Dirac gaugino from
grand gauge-Higgs unification

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Abstract

We show that models of the Dirac gaugino can naturally be embedded into a kind of the grand unified theory (GUT), the grand gauge-Higgs unification (gGHU) model, with the gauge group $SU(5) \times SU(5)/\mathbb{Z}_2$ on an $S^1/\mathbb{Z}_2$ orbifold. The supersymmetric gGHU is known to possess a light chiral adjoint supermultiplet after the GUT breaking, thank to the exchange symmetry of two $SU(5)$ groups. Identifying the ‘predicted’ adjoint fermion with the Dirac partner of the gaugino, we argue that the supersoft term, responsible for the Dirac gaugino mass, can be obtained from the supersymmetric Chern-Simons (CS) like term in the gGHU setup. Although the latter term does not respect the exchange symmetry, we propose a novel way to introduce its breaking effect within a consistent orbifold construction. We also give a concrete setup of fermion field contents (bulk and boundary-localized fermions) that induce the requisite CS-like term, and calculate its coefficient from the bulk profile of chiral fermion zero modes. Our gGHU setup may be regarded as an extra-dimensional realization of the Goldstone gaugino scenario that was proposed before as a solution to the problem of the adjoint scalar masses.

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1 Introduction

The standard model (SM) of the particle physics, with a possibly simple extension for the neutrino masses, is an extremely good phenomenological model. It basically explains the vast amounts of the experimental results below the TeV scale. Given the excellent phenomenological success, it may be suggestive to extrapolate the model to the very high energy region never reached by the experiments. Such a naïve extrapolation indicates $[1–3]$ that the quartic coupling of the Higgs field vanishes at an intermediate scale around $10^{11}$ GeV. It is interesting to assume that this is a footprint of the new physics beyond the SM. So far, two scenarios have been proposed as such candidates that predict the vanishing of the quartic coupling: the gauge-Higgs unification scenario $[4–7]$ and the Dirac gaugino scenario $[8–11]$.

In the former extra-dimensional scenario $[4–6]$, the electroweak (EW) gauge symmetry is broken via the so-called Hosotani mechanism $[12–15]$, in which a gauge field in higher dimensions gives rise to the zero mode in its extra-dimensional components that takes “nontrivial” vacuum expectation values (VEVs). In other words, the Higgs field is a part of the gauge field and thus has the vanishing self-coupling above the scale where the extra dimensions become visible. This can be expressed as a boundary condition, named the “gauge-Higgs condition” $[16, 17]$, on the renormalization group equation of the Higgs quartic coupling in the four-dimensional (4D) effective theory. It requires the coupling constant vanishing at the compactification scale, which is to be identified with the intermediate scale $[7]$.

In the latter supersymmetric (SUSY) scenario $[8, 9]$, adjoint chiral supermultiplets are introduced so that the gauginos are (pseudo-)Dirac fermions instead of Majorana. In the pure-Dirac limit, the $D$-term contribution to the quartic scalar couplings are canceled by the exchange of the scalar component of the adjoint multiplets. Then the above intermediate scale may be identified with the adjoint scalar mass scale $[10, 11]$. Aside from this intermediate scale scenario, the Dirac gaugino models have been studied also in the context of the TeV-scale SUSY, which features other attractive properties of the Dirac gaugino models, such as the “supersoftness” $[18]$ and the “supersafeness” $[19, 20]$. Given the null results for the signal beyond the SM at the LHC, the supersafeness property may be helpful for relaxing the constraints on the SUSY breaking scale $[21, 22]$. The origin of the supersoft operator, responsible for the Dirac mass term of the gauginos, and related problems were discussed in Refs. $[23–26]$. The issue of the $D$-term cancellation and the Higgs mass was also addressed, for instance in the minimal $R$-symmetric model $[27–29]$ and also in the next-to-minimal extension $[30]$.

The Dirac gaugino scenario is attractive as a low-energy effective theory, but it contains some nontrivial assumptions to be addressed if we try to construct a concrete UV completion. See Sect. $[2]$ for a brief review in this point. Among others, the required adjoint chiral superfields look less natural especially when we try to embed the Dirac gaugino models into a grand unified theory (GUT) $[31–35]$.

In this respect, there is an interesting class of GUT models that naturally “predicts” the presence of light adjoint chiral multiplet: it is (a version of) the grand gauge-Higgs unification (gGHU) model $[36, 39]$. In the gGHU scenario, we utilize the Hosotani mechanism to break

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$^5$Other versions of “gGHU” were proposed in several contexts in Refs. $[40–44]$ and also in Ref. $[45]$, where the
the Georgi-Glashow’s $SU(5)_G$ gauge symmetry [31], instead of the EW gauge symmetry. In this case, the adjoint Higgs field is identified with the zero mode of the extra-dimensional component of the gauge field. Since such a component has a flat potential at tree level, the position of the vacuum is determined by quantum corrections. In general the mass and potential of the zero modes would be much distorted by large radiative corrections of order of the compactification scale. In a supersymmetric version of the model, however, the mass of the zero mode will be of order of SUSY breaking scale $M_{SB}$, which can be much smaller than the compactification scale. Therefore the existence of the light adjoint chiral superfields is a generic prediction [38] of the gGHU models with supersymmetry. In this way, such models provide a natural starting point for constructing a satisfactory UV completion of the Dirac gaugino models.

The purpose of the present article is to show that the gGHU setup can give a good UV completion of the Dirac gaugino models. Specifically we will show that the operator responsible for the Dirac gaugino mass can be generated as a kind of supersymmetric Chern-Simons (CS) term [52]. Actually we will focus on its bosonic components and elaborate how its coefficient can be computed from a suitable choice of bulk and boundary-localized fermions and their mass parameters.

In principle one could add the requisite CS-like term to the starting five-dimensional (5D) theory by hand. A more interesting possibility is to start with a 5D theory without such term and to generate it radiatively. Actually in the present paper, we will be interested in the situation in which the requisite CS term is generated as the term representing anomaly inflow [53], and thus its coefficient can be determined through a profile of fermion zero modes spread in the 5D bulk. Alternatively we can calculate it by summing up massive Kaluza-Klein (KK) modes. Such calculation will be applicable even when no fermion zero mode is present, as we shall show in a separate publication.

Before going into detailed discussion, let us summarize here our gGHU setup for Dirac gaugino. For concreteness, we consider a 5D supersymmetric $SU(5)$ gGHU model compactified on an $S^1/\mathbb{Z}_2$ orbifold, with the compactification scale $1/R$ being the GUT scale $M_{GUT}$. The SUSY breaking scale $M_{SB}$ can be either the intermediate scale or a lower scale. We start with the bulk symmetry $SU(5)_1 \times SU(5)_2 \times \mathbb{Z}_2^{\mathbb{C}} \times U(1)_D$, where the $\mathbb{Z}_2^{\mathbb{C}}$ exchanges the two $SU(5)$ factors. The bulk $SU(5)_1 \times SU(5)_2$ symmetry is broken by the orbifold boundary conditions (BCs) down to its diagonal subgroup $SU(5)_V$, which is identified with the $SU(5)_G$. This duplicated structure is a source of adjoint zero modes in the gGHU setup [36, 37]. Notice also that the bulk gauge group contains a $U(1)_D$ factor, which is a basic ingredient for the Dirac gaugino models, as will be reviewed in Sect. 2. Correspondingly the CS-like term to be generated is related to a mixed anomaly between the $U(1)_D$ and $SU(5)$ gauge groups. Therefore we will refer to it as mixed CS-like term in the present paper.

This article is organized as follows. In the following two sections, brief reviews are given respectively of the Dirac gaugino and the gGHU scenarios. In Sect. 2, we summarize the basic

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SU(5) symmetry is broken by orbifold boundary conditions [46-51], while one utilizes the Hosotani mechanism to break the EW symmetry [40, 44] or to reduce the rank of unified gauge groups [45].

Phenomenological implications of the chiral adjoints at TeV scale were studied in Refs. [38, 39], where characteristic signatures to be observed in future collider experiments were also discussed.
assumptions of the Dirac gaugino models. We also comment on the issue of the lemon-twist (LT) operator and its proposed solution [25, 26]. In Sect. 3, we review some elements of the SU(5) gGHU models. Specifically we explain how adjoint zero modes arise in a model with SU(5)_1 × SU(5)_2 × Z_2^x. We also explain how to obtain incomplete GUT multiplets in the gGHU setup. In Sect. 4, we examine the Z_2 properties of the mixed CS-like term and explain how required Z_2 breaking can be incorporated in a consistent S^1/Z_2 orbifold. We also derive a concrete expression for the coefficient of the Dirac gaugino mass terms. The section 5 is devoted to summary and discussion. In Appendix A we summarize the field contents and the supersymmetric Lagrangian of our model; we also outline how the supersymmetric CS-like term is related to the supersoft term responsible for the Dirac gaugino mass terms. After a review on anomaly inflow on orbifold and its relation to the CS term in Appendix B, the detailed calculations of the mixed CS-like term are given in Appendix C using a simplified setup.

2 Elements of Dirac gaugino

Here we give a brief review on models of Dirac gaugino, recalling the basic assumptions behind the construction. We start with a supersymmetric model that contains an adjoint chiral superfield Φ^a for each gauge group G_A in the SM (A = 3, 2, 1),

\[ Φ^a(x, θ) = φ^a(x) + \sqrt{2}θ^{aα}ψ_α^a(x) + \cdots, \] (1)

where θ^{aα} is the superspace coordinate and the adjoint index of G_A is denoted by a. We assume that supersymmetry is broken by a nonvanishing D-term \langle D_D \rangle of a hidden-sector U(1)_D. Then the Dirac gaugino mass term can be obtained if we further assume that integration of messenger sector fields gives rise to the so-called supersoft operator [9, 18]

\[ L_4^\text{supersoft} = \frac{C_A g_A^2 g_D Λ}{2} \int d^2θ \sqrt{2} Φ^a W_α^a W^α_D. \] (2)

Here Λ is a mass scale at which the above operator is generated; W_α^a (W_D^α) are the field strength superfield of the SM gauge group G_A (the hidden-sector U(1)_D factor), respectively. In Eq. (2), we have put the coefficient C_A as well as the gauge coupling g_A and g_D of the gauge group G_A and U(1)_D respectively. Substituting the nonvanishing D-term, \langle W_D^α \rangle = θ^α \langle D_D \rangle, into the supersoft operator, we obtain a Dirac mass,

\[ m_{DA} = C_A g_A^2 g_D \langle D_D \rangle Λ, \] (3)

of the G_A-gaugino and the fermion component of Φ^a.

