On the solutions of the equation $x^3 + ax = b$ in $\mathbb{Z}_3^*$
with coefficients from $\mathbb{Q}_3$

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Abstract
In this paper we present the algorithm of finding the solutions of the equation $x^3 + ax = b$
in $\mathbb{Z}_3^*$ with coefficients from $\mathbb{Q}_3$.

AMS Subject Classification: 17Sxx.
Key words: $p$-adic numbers, solvability of equation, congruence.

1 Introduction
In the present time description of different structures of mathematics are studying
over field of $p$-adic numbers. In particular, now $p$-adic analysis is one of intensive
developing directions of mathematics. Numerous applications of $p$-adic numbers
found their own reflection in the theory of $p$-adic differential equations, $p$-adic theory
of probabilities, $p$-adic mathematical physics and others.

In the field of complex numbers it is well known fundamental Abel’s theorem
about insolvability in radicals of general equation of $n$-th degree ($n > 5$). In this field
square equation is solved by discriminant, for cubic equation there exist Cardano’s
formulas. In the field of $p$-adic numbers square equation no always has solution.
It is known the criteria of solvability of the equation $x^2 = a$ [3] and in [1] we can
find the solvability criteria for the equation $x^q = a$, where $q$ is an arbitrary natural
number.

The solvability criterion for the cubic equation $x^3 + ax = b$ in the field of $3 - adic$
numbers is different from the case $p > 3$.

In [6] criteria of solvability for the equation $x^3 + ax = b$ in $\mathbb{Z}_3^*$ with coefficients
from $\mathbb{Q}_3$ numbers with condition of $|a|_3 \neq \frac{1}{3}$ is studied.

Using the results of [6] in this paper we present the algorithm of finding the
solutions of the equation $x^3 + ax = b$ in $\mathbb{Z}_3^*$ with coefficients from $\mathbb{Q}_3$ for any $a$.

2 Preliminaries
Let $\mathbb{Q}$ be the field of rational numbers. Every rational number $x \neq 0$ can be
represented in the form $x = p^{\gamma(x)} \frac{n}{m}$, where $n, \gamma(x) \in \mathbb{Z}$, $m$ is a positive integer,
$(p, n) = 1$, $(p, m) = 1$ and $p$ is a fixed prime number. In the field $\mathbb{Q}$ we define a
norm by
$$|x|_p = \begin{cases} p^{-\gamma(x)}, & x \neq 0, \\ 0, & x = 0. \end{cases}$$

The norm $|x|_p$ is called a $p$-adic norm of $x$ and it satisfies so called the strong
triangle inequality. The completion of $\mathbb{Q}$ with respect to $p$-adic norm defines the
The cubic equation

Now let us study the main subject of our research – a cubic equation. Let we have

\[ p \neq 0 \]

where \( \gamma = \gamma(x) \in \mathbb{Z} \) and \( x_j \) are integers, \( 0 \leq x_j \leq p - 1 \), \( x_0 \neq 0 \) (\( j = 0, 1, \ldots \)).

\( p \)-Adic numbers \( x \), for which \( |x|_p \leq 1 \), are called integer \( p \)-adic numbers, and the set of these numbers is denoted by \( \mathbb{Z}_p \). Integers \( x \in \mathbb{Z}_p \), for which \( |x|_p = 1 \), are called units of \( \mathbb{Z}_p \), and their set is denoted by \( \mathbb{Z}_p^* \).

For any numbers \( a \) and \( m \) it is known the following

**Theorem 1** [2]. If \( (a, m) = 1 \), then a congruence \( ax \equiv b \) (mod \( m \)) has one and only one solution.

We also need the following

**Lemma 1** [1]. The following is true:

\[
\left( \sum_{i=0}^{\infty} x_i p^i \right)^q = x_0^q + \sum_{k=1}^{\infty} \left( q x_0^{q-1} x_k + N_k(x_0, x_1, \ldots, x_{k-1}) \right) p^k,
\]

where \( x_0 \neq 0 \), \( 0 \leq x_j \leq p - 1 \), \( N_1 = 0 \) and for \( k \geq 2 \)

\[
N_k = N_k(x_0, \ldots, x_{k-1}) = \sum_{\sum_{i=1}^{k-1} m_i = q, \sum_{i=1}^{k-1} m_i = k} q! \frac{x_0^{m_0} x_1^{m_1} \ldots x_{k-1}^{m_{k-1}}}{m_0! m_1! \ldots m_{k-1}!}.
\]

For \( q = 3 \) we have

\[
\left( \sum_{i=0}^{\infty} x_i p^i \right)^3 = x_0^3 + \sum_{k=1}^{\infty} \left( 3 x_0^2 x_k + N_k(x_0, x_1, \ldots, x_{k-1}) \right) p^k.
\]

For \( j \leq k \) we put

\[
P_k^j = P_k^j(x_0, x_1, \ldots, x_{j-1}) = \sum_{\sum_{i=0}^{k-1} m_i = 3, \sum_{i=1}^{j-1} m_i = k} \frac{6}{m_0! m_1! \ldots m_{j-1}!} x_0^{m_0} x_1^{m_1} \ldots x_{j-1}^{m_{j-1}}.
\]

Also the following is true:

\[
\left( \sum_{i=0}^{\infty} a_i p^i \right) \left( \sum_{j=0}^{\infty} x_j p^j \right) = \sum_{k=0}^{\infty} \left( \sum_{s=0}^{k} x_s a_{k-s} \right) p^k. \tag{1}
\]

### 3 The main result

Now let us study the main subject of our research – a cubic equation. Let we have the cubic equation

\[
y^3 + ry^2 + sy + t = 0.
\]

Note that, by replacing \( y = x - \frac{r}{3} \), the cubic equation \( y^3 + ry^2 + sy + t = 0 \) is taken to the next equation

\[
x^3 + ax = b. \tag{2}
\]

So, we will study equation (2) in \( \mathbb{Q}_p \), where \( x = p^{\gamma(x)}(x_0 + x_1 p + \ldots), \ a = p^{\gamma(a)}(a_0 + a_1 p + \ldots), \ b = p^{\gamma(b)}(b_0 + b_1 p + \ldots), \ x_j, a_j, b_j \in \{0, 1, \ldots, p-1\}, \ x_0, a_0, b_0 \neq 0, \ (j = 0, 1, \ldots) \).
In this paper we study the cubic equation over 3-adic numbers, i.e. \( a, b \in \mathbb{Q}_3 \) and \( x \in \mathbb{Z}_3^* \).

Putting the canonical form of \( a, b \) and \( x \) in (2), we get

\[
\left( \sum_{k=0}^{\infty} x_k 3^k \right)^3 + 3^{\gamma(a)} \left( \sum_{k=0}^{\infty} a_k 3^k \right) \left( \sum_{k=0}^{\infty} x_k 3^k \right) = 3^{\gamma(b)} \sum_{k=0}^{\infty} b_k 3^k .
\]

By Lemma 1 and equality (1), the equation (2) becomes

\[
x_0^3 + \sum_{k=1}^{\infty} (3x_0^2 x_k + N_k(x_0, x_1, \ldots, x_{k-1})) 3^k + 3^{\gamma(a)} \left( a_0 x_0 + \sum_{k=1}^{\infty} \left( \sum_{s=0}^{k} x_s a_{k-s} \right) 3^k \right) = 3^{\gamma(b)} \left( b_0 + \sum_{k=1}^{\infty} b_k 3^k \right) .
\]

(3)

**Proposition 1.** If one of the following conditions:

1) \( \gamma(a) = 0 \) and \( \gamma(b) < 0 \);
2) \( \gamma(a) > 0 \) and \( \gamma(b) > 0 \);
3) \( \gamma(a) > 0 \) and \( \gamma(b) < 0 \);
4) \( \gamma(a) < 0 \) and \( \gamma(b) = 0 \);
5) \( \gamma(a) < 0 \) and \( \gamma(b) > 0 \),

is fulfilled, then the equation (2) has not a solution in \( \mathbb{Z}_3^* \).

**Proof.** 1) Let \( \gamma(a) = 0 \) and \( \gamma(b) < 0 \). Multiplying the equation (3) by \( 3^{-\gamma(b)} \), we get the following congruence

\( b_0 \equiv 0 \pmod{3} \),

which is not correct. Consequently, the equation (2) has no solution in \( \mathbb{Z}_3^* \).

2) Let \( \gamma(a) > 0 \) and \( \gamma(b) > 0 \). Then from (3) it follows a congruence

\( x_0^3 \equiv 0 \pmod{3} \),

which has not a nonzero solution. Therefore, in \( \mathbb{Z}_3^* \) the equation (2) has not a solution.

In other cases we analogously get the congruences

\( b_0 \equiv 0 \pmod{3} \)

or

\( a_0 x_0 \equiv 0 \pmod{3} \),

a contradiction. Therefore, in \( \mathbb{Z}_3^* \) we have not a solution.

□

From the Proposition 1 we have that the cubic equation may have a solution if one of the following four cases

1) \( \gamma(a) = 0, \gamma(b) = 0 \),
2) \( \gamma(a) = 0, \gamma(b) > 0 \),
3) \( \gamma(a) < 0, \gamma(b) < 0 \),
4) \( \gamma(a) > 0, \gamma(b) = 0 \),

is hold.

