Some extensions of the Einstein-Dirac Equation

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Abstract: We considered an extension of the standard functional for the Einstein-Dirac equation where the Dirac operator is replaced by the square of the Dirac operator and a real parameter controlling the length of spinors is introduced. For one distinguished value of the parameter, the resulting Euler-Lagrange equations provide a new type of Einstein-Dirac coupling. We establish a special method for constructing global smooth solutions of a newly derived Einstein-Dirac system called the CL-Einstein-Dirac equation of type II (see Definition 3.1).

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1 Introduction

Let \((Q^{n,r}, \eta)\) be an n-dimensional (connected smooth) pseudo-Riemannian manifold, where the index \(r\) is the number of negative eigenvalues of the metric \(\eta\). Assume that \((Q^{n,r}, \eta)\) is space- and time-oriented and has a fixed spin structure [1]. For simplicity, we will often write \(Q\) to mean \(Q^{n,r}\). Let \(\Sigma(Q) = \Sigma(Q, \eta)\) denote the spinor bundle of \((Q^{n,r}, \eta)\) equipped with the \(\text{Spin}^+(n, r)\)-equivariant nondegenerate complex product \(\langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_{\eta}\), and let \((\cdot, \cdot) = \text{Re}\langle \cdot, \cdot \rangle\) denote the real part of \(\langle \cdot, \cdot \rangle\). Let \(\text{Ric} = \text{Ric}_{\eta}\) and \(S = S_{\eta}\) be the Ricci tensor and the scalar curvature of \((Q^{n,r}, \eta)\), respectively. Let \(D = D_{\eta}\) be the Dirac operator acting on sections \(\psi \in \Gamma(\Sigma(Q))\) of the spinor bundle \(\Sigma(Q)\). Then the standard functional for the Einstein-Dirac equation is given by

\[
W_1(\eta, \psi) = \int \left\{ aS_{\eta} + b + \epsilon \nu_1(\psi, \psi) - \epsilon ((\sqrt{-1})^rD_{\eta}\psi, \psi) \right\} \mu_\eta,
\]

where \(a, b, \epsilon, \nu_1 \in \mathbb{R}\), \(\epsilon \neq 0\), are real numbers and \(\mu_\eta\) is the volume form of \((Q^{n,r}, \eta)\). The Euler-Lagrange equations (called the Einstein-Dirac equation) are the Dirac equation

\[
(\sqrt{-1})^rD\psi = \nu_1 \psi
\]

and the Einstein equation

\[
a \left\{ \text{Ric} - \frac{S}{2} \eta \right\} - \frac{b}{2} \eta = \frac{\epsilon}{4} T_1
\]
coupled via a symmetric tensor field $T_1$,

$$T_1(X,Y) = \left((\sqrt{-1})^r \{ X \cdot \nabla_Y \psi + Y \cdot \nabla_X \psi \}, \psi \right),$$  \hspace{1cm} (1.4)

where $X, Y$ are vector fields on $Q^{n,r}$ and the dot “·” indicates the Clifford multiplication. Observe that the system (1.2)-(1.4) contains four differential operators, namely, the spin connection $\nabla$, the Dirac operator $D$, the Ricci tensor $	ext{Ric}$ and the scalar curvature $S$. The spin connection and the Dirac operator act on spinor fields and are operators of first-order, while the Ricci tensor and the scalar curvature are second-order operators acting on metrics. Therefore, it is natural to ask whether one can derive such Euler-Lagrange equations from the functional

$$W_2(\eta, \psi) = \int \left\{ aS_\eta + b + \epsilon \nu_2(\psi, \psi) - \epsilon((D_\eta \circ D_\eta)(\psi), \psi) \right\} \mu_\eta,$$

$$\nu_2 \in \mathbb{R},$$  \hspace{1cm} (1.5)

that generalize the system (1.2)-(1.4) and all the involved operators acting on spinor fields are of second-order. In Section 2 we will show that the answer of the question is positive and (1.5) yields in fact the following system (see Theorem 2.1):

$$D^2\psi = \nu_2\psi,$$  \hspace{1cm} (1.6)

$$a\left\{ \text{Ric} - \frac{S}{2} \eta \right\} - \frac{b}{2} \eta = \frac{\epsilon}{4} T_2,$$

where $T_2$ is a symmetric tensor field defined by

$$T_2(X,Y) = \left( X \cdot \nabla_Y (D\psi) + Y \cdot \nabla_X (D\psi), \psi \right)$$

$$+ (-1)^r \left( X \cdot \nabla_Y \psi + Y \cdot \nabla_X \psi, D\psi \right).$$  \hspace{1cm} (1.7)

In this paper the system (1.2)-(1.4) is called the \textit{classical Einstein-Dirac equation of type I} [5, 6, 7] and the system (1.6)-(1.7) the \textit{classical Einstein-Dirac equation of type II}.

Let us turn to another situation where a real parameter controlling the length of spinors is introduced. Let $\varphi = \varphi_\eta$ be a spinor field on $(Q^{n,r}, \eta)$ such that either $(\varphi, \varphi) > 0$ at all points or $(\varphi, \varphi) < 0$ at all points. Fix a shorthand notation

$$\varphi^k := (\sigma \varphi, \varphi^k \varphi), \hspace{1cm} \varphi^0 := \varphi,$$

where $k \in \mathbb{R}$ is a real number and $\sigma = \sigma_\varphi \in \mathbb{R}$ is a constant defined by

$$\sigma = 1 \text{ if } (\varphi, \varphi) > 0 \hspace{1cm} \text{and} \hspace{1cm} \sigma = -1 \text{ if } (\varphi, \varphi) < 0.$$

Combining the functional (1.1) with (1.5), we extend the spinorial part as

$$W(\eta, \varphi) = \int \left\{ aS_\eta + b + \epsilon \nu(\sigma \varphi^k, \varphi^k) - \epsilon(\sigma P_\eta(\varphi^k), \varphi^k) \right\} \mu_\eta,$$

$$\nu \in \mathbb{R},$$  \hspace{1cm} (1.8)

where $P_\eta = (\sqrt{-1})^r D_\eta$ or $P_\eta = D_\eta \circ D_\eta$, and look at the Euler-Lagrange equations derived from (1.8). We will show in Section 3 (see Theorem 3.1) that, when $k \neq \frac{-1}{2}$, the
Euler-Lagrange equations of (1.8) are actually equivalent to the system (1.2)-(1.4) or to
the system (1.6)-(1.7) depending on a choice of $P_\eta$. However, in the distinguished case
$k = -\frac{1}{2}$ in which the length $|\varphi^k| = \pm 1$ becomes constant, we are led to a new Einstein-
Dirac system, i.e.,

$$P_\eta \psi = f \psi,$$

$$a\left\{\text{Ric} - \frac{S}{2} \eta\right\} - \frac{c}{2} \eta = \frac{\epsilon}{4} T - \frac{\epsilon}{2} f \eta, \quad a, c, \epsilon \in \mathbb{R},$$

(1.9)

where $\psi$ is of constant length $|\psi| = \pm 1$ and $f : Q^{n,r} \to \mathbb{R}$ is a real-valued function and $T$
is a symmetric tensor field defined by

$$T(X, Y) = \left(\sigma(\sqrt{-1})^r \{X \cdot \nabla_Y \psi + Y \cdot \nabla_X \psi\}, \psi\right)$$

(1.10)

if $P_\eta = (\sqrt{-1})^r D_\eta$ and by

$$T(X, Y) = \sigma \left(X \cdot \nabla_Y (D\psi) + Y \cdot \nabla_X (D\psi), \psi\right)$$

$$+ \sigma(-1)^r \left(X \cdot \nabla_Y \psi + Y \cdot \nabla_X \psi, D\psi\right)$$

(1.11)

if $P_\eta = D_\eta \circ D_\eta$, respectively. The system (1.9)-(1.11) will be called the CL-Einstein-
Dirac equation of type I if $P_\eta = (\sqrt{-1})^r D_\eta$ and the CL-Einstein-Dirac equation of type II
if $P_\eta = D_\eta \circ D_\eta$, respectively ("CL" means the "constant length" of spinors). A non-
trivial spinor field $\psi$ on $(Q^{n,r}, \eta)$ is called a CL-Einstein spinor of type I (resp. type II) if
it satisfies the CL-Einstein-Dirac equation of type I (resp. type II). It will be pointed out
(see Remark 3.1) why one can not weaken the "constant length" condition for CL-Einstein
spinors.

Sections 4 and 5 of the paper are devoted to establishing a special method for con-
structing global (smooth) solutions of the CL-Einstein-Dirac equation of type II. The
essential idea of this construction is the fact that, under conformal change of metrics, the
CL-Einstein-Dirac equation of type II behave in a relatively stable way (more stable than
the CL-Einstein-Dirac equation of type I and both types of the classical Einstein-Dirac
equation). More precisely, we show in Section 4 that if $(Q^{n,r}, \eta)$ admits a non-trivial
spinor field $\psi$, called a reduced weakly parallel spinor, satisfying the differential equation
in Definition 4.3, then over the manifold $(Q^{n,r}, \eta)$ with conformally changed metric
$\tilde{\eta} = e^u \eta$ the pullback $\tilde{\psi}$ of $\psi$ becomes a CL-Einstein spinor of type II (see Theorem 4.2).
Parallel spinors [8] are trivial examples for reduced weakly parallel spinors. In Section 5
we will provide examples for reduced weakly parallel spinors that are not parallel spinors
(see Theorem 5.2).

