OPTIMAL CONTRACEPTION CONTROL FOR A NONLINEAR POPULATION MODEL WITH SIZE STRUCTURE AND A SEPARABLE MORTALITY

Rong Liu¹, Feng-Qin Zhang¹ and Yuming Chen¹,²,*

¹Department of Applied Mathematics, Yuncheng University
Yuncheng 044000, China
²Department of Mathematics, Wilfrid Laurier University
Waterloo, ON N2L 3C5, Canada

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Abstract. This paper is concerned with the problem of optimal contraception control for a nonlinear population model with size structure. First, the existence of separable solutions is established, which is crucial in obtaining the optimal control strategy. Moreover, it is shown that the population density depends continuously on control parameters. Then, the existence of an optimal control strategy is proved via compactness and extremal sequence. Finally, the conditions of the optimal strategy are derived by means of normal cones and adjoint systems.

1. Introduction. Wild animals are not only valuable natural resources but also important components of a healthy ecosystem. However, relatively high densities of small rodents (called vermin) can result in a considerable threat to crops and can even destroy the ecological balance. This makes it necessary for us to intervene the growth of the vermin. Usually, chemical drugs are applied to poison the vermin, which will pollute the environment and destroy the ecological system. Recently, decreasing the reproductive rate instead of increasing mortality has been suggested as a promising way for managing the impact of overabundant species (see, for example, [16]). Often, female sterilant is used to achieve this purpose. See [21, 28] on contraception control for models without stage structures and see [8, 9, 10, 14, 25] for optimal birth control on models with age structures.

Note that age is only a special kind of size and size is one of the most natural and important variables to describe population dynamics. Here by size we mean some indices displaying the physiological or statistical characteristics of population individuals. Sizes can be mass, length, diameter, volume, maturity, and so on. For instance, for some animals and most trees, their metabolic capacity is related to their surface area [27]. Further studies show that the amount of food obtained by individuals is proportional to their surface area and the cost of the metabolism is proportional to their surface area.
proportional to their volume [29]. Moreover, the size of the individual can determine it’s prey object and niche, thus it can determine the ability of intra-competition and inter-competition [5]. As a result, modelling population dynamics with size structure has been an active and fruitful theme in mathematical biology (to name a few, see [2, 4, 6, 7, 11, 12, 13, 15, 17, 18, 19, 20, 22, 23, 24, 26, 31]). Unfortunately, only a few papers deal with the control problem [2, 11, 12, 13, 15, 18, 19, 23, 31]. Moreover, all focus on optimal harvesting except Araneda et al. [2] on optimal harvesting time and He et al. [11] on optimal birth control.

To the best of our knowledge, so far there is no investigation on the optimal contraception control of size-structured population models. The purpose of this paper is to make some contribution in this direction. The remaining part of this paper is organized as follows. First, we propose the model in Section 2, followed by the existence of separable solutions in Section 3. The last two sections are devoted to the optimal control policy. The existence of a unique optimal policy is proved via compactness and extremal sequence in Section 4 while optimality conditions are derived in Section 5.

2. The model formulation. Our study is inspired by that of Kato [19], where the author investigated the optimal harvesting problem for the following nonlinear size-structured population model

\[
\begin{align*}
\frac{\partial u(x,t)}{\partial t} + \beta(V(x,t)u(x,t)) & + \mu(x,t)u(x,t) + \Phi(I(t))u(x,t) = -\alpha(t)u(x,t) + f(x,t), \\
V(0,t)u(0,t) & = \int_0^l \beta(x,t,J(t))u(x,t) \, dx, \\
u(x,0) & = u_0(x), \\
I(t) & = \int_0^l m(x)u(x,t) \, dx, \\
J(t) & = \int_0^l b(x)u(x,t) \, dx,
\end{align*}
\]

(1)

with the objective functional

\[
\text{Maximize } \int_0^T \int_0^l g(x)\alpha(t)u^\alpha(x,t)dxdt \quad \text{subject to } \alpha \in \mathcal{U},
\]

where \(\mathcal{U} = \{ \alpha \in L^\infty(0,T) : 0 \leq \alpha(t) \leq L, \text{ a.e. } t \in (0,T) \}\). Here \(u(x,t)\) is the population density of size \(x \in [0,l]\) at time \(t \in [0,T]\); \(l \in (0,\infty)\) is a maximal size and \(T\) is a given time; \(V(x,t)\) represents the growth rate depending on the individual’s size \(x\) and time \(t\); \(\beta(x,t,J(t))\) is the fertility rate depending on size \(x\), time \(t\) and the total population \(J(t)\) weighted by \(b(x)\); \(\mu(x,t)\) is the natural mortality rate and \(\Phi(I(t))\) stands for an external mortality rate which depends on the total population \(I(t)\) weighted by \(m(x)\) possibly due to the intra-competition such as the limitation of the habitat; \(\alpha(t)\) is a harvesting rate depending only on time \(t\) and \(f(x,t)\) stands for a certain inflow such as immigration rate.

To build our model, we assume that the female sterilant applied at any time is eaten completely by the vermin (including the male vermin) and at any time the vermin individual with the same size eats the same amount of the female sterilant. In a similar way as to develop (1), we propose the following model on contraception
control for nonlinear size-structured population dynamics,

\[
\begin{aligned}
\frac{\partial u(x,t)}{\partial t} + \frac{\partial (V(x,t)u(x,t))}{\partial x} &= f(x,t) - \mu(x,t)u(x,t) - \Phi(I(t))u(x,t), \\
V(0,t)u(0,t) &= \int_0^t \beta(x,t)m(x,t)[1 - \delta \alpha(x,t)]u(x,t)dx, \\
u(x,0) &= u_0(x), \\
I(t) &= \int_0^l b(x)u(x,t)dx,
\end{aligned}
\]  

where \( Q = [0,l] \times [0,T] \). Here the state variable \( u(x,t) \) represents the density of the vermin individuals of size \( x \) at time \( t \). All meanings of the parameters are exact to or similar as those for (1) except the following. \( \beta(x,t) \) is the natural birth rate of the vermin; \( \delta \alpha(x,t) \) is the infertility rate of the female individuals of size \( x \) at time \( t \); \( m(x,t) \) is the proportion of the female in individuals of size \( x \) at time \( t \); \( \alpha(x,t) \) is the control variable, which measures the average amount of female sterilant eaten by a single female individual of size \( x \) at time \( t \) and is supposed to belong to the set \( \Omega \) \( = \{ \alpha \in L^\infty(Q) : 0 \leq \alpha(x,t) \leq L \ a.e. \ (x,t) \in Q \} \).

In the coming discussion, we make the following assumptions.

