Exact renormalization-group analysis
of first order phase transitions
in clock models

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Abstract
We analyze the exact behavior of the renormalization group flow in one-dimensional
clock-models which undergo first order phase transitions by the presence of complex in-
teractions. The flow, defined by decimation, is shown to be single-valued and continuous
throughout its domain of definition, which contains the transition points. This fact is in
disagreement with a recently proposed scenario for first order phase transitions claiming
the existence of discontinuities of the renormalization group. The results are in partial
agreement with the standard scenario. However in the vicinity of some fixed points of the
critical surface the renormalized measure does not correspond to a renormalized Hamilton-
ian for some choices of renormalization blocks. These pathologies although similar to
Griffiths-Pearce pathologies have a different physical origin: the complex character of the
interactions. We elucidate the dynamical reason for such a pathological behavior: entire
regions of coupling constants blow up under the renormalization group transformation.
The flows provide non-perturbative patterns for the renormalization group behavior of
electric conductivities in the quantum Hall effect.

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The behavior of the renormalization group flow around first order transition points has been a controversial matter for years. The conventional scenario is based on a smooth behavior of the renormalization group flow and the existence of a discontinuity fixed point whose attraction domain contains the transition surface and has relevant exponents of the form $y = D_1[2]$, $D$ being the dimensionality of the system. The singularities associated with first order transitions are generated by infinite iterations of renormalization group transformations in the thermodynamic limit.

However, recently, some authors claimed to have numerical evidence of the failure of the conventional picture [3] [4]. They proposed a new scenario where the renormalization group flow is discontinuous at the first order transition surface. The ambiguity arises because the renormalization group transformation associates to any system on the transition surface as many different renormalized Gibbs measures as phases coexist at the transition.

More recently, van Enter, Fernández and Sokal [5] rigorously proved that the second picture turns to be false. Their result states that for systems with bounded fluctuating variables and absolutely summable real Hamiltonians, the renormalization group transformation, when properly defined by some blocking procedure, is single-valued and continuous in a domain of the space of parameters which includes first order transitions points. The only pathology which arises for some systems (e.g. Ising model in $D \geq 2$) is that for some renormalization group prescriptions (majority rule, decimation, etc.) the renormalization group transformation is not defined at all in a neighborhood of first order transition points, as previously pointed out by Griffiths-Pearce [6] and Israel [7]. The numerical results of ref. [3] might be understood as artifacts of the truncation of the local renormalized Hamiltonians used in the Monte Carlo renormalization group analysis [8]. The method used in ref. [4] does not rely on truncation of local interactions but assumes the existence of only one relevant perturbation at the fixed point [9] and in case of Ising model there are two: one associated to the temperature $T = 1/\beta$ and another to an external magnetic field $H$ [2][8].

In this note we analyze those issues in some exactly soluble models recently introduced by two of us [10]. The simplicity of the models allow us to analyze exact renormalization group flows in finite-dimensional coupling spaces. Although they exhibit very peculiar properties due to the presence of complex interactions, we can extract some rigorous lessons about the behavior of the renormalization group. The physical interest of those models comes from the relevance of their ($\sigma$-model) higher dimensional generalizations in statistical mechanics and quantum field theory ($\sigma$-models with similar complex couplings arise e.g. 1
in the effective description of the quantum Hall effect [11]). They can also be considered as one-dimensional versions of Chiral Potts models which have been a recent focus of interest [12].

The complex character of the interactions of the models makes possible the existence of first order phase transitions though they have only short range interactions (in fact nearest-neighbor), which is impossible for one-dimensional systems with real Hamiltonians (see ref. [13] for a recent review). Another unusual property of the models is that the correlation length at the (first-order) transition surface is infinite which implies a very peculiar behavior of finite size effects in the vicinity of the critical points [14].

Our results for the three and four-state clock models are in agreement with the conventional scenario for the renormalization group near first order critical points. The renormalization group transformation is single-valued and continuous throughout its domain of definition which contains transition points, and has discontinuity fixed points with the peculiarity that the correlation length at those fixed points is infinite ($\xi = \infty$).

