_k f = P_k f$ for all $k \in \mathbb{Z}$. On the other hand, if $m > 0$ the operator depends on the range of $k$;

$$P^m_k f = \begin{cases} 0 & \text{if } k < 0, \\ P_{\leq 0} f & \text{if } k = 0, \\ P_k f & \text{if } k > 0. \end{cases}$$

We also define the operator $\widetilde{P}^m_k$ similarly depending on $m$.

We make a further decomposition involving an angular variable as described in [20, Chapter IX, Section 4]. For each $l \in \mathbb{N}$ we consider an equally spaced set of points with grid length $2^{-l}$ on the unit sphere $S^2$, that is we fix a collection $\Omega_l := \{\xi^n_l\}_\nu$ of unit vectors that satisfy $|\xi^n_l - \xi^n_{l'}| \geq 2^{-l}$ if $\nu \neq \nu'$ and for each $\xi \in S^2$ there exists a $\xi^n_l$ such that $|\xi - \xi^n_l| < 2^{-l}$. Let $K^n_l$ denote the corresponding cone in the $\xi$-space whose central direction is $\xi^n_l$, i.e., $K^n_l = \{\xi : |\xi^n_l - \xi^n_{l'}| \leq 2 \cdot 2^{-l}\}$. Define $\rho^n_l = \rho(2^{l}(\xi^n_{l'} - \xi^n_l))$ and $\kappa^n_l = \rho^n_l \cdot (\sum \rho^n_{l'})^{-1}$. Then $\kappa^n_l$ is a smooth partition of unity subordinate to the covering of $\mathbb{R}^3 \setminus \{0\}$ with the cone $K^n_l$ such that each $\kappa^n_l$ is supported in $2K^n_l$ is homogeneous of degree 0 and satisfies $\sum_{\nu \in \Omega_l} \kappa^n_l = 1, \quad |\partial^n_l \kappa^n_l(\xi)| \leq A_\alpha 2^{\alpha l} |\xi|^{-|\alpha|}, \text{ for all } \xi \neq 0.$
Let \( \tilde{\kappa}_l \) be satisfying similar properties but having slightly bigger support such that \( \kappa^\prime_l \kappa_l = 1 \). We define fourier multiplier \( \tilde{K}_l f := \mathcal{F}^{-1}(\kappa^\prime_l \mathcal{F} f) \) and \( \tilde{K}_l^\prime f := \mathcal{F}^{-1}(\kappa^\prime_l \mathcal{F} f) \). Then we have \( I = \sum_{\nu} \tilde{K}_l \) and \( \tilde{K}_l^\prime = \tilde{K}_l^\prime \tilde{K}_l = \tilde{K}_l^\prime \).

### 2.2. Dirac operator and null structures

To exploit the null structure effectively, we need to diagonalize the equation (\ref{eq:1}). Following [7, 1] let us introduce the projection operators \( \Pi_\pm^m(D) \) with symbols

\[
\Pi_\pm^m(\xi) = \frac{1}{2} [I \pm \frac{1}{(\xi)_m} (\xi \cdot \alpha + m\beta)], \quad \text{where} \quad (\xi)_m = \begin{cases} (m^2 + |\xi|^2)^{\frac{1}{2}} & \text{if } m > 0, \\ |\xi| & \text{if } m = 0. \end{cases}
\]

Then we have the identity

\[
\alpha \cdot D + m\beta = \langle D \rangle_m (\Pi_+^m(D) - \Pi_-^m(D)).
\]

Using the relation we decompose the linear propagator into two parts which propagate to opposite directions

\[
U_m(t, D) = e^{-it(D)m} \Pi_+^m(D) - e^{it(D)m} \Pi_-^m(D),
\]

where \( e^{\pm it(D)m} \) is the Fourier multiplier operator given by \( \mathcal{F}_\pm (e^{\pm it(D)m}) \psi(\xi) = e^{\pm it(\xi)_m} \hat{\psi}(\xi) \). Indeed, this can be easily shown since the projection operators are idempotent

\[
\Pi_\pm^m(D) \Pi_\mp^m(D) = \Pi_\pm^m(D), \quad \text{and} \quad \Pi_\pm^m(D) \Pi_\mp^m(D) = 0. \tag{11}
\]

We denote briefly \( \psi_\pm := \Pi_\pm^m(D) \psi \) and split \( \psi = \psi_+ + \psi_- \). By applying the operators \( \Pi_\pm^m(D) \) to the equation (\ref{eq:1}) we obtain the following system of equations

\[
\begin{align*}
(-i\partial_t + \langle D \rangle_m)\psi_+ &= \Pi_+^m(D)[(V * \langle \psi, \beta \psi \rangle_{C^1}) \beta \psi], \quad \psi_+^0 \in H^s(\mathbb{R}^3) \\
(-i\partial_t - \langle D \rangle_m)\psi_- &= \Pi_-^m(D)[(V * \langle \psi, \beta \psi \rangle_{C^1}) \beta \psi], \quad \psi_-^0 \in H^s(\mathbb{R}^3)
\end{align*} \tag{12}
\]

where \( \psi_0^\pm = \Pi_\pm^m(D) \psi_0 \). The linear solutions to (12) are given by \( e^{-it(D)m} \psi_0^+ \) and \( e^{it(D)m} \psi_0^- \) respectively.

We investigate the nonlinear term by applying the projection operator \( \Pi_\pm^m(D) \). We decompose \( \langle \psi, \beta \psi \rangle \) as

\[
\langle \psi, \beta \psi \rangle = \langle \Pi_\pm^m(D) \psi, \beta \Pi_\pm^m(D) \psi \rangle + \langle \Pi_+^m(D) \psi, \beta \Pi_-^m(D) \psi \rangle + \langle \Pi_-^m(D) \psi, \beta \Pi_+^m(D) \psi \rangle.
\]

Observe that \( \overline{\alpha \beta}^T = \alpha \beta \), which implies \( \overline{\Pi_\pm^m(\xi)} = \Pi_\mp^m(\xi) \). We can change the order of \( \beta \) and \( \Pi_\pm^m(D) \) as follows:

\[
\beta \Pi_\pm^m(D) = \left( \Pi_\mp^m(D) \pm \frac{m\beta}{(D)_m} \right) \beta. \tag{13}
\]

Using these we compute the Fourier transform for \( \theta_1, \theta_2 \in \{+1, -1\} \)

\[
\mathcal{F}(\Pi_{\theta_1}^m(D) \psi_1, \beta \Pi_{\theta_2}^m(D) \psi_2)_{C^4}(\xi) = \mathcal{F}(\Pi_{\theta_1}^m(D) \psi_1, \left( \Pi_{\theta_2}^m(D) + \theta_2 \frac{m\beta}{(D)_m} \right) \beta \psi_2)_{C^4}(\xi) = \int \langle \Pi_{\theta_2}^m(\eta - \xi) \hat{\psi}_1(\eta), \beta \hat{\psi}_2(\eta - \xi) \rangle_{C^4} d\eta + \theta_2 \mathcal{F}(\Pi_{\theta_1}^m(D) \psi_1, \frac{m}{(D)_m} \psi_2)_{C^4}(\xi).
\]

We analyze the symbols from the bilinear operator and explain the role of null structure.
Lemma 2.1. For \( m \geq 0 \) the following holds true for \( \xi, \eta \in \mathbb{R}^3 \)
\[
|\Pi_m^+(\xi)\Pi_m^-(\eta)| \lesssim C_1 \angle(\xi, \eta) + mC_2 \left( \frac{1}{(m+1)} + \frac{1}{m} \right),
\]
\[
|\Pi_m^+(\xi)\Pi_m^-(\eta)| \lesssim C_1 \angle(-\xi, \eta) + mC_2 \left( \frac{1}{(m+1)} + \frac{1}{m} \right),
\]
where the constants \( C_1, C_2 \) are independent of \( m \). Here, \( \angle(\xi, \eta) \) denotes the smaller angle between two vectors \( \xi \) and \( \eta \).

Proof. See [7, Lemma 2] for \( m = 0 \) and [1, Lemma 2.1] for \( m > 0 \).

Hence, whenever the angle between \( \pm \xi \) and \( \eta \) is small, the structure of the bilinear form is significantly improved. For later use, we formulate the bound of symbols when the frequency and angular localization are concerned.

Lemma 2.2. Let \( m \geq 0, \theta_1, \theta_2 \in \{+1, -1\} \) and \( l \in \mathbb{N} \). Suppose \( \xi^{\nu}_l, \xi'^{\nu}_l \in \Omega_l \) with \( |	heta_1 \xi^{\nu}_l - \theta_2 \xi'^{\nu}_l| \leq 2^{-1} \) and \( v, w \in \mathbb{C}^4 \). Then,
1. if \( m = 0 \) and \( k_1, k_2 \in \mathbb{Z} \),
\[
|\langle \Pi_{\theta_1}^m(2^{k_1} \xi^{\nu}_l)v, \beta \Pi_{\theta_2}^m(2^{k_2} \xi'^{\nu}_l)w \rangle_{\mathbb{C}^4}| \lesssim 2^{-l} |\Pi_{\theta_1}^m(2^{k_1} \xi^{\nu}_l)v| |\Pi_{\theta_2}^m(2^{k_2} \xi'^{\nu}_l)w|,
\]
2. if \( m > 0 \) and \( k_1, k_2 \in \mathbb{N}_0 \)
\[
|\langle \Pi_{\theta_1}^m(2^{k_1} \xi^{\nu}_l)v, \beta \Pi_{\theta_2}^m(2^{k_2} \xi'^{\nu}_l)w \rangle_{\mathbb{C}^4}| \lesssim (2^{-l} + 2^{-m \min(k_1, k_2)}) |\Pi_{\theta_1}^m(2^{k_1} \xi^{\nu}_l)v| |\Pi_{\theta_2}^m(2^{k_2} \xi'^{\nu}_l)w|.
\]

Proof. Since the projection operators are idempotent (11), we may assume \( v = \Pi_{\theta_1}^m(2^{k_1} \xi^{\nu}_l)v \) and \( w = \Pi_{\theta_2}^m(2^{k_2} \xi'^{\nu}_l)w \). The identity (13) gives
\[
\langle \Pi_{\theta_1}^m(2^{k_1} \xi^{\nu}_l)v, \beta \Pi_{\theta_2}^m(2^{k_2} \xi'^{\nu}_l)w \rangle_{\mathbb{C}^4} = \langle \Pi_{\theta_2}^m(2^{k_2} \xi'^{\nu}_l)\Pi_{\theta_1}^m(2^{k_1} \xi^{\nu}_l)v, \beta w \rangle_{\mathbb{C}^4} + \theta_2 \frac{m}{2^{k_2}} \langle \Pi_{\theta_1}^m(2^{k_1} \xi^{\nu}_l)v, \Pi_{\theta_2}^m(2^{k_2} \xi'^{\nu}_l)w \rangle_{\mathbb{C}^4}.
\]
Then (14) and (15) follow from the Lemma 2.1 and Cauchy-Schwarz inequality.

