BOUNDS ON THE NUMBER OF RATIONAL POINTS OF
ALGEBRAIC HYPER_SURFACES OVER FINITE FIELDS, WITH
APPLICATIONS TO PROJECTIVE REED-MULLER CODES

DANIELE BARTOLI
Dipartimento di Matematica ed Informatica
Università degli Studi di Perugia
Via Vanvitelli 1, 06123 Perugia, Italy
and
Department of Mathematics
Ghent University
Krijgslaan 281, 9000 Ghent, Belgium

ADNEN SBOUI
Institut Préparatoire aux Études d’Ingénieurs d’El-Manar
Université Tunis El Manar, Campus universitaire El Manar
B.P.244 El Manar II - 2092 Tunis, Tunisia

LEO STORME
Department of Mathematics
Ghent University
Krijgslaan 281, 9000 Ghent, Belgium

(Communicated by Gary McGuire)

Abstract. We present bounds on the number of points in algebraic curves
and algebraic hypersurfaces in \( \mathbb{P}^n(\mathbb{F}_q) \) of small degree \( d \), depending on the
number of linear components contained in such curves and hypersurfaces. The
obtained results have applications to the weight distribution of the projective
Reed-Muller codes \( \text{PRM}(q,d,n) \) over the finite field \( \mathbb{F}_q \).

1. Introduction

Let \( \theta_n = (q^{n+1} - 1)/(q - 1) \).

Consider the set \( \mathbb{F}_q[X_0, \ldots, X_n]_d^h \) of all homogeneous polynomials of degree \( d \) over
the finite field \( \mathbb{F}_q \) of order \( q \) in the \( n+1 \) variables \( X_0, \ldots, X_n \). Consider also the
\( n \)-dimensional projective space \( \mathbb{P}^n(\mathbb{F}_q) \) over the finite field of order \( q \), and order the
points \( P_0, \ldots, P_{\theta_n - 1} \) of \( \mathbb{P}^n(\mathbb{F}_q) \) in a certain way, where we normalize the coordinates
of the points \( P_i \) by making the leftmost non-zero coordinate equal to one.

Then the \( d \)-th order \( q \)-ary projective Reed-Muller code \( \text{PRM}(q,d,n) \) is the image
of the map

\[
\Phi : \mathbb{F}_q[X_0, \ldots, X_n]_d^h \cup \{0\} \to \mathbb{F}_q^{\theta_n} : F(X_0, \ldots, X_n) \mapsto (F(P_0), \ldots, F(P_{\theta_n - 1})).
\]

2010 Mathematics Subject Classification: Primary: 51E20, 94B05; Secondary: 05B25.

Key words and phrases: Algebraic varieties, quadrics, small weight codewords, intersections,
projective Reed-Muller codes.

The first author acknowledges the support of the European Community under a Marie-Curie
Intra-European Fellowship (FACE project: No. 626511). The second author thanks the Depart-
ment of Mathematics of Ghent University for the financial support given to him.
The parameters of these linear codes are

- Length of PRM($q, d, n$) is $\theta_n$ (see [3]).
- The dimension of PRM($q, d, n$) is $k = \binom{n+d}{n}$, for $d < q$ ([3]), while for $d \leq n(q-1)$,
  $$k = \sum_{t=d \mod q-1}^{n+1} \left( \sum_{j=0}^{t} (-1)^j \binom{n+1}{j} \binom{t-jq+n}{t-jq} \right)$$
  (see [11]).
- Minimum distance of PRM($q, d, n$): $d = (q-s)q^{n-r-1}$, with $d - 1 = r(q - 1) + s, 0 \leq s < q - 1, d \leq n(q - 1)$ (see [11, 12]). For $d < q$, this reduces to $d = q^n - (d - 1)q^{n-1}$.

The non-zero codewords of minimum weight of PRM($q, d, n$) correspond to the algebraic hypersurfaces of degree $d$ having the largest number of points in $\mathbb{P}^n(\mathbb{F}_q)$.

For a more detailed introduction on the problem of determining the maximum number of points in varieties in projective spaces over finite fields, we refer the reader to [1, 2, 4].

For $d \leq q - 1$, it is known that they correspond to the algebraic hypersurfaces which are the union of $d$ hyperplanes passing through a common $(n-2)$-dimensional subspace of $\mathbb{P}^n(\mathbb{F}_q)$ [10].

A. Sboui also determined the second and third weight of PRM($q, d, n$), with $d < q$ and $d > 7$. He proved the following results [9, Theorem 3.10].

**Theorem 1.1.** (1) ([8]) The second weight of the code PRM($q, d, n$) is defined by the algebraic hypersurfaces $A_d^2$ of degree $d$ which are the union of $d$ hyperplanes, $d - 1$ of which meet in a common subspace of dimension $n - 2$ and with the $d$-th hyperplane not passing through this common subspace of dimension $n - 2$.

(2) The third weight of the code PRM($q, d, n$) is defined by the algebraic hypersurfaces $A_d^3$ of degree $d$ which are the union of $d$ hyperplanes, $d - 2$ of which meet in a common subspace $K_1$ of dimension $n - 2$, and where the last two hyperplanes $H_{d-1}$ and $H_d$ meet in a subspace $K_2$, different from $K_1$, such that $K_2$ is contained in exactly one of the $d - 2$ hyperplanes passing through $K_1$.

