Identification of Semiparametric Panel Multinomial Choice Models with Infinite-Dimensional Fixed Effects

Wayne Yuan Gao† and Ming Li‡

November 20, 2024

Abstract

This paper proposes a robust method for semiparametric identification and estimation in panel multinomial choice models, where we allow for infinite-dimensional fixed effects that enter into consumer utilities in an additively nonseparable way, thus incorporating rich forms of unobserved heterogeneity. Our identification strategy exploits multivariate monotonicity in parametric indexes, and uses the logical contraposition of an intertemporal inequality on choice probabilities to obtain identifying restrictions. We provide a consistent estimation procedure, and demonstrate the practical advantages of our method with Monte Carlo simulations and an empirical illustration on popcorn sales with the Nielsen data.
1 Introduction

This paper proposes a method for semiparametric identification and estimation in panel multinomial choice models, where we allow for infinite-dimensional fixed effects that enter into consumer utilities in an additively nonseparable manner. The proposed method also applies much more widely beyond panel multinomial choice models, and can be easily adapted to a wide range of models characterized by \emph{multi-index single-crossing conditions}, which we introduce later in this paper.

To fix ideas, we start with the following panel multinomial choice model:

$$y_{ijt} = 1 \left\{ u \left( X_{ijt}' \beta_0, A_{ij}, \epsilon_{ijt} \right) \geq \max_{k \in \{1, \ldots, J\}} u \left( X_{ikt}' \beta_0, A_{ik}, \epsilon_{ikt} \right) \right\}$$

where agent $i$’s utility from a candidate product $j$ at time $t$, represented by $u \left( X_{ijt}' \beta_0, A_{ij}, \epsilon_{ijt} \right)$, is taken to be a function of three components. The first is a linear index $X_{ijt}' \beta_0$ of observable characteristics $X_{ijt}$, which contains a finite-dimensional parameter of interest $\beta_0$ we will identify and estimate. The second term $A_{ij}$ is an infinite-dimensional fixed effect that can be heterogeneous across each agent-product combination. We emphasize that $X_{ijt}$ and $A_{ij}$ can be arbitrarily dependent. The last term $\epsilon_{ijt}$ is an idiosyncratic time-varying error term of arbitrary dimensions. The three components are then aggregated by an unknown utility function $u$ in an additively nonseparable way, with the only restriction being that each agent’s utility $u \left( X_{ijt}' \beta_0, A_{ij}, \epsilon_{ijt} \right)$ is increasing in its first argument. Each agent then chooses a certain product in a given time period, represented by $y_{ijt} = 1$, if and only if this product gives him the highest utility among all available products.

The infinite-dimensionality of the terms $u$, $A_{ij}$ and $\epsilon_{ij}$ and the additive nonseparability in their interactions jointly produce rich forms of unobserved heterogeneity. Across each agent-product combination $ij$, we are effectively allowing for nonparametric variations in agent utilities as functions of the index $X_{ijt}' \beta_0$, which proxies for the effects of complicated unobserved factors that influence choice behavior, such as brand loyalty, subtle flavors, and unique styles of products. Moreover, unrestricted heterogeneity in the distribution of the
error term $\epsilon_{ijt}$ is accommodated.

The generality of our setup encompasses many semiparametric (or parametric) panel multinominal choice models with scalar-valued fixed effects, scalar-valued error terms and various degrees of additive separability in the related literature, including the following standard formulation:

$$y_{ijt} = \mathbb{1} \left\{ X'_{ijt} \beta_0 + A_{ij} + \epsilon_{ijt} \geq \max_{k \in \{1, \ldots, J\}} \left( X'_{ikt} \beta_0 + A_{ik} + \epsilon_{ikt} \right) \right\}. $$

Relatively speaking, in this paper we are able to accommodate the infinite dimensionality of unobserved heterogeneity and the lack of additive separability in agent utility functions, under a standard time homogeneity assumption on the idiosyncratic error term that is widely adopted in the related literature.

Our key identification strategy exploits the standard notion of multivariate monotonicity in its contrapositive form. The idea is very simple and intuitive: whenever we observe a strict increase in the choice probabilities of a specific product from one period to another, by logical contraposition it cannot be possible that this product becomes worse while all other products become better over the two periods. Hence, by taking the logical contraposition of a carefully constructed inequality on conditional choice probabilities, we can obtain an identifying restriction on the index values free of all infinite-dimensional nuisance parameters, with which we construct a population criterion function.

Based on our identification result, we provide consistent two-step set (or point) estimators, together with a computational algorithm adapted to the technical challenges of our framework. The first stage takes the form of a standard nonparametric regression, where we estimate a collection of intertemporal differences in conditional choice probabilities. In the second stage, we numerically minimize our sample criterion function with the first-stage estimates plugged in. A highlight of our computation procedure is the adoption of a spherical-coordinate reparameterization of our criterion functions in terms of angles, which enables us to exploit a combination of topological, geometric and computational advantages. A simulation study is conducted to analyze the finite-sample performance of our method.
and the adequacy of our computational procedure for practical implementation.

We also provide an empirical illustration of our procedure, where we use the Nielsen data on popcorn sales in the United States to explore the effects of marketing promotion effects. The results show that our procedure produces estimates that conform well with economic intuition. For example, we find that special in-store displays boost sales not only through a direct promotion effect but also through the attenuation of consumer price sensitivity, a result that cannot be produced by other methods based on additive separability. Intuitively, marketing managers are more likely to promote products that they know consumers are more price and promotion sensitive to. Hence, the average effective price sensitivity of promoted products tend to be larger than those not promoted due to the selection effect. Given the nonadditive nature of such selection effects, estimators based on additive separability will be biased. In contrast, our method is robust to such confounding effects, thus producing more sensible estimates.

The validity of our key identification strategy (along with our estimation procedure) relies only on monotonicity in an index structure, and therefore it has wider applicability beyond panel multinomial choice models. We also introduce the multi-index single-crossing (MISC) condition framework, a general econometric framework under which our method can be applied. This framework encompasses the key ingredients of a large class of models, such as binary choice models with awareness, binary choice with endogeneity, dyadic network formation, bilateral matching, and endogenous censoring.

This paper builds upon and contributes to a large literature on semiparametric (and parametric) discrete choice models, dating back to McFadden (1974) and Manski (1975), and more specifically a recent branch of research that focuses on panel multinomial choice models. Our work is most closely related to the work by Pakes and Porter (2024), who also exploit weak monotonicity and time homogeneity, but restrict the effect of unobserved heterogeneity to be a scalar index that is additively separable from the index of observable characteristics. Another related paper is Shi, Shum, and Song (2018), who exploit cyclical
monotonicity of vector-valued functions in a fully additive panel multinomial choice model. Khan, Ouyang, and Tamer (2021) consider another additive model, but utilize the subsample of observations with time-invariant covariates along all products but one so as to leverage univariate monotonicity. Relatedly, the earlier work by Honoré and Kyriazidou (2000) also exploits univariate monotonicity when certain covariates across two periods are equal in a dynamic panel setting. Another recent paper by Chernozhukov, Fernández-Val, and Newey (2019) consider a model with an additive effect under an “on-the-diagonal” restriction (i.e., when covariates at two different time periods coincide). Our method is significantly different from and thus complementary to those proposed in these afore-cited papers.

At a more general level, our work can be related to and compared to semiparametric methods of identification and estimation in models that feature monotonicity in a single parametric index. A related class of estimators that leverage univariate monotonicity, known as maximum score or rank-order estimators, date back to a series of important contributions by Manski (1975, 1985, 1987), and are further investigated in Han (1987), Horowitz (1992), Abrevaya (2000), Honoré and Lewbel (2002) and Fox (2007). Despite the similarity in the reliance on monotonicity, the multinomial or multi-index nature of our current model induces a key difference from the single-index setting, leading to a significantly different method of estimation relative to rank-order estimators.

The rest of this paper is organized as follows. Section 2 introduces our main model specifications and assumptions. Section 2.2 presents our key identification strategy. In Section 3 we provide consistent estimators along with a computational procedure to implement it. Section 4 discusses the generalization of our method to the framework of multi-index single-crossing conditions. Section 5 and Section 6 switch back to our main panel multinomial model, for which we provide a simulation study and an empirical illustration using the Nielsen data.
2 Panel Multinomial Choice Model

2.1 Model and Assumptions

In this section we present a semiparametric panel multinomial choice model featured by infinite-dimensional unobserved heterogeneity and flexible forms of nonseparability, which we will use as the main model to illustrate our identification and estimation method.

Specifically, we consider the following model, which states that individual $i$ chooses product $j$ at time $t$ if and only if $i$ prefers product $j$ to all other alternatives at time $t$:

$$y_{ijt} = 1 \left\{ u \left( X'_{ijt} \beta_0, A_{ij}, \epsilon_{ijt} \right) > \max_{k \in \{1,\ldots,J\} \setminus \{j\}} u \left( X'_{ikt} \beta_0, A_{ik}, \epsilon_{ikt} \right) \right\}$$  \hspace{1cm} (1)

where:

- $i \in \{1, \ldots, N\}$ denotes $N$ individuals.
- $j \in \mathcal{J} := \{1, \ldots, J\}$ denotes $J$ choice alternatives, with $J$ products indexed by $1, \ldots, J$. In this paper, we take the number of products $J$ to be fixed.
- $t \in \{1, \ldots, T\}$ denotes $T \geq 2$ different time periods. In this paper, we consider a short-panel setting where $T$ is fixed.
- $X_{ijt}$ is $\mathbb{R}^d$-valued vector of observable characteristics specific to each agent-product-time tuple $ijt$. This could include, for example, buyer characteristics such as income level, product characteristics such as price and promotion status, as well as interaction and higher-order terms of those characteristics.
- $y_{ijt}$ is an observable binary variable, with $y_{ijt} = 1$ indicating that buyer $i$ chooses products $j$ at time $t$ and $y_{ijt} = 0$ indicating otherwise.
- $\beta_0 \in \mathbb{R}^d$ is a finite-dimensional unknown parameter of interest.
- $A_{ij}$ represents an $ij$-specific time-invariant unobserved heterogeneity term of arbitrary dimensions, which we will refer to as the ($ij$-specific) fixed effect.
• $\epsilon_{ijt}$ is an $ijt$-specific unobserved error term of arbitrary dimensions, which captures time-idiosyncratic utility shocks to product $j$ for agent $i$ at time $t$.

• $u$ is an unknown function, interpreted as a *utility function* that aggregates the parametric index $X'_{ijt}\beta_0$, the fixed effect $A_{ij}$ and the error term $\epsilon_{ijt}$ into a scalar representing agent $i$’s utility from choosing product $j$ at time $t$.

Now, we first present our main modeling assumptions, and will follow up with a discussion of these assumptions along with our model specification (1).

To economize on notation, we will from now on frequently refer to the collection of variables concatenated along product and time dimensions: $X_{it} := (X_{ijt})_{j=1}^J$, $X_i = (X_{it})_{t=1}^T$, $A_i := (A_{ij})_{j=1}^J$, $\epsilon_{it} = (\epsilon_{ijt})_{j=1}^J$ and $\epsilon_i = (\epsilon_{it})_{t=1}^T$. Recall that we defined $\delta_{ijt} := X'_{ijt}\beta_0$. The first assumption below imposes a monotonicity restriction on the utility function.

**Assumption 1** (Cross-Sectional Random Sampling). $(Y_i, X_i, A_i, \epsilon_i)$ is i.i.d. across $i \in \{1, ..., N\}$ with $N \to \infty$.

Assumption 1 is a standard assumption. Recall that the number of time periods $T$ is held fixed at a finite number, and we focus on a short panel setting with cross-sectional asymptotics.

**Assumption 2** (Monotonicity in the Index). For every realization of $(A_{ij}, \epsilon_{ijt})$, the real-valued mapping defined by $\tilde{\delta} \mapsto u(\tilde{\delta}, A_{ij}, \epsilon_{ijt})$ is weakly increasing in the scalar-valued argument $\tilde{\delta}$.

Essentially, Assumption 2 states that $u(X'_i\beta_0, A_{ij}, \epsilon_{ijt})$, individual $i$’s utility of choosing product $j$ at time $t$, is increasing in the parametric index $X'_{ijt}\beta_0$. Given the parametric index structure $X'_{ijt}\beta_0$, monotonicity itself is a very weak assumption. In the standard

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1It is worth noting that so far we have not made any explicit restriction on the structure of the spaces on which the arbitrary dimensional random elements $A_i$ and $\epsilon_i$ are defined, but implicit in our specification as well as Assumption 1 is the requirement that $(Y_i, X_i, A_i, \epsilon_i)$ be well-defined as random elements (measurable functions) on a sufficiently rich probability space $(\Omega, \mathcal{F}, P)$.

2It should be clarified that increasingness is without loss of generality given monotonicity, since the index $\delta_{ijt} = X'_{ijt}\beta_0$ contains an unknown parameter with unrestricted signs.
panel multinomial choice model with scalar-valued $A_{ij}$ and $\epsilon_{ij}$ along with additive $u$, i.e.,

$$u \left( X'_{ijt} \beta_0, A_{ij}, \epsilon_{ijt} \right) = X'_{ijt} \beta_0 + A_{ij} + \epsilon_{ijt},$$

Assumption 2 is trivially satisfied (with strictness) by construction.

