Calibration and Consistency of Adversarial Surrogate Losses

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Abstract

Adversarial robustness is an increasingly critical property of classifiers in applications. The design of robust algorithms relies on surrogate losses since the optimization of the adversarial loss with most hypothesis sets is NP-hard. But which surrogate losses should be used and when do they benefit from theoretical guarantees? We present an extensive study of this question, including a detailed analysis of the $\mathcal{H}$-calibration and $\mathcal{H}$-consistency of adversarial surrogate losses. We show that, under some general assumptions, convex loss functions, or the supremum-based convex losses often used in applications, are not $\mathcal{H}$-calibrated for important hypothesis sets such as generalized linear models or one-layer neural networks. We then give a characterization of $\mathcal{H}$-calibration and prove that some surrogate losses are indeed $\mathcal{H}$-calibrated for the adversarial loss, with these hypothesis sets. Next, we show that $\mathcal{H}$-calibration is not sufficient to guarantee consistency and prove that, in the absence of any distributional assumption, no continuous surrogate loss is consistent in the adversarial setting. This, in particular, proves that a claim presented in a COLT 2020 publication is inaccurate.\(^1\) Next, we identify natural conditions under which some surrogate losses that we describe in detail are $\mathcal{H}$-consistent for hypothesis sets such as generalized linear models and one-layer neural networks. We also report a series of empirical results with simulated data, which show that many $\mathcal{H}$-calibrated surrogate losses are indeed not $\mathcal{H}$-consistent, and validate our theoretical assumptions.

Keywords: Adversarial Robustness, Surrogate Losses, Calibration, Consistency.

1. Introduction

Complex multi-layer artificial neural networks trained on large datasets have been shown to form accurate learning models which have achieved a remarkable performance in several applications in recent years, in particular in speech and visual recognition tasks (Sutskever et al., 2014; Krizhevsky et al., 2012). However, these rich models are susceptible to imperceptible perturbations (Szegedy et al., 2013). A complex neural network may, for example, misclassify a traffic sign, as a result of a minor variation, which may be the presence of a small advertisement sticker on the sign. Such misclassifications can have dramatic consequences in practice, for example, for self-driving cars. These concerns have motivated the study of adversarial robustness, that is the design of classifiers

\(^1\) Calibration results there are correct modulo subtle definition differences, but the consistency claim does not hold.
that are robust to small $\ell_p$ norm input perturbations (Goodfellow et al., 2014; Madry et al., 2017; Tsipras et al., 2018; Carlini and Wagner, 2017). The standard 0/1 loss is then replaced with a more stringent adversarial loss, which requires a predictor to correctly classify an input point $x$ and also to maintain the same classification for all points at a small $\ell_p$ distance of $x$. But, can we devise efficient learning algorithms with theoretical guarantees for the adversarial loss?

Designing such robust algorithms requires resorting to appropriate surrogate losses as optimizing the adversarial loss is NP-hard for most hypothesis sets. A key property for surrogate adversarial losses is their consistency, that is, that exact or near optimal minimizers of the surrogate loss are also exact or near optimal minimizers of the original adversarial loss. The notion of consistency has been extensively studied in the case of the standard 0/1 loss or the multi-class setting (Zhang, 2004; Bartlett et al., 2006; Tewari and Bartlett, 2007; Steinwart, 2007). However, those results or proof techniques cannot be used to establish or characterize consistency in adversarial settings. This is because the adversarial loss of a predictor $f$ at point $x$ is inherently not just a function of $f(x)$ but also of its values around a neighborhood of $x$. As we shall see, the study of consistency is significantly more complex in the adversarial setting, with subtleties that have in fact led to some inaccurate claims made in prior work that we discuss later.

Consistency requires a property of the surrogate and the original losses to hold true for the family of all measurable functions. As argued by Long and Servedio (2013), the notion of $\mathcal{H}$-consistency which requires a similar property for the surrogate and original losses, but with the near or optimal minimizers considered on the restricted hypothesis set $\mathcal{H}$, is a more relevant and desirable property for learning. Long and Servedio (2013) gave examples of surrogate losses that are not $\mathcal{H}$-consistent when $\mathcal{H}$ is the class of all measurable functions but satisfy a condition namely, realizable $\mathcal{H}$-consistency when $\mathcal{H}$ is the class of linear functions. More recently, Zhang and Agarwal (2020) studied the notion of improper realizable $\mathcal{H}$-consistency of linear classes where the surrogate $\phi$ can be optimized over a larger class such as that of piecewise linear functions. Note that these works concern the standard 0/1 classification loss.

This motivates our main objective: an extensive study of the $\mathcal{H}$-consistency of adversarial surrogate losses, which is critical to the design of robust algorithms with guarantees in this setting. A more convenient notion in the study of $\mathcal{H}$-consistency is that of $\mathcal{H}$-calibration, which is a related notion that involves conditioning on the input point. $\mathcal{H}$-calibration often is a sufficient condition for $\mathcal{H}$-consistency in the standard classification settings (Steinwart, 2007). However, the adversarial loss presents new challenges and requires carefully distinguishing among these notions to avoid drawing false conclusions. As an example, the recent COLT 2020 paper of Bao et al. (2020), which presents a study of $\mathcal{H}$-calibration for the adversarial loss in the special case where $\mathcal{H}$ is the class of linear functions, concludes that the $\mathcal{H}$-calibrated surrogates they propose are $\mathcal{H}$-consistent. This is falsified as a by-product of our results, which further suggests that the adversarial setting is more complex and requires a more delicate analysis. At the same time, our work is inspired by the work of Bao et al. (2020) where the author propose a natural robust loss function and studied calibration and consistency of surrogates for optimizing it. However, the proposed loss function corresponds to the adversarial 0/1 loss only when the class $\mathcal{H}$ of functions comprises of linear classifiers. We on the other hand directly study the adversarial 0/1 loss and for hypothesis sets beyond linear classifiers.

In Section 3, we give a detailed analysis of the $\mathcal{H}$-calibration properties of several natural surrogate losses. We present a series of new negative results showing that, under some general assumptions, convex loss functions and supremum-based convex losses, that is losses defined as the supremum over a ball of a convex function, which are those commonly used in applications, are not
functions, including the calibrated supremum-based $\rho$ denoted by $B$. We will denote vectors as lowercase bold letters (e.g. $\mathbf{x}$).

2. Preliminaries

We will denote vectors as lowercase bold letters (e.g. $\mathbf{x}$). The $d$-dimensional $l_2$-ball with radius $r$ is denoted by $B_d^2(r) = \{ \mathbf{z} \in \mathbb{R}^d | \| \mathbf{z} \|_2 \leq r \}$. We denote by $\mathcal{X}$ the set of all possible examples. $\mathcal{X}$ is also sometimes referred to as the input space. The set of all possible labels is denoted by $\mathcal{Y}$. We will limit ourselves to the case of binary classification where $\mathcal{Y} = \{-1, 1\}$. Let $\mathcal{H}$ be a family of functions from $\mathbb{R}^d$ to $\mathbb{R}$. Given a fixed but unknown distribution $\mathcal{D}$ over $\mathcal{X} \times \mathcal{Y}$, the binary classification learning problem is then formulated as follows. The learner is asked to select a classifier $f^* \in \mathcal{H}$ that has the minimal generalization error with respect to the distribution $\mathcal{D}$. The generalization error of a classifier $f \in \mathcal{H}$ is defined by $R_{\ell_0}(f) = \mathbb{E}_{(x,y) \sim \mathcal{D}}[\ell_0(f(x), y)]$, where $\ell_0(f(x), y) = 1_{yf(x) \leq 0}$ is the standard $0/1$ loss. More generally, the $\ell$-risk of a classifier $f$ for a surrogate loss $\ell(f, x, y)$ is defined by

$$R_\ell(f) = \mathbb{E}_{(x,y) \sim \mathcal{D}}[\ell(f(x), y)].$$

Moreover, the minimal $(\ell, \mathcal{H})$-risk, which is also called the Bayes $(\ell, \mathcal{H})$-risk, is defined by $R_{\ell, \mathcal{H}}^* = \inf_{f \in \mathcal{H}} R_\ell(f)$. Our goal is to understand whether the minimization of the $\ell$-risk can lead to that of the generalization error. This motivates the definition of $\mathcal{H}$-consistency (or simply consistency) stated below.

**Definition 1 ($\mathcal{H}$-Consistency)** Given a hypothesis set $\mathcal{H}$, we say that a loss function $\ell_1$ is $\mathcal{H}$-consistent with respect to loss function $\ell_2$, if the following holds:

$$R_{\ell_1}(f_n) - R_{\ell_1, \mathcal{H}}^* \xrightarrow{n \to +\infty} 0 \implies R_{\ell_2}(f_n) - R_{\ell_2, \mathcal{H}}^* \xrightarrow{n \to +\infty} 0,$$

in Section 4, we study the $\mathcal{H}$-consistency of surrogate loss functions. We prove that, in the absence of distributional assumptions, many surrogate losses shown to be $\mathcal{H}$-calibrated in Section 3 are in fact not $\mathcal{H}$-consistent. Next, in contrast, we show that when the minimum of the surrogate loss is achieved within $\mathcal{H}$, under some general conditions, the $\rho$-margin ramp loss (see, for example, Mohri et al. (2018)) is $\mathcal{H}$-consistent for $\mathcal{H}$ being the linear hypothesis set, any non-decreasing and continuous $g$-based hypothesis set, or the ReLU-based function class. We then give similar $\mathcal{H}$-consistency guarantees for supremum-based surrogate losses based on a non-increasing auxiliary function, including the calibrated supremum-based $\rho$-margin ramp loss when $\mathcal{H}$ is the family of one-layer neural networks.

In Section 5, we report a series of empirical results on simulated data, which show that many $\mathcal{H}$-calibrated surrogate losses are indeed not $\mathcal{H}$-consistent, and justify our realizability assumptions. Overall, our analysis suggests that surrogate losses typically used in practice do not benefit from any guarantee and that minimizing such losses may not in fact lead to a more favorable adversarial loss. They also provide alternative surrogate losses with theoretical guarantees that can be useful to the design of algorithms in this setting.

We give a more detailed discussion of related work in Appendix A. We start with an introduction of some notation and key definitions (Section 2).
for all probability distributions and sequences of \( \{ f_n \}_{n \in \mathbb{N}} \subset \mathcal{H} \).

In the rest of the paper, the loss \( \ell_2 \) in the definition above will correspond to the 0/1 loss or the adversarial 0/1 loss depending on the context, \( \ell_1 \) to a surrogate loss for \( \ell_2 \). For a distribution \( \mathcal{P} \) over \( \mathcal{X} \times \mathcal{Y} \) with random variables \( X \) and \( Y \), let \( \eta_\mathcal{P} : \mathcal{X} \to [0,1] \) be a measurable function such that, for any \( x \in \mathcal{X}, \eta_\mathcal{P}(x) = \mathcal{P}(Y=1 \mid X=x) \). By the property of conditional expectation, we can rewrite (1) as \( \mathcal{R}_\ell(f) = \mathbb{E}_X[\mathcal{C}_\ell(f, x, \eta_\mathcal{P}(x))] \), where \( \mathcal{C}_\ell(f, x, \eta) \) is the generic conditional \( \ell \)-risk (or inner \( \ell \)-risk) defined as followed:

\[
\forall x \in \mathcal{X}, \forall \eta \in [0,1], \quad \mathcal{C}_\ell(f, x, \eta) = \eta \ell(f, x, +1) + (1 - \eta) \ell(f, x, -1).
\]

Moreover, the minimal inner \( \ell \)-risk on \( \mathcal{H} \) is denoted by \( \mathcal{C}_{\ell, \mathcal{H}}^*(f, x, \eta) = \inf_{f \in \mathcal{H}} \mathcal{C}_\ell(f, x, \eta) \). We also define, overloading the notation, the pseudo-minimal inner \( \ell \)-risk \( \mathcal{C}_{\ell, \mathcal{H}}^{**}(\eta) = \inf_{f \in \mathcal{H}, x \in \mathcal{X}} \mathcal{C}_\ell(f, x, \eta) \).

For convenience, we denote \( \Delta \mathcal{C}_{\ell, \mathcal{H}}(f, x, \eta) = \mathcal{C}_\ell(f, x, \eta) - \mathcal{C}_{\ell, \mathcal{H}}^*(\eta) \).

The notion of calibration for the inner risk is often a powerful tool for the analysis of \( \mathcal{H} \)-consistency (Steinwart, 2007). In this paper, we consider a uniform version of the notion of calibration.

**Definition 2 (Uniform \( \mathcal{H} \)-Calibration)** [Definition 2.15 in (Steinwart, 2007)] Given a hypothesis set \( \mathcal{H} \), we say that a loss function \( \ell_1 \) is uniformly \( \mathcal{H} \)-calibrated with respect to a loss function \( \ell_2 \) if, for any \( \epsilon > 0 \), there exists \( \delta > 0 \) such that for all \( \eta \in [0,1] \), \( f \in \mathcal{H}, x \in \mathcal{X} \), we have

\[
\mathcal{C}_{\ell_1}(f, x, \eta) < \mathcal{C}_{\ell_2, \mathcal{H}}^*(x, \eta) + \delta \implies \mathcal{C}_{\ell_2}(f, x, \eta) < \mathcal{C}_{\ell_2, \mathcal{H}}^*(x, \eta) + \epsilon.
\]

Steinwart (2007, Remark 2.14) points out that the excess risk of a surrogate loss \( \ell_1 \) can be upper bounded in terms of the excess risk of target loss \( \ell_2 \) with a function that is independent of the specific distribution \( \mathcal{P} \) if \( \ell_1 \) is uniformly calibrated with respect to \( \ell_2 \) under certain conditions. For convenience of proofs, we also introduce the Uniform Pseudo-\( \mathcal{H} \)-Calibration from Bao et al. (2020).

**Definition 3 (Uniform Pseudo-\( \mathcal{H} \)-Calibration)** [Definition 2 in (Bao et al., 2020)] Given a hypothesis set \( \mathcal{H} \), we say that a loss function \( \ell_1 \) is uniformly pseudo-\( \mathcal{H} \)-calibrated with respect to a loss function \( \ell_2 \) if, for any \( \epsilon > 0 \), there exists \( \delta > 0 \) such that for all \( \eta \in [0,1] \) and \( f \in \mathcal{H}, x \in \mathcal{X} \), we have

\[
\mathcal{C}_{\ell_1}(f, x, \eta) < \mathcal{C}_{\ell_2, \mathcal{H}}^*(\eta) + \delta \implies \mathcal{C}_{\ell_2}(f, x, \eta) < \mathcal{C}_{\ell_2, \mathcal{H}}^*(\eta) + \epsilon.
\]

Although the only difference between (4) and (5) is the definition of minimal inner risk: \( \mathcal{C}_{\ell_2, \mathcal{H}}^*(\eta) \) and \( \mathcal{C}_{\ell_2, \mathcal{H}}^*(x, \eta) \), in general, uniform pseudo-\( \mathcal{H} \)-calibration does not imply \( \mathcal{H} \)-consistency. However, as shown in Section 3, for the appropriate hypothesis sets \( \mathcal{H} \) and losses considered in this paper, the two definitions coincide, that is, for any \( x \in \mathcal{X} \), \( \mathcal{C}_{\ell_2, \mathcal{H}}^*(x, \eta) = \mathcal{C}_{\ell,H}^*(\eta) \) when \( \ell = \ell_1 \) and \( \ell_2 \), and thus we can make use of Definition 3 in the proofs. For simplicity, we are referring to Definition 2 (or Definition 3), when we later write \( \mathcal{H} \)-Calibration and \( \mathcal{H} \)-calibrated (or Pseudo-\( \mathcal{H} \)-Calibration and pseudo-\( \mathcal{H} \)-calibrated).

Steinwart (2007) points out that if \( \ell_1 \) is \( \mathcal{H} \)-calibrated with respect to \( \ell_2 \), then \( \mathcal{H} \)-consistency, that is condition (2), holds for any probability distribution verifying the additional condition of \( \mathcal{P} \)-minimizability (Steinwart, 2007, Definition 2.4). This result holds, in fact, under more general assumptions, as we will show later. Next, we introduce the notions of uniform calibration function (Steinwart, 2007), and uniform pseudo-calibration function.
In this paper, we aim to characterize surrogate losses satisfying $H$ with $\ell_2$.

Similarly, we define the adversarial generalization error and, similarly, the uniform pseudo-calibration function gives the maximal $\delta$.

Robust Classification. For simplicity, we are referring to Definition 4, when we later write calibration function or pseudo-calibration function.

Proposition 5 (Lemma 2.16 in (Steinwart, 2007)) Given a hypothesis set $H$, loss $\ell_1$ is uniformly $H$-calibrated (or uniformly pseudo-$H$-calibrated) with respect to $\ell_2$ if and only if its uniform calibration function $\delta$ satisfies $\delta(\epsilon) > 0$ (resp. its uniform pseudo-calibration function $\hat{\delta}(\epsilon) > 0$) for all $\epsilon > 0$.

For simplicity, we are referring to Definition 4, when we later write calibration function or pseudo-calibration function.

Robust Classification. In adversarially robust classification, the loss at $(x, y)$ is measured in terms of the worst loss incurred over an adversarial perturbation of $x$ within a ball of a certain radius in a norm. In this work we will consider perturbations in the $l_2$ norm $\| \cdot \|$. We will denote by $\gamma$ the maximum magnitude of the allowed perturbations. Given $\gamma > 0$, a data point $(x, y)$, a function $f \in H$, and a margin-based loss $\phi : \mathbb{R} \to \mathbb{R}_+$, we define the adversarial loss of $f$ at $(x, y)$ as

$$\hat{\phi}(f, x, y) = \sup_{x' : \|x - x'\| \leq \gamma} \phi(yf(x')).$$

The above naturally motivates supremum-based surrogate losses that are commonly used to optimize the adversarial 0/1 loss (Goodfellow et al., 2014; Madry et al., 2017; Shafahi et al., 2019; Wong et al., 2020). We say that a surrogate loss $\hat{\phi}(f, x, y)$ is supremum-based if it is of the form defined in (7). We say that the supremum-based surrogate is convex if the function $\phi$ in (7) is convex. When $\phi$ is non-increasing, the following equality holds (Yin et al., 2019):

$$\sup_{x' : \|x - x'\| \leq \gamma} \phi(yf(x')) = \phi\left( \inf_{x' : \|x - x'\| \leq \gamma} yf(x') \right).$$

Next we define the adversarial 0/1 loss as

$$\ell_\gamma(f, x, y) = \sup_{x' : \|x - x'\| \leq \gamma} \mathbb{1}_{yf(x') \leq 0} = \mathbb{1}_{\inf_{x' : \|x - x'\| \leq \gamma} yf(x') \leq 0}.$$

Similarly, we define the adversarial generalization error and the Bayes ($\ell_\gamma, H$)-risk as

$$\mathcal{R}_{\ell_\gamma}(f) = \mathbb{E}_{(x, y) \sim p} [\ell_\gamma(f, x, y)] \quad \text{and} \quad \mathcal{R}_{\ell_\gamma, H} = \inf_{f \in H} \mathcal{R}_{\ell_\gamma}(f).$$

In this paper, we aim to characterize surrogate losses satisfying $H$-consistency and $H$-calibration with $\ell_2 = \ell_\gamma$ and for the following natural hypothesis sets $H$:
Assume that for any \( \gamma \) and \( X \) calibrated under certain general conditions. Without loss of generality, in this section, we assume results with positive ones by identifying a family of quasi-concave even functions that are indeed calibrated under some broad assumptions. Next, we give a series of negative results showing that, under general assumptions, convex losses and supremum-based convex losses, which are typically used in practice for adversarial robustness, are not calibrated. We then complement these results with positive ones by identifying a family of quasi-concave even functions that are indeed calibrated under certain general conditions. Without loss of generality, in this section, we assume the input space to be \( \mathcal{X} = B^2_2(1) \) and \( \gamma \in (0, 1) \). Specifically, we assume the input space to be \( \mathcal{X} = \{ x \in \mathbb{R}^d \mid \gamma < \|x\|_2 \leq 1 \} \) when considering one-layer ReLU neural networks \( \mathcal{H}_{\text{NN}} \).

3. \( \mathcal{H} \)-Calibration

Calibration is a condition often used to prove consistency and is typically a first step in analyzing surrogate losses. Thus, in this section, we first present a detailed study of the calibration properties of several loss functions. We first prove the equivalence of \( \mathcal{H} \)-calibration and pseudo-\( \mathcal{H} \)-calibration under some broad assumptions. Next, we give a series of negative results showing that, under general assumptions, convex losses and supremum-based convex losses, which are typically used in practice for adversarial robustness, are not calibrated. We then complement these results with positive ones by identifying a family of quasi-concave even functions that are indeed calibrated under certain general conditions. Without loss of generality, in this section, we assume the input space to be \( \mathcal{X} = B^2_2(1) \) and \( \gamma \in (0, 1) \). Specifically, we assume the input space to be \( \mathcal{X} = \{ x \in \mathbb{R}^d \mid \gamma < \|x\|_2 \leq 1 \} \) when considering one-layer ReLU neural networks \( \mathcal{H}_{\text{NN}} \).

3.1. Equivalence of calibration definitions

We first show that the definitions of \( \mathcal{H} \)-calibration and pseudo-\( \mathcal{H} \)-calibration coincide for the hypothesis sets and losses considered in the paper.

**Theorem 6** [Equivalence of calibration definitions] Without loss of generality, let \( \mathcal{X} = B^2_2(1) \) and \( \gamma \in (0, 1) \). Then,

1. If \( \mathcal{H} \) satisfies: for any \( x \in \mathcal{X} \), there exists \( f \in \mathcal{H} \) such that \( \inf_{x' \mid \|x-x'\| \leq \gamma} f(x') > 0 \), and \( f \in \mathcal{H} \) such that \( \sup_{x' \mid \|x-x'\| \leq \gamma} f(x') < 0 \), then, for any \( x \in \mathcal{X} \), \( C_{\ell, \mathcal{H}, \gamma}(x, \eta) = C_{\ell, \mathcal{H}, \gamma}(\eta) \).

2. Let \( \phi \) be a margin-based loss. If \( \mathcal{H} \) satisfies: for any \( x \in \mathcal{X} \), \( \{ f(x) \mid f \in \mathcal{H} \} = \mathbb{R} \), then, for any \( x \in \mathcal{X} \), \( C_{\phi, \mathcal{H}, \gamma}(x, \eta) = C_{\phi, \mathcal{H}, \gamma}(\eta) \).

3. Let \( \phi_\rho(\ell) = \min\{1, \max\{0, 1 - \frac{\ell}{\rho}\}\} \) for a fixed \( \rho > 0 \) be the \( \rho \)-margin loss and \( \phi_\rho(f, x, y) = \sup_{x' \mid \|x-x'\| \leq \gamma} \phi_\rho(yf(x')) \) be the corresponding supremum-based loss. If \( \mathcal{H} \) satisfies: for any \( x \in \mathcal{X} \), there exists \( f \in \mathcal{H} \) such that \( \inf_{x' \mid \|x-x'\| \leq \gamma} f(x') > \rho \), and \( f \in \mathcal{H} \) such that \( \sup_{x' \mid \|x-x'\| \leq \gamma} f(x') < -\rho \), then, for any \( x \in \mathcal{X} \), \( C_{\phi_\rho, \mathcal{H}, \gamma}(x, \eta) = C_{\phi_\rho, \mathcal{H}, \gamma}(\eta) \).

The proof is deferred to Appendix C.1. Note that, by Definitions 2 and 3, when \( C_{\ell, \mathcal{H}, \gamma}(x, \eta) = C_{\ell, \mathcal{H}, \gamma}(\eta), \forall x \in \mathcal{X} \), if a loss function \( \ell \) is \( \mathcal{H} \)-calibrated with respect to \( \ell, \gamma \), then it is also \( \mathcal{H} \)-calibrated with respect to \( \ell \), since \( C_{\ell, \mathcal{H}, \gamma}(\eta) \leq C_{\ell, \mathcal{H}, \gamma}(x, \eta) \), \( \forall x \in \mathcal{X} \). As a result, we obtain the following.

**Corollary 7** Assume that for any \( x \in \mathcal{X} \), there exists \( f \in \mathcal{H} \) such that \( \inf_{x' \mid \|x-x'\| \leq \gamma} f(x') > 0 \), and \( f \in \mathcal{H} \) such that \( \sup_{x' \mid \|x-x'\| \leq \gamma} f(x') < 0 \), then \( \ell \) is not pseudo-\( \mathcal{H} \)-calibrated with respect to \( \ell, \gamma \), then \( \ell \) is also not \( \mathcal{H} \)-calibrated with respect to \( \ell \).
This result is most helpful for obtaining our negative results of $\mathcal{H}$-calibration in Section 3.2. Specifically, in order to prove that a loss function $\ell$ is not $\mathcal{H}$-calibrated with respect to $\ell_\gamma$, we only need to prove that $\ell$ is not pseudo-$\mathcal{H}$-calibrated with respect to $\ell_\gamma$, which helps simplify our proofs. Similarly, by Theorem 6 and Definitions 2 and 3, we can derive the following corollary, which is most helpful for obtaining our positive results of $\mathcal{H}$-calibration in Section 3.3.

**Corollary 8** Let $\phi$ be a margin-based loss, $\phi_\rho(t) = \min\{1, \max\{0, 1 - \frac{t}{\rho}\}\}$, $\rho > 0$ be the $\rho$-margin loss and $\tilde{\phi}_\rho(f, x, y) = \sup_{x' : |x - x'| \leq \gamma} \phi_\rho(yf(x'))$ be the corresponding supremum-based loss. Then,

1. If $\mathcal{H}$ satisfies: for any $x \in \mathcal{X}$, $\{f(x) : f \in \mathcal{H}\} = \mathbb{R}$, then $\phi$ is pseudo-$\mathcal{H}$-calibrated with respect to $\ell_\gamma$ if and only if $\phi$ is $\mathcal{H}$-calibrated with respect to $\ell_\gamma$.