2.3. Function spaces. In this subsection we introduce \( U^p, V^p \) function spaces. For the general theory, see e.g. [14, 9, 10].

Let \( 1 \leq p < \infty \). We call a finite set \( \{t_0, \ldots, t_J\} \) a partition if \(-\infty < t_0 < t_1 < \ldots < t_j \leq \infty \), and denote the set of all partitions by \( \mathcal{T} \). A corresponding step-function \( a : \mathbb{R} \to L^2(\mathbb{R}^3) \) is called \( U^p \)-atom if
\[
a(t) = \sum_{j=1}^{J} 1_{[t_{j-1}, t_j]}(t) f_j, \quad \sum_{j=1}^{J} \| f_j \|_{L^2(\mathbb{R}^3)} = 1, \quad \{t_0, \ldots, t_J\} \in \mathcal{T},
\]
and \( U^p \) is the atomic space. The norm is defined by
\[
\| u \|_{U^p} := \inf \left\{ \sum_{k=1}^{\infty} |\lambda_k| : u = \sum_{k=1}^{\infty} \lambda_k a_k, \text{ where } a_k \text{ are } U^p \text{-atoms and } \lambda_k \in \mathbb{C} \right\}.
\]
Further, let \( V^p \) be the space of all right-continuous \( v : \mathbb{R} \to L^2(\mathbb{R}^3) \) satisfying
\[
\| v \|_{V^p} := \sup_{\{t_0, \ldots, t_J\} \in \mathcal{T}} \left( \sum_{j=1}^{J} \| v(t_j) - v(t_{j-1}) \|_{L^2(\mathbb{R}^3)} \right)^{\frac{1}{p}}.
\]
with the convention \( v(t_j) = 0 \) if \( t_j = \infty \). Likewise, let \( V^p_{\text{loc}} \) denote the spaces of all functions \( v : \mathbb{R} \to L^2(\mathbb{R}^3) \) satisfying \( v(-\infty) = 0 \) and \( \| v \|_{V^p} < \infty \), equipped with the norm (16). We define \( V^p_{\text{loc}, \text{rc}} \) by the closed subspace of all right continuous \( V^p \) functions.
Lemma 2.3. Let $1 \leq p < q < \infty$ and $p'$ denote the Hölder conjugate.

1. $U^p, V^p, V^p_{\text{loc}}$ and $V^p_{\text{loc}, r\xi}$ are Banach spaces.
2. The embeddings $U^p \hookrightarrow V^p_{\text{loc}, r\xi} \hookrightarrow U^q \hookrightarrow L^\infty(\mathbb{R}; L^2)$ are continuous.
3. The embeddings $V^p \hookrightarrow V^q$ and $V^p \hookrightarrow V^q$ are continuous.
4. For $u \in U^p$, $u(-\infty) := \lim_{t \to -\infty} u(t)$ and $u(\infty) := \lim_{t \to \infty} u(t)$ exist.
5. (Duality) For $1 < p < \infty$, $\|u\|_{U^p} = \sup_{v \in V^{p'}} \langle v, u \rangle_{V^p', V^p}$ for free solutions to the ones for functions in atomic spaces.

Observe that the properties in Lemma 2.3 also hold for the adapted spaces $U^p$ and $V^p_{\text{loc}}$. By the atomic structure of $U^p$ we can relate the estimates for free solutions to the ones for functions in atomic spaces.

Lemma 2.5 (Transfer principle). Let $\theta_k \in \{+1, -1\}$ for $k = 1, \ldots, n$. Let $T : L^2 \to L^1_{\text{loc}}(\mathbb{R}^3; \mathbb{C})$ be a multi-linear operator satisfying that

$$\|T(e^{\theta_1 \text{i} t (D)} f_1, \ldots, e^{\theta_n \text{i} t (D)} f_n)\|_{L^p_{\xi} X} \leq C \prod_{k=1}^n \|f_k\|_{L^2}$$

for some $1 < p < \infty$ and a Banach space $X \subset L^1_{\text{loc}}(\mathbb{R}^3; \mathbb{C})$. Then,

$$\|T(u_1, \ldots, u_n)\|_{L^p_{\xi} X} \leq C \prod_{k=1}^n \|u_k\|_{U^p_{\theta_k (D)} m}.$$

3. Linear and bilinear estimates.

3.1. Estimates for free solutions. In this subsection we prove localized linear and bilinear estimates for free solutions to Klein-Gordon equation, $e^{\pm \text{i} t (D)} f$ with $f : \mathbb{R} \to \mathbb{C}^4$. First, let us introduce the Strichartz estimates.

Lemma 3.1 (Klein-Gordon Strichartz estimates). Let $m > 0$. Suppose $2 \leq p, q \leq \infty$, $\frac{2}{p} + \frac{2}{q} = \frac{3}{2}$ and $(p, q) \neq (2, \infty)$. Then

$$\|e^{\pm \text{i} t (D)} P_k^m f\|_{L^p_t L^q_x(\mathbb{R}^{1+3})} \lesssim \langle 2^k \rangle_m^{\frac{1}{2} - \frac{1}{q}} \|f\|_{L^2_\xi(\mathbb{R}^3)}$$

(17)

Proof. See [5].

Next, we introduce refinement of linear estimates after making the localization to cubes. For detailed explanation and proof, see [2, Lemma 3.1] and references therein. We obey the notation in [2]. For the convenience of reader we explicitly organize here. For $k \in \mathbb{Z}$ let us consider the lattice point $L_k = 2^k \mathbb{Z}^3$. Let $\eta : \mathbb{R} \to [0, 1]$ be an even smooth function supported in the interval $[-\frac{3}{2}, \frac{1}{2}]$ with the property that $\sum_{n \in \mathbb{Z}} \eta(\xi - n) = 1$ for $\xi \in \mathbb{R}$. Let $\gamma : \mathbb{R}^3 \to [0, 1], \gamma(\xi) = \eta(\xi_1) \eta(\xi_2) \eta(\xi_3)$, where $\xi = (\xi_1, \xi_2, \xi_3)$. For $k \in \mathbb{Z}$ and $n \in L_k$ let $\gamma_{k,n}(\xi) = \gamma(\frac{\xi - n}{2^k})$. Clearly, $\sum_{n \in L_k} \gamma_{k,n} \equiv 1$ on $\mathbb{R}^3$. We define the projection operator $\Gamma_{k,n}$ by $\Gamma_{k,n} f = F^{-1}(\gamma_{k,n} F f)$. Then
We claim that the kernel time convolution operator with the kernel $\tilde{T}$ uniformly in $(\text{Localized Strichartz estimates})$

Lemma 3.2 \hspace{1em} \text{Let} \, m \geq 0. \, \text{Suppose} \, \frac{1}{p} + \frac{1}{q} = \frac{1}{2} \, \text{with} \, 2 < p < \infty. \, \text{Then}

$$\sup_{k' \leq k \in \mathbb{Z}} 2^{-\frac{k}{2}} 2^{-\frac{k'}{p}} \left( \sum_{n \in L_{k'}} \| \Gamma_{k', n} \, P_k^m \, e^{\pm it(D)} \, f \|_{L^p_{t} L^q_x} \right)^{\frac{1}{2}} \lesssim \| f \|_{L^2(\mathbb{R}^3)}. \quad (18)$$

**Proof.** By orthogonality it suffices to show that

$$\| \Gamma_{k', n} \, P_{\leq 0} e^{\pm it(D)} \, f \|_{L^p_{t} L^q_x} \lesssim 2^{\frac{k'}{p}} \| f \|_{L^2(\mathbb{R}^3)}.$$\hspace{1em} (19)

This case follows from Bernstein’s inequality and Strichartz estimates (17)

$$\| \Gamma_{k', n} \, P_{\leq 0} e^{\pm it(D)} \, f \|_{L^p_{t} L^q_x} \lesssim 2^{\frac{k'}{p}} \| f \|_{L^2(\mathbb{R}^3)}.$$\hspace{1em} (18)

Since (19) is trivial if $m > 0$ and $k = 0$, from now we consider the remaining cases where $m > 0$ and $k \geq 1$, or $m = 0$ and $k \in \mathbb{Z}$.