\( (A_1) \): There exists \( \beta \in R_+ \supseteq [0, \infty) \) such that \( 0 \leq \beta(x,t) \leq \bar{\beta} \) a.e. \((x,t) \in Q \).

\( (A_2) \): \( \mu(\cdot,t) \in L^1_{\text{loc}}((0,l)) \) and \( \mu(x,t) \geq 0 \) a.e. \((x,t) \in Q \).

\( (A_3) \): \( V : [0,l] \times [0,T] \rightarrow R_+ \) is a bounded continuous function such that \( V(x,t) > 0 \) for \((x,t) \in Q \) and \( V(l,t) = 0 \) for \( t \in [0,T] \). Furthermore, there is a constant \( L_V \) such that

\[
|V(x_1,t) - V(x_2,t)| \leq L_V|x_1 - x_2|
\]

for all \( x_1, x_2 \in [0,l] \) and \( t \in [0,T] \).

\( (A_4) \): \( m, f \in L^\infty(Q) \) and \( 0 < m(x,t) < 1 \), \( 0 \leq \delta \alpha(x,t) < 1 \) for all \((x,t) \in Q \).

\( (A_5) \): \( b \in L^\infty(0,l) \) and there exists \( \bar{b} \) such that \( 0 < b(x) \leq \bar{b} \) for all \( x \in [0,l] \).

\( (A_6) \): \( u_0(x) \in L^1(0,l) \) and there exists \( \bar{u} \in R_+ \) such that \( 0 \leq u_0(x) \leq \bar{u} \).

\( (A_7) \): \( \Phi : R_+ \rightarrow R_+ \) is a continuous function and there exists \( \bar{\Phi} \in R_+ \) such that \( \Phi(I) \leq \bar{\Phi} \) for all \( I \in R_+ \). Furthermore, there exists an increasing function \( C_\Phi : R_+ \rightarrow R_+ \) such that

\[
|\Phi(I_1) - \Phi(I_2)| \leq C_\Phi(r)|I_1 - I_2| \quad \text{for} \quad I_1, I_2 \leq r.
\]

Note that (2) is a special case of [19, system (4.1)], which is a general model for size-structured population dynamics with time dependent birth and aging functions. By [19, Theorem 4.1], for each \( u_0 \in L^1_+ \), system (2) has a unique global solution \( u \in C([0,T];L^1_+) \). For the definition of solution, see Definition 3.3 in the next section.

Let \( u^\alpha \) be the solution of (2) corresponding to \( \alpha \in \Omega \). Similar to [13], we investigate the following optimization problem,

\[
\text{minimize} \quad J(\alpha) = \int_0^T \int_0^l [g(u^\alpha(x,t) - \bar{u}(x)) + h(t)u^\alpha(x,t)\alpha(x,t)]dxdt,
\]

where \( \bar{u}(x) \in L^\infty(0,l) \) is a given ideal distribution of the vermin population, that is, the maximal amount of the vermin which will not affect the crop’s growth. The function \( g(u^\alpha(x,t) - \bar{u}(x)) \) represents the proximity of the controlled variable to the ideal distribution while the function \( h(t)u^\alpha(x,t)\alpha(x,t) \) represents the costs of the contraception control, which include the costs of the female sterilant and the costs of the related labor in applying the sterilant. Therefore, an optimal control
policy is one that the density of the vermin falls to as close as possible to the ideal distribution and the control cost is as low as possible.

Though (2) has a unique solution for each \( u_0 \), this will not help us much in dealing with the optimal control problem. As in [19], we shall seek separable solutions to (2).

3. The separable solutions. In this section, we provide some properties of the solutions, which include boundedness and the continuous dependence of the population density on the control parameter. As in [12], we first introduce some definitions.

**Definition 3.1.** The unique solution \( x = \varphi(t; t_0, x_0) \) of the initial-value problem \( x'(t) = V(x, t), \ x(t_0) = x_0 \) is said to be a characteristic curve of (2). Let \( z(t) = \varphi(t; 0, 0) \) be the characteristic curve through \((0, 0)\) in the \(x-t\) plane.

**Definition 3.2.** The derivative of the function \( u(x, t) \) at \((x, t)\) along the characteristic curve \( \varphi \) is given by

\[
D_\varphi u(x, t) = \lim_{h \to 0} \frac{u(\varphi(t + h; t, x), t + h) - u(x, t)}{h}.
\]

For an arbitrary point \((x, t)\) in the first quadrant of the \(x-t\) plane such that \( x \leq z(t) \), that is, \( \varphi(t; t, x) \leq z(t) \), define the initial time \( \tau \) by \( \varphi(\tau; t, x) = x \) if and only if \( \varphi(\tau; t, x) = 0 \). Utilizing the characteristic curve technique as in [19], we can define the solution of (2) as follows.

**Definition 3.3.** A function \( u(x, t) \in C([0, T], L^1) \) is said to be a solution of (2) if it satisfies

\[
u(x, t) = \begin{cases} 
\frac{F(\varphi(x, t))}{V(0, \varphi(x, t))} + \int_0^t G_V(s, u(x, s))(\varphi(s; t, x))ds & \text{if } x \leq z(t), \\
u_0(\varphi(0; t, x)) + \int_0^t G_V(s, u(x, s))(\varphi(s; t, x))ds & \text{if } x > z(t),
\end{cases}
\]

where, for \( t \in [0, T] \) and \( \phi \in L^1 \),

\[
F(t, \phi) = \int_0^t \beta(x, t)m(x, t)[1 - \delta_\alpha(x, t)]\phi(x)dx,
\]

\[
G_V(t, \phi)(x) = \int_0^t b(x)\phi(x)dx + V_\varepsilon(x, t)\phi(x).
\]

As mentioned earlier, according to [19], we consider the following separable solution of (2)

\[
u(x, t) = \tilde{u}(x, t)g(t).
\]

Substituting (4) into (2) gives the following two subsystems about \( \tilde{u}(x, t) \) and \( g(t) \),

\[
\begin{align*}
&\frac{\partial \tilde{u}(x, t)}{\partial t} + \frac{\partial (V(x, t)\tilde{u}(x, t))}{\partial x} = \frac{f(x, t)}{y(t)} - \mu(x, t)\tilde{u}(x, t), \quad (x, t) \in Q, \\
&V(0, t)\tilde{u}(0, t) = \int_0^t \beta(x, t)m(x, t)[1 - \delta_\alpha(x, t)]\tilde{u}(x, t)dx, \quad t \in (0, T], \\
&\tilde{u}(x, 0) = u_0(x), \quad x \in [0, l], \\
&y'(t) + \Phi(\tilde{I}(t)y(t))g(t) = 0, \quad t \in [0, T], \\
&y(0) = 1, \\
&\tilde{I}(t) = \int_0^t b(x)\tilde{u}(x, t)dx, \quad t \in [0, T].
\end{align*}
\]