An intriguing property of Dirac gaugino models comes from the fact that the supersoft operator contains a trilinear coupling of the scalar component φ^a of Φ^a to the other scalar fields. This has two important consequences [18]. The first one is the supersoftness, that is, radiative corrections to a scalar mass become finite since usual logarithmic divergences are
canceled by the adjoint scalar loop. The second consequence is $D$-term cancellation, which means that the usual $D$-term contributions to the quartic scalar couplings are canceled by tree-level exchange of the adjoint scalar $\phi^a$.

Let us summarize the assumptions in Dirac gaugino models: (i) the presence of light adjoint chiral fields whose fermion components are the Dirac partner of the gauginos, (ii) the generation of the supersoft operator, and (iii) $D$-term SUSY breaking in the hidden-sector. As we see in Sect. 3, the assumption (i) can naturally be explained in a supersymmetric version of the gGHU setup. To discuss the assumption (ii) in such setup is the main purpose of the present work and is given in Sect. 4. As for the SUSY breaking, we just note that an example of dynamical SUSY breaking with a nonzero $D$-term is provided by the $SU(4) \times U(1)$ model in Ref. [54]; another example is Nambu–Jona-Lasinio type models of Refs. [55–57].

We add a comment on masses of the adjoint scalars $\phi^a = (\sigma^a + i\pi^a)/\sqrt{2}$. The supersoft operator (2) gives a mass $2m_D$ to the real part $\sigma^a$, but its pseudo-scalar partner $\pi^a$ remains massless [18]. One expects radiative corrections to their masses, except for the singlet. Phenomenologies with the chiral adjoints are quite different depending on whether there is a superpotential coupling to the Higgs doublets.

### 2.1 Problem of adjoint scalar mass and its solutions

The successful generation of the supersoft term (2) is not the end of the story: we should take care that unwanted terms are not generated at the same time. Among others, there is the so-called lemon-twist (LT) operator [23, 24],

$$L_{LT}^{4d} = \frac{1}{\Lambda^2} \int d^2 \theta \Phi^a \Phi^a W_D \Phi_D^a ,$$

which contributes to a $B$-term like, holomorphic mass term of $\phi^a$. This contribution, if present, decreases one eigenvalue of the mass squared of the scalar component $\phi^a$ to make it tachyonic. This is problematic especially when both the operators in Eqs. (2) and (4) are generated at one loop level.

A solution to this problem was proposed in Ref. [25, 26]: if the scalar component $\text{Im} \phi$ behaves as a (pseudo-)Goldstone field of a broken anomalous symmetry, then the LT term (4) is forbidden by the shift symmetry of the Goldstone mode, while the desired term (2) is still generated from the anomaly. Such scenario was called Goldstone gaugino scenario.

In this respect, it is interesting to note that extra-dimensional components of gauge fields are kinds of Nambu-Goldstone modes related to the breaking of 4D gauge symmetry on each point (4D slice) in the extra dimensions. This fact is clear in the lattice regularization or deconstruction [58] of the extra dimensions. As a result, if the adjoint chiral fields predicted in the gGHU setup are identified with the Dirac partner of the gauginos, the resulting model may be regarded as an extra-dimensional realization of the Goldstone gaugino scenario. We note that in the gGHU picture, the absence of the LT operator can be understood directly from the 5D gauge invariance.
3 Review of grand gauge-Higgs unification

The basic idea of the gGHU scenario is to break the unified gauge symmetry by the Hosotani mechanism. The gGHU, when applied to the SUSY SU(5), has intriguing properties: a natural realization of the doublet-triplet (DT) splitting and the prediction of light chiral adjoint fields. Here we will explain these properties, recalling some elements of orbifold construction for later purposes.

For definiteness, let us consider a five-dimensional SU(5) model with a simple Lagrangian

$$\mathcal{L}_5 = -\frac{1}{4} F_{MN}^a F_{aMN} + \bar{\Psi}_R (i \gamma_5 D_5 - m) \Psi_R ,$$

where $F_{MN}^a$, $\Psi_R$ and $D_M$ are respectively the field strength, a fermion field belonging to the $R$ representation and the covariant derivative acting on it. The 5D Lorentz and the adjoint indices are denoted by $M = (\mu, 5) = (0, 3, 5)$ and $a$ respectively. We will consider only the case without supersymmetry, but it is straightforward to supersymmetrize the whole setup by replacing the gauge (fermion) field with the vector (chiral) supermultiplet.

To realize the chiral fermions of the SM, we compactify the fifth dimension on an $S^1/\mathbb{Z}_2$ orbifold, which is a quotient space of a circle $S^1$, divided by the identification under the 5D parity $P_5 : x^5 \rightarrow -x^5$. Two fixed points are denoted by $x^5 p_\pi R$ ($p = 0, 1$). The circle with the radius $R$ can be regarded as a quotient of the covering space $\mathbb{R}$ divided by the translation $T : x^5 \rightarrow x^5 + 2\pi R$, and the product $P_5^\prime = TP_5$ generates the parity around $x^5 = x^5_1$; that is, $P_5^\prime : \pi R - x^5 \rightarrow \pi R + x^5$.

When the theory has another $\mathbb{Z}_2$ symmetry, the identification can be twisted; for instance if we choose a nontrivial element of the gauge group, $P_g$, as the generator of the additional $\mathbb{Z}_2$, the bulk gauge symmetry can be reduced by the orbifold BCs \([46–51]\). This can be understood by applying the identification to the fields\(^7\)

$$\{ A_M(x^5), \Psi_R(x^5) \} = P_5 P_g \{ A_M(x^5), \Psi_R(x^5) \}
= \{ (-)^M \bar{P}_g A_M(-x^5) \bar{P}_g^\dagger, \eta_\Psi \gamma_5 \rho_R[P_g] \Psi_R(-x^5) \} ,$$

where $(-)^M$ takes $+1(-1)$ for $M = \mu (5)$, and $\rho_R[P_g]$ denotes the matrix representation of $P_g$ on the fermion $\Psi_R$. A parallel discussion holds for $P_5^\prime$ with (generally different) $P_g^\prime$. Note that for each fermion $\Psi$, the sign factor $\eta_\Psi$ can be $+1$ or $-1$. A similar sign degrees of freedom $\eta_{5\Psi}$ and $\eta_T (= \eta_\Psi \eta_5)$ exist for the parity $P_5^\prime$ and the translation $T$ respectively. On the other hand, there is no such sign degree of freedom for the gauge field. The components of $A_\mu$ that commute with $P_g$ and $P_g^\prime$ have a zero mode and correspond to the low-energy gauge symmetry; the other components, not even functions of $x^5$ or $x^5' = x^5 - \pi R$, do not have zero modes and thus decouple from the low-energy theory. Similarly, the components of $A_5$ that \textit{anti}-commute with both $P_g$ and $P_g^\prime$ have zero modes. In the usual Hosotani mechanism, it is these zero modes that acquire a nontrivial VEV to break the gauge symmetry further.

\(^7\)Here $P_g$ is an abstract group element while $\bar{P}_g$ is the corresponding matrix in the defining representation.
3.1 Adjoint zero modes via diagonal embedding

To apply the Hosotani mechanism to the $SU(5)_G$ breaking, we need an adjoint zero mode of $A_5^a$. Due to the factor $(-)^M$, however, $A_5^a$ has the 5D parity opposite to $A_{5\mu}^a$, and so the $A_5^a$ does not have the zero mode in the (adjoint) component corresponding to the zero mode gauge fields $A_{5\mu}^a$ of the unbroken gauge group. Actually a way of realizing the adjoint zero modes is provided by the diagonal embedding method, which was developed in the context of the string theory [59–66] and applied to our field theoretical setup [36, 37].

For this purpose, we introduce two copies of the gauge group and suppose that there is a $\mathbb{Z}_2^{ex}$ symmetry that exchanges the two gauge fields, $A_M^{(1)}$ and $A_M^{(2)}$. We denote by $P_{ex}$ the generator of this $\mathbb{Z}_2^{ex}$:

$$P_{ex} : \quad A_M^{(1)}(x) \leftrightarrow A_M^{(2)}(x).$$

Then the orbifold BCs that give rise to the desired adjoint zero modes are given by the combined (or simultaneous) action of the 5D parity $P_5$ and the $\mathbb{Z}_2^{ex}$ exchange $P_{ex}$:

$$\left( A_M^{(1)}(x^5), \quad A_M^{(2)}(x^5) \right) = (-)^M \left( A_M^{(2)}(-x^5), \quad A_M^{(1)}(-x^5) \right)$$

around the first fixed point $x_0^5 = 0$, and similar ones around the other fixed point $x^5 = \pi R$. Defining the $\mathbb{Z}_2^{ex}$ eigenstates by $X^{(\pm)} \equiv (X^{(1)} \pm X^{(2)})/\sqrt{2}$, we see that $A_{5\mu}^{(+)}$ and $A_5^{(-)}$ have zero modes. This means that the gauge symmetry is reduced to the diagonal subgroup of the two gauge groups and that the $A_5^{(-)}$ zero mode behaves as an adjoint field under the remaining gauge symmetry. In this way, we obtain the adjoint scalar field that can be used to break the diagonal subgroup further.

Notice that, in this type of GHU scenario, the Higgs field which is unified with the gauge field is not the SM Higgs field, but the adjoint Higgs field that breaks the GUT gauge symmetry down to the SM one. After the $SU(5)_G$ breaking, a part of adjoint fields are absorbed via the Higgs mechanism, and more importantly, there appear the adjoint scalar fields of the SM gauge group, namely, the color octet, the weak triplet and the singlet fields, in the low-energy effective theory. As for the SM Higgs, we need a separate consideration as we review shortly.

