Reachability via Compositionality in Petri nets

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Abstract. We introduce a novel technique for checking reachability in Petri nets that relies on a recently introduced compositional algebra of nets. We prove that the technique is correct, and discuss our implementation. We report promising experimental results on some well-known examples.

Introduction

We introduce a novel technique for checking reachability in 1-bounded Petri nets. Our approach relies on a structural decomposition of nets, using the algebra of nets with boundaries developed in [1, 12, 18] and the algebra of labelled transition systems (LTS) originally developed in [10]. After explaining the intuitions and some motivating examples, we prove the technique correct, discuss our implementation and report on experimental results.

Many asynchronous systems are regular in their structure, in the sense that they can be considered as a suitable composition of several identical, communicating components. In many such systems, the communication between individual components can be characterised using relatively small (w.r.t. the size of the global state space) amounts of information, and as a consequence, the reachability of a particular global state can be checked locally. The algebra of nets with boundaries allows us to capture precisely how separate “component nets” communicate with each other.

Fig. 1: The net $B_4$ and a “cut” along its transition $t_2$.

To illustrate the ideas that underlie our approach we introduce the simple, well-known $1$-bounded buffer net, $B_n$, illustrated in the left part of Fig. 1. We wish to check whether the “opposite” marking is reachable—that is, the places in the lower row are to be marked and the places in the upper row are to be unmarked. Taking a global view, a simple calculation confirms that the length of the firing sequence necessary to reach the desired marking is quadratic in $n$ (see Fig. 8a). We will, instead, check for reachability locally, component-wise, so imagine that the net is “split” into two nets $N_0$ and $N_1$, sharing the transition $t_2$, as in the right part of Fig. 1.

Remark 1. Observe (1) that $N_0$ and $N_1$ can proceed independently to reach the desired local marking, only “synchronising” on $t_2$ and (2) the “synchronisation policy” is quite simple to describe. Indeed, $N_1$ can fire its local copy of $t_2$ an arbitrary number (including 0) of times during a successful computation; $N_0$ can reach its desired marking after firing its copy of $t_2$ at least twice, after which $t_2$ can be fired an arbitrary additional number of times. These two “policies” are clearly compatible, meaning that the entire net can reach its global desired marking.

\footnote{For example, see [7, Fig. 6].}
To make the above intuitions precise, we recall the algebra of nets introduced in [18]. We will use a non-standard graphical representation of nets, more suited for illustrating the operations of the algebra: $B_4$ is rendered with the alternative graphical notation in the left-most diagram of Fig. 2. Transitions are represented using undirected links and each link can be connected to an arbitrary number of ports. Each place has two ports: one for incoming transitions, illustrated with a triangle pointing into the place, and one for outgoing transitions, illustrated with a triangle pointing out of the place. Thus the pre-set of a transition is the set of places to which it is connected via their outgoing port, and its post-set is the set of places to which it is connected via their incoming port. Transitions can also be connected to boundary ports, which serve as an interface between nets with boundaries. The net $B_4$ can be expressed as the composition $\top; b_1; b_1; b_1; b_1; \bot$; the individual components $\top$, $b_1$ and $\bot$ are illustrated in Fig. 2. The operation ';' that composes two nets along a compatible, common boundary is defined formally in §1.1.

Each component net with boundaries, together with its initial marking and desired local marking, can be translated to a non-deterministic finite automaton (NFA), with states being the reachable markings, and transitions the boundary interactions observed when net transitions fire. The initial state is the initial marking and the final state is the desired marking. We illustrate this translation in Fig. 3. For example, in the translation of $b_1$, state 0 corresponds to the initial marking and state 1 to the desired complementary marking. The labels of transitions are, in general, pairs of binary strings $\alpha$ and $\beta$, written $\alpha/\beta$, representing interaction on the left ($\alpha$) and the right ($\beta$) boundaries. The concept of “interaction on a boundary” is important and we will explain it further below. To guarantee compositionality, we must use an underlying step firing semantics of nets, i.e. a transition in the NFA witnesses the firing of a (possibly empty) set of independent transitions within the component net. Returning to the translation of $b_1$: the 0/0 labelled NFA-transitions in state 0 and 1 witness the possibility of no behaviour (i.e. the empty set of net-transitions firing) with the 0/0 label signifying that no net-transitions connected to either boundary were fired. The NFA-transition $0/0 \rightarrow 1$ witnesses that the right hand side net-transition has fired and produced the desired marking. The fact that the fired transition is connected to the port on the right boundary is recorded by 1 in the transition label. The remaining NFA-transition is symmetric.

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Fig. 2: Obtaining $B_4$ as a composition of nets $\top$, $b_1$ and $\bot$.

Fig. 3: Translation to NFAs.

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All illustrations of automata were generated with GraphViz [http://www.graphviz.org](http://www.graphviz.org). For space-efficiency, transitions are annotated with sets: $\{x, y\}$, representing the existence of two transitions, labelled respectively $x$ and $y$. We use $*$ in the labels as shorthand for any choice of 0 and 1.
The principle of compositionality, proved in Theorem 4, is illustrated in Fig. 4: given two $b_1$ nets, we can obtain the NFA representing their (composite) behaviour in two ways: 1) compose two $b_1$ nets to form the net $b_1 \cdot b_1$, and then generate its NFA, or equivalently, 2) generate the two (identical) NFAs for each $b_1$ and compose them, using a variant of the product construction. Compositionality ensures that the diagram commutes, in other words, the global behaviour of the composition of the two nets is completely determined by the behaviour of the individual nets, when synchronised along their common boundary.

![Compositionality diagram](image)

**Fig. 4: Compositionality at work.**

The NFA generated for $b_n = b_1 \cdot \ldots \cdot b_1$ ($b_1$ composed $n$ times) has $2^n$ states, thus directly computing the automaton for $b_n$ is feasible only for small $n$. Fortunately, to generate a correct NFA of the composite net, it is sufficient to capture how each component net must interact on its boundaries in order to reach its local desired marking—its “synchronisation policy”. To do this, we close the NFA with respect to internal ($\epsilon$-) moves—those transitions labelled solely with 0s, signifying no interaction at the boundaries—to obtain an automaton with the same states, but with transitions being paths $a(0/0)^* \cdot a/b(0/0)^*b$. We then minimise the new NFA, obtaining a deterministic automaton (DFA), with an “error” state that is reached whenever an illegal (i.e. not in the behaviour of the underlying net) interaction is observed on the boundaries. This DFA minimally represents the entire behaviour (assuming that an observer may only observe traces) of the net, w.r.t. interactions on its boundaries.