In a way, Assumption 2 endows the parametric index $X'_{ijt} \beta_0$ and the parameter $\beta_0$ with economic interpretations. Under Assumption 2, $X'_{ijt} \beta_0$ may be considered as a quality measure of the match between agent $i$ and product $j$ based on their observable characteristics at time $t$, inducing a consequent interpretation of the parameter $\beta_0$ as representing how a certain change in a linear combination of observable characteristics may increase utilities for all agents from a certain product $j$, *ceteris paribus*. Hence, without Assumption 2, it would be hard to interpret $\beta_0$.

Moreover, the monotonicity restriction is imposed on the index $X'_{ijt} \beta_0$, but not directly on any specific observable characteristics in $X_{ijt}$: quadratic or higher-order polynomial terms as well as other nonlinear or non-monotone functions of observable characteristics may be included in $X_{ijt}$ whenever appropriate.

**Assumption 3** (Pairwise Time Homogeneity). The distributions of $\epsilon_{it}$ and $\epsilon_{is}$ conditional on $(X_{is}, X_{it}, A_i)$ are the same across any pair of periods $s \neq t \in \{1, \ldots, T\}$, i.e.,

$$\epsilon_{is} \mid (X_{is}, X_{it}, A_i) \sim \epsilon_{it} \mid (X_{is}, X_{it}, A_i).$$

Assumption 3, a multinomial extension of the group homogeneity assumption in Manski (1987), is also a standard assumption in the literature on panel multinomial choice models, such as in Shi, Shum, and Song (2018), Pakes and Porter (2024).\(^3\) Note that Assumption 3

\(^3\)In particular, Pakes and Porter (2024) investigate the following panel multinomial choice model:

$$y_{ijt} = 1 \left\{ g_j (X_{ijt}, \beta_0) + f_j (A_{ij}, \epsilon_{ijt}) > \max_{k \neq j} g_k (X_{ikt}, \beta_0) + f_k (A_{ik}, \epsilon_{ikt}) \right\}, \quad (2)$$

where the function $g_j$ produces a potentially nonlinear parametric index and $f_j$ aggregates fixed effects and idiosyncratic errors into a scalar value in a nonseparable way, while additive separability between the observable covariate index $g_j (X_{ijt}, \beta_0)$ and the unobserved heterogeneity index $f_j (A_{ij}, \epsilon_{ijt})$ is still maintained. Moreover, although the dimensions of $(A_{ij}, \epsilon_{ijt})$ are not restricted in Pakes and Porter (2024), their overall effect is taken to be represented by a scalar value, $f_j (A_{ij}, \epsilon_{ijt})$. We reiterate that our model (1) not only incorporates infinite-dimensionality in unobserved heterogeneity as captured by $A_{ij}$ and $\epsilon_{ijt}$, but also allows such heterogeneity to enter into agent utility functions in a fully nonseparable way.
concerns only the marginal distributions of $\epsilon_{it}$ in different periods, and we make no explicit assumptions on the serial dependence between $\epsilon_{is}$ and $\epsilon_{it}$.

As shown in the next subsection, Assumption 3 is the key condition underlying our identification strategy. Essentially, we rely on Assumption 3 to relate intertemporal changes in (a certain type of) conditional choice probabilities across two periods $s$ and $t$ to intertemporal changes in the parametric indexes of all products \( \left( X'_{ijss\beta_0} - X'_{ijst\beta_0} \right)_{j \in J} \), which produce identifying restrictions on the parameter $\beta_0$.

### 2.2 Key Identification Strategy

In this section, we present semiparametric identification results for model (2) under Assumptions 2-3. However, as will become clear later in this section, the underlying idea of our identification strategy applies more widely beyond panel multinomial choice models. See Section 4 for more details.

Our key identification strategy exploits the standard notion of multivariate monotonicity in its contrapositive form. As a reminder, we start with a standard definition of multivariate monotonicity, followed by a statement of its logical contraposition.

**Definition 1 (Multivariate Monotonicity).** A real-valued function $\psi : \mathbb{R}^J \rightarrow \mathbb{R}$ is said to be **weakly increasing** if, for any pair of vectors $\vec{\delta}$ and $\hat{\delta}$ in $\mathbb{R}^J$, if $\vec{\delta}_j \leq \hat{\delta}_j$ for every $j = 1, ..., J$, then $\psi(\vec{\delta}) \leq \psi(\hat{\delta})$.

**Remark 1 (Logical Contraposition).** The following is equivalent to Definition 1:

\[
\{ \psi(\vec{\delta}) > \psi(\hat{\delta}) \} \Rightarrow \text{NOT } \{ \vec{\delta}_j \leq \hat{\delta}_j \text{ for all } j = 1, ..., J \}.
\]

(3)

for any $\left( \vec{\delta}, \hat{\delta} \right)$, where “NOT” denotes the logical negation operator.

Our subsequent identification strategy will leverage heavily the simple contraposition of monotonicity (3), and our arguments proceed in three major steps. First, we define a multivariate monotone function in the form of conditional choice probabilities. Second, we construct an observable inequality based on the monotone function we define, effectively
producing the left-hand side of (3). Finally, we use the contraposition of monotonicity to obtain the right-hand side of (3), which will translate into identifying restrictions on the parameter $\beta_0$ via the indexes $\delta_{it} := (\delta_{ijt})_{j=1}^{J}$.

We now present our key identification strategy step by step. For the moment, we fix a particular product $j \in \{1, ..., J\}$, a pair of time periods $t \neq s \in \{1, ..., T\}$ and condition on a generic realization of the observable covariates in the two periods $t$ and $s$, i.e., $(X_{it}, X_{is}) = (\mathbf{X}, \mathbf{X}) \in \text{Supp}(X_{it}, X_{is})$.

**Step 1: Construction of a monotone function**

For each individual $i$, consider $i$’s choice probability of $j$ given $(X_{it}, A_i)$:

$$
\mathbb{E}[y_{ijt}|X_{it}, A_i] = \int \mathbb{1}\left\{u\left(X_{ijt}'\beta_0, A_{ij}, \epsilon_{ijt}\right) \geq \max_{k \neq j} u\left(X_{ikt}'\beta_0, A_{ik}, \epsilon_{ikt}\right)\right\} d\mathbb{P}(\epsilon_{it}|X_{it}, A_i)
$$

$$
= \int \mathbb{1}\left\{u(\delta_{ijt}, A_{ij}, \epsilon_{ijt}) \geq \max_{k \neq j} u(\delta_{ikt}, A_{ik}, \epsilon_{ikt})\right\} d\mathbb{P}(\epsilon_{it}|A_i)
$$

$$
=: \psi_j(\delta_{ijt}, (-\delta_{ikt})_{k \neq j}, A_i)
$$

(4)

where the second equality follows from the index definition $\delta_{ijt} = X_{ijt}'\beta_0$ and Assumption 3 (Conditional Time Homogeneity of Errors), which enables us to write $\psi_j$ without the time subscript $t$. Clearly, the monotonicity of the utility function $u$ in the index argument $\delta_{ijt}$ (Assumption 2) translates into the multivariate monotonicity of the function $\psi_j$ in the vector of indexes $(\delta_{ijt}, (-\delta_{ikt})_{k \neq j})^4$.

**Lemma 1.** $\psi_j(\cdot, A_i) : \mathbb{R}^J \rightarrow \mathbb{R}$ is weakly increasing, for any realized $A_i$.

In terms of economic interpretation, $\psi_j(\delta_{it}, A_i)$ summarizes each agent $i$’s conditional choice probability of product $j$ given $i$’s fixed effect $A_i$ as a function of the index vector $\delta_{it}$. Lemma 1 admits a simple interpretation: if a product $j$ becomes weakly better for agent $i$ (in terms of the index $\delta_{ijt}$), while all other products $k \neq j$ becomes weakly worse, then agent $i$’s choice probability of product $j$ should weakly increase.

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4We flip the signs of $(\delta_{ikt})_{k \neq j}$ purely for the ease of exposition: as discussed earlier, it is the monotonicity, not the exact direction of monotonicity, that matters in our analysis.
However, as $A_i$ is not observable, the conditional choice probability function $\psi_j (\cdot, A_i)$ is not directly identified from data in the short-panel setting under consideration here. In the next step, we construct an observable quantity based on $\psi_j$ by averaging out $A_i$.

**Step 2: Construction of an observable inequality**

Consider the following intertemporal difference in conditional choice probabilities:

$$\gamma_{j,t,s} (X, \bar{X}) := \mathbb{E} \left[ y_{ijt} - y_{ijs} \mid X_{it} = X, X_{is} = \bar{X} \right] \quad (5)$$

which is by construction directly identified from data.

Write $\delta := X \beta_0 \equiv \left( X_j \beta_0 \right)_{j=1}^J$ and similarly for $\delta_i$, and $X_{i,ts} := (X_{it}, X_{is})$. The following lemma translates the monotonicity of $\psi_j (\delta, A_i)$ into a restriction on the sign of the observable quantity $\gamma_{j,t,s} (X, \bar{X})$, effectively corresponding to an observable scalar inequality.

**Lemma 2.** $\{ \delta_j \leq \bar{\delta}_j \text{ and } \delta_k \geq \bar{\delta}_k \text{ for all } k \neq j \} \implies \{ \gamma_{j,t,s} (X, \bar{X}) \leq 0 \}$.

To see why Lemma 2 is true, rewrite $\gamma_{j,t,s} (X, \bar{X})$ as

$$\gamma_{j,t,s} (X, \bar{X}) = \mathbb{E} \left[ \mathbb{E} \left[ y_{ijt} - y_{ijs} \mid X_{i,ts} = (X, \bar{X}), A_i \right] \mid X_{i,ts} = (X, \bar{X}) \right]$$

$$= \mathbb{E} \left[ \mathbb{E} \left[ y_{ijt} \mid X_{it} = X, A_i \right] - \mathbb{E} \left[ y_{ijs} \mid X_{is} = X, A_i \right] \mid X_{i,ts} = (X, \bar{X}) \right]$$

$$= \int \left[ \psi_j \left( \delta_j, \left( -\delta_k \right)_{k \neq j}, A_i \right) - \psi_j \left( \bar{\delta}_j, \left( -\bar{\delta}_k \right)_{k \neq j}, A_i \right) \right] dP \left( A_i \mid X_{i,ts} = (X, \bar{X}) \right).$$

Whenever $\delta_j \leq \bar{\delta}_j$ and $\delta_k \geq \bar{\delta}_k$ for all $k \neq j$, by Lemma 1 we have

$$\psi_j \left( \delta_j, \left( -\delta_k \right)_{k \neq j}, A_i \right) - \psi_j \left( \bar{\delta}_j, \left( -\bar{\delta}_k \right)_{k \neq j}, A_i \right) \leq 0$$

for every possible realization of $A_i$. Consequently, the inequality will be preserved after integrating over $A_i$ cross-sectionally with respect to $P \left( A_i \mid X_{it} = X, X_{is} = \bar{X} \right)$, a potentially hugely complicated probability measure that we leave unspecified.

**Step 3: Derivation of the key identifying restriction**

We now take the logical contraposition of Lemma 2:
Proposition 1 (Key Identifying Restriction). Under Assumptions 2–3,
\[
\{\gamma_{j,t,s}(\mathbf{X}, \mathbf{X}) > 0\} \Rightarrow \text{NOT} \left\{\left(\mathbf{X}_j - \mathbf{X}_j\right)' \beta_0 \leq 0 \text{ and } \left(\mathbf{X}_k - \mathbf{X}_k\right)' \beta_0 \geq 0 \forall k \neq j\right\}
\] (6)
Recall that \(\delta_{ijt} = X'_{ijt} \beta_0\), so Proposition 1 follows immediately from Lemma 2 and defines an identifying restriction on \(\beta_0\) that is free of all unknown nonparametric heterogeneity terms \(u, A\) and \(\epsilon\). Proposition 1 is also very intuitive: if we observe an intertemporal increase in the conditional choice probability of product \(j\) from one period to another, it is impossible that product \(j\)’s index becomes worse, while all other products’ indexes become better.

The simple idea behind Proposition 1 is to leverage the contraposition of monotonicity in the index vector, which, apart from its simplicity, brings about robustness against the rich built-in forms of unobserved heterogeneity along with nonseparability. As the validity of this idea relies only on monotonicity in an index structure, it is applicable more widely beyond the panel multinomial choice settings we are currently considering. See Section 4 for a general framework under which the contraposition of monotonicity may be utilized.

Formulation of Population Criterion Functions

We now formulate a population criterion function based on Proposition 1. For every candidate parameter \(\beta \in \mathbb{R}^D\), we represent in Boolean algebra the right hand side of (6) in Proposition 1 by
\[
\lambda_j(\mathbf{X}, \mathbf{X}; \beta) := \prod_{k=1}^{J} \mathbb{1}_{\left\{(-1)^{1(k \neq j)} \left(\mathbf{X}_k - \mathbf{X}_k\right)' \beta \leq 0\right\}},
\] (7)
where \((-1)^{1(k \neq j)}\) takes the value \(-1\) for \(k \neq j\) and \(1\) for \(k = j\). Therefore, Proposition 1 can be written algebraically as: \(\gamma_{j,t,s}(\mathbf{X}, \mathbf{X}) > 0\) implies \(\lambda_j(\mathbf{X}, \mathbf{X}; \beta_0) \equiv 0\) for any \((\mathbf{X}, \mathbf{X})\). We now define the following criterion function by taking a cross-sectional expectation over the random realization of \((\mathbf{X}_{it}, \mathbf{X}_{is})\):
\[
Q_{j,t,s}(\beta) := \mathbb{E}\left[\mathbb{1}_{\{\gamma_{j,t,s}(\mathbf{X}_{it}, \mathbf{X}_{is}) > 0\}} \lambda_j(\mathbf{X}_{it}, \mathbf{X}_{is}; \beta)\right],
\] (8)
which is clearly nonnegative and minimized to zero at the true parameter value \(\beta_0\). Without normalization and further assumptions for point identification, there might be multiple values...
of $\beta_0$ that minimize $Q_{j,t,s}$ to zero.