However, there are some pathologies of the systems at the points of the coupling space where the renormalization group transformation is not defined. They arise because of the appearance of renormalized measures without renormalized Hamiltonians and by no means are related to the ambiguities of the discontinuous scenario. In these sense they are analogous to Griffiths-Pearce singularities. In the present case there is also a dynamical reason behind such a pathological behavior: the blow up of the corresponding renormalization group trajectories at infinite under a finite number of (fractional) iterations. The renormalization group flow generates a (local) one-parameter group of local transformations [15] but there is no global renormalization group. If we exclude the pathological region of parameters in the vicinity of first order transition points the renormalization group is globally defined. This standard dynamical behavior is the reason for such a pathological behavior. However, the physical origin of these pathologies is different from that of Griffiths-Pearce singularities. In this case is the existence of complex interactions interactions which makes possible the appearance of renormalized measures which vanish for a very large sets of configurations and cannot be described in terms of renormalized Hamiltonians. In the present models the pathological values of the couplings are non-generic and non universal. They depend on the renormalization group prescription and might be thought as artifacts of the choice of the prescription. However, we find pathologies for any prescription based on decimation.
The renormalization of systems with complex topological \( \theta \)-terms (\( \sigma \)-models, gauge theories, etc) has been always problematic from the numerical point of view, and semi-classical considerations based in instanton calculus cannot be taken for grant. The exact results obtained in our models shed some light in the possible (pathological) behavior of the renormalization group for those systems and provide a non-perturbative pattern which seems to pick up the essential behavior of \( \theta \)-terms under renormalization.

We consider the \( q \)-state clock of a classical spin variable \( \vec{s}_n \) fluctuating among the \( q \)-roots of unity,

\[
\vec{s}_n = (\cos \frac{2\pi p_n}{q}, \sin \frac{2\pi p_n}{q}) \quad p_n = 0, \ldots, q - 1
\]

in one-dimensional space with interacting Hamiltonian [10]

\[
\beta \mathcal{H} = -\sum_{n=1}^{N} (J \vec{s}_n \cdot \vec{s}_{n+1} - i\varepsilon \cdot (\vec{s}_n \times \vec{s}_{n+1}))
\]

\[
= -\sum_{n=1}^{N} \left( J \cos \frac{2\pi}{q} (p_n - p_{n+1}) + i\varepsilon \sin \frac{2\pi}{q} (p_n - p_{n+1}) \right),
\]

where \( \beta \) denotes as usual the inverse of the temperature. The eigenvalues of the transfer matrix \( T \)

\[
\lambda_k = \sum_{n=0}^{q-1} \exp \{ J \cos(2\pi n/q) + i\varepsilon \sin(2\pi n/q) - ik(2\pi n/q) \}
\]

(3)

cross each other an infinity number of times. The crossings of leading levels correspond to first order transition points with infinite correlation length [10]. In the case of 3-state clock model the three eigenvalues of the transfer matrix

\[
\lambda_0 = e^J + 2e^{-\frac{J}{2}} \cos(\theta), \quad \lambda_1 = e^J - 2e^{-\frac{J}{2}} \cos(\theta + \pi/3), \quad \lambda_2 = e^J - 2e^{-\frac{J}{2}} \cos(\theta - \pi/3)
\]

(4)

have leading level crossings at the transition temperatures \( \theta = (2m + 1)\pi/3 \), where \( \theta = \varepsilon\sqrt{3}/2 \). Single alternate decimation induces the following renormalization group transformation

\[
J' = \frac{2}{3} \log \left( \frac{e^{2J} + 2e^{-J}}{(e^{-2J} + 4e^{J} + 4e^{-J} \cos 3\theta)^{\frac{1}{2}}} \right), \quad \theta' = \arctan \frac{2e^J \sin \theta - e^{-J} \sin 2\theta}{2e^J \cos \theta + e^{-J} \cos 2\theta}.
\]

(5)

There is an additional free energy renormalization which is not relevant for the analysis of the critical behavior of correlation functions. The transformation (5) has to be understood as the projection of the renormalization group transformation on the plane of coupling constants \((J, \theta)\).
The flow is univocally defined everywhere and exhibits a very interesting dynamical behavior. There are not traces of the ambiguities associated to discontinuous scenarios. The critical surfaces $\theta = (2m + 1)\pi/3$ are invariant under renormalization group transformations as well as the subcritical ones $\theta = 2m\pi/3$. There are three kinds of fixed points (see Fig. 1)

| $F^n_\infty$ | $J = 0, \theta = 2n\pi/3$ | (stable) | infinite temperature |
|--------------|----------------------------|----------|----------------------|
| $F^\theta_0$ | $J = \infty$               | (unstable)| zero temperature     |
| $F^n_c$      | $J = \frac{2}{3}\log 2, \theta = (2n + 1)\pi/3$ | (saddle point) | transition temperature |

