AN UPPER BOUND ON THE LS-CATEGORY IN PRESENCE OF THE FUNDAMENTAL GROUP

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Abstract. We prove that
\[ \text{cat}_{LS} X \leq \left\lfloor \frac{\text{cd}(\pi_1(X)) + \dim X}{2} \right\rfloor \]
for every CW complex \( X \) where \( \text{cd}(\pi_1(X)) \) denotes the cohomological dimension of the fundamental group of \( X \). We obtain this as a corollary of the inequality
\[ \text{cat}_{LS} X \leq \left\lfloor \frac{\text{cat}_{LS}(u_X) + \dim X}{2} \right\rfloor \]
where \( u_X : X \to B\pi_1(X) \) is a classifying map for the universal covering of \( X \).

1. Introduction

The reduced Lusternik-Schnirelmann category (briefly LS-category) \( \text{cat}_{LS} X \) of a topological space \( X \) is the minimal number \( n \) such that there is an open cover \( \{U_0, \ldots, U_n\} \) of \( X \) by \( n + 1 \) contractible in \( X \) sets. We note that the LS-category is a homotopy invariant. The Lusternik-Schnirelmann category has many applications. Perhaps the most famous is the classical Lusternik and Schnirelmann theorem [4] which states that \( \text{cat}_{LS} M \) gives a low bound for the number of critical points on a manifold \( M \) of any smooth not necessarily Morse function. This theorem was used by Lusternik and Schnirelmann in their solution of Poincare’s problem on the existence of three closed geodesics on a 2-sphere [12]. In modern time the LS-category was used in the proof of the Arnold conjecture on symplectomorphisms [15].

The LS-category is a numerical homotopy invariant which is difficult to compute. Even to get a reasonable bound for \( \text{cat}_{LS} \) very often is a serious problem. In this paper we discuss only upper bounds. For nice spaces, such as CW complexes, it is an easy observation that \( \text{cat}_{LS} X \leq \dim X \). In the 40s Grossmann [10] (and independently in the 50s G.W. Whitehead [17] [4]) proved that for simply connected CW complexes \( \text{cat}_{LS} X \leq \dim X/2 \).

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In the presence of the fundamental group the LS-category can be equal to the dimension. In fact, $\text{cat}_{LS} X = \dim X$ if and only if $X$ is essential in the sense of Gromov. This was proven for manifolds in [11]. For general CW complexes we refer to Proposition 2.6 of this paper. We recall that an $n$-dimensional complex $X$ is called inessential if a map $u_X : X \to B\pi_1(X)$ that classifies its universal cover can be deformed to the $(n-1)$-skeleton $(B\pi_1(X))^{(n-1)}$. Otherwise, it is called essential. Typical examples of essential CW complexes are aspherical manifolds.

Yu. Rudyak conjectured that in the case of free fundamental group there should be the Grossmann-Whitehead type inequality at least for closed manifolds. There were partial results towards Rudyak’s conjecture [8],[16] until it was settled in [5]. Later it was shown in [6] (also see the followup [13]) that the Grossmann-Whitehead type estimate holds for complexes with the fundamental group having small cohomological dimension. Namely, it was shown that $\text{cat}_{LS} X \leq \text{cd}(\pi_1(X)) + \dim X/2$. Clearly, this upper bound is far from being optimal for fundamental groups with sufficiently large cohomological dimension. Indeed, for the product of an aspherical $m$-manifold $M$ with the complex projective space we have $\text{cat}_{LS}(M \times \mathbb{C}P^n) = m + n$ but our upper bound is $m + (m + 2n)/2 = \frac{3}{2}m + n$. Moreover, our bound quits to be useful for complexes with $\text{cd}(\pi_1(X)) \geq \dim X/2$. The desirable bound here is

$$\text{cat}_{LS} X \leq \frac{\text{cd}(\pi_1(X)) + \dim X}{2}.$$  

Such an upper bound was proven in [8] for the systolic category, a differential geometry relative of the LS-category. Nevertheless, for the classical LS-category a similar estimate was missing until now.

