Asymptotic wave-splitting in anisotropic linear acoustics

B. L. G. Jonsson and M. Norgren
Electromagnetic Engineering,
Royal Institute of Technology, Stockholm, Sweden
August 11, 2009

Abstract

Linear acoustic wave-splitting is an often used tool in describing sound-wave propagation through earth’s subsurface. Earth’s subsurface is in general anisotropic due to the presence of water-filled porous rocks. Due to the complexity and the implicitness of the wave-splitting solutions in anisotropic media, wave-splitting in seismic experiments is often modeled as isotropic. With the present paper, we have derived a simple wave-splitting procedure for an instantaneously reacting anisotropic media that includes spatial variation in depth, yielding both a traditional (approximate) and a ‘true amplitude’ wave-field decomposition. One of the main advantages of the method presented here is that it gives an explicit asymptotic representation of the linear acoustic-admittance operator to all orders of smoothness for the smooth, positive definite anisotropic material parameters considered here. Once the admittance operator is known we obtain an explicit asymptotic wave-splitting solution.

1 Introduction

The present paper derives an explicit asymptotic representation of the linear acoustic-admittance operator in an instantaneously reacting anisotropic me-
Wave-splitting, or wave-field decomposition, is a tool to decompose the wave-field into ‘up’- and ‘down’-going wave field constituents in configurations with a certain directionality [19, 11, 18, 24], as e.g., a seismic experiment for probing earth’s subsurface e.g., [23]. The wave-splitting procedure results in two one-way equations for the wave-field constituents. Wave-splitting has been used to model and analyze wave propagation in both inverse problems and migration models. The method of wave-splitting has a long history with a wide area of applications; an overview of some of the history is given in [8]. For the isotropic case wave-splitting has been used extensively to construct fast propagation methods [12, 7, 27, 35].

Recently, there has been an interest in methods that are almost frequency independent in calculation complexity based on wave-splitting [29, 25]. Wave-splitting has also given raise to algorithms to reconstruct material parameters, for example the generalized Bremmer coupling series approach and the downward continuation approach [11, 23, 30] see also [33]. Another application area of wave-splitting is in the context of boundary conditions and time-reversal mirrors [17, 6]. Wave-splitting methods have been implemented in several physically different contexts and for a range of constitutive relations: wave-splitting for wave equations [32, 34]. The electromagnetic equations are wave-field decomposed both for isotropic [22, 5, 24], anisotropic lossless (the spectral theoretical approach) [16] and wave-splitting has been extended to the homogeneous lossless stratified bi-anisotropic case [26, 20]. Wave-splitting methods have been applied to linear-elastodynamic equations as for propagation on beams see e.g., [15] as well as in the half-space in homogeneous stratified anisotropic media [9] and up-/down symmetric media [13].

One limitation to the present methods of wave-splitting is that it has been almost exclusively limited to isotropic media see e.g., [19, 31, 2, 8, 24]. The underlying reason for this is that the wave-splitting procedures can be reduced to solving a certain key equation for the linear acoustic-admittance. This equation is almost trivial to solve in the case of isotropic media where it reduces to a square root of a certain elliptic operator. However, in the heterogeneous anisotropic media, the equation for the acoustic admittance is a non-linear equation, an operator Riccati equation, and this equation has largely resisted explicit solutions. It has been shown in [13] that wave-splitting methods for the exceptional case of up-/down symmetric anisotropic
materials resemble the methods for the isotropic case.

The method developed in the present paper solves the above mentioned equation for the acoustic admittance operator in an inherently heterogeneous anisotropic and instantaneously reacting media, at the same time the method yields explicit asymptotic solutions. We expect that with the here developed methods, wave-splitting in anisotropic media can provide a starting point for wave-splitting applications in anisotropic media. We also expect that some of the isotropic media applications can be extended to anisotropic media.

Before this paper there was essentially one other method for wave-splitting that has been extended or developed for the anisotropic case: The spectral theoretical approach [9, 18]. This method is a constructive method, but it rests on a certain spectral projection of the so called linear acoustic systems matrix, which can be hard both to evaluate numerically and to extend the method to more general constitutive relations. The leading order term of the linear acoustic admittance operator was obtained by [18], for the case of a symmetric heterogeneous anisotropic instantaneously reacting media. Higher order terms did not appear in their work, and seems to be hard to obtain by the method presented there.

An often occurring approximation when working with wave-splitting is to ignore a certain lower order ‘vertical’ variation in the so called decomposition operator, see e.g., [11]. It is known that this lower order variation can be accounted for in an exact decomposition for the isotropic media [32, 36], the latter introduced the notation of ‘true amplitude one-way wave equations’. An alternative approach is to include the correction term in the sources of the problem [11, 7]. In the present paper we show that the explicit asymptotic admittance operator can include or ignore the vertical variation of the decomposition operator with minimal changes to the solution. We give both the solutions for both these cases in heterogeneous anisotropic media.

This paper is organized in seven sections and an appendix, where Section 2 contains the explicit limitations to the material parameters of the linear acoustic equation considered here, as well as a reformulation of the linear acoustics into a form more suitable for wave-splitting. The wave-splitting problem is posed in Section 3 and reformulated into find a solution of an operator Ricatti equation for the acoustic admittance. The section ends with a brief comparison with earlier wave-splitting methods and their limitations. Before proceeding with the splitting-procedure, we recall the basics of pseudodifferential operators with parameters in Section 4. The key element in this section is, apart from the introduction of symbols, the asymptotic
composition formula of symbols.

The explicit asymptotic pseudodifferential solution to the operator Riccati equation is derived in Section 5. The asymptotic acoustic-admittance solution is given to all orders of smoothness in terms of a recursive formula. The lowest order terms are given explicitly. The result is compared with the known result for special cases, and our result is in accordance with and generalizes these known results. The asymptotic expression for the solution expressed in the symbol of the acoustic admittance operator is presented in Section 6 together with certain smoothness claims. Section 7 contains conclusions and reflections on our result. In an appendix we have included a discussion on the underlying function spaces used in the wave-splitting procedure as well and the normalization freedom of wave-splitting solutions.