We follow the main stream of the proof in [2, Lemma 3.1]. Let $T_1$ be the operator on $L^2(\mathbb{R}^3)$ into $L^p_{t} L^q_x(\mathbb{R}^{1+3})$ defined by $T_1 = \Gamma_{k', n} \, P^m \, e^{\pm it(D)}$. The $T_1T_1^*$ is a space-time convolution operator with the kernel

$$K_{k', k; n}(t, x) = \int_{\mathbb{R}^3} e^{\pm it(\xi) + \xi \cdot x} r_x^{-2} \, f_k(\xi) \, \xi^{-2} \, d\xi.$$\hspace{1em} (20)

We claim that the kernel $K_{k', k; n}$ satisfies the following estimates

$$|K_{k', k; n}(t, x)| \lesssim 2^{k'} \, 2^k |t|^{-1}, \quad (20)$$

uniformly in $x$ and $n \in L_k$. Suppose for a moment (20) holds. By the standard $TT^*$ argument to prove (19) is equivalent to showing

$$\| T_1 T_1^* \|_{L^p_{t} L^q_x \to L^p_{t} L^q_x} \lesssim 2^{\frac{k'}{p}} \, 2^{\frac{k}{q}} \| T_1 \|_{L^{p'}_{t} L^{q'}_x \to L^{p'}_{t} L^{q'}_x} \quad (21)$$

where $p'$ is holder conjugate of $p$. From Young’s inequality and Plancherel’s theorem we have

$$\| K_{k', k; n}(t, \cdot) \ast F(t) \|_{L^p} \lesssim \| K_{k', k; n}(t, x) \|_{L^\infty} \| F(s) \|_{L^1}$$

$$\| K_{k', k; n}(t, \cdot) \ast F(t) \|_{L^2} \lesssim \| F \|_{L^\infty} \| K_{k', k; n}(t, \cdot) \|_{L^1} \| F(s) \|_{L^2} \lesssim \| F \|_{L^2}.$$\hspace{1em} (21)

By interpolating these two estimates we obtain for $q \geq 2$

$$\| K_{k', k; n}(t, \cdot) \ast F(t) \|_{L^2} \lesssim \| K_{k', k; n}(t, x) \|_{L^\infty} \frac{1}{2} \| F(s) \|_{L^q}.$$\hspace{1em} (21)

Then we estimate

$$\| T_1 T_1^* F \|_{L^p_{t} L^q_x} \lesssim \left\| \int_{\mathbb{R}} \| K_{k', k; n}(t-s, \cdot) \ast F(s) \|_{L^q_x} ds \right\|_{L^p_t}$$

$$\lesssim \int_{\mathbb{R}} \| K_{k', k; n}(t-s, \cdot) \|_{L^\infty} \| F(s) \|_{L^q} \| ds \|_{L^p_t}.$$\hspace{1em} (21)
Then Hardy-Littlewood-Sobolev inequality with $0 < \frac{2}{q} = \frac{1}{p} - \frac{1}{2} < 1$ implies (21). Finally we prove (20). Rescaling yields
\[ K_{k',k,n}(t,x) = 2^{3k} K_{k'-1,2-k,n}(2^k t, 2^k x) \]
Then it suffices to show that for $|\alpha| \sim 1$
\[ |K_{k'-1,1}(s,y)| \lesssim 2^{k'-k} |s|^{-1}, \tag{22} \]
where for $\langle \xi \rangle = (|\xi|^2 + 2^{-2k} m)^{\frac{1}{2}}$
\[ K_{k'-1,1}(s,y) = \int_{\mathbb{R}^3} e^{\pm i\langle \xi \rangle \kappa + iy \xi} \gamma_1^2(\xi) \gamma_{k'-1-\alpha}(\xi) d\xi. \]
By rotation, we may assume that $y = (0,0,|y|)$. We change of variables using spherical coordinates:
\[ K_{k'-1,1}(s,y) = \int_0^\pi \int_0^{2\pi} \int_0^\infty e^{i|y|r \cos \theta + s(r) \kappa} \zeta_{k',k}(\theta,\phi,r) \sin \theta r^2 d\theta d\phi dr, \]
where $\zeta_{k',k}$ is smooth cutoff function supported in a thickened spherical cap of size $2^{k'-k}$ located near unit sphere. The derivative of phase function with respect to $r$ is $|y| \cos \theta + s(r) \kappa$. Thus the worst case occurs when $0 < \theta \ll 1$ and $|y| \sim 2^k(2^k)^{-1} |s|$, otherwise since the derivative of phase function has a lower bound we can perform an integration by parts arbitrarily many times and get sufficient decay. So we only discuss the first case, where $\zeta_{k',k}(r)$ and $\zeta_{k',k}(\theta)$ are supported in an interval of length $\lesssim 2^{k'-k}$ in $\{ \frac{1}{4}, 4 \}$ and $[0, \pi)$ respectively and $|\partial_y \zeta_{k',k}| \lesssim 2^{k-k'}$. We integrate by parts with respect to $\theta$:
\[ K_{k'-1,1}(s,y) = \int_0^\pi \int_0^{2\pi} \left[ \frac{i}{|y|} e^{i|y|r \cos \theta + s(r) \kappa} \zeta_{k',k}(\theta,\phi,r) \right]_0^\pi rd\phi dr \]
\[ - i \frac{|y|}{|y|} \int_0^\pi \int_0^{2\pi} \int_0^\infty e^{i|y|r \cos \theta + s(r) \kappa} \partial_y \zeta_{k',k}(\theta,\phi,r) r d\theta d\phi dr \]
Then the support properties of $\zeta_{k',k}$ imply
\[ |K_{k'-1,1}(s,y)| \lesssim 2^{k'-k} |y|^{-1}, \]
which implies (22) since $2^k(2^k)^{-1} k_m \sim 1$. \hfill \Box

We prove bilinear estimates for free solutions which propagate to the same direction.

**Lemma 3.3.** Let $m \geq 0$. Suppose \( \begin{cases} k \in \mathbb{Z} \text{ and } k_1, k_2 \in \mathbb{N}, & \text{if } m > 0 \\ k, k_1, k_2 \in \mathbb{Z}, & \text{if } m = 0 \end{cases} \) and $2^k \ll 2^{k_1} \ll 2^{k_2}$. Then we have
\[ \| P_k \langle P_{k_1} e^{\pm i t D} m f \rangle P_{k_2} e^{\pm i t D} m g \|_{L_x^2(\mathbb{R}^{1+3})} \lesssim 2^k \| f \|_{L_x^2(\mathbb{R}^3)} \| g \|_{L_x^2(\mathbb{R}^3)}, \tag{23} \]
where $\langle f(x), g(x) \rangle_{\mathbb{R}^4} := \sum_{j=1}^4 f_j(x) g_j(x)$.

**Proof.** We may assume $\theta_1 = \theta_2 = +$. We make a localization by cubes of size $2^k$
\[ \| P_k \langle P_{k_1} e^{\pm i t D} m f \rangle P_{k_2} e^{\pm i t D} m g \|_{L_x^2(\mathbb{R}^{1+3})} \lesssim \sum_{|n_1| \sim 2^{k_1}, |n_1 + n_2| \sim 2^k} \| I_{n_1, n_2} \|_{L_{x,z}^2(\mathbb{R}^{1+3})}, \]
We estimate the Jacobi on \( \{ \eta, \xi \} \) since \(|\eta| \sim |\xi| \). We make the change of variable (24), we obtain
\[
I_{n_1,n_2}(t,x) = \int \int e^{ix \cdot (\xi + \eta) + it \sqrt{m + |\eta|^2 + \sqrt{m + |\xi|^2}}} 
\times \langle \varphi_{k_1}(\xi) \gamma_{k,n_1}(\xi) \hat{f}(\xi), \varphi_{k_2}(\eta) \gamma_{k,n_2}(\eta) \hat{g}(\eta) \rangle_{\mathbb{R}^4} d\eta d\xi.
\]

Here, we chosen almost disjoint sets \( A_i \) for \( i = 1, 2, 3 \) such that
\[
\mathbb{R}^3 = \bigcup_{i=1}^3 A_i \quad \text{and} \quad A_i \subset \{ \xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3 : |\xi| \sim |\xi_i| \}.
\]

We make the change of variable \((\eta, \xi_1) \mapsto (\xi + \eta, \sqrt{m + |\eta|^2 + \sqrt{m + |\xi|^2}}) = \zeta = (\xi_1, \cdots, \xi_4)\) with
\[
d\eta d\xi_1 = \left| \frac{\partial (\xi_1, \cdots, \xi_4)}{\partial (\xi, \eta_1, \eta_2, \eta_3)} \right|^{-1} d\zeta.
\]

We estimate the Jacobi on \( A_1 \)
\[
I_{1,n_2}^1 \lesssim \int \left\| \int e^{ix \cdot (\xi + \eta) + it \sqrt{m + |\eta|^2 + \sqrt{m + |\xi|^2}}} 
\times \langle \varphi_{k_1}(\xi) \gamma_{k,n_1}(\xi) \hat{f}(\xi), \varphi_{k_2}(\eta) \gamma_{k,n_2}(\eta) \hat{g}(\eta) \rangle_{\mathbb{R}^4} d\eta d\xi_1 \right\|_{L^2_{\xi_2}} d\xi_3.
\]

By applying Plancherel’s theorem with respect to variable \((x,t)\) and reversing the change of variables with (24), we obtain
\[
\left\| I_{n_1,n_2}^1 \right\|_{L^2_{\xi_2}(\mathbb{R}^{1+3})} \lesssim \int \left\| \sum_{\{ (\xi_2, \xi_3) \leq 2^k \}} \left\| \langle \varphi_{k_1}(\xi) \gamma_{k,n_1}(\xi) \hat{f}(\xi), \varphi_{k_2}(\eta) \gamma_{k,n_2}(\eta) \hat{g}(\eta) \rangle \right\|_{L^2_{\xi_2}} d\xi_3.
\]

We estimate by using Cauchy-Schwarz inequality
\[
\left\| I_{n_1,n_2}^1 \right\|_{L^2_{\xi_2}(\mathbb{R}^{1+3})} \lesssim 2^k \left\| \langle \varphi_{k_1}(\xi) \gamma_{k,n_1}(\xi) \hat{f}(\xi), \varphi_{k_2}(\eta) \gamma_{k,n_2}(\eta) \hat{g}(\eta) \rangle \right\|_{L^2_{\xi_2}} \lesssim 2^k \left\| P_{k_1}^{\mathbb{R}^m} f \right\|_{L^2(\mathbb{R}^3)} \left\| P_{k_2}^{\mathbb{R}^m} g \right\|_{L^2(\mathbb{R}^3)}.
\]

Then we finally have by orthogonality
\[
\left\| P_k (P_{k_1}^{\mathbb{R}^m} e^{it(D)} f, P_{k_2}^{\mathbb{R}^m} e^{it(D)} g) \right\|_{L^2_{\xi_2}(\mathbb{R}^{1+3})} \lesssim \sum_{|n_1 + n_2| \sim 2^k} 2^k \left\| P_{k_1}^{\mathbb{R}^m} f \right\|_{L^2(\mathbb{R}^3)} \left\| P_{k_2}^{\mathbb{R}^m} g \right\|_{L^2(\mathbb{R}^3)} \lesssim 2^k \left\| P_{k_1}^{\mathbb{R}^m} f \right\|_{L^2} \left\| P_{k_2}^{\mathbb{R}^m} g \right\|_{L^2},
\]
where we used Cauchy-Schwarz inequality with respect to \( n \) variable.
3.2. Estimates for functions in $V^2_{\pm(D),m}$. In this subsection, we prove linear and bilinear estimates in $V^2_{\pm(D),m}$ by transferring the ones for free solutions in the previous section.