The third class of fixed points $F^n_c$ are sitting on the transition curves and attract all transition points. There is one relevant direction at those points which is tangent to the renormalized trajectory

$$\theta = \arccos\left(\frac{e^{3J/2}}{2}\right) + (2m + 1)\frac{\pi}{3}$$

and flows towards the infinite temperature fixed points $F^n_\infty$. The correlation length vanishes at the infinite temperature fixed point $F^n_\infty$ and becomes infinite at the points $F^n_c$ and $F^\theta_0$ of the critical surface.

The only second order critical points where the continuum limit can be obtained by scaling are the zero temperature fixed points $F^\theta_0$. The existence of a line of fixed points leads to the existence of different quantum mechanical systems [16].

Another interesting aspect of the renormalization group are the critical exponents. The linearized renormalization group tranformations around the different fixed points are

$$L_{F^n_\infty} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad L_{F^n_c} = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix} \quad L_{F^\theta_0} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$ 

Therefore, the infinite temperature fixed points $F^n_\infty$ are stable and have two irrelevant directions ($\lambda_i = 2^{y_i} = 0, i = 1, 2$).

The second order transition points at zero temperature have two marginal perturbations ($y = 0$), one along the line of fixed points $J = 0$ and another tangent to a renormalized trajectory flowing away towards the other types of fixed points.

The fixed points on the first order transition surface $F^n_c$ are saddle points with one relevant direction ($\lambda_1 = 2^{y_1} = 2$) flowing along the renormalized trajectory (6) towards the the infinite temperature fixed points $F^n_\infty$ and $F^{n+1}_\infty$ and one irrelevant direction ($\lambda_2 = 2^{y_2} = 0$) along the critical line $\theta = (2n + 1)\pi/3$. The existence of one relevant perturbation with
$y = 1$ agrees with the picture advocated by the standard scenario for discontinuity fixed points of first order transitions except for the fact that the correlation length is infinite. In general, the number of relevant perturbations with critical dimension equal to the space dimension $D = 1$ has be equal to the number of phases minus one. In this case it implies that the critical exponent $\nu = 1$.

The only amazing behavior of the renormalization group flow arises for

$$J < J_0(\theta) = \frac{2}{3} \log 2 \cos(\theta - (2m + 1)\frac{\pi}{3}).$$

(8)

where some eigenvalues of the transfer matrix $T$ become negative. One single iteration of the renormalization group transformation maps any point of this region into a point of the region $J > J_0(\theta)$ where the transfer matrix is positive ($T > 0$), and the renormalization group transformation is singular at the transition points $J = -\frac{2}{3} \log 2, \theta = (2n + 1)\pi/3$ which are mapped by (5) into $J = \infty, \theta = (2n + 1)\pi/3$. Since our decimation procedure transforms the transfer matrix $T$ into $T^2$, which is always positive the corresponding renormalization group transformation maps the points of the region $J < J_0(\theta)$ into those of the region $J > J_0(\theta)$. The renormalized trajectory (6) is the borderline between those domains.

The singularity at the points $J = \infty, \theta = (2n + 1)\pi/3$ is due to the diagonal character of $T^2$. The vanishing of non-diagonal entries implies that the renormalized measure is not Gibbsian and does not correspond to any regular renormalized Hamiltonian.

In order to analyze a more natural flow for Hamiltonians with $T \not\geq 0$ we define a renormalization group transformation by double decimation (see Fig. 2), which leaves invariant the domain where $T$ is not non-positive. The corresponding renormalization group transformation

$$J'' = \frac{2}{3} \log \frac{e^{3J} + 6 + 2e^{-3J/2} \cos 3\theta}{3(4 \cosh^2 3J/2 + 1 + 4 \cosh 3J/2 \cos 3\theta)^{1/2}}$$

$$\theta'' = \arctan \frac{2 \cosh 3J/2 \sin \theta - \sin 2\theta}{2 \cosh 3J/2 \cos \theta + \cos 2\theta}$$

(9)

exhibits a similar global (universal) behavior for the $T > 0$ region $J > J_0(\theta)$ (see Fig. 1), and it is not singular at $J = -\frac{2}{3} \log 2, \theta = (2n + 1)\pi/3$. In fact, these points become fixed (unstable) points. However, the transformation (9) is again singular at the points satisfying that

$$e^{3J} + 6 + 2e^{-3J/2} \cos 3\theta = 0,$$

(10)
which are mapped into the infinite line $J = -\infty$.

