Existence and uniqueness of solutions for a class of fractional nonlinear boundary value problems under mild assumptions

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Abstract

We deal with the following Riemann–Liouville fractional nonlinear boundary value problem:

\[
\begin{cases}
D_0^\alpha v(x) + f(x, v(x)) = 0, & 2 < \alpha \leq 3, x \in (0, 1), \\
v(0) = v'(0) = v(1) = 0.
\end{cases}
\]

Under mild assumptions, we prove the existence of a unique continuous solution \( v \) to this problem satisfying

\[|v(x)| \leq cx^{\alpha - 1}(1 - x) \quad \text{for all } x \in [0, 1] \text{ and some } c > 0.\]

Our results improve those obtained by Zou and He (Appl. Math. Lett. 74:68–73, 2017).

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1 Introduction

Fractional differential equations have attracted great attention due to their ability to model various phenomena in applied sciences. The so-called fractional differential equations are specified by generalizing the standard integer-order derivative to arbitrary order. For more interesting theoretical results and scientific applications of fractional differential equations, we refer to the monographs of Diethelm [2] and Kilbas et al. [3] and references therein.

The existence, uniqueness, and global behavior of solutions for boundary value problems of fractional differential equations have been considered in several recent papers (see, e.g., [1, 4–9] and references therein).
Zou and He [1] investigated the problem

\begin{equation}
\begin{cases}
D^\alpha v(x) + f(x, v(x)) = 0, & 2 < \alpha \leq 3, x \in (0, 1), \\
v(0) = v'(0) = v(1) = 0,
\end{cases}
\end{equation}

where $D^\alpha$ denotes the standard Riemann–Liouville fractional derivative, and $f$ satisfies the following conditions:

(H1) $f \in C((0, 1) \times \mathbb{R}, \mathbb{R})$ and $\int_0^1 |f(x, 0)| \, dx < \infty$;

(H2) There exists $q \in C((0, 1), (0, \infty))$ such that

$$|f(x, v) - f(x, w)| \leq q(x)|v - w|, \quad \forall x \in (0, 1), v, w \in \mathbb{R},$$

and

$$0 < \int_0^1 q(x) \, dx < \infty.$$  \hspace{1cm} (1.2)

Let $L > 0$ be the minimum positive constant such that

$$\int_0^1 G_0(x, y)q(y)y^{\alpha-1}(1-y) \, dy \leq Lx^{\alpha-1}(1-x),$$  \hspace{1cm} (1.3)

where $G_0(x, y)$ is the Green’s function (given later in this paper) associated with problem (1.1). By using Banach’s contraction principle on some convenient Banach space they have obtained the following result.

**Theorem 1.1** Under assumptions (H1)–(H2) and $L < 1$, problem (1.1) has a unique solution in $C([0, 1])$.

Motivated by this result, we prove that the conclusion of Theorem 1.1 remains true under the following weaker assumptions:

(A1) $f \in C((0, 1) \times \mathbb{R}, \mathbb{R})$ and $\int_0^1 (1-x)^{\alpha-2}|f(x, 0)| \, dx < \infty$;

(A2) There exists $q \in C((0, 1), (0, \infty))$ such that

$$|f(x, v) - f(x, w)| \leq q(x)|v - w|, \quad \forall x \in (0, 1), v, w \in \mathbb{R},$$

and

$$0 < M_{q, \alpha} := \frac{1}{\Gamma(\alpha - 1)} \int_0^1 x^{\alpha-1}(1-x)^{\alpha-1}q(x) \, dx < \infty.$$  \hspace{1cm} (1.4)

**Remark 1.2** It is clear that conditions (H1)–(H2) imply (A1)–(A2).

Conversely, for $\beta \in [1, \alpha - 1)$, the function $f(x, v) := (1-x)^{-\beta}(1 + v)$ satisfies hypotheses (A1)–(A2) but not conditions (H1)–(H2). So assumptions (A1)–(A2) are weaker.

In this paper, for $\alpha \in [2, 3)$, we use the following notations:

- $h(x) := x^{\alpha-1}(1-x), x \in [0, 1]$.
- $G_0(x, y)$ denotes the Green’s function of the operator $\nu \mapsto -D^\alpha \nu$ with boundary conditions $\nu(0) = \nu'(0) = \nu(1)$. 
• $E := \{ a > 0 : \int_0^1 G_a(x, y)h(y)q(y) \, dy \leq ah(x), x \in [0, 1] \}$ (we will see that $E \neq \emptyset$).

