Tetrons - a possible Solution to the Family Problem

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Abstract

A model is presented, in which fermion and vector boson states are constructed from constituents (‘tetrons’). The model encodes all observed structures and phenomena of elementary particle physics in group theoretic items of the permutation group $S_4$. Details of the model like symmetry breaking, distribution of charges and mass generation are worked out. As a sideproduct a deeper understanding of parity violation is obtained.
1 Introduction

According to present ideas the observed elementary particles (leptons, quarks and vector bosons) are pointlike. Their mathematical description [1] as Dirac or Yang-Mills fields follows this philosophy. In the present paper I propose a model, in which they acquire an extension and are composite of more fundamental fields called tetrons.

2 The Model for Fermions

My aim is to shed light on the fermion spectrum of elementary particle physics, i.e. the 24 spin-\(\frac{1}{2}\)-states observed in nature and habitually denoted by

\[
\begin{align*}
\nu_e & \quad e & \quad u_{1,2,3} & \quad d_{1,2,3} \\
\nu_\mu & \quad \mu & \quad c_{1,2,3} & \quad s_{1,2,3} \\
\nu_\tau & \quad \tau & \quad t_{1,2,3} & \quad b_{1,2,3}
\end{align*}
\]

(1)

(2)

(3)

where the index 1,2,3 stands for quark color. These states arrange themselves in 3 families of

\[
\begin{align*}
N & \quad E & \quad U_{1,2,3} & \quad D_{1,2,3}
\end{align*}
\]

(4)

each consisting of two quadruplets of the form \((L, Q_{1,2,3})\) where I abbreviate quark states by the letter Q, up-quarks by U, down-quarks by D, leptons by L, neutrinos by N and e, \(\mu\) and \(\tau\) by E. Including spin (and righthanded neutrinos) there are altogether 48 degrees of freedom with a mass spectrum which ranges between about \(10^{-6}\) eV for the electron-neutrino [2] and 175 GeV for the top-quark [3].
The underlying dynamics of this system is nowadays usually supposed to be a local gauge theory with gauge group $U(1)_{B-L} \times SU(3)_c \times SU(2)_L \times SU(2)_R$ \cite{1}, $SU(4) \times SU(2)_L \times SU(2)_R$ \cite{5} or $SO(10)$ \cite{6} with 15, 21 or 45 gauge bosons respectively plus an as yet unknown mechanism which takes care of the family repetition.

I want to accommodate this system as bound states of smaller, more fundamental objects. The neatest idea \cite{8} is to consider states consisting of 4 identical particles (‘tetrons’) bound together by a new super strong, super short range interaction, whose charge and symmetry nature are not discussed at this point but will become transparent in the course of discussion. The tetrons transform under the permutation group $S_4$ \cite{7} and will be seen to yield the observed spectrum of quarks and leptons.

In a crude classical picture they may be assumed to sit on the corners of an equilateral tetrahedron with corresponding symmetry properties, i.e. invariance under the tetrahedron’s symmetry group $T_d$ which is isomorphic to the group $S_4$ of permutations of the set of the 4 tetrons which for convenience we numerate as 1,2,3,4. The group consists of 24 permutations which may be denoted as $\sigma = abcd : (1,2,3,4) \rightarrow (a,b,c,d)$. In the limit of perfect $S_4$-symmetry the 24 tetrahedron states $\phi_\sigma = |abcd>$ generated by these permutations are of course all identical.

In order to become different with different masses and charges there must be some sort of symmetry breaking, the simplest possibility being that the 4 originally identical tetrons together with their bound states differ from each other even when rotational symmetry is taken into account. In order to implement this there are several possibilities which will be described later. At the moment I simply assume that a symmetry breaking exists and want to show how the 24 different permutation states can be arranged in parallel
to the observed family structure.

The nonabelian group \( S_4 \) may be written as a semidirect product

\[
S_4 = Z_4 \circ Z_3 \circ Z_2
\]

where \( Z_n \) is the (abelian) symmetric group of \( n \) elements. The subgroup \( Z_3 \circ Z_2 \) of \( S_4 \) can be identified with the permutation group \( S_3 \) of 3 elements, whereas \( Z_4 \circ Z_2 \) is isomorphic to the dihedral group \( D_4 \).

The \( Z_3 \)-part of the decomposition (5) allows to divide the 24 elements of \( S_4 \) into 3 groups of 8 elements (´orbits´), which correspond to permutations which preserve a certain ordering (...1234123...+backward, ...2134...+backward and ...4231...+backward) and will essentially make up the 3 fermion families.

There are then two possibilities to associate the up-half \((N, U_{1,2,3})\) of a family to elements of \( S_4 \). One is based on the so-called Kleinsche Vierergruppe \( K \) (also a subgroup of \( S_4 \)). It is the smallest noncyclic group and isomorphic to \( Z_2 \times Z_2 \). Considered as a subgroup of \( S_4 \) it consists of the 3 even permutations \( 2143, 3412, 4321 \), where 2 pairs of numbers are interchanged, plus the identity. The other possibility is to take the \( Z_4 \) in eq. (5), i.e. to identify the up-half of the first family (where by \textit{first} I do not necessarily mean lightest, see later) with the 3 permutations \( 2341, 3412, 4123 \) plus the identity. The down-half is then associated to the ´backward running´ permutations \( 4321, 3214, 1234 \).

\[1\] All these groups have geometrical interpretations as symmetry groups of simple geometrical objects. For example, \( D_4 \) is the symmetry group of a square, \( S_3 \) of an equilateral triangle and \( K \) that of a rectangle (everything in 2 dimensions). \( S_4 \) itself is isomorphic to the symmetry group of a unilateral tetrahedron in 3 dimensions and as such may be considered as a subgroup of \( \text{SO}(3) \). The latter fact, however, will not be relevant (because we are going to completely break \( S_4 \)), until the point when parity violation of the weak interactions is discussed in the section on vector bosons.
\[2143\] and \[1432\]. In both cases one will have to look for a mechanism, which suppresses leptoquarks, i.e. unwanted lepton number violating processes like proton decay etc. This question will be followed in the section about vector bosons.

In table 1 there is a preliminary assignment of particle states to permutations of \( S_4 \). It should be noted, however, that this is only mnemonic, because it will later turn out that one has to use linear combinations of \( S_4 \)-states.

Another way of representing the above classification is by saying that 24 elements of \( S_4 \) are divided in 6 classes which transform under its subgroup \( S_3 \). In fact, \( S_3 \) consists of 3 even permutations denoted by \( id = I, II, III \), \( g_1 = \overline{III}, I, II \) and \( g_2 = \overline{III}, III, I \) and 3 odd permutations denoted by \( u_0 = \overline{III}, I, II \), \( u_1 = \overline{II}, III, I \) and \( u_2 = \overline{I}, III, II \). It can be further decomposed as \( S_3 \approx Z_3 \triangleleft S_2 \) where \( S_2 = Z_2 \) and \( Z_3 = \{id, g_1, g_2\} \) is the cyclic group of 3 elements and in geometrical terms corresponds to rotations by \( \pm 2\pi/3 \). This part of \( S_3 \) is assumed to account for the 3 observed fermion families, i.e. to be the family group while the \( S_2 \)-part will correspond to weak isospin, i.e. flip neutrino and electron.

The down-quadruplet \((E, D_{1,2,3})\) can then be represented by a single even element of \( S_3 \) and \((N, U_{1,2,3})\) by a single odd element, while the 4 states within a quadruplet correspond to the orbits of \( Z_4 \) in \( S_4 \). The situation is summarized in the following list:

- 1.family: path \( ...1234... \) corresponding to \( id = \overline{I, II, III} \) (+backward \( p_2 = \overline{III, I, II} \))
- 2.family: path \( ...2134... \) corresponding to \( p_1 = \overline{II, I, III} \) (+backward \( g_1 = \overline{III, I, II} \))
- 3.family path \( ...4231... \) corresponding to \( g_2 = \overline{II, III, I} \) (+backward \( g_2 = \overline{II, III, I} \))

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Table 1: Preliminary list of elements of $S_4$ ordered in 3 families. F are the forward oriented elements (corresponding to the $Z_4$ subgroup of $S_4$), and B the backward oriented ones. Together they give $D_4$. $z_i$ and $k_i$ denote the elements of $Z_4$ and $K$, respectively, and $(a \leftrightarrow b)$ a simple permutation where $a$ and $b$ are interchanged. An alternative assignment based on $K$ instead of $Z_4$ is possible where the rows $F_1 \leftrightarrow B_3$ and $F_3 \leftrightarrow B_1$ are interchanged. In any case permutations with a 4 at the last position form a $S_3$ subgroup of $S_4$ and may be thought of giving the set of lepton states. Note however, that the assignment of particle states is only mnemonic, because it will turn out that one has to use linear combinations of $S_4$-states. Furthermore, there is still an arbitrariness as to whether the 'first' family is the lightest or the heaviest, i.e. whether the identity permutation $1234$ corresponds to the top quark or to the electron and so on. This is simply due to the fact that I have not yet discussed the question of symmetry breaking, masses and charges, but will do so in section 4.
In geometrical terms the 3 families can be visualized as the 3 closed paths which can be drawn in a tetrahedron. The interactions among the tetrons in the i-th family runs along the path i, i=I,II or III. A given state \( \phi_\sigma = |abcd \rangle \) has thus bindings along the (open) path \( a \rightarrow b \rightarrow c \rightarrow d \).