The solvability criteria of the cubic equation for the cases 1), 2), 3) is given in [6]. The criteria for the case 4) \( \gamma(a) > 0, \gamma(b) = 0 \) is found only when \( \gamma(a) > 1 \), but
In this paper we present the algorithm of finding of the equation \( x^3 + ax = b \) for all cases.

**Theorem 2.** If \( \gamma(a) = \gamma(b) = 0 \) and \( a_0 = 1 \), then \( x \) to be a solution of the equation (2) in \( \mathbb{Z}_3^* \) if and only if the next congruences
\[
x_0^3 + a_0x_0 \equiv b_0 \pmod{3},
\]
\[
x_1a_0 + x_0a_1 + N_1(x_0) + M_1(x_0) \equiv b_1 \pmod{3},
\]
\[
x_k a_0 + x_{k-1}a_1 + \ldots + x_0a_k + x_0^2x_{k-1} + N_k(x_0, x_1, \ldots, x_{k-1}) + + M_k(x_0, x_1, \ldots, x_{k-1}) \equiv b_k \pmod{3}, \quad k \geq 2
\]
are fulfilled, where integers \( M_k(x_0, \ldots, x_{k-1}) \) are defined consequently from the following correlations
\[
x_0^3 + a_0x_0 = b_0 + M_1(x_0) \cdot 3,
\]
\[
x_1a_0 + x_0a_1 + N_1(x_0) = b_1 - M_1(x_0) + M_2(x_0, x_1) \cdot 3,
\]
\[
x_{k-1}a_0 + x_{k-2}a_1 + \ldots + x_0a_{k-1} + x_0^2x_{k-2} + N_{k-1}(x_0, x_1, \ldots, x_{k-2}) = = b_{k-1} - M_{k-1}(x_0, x_1, \ldots, x_{k-2}) + M_k(x_0, x_1, \ldots, x_{k-1}) \cdot 3, \quad k \geq 3.
\]

**Proof.** Let the conditions of the theorem are given, then from [6] we have necessity and sufficiency of existence of a solution of the equation (2).

Let
\[
x_0 + x_1 \cdot 3 + x_2 \cdot 3^2 + \ldots, \quad 0 \leq x_j \leq 2, \quad x_0 \neq 0, (j = 0, 1, \ldots)
\]
- a solution of the equation (2), then equality (3) becomes
\[
x_0^3 + \sum_{k=1}^{\infty} \left(3x_0^2x_k + N_k(x_0, x_1, \ldots, x_{k-1})\right)3^k + + a_0x_0 + \sum_{k=1}^{\infty} \left(\sum_{s=0}^{k} x_s a_{k-s}\right)3^k = b_0 + \sum_{k=1}^{\infty} b_k 3^k.
\]

So we have
\[
x_0^3 + a_0x_0 + (x_1a_0 + x_0a_1 + N_1(x_0)) \cdot 3 + + \sum_{k=2}^{\infty} \left(x_k a_0 + x_{k-1}a_1 + \ldots + x_0a_k + x_0^2x_{k-1} + N_k(x_0, x_1, \ldots, x_{k-1})\right)3^k = b_0 + \sum_{k=1}^{\infty} b_k 3^k,
\]

from which it follows necessity of the fulfilling of the congruences of the theorem.

If the equation (2) has a solution \( x \in \mathbb{Z}_3^* \), then from (3) it follows that
\[
x_0^3 + a_0x_0 \equiv b_0 \pmod{3}.
\]

Now let \( x \) is satisfied the congruences of the theorem. Since \( (a_0, 3) = 1 \), then by Theorem 1 there exist solutions \( x_k \) of the next congruences
\[
x_0^3 + a_0x_0 \equiv b_0 \pmod{3},
\]
\[
x_1a_0 + x_0a_1 + N_1(x_0) + M_1(x_0) \equiv b_1 \pmod{3},
\]
\[
x_k a_0 + x_{k-1}a_1 + \ldots + x_0a_k + x_0^2x_{k-1} + N_k(x_0, x_1, \ldots, x_{k-1}) + + M_k(x_0, x_1, \ldots, x_{k-1}) \equiv b_k \pmod{3}, \quad k \geq 2,
\]
where integers $M_k(x_0, \ldots, x_{k-1})$ are satisfied the conditions of the theorem. Then
\[
x_0^3 + \sum_{k=1}^{\infty} \left( 3x_0^2x_k + N_k(x_0, x_1, \ldots, x_{k-1}) \right) 3^k + a_0x_0 + \sum_{k=1}^{\infty} \left( \sum_{s=0}^{k} x_s a_{k-s} \right) 3^k =
\]
\[
= x_0^3 + a_0x_0 + (N_1 + x_0a_1 + a_0x_1)3^1 +
\]
\[
+ \sum_{k=2}^{\infty} \left( 3x_0^2x_{k-1} + N_k(x_0, x_1, \ldots, x_{k-1}) + x_0a_k + x_1a_{k-1} + \ldots + x_{k-1}a_1 + x_k a_0 \right) 3^k =
\]
\[
= b_0 + M_1(x_0) \cdot 3 + (b_1 - M_1(x_0) + M_2(x_0, x_1) \cdot 3) \cdot 3^1 +
\]
\[
+ \sum_{k=2}^{\infty} (b_k - M_k(x_0, x_1, \ldots, x_{k-1}) + M_{k+1}(x_0, x_1, \ldots, x_k) \cdot 3) \cdot 3^k = b_0 + \sum_{k=1}^{\infty} b_k 3^k.
\]
Therefore, we show that $x = \sum_{k=0}^{\infty} x_k 3^k$ is a solution of the equation (2). \(\square\)

Let us examine a case $\gamma(a) = 0, \gamma(b) > 0$ and get necessary and sufficient conditions for a solution of the equation (2).

**Theorem 3.** Let $\gamma(a) = 0, \gamma(b) = m > 0$ and $a_0 = 2$. Then $x$ to be a solution of the equation (2) in $\mathbb{Z}_3^*$ if and only if the next congruences
\[
x_0^3 + a_0 x_0 \equiv 0 \pmod{3},
\]
\[
x_1a_0 + x_0a_1 + N_1(x_0) + M_1(x_0) \equiv 0 \pmod{3},
\]
\[
x_ka_0 + x_{k-1}a_1 + \ldots + x_0a_k + x_0^2 x_{k-1} + N_k(x_0, x_1, \ldots, x_{k-1}) +
\]
\[
+ M_k(x_0, x_1, \ldots, x_{k-1}) \equiv 0 \pmod{3}, 2 \leq k \leq m - 1,
\]
\[
x_ka_0 + x_{k-1}a_1 + \ldots + x_0a_k + x_0^2 x_{k-1} + N_k(x_0, x_1, \ldots, x_{k-1}) +
\]
\[
+ M_k(x_0, x_1, \ldots, x_{k-1}) \equiv b_k \pmod{3}, k \geq m
\]
are fulfilled, where integers $M_k(x_0, x_1, \ldots, x_{k-1})$ are defined consequently from the following correlations
\[
x_0^3 + 2x_0 = M_1(x_0) \cdot 3,
\]
\[
x_1a_0 + x_0a_1 + N_1(x_0) = -M_1(x_0) + 3M_2(x_0, x_1),
\]
\[
x_ka_0 + x_{k-1}a_1 + \ldots + x_0a_k + x_0^2 x_{k-1} + N_k(x_0, x_1, \ldots, x_{k-1}) =
\]
\[
= -M_k(x_0, x_1, \ldots, x_{k-1}) + 3M_{k+1}(x_0, x_1, \ldots, x_k), 2 \leq k \leq m - 1,
\]
\[
x_ka_0 + x_{k-1}a_1 + \ldots + x_0a_k + x_0^2 x_{k-1} + N_k(x_0, x_1, \ldots, x_{k-1}) =
\]
\[
= b_{k-m} - M_k(x_0, x_1, \ldots, x_{k-1}) + 3M_{k+1}(x_0, x_1, \ldots, x_k), k \geq m.
\]

**Proof.** From [6] we have necessity and sufficiency of existence of a solution of the equation (2).

Let
\[
x = x_0 + x_1 \cdot 3 + x_2 \cdot 3^2 + \ldots, 0 \leq x_j \leq 2, x_0 \neq 0, (j = 0, 1, \ldots)
\]
- a solution of the equation (2), then equality (3) becomes
\[
x_0^3 + \sum_{k=1}^{\infty} \left( 3x_0^2x_k + N_k(x_0, x_1, \ldots, x_{k-1}) \right) 3^k +
\]

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from which it follows necessity of the fulfilling of the congruences of the theorem.

