Renewal-scaled solutions of the Kolmogorov forward equation for residual times

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Abstract

Let \( N(\tau) \) be a renewal process for independent holding times \( \{X_k\}_{k \geq 1} \), where \( \{X_k\}_{k \geq 0} \) are identically distributed with density \( p(x) \). If the associated residual time \( R(\tau) \) has a density \( u(x, \tau) \), its Kolmogorov forward equation is given by

\[
\frac{\partial}{\partial \tau} u(x, \tau) - \frac{\partial}{\partial x} u(x, \tau) = p(x)u(0, \tau), \quad x, \tau \in [0, \infty),
\]

with an initial holding time density \( u(x, 0) = u_0(x) \). We derive a measure-valued solution formula for the density of residuals times after an expected number of renewals occur. Solutions under this time scale are then shown to evolve continuously in the space of measures with the weak topology under mild conditions.

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1 Introduction

Let \( \{X_i\}_{i \geq 1} \) be a sequence of iid random variables in \( \mathbb{R}_+ = [0, \infty) \) called holding times, each having a density \( p(x) \in L^1(\mathbb{R}_+) \). Also define a random initial holding time \( X_0 \in \mathbb{R}_+ \), independent from \( \{X_i\}_{i \geq 1} \), but possibly distinct in law. The renewal process \( N(\tau) \) associated with \( \{X_i\}_{i \geq 0} \) counts the number of renewals up to time \( \tau \), and is given by

\[
N(\tau) = \sup \left\{ i \geq 0 : \sum_{j=0}^{i} X_i \leq \tau \right\}.
\]

Related to the renewal process is the residual time (also known as the forward recurrence time), defined as

\[
R(\tau) = \sum_{i=1}^{N(\tau)+1} X_i - \tau.
\]

At time \( \tau \), \( R(\tau) \) is the time remaining until the next renewal. Residual times are fundamental in renewal theory and queueing theory (see Ch. 9 of Cox [2] for a detailed introduction), and may be viewed as an extension of homogeneous Poisson processes, in which holding times, and consequently residual times, are exponentially distributed [9].

If \( R(\tau) \) has a differentiable density in time and space \( u(x, \tau) \in C^1(\mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}_+) \), then the Kolmogorov forward equation for residual times is given by

\[
\frac{\partial}{\partial \tau} u(x, \tau) - \frac{\partial}{\partial x} u(x, \tau) = p(x)u(0, \tau), \quad \tau, x \geq 0, \quad u(x, 0) = u_0(x).
\]

This equation has the explicit solution (see Pg. 63 of [1]) of

\[
u(x, \tau) = u_0(x + \tau) + \int_0^\tau p(x + \tau - s)\alpha(s)ds.\]

Here, \( \alpha(s) \) is the renewal density

\[
\alpha(\tau) = u_0(\tau) \ast \sum_{i=0}^{\infty} p^{*i}(\tau),
\]

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When a bulb breaks, it is immediately replaced by a new light bulb with a random run length distributed with respect whose probability has the expansion.

Next, we consider the probability that the particle at initial time $\tau$ is done by considering a particle at time $t$ specifically, we introduce the new time scale transforms, a method similar to [4]. In Section 2.1 we derive (1.3) through asymptotic expansions, and in Section 2.2, we derive (1.4) through Laplace transforms, a method similar to [4].

In this section will use an equivalent reformulation for renewal-scaled solutions continuously evolve under initial holding time $s$ with point masses.

The main novelty of this paper is a change of variables for (1.3) based on expected renewals, presented in Section 3. To derive (1.3), we will compute asymptotics for probabilities that a particle is in a small interval. Assume for $p^{(i)}$ denotes $i$-fold self-convolution (with the convention $p^{(0)} = 1$ and $p^{(1)} = p$). An important fact (see Sect. 9.12 in [10]) is that

$$A(\tau) := \mathbb{E}[N(\tau)] = \int_0^\tau \alpha(s) ds. \quad (1.6)$$

In Section 2.1 we derive (1.3) through asymptotic expansions, and in Section 2.2, we derive (1.4) through Laplace transforms, a method similar to [4].