All observed Fermions have Spin \( \frac{1}{2} \) and they have associated antifermions. These features are indispensable for any theory of elementary particles. How can they be accommodated in the present model? Concerning these questions I refer to a forthcoming paper [9] where it is argued that

- Assuming tetrons are given by complex scalar fields \( \chi(x) \), anti-tetrons are described by \( \chi^*(x) \).

- So far we have 24 tetrahedron states \( \phi_{abcd} \). Assuming a relativistic motion these states will appear in left and righthanded form \( \phi^L_\sigma \) and \( \phi^R_\sigma \). If the lefthanded live on a tetrahedron \( \vec{T} \) the righthanded will be shown to live on the 'anti-tetrahedron' \( -\vec{T} \) which is obtained from the original one by parity inversion \( P : \vec{x} \rightarrow -\vec{x} \) and is in fact the tetrahedron which completes \( \vec{T} \) to a cube. It is this cube, by the way, and its centre and axes, which will later be used to describe vector boson states (see section 5). Note that although most of the time I consider tetrahedrons at rest, the present framework will be suitable general to be applicable to relativistic tetrahedron states. It further turns out that states of even and odd permutations naturally live on opposite tetrahedrons \( \vec{T} \) and \( -\vec{T} \), so that it is in principle important to keep track of right and lefthanded components, as will be discussed in more detail in section 5. For simplicity of representation, however, I will not differentiate at the moment.
The question of the spin of the constituent tetrons will be further discussed in [9].

3 Alignment of States after Symmetry Breaking

As stated, to first approximation all 24 tetrahedron states are identical (symmetric limit). Although the symmetry breaking that makes them different can have various origins, it must have to do with a breaking of the new, superstrong interaction that keeps the 4 tetrons together.

In more concrete terms one may ask, what properties the tetrons need to form the 24 states and in particular to make them all different. I have considered various possibilities but will present here only the most appealing: namely, assuming non-identical tetrons, I demand that these appear in 4 different 'charge states' called \( \chi(q_i, x), i = 1, 2, 3, 4 \) fulfilling the following selection rules: In a tetrahedron bound state with 4 tetrons

- (A) each charge \( q_i \), which can take one of 4 possible values appears once and only once
- (B) the sum of charges vanishes, i.e. the bound states are singlets under the new superstrong interaction.

From (A) we directly get 24 product states

\[
\phi_\sigma(x_1, x_2, x_3, x_4) = \chi(q_{\sigma(1)}, x_1)\chi(q_{\sigma(2)}, x_2)\chi(q_{\sigma(3)}, x_3)\chi(q_{\sigma(4)}, x_4)
\]

(6)
corresponding to the 24 permutations \( \sigma \in S_4 \) and where I have used the notation \( \sigma : (1, 2, 3, 4) \to (\sigma(1), \sigma(2), \sigma(3), \sigma(4)) \). Problem: At this point the
24 states can still be transformed into each other by a suitable rotation of the tetrahedron (permutation of the $x_i$). In order to get them different I demand additionally

- (C) the existence of cores or "nuclei" with 4 different charges $Q_i$ in the centre of each tetron $i$, i.e. in the corners of the tetrahedron, which are surrounded by the charges $q_{\sigma(i)}$. The range of possible values of the $Q_i$ may be chosen identical to that of the $q_i$.

The wave function for the tetrahedrons then reads

$$\phi_\sigma(x_1, x_2, x_3, x_4) = \chi(q_{\sigma(1)}, Q_1, x_1) \chi(q_{\sigma(2)}, Q_2, x_2) \chi(q_{\sigma(3)}, Q_3, x_3) \chi(q_{\sigma(4)}, Q_4, x_4)$$

and, modulo rotations, there are now 24 different tetrahedron states indexed by $\sigma \in S_4$. Equivalently one may say, that a tetron is described by 2 quantum numbers $q$ and $Q$ fulfilling the above restrictions and selection rules.

Note that I have not yet specified the nature of these quantum numbers. At the moment they are just properties which distinguish the tetrons. Furthermore, in order that eq. holds, one is not really tied to the geometrical model of the tetrahedron. It would suffice to have objects which can be combined in all different ways with respect to 2 properties $q$ and $Q$. It is just for definiteness, that I am calling the 4-tetron bound states 'tetrahedrons'.

Using $\sigma(1) = a$, $\sigma(1) = b$, $\sigma(1) = c$ and $\sigma(1) = d$ one may rewrite the last equation in various forms:

$$\phi_\sigma = |abcd> = \phi_{abcd} = \chi(q_a, Q_1) \otimes \chi(q_b, Q_2) \otimes \chi(q_c, Q_3) \otimes \chi(q_d, Q_4)$$

where I have used the notation $\sigma = \overline{abcd} : 1234 \rightarrow \overline{abcd}$.

In some sense the tetrahedron is similar to a chemical molecule with nuclei $Q_i$ and wave function clouds $q_j$. Conditions (A) and (B) ensure that the lowest lying orbitals cannot be infinitely filled.
To understand the breaking more clearly, consider the simplified case of 2 clouds $q_+$ and $q_-$ surrounding 2 cores $Q_+$ and $Q_-$ and forming 2-core-2-cloud bound states, namely

$$\phi_{+-} = \chi(q_+, Q_+) \otimes \chi(q_-, Q_-)$$

(9)

and

$$\phi_{-+} = \chi(q_-, Q_+) \otimes \chi(q_+, Q_-)$$

(10)

and no others (i.e. assuming a modified selection rule that only bound states with 2 different clouds and 2 different cores exist). For state $\phi_{+-}$ cloud $q_+$ is nearer to core $Q_+$ whereas in $\phi_{-+}$ it is nearer to $q_-$, and analogously for cloud $q_-$.

Because of all charges being different (i.e. due to the breaking of permutation symmetry) the states $\phi_{+-}$ and $\phi_{-+}$ will not be degenerate. One of them (representing the neutrino) will be lower in mass than the other (representing the electron). In nonrelativistic perturbation theory one would generically have mass formulas

$$E_{+-} = E(q_+, Q_+) + E(q_-, Q_-)$$

$$+ < \phi_{+-} | V(q_+, q_-) + V(Q_+, Q_-) + V(q_+, Q_-) + V(q_-, Q_+)| \phi_{+-} >$$

(11)

$$E_{-+} = E(q_-, Q_+) + E(q_+, Q_-)$$

$$+ < \phi_{-+} | V(q_+, q_-) + V(Q_+, Q_-) + V(q_-, Q_-) + V(q_+, Q_+)| \phi_{-+} >$$

(12)

where $V(a, b)$ denotes the interaction between charge $a$ and $b$ and $E(a, b)$ denotes the lowest order energy eigenvalues of completely separated tetrons. Mass generation by breaking terms of the new interactions will be further addressed in the section 4.

Note that even though most of the time I consider tetrahedrons at rest, the framework eq. (8) is suitable general to be applicable to relativistic tetrahedron states by considering spacetime instead of space coordinates. Then, if one dislikes the geometrical notion of 'cores' and 'clouds', one may, on a somewhat more abstract level, consider the $\phi_{abcd}$ to be products of 16 complex fields $\chi_i^a$, $i, a = 1, 2, 3, 4$

$$\phi_{abcd} = \chi_1^a \chi_2^b \chi_3^c \chi_4^d$$

(13)

at spacetime point $x$. Compared to eq. (8) the role of the charges $Q$ ($q$) is played by the lower (upper) indices. If one looks at this equation, there is
one question immediately arising: how can the products of tetrons $\chi$ on the right hand side become Dirac spinors $\phi_{abcd}$? An answer to this question will be given in a forthcoming paper [9].