Now let \( x \) is satisfied the congruences of the theorem. Since \( (a_0, 3) = 1 \), then by Theorem 1 there exist solutions \( x_k \) of the next congruences

\[
x^3_0 + a_0x_0 \equiv 0 \pmod{3},
\]
\[
x_1a_0 + x_0a_1 + N_1(x_0) + M_1(x_0) \equiv 0 \pmod{3},
\]
\[
x_ka_0 + x_{k-1}a_1 + \ldots + x_0a_k + x_0^2x_{k-1} + N_k(x_0, x_1, \ldots, x_{k-1}) + M_k(x_0, x_1, \ldots, x_{k-1}) \equiv 0 \pmod{3}, 2 \leq k \leq m - 1,
\]
\[
x_ka_0 + x_{k-1}a_1 + \ldots + x_0a_k + x_0^2x_{k-1} + N_k(x_0, x_1, \ldots, x_{k-1}) + M_k(x_0, x_1, \ldots, x_{k-1}) \equiv b_{k-m} \pmod{3}, k \geq m,
\]

where integers \( M_k(x_0, \ldots, x_{k-1}) \) are satisfied the conditions of the theorem.

We have

\[
x^3_0 + \sum_{k=1}^{\infty} \left( 3x_0^3x_k + N_k(x_0, x_1, \ldots, x_{k-1}) \right) 3^k + a_0x_0 + \sum_{k=1}^{\infty} \left( \sum_{s=0}^{k} x_s a_{k-s} \right) 3^k =
\]
\[
= x^3_0 + a_0x_0 + (N_1 + x_0a_1 + a_0x_1) 3^1 + \sum_{k=2}^{\infty} \left( x_0^2x_{k-1} + N_k(x_0, x_1, \ldots, x_{k-1}) + x_0a_k + x_1a_{k-1} + \ldots + x_{k-1}a_1 + x_ka_0 \right) 3^k =
\]
\[
= M_1(x_0) \cdot 3 + (-M_1(x_0) + M_2(x_0, x_1) \cdot 3) \cdot 3^1 + \sum_{k=2}^{m-1} (-M_k(x_0, x_1, \ldots, x_{k-1}) + M_{k+1}(x_0, x_1, \ldots, x_1) \cdot 3) \cdot 3^k + \sum_{k=m}^{\infty} (b_{k-m} - M_k(x_0, x_1, \ldots, x_{k-1}) + M_{k+1}(x_0, x_1, \ldots, x_1) \cdot 3) \cdot 3^k = 3^m \left( b_0 + \sum_{k=1}^{\infty} b_k 3^k \right).
\]

Therefore, we show that \( x \) is a solution of the equation (2). \( \square \)

The following theorem gives necessary and sufficient conditions for a solution of the equation (2) for the case \( \gamma(a) < 0 \) and \( \gamma(b) < 0 \).

**Theorem 4.** Let

\[
\gamma(a) = \gamma(b) = -m < 0 \quad (m > 0).
\]
Then \( x \) to be a solution of the equation (2) in \( \mathbb{Z}_3^* \) if and only if the next congruences
\[
a_0x_0 \equiv b_0 \pmod{3},
\]
\[
x_ka_0 + x_{k-1}a_1 + \ldots + x_0a_k + M_k(x_0, x_1, \ldots, x_{k-1}) \equiv b_k \pmod{3}, \quad 1 \leq k \leq m - 1,
\]
\[
x_ma_0 + x_{m-1}a_1 + \ldots + x_0a_m + x_0^3 + M_m(x_0, x_1, \ldots, x_{m-1}) \equiv b_m \pmod{3},
\]
\[
x_{m+1}a_0 + x_ma_1 + \ldots + x_0a_{m+1} + M_{m+1}(x_0, x_1, \ldots, x_m) \equiv b_{m+1} \pmod{3},
\]
\[
x_ka_0 + x_{k-1}a_1 + \ldots + x_0a_k + x_0^2x_{k-m-1} + N_{k-m}(x_0, x_1, \ldots, x_{k-m-1}) +
\]
\[+M_k(x_0, x_1, \ldots, x_{k-1}) \equiv b_k \pmod{3}, \quad k \geq m + 2
\]
are fulfilled, where integers \( M_k(x_0, x_1, \ldots, x_{k-1}) \) are defined consequently from the equalities
\[
a_0x_0 = b_0 + M_1(x_0) \cdot 3,
\]
\[
x_ka_0 + x_{k-1}a_1 + \ldots + x_0a_k = b_k - M_k(x_0, x_1, \ldots, x_{k-1}) + 3M_{k+1}(x_0, x_1, \ldots, x_k), \quad 1 \leq k \leq m - 1,
\]
\[
x_ma_0 + x_{m-1}a_1 + \ldots + x_0a_m + x_0^3 = b_m - M_m(x_0, x_1, \ldots, x_{m-1}) + 3M_{m+1}(x_0, x_1, \ldots, x_m),
\]
\[
x_{m+1}a_0 + x_ma_1 + \ldots + x_0a_{m+1} = b_{m+1} - M_{m+1}(x_0, x_1, \ldots, x_m) + 3M_{m+2}(x_0, \ldots, x_{m+1}),
\]
\[
x_ka_0 + x_{k-1}a_1 + \ldots + x_0a_k + x_0^2x_{k-m-1} + N_{k-m}(x_0, x_1, \ldots, x_{k-m-1}) =
\]
\[= b_k - M_k(x_0, x_1, \ldots, x_{k-1}) + 3M_{k+1}(x_0, x_1, \ldots, x_k), \quad k \geq m + 2.
\]

**Proof.** Recall that the condition \( \gamma(a) = \gamma(b) = -m < 0 \) gives the solvability of the equation (2).

Multiplying (3) by \( 3^m \), we get
\[
3^m \left( x_0^3 + \sum_{k=1}^{\infty} \left( 3x_0^2x_k + N_k(x_0, x_1, \ldots, x_{k-1}) \right) 3^k \right) + a_0x_0 + \sum_{k=1}^{\infty} \left( \sum_{s=0}^{k} x_s a_{k-s} \right) 3^k =
\]
\[= a_0x_0 + \sum_{k=1}^{m-1} (x_ka_0 + x_{k-1}a_1 + \ldots + x_0a_k) 3^k +
\]
\[+(x_0^3 + x_ma_0 + x_{m-1}a_1 + \ldots + x_0a_m)3^m + (x_{m+1}a_0 + x_ma_1 + \ldots + x_0a_{m+1})3^{m+1} +
\]
\[+ \sum_{k=m+2}^{\infty} \left( x_0^2x_{k-m-1} + N_{k-m}(x_0, x_1, \ldots, x_{k-m-1}) + x_ka_0 + x_{k-1}a_1 + \ldots + x_0a_k \right) 3^k =
\]
\[= b_0 + \sum_{k=1}^{\infty} b_k 3^k,
\]
which deduces necessity of the fulfilling of the congruences of the theorem.

Since \( (a_0, 3) = 1 \), then there exist unique solutions \( x_k \) of the next congruences
\[
a_0x_0 \equiv b_0 \pmod{3},
\]
\[
x_ka_0 + x_{k-1}a_1 + \ldots + x_0a_k + M_k(x_0, x_1, \ldots, x_{k-1}) \equiv b_k \pmod{3}, \quad 1 \leq k \leq m - 1,
\]
\[
x_ma_0 + x_{m-1}a_1 + \ldots + x_0a_m + x_0^3 + M_m(x_0, x_1, \ldots, x_{m-1}) \equiv b_m \pmod{3},
\]
\[
x_{m+1}a_0 + x_ma_1 + \ldots + x_0a_{m+1} + M_{m+1}(x_0, x_1, \ldots, x_m) \equiv b_{m+1} \pmod{3},
\]
\[
x_ka_0 + x_{k-1}a_1 + \ldots + x_0a_k + x_0^2x_{k-m-1} + N_{k-m}(x_0, x_1, \ldots, x_{k-m-1}) +
\]
\[+M_k(x_0, x_1, \ldots, x_{k-1}) \equiv b_k \pmod{3}, \quad k \geq m + 2,
\]
where integers $M_k(x_0, x_1, \ldots, x_{k-1})$, are defined as in the statement of the theorem.

Then
\[
\begin{align*}
   a_0 x_0 &+ \sum_{k=1}^{m-1} (x_k a_0 + x_{k-1} a_1 + \ldots + x_0 a_k) 3^k + \\
   + (x_0^3 + x_m a_0 + x_{m-1} a_1 + \ldots + x_0 a_m) 3^m+
   + (x_{m+1} a_0 + x_m a_1 + \ldots + x_0 a_{m+1}) 3^{m+1}+
   + \sum_{k=m+2}^{\infty} (x_0^2 x_{k-m-1} + N_{k-m}(x_0, x_1, \ldots, x_{k-m-1}) + x_k a_0 + x_{k-1} a_1 + \ldots + x_0 a_k) 3^k = \\
   = b_0 + M_1(x_0) \cdot 3 + (b_1 - M_1(x_0) + M_2(x_0, x_1) \cdot 3) \cdot 3 + \\
   + \sum_{k=2}^{\infty} (b_k - M_k(x_0, x_1, \ldots, x_{k-1}) + M_{k+1}(x_0, x_1, \ldots, x_k) \cdot 3) \cdot 3^k = b_0 + \sum_{k=1}^{\infty} b_k 3^k.
\end{align*}
\]

Therefore, we show that $x = \sum_{k=0}^{\infty} x_k 3^k$ is a solution of the equation (2). \[\square\]

Examining various cases of $\gamma(a)$ and $\gamma(b)$ we need to study only the case $\gamma(a) > 0$ and $\gamma(b) = 0$. Because of appearance of uncertainty of a solution, we divide this case to $\gamma(a) > 1$ and $\gamma(a) = 1$.