More generally, fix any function $G : \mathbb{R} \to \mathbb{R}$ that is one-sided sign preserving, i.e., $G(z) > 0$ for $z > 0$ and $G(z) = 0$ for $z \leq 0$. For example, we can choose $G(z) = [z]_+$ where $[z]_+$ is the positive part function. Then, we define $Q^G_{j,t,s}$ as

$$Q^G_{j,t,s}(\beta) := \mathbb{E}[G(\gamma_{j,t,s}(X_{it},X_{is})) \lambda_j(X_{it},X_{is};\beta)],$$

which is also minimized to zero at the true parameter value $\beta_0$. The sign-preserving function $G$, if also set to be monotone, continuous or bounded, serves as a smoothing function that helps with the finite-performance of our estimators. We will provide more discussions on function $G$ in the next section, when we construct estimators based on the sample analog of the population criterion function defined here. It is worth pointing out that this smoothing function $G$ is built into the population criterion function as in (9), which is different from the usual technique where smoothing is only done in finite samples but not in the population. For notational simplicity, we suppress $G$ in $Q^G_{j,t,s}$ throughout this paper.

So far we have focused on a fixed product $j$ and a fixed pair of periods $(t,s)$, but in practice we may utilize the information across all products and all pairs of periods by defining the aggregated criterion function:

$$Q(\beta) := \sum_{j=1}^{J} \sum_{t \neq s}^{T} Q_{j,t,s}(\beta), \quad \text{for any } \beta \in \mathbb{R}^D.\quad (10)$$

We summarize our main identification result in the following theorem.

**Theorem 1** (Set Identification). Under model (1) and Assumptions 2–3,

$$\beta_0 \in B_0 := \{\beta \in \mathbb{R}^D : Q(\beta) = 0\}.\quad (11)$$

We will refer to $B_0$ as the identified set. In Appendix C, we provide sufficient conditions for point identification of $\beta_0$ up to scale normalization. However, since point identification, or lack thereof, is conceptually irrelevant to our key methodology, and as set identification and set estimation are becoming increasingly relevant in econometric theory as well as applied research, we will focus on set identification and estimation results in the main text. Of course, whenever the additional assumptions for point identification are satisfied in data,
the set estimator will shrink to a point asymptotically.

Relative to the well-known maximum-score criterion function as studied by Manski (1985, 1987) utilizing univariate monotonicity, the nonstandardness of our criterion function arises from a key difference of multivariate monotonicity from univariate monotonicity. To see this more clearly, consider the special case of a single-index setting \((J = 1)\), in which case the following equivalence relationship holds given the univariate monotonicity in the index:

\[
\{ \gamma(X_t, X_s) > 0 \} \Leftrightarrow \{ (X_t - X_s) \beta > 0 \},
\]

(12)

Such an “if-and-only-if” relationship is a unique feature of the single-index setting, which cannot be generalized to the multi-index setting with \(J \geq 2\), as the right hand side of (6),

\[
\text{NOT } \left\{ (X_j - X_j) \beta_0 \leq 0 \text{ and } (X_k - X_k) \beta_0 \geq 0 \text{ for all } k \neq j \right\},
\]

does not imply \(\gamma_{j,t,s}(X, X) \geq 0\) in the converse direction. This breaks the “if-and-only-if” relationship that the maximum-score criterion function in Manski (1985, 1987) is built upon, and thus the maximum-score estimator does not generalize to multi-index settings. The lack of “if-and-only-if” relationship in the multi-index setting equivalence leads to a key difference in the criterion functions, and consequently a different approach of estimation. Importantly, while the original maximum score criterion and estimator cannot be generalized to multi-index settings, our proposed procedure can be easily applied under a general econometric framework characterized by multi-index single-crossing conditions, which we introduce in Section 4.

\(^{5}\text{This arises naturally in binomial choice models with the characteristics of the outside option set to be zero. In this case, even though there are nominally two choice alternatives, choice behavior is completely determined by a single index based on the characteristics of the non-default option.}\)
3 Estimation and Computation

3.1 Two-Step Semiparametric Estimation

We construct our estimator as a semiparametric two-step M-estimator based on (10). The first stage of our procedure concerns with nonparametrically estimating the intertemporal differences in conditional choice probabilities of the following form

$$\gamma_{j,t,s}(\mathbf{X}, \mathbf{X}) = \mathbb{E} \left[ y_{ijt} - y_{ijs} | \mathbf{X}_{i,ts} = (\mathbf{X}, \mathbf{X}) \right]$$

for all on-support realizations \((\mathbf{X}, \mathbf{X})\), all pairs of periods \((t, s)\) and all products \(j\).\(^6\) We note that the first stage estimation includes the observable characteristics of all products \(J\). For example, when \(J = 3\) and \(D = 3\), there are \(3 \times 3 \times 2 = 18\) variables in the conditioned set of \(\gamma\). Given the potentially large number of regressors, one may want to use neural networks (Bach, 2017; Chen and White, 1999) or penalized sieves (Chen, 2013) for the first step estimation.

Given the first-stage estimators \(\hat{\gamma}_{j,t,s}\) and the smoothing function \(G\), in the second stage we numerically compute minimizers of the sample criterion function,

$$\hat{Q}(\beta) := \sum_{j=1}^{J} \sum_{t \neq s} \hat{Q}_{j,t,s}(\beta), \quad \hat{Q}_{j,t,s}(\beta) := \frac{1}{N} \sum_{i=1}^{N} G(\hat{\gamma}_{j,t,s}(\mathbf{X}_{i,ts})) \lambda_j(\mathbf{X}_{i,ts}; \beta).$$

Observing that the scale of \(\beta_0\) cannot be identified given that \(\lambda_j(\mathbf{X}_{i,ts}; \beta)\) consists of indicator functions of the form \(1 \{ (X_{ijt} - X_{ijs})' \beta \geq 0 \}\), we impose the following scale normalization \(\beta_0 \in \mathbb{S}^{D-1} := \{ v \in \mathbb{R}^D : \|v\| = 1 \}\). Following Chernozhukov, Hong, and Tamer (2007), we define the set estimator by

$$\hat{B}_\epsilon := \left\{ \beta \in \mathbb{S}^{D-1} : \hat{Q}(\beta) \leq \min_{\beta' \in \mathbb{S}^{D-1}} \hat{Q}(\beta') + \hat{c} \right\}$$

with \(\hat{c} := O_p(c_N \log N)\). We now introduce assumptions for the consistency of \(\hat{B}_\epsilon\).

Assumption 4 (First-Stage Estimation). For any \((j, t, s)\):

\(^6\)In practice, we only need to estimate \(\gamma_{j,t,s}\) for \((J - 1)\) products and \(\frac{1}{2}T(T - 1)\) ordered pairs of periods. The former is because conditional choice probabilities must sum to one across all \(J\) products, so we may easily compute the estimator for the last product from the other \((J - 1)\) estimates: \(\gamma_{t,s} = 1 - \sum_{j=1}^{J-1} \gamma_{j,t,s}\). The latter is because \(\gamma_{j,t,s} = -\gamma_{j,s,t}\) by construction, so we may estimate it for either \((t, s)\) or \((s, t)\). Notice, however, that each ordered pair \((t, s)\) or \((s, t)\) provides complementary identifying information, as \(\lambda(\mathbf{X}_{i,ts}; \beta)\) and \(\lambda(\mathbf{X}_{i,st}; \beta)\) do not admit such kind of deterministic relationship.
(i) \( \gamma_{j,t,s} \in \Gamma \), and \( \mathbb{P}(\hat{\gamma}_{j,t,s} \in \Gamma) \to 1 \), with \( \Gamma \) being a \( \mathbb{P} \)-Donsker class of functions in \( L_2(X) \) such that \( \sup_{\gamma_{j,t,s} \in \Gamma} \mathbb{E} |\gamma_{j,t,s}| < \infty \);

(ii) \( \|\hat{\gamma}_{j,t,s} - \gamma_{j,t,s}\|_2 := \sqrt{\int (\hat{\gamma}_{j,t,s}(X_{i,ts}) - \gamma_{j,t,s}(X_{i,ts}))^2 d\mathbb{P}(X_{i,ts})} = O_p(c_N) \) with \( c_N \searrow 0 \).

Through Assumption 4 we take as given the large set of theoretical results on nonparametric regression in the literature. Many kernel-based and sieve-based methods have been developed with different properties demonstrated under various sets of conditions. See Wasserman (2006) and Chen (2007) for more comprehensive surveys.

**Assumption 5** (Nice Smoothing Function). The one-sided sign-preserving function \( G : \mathbb{R} \to \mathbb{R}_+ \) is Lipschitz continuous with a finite Lipschitz constant.

Assumption 5 is not necessary for consistency per se given that our identification result is valid with any choice of the one-sided sign-preserving function \( G \), nevertheless we take \( G \) to be Lipschitz so as to simplify the proof.

To state the next assumption, we decompose each row (corresponding to each product) of \( X - \bar{X} \) as the product of its norm and its direction, i.e., \( X_{j,t} - \bar{X}_{j,t} \equiv r_j((X - \bar{X}) \cdot v_j(X - \bar{X})) \), where \( r_j(X - \bar{X}) := \|X_{j,t} - \bar{X}_{j,t}\| \), and \( v_j(X - \bar{X}) := (X_{j,t} - \bar{X}_{j,t}) / \|X_{j,t} - \bar{X}_{j,t}\| \) if \( X_{j,t} \neq \bar{X}_{j,t} \) while \( v_j(X - \bar{X}) := 0 \) if \( X_{j,t} = \bar{X}_{j,t} \).

**Assumption 6** (Continuous Distribution of Directions). The marginal distribution of \( v_j(X_{it} - X_{is}) \) has no mass point except possibly at \( 0 \) for each \( (j, t, s) \).

Assumption 6 is a technical assumption that ensures the continuity of the population criterion function. We note that Assumption 6 is fairly weak: it essentially requires that the directions of intertemporal differences in observable characteristics are continuously distributed on their own supports. In particular, this allows all but one dimensions of observable characteristics to be discrete.

With the above assumptions, we now establish the consistency of the set estimator \( \hat{B}_\delta \) based on Chernozhukov, Hong, and Tamer (2007).
Theorem 2 (Consistency). Under Assumptions 2–6, the set estimator \( \hat{B}_c \) is consistent in Hausdorff distance: 
\[
d_H (\hat{B}_c, B_0) = o_p(1),
\]
where 
\[
d_H (\hat{B}_c, B_0) := \max \{ \sup_{\beta \in \hat{B}_c} \inf_{\hat{\beta} \in B_0} \| \beta - \hat{\beta} \|, \sup_{\beta \in B_0} \inf_{\beta \in \hat{B}_c} \| \beta - \beta \| \}. 
\]
Furthermore, if \( \beta_0 \) is point-identified on \( S^{D-1} \), 
\[
\| \hat{\beta} - \beta_0 \| = o_p(1) \text{ for any } \hat{\beta} \in \hat{B} := \arg \min_{\hat{\beta} \in S^{D-1}} \hat{Q} (\hat{\beta}).
\]

3.2 Computation

Choice of the Smoothing Function \( G \)

Besides the requirement of Lipschitz continuity in Assumption 5, in practice we take \( G \) to be bounded from above by setting 
\[
G(z) = 2\Phi \left( \left| z \right| \right) - 1,
\]
where \( \Phi \) is the standard normal CDF. We now motivate our choice of \( G \).

Recall that our identification strategy is based on the logical implication of the event 
\[
\gamma_{j,t,s}(X, \bar{X}) > 0,
\]
so for identification purposes we are only interested in 
\[
1 \{ \gamma_{j,t,s}(X, \bar{X}) > 0 \},
\]
i.e., whether the event \( \gamma_{j,t,s}(X, \bar{X}) > 0 \) occurs, but not in the exact magnitude of 
\( \gamma_{j,t,s}(X, \bar{X}) \). However, when \( \gamma_{j,t,s}(X, \bar{X}) \) is close to zero, the estimator \( \hat{\gamma}_{j,t,s}(X, \bar{X}) \) is relatively more likely to have the wrong sign, so that the plug-in estimator 
\[
1 \{ \hat{\gamma}_{j,t,s}(X, \bar{X}) > 0 \}
\]
may induce an error of the size 1. Hence the smoothing by \( G(\cdot) \) helps down-weight the observations when \( \hat{\gamma}_{j,t,s}(X, \bar{X}) \) is close to zero and shrinks the magnitude of possible errors.

