Metallic Ratios in Primitive Pythagorean Triples:
Metals Means embedded in Pythagorean Triangles and other Right Triangles

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Lecture Link 1: https://youtu.be/LFW1saNOp20
Lecture Link 2: https://youtu.be/vBfVDaFnA2k
Lecture Link 3: https://youtu.be/raosniXwRhw
Lecture Link 4: https://youtu.be/74uF4sBqYjs
Lecture Link 5: https://youtu.be/Qh2B1tMl8Bk

Abstract

The Primitive Pythagorean Triples are found to be the purest expressions of various Metallic Ratios. Each Metallic Mean is epitomized by one particular Pythagorean Triangle. Also, the Right Angled Triangles are found to be more "Metallic" than the Pentagons, Octagons or any other (n²+4)gons. The Primitive Pythagorean Triples, not the regular polygons, are the prototypical forms of all Metallic Means.

Keywords: Metallic Mean, Pythagoras Theorem, Right Triangle, Metallic Ratio Triads, Pythagorean Triples, Golden Ratio, Pascal’s Triangle, Pythagorean Triangles, Metallic Ratio

A Primitive Pythagorean Triple for each Metallic Mean (Δn):

Author’s previous paper titled “Golden Ratio” cited by the Wikipedia page on “Metallic Mean” [1] & [2], among other works mentioned in the References, have already highlighted the underlying proposition that the Metallic Means are more closely associated with; and more holistically represented by the “Right Angled Triangles”, rather than Pentagon, Octagon or any other ‘n²+4’gon.

The mathematical correlations between different Metallic Ratios, their Geometric Substantiation, their close correspondence with Pythagorean Triples and p ≡ 1(mod 4) Primes, the intriguing Triads of Metallic Means, their presence in Pascal’s Triangle, all such intriguing aspects of Metallic Means have been recently introduced in the series of papers mentioned in the Reference [1] to [10], and in the Video Lecture Links mentioned above.

Now, let us see the cardinal aspects of these Metallic Ratios........
A Primitive Pythagorean Triple for each Metallic Mean ($\delta_n$)

**Fifth Metallic Mean:**

$$\delta_5 = \cot \left( \frac{\theta}{4} \right)$$

**Sixth Metallic Mean:**

$$\delta_6 = \cot \left( \frac{\theta}{4} \right)$$

**Twelfth Metallic Mean:**

$$\delta_{12} = \cot \left( \frac{\theta}{4} \right)$$

**Thirteenth Metallic Mean:**

$$\delta_{13} = \cot \left( \frac{\theta}{4} \right)$$

*Figure 1: A Primitive Pythagorean Triple for each Metallic Mean*
A Pythagorean Triple \((a, b, c)\) epitomizes one particular Metallic Ratio, if its Hypotenuse minus Longer Cathetus \((c - b)\) is equal to 1, 2 or 8.

If \((c - b) = 1\) or 2 or 8: such Pythagorean Triple \((a, b, c)\) manifests the \(n\)th Metallic Ratio \(\delta_n\):

where \(n = 2 \times \sqrt{\frac{c + b}{c - b}}\) and, the \(n\)th Metallic Mean \(\delta_n = \cot \frac{\theta}{4}\)

where \(\theta\) is the Smaller Acute Angle of the associated Pythagorean Triangle (with specific variations for first four Metallic Means!)

Figure 2: The Metallic Mean \(\delta_n\) in the associated Primitive Pythagorean Triple
Both the acute angles, both of the catheti as well as the Hypotenuse of such Pythagorean Triangle exhibit the corresponding Metallic Mean, as illustrated in above Figure.

Remarkably, the Hypotenuse $C$ of such Triangle:

$$C = (\delta_n + \frac{1}{\delta_n})^2 \times \left(\frac{c-b}{8}\right)$$

$$= (n^2 + 4) \times \left(\frac{c-b}{8}\right)$$

$$= \frac{L_n + L_{n-2}}{F_{n-1}} \times \left(\frac{c-b}{8}\right)$$

where $F_n$, $F_{n+1}$... is the Fibonacci-like Integer Sequence (Fibonacci Sequence for Golden Ratio, Pell Sequence for Silver Ratio, Bronze-Fibonacci Sequence for Bronze Ratio, etc.) and $L_n$, $L_{n+1}$... is the corresponding Lucas Sequence associated with that particular Metallic Mean.

The following Table 1 shows associated Pythagorean Triples for first few Metallic Means:

| n  | Associated Pythagorean Triple |
|----|-------------------------------|
| 1  | 3-4-5                         |
| 2  | None                          |
| 3  | 5-12-13                       |
| 4  | 3-4-5                         |
| 5  | 20-21-29                      |
| 6  | 3-4-5                         |
| 7  | 28-45-53                      |
| 8  | 8-15-17                       |
| 9  | 36-77-85                      |
| 10 | 5-12-13                       |
| 11 | 44-117-125                    |
| 12 | 12-35-37                      |
| 13 | 52-165-173                    |
| 14 | 7-24-25                       |
| 15 | 60-221-229                    |
| 16 | 16-63-65                      |
| 17 | 68-285-293                    |
| 18 | 9-40-41                       |
| 19 | 76-357-365                    |
| 20 | 20-99-101                     |
Noticeably, certain patterns can be observed from above table.

Consider the radical \((n^2 + 4)\) in the fractional expression of the \(n^{th}\) Metallic Mean: 
\[
\delta_n = \frac{n + \sqrt{n^2 + 4}}{2}
\]

The \(n^{th}\) Metallic Mean \(\delta_n\) is epitomized by such Primitive Pythagorean Triple whose Hypotenuse is Factor of this Radical \((n^2 + 4)\); with following sub-rules:

- **For \(n\) is Odd**: \((c - b) = 8\) and Hypotenuse of associated Pythagorean Triple: \(c = (n^2 + 4)\), and the Smaller Cathetus of associated Pythagorean Triple = \(4n\)

- **For \(n\) is Even and multiple of Four \((n=4x)\)**: \((c - b) = 2\) and Hypotenuse of associated Pythagorean Triple: \(c = (nx + 1) = \frac{(n^2+4)}{4}\); and the Smaller Cathetus of associated Pythagorean Triple = \(n\)

- **For \(n\) is Even but not Multiple of Four**: \((c - b) = 1\) and Hypotenuse of associated Pythagorean Triple: \(c = \frac{(n^2+4)}{8}\) and the Smaller Cathetus of associated Pythagorean Triple = \(\frac{n}{2}\), and stuff like that.