Note that the states of the NFA obtained from a net are 1-1 with the reachable markings of the underlying net; in general, this is not the case after $\epsilon$-closure and minimisation: the states of the minimal DFA merely capture the “protocol” the net must follow when interacting with its environment, in order to arrive at the desired marking. Indeed, for $b_n$, the resulting minimal DFA has $n + 2$ states. Of course, computing the minimisation of an NFA can be very expensive—in the worse case, triple exponential in the number of places of the original net—our strategy is thus roughly to decompose nets as far as possible (thereby only minimising small NFAs) and take advantage of any regular, repetitive structure in the net, via memoisation. As discussed, compositionality guarantees correctness—the fact that the square in Fig. 5, illustrating the process for $B_4$, commutes is a consequence of Theorems 7 and 9.

The applicability of our approach depends on finding “good” decompositions of nets. For $B_n \overset{\text{def}}{=} \top : b_n : \bot$, there are many potential decompositions: the optimal is the 1st decomposition in Fig. 6, which corresponds to the algebra term $(\top : (b_1 ; \ldots ; (b_1 : \bot) \ldots \))$. Indeed, the composition of $b_1$ and $\bot$ minimises to the trivial accepting automaton; Fig. 7 contains illustrative translation steps of the different decompositions of $B_4$. In (i) the composition of the automaton for $b_1$ is composed with the automaton for $\bot$: after minimisation we again obtain the automaton for $\bot$. Thus the

\[ (a, b) \overset{\alpha/\beta}{\rightarrow} (a', b') \text{ iff } \exists \gamma. \ a \overset{\alpha/\gamma}{\rightarrow} a' \land b \overset{\gamma/\beta}{\rightarrow} b'. \]

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All experiments were run on an Intel i7-2600 3.40GHz CPU, 16GB of RAM, running 64-bit Ubuntu Linux.
procedure reaches a fixed point after the first step, as illustrated in (ii). This fact formally captures the intuition about $N_1$ given in Remark 1. For this decomposition, memoisation guarantees that the composition and minimisation is performed only once. In particular, this means that checking reachability for $B_n$, given this decomposition, is linear in $n$. However, other decompositions do not lead to such good performance. In particular, consider the 2nd decomposition of Fig. 6 here, memoisation does not help (we obtain a different NFA composition after each step) and we must perform minimisation after each composition, as illustrated in steps (iii) and (iv) of Fig. 6

Our automated approach to deconstructing $B_n$ (discussed in §2.1) produces the 3rd (balanced) decomposition of Fig. 6. In this particular case we decompose by identifying a transition that connects two components of similar size. This decomposition, while not optimal, allows frequent use of memoisation, reducing the amount of computation. A table of running times for the construction of a minimal DFA for $B_n$, following the three decompositions of Fig. 6, is given in Fig. 8a.

We have illustrated how the operation ‘;’ allows decomposition of the net $B_n$ in order to exhibit its the regular structure. We will briefly consider a second example that illustrates the use of the second operation of the algebra, ‘⊗’. Consider the net in Fig. 9, where we want to check whether all the places can be marked; N.B this net is not 1-safe, but 1-boundedness means that a transition is blocked if there is a token present in its post-set. Our automated procedure constructs the decomposition illustrated in the right part of Fig. 9. In Fig. 13 we illustrate the steps involved in calculating the minimal DFA for $T_3$, and give a table of experimental results in Fig. 8b.

Structure of the paper. In §1 we study the foundations of our technique and prove it correct. In §2 we discuss our implementation and give additional experimental results. Connections with related work are in §3 and we conclude with directions for future research in §4. Due to space constraints, proofs and non-essential figures have been moved to the appendix.
Fig. 7: Translation of the decompositions in Fig. 6. (i),(ii) initial steps using the right decomposition; (iii), (iv) initial steps using the left decomposition; (v) final step using the balanced decomposition of $B_4$.

| n  | min # firing sequence | Time [s] | right | left | balanced |
|----|-----------------------|----------|-------|------|----------|
| 16 | 136 0.000 0.020 0.008 |
| 32 | 528 0.000 0.140 0.024 |
| 64 | 2080 0.000 1.108 0.172 |
| 128| 8256 0.000 12.597 2.954 |
| 256| 32896 0.000 - 74.737 |
| 65536| 2147516416 0.228 - |

Fig. 8: NFA construction times for $B_n$ and $T_n$.

(a) Time to construct minimal DFA for $B_n$ with the three decompositions illustrated in Fig. 6

(b) Time to construct minimal DFA for $T_n$, using the decomposition described in Fig. 9

Fig. 9: The net $T_3$, in traditional and alternative graphical notation, and its decomposition.
1 Nets with boundaries

In this section we give the theoretical underpinnings of our technique, harnessing the compositionality of the algebra of nets with boundaries in order to prove its correctness.

Notational conventions. For $n \in \mathbb{N}$ let $n = \{0, 1, \ldots, n-1\}$. We write $2^X$ for the powerset of $X$. We write $X + Y$ for the set $\{(x, 0) \mid x \in X\} \cup \{(y, 1) \mid y \in Y\}$. Given $U \subseteq 2^X$ and $V \subseteq 2^Y$ we write $U \uplus V = \{U \uplus V \mid U \in U, V \in V\} \subseteq 2^{X+Y}$. We identify binary strings $\alpha = \alpha_0\alpha_1 \ldots \alpha_{k-1}$ of length $k$ with subsets of $k$ in the obvious way: $\alpha_i = 1$ iff $i \in \alpha$.

Definition 2. A net with boundaries $N : k \to l$ is $(P, T, k, l, \varnothing-, \varnothing, \bullet-, \bullet)$ where:

- $P$ is the set of places, $T$ is the set of transitions
- $k, l \in \mathbb{N}$ are, respectively, the left and the right boundaries
- $\varnothing-, \varnothing : T \to 2^k$ give, respectively, the pre- and post-sets of each transition
- $\bullet- : T \to 2^l$ and $\bullet : T \to 2^l$ connect each transition to, resp., the left and the right boundary.