Next, it must be realized that the physical states are linear combinations of the product states eq. (7). Consider, for example, the 4 states generated by applying the subgroup $Z_4$ on the unit element of $S_4$, i.e. $|1234>$, $|2341>$, $|3412>$ and $|4123>$. They are naturally built into a singlet

$$\phi_{\nu_\tau} = \frac{1}{\sqrt{4}}[\phi_{1234} + \phi_{2341} + \phi_{3412} + \phi_{4123}]$$  \hspace{1cm} (14)

representing the $\tau$-neutrino and 3 nonsinglet combinations

$$\phi_{t_1} = \frac{1}{\sqrt{4}}[\phi_{1234} + i\phi_{2341} - \phi_{3412} - i\phi_{4123}]$$  \hspace{1cm} (15)

$$\phi_{t_2} = \frac{1}{\sqrt{4}}[\phi_{1234} - \phi_{2341} + \phi_{3412} - \phi_{4123}]$$  \hspace{1cm} (16)

$$\phi_{t_3} = \frac{1}{\sqrt{4}}[\phi_{1234} - i\phi_{2341} - \phi_{3412} + i\phi_{4123}]$$  \hspace{1cm} (17)

which are degenerate in energy, as shown in the next section, and will be used to represent the 3 color states of the top-quark.

The reason for considering these linear combinations instead of the simple product state eq. (7) is that they turn out to be eigenfunctions of the $U(1)_{B-L} \times SU(3)_c$ charge operators $\lambda_3$, $\lambda_8$ and $Y_{B-L}$.

To prove this, one first shows that the states (14)-(17) are eigenfunctions of permutation operators $R_0 = 1234$, $R_1 = 2341$, $R_2 = 3412$ and $R_3 = 4123$ and afterwards constructs the $U(1)_{B-L} \times SU(3)_c$ charges as linear combinations of the $R_i$. In fact, writing $\phi_{\nu_\tau} = (1, 0, 0, 0)$, $\phi_{t_1} = (0, 1, 0, 0)$ etc, the action of the $R_j$ on the states (14)-(17) is given by $R_0 = (1, 1, 1, 1)$, $R_1 = diag(1, -i, -1, i)$, $R_2 = diag(1, -1, 1, -1)$ and $R_3 = diag(1, i, -1, -i)$. Therefore, one has $Y_{B-L} = -\frac{1}{6}(3, -1, -1, -1) = -\frac{1}{6}(R_1 + R_2 + R_3)$ and similarly for $\lambda_3$ and $\lambda_8$. 

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Of course, at this point $Z_4$ is merely a discrete symmetry. From the Casimirs $\lambda_3$, $\lambda_8$ and $Y_{B-L}$ it should be completed to a global $U(1)_{B-L} \times SU(3)_c$ and afterwards in the limit when the tetrahedron shrinks to a point-like fermion an effective local gauge symmetry should be constructed.

Note that with the help of suitable permutations the above analysis can be extended to any other quark-lepton quadruplet. Note further that using $SU(4)$ instead of $U(1)_{B-L} \times SU(3)_c$ one would in general get leptoquarks besides gluon interactions and the $U(1)_{B-L}$-photon. These, however, can be shown to be forbidden in the present model. I shall come back to this point in section 5.

If one prefers to start with the Kleinsche Vierergruppe $K$ instead of $Z_4$ one has an alternative representation for $\phi_{\nu r}$ and $\phi_{ti}$

\[
\phi_{\nu r} = \frac{1}{\sqrt{4}} [\phi_{1234} + \phi_{2143} + \phi_{3412} + \phi_{4321}] \tag{18}
\]

\[
\phi_{t1} = \frac{1}{\sqrt{4}} [\phi_{1234} + \phi_{2143} - \phi_{3412} - \phi_{4321}] \tag{19}
\]

\[
\phi_{t2} = \frac{1}{\sqrt{4}} [\phi_{1234} - \phi_{2143} + \phi_{3412} - \phi_{4321}] \tag{20}
\]

\[
\phi_{t3} = \frac{1}{\sqrt{4}} [\phi_{1234} - \phi_{2143} - \phi_{3412} + \phi_{4321}] \tag{21}
\]

which are eigenstates of the permutation operators $K_0 = 1234 = (1,1,1,1)$, $K_1 = 2143 = (1,1,-1,-1)$, $K_2 = 3412 = (1,-1,1,-1)$ and $K_3 = 4321 = (1,-1,-1,1)$ and correspondingly a different representation for the Casimirs $\lambda_3 = \frac{1}{2}(K_1 - K_2)$, $\lambda_8 = \frac{1}{2\sqrt{3}}(K_1 + K_2 - 2K_3)$ and $Y_{B-L} = -\frac{1}{6}(3,-1,-1,-1) = -\frac{1}{6}(K_1 + K_2 + K_3)$. We shall see shortly whether $K$ oder $Z_4$ is preferable.

Having dealt with $Z_4$ (or $K$), one can treat the rest of $S_4 = Z_4 \circ Z_3 \circ Z_2$ in a similar fashion. Namely, for $S_2 = Z_2$ one may define 2 states

\[
\phi_{\pm} = \frac{1}{\sqrt{2}} [\phi_{13} \pm \phi_{31}] \tag{22}
\]
corresponding to eqs. (9) and (10) which are eigenstates of the generator \(3\mathbb{T}\) of \(S_2\) with eigenvalue +1 and -1, i.e. these states should be identified with the two partners of a weak isospin doublet (like the electron and its neutrino).  

One can easily construct a set of Pauli matrices \(\sigma_i, i = 1, 2, 3\) for the states eq. (22) by using \(3\mathbb{T}\) as \(\sigma_3\) and then defining creation and annihilation operators \(\sigma_+\phi_+ = \phi_+\) and \(\sigma_-\phi_- = \phi_-\) and from these \(\sigma_1 = \sigma_+ + \sigma_-\) and \(\sigma_2 = i(\sigma_+ - \sigma_-)\). This set of matrices is easily seen to obey SU(2) commutation relations and is going to generate the weak \(SU(2)\)-symmetry.

Combining eqs. (17) and (22) the lepton and quark states get 4 additional terms: neutrinos and up-type quarks with a positive, electrons and down-type quarks with a negative sign:

\[
\phi_{N/E} = \frac{1}{\sqrt{8}} (\phi_0^F \pm \phi_0^B)
\]

\[
\phi_0^F = \phi_{1234} + \phi_{2134} + \phi_{3412} + \phi_{4321}
\]

\[
\phi_0^B = \phi_{3214} + \phi_{4123} + \phi_{1432} + \phi_{2341}
\]

\[
\phi_{U_1/D_1} = \frac{1}{\sqrt{4}} (\phi_1^F \pm \phi_1^B)
\]

\[
\phi_1^F = \phi_{1234} - \phi_{3412}
\]

\[
\phi_1^B = \phi_{3214} - \phi_{1432}
\]

\[
\phi_{U_2/D_2} = \frac{1}{\sqrt{8}} (\phi_2^F \pm \phi_2^B)
\]

\[
\phi_2^F = \phi_{1234} - \phi_{2143} + \phi_{3412} - \phi_{4321}
\]

\[
\phi_2^B = \phi_{3214} - \phi_{4123} + \phi_{1432} - \phi_{2341}
\]

\footnote{I consider \(S_2\) to be the permutations of two objects called here 1 and 3 to be in agreement with table 1.}
\begin{align*}
\phi_{U_3/D_3} &= \frac{1}{\sqrt{4}}(\phi_3^F \pm \phi_3^B) \tag{26} \\
\phi_3^F &= \phi_{234} - \phi_{342} \\
\phi_3^B &= \phi_{243} - \phi_{423}
\end{align*}

In the language of molecular physics these are the symmetry adapted wave function of the dihedral group $D_4$. $D_4$ is of order 8. It contains the even elements of the Kleinsche Vierergruppe combined with the odd permutation $\bar{3}214 = (1 \leftrightarrow 3)$ and according to table 1 generates the first family.

