Some models of spin coherence and decoherence in storage rings

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Abstract

I present some simple exactly solvable models of spin diffusion caused by synchrotron radiation noise in storage rings. I am able to use standard stochastic differential equation and Fokker-Planck methods and I thereby introduce, and exploit, the polarization density. This quantity obeys a linear evolution equation of the Bloch type, which is, like the Fokker-Planck equation, universal in the sense that it is independent of the state of the system. I also briefly consider Bloch equations for other local polarization quantities derived from the polarization density. One of the models chosen is of relevance for some existing and proposed low energy electron (positron) storage rings which need polarization. I present numerical results for a ring with parameters typical of HERA and show that, where applicable, the results of my approach are in satisfactory agreement with calculations using SLIM. These calculations provide a numerical check of a basic tenet of the conventional method of calculating depolarization using the \( \vec{n} \)-axis. I also investigate the equilibrium behaviour of the spin ensemble when there is no synchrotron radiation. Finally, I summarize other results which I have obtained using the polarization density and which will be published separately.
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Introduction

This paper provides an introduction to the use of spin polarization transport equations of Fokker-Planck and Liouville type in electron (positron) and proton storage rings. As vehicles for this study I use two exactly solvable but simple configurations called Machine I and Machine II.  

Machine I is a smoothed planar ring with no vertical emittance and with spins lying in the machine plane which diffuse as the result of the stochastic nature of synchrotron radiation photon emission. It is therefore an extremely simple arrangement but it serves to introduce the concepts without the latter being obscured by unnecessary complication.

Machine II is similar to Machine I except that it contains a Siberian Snake and is therefore relevant to some existing and proposed low energy electron storage rings. By introducing the snake the equilibrium polarization is constrained to the machine plane together with the spins and it therefore becomes possible to compare the Fokker-Planck approach with the conventional method of calculating the spin depolarization rate and to comment on the validity of the latter.

A novel aspect of this work is the introduction of a phase space dependent quantity called the polarization density. This satisfies an evolution equation of the Bloch type which provides a causal azimuthal evolution of the polarization density. This linear equation is universal, like the Fokker-Planck equation, in the sense that it is independent of the state of the system. There is no equation which could provide a causal azimuthal evolution for the polarization vector of a given ensemble, even in the simple cases of machines I and II. Other local polarization quantities derived from the polarization density also obey evolution equations of the Bloch type, but these equations depend on the orbital state of the system.

The basic orbital formalism needed is introduced in section 1. Then in section 2 the first model, Machine I, is studied. This configuration was already used in a preliminary study of the rate of decoherence of spins (= ‘horizontal spin diffusion’) lying in the machine plane of such a machine. As explained in those reports, the aim was to estimate the difficulty of using a radial rf field to flip spins which had been previously polarized into the vertical direction by the Sokolov-Ternov effect. The approach adopted was to consider the effect of damped stochastic synchrotron motion on the development of the distribution of spins lying in the machine plane and some numerical results were given. In this paper I will fill in some details omitted in and develop some extra analytical tools.

The dynamics for Machine I can be studied for various initial spin-orbit distributions. In this paper I consider two scenarios, i.e. two stochastic processes. The approach is pedagogical, step by step and exhaustive and uses elementary methods for the solution of stochastic differential equations. In scenario 1 I study the phase space distribution for a starting distribution which is pointlike in both spin and orbit space. In scenario 2 I begin with a point like spin distribution but with an orbital distribution which is already in equilibrium. The resulting phase space evolutions are referred to as Process 1 and Process 2 respectively. By comparing the asymptotic spin distributions I discover that they are not unique but that in both cases there is no complete decoherence. I also study the effect of switching off the synchrotron radiation. Process 1 has some theoretical importance and Process 2 is more closely related to physical situations. Numerical results for a machine similar to the HERA electron ring are presented.

1In appendices C and D I briefly consider machines III and IV which are closely related to Machine I.
2For Bloch equations in spin diffusion problems, see [Abr61].
3But it is model dependent.
4For an earlier treatment, see also [Kou91].
In section 3 I consider Machine II, where in addition to the fields of Machine I the spin experiences the influence of a pointlike Siberian Snake [DK78]. The aim here is to discover to what extent a snake can, by its tendency to cancel spin perturbations, suppress decoherence. I consider three scenarios, the stochastic processes 3, 4, 5. Process 3 has the same initial spin-orbit distribution as in scenario 2 for Machine I and as one will see the spin exhibits transients for the first few orbital damping times. Process 4 has no transients and for Process 5 I set the initial local polarization parallel to the $\vec{n}$-axis [DK72, HH96] at each point in phase space. By comparing these processes one finds that in the presence of radiation the asymptotic polarization direction is not exactly parallel to the $\vec{n}$-axis. Furthermore in the presence of the snake there is complete depolarization.

All of the noise processes studied in this report are processes of Ornstein-Uhlenbeck type 5 and in particular they are Markovian diffusion processes. One can determine their statistical properties in an explicit form.

Finally, in an Epilogue, I summarize further results involving the polarization density. The inclusion of a detailed account here would make this paper too long. These aspects will appear in a separate article.

1 The orbital model

In this section I lay the foundations for the models to be discussed in sections 2 and 3.

The underlying mathematical model for machines I and II [BBHMR94a, BBHMR94b] comprises a three-dimensional spin-orbit system for the two longitudinal orbit variables and the angle describing the orientation of the spin in the machine plane. In contrast to discrete stochastic processes used occasionally in the literature my ‘time’ parameter $s$ is continuous, so that I can work with differential equations. Only synchrotron motion is considered and the influence of the much faster betatron oscillations is neglected as are the Stern-Gerlach forces (back reaction of the spin onto the orbit) and depolarizer fields [BHR94a, BHR94b, Hei96]. There are no vertical bends so that the design orbit is planar. Underlying machines I and II is the ‘smooth ring’ approximation. This takes advantage of the fact that the synchrotron tune is usually very small so that the optical functions can be averaged around the ring. Then the combined orbital motion is described by the following stochastic differential equation:

$$\frac{d}{ds} \left( \begin{array}{c} \sigma(s) \\ \eta(s) \end{array} \right) = \left( \begin{array}{c} 0 \\ \frac{\Omega_s}{\kappa} - 2 \cdot \frac{\alpha_s}{L} \end{array} \right) \cdot \left( \begin{array}{c} \sigma(s) \\ \eta(s) \end{array} \right) + \sqrt{\omega} \cdot \left( \begin{array}{c} 0 \\ \zeta(s) \end{array} \right),$$

where $s$ denotes the distance around the ring (the ‘azimuth’), $\sigma$ the distance to the centre of the bunch, $\eta$ the fractional energy deviation, and $\zeta$ simulates the noise due to the synchrotron radiation. Also $\alpha_s$ is the one turn synchrotron damping decrement and $\omega$ is the one turn averaged stochastic kick strength where in terms of the curvature $K_x$ of the design orbit in the horizontal plane [Jow85, BHMR91]:

$$\omega \equiv \left( |K_x|^3 \cdot C_2 \right)_{\text{average}}, \quad C_2 \equiv \frac{55 \cdot \sqrt{3}}{48} \cdot C_1 \cdot \Lambda \cdot \gamma_0^2, \quad C_1 \equiv \frac{2}{3} \cdot e^2 \cdot \frac{\gamma_0^4}{E_0}, \quad \Lambda \equiv \frac{\hbar}{m_0 c_0}. $$

5 to be explained in sections 2.2 and 3.2
6 The prime denotes the derivative w.r.t. $s$.
7 Note that in contrast to the notation in [BBHMR94a, BBHMR94b] I use the symbol $\eta$ instead of $p_{\sigma}$ and $\omega$ instead of $\tilde{\omega}$. 
Here $\gamma_0$ and $E_0$ denote the design values of the Lorentz factor and the energy, $c_0$ the vacuum velocity of light, and $e$ resp. $m_0$ denote the charge resp. rest mass of the electron. The stochastic averages of the kicks $\zeta(s)$ are

$$<\zeta(s_1) \cdot \zeta(s_2)> = \delta(s_1 - s_2), \quad <\zeta(s)> = 0.$$  

Thus the stochastic part of the synchrotron radiation is treated as a Gaussian white noise process. This is sufficient for my purposes since the characteristic time for the emission of a photon is very small compared with other time scales of the system. Finally, $\kappa$ is the compaction factor, $L$ is the length of the ring and $\Omega_s = 2\pi \cdot Q_s / L$ where $Q_s$ is the synchrotron tune. The ring is perfectly aligned so that in the smooth approximation the closed orbit and design orbit are identical. The vertical emittance is taken to be zero.

## 2 Machine I

### 2.1

For ‘Machine I’ the spin vectors are restricted to the horizontal (machine) plane so that the spin vector $\vec{\xi}$ can be described by a single phase angle $\psi$. Although spin is a quantum mechanical phenomenon, in high energy storage rings it can be treated at the semiclassical level using the Thomas-BMT equation \[\text{[Tho27, BMT59]}\]

$$\vec{\xi}' = \vec{\Omega}_I \wedge \vec{\xi}, \quad (2.1)$$

describing the precession of a classical spin $\vec{\xi}$ in electric and magnetic fields. Alternatively I can take $\vec{\xi}$ to be the spin expectation value of an electron in a pure state with spin along $\vec{\xi}/||\vec{\xi}||$. The precession vector $\vec{\Omega}_I$ is a function of the magnetic and electric fields and of the particle velocity and energy. As is usual in this context I now write $\vec{\Omega}_I$ as a sum of a piece $\vec{\Omega}_{I,0}$ accounting for the fields on the closed orbit and a piece $\vec{\Omega}_{osc}$ accounting for synchrotron motion with respect to the closed orbit, i.e.

$$\vec{\Omega}_I = \vec{\Omega}_{I,0} + \vec{\Omega}_{osc}.$$  

Thus the Thomas-BMT equation on the closed orbit takes the form:

$$\vec{\xi}' = \vec{\Omega}_{I,0} \wedge \vec{\xi}. \quad (2.2)$$

Since only motion in the horizontal plane need be considered one can write

$$\vec{\Omega}_{I,0} \equiv ||\vec{\Omega}_{I,0}|| \cdot \vec{e}_3, \quad \vec{\Omega}_{osc} \equiv \Omega_{osc} \cdot \vec{e}_3,$$

where $\vec{e}_3$ points normal to the machine plane. \[\text{[10b]}\] By averaging (‘smoothing’) over one turn one obtains:

$$||\vec{\Omega}_{I,0}|| = (2\pi \nu / L), $$

\[\text{[8]}\]Note that $\delta$ denotes Dirac’s delta function.

\[\text{[9]}\]The extension to full three-dimensional spin motion, i.e. the inclusion of vertical spin is briefly considered in appendices C and D.

\[\text{[10]}\]Two additional unit-vectors $\vec{e}_1, \vec{e}_2$ are radial resp. longitudinal w.r.t. the closed orbit. Moreover $\vec{e}_1, \vec{e}_2, \vec{e}_3$ constitute an orthonormal, right-handed dreibein on the closed orbit. This defines the ‘machine frame’.
where $\nu \equiv \gamma_0 \cdot (g-2)/2$ is the number of spin precessions per turn \cite{Cha81}. Spin motion will be calculated conveniently with respect to a dreibein of orthonormal axes $\vec{m}_{0,I}(s), \vec{l}_{0,I}(s), \vec{n}_{0,I}(s)$, which obey the Thomas-BMT equation on the closed orbit. By choice the vectors $\vec{m}_{0,I}, \vec{l}_{0,I}$ precess in the horizontal plane around the vertical dipole field according to (2.2). The vector $\vec{n}_{0,I} (= \vec{m}_{0,I} \wedge \vec{l}_{0,I}$) is vertical and therefore periodic in $s$ with period $L$ (i.e. 1-turn periodic) in the machine frame.\footnote{Thus $\vec{n}_{0,I}$ is the so-called ‘$\vec{n}_0$-axis’ of Machine I.} The orthonormal axes can be chosen as:

\[
\vec{m}_{0,I}(s) \equiv \sin(||\vec{\Omega}_{I,0}|| \cdot s) \cdot \hat{e}_1 - \cos(||\vec{\Omega}_{I,0}|| \cdot s) \cdot \hat{e}_2 ,
\]
\[
\vec{l}_{0,I}(s) \equiv \cos(||\vec{\Omega}_{I,0}|| \cdot s) \cdot \hat{e}_1 + \sin(||\vec{\Omega}_{I,0}|| \cdot s) \cdot \hat{e}_2 ,
\]
\[
\vec{n}_{0,I}(s) \equiv \hat{e}_3 .
\]

In dealing only with horizontal spin I introduce the spin phase angle $\psi$ by

\[
\vec{\xi} \equiv \frac{\hbar}{2} \cdot \left(\vec{m}_{0,I} \cdot \cos(\psi) + \vec{l}_{0,I} \cdot \sin(\psi)\right).
\]

By only including synchrotron oscillations and averaging (smoothing) the instantaneous precession rate over one turn the Thomas-BMT equation is equivalent to

\[
\psi' = \Omega_{\text{osc}} = \left(2\pi \nu/L\right) \cdot \eta .
\]

Thus $\psi$ only couples to and is only driven by $\eta$.

I also introduce the spin vector

\[
\vec{S} \equiv \frac{\hbar}{2} \cdot \begin{pmatrix} \cos(\psi) \\ \sin(\psi) \\ 0 \end{pmatrix}, \tag{2.3}
\]

describing the spin in the $(\vec{m}_{0,I}, \vec{l}_{0,I}, \vec{n}_{0,I})$-frame, so that the Thomas-BMT equation reads as\footnote{Note that $\vec{\xi}$, unlike $\vec{S}$, is the spin vector in an arbitrary frame. Thus (2.1), unlike (2.4a), is valid in an arbitrary frame.} \cite{Cha81}

\[
\vec{S}'(s) = \vec{W}_I \left(\eta(s)\right) \wedge \vec{S}(s) , \tag{2.4a}
\]

with

\[
\vec{W}_I(\eta) \equiv \frac{2\pi \nu}{L} \cdot \eta \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} . \tag{2.4b}
\]

Thus for Machine I one has to deal with the following three-component Langevin equation:

\[
\begin{pmatrix} \sigma'(s) \\ \eta'(s) \\ \psi'(s) \end{pmatrix} = \begin{pmatrix} 0 & -\kappa & 0 \\ \Omega_s^2 / \kappa & -2 \cdot \alpha_s / L & 0 \\ 0 & 2\pi \nu / L & 0 \end{pmatrix} \cdot \begin{pmatrix} \sigma(s) \\ \eta(s) \\ \psi(s) \end{pmatrix} + \sqrt{\omega} \cdot \begin{pmatrix} 0 \\ \zeta(s) \\ 0 \end{pmatrix} . \tag{2.5}
\]

One now sees that the noise not only acts on the orbit motion but also indirectly on the spin via its coupling to $\eta$. It is this coupling which will lead to the spin decoherence.
2.2 The Langevin equation for Machine I

2.2.1

With the abbreviations:
\[ a \equiv -\kappa, \quad b \equiv \Omega_s^2/\kappa = (4\pi^2Q_s^2)/(\kappa L^2), \quad c \equiv -2 \cdot \alpha_s/L, \quad d \equiv ||\vec{\Omega}_{I,0}|| = 2\pi\nu/L, \]
the Langevin equation (2.5) can be rewritten as
\[ d\vec{x}(s) = \mathbf{A}_I \cdot \vec{x}(s) \cdot ds + \mathbf{B} \cdot d\vec{\mathbf{W}}(s), \]
where
\[ \mathbf{A}_I \equiv \begin{pmatrix} 0 & a & 0 \\ b & c & 0 \\ 0 & d & 0 \end{pmatrix}, \quad \mathbf{B} \equiv \sqrt{\omega} \cdot \begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \vec{x} \equiv \begin{pmatrix} \sigma \\ \eta \\ \psi \end{pmatrix}, \]
with
\[ d\vec{\mathbf{W}}(s) \equiv \begin{pmatrix} d\mathcal{W}_1(s) \\ d\mathcal{W}_2(s) \\ d\mathcal{W}_3(s) \end{pmatrix}. \]

Here the \( \mathcal{W}_k(s) \) are Wiener processes [Gar85] related to the Gaussian white noise process \( \zeta(s) \) formally by:
\[ d\mathcal{W}_k(s) = \zeta(s) \cdot ds. \]

For a practical storage ring:
\[ a < 0, \quad b > 0, \quad c < 0, \quad d > 0, \quad \omega > 0. \]

Furthermore
\[ a \cdot b + c^2/4 < 0, \]
since \( \alpha_s \ll Q_s \). The inequalities (2.10), (2.11) are assumed throughout this paper. For the HERA electron ring running at about 27 GeV the values are approximately:
\[ Q_s \approx 0.06, \quad \alpha_s \approx 0.0032, \quad \kappa \approx 0.00069, \quad \omega \approx 2 \cdot 10^{-12} m^{-1}, \quad L \approx 6300 m, \quad d \approx 6.2 \cdot 10^{-2} m^{-1}, \]
so that one has:
\[ a \approx -6.9 \cdot 10^{-4}, \quad b \approx 5.2 \cdot 10^{-6} m^{-2}, \quad c \approx -1.0 \cdot 10^{-6} m^{-1}, \]
\[ d \approx 6.2 \cdot 10^{-2} m^{-1}, \quad \omega \approx 2.0 \cdot 10^{-12} m^{-1}, \quad L \approx 6.3 \cdot 10^3 m. \]

The orbital damping ‘time’ \( \tau_{damp} \) of the system is given by
\[ \tau_{damp} \equiv \frac{2}{c} = \frac{L}{\alpha_s}, \]
so that \( 1/\tau_{damp} \) is the ‘orbital damping rate’. \[ \text{[13]} \]
In particular I get
\[ \tau_{damp} \approx 2.0 \cdot 10^6 m, \]
which corresponds to about 310 turns or about 6.6 milliseconds.

Note that by (2.4b), (2.4):
\[ \vec{W}_I(\eta) \equiv d \cdot \eta \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \]

\[ \text{[13]} \text{For more details on } \tau_{damp}, \text{ see Appendix E.} \]
2.2.2

Because $\mathbf{A}$ and $\mathbf{B}$ are matrices which do not depend on $\vec{x}$ the Langevin equation (2.7) describes three-component processes of Ornstein-Uhlenbeck type [Gar85]. Thus the stochastic integrations involved in the solution of (2.7) can be either defined as Ito-integrations or Stratonovich-integrations and lead by both methods to the Fokker-Planck equation (2.22). The analogous situation holds for Machine II.

If $s_0$ denotes the starting azimuth of a process $\vec{x}(s)$ then $\vec{x}(s_0)$ is always assumed to be chosen so that $\vec{x}(s)$ is a Markovian diffusion process [Arn73, Gar85].

The three-component differential equation (2.7) has essentially only two nontrivial components. Writing (2.7) in more detail I get:

$$
\begin{align}
\frac{d\vec{z}}{ds} &= \mathbf{A}_{orb} \cdot \vec{z} \cdot ds + \mathbf{B}_{orb} \cdot d\vec{W}_{orb}(s), \\
\frac{d\psi}{ds} &= \eta(s) \cdot ds,
\end{align}
$$

(2.14)

where

$$
\vec{z} \equiv \begin{pmatrix} \sigma \\ \eta \end{pmatrix},
$$

(2.15)

and where:

$$
\mathbf{A}_{orb} \equiv \begin{pmatrix} 0 & a \\ b & c \end{pmatrix}, \quad \mathbf{B}_{orb} \equiv \sqrt{\omega} \cdot \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad \vec{W}_{orb}(s) \equiv \begin{pmatrix} W_1(s) \\ W_2(s) \end{pmatrix}.
$$

The two-dimensionality is seen by transforming $\psi$ linearly to a new variable:

$$
\tilde{\psi} \equiv \psi - \frac{d}{a} \cdot \sigma.
$$

(2.16)

Then my Langevin equation (2.7) is equivalent to

$$
\begin{align}
\frac{d\tilde{z}}{ds} &= \mathbf{A}_{orb} \cdot \tilde{z} \cdot ds + \mathbf{B}_{orb} \cdot d\tilde{W}_{orb}(s), \\
\frac{d\tilde{\psi}}{ds} &= 0,
\end{align}
$$

(2.17)

so that $\tilde{\psi}(s)$ is $s$-independent.

2.3 The Fokker-Planck equation for Machine I. Further properties of Machine I

2.3.1

I abbreviate the stochastic average of a function $f(\sigma, \eta, \psi)$ for a process $\vec{x}(s)$ by

$$
< f(\sigma(s), \eta(s), \psi(s)) >,
$$

so that:

$$
< f(\sigma(s), \eta(s), \psi(s)) > \equiv \int_{-\infty}^{+\infty} d\sigma \int_{-\infty}^{+\infty} d\eta \int_{-\infty}^{+\infty} d\psi \cdot w(\sigma, \eta, \psi; s) \cdot f(\sigma, \eta, \psi),
$$

(2.18)

where the ‘probability density’ $w$ characterizes the state of the system at azimuth $s$. From (2.18) follows:

$$
w(\sigma, \eta, \psi; s) = < \delta(\sigma - \sigma(s)) \cdot \delta(\eta - \eta(s)) \cdot \delta(\psi - \psi(s)) >,
$$

(2.19)

14For the special processes 1 and 2 considered in detail I choose $s_0 = 0$. 

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so that the probability density is nonnegative and normalized by:

\[ 1 = \int_{-\infty}^{+\infty} d\sigma \int_{-\infty}^{+\infty} d\eta \int_{-\infty}^{+\infty} d\psi \cdot w(\sigma, \eta, \psi; s). \quad (2.20) \]

One sees by (2.18), (2.20) that the domains of the variables \( \sigma, \eta, \psi \) are chosen to be \((-\infty, +\infty)\), i.e. the real numbers, and that the probability density obeys boundary conditions for each of the variables \( \sigma, \eta, \psi \) with \( w \to 0 \) for \( \sigma, \eta, \psi \to \pm \infty \). I call these ‘standard’ boundary conditions. Moreover I always assume that the stochastic averages of the functions of interest are finite. For the motivation of these boundary conditions see section 2.3.6.

The key quantity of interest when dealing with spin is the ‘polarization vector’ \( \vec{P}^w_{\text{tot}}(s) \). This is the stochastic average of the normalized spin vector, i.e. it is given by

\[
\vec{P}^w_{\text{tot}}(s) \equiv \frac{2}{\hbar} < \vec{S}(s) > = \frac{2}{\hbar} \int_{-\infty}^{+\infty} d\sigma \int_{-\infty}^{+\infty} d\eta \int_{-\infty}^{+\infty} d\psi \cdot w(\sigma, \eta, \psi; s) \cdot \vec{S},
\]

\[
= \int_{-\infty}^{+\infty} d\sigma \int_{-\infty}^{+\infty} d\eta \int_{-\infty}^{+\infty} d\psi \cdot w(\sigma, \eta, \psi; s) \cdot \begin{pmatrix} \cos(\psi) \\ \sin(\psi) \\ 0 \end{pmatrix},
\]

where:

\[
\vec{S}(s) \equiv \frac{\hbar}{2} \begin{pmatrix} \cos(\psi(s)) \\ \sin(\psi(s)) \\ 0 \end{pmatrix}.
\]

I define the ‘polarization’ as the norm \( ||\vec{P}_{\text{tot}}^w|| \) of the polarization vector.

### 2.3.2

The processes can be either described ‘directly’ by handling the stochastic averages <\( f \)> or ‘indirectly’ by ‘ensembles’ via the corresponding probabilities, e.g. the probability density \( w \). The latter obeys a Fokker-Planck equation and for the Langevin equation (2.7) the Fokker-Planck equation has the form [Gar83, Ris89]

\[
\frac{\partial w}{\partial s} = -a \cdot \eta \cdot \frac{\partial w}{\partial \sigma} - b \cdot \sigma \cdot \frac{\partial w}{\partial \eta} - d \cdot \eta \cdot \frac{\partial w}{\partial \psi} - c \cdot w - c \cdot \eta \cdot \frac{\partial w}{\partial \eta} + \frac{\omega}{2} \cdot \frac{\partial^2 w}{\partial \eta^2},
\]

where

\[
\mathcal{D} \equiv \mathcal{B} \cdot \mathcal{B}^T = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & 0 \end{pmatrix}.
\]

Therefore the Fokker-Planck equation can be written as

\[
\frac{\partial w}{\partial s} \equiv L_{\text{FP,orb}} w + L_{\text{FP,spin}} w,
\]

where I used the abbreviations

\[
L_{\text{FP,orb}} \equiv -a \cdot \eta \cdot \frac{\partial}{\partial \sigma} - b \cdot \sigma \cdot \frac{\partial}{\partial \eta} - c - c \cdot \eta \cdot \frac{\partial}{\partial \eta} + \frac{\omega}{2} \cdot \frac{\partial^2}{\partial \eta^2},
\]

\[
L_{\text{FP,spin}} \equiv -d \cdot \eta \cdot \frac{\partial}{\partial \psi}.
\]
The ‘orbital part’ \(w_{\text{orb}}\) of a probability density \(w\) is defined by

\[
w_{\text{orb}}(\sigma, \eta; s) \equiv \int_{-\infty}^{+\infty} d\psi \cdot w(\sigma, \eta, \psi; s) . \tag{2.23}
\]

Because \(w\) denotes the probability density of a process \(\vec{x}(s)\), one observes that \(w_{\text{orb}}\) is the probability density of the corresponding orbital process \(\vec{z}(s)\). Given an orbital function \(f(\sigma, \eta)\) one sees that its stochastic average is determined by \(w_{\text{orb}}\), i.e.

\[
< f(\sigma(s), \eta(s)) > = \int_{-\infty}^{+\infty} d\sigma \int_{-\infty}^{+\infty} d\eta \cdot w_{\text{orb}}(\sigma, \eta; s) \cdot f(\sigma, \eta) .
\]

Furthermore the orbital part of the probability density is normalized by

\[
1 = \int_{-\infty}^{+\infty} d\sigma \int_{-\infty}^{+\infty} d\eta \cdot w_{\text{orb}}(\sigma, \eta; s) , \tag{2.24}
\]

which follows from (2.20). Because \(w\) solves the Fokker-Planck equation (2.22), \(w_{\text{orb}}\) solves the ‘orbital Fokker-Planck equation’

\[
\frac{\partial w_{\text{orb}}}{\partial s} = L_{\text{FP},\text{orb}} w_{\text{orb}} . \tag{2.25}
\]

The ‘spin part’ \(w_{\text{spin}}\) of a probability density \(w\) is defined by

\[
w_{\text{spin}}(\psi; s) \equiv \int_{-\infty}^{+\infty} d\sigma \int_{-\infty}^{+\infty} d\eta \cdot w(\sigma, \eta, \psi; s) , \tag{2.26}
\]

and it is normalized by

\[
1 = \int_{-\infty}^{+\infty} d\psi \cdot w_{\text{spin}}(\psi; s) ,
\]

which follows from (2.20). Given a function \(f(\psi)\) depending only on \(\psi\) one sees that its stochastic average is determined by \(w_{\text{spin}}\), i.e.

\[
< f(\psi(s)) > = \int_{-\infty}^{+\infty} d\psi \cdot w_{\text{spin}}(\psi; s) \cdot f(\psi) .
\]

With the standard boundary conditions one can introduce via Fourier transformation a ‘characteristic function’, namely \(\Phi\) corresponding to \(w\), defined by [Gar85]:

\[
\Phi(\vec{u}; s) = \int_{-\infty}^{+\infty} dx_1 \int_{-\infty}^{+\infty} dx_2 \int_{-\infty}^{+\infty} dx_3 \cdot \exp(i \cdot \vec{u}^T \cdot \vec{x}) \cdot w(\vec{x}; s) , \tag{2.27}
\]

so that

\[
w(\vec{x}; s) = \frac{1}{8\pi^3} \cdot \int_{-\infty}^{+\infty} du_1 \int_{-\infty}^{+\infty} du_2 \int_{-\infty}^{+\infty} du_3 \cdot \exp(-i \cdot \vec{u}^T \cdot \vec{x}) \cdot \Phi(\vec{u}; s) . \tag{2.28}
\]

\footnote{Thus \(w_{\text{orb}}\) describes the orbital distribution.}
Since \( w \) fulfills the Fokker-Planck equation (2.22), one finds:

\[
\frac{\partial \Phi}{\partial s} = \sum_{j,k=1}^{3} A_{I,kj} \cdot u_k \cdot \frac{\partial \Phi}{\partial u_j} - \frac{1}{2} \sum_{j,k=1}^{3} D_{jk} \cdot u_j \cdot u_k \cdot \Phi .
\] (2.29)

Analogously, for the orbital part one defines:

\[
\Phi_{orb}(\vec{t}, s) \equiv \int_{-\infty}^{+\infty} d\sigma \int_{-\infty}^{+\infty} d\eta \cdot \exp(i \cdot \vec{t}^T \cdot \vec{z}) \cdot w_{orb}(\vec{z}; s) ,
\] (2.30)

from which follows

\[
w_{orb}(\vec{z}; s) = \frac{1}{4\pi^2} \cdot \int_{-\infty}^{+\infty} dt_1 \int_{-\infty}^{+\infty} dt_2 \cdot \exp(-i \cdot \vec{t}^T \cdot \vec{z}) \cdot \Phi_{orb}(\vec{t}, s) .
\] (2.31)

Because \( w_{orb} \) fulfills the orbital Fokker-Planck equation (2.25), I conclude:

\[
\frac{\partial \Phi_{orb}}{\partial s} = \sum_{j,k=1}^{2} A_{I,kj} \cdot t_k \cdot \frac{\partial \Phi_{orb}}{\partial t_j} - \frac{1}{2} \sum_{j,k=1}^{2} D_{jk} \cdot t_j \cdot t_k \cdot \Phi_{orb} .
\] (2.32)

2.3.6

My chosen boundary conditions (see (2.20) and the sentences following) are very natural for \( \sigma, \eta \). After all, the rms relative energy spread for the values (2.12) is about \( 10^{-3} \) and the rms bunch length is about 1 cm. On the contrary I will be dealing with spreads in \( \psi \) of order \( 2\pi \) or more so that at first sight it would seem unnatural to choose the domain \((-\infty, +\infty)\) for \( \psi \). Indeed, if one writes the spin vector \( \vec{S} \) in spherical coordinates as:

\[
\vec{S} = \frac{\hbar}{2} \cdot \begin{pmatrix} \cos(\psi) \cdot \sin(\theta) \\ \sin(\psi) \cdot \sin(\theta) \\ \cos(\theta) \end{pmatrix} ,
\] (2.33)

then by (2.3) one can identify \( \psi \) as the azimuthal angle, where the polar angle \( \theta \) equals \( \pi/2 \). Since the values \( \psi = 0 \) resp. \( \psi = 2\pi \) are identified it would then seem more appropriate to use a probability density \( w_{per} \) which fulfills periodic boundary conditions in \( \psi \):

\[
w_{per}(\sigma, \eta, \psi + 2\pi; s) = w_{per}(\sigma, \eta, \psi; s) .
\] (2.34)

The normalization condition (2.20) would then be replaced by

\[
1 = \int_{-\infty}^{+\infty} d\sigma \int_{-\infty}^{+\infty} d\eta \int_{0}^{2\pi} d\psi \cdot w_{per}(\sigma, \eta, \psi; s) .
\] (2.35)

However Process 2 considered in this section has, for \( s > 0 \), a Gaussian probability density. Furthermore Process 1 has, for \( s > 0 \), a probability density which is a combination of a Gaussian function and a delta function. \[\text{16}\] Thus it is more convenient to adopt boundary conditions which allow one to work with Gaussians as much as possible. Thus machines I and II are treated with the standard boundary conditions and the periodic boundary conditions are only mentioned in passing.

\[\text{16}\] Both processes have standard boundary conditions.
The orbital part $w_{\text{per,orb}}$ of $w_{\text{per}}$ is defined by:

$$w_{\text{per,orb}}(\sigma, \eta; s) \equiv \int_0^{2\pi} d\psi \cdot w_{\text{per}}(\sigma, \eta, \psi; s),$$  \hspace{1cm} (2.36)

and the spin part $w_{\text{per,spin}}$ of $w_{\text{per}}$ is defined by:

$$w_{\text{per,spin}}(\psi; s) \equiv \int_{-\infty}^{+\infty} d\sigma \int_{-\infty}^{+\infty} d\eta \cdot w_{\text{per}}(\sigma, \eta, \psi; s),$$  \hspace{1cm} (2.37)

which is normalized by (2.33) as:

$$1 = \int_0^{2\pi} d\psi \cdot w_{\text{per,spin}}(\psi; s).$$  \hspace{1cm} (2.38)

Note that $w_{\text{per,spin}}$ is periodic in $\psi$:

$$w_{\text{per,spin}}(\psi + 2\pi; s) = w_{\text{spin}}(\psi; s).$$  \hspace{1cm} (2.39)

The polarization vector is defined by:

$$\vec{P}_{\text{tot}}(s) = \int_{-\infty}^{+\infty} d\sigma \int_{-\infty}^{+\infty} d\eta \int_0^{2\pi} d\psi \cdot w_{\text{per}}(\sigma, \eta, \psi; s) \cdot \begin{pmatrix} \cos(\psi) \\ \sin(\psi) \\ 0 \end{pmatrix}. $$  \hspace{1cm} (2.40)

Given a process with standard boundary conditions with probability density $w$ and defining $w_{\text{per}}$ in one of the two following ways:

$$w_{\text{per}}(\sigma, \eta, \psi; s) \equiv \sum_{n=-\infty}^{\infty} w(\sigma, \eta, \psi + 2\pi \cdot n; s),$$  \hspace{1cm} (2.41)

$$w_{\text{per}}(\sigma, \eta, \psi; s) \equiv \frac{1}{2\pi} \cdot \int_{-\infty}^{+\infty} d\psi_1 \cdot w(\sigma, \eta, \psi_1; s)$$

$$+ \frac{\sqrt{3} \cdot \cos(\psi)}{2\pi} \cdot \int_{-\infty}^{+\infty} d\psi_1 \cdot \cos(\psi_1) \cdot w(\sigma, \eta, \psi_1; s)$$

$$+ \frac{\sqrt{3} \cdot \sin(\psi)}{2\pi} \cdot \int_{-\infty}^{+\infty} d\psi_1 \cdot \sin(\psi_1) \cdot w(\sigma, \eta, \psi_1; s),$$  \hspace{1cm} (2.42)

one observes that $w_{\text{per}}$ fulfills the above mentioned properties and solves the Fokker-Planck equation (2.22). Moreover one then finds that $w_{\text{per,orb}} = w_{\text{orb}}$.

The expression (2.42) is of special interest if semiclassical considerations come into play. In fact by adopting the spinning particle Wigner function formalism of [Str57, GV88, GV89] one originally deals with a Wigner function of the form (2.42) and its evolution equation, and one can then in turn try to construct the probability density $w$ and its underlying process, i.e. design a model like Machine I from quantum mechanics.