2 Linear acoustics and the systems matrix

The motion of sound waves propagation through earth’s subsurface is approximated by the linear acoustic equations [21, 11, 4, 10]. The linear acoustic equations are a linear system of equations in time and space which describe the spatial and temporal changes of the pressure, \( p \), and the particle velocity, \( v = (v_1, v_2, v_3) \). In the present paper we consider the linear acoustic equations under a time-Laplace transform, yielding the equations of motion in the form:

\[
\begin{align*}
\kappa(x)p(x, s) + \partial_j v_j(x, s) &= q(x, s), \\
\rho \partial_{jk}(x)v_k(x, s) + \partial_j p(x, s) &= f_j(x, s), \quad j = 1, 2, 3,
\end{align*}
\]

where we have used the summation notation over repeated index \( j, k \in \{1, 2, 3\} \). This kind of summation notation over repeated \( j, k \) indices is used throughout the paper. Here \( x = (x_1, x_2, x_3) \) is a point in space, \( \partial_k := \frac{\partial}{\partial x_k} \) and \( s \in \mathbb{C} \) is the one-sided Laplace transform coordinate dual to time. Zero initial conditions have been assumed as to make \( \partial_t \rightarrow s \). Causality of the motion is taken into account by requiring that the Laplace-domain quantities are bounded functions of position in all of space for all \( s \) such that Re \( s > 0 \). In addition we assume that

\[
| \arg s | < \frac{\pi}{2}.
\]

Furthermore, \( f_k \) is the volume source density of force, and \( q \) is the volume source density of injection rate.
The scalar compressibility, $\kappa$, and the volume density mass tensor, $\rho$, are the material coefficients in the equations. Sound waves in earth’s subsurface propagate through, e.g., water-filled porous rocks, hence $\rho$ is assumed to be a filled $3 \times 3$ tensor or equivalently that the medium is assumed to be anisotropic. Both material parameters are in the linear acoustic approximation assumed to be instantaneously reacting i.e., independent of $s$, and we assume that they satisfy the inequalities

$$0 < \kappa_0 \leq \kappa(x) \leq \kappa_1 < \infty, \forall x \in \mathbb{R}^3$$
$$0 < \rho_0 |\zeta|^2 \leq \rho_{jk}(x)\zeta_j\zeta_k \leq \rho_1 |\zeta|^2 < \infty, \forall \zeta \in \mathbb{R}^3, x \in \mathbb{R}^3.$$  

Hence, $\rho$ and its inverse are positive definite, $\alpha := \rho^{-1}$, and $\alpha_{33}(x) > 0$, which is used repeatedly throughout the paper. In the upcoming calculations we use $\alpha$ rather then $\rho$ in (2), that is we reformulate (2) as

$$sv_j(x, s) + \alpha_{jk}\partial_k p(x, s) = \alpha_{jk}f_k(x, s), \ j = 1, 2, 3.$$  

The last assumption on $\kappa$ and $\alpha$ (or $\rho$) is that they are in $C^\infty$ and that all derivatives of $\kappa, \alpha$ are bounded functions. This last assumption simplify the upcoming microlocal analysis.

We single out depth as the preferred direction of propagation, as for e.g., a seismic experiment on probing a subsurface, and choose a coordinate system where the $x_3$-axis is parallel with the depth direction, also called the ‘vertical’-axis. The wave-splitting procedure decomposes the wave-field into ‘up’- and ‘down’-going wave-field constituents with respect to the vertical axis. We reformulate (1) and (5) towards a form where we can apply this up/down splitting of the field. This rewriting has been given in e.g., [11, 9, 18], but since it is a short derivation we include it here for completeness. The equations are separated into two parts; the first part consists of (6) with $j = 1, 2$ describing how the ‘horizontal’ or ‘transverse’ velocity components $v_1, v_2$ depend on pressure. The second part consists of the remaining two equations in (1) and (6):

$$\kappa sp + \partial_3v_3 - s^{-1}\partial_\mu(\alpha_{\mu k}\partial_k p) = q - s^{-1}\partial_\mu(\alpha_{\mu k}f_k),$$
$$sv_3 + \alpha_{3k}\partial_{x_k} p = \alpha_{3k}f_k,$$  

where we have used (5) with $j \in \{1, 2\}$ to remove all occurrences of $v_1, v_2$ in (1) and that $\text{Re } s > 0$.

We use the repeated indices $\mu, \nu$ to indicate summation over 1, 2, just as we use the repeated indices $j,k$ to indicate summation over 1, 2, 3 throughout

5
the paper. Note in particular that the term \( \partial_\mu (\alpha_{\mu k} \partial_k p) \) contain derivatives with respect to \( x_3 \). An equivalent matrix formulation of (6) is

\[
(T \partial_3 + \hat{A}) F = \left( q - s^{-1} \partial_\mu (\alpha_{\mu k} f_k) \right) \alpha_{3 k} f_k \]

with \( F = (v_3, p)^T \) and

\[
T = \begin{pmatrix} 1 & -\partial_\mu (\alpha_{\mu 3'}) \\ 0 & \alpha_{33} \end{pmatrix}, \quad \hat{A} = \begin{pmatrix} 0 & \kappa s - s^{-1} \partial_\mu (\alpha_{\mu \nu} \partial_\nu \cdot) \\ s & \alpha_{3 \mu} \partial_\mu \end{pmatrix}.
\]

The desired form of (7) suitable for wave-splitting is obtained by freeing \( \partial_3 \) from \( T \). This is achieved by observing that the matrix operator \( T \) is invertible since \( \alpha_{33} \neq 0 \). We find

\[
(\partial_3 + A) F = N,
\]

with \( N = T^{-1} (q - s^{-1} \partial_\mu (\alpha_{\mu k} f_k), \alpha_{3 k} f_k)^T \), and the acoustic systems matrix, \( A \), is given by

\[
A = T^{-1} \hat{A} = \begin{pmatrix} \partial_\mu (\alpha_{\mu 3} \alpha_{33}^{-1}) & s \kappa - s^{-1} \partial_\mu (Q_{\mu \nu} \partial_\nu \cdot) \\ 0 & \alpha_{3 \mu} \partial_\mu \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix},
\]

in which \( Q_{\mu \nu} = \alpha_{\mu \nu} - \alpha_{\mu 3} \alpha_{33}^{-1} \alpha_{3 \nu}, \mu, \nu \in \{1, 2\} \). The 2 \( \times \) 2 matrix \( Q \) is positive definite since the 3 \( \times \) 3 matrix \( \alpha \) is positive definite. In fact, \( Q \) is the Schur complement of \( \alpha_{33} \) in \( \alpha \), see e.g., [14]. An explicit proof for a similar case is given below in Equations (37) and (38). See also [18].

3 Acoustic wave-splitting

In this section we state the wave-splitting problem for the equation (9), and discuss some existing solution methods and their limitations. We furthermore reformulate the wave-splitting problem into the problem of solving an operator Riccati equation.

