B to $\pi$ Form Factors in collinear factorization approach

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Abstract

The form factors for semi-leptonic $B$ decays, $\bar{B} \rightarrow \pi l \bar{\nu}_l$, are calculated under collinear factorization approach. The end-point divergences are regularized by a $\xi$-regularization, where $\xi$ means the collinear fraction of the spectator anti-quark of the $\bar{B}$ meson. The form factors are calculated up-to $O(\alpha_s/m_B)$. The complete $O(1/m_B)$ contributions from the $\pi$ meson are calculated explicitly by a collinear expansion method. A well-defined power expansion scheme is built such that the $O(1/m_B)$ contributions are about 30% of the leading order contributions. A small value $F_+(0) = 0.164$ is found. This confirms the SCET result $F_+(0) = 0.17$ from $B \rightarrow \pi\pi$ decays. The form factors are calculated for $0 \leq q^2 \leq 16$ GeV$^2$, where $q^2$ is the invariant mass of the lepton pair in $\bar{B} \rightarrow \pi l \bar{\nu}_l$. An extrapolation of the form factors to $q^2 > 16$ GeV$^2$ is made to obtain $|V_{ub}|^{-2} \int_0^{26.42\text{GeV}^2} dq^2 (d\Gamma/dq^2)_{th} = (5.71 \pm 0.91) \text{ps}^{-1}$. We determine $|V_{ub}| = (3.95 \pm 0.13_{\text{exp}} \pm 0.32_{\text{th}}) \times 10^{-3}$ from the world averaged branching ratio $Br(\bar{B} \rightarrow \pi l \bar{\nu}_l) = (1.36 \pm 0.09) \times 10^{-4}$.

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I. INTRODUCTION

$B$ factories have obtained precise results on Cabibbo-Kobayashi-Maskawa (CKM) matrix elements of the standard model (SM) [1]. The LHCb has already started running. More results on CKM matrix elements with higher accuracy are expected in near future [2]. Being an important element of CKM matrix elements, $|V_{ub}|$ still contains large uncertainties dominated by theoretical ones. For example, the $|V_{ub}|$ from inclusive $\bar{B} \to X_u l\bar{\nu}_l$ processes contains 10% uncertainties, where the 7% uncertainty comes from the 60 MeV uncertainty for $m_b$ and the other 3% uncertainty comes from experiments. The $|V_{ub}|$ from exclusive $\bar{B} \to \pi l\nu_l$ processes contains a $10 - 15\%$ uncertainty $[3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19]$ from the $B \to \pi$ form factor, $F_{B\pi}^{+}$, and a 6% uncertainty from experiments $[20, 21, 22]$. In addition, the $q^2$ spectrum of $\bar{B} \to \pi l\nu_l$ has been well constrained experimentally $[22]$.

To take full advantage of the experimental precision for exclusive $\bar{B} \to \pi l\nu_l$, it is necessary to pin down the theoretical uncertainties on $F_{B\pi}^{+}$ to few percentage level. However, it is still a difficult task. Currently, nonperturbative methods, including QCD light-cone sum rule (LCSR) $[3, 4, 5, 6, 7, 8, 9]$ and lattice QCD (LQCD) $[10, 11, 12, 13]$, are available. Innovative parameterization (PA) methods with model independent inputs from theories have been built $[14, 15, 16, 17, 18, 23]$.

Naive application of perturbative QCD (pQCD) to $F_{B\pi}^{+}$ needs to account for logarithmic or linear end-point divergences $[24, 25, 26, 27, 28]$. Different methods have been proposed for the end-point divergences $[26, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37]$. In principle, the pQCD method can give a model independent determination for the $|V_{ub}|$ and its precision can be improved order by order by perturbation theories. Based on factorization theorem $[38]$, the pQCD expression for the form factor can be written into a factorized formula in terms of hard scattering function and nonperturbative meson distribution amplitudes (DAs). The hard scattering function contains short distance contributions and can be calculated, perturbatively. The meson DAs contain long distance contributions and are universal. Once the meson DAs are determined from other processes, the factorization formula can make model independent predictions. However, due to large uncertainties associated with the pQCD form factors $[31, 32, 33, 34, 35]$, the pQCD method has not been applied to derive equally precise $|V_{ub}|$ as the other methods, such as, LCSR, LQCD, and PA. In this paper, we would like to improve the precision order of the pQCD form factors, such that one can
derive the $|V_{ub}|$ with equally theoretical uncertainties as the other approaches.

There are two compatible pQCD methods, the collinear factorization denoted by $C^f$, and the $k_T$ factorization, or, pQCD factorization, denoted by $k_T^f$. The $k_T^f$ has been widely used for $F_{B\pi}^{+}$. In $k_T^f$, end-point singularities are supposedly solved by the intervention of parton transverse momenta. However, the transverse parton momenta would induce large logarithms $\alpha_s \ln^2 k_T$ from higher loop corrections. In addition, there are also large logarithms $\alpha_s \ln x$ associated with subleading twist (twist-3) contributions. These large logarithms need to be re-summed to be Sudakov factors. The Sudakov factors are expected to suppress the contributions from end-point regions. In practical applications, some criteria for the Sudakov factors are needed. When $k_T^f$ is generalized to include subleading order contributions in the $1/m_B$ expansion, the subleading order, $O(1/m_B)$, corrections dominate over the leading order contributions. Intrinsic transverse degrees of freedom of the meson wave functions (for $B$ meson and pion) are needed to cure the ill behavior of the power expansion.

Unlike the complicated features and related issues of $k_T^f$, $C^f$ has a simple structure and is directly related to the parton model (PM). It is expected that, once the end-point singularity can be regularized within $C^f$, the $C^f$ formalism would be instructive for both theory and phenomenology. In the approach proposed by Akhoury, Sterman, and Yao (ASY), the heavy quark effective theory (HQET) and Sudakov re-summation were incorporated with $C^f$. The end-point divergent problem is solved in the ASY approach. However, a dynamical zero point was found and a small partial decay rate for $\bar{B} \to \pi l \nu_l$ was obtained.