2. If $\mathcal{H}$ satisfies: for any $x \in \mathcal{X}$, there exists $f \in \mathcal{H}$ such that $\inf_{x' : |x - x'| \leq \gamma} f(x') > \rho$, and $f \in \mathcal{H}$ such that $\sup_{x' : |x - x'| \leq \gamma} f(x') < -\rho$, then $\tilde{\phi}_\rho$ is pseudo-$\mathcal{H}$-calibrated with respect to $\ell_\gamma$ if and only if $\tilde{\phi}_\rho$ is $\mathcal{H}$-calibrated with respect to $\ell_\gamma$.

Therefore, for the hypothesis sets $\mathcal{H}$, under broad assumptions, we can provide alternative losses which are $\mathcal{H}$-calibrated with respect to $\ell_\gamma$ by considering losses that are pseudo-$\mathcal{H}$-calibrated with respect to $\ell_\gamma$.

In this paper, we will adopt the natural condition 1. of Theorem 6 for the hypothesis set, which is easily satisfied for any non-trivial class: for any $x \in \mathcal{X}$, there exists $f \in \mathcal{H}$ such that $\inf_{x' : |x - x'| \leq \gamma} f(x') = \inf_{|s| \leq 1} f(x + \gamma s) > 0$ and there exists $f \in \mathcal{H}$ such that $\sup_{x' : |x - x'| \leq \gamma} f(x') = \sup_{|s| \leq 1} f(x + \gamma s) < 0$. As an example, consider the class $\mathcal{H}_{\text{NN}}$ of one layer ReLU networks as described in Section 2. For any $x \in \mathcal{X}$ with $\|x\| = t > \gamma$, let $w_j = Wx$ and $u_j = \frac{A}{n}$, for $j = 1, \ldots, n$. Then, the following holds:

$$\forall s : \|s\| \leq 1, \quad w_j \cdot (x + \gamma s) = W(x \cdot x + \gamma (x \cdot s)) \geq W(\|x\|^2 - \gamma \|x\| \|s\|) \geq W(t - \gamma) > 0.$$

Therefore, we have

$$\inf_{|s| \leq 1} \sum_{j=1}^n u_j (w_j \cdot (x + \gamma s))_+ \geq \Lambda W(t - \gamma) > 0.$$

Similarly, taking $u_j = -\frac{A}{n}$ instead, for $j = 1, \ldots, n$, yields

$$\sup_{|s| \leq 1} \sum_{j=1}^n u_j (w_j \cdot (x + \gamma s))_- \leq -\Lambda W(t - \gamma) < 0.$$

### 3.2. Negative results

In this section, we aim to study that common losses are not calibrated with respect to $\ell_\gamma$. Note by Corollary 7, in order to prove that a loss $\ell$ is not $\mathcal{H}$-calibrated with respect to $\ell_\gamma$, we only need to prove that $\ell$ is not pseudo-$\mathcal{H}$-calibrated with respect to $\ell_\gamma$, as showed in our proofs of this section.

#### 3.2.1. Convex losses

We first study convex losses which are often used for standard binary classification problems. For a linear hypothesis set, $\mathcal{H} = \mathcal{H}_{\text{lin}}$, Bao et al. (2020, Corollary 9) showed that convex losses are not pseudo-$\mathcal{H}_{\text{lin}}$-calibrated for the adversarial 0/1 loss.
Theorem 9 (Bao et al. (2020)) If a margin-based loss $\phi: \mathbb{R} \rightarrow \mathbb{R}_+$ is convex, then it is not pseudo-$H_{\text{lin}}$-calibrated with respect to $\ell_\gamma$.

Note that this result would not imply that $\phi$ is not $H$-calibrated with respect to $\ell_\gamma$, since $H_{\text{lin}}$ does not satisfy condition 1. of Theorem 6. However, all of our results below hold under both $H$-calibration (Definition 2) and pseudo-$H$-calibration (Definition 3), since the hypothesis sets $H$ considered below all satisfy that condition. Actually, we give the proofs under the Definition 3 of pseudo-$H$-calibration, which, by Corollary 7, imply the negative results of $H$-calibration (4).

Our first main contribution is to extend the above result to a more general case when $H$ is the class of generalized linear models $H_g$ under both calibration definitions. In particular, we show that convex losses are not $H_g$-calibrated with respect to $\ell_\gamma$ for a non-decreasing and continuous function $g$ that satisfies $g(1 + \gamma) < G$ and $g(-1 - \gamma) > -G$ for some $G > 0$. Verifying this condition is straightforward for $G$ sufficiently large. It is obvious that $H_g$ with this condition on $g$ satisfy the condition 1. in Theorem 6 on $H$.

**Theorem 10** Let $g$ be a non-decreasing and continuous function such that $g(1 + \gamma) < G$ and $g(-1 - \gamma) > -G$. If a margin-based loss $\phi: \mathbb{R} \rightarrow \mathbb{R}_+$ is convex, then it is not $H_g$-calibrated with respect to $\ell_\gamma$.

The proof of Theorem 10 is included in Appendix C.2. The key in proving the above theorem is to analyze the pseudo-calibration function $\delta(\epsilon)$ as defined in (6). Naturally, this requires us to understand $\delta_{\phi, H_g}(\eta)$ that in turn depends on the worst case perturbation of a given data point according to $\phi$. To do so, we use the result of Awasthi et al. (2020) that characterizes such perturbations for the case where $g$ is the ReLU function. We extend the characterization to non-decreasing continuous functions, and as a result obtain the form of the pseudo-calibration function in Lemma 29. Requiring $\delta(\epsilon) > 0$ for an appropriate value of $\eta$, then leads to a natural condition on the function $\delta(\alpha_1, \alpha_2) = \frac{1}{2} \phi(g(\alpha_1) + \alpha_2) + \frac{1}{2} \phi(-g(\alpha_1) - \alpha_2)$. Notice that this function is solely determined by the value of $g(\alpha_1) + \alpha_2$. For $\phi$ to be calibrated, we obtain the condition that $\delta(\alpha_1, \alpha_2)$ should not achieve a minimum inside the set $A = \{g(\alpha_1) + \alpha_2: -g(\alpha_1) + \gamma \leq \alpha_2 \leq -g(\alpha_1 - \gamma)\}$. However, notice that $0 \in A$ and $g(\alpha_1) + \alpha_2 = 0$ implies that $\delta$ equals $\phi(0)$. Furthermore, due to convexity of $\phi$, $\phi(0) \leq \phi(\alpha_1, \alpha_2)$ thereby leading to a contradiction. As a special case, consider $(\cdot)_+$ which is non-decreasing and continuous. Then the condition $(-1 - \gamma)_+ = 0 > -G$ is trivially satisfied, leading to the following corollary.

**Corollary 11** Assume that $G > 1 + \gamma$. If a margin-based loss $\phi: \mathbb{R} \rightarrow \mathbb{R}_+$ is convex, then $\phi$ is not $H_{\text{relu}}$-calibrated with respect to $\ell_\gamma$.

While convex surrogates are natural for the 0/1 loss, the current practice in designing practical algorithms for the adversarial loss involves using convex supremum-based surrogates (Madry et al., 2017; Wong et al., 2020; Shafahi et al., 2019). We next investigate such losses.

### 3.2.2. Supremum-Based Convex Losses

We study losses of the type $\hat{\phi}(f, x, y) = \sup_{x' : \|x - x'\| \leq \epsilon} \phi(y f(x'))$, with $\phi$ convex, which are often used in practice as surrogates for the adversarial 0/1 loss. The following theorem presents a negative result for supremum-based convex surrogate losses for the broad class of hypothesis sets $H$ investigated in this section. Its proof is deferred to Appendix C.4.
Theorem 12  Let $H$ be a hypothesis set containing 0. Assume that for any $x \in X$, there exists $f \in H$ such that $\inf_{y:|x-x'| \leq \gamma} f(x') > 0$, and $f \in H$ such that $\sup_{|x-x'| \leq \gamma} f(x') < 0$. If a margin-based loss $\phi$ is convex and non-increasing, then the surrogate loss defined by $\hat{\phi}(f, x, y) = \sup_{x':|x-x'| \leq \gamma} \phi(yf(x'))$ is not $\mathcal{H}$-calibrated with respect to $\ell_\gamma$.

The theorem above provides theoretical evidence that the current practice of making neural networks adversarially robust via minimizing convex supremum-based surrogates may have serious deficiencies. This lack of a principled choice of the surrogate loss may also explain why in practice the adversarial accuracies that are achievable are much lower than the corresponding natural accuracies of the model (Madry et al., 2017). In general, optimizing non-calibrated or non-consistent surrogates could lead to undesirable solutions even under strong assumptions (such as the Bayes risk being zero). See Section 5, where we empirically demonstrate this in a variety of settings.

In contrast with Theorem 10, the challenge in proving the above theorem is that, since we are working with a general class of functions, we no longer can hope for a complete characterization of the worst-case adversarial perturbations around a given point $x$. In fact, this is a challenging problem even in the case of one-layer networks (Awasthi et al., 2020). This presents a difficulty in analyzing the pseudo-calibration function $\hat{\delta}(\epsilon)$. Our key insight (Lemma 33) is that the pseudo-calibration function can be characterized by two quantities $\overline{M}(f, x, \gamma) = \inf_{x':|x-x'| \leq \gamma} f(x')$, $\overline{M}(f, x, \gamma) = \sup_{x':|x-x'| \leq \gamma} f(x')$. Once this is achieved, we follow a strategy similar to that of the proof of Theorem 10, where the condition $\hat{\delta}(\epsilon) > 0$ corresponds to an appropriate convex function not achieving a minimum in a set that contains 0, thereby reaching a contradiction. By Theorem 12 and the fact that $0 \not\in \mathcal{H}_{NN}$, we can derive the following corollary for the class of one layer ReLU neural networks.

Corollary 13  If a margin-based loss $\phi$ is convex and non-increasing, then the surrogate loss defined by $\hat{\phi}(f, x, y) = \sup_{x':|x-x'| \leq \gamma} \phi(yf(x'))$ is not $\mathcal{H}_{NN}$-calibrated with respect to $\ell_\gamma$.

3.3. Positive results

In this section, we aim to provide alternative losses which could be calibrated with respect to $\ell_\gamma$. By Corollary 8, we first give general pseudo-calibration results of our hypothesis $\mathcal{H}_g$ and $\mathcal{H}_{NN}$, and then show that specific $\mathcal{H}_g$ with $G = +\infty$ and $\mathcal{H}_{NN}$ with $\Lambda = +\infty$ has corresponding true calibration results and then would also has consistency results under appropriate conditions in Section 4.

3.3.1. Characterization

In light of the negative results in Section 3.2, to find calibrated surrogate losses for adversarially robust classification, we need to consider non-convex ones. One possible candidate is the family of quasi-concave even losses introduced by Bao et al. (2020), which were shown to be pseudo-$\mathcal{H}_{lin}$-calibrated with respect to the adversarial 0/1 loss under certain assumptions.

Definition 14 (Bao et al. (2020)) A margin-based loss function $\phi$ is said to be quasi-concave even, if $\phi(t) + \phi(-t)$ is quasi-concave.

Theorem 15 (Bao et al. (2020)) Assume that a margin-based loss $\phi$ is bounded, non-increasing, and quasi-concave even. Let $B \overset{\text{def}}{=} \phi(1) + \phi(-1)$ and assume $\phi(-1) > \phi(1)$. Then $\phi$ is pseudo-$\mathcal{H}_{lin}$-calibrated with respect to $\ell_\gamma$ if and only if $\phi(\gamma) + \phi(-\gamma) > B$. 

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Consider increasing and quasi-concave even. This is stated formally below.

**Theorem 16** Let \( g \) be a non-decreasing and continuous function such that \( g(1 + \gamma) < G \) and \( g(-1 - \gamma) > -G \) for some \( G > 0 \). Let a margin-based loss \( \phi \) be bounded, continuous, non-increasing, and quasi-concave even. Assume that \( \phi(g(-1) - G) > \phi(G - g(-1)) \) and \( g(-1) + g(1) = 0 \). Then \( \phi \) is pseudo-\( \mathcal{H}_g \)-calibrated with respect to \( \ell_\gamma \) if and only if

\[
\phi(G - g(-1)) + \phi(g(-1) - G) = \phi(g(1) + G) + \phi(-g(1) - G) \\
\text{and} \quad \min\{\phi(\bar{A}) + \phi(-\bar{A}), \phi(A) + \phi(-A)\} > \phi(G - g(-1)) + \phi(-g(1) - G),
\]

where \( \bar{A} = \sup_{\alpha_1 \in [-1,1]} g(\alpha_1) - g(\alpha_1 - \gamma) \) and \( A = \inf_{\alpha_1 \in [-1,1]} g(\alpha_1) - g(\alpha_1 + \gamma) \).

The conditions in the Theorem above are if and only if and hence precisely characterize when quasi-concave even losses are pseudo-\( \mathcal{H}_g \)-calibrated. To interpret the conditions better, consider ReLU functions. In this case, the assumptions in Theorem 16 can be further simplified, since \( \bar{A} = \sup_{\alpha_1 \in [-1,1]} (\alpha_1)_+ - (\alpha_1 - \gamma)_+ = \gamma \) and \( A = \inf_{\alpha_1 \in [-1,1]} (\alpha_1)_+ - (\alpha_1 + \gamma)_+ = -\gamma \). As a result we get the following.

**Corollary 17** Assume that \( G > 1 + \gamma \). Let a margin-based loss \( \phi \) be bounded, continuous, non-increasing, and quasi-concave even. Assume that \( \phi(G) > \phi(-G) \). Then \( \phi \) is pseudo-\( \mathcal{H}_{\text{relu}} \)-calibrated with respect to \( \ell_\gamma \) if and only if

\[
\phi(G) + \phi(-G) = \phi(1 + G) + \phi(-1 - G) \quad \text{and} \quad \phi(\gamma) + \phi(-\gamma) = \phi(G) + \phi(-G).
\]

Theorem 16 is proved in Appendix C.3. We again use the characterization of the pseudo-calibration function as derived in Lemma 29. In Lemma 31 we further simplify the characterization to a set of three conditions that the surrogate loss must satisfy. Finally, we show that quasi-concave even losses satisfy them under the conditions of the Theorem. Along the way, building on the work of Bao et al. (2020), we establish several useful properties of quasi-concave even losses in Lemma 32.

### 3.3.2. Calibration

To demonstrate the applicability of Theorem 16, we consider a specific surrogate loss namely the \( \rho \)-margin loss \( \phi_\rho(t) = \min\{1, \max\{0, 1 - \frac{t}{\rho}\}\} \), \( \rho > 0 \), which is a generalization of the ramp loss (see, for example, Mohri et al. (2018)). Using Theorem 15, Theorem 16 and Corollary 17 in Section 3.3.1, we can conclude that the \( \rho \)-margin loss is pseudo-\( \mathcal{H}_g \)-calibrated under reasonable conditions for linear hypothesis sets and non-decreasing \( g \)-based hypothesis sets, since \( \phi_\rho(t) \) is bounded, non-increasing and quasi-concave even. This is stated formally below.

**Theorem 18** Consider \( \rho \)-margin loss \( \phi_\rho(t) = \min\{1, \max\{0, 1 - \frac{t}{\rho}\}\} \), \( \rho > 0 \). Then,

1. \( \phi_\rho \) is pseudo-\( \mathcal{H}_{\text{lin}} \)-calibrated with respect to \( \ell_\gamma \) if and only if \( \rho > \gamma \);

2. Given a non-decreasing and continuous function \( g \) such that \( g(1 + \gamma) < G \) and \( g(-1 - \gamma) > -G \). Assume that \( g(-1) + g(1) = 0 \). Then \( \phi_\rho \) is pseudo-\( \mathcal{H}_g \)-calibrated with respect to \( \ell_\gamma \) if and only if

\[
\phi_\rho(G - g(-1)) = \phi_\rho(g(1) + G) \quad \text{and} \quad \min\{\phi_\rho(\bar{A}), \phi_\rho(-\bar{A})\} > \phi_\rho(G - g(-1)),
\]

where \( \bar{A} = \sup_{\alpha_1 \in [-1,1]} g(\alpha_1) - g(\alpha_1 - \gamma) \) and \( A = \inf_{\alpha_1 \in [-1,1]} g(\alpha_1) - g(\alpha_1 + \gamma) \).
3. Assume that $G > 1 + \gamma$. Then $\phi_\rho$ is pseudo-$\mathcal{H}_{\text{relu}}$-calibrated with respect to $\ell_\gamma$ if and only if $G \geq \rho > \gamma$.

Specifically, $\mathcal{H}_g$ with the extra assumption $G = +\infty$ satisfy the condition 1. in Corollary 8, as a result we get the following.

**Corollary 19** Consider $\rho$-margin loss $\phi_\rho(t) = \min\{1, \max\{0, 1 - \frac{t}{\rho}\}\}$, $\rho > 0$. Then,

1. Given a non-decreasing and continuous function $g$. Assume that $G = +\infty$ and $g(-1) + g(1) \geq 0$. Then $\phi_\rho$ is $\mathcal{H}_g$-calibrated with respect to $\ell_\gamma$ if and only if $\min\{\phi_\rho(A), \phi_\rho(-A)\} > 0$, where $A = \sup_{\alpha \in [-1,1]} g(\alpha - 1) - g(\alpha + 1)$ and $\overline{A} = \inf_{\alpha \in [-1,1]} g(\alpha - 1) - g(\alpha + 1)$;

2. Assume that $G = +\infty$. Then $\phi_\rho$ is $\mathcal{H}_{\text{relu}}$-calibrated with respect to $\ell_\gamma$ if and only if $\rho > \gamma$.

Recall that in Theorem 13 we ruled out the possibility of finding $\mathcal{H}_{\text{NN}}$-calibrated supremum-based convex surrogate losses with respect to the adversarial 0/1 loss, where $\mathcal{H}_{\text{NN}}$ is the class of one layer neural networks. However, we show that the supremum-based $\rho$-margin loss is indeed pseudo-$\mathcal{H}$-calibrated and $\mathcal{H}$-calibrated. We state the pseudo-calibration result below and present the proof in Appendix C.5.

**Theorem 20** Consider $\rho$-margin loss $\phi_\rho(t) = \min\{1, \max\{0, 1 - \frac{t}{\rho}\}\}$, $\rho > 0$. If $\Lambda W(1 - \gamma) \geq \rho$, then the surrogate loss $\tilde{\phi}_\rho(f, x, y) = \sup_{x' \parallel ||x - x'|| \leq \gamma} \phi_\rho(yf(x'))$ is pseudo-$\mathcal{H}_{\text{NN}}$-calibrated with respect to $\ell_\gamma$.

Specifically, $\mathcal{H}_{\text{NN}}$ with the extra assumption $\Lambda = +\infty$ satisfy the condition 2. in Corollary 8, as a result we get the following.

**Corollary 21** Consider $\rho$-margin loss $\phi_\rho(t) = \min\{1, \max\{0, 1 - \frac{t}{\rho}\}\}$, $\rho > 0$. If $\Lambda = +\infty$, then the surrogate loss $\tilde{\phi}_\rho(f, x, y) = \sup_{x' \parallel ||x - x'|| \leq \gamma} \phi_\rho(yf(x'))$ is $\mathcal{H}_{\text{NN}}$-calibrated with respect to $\ell_\gamma$.

The results of this section suggest that the ramp loss and more generally quasi-concave even losses may be good surrogates for the adversarial 0/1 loss. However, calibration, in general, is not equivalent to consistency, our eventual goal. In the next section we study conditions under which we can expect these surrogates losses to be $\mathcal{H}$-consistent as well.

### 4. $\mathcal{H}$-Consistency

In this section, we study the $\mathcal{H}$-consistency of surrogate loss functions. The results of the previous section suggest that convex losses or supremum-based convex losses would not be $\mathcal{H}$-consistent. However, $\mathcal{H}$-calibrated quasi-concave even losses, such as the ramp loss present an intriguing possibility. In fact, the recent work of Bao et al. (2020) made a claim that since quasi-concave even losses are $\mathcal{H}_{\text{lin}}$-calibrated they are also $\mathcal{H}_{\text{lin}}$-consistent. We first present a result that falsifies this claim. In fact, our result stated below shows that without assumptions on the data distribution, no continuous margin based loss or a continuous supremum-based surrogate could be $\mathcal{H}_{\text{lin}}$-consistent.
4.1. Negative results

**Theorem 22** No continuous margin-based loss function $\phi$ is $\mathcal{H}_\text{lin}$-consistent with respect to $\ell_\gamma$.

This theorem is proved in Appendix C.6. In order to establish the theorem, we carefully design a distribution on the unit disk where the label of each example $x$ is first generated as $\text{sgn}(w^* \cdot x)$ and then flipped independently with a carefully chosen probability. It is crucial that this flipping probability is asymmetric thereby ensuring that for the resulting joint distribution, $w^*$ remains the optimal linear classifier according to $\ell_\gamma$, but any continuous surrogate is led astray to a classifier that is far from $w^*$. In particular, Theorem 22 contradicts the $\mathcal{H}$-consistency claim of Bao et al. (2020) for quasi-concave even losses when $\mathcal{H}$ is the family of linear functions. Furthermore, the theorem can be easily extended to rule out $\mathcal{H}$-consistency of supremum-based surrogates as well.

**Theorem 23** For continuous and non-increasing margin-based loss $\phi$, surrogates of the form

$$\tilde{\phi}(f, x, y) = \sup_{x' : \|x - x'\| \leq \gamma} \phi(yf(x'))$$

are not $\mathcal{H}_\text{lin}$-consistent with respect to $\ell_\gamma$.

**Proof** As shown by Awasthi et al. (2020), for a continuous and non-increasing margin-based loss $\phi$, when $f \in \mathcal{H}_\text{lin}$, the supremum-based surrogate loss can be expressed as follows:

$$\tilde{\phi}(f, x, y) = \sup_{x' : \|x - x'\| \leq \gamma} \phi(yf(x')) = \phi\left(\inf_{\|s\| \leq 1} (yf(x + \gamma s))\right) = \phi(\langle w \cdot x, y \rangle - \gamma) = \psi(\langle w \cdot x, y \rangle),$$

where $\psi(t) = \phi(t - \gamma)$ is also a continuous margin-based loss. In view of Theorem 22, we conclude that the supremum-based surrogate loss $\tilde{\phi}$ is also not $\mathcal{H}_\text{lin}$-consistent with respect to $\ell_\gamma$.

4.2. Positive results

In this section, we investigate the nature of the assumptions on the data distributions that may lead to $\mathcal{H}$-consistency of surrogate losses. We take inspiration from the work of Long and Servedio (2013) and Zhang and Agarwal (2020) who study $\mathcal{H}$-consistency for the standard $0/1$ loss. These studies establish consistency under a realizability assumption on the data distribution stated below that requires the Bayes ($\ell_0, \mathcal{H}$)-risk to be zero.

**Definition 24 ($\mathcal{H}$-realizability)** A distribution $\mathcal{P}$ over $X \times Y$ is $\mathcal{H}$-realizable if it labels points according to a deterministic model in $\mathcal{H}$, i.e., if $\exists f \in \mathcal{H}$ such that $\mathbb{P}_{(x,y) \sim \mathcal{P}}(\text{sgn}(f(x)) = y) = 1$.

Similar to $\mathcal{H}$-realizability, we will assume that, under the data distribution, the Bayes ($\ell_\gamma, \mathcal{H}$)-risk is zero. We show that the $\mathcal{H}$-calibrated losses studied in previous sections are $\mathcal{H}$-consistent under natural conditions along with the realizability assumption.
4.2.1. Non-supremum-based surrogates

**Theorem 25** Let $\mathcal{P}$ be a distribution over $\mathcal{X} \times \mathcal{Y}$ and $\mathcal{H}$ a hypothesis set for which $\mathcal{R}^*_{\epsilon, \gamma}$ is $0$. Let $\phi$ be a margin-based loss. If for $\eta \geq 0$, there exists $f^* \in \mathcal{H} \subseteq \mathcal{H}$ such that $\mathcal{R}_{\phi}(f^*) \leq \mathcal{R}^*_{\phi, \epsilon, \gamma} + \eta < +\infty$ and $\phi$ is $\mathcal{H}$-calibrated\(^2\) with respect to $\ell$, then for all $\epsilon > 0$ there exists $\delta > 0$ such that for all $f \in \mathcal{H}$ we have

$$\mathcal{R}_{\phi}(f) + \eta < \mathcal{R}^*_{\phi, \epsilon, \gamma} + \delta \implies \mathcal{R}^*_{\phi, \epsilon, \gamma} < \epsilon.$$

The proof of Theorem 25 is presented in Appendix C.7. Using Corollary 19 in Section 3.3.2 and Theorem 25 above, we immediately conclude that the calibrated $\rho$-margin loss in Section 3.3.2 is consistent with respect to $\ell$, for all distributions that satisfy our realizability assumptions.

**Theorem 26** Consider the $\rho$-margin loss $\phi_{\rho}(t) = \min(1, \max\{0, 1 - t/\rho\})$, $\rho > 0$. Then,

1. Let $g$ be a non-decreasing and continuous function. Assume $G = +\infty$ and $g(-1) + g(1) \geq 0$. Let $\overline{A} = \sup_{\alpha \in [-1, 1]} g(\alpha - 1) - g(\alpha + 1) - \gamma$ and $\underline{A} = \min\{\phi_{\rho}(\overline{A}), \phi_{\rho}(\underline{A})\}$. If $\min\{\phi_{\rho}(\overline{A}), \phi_{\rho}(\underline{A})\} > 0$, then $\mathcal{H}$ is $\mathcal{H}$-consistent with respect to $\ell$, for all distributions $\mathcal{P}$ over $\mathcal{X} \times \mathcal{Y}$ that satisfies $\mathcal{R}^*_{\epsilon, \gamma} = 0$ and there exists $f^* \in \mathcal{H}$ such that $\mathcal{R}_{\phi}(f^*) = \mathcal{R}^*_{\phi, \epsilon, \gamma} < \infty$.