Noteworthy here are the **Special Cases of First Four Metallic Ratios** : Noticeably, the same Pythagorean Triple \(3-4-5\) is associated with \(\delta_4, \delta_6\) and the Golden Ratio \(\delta_1\) or \(\phi\)

\[
\delta_1 \text{ or } \phi = \cot \left( \frac{180^0 - \text{Larger Acute Angle of } 3:4:5 \text{ Triple}}{4} \right)
\]

\[
\delta_4 = \cot \left( \frac{\text{Larger Acute Angle of } 3:4:5 \text{ Triple}}{4} \right)
\]

\[
\delta_6 = \cot \left( \frac{\text{Smaller Acute Angle of } 3:4:5 \text{ Triple}}{4} \right)
\]

Similarly, same Pythagorean Triple \(5-12-13\) is associated with \(\delta_3\) and \(\delta_{10}\)

\[
\delta_3 = \cot \left( \frac{\text{Larger Acute Angle of } 5:12:13 \text{ Triple}}{4} \right)
\]

\[
\delta_{10} = \cot \left( \frac{\text{Smaller Acute Angle of } 5:12:13 \text{ Triple}}{4} \right)
\]

From the Fifth Metallic Ratio onwards: \(\delta_n = \cot \left( \frac{\text{Smaller Acute Angle of } 3:4:5 \text{ Triple}}{4} \right)\)
And, $\delta_5$ onwards, each $\delta_n$ has its own distinctive associated Triple, no other Metallic Means have common associated Pythagorean Triples.

Moreover, the Silver Ratio $\delta_2$ has No associated Pythagorean Triple: as the Larger Acute Angle of possible Pythagorean Triple associated with Silver Ratio would be $(4 \arctan \delta_2 - 270^\circ)$ that equals Zero Degrees, moreover $(\delta_2 + \frac{1}{\delta_2})^2$ equals 8 and there is no Pythagorean Triple having Hypotenuse 1, 2 or 8. Another important reason for not having any Pythagorean Triple that may represent $\delta_2$ would become obvious in the subsequent part of this paper.

**Right Triangles are more “Metallic” than any ‘$n^2+4$’gon**

Golden Ratio in regular Pentagon and Silver Ratio in regular Octagon, but contrary to natural expectations, the Tridecagon could not substantiate the Bronze Ratio, or $\delta_5$ is not observed in 29-gon. While these $(n^2 + 4)$-gons could yield no more Metallic Ratios, the Right Angled Triangles are the newly discovered “Metallic Mines” waiting to be exploited....!

Each Metallic Ratio can be constructed geometrically with a special Right Angled Triangle. Any $n^{th}$ Metallic Mean can be represented by the Right Triangle having its catheti 1 and $\frac{n}{2}$. Hence, the right triangle with one of its catheti = 1 may substantiate any Metallic Mean, having its second cathetus = $\frac{n}{2}$, where $n = 1$ for Golden Ratio, $n = 2$ for Silver Ratio, $n = 3$ for Bronze Ratio, and so on. Such Right Triangle provides the precise value of $n^{th}$ Metallic Mean by the generalised formula:

$$\text{The } n^{th} \text{ Metallic Mean } (\delta_n) = \text{Hypotenuse} + \text{Cathetus } \frac{n}{2}$$

This Generalised Geometric Construction of all Metallic Ratios: cited by the Wikipedia in its page on “Metallic Mean” [1]. This generalised geometric substantiation of all Metallic Means was published in author’s previous paper titled Golden Ratio [2].

Note, for $n=1$ : the right Triangle that represents Golden Ratio has its Cathetus 1 longer than its Second Cathetus $\frac{n}{2}$

And, in case of $n=2$ : the Triangle representing Silver Ratio is an Isosceles Right Triangle.

Also, the Right Triangles those represent the First and Fourth Metallic Means: $\delta_1$ (that is Golden Ratio ($\phi$)) and $\delta_4$ (which equals $\phi^3$), are similar triangles. Hence, the abovementioned Special Cases are observed from First to Fourth Metallic Means. From $5^{th}$ Metallic Mean onwards, the generalised rules are applicable.
Figure 3: Generalised Geometric Construction of $n^{th}$ Metallic Mean $\delta_n$ with $1 : \frac{n}{2}$ Right Triangle

Such Right Triangle not only provides for the accurate geometric construction and precise fractional expression of any $n^{th}$ Metallic Mean, but its every geometric feature is the prototypical form of that Metallic Mean [2], and [4] to [7]. The characteristic geometry of such Right Triangle having its catheti $1$ and $\frac{n}{2}$ is resplendent with the corresponding $n^{th}$ Metallic Mean ($\delta_n$) embedded in its every geometric aspect.

For example, the remarkable expression of Golden Ratio in every geometric feature of $1:2:2\sqrt{5}$ triangle, including all its angles and side lengths, its ‘Incenter-Excenters Orthocentric system’, its Gergonne and Nagel triangles, and also the Nobbs points and the Gergonne line, various triangle centers as well as the Incircle of $1:2:2\sqrt{5}$ triangle, make this triangle the quintessential form of the Golden Ratio ($\phi$) and also of the fourth Metallic Mean ($\phi^3$). [2]

Moreover, such Fractional Expression Triangle is also the Limiting Triangle for the Pythagorean Triples formed with the Hypotenuses those equal the alternate terms of the Integer Sequence associated with that Metallic Mean ($\delta_n$). For example, the Pythagorean triples derived from Fibonacci series, approach the $1:2:2\sqrt{5}$ triangle’s proportions, as the series advances. Likewise, the Pythagorean triples having alternate Pell Numbers as their Hypotenuses, approach the $1:1:2\sqrt{2}$ triangle’s proportions, as the series advances, and so on.

Metallic Ratios and various Right Triangles:

Remarkably, all Metallic Ratios can be precisely expressed with various Right Angled Triangles. Three of the more intriguing such triangles are:

1) The abovementioned $1 : \frac{n}{2}$ Right Triangle
2) The $1 : \delta_n$ Right Triangle
3) The Primitive Pythagorean Triples associated with each Metallic Mean $\delta_n$

These three Right Angled Triangles not only epitomize the Metallic Ratios, but they also exhibit the classical geometric correspondence with each other, as illustrated below in Figure 4.
Three Right Triangles
associated with each Metallic Mean $\delta_n$

$\Psi = \arctan \frac{n}{2}$
$\omega = \arctan \frac{2}{n}$
$\alpha = \arctan \delta_n$
$\beta = \arctan \frac{1}{\delta_n}$

Metallic Ratio $\delta_n = \text{Hypotenuse} + \text{Cathetus} \left( \frac{n}{2} \right)$

Metallic Ratio $\delta_n = \text{Ratio between Two Catheti}$

$\theta = 4 \times \arctan \frac{1}{\delta_n}$

Pythagorean Triple
$\Phi = 4 \times \arctan (\delta_n) - 270^0$

$\Psi + 2\beta = 90^0$
$\omega + 2\alpha = 180^0$
$2\beta = \omega$
$\Psi + \beta = \alpha$
$\theta = 2\omega = 4\beta$
$\frac{\theta}{2} + 2\alpha = 180^0$
$2\Psi - \Phi = 90^0$