In this paper, we propose a different approach to solve the end-point divergent problem and avoid the problems in the ASY approach. The key solution is a $\xi$-regularization (denoted by $\xi^R$) which can effectively regularize the end-point divergences. The $\xi^R$ has been applied to effectively regularize the end-point divergences in twist-3 hard spectator and annihilation contributions in charmless hadronic $B$ decays. In this paper, we show that $\xi^R$ is also effective for $F_{B\pi}^{+}$. The suppression of end-point radiative corrections is provided by the $B$ meson distribution amplitude (DA). This provides a stronger suppression effect than any Sudakov factors. The extension of application range of $q^2$ is given by including subleading corrections in $1/m_B$ expansion. The complete $O(1/m_B)$ contributions from the $\pi$ meson side are calculated by using a collinear expansion (CE) method for exclusive processes, which is developed by Yeh.
contributions are solved by a simultaneous use of the $\xi^R$ and a non-constant twist-3 pseudo-scalar (PS) DA. The factorization of the $B \to \pi$ form factors has been shown valid under $C^{f_2}$ and soft collinear effective theory (SCET), respectively. There lacked explicit regularization methods for the end point divergences in these previous proofs. Our approach provides practical calculations for the form factors to show the factorization up to $O(\alpha_s/m_B)$. The non-constant twist-3 PS DA exists for a $\pi$ meson in its energetic state. On the other hand, a constant PS DA is usually used in the literature. As shown in [41], the constant PS DA is appropriate to describe a $\pi$ meson in its chiral state (or, a soft pion), but not an energetic pion. Unexpected large contributions associated with the constant PS DA are already noticed in $k_T^{f_2}$ and LCSR. If the constant PS DA corresponds to the soft pion state, then these large contributions from the constant pion PS DA can be identified as soft dominated contributions instead of hard dominated contributions as expected in the calculations performed in the $k_T^{f_2}$ and LCSR. This point of view of using the pion state corresponding to the DA to identify the kinds of contributions (soft or hard) is different from the traditional way of using the scaling of the relevant contributions. This provides another method to identify the considered contributions. The unexpected large contributions from the pion PS DA in $k_T^{f_2}$ and LCSR could be overestimated. Since end-point singularities at leading and subleading order in $1/m_B$ expansion can be effectively solved, the $F^{B\pi}_{+\cdot 0}$ are calculable under $C^f$. Another effect associated with $\xi^R$ is that, for $\bar{u} \sim O(1)$, the divergences from $\eta \sim O(\Lambda/m_B)$ can be regularized. This extends the application range of $C^f$ from small $q^2 \sim 0$ to moderate $q^2 \sim 16\, \text{GeV}^2$. Note that the relevant energy scale of $\alpha_s$ is set as $t = 1.65\sqrt{1 - q^2/m_B^2}$, which is about 1 GeV at $q^2 = 16\, \text{GeV}^2$ and the coupling constant $\alpha_s$ is about 0.48, or, $\alpha_s/\pi \simeq 0.16 < 1$. There are also possible subleading order contributions in $1/m_B$ expansion from the $B$ meson side. Only $1/m_B$ contributions from the two parton Fock state of the $B$ meson are calculated. In summary, we plan to make the following progresses in theory.

1. The $C^f$ is applicable for $F^{B\pi}_{+\cdot 0}$ at leading twist order by $\xi^R$.

2. $O(1/m_B)$ two parton contributions are shown calculable under $C^f$.

3. The twist-3 three parton corrections from the pion are explicitly calculated. This is a first result in the literature.
4. The complete twist-3 contributions are shown less than the twist-2 contributions for $0 \leq q^2 \leq 16 \text{ GeV}^2$. A well-defined power expansion is given.

The result is applied to extract $|V_{ub} F_{\pi}^{B}(0)|$ and $|V_{ub}|$ from the world averaged branching fraction $Br_{\text{SL}}^{\exp}$ for semi-leptonic decays $B \rightarrow \pi l\bar{\nu}_l$. A fitting method is used to determine the value of a parameter $\omega_B$ for the $B$ meson distribution amplitude. A parameterization form, $F_{\pi}^{C}(q^2)$, of $F_{\pi}^{B}(q^2)$ versus $q^2$ is given by a minimal $\chi^2$ fitting. Our analysis gives $|V_{ub}| = (3.95 \pm 0.13^{\text{exp}} \pm 0.32^{\text{th}}) \times 10^{-3}$ with experimental and theoretical errors. This agrees well with the world averaged value of $|V_{ub}|$, $|V_{ub}| = (3.95 \pm 0.35) \times 10^{-3}$ [45]. The fit form factor $F_{\pi}^{C}(q^2)$ predicts $F_{\pi}^{B}(0) = 0.16$ which confirms the founding $F_{\pi}^{\text{SCET}}(0) = 0.17$ by the soft collinear effective theory (SCET) from an analysis for $B \rightarrow \pi\pi$ decays [36].

The organization is as follows. $\xi_R^R$ is defined and shown effective for leading twist contributions in Section II. The comparison between the $\xi_R^R$ and the $k_T$-regularization (denoted by $k_T^R$ in this paper) is given in this Section. To generalize $C_f^R$ for higher twist contributions, the collinear expansion method is used in Section III. In Section IV the $F_{+,0}^{B\pi}$ are explicitly calculated up-to $O(\alpha_s/m_b)$. In Section V, the form factors are applied to determine $|V_{ub}|$ from experiments. Last two Sections are devoted for comparisons and discussions. Two Appendices are given.

II. LEADING TWIST $B \rightarrow \pi$ FORM FACTORS, END-POINT DIVERGENCES, AND $\xi$-REGULARIZATION

In this section, we first review how the end-point divergent problem of the leading twist $B \rightarrow \pi$ transition form factors can arise. We then define the $\xi_R^R$ and explain how it is effective for end-point singularity. The $B \rightarrow \pi$ form factors $F_{+,0}^{B\pi}(q^2)$ are defined by

$$
\langle \pi(p) | \bar{q} \gamma^\mu b | B(P_B) \rangle = 2F_{+,0}^{B\pi}(q^2)p_\pi^\mu + [F_{+,0}^{B\pi}(q^2) - F_{+}^{B\pi}(q^2) - F_{0}^{B\pi}(q^2)] \frac{(m_B^2 - m_\pi^2)}{q^2} q^\mu , \tag{1}
$$

where $q = P_B - p_\pi$. Another set of form factors, $f_{1,2}^{B\pi}$, is also used in literature. Their definitions are

$$
\langle \pi(p) | \bar{q} \gamma^\mu b | B(P_B) \rangle = f_{1}^{B\pi}(q^2)P_\pi^\mu + f_{2}^{B\pi}(q^2)p_\pi^\mu . \tag{2}
$$
\[ F_{+,0}^{B\pi}(q^2) = \frac{1}{2}(f_{11}^{B\pi}(q^2) + f_{22}^{B\pi}(q^2)), \]
\[ F_{0}^{B\pi}(q^2) = \frac{1}{2}((2 - \eta)f_{11}^{B\pi}(q^2) + \eta f_{22}^{B\pi}(q^2)), \]