2. If $\rho > \gamma$, then $\phi_{\rho}$ is $\mathcal{H}$-consistent with respect to $\ell$, for all distributions $\mathcal{P}$ over $\mathcal{X} \times \mathcal{Y}$ that satisfies $\mathcal{R}^*_{\epsilon, \gamma} = 0$ and there exists $f^* \in \mathcal{H}$ such that $\mathcal{R}_{\phi}(f^*) = \mathcal{R}^*_{\phi, \epsilon, \gamma} < \infty$.

4.2.2. Supremum-based surrogates

We can also extend the above to obtain $\mathcal{H}$-consistency of supremum-based convex surrogates. However we need the stronger condition that $\mathcal{R}_{\phi}$ is minimized exactly inside $\mathcal{H}$.

**Theorem 27** Given a distribution $\mathcal{P}$ over $\mathcal{X} \times \mathcal{Y}$ and a hypothesis set $\mathcal{H}$ such that $\mathcal{R}^*_{\epsilon, \gamma} = 0$. Let $\phi$ be a non-increasing margin-based loss. If there exists $f^* \in \mathcal{H} \subseteq \mathcal{H}$ such that $\mathcal{R}_{\phi}(f^*) = \mathcal{R}^*_{\phi, \epsilon, \gamma} < \infty$ and $\tilde{\phi}(x, y) = \sup_{x', y' \in [\mathcal{X} \times \mathcal{Y}]} \phi(y \mathcal{f}(x'))$ is $\mathcal{H}$-calibrated\(^3\) with respect to $\epsilon$, then for all $\epsilon > 0$ there exists $\delta > 0$ such that for all $f \in \mathcal{H}$ we have

$$\mathcal{R}_{\tilde{\phi}}(f) < \mathcal{R}^*_{\phi, \epsilon, \gamma} + \delta \implies \mathcal{R}_{\epsilon, \gamma}(f) < \mathcal{R}^*_{\epsilon, \gamma} + \epsilon.$$

The proof of Theorem 27 is presented in Appendix C.7. Again, when combined with Corollary 21 in Section 3.3.2 we conclude that the $\mathcal{H}_{\text{NN}}$-calibrated supremum-based $\rho$-margin loss is also $\mathcal{H}_{\text{NN}}$-consistent with respect to $\ell$, for all distributions that satisfy our realizability assumptions.

**Theorem 28** Consider the $\rho$-margin loss $\phi_{\rho}(t) = \min(1, \max\{0, 1 - t/\rho\})$, $\rho > 0$. If $\Lambda = +\infty$, then $\tilde{\phi}_{\rho}(x, y) = \sup_{x' \in [\mathcal{X} \times \mathcal{Y}]} \phi_{\rho}(y \mathcal{f}(x'))$ is $\mathcal{H}_{\text{NN}}$-consistent with respect to $\ell$, for all distributions $\mathcal{P}$ over $\mathcal{X} \times \mathcal{Y}$ such that: $\mathcal{R}^*_{\epsilon, \gamma, \mathcal{H}_{\text{NN}}} = 0$ and there exists $f^* \in \mathcal{H}_{\text{NN}}$ such that $\mathcal{R}_{\phi}(f^*) = \mathcal{R}^*_{\phi, \epsilon, \gamma} < \infty$.

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2. The theorem still holds if uniform $\mathcal{H}$-calibration is replaced by weaker non-uniform $\mathcal{H}$-calibration (Steinwart, 2007, Definition 2.7), since the proof only makes use of the weaker non-uniform property.

3. The theorem still holds if uniform $\mathcal{H}$-calibration is replaced by weaker non-uniform $\mathcal{H}$-calibration (Steinwart, 2007, Definition 2.7), since the proof only makes use of the weaker non-uniform property.
We prove the above theorems by building upon the framework of Steinwart (2007). The goal is to show that under the assumptions of the theorems, the adversarial 0/1 loss and the surrogate loss become $\mathcal{P}$-minimizable which is enough to show consistency. In Theorem 25 we use the fact that the adversarial 0/1 Bayes risk is zero to obtain $\mathcal{P}$-minimizability of the adversarial 0/1 loss. We also show that the framework of Steinwart (2007) can be extended to only require $\mathcal{P}$-minimizability of the surrogate up to an error of $\eta$, and the consistency guarantees degrade smoothly with $\eta$. To prove Theorem 27, we prove a general result (Lemma 39) that if the adversarial 0/1 Bayes risk is zero then $\mathcal{P}$-minimizability of a surrogate implies $\mathcal{P}$-minimizability of its supremum based counterpart.

5. Experiments

We present experiments on simulated data to support our theoretical findings. The goal is two fold. First, we empirically demonstrate that indeed calibrated surrogates in (Bao et al., 2020) may not be $\mathcal{H}$-consistent unless assumptions on the data distribution are made, even when $\mathcal{H}$ is the class of linear functions. This is consistent with our negative result in Theorem 22 and provides an empirical counterexample to the claim made in (Bao et al., 2020). Secondly, we study the necessity of the realizability assumptions that we make in Section 4.2 to establish $\mathcal{H}$-consistency of quasi-concave even surrogates. We generate data points $x \in \mathbb{R}^2$ on the unit circle and consider $\mathcal{H}$ to be linear models $\mathcal{H}_{\text{lin}}$. We denote $f(x) = w \cdot x$, $w = (\cos(t),\sin(t))^\top$, $t \in [0,2\pi)$, $f \in \mathcal{H}_{\text{lin}}$. All risks in the experiments are approximated by their empirical counterparts computed over $10^7$ i.i.d. samples from the distribution.

To demonstrate the need for assumptions on the data distribution for $\mathcal{H}$-consistency, we construct a scenario we call the **Unit Circle** case. We consider four surrogates: $\phi_{\text{hinge}}, \phi_{\text{ramp}}, \phi_{\text{sig}}$ and $\phi_{\log}$ defined in Appendix B. In general, we refer all of these surrogates as $\phi_{\text{sur}}$. We generate data points $x$ from the uniform distribution on the unit circle. Denote $x = (\cos(\theta),\sin(\theta))^\top$, $\theta \in [0,2\pi)$. Set the label of a point $x$ as follows: if $\theta \in (\frac{\pi}{2},\pi)$, then $y = -1$ with probability $\frac{3}{4}$ and $y = 1$ with probability $\frac{1}{4}$; if $\theta \in (0,\frac{\pi}{2})$ or $(\frac{3\pi}{2},2\pi)$, then $y = 1$; if $\theta \in (\pi,\frac{3\pi}{2})$, then $y = -1$. Set $\gamma = \sqrt{\frac{2}{3}}$.

In this case the optimal Bayes ($\ell_{\gamma,\mathcal{H}_{\text{lin}}}$)-risk $\mathcal{R}^*_\ell_{\gamma,\mathcal{H}_{\text{lin}}} \approx 0.5000 \neq 0$ and is achieved by $w_{\ell_{\gamma}} = (\cos(\theta),\sin(\theta))^\top$ with $\theta \approx 0.7855$. The results obtained by optimizing the different surrogate
Table 1: (a) Unit Circle; (b) Segments.

| \(\phi_{\text{sur}}\) | \(R_{\ell_1}(f^*)\) | \(\theta_{\phi_{\text{sur}}}\) | \(\mathcal{H}_{\text{lin}}\)-cal. | \(\mathcal{H}_{\text{lin}}\)-cons. | \(\phi_{\text{sur}}\) | \(R_{\ell_1}(f^*)\) | \(\theta_{\phi_{\text{sur}}}\) | \(\mathcal{H}_{\text{lin}}\)-cal. | \(\mathcal{H}_{\text{lin}}\)-cons. |
|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| \(\phi_{\text{hinge}}\) | 0.5257 | 0.1420 | \(\times\) | \(\times\) | \(\phi_{\text{hinge}}\) | 0.0781 | 0.6907 | 1.3548 | \(\times\) | \(\times\) |
| \(\phi_{\text{ramp}}\) | 0.5263 | 0.1288 | \(\checkmark\) | \(\times\) | \(\phi_{\text{ramp}}\) | 0.0781 | 0.3454 | 1.3548 | \(\checkmark\) | \(\times\) |
| \(\phi_{\text{sig}}\) | 0.5261 | 0.1320 | \(\checkmark\) | \(\times\) | \(\phi_{\text{sig}}\) | 0.0777 | 0.4247 | 1.3498 | \(\checkmark\) | \(\times\) |
| \(\phi_{\text{log}}\) | 0.5258 | 0.1414 | \(\times\) | \(\times\) | \(\phi_{\text{log}}\) | 0.0763 | 0.8078 | 1.3341 | \(\times\) | \(\times\) |
| \(\phi_1\) | 0.0111 | 0 | \(\frac{\pi}{6}\) | \(\times\) | \(\phi_1\) | 0 | 0 | \(\checkmark\) | \(\checkmark\) |
| \(\phi_2\) | 0 | 0 | 0 | \(\checkmark\) | \(\checkmark\) |

losses are in Table 1(a) and the plots for 1000 samples and 2000 samples are shown in Figure 1. Table 1(a) shows that neither calibrated nor non-calibrated (convex) surrogates are \(\mathcal{H}_{\text{lin}}\)-consistent with respect to \(\ell_{\gamma}\) for this distribution. Figure 1 shows that the classifiers obtained by optimizing the four surrogates are almost the same but deviate a lot from the optimal Bayes classifier for \(\ell_{\gamma}\). This shows that indeed calibrated surrogates may not be consistent, and contradicts Figure 12 of Bao et al. (2020). The discrepancy results from the incorrect calculation of the adversarial Bayes risk in (Bao et al., 2020).

Next, we justify the realizability assumptions made in Section 4.2 for obtaining \(\mathcal{H}\)-consistency of surrogate losses. In order to do this we construct a scenario that we call as the Segments case. Here, we consider six surrogates, the four studied above and two more surrogates \(\phi_1\) and \(\phi_2\) defined in Appendix B. The loss \(\phi_1\) is a convex loss and \(\phi_2\) is the \(\rho\)-margin ramp loss for some \(\rho > \gamma\). In general, we refer all of these surrogates to \(\phi_{\text{sur}}\). We show in Appendix B.2, \(\phi_{\text{hinge}}, \phi_{\text{log}}\) and \(\phi_1\) are not calibrated while \(\phi_{\text{ramp}}, \phi_{\text{sig}}\) and \(\phi_2\) are calibrated with respect to \(\ell_{\gamma}\).

We consider the following data distribution: \(\mathbb{P}(Y = 1) = \mathbb{P}(Y = -1) = \frac{1}{2}\), and \(X \mid Y = 1\) is the uniform distribution on the line segment \(\{ (\hat{\gamma}, z) \mid z \in [0, \sqrt{1 - \hat{\gamma}^2}] \}\) and \(X \mid Y = -1\) is the uniform distribution on the line segment \(\{ (-\hat{\gamma}, z) \mid z \in [-\sqrt{1 - \hat{\gamma}^2}, 0] \}\) where \(\hat{\gamma} = \gamma + \frac{\gamma - 100}{100} = 1.099\gamma\), \(\gamma \in (0, 1)\). Finally, we set \(\gamma = 0.1\). Let \(\mathbf{w}^* = (1, 0)^T\). It is easy to check that \(\mathbf{w}^*\) achieves the optimal adversarial Bayes risk \((\ell_{\gamma}, \mathcal{H}_{\text{lin}})\)-risk \(R_{\ell_{\gamma}, \mathcal{H}_{\text{lin}}}^* = 0\).

The results for six different surrogate losses are in Table 1(b). For \(\phi_{\text{hinge}}, \phi_{\text{ramp}}, \phi_{\text{sig}}\) and \(\phi_{\text{log}},\) the Bayes \((\phi_{\text{sur}}, \mathcal{H}_{\text{lin}})\)-risk \(R_{\phi_{\text{sur}}, \mathcal{H}_{\text{lin}}}^* \neq 0\). Table 1(b) shows that they are not \(\mathcal{H}_{\text{lin}}\)-consistent with respect to \(\ell_{\gamma}\). For \(\phi_1\) and \(\phi_2\), the Bayes \((\phi_{\text{sur}}, \mathcal{H}_{\text{lin}})\)-risk \(R_{\phi_{\text{sur}}, \mathcal{H}_{\text{lin}}}^* = 0\). Table 1(b) shows that \(\phi_1\) is not \(\mathcal{H}_{\text{lin}}\)-consistent (recall that \(\phi_1\) is not calibrated) but \(\phi_2\) is \(\mathcal{H}_{\text{lin}}\)-consistent for this distribution.

Hence even when \(R_{\ell_{\gamma}, \mathcal{H}_{\text{lin}}}^* = 0\), unless a condition is also imposed on \(R_{\phi_{\text{sur}}, \mathcal{H}_{\text{lin}}}^*\), one cannot expect consistency, thus justifying our realizability assumption. Note that \(R_{\phi_{\text{sur}}, \mathcal{H}_{\text{lin}}}^* = R_{\ell_{\gamma}, \mathcal{H}_{\text{lin}}}^* = 0\) is a special case satisfying the conditions in Theorem 25 when \(\eta = 0\). For this distribution, \(\phi_{\text{ramp}}\) is not \(\mathcal{H}_{\text{lin}}\)-consistent while \(\phi_2\) is \(\mathcal{H}_{\text{lin}}\)-consistent, although both are calibrated. We compare them in Figure 2, showing that minimizing \(\mathcal{H}_{\text{lin}}\)-consistent surrogate \(\phi_2\) minimizes the generalization error for large sample sizes but the same does not hold for \(\mathcal{H}_{\text{lin}}\) non-consistent surrogate \(\phi_{\text{ramp}}\).

6. Conclusion

We presented a detailed study of calibration and consistency for adversarial robustness. These results can help guide the design of algorithms for learning robust predictors, an increasingly important problem in applications. Our theoretical results show in particular that many of the surrogate losses typically used in practice do not benefit from any guarantee. Our empirical results further illustrate that in the context of a general example. Our results also show that some of the calibration

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4. Private Communication.
results presented in previous work do not bear any significance, since we prove that in fact they do not guarantee consistency. Instead, we give a series of positive calibration and consistency results for several families of surrogate functions, under some realizability assumptions.

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Appendix A. Related Work

The notions of calibration and consistency with respect to the 0/1 loss have been widely studied in the statistical learning theory literature to analyze the properties of surrogate losses (Zhang, 2004; Bartlett et al., 2006). Bartlett et al. (2006) showed that margin-based convex surrogates, that is mappings of the form \( f, x, y \rightarrow \phi(y f(x)) \), where \( f \) is a real-valued predictor and \( \phi: \mathbb{R} \rightarrow \mathbb{R}_+ \) a function differentiable at 0 with \( \phi'(0) < 0 \), are calibrated with respect to the class of all measurable functions. Extensions of calibration and consistency to multi-class settings have also been studied (Tewari and Bartlett, 2007). In the special case of the 0/1 loss and margin-based convex surrogates, calibration immediately implies consistency for the class of all measurable functions. One can then even derive quantitative bounds relating the excess \( \phi \)-risk to the excess 0/1 loss of any function \( f \) (Zhang, 2004; Bartlett et al., 2006).

The case of adversarial loss is more complex. This is because, in particular, the loss of a predictor \( f \) at point \( x \) does not just depend on its value \( f(x) \) at that point but also on its values in a neighborhood of \( x \). Steinwart (2007) proposed a general framework to study and characterize calibration and consistency, in particular via a calibration function. He also defined a \( \mathcal{P} \)-minimizability condition under which calibration implies consistency. But, while \( \mathcal{P} \)-minimizability holds for the 0/1 loss and margin-based convex surrogates over the class of all measurable functions, the condition does not hold in general for the adversarial loss. Our work borrows tools from the work of Steinwart (2007). However, to establish \( \mathcal{H} \)-consistency in the context of the adversarial loss, additional insights are needed and often stronger assumptions on the data distribution are required. These assumptions are captured in the notion of realizable \( \mathcal{H} \)-consistency that requires that the optimal risk of both the 0/1 loss and the surrogate loss being achieved inside the class \( \mathcal{H} \). Our positive results for \( \mathcal{H} \)-consistency rely on similar but weaker assumptions. Long and Servedio (2013) gave examples of surrogate losses that are not \( \mathcal{H} \)-consistent when \( \mathcal{H} \) is the class of all measurable functions but satisfy realizable \( \mathcal{H} \)-consistency when \( \mathcal{H} \) is the class of linear functions. Zhang and Agarwal (2020) studied the notion of improper realizable \( \mathcal{H} \)-consistency of linear classes where the surrogate \( \phi \) can be optimized over a larger class such as that of piecewise linear functions.

These notions of calibration and consistency are relatively unexplored for the robust 0/1 loss. Bao et al. (2020) recently initiated the study of these notions for the robust loss. In particular, the authors studied the \( \gamma \)-margin loss defined by: \( \ell_\gamma(f, x, y) = \mathbb{I}_{yf(x) \leq \gamma} \). This loss function coincides with the adversarial loss (only) in the special case where linear classifiers with adversarial perturbations measured in \( \ell_2 \) norm are considered. The authors showed that, when \( \mathcal{H} \) is linear, convex surrogates are not \( \mathcal{H} \)-calibrated and proposed a class of quasi-concave even \( \mathcal{H} \)-calibrated surrogates.

Our positive results for \( \mathcal{H} \)-calibration significantly extend those beyond linear hypothesis sets. More importantly, Bao et al. (2020) incorrectly concluded that \( \mathcal{H} \)-calibration of quasi-concave even surrogate losses implies their \( \mathcal{H} \)-consistency. Our negative results falsify this claim and in fact rule out the \( \mathcal{H} \)-consistency of a large class of surrogates, unless assumptions on the data distribution are imposed. Finally, while the results of Bao et al. (2020) do imply that quasi-concave even surrogates are \( \mathcal{H} \)-consistent with respect to the \( \ell_\gamma \) loss over the set of all measurable functions, this does not provide insights into the adversarial loss since the two losses only coincide for linear hypothesis sets.

There has also been recent works on theoretically understanding different aspects of adversarial robustness. Tsipras et al. (2018) give constructions under which every classifier with small 0/1 loss has a large adversarial 0/1 loss thereby pointing to a tension between the two criteria. This has been
tradeoff has been explored in subsequent work (Zhang et al., 2019; Carmon et al., 2019). Bubeck et al. (2018b), Bubeck et al. (2018a) and Awasthi et al. (2019) quantify computational bottlenecks in learning classifiers with small adversarial loss. There has also been a line of work analyzing the sample complexity of optimizing adversarial surrogate losses using notions of VC-dimension and Rademacher complexity appropriately extended to the adversarial case (Yin et al., 2019; Khim and Loh, 2018; Awasthi et al., 2020; Montasser et al., 2019; Cullina et al., 2018). Another recent line of concerns constructing computationally efficient adversarially robust classifiers for linear classifiers (Diakonikolas et al., 2020) and exploring the connections between adversarial learning and agnostic PAC learning (Montasser et al., 2020). Finally, an alternative adversarial setting has been theoretically studied in (Feige et al., 2015, 2018; Attias et al., 2018), where the adversary has at his disposal a finite set of perturbations for each input.
Appendix B. Details of Experiments

As shown by Bao et al. (2020), the adversarial 0/1 loss \( \ell_\gamma = \mathbb{1}_{yf(x) \leq \gamma} \) when \( f \in \mathcal{H}_{\text{lin}} \). In this experiment, we approximate \( R^*_{\ell_\gamma, \mathcal{H}_{\text{lin}}} \) over a grid. For surrogate losses, we approximate \( f^* = \arg\min_{f \in \mathcal{H}_{\text{lin}}} R_{\phi_{\text{sur}}}(f) \) over the same grid.

B.1. Definition of Surrogates

- Shifted Hinge loss: \( \phi_{\text{hinge}} = \max\{0, 1 - t + 0.2\} \);
- Shifted Ramp loss: \( \phi_{\text{ramp}} = \min\{1, \max\{0, 1-t+0.2\}\} \);
- Shifted Sigmoid loss: \( \phi_{\text{sig}} = \frac{1}{1+e^{-0.2}} \);
- Shifted Logistic loss: \( \phi_{\text{log}} = \log_2(1+e^{t-0.2}) \);
- One convex loss: \( \phi_1(t) = \max\{0, \frac{\gamma}{2} - t\} \); and
- \( \rho \)-margin loss: \( \phi_2(t) = \min\{1, \max\{0, 1 - \frac{t}{\hat{\gamma}}\}\} \) for \( \hat{\gamma} > \gamma \).

B.2. Theoretical Analysis of Surrogates

\( \phi_{\text{hinge}}, \phi_{\text{log}}, \) and \( \phi_1 \) are convex surrogates and thus are not calibrated with respect to \( \ell_\gamma \) by Corollary 9 of (Bao et al., 2020). However, \( \phi_{\text{ramp}}, \phi_{\text{sig}} \) and \( \phi_2 \) are quasi-concave even losses and calibrated with respect to \( \ell_\gamma \) since they satisfy the conditions in Theorem 11 of (Bao et al., 2020).

Note that \( \mathbb{E}_{(X,Y)}[\phi_2(Yw \cdot X)] = 0 \) if and only if \( w = (1,0)^T \). Therefore, \( \phi_2 \) is \( \mathcal{H}_{\text{lin}} \)-consistent for the distribution \text{Segments}. However, for \( w = (1,0)^T \) or \( w = (\cos(\theta), \sin(\theta))^T \) where \( \theta = \frac{\pi}{6} \), we have \( \mathbb{E}_{(X,Y)}[\phi_1(Yw \cdot X)] = 0 \). Note when \( w = (\cos(\theta), \sin(\theta))^T \) where \( \theta = \frac{\pi}{6} \), we have \( \mathbb{E}_{(X,Y)}[\ell_\gamma(Yw \cdot X)] \neq 0 \). Therefore, \( \phi_1 \) is not \( \mathcal{H}_{\text{lin}} \)-consistent for the distribution \text{Segments}.
Appendix C. Deferred Proofs

C.1. Proof of Theorem 6

Theorem 6 [Equivalence of calibration definitions] Without loss of generality, let $\mathcal{X} = B_2^d(1)$ and $\gamma \in (0, 1)$. Then,

1. If $\mathcal{H}$ satisfies: for any $x \in \mathcal{X}$, there exists $f \in \mathcal{H}$ such that $\inf_{x' : |x' - x| \leq \gamma} f(x') > 0$, and $f \in \mathcal{H}$ such that $\sup_{x' : |x' - x| \leq \gamma} f(x') < 0$, then, for any $x \in \mathcal{X}$, $C^*_{\gamma, \mathcal{H}}(x, \eta) = C^*_{\gamma, \mathcal{H}}(\eta)$.

2. Let $\phi$ be a margin-based loss. If $\mathcal{H}$ satisfies: for any $x \in \mathcal{X}$, $\{ f(x) : f \in \mathcal{H} \} = \mathbb{R}$, then, for any \( x \in \mathcal{X}, C^*_{\phi, \mathcal{H}}(x, \eta) = C^*_{\phi, \mathcal{H}}(\eta) \).

3. Let $\phi_\rho(t) = \min \{ 1, \max \{ 0, 1 - \frac{t}{\rho} \} \}$ for a fixed $\rho > 0$ be the $\rho$-margin loss and $\tilde{\phi}_\rho(f, x, y) = \sup_{x' : |x' - x| \leq \rho} yf(x')$ be the corresponding supremum-based loss. If $\mathcal{H}$ satisfies: for any $x \in \mathcal{X}$, there exists $f \in \mathcal{H}$ such that $\inf_{x' : |x' - x| \leq \gamma} f(x') > \rho$, and $f \in \mathcal{H}$ such that $\sup_{x' : |x' - x| \leq \gamma} f(x') < -\rho$, then, for any $x \in \mathcal{X}$, $C^*_{\phi_\rho, \mathcal{H}}(x, \eta) = C^*_{\phi_\rho, \mathcal{H}}(\eta)$.

Proof 1) First, note that for any $x \in \mathcal{X}$ and $f \in \mathcal{H}$, we cannot have both $\ell_\gamma(f, x, +1) = 0$ and $\ell_\gamma(f, x, -1) = 0$. In view of that, $C_{\ell_\gamma, \mathcal{H}}(x, \eta) = \eta \ell_\gamma(f, x, +1) + (1 - \eta) \ell_\gamma(f, x, -1) \geq \min \{ \eta, 1 - \eta \}$. By assumption, for any $x \in \mathcal{X}$, there exist $f_+ \in \mathcal{H}$ such that $\ell_\gamma(f_+, x, +1) = 1$ and $\ell_\gamma(f_+, x, -1) = 0$ and $f_- \in \mathcal{H}$ such that $\ell_\gamma(f_-, x, +1) = 0$ and $\ell_\gamma(f_-, x, -1) = 1$, that is $C_{\ell_\gamma, \mathcal{H}}(f_+, x, \eta) = \eta$ and $C_{\ell_\gamma, \mathcal{H}}(f_-, x, \eta) = 1 - \eta$. Thus, $\min \{ \eta, 1 - \eta \}$ is achieved and we have, for all $x \in \mathcal{X}$, $C^*_{\ell_\gamma, \mathcal{H}}(x, \eta) = \min \{ \eta, 1 - \eta \}$. This implies $C^*_{\ell_\gamma, \mathcal{H}}(\eta) = C^*_{\ell_\gamma, \mathcal{H}}(x, \eta) = \min \{ \eta, 1 - \eta \}$ for all $x \in \mathcal{X}$.

2) Given the assumption, for any $x \in \mathcal{X}$, we have $C^*_{\phi_\rho, \mathcal{H}}(x, \eta) = \inf_{f \in \mathcal{H}} \{ \eta \phi(f(x)) + (1 - \eta) \phi(-f(x)) \} = \inf_{u \in \mathbb{R}} \{ \eta \phi(u) + (1 - \eta) \phi(-u) \}$, which is independent of $x$. This implies $C^*_{\phi_\rho, \mathcal{H}}(\eta) = C^*_{\phi_\rho, \mathcal{H}}(x, \eta)$ for $x \in \mathcal{X}$.