A. Sboui also determined the configuration of $d$ ($< q$) hyperplanes $H_1, \ldots, H_d$ having the smallest number of points. This configuration is described in the following way: for every $1 \leq i, j \leq d$, $i \neq j$, we have $H_i \cap H_j = K_i^j$, where the spaces $K_i^j$ are $\left( \begin{array}{c} d \\ 2 \end{array} \right)$ subspaces of dimension $n - 2$, all distinct and meeting in a common subspace of dimension $n - 3$.

He furthermore proved that if $q > d(d - 1)/2$, then any algebraic hypersurface of degree $d$, not the union of $d$ hyperplanes, contains fewer points than any algebraic hypersurface which is the union of $d$ hyperplanes.

The consequence of this result is the following: the weight $w_m^i$ given by the minimal hyperplane arrangement is the highest weight of a codeword in PRM($q, d, n$) which can be given by any hyperplane arrangement. Moreover, for $q > d(d - 1)/2$, any algebraic hypersurface of degree $d$ in $\mathbb{P}^n(\mathbb{F}_q)$ containing an absolutely irreducible non-linear factor cannot correspond to a codeword with weight less than $w_m^i$, see [6].

If we suppress the condition $q > d(d - 1)/2$, Rodier and Sboui proved that the third highest number of points, so the third weight $w_3$, given by an $A_3^d$ hyperplane...
hypersurfaces and reed-muller codes 357

arrangement, is also reached by a hypersurface of degree \( d \) composed of \( d - 2 \) hyperplanes and one irreducible quadric hypersurface.

We continue the study of Sboui for \( d < 3 \sqrt{q} \). The goal of this paper is to determine results related to the question how many points an algebraic curve (resp. an algebraic hypersurface) over a finite field can have, depending on its number of lines (resp. hyperplanes).

In this way, also this article contributes to the determination of the weights of the \( d \)-th order \( q \)-ary projective Reed-Muller codes \( \text{PRM}(q,d,n) \), for small order \( d \) \([3,11]\).

2. algebraic plane curves

In this article, we will make regularly use of the Hasse-Weil bound for an absolutely irreducible algebraic curve defined over \( \mathbb{F}_q \).

Theorem 2.1 (Hasse and Weil). Let \( X \) be an absolutely irreducible algebraic curve of degree \( d \) defined over \( \mathbb{F}_q \), then

\[
|\#X - (q+1)| \leq (d-1)(d-2)\sqrt{q}.
\]

The following lemma presents an upper bound on the number of points of an algebraic plane curve, not containing a linear component defined over \( \mathbb{F}_q \). The lower bound \( q > 13 \) in the next lemma arises from the calculations for the upper bounds on the size of the algebraic hypersurface \( \Phi \) in Subsection 4.1. There we use that for \( q > 13, 2\sqrt{q} - 1/2 - q/2 < 0 \).

Lemma 2.2. Let \( C \) be an algebraic plane curve of degree \( d \) not containing a linear factor (line), with \( 2 \leq d \leq \sqrt{q} \) and \( q > 13 \), then \( \#C \leq B \), where

\[
B = \begin{cases} 
\frac{d-1}{2}(q+1) + 2\sqrt{q}, & \text{for } d \text{ odd,} \\
\frac{d}{2}(q+1), & \text{for } d \text{ even.}
\end{cases}
\]

Proof. We proceed by induction on \( d \).

The result is true for \( d = 2 \) and for \( d = 3 \). Assume that the formula is valid for all \( d' \leq d - 1 \).

(1) If \( C \) contains a conic \( C \), then \( C = C' \cup C, \#C = q+1 \), and we use the inequality \( \#C \leq \#C' + \#C \).

(1.1) If the degree \( d \) of \( C \) is even, we apply the induction hypothesis to \( C' \), and then \( \#C \leq \frac{d}{2}(q+1) \).

(1.2) If the degree \( d \) of \( C \) is odd, similarly we get \( \#C \leq \frac{d-3}{2}(q+1) + 2\sqrt{q} + q+1 \), which is equal to \( \frac{d-1}{2}(q+1) + 2\sqrt{q} \).

(2) If \( C \) contains an absolutely irreducible cubic curve \( C \); using the Hasse-Weil bound for an absolutely irreducible cubic curve (\( \#C \leq q+1 + 2\sqrt{q} \)), we prove the result by the same method as the case (1).

(3) We now study the case where \( C \) does not contain a conic nor an absolutely irreducible cubic curve. In this case, \( C \) only contains absolutely irreducible components of degree \( m \), \( 4 \leq m \leq d \). Let \( \ell \) be the number of absolutely irreducible components \( C_i \) of odd degree \( m_i \) in \( C \), and let \( k \) be the number of absolutely irreducible components \( C'_i \) of even degree \( n_i \) in \( C \), then \( C = (\cup_{i=1}^{\ell} C_i) \cup (\cup_{i=1}^{k} C'_i) \), with \( \sum_{i=1}^{\ell} m_i + \sum_{i=1}^{k} n_i = d \).

By the Hasse-Weil bound, we obtain

\[
\#C \leq \sum_{i=1}^{\ell} (q+1 + (m_i - 1)(m_i - 2)\sqrt{q}) + \sum_{i=1}^{k} (q+1 + (n_i - 1)(n_i - 2)\sqrt{q}).
\]
Now the following inequalities are valid, see also [13]:

(i) \( q + 1 + (m - 1)(m - 2)\sqrt{q} < \frac{(q+1)(m-1)}{2} + 2\sqrt{q} \), for \( m \) odd and \( 5 \leq m \leq \frac{\sqrt{q}}{2} \).