Further remarks on the periodic boundary conditions are made in sections 2.7.3 and Appendix D. For the effect of boundary conditions on Fokker-Planck equations, see also [Gar85] and for the effect on stochastic differential equations, see [GS71].
2.3.7

With the standard boundary conditions one can immediately write down a differential equation for the covariance matrix of any process running with Machine I. The covariance matrix of a process is defined by

\[
\sigma(s) \equiv \begin{pmatrix}
\sigma_{11}(s) & \sigma_{12}(s) & \sigma_{13}(s) \\
\sigma_{21}(s) & \sigma_{22}(s) & \sigma_{23}(s) \\
\sigma_{31}(s) & \sigma_{32}(s) & \sigma_{33}(s)
\end{pmatrix},
\]  

(2.43)

with

\[
\begin{align*}
\sigma_{11}(s) & \equiv \langle (\sigma(s) - \langle \sigma(s) \rangle)^2 \rangle = \langle \sigma(s)^2 \rangle - \langle \sigma(s) \rangle^2, \\
\sigma_{12}(s) & \equiv \langle (\sigma(s) - \langle \sigma(s) \rangle) \cdot (\eta(s) - \langle \eta(s) \rangle) \rangle \\
& = \langle \sigma(s) \cdot \eta(s) \rangle - \langle \sigma(s) \rangle \cdot \langle \eta(s) \rangle, \\
\sigma_{13}(s) & \equiv \langle \sigma(s) \cdot \psi(s) \rangle = \langle \sigma(s) \rangle \cdot \langle \psi(s) \rangle, \\
\sigma_{21}(s) & \equiv \sigma_{12}(s), \\
\sigma_{22}(s) & \equiv \langle \eta(s)^2 \rangle - \langle \eta(s) \rangle^2, \\
\sigma_{23}(s) & \equiv \langle \eta(s) \cdot \psi(s) \rangle = \langle \eta(s) \rangle \cdot \langle \psi(s) \rangle, \\
\sigma_{31}(s) & \equiv \sigma_{13}(s), \\
\sigma_{32}(s) & \equiv \sigma_{23}(s), \\
\sigma_{33}(s) & \equiv \langle \psi(s)^2 \rangle - \langle \psi(s) \rangle^2.
\end{align*}
\]

Clearly, from their definition the diagonal elements are always nonnegative. Furthermore the \(\sigma\) matrix is nonnegative definite\(^{{17}}\) and symmetric.

Because \(\vec{x}(s)\) is a process of Ornstein-Uhlenbeck type it may be shown by using the standard boundary conditions that the covariance matrix satisfies the following differential equation \cite{Van81}:

\[
\sigma' = A_I \cdot \sigma + \sigma \cdot A_I^T + D .
\]  

(2.44)

In component form \(2.44\) results in:

\[
\begin{align*}
\sigma'_{11} &= 2 \cdot a \cdot \sigma_{12} , \\
\sigma'_{12} &= a \cdot \sigma_{22} + b \cdot \sigma_{11} + c \cdot \sigma_{12} , \\
\sigma'_{22} &= 2 \cdot b \cdot \sigma_{12} + c \cdot \sigma_{22} + \omega , \\
\sigma'_{13} &= a \cdot \sigma_{23} + d \cdot \sigma_{12} , \\
\sigma'_{23} &= b \cdot \sigma_{13} + c \cdot \sigma_{23} + d \cdot \sigma_{22} , \\
\sigma'_{33} &= 2 \cdot d \cdot \sigma_{23} .
\end{align*}
\]

\(^{17}\)This means that for every three-component vector \(\vec{v}\) with real components one has the inequality:

\[
\sum_{j,k=1}^{3} \sigma_{jk} \cdot v_j \cdot v_k \geq 0 .
\]

If \(\sigma\) is nonsingular, then it is positive definite, i.e. the equal sign in the above inequality then only occurs for \(\vec{v} = 0\).
For the first moment vector $\langle \vec{x}(s) \rangle$ one gets the following differential equation:

$$\langle \vec{x}'(s) \rangle = A' \langle \vec{x}(s) \rangle .$$ (2.45)

The differential equations (2.44),(2.45) can be easily derived from (2.7). They are valid for all processes running with Machine I and are particularly useful for processes whose probability densities are determined only by the covariance matrix and the first moment vector such as processes 1 and 2. These equations also show that for every process running with Machine I the covariance matrix and the first moment vector depend smoothly on $s$.

For the orbital part one gets:

$$\sigma'_{\text{orb}} = A_{\text{orb}} \cdot \sigma_{\text{orb}} + A_{\text{orb}}^T \cdot D_{\text{orb}},$$ (2.46)

$$\langle \vec{z}'(s) \rangle = A_{\text{orb}} \cdot \langle \vec{z}(s) \rangle ,$$ (2.47)

where:

$$\sigma_{\text{orb}}(s) \equiv \left( \begin{array}{ccc} \sigma_{11}(s) & \sigma_{12}(s) \\
\sigma_{21}(s) & \sigma_{22}(s) \end{array} \right) , \quad D_{\text{orb}} \equiv B_{\text{orb}} \cdot B_{\text{orb}}^T = \left( \begin{array}{ccc} 0 & 0 \\
0 & \omega \end{array} \right) ,$$

and where $\sigma_{\text{orb}}$ denotes the ‘orbital covariance matrix’. Note that one has by (2.46):

$$\left( \det(\sigma_{\text{orb}}) \right)' = 2 \cdot c \cdot \det(\sigma_{\text{orb}}) + \omega \cdot \sigma_{11} .$$ (2.48)

Because one has for $j,k = 1, 2, 3$:

$$\langle x_j(s) \rangle = \int_{-\infty}^{+\infty} d\sigma \int_{-\infty}^{+\infty} d\eta \int_{-\infty}^{+\infty} d\psi \cdot w(\sigma, \eta, \psi; s) \cdot x_j = -i \cdot \left( \frac{\partial \Phi}{\partial u_j} (\vec{u}; s) \right)_{\vec{u}=0} ,$$

$$\langle x_j(s) \cdot x_k(s) \rangle = \int_{-\infty}^{+\infty} d\sigma \int_{-\infty}^{+\infty} d\eta \int_{-\infty}^{+\infty} d\psi \cdot w(\sigma, \eta, \psi; s) \cdot x_j \cdot x_k = -\left( \frac{\partial^2 \Phi}{\partial u_j \partial u_k} (\vec{u}; s) \right)_{\vec{u}=0} ,$$

the differential equations (2.44),(2.45) can be alternatively derived from (2.22) or from (2.29). Note that the differential equations (2.44),(2.45) in general do not hold for boundary conditions different from the standard boundary conditions. Since Machine II also gives rise to processes of Ornstein-Uhlenbeck type, relations analogous to equations (2.44) and (2.45) will apply.

2.3.8

In Machine I the orbital motion is not influenced by the spin motion (see (2.7)). Thus once the orbital motion of a process has been determined, finding the spin motion reduces to solving the stochastic differential equation (2.14b) for $\psi$. Equation (2.14b) for the process $\psi(s)$ is equivalent to the Thomas-BMT equation (2.4a) for the process $\vec{S}(s)$. The $s$-dependent vector $\vec{W}_I(\eta(s))$ is a stochastic process whose properties are determined by the process $\eta(s)$. With (2.4a) one has moulded the spin motion of a stochastic process into the stochastic motion of the spin vector.

---

18See also [Van81]. From the normalization (2.20) of $w$ it also follows by (2.28):

$$\Phi(\vec{u}=0; s) = 1 .$$

19The same is true for Machine II.
Instead of the spin variable $\psi$ this approach uses the variable $\vec{S}$ which together with the orbit variables constitutes a five-component spin-orbit vector

$$\begin{pmatrix}
\sigma \\
\eta \\
\vec{S}
\end{pmatrix},$$

whose fifth component vanishes in our case since the spin is horizontal. For the models studied in the present article the three-component vector $\vec{x}$ is more convenient.

### 2.4 The probability density of Process 1

#### 2.4.1

In this section I consider the outcome of scenario 1, which I call ‘Process 1’. It is denoted by $\vec{x}^{(1)}(s)$ and I abbreviate:

$$\vec{x}^{(1)}(s) \equiv \begin{pmatrix}
\sigma^{(1)}(s) \\
\eta^{(1)}(s) \\
\psi^{(1)}(s)
\end{pmatrix}.$$  

As explained in the Introduction this process corresponds to deterministic initial values which I abbreviate as

$$\vec{x}^{(1)}(0) = \begin{pmatrix}
\sigma^{(1)}(0) \\
\eta^{(1)}(0) \\
\psi^{(1)}(0)
\end{pmatrix} = \begin{pmatrix}
<\sigma^{(1)}(0)> \\
<\eta^{(1)}(0)> \\
<\psi^{(1)}(0)>
\end{pmatrix} \equiv \begin{pmatrix}
\sigma_0 \\
\eta_0 \\
\psi_0
\end{pmatrix},$$  

where $\sigma_0, \eta_0, \psi_0$ denote arbitrary, but fixed, real numbers. The process $\vec{x}^{(1)}(s)$ and the orbital process

$$\vec{z}^{(1)}(s) \equiv \begin{pmatrix}
\sigma^{(1)}(s) \\
\eta^{(1)}(s)
\end{pmatrix}$$
are Markovian diffusion processes.

My main task in this section is to find the corresponding probability density, $w_1$. It is easily shown that

$$\exp(\mathcal{A}_{orb} \cdot s) = \frac{i}{2 \lambda} \cdot \begin{pmatrix}
g_1(s) & g_2(s) \\
-\lambda g_1(s) & -g_3(s)
\end{pmatrix}.$$  

where

$$g_1(s) \equiv \lambda_2 \cdot \exp(\lambda_1 \cdot s) - c.c. = i \cdot \exp(c \cdot s/2) \cdot [c \cdot \sin(\lambda \cdot s) - 2 \cdot \lambda \cdot \cos(\lambda \cdot s)],$$

$$g_2(s) \equiv \exp(\lambda_1 \cdot s) - c.c. = 2i \cdot \sin(\lambda \cdot s) \cdot \exp(c \cdot s/2),$$

$$g_3(s) \equiv \lambda_1 \cdot \exp(\lambda_1 \cdot s) - c.c. = g'_2(s) = i \cdot \exp(c \cdot s/2) \cdot [c \cdot \sin(\lambda \cdot s) + 2 \cdot \lambda \cdot \cos(\lambda \cdot s)],$$

\(^{(2.51)}\)
and where $\lambda_1, \lambda_2$ are the eigenvalues of the matrix $\mathbf{A}_{\text{orb}}$ and are given by

$$
\lambda_1 \equiv i \cdot \sqrt{-a \cdot b - c^2/4 + \frac{c}{2}} \equiv i \cdot \lambda + \frac{c}{2}, \quad \lambda_2 \equiv \lambda^*.
$$

Note that the ‘orbital tune’ is given by

$$
Q_{\text{orb}} \equiv \frac{\lambda \cdot L}{2 \cdot \pi},
$$
reflecting the well known fact that the damping causes a small shift in the orbital tune away from $Q_s$ via the term $c^2/4$. Note also that $\lambda > 0$, $\lambda_1 \cdot \lambda_2 > 0$, which follows from (2.10), (2.11). If one specifies the constants according to (2.12), one gets

$$
\lambda \approx 6.0 \cdot 10^{-5} \text{ m}^{-1}.
$$

The first moment vector of Process 1 reads by (2.45), (2.49) as:

$$
< \vec{x}^{(1)}(s) > = \begin{pmatrix}
< \sigma^{(1)}(s) > \\
< \eta^{(1)}(s) > \\
< \psi^{(1)}(s) >
\end{pmatrix} = \exp(\mathbf{A}_r \cdot s) \cdot < \vec{x}^{(1)}(0) >
$$

$$
= \begin{pmatrix}
\frac{i}{2 \lambda} \cdot (\sigma_0 \cdot g_1(s) - a \cdot \eta_0 \cdot g_2(s)) \\
\frac{i}{2 \lambda} \cdot (-b \cdot \sigma_0 \cdot g_2(s) - \eta_0 \cdot g_3(s)) \\
\psi_0 - \frac{d}{a} \cdot \sigma_0 + \frac{i d}{2 a \lambda} \cdot (\sigma_0 \cdot g_1(s) - a \cdot \eta_0 \cdot g_2(s))
\end{pmatrix}, \quad (2.52)
$$

One sees by (2.52) that $< \sigma^{(1)}(s) >$ and $< \eta^{(1)}(s) >$ damp away with the orbital damping rate $1/\tau_{\text{damp}}$.

Coming to the covariance matrix $\mathbf{\sigma}_1$ of Process 1, one finds by the deterministic initial values (2.49):

$$
\mathbf{\sigma}_1(0) = 0. \quad (2.53)
$$

Therefore the differential equation (2.44) for $\mathbf{\sigma}_1$ is solved by:

$$
\mathbf{\sigma}_1(s) = \int_0^s ds_1 \cdot \exp(\mathbf{A}_r \cdot s_1) \cdot \mathbf{D} \cdot \exp(\mathbf{A}_r^T \cdot s_1)
$$

$$
= -\frac{\omega}{8 \cdot \lambda^2} \begin{pmatrix}
2 \cdot a^2 \cdot g_4(s) & a \cdot g_2^2(s) & 2 \cdot a \cdot d \cdot g_4(s) \\
2 \cdot a \cdot g_2^2(s) & 2 \cdot g_5(s) & d \cdot g_2^2(s) \\
2 \cdot a \cdot d \cdot g_4(s) & d \cdot g_2^2(s) & 2 \cdot d^2 \cdot g_4(s)
\end{pmatrix}, \quad (2.54)
$$

where

$$
g_4(s) \equiv \int_0^s ds_1 \cdot g_2^2(s_1)
$$

$$
= -\frac{1}{abc} \cdot \exp(c \cdot s) \cdot [c \cdot \lambda \cdot \sin(2\lambda \cdot s) - c^2 \cdot \sin^2(\lambda \cdot s) - 2 \cdot \lambda^2] - \frac{2\lambda^2}{abc}, \quad (2.55)
$$

$$
g_5(s) \equiv \int_0^s ds_1 \cdot g_3^2(s_1)
$$

$$
= \exp(c \cdot s) \cdot [-2 \cdot \lambda^2/c - \lambda \cdot \sin(2\lambda \cdot s) - c \cdot \sin^2(\lambda \cdot s)] + 2 \cdot \lambda^2/c. \quad (2.56)
$$

21The symbol $^*$ denotes complex conjugation.
2.4.2

Because Process 1 is initially deterministic, one finds by (2.19), (2.49):
\[ w_1(\sigma, \eta, \psi; 0) = \delta(\sigma - \sigma_0) \cdot \delta(\eta - \eta_0) \cdot \delta(\psi - \psi_0) . \]  

From this follows by (2.23):
\[ w_{1,orb}(\sigma, \eta; 0) = \delta(\sigma - \sigma_0) \cdot \delta(\eta - \eta_0) , \]  

where \( w_{1,orb} \) denotes the orbital part of \( w_1 \). Denoting the characteristic function of Process 1 by \( \Phi_1 \) I obtain via (2.27), (2.57):
\[ \Phi_1(\vec{u}; 0) = \exp \left( i \cdot \vec{u}^T \cdot <\vec{x}^{(1)}(0)> \right) . \]  

Equations (2.29), (2.59) pose an initial value problem and it is easily checked by substitution and by using (2.44), (2.45) that its solution is given by:
\[ \Phi_1(\vec{u}, s) = \exp \left( -\frac{1}{2} \cdot \sum_{j,k=1}^{3} \sigma_{1,jk}(s) \cdot u_j \cdot u_k + i \cdot \vec{u}^T \cdot <\vec{x}^{(1)}(s)> \right) . \]
By (2.54) one observes:

\[ \sigma_{1,13}(s) = \frac{d}{a} \cdot \sigma_{1,11}(s), \quad \sigma_{1,23}(s) = \frac{d}{a} \cdot \sigma_{1,12}(s), \quad \sigma_{1,33}(s) = \frac{d^2}{a^2} \cdot \sigma_{1,11}(s), \] (2.61)

so that from (2.28), (2.60) it follows that:

\[ w_1(\sigma, \eta, \psi; s) = w_{1,\text{orb}}(\sigma, \eta; s) \cdot \delta \left( \psi - \psi_0 - \frac{d}{a} \cdot (\sigma - \sigma_0) \right). \] (2.62)

By (2.60), (2.61) the characteristic function \( \Phi_{1,\text{orb}} \) corresponding to \( w_{1,\text{orb}} \) (see (2.30)) reads as:

\[ \Phi_{1,\text{orb}}(\vec{t}, s) = \exp \left( -\frac{1}{2} \cdot \sum_{j,k=1}^{2} \sigma_{1,\text{orb},jk}(s) \cdot t_j \cdot t_k + i \cdot \vec{t}^T \cdot <z^{(1)}(s)> \right), \] (2.63)

where \( \sigma_{1,\text{orb}} \) denotes the orbital covariance matrix of Process 1, which by (2.54) reads as:

\[ \sigma_{1,\text{orb}}(s) = \left( \begin{array}{cc} \sigma_{1,11}(s) & \sigma_{1,12}(s) \\ \sigma_{1,21}(s) & \sigma_{1,22}(s) \end{array} \right) = \frac{-\omega}{8 \cdot \lambda^2} \cdot \left( \begin{array}{cc} 2 \cdot a^2 \cdot g_4(s) & a \cdot g_3^2(s) \\ a \cdot g_3^2(s) & 2 \cdot g_5(s) \end{array} \right). \] (2.64)

By inserting the expression for \( \Phi_{1,\text{orb}} \) into (2.31) one sees that \( w_{1,\text{orb}} \) is Gaussian \(^2\) in \( \sigma, \eta \) of the form

\[ w_{1,\text{orb}}(\sigma, \eta; s) = \frac{1}{2\pi} \cdot \det(\sigma_{1,\text{orb}}(s))^{-1/2} \cdot \exp \left( -\frac{1}{2} \cdot \left( \begin{array}{cc} \sigma - <\sigma^{(1)}(s)> \\ \eta - <\eta^{(1)}(s)> \end{array} \right)^T \cdot \sigma_{1,\text{orb}}^{-1}(s) \cdot \left( \begin{array}{c} \sigma - <\sigma^{(1)}(s)> \\ \eta - <\eta^{(1)}(s)> \end{array} \right) \right), \] (2.65)

if \( \sigma_{1,\text{orb}}(s) \) is nonsingular.

I now show that \( \sigma_{1,\text{orb}}(s) \) is nonsingular for \( s > 0 \). By (2.53) \( \det(\sigma_{1,\text{orb}}(0)) \) vanishes, so that one obtains via (2.48):

\[ \det(\sigma_{1,\text{orb}}(s)) = \omega \cdot \int_0^s ds_1 \cdot \exp \left( 2 \cdot c \cdot (s - s_1) \right) \cdot \sigma_{11}(s_1), \] (2.66)

which by (2.64) simplifies to:

\[ \det(\sigma_{1,\text{orb}}(s)) = \frac{-a^2 \cdot \omega^2}{4 \lambda^2} \cdot \int_0^s ds_1 \cdot \exp \left( 2 \cdot c \cdot (s - s_1) \right) \cdot g_4(s_1). \] (2.67)

By (2.54) one sees that \( g_4^2 \) is nonpositive so that by (2.53) \( g_4(s) \) is nonpositive and monotonically decreasing for increasing \( s \). Also one obtains by (2.53):

\[ g_4(0) = g_4'(0) = g_4''(0) = 0, \quad g_4'''(0) = -8 \cdot \lambda^2 < 0. \] (2.68)

By the above mentioned properties of \( g_4 \) it is clear that \( g_4(s) < 0 \) for \( s > 0 \) so that by (2.67) \( \det(\sigma_{1,\text{orb}}(s)) \) is positive for \( s > 0 \). Hence the orbital covariance matrix (2.64) is nonsingular for \( s > 0 \) so that in fact \( w_{1,\text{orb}} \) is Gaussian for \( s > 0 \).

\(^2\)I take the usual definition of ‘Gaussian’, which implies that the covariance matrix is nonsingular.
Finally from (2.64) and for \(s > 0\) it follows that:

\[
\sigma_{1,orb}^{-1}(s) = -\text{det}\left(\sigma_{1,orb}(s)\right)^{-1} \cdot \frac{\omega}{8 \cdot \lambda^2} \cdot \begin{pmatrix}
2 \cdot g_5(s) & -a \cdot g_2^2(s) \\
-a \cdot g_2^2(s) & 2 \cdot a^2 \cdot g_4(s)
\end{pmatrix}^{-1}.
\]

Now I have made the probability density \(w_1\) of Process 1 explicit. It is defined by (2.62), where \(w_{1,orb}\) is given for \(s = 0\) by (2.58) and for \(s > 0\) by (2.65). The probability density \(w_1\) fulfills the Fokker-Planck equation (2.22) and the normalization condition (2.20). One sees that \(w_1\) factors for \(s > 0\) into a Gaussian function and a delta function. However \(w_1\) is not Gaussian because, as follows from (2.61), the covariance matrix of Process 1 is singular. One thus observes the rather unusual feature that not every process running with Machine I has a nonsingular covariance matrix.

As mentioned in section 2.2.2 the three-component differential equation (2.7) has only two nontrivial components. This is reflected by Process 1 because the probability density (2.62) can be written as:

\[
w_1(\sigma, \eta, \psi; s) = w_{1,orb}(\sigma, \eta; s) \cdot \delta\left(\tilde{\psi} - <\tilde{\psi}^{(1)}(0)>\right),
\]

where:

\[
\tilde{\psi} \equiv \psi - \frac{d}{a} \cdot \sigma, \quad \tilde{\psi}^{(1)}(s) \equiv \psi^{(1)}(s) - \frac{d}{a} \cdot \sigma^{(1)}(s).
\]

### 2.5 Further properties of Process 1 and the transition probability density for Machine I

#### 2.5.1

For \(s > 0\) the spin part \(w_{1,spin}\) of \(w_1\) has by (2.26), (2.62), (2.65) the form:

\[
w_{1,spin}(\psi; s) = \int^{+\infty}_{-\infty} d\sigma \int^{+\infty}_{-\infty} d\eta \cdot w_1(\sigma, \eta, \psi; s)
\]

\[
= \left(2 \pi \cdot \sigma_{1,33}(s)\right)^{-1/2} \cdot \exp\left(-\frac{\left(\psi - <\tilde{\psi}^{(1)}(s)>\right)^2}{2 \cdot \sigma_{1,33}(s)}\right),
\]

and for \(s = 0\) one has by (2.26), (2.57):

\[
w_{1,spin}(\psi; 0) = \delta(\psi - \psi_0).
\]

With (2.69), (2.70) one can easily calculate the polarization vector for Process 1 in the \((\vec{m}_{0,1}, \vec{l}_{0,1}, \vec{n}_{0,1})\)-frame:

\[
\vec{P}_{tot}^{w_1}(s) = \frac{2}{\hbar} \cdot <\tilde{S}^{(1)}(s)> = \int^{+\infty}_{-\infty} d\sigma \int^{+\infty}_{-\infty} d\eta \int^{+\infty}_{-\infty} d\psi \cdot w_1(\sigma, \eta, \psi; s) \cdot \begin{pmatrix}
\cos(\psi) \\
\sin(\psi)
\end{pmatrix}
\]

\[
= \int^{+\infty}_{-\infty} d\psi \cdot w_{1,spin}(\sigma, \eta, \psi; s) \cdot \begin{pmatrix}
\cos(\psi) \\
\sin(\psi)
\end{pmatrix}
\]

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\[
\begin{align*}
&= \exp(-\sigma_{1,33}(s)/2) \cdot \begin{pmatrix}
\cos\left(<\psi^{(1)}(s)>\right) \\
\sin\left(<\psi^{(1)}(s)>\right) \\
0
\end{pmatrix},
\end{align*}
\]  

(2.71)

where:

\[
\tilde{S}^{(1)}(s) \equiv \frac{\hbar}{2} \cdot \begin{pmatrix}
\cos(\psi^{(1)}(s)) \\
\sin(\psi^{(1)}(s)) \\
0
\end{pmatrix}.
\]

The polarization is thus

\[
||\tilde{P}_{\text{tot}}^{w_1}(s)|| = \exp\left(-\sigma_{1,33}(s)/2\right)
\]

(2.72)

and is consistent with the requirement that

\[
||\tilde{P}_{\text{tot}}^{w_1}(0)|| = 1.
\]

2.5.2

For the far future, i.e. for \(s = +\infty\), the covariance matrix has by (2.54) the form

\[
\mathcal{A}_1(+\infty) = \begin{pmatrix}
\sigma_{\sigma}^2 & 0 & \frac{d}{a} \cdot \sigma_{\sigma}^2 \\
0 & \sigma_{\eta}^2 & 0 \\
\frac{d}{a} \cdot \sigma_{\sigma}^2 & 0 & \sigma_{\psi}^2
\end{pmatrix},
\]

where

\[
\sigma_{\sigma}^2 \equiv \frac{\omega \cdot a}{2bc} > 0, \quad \sigma_{\eta}^2 \equiv -\frac{\omega}{2c} = -\frac{\omega \cdot L}{4\alpha_s} = -\frac{b}{a} \cdot \sigma_{\sigma}^2 > 0, \quad \sigma_{\psi}^2 \equiv \frac{d^2}{a^2} \cdot \sigma_{\sigma}^2 = \frac{\nu^2 \cdot \sigma_{\eta}^2}{Q_s^2} > 0.
\]

The equilibrium first moment vector reads as:

\[
<\tilde{x}^{(1)}(+\infty)> = (0, 0, \psi_0 - \frac{d}{a} \cdot \sigma_0)^T,
\]

which follows from (2.52). Therefore, using (2.62), (2.65), the probability density \(w_1\) at \(s = +\infty\) reads as

\[
w_1(\sigma, \eta, \psi; +\infty) = w_{1, \text{orb}}(\sigma, \eta; +\infty) \cdot \delta\left(\psi - \psi_0 - \frac{d}{a} \cdot (\sigma - \sigma_0)\right),
\]

(2.73)

\(\text{If one specifies the constants according to (2.12) one gets}
\]

\[
\sigma_{\eta}^2 \approx 1.0 \cdot 10^{-6}, \quad \sigma_{\sigma}^2 \approx 0.00013 \, m^2, \quad \sigma_{\psi}^2 \approx 1.06.
\]

By this one also finds that \(\tilde{\psi}\) is quite different from \(\psi\), because

\[
\tilde{\psi} - \psi = -\frac{d}{a} \cdot \sigma \approx -\frac{d}{a} \cdot \sigma_{\sigma} = \sigma_{\psi} \approx 1.03.
\]

\(\text{20}\)
average orbital action versus N

Figure 2: The stochastic average $\langle J^{(1)}_{\text{orb}}(NL) \rangle$ of the orbital action variable for the first 500 turns of Process 1 assuming the HERA values (2.12) with $\sigma_0 = \eta_0 = 0$

where

$$w_{1,\text{orb}}(\sigma, \eta; +\infty) \equiv \frac{1}{2\pi \cdot \sigma_0 \cdot \sigma_\eta} \cdot \exp\left(-\sigma^2/2\sigma_\sigma^2 - \eta^2/2\sigma_\eta^2\right) \equiv w_{\text{norm}}(\sigma, \eta).$$

One thus sees that Process 1 reaches equilibrium, i.e. for $s \to +\infty$ it approaches a stationary state determined by (2.73). Because $\sigma_1(+\infty)$ is singular this stationary state is not Gaussian. In fact it is factored by (2.73) into a Gaussian function and a delta function just as at finite $s$.

The polarization vector of Process 1 for $s = +\infty$ takes the form

$$\tilde{P}^{w_1}_{\text{tot}}(+\infty) = \exp\left(-\sigma_{1,33}(+\infty)/2\right) \cdot \begin{pmatrix}
\cos(\langle \psi^{(1)}(+\infty) \rangle) \\
\sin(\langle \psi^{(1)}(+\infty) \rangle) \\
0
\end{pmatrix}
= \exp\left(-\frac{d^2 \cdot \sigma_\sigma^2}{2a^2}\right) \cdot \begin{pmatrix}
\cos(\psi_0 - (d \cdot \sigma_0)/a) \\
\sin(\psi_0 - (d \cdot \sigma_0)/a) \\
0
\end{pmatrix}, \quad (2.75)
where I used (2.71). Then the polarization of Process 1 is given at $s = +\infty$ by

$$||\vec{P}_{w1}^{\text{tot}}(+\infty)|| = \exp(-d^2 \cdot \sigma_a^2 / 2a^2).$$

(2.76)

So for Process 1 the polarization does not decay completely, i.e. there is no complete spin decoherence! If one specifies the constants according to (2.12) one gets

$$||\vec{P}_{w1}^{\text{tot}}(+\infty)|| \approx 0.59.$$  

(2.77)

So one gets 59% equilibrium polarization, i.e. only a moderate spin decoherence as already pointed out in [BBHMR94a, BBHMR94b]. The detailed $s$-dependence is shown in figure 1 where one sees that the polarization reaches its asymptotic value after a few $\tau_{\text{damp}}$. Careful inspection of the curve reveals a small ripple at twice the synchrotron frequency. Furthermore one can show that $\sigma_{1,33}$ approaches its equilibrium value on the scale of half the orbital damping time $\tau_{\text{damp}}$.

Conventional wisdom has suggested that $\sigma_\psi$ should increase asymptotically like $\sqrt{s}$ as for any simple diffusion process. This is not the case as one has just seen. However, for the simpler two-dimensional pure diffusion problem for $\eta$ and $\psi$ without synchrotron oscillations the $\sqrt{s}$ growth does emerge and for HERA it would result in a complete decoherence after a few orbital damping times. So synchrotron motion is an essential ingredient.

2.5.3

In the absence of synchrotron radiation ($c = \omega = 0$) the orbital equations of motion (2.17a) reduce to Hamiltonian equations of motion for the Hamiltonian

$$H_{\text{orb}} \equiv -\frac{b}{2} \cdot \sigma^2 + \frac{a}{2} \cdot \eta^2.$$

The Poisson bracket relation for $\sigma$ and $\eta$ is:

$$\{\sigma, \eta\} = 1.$$

Introducing the abbreviations

$$\lambda_0 \equiv \sqrt{-a \cdot b},$$

one gets:

$$Q_s = \frac{\lambda_0 \cdot L}{2 \cdot \pi},$$

and the ‘orbital action’ variable reads as

$$J_{\text{orb}} \equiv -\frac{L}{2 \cdot \pi \cdot Q_s} \cdot H_{\text{orb}} = \sqrt{-\frac{b}{4a} \cdot \sigma^2} + \sqrt{-\frac{a}{4b} \cdot \eta^2}.$$

The corresponding orbital phase variable $\phi$ is defined by:

$$\sigma = \left(-\frac{a}{b}\right)^{1/4} \cdot \sqrt{2J_{\text{orb}}} \cdot \cos(\phi), \quad \eta = \left(-\frac{b}{a}\right)^{1/4} \cdot \sqrt{2J_{\text{orb}}} \cdot \sin(\phi).$$
Then
\[ \{ \phi, J_{\text{orb}} \} = 1, \]
so that \( J_{\text{orb}}, \phi \) are action-angle variables for the Hamiltonian \( H_{\text{orb}} \). In the presence of radiation the average action for Process 1 takes the form
\[
<J_{\text{orb}}^{(1)}(s)> = \sqrt{-\frac{b}{4a}} \cdot <\left(\sigma^{(1)}(s)\right)^2 > + \sqrt{-\frac{a}{4b}} \cdot <\left(\eta^{(1)}(s)\right)^2 >
\]
\[
= \sqrt{-\frac{b}{4a}} \cdot \left[ <\left(\sigma^{(1)}(s)\right)^2 > - \frac{a}{b} \cdot <\left(\eta^{(1)}(s)\right)^2 > \right]
\]
\[
= \sqrt{-\frac{1}{4 \cdot \lambda^2}} \cdot \frac{b}{4a} \cdot \left( a^2 \omega \cdot g_4(s) + [\sigma_0 \cdot g_1(s) - a \cdot \eta_0 \cdot g_2(s)]^2 - \frac{a \omega}{b} \cdot g_5(s) \right.
\]
\[
- \frac{a}{b} \cdot [b \cdot \sigma_0 \cdot g_2(s) + \eta_0 \cdot g_3(s)]^2 \right).
\]
Note that the equilibrium value \(< J_{\text{orb}}^{(1)}(+\infty) >\) is independent of \( \sigma_0, \eta_0 \):  
\[
<J_{\text{orb}}^{(1)}(+\infty) > = \sqrt{-\frac{a}{b}} \cdot \sigma_{\eta}^2.
\]
If one chooses \( \sigma_0 = \eta_0 = 0 \) one gets:
\[
<J_{\text{orb}}^{(1)}(s) > = -\omega \cdot \frac{4}{c \cdot \lambda^2} \cdot \sqrt{-\frac{a}{b}} \cdot \left( -\exp(c \cdot s) \cdot [2 \cdot \lambda^2 + c^2 \cdot \sin^2(\lambda \cdot s)] + 2 \cdot \lambda^2 \right).
\]
To illustrate the influence of the synchrotron radiation on the orbital motion of Process 1, I display this \(< J_{\text{orb}}^{(1)}(s) >\) in figure 2 for the first 500 turns, where I assume the HERA values (2.12) and \( \sigma_0 = \eta_0 = 0 \). The stochastic average \(< J_{\text{orb}}^{(1)}(s) >\) reaches its asymptotic level after a few \( \tau_{\text{damp}} \) and with these parameters the \( \sin^2(\lambda \cdot s) \) term gives a negligible contribution. Note that with large \( \sigma_0 \) and \( \eta_0 \) the curve could approach \(< J_{\text{orb}}^{(1)}(+\infty) >\) from above.

In the radiationless case, i.e. in the limit, where \( c, \omega \to 0 \), the Fokker-Planck equation (2.22) reduces to the Liouville equation:
\[
\frac{\partial w}{\partial s} = \{ H_{\text{orb}}, w \}. \quad (2.78)
\]

2.5.4

Because Process 1 has deterministic initial values, its probability density determines the transition probability density \( w_{I,\text{trans}} \) of all processes with standard boundary conditions, as shown below. In turn for every such process the probability density obeys for \( s_1 \leq s \):
\[
w(\sigma, \eta, \psi; s) = \int_{-\infty}^{+\infty} d\sigma_1 \int_{-\infty}^{+\infty} d\eta_1 \int_{-\infty}^{+\infty} d\psi_1 \cdot w_{I,\text{trans}}(\sigma, \eta, \psi, s|\sigma_1, \eta_1, \psi_1; s_1) \cdot w(\sigma_1, \eta_1, \psi_1; s_1).
\]

\[\text{From section 2.9 it is clear that every process running with Machine I has this equilibrium average value of } J_{\text{orb}}.\]
In particular the transition probability density is nonnegative and normalized by:

\[ 1 = \int_{-\infty}^{+\infty} d\sigma \int_{-\infty}^{+\infty} d\eta \int_{-\infty}^{+\infty} d\psi \cdot w_{I,\text{trans}}(\sigma, \eta, \psi; s|\sigma_1, \eta_1, \psi_1; s_1). \tag{2.80} \]

In this case one has:

\[ w_{I,\text{trans}}(\sigma, \eta, \psi; s|\sigma_0, \eta_0, \psi_0; 0) = w_1(\sigma, \eta, \psi; s). \]

Because the Langevin equation \((2.7)\) is \(s\)-independent, the transition probability density obeys:

\[ w_{I,\text{trans}}(\sigma, \eta, \psi; s|\sigma_0, \eta_0, \psi_0; s_1) = w_{I,\text{trans}}(\sigma, \eta, \psi; s-s_1|\sigma_0, \eta_0, \psi_0; 0), \tag{2.81} \]

i.e.:

\[ w_{I,\text{trans}}(\sigma, \eta, \psi; s|\sigma_0, \eta_0, \psi_0; s_1) = w_1(\sigma, \eta, \psi; s-s_1), \]

where \(s_1 \leq s\). From this it finally follows by \((2.62), (2.83)\) that:

\[ w_{I,\text{trans}}(\sigma, \eta, \psi; s|\sigma_0, \eta_0, \psi_0; s_1) = w_{1,\text{orb}}(\sigma, \eta; s-s_1) \cdot \delta\left(\psi - \psi_0 - \frac{d}{a} (\sigma - \sigma_0)\right). \tag{2.82} \]

Note that the transition probability density is only defined for \(s_1 \leq s\). It also fulfills:

\[
\frac{\partial w_{I,\text{trans}}}{\partial s} = -a \cdot \eta \cdot \frac{\partial w_{I,\text{trans}}}{\partial \sigma} - b \cdot \sigma \cdot \frac{\partial w_{I,\text{trans}}}{\partial \eta} - d \cdot \eta \cdot \frac{\partial w_{I,\text{trans}}}{\partial \psi} - c \cdot w_{I,\text{trans}} - c \cdot \eta \cdot \frac{\partial w_{I,\text{trans}}}{\partial \eta} + \frac{\omega}{2} \cdot \frac{\partial^2 w_{I,\text{trans}}}{\partial \eta^2},
\] \[ \tag{2.83} \]

and the following initial condition:

\[ w_{I,\text{trans}}(\sigma, \eta, \psi; s_1|\sigma_1, \eta_1, \psi_1; s_1) = \delta(\sigma - \sigma_1) \cdot \delta(\eta - \eta_1) \cdot \delta(\psi - \psi_1). \tag{2.84} \]

One sees by \((2.7)\) that the probability density has a causal azimuthal evolution, i.e. \(w(\sigma, \eta, \psi; s_1)\) determines \(w\) at a later azimuth \(s\). The transition probability density \(w_{I,\text{trans}}\) is independent of the process, and is hence a Green function for the Fokker-Planck equation \((2.22)\) corresponding to the standard boundary conditions.