Consider a linear invertible composition operator \( \mathcal{L} = (\mathcal{L}^+, \mathcal{L}^-) \), a 2 \( \times \) 2 matrix of operators, where \( \mathcal{L}^\pm \) is the 2 \( \times \) 1 columns of operators, which maps the up- and down-ward components, \( W = (u_+, u_-)^T \), to the wave-field, by

\[
F = \mathcal{L} W.
\]

\[\textsuperscript{1}\] Underlying spaces and the normalization freedom of the solution is discussed in the appendix.
We reformulate (9) in terms of \( W \) to find
\[
\mathcal{L}(\partial_3 + \mathcal{G})W = N - (1 - \eta)(\partial_3 \mathcal{L})W, \tag{12}
\]
where we require \( \mathcal{L} \) and \( \mathcal{G} \) to satisfy
\[
\eta(\partial_3 \mathcal{L}) + \mathcal{A} \mathcal{L} = \mathcal{L} \mathcal{G}, \quad \mathcal{G} = \begin{pmatrix} \mathcal{G}^+ & 0 \\ 0 & \mathcal{G}^- \end{pmatrix} \tag{13}
\]
for some scalar operators \( \mathcal{G}^\pm \). Here \( \eta \) takes the value of either zero or one. The term \( \partial_3 \mathcal{L} \) in (12) and (13) is below shown to be a lower order correction term as compared to the leading order of \( \mathcal{L} \mathcal{G} \), e.g., the operator is smoothing as compared with \( \mathcal{L} \mathcal{G} \). Since \( \partial_3 \mathcal{L} \) is smoothing, it is common to ignore it, e.g., \( \eta = 0 \), in the wave-splitting procedure. It is accounted for in this case through a correction term in the sources. An alternative approach is to consider the case with \( \eta = 1 \) and adjust \( \mathcal{L} \) with respect to these perturbations. Below we find solutions \( \mathcal{L}_\eta, \mathcal{G}_\eta \) of (13) for both cases.

There is a family of solutions \( \mathcal{L}_\eta \) which solves (13), these solutions are related to each other by a ‘normalization’ of \( \mathcal{L}_\eta \). A discussion of this normalization freedom can be found in the appendix.

To find a solution to (13) we make the ansatz that the columns of the composition operator \( \mathcal{L}_\eta^\pm \) can be written as \( \mathcal{L}_\eta^\pm = (\mathcal{Y}_\eta^\pm, 1)^T \) where \( \mathcal{Y}_\eta^\pm \) is the acoustic admittance operators, it is also called the acoustic generalization of the Dirichlet-to-Neumann operator. Inserting this ansatz into (13) and eliminating the occurrences of \( \mathcal{G}_\eta^\pm \) we obtain the following operator Riccati equation
\[
\mathcal{Y}_\eta^\pm \mathcal{A}_{21} \mathcal{Y}_\eta^\pm + \mathcal{Y}_\eta^\pm \mathcal{A}_{22} - \mathcal{A}_{11} \mathcal{Y}_\eta^\pm - \mathcal{A}_{12} - \eta \partial_3 \mathcal{Y}_\eta^\pm = 0, \quad \eta \in \{0, 1\} \tag{14}
\]
together with the relation for \( \mathcal{G}_\eta^\pm \) as
\[
\mathcal{G}_\eta^\pm = \mathcal{A}_{21} \mathcal{Y}_\eta^\pm + \mathcal{A}_{22}. \tag{15}
\]
Solving the equation (14) yields a pair of solutions which are used to find the solution to the wave-splitting problem, i.e., once \( \mathcal{Y}_\eta^\pm \) is determined we find \( \mathcal{L}_\eta, \mathcal{G}_\eta \) such that (13) is satisfied.

Before proceeding to find an asymptotic solution to (14) let us shortly discuss the existing methods for solving the wave-splitting problem, i.e., to find \( \mathcal{L}_\eta, \mathcal{G}_\eta \) such that (13) is satisfied for special cases of \( \mathcal{A} \). For \( \eta = 0 \) and
the cases when $A_{11}$ and $A_{22}$ vanish is the most well known case and we find solutions to (14) as

$$Y_{\eta=0,iso}^{\pm} = \pm A_{21}^{-1}(A_{21}A_{12})^{1/2} = \pm s^{-1}\alpha_{33}^{-1}(s^2\kappa - \partial_\mu Q_{\mu\nu}\partial_\nu)^{1/2},$$

resulting in

$$G_{\eta=0,iso}^{\pm} = A_{21}Y_{\eta=0,iso}^{\pm} = \pm (\alpha_{33}^{-1}(s^2\kappa - \partial_\mu Q_{\mu\nu}\partial_\nu))^{1/2},$$

since $A_{22} = 0$. This case includes the isotropic media case, where $\alpha_{jk}(x) = \rho_{iso}^{-1}(x)\delta_{jk}$ and the up/down symmetric case \[13\] i.e., $\alpha_{3\mu} = 0 = \alpha_{\mu3}$ for $\mu = 1, 2$. Note that the square root of $A_{21}A_{12}$ is a square root of an elliptic operator with parameter, since $Q$ is a positive definite matrix, $\kappa > 0$, $|\text{arg } s| < \pi/2$, and $\text{Re } s > 0$.

For the case of homogeneous material parameters, e.g., $\kappa$ and $\alpha$ independent of $x$ one can find solutions to the wave-splitting problem through diagonalization of the $A$ in spatial Fourier domain, in this case $\partial_3 L_\eta = 0$.

A spectral theoretical approach to wave-splitting for $\eta = 0$ resulting in a splitting-matrix, $\mathcal{B}$, has been considered by \[9\] for stratified layers of homogeneous anisotropic media with instantaneous reaction, it was extended into heterogeneous anisotropic instantaneously reacting media in \[18\]. This rather complex method yields, in a constructive way, the existence of the wave-splitting. The method constructs the wave-splitting through essentially a spectral projector of the acoustic systems matrix. In the linear acoustics, \[18\] found the leading order term of the splitting operator, $\mathcal{B}$, and hence of the admittance operator. A solution, is obtained through the equation

$$Y_0^{\pm} = B_2^{-1}((\pm I - B_{22})$$

where $B = \frac{1}{i\pi} \int_{\mathbb{R}} (A - \lambda)^{-1} \text{d}\lambda$ is a $2 \times 2$ matrix of operators with elements denoted by $B_{\mu\nu}$, $\mu, \nu = 1, 2$. The existence of the splitting matrix was shown under the additional requirement $\alpha(x)$ is a symmetric positive definite matrix. There are two main restrictions of the above outlined method. The first and more serious restriction is that the splitting matrix is complex to calculate explicitly, in the sense that it is derived from a spectral projection of a non-self adjoint operator. The second restriction is the requirement that $\alpha$ is symmetric limiting the range of material parameters that can be considered.