2 Coupling of the square of the Dirac operator to the Ein-
stein equation

We first recall the process of obtaining the classical Einstein-Dirac equation of type I in
pseudo-Riemannian signature [6, 7]. Applying the process to the behaviour of the square
of the Dirac operator under change of metrics, we then derive the classical Einstein-Dirac equation of type II.

Let $h$ be a symmetric $(0,2)$-tensor field on $(Q^n, \eta)$, and let $H$ be the $(1,1)$-tensor field induced by $h$ via $h(X,Y) = \eta(X,H(Y))$. Then the tensor field $\tilde{\eta}$ defined by

$$\tilde{\eta}(X,Y) = \eta(X,e^H(Y)) = \eta(e^{\frac{h}{2}}(X),e^{\frac{h}{2}}(Y))$$ \hspace{1cm} (2.1)$$

is a pseudo-Riemannian metric of the same index $r$. Let $K := e^{\frac{H}{2}}$ and let $\Lambda$ be the $(1,2)$-tensor field defined by

$$2\eta(\Lambda(X,Y), Z) = \eta\left(Z, K\{((\nabla^\eta_{K^{-1}(X)}K^{-1})(Y)) - K\{((\nabla^\eta_{K^{-1}(Y)}K^{-1})(X))\right\}$$

$$+ \eta\left(Y, K\{((\nabla^\eta_{K^{-1}(Z)}K^{-1})(X)) - K\{((\nabla^\eta_{K^{-1}(X)}K^{-1})(Z))\right\}$$

$$+ \eta\left(X, K\{((\nabla^\eta_{K^{-1}(Z)}K^{-1})(Y)) - K\{((\nabla^\eta_{K^{-1}(Y)}K^{-1})(Z))\right\}.$$

Then the Levi-Civita connections $\nabla^\pi$ and $\nabla^\eta$ are related by

$$\nabla^\pi_{K^{-1}(X)}(K^{-1}(Y)) = K^{-1}\left(\nabla^\eta_{K^{-1}(X)}Y\right) + K^{-1}\{\Lambda(X,Y)\}. \hspace{1cm} (2.2)$$

Let $\tilde{K} : \Sigma(Q)^\pi \rightarrow \Sigma(Q)^\eta$ be a natural isomorphism preserving the inner product of spinors and the Clifford multiplication with

$$\langle \tilde{K}(\varphi), \tilde{K}(\psi) \rangle_\eta = \langle \varphi, \psi \rangle_\pi, \quad (KX) \cdot (\tilde{K}\psi) = \tilde{K}(X \cdot \psi) \hspace{1cm} (2.3)$$

for all $X \in \Gamma(T(Q))$, $\varphi, \psi \in \Gamma(\Sigma(Q)^\pi)$, where the dot "." in the latter relation indicates the Clifford multiplication with respect to $\eta$ and $\tilde{\eta}$, respectively. Let $(E_1, \ldots, E_n)$ be a local $\eta$-orthonormal frame field on $(Q^n, \eta)$. For shortness we introduce the notation $\chi(i) := \eta(E_i, E_i)$ and $\chi(i_1 \ldots i_s) := \chi(i_1) \chi(i_2) \cdot \chi(i_s)$ for $1 \leq s \leq n$. Then, because of (2.2), the spinor derivatives $\nabla^\eta, \nabla^\pi$ are related by [4]

$$\tilde{K} \circ \nabla^\pi_{K^{-1}(E_j)} \circ (\tilde{K})^{-1}(\psi) = \nabla^\eta_{K^{-1}(E_j)}\psi + \frac{1}{4} \sum_{k,l=1}^{n} \chi(kl) \Lambda_{jkl} E_k \cdot E_l \cdot \psi, \hspace{1cm} (2.4)$$

where $\Lambda_{jkl} := \eta(\Lambda(E_j, E_k, E_l)$, and the Dirac operators $D_\eta$, $D_\pi$ by

$$\{\tilde{K} \circ D_\pi \circ (\tilde{K})^{-1}\}(\psi)$$

$$= \sum_{i=1}^{n} \chi(i)E_i \cdot \nabla^\eta_{K^{-1}(E_i)}\psi + \frac{1}{4} \sum_{j,k,l=1}^{n} \chi(jkl) \Lambda_{jkl} E_j \cdot E_k \cdot E_l \cdot \psi$$

$$= \sum_{i=1}^{n} \chi(i)E_i \cdot \nabla^\eta_{K^{-1}(E_i)}\psi - \frac{1}{2} \sum_{j,k=1}^{n} \chi(jk) \Lambda_{jkk} E_k \cdot \psi$$

$$+ \frac{1}{2} \sum_{j<k<l}^{n} \chi(jkl)(\Lambda_{jkl} + \Lambda_{klj} + \Lambda_{ljk}) E_j \cdot E_k \cdot E_l \cdot \psi.$$ \hspace{1cm} (2.5)
In order to compute the infinitesimal variation of the Dirac operator, we consider an one-parameter family of metrics of index \( r \),

\[
\eta_t(X, Y) := \eta(X, e^{tH}(Y)) = \eta(e^{\frac{it}{\sqrt{r}}}X, e^{\frac{it}{\sqrt{r}}}Y), \quad \eta_0 := \eta, \quad t \in \mathbb{R},
\]

which is generated by a symmetric (0,2)-tensor field \( h \) on \( (Q^{n,r}, \eta) \). Let \( \Lambda_t \) be the (1,2)-tensor in (2.2) determined by the pair \( (\nabla^h, \nabla^\eta) \) of the Levi-Civita connections (with \( K_t = e^{\frac{it}{\sqrt{r}}} \)). Let \( \Omega_t \) be a 3-form generated by the tensor \( \Lambda_t \) via

\[
\Omega_t(X, Y, Z) = \eta(\Lambda_t(X, Y), Z) + \eta(\Lambda_t(Y, Z), X) + \eta(\Lambda_t(Z, X), Y).
\]

Then direct computations show:

**Lemma 2.1**

\[
\begin{align*}
\lim_{t \to 0} \frac{d}{dt} \left\{ \Lambda_t(X, Y) - \Lambda_t(Y, X) \right\} &= -\frac{1}{2}(\nabla^\eta_X H)(Y) + \frac{1}{2}(\nabla^\eta_Y H)(X), \\
\lim_{t \to 0} \frac{d}{dt} \eta(\Lambda_t(X, Y), Z) &= \frac{1}{2} \eta((\nabla^\eta_Y H)(X), Z) - \frac{1}{2} \eta((\nabla^\eta_Z H)(X), Y), \\
\lim_{t \to 0} \frac{d}{dt} \Omega_t(X, Y, Z) &= 0.
\end{align*}
\]

Applying Lemma 2.1 to (2.5), we arrive at the variation formula of the Dirac operator:

\[
\lim_{t \to 0} \frac{d}{dt} \left\{ \widehat{K}_t \circ D_\eta \circ (\widehat{K}_t)^{-1} \right\}(\psi) = -\frac{1}{2} \sum_{j=1}^{n} \chi(j) h(E_j) \cdot \nabla^\eta_{E_j} \psi - \frac{1}{4} \text{div}_\eta(h) \cdot \psi + \frac{1}{4} \text{grad}_\eta(\text{Tr}_\eta(h)) \cdot \psi.
\]

Recall [1] that for the standard complex product \( \langle \cdot, \cdot \rangle \) on the spinor bundle \( \Sigma(Q) \), the relation

\[
\langle X \cdot \varphi, \psi \rangle + (-1)^r \langle \varphi, X \cdot \psi \rangle = 0
\]

holds for all vector fields \( X \) and for all spinor fields \( \varphi, \psi \). Taking the real part of (2.9) gives some simple but crucial identities:

\[
\begin{align*}
((\sqrt{-1})^r X \cdot \psi, \psi) &= 0, \\
(X \cdot \psi, Y \cdot \psi) &= (-1)^r \eta(X, Y)(\psi, \psi), \\
(X \cdot Y \cdot \psi, \psi) &= -\eta(X, Y)(\psi, \psi).
\end{align*}
\]

Let Sym(0,2) denote the space of all symmetric (0,2)-tensor fields on \( (Q^{n,r}, \eta) \), and let \( ((\cdot, \cdot)) = ((\cdot, \cdot))_\eta \) denote the naturally induced metric on the space Sym(0,2). Denote by
\[ \psi_{\eta} = (\hat{K}_t)^{-1}(\psi) \in \Gamma(\Sigma(Q)_{\eta}) \] the pullback of \( \psi = \psi_{\eta} \in \Gamma(\Sigma(Q)_{\eta}) \) via natural isomorphism \( \hat{K}_t \) (see (2.3)). Then (2.8) and (2.10) together give the formula (1.4) for the first type energy-momentum tensor \( T_1 : \)

\[
\left. \frac{d}{dt} \right|_{t=0} \left( (\sqrt{-1})^r D_{\eta} \psi_{\eta} \ , \ \psi_{\eta} \right) = -\frac{1}{4}((T_1 \ , \ h)) , \tag{2.13}
\]

where

\[
T_1(X,Y) = \left( (\sqrt{-1})^r \{X \cdot \nabla^\eta \psi + Y \cdot \nabla^\eta \psi\}, \ \psi \right) . \tag{2.14}
\]

Moreover, using (2.8) and (2.9) and noting that \( (\sqrt{-1})^r D_{\eta} \) is symmetric with respect to the \( L^2 \)-product, we can derive the formula (1.7) for the second type energy-momentum tensor \( T_2 \).