Additionally, we assume that for any $t \neq t' \in T$, $\bullet t \cap \bullet t' = \varnothing$ and $\bullet t \cap \bullet t' = \varnothing$. Ordinary Petri nets can be considered as nets $N : 0 \to 0$ with no boundaries.

We must use step semantics of nets instead of the more common interleaving semantics to guarantee compositionality; we will illustrate this in Remark 3. Let $^\circ t \overset{\text{def}}{=} t \cup t$. Transitions $t \neq t' \in T$ are said to be independent when $^\circ t \cap ^\circ t' = \varnothing$. A set $U \subseteq T$ is said to be mutually independent (MI) when for all $u \neq u' \in U$, $u$ and $u'$ are independent. For sets of transitions $U \subseteq T$ we will abuse notation and write $^\circ U = \bigcup_{u \in U} ^\circ u$, and similarly for $U^\circ, \bullet U$ and $U^\bullet$.

Each net with boundaries $N : k \to l$ determines an LTS where each transitions witness the step semantics of the underlying net, originally described by Katis et al [2]. For the 1-bounded case, the labels are pairs of binary strings of length $k$ and $l$, respectively. The states are markings of $N$, denoted by $[N]_X$, where $X \subseteq \mathcal{P}$. The transition relation is defined:

$[N]_X \xrightarrow{\alpha/\beta}[N]_{X'} \iff \exists \text{MI } U \subseteq T, ^\circ U \subseteq X, U^\circ \cap X = \varnothing, X' = (X \setminus ^\circ U) \cup U^\circ, \bullet U = \alpha, U^\bullet = \beta$

1.1 Composition of nets with boundaries

Suppose that $N : k \to l$ and $M : l \to m$ are nets with boundaries. A synchronisation is a pair $(U, V)$ where $U \subseteq T_N$ and $V \subseteq T_M$ are MI sets of transitions, with $U^\bullet = \bullet V$. Given synchronisations $(U, V)$ and $(U', V')$, we say $(U, V) \subseteq (U', V')$ exactly when $U \subseteq U'$ and $V \subseteq V'$. The trivial synchronisation is $(\varnothing, \varnothing)$. A synchronisation $(U, V)$ is said to be minimal when it is non trivial and, for all synchronisations $(U', V')$, if $(U', V') \subseteq (U, V)$ then $(U', V')$ is trivial. The set of minimal synchronisations of $N$ and $M$ is denoted $\text{sync}_{\text{min}}(N, M)$. The composed net $N : M : k \to m$ has:

- $P_N + P_M$ as its set of places,
- $\text{sync}_{\text{min}}(M, N)$ as its set of transitions. Given $(U, V) \in \text{sync}_{\text{min}}(M, N)$ we let $^\circ (U, V) \overset{\text{def}}{=} ^\circ U \uplus ^\circ V, (U, V)^\circ \overset{\text{def}}{=} ^\circ U \uplus (U, V)^\circ \overset{\text{def}}{=} ^\circ U$ and $(U, V)^\bullet \overset{\text{def}}{=} (U, V)^\circ$.

Examples of compositions of the net $B_n : 0 \to 0$ are given in Figs. 2 and 6. Another example is given in Fig. 10 with the resulting transition arising from the minimal synchronisation $\{(t_1, t_2), \{t_3\}\}$.

Remark 3. The example in Fig. 10 illustrates the necessity for step semantics in order for compositionality to hold. Indeed, in the composition $N_0; N_1$ we have the transition $[N_0; N_1]_{\{0\}} \overset{\text{def}}{=} [N_0; N_1]_{\{1\}}$ that witnesses the firing of its transition. This transition decomposes into $[N_0]_{\{0\}} \overset{\text{def}}{=} [N_0]_{\{1\}}$ and $[N_1]_{\{0\}} \overset{\text{def}}{=} [N_1]_{\{1\}}$. The first of these requires the simultaneous firing of $t_1$ and $t_2$ in $N_0$; thus if we had considered interleaving semantics then compositionality would fail in this example.

\(^5\) That is, at most one transition can be connected to any place on the boundary. This assumption allows us to simplify the definition of composition of nets; for the more general case see [2].
The next result is a special case of Theorem 3.6, where a more general algebra of nets is considered. We will rely on this to prove the correctness of our technique in Theorems 7 and 9.

**Theorem 4 (Compositionality).** Suppose that \( N : k \to l \) and \( M : l \to m \) are nets with boundaries. The following holds for all \( X, X' \subseteq P_N, Y, Y' \subseteq P_M, \alpha \in \{0, 1\}^k \) and \( \beta \in \{0, 1\}^m \):

\[
[N ; M]_{X \otimes Y} \xrightarrow{\alpha / \beta} \big[ N ; M \big]_{X' \otimes Y'} \iff \exists \gamma \in \{0, 1\}^l. [N]_X \xrightarrow{\alpha / \gamma} [N]_X \land [M]_Y \xrightarrow{\gamma / \beta} [M]_Y.
\]

\[\square\]

The conclusion of Theorem 4 implies that, for instance, bisimilarity is a congruence w.r.t. \( \equiv \). For the purposes of reachability checking, traces are sufficient.

**Corollary 5.** There exists a trace \([N ; M]_{X \otimes Y} \xrightarrow{\alpha_1 / \beta_1} \ldots \xrightarrow{\alpha_r / \beta_r} [N ; M]_{X' \otimes Y'}\) iff there exist traces \([N]_X \xrightarrow{\alpha_1 / \gamma_1} \ldots \xrightarrow{\alpha_r / \gamma_r} [N]_X\) and \([M]_Y \xrightarrow{\gamma_1 / \beta_1} \ldots \xrightarrow{\gamma_r / \beta_r} [M]_Y\).

\[\square\]

In particular, to check for reachability in a composed net, it suffices to find computations in the components that agree on their shared boundary.