The main novelty of this paper is a change of variables for (1.3) based on expected renewals, presented in Section 3. Specifically, we introduce the new time scale $t(\tau) = A(\tau)$. Using the $t$ time scale is a natural option for several scenarios in queuing theory. For an example, consider a factory lit by a large collection of light bulbs, running simultaneously. When a bulb breaks, it is immediately replaced by a new light bulb with a random run length distributed with respect to $p(x)$. If the initial distribution of run lengths at time zero for the $N$ bulbs is approximately $u_0(x)$, then $u(x, \tau)$ estimates remaining run lengths at time $\tau$. What, then, is the distribution of remaining run lengths after $N\tau$ light bulbs have been replaced? The answer to this question is provided through the renewal-scaled solution formula (3.6). Given the expansive history of renewal theory, the author expected to find (3.6) in the literature, but was unable to do so after a thorough search.

Both the change of time scale and method for renewal-scaled solutions are motivated by the work of Menon, Niethammer, and Pego in [8], who investigated a wide range of clustering events generalizing the 1D Allen-Cahn equation in mathematical physics. The measure-valued space of clusters was shown to be continuous in time in the space of probability measures $\mathcal{P}(\mathbb{R}^\tau)$ through using an intrinsic time scale based on the number of clusters in the system. In our case, we define measure-valued solutions of (1.3) in the $t$ time scale which evolve continuously in $\mathcal{P}(\mathbb{R}^\tau)$ under a wide range of holding times. We conclude with an example in Section 3.3, which illustrates with how renewal-scaled solutions continuously evolve under initial holding times with point masses.

### 2 Strong solutions

#### 2.1 Derivation of forward equation

In this section will use an equivalent reformulation for $R(\tau)$ from the perspective of a random dynamical system. This is done by considering a particle at time $\tau = 0$ randomly placed on $\mathbb{R}^+$, with its initial position $R(0)$ given by the first holding time $X_0$. The particle drifts toward the origin at unit speed until it reaches the origin at time $\tau = X_0$. At this time, it is immediately reassigned to $R(X_0) = X_1 \in \mathbb{R}^+$, distributed with respect to $p(x)$. The particle proceeds as before, moving toward the origin and being reassigned upon its arrival, with $R(\sum_{j=1}^{k-1} X_j) = X_k$, and $\{X_k\}_{k \geq 1}$ iid (see Figure 2.1). It is clear that the position of the particle at time $\tau$ corresponds with the residual time $R(\tau)$ under holding times $\{X_k\}_{k \geq 1}$.

To derive (1.3), we will compute asymptotics for probabilities that a particle is in a small interval. Assume for now that $p(x)$ is continuous, and $u(x, \tau)$ is differentiable in both $x$ and $\tau$. Then an asymptotic expansion for the probability that a particle is in $[x, x + \Delta x]$ at time $\tau$ is

$$\mathbb{P}(R(\tau) \in [x, x + \Delta x]) = u(x, \tau)\Delta x + \frac{1}{2}\partial_x u(x, \tau)(\Delta x)^2 + o((\Delta x)^2). \quad (2.1)$$

Next, we consider the probability that the particle at initial time $\tau - \Delta x$ is contained in $[x, x + \Delta x]$ at time $\tau$. This can occur in two ways. The simplest case is when the particle is initially in $[x + \Delta x, x + 2\Delta x]$. Denote this event $A$, whose probability has the expansion

$$\mathbb{P}(A) = \mathbb{P}(R(\tau - \Delta x) \in [x + \Delta x, x + 2\Delta x]) = u(x, \tau)\Delta x - \partial_t u(x, \tau)(\Delta x)^2 + \frac{3}{2}\partial_x u(x, \tau)(\Delta x)^2 + o((\Delta x)^2). \quad (2.2)$$