**Corollary 3.4.** Let $m \geq 0$. Let $k', k, l \in \mathbb{Z}$, $k' \leq k$ and $2 < p < \infty$. Then we have for $\psi \in V^2_{\pm(D),m}$

\[
\left( \sum_\nu \sum_{n \in L_{k'}} \| \Gamma'_{\nu} \Gamma_{k',n} P^m_k \psi \|^2_{L^p_t L^2_x} \right)^{\frac{1}{2}} \lesssim 2^\frac{k}{2} 2^{\frac{k'}{p}} \| P^m_k \psi \|_{V^2_{\pm(D),m}},
\]

(25)

**Proof.** From Lemma 2.5 and (19) we have

\[
\| \Gamma_{k',n} P^m_k \psi \|_{L^p_t L^2_x} \lesssim 2^\frac{k}{2} 2^\frac{k'}{p} \| \psi \|_{U^p_m(O_m)},
\]

where in the second inequality we used embedding for $p > 2$. We can easily check (26) also holds for operator $\Gamma'_{k',n} P^m_k$ having slightly larger support, from which we induce

\[
\| \Gamma'_{k',n} P^m_k \psi \|_{L^p_t L^2_x} \lesssim 2^\frac{k}{2} 2^\frac{k'}{p} \| \psi \|_{V^2_{\pm(D),m}}.
\]

Note that the projection operators into cube $\{ \Gamma_{k',n} \}$, $n \in L_{k'}$ are almost disjoint with respect to $n$ and so are the $\{ \Gamma'_{\nu} \}$, $\nu \in \Omega_l$ with respect to $\nu$. Thus, we obtain from the definition of $V^2_{\pm(D)}$

\[
\left( \sum_\nu \sum_{n \in L_{k'}} \| \Gamma'_{\nu} \Gamma_{k',n} P^m_k \psi \|^2_{L^p_t L^2_x} \right)^{\frac{1}{2}} \lesssim 2^\frac{k}{2} 2^\frac{k'}{p} \left( \sum_\nu \sum_{n \in L_{k'}} \| \Gamma'_{\nu} \Gamma_{k',n} P^m_k \psi \|^2_{V^2_{\pm(D),m}} \right)^{\frac{1}{2}} \lesssim 2^\frac{k}{2} 2^\frac{k'}{p} \| P^m_k \psi \|_{V^2_{\pm(D),m}}.
\]

\[\square\]

We obtain bilinear estimates in $V^2_{\pm(D)}$ from the above linear one.

**Corollary 3.5.** Let $m \geq 0$, $\theta_1, \theta_2 \in \{+1, -1\}$ and $k_1, k_2 \in \mathbb{Z}$. Then for $\psi_i \in V^2_{\theta_i(D)}$ satisfying $\Pi^m_k(D) P^m_k \psi_i = \psi_i$, $i = 1, 2$, we have for $0 < s < \frac{1}{2}$

\[
\| \langle \psi_1, \beta \psi_2 \rangle_{C^4} \|_{L^2(R^{1+3})} \lesssim 2^{sk_1, 2sk_2} 2^{(1-2s) \min(k_1, k_2)} \| \psi_1 \|_{V^2_{\theta_1(D),m}} \| \psi_2 \|_{V^2_{\theta_2(D),m}}.
\]

(27)

**Proof.** Letting $k' = k$ in (25) gives

\[
\| P^m_k \psi \|_{L^p_t L^2_x} \lesssim 2^\frac{2k}{p} \| P^m_k \psi \|_{V^2_{\pm(D),m}},
\]

for $2 < p, q < \infty$ satisfying $\frac{1}{p} + \frac{1}{q} = \frac{1}{2}$. Using this, we have

\[
\| \langle \psi_1, \beta \psi_2 \rangle_{C^4} \|_{L^2(R^{1+3})} \lesssim \| \psi_1 \|_{L^p_t L^2_x} \| \psi_2 \|_{L^q_t L^2_x} \lesssim 2^\frac{3k}{2} 2^{1-\frac{3}{2}k_2} \| \psi_1 \|_{V^2_{\theta_1(D),m}} \| \psi_2 \|_{V^2_{\theta_2(D),m}},
\]

which implies (27) if we let $\frac{3}{2} = s$.

\[\square\]

By making use of the null structure in the Dirac equations, we improve the above bilinear estimates when high-high interaction makes the low frequency. First, we consider the case $\theta_1 \theta_2 = 1$. 

Proposition 3.7. Let $m \geq 0$, $\theta_1 \theta_2 = 1$ and \[ k \in \mathbb{Z} \text{ and } k_1, k_2 \in \mathbb{N}, \quad \text{if } m > 0, \]
\[ k, k_1, k_2 \in \mathbb{Z}, \quad \text{if } m = 0. \]
Suppose $2^k \ll 2^{k_1} \sim 2^{k_2}$. Then for $\psi_i \in V^2_{\theta_i(D)}$ satisfying $\Pi^m_{\theta_i(D)} P^m_{k_i} \psi_i = \psi_i$, $i = 1, 2$, we have

\[ \| P_k \langle \psi_1, \beta \psi_2 \rangle \|_{L^4(\mathbb{R}^{1+3})} \lesssim \begin{cases} \begin{aligned} & 2^{k} \| \psi_1 \|_{V^2_{\theta_1(D)m}} \| \psi_2 \|_{V^2_{\theta_2(D)m}}, \quad \text{if } m > 0 \text{ and } k \leq 0, \\ & 2^{\frac{k}{2}} 2^{k_1} \| \psi_1 \|_{V^2_{\theta_1(D)m}} \| \psi_2 \|_{V^2_{\theta_2(D)m}}, \quad \text{otherwise.} \end{aligned} \end{cases} \]

(28)

Proof. We decompose the function $\psi_1$ and $\psi_2$ in the left-hand side by applying the cubic localization $\Gamma_{k,n}$, $n \in L_k$ and cone localization $K_{\nu}^\nu$ whose central direction is $\nu \in \Omega_l$, with the relation $2^{-l} \sim 2^{k_{\text{max}}(k_1, k_2)}$. After the localization, we observe from the condition $2^k \ll 2^{k_1} \sim 2^{k_2}$ that the summation on cone actually runs over $|n - n'| \lesssim k$ and the summation on cone over $|\theta_1 \xi_{l'} - \theta_2 \xi_{l''}| = |\xi_1 - \xi_2| \sim 2^{-l}$. Otherwise, the left-hand side is zero.

\[ \| P_k \langle \psi_1, \beta \psi_2 \rangle \|_{L^4(\mathbb{R}^{1+3})} \lesssim \sum_{n,n', \in L_k} \sum_{\nu, \nu' \in \Omega_l} \| P_k \langle K_{\nu}^\nu \Gamma_{k,n} \psi_1, \beta K_{\nu'}^{\nu'} \Gamma_{k,n'} \psi_2 \rangle \|_{L^4(\mathbb{R}^{1+3})}. \]

Now, we consider the case when $m = 0$. We estimate the localized $L^2$ norm using the Plancherel identity and Lemma 2.2.

\[ \| P_k \langle \psi_1, \beta \psi_2 \rangle \|_{L^2(\mathbb{R}^{1+3})} \lesssim \sum_{n,n', \in L_k} \sum_{\nu, \nu' \in \Omega_l} 2^{-l} \| K_{\nu}^\nu \Gamma_{k,n} \psi_1 \|_{L^4(\mathbb{R}^{1+3})} \| K_{\nu'}^{\nu'} \Gamma_{k,n'} \psi_2 \|_{L^4(\mathbb{R}^{1+3})}. \quad (29) \]

Applying the Cauchy-Schwarz inequality with respect to $n, n'$ and $\nu, \nu'$ and then using (25) we finally get

\[ \| P_k \langle \psi_1, \beta \psi_2 \rangle \|_{L^2(\mathbb{R}^{1+3})} \lesssim 2^{k-k_1} 2^{\frac{k}{2}} 2^{k_1} \| \psi_1 \|_{V^2_{\theta_1(D)m}} \| \psi_2 \|_{V^2_{\theta_2(D)m}}. \]

The same argument as above applies to the case when $m > 0$ once the constant in (29) is changed into $(2^{-l} + 2^{-k_1})$. \qed

Next we consider the case $\theta_1 \theta_2 = -1$, which follows from the bilinear estimates in Lemma 3.3.