3) By definition of the loss function, for any $x \in \mathcal{X}$ and $f \in \mathcal{H}$, we have

$$C^*_{\phi_\rho, \mathcal{H}}(f, x, \eta) = \eta \rho_\rho(\overline{M}(f, x, \gamma)) + (1 - \eta) \rho_\rho(-\overline{M}(f, x, \gamma)),$$

where $\overline{M}(f, x, \gamma) = \inf_{x' : |x' - x| \leq \gamma} f(x')$ and $M(f, x, \gamma) = \sup_{x' : |x' - x| \leq \gamma} f(x')$. Now, we must have either $\rho_\rho(\overline{M}(f, x, \gamma)) = 0$ or $\rho_\rho(-\overline{M}(f, x, \gamma)) = 1$. Otherwise, we would have $\overline{M}(f, x, \gamma) > 0$ and $-\overline{M}(f, x, \gamma) > 0$, but since $\overline{M}(f, x, \gamma) \leq \overline{M}(f, x, \gamma)$, the first inequality would imply $\overline{M}(f, x, \gamma) > 0$, which would contradict the second inequality. In view of that, the lower bound $C^*_{\phi_\rho, \mathcal{H}}(f, x, \eta) \geq \min \{ \eta, 1 - \eta \}$ holds.

By assumption, for any $x \in \mathcal{X}$, there exists $f_+$ such that $\rho_\rho(\overline{M}(f_+, x, \gamma)) = 0$ and $\rho_\rho(-\overline{M}(f_+, x, \gamma)) = 1$, that is $C^*_{\phi_\rho, \mathcal{H}}(f_+, x, \eta) = 1 - \eta$, and $f_+$ such that $\rho_\rho(\overline{M}(f_+, x, \gamma)) = 1$ and $\rho_\rho(-\overline{M}(f_+, x, \gamma)) = 0$, that is $C^*_{\phi_\rho, \mathcal{H}}(f_+, x, \eta) = \eta$. Thus, the lower bound is reached and, for any $x \in \mathcal{X}$, we have $C^*_{\phi_\rho, \mathcal{H}}(x, \eta) = \min \{ \eta, 1 - \eta \}$. This implies $C^*_{\phi_\rho, \mathcal{H}}(\eta) = C^*_{\phi_\rho, \mathcal{H}}(x, \eta) = \min \{ \eta, 1 - \eta \}$ for any $x \in \mathcal{X}$.

C.2. Proof of Theorem 10

As shown by (9) and Awasthi et al. (2020), for $f \in \mathcal{H}_2$, the adversarial 0/1 loss has the equivalent form

$$\ell_{\gamma}(f, x, y) = \mathbb{1}_{y \geq \gamma} \inf_{x' : |x' - x| \leq \gamma} (yg(w \cdot x') + by) \leq 0 = \mathbb{1}_{y \geq \gamma} \inf_{x' : |x' - x| \leq \gamma} (yg(w \cdot x' - y) + by) \leq 0 = \mathbb{1}_{y \geq \gamma} (yg(w \cdot x' - y) + by) \leq 0. \quad (10)$$
Define $\mathcal{F}_1 = \{x \to w \cdot x \mid \|w\| = 1\}$ and $\mathcal{F}_2 = \{x \to b \mid |b| \leq G\}$. Note that for any $f \in \mathcal{H}_g$ and $x \in \mathcal{X}$, there exist $\alpha_1 \in \mathcal{A}_{\mathcal{F}_1}$ and $\alpha_2 \in \mathcal{A}_{\mathcal{F}_2}$ such that $f(x) = g(\alpha_1) + \alpha_2$, where $\mathcal{A}_{\mathcal{F}_1} \overset{\text{def}}{=} \{f_1(x) \mid f_1 \in \mathcal{F}_1, x \in \mathcal{X}\}$ and $\mathcal{A}_{\mathcal{F}_2} \overset{\text{def}}{=} \{f_2(x) \mid f_2 \in \mathcal{F}_2, x \in \mathcal{X}\}$. Therefore, we can rewrite (10) as

$$\ell_\gamma(\alpha_1, \alpha_2, y) = \mathbb{I}_{y g(\alpha_1 - \gamma) + \alpha_2 y \leq 0}.$$

Similarly, we can rewrite the inner risk and pseudo-minimal inner risk of $\ell_\gamma$ and $\phi$ as

$$C_{\ell_\gamma}(\alpha_1, \alpha_2, \eta) = \eta \ell_\gamma(\alpha_1, \alpha_2, 1) + (1 - \eta) \ell_\gamma(\alpha_1, \alpha_2, -1), \quad C_{\ell_\gamma, \mathcal{H}_g}^\star(\eta) = \inf_{\alpha_1 \in \mathcal{A}_{\mathcal{F}_1}, \alpha_2 \in \mathcal{A}_{\mathcal{F}_2}} C_{\ell_\gamma}(\alpha_1, \alpha_2, \eta),$$

$$\Delta C_{\ell_\gamma, \mathcal{H}_g}(\alpha_1, \alpha_2, \eta) = C_{\ell_\gamma}(\alpha_1, \alpha_2, \eta) - C_{\ell_\gamma, \mathcal{H}_g}^\star(\eta), \quad \Delta C_{\phi, \mathcal{H}_g}(\alpha_1, \alpha_2, \eta) = C_\phi(\alpha_1, \alpha_2, \eta) - C_{\phi, \mathcal{H}_g}^\star(\eta).$$

Next, we characterize the pseudo-calibration function of losses ($\phi, \ell_\gamma$) given hypothesis set $\mathcal{H}_g$.

**Lemma 29** Given a non-decreasing and continuous function $g$ such that $g(1 + \gamma) < G$ and $g(-1 - \gamma) > -G$. For a margin-based loss $\phi$ and hypothesis set $\mathcal{H}_g$, the pseudo-calibration function of losses ($\phi, \ell_\gamma$) is

$$\tilde{\delta}(\epsilon) = \inf_{\eta \in [0, 1]} \tilde{\delta}(\epsilon, \eta),$$

where

$$\tilde{\delta}(\epsilon, \eta) = \begin{cases} +\infty & \text{if } \epsilon > \max\{\eta, 1 - \eta\}, \\ \inf_{\alpha_1 \in \mathcal{A}_{\mathcal{F}_1}, \alpha_2 \in \mathcal{A}_{\mathcal{F}_2}} \Delta C_{\phi, \mathcal{H}_g}(\alpha_1, \alpha_2, \eta) & \text{if } |2\eta - 1| < \epsilon \leq \max\{\eta, 1 - \eta\}, \\ \inf_{\alpha_1 \in \mathcal{A}_{\mathcal{F}_1}, \alpha_2 \in \mathcal{A}_{\mathcal{F}_2}} \Delta C_{\phi, \mathcal{H}_g}(\alpha_1, \alpha_2, \eta) & \text{if } \epsilon \leq |2\eta - 1|. \end{cases}$$

**Proof** The inner $\ell_\gamma$-risk is

$$C_{\ell_\gamma}(\alpha_1, \alpha_2, \eta) = \eta \mathbb{I}_{g(\alpha_1 - \gamma) + \alpha_2 \leq 0} + (1 - \eta) \mathbb{I}_{g(\alpha_1 + \gamma) + \alpha_2 \geq 0}$$

$$= \begin{cases} 1 & \text{if } -g(\alpha_1 + \gamma) \leq \alpha_2 \leq -g(\alpha_1 - \gamma), \\ \eta & \text{if } \alpha_2 < -g(\alpha_1 + \gamma), \\ 1 - \eta & \text{if } \alpha_2 > -g(\alpha_1 - \gamma). \end{cases}$$

Since $-G < -g(1 + \gamma)$ and $G > -g(-1 - \gamma)$, the pseudo-minimal inner $\ell_\gamma$-risk is

$$C_{\ell_\gamma, \mathcal{H}_g}^\star(\eta) = \min\{\eta, 1 - \eta\}.$$

Then, it can be computed that

$$\Delta C_{\ell_\gamma, \mathcal{H}_g}(\alpha_1, \alpha_2, \eta) = \begin{cases} \max\{\eta, 1 - \eta\} & \text{if } -g(\alpha_1 + \gamma) \leq \alpha_2 \leq -g(\alpha_1 - \gamma), \\ |2\eta - 1| \mathbb{I}_{(2\eta - 1)(\alpha_2 + g(\alpha_1 + \gamma)) \leq 0} & \text{if } \alpha_2 > -g(\alpha_1 - \gamma) \text{ or } \alpha_2 < -g(\alpha_1 + \gamma). \end{cases}$$
By definition, for a fixed \( \eta \in [0, 1] \),

\[
\tilde{\delta}(\epsilon, \eta) = \inf_{\alpha_1 \in \mathcal{A}_{\mathcal{F}_1}, \alpha_2 \in \mathcal{A}_{\mathcal{F}_2}} \{ \Delta C_{\phi, \mathcal{H}_g} (\alpha_1, \alpha_2, \eta) \mid \Delta C_{\phi, \mathcal{H}_g} (\alpha_1, \alpha_2, \eta) \geq \epsilon \}.
\]

If \( \epsilon > \max\{\eta, 1 - \eta\} \), then for all \( \alpha_1 \in \mathcal{A}_{\mathcal{F}_1}, \alpha_2 \in \mathcal{A}_{\mathcal{F}_2}, \Delta C_{\phi, \mathcal{H}_g} (\alpha_1, \alpha_2, \eta) < \epsilon \), which implies that \( \tilde{\delta}(\epsilon, \eta) = \infty \). If \([2\eta - 1] < \epsilon \leq \max\{\eta, 1 - \eta\} \), then \( \Delta C_{\phi, \mathcal{H}_g} (\alpha_1, \alpha_2, \eta) \geq \epsilon \) is achieved when \( -g(\alpha_1 + \gamma) \leq \alpha_2 \leq -g(\alpha_1 - \gamma) \), which leads to \( \tilde{\delta}(\epsilon, \eta) = \inf_{\alpha_1 \in \mathcal{A}_{\mathcal{F}_1}, \alpha_2 \in \mathcal{A}_{\mathcal{F}_2}} \Delta C_{\phi, \mathcal{H}_g} (\alpha_1, \alpha_2, \eta) \). If \( \epsilon \leq |2\eta - 1| \), then \( \Delta C_{\phi, \mathcal{H}_g} (\alpha_1, \alpha_2, \eta) \geq \epsilon \) is achieved when \( -g(\alpha_1 + \gamma) \leq \alpha_2 \leq -g(\alpha_1 - \gamma) \) or \( (2\eta - 1)(\alpha_2 + g(\alpha_1 + \gamma)) \leq 0 \). Therefore, \( \tilde{\delta}(\epsilon, \eta) = \inf_{\alpha_1 \in \mathcal{A}_{\mathcal{F}_1}, \alpha_2 \in \mathcal{A}_{\mathcal{F}_2}} \Delta C_{\phi, \mathcal{H}_g} (\alpha_1, \alpha_2, \eta) \).

Note that in our setting, \( \alpha_1 \in [-1, 1] \) and \( \alpha_2 \in [-G, G] \). Therefore \( g(\alpha_1) + \alpha_2 \in [g(-1) - G, g(1) + G] \), since \( g \) is continuous. Then,

\[
\{ g(\alpha_1) + \alpha_2 : \alpha_1 \in \mathcal{A}_{\mathcal{F}_1}, \alpha_2 \in \mathcal{A}_{\mathcal{F}_2}, -g(\alpha_1 + \gamma) \leq \alpha_2 \leq -g(\alpha_1 - \gamma) \} = \left[ \inf_{\alpha_1 \in [-1, 1]} g(\alpha_1) - g(\alpha_1 + \gamma), \sup_{\alpha_1 \in [-1, 1]} g(\alpha_1) - g(\alpha_1 - \gamma) \right],
\]

\[
\{ g(\alpha_1) + \alpha_2 : \alpha_1 \in \mathcal{A}_{\mathcal{F}_1}, \alpha_2 \in \mathcal{A}_{\mathcal{F}_2}, -g(\alpha_1 - \gamma) \leq \alpha_2 \leq -g(\alpha_1 + \gamma) \} = [g(-1) - G, \sup_{\alpha_1 \in [-1, 1]} g(\alpha_1) - g(\alpha_1 - \gamma)],
\]

\[
\{ g(\alpha_1) + \alpha_2 : \alpha_1 \in \mathcal{A}_{\mathcal{F}_1}, \alpha_2 \in \mathcal{A}_{\mathcal{F}_2}, \alpha_2 \geq -g(\alpha_1 + \gamma) \} = \left[ \inf_{\alpha_1 \in [-1, 1]} g(\alpha_1) - g(\alpha_1 + \gamma), g(1) + G \right].
\]

Since \( g \) is non-decreasing, we have \( g(\alpha_1) - g(\alpha_1 + \gamma) \leq 0 \) and \( g(\alpha_1) - g(\alpha_1 - \gamma) \geq 0 \) for any \( \alpha_1 \in [-1, 1] \). Also, \( -g(\alpha_1) \in [-g(1), -g(-1)] \subset [-G, G] \). Therefore,

\[
0 \in \{ g(\alpha_1) + \alpha_2 : \alpha_1 \in \mathcal{A}_{\mathcal{F}_1}, \alpha_2 \in \mathcal{A}_{\mathcal{F}_2}, -g(\alpha_1 + \gamma) \leq \alpha_2 \leq -g(\alpha_1 - \gamma) \} \, \quad \text{(12)}
\]

**Theorem 10** Let \( g \) be a non-decreasing and continuous function such that \( g(1 + \gamma) < G \) and \( g(-1 - \gamma) > -G \). If a margin-based loss \( \phi : \mathbb{R} \to \mathbb{R}_+ \) is convex, then it is not \( \mathcal{H}_g \)-calibrated with respect to \( \ell_\gamma \).

**Proof** Suppose that \( \phi \) is pseudo-\( \mathcal{H}_g \)-calibrated with respect to \( \ell_\gamma \). By Proposition 5, \( \phi \) is pseudo-\( \mathcal{H}_g \)-calibrated with respect to \( \ell_\gamma \) if and only if its pseudo-calibration function \( \tilde{\delta} \) satisfies \( \tilde{\delta}(\epsilon) > 0 \) for all \( \epsilon > 0 \) and \( \eta \in [0, 1] \). By lemma 29, take \( \eta = \frac{1}{2} \), we obtain

\[
\inf_{\alpha_1 \in \mathcal{A}_{\mathcal{F}_1}, \alpha_2 \in \mathcal{A}_{\mathcal{F}_2} : -g(\alpha_1 + \gamma) \leq \alpha_2 \leq -g(\alpha_1 - \gamma)} \Delta C_{\phi, \mathcal{H}_g} (\alpha_1, \alpha_2, \frac{1}{2}) > 0
\]

which is equivalent to

\[
\inf_{\alpha_1 \in \mathcal{A}_{\mathcal{F}_1}, \alpha_2 \in \mathcal{A}_{\mathcal{F}_2} : -g(\alpha_1 + \gamma) \leq \alpha_2 \leq -g(\alpha_1 - \gamma)} C_{\phi} (\alpha_1, \alpha_2, \frac{1}{2}) > \inf_{\alpha_1 \in \mathcal{A}_{\mathcal{F}_1}, \alpha_2 \in \mathcal{A}_{\mathcal{F}_2}} C_{\phi} (\alpha_1, \alpha_2, \frac{1}{2}). \quad \text{(13)}
\]

By the definition of inner risk,

\[
C_{\phi} (\alpha_1, \alpha_2, \frac{1}{2}) = \frac{1}{2} \phi(g(\alpha_1) + \alpha_2) + \frac{1}{2} \phi(-g(\alpha_1) - \alpha_2).
\]
Define $\bar{\phi}(\alpha_1, \alpha_2) = \phi(g(\alpha_1) + \alpha_2) + \phi(-g(\alpha_1) - \alpha_2)$. By Jensen’s inequality, $\phi(0) \leq \frac{1}{2} \bar{\phi}(\alpha_1, \alpha_2)$ for all $\alpha_1 \in \mathcal{A}_F$, $\alpha_2 \in \mathcal{A}_S$.

Since $C_\phi(\alpha_1, \alpha_2, \frac{1}{2}) = \frac{1}{2} \bar{\phi}(\alpha_1, \alpha_2)$ and (12), we obtain

$$\inf_{\alpha_1 \in \mathcal{A}_F, \alpha_2 \in \mathcal{A}_S ; -g(1+\gamma) \leq \alpha_2 \leq g(1-\gamma)} C_\phi(\alpha_1, \alpha_2, \frac{1}{2}) = \inf_{\alpha_1 \in \mathcal{A}_F, \alpha_2 \in \mathcal{A}_S} C_\phi(\alpha_1, \alpha_2, \frac{1}{2}) = \phi(0),$$

contradicting (13). Therefore, $\phi$ is not pseudo-$\mathcal{H}_g$-calibrated with respect to $\ell_\gamma$. By Corollary 7, $\phi$ is also not $\mathcal{H}_g$-calibrated with respect to $\ell_\gamma$.

**C.3. Proof of Theorem 16**

Following the notations in Appendix C.2, we first give equivalent conditions of pseudo-calibration based on inner risk of $\phi$ and $\mathcal{H}_g$.

**Lemma 30** Let $\phi$ be a non-decreasing and continuous function $g$ such that $g(1+\gamma) < G$ and $g(-1-\gamma) > -G$. Let $\phi$ be a margin-based loss. Then $\phi$ is pseudo-$\mathcal{H}_g$-calibrated with respect to $\ell_\gamma$ if and only if

$$\inf_{\alpha_1 \in \mathcal{A}_F, \alpha_2 \in \mathcal{A}_S \colon -g(1+\gamma) \leq \alpha_2 \leq g(1-\gamma)} C_\phi(\alpha_1, \alpha_2, \frac{1}{2}) > \inf_{\alpha_1 \in \mathcal{A}_F, \alpha_2 \in \mathcal{A}_S} C_\phi(\alpha_1, \alpha_2, \frac{1}{2}),$$

and

$$\inf_{\alpha_1 \in \mathcal{A}_F, \alpha_2 \in \mathcal{A}_S \colon 0 \leq \alpha_2 \leq g(1-\gamma)} C_\phi(\alpha_1, \alpha_2, \frac{1}{2}) > \inf_{\alpha_1 \in \mathcal{A}_F, \alpha_2 \in \mathcal{A}_S} C_\phi(\alpha_1, \alpha_2, \frac{1}{2})$$

for all $\eta \in [0, \frac{1}{2})$.

**Proof** Let $\hat{\phi}$ be the pseudo-calibration function of $(\phi, \ell_\gamma)$ for hypothesis sets $\mathcal{H}_g$. By Lemma 29, $\hat{\phi}(\epsilon) = \inf_{\eta \in [0, 1]} \hat{\phi}(\epsilon, \eta)$, where

$$\hat{\phi}(\epsilon, \eta) = \begin{cases} +\infty & \text{if } \epsilon > \max\{\eta, 1-\eta\}, \\ \inf_{\alpha_1 \in \mathcal{A}_F, \alpha_2 \in \mathcal{A}_S ; -g(1+\gamma) \leq \alpha_2 \leq g(1-\gamma)} \Delta C_{\phi, \mathcal{H}_g}(\alpha_1, \alpha_2, \eta) & \text{if } |2\eta - 1| < \epsilon \leq \max\{\eta, 1-\eta\}, \\ \inf_{\alpha_1 \in \mathcal{A}_F, \alpha_2 \in \mathcal{A}_S ; -g(1+\gamma) \leq \alpha_2 \leq g(1-\gamma)} \Delta C_{\phi, \mathcal{H}_g}(\alpha_1, \alpha_2, \eta) & \text{if } \epsilon \leq |2\eta - 1|. \end{cases}$$

By Proposition 5, $\phi$ is pseudo-$\mathcal{H}_g$-calibrated with respect to $\ell_\gamma$ if and only if its pseudo-calibration function $\hat{\phi}$ satisfies $\hat{\phi}(\epsilon) > 0$ for all $\epsilon > 0$. This is equivalent to $\hat{\phi}(\epsilon, \eta) > 0$ for all $\epsilon > 0$ and $\eta \in [0, 1]$. For $\eta = \frac{1}{2}$, we have

$$\hat{\phi}(\epsilon, \frac{1}{2}) > 0 \text{ for all } \epsilon > 0$$

$$\iff \inf_{\alpha_1 \in \mathcal{A}_F, \alpha_2 \in \mathcal{A}_S \colon -g(1+\gamma) \leq \alpha_2 \leq g(1-\gamma)} C_\phi(\alpha_1, \alpha_2, \frac{1}{2}) > \inf_{\alpha_1 \in \mathcal{A}_F, \alpha_2 \in \mathcal{A}_S} C_\phi(\alpha_1, \alpha_2, \frac{1}{2}).$$
For $1 \geq \eta > \frac{1}{2}$, we have $|2\eta - 1| = 2\eta - 1$, $\max\{\eta, 1 - \eta\} = \eta$, and

\[
\inf_{\alpha_1 \in A_{\phi_1}, \alpha_2 \in A_{\phi_2} : -g(\alpha_1 + \gamma) \leq \alpha_2 \leq -g(\alpha_1 - \gamma) \text{ or } (2\eta - 1)(\alpha_2 + g(\alpha_1 + \gamma)) \leq 0} \Delta C_{\phi, g_\gamma}(\alpha_1, \alpha_2, \eta) \\
= \inf_{\alpha_1 \in A_{\phi_1}, \alpha_2 \in A_{\phi_2} : \alpha_2 \leq -g(\alpha_1 - \gamma)} \Delta C_{\phi, g_\gamma}(\alpha_1, \alpha_2, \eta).
\]

Therefore, $\delta(\epsilon, \eta) > 0$ for all $\epsilon > 0$ and $\eta \in \left(\frac{1}{2}, 1\right]$ if and only if

\[
\begin{cases}
\inf_{\alpha_1 \in A_{\phi_1}, \alpha_2 \in A_{\phi_2} : -g(\alpha_1 + \gamma) \leq \alpha_2 \leq -g(\alpha_1 - \gamma)} C_{\phi}(\alpha_1, \alpha_2, \eta) > \inf_{\alpha_1 \in A_{\phi_1}} C_{\phi}(\alpha_1, \alpha_2, \eta) & \text{for } \eta \in \left(\frac{1}{2}, 1\right] \text{ s.t. } 2\eta - 1 < \epsilon \leq \eta, \\
\inf_{\alpha_1 \in A_{\phi_1}, \alpha_2 \in A_{\phi_2} : \alpha_2 \leq -g(\alpha_1 - \gamma)} C_{\phi}(\alpha_1, \alpha_2, \eta) > \inf_{\alpha_1 \in A_{\phi_1}, \alpha_2 \in A_{\phi_2}} C_{\phi}(\alpha_1, \alpha_2, \eta) & \text{for } \eta \in \left(\frac{1}{2}, 1\right] \text{ s.t. } \epsilon \leq 2\eta - 1,
\end{cases}
\]

for all $\epsilon > 0$, which is equivalent to

\[
\begin{cases}
\inf_{\alpha_1 \in A_{\phi_1}, \alpha_2 \in A_{\phi_2} : -g(\alpha_1 + \gamma) \leq \alpha_2 \leq -g(\alpha_1 - \gamma)} C_{\phi}(\alpha_1, \alpha_2, \eta) > \inf_{\alpha_1 \in A_{\phi_1}} C_{\phi}(\alpha_1, \alpha_2, \eta) & \text{for } \eta \in \left(\frac{1}{2}, 1\right] \text{ s.t. } \epsilon \leq \frac{\epsilon + 1}{2}, \\
\inf_{\alpha_1 \in A_{\phi_1}, \alpha_2 \in A_{\phi_2} : \alpha_2 \leq -g(\alpha_1 - \gamma)} C_{\phi}(\alpha_1, \alpha_2, \eta) > \inf_{\alpha_1 \in A_{\phi_1}, \alpha_2 \in A_{\phi_2}} C_{\phi}(\alpha_1, \alpha_2, \eta) & \text{for } \eta \in \left(\frac{1}{2}, 1\right] \text{ s.t. } \frac{\epsilon + 1}{2} \leq \eta,
\end{cases}
\]

for all $\epsilon > 0$. We observe that

\[
\begin{cases}
\eta \in \left(\frac{1}{2}, 1\right] \quad \epsilon \leq \frac{\epsilon + 1}{2}, \quad \epsilon > 0 \quad \Rightarrow \quad \left\{ \frac{1}{2} < \eta \leq 1 \right\}, \quad \text{and} \\
\eta \in \left(\frac{1}{2}, 1\right] \quad \frac{\epsilon + 1}{2} \leq \eta, \quad \epsilon > 0 \quad \Rightarrow \quad \left\{ \frac{1}{2} < \eta \leq 1 \right\}, \quad \text{and}
\end{cases}
\]

\[
\inf_{\alpha_1 \in A_{\phi_1}, \alpha_2 \in A_{\phi_2} : -g(\alpha_1 + \gamma) \leq \alpha_2 \leq -g(\alpha_1 - \gamma)} C_{\phi}(\alpha_1, \alpha_2, \eta) \geq \inf_{\alpha_1 \in A_{\phi_1}, \alpha_2 \in A_{\phi_2} : \alpha_2 \leq -g(\alpha_1 - \gamma)} C_{\phi}(\alpha_1, \alpha_2, \eta) \quad \text{for all } \eta.
\]

Therefore we reduce the above condition (15) as

\[
\inf_{\alpha_1 \in A_{\phi_1}, \alpha_2 \in A_{\phi_2} : \alpha_2 \leq -g(\alpha_1 - \gamma)} C_{\phi}(\alpha_1, \alpha_2, \eta) > \inf_{\alpha_1 \in A_{\phi_1}, \alpha_2 \in A_{\phi_2}} C_{\phi}(\alpha_1, \alpha_2, \eta) \quad \text{for all } \eta \in \left(\frac{1}{2}, 1\right]. \quad (16)
\]