The analysis of the renormalization group flow for $e^{3J} + 6 + 2e^{-3J/2} \cos 3\theta < 0$ requires an analytic extension of the model to the complex plane of the coupling constant $J$ [17]. The only really pathological points are those of the curve (10) which can be characterized by the fact that the diagonal elements of $T^3$ vanish. In such a case the renormalized measure vanishes for a large set of clock variables configurations and cannot be described in terms of Gibbsian positive weights associated to a renormalized Hamiltonian.

Therefore, the pathological behavior of systems without non-negative transfer matrix can be improved but not completely cured by this change of the blocking procedure. However, we remark that the set of pathological Hamiltonians is non-generic in the space of couplings constants: it has codimension one or two depending on the blocking procedure.

The universal behavior for $J > J_0(\theta)$ also holds for the continuous flow associated to the transformation $T \to T^t$ which is only valid for $T > 0$. The integration of the corresponding differential equations

$$
\dot{J} = \frac{2}{3}e^{-J} \lambda_0 \log \lambda_0 + \frac{2}{3}(1 + \frac{2e^{-3J/2}}{\cos \theta}) \cot \theta \dot{\theta} - \frac{e^{J/2} + 2e^{-J} \cos \theta}{3\sqrt{3} \sin \theta} (\lambda_1 \log \lambda_1 - \lambda_2 \log \lambda_2)
$$

$$
\dot{\theta} = \frac{\sqrt{3}}{18}e^{J/2}(3 \cos \theta (\lambda_1 \log \lambda_1 - \lambda_2 \log \lambda_2) - \sqrt{3} \sin \theta (2\lambda_0 \log \lambda_0 - \lambda_1 \log \lambda_1 - \lambda_2 \log \lambda_2))
$$

(11)

yields a continuous flow with the same global characteristics and the same picture for first order transition points (see Fig. 1).

The behavior of the renormalization group is essentially changed by the presence of an external real magnetic field $H$. When $H \neq 0$ there are not first order transition points and Lee-Yang theorem holds in this case. The renormalization group can be exactly analyzed by introducing a new coupling which breaks the internal rotation symmetry of the clock model. The only first order critical points $F_c$ belong to the surface $H = 0$ and a new relevant direction appears at $F_c^n$ along the renormalized trajectory associated to the external magnetic perturbation.

In the case of the 4-state model whose critical behavior was analyzed in [10], the renormalization group transformation induced by decimation is not well defined unless we introduce a new interaction term of the form $J_1 \cos \pi (p_n - p_{n+1})$ in the Hamiltonian. Therefore we need to consider a new family of models with Hamiltonians

$$
\beta \mathcal{H}' = -\sum_{n=1}^{N} \left( J \cos \frac{\pi}{2} (p_n - p_{n+1}) + J_1 \cos \pi (p_n - p_{n+1}) + i\varepsilon \sin \frac{\pi}{2} (p_n - p_{n+1}) \right).
$$

(12)
The eigenvalues of the transfer matrix are given by
\[
\begin{align*}
\lambda_0 &= 2e^{J_1} \cosh J + 2e^{-J_1} \cos \epsilon \\
\lambda_1 &= 2e^{J_1} \sinh J + 2e^{-J_1} \sin \epsilon \\
\lambda_2 &= 2e^{J_1} \cosh J - 2e^{-J_1} \cos \epsilon \\
\lambda_3 &= 2e^{J_1} \sinh J - 2e^{-J_1} \sin \epsilon
\end{align*}
\] (13)
and crossings of leading eigenvalues correspond again to first order transition points with infinite correlation length.