• $M := \inf E$. \hfill (1.5)

We will prove that $M$ is a positive constant satisfying the following range estimation:

$$M_{q,a+1} \leq M \leq M_{q,a}. \hfill (1.6)$$

• For $a \in \mathbb{R}$, $a^* := \max(a, 0)$.

• $C_h([0,1]) := \{ v \in C([0, 1]) : \text{there is } \sigma > 0 \text{ such that } |v(x)| \leq \sigma h(x), x \in [0, 1] \}$.

In the next remark, we list some properties of elements of $C_h([0,1])$.

Remark 1.3

(i) $C_h([0,1])$ is a Banach space equipped with the following $h$-norm:

$$\|v\|_h := \inf \{ \sigma > 0 : |v(x)| \leq \sigma h(x), x \in [0, 1] \} = \sup_{x \in (0,1)} \frac{|v(x)|}{h(x)}. \hfill (1.7)$$

(ii) $v \in C_h([0,1])$ if and only if $v = h\varphi$, where $\varphi$ is a bounded continuous function in $(0, 1)$.

Our main result is the following:

**Theorem 1.4** Assume that (A1) and (A2) hold. If $M < 1$, then problem (1.1) has a unique solution $v$ in $Ch([0,1])$. In addition, for any $v_0 \in C_h([0,1])$, the iterative sequence $v_k(x) := \int_0^1 G_a(x, y)f(y, v_{k-1}(y)) \, dy$ converges to $v$ with respect to the $h$-norm, and we have

$$\|v_k - v\|_h \leq \frac{M^k}{1-M} \|v_1 - v_0\|_h. \hfill (1.8)$$

Our paper is organized as follows. In Sect. 2, we improve the estimates on Green’s function $G_a$ obtained in [1, Lemma 2.2]. This allows us to obtain the range estimation (1.6). Our main result is proved in Sect. 3. Some examples and approximations are given at the end.

2 Preliminaries

**Definition 2.1** ([3]) Let $f : (0, \infty) \to \mathbb{R}$ be a measurable function.

(i) The Riemann–Liouville fractional integral of order $\gamma > 0$ for $f$ is defined as

$$I^\gamma f(x) := \frac{1}{\Gamma(\gamma)} \int_0^x (x - y)^{\gamma-1} f(y) \, dy,$$

where $\Gamma$ is the Euler gamma function.

(ii) The Riemann–Liouville fractional derivative of order $\gamma > 0$ for $f$ is defined as

$$D^\gamma f(x) := \frac{1}{\Gamma(n-\gamma)} \left( \frac{d}{dx} \right)^n \int_0^x (x - y)^{n-\gamma-1} f(y) \, dy,$$

where $n = [\gamma] + 1$, and $[\gamma]$ is the integer part of $\gamma$. 
By [10, Lemma 2.2] the Green’s function associated with problem (1.1) is given by

\[ G_\alpha(x,y) = \begin{cases} \frac{1}{\Gamma(\alpha)} x^{\alpha-1} (1-y)^{\alpha-1} \quad & \text{for } 0 \leq y \leq x \leq 1, \\ \frac{1}{\Gamma(\alpha)} x^{\alpha-1} (1-y)^{\alpha-1} \quad & \text{for } 0 \leq x \leq y \leq 1. \end{cases} \]  

(2.1)

**Lemma 2.2** The Green’s function \( G_\alpha(x,y) \) has the following properties:

(i) \( G_\alpha(x,y) \) is a nonnegative continuous function on \([0,1] \times [0,1]\).

(ii) For all \( x, y \in [0,1] \), we have

\[ H_\alpha(x,y) \leq G_\alpha(x,y) \leq (\alpha - 1)H_\alpha(x,y), \]

where \( H_\alpha(x,y) := \frac{1}{\Gamma(\alpha)} x^{\alpha-2} (1-y)^{\alpha-2} \min(x,y)(1-\max(x,y)) \).