(ii) \( q + 1 + (n - 1)(n - 2)\sqrt{q} < \frac{(q+1)n}{2} \), for \( 4 \leq n \leq \frac{\sqrt{q}}{2} + 1 \).

Using (i) and (ii), \( \#C \leq \sum_{i=1}^{\ell} \left( \frac{(q+1)(m-1)}{2} + 2\sqrt{q} \right) + \sum_{i=1}^{k} \left[ \frac{(q+1)n}{2} \right] \), so

\[
\#C \leq \frac{q+1}{2} \left( \sum_{i=1}^{\ell} m_i + \sum_{i=1}^{k} n_i \right) + \ell \left( 2\sqrt{q} - \frac{q+1}{2} \right).
\]

Since \( 2\sqrt{q} - \frac{q+1}{2} < 0 \), we conclude with the two following cases:

a) If \( d \) is even, \( \ell \) is also even. From inequality (1), \( \#C \leq \frac{d(q+1)}{2} + \ell(2\sqrt{q} - \frac{q+1}{2}) \), and so \( \#C \leq \frac{d(q+1)}{2} \).

b) If \( d \) is odd, \( \ell \) is also odd. From inequality (1), \( \#C \leq \frac{(d-1)(q+1)}{2} + 2\sqrt{q} + (\ell - 1)(2\sqrt{q} - \frac{q+1}{2}) \), and so \( \#C \leq \frac{(d-1)(q+1)}{2} + 2\sqrt{q} \).

**Remark 1.** The bounds of Lemma 2.2 are sharp for \( d \) even. For \( d \leq 2(q-1) \) even, it is possible to construct \( d/2 \) pairwise disjoint conics. For \( q \) odd, the conics \( C_c : X_0^2 - c d_1 X_1^2 + c X_2^2 = 0 \), with \( c \in \mathbb{F}_q^* \) and with \( d_1 \) a non-square in \( \mathbb{F}_q \), are \( q-1 \) pairwise disjoint conics in PG(2, \( q \)) [14, Remark 4.2]. Similarly, for \( q \) even, in [15, Remark 3], it is proven that the conics \( C_f : X_0^2 + f d_1 X_2^2 + f X_2^2 + X_1 X_2 = 0 \), are pairwise disjoint when \( f \in \mathbb{F}_q^* \) and \( \text{Tr}(d_1) = 1 \) with \( \text{Tr} \) the trace function from \( \mathbb{F}_q \) to \( \mathbb{F}_2 \).

For \( d \) even, using \( d/2 \) of these pairwise disjoint conics, an algebraic curve of degree \( d \) containing \( (q+1)/2 \) points can be constructed.

**Lemma 2.3.** Let \( C \) be an algebraic plane curve of degree \( d \) in \( \mathbb{P}^2(\mathbb{F}_q) \), such that \( 2 \leq d \leq \frac{\sqrt{q}}{2} \) and \( q > 13 \). If \( C \) contains at most \( r \) different lines defined over \( \mathbb{F}_q \), then \( \#C \leq B_r \), where

\[
B_r = \left\{ \begin{array}{ll}
\left( \frac{d+r}{2} \right) q + \frac{d-r}{2} + 1, & d - r \text{ even,} \\
\left( \frac{d+r-1}{2} \right) q + 2\sqrt{q} + \frac{d-r+1}{2}, & d - r \text{ odd.}
\end{array} \right.
\]

**Proof.** We proceed by induction on \( d \). For \( d = 2 \), the only possible values for \( r \) are 0, 1, or 2, and the result is true.

Assume that the result is true for all the degrees less than or equal to \( d - 1 \). Let \( C \) be an algebraic curve of degree \( d \) containing at most \( r \) lines (\( 0 \leq r \leq d \)).

1) For \( r = 0 \), \( C \) does not contain any line, so from Lemma 2.2: \( \#C \leq B \), where

\[
B = \left\{ \begin{array}{ll}
\frac{d}{2}(q+1), & d \text{ even,} \\
\frac{d-1}{2}(q+1) + 2\sqrt{q}, & d \text{ odd,}
\end{array} \right.
\]

which satisfies \( \#C \leq B_0 \).

2) Assume that \( C \) contains only one line (\( r = 1 \)), then \( C = C' \cup \Delta \), \( \Delta \) a line and \( C' \) an algebraic curve of degree \( d - 1 \). Then \( \#C \leq \#C' + \#\Delta \); we apply Lemma 2.2 to \( C' \), and we obtain that \( \#C \leq B \), where

\[
B = \left\{ \begin{array}{ll}
\frac{d-1}{2}(q+1) + q + 1, & d \text{ odd,} \\
\frac{d-2}{2}(q+1) + 2\sqrt{q} + q + 1, & d \text{ even,}
\end{array} \right.
\]

which is equal to \( B_1 \).

3) Assume that \( C \) contains \( r \) different lines (\( r \geq 2 \)), then \( C = C' \cup \Delta \), with \( \Delta \) a line and \( C' \) an algebraic curve of degree \( d - 1 \). Then \( \#C \leq \#C' + \#\Delta - 1 \), because
Theorem 3.1. \#C \leq \#C' + \#\Delta - \#(C' \cap \Delta) and \#(C' \cap \Delta) \geq 1 considering that C' contains at least one line. So \#C' \leq B, where

\[
B = \begin{cases} 
\frac{d+1+r-1}{2} q + \frac{d-r+1}{2} + 1, & d - r = (d - 1) - (r - 1) \text{ even}, \\
\frac{d+1+r-1}{2} q + \frac{d-r+1}{2} + 2\sqrt{q} + q + 1 - 1, & d - r = (d - 1) - (r - 1) \text{ odd}, 
\end{cases}
\]

which is equal to \( B_r \).

\[\square\]

Theorem 2.4. Let \( C \) be an algebraic plane curve of degree \( d \) over \( \mathbb{F}_q \), such that \( 2 \leq d \leq \frac{\sqrt{q}}{2} \) and \( q > 13 \). If \#C > B_{r-1}, then C contains at least \( r \) different lines defined over \( \mathbb{F}_q \).