![Figure 2.1: A dynamical system representation](image-url)
The second case is when a particle begins in \([0, \Delta x]\), reaches the origin, and is reassigned before drifting into \([x, x + \Delta x]\) at time \(\tau\). Denote this event \(B\). We may partition this event according to the particle’s initial position \(s \in [0, \Delta x]\), and how many times the particle jumps. Then

\[
\begin{align*}
\mathbb{P}(B) &= \int_0^{\Delta x} u(s, \tau - \Delta x)(\mathbb{P}((1 \text{ jump}) \cap B|R(\tau - \Delta x) = s) + \mathbb{P}((> 1 \text{ jump}) \cap B|R(\tau - \Delta x) = s))ds \\
&= \int_0^{\Delta x} u(s, \tau - \Delta x) \left( \int_{s+\Delta x}^{s+2\Delta x} p(y)dy \right) ds + o((\Delta x)^2) = u(0, \tau)p(x)(\Delta x)^2 + o((\Delta x)^2).
\end{align*}
\]

(2.3)

Since \(\mathbb{P}(R(\tau) \in [x, x + \Delta x]) = \mathbb{P}(A) + \mathbb{P}(B)\), we compare \((\Delta x)^2\) terms in (2.1), (2.2), and (2.3) to arrive at (1.3).

### 2.2 Classical solutions

In this section, we will derive the solution formula (1.4) for initial conditions in the space

\[ C = \{ u \in C^1(\mathbb{R}^+ \cap L^1(\mathbb{R}^+): u > 0, \|u\|_1 = 1, u(x) \to 0 \text{ as } x \to \infty \}. \]

(4.2)

**Definition 1.** Let \(p(x) \in C\). A function \(u(x, \tau) \in C^1(\mathbb{R}^+ \times \mathbb{R}^+; \mathbb{R}^+\) with \(u(x, 0) \in C\) which solves (1.3) is a strong solution.

Our first step in finding an explicit strong solution formula is integrating (1.3) along characteristics \(x(\tau) = x_0 - \tau\), which gives us the integral form of (1.3)

\[
u(x, \tau) = u_0(x + \tau) + \int_0^\tau p(x + \tau - s)u(0, s)ds.\]

(5.2)

By comparing (1.4) and (2.5), we see that finding an explicit solution formula is equivalent to showing \(u(0, \tau) = \alpha(\tau)\). This fact follows immediately by simply setting \(x = 0\) in (2.5), from which we obtain the well-known renewal equation whose solution is \(\alpha(\tau)\). However, we derive here the explicit formula for \(\alpha(\tau)\), as some of the intermediate formulas produced will be of use in later sections.

Toward this end, we apply the Laplace transform

\[
u(q, \tau) := \int_{\mathbb{R}^+} e^{-qt} u(x, \tau)dx, \quad q \in \mathbb{C}_+,
\]

(6.2)

to both sides of (1.3) to yield

\[
\frac{d}{d\tau} \nu - q\nu = (\rho(q) - 1)\nu(0, s).
\]

(7.2)

This may easily be solved for \(\nu\), with

\[
\nu(q, \tau) = e^{q\tau} \nu_0(q) + (\rho(q) - 1) \int_0^\tau e^{q(\tau - s)}\nu(0, s)ds.
\]

(8.2)

This solution is similar to (1.4), but now we can use Laplace inversion to extract a formula for \(u(0, s)\). Since we are taking the Laplace transform of a probability density, \(|\nu(q, \tau)| < 1\) for \(\tau > 0, q \in \mathbb{C}_+\). Thus \(\nu(q, \tau) e^{-qt} \to 0\) as \(\tau \to \infty\). We then can obtain, from the \(\tau \to \infty\) limit of (2.8),

\[
\frac{1}{1 - \rho(q)} \nu_0(q) = \int_{\mathbb{R}^+} e^{-qs} \alpha(s)ds = [\nu(0, \cdot)](q).
\]

(9.2)

We now have a formula for \(\nu(0, \tau)\) based on the initial data. Notice that the left hand side is a Laplace transform in the spatial variable, whereas the right hand side is a Laplace transform of \(u(0, \tau)\) in the time variable.