**Proof** It is obvious that (i) holds. Now we prove (ii). From (2.1), for all \( x, y \in (0,1) \), we have

\[ \Gamma(\alpha)G_\alpha(x,y) = x^{\alpha-1} (1-y)^{\alpha-1} - (x-y)^{\alpha-1} \]

(2.3)

\[ = x^{\alpha-1} (1-y)^{\alpha-1} \left( 1 - \left( \frac{x-y}{x(1-y)} \right)^{\alpha-1} \right). \]

(2.4)

Since for \( \lambda > 0 \) and \( t \in [0,1] \),

\[ \min(1,\lambda)(1-t) \leq 1 - t^\lambda \leq \max(1,\lambda)(1-t), \]

we deduce that

\[ 1 - \frac{(x-y)^\lambda}{x(1-y)} \leq 1 - \left( \frac{(x-y)^\lambda}{x(1-y)} \right)^{\alpha-1} \leq (\alpha - 1) \left( 1 - \frac{(x-y)^\lambda}{x(1-y)} \right). \]

Using this fact and (2.4), we obtain

\[ x(1-y) - (x-y)^\lambda \leq \frac{\Gamma(\alpha)G_\alpha(x,y)}{x^{\alpha-2}(1-y)^{\alpha-2}} \leq (\alpha - 1)(x(1-y) - (x-y)^\lambda). \]

Hence estimates (2.2) follow from

\[ x(1-y) - (x-y)^\lambda = \min(x,y)(1-\max(x,y)). \]

\[ \square \]

**Remark 2.3** In [1, Lemma 2.2], the authors stated that for all \( x, y \in [0,1] \),

(i) \( x^{\alpha-1}(1-x)y(1-y)^{\alpha-1} \leq \Gamma(\alpha)G_\alpha(x,y) \leq (\alpha - 1)y(1-y)^{\alpha-1} \),

(ii) \( x^{\alpha-1}(1-x)y(1-y)^{\alpha-1} \leq \Gamma(\alpha)G_\alpha(x,y) \leq (\alpha - 1)x^{\alpha-1}(1-x) \).

Note that since for all \( x, y \in [0,1] \),

\[ xy \leq \min(x,y) \text{ and } (1-x)(1-y) \leq (1-\max(x,y)), \]

we get

\[ x^{\alpha-1}(1-x)y(1-y)^{\alpha-1} \leq \Gamma(\alpha)H_\alpha(x,y) \leq \min(x^{\alpha-1}(1-x), y(1-y)^{\alpha-1}). \]

Combining this fact with (2.2), we immediately obtain inequalities (i) and (ii).
Therefore estimates (2.2) improve those stated in [1, Lemma 2.2].

**Lemma 2.4** Let \( q \in C((0,1), [0, \infty)) \) and assume that \( 0 < M_{q,a} < \infty \). Then

\[
M_{q,a+1} \leq M \leq M_{q,a},
\]

where \( M \) is the constant defined by (1.5).

**Proof** Let

\[
E = \left\{ a > 0 : \int_0^1 G_a(x,y)h(y)q(y) \, dy \leq ah(x), x \in [0,1] \right\},
\]

where \( h(x) := x^{\alpha - 1} (1 - x), x \in [0,1] \).

By (2.2) we obtain

\[
\int_0^1 G_a(x,y)h(y)q(y) \, dy \leq \frac{1}{\Gamma(\alpha - 1)} x^{\alpha - 2} \int_0^1 y^{\alpha - 1} (1 - y)^{\alpha - 1} \min(x,y)(1 - \max(x,y))q(y) \, dy \leq M_{q,a} h(x).
\]

It follows that \( E \neq \emptyset \) and \( M \leq M_{q,a} \), where \( M := \inf E \).