where \( \eta = 1 - q^2/m_B^2 \). Under \( q^2 \to 0 \), \( F_{+,0}^{B\pi}(q^2) = F_{0}^{B\pi}(q^2) \). At maximal recoil limit (the energetic limit for the \( \pi \) meson), the form factors \( F_{+,0}^{B\pi}(q^2) \) and \( F_{0}^{B\pi}(q^2) \) becomes identical to cancel the \( q^2 \to 0 \) pole in Eq. (1). The \( B \) meson’s momentum \( P_B \) is defined in the \( B \) meson's rest frame \( P_B^\mu = (m_B,0,0,0) = (P_B^+, P_B^-, 0_\perp) \) with \( P_B^+ = P_B^- = m_B/\sqrt{2} \). The \( \pi \) meson momentum \( p_\pi \) is written as \( p_\pi^\mu = \frac{1}{2}(\eta m_B, 0, 0, \eta m_B) = (p_\pi^+, 0, 0_\perp) \) where \( p_\pi^+ = \eta m_B/\sqrt{2} \). \( q = P_B - p_\pi = (m_B(1 - \eta/2), 0, 0, -\eta m_B/2) = (q^+, q^-, 0_\perp) \) where \( q^+ = (1 - \eta)m_B/\sqrt{2} \) and \( q^- = m_B/\sqrt{2} \). The light-cone coordinate system will be used in this work. Under \( C_I \), partons carry collinear fractions of external meson momenta. The spectator anti-quark of the \( \bar{B} \) meson carries a momentum \( l_{sp}^\mu = (0, l_{sp}^-, 0_\perp) \) where \( l_{sp}^- = \xi P_B^- = \xi m_B/\sqrt{2} \). The \( b \) quark’s momentum is defined as \( P_b^\mu = (P_b^+, P_b^-, 0_\perp) \) with \( P_b^- = \xi P_B^- = \xi m_B/\sqrt{2} \) and \( P_b^+ = m_b^2/(\sqrt{2}\xi m_B) \). The \( b \) quark is defined on-shell. This is different from the usual treatment in the literature that the \( b \) quark is assumed almost on-shell, \( P_b^2 \simeq m_b^2 \). The quark \( q \) inside the \( \pi \) meson is defined to carry a momentum \( l_q = (l_q^+, 0^-, 0_\perp) \) with \( l_q^+ = u\eta m_B/\sqrt{2} \). The anti-quark \( \bar{q} \) inside the \( \pi \) meson has a momentum \( l_{\bar{q}} = (l_{\bar{q}}^+, 0^-, 0_\perp) \) with \( l_{\bar{q}}^+ = \bar{u}\eta m_B/\sqrt{2} \). \( \xi \) and \( u \) are momentum fraction variables and \( \bar{\xi} = 1 - \xi \) and \( \bar{u} = 1 - u \). \( E = m_B/\sqrt{2} \) is used in the following text.

Under large recoil condition, \( \eta \to 1 \), the \( \pi \) meson has an energetic momentum \( p_\pi^+ \simeq m_B/\sqrt{2} \gg m_\pi \). We assume that the virtual gluon carries a hard-collinear energy scale. The PQCD is applicable because the involved strong coupling constant \( \alpha_s(t) \) at the interaction energy scale \( t = \sqrt{\eta\Lambda_h m_B} \) with \( \Lambda_h = 0.5 \) GeV is around 0.3. Under \( C_I \), the twist-2 contribution to the matrix element \( \langle \pi|\bar{q}\gamma^\mu b|\bar{B} \rangle \) is written as

\[ M_{tw2,\mu} = f_\pi f_B \frac{\pi\alpha_s(t)C_F}{N_c} \int_0^1 d\xi \phi_{B\pi}^{tw2}(\xi) \int_0^1 du \phi_{\pi}^P(u) H^{tw2,\mu}(\xi, u), \]

where \( H^{tw2,\mu}(\xi, u) = H^{(a),tw2,\mu}(\xi, u) + H^{(b),tw2,\mu}(\xi, u) \) denote the hard scattering functions for the lowest order Feynman diagrams depicted in Fig. 1(a) and (b). \( \phi_{B\pi}^{tw2}(\xi) \) and \( \phi_{\pi}^P(u) \) are the \( B \) meson’s and \( \pi \) meson’s leading twist LCDA, respectively. \( f_B \) and \( f_\pi \) are the \( B \) meson’s and \( \pi \) meson’s decay constants. \( C_F = (N_c^2 - 1)/(2N_c) \) and \( N_c \) are color factors. \( \alpha_s \)
is the strong coupling constant. \( H^{(a,b),tw2,\mu}(\xi, u) \) are written as

\[
H^{(a),tw2,\mu}(\xi, u) = \frac{1}{\eta \bar{u} \xi E^2} \left( P^\mu_B - \frac{1}{\eta} p^\mu_\pi \right),
\]

(6)

\[
H^{(b),tw2,\mu}(\xi, u) = \frac{(\bar{\xi} + \eta \bar{u})}{\eta \bar{u} (\xi - \eta \bar{u}) E^2 p^\mu_\pi}.
\]

(7)

One can arrive at the form factors \( f_{1,2}^{B\pi,tw2} \)

\[
f_{1}^{B\pi,tw2}(q^2) = \frac{\pi \alpha_s(t) C_F}{E^2} \frac{f_\pi f_B}{N_c} \int d\xi \phi_B^{tw2}(\xi) \int du \phi_\pi^P(u) \left[ \frac{1}{\eta \bar{u}} \right],
\]

(8)

\[
f_{2}^{B\pi,tw2}(q^2) = \frac{\pi \alpha_s(t) C_F}{E^2} \frac{f_\pi f_B}{N_c} \int d\xi \phi_B^{tw2}(\xi) \int du \phi_\pi^P(u) \left[ \frac{(1 + \eta) \eta \bar{u} - (1 + \eta) \xi + \eta}{\eta^2 \bar{u} (\xi - \eta \bar{u})} \right],
\]

(9)

where \( \bar{\eta} = 1 - \eta \) and \( \bar{u} = 1 - u \). The form factors \( F_{+,0}^{B\pi,tw2} \) from \( f_{1,2}^{B\pi,tw2} \) by Eqs.(8,11) are written as

\[
F_{+}^{B\pi,tw2}(q^2) = \frac{\pi \alpha_s(t) C_F}{2 E^2} \frac{f_\pi f_B}{N_c} \int d\xi \phi_B^{tw2}(\xi) \int du \phi_\pi^P(u) \left[ \frac{\eta - \xi + \eta \bar{u}}{\eta^2 \bar{u} (\xi - \eta \bar{u})} \right],
\]