On the other hand, when \( \gamma_{j,t,s}(X, \bar{X}) \) is positive and large so that 
\[
1 \{ \gamma_{j,t,s}(X, \bar{X}) > 0 \}
\]
can be estimated well, we do not care much about the magnitude of \( \gamma_{j,t,s}(X, \bar{X}) \), which does not provide additional identifying information. By setting \( G \) to be bounded from above, we dampen the effects of large \( \gamma_{j,t,s}(X, \bar{X}) \) at the same time, so that the numerical maximization of \( \hat{Q} \) is not too sensitive to potential large but redundant variations in \( \hat{\gamma}_{j,t,s}(X, \bar{X}) \).
Angle-Space Reparameterization of $S^{D-1}$

In the optimization of $\hat{Q}(\beta)$ over $\beta \in S^{D-1}$, we work with a reparameterization of $S^{D-1}$ with $(D - 1)$ angles in spherical coordinates\(^7\). Specifically, define the angle space $\Theta$ by

$$\Theta := [-\pi, \pi) \times \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]^{D-2},$$

and the transformation $\theta \mapsto \beta(\theta)$ by standard spherical coordinate transformation, we now instead solves the optimization of $\hat{Q}(\beta(\theta))$ over $\Theta$, which we further equip with its natural geodesic metric $\rho_{\Theta}(\theta, \bar{\theta}) := \arccos(\beta(\theta) \beta(\bar{\theta}))$, which is strongly equivalent\(^8\) to the (imported) Euclidean distance $\|\beta(\theta) - \beta(\bar{\theta})\|$.

This reparameterization $(\Theta, \rho_{\Theta})$ enables us to exploit the compactness and convexity of the parameter space $\Theta = [-\pi, \pi) \times \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]^{D-2}$, which takes the form of a hyper-rectangle. First, $(\Theta, \rho_{\Theta})$ preserves all topological structure of the unit sphere, and particularly inherits the compactness of $(S^{D-1}, \|\cdot\|)$, automatically satisfying the compactness condition usually imposed for extremum estimation and making it numerically feasible to initiate a grid on the whole parameter space. Second, while the unit sphere $S^{D-1}$ is not convex, the new parameter space $\Theta$ becomes convex algebraically, making it computationally easy to define bisection points in the parameter space. Third, it also preserves the geometric structures of the sphere, including for instance the obvious observation that $-\pi$ and $\pi$ in the first coordinate of $\Theta$ should be treated as exactly the same point, or more rigorously, $\rho_{\Theta}((\pi - \epsilon, \theta_2, ..., \theta_{D-1}), (\pi, \theta_2, ..., \theta_{D-1})) \to 0$ as $\epsilon \to 0$. This seemingly trivial property is nevertheless important in defining and interpreting whether certain parameter estimates converge asymptotically or not.

---

\(^7\)The idea and the motivations for using the angle-space reparameterization were also found in Manski and Thompson (1986), who however used only one angle parameter.

\(^8\)Two metrics $d_1$ and $d_2$ defined on some non-empty set $X$ are strongly equivalent if and only if there exist positive constants $c_1$ and $c_2$ such that for every $x, y \in X$, we have $c_1 d_1 (x, y) \leq d_2 (x, y) \leq c_2 d_1 (x, y)$.
An Adaptive-Grid Algorithm

With the angle reparameterization, we seek to numerically compute a conservative rectangular enclosure of $\arg\min \hat{Q}(\theta)$, deploying a bisection-style grid-search algorithm that recursively shrinks and refines an adaptive grid to any pre-chosen precision (as defined by $\rho_\Theta$). Unlike gradient-based local optimization algorithms, our adaptive grid algorithm handles well the built-in discreteness in our sample criterion function, which has zero derivative almost everywhere, while maintains global initial coverage over the whole parameter space.

While a brute-force global search algorithm is the safest choice if the dimension of product characteristics $D$ is relatively small, our adaptive-grid algorithm performs significantly faster. The essential structure of our algorithm is laid out as follows, with a corresponding illustration in Figure 1.

Step 1: Initialize a global grid $\Theta^{(1)}$ of some chosen size $M_0^{D-1}$ on $\Theta$.

Step 2: Compute $\hat{Q}(\theta)$ for each $\theta \in \Theta^{(1)}$, and select all points in $\Theta^{(1)}$ with a criterion value below the $\alpha$th-quantile in $\hat{Q}\left(\Theta^{(1)}\right) := \{\hat{Q}(\theta) : \theta \in \Theta^{(1)}\}$ into

$$\Theta^{(1)} := \{\theta \in \Theta^{(1)} : \hat{Q}(\theta) \leq \text{quantile}_\alpha\left(\hat{Q}\left(\Theta^{(1)}\right)\right)\}.$$ 

Step 3: Take the enclosing rectangle of $\Theta^{(1)}$, by defining $\theta_d^{(1)} := \min^* \Theta_d^{(1)}$ and $\bar{\theta}_d^{(1)} := \max^* \Theta_d^{(1)}$, where $\Theta_d^{(1)} := \{\theta_d : \theta \in \Theta^{(1)}\}$ for each $d = 1, ..., D - 1$ and the operator $\min^*$ and $\max^*$ have standard definitions of min and max except for the first dimension $d = 1$. For the first dimension, it is necessary to account for the underlying spherical geometry and the periodicity of angles, i.e. $\theta_1 + 2\pi \equiv \theta_1$ and in particular $-\pi \equiv \pi$. This, however, is largely a programming nuisance: whenever $\Theta_1^{(1)} \subseteq \Theta_1^{(1)}$ crosses over at $-\pi$ and $\pi$, we can add $2\pi$ to
every $\theta_1 \in \Theta_1^{(1)}$ and obtain lower and upper bounds of $\Theta_1^{(1)} + 2\pi$, as illustrated in Figure 1.

Step 4: We initialize a refined grid $\Theta^{(2)}$ on $\Theta^{(1)} := \times_{d=1}^{D-1} [\theta^{(1)}_d, \bar{\theta}^{(1)}_d]$ of size $M_0^{D-1}$.

Step 5: Reiterate until refinement stops (falls below a certain numerical precision).

Note that the above is simply a sketch of our algorithm. To be conservative, we add in buffers at each step of refinement, keep track of both outer and inner boundaries of the lower-quantile set $\Theta^{(m)}$, and make sure that the minimizers of the criterion functions at all computed points are indeed enclosed by the set returned in the end. We find the current algorithm to be conservative and perform reasonably well in our simulations.

4 General Econometric Framework of Multi-Index Single-Crossing Conditions

Our key identification strategy, and consequently the associated estimation method, apply much more widely beyong panel multinomial choice models. We now introduce a general econometric framework defined by multi-index single-crossing (MISC) conditions, and show how our proposed methods can be exploited in a wide range of models nested under the MISC condition framework.

Formally, let $(y_i, X_i)_{i=1}^n$ be a random sample of data with $X_i$ distributed on the support $\mathcal{X} \subseteq \mathbb{R}^{d_x}$ and $y_i$ distributed on $\mathcal{Y} \subseteq \mathbb{R}^{d_y}$. Let $h_0 : \mathcal{X} \rightarrow \mathbb{R}$ denote a functional of the conditional distribution of $y_i$ given $X_i$ that is directly identified from data. For each of $j = 1, ..., J \in \mathbb{N}$, let $\phi_j : \mathcal{X} \rightarrow \mathbb{R}^{d_w}$ be some known transformation of $X_i$ for $j = 1, ..., J$, and define $W_{ij} := \phi_j (X_i)$ with $W_i := (W_{i1}, ..., W_{ij})$. Let $\theta_{0j} \in \Theta_j \subseteq \mathbb{R}^{d_{theta_j}}$ be an unknown finite-dimensional parameter and write $\theta_0 := (\theta_{01}', ..., \theta_{0J}')' \in \Theta := \times_{j=1}^J \Theta_j$.

---

9Our algorithm relies heavily on the compactness and convexity of the angle space $\Theta$. Compactness allows us to start with a global grid over the whole parameter space for initial evaluations of the sample criterion function. At each step of recursion, the convexity of $\Theta$ enables us to conveniently refine the grid by separately cutting each coordinate of $\Theta^{(m)}$ into smaller pieces through simple division.
Definition 2 \textit{(Multi-Index Single-Crossing Condition).} We say that $(h_0, \theta_0)$ satisfy the \textit{(weak) multi-index single-crossing condition} if, for any realization $x \in \mathcal{X}$ and $w = \phi(x)$, we have

\begin{align*}
w_j' \theta_0j &\geq 0, \forall j = 1, \ldots, J \Rightarrow h_0(x) \geq 0, \\
w_j' \theta_0j &\leq 0, \forall j = 1, \ldots, J \Rightarrow h_0(x) \leq 0.
\end{align*}

The condition is said to be strict if the inequalities on the right hand side of (15) are strict.

In words, if the $J$ parametric indexes $w_1' \theta_{01}$, $w_2' \theta_{02}$, ... and $w_J' \theta_{0J}$ are all positive, then the functional $h_0$ must be positive; if the $J$ indexes are all zero, then $h_0$ must be zero; if the $J$ indexes are all negative, then $h_0$ must be negative. Essentially, the MISC condition provides a parsimonious way to semiparametrically model how the multiple economic factors jointly affect a certain statistic of the relevant economic outcome. The MISC condition basically requires that, if all the relevant factors reach certain threshold, then the outcome statistic must also reach certain threshold. Such kind of requirement is often very easy to obtain in an economic or econometric model: while multiple factors in an economic model may interact with each other in potentially complicated manners and there might be many configurations of the factors that lead to ambiguous theoretical predictions, there are also often simple configurations that we understand reasonably well. Hence, the MISC condition imposes relatively weak requirements on the economic or econometric model of a specific problem, and thus provides a relatively general framework for semiparametric econometric analysis, where most of the modeling ingredients can be left nonparametric beyond the parametric indexes that encode different economic factors in a certain economic problem.

Clearly, the panel multinomial choice model considered in previous sections falls under the MISC condition framework. Specifically, focusing on a pair of time periods $(t, s)$ and a particular product $j_0$ for illustration, define $\theta_{0j} := \beta_{0j}$, $h_0(X_i) := \gamma_{j_0, ts}(X_i)$, $W_{ij_0} := X_{ij_0t} - X_{ij_0s}$ and $W_{ij} = -(X_{ijt} - X_{ij_s})$ for $j \neq j_0$. Then the MISC conditions (15) are satisfied under model 1 and Assumptions 2-3.
We now provide a few more examples of models nested in the MISC condition framework.

**Example 1** (Binary Choice with Awareness). Consider the following binary choice model

\[
y_i = \mathbb{1}\left\{X_{i1}'\theta_{01} \geq u_i\right\} \cdot \mathbb{1}\left\{X_{i2}'\theta_{02} \geq v_i\right\}
\]

where \(y_i\) denotes whether consumer \(i\) purchases a certain, \(X_{i1}\) denotes a vector of covariates that influence the consumer’s utility from a product, and \(X_{i2}\) denotes a vector of covariates that influence the consumer’s awareness of the product (such as advertising). Here \(J = 2\), \(X_i := (X_{i1}, X_{i2})\), \(W_{i1} := X_{i1}\), and \(W_{i2} := X_{i2}\). Let the functional \(h_0\) be defined by \(h_0(x) := \mathbb{E}[y_i|X_i = x] - \frac{1}{4}\). Then, under the conditional median restrictions \(\text{med}(u_i|X_i) = \text{med}(v_i|X_i) = 0\) and the conditional independence restriction \(u_i \perp v_i|X_i\), it can be shown that

\[
X_{i1}'\theta_{01} > 0, \; X_{i2}'\theta_{02} > 0 \quad \Rightarrow \quad h_0(X_i) > 0,
\]

\[
X_{i1}'\theta_{01} < 0, \; X_{i2}'\theta_{02} < 0 \quad \Rightarrow \quad h_0(X_i) < 0,
\]

again satisfying the MISC condition.

**Example 2** (Binary Choice with Endogeneity). Consider the binary choice model

\[
Y_i = \mathbb{1}\left\{W_i'\beta_0 \geq \epsilon_i\right\},
\]

and suppose that one component of \(W_i\), say, \(W_{i1}\) is endogenous. Suppose that there exist exogenous instrumental variables \(Z_i\) and define \(\xi_i := W_{i1} - Z_i'\gamma_0\) as the residual from the reduced-form linear projection of \(W_{i1}\) on \(Z_i\). Assume that the endogeneity between \(\epsilon_i\) and \(W_{i1}\) is captured by the following control function

\[
\text{med}(\epsilon_i|Z_i, \xi_i) = \lambda(\alpha_0 \xi_i),
\]

where \(\lambda\) is an unknown increasing function with location normalization \(\lambda(0) = 0\), and the sign parameter \(\alpha_0 \in \{-1, 1\}\) controls the direction of the monotonicity. The above can be viewed as an adaption of the binary choice model that merges the conditional median restriction in Manski (1975) with the control function approach in Blundell and Powell (2004): here we only impose the control function restriction on the conditional median instead of the
whole distribution as in Blundell and Powell (2004). Then, writing \( Z_i := (W_{i1}, Z_i) \) and \( \gamma_0 := (-\alpha_0, \alpha_0 \gamma_0)' \), we have

\[
W'_i\beta_0 > 0, \quad Z'_i\gamma_0 > 0 \quad \Rightarrow \quad \mathbb{E}[Y_i | W_i, Z_i] > \frac{1}{2}.
\]

and its “<” counterpart, which can be viewed as a MISC condition with \( K = 2, h_0(W_i, Z_i) := \mathbb{E}\left[Y_i - \frac{1}{2} \mid W_i, Z_i\right], \phi_1(W_i, Z_i) := W_i, \phi_2(W_i, Z_i) := (W_{i1}, Z_i), \theta_{0,1} := \beta_0, \) and \( \theta_{0,2} := \gamma_0 \).