To perform Laplace inversion on (2.9), observe that \(p\) is a probability density, so that \(|\rho(q)| < 1\) for \(q \in \mathbb{C}_+\). Thus, we can express the left hand side of (2.9) as the geometric series

\[
[\nu(0, \cdot)](q) = \nu_0(q) \sum_{i=0}^\infty (\rho(q))^i.
\]

(10.2)

From the convolution formula, our renewal density has an explicit expression

\[
u(0, \tau) = u_0(\tau) * \sum_{i=0}^\infty \rho^{(i)}(\tau),
\]

(11.2)

which is equivalent to \(\alpha(\tau)\). Since \(\alpha(t)\) is locally integrable (Th. 3.18 of [6]), (1.4) is well-defined, \(u(x, \tau)\) is differentiable in both spatial and time variables, and for fixed \(\tau \geq 0, u(x, \tau) \to 0\) as \(x \to \infty\). Finally, we may show that \(u(\tau, x)\) is a probability density for each \(\tau \geq 0\) integrating (1.3) with respect to the \(x\) variable. We summarize our findings in
Theorem 2. Let \( p(x) \in C(\mathbb{R}^+) \), and \( u_0(x) \in \mathcal{C} \). Then there exists a unique strong solution \( u(x, \tau) \) of (1.3) with \( u(x, 0) = u_0(x) \) given by (1.4). For each \( \tau > 0 \), \( u(\cdot, t) \in \mathcal{C} \).

3 Rescaling by expected renewals

In preparation for defining measure-valued solutions in the next section, we’ll first reformulate (1.3) in terms of probability measures \( \mathcal{P}(\mathbb{R}^+) \). In one dimension a probability measure \( p \in \mathcal{P} \) may be identified with its cumulative distribution function \( G_p(x) = \mu([0, x)) \). In the future, when no confusion arises, we will often write \( G_\mu \in \mathcal{P}(\mathbb{R}^+) \). The Laplace transform of a measure \( G \) is then defined as

\[
\hat{G}(q) = \int_0^\infty e^{-qy}G(dx), \quad q \in \mathbb{C}^+.
\]  

(3.1)

If a measure \( G(x) \) admits a density \( G(dx) = g(x)dx \), then \( \hat{G}(q) = \hat{g}(q) \).

For \( p(x), u_0(x) \in \mathcal{C} \), (2.7) may be rewritten in terms of measures as

\[
\frac{d}{d\tau} \hat{U} - qU = (1 - \hat{P}(q))\alpha(\tau),
\]  

(3.2)

where \( dU_\tau(x) = u(x, \tau)dx \) and \( P(dx) = p(x)dx \). Since \( p \) and \( u_0 \) are strictly positive and continuous, so is \( \alpha(\tau) \), and thus \( t = A(\tau) \) is strictly increasing and differentiable, with \( \frac{dt}{d\tau} = \alpha(\tau) \). We may now transform (3.2) as

\[
\frac{1}{\alpha(\tau)} \frac{d}{d\tau}(e^{-q\tau}\hat{U}_\tau) = (1 - \hat{P}(q))e^{-q\tau}.
\]

From the chain rule, we now convert to the \( t \) time scale, with

\[
\frac{d}{dt}(e^{-q\tau(t)}\hat{F}_t) = (1 - \hat{P}(q))e^{-q\tau(t)},
\]  

(3.3)

where we now define \( \hat{F}_t(x) = U_{\tau(t)}(x) \). This in turn gives us the solution formula

\[
\hat{F}_t(q)e^{-q\tau(t)} - \hat{F}_0(q) = (1 - \hat{P}(q)) \int_0^t e^{-q\tau(s)}ds.
\]  

(3.4)