Given the probability density \(w\) and the transition probability density \(w_{I,\text{trans}}\) the 'joint probability density' \(w_{\text{joint}}\) of the process is defined as:

\[ w_{\text{joint}}(\sigma, \eta, \psi; s; \sigma_1, \eta_1, \psi_1; s_1) \equiv w_{I,\text{trans}}(\sigma, \eta, \psi; s|\sigma_1, \eta_1, \psi_1; s_1) \cdot w(\sigma_1, \eta_1, \psi_1; s_1). \tag{2.85} \]

Note that the joint probability density is only defined for \(s_1 \leq s\) and it is used in section 2.9.4.

**2.5.5**

Statements analogous to those in the previous section can be made about the orbital part of Machine I. Thus all of the statements in section 2.5.4 are valid when \(w\) is replaced by \(w_{\text{orb}}\) and the variable \(\psi\) is omitted. In particular the orbital transition probability density \(w_{\text{orb,trans}}\) for all processes with standard boundary conditions reads as:

\[ w_{\text{orb,trans}}(\sigma, \eta; s|\sigma_0, \eta_0; s_1) = w_{1,\text{orb}}(\sigma, \eta; s-s_1), \tag{2.86} \]

where \(s_1 \leq s\).
where \( w_{1,\text{orb}} \) is given by (2.58), (2.63) and where \( s_1 \leq s \). For the orbital part of a probability density one obtains for \( s_1 \leq s \):

\[
w_{\text{orb}}(\sigma, \eta; s) = \int_{-\infty}^{+\infty} d\sigma_1 \int_{-\infty}^{+\infty} d\eta_1 \ w_{\text{orb},\text{trans}}(\sigma, \eta; s|\sigma_1, \eta_1; s_1) \cdot w_{\text{orb}}(\sigma_1, \eta_1; s_1). \tag{2.87}
\]

In particular the orbital transition probability density is nonnegative and normalized by:

\[
1 = \int_{-\infty}^{+\infty} d\sigma \int_{-\infty}^{+\infty} d\eta \cdot w_{\text{orb},\text{trans}}(\sigma, \eta; s|\sigma_1, \eta_1; s_1). \tag{2.88}
\]

Note that the orbital transition probability density is only defined for \( s_1 \leq s \). The orbital transition probability density fulfills:

\[
\frac{\partial w_{\text{orb},\text{trans}}}{\partial s} = L_{FP,\text{orb}} w_{\text{orb},\text{trans}}, \tag{2.89}
\]

and the following initial conditions:

\[
w_{\text{orb},\text{trans}}(\sigma, \eta; s_1|\sigma_1, \eta_1; s_1) = \delta(\sigma - \sigma_1) \cdot \delta(\eta - \eta_1). \tag{2.90}
\]

Note that:

\[
w_{\text{orb},\text{trans}}(\sigma, \eta; s = +\infty|\sigma_0, \eta_0; s_1) = w_{1,\text{orb}}(\sigma, \eta; s = +\infty) = w_{\text{norm}}(\sigma, \eta). \tag{2.91}
\]

One sees by (2.87) that the orbital probability density has a causal azimuthal evolution, i.e. \( w_{\text{orb}}(\sigma, \eta; s_1) \) determines \( w_{\text{orb}} \) at a later azimuth \( s \). The orbital transition probability density \( w_{\text{orb},\text{trans}} \) is independent of the process, and is hence a Green function for the orbital Fokker-Planck equation (2.25) corresponding to the standard boundary conditions.

Given \( w_{\text{orb}} \) and the orbital transition probability density the ‘orbital joint probability density’ \( w_{\text{orb},\text{joint}} \) is defined as:

\[
w_{\text{orb},\text{joint}}(\sigma, \eta; s; \sigma_1, \eta_1; s_1) \equiv w_{\text{orb},\text{trans}}(\sigma, \eta; s|\sigma_1, \eta_1; s_1) \cdot w_{\text{orb}}(\sigma_1, \eta_1; s_1). \tag{2.92}
\]

Note that the orbital joint probability density is only defined for \( s_1 \leq s \) and it will be used in Appendix E.

### 2.6 The probability density of Process 2

#### 2.6.1

Although Process 1 has led to most of the methods needed for problems of this kind it is too idealized; in an electron storage ring it is not possible to have an initial state with deterministic orbital values, i.e. sharp orbital values at the initial azimuth \( s = 0 \), and complete polarization at the same azimuth since an injected beam or a beam at orbital equilibrium always occupies a nonzero phase space volume.

Therefore in this section I consider another process, called ‘Process 2’, running with Machine I. It solves the Langevin equation (2.7) and fulfills the standard boundary conditions. It is denoted by \( \vec{x}^{(2)}(s) \) and I abbreviate:

\[
\vec{x}^{(2)}(s) \equiv \begin{pmatrix} \sigma^{(2)}(s) \\ \eta^{(2)}(s) \\ \psi^{(2)}(s) \end{pmatrix}.
\]
However, in contrast to Process 1 the orbital variables are not deterministic at \( s = 0 \) but have a Gaussian distribution with the ‘equilibrium’ probability density \( w_{\text{norm}}(\sigma, \eta) \), defined in (2.74). It describes an initial situation with complete polarization and orbital equilibrium. Denoting the probability density of Process 2 by \( w_{2} \) one therefore has by (2.19), (2.23):

\[
w_{2}(\sigma, \eta, \psi; 0) = w_{\text{norm}}(\sigma, \eta) \cdot \delta(\psi - \psi_0).
\]

(2.93)

This fulfills (2.20) and by applying the orbital transition probability density one gets via (2.87) the expected result that Process 2 is at ‘orbital equilibrium’, i.e. the orbital part \( w_{2,\text{orb}} \) of \( w_{2} \) has the form

\[
w_{2,\text{orb}} = w_{\text{norm}}.
\]

(2.94)

Thus for Process 2 the orbital variables remain in equilibrium, i.e. the orbital process

\[
\vec{z}^{(2)}(s) \equiv \left( \begin{array}{c} \sigma^{(2)}(s) \\ \eta^{(2)}(s) \end{array} \right)
\]

is stationary. Note also that \( \vec{x}^{(2)}(s) \) and \( \vec{z}^{(2)}(s) \) are Markovian diffusion processes.

### 2.6.2

To obtain the probability density of Process 2 in explicit form I again use the characteristic function. Because of the initial conditions (2.93) one gets:

\[
\Phi_{2}(\vec{u}; 0) = \exp\left(-\frac{1}{2} \cdot \sigma_{\sigma}^{2} \cdot u_{1}^{2} - \frac{1}{2} \cdot \sigma_{\eta}^{2} \cdot u_{2}^{2} + i \cdot u_{3} \cdot \psi_{0}\right).
\]

(2.95)

Equations (2.29), (2.95) pose an initial value problem and it is easily checked by substitution and by (2.44), (2.45) that its solution is given by:

\[
\Phi_{2}(\vec{u}; s) = \exp\left(-\frac{1}{2} \cdot \sum_{j,k=1}^{3} \sigma_{2,jk} \cdot u_{j} \cdot u_{k} + i \cdot \vec{u}^{T} \cdot <\vec{x}^{(2)}(s)>\right),
\]

(2.96)

where \( \sigma_{\sigma} \) denotes the covariance matrix of Process 2. In addition (2.44), (2.45) lead to

\[
<\vec{x}^{(2)}(s)> = (0, 0, \psi_{0})^{T},
\]

(2.97)

and:

\[
\sigma_{2}(s) = \exp(\mathcal{A}_{f} \cdot s) \cdot \sigma_{2}(0) \cdot \exp(\mathcal{A}^{T}_{f} \cdot s) + \int_{0}^{s} ds_{1} \cdot \exp(\mathcal{A}_{f} \cdot s_{1}) \cdot \mathcal{D} \cdot \exp(\mathcal{A}^{T}_{f} \cdot s_{1})
\]

\[
= \begin{pmatrix}
\sigma_{\sigma}^{2} & 0 & \frac{d}{\delta} \cdot \sigma_{\sigma}^{2} \cdot [1 - \frac{i}{\delta} \cdot g_{1}(s)] \\
0 & \sigma_{\eta}^{2} & \frac{i \cdot db}{2 \alpha \lambda} \cdot \sigma_{\sigma}^{2} \cdot g_{2}(s) \\
\frac{d}{\delta} \cdot \sigma_{\sigma}^{2} \cdot [1 - \frac{i}{\delta} \cdot g_{1}(s)] & \frac{i \cdot db}{2 \alpha \lambda} \cdot \sigma_{\sigma}^{2} \cdot g_{2}(s) & \frac{d^{2}}{\delta \alpha \lambda} \cdot \sigma_{\sigma}^{2} \cdot [2 \cdot \lambda - \frac{i}{\delta} \cdot g_{1}(s)]
\end{pmatrix}.
\]

(2.98)
Figure 3: Polarization $||\vec{P}_{\text{tot}}(NL)||$ of Process 2 for the first 1000 turns assuming the HERA values (2.12).

Inserting the explicit form of $\Phi_2$ into (2.28) one finds that $w_2$ is given for $s > 0$ by

$$w_2(\sigma, \eta, \psi; s) = \sqrt{(2\pi)^{-3} \cdot \det(\sigma_2(s))^{-1} \cdot \exp \left[ -\frac{1}{2} \cdot \begin{pmatrix} \sigma & \eta \\ \eta & \psi - \psi_0 \end{pmatrix}^T \cdot \sigma_2^{-1}(s) \cdot \begin{pmatrix} \sigma \\ \eta \\ \psi - \psi_0 \end{pmatrix} \right]},$$

(2.99)

so that $w_2$ is Gaussian for $s > 0$ because the covariance matrix $\sigma_2$ is nonsingular for $s > 0$.\footnote{Its determinant has the form:

$$\det(\sigma_2(s)) = \frac{a \cdot d^2 \cdot \omega^3}{16 \cdot b \cdot c^2 \cdot \lambda^2} \cdot g_4(s),$$

which is positive for $s > 0$ because, as shown in section 2.4.2, $g_4(s)$ is negative for $s > 0.$}

By (2.93), (2.99) $w_2$ fulfills the normalization condition (2.20).
Figure 4: Polarization $||\vec{P}_{tot}^{wz}(NL)||$ of Process 2a for the first 1000 turns assuming the HERA values (2.12), except that $c, \omega \to 0$ with $\omega/c = \text{const} = -2 \cdot \sigma_\eta^2 \approx -2.0 \cdot 10^{-6}$.

2.6.3

For $s = +\infty$ the first moment vector (2.97) reads as:

$$<x(2)(+\infty)> = (0, 0, \psi_0)^T,$$

and the covariance matrix has the form:

$$\Sigma_2(+\infty) = \begin{pmatrix}
\sigma_\sigma^2 & 0 & (d \cdot \sigma_\sigma^2)/a \\
0 & \sigma_\eta^2 & 0 \\
(d \cdot \sigma_\sigma^2)/a & 0 & (2d^2 \cdot \sigma_\sigma^2)/a^2
\end{pmatrix}.$$

So Process 2 also reaches equilibrium. However, one sees that processes 1 and 2 approach different stationary states for $s \to +\infty$, so that Machine I has no unique equilibrium state. In particular the equilibrium state for Process 2 is Gaussian whereas for Process 1 it is not.
2.6.4

With (2.93), (2.99) one can easily calculate the polarization vector for Process 2 in the 
\( (\vec{m}_0, l_0, \vec{n}_0) \)-frame:

\[
\vec{P}_{\text{tot}}^w(s) = \frac{2}{\hbar} < \vec{S}^{(2)}(s) > = \int_{-\infty}^{+\infty} d\sigma \int_{-\infty}^{+\infty} d\eta \int_{-\infty}^{+\infty} d\psi \cdot w_2(\sigma, \eta, \psi; s) \cdot \begin{pmatrix} \cos(\psi) \\ \sin(\psi) \\ 0 \end{pmatrix}
\]

\[
= \exp(-\sigma_{2,33}(s)/2) \cdot \begin{pmatrix} \cos(\psi_0) \\ \sin(\psi_0) \\ 0 \end{pmatrix},
\]

(2.100)

where:

\[
\vec{S}^{(2)}(s) \equiv \frac{\hbar}{2} \cdot \begin{pmatrix} \cos(\psi^{(2)}(s)) \\ \sin(\psi^{(2)}(s)) \\ 0 \end{pmatrix}.
\]

The polarization is then

\[
||\vec{P}_{\text{tot}}^w(s)|| = \exp(-\sigma_{2,33}(s)/2),
\]

(2.101)

and of course

\[
||\vec{P}_{\text{tot}}^w(0)|| = 1.
\]

Comparing (2.71), (2.100) one sees that the polarization vectors of processes 1 and 2 are different.

2.6.5

The polarization vector of Process 2 for \( s = +\infty \) takes the form

\[
\vec{P}_{\text{tot}}^w(+\infty) = \exp(-\sigma_{2,33}(+\infty)/2) \cdot \begin{pmatrix} \cos(\psi_0) \\ \sin(\psi_0) \\ 0 \end{pmatrix} = \exp(-\frac{d^2 \cdot \sigma_\alpha^2}{a^2}) \cdot \begin{pmatrix} \cos(\psi_0) \\ \sin(\psi_0) \\ 0 \end{pmatrix}.
\]

The polarization of Process 2 at \( s = +\infty \), i.e. the equilibrium value of the polarization, is therefore given by

\[
||\vec{P}_{\text{tot}}^w(+\infty)|| = \exp(-\frac{d^2 \cdot \sigma_\alpha^2}{a^2}).
\]

(2.102)

So also for Process 2 the polarization does not decay completely, i.e. there is no complete spin decoherence. If one specifies the constants according to (2.12) one gets

\[
||\vec{P}_{\text{tot}}^w(+\infty)|| \approx 0.35,
\]

i.e. one gets 35% equilibrium polarization, which is almost a factor of two smaller than for Process 1.
One sees by (2.98) that $\sigma_{2,33}(0) = 0$ as required and that $\sigma_{2,33}(s)$ approaches its equilibrium value on the scale of the orbital damping time $\tau_{damp}$, so that the equilibrium polarization is approached more slowly than for Process 1.

The polarization of Process 2 is displayed for the HERA values (2.12) in figure 3 for the first 1000 turns where one sees that in contrast to Process 1 the spin equilibrium is reached only after strong oscillations at the synchrotron frequency. The reason for the difference is clear. In Process 2 the short time behaviour is dominated by synchrotron motion and the beam has a prepared energy spread. The damping and diffusion act on a longer time scale. But in Process 1 there is no initial energy spread.

It is also interesting to study how the polarization would behave when starting with the equilibrium orbital distribution but with no synchrotron radiation. I call this ‘Process 2a’. One could use a solution based on the first three terms on the rhs of (2.22) but it is more convenient to use the result (2.100) in the limit where $c, \omega \to 0$ with $\omega/c = \text{const} \approx -2.0 \cdot 10^{-6}$. In this case the orbital phase space distribution remains unaltered but the damping and diffusion forces have been turned off. The resulting polarization is displayed in figure 4 where one sees that the polarization never reaches equilibrium and continues to oscillate strongly at the synchrotron frequency. So although the orbital distributions for processes 2 and 2a are identical the spins behave very differently owing to the very different ‘hidden’ orbital dynamics. In Process 2 the spin motion is irreversible. In Process 2a the spins tend to ‘remember’ their initial distribution.

This figure gives an impression of what could happen if one were considering protons and is reminiscent of the long term polarization oscillations in figure 9 in [HH96]. In Appendix D I consider the nature of the equilibrium distribution for $\psi$ in the radiationless case in more detail.

This completes the detailed account of the analytical derivation of the results for Machine I discussed in [BBHMR94a, BBHMR94b]. I now continue with further developments of the subject.

### 2.7 The polarization density and its Bloch equation for Machine I

#### 2.7.1

In this section I introduce the concept of ‘polarization density’.

Given a process with probability density $w$, the polarization density $P^w$ in the $(\vec{m}_0, I, \vec{l}_0, I, \vec{n}_0, I)$-frame is defined by

$$
\vec{P}^w(\sigma, \eta, s) = \int_{-\infty}^{+\infty} d\psi \cdot w(\sigma, \eta, \psi; s) \cdot \frac{2}{\hbar} \cdot \vec{S}
$$

$$
= \int_{-\infty}^{+\infty} d\psi \cdot w(\sigma, \eta, \psi; s) \cdot \begin{bmatrix}
\cos(\psi) \\
\sin(\psi) \\
0
\end{bmatrix}.
$$

One easily sees by (2.21) that the polarization vector $\vec{P}_{tot}$ satisfies

$$
\vec{P}_{tot}(s) = \int_{-\infty}^{+\infty} d\sigma \int_{-\infty}^{+\infty} d\eta \cdot \vec{P}^w(\sigma, \eta; s),
$$

Note that in this limit there is a very small shift in $\lambda$. 

30
hence the name ‘polarization density’. $\vec{P}^w(\sigma, \eta; s)$ describes the contribution to the polarization vector from a point in the orbital phase space. The standard boundary conditions of $w$ are taken into account in (2.103) via the integration range of $\psi$. Also one sees by (2.104) and the finiteness of the polarization vector that the polarization density obeys standard boundary conditions in the variables $\sigma, \eta$.

Note that by (2.62), (2.103) the polarization density for Process 1 is given by:

$$\vec{P}^{w1}(\sigma, \eta; s) = w_{1,orb}(\sigma, \eta; s) \cdot \begin{pmatrix} \cos\left(\psi_0 + (d \cdot (\sigma - \sigma_0))/a\right) \\ \sin\left(\psi_0 + (d \cdot (\sigma - \sigma_0))/a\right) \\ 0 \end{pmatrix}.$$  (2.105)

Using the Fokker-Planck equation (2.22) one finds that the polarization density obeys the following equation of the Bloch type:

$$\frac{\partial \vec{P}^w}{\partial s} = L_{FP,orb} \vec{P}^w + \vec{W}_I \wedge \vec{P}^w,$$

which follows from (2.22) by partial integration. Explicitly one has

$$\frac{\partial \vec{P}^w}{\partial s} = -a \cdot \eta \cdot \frac{\partial \vec{P}^w}{\partial \sigma} - b \cdot \sigma \cdot \frac{\partial \vec{P}^w}{\partial \eta} + \vec{W}_I \wedge \vec{P}^w - c \cdot \vec{P}^w - c \cdot \eta \cdot \frac{\partial \vec{P}^w}{\partial \eta} + \frac{\omega}{2} \cdot \frac{\partial^2 \vec{P}^w}{\partial \eta^2}.$$  (2.106)

The radiationless Bloch equation reads as:

$$\frac{\partial \vec{P}^w}{\partial s} = -a \cdot \eta \cdot \frac{\partial \vec{P}^w}{\partial \sigma} - b \cdot \sigma \cdot \frac{\partial \vec{P}^w}{\partial \eta} + \vec{W}_I \wedge \vec{P}^w.$$  (2.107)

There is an obvious connection between the radiationless Bloch equation (2.107) and the Thomas-BMT equation (2.4a). In fact the $s$-dependent vector:

$$\vec{P}^w(\sigma(s), \eta(s); s)$$  (2.108)

fulfills (2.4a) because in the absence of radiation $\sigma(s), \eta(s)$ fulfill the following equations of motion:

$$\sigma'(s) = a \cdot \eta(s), \quad \eta'(s) = b \cdot \sigma(s).$$  (2.109)

One easily sees that this connection between the radiationless Bloch equation and the Thomas-BMT equation also holds if $\vec{P}^w$ in (2.108) is replaced by any other quantity obeying (2.107). An analogous connection holds for Machine II.

2.7.2

Just as for the Fokker-Planck equation, the Bloch equation (2.106) for the polarization density also has a causal azimuthal evolution, i.e. an initial polarization density $\vec{P}^w(\sigma, \eta; s_0)$ determines...
\( \vec{P}^w \) at a later azimuth \( s \). In particular there exists a function \( P_I(\sigma, \eta; s|\sigma_1, \eta_1; s_1) \), which is a 3 × 3-matrix fulfilling for \( s_1 \leq s \):

\[
\vec{P}^w(\sigma, \eta; s) = \int_{-\infty}^{+\infty} d\sigma_1 \int_{-\infty}^{+\infty} d\eta_1 \cdot P_I(\sigma, \eta; s|\sigma_1, \eta_1; s_1) \cdot \vec{P}^w(\sigma_1, \eta_1; s_1),
\]

so that \( P_I \) ‘transports’ a polarization density from one azimuth to another. Note that \( P_I \) is only defined for \( s_1 \leq s \). The function \( P_I \) is derived from the transition probability density \( w_{I,trans} \) and it may be shown that it can be written as:

\[
P_I(\sigma, \eta; s|\sigma_1, \eta_1; s_1) \equiv w_{orb,trans}(\sigma, \eta; s|\sigma_1, \eta_1; s_1) \cdot R_I(\sigma, \sigma_1),
\]

where the 3 × 3-matrix \( R_I \) has the form

\[
R_I(\sigma, \sigma_1) \equiv \begin{pmatrix}
\cos\left(\frac{d \cdot (\sigma - \sigma_1)}{a}\right) & -\sin\left(\frac{d \cdot (\sigma - \sigma_1)}{a}\right) & 0 \\
\sin\left(\frac{d \cdot (\sigma - \sigma_1)}{a}\right) & \cos\left(\frac{d \cdot (\sigma - \sigma_1)}{a}\right) & 0 \\
0 & 0 & 1
\end{pmatrix}.
\]

Using (2.106), (2.110) one finds that \( P_I \) fulfills the following equation:

\[
\frac{\partial P_I}{\partial s} = L_{FP,orb} P_I + W_I \cdot P_I,
\]

where

\[
W_I \equiv \begin{pmatrix}
0 & -d \cdot \eta & 0 \\
d \cdot \eta & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}.
\]

Also one finds that \( P_I \) fulfills the following initial conditions:

\[
P_I(\sigma, \eta; s|\sigma_1, \eta_1; s_1) = \delta(\sigma - \sigma_1) \cdot \delta(\eta - \eta_1) \cdot \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}.
\]

One sees that \( P_I \) is a Green function for the Bloch equation (2.106) corresponding to the standard boundary conditions.

In the radiationless case the orbital transition probability density \( w_{orb,trans} \) modifies to \( w_{orb,trans,rad} \), where:

\[
w_{orb,trans,rad}(\sigma, \eta; s|\sigma_1, \eta_1; s_1) = w_{orb,trans,rad}(\vec{z}; s|\vec{z}_1; s_1) = \delta(\vec{z} - \exp((s - s_1) \cdot A_{orb,rad}) \cdot \vec{z}_1),
\]

with

\[
A_{orb,rad} \equiv \begin{pmatrix}
0 & a & 0 \\
b & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}, \quad \vec{z}_1 \equiv \begin{pmatrix}
\sigma_1 \\
\eta_1
\end{pmatrix}.
\]

This follows from (2.58), (2.63), (2.86). Thus \( P_I \) modifies in this limit to \( P_{I,rad} \) with:

\[
P_{I,rad}(\sigma, \eta; s|\sigma_1, \eta_1; s_1) \equiv w_{orb,trans,rad}(\sigma, \eta; s|\sigma_1, \eta_1; s_1) \cdot R_I(\sigma, \sigma_1).
\]

\[\text{29Note that Process 1 is deterministic in this limit.}\]
2.7.3

With the periodic boundary conditions discussed in section 2.3.6 one can express the polarization density in terms of $w_{\text{per}}$. By (2.103) the polarization density reads for the two forms (2.41), (2.42), of $w_{\text{per}}$ as:

\[
\vec{P}_w(\sigma, \eta; s) = \int_0^{2\pi} d\psi \cdot w_{\text{per}}(\sigma, \eta, \psi; s) \cdot \begin{pmatrix} \cos(\psi) \\ \sin(\psi) \\ 0 \end{pmatrix}, \tag{2.113}
\]

\[
\vec{P}_w(\sigma, \eta; s) = \frac{2}{\sqrt{3}} \cdot \int_0^{2\pi} d\psi \cdot w_{\text{per}}(\sigma, \eta, \psi; s) \cdot \begin{pmatrix} \cos(\psi) \\ \sin(\psi) \\ 0 \end{pmatrix}. \tag{2.114}
\]

2.7.4

Having defined the polarization density I now introduce the ‘local polarization vector’ $\vec{P}_w$ defined by

\[
\vec{P}_w(\sigma, \eta; s) \equiv w_{\text{orb}}(\sigma, \eta, \psi; s) \cdot \vec{P}_{\text{loc}}^w(\sigma, \eta; s), \tag{2.115}
\]

and the ‘local polarization’ defined by its norm $||\vec{P}_w||$. Obviously

\[
\vec{P}_w(\sigma, \eta; s) = \int_{-\infty}^{+\infty} d\sigma \int_{-\infty}^{+\infty} d\eta \cdot w_{\text{orb}}(\sigma, \eta; s) \cdot \vec{P}_{\text{loc}}^w(\sigma, \eta; s). \tag{2.116}
\]

$\vec{P}_w$ is simply the spin polarization for an infinitesimal packet of orbital phase space.

Clearly, I restrict myself to situations where the polarization density vanishes if $w_{\text{orb}}$ vanishes and where $0 \leq ||\vec{P}_w|| \leq 1$. The direction $\vec{P}_w$ of the local polarization is defined by

\[
\vec{P}_w(\sigma, \eta; s) \equiv ||\vec{P}_w|| \cdot \vec{P}_{\text{loc}}^w. \tag{2.116}
\]

Hence by (2.103) the local polarization vector and the local polarization direction for Process 1 read as:

\[
\vec{P}_{\text{loc}}^w(\sigma, \eta; s) = \vec{P}_{\text{dir}}^w(\sigma, \eta; s) = \begin{pmatrix} \cos\left(\psi_0 + (d \cdot (\sigma - \sigma_0))/a\right) \\ \sin\left(\psi_0 + (d \cdot (\sigma - \sigma_0))/a\right) \\ 0 \end{pmatrix},
\]

and the local polarization is:

\[
||\vec{P}_{\text{loc}}^w(\sigma, \eta; s)|| = 1. \tag{2.117}
\]

So for Process 1 each point in phase space is fully polarized and the 59% is simply due to the spread in $\vec{P}_{\text{dir}}^w$, not due to the value of $||\vec{P}_{\text{loc}}^w||$.

2.7.5

By (2.23), (2.100), (2.113) the local polarization vector obeys the following evolution equation of Bloch type:

\[
\frac{\partial \vec{P}_w}{\partial s} = \underbrace{-a \cdot \eta \cdot \frac{\partial \vec{P}_w}{\partial \sigma} - b \cdot \sigma \cdot \frac{\partial \vec{P}_w}{\partial \eta}}_{\text{radiationless part}} + \vec{W}_I \wedge \vec{P}_w - c \cdot \eta \cdot \frac{\partial \vec{P}_w}{\partial \eta} \underbrace{.}_{\text{damping term}}
\]
which depends on \( w_{orb} \) and is therefore not universal. But for processes at orbital equilibrium, i.e. if \( w_{orb} = w_{norm} \), this simplifies by \( (2.74) \), to:

\[
\begin{align*}
\frac{\partial \vec{P}_{\text{loc}}^w}{\partial s} &= -a \cdot \eta \cdot \frac{\partial \vec{P}_{\text{loc}}^w}{\partial \sigma} - b \cdot \sigma \cdot \frac{\partial \vec{P}_{\text{loc}}^w}{\partial \eta} + \vec{W}_I \wedge \vec{P}_{\text{loc}}^w + c \cdot \eta \cdot \frac{\partial \vec{P}_{\text{loc}}^w}{\partial \eta} + \frac{\omega}{2} \cdot \frac{\partial^2 \vec{P}_{\text{loc}}^w}{\partial \eta^2},
\end{align*}
\]

(2.119)

which provides a causal azimuthal evolution, because \( (2.106) \) for the polarization density does. Note that the damping terms in equations \( (2.106), (2.119) \) have different structures. However, as seen by \( (2.107), (2.118) \), in the absence of radiation the local polarization vector obeys the same radiationless Bloch equation as the polarization density. Furthermore \( ||\vec{P}_{\text{loc}}^w|| \) fulfills the Liouville equation \( (2.78) \) in that case.

2.7.6

By \( (2.110), (2.118) \) the local polarization direction obeys the following evolution equation of Bloch type:

\[
\frac{\partial \vec{P}_{\text{dir}}^w}{\partial s} = -a \cdot \eta \cdot \frac{\partial \vec{P}_{\text{dir}}^w}{\partial \sigma} - b \cdot \sigma \cdot \frac{\partial \vec{P}_{\text{dir}}^w}{\partial \eta} + \vec{W}_I \wedge \vec{P}_{\text{dir}}^w - c \cdot \eta \cdot \frac{\partial \vec{P}_{\text{dir}}^w}{\partial \eta} + \omega \cdot \left( \frac{1}{w_{orb}} \cdot \frac{\partial w_{orb}}{\partial \eta} - \frac{\partial \vec{P}_{\text{dir}}^w}{\partial \eta} + \frac{1}{||\vec{P}_{\text{loc}}^w||} \cdot \frac{\partial ||\vec{P}_{\text{loc}}^w||}{\partial \eta} \right) 
\]

(2.120)

which depends on \( w_{orb} \) and \( ||\vec{P}_{\text{loc}}^w|| \) and is therefore not universal. It is also nonlinear in \( \vec{P}_{\text{dir}}^w \).

As seen by \( (2.107), (2.120) \), in the absence of radiation the local polarization direction obeys the same radiationless Bloch equation as the polarization density.

2.7.7

The chief virtue of the polarization density stems from the fact that it satisfies a universal and linear differential equation (of the Bloch type). In the case of Machine I this equation is given by \( (2.106) \). Furthermore this equation provides a causal azimuthal evolution but this feature is not as important as universality and linearity.

One has seen that the local polarization vector and its direction also obey Bloch equations but that these equations are not universal. Furthermore the equation for the local polarization direction is in general nonlinear (see \( (2.120) \)). Clearly, in contrast to the full Fokker-Planck equation \( (2.22) \), all these Bloch equations enable one to study average spin behaviour without having to look closely at the \( \psi \) distribution \( w_{\text{spin}} \). The polarization density, the local polarization vector and its direction only depend on orbital variables and the effects of radiation are contained in damping and diffusion terms of \( (2.106), (2.118), (2.120) \) which are associated with the orbital Fokker-Planck operator \( L_{FP,_{\text{orb}}} \). Indeed, it is no accident that Bloch equations emerge for Machine I. See section 5.

I make further comments about Bloch equations in section 2.8.4.
2.8 The polarization properties of Machine I for G-processes

2.8.1

In this section I consider a special class of processes running with Machine I and one aim is to consider the azimuthal evolution of the polarization vector.

I consider processes running with Machine I which have a general Gaussian probability density in $\sigma, \eta, \psi$ for $s > s_0$. Thus for $s > s_0$:

$$w(\sigma, \eta, \psi; s) = w(\vec{x}; s) = (2\pi)^{-3/2} \cdot \det(\sigma(s))^{-1/2} \cdot \exp\left[-\frac{1}{2} \cdot (\vec{x} - <\vec{x}(s)>)^T \cdot \sigma^{-1}(s) \cdot (\vec{x} - <\vec{x}(s)>)\right],$$

(2.121)

where $<\vec{x}(s)>)$ and $\sigma$ denote the first moment vector and covariance matrix of the process. I call these 'G-processes'. Hence Process 2 is a G-process. By (2.27), (2.121) the characteristic function $\Phi$ of a G-process reads as:

$$\Phi(\vec{u}; s) = \exp\left(-\frac{1}{2} \cdot \sum_{j,k=1}^{3} \sigma_{jk}(s) \cdot u_j \cdot u_k + i \cdot \vec{u}^T \cdot <\vec{x}(s)>)\right).$$

(2.122)

Because the first moment vector and the covariance matrix depend continuously on $s$, (2.122) holds even at $s = s_0$, so that the characteristic function is especially convenient for those G-processes whose covariance matrix is singular at $s = s_0$.

Due to (2.26) the spin part $w_{spin}$ of $w$ is also Gaussian for $s > s_0$, i.e.:

$$w_{spin}(\psi; s) \equiv \left(2\pi \cdot \sigma_{33}(s)\right)^{-1/2} \cdot \exp\left(-\frac{(\psi - <\psi(s)>)^2}{2 \cdot \sigma_{33}(s)}\right).$$

(2.123)

The corresponding polarization vector has the simple form:

$$\vec{P}_{w_{tot}}(s) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} d\eta \int_{-\infty}^{+\infty} d\psi \cdot w(\sigma, \eta, \psi; s) \cdot \begin{pmatrix} \cos(\psi) \\ \sin(\psi) \\ 0 \end{pmatrix} = \exp(-\sigma_{33}(s)/2) \cdot \begin{pmatrix} \cos(<\psi(s)> ) \\ \sin(<\psi(s)>) \\ 0 \end{pmatrix},$$

(2.124)

and the polarization:

$$||\vec{P}_{w_{tot}}(s)|| = \exp(-\sigma_{33}(s)/2).$$

(2.125)

Note that (2.124), (2.123) hold even at $s = s_0$ because $\sigma_{33}(s)$ and $<\psi(s)>)$ depend continuously on $s$.

---

30I allow the covariance matrix of a G-process to be possibly singular at the starting azimuth $s = s_0$. 
2.8.2

By (2.47) the stochastic averages $<\sigma(s)>, <\eta(s)>$ of the orbital variables have a causal azimuthal evolution, i.e. they are determined by the initial values $<\sigma(s_0)>, <\eta(s_0)>$:

$$
\left(\begin{array}{c}
<\sigma(s)>
\\
<\eta(s)>
\end{array}\right) = \exp\left(\mathbf{A}_{orb} \cdot (s-s_0)\right) \cdot \left(\begin{array}{c}
<\sigma(s_0)>
\\
<\eta(s_0)>
\end{array}\right).
$$

(2.126)

However, for the spin vector such a causal behaviour does not prevail and this already shows up for G-processes.