**Example 3** (Dyadic Network Formation). Consider the following dyadic network formation model studied in Gao, Li, and Xu (2023), which generalizes the one studied in Graham (2017):

\[
\mathbb{E}[y_{ij} | X_i, X_j, A_i, A_j] = \psi\left(w(X_i, X_j)'\theta_0, A_i, A_j\right).
\]

Here \( y_{ij} \) is a binary outcome indicating whether individuals \( i \) and \( j \) are linked in an undirected network, \( X_i \) and \( X_j \) are the individuals’ observable covariates, \( w(X_i, X_j) \) is a known pairwise transformation of individual covariates (with the leading example being \( w_h(X_i, X_j) := |X_{ih} - X_{jh}| \) for each coordinate \( h = 1, ..., d_x \), \( A_i \) and \( A_j \) are unobserved individual degree heterogeneity terms, and \( \psi : \mathbb{R}^3 \rightarrow \mathbb{R} \) is an unknown function assumed to be multivariate increasing in all its three arguments. Specifically, fixing a particular pair of individuals \( \bar{i} \) and \( \bar{j} \) and two generic realizations \( \bar{x}, \bar{x'} \) of \( X_i \), it can be shown that, with

\[
w := w(X_i, X_j') - w(X_i, X_j), \quad w := w(X_i', X_i) - w(X_i, X_i'), \quad h_0(\bar{x}, \bar{x'}) := \max\left(0, \mathbb{E}\left[|y_{\bar{i}k} - y_{\bar{j}k}| \mid X_k = \bar{x}\right]\right) \mathbb{E}\left[|y_{\bar{i}k} - y_{\bar{j}k}| \mid X_k = \bar{x'}\right] - \max\left(0, \mathbb{E}\left[|y_{\bar{j}k} - y_{\bar{i}k}| \mid X_k = \bar{x}\right]\right) \mathbb{E}\left[|y_{\bar{j}k} - y_{\bar{i}k}| \mid X_k = \bar{x'}\right]
\]

the weak MISC condition is satisfied (under quite mild additional conditions):

\[
w'\theta_0 > 0, w'\theta_0 > 0 \quad \Rightarrow \quad h_0(\bar{x}, \bar{x}) \geq 0,
\]

\[
w'\theta_0 < 0, w'\theta_0 < 0 \quad \Rightarrow \quad h_0(\bar{x}, \bar{x}) \leq 0.
\]

**Example 4** (Censored Monotone Transformation Model with Endogeneity). The approach proposed in Example 2 above can also be adapted to the following censored monotone trans-
formation model with endogeneity

\[ Y_i = \max \left\{ \phi \left( W_i' \beta_0, \epsilon_i \right), 0 \right\}, \]

where one component of the observed covariates, \( W_{i1} \), is endogenous, and \( \phi \) is an unknown bivariate increasing function. This model generalizes the usual censored regression model \( Y_i = \max \left\{ W_i' \beta_0 + \epsilon_i, 0 \right\} \), say, in Blundell and Powell (2007), by incorporating a flexible unknown monotone transformation \( \phi \) with non-additive error term. Since \( \beta_0, \epsilon_i \) and \( \phi \) are all unknown, one may normalize \( \phi(0,0) = 0 \). By the equivariance of (conditional) quantiles under monotone transformations, we have

\[ \text{med} (Y_i | W_i, Z_i) = \max \left\{ \phi \left( W_i' \beta_0, \text{med} (\epsilon_i | W_i, Z_i) \right), 0 \right\}. \]

Similar to Example 2, define \( \xi_i := W_{i1} - Z_i' \gamma_0 \) as the residual from the reduced-form linear projection of \( W_{i1} \) on the instrumental variables \( Z_i \), and assume that the endogeneity between \( \epsilon_i \) and \( W_{i1} \) is captured by the control function \( \text{med} (\epsilon_i | W_i, Z_i) = \text{med} (\epsilon_i | Z_i, \xi_i) = \lambda (\alpha_0 \xi_i) \), where \( \lambda \) being an increasing function with normalization \( \lambda (0) = 0 \). Writing \( \Xi_i := (W_{i1}, Z_i) \) and \( \tau_0 := (\alpha_0, -\alpha_0 \gamma_0)' \), we have

\[ W_i' \beta_0 > 0, \quad \Xi_i' \tau_0 > 0 \quad \Rightarrow \quad \text{med} (Y_i | W_i, Z_i) > 0. \]
\[ W_i' \beta_0 \leq 0, \quad \Xi_i' \tau_0 \leq 0 \quad \Rightarrow \quad \text{med} (Y_i | W_i, Z_i) = 0. \]

which can be viewed as a MISC condition with a weak “≤” side and \( h_0 (X_i) := \text{med} (Y_i | W_i, Z_i) \) given by the conditional median instead of conditional expectation in Example 2.

Given the MISC conditions (15), taking the logical contraposition again yields identifying restrictions, which can be encoded algebraically in a similar manner as in (6) and (9). Specifically, let \( G \) again be a one-sided sign-preserving function as in (9), and define

\[ \lambda_j (W_i; \theta) := \prod_{k=1}^J 1 \left\{ W_{ij} \theta_j \leq 0 \right\}. \]

Proposition 2. Under condition (15), we have

\[ h_0 (X_i) > 0 \Rightarrow \text{NOT} \left\{ W_{ij} \theta_j \leq 0 \ \forall j \right\}, \]
\[ h_0(X_i) > 0 \Rightarrow \text{NOT} \left\{ W_{ij} \theta_j \geq 0 \ \forall j \right\}. \]

Furthermore, with \( Q^G := Q^G_+ + Q^G_- \) and

\[
Q^G_+ (\beta) := \mathbb{E} [G (h_0 (X_i)) \lambda_j (W_i; \beta)], \\
Q^G_- (\beta) := \mathbb{E} [G (-h_0 (X_i)) \lambda_j (-W_i; \beta)],
\]  \( \text{(16)} \)

we have \( Q^G (\beta) \geq Q^G (\beta_0) = 0. \)

Proposition 2 generalizes our key identification result (Theorem 1). Notice that Proposition 2 applies to all functionals \( h_0 \) on the conditional distribution of \( y_i \) given \( X_i \) that satisfy the MISC conditions.

One could also proceed with the two-step estimation procedure described in Section 3. Given a first-stage nonparametric estimator \( \hat{h} \) of \( h_0 \), we can estimate \( \beta_0 \) (or the identified set) based on the sample criterion \( \hat{Q}^G := \hat{Q}^G_+ + \hat{Q}^G_- \) with

\[
\hat{Q}^G_+ (\beta) := \frac{1}{n} \sum_{i=1}^{n} G \left( \hat{h} (X_i) \right) \lambda_j (W_i; \beta)
\]

and \( \hat{Q}^G_- \) similarly defined. Note again, that the maximum score estimator, which only applies to the single-index setting, cannot be similarly generalized in this manner.

5 Simulation

We now switch back to the panel multinomial choice model introduced in Section 2, which is our main model of interest in this paper, and examine the finite sample performance of our proposed estimator. For each DGP, we run \( M = 1,000 \) simulations of model (1) with the following utility specification:

\[
u(X_{ijt}', \beta_0, A_{ij}, \epsilon_{ijt}) = A_{i0} (X_{ijt}' \beta_0 + A_{ij}) + \epsilon_{ijt},\]

where \( A_{i0} \) is an unobserved scale fixed effect that captures agent-level heteroskedasticity in utilities, and \( A_{ij} \) is an unobserved location shifter specific to each agent-product pair. The ability to deal with nonlinear dependence caused by the unobservable fixed effects in a
robust way differentiates our method from others. To allow for such dependence, we generate correlation between the observable characteristics $X_i$ and the fixed effects $A_i$ via a latent variable $Z_i$.

Furthermore, we set $\beta_0 = (2, 1, ..., 1)' \in \mathbb{R}^D$ and $\beta_0 = \beta_0 / \|\beta_0\|$, and draw $\epsilon_{ijt} \sim TIEV(0, 1)$. To summarize, for each of the $M = 1,000$ simulations we first generate $(\beta_0, X_{it}, A_i, \epsilon_{it})$ for all $it$ combinations. Then we calculate the individual choice $Y$ matrix according to model (1). Next, we compute $\hat{\beta}$ from the simulated observable data of $(X, Y)$, where for the first stage we use nonparametric regression with second-order polynomial basis functions with $\ell_1$-regularization and 10-fold cross validation to estimate $\gamma$ (thus $G(\gamma)$), and for the second stage we apply the adaptive-grid algorithm detailed in Section 3.2 to find $\hat{\beta}$.

Finally, we evaluate the performance of $\hat{\beta}$ against the true $\beta_0$.

Baseline Results

For the baseline configuration we set $N = 10,000, D = 3, J = 3, T = 2$. Since in this case the sufficient conditions for point identification are satisfied, any point from the argmin set $\hat{B}_b := \arg\min_{\beta \in \mathbb{R}^{D-1}} \hat{Q}_b(\beta)$ is a consistent estimator of $\beta_0$ for each round of simulation $b = 1, ..., M$. Specifically, we define

$$\hat{\beta}_{b,d}^u := \max \hat{B}_{b,d}, \quad \hat{\beta}_{b,d}^l := \min \hat{B}_{b,d}, \quad \hat{\beta}_{b,d}^m := \frac{1}{2} \left( \hat{\beta}_{b,d}^u + \hat{\beta}_{b,d}^l \right),$$

where $\hat{\beta}_{b,d}^u$, $\hat{\beta}_{b,d}^l$, and $\hat{\beta}_{b,d}^m$ represent the maximum, minimum, and middle point along dimension $d$ for each round of simulation $b$ of the argmin set $\hat{B}$, respectively.

Table 1 summarizes our baseline results. In the first row we use the middle point $\hat{\beta}_b^m$ along each dimension of $\hat{B}$ to calculate the average bias. The bias is very small across all three dimensions with a magnitude between -.0067 and .0008. The next two rows show the biases in estimating $\beta_{0,d}$ using $\hat{\beta}_{b,d}^u$ and $\hat{\beta}_{b,d}^l$ respectively and the biases are again close to zero.

---

**Appendix D**: We provide additional simulation results. Specifically, we first present a graphical illustration of the identified set $B_0$ based on the population criterion (10). Second, we inspect how our estimator performs without point identification. Lastly, we vary $(D, J, T)$ to examine how robust our method is against various simulation specifications.
Table 1: Baseline Performance

| mid bias | \( \frac{1}{M} \sum_{b=1}^{M} (\hat{\beta}_{b,d}^m - \beta_{0,d}) \) | .0008 | .0003 | -.0067 |
| upper bias | \( \frac{1}{M} \sum_{b=1}^{M} (\hat{\beta}_{b,d}^u - \beta_{0,d}) \) | .0083 | .0085 | .0046 |
| lower bias | \( \frac{1}{M} \sum_{b=1}^{M} (\hat{\beta}_{b,d}^l - \beta_{0,d}) \) | -.0068 | -.0080 | -.0180 |
| mean(u-l) | \( \frac{1}{M} \sum_{b=1}^{M} (\hat{\beta}_{b,d}^u - \hat{\beta}_{b,d}^l) \) | .0152 | .0165 | .0226 |
| standard deviation | \( \sqrt{\frac{1}{M} \sum_{b=1}^{M} (\hat{\beta}_{b,d}^m - \overline{\beta}_{b,d}^m)^2} \) | .0302 | .0311 | .0447 |
| root MSE (by coordinate) | \( \left( \frac{1}{M} \sum_{b=1}^{M} (\hat{\beta}_{b,d}^m - \beta_{0,d})^2 \right)^{1/2} \) | .0293 | .0300 | .0438 |
| root MSE (whole vector) | \( \left( \frac{1}{M} \sum_{b=1}^{M} \|\hat{\beta}_{b}^m - \beta_0\|^2 \right)^{1/2} \) | .0606 |
| mean norm deviations (MND) | \( \frac{1}{M} \sum_{b=1}^{M} \|\hat{\beta}_{b}^m - \beta_0\| \) | .0525 |

The fourth row measures the average width of the set estimator \( \hat{B} \) along each dimension. It is relatively tight compared to the magnitude of \( \beta_0 \). The fifth and sixth row summarizes the standard deviation and root mean squared error for each coordinate of \( \hat{\beta}^m \). In the second part of Table 1 we report vector rMSE and MND based on \( \hat{\beta}^m \), and our method performs very well.

**Results Varying N**

Next, we vary \( N \) while maintaining \( D = 3, J = 3, T = 2 \) to show how our method performs under different sample sizes. In addition to the baseline setup with \( N = 10,000 \), we calculate mean absolute deviation (MAD), average size of the estimated set, rMSE and MND for \( N = 4,000 \) and \( N = 1,000 \). Results are summarized in Table 2.

Table 2 provides numerical evidence that a larger \( N \) helps with overall performance. The sum of absolute bias decreases from .0520 to .0077 when \( N \) increases from 1,000 to 10,000. The average size of the estimated sets, rMSE, and MND follow a similar pattern. However, even with a relatively small \( N = 1,000 \), our result is still informative and accurate, with the rMSE and MND being equal to .1320 and .1128, respectively. We note that here the total
Table 2: Performance under Varying $N$

| $N$       | $\sum_d |bias_d|$ | $\sum_d \text{mean}(u-l)_d$ | rMSE  | MND  |
|-----------|----------------|-----------------------------|-------|------|
| 10,000    | .0077          | .0543                       | .0606 | .0525|
| 4,000     | .0135          | .0863                       | .0762 | .0662|
| 1,000     | .0520          | .1613                       | .1320 | .1128|
| ($N=1,000$) | ($N=1,000$) | ($N=1,000$) | rMSE$_{1000}$ | MND$_{1000}$ |
| 10,000    | 3.16           | 2.15                        | 2.18  | 2.15 |
| 4,000     | 2.00           | 1.59                        | 1.73  | 1.70 |

number of time periods $T$ is set to a minimum of 2. Our method can extract information from each of the $T (T-1)$ ordered pairs of time periods, thus a larger $T$ would in theory improve the performance of our estimators. See Appendix D for more simulation results with larger $T$.