(10)

\[
F_{0}^{B\pi,tw2}(q^2) = \frac{\pi \alpha_s(t) C_F}{2 E^2} \frac{f_\pi f_B}{N_c} \int d\xi \phi_B^{tw2}(\xi) \int du \phi_\pi^P(u) \left[ \frac{\eta + (1 - 2\eta) \xi - (1 - 2\eta) \eta \bar{u}}{\eta^2 \bar{u} (\xi - \eta \bar{u})} \right]
\]

(11)

\( \phi_\pi(u) \) is linear in \( u \). \( f_{1,2}^{B\pi,tw2} \) are finite. However, if \( (\xi - \eta \bar{u}) \) is approximated to be \( (-\eta \bar{u}) \), then \( f_{2}^{B\pi,tw2} \) becomes logarithmic divergence at \( \bar{u} \to 0 \). This is the end-point divergent problem of the \( B \to \pi \) transition form factors. The key point is that, without loss of generality, the denominator of the internal \( b \) quark propagator is approximately to be, \( (P_b + k)^2 - m_b^2 \approx m_B^2 (\xi - \eta \bar{u}) + O(\Lambda_{QCD}^2) \) and \( (\Lambda_{QCD}^2) \) where \( \Lambda = m_B - m_b \). \( \xi \simeq O(\Lambda^2/m_B) \) and \( \Lambda = \Lambda_{QCD}^2 \). The error terms are of next-next-to-leading order in \( 1/m_B^2 \) expansion. In the end-point region of \( \bar{u}, \bar{u} \sim O(\Lambda_{QCD}/m_B) \), \( \xi \) is as large as \( \eta \bar{u} \) for \( \eta \sim O(1) \). \( \xi \) retained in \( (P_b + k)^2 - m_b^2 \) is necessary.

Akhoury, Sterman, and Yao (ASY) [26] proposed an approach to combine the heavy quark effective theory (HQET), a resummation of Sudakov double logarithms, and \( C_f \) for exclusive processes. In the ASY approach, the \( b \) quark of the \( B \) meson is almost static at the energy scale much less than the \( b \) quark’s mass, \( m_b \). The exchanged gluon in the Feynman diagram in Fig. 7(b) carries soft energy. The \( b \) quark inside the \( B \) meson can be effectively described by an effective field, \( h_v \), with \( v = P_B/m_B \) of HQET. According to HQET, the light degrees of freedom of the \( B \) meson are identified as a brown muck such that only soft interactions can
Figure 1: One gluon exchanged Feynman diagrams for the hard scattering functions $H^\mu$ of the $C^I$ amplitude for the $\bar B \to \pi l \bar \nu_l$ decays. The external meson states $|\bar B\rangle$ and $\langle \pi |$ are not shown. The cross vertex denotes the vector operator $\gamma^\mu$. The diagram (a) describes the one gluon exchanged process between the bottom quark $b$ and the spectator anti-quark $\bar q$ of the $\bar B$ meson. The diagram (b) describes the one gluon exchanged process between the quark $q$ and the anti-quark $\bar q$ of the $\pi$ meson.

Exist between the spectator anti-quark and the $b$ quark inside the $\bar B$ meson. There needs a subtraction operation to separate long distance and short distance contributions from the $B$ meson side. The subtraction operation is equivalent to neglect $\xi$ in the denominator and $\bar \xi$ in the numerator of the $H^{(b),tw2,\mu}$ in Eq.(7). It is obvious that once $\xi$ and $\bar \xi$ are subtracted from the $H^{(b),tw2,\mu}$ term, the end-point singularity in $f^{B\pi,tw2}$ is solved. When loop corrections are concerned, there would arise large infrared (IR) logarithms from the internal $b$ quark line in the Feynman diagram in Fig. 1(b). A resummation over the large IR logarithms gives Sudakov form factors. Although the ASY approach can successfully solve the end-point divergent problem, there exist two problems. The first one is that there is a dynamical zero point at $\eta = 1/2$ in the $|f_1 + f_2|$. We find that the dynamical zero point is due to the neglect of the $\bar \xi$ factor in the $H^{(b),tw2,\mu}$ term. Refer to Eq.(7). The $\bar \xi$ factor is recovered as one include the dynamical contributions from the $b$ quark. This can be understood by referring to Appendix A. The dynamical zero point is avoided in $\xi^R$. The second is that small partial rates for $B \to \pi l \bar \nu_l$ were predicted in the ASY approach. This implies that the ignored contributions in the ASY approach are important. In $\xi^R$, the ignored contributions in the ASY approach are recovered. See Section V for this point. In our approach, two problems in the ASY approach are solved.

In $k^2_T$, the divergent term $(\eta \bar u)$ is replaced by $(k^2_T + \eta \bar u E)^{-1}$. While the transverse momenta of partons are remained in the parton propagators, the partons become off-shell because only collinear and transverse momenta are kept. Off-shell partons may radiate infinite soft gluons as they pass through space before they compose into external mesons.
Re-summation of soft gluon radiations results in Sudakov factors, which are expected to suppress the soft radiations from the end-point region. However, some criteria are required. This is an uncertainty of $k_f^f$.

$\xi$ is of order $O(\Lambda/E)$ that the $B$ meson distribution amplitude $\phi_B(\xi)$ has a peak at $\xi \sim O(\Lambda/E)$. In the end-point region of $\bar{u}, \bar{u} \sim O(\Lambda/E)$, $\phi_\pi^{p}(u) \sim O(\Lambda/E)$. The overall scale of the leading part of the form factor $f_2^{B\pi,tw2}$ is counted as $O(1/E^2)$ times the decay constants $f_\pi f_B$. Similarly, $f_1^{B\pi,tw2}$ is dominated by large $\bar{u} \sim O(1)$ and is counted as $O(1/E^2)$ times the decay constants $f_\pi f_B$. Although $f_1^{B\pi,tw2}$ and $f_2^{B\pi,tw2}$ receive contributions from different configuration regions, they are of the same order. Accordingly, contributions of subleading order in $1/m_B$ expansion should be included. They are calculated in next Section.