As for fermions, we introduce a $\mathbb{Z}_2^{ex}$ pair of bulk fermions: $\Psi^{(1)}_{(R_1,R_2)}$ belonging to $R_1$ ($R_2$) representation of the first (second) gauge group, and its $\mathbb{Z}_2^{ex}$ partner $\Psi^{(2)}_{(R_2,R_1)}$. The BCs for them are given by

$$\left( \Psi^{(1)}_{(R_1,R_2)}(x^5), \quad \Psi^{(2)}_{(R_2,R_1)}(x^5) \right) = \eta_5 \gamma_5 \left( \Psi^{(2)}_{(R_2,R_1)}(-x^5), \quad \Psi^{(1)}_{(R_1,R_2)}(-x^5) \right),$$

and similar ones for $P_5'$. We summarize in Table I the parity eigenvalues of each field under the $P_5P_{ex}$ and $P_5'P_{ex}$. Note that the BCs of $\Psi^{(\pm)}_L$ are the same as those of $\Psi^{(\pm)}_R$ with the opposite $P_{ex}$ parity. Consequently, for instance, when $\Psi^{(+)}_L$ has the zero mode, $\Psi^{(-)}_R$ also does. Hereafter we set $R_2 = 1$ for simplicity. In this case, these fields $\Psi^{(+)}_L$ and $\Psi^{(-)}_R$ belong to the same representation under the remaining gauge symmetry. Therefore a bulk fermion in the present setup gives rise to zero modes in a vector-like representation with the opposite $P_{ex}$ parity. The chiral fermions, such as the SM fermions, may be put on the boundaries.
| \( A_\mu^{(+)} \) | \( A_\mu^{(-)} \) | \( A_5^{(+)} \) | \( A_5^{(-)} \) | \( \Psi_R^{(+)} \) | \( \Psi_R^{(-)} \) | \( \Psi_L^{(+)} \) | \( \Psi_L^{(-)} \) |
|---|---|---|---|---|---|---|---|
| (+, +) | (−, −) | (−, −) | (+, +) | (ηₚ, ̄ηₚ) | (−ηₚ, − ̄ηₚ) | (−ηₚ, − ̄ηₚ) | (ηₚ, ̄ηₚ) |

### 3.2 Doublet-triplet splitting and gauge coupling unification

A striking feature of the present gGHU scenario is that the DT splitting can be naturally realized even in an SU(5) model \[38\]. This is possible on a specific vacuum where the Wilson line \( W \), the order parameter of the SU(5) breaking, is given, in the defining representation, by

\[
W = \mathcal{P} \exp \left( \frac{i g}{\sqrt{2}} \int A_5^{(-)} dx^5 \right) = \text{diag}(1, 1, 1, -1, -1),
\]

where \( \mathcal{P} \) denotes the path-ordering and \( g \) is the 5D gauge coupling constant. Since the unity in \( W \) corresponds to the trivial vacuum with vanishing \( \langle A_5 \rangle \), the above \( W \) corresponds to the (inversely) missing VEV \[67–73\]; schematically, it is \( \langle A_5 \rangle = \text{diag}(0, 0, 0, v, v) \) with \( v \neq 0 \).

Notice that usually the traceless condition of the SU(5) generators forbids this type of missing VEV; in the present case, it is allowed since \( W \) is an element of the SU(5) group, not the algebra.

For later purpose, let us introduce some notation for a diagonal \( W \): we write its diagonal component as \( w = e^{a_\pi i} \). Consequently, we can express the SU(5)\( _G \)-breaking VEV (10) by stating that \( e^{a_\pi i} = +1 \) for the SU(3) and \( e^{a_\pi i} = -1 \) for the SU(2) subgroups.

The Wilson line (10) can be used to realize the DT splitting. To see this, we introduce a pair of the bulk Higgs fields, \( H^{(1)}_{(5,1)} \) and \( H^{(2)}_{(1,5)} \), in the SU(5)\( _1 \times SU(5)\_2 \) setup. Naïvely the missing VEV contributes to the doublet mass instead of the triplet mass. Instead, assigning an antiperiodic BC to the bulk Higgs multiplets, namely, \( \eta_T = \eta_H \eta_H' = -1 \), we obtain the zero mode only in the doublet component. In this way we can naturally realize the correct pattern of SU(5) gauge symmetry breaking and the DT splitting if the expectation value of the Wilson line is given by Eq. (10).

We note that the required value (10) respects the \( Z^e_x \) exchange symmetry of SU(5)\( _1 \times SU(5)\_2 \): the \( P_{ex} \) transformation flips the sign of \( A_5^{(-)} \), which transforms \( W \) to its complex conjugate \( W^* \). This will guarantee that the vacuum is an extremum of dynamically generated effective potential without fine-tuning, although we will not attempt to analyze it here.

Another remark concerns the gauge coupling unification, which is realized in the minimal SUSY SM but is ruined by the adjoint chiral supermultiplets and/or by a deviation of the SUSY-breaking scale \( M_{SB} \) from TeV scale. In this respect, note that the above mechanism for the DT splitting can also generate a mass splitting in a bulk hypermultiplet other than

\[8\] Alternatively we can gauge away the Wilson line \( W \), so that the BC at \( x_1^5 \) is modified to \( P'_g W \).
the \((5, 1)\) and \((1, 5)\) Higgs fields. This allows us to introduce (vector-like) \(SU(5)\) incomplete multiplets in the 4D effective theory. For instance, a \(\mathbb{Z}_2^{ex}\) pair of periodic 10 superfields give vector-like pairs whose quantum numbers of the SM gauge group are the same as the right-handed up quark and the right-handed charged lepton respectively. The gauge coupling unification can be recovered by a suitable choice of additional bulk superfields. A concrete example was given in Ref. \[38\].

4 Chern-Simons term in grand gauge-Higgs unification

As we reviewed in the previous section, the grand GHU model, a supersymmetric \(SU(5)_1 \times SU(5)_2\) theory on \(S^1/\mathbb{Z}_2\) orbifold, is a natural starting point for constructing models of Dirac gaugino. It “predicts” the light chiral adjoint \(\Phi^a\), a chiral supermultiplets \(\Phi^a\) in the adjoint representation of the SM gauge group. We can identify its fermionic component as a Dirac partner of each gaugino. Then the next task is to generate the supersoft operator \(\mathcal{O}\) in the gGHU setup. For this purpose, we will be interested in a particular bosonic term, \(\phi^a F^a_{\mu\nu} \tilde{F}^D_{\mu\nu}\), contained in that operator. A straightforward supersymmetrization \[52\] will lead to the desired operator, as we sketch in Appendix A.2.

We consider a 5D \(SU(5)_1 \times SU(5)_2 \times U(1)_D\) gauge theory, extending the gGHU to include the \(U(1)_D\) gauge group responsible for the mediation of SUSY breaking; we assume that the \(U(1)_D\) gauge field is \(\mathbb{Z}_2^{ex}\)-even so that it has a zero mode, and denote its field strength by \(F^D_{\mu\nu}\). Notice that the supersoft operator \(\mathcal{O}\) contains a term of the form \(\phi^a F^a_{\mu\nu} \tilde{F}^D_{\mu\nu}\), where \(\phi^a = (\sigma^a + i\pi^a)/\sqrt{2}\) is the adjoint scalar. In the gGHU setup, the pseudo-scalar component \(\pi^a\) arises from the fifth component of the gauge field, \(A^{(-)}_5\), while the 4D gauge fields are from the zero modes of \(A^{(+)}_5\). In this way, we are led to the idea that the 5D counterpart of the supersoft operator \(\mathcal{O}\) in the gGHU scenario is given by a mixed CS-like term

\[
A^{a(-)}_5 F^{a(+)}_{\mu\nu} \tilde{F}^D_{\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} A^{a(-)}_5 F^{a(+)}_{\mu\nu} F^D_{\rho\sigma},
\]

To discuss how this term can be generated, we define the CS-like term of \(SU(5)_1^2 \times U(1)_D\) by

\[
\mathcal{O}^{(i)} = A^{a(i)}_5 F^{a(i)}_{\mu\nu} \tilde{F}^D_{\mu\nu},
\]

where \(i = 1, 2\). The desired operator \(\mathcal{O}^{(-)}\) is contained in a combination

\[
\mathcal{O}^{(-)} = \mathcal{O}^{(1)} - \mathcal{O}^{(2)} = A^{a(-)}_5 F^{a(+)}_{\mu\nu} \tilde{F}^D_{\mu\nu} + A^{a(+)}_5 F^{a(-)}_{\mu\nu} \tilde{F}^D_{\mu\nu},
\]

where the last term on the right-hand side contains at least one massive KK mode and thus decouples from the low-energy effective theory. We see from the relation \(\mathcal{O}^{(-)} = \mathcal{O}^{(1)} - \mathcal{O}^{(2)}\) requires asymmetry between \(SU(5)_1\) and \(SU(5)_2\), that is, a sort of breaking effect of the \(\mathbb{Z}_2^{ex}\) invariance.

Now, it is important to notice that the desired operator \(\mathcal{O}^{(-)}\) is odd under the 5D parity; it is also odd under the \(\mathbb{Z}_2^{ex}\). To generate such term, we should break the \(\mathbb{Z}_2^{ex}\) invariance as well as the \(\mathbb{Z}_2^{5d}\) parity of the bulk Lagrangian.
Table 2: Field contents of the fermion sector relevant for calculating the mixed CS-like term: see Table 3 for the supersymmetric setup.

| bulk fields | $SU(5)_1 \times SU(5)_2 \times SU(5)_G \times U(1)_D \times P_5$ |
|-------------|---------------------------------------------------------------|
| $\Psi^{(1)}_L(x)\Psi^{(2)}_L(x)$ | $1 \times R \times R \times qD \times +$ |
| $\Psi^{(1)}_R(x)$ | $(1, R) \times R \times qD \times +$ |
| $\Psi^{(2)}_R(x)$ | $(1, R) \times R \times qD \times -$ |

| boundary fields | localized position | $SU(5)_G \times U(1)_D \times P_{ex}$ |
|-----------------|------------------|-----------------------------------|
| $\psi^{p=0}_L(x)$ | $x^5 = 0$ | $R \times qD \times -$ |
| $\psi^{p=1}_L(x)$ | $x^5 = \pi R$ | $R \times qD \times -$ |

One may wonder whether the breaking of the $\mathbb{Z}_{2}^{5d} = \{1, P_5\}$ or the $\mathbb{Z}_{2}^{ex} = \{1, P_{ex}\}$ would be unacceptable for a consistent construction of orbifold. Actually it is not the full $\mathbb{Z}_{2}^{5d} \times \mathbb{Z}_{2}^{ex}$ invariance but its diagonal subgroup $\mathbb{Z}_{2}^{comb} = \{1, P_5P_{ex}\}$ that is required from the consistency of the theory. In other words, we require only the invariance under the simultaneous action of the $P_5$ and the $P_{ex}$.

4.1 Model: fermion sector

With the above in mind, let us present our concrete setup for generating the CS-like term (13), through anomaly inflow mechanism sketched in Appendix B. In discussing such anomaly inflow, we can focus on the gauge-fermion sector of the model, whose supersymmetric form is summarized in Appendix A.

