THE GLASSEY CONJECTURE FOR NONTRAPPING OBSTACLES

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ABSTRACT. We verify the 3-dimensional Glassey conjecture for exterior domain \((M, g)\), where the metric \(g\) is asymptotically Euclidean, provided that certain local energy assumption is satisfied. The radial Glassey conjecture exterior to a ball is also verified for dimension three or higher. The local energy assumption is satisfied for many important cases, including exterior domain with nontrapping obstacles and flat metric, exterior domain with star-shaped obstacle and small asymptotically Euclidean metric, as well as the nontrapping asymptotically Euclidean manifolds \((\mathbb{R}^n, g)\).

1. Introduction

The purpose of this paper is to show how local energy estimates for certain linear wave equations involving asymptotically Euclidean perturbations of the standard Laplacian lead to optimal global and long time existence theorems for the corresponding small amplitude nonlinear wave equations with power nonlinearities in the derivatives, which is also known as the Glassey conjecture, see [11] and references therein. For spatial dimension three, we could prove global and almost global existence. However, for dimension four and higher, the current technology could only apply for the radial case, and we obtain existence results with certain lower bound of the lifespan, which is sharp in general. The general case is still open, even for the Minkowski spacetime, when the spatial dimension is four or higher.

Let us start by describing the asymptotically Euclidean manifolds \((M, g)\), where \(M = \mathbb{R}^n \setminus K\) with smooth and compact obstacle \(K\) and \(n \geq 3\). Without loss of generality, when \(K\) is nonempty, we may assume the origin lies in the interior of \(K\) and \(K \subset B_1 = \{ x \in \mathbb{R}^n : |x| < 1 \}\). By asymptotically Euclidean, we mean that

\[
(H1)\quad g = g_0 + g_1(r) + g_2(x), \quad g = g_{ij}(x)dx^idx^j = \sum_{i,j=1}^{n} g_{ij}(x)dx^idx^j
\]

where \((g_{ij})\) is uniformly elliptic, \((g_{0,ij}) = \text{Diag}(1,1,\cdots,1)\) is the standard Euclidean metric, the first perturbation \(g_1\) is radial, and

\[
(H1.1)\quad \sum_{j,k} \| \partial_x^\alpha g_{i,j,k} \|_{L^1+|\alpha|-1}^{i-1} \lesssim 1, \quad i = 1, 2, \forall \alpha.
\]
Here, we say $g_1$ is radial, if, when writing out the metric $g$, with $g_2 = 0$, in polar coordinates $x = r \omega$ with $r = |x|$ and $\omega \in S^{n-1}$, we have

$$g = g_0 + g_1 = \tilde{g}_{11}(r)dr^2 + \tilde{g}_{22}(r)r^2 d\omega^2.$$  

In this form, the assumption (H1.1) for $g_1$ is equivalent to the following requirement

(\text{H1.2}) \quad \|\partial_x^2 (\tilde{g}_{11} - 1, \tilde{g}_{22} - 1)\|_{L^\infty_x L^2} \lesssim 1.

When $g = g_0 + \delta (g_1 + g_2)$ with a small parameter $\delta$, we call it a small perturbation.

We shall consider Dirichlet-wave equations on $(M, g)$,

$$\begin{cases}
\Box_g u \equiv (\partial^2_t - \Delta_g) u = F, \quad x \in M, t > 0 \\
u(t, x) = 0, \quad x \in \partial M, t > 0 \\
u(0, x) = \phi(x), \partial_t u(0, x) = \psi(x),
\end{cases}$$

(1.1)

where $\Delta_g$ is the Laplace-Beltrami operator associated with $g$.

Now we can state the local energy assumption that we shall make

**Hypothesis 2.** For any given $R > 1$, we have

(\text{H2}) \quad \|(\partial u, u)\|_{L^2_x L^2_t (B_R)} \leq C \left(\|\phi\|_{H^1} + \|\psi\|_{L^2} + \|F\|_{L^2_x L^2_t}\right),

for any solutions to (1.1) with data $(\phi, \psi)$ and the forcing term $F(t, x)$ vanishes for $|x| > R$, and $\phi|_{\partial M} = 0$. Here the constant $C$ may be dependent on the obstacle $\mathcal{K}$, and $R$.

Let us review some important cases where the assumption (H2) is valid. First of all, when $g_1 = g_2 = 0$, it is true for any nontrapping obstacle $\mathcal{K}$. In which case, we have

$$\|(\partial u(t), u(t))\|_{L^2_x L^2_t (B_R)} \leq \alpha(t) \left(\|\phi\|_{H^1} + \|\psi\|_{L^2}\right)$$

with $\alpha(t) \lesssim (t)^{-(n-1)} \in L^1_t \cap L^\infty_t$, for any homogeneous solutions to (1.1) with data $(\phi, \psi)$ supported in $B_R$. See [17, 24] and references therein. For the case where $g$ is a compact perturbation of $g_0$, and $M$ is assumed to be nontrapping with respect to the metric, one also has (H2) for the Dirichlet-wave equation for all $n \geq 3$ ([32, 2]). For general nontrapping asymptotically Euclidean manifolds without obstacles, it is also known to be true ([1]), at least when (H1.1) is replaced by

(\text{H1.1'}) \quad \sum_{j,k} \|\partial_x^2 \partial_t^{i+j} u\|_{L^\infty_x L^2_t} \lesssim 1, \quad i = 1, 2,

for some $\delta > 0$. At last, if $g$ is a small asymptotically Euclidean metric perturbation, and the obstacle is star-shaped (that is, $\mathcal{K} = \{r \omega : 0 \leq r \leq \gamma(\omega) < 1, \omega \in S^{n-1}\}$), for some smooth positive function $\gamma$, it is essentially proved in [19] that we still have (H2).

Having described the main assumptions about the linear problem, let us now turn to the nonlinear equations. Let $n \geq 3$, $p > 1$, we consider the following nonlinear wave equations,

$$\begin{cases}
\Box_g u = \sum_{\alpha=0}^n a_\alpha(u) \partial_\alpha u p = F_p(u, \partial_t u), \quad x \in M \\
u(t, x) = 0, \quad x \in \partial M, t > 0 \\
u(0, x) = \phi(x), \partial_t u(0, x) = \psi(x),
\end{cases}$$

(1.2)

for given smooth functions $a_\alpha$, as well as the radial problems (with $g_2 = 0$, $\mathcal{K} = B_1$)

$$\begin{cases}
\Box_g u = a |\partial_\alpha u|^p + b |\nabla u|^p \equiv G_p(u, \partial_t u), \quad x \in M \\
u(t, x) = 0, \quad |x| = 1, t > 0 \\
u(0, x) = \phi(x), \partial_t u(0, x) = \psi(x),
\end{cases}$$

(1.3)
for given constants $a$, $b$. We will study the long time existence of such problems with small enough initial data, according to certain norm.

For this problem posed on the Minkowski space-time, it is conjectured that the critical exponent $p$ for the problem, to admit global solutions with small, smooth initial data with compact support is

$$p_c = 1 + \frac{2}{n-1}$$

in [7] (see also [25, 23]). The conjecture was verified in dimension $n = 2, 3$ for general data (9) and [33] independently, as well as the radial case in [26] for $n = 3$. For the radial data, the existence results with sharp lifespan for any $p \in (1, 1 + 2/(n-2))$ was recently proved in [11] (see also [6] for the critical case $n = 2$ and $p = 3$), which particularly verified the Glassey conjecture in the radial case. On the other hand, for higher dimension $n \geq 4$, the blow up results (together with an explicit upper bound of the lifespan) for (1.2), with $F_p(u, \partial_t u) = |\partial_t u|^p$ and $p \leq p_c$, were obtained in [36, 37] when $g$ is a compact metric perturbation. Recently, in [34], the author extended the existence results in [9, 33, 11] to the setting with small space-time dependent asymptotically flat perturbation of the metric on $\mathbb{R}^n$ with $n \geq 3$.

We can now state our main results. The first result is about the problem (1.2) with general data, which verifies the 3-dimensional Glassey conjecture in exterior domains, with asymptotically Euclidean metric perturbation, under the local energy assumption.