I have made the assignments in such a way that isospin partners are obtained by changing the sign of the odd permutation functions. This is in accord with eq. (22) but if we do so in the general case, the $Z_4$-functions eqs. (14)-(17) are out of the game, because $\phi_{t_3}$ and $\phi_{t_3}$ would have to be isospin partners. Nevertheless, it is possible to save the $Z_4$-scenario by choosing a somewhat different assignment in eqs. (23)-(26) and then demanding that the germ of weak SU(2) does not actually correspond to the permutation $\bar{3}214 = (1 \leftrightarrow 3)$ as in eq. (22) but to the transitions $\phi_N \leftrightarrow \phi_E$ and $\phi_{U_i} \leftrightarrow \phi_{D_i}$. Since this is somewhat artificial I think that the $K$ is the better option.

Finally there is a state-mixing due to the family group $Z_3 \subset S_3$ which means that instead of $...1234123...+$backward, $...2134...+$backward and $...4231...+$backward of section 2 the 3 families correspond to

\begin{align*}
\phi_A &= \frac{1}{\sqrt{3}}[\phi_{id} + \phi_{g_1} + \phi_{g_2}] \\
\phi_B &= \frac{1}{\sqrt{3}}[\phi_{id} + \alpha \phi_{g_1} + \alpha^2 \phi_{g_2}] \\
\phi_C &= \frac{1}{\sqrt{3}}[\phi_{id} + \alpha^2 \phi_{g_1} + \alpha \phi_{g_2}] \tag{27}
\end{align*}

where $\alpha = \exp(2i\pi/3)$ and $id, g_1$ and $g_2$ denote the elements of $Z_3$, i.e. even permutations of $S_3$, $id = \bar{1}23\bar{3}12 = \bar{3}12$ and $g_1 = \bar{1}321$. In the notation of
eq. (7) one has $\phi_\sigma = \chi(q_\sigma(1), Q_1) \otimes \chi(q_\sigma(2), Q_2) \otimes \chi(q_\sigma(3), Q_3)$ for $\sigma \in S_3$, e.g. $\phi_{g_1} = \chi(q_2, Q_1) \otimes \chi(q_3, Q_2) \otimes \chi(q_1, Q_3)$.

The reason to form the combinations eq. (27)-(27) is that they are idempotent and orthogonal eigenstates of the ”charge operators” $g_1$ and $g_2$, because of the following relations

\begin{align*}
g_1 \phi_A & = \phi_A \\
g_1 \phi_B & = \alpha^2 \phi_B \\
g_1 \phi_C & = \alpha \phi_C \\
g_2 \phi_A & = \phi_A \\
g_2 \phi_B & = \alpha \phi_B \\
g_2 \phi_C & = \alpha^2 \phi_C
\end{align*}

(28)

which follow easily from the properties $g_1^2 = g_2$, $g_1 g_2 = id$ and $g_2^2 = g_1$.

Extending this to $S_3$ one gets the 6 expressions

\begin{align*}
\phi_{\nu_e} & = \frac{1}{\sqrt{6}} \left[ \phi_{id} + \phi_{g_1} + \phi_{g_2} + \phi_{u_0} + \phi_{u_1} + \phi_{u_2} \right] \\
\phi_e & = \frac{1}{\sqrt{6}} \left[ \phi_{id} + \phi_{g_1} + \phi_{g_2} - \phi_{u_0} - \phi_{u_1} - \phi_{u_2} \right] \\
\phi_{\nu_a} & = \frac{1}{\sqrt{12}} \left[ 2 \phi_{id} - \phi_{g_1} - \phi_{g_2} + 2 \phi_{u_0} - \phi_{u_1} - \phi_{u_2} \right] \\
\phi_{\mu} & = \frac{1}{\sqrt{12}} \left[ 2 \phi_{id} - \phi_{g_1} - \phi_{g_2} - 2 \phi_{u_0} + \phi_{u_1} + \phi_{u_2} \right] \\
\phi_{\nu_e} & = \frac{1}{2} \left[ \phi_{g_1} - \phi_{g_2} - \phi_{u_1} + \phi_{u_2} \right] \\
\phi_{e} & = \frac{1}{2} \left[ - \phi_{g_1} - \phi_{g_2} - \phi_{u_1} + \phi_{u_2} \right]
\end{align*}

(29)

for the leptons.

With these building blocks in mind one may write down the complete formula for each fermion as a specific linear combination of the 24 product states eq.
\[ \phi^J = \sum_{a,b,c,d=1}^{24} \lambda^J_{abcd} |abcd > \]  

(30)

where \( J = \nu_e, e, ... \) numbers the 24 elementary fermions. The representations \( A_1, A_2 \) and \( E \) of \( S_4 \) with dimension 1, 1 and 2 respectively are reserved for leptons, whereas the 3-dimensional representations \( T_1 \) and \( T_2 \) are used for up- and down-type quarks, respectively (for more details of the \( S_4 \)-representations see the appendix). The \( \tau \)-neutrino, for example, then corresponds to the overall singlet

\[
\phi_{\nu_\tau} = \frac{1}{\sqrt{2^4}} \{(\phi_{1234} + \phi_{2143} + \phi_{3412} + \phi_{4321} + \phi_{2314} + \phi_{1432} + \phi_{3214} + \phi_{1243}) + \phi_{2341} + \phi_{2314} + \phi_{3241} + \phi_{1342} + \phi_{3124} + \phi_{1324} + \phi_{3421} + \phi_{4312} + \phi_{3241} + \phi_{4123} \} \]  

(31)

and the wave function for the \( \tau \)-lepton is given by

\[
\phi_\tau = \frac{1}{\sqrt{2^4}} \{(\phi_{1234} + \phi_{2143} + \phi_{3412} + \phi_{4321} - \phi_{2314} - \phi_{1432} - \phi_{4123}) - \phi_{2134} - \phi_{1243} - \phi_{3421} - \phi_{4312} + \phi_{1342} + \phi_{1324} + \phi_{4213} - \phi_{2413} - \phi_{2431} - \phi_{4213} - \phi_{2314} - \phi_{3124} + \phi_{2314} + \phi_{4132} + \phi_{1423} + \phi_{4132} + \phi_{1423} \} \]  

(32)

whereas the second color component of the strange quark reads

\[
\phi_{s2} = \frac{1}{\sqrt{8}} \{(\phi_{1234} - \phi_{2143} + \phi_{3412} - \phi_{4321} + \phi_{1432} + \phi_{2314} - \phi_{2134} - \phi_{4312}) \} \]  

(33)

This is obtained from the 22 matrix element of \( T_1 \). Its weak isospin partner \( \phi_{c2} \) (= the 22 matrix element of \( T_2 \)) is obtained by changing the signs of the odd permutations in eq. (33).
4 A phenomenological Approach to Masses and Charges

As stated, to first approximation all 24 states are identical (symmetric limit). In this limit all masses are equal. Since the symmetry breaking mechanism must have to do with the (new) interaction that keeps the 4 tetrons together, the interaction must have a \( S_4 \)-breaking part \( H_X \) and (using for simplicity nonrelativistic framework) in first order perturbation theory the masses would be calculable as

\[
m_{abcd} = < abcd | H_X | abcd >
\] (34)

One could then use the charge eigenstates eq. (30) to calculate the matrix elements eq. (34). If one further assumes that

\[
H_X = V(x_1, x_2, x_3, x_4) = V_{12} + V_{13} + V_{14} + V_{23} + V_{24} + V_{34}
\] (35)

with \( V_{ij} = q_i q_j V_0(|x_i - x_j|) \), one can easily calculate

\[
m_J = < \phi^J | V | \phi^J > = \sum_{abcd,a'b'c'd'} \lambda^*_{abcd} \lambda_{a'b'c'd'} < abcd | V | a'b'c'd' >
\] (36)

with

\[
< abcd | V | a'b'c'd' > = \delta_{cc'} \delta_{dd'} q_1 q_2 [\delta_{aa'} \delta_{bb'} J_{ab}^C + \delta_{ab'} \delta_{ba'} J_{ab}^A] + \delta_{bb'} \delta_{dd'} q_1 q_3 [\delta_{aa'} \delta_{cc'} J_{ac}^C + \delta_{ac'} \delta_{cc'} J_{ac}^A] + \delta_{bb'} \delta_{cc'} q_1 q_4 [\delta_{aa'} \delta_{dd'} J_{ad}^C + \delta_{ad'} \delta_{dd'} J_{ad}^A] + \delta_{bb'} \delta_{cc'} q_2 q_3 [\delta_{aa'} \delta_{cc'} J_{bc}^C + \delta_{bc'} \delta_{cc'} J_{bc}^A] + \delta_{aa'} \delta_{dd'} q_2 q_4 [\delta_{bb'} \delta_{dd'} J_{bd}^C + \delta_{bd'} \delta_{dd'} J_{bd}^A] + \delta_{aa'} \delta_{bb'} q_3 q_4 [\delta_{cc'} \delta_{dd'} J_{cd}^C + \delta_{cd'} \delta_{dd'} J_{cd}^A] (37)
\]
$J^A$ and $J^C$ are generalizations of exchange and Coulomb integrals

\[
J^C_{ab} = \int dx_i^4 dx_j^4 V_0(|x_i - x_j|) \phi^*_a(x_i) \phi^*_b(x_j) \phi^*_a(x_i) \phi^*_b(x_j)
\]

\[
J^C_{ab} = \int dx_i^4 dx_j^4 V_0(|x_i - x_j|) \phi^*_a(x_i) \phi^*_b(x_j) \phi^*_a(x_i) \phi^*_b(x_j)
\]

They are symmetric in the interchange of a and b, so that besides the 4 unknowns $q_i$ one has 20 unknown integrals.