Splitting for $\eta = 1$ has to some extent been considered in frequency domain see e.g., \[36\] but it is most extensively studied in the so called time-domain wave-splitting procedures see e.g., \[8\] and references therein. Both approaches consider isotropic media.
4 Pseudodifferential preliminaries

Our solution to the wave-splitting problem is based on the theory of microlocal analysis, in this case the theory of pseudodifferential operators. Before we enter into the details of finding such a solution, we need some preliminary notation and basic facts about pseudodifferential operators with parameters. We follow the notations of [28].

A point \( x = (x_1, x_2, x_3) \) in space will be represented by a transverse part \( x_\perp \) and a vertical part \( x_3 \), e.g., \( x = (x_\perp, x_3) \). Let the transverse spatial Fourier transform be given as

\[
\hat{u}(\xi, x_3, s) = \int_{\mathbb{R}^2} e^{-i\xi \cdot x_\perp} u(x_\perp, x_3, s) \, dx_\perp,
\]

where \( \xi = (\xi_1, \xi_2) \) is the Fourier dual to \( x_\perp \). Since \( A_{\mu\nu} \) for all \( \mu, \nu = 1, 2 \) are differential operators with smooth coefficients, we find the left-symbol, \( a_{12} \), of \( A_{12} \) by

\[
a_{12} = s\kappa + s^{-1}Q_{\mu\nu}\xi_\mu \xi_\nu - s^{-1}(\partial_\mu Q_{\mu\nu})i\xi_\nu.
\]

Resulting in

\[
a_{11} = i\xi_\mu \alpha_3 \alpha_{33}^{-1} + (\partial_\mu (\alpha_3 \alpha_{33}^{-1})), \quad a_{21} = s\alpha_{33}^{-1}, \quad a_{22} = i\xi_\mu \alpha_3 \alpha_{33}^{-1}.
\]

The above symbols are symbols with a parameter, \( s \in \mathbb{C} \). Since \( s \) corresponds to the time derivative, it is necessary to consider the parameter \( s \) as if it is of the same order as \( \xi \).

The symbols are decomposed into order of homogeneity in \( s, \xi \). To illustrate this concept, we separate \( a_{12} \) into its poly-homogeneous components in \( s, \xi \): \( a_{12} = a_{12;1} + a_{12;0} \), and similarly for the other components of \( a \). The explicit form for each of the poly-homogeneous terms of \( a \) is

\[
a_{11,1} = i\xi_\mu \alpha_3 \alpha_{33}^{-1}, \quad a_{11,0} = \partial_\mu (\alpha_3 \alpha_{33}^{-1}) \quad a_{11,0} = \partial_\mu (\alpha_3 \alpha_{33}^{-1})
\]

\[
a_{22,1} = i\xi_\mu \alpha_3 \alpha_{33}^{-1}, \quad a_{21,1} = s\alpha_{33}^{-1}, \quad a_{21,1} = s\alpha_{33}^{-1}
\]

\[
a_{12,1} = s\kappa + s^{-1}Q_{\mu\nu}\xi_\mu \xi_\nu, \quad a_{12,0} = -s^{-1}\partial_\mu Q_{\mu\nu}i\xi_\nu.
\]

The composition formula for symbols see e.g., [28] is formulated as follows: Let \( P \) and \( Q \) be two (properly supported) pseudodifferential operators in \( \mathbb{R}^2 \) with corresponding symbols \( q(x_\perp, \xi) \), \( p(x_\perp, \xi) \) respectively. The composition
formula yielding the symbol, \( r \), corresponding to \( R = PQ \), is given by the asymptotic relation \([28, I.3.6], [3, I.1.5]\)

\[
r(x_\perp, \xi) \sim \sum_\beta \frac{1}{\beta!} \left[ \partial_\xi^\beta p(x_\perp, \xi) \right] \left[ \left( \frac{1}{i} \partial_{x_\perp} \right)^\beta q(x_\perp, \xi) \right],
\]

(25)

where \( \beta \) is the multi-index \( \beta \in \mathbb{N}^2 \). Here \( \beta = (\beta_1, \beta_2) \), \( \beta! = \beta_1! \beta_2! \), and \( \partial_\xi^\beta = \partial_{\xi_1}^{\beta_1} \partial_{\xi_2}^{\beta_2} \).

The correspondence between a symbol, \( r \), and its operator, \( R \), acting upon some function \( u \) is

\[
(Ru)(x_\perp) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} e^{ix_\perp \cdot \xi} r(x_\perp, \xi) \hat{u}(\xi) \, d\xi.
\]

(26)

Note that partial differential operators with smooth coefficients are properly supported, and by a uniformity argument this property extend to partial differential operators with parameters, it is also true for each given branch of the square root of elliptical operators with parameters.

5 Asymptotics of the acoustic admittance operator

In this section we derive the asymptotic expansion of the admittance operator \( \mathcal{Y} \). This is rather long recursive expressions, where the next term is expressed in terms of all earlier terms. Once we derived the general expression we compare the derived result with existing results for a pair of particular cases. Henceforth we suppress the index \( \eta \) for readability.

Starting from the operator Ricatti equation

\[
\mathcal{Y}(A_{21} \mathcal{Y} + A_{22}) - A_{11} \mathcal{Y} - A_{12} - \eta (\partial_3 \mathcal{Y}) = 0, \quad \eta \in 0, 1
\]

(27)

we are interested in deriving the asymptotic expansion of the symbol \( y \) corresponding to a solution \( \mathcal{Y} \) of (27). We expect to get two (separate) solutions \( \mathcal{Y}^+ \) and \( \mathcal{Y}^- \) from (27). The normalization freedom of \( \mathcal{L} \) imply a normalization freedom of \( \mathcal{Y} \) and hence we expect an infinite number of solutions in two equivalence classes, see Appendix [A]. However, for our purpose it is sufficient to find one pair of solutions to (27), e.g., \( \mathcal{Y}^\pm \).