The other operation on nets with boundaries is \( \otimes \), which can be understood as a parallel composition of nets. Given \( N : k \to l \) and \( M : m \to n \), \( M \otimes N : k + m \to l + n \) has:

- \( P_N + P_M \) as its set of places,
- \( T_N + T_M \) as its set of transitions. \( \circ (t, 0) \) \( \overset{df}{=} \{ (p, 0) \mid p \in \circ t \} \), \( \circ (t, 1) \) \( \overset{df}{=} \{ (p, 1) \mid p \in \circ t \} \), and similarly for \( (t, 0)^\circ \) and \( (t, 1)^\circ \). Instead \( t^* (t, 0) = t^* \) while \( t^* (t, 1) = \{ k + i \mid i \in t^* \} \); similarly \( (t, 0)^* = t^* \) and \( (t, 1)^* = \{ l + i \mid i \in t^* \} \).

Compositionality also holds w.r.t. \( \otimes \): \([M \otimes N]_{X + Y} \xrightarrow{\alpha_1 / \beta_1} \ldots \xrightarrow{\alpha_r / \beta_r} [M \otimes N]_{X' + Y'}\) iff \([M]_X \xrightarrow{\alpha_1 / \beta_1} [M]_X\) and \([N]_Y \xrightarrow{\beta_1 / \beta_2} [N]_Y\). Due to space constraints we omit the details here; they are straightforward as there is no interaction between the two nets.

### 1.2 From nets with boundaries to NFAs

By an NFA with boundaries \( A : k \to l \) we mean an NFA \( A \) with set of labels \( \{0, 1\}^k \times \{0, 1\}^l \), written \( \alpha / \beta \), where \( \alpha \in \{0, 1\}^k \) and \( \beta \in \{0, 1\}^l \). Given NFA with boundaries \( A : k \to l \) and \( B : l \to m \), the NFA with boundaries \( A \otimes B : k \to m \) is obtained by a variant of the product construction where \( (x, y) \xrightarrow{\alpha / \beta} (x', y') \) if there exists \( \gamma \in \{0, 1\}^l \) such that \( x \xrightarrow{\alpha / \gamma} x' \) and \( y \xrightarrow{\beta / \gamma} y' \). Given NFA with boundaries \( A : k \to l \) and \( B : m \to n \), the NFA with boundaries \( A \otimes B : k + m \to l + n \) is obtained via another variant of the product construction: here \( (x, y) \xrightarrow{\alpha_1 / \beta_1} (x', y') \) if \( x \xrightarrow{\alpha_1 / \beta_1} x' \) and \( y \xrightarrow{\beta_1 / \beta_2} y' \). The algebra of automata with boundaries described above is an instance of \( \text{Span(Graph)} \).

Given a net with boundaries \( N : k \to l \), and non-empty sets \( X, Y \subseteq 2^{2^P_N} \) of, respectively, initial and final markings, we can consider its labelled transition system as an NFA, written \( \text{NFA}(N, X, Y) \), that has initial states \( X \) and final states \( Y \). If \( N : k \to l \) does not have any places then \( \text{NFA}(N, \{ \emptyset \}, \{ \emptyset \}) \) has exactly one state, which is an accept state (see NFA for \( T, \perp \) in Fig. 3). The following is immediate.

**Proposition 6.** Given \( N : k \to l \), initial and final markings \( X, Y \), a marking in \( Y \) is reachable from a marking in \( X \) iff \( L(\text{NFA}(N, X, Y)) \neq \emptyset \). \( \square \)
We also have the following as an immediate consequence of Theorem 4:

\[ \text{NFA}(N : M : k \to m, X \cup X', Y \cup Y') \cong (\text{NFA}(N : k \to l, X, Y)) ; (\text{NFA}(M : l \to m, X', Y')) \]

and in particular the two automata accept the same language.

1.3 Weak closure and minimisation

Hiding internal computations in individual component nets is crucial for the performance of our technique. The procedure is akin to the \( \tau \)-reflexive-transitive closure of an LTS \( L' \), which yields an LTS \( L' \) on which bisimilarity agrees with weak-bisimilarity on \( L \), in the sense of Milner [13].

Let \( \epsilon_{k,l} = 0^k/0^l \). Sometimes we will write simply \( \epsilon \) when \( k \) and \( l \) are clear from the context. Notice that given any net \( N : k \to l \), for each marking \( X \) there is a transition \([N]_X \epsilon_{k,l} [N]_X\) that arises from firing the empty set of net-transitions. In general, transitions \([N]_X \epsilon_{k,l} [N]_X\) witness the firing of “internal” net-transitions in \( N \), i.e. those that are not connected to any boundary port.

The weak transition system induced by \( N : k \to l \) has transitions:

\[ [N]_X \xrightarrow{\alpha/\beta} [N]_X', \quad \Leftrightarrow \quad \exists X'' : [N]_X (\epsilon_{k,l})^* [N]_{X''}, \quad [N]_{X''} \xrightarrow{\alpha/\beta} [N]_{X''}, \quad [N]_{X''} (\epsilon_{k,l})^* [N]_X, \quad (1) \]

Note that the above notion of weak transition differs from that considered in [2] but is close to the weak transitions of [17].

**Theorem 7 (Compositionality w.r.t. weak semantics).** Suppose that \( N : k \to l \) and \( M : l \to m \) are nets with boundaries. Then for all \( X, X' \subseteq P_N, Y, Y' \subseteq P_M, \alpha \in \{0,1\}^k, \beta \in \{0,1\}^m \):

(i) if \( [N]_X \alpha/\beta [M]_X' \xrightarrow{\gamma} [N]_Y \), then \( \exists p, q \in \mathbb{N}, \gamma_i, \gamma_j \in \{0,1\}^l \) for \( 1 \leq i \leq p \) and \( 1 \leq j \leq q \)

\[ [N]_X \xrightarrow{0^k/\gamma_j} \cdots \xrightarrow{0^k/\gamma_2} \xrightarrow{0^k/\gamma_1} \cdots \xrightarrow{0^k/\gamma_1} [N]_{X'}, \quad \text{and} \quad [M]_Y \gamma_j^{\beta_2} \cdots \gamma_2^{\beta_1} \gamma_1^0 \xrightarrow{\gamma} [M]_{Y'}. \]

(ii) if \( [N]_X \alpha/\beta [N]_X \), then \( [N]_X \xrightarrow{\gamma} [N]_Y \), for some \( \gamma \in \{0,1\}^l \) then \( [N]_X \alpha/\beta [M]_X' \xrightarrow{\gamma} [N]_Y [M]_{X'}. \)

Given an NFA with boundaries \( A : k \to l \), let \( \epsilon \text{min}(A) : k \to l \) denote the DFA obtained by \( \epsilon_{k,l} \)-closure and minimisation.

