A Note on the Geometry of Certain Classes of Linear Operators

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Abstract
In this note we introduce a new technique to answer an issue posed in Fávaro et al. (Bull Braz Math Soc 51:27–46, 2020) concerning geometric properties of the set of non-surjective linear operators. We also extend and improve a related result from the same paper.

Keywords Spaceability · Lineability · Sequence spaces

Mathematics Subject Classification 46B87 · 15A03 · 47B37 · 47L05

1 Introduction

In 1872 Weierstrass constructed an example of a nowhere differentiable continuous function from [0, 1] on \( \mathbb{R} \). This non intuitive result, now known as the Weierstrass Monster, was pushed further in 1966, when Gurariy constructed an infinite-dimensional subspace formed, except for the null vector, by continuous nowhere differentiable functions. In 1995, Rodríguez-Piazza (1995) proved in that every separable Banach space is isometric to a space of continuous nowhere differentiable functions and, in 2004, Aron et al. (2005) investigated similar problems in other settings, initiating the field of research known as “lineability”: the idea is to look for linear structures inside exotic subsets of vector spaces. If \( V \) is a vector space and \( \alpha \) is a cardinal number, a
subset $A$ of $V$ is called $\alpha$-lineable in $V$ if $A \cup \{0\}$ contains an $\alpha$-dimensional linear subspace $W$ of $V$. When $V$ has a topology and the subspace $W$ can be chosen to be closed, we say that $A$ is spaceable. We refer to the book (Aron et al. 2016) for a general panorama of the subject. Recall that by dimension of a vector space we mean the cardinality of any of its Hamel bases.

As a matter of fact, with the development of the theory, it was observed that positive results of lineability were quite common, although general techniques are, in general, not available. Towards a more demanding notion of linearity, Fávaro, Pellegrino and Tomáz introduced a more involved geometric concept: let $\alpha$, $\beta$ and $\lambda$ be cardinal numbers, with $\alpha < \beta \leq \lambda$, and let $V$ be a vector space such that $\dim V = \lambda$. A subset $A$ of $V$ is called $(\alpha, \beta)$-lineable if, for every subspace $W_{\alpha} \subset V$ such that $\dim W_{\alpha} = \alpha$ and $W_{\alpha} \subset A \cup \{0\}$ there is a subspace $W_{\beta} \subset V$ with $\dim W_{\beta} = \beta$ and $W_{\alpha} \subset W_{\beta} \subset A \cup \{0\}$. When $V$ is a topological vector space, we shall say that $A$ is $(\alpha, \beta)$-spaceable when the subspace $W_{\beta}$ can be chosen to be closed.

A well-known technique in lineability is the “mother vector technique”: it consists of choosing a vector $v$ in the set $A$ and generating a subspace $W \subset A \cup \{0\}$ containing “copies” of $v$. However, in general, the vector $v$ does not belong to the generated subspace (see, for instance, Pellegrino and Teixeira 2009). Constructing a vector space of prescribed dimension and containing an arbitrary given vector is a rather more involved problem, which is probably another motivation of this more strict approach to lineability.

Under this new perspective, several simple problems, from the point of view of ordinary lineability, gain more subtle contours. For instance, it is obvious that the set of continuous linear operators $u : \ell_p \to \ell_q$ that are non-surjective is $c$-spaceable (here and henceforth $c$ denotes the continuum). In fact, if $\pi_1 : \ell_q \to \mathbb{K}$ is the projection at the first coordinate, just consider the collection of continuous and non-surjective linear operators for which $\pi_1 \circ u \equiv 0$. Therefore, only $(\alpha, \beta)$-lineability matters in this framework.

In this note we answer a question posed in Fávaro et al. (2020) on the $(1, c)$-lineability of a certain set of non surjective functions. Our solution uses a technique that, to the best of the authors’ knowledge, is new.