We introduce a $\mathbb{Z}_{2}^{ex}$ pair of bulk hypermultiplets, whose fermionic components can be represented by a 5D Dirac fermion $\Psi^{(1)}_{R(1)\bar{R}}$ and its $\mathbb{Z}_{2}^{ex}$ partner $\Psi^{(2)}_{R(1)\bar{R}}$. Here $R$ is an $SU(5)$ representation such as $\bar{5}, 10$. We refer to them as messenger multiplet. These messenger fermions contain zero modes in $\Psi^{(1)}_{L(\bar{R})\bar{R}}$ and $\Psi^{(2)}_{R(\bar{R})\bar{R}}$. We also add the boundary-localized field $\psi^{p}_L(x)$ on each boundary at $x^5 = x^5_p$ that belongs to the representation $R$ of the diagonal $SU(5)$. The $U(1)_D$ charges of these fermions are commonly set to $qD$. The field contents in the fermion sector is summarized in Table 2, where quantum numbers of the boundary-localized fields are shown only for the unbroken symmetries.

The 5D Lagrangian for these fermions is given by $L_\Psi = L_\Psi^\text{bulk} + L_\Psi^\text{boundary}$, where

\begin{align}
L_\Psi^\text{bulk} &= \sum_{i=1,2} \overline{\Psi}^{(i)}(i\not{D} - \gamma_5 D_5 - m_i) \Psi^{(i)}, \\
L_\Psi^\text{boundary} &= \sum_{p=0,1} \left\{ \overline{\psi}^p \not{D} \psi^p - \sqrt{\mu_p} \left( \overline{\psi}^p \Psi^{(1)}_R + c.c. \right) \right\} 2\delta(x^5 - x^5_p).
\end{align}

\textsuperscript{9}Our Lagrangian is normalized when integrated over the interval $[0, \pi R]$. Therefore in Eq. (15), we put a factor of 2 in front of the delta functions.
Here $D_M$ is the covariant derivative with respect to $SU(5)_1 \times SU(5)_2 \times U(1)_D$, and $m_{i=1,2}$ are bulk mass parameters. With a choice $m_1 = -m_2 = m$, the bulk fermion mass terms take the form

$$-\mathcal{L}_{\text{mass}}^{\text{bulk}} = m_- \left\{ \bar{\Psi}^{(1)} \psi^{(1)} - \bar{\Psi}^{(2)} \psi^{(2)} \right\} = m_- \left\{ \bar{\Psi}^{(+)} \psi^{(-)} + \bar{\Psi}^{(-)} \psi^{(+)} \right\}. \quad (16)$$

For notational simplicity we will denote $m_-$ just by $m$, except in Appendix where a common piece $m_+$ of the bulk mass parameters is also added.

Some remarks are in order. Our choice of the bulk mass parameters, $m_1 = -m_2 = m$, explicitly breaks the $\mathbb{Z}^{ex}_2$ invariance. Note that we are considering a mass term that is constant in the fifth dimension instead of the “usual” kink mass term. Consequently the bulk messenger mass term (16) breaks the $\mathbb{Z}_2^{ex}$ as well as the $\mathbb{Z}_2^{d}$ while keeping the simultaneous $\mathbb{Z}_2^{\text{comb}}$, that is,

$$\mathbb{Z}_2^{d} \times \mathbb{Z}_2^{ex} \rightarrow \mathbb{Z}_2^{\text{comb}}. \quad (17)$$

Therefore such a bulk mass term is allowed in a consistent orbifold construction. We assume that the bulk mass term for the messenger fields is the unique source of the $\mathbb{Z}_2$ breaking (17), so that the bulk mass parameter $m$ characterizes its breaking effects. We expect that the required CS-like term, if generated successfully, is proportional to a power of this mass parameter. We will take $m$ as a free parameter.

The second remark concerns fermion zero modes. In the absence of our bulk mass term, the messenger fermions contain zero modes in $\Psi^{(+)}_L$ and $\Psi^{(-)}_R$ components. Once we add the bulk mass term, however, the would-be massless modes will acquire a mass as can be seen from the second form in Eq. (16).

The absence of the fermion zero modes will complicate the following discussion of generating the CS-like term. One can still calculate the effective action by integrating out the heavy messenger fields, which we will not do here in the present paper. Instead, we focus on the possibility of determining the CS-like term through chiral anomaly induced by fermion zero modes. This can be achieved by introducing boundary fields, $\psi^{p}_{L}$, that have bulk-boundary mixing mass terms with one of the zero modes, $\Psi^{(-)}_R$, as in Eq. (15). The total anomaly can be canceled by introducing additional fields, as we shall discuss shortly in Sect. 4.3.

We note that for the existence of a fermion zero mode, it is enough to add a single boundary fermion, $\psi^0_{L}$ at $x^5 = 0$ or $\psi^1_{L}$ at $x^5 = \pi R$, but we consider adding both for definiteness.

### 4.2 Mixed CS-like term and Dirac gaugino mass

Given the Lagrangian as above, we analyze the bulk equations of motion (EOMs) and boundary conditions to find the KK spectrum and wavefunctions. In the present paper, we confine ourselves to the limiting case where the bulk-boundary mixing masses are very large, $\mu_p \gg m$ and $\mu_p \gg 1/R$. In this case, the two zero modes are dominantly contained in $\Psi^{(1)}_L$ and $\Psi^{(2)}_L$,

$$\Psi^{(i)}_L(x^\mu, x^5) = \psi^{(i)}_{L0}(x) \xi^{(i)}_{L0}(x^5) + \cdots, \quad \xi^{(i)}_{L0}(x^5) \propto e^{m_i x^5}, \quad (18)$$
which are localized, by our choice \( m_1 = -m_2 = m \), to the opposite boundaries.

Using the profiles (18) of the zero modes, we can calculate the coefficient of the effective mixed CS-like term. We defer detailed calculations to Appendix C, using a simplified setup with \( SU(5) \) gauge groups replaced by \( U(1) \)'s. The result for \( SU(5)_1 \times SU(5)_2 \times U(1)_D \) model is given by Eq. (64). With a straightforward modification of group-theoretical factors, the result for the \( SU(5)_1 \times SU(5)_2 \times U(1)_D \) model is given by

\[
\mathcal{L}^{4d}_{CS} = \frac{2 T(R) g_G^2 q_D g_D}{16 \pi^2} \pi R f(m \pi R) \ A_5^{a(-)} F_{\mu \nu}^{a(+)} F_{\mu \nu}^{D}, \quad f(z) = \frac{1}{\tanh z} - \frac{1}{z}, \quad (19)
\]

where \( g_G \) and \( g_D \) are the 4D gauge coupling constants of \( SU(5)_G \times U(1)_D \), and \( q_D \) is the \( U(1)_D \) charge of the messenger fermion fields. The group-theoretical factor \( T(R) \) is defined by \( \text{tr}(t^a_R t^b_R) = T(R) \delta^{ab} \), using the \( SU(5) \) generators \( t^a_R \) in the representation \( R \).

The dependence on the bulk mass parameter \( m \) is contained in the function \( f(z) \), which is approximated by \( f(m \pi R) \sim (m \pi R)/3 \) for a small \( m \), while it approaches 1 for a large \( m \). Therefore the coefficient is typically of order of the inverse of the compactification scale, while it is suppressed for a small \( m \), as is expected. Thus we conclude that the mixed CS-like term responsible for the Dirac gaugino mass is actually generated in our setup of a supersymmetric gGHU model.

Now, we can match the above result to the supersoft operator (2) with a coefficient \( C_A \) for the gauge group \( G_A \). Identifying the mass scale \( \Lambda \) in Eq. (2) with the compactification scale \( M_{\text{GUT}} = 1/R \), we find that the coefficient of the supersoft operator is given by

\[
C_A = \frac{2 T(R) q_D}{16 \pi} f(m \pi R). \quad (20)
\]

We see the coefficients are universal for the SM gauge group: \( A = 3 \) for \( SU(3) \), \( A = 2 \) for \( SU(2) \) and \( A = 1 \) for \( U(1) \). This feature is specific to the present limiting case of large bulk-boundary mixing, \( \mu_\rho \gg m \) and \( \mu_\rho \gg 1/R \), where the coefficients become independent of the Wilson line \( a \): \( e^{a \pi i} = +1 \) for the \( SU(3) \) and \( e^{a \pi i} = -1 \) for the \( SU(2) \).

In passing, we give an order estimate of the resulting Dirac gaugino mass scale (3). For a rough estimate, we set gauge couplings to \( \mathcal{O}(1) \). Taking \( M_{\text{GUT}} = 1/R = \mathcal{O}(10^{16} \, \text{GeV}) \) and \( \sqrt{\langle D_D \rangle} = \mathcal{O}(10^{12} \, \text{GeV}) \) as a reference value, we have

\[
m_D \approx 10^3 \, \text{GeV} \times \left( \frac{\sqrt{\langle D_D \rangle}}{10^{12} \, \text{GeV}} \right)^2 \left( \frac{m}{10^{12} \, \text{GeV}} \right) \quad (21)
\]

for \( m \ll M_{\text{GUT}} \). Thus the gaugino mass of TeV scale is possible for a moderate choice of parameters. On the other hand, it has the upper bound for a fixed value of SUSY-breaking VEV \( \langle D_D \rangle \), since the dependence of the bulk mass parameter is saturated for \( m \lesssim M_{\text{GUT}} \). Therefore \( m_D \) can be of the intermediate scale only for a sufficiently large value of \( \langle D_D \rangle \): for instance, \( m_D = \mathcal{O}(10^{11} \, \text{GeV}) \) for \( \sqrt{\langle D_D \rangle} = \mathcal{O}(10^{14} \, \text{GeV}) \).
4.3 Comments on other anomalies and CS-like terms

Up to now, we have focused on the mixed CS-like term (11), induced from the mixed anomaly of $SU(5)^(-)SU(5)^{(+)2}U(1)_D$ spread in the bulk. Here we make some comments on other anomalies and CS-like terms.

In the above setup, other anomalies do not vanish even in the 4D effective theory, including the cubic $SU(5)^{(+)2}$ anomaly in particular. As usual, we choose the matter content so that the 4D effective gauge symmetries are anomaly free. We can always cancel the $U(1)_D$ anomalies by adding $SU(5)$-singlets, or, by a suitable choice of the $U(1)_D$ charges. For the cancellation of the $SU(5)$ anomaly, we can introduce the “vector-like” partners, $\psi^p_R(R, -, q_D)$, for the (chiral) boundary fermions. In this case, however, we have to assume that these partners have only small mixing to remaining fields, not to disturb the above discussions. A more radical, interesting possibility is to identify the boundary fields as the SM matter fields: Namely, we introduce the boundary fields in the $\bar{5}$ and 10 representations. The representation $R$ of the messenger multiplets can be either $\bar{5}$ and 10. The mixing between bulk and boundary fields could play some roles for generating the structure of the Yukawa matrices. We leave this possibility as a future work.