On the other hand, using again (2.2) and that

\[
\min(x,y)(1 - \max(x,y)) \geq xy(1 - x)(1 - y) \quad \text{for} \quad x, y \in [0,1],
\]

we deduce that for any \( a \in E \),

\[
ah(x) \geq \frac{1}{\Gamma(\alpha)} x^{\alpha - 2} \int_0^1 y^{\alpha - 1} (1 - y)^{\alpha - 1} \min(x,y)(1 - \max(x,y))q(y) \, dy \geq \frac{1}{\Gamma(\alpha)} x^{\alpha - 2} \int_0^1 y^{\alpha - 1} (1 - y)^{\alpha - 1} xy(1 - x)(1 - y)q(y) \, dy = h(x) M_{q,a+1}.
\]

Hence for each \( a \in E \),

\[
a \geq M_{q,a+1}.
\]

Therefore \( M \geq M_{q,a+1} \), that is, \( M \in [M_{q,a+1}, M_{q,a}] \). \( \square \)

**Remark 2.5** From Lemma 2.4 it is obvious that if \( M_{q,a} < 1 \), then

\( M := \inf E < 1 \). Note that the inequality \( M_{q,a} < 1 \) can be verified for a large class of functions \( q \), including the singular cases. For example, let

\[
B(a,b) := \int_0^1 t^{a - 1} (1 - t)^{b - 1} \, dt \quad \text{for} \quad a > 0 \quad \text{and} \quad b > 0.
\]

Then by using MATLAB we obtain

(i) If \( q \in C((0,1)) \) with \( q > 0 \) and \( \|q\|_\infty \leq 1 \), then

\[
M_{q,a} \leq \frac{B(a,a)}{\Gamma(a - 1)} < 1.
\]
(ii) If \( q(x) := (1 - x)^{-\frac{\alpha}{2}} \), then
\[
M_{\alpha} = \frac{B(\alpha, \frac{\alpha}{2})}{\Gamma(\alpha - 1)} < 1.
\]

(iii) If \( q(x) := x^{-\frac{\alpha}{2}} (1 - x)^{-\frac{\alpha}{2}} \), then
\[
M_{\alpha} = \frac{B(\frac{2\alpha}{3}, \frac{\alpha}{2})}{\Gamma(\alpha - 1)} < 1.
\]

3 Existence and uniqueness
We need the following useful lemma.

Lemma 3.1 Let \( 2 < \alpha < 3 \), and let \( \psi \) be a function such that \( x \to (1 - x)^{\alpha - 1} \psi(x) \in C((0, 1)) \cap L^1((0, 1)) \). Then the unique continuous solution of the problem

\[
\begin{aligned}
\mathcal{D}^\alpha v(x) &= -\psi(x), & x \in (0, 1), \\
v(0) = v'(0) = v(1) &= 0,
\end{aligned}
\tag{3.1}
\]

is given by

\[
V\psi(x) := \int_0^1 G_\alpha(x, y)\psi(y) \, dy.
\]

Proof Let \( \psi \) be a function such that \( x \to (1 - x)^{\alpha - 1} \psi(x) \in C((0, 1)) \cap L^1((0, 1)) \). Since by Lemma 2.2, \( G_\alpha(x, y) \) belongs to \( C([0, 1] \times [0, 1]) \) with
\[
0 \leq G_\alpha(x, y) \leq \frac{1}{\Gamma(\alpha - 1)} (1 - y)^{\alpha - 1},
\]
we deduce by the dominated convergence theorem that \( V\psi \in C([0, 1]) \) and \( V\psi(0) = V\psi(1) = 0 \). Therefore \( I^{3-\alpha}(V|\psi|) \) is bounded on \([0, 1]\). By Fubini's theorem we obtain
\[
I^{3-\alpha}(V\psi)(x) = \frac{1}{\Gamma(3 - \alpha)} \int_0^x (x - y)^{2-\alpha} V\psi(y) \, dy
\]
\[
= \int_0^1 K(x, r)\psi(r) \, dr,
\]
where \( K(x, r) := \frac{1}{\Gamma(3-\alpha)} \int_0^x (x - y)^{2-\alpha} G_\alpha(y, r) \, dy. \)

Simple calculation gives
\[
K(x, r) = \frac{1}{2} x^2 (1 - r)^{\alpha - 1} - \frac{1}{2} (x - r)^2.
\]

Hence, for \( x \in (0, 1) \), we have
\[
I^{3-\alpha}(V\psi)(x) = \frac{x^2}{2} \int_0^1 (1 - r)^{\alpha - 1} \psi(r) \, dr - \frac{1}{2} \int_0^x (x - r)^2 \psi(r) \, dr.
\]
This implies that
\[
\frac{d^3}{dx^3} (I^{3-a}(V\varphi))(x) = -\varphi(x).
\]

Now, since for each \(y \in (0,1)\),
\[
\lim_{x \to 0} \frac{G_\alpha(x,y)}{x} = 0 \quad \text{and} \quad \frac{1}{\Gamma(\alpha - 1)} (1 - y)^{\alpha - 1},
\]

by the dominated convergence theorem we obtain \((V\varphi)'(0) = 0\).