Since \( \tau(t) \to \infty \) as \( t \to \infty \), the limit of (3.4) is then

\[
\hat{F}_0(q) = (1 - \hat{P}(q)) \int_0^\infty e^{-q\tau(s)}ds,
\]  

(3.5)

and consequently \( \hat{F}_t \) takes the simple form

\[
\hat{F}_t(q) = (1 - \hat{P}(q)) \int_t^\infty e^{q(\tau(t) - \tau(s))}ds.
\]  

(3.6)

3.1 Extension to measure valued solutions

The solution formula (3.6) has only been defined for the strict class of functions \( \mathcal{C} \), but its form suggests that we may directly extend solutions for measures \( F_0, P \in \mathcal{P}(\mathbb{R}^+) \). To do so, however, we require a proper extension for \( A(\tau) \), since the trace \( \alpha(\tau) = u(0, \tau) \) assumes solutions have a density. Nonetheless, we can use formula (2.11) to define

\[
A = F_0 * \sum_{i=0}^\infty P^{*^{(i)}},
\]  

(3.7)

where convolution of the two measures \( \mu \) and \( \nu \) is defined as

\[
\mu * \nu(E) = \int_{\mathbb{R}^+} \int_{\mathbb{R}^+} 1_E(x+y)d\mu(x)d\nu(y).
\]  

(3.8)

For a sequence of measures \( f_n \to \mu \), and \( g_n \to \nu \) converging weakly in \( \mathbb{R}^+ \), then by Prokhorov’s theorem, \( f_n \) and \( g_n \) are tight, and \( f_n * g_n \to \mu * \nu \) weakly. We may use this to show that \( A(\tau) \) is nondecreasing for arbitrary measures \( F_0, P \in \mathcal{P}(\mathbb{R}^+) \) by regularization.
Even with a proper notion of $A(\tau)$, still more is required to make (3.6) well-defined for measures. This is because $A(\tau)$ may be constant on an interval, or have jumps, both of which prohibit an inverse with a domain defined on all of $\mathbb{R}^+$. To address this, we use a generalized notion of inverse for nondecreasing functions. For a nondecreasing function $f : \mathbb{R}^+ \rightarrow \mathbb{R}^+$, define the generalized inverse $f^\dagger : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ by
\[
f^\dagger(q) = \inf\{s \in \mathbb{R}^+ | f(s) > q\}.
\] (3.9)
When $f$ is strictly increasing and continuous, the usual inverse $f^{-1}$ and generalized inverse $f^\dagger$ coincide. For a generic distribution $A(\tau)$, we define
\[
\tau(t) = A^\dagger(t).
\] (3.10)

**Definition 3.** Let $P \in \mathcal{P}(\mathbb{R}^+)$. A map $(F_0, t) \mapsto F_t \in \mathcal{P}(\mathbb{R}^+)$ defined through (3.6) is a renewal-scaled solution to (1.3), where $A(t)$ and $\tau(t)$ are defined by (3.7) and (3.10).

### 3.2 Properties of renewal-scaled solutions

Here we will demonstrate that the extension of the solution for strong solutions to include measure-valued initial data and holding times is natural in the sense that measure-valued renewal-scaled solutions are weak limits of strong solutions in the $t$ time scale. We also show that our rescaling has the effect of smoothing point masses approaching the origin, as weak solutions evolve continuously so long as $A^\dagger(t)$ is strictly increasing.

**Theorem 4.** Let $P \in \mathcal{P}(\mathbb{R}^+)$ be nondefective (meaning $P(0) < 1$). The following hold:

1. Let $u^n(x, \tau)$ be a sequence of strong solutions with initial conditions $u^n_0(x) \in C$ and holding time densities $p^n(x) \in C$, and let $F^n(t)$ be the corresponding renewal-scaled solutions. Suppose $F_1$ is renewal-scaled solution for initial data $F_0 \in \mathcal{P}(\mathbb{R}^+)$ and holding time measure $P \in \mathcal{P}(\mathbb{R}^+)$. Then if $u^n_0 \rightarrow F_0$ and $p^n \rightarrow P$ weakly, then $F^n_t \rightarrow F_t$ weakly at all points of continuity of $A^\dagger(t)$.