An alternate single decimation defines the following renormalization group transformation
\[
\begin{align*}
J' &= \frac{1}{2} \log \frac{1 + e^{4J_1} \cosh 2J}{e^{4J_1} + \cos 2\epsilon} \\
J'_1 &= \frac{1}{4} \log \frac{(1 + e^{-4J_1} \cos 2\epsilon)(1 + e^{4J_1} \cosh 2J)}{2 \cosh 2J + 2 \cos 2\epsilon} \\
\epsilon' &= \frac{1}{2} \arccos \frac{1 + \cosh 2J \cos 2\epsilon}{\cosh 2J + \cos 2\epsilon}.
\end{align*}
\] (14)
In this case a new problem arises. The transformation (14) is not defined for couplings of the domain
\[
B_2 = \{(J, J_1, \epsilon); e^{4J_1} < -\cos 2\epsilon, \ \epsilon > \frac{\pi}{4}\},
\] (Black Hole) (15)
in terms of renormalized hamiltonians of the type (12) with real couplings \(J, J_1, \epsilon\). However, there exist an analytic extension of \(\mathcal{H}'\) to the complex planes of \(J\) and \(J_1\) that preserves the hermitian character of the transfer matrix; and the renormalized hamiltonians of the points of \(B_2\) belongs to such a class of hamiltonians [17]. Thus, the only real pathology arises for couplings in the border of the black hole
\[
\partial B_2 = \{(J, J_1, \epsilon); e^{4J_1} = -\cos 2\epsilon, \ \epsilon > \frac{\pi}{4}\}.
\]
Those pathological points are non-generic and vary with the definition of the renormalization group transformation. A common characteristic of those points is that they yield renormalized measures which vanish for a large set of clock variables configurations and cannot be described in terms of a Gibbs measure associated to a renormalized Hamiltonian.

Apart from such a pathological zone the renormalization group is well defined everywhere and none ambiguity arises at first order transition surfaces. Because of the periodicity in \(\epsilon\) it suffices to consider the region \(0 \leq \epsilon \leq \frac{\pi}{2}\). The critical surfaces in that region are
\[
\begin{align*}
e^{-J+2J_1} &= -\sqrt{2} \cos(\epsilon + \frac{\pi}{4}) \quad \frac{\pi}{4} < \epsilon < \frac{\pi}{2} \\
J &< 2J_1, \quad \epsilon = \frac{\pi}{2}
\end{align*}
\] (16a)
(16b)
and the renormalization group fixed points are listed below (see also Fig. 3)

| Fixed Point | Conditions | Temperature  |
|-------------|------------|-------------|
| $F_\infty$ | $J_1 = J = \varepsilon = 0$ | infinite temperature |
| $F_0$ | $J = J_1 = \infty$ | unstable, zero temperature |
| $F^\varepsilon_0$ | $J = \infty, J_1 = \frac{1}{4}\log\cos\varepsilon, 0 < \varepsilon < \pi/4$ | zero temperature |
| $F_c$ | $J = \frac{1}{2}\log 3, J_1 = \frac{1}{4}\log 3, \varepsilon = \pi/2$ | transition temperature |
| $F^*_c$ | $J = \infty, J_1 = -\infty, \varepsilon = \pi/4$ | zero temperature |
| $F^\dagger_c$ | $J = 0, J_1 = \infty, \varepsilon = \pi/2$ | zero temperature |
| $F^\dagger$ | $J = 0, J_1 = \infty, \varepsilon = 0$ | zero temperature |

Once again positivity provides a useful information about the renormalization group flow. The domain where the transfer matrix is positive ($T > 0$) is located below the surface

$$J_0(J_1, \varepsilon) = \max\{\text{arccosh}(e^{-2J_1}\cos\varepsilon), \text{arcsinh}(e^{-2J_1}\sin\varepsilon)\}.$$  \hspace{1cm} (17)

The points above such a surface $J < J_0(J_1, \varepsilon)$ where $T$ is not non-positive are mapped by one single iteration of the renormalization group transformation into points below the surface, $J > J_0(J_1, \varepsilon)$, by the same reason that in the case of the 3-state clock model. A double iteration would cure this anomaly but then the domain of definition of the renormalization group transformation in terms of Hamiltonians of the type (12) with real couplings $J, J_1, \varepsilon$ is smaller than that of a single alternate decimation (14). The renormalization group flow for the region where $T$ is not non-positive $J < J_0$ will be discussed in [17]. Let us concentrate here on the region with positive transfer matrix $J > J_0$. On the first order transition surface the only fixed points are $F_c, F^*_c$ and $F^\dagger_c$. Although the attraction domain of all those points contains transition points, only $F_c$ has one eigenvalue 2 ($y = 1$) of the linearized renormalization group transformation as required in the standard Nienhuis-Nauenberg picture for discontinuity fixed points. In our case it implies that the critical exponent $\nu = 1$.