Proof. We can deduce the result directly from Lemma 2.3 by rewriting the bound \( B_r \) for \( r - 1 \). Indeed,

\[
B_{r-1} = \begin{cases} 
\frac{d+r-1}{2} q + \frac{d-r+1}{2} + 1, & d - r \text{ odd}, \\
\frac{d+r-2}{2} q + \frac{d-r+1}{2} + 2\sqrt{q}, & d - r \text{ even}, 
\end{cases}
\]

is an upper bound on the size of such algebraic plane curves containing at most \( r - 1 \) different lines defined over \( \mathbb{F}_q \). The result follows from the fact that the lower bound \( B_{r-1} \), for constant \( d \), is an increasing function in \( r \), i.e., \( B_r > B_{r-1} \), independent of the parity of \( d - r \).

\[\square\]

3. The existence of a \( d \)-secant not containing any singular point of the algebraic hypersurface \( \Phi \)

The goal is now to use the preceding upper bounds on the number of points of algebraic curves of degree \( d \) to determine upper bounds on the number of points of algebraic hypersurfaces \( \Phi \) of degree \( d < \sqrt{q} \) in \( \mathbb{P}^n(\mathbb{F}_q) \) containing exactly \( r - 1 \) different linear components \( H_1, \ldots, H_{r-1} \) defined over \( \mathbb{F}_q \). To derive these upper bounds, we need to have a particular \( d \)-secant to this algebraic hypersurface \( \Phi \). The following theorems prove the existence of such a particular \( d \)-secant to \( \Phi \).

Theorem 3.1. Consider an absolutely irreducible algebraic hypersurface \( \Phi : F(X_0, \ldots, X_n) = 0 \) in \( \mathbb{P}^n(\mathbb{F}_q) \), where \( \deg F = d, d < \sqrt{q} \), then through a point of \( \mathbb{P}^n(\mathbb{F}_q) \) not on \( \Phi \), there is a line of \( \mathbb{P}^n(\mathbb{F}_q) \) not containing a singular point of \( \Phi \).

Proof. Assume that \((0, \ldots, 0, 1) \notin \Phi \). Consider the system of equations

\[
\begin{align*}
F(X_0, \ldots, X_n) &= 0, \\
\frac{\partial F}{\partial X_0}(X_0, \ldots, X_n) &= 0, \\
&\vdots \\
\frac{\partial F}{\partial X_n}(X_0, \ldots, X_n) &= 0,
\end{align*}
\]

defining the singular points of \( \Phi \).

For at least one value \( i \in \{0, \ldots, n\} \), \( \frac{\partial F}{\partial X_i}(X_0, \ldots, X_n) \neq 0 \), or else \( F(X_0, \ldots, X_n) \) is a \( p \)-th power, where \( p \) is the characteristic of \( \mathbb{F}_q \), but this contradicts the absolutely irreducibility of \( \Phi \).

Assume that \( \frac{\partial F}{\partial X_0}(X_0, \ldots, X_n) \neq 0 \). To find an upper bound on the number of singular points of \( \Phi \), we calculate the resultant \( R(F, \frac{\partial F}{\partial X_0}) \) of \( F(X_0, \ldots, X_n) \) and \( \frac{\partial F}{\partial X_0}(X_0, \ldots, X_n) \) with respect to \( X_0 \).

Assume that \( R(F, \frac{\partial F}{\partial X_0}) \) has degree \( e \leq d(d - 1) \). Note that the resultant \( R(F, \frac{\partial F}{\partial X_0}) \) cannot be identically zero, or else \( \frac{\partial F}{\partial X_0} \) divides \( F \), which is impossible.
This resultant $R(F, \frac{\partial F}{\partial x_0})$ is a homogeneous polynomial of degree $e < q^{2/3}$ in the variables $(X_0, \ldots, X_{n-1})$, so defines an algebraic hypersurface of degree $e$ in $\mathbb{P}^{n-1}(\mathbb{F}_q)$. Hence, $|R(F, \frac{\partial F}{\partial x_0})| \leq e q^{n-2} + \theta_{n-3}$ [10].

A point $(x_0, \ldots, x_{n-1})$ of the algebraic hypersurface $R(F, \frac{\partial F}{\partial x_0}) = 0$ in $\mathbb{P}^{n-1}(\mathbb{F}_q)$ defines a line $\langle (x_0, \ldots, x_{n-1}, 0), (0, \ldots, 0, 1) \rangle$ in $\mathbb{P}^n(\mathbb{F}_q)$ through $(0, \ldots, 0, 1)$ containing an intersection point of $\Phi : F(X_0, \ldots, X_n) = 0$ and $\frac{\partial F}{\partial X_0}(X_0, \ldots, X_n) = 0$. The singular points of $\Phi$ must belong to these lines.

