Carlitz $q$-Bernoulli Numbers and $q$-Stirling Numbers

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Abstract. In this paper, we consider Carlitz $q$-Bernoulli numbers and $q$-stirling numbers of the first and the second kind. From the properties of $q$-stirling numbers, we derive many interesting formulae associated with Carlitz $q$-Bernoulli numbers. Finally, we will prove

$$\beta_{n,q} = \sum_{m=0}^{n} \sum_{k=0}^{n} \frac{1}{(1-q)^{n+m-k}} \sum_{d_0 + \cdots + d_k = n-k} q^{\sum_{i=0}^{k} d_i} s_1(q, k, m)(-1)^{n-m} \frac{m+1}{[m+1]_q},$$

where $\beta_{n,q}$ are called Carlitz $q$-Bernoulli numbers.

1. Introduction

Let $p$ be a fixed prime number. Throughout this paper, $\mathbb{Z}_p$, $\mathbb{Q}_p$, $\mathbb{C}$ and $\mathbb{C}_p$ will, respectively, denote the ring of $p$-adic rational integers, the field of $p$-adic rational numbers, the complex number field, and the completion of algebraic closure of $\mathbb{Q}_p$. For a fixed positive integer with $(p, d) = 1$, let

$$X = X_d = \lim_{N \to \infty} \mathbb{Z}/dp^N \mathbb{Z}, \quad X_1 = \mathbb{Z}_p,$$

$$X^* = \bigcup_{0 < a < dp, (a, p) = 1} a + dp\mathbb{Z}_p,$$

$$a + dp^N \mathbb{Z}_p = \{ x \in X \mid x \equiv a \pmod{dp^N} \},$$

where $a \in \mathbb{Z}$ lies in $0 \leq a < dp^N$, see [1-21]. The $p$-adic absolute value in $\mathbb{C}_p$ is normalized so that $|p|_p = 1/p$. When one talks of $q$-extension, $q$ is variously considered as an indeterminate, a complex number $q \in \mathbb{C}$ or a $p$-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}_p$, then we assume $|q-1|_p < p^{-\frac{1}{p-1}}$, so that $q^x = \exp(x \log q)$ for $|x|_p \leq 1$.

We use the notation $[x]_q = [x : q] = \frac{1 - q^x}{1 - q}$. For $f \in C^{(1)}(\mathbb{Z}_p) = \{ f \mid f' \in C(\mathbb{Z}_p) \}$, let us start with the expressions

$$\frac{1}{[p^N]_q} \sum_{0 \leq j < p^N} q^j f(j) = \sum_{0 \leq j < p^N} f(j) \mu_q(j + p^N \mathbb{Z}_p),$$

representing $q$-analogue of Riemann sums for $f$. The $p$-adic $q$-integral of a function $f \in C^{(1)}(\mathbb{Z}_p)$ is defined by

$$\int_X f(x) d\mu_q(x) = \int_{\mathbb{Z}_p} f(x) d\mu_q(x) = \lim_{N \to \infty} \frac{1}{[p^N]_q} \sum_{x=0}^{p^N-1} f(x) q^x, \quad \text{see [8]},$$

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For \( f \in C^{(1)}(\mathbb{Z}_p) \), it is easy to see that,
\[
| \int_{\mathbb{Z}_p} f(x) d\mu_q(x) |_p \leq p \| f \|_1, \quad \text{see} \ [6 - 14],
\]
where \( \| f \|_1 = \sup \left\{ |f(0)|_p, \sup_{x \neq y} \left| \frac{f(x) - f(y)}{x - y} \right|_p \right\} \). If \( f_n \to f \) in \( C^{(1)}(\mathbb{Z}_p) \), namely \( \| f_n - f \|_1 \to 0 \), then
\[
\int_{\mathbb{Z}_p} f_n(x) d\mu_q(x) \to \int_{\mathbb{Z}_p} f(x) d\mu_q(x), \quad \text{see} \ [6 - 10].
\]

The \( q \)-analogue of binomial coefficient was known as
\[
\begin{align*}
\left[ \frac{x}{n} \right]_q &= \left[ \frac{x}{n} \right] + q^x \left[ \frac{x}{n} \right]_q = q^{x-n} \left[ \frac{x}{n-1} \right] + \left[ \frac{x}{n} \right], \quad \text{cf.} \ [6, 10].
\end{align*}
\]