By (2.124) one sees that two G-processes which have the same values for $<\psi(s_0)>$ and $\sigma_{33}(s_0)$ have the same initial polarization vector. However it does not follow from this that both processes have the same polarization vector for $s > s_0$, because by using the differential equations (2.44), (2.45) and by using the freedom of choice of $<\sigma(s_0)>, <\eta(s_0)>, \sigma_{11}(s_0), \sigma_{12}(s_0), \sigma_{13}(s_0), \sigma_{22}(s_0), \sigma_{23}(s_0)$ one easily finds that the two processes in general have different polarization vectors for $s > s_0$. This holds even if both processes are in the same orbital state, i.e. have the same $w_{orb}$. As an example I compare the initial conditions:

$$
<\sigma(s_0)> = <\eta(s_0)> = <\psi(s_0)> = 0, \quad \sigma_{11}(s_0) = \sigma_{\sigma}^2, \quad \sigma_{22}(s_0) = \sigma_{\eta}^2, \quad \sigma_{33}(s_0) = 2 \cdot (d^2/a^2) \cdot \sigma_{\sigma}^2, \quad \sigma_{12}(s_0) = \sigma_{13}(s_0) = \sigma_{23}(s_0) = 0,
$$

(2.127)

with the initial conditions:

$$
<\sigma(s_0)> = <\eta(s_0)> = <\psi(s_0)> = 0, \quad \sigma_{11}(s_0) = \sigma_{\sigma}^2, \quad \sigma_{22}(s_0) = \sigma_{\eta}^2, \quad \sigma_{33}(s_0) = 2 \cdot (d^2/a^2) \cdot \sigma_{\sigma}^2, \quad \sigma_{13}(s_0) = (d/a) \cdot \sigma_{\sigma}^2, \quad \sigma_{12}(s_0) = \sigma_{23}(s_0) = 0,
$$

(2.128)

where each set of initial conditions defines a specific G-process. Both processes are in orbital equilibrium, so they are in the same orbital state with $w_{orb} = w_{norm}$. In particular they have the same orbital stochastic averages $<\sigma(s)> = <\eta(s)> = 0$. Also both processes have same initial polarization vector:

$$
\vec{P}_{tot}^w(s_0) = \exp\left(-\frac{d^2 \cdot \sigma_{\sigma}^2}{a^2}\right) \cdot \mathbf{1} = \left(\begin{array}{c}1 \\
0 \\
0\end{array}\right).
$$

(2.129)

However using (2.44) one quickly finds that the polarization vectors for the two processes evolve in different ways. In particular the equilibrium polarization $||\vec{P}_{tot}^w(+\infty)||$ is $\exp(-(2 \cdot d^2 \cdot \sigma_{\sigma}^2)/a^2)$ for the process (2.127) and it is $\exp(-(d^2 \cdot \sigma_{\sigma}^2)/(a^2))$ for the process (2.128).

Hence the initial values $<\sigma(s_0)> = <\eta(s_0)> = \vec{P}_{tot}^w(s_0)$ do not determine the future behaviour of the polarization vector. In particular there exists no differential equation for the azimuthal evolution of the polarization vector, which could provide such a causal azimuthal evolution. Indeed by differentiating (2.124) and by using (2.44), (2.45) one obtains for a G-process the differential equation:

$$
\left(\vec{P}_{tot}^w(s)\right)' = d \cdot <\eta(s)> \cdot \left(\begin{array}{c}0 \\
0 \\
1\end{array}\right) \wedge \vec{P}_{tot}^w(s) - d \cdot \sigma_{23}(s) \cdot \vec{P}_{tot}^w(s),
$$

(2.130)

Note that (2.124) not only holds for G-processes.

Note also that process (2.128), unlike process (2.127), is stationary.
which at first sight appears to be an appropriate evolution equation for the polarization vector. However it is not a universal equation because $\sigma_{23}(s)$ depends on the process, confirming the above conclusions. Note also that by (2.44), (2.124), one can write (2.130) as:

$$
\left( \vec{P}_w^{\text{tot}}(s) \right)' = \mathbf{d} \cdot <\eta(s)> \cdot \left( \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \wedge \vec{P}_w^{\text{tot}}(s) \right) + \frac{\vec{P}_w^{\text{tot}}(s)}{2 \cdot ||\vec{P}_w^{\text{tot}}(s)||^2} \cdot \left( ||\vec{P}_w^{\text{tot}}(s)||^2 \right)' \tag{2.131}
$$

Concluding this section I have seen that, at least without further approximation, no (Bloch) equation for the polarization vector exists in Machine I which would provide a causal evolution for the stochastic average:

$$
\left( \begin{pmatrix} <\sigma(s)> \\ <\eta(s)> \\ (\hbar/2) \cdot \vec{P}_w^{\text{tot}}(s) \end{pmatrix} \right)
$$

of the five-component spin-orbit vector. However there is a universal Bloch equation (2.100) giving a causal azimuthal evolution of the polarization density.

2.8.3

To calculate the local polarization quantities of a G-process one first observes for $s > s_0$ that:

$$
\int_{-\infty}^{+\infty} d\psi \cdot w(\sigma, \eta, \psi; s) \cdot \exp(i \cdot \psi) = w_{\text{orb}}(\sigma, \eta; s)
$$

$$
\cdot \exp\left( -\frac{2 \cdot i \cdot \sigma_{\text{inv,13}}(s) \cdot (\sigma - <\sigma(s)>)}{2 \cdot \sigma_{\text{inv,33}}(s)} + \frac{2 \cdot i \cdot \sigma_{\text{inv,23}}(s) \cdot (\eta - <\eta(s)>)}{2 \cdot \sigma_{\text{inv,33}}(s)} + 1 \right)
$$

$$
\cdot \exp\left( i \cdot <\psi(s)> \right),
$$

where $\sigma_{\text{inv}}$ denotes the inverse of the covariance matrix. From this follows for $s > s_0$:

$$
\vec{P}_w^{\text{tot}}(\sigma, \eta; s) = w_{\text{orb}}(\sigma, \eta; s) \cdot \exp\left( -\frac{1}{2 \cdot \sigma_{\text{inv,33}}(s)} \right)
$$

$$
\left( \begin{pmatrix} \cos\left( -\frac{\sigma_{\text{inv,13}}(s)}{\sigma_{\text{inv,33}}(s)} \cdot (\sigma - <\sigma(s)> - \frac{\sigma_{\text{inv,23}}(s)}{\sigma_{\text{inv,33}}(s)} \cdot (\eta - <\eta(s)> - <\psi(s)> \right) \\ \sin\left( -\frac{\sigma_{\text{inv,13}}(s)}{\sigma_{\text{inv,33}}(s)} \cdot (\sigma - <\sigma(s)> - \frac{\sigma_{\text{inv,23}}(s)}{\sigma_{\text{inv,33}}(s)} \cdot (\eta - <\eta(s)> - <\psi(s)> \right) \\ 0 \end{pmatrix} \right).
$$

Therefore the local polarization vector reads for $s > s_0$ as:

$$
\vec{P}_w^{\text{tot}}(\sigma, \eta; s) = \exp\left( -\frac{1}{2 \cdot \sigma_{\text{inv,33}}(s)} \right)
$$

$$
\left( \begin{pmatrix} \cos\left( -\frac{\sigma_{\text{inv,13}}(s)}{\sigma_{\text{inv,33}}(s)} \cdot (\sigma - <\sigma(s)> - \frac{\sigma_{\text{inv,23}}(s)}{\sigma_{\text{inv,33}}(s)} \cdot (\eta - <\eta(s)> - <\psi(s)> \right) \\ \sin\left( -\frac{\sigma_{\text{inv,13}}(s)}{\sigma_{\text{inv,33}}(s)} \cdot (\sigma - <\sigma(s)> - \frac{\sigma_{\text{inv,23}}(s)}{\sigma_{\text{inv,33}}(s)} \cdot (\eta - <\eta(s)> - <\psi(s)> \right) \\ 0 \end{pmatrix} \right).
$$

(2.133)
and the local polarization for \( s > s_0 \) is:

\[
||\vec{P}_{\text{loc}}^w(\sigma, \eta; s)|| = \exp\left( -\frac{1}{2 \cdot \sigma_{\text{inv},33}(s)} \right).
\] (2.134)

One sees that for \( s > s_0 \) the local polarization of every G-process is uniform across phase space and that \( 0 \leq ||\vec{P}_{\text{loc}}^w|| \leq 1 \). Of course, the polarization density (2.132) obeys the Bloch equation (2.106).

As an example one gets for Process 2:

\[
\vec{P}_{w2}(\sigma, \eta; s) = w_{\text{norm}}(\sigma, \eta) \cdot \exp\left( \frac{d^2 \cdot \omega}{8 \lambda^2} \cdot g_4(s) \right)
\]

\[
\begin{pmatrix}
\cos\left( \frac{d}{2 \cdot \lambda} \cdot \left[ 2 \cdot \lambda - i \cdot g_1(s) \right] \cdot \sigma - \frac{id}{2 \lambda} \cdot g_2(s) \cdot \eta + \psi_0 \right)

\sin\left( \frac{d}{2 \cdot \lambda} \cdot \left[ 2 \cdot \lambda - i \cdot g_1(s) \right] \cdot \sigma - \frac{id}{2 \lambda} \cdot g_2(s) \cdot \eta + \psi_0 \right)

0
\end{pmatrix},
\] (2.135)

from which follows:

\[
||\vec{P}_{\text{loc}}^{w2}(\sigma, \eta; s)|| = \exp\left( \frac{d^2 \cdot \omega}{8 \lambda^2} \cdot g_4(s) \right).
\] (2.136)

Thus the local polarization of Process 2 starts from the value 1 at \( s = 0 \) and decreases monotonically with increasing azimuth. It approaches the following equilibrium value:

\[
||\vec{P}_{\text{loc}}^{w2}(\sigma, \eta; +\infty)|| = \exp\left( -\frac{d^2 \cdot \sigma^2}{2 \alpha^2} \right).
\]

With the HERA values (2.12) the local polarization value for the equilibrium of Process 2 is 0.59. Contrast this with (2.117).

2.8.4

For G-processes at orbital equilibrium, i.e. for \( w_{\text{orb}} = w_{\text{norm}} \), the Bloch equation (2.120) simplifies by (2.74), (2.134) to:

\[
\frac{\partial \vec{P}_{\text{dir}}^w}{\partial s} = -a \cdot \eta \cdot \frac{\partial \vec{P}_{\text{dir}}^w}{\partial \sigma} - b \cdot \sigma \cdot \frac{\partial \vec{P}_{\text{dir}}^w}{\partial \eta} + \vec{W}_I \wedge \vec{P}_{\text{dir}}^w + c \cdot \eta \cdot \frac{\partial \vec{P}_{\text{dir}}^w}{\partial \eta},
\] (2.137)

which provides a causal azimuthal evolution, because (2.119) ensures this for the local polarization vector and because the local polarization is uniform. Note that the damping terms in equations (2.106), (2.137) have different structures and that there is no diffusion term in (2.137).

\footnote{In fact \( \sigma_{\text{inv},33}(s) \) is positive definite for \( s > s_0 \) because \( \sigma(s) \) is. From this it follows that: \( \sigma_{\text{inv},33}(s) > 0 \) for \( s > s_0 \), which proves the assertion.}
2.9 Miscellaneous equilibrium properties of Machine I

2.9.1

With the examples of processes 1, 2 one has already seen that Machine I has no unique equilibrium state. Therefore in this section I study the asymptotic behaviour of arbitrary processes running with Machine I.

To come to that I conclude first of all from (2.79), (2.82), (2.86), (2.91):

\[
\begin{align*}
\int_{-\infty}^{+\infty} d\sigma_0 \int_{-\infty}^{+\infty} d\eta_0 \int_{-\infty}^{+\infty} d\psi_0 \cdot w(\sigma_0, \eta_0, \psi_0; s_0) \cdot w(\sigma_0, \eta_0, \psi_0; s_0) \\
\int_{-\infty}^{+\infty} d\sigma_0 \int_{-\infty}^{+\infty} d\eta_0 \int_{-\infty}^{+\infty} d\psi_0 \cdot w_{\text{orb}, \text{trans}}(\sigma, \eta; +\infty | \sigma_0, \eta_0; s_0) \\
\int_{-\infty}^{+\infty} d\sigma_0 \int_{-\infty}^{+\infty} d\eta_0 \int_{-\infty}^{+\infty} d\psi_0 \cdot w(\sigma_0, \eta_0, \psi_0; s_0) \\
\int_{-\infty}^{+\infty} d\sigma_0 \int_{-\infty}^{+\infty} d\eta_0 \int_{-\infty}^{+\infty} d\psi_0 \cdot w(\sigma_0, \eta_0, \psi_0; s_0) \\
\end{align*}
\]

The equilibrium probability density (2.138) is not the same for every process, reflecting the fact that Machine I has no unique equilibrium state. From (2.138), (2.139) it is also clear that only average information about the initial state is needed in order to determine the corresponding equilibrium state, i.e. for a given equilibrium state there are many different initial states which all approach the same equilibrium.

From (2.139) it follows that \( w_{\text{aver}} \) is normalized:

\[
\int_{-\infty}^{+\infty} d\psi \cdot w_{\text{aver}}(\psi; s) = \int_{-\infty}^{+\infty} d\sigma \int_{-\infty}^{+\infty} d\eta \cdot w(\sigma, \eta, \psi + \frac{d}{a} \cdot \sigma; s) .
\] (2.139)

\[ w_{\text{orb}}(\sigma, \eta; +\infty) = w_{\text{norm}}(\sigma, \eta) \cdot \int_{-\infty}^{+\infty} d\psi \cdot w_{\text{aver}}(\psi - \frac{d}{a} \cdot \sigma; s_0) = w_{\text{norm}}(\sigma, \eta) .
\] (2.141)

This confirms again that every process running with Machine I leads to the same orbital equilibrium characterized by \( w_{\text{norm}} \).

\[34\] For example the \( \eta \)-dependence of the initial state is completely integrated out.
The equilibrium probability density (2.138) not only fulfills the Fokker-Planck equation (2.22) but also the radiationless Fokker-Planck equation

\[
\frac{\partial w}{\partial s} = -a \cdot \eta \cdot \frac{\partial w}{\partial \sigma} - b \cdot \sigma \cdot \frac{\partial w}{\partial \eta} - d \cdot \eta \cdot \frac{\partial w}{\partial \psi}.
\]

So at equilibrium the damping and diffusion balance each other and the Fokker-Planck equation effectively reduces to a Liouville equation, i.e. at equilibrium one effectively has a Hamiltonian flow of the probability density. Furthermore, since Machine I is smooth, the asymptotic \( w \) is independent of \( s \) so that \( \partial w/\partial s = 0 \).

For the orbital part one gets analogously:

\[
\frac{\partial w_{\text{orb}}}{\partial s} = -a \cdot \eta \cdot \frac{\partial w_{\text{orb}}}{\partial \sigma} - b \cdot \sigma \cdot \frac{\partial w_{\text{orb}}}{\partial \eta} = \{H_{\text{orb}}, w_{\text{orb}}\} = 0.
\]

Using the fact that the orbital equilibrium is unique with \( w_{\text{orb}} = w_{\text{norm}} \), one thus has:

\[
\{H_{\text{orb}}, w_{\text{norm}}\} = 0.
\]

This relation is obviously fulfilled because:

\[
w_{\text{norm}}(\sigma, \eta) = \frac{1}{2\pi \cdot \sigma_{\eta} \cdot \sigma} \cdot \exp\left(-\frac{H_{\text{orb}}}{a \cdot \sigma_{\eta}^{2}}\right).
\]

For more details on the Hamiltonian description, see Appendix D.

2.9.2

Having obtained a tractable formula for the equilibrium states of Machine I one can now consider the equilibrium polarization properties. The spin part of the equilibrium probability density has the form:

\[
w_{\text{spin}}(\psi; +\infty) = \int_{-\infty}^{+\infty} d\sigma \int_{-\infty}^{+\infty} d\eta \cdot w_{\text{norm}}(\sigma, \eta) \cdot w_{\text{aver}}(\psi - \frac{d}{a} \cdot \sigma; s_0)
\]

\[
= -\frac{a}{d} \cdot \int_{-\infty}^{+\infty} d\eta \int_{-\infty}^{+\infty} d\psi_1 \cdot w_{\text{norm}}\left(\frac{a}{d} \cdot (\psi - \psi_1), \eta\right) \cdot w_{\text{aver}}(\psi_1; s_0).
\]

\[
= -\frac{a}{d} \cdot \int_{-\infty}^{+\infty} d\psi_1 \cdot w_{\text{norm, red}}\left(\frac{a}{d} \cdot (\psi - \psi_1)\right) \cdot w_{\text{aver}}(\psi_1; s_0),
\]

where I introduced the abbreviation

\[
w_{\text{norm, red}}(\sigma) \equiv \int_{-\infty}^{+\infty} d\eta \cdot w_{\text{norm}}(\sigma, \eta) = (2\pi)^{-1/2} \cdot \sigma_{\eta}^{-1} \cdot \exp\left(-\sigma^2/2\sigma_{\eta}^2\right).
\]

To determine the equilibrium polarization vector I introduce the auxiliary constant

\[
h^w \equiv |h^w| \cdot \exp(i \cdot \chi^w) \equiv \int_{-\infty}^{+\infty} d\psi \cdot w_{\text{spin}}(\psi; +\infty) \cdot \exp(i \cdot \psi)
\]

\[
= -\frac{a}{d} \cdot \int_{-\infty}^{+\infty} d\psi_1 \int_{-\infty}^{+\infty} d\psi \cdot w_{\text{norm, red}}\left(\frac{a}{d} \cdot (\psi - \psi_1)\right) \cdot w_{\text{aver}}(\psi_1; s_0) \cdot \exp(i \cdot \psi)
\]

\[
= -\frac{a}{d} \cdot (2\pi)^{-1/2} \cdot \sigma_{\eta}^{-1} \cdot \int_{-\infty}^{+\infty} d\psi_1 \int_{-\infty}^{+\infty} d\psi \cdot \exp\left(-\frac{a^2}{2d^2\sigma_{\eta}^{2}} \cdot (\psi - \psi_1)^2\right)
\]

\[
\cdot w_{\text{aver}}(\psi_1; s_0) \cdot \exp(i \cdot \psi)
\]

\[
= \exp\left(-\frac{d^2 \cdot \sigma_{\eta}^{2}}{2a^2}\right) \cdot \int_{-\infty}^{+\infty} d\psi_1 \cdot w_{\text{aver}}(\psi_1; s_0) \cdot \exp(i \cdot \psi_1).
\]

(2.143)
By (2.21), (2.143) the equilibrium polarization is given by
\[
\|P_{\text{tot}}^w(+\infty)\|^2 = \int_{-\infty}^{+\infty} d\psi \cdot w_{\text{span}}(\psi; +\infty) \cdot \cos(\psi)^2
\]
\[
+ \int_{-\infty}^{+\infty} d\psi \cdot w_{\text{span}}(\psi; +\infty) \cdot \sin(\psi)^2
\]
\[
= \int_{-\infty}^{+\infty} d\psi \cdot w_{\text{span}}(\psi; +\infty) \cdot \exp(i \cdot \psi)^2 = |h_w^1|^2
\]
\[
= \exp\left(-\frac{d^2 \cdot \sigma^2}{2a^2}\right) \int_{-\infty}^{+\infty} d\psi \cdot w_{\text{aver}}(\psi; s_0) \cdot \exp(i \cdot \psi)^2. \tag{2.144}
\]
An interesting application of (2.144) is that it allows the determination of the maximum equilibrium polarization possible for Machine I. First of all one observes by (2.139), (2.140) that
\[
\text{Process 1 is not the only possible process having this equilibrium value. Another example is given by Process 1, which is fully ordered at the start. However Process 1 reaches this value (see (2.76)). So if one specifies the constants according to (2.12), then no process running with Machine I has an equilibrium polarization greater than the 0.59 of Process 1.}
\]
This is not surprising since Process 1 is fully ordered at the start. However Process 1 is not the only possible process having this equilibrium value. Another example is given by the stationary process with the probability density:
\[
w(\sigma, \eta, \psi; s) = w_{\text{norm}}(\sigma, \eta) \cdot \delta(\psi - \frac{d}{a} \cdot \sigma). \tag{2.145}
\]
Coming to the local polarization quantities at equilibrium, I conclude from (2.138), (2.143):
\[
P_{\text{tot}}^w(\sigma, \eta; +\infty) = w_{\text{norm}}(\sigma, \eta) \cdot \int_{-\infty}^{+\infty} d\psi \cdot w_{\text{aver}}(\psi - \frac{d}{a} \cdot \sigma; s_0) \cdot \begin{pmatrix} \cos(\psi) \\ \sin(\psi) \end{pmatrix}
\]
\[
= \Re \left\{ w_{\text{norm}}(\sigma, \eta) \cdot \int_{-\infty}^{+\infty} d\psi \cdot w_{\text{aver}}(\psi - \frac{d}{a} \cdot \sigma; s_0) \cdot \begin{pmatrix} 1 \\ -i \end{pmatrix} \cdot \exp(i \cdot \psi) \right\}
\]
\[
= \Re \left\{ w_{\text{norm}}(\sigma, \eta) \cdot \int_{-\infty}^{+\infty} d\psi \cdot w_{\text{aver}}(\psi; s_0) \cdot \begin{pmatrix} 1 \\ -i \end{pmatrix} \cdot \exp\left(i \cdot (\psi + \frac{d}{a} \cdot \sigma)\right) \right\}
\]
\[
= \Re \left\{ w_{\text{norm}}(\sigma, \eta) \cdot \begin{pmatrix} 1 \\ -i \end{pmatrix} \cdot \exp(i \cdot \frac{d}{a} \cdot \sigma \cdot \int_{-\infty}^{+\infty} d\psi \cdot w_{\text{aver}}(\psi; s_0) \cdot \exp(i \cdot \psi) \right\}
\]
\[
= \Re \left\{ w_{\text{norm}}(\sigma, \eta) \cdot \begin{pmatrix} 1 \\ -i \end{pmatrix} \cdot \exp(i \cdot \frac{d}{a} \cdot \sigma \cdot h_w^1 \cdot P_{\text{max}}^{-1} \right\}
\]
\[
\begin{align*}
\text{Having obtained the simple form (2.146) of the equilibrium polarization density one can now also write down the other equilibrium polarization quantities:}
\end{align*}
\]
\[
\begin{align*}
\vec{P}_\text{loc}(\sigma, \eta; +\infty) &= |h^w| \cdot P_{\text{max}}^{-1} \cdot w_{\text{norm}}(\sigma, \eta) \cdot \frac{1}{a} \cdot \exp(i \cdot \frac{d}{a} \cdot \sigma) \cdot \exp(i \cdot \chi^w) \\
\vec{P}_\text{dir}(\sigma, \eta; +\infty) &= |h^w| \cdot P_{\text{max}}^{-1} \cdot w_{\text{norm}}(\sigma, \eta) \cdot \begin{pmatrix}
\cos((d \cdot \sigma)/a + \chi^w) \\
\sin((d \cdot \sigma)/a + \chi^w) \\
0
\end{pmatrix}, \\
\vec{P}_\text{tot}(+\infty) &= |h^w| \cdot \begin{pmatrix}
\cos(\chi^w) \\
\sin(\chi^w) \\
0
\end{pmatrix}.
\end{align*}
\]
Thus one has found that the polarization quantities at equilibrium are characterized by the complex constant \(h^w\), which is easily determined (see (2.139), (2.143)) if one knows the initial state:
\[
\begin{align*}
\text{Note that by (2.146), (2.147), (2.148) one finds that at equilibrium the polarization density, the local polarization vector and its direction, besides fulfilling their Bloch equations (2.106), (2.119), (2.120), also fulfill the radiationless Bloch equation (2.107). Since Machine I is smooth the asymptotic polarization quantities are independent of s. So } \partial \vec{P}^w/\partial s = \partial \vec{P}_\text{loc}^w/\partial s = \partial \vec{P}_\text{dir}^w/\partial s = 0.
\end{align*}
\]

2.9.3

Now I apply the differential equations (2.44), (2.43) to find the first and second moments for the equilibrium of an arbitrary process running with Machine I. First of all I get
\[
\begin{align*}
\sigma_{11}(+\infty) &= \sigma^2, \\
\sigma_{22}(+\infty) &= \sigma^2, \\
\sigma_{12}(+\infty) &= \sigma_{21}(+\infty) = 0,
\end{align*}
\]
which follows from (2.141).

Applying (2.44) one then gets:
\[
\begin{align*}
0 &= \sigma_{13}(+\infty) = a \cdot \sigma_{23}(+\infty) + d \cdot \sigma_{12}(+\infty) = a \cdot \sigma_{23}(+\infty), \\
0 &= \sigma_{23}(+\infty) = b \cdot \sigma_{13}(+\infty) + c \cdot \sigma_{23}(+\infty) + d \cdot \sigma_{22}(+\infty), \\
&= b \cdot \sigma_{13}(+\infty) + d \cdot \sigma_{22}(+\infty) = b \cdot \sigma_{13}(+\infty) + d \cdot \sigma^2_n.
\end{align*}
\]
where I also used:

\[ 0 = \sigma'(\infty), \]

which follows from the fact that every process running with Machine I approaches equilibrium. Also from (2.44) it follows that:

\[ 0 = \sigma'_{33} + \frac{d^2}{a^2} \cdot \sigma'_{11} - \frac{2d}{a} \cdot \sigma'_{13}, \]

i.e.:

\[ \sigma_{33}(s) + \frac{d^2}{a^2} \cdot \sigma_{11}(s) - \frac{2d}{a} \cdot \sigma_{13}(s) = \sigma_{33}(s_0) + \frac{d^2}{a^2} \cdot \sigma_{11}(s_0) - \frac{2d}{a} \cdot \sigma_{13}(s_0). \]

Hence the equilibrium covariance matrix has the form

\[
\sigma(+\infty) = \begin{pmatrix}
\sigma^2_\sigma & 0 & (d \cdot \sigma^2_\sigma)/a \\
0 & \sigma^2_\sigma & 0 \\
(d \cdot \sigma^2_\sigma)/a & 0 & (d^2/a^2) \cdot \sigma^2_\sigma + \sigma_{33}(s_0) + (d^2/a^2) \cdot \sigma_{11}(s_0) - (2d/a) \cdot \sigma_{13}(s_0)
\end{pmatrix}.
\]

(2.150)

This is the equilibrium covariance matrix for an arbitrary process running with Machine I. One sees that the \((33)\)-element is simply determined by the initial covariance matrix. Stating it differently: two processes running with Machine I have equilibrium covariance matrices which can only differ by the \((33)\)-element. Of course, the equilibrium covariance matrices of processes 1 and 2 have the form (2.150). Also one finds that the equilibrium covariance matrices of the processes defined by (2.127), (2.128) are different, confirming the results of section 2.8.2.

By (2.16) one has:

\[
\sigma_{33}(s) + \frac{d^2}{a^2} \cdot \sigma_{11}(s) - \frac{2d}{a} \cdot \sigma_{13}(s) = \left( \tilde{\psi}(s) - \psi(s) \right)^2,
\]

so that by the nonnegativity of this expression and by (2.150) the minimum value possible for \(\sigma_{33}(+\infty)\) is given by \((d^2/a^2) \cdot \sigma^2_\sigma\). Note that the determinant of (2.150) only vanishes in this case, i.e. the equilibrium covariance matrix is singular if and only if

\[ 0 = \sigma_{33}(s_0) + \frac{d^2}{a^2} \cdot \sigma_{11}(s_0) - \frac{2d}{a} \cdot \sigma_{13}(s_0). \]

An example is Process 1. Another example is given by the process with the probability density (2.145).

Using (2.45) one easily finds the equilibrium first moment vector of an arbitrary process \(\bar{x}(s)\) running with Machine I:

\[
<\bar{x}(+\infty)> = \left( 0, 0, <\psi(s_0) > - \frac{d}{a} <\sigma(s_0) > \right)^T.
\]

(2.151)

The equilibrium first moment vector obeys:

\[
A_x \cdot <\bar{x}(+\infty)> = 0,
\]

(2.152)
which of course can be concluded directly from (2.45) even without knowing the explicit form (2.151). Note that \(<\vec{x}(+\infty)\rangle\) is not uniquely determined by (2.152), because the determinant of \(A_I\) vanishes (see (2.8)). Thus the nonuniqueness of the equilibrium state of Machine I follows from the singular nature of the matrix \(A_I\).

For G-processes (see section 2.8) the equilibrium states are just determined by the two numbers \(<\psi(+\infty)\rangle\) resp. \(\sigma_{33}(+\infty)\) so that the family of those equilibrium states is two-parametric. 

2.9.4

A stationary process fulfills the condition of ‘detailed balance’ [Gar85], if one has:

\[
\begin{align*}
\text{w}_{\text{joint}}(\vec{x};s;\vec{x}_1;\sigma) &= \text{w}_{\text{joint}}(\varepsilon \cdot \vec{x};\varepsilon \cdot \vec{x}_1;\sigma \cdot \vec{x};\sigma_1),
\end{align*}
\]

where \(\text{w}_{\text{joint}}\) denotes the joint probability density and where the matrix

\[
\varepsilon \equiv \begin{pmatrix} \varepsilon_1 & 0 & 0 \\ 0 & \varepsilon_2 & 0 \\ 0 & 0 & \varepsilon_3 \end{pmatrix}
\]

defines the time reversal operation with:

\[
\varepsilon_1^2 = \varepsilon_2^2 = \varepsilon_3^2 = 1.
\]

Using the probability density and the transition probability density this can be expressed via (2.83) by:

\[
\text{w}_{I,\text{trans}}(\vec{x};s|\vec{x}_1;\sigma) \cdot \text{w}(\vec{x}_1;\sigma) = \text{w}_{I,\text{trans}}(\varepsilon \cdot \vec{x};\varepsilon \cdot \vec{x}_1;\sigma \cdot \vec{x};\sigma_1) \cdot \text{w}(\varepsilon \cdot \vec{x};\sigma).
\]

The condition of detailed balance roughly means that for the stationary process described by \(w\) each possible transition

\[(\vec{x}_1;\sigma) \rightarrow (\vec{x};\sigma)\]

is balanced by the ‘reverse’ transition:

\[(\varepsilon \cdot \vec{x};\sigma) \rightarrow (\varepsilon \cdot \vec{x}_1;\sigma).\]

Choosing the matrix \(\varepsilon\) so that

\[
1 = \varepsilon_1 = \varepsilon_3 = -\varepsilon_2,
\]

I will show for Machine I that every stationary process fulfills the condition of detailed balance. By (2.138) the probability density of a stationary process can be written in the form

\[
\text{w}(\sigma,\eta,\psi;\sigma) = \text{w}_{\text{norm}}(\sigma,\eta) \cdot \text{w}_{\text{aver}}(\psi - \frac{d}{a} \cdot \sigma;\sigma_0),
\]

\footnote{In particular the family of stationary G-processes is two-parametric.}

\footnote{Thus I choose \(\eta\) as a ‘velocity’ variable.}
where \( w_{\text{aver}} \) is given by (2.139). Thus by using (2.82), (2.85), (2.86), (2.92), (2.154) the joint probability density of a stationary process can be written as:

\[
w_{\text{joint}}(\sigma, \eta, \psi; \sigma_1, \eta_1, \psi_1; s_1) = w_{\text{orb, joint}}(\sigma, \eta; s; \sigma_1, \eta_1; s_1) \cdot \delta\left(\psi_1 - \psi - \frac{d}{a} \cdot (\sigma - \sigma_1)\right) \cdot w_{\text{aver}}(\psi_1 - \frac{d}{a} \cdot \sigma_1; s_0),
\]  

(2.155)

where \( w_{\text{orb, joint}} \) denotes the orbital joint probability density, which via (2.92), (2.154) is given by:

\[
w_{\text{orb, joint}}(\sigma, \eta; s; \sigma_1, \eta_1; s_1) = w_{\text{orb, trans}}(\sigma, \eta; s|\sigma_1, \eta_1; s_1) \cdot w_{\text{norm}}(\sigma_1, \eta_1).
\]  

(2.156)

From (2.74), (2.86), (2.156) follows:

\[
w_{\text{orb, joint}}(\sigma, \eta; s; \sigma_1, \eta_1; s_1) = w_{\text{orb, joint}}(\sigma_1, -\eta_1; s; \sigma, -\eta; s_1).
\]  

(2.157)

Combining (2.155), (2.157) one observes that (2.153) holds so that I have proven that for Machine I every stationary process fulfills the condition of detailed balance.

For Machine I, as for many stochastic systems whose stationary states obey detailed balance, the ‘Onsager relations’ [Gar85] hold. By these relations the covariance matrix \( \sigma \) for a stationary process fulfills:

\[
\varepsilon \cdot A_I \cdot \sigma = \left(\varepsilon \cdot A_I \cdot \sigma\right)^T.
\]  

(2.158)

In fact, the covariance matrix of a stationary process is the equilibrium covariance matrix and the latter, given by (2.150), fulfills the Onsager relations (2.158). This proves that the Onsager relations hold for every stationary process of Machine I.

### 2.10 Resonant spin flip

The study of Machine I was originally motivated by a wish to know whether it is possible to flip vertical polarization from up to down by perturbing the spins with an oscillating radial magnetic field running at a frequency close to \( \nu \) and hence almost in resonance with the precession of the spin basis \( \vec{m}_{0,I}(s), \vec{l}_{0,I}(s), \vec{n}_{0,I}(s) \). My calculation suggests that for the smoothed machine the horizontal spin components would partially decohere within a few orbital damping times. Perhaps in reality there would be complete decoherence [BBHMR94a, BBHMR94b]. In any case it looks as if the spin flip procedure should be completed within a fraction of the orbital damping time. However it must be borne in mind that I have neglected the oscillating field in my calculations. I am pursuing this topic further.

### 2.11 Recapitulation of Machine I

Although the stochastic spin-orbit system of Machine I is very simple it has served to illustrate the application of standard stochastic differential equation theory to such systems. Moreover, this study is a useful introduction to the treatment of a more complicated system, namely Machine II. To orient the reader I recapitulate the main results here:

- For Machine I all processes reach equilibrium.
• However the equilibrium is not unique.

• There is no equation which provides a causal azimuthal evolution for the stochastic average of the five-component spin-orbit vector, i.e. there is no appropriate Bloch equation for the polarization vector in Machine I.

• But there is a universal Bloch equation which provides a causal azimuthal evolution for the polarization density.

• The Bloch equations for the local polarization vector and its direction provide causal azimuthal evolution only under certain circumstances.

• The value of the local polarization is uniform across phase space for Process 1 and G-processes (e.g. Process 2).

• There is an upper limit to the equilibrium polarization, namely the equilibrium value of Process 1.

3 Machine II

3.1

In section 2 I studied the spin distribution w.r.t. a pair of (usually) nonperiodic vectors \((\vec{m}_0, \vec{l}_0)\) lying in the horizontal plane and found that this distribution always reaches equilibrium. However, this is not an equilibrium w.r.t. the directions \(\vec{e}_1, \vec{e}_2\). On the contrary, the equilibrium spin direction on the closed orbit is the \(\vec{n}_0\)-axis, i.e. that solution \(\vec{n}_0\) to the Thomas-BMT equation on the closed orbit which is 1-turn periodic in the machine frame and in Machine I this is vertical. Thus it would be interesting to use my formalism to study spin diffusion w.r.t. an equilibrium spin direction which is also constrained to lie in the horizontal plane. This can be arranged by including a Siberian Snake \([DK78]\) in the smoothed optic of Machine I to create ‘Machine II’. Siberian snakes are devices that rotate a spin on the closed orbit by an angle of \(\pi\) around a fixed axis which usually lies in the horizontal plane. With such a snake the \(\vec{n}_0\)-axis is horizontal.