**Theorem 5.** Let $\gamma(a) = 2$, $\gamma(b) = 0$ and $(b_0, b_1) = (1, 0)$ or $(2, 2)$. Then $x$ to be a solution of the equation (2) in $\mathbb{Z}_3^*$ if and only if he next congruences
\[
\begin{align*}
   x_0^3 &\equiv b_0 \pmod{3}, \\
   x_0^3 &\equiv b_0 + b_1 \cdot 3 \pmod{9}, \\
   x_0^2 x_1 + x_0 a_0 + M_1(x_0) &\equiv b_2 \pmod{3}, \\
   x_0^2 x_2 + P_2(x_0, x_1) + x_1 a_0 + x_0 a_1 + x_0 x_1^2 + M_2(x_0, x_1) &\equiv b_3 \pmod{3}, \\
   x_0^2 x_{k-1} + P_{k-1}(x_0, x_1, \ldots, x_{k-2}) + 2 x_0 x_1 x_{k-2} + x_{k-2} a_0 + x_{k-3} a_1 + \ldots + x_0 a_{k-2} + \\
   + M_{k-1}(x_0, x_1, \ldots, x_{k-2}) &\equiv b_k \pmod{3}, \quad k \geq 4
\end{align*}
\]

are fulfilled, where integers $M_k(x_0, x_1, \ldots, x_{k-1})$ are defined from the equalities
\[
\begin{align*}
   x_0^3 &= b_0 + b_1 \cdot 3 + M_1(x_0) \cdot 9, \\
   x_0^2 x_1 + x_0 a_0 &\equiv b_2 - M_1(x_0) + 3M_2(x_0, x_1), \\
   x_0^2 x_2 + P_2(x_0, x_1) + x_1 a_0 + x_0 a_1 + x_0 x_1^2 &\equiv b_3 - M_2(x_0, x_1) + 3M_3(x_0, x_1, x_2), \\
   x_0^2 x_{k-1} + P_{k-1}(x_0, x_1, \ldots, x_{k-2}) + 2 x_0 x_1 x_{k-2} + x_{k-2} a_0 + x_{k-3} a_1 + \ldots + x_0 a_{k-2} = \\
   = b_k - M_{k-1}(x_0, x_1, \ldots, x_{k-2}) + 3M_k(x_0, x_1, \ldots, x_{k-1}), \quad k \geq 4.
\end{align*}
\]

**Proof.** Again from [6] we have necessity and sufficiency of existence of a solution of the equation (2).

If
\[
x = x_0 + x_1 \cdot 3 + x_2 \cdot 3^2 + \ldots, \quad 0 \leq x_j \leq 2, \quad x_0 \neq 0, (j = 0, 1, \ldots)
\]

- a solution of the equation (2), then equality (3) becomes
\[
x_0^3 + \sum_{k=1}^{\infty} (3x_0^2 x_k + N_k(x_0, x_1, \ldots, x_{k-1})) 3^k +
\]
\[+3^m \left( a_0 x_0 + \sum_{k=1}^{\infty} \left( \sum_{s=0}^{k} x_s a_{k-s} \right) 3^k \right) = b_0 + b_1 \cdot 3 + \sum_{k=2}^{\infty} b_k 3^k.\]

Since \( N_k(x_0, x_1, \ldots, x_{k-1}) \), \( n \in \mathbb{N} \) depend only on \( x_0, x_1, \ldots, x_{k-1} \), it is easy to check that \( N_k(x_0, x_1, \ldots, x_{k-1}) \) can be written

\[N_2(x_0, x_1) = 3x_0 x_1^2,\]

\[N_k(x_0, x_1, \ldots, x_{k-1}) = P_k^{k-1}(x_0, x_1, \ldots, x_{k-2}) + 6x_0 x_1 x_{k-1}, \quad k \geq 3,\]

and we have

\[x_0^3 + x_0^2 x_1^2 + (x_0^2 x_2 + x_0 x_1^2) 3^2 + \sum_{k=3}^{\infty} (3x_0^2 x_k + P_k^{k-1}(x_0, x_1, \ldots, x_{k-2}) + 6x_0 x_1 x_{k-1}) 3^k + \]

\[+3^m \left( a_0 x_0 + \sum_{k=1}^{\infty} \left( \sum_{s=0}^{k} x_s a_{k-s} \right) 3^k \right) = b_0 + b_1 \cdot 3 + \sum_{k=2}^{\infty} b_k 3^k. \tag{4}\]

Since \( m = 2 \) the equality (4) becomes

\[x_0^3 + (x_0^2 x_1 + x_0 a_0) 3^2 + (x_0^2 x_2 + P_3^2(x_0, x_1) + x_1 a_0 + x_0 a_1 + x_0 x_1^2) 3^3 + \]

\[+ \sum_{k=1}^{\infty} (x_0^2 x_{k-1} + P_k^{k-1}(x_0, x_1, \ldots, x_{k-2}) + 2x_0 x_1 x_{k-2} + x_{k-2} a_0 + x_{k-3} a_1 + \ldots + x_{a_{k-2}}) 3^k = \]

\[= b_0 + b_1 \cdot 3 + \sum_{k=2}^{\infty} b_k 3^k,\]

from which it follows necessity of the fulfilling of the congruences of the theorem.

Let \( x \) is satisfied the congruences of the theorem. Since \( (a_0, 3) = 1 \), then by Theorem 1 there exist solutions \( x_k \) of the next congruences

\[x_0^3 \equiv b_0 (mod 3),\]

\[x_0^3 \equiv b_0 + b_1 \cdot 3 \quad (mod 9),\]

\[x_0^2 x_1 + x_0 a_0 + M_1(x_0) \equiv b_2 (mod 3),\]

\[x_0 x_2 + P_3^2(x_0, x_1) + x_1 a_0 + x_0 a_1 + x_0 x_1^2 + M_2(x_0, x_1) \equiv b_3 (mod 3),\]

\[x_0^2 x_{k-1} + P_k^{k-1}(x_0, x_1, \ldots, x_{k-2}) + 2x_0 x_1 x_{k-2} + x_{k-2} a_0 + x_{k-3} a_1 + \ldots + x_{a_{k-2}} + \]

\[+ M_{k-1}(x_0, x_1, \ldots, x_{k-2}) \equiv b_k (mod 3), \quad k \geq 4\]

where integers \( M_k(x_0, x_1, \ldots, x_{k-1}) \), are defined in the statement of the theorem.

So, we have

\[x_0^3 + (x_0^2 x_1 + x_0 a_0) 3^2 + (x_0^2 x_2 + P_3^2(x_0, x_1) + x_1 a_0 + x_0 a_1 + x_0 x_1^2) 3^3 + \]

\[+ \sum_{k=1}^{\infty} (x_0^2 x_{k-1} + P_k^{k-1}(x_0, x_1, \ldots, x_{k-2}) + 2x_0 x_1 x_{k-2} + x_{k-2} a_0 + x_{k-3} a_1 + \ldots + x_{a_{k-2}} 3^k = \]

\[= b_0 + b_1 \cdot 3 + M_1(x_0) \cdot 9 + (b_2 - M_1(x_0) + 3M_2(x_0, x_1)) 9 + \]

\[+ \sum_{k=3}^{\infty} (b_k - M_{k-1}(x_0, x_1, \ldots, x_{k-2}) + 3M_k(x_0, x_1, \ldots, x_{k-1})) \cdot 3^k\]
Therefore, we checked that \( x \) is a solution of the equation (2).

\[ x^3 = b_0 + b_1 \cdot 3 + \sum_{k=2}^{\infty} b_k 3^k. \]

**Theorem 6.** Let \( \gamma(a) = 3, \gamma(b) = 0 \) and \( (b_0, b_1) = (1, 0) \) or \( (2, 2) \). Then \( x \) to be a solution of the equation (2) in \( \mathbb{Z}_3^* \) if and only if he next congruences

\[
\begin{align*}
x_0^3 &\equiv b_0 \pmod{3}, \\
x_0^2 &\equiv b_0 + b_1 \cdot 3 \pmod{9}, \\
x_0^2 x_1 + M_1(x_0) &\equiv b_2 \pmod{3}, \\
x_0^2 x_2 + P_3^2(x_0, x_1) + x_0 a_0 + x_0 x_1^2 + M_2(x_0, x_1) &\equiv b_3 \pmod{3}, \\
x_0^2 x_{k-1} + P_k^{k-1}(x_0, x_1, \ldots, x_{k-2}) + 2 x_0 x_1 x_{k-2} + x_{k-3} a_0 + x_{k-4} a_1 + \ldots + x_o a_{k-3} + M_{k-1}(x_0, x_1, \ldots, x_{k-2}) &\equiv b_k \pmod{3}, \quad k \geq 4
\end{align*}
\]

are fulfilled, where integers \( M_k(x_0, x_1, \ldots, x_{k-1}) \) are defined from the equalities

\[
\begin{align*}
x_0^3 &= b_0 + b_1 \cdot 3 + M_1(x_0) \cdot 9, \\
x_0^2 x_1 &= b_2 - M_1(x_0) + 3 M_2(x_0, x_1), \\
x_0^2 x_2 + P_3^2(x_0, x_1) + x_0 a_0 + x_0 x_1^2 &= b_3 - M_2(x_0, x_1) + 3 M_3(x_0, x_1, x_2), \\
x_0^2 x_{k-1} + P_k^{k-1}(x_0, x_1, \ldots, x_{k-2}) + 2 x_0 x_1 x_{k-2} + x_{k-3} a_0 + x_{k-4} a_1 + \ldots + x_o a_{k-3} &\equiv 0, \\
&= b_k - M_{k-1}(x_0, x_1, \ldots, x_{k-2}) + 3 M_{k-1}(x_0, x_1, \ldots, x_k) + 3 M_k(x_0, x_1, \ldots, x_{k-1}) \equiv b_k \pmod{3}, \quad k \geq 4.
\end{align*}
\]