**Lemma 2.2** Let \( U \) be an open subset of \( Q^{n,r} \) with compact closure, and let \( h \) be a symmetric tensor field with support in \( U \). Then for any spinor field \( \psi \) on \( (Q^{n,r}, \eta) \), we have

\[
\left. \frac{d}{dt} \right|_{t=0} \int_U \left( (D_{\eta} \circ D_{\eta})(\psi_{\eta}), \ \psi_{\eta} \right) \mu_{\eta} = -\frac{1}{4} \int_U \left( (T_2 \ , \ h) \right) \mu_{\eta}, \tag{2.15}
\]

where

\[
T_2(X,Y) = \left( X \cdot \nabla^\eta (D_{\eta} \psi) + Y \cdot \nabla^\eta (D_{\eta} \psi), \ \psi \right) + (-1)^r \left( X \cdot \nabla^\eta \psi + Y \cdot \nabla^\eta \psi, \ D_{\eta} \psi \right). \tag{2.16}
\]

**Proof.** Letting \( D = D_{\eta} \) and \( \psi = \psi_{\eta} \), we compute

\[
\left. \frac{d}{dt} \right|_{t=0} \int_U \left( (D_{\eta} \circ D_{\eta})(\psi_{\eta}), \ \psi_{\eta} \right) \mu_{\eta} = \int_U \left( \frac{d}{dt} \right|_{t=0} (\hat{K}_t D_{\eta})(D \psi)_{\eta}, \ \psi \right) \mu_{\eta} + \int_U \left( D_{\eta} \left( \frac{d}{dt} \right|_{t=0} (\hat{K}_t D_{\eta})(\psi_{\eta}) \right), \ \psi \right) \mu_{\eta}
\]

\[
= \int_U \left( -\frac{1}{2} \sum_{j=1}^n \chi(j) h(E_j) \cdot \nabla^\eta_{E_j} (D \psi) - \frac{1}{4} \text{div}_{\eta}(h) \cdot (D \psi) + \frac{1}{4} \text{grad}_{\eta}(\text{Tr}_{\eta}(h)) \cdot (D \psi), \ \psi \right) \mu_{\eta}
\]

\[
+ \int_U \left( (\sqrt{-1})^{3r} \left\{ -\frac{1}{2} \sum_{j=1}^n \chi(j) h(E_j) \cdot \nabla^\eta_{E_j} \psi - \frac{1}{4} \text{div}_{\eta}(h) \cdot \psi
\right.
\]

\[
\left. + \frac{1}{4} \text{grad}_{\eta}(\text{Tr}_{\eta}(h)) \cdot \psi \right\}, \ (\sqrt{-1})^{r} D_{\eta} \psi \right) \mu_{\eta}
\]

\[
= -\frac{1}{2} \int_U \left( \sum_{i=1}^n \chi(i) h(E_i) \cdot \nabla^\eta_{E_i} (D \psi), \ \psi \right) \mu_{\eta} - \frac{(-1)^r}{2} \int_U \left( \sum_{i=1}^n \chi(i) h(E_i) \cdot \nabla^\eta_{E_i} \psi, \ D \psi \right) \mu_{\eta}
\]

\[
= -\frac{1}{4} \int_U \left( (T_2, h) \right) \mu_{\eta} .
\]
We further need to recall the well-known formulas for the variation of the volume form and the scalar curvature, which one easily obtain from (2.6) and from the pseudo-Riemannian version of the second formula in Proposition 2.2 of [7].

**Lemma 2.3** (see [3]) Let \( U \) be an open subset of \( Q^{n,r} \) with compact closure, and let \( h \) be a symmetric tensor field with support in \( U \). Then we have

\[
\frac{d}{dt} \mu_{\eta} \bigg|_{t=0} = \frac{1}{2} \left( \eta, h \right) \mu_{\eta},
\]

\[
\frac{d}{dt} \left( \int_U S_{\eta} \mu_{\eta} \right) \bigg|_{t=0} = - \int_U \left( \left( \text{Ric}_{\eta}, h \right) \right) \mu_{\eta}.
\]

Making use of Lemma 2.2 and 2.3 and following the proof of Theorem 2.1 of [6], we now establish the main result of this section.

**Theorem 2.1** Let \( Q^{n,r} \) be a pseudo-Riemannian spin manifold. Fix the notation \( P_\eta \) to mean either \( P_\eta = (\sqrt{-1})^r D_\eta \) or \( P_\eta = D_\eta \circ D_\eta \). Then, a pair \((\eta_0, \psi_0)\) is a critical point of the Lagrange functional

\[
W(\eta, \psi) = \int_U \left\{ aS_\eta + b + \epsilon \nu (\psi_\eta, \psi_\eta) - \epsilon (P_\eta(\psi), \psi_\eta) \right\} \mu_{\eta}, \quad a, b, \epsilon, \nu \in \mathbb{R}, \quad \epsilon \neq 0,
\]

for all open subsets \( U \) of \( Q^{n,r} \) with compact closure if and only if \((\eta_0, \psi_0)\) is a solution of the following system of differential equations:

\[
P_\eta(\psi) = \nu \psi \quad \text{and} \quad a \left\{ \text{Ric}_\eta - \frac{1}{2} S_{\eta} \right\} - \frac{b}{2} \eta = \frac{\epsilon}{4} T,
\]

(2.16)

where \( T \) is a symmetric tensor field defined by (2.14) or by (2.15) depending on a choice of \( P_\eta \).

We close the section with generalizing Definition 2.1 and 3.1 of [6].

**Definition 2.1** (i) A non-trivial spinor field \( \psi \) on \((Q^{n,r}, \eta)\), \( n \geq 3 \), is called an **Einstein spinor of type I** for the eigenvalue \((\sqrt{-1})^3 r \nu_1, \nu_1 \in \mathbb{R} \), if it is a solution of the system (1.2)-(1.4).

(ii) A non-trivial spinor field \( \psi \) on \((Q^{n,r}, \eta)\), \( n \geq 3 \), is called an **Einstein spinor of type II** for the eigenvalue \( \nu_2 \in \mathbb{R} \) if it is a solution of the system (1.6)-(1.7).

**Definition 2.2** Assume that \( a(n - 2)S + bn \) \( (a, b \in \mathbb{R}) \) does not vanish at any point of \((Q^{n,r}, \eta)\), \( n \geq 3 \). A non-trivial spinor field \( \psi \) on \((Q^{n,r}, \eta)\) is called a **weak Killing spinor** (shortly, WK-spinor) with WK-number \((\sqrt{-1})^3 r \nu_1 \neq 0, \nu_1 \in \mathbb{R} \), if \( \psi \) is a solution of the differential equation

\[
\nabla_X \psi = (\sqrt{-1})^3 r \beta(X) \cdot \psi + n a(X) \psi + X \cdot \alpha \cdot \psi,
\]

(2.17)
where $\alpha$ is a 1-form and $\beta$ is a symmetric tensor field defined by
$$
\alpha = \frac{a(n-2)\,dS}{2(n-1)\{a(n-2)S + bn\}}
$$
and
$$
\beta = \frac{2\,\nu_1}{a(n-2)S + bn} \left\{ a \left\{ \text{Ric} - \frac{1}{2} S \eta \right\} - \frac{b}{2} \eta \right\},
$$
respectively.

**Remark 2.1** As in the Riemannian case (see Theorem 3.1 of [6]), any pseudo-Riemannian WK-spinor $\psi$ with positive length $(\psi, \psi) > 0$ (resp. negative length $(\psi, \psi) < 0$) becomes an Einstein spinor of type I: Since
$$
d\left( \frac{(\psi, \psi)}{a(n-2)S + bn} \right) = 0,
$$
it follows that
$$
\frac{(\psi, \psi)}{a(n-2)S + bn}
$$
is constant on $Q^{n,r}$. One verifies easily that the equations (1.2)-(1.4) are indeed satisfied with
$$
\epsilon = - \frac{a(n-2)S + bn}{\nu_1 (\psi, \psi)}.
$$

**Remark 2.2** Evidently, the solution space of the type I classical Einstein-Dirac equation is a subspace of that of the type II classical Einstein-Dirac equation. Hence it is of interest to find such Einstein spinors of type II that are not Einstein spinors of type I: Let $(Q^{n,r}, \eta)$ admit a spinor field $\psi$ satisfying the differential equation [2]
$$
\nabla_X \psi = - (\sqrt{-1})^{3r+1} \frac{\nu_1}{n} X \cdot \psi.
$$
Then the metric $\eta$ is necessarily Einstein with scalar curvature
$$
S = (-1)^{r+1} \frac{4(n-1)\nu_1^2}{n}.
$$
If we choose the parameters $a$ and $b$ so as to be related by
$$
b = - \frac{a(n-2)}{n} S = (-1)^r \frac{4a(n-1)(n-2)\nu_1^2}{n^2},
$$
then $\psi$ satisfies (1.6)-(1.7) with
$$
\nu_2 = (-1)^{r+1} \nu_1^2 \quad \text{and} \quad a \left\{ \text{Ric} - \frac{S}{2} \eta \right\} - \frac{b}{2} \eta = \frac{\epsilon}{4} T_2 = 0.
$$
However, $\psi$ does not satisfy (1.2)-(1.4) in general.
3 Derivation of the CL-Einstein-Dirac equations

Let \( \varphi = \varphi_\eta \) be a spinor field on \((Q^{n,r}, \eta)\) such that either \((\varphi, \varphi) > 0\) at all points or \((\varphi, \varphi) < 0\) at all points. We use the simplifying notation
\[
\varphi^k := (\sigma \varphi, \varphi)^k \varphi, \quad k \in \mathbb{R},
\]
where \( \sigma = \sigma_\varphi \in \mathbb{R} \) is a constant defined by
\[
\sigma = 1 \text{ if } (\varphi, \varphi) > 0 \quad \text{and} \quad \sigma = -1 \text{ if } (\varphi, \varphi) < 0.
\]

Via direct computations, one verifies easily the following variation formulas.