### 2 Lineability vs Injectivity and Surjectivity

Lineability properties of the sets of injective and surjective continuous linear operators between classical sequence spaces were recently investigated by Aron et al. (2018) and Diniz et al. (2020). In Fávaro et al. (2020, Theorem 3.1) the authors investigated more subtle geometric properties in the setting of non injective continuous linear operators by proving that if $p, q \geq 1$ and

$$A := \left\{ u : \ell_p \to \ell_q : u \text{ is linear, continuous and non injective} \right\}, \quad (2.1)$$

then $A$ is $(1, c)$-lineable. In the same paper the authors pose a question on the $(1, c)$-lineability of the set...
\[ D := \{ u : \ell_p \to \ell_q : \text{u is linear, continuous and non-surjective}\}. \quad (2.2) \]

In this section we shall show that \( D \) is \((1, c)\)-lineable and, as a matter of fact, our technique works in a more general environment of sequence spaces. We shall say that a Banach sequence space \( E \) of \( X \)-valued sequences where \( X \) is a Banach space is \textit{reasonable} if \( c_{00}(X) \subset E \) and, for all \( x = (x_j)_{j=1}^\infty \in E \) and \( (\alpha_j)_{j=1}^\infty \in \ell_\infty \), we have \( (\alpha_j x_j)_{j=1}^\infty \in E \) with

\[
\| (\alpha_j x_j)_{j=1}^\infty \| \leq \| (\alpha_j)_{j=1}^\infty \| \| (x_j)_{j=1}^\infty \|.
\]

The class of reasonable sequence spaces includes various classical sequence spaces. For instance, for \( 1 < p < \infty \), the \( \ell_p(X) \) spaces of \( p \)-summable sequences, the \( \ell^w_p(X) \) spaces of weakly \( p \)-summable sequences and the \( \ell^u_p(X) \) spaces of unconditionally \( p \)-summable sequences are reasonable sequence spaces. The spaces \( \ell_\infty(X), c_0(X), \) of bounded and null sequences, respectively and the Lorentz spaces \( \ell(w,p)(X) \) are also reasonable sequence spaces.

Our result reads as follows:

\textbf{Theorem 2.1} Let \( V \neq \{0\} \) be a normed vector space and \( X \neq \{0\} \) be a Banach space. Let \( E \) be a reasonable sequence space of \( X \)-valued sequences. The set

\[ DV,E = \{ u : V \to E : u \text{ is linear, continuous and non-surjective} \} \]

is \((1, c)\)-lineable.

\section{The Proof}

For a fixed \( v \in DV,E\setminus\{0\} \), let

\[ \mathbb{N}_v = \{ k \in \mathbb{N} : \pi_k \circ v \neq 0 \}, \]

where

\[ \pi_k : E \to X \]
\[ (x_j)_{j=1}^\infty \mapsto x_k \]

is the \( k \)-th projection over \( X \). If \( \mathbb{N}_v \) is a proper subset of \( \mathbb{N} \), the proof is simple. In fact, if \( j_0 \in \mathbb{N}\setminus\mathbb{N}_v \), since \( c_{00}(X) \subset E \), it is obvious that the subspace

\[ N := \{ u : V \to E : u \text{ is linear, continuous and } \pi_{j_0} \circ u \equiv 0 \} \]

is contained in \( DV,E \) and it is also plain that \( v \in N \). We will prove that \( \dim N \geq c \).

Since \( v \) is not identically zero, there exists \( x_0 \in V \) such that \( v(x_0) = w_0 \neq 0 \). By the
Hahn–Banach Theorem, there is a continuous linear functional \( \varphi : E \to \mathbb{K} \), such that \( \varphi (w_0) = 1 \). Fixing \( a \in X \setminus \{0\} \), for each \( k \in \mathbb{N} \), let us define

\[
w_k = (0, \ldots, 0, a, 0, 0, \ldots) \in E
\]

and consider the linear operators \( T_k : E \to E \) given by

\[
T_k (w) = \varphi (w) \frac{w_k}{\|w_k\|}.
\]

Obviously, the operators \( T_k \) are continuous and

\[
\|T_k\| = \sup_{\|w\| \leq 1} \left| \varphi (w) \frac{w_k}{\|w_k\|} \right| = \sup_{\|w\| \leq 1} |\varphi (w)| = \|\varphi\|.
\]

Hence, the operators \( S_k = T_k \circ v \) are continuous too and

\[
\|S_k\| = \|T_k \circ v\| \leq \|T_k\| \|v\| = \|\varphi\| \|v\|.
\]