For $\frac{1}{2} > \eta \geq 0$, we have $|2\eta - 1| = 2\eta - 1$, $\max\{\eta, 1 - \eta\} = 1 - \eta$, and

\[
\inf_{\alpha_1 \in A_{\phi_1}, \alpha_2 \in A_{\phi_2} : -g(\alpha_1 + \gamma) \leq \alpha_2 \leq -g(\alpha_1 - \gamma) \text{ or } (2\eta - 1)(\alpha_2 + g(\alpha_1 + \gamma)) \leq 0} \Delta C_{\phi, g_\gamma}(\alpha_1, \alpha_2, \eta) \\
= \inf_{\alpha_1 \in A_{\phi_1}, \alpha_2 \in A_{\phi_2} : \alpha_2 \leq -g(\alpha_1 + \gamma)} \Delta C_{\phi, g_\gamma}(\alpha_1, \alpha_2, \eta).
\]

Therefore, $\delta(\epsilon, \eta) > 0$ for all $\epsilon > 0$ and $\eta \in \left[0, \frac{1}{2}\right)$ if and only if

\[
\begin{cases}
\inf_{\alpha_1 \in A_{\phi_1}, \alpha_2 \in A_{\phi_2} : -g(\alpha_1 + \gamma) \leq \alpha_2 \leq -g(\alpha_1 - \gamma)} C_{\phi}(\alpha_1, \alpha_2, \eta) > \inf_{\alpha_1 \in A_{\phi_1}} C_{\phi}(\alpha_1, \alpha_2, \eta) \quad \text{for } \eta \in \left(0, \frac{1}{2}\right) \text{ s.t. } 1 - 2\eta < \epsilon \leq 1 - \eta, \\
\inf_{\alpha_1 \in A_{\phi_1}, \alpha_2 \in A_{\phi_2} : \alpha_2 \leq -g(\alpha_1 + \gamma)} C_{\phi}(\alpha_1, \alpha_2, \eta) > \inf_{\alpha_1 \in A_{\phi_1}, \alpha_2 \in A_{\phi_2}} C_{\phi}(\alpha_1, \alpha_2, \eta) \quad \text{for } \eta \in \left(0, \frac{1}{2}\right) \text{ s.t. } \epsilon \leq 1 - 2\eta,
\end{cases}
\]
for all \( \epsilon > 0 \), which is equivalent to
\[
\begin{align*}
\inf_{\alpha_1 \in \mathcal{A}_1, \alpha_2 \in \mathcal{A}_2; \, -g(\alpha_1 + \gamma) \leq \alpha_2 \leq -g(\alpha_1 - \gamma)} C_\phi(\alpha_1, \alpha_2, \eta) > \inf_{\alpha_1 \in \mathcal{A}_1, \, \alpha_2 \in \mathcal{A}_2} C_\phi(\alpha_1, \alpha_2, \eta) \quad \text{for} \quad \eta \in \left[0, \frac{1}{2}\right) \, \text{s.t.} \, \frac{1-\epsilon}{2} < \eta \leq 1 - \epsilon,
\end{align*}
\]
for all \( \epsilon > 0 \). We observe that
\[
\left\{ \eta \in \left[0, \frac{1}{2}\right] \mid \frac{1-\epsilon}{2} < \eta \leq 1 - \epsilon, \epsilon > 0 \right\} = \left\{ 0 \leq \eta < \frac{1}{2} \right\}, \quad \text{and}
\]
\[
\left\{ \eta \in \left[0, \frac{1}{2}\right] \mid \eta \leq \frac{1-\epsilon}{2}, \epsilon > 0 \right\} = \left\{ 0 \leq \eta < \frac{1}{2} \right\}, \quad \text{and}
\]
\[
\inf_{\alpha_1 \in \mathcal{A}_1, \alpha_2 \in \mathcal{A}_2; \, -g(\alpha_1 + \gamma) \leq \alpha_2 \leq -g(\alpha_1 - \gamma)} C_\phi(\alpha_1, \alpha_2, \eta) \geq \inf_{\alpha_1 \in \mathcal{A}_1, \alpha_2 \in \mathcal{A}_2} C_\phi(\alpha_1, \alpha_2, \eta) \quad \text{for all} \quad \eta.
\]
Therefore we reduce the above condition (17) as
\[
\inf_{\alpha_1 \in \mathcal{A}_1, \alpha_2 \in \mathcal{A}_2; \, \alpha_2 \leq -g(\alpha_1 + \gamma)} C_\phi(\alpha_1, \alpha_2, \eta) > \inf_{\alpha_1 \in \mathcal{A}_1, \alpha_2 \in \mathcal{A}_2} C_\phi(\alpha_1, \alpha_2, \eta) \quad \text{for all} \quad \eta \in \left[0, \frac{1}{2}\right).
\]  
(18)

To sum up, by (14), (16) and (18), we conclude the proof.

Define \( \overline{A} = \sup_{\alpha_1 \in [-1,1]} g(\alpha_1) - g(\alpha_1 - \gamma) \) and \( \underline{A} = \inf_{\alpha_1 \in [-1,1]} g(\alpha_1) - g(\alpha_1 + \gamma) \). Since \( g \) is non-decreasing, we have \( \overline{A} \geq 0 \) and \( \underline{A} \leq 0 \). Note the inner risk \( C_\phi(\alpha_1, \alpha_2, \eta) \) only depends on \( t = g(\alpha_1) + \alpha_2 \) and \( \eta \). Therefore, we can rewrite the inner risk and pseudo-minimal inner risk of \( \phi \) as
\[
C_\phi(t, \eta) = \eta \phi(t) + (1 - \eta)\phi(-t), \quad C_{\phi, \mathcal{G}_G}(\eta) = \inf_{g(-1) - G \leq t \leq g(-1) + G} C_\phi(t, \eta),
\]
By (11), Lemma 30 is equivalent to Lemma 31.

**Lemma 31** Given a non-decreasing and continuous function \( g \) such that \( g(1 + \gamma) < G \) and \( g(-1 - \gamma) > -G \). Let \( \phi \) be a margin-based loss. Then \( \phi \) is pseudo-\( \mathcal{H}_G \)-calibrated with respect to \( \ell \), if and only if
\[
\inf_{\Delta \leq \overline{A}} C_\phi(t, \frac{1}{2}) > \inf_{g(-1) - G \leq t \leq g(-1) + G} C_\phi(t, \frac{1}{2}), \quad \text{and}
\]
\[
\inf_{g(-1) - G \leq t \leq g(-1) + G} C_\phi(t, \eta) > \inf_{g(-1) - G \leq t \leq g(-1) + G} C_\phi(t, \eta) \quad \text{for all} \quad \eta \in \left(\frac{1}{2}, 1\right], \quad \text{and}
\]
\[
\inf_{\Delta \leq \overline{A}} C_\phi(t, \eta) > \inf_{g(-1) - G \leq t \leq g(-1) + G} C_\phi(t, \eta) \quad \text{for all} \quad \eta \in \left[0, \frac{1}{2}\right).
\]

Lemma 32 concludes some results that will be useful in the proof of Theorem 16.

**Lemma 32** Let \( \phi \) be a margin-based loss. If \( \phi \) is bounded, continuous, non-increasing, quasi-concave even, and assume \( \phi(g(-1) - G) > \phi(G - g(-1)) \), \( g(-1) + g(1) \geq 0 \), then
1. The inner $\phi$-risk $\mathcal{C}_\phi(t, \eta)$ is quasi-concave in $t \in \mathbb{R}$ for all $\eta \in [0, 1]$.

2. $\phi(t) + \phi(-t)$ is non-increasing in $t$ when $t \geq 0$.

3. For $l, u \in \mathbb{R}$ ($l \leq u$), $\inf_{t \in [l, u]} \mathcal{C}_\phi(t, \eta) = \min \{\mathcal{C}_\phi(l, \eta), \mathcal{C}_\phi(u, \eta)\}$ for all $\eta \in [0, 1]$.

4. For all $\eta \in (\frac{1}{2}, 1]$, $\mathcal{C}_\phi(t, \eta)$ is non-increasing in $t$ when $t \geq 0$.

5. For all $\eta \in (\frac{1}{2}, 1]$, $\mathcal{C}_\phi(g(-1) - G, \eta) > \mathcal{C}_\phi(g(1) + G, \eta)$.

6. For all $\eta \in [0, \frac{1}{2})$, $\mathcal{C}_\phi(t, \eta)$ is non-decreasing in $t$ when $t \leq 0$.

7. For all $\eta \in [0, \frac{1}{2})$, $\mathcal{C}_\phi(g(-1) - G, \eta) < \mathcal{C}_\phi(g(1) + G, \eta)$ if and only if $\phi(G - g(-1)) + \phi(g(-1) - G) = \phi(g(1) + G) + \phi(-g(1) - G)$.

**Proof** Part 1, 2, 4 of Lemma 32 are stated in Lemma 13 of (Bao et al., 2020). Part 3 is a corollary of Part 1 by the characterization of continuous and quasi-convex functions in (Boyd and Vandenberghe, 2014).

Consider Part 5. For $\eta \in (\frac{1}{2}, 1]$,
\[
\mathcal{C}_\phi(g(-1) - G, \eta) - \mathcal{C}_\phi(g(1) + G, \eta)
\geq \mathcal{C}_\phi(g(-1) - G, \eta) - \mathcal{C}_\phi(G - g(-1), \eta)
\]
\[
=(2\eta - 1)(\phi(g(-1) - G) - \phi(G - g(-1)))
\]
\[
> 0.
\]

Consider Part 6. For $\eta \in [0, \frac{1}{2})$, and $\alpha_1, \alpha_2 \leq 0$. Suppose that $\alpha_1 < \alpha_2$, then
\[
\phi(\alpha_1) - \phi(-\alpha_1) - \phi(\alpha_2) + \phi(-\alpha_2)
\geq \phi(\alpha_2) - \phi(-\alpha_2) - \phi(\alpha_2) + \phi(-\alpha_2)
\]
\[
= 0,
\]

since $\phi$ is non-increasing.

By Part 2 of Lemma 32, $\phi(t) + \phi(-t)$ is non-decreasing in $t$ when $t \leq 0$.

Therefore, for $\eta \in [0, \frac{1}{2})$,
\[
\mathcal{C}_\phi(\alpha_1, \eta) - \mathcal{C}_\phi(\alpha_2, \eta)
\]
\[
=(\phi(\alpha_1) - \phi(-\alpha_1) - \phi(\alpha_2) + \phi(-\alpha_2))\eta + \phi(-\alpha_1) - \phi(-\alpha_2)
\]
\[
\leq (\phi(\alpha_1) - \phi(-\alpha_1) - \phi(\alpha_2) + \phi(-\alpha_2))\frac{1}{2} + \phi(-\alpha_1) - \phi(-\alpha_2)
\]
\[
=\frac{1}{2}(\phi(\alpha_1) + \phi(-\alpha_1) - \phi(\alpha_2) - \phi(-\alpha_2))
\]
\[
\leq 0.
\]

Consider Part 7. Since $\phi$ is non-increasing, we have
\[
\phi(g(-1) - G) - \phi(G - g(-1)) + \phi(-g(1) - G) - \phi(g(1) + G)
\geq \phi(g(-1) - G) - \phi(G - g(-1)) + \phi(g(1) + G) - \phi(g(1) + G)
\]
\[
= \phi(g(-1) - G) - \phi(G - g(-1))
\]
\[
> 0.
\]
\[\iffalse \text{Suppose } \phi(G - g(-1)) + \phi(g(-1) - G) = \phi(g(1) + G) + \phi(-g(1) - G), \text{ then for } \eta \in [0, \frac{1}{2}], \]
\[C_\phi(g(-1) - G, \eta) - C_\phi(g(1) + G, \eta) \]
\[= (\phi(g(-1) - G) - \phi(G - g(-1)) + \phi(-g(1) - G) - \phi(g(1) + G))\eta \]
\[+ \phi(G - g(-1)) - \phi(-g(1) - G) \]
\[< (\phi(g(-1) - G) - \phi(G - g(-1)) + \phi(-g(1) - G) - \phi(g(1) + G)) \frac{1}{2} \]
\[+ \phi(G - g(-1)) - \phi(-g(1) - G) \]
\[= \frac{1}{2}(\phi(G - g(-1)) + \phi(-g(1) - G) - \phi(g(1) + G) - \phi(-g(1) - G)) \]
\[= 0. \]
\[\implies \text{Suppose } C_\phi(g(-1) - G, \eta) < C_\phi(g(1) + G, \eta) \text{ for } \eta \in [0, \frac{1}{2}], \text{ then } \]
\[C_\phi(g(-1) - G, \eta) - C_\phi(g(1) + G, \eta) \]
\[= (\phi(g(-1) - G) - \phi(G - g(-1)) + \phi(-g(1) - G) - \phi(g(1) + G))\eta \]
\[+ \phi(G - g(-1)) - \phi(-g(1) - G) \]
\[< 0 \]
\[\text{for } \eta \in [0, \frac{1}{2}). \text{ By taking } \eta \to \frac{1}{2}, \text{ we have } \]
\[\frac{1}{2}(\phi(G - g(-1)) + \phi(g(-1) - G) - \phi(g(1) + G) - \phi(-g(1) - G)) \]
\[= (\phi(g(-1) - G) - \phi(G - g(-1)) + \phi(-g(1) - G) - \phi(g(1) + G)) \frac{1}{2} \]
\[+ \phi(G - g(-1)) - \phi(-g(1) - G) \]
\[\leq 0. \]
By Part 2 of Lemma 32, we have
\[\phi(G - g(-1)) + \phi(g(-1) - G) - \phi(g(1) + G) - \phi(-g(1) - G) \]
\[\geq \phi(g(1) + G) + \phi(-g(1) - G) - \phi(g(1) + G) - \phi(-g(1) - G) \]
\[= 0. \]
Therefore, we obtain \(\phi(G - g(-1)) + \phi(g(-1) - G) - \phi(g(1) + G) - \phi(-g(1) - G) = 0\), namely \(\phi(G - g(-1)) + \phi(g(-1) - G) = \phi(g(1) + G) + \phi(-g(1) - G)\).

**Theorem 16** Let \(g\) be a non-decreasing and continuous function such that \(g(1 + \gamma) < G\) and \(g(-1 - \gamma) > -G\) for some \(G > 0\). Let a margin-based loss \(\phi\) be bounded, continuous, non-increasing, and quasi-concave even. Assume that \(\phi(g(-1) - G) > \phi(G - g(-1))\) and \(g(-1) + g(1) \geq 0\). Then \(\phi\) is pseudo-\(\mathcal{H}_g\)-calibrated with respect to \(\ell_2\) if and only if
\[\phi(G - g(-1)) + \phi(g(-1) - G) = \phi(g(1) + G) + \phi(-g(1) - G) \]
and \(\min\{\phi(\overline{A}) + \phi(-\overline{A}), \phi(\overline{A}) + \phi(-\overline{A})\} > \phi(G - g(-1)) + \phi(g(-1) - G),\)
where \(\overline{A} = \sup_{\alpha \in [-1,1]} g(\alpha) - g(\alpha - \gamma)\) and \(\overline{A} = \inf_{\alpha \in [-1,1]} g(\alpha) - g(\alpha + \gamma)\).
Proof By Lemma 31, $\phi$ is pseudo-$\mathcal{H}_g$-calibrated with respect to $\ell_\gamma$ if and only if

$$\inf_{A \in \mathcal{A}} C_\phi(t, \frac{1}{2}) > \inf_{g(-1) - G \leq t \leq g(1) + G} C_\phi(t, \frac{1}{2}),$$

and

$$\inf_{g(-1) - G \leq t \leq g(1) + G} C_\phi(t, \eta) > \inf_{g(-1) - G \leq t \leq g(1) + G} C_\phi(t, \eta) \text{ for all } \eta \in (\frac{1}{2}, 1],$$

and

$$\inf_{A \in \mathcal{A}} C_\phi(t, \eta) > \inf_{g(-1) - G \leq t \leq g(1) + G} C_\phi(t, \eta) \text{ for all } \eta \in [0, \frac{1}{2}).$$

Suppose that $\phi$ is pseudo-$\mathcal{H}_g$-calibrated with respect to $\ell_\gamma$. Since for $\eta \in [0, \frac{1}{2}),$

$$\inf_{A \in \mathcal{A}} C_\phi(t, \eta) = \min\{C_\phi(A, \eta), C_\phi(g(1) + G, \eta)\} \quad \text{(Part 3 of Lemma 32)}$$

$$\inf_{g(-1) - G \leq t \leq g(1) + G} C_\phi(t, \eta) = \min\{C_\phi(g(-1) - G, \eta), C_\phi(g(1) + G, \eta)\} \quad \text{(Part 3 of Lemma 32)}$$

we have $C_\phi(g(-1) - G, \eta) < C_\phi(g(1) + G, \eta)$, otherwise $\inf_{A \in \mathcal{A}} C_\phi(t, \eta) \leq C_\phi(g(1) + G, \eta) = \inf_{g(-1) - G \leq t \leq g(1) + G} C_\phi(t, \eta)$. By Part 7 of Lemma 32, $\phi(G - g(-1)) + \phi(g(-1) - G) = \phi(g(1) + G) + \phi(-g(1) - G)$.

Also,

$$\frac{1}{2} \min\{\phi(A) + \phi(-A), \phi(A) + \phi(-A)\}$$

$$= \inf_{A \in \mathcal{A}} C_\phi(t, \frac{1}{2}) \quad \text{(Part 3 of Lemma 32)}$$

$$> \inf_{g(-1) - G \leq t \leq g(1) + G} C_\phi(t, \frac{1}{2}) \quad \text{(Lemma 31)}$$

$$= \frac{1}{2} \min\{\phi(G - g(-1)) + \phi(g(-1) - G), \phi(g(1) + G) + \phi(-g(1) - G)\} \quad \text{(Part 3 of Lemma 32)}$$

$$= \frac{1}{2}(\phi(G - g(-1)) + \phi(-g(1) - G))$$

Now for the other direction, assume that $\phi(G - g(-1)) + \phi(g(-1) - G) = \phi(g(1) + G) + \phi(-g(1) - G)$ and $\min\{\phi(A) + \phi(-A), \phi(A) + \phi(-A)\} > \phi(G - g(-1)) + \phi(g(-1) - G)$. Similarly,

$$\inf_{A \in \mathcal{A}} C_\phi(t, \frac{1}{2})$$

$$= \frac{1}{2} \min\{\phi(A) + \phi(-A), \phi(A) + \phi(-A)\}$$

$$> \frac{1}{2}(\phi(G - g(-1)) + \phi(-g(1) - G))$$

$$= \frac{1}{2} \min\{\phi(G - g(-1)) + \phi(g(-1) - G), \phi(g(1) + G) + \phi(-g(1) - G)\}$$

$$= \inf_{g(-1) - G \leq t \leq g(1) + G} C_\phi(t, \frac{1}{2}).$$
For $\eta \in \left(\frac{1}{2}, 1\right]$, 
\[
\inf_{g(-1)-G \leq t \leq A} C_\phi(t, \eta) = \min \{C_\phi(g(-1) - G, \eta), C_\phi(A, \eta)\} \quad \text{(Part 3 of Lemma 32)}
\]
\[
\inf_{g(-1)-G \leq g(1)+G} C_\phi(t, \eta) = \min \{C_\phi(g(-1) - G, \eta), C_\phi(g(1) + G, \eta)\} \quad \text{(Part 3 of Lemma 32)}
\]
\[
= C_\phi(g(1) + G, \eta) \quad \text{(Part 5 of Lemma 32)}
\]
Since $\phi$ is non-increasing, we have 
\[
\phi(g(1) - G) - \phi(g(1) + G) + \phi(-A) - \phi(-A) = 0.
\]
Then for $\eta \in \left(\frac{1}{2}, 1\right]$, 
\[
\begin{align*}
C_\phi(A, \eta) - C_\phi(g(1) + G, \eta) 
&= (\phi(A) - \phi(-A) + \phi(g(1) - G) - \phi(g(1) + G)) \eta + \phi(g(1) + G) - \phi(g(1) - G) \\
&\geq (\phi(A) - \phi(-A) + \phi(g(1) - G) - \phi(g(1) + G)) \frac{1}{2} + \phi(-A) - \phi(g(1) - G) \\
&= \frac{1}{2} (\phi(A) + \phi(-A) - \phi(g(1) - G) - \phi(g(1) + G)) \\
&> 0.
\end{align*}
\]
Again, by Part 5 of Lemma 32, for all $\eta \in \left(\frac{1}{2}, 1\right]$, $C_\phi(g(-1) - G, \eta) - C_\phi(g(1) + G, \eta) > 0$.
As a result, for $\eta \in \left(\frac{1}{2}, 1\right]$, 
\[
\inf_{g(-1)-G \leq t \leq A} C_\phi(t, \eta) - \inf_{g(-1)-G \leq g(1)+G} C_\phi(t, \eta) 
= \min \{C_\phi(g(-1) - G, \eta), C_\phi(g(1) + G, \eta), C_\phi(A, \eta) - C_\phi(g(1) + G, \eta)\} \\
> 0.
\]
Finally, for $\eta \in [0, \frac{1}{2})$, by Part 7 of Lemma 32, we have $C_\phi(g(-1) - G, \eta) < C_\phi(g(1) + G, \eta)$ and 
\[
\begin{align*}
\inf_{A \leq t \leq g(1)+G} C_\phi(t, \eta) &= \min \{C_\phi(A, \eta), C_\phi(g(1) + G, \eta)\} \quad \text{(Part 3 of Lemma 32)} \\
\inf_{g(-1)-G \leq g(1)+G} C_\phi(t, \eta) &= \min \{C_\phi(g(-1) - G, \eta), C_\phi(g(1) + G, \eta)\} \quad \text{(Part 3 of Lemma 32)} \\
&= C_\phi(g(1) - G, \eta).
\end{align*}
\]
Since $\phi(A) + \phi(-A) > \phi(G - g(-1)) + \phi(g(-1) - G)$ and $\phi$ is non-increasing, we have 
\[
\begin{align*}
\phi(G - g(-1)) - \phi(g(-1) - G) + \phi(A) - \phi(-A) \\
= \phi(G - g(-1)) - \phi(-A) + \phi(A) - \phi(g(-1) - G) \\
< \phi(A) - \phi(g(-1) - G) + \phi(A) - \phi(g(-1) - G) \\
= 2(\phi(A) - \phi(g(-1) - G)) \\
\leq 0.
\end{align*}
\]
Then for $\eta \in [0, \frac{1}{2})$,
\[
\begin{align*}
\mathcal{C}_\phi(A, \eta) - \mathcal{C}_\phi(g(-1) - G, \eta) &= (\phi(A) - \phi(-A) + \phi(G - g(-1)) - \phi(g(-1) - G)) \eta + \phi(-A) - \phi(G - g(-1)) \\
&\geq (\phi(A) - \phi(-A) + \phi(G - g(-1)) - \phi(g(-1) - G)) \frac{1}{2} + \phi(-A) - \phi(G - g(-1)) \\
&= \frac{1}{2} (\phi(A) + \phi(-A) - \phi(g(-1) - G) - \phi(G - g(-1))) > 0.
\end{align*}
\]

Therefore,
\[
\inf_{A \leq g(1) + G} \mathcal{C}_\phi(t, \eta) > \inf_{g(-1) - G \leq g(1) + G} \mathcal{C}_\phi(t, \eta) \text{ for all } \eta \in [0, \frac{1}{2}).
\]

\section*{C.4. Proof of Theorem 12}

We first characterize the pseudo-calibration function of losses $(\ell, \ell_\gamma)$ given a hypothesis set $\mathcal{H}$.

\textbf{Lemma 33} \textit{Given a hypothesis set $\mathcal{H}$. Assume for any $x \in \mathcal{X}$, there exists $f \in \mathcal{H}$ such that $\inf_{|x' - x| \leq \gamma} f(x') > 0$, and $f \in \mathcal{H}$ such that $\sup_{|x' - x| \leq \gamma} f(x') < 0$. For a surrogate loss $\ell$, the pseudo-calibration function of losses $(\ell, \ell_\gamma)$ is $\hat{\delta}(\epsilon) = \inf_{\eta \in [0,1]} \hat{\delta}(\epsilon, \eta)$, where

\[
\hat{\delta}(\epsilon, \eta) = \begin{cases} 
+\infty & \text{if } \epsilon \geq \max\{\eta, 1 - \eta\}, \\
\inf_{f \in \mathcal{H}, \inf_{x \in \mathcal{X}} \frac{M(f, x, \gamma)^2}{\sup_{x' \in \mathcal{X}} |f(x')|} \geq 0} \Delta \mathcal{C}_{\ell, \gamma}(f, x, \eta) & \text{if } |2\eta - 1| < \epsilon \leq \max\{\eta, 1 - \eta\}, \\
\inf_{f \in \mathcal{H}, \inf_{x \in \mathcal{X}} \frac{M(f, x, \gamma)^2}{\sup_{x' \in \mathcal{X}} |f(x')|} \geq 0, \sup_{|x' - x| \leq \gamma} f(x') < 0} \Delta \mathcal{C}_{\ell, \gamma}(f, x, \eta) & \text{if } \epsilon \leq |2\eta - 1|,
\end{cases}
\]

and $M(f, x, \gamma) = \inf_{x' \in \mathcal{X}, \inf_{x' |x' - x| \leq \gamma} f(x') \geq 0} f(x')$, $\overline{M}(f, x, \gamma) = \sup_{x' \in \mathcal{X}, \inf_{x' |x' - x| \leq \gamma} f(x') < 0} f(x')$.