**Theorem 1.1.** Let $n = 3$, $\mathcal{K}$ be empty or smooth and compact obstacles, and $p > 2$. Consider the problem (1.2) on $(M, g)$ satisfying (H1) and (H2). There exists a small positive constant $\varepsilon_0$, such that the problem (1.2) has a unique global solution satisfying $u \in C([0, \infty); H^2(M)) \cap C^1([0, \infty); H^1(M))$, whenever the initial data satisfy the compatibility conditions of order 2, and

$$\sum_{|\alpha| \leq 2} \| (\nabla, \Omega)^{\alpha} (\nabla \phi, \psi) \|_{L^2(M)} = \varepsilon \leq \varepsilon_0, \| \phi \|_{L^2(M)} < \infty.$$  

Moreover, when $p = 2$, there exists some $c > 0$, so that we have unique solution satisfying $u \in C([0, T_e]; H^2(M)) \cap C^1([0, T_e]; H^1(M))$, with $T_e = \exp(c/\varepsilon)$.

Here, by the compatibility conditions of order 2, we mean that

$$\psi(x) = 0, \psi(x) = 0, \Delta_y \phi + F_p(\phi, \psi) = 0$$

for any $x \in \partial M$. In general, we see from the equation (1.2) that, formally, there exist $\Phi_k$ such that

$$\partial_t^k u(0, x) = \Phi_k(J_k \phi, J_{k-1} \psi)$$

for $x \in M$, where $J_k f = \nabla^{\leq k} f$. Then the compatibility conditions of order $k$ is precisely $\Phi_k(J_k \phi, J_{j-1} \psi)(x) = 0$ for any $x \in \partial M$ and $0 \leq j \leq k$.

In particular, as special cases, we have the following corollaries, for which, as we have recalled, it is known that (H1) and (H2) is true. See [17, 24], [1] and Lemma 3.1 for the corresponding local energy estimates.

**Corollary 1.** Let $g = g_0$ and $\mathcal{K}$ be empty or a nontrapping obstacle, then the 3-dimensional Glassey conjecture is true.

**Corollary 2.** Let $M = \mathbb{R}^3$ and $g$ be a nontrapping asymptotically Euclidean perturbation of the flat metric ((H1) with (H1.1')), then the 3-dimensional Glassey conjecture is true.
Corollary 3. Let \( g \) be a small, asymptotically Euclidean perturbation of the flat metric, and \( K \) be a star-shaped obstacle, then the 3-dimensional Glassey conjecture is true.

Turning to the problem (1.3) with radial data, we can prove the long time existence of the radial solutions, in spirit of [11], where the lower bound of the lifespan is sharp in general ([36, 37]). To simplify the exposition, we will only prove a weaker version of the long time existence theorem, comparing that of [11]. It is not hard to see that our proof could be easily adapted to prove the same set of Theorems 1.1-1.3 (with \( n \geq 3 \)) in [11], we leave the details to the reader.

Theorem 1.2. Let \( n \geq 3 \), \( p > p_c = 1 + 2/(n - 1) \), \( K = B_1 \), \( g_2 = 0 \), \((M,g)\) satisfying (H1) and (H2). Consider the problem (1.3) with radial data, there exists a small positive constant \( \varepsilon_0 \), such that the problem has a unique global radial solution satisfying \( u \in C([0, \infty); H^2(M)) \cap C^1([0, \infty); H^1(M)) \), whenever the initial data satisfy the boundary conditions of order 1, and

\[
(1.6) \quad \sum_{|\alpha| \leq 1} \| \nabla^\alpha (\nabla \phi, \psi) \|_{L^2(M)} = \varepsilon \leq \varepsilon_0, \quad \| \phi \|_{L^2(M)} < \infty.
\]

Moreover, when \( p \leq p_c \), there exist some \( c > 0 \), so that we have unique radial solutions satisfying \( u \in C([0, T_c]; H^2(M)) \cap C^1([0, T_c]; H^1(M)) \), with \( T_c = \exp(c \varepsilon^{1-p}) \) for \( p = p_c \) and \( T_c = c \varepsilon^{2/(p-1)/(n-1)(p-1)-2} \) for \( 1 < p < p_c \).

As before, it is clear that Theorem 1.2 applies for the flat or small asymptotically Euclidean metric, in the domain exterior to a ball.

Corollary 4. Let \( g = g_0 \) and \( K = B_1 \), then the radial Glassey conjecture is true, for dimension \( n \geq 3 \).

Corollary 5. Let \( g \) be a small, radial, asymptotically Euclidean perturbation of the flat metric, and \( K = B_1 \), then the radial Glassey conjecture is true, for dimension \( n \geq 3 \).

Remark 1. Comparing the current Theorem 1.2 with Theorem 1.1 in [11], we remove a technical restriction \( p < 1 + 2/(n - 2) \), which, in \( \mathbb{R}^n \), is partly due to the \( H^2 \) regularity. The reason, for us to avoid the restriction in the case of exterior domain, is that we have radial Sobolev embedding \( H^1 \subset L^\infty \) (see Lemma 2.1), which is not true in \( \mathbb{R}^n \).

As in [11] and [34], one of the main ingredients in the proof is the local energy estimates with variable coefficients, in spirit of [19, 10]. The local energy estimates first appeared in [22], which are often called Morawetz estimates. By now there is an extensive literature devoted to this topic and its applications; without being exhaustive we mention [30, 15, 27, 13, 2, 14, 29, 12, 18, 19, 20, 8, 28, 21, 31, 16]. Based on (H1) and (H2), we could prove the following version of the local energy estimates. See (1.14) for the notations.

Theorem 1.3. For \((M,g)\) satisfying (H1), there exists \( R_0 \geq 4 \), which depends on \( g \), such that for any solutions to (1.1) with \( \phi|_{\partial M} = 0 \), we have \( u \in C([0, \infty); H^1(M)) \), and

\[
(1.7) \quad \| u \|_{LE \cap E} \lesssim \| \phi \|_{H^1} + \| \psi \|_{L^2} + \| F \|_{LE \cdot E} + \| L_2 \cdot E \}
\]

provided that (H2) is true with some \( R \geq 4R_0 \).
To prove the existence results, as usual, we need to prove higher order local energy estimates, that is,

**Proposition 1.4** (Higher order local energy estimates). For \((M, g)\) satisfying \((H1)\), there exists \(R_1 \geq R_0\) such that, we have

\[
\|u\|_{LE, \cap E_k} \lesssim \sum_{|\alpha| \leq k} \|\nabla, \Omega^{\alpha} (\nabla \phi, \psi)\|_{L^2} + \|Z_{\alpha} F\|_{LE^{\alpha}, L^1_k L^2_2} + \sum_{|\gamma| \leq k - 1} \|\partial^\gamma F(0, x)\|_{L^2_2} + \|\partial^\gamma F(\mathcal{L}^\alpha \cap L^2_2)_{2n}(B_{2R})\|
\]
for any solutions to \((1.1)\) satisfying compatibility condition of order \(k\), provided that we have \((H2)\) with some \(R \geq 4R_1\).

Here, once again, we see from the equation \((1.1)\) that, formally, there exist \(\Phi_k\) such that

\[
\partial^k_\xi u(x, 0) = \Phi_k(J_k \phi, J_{k-1} \psi, J_{k-2} F)
\]
for \(x \in M\), where \(J_k F(x) = \partial^{\leq k} F(0, x)\). Then the compatibility conditions of order \(k\) is precisely \(\Phi_k(J_j \phi, J_{j-1} \psi, J_{j-2} F)(x) = 0\) for any \(x \in \partial M\) and \(0 \leq j \leq k\).

For the existence results with \(p \leq p_c\), we will also require a relation between the KSS type estimates \([13, 12, 19]\) and the local energy estimates. Basically, it is known that, the local energy norm, together with the energy norm, could control the KSS-type norm, see, e.g., \([19, 34]\) Lemma 3.4. Moreover, we observe here that a dual version also holds.

**Lemma 1.5.** For any \(\mu \in [0, 1/2]\), there are positive constants \(C_\mu\) and \(C\), independent of \(T \geq 2\), such that

\[
\|\partial^\mu u\|_{L^2_2 \cap L^2_2} + \|r^{-1/2} u\|_{L^2_2 \cap L^2_2} \leq C(\ln T)^{1/2} \|u\|_{LE, \cap E([0, T] \times M)}.
\]

Moreover, we have

\[
\|F\|_{LE^{\alpha}, L^1_k L^2_2} \leq C(\ln T)^{1/2} \|F\|_{L^2_2 \cap L^2_2}.
\]

This paper is organized as follows. In the next section, we recall some Sobolev type estimates, in relation with trace theorem and Hardy’s inequality. In Section 3, we give the proof of the local energy estimates Theorem 1.3, and Proposition 1.4, based on \((H1)\) and \((H2)\), as well as a relation between the local energy estimates and KSS type estimates, Lemma 1.5. In the fourth section, we give the proof of the three dimensional Glassey conjecture, following the approach of \([11, 34]\), adapted in the setting of exterior domains. In the last section, we outline the proof for the radial Glassey conjecture.