We do not undertake to examine this further, because apart from the many unknown integrals there is the additional uncertainty about the validity of the nonrelativistic approach. The nonrelativistic picture with a Hamiltonian of the form $H = \sum_i E_i^i + H_X$ with $H_X = V = \sum_{ij} V(|x_i - x_j|)$ is qualitatively nice to give a good overview what states exist. However, it is very unlikely that it works quantitatively correct.

Instead I shall analyze the mass matrix of the fermions as it comes out from their description of the form of linear combinations of tetrahedron states eq. (30). A complete analysis of these states would involve a 24 times 24 mass matrix. We shall not attempt this here but to get an intuition about what can be achieved we split the problem into 3 pieces

- Mass splittings among the weak isospin dublets: in the present model weak isospin states like $\nu_e$ and $e$ are described by $S_2$ wave functions $\phi_{\pm}$ eq. (22) which are interaction eigenstates with mass matrix

\[
m_{\pm} = \frac{1}{2} \begin{pmatrix}
1 & \pm 1 \\
1 & \pm 1
\end{pmatrix}
\begin{pmatrix}
m_{11} & m_{12} \\
2m_{21} & m_{22}
\end{pmatrix}
\begin{pmatrix}
1 & \pm 1
\end{pmatrix}
\]

(40)

It is then easy to accommodate a mass structure $m_e \gg m_{\nu_e}$ namely with a "democratic" mass matrix $m_{11} = m_{12} = m_{21} = m_{22}$.  

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• Mass splittings among the families: in the present model family states like $e, \mu$ and $\tau$ are described by $Z_3$ wave functions $\phi_{A,B,C}$ eq. (27), which are interaction eigenstates with mass matrix

$$m_{A,B,C} = \frac{1}{3} \begin{pmatrix} 1, 1, \alpha, \alpha^2 & 1, \alpha^2, \alpha^* \end{pmatrix} \begin{pmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{pmatrix} \begin{pmatrix} 1 \\ 1, \alpha, \alpha^2 \\ 1, \alpha^2, \alpha^* \end{pmatrix}$$

(41)

Using the algebraic identity $1 + \alpha + \alpha^2 = 0$ which implies $Im(\alpha) = -Im(\alpha^2)$ and assuming the mass matrix to be symmetrical, it is easy to show that

$$m_A = \frac{1}{3}(tr(m) + 2m_n)$$

(42)

$$m_B = m_C = \frac{1}{3}(tr(m) - m_n)$$

(43)

where $m_n = m_{12} + m_{13} + m_{23}$ and one immediately sees that by choosing $tr(m) = m_n$ one can obtain a heavy family with $m_\tau = m_A$ and two light ones $m_B = m_C = 0$.

• Mass splittings among the quarks and leptons: in the present model quark and lepton states $E$ and $D$ are distinguished by their behavior under $Z_4$ (or $K$) and therefore described by the wave functions $\phi_E$ and $\phi_{D_1,2,3}$, eqs. (14)-(17) (or (18)-(21)) which are interaction eigenstates with mass matrix

$$m_{E,D_1,D_2,D_3} = \frac{1}{4} \begin{pmatrix} 1, 1, 1, 1 & 1, -i, -1, i & 1, -1, 1, -1 & 1, i, -1, -i \end{pmatrix}$$

$$\begin{pmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \\ m_{41} & m_{42} & m_{43} & m_{44} \end{pmatrix} \begin{pmatrix} 1, 1, 1, 1 \\ 1, i, -1, -i \\ 1, -1, 1, -1 \\ 1, -i, -1, i \end{pmatrix}$$

(44)
The requirement \( m_{D_1} = m_{D_2} = m_{D_3} \) of quark masses being color independent directly leads to

\[
m_E = \frac{1}{4} (\text{tr}(m) + 6m_n)
\]

(45)

\[
m_{D_1} = m_{D_2} = m_{D_3} = \frac{1}{4} (\text{tr}(m) - 2m_n)
\]

(46)

where \( m_n = m_{13} + m_{24} \) and one is this way able to accommodate any quark lepton mass ratio one likes. For \( K \) instead of \( Z_4 \) as symmetry group one obtains a result which is formally identical.

Notes added:
1. Although this approach knows nothing about the true nature of the tetron interactions and relies solely on the symmetry properties of the tetrahedrons, it can in principle be used to calculate all mass ratios of the fermion spectrum. The basic mass scale is of course set by the strength of the new interaction, but mass ratios can be inferred from symmetry principles.
2. Almost identical results can be obtained on the basis of the approach eq. (34), if one supposes that the matrix elements transform according to a representation of \( S_4 \) or \( S_3 \) as described in the appendix. This will not be discussed here.

The above procedure for masses may be applied as well to standard model fermion charges, provided one assumes that each charge \( C \) corresponds to an operator \( C_{op} \) acting on the linear combinations of states \( \phi^J \) eq. (33) as

\[
C_{op} \phi^J = C_J \phi^J
\]

(47)

In the left-right symmetric standard model \( U(1)_{B-L} \times SU(3)_c \times SU(2)_L \times SU(2)_R \) there are the following charges: \( Y_{B-L} \) (B-L charge), \( C_8 \) and \( C_3 \) (color charges) and \( T_{3L} \) and \( T_{3R} \) for weak isospin.
• Starting with weak isospin doublets and using that weak isospin states are described in the present model by $S_2$ wave functions $\phi_\pm$ (eq. (22)) with eigenvalues $\pm \frac{1}{2}$ we require
\[
\pm \frac{1}{2} = \frac{1}{2} \begin{pmatrix} 1 & \pm 1 \end{pmatrix} \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} \begin{pmatrix} 1 \\ \pm 1 \end{pmatrix}
\]
(48)
We therefore find that
\[
T_{3op} = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}
\]
(49)
and the analysis is identical for left and right-handed fermions.

• Going next to $B-L$ and to the color charges we have for the eigenvalues of a quark-lepton quadruplet $L$ and $Q$ in $Z_4$ the following condition:
\[
C(L, Q_1, Q_2, Q_3) = \frac{1}{4} \begin{pmatrix} 1,1,1,1 & 1,-i,-1,i & 1,-1,1,-1 & 1,i,-1,-i \\ c_{11} & c_{12} & c_{13} & c_{14} \\ c_{21} & c_{22} & c_{23} & c_{24} \\ c_{31} & c_{32} & c_{33} & c_{34} \\ c_{41} & c_{42} & c_{43} & c_{44} \end{pmatrix} \begin{pmatrix} 1,1,1,1 \\ 1,i,-1,-i \\ 1,-1,1,-1 \\ 1,-i,-1,i \end{pmatrix}
\]
(50)
where on the left hand side there are the standard model values $C_8(L) = 0$, $C_8(Q_1) = 1$, $C_8(Q_2) = -2$, $C_8(Q_3) = 1$ and similarly $C_3(L) = 0$, $C_3(Q_1) = 1$, $C_3(Q_2) = 0$, $C_3(Q_3) = -1$ and $Y_{B-L}(L) = -\frac{1}{2}$, $Y_{B-L}(Q_1) = Y_{B-L}(Q_2) = Y_{B-L}(Q_3) = \frac{1}{6}$. With a little algebra one
obtains

\[ Y_{B-L} = -\frac{1}{3} \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} \] (51)

\[ C_3 = \frac{1}{2} \begin{pmatrix} 0 & 1 & 0 & 1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ -1 & 0 & -1 & 0 \end{pmatrix} \] (52)

\[ C_8 = \frac{1}{2} \begin{pmatrix} 0 & 1 & -2 & 1 \\ 1 & 0 & 1 & -2 \\ -2 & 1 & 0 & 1 \\ 1 & -2 & 1 & 0 \end{pmatrix} \] (53)