\textbf{Proof} Let
\[
\overline{M}(f, x, \gamma) := \inf_{x' |x' - x| \leq \gamma} f(x'),
\]

and
\[
\underline{M}(f, x, \gamma) := -\sup_{x' |x' - x| \leq \gamma} f(x').
\]

The inner $\ell_\gamma$-risk is
\[
\mathcal{C}_{\ell_\gamma}(f, x, \eta) = \eta \mathbb{I}_{\{\overline{M}(f, x, \gamma) \geq 0\}} + (1 - \eta) \mathbb{I}_{\{\overline{M}(f, x, \gamma) < 0\}}
\]
\[
\begin{cases} 
1 & \text{if } \overline{M}(f, x, \gamma) \leq 0 \leq \overline{M}(f, x, \gamma), \\
\eta & \text{if } \overline{M}(f, x, \gamma) < 0, \\
1 - \eta & \text{if } \overline{M}(f, x, \gamma) > 0.
\end{cases}
\]

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Since $\mathcal{H}$ satisfies the condition that for any $x \in X$, there exists $f \in \mathcal{H}$ such that $\underline{M}(f, x, \gamma) > 0$, and $f \in \mathcal{H}$ such that $\overline{M}(f, x, \gamma) < 0$, the pseudo-minimal inner $\ell_\gamma$-risk is

$$ C_{\ell_\gamma}(\eta) = \min\{\eta, 1 - \eta\}. $$

Then, it can be computed that

$$ \Delta C_{\ell_\gamma}(f, x, \eta) = \begin{cases} \max\{\eta, 1 - \eta\} & \text{if } M(f, x, \gamma) \leq 0 \leq \overline{M}(f, x, \gamma), \\ |2\eta - 1|\mathbb{I}_{(2\eta - 1)(\underline{M}(f, x, \gamma)) \leq 0} & \text{if } M(f, x, \gamma) > 0 \text{ or } \overline{M}(f, x, \gamma) < 0. \end{cases} $$

By definition, for a fixed $\eta \in [0, 1]$, we have

$$ \delta(\epsilon, \eta) = \inf_{f \in \mathcal{H}, x \in X} \{\Delta C_{\ell_\gamma}(f, x, \eta) \mid \Delta C_{\ell_\gamma}(f, x, \eta) \geq \epsilon\} $$

If $\epsilon > \max\{\eta, 1 - \eta\}$, then for all $f \in \mathcal{H}$, $x \in X$, $\Delta C_{\ell_\gamma}(f, x, \eta) < \epsilon$, which implies that $\delta(\epsilon, \eta) = \infty$. If $|2\eta - 1| < \epsilon \leq \max\{\eta, 1 - \eta\}$, then $\Delta C_{\ell_\gamma}(f, x, \eta) \geq \epsilon$ is achieved when $M(f, x, \gamma) \leq 0 \leq \overline{M}(f, x, \gamma)$, which leads to $\delta(\epsilon, \eta) = \sup_{f \in \mathcal{H}, x \in X} M(f, x, \gamma) \geq \epsilon$. If $\epsilon \leq |2\eta - 1|$, then $\Delta C_{\ell_\gamma}(f, x, \eta) \geq \epsilon$ is achieved when $M(f, x, \gamma) \leq 0 \leq \overline{M}(f, x, \gamma)$ or $(2\eta - 1)(\underline{M}(f, x, \gamma)) \leq 0$. Therefore, $\delta(\epsilon, \eta) = \sup_{f \in \mathcal{H}, x \in X} M(f, x, \gamma) \geq \epsilon$. $\Box$

**Theorem 12** Let $\mathcal{H}$ be a hypothesis set containing 0. Assume that for any $x \in X$, there exists $f \in \mathcal{H}$ such that $\inf_{|x'| - x| \leq \gamma} f(x') > 0$, and $f \in \mathcal{H}$ such that $\sup_{|x'| - x| \leq \gamma} f(x') < 0$. If $\tilde{\phi}$ is convex and non-increasing, then the surrogate loss defined by $\tilde{\phi}(f, x, y) = \sup_{x' : |x'| - x| \leq \gamma} \phi(y f(x'))$ is not $\mathcal{H}$-calibrated with respect to $\ell_\gamma$.

**Proof** Suppose that $\tilde{\phi}$ is pseudo-$\mathcal{H}$-calibrated with respect to $\ell_\gamma$. By Proposition 5, $\tilde{\phi}$ is pseudo-$\mathcal{H}$-calibrated with respect to $\ell_\gamma$, and if and only if its pseudo-calibration function $\delta$ satisfies $\delta(\epsilon) > 0$ for all $\epsilon > 0$, which leads to $\delta(\epsilon, \eta) > 0$ for all $\epsilon > 0$ and $\eta \in [0, 1]$. By Lemma 33, take $\eta = \frac{1}{2}$, we obtain

$$ \inf_{f \in \mathcal{H}, x \in X} C_{\ell_\gamma}(f, x, \frac{1}{2}) > 0, $$

which is equivalent to

$$ \inf_{f \in \mathcal{H}, x \in X} C_{\ell_\gamma}(f, x, \frac{1}{2}) > \inf_{f \in \mathcal{H}, x \in X} C_{\ell_\gamma}(f, x, \frac{1}{2}), $$

where $\underline{M}(f, x, \gamma) = \inf_{|x'| - x| \leq \gamma} f(x')$, $\overline{M}(f, x, \gamma) = \sup_{|x'| - x| \leq \gamma} f(x')$. As shown by Awasthi et al. (2020), $\tilde{\phi}$ has the equivalent form

$$ \tilde{\phi}(f, x, y) = \phi \left( \inf_{|x'| - x| \leq \gamma} (y f(x')) \right). $$

By the definition of inner risk,

$$ C_{\tilde{\phi}}(f, x, \frac{1}{2}) = \frac{1}{2}(\phi(\overline{M}(f, x, \gamma)) + \phi(-\overline{M}(f, x, \gamma))). $$
Since $\phi$ is convex, by Jensen’s inequality,
\[
C_\hat{\phi}(f, x, \frac{1}{2}) \geq \phi\left(\frac{1}{2}M(f, x, \gamma) - \frac{1}{2}\overline{M}(f, x, \gamma)\right) = \phi\left(\frac{1}{2}(M(f, x, \gamma) - \overline{M}(f, x, \gamma))\right) \geq \phi(0),
\]
where the last inequality used the fact that
\[
\frac{1}{2}(M(f, x, \gamma) - \overline{M}(f, x, \gamma)) \leq 0
\]
and $\phi$ is non-increasing. For $f = 0$, we have $M(f, x, \gamma) = \overline{M}(f, x, \gamma) = 0$ and by (20),
\[
C_\hat{\phi}(f, x, \frac{1}{2}) = \frac{1}{2}(\phi(0) + \phi(0)) = \phi(0).
\]
Furthermore, when $M(f, x, \gamma) = \overline{M}(f, x, \gamma) = 0$, $M(f, x, \gamma) \leq 0 \leq \overline{M}(f, x, \gamma)$ is satisfied. Therefore, we obtain
\[
\inf_{f \in \mathcal{H}, x \in X: M(f, x, \gamma) \leq 0 \leq \overline{M}(f, x, \gamma)} C_\hat{\phi}(f, x, \frac{1}{2}) = \inf_{f \in \mathcal{H}, x \in X} C_\hat{\phi}(f, x, \frac{1}{2}) = \phi(0),
\]
where the minimum can be achieved by $f = 0$, contradicting (19). Therefore, $\hat{\phi}$ is not pseudo-$\mathcal{H}$-calibrated with respect to $\ell_\gamma$. By Corollary 7, $\phi$ is not $\mathcal{H}$-calibrated with respect to $\ell_\gamma$.

\section{Proof of Theorem 20}

As with the Proof of Theorem 16, we first give the equivalent conditions of pseudo-calibration based on inner risk of $\phi$ and $\mathcal{H}_{\text{NN}}$.

\textbf{Lemma 34} \textit{Given a hypothesis set $\mathcal{H}$. Assume for any $x \in X$, there exists $f \in \mathcal{H}$ such that $\inf_{||x' - x|| \leq \gamma} f(x') > 0$, and $f \in \mathcal{H}$ such that $\sup_{||x' - x|| \leq \gamma} f(x') < 0$. Let $\ell$ be a surrogate loss function. Then $\ell$ is pseudo-$\mathcal{H}$-calibrated with respect to $\ell_\gamma$ if and only if}
\[
\inf_{f \in \mathcal{H}, x \in X: M(f, x, \gamma) \leq 0 \leq \overline{M}(f, x, \gamma)} C_\ell(f, x, \frac{1}{2}) > \inf_{f \in \mathcal{H}, x \in X} C_\ell(f, x, \frac{1}{2}), \text{ and}
\]
\[
\inf_{f \in \mathcal{H}, x \in X: M(f, x, \gamma) \leq 0 \leq \overline{M}(f, x, \gamma)} C_\ell(f, x, \eta) > \inf_{f \in \mathcal{H}, x \in X} C_\ell(f, x, \eta) \text{ for all } \eta \in \left(\frac{1}{2}, 1\right], \text{ and}
\]
\[
\inf_{f \in \mathcal{H}, x \in X: M(f, x, \gamma) \leq 0 \leq \overline{M}(f, x, \gamma)} C_\ell(f, x, \eta) > \inf_{f \in \mathcal{H}, x \in X} C_\ell(f, x, \eta) \text{ for all } \eta \in \left[0, \frac{1}{2}\right).
\]

\textit{where $M(f, x, \gamma) = \inf_{||x' - x|| \leq \gamma} f(x')$, $\overline{M}(f, x, \gamma) = \sup_{||x' - x|| \leq \gamma} f(x')$.}

\textbf{Proof} Let $\hat{\delta}$ be the pseudo-calibration function of $(\ell, \ell_\gamma)$ for the hypothesis set $\mathcal{H}$. By Lemma 33, $\hat{\delta}(\epsilon) = \inf_{\eta \in [0, 1]} \delta(\epsilon, \eta)$, where
\[
\hat{\delta}(\epsilon, \eta) = \begin{cases} +\infty & \text{if } \epsilon > \max\{\eta, 1 - \eta\}, \\ \inf_{f \in \mathcal{H}, x \in X: M(f, x, \gamma) \leq 0 \leq \overline{M}(f, x, \gamma)} \Delta C_{\ell, \mathcal{H}}(f, x, \eta) & \text{if } |2\eta - 1| < \epsilon \leq \max\{\eta, 1 - \eta\}, \\ \inf_{f \in \mathcal{H}, x \in X: M(f, x, \gamma) \leq 0 \leq \overline{M}(f, x, \gamma) \text{ or } (2\eta - 1)(\overline{M}(f, x, \gamma) - M(f, x, \gamma)) \leq 0} \Delta C_{\ell, \mathcal{H}}(f, x, \eta) & \text{if } \epsilon \leq |2\eta - 1|. \end{cases}
\]
By Proposition 5, $\ell$ is pseudo-$\mathcal{H}$-calibrated with respect to $\ell_\gamma$ if and only if its pseudo-calibration function $\tilde{\delta}$ satisfies $\tilde{\delta}(\epsilon) > 0$ for all $\epsilon > 0$. This is equivalent to $\tilde{\delta}(\epsilon, \eta) > 0$ for all $\epsilon > 0$ and $\eta \in [0, 1]$. For $\eta = \frac{1}{2}$, we have

$$\tilde{\delta}(\epsilon, \frac{1}{2}) > 0 \quad \text{for all } \epsilon > 0 \iff \inf_{f \in \mathcal{F}, x \in \mathcal{X}} \mathcal{C}_\ell(f, x, \frac{1}{2}) > \inf_{f \in \mathcal{F}, x \in \mathcal{X}} \mathcal{C}_\ell(f, x, \frac{1}{2}). \quad (21)$$

For $1 \geq \eta > \frac{1}{2}$, we have $|2\eta - 1| = 2\eta - 1, \max\{\eta, 1 - \eta\} = \eta$, and

$$\inf_{f \in \mathcal{F}, x \in \mathcal{X}} C_\ell(f, x, \eta) \quad \text{for all } \eta \in \left(\frac{1}{2}, 1\right] \text{ if and only if}$$

$$\inf_{f \in \mathcal{F}, x \in \mathcal{X}} C_\ell(f, x, \eta) > \inf_{f \in \mathcal{F}, x \in \mathcal{X}} C_\ell(f, x, \eta) \quad \text{for all } \eta \in \left(\frac{1}{2}, 1\right] \text{ such that } 2\eta - 1 < \epsilon \leq \eta,$$

$$\text{for all } \epsilon > 0, \text{ which is equivalent to}$$

$$\inf_{f \in \mathcal{F}, x \in \mathcal{X}} C_\ell(f, x, \eta) > \inf_{f \in \mathcal{F}, x \in \mathcal{X}} C_\ell(f, x, \eta) \quad \text{for all } \eta \in \left(\frac{1}{2}, 1\right] \text{ such that } \epsilon \leq 2\eta - 1,$$

for all $\epsilon > 0$. We observe that

$$\left\{ \eta \in \left(\frac{1}{2}, 1\right] \left| \frac{1}{2} < \eta \leq \frac{1}{2} + \frac{1}{2}, \epsilon > 0 \right. \right\} = \left\{ \frac{1}{2} < \eta \leq 1 \right\},$$

$$\text{and}$$

$$\left\{ \eta \in \left(\frac{1}{2}, 1\right] \left| \frac{1}{2} + \frac{1}{2} \leq \eta, \epsilon > 0 \right. \right\} = \left\{ \frac{1}{2} < \eta \leq 1 \right\},$$