The $\pi$ meson’s twist-2 DA $\phi_\pi^{p}(u)$ is modeled to be its asymptotic form $\phi_\pi^{p}(u) = 6u\bar{u}$ by neglecting its scale dependence. This is fulfilled for the precision of $O(\alpha_s)$. The $B$ meson’s twist-2 DA $\phi_B^{tw2}(\xi)$ is assumed to be modeled as

$$
\phi_B^{tw2(I)}(\xi) = \frac{m_B^2 \xi}{\omega_B^2} \exp\left(-\frac{m_B}{\omega_B} \xi\right)
$$

which satisfies the following conditions

$$
\int_0^\infty d\xi \phi_B^{tw2(I)}(\xi) = 1, \quad \int_0^\infty d\xi \frac{\phi_B^{tw2(I)}(\xi)}{\xi} = \frac{m_B}{\lambda_B}.
$$

The value of $\lambda_B$ satisfies the condition $6\lambda_B \leq 4\bar{\Lambda}$ with $\bar{\Lambda} = m_B - m_b$. Different models for the $\phi_B^{tw2}(\xi)$ have only tiny differences if they are constrained by Eq. (13) [47]. We note that the integration range over $\xi$ in the calculation of the form factors is $[0, 1]$ instead of $[0, \infty]$. The difference between these two integration ranges is equal to

$$
\Delta = \int_0^\infty d\xi \phi_B^{tw2, I}(\xi) - \int_0^1 d\xi \phi_B^{tw2, I}(\xi) = \left(\frac{m_B}{\omega_B} + 1\right) \exp\left[-\frac{m_B}{\omega_B}\right]
$$

$$
\simeq 1.29 \times 10^{(-4)},
$$

where the last estimated number is calculated by using $\omega_B = 0.46$ GeV and $m_B = 5.28$ GeV. $\Delta$ is negligible.

To distinguish from the $k_T^f$ for end-point divergences, we name the retain of $\xi$ in the internal $b$ quark propagator as the $\xi$-regularization, $\xi^R$. It is instructive to see how $\xi^R$ works
for end-point divergences. Let’s consider the $\eta \to 1$ limit of $F_{+}^{B\pi,tw}(q^2)$. Except of the relevant parameters, the most singular part is the integrations

$$\int_{0}^{1} d\xi \phi_{B}^{tw2}(\xi) \int_{0}^{1} du \phi_{\pi}^{P}(u) \frac{1}{\xi \bar{u}(\xi - u)}.$$ 

It is seen that once $\xi$ is neglected in $(\xi - \bar{u})^{-1}$, an end-point divergence arise to be $\log(\bar{u})$. The retain of $\xi$ results in, for the integration over $u$,

$$\int_{0}^{1} du \phi_{\pi}^{P}(u) \frac{1}{\bar{u}(\xi - \bar{u})} = 6(1 + \xi \ln \frac{\xi}{\bar{u}} - i\pi \xi).$$

One can observe that the original logarithmic divergence $\log \bar{u}$ is indeed regularized by $\xi$, effectively. The further integration over $\xi$ is analytic because the end-points for $\xi \to 0$ or $\bar{\xi} \to 0$ are prevented by the $B$ meson distribution amplitude $\phi_{B}^{tw2}(\xi)$. For example, the integration for the most singular term is analytical as

$$\int_{0}^{1} d\xi \phi_{B}^{tw2(I)}(\xi) \frac{\xi}{\xi \ln \frac{\xi}{\bar{u}}} = -[1 - e^{-a} + (a - 1)(\gamma_E + \Gamma(a) + \ln a)],$$

where $a = m_{B}/\omega_{B}$, $\gamma_E$ is the Euler number, and $\Gamma(a)$ is the Gamma function. Another widely used model for $\phi_{B}^{tw2}(\xi)$ has the form

$$\phi_{B}^{tw2(II)}(\xi) = \frac{N_{B}\xi^2(1 - \xi)^2}{(\xi^2 + \epsilon_{B}(1 - \xi)^2)^2},$$

where $N_{B}$ and $\epsilon_{B}$ are determined by Eq. (13). Although the integration can not be expressed explicitly, the result is also analytic because the $\phi_{B}^{tw2,II}$ has a stronger suppression effect on the end-points, $\xi, \bar{\xi} \to 0$. Of course, the final result would depend on the model for $\phi_{B}^{tw2}(\xi)$.

This model dependence is due to our rare knowledge for the $B$ meson. However, these two models give almost the same numerical results for $F_{+}^{B\pi,tw2}$ in $\xi^{R}$. For example, we can use $\omega_{B} = 0.46$ GeV to obtain

$$F_{+}^{B\pi,tw2,(I)}(0) = 0.110,$$
$$F_{+}^{B\pi,tw2,(II)}(0) = 0.104,$$  \hspace{1cm} (14)

where $F_{+}^{B\pi,tw2,(I)}(0)$ and $F_{+}^{B\pi,tw2,(II)}(0)$ are calculated by using $\phi_{B}^{(I)}(\xi)$ and $\phi_{B}^{(II)}(\xi)$, respectively. $N_{B} = 0.1536$ and $\epsilon_{B} = 0.0061$ are used in the calculation for $F_{+}^{B\pi,tw2,(II)}(0)$.

This shows explicitly that the $\xi^{R}$ indeed regularizes the end-point divergence $\log \bar{u}$. The suppression of contributions from the end-point region in $\xi^{R}$ is provided by the $B$ meson's DA, $\phi_{B}^{tw2}(\xi)$. The success of $\xi^{R}$ implies that the dynamical role of the spectator (anti-)quark
of the $B$ meson is important in the studies of heavy to light processes. This is contrary to the heavy to heavy processes, such as $B \to D l\nu$, in which heavy quark symmetry is useful. In the heavy quark infinite limit of the heavy to heavy processes, the light degrees of freedom of the heavy mesons remained independent of the $h_v \to h_v' l\nu$ process, described by a Isgur-Wise function, $\zeta(v \cdot v')$. $h_v(h_v')$ are effective fields for the $b$ and $c$ quarks of the $B$ meson and the $D$ meson, respectively. If the same picture applies for $\bar{B} \to \pi l\nu$ processes, then the $b \to ul\nu$ transition should be in a similar condition as that of $h_v \to h_v' l\nu$ process. This means that the final state $\pi$ meson should be in its soft pion state, at which the pQCD method is inapplicable in principle. In fact, the light degrees of freedom of the $\bar{B}$ meson would experience violate fluctuations during the $b \to ul\nu$ transition proceeds. This implies the applicability of pQCD [48]. The heavy quark symmetry is inapplicable in $\bar{B} \to \pi l\bar{\nu}_l$ decays. We will see later that the intervention of the $\xi$ variable becomes a straightforward step under CE. See the next Section.