Figure 4: Three Right Angled Triangles personifying Metallic Means
arctan $\frac{n}{2}$ is the Complimentary Angle of $2 \arctan \frac{1}{\delta_n}$

arctan $\frac{2}{n}$ is the Supplementary Angle of $2 \arctan \delta_n$

$2 \arctan \frac{1}{\delta_n} = \arctan \frac{2}{n}$

$\arctan \frac{1}{\delta_n} + \arctan \frac{n}{2} = \arctan \delta_n$

Noticeably, the $1:2:\sqrt{5}$ triangle, which is the $1: \frac{n}{2}$ Right Triangle for Golden Ratio, exhibits the similar but inverse relationship with associated $3$-$4$-$5$ Pythagorean Triple. The angle $36.87^\circ$ of $3$-$4$-$5$ triangle is the complementary angle for twice the $26.565^\circ$ angle of $1:2:\sqrt{5}$ triangle, while the angle $53.13^\circ$ of $3$-$4$-$5$ triangle is the supplementary angle for twice the $63.435^\circ$ angle of $1:2:\sqrt{5}$ triangle. Further, two smaller acute angles of the two triangles, $26.565^\circ$ and $36.87^\circ$, add up to angle $63.435^\circ$ of $1:2:\sqrt{5}$ triangle, while the angle $53.13^\circ$ of Pythagorean triple is twice the $26.565^\circ$ angle of $1:2:\sqrt{5}$ triangle. The classical geometric relationship between these two right triangles has been described in detail in the work mentioned in References [2] and [6].

Moreover,

**Smaller Acute Angle** of associated Pythagorean Triple $= 2 \arctan \frac{2}{n} = 4 \arctan \frac{1}{\delta_n}$

Hence remarkably, the Doubling of the Smaller Acute Angle of the $1: \frac{n}{2}$ Right Triangle for any $n^{th}$ Metallic Mean ($\delta_n$) produces the associated Pythagorean Triple, while Halving that Smaller Acute Angle produces the $1: \delta_n$ right triangle. (This is another reason why Silver Ratio $\delta_2$ has No associated Pythagorean Triple, as doubling the acute angle of $1$-$1$-$\sqrt{2}$ Isosceles Triangle would be $90^\circ$.)

**Right Angled Triangles and the “TRIADS” Of Metallic Means:**

If $K$, $m$ and $n$ are three positive integers such that $n$ is the smallest of the three integers and $\frac{mn + 4}{m - n} = k$

then, it is observed that

$$\frac{\delta_m \times \delta_n + 1}{\delta_m - \delta_n} = \delta_k$$

where $\delta_k$, $\delta_m$ and $\delta_n$ are the $k^{th}$, $m^{th}$ and $n^{th}$ Metallic Means respectively.

This explicit formula, among several other formulae those give the precise mathematical relations between different Metallic Means, has been recently published in the work mentioned in Reference [3].

The abovementioned explicit formula gives the “Triads” of Metallic Means as $[\delta_n, \delta_m, \delta_k]$
If \( \frac{mn + 4}{m - n} = k \) and \( \frac{kn + 4}{k - n} = m \n\)

then, \( \frac{\delta_m \times \delta_n + 1}{\delta_m - \delta_n} = \delta_k \) and also \( \frac{\delta_k \times \delta_n + 1}{\delta_k - \delta_n} = \delta_m \)

Moreover,

\[
\frac{km - 4}{k + m} = n \quad \text{and} \quad \frac{\delta_k \times \delta_m - 1}{\delta_k + \delta_m} = \delta_n
\]

For example, if \( n=6 \), the three integers 6, 11 and 14 satisfy the prerequisite \( \frac{mn + 4}{m - n} = k \); Hence, the three Metallic means \( \delta_6, \delta_{11}, \delta_{14} \) form a **Triad** \( [\delta_6, \delta_{11}, \delta_{14}] \) such that:

\[
\frac{\delta_{11} \times \delta_6 + 1}{\delta_{11} - \delta_6} = \delta_{14} \quad \text{and also} \quad \frac{\delta_{14} \times \delta_6 + 1}{\delta_{14} - \delta_6} = \delta_{11} \quad \text{Also}, \quad \frac{\delta_{14} \times \delta_{11} - 1}{\delta_{14} + \delta_{11}} = \delta_6
\]

Several properties of these Triads are described in the works mentioned in References, like presence of these Triads of Metallic Means in Pascal’s Triangle [8], geometric substantiation of these Triads [7], etc.

However, more interestingly, the Pythagorean Triples associated with three Metallic Means in a TRIAD unveil certain patterns those are peculiar to these TRIADS. Consider three Pythagorean Triples associated with three Metallic Means in a TRIAD \( [\delta_n, \delta_m, \delta_k] \). Remarkably, the product of Hypotenuses of two Triples associated with \( \delta_n \) and \( \delta_m \) equals the Hypotenuse of the Triple for \( \delta_k \): \( C_n \times C_m = C_k \)

For instance, consider the TRIAD \( [\delta_6, \delta_7, \delta_{46}] \):

and the associated Triples are: 3-4-5, 28-45-53, 23-264-265

Likewise,

\[
\text{TRIAD} \ [\delta_{14}, \delta_{16}, \delta_{114}] : 7-24-25, 16-63-65, 57-1624-1625
\]

\[
\text{TRIAD} \ [\delta_6, \delta_{11}, \delta_{14}] : 3-4-5, 44-117-125, 7-24-25
\]

\[
\text{TRIAD} \ [\delta_{12}, \delta_{13}, \delta_{160}] : 12-35-37, 52-165-173, 160-6399-6401
\]

Remarkably, these Triads of Metallic Means can be represented geometrically with various Right Triangles, as shown below. For instance, the **Triad** \( [\delta_n, \delta_m, \delta_k] \) is illustrated geometrically, in form of with \( \frac{n}{2} \) Right Triangles, in following **Figure 5**.
1: \( \frac{n}{2} \) Right Triangles for the “TRIAD” of Metallic Means

![Diagram of right triangles](image)

Angle BAC = \( \Phi_1 \)
Angle BAD = \( \Phi_2 \)
Angle BAE = \( \Phi_3 \)

Triangle ABC represents: \( \delta_n \)
Triangle ABD represents: \( \delta_m \)
Triangle ABE represents: \( \delta_k \)

Remarkably, in above Figure 5: if the three Metallic Means \( \delta_n, \delta_m \) and \( \delta_k \) constitute a Triad as \( \frac{mn + 4}{m - n} = k \), then it can be observed:

\[ \theta_1 = \theta_2 + \theta_3 \quad (\text{and also} \quad \Phi_1 + 90^0 = \Phi_2 + \Phi_3) \]

In other words,

\[ \arctan \ \frac{2}{n} = \arctan \ \frac{2}{m} + \arctan \ \frac{2}{k} \]

Solving it gives the correlations: \( \frac{mn + 4}{m - n} = k \) and \( \frac{km + 4}{k - n} = m \)

And also \( \frac{km - 4}{k + m} = n \)

And hence satisfy the prerequisite for \( \frac{\delta_m \times \delta_n + 1}{\delta_m - \delta_n} = \delta_k \) and \( \frac{\delta_k \times \delta_n + 1}{\delta_k - \delta_n} = \delta_m \)

And also \( \frac{\delta_k \times \delta_m - 1}{\delta_k + \delta_m} = \delta_n \)
Moreover, the Hypotenuses of these Triad-Triangles: \( H_n, H_m \) and \( H_k \) in above Figure 5 exhibit following relations.

\[
\frac{H_k}{H_m} = \frac{2}{m-n} \times H_n
\]

Simplifying this, we get

\[
\frac{k^2 + 4}{m^2 + 4} = \frac{n^2 + 4}{(m-n)^2}
\]

And solving it gives \( k = \frac{mn + 4}{m-n} \); which is the prerequisite for formation of a Triad \([\delta_n, \delta_m, \delta_k]\).