To prove the uniqueness, let \(v, w \in C([0,1])\) be two solutions of problem (3.1) and set \(\theta := v - w\). Then \(\theta \in C([0,1])\), and we have
\[
\begin{aligned}
D^a \theta(x) &= 0, \quad x \in (0,1), \\
\theta(0) &= \theta'(0) = \theta(1) = 0.
\end{aligned}
\]

By [3, Corollary 2.1] there exist \(c_1, c_2, c_3 \in \mathbb{R}\) such that
\[
\theta(x) = c_1 x^{\alpha - 1} + c_2 x^{\alpha - 2} + c_3 x^{\alpha - 3}.
\]

Applying the boundary conditions, we obtain \(c_3 = c_2 = c_1 = 0\), that is, \(v = w\). \(\square\)

**Remark 3.2** The conclusion of Lemma 3.1 remains true for \(\alpha = 3\).

**Proof of Theorem 1.4** Assume that (A1) and (A2) hold and \(M < 1\), where \(M\) is given by (1.5). Let us prove that problem (1.1) has a unique solution \(v\) in \(C_h([0,1])\). In addition, for any \(v_0 \in C_h([0,1])\), the iterative sequence \(v_k(x) := \int_0^1 G_\alpha(x,y)f(y,v_{k-1}(y))\,dy\) converges to \(v\) with respect to the \(h\)-norm, and we have
\[
\|v_k - v\|_h \leq \frac{M^k}{1 - M} \|v_1 - v_0\|_h.
\]

To this end, define the operator \(T\) by
\[
Tv(x) = \int_0^1 G_\alpha(x,y)f(y,v(y))\,dy, x \in [0,1], v \in C_h([0,1]).
\]

We claim that \(T\) is a contraction operator from \((C_h([0,1]), \|\cdot\|_h)\) into itself. Let \(v \in C_h([0,1])\), and let \(\sigma > 0\) be such that \(|v(x)| \leq \sigma h(x)\) for all \(x \in [0,1]\).

Since by Lemma 2.2(ii), \(0 \leq G_\alpha(x,y) \leq \frac{1}{\Gamma(\alpha - 1)} (1 - y)^{\alpha - 2}\), it follows from (A2) that
\[
|G_\alpha(x,y)f(y,v(y))| \leq \frac{1}{\Gamma(\alpha - 1)} (1 - y)^{\alpha - 2} \left( |f(y,v(y)) - f(y,0)| + |f(y,0)| \right)
\leq \frac{1}{\Gamma(\alpha - 1)} (1 - y)^{\alpha - 2} \left( q(y)|v(y)| + |f(y,0)| \right)
\leq \frac{1}{\Gamma(\alpha - 1)} (\sigma y^{\alpha - 1}(1 - y)^{\alpha - 1} q(y) + (1 - y)^{\alpha - 2})|f(y,0)|).
\]
Since $G_{\alpha}(x, y)$ is continuous on $[0, 1] \times [0, 1]$, by (A1)–(A2) and the dominated convergence theorem we deduce that $Tv \in C([0, 1])$.

Furthermore, from Lemma 2.2(ii) we have
\[
0 \leq G_{\alpha}(x, y) \leq \frac{1}{\Gamma(\alpha - 1)} h(x)(1 - y)^{\alpha - 2}.
\]

Hence by using (3.3) and similar arguments as before we obtain
\[
|Tv(x)| \leq \left[ \sigma M_{q, \alpha} + \frac{1}{\Gamma(\alpha - 1)} \int_0^1 (1 - y)^{\alpha - 2} |f(y, 0)| \, dy \right] h(x),
\]
and thus $T(C_h([0, 1])) \subset C_h([0, 1])$.

