The synchronizability of highly clustered scale-free networks

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(Dated: March 23, 2022)

In this letter, we consider the effect of clustering coefficient on the synchronizability of coupled oscillators located on scale-free networks. The analytic result for the value of clustering coefficient aiming at a highly clustered scale-free network model, the Holme-Kim is obtained, and the relationship between network synchronizability and clustering coefficient is reported. The simulation results strongly suggest that the more clustered the network, the poorer its synchronizability.

PACS numbers: 89.75.-k, 05.45.Xt

Many social, biological, and communication systems can be properly described as complex networks with nodes representing individuals and edges mimicking the interactions among them [1, 2, 3]. Examples are numerous: these include the Internet, the World Wide Web, social networks, metabolic networks, food webs, and many others [4, 5, 6]. Recent empirical studies indicate that the networks in various fields have some common characteristics, the most important of which are called small-world effect [8] and scale-free property [9]. Networks of small-world effect have small average distance as random networks and large clustering coefficient as regular ones. And the scale-free property means the degree distribution of networks obeys the power-law form.

One of the ultimate goals of researches on complex networks is to understand how the structure of complex networks affects the dynamical process taking place on them, such as traffic flow [10, 11, 12, 13], epidemic spread [14, 15, 16], cascading behavior [17, 18, 19], and so on. In this letter, we concentrate on the synchronizability, which is observed in a variety of natural, social, physical and biological systems [20, 21, 22]. The large networks of coupled dynamical systems that exhibit synchronized state are subjects of great interest. Previous studies mainly focus on the Watts-Strogatz [8] networks and Barabási-Albert [9] networks, which have demonstrated that scale-free and small-world networks are much easier to synchronize than regular lattices [23, 24, 25, 26]. Since many real-life networks are scale-free small-world networks, and there are already some models that can simultaneously reproduce the small-world and scale-free characteristics [27, 28, 29, 30], we will investigate the relationship between network synchronizability and clustering coefficient based on HK model.

As a remark, previous studies mainly concentrate on how the average distance and heterogeneity of degree/betweenness distribution affect the network synchronizability [24, 25, 31, 32, 33, 34, 35], while there are few systemic works about the effect of clustering coefficient. Although there are a number of highly scale-free models, the HK model is a typical one who has tunable clustering coefficient thus provides us a good researching stage. This is the reason why we choose HK model as our theoretic template.

The HK network is generated by the following processes:

(1) In each step, $m$ edges are added in the networks, and $t$ is a discrete parameter which denotes the global time that the system totally goes.

(2) First an edge is added with the probability $\Pi(k_i)$,

$$\Pi(k_i) = \frac{k_i}{\sum_j k_j}. \quad \text{(1)}$$

(3) Then, in the following $m-1$ time steps, do a PA (preferential attachment) step with the probability $p$ or a TF (triad formation) step with the probability $1-p$ (see Ref.[27] for details).

By using the rate-equation [36], one can obtain the evolution of nodes’ degree as follows:

$$\frac{\partial k_i(t)}{\partial t} = \{1 - \frac{k_i(t-1)}{\sum_{j=1}^{t-1} k_j(t-1)}\}^{m_{PA}} + \{1 - \frac{\sum_{l \in \Omega_i} k_l(t-1)k_l(t-1)}{\sum_{j=1}^{t-1} k_j(t-1)}\}^{m_{TF}}. \quad \text{(2)}$$

where $\Omega_i$ denotes the set of neighbors of node $i$, $k_i(t)$ is the degree of node $i$ at time step $t$. In the above formula, $m_{TF}$ is the number of edges that is connected following the rule of triad formation in each step while
where $A$ is a quadratic polynomial of $p$. Clearly, $p(k)=k^{-3}$ in the limit case $N \to \infty$.

The clustering coefficient of the whole network is the average of $c_i$ over all nodes $i$, where $c_i$ is the ratio between the number of edges among node $i$’s neighbors which is denoted by $n_i$ and the total possible number. Then,

$$c_i = \frac{2n_i}{k_i(k_i-1)}.$$  

Using the rate-equation approach \cite{37}, the detailed expression of $c(k)$, which denotes the average clustering coefficient over all the $k$-degree nodes, should be as follows:

$$c(k) = \frac{1}{k(k+1)} \left( \frac{2m_P A m_{TF}}{m} (k-m) + \frac{1}{m} \left( \frac{2m_P A m_{TF}}{m} (k-m) + \frac{m_{PA} (m_{PA}-1) \ln N}{16m} k^2 \right) \right).$$  

Here we assume that $m_0$ (the number of initial nodes) is equal to $m$ (the edges added each time step). The three items in the right side in the above expression is got from three mechanism shown in Fig. 1.