**Remark 8.** Recall that any ordinary net \( N \) can be considered as a net with boundaries \( N : 0 \to 0 \). Now \( \epsilon \text{min}(\text{NFA}(N, X, Y)) : 0 \to 0 \) is one of two DFAs: the DFA with one accept state (if a marking in \( Y \) is reachable from some marking in \( X \)) and the DFA with one non-accept state (if no markings in \( Y \) are reachable from any marking in \( X \)).

Given an ordinary Petri net \( N \), initial markings \( X \) and final markings \( Y \), a simple but extremely inefficient way of checking the reachability of a marking is thus to directly compute \( \epsilon \text{min}(\text{NFA}(N, X, Y)) \) and check whether the single state in the resulting DFA is an accept state. Our technique for checking reachability is based on computing this DFA using a structural decomposition of \( N \), which, when combined with memoisation, can result in fast execution times.

1.4 Correctness

Here we give a formal account of our technique and prove it correct, using the previous results in this section. A *wiring expression* is a syntactic term formed from the following grammar

\[ T ::= x \mid T ; T \mid T \otimes T \]

where the leaves \( x \) are variables. A *variable assignment* \( \mathcal{V} \) is a map that takes variables to nets with boundaries. Given a pair \((t, \mathcal{V})\) of a wiring expression \( t \) and variable assignment \( \mathcal{V} \), its semantics \( [[t]]_\mathcal{V} \) is a net with boundaries, defined recursively in the obvious way: \( [[x]]_\mathcal{V} \overset{\text{def}}{=} \mathcal{V}(x) \),
The function \(\text{trans}(\cdot)\) is the formalisation of our approach, taking a wiring decomposition, together with initial and final markings to a minimal DFA. Sets of markings of the leaf nets given by \(\mathcal{I}\) and \(\mathcal{F}\) can be combined to form a set of markings \(\text{mrk}(t)_x\) of \(\llbracket t \rrbracket_v\) in an obvious way: \(\text{mrk}(x)_x \equiv \mathcal{I}(x)\), \(\text{mrk}(t ; t')_I \equiv \text{mrk}(t)_I \cup \text{mrk}(t')_I\), \(\text{mrk}(t \otimes t')_I \equiv \text{mrk}(t)_I \ominus \text{mrk}(t')_I\) (and similarly for \(\mathcal{F}\)).

**Theorem 9 (Correctness).** Suppose \((t, V)\) is a wiring decomposition of \(N : k \to l, I\) initial markings and \(F\) final markings. Then \(\text{trans}(t)_{(V, I, F)} \equiv \text{def} \cdot \text{min}(\text{NFA}(\llbracket V \rrbracket, I(x), F(x)))\). \(\Box\)

An example application of Theorem 9 is the commutativity of the diagram in Fig. 1.

Note that we have not discussed how to obtain a wiring decomposition, starting from a net \(N : k \to l\). As demonstrated in Fig. 8a, different decompositions result in markedly different performance. Our automated procedure for obtaining a decomposition is described in 2.1.

# 2 Implementation and experimental results

Our implementation has been written in Haskell, and is available for download.\(^6\) The high level view of our algorithm is:

1. As input, take an ordinary marked net \(N\) (considered as a net with boundaries \(N : 0 \to 0\)) and a target marking, given place-wise, to be checked for reachability. Concretely, each place is labelled with ‘Yes,’ (token must be present) ‘No’ (token must be absent) or ‘Don’t care.’
2. Using an automatic decomposition procedure (described in §2.1), we decompose the net, obtaining a wiring decomposition (as introduced in §1.4) enhanced with additional information to enable memoisation.
3. Taking advantage of memoisation—to eliminate duplicate computations—traverse the wiring decomposition tree to compute \(\text{trans}(\cdot)\):
   (a) At leaves, we have (typically, small) nets with boundaries, and the local desired marking. We use the procedure described in §1.2 to generate the NFA that corresponds to the net and apply \(\epsilon\)-closure and minimisation, described in §2.2.
   (b) At a composition node, we generate the NFAs corresponding to each sub-tree, and compose them using the variant of product-construction discussed in §1 finally \(\epsilon\)-closing and minimising the resulting NFA.
   (c) At a tensor node, we generate the NFAs corresponding to each sub-tree, combine them using the standard product construction on NFAs, and perform minimisation.

The experimental results given in Figs. 8 and 12b are given for pre-constructed decompositions, that is, only step 3 of the algorithm is performed. The results in Fig. 12a were obtained using the implementation of the full algorithm.

\(^6\) [http://users.ecs.soton.ac.uk/os1v07/ICALP13](http://users.ecs.soton.ac.uk/os1v07/ICALP13)
2.1 Decomposer

Our net decomposition algorithm attempts to find decompositions via two simple approaches: first we look for a net-transition that, when removed, results in two disconnected nets. If many such transitions exist then we take the one that results in the most balanced (in number of places) decomposition. An example is the balanced decomposition in Fig. 6. If such a transition cannot be found, we look for a place that, once removed, results in two disconnected nets. This results in a ';' node (that results from removing the place) followed by a '⊗' node (that composes the two disconnected nets). Again, if many such places exist, we look for one that results in the most balanced decomposition. An example of this decomposition strategy is the decomposition in Fig. 9. Both searches are quadratic in the size of the net. If neither a suitable transition nor place is found, we remove a place that results in the smallest boundary, after decomposition. The time taken to decompose the net $T_n$ is given in Fig. 12a; in this example the time to decompose the net dominates. Note that, given a net, a decomposition must be computed (or given as input) only once, whence different various initial markings and desired markings can be considered.

2.2 NFA $\epsilon$-closure and minimisation

Our approach relies on ignoring internal computations to reduce the state space to be explored. To produce minimal DFAs for an input NFA, we apply epsilon closure, and minimisation, as detailed in §1.3. We perform epsilon closure through a variant of the subset-construction on NFAs, which constructs the NFA of sets of states reachable through $\epsilon$- or standard transitions, starting from the $\epsilon$-closure of the initial states of the input NFA. To perform minimisation we employ the well-known algorithm of Brzozowski [3].