Proposition 3.7. Let $m \geq 0$, $\theta_1 \theta_2 = -1$ and \[ k \in \mathbb{Z} \text{ and } k_1, k_2 \in \mathbb{N}, \quad \text{if } m > 0, \]
\[ k, k_1, k_2 \in \mathbb{Z}, \quad \text{if } m = 0. \]
Suppose $2^k \ll 2^{k_1} \sim 2^{k_2}$. Then for $\psi_i \in V^2_{\theta_i(D)}$ satisfying $\Pi^m_{\theta_i(D)} P^m_{k_i} \psi_i = \psi_i$, $i = 1, 2$, we have

\[ \| P_k \langle \psi_1, \beta \psi_2 \rangle \|_{L^2(\mathbb{R}^{1+3})} \lesssim 2^k \| \psi_1 \|_{V^2_{\theta_1(D)m}} \| \psi_2 \|_{V^2_{\theta_2(D)m}}. \quad (30) \]

Proof. We suffice to show that for $\psi_1 \in V^2_{\theta_1(D)m}, \psi_2 \in V^2_{\theta_2(D)m}$

\[ \| P_k \langle P^m_{k_1} \psi_1, \beta P^m_{k_2} \psi_2 \rangle \|_{L^2(\mathbb{R}^{1+3})} \lesssim 2^k \| \psi_1 \|_{V^2_{\theta_1(D)m}} \| \psi_2 \|_{V^2_{\theta_2(D)m}}. \]
Let $e^{it(D)}\psi = e^{\mp it(D)}\tilde{\psi}$, we have $\|\psi\|_{V^2_{\pm}(D)} = \|\tilde{\psi}\|_{V^2_{\mp}(D)}$. Thus it further reduces to showing that

$$\|P_k\langle P_{k_1}^m \psi_1, \beta P_{k_1}^m \psi_2 \rangle_{\mathbb{R}^4}\|_{L^2_t(L^{1+3/2})} \lesssim 2^k \|\psi_1\|_{V^2_{\pm}(D)} \|\psi_2\|_{V^2_{\mp}(D)},$$

where $\langle \psi, \phi \rangle_{\mathbb{R}^4} = \sum_{i=1}^4 \psi_i \phi_i$. This follows from a slight modification of the proof in [13, Proposition 2.5] by using (23). We provide an explicit proof in Appendix for convenience of readers.

Collecting all the cases we prove above, we arrange them in the following Corollary.

**Corollary 3.8.** Let $m \geq 0$ and $\theta_1, \theta_2 \in \{-1, 1\}$. Suppose $k, k_1, k_2 \in \mathbb{Z}$ satisfy $2^k \lesssim 2^{k_1} \sim 2^{k_2}$. For $\psi_i \in V^2_{\theta_i}(D)$ satisfying $\Pi^m_{\theta_i}(D) P_{k_1}^m \psi_i = \psi_i$, $i = 1, 2$ we have

$$\|P_k\langle \psi_1, \beta \psi_2 \rangle_{\mathbb{C}^4}\|_{L^2_t(L^{1+3/2})} \lesssim B_m(k) \|\psi_1\|_{V^2_{\theta_1}} \|\psi_2\|_{V^2_{\theta_2}},$$

where $B_m(k) = \begin{cases} 2^k & \text{if } m > 0 \text{ and } k \leq 0, \\ 2^k & \text{otherwise}. \end{cases}$

**Proof.** The only remaining case we need to prove is when $m > 0$ and $\min(k_1, k_2) = 0$. By (3.1) and Lemma 2.5 we have for $i = 1, 2$

$$\|\psi_i\|_{L^1_t L^3_x} \lesssim \|\psi_i\|_{U^4_{\pm}(D)},$$

where we used $2^{k_1}, 2^{k_2} \lesssim 1$. Then we apply Bernstein inequality

$$\|P_k\langle \psi_1, \beta \psi_2 \rangle_{\mathbb{C}^4}\|_{L^2_t(L^{1+3/2})} \lesssim 2^k \|P_k\langle \psi_1, \beta \psi_2 \rangle_{\mathbb{C}^4}\|_{L^2_t L^2_t(L^{1+3/2})} \lesssim 2^k \|\psi_1\|_{L^1_t L^2_x} \|\psi_2\|_{L^1_t L^2_x},$$

which implies the desired result from (32) since $V^2_{\pm}(D) \hookrightarrow U^4_{\pm}(D)$.

**4. Proof of main results.** We introduce the resolution space $X^s_{m, \pm}$ corresponding to the Sobolev spaces of regularity $s$. We first define $X^s_{m, \pm}$ on the projection space by the space of functions in $C(\mathbb{R}, H^s_m(\mathbb{R}^3; \mathbb{C}^4))$ such that

$$X^s_{m, \pm} := \left\{ \|\psi\|_{X^s_{m, \pm}} := \left( \sum_{k \in \mathbb{Z}} 2^{sk} \|P_k^m \psi\|_{V^2_{\pm}(D)}^2 \right)^{1/2} < \infty \right\}.$$

For given $\psi \in H^s_m(\mathbb{R}^3)$ we decompose $\psi = \Pi^m_+(D)\psi + \Pi^m_-(D)\psi =: \psi_+ + \psi_-$ and define $X^s_m$ by

$$\psi \in X^s_m \iff (\psi_+, \psi_-) \in X^s_{m,+} \times X^s_{m,-} \text{ and } \|\psi\|_{X^s_m} := \|\psi_+\|_{X^s_{m,+}} + \|\psi_-\|_{X^s_{m,-}}.$$ 

It suffices to consider positive times. We will represent a solution $(\psi_+, \psi_-)$ to the system (12) using the Duhamel’s formula on $[0, \infty]$.

$$\psi_+(t) = e^{-it(D)} = \Pi^m_+(D)\psi_0 + \sum_{\theta_1, \theta_2, \theta_3 \in \{\pm\}} \int_0^t e^{-i(t-t')(D)} \Pi^m_+(D) \left( V_{\gamma} \ast \langle \psi_{\theta_1}, \beta \psi_{\theta_2} \rangle \beta \psi_{\theta_3} \right)(t') dt'$$

$$\psi_-(t) = e^{it(D)} = \Pi^m_-(D)\psi_0 + \sum_{\theta_1, \theta_2, \theta_3 \in \{\pm\}} \int_0^t e^{i(t-t')(D)} \Pi^m_-(D) \left( V_{\gamma} \ast \langle \psi_{\theta_1}, \beta \psi_{\theta_2} \rangle \beta \psi_{\theta_3} \right)(t') dt',$$

(33)
whenever the initial data satisfies $\| \psi_0 \|_H^* \leq \delta$.

For all $\psi_0 \in H^2(\mathbb{R}^3)$ we immediately have

$$\| e^{-it(D)} \Pi_m^m(D) \psi_0 \|_{X^s_{m,+}} + \| e^{it(D)} \Pi_m^m(D) \psi_0 \|_{X^s_{m,-}} \approx \| \psi_0 \|_{H^s_m}. \quad (34)$$

Next we study the nonlinear part. We consider more general situations: Let $s > 0$. For all $\psi_i \in X_m^s$ such that $\psi_i = \Pi_m^m(D) \psi_i$ for $i = 1, 2, 3$ we claim that

$$\left\| \int_0^t e^{i(t-t') (D)} \Pi_m^m(D) \left( V_\gamma (\psi_1, \beta \psi_2) \beta \psi_3 \right)(t') dt' \right\|_{X^s_{m, \pm}} \lesssim \prod_{i=1}^3 \| \psi_i \|_{X^s_{m, \epsilon_i}}. \quad (35)$$

For a moment we assume (35) and prove the Theorem 1.2. Let $T(\psi)$ denote the nonlinear operator defined by the sum of the right hand sides of (33). Then from (34) and (35) we conclude that

$$\| T(\psi) \|_{X^s_m} \lesssim \delta + \| \psi \|_{X^s_{m'}}, \quad (36)$$

$$\| T(\psi) - T(\phi) \| \lesssim \| \psi - \phi \|_{X^s_{m'}} \| \psi \|_{X^s_{m'}}. \quad (37)$$

Indeed, the second one follows by taking $\psi_1 = \psi - \phi, \psi_2 = \psi, \psi_3 = \phi$ or $\psi_1 = \psi, \psi_2 = \psi - \phi, \psi_3 = \phi$ or $\psi_1 = \psi, \psi_2 = \phi, \psi_3 = \psi - \phi$ in (35). Theorem 1.2 now follows from the standard approach via the contraction mapping principle. In particular, the scattering claim follows from (35) and the fact that functions in $U^2$ have a limit at $\infty$, once we define a scattering state $\psi_{sc}$ by

$$\psi_{sc} := \sum_{\theta \in \{+, -\}} \left( \Pi_m^m(D) \psi_0 + \lim_{t \to \infty} \int_0^t e^{i \theta \theta'(D)} \Pi_m^m(D) \left( V_\gamma (\psi_1, \beta \psi_2) \beta \psi_3 \right)(t') dt' \right).$$