In this paper we prove the desirable upper bound. We obtain such a bound as a corollary of the following inequality

$$\text{cat}_{LS} X \leq \left\lceil \frac{\text{cat}_{LS}(u_X) + \dim X}{2} \right\rceil$$

where $u_X : X \to B\pi_1(X)$ is a classifying map for the universal covering of $X$. We note that this inequality gives a meaningful upper bound on the LS-category for complexes with any fundamental group. Also we note that the new upper bound gives the optimal estimate for the above example $M \times \mathbb{C}P^n$, the product of an aspherical manifold and the complex projective space. Namely,

$$\text{cat}_{LS}(M \times \mathbb{C}P^n) \leq \left\lceil (m + (m+2n))/2 \right\rceil = m + n.$$
2. Preliminaries

The proof of the new upper bound for $\text{cat}_{\text{LS}} X$ is based on a further modification of the Kolmogorov-Ostrand multiple cover technique [5]. That technique was extracted by Ostrand from the work of Kolmogorov on the 13th Hilbert problem [14]. Also in this paper we make use of the following well-known fact.

**Proposition 2.1.** Let $f : X \to Y$ be a homotopy domination. Then $\text{cat}_{\text{LS}} Y \leq \text{cat}_{\text{LS}} X$.

**Proof.** Let $s : Y \to X$ be a left homotopy inverse to $f$, i.e. $f \circ s \sim 1_Y$. Let $U_0, \ldots, U_k$ be an open cover of $X$ by sets contractible in $X$. One can easily check that $s^{-1}(U_0), \ldots, s^{-1}(U_k)$ is an open cover by sets contractible in $Y$. □

Let $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$ be a family of sets in a topological space $X$. The *multiplicity* of $\mathcal{U}$ (or the *order*) at a point $x \in X$, denoted $\text{Ord}_x \mathcal{U}$, is the number of elements of $\mathcal{U}$ that contain $x$. A family $\mathcal{U}$ is a cover of $X$ if $\text{Ord}_x \mathcal{U} \neq 0$ for all $x$.

**Definition 2.2.** A family $\mathcal{U}$ of subsets of $X$ is called a $k$-cover, $k \in \mathbb{N}$ if every subfamily of $\mathcal{U}$ that consists of $k$ sets forms a cover of $X$.

The following is obvious (see [5]).

**Proposition 2.3.** A family $\mathcal{U}$ that consists of $m$ subsets of $X$ is an $(n + 1)$-cover of $X$ if and only if $\text{Ord}_x \mathcal{U} \geq m - n$ for all $x \in X$.

Let $K$ be a simplicial complex. By the definition the dual to the $m$-skeleton $K^{(m)}$ is a subcomplex $L = L(K, m)$ of the barycentric subdivision $\beta K$ that consists of simplices of $\beta K$ which do not intersect $K^{(m)}$. Note that $\beta K$ is naturally embedded in the join product $K^{(n)} \ast L$.

Then the following is obvious:

**Proposition 2.4.** For any $n$-dimensional complex $K$ the complement $K \setminus K^{(m)}$ to the $m$-skeleton is homotopy equivalent to an $(n - m - 1)$-dimensional complex $L$.

**Proof.** The complex $L$ is the dual to $K^{(m)}$. Clearly, $\dim L = n - m - 1$. The complement $K \setminus K^{(m)}$ can be deformed to $L$ along the field of intervals defined by the embedding $\beta K \subset K^{(n)} \ast L$. □

Let $f : X \to Y$ be a continuous map. We recall that the LS-category of $f$, $\text{cat}_{\text{LS}} f$ is the smallest number $k$ such that $X$ can be covered by
$k + 1$ open sets $U_0, \ldots, U_k$ such that the restriction $f|_{U_i} : U_i \to Y$ of $f$ to each of them is null-homotopic. Clearly,

$$\text{cat}_{LS} f \leq \text{cat}_{LS} X, \text{cat}_{LS} Y.$$ 

We denote by $u_X : X \to B\pi$, $\pi = \pi_1(X)$, a map that classifies the universal covering $p : \tilde{X} \to X$ of $X$. Thus, $p$ is the pull-back of the universal covering $q : E\pi \to B\pi$. Here $B\pi$ is any aspherical CW complex with the fundamental group $\pi$. Thus, any map $u : X \to B\pi$ that induces an isomorphism of the fundamental groups is a classifying map.