Moreover, entire geometry of such Triad-Triangles is resplendent with the precise correlation among the three Metallic Means \( \delta_n, \delta_m \) and \( \delta_k \). Following intriguing relations are observed in above Figure 5. Consider the larger acute angles of the three triangles;

\[
\Phi_2 + \Phi_3 = 2 \arctan \delta_n
\]

\[
\Phi_2 - \Phi_1 = 2 \arctan \frac{1}{\delta_k}
\]

\[
\Phi_3 - \Phi_1 = 2 \arctan \frac{1}{\delta_m}
\]

Similarly, \( \arctan \frac{1}{\delta_n} = \arctan \frac{1}{\delta_m} + \arctan \frac{1}{\delta_k} \)

simplifying which we get:

\[
\frac{km - 4}{k + m} = n \quad \text{and} \quad \frac{\delta_k \times \delta_m - 1}{\delta_k + \delta_m} = \delta_n
\]

And,

\[
\arctan \frac{1}{\delta_n} + \arctan \frac{1}{\delta_m} + \arctan \frac{1}{\delta_k} = 2 \arctan \frac{1}{\delta_n} = (\theta_1) = [\theta_2 + \theta_3]
\]

\[
= \arctan \frac{2}{n} = \left[ 180^0 - (\Phi_2 + \Phi_3) \right]
\]

= Half of the Smaller Acute Angle of the Pythagorean Triple for \( \delta_n \).
Similarly,

\[
\arctan \delta_n + 90^0 = \arctan \delta_m + \arctan \delta_k
\]

And,

\[
\arctan \delta_n + \arctan \delta_m + \arctan \delta_k = \Phi_1 + 180^0 = (\Phi_2 + \Phi_3) + 90^0
\]

Moreover, these Triads of Metallic Means can also be represented with 1:δ_n Right Triangles, as shown below.

1: δ_n Right Triangles for the “TRIAD” of Metallic Means

![Diagram of right triangles representing metallic means](image)

\[
\beta_1 = \beta_2 + \beta_3 \quad \text{which is same as:}
\]

\[
\arctan \frac{1}{\delta_n} = \arctan \frac{1}{\delta_m} + \arctan \frac{1}{\delta_k}
\]
Similarly, \( \alpha_1 + 90^0 = \alpha_2 + \alpha_3 \)

Also,
\[
\begin{align*}
\alpha_3 - \alpha_1 &= \beta_2 = \arctan \frac{1}{\delta_m} \\
\alpha_2 - \alpha_1 &= \beta_3 = \arctan \frac{1}{\delta_k}
\end{align*}
\]

which are same as:
\[
\arctan \delta_k - \arctan \delta_n = \arctan \frac{1}{\delta_m}
\]
\[
\arctan \delta_m - \arctan \delta_n = \arctan \frac{1}{\delta_k}
\]

and these are nothing but the Trigonometric Forms of the equations:
\[
\frac{\delta_k \times \delta_n + 1}{\delta_k - \delta_n} = \delta_m \quad \text{and} \quad \frac{\delta_m \times \delta_n + 1}{\delta_m - \delta_n} = \delta_k
\]

Moreover, consider above Figures 5 and 6 together. An intriguing correspondence is observed between the Triad-Triangles in those two figures.

\[
\begin{align*}
\Phi_2 + \Phi_3 &= 2\alpha_1 \\
\Phi_2 - \Phi_1 &= 2\beta_3 \\
\Phi_3 - \Phi_1 &= 2\beta_2 \\
\beta_1 + \beta_2 + \beta_3 &= (\theta_1) = [\theta_2 + \theta_3] = \arctan \frac{2}{n} = \left[ 180^0 - (\Phi_2 + \Phi_3) \right]
\end{align*}
\]

Also,
\[
\alpha_1 + \alpha_2 + \alpha_3 = (\Phi_1 + 180^0) = \left[ (\Phi_2 + \Phi_3) + 90^0 \right]
\]
Another Couple of Right Triangles for Metallic Means:

It is worth mentioning here that a couple of other Right Triangles, having their Catheti equal to two terms of associated Fibonacci Sequences and corresponding Lucas Sequences, also epitomize the $\delta_n$th Metallic Mean $\delta_n$.

Consider a Right Triangle having its catheti $G_m$ and $G_{m+k}$ which are the $m^{th}$ and $(m+k)^{th}$ terms of the Integer Sequence associated with the $n^{th}$ Metallic Mean $\delta_n$, like Fibonacci Sequence for Golden Ratio, Pell Sequence for Silver Ratio, and so on. And, consider another Right Triangle having its catheti $L_m$ and $L_{m+k}$ which are the $m^{th}$ and $(m+k)^{th}$ terms of the corresponding Lucas Sequence, provided $k$ is an Odd positive integer. These two Right Triangles synergize with each other to impart the phenomenal expression of corresponding Metallic Mean $\delta_n$, as illustrated in Figures 7, 8 and 9 below.

Note in Figures 7 below: Catheti 1 and 2 are Second and Third Fibonacci Numbers, and Catheti 3 and 4 are corresponding Lucas Numbers. Hence, the $1:2:\sqrt{5}$ and $3-4-5$ Right Triangles synergize to produce Golden Ratio $\varphi$:

$$63.435^0 + 53.13^0 = 116.565^0 = 2 \arctan \varphi$$

And, this can be generalised as follows: Note $k$ must be Odd Integer in following Figure 8:

$$26.565^0 + 36.87^0 = 63.435^0 = 2 \arctan \frac{1}{\varphi}$$

Figures 7: Golden Ratio in the Amalgam of $1:2:\sqrt{5}$ and $3-4-5$ Right Triangles
Deviation of all Four Acute Angles from arctangent of K\textsuperscript{th} power of Metallic Mean \( = \tan^{-1} \frac{1}{(\delta_n)^{2m+k}} \)

\[\theta_1 + \psi_1 = 2 \arctan (\delta_n)^k\]

\[\theta_2 + \psi_2 = 2 \arctan \left( \frac{1}{\delta_n} \right)^k\]

**Figures 8**: The \( n \textsuperscript{th} \) Metallic Mean \( \delta_n \) in the Amalgam of Two Right Triangles

**Merger of Two Triangles along the Common Hypotenuse**

\[\text{Sum of Longer Catheti} = (\delta_n)^k\]

\[\text{Sum of Shorter Catheti} = \phi\]

**Figures 9**: The \( n \textsuperscript{th} \) Metallic Mean \( \delta_n \) in the Amalgam of Two Right Triangles
The geometric correspondence between such couple of Right Triangles is described in detail in the works mentioned in References [2] and [6].