This layout is of great practical interest for some existing or proposed electron storage rings (e.g. the MIT-Bates, Amps and BTCF rings) \(^{37}\) where horizontal spin polarization is required at the interaction points but whose energy is too low for a useful Sokolov-Ternov \([ST64]\) polarization rate to be achieved. These rings use Siberian Snakes to ensure that the \(\vec{n}_0\)-axis lies in the horizontal plane. A polarized electron beam is injected with its polarization vector parallel to the \(\vec{n}_0\)-axis at the injection point and as well as determining the \(\vec{n}_0\)-axis the snakes are supposed to suppress the spin diffusion that one naively expects when spins lie in the horizontal plane so that useful polarization lifetimes can be achieved. \(^{38}\)

In this section I use my model to calculate the spin decoherence in the presence of a pointlike snake whose rotation axis is radial \(^{39}\). As we will see, this will allow a comparison

\(^{37}\)See \([Bar96]\) and the reference list therein.

\(^{38}\)But recall from Machine I that it is not so clear that there will be complete depolarization.

\(^{39}\)In the language of \([Mon84]\) this is a ‘type 2’ snake. See section 6.3 thereof. The depolarization time determined below (see eq. (3.64)) would be the same if the rotation axis were longitudinal.
with calculations using the SLIM formalism [Cha81] which is based on a linearized description of spin motion and will allow a basic assumption underlying conventional treatments [DK72, DK73, Man87, BHMR91] to be checked from scratch - at least for this model.

The Thomas-BMT equation for Machine II reads as

$$\vec{\xi}' = \vec{\Omega}_{II} \wedge \vec{\xi},$$

(3.1)

with

$$\vec{\Omega}_{II} \equiv \vec{\Omega}_{II,0} + \vec{\Omega}_{osc},$$

$$\vec{\Omega}_{II,0} \equiv \Omega_{II,0,\text{dipole}} \cdot \vec{e}_3 + \Omega_{II,0,\text{snake}} \cdot \delta_{L,\text{per}} \cdot \vec{e}_1,$$

$$\Omega_{II,0,\text{dipole}} \equiv ||\vec{\Omega}_{I,0}|| = d,$$

(3.2)

where the machine frame dreibein $\vec{e}_1, \vec{e}_2, \vec{e}_3$ has the same meaning as for Machine I. Here $\Omega_{II,0,\text{dipole}}, \Omega_{II,0,\text{snake}}$ are constant, $\delta_{L,\text{per}}(s)$ denotes the periodic delta function with period $L \equiv \frac{40}{47}$ and $\vec{\Omega}_{osc}$ is defined in section 2.1. The snake is located at $s = 0$ and:

$$\Omega_{II,0,\text{snake}} = \pi.$$

The $\vec{n}_0$-axis for Machine II is given in Appendix A and reads as:

$$\vec{n}_{0,II}(s) \equiv \cos\left(g_6(s)\right) \cdot \vec{e}_1 + \sin\left(g_6(s)\right) \cdot \vec{e}_2,$$

(3.3)

with

$$g_6(s) \equiv d \cdot \left(s - \frac{L}{2} - L \cdot \mathcal{G}(s/L)\right),$$

where the step function $\mathcal{G}$ is defined by:

$$\mathcal{G}(s) \equiv N \quad \text{if } N < s < N + 1,$$

and where $N$, as always in this section, denotes an integer. Note that $\vec{n}_{0,II}(s)$ is 1-turn periodic in the machine frame. I also define the vectors $\vec{m}_{0,II}(s), \vec{l}_{0,II}(s)$:

$$\vec{l}_{0,II}(s) \equiv \theta_{2L,\text{per}}(s) \cdot \vec{e}_3,$$

$$\vec{m}_{0,II}(s) \equiv \vec{l}_{0,II}(s) \wedge \vec{n}_{0,II}(s),$$

(3.4)

where $\theta_{2L,\text{per}}$ denotes a 2-turn periodic step function and where the step is located at the snake. Explicitly one has:

$$\theta_{2L,\text{per}}(s) \equiv \begin{cases} 1 & \text{if } 2NL < s < (2N + 1)L \\ -1 & \text{if } (2N + 1)L < s < 2NL + 2L \end{cases}.$$
Note that:

\[ \theta_{2L,\text{per}}(s) = (-1)^{G(s/L)}. \]

The vectors \( \vec{m}_{0,II}(s), \vec{l}_{0,II}(s) \) are 2-turn periodic in \( s \) in the machine frame. The vectors \( \vec{n}_{0,II}, \vec{m}_{0,II} \) precess in the horizontal plane around the vertical dipole field.

As before I deal only with horizontal spin and therefore define the phase angle \( \psi \) by

\[ \vec{\xi} \equiv \frac{\hbar}{2} \left( \vec{n}_{0,II} \cdot \cos(\psi) + \vec{m}_{0,II} \cdot \sin(\psi) \right). \]

Hence the Thomas-BMT equation (3.1), reexpressed in the \( (\vec{n}_{0,II}, \vec{m}_{0,II}, \vec{l}_{0,II}) \)-frame, is equivalent to

\[ \psi' = (2\pi \nu / L) \cdot \eta \cdot \theta_{2L,\text{per}} = d \cdot \eta \cdot \theta_{2L,\text{per}} = \eta \cdot \vec{d}, \]

where:

\[ \vec{d}(s) \equiv d \cdot \theta_{2L,\text{per}}(s). \quad (3.5) \]

I also introduce the spin vector

\[ \vec{S} \equiv \frac{\hbar}{2} \begin{pmatrix} \cos(\psi) \\ \sin(\psi) \\ 0 \end{pmatrix}, \]

describing the spin in the \( (\vec{n}_{0,II}, \vec{m}_{0,II}, \vec{l}_{0,II}) \)-frame. Then, as shown in Appendix A, the Thomas-BMT equation reads as

\[ \vec{S}'(s) = \vec{W}_{II}(\eta(s); s) \wedge \vec{S}(s), \quad (3.6) \]

with

\[ \vec{W}_{II}(\eta; s) = \vec{d}(s) \cdot \eta \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \quad (3.7) \]

### 3.2 The Langevin equation and the Fokker-Planck equation for Machine II

#### 3.2.1

The Langevin equation for Machine II is given by

\[ d\vec{x}(s) = A_{II} \cdot \vec{x}(s) \cdot ds + \vec{B} \cdot d\vec{W}(s), \quad (3.8) \]

where

\[ A_{II} \equiv \begin{pmatrix} 0 & a & 0 \\ b & c & 0 \\ 0 & d & 0 \end{pmatrix}. \]

---

41Because \( \vec{m}_{0,II}(s), \vec{l}_{0,II}(s) \) are 2-turn periodic in \( s \) in the machine frame, the fractional part of the closed-orbit spin tune equals 1/2. This is the trademark of a snake [Mon84].
Because $A_{II}$ and $B$ are matrices independent of $\vec{x}$, the Langevin equation (3.8) describes three-component processes of Ornstein-Uhlenbeck type \cite{Gar85}. If $s_0$ denotes the starting azimuth of a process $\vec{x}(s)$ then as for Machine I $\vec{x}(s_0)$ is always assumed to be chosen so that $\vec{x}(s)$ is a Markovian diffusion process. \footnote{For the special processes 3,4 and 5 considered in detail I choose the starting azimuth as $s = 0$.}

### 3.2.2

The Fokker-Planck equation corresponding to the Langevin equation (3.8) has the form \cite{Gar85, Ris89}

$$\frac{\partial w}{\partial s} = -\sum_{j,k=1}^{3} \frac{\partial}{\partial x_j} (A_{II,jk} \cdot x_k \cdot w) + \frac{1}{2} \sum_{j,k=1}^{3} \frac{\partial^2}{\partial x_j \partial x_k} (D_{jk} \cdot w) .$$

Therefore the Fokker-Planck equation can be written

$$\frac{\partial w}{\partial s} = -a \cdot \eta \cdot \frac{\partial w}{\partial \sigma} - b \cdot \sigma \cdot \frac{\partial w}{\partial \eta} + d \cdot \eta \cdot \frac{\partial w}{\partial \psi} - c \cdot w - c \cdot \eta \cdot \frac{\partial w}{\partial \eta} + \frac{\omega}{2} \cdot \frac{\partial^2 w}{\partial \eta^2} .$$

\[\equiv L_{FP,orb} w + L_{FP,II,spin} w ,\tag{3.9}\]

where I used the abbreviation

$$L_{FP,II,spin} \equiv -d \cdot \eta \cdot \frac{\partial}{\partial \psi} .$$

As for Machine I use standard boundary conditions in all three variables $\sigma, \eta, \psi$ so that the probability densities of the processes considered are normalized by (2.20).

By the Fokker-Planck equation (3.9), the characteristic function $\Phi$ corresponding to a probability density $w$ (see (2.27)) obeys:

$$\frac{\partial \Phi}{\partial s} = \sum_{j,k=1}^{3} A_{II,jk} \cdot u_k \cdot \frac{\partial \Phi}{\partial u_j} - \frac{1}{2} \cdot \sum_{j,k=1}^{3} D_{jk} \cdot u_j \cdot u_k \cdot \Phi .$$

\[\text{(3.10)}\]

### 3.3 The polarization density and its Bloch equation for Machine II

The polarization density is defined in the same way as for Machine I (see (2.103)). From the Fokker-Planck equation (3.9) one obtains:

$$\frac{\partial \vec{P}^w}{\partial s} = L_{FP,orb} \vec{P}^w + \vec{W}_{II} \wedge \vec{P}^w ,$$

which is the Bloch equation for Machine II w.r.t. the ($\vec{n}_{0,II}, \vec{m}_{0,II}, \vec{l}_{0,II}$)-frame. Writing it out explicitly one gets

$$\frac{\partial \vec{P}^w}{\partial s} = -a \cdot \eta \cdot \frac{\partial \vec{P}^w}{\partial \sigma} - b \cdot \sigma \cdot \frac{\partial \vec{P}^w}{\partial \eta} + \vec{W}_{II} \wedge \vec{P}^w - c \cdot \vec{P}^w - c \cdot \eta \cdot \frac{\partial \vec{P}^w}{\partial \eta} + \omega \cdot \frac{\partial^2 \vec{P}^w}{\partial \eta^2} .$$

\[\text{(3.11)}\]

The radiationless Bloch equation underlying Machine II reads as:

$$\frac{\partial \vec{P}^w}{\partial s} = -a \cdot \eta \cdot \frac{\partial \vec{P}^w}{\partial \sigma} - b \cdot \sigma \cdot \frac{\partial \vec{P}^w}{\partial \eta} + \vec{W}_{II} \wedge \vec{P}^w .$$

\[\text{(3.12)}\]
3.4 Further properties of Machine II

3.4.1

With the standard boundary conditions one can immediately write down a differential equation for the covariance matrix $\sigma$ of any process running with Machine II:

$$\sigma' = A_{II} \cdot \sigma + \sigma \cdot A_{II}^T + D.$$  (3.13)

Writing out the components results in:

$$\sigma'_{11} = 2 \cdot a \cdot \sigma_{12},$$
$$\sigma'_{12} = a \cdot \sigma_{22} + b \cdot \sigma_{11} + c \cdot \sigma_{12},$$
$$\sigma'_{22} = 2 \cdot b \cdot \sigma_{12} + 2 \cdot c \cdot \sigma_{22} + \omega,$$
$$\sigma'_{13} = a \cdot \sigma_{23} + \hat{d} \cdot \sigma_{12},$$
$$\sigma'_{23} = b \cdot \sigma_{13} + c \cdot \sigma_{23} + \hat{d} \cdot \sigma_{22},$$
$$\sigma'_{33} = 2 \cdot \hat{d} \cdot \sigma_{23}.$$  (3.18)

For the first moment vector one gets the following differential equation:

$$<\vec{x}'(s)> = A_{II}(s) \cdot <\vec{x}(s)>.$$  (3.14)

The differential equations (3.13), (3.14) can be easily derived from any of the equations (3.8), (3.9) or (3.10) in analogy with (2.44), (2.45).

3.4.2

In this section I consider further properties arising for processes which are at orbital equilibrium, i.e. for which $w_{orb} = w_{norm}$. Firstly:

$$<\sigma(s)> = <\eta(s)> = 0.$$  (3.15)

Then from (3.14), (3.13) one gets:

$$<\psi'(s)> = 0.$$  (3.16)

From (3.13), (3.16) follows:

$$<\vec{x}(s)> = (0, 0, <\psi(s_0)>)^T.$$  (3.17)

The orbital matrix elements of the covariance matrix read as:

$$\sigma_{11} = \sigma^2, \quad \sigma_{12} = \sigma_{21} = 0, \quad \sigma_{22} = \sigma^2,$$
so that the determinant of the covariance matrix is:

$$\det(\sigma) = \sigma^2 \cdot \sigma^2 \cdot \sigma_{33} - \sigma^2 \cdot \sigma_{23}^2 - \sigma^2 \cdot \sigma^2.$$  (3.19)

From this follows by using the differential equation (3.13):

$$\left(\det(\sigma)\right)' = -2 \cdot c \cdot \sigma^2 \cdot \sigma_{23}^2.$$  (3.20)

---

43The statements of section 3.4.2 are only valid for processes at orbital equilibrium.
so that:

$$\det(\sigma(s)) = \det(\sigma(s_0)) - 2 \cdot c \cdot \sigma_\sigma^2 \cdot \int_{s_0}^{s} ds_1 \cdot \sigma_{23}^2(s_1).$$  \hspace{1cm} (3.21)$$

Also one obtains by (3.13), (3.18):

$$\left( \begin{array}{c} \sigma'_{13} \\ \sigma'_{23} \end{array} \right) = \Delta_{orb} \cdot \left( \begin{array}{c} \sigma_{13} \\ \sigma_{23} \end{array} \right) + \sigma_\eta \cdot \left( \begin{array}{c} 0 \\ \hat{d} \end{array} \right),$$  \hspace{1cm} (3.22)$$

$$\sigma'_{33} = 2 \cdot \hat{d} \cdot \sigma_{23}.$$  \hspace{1cm} (3.23)$$

Note that (3.22), (3.23) are formally solved by:

$$\left( \begin{array}{c} \sigma_{13}(s) \\ \sigma_{23}(s) \end{array} \right) = \exp(\Delta_{orb} \cdot (s - s_0)) \cdot \left( \begin{array}{c} \sigma_{13}(s_0) \\ \sigma_{23}(s_0) \end{array} \right) + \sigma_\eta \cdot \int_{s_0}^{s} ds_1 \cdot \exp(\Delta_{orb} \cdot (s - s_1)) \cdot \left( \begin{array}{c} 0 \\ \hat{d}(s_1) \end{array} \right),$$  \hspace{1cm} (3.24)$$

By (3.17), (3.18), (3.24) one sees that the first moment vector and the covariance matrix depend continuously on $s$. However the dependence is not smooth because the discontinuity of $\hat{d}(s)$ at $s = NL$ causes (see (3.22)) $\sigma'_{23}(s)$ to be discontinuous at $s = NL$ ($N$-integer).

In addition to the Bloch equation for the polarization density, at orbital equilibrium a Bloch equation for the local polarization vector holds. In fact from (2.74), (2.115), (3.11) follows:

$$\frac{\partial \vec{P}_{loc}^w}{\partial s} = -a \cdot \eta \cdot \frac{\partial \vec{P}_{loc}^w}{\partial \sigma} - b \cdot \sigma \cdot \frac{\partial \vec{P}_{loc}^w}{\partial \eta} + \vec{W}_I \land \vec{P}_{loc}^w + c \cdot \eta \cdot \frac{\partial \vec{P}_{loc}^w}{\partial \eta} + \frac{\omega}{2} \cdot \frac{\partial^2 \vec{P}_{loc}^w}{\partial \eta^2}. \hspace{1cm} (3.25)$$

### 3.4.3

In the remainder of section 3 I study three different G-processes at orbital equilibrium which I call processes 3, 4, 5. \(^{44}\) For Machine II G-processes are defined as for Machine I; see section 2.8. Moreover by (3.18) the inverse $\sigma_{inv}$ of the covariance matrix fulfills:

$$\sigma_{inv,13} = -\frac{\sigma_\eta^2 \cdot \sigma_{13}}{\det(\sigma)}, \hspace{1cm} \sigma_{inv,23} = -\frac{\sigma_\sigma^2 \cdot \sigma_{23}}{\det(\sigma)}, \hspace{1cm} \sigma_{inv,33} = \frac{\sigma_\sigma^2 \cdot \sigma_\eta^2}{\det(\sigma)}. \hspace{1cm} (3.26)$$

Using sections 2.8 and 3.4.2 one finds:

$$\vec{P}_{tot}^w(s) = \exp(-\sigma_{33}(s)/2) \cdot \begin{pmatrix} \cos(<\psi(s_0)>), \\ \sin(<\psi(s_0)>), \\ 0 \end{pmatrix}, \hspace{1cm} (3.27)$$

$$||\vec{P}_{tot}^w(s)|| = \exp(-\sigma_{33}(s)/2), \hspace{1cm} (3.28)$$

$$||\vec{P}_{loc}^w(\sigma, \eta; s)|| = \exp(-\frac{\det(\sigma)}{2 \cdot \sigma_\sigma^2 \cdot \sigma_\eta^2}), \hspace{1cm} (3.29)$$

\(^{44}\)The statements of section 3.4.3 are only valid for G-processes at orbital equilibrium.
\[
\vec{P}_{\text{dir}}(\sigma, \eta; s) = \begin{pmatrix}
\cos\left(\frac{\sigma_{13}(s)}{\sigma^2} \cdot \sigma + \frac{\sigma_{23}(s)}{\sigma^2} \cdot \eta + <\psi(s_0)>\right) \\
\sin\left(\frac{\sigma_{13}(s)}{\sigma^2} \cdot \sigma + \frac{\sigma_{23}(s)}{\sigma^2} \cdot \eta + <\psi(s_0)>\right) \\
0
\end{pmatrix},
\]

(3.30)

\[
\vec{P}_w(\sigma, \eta; s) = w_{\text{norm}}(\sigma, \eta) \cdot ||\vec{P}_{\text{loc}}(\sigma, \eta; s)|| \cdot \vec{P}_{\text{dir}}(\sigma, \eta; s),
\]

(3.31)

\[
\Phi(\vec{u}; s) = \exp\left(-\frac{1}{2} \cdot \sum_{j,k=1}^{3} \sigma_{jk}(s) \cdot u_j \cdot u_k + i <\psi(s_0)> \cdot u_3\right),
\]

(3.32)

where I also used (2.122), (3.26). Because the first moment vector and the covariance matrix depend continuously on \(s\), one observes that the quantities in (3.27), (3.28), (3.29), (3.30), (3.31), (3.32) depend continuously on \(s\) so that these equations even hold at \(s = s_0\). Of course \(0 \leq ||\vec{P}_{\text{loc}}|| \leq 1\) and as with all G-processes \(||\vec{P}_{\text{loc}}||\) is uniform across phase space.

In addition to the Bloch equation (3.11) for the polarization density and the Bloch equation (3.25) for the local polarization vector a Bloch equation for the local polarization direction holds. In fact from (2.116), (3.25), (3.29) follows:

\[
\frac{\partial \vec{P}_{\text{dir}}}{\partial s} = -a \cdot \eta \cdot \frac{\partial \vec{P}_{\text{dir}}}{\partial \sigma} - b \cdot \sigma \cdot \frac{\partial \vec{P}_{\text{dir}}}{\partial \eta} + \bar{W}_{II} \wedge \vec{P}_{\text{dir}} + c \cdot \eta \cdot \frac{\partial \vec{P}_{\text{dir}}}{\partial \eta}.
\]

(3.33)

Naturally one could describe processes 3, 4 and 5 with the aid of the Bloch equations (3.11), (3.25), (3.33) which one would solve by standard methods (e.g. using Green functions or method of characteristics) but for such simple G-processes the first and second moments are easily obtained, so that on this occasion the Bloch equations are not needed.

### 3.5 The probability density of Process 3

#### 3.5.1

Now I consider the process \(\vec{x}^{(3)}(s)\), called ‘Process 3’ and I abbreviate:

\[
\vec{x}^{(3)}(s) \equiv \begin{pmatrix} \sigma^{(3)}(s) \\ \eta^{(3)}(s) \\ \psi^{(3)}(s) \end{pmatrix}.
\]

This process is characterized by the following three conditions:

- It is a G-process at orbital equilibrium and its starting azimuth is \(s_0 = 0\).
- The initial local polarization direction is parallel to the \(\vec{n}_0\)-axis of Machine II.
- Its initial local polarization is 1.

By the second and third conditions one has:

\[
||\vec{P}_{\text{loc}}(\sigma, \eta; 0)|| = 1, \quad \vec{P}_{\text{dir}}(\sigma, \eta; 0) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},
\]

(3.34)
where the probability density of Process 3 is denoted by \( w_3 \). Thus one has by (3.29), (3.30):
\[
<\psi(3)(0)> = \psi_{0,m}, \quad (3.35)
\]
\[
\sigma_{3,13}(0) = \sigma_{3,23}(0) = 0, \quad (3.36)
\]
\[
\det(\sigma_3(0)) = 0, \quad (3.37)
\]
where the covariance matrix of Process 3 is denoted by \( \sigma_3 \) and where:
\[
\psi_{0,m} = 2\pi \cdot m, \quad (3.38)
\]
\( m \) being an arbitrary integer. From the first condition on Process 3 and from (3.19), (3.36), (3.37) it follows that:
\[
\sigma_{3,33}(0) = 0. \quad (3.39)
\]
Also from (3.17), (3.35) follows:
\[
<\vec{x}^{(3)}(s)> = <\vec{x}^{(3)}(0)> = (0, 0, \psi_{0,m})^T. \quad (3.40)
\]
By (3.32), (3.35) one obtains:
\[
\Phi_3(\vec{u}; s) = \exp\left(-\frac{1}{2} \cdot \sum_{j,k=1}^{3} \sigma_{3,jk}(s) \cdot u_j \cdot u_k + i \cdot \psi_{0,m} \cdot u_3\right), \quad (3.41)
\]
where \( \Phi_3 \) denotes the characteristic function corresponding to \( w_3 \) and which for \( s = 0 \) simplifies by (3.18), (3.36), (3.39) to:
\[
\Phi_3(\vec{u}; 0) = \exp\left(-\frac{1}{2} \cdot \sigma_3^2 \cdot u_1^2 - \frac{1}{2} \cdot \sigma_2^2 \cdot u_2^2 + i \cdot \psi_{0,m} \cdot u_3\right). \quad (3.42)
\]
From (2.28), (3.42) it follows that the initial probability density of Process 3 takes the expected form:
\[
w_3(\sigma, \eta, \psi; 0) = w_{\text{norm}}(\sigma, \eta) \cdot \delta(\psi - \psi_{0,m}). \quad (3.43)
\]
By (2.28), (3.41) one sees that \( w_3 \) is Gaussian if \( \sigma_3 \) is nonsingular. Because the integral:
\[
\int_0^s ds_1 \cdot \sigma_{3,23}^2(s_1)
\]
is positive for \( s > 0 \), the determinant of the covariance matrix of Process 3 is positive for \( s > 0 \). Thus for \( s > 0 \) the probability density is Gaussian, confirming that Process 3 is a G-process, and one obtains for \( s > 0 \):
\[
w_3(\sigma, \eta, \psi; s) = \sqrt{(2\pi)^{-3} \cdot \det(\sigma_3(s))^{-1}} \cdot \exp\left[-\frac{1}{2} \cdot \begin{pmatrix} \sigma \\ \eta \\ \psi - \psi_{0,m} \end{pmatrix}^T \cdot \sigma_3^{-1}(s) \cdot \begin{pmatrix} \sigma \\ \eta \\ \psi - \psi_{0,m} \end{pmatrix}\right]. \quad (3.44)
\]

\(^{45}\)The physical properties of Process 3 are independent of the value of \( m \), as will become clear below. Thus without loss of generality one could set \( m = 0 \). The same holds for processes 4 and 5.
\(^{46}\)This can be concluded from:
\[
\sigma'_{3,23}(0) = d \cdot \sigma_2^2 > 0,
\]
which follows from (3.22), (3.30).
By (3.43), (3.44) $w_3$ fulfills the normalization condition (2.20). Also it follows for the orbital part of $w_3$ that:

$$w_{3,\text{orb}} = w_{\text{norm}},$$

confirming that Process 3 is at orbital equilibrium. Note that $\vec{x}^{(3)}(s)$ is a Markovian diffusion process.

### 3.5.2

Now I continue the calculation of the covariance matrix $\sigma_3$ of Process 3 and by (3.13) this is basically an integration problem. In this section I determine the matrix elements $\sigma_{3,13}, \sigma_{3,23}$ which fulfill the differential equation (3.22). To do this I first obtain a 2-turn periodic special solution (denoted by $(g_7, g_8)$) of (3.22), i.e.

$$
\begin{pmatrix}
  g'_7 \\
  g'_8
\end{pmatrix} = A_{\text{orb}} \cdot
\begin{pmatrix}
  g_7 \\
  g_8
\end{pmatrix} + \sigma^2 \eta \cdot
\begin{pmatrix}
  0 \\
  \hat{d}
\end{pmatrix},
\tag{3.45}
$$

Because the difference of two such solutions solves the homogeneous equation corresponding to (3.44) one observes that $(g_7, g_8)$ is unique since the 2-turn periodic solution to the homogeneous equation vanishes. I can then write:

$$
\begin{pmatrix}
  \sigma_{3,13}(s) \\
  \sigma_{3,23}(s)
\end{pmatrix} =
\begin{pmatrix}
  g_7(s) \\
  g_8(s)
\end{pmatrix} +
\begin{pmatrix}
  g_9(s) \\
  g_{10}(s)
\end{pmatrix},
\tag{3.46}
$$

where $(g_9, g_{10})$ is a solution of the corresponding homogeneous equation, i.e.

$$
\begin{pmatrix}
  g'_9 \\
  g'_{10}
\end{pmatrix} = A_{\text{orb}} \cdot
\begin{pmatrix}
  g_9 \\
  g_{10}
\end{pmatrix}.
\tag{3.47}
$$

With (3.47) one finds that after a few orbital damping times $g_9, g_{10}$ fade away so that by (3.46) $\sigma_{3,13}(s), \sigma_{3,23}(s)$ become 2-turn periodic in $s$. Hence after a few orbital damping times one gets:

$$
\begin{pmatrix}
  \sigma_{3,13}(s) \\
  \sigma_{3,23}(s)
\end{pmatrix} \approx
\begin{pmatrix}
  g_7(s) \\
  g_8(s)
\end{pmatrix}.
\tag{3.48}
$$

One finds that $(g_7, g_8)$ is given explicitly by:

$$
\begin{align*}
  g_7(s) &= \frac{\hat{d}(s) \cdot \sigma^2}{2 \cdot \lambda \cdot b} \cdot 
  \left( i \cdot g_1(s - L \cdot \mathcal{G}(s/L)) \cdot g_{11} \cdot \exp(-c \cdot L/2) \\
  &\quad + i \cdot g_1(s - L \cdot \mathcal{G}(s/L) - L) \cdot g_{11} \cdot \exp(c \cdot L/2 - 2 \cdot \lambda) \right), \\
  g_8(s) &= -\frac{\hat{d}(s) \cdot \sigma^2}{2 \cdot \lambda} \cdot
  \left( i \cdot g_2(s - L \cdot \mathcal{G}(s/L)) \cdot g_{11} \cdot \exp(-c \cdot L/2) \\
  &\quad + i \cdot g_2(s - L \cdot \mathcal{G}(s/L) - L) \cdot g_{11} \cdot \exp(c \cdot L/2) \right),
\end{align*}
\tag{3.49}
$$

\[47\]This follows from the form of the matrix $A_{\text{orb}}$.  

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where
\[ g_{11} \equiv \frac{1}{\cosh(c \cdot L/2) + \cos(\lambda \cdot L)} . \]

This can be checked by showing that the expressions in (3.49) solve (3.43) and are 2-turn periodic in \( s \). Using (3.36), (3.46) and (3.49) to fix \((g_9, g_{10})\) at \( s = 0 \) one then obtains:

\[
\frac{i \cdot d \cdot g_{11} \cdot \sigma_{\eta}^2}{2 \cdot \lambda \cdot b} \begin{pmatrix}
g_9(s) \\
g_{10}(s)
\end{pmatrix} = \begin{pmatrix}
g_1(s) \cdot [\sinh(c \cdot L/2) + \cos(\lambda \cdot L)] - g_1(s - L) \cdot \exp(c \cdot L/2) \\
-b \cdot g_2(s) \cdot [\sinh(c \cdot L/2) + \cos(\lambda \cdot L)] + b \cdot g_2(s - L) \cdot \exp(c \cdot L/2)
\end{pmatrix},
\]

(3.50)

and it is simple to confirm that the expression in (3.50) solves (3.47). Combining (3.46), (3.49), (3.50) one finally has the explicit forms:

\[
\sigma_{3,13}(s) = \frac{\sigma_{\eta}^2}{2 \cdot \lambda \cdot b} \cdot \left( i \cdot \hat{d}(s) \cdot g_1(s - L \cdot \mathcal{G}(s/L)) \cdot g_{11} \cdot \exp(-c \cdot L/2) + i \cdot \hat{d}(s) \cdot g_2(s - L \cdot \mathcal{G}(s/L) - L) \cdot g_{11} \cdot \exp(c \cdot L/2) - 2 \cdot \lambda \cdot \hat{d}(s) - i \cdot g_1(s) \cdot g_{11} \cdot \exp(-c \cdot L/2) - i \cdot g_1(s - L) \cdot g_{11} \cdot \exp(c \cdot L/2) + i \cdot g_1(s) \right),
\]

\[
\sigma_{3,23}(s) = \frac{\sigma_{\eta}^2}{2 \cdot \lambda} \cdot \left( -i \cdot \hat{d}(s) \cdot g_2(s - L \cdot \mathcal{G}(s/L)) \cdot g_{11} \cdot \exp(-c \cdot L/2) - i \cdot \hat{d}(s) \cdot g_2(s - L \cdot \mathcal{G}(s/L) - L) \cdot g_{11} \cdot \exp(c \cdot L/2) + i \cdot g_2(s) \cdot g_{11} \cdot \exp(-c \cdot L/2) + i \cdot g_2(s - L) \cdot g_{11} \cdot \exp(c \cdot L/2) - i \cdot g_2(s) \right).
\]

(3.51)

This can be checked by showing that the expressions (3.51) solve (3.22) and obey:

\[
\lim_{0<s\to0} [\sigma_{3,13}(s)] = \lim_{0<s\to0} [\sigma_{3,23}(s)] = 0.
\]

Note also that for \( 0 \leq s \leq L \) one has:

\[
\sigma_{3,13}(s) = \sigma_{2,13}(s), \quad \sigma_{3,23}(s) = \sigma_{2,23}(s),
\]

which also follows from the fact that processes 2 and 3 are identical for \( 0 \leq s \leq L \).

3.5.3

Coming finally to \( \sigma_{3,33} \) I first of all get from (3.22), (3.23), (3.39):

\[
\sigma_{3,33}(s) = \sigma_{3,33}(0) + \int_0^s ds_1 \cdot \sigma'_{3,33}(s_1) = \int_0^s ds_1 \cdot \sigma'_{3,33}(s_1) = 2 \cdot \int_0^s ds_1 \cdot \sigma_{3,23}(s_1) \cdot \hat{d}(s_1),
\]

(3.52)
\[ \sigma_{3,33}(s) = g_{12}(s) + g_{13}(s), \quad (3.53) \]

where

\[ g_{12}(s) = 2 \cdot \int_0^s ds_1 \cdot g_8(s_1) \cdot \hat{d}(s_1), \quad g_{13}(s) = 2 \cdot \int_0^s ds_1 \cdot g_{10}(s_1) \cdot \hat{d}(s_1). \quad (3.54) \]

By straightforward integrations one then obtains:

\[ g_{14}(s) = g_{14}(s) + s \cdot g_{15}, \quad (3.55) \]
\[ g_{16}(s) = \frac{i \cdot \hat{d} \cdot \sigma^2}{\lambda} \cdot \left( \frac{1 - \exp(\lambda_1 \cdot L)}{1 + \exp(\lambda_1 \cdot L)} \cdot g_{16}(s) + \text{c.c.} \right). \quad (3.56) \]

where:

\[ g_{14}(s) = \frac{i \cdot \hat{d} \cdot \sigma^2}{\lambda \cdot a \cdot b} \cdot \left( g_1(s - G(s/L)) \cdot \exp(-c \cdot L/2) + g_1(s - G(s/L) - L) \cdot \exp(c \cdot L/2) - g_1(0) \cdot \exp(-c \cdot L/2) - g_1(-L) \cdot \exp(c \cdot L/2) \right) - [s - G(s/L)] \cdot g_{15}, \quad (3.57) \]
\[ g_{15} = 2 \cdot \frac{d^2 \cdot g_{11} \cdot \sigma^2}{a \cdot b \cdot L \cdot \lambda} \cdot \left( 2 \cdot \lambda \cdot \sinh(c \cdot L/2) - c \cdot \sin(\lambda \cdot L) \right), \quad (3.58) \]
\[ g_{16}(s) = \frac{1 \cdot \hat{d}(s) \cdot \exp(\lambda_1 \cdot s) - d}{\lambda_1} + \frac{2 \cdot \hat{d}}{\lambda_1} \cdot \exp(\lambda_1 \cdot L) \cdot \frac{1 - \exp\left( 2 \cdot L \cdot \lambda_1 \cdot G(s/2L) \right)}{1 + \exp(\lambda_1 \cdot L)} + \frac{2 \cdot d}{\lambda_1} \cdot [G(s/2L + 1/2) - G(s/2L)] \cdot \exp\left( \lambda_1 \cdot (2L \cdot G(s/2L) + L) \right). \quad (3.59) \]

One sees by (3.53) that after a few orbital damping times \( g_{16} \) becomes constant so that by (3.56) \( g_{13} \) becomes constant, too. Also one observes that \( g_{14}(s) \) is 1-turn periodic in \( s \) because the function \( G(s/L) - s \) is 1-turn periodic in \( s \). Hence after a few orbital damping times one gets:

\[ \sigma_{3,33}(s) \approx g_{13}(+\infty) + g_{14}(s) + s \cdot g_{15}, \quad (3.60) \]

so that \( \sigma_{3,33}(s) \) quickly splits up additively into a term increasing linearly with \( s \) plus a term 1-turn periodic in \( s \). The expression (3.53) can be checked by showing that it solves (3.28) and obeys:

\[ \lim_{0<s \to 0} \left[ \sigma_{3,33}(s) \right] = 0. \]

This completes the calculation of the covariance matrix, whose matrix elements are given by (3.18), (3.51), (3.53). Note that due to \( G(s) \) the functions \( \sigma_{3,13}(s), \sigma_{3,23}(s), \sigma_{3,33}(s) \) at first sight are undetermined at the points where \( s/L = \text{integer} \). Nevertheless due to their continuity in \( s \) (see section 3.4.2) they are well defined at these points.