The following proposition is proven in [7], Proposition 4.3.

**Proposition 2.5.** A classifying map $u_X : X \to B\pi$ of the universal covering of a CW complex $X$ can be deformed into the $d$-skeleton $B\pi^{(d)}$ if and only if $\text{cat}_{LS}(u_X) \leq d$.

The following proposition for closed manifolds was proven by Katz and Rudyak [11], although it was already known to Berstein in a different equivalent formulation [1].

**Proposition 2.6.** For an $n$-dimensional CW complex $X$, $\text{cat}_{LS} X = n$ if and only if $X$ is essential.

**Proof.** Suppose that $X$ is essential. By Proposition 2.5 we obtain that $\text{cat}_{LS}(u_X) > n - 1$. Thus, $\text{cat}_{LS} X \geq \text{cat}_{LS}(u_X) \geq n$ and, since $\text{dim} X = n$, $\text{cat}_{LS} X = n$.

The implication in the other direction can be derived from the proof of Theorem 4.4 in [7]. Here we give the sketch of the proof. Let $u_X : X \to B\pi^{(n-1)}$ be a classifying map. To prove the inequality $\text{cat}_{LS} X \leq n - 1$ it suffices to show that the Ganea-Schwarz fibration $p_n^X : G_{n-1}(X) \to X$ admits a section. Since the fiber of the Ganea-Schwarz fibration $p_n^\pi$ is $(n - 1)$-connected, the map $u_X$ admits a lift $f : X \to G_{n-1}(B\pi)$. Then the map $p'$ in the pull-back diagram

$$
\begin{array}{ccc}
G_{n-1}(X) & \longrightarrow & Z \\
q \downarrow & & \downarrow u_X' \\
X & \longrightarrow & G_{n-1}(B\pi)
\end{array}
$$

admits a section $s : X \to Z$. Here $p_n^X = p' \circ q$. Since $X$ is $n$-dimensional, to show that $s$ has a lift with respect to $q$ it suffices to prove that the homotopy fiber $F$ of the map $q$ is $(n - 1)$-connected. Note that the homotopy exact sequence of the fibration

$$
F \rightarrow (p_n^{X})^{-1}(x_0) \xrightarrow{u'_X} (p_n^{B\pi})^{-1}(y_0)
$$
where \( u' \) is the restriction of \( u' \circ q \) to the fiber \( (p_{n-1}^X)^{-1}(x_0) \) coincides with the homotopy exact sequence of the fibration

\[
F \rightarrow *_{n}(M) \xrightarrow{\ast \Omega(u \circ X)} *_{n}(B \pi)
\]

obtained from the loop map \( \Omega(u \circ X) \) turned into a fibration by taking the iterated join product. Since \( \pi_0(\Omega u \circ X) = 0 \), we obtain \( \pi_i(*_{n} \Omega u \circ X) = 0 \) for \( i \leq n - 1 \) (see Proposition 2.4 [7]) and hence \( \pi_i(F) = 0 \) for \( i \leq n - 1 \). \( \square \)

3. Multiple covers of polyhedra

For a point \( x \in X \) in a CW complex \( X \) by \( d(x) \) we denote the dimension of the open cell \( e \) containing \( x \). We call a subset \( A \subset X \) in a CW complex \( X \) \( r \)-deformable if \( A \) can be deformed in \( X \) to the \( r \)-skeleton \( X^{(r)} \). A deformation \( H : A \times I \rightarrow X \) to the 0-skeleton \( X^{(0)} \) is called monotone if \( d(H(x,t)) \) is monotonically decreasing function of \( t \) for all \( x \in A \).

**Proposition 3.1.** Let \( X \) be a connected simplicial complex of dimension \( \leq n(r+1) - 1 \). Then for any \( m \geq n \) there exists an open cover \( \mathcal{U} = \{ U_1, \ldots, U_m \} \) of \( X \) by \( r \)-deformable sets such that \( \text{Ord}_x \mathcal{U} \geq m - k + 1 \) for every \( k \leq n \) and all \( x \in X^{(k(r+1)-1)} \). Equivalently, the restriction of \( \mathcal{U} \) to the \((k(r+1)-1)\)-skeleton is a k-cover.