To translate (27) into an equation for symbols, we repeatedly use the composition relation (25). Let’s denote the symbol corresponding to \( \mathcal{Y} \) by \( y \).
We start with the middle part of (27): $\mathcal{R}_1 := -A_{11} \mathcal{Y} - A_{12}$, its symbol, $r_1$, has the asymptotic expansion

$$r_1 \sim -i \xi_\mu \alpha_{\mu 3} \alpha_{33}^{-1} y - \partial_\mu (\alpha_{\mu 3} \alpha_{33}^{-1} y) - s \kappa - s^{-1} Q_{\mu \nu} \xi_\mu \xi_\nu - s^{-1} \partial_\mu Q_{\mu \nu} i \xi_\nu$$  \hspace{1cm} (28)

after simplification. The simple form is due to that $A_{11}$ is a first order differential operator. Similarly we study the first part of (27) and denote this term $\mathcal{R}_2 := \mathcal{Y} (A_{21} \mathcal{Y} + A_{22})$, its corresponding symbol, $r_2$, can be obtained directly from (25). Once again due to the simple form of $A_{21}$, we find that

$$r_2 \sim \sum_\beta \frac{1}{\beta !} (\partial^{\beta}_\xi y) (\frac{1}{i} \partial_\mu)^\beta (s \alpha_{33}^{-1} y + i \xi_\mu \alpha_{\mu 3} \alpha_{33}^{-1})$$ \hspace{1cm} (29)

Solving the operator Riccati equation on a symbol level is equivalent to solving the equation

$$r_1 + r_2 - \eta \partial_3 y = 0$$ \hspace{1cm} (30)

or equivalently solving

$$\sum_\beta \frac{1}{\beta !} (\partial^{\beta}_\xi y) (\frac{1}{i} \partial_\mu)^\beta (s \alpha_{33}^{-1} y + i \xi_\mu \alpha_{\mu 3} \alpha_{33}^{-1}) - i \xi_\mu \alpha_{\mu 3} \alpha_{33}^{-1} y - \partial_\mu (\alpha_{\mu 3} \alpha_{33}^{-1} y) - s \kappa - s^{-1} Q_{\mu \nu} \xi_\mu \xi_\nu + s^{-1} \partial_\mu Q_{\mu \nu} i \xi_\nu - \eta \partial_3 y \sim 0$$ \hspace{1cm} (31)

with respect to $y$. We observe that the ansatz $y = y_0 + y_{-1} + y_{-2} + \ldots$, where each term, $y_{-n}$ is poly-homogeneous in $\xi$, $s$ of order $-n$, yields a consistent solution of (31). With this observation, it is also clear that the term $\partial_3 y$ is a lower order term, see also Section 6.

Inserting the expansion of $y$ into (31) enable us to solve the equation in an iterative manner, first we collect all 1:st order terms:

$$s \alpha_{33}^{-1} y_0^2 + i \xi_\mu \alpha_{33}^{-1} (\alpha_{3 \mu} - \alpha_{\mu 3}) y_0 \sim s \kappa + s^{-1} Q_{\mu \nu} \xi_\mu \xi_\nu$$ \hspace{1cm} (32)

with solutions

$$y_0^\pm \sim s^{-1} (\frac{1}{2} i \xi_\mu (\alpha_{3 \mu} - \alpha_{\mu 3}) \pm \gamma_1)$$ \hspace{1cm} (33)

where

$$\gamma_1 := \alpha_{33}^{1/2} (s^2 \kappa + \tilde{Q}_{\mu \nu} \xi_\mu \xi_\nu)^{1/2}$$ \hspace{1cm} (34)
with
\[
\tilde{Q}_{\mu\nu}\xi_\mu\xi_\nu = (Q_{\mu\nu} - \frac{1}{4}(\alpha_{3\mu} - \alpha_{\mu3})\alpha_{33}^{-1}(\alpha_{3\nu} - \alpha_{\nu3}))\xi_\mu\xi_\nu
\] (35)
\[
= (\alpha_{\mu\nu} - \frac{1}{4}(\alpha_{\mu3} + \alpha_{3\mu})\alpha_{33}^{-1}(\alpha_{\nu3} + \alpha_{3\nu}))\xi_\mu\xi_\nu. 
\] (36)

Note that $\tilde{Q}_{\mu\nu}$ is positive definite, this follows from the assumption that $\alpha$ is positive definite and the observation that for any vector $\xi \in \mathbb{R}^2$ we can find a vector $\zeta \in \mathbb{R}^3$ such that
\[
2\tilde{Q}_{\mu\nu}\xi_\mu\xi_\nu = \alpha_{jk}\zeta_j\zeta_k
\] (37)
and since $\alpha > 0$ we find that $\tilde{Q} > 0$. The relation between $\xi$ and $\zeta$ is
\[
\zeta = (\xi_1, \xi_2, -\frac{1}{2}\alpha_{33}^{-1}(\alpha_{3\mu} + \alpha_{\mu3})\xi_\mu).
\] (38)

This proof is similar to the proof that if $\alpha > 0$ then $Q > 0$ given in [18].

The zero order terms in (31) are:
\[
\alpha_{33}^{-1}y_{-1}(2sy_0^\pm + i\xi_\mu(\alpha_{3\mu} - \alpha_{\mu3})) + \sum_{|\beta|=1} \frac{1}{\beta!}(\partial_\beta^3 y_0^\pm)(\frac{1}{i}\partial_x)\beta(s\alpha_{33}^{-1}y_0^\pm + i\xi_\mu\alpha_{3\mu}\alpha_{\mu3}^{-1})
\]
\[
- \partial_\mu(\alpha_{\mu3}\alpha_{33}^{-1}y_0^\pm) - \eta\partial_3y_0^\pm \sim -s^{-1}\partial_\mu Q_{\mu\nu}i\xi_\nu. 
\] (39)

Once we observe that $2sy_0^\pm + i\xi_\mu(\alpha_{3\mu} - \alpha_{\mu3}) = \pm 2\gamma_1$ and recall that $\gamma_1$ is the first order symbol of a square root of a second order elliptic operator with parameters since $|\arg s| < \pi/2$ and Re $s > 0$ and hence invertible [28]. We find that the next order term, $y_{-1}$ has the asymptotic behavior:
\[
y_{-1}^\pm \sim \frac{\alpha_{33}}{2\gamma_1}\left\{ -s^{-1}\partial_\mu Q_{\mu\nu}i\xi_\nu + \partial_\mu(\alpha_{\mu3}\alpha_{33}^{-1}y_0^\pm) + \eta\partial_3y_0^\pm
\]
\[
- \sum_{|\beta|=1} \frac{1}{\beta!}(\partial_\beta^3 y_0^\pm)(\frac{1}{i}\partial_x)\beta(s\alpha_{33}^{-1}y_0^\pm + i\xi_\mu\alpha_{3\mu}\alpha_{\mu3}^{-1}) \right\}. 
\] (40)

We proceed similarly for each poly-homogeneous order in (31) and can thus from the $-n$:th order terms find the $y_{-n-1}^\pm$ solution for $n > 0$. The

---

2Recall that we always sum over 1, 2, 3 for repeated the repeated indices $j, k$, and over 1, 2 for the repeated indices $\mu, \nu$. 