A notable implementation detail is that we use a variant of Reduced Ordered Binary Decision Diagrams (ROBDD, commonly written as BDD) to encode the transition relation of the NFA—the labels of our transitions are binary strings and thus any state $x \in X$ gives rise to a function $\{0,1\}^{k+l} \rightarrow \mathcal{P}(X)$. Traditionally, BDDs are used to provide compact representations for functions $\{0,1\}^n \rightarrow \{0,1\}$, but we found it a straightforward exercise to generalise from the boolean algebra of the booleans to the boolean algebra of subsets.

2.3 Experimental results and discussion

In addition to the results in Fig. 8 we considered a standard net encoding of the dining philosopher problem. Given the nets in Fig. 11, let $Ph_{Row_1} \overset{def}{=} (ph; fk)$, $Ph_{Row_{k+1}} \overset{def}{=} (ph; (fk; Ph_{Row_k}))$. Then a table of $n$ dining philosophers can be obtained as:

$$Ph_n \overset{def}{=} d_3 : ((i_3 \otimes Ph_{Row_n}) ; e_3) \quad (\text{see Fig. 14}).$$

Running times, when checking for deadlock in $Ph_n$, are given in Fig. 12b. The slow growth w.r.t. $n$ illustrates the fact that our technique works well when a fixed point is quickly reached when traversing a wiring decomposition, for example, the right decomposition of $B_n$ in Fig. 6 reaches a fixed point after one ';' node in the wiring decomposition. The fixed point for (2) is reached when calculating $Ph_{Row_3}$: the resulting minimal DFA has 10 states, as shown in Fig. 15. Intuitively, this means that while one can distinguish between 1, 2 and $\geq 3$ philosophers via interaction on the boundary, all $Ph_{Row_k}$ reduce to the same minimal DFA for $k \geq 3$. Our procedure takes advantage of this: memoisation of compositions means that we minimise only once.

Many nets are not amenable to efficient decomposition and are unsuitable for our technique. For instance, our implementation performs poorly when input nets are cliques, nets where every place is connected to every other by a transition, or in general, on “densely connected” nets. One reason why our technique is infeasible for such nets is because two factors influence the size of the generated NFA from a net $N : k \rightarrow l$: (i) the number of places—if $N$ has $n$ places, this can translate to potentially $2^n$ NFA-states, and (ii) the size of the net boundaries, since it implies an alphabet of size up to $2^{(k+l)}$. In fact, even with hand constructed decompositions, our implementation fails to terminate even for very small cliques due to large boundaries in any decomposition.
Fig. 11: Component nets of philosopher decomposition.

| n    | deconstruction [s] | NFA generation [s] | n    | time [s] |
|------|--------------------|--------------------|------|----------|
| 4    | 0.052              | 0.008              | 1    | 2.072    |
| 5    | 0.240              | 0.008              | 4    | 3.844    |
| 6    | 1.108              | 0.004              | 16   | 3.924    |
| 7    | 5.104              | 0.008              | 64   | 3.920    |
| 8    | 23.261             | 0.008              | 128  | 3.908    |
| 9    | 103.106            | 0.012              | 256  | 3.908    |
| 10   | 451.628            | 0.012              | 1024 | 3.684    |

(a) Time to deconstruct $T_n$ (as per Fig. 9) and generate the minimal DFA.
(b) Time to generate minimal DFA for $P_{hn}$, defined in (2).

Fig. 12: Example NFA construction times for $B_n$ and $T_n$.

3 Related work

Algebras of acts and automata. The algebra of automata with boundaries used in this paper is an instance of the algebra of Span(Graph) [10], developed by R.F.C. Walters and collaborators: in fact, a translation from nets to this algebra was already present in [9]. The goal of the more recent work [1,2,18] was to lift this algebra to the level of nets in a compositional way, study the resulting behavioural equivalences and explore connections with process algebra. A theme of our work is to ignore state and focus on external interactions: here we were inspired by the ideas of Milner [13]. Conceptually related approaches in semantics of programming languages include [8,15].

Reachability in bounded, finite state Petri nets is a widely-studied problem and there are several well-known approaches to mitigating the impact of state-explosion (it follows from [4] that the problem is PSPACE-complete.) Due to space constraints we are able to offer only cursory overviews and comparisons of techniques that are most related to our approach. A well-known technique is partial order reduction: in a seminal paper, McMillan [12] used the unfolding construction [14] in order to analyse reachability in Petri nets by generating finite complete prefixes, that is, initial parts of unfoldings that suffice for reachability. The algorithm to compute the finite complete prefix was later improved [7,11]. Unfoldings (and finite complete prefixes) carry more information about the computations of nets than merely reachability, for instance, allowing LTL model checking [5]. For an overview of the extensive field see [6]. A finite complete prefix must be constructed prior to a reachability analysis, analogously to our construction of a wiring decomposition prior to translation. Because of the different nature of the two approaches, it is difficult to offer a thorough analysis of the relative performance of the two approaches: on some of the examples we have considered the performance of our implementation is competitive (compare Fig. 8a with [7, Table 1].)

Another technique, known as symmetry reduction [16,19], exploits symmetries in the state space: the goal is, roughly, to build a reduced reachability graph in order to visit only one representative from each orbit. Our use of memoisation is similar in spirit to symmetry reduction, since we only need to translate any particular wiring decomposition once.

In experiments ($B_n$, $T_n$, $P_{hn}$ and others) our implementation often performs well in identifying unreachable configurations; this is because in many systems the reasons for a configuration being
unreachable are “local”. Here our approach contrasts with techniques such as unfolding or symmetry reduction where (efficient representations of) explicit reachability graphs are constructed.

4 Conclusions and future work

We have introduced a new technique for reachability in bounded Petri nets, based on (i) structural decomposition using a recently developed compositional algebra and (ii) avoiding state explosion by focusing only on interactions between component nets, forgetting internal state. Our technique depends on finding efficient decompositions and works best when the computation reaches fixpoints w.r.t. interactions on boundaries in composed systems, as illustrated in the examples that we have highlighted. We have proved that the technique is correct, implemented it and performed a number of experiments. Finally, we have developed and implemented an algorithm for automatic decomposition of nets that performs adequately on a number of examples.