Now such a line $\langle (x_0, \ldots, x_{n-1}, 0), (0, \ldots, 0, 1) \rangle$ shares at most $d$ points with $\Phi$, so there are at most $d(e q^{n-2} + \theta_{n-3}) \leq d(d - 1)q^{n-2} + \theta_{n-3}$ intersection points of $\Phi : F(X_0, \ldots, X_n) = 0$ with $\frac{\partial F}{\partial X_0}(X_0, \ldots, X_n) = 0$. We are only interested in the singular intersection points of $\Phi$ with $\frac{\partial F}{\partial X_0}(X_0, \ldots, X_n) = 0$, so, by Bézout’s theorem, we can reduce the upper bound to $d(d - 1)q^{n-2} + \theta_{n-3}/2$.

Since $d < \sqrt[3]{q}$, $((d - 1)d^2 q^{n-2} + d \theta_{n-3})/2 < \theta_{n-1}$, so through $(0, \ldots, 0, 1)$, there is a line not containing a singular point of $\Phi$.

**Theorem 3.2.** Consider an algebraic hypersurface $\Phi$ in $\mathbb{P}^n(\mathbb{F}_q)$, where $\deg F = d$, $d < \sqrt[3]{q}$, reducible into the absolutely irreducible components $\Phi_1, \ldots, \Phi_s$ of degrees $d_1, \ldots, d_s$, where $r - 1$ of them are pairwise distinct hyperplanes defined over $\mathbb{F}_q$, then through a point not on $\Phi$, there is a line of $\mathbb{P}^n(\mathbb{F}_q)$ not containing a singular point of any of the components $\Phi_i$, $i = 1, \ldots, s$, and intersecting the $r - 1$ hyperplanes of $\Phi$ in $r - 1$ distinct points.

**Proof.** Here $d_1 + \cdots + d_s = d$, so $\sum_{i=1}^s d_i^3 + 3 \sum_{i,j=1; i \neq j}^s d_i^2 d_j = d^3$, and consequently $\sum_{i=1}^s (d_i - 1) d_i^2 \leq d^3$.

Hence, using the bound of the proof of the preceding theorem, we obtain at most

$$\sum_{i=1}^s (d_i - 1) d_i^2 q^{n-2} + d_i \theta_{n-3} \leq \frac{d^3 q^{n-2} + d \theta_{n-3}}{2}$$

distinct singular points on the absolutely irreducible components of $\Phi$.

We also need to avoid the, at most, $\binom{r - 1}{2} \theta_{n-2} \leq \binom{d}{2} \theta_{n-2}$ different intersection points of the $r - 1$ hyperplanes contained in $\Phi$.

Hence, we need to avoid at most

$$\frac{d^3}{2} q^{n-2} + \frac{d^2}{2} \theta_{n-2} < q^{n-1} + \cdots + q + 1$$

points, so indeed, there is a line through every point not on $\Phi$ containing none of the singular points of any of the components of $\Phi$, and intersecting the $r - 1$ hyperplanes of $\Phi$ in $r - 1$ distinct points. $\square$

4. **Hypersurfaces in $\mathbb{P}^n(\mathbb{F}_q)$**

4.1. **Determination of the upper bounds.** We now wish to obtain an upper bound on the number of points in $\mathbb{P}^n(\mathbb{F}_q)$ belonging to an algebraic hypersurface $\Phi$ of degree $d < \sqrt[3]{q}$, containing $r - 1$ distinct hyperplanes defined over $\mathbb{F}_q$.

By Theorem 3.2, we know that there exists a line $\ell$ in $\mathbb{P}^n(\mathbb{F}_q)$, not contained in $\Phi$, not containing any of the singular points of $\Phi$, and intersecting the $r - 1$ hyperplanes of $\Phi$ in $r - 1$ distinct points. Assume that $P_1, \ldots, P_d$ are the $d$ points of $\ell$ belonging to $\Phi$, counted according to the intersection multiplicities of $\ell$ and $\Phi$ in their intersection points. We can partition these points $P_1, \ldots, P_d$ into the following sets: (1) the set of points $\{P_1, \ldots, P_{r-1}\}$ belonging to the linear components of $\Phi$,
(2) the set of points \( \{ P_r, \ldots, P_m \} \) all belonging to \( \mathbb{P}^n(F_q) \) and belonging to the non-linear components of \( \Phi \), and (3) the set of points \( \{ P_{m+1}, \ldots, P_d \} \) not belonging to \( \mathbb{P}^n(F_q) \), but to an extension of \( \mathbb{P}^n(F_q) \), and again belonging to the non-linear components of \( \Phi \).

Assume that exactly \( m' \leq m \) of the points \( P_1, \ldots, P_m \) are distinct.

We form this partition for the following reasons. First of all, the points \( P_1, \ldots, P_{r-1} \) belong to a linear component, i.e. a hyperplane, of \( \Phi \), hence, they belong to \( \theta_{n-2} \) lines of \( \Phi \) in these hyperplanes. Secondly, the points \( P_i, i \in \{ r, \ldots, m \} \), belong to a non-linear component of \( \Phi \) of degree \( d_i > 1 \). Here, \( P_i \) is a non-singular point of this non-linear component, so has a tangent hyperplane \( \Pi_i \) to this component. All the lines of \( \Phi \) through \( P_i \) lie in this tangent hyperplane \( \Pi_i \). This tangent hyperplane is not contained in \( \Phi \), so contains at most \( d_i q^{n-2} + \theta_{n-3} \) points of \( \Phi \) \cite{10}, so this implies that \( P_i \) belongs to at most \( d_i q^{n-3} + \theta_{n-4} \) lines of \( \Phi \).