As in [19], we define a solution to the subsystems (5) and (6) as follows.
Definition 3.4. Let $\alpha \in L^\infty(Q)$. A pair of functions $(\tilde{u}(x,t), y(t))$ with $\tilde{u} \in C([0,T], L^1)$ and $y \in C([0,T], R_+)$ is said to be a solution of the subsystems (5) and (6) if it satisfies the following two equations

$$
\tilde{u}(x,t) = \begin{cases} 
\frac{F(\tau, \tilde{u}(\tau , \cdot))}{V(0,\tau)} + \int_{\tau}^{t} G_y(\cdot, \tilde{u}(s, \cdot))(\varphi(s; t, x))ds, & x \leq \tilde{z}(t) \\
u_0(\varphi(0; t, x)) + \int_{0}^{t} G_y(\cdot, \tilde{u}(s, \cdot))(\varphi(s; t, x))ds, & x > \tilde{z}(t)
\end{cases}
$$

$$
y(t) = \exp \left\{ - \int_{0}^{t} \Phi(\tilde{I}(s)y(s))ds \right\},
$$

(7)

where

$$
\tilde{I}(s) = \int_{0}^{s} b(x) \tilde{u}(x, s)dx,
$$

$$
G_y(t, \phi)(x) = -\mu(x,t)\phi(x) - V_x(x,t)\phi(x) + \frac{f(x,t)}{y(t)}.
$$

Theorem 3.5. Let assumptions $(A_1)$ – $(A_7)$ hold. Then, for any $\alpha \in \Omega$, subsystems (5) and (6) have a unique solution $(\tilde{u}^\alpha(x,t), y^\alpha(t))$, which is non-negative.

Proof. Denote $\exp\{-\tilde{\Phi}T\} > 0$ by $\theta$ and let $A = \{h \in C([0, T] : \theta \leq h(t) \leq 1 \text{ for } t \in [0, T]\}$. For any $\lambda > 0$, define an equivalent norm in $C([0, T])$ by

$$
\|h\|_\lambda = \sup_{t \in [0, T]} e^{-\lambda t} |h(t)| \quad \text{for } h \in C([0, T]).
$$

The result is established by applying the Banach fixed point theorem.

Firstly, by (6), we have $y(t) = \exp\{-\int_{0}^{t} \tilde{\Phi}(\tilde{I}(s)y(s))ds\} \geq \theta$, which means that $y(t) \in A$. By [19, Theorem 4.1], we have that, for fixed $y(t) \in A$, subsystem (5) has a unique non-negative solution $\tilde{u}^\alpha(x,t) \in L^\infty(Q)$, which satisfies

$$
\|\tilde{u}^\alpha(\cdot , t)\|_{L^1} \leq e^{(\beta + 2L\nu)t} \|u_0\|_{L^1} + \int_{0}^{t} e^{(\beta + 2L\nu)(t-s)} \left\| \frac{f(\cdot, s)}{y(s)} \right\|_{L^1} ds
$$

$$
\leq e^{(\beta + 2L\nu)t} \|u_0\|_{L^1} + \int_{0}^{t} e^{(\beta + 2L\nu)(t-s)} \left\| \frac{f(\cdot, s)}{y(s)} \right\|_{L^1} ds
$$

$$
\leq e^{(\beta + 2L\nu)t} \left( \|u_0\|_{L^1} + \frac{\|f(\cdot, \cdot)\|_{L^Q}}{\delta} \right) \equiv r_0.
$$

(8)

Secondly, let $\tilde{I}^\alpha(t) = \int_{0}^{t} b(x) \tilde{u}^\alpha(x,t)dx$. For fixed $\tilde{I}^\alpha$, we define a mapping $A$ on $A$ by

$$
[Ah](t) = e^{-\int_{0}^{t} \tilde{\Phi}(\tilde{I}^\alpha h(s))ds} \quad \text{for } h \in A.
$$

It’s easy to see that $\theta \leq [Ah]\|_{L^1} \leq 1$, so $A$ is a map from $A$ to $A$. Moreover, by (8) and $(A_5)$, we can get

$$
|\tilde{I}^\alpha(t)| = \left| \int_{0}^{t} b(x) \tilde{u}^\alpha(x,t)dx \right| \leq \int_{0}^{t} |b(x)| \|\tilde{u}^\alpha(x,t)| \, dx \leq \bar{b} r_0 \triangleq r_1.
$$

(9)

Then, for any $h_1, h_2 \in A$, we have

$$
\|(Ah_1)(t) - (Ah_2)(t)\|_\lambda = \sup_{t \in (0, T)} \left[ e^{-\lambda t} |(Ah_1)(t) - (Ah_2)(t)| \right]
$$
Define a map such that

\[ A \]

By the Banach fixed point theorem, there exists a unique \( \tilde{\lambda} > C \). Thus, choosing \( y \)

\[ y \]

Then, for any \( t \)

\[ t \]

Thirdly, by [19, Lemma 2.4], for any \( y_1, y_2 \in A \), there exists a constant \( M > 0 \) such that

\[
\| \tilde{y}_{y_1}(\cdot,t) - \tilde{y}_{y_2}(\cdot,t) \|_{L^1} \leq M \int_0^t |y_1(s) - y_2(s)| \, ds \quad \text{for all } t \in [0,T]
\]

and

\[
e^{-\lambda t} \| \tilde{y}_{y_1}(\cdot,t) - \tilde{y}_{y_2}(\cdot,t) \|_{L^1} \leq \frac{M}{\lambda} |y_1 - y_2| \quad \text{for all } t \in [0,T].
\]

Define a map \( B : A \to A \) by

\[
(By)(t) = \tilde{y}(t) \quad \text{for } y \in A.
\]

Then, for any \( y_1, y_2 \in A \), by (9) and (10), we obtain

\[
\| (By_1)(t) - (By_2)(t) \|_{L^1} \leq H_1 + H_2
\]

For \( H_2 \), we consider \( \exp\{-\lambda t\} \int_0^t \| \tilde{I}^{y_1}(s) - \tilde{I}^{y_2}(s) \|_{L^1} \, ds \). Since

\[
\tilde{I}^{y_1}(s) - \tilde{I}^{y_2}(s) = \int_0^s b(x) \tilde{u}^{y_1}(x,s) \, ds - \int_0^s b(x) \tilde{u}^{y_2}(x,s) \, ds
\]

\[
\leq \int_0^s |b(x)| \cdot |\tilde{u}^{y_1}(x,s) - \tilde{u}^{y_2}(x,s)| \, ds
\]

\[
\leq \tilde{b} \| \tilde{u}^{y_1}(\cdot,s) - \tilde{u}^{y_2}(\cdot,s) \|_{L^1},
\]

we have

\[
e^{-\lambda t} \int_0^t \| \tilde{I}^{y_1}(s) - \tilde{I}^{y_2}(s) \|_{L^1} \, ds \leq \tilde{b} e^{-\lambda t} \int_0^t \| \tilde{u}^{y_1}(\cdot,s) - \tilde{u}^{y_2}(\cdot,s) \|_{L^1} \, ds
\]
Theorem 3.6. Let assumptions immediately produces the following result.