$$\inf_{f \in \mathcal{F}, x \in \mathcal{X}} C_\ell(f, x, \eta) \geq \inf_{f \in \mathcal{F}, x \in \mathcal{X}} C_\ell(f, x, \eta) \quad \text{for all } \eta,$$

```markdown
Therefore we reduce the above condition (22) as

$$\inf_{f \in \mathcal{F}, x \in \mathcal{X}} C_\ell(f, x, \eta) > \inf_{f \in \mathcal{F}, x \in \mathcal{X}} C_\ell(f, x, \eta) \quad \text{for all } \eta \in \left(\frac{1}{2}, 1\right]. \quad (23)$$

For $\frac{1}{2} > \eta \geq 0$, we have $|2\eta - 1| = 1 - 2\eta, \max\{\eta, 1 - \eta\} = 1 - \eta$, and

$$\inf_{f \in \mathcal{F}, x \in \mathcal{X}} C_\ell(f, x, \eta) > \inf_{f \in \mathcal{F}, x \in \mathcal{X}} C_\ell(f, x, \eta) \quad \text{for all } \eta \in \left[0, \frac{1}{2}\right) \text{ such that } 2\eta < \epsilon \leq 1 - \eta,$$

$$\inf_{f \in \mathcal{F}, x \in \mathcal{X}} C_\ell(f, x, \eta) > \inf_{f \in \mathcal{F}, x \in \mathcal{X}} C_\ell(f, x, \eta) \quad \text{for all } \eta \in \left[0, \frac{1}{2}\right) \text{ such that } \epsilon \leq 1 - 2\eta,$$

Therefore, $\tilde{\delta}(\epsilon, \eta) > 0$ for all $\epsilon > 0$ and $\eta \in \left[0, \frac{1}{2}\right)$ if and only if

$$\inf_{f \in \mathcal{F}, x \in \mathcal{X}} C_\ell(f, x, \eta) > \inf_{f \in \mathcal{F}, x \in \mathcal{X}} C_\ell(f, x, \eta) \quad \text{for all } \eta \in \left[0, \frac{1}{2}\right) \text{ such that } 1 - 2\eta < \epsilon \leq 1 - \eta,$$

$$\inf_{f \in \mathcal{F}, x \in \mathcal{X}} C_\ell(f, x, \eta) > \inf_{f \in \mathcal{F}, x \in \mathcal{X}} C_\ell(f, x, \eta) \quad \text{for all } \eta \in \left[0, \frac{1}{2}\right) \text{ such that } \epsilon \leq 1 - 2\eta,$$

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for all $\epsilon > 0$, which is equivalent to
\[
\begin{cases}
\inf_{f \in \mathcal{X} : \tilde{M}(f, x, \gamma) \leq 0} C(f, x, \eta) > \inf_{f \in \mathcal{X} : M(f, x, \gamma) \leq 0} C(f, x, \eta) & \text{for all } \eta \in [0, \frac{1}{2}) \text{ such that } \frac{1-\epsilon}{2} < \eta \leq 1 - \epsilon, \\
\inf_{f \in \mathcal{X} : \tilde{M}(f, x, \gamma) \leq 0} C(f, x, \eta) > \inf_{f \in \mathcal{X} : M(f, x, \gamma) \geq 0} C(f, x, \eta) & \text{for all } \eta \in [0, \frac{1}{2}) \text{ such that } \eta \leq \frac{1-\epsilon}{2},
\end{cases}
\]
(24)
for all $\epsilon > 0$. We observe that
\[
\left\{ \eta \in \left[0, \frac{1}{2}\right) \mid \frac{1-\epsilon}{2} < \eta \leq 1 - \epsilon, \epsilon > 0 \right\} = \left\{ 0 \leq \eta < \frac{1}{2} \right\}, \text{ and}
\]
\[
\left\{ \eta \in \left[0, \frac{1}{2}\right) \mid \eta \leq \frac{1-\epsilon}{2}, \epsilon > 0 \right\} = \left\{ 0 \leq \eta < \frac{1}{2} \right\}, \text{ and}
\]
\[
\inf_{f \in \mathcal{X} : \tilde{M}(f, x, \gamma) \leq 0} C(f, x, \eta) \geq \inf_{f \in \mathcal{X} : M(f, x, \gamma) \geq 0} C(f, x, \eta) \text{ for all } \eta \in \left[0, \frac{1}{2}\right).
\]
Therefore we reduce the above condition (24) as
\[
\inf_{f \in \mathcal{X} : \tilde{M}(f, x, \gamma) \geq 0} C(f, x, \eta) > \inf_{f \in \mathcal{X} : M(f, x, \gamma) \geq 0} C(f, x, \eta) \text{ for all } \eta \in \left[0, \frac{1}{2}\right). \tag{25}
\]
To sum up, by (21), (23) and (25), we conclude the proof.

\begin{flushright}
\textbf{■}
\end{flushright}

**Theorem 20** Consider $\rho$-margin loss $\phi_{\rho}(t) = \min\left\{1, \max\left\{0, 1 - \frac{t}{\rho}\right\}\right\}$, $\rho > 0$. If $\Lambda W(1 - \gamma) \geq \rho$, then the surrogate loss $\tilde{\phi}_{\rho}(f, x, y) = \sup_{x' : ||x - x'|| \leq \gamma} \phi_{\rho}(y f(x'))$ is pseudo-$\mathcal{H}_{\text{NN}}$-calibrated with respect to $\ell_{\gamma}$.

\begin{flushleft}
\textbf{Proof}
\end{flushleft} By Lemma 34, $\tilde{\phi}_{\rho}$ is pseudo-$\mathcal{H}_{\text{NN}}$-calibrated with respect to $\ell_{\gamma}$ if and only if
\[
\inf_{f \in \mathcal{X} : \tilde{M}(f, x, \gamma) \leq 0} C_{\tilde{\phi}_{\rho}}(f, x, \frac{1}{2}) > \inf_{f \in \mathcal{X} : M(f, x, \gamma) \leq 0} C_{\tilde{\phi}_{\rho}}(f, x, \frac{1}{2}), \text{ and}
\]
\[
\inf_{f \in \mathcal{X} : \tilde{M}(f, x, \gamma) \geq 0} C_{\tilde{\phi}_{\rho}}(f, x, \eta) > \inf_{f \in \mathcal{X} : M(f, x, \gamma) \geq 0} C_{\tilde{\phi}_{\rho}}(f, x, \eta) \text{ for all } \eta \in \left(\frac{1}{2}, 1\right], \text{ and}
\]
\[
\inf_{f \in \mathcal{X} : \tilde{M}(f, x, \gamma) \geq 0} C_{\tilde{\phi}_{\rho}}(f, x, \eta) > \inf_{f \in \mathcal{X} : M(f, x, \gamma) \geq 0} C_{\tilde{\phi}_{\rho}}(f, x, \eta) \text{ for all } \eta \in \left[0, \frac{1}{2}\right).
\]
where $M(f, x, \gamma) = \inf_{x' : ||x - x'|| \leq \gamma} f(x')$, $\tilde{M}(f, x, \gamma) = \sup_{x' : ||x - x'|| \leq \gamma} f(x')$. As shown by Awasthi et al. (2020), $\tilde{\phi}_{\rho}$ has the equivalent form
\[
\tilde{\phi}_{\rho}(f, x, y) = \phi_{\rho}\left(\inf_{x' : ||x - x'|| \leq \gamma} (y f(x'))\right).
\]
The inner $\tilde{\phi}_{\rho}$-risk is
\[
C_{\tilde{\phi}_{\rho}}(f, x, \eta) = \eta \phi_{\rho}(\tilde{M}(f, x, \gamma)) + (1 - \eta)\phi_{\rho}(-\tilde{M}(f, x, \gamma)).
\]
Next we analyze three cases:
• When $\eta = \frac{1}{2}$, since $\phi_\rho$ is non-increasing,

$$
\inf_{f \in \mathcal{F}, x \in \mathcal{X}} C_{\phi_\rho}(f, x, \frac{1}{2}) \leq \inf_{f \in \mathcal{F}, x \in \mathcal{X}} \frac{1}{2} \phi_\rho(M(f, x, \gamma)) + \frac{1}{2} \phi_\rho(-M(f, x, \gamma)) \\
\geq \frac{1}{2} \phi_\rho(0) + \frac{1}{2} \phi_\rho(0) = 1.
$$

Take $f = 0 \in \mathcal{H}_{NN}$, then $M(f, x, \gamma) = M(f, x, \gamma) = 0$, $C_{\phi_\rho}(f, x, \frac{1}{2}) = \frac{1}{2} \phi_\rho(0) + \frac{1}{2} \phi_\rho(0) = \phi_\rho(0) = 1$. Therefore

$$
\inf_{f \in \mathcal{F}, x \in \mathcal{X}} C_{\phi_\rho}(f, x, \frac{1}{2}) = 1. 
$$

Let $x \in \mathcal{X}$ such that $||x|| = 1$, $w_j = Wx$, $u_j = \frac{A}{j}$, $j = 1, \ldots, n$. Then for any $s \in \{s : ||s|| \leq 1\}$, $w_j \cdot (x + s) = W(x \cdot x + s) \geq W(||x||^2 - \gamma ||x|| ||s||) \geq W(1 - \gamma) > 0$. Therefore, we obtain $M(f, x, \gamma) = \inf_{||s|| \leq 1} \sum_{j=1}^n u_j (w_j \cdot (x + s)) \geq \Lambda W(1 - \gamma) > 0$ and $-M(f, x, \gamma) \leq -M(f, x, \gamma) < 0$. Then $C_{\phi_\rho}(f, x, \frac{1}{2}) = \frac{1}{2} \phi_\rho(M(f, x, \gamma)) + \frac{1}{2} \phi_\rho(-M(f, x, \gamma)) \leq \frac{1}{2} \phi_\rho(\Lambda W(1 - \gamma)) + \frac{1}{2} \times 1 < 1$. Therefore

$$
\inf_{f \in \mathcal{F}, x \in \mathcal{X}} C_{\phi_\rho}(f, x, \frac{1}{2}) = \frac{1}{2} \phi_\rho(\Lambda W(1 - \gamma)) + \frac{1}{2} < 1 = \inf_{f \in \mathcal{F}, x \in \mathcal{X}} C_{\phi_\rho}(f, x, \frac{1}{2}). 
$$

(26)

• When $\eta \in \left(\frac{1}{2}, 1\right]$, since $\phi_\rho$ is non-increasing and $-M(f, x, \gamma) \leq -M(f, x, \gamma)$,

$$
\inf_{f \in \mathcal{F}, x \in \mathcal{X}} C_{\phi_\rho}(f, x, \eta) \leq \inf_{f \in \mathcal{F}, x \in \mathcal{X}} \eta \phi_\rho(M(f, x, \gamma)) + (1 - \eta) \phi_\rho(-M(f, x, \gamma)) \\
\leq \inf_{f \in \mathcal{F}, x \in \mathcal{X}} \eta \phi_\rho(M(f, x, \gamma)) + (1 - \eta) \phi_\rho(-M(f, x, \gamma)) \\
+ (1 - \eta)(\phi_\rho(-M(f, x, \gamma)) - \phi_\rho(-M(f, x, \gamma))) \\
\leq \inf_{f \in \mathcal{F}, x \in \mathcal{X}} \eta \phi_\rho(M(f, x, \gamma)) + (1 - \eta) \phi_\rho(-M(f, x, \gamma)) \\
\geq \eta.
$$

Let $x \in \mathcal{X}$ such that $||x|| = 1$, $w_j = Wx$, $u_j = \frac{A}{j}$, $j = 1, \ldots, n$. Then for any $s \in \{s : ||s|| \leq 1\}$, $w_j \cdot (x + s) = W(x \cdot x + s) \geq W(||x||^2 - \gamma ||x|| ||s||) \geq W(1 - \gamma) > 0$. Since $\Lambda W(1 - \gamma) \geq \rho$, we obtain $M(f, x, \gamma) = \inf_{||s|| \leq 1} \sum_{j=1}^n u_j (w_j \cdot (x + s)) \geq \Lambda W(1 - \gamma) \geq \rho$ and $-M(f, x, \gamma) \leq -M(f, x, \gamma) \leq -\rho$. Then

$$
C_{\phi_\rho}(f, x, \eta) = \eta \phi_\rho(M(f, x, \gamma)) + (1 - \eta)(\phi_\rho(-M(f, x, \gamma))) = \eta \times 0 + (1 - \eta) \times 1 = 1 - \eta.
$$

Therefore

$$
\inf_{f \in \mathcal{F}, x \in \mathcal{X}} C_{\phi_\rho}(f, x, \eta) \leq 1 - \eta < \eta = \inf_{f \in \mathcal{F}, x \in \mathcal{X}} C_{\phi_\rho}(f, x, \eta). 
$$

(27)
• When $\eta \in [0, \frac{1}{2})$, since $\phi_\rho$ is non-increasing and $M(f, x, \gamma) \leq \overline{M}(f, x, \gamma)$,
\[
\inf_{f \in \mathcal{F}, x \in X: M(f, x, \gamma) \geq 0} C_{\phi_\rho}(f, x, \eta) \\
= \inf_{f \in \mathcal{F}, x \in X: M(f, x, \gamma) \geq 0} \eta \phi_\rho(M(f, x, \gamma)) + (1 - \eta) \phi_\rho(-M(f, x, \gamma)) \\
= \inf_{f \in \mathcal{F}, x \in X: M(f, x, \gamma) \geq 0} \eta \phi_\rho(M(f, x, \gamma)) + (1 - \eta) \phi_\rho(M(f, x, \gamma)) + \eta(\phi_\rho(M(f, x, \gamma)) - \phi_\rho(M(f, x, \gamma))) \\
\geq \inf_{f \in \mathcal{F}, x \in X: M(f, x, \gamma) \geq 0} \eta \phi_\rho(M(f, x, \gamma)) + (1 - \eta) \phi_\rho(-M(f, x, \gamma)) \\
\geq 1 - \eta.
\]

Let $x \in X$ such that $|x| = 1$, $w_j = Wx$, $u_j = -\frac{x}{\|x\|}$, $j = 1, \ldots, n$. Then for any $s \in \{s: \|s\| \leq 1\}$, $w_j \cdot (x + \gamma s) = W(x \cdot x + \gamma(x \cdot s)) \geq W(\|x\|^2 - \gamma \|x\| \|s\|) \geq W(1 - \gamma) > 0$. Since $\Lambda W(1 - \gamma) \geq \rho$, we obtain \(M(f, x, \gamma) = \inf_{s \in \mathcal{S}} \sum_{j=1}^{n} u_j (w_j \cdot (x + \gamma s)) \leq \Lambda W(1 - \gamma) \leq -\rho\) and $M(f, x, \gamma) \leq \overline{M}(f, x, \gamma) \leq -\rho$. Then $C_{\phi_\rho}(f, x, \eta) = \eta \phi_\rho(M(f, x, \gamma)) + (1 - \eta) \phi_\rho(-M(f, x, \gamma)) = \eta \times 1 + (1 - \eta) \times 0 = \eta$. Therefore
\[
\inf_{f \in \mathcal{F}, x \in X} C_{\phi_\rho}(f, x, \eta) \leq \eta < 1 - \eta \leq \inf_{f \in \mathcal{F}, x \in X: M(f, x, \gamma) \geq 0} C_{\phi_\rho}(f, x, \eta).
\] (28)

To sum up, by (26), (27) and (28), we conclude the proof. \[
\]

C.6. Proof of Theorem 22

Theorem 22 No continuous margin-based loss function $\phi$ is $\mathcal{H}_{\text{lin}}$-consistent with respect to $\ell_\gamma$.

Proof Let $x$ follow the uniform distribution on the unit circle. Denote $x = (\cos(\theta), \sin(\theta))^T, \theta \in [0, 2\pi)$ and $f(x) = \mathbf{w} \cdot x$, $\mathbf{w} = (\cos(t), \sin(t))^T, t \in [0, 2\pi)$, $f \in \mathcal{F}_{\text{lin}} = \{x \to \mathbf{w} \cdot x | \|\mathbf{w}\|_2 = 1\}$. We set the label of a point $x$ as follows: if $\theta \in (\sigma, \pi)$, where $\sigma \in (0, \pi)$, then set $y = -1$ with probability $\frac{1}{2}$ and $y = 1$ with probability $\frac{1}{2}$; if $\theta \in (0, \sigma)$ or $(\sigma + \pi, 2\pi)$, then set $y = 1$; if $\theta \in (\pi, \sigma + \pi)$, then set $y = -1$.

Let $\eta: X \to [0, 1]$ be a measurable function such that $\eta(X) = \mathbb{P}(Y = 1 | X)$. For $\ell_\gamma(\tau) = 1_{\tau \leq \gamma}$, we want to solve
\[
\mathcal{R}^*_{\ell_\gamma, \mathcal{H}_{\text{lin}}} = \min_{f \in \mathcal{F}_{\text{lin}}} \mathcal{R}_{\ell_\gamma}(f) = \min_{f \in \mathcal{F}_{\text{lin}}} \mathbb{E}_X[\ell_\gamma(f(X))\eta + \ell_\gamma(-f(X))(1 - \eta)].
\]

Let $\eta': \Theta \to [0, 1]$ be a measurable function such that $\eta' = \mathbb{P}(Y = 1 | \Theta)$, $\Theta \sim \mathcal{U}(0, 2\pi)$. In our example, we have
\[
\eta' = \begin{cases} 
\frac{1}{4} & \theta \in (\sigma, \pi) \\
1 & \theta \in (0, \sigma) \text{ or } \theta \in (\sigma + \pi, 2\pi) \\
0 & \theta \in (\pi, \sigma + \pi) 
\end{cases}
\]

Therefore we obtain
\[
\mathcal{R}^*_{\ell_\gamma, \mathcal{H}_{\text{lin}}} = \min_{\theta \in [0, 2\pi]} \mathbb{E}_\Theta[\ell_\gamma(\cos(\Theta - t))\eta' + \ell_\gamma(-\cos(\Theta - t))(1 - \eta')].
\]

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\[ \ell_r(t) = \frac{1}{2\pi} \min_{\gamma \in [0, 2\pi]} \int_{\gamma}^\pi \frac{1}{4} \ell_r(\cos(\theta-t)) + \frac{3}{4} \ell_r(-\cos(\theta-t)) \, d\theta + \int_{\pi-\gamma}^\sigma \ell_r(\cos(\theta-t)) \, d\theta \]
\[ + \int_{\pi-\gamma}^\sigma \ell_r(-\cos(\theta-t)) \, d\theta \]
\[ \ell_r(t) = \frac{1}{2\pi} \min_{\gamma \in [0, 2\pi]} \int_{\gamma}^\pi \frac{1}{4} \ell_r(\cos(\theta-t)) + \frac{3}{4} \ell_r(-\cos(\theta-t)) \, d\theta + \int_0^\sigma \ell_r(\cos(\theta-t)) \, d\theta \]
\[ + \int_0^\sigma \ell_r(-\cos(\theta-t)) \, d\theta + \int_{\pi-\gamma}^\sigma \ell_r(\cos(\theta-t)) \, d\theta \]
\[ \ell_r(t) = \frac{1}{2\pi} \min_{\gamma \in [0, 2\pi]} \int_{\gamma}^\pi \frac{1}{4} \ell_r(\cos(\theta-t)) \, d\theta + \int_{\sigma-\pi}^\sigma \frac{7}{4} \ell_r(\cos(\theta-t)) \, d\theta + \int_0^\sigma \frac{7}{4} \ell_r(\cos(\theta-t)) \, d\theta \]
\[ + \int_0^\sigma \frac{1}{4} \ell_r(\cos(\theta-t)) \, d\theta \]
\[ \ell_r(t) = \frac{1}{2\pi} \min_{\gamma \in [0, 2\pi]} \int_{\gamma}^\pi \frac{1}{4} \ell_r(\cos(\theta-t)) \, d\theta + \int_{\sigma-\pi}^\sigma \frac{7}{4} \ell_r(\cos(\theta-t)) \, d\theta \]
\[ + \int_0^\sigma \frac{7}{4} \ell_r(\cos(\theta-t)) \, d\theta \]
\[ \ell_r(t) = \frac{1}{2\pi} \min_{\gamma \in [0, 2\pi]} \int_{\gamma}^\pi \frac{1}{4} \ell_r(\cos(\theta-t)) \, d\theta + \int_{\sigma-\pi}^\sigma \frac{7}{4} \ell_r(-\cos(\theta-t+\sigma)) \, d\theta \]
\[ \ell_r(t) = \frac{1}{2\pi} \min_{\gamma \in [0, 2\pi]} \int_{\gamma}^\pi \frac{1}{4} \ell_r(\cos(\theta-t)) \, d\theta + \frac{7}{4} \ell_r(-\cos(\theta+\sigma)) \, d\theta \]
\[ \ell_r(t) = \frac{1}{2\pi} \min_{\gamma \in [0, 2\pi]} \int_{-\pi t}^{-\pi t} \frac{1}{4} \ell_r(\cos(\theta)) \, d\theta + \frac{7}{4} \ell_r(-\cos(\theta+\sigma)) \, d\theta \]
\[ \int_{-\pi t}^{-\pi t} \frac{1}{4} \ell_r(\cos(\theta)) \, d\theta \]
\[ = \int_{-\pi t}^{-\pi t} \frac{1}{4} \ell_r(\cos(\theta)) \, d\theta + \int_{-\pi t}^{-\pi t} \frac{7}{4} \ell_r(-\cos(\theta+\sigma)) \, d\theta \]
\[ = 2\pi - \frac{23}{8} \sigma + \frac{7}{4} \pi t \geq 2\pi - 2\sigma \]
\[
\text{where the equality is achieved when } t = \frac{\sigma}{2}.
\]
\[
\text{Take } \gamma = \cos\left(\frac{\sigma}{2}\right) \in (0, 1). \text{ For } \sigma \in (0, \frac{\pi}{2}], \text{ we analyze six cases:}
\]
\[
\text{• When } -t \in [-\frac{3\sigma}{2}, -\frac{\sigma}{2}],
\]
\[
\int_{-\pi t}^{-\pi t} \frac{1}{4} \ell_r(\cos(\theta)) \, d\theta + \frac{7}{4} \ell_r(-\cos(\theta+\sigma)) \, d\theta
\]
\[
= 2\pi - \frac{23}{8} \sigma + \frac{7}{4} \pi t \geq 2\pi - 2\sigma
\]
\[
\text{where the equality is achieved when } t = \frac{\sigma}{2}.
\]
\[
\text{• When } -t \in [-\frac{\sigma}{2}, \frac{\sigma}{2}],
\]
\[
\int_{-\pi t}^{-\pi t} \frac{1}{4} \ell_r(\cos(\theta)) \, d\theta + \frac{7}{4} \ell_r(-\cos(\theta+\sigma)) \, d\theta
\]
\[
= 2\pi - \frac{15}{8} \sigma - \frac{1}{4} \pi t \geq 2\pi - 2\sigma
\]
\[
\text{where the equality is achieved when } t = \frac{\sigma}{2}.
\]
• When \(-t \in \left[ \frac{\alpha}{2}, -\frac{3\alpha}{2} + \pi \right],\)
  \[
  \int_{-t}^{\pi - t} \frac{1}{4} \cos(\theta) \leq \gamma + \frac{7}{4} \cos(\theta + \sigma) \leq \gamma \ d\theta \\
  = \int_{-t}^{-\frac{3\alpha}{2} + \pi} \frac{1}{4} + \frac{7}{4} \ d\theta \ + \int_{-\frac{3\alpha}{2} + \pi}^{\pi - t} \frac{1}{4} + \frac{7}{4} \ d\theta \\
  = 2\pi - \frac{7}{4}\sigma .
  \]

• When \(-t \in \left[ -\frac{3\alpha}{2} + \pi, -\frac{\alpha}{2} + \pi \right],\)
  \[
  \int_{-t}^{\pi - t} \frac{1}{4} \cos(\theta) \leq \gamma + \frac{7}{4} \cos(\theta + \sigma) \leq \gamma \ d\theta \\
  = \int_{-t}^{-\frac{3\alpha}{2} + \pi} \frac{1}{4} + \frac{7}{4} \ d\theta \ + \int_{-\frac{3\alpha}{2} + \pi}^{\pi - t} \frac{1}{4} + \frac{7}{4} \ d\theta \\
  = \frac{\pi}{4} + \frac{7}{8}\sigma - \frac{7}{4} t \geq 2\pi - \frac{7}{4}\sigma 
  \]

where the equality is achieved when \(t = \frac{3\alpha}{2} - \pi\).

• When \(-t \in \left[ -\frac{\alpha}{2} + \pi, \frac{\alpha}{2} + \pi \right],\)
  \[
  \int_{-t}^{\pi - t} \frac{1}{4} \cos(\theta) \leq \gamma + \frac{7}{4} \cos(\theta + \sigma) \leq \gamma \ d\theta \\
  = \int_{-t}^{-\frac{3\alpha}{2} + \pi} \frac{1}{4} + \frac{7}{4} \ d\theta \ + \int_{-\frac{3\alpha}{2} + \pi}^{\pi - t} \frac{7}{4} \ d\theta \\
  = \frac{9\pi}{4} - \frac{1}{8}\sigma + \frac{1}{4} t \geq 2\pi - \frac{1}{4}\sigma 
  \]

where the equality is achieved when \(t = -\frac{\alpha}{2} - \pi\).

• When \(-t \in \left[ \frac{\alpha}{2} + \pi, -\frac{3\alpha}{2} + 2\pi \right],\)
  \[
  \int_{-t}^{\pi - t} \frac{1}{4} \cos(\theta) \leq \gamma + \frac{7}{4} \cos(\theta + \sigma) \leq \gamma \ d\theta \\
  = \int_{-t}^{-\frac{3\alpha}{2} + 2\pi} \frac{1}{4} + \frac{7}{4} \ d\theta \ + \int_{-\frac{3\alpha}{2} + 2\pi}^{\pi - t} \frac{1}{4} + \frac{7}{4} \ d\theta \\
  = 2\pi - \frac{1}{4}\sigma 
  \]

Similarly for \(\sigma \in \left[ \frac{\alpha}{2}, \pi \right],\) we analyze six cases:

• When \(-t \in \left[ -\frac{3\alpha}{2}, -\frac{\alpha}{2} - \pi \right],\)
  \[
  \int_{-t}^{\pi - t} \frac{1}{4} \cos(\theta) \leq \gamma + \frac{7}{4} \cos(\theta + \sigma) \leq \gamma \ d\theta \\
  = \int_{-t}^{-\frac{3\alpha}{2}} \frac{1}{4} + \frac{7}{4} \ d\theta \ + \int_{-\frac{3\alpha}{2}}^{\pi - t} \frac{3\alpha}{2} + \pi \frac{7}{4} \ d\theta \\
  = \frac{7}{4}\pi - \frac{11}{4}\sigma + 2t \geq \frac{15}{4}\pi - \frac{15}{4}\sigma 
  \]

where the equality is achieved when \(t = \pi - \frac{\alpha}{2}\).
• When \(-t \in \left[\frac{\sigma}{2} - \pi, -\frac{\sigma}{2}\right]\),
\[
\int_{-t}^{\pi-t} \frac{1}{4} 11_{\cos(\theta) \leq \gamma} + \frac{7}{4} 11_{-\cos(\theta+\sigma) \leq \gamma} \, d\theta = \int_{-t}^{-\frac{\sigma}{2} + \pi} \frac{1}{4} \, d\theta + \int_{-\frac{\sigma}{2} + \pi}^{\pi-t} \frac{7}{4} \, d\theta + \int_{\pi-t}^{\frac{\sigma}{2}} \frac{1}{4} \, d\theta = 2\pi - \frac{23}{8} \sigma + \frac{7}{4} \, t \geq 2\pi - 2\sigma
\]
where the equality is achieved when \(t = \frac{\sigma}{2}\).

• When \(-t \in \left[-\frac{3\sigma}{2}, -\frac{3\sigma}{2} + \pi\right]\),
\[
\int_{-t}^{\pi-t} \frac{1}{4} 11_{\cos(\theta) \leq \gamma} + \frac{7}{4} 11_{-\cos(\theta+\sigma) \leq \gamma} \, d\theta = \int_{-t}^{-\frac{\sigma}{2} + \pi} \frac{7}{4} \, d\theta + \int_{-\frac{\sigma}{2} + \pi}^{\pi-t} \frac{1}{4} \, d\theta + \int_{\pi-t}^{-\frac{\sigma}{2} + \pi} \frac{1}{4} \, d\theta = 2\pi - \frac{15}{8} \sigma - \frac{1}{4} \, t \geq 2\pi - 2\sigma
\]
where the equality is achieved when \(t = \frac{\sigma}{2}\).

• When \(-t \in \left[-\frac{3\sigma}{2} + \pi, \frac{\sigma}{2}\right]\),
\[
\int_{-t}^{\pi-t} \frac{1}{4} 11_{\cos(\theta) \leq \gamma} + \frac{7}{4} 11_{-\cos(\theta+\sigma) \leq \gamma} \, d\theta = \int_{-t}^{-\frac{\sigma}{2} + \pi} \frac{1}{4} \, d\theta + \int_{-\frac{\sigma}{2} + \pi}^{\pi-t} \frac{1}{4} \, d\theta = -\frac{\pi}{4} + \frac{3}{4} \sigma - 2\pi \geq \frac{9}{4} \pi - \frac{9}{4} \sigma
\]
where the equality is achieved when \(t = \frac{3\sigma}{2} - \pi\).

• When \(-t \in \left[\frac{\sigma}{2}, -\frac{\sigma}{2} + \pi\right]\),
\[
\int_{-t}^{\pi-t} \frac{1}{4} 11_{\cos(\theta) \leq \gamma} + \frac{7}{4} 11_{-\cos(\theta+\sigma) \leq \gamma} \, d\theta = \int_{-t}^{-\frac{\sigma}{2} + \pi} \frac{7}{4} \, d\theta + \int_{-\frac{\sigma}{2} + \pi}^{\pi-t} \frac{1}{4} \, d\theta = \frac{7\pi}{4} + \frac{1}{8} \sigma - \frac{1}{4} \, t \geq \frac{7\pi}{4} - \frac{1}{4} \sigma
\]
where the equality is achieved when \(t = -\frac{\sigma}{2}\).

• When \(-t \in \left[-\frac{\sigma}{2} + \pi, -\frac{3\sigma}{2} + 2\pi\right]\),
\[
\int_{-t}^{\pi-t} \frac{1}{4} 11_{\cos(\theta) \leq \gamma} + \frac{7}{4} 11_{-\cos(\theta+\sigma) \leq \gamma} \, d\theta = \int_{-t}^{-\frac{\sigma}{2} + \pi} \frac{1}{4} \, d\theta + \int_{-\frac{\sigma}{2} + \pi}^{\pi-t} \frac{7}{4} \, d\theta = \frac{9}{4} \pi - \frac{1}{8} \sigma + \frac{1}{4} \, t \geq \frac{7\pi}{4} + \frac{1}{4} \sigma
\]
where the equality is achieved when \( t = \frac{3\sigma}{2} - 2\pi \).

Therefore for \( \sigma \in (0, \pi) \),

\[
\min_{t \in (0,2\pi)} \int_{-t}^{\pi-t} \frac{1}{4} \frac{1}{\cos(\theta) \leq t} + \frac{7}{4} \frac{1}{\cos(\theta + \sigma) \leq t} \ d\theta = 2\pi - 2\sigma
\]

where the equality is achieved when \( t = \frac{\sigma}{2} \). Therefore

\[
R_{\ell_\gamma, H_{lin}}^* = \frac{1}{2\pi} (2\pi - 2\sigma) = 1 - \frac{\sigma}{\pi},
\]

where the unique Bayes classifier satisfies \( t^*_1 = \frac{\sigma}{2} \).

For continuous margin-based loss \( \phi \), by (29) we have

\[
R_{\phi, H_{lin}}^* = \frac{1}{2\pi} \min_{t \in [0,2\pi]} \int_{-t}^{\pi-t} \frac{1}{4} \phi(\cos(\theta - t)) d\theta + \int_{0}^{\pi} \frac{7}{4} \phi(\sin(\theta - t)) d\theta
\]

\[
= \frac{1}{2\pi} \min_{t \in [0,2\pi]} \int_{-t}^{\pi-t} \frac{1}{4} \phi(\cos(\theta)) + \frac{7}{4} \phi(-\cos(\theta + \sigma)) d\theta.
\]

If \( t^* = \frac{\sigma}{2} \) is the minimizer of \( g(t) = \int_{-t}^{\pi-t} \frac{1}{4} \phi(\cos(\theta)) + \frac{7}{4} \phi(-\cos(\theta + \sigma)) d\theta \), \( t \in [0,2\pi] \), since \( \frac{\sigma}{2} \) is not at the boundary of \([0,2\pi]\), we need

\[
g\left(\frac{\sigma}{2}\right) = 0.
\]

Since \( \phi \) is continuous, by Leibniz Integral Rule, we have

\[
g'\left(\frac{\sigma}{2}\right) = -\frac{1}{4} \phi\left(\cos\left(\pi - \frac{\sigma}{2}\right)\right) - \frac{7}{4} \phi\left(-\cos\left(\pi + \frac{\sigma}{2}\right)\right) + \frac{1}{4} \phi\left(\cos\left(\frac{\sigma}{2}\right)\right) + \frac{7}{4} \phi\left(-\cos\left(\frac{\sigma}{2}\right)\right)
\]

\[
= -\frac{1}{4} \phi\left(-\cos\left(\frac{\sigma}{2}\right)\right) - \frac{7}{4} \phi\left(\cos\left(\frac{\sigma}{2}\right)\right) + \frac{1}{4} \phi\left(\cos\left(\frac{\sigma}{2}\right)\right) + \frac{7}{4} \phi\left(-\cos\left(\frac{\sigma}{2}\right)\right)