Consider this line \( \ell \) and consider all \( \theta_{n-2} \) planes of \( \mathbb{P}^n(F_q) \) through \( \ell \). Denote them by \( H_1, \ldots, H_{\theta_{n-2}} \). Assume that \( H_i \cap \Phi \) contains \( x_i \) lines defined over \( F_q \). It is possible that some of these \( x_i \) lines coincide. This will be no problem for the future arguments. To have a correct upper bound on the number of points of the algebraic hypersurfaces and Reed-Muller codes 361

\[
\# \Phi \leq \sum_{i=1}^{k} \left[ \frac{d + x_i - 1}{2} q + \frac{d - x_i + 1}{2} + 2\sqrt{q} - m' \right] + \sum_{i=k+1}^{\theta_{n-2}} \left[ \frac{d + x_i}{2} q + \frac{d - x_i}{2} + 1 - m' \right] + m',
\]

\[
\leq \frac{q - 1}{2} \left( \sum_{i=1}^{\theta_{n-2}} x_i \right) + \frac{dq}{2} \theta_{n-2} + \frac{d}{2} \theta_{n-2} + \theta_{n-2}
\]

\[
+ m'(1 - \theta_{n-2}) + k \left( 2\sqrt{q} - \frac{1}{2} - \frac{q}{2} \right).
\]

The last term is negative for \( q > 13 \).

We now discuss separately the two cases \( d - r + 1 \) odd and \( d - r + 1 \) even.

**Case 1: assume that \( d - r + 1 \) is odd.**

At least \( k \geq \theta_{n-2} - (m' - r + 1)(d - r + 1)q^{n-3} + \theta_{n-4} \geq \theta_{n-2} - (m - r + 1)(d - r + 1)q^{n-3} + \theta_{n-4} \) planes through \( \ell \) have exactly \( r - 1 \) lines of \( \Phi \), for the points of \( \ell \cap \Phi \) lying on the non-linear components of \( \Phi \) lie in total on at most \( (m' - r + 1)(d - r + 1)q^{n-3} + \theta_{n-4} \leq (m - r + 1)(d - r + 1)q^{n-3} + \theta_{n-4} \) lines of \( \Phi \). For these planes through \( \ell \), the only lines of \( \Phi \) in these planes arise from the intersections of these planes with the linear components of \( \Phi \). So for these planes, \( d - x_i = d - r + 1 \) is odd. Hence,

\[
\# \Phi \leq \left( \frac{q - 1}{2} \right) [(m - r + 1)((d - r + 1)q^{n-3} + \theta_{n-4})] + \left( \frac{q - 1}{2} \right)(r - 1)\theta_{n-2}
\]

\[
+ \frac{d}{2} (\theta_{n-1} - 1) + \frac{d}{2} \theta_{n-2} + \theta_{n-2} + m'(1 - \theta_{n-2})
\]

\[
+ (\theta_{n-2} - (m - r + 1)((d - r + 1)q^{n-3} + \theta_{n-4})) \left( 2\sqrt{q} - \frac{1}{2} - \frac{q}{2} \right).
\]
This reduces to
\[
\#\Phi \leq \left(\frac{r + d - 2}{2}\right) q^{n-1} + ((m - r + 1)(d - r + 1) + d + 2\sqrt{q})q^{n-2} \nonumber \\
- (m + r - 1)(d - r + 1) q^{n-3} + \left(\frac{2m - 2r + 1}{2}\right) \left(\frac{q^{n-3} - 1}{2}\right) - \frac{r - 1}{2} \nonumber \\
+ d\theta_{n-3} - \frac{d}{2} + \theta_{n-3} + \left(\frac{m - r}{2}\right) \theta_{n-4} \nonumber \\
+(q^{n-3} - (m - r + 1)(d - r + 1)q^{n-3} - (m - r)\theta_{n-4}) \left(2\sqrt{q} - \frac{1}{2}\right),
\]
\[
\#\Phi \leq \left(\frac{r + d - 2}{2}\right) q^{n-1} + ((m - r + 1)(d - r + 1) + d + 2\sqrt{q})q^{n-2} \nonumber \\
+ \left(- \frac{(m - r + 1)(d - r + 1)}{2} + \frac{2m - 2r}{2} + 1 + d + 2\sqrt{q}\right) q^{n-3} \nonumber \\
-(m - r + 1)(d - r + 1) \left(2\sqrt{q} - \frac{1}{2}\right) q^{n-3} \nonumber \\
+ \left(d + 1 + \frac{m - r}{2} - (m - r) \left(2\sqrt{q} - \frac{1}{2}\right)\right) \theta_{n-4} - \frac{2m - r + d}{2},
\]
\[
\#\Phi \leq \left(\frac{r + d - 2}{2}\right) q^{n-1} + ((m - r + 1)(d - r + 1) + d + 2\sqrt{q})q^{n-2} \nonumber \\
+ \left(- \frac{(m - r + 1)(d - r + 1)}{2} + \frac{2m - 2r}{2} + 1 + d + 2\sqrt{q}\right) q^{n-3} \nonumber \\
-(m - r + 1)(d - r + 1) \left(2\sqrt{q} - \frac{1}{2}\right) q^{n-3} \nonumber \\
+ (d + 2\sqrt{q} - (m - r + 1)(2\sqrt{q} - 1)) \theta_{n-4} - \frac{2m - r + d}{2}.
\]