**Lemma 3.1** Let \( U \) be an open subset of \((Q^{n,r}, \eta)\) with compact closure, and let \( \varphi_c \) be a spinor field with support in \( U \). Then we have
\[
(i) \quad \frac{d}{dt} \bigg|_{t=0} ((\varphi + t\varphi_c)^k, (\varphi + t\varphi_c)^k) = 2(2k + 1)(\sigma \varphi, \varphi)^{2k}(\sigma \varphi, \varphi_c),
\]
\[
(ii) \quad \frac{d}{dt} \bigg|_{t=0} \int_U \left( \sigma P_\eta ((\varphi + t\varphi_c)^k, (\varphi + t\varphi_c)^k) \right) \mu_\eta
\]
\[
= 4k \int_U \left( \sigma P_\eta ((\sigma \varphi, \varphi)^k \varphi, (\sigma \varphi, \varphi)^{k-1} \varphi \varphi_c) \right) \mu_\eta
\]
\[
+ 2 \int_U \left( \sigma P_\eta ((\sigma \varphi, \varphi)^k \varphi, (\sigma \varphi, \varphi)^k \varphi_c) \right) \mu_\eta,
\]
where \( P_\eta = (\sqrt{-1})^r D_\eta \) or \( P_\eta = D_\eta \circ D_\eta \).

**Theorem 3.1** Let \( Q^{n,r} \) be a pseudo-Riemannian spin manifold. Consider the Lagrange functional
\[
W(\eta, \varphi) = \int_U \left\{ a S_\eta + b + \epsilon \nu (\sigma \varphi^k, \varphi^k) \eta - \epsilon (\sigma P_\eta (\varphi^k, \varphi^k) \eta \right\} \mu_\eta
\]
over open subsets \( U \) of \( Q^{n,r} \) with compact closure, where \( a, b, k, \epsilon, \nu \in \mathbb{R}, \epsilon \neq 0 \), are real numbers.

(i) In case of \( 2k + 1 \neq 0 \), a pair \((\eta^*, \varphi^*)\) is a critical point of \( W(\eta, \varphi) \) for all open subsets \( U \) of \( Q^{n,r} \) with compact closure if and only if \((\eta^*, \varphi^*)\) is a solution of the following system of differential equations:
\[
P_\eta(\varphi^k) = \nu \varphi^k \quad \text{and} \quad a \left\{ \text{Ric}_\eta - \frac{1}{2} S_\eta \eta \right\} - \frac{b}{2} \eta = \frac{\epsilon}{4} T, \quad (3.1)
\]
where $T$ is a symmetric tensor field defined by

$$T(X,Y) = T_1(X,Y) = \left(\sigma (\sqrt{-1})^r \{ X \cdot \nabla_Y \varphi^k + Y \cdot \nabla_X \varphi^k \}, \varphi^k \right)$$  \hspace{1cm} (3.2)

if $P_\eta = (\sqrt{-1})^r D_\eta$ and defined by

$$T(X,Y) = T_2(X,Y) = \sigma \left( X \cdot \nabla_Y (D_\eta \varphi^k) + Y \cdot \nabla_X (D_\eta \varphi^k), \varphi^k \right)$$

$$+ \sigma(-1)^r \left( X \cdot \nabla_Y \varphi^k + Y \cdot \nabla_X \varphi^k, D_\eta \varphi^k \right)$$  \hspace{1cm} (3.3)

if $P_\eta = D_\eta \circ D_\eta$, respectively.

(ii) In case of $2k+1 = 0$, a pair $(\eta^*, \varphi^*)$ is a critical point of $W(\eta, \varphi)$ for all open subsets $U$ of $Q^{n,r}$ with compact closure if and only if $(\eta^*, \varphi^*)$ is a solution of the following system of differential equations:

$$P_\eta(\varphi^k) = f \varphi^k$$ \hspace{1cm} (3.4)

and

$$a \left\{ \text{Ric}_\eta - \frac{1}{2} S_\eta \right\} - \frac{b + \nu}{2} \eta = \frac{e}{4} T - \frac{e}{2} f \eta,$$ \hspace{1cm} (3.5)

where $f : Q^{n,r} \rightarrow \mathbb{R}$ is a real-valued function and $T$ is a symmetric tensor field defined by (3.2) or by (3.3) depending on a choice of $P_\eta$.

**Proof.** Let $h$ be a symmetric tensor field with support in $U$, and let $\varphi_c$ be a spinor field with support in $U$. Let $\eta_t$ be an one-parameter family of metrics in (2.6). Using Lemma 3.1, we compute at $t = 0$:

$$\frac{d}{dt} W(\eta_t, \varphi + t\varphi_c) = \frac{d}{dt} W(\eta_t, \varphi) + \frac{d}{dt} W(\eta_t, \varphi + t\varphi_c)$$

$$= \frac{d}{dt} \int_U a S_{\eta_t} \mu_{\eta_t} + \frac{d}{dt} \int_U a S_{\eta_t} \mu_{\eta_t} + \frac{d}{dt} \int_U b \mu_{\eta_t} + \frac{d}{dt} \int_U \epsilon \nu(\sigma \varphi^k, \varphi^k) \mu_{\eta_t}$$

$$- \frac{d}{dt} \int_U \epsilon(\sigma P_{\eta_t}(\varphi^k), \varphi^k) \mu_{\eta_t} - \frac{d}{dt} \int_U \epsilon(\sigma P_{\eta_t}(\varphi^k), \varphi^k) \mu_{\eta_t}$$

$$+ \frac{d}{dt} \int_U \epsilon(\sigma (\varphi + t\varphi_c)^k, \varphi^k) \mu_{\eta_t} - \frac{d}{dt} \int_U \epsilon(\sigma P_{\eta_t}(\varphi + t\varphi_c)^k, \varphi^k) \mu_{\eta_t}$$

$$= \int_U \left( \left( \left( - a \text{Ric}_\eta + \frac{a}{2} S_\eta \eta + \frac{b}{2} \eta + \frac{e}{4} T + \frac{\epsilon \nu}{2} (\sigma \varphi^k, \varphi^k) \eta - \frac{e}{2} (\sigma P_{\eta_t}(\varphi^k), \varphi^k) \eta, h \right) \right) \right) \mu_{\eta_t}$$

$$+ \int_U \left( 2\epsilon (2k + 1)(\sigma \varphi^k, \varphi^k) - 4\epsilon k (\sigma \varphi, \varphi)^{-1} (\sigma P_{\eta_t}(\varphi^k), \varphi^k) \cdot \sigma \varphi$$

$$- 2\epsilon (\sigma \varphi, \varphi^k) \cdot \sigma P_{\eta_t}(\varphi^k), \varphi_c \right) \mu_{\eta_t}. $$
It follows that a pair \((\eta^*, \varphi^*)\) is a critical point of the functional \(W(\eta, \varphi)\) for all open subsets \(U\) of \(Q^{n,r}\) with compact closure if and only if it is a solution of the equations

\[
\frac{\epsilon}{4} T = a \text{Ric}_\eta - \frac{a}{2} S_\eta - \frac{b}{2} \eta - \frac{\epsilon \nu}{2} (\sigma \varphi^k, \varphi^k) \eta + \frac{\epsilon}{2} (\sigma P_\eta(\varphi^k), \varphi^k) \eta
\]  

(3.6)

and

\[
P_\eta(\varphi^k) = -2k(\sigma \varphi, \varphi)^{-2k-1} (\sigma P_\eta(\varphi^k), \varphi^k) \varphi^k + \nu(2k + 1) \varphi^k. \tag{3.7}
\]

Inner product of (3.7) with \(\sigma \cdot \varphi^k\) gives

\[
0 = (2k + 1) \left\{ (\sigma P_\eta(\varphi^k), \varphi^k) - \nu(\sigma \varphi^k, \varphi^k) \right\}, \tag{3.8}
\]

and so, in case of \(2k + 1 \neq 0\), (3.6)-(3.8) imply part (i) of the theorem. Now we consider the other case \(2k + 1 = 0\). In this case, \((\sigma \varphi^k, \varphi^k) = (\sigma \varphi, \varphi)^{2k+1} = 1\) and hence (3.7) gives

\[
P_\eta(\varphi^k) = f \varphi^k \tag{3.9}
\]

with \(f := (\sigma P_\eta(\varphi^k), \varphi^k)\). Thus, (3.6) and (3.9) together prove part (ii) of the theorem. \(\Box\)

We observe that the system (3.1)-(3.3) is not new and is in fact equivalent to the classical system (2.16). We therefore focus our attention on the system (3.4)-(3.5) which is a new Einstein-Dirac system.