**Metallic Means and Pythagorean Primes: The “Prime Families” of Metallic Means:**

From the close correspondence between Metallic Means and Pythagorean Triples described in this paper, it becomes obvious that various Metallic Means are also closely associated with different Pythagorean Primes.

Consider the radical \((n^2 + 4)\) in the Fractional expression of the \(n^{th}\) Metallic Mean \((\delta_n)\). By Fermat’s Theorem on Sums of Two Squares, this radical is an integer multiple of a prime of the form \(p \equiv 1(\text{mod } 4)\). The Greatest Prime Factor (i.e. the Largest Prime Divisor) of this radical \((n^2 + 4)\) is a Pythagorean Prime, as shown below in Table 2.

| \(n\) | \(n^2 + 4\) | Greatest Prime Factor of \((n^2 + 4)\) : A Pythagorean Prime |
|-------|-------------|-------------------------------------------------------------|
| 1     | 5           | 5                                                           |
| 2     | 8           | 2 : Exception, Not \(p \equiv 1(\text{mod } 4)\)            |
| 3     | 13          | 13                                                          |
| 4     | 20          | 5                                                           |
| 5     | 29          | 29                                                          |
| 6     | 40          | 5                                                           |
| 7     | 53          | 53                                                          |
| 8     | 68          | 17                                                          |
| 9     | 85          | 17                                                          |
| 10    | 104         | 13                                                          |
| 11    | 125         | 5                                                           |
| 12    | 148         | 37                                                          |
| 13    | 173         | 173                                                         |
Table 2: The Greatest Prime Factors of the radical \((n^2 + 4)\)

| n  | \(n^2 + 4\) | \(\text{GPF} = \text{Greatest Prime Factor}\) |
|----|-------------|---------------------------------------------|
| 14 | 200         | 5                                           |
| 15 | 229         | 229                                         |
| 16 | 260         | 13                                          |

It is noticeable from above table that multiple values of \(n\) exhibit common Greatest Prime Factor of \((n^2 + 4)\).

For example, for 3\textsuperscript{rd}, 10\textsuperscript{th} and 16\textsuperscript{th} Metallic Means, the common Greatest Prime Factor of the radical \((n^2 + 4)\) is 13.

Also, for 1\textsuperscript{st}, 4\textsuperscript{th}, 6\textsuperscript{th}, 11\textsuperscript{th} Metallic Means, the common Greatest Prime Factor of the radical \((n^2 + 4)\) is 5.

Hence, the different Metallic Means can be classified into various groups corresponding to the Greatest Prime Factor (GPF) of the radical \((n^2 + 4)\).

This GPF is necessarily a Pythagorean Prime (4\(x + 1\)), as shown below in Table 3.

\[\text{Table 3 : Prime Families of Metallic Means}\]

| Greatest Prime Factor (GPF) of \([n^2+4]\) | \(\mathbb{N}_s\) of the associated Metallic Means \((\delta_n)\) |
|-------------------------------------------|-------------------------------------------------------------|
| 5                                        | 1, 4, 6, 11                                                 |
| 13                                       | 3, 10, 16, 29                                               |
| 17                                       | 8, 9, 26,                                                   |
| 29                                       | 5, 24, 34                                                   |
| 37                                       | 12, 25, 49                                                  |
| 41                                       | 18, 23, 59                                                  |
Noticeably, as described in previous section: the Hypotenuse of associated Pythagorean Triple is a factor of 
\((n^2+4)\), and the associated Pythagorean Primes, as shown in Table 3 are the Greatest Prime Factors of 
\((n^2+4)\). Note: the 8th and the 9th Metallic Means both have Pythagorean Prime 17 as the GPF of their 
\((n^2+4)\), however they have different associated Primitive Pythagorean Triples, as shown in Table 1.

Moreover, consider the Triads of Metallic Means \([\delta_n, \delta_m, \delta_k]\). Noticeably Two of the Three Metallic Ratios forming such Triad, belong to same Prime Family i.e. Two of the Three Metallic Ratios have common Greatest Prime Factors of their respective \((n^2+4)\) radicals.

For example, consider the Triad \([\delta_4, \delta_5, \delta_{24}]\) : \(\delta_5\), and \(\delta_{24}\) belong to the 29 prime family. Similarly, consider the Triad \([\delta_5, \delta_6, \delta_{34}]\) : \(\delta_5\), and \(\delta_{34}\) belong to the 29 prime family.

Consider another example, the Triad \([\delta_2, \delta_3, \delta_{10}]\) : \(\delta_3\), and \(\delta_{10}\) belong to the 13 prime family. Similarly, consider the Triad \([\delta_3, \delta_4, \delta_{16}]\) : \(\delta_3\), and \(\delta_{16}\) belong to the 13 prime family.

Moreover, beside the Greatest Prime Factors indicated in above Table, multiple Pythagorean Primes constitute the factors of various \((n^2+4)\) radicals :

\[
(59)^2 + 4 = 41 \times 17 \times 5
\]

Likewise, \((49)^2 + 4 = 37 \times 13 \times 5\); and so on.

Remarkably, the Metallic Means belonging to same Prime Family exhibit very distinctive relations among themselves, as shown below.

The intriguing correlations among various Metallic Means, as illustrated by a couple of examples below, indicate that several more mathematical formulae can be generated to describe the precise relations between different Metallic Ratios.
For instance, consider the Prime Family of Metallic Means associated with the Pythagorean Prime $13: \delta_3, \delta_{10}$ and $\delta_{16}$.

Metallic Means associated with the Pythagorean Prime $13: \delta_3, \delta_{10}$ and $\delta_{16}$

For instance, consider the Prime Family of Metallic Means associated with the Pythagorean Prime $13: \delta_3, \delta_{10}$ and $\delta_{16}$.

$$\begin{align*}
10 + 3 & = 13 = 16 - 3 \\
\delta_{10} & \quad \delta_3 & \quad \delta_{16}
\end{align*}$$

Hence, the Pythagorean Prime of this Family $13 = \frac{3^2 + 4}{1}$. Note the Digit 1 in Denominator.