In any case we can choose a set of boundary matter fields so as to cancel the cubic $SU(5)^{(+)2}$ anomaly. Even after the total anomaly is canceled, the corresponding CS-like term might be generated so as to cancel the anomaly in the bulk, in a similar manner in which the desired term (11) is generated. However, such CS-like term, even if generated, decouples from the 4D effective theory since it involves at least one field that has no zero mode.

Meanwhile, there is a CS-like term that does not decouple from the low-energy,

$$\text{tr} \left[ A_5^{(-)} F^{(+)} F^{(+)} \right].$$

This term will be generated even if there is no net anomaly for $SU(5)^(-)SU(5)^{(+)2}$. Generically we expect that its effects to the low-energy effective theory will be small as it is a higher-dimensional operator. (as far as the $D$-term of $SU(5)^{(+)2}$ does not have a large VEV). There is a possible exception, however. Recall from Sect. 2 that the pseudo-scalar component of the adjoint chiral multiplet do not get a mass from the SUSY breaking. In particular the hypercharge component of $A_5^{(-)}$, being a singlet under the SM gauge group, remains massless. Since the above term (22) contains axion-like couplings of the hypercharge component of $A_5^{(-)}$ to the $SU(3) \times SU(2) \times U(1)$ gauge field strengths in the SM, it would be interesting to examine whether such a component of $A_5^{(-)}$ can play a role of an axion-like field.

5 Summary and Discussion

In this article, we have shown that the grand gauge-Higgs unification model is a good starting point for constructing the Dirac gaugino models; the light adjoint chiral superfields predicted

\footnote{This is true if the adjoint chiral multiplets have no superpotential coupling to the Higgs fields.}
in the gGHU play a role of the Dirac partner of the gauginos, and the supersoft term \( (2) \) can be obtained as a sort of the supersymmetric CS term in the 5D setup. We have presented a concrete setup of field contents and calculated the coefficient of the mixed CS-like term from the profile of chiral fermion zero modes. The same result (and some generalization) can be obtained by summing up the massive fermion KK modes, as we shall show in a separate publication.

Our model may be regarded as an extra-dimensional realization of the Goldstone gaugino scenario, proposed before as a solution to the adjoint scalar mass problem in a generic model of Dirac gauginos. In our present approach, the absence of the LT operator \( (4) \) follows directly from the 5D gauge invariance. We also note that our model based on the supersymmetric gGHU supply a natural GUT completion of the Goldstone gaugino scenario.

A nontrivial point in our construction is the \( \mathbb{Z}_2^\text{bd} \times \mathbb{Z}_2^\text{ex} \) properties of the desired CS-like term: it is not invariant under the 5D parity \( P_5 \) nor the exchange \( P_{\text{ex}} \) of the two \( SU(5) \) groups. To incorporate such \( \mathbb{Z}_2 \) breaking effect within a consistent orbifold compactification, we introduce the bulk fermion mass term \( (16) \) that is \( P_5 \)-odd and \( P_{\text{ex}} \)-odd while invariant under the simultaneous action of \( P_5 \) and \( P_{\text{ex}} \). Consequently the calculated coefficient of the mixed CS-like term, a function of the bulk mass parameter \( m \), vanishes in the limit \( m \rightarrow 0 \), and can be parametrically small since it expresses the explicit breaking \( (17) \) of the \( \mathbb{Z}_2 \) symmetries.

Specifically we calculated the coefficient though anomaly inflow induced by fermion zero modes. Since the \( \mathbb{Z}_2 \) breaking by the bulk fermion mass term removes the (would-be) fermion zero modes, we put boundary-localized fermions with bulk-boundary mixing masses \( \mu_{p=0,1} \). The boundary fields have an effect of changing the boundary conditions for the bulk fields especially when the mixing masses are large. Interestingly, the obtained CS-like term becomes independent of the Wilson line \( a \) in this limit. The reason for the \( a \)-independence is that the fermion zero modes are dominantly contained in the bulk fields, and the charge density \( \rho(x^5) \) of such zero modes is not affected by the Wilson-line phase \( e^{\text{i}ax^5/R} \). A phenomenological implication is that the Dirac gaugino masses (at the GUT scale) are predicted to be universal, that is, common for gluino, wino and bino.

Note that in the gGHU, the \( SU(5)_G \) gauge symmetry is broken by the Wilson-line VEV \( (10) \): \( e^{\text{i}a_3} = +1 \) for \( SU(3) \) and \( e^{\text{i}a_2} = -1 \) for \( SU(2) \). Therefore the above universality is not trivial at all, and is specific to the case with large bulk-boundary mixing masses \( \mu_p \). It is interesting to extend the present work to more general cases with a finite \( \mu_p \) or the case without the boundary-localized fermion, where nontrivial \( a \)-dependence is expected.

Phenomenologically it is very important whether the Dirac gaugino masses are universal or not, both in the TeV scale scenario and in the intermediate scale one that we mentioned in Introduction. For instance, to estimate the proton decay rate, we have to know the gaugino mass spectrum so as to determine the unification scale very accurately. We hope to report on this point in the near future. As for the proton decay, its rate will depend on models of the flavor, as was discussed in Ref. \[38\]; it depends on where the first generation of quarks and leptons reside in the extra dimensions. A further study on these points will be desired.

An important assumption of the present work is that the bulk mass term of the messenger multiplets is the unique source of the \( \mathbb{Z}_2 \) breaking \( (17) \). In general, once the \( \mathbb{Z}_2 \) symmetry is
broken by one sector, one could introduce a similar $\mathbb{Z}_2$-breaking effect in the other sectors. This
includes a bare CS-like term, and a $\mathbb{Z}_2^{ex} \times \mathbb{Z}_2^{5d}$-breaking mass term for the Higgs hypermultiplets.
Such term would contribute to a mass of the Higgs doublets, destabilizing the EW scale. A
clever model building will be necessary for the Higgs sector not to couple to the $\mathbb{Z}_2$-breaking
messenger sector: such coupling, if exist, should be suppressed sufficiently.

Another issue related to the discussion in Sect. 3.2 is how the $\mathbb{Z}_2$ breaking affects the correct
pattern of the Wilson-line VEV \( \langle 10 \rangle \): a slight shift would spoil the DT splitting. Actually this
does not happen as can be seen from Eq. (49): the KK mass spectrum of the bulk fermions
is symmetric under the sign flip of the Wilson line, \( a \to -a \). Consequently the Wilson-line
VEV \( \langle 10 \rangle \) is stable against the $\mathbb{Z}_2$ breaking. Note that this is true even for a finite $\mu_p$ case.

Finally we speculate about a possible origin of the proposed $\mathbb{Z}_2$-breaking by the bulk
messenger mass term. An idea is to apply the diagonal embedding method to the $U(1)_D$ gauge
group: we introduce $U(1)_{D1} \times U(1)_{D2}$ gauge group and identify its $\mathbb{Z}_2^{ex}$-even combination as
the $U(1)_D$ of the present model. By denoting the odd combination by $U(1)_M$ and its gauge
field by $B_M^{(-)}$, we suppose that the extra-dimensional component $B_M^{(-)}$, or its real scalar SUSY
partner, develops a nonvanishing VEV via some mechanism. Then such a VEV will generate
a fermion mass for $U(1)_M$-charged multiplets, but not for $U(1)_M$-neutral fields. This could
explain our assumptions, the presence of the bulk messenger mass term and the absence of
the bulk Higgs mass term. This possibility might deserve further study.

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A Note on supersymmetric Lagrangian

For completeness, we present the supersymmetric Lagrangian of our gGHU setup for generating
the mixed CS-like term, recalling the 4D superfield formalism for the 5D supersymmetric
Lagrangian \[ \text{[52]} \]. As is well-known, a 5D vector supermultiplet (in the adjoint representation)
contains a 5D vector $A_M^a$ and a real scalar $\Sigma^a$ as the bosonic part. The correspondence to
4D $N = 1$ superfields is that the scalar component of a 4D chiral superfield $\Phi^a(x^M, \theta)$ is
given by $\phi^a(x^M) = (\Sigma^a + iA_5^a) / \sqrt{2}$. We identify their zero modes with the scalar components,
$\phi^a = (\sigma^a + i\pi^a) / \sqrt{2}$, contained in Eq. (1).

For the $SU(5)_1 \times SU(5)_2$, we introduce a $\mathbb{Z}_2^{ex}$ pair of 5D vector supermultiplets, $V^{(i)}(x^M, \theta, \bar{\theta})$
and $\Phi_{ad}^{(i)}(x^M, \theta)$ \( (i = 1, 2) \), where the $i$-th fields belong to the adjoint representation of the
$SU(5)_i$. We have put the subscript “ad” to distinguish the adjoint chiral multiplets from the
hypermultiplets below. The orbifold boundary conditions at $x^5 = 0$ are given by

\[ V^{(1)}(x^\mu, -x^5, \theta, \bar{\theta}) = + V^{(2)}(x^\mu, x^5, \theta, \bar{\theta}), \quad \Phi_{ad}^{(1)}(x^\mu, -x^5, \theta) = - \Phi_{ad}^{(2)}(x^\mu, x^5, \theta), \quad (23) \]
Table 3: Field contents of the supersymmetric gGHU setup for generating the supersoft operators in the Dirac gaugino models: the gauge vector multiplets and matter chiral multiplets as well as Higgs hypermultiplets are omitted.

| bulk fields       | $SU(5)_1 \times SU(5)_2$ | $SU(5)_G$ | $U(1)_D$ | $P_5$ |
|-------------------|---------------------------|-----------|----------|-------|
| $\Phi_m^{(1)}(x^M, \theta)$ | (R, 1)                  | R         | $+q_D$   | +     |
| $\Phi_m^{(2)}(x^M, \theta)$ | (1, R)                  | R         | $+q_D$   | +     |
| $\Phi_m^{c(1)}(x^M, \theta)$ | (R̄, 1)                  | R̄        | $-q_D$   | -     |
| $\Phi_m^{c(2)}(x^M, \theta)$ | (1, R̄)                  | R̄        | $-q_D$   | -     |

| boundary fields   | localized position | $SU(5)_G$ | $U(1)_D$ | $P_{ex}$ |
|-------------------|-------------------|-----------|----------|---------|
| $\phi_{p=0}(x^\mu, \theta)$ | $x_0^5 = 0$       | R         | $+q_D$   | -       |
| $\phi_{p=1}(x^\mu, \theta)$ | $x_1^5 = \pi R$   | R         | $+q_D$   | -       |

while those at $x_1^5 = \pi R$ will be affected when we gauge away the Wilson line [10].