Moreover, for \( r = 0 \) we may assume that each set \( U_i \) is monotone \( r \)-deformable.

**Proof.** It suffices to prove the Proposition for complexes with \( \dim X = n(r+1) - 1 \). We do it by induction on \( n \). For \( n = 1 \) the statement is obvious. Suppose that it holds true for \( n - 1 \geq 1 \). We prove it for \( n \) by induction on \( m \). First we establish the base of induction by proving the statement for \( m = n \). By the external induction applied to \( X^{(n-1)(r+1)-1} \) with \( m = n - 1 \) there is an open cover \( \mathcal{U} = \{ U_1, \ldots, U_{n-1} \} \) of \( X^{(n-1)(r+1)-1} \) such that each \( U_i \) is \( r \)-deformable and \( \text{Ord}_x \mathcal{U} \geq (n-1) - k + 1 = n - k \) for all \( x \in X^{(k(r+1)-1)} \). We can enlarge each \( U_i \) to a \( r \)-deformable open in \( X \) set \( U'_i \subset X \).

Let \( G = \bigcup_{i=1}^{n-1} U'_i \). Since the complement \( X \setminus X^{(n-1)(r+1)-1} \) is homotopy equivalent to a \( r \)-dimensional complex (see Proposition 2.4), \( Z_0 = X \setminus G \) is \( r \)-deformable. Since \( Z_0 \) is closed, we can find an open enlargement \( W_0 \) to an \( r \)-deformable set whose closure does not intersect \( X^{(n-1)(r+1)-1} \). Thus, the cover \( \{ U'_1, \ldots, U'_{n-1}, W_0 \} \) satisfies the condition of Proposition for \( k = n \).

Consider the set

\[
Z_1 = \{ x \in X^{(n-1)(r+1)-1} \mid \text{Ord}_x \mathcal{U} = 1 \}.
\]
Clearly, \(Z_1\) is closed. By the induction assumption \(Z_1\) does not intersect the skeleton \(X^{((n-2)(r+1)-1)}\). Since the complement, 
\[
X^{((n-1)(r+1)-1)} \setminus X^{((n-2)(r+1)-1)}
\]
is homotopy equivalent to an \(r\)-dimensional complex, \(Z_1\) is \(r\)-deformable in \(X^{((n-1)(r+1)-1)}\). Let \(W_1\) be an enlargement of \(Z_1\) to an open \(r\)-deformable in \(X\) sets such that the closure \(\bar{W}_1\) does not intersect \(\bar{W}_0 \cup X^{(n-2)(r+1)-1}\). Note that the cover \(\{U_1', \ldots, U_{n-1}', W_0 \cup W_1\}\) satisfies the condition of Proposition with \(k = n\) and \(k = n-1\).

Next we consider
\[
Z_2 = \{x \in X^{((n-2)(r+1)-1)} \mid \text{Ord}_x U = 2\}
\]
and similarly define an open set \(W_2\) and so on up to \(W_{n-1}\). By the construction each set \(W_i\) is \(r\)-deformable and the closures \(\bar{W}_i\) are disjoint. Therefore, the union \(U_n' = W_0 \cup \cdots \cup W_{n-1}\) is \(r\)-contractible. Then the cover \(U_0' \cup \cdots \cup U_n'\) satisfies all the conditions of Proposition for all \(k \leq n\).

The proof of the inductive step is very similar to the above. Assume that the statement of Proposition holds for \(n\) and \(m-1 \geq n\). We prove it for \(n\) and \(m\). Let \(\mathcal{U} = \{U_1, \ldots, U_{m-1}\}\) be an open cover of \(X\) by \(r\)-deformable sets such that for any \(k \leq n\) the restriction of \(\mathcal{U}\) to \(X^{(k(r+1)-1)}\) is a \(k\)-cover. Thus, \(\text{Ord}_x \mathcal{U} \geq (m-1) - n + 1 = m - n\) for all \(x\). Let
\[
Z_0 = \{x \in X \mid \text{Ord}_x \mathcal{U} = m - n\}.
\]
By the induction assumption \(Z_0 \cap X^{((n-1)(r+1)-1)} = \emptyset\). Thus, \(Z_0\) is \(r\)-deformable in \(X\). We consider an open \(r\)-deformable neighborhood \(W_0\) of \(Z_0\) with \(W_0 \cap X^{((n-1)(r+1)-1)} = \emptyset\).