The renormalization group is well defined on the domain of the transition surface with $J < J_0$. Transition points with $J < 2J_1$ in the plane $\varepsilon = \pi/2$ are attracted by $F^\dagger_c$, transition points with $J = 2J_1$ move towards the fixed point $F^*_c$ along the renormalized trajectory defined by the edge $J = 2J_1, \varepsilon = \pi/2$ of the transition surfaces (16a) and (16b), and the remainder points of the transition surface with $J > 2J_1$ flow towards the fixed point $F^*_c$. A similar behavior is observed for points below the transition surface with $J > 2J_1$, which reach the black hole domain $B_2$ after a finite number of iterations. This is
one of the reasons why the (complex) region $B_{2n}$ (black hole) for $T^{2n}$ grows with $n$. The new pathological points for $T^{2n}$ are points that already reached the region $B_2$ in less than $n$ iterations of $T^2$.

On the other hand, the continuous flow defined for systems with $T > 0$ by the equations

\[
\begin{align*}
\dot{J} &= \frac{1}{4} (\lambda_1 \log \lambda_1 + \lambda_3 \log \lambda_3) e^{J_1} \cosh J - \frac{1}{4} (\lambda_0 \log \lambda_0 + \lambda_2 \log \lambda_2) e^{J_1} \sinh J \\
\dot{J}_1 &= \frac{(\lambda_0 \log \lambda_0 + \lambda_2 \log \lambda_2)}{8e^{J_1} \cosh J} - \frac{(\lambda_0 \log \lambda_0 - \lambda_2 \log \lambda_2)}{8e^{-J_1} \cos \varepsilon} - \varepsilon \tan \frac{\varepsilon}{2} - J \tanh \frac{J}{2} \\
\dot{\varepsilon} &= \frac{e^{J_1}}{4} (\cos \varepsilon (\lambda_1 \log \lambda_1 - \lambda_3 \log \lambda_3) - \sin \varepsilon (\lambda_0 \log \lambda_0 - \lambda_2 \log \lambda_2))
\end{align*}
\]

is defined everywhere below the surface (17). In particular, it is well defined in the pathological points of $\partial B_2$. The only pathological property of the points of the domain $B_2$ is that they have been swallowed up by the line $J = \infty$, $J_1 = -\infty$, $\varepsilon \in (\pi/4, \pi/2]$ in a finite time $t < 2$. This feature explains why there is a restriction on the domain of definition of $T^2$. In fact, all the points below the transition surface will reach such a line in a finite time, which means that any given point of such a domain does not belong to the domain of $T^n$ for $n$ large enough. The points at the transition surface (16a) with $J > 2J_1$, $J > J_0$ are also attracted by the fixed point $F^*_c$ but in an infinite time (The intersection curve of the critical surface and $J = J_0$ is the renormalized trajectory joining $F^*_c$ and $F_c$ (see Fig. 3)). Therefore, the renormalization group transformation is well defined in such points for any kind of decimation blocking.

The pathologies observed so far in the definition of the renormalization group transformation can be then understood as a pure consequence of the dynamical behavior of a renormalization group flow with a very strong attractive domain. For any continuous flow with such a strong domain attractor the corresponding (local) one-parameter group of local transformations cannot be implemented to a global group of transformations [15]. In the present case the transformation $T^t$ cannot be globally defined for any value of $t \neq 0$. However, if we exclude the region below the transition surface (16a) the renormalization group flow defines a global one-parameter group of transformations for any value of $t \in \mathbb{R}$.