A mappping on \((A, \nu)\) which is non-negative and bounded, and can be written as

\[ \lambda > 0 \]

Choose \(\lambda > 0\) such that \(\exp(C_\Phi(r_1)r_1T)C_\Phi(r_1)M \bar{b} < 1\). Then \(\mathcal{B}\) is a contraction mapping on \((A, \parallel \cdot \parallel_\lambda)\). Therefore, by the Banach fixed point theorem again, \(\mathcal{B}\) owns a unique fixed point on \(A\).

In summary, we have proved that subsystems (5) and (6) have a unique solution \((u^y(x, t), y(t))\), which is non-negative and uniformly bounded. \(\square\)

Combining Theorem 3.5 with the fact that (2) has a unique solution for each \(u_0\) immediately produces the following result.

**Theorem 3.6.** Let assumptions \((A_1)-(A_7)\) hold. Then (2) has a unique solution, which is non-negative and bounded, and can be written as \(u(x, t) = \bar{u}^y(x, t)y(t)\), where \((\bar{u}^y(x, t), y(t))\) is a solution of (5) and (6).

To conclude this section, we prove that the population density depends continuously on the control parameter \(\alpha\).

**Theorem 3.7.** Assume that \((A_1)-(A_7)\) hold. Then the solution \(u^\alpha(x, t)\) of (2) is continuous with respect to the control variable \(\alpha\), that is, there exists a positive constant \(B\) such that, for any \(t \in [0, T]\), \(\alpha_1, \alpha_2 \in \Omega\), we have

\[ \parallel u_1(\cdot, t) - u_2(\cdot, t) \parallel_{L^1} \leq B \int_0^t \parallel \alpha_1(\cdot, s) - \alpha_2(\cdot, s) \parallel_{L^1} ds, \]

where \(u_1\) and \(u_2\) are the solutions of (2) corresponding to \(\alpha_1\) and \(\alpha_2 \in \Omega\), respectively.

**Proof.** Since \(u_1\) and \(u_2\) are solutions of (2) corresponding to \(\alpha_1\) and \(\alpha_2 \in \Omega\), respectively, by Theorem 3.6, we have

\[ u_1(x, t) = \bar{u}^{\alpha_1}(x, t)y_1(t) \quad \text{and} \quad u_2(x, t) = \bar{u}^{\alpha_2}(x, t)y_2(t). \]

From (9), we know that \(|\tilde{I}(s)| \leq r_1\). Then, by assumption \((A_7)\) and (7), we get

\[ \parallel y_1(t) - y_2(t) \parallel \leq e^{-\lambda t} \int_0^t M \bar{b} \parallel y_1(s) - y_2(s) \parallel_\lambda ds \]

\[ \leq \frac{M \bar{b}}{\lambda^2} \parallel y_1(t) - y_2(t) \parallel_\lambda. \]
Applying the Gronwall’s lemma produces
\[
|y_1(t) - y_2(t)| \leq M \int_0^t \| \tilde{u}^{y_1}(\cdot, s) - \tilde{u}^{y_2}(\cdot, s) \|_{L^1} ds,
\]
where \( M = \exp\{C_\Phi(r_1) r_1 T \} TC_\Phi^2(r_1) r_1 \tilde{b} + C_\Phi(r_1) r_1 \tilde{b} \).

It’s easy to show that \( y(t) \leq 1 \). Thus, by (4) and (8), we obtain
\[
\| u_1(\cdot, t) - u_2(\cdot, t) \|_{L^1} = \| \tilde{u}^{y_1}(\cdot, t) y_1(t) - \tilde{u}^{y_2}(\cdot, t) y_2(t) \|_{L^1} \\
\leq \| \tilde{u}^{y_1}(\cdot, t) - \tilde{u}^{y_2}(\cdot, t) \|_{L^1} + r_0 |y_1(t) - y_2(t)|.
\]
Note that \( \tilde{u}^{y_1}(x, t) \) and \( \tilde{u}^{y_2}(x, t) \) are solutions of (5) corresponding to \( \alpha_1 \) and \( \alpha_2 \in \Omega \), respectively. Treating \( \tilde{u}^{y_1}(x, t) \) and \( \tilde{u}^{y_2}(x, t) \) in the same manner as that in obtaining (8), we have
\[
e^{-\beta t} \| \tilde{u}^{y_1}(\cdot, t) - \tilde{u}^{y_2}(\cdot, t) \|_{L^1} \\
\leq \int_0^t e^{-\beta s} \left\| \frac{f(\cdot, s)}{y_1(\cdot, s)} - \frac{f(\cdot, s)}{y_2(\cdot, s)} \right\|_{L^1} ds + \delta r_0 \int_0^t e^{-\beta s} \| \alpha_1(\cdot, s) - \alpha_2(\cdot, s) \|_{L^1} ds \\
+ L \int_0^t e^{-\beta s} \| \tilde{u}^{y_1}(\cdot, s) - \tilde{u}^{y_2}(\cdot, s) \|_{L^1} ds
\]
\[
\leq \int_0^t e^{-\beta s} \left\| \frac{f(\cdot, s)}{y_1(\cdot, s) - y_2(\cdot, s)} \right\|_{L^1} ds + \delta r_0 \int_0^t e^{-\beta s} \| \alpha_1(\cdot, s) - \alpha_2(\cdot, s) \|_{L^1} ds \\
+ L \int_0^t e^{-\beta s} \| \tilde{u}^{y_1}(\cdot, s) - \tilde{u}^{y_2}(\cdot, s) \|_{L^1} ds
\]
\[
\leq \left( \frac{M}{\beta^2} \| f \|_{L^1(Q)} + L \right) \int_0^t e^{-\beta s} \| \tilde{u}^{y_1}(\cdot, s) - \tilde{u}^{y_2}(\cdot, s) \|_{L^1} ds \\
+ \delta r_0 \int_0^t e^{-\beta s} \| \alpha_1(\cdot, s) - \alpha_2(\cdot, s) \|_{L^1} ds
\]
\[
\triangleq M_1 \int_0^t e^{-\beta s} \| \tilde{u}^{y_1}(\cdot, s) - \tilde{u}^{y_2}(\cdot, s) \|_{L^1} ds + \delta r_0 \int_0^t e^{-\beta s} \| \alpha_1(\cdot, s) - \alpha_2(\cdot, s) \|_{L^1} ds.
\]
Using Gronwall’s lemma again gives us
\[
e^{-\beta t} \| \tilde{u}^{y_1}(\cdot, t) - \tilde{u}^{y_2}(\cdot, t) \|_{L^1} \\
\leq M_2 \delta r_0 \int_0^t e^{M_1(t-s)} \left\| \int_0^s e^{-\beta \tau} \| \alpha_1(\cdot, \tau) - \alpha_2(\cdot, \tau) \|_{L^1} d\tau \right\|_{L^1} ds
\]
\[
+ \delta r_0 \int_0^t e^{-\beta s} \| \alpha_1(\cdot, s) - \alpha_2(\cdot, s) \|_{L^1} ds.
\]
The required result follows immediately from (13)–(15) and hence the proof is complete.
4. Existence of optimal control policy. The purpose of this section is to prove the existence of the optimal control policy.