Next we consider the closed set
\[
Z_1 = \{x \in X^{((n-1)(r+1)-1)} \mid \text{Ord}_x \mathcal{U} = m - n + 1\}.
\]
By the induction assumption \(Z_1\) does not intersect \(X^{((n-2)(r+1)-1)}\). As above, we define a \(r\)-deformable set \(W_1\) with
\[
W_1 \cap (\bar{W}_0 \cup X^{((n-2)(r+1)-1)}) = \emptyset
\]
and so on. We define \(U_m = W_0 \cup \cdots \cup W_{n-1}\). Then the condition of Proposition is satisfied for all \(k\) with \(\mathcal{U}' = \{U_1, \ldots, U_{m-1}, U_m\}\).

Now we revise our proof for \(r = 0\) in order to verify the extra condition of Proposition. In the proof of the base of induction on \(m\) the enlargements \(U_i'\) can be chosen monotone deformable to \(U_i\). Hence, each \(U_i'\) is monotone 0-deformable. Since \(W_0\) lives in the complement to the \((n-1)\)-skeleton, it is monotone 0-deformable. The set \(W_1\) can be chosen monotone deformable to the monotone 0-deformable set \(W_1 \cap X^{(n-1)} \subset X^{(n-1)} \setminus X^{(n-2)}\). Thus, \(W_1\) is monotone 0-deformable.
and so on. As the result we obtain that the set $U'_n = W_0 \cup \cdots \cup W_{n-1}$ is monotone 0-deformable. In the proof of inductive step the same argument shows that the set $U_m = W_0 \cup \cdots \cup W_{n-1}$ is monotone 0-deformable. □

3.1. **Borel construction.** Let a group $\pi$ act on spaces $X$ and $E$ with the projections onto the orbit spaces $q_X : X \to X/\pi$ and $q_E : E \to E/\pi = B$. Let $q_{X \times E} : X \times E \to X \times E = (X \times E)/\pi$ denote the projection onto the orbit space of the diagonal action of $\pi$ on $X \times E$. Then there is a commutative diagram called the Borel construction [2]:

$$
\begin{array}{ccc}
X & \xleftarrow{pr_X} & X \times E & \xrightarrow{pr_2} & E \\
q_X \downarrow & & q \downarrow & & q_E \downarrow \\
X/\pi & \xleftarrow{p_{XE}} & X \times E/\pi = (X \times E)/\pi & \xrightarrow{p_X} & B.
\end{array}
$$

If $\pi$ is discrete and the actions are free and proper, then all projections in the diagram are locally trivial bundles with the structure group $\pi$. Then the fiber of $p_X$ is homeomorphic to $X$ and the fiber of $p_E$ is homeomorphic to $E$. For any invariant subset $Q \subset X$ the map $p_X$ defines the pair of bundles $p_X : (X \times E, Q \times E) \to B$ with the stratified fiber $(X, Q)$ and the structure group $\pi$.

If $X/\pi$ and $B$ are CW complexes for proper free actions of discrete group $\pi$, their CW structures define a natural CW structure on $X \times E/\pi = (X \times E)/\pi$ as follows: First, $X$ and $E$ being covering spaces inherit CW structures from $X/\pi$ and $B$ respectively. Since the diagonal action of $\pi$ on $X \times E$ preserves the product CW complex structure on $X \times E$ and takes cells to cells homeomorphically, the orbit space $X \times E/\pi$ receives the induced CW complex structure.