Finally, we numerically investigate the speed of convergence when we increase $N$ from 1,000 to 4,000 and 10,000 in the second part of Table 2. Compared with the case of $N_0 = 1,000$, the relative ratios of rMSE are 1.73 for $N = 4,000$ and 2.18 for $N = 10,000$, both of which lie between $(N/N_0)^{1/3}$ and $(N/N_0)^{1/2}$. A similar pattern is also found for calculations based on MND. These results indicate that our estimator achieves a convergence rate slower than the $N^{-1/2}$ but faster than the $N^{-1/3}$ rate.

6 Empirical Application

6.1 Data and Methodology

We now provide an empirical application of the panel multinomial choice model and the estimation method we propose, using the Nielsen Retail Scanner Data on popcorn sales to explore the effects of display promotion effects. The data contains weekly information on
store-level price, sales and display promotion status generated by about 35,000 participating retail stores with point-of-sale systems across the United States.

Among a huge variety of products we choose to focus on popcorn for two reasons. First, purchases of popcorn are more likely to be driven by temporary urges of consumption without too much dynamic planning. Second, there is good variation in the display promotion status of popcorn, which enables us to estimate how important special in-store displays affect consumer’s purchase decisions.

We aggregate the store level data to the $N = 205$ designated market area (DMA) level for year 2015. We focus on the top 3 brands ranked by market share, aggregate the rest into a fourth product “all other products”, and allow an outside option of “no purchase”. We calculate the dependent variable “market share” for each of the $J = 5$ brands. The observed product characteristics include price, promotion status and their interaction term.\footnote{We calculate Price$_{ijt}$ as the weighted average unit price of all UPCs of the brand $j$ in DMA $i$ during week $t$. In the data we find two variables related to promotion: display and feature. Due to their similarity, we calculate Promo$_{ijt}$ as (feature/display)$_{ijt}$. The interaction term Price$_{ijt} \times$ Promo$_{ijt}$ is included in $X$ to show the effect of promotion on consumers’ price elasticity.}

Notationally, $i$ denotes each of the $N = 205$ DMAs, $j$ represents each of the $J = 5$ brands, and $t$ indexes each of the $T = 52$ weeks in 2015. The summary statistics of these variables are provided in Table 3.

To describe the methodology, we use the observed DMA-level market shares as an estimate of $s_{ijt} = \mathbb{E} [y_{ijt} | X_{it}, A_i]$. Under Assumption 3, we first estimate

$$
\mathbb{E} [s_{ijt} - s_{ijs} | X_{i,t,s}] = \int (\mathbb{E} [y_{ijt} | X_{it}, A_i] - \mathbb{E} [y_{ijs} | X_{is}, A_i]) d\mathbb{P} (A_i | X_{i,t,s}) .
$$

\begin{table}[h]
\centering
\caption{Empirical Application: Summary Statistics}
\begin{tabular}{lccccc}
\hline
 & mean & s.d. & min & max \\
\hline
DMA-level Market Share $s_{ijt}$ & 25.00\% & 21.59\% & 0.07\% & 96.69\% \\
Price$_{ijt}$ & .4924 & .1803 & .1094 & 1.3587 \\
Promo$_{ijt}$ & .0282 & .0377 & .0000 & .5000 \\
Price$_{ijt} \times$ Promo$_{ijt}$ & .0136 & .0203 & .0000 & .4505 \\
\hline
\end{tabular}
\end{table}
Table 4: Empirical Application: Estimation Results

|               | $\hat{\beta}_c = 0$ | $[\hat{\beta}_l, \hat{\beta}_u]_{\hat{c} = 0}$ | $\hat{\beta}_c = .14$ | $[\hat{\beta}_l, \hat{\beta}_u]_{\hat{c} = .14}$ |
|---------------|----------------------|-----------------------------------------------|------------------------|-----------------------------------------------|
| Price$_{ijt}$ | -.9565               | [-.9572, -.9557]                            | -.9530                 | [-.9646, -.9415]                            |
| Promo$_{ijt}$ | .1573                | [.1521, .1626]                               | .1700                  | [.1242, .2159]                              |
| Price$_{ijt} \times$ Promo$_{ijt}$ | .2457               | [.2395, .2519]                               | .2324                  | [.1515, .3134]                             |

Table 5: Empirical Illustration: Comparison of Results

|               | $\hat{\beta}_c = .14$ | $\hat{\beta}_{CyclicMono}$ | $\hat{\beta}_{OLS}$ | $\hat{\beta}_{OLS-FE}$ | $\hat{\beta}_{MLogit-FE}$ |
|---------------|------------------------|-----------------------------|----------------------|--------------------------|----------------------------|
| Price$_{ijt}$ | -.9530                 | -.3781                      | .0240                | -.3803                   | -.8511                     |
| Promo$_{ijt}$ | .1700                  | -.0567                      | .5760                | .5978                    | .4589                      |
| Price$_{ijt} \times$ Promo$_{ijt}$ | .2324 | .9240 | -.8171 | -.7057 | -.2552 |

Specifically, we nonparametrically regress $(s_{ijt} - s_{ijs})$ on the second-order polynomial basis functions of $X_{i,ts}$ with $\ell_1$-regularization and 10-fold cross validation, and obtain an estimator $\hat{\gamma}_j$ of $\gamma_j (X, \bar{X}) = \mathbb{E} [s_{ijt} - s_{ijs} | X_{i,ts} = (\bar{X}, \bar{X})]$. Then, we plug $\hat{\gamma}$ into our second-stage algorithm and compute the (approximate) argmin set $\hat{B}_c$.

6.2 Results and Discussion

We report our estimation results in Table 4. $[\hat{\beta}_l, \hat{\beta}_u]_{\hat{c}}$ corresponds to the lower and upper bounds of the (approximate) argmin set $\hat{B}_c$, while $\hat{\beta}_c^m := \frac{1}{2} (\hat{\beta}_l + \hat{\beta}_u)$ represents to the middle point. We show both the exact argmin set ($\hat{c} = 0$) and the approximate argmin set with $\hat{c} = .1 \times N^{-\frac{1}{4}} \log (N) \approx .14$ for $N = 205$. The estimated coefficients for Price (negative) and Promo (positive) are economically intuitive.

The most interesting result is the positive estimated coefficient on the interaction term Price$_{ijt} \times$ Promo$_{ijt}$. An intuitive explanation for the positive sign is that by displaying certain products in front rows, consumers no longer see their price tags adjacent to those of their competitors, and thus become less price-sensitive for these specially promoted products.
Furthermore, we compare our $\hat{\beta}^m$ with the estimates obtained through four other methods, i.e. Cyclic Monotonicity (CM) based on Shi, Shum, and Song (2018)\textsuperscript{13}, OLS, OLS with scalar-valued fixed effects (OLS-FE) and the multinomial logit with fixed effects (MLogit-FE). Results (normalized to $S^{D-1}$) are summarized in Table 5.

The OLS estimator for Price is a positive .0240, which is counterintuitive. Moreover, displaying the product at the front row of the store will likely make consumers less price sensitive, suggesting a positive coefficient for $\text{Price} \times \text{Promo}$. However, the estimated coefficients for the interaction term using OLS, OLS-FE and MLogit-FE are all negative. Finally, the CM-based estimator for Promo is a negative -.0567, invalidating the effects of in store promotions. In addition, the CM-based estimator for $\text{Price} \times \text{Promo}$ is a large .9240, which can easily make the effective coefficient for Price (i.e., $\hat{\beta}_{\text{Price}} + \hat{\beta}_{\text{Price} \times \text{Promo}}$) positive.

We regard the contrast between our result and the results obtained in these alternative methods as an empirical illustration that by accommodating more flexible forms of unobserved heterogeneity through the arbitrary dimensional fixed effects that are allowed to enter into consumers’ utility functions in an additively nonseparable way, our method is able to produce economically more reasonable results.

\textbf{6.3 A Possible Explanation via Monte-Carlo Simulations}

We provide a possible explanation here to the empirical findings in Table 5 via simulations. Recall that “Promo” captures whether a product gains increased exposure by being highlighted by stores. We argue that the negative estimated coefficients obtained in traditional methods in Table 5 for $\text{Price}_{ijt} \times \text{Promo}_{ijt}$ may be caused by a positive correlation between display promotion and an unobserved index sensitivity term.

Specifically, suppose the utility function can be written as

$$u_{ijt} = A_{ij} \times (X'_{ijt} \hat{\beta}_0) + \epsilon_{ijt}, \quad (17)$$

where $X_{ijt}$ contains Price, Promo, and $\text{Price} \times \text{Promo}$, $A_{ij}$ is the $ij$—specific fixed effect which

\textsuperscript{13}We used 2-week cycles for all available weeks in the data for the CM method.
Table 6: Percentage of Correct Signs of Estimated Coefficients

| α     | $\hat{\beta}_m$ | $\hat{\beta}_{\text{CyclicMono}}$ | $\hat{\beta}_{\text{OLS}}$ | $\hat{\beta}_{\text{OLS-FE}}$ | $\hat{\beta}_{\text{MLogit-FE}}$ |
|-------|-----------------|-------------------------------|------------------|-----------------|-----------------|
| .15   | 92.40%          | 0.00%                         | 0.00%            | 0.00%           | 27.40%          |
| .30   | 86.00%          | 0.00%                         | 0.00%            | 0.00%           | 0.20%           |
| .50   | 74.10%          | 0.00%                         | 0.00%            | 0.00%           | 0.00%           |

may capture index sensitivity (which can be thought as inversely related to unobserved brand loyalty), and $\epsilon_{ijt}$ is the exogenous random shock. Suppose $A_{ij}$ and $\text{Promo}_{ijt}$ are positively correlated, which is reasonable because marketing managers with their expertise are more likely to promote products to which consumers are more price and promotion sensitive. Thus, traditional estimation methods that base on linearity would be unable to detect such pattern and wrongly attribute the effect on price elasticities from $A_{ij}$ to Promo.

To provide some numerical evidence of the claim, we run the following Monte Carlo simulation. We let $\beta_0 = (-4, 2, 2)^\prime$, $Z \sim \mathcal{U}[0, 1]$, $A_{ij} = Z + 1$, and $\epsilon_{ijt} \sim \text{TIEV}(0, 1)$. For $X_{ijt}$ vector, we draw $X_{ijt,1} \sim \mathcal{U}[0, 4]$ and $W \sim \mathcal{U}[0, 1]$, and let $X_{ijt,2} = (1 - \alpha) \times W + \alpha \times Z$ and $X_{ijt,3} = X_{ijt,1} \times X_{ijt,2}$. We emphasize that $X_{ijt,2}$ (Promo) is positively correlated with $A_{ij}$ through $Z$, with $\alpha$ measuring the strength of the correlation. We consider three values of $\alpha$: .15, .3, and .5.

We run 1,000 simulations for each of the five methods in Table 5 to estimate $\beta_0$. To replicate the data structure of the empirical exercise, we set $N = 205$, $D = 3$, $J = 4$, and $T = 10$. We report in Table 6 the percentage of simulations that the corresponding method is able to generate correct signs for all coordinates of $X_{ijt}$.

The percentages that our proposed method is able to generate correct signs for all coordinates of $X_{ijt}$ for $\alpha = .15$, .3, and .5 are 92.40%, 86.00%, and 74.10%, respectively. The accuracy of the estimator is negatively affected by the correlation between $X_{ijt,2}$ (Promo) and $A_{ij}$ (multiplicative fixed effect). None of the other methods in Table 6 generates estimates of $\beta_0$ with correct signs. It is worth mentioning that the CM-based method requires $A_{ij}$
entering the utility function linearly, which is violated in (17). Apparently, all these other models, due to their additive separable structure, completely ignore the positive dependence between the Promo and the multiplicative fixed effect $A_{ij}$, thus producing biases in their estimates.

Intuitively, since products with larger $A_{ij}$ are more likely to be promoted ($X_{ijt,2} = 1$) by the selection of marketing managers, the average effective price sensitivity of promoted products tends to be larger than those products not promoted. This drives those estimators that ignore such confounding selection effects to produce a negative coefficient on the interaction term in Table 5. In contrast, our method handles such non-additive dependence between observable characteristics and unobserved fixed effects well, illustrating its robustness in these models.

Acknowledgments

We are grateful to Xiaohong Chen, Peter Phillips, and Phil Haile for their invaluable advice and encouragement. We thank Don Andrews, Isaiah Andrews, Tim Armstrong, Tim Christensen, Ben Connault, Francis Diebold, Bo Honoré, Joel Horowitz, Yuichi Kitamura, Patrick Kline, Lixiong Li, Yuan Liao, Charles Manski, Aviv Nevo, Matt Seo, Xiaoxia Shi, Frank Schorfheide, Elie Tamer, Ed Vytlacil, Rui Wang, Sheng Xu and seminar participants at Georgetown, UCSD, Berkeley, UCL, Northwestern, UW-Madison, UPenn, NYU, Princeton, HKU, CUHK, SMU, LSE, KU Leuven, Sciences Po, PSU, Harvard-MIT, Microeconometrics Class of 2019 Conference (Duke) and 2020 Winter Meeting of the Econometric Society for helpful comments. We thank Chuyue Tian for excellent research assistance.