In future work we plan to improve our decomposition algorithm and characterise the class of nets to which our approach is suited. Additionally, by using the full algebra \([2][18]\) of nets, in particular, the possibility of connecting several transitions to the same boundary port, we hope to alleviate some of the problems identified in \(\text{§2}\). We also plan to generalise our approach to other models: for example by examining symbolic representations of the algebras of P/T nets in \([1][2]\) we hope to extend our technique to coverability.

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Appendix

In order to prove compositionality we first need to prove a small, technical lemma.

**Lemma 10.** Suppose that $N : k \to l$ and $M : l \to m$ are nets with boundaries and $(U,V)$ is a non-trivial synchronisation. Then there exists a mutually independent family $\{(U_i,V_i)\}_{i \in I}$ of minimal synchronisations with $U = \bigcup_{i \in I} U_i$ and $V = \bigcup_{i \in I} V_i$.

**Proof.** We argue by induction on $|U + V|$. If $(U,V)$ is minimal then the singleton family $\{(U,V)\}$ satisfies the requirements. Otherwise there exists a minimal synchronisation $(U',V') \subseteq (U,V)$. Now since there is at most one transition connected to each point on the boundary, we have $U^* \cap (U \setminus U') = \emptyset$ and, similarly, $V^* \cap (V \setminus V') = \emptyset$. Since $U^* = \bullet V$, we must also have $(U \setminus U')^* = \bullet (V \setminus V')$ and thus $(U \setminus U', V \setminus V')$ is a synchronisation. By the inductive hypothesis, there exists a mutually independent family $\{(U_i,V_i)\}_{i \in I}$, and so $\{(U',V')\} \cup \{(U_i,V_i)\}_{i \in I}$ fulfills the requirements.

**Proof of Theorem 4.**

Proof. $(\Rightarrow)$ If $[N : M]_{X \xrightarrow{\alpha/\beta} Y, X' \xrightarrow{\gamma/\delta} Y'}$ then there exists mutually independent set of minimal synchronisations $W \subseteq \text{sync}_{\text{min}}(N,M)$ with $\bullet W = \alpha$ and $\alpha^* = \beta$. Consider $U_{\text{def}} \bigcup_{(X,Y) \in W} X \subseteq T_N$ and $V_{\text{def}} \bigcup_{(X,Y) \in W} Y \subseteq T_M$. Since each $(X,Y) \in W$ is a synchronisation, we have $X^* = \bullet Y$ and so $U^* = \bullet V$. By definition, in each $(X,Y) \in W$, $X$ and $Y$ are mutually independent in, respectively, $N$ and $M$. Since $W$ is mutually independent, if $(X,Y) \neq (X',Y') \in W$ we have $\circ(X,Y) \circ \cap \circ(X',Y') = \emptyset$, so $(\circ X^o + \circ Y^o) \cap (\circ X'^o + \circ Y'^o) = \emptyset$ and thus both $\circ X^o \cap \circ Y^o = \emptyset$ and $\circ Y^o \cap \circ Y'^o = \emptyset$. It follows that $U$ and $V$ are mutually independent in $N$ and $M$, respectively, and letting $\gamma_{\text{def}} = U^* = \bullet V$ we have $[N : M]_{X \xrightarrow{\alpha/\gamma} Y, X' \xrightarrow{\gamma/\delta} Y'}$ as required.

$(\Leftarrow)$ If $[N : M]_{X \xrightarrow{\alpha/\gamma} Y, X' \xrightarrow{\gamma/\delta} Y'}$ then there exists mutually independent $U \subseteq T_N$ with $U^* = \gamma$, and mutually independent $V \subseteq T_M$ with $V^* = \gamma$, $V^* = \beta$. In particular, $(U,V)$ is a synchronisation and so, using the conclusion of Lemma 10 there exists a mutually independent family $\{(U_i,V_i)\}_{i \in I}$ of minimal synchronisations with $U = \bigcup_i U_i$ and $V = \bigcup_i V_i$. This family witnesses the transition $[N : M]_{X \xrightarrow{\alpha/\beta} Y, X' \xrightarrow{\gamma/\delta} Y'}$.

**Proof of Corollary 5.**

Proof. Simple induction on $p$, using the conclusion of Theorem 4.

**Lemma 11.** Suppose that $N : k \to l$ and $M : l \to m$ are nets with boundaries. If there is a trace

$[N : M]_{X \xrightarrow{\alpha_1} Y, X' \xrightarrow{\beta_1} Y'}$

then there exists $p \in \mathbb{N}$, $\gamma_i \in \{0,1\}$ for $1 \leq i \leq p$ and traces

$[N]_{X \xrightarrow{\alpha_1/\gamma_1} Y, \cdots, \alpha_n/\gamma_n} [N]_{X'}$

and

$[M]_{Y \xrightarrow{\gamma_1/0^{\alpha_1}} Y_1, \cdots, \gamma_n/0^{\alpha_n}} [M]_{Y'}$.

**Proof.** Induction on the length of the trace, using the conclusion of Theorem 4.
Fig. 13: Steps involved in translating $T_3$ to an NFA.
Proof of Theorem 7

Proof. (i) Suppose that \([N : M]_{\mathcal{X}_{\mathcal{U}_{\mathcal{V}}}} \overset{\alpha/\beta}{\Rightarrow} [N : M]_{\mathcal{X}_{\mathcal{U}_{\mathcal{V}}}}\), for some \(\alpha \in \{0, 1\}^k\) and \(\beta \in \{0, 1\}^n\). Then, by definition, there exist \(X'' \cup Y''\), \(X''' \cup Y'''\) with

\[
[N : M]_{\mathcal{X}_{\mathcal{U}_{\mathcal{V}}}} \overset{\alpha/\beta}{\Rightarrow} [N : M]_{\mathcal{X}_{\mathcal{U}_{\mathcal{V}}}} \overset{\alpha/\beta}{\Rightarrow} [N : M]_{\mathcal{X}_{\mathcal{U}_{\mathcal{V}}}} \overset{\alpha/\beta}{\Rightarrow} [N : M]_{\mathcal{X}_{\mathcal{U}_{\mathcal{V}}}},
\]

Now we use the conclusions of Lemma 11 and Theorem 4 to obtain the required traces.