4.1. Proof of (35). Let $s > 0$ and $\psi_i \in X_m^s$ such that $\psi_i = \Pi_m^m(D) \psi_i$ for $i = 1, 2, 3$. We compute square of the left side norm in (35) using the duality in Lemma 2.3

$$\left\| \int_0^t e^{i(t-t') (D)} \Pi_m^m(D) \left( V_\gamma (\psi_1, \beta \psi_2) \beta \psi_3 \right)(t') dt' \right\|_{X^s_{m, \pm}}^2$$

$$= \sum_{k_4 \in \mathbb{Z}} 2^{sk_4} \sup_{\psi_4 \in L^2_{X^s_{m, \pm}(\mathbb{R}^3)}} \left| \int e^{i(t-t') (D)} \Pi_m^m(D) \left( V_\gamma (\psi_1, \beta \psi_2) \beta \psi_3 \right)(t') dt' \right|^2$$

$$= \sum_{k_4 \in \mathbb{Z}} 2^{sk_4} \sup_{\psi_4 \in L^2_{X^s_{m, \pm}(\mathbb{R}^3)}} \left| \int V_\gamma (\psi_1, \beta \psi_2) \beta \psi_3, \Pi_m^m(D) \psi_4 \right|^2.$$
\[
\lesssim \sum_{k_4 \in \mathbb{Z}} 2^{2sk_4} \sup_{\|\psi_{4,k_4}\|_{V^2_{\theta_4}(D)}^2 = 1} \left( \sum_{k,k_1,k_2,k_3 \in \mathbb{Z}} \|P_k V_\gamma \ast (\psi_{1,k_1}, \beta \psi_{2,k_2})\|_{L^2_{t,x}}^2 \right) \\
\times \|P_k (\beta \psi_{3,k_3}, \psi_{4,k_4})\|_{L^2_{t,x}}^2 \right)^2
\]
\[
\lesssim \sum_{k_4 \in \mathbb{Z}} 2^{2sk_4} \sup_{\|\psi_{4,k_4}\|_{V^2_{\theta_4}(D)}^2 = 1} \left( \sum_{k,k_1,k_2,k_3 \in \mathbb{Z}} 2^{-\gamma k} (2^k)^{\gamma - 2} \|P_k (\psi_{1,k_1}, \beta \psi_{2,k_2})\|_{L^2_{t,x}}^2 \right) \\
\times \|P_k (\beta \psi_{3,k_3}, \psi_{4,k_4})\|_{L^2_{t,x}}^2 \right)^2
\]
\[
:= I_1 + I_2 + I_3,
\]
where \( I_1 = \sum_{2^{k_2+2k_3} \leq 2^{k_4}} I_2 = \sum_{2^{k_1} \geq 2^{k_2}} \) and \( I_3 = \sum_{2^{k_1} \leq 2^{k_2}} \).

It suffices to consider low regularity, so in the following proof the range of \( s \) is restricted to \( 0 < s < \min\left(\frac{2}{\gamma}, \frac{2}{7}\right) \). Once this assumption is given we have for all \( k \in \mathbb{Z} \)
\[
2^{(2-2s-\gamma)k} (2^k)^{\gamma - 2} \leq 1,
\]
which will be repeatedly used in the following computation.

We remark that all the estimates below will be obtained in \( V^2 \) space where \( U^2 \) space is continuously embedded. We actually show stronger ones than we need.

4.1.1. Estimates for \( I_1 \). We further divide the case \( I_1 \leq I_{11} + I_{12} + I_{13} \) where \( I_{11} := \sum_{2^{k_2+2k_3} \leq 2^{k_2}} I_{12} := \sum_{2^{k_1} \geq 2^{k_2}} \) and \( I_{13} := \sum_{2^{k_1} \leq 2^{k_2}} \).

We estimate \( I_{11} \) by applying (31)
\[
I_{11} \lesssim \sum_{k_4 \in \mathbb{Z}} 2^{2sk_4} \left( \sum_{k \in \mathbb{Z}, 2^{k} \leq 2^{k_{1}+2k_2}} 2^{-\gamma k} (2^k)^{\gamma - 2} \left[ B_m(k, k_1, k_2) \right] \right) \\
\times \left[ \|\psi_{1,k_1}\|_{V^2_{\theta_1}(D)} \|\psi_{2,k_2}\|_{V^2_{\theta_2}(D)} \|\psi_{3,k_3}\|_{V^2_{\theta_3}(D)} \right]^2
\]
\[
\lesssim \sum_{k_3 \in \mathbb{Z}} 2^{2sk_3} \|\psi_{3,k_3}\|_{V^2_{\theta_3}(D)} \left( \sum_{k \in \mathbb{Z}, 2^{k} \leq 2^{k_{1}} \leq 2^{k_{1}+2k_2}} 2^{-\gamma k} (2^k)^{\gamma - 2} \left[ B_m(k, k_1, k_2) \right] \right) \\
\times 2^{sk_1} \|\psi_{1,k_1}\|_{V^2_{\theta_1}(D)} \|\psi_{2,k_1}\|_{V^2_{\theta_2}(D)} \right)^2.
\]

For the massless case \( (m = 0) \), we have
\[
I_{11} \lesssim \|\psi\|_{X^2_{m,\theta_3}} \left( \sum_{2^{k_{2}+k_{1}}} 2^{(2-\gamma-2s)k} (2^k)^{\gamma - 2} 2^{2s(k-k_1)} \right) \\
\times 2^{sk_1} \|\psi_{1,k_1}\|_{V^2_{\theta_1}(D)} \|\psi_{2,k_1}\|_{V^2_{\theta_2}(D)} \right)^2
\]
\[
\lesssim \|\psi\|_{X^2_{m,\theta_3}} \left( \sum_{2^{k_{2}+k_{1}}} 2^{(2-\gamma-2s)(k-k_1)} 2^{sk_1} \|\psi_{1,k_1}\|_{V^2_{\theta_1}(D)} \|\psi_{2,k_1}\|_{V^2_{\theta_2}(D)} \right)^2
\]
\[
\lesssim \|\psi\|_{X^2_{m,\theta_1}} \|\psi_2\|_{X^2_{m,\theta_2}} \|\psi_3\|_{X^2_{m,\theta_3}},
\]
where we used \( 0 \leq \gamma < 2, 0 < s < \frac{2}{7} \) and Cauchy-Schwarz inequality.

For the massive case \( (m > 0) \), we further split the summation over \( k \). We obtain from (31)
\[
I_{11} \lesssim \|\psi\|_{X^2_{m,\theta_3}} \left( \sum_{2^{k_{2}+k_{1}}} 2^{(2-\gamma-2s)k} + \sum_{2^{k_{2}+k_{1}}} 2^{2s(k-k_1)} \right)
\]
4.1.2. Estimates for the bilinear estimates (31) only with the second case regardless of \( m \), with summation over \( m > k \), the support condition. We estimate using (31) and (27) assumption 0 where in the first inequality we used (39) and in the second inequality we used the assumption 0 if \( k > 1 \) to bound the low frequency part near zero. We note that if \( 2^k \sim 2^{k_1} >> 2^{k_2} \), from which we have \( k > 0 \) if \( m > 0 \). Thus, using (31) we estimate

\[
I_{12} \lesssim \sum_{k_4 \in \mathbb{Z}} 2^{sk_4} \left( \sum_{k \in \mathbb{Z}, 2^{k_1} - 2^{k_2} > 2^{k_3}} 2^{-\gamma k_1 (2^{k_1})^{\gamma-2} 2^{k_2} (1-s)k_2} \right)
\]

\[
\times \left\| \psi_1, k_1 \right\|_{V^2_{k_1}(D)_{m}} \left\| \psi_2, k_2 \right\|_{V^2_{k_2}(D)_{m}} 2^k \left\| \psi_3, k_3 \right\|_{V^2_{k_3}(D)_{m}} \right)^2
\]

\[
\lesssim \sum_{k_3 \in \mathbb{Z}} 2^{sk_3} \left( \sum_{k_1 \in \mathbb{Z}, k_2 > 2^{k_3}} 2^{(2-\gamma-2s)k_1 (2^{k_1})^{\gamma-2} (1-2s)(k_2-k_1)} \right)
\]

\[
\times \left\| \psi_1, k_1 \right\|_{V^2_{k_1}(D)_{m}} \left\| \psi_2, k_2 \right\|_{V^2_{k_2}(D)_{m}} 2^{sk_2} \left\| \psi_3, k_3 \right\|_{V^2_{k_3}(D)_{m}} \right)^2
\]

since \( 2^{(2-\gamma-2s)k_1 (2^{k_1})^{\gamma-2} \leq 1 \) from (39) and \( 0 < s < \frac{1}{2} \).

\( I_{13} \) can be similarly bounded as \( I_{12} \) by just exchanging the role of \( \psi_1 \) and \( \psi_2 \).

4.1.2. Estimates for \( I_2 \). In this case, we have \( 2^k \sim 2^{k_4} >> 2^{k_3} \). We note that if \( m > 0 \) the summation over \( k \) actually runs over \( k > 0 \) since \( 2^{k_3} > 1 \). Thus, we use the bilinear estimates (31) only with the second case regardless of \( m \).

Consider \( I_{21} \). In this range, it holds that \( 2^{k_1} \sim 2^{k_2} \gg 2^{k_4} \gg 2^{k_3} \) from the support condition. We estimate using (31) and (27)

\[
I_{21} \lesssim \sum_{k_4 \in \mathbb{Z}} 2^{sk_4} \left( \sum_{k \in \mathbb{Z}, 2^{k_1} - 2^{k_2} > 2^{k_3}} 2^{(1-\gamma)k (2^{k_1})^{\gamma-2} 2^{k_2} (1-2s)k_2} \right)
\]

\[
\times \left\| \psi_1, k_1 \right\|_{V^2_{k_1}(D)_{m}} \left\| \psi_2, k_2 \right\|_{V^2_{k_2}(D)_{m}} \right)^2
\]

\[
\lesssim \sum_{k_3 \in \mathbb{Z}} 2^{sk_3} \left( \sum_{k_1 \in \mathbb{Z}, k_2 > 2^{k_3}} 2^{(1-s)k_1 (2^{k_1})^{\gamma-2} 2^{k_2} (1-2s)(k_2-k_1)} \right)
\]

\[
\times \left\| \psi_1, k_1 \right\|_{V^2_{k_1}(D)_{m}} \left\| \psi_2, k_2 \right\|_{V^2_{k_2}(D)_{m}} \right)^2
\]

since \( 2^{(1-\gamma-2s)k_1 (2^{k_1})^{\gamma-2} \leq 1 \) from (39) and \( 0 < s < \frac{1}{2} \).
Next consider $I_{22}$. In this range we have $2^k \sim 2^{k_1} \sim 2^{k_2}$. We estimate using (27)

$$I_{22} \lesssim \sum_{k_4 \in \mathbb{Z}} 2^{2sk_4} \left( \sum_{k_3 \in \mathbb{Z}, 2^k \sim 2^{k_1} \sim 2^{k_2}} 2^{-\gamma k_3} 2^{(1-s)k_3} 2^{sk_4} \left\| \psi_{1,k_4} \right\|_{V_{\theta_1(D),m}^2} \left\| \psi_{2,k_2} \right\|_{V_{\theta_2(D),m}^2} \left\| \psi_{3,k_3} \right\|_{V_{\theta_3(D),m}^2} \right)^2 \right).$$

$$\lesssim \sum_{k_4 \in \mathbb{Z}} 2^{2sk_4} \left( \sum_{2^{k_3+2k_2} \sim 2^{k_3+2k_4}} 2^{2sk_4} \left\| \psi_{1,k_4} \right\|_{V_{\theta_1(D),m}^2} \left\| \psi_{2,k_2} \right\|_{V_{\theta_2(D),m}^2} \left\| \psi_{3,k_3} \right\|_{V_{\theta_3(D),m}^2} \right)^2 \right).$$