Using \( K \) instead of \( Z_4 \) the same form is obtained for \( Y_{B-L} \) and \( C_8 \) but \( C_3 \) is modified to read:

\[ C_3 = \frac{1}{2} \begin{pmatrix} 0 & 1 & 0 & -1 \\ 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 1 \\ -1 & 0 & 1 & 0 \end{pmatrix} \] (54)

These charge operators can be extended to 24 times 24 matrices acting on the fermion wave functions eq. (30). They cannot be traced to charges of a single tetron, but correspond to properties of the tetrahedron wave functions \( \phi_\sigma \). As will be shown, there is a direct relation to the construction of vector boson states in the next section.
5 The Model for Vector Bosons

Vector Bosons are bound states of 8 tetrons which arise when 2 of the fermion-tetrahedrons \( \phi_\sigma = \phi_{abcd} \) and \( \phi_{\sigma'} = \phi_{a'b'c'd'} \) of eq. (8) approach each other and a and a', b and b', c and c' and d and d' interact in such a way that a cube is formed. More precisely, I am talking about a tetrahedron and an anti-tetrahedron, where a and a', b and b', c and c' and d and d' sit on opposite corners of the cube. Note that we are dealing with Spin 1 objects here because of the spin \( \frac{1}{2} \) nature of the tetrahedrons discussed in section 2. When forming fermion-antifermion products one should therefore in principle write \( \bar{\phi}_{\sigma'} \gamma_\mu \phi_\sigma \). To keep things manageable, however, the Dirac structure will not be made explicit in the formulas below. Instead I shall simply write \( \phi_{\sigma'}^* \phi_\sigma \). Also, as stressed before, the geometrical picture of tetrahedrons and cubes should not be taken too literally. It is just a memo for the behavior of the permutation group \( S_4 \).

In the symmetric limit, in which all tetrons are identical, there are \( 24 \times 24 \) identical cube states. When the symmetry is broken with charges \( q_a, q_b, Q_a, Q_b, \ldots \), one has \( 24^2 \) different states generated by applying the elements of \( S_4 \times S_4 \) on \( \phi_{\sigma'}^* \phi_\sigma \). One is thus confronted with a lot of different tetrahedron-antitetrahedron bound states which mediate a lot of interactions like inter family interactions, leptoquarks and so on, which one does not want.

In order to reduce the number of vector boson states there must be a mechanism which in the process of vector boson formation from fermions (i.e. the tetrahedron and anti-tetrahedron approaching each other) restores parts of the symmetry like, for example, the \( Z_3 \)-(family)-part, so that there are no interaction particles associated with \( Z_3 \). Instead, \( Z_3 \) must become a symmetry of the cube which transforms e.g. the \( W^+ \) arising from \( \tau^+ + \nu_\tau \) to a \( W^+ \) decaying into \( e^+ + \nu_e \). A similar rule holds for the transformations.
between quarks and leptons forbidding the existence of leptoquarks, i.e. lepton number violating processes, and at the same time allowing $W^+$-mediated transitions e.g. between $\bar{b} + t$ and $\tau^+ + \nu_\tau$.

The background behind this is a nonexistence

**THEOREM**: The following statements are equivalent:

- the weak interactions are universal
- leptoquark and interfamily interactions do not exist

This theorem seems quite obvious on an abstract level and in particular in the present model because one needs interfamily and leptoquark transitions as symmetries and not as interactions. It should be noted however that in principle the symmetries of vector bosons under tetron transformations could be larger than given by the theorem.

Later I am going to explicitly identify which transitions are symmetries for vector bosons and which ones are not. This will be done by considering step by step the possible fermion-antifermion states and comparing them to the observed vector bosons.

Before doing this, let me discuss in some detail, what might be the cause of the increased symmetry, i.e. the universality of the weak interaction, on the tetron level. One may speculate that it is caused by a partwise annihilation of tetron charge clouds $q_i$ so that some corners of the cube become indistinguishable and the cube symmetrical under $Z_3$ and leptoquark transformations. As soon as this happens, these symmetries can be interpreted as ordinary rotation symmetries in space (e.g. for $Z_3$: $\pm 2\pi/3$ rotations about the 3 body

---

One may even consider the possibility of the annihilation being complete. In that case one is left with 24 products $\phi_i^* \phi_{i\sigma}$ to form vector boson states. Due to $\phi_i^* \phi_{i\sigma} = [\phi_i^* \phi_{i\sigma}]^* = [\phi_i^* \phi_{i\sigma-1}]^*$ this corresponds to 24 real degrees of freedom.
diagonals of the cube). This annihilation may affect, for example, the 'front' triangle of the tetrahedron which is approaching a corresponding 'front' of an anti-tetrahedron. Stripped off of 3 of their clouds, e.g. $q_a, q_b$ and $q_d$ the tetrahedron will then look like $\chi(q_1) \otimes \chi(q_2) \otimes \chi(q_3) \otimes \chi(q_4)$ and similarly for the anti-tetrahedron and the resulting cube will be symmetric under an $S_3$ and in particular under interchanges ($Q_1 \leftrightarrow Q_2$) and ($Q_1 \leftrightarrow Q_4$) which according to table 1 transform the families into each other. Note however, that this picture is at best heuristic as was the assignment of particle states in table 1. The point is that according to sect. 3 fermion states are linear combinations of permutation states and this complicates the situation. Therefore, instead of giving handwaving arguments I will now consider explicitly the possible fermion-antifermion states.

I start with the electroweak sector and want to represent $W^\pm$ and $Z$ by suitable combinations $\phi^a_{\alpha \beta \gamma \delta} \phi^{bcde}$. A simple formula can be obtained from the wave functions eq. (22):

\[
W^- = '\ \bar{l} \gamma_\mu l' = \phi^+_+ \phi^- = \\
[\chi^*(q'_1, Q'_1) \otimes \chi^*(q'_3, Q'_3) + \chi^*(q'_1, Q'_3) \otimes \chi^*(q'_3, Q'_1)] \\
\otimes [\chi(q_1, Q_1) \otimes \chi(q_3, Q_3) - \chi(q_1, Q_3) \otimes \chi(q_3, Q_1)]
\]

and

\[
Z = '\ \bar{l} \gamma_\mu \nu l - \bar{l} \gamma_\mu l' = |\phi_+|^2 - |\phi_-|^2
\]

where by $Z$ I actually do not mean the physical $Z$-boson but what is usually called $W_3$. I will from now on leave out the primes in the arguments implicitly understanding that wherever a wave functions with an asterics appears it should get primed arguments. Then, under very moderate assumptions on the product $\otimes$, the above formulae may be simplified to

\[
W^\pm = |\chi(q_1, Q_1) \chi(q_3, Q_3)|^2 - |\chi(q_1, Q_3) \chi(q_3, Q_1)|^2 \\
\pm [\chi^*(q_1, Q_1) \chi^*(q_3, Q_3) \chi(q_1, Q_3) \chi(q_3, Q_1) - c.c.]
\]

\[
Z = 2[\chi^*(q_1, Q_1) \chi^*(q_3, Q_3) \chi(q_1, Q_3) \chi(q_3, Q_1) + c.c.]
\]

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These results can easily be generalized to make the universality of $W^\pm$ and $Z$ on the family level explicit. Namely, using the representation (58) of the wave functions and the $Z_3$-(family)-symmetry of the weak interactions implies

$$W^- = \{|\phi_{id}|^2 - |\phi_{u0}|^2 + \phi_{id}^* \phi_{u0} - \phi_{u0}^* \phi_{id}\}$$

$$= \{|\phi_{g1}|^2 - |\phi_{u1}|^2 + \phi_{g1}^* \phi_{u1} - \phi_{u1}^* \phi_{g1}\}$$

$$= \{|\phi_{g2}|^2 - |\phi_{u2}|^2 + \phi_{g2}^* \phi_{u2} - \phi_{u2}^* \phi_{g2}\}$$

$$Z = 2[\phi_{id}^* \phi_{u0} + \phi_{u0}^* \phi_{id}]$$

$$= 2[\phi_{g1}^* \phi_{u1} + \phi_{u1}^* \phi_{g1}]$$

$$= 2[\phi_{g2}^* \phi_{u2} + \phi_{u2}^* \phi_{g2}]$$

(59)

on the basis of eqs. (29) for $S_3$. Furthermore, the validity of the nonexistence theorem, i.e. absence of interfamily interactions, can be explicitly verified by showing that terms like $\bar{\nu}_e \mu$, $\bar{\nu}_\mu \tau$ and $\bar{\nu}_\tau e$ which represent the $S_3$-symmetric form of interfamily interactions identically vanish. A possible generation mixing may also be incorporated.