Notice that \( S_k (x_0) = w_k / \|w_k\| \) and, consequently,

\[
\pi_{j_0+k} \circ S_k \neq 0.
\]

Hence, \( S_k \in N \) for each \( k \in \mathbb{N} \). It is obvious that the set

\[
\{S_k : k \in \mathbb{N}\} \subset N
\]

is linearly independent. Define

\[
\Psi : \ell_1 \to \mathcal{L} (V; E) \\
(a_n)_{n=1}^\infty \mapsto \sum_{n=1}^\infty a_n S_n.
\]

Since \( \Psi \) is well-defined, linear and injective we have

\[
dim \Psi (\ell_1) = \epsilon
\]

and since \( \Psi (\ell_1) \subset N \) the proof of the case \( \mathbb{N}_v \neq \mathbb{N} \) is done.

Now, let us suppose that \( \mathbb{N}_v = \mathbb{N} \). By (2.3) we know that, for each \( (\alpha_n)_{n=1}^\infty \in \ell_\infty \),

\[
S_{(\alpha_n)_{n=1}^\infty}^V : V \to E \\
x \mapsto (\alpha_n (v (x))_{n=1}^\infty
\]

is a well-defined continuous linear operator. It is plain that \( S_{(\alpha_n)_{n=1}^\infty}^V \in D_{V,E} \) whenever \( (\alpha_n)_{n=1}^\infty \) is a sequence in \( \ell_\infty \) having some null entry (because, if \( \alpha_i = 0 \), then the

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i-th coordinate of \( S^v_{(\alpha_n)_{n=1}}^\infty \) \((x)\) is zero for all \( x \in V \). Let us consider, therefore, 
\((\alpha_n)_{n=1}^\infty \in \ell_\infty\) such that \( \alpha_n \neq 0 \) for all \( n \in \mathbb{N} \) and fix \( w = (w_n)_{n=1}^\infty \in E \setminus v(V) \). Since 
\((\alpha_n w_n)_{n=1}^\infty \in E \), we have

\[
(\alpha_n w_n)_{n=1}^\infty \in E \setminus S^v_{(\alpha_n)_{n=1}}^\infty(V).
\]

In fact, if there were \( x \in V \) such that \( S^v_{(\alpha_n)_{n=1}}^\infty (x) = (\alpha_n w_n)_{n=1}^\infty \), we would have

\[
(\alpha_n w_n)_{n=1}^\infty = (\alpha_n (v(x))_n)_{n=1}^\infty
\]

and, since \( \alpha_n \neq 0 \) for all \( n \in \mathbb{N} \), we would have \( v(x) = w \), which is impossible. Hence, \( S^v_{(\alpha_n)_{n=1}}^\infty \in D_{V,E} \).

Now consider the linear map

\[
\Lambda : \ell_\infty \to \mathcal{L}(V;E)
\]

\[
\Lambda((\mu_n)_{n=1}^\infty) = S^v_{(\mu_n)_{n=1}}^\infty.
\]

We have just proved that \( S^v_{(\alpha_n)_{n=1}}^\infty \in D_{V,E} \) for all \( (\alpha_n)_{n=1}^\infty \in \ell_\infty \); thus

\[
\Lambda(\ell_\infty) \subset D_{V,E}.
\]

Note that \( \Lambda \) is injective. In fact, if \((\mu_n)_{n=1}^\infty \in \ell_\infty \) and \( \Lambda((\mu_n)_{n=1}^\infty) = 0 \), since \( \mathbb{N}_v = \mathbb{N} \), it follows that, for all \( k \in \mathbb{N} \), there is \( x^{(k)} \in V \) such that \( (v(x^{(k)}))_k \neq 0 \). However, the \( k \)-th coordinate of \( S^v_{(\mu_n)_{n=1}}^\infty(x^{(k)}) \) is \( \mu_k(v(x^{(k)}))_k \), which must be null and, consequently, \( \mu_k = 0 \) for all \( k \) and \((\mu_n)_{n=1}^\infty = 0 \).