4.1.3. Estimates for $I_3$. As we did in the previous section for $I_2$, we may assume $k > 0$ if $m > 0$. We divide the case $I_3 \leq I_{31} + I_{32} + I_{33}$, where $I_{31} := \sum_{2^{k_1} \sim 2^{k_2}}$, $I_{32} := \sum_{2^{k_1} \sim 2^{k_2}}$ and $I_{33} := \sum_{2^{k_1} \sim 2^{k_2}}$.

Consider $I_{31}$. We estimate using (31)

$$I_{31} \lesssim \sum_{k_4 \in \mathbb{Z}} 2^{2sk_4} \left( \sum_{k_3 \in \mathbb{Z}, 2^k \sim 2^{k_1} \sim 2^{k_2}} 2^{-\gamma k_3} 2^{(1-s)k_3} 2^{sk_4} \left\| \psi_{1,k_4} \right\|_{V_{\theta_1(D),m}^2} \left\| \psi_{2,k_2} \right\|_{V_{\theta_2(D),m}^2} \right)^2 \right).$$

Next, we consider $I_{32}$. In this range, we have $2^k \sim 2^{k_1} \sim 2^{k_3}$. We estimate

$$I_{32} \lesssim \sum_{k_4 \in \mathbb{Z}} 2^{2sk_4} \left( \sum_{k_3 \in \mathbb{Z}, 2^k \sim 2^{k_1} \sim 2^{k_2}} 2^{-\gamma k_3} 2^{(1-s)k_3} 2^{sk_4} \left\| \psi_{1,k_4} \right\|_{V_{\theta_1(D),m}^2} \left\| \psi_{2,k_2} \right\|_{V_{\theta_2(D),m}^2} \right)^2 \right).$$
Our aim is to prove that for $t \in \mathbb{R}$ which is equivalent to $s < 0$. Assuming that (42) holds, the claim follows since the validity of (40) implies any resolution space $\tilde{x}_n$ nonlinear term estimate (36) essential to occur the contraction cannot be true for potential satisfy this property. We provide the ill-posed result which shows the potential $V$ let $Proposition 5.1.$

In this section, we consider the supercritical range where the inequality we define the annulus $W_{\lambda} = \{ \xi \in \mathbb{R}^3 : \lambda \leq |\xi| \leq 2\lambda \}$. Let $\phi : \mathbb{R}^3 \to \mathbb{C}$ be the inverse Fourier transform of the characteristic function $\chi_{W_{\lambda}}$, and we choose $\psi = (\phi, 0, 0, 0)$. Obviously, $\|\psi\|_{H_m^s} \lesssim \lambda^{2s}. $ Next, we consider $N(t, \xi) := \mathcal{F}_x \left( \int_0^t U_m(t-\tau, D) (V \ast \langle U_m(\tau, D) \psi, \beta U_m(\tau, D) \psi \rangle \beta U_m(\tau, D) \psi) (\tau) d\tau \right).$ Our aim is to prove that for $t = \epsilon \lambda^{-1}$ with $0 < \epsilon \ll 1$ and $\xi \in W_{\lambda},$

$$|N(t, \xi)| \gtrsim |t|^{\lambda^2} = \lambda_{\xi}^3. $$

Assuming that (42) holds, the claim follows since the validity of (40) implies $\epsilon \lambda^{x+\frac{3}{2}} \lesssim \|\langle \xi \rangle^s N(t, \xi)\|_{L^2(\mathbb{R}^3)} \lesssim \lambda^{3x+\frac{3}{2}},$ which is equivalent to $\epsilon \lesssim \lambda^{2s}$ for fixed $s > 0$. This can hold as $\lambda \to \infty$ only if $s \geq 0$. Hence, it suffices to show (42).
We compute the Fourier transform

\[ N(t, \xi) \approx \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_0^t U_m(t - \tau, \xi) \hat{V}(\eta)(U_m(\tau, \eta - \sigma) \hat{\psi}(\eta - \sigma), \beta U_m(-\tau, -\sigma) \hat{\psi}(-\sigma)) \times \beta U_m(\tau, \xi - \eta) \hat{\psi}(\xi - \eta) d\tau d\sigma d\eta. \]

Let us denote \( j \)th component of \( x \in \mathbb{C}^4 \) by \( x_j \) and the \((i, j)\) entry of \( 4 \times 4 \) matrix \( A \) by \( A_{ij} \). Putting \( \hat{\psi} = (\chi_{W_\lambda}, 0, 0, 0) \) we compute

\[ |N(t, \xi)| \approx |[N(t, \xi)]_j| \]

\[ \geq \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \hat{V}(\eta) \times \int_0^t \text{Re} \left( [U_m(\tau, \eta - \sigma)^T \beta U_m(-\tau, -\sigma)]_{11} [U_m(t - \tau, \xi) \beta U_m(\tau, \xi - \eta)]_{11} \right) d\tau \]

\[ \times \chi_{W_\lambda}(\eta - \sigma) \chi_{W_\lambda}(\sigma) \chi_{W_\lambda}(\xi - \eta) d\sigma d\eta. \]

From Lemma 6.1 we find the integration over \( \tau \) is bounded below for \( \lambda \gg 1 \)

\[ \int_0^t \text{Re} \left( [U_m(\tau, \eta - \sigma)^T \beta U_m(-\tau, -\sigma)]_{11} [U_m(t - \tau, \xi) \beta U_m(\tau, \xi - \eta)]_{11} \right) d\tau \gtrsim t. \]

Since \( \hat{V} \) is positive we finally obtain

\[ |N(t, \xi)| \gtrsim |t| \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \hat{V}(\eta) \chi_{W_\lambda}(\eta - \sigma) \chi_{W_\lambda}(\sigma) \chi_{W_\lambda}(\xi - \eta) d\eta d\sigma \gtrsim |t| \lambda^4. \]

6. Appendix.

Lemma 6.1. Let \( m \geq 0 \), \( \xi, \eta \in \mathbb{R}^3 \) and \( t, \tau \in \mathbb{R} \). Suppose \( |\xi|, |\eta| \sim \lambda \) and \( |\tau|, |t| \leq \epsilon \lambda^{-1} \) with \( 0 < \epsilon \ll 1 \). Then we have for sufficiently large \( \lambda \gg 1 \)

\[ \text{Re}[U_m(\tau, \xi) \beta U_m(t, \eta)]_{11} \gtrsim 1 \quad \text{and} \quad \text{Im}[U_m(\tau, \xi) \beta U_m(t, \eta)]_{11} \lesssim \lambda^{-1}, \]

\[ \text{Re}[U_m(\tau, \xi)^T \beta U_m(t, \eta)]_{11} \gtrsim 1 \quad \text{and} \quad \text{Im}[U_m(\tau, \xi)^T \beta U_m(t, \eta)]_{11} \lesssim \lambda^{-1}, \]