### III. COLLINEAR EXPANSION

In this section, we describe how CE can be applied to derive twist-3 two parton and twist-3 three parton contributions. In this work, we consider only the twist-3 three parton contributions from the $\pi$ meson. The three parton contributions from the $B$ meson are complicate and left to other places. We begin with the amplitude for the diagrams in Fig. 2.

$$M^\mu = \int \frac{d^4l_B}{(2\pi)^4} \int \frac{d^4l}{(2\pi)^4} \text{Tr}[H^\mu(l_B, l)\Phi_B(l_B)\Phi_\pi(l)]$$

$$+ \int \frac{d^4l_B}{(2\pi)^4} \int \frac{d^4l}{(2\pi)^4} \int \frac{d^4l'}{(2\pi)^4} \text{Tr}[H'_\mu(l_B, l, l')\Phi_B(l_B)\Phi'_\pi(l, l')]$$

$$+ \cdots ,$$

where the first line corresponds to the four parton interactions (for the diagram depicted in Fig. 2(a)), and the second line represent five parton interactions (for the diagrams depicted in Fig. 2(b)). The higher Fock state interactions are neglected. The $H^\mu(l_B, l)$, and $H'_\mu(l_B, l, l')$ describe the parton interactions in the hard scattering center. The $\Phi_B(l_B), \Phi_\pi(l), \Phi'_\pi(l, l')$,
The factorization of the amplitude into partonic and hadronic parts is composed of three steps, the factorization of loop parton momenta, the color index factorization, and spin index factorization. The factorization of loop parton momenta is performed by means of a Taylor expansion for the partonic part and followed by relevant integral transformations. The color index factorization and the spin index factorization are similar and can be done by color algebra and Fierz identities, respectively. These three steps are shown explicitly below.

Once the partonic part is separated from the hadronic part of the amplitude, it is an important task to examine whether the partonic part suffers from soft divergences. This is the proof of the factorization theorem. The $O(\alpha_s)$ analysis for the twist-2 and twist-3 two parton contributions for $B \to \pi$ form factors has been given in [49]. The factorizability of $B \to \pi$ form factors under $C^I$ is shown valid up to twist-3 order under two parton approx-
imation. An all order proof of the factorization theorem for $B \to \pi$ form factors is shown valid at twist-2 order in SCET \[50\]. The factorizability of the twist-3 three parton contributions to the $B \to \pi$ form factors is still inaccessible. We assume that this is also valid in this work. A simple analysis shows that the twist-3 three parton contributions are also factorizable in $C^f$. However, the complete analysis is tedious and very technical. We skip this part and leave it for other place. In next section, the twist-3 three parton contributions from the pion are calculated based on this assumption. The analysis of the factorizability for the subleading twist contributions from the $B$ meson is more complicate. The complete $O(\alpha_s)$ factorization analysis also needs a lot of space and is not appropriate to be given here. Similarly, we also assume that the factorization is also valid for the subleading twist contributions from the $B$ meson. The calculations of the relevant quantities are based on this assumption. At order $O(\alpha_s)$, the explicit results given in the next section for the considered contributions confirm the factorizability. However, a complete analysis for the subleading twist contributions is important to make sure that the calculations given in this paper are perturbatively meaningful.

In Eq.(15), the loop parton momenta of the partons of the $B$ meson, $l_B$, is defined to flow from the $B$ mesons into the scattering center, and those momenta $l$ and $l'$ are defined to flow from the scattering center into the $\pi$ meson. To perform collinear expansion, the momenta, $l, l', l_B$ are found convenient to separate their on-shell part from their off-shell parts. For example, $l$ can be written as

$$l^\mu = \hat{l}^\mu + \frac{l^2 + l_\perp^2}{2n \cdot l} n^\mu + l_\perp^\mu, \quad (19)$$

where $l^2$ denotes the virtuality of $l$, $l_\perp^\mu$ is the transverse momentum, and $l_\perp^\mu l_{\perp,\mu} = -l_\perp^2$. $\hat{l}^2 = n^2 = 0$. $n$ is an auxiliary light-like vector. Under $C^f$, only collinear partons can involve in hard scattering center. The most important contributions should come from collinear momentum configuration according to the power counting rules \[41, 43\]. If $l$ represents the momentum of a collinear parton, i.e., $l$ is a collinear momentum, then $l$ has a limited (transverse) virtuality, $l^2 \sim l_\perp^2 \sim O(\Lambda^2)$. In $C^f$, the partons external to the hard scattering center are on-shell. According to Eq.(19), the on-shell parton can have a momentum $\hat{l}^\mu$ with $\hat{l}^2 = 0$, or,

$$l^\mu_L = \hat{l}^\mu + \frac{l_\perp^2}{2n \cdot l} n^\mu + l_\perp^\mu, \quad (20)$$

$$l_\perp^2 = 0.$$
The $\hat{l}^\mu$ is the collinear component of $l^\mu$ and is used for collinear partons in $C^f$. The $l^\mu_L$ has non-collinear momenta and is usually used in the $k_T^f$. It is noted that $n \cdot \hat{l} = uE$ with $u$ the momentum fraction and $E$ the energy scale. For $u \sim O(1)$, the $n^\mu$ term of $l^\mu_L$ is suppressed than the $\hat{l}^\mu$ and $l^\mu_T$ terms. For $u \sim O(\Lambda/E)$, three parts of $l^\mu_L$ are equally important. There requires some cares for the use of $l^\mu_L$ in the $k_f T$. Especially, it is important in the end-point region. We argue that the $n^\mu$ term of $l^\mu_L$ should be kept in $k_f T$. The calculations for $B \to \pi$ form factors in $k_f T$ should be rechecked.

The CE is able to derive higher twist contributions from higher Fock state, non-collinear components of parton momentum, and miss-matched spin states. The higher twist contributions can be factorized into partonic and hadronic parts, which are separately gauge invariant. The parton model interpretation for higher twist contributions is similar to the leading twist contributions.

The CE may confront with loop expansion (LE). We choose our strategy [41, 43] to firstly perform the LE for the amplitudes and then to use CE. Changing the application order of these two expansions would not make differences. The parton functions are expanded in LE as

$$H^\mu(l_B, l) = H^{(1)}(l_B, l) + O(\alpha_s^2),$$
$$H^\nu_{\nu}(l_B, l, l') = H^{(1)}(l_B, l, l') + O(\alpha_s^2),$$

where $H^{(1)}(l_B, l)$ and $H^{(1)}_{\nu}(l_B, l, l')$ are of order $O(\alpha_s)$. In this paper, we only consider $O(\alpha_s)$ contributions and assume $O(\alpha_s^2)$ not important.