Note that instead of eq. (29) one should of course use the full $S_4$-combinations eq. (30) as factors to be used in the representation of vector bosons. In this case the restrictions from the nonexistence theorem become more complicated in shape, but the principle does not change.

In a similar fashion as eq. (56) one can use the representation of leptons and quarks eqs. (14)-(17) for $Z_4$ or eq. (18)-(21) for the Kleinsche Vierergruppe to write down formulae for the gluons and the photon. The photon, or more precisely, the $U(1)_{B-L}$ gauge boson, is given by

$$\gamma = 13\bar{\nu}_l \gamma_\mu \nu_l - \bar{q}_1 \gamma_\mu q_1 - \bar{q}_2 \gamma_\mu q_2 - \bar{q}_3 \gamma_\mu q_3$$

(61)

A straightforward calculation yields

$$\gamma = f_0^*(f_1 + f_2 + f_3) + f_1^* f_2 + f_1^* f_3 + f_2^* f_3 + c.c.$$

(62)
where \( f_0 = \phi_{1234}, f_1 = \phi_{2341}, f_2 = \phi_{3412} \) and \( f_3 = \phi_{4123} \) for \( Z_4 \). In the case of \( K \) the result eq. (62) is formally identical with \( f_1 \) and \( f_3 \) replaced by 
\[ f_1 = \phi_{2143} \quad \text{and} \quad f_3 = \phi_{4321}. \]

One may compare this result with the expression for the \( U(1)_{B-L} \)-photon obtained in the weak sector
\[
\gamma = '\bar{\nu}_l \gamma_\mu \nu_t + \bar{\nu}_t \gamma_\mu \nu_l
\]
\[ = 2[\phi_{id}^* \phi_{id} + \phi_{wu}^* \phi_{wu}] \quad (63)\]
to see that the full \( S_4 \)-expression for the photon will contain the terms eq. (62) together with corresponding odd \((1 \leftrightarrow 3)\) contributions.

Next the diagonal gluons are given by
\[
g_3 = '\bar{q}_1 \gamma_\mu q_1 - \bar{q}_3 \gamma_\mu q_3
\]
\[ = \frac{1}{2} \left[ \mp f_0^* (f_1 - f_3) - f_1^* f_2 \mp f_2^* f_3 \pm \text{c.c.} \right] \quad (64)\]
\[
g_8 = '\bar{q}_2 \gamma_\mu q_2 - \bar{q}_1 \gamma_\mu q_1 - \bar{q}_3 \gamma_\mu q_3
\]
\[ = \frac{1}{2} \left[ f_0^* (2 f_2 - f_1 - f_3) + 2 f_1^* f_3 - f_1^* f_2 - f_2^* f_3 \right] + \text{c.c.} \quad (65)\]

where the upper sign in eq. (64) corresponds to \( Z_4 \) and the lower to \( K \). The results eqs. (62), (64) and (65) can be directly compared to the charge matrices eqs. (51)-(54) obtained at the end of the last section.

Analogously one gets the nondiagonal gluons like
\[
g_{13} = '\bar{q}_1 \gamma_\mu q_3 + \bar{q}_3 \gamma_\mu q_1
\]
\[ = \frac{1}{4} \left[ \left| f_0 \right|^2 - \left| f_1 \right|^2 + \left| f_2 \right|^2 - \left| f_3 \right|^2 + [f_1^* f_3 - f_0^* f_2 \pm \text{c.c.}] \right] \quad (66)\]
\[
g_{31} = '\bar{q}_1 \gamma_\mu q_3 - \bar{q}_3 \gamma_\mu q_1
\]
\[ = \frac{1}{2} \left[ \pm f_1^* f_2 - f_2^* f_3 - f_0^* f_1 + f_0^* f_3 \pm \text{c.c.} \right] \quad (67)\]
where again the upper sign is for $Z_4$ and the lower for $K$.

Unfortunately on this level leptoquarks $\bar{l}\gamma\mu q_i$ exist as well. For example, one has

\[
LQ_{02} = \frac{1}{2} [\bar{l}\gamma\mu q_2 + \bar{q}_2\gamma\mu l]' = \frac{1}{4} \{ |f_0|^2 - |f_1|^2 + |f_2|^2 - |f_3|^2 - |f_1^* f_3 - f_0^* f_2 + c.c. | \} \quad (68)
\]

\[
LQ_{20} = \frac{1}{2} [\bar{l}\gamma\mu q_2 - \bar{q}_2\gamma\mu l]' = \frac{1}{2} [ f_1^* f_2 - f_2^* f_3 - f_0^* f_1 - f_0^* f_3 - c.c. ] \quad (69)
\]

In order that they vanish in accord with the nonexistence theorem and with phenomenology one has to impose the following additional constraints

\[
|f_1 + f_2|^2 = |f_0 + f_3|^2 \quad (70)
\]

\[
|f_1 + f_3|^2 = |f_0 + f_2|^2 \quad (71)
\]

\[
|f_2 + f_3|^2 = |f_0 + f_1|^2 \quad (72)
\]

\[
f_0^* f_1 - f_1^* f_2 - f_1^* f_3 - c.c. = 0 \quad (73)
\]

\[
f_0^* f_2 + f_1^* f_2 - f_2^* f_3 - c.c. = 0 \quad (74)
\]

\[
f_0^* f_3 + f_1^* f_3 + f_2^* f_3 - c.c. = 0 \quad (75)
\]

A straightforward calculation then yields $\bar{l}\gamma\mu q_1 = \bar{l}\gamma\mu q_2 = \bar{l}\gamma\mu q_3 = 0$ which is the desired result. Note that the absence of leptoquarks indicates that the effective gauge group is not $SU(4)$ but $U(1)_{B-L} \times SU(3)_c$. There is then no room in this model for exotic grand unified vector bosons.

Imposing the constraints on the other vector bosons allows to represent them in a somewhat more convenient form. For example, for the photon one obtains

\[
\gamma = 3 |f_0|^2 - |f_1|^2 - |f_2|^2 - |f_3|^2 + [2f_0^* (f_1 + f_2 + f_3) + c.c.] \quad (76)
\]
One may extend this analysis step by step to $D_4$- and then to $S_4$-functions. Applying symmetry under $3214 = (1 \leftrightarrow 3)$ (plus $Z_3$-family transformations) yields gluons and photon whereas symmetry under even permutations $g \in A_4$ yields the weak gauge bosons.

In summary we have analyzed the possible interactions between tetrahedron and antitetrahedron states and shown which transformations should be symmetries of the resulting vector boson state and which ones should not. As a consequence, the $24^2 S'_4 \times S_4$ tetrahedron-antitetrahedron states get reduced to

- 4 states $\gamma$, $W^\pm$ and $Z$ generated by

$$S'_4 / A'_4 \times S_4 / (Z_4 \circ Z_3) = Z'_2 \times Z_2 \quad (77)$$

where $A_4 \subset S_4$ is the group of even permutations in $S_4$. The first factor in (77) is due to the symmetry under even transformation / symmetry breaking under odd transformations whereas the second factor corresponds to the 12 orbits in the family and color group. In more concrete terms the resulting 4 real degrees of freedom are given by

$$|\phi_{1234}|^2$$
$$|\phi_{3214}|^2$$
$$\phi^*_{1234} \phi_{3214} = [\phi^*_{3214} \phi_{1234}]^* \quad (78)$$

objects, which have already appeared, in a somewhat different notation, in eqs. (55) and (60). Note that the 'singlet'

$$|\phi_{id}|^2 + |\phi_{uo}|^2 = |\phi_{1234}|^2 + |\phi_{3214}|^2 \quad (79)$$

corresponds to the photon. As we have seen, for the photon this is not the whole story, because the factorization of the photon interaction takes place both in the weak and the strong sector.
• 16 gluon and leptoquark states generated either by $K' \times K$ or $Z'_4 \times Z_4$

and in more concrete terms given by $f^*_i f_j$, precisely the objects which appear in the above equations for photons and gluons.

