The Quantum Spherical Spin Glass Model: A Limitation to Static Approximation.

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In this work, we confront the static approximate with an exact solution in the quantum spherical p-spin interaction model \((p \to \infty \text{ and } p = 2)\). On the one hand, this study indicates that the static approximate corresponds to exact solution in the cases \(p \to \infty \text{ and } p = 2\) in the classic regime. On the other, the static approximate differs from the exact solution for \(p = 2\) in the quantum regime.

I. INTRODUCTION

The term spin glass appeared to designate a class of metallic alloys which are formed from noble metal ions weakly diluted in transition metal magnetic. One reason for the name of spin glass is because the magnetic moments of these alloys present a locally fixed orientation without any periodic ordering. Conceptually, it reminds amorphous structures as conventional glass. In other words, the spin glass phase can be understood as a set of spins exhibiting a phase frozen at low temperatures. However, unlike the cases ferromagnets and antiferromagnets, lacks long range order.

The first theoretical model developed to study the existence of a spin glass phase was proposed Edwards and Anderson. Subsequently, a version of the infinite-range Edwards-Anderson model was developed by Sherrington and Kirkpatrick \((\text{SK model})\). Unlike pair interactions, proposed in the two previous models, Crisanti and Sommers developed a model that generalizes the interactions for \(p\) spins, know as classical spherical (CS) \(p\)-spin interaction spin glass model.

However, at low temperature, experimental evidence suggests that quantum effects are significant. Starting addition, quantum theoretical models were developed to describe the spin glass phase at this temperature regime. In particular, we cite the quantum spherical (QS) \(p\)-spin glass model \(10,15\). Based on previous model, Cugliandolo, Grempel and Santos conducted a formal study for QS \(p\)-spin interaction model. Unfortunately, it was not possible to treat this model analytically for a generic value of \(p\). Thus, after performing the replica method, the authors use a static approximation (SA) \(11\). However, Menezes and Theumann show that the effective action of the quantum spherical model for \(p = 2\) is invariant over a generalized form of Becchi-Rouet-Stora-Tyutin super-symmetry, and thus a result via annealed average is exact for this model.

Within this context, in this contribution, we compare the static approximation with the exact solution for the QS \(p\)-spin interaction model in two particular cases: (i) \(p = 2\) and (ii) \(p \to \infty\). The objective of this study is to quantify how good is the static approximation for the values of \(p\) indicated. For this purpose, follow the standard procedure of replica method where we limit the hypothesis of replica symmetry - that is sufficient for this model. Our results suggest that, on the one hand, the limit \(p \to \infty\) the static approximation is exact. On the other hand, the case \(p = 2\) the static approximation is exact in the classic limit but is not good in the quantum limit.

This paper is organized as follows: In Sec.II and IIIA, respectively, we review the QS \(p\)-spin interaction model and exact solution for \(p = 2\). In Sec. IIIB, we show the exact solution for \(p \to \infty\). In Sec. IV, we solve the QS model using static approximation for \(p = 2\) and \(p \to \infty\). In Sec. V, we compare the exact solution with the result obtained via static approximation. Finally, in Sec. VI, we discuss the limitation of the result obtained via static approximation for QS \(p\)-spin interaction model.

II. MODEL

The hamiltonian for QS \(p\)-spin interaction model is given by

\[
\hat{H} = \frac{1}{2I} \sum_{i=1}^{N} \hat{P}_i^2 - \sum_{1 \leq i_1 < \cdots < i_p \leq N} J_{i_1 \cdots i_p} \hat{S}_{i_1} \cdots \hat{S}_{i_p} + \mu \sum_{i=1}^{N} \hat{S}_i^2 \tag{1}
\]

where the spin operators have continuous eigenvalues \(S_i \in (-\infty, \infty)\), \(\hat{P}_i\) is the momentum operator canonically conjugated to \(\hat{S}_i\) (satisfies the relation \([\hat{S}_k, \hat{P}_l] = i\delta_{kl}\)), \(I\) is the moment of inertia of the spins (quantum rotors), \(\mu\) is the lagrange multiplier (mean spherical constraint \(\sum_{i=1}^{N} \langle \hat{S}_i^2 \rangle = N\)) and \(J_{i_1 \cdots i_p}\) are the elements of a random symmetric matrix (distributed according to a Gaussian with zero mean and variance \(\sigma^2 = J^2 p!/(2N^{p-1})\)). Since the distribution is the same for any set of \(p\) spins, this is equivalent to the infinite range (mean field) - as in the SK model.