**Definition 3.1** A non-trivial spinor field \(\psi\) on \((Q^{n,r}, \eta), n \geq 3\), is called a **CL-Einstein spinor of type I** (resp. **type II**) with characteristic function \(f\) if it is of constant length \(|\psi| = \pm 1\) and satisfies the system (1.9) and (1.10) (resp. (1.9) and (1.11)).

**Remark 3.1** Let \(\varphi\) be a spinor field on \((Q^{n,r}, \eta)\) such that either \((\varphi, \varphi) > 0\) at all points or \((\varphi, \varphi) < 0\) at all points. Let \(T_1\) and \(T_2\) be symmetric tensor fields induced by \(\varphi\) as in (1.10) and (1.11), respectively. Then, via direct computations, one finds that

\[
\text{div}(T_1)(X) = \sigma \sum_{i=1}^{n} \chi(i)(\nabla_{E_i}T_1)(E_i, X) \tag{3.10}
\]

and

\[
\text{div}(T_2)(X) = \sigma \left( \nabla_X (D^2 \varphi), \varphi \right) - \sigma \left( \nabla_X \varphi, (\sqrt{-1})^r D \varphi \right) - \sigma \left( (\sqrt{-1})^r X \cdot D^2 \varphi, \varphi \right)
\]

and

\[
\text{div}(T_2)(X) = \sigma \left( \nabla_X (D^2 \varphi), \varphi \right) - \sigma \left( \nabla_X \varphi, D^2 \varphi \right) - \sigma \left( X \cdot D^3 \varphi, \varphi \right) - \sigma \left( X \cdot D^2 \varphi, D \varphi \right). \tag{3.11}
\]

(i) If \((\sqrt{-1})^r D \varphi = f_1 \varphi\) for some function \(f_1 : Q^{n,r} \rightarrow \mathbb{R}\) and \(\varphi\) is of constant length \(|\varphi| = \pm 1\), then

\[
\text{div}(T_1)(X) = 2 df_1(X)(\sigma \varphi, \varphi) = 2 df_1(X),
\]
and so
\[
\text{div} \left( \frac{1}{4} T_1 - \frac{f_1}{2} \eta \right) = 0, 
\]
which is required by the Einstein equation in (1.9).

(ii) Similarly, if \( D^2 \varphi = f_2 \varphi \) for some function \( f_2 : Q^{n,r} \to \mathbb{R} \) and \( \varphi \) is of constant length \( |\varphi| = \pm 1 \), then
\[
\text{div}(T_2)(X) = 2 df_2(X)(\sigma \varphi, \varphi) = 2 df_2(X),
\]
and so
\[
\text{div} \left( \frac{1}{4} T_2 - \frac{f_2}{2} \eta \right) = 0. 
\]

From (3.12)-(3.13) we see that the Einstein equation
\[
\{ \text{Ric} - \frac{S}{2} \} \eta - \frac{c}{2} \eta = \frac{\epsilon}{4} T_1 - \frac{\epsilon}{2} f_1 \eta
\]
of the CL-Einstein-Dirac equation (1.9) has a natural coupling structure. However, we should note that neither (3.12) nor (3.13) holds in general, unless \((\varphi, \varphi)\) is of constant length.

We can rewrite the CL-Einstein-Dirac equation of type I
\[
(\sqrt{-1})^r D\psi = f_1 \psi, 
\]
(3.14)
\[
a \{ \text{Ric} - \frac{S}{2} \} \eta - \frac{c}{2} \eta = \frac{\epsilon}{4} T_1 - \frac{\epsilon}{2} f_1 \eta, 
\]
(3.15)
where
\[
T_1(X,Y) = \left( \sigma(\sqrt{-1})^r \{ X \cdot \nabla_Y \psi + Y \cdot \nabla_X \psi \}, \psi \right),
\]
(3.16)
in an equivalent form: Since contracting both sides of (3.15) gives
\[
\epsilon(n-1)f_1 = a(n-2)S + cn, 
\]
(3.17)
one checks that the system (3.14)-(3.15) is actually equivalent to the system
\[
\epsilon(\sqrt{-1})^r D\psi = \left\{ \frac{a(n-2)}{n-1} S + \frac{cn}{n-1} \right\} \psi
\]
(3.18)
and
\[
a \{ \text{Ric} - \frac{S}{2(n-1)} \} \eta + \frac{c}{2(n-1)} \eta = \frac{\epsilon}{4} T_1.
\]
(3.19)
Since the system (3.18)-(3.19) is similar to the classical Einstein-Dirac equation of type I, we are led to an analogue of the WK-equation in Definition 2.2.

**Definition 3.2** A non-trivial spinor field \( \psi \) on \((Q^{n,r}, \eta), n \geq 3 \), is called a **WW-spinor** if \( \psi \) satisfies the differential equation
\[
\nabla_X \psi = (\sqrt{-1})^{3r} \left( - \frac{2a}{c} \right) \left\{ \text{Ric}(X) - \frac{S}{2(n-1)} X + \frac{c}{2a(n-1)} X \right\} \cdot \psi
\]
(3.20)
for some constants $\epsilon, a, c \in \mathbb{R}$, $\epsilon \neq 0, a \neq 0$, and for all vector fields $X$.

Note that if the scalar curvature $S$ of $(Q^{n,r}, \eta)$ is constant, then the WW-equation (3.20) is equivalent to the WK-equation (2.17). Because of (2.10), the length $|\psi|$ of any WW-spinor $\psi$ is constant. It follows that, by rescaling the length $|\psi|$ if necessary, one may assume without loss of generality that any WW-spinor $\psi$ is of unit length $|\psi| = \pm 1$ or of zero length $|\psi| = 0$. As any WK-spinor of positive (resp. negative) length is an Einstein spinor of type I, one then checks that any WW-spinor $\psi$ of unit length is a CL-Einstein spinor of type I.

4 Constructing solutions of the CL-Einstein-Dirac equation of type II

Let $\eta_1$ and $\eta_2, \eta_2 = e^u \eta_1$, be conformally equivalent metrics on $Q^{n,r}$. By (2.3) there are natural isomorphisms $j : T(Q) \rightarrow T(Q)$ and $j : \Sigma(Q)_{\eta_1} \rightarrow \Sigma(Q)_{\eta_2}$ preserving the inner products of vectors and spinors as well as the Clifford multiplication:

$$\eta_2(jX, jY) = \eta_1(X, Y), \quad \langle j\varphi_1, j\varphi_2 \rangle_{\eta_2} = \langle \varphi_1, \varphi_2 \rangle_{\eta_1},$$

$$(jX) \cdot (j\varphi) = j(X \cdot \varphi), \quad X, Y \in \Gamma(T(Q)), \quad \varphi, \varphi_1, \varphi_2 \in \Gamma(\Sigma(Q)_{\eta_1}).$$

Denote by $\overline{X} := j(X)$ and $\overline{\varphi} := j(\varphi)$ the corresponding vector fields and spinor fields on $(Q^{n,r}, \eta_2)$, respectively. Then, for any spinor field $\psi$ on $(Q^{n,r}, \eta_1)$, we have

$$\nabla_{\overline{X}} \overline{\psi} = e^{-\frac{u}{2}} \nabla^{\eta_1}_X \psi - \frac{1}{4} \eta_2(\overline{X}, \text{grad}_{\eta_2}(u)) \psi - \frac{1}{4} \overline{X} \cdot \text{grad}_{\eta_2}(u) \cdot \overline{\psi},$$

$$D_{\eta_2} \overline{\psi} = e^{-\frac{u}{2}} D_{\eta_1} \psi + \frac{n-1}{4} \text{grad}_{\eta_2}(u) \cdot \overline{\psi},$$

$$(D_{\eta_2} \circ D_{\eta_2}) \overline{\psi} = e^{-u} (D_{\eta_1} \circ D_{\eta_1}) \psi - \frac{1}{2} e^{-\frac{u}{2}} \text{grad}_{\eta_2}(u) \cdot D_{\eta_1} \psi$$

$$- \frac{n-1}{2} e^{-u} \text{grad}_{\eta_2}(u) \overline{\psi} + \frac{(n-1)^2}{16} |du|_{\eta_2}^2 \overline{\psi} + \frac{n-1}{4} \Delta_{\eta_2}(u) \overline{\psi}.$$ 

Now consider a special class of spinors.