With (3.18), (3.40), (3.51), (3.53) I have determined the first moment vector and the covariance matrix of Process 3 so that the probability density is fixed.
From \((3.27),(3.35)\) follows:

\[
\vec{P}_{\text{tot}}^{w3}(s) = \exp(-\sigma_{3,33}(s)/2) \cdot \begin{pmatrix}
1 \\
0 \\
0
\end{pmatrix}.
\] (3.61)

By \((3.53),(3.61)\) the polarization of Process 3 is given by

\[
||\vec{P}_{\text{tot}}^{w3}(s)|| = \exp\left(-\frac{\sigma_{3,33}(s)}{2}\right) \cdot \exp\left(-\frac{g_{13}(s)}{2}\right) \cdot \exp\left(-\frac{g_{14}(s)}{2}\right) \cdot \exp\left(-\frac{s \cdot g_{15}}{2}\right).
\] (3.62)

By \((3.62)\) one sees that after a few orbital damping times the polarization reads as:

\[
||\vec{P}_{\text{tot}}^{w3}(s)|| \approx \exp\left(-\frac{g_{13}(s)}{2}\right) \cdot \exp\left(-\frac{g_{14}(s)}{2}\right) \cdot \exp\left(-\frac{s \cdot g_{15}}{2}\right).
\]

At long times this is the product of a factor which has the period of the ring and an exponentially decaying factor. Because the factor \(\exp(-g_{13}(s)/2)\) is not constant from the beginning, \(||\vec{P}_{\text{tot}}^{w3}||\) contains a 'transient' contribution which later damps away.

Due to the factor \(\exp(-s \cdot g_{15}/2)\) one observes:

\[
\lim_{s \to +\infty} [\sigma_{3,33}(s)] = +\infty,
\]

i.e. one has complete spin decoherence of Process 3:

\[
||\vec{P}_{\text{tot}}^{w3}(+\infty)|| = 0.
\]

I define the depolarization rate as:

\[
\frac{1}{\tau_{\text{spin}}} \equiv \frac{g_{15}}{2} > 0.
\] (3.63)

If one specifies the constants according to \((2.12)\) one gets

\[
\tau_{\text{spin}} \approx 7.6 \cdot 10^7 m,
\] (3.64)

which corresponds to 12000 turns, i.e. about 260 milliseconds. If on the other hand \(Q_{s}\) (and therefore \(\lambda\)) were close to half of an integer \(g_{15}\) would, because of its factor \(g_{11}\), become very large and \(\tau_{\text{spin}}\) would be very small. This is exactly what one expects when sitting close to a spin-orbit resonance \([3HMR91]\).

Process 3 is a rough model for the behaviour of the polarization after injection into the rings mentioned in section 3.1 and it is therefore interesting to study the transient behaviour in \((3.62)\). To come to that I study the complicated azimuthal dependence of the polarization 'turn by turn', i.e. I consider its behaviour with increasing number of turns and thus investigate the sequence \(\sigma_{3,33}(2NL)\), where \(N\) is a nonnegative integer. My 'observation point' is at the azimuth \(s = 2NL\), i.e. at the snake after every second turn. \[48\]

First of all one gets

\[
\sigma_{3,33}(2NL) = g_{13}(2NL) + g_{14}(2NL) + 2NL \cdot g_{15}.
\] (3.65)

\[48\]I consider the sequence \(\sigma_{3,33}(2NL)\) instead of \(\sigma_{3,33}(NL)\) because of mathematical convenience.
Figure 5: The main term $\sigma_{3,33,\text{main}}(2NL)$ of $\sigma_{3,33}(2NL)$ for the first 1000 turns of Process 3 assuming the HERA values (2.12)

This can be simplified because $g_{14}(s)$ is 1-turn periodic in $s$ so that:

$$g_{14}(2NL) = g_{14}(0) .$$

Also one has by (3.57):

$$g_{14}(0) = 0 ,$$

so that (3.63) simplifies to:

$$\sigma_{3,33}(2NL) = \sigma_{3,33,\text{inter}}(2NL) + \sigma_{3,33,\text{main}}(2NL) ,$$

where

$$\sigma_{3,33,\text{main}}(s) \equiv s \cdot g_{15} , \quad \sigma_{3,33,\text{inter}}(s) \equiv g_{13}(s) .$$

Inserting (3.66) into (3.62) yields:

$$||\vec{P}_{\text{tot}}(2NL)|| = \exp\left(-\sigma_{3,33,\text{inter}}(2NL)/2\right) \cdot \exp\left(-\sigma_{3,33,\text{main}}(2NL)/2\right) .$$

$\text{inter} \equiv \text{‘intermediate’}$ which expresses that it affects the polarization only at the beginning of Process 3.
Figure 6: The intermediate term \( \sigma_{3,33,\text{inter}}(2NL) \) of \( \sigma_{3,33}(2LN) \) for the first 1000 turns of Process 3 assuming the HERA values (2.12).

All transient behaviour is contained in the first factor on the rhs of (3.68) and as one has seen this term converges to a finite value after a few orbital damping times. Unfortunately, polarimeters are usually not fast enough to measure the transients. To look at \( \sigma_{3,33,\text{inter}}(2NL) \) in more detail I use (3.59) to calculate:

\[
\frac{g_{16}(2NL)}{g_{13}(2NL)} = \frac{d}{\lambda_1} \cdot \exp(\lambda_1 \cdot 2NL) - \frac{d}{\lambda_1} \cdot \exp(\lambda_1 \cdot L) \cdot \frac{1 - \exp(\lambda_1 \cdot 2NL)}{1 + \exp(\lambda_1 \cdot L)}
\]

\[
= \frac{d}{\lambda_1} \cdot \left( \exp(\lambda_1 \cdot 2NL) \cdot \left[ 1 - \frac{2 \cdot \exp(\lambda_1 \cdot L)}{1 + \exp(\lambda_1 \cdot L)} \right] - 1 + \frac{2 \cdot \exp(\lambda_1 \cdot L)}{1 + \exp(\lambda_1 \cdot L)} \right)
\]

\[
= \frac{d}{\lambda_1} \cdot \frac{1 - \exp(\lambda_1 \cdot L)}{1 + \exp(\lambda_1 \cdot L)} \cdot [\exp(\lambda_1 \cdot 2NL) - 1], \tag{3.69}
\]

from which it follows by (3.56) that:

\[
\sigma_{3,33,\text{inter}}(2NL) = g_{13}(2NL) = \frac{i \cdot d \cdot \sigma_\eta^2}{\lambda} \cdot \frac{1 - \exp(\lambda_1 \cdot L)}{1 + \exp(\lambda_1 \cdot L)} \cdot g_{16}(2NL) + \text{c.c.}
\]

\[
= \frac{i \cdot d^2 \cdot \sigma_\eta^2}{\lambda \cdot \lambda_1} \cdot \left( \frac{1 - \exp(\lambda_1 \cdot L)}{1 + \exp(\lambda_1 \cdot L)} \right)^2 \cdot [\exp(\lambda_1 \cdot 2NL) - 1] + \text{c.c.}
\]
Figure 7: Polarization $||\vec{F}_{tot}^{\mu \nu}(2NL)||$ of Process 3 for the first 1000 turns assuming the HERA values $<2.12>$. 

$$
= \frac{d^2 \cdot \sigma_3^2}{\lambda} \left( i \cdot g_{17} \cdot [1 - \exp(c \cdot NL) \cdot \cos(\lambda \cdot 2NL)] \\
- g_{18} \cdot \exp(c \cdot NL) \cdot \sin(\lambda \cdot 2NL) \right) \right),
$$

where

$$
g_{17} \equiv -\frac{i}{a \cdot b} \cdot [\cosh(c \cdot L/2) + \cos(\lambda \cdot L)]^{-2} \\
\cdot \left( \lambda \cdot \cosh(c \cdot L) + \lambda \cdot \cos(\lambda \cdot 2L) - 2 \cdot \lambda - 2 \cdot c \cdot \sinh(c \cdot L/2) \cdot \sin(\lambda \cdot L) \right),
$$

$$
g_{18} \equiv -\frac{1}{2 \cdot a \cdot b} \cdot [\cosh(c \cdot L/2) + \cos(\lambda \cdot L)]^{-2} \\
\cdot \left( c \cdot \cosh(c \cdot L) + c \cdot \cos(\lambda \cdot 2L) - 2 \cdot c + 8 \cdot \lambda \cdot \sinh(c \cdot L/2) \cdot \sin(\lambda \cdot L) \right).
$$

For the HERA values $<2.12>$ the main term $\sigma_{3,33,\text{main}}(2NL)$ and the intermediate term $\sigma_{3,33,\text{inter}}(2NL)$ are displayed in figure 5 and figure 6 for the first 1000 turns. One sees by comparing figure 5 with figure 6 that for the first few hundreds of turns the intermediate
term dominates the polarization. However the intermediate term is so small that it never seriously degrades the polarization. The strongest effect is a degradation of the polarization value to 0.93 after 8 turns. This can be seen in figure 7 where the polarization is displayed for the first 1000 turns. The polarization is also displayed for the first 20000 turns in figure 8. Finally in figure 9 the polarization is displayed for the limiting case, where $c, \omega \rightarrow 0$ with $\omega/c = \text{const} = -2 \cdot \sigma_\eta^2 \approx -2.0 \cdot 10^{-6}$. I call this ‘Process 3a’. Figure 9 shows that in the absence of radiation and when only synchrotron motion is considered the snake holds the polarization within narrow limits. This is a kind of spin echo effect [Abr61].

Comparing figure 7 and figure 8 with figure 3 one also sees that for the first few thousands of turns the polarization of Process 3 is much larger than for Process 2 showing that as expected the snake can strongly suppress oscillations in the spin distribution. However, and this is at first unexpected when recalling the equilibrium reached by Process 2, in the end there is complete decoherence for Process 3. But, on the other hand, one should not be surprised when one recalls that the calculations with SLIM [Cha81] for a perfectly aligned flat ring with a pointlike radial snake in which only spin diffusion due to synchrotron motion generated in the arcs is included, also predict complete depolarization [Bar97]. Similar calculations also show that spin diffusion due to horizontal betatron motion in the arcs is very much less than that due to synchrotron motion.
3.5.5

Another illustration of the transient behaviour of Process 3 is provided by calculating the polarization after one turn. During the first turn Process 3 is identical to Process 2 so that by (2.101) one gets:

$$||\vec{P}_{tot}^{w3}(L)|| = ||\vec{P}_{tot}^{w2}(L)|| = \exp\left(-\sigma_{2,33}(L)/2\right) = \exp\left(-\frac{d^2}{2a^2\lambda} \cdot \sigma^2 \cdot \left[2 \cdot \lambda - i \cdot g_1(L)\right]\right).$$

Thus due to its transient behaviour the polarization of Process 3 behaves during the first turn as if it decays exponentially with the naive depolarization time given by

$$\frac{2a^2\lambda L}{d^2 \cdot \sigma^2} \cdot \left(2 \cdot \lambda - i \cdot g_1(L)\right)^{-1},$$

which is quite different from $\tau_{\text{spin}}$. In fact assuming the HERA values (2.12) it results in

$$\frac{2a^2\lambda L}{d^2 \cdot \sigma^2} \cdot \left(2 \cdot \lambda - i \cdot g_1(L)\right)^{-1} \approx 85000 \text{ m}.$$
By (3.19), (3.29), (3.30), (3.31), (3.40) the polarization density, the local polarization, and the direction of the local polarization of Process 3 read as:

$$||\vec{P}_{loc}^w(\sigma, \eta; s)|| = \exp \left( -\frac{\det(\sigma_3)}{2 \cdot \sigma_2^2 \cdot \sigma_1^2} \right) \cdot \exp \left( \frac{\sigma_{3,13}(s)}{2 \cdot \sigma_2^2} + \frac{\sigma_{3,23}(s)}{2 \cdot \sigma_1^2} \right), \quad (3.71)$$

$$\vec{P}_{dir}^w_3(\sigma, \eta; s) = \begin{pmatrix} \cos \left( \frac{\sigma_{3,13}(s)}{\sigma_2^2} \cdot \sigma + \frac{\sigma_{3,23}(s)}{\sigma_1^2} \cdot \eta \right) \\ \sin \left( \frac{\sigma_{3,13}(s)}{\sigma_2^2} \cdot \sigma + \frac{\sigma_{3,23}(s)}{\sigma_1^2} \cdot \eta \right) \\ 0 \end{pmatrix}, \quad (3.72)$$

$$\vec{P}_3^w(\sigma, \eta; s) = w_{norm}(\sigma, \eta) \cdot ||\vec{P}_{loc}^w(\sigma, \eta; s)|| \cdot \vec{P}_{dir}^w(\sigma, \eta; s). \quad (3.73)$$

The local polarization starts from the value 1 at $s = 0$ and decreases towards its vanishing equilibrium value.

With the local polarization quantities at hand one can reconsider the transient behaviour of Process 3 in more detail. After a few orbital damping times the transient behaviour disappears so that the local polarization quantities acquire certain periodicity properties w.r.t. $s$. In fact from section 3.5.2 it is clear that on this time scale $\sigma_{3,13}(s), \sigma_{3,23}(s)$ become 2-turn periodic in $s$ and change sign from turn to turn:

$$g_7(s) = -g_7(s + L), \quad g_8(s) = -g_8(s + L).$$

Hence by (3.72) the local polarization direction becomes 2-turn periodic in $s$ in the $(\vec{n}_{0,II}, \vec{m}_{0,II}, \vec{l}_{0,II})$-frame. Since in the machine frame $\vec{m}_{0,II}$ also changes sign from turn to turn, the local polarization direction becomes 1-turn periodic in the machine frame after a few orbital damping times. Note that at the phase space point where $\sigma = \sigma_\sigma$ and $\eta = \sigma_\eta$ and for the HERA values (2.12) the $\vec{n}_0$-axis deviates at the snake by about 200 milliradians from the asymptotic local polarization direction.

Moreover in section 3.5.3 I observed that $\sigma_{3,33}(s)$ quickly splits up additively into a term increasing linearly with $s$ plus a term 1-turn periodic in $s$. Hence by (3.71) the local polarization factors into an exponentially decaying part and a part 1-turn periodic in $s$. [50]

I now round off section 3 by considering two processes with contrasting transient behaviour, one of which shows no transients and one which will turn out to illustrate very nicely the validity of a tenet at the basis of the standard method of calculating the rate of depolarization. Both processes are G-processes at orbital equilibrium.

### 3.6 The probability density of Process 4

#### 3.6.1

For Process 3 I found that during the first few orbital damping times transient behaviour prevents an exponential decay of the polarization. Knowing this it is now simple to define a modification of Process 3 which shows no transient behaviour of the polarization properties at any $s$ and which from the beginning has those periodicity properties which Process 3 acquires.

[50] In fact one can use these properties of the local polarization and its direction as the definition of a transient free process.
only after a few orbital damping times. For this process $\vec{x}^{(4)}(s)$, called ‘Process 4’, the exponential decay of the polarization shows up right from the beginning because no transient terms destroy the exponential structure. I abbreviate:

$$\vec{x}^{(4)}(s) \equiv \begin{pmatrix} \sigma^{(4)}(s) \\ \eta^{(4)}(s) \\ \psi^{(4)}(s) \end{pmatrix}.$$  

Process 4 is characterized by the following four conditions:

- It is a G-process at orbital equilibrium and its starting azimuth is $s_0 = 0$.
- The direction of the local polarization is 2-turn periodic in $s$ in the $(\vec{n}_{0,II}, \vec{m}_{0,II}, \vec{l}_{0,II})$-frame and 1-turn periodic in the machine frame.
- The direction of the local polarization on the closed orbit is parallel to the $\vec{n}_0$-axis of Machine II.
- Its initial local polarization is 1.

The second condition ensures that the process is free of transients.

By the third condition one has:

$$\vec{P}_{w4}^{\text{dir}}(0, 0; s) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},$$

where the probability density of Process 4 is denoted by $w_4$. Thus one has by (3.30), (3.74):

$$< \psi^{(4)}(0) > = \psi_{0,m},$$

where $\psi_{0,m}$ is given by (3.38). From (3.17), (3.75) follows:

$$< \vec{x}^{(4)}(s) >= < \vec{x}^{(4)}(0) >= (0, 0, \psi_{0,m})^T.$$  

Also, due to the second condition on Process 4 one observes by (3.30) that $\sigma_{4,13}(s), \sigma_{4,23}(s)$ are 2-turn periodic in $s$, where $\sigma_4$ denotes the covariance matrix of Process 4. Because $\sigma_{4,13}(s), \sigma_{4,23}(s)$ obey (3.22) and are 2-turn periodic in $s$, one concludes by section 3.5.2 that:

$$\begin{pmatrix} \sigma_{4,13}(s) \\ \sigma_{4,23}(s) \end{pmatrix} = \begin{pmatrix} g_7(s) \\ g_8(s) \end{pmatrix},$$

where $g_7, g_8$ are given by (3.49).

Coming to $\sigma_{4,33}$ one first of all gets by using (3.23), (3.54), (3.55), (3.77):

$$\begin{align*}
\sigma_{4,33}(s) &= \sigma_{4,33}(0) + \int_0^s ds_1 \cdot \sigma'_{4,33}(s_1) = \sigma_{4,33}(0) + 2 \cdot \int_0^s ds_1 \cdot \sigma_{4,23}(s_1) \cdot \hat{d}(s_1) \\
&= \sigma_{4,33}(0) + 2 \cdot \int_0^s ds_1 \cdot g_8(s_1) \cdot \hat{d}(s_1) \\
&= \sigma_{4,33}(0) + g_{14}(s) + s \cdot g_{15} = \sigma_{4,33}(0) + g_{14}(s) + \frac{2 \cdot s}{\tau_{\text{spin}}},
\end{align*}$$

(3.78)
so that $\sigma_{4,33}(s)$ separates additively into a part 1-turn periodic in $s$ and a part linear in $s$. Now I have exploited the first three conditions on Process 4 and to fix $\sigma_{4,33}$ I now impose the fourth condition which by (3.29) reads as:

$$\det(\sigma_{4}(0)) = 0.$$  \hfill (3.79)

From (3.19), (3.77), (3.79) follows:

$$\sigma_{4,33}(0) = \frac{\sigma_{4,13}(0)}{\sigma_{\sigma}} + \frac{\sigma_{4,23}(0)}{\sigma_{\eta}} = \frac{g_{7}^{2}(0)}{\sigma_{\sigma}} + \frac{g_{8}^{2}(0)}{\sigma_{\eta}} ,$$  \hfill (3.80)

which fixes $\sigma_{4,33}(0)$. Inserting this into (3.78) yields:

$$\sigma_{4,33}(s) = \frac{g_{7}^{2}(0)}{\sigma_{\sigma}} + \frac{g_{8}^{2}(0)}{\sigma_{\eta}} + g_{14}(s) + s \cdot g_{15} = \frac{g_{7}^{2}(0)}{\sigma_{\sigma}} + \frac{g_{8}^{2}(0)}{\sigma_{\eta}} + g_{14}(s) + \frac{2 \cdot s}{\tau_{\text{spin}}} .$$  \hfill (3.81)

With (3.18), (3.76), (3.77), (3.81) I have determined the first moment vector and the covariance matrix of Process 4. By (3.32), (3.75) the characteristic function $\Phi_{4}$ corresponding to $w_{4}$ reads
as:

\[ \Phi_4(\vec{u}; s) = \exp\left( -\frac{1}{2} \cdot \sum_{j,k=1}^{3} \sigma_{4,jk}(s) \cdot u_j \cdot u_k + i \cdot \psi_{0,m} \cdot u_3 \right). \]  

(3.82)

Therefore to prove that the above construction constitutes a G-process I just have to show that the covariance matrix is nonsingular for \( s > 0 \). This can be done in analogy to Process 3. In fact from (3.21), (3.79) it follows that:

\[ \det(\sigma_4(s)) = -2 \cdot c \cdot \sigma_{\sigma}^2 \cdot \int_{0}^{s} ds_1 \cdot \sigma_{4,23}^2(s_1), \]  

(3.83)

and by (3.77) one has

\[ \sigma_{4,23}(0) \neq 0. \]

Hence by (3.83) \( \sigma_4 \) is nonsingular for \( s > 0 \), confirming that Process 4 is a G-process. Thus the probability density reads for \( s > 0 \) as:

\[ w_4(\sigma, \eta, \psi; s) = \sqrt{(2\pi)^{-3} \cdot \det(\sigma_4(s))^{-1}} \cdot \exp\left[ -\frac{1}{2} \cdot \left( \begin{array}{c} \sigma \\ \eta \\ \psi - \psi_{0,m} \end{array} \right)^T \sigma_4^{-1}(s) \cdot \left( \begin{array}{c} \sigma \\ \eta \\ \psi - \psi_{0,m} \end{array} \right) \right], \]  

(3.84)

and by using (2.28),(3.82) and the expressions for the first moment vector and the covariance matrix it reads at \( s = 0 \) as:

\[ w_4(\sigma, \eta, \psi; 0) = w_{\text{norm}}(\sigma, \eta) \cdot \delta(\psi - \frac{\sigma_{4,13}(0)}{\sigma_{\sigma}^2} \cdot \sigma - \frac{\sigma_{4,23}(0)}{\sigma_{\eta}^2} \cdot \eta - \psi_{0,m}). \]  

(3.85)

By (3.84), (3.85) \( w_4 \) fulfills the normalization condition (2.20) and the orbital part of \( w_4 \) obeys:

\[ w_{4,\text{orb}} = w_{\text{norm}}, \]

confirming that Process 4 is at orbital equilibrium. Note that \( \vec{x}^{(4)}(s) \) is a Markovian diffusion process.

### 3.6.2

By (3.27),(3.76) the polarization vector reads as:

\[ \vec{P}_{\text{tot}}^w(s) = \exp(-\sigma_{4,33}(s)/2) \cdot \left( \begin{array}{c} 1 \\ 0 \\ 0 \end{array} \right), \]  

(3.86)

so that by (3.78) the polarization is:

\[ ||\vec{P}_{\text{tot}}^w(s)|| = \exp(-\sigma_{4,33}(s)/2) = \exp(-\sigma_{4,33}(0)/2 - g_{14}(s)/2) \cdot \exp(-s/\tau_{\text{spin}}), \]

(3.87)

where \( \sigma_{4,33}(0) \) is given by (3.80). Thus Process 4 exhibits complete spin decoherence:

\[ ||\vec{P}_{\text{tot}}^w(\infty)|| = 0. \]  

(3.88)
Figure 11: Polarization $||\vec{P}_{\text{tot}}^{4a}(2NL)||$ of Process 4a for the first 1000 turns assuming the HERA values $2.12$, except that $c, \omega \to 0$ with $\omega/c = \text{const} = -2 \cdot \sigma_\eta^2 \approx -2.0 \cdot 10^{-6}$.

Note that by (3.87) the initial polarization of Process 4 is not complete, so that the initial direction of the local polarization is not uniform. This will be confirmed below.

By (3.19), (3.29), (3.77), (3.78) the local polarization reads as:

$$||\vec{P}_{\text{loc}}^{4}(\sigma, \eta; s)|| = \exp\left(-\frac{\det(\vec{g}_4(s))}{2 \cdot \sigma_\sigma^2 \cdot \sigma_\eta^2}\right) = \exp\left(-\frac{\sigma_{4,33}(s)}{2}\right) \cdot \exp\left(\frac{\sigma_{4,13}^2(s)}{2 \cdot \sigma_\sigma^2} + \frac{\sigma_{4,23}^2(s)}{2 \cdot \sigma_\eta^2}\right)$$

$$= \exp\left(-\frac{s}{\tau_{\text{spin}}}\right) \cdot \exp\left(\frac{\sigma_{4,13}^2(s)}{2 \cdot \sigma_\sigma^2} + \frac{\sigma_{4,23}^2(s)}{2 \cdot \sigma_\eta^2} - \frac{\sigma_{4,33}(0)}{2} - \frac{g_{14}(s)}{2}\right)$$

$$= \exp\left(-\frac{s}{\tau_{\text{spin}}}\right) \cdot \exp\left(\frac{g_7^2(s)}{2 \cdot \sigma_\sigma^2} + \frac{g_8^2(s)}{2 \cdot \sigma_\eta^2} - \frac{\sigma_{4,33}(0)}{2} - \frac{g_{14}(s)}{2}\right).$$

As for every G-process the local polarization of Process 4 is uniform. It starts from the value 1 at $s = 0$ and decreases towards its vanishing equilibrium value.
By (3.30), (3.75), (3.77) the direction of the local polarization reads as:

\[
\vec{P}_{\text{dir}}^{w_4}(\sigma, \eta; s) = \begin{pmatrix}
\cos \left( \frac{g_7(s)}{\sigma_2^2} \cdot \sigma + \frac{g_8(s)}{\sigma_2^2} \cdot \eta \right) \\
\sin \left( \frac{g_7(s)}{\sigma_2^2} \cdot \sigma + \frac{g_8(s)}{\sigma_2^2} \cdot \eta \right) \\
0
\end{pmatrix},
\]

which is 2-turn periodic in \(s\). Also it is 1-turn periodic in the machine frame. By (3.31), (3.89), (3.90) the polarization density is given by:

\[
\vec{P}^{w_4}(\sigma, \eta; s) = w_{\text{norm}}(\sigma, \eta) \cdot \exp \left( -\frac{s}{\tau_{\text{spin}}} \right) \cdot \exp \left( \frac{g_7^2(s)}{2 \cdot \sigma_2^2} + \frac{g_8^2(s)}{2 \cdot \sigma_2^2} - \frac{\sigma_{4,33}(0)}{2} - \frac{g_{14}(s)}{2} \right) \cdot
\]

\[
\begin{pmatrix}
\cos \left( \frac{g_7(s)}{\sigma_2^2} \cdot \sigma + \frac{g_8(s)}{\sigma_2^2} \cdot \eta \right) \\
\sin \left( \frac{g_7(s)}{\sigma_2^2} \cdot \sigma + \frac{g_8(s)}{\sigma_2^2} \cdot \eta \right) \\
0
\end{pmatrix}.
\]

The observed periodicity properties of the local polarization quantities of Process 4 show the lack of any transient behaviour of Process 4.

### 3.6.3

To compare processes 3 and 4 in more detail one observes by (3.48), (3.77) that after a few orbital damping times one gets:

\[
\sigma_{4,13}(s) \approx \sigma_{3,13}(s), \quad \sigma_{4,23}(s) \approx \sigma_{3,23}(s).
\]

Applying this to (3.30) one sees that on this time scale the direction of the local polarization of Process 4 becomes the same as that of Process 3.

Also by (3.53), (3.81) one observes that:

\[
\sigma_{4,33}(s) - \sigma_{3,33}(s) \approx \sigma_{4,33}(0) - g_{13}(+\infty) = \sigma_{4,33}(0) - \frac{i \cdot d^2 \cdot \lambda}{\sigma_2^2} \cdot \sigma_{4,17}.
\]

Combining (3.19), (3.92), (3.93) one gets:

\[
\det \left( \sigma_4(s) \right) - \det \left( \sigma_3(s) \right) \approx \sigma_2^2 \cdot \sigma_2^2 \cdot \left[ \sigma_{4,33}(s) - \sigma_{3,33}(s) \right]
\approx \sigma_2^2 \cdot \sigma_2^2 \cdot \sigma_{4,33}(0) - \frac{i \cdot d^2 \cdot \sigma_2^2 \cdot \sigma_2^2}{\lambda} \cdot g_{17},
\]

so that by (3.29), (3.94) one gets after some orbital damping times:

\[
||\vec{P}^{w_4}(\sigma, \eta; s)|| \approx \exp \left( -\frac{\sigma_{4,33}(0)}{2} + \frac{i \cdot d^2}{2 \cdot \lambda} \cdot g_{17} \right) \cdot ||\vec{P}^{w_3}_{\text{loc}}(\sigma, \eta; s)||.
\]

Hence after a few orbital damping times one finds that the local polarization of Process 3 is proportional to the local polarization of Process 4. One thus sees that the local polarization, as well as the polarization, is the product of a 1-turn periodic factor and an exponentially decaying
factor with the same depolarization rate $1/\tau_{\text{spin}}$ as was observed for the long term behaviour of Process 3. Furthermore the polarization densities are proportional:

$$\vec{P}^{w_4}(\sigma, \eta; s) \approx \exp\left(-\frac{\sigma_{33}(0)}{2} + \frac{i \cdot d^2 \cdot \sigma^2}{2 \cdot \lambda} \cdot g_{17}\right) \cdot \vec{P}^{w_3}(\sigma, \eta; s).$$  (3.96)

For the HERA values (2.12) the polarization of Process 4 is displayed in figure 10 for the first 20000 turns. In figure 11 the polarization is displayed for the limiting case, where $c, \omega \to 0$ with $\omega/c = \text{const} = -2 \cdot \sigma^2 \approx -2.0 \cdot 10^{-6}$. I call this ‘Process 4a’. Note that the polarization is constant from turn to turn, i.e. 1-turn periodic. I return to this point at the end of section 3.

3.6.4

One easily observes that all processes which fulfill the first three conditions for Process 4 are devoid of transient behaviour. In fact one observes for those processes:

$$||\vec{P}^w_{\text{loc}}(\sigma, \eta; s)|| = \exp\left(-\frac{s}{\tau_{\text{spin}}} \right) \cdot \exp\left(\frac{g_{7}^2(s)}{2 \cdot \sigma^2} + \frac{g_{8}^2(s)}{2 \cdot \sigma^2} - \frac{\sigma_{33}(0)}{2} - \frac{g_{14}(s)}{2}\right),$$

$$\vec{P}^w_{\text{dir}}(\sigma, \eta; s) = \begin{pmatrix} \cos \left(\frac{g_{7}(s)}{\sigma^2} \cdot \sigma + \frac{g_{8}(s)}{\sigma^2} \cdot \eta\right) \\ \sin \left(\frac{g_{7}(s)}{\sigma^2} \cdot \sigma + \frac{g_{8}(s)}{\sigma^2} \cdot \eta\right) \\ 0 \end{pmatrix}.$$ 

In this class of processes Process 4 is the one with the largest polarization, i.e. the polarization $||\vec{P}^w_{\text{tot}}||$ of those processes obeys:

$$||\vec{P}^w_{\text{tot}}(s)|| \leq ||\vec{P}^{w_4}_{\text{tot}}(s)||.$$

3.7 The probability density of Process 5

3.7.1

In this section I consider the process $\vec{x}^{(5)}(s)$, called ‘Process 5’, whose initial state was used in several studies of the past [DK72, BHMR91]. I abbreviate:

$$\vec{x}^{(5)}(s) \equiv \begin{pmatrix} \sigma^{(5)}(s) \\ \eta^{(5)}(s) \\ \psi^{(5)}(s) \end{pmatrix}.$$ 

Process 5 is characterized by the following three conditions:

- It is a G-process at orbital equilibrium and its starting azimuth is $s_0 = 0$.

- The direction of the initial local polarization is given by the $\vec{n}$-axis of Machine II [DK72, HH96].

- Its initial local polarization is 1.
For my two-dimensional orbital phase space the $\vec{n}$-axis, denoted by $\vec{n}_{II}$ (see Appendix B), is a unit-vector periodic solution of the radiationless Bloch equation (B.1) satisfying the condition:

$$\vec{n}_{II}(\sigma, \eta; s) = \vec{n}_{II}(\sigma, \eta; s + L),$$

in the machine frame and it is a key component of the standard method for calculating the rate of depolarization by perturbation theory in real rings. In the absence of radiation an ensemble, for which the local polarization direction is parallel to the $\vec{n}$-axis and for which the local polarization equals 1, remains in this state. See the discussion in [HH96]. In the presence of radiation it is usually assumed [DK73, Man87, BHMR91] that the local polarization direction remains parallel to the $\vec{n}$-axis at each point in phase space and it is this assumption that provides a way to calculate the depolarization rate for real rings using perturbation theory. In this section I will use my exact analytical methods to check these assumptions for Machine II and to show analytically that in the absence of radiation an ensemble initially polarized along the $\vec{n}$-axis is in spin equilibrium.

Process 5 differs from Process 4 by the second condition. By the first condition one has via (3.29), (3.30) by which one gets:

$$\frac{1}{\sqrt{\det(\sigma)}} \frac{d(\verepsilon, \sigma, \xi)}{\sqrt{\sigma}} \cdot \sigma = \frac{\sigma_{51}(s)}{\sigma_{52}(s)} \cdot \sigma + \frac{\sigma_{23}(s)}{\sigma_{24}(s)} \cdot \eta - \frac{\sigma_{12}(s)}{\sigma_{13}(s)} \cdot \sigma + \frac{\sigma_{23}(s)}{\sigma_{24}(s)} \cdot \eta + \frac{\sigma_{12}(s)}{\sigma_{13}(s)} \cdot \sigma + \frac{\sigma_{23}(s)}{\sigma_{24}(s)} \cdot \eta + \frac{\sigma_{12}(s)}{\sigma_{13}(s)} \cdot \sigma + \frac{\sigma_{23}(s)}{\sigma_{24}(s)} \cdot \eta + \frac{\sigma_{12}(s)}{\sigma_{13}(s)} \cdot \sigma + \frac{\sigma_{23}(s)}{\sigma_{24}(s)} \cdot \eta + \frac{\sigma_{12}(s)}{\sigma_{13}(s)} \cdot \sigma + \frac{\sigma_{23}(s)}{\sigma_{24}(s)} \cdot \eta$$

where $\sigma$ denotes the covariance matrix of Process 5. Applying the second condition on Process 5 one gets:

$$\vec{P}_{\text{dir}}(\sigma, \eta; s) = \lim_{0<s \to 0} [\vec{n}_{II}(\sigma, \eta; s)]$$

where $\vec{n}_{II}$ denotes the $\vec{n}$-axis in the $(\vec{n}_{0,II}, \vec{m}_{0,II}, \vec{l}_{0,II})$-frame which is given by (B.7). From (3.93), (3.100), (B.7) follows:

$$\sigma_{51}(0) = \sigma_{52}(0) \cdot \lim_{0<s \to 0} [\sigma_{19}(s)] = 0,$$

$$\sigma_{52}(0) = \sigma_{52}(0) \cdot \lim_{0<s \to 0} [\sigma_{20}(s)] = -\frac{d \cdot \sigma_{12}(0)}{\lambda_{0}} \cdot \frac{\sin(\lambda_{0} \cdot L)}{1 + \cos(\lambda_{0} \cdot L)}.$$
and:

\[ < \psi^{(5)}(0) > = \psi_{0,m}, \]  

(3.102)

where \( \psi_{0,m} \) is given by (3.38). With (3.102) one can simplify (3.99) to:

\[
\vec{P}^{\text{dir}}_{\text{w5}}(\sigma, \eta; s) = \begin{pmatrix}
\cos \left( \frac{\sigma_{5,13}(s)}{\sigma_\sigma^2} \cdot \sigma + \frac{\sigma_{5,23}(s)}{\sigma_\eta^2} \cdot \eta \right) \\
\sin \left( \frac{\sigma_{5,13}(s)}{\sigma_\sigma^2} \cdot \sigma + \frac{\sigma_{5,23}(s)}{\sigma_\eta^2} \cdot \eta \right)
\end{pmatrix},
\]  

(3.103)

so that by (3.31), the polarization density reads as:

\[
\vec{P}^{\text{w5}}(\sigma, \eta; s) = w_{\text{w5}}(\sigma, \eta) \cdot ||\vec{P}^{\text{loc}}_{\text{w5}}(\sigma, \eta; s)|| \cdot \begin{pmatrix}
\cos \left( \frac{\sigma_{5,13}(s)}{\sigma_\sigma^2} \cdot \sigma + \frac{\sigma_{5,23}(s)}{\sigma_\eta^2} \cdot \eta \right) \\
\sin \left( \frac{\sigma_{5,13}(s)}{\sigma_\sigma^2} \cdot \sigma + \frac{\sigma_{5,23}(s)}{\sigma_\eta^2} \cdot \eta \right)
\end{pmatrix}.
\]  

(3.104)

From (3.97), (3.102) follows:

\[ < \vec{x}^{(5)}(s) > = (0, 0, \psi_{0,m})^T. \]  

(3.105)

Now I have exploited the first two conditions on Process 5 and one sees that they do not fix \( \sigma_{5,33} \). Therefore I impose the third condition, namely:

\[ ||\vec{P}^{\text{loc}}_{\text{w5}}(\sigma, \eta; 0)|| = 1, \]

so that by (3.98) one has:

\[ \det \left( \sigma_5(0) \right) = 0. \]  

(3.106)

By (3.19) this leads to:

\[ \sigma_{5,33}(0) = \frac{\sigma_{5,13}^2(0)}{\sigma_\sigma^2} + \frac{\sigma_{5,23}^2(0)}{\sigma_\eta^2}, \]  

(3.107)

and from (3.101), (3.107) follows:

\[ \sigma_{5,33}(0) = \frac{d^2 \cdot \sigma_\eta^2}{\lambda_0^2} \cdot \left( \frac{\sin(\lambda_0 \cdot L)}{1 + \cos(\lambda_0 \cdot L)} \right)^2. \]  

(3.108)

With (3.22), (3.23), (3.101), (3.108) one has an initial value problem which determines \( \sigma_{5,13}, \sigma_{5,23}, \sigma_{5,33} \).