In summary, besides the above dynamical system explanation, we have shown three sources of pathological behavior of the renormalization group, all of them motivated by the complex nature of the interacting terms of the Hamiltonian:
i) The presence of negative eigenvalues in the transfer matrix yields an odd behaviour under decimation renormalization group transformations which generates pathologies. In particular, a continuum renormalization group cannot be implemented in that case.

ii) In the 4-states model there are some values of the parameters (region $B_2$) where the transfer matrix is positive but the renormalization group is not defined. This can happen because the correspondence between eigenvalues of $T^2$ and the parameters of the renormalized Hamiltonian is not analytic.

iii) It is possible to extend the domain of definition of the renormalization group transformation by analytic continuation in the space of coupling parameters [17]. In this way the set of pathological Hamiltonians can be reduced to non-generic hypersurfaces with non-trivial codimension in the space of coupling parameters (e.g. the boundary of $B_2$).

In any case, although the Hamiltonians of these models are complex the Boltzmann weights can be paired to give rise to real contributions. The main difference with standard systems with real couplings is that some of the weights might vanish. However, the configurations with vanishing contributions are non-generic and should not be confused with the pathological renormalized measures, where the configurations with vanishing contributions become generic (e.g. the transfer matrix has null all non-diagonal entries).

One might conjecture that the pathological behavior of renormalization group transformation observed in higher dimensional systems is generated by similar dynamical properties. In fact, the same behavior occurs in the renormalization group flow of the $q$-state clock models with Hamiltonians

$$
\beta H' = - \sum_{n=1}^{N,[q/2]} \sum_{r=1}^q \left( J_r \cos \frac{2r\pi}{q} (p_n - p_{n+1}) + i\varepsilon_r \sin \frac{2r\pi}{q} (p_n - p_{n+1}) \right)
$$

which are the natural generalization of the systems (2).

Finally, we remark that the renormalization group flow of the $q = 3$ model (Fig. 1) is similar to the expected flow of dissipative $\sigma_{xx}$ and Hall $\sigma_{xy}$ conductivities in the quantum Hall effect when described in terms of a $\sigma$-model [18]. Although the clock model can be considered as a one-dimensional discretization of such a $\sigma$-model, it was not expected to belong to the same renormalization class and provide an exact non-perturbative pattern for the renormalization group behavior of those conductivities (We thank Alexei Morozov for pointing out such a connection).

The most remarkable property of this pattern is that in the thermodynamic limit the dissipative conductivity would vanish and the Hall conductivity would be quantized.
as it is observed in the quantum Hall effect (see fig.1 of the first and third papers of Ref. [18]). These facts raise the conjecture that a solution of the strong CP-problem of QCD might be found in a similar way in terms of a renormalization flow of the $\theta$ parameter towards quantized (unobservable) values $\theta = 2\pi n$. In the continuum limit all values of $\theta$ are consistent but in a given (perhaps fundamental) discretisation the topological CP-breaking couplings flow in the infrared to the quantized CP-preserving values $2\pi n$.

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Figure Captions

Fig. 1.: Renormalization group flow for the 3-state model. The renormalization group transformation is universal for $J > J_0(J_1, \varepsilon)$ ($T > 0$). The renormalization group trajectories are integral curves of the continuous flow defined by equation (11). Besides the zero temperature fixed points $F_0^\theta$, which are not displayed in the picture, there are two types of fixed points: infinite temperature fixed points $F_\infty^n (J = 0, \theta = 2n\pi/3)$, and discontinuity fixed points $F_c^n (J = \frac{2}{3} \log 2, \theta = (2n+1)\pi/3)$. The later behave as attractors for the transition points on the lines $\theta = (2n+1)\pi/3$. In the domain $J < J_0(J_1, \varepsilon)$ where $T < 0$, the renormalization group is non-universal. In fact, it becomes singular at the points $F_n$ with $J = -\frac{2}{3} \log 2, \theta = (2n+1)\pi/3$ and the curve $e^{3J} + 6 + 2e^{-3J/2} \cos 3\theta = 0$ (dashed curve) for single and double decimations, respectively.

Fig. 2.: Single and double decimations.

Fig. 3.: Renormalization group flow for 4-state model. Shadowed surfaces correspond to transition points satisfying the positivity condition $T > 0$. For simplicity, we only show the spectrum of fixed points and renormalized trajectories of the renormalization group transformation. The boundary of the domain $B_2$, $\partial B_2 = \{(J, J_1, \varepsilon), e^{4J_1} = -\cos 2\varepsilon, \varepsilon > \frac{\pi}{4}\}$ where the renormalization group transformation is not defined can be seen on the left back corner of the $(\varepsilon, e^{-4J_1}, 0)$ plane.