Self-equilibrated Tapered Three-stage Tensegrity Mast

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Abstract. Investigation of tensegrity structures for the space application is ongoing owing to the characteristics of being lightweight and flexible. Tensegrity structures consist of struts and cables are self-stressed and stable under gravitational loading. Form-finding is an important process to obtain the configuration of tensegrity structures that are in self-equilibrated state. Form-finding of tensegrity structures involves a complex computational strategy in solving the geometrical and forces of the structures. This paper aims to form-finding for a tapered three-stage tensegrity mast. The form-finding strategy involves the assemblage of the tensegrity mast, establishment of equilibrium equations and determination of one possible set of coefficient beta. Several cases of configurations with various twist angles with range of 20°-40° are investigated. A configuration with 9 struts and 42 cables satisfying the material elastic condition was successfully found. The scalable self-equilibrated tensegrity mast is recommended for space applications.

1. Introduction

Tensegrity is an amazing system with characteristics of being lightweight, self-stressed, flexible and controllable. The design of tensegrity is different from the conventional ways with continuous transmission of compression. In tensegrity (Figure 1), the tensional network is assembled in order to support the floated compression. Most importantly, the structure can maintain its shape via self-equilibrium without any supports under gravitational loading. There is also a potential to convert local pressure into global deformation and subsequently seek for another balanced configuration in tensegrity.

Over the past decade, extensive investigations on tensegrity structures have been performed. Deployable and transformation capabilities of tensegrity structures have been utilized in active deployable structures (particularly for space engineering) as well as the robotic and automation community in the recent years. A design of a deployable tensegrity hollow rope footbridge with three actuation schemes were explored by Rhode-Barbarigos, et al. [1]. The crawling ability of a six-strut tensegrity robot was affirmed experimentally by Shibata, et al. [2]. Several methods to move the tensegrity robot have been highlighted: adopting shape memory alloy coil or motor driven wire as cables, pneumatic cylinder as struts or employing actuator along the cable connectivity to two specific struts.
For the space applications, deployable tensegrity in ring shape as reflectors for small satellites was suggested by Tibert and Pellegrino [3]. The tensegrity reflector has lower construction cost and higher precision in geometrical assemblage. R. Skelton recently proposed tensegrity technology in a design of a robotic system that allows growth of a habitat in space [4]. The design aims to solve several space travel problems specifically the growth, radiation protection, and the gravity problems.

There are many established form-finding strategies on tensegrity structures which are reviewed by Tibert and Pallegrino [5]. Li et al. [6] presented Monte Carlo method whereas Gan et al. [7] proposed a combination of numerical technique and genetic algorithm for form-finding of irregular tensegrity structures. Zhang and Ohsaki [8] used fictitious materials properties in a nonlinear programming approach for form-finding of tensegrity structures. Form-finding is an important process to obtain the configuration of tensegrity structures that are in self-equilibrated state. Form-finding of tensegrity structures involves a complex computation on the geometrical and forces of the structures, and the study generally remains open. This paper aims to form-finding of a tapered three-stage tensegrity mast, with a thought that the technology could be applied as space structures.

The remainder of the paper is organized as follows: Section 2 presents the search strategies for form-finding of self-equilibrated configuration of the tapered three-stage tensegrity mast. Section 3 presents form-finding results especially the self-equilibrated configuration and the axial forces of the mast. Lastly, concluding remarks are presented in Section 4.

2. Form-finding Computational Strategy
2.1. Description of Tensegrity Mast

Triangular cell is chosen for the tensegrity mast, owing to its simplicity of application. Figure 2 shows the triangular cell consisting of three struts and nine cables. Three cables connect the lower (as well as upper) end of struts to form lower (as well as upper) triangle surface. The remaining three cables connect the lower and upper end of struts vertically. The triangular cell is prestressed after the upper and lower triangles are rotated by an angle termed twist angle, alpha $\alpha$ and the above-mentioned vertical cables become diagonal cables.
Tensegrity can be classified into class 1, 2 and $k$ relying on the numbers of strut at a node. A class 1 tensegrity mast is chosen for the study. Specifically the mast is assembled with the principle of discontinuous struts connected solely by cables. The tensegrity mast with three triangular cells are assembled according to the chart as shown in Figure 3. The lower and upper triangular surfaces of $i^{th}$ ($i = 1, 2, 3$) triangular cell are denoted as $L_iB$ and $L_iT$, respectively. The triangular cell that is stacked up on the current $i^{th}$ triangular cell is denoted as the $(i+1)^{th}$ cell. All the elements are classified into specific groups as shown in Figure 3.

During the stacking of the tensegrity mast, the odd-numbered $i^{th}$ triangular cell is twisted in clockwise direction whereas the triangular cell with even-number is twisted in counterclockwise direction. The horizontal cables forming the upper and lower triangular surfaces (i.e. the interface of two triangular cells) are replaced by six saddle cables which are eventually arranged in hexagonal pattern. The height of N-stage tensegrity mast is not equal to total height of three triangular cells due to the saddle height $h$.  