**Definition 4.1** A non-trivial spinor field $\psi$ on $(Q^{n,r}, \eta)$, $n \geq 3$, is called a weakly $T$-parallel spinor with conformal factor $u$ if it is of constant length $|\psi| = \pm 1$ and the equation

$$\nabla_X \psi = -\frac{1}{4} du(X) \psi - \frac{1}{4} \beta(X) \cdot \text{grad}(u) \cdot \psi$$

holds for all vector fields $X$, for a symmetric $(1,1)$-tensor field $\beta$ with

$$\text{Tr}(\beta) = n,$$
and for a real-valued function $u : Q^{n,r} \rightarrow \mathbb{R}$ such that $|du|$ has no zeros on an open dense subset of $Q^{n,r}$.

Note that if $\psi$ is a parallel spinor on $(Q^{n,r}, \eta_1)$, then the pullback $\overline{\psi}$ of $\psi$ is a weakly T-parallel spinor on $(Q^{n,r}, \eta_2)$ with $\beta = \text{id}$ the identity map. In the following, we identify via the metric $\eta$ any exact 1-form $"du"$ with the vector field $\text{grad}(u)$ and $(1,1)$-tensor field $\beta$ with the induced $(0,2)$-tensor field $\beta(X,Y) = \eta(X, \beta(Y))$.

**Proposition 4.1** Let $(Q^{n,r}, \eta)$ admit a weakly T-parallel spinor $\psi$ solving the equation (4.4). Then we have

(i) $\beta(du) = du$,

(ii) $\nabla_{du} \psi = 0$,

(iii) $D\psi = \frac{n-1}{4} du \cdot \psi$,

(iv) $D^2\psi = \left\{ \frac{(n-1)^2}{16}|du|^2 + \frac{n-1}{4} \Delta u \right\} \psi$, where $\Delta := -\text{div} \circ \text{grad}$,

(v) $S = \frac{1}{4} \{ (n-1)^2 + 1 - |\beta|^2 \}|du|^2 + (n-1)\Delta u$.

**Proof.** Since $(\sigma \psi, \psi) = 1$ is constant and $\beta$ is symmetric,

$$0 = \sigma(\nabla_X \psi, \psi) = -\frac{1}{4} du(X) + \frac{1}{4} \eta(\beta(X), \text{grad}(u)) = -\frac{1}{4} du(X) + \frac{1}{4} \eta(X, \beta(du)),$$

which proves part (i). Using (ii)-(iii), we compute

$$D^2\psi = \frac{n-1}{4} D(du \cdot \psi) = \frac{n-1}{4} \Delta(u) \psi - \frac{n-1}{4} \nabla_{du} \psi - \frac{n-1}{4} du \cdot D\psi$$

$$= \left\{ \frac{n-1}{4} \Delta u + \frac{(n-1)^2}{16}|du|^2 \right\} \psi,$$

which proves part (iv). Substituting (iv) and (4.4) into the Schrödinger-Lichnerowicz formula $D^2\psi = \Delta \psi + S$ gives, one proves part (v).

**Remark 4.1** It is remarkable that when $Q^{n,r}$ is a closed manifold, the function $f_2 = \frac{(n-1)^2}{16}|du|^2 + \frac{n-1}{4} \Delta u$ in part (iv) of Proposition 4.1 cannot be constant: Suppose $f_2$ is a constant and hence an eigenvalue of $D^2$. Then $f_2$ must be equal to a "positive" constant $\lambda^2$ and for metric $\eta_1 := e^{-u} \eta$, we have $\Delta_{\eta_1}(u) = \frac{n-3}{4}|du|^2_{\eta_1} + \frac{n-1}{4} \lambda^2 e^u$. The last relation is however a contradiction, since the left-hand side becomes zero after integration.

Let $\psi$ be a weakly T-parallel spinor on $(Q^{n,r}, \eta)$ solving the equation (4.4). Then, a direct computation gives
\[ T_2(X, Y) = \frac{\epsilon \sigma}{4} \left( X \cdot \nabla_Y (D\psi) + Y \cdot \nabla_X (D\psi), \psi \right) \]
\[ + \frac{\epsilon \sigma}{4} (-1)^r \left( X \cdot \nabla_Y \psi + Y \cdot \nabla_X \psi, D\psi \right) \]
\[ = \frac{\epsilon \sigma (n-1)}{16} \left( X \cdot \nabla_Y (du \cdot \psi) + Y \cdot \nabla_X (du \cdot \psi), \psi \right) \]
\[ + \frac{\epsilon \sigma (n-1)}{64} (-1)^r \left( X \cdot \nabla_Y \psi + Y \cdot \nabla_X \psi, du \cdot \psi \right) \]
\[ = -\frac{\epsilon (n-1)}{8} \eta(X, \nabla_Y du) - \frac{\epsilon (n-1)}{16} du(X) du(Y) + \frac{\epsilon (n-1)}{16} |du|^2 \beta(X, Y). \]

Guided by the last computation, one immediately proves:

**Theorem 4.1** Let \( \psi \) be a weakly T-parallel spinor on \( (Q^{n,r}, \eta) \) such that \( \beta \) and \( u \) are related to the Ricci tensor and the scalar curvature of \( (Q^{n,r}, \eta) \) by

\[ |du|^2 \beta(X, Y) = \frac{4}{n-2} \left\{ \text{Ric}(X,Y) - \frac{1}{2} S \eta(X,Y) \right\} - \frac{2c}{a(n-2)} \eta(X,Y) \]
\[ + 2 \eta(X, \nabla_Y (du)) + du(X) du(Y) \]
\[ + \left\{ \frac{n-1}{2} |du|^2 + 2 \square u \right\} \eta(X,Y), \tag{4.5} \]

where \( a, c \in \mathbb{R}, a \neq 0 \), are real numbers. Then \( \psi \) becomes a solution of the CL-Einstein-Dirac equation of type II (i.e., the system (1.9) and (1.11)), where the characteristic function \( f \) is given by

\[ f = \frac{(n-1)^2}{16} |du|^2 + \frac{n-1}{4} \triangle u \]
and the parameter $\epsilon$ should be chosen to satisfy

$$\epsilon = \frac{4a(n-2)}{n-1}. $$

**Definition 4.2** A non-trivial spinor field $\psi$ on $(Q^{n,r}, \eta)$, $n \geq 3$, is called a *weakly parallel spinor* (shortly, WP-spinor) with conformal factor $u$ if it is a weakly $T$-parallel spinor with conformal factor $u$ and satisfies (4.5) for some constants $a, c \in \mathbb{R}$, $a \neq 0$.

**Definition 4.3** A non-trivial spinor field $\psi$ on $(Q^{n,r}, \eta)$, $n \geq 3$, is called a *reduced weakly parallel spinor* (shortly, reduced WP-spinor) with conformal factor $u$ if it is of constant length $|\psi| = \pm 1$ and the differential equation

$$|du|^2 \nabla_X \psi = -\frac{1}{n-2}\left\{\text{Ric}(X) - \frac{S}{n}X\right\} \cdot du \cdot \psi \quad (4.6)$$

holds for all vector fields $X$ and for a real-valued function $u : Q^{n,r} \rightarrow \mathbb{R}$ with such properties that $|du|$ has no zeros on an open dense subset of $Q^{n,r}$ and $e^u$ is proportional to the scalar curvature $S$, i.e.,

$$S = c^* e^u, \quad c^* \in \mathbb{R}. \quad (4.7)$$

Note that (4.6) generalizes the equation $\nabla_X \psi = 0$ for parallel spinors and that any reduced WP-spinor $\psi$ is a harmonic spinor $D\psi = 0$. Applying (4.6) to $0 = \sigma \cdot |du|^2(\nabla_X \psi, \psi)$, one shows:

**Proposition 4.2** Let $(Q^{n,r}, \eta)$ admit a reduced WP-spinor $\psi$ with conformal factor $u$. Then

$$\nabla_{du} \psi = 0 \quad \text{and} \quad \text{Ric}(du) = \frac{S}{n} du.$$

We are going to prove that the equation (4.5) for WP-spinors is conformally equivalent to the equation (4.6) for reduced WP-spinors. Consider conformally equivalent metrics $\eta_2 = e^u \eta_1$ on $Q^{n,r}$. Let $(F_1, \ldots, F_n)$ be a local $\eta_1$-orthonormal frame field on $Q^{n,r}$. Then $(\overline{F}_1 := e^{-\frac{u}{2}} F_1, \ldots, \overline{F}_n := e^{-\frac{u}{2}} F_n)$ is $\eta_2$-orthonormal. Since the Ricci tensors $\text{Ric}_{\eta_2}$ and $\text{Ric}_{\eta_1}$ are related by

$$\text{Ric}_{\eta_2}(\overline{F}_i, \overline{F}_j) - e^{-u} \text{Ric}_{\eta_1}(F_i, F_j)$$

$$= -\frac{n-2}{2} \eta_2(F_i, \nabla_{\overline{F}_j} (\text{grad}_{\eta_2} u)) - \frac{n-2}{4} du(\overline{F}_i) du(\overline{F}_j)$$

$$+ \frac{1}{2} \triangle_{\eta_2}(u) \eta_2(\overline{F}_i, \overline{F}_j) + \frac{n-2}{4} |du|_{\eta_2}^2 \eta_2(\overline{F}_i, \overline{F}_j)$$

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and the scalar curvatures $S_{\eta_2}$ and $S_{\eta_1}$ by

$$S_{\eta_2} - e^{-u} S_{\eta_1} = (n-1) \triangle_{\eta_2}(u) + \frac{(n-1)(n-2)}{4} |du|^2_{\eta_2},$$

we have in particular the following formula.