**Proof.** Analogously to the proof of the Theorem 5.

Similarly to the Theorem 5, it is proved the following

**Theorem 7.** Let \( \gamma(a) = m \geq 4, \gamma(b) = 0 \) and \( (b_0, b_1) = (1, 0) \) or \( (2, 2) \). Then \( x \) to be a solution of the equation (2) in \( \mathbb{Z}_3^* \) if and only if he next congruences

\[
\begin{align*}
x_0^3 &\equiv b_0 \pmod{3}, \\
x_0^2 &\equiv b_0 + b_1 \cdot 3 \pmod{9}, \\
x_0^2 x_1 + M_1(x_0) &\equiv b_2 \pmod{3}, \\
x_0^2 x_2 + P_3^2(x_0, x_1) + x_0 a_0 + x_0 x_1^2 + M_2(x_0, x_1) &\equiv b_3 \pmod{3}, \\
x_0^2 x_{k-1} + P_k^{k-1}(x_0, x_1, \ldots, x_{k-2}) + 2 x_0 x_1 x_{k-2} + x_{k-3} a_0 + x_{k-4} a_1 + \ldots + x_o a_{k-3} + P_m^{m-1}(x_0, x_1, \ldots, x_{m-2}) + 2 x_0 x_1 x_{m-2} + x_0 a_0 &\equiv b_m \pmod{3}, \\
x_0^2 x_{k-1} + P_k^{k-1}(x_0, x_1, \ldots, x_{k-2}) + 2 x_0 x_1 x_{k-2} + x_{k-3} a_0 + x_{k-4} a_1 + \ldots + x_o a_{k-3} &\equiv b_k \pmod{3}, \quad k \geq m + 1
\end{align*}
\]

are fulfilled, where integers \( M_k(x_0, x_1, \ldots, x_{k-1}) \) are defined from the equalities

\[
\begin{align*}
x_0^3 &= b_0 + b_1 \cdot 3 + M_1(x_0) \cdot 9, \\
x_0^2 x_1 &= b_2 - M_1(x_0) + 3 M_2(x_0, x_1), \\
x_0^2 x_2 + P_3^2(x_0, x_1) + x_0 a_0 + x_0 x_1^2 &= b_3 - M_2(x_0, x_1) + 3 M_3(x_0, x_1, x_2), \\
x_0^2 x_{k-1} + P_k^{k-1}(x_0, x_1, \ldots, x_{k-2}) + 2 x_0 x_1 x_{k-2} + x_{k-3} a_0 + x_{k-4} a_1 + \ldots + x_o a_{k-3} &\equiv 0, \\
&= b_k - M_{k-1}(x_0, x_1, \ldots, x_{k-2}) + 3 M_{k-1}(x_0, x_1, \ldots, x_k) + 3 M_k(x_0, x_1, \ldots, x_{k-1}) \equiv b_k \pmod{3}, \quad k \geq m + 1
\end{align*}
\]
\[
x_0^2x_{m-1} + P_m^{m-1}(x_0, \ldots, x_{m-2}) + 2x_0x_1x_{m-2} + x_0a_0 =
\]
\[
= b_m - M_{m-1}(x_0, \ldots, x_{m-2}) + 3M_m(x_0, \ldots, x_{m-1}),
\]
\[
x_0^2x_{k-1} + P_k^{k-1}(x_0, x_1, \ldots, x_{k-2}) + 2x_0x_1x_{k-2} + x_k a_0 + \ldots + x_0 a_{k-m} =
\]
\[
= b_k - M_{k-1}(x_0, x_1, \ldots, x_{k-2}) + 3M_k(x_0, x_1, \ldots, x_{k-1}), \quad k \geq m + 1.
\]

Now we consider the equality (3) with \( \gamma(a) = 1, \gamma(b) = 0 \). Then we get
\[
x_0^3 + \sum_{k=1}^{\infty} \left( 3x_0^2x_k + N_k(x_0, x_1, \ldots, x_{k-1}) \right) 3^k + 3 \left( a_0x_0 + \sum_{k=1}^{\infty} \left( \sum_{s=0}^{k} x_s a_{k-s} \right) 3^k \right) =
\]
\[
= x_0^3 + a_0x_0 \cdot 3 + \sum_{k=2}^{\infty} \left( x_0^2x_{k-1} + N_k(x_0, x_1, \ldots, x_{k-1}) \right) 3^k +
\]
\[
+ \sum_{k=2}^{\infty} (x_{k-1}a_0 + x_{k-2}a_1 + \ldots + x_0a_{k-1}) 3^k = b_0 + \sum_{k=1}^{\infty} b_k 3^k. \quad (5)
\]

For \( k \geq 1, s \geq 1, i \leq j - 1 \) we denote
\[
A_0 = x_0^2 + a_0,
\]
\[
A_k = \frac{A_{k-1}}{3} + a_k + R_k, \quad R_k = \sum_{j=0}^{k} x_j x_{k-j}, \quad k \geq 1,
\]
\[
N_j^i = \begin{cases} 
\frac{N_j^{j-1}}{3}, & j = 3s - 1, \\
\frac{N_j^{j-1}}{3} + x_j^3, & j = 3s, \\
\frac{N_j^{j-1} - x_j^3}{3}, & j = 3s + 1,
\end{cases}
\]
\[
S_j^i = \begin{cases} 
\frac{P_j^{j-1}}{3}, & j = 3s - 1, \\
\frac{P_j^{j-1}}{3} + x_j^3, & j = 3s, \\
\frac{P_j^{j-1} - x_j^3}{3}, & j = 3s + 1.
\end{cases}
\]

**Theorem 8.** Let \( \gamma(a) = 1, \gamma(b) = 0 \) and \( x \in \mathbb{Z}_3^* \) to be so that \( A_0 = x_0^2 + a_0 \equiv 0 \pmod{3} \). Then \( x \) to be a solution of the equation (2) in \( \mathbb{Z}_3^* \) if and only if the congruences
\[
x_0^3 \equiv b_0 \pmod{3},
\]
\[
x_0a_0 + M_1(x_0) \equiv b_1 \pmod{3},
\]
\[
(x_0^2 + a_0)x_{k-1} + N_k(x_0, x_1, \ldots, x_{k-2}) + x_{k-2}a_1 + \ldots + x_0a_{k-1} +
\]
\[
+ M_k(x_0, \ldots, x_{k-2}) \equiv b_k \pmod{3}, \quad k \geq 2
\]
are faithfully, where
\[
M_1(x_0) = \frac{x_0^3 - b_0}{3}
\]
and integers \( M_k(x_0, \ldots, x_{k-2}), (k \geq 2) \) are defined from the equalities
\[
x_0a_0 + M_1(x_0) = b_1 + M_2(x_0) \cdot 3,
\]
\[(x_0^2 + a_0)x_{k-1} + N'_k(x_0, x_1, \ldots, x_{k-2}) + x_{k-2}a_1 + \ldots + x_0a_{k-1} + M_k(x_0, \ldots, x_{k-2}) = b_k + M_{k+1}(x_0, \ldots, x_{k-1}) \cdot 3, \quad k \geq 2.\]

**Proof.** Let \( x \in \mathbb{Z}_3^* \) be a solution of the equation (2), then we have

\[
x_0^3 + \sum_{k=1}^{\infty} (3x_0^2x_k + N_k(x_0, x_1, \ldots, x_{k-1})) 3^k + 3 \left( a_0x_0 + \sum_{k=1}^{\infty} \left( \sum_{s=0}^{k} x_sa_{k-s} \right) 3^k \right) =
\]

\[
= x_0^3 + a_0x_0 \cdot 3 + \sum_{k=2}^{\infty} (x_0^2x_{k-1} + N_k(x_0, x_1, \ldots, x_{k-1})) 3^k +
\]

\[
+ \sum_{k=2}^{\infty} (x_{k-1}a_0 + x_{k-2}a_1 + \ldots + x_0a_{k-1}) 3^k = x_0^3 + a_0x_0 \cdot 3 +
\]

\[
+ \sum_{k=2}^{\infty} (x_0^2 + a_0)x_{k-1} + N'_k(x_0, x_1, \ldots, x_{k-2}) + x_{k-2}a_1 + \ldots + x_0a_{k-1}) 3^k =
\]

\[
= b_0 + \sum_{k=1}^{\infty} b_k 3^k.
\]

Therefore, the congruences of the theorem are fulfilled. Let a following system of the congruences

\[
x_0^3 \equiv b_0 \pmod{3},
\]

\[
x_0a_0 + M_1(x_0) \equiv b_1 \pmod{3},
\]

where \( M_1(x_0) = \frac{x_0^2 - b_0}{3} \), has a solution \( x_0 \). Then denote by \( M_2(x_0) \) the number satisfying the equality \( 3M_2(x_0) = x_0a_0 + M_1(x_0) - b_1 \).