3.7.2

Coming to the calculation of \( \sigma_{5,13}, \sigma_{5,23} \) for \( s > 0 \) one has by (3.22), (3.101):

\[
\begin{pmatrix}
\sigma_{5,13}(s) \\
\sigma_{5,23}(s)
\end{pmatrix}
= \begin{pmatrix}
\sigma_{5,13}(s) \\
\sigma_{5,23}(s)
\end{pmatrix}
+ \begin{pmatrix}
g_{21}(s) \\
g_{22}(s)
\end{pmatrix}
= \begin{pmatrix}
g_7(s) \\
g_8(s) \\
g_9(s) \\
g_{10}(s) \\
g_{21}(s) \\
g_{22}(s)
\end{pmatrix},
\]

(3.109)
Figure 12: Difference $||\vec{P}_{tot}^{\mu_5}(2NL)|| - ||\vec{P}_{tot}^{\mu_4}(2NL)||$ of the polarization of processes 4,5 for the first 1000 turns assuming the HERA values (2.12) 

where:

$$g_{21}(s) = \frac{i \cdot a \cdot d \cdot \sigma_{2}^{2}}{2 \cdot \lambda_0 \cdot \lambda} \cdot \frac{\sin(\lambda_0 \cdot L)}{1 + \cos(\lambda_0 \cdot L)} \cdot g_2(s),$$

$$g_{22}(s) = \frac{i \cdot d \cdot \sigma_{2}^{2}}{2 \cdot \lambda_0 \cdot \lambda} \cdot \frac{\sin(\lambda_0 \cdot L)}{1 + \cos(\lambda_0 \cdot L)} \cdot g_3(s).$$

(3.110)

This can be checked by showing that the expression in (3.109) obeys (3.22), (3.101). With (3.109) one finds that after a few orbital damping times $\sigma_{5,13}(s), \sigma_{5,23}(s)$ become 2-turn periodic in $s$ with:

$$\left( \begin{array}{c} \sigma_{5,13}(s) \\ \sigma_{5,23}(s) \end{array} \right) \approx \left( \begin{array}{c} g_7(s) \\ g_8(s) \end{array} \right).$$

(3.111)

By (3.48), (3.92), (3.111) one sees that the asymptotic local polarization direction of processes 3,4,5 are the same. Coming finally to $\sigma_{5,33}$ I first of all get from (3.23), (3.52), (3.109):

$$\sigma_{5,33}(s) = \sigma_{5,33}(0) + \int_{0}^{s} ds_1 \cdot \sigma_{5,33}'(s_1) = \sigma_{5,33}(0) + 2 \cdot \int_{0}^{s} ds_1 \cdot \sigma_{5,23}(s_1) \cdot \hat{d}(s_1)$$

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\[ \sigma_{5,33}(0) + 2 \cdot \int_0^s ds_1 \cdot \sigma_{3,23}(s_1) \cdot \hat{d}(s_1) + \frac{i \cdot d \cdot \sigma^2}{\lambda_0 \cdot \lambda} \cdot \sin(\lambda_0 \cdot L) \cdot \int_0^s ds_1 \cdot g_3(s_1) \cdot \hat{d}(s_1) \]
\[ \equiv \sigma_{5,33}(0) + \sigma_{3,33}(s) + g_{23}(s) , \tag{3.112} \]

where
\[ g_{23}(s) = \frac{i \cdot d \cdot \sigma^2}{\lambda_0 \cdot \lambda} \cdot \sin(\lambda_0 \cdot L) \cdot \lambda_1 \cdot g_{16}(s) + c.c. \tag{3.113} \]

This can be checked by showing that the expression in (3.112) obeys (3.23), (3.108). With (3.18), (3.105), (3.109), (3.112) I have determined the first moment vector and the covariance matrix of Process 5. By (3.32), (3.102) the characteristic function \( \Phi_5 \) corresponding to \( w_5 \) reads as:
\[ \Phi_5(\vec{u}; s) = \exp \left( -\frac{1}{2} \cdot \sum_{j,k=1}^3 \sigma_{5,jk}(s) \cdot u_j \cdot u_k + i \cdot \psi_{0,m} \cdot u_3 \right) . \tag{3.114} \]

Therefore to prove that the above construction constitutes a G-process I just have to show that the covariance matrix is nonsingular for \( s > 0 \). This can be done analogously to processes 3 and 4. In fact it follows from (3.21), (3.106) that:
\[ \det(\sigma_5(s)) = -2 \cdot c \cdot \sigma^2 \cdot \int_0^s ds_1 \cdot \sigma_{5,23}(s_1) , \tag{3.115} \]
and by (3.109) one has
\[ \sigma_{5,23}(0) \neq 0 . \]

Hence by (3.113) \( \sigma_5 \) is nonsingular for \( s > 0 \), confirming that Process 5 is a G-process. Thus for \( s > 0 \) the probability density reads as:
\[ w_5(\sigma, \eta, \psi; s) = \sqrt{(2\pi)^{-3} \cdot \det(\sigma_5(s))^{-1}} \cdot \exp \left[ -\frac{1}{2} \cdot \begin{pmatrix} \sigma & \eta \\ \eta & \psi - \psi_{0,m} \end{pmatrix}^T \cdot \sigma_5^{-1}(s) \cdot \begin{pmatrix} \sigma \\ \eta \\ \psi - \psi_{0,m} \end{pmatrix} \right] , \tag{3.116} \]
and by using (2.28),(3.114) and the expressions for the first moment vector and the covariance matrix it reads at \( s = 0 \) as:
\[ w_5(\sigma, \eta, \psi; 0) = w_{\text{norm}}(\sigma, \eta) \cdot \delta(\psi - \sigma_{5,23}(0) \cdot \eta - \psi_{0,m}) . \tag{3.117} \]

By (3.116), (3.117) \( w_5 \) fulfills the normalization condition (2.20) and the orbital part of \( w_5 \) obeys:
\[ w_{5,\text{orb}} = w_{\text{norm}} , \]
confirming that Process 5 is at orbital equilibrium. Note that \( \vec{x}^{(5)}(s) \) is a Markovian diffusion process.
Figure 13: Angle in radians between the local polarization directions of processes 4,5 at the snake for the phase space point where $\sigma = \sigma_\sigma$ and $\eta = \sigma_\eta$ versus the number $N$ of turns assuming the HERA values (2.12)

3.7.3

By (3.27), (3.102) the polarization vector reads as:

$$\vec{P}_{tot}^{w_5}(s) = \exp\left(-\frac{\sigma_{5,33}(s)}{2}\right) \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},$$

(3.118)

so that by (3.112) the polarization is:

$$||\vec{P}_{tot}^{w_5}(s)|| = \exp\left(-\frac{\sigma_{5,33}(s)}{2}\right) = \exp\left(-\frac{\sigma_{5,33}(0)}{2} - \frac{\sigma_{3,33}(s)}{2} - g_{23}(s)/2\right).$$

(3.119)

Thus one has complete spin decoherence of Process 5:

$$||\vec{P}_{tot}^{w_5}(+\infty)|| = 0.$$

(3.120)

Since $g_{16}(s)$ becomes constant after a few orbital damping times, $g_{23}(s)$ becomes constant too. Hence I get:

$$\sigma_{5,33}(s) \approx \sigma_{5,33}(0) + \sigma_{3,33}(s) + g_{23}(+\infty)$$

$$\approx \sigma_{5,33}(0) + g_{13}(+\infty) + g_{14}(s) + s \cdot g_{15} + g_{23}(+\infty),$$

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i.e. after a few orbital damping times $\sigma_{5,33}(s)$ splits up additively into a term increasing linearly with $s$ plus a term 2-turn periodic in $s$.

To study the azimuthal dependence of the polarization ‘turn by turn’, I consider the sequence $\sigma_{5,33}(2NL)$.

First of all one gets by (3.112):

$$\sigma_{5,33}(2NL) = \sigma_{5,33}(0) + \sigma_{3,33}(2NL) + g_{23}(2NL) .$$  \hfill (3.121)

This can be simplified by calculating

$$g_{23}(2NL) = \frac{i \cdot d^2 \cdot g_{11} \cdot \sigma_0^2}{2 \cdot \lambda_0 \cdot \lambda} \cdot \frac{\sin(\lambda_0 \cdot L)}{1 + \cos(\lambda_0 \cdot L)} \cdot [g_2(2NL) - 4 \cdot i \cdot \exp(c \cdot L \cdot (N + 1/2)) \cdot \sin(\lambda \cdot L) \cdot \cos(\lambda \cdot 2NL) - \exp(c \cdot L) \cdot g_2(2NL)] + 4 \cdot i \cdot \exp(c \cdot L/2) \cdot \sin(\lambda \cdot L)] ,$$  \hfill (3.122)

where I used (3.69). This converges after a few orbital damping times to a constant whereas $\sigma_{3,33}(2NL)$ grows exponentially. It is now clear that processes 3, 4 and 5 all have the same depolarization time $\tau_{\text{spin}}$ and the same asymptotic local polarization direction. \[\square\]

A plot of the azimuthal dependence of the polarization for Process 5 for the HERA values (2.12) is visually indistinguishable from the plot in figure 10 for Process 4 and is therefore not presented here. \[\square\] However the polarizations of the two processes are not identical as can be seen in figure 12 where I plot the difference. This is tiny and shows oscillating transient behaviour which must be due to transient behaviour in Process 5 since Process 4 is transient free. In figure 13 I plot the azimuthal dependence of the angle between the local polarization direction of processes 4 and 5 at the phase space point where $\sigma = \sigma_\sigma$ and $\eta = \eta$. One again sees oscillating transient behaviour with an angle difference of typically 0.05 milliradians whereas the angle between the $\vec{n}_0$-axis and the $\vec{n}$-axis at the same point is about 200 milliradians.

The interpretation of these findings is straightforward; although, as discussed above, the $\vec{n}$-axis should give the local polarization direction in the absence of radiation, this is no longer exactly true in the presence of radiation. We have already seen for Process 3 that when the initial polarization direction deviates typically by 200 milliradians from the asymptotic local polarization direction, polarization fluctuations of several percent occur which eventually damp away while at the same time the local polarization direction approaches its asymptotic distribution. A similar thing happens with Process 5 except that the difference between the $\vec{n}$-axis and the asymptotic local polarization direction is very small so that the polarization fluctuations are correspondingly small. It can be shown using (3.90), (3.102) that the angle between the $\vec{n}$-axis and the asymptotic local polarization direction at the phase space point where $\sigma = \sigma_\sigma$ and $\eta = \sigma_\eta$ is given at the snake by:

$$\frac{1}{\sigma_\eta} \cdot g_8(0) - \frac{1}{\sigma_\sigma} \cdot g_7(0) + \sigma_\eta \cdot \lim_{0 < s \to 0} [g_{20}(s)] ,$$

which vanishes in the limit where $c$ and $\omega$ go to zero with $\omega/c = \text{const} = -2 \cdot \sigma_\eta^2 \approx -2.0 \cdot 10^{-6}$ and which for small $c$ is given approximately by:

$$\frac{d \cdot c \cdot \sigma_\eta}{2 \cdot \lambda_0^2} \cdot \frac{\lambda_0 \cdot L \sin(\lambda_0 \cdot L)}{1 + \cos(\lambda_0 \cdot L)} .$$

\[53\] One can easily show that this not only holds for processes 3, 4 and 5 but for all G-processes at orbital equilibrium.

\[54\] Note that the initial polarization of processes 4 and 5 is not complete because the initial direction of the local polarization is not uniform - see (3.90), (3.103).
Thus I have shown that the \( \vec{n} \)-axis does not describe the direction of the local polarization in Machine II in the presence of radiation. However, the relative difference in the directions is extremely small and can, for practical purposes be ignored for Machine II.

The depolarization rate obtained using SLIM \cite{Cha81} for a real perfectly aligned flat HERA lattice with a pointlike radial snake and when only spin diffusion due to synchrotron motion generated in the arcs is included, is in satisfactory agreement with \( \tau_{\text{spin}}^{-1} \) [Bar97].

As one can see from (B.8), the SLIM approximation to the \( \vec{n} \)-axis is quite good near phase space points where \( \sigma = \sigma_\sigma \) and \( \eta = \sigma_\eta \). However, it becomes progressively worse towards the edges of phase space.

In the absence of radiation the local polarization direction should be parallel to the \( \vec{n} \)-axis and the polarization of an ensemble in orbital equilibrium set up in this state should be constant from turn to turn. This can be confirmed analytically by putting \( c \) and \( \omega \) to zero with \( \omega/c = \text{const} = -2 \cdot \sigma_\eta^2 = -2 \cdot 10^{-6} \), in which limit Process 5 modifies to ‘Process 5a’. Using (3.112), (3.118) the polarization at the snake is then given by:

\[
||\vec{P}_{\text{tot}}(NL)|| = \exp\left(-\frac{d^2 \cdot \sigma_\eta^2 \cdot \sin^2(\lambda_0 \cdot L)}{2\lambda_0 \cdot (1 + \cos(\lambda_0 \cdot L))^2}\right),
\]

which is indeed constant from turn to turn. Also the local polarization is constant from turn to turn. For the HERA values (2.12) this expression gives about 0.98. It also coincides with the constant polarization seen in Process 4a. One can now see why the polarization of Process 4a is constant from turn to turn; if, as in Process 4, I require that the local polarization direction is already asymptotic at \( s = 0 \) and switch off the radiation, this local polarization direction must coincide with the \( \vec{n} \)-axis since it now fulfills the characteristic properties of the \( \vec{n} \)-axis. Then, just as in Process 5a, the polarization must be constant from turn to turn and Process 4a is identical with Process 5a.

### 3.8 The polarization density and spin matching for Machine II

Although the synchrotron radiation parameters \( c \) and \( \omega \) are \( s \)-independent the present formalism can be used to analyze more complex rings. For example, the Green function for the polarization density, in particular the radiationless Green function, can be used to analyze the effect of ‘lumped’ radiators such as asymmetric wigglers \cite{Mon84}. If, for example, a wiggler is placed diametrically opposite the snake, then one expects on the basis of standard ‘spin matching’ concepts that the spin diffusion due to the extra excitation of \( \eta \) would be almost cancelled [BKRRS85]. This can be made quantitative by considering the ‘\( \mathcal{G} \)’ matrix’, the \( 2 \times 6 \) spin-orbit coupling matrix of the SLIM formalism. Writing this in the form

\[
\mathcal{G} = (g_\sigma, g_z, g_s),
\]

where the \( g \)'s are \( 2 \times 2 \) matrices it is simple to show \cite{BHR94b} that \( g_s \), for the interval from \( s = L/2 \) to \( s = 3L/2 \) in Machine II is:

\[
g_s(3L/2; L/2) = 2 \cdot d \cdot \left(\cos(\lambda_0 \cdot L/2) - 1\right) \cdot \begin{pmatrix} \frac{1}{\lambda_0} \cdot \cos(\lambda_0 \cdot L/2) & \frac{1}{\lambda_0} \cdot \sin(\lambda_0 \cdot L/2) \\ 0 & 0 \end{pmatrix}.
\]

(3.123)

Asymmetric wigglers can be designed so that the overall spin phase advance is zero and so that no dispersion is generated overall.
The nonzero matrix elements of $g_s$ vanish as $\lambda_0$ goes to zero and then in linear approximation a spin travelling from $s = L/2$ to $s = 3L/2$ on any synchrotron orbit is unperturbed overall. This interval is then said to be ‘spin transparent’ and the depolarization due to the wiggler is cancelled. For the HERA values (2.12) $g_s$ does not exactly vanish but since $Q_s$ is small $g_s$ is still small enough to ensure that the depolarization due to the wiggler should be largely suppressed.

However, in realistic machines like HERA, there is still excitation in the remainder of the ring. So even if the radiation power from the wiggler were dominant, the radiation in the remainder of the ring would still cause depolarization on the ‘time’ scale $\tau_{\text{spin}}$. These conjectures are confirmed by numerical calculations with SLIM [Bar97].

Spin transparency can be discussed using the polarization density. Since in this section I am only considering the radiation from the wiggler, this condition can be investigated using the polarization density for the radiationless case. The causality properties of the azimuthal evolution for Machine II are the same as for Machine I, so that in particular one has a Green function for the polarization density. For the radiationless case the Green function $R_{II,nrad}$ for the radiationless Bloch equation (3.12) is given by:

$$P_{II,nrad}(\sigma, \eta; s|\sigma_1, \eta_1; s_1) \equiv w_{\text{orb,trans,nrad}}(\sigma, \eta; s|\sigma_1, \eta_1; s_1) \cdot R_{II,nrad}(\sigma_1, \eta_1; s_1),$$

where $w_{\text{orb,trans,nrad}}$ is given by (2.112) and where:

$$R_{II,nrad} \equiv \begin{pmatrix}
\cos(\rho) & -\sin(\rho) & 0 \\
\sin(\rho) & \cos(\rho) & 0 \\
0 & 0 & 1
\end{pmatrix},$$

with:

$$\rho(\sigma_1, \eta_1; s_1) \equiv \frac{\sigma_1}{a} \cdot \rho_1(s; s_1) + \frac{\eta_1}{\lambda_0} \cdot \rho_2(s; s_1),$$

$$\rho_1(s; s_1) \equiv \frac{d}{1 + \cos(\lambda_0 \cdot L)} \cdot \left\{ \cos\left(\lambda_0 \cdot \left[2 \cdot L \cdot G(s/2L) - s_1\right]\right) + \cos\left(\lambda_0 \cdot \left[2 \cdot L \cdot G(s/2L) + L - s_1\right]\right) - \left(\cos\left(\lambda_0 \cdot \left[2 \cdot L \cdot G(s/2L) + L - s_1\right]\right) + \cos\left(\lambda_0 \cdot \left[2 \cdot L \cdot G(s/2L) - s_1\right]\right)\right) \right\}$$

$$+ \left(\cos\left(\lambda_0 \cdot \left[2 \cdot L \cdot G(s/2L) - s_1\right]\right) + \cos\left(\lambda_0 \cdot \left[2 \cdot L \cdot G(s/2L) + L - s_1\right]\right)\right) \right\},$$

$$\rho_2(s; s_1) \equiv \frac{d}{1 + \cos(\lambda_0 \cdot L)} \cdot \left\{ \sin\left(\lambda_0 \cdot \left[2 \cdot L \cdot G(s/2L) - s_1\right]\right) + \sin\left(\lambda_0 \cdot \left[2 \cdot L \cdot G(s/2L) + L - s_1\right]\right) - \left(\sin\left(\lambda_0 \cdot \left[2 \cdot L \cdot G(s/2L) + L - s_1\right]\right) + \sin\left(\lambda_0 \cdot \left[2 \cdot L \cdot G(s/2L) - s_1\right]\right)\right) \right\}$$

$$+ \left(\sin\left(\lambda_0 \cdot \left[2 \cdot L \cdot G(s/2L) + L - s_1\right]\right) + \sin\left(\lambda_0 \cdot \left[2 \cdot L \cdot G(s/2L) - s_1\right]\right)\right) \right\},$$

$$\left(\cos\left(\lambda_0 \cdot \left[2 \cdot L \cdot G(s/2L) - s_1\right]\right) + \cos\left(\lambda_0 \cdot \left[2 \cdot L \cdot G(s/2L) + L - s_1\right]\right)\right) \right\},$$

$$\left(\cos\left(\lambda_0 \cdot \left[2 \cdot L \cdot G(s/2L) + L - s_1\right]\right) + \cos\left(\lambda_0 \cdot \left[2 \cdot L \cdot G(s/2L) - s_1\right]\right)\right) \right\},$$

$$+ \left(\cos\left(\lambda_0 \cdot \left[2 \cdot L \cdot G(s/2L) - s_1\right]\right) + \cos\left(\lambda_0 \cdot \left[2 \cdot L \cdot G(s/2L) + L - s_1\right]\right)\right) \right\},$$

$$+ \left(\cos\left(\lambda_0 \cdot \left[2 \cdot L \cdot G(s/2L) + L - s_1\right]\right) + \cos\left(\lambda_0 \cdot \left[2 \cdot L \cdot G(s/2L) - s_1\right]\right)\right) \right\},$$

$$+ \left(\cos\left(\lambda_0 \cdot \left[2 \cdot L \cdot G(s/2L) - s_1\right]\right) + \cos\left(\lambda_0 \cdot \left[2 \cdot L \cdot G(s/2L) + L - s_1\right]\right)\right) \right\},$$

$$+ \left(\cos\left(\lambda_0 \cdot \left[2 \cdot L \cdot G(s/2L) + L - s_1\right]\right) + \cos\left(\lambda_0 \cdot \left[2 \cdot L \cdot G(s/2L) - s_1\right]\right)\right) \right\}. \quad (3.124)$$

These conjectures are confirmed by numerical calculations with SLIM [Bar97].
Moreover:
\[
\tilde{P}^w(\sigma, \eta; s) = \int_{-\infty}^{+\infty} d\eta_1 \int_{-\infty}^{+\infty} d\eta_1 \cdot P_{II,nrad}(\sigma, \eta; \sigma_1, \eta_1; s_1) \cdot \tilde{P}^w(\sigma_1, \eta_1; s_1), \tag{3.129}
\]
where \(\tilde{P}^w\) is an arbitrary solution of (3.12). Therefore, and by using the connection (see section 2.7.1) between (3.12) and the Thomas-BMT equation (3.6), one finds for a Thomas-BMT solution \(\tilde{S}(s)\) on a given orbit \(\sigma(s), \eta(s)\):
\[
\tilde{S}(s) = R_{II,nrad}(\sigma(s_1), \eta(s_1); s; s_1) \cdot \tilde{S}(s_1)
\]
\[
= \begin{pmatrix}
\cos\left(\rho(\sigma(s_1), \eta(s_1); s; s_1)\right) & -\sin\left(\rho(\sigma(s_1), \eta(s_1); s; s_1)\right) & 0 \\
\sin\left(\rho(\sigma(s_1), \eta(s_1); s; s_1)\right) & \cos\left(\rho(\sigma(s_1), \eta(s_1); s; s_1)\right) & 0 \\
0 & 0 & 1
\end{pmatrix} \cdot \tilde{S}(s_1), \tag{3.130}
\]
where by (3.126):
\[
\rho(\sigma(s_1), \eta(s_1); s; s_1) = \frac{\sigma(s_1)}{\lambda_0} \cdot \rho_1(s; s_1) + \frac{\eta(s_1)}{\lambda_0} \cdot \rho_2(s; s_1). \tag{3.131}
\]
Note that, due to the absence of radiation, \(\sigma(s), \eta(s)\) fulfill (2.109). By (3.130) \(\tilde{S}(s_1)\) evolves into \(\tilde{S}(s)\) via the spin transfer matrix \(R_{II,nrad}\). For \(s_1 = L/2, s = 3L/2\) one has by (3.127), (3.128):
\[
\rho_1(3L/2; L/2) = 2 \cdot d \cdot \cos(\lambda_0 \cdot L/2) \cdot \left(\cos(\lambda_0 \cdot L/2) - 1\right),
\]
\[
\rho_2(3L/2; L/2) = 2 \cdot d \cdot \sin(\lambda_0 \cdot L/2) \cdot \left(\cos(\lambda_0 \cdot L/2) - 1\right).
\]
\tag{3.132}
The matrix \(g(s) = (3L/2; L/2)\) of the linearized formalism (see (3.123)) can now be written via (3.130), (3.131), (3.132) as:
\[
g(s)(3L/2; L/2) = \begin{pmatrix}
\rho_1(3L/2; L/2)/a & \rho_2(3L/2; L/2)/\lambda_0 \\
0 & 0
\end{pmatrix}. \tag{3.133}
\]
By (3.133) the condition for spin transparency is equivalent to:
\[
\rho_1(3L/2; L/2) = \rho_2(3L/2; L/2) = 0. \tag{3.134}
\]
In this case the matrix \(R_{II,nrad}(\sigma(s_1), \eta(s_1); 3L/2; L/2)\) reduces, see (3.130) and (3.131), to the unit matrix so that for Machine II the spin transparency condition of the linearized formalism even applies to large deviations of the spin vector from the \(\tilde{n}_0\)-axis. By (3.124), (3.125), (3.126), (3.134) one then gets:
\[
\tilde{P}^w(\tilde{z}; 3L/2) = \tilde{P}^w\left(\exp(-L \cdot A_{\text{orb,nrad}}) \cdot \tilde{z}; L/2\right), \tag{3.135}
\]
so that via (2.21):
\[
\tilde{P}^w_{\text{tot}}(3L/2) = \tilde{P}^w_{\text{tot}}(L/2). \tag{3.136}
\]
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4 Summary

The spin depolarization rate in an electron storage ring is usually calculated using standard algorithms like SLIM or a Monte-Carlo tracking program such as SITROS \cite{Kew83,Boe94}. SLIM exploits the Derbenev-Kondratenko (DK) formula \footnote{Other algorithms exploiting the DK approach, but at higher order, are SMILE \cite{Man87} and Sodom \cite{Yok92}.} and a first order perturbation theory.

Methods based on the DK formalism are only applicable once various transient phenomena have damped away. In this paper I have shown how one can, instead, apply standard Fokker-Planck methods, but to very simple model rings. In the process I introduced the polarization density, a quantity which obeys a universal evolution equation of the Bloch type. This is a linear equation valid for arbitrary spin distributions and can therefore be used far from orbital and/or spin equilibrium. I also introduced the local polarization vector and its direction. Both obey Bloch type evolution equations which depend on the orbital state and are therefore not universal.

In the numerical part of this report I have used the parameters (2.12) of a typical ring, namely the HERA electron ring, to study the short and long term behaviour of nine different scenarios, the stochastic Processes 1,2,3,4 and 5 and the noise free processes 2a, 3a, 4a and 5a. It is found that Processes 1 and 2 do not lead to complete spin decoherence and the corresponding equilibrium polarization is calculated. However in the presence of a Siberian Snake (processes 3,4 and 5) there is complete spin decoherence, a result which is to some extent counter intuitive given that snakes tend to stabilize the polarization. The asymptotic depolarization rates agree with SLIM estimates and in the presence of radiation the asymptotic local polarization direction is not exactly parallel to the $\vec{n}$-axis. The radiationless Green function for the polarization density offers another tool for studying spin matching. In processes 2a and 3a the polarization oscillates indefinitely but in processes 4a and 5a, the spin distribution begins in equilibrium and remains there.

The models considered here are extremely simple but in the next section I indicate how the polarization density (and the local polarization vector and its direction) can be used in realistic cases.

5 Epilogue

This paper is the first part of a paper prepared in 1995 but not distributed. In the second part I demonstrate that the polarization density defined on a six-dimensional orbital phase space for full three-dimensional spin motion also obeys a universal evolution equation of Bloch type with the same linear structure as equations (2.106), (3.11), namely

$$\frac{\partial \vec{P}_w}{\partial s} = L_{FP,orb,gen} \vec{P}_w + \vec{W}_{gen} \wedge \vec{P}_w,$$

where $L_{FP,orb,gen}$ denotes the orbital Fokker-Planck operator and where the vector $\vec{W}_{gen}$ is determined by the Thomas-BMT equation. In the more general formulation I use periodic boundary conditions for the spin phase. \footnote{See section 2.3.6.} The polarization density is appropriate for use with arbitrarily complicated rings. The local polarization vector and its direction are also useful tools (especially at orbital equilibrium) and by using these and their Bloch equations I obtain...
an expression for the depolarization rate in terms of the azimuthal and phase space average of \( ||\partial \vec{P}_{dir}/\partial \eta||^2 \cdot |K|^3 \) which generalizes the average of \( ||\partial \vec{n}/\partial \eta||^2 \cdot |K|^3 \) in equation 6.2 in [DK73]. By calculating the depolarization rate for Machine II in terms of \( ||\partial \vec{P}_{dir}/\partial \eta||^2 \cdot |K|^3 \) I reproduce the \( 1/\tau_{\text{spin}} \) of (3.63). In fact the derivation of the Bloch equation for the general polarization density constitutes a classical construction for the pure depolarization part of equation 2 in [DK75] which was obtained by semiclassical methods. Using the polarization density one can estimate the angle between the \( \vec{n} \)-axis and the true local polarization direction in practical rings. The Sokolov-Ternov process can be included by adding in parts of the Baier-Katkov-Strakovenko expression [BKS70] in an obvious way. One then has many of the terms in equation 2 in [DK75]. The remaining terms can only be obtained by a full semiclassical treatment of the radiation process and this work is in progress and will be published at a later date together with the classical work on the general polarization density.

The use of the polarization density obviates the need to begin the calculation of the depolarization rate by first diagonalizing the the combined spin-orbit Hamiltonian as in [DK73].

### Appendix A

#### A.1

In this appendix I show that the Thomas-BMT equation (3.6) for Machine II, defined in the rotating frame \( \vec{n}_{0,II}, \vec{m}_{0,II}, \vec{l}_{0,II} \) of section 3.1, follows from (3.1). First of all I have to show that the vectors \( \vec{m}_{0,II}(s), \vec{l}_{0,II}(s), \vec{n}_{0,II}(s) \) solve the closed orbit Thomas-BMT equation:

\[
\vec{\xi}'(s) = \vec{\Omega}_{II,0}(s) \times \vec{\xi}(s),
\]

where by section 3.1:

\[
\vec{\Omega}_{II,0}(s) \equiv d \cdot \vec{e}_3 + \pi \cdot \delta_{L,\text{per}}(s) \cdot \vec{e}_1.
\]

I call the region outside the snake ‘the arcs’ and there (A.1) reduces to a precession around \( \vec{e}_3 \), so that on the closed orbit and in the machine frame the spin transfer matrix in the arcs reads as:

\[
\mathbf{M}_{\text{arc}}(s_2, s_1) \equiv \begin{pmatrix}
\cos\left(d \cdot (s_2 - s_1)\right) & -\sin\left(d \cdot (s_2 - s_1)\right) & 0 \\
\sin\left(d \cdot (s_2 - s_1)\right) & \cos\left(d \cdot (s_2 - s_1)\right) & 0 \\
0 & 0 & 1
\end{pmatrix},
\]

i.e. in the arcs:

\[
\mathbf{M}_{\text{arc}}(s_2, s_1) \cdot \vec{\xi}(s_1) = \vec{\xi}(s_2).
\]

To get the spin transfer matrix for the snake I first write down (A.1) for the snake which results in

\[
\vec{\xi}'(s) = \pi \cdot \delta_{L,\text{per}}(s) \cdot \left( \vec{e}_1 \times \vec{\xi}(s) \right) \equiv \delta_{L,\text{per}}(s) \cdot \mathbf{M}_0 \cdot \vec{\xi}(s),
\]

Note that \( K \) denotes the design orbit curvature, e.g. \( K = K_x \) (see section 1).
where in the machine frame $M_0$ reads as:

$$M_0 \equiv \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\pi \\ 0 & \pi & 0 \end{pmatrix}.$$  

Because the matrix $M_0$ is $s$-independent in the machine frame one easily finds that in this frame the spin transfer matrix for the snake is given on the closed orbit by:

$$M_{\text{snake}} = \exp(M_0) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$  

(A.3)

The $\vec{n}_0$-axis of Machine II is given by:

$$\vec{n}_{0,\text{II}}(s) \equiv \cos[d \cdot (s - L/2 - L \cdot \mathcal{G}(s/L))] \cdot \vec{e}_1 + \sin[d \cdot (s - L/2 - L \cdot \mathcal{G}(s/L))] \cdot \vec{e}_2.$$  

(A.4)

It obviously solves (A.1) in the arcs and is 1-turn periodic in the machine frame. To check if it solves (A.1) also at the snake I note that

$$\lim_{0<s\to0} [\vec{n}_{0,\text{II}}(s)] = \cos(d \cdot L/2) \cdot \vec{e}_1 - \sin(d \cdot L/2) \cdot \vec{e}_2,$$

$$\lim_{0>s\to0} [\vec{n}_{0,\text{II}}(s)] = \cos(d \cdot L/2) \cdot \vec{e}_1 + \sin(d \cdot L/2) \cdot \vec{e}_2,$$

from which follows by (A.3):

$$\lim_{0<s\to0} [\vec{n}_{0,\text{II}}(s)] = M_{\text{snake}} \cdot \lim_{0>s\to0} [\vec{n}_{0,\text{II}}(s)],$$

so that in fact $\vec{n}_{0,\text{II}}(s)$ solves (A.1) at the snake. Thus $\vec{n}_{0,\text{II}}(s)$ is a unit-vector solution of (A.1), which is 1-turn periodic in the machine frame, i.e. $\vec{n}_{0,\text{II}}$ is the $\vec{n}_0$-axis of Machine II.