**Figure 2. Triangular cell**

**Figure 3. Element connectivity**
2.2. Equilibrium Equations

The connections for the struts and cables of the tensegrity mast are assumed to be pinned connected. Hence, the equilibrium equations of a tensegrity mast could be derived based on the basic formulations of pin-jointed structures.

The tensegrity mast in this study consists of \( m \) elements, \( n \) nodes, \( n_c \) constraints and \( n_u (3n-n_c) \) unconstrained degree of freedoms. Consider an element of the tensegrity mast \( k \) connecting node \( i \) and \( j \) \((i < j)\) in a Cartesian coordinate system \( O-XYZ \). The coordinates of node \( i \) and \( j \) are denoted as \( x_i \) and \( x_j \), respectively. The length of element \( k \) is evaluated using the following expression:

\[
L_k = \sqrt{(x_j - x_i)^T (x_j - x_i)}
\]

Considering equilibrium of forces at all nodes, the following equation could be obtained:

\[
F = Bn
\]

where \( B \) is a \( n_u \times m \) matrix consisting of directional cosines \( \lambda \), \( F \) is a vector of nodal forces with size \( n_u \) and \( n \) is a vector of axial forces with size \( m \). The directional cosine \( \lambda \) of an element \( k \) is expressed:

\[
\lambda = \frac{(x_j - x_i)}{L_k}
\]

The Moore-Penrose generalized inverse is used to obtain the generalized inverse of the matrix \( B \) with rank deficiency (i.e. rank < \( n_u \)). For the case of self-equilibrated state without external forces (i.e. \( f = 0 \)), by making use of Moore-Penrose generalized inverse, the solution for the axial force vector \( n \) for the tensegrity mast is given as:

\[
n = (I_m - B^T B)\beta
\]

where \( \beta \) is a vector of arbitrary coefficient of size \( m \).

An optimization process to find one possible optimal combination of the coefficients \( \beta_i \) is performed with the following optimization problem:

\[
\min f(x) = -\beta_1 \times \beta_2 \times \ldots \times \beta_i
\]

where \( i \) is the numbers of self-stress mode.

A combination of \( \beta_i \) is used to determine the possible axial force \( n \) in an iterative way that satisfy the following inequality constraints which comes from the elastic condition of the material properties.:

\[
0 \leq n_i \leq \sigma_c A_c
\]

\[
-\frac{\pi^2 E_s I_s}{L_s^2} \leq n_s \leq \sigma_s A_s
\]

where \( n_c,\sigma_c, A_c \) are axial forces, yield stress and cross sectional area for cable elements, respectively; \( n_u, E_u, I_u, L_u, \sigma_u, A_u \) are axial forces, Young modulus, moment inertia of section, element current length, yield stress and cross sectional area for strut elements, respectively. In this study, all strut elements have circular cross sectional area.

2.3 Form-Finding Algorithm

The flowchart in Figure 4 shows the form-finding algorithm of the tapered three stage tensegrity mast. The form-finding algorithm begins with the preparation of geometrical input of the three-stage class 1 tensegrity mast with triangular cells. Equation for vector of element axial force (Equation 4) satisfying self-equilibrium condition is established. An initial estimate for coefficients \( \beta_i \) through the solution of the optimization problem as shown in Equation (5). It is noted that, only one possible combination of
coefficients $\beta_1$, $\beta_2$, and $\beta_3$ which satisfies the inequality constraints as shown in Equations (6a and b) is chosen through an iterative process. If the combination of coefficient $\beta_i$ does not satisfy Equation (6a and b), a new trial by new nodal coordinates of the tensegrity mast is given and the algorithm is repeated as shown in Figure 4. The form-finding algorithm seek for a possible configuration of three-stage tensegrity mast that is in self-equilibrated state. By using Equation 4, the initial axial forces for the tensegrity mast could be obtained with the chosen combination of the coefficients $\beta_i$.

### Figure 4. Form-finding algorithm

#### 3. Results and Discussions

This section presents the initial parameters for the form-finding process as well as the form-finding results such as the topology and the initial forces of the self-equilibrated tensegrity mast.

#### 3.1. Geometrical and Material Parameters

The tapered three-stage tensegrity mast with aspect ratio (width/height) of 1:3.3 is modelled with the base width and top width measuring 300 mm and 100 mm, respectively. The triangular cell with nominal height of 450 mm is used. The saddle depth between two triangular cells is 180 mm. The tensegrity mast of 9 struts and 42 cables is assembled. Table 1 shows the material properties for the tensegrity mast. Three self-equilibrium stress modes are obtained.

| Material properties          |  |  |  |  |
|-----------------------------|--|---|---|---|
| Elastic Modulus, $E$ (MPa)  | 200 000 |  |  |  |
| Yield Stress, $\sigma_s$ (MPa) | 250 |  |  |  |
| Density, $\gamma$ (kg/mm$^3$) | 7.85 x 10$^6$ |  |  |  |
| Strut Cross sectional Area, $A_s$ (mm$^2$) | 113.10 |  |  |  |
| Cable Cross sectional Area, $A_c$ (mm$^2$) | 19.63 |  |  |  |
3.2. Self-equilibrated Tensegrity Mast
In the form-finding of self-equilibrated tensegrity mast, twist angle of each triangular cell with range of 20°-40°, with 5° increment are investigated. Figure 5 shows the number of slackened cables after the application of various twist angles to the tensegrity mast. Significant increment in the number of slackened cables is observed in the cases where the twist angle is 30° and onwards. Since the case of the tensegrity mast with the zero slackened cables represents the solution to the self-equilibrated configuration, configuration with twist angle 25° is one of the possible tensegrity mast at self-equilibrated state.