Next, for any $v, w \in C_h([0, 1])$, by using (A2) we obtain that for $x \in [0, 1]$,
\[
|Tv(x) - Tw(x)| \leq \int_0^1 G_{\alpha}(x, y) \left| f(y, v(y)) - f(y, w(y)) \right| \, dy
\]
\[
\leq \int_0^1 G_{\alpha}(x, y) q(y) \left| v(y) - w(y) \right| \, dy
\]
\[
\leq \|v - w\|_h \int_0^1 G_{\alpha}(x, y) q(y) h(y) \, dy
\]
\[
\leq M \|v - w\|_h h(x).
\]

Hence
\[
\|Tv - Tw\|_h \leq M \|v - w\|_h.
\]

Since $M < 1$, $T$ becomes a contraction operator in $C_h([0, 1])$. So there exists a unique $v \in C_h([0, 1])$ satisfying
\[
v(x) = \int_0^1 G_{\alpha}(x, y) f(y, v(y)) \, dy, \quad x \in (0, 1).
\]

It remains to prove that $v$ is a solution of problem (1.1). Indeed, it is clear that $x \to (1 - x)^{\alpha - 1} f(x, v(x)) \in C((0, 1))$. Next, by using (A2) and $v \in C_h([0, 1])$ we obtain
\[
\left| (1 - x)^{\alpha - 1} f(x, v(x)) \right| \leq (1 - x)^{\alpha - 1} \left( |f(x, v(x)) - f(x, 0)| + |f(x, 0)| \right)
\]
\[
\leq (1 - x)^{\alpha - 1} \left( q(x) |v(x)| + |f(x, 0)| \right)
\]
\[
\leq \sigma x^{\alpha - 1} (1 - x)^{\alpha - 1} q(x) + (1 - x)^{\alpha - 2} |f(x, 0)|.
\]

Therefore by (A1) and (A2) it follows that $x \to (1 - x)^{\alpha - 1} f(x, v(x)) \in L^1((0, 1))$. Hence from Lemma 3.1 we derive that $v$ is a solution of problem (1.1).

Finally, it is well known that for any $v_0 \in C_h([0, 1])$, the iterative sequence $v_k(x) := \int_0^1 G_{\alpha}(x, y) f(y, v_{k-1}(y)) \, dy$ converges to $v$, and we have
\[
\|v_k - v\|_h \leq \frac{M^k}{1 - M} \|v_1 - v_0\|_h.
\]
Example 3.3 Let $2 < \alpha \leq 3$. Consider the problem

$$
\begin{aligned}
D^\alpha v(x) + q(x) \cos v &= 0, \quad x \in (0, 1), \\
v(0) = v'(0) = v(1) &= 0,
\end{aligned}
$$

(3.5)

where $q \in C((0, 1))$ with $q > 0$ and $\|q\|_\infty \leq 1$. Let $f(x, v) := q(x) \cos v$ for $(x, v) \in (0, 1) \times \mathbb{R}$. We have $f \in C((0, 1) \times \mathbb{R}, \mathbb{R})$ and

$$
\int_0^1 (1-x)^{\alpha - 2} |f(x, 0)| \, dx \leq \|q\|_\infty \int_0^1 (1-x)^{\alpha - 2} \, dx < \infty.
$$

So assumption (A1) is verified.

On the other hand, since $\cos v$ is a Lipschitz function, we obtain

$$|f(x, v) - f(x, w)| \leq q(x)|v - w|, \quad x \in (0, 1), v, w \in \mathbb{R}.$$

By Lemma 2.4 and Remark 2.5(i) we have

$$0 < M \leq M_{q, \alpha} = \frac{\|q\|_\infty}{\Gamma(\alpha - 1)} \int_0^1 \frac{x^{\alpha - 1}(1-x)^{\alpha - 1}}{\Gamma(\alpha - 1)} \, dx < 1.$$

Hence by Theorem 1.4 problem (3.5) has a unique solution $v \in C_b([0, 1])$.

Example 3.4 Let $2 < \alpha \leq 3$ and consider the singular problem

$$
\begin{aligned}
D^\alpha v(x) + (1-x)^{-\frac{\alpha}{2}} (1 + \sin v) &= 0, \quad x \in (0, 1), \\
v(0) = v'(0) = v(1) &= 0,
\end{aligned}
$$

(3.6)

In this case, we have $f(x, v) = (1-x)^{-\frac{\alpha}{2}} (1 + \sin v)$ for $(x, v) \in (0, 1) \times \mathbb{R}$. So $f \in C((0, 1) \times \mathbb{R}, \mathbb{R})$ and $\int_0^1 (1-x)^{\alpha - 2} |f(x, 0)| \, dx = \int_0^1 (1-x)^{\frac{\alpha}{2} - 2} \, dx < \infty$, that is, assumption (A1) is satisfied.