Now I consider $\vec{l}_{0,\text{II}}$ which is defined by:

$$\vec{l}_{0,\text{II}}(s) \equiv \theta_{2L,\text{per}}(s) \cdot \vec{e}_3.$$  

One easily finds that $\vec{l}_{0,\text{II}}(s)$ solves (A.1) in the arcs. To check if it solves (A.1) also at the snake I calculate by (A.3):

$$\lim_{0<s\to0} [\vec{l}_{0,\text{II}}(s)] = \vec{e}_3 = -M_{\text{snake}} \cdot \vec{e}_3 = M_{\text{snake}} \cdot \lim_{0>s\to0} [\vec{l}_{0,\text{II}}(s)],$$

so that in fact $\vec{l}_{0,\text{II}}(s)$ solves (A.1) at the snake. Thus I have shown that the dreibein $\vec{n}_{0,\text{II}}(s)$, $\vec{m}_{0,\text{II}}(s)$, $\vec{l}_{0,\text{II}}(s)$ solves (A.1), where

$$\vec{m}_{0,\text{II}} \equiv \vec{l}_{0,\text{II}} \wedge \vec{n}_{0,\text{II}}.$$  

A.2

Now one can derive (3.6) from (3.1). First of all I define

$$\vec{S} \equiv \begin{pmatrix} \vec{n}_{0,\text{II}}^T \cdot \vec{\xi} \\ \vec{m}_{0,\text{II}}^T \cdot \vec{\xi} \\ \vec{l}_{0,\text{II}}^T \cdot \vec{\xi} \end{pmatrix},$$  

(A.5)
so that (3.1) is equivalent to the following equation for $\mathbf{S}$:

$$\mathbf{S}' = \mathbf{W}_{II} \wedge \mathbf{S},$$  \hfill (A.6)

with

$$\mathbf{W}_{II} \equiv \left( n_{0,II}^{T} \cdot [\mathbf{\Omega}_{II} - \mathbf{U}] \right) \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \left( m_{0,II}^{T} \cdot [\mathbf{\Omega}_{II} - \mathbf{U}] \right) \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \left( l_{0,II}^{T} \cdot [\mathbf{\Omega}_{II} - \mathbf{U}] \right) \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

where [BHR94a, BHR94b].

$$\mathbf{U} \equiv \frac{1}{2} \cdot [\mathbf{n}_{0,II} \wedge \mathbf{n}_{0,II}' + \mathbf{m}_{0,II} \wedge \mathbf{m}_{0,II}' + \mathbf{l}_{0,II} \wedge \mathbf{l}_{0,II}'] = \mathbf{\Omega}_{II,0}.$$ Therefore

$$\mathbf{W}_{II} = d \cdot \eta \cdot \left( \bar{e}_{3}^{T} \cdot \mathbf{l}_{0,II} \right) \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = d \cdot \eta \cdot \theta_{2L,per} \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$ Hence I have shown that (3.6) follows from (3.1).

**Appendix B**

**B.1**

In this appendix I calculate the $\mathbf{n}$-axis of Machine II. As mentioned in section 3.7.1 the $\mathbf{n}$-axis is denoted by $\mathbf{n}_{II}$ and it obeys the following radiationless Bloch equation:

$$\frac{\partial \mathbf{n}_{II}}{\partial s} = -a \cdot \eta \cdot \frac{\partial \mathbf{n}_{II}}{\partial \sigma} - b \cdot \sigma \cdot \frac{\partial \mathbf{n}_{II}}{\partial \eta} + \mathbf{\Omega}_{II} \wedge \mathbf{n}_{II}. \hfill (B.1)$$

In fact, (B.1) holds because in the $(\mathbf{n}_{0,II}, \mathbf{m}_{0,II}, \mathbf{l}_{0,II})$-frame the $\mathbf{n}$-axis obeys the radiationless Bloch equation (3.12).

I begin by assuming that the $\mathbf{n}$-axis is horizontal, so that I make the ansatz:

$$\mathbf{n}_{II} \equiv \cos(f_{II}) \cdot \bar{e}_{1} + \sin(f_{II}) \cdot \bar{e}_{2}, \hfill (B.2)$$

where $f_{II}(\sigma, \eta; s)$ needs to be determined. Then with the ansatz that $f_{II}$ is linear in $\sigma, \eta$ one finally gets

$$f_{II}(\sigma, \eta; s) = g_{6}(s) + \sigma \cdot g_{19}(s) + \eta \cdot g_{20}(s), \hfill (B.3)$$

where $g_{6}$ is defined in section 3.1 and where:

$$g_{19}(s) = \frac{d \cdot b}{\lambda_{0}^{2}} \cdot \frac{1}{1 + \cos(\lambda_{0} \cdot L)} \cdot \left[ \cos\left(\lambda_{0} \cdot [s - L \cdot G(s/L)]\right) \right.

+ \cos\left(\lambda_{0} \cdot [s - L \cdot G(s/L)]\right) - \cos(\lambda_{0} \cdot L) - 1 \right],$$

$$g_{20}(s) = \frac{d}{\lambda_{0}} \cdot \frac{1}{1 + \cos(\lambda_{0} \cdot L)} \cdot \left[ \sin\left(\lambda_{0} \cdot [s - L \cdot G(s/L)]\right) + \sin\left(\lambda_{0} \cdot [s - L \cdot G(s/L)]\right) \right]. \hfill (B.4)$$

$^{59}$The $\mathbf{n}$-axis for Machine I is given by $\bar{e}_{3}$. 82
B.2

In this section I show that \( \tilde{n}_{II} \), as defined by (B.2), (B.3), (B.4) fulfills all properties of an \( \tilde{n} \)-axis.

Inserting (B.2) into (B.1) gives the following equation in the arcs:

\[
\frac{\partial f_{II}}{\partial s} = -a \cdot \eta \cdot \frac{\partial f_{II}}{\partial \sigma} - b \cdot \sigma \cdot \frac{\partial f_{II}}{\partial \eta} + d \cdot \eta + d. \tag{B.5}
\]

Also by (B.2), (B.3) one finds that \( \tilde{n}_{II} \) depends smoothly on the variables \( \sigma, \eta \) so that (B.1) reads at the snake as:

\[
\frac{\partial \tilde{n}_{II}}{\partial s} = \pi \cdot \delta_{L,\text{per}} \cdot \left( \tilde{e}_1 \wedge \tilde{n}_{II} \right). \tag{B.6}
\]

Note that for \( 0 < s < L \) \( f_{II} \) reduces by (B.3), (B.4) to:

\[
f_{II}(\sigma, \eta; s) = d \cdot (s - L/2) + \frac{d \cdot b \cdot \sigma}{\lambda_0} \cdot \frac{1}{1 + \cos(\lambda_0 \cdot L)} \cdot \left[ \cos\left(\lambda_0 \cdot (s - L)\right) + \cos(\lambda_0 \cdot s) - \cos(\lambda_0 \cdot L) - 1 \right] + \frac{d \cdot \eta}{\lambda_0} \cdot \frac{1}{1 + \cos(\lambda_0 \cdot L)} \cdot \left[ \sin\left(\lambda_0 \cdot (s - L)\right) + \sin(\lambda_0 \cdot s) \right].
\]

It is easily checked that this expression solves (B.3). One also observes by (B.3), (B.4) that:

\[
\lim_{0 < s \to 0} [f_{II}(\sigma, \eta; s)] = -\lim_{0 > s \to 0} [f_{II}(\sigma, \eta; s)].
\]

From this follows by (A.3), (B.2):

\[
\lim_{0 < s \to 0} [\tilde{n}_{II}(\sigma, \eta; s)] = M_{\text{snake}} \cdot \lim_{0 > s \to 0} [\tilde{n}_{II}(\sigma, \eta; s)],
\]

so that \( \tilde{n}_{II} \) obeys (B.6) at the snake. One also sees that \( f_{II}(\sigma, \eta; s) \), given by (B.3), is 1-turn periodic in \( s \).

Thus I have shown that \( \tilde{n}_{II}(\sigma, \eta; s) \), given by (B.2), (B.3), is a unit-vector solution of the radiationless Bloch equation (B.1), 1-turn periodic in \( s \) in the machine frame. It is thus the vector field \( \tilde{n} \) of [DK72, HH96].

Note that for \( \sigma = \eta = 0 \) one gets:

\[
\tilde{n}_{II}(\sigma = 0, \eta = 0; s) = \tilde{n}_{0,II}(s).
\]

One also observes that with this ansatz a singularity in \( \tilde{n}_{II} \) occurs if the fractional part of the orbital tune \( Q_s = (\lambda_0 \cdot L)/(2\pi) \) equals 1/2, i.e. if one is at a spin-orbit resonance.

B.3

Denoting the \( \tilde{n} \)-axis in the \( (\tilde{n}_{0,II}, \tilde{m}_{0,II}, \tilde{l}_{0,II}) \)-frame by \( \tilde{n}_{II} \) one observes by (B.3), (B.4), (B.2), (B.3):

\[
\tilde{n}_{II}(\sigma, \eta; s) = \begin{pmatrix}
\tilde{n}_{0,II}(s) \cdot \tilde{n}_{II}(\sigma, \eta; s) \\
\tilde{m}_{0,II}(s) \cdot \tilde{n}_{II}(\sigma, \eta; s) \\
\tilde{l}_{0,II}(s) \cdot \tilde{n}_{II}(\sigma, \eta; s)
\end{pmatrix} = \begin{pmatrix}
\cos(f_{II}(\sigma, \eta; s) - g_6(s)) \\
\theta_{2L,\text{per}}(s) \cdot \sin(f_{II}(\sigma, \eta; s) - g_6(s)) \\
0
\end{pmatrix}
\]
\[
\begin{pmatrix}
\cos\left(g_{19}(s) \cdot \sigma + g_{20}(s) \cdot \eta \right) \\
\theta_{2L,\text{per}}(s) \cdot \sin\left(g_{19}(s) \cdot \sigma + g_{20}(s) \cdot \eta \right) \\
0
\end{pmatrix}.
\]

The corresponding formula obtained from the SLIM formalism, which linearizes spin motion and is therefore only applicable for small angles between the \(\vec{n}\)-axis and the \(\vec{n}_0\)-axis, is \([\text{BHR92, BHR94b}]:\)

\[
\hat{n}_{II, \text{SLIM}}(\sigma, \eta; s) = \begin{pmatrix} 1 \\ \theta_{2L,\text{per}}(s) \cdot \left(g_{19}(s) \cdot \sigma + g_{20}(s) \cdot \eta \right) \\ 0 \end{pmatrix}.
\]

From the above it is clear that \(\hat{n}_{II}\) obeys the radiationless Bloch equation (3.12).

\section*{Appendix C}

In this Appendix I briefly reconsider Machine I by extending it to ‘Machine III’ which is obtained by including the nonhorizontal component of the spin vector, i.e. by considering the full three-dimensional spin motion. For horizontal spin the dynamics of Machine III is the same as for Machine I.

As for Machine I I denote the spin vector in the \((\vec{m}_0, \vec{t}_0, \vec{n}_0, \vec{n}_0, \vec{l}_0, I)\)-frame by \(\vec{S}\). To cover the full three-dimensional spin motion one can employ, as mentioned in section 2.3.6, spherical coordinates, so that:

\[
\vec{S} \equiv \frac{\hbar}{2} \cdot \begin{pmatrix} \cos(\psi) \cdot \sin(\theta) \\ \sin(\psi) \cdot \sin(\theta) \\ \cos(\theta) \end{pmatrix},
\]

where \(\psi, \theta\) denote the azimuthal and polar angles. The horizontal spin vector (2.3) can be obtained from (C.1) by setting \(\theta = \pi/2\). Thus for \(\theta = \pi/2\) Machine III effectively reduces to Machine I.

In dealing with 4 variables, the processes to be studied for Machine III are denoted by \(\vec{y}(s)\), where:

\[
\vec{y}(s) \equiv \begin{pmatrix} \sigma(s) \\ \eta(s) \\ \psi(s) \\ \theta(s) \end{pmatrix},
\]

i.e. compared with \(\vec{x}(s)\) they have an additional component. The Langevin equation for Machine III is then defined by:

\[
d\vec{y}(s) = \vec{A} \cdot \vec{y}(s) \cdot ds + \vec{B} \cdot d\vec{W}(s),
\]

where

\[
\vec{A} \equiv \begin{pmatrix} 0 & a & 0 & 0 \\ b & c & 0 & 0 \\ 0 & d & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \vec{B} \equiv \sqrt{\omega} \cdot \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},
\]
Here the $\dot{W}_k(s)$ are Wiener processes. One sees by the behaviour of the $\theta$-variable in the Langevin equation (C.3) that Machine III indeed describes the same dynamics as Machine I. Therefore the Fokker-Planck equation corresponding to the Langevin equation (C.3) is identical with the Fokker-Planck equation (2.22) for Machine I.

For Machine III I adopt standard boundary conditions in all four variables $\sigma, \eta, \psi, \theta$ so that: $\dot{w} \to 0$ for $\sigma, \eta, \psi, \theta \to \pm \infty$, where $\dot{w}$ denotes the probability density. Thus the probability density is normalized by:

$$1 = \int_{-\infty}^{+\infty} d\sigma \int_{-\infty}^{+\infty} d\eta \int_{-\infty}^{+\infty} d\psi \int_{-\infty}^{+\infty} d\theta \cdot \dot{w}(\sigma, \eta, \psi, \theta; s).$$

(C.4)

The polarization vector is defined by:

$$\vec{P}^w_{\text{tot}}(s) \equiv \int_{-\infty}^{+\infty} d\sigma \int_{-\infty}^{+\infty} d\eta \int_{-\infty}^{+\infty} d\psi \int_{-\infty}^{+\infty} d\theta \cdot \dot{w}(\sigma, \eta, \psi, \theta; s) \cdot \begin{pmatrix} \cos(\psi) \cdot \sin(\theta) \\ \sin(\psi) \cdot \sin(\theta) \\ \cos(\theta) \end{pmatrix}.$$  

(C.5)

The polarization density is defined by:

$$\vec{P}^w(\sigma, \eta; s) \equiv \int_{-\infty}^{+\infty} d\psi \int_{-\infty}^{+\infty} d\theta \cdot \dot{w}(\sigma, \eta, \psi, \theta; s) \cdot \begin{pmatrix} \cos(\psi) \cdot \sin(\theta) \\ \sin(\psi) \cdot \sin(\theta) \\ \cos(\theta) \end{pmatrix}.$$  

(C.6)

Using the Fokker-Planck equation (2.22) one observes by (C.6) that the polarization density fulfills the same Bloch equation (2.106) as for Machine I.

For a process with only horizontal spin the probability density has the form:

$$\dot{w}(\sigma, \eta, \psi, \theta; s) = w(\sigma, \eta, \psi; s) \cdot \delta(\theta - \pi/2 + 2\pi N),$$  

(C.7)

where $N$ denotes an integer and where $w$ denotes the probability density which arises if the process would be described in the framework of Machine I. Note that $\dot{w}$, as given by (C.7), fulfills the Fokker-Planck equation (2.22).

## Appendix D

### D.1

In section 2.9 I observed that the equilibrium behaviour of Machine I in the presence of radiation resembles a Hamiltonian flow. In this appendix I now briefly reconsider Machine I by switching off the radiation effects to obtain a real Hamiltonian flow. I call this model ‘Machine IV’. The aim is to investigate the existence of an equilibrium spin distribution and to do that I adopt the usual approach of describing the system in terms of action-angle variables. Since I use a
canonical formalism I need an even number of variables. Hence I supplement \( \sigma, \eta, \psi \) by a fourth variable \( J \), canonically conjugate to \( \psi \), which is defined by:

\[
J \equiv \frac{\hbar}{2} \cdot \cos(\theta) ,
\]

where \( \theta \) denotes the polar angle in the spherical coordinate expression (2.33) of the spin vector \( \vec{S} \). Thus for Machine IV the spin vector in the \((\vec{m}_{0,1}, \vec{l}_{0,1}, \vec{n}_{0,1})\)-frame is parametrized as:

\[
\vec{S} \equiv \begin{pmatrix}
\sqrt{\frac{\hbar^2}{4} - J^2} \cdot \cos(\psi) \\
\sqrt{\frac{\hbar^2}{4} - J^2} \cdot \sin(\psi) \\
J
\end{pmatrix} .
\]  

(H.2)

Hence, as for Machine III, I consider the full three-dimensional spin motion, i.e. the nonhorizontal component of the spin vector is included. One sees that the additional variable describes the projection of the spin onto the vertical direction so that for processes running with Machine I one has: \( J = 0 \). But owing to the field geometry in Machine I one could have carried through the analysis with nonzero \( J \) if I had only been interested in the \( \psi \) distribution. The Poisson brackets are defined by [DK73, Yok86, BHR94a, BHR94b] [61]:

\[
1 = \{ \sigma, \eta \} = \{ \psi, J \} ,
\]

\[
0 = \{ \sigma, \psi \} = \{ \sigma, J \} = \{ \eta, \psi \} = \{ \eta, J \} ,
\]

and the Hamiltonian reads as:

\[
H_a \equiv -\frac{b}{2} \cdot \sigma^2 + a \cdot \eta^2 + d \cdot \eta \cdot J \equiv H_{orb} + H_{spin} ,
\]

where \( H_{orb} \) is defined in section 2.5 and:

\[
H_{spin} \equiv d \cdot \eta \cdot J .
\]

Due to the absence of radiation the Langevin equation for Machine IV reduces to the following canonical equations of motion:

\[
\sigma' = a \cdot \eta + d \cdot J , \quad \eta' = b \cdot \sigma , \quad \psi' = d \cdot \eta , \quad J' = 0 .
\]

One sees that \( H_{spin} \) contributes a (very small) Stern-Gerlach term [BHR94a, BHR94b] appearing in the first identity of (D.5) which was neglected for Machine I because this Stern-Gerlach effect vanishes at \( J = 0 \).

Thus Machine III, unlike Machine IV, is lacking the Stern-Gerlach force and Machine IV, unlike Machine III, is lacking radiation effects. Both machines involve the full three-dimensional spin motion but they parametrize it in different ways.

\[\text{(D.5)}\]

\[\text{Note that for Machine III (see Appendix C) the variable } \theta \text{ is used instead of } J.\]

\[\text{Note that because of (D.1), (D.3) I use units in Appendix D where } \hbar \text{ has the dimension of length BHR94a, BHR94b}.\]
D.2

Coming to the probability density, I now adopt boundary conditions for the variables $\psi, J$ which are natural for their role as spherical coordinates. In particular I adopt for $\psi$ the periodic boundary conditions mentioned in section 2.3.6. The probability density in the present case also depends on $J$ and I denote it by $w_{\text{mod,}a}$. The normalization condition (2.35) for $w_{\text{per}}$ translates by using (D.1) into:

$$1 = \int_{-\infty}^{+\infty} d\sigma \int_{-\infty}^{+\infty} d\eta \int_0^{2\pi} d\psi \int_{-\bar{h}/2}^{+\bar{h}/2} dJ \cdot w_{\text{mod,}a}(\sigma, \eta, \psi, J; s).$$  \hspace{1cm} (D.6)

For horizontal spin $w_{\text{mod,}a}$ has the form:

$$w_{\text{mod,}a}(\sigma, \eta, \psi, J; s) \equiv w_{\text{per}}(\sigma, \eta, \psi; s) \cdot \delta(J).$$  \hspace{1cm} (D.7)

Due to the absence of radiation the Fokker-Planck equation for Machine IV is the Liouville equation for the phase space evolution associated with the Hamiltonian (D.4):

$$0 = \left( \frac{d}{ds} \right)_{\text{tot}} w_{\text{mod,}a} = \frac{\partial w_{\text{mod,}a}}{\partial s} + \{w_{\text{mod,}a}, H_a\}. \hspace{1cm} (D.8)$$

The total derivative is zero for a Hamiltonian flow \cite{Gol80}.

D.3

To investigate the matter of equilibrium with the help of the evolution equation (D.8) I transform to action-angle variables. To come to these I first replace $\sigma, \eta$ by the orbital variables $\phi, J_{\text{orb}}$ defined in section 2.5. Thus I treat my system using the variables $\phi, J_{\text{orb}}, \psi, J$ with the Poisson brackets:

$$1 = \{\phi, J_{\text{orb}}\} = \{\psi, J\},$$

$$0 = \{\phi, \psi\} = \{\phi, J\} = \{J_{\text{orb}}, \psi\} = \{J_{\text{orb}}, J\}. \hspace{1cm} (D.9)$$

The Hamiltonian (D.4) transforms into

$$H_b \equiv -\sqrt{-ab} \cdot J_{\text{orb}} - d \cdot \left(\frac{b}{a}\right)^{1/4} \cdot \sqrt{2J_{\text{orb}}} \cdot \sin(\phi) \cdot J. \hspace{1cm} (D.10)$$

This Hamiltonian is not yet in action-angle form since it still contains the phase $\phi$. The probability density $w_{\text{mod,}b}$ for the variables $\phi, J_{\text{orb}}, \psi, J$ reads as:

$$w_{\text{mod,}b}(\phi, J_{\text{orb}}, \psi, J; s) \equiv w_{\text{mod,}a}(\sigma, \eta, \psi, J; s). \hspace{1cm} (D.11)$$

Note that $w_{\text{mod,}b}$ is periodic in $\phi, \psi$ with period $2\pi$. For the final step in obtaining the action-angle variables I use the spin-orbit action-angle formalism of \cite{DK73, Yok86, BHR}. This allows one to perform the following canonical transformation \cite{62}

$$\phi, J_{\text{orb}}, \psi, J \rightarrow \phi_{\text{new}}, J_{\text{orb,new}}, \psi_{\text{new}}, J_{\text{new}}:

\begin{align*}
\phi_{\text{new}} &\equiv \phi + d \cdot (a^3 \cdot b)^{-1/4} \cdot (2J_{\text{orb}})^{-1/2} \cdot \cos(\phi) \cdot J, \\
J_{\text{orb,new}} &\equiv J_{\text{orb}} + d \cdot (a^3 \cdot b)^{-1/4} \cdot \sqrt{2J_{\text{orb}}} \cdot \sin(\phi) \cdot J, \\
\psi_{\text{new}} &\equiv \psi + d \cdot (a^3 \cdot b)^{-1/4} \cdot \sqrt{2J_{\text{orb}}} \cdot \cos(\phi), \\
J_{\text{new}} &\equiv J.
\end{align*} \hspace{1cm} (D.12a-d)$$

\cite{62}See equations 4.30-32 in \cite{Yok86}.
whereby the terms containing $J$ in (D.12a), (D.12b) are due to Stern-Gerlach effects and are very small so that in effect the new orbital variables are numerically very close to the original orbital variables. The new variables have the following Poisson brackets

$$
1 = \{\phi_{\text{new}}, J_{\text{orb, new}}\} = \{\psi_{\text{new}}, J_{\text{new}}\},
0 = \{\phi_{\text{new}}, \psi_{\text{new}}\} = \{\phi_{\text{new}}, J_{\text{new}}\} = \{J_{\text{orb, new}}, \psi_{\text{new}}\} = \{J_{\text{orb, new}}, J_{\text{new}}\}.
$$

(D.13)

The Hamiltonian $H_b$ transforms into:

$$
H_c \equiv -\sqrt{-ab} \cdot J_{\text{orb, new}},
$$

(D.14)

and since it now only contains an action I finally have the desired form. Note that this Hamiltonian does not contain the spin action $J_{\text{new}}$. Note also that $\psi_{\text{new}}$ is identical to $\tilde{\psi}$ in section 2.2. Thus one sees that this canonical transformation which has removed the spin dependence from the Hamiltonian is equivalent to the reduction from a three-dimensional problem to a two-dimensional problem observed by (2.17b). This reduction reflects the nonuniqueness of the equilibrium state already observed in section 2 in the presence of radiation effects. Denoting the probability density in these variables by $w_{\text{mod, c}}$ one gets:

$$
w_{\text{mod, c}}(\phi_{\text{new}}, J_{\text{orb, new}}, \psi_{\text{new}}, J_{\text{new}}; s) = w_{\text{mod, b}}(\phi, J_{\text{orb}}, \psi, J; s).
$$

(D.15)

Note that $w_{\text{mod, c}}$ is periodic in $\phi_{\text{new}}, \psi_{\text{new}}$ with period $2\pi$. The corresponding Liouville equation reads as:

$$
\frac{\partial w_{\text{mod, c}}}{\partial s} = \{H_c, w_{\text{mod, c}}\}.
$$

(D.16)

D.4

Having obtained action-angle variables one now can discuss equilibrium. I define ‘equilibrium’ to mean that $\partial w_{\text{mod, c}}/\partial s$ is zero. Then by (D.14) the Poisson bracket $\{H_c, w_{\text{mod, c}}\}$ vanishes. Since $H_c$ is independent of $\phi_{\text{new}}$ the probability density $w_{\text{mod, c}}$ must be independent of $\phi_{\text{new}}$. However, since (D.14) does not contain $J_{\text{new}}$ the probability density $w_{\text{mod, c}}$ can still depend on $\psi_{\text{new}}$. The interpretation of this is that since the $\psi_{\text{new}}$ for each particle is constant (see (2.17b)), the $\psi_{\text{new}}$ distribution does not change as $s$ increases and is therefore in equilibrium. So although I have a complete transformation to action-angle variables the special form for the Hamiltonian (D.14) means that by insisting on equilibrium one cannot say very much about the $\psi_{\text{new}}$ distribution except that $w_{\text{mod, c}}$ has the form

$$
w_{\text{mod, c}}(\phi_{\text{new}}, J_{\text{orb, new}}, \psi_{\text{new}}, J_{\text{new}}; s) = w_{\text{mod, c}}(J_{\text{orb, new}}, \psi_{\text{new}}, J_{\text{new}}),
$$

(D.17)

so that $w_{\text{mod, c}}$ neither depends on $\phi_{\text{new}}$ nor on $s$. The dependence of $w_{\text{mod, c}}$ on $\psi_{\text{new}}$ reflects the nonuniqueness of the equilibrium state already observed in section 2 in the presence of radiation effects. If on the contrary the Hamiltonian had contained $J_{\text{new}}$ the $\psi_{\text{new}}$ distribution would have had to be uniform. In the case of horizontal spin one has $J = 0$ so that by (D.12a) one has: $J_{\text{new}} = 0$. Then (D.17) simplifies to:

$$
w_{\text{mod, c}}(J_{\text{orb, new}}, \psi_{\text{new}}, J_{\text{new}}) \equiv w_{\text{rest}}(J_{\text{orb, new}}, \psi_{\text{new}}) \cdot \delta(J_{\text{new}}),
$$

(D.18)

where $w_{\text{rest}}$ neither depends on $\phi_{\text{new}}, J_{\text{new}}$ nor on $s$.\footnote{The above action-angle formalism neglects higher orders in $\hbar$ in a specific way which is made use of in (D.13).}
Appendix E

E.1

The $s$-dependence of the first moment of the variables $\sigma, \eta$ allows one to define a damping time for every process. It turns out that this ‘orbital damping time’ is independent of the process. It is the same for machines I and II since they have the same orbital equations of motion.

Also it is shown how the orbital damping time is involved in the orbital correlation matrix (defined below).

E.2

For a given process $\vec{x}(s) = (\sigma(s), \eta(s), \psi(s))^T$ the stochastic averages of the orbital variables are given via (2.47), (2.50) by:

$$<\vec{z}(s)> = \exp\left(A_{\text{orb}} \cdot (s - s_1)\right) \cdot <\vec{z}(s_1)>$$

$$= \frac{i}{2 \cdot \lambda} \cdot \left( \begin{array}{ccc} g_1(s - s_1) & -a \cdot g_2(s - s_1) & -b \cdot g_2(s - s_1) \\ -b \cdot g_2(s - s_1) & g_2(s - s_1) & g_3(s - s_1) \end{array} \right) \cdot <\vec{z}(s_1)> .$$  \hspace{1cm} (E.1)

One observes by (2.51), (E.1) that $<\vec{z}(s)>$ contains the exponentially decreasing factor $\exp(c \cdot s/2)$ and that the remaining factors are periodic in $s$ with period $2 \cdot \pi/\lambda$. I therefore define the orbital damping time, denoted as $\tau_{\text{damp}}$, by:

$$\tau_{\text{damp}} \equiv -\frac{2}{c} = \frac{L}{\alpha_s} .$$  \hspace{1cm} (E.2)

One sees that $\tau_{\text{damp}}$ is independent of the process.

E.3

The ‘orbital correlation matrix’ $K_{\text{orb}}$ is defined by [Gar85]:

$$K_{\text{orb}}(s; s_1) \equiv \left( \begin{array}{cc} k_{\text{orb},11}(s; s_1) & k_{\text{orb},12}(s; s_1) \\ k_{\text{orb},21}(s; s_1) & k_{\text{orb},22}(s; s_1) \end{array} \right) ,$$  \hspace{1cm} (E.3)

where

$$k_{\text{orb},11}(s; s_1) \equiv <\sigma(s) \cdot \sigma(s_1) > - <\sigma(s) > \cdot <\sigma(s_1) > ,$$

$$k_{\text{orb},12}(s; s_1) \equiv <\sigma(s) \cdot \eta(s_1) > - <\sigma(s) > \cdot <\eta(s_1) > ,$$

$$k_{\text{orb},21}(s; s_1) \equiv <\eta(s) \cdot \sigma(s_1) > - <\eta(s) > \cdot <\sigma(s_1) > ,$$

$$k_{\text{orb},22}(s; s_1) \equiv <\eta(s) \cdot \eta(s_1) > - <\eta(s) > \cdot <\eta(s_1) > .$$  \hspace{1cm} (E.4)

Using the orbital joint probability density $w_{\text{orb, joint}}$ given by (2.92) one finds for $s_1 \leq s$:

$$<\sigma(s) \cdot \sigma(s_1)> = \int_{-\infty}^{+\infty} d\sigma \int_{-\infty}^{+\infty} d\sigma_1 \int_{-\infty}^{+\infty} d\eta_1 \cdot \sigma \cdot \sigma_1 \cdot w_{\text{orb, joint}}(\sigma, \eta; s; \sigma_1, \eta_1; s_1) ,$$

$$<\sigma(s) \cdot \eta(s_1)> = \int_{-\infty}^{+\infty} d\sigma \int_{-\infty}^{+\infty} d\sigma_1 \int_{-\infty}^{+\infty} d\eta_1 \cdot \sigma \cdot \eta_1 \cdot w_{\text{orb, joint}}(\sigma, \eta; s; \sigma_1, \eta_1; s_1) ,$$

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\[ \langle \eta(s) \cdot \sigma(s_1) \rangle = \int_{-\infty}^{+\infty} d\sigma \int_{-\infty}^{+\infty} d\sigma_1 \int_{-\infty}^{+\infty} d\eta \int_{-\infty}^{+\infty} d\eta_1 \cdot \eta \cdot \sigma_1 \cdot w_{\text{orb, joint}}(\sigma, \eta; s; \sigma_1, \eta_1; s_1), \]

\[ \langle \eta(s) \cdot \eta(s_1) \rangle = \int_{-\infty}^{+\infty} d\sigma \int_{-\infty}^{+\infty} d\sigma_1 \int_{-\infty}^{+\infty} d\eta \int_{-\infty}^{+\infty} d\eta_1 \cdot \eta_1 \cdot \eta_1 \cdot w_{\text{orb, joint}}(\sigma, \eta; s; \eta_1, \eta_1; s_1). \]

(E.5)

Because of (2.47), (2.89), (2.92), (E.4), (E.5) one has for \( s_1 \leq s \):

\[ \frac{\partial K_{\text{orb}}(s; s_1)}{\partial s} = A_{\text{orb}} \cdot K_{\text{orb}}(s; s_1). \]  

(E.6)

Also by (2.43), (E.4) one has:

\[ K_{\text{orb}}(s_1; s_1) = \sigma_{\text{orb}}(s_1), \]

(E.7)

where \( \sigma_{\text{orb}} \) denotes the orbital covariance matrix. From (E.6), (E.7) it follows for \( s_1 \leq s \) that:

\[ K_{\text{orb}}(s; s_1) = \exp \left( A_{\text{orb}} \cdot (s - s_1) \right) \cdot \sigma_{\text{orb}}(s_1), \]

(E.8)

and from (2.50),(E.8) I have for \( s_1 \leq s \):

\[ K_{\text{orb}}(s; s_1) = \frac{i}{2 \cdot \lambda} \cdot \begin{pmatrix} g_1(s - s_1) & -a \cdot g_2(s - s_1) \\ -b \cdot g_2(s - s_1) & -g_3(s - s_1) \end{pmatrix} \cdot \sigma_{\text{orb}}(s_1). \]  

(E.9)

Then by (2.51),(E.9) one finds for \( s_1 \leq s \) that the matrix elements of the orbital correlation matrix contain the exponentially decreasing factor \( \exp(c \cdot s/2) \). The remaining factors are periodic in \( s \) with period \( 2 \cdot \pi/\lambda \). Therefore \( \tau_{\text{damp}} \) is not only the orbital damping time, but also plays the role of an 'orbital correlation time'.

**Guide for the reader**

Please note the following conventions used in this paper:

- The modulus of a real or complex number \( v \) is denoted by \(|v|\). The real part of a complex number \( v \) is denoted by \( \Re\{v\} \).

- The transpose of a matrix is denoted by \( ^T \).

- The symbol \( \cdot \) denotes either matrix multiplication or scalar multiplication of matrices (this includes the multiplication of scalars).

- Objects \( \vec{v} \) which are denoted with an arrow (e.g. \( \vec{x}, \vec{z} \)) are column vectors, i.e. \( n \times 1 \) matrices. Thus \( \vec{v} = (v_1, ..., v_n)^T \), where \( v_1, ..., v_n \) are the components of \( \vec{v} \). The norm \( ||\vec{v}|| \) of a vector \( \vec{v} \) is defined by \( ||\vec{v}|| \equiv \sqrt{v_1^2 + ... + v_n^2} \).

- The vector product is denoted by \( \wedge \).

- A necessary ingredient of a Gaussian probability density, is that the resulting covariance matrix is nonsingular.
The starting azimuth of a process is denoted by $s_0$. Thus the domain of the azimuthal variable is given by $[s_0, +\infty)$. For processes 1,2,3,4 and 5 I have chosen $s_0 = 0$, i.e. the domain is given by the nonnegative real numbers.

The following table helps to find some of the main results on processes 1,2,3,4 and 5:

| name of the process                              | Process 1   | Process 2   | Process 3   | Process 4   | Process 5   |
|------------------------------------------------|--------------|--------------|--------------|--------------|--------------|
| Langevin equation                              | (2.7)        | (2.7)        | (3.8)        | (3.8)        | (3.8)        |
| Fokker-Planck equation                         | (2.22)       | (2.22)       | (3.3)        | (3.3)        | (3.3)        |
| probability density                            | (2.62)       | (2.93)       | (3.43)       | (3.84)       | (3.116)      |
| characteristic function                        | (2.60)       | (2.96)       | (3.41)       | (3.82)       | (3.114)      |
| Bloch equation for the polarization density    | (2.106)      | (2.106)      | (3.11)       | (3.11)       | (3.11)       |
| polarization density                           | (2.105)      | (2.135)      | (3.73)       | (3.91)       | (3.104)      |
| polarization vector                            | (2.71)       | (2.100)      | (3.61)       | (3.86)       | (3.118)      |
| complete decoherence of spin                   | no           | no           | yes          | yes          | yes          |

The following table helps to find some of the main abbreviations:

| $g_1(s), g_2(s), g_3(s), g_4(s), g_5(s)$ | section 2.4 |
| $\lambda$                                | section 2.4  |
| $\lambda_0, \sigma_\sigma, \sigma_\eta, \sigma_\psi$ | section 2.5 |
| $G(s), \delta_{L,per}(s), \theta_{2L,per}(s), d(s)$ | section 3.1 |
| $\bar{m}_{0,II}(s), \bar{m}_{0,II}(s), \bar{l}_{0,II}(s)$ | section 3.1 |
| $g_6(s)$                                  | section 3.1  |
| $g_7(s), g_8(s), g_9(s), g_{10}(s), g_{11}, g_{12}(s)$ | section 3.5 |
| $g_{13}(s), g_{14}(s), g_{15}, g_{16}(s), g_{17}, g_{18}$ | section 3.7 |
| $g_{21}(s), g_{22}(s), g_{23}(s)$         | section 3.7  |
| $g_{19}(s), g_{20}(s)$                    | Appendix B   |

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I use this opportunity to mention some typing errors on page 7 of [BBHMR94b].
Line 10: replace $p_{\sigma}$ by $\sigma_{p_{\sigma}}$. Line 11: replace $\sigma_{p_{\sigma}}$ by $\sigma_{\psi}$. Line 31: replace $\sigma_{p_{\sigma}}$ by $\sigma_{\psi}$.
Corresponding corrections should be made in [BBHMR94a].

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