![Figure 5. Number of slackened cables](image)

Table 2 and Figure 6 show the nodal coordinates and member connectivity for the initial and final configurations for the tensegrity mast. It is emphasized that the obtained solution for the tensegrity mast represents one possible self-equilibrated configuration, and it could be more to be discovered.

| Node | Initial configuration (Twist angle 0°) | Final configuration (Twist angle 25°) | Members | i,j, type of elements (i.e. 1 is strut, 2 is cable) |
|------|---------------------------------------|--------------------------------------|---------|--------------------------------------------------|
| 1    | -150.00, -86.60, 0.00                 | -150.00, -86.60, 0.00                | 1       | 18 35 1,6,1 5,9,2 8,11,2                        |
| 2    | 150.00, -86.60, 0.00                  | 150.00, -86.60, 0.00                | 2       | 19 36 2,4,1 9,6,2 9,12,2                        |
| 3    | 0.00, 173.21, 0.00                    | 0.00, 173.21, 0.00                  | 3       | 20 37 3,5,1 6,7,2 7,13,2                        |
| 4    | -104.55, -60.36, 450.00               | -120.26, -10.52, 450.00             | 4       | 21 38 7,12,1 7,4,2 8,14,2                       |
| 5    | 104.55, -60.36, 450.00                | 69.24, -98.89, 450.00               | 5       | 22 39 8,10,1 10,14,2 9,15,2                     |
| 6    | 0.00, 120.72, 450.00                  | 51.02, 109.41, 450.00               | 6       | 23 40 9,11,1 14,11,2 4,10,2                     |
| 7    | -122.73, 70.86, 270.00               | -81.28, 116.08, 270.00              | 7       | 24 41 13,18,1 11,15,2 5,11,2                     |
| 8    | 0.00, -141.71, 270.00                | -59.89, -128.43, 270.00             | 8       | 25 42 14,16,1 15,12,2 6,12,2                     |
| 9    | 122.73, 70.86, 270.00                | 141.17, 12.35, 270.00               | 9       | 26 43 15,17,1 12,13,2 13,16,2                     |
| 10   | -77.28, 44.61, 720.00                | -77.28, 44.61, 720.00               | 10      | 27 44 1,2,2 13,10,2 14,17,2                     |
| 11   | 0.00, -89.23, 720.00                 | 0.00, -89.23, 720.00                | 11      | 28 45 2,3,2 1,4,2 15,18,2                        |
| 12   | 77.28, 44.61, 720.00                 | 77.28, 44.61, 720.00                | 12      | 29 46 3,1,2 2,5,2 10,16,2                        |
| 13   | 0.00, 110.22, 540.00                 | 0.00, 110.22, 540.00                | 13      | 30 47 16,17,2 3,6,2 11,17,2                      |
| 14   | -95.45, -55.11, 540.00               | -95.45, -55.11, 540.00             | 14      | 31 48 17,18,2 1,7,2 12,18,2                      |
| 15   | 95.45, -55.11, 540.00                | 95.45, -55.11, 540.00               | 15      | 32 49 18,16,2 2,8,2 1,8,2                        |
3.3. Initial Axial Forces

The tensegrity mast was successfully searched when the axial forces of the mast lie between the upper and lower axial force limits (see equation 6a and b). Calculation of axial forces as in Equation 4 depends of the magnitude of the set of coefficients beta $\beta_i$ ($i=1,2,3$). Figure 7 shows the initial axial forces (pre-stressed force) of the mast with the determined value for coefficients $\beta$ are -12146, -5931, -6689. It is noted that all the cables are in tension and struts in compression. No axial forces of the mast exceed the upper axial force limits (i.e. 4610 N and -9050 N for cables and struts, respectively).

4. Concluding Remarks

This paper presents the form-finding of a tapered three-stage Class 1 tensegrity mast that is at self-equilibrated state. The effort of form-finding through various trial cases involving different twist angles are presented. The results of slackened cables are used to obtain the solution to the self-equilibrated
The determination of the coefficients $\beta$ is a challenge especially the class 1 masts are to be searched. The axial forces of the searched tensegrity mast lie within the upper and lower axial force limit which satisfy the material elastic conditions. The self-equilibrated tensegrity mast is recommended for the space applications such as deployable structures or tensegrity robots.

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