---
solutions are expressed in terms of $y_{-n}^\pm, \ldots, y_0^\pm$. Its explicit expression is

$$
y_{-n-1}^\pm = \pm \frac{\alpha_{33}}{2\gamma_1} \left\{ \partial_\mu (\alpha_{\mu 3} \alpha_{33}^{-1} y_{-n}^\pm) + \eta \partial_3 y_{-n}^\pm - s \alpha_{33}^{-1} \sum_{j+k=-n-1 \atop -n \leq j,k \leq -1} y_j^\pm y_k^\pm ight. \\
left. - \sum_{k=1}^{n+1} \sum_{|\beta|=k} \frac{1}{\beta!} \sum_{j+m=k-n-1 \atop -n \leq j,m \leq 0} (\partial_\beta y_j^\pm) \left( \frac{1}{1} \partial x \right) (s \alpha_{33}^{-1} y_m^\pm + \delta_{0n} \xi \alpha_{33} \alpha_{33}^{-1}) \right\}. \tag{41}
$$

With the results in (33), (40) and (41) we have thus obtained an asymptotic representation of $y^\pm$ and hence of $Y^\pm$. This solution solves the operator Riccati equation (27).

Comparing with the case of a symmetric $\alpha = \alpha(x)$ as detailed in [18] we find that their leading order term agrees with our leading order term for the case when $\eta = 0$. Furthermore, for the simpler isotropic case the above result agrees with the leading order term reported in [23].

6 Claim of lower order

In this section we use the above derived acoustic admittance symbol to express the respective (matrix) symbols of the wave-splitting solution. We also show the claim that $\partial_3 \mathcal{L}$ is smoother than $\mathcal{L} \mathcal{G}$.

The above introduce ansatz $\mathcal{L}^\pm = (\mathcal{Y}^\pm, I)^T$ does indeed give us a solution to the splitting problem. Starting from (15) we find that the diagonal terms in $\mathcal{G}$ have the symbol, $g^\pm$ with poly-homogeneous expansion

$$
g^\pm = \alpha_{33}^{-1} (s y^\pm + i \xi \alpha_{33} \mu) \sim \alpha_{33}^{-1} (s y_0^\pm + i \xi \alpha_{33} \mu) + \alpha_{33}^{-1} s (y_{-1}^\pm + y_{-2}^\pm + \ldots) = g_1^\pm + g_0^\pm + g_{-1}^\pm + \cdots, \tag{42}
$$

where

$$
g_{-1}^\pm = \alpha_{33}^{-1} (s y_0^\pm + i \xi \alpha_{33} \mu), \quad g_{-n}^\pm = \alpha_{33}^{-1} s y_{-n-1}^\pm, \quad n = 0, 1, 2, \ldots \tag{43}
$$

Hence $g^\pm$ is expressed as a poly-homogeneous sum with leading order term of poly-homogeneous order one.

Similarly we find that the composition operator $\mathcal{L}$ with symbol, $\ell$, of the form:

$$
\ell = \begin{pmatrix} y^+ & y^- \\ 1 & 1 \end{pmatrix} \sim \ell_0 + \ell_{-1} + \ell_{-2} + \cdots, \tag{44}
$$
where
\[
\ell_0 = \begin{pmatrix} y_0^+ & y_0^- \\ 1 & 1 \end{pmatrix}, \quad \ell_{-n} = \begin{pmatrix} y_{-n}^+ & y_{-n}^- \\ 0 & 0 \end{pmatrix}, \quad n = 0, 1, 2 \ldots
\] (45)

We claimed in Section 3 that the term $\partial_3 \mathcal{L}$ is of lower order than $\mathcal{L}G$, we are now in the position to show this claim. We do this by explicitly giving the symbols of $\partial_3 \mathcal{L}$ and $\mathcal{L}G$. The symbol of $\partial_3 \mathcal{L}$ can be written as
\[
\partial_3 \ell \sim \begin{pmatrix} \partial_3 y^+ & \partial_3 y^- \\ 0 & 0 \end{pmatrix},
\] (46)

where
\[
\partial_3 y^\pm \sim s^{-1} \left( -\frac{1}{2} \xi_\mu \partial_3 (\alpha_{3\mu} - \alpha_{\mu 3}) \pm \frac{1}{2\gamma_1} (s^2 \partial_3 \kappa + (\partial_3 \tilde{Q}_{\mu\nu}) \xi_\mu \xi_\nu) \right) + \text{terms of order } -1 \text{ and lower.}
\] (47)

The symbol, $p$, of $P := \mathcal{L}G$ is
\[
p \sim \sum_\beta (\partial_\xi^\beta \ell)(\frac{1}{i} \partial_\xi)^\beta g \sim \ell_0 g_1 + \ell_{-1} g_1 + \ell_0 g_0 + \sum_{|\beta|=1} (\partial_\xi^\beta \ell_0)(\frac{1}{i} \partial_\xi)^\beta g_1 + \text{terms of order } -1 \text{ and lower.}
\] (48)

where $\ell_m$ is the $2 \times 2$ matrix see (44) and $g_m = \text{diag}(g_m^+, g_m^-)$, with $g_m^\pm$ for $m \in 1, 0, -1, -2, \ldots$ as given in (42).

Comparing the result for the respective leading orders, of $\partial_3 \ell$ and $p$, we find that $\partial_3 \ell_0$ is a matrix containing the terms $(\partial_3 y^\pm, 0)$ while the leading expression for $p$ is $\ell_0 g_1$ with terms $(y_0^+ g_1^+, g_1^-)$. We find the claim to be shown, $\mathcal{L}G$ has leading order symbol of order one whereas $\partial_3 \mathcal{L}$ has leading order symbol of order zero in the poly-homogeneous expansion of the symbols in $s, \xi$. Hence, $\partial_3 \mathcal{L}$ is a smoother term as compared with $\mathcal{L}G$ as claimed.

7 Conclusion

The main advantage with the above procedure to derive $Y^\pm$ as compared to the spectral theoretical approach is that it gives an explicit asymptotic representation of $Y^\pm$ through its asymptotic symbol expansion $y \sim y_0 + y_{-1} + \ldots$ as given above. The method yields, as far as the authors know, the first explicit form of $y_{-n}$ for $n > 0$ in anisotropic media.
The presented method is not limited to material parameters for which certain spectral properties of $\mathcal{A}$ exist, in our method it suffices to consider material parameters $\kappa(x), \alpha(x)$ such that $\alpha_{33}$ and $s^2\kappa + \partial_{x_{\mu}} \tilde{Q}_{\mu\nu} \partial_{x_{\nu}}$ have well defined inverses and square roots.

The flexibility of the presented approach is shown by the possibility to include the often neglected term due to the vertical derivatives of the composition operator with minimal changes to the derivation procedure. The simplicity of the method indicates that it might be extended to the case of the electromagnetic equations, possibly with anisotropic conductivity, as well as to linear elasticity.