2. For any $P, F_0 \in \mathcal{P}(\mathbb{R}^+)$, $F_t$ is a probability measure for all $t \geq 0$.

3. If $A^\dagger(t)$ is strictly increasing, then for $t > 0$, the map $t \mapsto F_t$ is continuous in $\mathcal{P}(\mathbb{R}^+)$ under the weak topology.

**Remark 5.** Since $C$ is dense in $\mathcal{P}(\mathbb{R}^+)$, for any $F_0, P \in \mathcal{P}(\mathbb{R}^+)$ we can always find $u^n, p^n \in C$ with $u^n_0 \rightarrow F_0$ and $p^n \rightarrow P$ weakly. Thus, for points of continuity of $A^\dagger$, we can define renewal-scaled solutions through weak limits of strong solutions.

**Remark 6.** The possible nonconvergence at jump points of $A^\dagger$ is essentially due to the multiple ways that one can define a generalized inverse for increasing functions. In our definition, inverses of caglad functions remain caglad, whereas using the traditional definition of a quantile from statistics, for instance, would instead give caglad functions.

**Remark 7.** We note that for statement (3), there are a variety of sufficient $P$ and $F_0$ which produce strictly increasing $A^\dagger(t)$ for $t > 0$. One such sufficient condition, for instance, is if $P(x)$ is strictly increasing on an interval about the origin. The point here is that jumps in the solution only occur when $A(\tau)$ is constant in an interval. As we will see in Section 3.3, point masses arriving at the origin (corresponding to jumps in $A(\tau)$) are continuously redistributed in the $t$ time scale.

**Proof.** (1) Renewal-scaled solutions for $(F^n_0, P^n)$ satisfy
\[
F^n_t(q) = (1 - p^n(q)) \int_t^\infty e^{q(t - s)} ds.
\] (3.11)
To show weak convergence, it is enough to show $F^n_t(q) \rightarrow \bar{F}_t(q)$. We first wish to show that $A^n \rightarrow A$ weakly. This follows from the continuity theorem [3, XIII.1], which states that weak convergence of locally finite measures is equivalent to pointwise convergence of their Laplace transforms. From (2.9),
\[
\bar{A}^n(q) = \frac{F^n_0(q)}{1 - p^n(q)} \rightarrow \frac{F_0(q)}{1 - P(q)} = \bar{A}(q).
\] (3.12)
Weak equivalence of measures implies that cumulative functions $A^n(t)$ converge to $A(t)$ at all continuity points of $A(t)$. Now, for all points of continuity of $\tau(t) = A^\dagger(t)$, $\lim_{n \rightarrow \infty}(A^n)^\dagger(t) = A^\dagger(t)$ [8, Lemma 3.1]. As the generalized inverse is a nondecreasing function, its discontinuity set is countable, and thus $\tau^n(t) \rightarrow \tau(t)$ almost everywhere.

What remains is to show convergence of the integrals
\[
\int_t^\infty e^{-q\tau^n(s)} ds \rightarrow \int_t^\infty e^{-q\tau(s)} ds.
\] (3.13)
Here, we will use the dominated convergence theorem. This requires bounding the integrand $e^{-q\tau^n(s)}$, which we obtain from several steps:

(i) Observe that there exist $M > 0$ and a positive integer $N$ such that $M$ is a point of continuity for $P$, and $P^n(M) > \frac{1}{2}$ for $n > N$. Define truncated holding times which are restricted to $[0, M]$, with cumulative functions

$$\tilde{P}^n(x) = P^n(x)/P^n(M), \quad \tilde{P}(x) = P(x)/P(M), \quad x \in [0, M]. \tag{3.14}$$

Denote $\tilde{P}^n$ and $\tilde{P}$ as random variables distributed with respect to $\tilde{P}^n$ and $\tilde{P}(x)$, respectively. Then clearly $\tilde{P}^n \to \tilde{P}$ weakly, and since $P^n(x) \leq P^n(x)$, it follows that $A^n(t) \leq \tilde{A}^n(t)$.