Thus the clustering coefficient $C$ can be solved as a function of the free parameter $p$,

$$c(p) = \int_{k_{min}}^{k_{max}} c(k) P(k) dk.$$  

In the above formula, $k_{max} \to 2m \sqrt{N}$ and $k_{min} = m$ \cite{38}. For $p \in [0,1]$, $c(p)$ can be simplified in linear approximation.

$$c(p) = B(m,N) + c(m,N)p.$$  

The extensive simulation results with different $p$ and $c$ for networks of different sizes strongly support the analytic results, especially in the larger-size networks, as shown in Fig. 2.

Other simulations have been done about the variation of average path length $l$ and standard deviation of degree distribution $\sigma$. The results show that both the average path length $l$ and standard deviation of degree distribution $\sigma$ behave slightly variation with vary $p$. The intuitionistic explanation is quite easily understood that $\sigma$ is directly related with degree distribution, that is to say, it is determined totally by the degree distribution, and hardly will the degree distribution vary when is $N$ large enough. So we have the reason to neglect the effect of the two structural properties on synchronizability.

Here, we concern the system of linear coupled limit-cycle oscillators on HK networks. Describing the state of the $i$th oscillator by $x_i$, the equations of motion governing
the dynamics of the $N$ coupled oscillators are:

$$\dot{x}_i = F(x_i) + K \sum_{j=1}^{N} M_{ij} G(x_j), \quad (9)$$

where $\dot{x}_i = F(x_i)$ characterizes the dynamics of individual oscillators, $G(x_j)$ is the output function, $K$ denotes the coupling strength and the $N \times N$ coupling matrix $M$ is

$$M_{ij} = \begin{cases} -k_i, & \text{for } i = j \\ 1, & \text{for } j \in \Lambda_i \\ 0, & \text{otherwise.} \end{cases} \quad (10)$$

In the above expression, $\Lambda_i$ denotes the neighbors of node $i$. Because of the negative semidefinition and the zero sum of each row of the matrix, all its eigenvalues are nonpositive real values and the biggest eigenvalue $\lambda_0$ is always zero. Thus the eigenvalues can be ranked as $\lambda_0 \geq \lambda_1 \geq \ldots \geq \lambda_N$, and $\lambda_0 = \lambda_1 = 0$ if and only if the network is disconnected.

In our coupled dynamic network, all the oscillators are identical and the same output function is used, the coupling fashion ensures the synchronization manifold is an invariant manifold and the nodes can be well approximated near the synchronous state by a linear operator. Under these conditions, the eigenratio $R = \frac{\lambda_2}{\lambda_N}$ can be used to measure the network synchronizability: the smaller it is, the stronger the synchronizability [25, 33, 34, 35, 39, 40, 41, 42, 43, 44, 45, 46].

We take only synchronizability $R(p)$ and clustering coefficient $c(p)$ into consideration. Having simulated $R(p)$ for different configurations versus $p$ (see Figure 3), we know that $R(p)$ is positively correlated with $p$, and so is $c(p)$ although it is not completely linear with $p$. Besides, we can easily find that when $p$ is not very large, the curve is approximately linear (see Figure 4).

Furthermore, we report the relationship between synchronizability $R(p)$ and clustering coefficient $c(p)$ as shown in Fig. 5. From the above figure, one important fact the curve reveals is exponential growing tendency. The function of fitted curve can be set as follows:

$$R(c) = A(m, N) + B(m, N)e^{c/m} \quad (11)$$

The terms $A(m, N), B(m, N)$, and $T(m, N)$ can be obtained by simulation. Here the simulation averaged over 50 different realizations is to measure the effect of random fluctuation of degree distribution, since the degree distribution can both affect $l$ and $\sigma$. In our simulation, due to the average effect, the two parameters, $l$ and $\sigma$ do not vary significantly so that we can only focus on clustering coefficient exclusively which causes the change of synchronizability.

Compared to the previous works, the advantages of HK model is that the clustering coefficient can be tuned while the other structural properties are almost kept fixed. Therefore, combining the behaviors of $c(p)$ versus $p$ and $R(p)$ versus $p$, we obtain the relationship between synchronizability and clustering coefficient. Fig. 4 demonstrates that the larger the clustering coefficient, the poorer the synchronizability. Due to that the synchronizability $R$ is determined by the ratio of maximal and minimal eigenvalues of coupling matrix which is exclusively related to the network topology, moreover $c$ is the only varied topological property, thus clustering coef-
FIG. 5: Synchronizability $R$ measured by $\frac{\lambda_c}{\lambda_2}$ as a function of clustering coefficient $c$ in the case of $N = 1800$, and $m = m_0 = 10$. The fitted curve follows the tendency of exponential growth.