\]

\[
= \frac{3}{2} \phi\left(-\cos\left(\frac{\sigma}{2}\right)\right) - \frac{3}{2} \phi\left(\cos\left(\frac{\sigma}{2}\right)\right).
\]

Thus if \( t^* = \frac{\sigma}{2} \) is the minimizer of \( R_{\phi, H_{lin}}^* \), we need \( \phi \) satisfies

\[
\phi\left(-\cos\left(\frac{\sigma}{2}\right)\right) = \phi\left(\cos\left(\frac{\sigma}{2}\right)\right).
\]

Therefore, if \( \phi \) is \( H_{lin} \)-consistent with respect to \( \ell_\gamma \), we need \( \phi \) satisfies (31) for any \( \sigma \in (0, \pi) \). Namely \( \phi \) satisfies

\[
\phi(-\tau) = \phi(\tau), \quad \tau \in [-1, 1] .
\]

Note in our example, \( \tau \in [-1, 1] \), \( \phi \) is continuous. We obtain that if \( \phi \) is \( H_{lin} \)-consistent with respect to \( \ell_\gamma \), \( \phi \) must be even function in \([-1, 1]\). Next we claim that if \( \phi \) is even function in \([-1, 1]\), \( \phi \) is not \( H_{lin} \)-consistent with respect to \( \ell_\gamma \). Indeed, for the distribution \( y = 1 \) if \( \theta \in (0, \pi) \) and \( y = -1 \) if \( \theta \in (\pi, 2\pi) \), we have

\[
R_{\phi, H_{lin}}^* = \frac{1}{2\pi} \min_{t \in [0,2\pi]} \int_{0}^{\pi} \phi(\cos(\theta - t)) + \int_{0}^{2\pi} \phi(-\cos(\theta - t)) d\theta
\]

\[
= \frac{1}{2\pi} \min_{t \in [0,2\pi]} \int_{0}^{\pi} \phi(\cos(\theta - t)) d\theta
\]

\[
= \frac{1}{2\pi} \min_{t \in [0,2\pi]} \int_{-t}^{\pi-t} \phi(\cos(\theta)) d\theta.
\]
Note that when $\phi$ is even function in $[-1, 1]$, $h(t) = \int_{-1}^{\pi/2} \phi(\cos(\theta)) \, d\theta$ satisfies

$$h'(t) = -\phi(-\cos(t)) + \phi(\cos(t)) = 0, \quad t \in [0, 2\pi].$$

Thus $h(t)$ is a constant for $t \in [0, 2\pi]$ and $R^*_{\phi, \mathcal{H}_{\text{lin}}}$ can be attained for any classifier $t \in [0, 2\pi]$. However, $R^*_{\ell_\gamma, \mathcal{H}_{\text{lin}}}$ cannot be attained for any classifier $t \in [0, 2\pi]$ with respect to this distribution. Therefore when $\phi$ is even function in $[-1, 1]$, $\phi$ is not $\mathcal{H}_{\text{lin}}$-consistent with respect to $\ell_\gamma$. By the claim, any continuous loss is not $\mathcal{H}_{\text{lin}}$-consistent with respect to $\ell_\gamma$. 

\section*{C.7. Proof of Theorem 25 and Theorem 27}

Since the proofs adopt some results of (Steinwart, 2007), we introduce the notation used in (Steinwart, 2007) to make the proofs more clear. In this section, we denote the loss $\ell(f, x, y)$ defined on a particular hypothesis set $\mathcal{H}$ as $\ell_{\mathcal{H}}(f, x, y)$. For a joint distribution $\mathcal{P}$ over $X \times Y$, the corresponding conditional distribution and marginal distribution are denoted as $\mathcal{P}(\cdot|x)$ and $\mathcal{P}_X$ respectively. In (Steinwart, 2007), given a distribution $\mathcal{P}$ over $X \times Y$, the $\ell_{\mathcal{H}}$-risk and the inner $\ell_{\mathcal{H}}$-risk of a classifier $f \in \mathcal{H}$ for the loss $\ell_{\mathcal{H}}$ are denoted by

$$R_{\ell_{\mathcal{H}}, \mathcal{P}}(f) = \mathbb{E}_{(x,y) \sim \mathcal{P}} \left[ \ell_{\mathcal{H}}(f, x, y) \right], \quad C_{\ell_{\mathcal{H}}, \mathcal{P}(\cdot|x), x}(f) = \mathbb{E}_{y \sim \mathcal{P}_X} \left[ \ell_{\mathcal{H}}(f, x, y) \right].$$

Accordingly, the minimal $\ell_{\mathcal{H}}$-risk and minimal inner $\ell_{\mathcal{H}}$-risk are denoted by $R^*_{\ell_{\mathcal{H}}, \mathcal{P}}$ and $C^*_{\ell_{\mathcal{H}}, \mathcal{P}(\cdot|x), x}$. For convenience, we will alternately use the notations of risk and inner risk presented above and Section 2 for the proofs. Next, we introduce the $\mathcal{P}$-minimizability proposed in (Steinwart, 2007).

**Definition 35 (\(\mathcal{P}\)-minimizability)** Given a distribution $\mathcal{P}$ over $X \times Y$ and a hypothesis set $\mathcal{H}$. We say that loss $\ell_{\mathcal{H}}(f, x, y)$ is $\mathcal{P}$-minimizable if for all $\epsilon > 0$ there exists $f_{\epsilon} \in \mathcal{H}$ such that for all $x \in X$ we have

$$C_{\ell_{\mathcal{H}}, \mathcal{P}(\cdot|x), x}(f_{\epsilon}) < C^*_{\ell_{\mathcal{H}}, \mathcal{P}(\cdot|x), x} + \epsilon.$$

The following lemmas are useful in the proofs of Theorem 25 and Theorem 27.

**Lemma 36** Given a distribution $\mathcal{P}$ over $X \times Y$ and a hypothesis set $\mathcal{H}$. Let $\phi$ be a margin-based loss. Then $\phi_{\text{all}}$ is $\mathcal{P}$-minimizable. If there exists $f^* \in \mathcal{H} \subset \mathcal{H}_{\text{all}}$ such that $R^*_{\phi_{\text{all}}, \mathcal{P}} = R_{\phi_{\text{all}}, \mathcal{P}}(f^*)$, then $\phi_{\text{all}}$ is also $\mathcal{P}$-minimizable in the almost surely sense.

**Proof** By Theorem 3.2 of (Steinwart, 2007), since $C^*_{\phi_{\text{all}}, \mathcal{P}(\cdot|x), x} < \infty$ for all $x \in X$, $\phi_{\text{all}}$ is $\mathcal{P}$-minimizable. Therefore, by Lemma 2.5 of (Steinwart, 2007), we have

$$R^*_{\phi_{\text{all}}, \mathcal{P}} = \int_X C^*_{\phi_{\text{all}}, \mathcal{P}(\cdot|x), x} \, d\mathcal{P}_X(x).$$

Then by the assumption,

$$\int_X C_{\phi_{\text{all}}, \mathcal{P}(\cdot|x), x}(f^*) \, d\mathcal{P}_X(x) = R_{\phi_{\text{all}}, \mathcal{P}}(f^*) = R^*_{\phi_{\text{all}}, \mathcal{P}} = \int_X C^*_{\phi_{\text{all}}, \mathcal{P}(\cdot|x), x} \, d\mathcal{P}_X(x).$$

Since

$$C^*_{\phi_{\text{all}}, \mathcal{P}(\cdot|x), x} \leq C_{\phi_{\text{all}}, \mathcal{P}(\cdot|x), x}(f^*),$$

we have

$$\int_X C_{\phi_{\text{all}}, \mathcal{P}(\cdot|x), x}(f^*) \, d\mathcal{P}_X(x) \leq \int_X C^*_{\phi_{\text{all}}, \mathcal{P}(\cdot|x), x} \, d\mathcal{P}_X(x).$$

Therefore,

$$R_{\phi_{\text{all}}, \mathcal{P}}(f^*) = \int_X C_{\phi_{\text{all}}, \mathcal{P}(\cdot|x), x}(f^*) \, d\mathcal{P}_X(x) \leq \int_X C^*_{\phi_{\text{all}}, \mathcal{P}(\cdot|x), x} \, d\mathcal{P}_X(x).$$

Thus, $\phi_{\text{all}}$ is also $\mathcal{P}$-minimizable in the almost surely sense.
for almost all \( x \in X \),
\[
C_{\phi_{\text{all}}}^* P(x) = C_{\phi_{\text{all}}} P(x) (f^*).
\]
As a result, for all \( \epsilon > 0 \), there exists an \( f^* \in \mathcal{H} \) such that for almost all \( x \in X \) we have
\[
C_{\phi_{\text{all}}} P(x) (f^*) < C_{\phi_{\text{all}}}^* P(x) + \epsilon \leq C_{\phi_{\text{all}}}^* P(x) + \epsilon.
\]
This completes the proof. 

\begin{lemma} \label{lem:calibration}
Given a distribution \( P \) over \( X \times \frac{1}{2} \) and a hypothesis set \( \mathcal{H} \). Let \( \phi \) be a margin-based loss. If for \( \eta \geq 0 \), there exists \( f^* \in \mathcal{H} \) such that \( R_{\phi_{\text{all}}} P (f^*) \leq R_{\phi_{\text{all}}}^* P + \eta \), then \( \phi_{\text{all}} \) satisfies
\[
\int_X C_{\phi_{\text{all}}} P(x) dP_X(x) \leq R_{\phi_{\text{all}}}^* P \leq \int_X C_{\phi_{\text{all}}}^* P(x) dP_X(x) + \eta.
\]
\end{lemma}

\begin{proof}
By Lemma \ref{lem:calibration}, \( \phi_{\text{all}} \) is \( P \)-minimizable. Then by Lemma 2.5 of \cite{steinwart2007}, we have
\[
R_{\phi_{\text{all}}}^* P = \int_X C_{\phi_{\text{all}}}^* P(x) dP_X(x).
\]
Therefore,
\[
R_{\phi_{\text{all}}}^* P \leq R_{\phi_{\text{all}}} P (f^*) \leq \int_X C_{\phi_{\text{all}}}^* P(x) dP_X(x) + \eta \leq \int_X C_{\phi_{\text{all}}}^* P(x) dP_X(x) + \eta.
\]
Also,
\[
\int_X C_{\phi_{\text{all}}} P(x) dP_X(x) \leq \int_X \inf_{f \in \mathcal{H}} C_{\phi_{\text{all}}} P(x) (f) dP_X(x)
\]
\[
\leq \inf_{f \in \mathcal{H}} \int_X C_{\phi_{\text{all}}} P(x) (f) dP_X(x) = R_{\phi_{\text{all}}}^* P.
\]
\end{proof}

\begin{lemma} \label{lem:calibration2}
Given a distribution \( P \) over \( X \times \frac{1}{2} \) with random variables \( X \) and \( Y \) and a hypothesis set \( \mathcal{H} \) such that \( R_{\ell_{\gamma}, \mathcal{H}} = R_{\ell_{\gamma}} (f^*) = 0 \), where \( f^* \in \mathcal{H} \) achieves the Bayes risk. Then \( f^* \) correctly classify \( x \in X \) in the almost surely sense and for almost all \( x \in X \), any \( x' \in \{ x': \| x' - x \| \leq \gamma \} \) has same label as \( x \).
\end{lemma}

\begin{proof}
Since \( R_{\ell_{\gamma}, \mathcal{H}} = R_{\ell_{\gamma}}, 0 \), the distribution \( P \) is \( \mathcal{H} \)-realizable. Therefore \( \mathbb{P}(Y = 1 | X = x) = 1 \) or 0. Thus
\[
C_{\ell_{\gamma}, \mathcal{H}}(x) (f) = \begin{cases} \sup_{x': \| x' - x \| \leq \gamma} \mathbb{I} (f(x') \leq 0), & \text{if } \mathbb{P}(Y = 1 | X = x) = 1, \\ \sup_{x': \| x' - x \| \leq \gamma} \mathbb{I} (-f(x') \leq 0), & \text{if } \mathbb{P}(Y = 1 | X = x) = 0, \end{cases}
\]
Since \( R_{\ell_{\gamma}, \mathcal{H}} (f^*) = R_{\ell_{\gamma}} (f^*) = 0 \), we have \( C_{\ell_{\gamma}, \mathcal{H}}(x) (f^*) = 0 \) for almost all \( x \in X \). When \( \mathbb{P}(Y = 1 | X = x) = 1 \), we obtain
\[
\sup_{x': \| x' - x \| \leq \gamma} \mathbb{I} (f^*(x')) = 0 \implies f^*(x') > 0 \text{ for any } x' \in \{ x': \| x' - x \| \leq \gamma \},
\]
(33)
When \( P(Y = 1|X = x) = 0 \), we obtain

\[
\sup_{x' : \|x - x'\| \leq \gamma} P(-f^*(x') \leq 0) = 0 \implies f^*(x') < 0 \text{ for any } x' \in \{x': \|x' - x\| \leq \gamma\}. \tag{34}
\]

Thus \( f^*(x) > 0 \) when \( P(Y = 1|X = x) = 1 \) and \( f^*(x) < 0 \) when \( P(Y = 1|X = x) = 0 \) for almost all \( x \in \mathcal{X} \). Therefore \( f^* \) correctly classify \( x \in \mathcal{X} \) in the almost surely sense. Furthermore, by (33) and (34), for almost all \( x \in \mathcal{X} \), any \( x' \in \{x': \|x' - x\| \leq \gamma\} \) has same label as \( x \).

**Lemma 39** Given a distribution \( P \) over \( \mathcal{X} \times \mathcal{Y} \) and a hypothesis set \( \mathcal{H} \) such that \( \mathcal{R}_{\mathcal{H}, \mathcal{X}}^* = 0 \). Let \( \phi \) be a margin-based loss and \( \tilde{\phi}(f, x, y) = \sup_{x' : \|x - x'\| \leq \gamma} \phi(y f(x')) \). If \( \phi_{\mathcal{H}} \) is \( P \)-minimizable in the almost surely sense, then \( \tilde{\phi}_{\mathcal{H}} \) is also \( P \)-minimizable in the almost surely sense.

**Proof** As shown by Awasthi et al. (2020), \( \tilde{\phi} \) has the equivalent form

\[
\tilde{\phi}(f, x, y) = \phi \left( \inf_{x' : \|x - x'\| \leq \gamma} (y f(x')) \right).
\]

Since \( \mathcal{R}_{\mathcal{H}, \mathcal{X}}^* = \mathcal{R}_{\mathcal{H}, \mathcal{X}}^* = 0 \), the distribution \( P \) is \( \mathcal{H} \)-realizable. Therefore \( P(Y = 1|X = x) = 1 \) or 0. Thus

\[
\mathcal{C}_{\phi_{\mathcal{H}}, P(|\mathcal{X})}(f)(x) = \begin{cases} 
\phi(f(x)), & \text{if } P(Y = 1|X = x) = 1, \\
\phi(-f(x)), & \text{if } P(Y = 1|X = x) = 0,
\end{cases}
\]

Note \( \tilde{\phi}(f, x, 1) = \phi \left( \inf_{x' : \|x - x'\| \leq \gamma} f(x') \right) = \phi \left( \inf_{x': |x' - x| \leq \gamma} f(x') \right) \), where WLOG we assume that \( f \) is continuous and \( m_{f, x} \in \{x': |x' - x| \leq \gamma\} \) is the point such that \( \min_{x': |x' - x| \leq \gamma} f(x') = f(m_{f, x}) \). Similarly \( \tilde{\phi}(f, x, -1) = \phi \left( -\sup_{x' : \|x - x'\| \leq \gamma} f(x') \right) = \phi \left( -f(M_{f, x}) \right) \), where WLOG we assume that \( f \) is continuous and \( M_{f, x} \in \{x': |x - x'| \leq \gamma\} \) is the point such that \( \max_{x': |x - x'| \leq \gamma} f(x') = f(M_{f, x}) \). Then for \( \tilde{\phi}_{\mathcal{H}} \), we have

\[
\mathcal{C}_{\phi_{\mathcal{H}}, P(|\mathcal{X})}(f^*)(x) = \begin{cases} 
\phi(f(m_{f, x})) , & \text{if } P(Y = 1|X = x) = 1, \\
\phi(-f(M_{f, x})) , & \text{if } P(Y = 1|X = x) = 0,
\end{cases}
\]

Since \( \phi_{\mathcal{H}} \) is \( P \)-minimizable in the almost surely sense, by the definition for all \( \epsilon > 0 \), there exists an \( f^* \in \mathcal{H} \) such that for almost all \( x \in \mathcal{X} \) we have

\[
\mathcal{C}_{\phi_{\mathcal{H}}, P(|\mathcal{X})}(f^*) < \mathcal{C}_{\phi_{\mathcal{H}}, P(|\mathcal{X})}(x) + \epsilon.
\]

When \( P(Y = 1|X = x) = 1 \), we obtain

\[
\mathcal{C}_{\phi_{\mathcal{H}}, P(|\mathcal{X})}(f^*) = \phi(f^*(m_{f^*, x})) = \mathcal{C}_{\phi_{\mathcal{H}}, P(|\mathcal{X})}(m_{f^*, x}) + f^* < \mathcal{C}_{\phi_{\mathcal{H}}, P(|\mathcal{X})}(m_{f^*, x}) + \epsilon \leq \mathcal{C}_{\phi_{\mathcal{H}}, P(|\mathcal{X})}(x) + \epsilon,
\]

where we used the fact that \( m_{f^*, x} \) satisfies \( P(Y = 1|X = m_{f^*, x}) = 1 \) by Lemma 38 and \( \phi \) is non-increasing. Similarly, when \( P(Y = 1|X = x) = 0 \), we obtain

\[
\mathcal{C}_{\phi_{\mathcal{H}}, P(|\mathcal{X})}(f^*) = \phi(-f^*(M_{f^*, x})) = \mathcal{C}_{\phi_{\mathcal{H}}, P(|\mathcal{X})}(M_{f^*, x}) + f^* < \mathcal{C}_{\phi_{\mathcal{H}}, P(|\mathcal{X})}(M_{f^*, x}) + \epsilon \leq \mathcal{C}_{\phi_{\mathcal{H}}, P(|\mathcal{X})}(x) + \epsilon.
\]
where we used the fact that $M_{f^*,x}$ satisfies $P(Y = 1|X = M_{f^*,x}) = 0$ by Lemma 38 and $\phi$ is non-increasing. Above all, for all $\epsilon > 0$, there exists an $f^* \in \mathcal{H}$ such that for almost all $x \in X$ we have

$$C_{\phi|X,Y}(f^*)(x) < C_{\phi|X,Y}(f^* + \epsilon).$$

We modify Theorem 2.8 of (Steinwart, 2007), whose proof is very similar.

**Theorem 40** Given a distribution $P$ over $X \times Y$ and a hypothesis set $\mathcal{H}$. Let $\ell_1: \mathcal{H} \times X \times Y \rightarrow [0, \infty]$, $\ell_2: \mathcal{H} \times X \rightarrow [0, \infty]$ be two losses defining on $\mathcal{H}$ such that $\mathcal{R}_{\ell_1,P}^* = \int_X C_{\ell_1,P}(f^*|X) \ dP_X(x) < \infty$ and $\int_X C_{\ell_2,P}(f^*|X) \ dP_X(x) \leq \int_X C_{\ell_2,P}(f|X) \ dP_X(x) + \eta < \infty$ for $\eta \geq 0$. Furthermore assume that there exist a function $b \in L_1^2(P_X)$ and measurable functions $\delta(\epsilon, \cdot): X \rightarrow (0, \infty)$, $\epsilon > 0$, such that

$$C_{\ell_1,P}(f^*|X) < C_{\ell_1,P}(f|X) + b(x)$$

and

$$C_{\ell_2,P}(f^*|X) < C_{\ell_2,P}(f|X) + \delta(\epsilon, x) \quad \implies \quad C_{\ell_1,P}(f^*|X) < C_{\ell_1,P}(f|X) + \epsilon$$

for all $x \in X$, $\epsilon > 0$ and $f \in \mathcal{H}$. Then for all $\epsilon > 0$ there exists $\delta > 0$ such that for all $f \in \mathcal{H}$ we have

$$\mathcal{R}_{\ell_2,P}(f) + \eta < \mathcal{R}_{\ell_2,P}(f^*) + \epsilon \quad \implies \quad \mathcal{R}_{\ell_1,P}(f) < \mathcal{R}_{\ell_1,P}(f^*) + \epsilon.$$

**Proof** Define $C_{1,x}(f) = C_{\ell_1,P}(f|X) - C_{\ell_1,P}(f^*|X)$ and $C_{2,x}(f) = C_{\ell_2,P}(f|X) - C_{\ell_2,P}(f^*|X)$ for $x \in X$, $f \in \mathcal{H}$. For a fixed $\epsilon > 0$, define $h(x) = \delta(\epsilon, x)$, $x \in X$. Then for all $x \in X$ and $f \in \mathcal{H}$ such that $C_{1,x}(f) \geq \epsilon$, we have $C_{2,x}(f) \geq h(x)$. Therefore,

$$\mathcal{R}_{\ell_2,P}(f) - \mathcal{R}_{\ell_2,P}(f^*) + \eta \geq \mathcal{R}_{\ell_2,P}(f) - \int_X C_{\ell_2,P}(f^*|X) \ dP_X(x)$$

$$= \int_X C_{2,x}(f) \ dP_X(x) \geq \int_{C_{1,x}(f) \geq \epsilon} h(x) \ dP_X(x),$$

for all $f \in \mathcal{H}$. Furthermore, since $h(x) > 0$ for all $x \in X$, the measure $\nu = bP_X$ is absolutely continuous with respect to $\mu = hP_X$, and thus there exists $\delta > 0$ such that $\nu(A) < \epsilon$ for all measurable $A \subset X$ with $\mu(A) < \delta$. Therefore, for $f \in \mathcal{H}$ with $\mathcal{R}_{\ell_2,P}(f) - \mathcal{R}_{\ell_2,P}(f^*) + \eta < \delta$ and $A = \{x \in X, C_{1,x}(f) \geq \epsilon\}$, we obtain

$$\mathcal{R}_{\ell_1,P}(f) - \mathcal{R}_{\ell_1,P}(f^*) = \int_{C_{1,x}(f) \geq \epsilon} C_{1,x}(f) \ dP_X(x) + \int_{C_{1,x}(f) < \epsilon} C_{1,x}(f) \ dP_X(x)$$

$$\leq \int_A b(x) \ dP_X(x) + \epsilon < 2\epsilon.$$

**Theorem 25** Let $P$ be a distribution over $X \times Y$ and $\mathcal{H}$ a hypothesis set for which $\mathcal{R}_{\ell_{\gamma,P}}^* = 0$. Let $\phi$ be a margin-based loss. If for $\eta \geq 0$, there exists $f^* \in \mathcal{H} \in \mathcal{H}$ such that $\mathcal{R}_{\phi}(f^*) \leq \mathcal{R}_{\phi}(f^* + \epsilon) + \eta < +\infty$ and $\phi$ is $\mathcal{H}$-calibrated\(^5\) with respect to $\ell_\gamma$, then for all $\epsilon > 0$ there exists $\delta > 0$ such that for all $f \in \mathcal{H}$ we have

$$\mathcal{R}_{\phi}(f) + \eta < \mathcal{R}_{\phi}(f^*) + \delta \quad \implies \quad \mathcal{R}_{\ell_\gamma}(f) < \mathcal{R}_{\ell_\gamma}(f^*) + \epsilon.$$

\(^5\) The theorem still holds if uniform $\mathcal{H}$-calibration is replaced by weaker non-uniform $\mathcal{H}$-calibration (Steinwart, 2007, Definition 2.7), since the proof only makes use of the weaker non-uniform property.
Using the notations in Section 2, we can rewrite (35) as

\[ 0 \leq \int_X C^+_{\ell_\mathcal{X},\mathcal{P}(|x|),x} d\mathcal{P}_X(x) \leq \mathcal{R}^*_{\ell_\mathcal{X},\mathcal{P}} = 0. \]

By Lemma 37, \( \phi_{\mathcal{H}} \) satisfies

\[ \int_X C^+_{\phi_{\mathcal{H}},\mathcal{P}(|x|),x} d\mathcal{P}_X(x) \leq \mathcal{R}^*_{\phi_{\mathcal{H}},\mathcal{P}} \leq \int_X C^+_{\phi_{\mathcal{H}},\mathcal{P}(|x|),x} d\mathcal{P}_X(x) + \eta < \infty. \]

Since for all \( x \in \mathcal{X} \) and \( f \in \mathcal{H} \), \( C_{\ell_\mathcal{X},\mathcal{P}(|x|),x}(f) \leq 1 \), we obtain

\[ C_{\ell_\mathcal{X},\mathcal{P}(|x|),x}(f) \leq C^+_{\ell_\mathcal{X},\mathcal{P}(|x|),x} + 1. \]

Also, since \( \phi \) is \( \mathcal{H} \)-calibrated with respect to \( \ell_\gamma \), for all \( x \in \mathcal{X} \), \( \epsilon > 0 \) and \( f \in \mathcal{H} \), there exists \( \delta > 0 \) such that

\[ C_{\phi_{\mathcal{H}},\mathcal{P}(|x|),x}(f) < C^+_{\phi_{\mathcal{H}},\mathcal{P}(|x|),x} + \delta \implies C_{\ell_\mathcal{X},\mathcal{P}(|x|),x}(f) < C^+_{\ell_\mathcal{X},\mathcal{P}(|x|),x} + \epsilon. \]

Therefore by Theorem 40, for all \( \epsilon > 0 \) there exists \( \delta > 0 \) such that for all \( f \in \mathcal{H} \) we have

\[ \mathcal{R}_{\phi_{\mathcal{H}},\mathcal{P}}(f) + \eta < \mathcal{R}^*_{\phi_{\mathcal{H}},\mathcal{P}} + \delta \implies \mathcal{R}_{\ell_\mathcal{X},\mathcal{P}}(f) < \mathcal{R}^*_{\ell_\mathcal{X},\mathcal{P}} + \epsilon. \]  

Using the notations in Section 2, we can rewrite (35) as

\[ \mathcal{R}_\phi(f) + \eta < \mathcal{R}^*_{\phi_{\mathcal{H}}} + \delta \implies \mathcal{R}_{\ell_\gamma}(f) < \mathcal{R}^*_{\ell_\gamma}. \]

**Theorem 27**  
**Given a distribution \( \mathcal{P} \) over \( \mathcal{X} \times \mathcal{Y} \) and a hypothesis set \( \mathcal{H} \) such that \( \mathcal{R}^*_{\ell_\gamma,\mathcal{P}} = 0 \). Let \( \phi \) be a non-increasing margin-based loss. If there exists \( f^* \in \mathcal{H} \) such that \( \mathcal{R}_\phi(f^*) = \mathcal{R}^*_{\phi_{\mathcal{H}},\mathcal{P}} < \infty \) and \( \tilde{\phi}(f, x, y) = \sup_{x' \in \mathcal{X}} \phi(yf(x')) \) is \( \mathcal{H} \)-calibrated with respect to \( \ell_\gamma \), then for all \( \epsilon > 0 \) there exists \( \delta > 0 \) such that for all \( f \in \mathcal{H} \) we have

\[ \mathcal{R}_{\tilde{\phi}}(f) < \mathcal{R}^*_{\phi_{\mathcal{H}}} + \delta \implies \mathcal{R}_{\ell_\gamma}(f) < \mathcal{R}^*_{\ell_\gamma} + \epsilon. \]

**Proof**  
By Lemma 36 and Lemma 39, \( \tilde{\phi}_{\mathcal{H}} \) is \( \mathcal{P} \)-minimizable in the almost surely sense. Then for any \( n \in \mathbb{N} \), there exists an \( f_n^* \in \mathcal{H} \) such that for almost all \( x \in \mathcal{X} \) we have

\[ C_{\phi_{\mathcal{H}},\mathcal{P}(|x|),x}(f_n^*) < C^+_{\phi_{\mathcal{H}},\mathcal{P}(|x|),x} + \frac{1}{n}. \]

Therefore

\[ \mathcal{R}^*_{\phi_{\mathcal{H}},\mathcal{P}} \leq \int_X C_{\phi_{\mathcal{H}},\mathcal{P}(|x|),x}(f_n^*) d\mathcal{P}_X(x) \leq \int_X C^+_{\phi_{\mathcal{H}},\mathcal{P}(|x|),x} d\mathcal{P}_X(x) + \frac{1}{n} \]

\[ \leq \inf_{f \in \mathcal{H}} \int_X C^+_{\phi_{\mathcal{H}},\mathcal{P}(|x|),x}(f) d\mathcal{P}_X(x) + \frac{1}{n} \leq \mathcal{R}^*_{\phi_{\mathcal{H}},\mathcal{P}} + \frac{1}{n}. \]

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6. The theorem still holds if uniform \( \mathcal{H} \)-calibration is replaced by weaker non-uniform \( \mathcal{H} \)-calibration (Steinwart, 2007, Definition 2.7), since the proof only makes use of the weaker non-uniform property.
By taking $n \to \infty$, we obtain

$$R^*_{\tilde{\phi}_\gamma, P} = \int_X \phi^*_{\gamma, P}(\cdot|x) \, dP_X(x).$$

Since $R^*_{\ell_{\gamma, 0}, P} = R^*_{\ell_{\gamma, 0}} = 0$, we obtain

$$0 \leq \int_X \phi^*_{\ell_{\gamma, 0}}(\cdot|x) \, dP_X(x) \leq R^*_{\ell_{\gamma, 0}, P} = 0.$$

Since for all $x \in X$ and $f \in \mathcal{H}$, $C_{\ell_{\gamma, 0}}(\cdot|x, x(f) \leq 1$, we obtain

$$C_{\ell_{\gamma, 0}}(\cdot|x, x(f) \leq C^*_{\ell_{\gamma, 0}}(\cdot|x, x + 1.$$

Also, since $\tilde{\phi}$ is $\mathcal{H}$-calibrated with respect to $\ell_{\gamma}$, for all $x \in X$, $\epsilon > 0$ and $f \in \mathcal{H}$, there exists $\delta > 0$ such that

$$C_{\ell_{\gamma}, P}(\cdot|x, x(f) < C^*_{\ell_{\gamma}, P}(\cdot|x, x + \delta.$$

Therefore by Theorem 40 ($\eta = 0$ here), for all $\epsilon > 0$ there exists $\delta > 0$ such that for all $f \in \mathcal{H}$ we have

$$R^*_{\phi_{\gamma}, P}(f) < R^*_{\phi_{\gamma}, P} + \delta \implies R_{\ell_{\gamma}, P}(f) < R^*_{\ell_{\gamma}, P} + \epsilon.$$  \hspace{1cm} (36)

Using the notations in Section 2, we can rewrite (36) as

$$R_{\phi}(f) + \eta < R^*_{\phi, P} + \delta \implies R_{\ell_{\gamma}}(f) < R^*_{\ell_{\gamma}, P} + \epsilon.$$