(ii) If \([N]_X \overset{\alpha/\beta}{\Rightarrow} [N]_X\) and \([M]_Y \overset{\gamma/\delta}{\Rightarrow} [M]_Y\), for some \(\alpha \in \{0, 1\}^k\), \(\beta \in \{0, 1\}^m\), \(\gamma \in \{0, 1\}^l\), then there exist \(pN, qN, pM, qM \in N\), \(X'', Y'' \subseteq P_N\), \(Y'' \subseteq P_M\) and traces

\[
[N]_X \overset{\alpha/\beta}{\Rightarrow} [N]_X \overset{\gamma/\delta}{\Rightarrow} [N]_X \overset{\epsilon}{\Rightarrow} [N]_X,
\]

\[
[M]_Y \overset{\epsilon}{\Rightarrow} [M]_Y \overset{\gamma/\delta}{\Rightarrow} [M]_Y \overset{\epsilon}{\Rightarrow} [M]_Y,
\]

Now, using the fact that each net in any marking can make \(\epsilon\) transition and remain in the same marking (witnessing the firing of the empty set of transitions), we can use Theorem 4 to obtain:

\[
[N : M]_{\mathcal{X}_{\mathcal{U}_{\mathcal{V}}}} \overset{\epsilon}{\Rightarrow} [N : M]_{\mathcal{X}_{\mathcal{U}_{\mathcal{V}}}} \overset{\epsilon}{\Rightarrow} [N : M]_{\mathcal{X}_{\mathcal{U}_{\mathcal{V}}} \cup \mathcal{X}_{\mathcal{U}_{\mathcal{V}}}} \overset{\epsilon}{\Rightarrow} [N : M]_{\mathcal{X}_{\mathcal{U}_{\mathcal{V}}} \cup \mathcal{X}_{\mathcal{U}_{\mathcal{V}}}},
\]

and thus \([N : M]_{\mathcal{X}_{\mathcal{U}_{\mathcal{V}}}} \overset{\alpha/\beta}{\Rightarrow} [N : M]_{\mathcal{X}_{\mathcal{U}_{\mathcal{V}}}}\), as required. \square

Proof of Theorem 8

Proof. We prove this by structural induction on \(t\). The base case, when \(t\) is a variable, trivially holds. The interesting inductive case is \(t \mapsto t'\). We must show that \(\epsilon_{\min}(\text{trans}(t)_{(\mathcal{V}_{\mathcal{I}} \cup \mathcal{V}_{\mathcal{F}})} \cup \text{trans}(t')_{(\mathcal{V}_{\mathcal{I}} \cup \mathcal{V}_{\mathcal{F}})})\) (\(\dagger\)) is isomorphic to \(\epsilon_{\min}(\text{NFA}([t : t']_{\mathcal{V}}, \text{mrk}(t : t')_{\mathcal{I}}, \text{mrk}(t : t')_{\mathcal{F}}))\). Using the definitions of \(\llbracket t \rrbracket_{\mathcal{V}}\) and \(\text{mrk}(\cdot)\):

\[
\epsilon_{\min}(\text{NFA}([t : t']_{\mathcal{V}}, \text{mrk}(t : t')_{\mathcal{I}}, \text{mrk}(t : t')_{\mathcal{F}})) = \epsilon_{\min}(\text{NFA}([t]_{\mathcal{V}}, \text{mrk}(t)_{\mathcal{I}} \cup \text{mrk}(t')_{\mathcal{I}}, \text{mrk}(t)_{\mathcal{F}} \cup \text{mrk}(t')_{\mathcal{F}})) \quad (3)
\]
The inductive hypothesis gives us that

\[ \text{trans}(t)_{(V,F)} \cong \epsilon_{\min}(\text{NFA}(\llbracket t \rrbracket_V, \text{mrk}(t)_I, \text{mrk}(t)_F)) \]  

(4)

and

\[ \text{trans}(t')_{(V,F)} \cong \epsilon_{\min}(\text{NFA}(\llbracket t' \rrbracket_V, \text{mrk}(t')_I, \text{mrk}(t')_F)) \]  

(5)

Substituting (4) and (5) in (†), and using (3), our task reduces to showing that:

\[ \epsilon_{\min}((\epsilon_{\min}(\text{NFA}(\llbracket t \rrbracket_V, \text{mrk}(t)_I, \text{mrk}(t)_F)) ; \epsilon_{\min}(\text{NFA}(\llbracket t' \rrbracket_V, \text{mrk}(t')_I, \text{mrk}(t')_F))) \cong \epsilon_{\min}(\text{NFA}(\llbracket t \rrbracket_V ; \llbracket t' \rrbracket_V, \text{mrk}(t)_I \uplus \text{mrk}(t')_I, \text{mrk}(t)_F \uplus \text{mrk}(t')_F)) \]  

(6)

To do this, it is sufficient to show that

\[ \epsilon_{\text{cl}}(\epsilon_{\min}(\text{NFA}(\llbracket t \rrbracket_V, \text{mrk}(t)_I, \text{mrk}(t)_F)) ; \epsilon_{\min}(\text{NFA}(\llbracket t' \rrbracket_V, \text{mrk}(t')_I, \text{mrk}(t')_F))) \]  

(7)

and

\[ \epsilon_{\text{cl}}(\text{NFA}(\llbracket t \rrbracket_V ; \llbracket t' \rrbracket_V, \text{mrk}(t)_I \uplus \text{mrk}(t')_I, \text{mrk}(t)_F \uplus \text{mrk}(t')_F)) \]  

(8)

recognise the same language, where \( \epsilon_{\text{cl}}(-) \) means \( \epsilon \)-closure. But (7) recognises the same language as

\[ \epsilon_{\text{cl}}(\text{NFA}(\llbracket t \rrbracket_V, \text{mrk}(t)_I, \text{mrk}(t)_F) ; \text{NFA}(\llbracket t' \rrbracket_V, \text{mrk}(t')_I, \text{mrk}(t')_F)) \]  

(9)

and now the translation between paths in (8) and (9) follows directly from the conclusion of Theorem [7] □