A.1 Supersymmetric Lagrangian for messenger multiplets

Here we focus on the supersymmetric extension of the Lagrangians given in Eqs. (14)–(15); as for the gauge sector, we will discuss the 5D supersymmetric CS term in the next subsection.

In general, a 5D bulk hypermultiplet consists of a pair of 4D $\mathcal{N} = 1$ chiral superfields,

$$
\Phi_m(x^M, \theta) = \phi_m(x^M) + \sqrt{2}\theta^a\psi_{ma}(x^M) + \cdots,
$$

$$
\Phi_m^c(x^M, \theta) = \phi_m^c(x^M) + \sqrt{2}\theta^a\psi_{ma}^c(x^M) + \cdots,
$$

where higher components in superspace coordinates have been omitted. In our case, the messenger fermion fields $\Psi^{(i)}(i = 1, 2)$ presented in Sect. 4 are contained in a $\mathbb{Z}_2$ pair of bulk hypermultiplets, $\Phi^{(i)}_m$ and $\Phi^{c(i)}_m$, with the identification

$$
\Psi^{(i)} = \begin{pmatrix} \Psi^{(i)}_L, \Psi^{(i)}_R \end{pmatrix}^T = \begin{pmatrix} \psi^{(i)}_m, \psi^{c(i)}_m \end{pmatrix}^T.
$$

Their representations under $SU(5)_1 \times SU(5)_2 \times U(1)_D$ are summarized in Table 3. At each boundary $x^5 = x_5^p = 0, 1$, we put the 4D chiral superfield $\phi_{p}$, whose fermion component is the boundary-localized fermion $\psi_{p}$. In the Table, the representations under the unbroken gauge symmetry are shown for these boundary-localized fields.

The bulk hypermultiplets satisfy the orbifold boundary conditions at $x_5^p$,

$$
\Phi^{(1)}_m(x^\mu, x_5^p - x^5, \theta) = + \Phi^{(2)}_m(x^\mu, x_5^p + x^5, \theta),
$$

$$
\Phi^{c(1)}_m(x^\mu, x_5^p - x^5, \theta) = - \Phi^{c(2)}_m(x^\mu, x_5^p + x^5, \theta),
$$

which lead to the zero modes in $\Phi^{(+)}_m$ and $\Phi^{(-)}_m$ before we add the bulk mass term and switch on the $SU(5)$-breaking by the Wilson line [10].
The supersymmetric action for the bulk messenger hypermultiplets is given by
\begin{align}
S_{\text{bulk}} &= \sum_{i=1,2} \int d^5x d^4\theta \left( \Phi^{(i)\dagger}_m e^{-V} \Phi^{(i)}_m + \Phi^{(i)}_m e^{+V} \Phi^{(i)\dagger}_m \right) \\
& \quad + \sum_{i=1,2} \int d^5x \left\{ d^2\theta \Phi^{(i)}_m \left( \tilde{D}_5 + m_i \right) \Phi^{(i)\dagger}_m + \text{c.c.} \right\},
\end{align}
where \( \tilde{D}_5 = D_5 - \Sigma/\sqrt{2} \) is the 5th component of the gauge covariant derivative that contains the real scalar \( \Sigma \). The bulk mass parameters \( m_i \) are taken to be \( m_1 = -m_2 = m \) as we discussed in Sect. 4.

The supersymmetric action that contains the boundary Lagrangian (15) is given by
\begin{align}
S_{\text{boundary}} &= \int d^4x \sum_{p=0,1} \left[ d^4\theta \phi^p e^{-V} \phi^p + \left\{ d^2\theta \sqrt{\mu_p} \phi^p \Phi^{c(-)}_m + \text{c.c.} \right\} \right]_{x^5=x^5_p}. (29)
\end{align}
In the limit of \( \mu_p \to \infty \), the bulk-boundary mixing terms force the \( \Phi^{c(i)}_m \) to obey Dirichlet boundary conditions, leaving zero modes in the \( \Phi^{(i)}_m \). We will see this explicitly in Appendix C.2.

A.2 Matching supersoft term to 5D supersymmetric CS term

Here we describe how the supersoft term (2) written in terms of 4D \( \mathcal{N} = 1 \) superspace language can be related to the bosonic CS-like term (11) in the 5D Lagrangian. For this purpose, we recall that the supersymmetric CS term in the 5D Lagrangian is given in terms of 4D \( \mathcal{N} = 1 \) superspace notation of Ref. [52] by
\begin{align}
\mathcal{L}^{5d}_{\text{CS}} &= \int d^2\theta \left\{ \sqrt{2} \Phi W_\alpha W^\alpha - \frac{2}{3} \left( \partial_5 V D_\alpha V - V D_\alpha \partial_5 V \right) W^\alpha \right\} + \text{H.c.} \\
& \quad - \int d^4\theta \frac{4}{3} \left[ \partial_5 V - \frac{1}{\sqrt{2}} \left( \Phi + \Phi^\dagger \right) \right]^3 \\
& = -\frac{1}{2} \epsilon^{LMNPQ} A_L F_{MN} F_{PQ} + \Sigma F^{MN} F_{MN} + 2 \Sigma \partial_M \Sigma \partial^M \Sigma + \cdots, (30)
\end{align}
where \( x^5 \) dependencies are implicit, and only the bosonic part is shown in the second expression. Although the above expression is for a single gauge group, it is straightforward to include mixed terms of several gauge groups, like those in the supersoft term (2).

Upon the reduction to 4D Lagrangian, there appear the same zero mode wavefunctions in the supersoft and the supersymmetric CS terms. We see that the coefficient of the 4D supersoft operator (2) can be read off from the corresponding 5D term of the form
\begin{align}
-\frac{1}{2} \epsilon^{LMNPQ} A_L F_{MN} F_{PQ} &= -A_5 F_{\mu\nu} \tilde{F}^{\mu\nu}. (31)
\end{align}
Note that there appears a factor 2 when we extend the above to the mixed CS term, as in Eq. (11), but the same is true in both hand sides of Eq. (30). We only have to take care of the normalization of the gauge couplings involved.
B Anomaly inflow and deformed Chern-Simons term

Here we recall some properties of anomalies on an $S^1/Z_2$ orbifold and the relation to the CS term [74,75]. This includes some preliminaries for a discussion in Appendix C.3. In this Appendix, we set the fermion charge to 1 for simplicity.

Consider a theory on an $S^1/Z_2$ orbifold with two fixed points $x^5_0 = 0$ and $x^5_1 = \pi R$, and suppose that the theory possesses a chiral fermion zero mode localized to one boundary at $x^5_0$. Then the anomaly calculated from the fermion zero mode is localized to $x^5_0$. On the general ground, however, it can be shown [74,75] that the anomalies should be localized equally on two boundaries $x^5_{p=0,1}$, implying that there should be an anomaly inflow from $x^5_0$ to $x^5_1$. Actually such an inflow is induced by the CS term: It contains the gauge field $A_5$ explicitly, and its gauge variation $\delta_g$, being a total derivative, results in the surface terms. These surface terms take the same form as the localized anomaly, and thus express the requisite inflow.

More generally one can consider a theory with a fermion zero mode spread in the bulk with a charge density $\rho(x^5)$. Then 4D current divergence suffers from anomaly at each 4D slice in the bulk,

$$\delta_g \Gamma_{\text{ZM}} = \int d^4x \int_{x^5_0}^{x^5_1} dx^5 \rho(x^5) \frac{g^2}{16\pi^2} \alpha(x^5) F_{\mu\nu} \tilde{F}^{\mu\nu},$$

where $\alpha$ is a transformation parameter of the fermion field. In this case, the required anomaly inflow can be induced by a term

$$\Gamma_{\text{CS}} = \int d^4x \int_{x^5_0}^{x^5_1} dx^5 u(x^5) \frac{g^3}{16\pi^2} A_5 F_{\mu\nu} \tilde{F}^{\mu\nu},$$

where $u(x^5)$ is an $x^5$-dependent coefficient function. The gauge variation, $\delta_g A_5 = (1/g) \partial_5 \alpha(x^5)$, gives, after partial integration,

$$\delta_g \Gamma_{\text{CS}} = - \int d^4x \int_{x^5_0}^{x^5_1} dx^5 \partial_5 u(x^5) \frac{g^2}{16\pi^2} \alpha F_{\mu\nu} \tilde{F}^{\mu\nu} + \int d^4x \left[ u(x^5) \frac{g^2}{16\pi^2} \alpha F_{\mu\nu} \tilde{F}^{\mu\nu} \right]_{x^5_0}^{x^5_1}.$$  

We see that the spread anomaly (32) in the bulk can be canceled if we require $\partial_5 u(x^5) = \rho(x^5)$, supplemented with the boundary conditions

$$u(x^5_1) = - u(x^5_0) = \frac{1}{2} \mathcal{A} \equiv \frac{1}{2} \int_{x^5_0}^{x^5_1} dx^5 \rho(x^5),$$

where $\mathcal{A}(=1)$ is the coefficient of a total anomaly in the 4D theory.

The CS term (33) is a position-dependent term in the 5D Lagrangian, called “deformed” Chern-Simons term [76]. In Appendix C.3, we shall discuss a similar term in the gGHU setup.

\footnote{This anomaly, induced by a single Weyl fermion, is one-half of the one induced by a Dirac fermion.}
The above discussion also makes it clear that the coefficient of CS term on $S^1/Z_2$ orbifold is not necessarily quantized, in contrast to the CS terms on a space without boundary.

The CS term on a consistent $S^1/Z_2$ orbifold should be regarded as a $P_5$-even term in the 5D Lagrangian. To see this point, we go “upstairs”, i.e., we work on the covering space $S^1$ with the coordinates $-\pi R < y < \pi R$. Then we can regard the CS term (33) as a $P_5$-even term by extending the coefficient function $u(y)$ as an odd function of $y$. For instance, a constant CS term on the $S^1/Z_2$ is actually accompanied by $u(y) = (1/2)\varepsilon(y)$, where $\varepsilon(y) = +1$ for $0 < y < \pi R$ and $\varepsilon(y) = -1$ for $-\pi R < y < 0$. The situation is the same as for the kink mass term, where the $P_5$-odd operator $\bar{\Psi}\Psi$ is accompanied by a $P_5$-odd function $m(y) = m\varepsilon(y)$.