Thus, we have
\[
\int_{\mathbb{Z}_p} \left[ \frac{x}{n} \right] d\mu_q(x) = \frac{(-1)^n}{[n+1]_q} q^{n+1-(\frac{n+1}{2})}, \quad \text{If} \ f(x) = \sum_{k \geq 0} a_k q \left[ \frac{x}{k} \right] \text{is the } \ q \text{-anologue of Mahler series of strictly differentiable function } f, \text{then we see that}
\]
\[
\int_{\mathbb{Z}_p} f(x) d\mu_q(x) = \sum_{k \geq 0} a_k q \frac{(-1)^k}{[k+1]_q} q^{k+1-(\frac{k+1}{2})}.
\]

Carlitz \( q \)-Bernoulli numbers \( \beta_{k,q}(= \beta_k(q)) \) can be determined inductively by
\[
\beta_{0,q} = 1, \quad q(q\beta+1)^k - \beta_{k,q} = \begin{cases} 1 & \text{if } k = 1 \\ 0 & \text{if } k > 1, \end{cases}
\]
with the usual convention of replacing \( \beta^n \) by \( \beta_{1,q} \), (see [2, 3, 4]). In this paper, we study the \( q \)-stirling numbers of the first and the second kind. From these \( q \)-stirling numbers, we derive some interesting \( q \)-stirling numbers identities associated with Carlitz \( q \)-Bernoulli numbers. Finally we will prove the following formula :
\[
\beta_{n,q} = \sum_{m=q}^n \sum_{k=m} \frac{1}{(1-q)^{n-m-k}} \sum_{d_0 + \cdots + d_k = n-k} q^{\sum_{i=0}^k id_i} s_{1,q}(k,m)(-1)^{n-m} \frac{m+1}{[m+1]_q},
\]
where \( s_{1,q}(k,m) \) is the \( q \)-stirling number of the first kind.

2. \( q \)-Stirling numbers and Carlitz \( q \)-Bernoulli numbers
For \( m \in \mathbb{Z}_+ \), we note that
\[
\beta_{m,q} = \int_{\mathbb{Z}_p} [x]^m d\mu_q(x) = \int_X [x]^m d\mu_q(x).
\]
From this formula, we derive
\[
\beta_{0,q} = 1, \quad q(q\beta + 1)^k - \beta_{k,q} = \begin{cases} 1 & \text{if } k = 1 \\ 0 & \text{if } k > 1, \end{cases}
\]
with the usual convention of replacing \( \beta^i \) by \( \beta_{i,q} \). By the simple calculation of \( p \)-adic \( q \)-integral on \( \mathbb{Z}_p \), we see that
\[
\beta_{n,q} = \frac{1}{(1-q)^n} \sum_{i=0}^n \binom{n}{i} (-1)^i \frac{i+1}{[i+1]_q},
\]
where \( \binom{n}{i} = \frac{n!}{i!(n-i)!} = \frac{n(n-1)\cdots(n-i+1)}{i!} \). Let \( F(t) \) be the generating function of Carlitz \( q \)-Bernoulli numbers. Then we have
\[
F(t) = \sum_{n=0}^{\infty} \beta_{n,q} \frac{t^n}{n!} = \sum_{n=0}^{\infty} \lim_{\rho \to \infty} \frac{1}{[\rho]_q} \sum_{x=0}^{p^\rho-1} q^x e^{[x]_q t} \tag{2}
\]
\[
= \frac{1}{(1-q)^n} \sum_{k=0}^{\infty} \binom{n}{k} \frac{k+1}{[k+1]_q} (-1)^k \frac{t^n}{n!} \]
\[
= e^{1-q} \sum_{k=0}^{\infty} \frac{(-1)^k}{(1-q)^k [k+1]_q k!} \frac{t^k}{k!}
\]
From (2) we note that,
\[
F(t) = e^{1-q} + e^{1-q} \sum_{k=1}^{\infty} \frac{(-1)^k}{(1-q)^{k-1} [1-q^{k+1}]_q} \frac{t^k}{k!}
\]
\[
+ e^{1-q} \sum_{k=1}^{\infty} \frac{(-1)^k}{(1-q)^{k-1} [1-q^{k+1}]_q} \frac{t^k}{k!}
\]
\[
= -t \sum_{n=0}^{\infty} q^{2n} e^{[n]_q t} + (1-q) \sum_{n=0}^{\infty} q^n e^{[n]_q t}.
\]
Therefore we obtain the following:

**Lemma 1.** Let \( F(t) = \sum_{n=0}^{\infty} \int_{\mathbb{Z}_p} [x]^n d\mu_q(x) \frac{t^n}{n!} \). Then we have
\[
F(t) = -t \sum_{n=0}^{\infty} q^{2n} e^{[n]_q t} + (1-q) \sum_{n=0}^{\infty} q^n e^{[n]_q t}.
\]
The $q$-Bernoulli polynomials in the variable $x$ in $\mathbb{C}_p$ with $|x|_p \leq 1$ are defined by
\[
\beta_{n,q}(x) = \int_{\mathbb{Z}_p} [x + t]^n_q \, dq(t) = \int_X [x + t]^n_q \, dq(x).
\]

(4)

Thus we have
\[
\int_{\mathbb{Z}_p} [x + t]^n_q \, dq(t) = \sum_{k=0}^{n} \binom{n}{k} [x]^{n-k}_q \int_{\mathbb{Z}_p} [t]^k_q \, dq(t)
= \sum_{k=0}^{n} \binom{n}{k} [x]^{n-k}_q \beta_{k,q} = (q^x \beta + [x]_q)^n.
\]

From (4) we derive
\[
\int_{\mathbb{Z}_p} [x + t]^n_q \, dq(t) = \beta_{n,q}(x) = \frac{1}{(1-q)^n} \sum_{k=0}^{n} \binom{n}{k} (-1)^k q^{kx} \frac{k+1}{[k+1]_q}.
\]

(5)

Let $F(t, x)$ be the generating function of $q$-Bernoulli polynomials. By (5) we see that
\[
F(t, x) = \sum_{n=0}^{\infty} \beta_{n,q}(x) \frac{t^n}{n!} = e \frac{1-q}{1-q} \sum_{k=0}^{\infty} \frac{1}{(1-q)^k q^{kx} (-1)^k \frac{k+1}{[k+1]_q} t^k}.
\]

(6)

From (6) we note that
\[
F(t, x) = -t \sum_{n=0}^{\infty} q^{2n+x} e^{[n+x]_q t} + (1-q) \sum_{n=0}^{\infty} q^n e^{[n+x]_q t}.
\]

(7)

By (4) and (7), we easily see that
\[
[m]_q^{k-1} \sum_{i=0}^{m-1} q^i \beta_{k,q} \frac{x + i}{m} = \beta_{k,q}(x), \quad m \in \mathbb{N}, k \in \mathbb{Z}_+.
\]

(8)

If we take $x = 0$ in (8), then we have
\[
[n]_q \beta_{n,q} = \sum_{k=0}^{m} \binom{m}{k} \beta_{k,q} \sum_{j=0}^{n-1} q^{j(k+1)} [j]_q^{n-k}.
\]

By (2), (6) and (7), we see that
\[
- \sum_{i=0}^{\infty} q^{2i+n} e^{[n+l]_q t} + \sum_{i=0}^{\infty} q^{2i+l} e^{[l]_q t} = \sum_{m=1}^{\infty} \frac{1}{m! \sum_{i=0}^{\infty} q^{2i+l} [m]_q^{m-1} t^{m-1}}.
\]

(9)
Note that \( \sum_{l=0}^{\infty} q^{2l+n}e^{[n+l]q^t} + \sum_{l=0}^{\infty} q^{2l}e^{[l]q^t} = \frac{1}{t}(F(t, n) - F(t)). \) Thus, we have

\[
\sum_{m=0}^{\infty} (\beta_{m,q}(n) - \beta_{m,q}) \frac{t^m}{m!} = \sum_{m=0}^{\infty} (m \sum_{l=0}^{n-1} q^{2[l]q^{m-1}}) \frac{t^m}{m!}, \tag{10}
\]

By comparing the coefficients on both sides in (10), we see that

\[
\beta_{m,q}(n) - \beta_{m,q} = m \sum_{l=0}^{n-1} q^{2[l]q^{m-1}}. \tag{11}
\]

Therefore we obtain the following:

**Proposition 2.** For \( m, n \in \mathbb{N} \), we have

\[
(q - 1) \sum_{l=0}^{n-1} q^{l[q]} + \sum_{l=0}^{n-1} q^{l[q]} = \frac{1}{m} \sum_{l=0}^{m-1} \binom{m}{l} [n]_q^{m-l} q^{n[l]q \beta_{l,q}} + (q^{mn} - 1) \beta_{m,q}.
\]