where the implicit constants depend only on \( \epsilon \) and \( m \).
Proof. For $|\xi| \sim \lambda$ and $\tau \leq \epsilon \lambda^{-1}$ with $0 < \epsilon \ll 1$, we have
\[
\cos (\tau (\xi)_m) \gtrsim 1 \quad \text{and} \quad |(\xi)_m^{-1} \sin (\tau (\xi)_m)| \leq \epsilon \lambda^{-1}.
\]
We directly compute from (9) and (2)
\[
\text{Re}[U_m(\tau, \xi) \beta U_m(t, \eta)]_{11} = \cos (\tau (\xi)_m) \cos (t(\eta)_m) + 2(\xi)_m^{-1} \sin (\tau (\xi)_m) \langle \eta \rangle_m^{-1} \sin (t(\eta)_m),
\]
\[
\text{Im}[U_m(\tau, \xi) \beta U_m(t, \eta)]_{11} = -\cos (\tau (\xi)_m) \langle \eta \rangle_m^{-1} \sin (t(\eta)_m) - \cos (t(\eta)_m) (\xi)_m^{-1} \sin (\tau (\xi)_m).
\]
Step 1: Let $P = \sum_{k_3 \in F} P_{k_3}^m$, with a finite set $F$ of integer $k_3$ of size $2^{k_3} \sim 2^{k_1} \sim 2^{k_2}$, such that $PP_i^m = P_i^m$ for $i = 1, 2$. We claim first that
\[
\|P_{\leq k}(P \psi, P \psi)_{R^*}\|_{L^2} \lesssim 2^k \|\psi\|^2_{V(\beta)_m} \quad (44)
\]
for any $2^k \lesssim 2^{k_1}$ and real-valued $\psi \in V(\beta)_m$. To prove (44), let $\varphi_k = (F^{-1} \varphi_{\leq k+1})^2$. Then, $\omega_k \geq 0$ and we have the pointwise bound
\[
\varphi_{\leq k} \lesssim \varphi_{\leq k+1} * \varphi_{\leq k+1} \lesssim \varphi_{\leq k+2}
\]
on the Fourier side, which implies
\[
\|P_{\leq k}(P \psi, P \psi)_{R^*}\|_{L^2} \lesssim \|\omega_k * (P \psi, P \psi)_{R^*}\|_{L^2} \lesssim \|P_{\leq k+2}(P \psi, P \psi)_{R^*}\|_{L^2}.
\]
So it suffices to show that
\[
\|\omega_k * (P \psi, P \psi)_{R^*}\|_{L^2} \lesssim 2^k \|\psi\|^2_{V(\beta)_m}. \quad (45)
\]
For real valued $f$, the quantity
\[
n(f) := \|T(f)\|_{L^1(R^3)}, \quad \text{for} \ T(f) = \left(\omega_k * \langle P f, P f\rangle_{R^*}\right)^{\frac{1}{2}},
\]
is subadditive. Indeed, since $\langle f, f \rangle_{R^*} = |f|^2$ we estimate using Cauchy-Schwarz inequality
\[
T^2(f + g)(x) = \int_{R^3} \omega_k(x - y) \langle Pf(y) + Pg(y), Pf(y) + Pg(y)\rangle_{R^*} dy
\]
\[
= \int_{R^3} \omega_k(x - y) \langle Pf(y) + Pg(y), Pf(y)\rangle_{R^*} dy
\]
\[
+ \int_{R^3} \omega_k(x - y) \langle Pf(y) + Pg(y), Pg(y)\rangle_{R^*} dy
\]
\[
\leq T(f + g)(x)T(f)(x) + T(f + g)(x)T(g)(x),
\]
which implies $T(f + g) \leq T(f) + T(g)$. From this it follows that
\[
n(f + g) \leq \|T(f) + T(g)\|_{L^2} \leq n(f) + n(g).
\]
Also, we obviously have \( n(cf) = |c|n(f) \) for all \( c \in \mathbb{C} \). Due to (23) we have
\[
\|n(e^{-it(D)^m}f)\|_{L^2_t} \lesssim \|\omega_k * (e^{-it(D)^m}f, e^{-it(D)^m}f)_{\mathbb{R}^d}\|_{L^2_tL^2_x} \lesssim 2^{\frac{k}{2}}\|f\|_{L^2_x}
\] for all \( f \in L^2(\mathbb{R}^3) \).

Let \( \psi \in U^4_{(D)_m} \) be with atomic decomposition
\[
\psi = \sum_j c_j a_j, \quad \text{such that} \quad \sum_j |c_j| \leq 2\|\psi\|_{U^4_{(D)_m}}, \quad \text{and} \quad U^4_{(D)_m} \text{-atoms } a_j.
\]
We have
\[
\|n(\psi)\|_{L^4_t} \leq \sum_j |c_j|\|n(a_j)\|_{L^4_t} \lesssim 2^{\frac{k}{2}}\|\psi\|_{U^4_{(D)_m}}, \tag{47}
\]
provided that for any \( U^4_{(D)_m} \)-atom \( a \) the estimate
\[
\|n(a)\|_{L^4_t} \lesssim 2^{\frac{k}{2}}
\]
holds true. Indeed, let \( a(t) = \sum_k 1_{I_k}(t)e^{-it(D)^m}\varphi_k \), for some partition \( (I_k) \) of \( \mathbb{R} \) and \( \varphi_k \in L^2(\mathbb{R}^3) \) satisfying \( \sum_k \|\varphi_k\|_{L^2_x} \leq 1 \). Then,
\[
\|n(a)\|_{L^4_t} \leq \left( \sum_k \|1_{I_k}(t)e^{-it(D)^m}\varphi_k\|_{L^4_t} \right) \lesssim \left( \sum_k \|n(e^{-it(D)^m}\varphi_k)\|_{L^4_t}^4 \right)^{\frac{1}{4}} \lesssim 2^{\frac{k}{2}} \left( \sum_k \|\varphi_k\|_{L^2_x} \right) \lesssim 2^{\frac{k}{2}},
\]
where we used (46) in the third inequality, which completes the proof of (47). This implies
\[
\|\omega_k * (P\psi, P\psi)_{\mathbb{R}^d}\|_{L^2(\mathbb{R}^{1+3})} = \|n(\psi(t))\|_{L^2_t}^2 \lesssim 2^k\|\psi\|_{U^4_{(D)_m}}^2 \lesssim 2^k\|\psi\|_{V^4_{(D)_m}}^2,
\]
where we used \( V^2_{(D)_m} \hookrightarrow U^4_{(D)_m} \). Hence the claim (44) is established.

**Step 2:** Let \( \phi_i := P_{k_i}^m\phi_i, \ i = 1, 2 \). We may assume \( \|\phi_i\|_{V^2_{(D)_m}} = 1 \). The functions \( \phi_\pm := \phi_1 \pm \phi_2 \) satisfy \( \phi_\pm = P\phi_\pm, \|\phi_\pm\|_{V^2_{(D)_m}} \leq 1 \) and
\[
\text{Re}(\langle \phi_1, \phi_2 \rangle_{\mathbb{R}^d}) = \frac{1}{2} \left( \langle \text{Re} \phi_+, \text{Re} \phi_+ \rangle_{\mathbb{R}^d} - \langle \text{Re} \phi_-, \text{Re} \phi_- \rangle_{\mathbb{R}^d} \right.
\]
\[
+ \langle \text{Im} \phi_+, \text{Im} \phi_+ \rangle_{\mathbb{R}^d} - \langle \text{Im} \phi_-, \text{Im} \phi_- \rangle_{\mathbb{R}^d} \right),
\]
\[
\text{Im}(\langle \phi_1, \phi_2 \rangle_{\mathbb{R}^d}) = \text{Re}(-i\langle \phi_1, \phi_2 \rangle_{\mathbb{R}^d}).
\]
We have
\[
\|P_{\leq k}(\phi_1, \phi_2)_{\mathbb{R}^4}\|_{L^2(\mathbb{R}^{1+3})} \leq \|P_{\leq k} \text{Re}(\phi_1, \phi_2)_{\mathbb{R}^4}\|_{L^2(\mathbb{R}^{1+3})} + \|P_{\leq k} \text{Re}(-i\phi_1, \phi_2)_{\mathbb{R}^4}\|_{L^2(\mathbb{R}^{1+3})} \tag{48}
\]
Since \( P\text{Re} \phi_\pm = \text{Re} \phi_\pm \) and \( P\text{Im} \phi_\pm = \text{Im} \phi_\pm \) are real-valued, the estimate (44) yields
\[
\|P_{\leq k}(\phi_1, \phi_2)_{\mathbb{R}^4}\|_{L^2(\mathbb{R}^{1+3})} \lesssim 2^k \left( \|\phi_+\|_{V^2_{(D)_m}}^2 + \|\phi_-\|_{V^2_{(D)_m}}^2 \right) \lesssim 2^k.
\]
We can similarly bound the second term in (48) once we set $\tilde{\phi}_\pm = -i\phi_1 \pm \phi_2$. Thus we finally obtain

$$\|P_{\leq k}(P_{k_1}^m \psi_1, P_{k_2}^m \psi_2)_{R^4}\|_{L^2(R^{1+3})} \lesssim 2^k \|P_{k_1}^m \psi_1\|_{L^3_{(0)}(R^3)} \|P_{k_2}^m \psi_2\|_{L^3_{(0)}(R^3)},$$

which completes the proof since $P_k = P_{\leq k+1} - P_{\leq k}$.

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REFERENCES

[1] I. Bejenaru and S. Herr, The cubic Dirac equation: small initial data in $H^1(R^3)$, Comm. Math. Phys., 335 (2015), 43–82.

[2] I. Bejenaru and S. Herr, On global well-posedness and scattering for the massive Dirac-Klein-Gordon system, J. Eur. Math. Soc. (JEMS), 19 (2017), 2445–2467.

[3] J. M. Chadam and R. T. Glassey, On the Maxwell-Dirac equations with zero magnetic field and their solution in two space dimensions, J. Math. Anal. Appl., 53 (1976), 495–507.

[4] S. Machihara and K. Tsutaya, Scattering theory for the Dirac equation with a non-local term, J. Math. Anal. Appl., 398 (2013), 917–941.

[5] S. Herr and C. Yang, Critical well-posedness and scattering results for fractional Hartree-type equations in a critical space, Ann. Inst. H. Poincaré Anal. Non Linéaire, 26 (3) (2009) 917–941.

[6] S. Herr and A. Tesfahun, Small data scattering for semi-relativistic equations with Hartree type nonlinearity, J. Differential Equations, 259 (2015), 5510–5532.

[7] S. Herr and C. Yang, Critical well-posedness and scattering results for fractional Hartree-type equations, Differential Integral Equations, 31 (2018), 701–714.

[8] H. Koch, D. Tataru and M. Vişan, Dispersive Equations and Nonlinear Waves. Generalized Korteweg-De Vries, Nonlinear Schrödinger, Wave and Schrödinger Maps, Birkhäuser/Springer, 2014.

[9] E. Lenzmann, Well-posedness for semi-relativistic Hartree equations of critical type, Mathematical Physics, Analysis and Geometry, 10 (2007), 43–64.

[10] E. M. Stein, Harmonic Analysis: Real-variable Methods, Orthogonality, and Oscillatory Integrals, vol. 43 of Princeton Mathematical Series, Princeton University Press, Princeton, NJ, 1993, With the assistance of Timothy S. Murphy, Monographs in Harmonic Analysis, III.

[11] A. Tesfahun, Small data scattering for cubic Dirac equation with Hartree type nonlinearity in $R^{1+3}$, ArXiv e-prints.
[22] C. Yang, Small data scattering of semirelativistic hartree equation, *Nonlinear Analysis*, **178** (2019), 41–55.

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