We note also that there are two kinds of the CS terms in the gGHU setup in Sect 4. The first type of the CS terms are the usual one that are $P_{ex}$-even as well as $P_5$-even in the above sense. The other ones are $P_{ex}$-odd and $P_5$-odd CS terms. Adding a constant to the $u(y)$ over the $S^1$ corresponds to the latter terms accompanied by the $P_5$-even extension of the $u(y)$. It would be interesting if the quantization condition can be discussed for such terms.

### C Calculations in a simplified setup

In this Appendix, we show some calculations in a simplified setup: a five-dimensional $U(1)_1 \times U(1)_2 \times \mathbb{Z}_2^{ex} \times U(1)_D$ model. Table 4 shows the fermion field contents relevant for the calculation of the mixed CS-like term. We introduce a $\mathbb{Z}_2^{ex}$ pair of bulk fermions $\Psi^{(1)}(q, 0, q_D)$ and $\Psi^{(2)}(0, q, q_D)$, where the $U(1)$ charges $(Q_1, Q_2, Q_D)$ are shown. The BCs around $x_p^5 = p\pi R$ ($p = 0, 1$) are $\Psi^{(1)}(x_p^5 - x^5) = -\gamma_5\Psi^{(2)}(x_p^5 + x^5)$. When each 5D Dirac fermion is decomposed into two Weyl spinors, $\gamma_5\Psi^{(i)}_{R/L} = \pm \Psi^{(i)}_{R/L}$, the components that have zero modes are $\Psi^{(+)}_L$ and $\Psi^{(-)}_R$. In addition, a left-handed fermion $\psi^{p=0,1}_L(q, -, q_D)$ is put at each boundary. Note that for the boundary-localized fermions, only the quantum numbers of the unbroken symmetry $(U(1)_V, \mathbb{Z}_2^{ex}, U(1)_D)$ are given, since the $U(1)_A$ corresponding to $Q_1 - Q_2$ is broken there.

| bulk fields | $U(1)_1 \times U(1)_2$ | $U(1)_V \times U(1)_A$ | $U(1)_D$ | $P_3$ |
|------------|------------------------|------------------------|-----------|-------|
| $\Psi^{(1)}_L(x^M)$ | $(q, 0)$ | $(q, +q)$ | $q_D$ | + |
| $\Psi^{(2)}_L(x^M)$ | $(0, q)$ | $(q, -q)$ | $q_D$ | + |
| $\Psi^{(1)}_R(x^M)$ | $(q, 0)$ | $(q, +q)$ | $q_D$ | - |
| $\Psi^{(2)}_R(x^M)$ | $(0, q)$ | $(q, -q)$ | $q_D$ | - |

| boundary fields | localized position | $U(1)_V$ | $U(1)_D$ | $P_{ex}$ |
|-----------------|--------------------|-----------|-----------|----------|
| $\psi^{p=0}_L(x^\mu)$ | $x_0^5 = 0$ | $q$ | $q_D$ | - |
| $\psi^{p=1}_L(x^\mu)$ | $x_1^5 = \pi R$ | $q$ | $q_D$ | - |
The coefficients $c$Here we have introduced the Wilson-line phase, $\xi_{x^5}$.

The relevant bulk Lagrangian is given by Eq. (14), with a modification

$$D_M = \partial_M - ig \left( Q_1 A_M^{(1)} + Q_2 A_M^{(2)} \right) - ig_{5D} Q_D B_M$$

$$= \partial_M - i \frac{g}{\sqrt{2}} (Q_1 + Q_2) A_M^{(+)} - i \frac{g}{\sqrt{2}} (Q_1 - Q_2) A_M^{(-)} - ig_{5D} Q_D B_M,$$

where $g_{5D}$ is the gauge coupling constant of the $U(1)_D$ gauge field $B_M$. Note that the gauge couplings of $A_M^{(\pm)} = (A_M^{(1)} \pm A_M^{(2)})/\sqrt{2}$ are normalized to be $g/\sqrt{2}$. We introduce the general bulk mass terms allowed by the gauge symmetry and the $Z_2^{\text{comb}}$-twist $P^5 P_{ex}$:

$$- \mathcal{L}_\text{bulk mass} = m_+ \left\{ \bar{\Psi}^{(1)} \Psi^{(1)} + \bar{\Psi}^{(2)} \Psi^{(2)} \right\} + m_- \left\{ \bar{\Psi}^{(1)} \Psi^{(1)} - \bar{\Psi}^{(2)} \Psi^{(2)} \right\}.$$

The $P_{ex}$-even mass must have a kink profile, which we take to be a constant value $m_+$ in the fundamental region $0 < x^5 < \pi R$. On the other hand, the $P_{ex}$-odd mass $m_-$ is a constant over $S^1$. Working in the fundamental region of $S^1/Z_2$, we have $m_1 = m_+ + m_-$ and $m_2 = m_+ - m_-$. The bulk EOMs for the Weyl spinor fields $\Psi_{x=L,R}^{(i)}(x^\mu, x^5)$ are

$$i \partial_5 \Psi_{R}^{(i)} = (-D_5 + m_i) \Psi_{L}^{(i)}$$

$$i \partial_5 \Psi_{L}^{(i)} = (+D_5 + m_i) \Psi_{R}^{(i)}.$$  

We decompose these fields by using the mode functions with a 4D mass $M$, $\xi_{x=L,R}^{(i)}(x^5)$, satisfying

$$M \xi_{R}^{(i)}(x^5) = (-D_5 + m_i) \xi_{L}^{(i)}(x^5)$$

$$M \xi_{L}^{(i)}(x^5) = (+D_5 + m_i) \xi_{R}^{(i)}(x^5),$$

from which we have

$$M^2 \xi_{x}^{(i)}(x^5) = (-D_5^2 + m_i^2) \xi_{x}^{(i)}(x^5).$$

The general solutions are given by

$$\xi_{x}^{(i)}(x^5) = e^{ia x^5/R} \left( c_{\chi x}^{(i)} e^{\sqrt{m_i^2-M^2} x^5} + c_{\chi x}^{(i)} e^{-\sqrt{m_i^2-M^2} x^5} \right).$$

Here we have introduced the Wilson-line phase, $a_1 = -a_2 = a$, where

$$a \equiv R \frac{g q}{\sqrt{2}} \langle A_5^{(-)} \rangle.$$}

The coefficients $c_{Lx}^{(i)}$ and $c_{Rx}^{(i)}$ ($\ell = 1, 2$) are related with each other via the EOMs (40) and (41).

The KK spectrum is determined from the BCs, which are affected by the boundary fields. At each boundary $x^5_{p=0,1}$, we introduce a bulk-boundary mixing mass term [15], where $X^{(\pm)} \equiv (X^{(1)} \pm X^{(2)})/\sqrt{2}$. Then the EOM of $\Psi_{R}^{(-)}$ is modified into

$$i \partial_5 \Psi_{R}^{(-)} = - \partial_5 \Psi_{L}^{(-)} + \sqrt{\mu_p} \psi_{p}^{(-)} 2 \delta(x^5 - x_{p}^5) + \cdots,$$
where we omit terms finite at \( x^5 = x^5_p \). Integrating it over a tiny region \([x^5_p - \epsilon, x^5_p + \epsilon]\), we get
\[
(-1)^p \Psi_L^{(-)}(x^{5\epsilon}_p) = \sqrt{\mu_p} \psi^p_L . \quad (x^{5\epsilon}_p \equiv x^5_p + (-1)^p \epsilon)
\] (46)

Now, we find from the KK decomposition, with the help of the EOM of \( \bar{\psi}^p_L \), the \( P_5 \)-oddness of \( \Psi^{(-)}_L(x^5_p + \epsilon) = -\Psi^{(-)}_L(x^5_p - \epsilon) \), and the continuity of \( \Psi^{(-)}_R \), that
\[
(-1)^p \xi^{(-)}_L(x^{5\epsilon}_p) = \frac{\mu_p}{M} \xi^{(-)}_R(x^{5\epsilon}_p) \sim \frac{\mu_p}{M} \xi^{(-)}_R(x^{5\epsilon}_p) .
\] (47)

Later we shall treat the massless modes separately since this condition is ill-defined at \( M = 0 \). Working in the fundamental region \( 0 \leq x^5 \leq \pi R \), we take the limit \( \epsilon \to 0 \) hereafter.

We are particularly interested in the limit \( \mu_p \gg 1/R \) (and also \( \mu_p \gg m_i \)). In this large \( \mu_p \) limit, we see from Eq. (47) that the lower-lying KK modes of \( \Psi^{(-)}_R \) (those with \( M \ll \mu_p \)) effectively obey the Dirichlet BC, while the BC of \( \Psi^{(-)}_L \) is determined by the EOMs. As for \( \Psi^{(+)} \), there are no bulk-boundary mixing mass, and their BCs correspond to taking \( \mu_p = 0 \). Recalling that \( \Psi^{(+)} \) obeys the BCs opposite to those of \( \Psi^{(-)} \), we see that
\[
\Psi^{(+)}_R(x^5_p) = 0 .
\] (48)

Inserting the general solutions (43) into the four BCs (47) and (48), we have four equations which are linear and homogeneous in the four parameters \( \xi^{(i)}_{L\ell} \). Then, nontrivial solutions are obtained for specific values of the KK mass \( M \) that make the determinant of the 4\( \times \)4 coefficient matrix of the simultaneous equations vanishing. A straightforward calculation shows that the KK mass spectrum \( M = M_n \) is given by the zeros of the function,
\[
N(M; a) = \frac{8\omega_1 \omega_2}{M^2} \left[ \cos(2\pi a) - N_c(M) \right] ,
\] (49)
where we have defined \( \omega_i(M) = \sqrt{M^2 - m_i^2} \) (\( i = 1, 2 \)) and
\[
N_c(M) = \cos (\omega_1 \pi R) \cos (\omega_2 \pi R)
+ \frac{\mu_0 - \mu_1}{\omega_1} \sin (\omega_1 \pi R) \cos (\omega_2 \pi R) + \frac{\mu_0 - \mu_1}{\omega_2} \cos (\omega_1 \pi R) \sin (\omega_2 \pi R)
- \frac{(\omega_1^2 + \omega_2^2) - [2\mu_0 - (m_1 + m_2)] [2\mu_1 - (m_1 + m_2)]}{2\omega_1 \omega_2} \sin (\omega_1 \pi R) \sin (\omega_2 \pi R) .
\] (50)

The spectrum in the KK tower can be read off from the last parenthesis in Eq. (49), while the overall factor should be treated carefully, especially for \( M = 0 \), since the BCs may be ill-defined.