**Lemma 3.2.** Let $\tilde{X}$ be the universal covering of an $n$-dimensional simplicial complex $X$ with the fundamental group $\pi = \pi_1(X)$. Suppose that $\tilde{X}$ admits a classifying map to a $d$-dimensional simplicial complex $B$, $\pi_1(B) = \pi$. Let $E$ be the universal covering of $B$. Then for the $n$-skeleton

$$
cat_{LS}(\tilde{X} \times E)^{(n)} \leq \left\lceil \frac{d + n}{2} \right\rceil
$$

where the CW complex structure on $\tilde{X} \times E$ is defined by the simplicial complex structures on $X$ and $B$. 
Proof. Denote by $K = \tilde{X} \times \pi E$. Since $(\tilde{X} \times E)^{(n)} = \bigcup_j \tilde{X}^{(n-j)} \times E^{(j)}$, we have

$$K^{(n)} = \bigcup_{j=0}^{d} \tilde{X}^{(n-j)} \times \pi E^{(j)}.$$ 

We show that $\text{cat}_{1S} K^{(n)} \leq d + \ceil{\frac{n-d}{2}} = \ceil{\frac{d+n}{2}}$.

Let $m = \ceil{(d+n)/2} + 1$. We apply Proposition 3.1 to $B$ with $r = 0$ to obtain an open cover $\mathcal{U} = \{U_1, \ldots, U_m\}$ by monotone 0-deformable in $B$ sets with $\text{Ord}_x \mathcal{U} \geq m - j$ for $x \in B^{(j)}$. We note that we apply Proposition 3.1 here with $n = d + 1$ and we need to be sure that our $m \geq d + 1$ which is satisfied. The substitution $i = k - 1$ helps to see the inequality $\text{Ord}_x \mathcal{U} \geq m - i$ for $x \in B^{(j)}$ for $x \in B^{(j)}$.

Next we observe that $2m - 1 \geq \dim X$. Hence we can apply Proposition 3.1 to get an open cover $\mathcal{V} = \{V_1, \ldots, V_m\}$ of $X$ by 1-deformable in $X$ sets such that the restriction of $\mathcal{V}$ to $X^{(2j-1)}$ is a $j$-cover, $j = 1, \ldots, k$, where $k$ be the smallest integer satisfying the inequality $n \leq 2k - 1$.

For every $i \leq m$ we define

$$W_i = p_E^{-1}(V_i) \cap p_X^{-1}(U_i).$$

We claim that the collection of sets $\{W_1, \ldots, W_m\}$ covers $K^{(n)}$. Let $x \in \tilde{X}^{(n-j)} \times \pi E^{(j)}$. Then the point $p_X(x) \in B^{(j)}$ is covered by at least $m - j$ sets $U_{k_1}, \ldots, U_{k_{m-j}} \in \mathcal{U}$. Since $\mathcal{V}$ restricted to $X^{(2(m-j)-1)}$ is a $(m-j)$-cover, the sets $V_{k_1}, \ldots, V_{k_{m-j}}$ cover $X^{(2(m-j)-1)}$. Note that $2(m-j) - 1 \geq d + n + 2 - 2j - 1 \geq n - j$. Therefore, the point $p_E(x) \in X^{(n-j)}$ is covered by $V_{k_s}$ for some $s \in \{1, \ldots, m - j\}$. Hence, $x \in W_{k_s}$.

We note that $W_i = Q_i \times \pi P_i \subset \tilde{X} \times \pi E$ where $P_i = q_B^{-1}(U_i)$ and $Q_i = q_X^{-1}(V_i)$. Thus, its intersection with $K^{(n)}$ can be written as

$$W_i(n) = W_i \cap K^{(n)} = \bigcup_j Q_i(n-j) \times \pi P_i(j)$$

where $P_i(k) = P_i \cap E^{(k)}$ and $Q_i(\ell) = Q_i \cap \tilde{X}^{(\ell)}$.

To complete the proof we show that each set $W_i(n)$ is contractible in $K^{(n)}$. We consider a monotone deformation $h_t : U_i \to B$ of $U_i$ to $B^{(0)}$. Let $\tilde{h}_t : P_i \to E$ be the lifting of $h_t$. Thus, $\tilde{h}_t$ is a $\pi$-equivariant deformation of $P_i$ to $E^{(0)}$. Then $1_{\tilde{X}} \times h_t : \tilde{X} \times P_i \to \tilde{X} \times E$ is a $\pi$-equivariant deformation and, hence, it defines a deformation of the orbit space $\tilde{h}_t : \tilde{X} \times \pi P_i \to K$ which is a lift of $h_t$ with respect to $p_{\tilde{X}}$. Since each skeleton $\tilde{X}^{(\ell)}$ is $\pi$-invariant, the deformation $\tilde{h}_t$ preserves the filtration of the fibers $\tilde{X}$ of the bundle $p_{\tilde{X}}$ by the skeleta. By the same
may assume that $h_t$ moves the set $Q_i(n-j) \times \pi P_i$ within $Q_i(n-j) \times \pi B$. Since $h_t$ is monotone, $\tilde{h}_t$ moves $Q_i(n-j) \times \pi P^d$ within $Q_i(n-j) \times \pi B^d \subset K^{(n)}$ for all $j$. Thus, $\tilde{h}_t$ deforms $W_i(n)$ within $K^{(n)}$ to the set
\[ Q_i \times \pi E^{(0)} \subset \tilde{X} \times \pi E^{(0)} = p^{-1}_X(B^{(0)}) \cong \bigcup_{b \in B^{(0)}} \tilde{X}. \]