III. EXACT SOLUTION

Solve the system means, in the context of this work, to find an analytical form for the grand thermodynamical potential, in the limit of replica’s number \(n \to 0\) and for a macroscopical system \((N \to \infty)\), from which can be
obtained physical properties of the system. For this end, we solve integrals that appear in the replicate partition function and obtaining explicit forms to determine the fields (in the context of field theory) from the saddle point equations.

To obtain the partition function of the model we follow the Feynman’s prescription to path integrals with the hamiltonian (1). The connection with thermodynamics is made through the grand thermodynamic potential \( \Omega (\propto \log Z(J)) \). We can make an analogy between the parameter \( \mu \) of spherical constraint and the chemical potential. The randomness is treated realizing the configurational average in \( \log Z(J) \) (quenched), in order to accomplish it we use the replica method. Finally, to uncouple the imaginary times (from the Feynman’s integral path) we introduce the Fourier series for quantities dependent on these times. After accomplish all these steps, the grand thermodynamic potential becomes

\[
\frac{\beta \Omega}{N} = \text{lim}_{n \to 0} \frac{1}{n} G[q(\omega_m)],
\]

(2)

where

\[
G[q(\omega_m)] = -\frac{J^2}{4} \sum_{\alpha \nu} \int_0^{\beta} d\tau \left[ \sum_m e^{-i \omega_m \tau} q_{\alpha \nu}(\omega_m) \right]^p
\]

\[-\frac{1}{2} \left( \sum_m \log |D_m| + \sum_{\alpha \nu} \lambda_m q_{\alpha \nu}(\omega_m) \right),
\]

(3)

\( D_m = \det( q(\omega_m) ) \) and \( \lambda_m = I \omega_m^2 \beta + 2 \beta \mu \). The term \( q(\omega_m) \) is the overlap’s matrix between replicas, \( \omega_m = 2\pi m/\beta \) are the Matsubara frequencies for bosons (since the variables has one commutation algebra). The fields \( q_{\alpha \nu}(\omega_m) \) are determined by \( \delta G[q(\omega_m)]/\delta q_{\alpha \nu}(\omega_m) = 0 \) in the Eq. (4), results

\[
\frac{\beta J^2 p}{2} \int_0^{\beta} d\tau e^{i \omega_m \tau} q^{p-1}_{\alpha \nu}(\tau) + [q^{-1}(\omega_m)]_{\alpha \nu} = \lambda_m \delta_{\alpha \nu}.
\]

(4)

In the next section, we study the Eq. (4) for two specific cases: \( p = 2 \) and \( p \to \infty \).

A. Case \( p = 2 \)

In the case \( p = 2 \), the replica symmetric ansatz, parametrizing \( q_{\alpha \nu}(\omega_m) = (q_0(\omega_m) - q(\omega_m)) \delta_{\alpha \nu} + q(\omega_m) \), is sufficient for a complete description of the model. This hypophysicaly allows us obtain explicitly the elements of the inverse matrix \( q^{-1}(\omega_m) \).

The \( q(\omega_m) = 0 \) solution always is valid (paramagnetic solution) for the saddle point equations. With this solution we obtain a quadratic equation for \( q_0(\omega_m) \), whose solution is

\[
q_0(\omega_m) = \frac{\lambda_m - \sqrt{\lambda_m^2 - 4(\beta J)^2}}{2(\beta J)^2}.
\]

(5)

The spherical constrain can be write as \( \sum_m q_0(\omega_m) = 1 \), and this sum over frequencies can be solved using a standard procedure with integrals in the complex plane. This approach leads us to

\[
\frac{1}{2\pi J^2} \int_{-\infty}^{+\infty} dx H(x) \coth \left( \frac{\beta x}{2} \right) = 1,
\]

(6)

where \( H(x) = \sqrt{4J^2 - (2\mu - I x^2)^2} \) and \( L_{\pm} = \sqrt{2(\mu \pm J)} \). The integral over real variable \( x \) is well defined if \( \mu \geq J \). Thus, \( \mu \) “sticks” at the value \( \mu_c = J \) below a certain temperature \( T_c \) (critical temperature) obtained in Eq. (6) by siting \( \mu = J \) for each fixed \( I \). The critical value \( I_{\text{Exact}}^{\text{Exact}} \) (where \( T_c = 0 \) is obtained analytically as

\[
1/J_{\text{Exact}}^{\text{Exact}} = \frac{9\beta^2}{16} \approx 5.5.
\]

B. Case \( p \to \infty \)

Now, we treat the model in the limit \( p \to \infty \). For a classical Ising spin case, this limits corresponds to random energy model and a quantum model with Ising spins in the presence of a transverse field is treated by Obuchi, Nishimori and Sherington.