**Lemma 4.1**

$$\text{Ric}_{\eta_2}(\overline{F}_i, \overline{F}_j) - \frac{1}{2} S_{\eta_2} (\overline{F}_i, \overline{F}_j) =$$

$$e^{-u} \left\{ \text{Ric}_{\eta_1}(F_i, F_j) - \frac{1}{2} S_{\eta_1} (F_i, F_j) \right\}$$

$$- \frac{n-2}{2} \eta_2(\overline{F}_i, \nabla^\eta_2 (\nabla_\eta_2 u)) - \frac{n-2}{4} du(\overline{F}_i) du(\overline{F}_j)$$

$$- \frac{n-2}{2} \mu_{\eta_2} (u) \eta_2(\overline{F}_i, \overline{F}_j) - \frac{(n-2)(n-3)}{8} |du|^2_{\eta_2} \eta_2(\overline{F}_i, \overline{F}_j).$$

**Theorem 4.2** A non-trivial spinor field $\psi$ on $(Q^{n,r}, \eta_1)$ is a reduced WP-spinor with conformal factor $u$ if and only if the pullback $\overline{\psi}$ of $\psi$ is a WP-spinor on $(Q^{n,r}, \eta_2 = e^u \eta_1)$ with conformal factor $u$.

**Proof.** We first prove the necessity. Let $\psi$ be a reduced WP-spinor on $(Q^{n,r}, \eta_1)$ with conformal factor $u$. In the notation of (4.1), we have

$$|du|^2_{\eta_2} \nabla^\eta_2 \overline{\psi}$$

$$= - \frac{1}{n-2} e^{-u} \left\{ \text{Ric}_{\eta_1}(X) - \frac{1}{n} S_{\eta_1} X \right\} \cdot \nabla_{\eta_2} (u) \cdot \overline{\psi}$$

$$- \frac{1}{4} |du|^2_{\eta_2} \eta_2(\overline{X}, \nabla_{\eta_2} (u)) \overline{\psi} - \frac{1}{4} |du|^2_{\eta_2} \overline{X} \cdot \nabla_{\eta_2} (u) \cdot \overline{\psi}$$

and hence

$$\nabla^\eta_2 \overline{\psi} = - \frac{1}{4} \eta_2(\overline{X}, \nabla_{\eta_2} (u)) \overline{\psi} - \frac{1}{4} \gamma(\overline{X}) \cdot \nabla_{\eta_2} (u) \cdot \overline{\psi}, \quad (4.8)$$

where $\gamma$ is a symmetric tensor field defined by

$$|du|^2_{\eta_2} \gamma(\overline{X}, \overline{Y})$$

$$= \frac{4}{n-2} e^{-u} \left\{ \text{Ric}_{\eta_1}(X,Y) - \frac{1}{n} S_{\eta_1} \eta_1(X,Y) \right\} + |du|^2_{\eta_2} \eta_1(X,Y). \quad (4.9)$$
On the other hand, using Lemma 4.1, we compute
\[ \Phi(X, Y) := \frac{4}{n-2} \left\{ \text{Ric}_{\eta_2}(X, Y) - \frac{1}{2} S_{\eta_2}(X, Y) \right\} - \frac{2c}{a(n-2)} \eta_2(X, Y) \]
\[ + 2 \eta_2(\nabla_{\eta_2} \text{grad}_{\eta_2}(u)) + du(\nabla_{\eta_2} X) du(Y) \]
\[ + \left\{ \frac{n-1}{2} |du|_{\eta_2}^2 + 2 \Delta_{\eta_2}(u) \right\} \eta_2(X, Y) \]
\[ = \frac{4 e^{-u}}{n-2} \left\{ \text{Ric}_{\eta_1}(X, Y) - \frac{1}{2} S_{\eta_1}(X, Y) - \frac{ce^u}{2a} \eta_1(X, Y) \right\} + |du|_{\eta_2}^2 \eta_1(X, Y). \]

Choose the parameters \( a, c \in \mathbb{R} \) such that the constant \( c^* \) in (4.7) satisfies
\[ c^* = -\frac{cn}{a(n-2)}. \]

Then \( S_{\eta_1} = -\frac{cn}{a(n-2)} e^u \) and
\[ \Phi(X, Y) = |du|_{\eta_2}^2 \gamma(X, Y). \quad (4.10) \]

From (4.8)-(4.10), we conclude that \( \tilde{\psi} \) is a weakly T-parallel spinor on \((Q^{n,r}, \eta_2 = e^u \eta_1)\) satisfying (4.5), i.e., \( \tilde{\psi} \) is a WP-spinor. In order to prove the sufficiency, we reverse the process of the proof for the necessity: Let \( \psi \) be a WP-spinor on \((Q^{n,r}, \eta_2 = e^u \eta_1)\). Then we have
\[ |du|_{\eta_2}^2 \beta(X, Y) \]
\[ = \frac{4 e^{-u}}{n-2} \left\{ \text{Ric}_{\eta_1}(X, Y) - \frac{1}{2} S_{\eta_1}(X, Y) - \frac{ce^u}{2a} \eta_1(X, Y) \right\} + |du|_{\eta_2}^2 \eta_1(X, Y). \]

Contracting both sides of this equation gives
\[ S_{\eta_1} = -\frac{cn}{a(n-2)} e^u. \]

Using (4.1), one verifies that \( \psi \) satisfies the equation (4.6) indeed. \( \square \)

5 An existence theorem for WK-spinors and that for reduced WP-spinors

We show that every parallel spinor may evolve to a WK-spinor (resp. a reduced WP-spinor). We give a description for the evolution in a more general way than that given in Section 5 of [7].

Let \((M^n, g_M)\) be a Riemannian manifold, and let \((\mathbb{R}, g_{\mathbb{R}})\) be the real line with the standard metric. Let \((Q^{n+1} = M^n \times \mathbb{R}, \eta_1 = g_M + \chi(n+1)g_{\mathbb{R}}), \chi(n+1) = \pm 1, \) be the
pseudo-Riemannian product manifold. We will write \( g_\mathbb{R} = dt \otimes dt \) using the standard coordinate \( t \in \mathbb{R} \) and regard \( \eta_1 \) as a reference metric on \( Q^{n+1} \). Let \((F_1, \ldots, F_n)\) denote a local \( \eta_1 \)-orthonormal frame field on \((M^n, g_M)\) as well as its lift to \((Q^{n+1}, \eta_1)\). Let \( F_{n+1} = \frac{d}{dt} \) denote the unit vector field on \((\mathbb{R}, g_\mathbb{R})\) as well as the lift to \((Q^{n+1}, \eta_1)\). We consider a doubly warped product of \( g_M \) and \( g_\mathbb{R} \):

\[
\eta_2 = A^2 \left( \sum_{i=1}^n F_i \otimes F_i \right) + \chi(n + 1)B^2 dt \otimes dt, \quad (5.1)
\]

where \( A = A(t), B = B(t) : \mathbb{R} \to \mathbb{R} \) are positive functions on \( \mathbb{R} \) and \( \{F_i = \eta_1(F_i, \cdot)\} \) is the dual frame field of \( \{F_i\} \). Let \( g_{M_t} \) be the metric on slice \( M_t := M^n \times \{t\}, t \in \mathbb{R} \), of the foliation \((Q^{n+1} = M^n \times \mathbb{R}, \eta_1)\) induced by the reference metric \( \eta_1 \), and let \( \nabla^{g_{M_t}} \) be the Levi-Civita connection. Then the Levi-Civita connection \( \nabla^{\eta_2} \) of \((Q^{n+1}, \eta_2)\) is related to \( \nabla^{g_{M_t}} \) by

\[
\nabla^{\eta_2}_{\mathcal{T}^i} F_j = A^{-2} \nabla^{g_{M_t}}_{F_i} F_j - \chi(n + 1) \delta_{ij} B^{-2} A^{-1} A_t F_{n+1}, \quad (5.2)
\]

\[
\nabla^{\eta_2}_{\mathcal{T}^n} F_j = \nabla^{\eta_2}_{\mathcal{T}^n_{n+1}} F_{n+1} = 0, \quad 1 \leq i, j \leq n, \quad (5.3)
\]

where \((\mathcal{T}^1 := A^{-1}F_1, \ldots, \mathcal{T}^n := A^{-1}F_n, \mathcal{T}^n_{n+1} := B^{-1}F_{n+1})\) is a \( \eta_2 \)-orthonormal frame field and \( A_t \) indicates the derivative \( A_t = dA(F_{n+1}) \). The second fundamental form \( \Theta_{\eta_2} = -\nabla^{\eta_2} F_{n+1} \) of slice \( M_t \) is expressed as

\[
\Theta_{\eta_2}(\mathcal{T}^j) = -B^{-1} A^{-1} A_t \mathcal{T}^j, \quad 1 \leq j \leq n. \quad (5.4)
\]