Using Theorem 1, we have existence of solutions \( x_k \) of the following congruences

\[
(x_0^2 + a_0)x_{k-1} + N'_k(x_0, x_1, \ldots, x_{k-2}) + x_{k-2}a_1 + \ldots + x_0a_{k-1} +
\]

\[
+ M_k(x_0, \ldots, x_{k-2}) \equiv b_k \pmod{3}, \quad k \geq 2,
\]

where integers \( M_k(x_0, \ldots, x_{k-2}), (k \geq 3) \) are defined from the equalities

\[
(x_0^2 + a_0)x_{k-1} + N'_k(x_0, x_1, \ldots, x_{k-2}) + x_{k-2}a_1 + \ldots + x_0a_{k-1} +
\]

\[
+ M_k(x_0, \ldots, x_{k-2}) = b_k + M_{k+1}(x_0, \ldots, x_{k-1}) \cdot 3, \quad k \geq 2.
\]

The next chain of equalities

\[
x_0^3 + a_0x_0 \cdot 3 + \sum_{k=2}^{\infty} (x_0^2 + a_0)x_{k-1} + N'_k(x_0, x_1, \ldots, x_{k-2}) + x_{k-2}a_1 + \ldots + x_0a_{k-1}) 3^k =
\]

\[
= b_0 + M_1(x_0) \cdot 3 + (b_1 - M_1(x_0) + M_2(x_0) \cdot 3) \cdot 3 +
\]

\[
+ \sum_{k=2}^{\infty} (b_k - M_k(x_0, x_1, \ldots, x_{k-2}) + M_{k+1}(x_0, x_1, \ldots, x_{k-1}) \cdot 3) \cdot 3^k = b_0 + \sum_{k=1}^{\infty} b_k 3^k,
\]

shows that \( x \) is a solution of the equation (2). \[ \square \]

From the proof of Theorem 6 it is easy to see that if \( A_0 = x_0^2 + a_0 \equiv 0 \pmod{3} \), then we have the following congruences and appropriate equalities
Lemma 2. Let $\gamma(a) = 1$, $\gamma(b) = 0$ and $x \in \mathbb{Z}_3^*$ to be so that $A_{k-j} \equiv 0 \pmod{3}$, $1 \leq j \leq k$, $A_k \not\equiv 0 \pmod{3}$ for some fixed $k$. If $x$ be a solution of the equation (2), then it is true the following system of the congruences

\begin{align*}
x_0^3 &\equiv b_0 \pmod{3}, \\
x_0a_0 + M_1(x_0) &\equiv b_1 \pmod{3}, \\
x_{j-1}a_j + x_{j-2}a_{j+1} + \ldots + x_0a_{j-1} + S^j_{2j} + M_{2j}(x_0, x_1, \ldots, x_{j-1}) &\equiv b_{2j} \pmod{3}, \quad (7) \\
(A_j - x_0x_j)x_j + x_{j-1}a_{j+1} + x_{j-2}a_{j+2} + \ldots + x_0a_{j+2} + S^j_{2j+1} &+ M_{2j+1}(x_0, x_1, \ldots, x_{j-1}) \equiv b_{2j+1} \pmod{3}, \\
A_{k}x_{k+i} + x_{k+i-1}a_{k+1} + x_{k+i-2}a_{k+2} + \ldots + x_0a_{2k+i} + S^{k+i}_{2k+1+i} &+ M_{2k+1+i}(x_0, x_1, \ldots, x_{k+i-1}) \equiv b_{2k+1+i} \pmod{3},
\end{align*}

where $1 \leq j \leq k$ and integers $M_k(x_0, \ldots, x_{k-2})$ are defined from the equalities

\begin{align*}
3 \cdot M_1(x_0) &= x_0^3 - b_0, \\
3 \cdot M_2(x_0) &= x_0a_0 + M_1(x_0) - b_1, \\
3 \cdot M_{2j+1}(x_0, \ldots, x_{j-1}) &= x_{j-1}a_j + x_{j-2}a_{j+1} + \ldots + x_0a_{j-1} + S^j_{2j} + M_{2j}(x_0, \ldots, x_{j-1}) - b_{2j}, \\
3 \cdot M_{2j+2}(x_0, x_1, \ldots, x_j) &= (A_j - x_0x_j)x_j + x_{j-1}a_{j+1} + x_{j-2}a_{j+2} + \ldots + x_0a_{j+2} + S^j_{2j+1} + M_{2j+1}(x_0, x_1, \ldots, x_{j-1}) - b_{2j+1}, \quad (8) \\
3 \cdot M_{2k+2+i}(x_0, x_1, \ldots, x_{k+i}) &= A_kx_{k+i} + x_{k+i-1}a_{k+1} + x_{k+i-2}a_{k+2} + \ldots + x_0a_{2k+i} + S^{k+i}_{2k+1+i} + M_{2k+1+i}(x_0, x_1, \ldots, x_{k+i-1}) - b_{2k+1+i}.
\end{align*}

Proof. We shall prove Theorem by induction. Let $k = 1$, i.e.

\begin{align*}
A_0 &= x_0^2 + a_0 \equiv 0 \pmod{3}, \\
A_1 &= \frac{A_0}{3} + a_1 + 2x_0x_1 \not\equiv 0 \pmod{3},
\end{align*}

where $x_0, a_0, a_1$ are integers.
then the system of the congruences (6) are true. Note that $S_2^1 = 0$, $S_3^1 = x_1^3$.

From (5) it is easy to get

$$\frac{A_0}{3} x_{t-2} + x_{t-2}a_1 + x_{t-3}a_2 + \ldots + x_0a_{t-1} + S_t^{t-2} + 2x_0x_1x_{t-2} + M_t(x_0, \ldots, x_{t-3}) \equiv b_t \pmod{p}, \quad t \geq 4.$$ 

Therefore,

$$A_1x_{t-2} + x_{t-3}a_2 + \ldots + x_0a_{t-1} + S_t^{t-2} + M_t(x_0, \ldots, x_{t-3}) \equiv b_t \pmod{3}, \quad t \geq 4,$$

where

$$3 \cdot M_{i+1}(x_0, \ldots, x_{t-2}) = A_1x_{t-2} + x_{t-3}a_2 + \ldots + x_0a_{t-1} + S_t^{t-2} + M_t(x_0, \ldots, x_{t-3}) - b_t, \quad t \geq 4.$$

Obviously, the statement of Lemma is true for $k = 1$, i.e. for $i = t - 3$.

Let $k = 2$, i.e. $A_0 \equiv 0 \pmod{3}$, $A_1 \equiv 0 \pmod{3}$. $A_2 = \frac{A_1}{3} + a_2 + x_1^3 + 2x_0x_2 \neq 0 \pmod{3}$, then from the equalities (9) it follows that the following congruences are be added to the system (6):

e) $x_1a_2 + x_0a_3 + S_2^3 + M_4(x_0, x_1) \equiv b_4 \pmod{3}$, it follows

$$3 \cdot M_5(x_0, x_1) = x_1a_2 + x_0a_3 + S_2^3 + M_4(x_0, x_1) - b_4;$$

f) $\frac{A_1}{3} x_2 + x_2a_2 + x_1a_3 + x_0a_4 + S_5^3 + M_5(x_0, x_1) \equiv b_5 \pmod{3}$, it follows

$$3 \cdot M_6(x_0, x_1, x_2) = \frac{A_1}{3} x_2 + x_2a_2 + x_1a_3 + x_0a_4 + S_5^3 + M_5(x_0, x_1) - b_5;$$

h) $\frac{A_1}{3} x_{t-2} + x_{t-2}a_2 + x_{t-3}a_3 + \ldots + x_0a_t + S_{t+1}^{t-1} + M_{t+1}(x_0, x_1, \ldots, x_{t-3}) \equiv b_{t+1} \pmod{3},$

where $t \geq 5$ and $M_{t+2}(x_0, x_1, \ldots, x_{t-2})$ are defined by equalities

$$3 \cdot M_{t+2}(x_0, x_1, \ldots, x_{t-2}) = \frac{A_1}{3} x_{t-2} + x_{t-2}a_2 + x_{t-3}a_3 + \ldots + x_0a_t +$$

$$+ S_{t+1}^{t-1} + M_{t+1}(x_0, x_1, \ldots, x_{t-3}) - b_{t+1}.$$

Since $S_2^3 = 0$, $S_3^3 = 0$, $S_5^3 = x_0x_2^2 + x_1^2x_2$, $S_{t+1}^{t-1} = S_{t+1}^{t-2} + x_1^2x_{t-2} + 2x_0x_2x_{t-2}$, we denote by $i = t - 4$ and have

e) $x_1a_2 + x_0a_3 + M_4(x_0, x_1) \equiv b_4 \pmod{3},$

f) $(A_2 - x_0x_2)x_2 + x_1a_3 + x_0a_4 + M_5(x_0, x_1) \equiv b_5 \pmod{3},$

h) $A_2x_{i+2} + x_{i+1}a_3 + x_ia_4 + \ldots + x_0a_{i+4} + S_{i+2}^{i+2} + M_{i+5}(x_0, x_1, \ldots, x_{i+1}) \equiv b_{i+5} \pmod{3},$ where

$$3 \cdot M_5(x_0, x_1) = x_1a_2 + x_0a_3 + M_4(x_0, x_1) - b_4,$$

$$3 \cdot M_6(x_0, x_1, x_2) = (A_2 - x_0x_2)x_2 + x_1a_3 + x_0a_4 + M_5(x_0, x_1) - b_5.$$

$$3 \cdot M_{6+i}(x_0, x_1, \ldots, x_{i+2}) = A_2x_{i+2} + x_{i+1}a_3 + \ldots + x_0a_{i+4} +$$

$$+ S_{i+2}^{i+2} + M_{i+5}(x_0, x_1, \ldots, x_{i+1}) - b_{i+5}.$$

So we showed that the statement of Lemma is true for $k = 2$. 

