On stability in dynamical Prisoner’s dilemma game with non-uniform interaction rates

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Abstract
Stability of evolutionary dynamics of non-repeated Prisoner’s Dilemma game with non-uniform interaction rates [1], via benefit and cost dilemma is studied. Moreover, the stability condition \((\frac{t+c}{b+c})^2 < r_1 r_3\) is derived in case of coexistence between cooperators and defectors. If \(r_1, r_3 \rightarrow \infty\) cooperation is the dominant strategy and defectors can no longer exploit cooperators.

Key words: Prisoner’s Dilemma game, non-uniform interaction rates and Dynamical Prisoner’s Dilemma game.

Introduction
Evolutionary game theory[3] was first introduced by John Maynard Smith in 1973[2]. He invented the important concept of an evolutionarily stable strategy (ESS) that resist invasion of other strategies in infinitely large populations. Prisoner’s dilemma (PD) game[4] studies cooperation between unrelated individuals. In non-repeated PD, individuals are either cooperators or defectors, acting accordingly whenever two of them interact. They both receive \(R\) upon mutual cooperation and \(P\) upon mutual defection. A defector exploiting a cooperator gets an amount \(T\) and the exploited cooperator receives \(S\), such that
T>R>P>S. So, it is best to defect, regardless of the opponent’s decision, which in turns makes cooperators unable to resist invasion by defectors.

In this paper, we study the effect of non-uniform interaction rates on evolutionary dynamics of non-repeated PD game [1], via benefit and cost dilemma. In section 1, we present the classical approach of the replicator equation of PD with uniform interaction rate between any two individuals. This rate does not depend on the strategies of these individuals. In section 2, we assume that the interaction rates are not uniform. For example, players who use the same strategy might interact more frequently than players who use different strategies. Non-uniform interaction rates lead to non-linear fitness functions and therefore allow richer dynamics than the classical replicator equation, which is based on linear fitness functions. If strategy D is a strict Nash equilibrium[5], then it remains uninvadable for positive non-uniform interaction rates. If D dominates C then non-uniform interaction rates can introduce a pair of interior equilibria; one of them is stable the other one is unstable. If C and D coexist, then non-uniform interaction rates cannot change the qualitative dynamics, but alter the location of the stable equilibrium.

1 Dynamical PD game with uniform interaction rate

Consider PD game between two players each having two strategies C (cooperation) and D (defection) and Cooperator pay a benefit $b$ at a cost $c$ to his Cooperator opponent, whereas Defector opponent only receive a benefit $b$ [6]. Then we have payoff matrix

$$
\begin{bmatrix}
  b - c & -c \\
  b & 0
\end{bmatrix}
$$

(1)
where, $b > c > 0$. We denote by $x$ and $y$ the frequency of individuals adopting strategy $C$ and $D$, respectively. We have $x + y = 1$.

In case of uniform interaction rate between any two individuals. We assume that interaction rate is independent of their strategies. The selection dynamics can be described by replicator equation [7]

$$\dot{x} = x(1 - x)(f_C - f_D)$$

(2)

The fitness of $C$ and $D$ players are linear functions of $x$ and $y$, given by

$$f_C = (b - c)x - cy, \quad f_D = bx.$$ 

(3)

The entire population will eventually consist of Defectors, since $b > b - c$ and $0 > -c$. The only stable equilibrium is $x = 0$. $D$ is a strict Nash equilibrium, and therefore an evolutionarily stable strategy (ESS).

2 Stability in dynamical PD game with non-uniform interaction rates

Suppose that an interaction between two players depends on their strategies. Let $r_1, r_2, r_3$ be reaction rates between each two players and $r_1, r_2, r_3 > 0$. Where, $r_1$ is an interaction rate of Cooperator with another Cooperator, $r_2$ is an interaction rate of Cooperator with Defector, and $r_3$ is an interaction rate of Defector with another Defector.

For uniform interaction rates $r_1 = r_2 = r_3$, the strategy $D$ dominates the strategy $C$. Hence, eventually the entire population will consist of defectors.

However, if players only interact with opponents of the same strategy, then cooperators cannot be exploited by defectors. In this case, where $r_2 = 0$ and $r_1, r_3 > 0$, cooperation is the dominant strategy, because $b - c > 0$. Assume $r_2 \neq 0$ which means that cooperators and defectors do interact. Without loss
of generality, assume that \( r_2 = 1 \).

The fitness of individuals is determined by the average payoff over a large number of interactions. Therefore, the fitness of C and D players are non-linear functions of \( x \) and \( y \), given by

\[
C = \frac{(b - c)r_1x - cy}{r_1x + y}, \quad D = \frac{bx}{x + r_3y}
\]

(4)

The replicator equations can be reduced to

\[
\dot{x} = x(1 - x)(f_C - f_D)
\]

(5)

where,

\[
f_C - f_D = \frac{((b - c)r_1x - cy)(x + r_3y) - (bx)(r_1x + y)}{(r_1x + y)(x + r_3y)}
\]

The equilibrium points are either on the boundary or in the interior. The boundary points \( x = 0 \) is stable equilibrium while \( x = 1 \) is unstable equilibrium. If \( f_C - f_D = 0 \) and \( x + y = 1 \) then we have two interior equilibrium are given by

\[
x^* = \frac{-(\alpha - 2\gamma) \pm \sqrt{\alpha^2 - 4\beta\gamma}}{2(\beta + \gamma - \alpha)}
\]

(6)

where,

\[
\alpha = r_1r_3(b - c) - (b + c), \quad \beta = -r_1c, \quad \gamma = -r_3c
\]

\[
\alpha^2 > 4\beta\gamma, \quad \beta + \gamma < \alpha, \quad \alpha > 2\gamma, \quad \alpha > 2\beta
\]

In case of non-uniform interaction rate, the selection dynamics depend on the size of \( r_1r_3 \) relative to \( \rho^2 \).[1] where,
\[ \rho = \frac{b + c}{b - c} \quad \text{and} \quad \rho > 1. \quad (7) \]

If \( r_1 r_3 < \left( \frac{b + c}{b - c} \right)^2 \) then defection is the dominant strategy. When \( r_1 r_3 = \left( \frac{b + c}{b - c} \right)^2 \), we have the bifurcation [8] point at

\[ x^* = \frac{\sqrt{r_1}}{\sqrt{r_1} + \sqrt{r_3}} \quad (8) \]

If \( r_1 r_3 > \left( \frac{b + c}{b - c} \right)^2 \) the two interior equilibria (6) move symmetrically away from the bifurcation point (8) toward the end points.

As \( r_1 \) and \( r_3 \) increase the stable interior equilibrium point moves from the bifurcation point (8) closer toward 1. Whereas, unstable interior equilibrium point moves from the bifurcation point (8) closer toward 0. As \( r_1, r_3 \rightarrow \infty \) we recover the case where \( r_2 = 0 \) and cooperation is the dominant strategy as defectors can no longer exploit cooperators.

3 Conclusion

For the non-repeated PD game, coexistence between cooperators and defectors is possible if the ratio of homogeneous \( r_1, r_3 \) over heterogeneous \( r_2 \) interaction rates exceeds a critical value. If \( r_1 = r_3 = \rho \), then the pair of equilibria approaches cooperator frequency of \( x = \frac{1}{2} \) which is entirely independent of the payoff matrix, as long as \( b > b - c > 0 > -c \). Both equilibria are stable, one consists of defectors alone, and the other consists of a mixture of defectors and cooperators.

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