**1.1 Notations.** Finally we close this section by listing the notations.

- \((x^0, x^1, \ldots, x^n) = (t, x) \in \mathbb{R}^{1+n}\), and \(\partial_i = \partial/\partial x^i\), \(0 \leq i \leq n\), with the abbreviations \(\partial = (\partial_0, \partial_1, \cdots, \partial_n) = (\partial, \nabla)\). \(\partial^\alpha = \partial_0^{\alpha_0} \cdots \partial_n^{\alpha_n}\) with multi-indices \(\alpha, \beta \in \mathbb{N}^{n+1}\).
- \(A \lesssim B\) means that \(A \leq CB\) where the constant \(C\) may change from line to line.
• The vector fields to be used will be labeled as
  \[ Y = (Y_1, \cdots, Y_{n(n+1)/2}) = (\nabla, \Omega, Z = (\partial_t, Y) \]
with rotational vector fields \( \Omega_{ij} = x_i \partial_j - x_j \partial_i, 1 \leq i < j \leq n \).
• With Dirichlet boundary condition, we define \( \dot{H}^1_0(M) \) as the closure of \( f \in C_0^\infty(M) \), with respect to the norm
  \[ \|f\|_{\dot{H}^1_0(M)} = \|\nabla f\|_{L^2(M)}. \]
When \( M = \mathbb{R}^n \), \( \dot{H}^1 \) means the closure of \( C_0^\infty \) with respect to the \( \dot{H}^1 \) norm.
• The space \( l^q_t(A) \) \( (1 \leq q \leq \infty) \) means
  \[ \|u\|_{l^q_t(A)} = \|\langle \Phi_j(x)u(t,x)\rangle\|_{l^q_t(A)} = \|\langle 2^{j\beta} \Phi_j(x)u(t,x)\rangle\|_{l^q_t(A)}, \]
for a partition of unity subordinate to the (inhomogeneous) dyadic (spatial) annuli, \( \sum_{j \geq 0} \Phi_j(x) = 1 \). Typical choice could be a radial, nonnegative \( \Phi_0(x) \in C_0^\infty \) with value 1 for \( |x| \leq 1 \), and 0 for \( |x| \geq 2 \), and \( \Phi_j(x) = \Phi(2^{-j}x) - \Phi(2^{1-j}x) \) for \( j \geq 1 \).
• \( \|\cdot\|_{E_m} \) is the energy norm of order \( m \geq 0 \),
  \[ \|u\| = \|u\|_{E_0} = \|\partial_t u\|_{L^2_t L^2_x(\mathbb{R} \times M)}, \|u\|_{E_m} = \sum_{|\alpha| \leq m} \|Z^\alpha u\|_E. \]
Also, we use \( \|\cdot\|_{E_E} \) to denote the local energy norm
  \[ \|u\|_{E_E} = \|\partial_t u\|_{L^{1/2}_t L^2_x(\mathbb{R} \times M)} + \|u/r\|_{L^{1/2}_t L^2_x(\mathbb{R} \times M)} \].
On the basis of the local energy norm, we can similarly define \( \|u\|_{E_E m} \), and the dual norm \( E^*_E = L^{1/2}_t L^2_x(\mathbb{R} \times M) \).
• \( \|u\|_{X+Y} = \inf_{u = u_1 + u_2} (\|u_1\|_X + \|u_2\|_Y) \)
• Let \( \beta(x) \in C_0^\infty \) such that \( \beta = 1 \) for \( |x| \leq R \) and vanishes for \( |x| \geq 2R \). Based on \( \beta \), we set \( \beta_1(x) = \beta(x/R), \beta_2(x) = \beta(2x) \),
  \[ \tilde{g} = \beta(4x)g_0 + (1 - \beta(4x))g = g_0 + (1 - \beta(4x))(g_1 + g_2), \]
which agrees with \( g \) for \( |x| \geq R/2 \) and \( g_0 \) for \( |x| \leq R/4 \).

2. Sobolev-type estimates

In this section, we recall several Sobolev type estimates in relation with the trace theorem and Hardy’s inequality. At first, we have the following trace theorem (see Lemma 2.2 in \[11\], (1.3), (1.7) in \[5\] and references therein)

**Lemma 2.1.** \( n \geq 2 \) and \( 1/2 < s < n/2 \), then
  \[ \|r^{(n-1)/2}(u\omega)\|_{L^2} \lesssim \|u\|_{L^2(|x| \geq r)} + \|\nabla u\|_{L^2(|x| \geq r)}. \]
We will also need the following variant of the Sobolev embeddings.

**Lemma 2.2.** \( n \geq 2 \). For any \( m \in \mathbb{R} \) and \( k \geq n/2 - n/q \) with \( q \in [2, \infty) \), we have
  \[ \|r^{(n-1)(1/2-1/q)+m} u\|_{L^2(M)} \lesssim \sum_{|\alpha| \leq k} \|r^m Y^\alpha u\|_{L^2(M)}. \]
Moreover, we have
  \[ \|r^{(n-1)/2+m} u\|_{L^\infty(M)} \lesssim \sum_{|\alpha| \leq (n+2)/2} \|r^m Y^\alpha u\|_{L^2(M)}, \]
where \( |\alpha| \) stands for the integer part of \( a \).
When $M = \mathbb{R}^n$, it is precisely Lemma 2.2 in [34] (see also Lemma 3.1 in [16]). The general results, for the exterior domain, then follow from a simple cutoff argument and the classical Sobolev embedding.

When dealing with (1.2), we need to have a local control of $u$, from $\nabla u$, which is achieved by the Hardy inequality.

**Lemma 2.3** (Hardy’s inequality). Let $n \geq 3$ and $M = \mathbb{R}^n \setminus K$ with smooth and compact $K$. Then for any $u \in \dot{H}^1_D(M)$, we have

\begin{equation}
\|u/r\|_{L^2(M)} \lesssim \|\nabla u\|_{L^2(M)}.
\end{equation}

**Proof.** It is classical, see [4, 3] and references therein. For reader’s convenience, we give an explicit proof in the case of star-shaped obstacle here. By density, it suffices to prove (2.4) for $u \in C_0^\infty(M)$. For this special case, the inequality is just a consequence of integration by parts:

\[
\int_{r=\gamma(\omega)}^{\infty} |u/r|^2 r^{n-1} dr = \frac{1}{n-2} \int_{\gamma(\omega)}^{\infty} u^2 \partial_r r^{n-2} dr = \frac{1}{n-2} \int_{\gamma(\omega)}^{\infty} u^2 r^{n-2} dr - \frac{2}{n-2} \int_{\gamma(\omega)}^{\infty} \partial_r u dr \\
\leq \frac{2}{n-2} \left( \int_{\gamma(\omega)}^{\infty} |u/r|^2 r^{n-1} dr \right)^{1/2} \left( \int_{\gamma(\omega)}^{\infty} |\partial_r u|^2 r^{n-1} dr \right)^{1/2},
\]

which, after integrating with respect to $\omega$, yields (2.4). \qed

As a direct consequence, we have

**Proposition 2.4.** Let $n = 3$ and $u \in \dot{H}^1_D(M) \cap \dot{H}^2(M)$, we have

\begin{equation}
\|u\|_{L^\infty(M)} \lesssim \sum_{|\alpha| \leq 1} \|\nabla^{\alpha} u\|_{L^2(M)}
\end{equation}

**Proof.** Viewing $(1 - \beta)u$ as function in $\mathbb{R}^n$, by Sobolev embedding $H^2(M \cap B_{2R}) \subset L^\infty(M \cap B_{2R})$, and $H^1 \cap \dot{H}^2 \subset L^\infty(\mathbb{R}^n)$, we have

\[
\|u\|_{L^\infty} \leq \|\beta u\|_{L^\infty} + \|(1 - \beta)u\|_{L^\infty} \\
\leq \|\beta u\|_{H^2} + \|(1 - \beta)u\|_{H^1 \cap \dot{H}^2} \\
\leq \|u\|_{L^2(B_{2R})} + \sum_{|\alpha| \leq 1} \|\nabla^{\alpha} u\|_{L^2(M)} \\
\leq \sum_{|\alpha| \leq 1} \|\nabla^{\alpha} u\|_{L^2(M)} ,
\]

where in the last step, we used Hardy’s inequality. \qed

3. Local energy estimates

In this section, we give the proof of the local energy estimates Theorem 1.3, and Proposition 1.4, based on (H1) and (H2), as well as a relation between the local energy estimates and KSS type estimates, Lemma 1.5.
3.1. Local energy estimates with variable coefficients. To begin, let us recall a local energy estimates with variable coefficients, which is essentially obtained in [19] (see also [34] Lemma 3.1, as well as [20, 10, 11, 35]).