The first step is to expand $H^{(1)}(l_B, l)$ and $H^{(1)}_{\nu}(l_B, l, l')$ with respect to the collinear momenta, $\hat{l}_B$, $\hat{l}$, and $\hat{l}'$, by a Taylor expansion

$$H^{(1)}(l_B, l) = H^{(1)}(\hat{l}_B, \hat{l}) + \sum_{k=l,l_B} \frac{\partial H^{(1)}(l_B, l)}{\partial l^\nu} \{ k=\hat{k}, k=l, l_B \} (l - \hat{l}^\nu + \cdots),$$

$$H^{(1)}_{\nu}(l_B, l, l') = H^{(1)}_{\nu}(\hat{l}_B, \hat{l}, \hat{l}') + \sum_{k=l,l',l_B} \frac{\partial H^{(1)}_{\nu}(l_B, l, l')}{\partial \eta^\nu} \{ k=\hat{k}, k=l, l_B, l' \} (k - \hat{k})^\eta + \cdots.$$
the above first expansion series and the first two terms in the above second expansion series. By substituting them into the original convolution integrations, one can obtain

\[
M^\mu = \int d\xi \int du \text{Tr}[H^{(1),\mu}(\xi, u)\phi_B(\xi)\phi_\pi(u)]
\]

\[
+ \int d\xi \int du \int du' \text{Tr}[H^{(1),\mu}_\nu(\xi, u, u')\phi_B(\xi)\phi_\pi^\nu(u, u')]
\]

\[
+ \int d\xi \int du \int du' \text{Tr}[H^{(1),\mu}_{\nu\eta}(\xi, u, u')\phi_B(\xi)\omega^\eta_{\nu\eta}\phi_\pi^\nu(u, u')]
\]

\[+ \cdots,
\]

where \(\omega^\eta_{\nu\eta} = g^\eta_{\nu\eta} - n_{\nu\eta}\). Two light-like auxiliary vectors are used \(n^\mu = (\bar{n}^+, \bar{n}^-, \bar{n}^\perp) = (1, 0, 0)\), \(\nu^\mu = (n^+, n^-, n^\perp) = (0, 1, 0)\). The hard scattering functions \(H^{(1),\mu}(\xi, u), H^{(1),\mu}_\nu(\xi, u, u'), H^{(1),\mu}_{\nu\eta}(\xi, u, u')\) are defined by low energy theorems

\[
H^{(1),\mu}(\xi, u) = H^{(1),\mu}(\hat{1}_B, \hat{1}) \tag{26}
\]

\[
H^{(1),\mu}_\nu(\xi, u, u') = H^{(1),\mu}(\hat{1}_B, \hat{1}, \hat{1}') \tag{27}
\]

\[
H^{(1),\mu}_{\nu\eta}(\xi, u, u') = \left(\frac{\partial H^{(1),\mu}}{\partial l'^\eta} - \frac{\partial H^{(1),\mu}}{\partial l'_\eta}\right)|_{(l_B = \hat{1}_B, l = \hat{1}, l' = \hat{1}')} \tag{28}
\]

The meson functions are defined as

\[
\phi_B(\xi) = \int \frac{d^4l_B}{(2\pi)^4} \int d^4z \delta(\xi - \frac{l_B}{p_B})e^{il_B\cdot z}\langle 0|\bar{q}(z)b(0)|B\rangle, \tag{29}
\]

\[
\phi_\pi(u) = \int \frac{d^4l}{(2\pi)^4} \int d^4z \delta(u - \frac{l}{p_{\pi}})e^{il\cdot z}\langle \pi|\bar{q}(z)q(0)|0\rangle, \tag{30}
\]

\[
\phi_\pi^\nu(u, u') = \int \frac{d^4l}{(2\pi)^4} \int \frac{d^4l'}{(2\pi)^4} \int d^4z \int d^4z' e^{il\cdot z}e^{i(l'\cdot - z')}
\]

\[
\times \delta(u - \frac{l}{p_{\pi}})\delta(u' - \frac{l'}{p_{\pi}})\langle \pi|\bar{q}(z)(-g)A^\nu(z')q(0)|0\rangle, \tag{31}
\]

\[
\phi_{\nu\eta}^\nu(u, u') = \int \frac{d^4l}{(2\pi)^4} \int \frac{d^4l'}{(2\pi)^4} \int d^4z \int d^4z' e^{il\cdot z}e^{i(l'\cdot - z')}
\]

\[
\times \delta(u - \frac{l}{p_{\pi}})\delta(u' - \frac{l'}{p_{\pi}})\langle \pi|\bar{q}(z)(-ig)G_{\nu\eta}(z')q(0)|0\rangle. \tag{32}
\]

By using the identities

\[
\delta(u - \frac{l}{p_{\pi}}) = \int \frac{d\lambda}{(2\pi)} e^{i\lambda(u - \frac{l}{p_{\pi}})}, \tag{35}
\]

\[
\int \frac{d^4l}{(2\pi)^4} \int d^4z e^{il\cdot (z - \frac{\lambda}{E}n)} = \delta^{(4)}(z - \frac{\lambda}{E}n), \tag{36}
\]

15
and their similarities for the corresponding integrations, the meson functions become

\[ \phi_B(\xi) = \int \frac{d\lambda}{(2\pi)} e^{i\lambda \xi} (0|\bar{q}(\frac{\lambda}{E})n)b(0)|B \rangle, \tag{37} \]

\[ \phi_\pi(u) = \int \frac{d\lambda}{(2\pi)} e^{i\lambda u}(\pi|\bar{q}(\frac{\lambda}{E})n)q(0)|0 \rangle, \tag{38} \]

\[ \phi^\nu_\pi(u, u') = \int \frac{d\lambda}{(2\pi)} \int \frac{d\lambda'}{(2\pi)} e^{i\lambda u}e^{i\lambda'(u'-u)}(\pi|\bar{q}(\frac{\lambda}{E})n)(-g)A^\nu(\frac{\lambda}{E})n)q(0)|0 \rangle, \tag{39} \]

\[ \phi^{\alpha
u}_\pi(u, u') = \int \frac{d\lambda}{(2\pi)} \int \frac{d\lambda'}{(2\pi)} e^{i\lambda u}e^{i\lambda'(u'-u)}(\pi|\bar{q}(\frac{\lambda}{E})n)(-ig)G^{\mu\nu}(\frac{\lambda}{E})n)q(0)|0 \rangle. \tag{40} \]

where \( G^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \). We note that the coupling \( g \) is absorbed into the meson functions, \( \phi^\nu_\pi \) and \( \phi^{\alpha
u}_\pi \). The three parton contributions are then counted as the same order of the two parton ones. This is different from the counting rule in \( k_T^2 \) \([31]\), in which the three parton contributions are counted as one more \( O(\alpha_s) \) order than the two parton ones.