A Non-uniqueness of wave-splitting solutions

To formalize our discussion of the normalization freedom of wave-splitting solutions we introduce the function space $\mathcal{H}^n := (\mathcal{H}^n(\mathbb{R}^3, \mathbb{C}), \mathcal{H}^{n+1}(\mathbb{R}^3, \mathbb{C}))^T$, where $\mathcal{H}^n$ is the $n$:th order Sobolev space of square integrable functions, $n \in \mathbb{R}$, see e.g., [3].

The normalization freedom implicitly appear in the first step in phrasing the wave-splitting problem:

$$F = \mathcal{L} W.$$  \hfill (49)

Here we need to specify to what spaces $F$ and $W$ belongs. Let’s start with the case outlined in the paper. We have implicitly assumed that the sources $N$ in (9) belong to $\mathcal{H}^{n-1}$, for some $n \in \mathbb{R}$ and hence, due to the derivatives in $\mathcal{A}$, we find that $F \in \mathcal{H}^n$. The obtained $\mathcal{L}$ with symbol given in (44) maps $\mathcal{H}^{n+1}$ to $\mathcal{H}^n$, and the obtained wave-field constituents satisfy $W \in \mathcal{H}^{n+1}$. Indeed if we use the notation $\mathcal{L}_n$ to indicate that $\mathcal{L}_n : \mathcal{H}^n \mapsto \mathcal{H}^{n-1}$ and similarly for $\mathcal{G}$ we can rephrase the requirements on $\mathcal{L}, \mathcal{G}$ as: find $\mathcal{L}, \mathcal{G}$ such that

$$\eta(\partial_3 \mathcal{L}_{n+1}) + \mathcal{A} \mathcal{L}_{n+1} = \mathcal{L}_{n} \mathcal{G}_{n+1}, \ \eta \in \{0, 1\}$$  \hfill (50)

with $\mathcal{G}_{n+1}$ diagonal.

To make the normalization freedom apparent we replace (49) with $F = \tilde{\mathcal{L}}_{n+1} \tilde{W}$, where

$$\tilde{\mathcal{L}}_m = \mathcal{L}_m \mathcal{N}_m^{-1}, \ \tilde{W} = \mathcal{N}_{n+1} W, \ m, n \in \mathbb{R}$$  \hfill (51)

for any invertible operator $\mathcal{N}_m$ from $\mathcal{H}^m$ to some desired space $\mathcal{W}_m$ for some $m \in \mathbb{R}$.
The natural question arises: what are the requirements on \( \mathcal{N} \) so that \( \tilde{\mathcal{L}} \) and its corresponding \( \tilde{\mathcal{G}} \) is a wave-splitting solution to (11)-(13). These operators \( \mathcal{N} \) represents the normalization freedom of the solution of the wave-splitting problem. It is rather straightforward to implicitly characterize which \( \mathcal{N} \) that provides a wave-splitting solution, e.g., we substitute \( \mathcal{L}, \mathcal{G} \) in (51) into (12) and (13) to find

\[
\eta(\partial_3 \tilde{\mathcal{L}}_{n+1}) + A \tilde{\mathcal{G}}_{n+1} \tilde{W} = \tilde{\mathcal{L}}_n [\mathcal{N}_n \mathcal{G}_{n+1} \mathcal{N}_{n+1}^{-1} - \eta(\partial_3 \mathcal{N}_{n+1}) \mathcal{N}_{n+1}^{-1}] \tilde{W} = \tilde{\mathcal{L}}_n \tilde{\mathcal{G}} \tilde{W},
\]

(52)

where we have identified \( \tilde{\mathcal{G}}_{n+1} := \mathcal{N}_n \mathcal{G}_{n+1} \mathcal{N}_{n+1}^{-1} - \eta(\partial_3 \mathcal{N}_{n+1}) \mathcal{N}_{n+1}^{-1} \). The requirement on \( \mathcal{N}_n \) is that \( \tilde{\mathcal{G}}_{n+1} \) remains diagonal for diagonal \( \mathcal{G}_{n+1} \). In addition as an implicit requirement on \( \mathcal{N} \), for \( \tilde{W} \in W_{n+1} \) we require that \( (\partial_3 \mathcal{N}_{n+1}) \mathcal{N}_{n+1}^{-1} \tilde{W} \in W_n \), as indicated in the restriction of the \( \tilde{\mathcal{L}}_{n+1} \) term to \( \tilde{\mathcal{L}}_n \) in front of \( (\partial_3 \mathcal{N}_{n+1}) \mathcal{N}_{n+1}^{-1} \) in (52).

The normalization freedom above is rather large, there are several diagonal (and anti-diagonal) \( \mathcal{N} \) which satisfy the above conditions, e.g., \( \mathcal{N} = \text{diag}(m, m'), m, m' \in \mathbb{R} \) and \( \mathcal{N} = \text{diag}((\mathcal{Y}^+)^p, (\mathcal{Y}^-)^{p'}) \), \( p, p' \in \mathbb{R} \). The normalization freedom was used in earlier wave-splitting papers, for the isotropic case [11] utilize the normalization freedom to derive wave-splitting solutions with self-adjoint admittance operator. Normalization was also discussed in the anisotropic case [18] where the spaces \( H^n \) also appeared.

One case of some interest is the normalization that switch from the acoustic-admittance representation to the acoustic-impedance representation. Here we are interested in a normalization operator

\[
\mathcal{N} = \text{diag}(\mathcal{Y}^+, \mathcal{Y}^-),
\]

(53)

and \( \tilde{W} \in H^n \). Thus \( \mathcal{N}_n : H^n \mapsto H^{n-1} \). The explicit form of the symbol of \( \mathcal{Y}^\pm \) ensure the implicit assumption that \( (\partial_3 \mathcal{N}_{n+1}) \mathcal{N}_{n+1}^{-1} \tilde{W} \in H^n \) and that \( \tilde{\mathcal{L}}_{n+1} : H^n \mapsto H^n \). We obtain

\[
\tilde{\mathcal{L}} = \begin{pmatrix} I & I \\ (\mathcal{Y}^+)^{-1} & (\mathcal{Y}^-)^{-1} \end{pmatrix}
\]

(54)

and

\[
\tilde{\mathcal{G}} = \begin{pmatrix} (\mathcal{Y}^+)^{-1} \mathcal{G}_+(\mathcal{Y}^+) - \eta(\partial_3 (\mathcal{Y}^+)^{-1}) \mathcal{Y}^+ \\ 0 \\ (\mathcal{Y}^-)^{-1} \mathcal{G}_-(\mathcal{Y}^-) - \eta(\partial_3 (\mathcal{Y}^-)^{-1}) \mathcal{Y}^- \end{pmatrix}.
\]
References

[1] L. M. Brekhovskikh and O. A. Godin. *Acoustics of Layered Media I*. Springer-Verlag, Berlin, 1990.