Observe that, choosing \((\lambda_n)_{n=1}^\infty = (1, 1, 1, \ldots) \in \ell_\infty \), then \( v = \Lambda((\lambda_n)_{n=1}^\infty) \in \Lambda(\ell_\infty) \). Since \( \Lambda \) is injective, we have

\[
\dim(\Lambda(\ell_\infty)) = \dim(\ell_\infty) = c,
\]

and the proof is done.

### 4 Final Remarks

We finish this note by showing that the previous technique gives us the following improvement of Fávaro et al. (2020, Theorem 3.1):

**Theorem 4.1** Let \( V \neq \{0\} \) be a normed vector space and \( X \neq \{0\} \) be a Banach space. Let \( E \) be a Banach sequence space such that \( c_{00}(X) \subset E \). If

\[
A_{V,E} := \{ u : V \to E : u \text{ is linear, continuous and non-injective} \} \neq \{0\}
\]

then \( A_{V,E} \) is \((1, \beta)\)-spaceable, where \( \beta = \max\{c, \dim X\} \).
For a fixed $v \in A_{V,E} \setminus \{0\}$, let $x_0, y_0 \in V$, with $x_0 \neq y_0$, be such that
\[
\nu(x_0) = \left(\nu(x_0)\right)_{n=1}^{\infty} = \left(\nu(y_0)\right)_{n=1}^{\infty} = \nu(y_0).
\]
It is obvious that the subspace

\[
M := \{u : V \to E : u \text{ is linear, continuous and } u(x_0) = u(y_0)\}
\]
is contained in $A_{V,E}$ and $v \in M$. It is sufficient to show that $M$ is closed and $\dim M \geq \beta$. Since $v$ is not identically zero, there exists $\xi_0 \in V$ such that $v(\xi_0) = w_0 \neq 0$. By the Hahn–Banach Theorem, there is a continuous linear functional $\varphi : E \to \mathbb{K}$, such that $\varphi(w_0) = 1$. Let $\{a_\gamma : \gamma \in \Gamma\}$ be a Hamel basis of $X$. For each $\gamma \in \Gamma$ and each $k \in \mathbb{N}$, let us define

\[
w^\gamma_k = (0, \ldots, 0, a_\gamma, 0, 0, \ldots) \in E
\]
and consider the linear operators $T^\gamma_k : E \to E$ given by

\[
T^\gamma_k(w) = \varphi(w) \frac{w^\gamma_k}{\|w^\gamma_k\|}.
\]
Obviously, the operators $T^\gamma_k$ are continuous and, thus, the operators $R^\gamma_k = T^\gamma_k \circ v$ are continuous too. Notice that $R^\gamma_k(\xi_0) = w^\gamma_k / \|w^\gamma_k\|$ and, consequently,

\[
\pi_k \circ R^\gamma_k \neq 0
\]
and thus, $R^\gamma_k \in M$ for each $k \in \mathbb{N}$. Let us see that

\[
\{ R^\gamma_k : k \in \mathbb{N}, \gamma \in \Gamma \} \subset M
\]
is linearly independent. In fact, let $(k_1, \gamma_1), \ldots, (k_n, \gamma_n) \in \mathbb{N} \times \Gamma$ be pairwise distinct and let $\lambda_1, \ldots, \lambda_n \in \mathbb{K}$ such that

\[
\lambda_1 R^\gamma_{k_1} + \cdots + \lambda_n R^\gamma_{k_n} = 0.
\]
Then

\[
0 = \lambda_1 R^\gamma_{k_1}(\xi_0) + \cdots + \lambda_n R^\gamma_{k_n}(\xi_0) = \lambda_1 \frac{w^\gamma_{k_1}}{\|w^\gamma_{k_1}\|} + \cdots + \lambda_n \frac{w^\gamma_{k_n}}{\|w^\gamma_{k_n}\|}
\]
and it is plain that $\lambda_1 = \cdots = \lambda_n = 0$ if $k_i \neq k_j$ whenever $i \neq j$, with $i, j \in \{1, \ldots, n\}$. With no loss of generality, assuming that $k = k_1 = \cdots = k_{p_1}$, $p_1 \leq n$, and $k_i \neq k$ if $i > p_1$, we have that the $k$-th coordinate of
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\[
\begin{align*}
\lambda_1 \frac{w_{\gamma_1}}{w_{k_1}} + \cdots + \lambda_n \frac{w_{\gamma_n}}{w_{k_n}}
\end{align*}
\]
is

\[
0 = \frac{\lambda_1}{w_{\gamma_1}} a_{\gamma_1} + \cdots + \frac{\lambda_p}{w_{\gamma_p}} a_{\gamma_p}
\]