Since $V_i$ is 1-deformable in $X$, so is $Q_i$ in $\tilde{X}$. Since $\tilde{X}$ is simply connected, $Q_i$ is contractible in $\tilde{X}$. Thus, we obtain that the set
\[ Q_i \times \pi E^{(0)} \cong \bigcup_{b \in B^{(0)}} Q_i \subset \bigcup_{b \in B^{(0)}} \tilde{X} \]
is 0-deformable in $\tilde{X} \times \pi E^{(0)} \subset K^{(n)}$. Therefore, $W_i(n)$ is 0-deformable in $K^{(n)}$. Since $K$ is connected, $W_i(n)$ is contractible in $K^{(n)}$.

Thus, $\text{cat}_{LS} K^{(n)} \leq m - 1 = \lceil \frac{n+d}{2} \rceil$. \hfill \Box

4. Main Result

**Theorem 4.1.** For every simplicial complex $X$ there is the inequality
\[ \text{cat}_{LS} X \leq \left\lfloor \frac{\text{cat}_{LS}(u_X) + \dim X}{2} \right\rfloor \]
where $u_X : X \to B\pi$ is a classifying map for the universal cover of $X$.

**Proof.** Let $\dim X = n$ and $\text{cat}_{LS}(u_X) = d$. In the proof we use the notations $B = B\pi$, $B^d = B\pi^{(d)}$ and $E = E\pi$, $E^d = E\pi^{(d)}$. By Proposition 2.5 we may assume that the map $u_X$ lands in $B^d$. Consider the diagram generated by the Borel construction
\[
\begin{array}{cccccc}
X & \xleftarrow{p_E} & \tilde{X} \times \pi E & \xrightarrow{p_{\tilde{X}}} & B \\
\uparrow & & \updownarrow & & \updownarrow \\
X & \xleftarrow{p_{E^d}} & \tilde{X} \times \pi E^d & \xrightarrow{p_{\tilde{X}^d}} & B^d \\
\end{array}
\]
Since $E$ is contractible, the map $p_E$ is a homotopy equivalence. Let $g$ be its homotopy inverse. Applying the homotopy lifting property we may assume that $g$ is a section of $p_E$. Then the map $p_{\tilde{X}}$ is homotopic to $p_{\tilde{X}} \circ g \circ p_E$. Note that the map $p_{\tilde{X}} \circ g : X \to B$ is a classifying map for $\tilde{X}$. Thus, it is homotopic to the map $u_X : X \to B$ whose image is in $B^d$. Therefore, $p_{\tilde{X}} : \tilde{X} \times \pi E \to B$ is homotopic to a map with image in $B^d$. Let $p_t : \tilde{X} \times \pi E \to B$ be such a homotopy. Thus, $p_0 = p_{\tilde{X}}$ and $p_1(\tilde{X} \times \pi E) \subset B^d$. Let $\bar{p}_t : \tilde{X} \times \pi E \to \tilde{X} \times \pi E$ be the lift of $p_t$ with $\bar{p}_0 = id$. Then $\bar{p}_1(\tilde{X} \times \pi E) \subset \tilde{X} \times \pi E^d$. 
First, we note that \( s = \bar{p}_1 \circ g : X \to \tilde{X} \times_{\pi} E^d \) is a homotopy section of \( p_{E^d} \). Indeed, the homotopy \( h_t = p_{E} \circ \bar{p}_t \circ g : X \to X \) is joining \( h_0 = p_{E} \circ \bar{p}_0 \circ g = p_{E} \circ g = 1_X \) with \( h_1 = p_{E} \circ \bar{p}_1 \circ g = p_{E^d} \circ \bar{p}_1 \circ g = p_{E^d} \circ s \).