Lemma 4.1. Let assumptions (A1)–(A7) hold. Then \( \{ I^\alpha(t) : \alpha \in \Omega \} \) is a relatively compact set in \( L^2(0,T) \), where \( I^\alpha(t) = \int_0^1 b(x) u^\alpha(x,t) dx \).

Proof. First, we show that \( \frac{dI^\alpha(t)}{dt} \) is uniformly bounded about the control variable \( \alpha \in \Omega \). Note that \( \frac{dI^\alpha(t)}{dt} = \int_0^1 b(x) \frac{\partial u^\alpha(x,t)}{\partial t} dx \). Multiplying (2) by \( b(x) \) and integrating on \((0,t)\), we obtain

\[
\int_0^t b(x) \frac{\partial u^\alpha(x,t)}{\partial t} dx = \int_0^t b(x)[f(x,t) - \mu u^\alpha(x,t) - \Phi(I^\alpha(t))u^\alpha(x,t)] dx - \int_0^t b(x) \frac{\partial(V(x,t)u^\alpha(x,t))}{\partial x} dx \leq I_1(t) + I_2(t).
\]

By the assumptions and Theorem 3.6, we know that \( I_1 \) is uniformly bounded about the control variable \( \alpha \in \Omega \). For \( I_2 \), by the second equation of (2), we obtain

\[
I_2(t) = \int_0^t b(x) \frac{\partial(V(x,t)u^\alpha(x,t))}{\partial x} dx = V(0,t)u^\alpha(0,t)b(0) + \int_0^t b(x)V(x,t)u^\alpha(x,t) dx = b(0) \int_0^t m(x,t) \beta(x,t)[1 - \delta \alpha(x,t)]u^\alpha(x,t) dx + \int_0^t b(x)V(x,t)u^\alpha(x,t) dx.
\]

Using the assumptions, we know that \( I_2 \) is also uniformly bounded about the control variable \( \alpha \in \Omega \). In summary, we have proved that \( \frac{dI^\alpha(t)}{dt} \) is uniformly bounded about the control variable \( \alpha \in \Omega \).

Next, we use Fréchet-Kolmogorov guidelines (see, for example, [30]) to prove that \( \{ I^\alpha(t) : \alpha \in \Omega \} \) is a relatively compact set in \( L^2(0,T) \). For convenience, we denote \( I^\alpha(t) = 0 \) if \( t < 0 \) or \( t > T \). Then \( I^\alpha(t) \) is continuous on \( R \). To apply Fréchet-Kolmogorov guidelines, we need to verify the following three things.

1° Uniform boundedness of \( I^\alpha(t) \) about \( \alpha \). This is easy to see since \( I^\alpha(t) = \int_0^1 b(x) u^\alpha(x,t) dx \).

2° \( \lim_{t \to 0} \int_0^T [I^\alpha(s + t) - I^\alpha(s)]^2 ds = 0 \). In fact, this follows from the uniform boundedness of \( \frac{dI^\alpha(t)}{dt} \) about \( \alpha \in \Omega \) and

\[
\int_0^T [I^\alpha(s + t) - I^\alpha(s)]^2 ds = \int_0^T \left[ \int_s^{s+t} \frac{dI^\alpha(r)}{dr} dr \right]^2 ds \\
\leq \int_0^T \left[ \int_s^{s+t} \left( \frac{dI^\alpha(r)}{dr} \right)^2 dr \right] ds.
\]
Assume that Theorem 4.2. This completes the proof.

Theorem 4.2. Assume that \((A_1)-(A_7)\) hold, and \(g, h : R \rightarrow R_+\) are nonnegative continuous convex functions. Then the control problem (2)–(3) has at least one solution.

Proof. Denote \(d = \inf_{\alpha \in \Omega} J(\alpha)\). With the conclusion of Theorem 3.7, it follows from Theorem 3.6 that \(0 \leq d < \infty\).

Let \(\{\alpha_n : n \geq 1\}\) be a minimizing sequence of \(J(\alpha)\) in \(\Omega\) satisfying

\[
d \leq J(\alpha_n) < d + \frac{1}{n} \quad \text{for all } n \geq 1.
\]

Since \(\{u^{\alpha_n}\}\) is uniformly bounded about the control variable \(\alpha \in \Omega\), there exists a subsequence of \(\{\alpha_n\}\), still denoted by \(\{\alpha_n\}\), such that \(u^{\alpha_n} \rightharpoonup u^*\) weakly in \(L^2(Q)\) as \(n \rightarrow \infty\) for some \(u^* \in L^2(Q)\). For \(\{u^{\alpha_n}\}\), it follows from the Mazur Theorem (see [1]) that there exists the following convex combination of \(\{u^{\alpha_n}\}\),

\[
\tilde{u}_n = \sum_{i=n+1}^{k_n} \lambda_i^n u^{\alpha_i}, \quad \lambda_i^n \geq 0, \quad \sum_{i=n+1}^{k_n} \lambda_i^n = 1, \quad k_n \geq n + 1,
\]

such that \(\tilde{u}_n \rightharpoonup u^*\) in \(L^2(Q)\) as \(n \rightarrow \infty\). It follows from Lemma 4.1 that there exists a subsequence of \(\{\alpha_n\}\), still denoted by \(\{\alpha_n\}\), such that \(I^{\alpha_n} \rightharpoonup I^*\) as \(n \rightarrow \infty\) and \(I^{\alpha_n}(t) \rightharpoonup I^*(t)\) for almost every \(t \in (0, T)\). Consequently, \(I^*(t) = \int_0^T b(x) u^*(x, t) dx\).