Lemma 3.1. Let \( n \geq 3 \) and \( M = \mathbb{R}^n \). Consider the linear problem \( \Box_g u = F \) with \( g(x) = g_0 + \delta q_1(x) \) satisfying (H1.1). Then there exists a constant \( \delta_0 \), such that for any \( 0 \leq \delta \leq \delta_0 \), we have the following local energy estimates,

\[
\|u\|_{LE \cap E} \lesssim \|\Box u(0)\|_{L^2} + \|F\|_{LE^* + L^1 L^2}.
\]

In addition, the same results apply for solutions to (1.1), when \( M = \mathbb{R}^n \setminus K \) with star-shaped \( K \).

Note that compared with [34] Lemma 3.1, we have relaxed the condition on the decay rate of asymptotically Euclidean metric a little bit. Notice that by the assumption, we have

\[
\Box_g = \Box - r_0^{ij}(x) \partial_i \partial_j + r_1^i(x) \partial_j,
\]

where

\[
\|\Box^{a} r_0(x)\|_{L^2_x} \lesssim \delta, \quad \|\Box^{a} r_1\|_{L^2_x} \lesssim \delta, \quad \forall \alpha.
\]

With this observation, it is not hard to see that basically the same proof for [34] Lemma 3.1 apply in the current situation. In the case of star-shaped obstacle, we need only to observe further that the boundary term will be nonnegative and can be disregarded, see e. g. P197 of [19]. We omit the details here.

3.2. Local energy estimates in exterior domain. With Lemma 3.1 at hand, we could give the proof of Theorem 1.3. First of all, by Duhamel’s principle, it suffices to prove

\[
\|u\|_{LE \cap E} \lesssim \|\phi\|_{H^2_D} + \|\psi\|_{L^2} + \|F\|_{LE^*}
\]

for solutions to (1.1). We divide the proof into three steps: controlling the local part, the local energy, and the energy.

3.2.1. Controlling the local part. At first, we notice that it is possible to choose \( R_0 \geq 4 \) large enough such that, \( \tilde{g} \), as defined in (1.15), satisfies the condition in Lemma 3.1 when \( R \geq R_0 \). Fixing the choice, we see that, on \( \mathbb{R}^n \),

\[
\|u\|_{LE \cap E} \lesssim \|\partial u(0)\|_{L^2} + \|\Box u\|_{LE^* + L^1 L^2}.
\]

Now, we define \( u_1 \) as the solution of the Dirichlet-wave equation with data \( (\beta_1 \phi, \beta_1 \psi) \) and forcing term \( \beta_1 F \), and \( u_2 = u - u_1 \).

For \( u_1 \), we have trivially

\[
\|(\partial u_1, u_1)\|_{L^2_x L^2_t(B_R)} \lesssim \|\beta_1 \phi\|_{H^2} + \|\beta_1 \psi\|_{L^2} + \|\beta_1 F\|_{L^2_x L^2_t} \lesssim \|\phi\|_{H^2} + \|\psi\|_{L^2} + \|F\|_{LE^*},
\]

by (H2) with \( 4R \) and the Hardy inequality (2.4).

To estimate \( u_2 \), we introduce \( u_0 \) as the solution of the Cauchy problem in \( \mathbb{R}^n \)

\[
\Box \beta u_0 = (1 - \beta_1)F, u_0(0, x) = (1 - \beta_1)\phi, \partial_t u_0(0, x) = (1 - \beta_1)\psi.
\]

For \( u_0 \), we know from (3.3) that,

\[
\|u_0\|_{LE} \lesssim \|\phi\|_{H^2_D} + \|\psi\|_{L^2} + \|F\|_{LE^*}.
\]

Now, let \( w = u_2 - (1 - \beta)u_0 \), noticing that

\[
\Box_g[(1 - \beta)u_0] = \Box_g[(1 - \beta)u_0] = (1 - \beta)\Box u_0 + [\Delta_g, \beta]u_0 = (1 - \beta_1)F + [\Delta_g, \beta]u_0,
\]

we have

\[
\Box_g[(1 - \beta)u_0] \lesssim \Box_g[(1 - \beta)u_0] \lesssim \Box u_0 + [\Delta_g, \beta]u_0.
\]

Finally, we can assume
After integration in time, we get for any $T$ which is controlled by the right hand side of (3.2.2), viewing it as solution of the Cauchy problem, we get from (3.2.2),

\[ \left\| (\partial_t, u) \right\|_{L^2_tL^2_x(B_R)} \lesssim \left\| (\beta, \Delta_g)u_0 \right\|_{L^2_tL^2_x} \lesssim \left\| u_0 \right\|_{LE} . \]

Recalling $u = u_1 + u_2 = u_1 + w + (1 - \beta)u_0$, (3.4)-(3.6), we arrived at

\[ \left\| (\partial_t, u) \right\|_{L^2_tL^2_x(B_R)} \lesssim \left\| \phi \right\|_{H^1} + \left\| \psi \right\|_{L^2} + \left\| F \right\|_{LE^*} . \]

### 3.2.2. Controlling the local energy

Turing to the full local energy estimates, we divide $u$ into $\beta_2u + (1 - \beta_2)u$. For $(1 - \beta_2)u$, due to the support property, and $\tilde{g}$ agrees with $g$ for $|x| \geq R/2$, we observe that

\[ \Box_g(1 - \beta_2)u = \Box_g(1 - \beta_2)u = (1 - \beta_2)F + [\Delta_{g^*}, \beta_2]u . \]

Viewing it as solution of the Cauchy problem, we get from (3.3) that

\[ \left\| u \right\|_{LE} \lesssim \left\| \beta_2u \right\|_{LE} + \left\| (1 - \beta_2)u \right\|_{LE} \lesssim \left\| \partial_t u(0) \right\|_{L^2_x} + \left\| F \right\|_{LE^*} + \left\| (\partial_t, u) \right\|_{L^2_tL^2_x(B_R)} \]

which is controlled by the right hand side of (3.2), by (3.7). That is, we have proved

\[ \left\| u \right\|_{LE} \lesssim \left\| \partial_t u(0) \right\|_{L^2_x} + \left\| F \right\|_{LE^*} , \]

which is local energy part of (3.2).

### 3.2.3. Controlling the energy

It remains to control the energy norm in (3.2). For this, we introduce a modified energy norm

\[ A(t) = \left( \int_M u^2(t, x) + g^{ij}(x)\partial_iu(t, x)\partial_ju(t, x) \frac{1}{2} \sqrt{|g|} dx \right)^{1/2} , \]

where $|g|$, $g^{ij}$ are the determinant and inverse matrix to the matrix $g_{ij}$. From geometrical point of view, it is a natural definition of the energy. By our assumption, it is equivalent to the classical energy norm $E$. For $A(t)$, we know from the definition, after integration by parts and noticing that $\partial_t u|_{\partial M} = 0$, that

\[ \frac{dA(t)^2}{dt} = \int_M u_t F \sqrt{|g|} dx . \]

After integration in time, we get for any $T$,

\[ A^2(T) \leq A^2(0) + \int_0^T \int_M \left| u_t F \right| \sqrt{|g|} dx dt \lesssim \left\| \partial_t u(0) \right\|_{L^2_x}^2 + \left\| u \right\|_{LE} \left\| F \right\|_{LE^*} . \]

Applying (3.8), we know that

\[ \left\| \partial u(T) \right\|_{L^2_x} \lesssim A^2(T) \lesssim \left\| \partial u(0) \right\|_{L^2_x}^2 + \left\| F \right\|_{LE^*}^2 . \]

and so

\[ \left\| u \right\|_{LE \cap E} \lesssim \left\| \partial u(0) \right\|_{L^2_x} + \left\| F \right\|_{LE^*} , \]

which is (3.2). This completes the proof of Theorem 1.3.
3.3. Higher order estimates. In this subsection, we give the proof of the higher order local energy estimates, Proposition 1.4, based on Theorem 3.1.