On the other hand, we clearly have

$$|f(x, v) - f(x, w)| \leq q(x)|v - w|, \quad x \in (0, 1), v, w \in \mathbb{R},$$

where $q(x) := (1-x)^{-\frac{\alpha}{2}}$.

From Lemma 2.4 and Remark 2.5(ii) we deduce that

$$0 < M \leq M_{q, \alpha} = \frac{1}{\Gamma(\alpha - 1)} \int_0^1 \frac{x^{\alpha - 1}(1-x)^{\frac{\alpha}{2} - 1}}{\Gamma(\alpha - 1)} \, dx < 1.$$

Hence by Theorem 1.4 this problem has a unique solution $v \in C_b([0, 1])$. In particular, for $\alpha = \frac{5}{2}$, the unique solution is approximated (see Fig. 1) by the iterative sequence $v_k(x) := \int_0^1 G_\frac{5}{2} (x, y)(1-y)^{-\frac{\alpha}{2}} (1 + \sin(v_{k-1}(y))) \, dy$ with $v_0(x) = x^{\frac{\alpha}{2}} (1-x), x \in [0, 1]$.
Example 3.5 Consider the problem

\[
\begin{cases}
D^{\frac{5}{2}}v(x) + x^{-\frac{5}{2}}(1 - x)^{-\frac{5}{4}}(1 + v) = 0, & x \in (0, 1), \\
v(0) = v'(0) = v(1) = 0.
\end{cases}
\]  

(3.7)

As in Example 3.4, we verify that assumptions (A1) and (A2) are satisfied. Therefore by Theorem 1.4 problem (3.7) has a unique solution \( v \in C_{\delta}([0, 1]) \), and the iterative sequence defined by \( v_0(x) := x^{\frac{5}{2}}(1 - x), \ x \in [0, 1] \), and

\[
v_k(x) := \int_{0}^{1} G_{\frac{5}{2}}(x, y)y^{-\frac{5}{2}}(1 - y)^{-\frac{5}{4}}(1 + v_{k-1}(y)) \, dy
\]

cconverges to \( v \) (see Fig. 2).
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Authors’ contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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References
1. Zou, Y., He, G.: On the uniqueness of solutions for a class of fractional differential equations. Appl. Math. Lett. 74, 68–73 (2017)
2. Diethelm, K.: The Analysis of Fractional Differential Equations. Lecture Notes in Mathematics, vol. 2004. Springer, Berlin (2010)
3. Kilbas, A.A., Srivastava, H.M., Trujillo, J.J.: Theory and Applications of Fractional Differential Equations. North-Holland Mathematics Studies, vol. 204. Elsevier, Amsterdam (2006)
4. Atangana, A., Akgül, A.: Analysis of new trends of fractional differential equations. In: Fractional Order Analysis, pp. 91–111. Wiley, USA (2020)
5. Bai, Z.: On positive solutions of a nonlocal fractional boundary value problem. Nonlinear Anal. 72(2), 916–924 (2010)
6. Bai, Z., Zhang, S., Sun, S., Yin, C.: Monotone iterative method for fractional differential equations. Electron. J. Differ. Equ. 2016, 6 (2016)
7. Cui, Y.: Uniqueness of solution for boundary value problems for fractional differential equations. Appl. Math. Lett. 51, 48–54 (2016)
8. Liang, S., Zhang, J.: Positive solutions for boundary value problems of nonlinear fractional differential equation. Nonlinear Anal. 71(11), 5545–5550 (2009)
9. Zhang, X., Liu, L., Wu, Y.: The uniqueness of positive solution for a fractional order model of turbulent flow in a porous medium. Appl. Math. Lett. 37, 26–33 (2014)
10. Zhang, X., Liu, L., Wu, Y.: Multiple positive solutions of a singular fractional differential equation with negatively perturbed term. Math. Comput. Model. 55(3–4), 1263–1274 (2012)