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Let the system of congruences (7)-(8) is true for $k$. By the induction hypothesis for $1 \leq j \leq k$ we have
\[
x_0^3 \equiv b_0 \pmod{3},
\]
\[
x_0a_0 + M_1(x_0) \equiv b_1 \pmod{3},
\]
\[
x_{j-1}a_j + x_{j-2}a_{j+1} + \ldots + x_0a_{2j-1} + S_{2j}^i + M_{2j}(x_0, x_1, \ldots, x_{j-1}) \equiv b_{2j} \pmod{3},
\]
\[\begin{align*}
(A_j - x_0x_j)x_j + x_{j-1}a_{j+1} + x_{j-2}a_{j+2} + \ldots + x_0a_{2j} + S_{2j+1}^j + \\
+ M_{2j+1}(x_0, x_1, \ldots, x_{j-1}) \equiv b_{2j+1} \pmod{3}.
\end{align*}\]

Since $A_k \equiv 0 \pmod{3}$, then from the congruences
\[
A_kx_{k+i} + x_{k+i-1}a_{k+1} + x_{k+i-2}a_{k+2} + \ldots + x_0a_{2k+i} + S_{2k+1}^{k+i+1} + \\
+ M_{2k+1+i}(x_0, x_1, \ldots, x_{k+i-1}) \equiv b_{2k+1+i} \pmod{3},
\]
we derive
\[
x_k^3 \equiv b_0 \pmod{3},
\]
\[
x_0a_0 + M_1(x_0) \equiv b_1 \pmod{3},
\]
\[
x_{j-1}a_j + x_{j-2}a_{j+1} + \ldots + x_0a_{2j-1} + S_{2j}^i + M_{2j}(x_0, x_1, \ldots, x_{j-1}) \equiv b_{2j} \pmod{3},
\]
\[\begin{align*}
(A_j - x_0x_j)x_j + x_{j-1}a_{j+1} + x_{j-2}a_{j+2} + \ldots + x_0a_{2j} + S_{2j+1}^j + \\
+ M_{2j+1}(x_0, x_1, \ldots, x_{j-1}) \equiv b_{2j+1} \pmod{3}.
\end{align*}\]

It is easy to check that
\[
S_{2k+3}^{k+2} = S_{2k+3}^{k+1} + R_{k+1}x_{k+1} - x_0x_{k+1},
\]
\[
S_{2k+i+3}^{k+i+2} = S_{2k+i+3}^{k+i+1} + R_{k+1}x_{k+i+1},
\]
\[
i \geq 1.
\]

By these correlations we deduce
\[
A_k \frac{x_{k+1} + x_{k+1}a_{k+1} + x_{k+1}a_{k+2} + \ldots + x_0a_{2k+2} + S_{2k+3}^{k+2}}{3} + M_{2k+3}(x_0, x_1, \ldots, x_k) =
\]
\[
= A_k \frac{x_{k+1} + x_{k+1}a_{k+1} + x_{k+1}a_{k+2} + \ldots + x_0a_{2k+2} + S_{2k+3}^{k+1} + R_{k+1}x_{k+1} - \\
- x_0x_{k+1}^2 + M_{2k+3}(x_0, x_1, \ldots, x_k) = (A_k + a_{k+1} + R_{k+1} - x_0x_{k+1})x_{k+1} + \\
x_ka_{k+2} + \ldots + x_0a_{2k+2} + S_{2k+3}^{k+1} + M_{2k+3}(x_0, x_1, \ldots, x_k) =
\]
\[
= (A_k - x_0x_{k+1})x_{k+1} + x_{k+1}a_{k+2} + \ldots + x_0a_{2k+2} + S_{2k+3}^{k+1} + M_{2k+3}(x_0, x_1, \ldots, x_k).
\]

For $i \geq 1$ we get
\[
A_k \frac{x_{k+1+i} + x_{k+1+i}a_{k+1} + \ldots + x_0a_{2k+i+2} + S_{2k+i+3}^{k+i+2}}{3} + M_{2k+i+3}(x_0, x_1, \ldots, x_{k+i}) =
\]
\[
= A_k \frac{x_{k+1+i} + x_{k+1+i}a_{k+1} + x_{k+1+i}a_{k+2} + \ldots + x_0a_{2k+i+2} + S_{2k+i+3}^{k+i+1} + R_{k+1}x_{k+1} + \\
+ M_{2k+i+3}(x_0, x_1, \ldots, x_{k+i}) = (A_k + a_{k+1} + R_{k+1})x_{k+1+i} + \\
x_k+i+1a_{k+1} + \ldots + x_0a_{2k+i+2} + S_{2k+i+3}^{k+i+1} + M_{2k+i+3}(x_0, x_1, \ldots, x_{k+i}) =
\]
$$= A_{k+1}x_{k+1+i} + x_{k+i}a_{k+2} + \ldots + x_0a_{2k+i+2} + S_{2k+i+3}^{k+i+1} + M_{2k+i+3}(x_0, x_1, \ldots, x_{k+i}).$$

Consequently, we have

$$x_k a_{k+1} + x_{k-1}a_{k+2} + \ldots + x_0a_{2k+1} + S_{2(k+1)}^{k+i} + M_{2(k+1)}(x_0, \ldots, x_k) \equiv b_{2(k+1)}(mod\ 3),$$

$$(A_{k+1}-x_0x_{k+1})x_{k+1} + x_{k+i}a_{k+2} + \ldots + x_0a_{2k+2} + S_{2k+3}^{k+i+1} + M_{2k+3}(x_0, \ldots, x_k) \equiv b_{2k+3}(mod\ 3),$$

$$A_{k+1} x_{k+1+i} + x_{k+i}a_{k+2} + \ldots + x_0a_{2k+i+2} + S_{2k+i+3}^{k+i+1} +$$

$$+ M_{2k+i+3}(x_0, x_1, \ldots, x_{k+i}) \equiv b_{2k+i+3}(mod\ 3),\quad i \geq 1.$$

So we established that the system of congruences (7)-(8) is true for $k + 1$. □

**Theorem 9.** Let $\gamma(a) = 1, \gamma(b) = 0$ and $x \in \mathbb{Z}_3^*$ to be such that $A_{k-j} \equiv 0 (mod\ 3), 1 \leq j \leq k, A_k \neq 0 (mod\ 3)$ for some fixed $k (k \geq 1)$. Then $x$ to be a solution of the equation (2) in $\mathbb{Z}_3^*$ if and only if the system of the congruences

$$x_0^3 \equiv b_0 (mod\ 3),$$

$$x_0a_0 + M_1(x_0) \equiv b_1 (mod\ 3),$$

$$x_{j-1}a_j + x_{j-2}a_{j+1} + \ldots + x_0a_{2j-1} + S_{2j}^j + M_2(x_0, x_1, \ldots, x_{j-1}) \equiv b_{2j}(mod\ 3),$$

$$(A_j - x_0x_j)x_j + x_{j-1}a_{j+1} + x_{j-2}a_{j+2} + \ldots + x_0a_{2j} + S_{2j+1}^j +$$

$$+ M_{2j+1}(x_0, x_1, \ldots, x_{j-1}) \equiv b_{2j+1}(mod\ 3),$$

has a solution, where $1 \leq j \leq k$ and integers $M_k(x_0, x_1, \ldots, x_{k-1})$ are defined from the equalities

$$3 \cdot M_1(x_0) = x_0^3 - b_0,$$

$$3 \cdot M_2(x_0) = x_0a_0 + M_1(x_0) - b_1,$$

$$3 \cdot M_{2j+1}(x_0, \ldots, x_{j-1}) = x_{j-1}a_j + x_{j-2}a_{j+1} + \ldots + x_0a_{2j-1} + S_{2j}^j + M_2(x_0, \ldots, x_{j-1}) - b_{2j},$$

$$3 \cdot M_{2j+2}(x_0, x_1, \ldots, x_j) = (A_j - x_0x_j)x_j + x_{j-1}a_{j+1} + x_{j-2}a_{j+2} + \ldots + x_0a_{2j} +$$

$$+ S_{2j+1}^j + M_{2j+1}(x_0, x_1, \ldots, x_{j-1}) - b_{2j+1}.$$

**Proof.** Necessity. If the equation (2) has a solution $x \in \mathbb{Z}_3^*$, i.e.

$$x = x_0 + x_1 \cdot 3 + x_2 \cdot 3^2 + \ldots,\quad 0 \leq x_j \leq 2,\; x_0 \neq 0,$$

then by Lemma 2 the system of the congruences (7) are true and $\{x_0, x_1, \ldots, x_k\}$ is a solution of this system.