(ii) Since $\tilde{P}^n$ is supported on a finite interval, it has a finite first and second moment. Thus allows us to apply Lorden’s estimate [7] to obtain

$$\tilde{A}^n(\tau) \leq \frac{\tau}{E[\tilde{P}^n]} + \frac{E[(\tilde{P}^n)^2]}{E[\tilde{P}^n]^2} + 1. \tag{3.15}$$

Note that we have added 1 to Lorden’s original estimate to account for a possibly distinct initial holding time.

(iii) Since $\tilde{P}^n$ are all supported in $[0, M]$, it follows that weak convergence to $\tilde{P}$ implies the convergence of moments. From this and (3.15) it is straightforward to show that there exist $a, b > 0$ such that for all positive $n$,

$$A^n(\tau) \leq \tilde{A}^n(\tau) \leq a\tau + b \tag{3.16}$$

$$\Rightarrow \tau^n(t) \geq \frac{t - b}{a}. \tag{3.17}$$

Thus, integrands in the left hand side of (3.13) are dominated by an exponentially decaying function, which proves (3.13) and thus part (1).

Part (2) follows immediately from the weak convergence shown in part (1), since $F^n$ are all probability measures.

For part (3), note that if $A(\tau)$ is strictly increasing, then $\tau(t)$ is continuous. From (3.6), it is clear that $\tilde{F}_t(q)$ is continuous in $t$, which in turn implies that $F_t$ evolves continuously under the weak topology in the space of measures.

\[\square\]

### 3.3 An example of renewal-scaled solutions with point mass initial conditions

To illustrate how renewal-scaled solutions behave under jumps in initial data, we conclude with an example with monodisperse initial conditions. Specifically, let the initial measure data satisfy $F_0 = 1_{(1, \infty)}(x)$ and $p(x) = e^{-x}$. Then $L(p(x)) = \frac{1}{q+1}, F_0(q) = e^{-q}$, and using (2.9),

$$A(q) = \frac{q}{q+1} e^{-q} = e^{-q} + \frac{e^{-q}}{q}. \tag{3.18}$$

For $t > 0$, Laplace inversion then gives

$$A(\tau) = \begin{cases} 0 & \tau \in [0, 1) \\ \frac{1}{\tau} & \tau \geq 1 \end{cases}, \tag{3.19}$$

which has a generalized inverse of

$$\tau(t) = A^{-1}(t) = \begin{cases} 1 & t \in (0, 1) \\ t & t > 1 \end{cases}. \tag{3.20}$$

Substitution of these variables into (3.6), for $t \in (0, 1]$, yields

$$\tilde{F}_t(q) = e^q \left( \frac{q}{q+1} \int_1^\infty e^{-q\tau(s)}ds \right) = e^q \left( \frac{q}{q+1} \right) \left( (1 - t)e^{-q} + \int_1^\infty e^{-qs}ds \right) = \frac{t}{q+1} + (1 - t).$$

Taking inverse Laplace transforms then gives us the solution $F_t(dx) = (1 - t)\delta_0(dx) + te^{-x}dx$ for $t \in (0, 1]$. By a similar calculation, we can show that $F_t(dx) = e^{-x}dx$ for $t > 1$.

In the $t$ time scale, initial conditions with a support not containing the origin will immediately jump through translation. For our example, the point mass jumps to the origin, and is then continuously reassigned to $[0, \infty)$ according to $p(x)$ at a constant rate of one. After the reassignment of the entire point mass, the solution, now a density, is stationary, since $u(x, \tau) = e^{-x}$ is a solution to (1.3) with $p(x) = u_0(x) = e^{-x}$ (one can see, in fact, that this is a manifestation of the memoryless property of exponential distributions).
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5 Conflicts of Interest

The author declares no conflict of interest.

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