Using that \(m \geq r - 1, d \geq r - 1, \) and \(d \geq m,\) and by cancelling the negative terms \(- \frac{(m - r + 1)(d - r + 1)}{2} q^{n-3}, -(m - r + 1)(d - r + 1)(2\sqrt{q} - \frac{1}{2})q^{n-3}, -(m - r + 1)(\sqrt{q} - 1) \theta_{n-4},\) and \(- \frac{2m - r + d}{2},\) this leads to the upper bound
\[
\#\Phi \leq \left(\frac{r + d - 2}{2}\right) q^{n-1} + ((d - r + 1)^2 + d + 2\sqrt{q})q^{n-2} \nonumber \\
+ (2\sqrt{q} + 2d - r + 1)q^{n-3} + (d + 2\sqrt{q}) \theta_{n-4}.
\]

**Case 2: assume that \(d - r + 1\) is even.**

At least \(\theta_{n-2} - (m - r + 1)(d - r + 1)q^{n-3} + \theta_{n-4}\) planes through \(\ell\) have exactly \(r - 1\) lines of \(\Phi,\) so for these planes \(d - x_i = d - r + 1\) is even. Hence,
\[
\#\Phi \leq \left(\frac{q - 1}{2}\right) [(m - r + 1)((d - r + 1)q^{n-3} + \theta_{n-4})] + \left(\frac{q - 1}{2}\right)(r - 1)\theta_{n-2} \nonumber \\
+ \frac{d}{2}(\theta_{n-1} - 1) + \frac{d}{2}\theta_{n-2} + \theta_{n-2} + m'(1 - \theta_{n-2}).
\]

This reduces to
\[
\#\Phi \leq \left(\frac{d + r - 1}{2}\right) q^{n-1} + \left(\frac{(m - r + 1)(d - r + 1)}{2} + d + 1\right) q^{n-2}
\]
Using that $m \geq r - 1$, $d \geq r - 1$, and $d \geq m$, and omitting the negative terms $-\frac{(m-r+1)(d-r+1)}{2}q^{n-3}$ and $-\frac{m-r}{2}$, this leads to the upper bound

$$\#\Phi \leq \left(\frac{d+1-r-2}{2}\right)q^{n-1} + \left(\frac{(d-r+1)^2}{2} + d + 1\right)q^{n-2}$$

$$+ \left(\frac{3d-r+3}{2}\right)q^{n-3} + (d+1)\theta_{n-4} - \frac{r+d}{2},$$

This leads to the following theorem.

**Theorem 4.1.** Let $\Phi$ be an algebraic hypersurface of degree $d < \sqrt[q]{q}$ in $\mathbb{P}^n(\mathbb{F}_q)$, containing exactly $r - 1$ hyperplanes defined over $\mathbb{F}_q$, then

(1)

$$\#\Phi \leq \left(\frac{r+d-2}{2}\right)q^{n-1} + ((d-r+1)^2 + d + 2\sqrt{q})q^{n-2}$$

$$+ (2\sqrt{q} + 2d - r + 1)q^{n-3} + (d + 2\sqrt{q})\theta_{n-4},$$

when $d - r + 1$ is odd,

(2)

$$\#\Phi \leq \left(\frac{d+r-1}{2}\right)q^{n-1} + \left(\frac{(d-r+1)^2}{2} + d + 1\right)q^{n-2}$$

$$+ \left(\frac{3d-r+3}{2}\right)q^{n-3} + (d+1)\theta_{n-4} - \frac{r+d}{2},$$

when $d - r + 1$ is even.

4.2. **Sharpness of the bounds.** We now investigate the sharpness of the bounds. We again split up the discussion into the cases $d - r + 1$ odd or even.

**Case 1:** assume that $d - r + 1$ is odd.

Let $\Phi$ be an algebraic hypersurface consisting of $r - 1$ hyperplanes, $(d-r-2)/2$ absolutely irreducible quadrics, and one absolutely irreducible cubic hypersurface. The hyperplanes contain $\theta_{n-1}$ points, the quadrics and the cubic hypersurface roughly $q^{n-1}$ points [5], so their union contains approximately $(d+r-2)q^{n-1}$ points in $\mathbb{P}^n(\mathbb{F}_q)$.

**Case 2:** assume that $d - r + 1$ is even.

Let $\Phi$ be an algebraic hypersurface consisting of $r - 1$ hyperplanes and $(d - r + 1)/2$ absolutely irreducible quadrics, then their union contains approximately $(d+r-1)q^{n-1}$ points in $\mathbb{P}^n(\mathbb{F}_q)$.

Both cases show that it is possible to explicitly construct an algebraic hypersurface containing $r - 1$ hyperplanes and having a number of points in $\mathbb{P}^n(\mathbb{F}_q)$ which differs from the upper bounds of Theorem 4.1, at most by a term roughly equal to $d^2q^{n-2}$. This difference is small with respect to the number of points of these hypersurfaces and therefore we can say that the bounds of Theorem 4.1 are sharp.
5. Applications to projective Reed-Muller codes

As indicated in the title of this article and in the introduction, the results of Theorem 4.1 give us new results on the small weight codewords of the corresponding $d$-th order $q$-ary projective Reed-Muller code $\text{PRM}(q,d,n)$.