We may assume that \( B \) is a simplicial complex. Denote by \( K = \tilde{X} \times_{\pi} E^d \). We consider the CW complex structure on \( K \) defined by the simplicial complex structures on \( X \) and \( B \). Next we show that the restriction \( (p_{E^d})|_K \) of \( K \) is a homotopy domination. Since \( \dim X = n \), there is a homotopy \( s_t : X \to K \) with \( s_0 = s \) and \( s_1(X) \subset K^{(n)} \). Then the homotopy \( q_t = p_{E^d} \circ s_t : X \to X \) joins \( q_0 = p_{E^d} \circ s = 1_X \) with \( q_1 = p_{E^d} \circ s_1 = (p_{E^d})|_K \circ s_1 \).

Therefore, by Proposition 2.1, \( \text{cat}_{LS} X \leq \text{cat}_{LS} K^{(n)} \). Lemma 3.2 implies

\[
\text{cat}_{LS} X \leq \left\lceil \frac{d + n}{2} \right\rceil.
\]

\[\square\]

**Corollary 4.2.** For any CW complex \( X \),

\[
\text{cat}_{LS} X \leq \left\lfloor \frac{\text{cd}(\pi_1(X)) + \dim X}{2} \right\rfloor.
\]

**Proof.** We note that every CW complex is homotopy equivalent to a simplicial complex of the same dimension. By the Eilenberg-Ganea theorem \( \pi = \pi_1(X) \) has a classifying complex \( B\pi \) of dimension equal \( \text{cd}(\pi) \) whenever \( \text{cd}(\pi) \neq 2 \) (see [3]). Thus, If \( \text{cd}(\pi) \neq 2 \), the result immediately follows from Theorem 4.1.

In the case when \( \text{cd}(\pi) = 2 \) one can find a classifying complex \( B\pi \) of dimension three [3]. Then Obstruction Theory implies that there is a map \( r : B\pi \to B\pi^{(2)} \) which is the identity on the 1-skeleton. It is easy to check that \( r \) induces an isomorphism of the fundamental groups: Obviously it is surjective and the kernel of \( r_* : \pi_1(B) \to \pi_1(B\pi^{(2)}) \) is trivial. In particular, its composition with a classifying map \( r \circ u_X : X \to B\pi^{(2)} \) is a classifying map and we can apply Theorem 4.1 to it. \[\square\]

In the proof of the main result we applied our technical proposition (Proposition 3.1) with \( r = 0 \) and \( r = 1 \). Using Proposition 3.1 with \( r = 0 \) and arbitrary \( r > 0 \) brings the following

**Lemma 4.3.** Suppose that \( \tilde{X} \) the universal covering of an \( n \)-dimensional simplicial complex \( X \) with the fundamental group \( \pi = \pi_1(X) \) is \( r \)-connected. Assume that \( \tilde{X} \) admits a classifying map to \( d \)-dimensional
complex $B$, $\pi_1(B) = \pi$. Let $E$ be the universal covering of $B$. Then
\[
\text{cat}_{LS}(\widetilde{X} \times \pi E)^{(n)} \leq \left\lceil \frac{rd + n}{r + 1} \rightceil.
\]

This Lemma brings the following generalization of Theorem 4.1.

**Theorem 4.4.** For every simplicial complex $X$ with $r$-connected universal cover $\widetilde{X}$ there is the inequality
\[
\text{cat}_{LS} X \leq \left\lceil \frac{r \text{ cat}_{LS}(u_X) + \dim X}{r + 1} \right\rceil
\]
where $u_X : X \to B\pi$ is a classifying map for the universal cover of $X$.

**Corollary 4.5.** For any CW complex $X$ with $r$-connected universal covering $\widetilde{X}$,
\[
\text{cat}_{LS} X \leq \left\lceil \frac{r \text{ cd} (\pi_1(X)) + \dim X}{r + 1} \right\rceil.
\]

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