Furthermore, the Ricci tensor \( \text{Ric}_{\eta_2} \) and the scalar curvature \( S_{\eta_2} \) of \((Q^{n+1}, \eta_2)\) are related to the Ricci tensor \( \text{Ric}_{g_{M_t}} \) and the scalar curvature \( S_{M_t} \) of slice \((M_t, g_M)\) by

\[
\text{Ric}_{\eta_2}(\mathcal{T}^i, \mathcal{T}^j) = A^{-2} \text{Ric}_{g_{M_t}}(F_i, F_j) - \chi(n + 1)(n - 1)B^{-2} A^{-2} A_t A_t \delta_{ij}
\]

\[
\quad + \chi(n + 1)\{B^{-3} A^{-1} B_t A_t - B^{-2} A^{-1} A_{tt}\} \delta_{ij}, \quad (5.5)
\]

\[
\text{Ric}_{\eta_2}(\mathcal{T}^n_{n+1}, \mathcal{T}^n_{n+1}) = nB^{-2} A^{-1}(B^{-1} B_t A_t - A_{tt}), \quad (5.6)
\]

\[
\text{Ric}_{\eta_2}(\mathcal{T}^i, \mathcal{T}^n_{n+1}) = 0, \quad (5.7)
\]

\[
S_{\eta_2} = A^{-2} S_{g_{M_t}} - \chi(n + 1)n(n - 1)B^{-2} A^{-2} A_t A_t
\]

\[
\quad + \chi(n + 1)2n\{B^{-3} A^{-1} B_t A_t - B^{-2} A^{-1} A_{tt}\}, \quad (5.8)
\]

where \( A_{tt} = (A_t)_t \) indicates the second derivative. From now on, we are interested in a special case that the warping functions \( A \) and \( B \) are related by

\[
B = (A^p)_t = pA^{p-1}A_t, \quad p \neq 0 \in \mathbb{R}. \quad (5.9)
\]
Definition 5.1 A doubly warped product (5.1) is called a \((Y)\)-warped product of \((M^n, g_M)\) and \((\mathbb{R}, g_\mathbb{R})\) with warping function \(A\) and \((Y)\)-factor \(p\) if the relation (5.9) is satisfied for some constant \(p \neq 0 \in \mathbb{R}\).

Proposition 5.1 Let \((Q^{n+1} = M^n \times \mathbb{R}, \eta_2)\) be a \((Y)\)-warped product of \((M^n, g_M)\) and \((\mathbb{R}, g_\mathbb{R})\) with warping function \(A\) and \((Y)\)-factor \(p\). Then the formulas (5.4)-(5.8) simplify to

\(i\) \(\Theta_{\eta_2} (\overline{F}_i, \overline{F}_j) = -p^{-1} A^{-p} \delta_{ij}, \quad 1 \leq i, j \leq n,\)

\(ii\) \(\text{Ric}_{\eta_2} (\overline{F}_i, \overline{F}_j) = A^{-2} \text{Ric}_{g_M} (F_i, F_j) + \chi (n+1) (p-n) p^{-2} A^{-2p} \delta_{ij},\)

\(iii\) \(\text{Ric}_{\eta_2} (\overline{F}_{n+1}, \overline{F}_{n+1}) = n(p-1) p^{-2} A^{-2p},\)

\(iv\) \(\text{Ric}_{\eta_2} (\overline{F}_i, \overline{F}_{n+1}) = 0,\)

\(v\) \(S_{\eta_2} = A^{-2} S_{g_M} + \chi (n+1) n(2p-n-1) p^{-2} A^{-2p}.\)

An argument similar to that of Proposition 5.1 of [7] shows:

Proposition 5.2 Let \((Q^{n+1} = M^n \times \mathbb{R}, \eta_2)\) be a \((Y)\)-warped product of \((M^n, g_M)\) and \((\mathbb{R}, g_\mathbb{R})\) with warping function \(A\) and \((Y)\)-factor \(\frac{n}{2}\). Assume that \((M^n, g_M)\) is Ricci-flat. Then the weak Killing equation (2.17), in case of \(b = 0\), is equivalent to the system of differential equations

\(\nabla^g_{\mathcal{V}} \psi = 0 \quad \text{and} \quad \nabla^\eta_{\mathcal{F}_{n+1}} \psi = - (\sqrt{-1})^3 \nu_1 \mathcal{F}_{n+1} \cdot \psi + \frac{1}{2} \text{Tr}_{g_M} (\Theta_{\eta_2}) \psi,\)

where \(\mathcal{V}\) is an arbitrary vector field on \(Q^{n+1}\) with \(\eta_2(\mathcal{V}, \mathcal{F}_{n+1}) = 0.\)

Proposition 5.3 Let \((Q^{n+1} = M^n \times \mathbb{R}, \eta_2)\) be a \((Y)\)-warped product of \((M^n, g_M)\) and \((\mathbb{R}, g_\mathbb{R})\) with warping function \(A\) and \((Y)\)-factor \(\frac{n+1}{2}\). Assume that \((M^n, g_M)\) is Ricci-flat. Then the reduced WP-equation in Definition 4.3 (in case that we set \(u = - \log A\)) is equivalent to the system of differential equations

\(\nabla^g_{\mathcal{V}} \psi = 0 \quad \text{and} \quad \nabla^\eta_{\mathcal{F}_{n+1}} \psi = \frac{1}{2} \text{Tr}_{g_M} (\Theta_{\eta_2}) \psi,\)

where \(\mathcal{V}\) is an arbitrary vector field on \(Q^{n+1}\) with \(\eta_2(\mathcal{V}, \mathcal{F}_{n+1}) = 0.\)
Proof. Since \( u = -\log A \), we have
\[
|du|_{\eta_2}^2 = \chi(n + 1) p^{-2} A^{-2p},
\]
\[
\text{grad}_{\eta_2}(u) = -\chi(n + 1) p^{-1} A^{-p} F_{n+1}.
\]
Moreover, by part (v) of Proposition 5.1, the scalar curvature \( S_{\eta_2} = 0 \) vanishes. Thus the reduced WP-equation becomes
\[
\nabla_{\eta_2}^2 \psi = -\frac{1}{n-1} \text{Ric}_{\eta_2}(V) \cdot \frac{\text{grad}_{\eta_2}(u)}{|du|_{\eta_2}^2} \cdot \psi
\]
\[
= -\frac{p}{n-1} A^p \text{Ric}_{\eta_2}(V) \cdot F_{n+1} \cdot \psi
\]
\[
= -\chi(n + 1) \frac{1}{n+1} A^{\frac{n+1}{2}} V \cdot F_{n+1} \cdot \psi
\]
(5.10)
and
\[
\nabla_{F_{n+1}}^2 \psi = -\frac{1}{n-1} \text{Ric}_{\eta_2}(F_{n+1}) \cdot \frac{\text{grad}_{\eta_2}(u)}{|du|_{\eta_2}^2} \cdot \psi
\]
\[
= -\frac{n}{n+1} A^{-\frac{n+1}{2}} \psi = \frac{1}{2} \text{Tr}_{g_M}(\Theta_{\eta_2}) \psi.
\]
(5.11)
On the other hand,
\[
\nabla_{V}^2 \psi = \nabla_{V}^{g_M} \psi + \chi(n + 1) \frac{1}{2} \Theta_{\eta_2}(V) \cdot F_{n+1} \cdot \psi
\]
\[
= \nabla_{V}^{g_M} \psi - \chi(n + 1) \frac{1}{n+1} A^{\frac{n+1}{2}} V \cdot F_{n+1} \cdot \psi.
\]
(5.12)
From (5.10)-(5.12) we conclude the proof. \( \square \)

Following a standard argument in the proof of Proposition 5.2 and Theorem 5.1 of [7] in pseudo-Riemannian signature, we now establish the following existence theorems.

Theorem 5.1 Let \( Q^{n+1} = M^n \times \mathbb{R}, \eta_2 \) be a \((Y)\)-warped product of \((M^n, g_M)\) and \((\mathbb{R}, g_\mathbb{R})\) with \((Y)\)-factor \( n+1 \). If \((M^n, g_M)\) admits a parallel spinor, then for any real number \( \lambda_Q \in \mathbb{R} \neq 0 \), \((Q^{n+1}, \eta_2)\) admits a WK-spinor to WK-number \( (\sqrt{-1})^r \lambda_Q \), where \( r = 0 \) if \( \chi(n + 1) = 1 \) and \( r = 1 \) if \( \chi(n + 1) = -1 \), respectively.

Theorem 5.2 Let \( Q^{n+1} = M^n \times \mathbb{R}, \eta_2 \) be a \((Y)\)-warped product of \((M^n, g_M)\) and \((\mathbb{R}, g_\mathbb{R})\) with \((Y)\)-factor \( \frac{n+1}{2} \). If \((M^n, g_M)\) admits a parallel spinor, then \((Q^{n+1}, \eta_2)\) admits a reduced WP-spinor that is not a parallel spinor.

Theorem 5.1 above improves Theorem 5.1 of [7], since \((Y)\)-warped products of \((M^n, g_M)\) and \((\mathbb{R}, g_\mathbb{R})\) with \((Y)\)-factor \( \frac{n+1}{2} \) essentially generalize the metrics in Lemma 5.3 of [7].
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