As is traditional, part of the difficulty to prove higher order estimates for exterior domain is due to the fact that the vector fields do not preserve the boundary condition \( u|_{\partial M} = 0 \) in general.

Despite of the difficulty, we notice that \( \partial_t \) preserves the boundary condition and commutates with the equation. As a consequence, provided the solution to (1.1) satisfies the compatibility condition of order \( k \), by Theorem 1.3, we have

\[
(3.10) \quad \sum_{0 \leq j \leq k} \| \partial_t^j u \|_{LE \cap E} \lesssim \sum_{|\alpha| \leq k} \| \nabla^\alpha (\nabla \phi, \psi) \|_{L^2} + \sum_{|\gamma| \leq k-1} \| \partial^\gamma F(0, x) \|_{L^2} \\
+ \sum_{0 \leq j \leq k} \| \partial_t^j F \|_{LE^* + L^1_t L^2_x}.
\]

Here, we have expressed the initial data of \( \partial_t^j u \), through the equation (1.1), by the combination of \( \nabla^\alpha \phi, \nabla^\alpha \psi \) and \( \partial^\gamma F(0, x) \).

To extend the vector field from \( \partial_t \) to \( Z \), we observe first

\[
\| Z^\alpha u \|_{LE \cap E} \lesssim \| Z^\alpha \beta_2 u \|_{LE \cap E} + \| Z^\alpha (1 - \beta_2) u \|_{LE \cap E}.
\]

For the second term, \( \| Z^\alpha (1 - \beta_2) u \|_{LE \cap E} \), notice that

\[
\square_g Z^\alpha (1 - \beta_2) u = \square_g Z^\alpha (1 - \beta_2) u - [\square_g, Z^\alpha] (1 - \beta_2) u - Z^\alpha [\square_g, \beta_2] u + Z^\alpha (1 - \beta_2) F.
\]

For \([\square_g, Z^\alpha]\), by (H1.1), we know that, for any given \( \delta > 0 \), there exists \( R_1 \geq R_0 \), such that for \( R \geq R_1 \), there exists \( c_i(x) \) such that

\[
\| [\square_g, Z^\alpha] v \| \leq c_1(x) \sum_{|\gamma| \leq |\alpha|} |Z^\gamma \partial v| + c_2(x) \sum_{|\gamma| \leq |\alpha|} |Z^\gamma v|,
\]

with \( \| c_i(x) \|_{L^\infty} \leq \delta \). Here, we used the fact that the first perturbation is radial, which commutates with the rotational vector fields \( \Omega \).

Applying Lemma 3.1, together with these information,

\[
\| Z^\alpha (1 - \beta_2) u \|_{LE \cap E} \lesssim \sum_{|\gamma| \leq |\alpha|} \| (\nabla, \Omega)^\gamma (\nabla \phi, \psi) \|_{L^2} + \sum_{|\gamma| \leq |\alpha|-1} \| Z^\gamma F(0, x) \|_{L^2} \\
+ \| [\square_g, Z^\alpha] (1 - \beta_2) u \|_{LE^* + L^1_t L^2_x} + \sum_{|\gamma| \leq |\alpha|-1} \| \partial^\gamma u \|_{L^2_t L^2_x(B_R)} \\
+ \| Z^\alpha (1 - \beta_2) F \|_{LE^* + L^1_t L^2_x} \\
\lesssim \sum_{|\gamma| \leq |\alpha|} \| (\nabla, \Omega)^\gamma (\nabla \phi, \psi) \|_{L^2} + \sum_{|\gamma| \leq |\alpha|-1} \| Z^\gamma F(0, x) \|_{L^2} \\
+ \delta \sum_{|\gamma| \leq |\alpha|} \| Z^\gamma (1 - \beta_2) u \|_{LE} + \sum_{|\gamma| \leq |\alpha|-1} \| \partial^\gamma u \|_{L^2_t L^2_x(B_R)} \\
+ \sum_{|\gamma| \leq |\alpha|} \| Z^\gamma F \|_{LE^* + L^1_t L^2_x}.
\]
Summing over $|\alpha| \leq k$ and setting $\delta$ small enough to be absorbed by the left, we conclude that
\[
\|u\|_{L^r(\Omega)} \leq \|\beta_2 u\|_{L^r(\Omega)} + \|(1-\beta_2) u\|_{L^r(\Omega)} \\
\lesssim \sum_{|\gamma| \leq k} \|\nabla(\nabla, \Omega)\nabla \psi\|_{L^r(\Omega)} + \|Z^n F\|_{L^r(\Omega)} + \|Z^n F\|_{L^r(\Omega)} + \sum_{|\gamma| \leq k} \|\nabla^\gamma u\|_{L^r(\Omega)}.
\]
(3.11)

To complete the proof of Proposition 1.4, it suffices to give the control of the last term in (3.11).

3.3.1. Controlling the local part. Let us prove Proposition 1.4, by (3.10), (3.11), and induction.

The case $k = 0$ follows from Theorem 1.3. Assume it is true for some $k = j \geq 0$, then for $k = j + 1$, since the problem satisfies the compatibility condition of order $j + 1$, we have the compatibility condition of order $j$ for $w = \partial_t u$, and
\[
\Box_g w = \partial_t F, w|_{\partial M} = 0, w(0, x) = \psi, \partial_t w(0, x) = \Delta_g \theta + F(0, x).
\]
At first, we give the estimate of
\[
\|\nabla^\gamma \partial^2 u\|_{L^2(\Omega)} \lesssim \|\nabla^\gamma \partial w\|_{L^2(\Omega)} + \|\nabla^\gamma \partial^2 u\|_{L^2(\Omega)} + \|\nabla^\gamma \partial^2 u\|_{L^2(\Omega)},
\]
where in the last inequality, we used the equation (1.1).

In conclusion, we get
\[
\|\nabla^\gamma \partial^2 u\|_{L^2(\Omega)} \lesssim \|\nabla^\gamma \partial w\|_{L^2(\Omega)} + \|\nabla^\gamma \partial^2 u\|_{L^2(\Omega)} + \|\nabla^\gamma \partial^2 u\|_{L^2(\Omega)} + \sum_{|\gamma| \leq j + 1} \|\nabla^\gamma u\|_{L^2(\Omega)} \lesssim \|w\|_{L^2(\Omega)} + \|\nabla^\gamma \partial^2 u\|_{L^2(\Omega)} + \sum_{|\gamma| \leq j + 1} \|\nabla^\gamma u\|_{L^2(\Omega)}.
\]
(3.12)

By the induction assumption, Proposition 1.4 with $k = j$, we have
\[
\sum_{|\gamma| \leq j + 1} \|\nabla^\gamma u\|_{L^r(\Omega)} \lesssim \|w\|_{L^r(\Omega)} + \sum_{|\gamma| = j} \|\nabla^\gamma \partial^2 u\|_{L^r(\Omega)} + \sum_{|\gamma| = j} \|\nabla^\gamma F\|_{L^r(\Omega)} + \sum_{|\gamma| = j} \|\nabla^\gamma F\|_{L^r(\Omega)}.
\]
Then, by (3.11) with $k = j + 1$, $\|u\|_{LE^{j+1}}$ is controlled by

$$
\sum_{|\gamma| \leq j+1} \|\langle \nabla, \Omega \rangle (\nabla \phi, \psi)\|_{L^2} + \|Z^\gamma F\|_{LE^jL^2} + \sum_{|\gamma| \leq j} \|Z^\gamma F(0, x)\|_{L^2}
$$

$$
+ \sum_{|\gamma| \leq j+1} \|\partial^\gamma u\|_{LE \cap E(B_R)}
\lesssim \sum_{|\gamma| \leq j+1} \|\langle \nabla, \Omega \rangle (\nabla \phi, \psi)\|_{L^2} + \|Z^\gamma F\|_{LE^jL^2} + \sum_{|\gamma| \leq j} \|Z^\gamma F(0, x)\|_{L^2}
$$

$$
+ \sum_{|\gamma| \leq j} \|\partial^\gamma F\|_{(L^\infty \cap L^2)(B_{2R})}.
$$

This completes the proof of Proposition 1.4.