### C.2 Limit of large bulk-boundary mixing

Here we consider some limiting cases; we turn off the Wilson line, \( a = 0 \), for simplicity.
First, let us consider the limit \( \mu_p = 0 \) so that boundary fields decouple from the bulk ones. In this case, we can explicitly check that in the presence of the \( P_{\text{ex}} \)-odd mass term, \( m_\pm = (m_1 - m_2)/2 \neq 0 \), the bulk fermions have no massless mode. Indeed, the EOMs (10) and (11) tell us that the zero modes should have a profile
\[
\xi^{(i)}_{\ell 0}(x^5) = c^{(i)}_{L,R} e^{\pm m_i x^5}, \quad \xi^{(i)}_{R0}(x^5) = c^{(i)}_{L,R} e^{-m_i x^5}.
\] (51)
At \( x^5 = x^5_p \) \( (p = 0, 1) \), the \( P_5 \) \( P_{\text{ex}} \)-odd fields \( \Psi^{(+)}_R \) and \( \Psi^{(-)}_L \) obey the Dirichlet BCs
\[
c^{(1)}_R e^{-m_1 x^5_p} + c^{(2)}_R e^{-m_2 x^5_p} = 0, \quad c^{(1)}_L e^{m_1 x^5_p} - c^{(2)}_L e^{m_2 x^5_p} = 0.
\] (52)
which forces \( c^{(i)}_\chi = 0 \) (if \( m_1 \neq m_2 \)).

We are mainly interested in the opposite limit in which the bulk-boundary mixing masses \( \mu_p = 0, 1 \) are much larger than the compactification scale \( 1/R \) and the bulk mass parameter \( m_i \). In this limit, the BCs of \( \psi^{(-)} \) changes effectively so that both of \( \psi^{(+)}_R \) and \( \psi^{(-)}_R \) obey the Dirichlet BCs:
\[
c^{(1)}_R e^{-m_1 x^5_p} + c^{(2)}_R e^{-m_2 x^5_p} = 0, \quad c^{(1)}_L e^{m_1 x^5_p} - c^{(2)}_L e^{m_2 x^5_p} = 0,
\] (53)
which forces \( c^{(i)}_R = 0 \), but \( c^{(i)}_L \) remains unconstrained. This means that there appear two zero modes \( \xi^{(1)}_{L0} \propto e^{m_1 x^5} \) and \( \xi^{(2)}_{L0} \propto e^{m_2 x^5} \), which behave differently for \( m_1 \neq m_2 \). In particular, when \( m_1 m_2 < 0 \), namely \( |m_+| < |m_-| \), the zero mode \( \xi^{(2)}_{L0} \) is localized towards the boundary opposite to the one around which \( \xi^{(1)}_{L0} \) is localized.

In the above limit \( \mu_p \gg 1/R \) and \( \mu_p \gg |m_i| \), Eq. (19) reduces to
\[
N(M; a) \sim \frac{16 \mu_0 \mu_1}{M^2} \sin(\omega_1 \pi R) \sin(\omega_2 \pi R),
\] (54)
from which we see that the nonzero KK masses are given by \( M_n^2 = m_i^2 + (n/R)^2 \) \( (n = 1, 2, \ldots) \). We note that this is true even in the presence of the Wilson line: The KK spectrum becomes independent of the Wilson line in this large \( \mu_p \) limit.

If we consider the further limit \( |m_i| \gg 1/R \), all the nonzero KK mass \( M_n \) become much larger than the compactification scale. Then we can integrate out the KK modes within the 5D picture, which results in the effective CS-like term \( [74, 75] \). Away from such limit, it may not be appropriate to integrate out KK modes within the 5D theory; the CS-like term may be ill-defined from the 5D point of view. In the 4D effective theory, however, we can still integrate out KK modes, which results in the effective term \( [2] \) for the corresponding zero modes.

### C.3 Calculating the coefficient of CS-like term

Here we calculate the coefficient of the mixed CS-like term \( [11] \) in \( m_+ = 0 \) case. For notational simplicity we denote \( m_- \) just as \( m \) hereafter. In the simplified setup, we can directly calculate it without using Eq. (13). We define the \( U(1)^{(-)} \) current \( J_M^{(-)} \) normalized according to
\[
S_{\text{int}} = \int d^4x \int_0^{\pi R} dx^5 \frac{g}{\sqrt{2}} A_M^{(-)} J_M^{(-)} + \cdots.
\] (55)
Then, with the normalized zero mode wavefunctions

\[ \xi_{L0}^{(i)}(x^5) = \sqrt{\frac{m}{\sinh(m\pi R)}} e^{\pm m(x^5 - \frac{\pi R}{2})}, \quad (56) \]

the four-divergence of the \( U(1)^{(-)} \) current spreads in the bulk according to\(^\text{12}\)

\[ \rho(x^5) = -q |\xi_{L0}^{(1)}(x^5)|^2 + q |\xi_{L0}^{(2)}(x^5)|^2 = \frac{2qm}{\sinh(m\pi R)} \sinh \left[ 2m \left( \frac{\pi R}{2} - x^5 \right) \right]. \quad (57) \]

The mixed CS-like term is to be generated to cancel this spread anomaly.

Assuming a background gauge field homogeneous with respect to the fifth direction, let us write the requisite CS-like term in the form

\[ \Gamma_{CS} = \frac{2(qg/\sqrt{2})^2q\delta_5D}{16\pi^2} \int d^4x \int_0^{\pi R} dx^5 \rho(x^5) A_5^{(-)} F^{(+)}_{\mu\nu} \tilde{F}^{\mu\nu}_{D}(x^5), \quad (58) \]

where \( \rho(x^5) \) represents a possible \( x^5 \) dependence. Note that we have put a factor 2 for the mixed CS-like term.

The coefficient function \( \rho(x^5) \) can be determined in several ways. Our method here is inspired by the fact that the distribution \( \rho(x^5) \) is odd under the reflection around the midpoint \( x^5 = \frac{\pi R}{2} \): We first pick up \( 0 < y < \frac{\pi R}{2} \) and consider an inflow from a point \( x^5 = y \) on the one side to the point \( x^5 = \pi R - y \) on the opposite side. Such an inflow is induced by the CS-like term restricted to this interval \( [y, \pi R - y] \equiv I_y \),

\[ L_{CS}[I_y] = \int_0^{\pi R} dx^5 \Theta_y(x^5) A_5^{(-)} F^{(+)}_{\mu\nu} \tilde{F}^{\mu\nu}_{D}(x^5). \quad (59) \]

Here \( \Theta_y(x^5) \) is a rectangular support function of the interval \( I_y \) and is given by

\[ \Theta_y(x^5) = \begin{cases} 1 & \text{for } x^5 \in [y, \pi R - y] \\ 0 & \text{otherwise} \end{cases}. \quad (60) \]

Then the gauge variation \( \delta_g \) with a parameter \( \alpha(x^5) \) gives, after the partial integration, an inflow from the point \( y \) to \( \pi R - y \) as

\[ \delta_g L_{CS}[I_y] = \int_0^{\pi R} dx^5 \Theta_y(x^5) \left( \frac{\sqrt{2}}{g} \partial_5 \alpha \right) F^{(+)}_{\mu\nu} \tilde{F}^{\mu\nu}_{D}(x^5) = \frac{\sqrt{2}}{g} \left[ \alpha F^{(+)}_{\mu\nu} \tilde{F}^{\mu\nu}_{D} \right]_{x^5=y}. \quad (61) \]

Since the anomaly spreads as \( \rho(x^5) \), it is canceled by integrating Eq. (59) with the weight \( \rho(x^5) \) as

\[ \int_0^{\pi R} dy \rho(y) L_{CS}[I_y] = \int_0^{\pi R} dy \rho(y) \int_0^{\pi R} dx^5 \Theta_y(x^5) A_5^{(-)} F^{(+)}_{\mu\nu} \tilde{F}^{\mu\nu}_{D}(x^5) = \int_0^{\pi R} dx^5 q u(x^5) A_5^{(-)} F^{(+)}_{\mu\nu} \tilde{F}^{\mu\nu}_{D}(x^5). \quad (62) \]

\(^{12}\)An extra minus sign comes from \( \gamma_5 = -1 \) for left-handed zero modes.
Now, the integration can be carried out for the function $\rho(x^5)$ in Eq. (57) with the result

\[
u(x^5) = \int_0^{\frac{\pi R}{2}} dy \frac{1}{q} \rho(y) \Theta_y(x^5) = \frac{2 \sinh (mx^5) \sinh [m(\pi R - x^5)]}{\sinh(m\pi R)}. \tag{63}\]

In this way we obtain the mixed CS-like term, Eq. (58) with the coefficient function (63).

In view of the preliminary discussion given in Appendix B, an alternative way to determine $u(x^5)$ would be to solve $\partial_5 u(x^5) = \rho(x^5)/q$, with a suitable boundary condition. It appears that the absence of the surface terms requires $u(x^5) = 0$ at each boundary, since we are considering a theory with no net anomaly. This is not the case, however. Since we are considering the gauge variation of the $U(1)^(-)$, which is broken by the orbifold BCs, the parameter $\alpha(x^5)$ vanishes at the boundaries. Consequently no boundary condition is required and thus we are left with an undetermined integration constant for $u(x^5)$. Note that such an integration constant would correspond to adding a bare mixed CS-like term.

Finally we compute the corresponding term in the 4D effective theory. To this end, we keep only the zero mode part of each field. The zero mode wavefunctions, $1/\sqrt{\pi R}$, relate the 5D gauge coupling constants to the 4D ones, $g_G = g/\sqrt{2\pi R}$ for the unified gauge coupling constant and $g_D = g_5D/\sqrt{\pi R}$ for $U(1)_D$. Thus integrating the 5D term over $0 \leq x^5 \leq \pi R$ yields the 4D term

\[
L_{CS}^{4d} = \frac{2q^2 g_G^2 g_D^2 g_D}{16\pi^2} \pi R f(m\pi R) A_5^{(-)} F^{(+)}_{\mu\nu} \tilde{F}^{\mu\nu}_D(x), \tag{64}
\]

where all the fields are 4D ones, and the coefficient function $f(z)$ is defined by

\[
f(z) \equiv \frac{1}{\tanh z} - \frac{1}{z} = \left\{ \begin{array}{ll} z/3 & \text{for } z \ll 1 \\ 1 & \text{for } z \gg 1 \end{array} \right.. \tag{65}\]

We recall that the above result is obtained from the messenger fermion multiplet $\Psi(1)(q,0,q_D)$ and $\Psi(2)(0,q,q_D)$ of the $U(1)_1 \times U(1)_2 \times U(1)_D$. After generalizing to $SU(5)_1 \times SU(5)_2 \times U(1)_D$ case, we obtain the announced result (19).

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