Here, the $3^{rd}$ and $(3 + 1)^{th}$ Metallic Means give the precise value of $16^{th}$ Mean: $\frac{\delta_4 \times \delta_3 + 1}{\delta_4 - \delta_3} = \delta_{16}$

Similarly, the $3^{rd}$ and $(3 - 1)^{th}$ Metallic Means give the precise value of $10^{th}$ Mean: $\frac{\delta_3 \times \delta_2 + 1}{\delta_3 - \delta_2} = \delta_{10}$

Moreover, $\frac{13 - 1}{2} = 6$ : this 6 forms a new Triad with members 10 and 16 of the family: $[\delta_6, \delta_{10}, \delta_{16}]$:

$$\begin{align*}
\frac{\delta_{16} \times \delta_6 + 1}{\delta_{16} - \delta_6} & = \delta_{10} \\
\frac{\delta_{10} \times \delta_6 + 1}{\delta_{10} - \delta_6} & = \delta_{16}
\end{align*}$$

Further, beyond 3, 10 and 16, more members would be added in the family of Pythagorean Prime 17, next member in the family would be 29 which equals the sum of all smaller members in the family; $3 + 10 + 16 = 29$ and, the Prime 13 = 29 – 16

And,

$$2 \times 16 + 3 + 1 = 36 : \text{this 36 forms a new Triad with members 16 and 29 of the family } [\delta_{16}, \delta_{29}, \delta_{36}]$$

$$\begin{align*}
\frac{\delta_{36} \times \delta_{16} + 1}{\delta_{36} - \delta_{16}} & = \delta_{29} \\
\frac{\delta_{29} \times \delta_{16} + 1}{\delta_{29} - \delta_{16}} & = \delta_{36}
\end{align*}$$
Likewise, consider another example for illustration and comparison. The Metallic Means associated with the Pythagorean Prime $17 : \delta_8, \delta_9$ and $\delta_{26}$.

\[ \begin{align*}
8 + 9 &= 17 &= 26 - 9 \\
\delta_8 &\quad \delta_9 &\quad \delta_{26}
\end{align*} \]

And, the Prime $17 = \frac{9^2 + 4}{5}$ : Note the Digit 5 in Denominator.

Here, the $9^{th}$ and $(9 + 5)^{th}$ Metallic Means give the precise value of $26^{th}$ Mean: \[ \frac{\delta_8 \times \delta_{14} + 1}{\delta_{14} - \delta_9} = \delta_{26} \]

Similarly, the $9^{th}$ and $(9 - 5)^{th}$ Metallic Means give the precise value of $8^{th}$ Mean: \[ \frac{\delta_8 \times \delta_4 + 1}{\delta_8 - \delta_4} = \delta_8 \]

Moreover, \[ \frac{17 - 5}{2} = 6 \] : this 6 forms a new Triad with members 8 and 26 of the family: $[\delta_8, \delta_9, \delta_{26}]$

Further, beyond 8, 9 and 26, more members would be added in the family of Pythagorean Prime 17, like next member in the family would be 43 which equals the sum of all smaller members in the family;

$8 + 9 + 26 = 43$ \quad and, \quad the Prime $17 = 43 - 26$

And,

\[ 2 \times 26 + 9 + 5 = 66 \] : this 66 forms a new Triad with members 26 and 43 of the family: $[\delta_{26}, \delta_{43}, \delta_{66}]$

Such several distinctive correlations are observed among the Metallic Means belonging to the same Pythagorean Prime Families, and these correlations are bound to generate more such intriguing mathematical formulae, which may provide the precise relations between different Metallic Ratios.
3, 6 and 9 in the Realm of Metallic Means: Triangles, Triads, Triples, and now 3, 6, 9!

The proponents of Vortex Based Mathematics will continue to make irrational claims, and their opponents will continue to debunk them on grounds of the Base-10 Number System. Let the both camps do their jobs with missionary zeal!

Author’s objective is just to appreciate the beauty of numbers and the special attributes of the digits 3, 6 and 9, especially their unique patterns in the realm of Metallic Means.

Consider the Triads of Metallic Means \([\delta_n, \delta_m, \delta_k]\) with various integer values of \(n\), shown below in Table 4.

Noticeably, in the following Table 4:

If \(n\) is NOT multiple of 3, the alternate values of \(m\) and \(k\) have their digital roots 3, 6, or 9.

And, if \(n\) is multiple of 3: **None** of the associated \(m_s\) and \(k_s\) have their digital roots 3, 6, or 9.

Hence, only one of the \(n\), \(m\) and \(k\) values in a Triad can have the Digital Root 3, 6 or 9.

Table 4: “Triads” of Metallic Means formed by the First Ten Metallic Means:

| \(n\) | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
|---|---|---|---|---|---|---|---|
| \(m\) | 2 | 6 |   |   |   |   |   |
| \(k\) | 6 | 2 |   |   |   |   |   |

\(n\) = 2, 2, 2, 2, 2, 2, 2, 2, 2, 2

Alternate \(m_s\) and \(k_s\) have their digital roots 3, 6, or 9.

| \(n\) | 3 | 3 | 3 | 3 | 3 | 3 | 3 |
|---|---|---|---|---|---|---|---|
| \(m\) | 4 | 16 |   |   |   |   |   |
| \(k\) | 16 | 4 |   |   |   |   |   |

For \(n = 3, 6, 9\):

None of the associated \(m_s\) and \(k_s\) have their digital roots 3, 6, or 9.
None of the associated $m_s$ and $k_s$ have their digital roots 3, 6, or 9.

None of the associated $m_s$ and $k_s$ have their digital roots 3, 6, 9.
More remarkably, the number of Triads formed (or the numbers of $m_s$ and $k_s$) increase noticeably for $n = 6$ and 9.

For Even $n_s$ : the number of Triads exhibit noticeable rise at $n = 6, 16, 26, 36$…….and so on.

For Odd $n_s$ : the number of Triads exhibit noticeable rise at $n = 9, 19, 29$…….and so on.

Moreover, it can be noticed from above Table : if $n$ is **multiple of 3**, the **Digital Root of** $|k − m|$ is 3, 6 or 9.

And, if $n$ is **NOT** multiple of 3, the Digital Root of **NONE** of the $|k − m|$ value is 3, 6 or 9.
And, if \( n \) is NOT multiple of 3, the Digital Roots of the \( |k - m| \) value are 1,2,4,5,7,8.........

Further, many more such intriguing patterns are embedded in the domain of Metallic Means.

For illustration, consider following couple of examples, based upon the formula

\[
\delta_m \times \delta_n + 1 \over \delta_m - \delta_n = \delta_k
\]

Consider the Triads of Metallic Means formed with \( n = 6 \), and the values of \( (m-n) \), as shown below.

|   | 6 | 6 | 6 | 6 | 6 | 6 | 6 |
|---|---|---|---|---|---|---|---|
| m | 7 | 8 | 10 | 11 | 14 | 16 | 26 | 46 |
| m-n | 1 | 2 | 4 | 5 | 8 | 10 | 20 | 40 |

Note the bottom row in above table which contains the values of \( (m-n) \).

The numbers in this \( (m-n) \) row exhibit typical 1 : 2 : 4 : 5 : 8 : 10 \( \times \) ( 1 : 2 : 4 ) pattern, and remarkably the numbers 3, 6 and 9 are conspicuous by their absence from this row!

For Even \( n \) : the \( (m-n) \) values exhibit typical 1 : 2 : 4 : 5 : 8 : 10 \( \times \) ( 1 : 2 : 4 ) pattern.