3.4. A relation between KSS type norm and local energy norm. In this subsection, we give a proof of Lemma 1.5. Since (1.9) and (1.10) are classical, we give only the proof of the dual version, that is, (1.11) and (1.12).

As usual, we use a cutoff argument [13]. Let $F_1 = F\chi_{|x| \leq T}$ and $F_2 = F - F_1$.

$$
\|F\|_{LE^jL^2} \lesssim \|F_1\|_{LE^j} + \|F_2\|_{L^1L^2}
$$

$$
\lesssim \|2^{j/2} F_1(t, x) \Phi_j(x)\|_{L^2} + T^{-1/2} \|x\|^{1/2} F_2\|_{L^2}
$$

$$
\lesssim \|2^{j/2} F_1(t, x) \Phi_j(x)\|_{L^2} + \|x\|^{1/2} F_2\|_{L^2}
$$

$$
\lesssim (\ln T)^{1/2} \|F\|_{L^2} (L^2).$$

Similarly,

$$
\|F\|_{LE^jL^2} \lesssim \|2^{j/2} F_1(t, x) \Phi_j(x)\|_{L^2} + T^{-1/2} \|x\|^{1/2} F_2\|_{L^2}
$$

$$
\lesssim \|2^{j/2} F_1(t, x) \Phi_j(x)\|_{L^2} + \|x\|^{1/2} F_2\|_{L^2}
$$

$$
\lesssim T^{1/2} \|F\|_{L^2} (L^2).$$

This completes the proof.

4. Glaesey conjecture with dimension 3

In this section, we will prove Theorem 1.1, mainly based on Lemma 2.2 and Proposition 1.4, for given $\phi$ and $\psi$ such that (1.4) and (1.5) are satisfied.

As usual, we shall use iteration to give the proof. We set $u_0 \equiv 0$ and recursively define $u_{k+1} (k \geq 0)$ be the solution to the linear equation

$$
\Box u_{k+1} = F_p(u_k, \partial_t u_k), u_{k+1}(t, x) = 0, u_{k+1}(0, x) = \phi(x), \partial_t u_{k+1}(0, x) = \psi(x).
$$

Note that the compatibility condition (1.5) ensures that, we still have the compatibility condition of order 2 for $u_{k+1}$, and we can apply Proposition 1.4 with $k = 2$.

Boundedness: By the smallness condition (1.4) on the data and the equation, we know from the definition of $F_p$ that, for $|\alpha| \leq 1$,

$$|Z^\alpha F_p(u_k, \partial_t u_k)(t, x)| \leq C(\|u_k\|_{L^\infty}) \|\partial_t u_k\|^{p-1} (|Z u_k| + |\nabla, \Omega| \partial u_k| + |\partial u_k|^p),$$

$$|\partial^\alpha F_p(u_k, \partial_t u_k)(t, x)| \leq C(\|u_k\|_{L^\infty}) \|\partial_t u_k\|^{p-1} (|\partial u_k|^2 + |\nabla u_k| + |\partial u_k|^p),$$

where $C(t)$ is a continuous increasing function. By Proposition 2.4, we know that

$$
\|u\|_{L^\infty_t L^p_x(M)} + \|Z u\|_{L^\infty_t L^p_x(M)} \lesssim \|u\|_{E_2},
$$

(4.1)
and so for \( \varepsilon \) small enough, we have
\[
\sum_{|\alpha| \leq 1} \| Z^\alpha F_p(u_k, \partial_t u_k)(0, \cdot) \|_{L^2_x} \lesssim C_\varepsilon \varepsilon^{p-1}(\varepsilon + \varepsilon^p) \lesssim \varepsilon
\]
\[
\sum_{|\alpha| \leq 1} \| \partial^\alpha F_p(u_k, \partial_t u_k) \|_{(L^2 \cap L^\infty)(B_2 \cap E_2)} \lesssim \tilde{C}(\| u_k \|_{E_2}) \| u_k \|_{E_2}^{p-1} \| u_k \|_{L^2_x \cap E_2},
\]
for some continuous increasing function \( \tilde{C}(t) \).

With the above estimates, it follows from Proposition 1.4 with \( k = 2 \) that there is a universal constant \( C_1 \) so that \( \| u_1 \|_{L^2_x \cap E_2} \leq C_1 \varepsilon \), and
\[
\| u_{k+1} \|_{L^2_x \cap E_2} \leq C_1 \varepsilon + C_1 \sum_{|\alpha| \leq 1} \| Z^\alpha F_p(u_k, \partial_t u_k) \|_{L^1_t L^2_x} + \tilde{C}(\| u_k \|_{E_2}) \| u_k \|_{E_2}^{p-1} \| u_k \|_{L^2_x \cap E_2}.
\]

We shall argue inductively to prove that
\[
\| u_{k+1} \|_{L^2_x \cap E_2} \leq 3C_1 \varepsilon,
\]
when \( \varepsilon \leq \varepsilon_0 \) and \( \varepsilon_0 \) is small enough, for all \( k \geq 0 \). By the above, it suffices to show
\[
\sum_{|\alpha| \leq 2} \| Z^\alpha F_p(u, \partial_t u) \|_{L^1_t L^2_x} \leq \varepsilon,
\]
for any \( u \) with \( \| u \|_{L^2_x \cap E_2} \leq 3C_1 \varepsilon \leq 1 \).

Notice that there exist smooth functions \( b_i, 1 \leq i \leq 5 \), such that
\[
Z^{\leq 2} F_p(u, \partial_t u) = b_1(u) |\partial u|^{p-1} Z^{\leq 2} \partial u + b_2(u) |\partial u|^{p-2} (Z^{\leq 1} \partial u)^2 + b_3(u) |\partial u|^{p-1} Z u Z^{\leq 1} \partial u + b_4(u) |\partial u|^p Z u Z u + b_5(u) |\partial u|^p Z^2 u.
\]

By Lemma 2.2,
\[
|\partial u| \lesssim \frac{\| u \|_{E_2}}{\langle r \rangle}, \quad |Z u| \lesssim \| u \|_{E_2}.
\]

By the boundedness of \( u \) (4.1), smoothness of \( b_1 \) and \( 4.5 \), we see that
\[
|Z^{\leq 2} F_p(u, \partial_t u)| \lesssim |\partial u|^{p-1} (|Z^{\leq 2} \partial u| + |Z^{\leq 2} u|/\langle r \rangle) + |\partial u|^{p-2} |Z^{\leq 1} \partial u|^2.
\]

The first term can be dealt with as follows, by (4.5), Lemma 2.2, and the fact that \( p > 2 \),
\[
\| |\partial u|^{p-1} (|Z^{\leq 2} \partial u| + |Z^{\leq 2} u|/\langle r \rangle) \|_{L^1_t L^2_x} \lesssim \| \langle r \rangle |\partial u| \|_{L^2_x \cap E_2}^{p-2} \| \langle r \rangle^{(3-p)/2} \partial u \|_{L^1_t L^2_x} \| \langle r \rangle^{-(p-1)/2} \| \langle r \rangle \| Z^{\leq 2} \partial u \|_{L^2_x \cap E_2}^{p-1} \| u \|_{E_2}^{p-2} \| u \|_{L^2_x \cap E_2}^{p-2} \| u \|_{L^2_x \cap E_2}^{p-2} \| u \|_{L^2_x \cap E_2}^{p-2} \| u \|_{L^2_x \cap E_2}^{p-2}.
\]

Similarly, for the second term, we get
\[
\| |\partial u|^{p-2} |Z^{\leq 1} \partial u|^2 \|_{L^1_t L^2_x} \lesssim \| \langle r \rangle |\partial u| \|_{L^2_x \cap E_2}^{p-2} \| \langle r \rangle^{-(p-2)/2} Z^{\leq 1} \partial u \|_{L^2_x \cap E_2}^2 \lesssim \| u \|_{E_2}^{p-2} \| \langle r \rangle^{-(p-2)/2-1/2} Z^{\leq 2} \partial u \|_{L^2_x \cap E_2}^2 \lesssim \| u \|_{E_2}^{p-2} \| u \|_{E_2}^{p-2}.
\]
In conclusion, we see that there exists a constant $C_2$ such that
\begin{equation}
\sum_{|\alpha| \leq 2} \| Z^\alpha F_p(u, \partial_t u) \|_{L_2^1 L_\infty^2} \leq C_2 \| u \|_{L_2^2}^{p-1} \leq C_2 (3C_1 \varepsilon)^p \leq \varepsilon
\end{equation}
for $\varepsilon \leq \varepsilon_0$ with
\[ C_2 (3C_1)^p \varepsilon_0^{p-1} \leq 1. \]
This finishes the proof of (4.4) and so is the uniform boundedness (4.3).