Define the function sequence of the control variable as

\[
\tilde{\alpha}_n(x, t) = \begin{cases} \sum_{i=n+1}^{k_n} \lambda_i^n u^{\alpha_i}(x, t) & \text{if } \sum_{i=n+1}^{k_n} \lambda_i^n u^{\alpha_i}(x, t) \neq 0, \\
0 & \text{if } \sum_{i=n+1}^{k_n} \lambda_i^n u^{\alpha_i}(x, t) = 0.
\end{cases}
\]

(17)

For any \(n + 1 \leq i \leq k_n, 0 \leq \alpha_i(x, t) \leq L\) since \(\alpha_i(x, t) \in \Omega\). Combining this with \(\lambda_i^n \geq 0\) and \(u^{\alpha_i}(x, t) \geq 0\), we have

\[
0 \leq \sum_{i=n+1}^{k_n} \lambda_i^n u^{\alpha_i}(x, t) \alpha_i(x, t) \leq L \sum_{i=n+1}^{k_n} \lambda_i^n u^{\alpha_i}(x, t),
\]

which implies that \(\tilde{\alpha}_n \in \Omega\).

From the boundedness of \(\{\tilde{\alpha}_n\}\) and the weak compactness of the bounded sequence, we obtain that there exists a subsequence of \(\{\tilde{\alpha}_n\}\), still denoted by \(\{\tilde{\alpha}_n\}\), such that \(\tilde{\alpha}_n \rightharpoonup \alpha^*\) weakly in \(L^2(Q)\) as \(n \rightarrow \infty\).
Because \( u^{\alpha_i} \) is the solution of (2) corresponding to \( \alpha = \alpha_i \in \Omega \), we have
\[
\begin{cases}
D_\varphi u^{\alpha_i}(x,t) = f(x,t) \\
-\mu(x,t) + V_x(x,t) + \Phi(I^{\alpha_i}(t))u^{\alpha_i}(x,t), & (x,t) \in Q, \\
V(0,t)u^{\alpha_i}(0,t) = \int_0^l \beta(x,t)m(x,t)[1 - \delta \alpha_i(x,t)]u^{\alpha_i}(x,t)dx, & t \in (0,T], \\
u^{\alpha_i}(x,0) = u_0(x), & x \in [0,l], \\
I^{\alpha_i}(t) = \int_0^t b(x)u^{\alpha_i}(x,t)dx, & t \in [0,T].
\end{cases}
\tag{18}
\]

By (16)–(18), we get
\[
\begin{cases}
D_\varphi \tilde{u}_n(x,t) = f(x,t) - [\mu(x,t) + V_x(x,t)]\tilde{u}_n \\
-\sum_{i=n+1}^{k_n} \lambda_i^n \Phi(I^{\alpha_i}(t))u^{\alpha_i}(x,t), \\
V(0,t)\tilde{u}_n(0,t) = \int_0^l \beta(x,t)m(x,t)[1 - \delta \tilde{\alpha}_n(x,t)]\tilde{u}_n(x,t)dx, \\
\tilde{u}_n(x,0) = u_0(x), \\
I^{\alpha_i}(t) = \int_0^t b(x)u^{\alpha_i}(x,t)dx.
\end{cases}
\tag{19}
\]

Since \( I^{\alpha_i}(t) \to I^*(t) \) as \( n \to \infty \), we obtain
\[
\sum_{i=n+1}^{k_n} \lambda_i^n \Phi(I^{\alpha_i}(t))u^{\alpha_i}(x,t) \to \Phi(I^*(t))u^*(x,t).
\]

Letting \( n \to \infty \) in (19) and using the continuity of \( \Phi \) yield
\[
\begin{cases}
D_\varphi u^*(x,t) = f(x,t) - [\mu(x,t) + V_x(x,t) + \Phi(I^*(t))]u^*(x,t), & (x,t) \in Q, \\
V(0,t)u^*(0,t) = \int_0^l \beta(x,t)m(x,t)[1 - \delta \alpha^*(x,t)]u^*(x,t)dx, & t \in (0,T], \\
u^*(x,0) = u_0(x), & x \in (0,l], \\
I^*(t) = \int_0^t b(x)u^*(x,t)dx, & t \in [0,T],
\end{cases}
\]

which implies that \( u^*(x,t) = u^{\alpha^*}(x,t) \) a.e. \((x,t) \in Q\) and \( I^*(t) = I^{\alpha^*}(t) \).

Next, we show that the control \( \alpha^* \in \Omega \) is an optimal policy. On the one hand, since \( d \leq J(\alpha_i) < d + \frac{1}{n} \) for any \( \alpha_i \in \Omega \), we have
\[
d \leq \sum_{i=n+1}^{k_n} \lambda_i^n J(\alpha_i) < d + \frac{1}{n}.
\]

Letting \( n \to \infty \) gives us
\[
\sum_{i=n+1}^{k_n} \lambda_i^n J(\alpha_i) \to d.
\]

On the other hand, by the definition of \( J(\cdot) \), (16), and (17), we have
\[
\begin{align*}
\sum_{i=n+1}^{k_n} \lambda_i^n J(\alpha_i) & = \sum_{i=n+1}^{k_n} \lambda_i^n \int_0^T \int_0^l g(u^{\alpha_i}(x,t) - \overline{u}(x)) + h(t)\alpha_i(x,t)u^{\alpha_i}(x,t)dxdt \\
& \geq \int_0^T \int_0^l g \left( \sum_{i=n+1}^{k_n} \lambda_i^n u^{\alpha_i}(x,t) - \overline{u}(x) \right)
\end{align*}
\]
Letting 

\[ J = \int_0^T g(u^*(x, t) - \bar{u}(x)) + h(t)\alpha^*(x, t)u^*(x, t) dt \]

for sufficiently small \( \varepsilon > 0 \), we get from Theorem 3.6 and assumption \( (A_7) \) that 

\[ \frac{1}{\varepsilon}[u_{\alpha^*} - u^*] \rightarrow z \quad \text{as} \quad \varepsilon \rightarrow 0, \]

Therefore, \( J(\alpha^*) = d = \inf_{\alpha \in \Omega} J(\alpha) \), which means that \( \alpha^* \) is an optimal control policy for the control problem (2)–(3). This completes the proof. \( \square \)

5. Optimality conditions. The purpose of this section is to derive the first-order necessary conditions of optimality in the form of an Euler-Lagrange system.