By hypothesis, \( k_1 = \cdots = k_{p_1} \) implies \( \gamma_1, \ldots, \gamma_{p_1} \) pairwise distinct and, therefore, \( a_{\gamma_1}, \ldots, a_{\gamma_{p_1}} \) are linearly independent. Hence, \( \lambda_1 = \cdots = \lambda_{p_1} = 0 \). Again, with no loss of generality, assuming that \( k = k_{p_1+1} = \cdots = k_{p_2}, p_2 \leq n \), and \( k_i \neq k \) if \( i > p_2 \), we have that the \( k \)-th coordinate of

\[
\begin{align*}
\lambda_{p_1+1} \frac{w_{\gamma_{p_1+1}}}{w_{k_{p_1+1}}} + \cdots + \lambda_n \frac{w_{\gamma_n}}{w_{k_n}}
\end{align*}
\]
is

\[
0 = \frac{\lambda_{p_1+1}}{w_{\gamma_{p_1+1}}} a_{\gamma_{p_1+1}} + \cdots + \frac{\lambda_{p_2}}{w_{\gamma_{p_2}}} a_{\gamma_{p_2}}
\]

By hypothesis, \( k_{p_1+1} = \cdots = k_{p_2} \) implies \( \gamma_{p_1+1}, \ldots, \gamma_{p_2} \) pairwise distinct and, therefore, \( a_{\gamma_{p_1+1}}, \ldots, a_{\gamma_{p_2}} \) are linearly independent. Hence, \( \lambda_{p_1+1} = \cdots = \lambda_{p_2} = 0 \). Proceeding in this way, after finitely many steps, either we get \( \lambda_1 = \cdots = \lambda_n = 0 \) or we obtain \( m < n \) such that \( \lambda_1 = \cdots = \lambda_m = 0 \) and \( k_{m+1}, \ldots, k_n \) are pairwise distinct. So we have

\[
0 = \frac{\lambda_{m+1}}{w_{\gamma_{m+1}}} a_{\gamma_{m+1}} + \cdots + \frac{\lambda_n}{w_{\gamma_n}} a_{\gamma_n}
\]

and, as we know, this implies \( \lambda_{m+1} = \cdots = \lambda_n = 0 \). Notice that, at this moment, we have shown that

\[
\dim M \geq \max \{ \aleph_0, \dim X \}.
\]  

(4.1)

Supposing that \( u \) lies in the closure of \( M \), let \( (u_n)_{n=1}^{\infty} \) be a sequence in \( M \) such that \( \lim_{n \to \infty} u_n = u \) (we are considering, as usual, the canonical sup norm in the set of continuous linear operators \( \mathcal{L}(V; E) \) from \( V \) to \( E \)). Since

\[
u(x_0) = \lim_{n \to \infty} u_n (x_0) = \lim_{n \to \infty} u_n (\gamma_0) = u (\gamma_0),
\]

we conclude that \( u \in M \). Thus, \( M \) is closed in \( \mathcal{L}(V; E) \) and \( \dim M \geq c \). Since \( c > \aleph_0 \), from here and (4.1) we get \( \dim M \geq \beta \).
We recall that, as commented in Fávaro et al. (2020) it is not true that \((1, c)\)-lineability is inherited by inclusions. So, the following result, which is proved by combination of the previous techniques, should be noticed.

**Theorem 4.2** Let \(V, X\) and \(E\) be as in Theorem 2.1. The set

\[ C_{V,E} := \{ u : V \to E : u \text{ is linear, continuous, non-surjective and non-injective} \} \]

is \((1, c)\)-lineable.

Finally, we can see that the developed techniques do not answer the questions asked in the context of \((\alpha, \beta)\)-lineability/spaceability for \(\alpha \geq 2\). So, we consider this an interesting open problem. In addition, another interesting problem is to verify whether it is possible to infer \((1, c)\)-spaceability instead \((1, c)\)-lineability in Theorem 2.1.

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