Similar proof will give us the convergence of the sequence $\{u_k\}$
\[ \| u_{k+1} - u_k \|_{L_2^2} \leq C_1 \| F_p(u_k) - F_p(u_{k-1}) \|_{L_2^1 L_\infty^2} \leq \frac{1}{2} \| u_k - u_{k-1} \|_{L_2^2} \]
provided that $\varepsilon_0$ is small enough.

Together with the uniform boundedness (4.3), we find an unique global solution $u \in L_{t,x}^\infty H^3 \cap Lip_t H^2$ with $\| u \|_{L_2^2} \leq 3C_1 \varepsilon$. Strictly speaking, to complete the proof, we need also to prove the regularity of the solution $u \in C_t H^3 \cap C_t^1 H^2$. As it is standard, we omit details here, and refer the reader to the end of Section 4 in [34] or [11] P533.

For the remaining case, $p = 2$, we need only to notice that by Lemma 1.5 and Proposition 1.4 with $k = 2$, we have for $T \geq 2$
\[ \| u \|_{L_2^2 \cap E_2} + (\ln T)^{-1/2} \| Z^\gamma \partial_t u \|_{L_2^1 (L_2^{1/2} L_2^2)} \]
\[ \lesssim \sum_{|\alpha| \leq 2} \| (\nabla, \Omega)^\alpha (\nabla \phi, \psi) \|_{L_2^2} + (\ln T)^{1/2} \| Z^\alpha F \|_{L_2^1 (L_2^{1/2} L_2^2)} \]
\[ + \sum_{|\gamma| \leq 1} \| Z^\gamma F(0, x) \|_{L_2^2} + \| \partial^\gamma F \|_{(L_2^\infty \cap L_2^1) L_2^2(B_{2R})} \]
for solutions to (1.1) in $[0, T] \times M$. See also the end of Section 5.2.

5. Radial Glassey conjecture

In this section, we outline the proof for Theorem 1.2, by using Lemma 2.1, Proposition 1.4, and Lemma 1.5.

We set $u_0 \equiv 0$ and recursively define $u_{k+1}$ to be the solution to the linear equation
\begin{equation}
\Delta u_{k+1} = G_p(u_k, \partial_t u_k), u_{k+1} |_{x \in \partial B_1} = 0, u_{k+1}(0, x) = \phi, \partial_t u_{k+1}(0, x) = \psi.
\end{equation}
By assumption, $u_k$ are radial functions.

5.1. Global existence. Recall Lemma 2.1, $M = \{|x| > 1\}$, where $r \sim \langle r \rangle$ and the fact that $u$ is radial, we have
\begin{equation}
\| \langle r \rangle^{n-1/2} \partial_t u \|_{L_\infty^\infty(M)} \lesssim \| u \|_{E_1}.
\end{equation}

By the smallness condition (1.6) on the data and the equation, we know from the definition of $G_p$ that, for $\varepsilon$ small enough, we have
\[ \| G_p(u_k, \partial_t u_k)(0, \cdot) \|_{L_2^2} \lesssim \varepsilon \lesssim \varepsilon
\]
\[ \| G_p(u_k, \partial_t u_k) \|_{L_2^2(B_{2R})} \lesssim \| u_k \|_{E_1} \| u_k \|_{L_2^2} \lesssim \varepsilon^{p-1} \| u_k \|_{L_2^2} \lesssim \| u_k \|_{L_2^2} \lesssim \varepsilon.
\]

With the above estimates, it follows from Proposition 1.4 with $k = 1$ that there is a universal constant $C_2$ so that $\| u_1 \|_{L_2^2} \leq C_2 \varepsilon$, and
\begin{equation}
\| u_{k+1} \|_{L_2^2} \leq C_2 \varepsilon + C_2 \sum_{|\alpha| \leq 1} \| \partial^\alpha G_p(u_k, \partial_t u_k) \|_{L_2^2} + C_2 \| u_k \|_{E_1} \| u_k \|_{L_2^2}.
\end{equation}
As in Section 4, for global existence, let us give the proof of the uniform boundedness of the iteration series $u_k$ with respect to $LE_1 \cap E_1$, which could be reduced to the proof of

$$\|u\|_{LE_1 \cap E_1} \leq 3C_2 \varepsilon \Rightarrow \|G_p(u, \partial_t u)\|_{LE_1} \leq \varepsilon,$$

for radial $u$ and small enough $\varepsilon$. In fact, $\|G_p(u, \partial_t u)\|_{LE_1}$ is controlled as follows,

$$\|G_p(u, \partial_t u)\|_{LE_1} = \|\partial^{\leq 1} G_p(u, \partial_t u)\|_{l^{1/2}_i L^2_x L^2_t} \lesssim \|\partial u\|_{l^{1/2}_i L^2_x L^2_t} \lesssim \|\partial (r)^{(n-1)/2} \partial u\|_{L^\infty_T} \|\partial^{(-n-1)(p-1)/2} \partial u\|_{l^{1/2}_i L^2_x L^2_t} \lesssim \|u\|_{E_1} \|\partial^{(-n-1)(p-1)/2} \partial u\|_{l^{1/2}_i L^2_x L^2_t} \lesssim \|u\|_{E_1} \|\partial^{\leq 1} \partial u\|_{L^2_x L^2_T} \lesssim \varepsilon^p$$

provided that $(n-1)(p-1)/2 > 1$, that is $p > p_c$. Then by choosing $\varepsilon > 0$ small enough, we get (5.4).

5.2. Long time existence. For $p \leq p_c$, we recall that it is known that, the local energy norm, together energy norm, could control the KSS-type norm, i.e., Lemma 1.5. Then, by (5.3) with $k = 1$, there exist $C_3 \geq C_2$, such that we have for $T \geq 2$,

$$\|u_{k+1}\|_{LE_1 \cap E_1 + (\ln T)^{-1/2} \|\partial^{\leq 1} \partial u_{k+1}\|_{l^{1/2}_i (L^2_x L^2_t)} + T^{\mu-1/2} \|\partial^{\leq 1} \partial u_{k+1}\|_{L^\infty_x (L^2_x L^2_t)}} \leq C_3 \varepsilon + C_3 \sum_{|\alpha| \leq 1} (\ln T)^{1/2} \|\partial^\alpha G_1\|_{l^{1/2}_i (L^2_x L^2_t)} + T^{1/2-\mu} \|\partial^\alpha G_2\|_{L^\infty_x (L^2_x L^2_t)}$$

for solutions to (5.1) and any $G_p(u_k, \partial_t u_k) = G_1 + G_2$.

With this, we could easily adapt the proof for $p \leq p_c$. When $p = p_c$, as before, we need to prove

$$\|u\|_{LE_1 \cap E_1 + (\ln T)^{-1/2} \|\partial^{\leq 1} \partial u\|_{l^{1/2}_i (L^2_x L^2_t)}} \leq 3C_3 \varepsilon \Rightarrow \|\partial^{\leq 1} G_p\|_{l^{1/2}_i (L^2_x L^2_t)} \lesssim \varepsilon^p \ln T \lesssim \varepsilon^p T^{1/2},$$

for $p = p_c = 1 + 2/(n-1)$, radial $u$ and small enough $\varepsilon$. In fact,

$$\|\partial^{\leq 1} G_p\|_{l^{1/2}_i (L^2_x L^2_t)} \lesssim \|\partial u\|_{l^{1/2}_i (L^2_x L^2_t)} \lesssim \|\partial (r)^{(n-1)/2} \partial u\|_{L^\infty_T} \|\partial^{(n-1)(p-1)/2} \partial u\|_{l^{1/2}_i (L^2_x L^2_t)} \lesssim \|u\|_{E_1} \|\partial^{\leq 1} \partial u\|_{L^2_x L^2_T} \lesssim \varepsilon^p \ln T \lesssim \varepsilon^p \ln T \lesssim \varepsilon^p T^{1/2}.$$

With (5.5) at hand, it will be essentially easy to prove almost global existence, by choosing $T = \exp(c \varepsilon^{-1/\mu})$ with small enough $c$, such that $\varepsilon \ln T \leq c \varepsilon \ll \varepsilon$.