Lemma 5.1. Under the assumption \( (A_7) \), for each \( \alpha \in \Omega \) and \( v \in L^\infty(Q) \) such that \( \alpha + \varepsilon v \in \Omega \) for sufficiently small \( \varepsilon > 0 \), we have

\[ \frac{1}{\varepsilon}[u^\alpha + \varepsilon v - u^\alpha] \rightarrow z \quad \text{as} \quad \varepsilon \rightarrow 0, \]

where \( z \in L^\infty(Q) \) is the solution of the following system

\[
\begin{align*}
D_\varepsilon z(x, t) &= -(\mu(x, t) + V_\varepsilon(x, t) + \Phi(I^\alpha(t)))z(x, t) - u^\alpha(x, t)\Phi'(I^\alpha(t))Z(t) \\
V(0, t)z(0, t) &= \int_0^t \beta(x, t)m(x, t)[1 - \delta\alpha(x, t)]z(x, t) dx, \\
&\quad - \int_0^t \delta\beta(x, t)m(x, t)v(x, t)u^\alpha(x, t) dx, \\
z(x, 0) &= 0, \\
Z(t) &= \int_0^t b(x)z(x, t) dx.
\end{align*}
\]

Proof. The existence and uniqueness of solution to (20) can be established by a similar way as that in the proofs of Theorem 3.5 and Theorem 3.6. According to [1, Lemma 3.13], \( \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon}[u^\alpha + \varepsilon v - u^\alpha] \) makes sense. Since \( u^\varepsilon \triangleq u^\alpha + \varepsilon v \) and \( u^\alpha \) are solutions of (2) corresponding to \( \alpha + \varepsilon v \) and \( \alpha \), respectively, \( \frac{1}{\varepsilon}[u^\varepsilon - u^\alpha] \) must be the solution of the following system

\[
\begin{align*}
D_\varepsilon \frac{1}{\varepsilon}[u^\varepsilon - u^\alpha] &= -(\mu(x, t) + V_\varepsilon(x, t) + \Phi(I^\alpha(t)))\frac{1}{\varepsilon}[u^\varepsilon - u^\alpha] - \frac{1}{\varepsilon}[\Phi(I^\varepsilon(t))u^\varepsilon - \Phi(I^\alpha(t))u^\alpha], \\
V(0, t)\frac{1}{\varepsilon}[u^\varepsilon - u^\alpha](0, t) &= \int_0^t \beta m[(u^\varepsilon - u^\alpha) - (\delta(\alpha + \varepsilon v)u^\varepsilon - \delta\alpha u^\alpha)] dx, \\
\frac{1}{\varepsilon}[u^\alpha + \varepsilon v - u^\alpha](x, 0) &= 0, \\
\frac{1}{\varepsilon}[I^\varepsilon(t) - I^\alpha(t)] &= \int_0^t b(x)\frac{1}{\varepsilon}(u^\varepsilon - u^\alpha) dx.
\end{align*}
\]

Letting \( \varepsilon \rightarrow 0 \), we get from Theorem 3.6 and assumption \( (A_7) \) that

\[ \frac{1}{\varepsilon}[\Phi(I^\varepsilon(t))u^\varepsilon(x, t) - \Phi(I^\alpha(t))u^\alpha(x, t)] \]

\[ = \frac{1}{\varepsilon}[\Phi(I^\varepsilon(t)) - \Phi(I^\alpha(t))]u^\varepsilon(x, t) + \frac{1}{\varepsilon}\Phi(I^\alpha(t))[u^\varepsilon(x, t) - u^\alpha(x, t)] \]

\[ \rightarrow \Phi'(I^\alpha(t))u^\alpha(x, t)Z(t) + \Phi(I^\alpha(t))z(x, t). \]
Similarly, we have
\[ \frac{1}{\varepsilon} [\delta(\alpha(x,t) + \varepsilon v(x,t))u^{\alpha + \varepsilon v}(x,t) - \delta \alpha(x,t)u^{\alpha}(x,t)] \]
\[ \rightarrow \delta u^{\alpha}(x,t)v(x,t) + \delta \alpha(x,t)z(x,t) \]
as \varepsilon \rightarrow 0. Passing to the limit as \varepsilon \rightarrow 0 produces the required result. ∎

Next we consider the following adjoint system of (2),
\[ \begin{align*}
D_{\cdot} \xi(x,t) &= \left[ \mu + \Phi(I^\alpha(x,t)) \right] \xi(x,t) - g'(u^*(x,t) - \bar{u}(x)) \\
&\quad - h(t)\alpha^*(x,t) - \beta(x,t)m(x,t)[1 - \delta \alpha^*(x,t)]\xi(0,t) \\
&\quad + b(x)\Phi(I^\alpha(x,t)) \int_0^1 u^* \xi dx, \\
\xi(l,t) &= \xi(x,T) = 0, \quad (x,t) \in Q. \tag{21}
\end{align*} \]

Treating (21) in the same manner as that done in Theorem 3.5–Theorem 3.7, we can get the following result with the proof being omitted.

**Theorem 5.2.** Let assumptions (A1)–(A7) hold. For each \( \alpha \in \Omega \), the adjoint system (21) has a unique bounded solution \( \xi^\alpha \in L^\infty(Q) \). Moreover, there exists a positive constant \( B_1 \) such that
\[ \| \xi_1 \|_{L^1(Q)} \leq B_1 \| \alpha_1 - \alpha_2 \|_{\infty}, \]
where \( \xi_1 \) and \( \xi_2 \) are solutions of (21) corresponding to \( \alpha_1 \) and \( \alpha_2 \in \Omega \), respectively.

Now we are ready to present the optimality conditions.

**Theorem 5.3.** Let \( \alpha^*(x,t) \) be an optimal policy for the contraception control problem (2)–(3). Under the conditions of Theorem 3.7, if \( g' \) is bounded, then
\[ \alpha^*(x,t) = \begin{cases} 0 & \text{if } \delta \beta(x,t)m(x,t)\xi(0,t) < h(t), \\ L & \text{if } \delta \beta(x,t)m(x,t)\xi(0,t) > h(t), \end{cases} \]
where \( \xi(x,t) \) is the solution of the adjoint system (21).