There is one item which has been spared out so far: the question of spin and the related topic of parity violation. In fact, most of the preceding analysis can be carried out for lefthanded and righthanded fermions separately. This way one naturally ends up with separate weak bosons for left and right. For the weak interactions this is okay, because we are expecting an effective $SU(2)_L \times SU(2)_R$ gauge theory. Photon and gluons, however, behave differently. They are identical for left- and righthanded fermions, i.e. they interact vectorlike. The question then immediately arises: what makes them peculiar?

To answer this question it is not necessary to know all about the spin properties of the tetrons themselves. What one has to realize is, that wave functions $\phi_u$ for odd permutations $u \in S_4$ naturally have opposite helicity to wave functions $\phi_g$ for odd permutations $g \in S_4$ and therefore should enter the linear combinations for fermions of definite helicity with a parity operator in front (assuming relativistic motion). The point is that, considered as an $O(3)$-transformation, an odd permutation has $\text{det}(u) = -1$ and therefore intrinsically contains a parity transformation which has to be taken care of. For definiteness one may consider the $K$ wave functions eqs. (18)-(21). Eq. (18) has to be modified according to

$$\phi_{\nu\tau} = \frac{1}{\sqrt{4}} \left[ \phi_{1234} + P \phi_{2341} + \phi_{3412} + P \phi_{4123} \right]$$

In the simple case eq. (22) of just 2 isospin partners $\phi_\pm$ the four helicity
states would be given by

\[
\phi_{+,L} = \frac{1}{\sqrt{2}} [\phi_{13} + P\phi_{31}] \\
\phi_{-,L} = \frac{1}{\sqrt{2}} [\phi_{13} - P\phi_{31}] \\
\phi_{+,R} = \frac{1}{\sqrt{2}} [P\phi_{13} + \phi_{31}] \\
\phi_{-,R} = \frac{1}{\sqrt{2}} [P\phi_{13} - \phi_{31}]
\] (81-84)

Similarly, in the general case eq. (30)ff of $S_4$-functions left handed fermion states will be generically of the form

\[
\phi_L = g_\phi + Pu_\phi
\] (85)

i.e. linear combinations of even and $P$-odd contributions, whereas right handed fermions will have the form

\[
\phi_R = Pg_\phi + u_\phi
\] (86)

Applying an even transformation $g \in S_4$ to the spin-1 object

\[
\bar{\phi}_L \gamma_\mu \gamma_5 \phi_L = \bar{g}_\phi \gamma_\mu g_\phi + Pu_\phi \gamma_\mu Pu_\phi + \bar{g}_\phi \gamma_\mu Pu_\phi + P\bar{u}_\phi \gamma_\mu g_\phi
\]

\[
\rightarrow_g \bar{g}_\phi' \gamma_\mu g_\phi' + Pu_\phi' \gamma_\mu Pu_\phi' + \bar{g}_\phi' \gamma_\mu Pu_\phi' + P\bar{u}_\phi' \gamma_\mu g_\phi'
\] (87)

where this time I have included the bar and the $\gamma$-matrix, does not change the helicity, i.e. a weak vector boson $W_L$ which according to eq. (77) is obtained by even permutations as symmetry transformations remains left-handed. There is no relation between $\bar{\phi}_L \gamma_\mu \phi_L$ and $\bar{\phi}_R \gamma_\mu \phi_R$, i.e. $W_L$ and $W_R$ are completely independent objects, i.e. both of the standard form eq. (60) with index L and R, respectively.

In contrast, applying an odd transformation $u \in S_4$ like $3214 = (1 \leftrightarrow 3)$,
which is the symmetry under which the gluons and the photon remain invariant, one ends up with a right handed vector boson

\[
\bar{\phi}_L \gamma_\mu \phi_L = g_\phi \gamma_\mu g_\phi + \bar{P} u_\phi \gamma_\mu P u_\phi + \bar{g}_\phi \gamma_\mu P u_\phi + \bar{P}_u g_\phi \gamma_\mu P u_\phi + \bar{P} g_\phi \gamma_\mu P u_\phi + \bar{P}_u g_\phi \gamma_\mu u_\phi
\]

This means imposing this symmetry instantly leads to a vectorlike structure of photon and gluons.

6 Conclusions

In conclusion we have here a scheme which accommodates all observed fermions and vector bosons. In addition it relates the number of these states in an obvious and natural way to the number of space dimensions, because the tetrahedron with 4 fundamental constituents is the minimal complex to build up 3 dimensions. With 3 constituents, for example, one would live in a 2-dimensional world with bound states of triangles which would yield only 6 bound permutation states instead of 24.

As we have seen, there are several other respects like parity violation and mass hierarchies in which the present model goes a step further in understanding than standard gauge theories. The real challenge will of course be to understand the nature of the tetron interactions and to write them in a renormalizable form.

About possible experimental tests: in the low energy limit the tetrahedron goes over into a point, i.e. it becomes an ordinary pointlike fermion. Increasing the energy one should be able to dissolve its spatial extension which will show up e.g. in the form of non-Dirac-like form factors. The question
then is: how small are the extensions of the tetrahedron? This question is
difficult to answer and is related to the problem of the strength and nature
of the superstrong interaction which binds the tetrons together. Certainly
there will be a correspondence between the tetrahedron extension and the
new coupling constant which is a new fundamental constant of nature from
which most of the couplings and masses known in particle physics will be
derived.

7 Appendix: Group and Representation Structure of \( T_d = S_4 \)

\( S_4 \) is the group of permutations of 4 objects and isomorphic to the symmetry
group \( T_d \) of an equilateral tetrahedron. Among the subgroups of \( S_4 \) there are
\( A_4, Z_3, S_2, S_3, Z_4 \) and the so-called \textit{Kleinsche Vierergruppe} \( K \) consisting of
permutations \( 1234, 2143, 3412 \) and \( 4321 \) which is isomorphic to \( Z_2 \times Z_2 \). It is
the smallest noncyclic group and describes the symmetries of a rectangle
in 2 dimensions. \( Z_3 \) is the cyclic group of 3 elements or equivalently the
subgroup of \( S_3 \) corresponding to the even permutations or equivalently the
group generated by \( \pm 2\pi/3 \) rotations of the plane.

\( S_4 \) can be written as \( Z_4 \odot S_3 \) where \( S_3 = Z_3 \odot S_2 \) is the point symmetry group
of an equilateral triangle. Correspondingly there are 4 orbits of \( S_3 \) and 6
orbits of \( Z_4 \) in \( S_4 \). In the present model the 4 degrees of \( Z_4 \) correspond to a
lepton and a quark with three colors. \( Z_3 \) is the family group and \( S_2 \) roughly
corresponds to weak isospin.

A short review of the representations of \( S_4 \) is in order: for a finite group the
number of irreducible representations is given by the number of conjugacy
classes (=5 in this case). Apart from the trivial representation $A_1$ there is
the totally antisymmetric representation $A_2$, which assigns 1 to all even and
-1 to every odd permutation. $A_1$ and $A_2$ are of dimension 1. There are two 3-
dimensional representations usually called $T_1$ and $T_2$, $T_1$ describing the action
of the permutation group on the tetrahedron. $T_2$ differs from $T_1$ by a minus
sign for odd permutations. Finally there is a 2-dimensional representation
$E$ which may be considered as the trivial extension of a corresponding 2-
dimensional representation of $S_3$.

One may use the representations $E$ and $T_1$ in the context of the symmetry
breaking, because in first order approximation the representation matrices
may be considered as state vectors (in the case of $E$ an average 'family state
vector') of the fermions. This has to do with the fact that the permutations
(and its representations) may be considered as ladder operators which gen-
erate all states $abcd$ from the ground state $1234$. These unbroken states can
then be used, in the spirit of first order perturbation theory, to derive mass
and charge relations for the fermions. (I have not used this approach in the
present paper, because I found a more elegant way to derive these relations.)

The covering group $\tilde{S}_4$ of $S_4$ is embedded in $SU(2)$ just as $S_4$ is in $SO(3)$.
Apart from those representations which are extensions from $S_4$ to $\tilde{S}_4$ it has
the following representations: a 2-dimensional representation $G_1$ which is
the restriction of the fundamental representation of $SU_2$ to the symmetry
transformations of a tetrahedron in much the same way as $T_1$ is the restriction
of the fundamental representation of $SO_3$. Then there is a representations
$G_2$ obtained from $G_1$ like $T_2$ is obtained from $T_1$ and a 4-dimensional (spin
3/2) representation $H$. 
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