Sufficiency. Let the system of the congruences (7) have a solution $x_0, x_1, \ldots, x_k$. Then by a condition $(A_3, 3) = 1$ and Theorem 1 we have an existence of solutions $x_{k+i}$ of the following congruences

$$A_k x_{k+i} + x_{k+i-1}a_{k+1} + x_{k+i-2}a_{k+2} + \ldots + x_0a_{2k+i} + S_{2k+i+3}^{k+i+1} +$$

$$+ M_{2k+i+3}(x_0, x_1, \ldots, x_{k+i-1}) \equiv b_{2k+i+3}(mod\ 3),$$

where $i \geq 1$ and integers $M_{2k+i+2}(x_0, x_1, \ldots, x_{k+i})$ are defined recurrently from the following equalities

$$3 \cdot M_{2k+i+2}(x_0, x_1, \ldots, x_{k+i}) = A_k x_{k+i} + x_{k+i-1}a_{k+1} + x_{k+i-2}a_{k+2} + \ldots + x_0a_{2k+i} +$$

$$+ S_{2k+i+3}^{k+i+1} + M_{2k+i+3}(x_0, x_1, \ldots, x_{k+i-1}) - b_{2k+i+3}.$$
Then we have
\[
x_0^3 + a_0 x_0 \cdot 3 + \sum_{j=1}^{k} (x_{j-1} a_j + x_{j-2} a_{j+1} + \ldots + x_0 a_{2j-1} + S_{2j}^i) 3^{2j} +
\]
\[
+ \sum_{j=1}^{k} ((A_j - x_0 x_j) x_j + x_{j-1} a_{j+1} + x_{j-2} a_{j+2} + \ldots + x_0 a_{2j+1} + S_{2j+1}^i) 3^{2j+1} +
\]
\[
+ \sum_{i=1}^{\infty} (A_k x_{k+i} + x_{k+i-1} a_{k+1} + x_{k+i-2} a_{k+2} + \ldots + x_0 a_{2k+i} + S_{2k+i}^i) \cdot 3^{2k+i} =
\]
\[
= b_0 + M_1(x_0) \cdot 3 + (b_1 - M_1(x_0) + M_2(x_0) \cdot 3) 3^j +
\]
\[
+ \sum_{j=1}^{k} (b_{2j} - M_{2j}(x_0, x_1, \ldots, x_{j-1}) + M_{2j+1}(x_0, x_1, \ldots, x_{j-1}) \cdot 3) 3^{2j+1} +
\]
\[
+ \sum_{j=1}^{k} (b_{2j+1} - M_{2j+1}(x_0, x_1, \ldots, x_{j-1}) + M_{2j+2}(x_0, x_1, \ldots, x_j) \cdot 3) 3^{2j+1} +
\]
\[
+ \sum_{i=1}^{\infty} (b_{2k+i+1} - M_{2k+i+1}(x_0, x_1, \ldots, x_{k+i-1}) + M_{2k+i+2}(x_0, x_1, \ldots, x_{k+i}) \cdot 3) 3^{2k+i} =
\]
\[
= b_0 + \sum_{j=1}^{\infty} b_j 3^j.
\]

So we checked that \( x \), which is satisfied the conditions of the theorem, is a solution of the equation (2). \( \square \)

The next theorem complete the existence of a solution for the equation (2).

**Theorem 10.** Let \( \gamma(a) = 1 \), \( \gamma(b) = 0 \) and \( x \in \mathbb{Z}_3^* \) to be so that \( A_k \equiv 0 (mod 3) \) for all \( k \in \mathbb{N} \). Then \( x \) to be a solution of the equation (2) in \( \mathbb{Z}_3^* \) if and only if the system of the congruences

\[
x_0^3 \equiv b_0 (mod 3),
\]
\[
x_0 a_0 + M_1(x_0) \equiv b_1 (mod 3),
\]
\[
x_{j-1} a_j + x_{j-2} a_{j+1} + \ldots + x_0 a_{2j-1} + S_{2j}^i + M_{2j}(x_0, x_1, \ldots, x_{j-1}) \equiv b_{2j} (mod 3),
\]
\[
(A_j - x_0 x_j) x_j + x_{j-1} a_{j+1} + x_{j-2} a_{j+2} + \ldots + x_0 a_{2j} + S_{2j+1}^i +
\]
\[
+ M_{2j+1}(x_0, x_1, \ldots, x_{j-1}) \equiv b_{2j+1} (mod 3),
\]

has a solution, where \( j \geq 1 \) and integers \( M_k(x_0, x_1, \ldots, x_{k-1}) \) are defined from the equalities

\[
3 \cdot M_1(x_0) = x_0^3 - b_0,
\]
\[
3 \cdot M_2(x_0) = x_0 a_0 + M_1(x_0) - b_1,
\]
\[
3 \cdot M_{2j+1}(x_0, x_1, \ldots, x_{j-1}) = x_{j-1} a_j + x_{j-2} a_{j+1} + \ldots + x_0 a_{2j-1} + S_{2j}^i + M_{2j}(x_0, \ldots, x_{j-1}) - b_{2j},
\]
\[
3 \cdot M_{2j+2}(x_0, x_1, \ldots, x_j) = (A_j - x_0 x_j) x_j + x_{j-1} a_{j+1} + x_{j-2} a_{j+2} + \ldots + x_0 a_{2j} +
\]
\[
+ S_{2j+1}^i + M_{2j+1}(x_0, x_1, \ldots, x_{j-1}) - b_{2j+1}.
\]

**Proof.** Necessity. If the equation (2) has a solution \( x \in \mathbb{Z}_3^* \), i.e.

\[
x = x_0 + x_1 \cdot 3 + x_2 \cdot 3^2 + \ldots, \quad 0 \leq x_j \leq 2, \quad x_0 \neq 0,
\]

then...
then by Lemma 2 the system of the congruences (7) are true and \{x_0, x_1, \ldots, x_k\} is a solution of this system.

**Sufficiency.** Let the system of the congruences (7) have a solution \(x_0, x_1, \ldots, x_k\). Then by a condition \(A_i x_i = b_i\) and Theorem 1 we have an existence of solutions \(x_{k+i}\) of the following congruences

\[
A_k x_{k+i} + x_{k+i-1} a_{k+1} + x_{k+i-2} a_{k+2} + \ldots + x_0 a_{2k+i} + S_{2k+1+i}^{k+i}
+ M_{2k+1+i}(x_0, x_1, \ldots, x_{k+i-1}) \equiv b_{2k+1+i} (\text{mod} 3),
\]

where \(i \geq 1\) and integers \(M_{2k+i+2}(x_0, x_1, \ldots, x_{k+i})\) are defined recurrently from the following equalities

\[
3 \cdot M_{2k+2+i}(x_0, x_1, \ldots, x_{k+i}) = A_k x_{k+i} + x_{k+i-1} a_{k+1} + x_{k+i-2} a_{k+2} + \ldots + x_0 a_{2k+i} + S_{2k+1+i}^{k+i}
+ M_{2k+1+i}(x_0, x_1, \ldots, x_{k+i-1}) - b_{2k+1+i}.
\]

Then we have

\[
x_0^3 + a_0 x_0 \cdot 3 + \sum_{j=1}^{\infty} (x_{j-1} a_j + x_{j-2} a_{j+1} + \ldots + x_0 a_{2j-1} + S_{2j}^j) 3^{2j} +
\]

\[
+ \sum_{j=1}^{\infty} ((A_j - x_0 x_j) x_j + x_{j-1} a_{j+1} + x_{j-2} a_{j+2} + \ldots + x_0 a_{2j} + S_{2j}^j) 3^{2j+1} =
\]

\[
= b_0 + M_1(x_0) \cdot 3 + (b_1 - M_1(x_0) + M_2(x_0) \cdot 3) \cdot 3 +
\]

\[
+ \sum_{j=1}^{\infty} (b_{2j} - M_{2j}(x_0, x_1, \ldots, x_{j-1}) + M_{2j+1}(x_0, x_1, \ldots, x_{j-1}) \cdot 3) 3^{2j} +
\]

\[
+ \sum_{j=1}^{\infty} (b_{2j+1} - M_{2j+1}(x_0, x_1, \ldots, x_{j-1}) + M_{2j+2}(x_0, x_1, \ldots, x_{j}) \cdot 3) 3^{2j+1} =
\]

\[
= b_0 + \sum_{j=1}^{\infty} b_j 3^j,
\]

consequently, \(x\) is a solution of the equation \((2)\). \(\square\)

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