Similarly, for $1 < p < p_c$, we need to prove, with $\mu = (n-1)(p-1)/4 \in (0, 1/2)$, for radial $u$ and solutions to (5.1),

$$\|u\|_{LE_1 \cap E_1 + T^{1/2-\mu} \|\partial^{\leq 1} \partial u\|_{l^{1/2}_i (L^2_x L^2_t)}} \leq 3C_3 \varepsilon \Rightarrow \|\partial^{\leq 1} G_p\|_{l^{1/2}_i (L^2_x L^2_t)} \lesssim \varepsilon^p T^{1/2-\mu}.$$

radial \( u \) and small enough \( \varepsilon \). We could apply the same kind proof here

\[
\|\partial^{\leq 1} G_p\|_{L^2} \lesssim \|\partial u\|_{L^2}^{p-1} \partial u \|_{L^2} \lesssim \|\partial u\|_{L^2} \|\partial u\|_{L^2} \lesssim \|u\|_{L^2}^{p-1} \|\partial u\|_{L^2} \lesssim \varepsilon^{pT^{1/2-\mu}}.
\]

With (5.6), the long time existence could be proved, by choosing

\[
T = c\varepsilon^{2(p-1/[(n-1)(p-1)]-2)}
\]

with small enough \( c \), such that \( \varepsilon^{pT^{1-2\mu}} \leq c^{1-2\mu} \varepsilon \ll \varepsilon \).

References

[1] J.-F. Bony, D. H"{a}fner, The semilinear wave equation on asymptotically Euclidean manifolds. Comm. Partial Differential Equations 35 (2010), no. 1, 23-67. MR2748617

[2] N. Burq: Global Strichartz estimates for nontrapping geometries: About an article by H. Smith and C. Sogge, Comm. Partial Differential Equations 28 (2003), 1675–1683. MR2001179

[3] J. Chabrowski, M. Willem, Hardy’s inequality on exterior domains , Proc. Amer. Math. Soc. 134 (2006), 1019–1022. MR2196033

[4] F. Colin, Hardys inequality in unbounded domains, Topol. Methods Nonlinear Anal. 17 (2001), no. 2, 277-284. MR1868901

[5] D. Fang, C. Wang, Weighted Strichartz estimates with angular regularity and their applications. Forum Math., 23 (2011), no. 1, 181–205. MR2769870

[6] D. Fang, C. Wang, Almost global existence for some semilinear wave equations with almost critical regularity, Comm. Partial Differential Equations, 38 (2013), 1467–1491.

[7] R. T. Glassey, MathReview to “Global behavior of solutions to nonlinear wave equations in three space dimensions” of Sideris, Comm. Partial Differential Equations (1983).

[8] K. Hidano, J. Metcalfe, H. F. Smith, C. D. Sogge, Y. Zhou, On abstract Strichartz estimates and the Strauss conjecture for nontrapping obstacles, Trans. Amer. Math. Soc. 362 (2010), no. 5, 2789–2809. MR2584618

[9] K. Hidano, K. Tsutaya, Global existence and asymptotic behavior of solutions for nonlinear wave equations, Indiana Univ. Math. J., 44 (1995), 1273–1305. MR1386769

[10] K. Hidano, C. Wang, K. Yokoyama, On almost global existence and local well-posedness for some 3-D quasi-linear wave equations, Adv. Differential Equations 17 (2012), no. 3-4, 267–306. MR2919103

[11] K. Hidano, C. Wang, K. Yokoyama, The Glassey conjecture with radially symmetric data. J. Math. Pures Appl. (9) 98 (2012), no. 5, 518–541. MR2980460

[12] K. Hidano, K. Yokoyama, A remark on the almost global existence theorems of Keel, Smith and Sogge. Funkcial. Ekvac. 48 (2005), no. 1, 1–34. MR2154375

[13] M. Keel, H. Smith, C. D. Sogge, Almost global existence for some semilinear wave equations, Dedicated to the memory of Thomas H. Wolff. J. Anal. Math. 87 (2002), 265–279. MR1945285

[14] M. Keel, H. F. Smith, C. D. Sogge, Almost global existence for quasilinear wave equations in three space dimensions, J. Amer. Math. Soc. 17 (2004), no. 1, 109–153 MR2015331

[15] C. E. Kenig, G. Ponce, L. Vega, On the Zakharov and Zakharov-Schulman systems, J. Funct. Anal. 127 (1995), 204–234. MR1308623

[16] H. Lindblad, J. Metcalfe, C. D. Sogge, M. Tohaneanu, C. Wang, The Strauss conjecture on Kerr black hole backgrounds. Math. Ann. to appear. arXiv:1304.4145.

[17] R. B. Melrose: Singularities and energy decay in acoustical scattering, Duke Math. J. 46 (1979), 43–59. MR0523601

[18] J. Metcalfe, C. D. Sogge, Hyperbolic trapped rays and global existence of quasilinear wave equations, Invent. Math. 159 (2005), no. 1, 75117. MR2142333

[19] J. Metcalfe, C. D. Sogge, Long-time existence of quasilinear wave equations exterior to star-shaped obstacles via energy methods, SIAM J. Math. Anal. 38 (2006), no. 1, 188–209. MR2217314

[20] J. Metcalfe, D. Tataru, Global parametrices and dispersive estimates for variable coefficient wave equations, Math. Ann. 353 (2012), no. 4, 1183–1237. MR2944027
[21] J. Metcalfe, D. Tataru, M. Tohaneanu, Price’s law on nonstationary space-times. Adv. Math. 230 (2012), no. 3, 995–1028. MR2921169
[22] C. S. Morawetz, Time decay for the nonlinear Klein-Gordon equations, Proc. R. Soc. Lond. Ser. A 306 (1968), 291–296. MR0234136
[23] M. A. Rammaha, Finite-time blow-up for nonlinear wave equations in high dimensions. Comm. Partial Differential Equations 12 (1987), no. 6, 677–700. MR0879355
[24] J. Ralston: Note on the decay of acoustic waves, Duke Math. J. 46 (1979), 799–804. MR0552527
[25] J. Schaeffer, Finite-time blow-up for \( u_{tt} - \Delta u = H(u, u_t) \). Comm. Partial Differential Equations 11 (1986), no. 5, 513–543. MR0829595
[26] T. C. Sideris, Global behavior of solutions to nonlinear wave equations in three dimensions, Comm. Partial Differential Equations 8 (1983), 1291–1323. MR0711440
[27] H. F. Smith, C. D. Sogge, Global Strichartz estimates for nontrapping perturbations of the Laplacian, Comm. Partial Differential Equations 25 (2000), 2171–2183. MR1789924
[28] C. D. Sogge, C. Wang, Concerning the wave equation on asymptotically Euclidean manifolds. J. Anal. Math. 112 (2010), no. 1, 1–32. MR2762995
[29] J. Sterbenz, Angular regularity and Strichartz estimates for the wave equation, Int. Math. Res. Not. (2005), 187–231. With an appendix by I. Rodnianski. MR2128434
[30] W. A. Strauss, Dispersal of waves vanishing on the boundary of an exterior domain. Comm. Pure Appl. Math. 28 (1975), 265–278. MR0367461
[31] D. Tataru, Local decay of waves on asymptotically flat stationary space-times. Amer. J. Math. 135 (2013), no. 2, 361–401. MR3038715
[32] M. Taylor: Grazing rays and reflection of singularities of solutions to wave equations, Comm. Pure Appl. Math. 29 (1976), 1–38. MR0397175
[33] N. Tzvetkov, Existence of global solutions to nonlinear massless Dirac system and wave equation with small data, Tsukuba J. Math. 22 (1998), no. 1, 193–211. MR1637692
[34] C. Wang, The Glassey conjecture on asymptotically flat manifolds. arXiv:1306.6254
[35] C. Wang, X. Yu, Global existence of null-form wave equations on small asymptotically Euclidean manifolds. arXiv:1207.5218.
[36] Y. Zhou, Blow up of solutions to the Cauchy problem for nonlinear wave equations, Chinese Ann. Math. Ser. B, 22 (2001), no. 3, 275–280. MR1845748
[37] Y. Zhou, W. Han, Blow-up of solutions to semilinear wave equations with variable coefficients and bounda, J. Math. Anal. Appl. 374 (2011), no. 2, 585–601. MR2729246

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