[2] J. Cao. *Applications of 3D domain wave splitting to direct and inverse scattering*. PhD thesis, Royal Institute of Technology, Stockholm, Sweden, 1998.

[3] H. O. Cordes. *The Technique of Pseudodifferential Operators*. London Mathematical Society Lecture Note Series 202. Cambridge University Press, Cambridge, 1995.

[4] A. Doicu, Y. Eremin, and T. Wriedt. *Acoustic & Electromagnetic Scattering Analysis*. Academic Press, San Diego, 2000.

[5] I. Egorov, G. Kristensson, and V. H. Weston. Transient electromagnetic wave propagation in laterally discontinuous, dispersive media. *Wave Motion*, 33(1):67–77, 2001.

[6] L. Fishman, B. L. G. Jonsson, and M. V. de Hoop. Time reversal mirrors and cross correlation functions in acoustic wave propagation, mathematical modeling of wave phenomena. In *3rd Conference on Mathematical Modeling on Wave Phenomena*, volume 1106 of *AIP Conf. Proceeding*, pages 183–202, 2009.

[7] M. Gustafsson. The Bremmer series for a multi-dimensional acoustic scattering problem. *J. Phys. A: Math. Gen.*, 33(9-10):1921–32, 2000.

[8] S. He et al. *Time domain wave-splitting and inverse problems*. Oxford University Press, Oxford, 1998.

[9] B. P. de Hon. *Transient Cross-Borehole Elastodynamic Signal Transfer Through a Horizontally Stratified Anisotropic Formation*. PhD thesis, Technische Universiteit, Delft, Holland, 1996.

[10] A. T. de Hoop. *Handbook of Radiation and Scattering of Waves*. Academic Press, Kent, 1995.

[11] M. V. de Hoop. Generalization of the Bremmer coupling series. *J. Math. Phys.*, 37(7):3246–3282, 1996.
[12] M. V. de Hoop and A. K. Gautesen. Uniform asymptotic expansion of the generalized Bremmer series. *SIAM J. Appl. Math.*, 60(4):1302–1329, 2000.

[13] M. V. de Hoop and A. T. de Hoop. Elastic wave up/down decomposition in inhomogeneous and anisotropic media: An operator approach and its approximations. *Wave Motion*, 20:57–82, 1994.

[14] R. A. Horn and C. A. Johnson. *Matrix Analysis*. Cambridge University Press, Cambridge, 1985.

[15] M. Johansson, P. D. Folkow, and P. Olsson. Dispersion free wave-splitting for structural elements. *Comput. Struct.*, 84(7):514–527, 2006.

[16] B. L. G. Jonsson. Wave splitting of Maxwell’s equations with anisotropic heterogeneous constitutive relations. *Inverse Probl. Imag.*, 3(3):405–452, 2009, math-ph/0809.0789.

[17] B. L. G. Jonsson, M. Gustafsson, V.H. Weston, and M. V. de Hoop. Retrofocusing of acoustic wave fields by iterated time reversal. *SIAM J. of Appl. Math.*, 64(6):1954–86, 2004.

[18] B. L. G. Jonsson and M. V. de Hoop. Wave field decomposition in anisotropic fluids: A spectral theory approach. *Acta Appl. Math.*, 62(2):117–171, 2001.

[19] G. Kristensson and R.J. Kruger. Direct and inverse scattering in the time domain for a dissipative wave equation. I. Scattering operators. *J. Math. Phys.*, 27(6):1667–1682, 1986.

[20] G. Kristensson, S. Poulsen, and S. Rikte. Propagators and scattering of electromagnetic waves in planar bianisotropic slabs — an application to frequency selective structures. *PIER*, 48:125, 2004.

[21] L. D. Landau and E. M. Lifshitz. *Fluid Mechanics*. Pergamon Press, Oxford, 3:rd edition, 1966.

[22] J. Lundstedt and S. He. Time domain direct and inverse problems for a nonuniform LCRG line with internal sources. *IEEE T. Electromagn. C*, 39(2):79–88, 1997.
[23] A. E. Malcolm and M. V. de Hoop. A method for inverse scattering based on the generalized bremmer coupling series. *Inverse Problems*, 21:1137–1167, 2005.

[24] A. Morro. One-way propagation in electromagnetic materials. *Math. Comput. Model.*, 39(11-12):1221–29, 2004.

[25] J. Popovic. A fast method for solving the Helmholtz equation based on wave splitting. Licentiate thesis, Royal Institute of Technology, Stockholm, 2009-06-12.

[26] S. Rikte, G. Kristensson, and M. Andersson. Propagation in bian-isotropic media — reflection and transmission. *IEE Proc.-Microw. Antennas Propag.*, 148(1):29–36, 2001.

[27] J. H. Le Rousseau and M. V. de Hoop. Generalized-screen approximation and algorithm for the scattering of elastic waves. *Q. J. Mech. Appl. Math.*, 56(1):1–33, 2003.

[28] M. A. Shubin. *Pseudodifferential Operators and Spectral Theory*. Springer-Verlag, Berlin, 1987.

[29] C. C. Stolk. A fast method for linear waves based on geometrical optics. *SIAM J. Numer. Anal*, 47(2):1108–1194, 2009.

[30] C. C. Stolk and M. V. de Hoop. Seismic inverse scattering in the downward continuation approach. *Wave Motion*, 43:579–598, 2006.

[31] M. J. N. van Stralen. *Directional decomposition of electromagnetic and acoustic wave-fields*. PhD thesis, Technische Universiteit, Delft, Holland, 1997.

[32] V. H. Weston. Invariant imbedding for the wave equation in three dimensions and the applications to the direct and inverse problems. *Inverse Problems*, 6(6):1075–1105, 1990.

[33] V. H. Weston. Time-domain wave splitting of Maxwell’s equations. *J. Math. Phys.*, 34(4):1370–1392, 1993.

[34] V. H. Weston and B. L. G. Jonsson. Wave front layer stripping approach to inverse scattering for the wave equation. *J. Math. Phys.*, 43(10):5045–59, 2002.
[35] R.-S. Wu, X.-B. Xie, and X.-Y. Wu. One-way and one-return approximations (De Wolf approximation) for fast elastic wave modeling in complex media. *Adv. Geophys.*, 48:265–322, 2007.

[36] Y. Zhang, G. Zhang, and N. Bleistein. True amplitude wave equation migration arising from true amplitude one-way wave equations. *Inverse Problems*, 19(5):1113–1138, 2003.