The non-zero codewords of minimum weight of $\text{PRM}(q,d,n)$ correspond to the algebraic hypersurfaces of degree $d$ having the largest number of points in $\mathbb{P}^n(F_q)$. Hence, the preceding results indicate that for $d < \sqrt[3]{q}$, the smaller the weight of a codeword in $\text{PRM}(q,d,n)$, the larger the number of hyperplanes in the algebraic hypersurface of degree $d$ in $\mathbb{P}^n(F_q)$ corresponding to this codeword. In this way, these results extend for $d < \sqrt[3]{q}$ the results of [6].

In [6], the authors proved that if $q > d(d - 1)/2$, then any algebraic hypersurface of degree $d$, not the union of $d$ hyperplanes, contains fewer points than any algebraic hypersurface which is the union of $d$ hyperplanes. The consequence of this result for the corresponding code $\text{PRM}(q,d,n)$ is such that: the weight $w_m$ given by the minimal hyperplane arrangement is the highest weight codeword in $\text{PRM}(q,d,n)$ which can be given by any hyperplane arrangement. Moreover, for $q > d(d - 1)/2$, any hypersurface of degree $d$ containing an irreducible non-linear factor cannot correspond to a weight less than $w_m$.

We managed to extend this result by allowing also non-linear components in our results on the sizes of algebraic hypersurfaces. We state the corresponding results for the weights of the codewords of the $d$-th order $q$-ary projective Reed-Muller code $\text{PRM}(q,d,n)$.

**Theorem 5.1.** Let $c$ be a non-zero codeword of the $d$-th order $q$-ary projective Reed-Muller code $\text{PRM}(q,d,n)$, $d < \sqrt[3]{q}$, of weight

\begin{equation}
\begin{aligned}
    w(c) &< q^n - \left( \frac{r + d - 4}{2} \right) q^{n-1} - \left( (d - r + 1)^2 + d - 1 + 2\sqrt{q} \right) q^{n-2} \\
    &\quad - (2d - r + 2\sqrt{q}) q^{n-3} - (d - 1 + 2\sqrt{q}) \theta_{n-4},
\end{aligned}
\end{equation}

when $d - r + 1$ is odd,

\begin{equation}
\begin{aligned}
    w(c) &< q^n - \left( \frac{d + r - 3}{2} \right) q^{n-1} - \left( \frac{(d - r + 1)^2 + d}{2} \right) q^{n-2} \\
    &\quad - \left( \frac{3d - r + 1}{2} \right) q^{n-3} - d\theta_{n-4} + \frac{r + d}{2},
\end{aligned}
\end{equation}

when $d - r + 1$ is even,

then $c$ corresponds to an algebraic hypersurface of degree $d$ in $\mathbb{P}^n(F_q)$, containing at least $r$ hyperplanes defined over $F_q$.

**References**

[1] A. Couvreur, An upper bound on the number of rational points of arbitrary projective varieties over finite fields, preprint, arXiv:1409.7544v1

[2] S. R. Ghorpade and G. Lachaud, Étale cohomology, Lefschetz theorems and number of points of singular varieties over finite fields, *Moscow Math. J.*, 2 (2002), 589–631.

[3] G. Lachaud, The parameters of projective Reed-Muller codes, *Discrete Math.*, 81 (1990), 217–221.

[4] G. Lachaud and R. Rolland, An overview of the number of points of algebraic sets over finite fields, preprint, arXiv:1405.3027v2
[5] S. Lang and A. Weil, Number of points of varieties in finite fields, *Amer. J. Math.*, 76 (1954), 819–827.

[6] F. Rodier and A. Sboui, Les Arrangements Minimaux et Maximaux d’Hyperplans dans $\mathbb{P}^n(\mathbb{F}_q)$, *C. R. Acad. Sc. Paris Ser. I*, 344 (2007), 287–290.

[7] F. Rodier and A. Sboui, Highest numbers of points of hypersurfaces and generalized Reed-Muller codes, *Finite Fields Appl.*, 14 (2008), 816–822.

[8] A. Sboui, Second highest number of points of hypersurfaces in $\mathbb{P}^n_q$, *Finite Fields Appl.*, 13 (2007), 444–449.

[9] A. Sboui, Special numbers of rational points on hypersurfaces in the $n$-dimensional projective space over a finite field, *Discrete Math.*, 309 (2009), 5048–5059.

[10] J.-P. Serre, Lettre à M. Tsfasman du 24 Juillet 1989, in Journées Arithmétiques de Luminy 17-21 Juillet 1989, *Astérisque*, 198 (1991), 351–353.

[11] A. B. Sørensen, Projective Reed-Muller codes, *IEEE Trans. Inf. Theory*, 37 (1991), 1567–1576.

[12] A. B. Sørensen, On the number of rational points on codimension-1 algebraic sets in $\mathbb{P}^n(\mathbb{F}_q)$, *Discrete Math.*, 135 (1994), 321–334.

[13] L. Storme and J. A. Thas, MDS codes and arcs in $PG(n, q)$ with $q$ even: An improvement of the bounds of Brue, Thas and Blokhuis, *J. Combin. Theory Ser. A*, 62 (1993), 139–154.

[14] L. Storme and H. Van Maldeghem, Cyclic arcs in $PG(2, q)$, *J. Algebraic Combin.*, 3 (1994), 113–128.

[15] L. Storme and H. Van Maldeghem, Arcs fixed by a large cyclic group, *Atti Sem. Mat. Fis. Univ. Modena*, XLIII (1995), 273–280.

Received July 2014; revised June 2015.

E-mail address: daniele.bartoli@dmi.unipg.it

E-mail address: adn.sboui@gmail.com

E-mail address: ls@cage.ugent.be