Now we consider the \( q \)-analogue of Jordan factor as follows:

\[
[x]_{k,q} = [x]_q[x - 1]_q \cdots [x - k + 1]_q = \frac{(1 - q^x)(1 - q^{x-1}) \cdots (1 - q^{x-k+1})}{(1 - q)^k}.
\]

The \( q \)-binomial coefficient is defined by

\[
\binom{n}{k}_q = \frac{[n]_q!}{[k]_q! [n-k]_q!} = \frac{(1 - q^n)(1 - q^{n-1}) \cdots (1 - q^{n-k+1})}{(1 - q)(1 - q^2) \cdots (1 - q^k)}, \tag{12}
\]

where \([n]_q! = [n]_q[n-1]_q \cdots [2]_q[1]_q\). The \( q \)-binomial formulas are known as

\[
\prod_{i=1}^{n} (a + bq^{i-1}) = \sum_{k=0}^{n} \binom{n}{k}_q q^{\binom{k}{2}} a^{n-k} b^k, \tag{13}
\]

and

\[
\prod_{i=1}^{n} (1 - bq^{i-1})^{-1} = \sum_{k=0}^{n} \binom{n+k-1}{k}_q b^k.
\]

The \( q \)-Stirling numbers of the first kind \( s_{1,q}(n, k) \) and the second kind \( s_{2,q}(n, k) \) are defined as

\[
[x]_{n,q} = q^{-\binom{n}{2}} \sum_{l=0}^{n} s_{1,q}(n, l) [x]_q^l, \quad n = 0, 1, 2, \cdots, \tag{14}
\]

and

\[
[x]_q^n = \sum_{k=0}^{n} q^{\binom{k}{2}} s_{2,q}(n, k) [x]_{k,q}, \quad n = 0, 1, 2, \cdots, \text{ see } [2, 3, 6]. \tag{15}
\]
The values \( s_1,q(n,1), \ n = 1,2,3, \cdots \), and \( s_2,q(n,2), \ n = 2,3, \cdots \), may be deduced from the following recurrence relation:

\[
s_1,q(n,k) = s_1,q(n-1,k-1) - [n-1]_q s_1,q(n-1,k), \quad \text{see [2, 3, 6]},
\]

for \( k = 1,2, \cdots, n, \ n = 1,2, \cdots \), with initial conditions \( s_1,q(0,0) = 1, s_1,q(n,k) = 0 \) if \( k > n \). For \( k = 1 \), it follows that

\[
s_1,q(n,1) = -[n-1]_q s_1,q(n-1,1), \ n = 2,3, \cdots,
\]

and since \( s_1,q(1,1) = 1 \), we have \( s_1,q(n,1) = (-1)^{n-1}[n-1]_q !, \ n = 1,2,3, \cdots \). The recurrence relation for \( k = 2 \) reduce to \( s_1,q(n,2) + [n-1]_q s_1,q(n-1,2) = (-1)^{n-2}[n-2]_q !, \ n = 3,4, \cdots \). By simple calculation, we easily see that

\[
\frac{(-1)^{n+1}s_1,q(n+1,2)}{[n]_q !} - \frac{(-1)^n s_1,q(n,2)}{[n-1]_q !} = (-1)^{n+1} \frac{s_1,q(n+1,2) - [n]_q s_1,q(n,2)}{[n]_q !} = (-1)^{n+1} \frac{(-1)^{n+1}[n-1]_q !}{[n]_q !} = \frac{1}{[n]_q}, \ n = 2,3,4, \cdots.
\]

Thus we have

\[
\frac{(-1)^n s_1,q(n,2)}{[n-1]_q !} = \sum_{k=1}^{n-1} \frac{1}{[k]_q}.
\]

This is equivalent to \( s_1,q(n,2) = (-1)^n [n-1]_q ! \sum_{k=1}^{n-1} \frac{1}{[k]_q} \). It is easy to see that

\[
\sum_{m=1}^{n} (-1)^{m+1} q^{\left(m+1\right)} \left[\begin{array}{c}
m+1 \\m+1
\end{array}\right] (m+1) \sum_{q=1}^{m} \frac{1}{[k]_q} = \sum_{k=1}^{n} (-1)^{k+1} q^{\left(k+1\right)} \left[\begin{array}{c}
n \\k
\end{array}\right] \frac{\left[\begin{array}{c}
k \\k
\end{array}\right]}{[k]_q}.
\]