For Odd \( n \) : the pattern based upon product of the Prime Factors of \( (n^2 + 4) \) is observed.

In either case, the integers 3, 6 and 9 are conspicuous by their absence from these \( (m-n) \) or \( (k-n) \) values.

But, what’s about integer 7 ? Consider another example with \( n = 34 \), as shown below.

|   | 34 | 34 | 34 | 34 | 34 | 34 | 34 | 34 | 34 | 34 | 34 |
|---|----|----|----|----|----|----|----|----|----|----|----|
| m | 35 | 36 | 38 | 39 | 42 | 44 | 54 | 63 | 74 | 92 | 150 |
| m-n | 1 | 2 | 4 | 5 | 8 | 10 | 20 | 29 | 40 | 58 | 116 |

As any \( n^{th} \) Metallic Mean \( \delta_n \) can give the precise values of other Metallic Means \( \delta_m \) and \( \delta_k \) by the formula:

\[
\delta_m \times \delta_n + 1 \over \delta_m - \delta_n = \delta_k ; \text{ with } k_{\text{max}} = m_{\text{max}} = n^2 + n + 4 ;
\]

Hence, \( (m-n)_{\text{max}} = (n^2 + 4) \) which is the all important Radical in the fractional expression of \( n^{th} \) Metallic Mean \( \delta_n \)

In above table with \( n = 34 \), \( (m-n)_{\text{max}} = 1160 \)

Consider the Prime Factorization of this \( (m-n)_{\text{max}} : 1160 = 1 \times 2 \times 2 \times 2 \times 5 \times 29 \)
Note the bottom row \((m-n)\) in above table with \(n = 34\), the numbers in the \((m-n)\) row exhibit the characteristic pattern based upon these factors 1, 2, 5 and 29. Numbers in this \((m-n)\) row are the \(1 : 2 : 4 : 5 : 8 : 10 \times (1 : 2 : 4)\) multiples of the prime factors 1, 2, 5 and 29.

Noticingly, the integer \(7\) is present not directly as \((m-n)\), but it's present only as the Digital Roots of certain \((m-n)\) values; for instance the red shaded number 232 in above example. Such presence of \(7\) as Digital Root of \((m-n)\) values is observed with \(n = 8, 10, 11, 14, 16, 22, 26, 29, 34, 36, 39,\) and so on. Remarkably, with \(n = 26, 36, 39,\) etc. multiple \((m-n)\) values are found to have their digital root \(Seven\).

However, the integers \(3, 6\) and \(9\) are invariably missing from this \((m-n)\) pattern, they are neither present directly as \((m-n)\) value, nor as the digital root of any \((m-n)\) or \((k-n)\) values.

As mentioned above, if \(n\) is multiple of 3, the Digital Root of \(|k - m|\) is 3, 6 or 9. And, if \(n\) is NOT multiple of 3, the Digital Roots of the \(|k - m|\) value are 1, 2, 4, 5, 7, 8, ....... Yes, the \(|k - m|\) pattern directly includes \(7\): for example the Triads like \(\delta_2, \delta_3, \delta_10\) where \(|k - m|\) equals 7 (and also as the Digital Root of \(|k - m|\) values, in Triads like \(\delta_6, \delta_8, \delta_60\)).

The more remarkable aspect of Integer \(7\) in the realm of Metallic Ratios can be observed in the Triads of Metallic Means with \(n = 7\) and Multiples of \(7\).

If \(n = 7, 14,\) etc.: the values of \(\frac{k}{n}\) and \(\frac{k + m}{n}\) exhibit a very characteristic pattern.

For instance, consider the Triad \([\delta_7, \delta_8, \delta_60]\)

\[\frac{k}{n} = \frac{60}{7} = 8.571428571428571428571428571428571428571428\]  
\[\frac{k + m}{n} = \frac{60 + 8}{7} = 9.71428571428571428571428571428571428\]

Note the Digits in Decimal Places: Numbers \(3, 6\) and \(9\) are conspicuous by their absence from the Repeating Pattern of \(571428\).

Exactly similar pattern is observed in all Triads with \(n = 14\):

like \([\delta_{14}, \delta_{15}, \delta_{214}]\); \([\delta_{14}, \delta_{16}, \delta_{114}]\); \([\delta_{14}, \delta_{18}, \delta_{64}]\); \([\delta_{14}, \delta_{19}, \delta_{54}]\); \([\delta_{14}, \delta_{24}, \delta_{34}]\); etc.

Consider the Triad \([\delta_{14}, \delta_{16}, \delta_{114}]\)

\[\frac{114}{14} = 8.1428571428571428571428571428571428\]
\[\frac{114 + 16}{14} = 9.28571428571428571428571428571428\]
And, exactly same are patterns in the Triads with \( n = 28, 35, 56 \), etc. among other multiples of Seven; but **NOT** with \( n = 21, 42 \), etc. where Digital Root of \( n \) is multiple of Three.

Moreover, consider the Triads of Metallic Ratios with \( n = 3 \) or \( 6 \):

For instance the Triad \([ \delta_3, \delta_4, \delta_{16} ]\) with \( n = 3 \)

\[
\frac{k}{n} = \frac{16}{3} = 5.33333333333333333333333333333333333333333
\]

\[
\frac{k + m}{n} = \frac{16 + 4}{3} = 6.66666666666666666666666666666666666666666
\]

Note the **3**s and **6**s in the Decimal places.

Similarly, the Triads with \( n = 6 \), like \([ \delta_6, \delta_{11}, \delta_{14} ]; [ \delta_6, \delta_{10}, \delta_{16} ]; [ \delta_6, \delta_{8}, \delta_{26} ]; [ \delta_6, \delta_{7}, \delta_{46} ]; \) etc.

\[
\frac{k}{n} = \frac{14}{6} = 2.33333333333333333333333333333333333333333
\]

\[
\frac{k + m}{n} = \frac{14 + 11}{6} = 4.166666666666666666666666666666666666666666
\]

Further, the Triads of Metallic Ratios with \( n = 9 \) and the multiples of nine \( 9 \) exhibit their own characteristic patterns.

The point is that the Numbers \( 3, 6 \) and \( 9 \) exhibit their very peculiar and distinctive attributes, in the dominion of Metallic Numbers.

Moreover, the idiosyncrasy of \( 3, 6 \) and \( 9 \) is exhibited in several other such patterns of Metallic Means and their Triads.

For instance, if \( n \) is multiple of \( 3 \): the digital roots of \([ (m-n) + (k-n) ]\) are invariably \( 4, 5, 4, 5 \)....