A Proof of Theorem 2

We first prove two lemmas before formally proving Theorem 2.

Lemma 3. $Q : S^{d-1} \rightarrow \mathbb{R}_+ \text{ is continuous.}$
Proof. Recalling that $v_k (X - X) = X_k - X_k/\|X_k - X\|$ whenever $X_k \neq X_k$, while $v_k (X - X) = 0$ when $X_k = X_k$, we have

$$G (\gamma_{j,t,s} (X_{i,t,s})) \lambda_j (X_{i,t,s}; \beta) = G (\gamma_{j,t,s} (X_{i,t,s})) \prod_{k=1}^J \mathbb{1} \{ (-1)^{1 \{k=j\}} (X_{ikt} - X_{iks}) \beta \geq 0 \}$$

$$= G (\gamma_{j,t,s} (X_{i,t,s})) \prod_{k=1}^J \mathbb{1} \{ (-1)^{1 \{k=j\}} v_k (X_{it} - X_{is}) \beta \geq 0 \}$$

which is continuous in $\beta$ with probability one, since $v_k (X_{it} - X_{is})$ has no mass point except possibly at 0, in which case the indicator degenerates to a constant over $\beta \in \mathbb{S}^{d-1}$. Since $X_{i,t,s}$ is i.i.d. across $i$, $\mathbb{S}^{d-1}$ is compact, and the indicator function is bounded, all conditions for Lemma 2.4 in Newey and McFadden (1994) are satisfied, by which we conclude that $Q = \sum_{j,t,s} Q_{j,t,s}$ is continuous on $\mathbb{S}^{d-1}$. \hfill \Box

**Lemma 4.** Under Assumptions 1, 5 and 6, $\sup_{\beta \in \mathbb{S}^{d-1}} |\hat{Q} (\beta) - Q (\beta)| = O_p (c_N)$.

**Proof.** We first prove the convergence of $\hat{Q}_{j,t,s} (\beta)$ to $Q_{j,t,s} (\beta)$ for each $(j, t, s)$. For each generic deterministic function $\tilde{\gamma}_{j,t,s}$, define

$$Q_{j,t,s} (\beta, \tilde{\gamma}) := \mathbb{E} [G (\tilde{\gamma}_{j,t,s} (X_{i,t,s})) \lambda_j (X_{i,t,s}; \beta)],$$

$$\hat{Q}_{j,t,s} (\beta, \tilde{\gamma}) := \frac{1}{n} \sum_{i=1}^n G (\tilde{\gamma}_{j,t,s} (X_{i,t,s})) \lambda_j (X_{i,t,s}; \beta).$$

so that $\hat{Q}_{j,t,s} (\beta) = \hat{Q}_{j,t,s} (\beta, \tilde{\gamma}_{j,t,s})$ and $Q_{j,t,s} (\beta) = Q_{j,t,s} (\beta, \gamma)$. For notational simplicity we suppress the subscript $(j, t, s)$ for the moment.

Defining $Q := \{ G (\tilde{\gamma} (X)) \lambda (X_{i,t,s}; \beta) : \tilde{\gamma} \in \Gamma, \beta \in \mathbb{S}^{d-1} \}$, we first argue that $Q$ is a $\mathbb{P}$-Donsker class based on Van Der Vaart and Wellner (1996). First, it is easy to show by Assumption 5 that $G (0) = 0$, which together with the Lipschitz continuity of $G$, we have $\mathbb{E} [G^2 (\tilde{\gamma} (X))] \leq M \mathbb{E} [\tilde{\gamma}^2 (X)] < \infty$ and $\mathbb{E} |G (\tilde{\gamma} (X_i))| \leq \mathbb{E} |\tilde{\gamma} (X_i)| \leq \sup_{\tilde{\gamma} \in \Gamma} \mathbb{E} |\tilde{\gamma} (X_i)| < \infty$. Then, as $\Gamma$ is $\mathbb{P}$-Donsker, $G \circ \tilde{\gamma}$ must also be $\mathbb{P}$-Donsker. Second, recall that $\lambda (X_{i,t,s}; \beta)$ is the product of indicators of half planes, while the collection of $\mathbb{1} \{ (X_k - X_k)' \beta \geq 0 \}$ over $\beta \in \mathbb{S}^{d-1}$ is a well-known VC Class of functions (sets) and is thus $\mathbb{P}$-Donsker. Finally, since the indicator function is uniformly bounded and $\sup_{\tilde{\gamma} \in \Gamma} \mathbb{E} |G (\tilde{\gamma} (X_i))| < \infty$, we conclude
that \( Q \) is also \( \mathbb{P} \)-Donsker:

\[
\sup_{\beta \in \mathbb{S}^{d-1}} \sup_{\tilde{\gamma} \in \Gamma} |\hat{Q}(\beta, \tilde{\gamma}) - Q(\beta, \tilde{\gamma})| = O_p\left(N^{-\frac{1}{2}}\right).
\]

(18)

Next, by Assumption 4, we have

\[
\sup_{\beta \in \mathbb{S}^{d-1}} |Q(\beta, \tilde{\gamma}) - Q(\beta, \gamma)| \leq \sup_{\beta \in \mathbb{S}^{d-1}} \int \left| G(\tilde{\gamma}(X)) - G(\gamma(X)) \right| \lambda_j(X; \beta) d\mathbb{P}(X)
\]

\[
\leq M \sqrt{\int (\tilde{\gamma}(X) - \gamma(X))^2 d\mathbb{P}(X)} = O_p(c_N)
\]

(19)

by Lipschitz continuity of \( G \), \(|\lambda_j| \leq 1\) and Cauchy-Schwarz. Combining (18) and (19), we have

\[
\sup_{\beta \in \mathbb{S}^{d-1}} |\hat{Q}(\beta, \tilde{\gamma}) - Q(\beta, \gamma)| \leq \sup_{\beta \in \mathbb{S}^{d-1}} \sup_{\tilde{\gamma} \in \Gamma} |\hat{Q}(\beta, \tilde{\gamma}) - Q(\beta, \tilde{\gamma})| + \sup_{\beta \in \mathbb{S}^{d-1}} |\hat{Q}(\beta, \tilde{\gamma}) - Q(\beta, \tilde{\gamma})| = O_p\left(N^{-\frac{1}{2}}\right) + O_p(c_N) = O_p(c_N)
\]

since \( N^{-\frac{1}{2}} = O_p(c_N) \) for nonparametric estimators. Summing over all \((j, t, s)\), we have

\[
\sup_{\beta \in \mathbb{S}^{d-1}} |\hat{Q}(\beta) - Q(\beta)| = O_p(c_N).
\]

**Main Proof of Theorem 2**

*Proof.* We verify Condition C.1 in Chernozhukov, Hong, and Tamer (2007, CHT thereafter) so as to apply their Theorem 3.1. Condition C.1(a) on the nonemptiness and compactness of parameter space is satisfied given Theorem 1. Condition C.1(b) on the continuity of the population criterion function \( Q \) is proved by Lemma 3. Condition C.1(c) on measurability of the sample criterion function is satisfied by its construction. Condition C.1(d)(e) regarding the uniform convergence of \( Q_n \) are satisfied by Lemma 4. Hence Theorem 3.1.(1) in CHT implies the Hausdorff consistency of \( \hat{B} \). The consistency of the point estimator under the additional assumption of point identification (i.e., \( B_0 \) is a singleton) follows from Theorem 3.2 of CHT.

\( \square \)
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B Pairwise Time Homogeneity of Errors

As mentioned in Section 2, Assumption 3 is stronger than necessary, and our identification strategy carries over under the weaker Assumption 3’, which requires that \( \epsilon_{it} \sim \epsilon_{is} | (X_{i,ts}, A_i) \). To see why Proposition 1 still holds, consider:

\[
\mathbb{E} [y_{ijt} - y_{ijs} | X_{i,ts} = (X, X), A_i]
\]

\[
= \int \mathbb{1} \left\{ u \left( \tilde{\delta}_j, A_{ij}, \epsilon_{ijt} \right) \geq \max_{k \neq j} u \left( \tilde{\delta}_k, A_{ik}, \epsilon_{ik} \right) \right\} d \mathbb{P} (\epsilon_{it} | X_{i,ts} = (X, X), A_i)
\]

\[
- \int \mathbb{1} \left\{ u \left( \tilde{\delta}_j, A_{ij}, \epsilon_{ijs} \right) \geq \max_{k \neq j} u \left( \tilde{\delta}_k, A_{ik}, \epsilon_{iks} \right) \right\} d \mathbb{P} (\epsilon_{is} | X_{i,ts} = (X, X), A_i)
\]

\[
= \int \mathbb{1} \left\{ u \left( \tilde{\sigma}_j, A_{ij}, \tilde{\epsilon}_{ij} \right) \geq \max_{k \neq j} u \left( \tilde{\sigma}_k, A_{ik}, \tilde{\epsilon}_{ik} \right) \right\} d \mathbb{P} (\tilde{\epsilon}_i | X_{i,ts} = (X, X), A_i)
\]

\[
- \int \mathbb{1} \left\{ u \left( \tilde{\sigma}_j, A_{ij}, \tilde{\epsilon}_{ij} \right) \geq \max_{k \neq j} u \left( \tilde{\sigma}_k, A_{ik}, \tilde{\epsilon}_{ik} \right) \right\} d \mathbb{P} (\tilde{\epsilon}_i | X_{i,ts} = (X, X), A_i)
\]

\[
= \int \left[ \mathbb{1} \left\{ u \left( \tilde{\sigma}_j, A_{ij}, \tilde{\epsilon}_{ij} \right) \geq \max_{k \neq j} u \left( \tilde{\sigma}_k, A_{ik}, \tilde{\epsilon}_{ik} \right) \right\} \right] d \mathbb{P} (\tilde{\epsilon}_i | X_{i,ts} = (X, X), A_i)
\]

where \( \tilde{\sigma} = X \beta_0, \tilde{\delta} = X \beta_0, \) and \( \tilde{\epsilon}_i \) denotes generic realizations of \( \epsilon_{it} \) and \( \epsilon_{is} \) conditional on \( X_{i,ts} = (X, X) \) and \( A_i \). Notice that the second equality follows from the assumption that \( \epsilon_{it} \sim \epsilon_{is} | (X_{i,ts}, A_i) \).

Again, if \( \tilde{\sigma}_j \leq \tilde{\sigma}_j \) and \( \tilde{\sigma}_k \geq \tilde{\sigma}_k \) for all \( k \neq j \), the bracketed term in the last line of the displayed equation above must be nonpositive for all realizations of \( A_i \) and \( \tilde{\epsilon}_i \), so that

\[
\mathbb{E} [y_{ijt} - y_{ijs} | X_{it} = X, X_{is} = X, A_i] \leq 0 \]

for all realizations of \( A_i \), which further implies that

\[
\gamma_{j,t,s} (X, X) = \mathbb{E} \left[ \mathbb{E} [y_{ijt} - y_{ijs} | X_{i,ts} = (X, X), A_i] | X_{i,ts} = (X, X) \right] \leq 0.
\]
Taking the logical contraposition again gives Proposition 1.

C Sufficient Conditions for Point Identification

In this section, we prove sufficient conditions for the point identification of $\beta_0$. For simplicity of notation, we fix $T = 2$. We first need to impose an assumption of strict multivariate monotonicity on the function $\psi_j$ defined in (4).

**Assumption 7** (Strict Monotonicity of $\psi_j$). For any realized $A_i$, the function $\psi_j(\cdot, A_i) : \mathbb{R}^J \rightarrow \mathbb{R}$ is strictly increasing, i.e., if $\delta_j > \delta_j$ for all $j$, then $\psi(\delta, A_i) > \psi(\delta, A_i)$.

We note that Assumption 7 is implied by a stronger version of Assumption 2 together with an additional condition on the support of $u$ given $(X_i, A_i)$.

**Assumption 1’** (Strict Monotonicity of $u$). $u(\delta_{ijt}, A_{ij}, \epsilon_{ijt})$ is strictly increasing in the index $\delta_{ijt}$, for every realization of $(A_{ij}, \epsilon_{ijt})$.

**Assumption 1”** (Overlapping Supports). Conditional on any realization of $X_i$ and $A_i$, we have $\bigcap_{j=1}^J \text{int}(\text{Supp}(u(X'_{ijt}\beta_0, A_{ij}, \epsilon_{ijt}))) \neq \emptyset$.

In particular, Assumption 1” is directly implied by the assumption of $\text{Supp}(u(X'_{ijt}\beta_0, A_{ij}, \epsilon_{ijt})) = \mathbb{R}$ conditional on any realization of $X_i$ and $A_i$, which is again satisfied in additive panel multinomial choice models with scalar fixed effects a la $u(X'_{ijt}\beta_0, A_{ij}, \epsilon_{ijt}) = X'_{ijt}\beta_0 + A_{ij} + \epsilon_{ijt}$ under the assumption of $\text{Supp}(\epsilon_{ijt}|X_i, A_i) = \mathbb{R}$ as commonly imposed in the literature.