From this, we derive

\[
\sum_{k=1}^{n} (-1)^{k+1} q^{\left(k+1\right)} \left[\begin{array}{c}
n \\k
\end{array}\right] \frac{1}{[k]_q} = \sum_{k=1}^{n} (-1)^{k+1} q^{\left(k+1\right)} \left[\begin{array}{c}
n \\k
\end{array}\right] \frac{\left[\begin{array}{c}
k \\k
\end{array}\right]}{[k]_q} = \frac{q^n}{[n]_q} \sum_{k=1}^{n} (-1)^{k+1} q^{\left(k+1\right)} \left[\begin{array}{c}
n \\k
\end{array}\right] = \frac{q^n}{[n]_q}.
\]

Note that \( \sum_{k=1}^{n} (-1)^{k+1} q^{\left(k+1\right)} \left[\begin{array}{c}
n \\k
\end{array}\right] = - \sum_{k=0}^{n} (-1)^k q^{\left(k\right)} \left[\begin{array}{c}
n \\k
\end{array}\right] + 1 = 1. \) Thus, we have

\[
\sum_{k=1}^{n} (-1)^{k+1} q^{\left(k+1\right)} \left[\begin{array}{c}
n \\k
\end{array}\right] = \sum_{k=1}^{n-1} (-1)^{k+1} q^{\left(k+1\right)} \left[\begin{array}{c}
k \\k
\end{array}\right] \frac{\left[\begin{array}{c}
k \\k
\end{array}\right]}{[k]_q} + \frac{q^n}{[n]_q}.
\]
Continuing this process, we see that
\[\sum_{k=1}^{n} (-1)^{k+1} q^{\frac{k+1}{2}} \frac{n}{k} \frac{[k]}{[q]^k} = \sum_{k=1}^{n} q^n.\]

The $p$-adic $q$-gamma function is defined as $\Gamma_{p,q}(n) = (-1)^n \prod_{(j,p)=1}^n [j]_q$. For all $x \in \mathbb{Z}_p$, we have $\Gamma_{p,q}(x+1) = E_{p,q}(x) \Gamma_{p,q}(x)$, where $E_{p,q}(x) = \begin{cases} -[x]_q & \text{if } |x|_p = 1 \\ -1 & \text{if } |x|_p < 1. \end{cases}$

Thus, we easily see that
\[\log \Gamma_{p,q}(x+1) = \log E_{p,q}(x) + \log \Gamma_{p,q}(x). \tag{16}\]

From the differentiating on both sides in (16), we derive
\[\frac{\Gamma'_{p,q}(x+1)}{\Gamma_{p,q}(x+1)} - \frac{\Gamma'_{p,q}(x)}{\Gamma_{p,q}(x)} + \frac{E'_{p,q}(x)}{E_{p,q}(x)}.\]

Continuing this process, we have
\[\frac{\Gamma'_{p,q}(x)}{\Gamma_{p,q}(x)} = \left(\sum_{j=1}^{x-1} \frac{q^j}{[j]_q}\right) \frac{\log q}{q-1} + \frac{\Gamma'_{p,q}(1)}{\Gamma_{p,q}(1)}.
\]

The classical Euler constant is known as $\gamma = \frac{\Gamma'(1)}{\Gamma(1)}$. In [15], Koblitz defined the $p$-adic $q$-Euler constant as
\[\gamma_{p,q} = -\frac{\Gamma'_{p,q}(1)}{\Gamma_{p,q}(1)}.\]

Therefore, we obtain the following:

**Theorem 3.** For $x \in \mathbb{Z}_p$, we have
\[\sum_{k=1}^{x-1} (-1)^{k+1} q^{\frac{k+1}{2}} \frac{x-1}{k} \frac{[k]}{[q]^k} = \frac{q-1}{\log q} \left(\frac{\Gamma'_{p,q}(x)}{\Gamma_{p,q}(x)} - \gamma_{p,q}\right).\]

From (5), (12), (14) and (15), we derive the following theorem:

**Theorem 4.** For $n, k \in \mathbb{Z}_+$, we have
\[\beta_{n,q} = \frac{1}{(1-q)^n} \sum_{l=0}^{n} \binom{n}{l} (-1)^l \sum_{k=0}^{l} (q-1)^k \sum_{q^m=0}^{k} s_{1,q}(k,m) \beta_{m,q},\]

where $s_{1,q}(k,m)$ is the $q$-Stirling number of the first kind.
By simple calculation, we easily see that
\[ q^{nt} = ([t]_q(q - 1) + 1)^n = \sum_{m=0}^{n} \binom{n}{m}(-1)^m(1 - q)^m[t]_q^m = \sum_{k=0}^{n} (q - 1)^k q^{\frac{n}{k}} \left[ m \right]_q [t]_{k,q} \]
\[ = \sum_{k=0}^{n} (q - 1)^k \sum_{m=0}^{k} s_{1,q}(k, m)[t]_q^m = \sum_{m=0}^{n} \left( \sum_{k=m}^{n} (q - 1)^k \left[ m \right]_q s_{1,q}(k, m) \right) [t]_q^m. \]
Thus we note
\[ \int_{\mathbb{Z}_p} q^{nt} d\mu_q(t) = \sum_{m=0}^{n} \left( \sum_{k=m}^{n} (q - 1)^k \left[ m \right]_q s_{1,q}(k, m) \right) \beta_{m,q}. \tag{17} \]
From the definition of $p$-adic $q$-integral on $\mathbb{Z}_p$, we also derive
\[ \int_{\mathbb{Z}_p} q^{nt} d\mu_q(t) = \sum_{m=0}^{n} \binom{n}{m} (q - 1)^m \beta_{m,q}. \tag{18} \]
By comparing the coefficients on the both sides of (17) and (18), we see that
\[ \binom{n}{m} (q - 1)^m = \sum_{k=m}^{n} (q - 1)^k \left[ m \right]_q s_{1,q}(k, m). \]
Therefore we obtain the following:

**Theorem 5.** For $n \in \mathbb{N}, m \in \mathbb{Z}_+$, we have
\[ \binom{n}{m} = \sum_{k=m}^{n} (q - 1)^{-m+k} \left[ m \right]_q s_{1,q}(k, m). \]

From Theorem 5, we can also derive the following interesting formula for $q$-Bernoulli numbers:

**Theorem 6.** For $n \in \mathbb{Z}_+$, we have
\[ \beta_{n,q} = \frac{1}{(1 - q)^n} \sum_{m=0}^{n} \left( \sum_{k=m}^{n} (q - 1)^{-m+k} \left[ m \right]_q s_{1,q}(k, m) \right) (-1)^m \frac{m + 1}{[m + 1]_q}. \]

From the definition of $q$-binomial coefficient, we easily derive
\[ \left[ \frac{x + 1}{n} \right]_q = \left[ \frac{x}{n - 1} \right]_q + q^x \left[ \frac{x}{n} \right]_q = q^{x-n} \left[ \frac{x}{n - 1} \right]_q + \left[ \frac{x}{n} \right]_q. \tag{19} \]
By (19), we see that
\[ \int_{\mathbb{Z}_p} \left[ \frac{x}{n} \right]_q d\mu_q(x) = \frac{(-1)^n}{[n + 1]_q} q^{n+1-n \choose 2}. \tag{20} \]
From the definition of \( q \)-Stirling number of the first kind, we also note that

\[
\int_{\mathbb{Z}_p} \left[ x \right]_n q d\mu_q(x) = [n]_q! \int_{\mathbb{Z}_p} \left[ x \right]_n q d\mu_q(x) = q^{-\left( \right)} \sum_{k=0}^{n} s_{1,q}(n,k) \beta_{k,q}. \tag{21}
\]

By using (20), (21), we see

\[
(-1)^n \frac{[n]_q!}{[n+1]_q} = \sum_{k=0}^{n} s_{1,q}(n,k) \beta_{k,q}. \tag{22}
\]

From (15) and (21), we derive

\[
\beta_{n,q} = q \sum_{k=0}^{n} s_{2,q}(n,k)(-1)^k \frac{[k]_q!}{[k+1]_q}.
\]

Therefore we obtain the following:

**Theorem 7.** For \( n \in \mathbb{Z}_+ \), we have

\[
\beta_{n,q} = q \sum_{k=0}^{n} s_{2,q}(n,k)(-1)^k \frac{[k]_q!}{[k+1]_q},
\]

where \( s_{2,q}(n,k) \) is the \( q \)-Stirling number of the second kind.