Assuming replica symmetry in Eq. (4), we see that \( q(\omega_m) = 0 \) is always a solution. But, when \( p \to \infty \) it follows that \( pq_{p-1}(\tau) \to 0 \) (if \( 0 \leq q(\tau) < 1 \)) so \( q(\omega_m) = 0 \).

For \( pq_{p-1}(\tau) \to 1 \) (if \( q(\tau) = 1 \)) and then \( q(\omega_m) \to \infty \) (not physical solution) or \( q_0(\omega_m) = q(\omega_m) = 1 \) which cannot occur at a finite temperature. Therefore, within the limits \( p \to \infty \) the system presents only the paramagnetic phase. Setting \( q(\omega_m) = 0 \) in the saddle point Eq. (4) for \( q_0(\omega_m) \) (with \( \alpha = \nu \)), we get

\[
\frac{\beta J^2 p}{2} \int_0^{\beta} d\tau e^{i \omega_m \tau} q^{p-1}_{\alpha \nu}(\tau) + \frac{1}{q_0(\omega_m)} = \lambda_m.
\]

(7)

Thus for \( p \to \infty \), we have \( pq_{p-1}(\tau) \to 0 \) (if \( q_0(\tau) < 1 \)), thus \( q_0(\omega_m) = (I \omega_m^2 \beta + 2(\beta \mu)^{-1})^{-1} \).

Using the spherical constraint in the form \( \sum_m q_0(\omega_m) = 1 \), and the standard procedure to evaluate sum over frequencies (like made in Sec. [IIIA]), where the integration in the complex plane is realized with help from residue theorem, we find

\[
\frac{1}{2} \sqrt{\frac{1}{2\mu} I} \coth \left( \beta \sqrt{\frac{\mu}{2I}} \right) = 1.
\]

(8)

This equation allows us get \( \mu(T) \) as a function of the temperature for each fixed \( I \). So, \( \mu = 0 \) is not allowed, i.e., \( T_c = 0 \) for all \( I \).

IV. STATIC APPROXIMATION

To employ the formalism of Feynman integrals path, we obtain a functional integral dependent on the ima-
ginary time ($\tau$) and its associated fields have the same dependence. This adds a great degree of difficulty, the analytical view point, to advance in the problem for any $p$ (see Eq. (3) for example).

In order to circumvent this problem, Bray and Moore suggested an approximate method to resolve it. Their method, which is referred as the static approximation (SA), is to neglect the (imaginary) time dependence of the order parameters (fields).

Following the proposal of Obuchi, Nishimori and Sherrington to implement the static approximation (combined with RS ansatz), we get to the the grand thermodynamic potential $\Omega_{SA}$ as

\[
\frac{\beta \Omega_{SA}}{N} = -\frac{(\beta J)^2}{4} (q_o^p - q^p) - \frac{1}{2} \log (q_o - q) - \frac{q}{2(q_o - q)} + \log \left[ 2 \sinh \left( \frac{\beta}{2} \sqrt{\frac{2\mu}{T}} \right) \right] - \frac{1}{2} \log (2\beta\mu) + \beta \mu q_o.
\] (9)

It is analogous to Eq. (2) and (3), where $\Omega_{SA}$ denotes the grand canonical potential over SA. Now, the saddle point equations $\partial \Omega_{SA}/\partial q_o = \partial \Omega_{SA}/\partial q = 0$ together to mean spherical condition $\partial \Omega_{SA}/\partial \mu = N$ leads us to similar Eq. (4), more precisely

\[
\left\{ \begin{array}{c}
\frac{(\beta J)^2}{2} pq_o^{p-1} - \frac{q}{(q_o - q)^2} = 0; \\
\frac{(\beta J)^2}{2} pq_0^{p-1} - \frac{q_o - q}{(q_o - q)^2} = 2\beta\mu; \\
1 + \frac{1}{2\mu} - \frac{q_o}{\sqrt{\mu T}} \coth \left( \frac{\beta}{2} \sqrt{\frac{2\mu}{T}} \right) = 0.
\end{array} \right.
\] (10)

In the next section we explore solutions of the system (10) in the same cases (particular values of $p$) treated in Sec. III.