**Lemma 5.** Assumptions 1’ and 1” imply Assumption 7.

Finally, we impose the following assumption on $\Delta X_i$, with $\Delta X_{ij} := X_{ij1} - X_{ij2}$ for all individual $i$ and product $j$ across period 1 and period 2.

**Assumption 8** (Full-Directional Support of $\Delta X_i$). Suppose either (a) or (b) is true:

(a) $0 \in \text{int}(\text{Supp}(\Delta X_i))$. 

(b) There exists some \( k \in \{1, \ldots, d_x \} \) such that \( \beta_k^0 \neq 0 \) and \( \text{Supp} \left( \Delta X_k \big| \Delta X_{il}, l \neq j \right) = \mathbb{R} \) for all \( j \in \{1, \ldots, J\} \). Furthermore, for all \( j \in \{1, \ldots, J\} \), \( \text{Supp} \left( \Delta X_{ij} | \Delta X_{il}, l \neq j \right) \) is not contained in a proper linear subspace of \( \mathbb{R}^{d_x} \).

Assumption 8(a) is satisfied when \((X_{ij})\) is continuous random vector. On the other hand, Assumption 8(b) can accommodate discrete regressors generally, but requires one continuous covariate with large support. Assumption 8 ensures that \( \Delta X_{ij}' \beta_0 > 0 \) and \( \Delta X_{ik}' \beta_0 < 0 \) for all \( k \neq j \) hold simultaneously with strictly positive probability.

**Theorem 3** (Point Identification). **Under Assumptions 1, 3, 7 and 8, \( \beta_0 \) is point identified on \( S^{D-1} \).**

**Proof.** Recall first that

\[
\gamma_j \left( \mathbf{X}, \mathbf{X} \right) = \int \left[ \psi_j \left( \delta_j, \left( -\delta_k \right)_{k \neq j} \right), \mathbf{A}_i \right] - \psi_j \left( \hat{\delta}_j, \left( -\hat{\delta}_k \right)_{k \neq j} \right), \mathbf{A}_i \right] \right] \text{d}\mathbb{P} \left( \mathbf{A}_i | \mathbf{X}_i = \left( \mathbf{X}, \mathbf{X} \right) \right). 
\]

Hence, under Assumption 7, we have

\[
\tilde{\delta}_j < \hat{\delta}_j \text{ and } \tilde{\delta}_k > \hat{\delta}_k \text{ for all } k \neq j \quad \Rightarrow \quad \gamma_{j,t,s} \left( \mathbf{X}, \mathbf{X} \right) > 0, \quad (20) 
\]
since \( \psi_j \left( \tilde{\delta}_j, \left( -\tilde{\delta}_k \right)_{k \neq j} \right), \mathbf{A}_i \right] < \psi_j \left( \hat{\delta}_j, \left( -\hat{\delta}_k \right)_{k \neq j} \right), \mathbf{A}_i \right] \) for every realization of \( \mathbf{A}_i \). Together with Assumption 8, we deduce that

\[
\mathbb{P} \left\{ \gamma_{j,t,s} \left( \mathbf{X}_i \right) > 0 \right\} \geq \mathbb{P} \left\{ \Delta X_{ij}' \beta_0 > 0 \wedge \Delta X_{ik}' \beta_0 < 0, \forall k \neq j \right\} > 0. 
\]

Now for any \( \beta \in S^{D-1} \setminus \{ \beta_0 \} \), define for any product \( j \),

\[
H_j (\beta) := \left\{ \mathbf{v} \in \text{Supp} \left( \Delta \mathbf{X}_i \right) : v_j' \beta < v_j' \beta_0, \wedge v_k' \beta_0 < 0 < v_k' \beta, \forall k \neq j \right\}. 
\]

As \( \beta \neq \beta_0 \), by Assumption 8 we know that

\[
\mathbb{P} \left( \Delta \mathbf{X}_i \in H_j (\beta) \right) > 0. \quad (21) 
\]

Moreover, for any realization of \( \mathbf{X}_i \), s.t. \( \Delta \mathbf{X}_i \in H_j (\beta) \), we must have: (i) \( \gamma_{j,t,s} \left( \mathbf{X}_i \right) > 0 \) by (20), and (ii):

\[
\lambda_j \left( \Delta \mathbf{X}_i, \beta \right) = \prod_{k=1}^{J} \mathbb{1} \left\{ (-1)^{1 \{k=j\}} \Delta X_{ik}' \beta \geq 0 \right\} = 1 
\]
so that \( G(\gamma_j(X_i)) \lambda_j(\Delta X_i, \beta) = G(\gamma_j(X_i)) > 0 \) for all such \( X_i \). Hence,

\[
\mathbb{E}[G(\gamma_j(X_i))|\Delta X_i \in H_j(\beta)] > 0.
\] (22)

Combining (21) and (22), we have:

\[
Q_j(\beta) = \mathbb{E}[G(\gamma_j(X_i)) \lambda_j(\Delta X_i, \beta)]
\]
\[
\geq \mathbb{E}[G(\gamma_j(X_i)) \lambda_j(\Delta X_i, \beta) 1\{\Delta X_i \in H_j(\beta)\}]
\]
\[
= \mathbb{E}[G(\gamma_j(X_i)) 1\{\Delta X_i \in H_j(\beta)\}]
\]
\[
= \mathbb{E}[G(\gamma_j(X_i))|\Delta X_i \in H_j(\beta)] \mathbb{P}(\Delta X_i \in H_j(\beta))
\]
\[
> 0 = Q_j(\beta_0).
\]

\[\square\]

D Additional Simulation Results

D.1 Adaptive-Grid Computation Algorithm

In this section, we illustrate a typical output of our second-step computation algorithm based on the adaptive-grid search over the angle space, and show that the algorithm works well. For this purpose we consider a simplified DGP without fixed effect \( A_{ij} \). We draw each of \( X_{ijt,d} \) independently across each dimension \( d \) from the standard normal distribution, and set the distribution of the idiosyncratic shock to be \( \epsilon_{ijt} \sim TIEV(0, 1) \), so that we can skip the first-step estimation and directly calculate the true conditional choice probability. Note that the conditions for point identification of \( \beta_0 \) are satisfied. Because we are only seeking to illustrate the validity of the algorithm itself, we set \( N \) to be large with \( N = 10^7 \) and \( D = 3, J = 3, T = 2 \). Then we apply our adaptive-grid algorithm to search for \( \beta_0 \).

Figure 2 shows how our computational algorithm works in finding the true unknown \( \theta_0 \), the angle representation of the true \( \beta_0 \) in the \( \Theta \) space. The horizontal and vertical axes correspond to the two polar coordinates that are associated with \( S^2 \). The blue dots
represent the points that our algorithm searches over but find not to be minimizers of the sample criterion $\hat{Q}$. The black box indicates the area that the minimizers for the sample criterion $\hat{Q}$ lie within, or more precisely, a rectangular enclosure of the numerical argmin set. The big black dot stands for the true parameter value $\theta_0 = (0.4205, 0.4636)'$.

It is evident from Figure 2 that our adaptive-grid algorithm is able to correctly locate an area that covers the true $\theta_0$, which lies within the small black box representing the estimated set of $\hat{\theta}$, demonstrating the efficacy of the algorithm. Besides, it is worth mentioning that our algorithm computes reasonably fast, as it first performs a rough search on the whole unit sphere $S^2$, then focuses on the area where the minimizers are most likely to lie. In the last few rounds of search, the algorithm evaluates the criterion function $\hat{Q}$ on a relatively small area of points shown by those blue and red dots in Figure 2 until the desired level of accuracy is achieved.

For a more transparent representation, we translate the angles $\theta$ in the polar coordinates into unit vectors $\beta$ on the unit sphere $S^2$ and show it in Figure 3, which is now plotted on $S^2 \subseteq \mathbb{R}^3$. Again the blue dots represent the points that do not achieve the minimum of $\hat{Q}$; the black box shows an enclosing set of the minimizers of $\hat{Q}$. The big black dot represents the...
true parameter value $\beta_0$, which resides inside the black box of the minimizers of $\hat{Q}$. Figure 3 illustrates that our computation algorithm is able to locate a tight area around $\beta_0$.

### D.2 Estimation without Point Identification

We now investigate the performance of our estimator when point identification fails. To make things comparable, we fix $(N, D, J, T)$ as in the baseline case, but modify the configuration in two different ways. We maintain the point identification in one setting but lose the point identification in the other.\footnote{Specifically, we set $Z_i \sim \mathcal{U}[-\sqrt{3}, \sqrt{3}]$, $X_{ijt,1} \sim \mathcal{U}[-1, 1]$, $X_{ijt,2} = Z_i + \mathcal{N}(0, 6)$, and $X_{ijt,3} \sim \mathcal{N}(0, 1)$ for the point identified case. For the DGP without point identification, we let $Z_i \sim \mathcal{U}[-\sqrt{3}, \sqrt{3}]$, $X_{ijt,1} \sim \mathcal{U}[-1, -0.9, ... , 0.9, 1]$, $X_{ijt,2} = Z_i + \mathcal{U}[-\sqrt{6}, \sqrt{6}]$, and $X_{ijt,3} \sim \mathcal{U}[-1, 1]$.} We deliberately control the location and scale of each variable to be comparable across the two configurations, with the only differences being the presence of discreteness and boundedness of supports. When point identification fails, we compute the set estimator $\hat{B}_c$ of (13) with $\hat{c} > 0$. Table 7 contains simulation results under the two

![Figure 3: The Argmin Set in $S^2$](image)
Table 7: Performance with and without Point ID: Further Examination

| point ID ? | ˆc   | rMSE | MND |
|------------|------|------|-----|
|            |      | ˆβm  | ˆβu | ˆβl |
| (i) yes    | -    | .0570| .0604| .0630| .0497| .0541| .0565 |
| (ii) no    | .01  | .0585| .0605| .0640| .0505| .0538| .0566 |
|            | .1   | .0514| .0732| .0841| .0438| .0692| .0780 |
|            | 1    | .0406| .1997| .2375| .0384| .1988| .2362 |

configurations, with different choices of ˆc when point identification fails. 15

In Table 7, we calculate the rMSE and MND of the upper bound ˆβu, the lower bound ˆβl and the middle point ˆβm of the (approximate) argmin sets ˆBc (with ˆc = 0 under point identification and three choices of ˆc under partial identification) with respect to the true normalized parameter β0. Across rows in (i) and (ii), we see that the lack of point identification does negatively affect the performance of our estimates, but the impact is limited to a moderate degree. Within rows in (ii), we observe that, as expected, a more conservative choice of the constant ˆc worsens performances of the upper and lower bounds by enlarging the estimated sets; in the meanwhile, it appears that the size (and the performance) of our estimator based on ˆβm is not terribly sensitive to the choice of ˆc.

D.3 Results Varying D, J, T

In this section, we show how our estimator performs under different (D, J, T). We maintain N = 10,000 as in the baseline configuration. We draw Zi ∼ N(0, 1) and construct A and

15Specifically, noting that cN log N ≤ N−1/4 log N ≈ 0.92 ≤ 1 for N = 10,000, we set ˆc = 0.01, 0.1 and 1, respectively.
$X$ according to the following specifications:

$$ A_{ij} \sim \begin{cases} 
0, & j = 1, \\
[Z_i]_+, & j = 2, \\
U[-0.25,0.25], & j = 3, \ldots, J, 
\end{cases} \quad X_{ijt,d} \sim \begin{cases} 
U[-1,1], & d = 1, \\
Z_i + \mathcal{N}(0,6), & d = 2, \\
\mathcal{N}(0,1), & d = 3, \ldots, D, 
\end{cases}$$

which coincides with the baseline model at $D = 3$, $J = 3$. We emphasize that in all configurations we allow for nonlinear dependence between $A$ and $X$ via the latent variable $Z_i$.

We report in Table 8 the performance of our estimators for each of the corresponding configurations across all $M = 1,000$ simulations.

| Table 8: Performance Varying $D, J, T$ |
|----------------------------------------|
| rMSE | $J = 3$ | $J = 4$ | $J = 3$ | $J = 4$ |
|------|---------|---------|---------|---------|
|      | $T = 2$ | $T = 4$ | $T = 2$ | $T = 4$ |
| $D = 3$ | .0594 | .0407 | .1316 | .0996 |
| $D = 4$ | .0716 | .0488 | .1453 | .1068 |

| MND | $J = 3$ | $J = 4$ | $J = 3$ | $J = 4$ |
|-----|---------|---------|---------|---------|
|     | $T = 2$ | $T = 4$ | $T = 2$ | $T = 4$ |
| $D = 3$ | .0522 | .0357 | .1142 | .0877 |
| $D = 4$ | .0650 | .0467 | .1325 | .0984 |

From Table 8 we find a larger $T$ improves the performance of our estimator, which is arguably more practically relevant given the increasing availability of long panel data. The improvement in performance with larger $T$ is because our method can extract more information from $T \times (T-1)$ ordered pairs of time periods which effectively increase the total number of observations. We also find that increase in $D$ or $J$ adversely affects the performance of our estimator, which is expected because more information is required to estimate more covariates or deal with more alternatives. For example, when $J$ is 3 and $T$ is 4, an increase in the dimension of product characteristics $D$ from 3 to 4 increases the rMSE from .0407 to .0488. The change in performance for increasing $J$ is more significant. For
instance, when $D = 4$ and $T = 4$, an increase in $J$ from 3 to 4 increases the MND from .0467 to .0984.