It is easy to see that

\[
\left[ \frac{n}{k} \right]_q = \sum_{d_0 + \cdots + d_k = n-k} q^{\sum_{i=0}^{k} i d_i}. \tag{23}
\]

By Theorem 4, we have the following:

**Theorem 8.** For \( n \in \mathbb{Z}_+ \), we have

\[
\beta_{n,q} = \sum_{m=0}^{n} \sum_{k=m}^{n} \frac{1}{(1-q)^{n+m-k}} \sum_{d_0 + \cdots + d_k = n-k} q^{\sum_{i=0}^{k} i d_i} s_{1,q}(k,m)(-1)^{n-m} \frac{m+1}{[m+1]_q},
\]

where \( s_{1,q}(k,m) \) is the \( q \)-Stirling number of the first kind.

**References**

[1] C. Adiga, N. Anitha, *On some continued fractions of Ramanujan*, Adv. Stud. Contemp. Math., 12(1)(2006), 155–162.

[2] L. Carlitz, *q-Bernoulli numbers and polynomials*, Duke Math. J., 15(1948), 987–1000.
[3] L. Carlitz, *q-Bernoulli and Eulerian numbers*, Trans. Amer. Math. Soc., 76(1954), 332–350.

[4] A.S. Hegazi, M. Mansour, *A note on q-Bernoulli numbers and polynomials*, J. Nonlinear Math. Phys., 13(2006), 9-18.

[5] T. Kim, C. Adiga, *On the q-analogue of gamma functions and related inequalities*, JIPAM. J. Inequal. Pure Appl. Math., 6(4)(2005), Article 118, 4 pp. (electronic).

[6] T. Kim, *q-Volkenborn integration*, Russ. J. Math. Phys., 9(3)(2002), 288–299.

[7] T. Kim, *On the analogs of Euler numbers and polynomials associated with p-adic q-integral on Zp at q = −1*, J. Math. Anal. Appl., 331(2)(2007), 779–792.

[8] T. Kim, *On a q-analogue of the p-adic log gamma functions and related integrals*, J. Number Theory, 76(2)(1999), 320–329.

[9] T. Kim, *On p-adic q-L-functions and sums of powers*, Discrete Math., 252(1-3)(2002), 179–187.

[10] T. Kim, S.-D. Kim, D.-W. Park, *On uniform differentiability and q-Mahler expansions*, Adv. Stud. Contemp. Math., 4(1)(2001), 35–41.

[11] T. Kim, *A note on the q-multiple zeta function*, Adv. Stud. Contemp. Math., 8(2)(2004), 111–113.

[12] T. Kim, *Sums of powers of consecutive q-integers*, Adv. Stud. Contemp. Math., 9(1)(2004), 15–18.

[13] T. Kim, *A note on p-adic invariant integral in the rings of p-adic integers*, Adv. Stud. Contemp. Math., 13(1)(2006), 95–99.

[14] T. Kim, *A note on some formulas for the q-Euler numbers and polynomials*, Proceedings of the Jangjeon Mathematical Society, 9(2)(2006), 227–232.

[15] N. Koblitz, *q-extension of the p-adic gamma function*, Trans. Amer. Math. Soc., 260(2)(1980), 449–457.

[16] H. Ozden, Y. Simsek, I. N. Cangul, *A note on p-adic q-Euler measure*, Adv. Stud. Contemp. Math., 14(2)(2007), 233–239.

[17] M. Schork, *Ward’s “calculus of sequences”, q-calculus and the limit q → −1*, Adv. Stud. Contemp. Math., 13(2)(2006), 131–141.

[18] Y. Simsek, *On p-adic twisted q – L-functions related to generalized twisted Bernoulli numbers*, Russ. J. Math. Phys., 13(3)(2006), 340–348.

[19] Y. Simsek, *Generalized Dedekind sums associated with the Abel sum and the Eisenstein and Lambert series*, Adv. Stud. Contemp. Math., 9(2)(2004), 125–137.

[20] Y. Simsek, *Twisted (h, q)-Bernoulli numbers and polynomials related to twisted (h, q)-zeta function and L-function*, J. Math. Anal. Appl., 324(2)(2006), 790–804.

[21] H.M. Srivastava, T. Kim, Y. Simsek, *q-Bernoulli numbers and polynomials associated with multiple q-zeta functions and basic L-series*, Russ. J. Math. Phys., 12(2)(2005), 241–268.