A. Case $p = 2$

Replacing $p = 2$ in the saddle point equation (10) is relatively simple. To note that $q = 0$ is always a solution, this one characterizes the paramagnetic phase and it is sufficient to describe the phase diagram. So, with $q = 0$ in the second Eq. (10) we come to a quadratic equation in the variable $q_o$, whose solution is given by

\[
q_o = \frac{(\mu)}{(\frac{\beta J}{2})} - \sqrt{\left( \frac{\mu}{2} \right)^2 - 1},
\] (11)

which is analogous to Eq. (5) for $\omega_m = 0$. Furthermore, in order that $q_o$ is real, is necessary that the condition is satisfied $\mu \geq J$. Thus, we conclude $\mu_{SA}^p = J$ for $T \leq T_{c}^{SA}$, where $T_{c}^{SA}$ is the critical temperature below which $\mu$ sticks at $J$ over the SA. This is the same value from sticks for $\mu$ in the Sec. IIIA when we treat the model exactly.

In the critical temperature $\mu = \mu_{c}^{SA} = J$, and then, sitting it in Eq. (11) we get $q_o = q_{o}^{c} = T_{c}^{SA}/J$. So, with this information in the spherical condition (third Eq. (11)) we obtain the equation which gives us $T_{c}^{SA}$ for each fixed $I$ and, consequently, the phase diagram as

\[
\frac{1}{2} \left( \frac{T_{c}^{SA}}{J} \right) = 1 - \frac{1}{2} \sqrt{\frac{1}{2J I} \coth \left( \frac{J}{T_{c}^{SA}} \right) \frac{1}{2J I}}
\] (12)

Finally, the critical value of $I (I_{c}^{SA})$ can be obtained analytically taking the limit $T_{c}^{SA} \rightarrow 0$ in Eq. (12), where we get $1/J I_{c}^{SA} = 8$.

B. Case $p \rightarrow \infty$

In this section, we approach the QS $p$-spin interaction model in the limit $p \rightarrow \infty$ over SA. In this case, the model is presented extremely simple in the analytical view point, it is found only in paramagnetic phase as in Sec. IIIA. Indeed, the first Eq. (11) always admits as solution $q = 0$, characterizing the paramagnetic phase. But, if we take the limit $p \rightarrow \infty$ it follows that $pq_o^{p-1} \rightarrow 0$ (if $0 \leq q < 1$) whence $q = 0$. Additionally, $pq_o^{p-1} \rightarrow \infty$ (if $q = 1$), thence $q \rightarrow \infty$ (not physical solution) or $q_o = q = 1$ that may not occur for a finite temperature. Therefore, $q = 0$ is the only admissible solution (paramagnetic solution). Now, we return in the second Eq. (10) with $q = 0$ and we obtain a similar Eq. (7) presented in Sec. IIIA

\[
\left( \frac{\beta J}{2} \right)^2 pq_o^{p-1} - 2\beta\mu q_o + 1 = 0.
\] (13)

Now, taking the limit $p \rightarrow \infty$, we have $pq_o^{p} \rightarrow 0$ (if $q_o < 1$) this leads us to $q_o = 0$. Therefore, with $q_o = 0$, in the third Eq. (10), we arrive

\[
\frac{1}{2} \sqrt{\frac{1}{2J I} \coth \left( \frac{\mu}{2J I} \right)} = 1,
\] (14)

This equation is exactly the same Eq. (5) obtained with the exact treatment.

V. STATIC APPROXIMATION X EXACT SOLUTION

In the two previous sections we indicate the exact and approximate (SA) solutions for the QS p-spin interaction model, respectively.

To compare the solutions obtained for $p = 2$, we find the numerical solution of the Eq. (3) and (12) – see Fig. 1

According to this phase diagram, we can see that the static approximation is a good approximation in the classical limit (small $1/JI$), however, it does not adequately describe our model when the quantum effect becomes relevant. Additionally, we can check that the critical values ($I_{c}$), both exact and approximate, agree well with
VI. CONCLUSIONS

In this work, we have compared the static approximation with exact solution for the quantum spherical $p$-spin interaction model in two particular cases: $p = 2$ and $p \to \infty$.

For $p = 2$, our results indicate that the correspondence between the static approximation and the exact solution depends directly on the value of the moment of inertia $I$. In other words, when $I \to \infty$ (classic limit) the static approximation is exact. However, for $I \to 0$ the exact solution differs the static approximation. This discrepancy is linked to the fact that to obtain the exact critical temperature of the system was necessary to accomplish a sum over all frequencies, while in the static approximation is considered single frequency $\omega_m = 0$. This limitation to static approximation in the quantum regime was also observed in another model (SK model in a transverse field).

For $p \to \infty$, we find that the static approximation is exact. This result is in agreement with the obtained to $p$-spin interaction model with ferromagnetic bias and transverse field.

Finally, our results suggest that static approximation is valid starting point to treat quantum spin glass models. However, it needs improvement to completely describe this class of models.

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