**Proof.** For any element of the tangent cone, \( v \in T_\Omega(\alpha^*) \) (see [3]), we have \( \alpha^\varepsilon \triangleq \alpha^* + \varepsilon v \in \Omega \) for sufficiently small \( \varepsilon > 0 \). Let \( u^\varepsilon \) and \( u^* \) be the solutions of (2) corresponding to \( \alpha = \alpha^\varepsilon \) and \( \alpha^* \), respectively. By the optimality of \( \alpha^* \), we have
\[ \int_0^T \int_0^l [g(u^\varepsilon(x,t) - \bar{u}(x)) + h(t)\alpha^\varepsilon(x,t)u^\varepsilon(x,t)]dxdt \geq \int_0^T \int_0^l [g(u^*(x,t) - \bar{u}(x)) + h(t)\alpha^*(x,t)u^*(x,t)]dxdt. \]

Consequently, by Lemma 5.1 and the conditions of the theorem, we have
\[ 0 \leq \int_0^T \int_0^l [g'(u^*(x,t) - \bar{u}(x)) + h(t)\alpha^*(x,t)]z(x,t)dxdt \]
\[ \quad + \int_0^T \int_0^l h(t)e(x,t)u^*(x,t)dxdt, \]
where \( z(x,t) \) is the solution of (20) with \( \alpha(x,t) \) and \( u^\alpha(x,t) \) being replaced by \( \alpha^*(x,t) \) and \( u^*(x,t) \), respectively.

We now prove that
\[ \int_0^T \int_0^l \delta \beta muv^* \xi(0,t)dxdt = - \int_0^T \int_0^l [g'(u^*(x,t) - \bar{u}(x)) + h(t)\alpha^*]zdxdt. \]
In fact, multiplying the first equation of (21) by $z(x,t)$ and integrating on $Q$, we obtain that the left side of the resultant is
\[
\int_0^T \int_0^t \xi_t(x,t) z(x,t) \, dx \, dt + \int_0^T \int_0^t V(x,t) \xi_x(x,t) z(x,t) \, dx \, dt
\]
\[
= - \int_0^T \int_0^t z_t \xi \, dx \, dt - \int_0^T \int_0^t V z_x \xi \, dx \, dt - \int_0^T \int_0^t V z \xi \, dx \, dt
\]
\[
= - \int_0^T \int_0^t (z_t + V z_x) \xi \, dx \, dt - \int_0^T \int_0^t V z \xi \, dx \, dt
\]
\[
- \int_0^T \int_0^t \beta(x,t) m(x,t) [1 - \delta \alpha^*(x,t)] z(x,t) \xi(0,t) \, dx \, dt
\]
\[
+ \int_0^T \int_0^t \delta \beta(x,t) m(x,t) v(x,t) u^*(x,t) \xi(0,t) \, dx \, dt
\]
and the right hand side of the resultant is
\[
\int_0^T \int_0^t \mu z \xi \, dx \, dt + \int_0^T \int_0^t \left[ b(x) \Phi'(I^\alpha^*(t)) \int_0^t u^*(r,t) \xi(r,t) \, dr \right] z(x,t) \, dx \, dt
\]
\[
+ \int_0^T \int_0^t \Phi(I^\alpha^*(t)) z \xi \, dx \, dt - \int_0^T \int_0^t \int_0^t \left[ g'(u^*(x,t) - \bar{u}(x)) + h(t) \alpha^*(x,t) \right] z \, dx \, dt
\]
\[
- \int_0^T \int_0^t \beta(x,t) m(x,t) [1 - \delta \alpha^*(x,t)] z(x,t) \xi(0,t) \, dx \, dt.
\]
Therefore, we have
\[
\int_0^T \int_0^t \left[ z_t(x,t) + (V(x,t) z(x,t))_x \right] \xi(x,t) \, dx \, dt
\]
\[
= - \int_0^T \int_0^t \mu z \xi \, dx \, dt - \int_0^T \int_0^t \int_0^t \left[ g'(u^*(x,t) - \bar{u}(x)) + h(t) \alpha^*(x,t) \right] z \, dx \, dt
\]
\[
- \int_0^T \int_0^t \int_0^t \left[ b(x) \Phi'(I^\alpha^*(t)) \int_0^t u^*(r,t) \xi(r,t) \, dr \right] z \, dx \, dt
\]
\[
+ \int_0^T \int_0^t \delta \beta(x,t) m(x,t) v(x,t) u^*(x,t) \xi(0,t) \, dx \, dt
\]
\[
- \int_0^T \int_0^t \Phi(I^\alpha^*(t)) z \xi \, dx \, dt.
\]
Next, multiplying the first equation of (20) by $\xi(x,t)$ with $\alpha(x,t)$ and $u^\alpha(x,t)$ being replaced by $\alpha^*(x,t)$ and $u^*(x,t)$, respectively, and integrating on $Q$, we have
\[
\int_0^T \int_0^t \left[ z_t(x,t) + (V(x,t) z(x,t))_x \right] \xi(x,t) \, dx \, dt
\]
\[
= - \int_0^T \int_0^t \mu(x,t) z(x,t) \xi(x,t) \, dx \, dt
\]
\[
- \int_0^T \int_0^t \Phi(I^\alpha^*(t)) z(x,t) \xi(x,t) \, dx \, dt
\]
\[-\int_0^T \int_0^l \Phi'(I^{x^*}(x,t))u^*(x,t)\xi(x,t)Z(t)dxdt.\]

By (22) and (23), we obtain
\[
\int_0^T \int_0^l \delta[\beta mnu^*](x,t)\xi(0,t)dxdt = -\int_0^T \int_0^l \left[ g'(u^*(x,t) - \bar{u}(x)) + h(t)\alpha^* \right]u^*(x,t)dxdt.
\]

It follows that
\[
\int_0^T \int_0^l [\delta\beta(x,t)m(x,t)\xi(0,t) - h(t)]v(x,t)u^*(x,t)dxdt \leq 0
\]
for every \(v \in T_\Omega(\alpha^*)\). Thus \(\delta\beta(x,t)m(x,t)\xi(0,t) - h(t)\) \(u^*(x,t) \in N_\Omega(\alpha^*)\) (see [3]), where \(N_\Omega(\alpha^*)\) is the normal cone of \(\Omega\) at \(\alpha^*\), which implies the conclusion of this theorem.

The conditions in Theorem 5.3 for the optimal policy provide a threshold condition, which depends on \(\delta\beta(x,t)m(x,t)\xi(0,t)\). Roughly speaking, if this quantity is less than 1, that is, if the price rate for contraception control is very large and the effective birth rate \(\beta(x,t)m(x,t)\) is very small, then there is no need to apply sterilant; otherwise, one needs to apply the sterilant to its maximum. However, since we do not have a clear biological meaning of the solution \(\xi\) to the adjoint system, it is hard to give a precise interpretation of the above threshold condition.

To conclude this paper, we mention that in practical application, we first need to fit the parameters of the model by the observation data and to determine the functions of \(g\) and \(h\). Then, combing (2) and (3) with (18), we can calculate the optimal contraception policy, optimal population density and optimal index. But this is a challenging problem of computational mathematics. We will leave it to future study.

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E-mail address: 18042318180@163.com
E-mail address: zhafq@263.net
E-mail address: ychen@wlu.ca