And, if \( n \) is **NOT** multiple of \( 3 \): the digital roots of \([ (m-n) + (k-n) ]\) are invariably \( 3, 6, \) or \( 9 \), as shown below.
For example,

consider $n = 30$ (digital root of $n$ is 3)

| $n$  | 30 | 30 | 30 | 30 | 30 | 30 | 30 |
|------|----|----|----|----|----|----|----|
| $m$  | 31 | 32 | 34 | 38 | 143| 256| 482|
| $k$  | 934| 482| 256| 143| 38 | 34 | 32 |
| $m-n$| 1  | 2  | 4  | 8  | 113| 226| 452|
| $k-n$| 904| 452| 226| 113| 8  | 4  | 2  |
| Digital Root of $(m-n) + (k-n)$ | 5  | 4  | 5  | 4  | 4  | 5  | 4  | 5 |

Likewise, consider $n = 29$ (digital root of $n$ is other than 3, 6 or 9)

| $n$  | 29 | 29 | 29 | 29 | 29 | 29 |
|------|----|----|----|----|----|----|
| $m$  | 30 | 34 | 42 | 94 | 198| 874|
| $k$  | 874| 198| 94 | 42 | 34 | 30 |
| $m-n$| 1  | 5  | 13 | 65 | 169| 845|
| $k-n$| 845| 169| 65 | 13 | 5  | 1  |
| Digital Root of $(m-n) + (k-n)$ | 9  | 3  | 6  | 6  | 3  | 9  |
Triangles, Triads, Triples and 3, 6, 9: Finally the “Triangular Numbers” in $[\delta_n, \delta_m, \delta_k]$: 

The abovementioned Triads of Metallic Ratios are also found in the Pascal’s Triangle. The renowned Pascal’s Triangle, having the binomial coefficients in its rows and the Fibonacci Numbers in its shallow diagonals, also substantiates the Triads of Metallic Means in an idiosyncratic manner. Take $m$ and $n$ from the Natural Number Diagonal of Pascal’s Triangle, then the Triangular Number Diagonal yields the value of $k$ for the Triad $[\delta_n, \delta_m, \delta_k]$, as illustrated below in Figures 10, 11 and 12. The details however are described in the work mentioned in Reference [8].

\[(m-n) = 1\]

$m$ & $n$ are Consecutive Integers on
Natural Numbers Diagonal.

Then, the Adjacent Integer $X$ from
Triangular Numbers Diagonal gives value of $k$:

\[n = \binom{n}{1}, \quad m = \binom{n+1}{1}\]

then, $X = \binom{n+1}{2}$

and $k = 2X + 4$ as shown here

**Figure 10:** TRIADS of Metallic Means with $m-n=1$, and calculating the value of $k$ from Pascal’s Triangle
(m-n) = 2 if n is \( \binom{n}{1} \) and m is \( \binom{n+2}{1} \); then x is \( \binom{n+1}{2} \) and y is \( \binom{n+2}{2} \), \[ k = \frac{(x+y)+3}{2} \]

**Figure 11:** TRIADS of Metallic Means with \( m-n=2 \), and calculating the value of k from Pascal’s Triangle

\[ 6 = \frac{(3+6)+3}{2} \]

**Triad \([\delta_2, \delta_4, \delta_6]\)**

\[ 42 = \frac{(36+45)+3}{2} \]

**Triad \([\delta_8, \delta_{10}, \delta_{42}]\)**

\[ 86 = \frac{(78+91)+3}{2} \]

**Triad \([\delta_{12}, \delta_{14}, \delta_{86}]\)**

\( (m-n) = 4 \) \n
w, x, y and z are the Triangular Numbers from \( \binom{n+1}{2} \) to \( \binom{m+2}{2} \)

\[ k = \frac{(w+x+y+z) - 2}{8} \]

i.e. all Triangular Numbers between n and m

\[ 4 = \frac{(3+6+10+15) - 2}{8} \]

**Triad \([\delta_2, \delta_4, \delta_6]\)**

\[ 25 = \frac{(36+45+55+66) - 2}{8} \]

**Triad \([\delta_8, \delta_{12}, \delta_{25}]\)**

\( (m-n) = 4 \), and calculating the value of k from Pascal’s Triangle

\[ 36 = \frac{(55+66+78+91) - 2}{8} \]

**Figure 12:** TRIADS of Metallic Means with \( m-n=4 \), and calculating the value of k from Pascal’s Triangle
The Mathematical Correlations between various Metallic Means:

On the last note, it is worth mentioning here that all empirical formulae those provide the precise mathematical relations between different Metallic Means, including the abovementioned intriguing formula

\[
\delta_m \delta_n + 1 = \delta_k ;
\]

are introduced and elaborated in the work mentioned in Reference [3].

The brief view of those other formulae is as follows:

Different “Odd Powers” of a Metallic Ratio yield the precise values of various Metallic Means by the following formulae;

Consider \( \delta_n \) as any \( n^{th} \) Metallic Ratio.

And let \( G_1, G_2, G_3, \ldots \) be the Integer Sequence associated with this Metallic Mean \( \delta_n \), for example: Fibonacci Sequence for Golden Ratio (\( \delta_1 \)), Pell Sequence for Silver Ratio (\( \delta_2 \)), and so on.

And, let \( L_1, L_2, L_3, \ldots \) be the Lucas Sequence associated with this Metallic Mean \( \delta_n \), for example: the Lucas Sequence 1, 3, 4, 7, 11, \ldots for Golden Ratio (\( \delta_1 \)), and the Pell-Lucas Sequence 2, 6, 14, 34, 82, \ldots for Silver Ratio (\( \delta_2 \)), and so on.

Let \( m \) be an Odd Positive Integer. The “Odd Powers” of a Metallic Ratio \( \delta_n \) can give the precise values of various Metallic Means by the following three formulae;

**Formula 1:**

\[
\arctan (\delta_n)^m + \arctan (\delta_n)^{m+2} = 2 \times \arctan (\delta_k) ; \quad \text{where} \quad k = 2 \times G_{(m+1)}
\]

**Formula 2:**

\[
\arctan (\phi)^m + \arctan (\phi)^{m+6} = 2 \times \arctan (\delta_k) ; \quad \text{where} \quad k = F_{(m+3)}
\]

**Formula 3:**

\[
\arctan (\delta_n)^m + \arctan (\delta_n)^{3m} = 2 \times \arctan (\delta_k) ; \quad \text{where} \quad k = 2 \times L_{(m)}
\]

**Formulae 1 and 3** are applicable for all Metallic Ratios, while **Formula 2** is applicable only in case of Golden Ratio (\( \delta_1 \) or \( \phi \)).

All these formulae are elaborated with appropriate examples in the work mentioned in Reference [3].

**Conclusion:**

This paper introduced and elaborated the purest expressions of various Metallic Ratios in *Primitive Pythagorean Triples*. Each Metallic Mean, except for the Second Mean, is shown to be epitomized by one particular *Primitive Pythagorean Triple*. This was the Prime Objective of this paper.

Also, various *Right Angled Triangles* are shown to represent the Metallic Ratios more holistically than the traditionally considered \((n^2+4)\)gons. This paper elaborated the concept of such special *Right Angled Triangles* those precisely epitomize the Metallic Means.
This work also gave a view of the Triads of Metallic Means, their correspondence with Pythagorean Primes, Pascal’s Triangle, Numbers 3, 6, 9, etc.

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