The second term in Eq. (25) is related to gauge phase factors

\[
\int d\xi \int du \int du' \text{Tr}[H^{(1),\mu}_\nu(\xi, u, u')\phi_B(\xi)\phi^\nu_\pi(u, u')]
\]

\[
= \int d\xi \int du' \text{Tr}[H^{(1),\mu}(\xi, u')\phi_B(\xi)\phi_{\pi,n-A}(u') - \int d\xi \int du \text{Tr}[H^{(1),\mu}(\xi, u)\phi_B(\xi)\phi_{\pi,n-A}(u)]
\]

\[
+ O(A_\perp),
\]

where

\[
\phi_{\pi,n-A}(u) = \int \frac{d\lambda}{(2\pi)} e^{i\lambda u}(\pi|\bar{q}(\frac{\lambda}{E})n)(-ig) \int_0^\infty d\eta n \cdot A^\alpha(n)T^\alpha q(0)|0 \rangle. \tag{42} \]

\( O(A_\perp) \) denote the terms composed of gauge fields with a transversal polarization and are identified as sub-leading twist contributions. The gauge phase factors are absorbed into the first term in Eq. (25). In covariant gauge, there are infinite number of similar gauge phase factor terms from higher order Feynman diagrams. Their treatments are similar to the above and skipped here. The result becomes

\[ M^\mu = \int d\xi \int du \text{Tr}[H^{(1),\mu}(\xi, u)\phi_B(\xi)\phi_\pi(u)] \tag{43} \]

\[ + \int d\xi \int du \int du' \text{Tr}[H^{(1),\mu}_\nu(\xi, u, u')\phi_B(\xi)\phi^{\alpha
u}_\pi(u, u')] + \cdots. \]

The color index factorization are performed in the following way,

\[ \text{Tr}_c[H^{(1),\mu}(\xi, u)\phi_B(\xi)\phi_\pi(u)] = [H^{(1),\mu}(\xi, u)]_{ij,k}[\phi_B(\xi)]_{kl}[\phi_\pi(u)]_{lj}, \tag{44} \]

\[ \text{Tr}_c[H^{(1),\mu}_\nu(\xi, u, u')\phi_B(\xi)\phi^{\alpha\nu}_\pi(u, u')] = [H^{(1),\mu\nu}_\nu(\xi, u, u')]_{ij,km}[\phi_B(\xi)]_{jk}[\phi^{\alpha\nu}_\pi(u, u')]_{jm]. \tag{45} \]
where $i, j, k, l$ are color indices in the fundamental representation and $b$ is the color index in the adjoint presentation. $[H^{(1),\mu}(\xi, u)]_{ij,kl}$ and $[H^{(1),\mu;b}(\xi, u, u')]_{ij,kl}$ are expanded in terms of color factors

$$[H^{(1),\mu}(\xi, u)]_{ij,kl} = H^{(1),\mu}(\xi, u) \left( \frac{1}{N_c^2} \delta_{ij} \delta_{kl} + (T^a)_{ij} (T^a)_{kl} + \cdots \right), \quad (46)$$

$$[H^{(1),\mu;b}(\xi, u, u')]_{ij,kl} = H^{(1),\mu;b}(\xi, u, u') \left( \frac{1}{N_c^2} \delta_{ij} \delta_{kl} (T^b)_{im} + (T^a)_{ij} (T^a)_{kl} (T^b)_{lm} + \cdots \right). \quad (47)$$

For $O(\alpha_s)$ Feynman diagrams depicted in Fig. 1, only the second terms in the right hand side of Eq.(46,47) can contribute. They can be simplified by the color algebra

$$\delta_{ik} \delta_{jl} \frac{C_F}{N_c} \phi_B(\xi)[\phi(u)] = \frac{C_F}{N_c} \phi_B(\xi) \phi_\pi(u). \quad (50)$$

Similarly, Eq.(49) is applied to Eq.(45) to have

$$\delta_{ik} \delta_{jl} \frac{C_F}{N_c} (T^b)_{im} \phi_B(\xi) \phi_\pi(u, u') = \frac{C_F}{N_c} \phi_B(\xi) (T^b \phi_\pi^{\mu\nu}(u, u')). \quad (51)$$

Before the spin index factorization is performed, it is necessary to eliminate all terms in the $H$ functions which may lead to higher twist contributions under the equation of motion for the quark. The equations of motion of light quarks (assumed mass-less), whose momentum is $l^\mu$ in the collinear region $l^\mu = (l^+, l^-, l_\perp) \sim (E, \Lambda/E, \Lambda)$, are equivalently to the following identities

$$\frac{i \not{l} \not{n}}{l^2 + i\epsilon} \not{n} = \frac{i \not{l}}{l^2 + i\epsilon} (l - \hat{l})^\alpha (i\gamma_\alpha) \frac{i \not{n}}{2n \cdot l + i\epsilon} \not{n} = \omega_\alpha^\mu \left[ \frac{i \not{l}}{l^2 + i\epsilon} l^\alpha \right] \left[ (i\gamma_\alpha) \frac{i \not{n}}{2n \cdot l + i\epsilon} \not{n} \right]. \quad (52)$$

Since the propagator (the special propagator)

$$\frac{i \not{n}}{2n \cdot l + i\epsilon}$$

does not propagate, its associated terms are absorbed into the hard scattering functions (the last square bracket term in the last term of Eq.(52)). The meanings of the above identities
are as follows. If there is one factor \( \not p \) in the hard scattering functions contact with the long distance part of the quark parton propagator of the \( \pi \) meson, then the result is to extract one short distance part of the propagator and a vertex \( i\gamma_\alpha \) with a non-collinear momentum factor \( (l - \hat{l})^\alpha \). The non-collinear momentum factor will be absorbed by the \( \pi \) meson function, \( \phi_\pi(u) \), to have

\[
(l - \hat{l})^\alpha \phi_\pi(u) = \omega_\alpha \int \frac{d\lambda}{2\pi} \int \frac{d\eta}{2\pi} e^{i\lambda u} e^{in(u' - u)} \langle \pi| \bar{q}(\frac{n}{E}\lambda) i\partial^\alpha'(\frac{n}{E}n)q(0)|0 \rangle
\]

\[
= \omega_\alpha \phi_\pi^{\alpha'}(u, u').
\]

The similar case arises when the \( \not p \) in the hard scattering functions contact with the short distance part of the quark parton propagator of the \( \pi \) meson, the result is

\[
\frac{i}{2n \cdot l + i\epsilon} \not p = \frac{i}{l^2 + i\epsilon} (-gA^\alpha)(i\gamma_\alpha) \frac{i}{2n \cdot l + i\epsilon} \not p
\]

\[
= \omega_\alpha \left[ \frac{i}{l^2 + i\epsilon} (-gA^\alpha') \right] \left[ (i\gamma_\alpha) \frac{i}{2n \cdot l + i\epsilon} \not p \right] + \cdots ,
\]

where dots denotes the term would be absorbed by the gauge phase factor of the matrix element. The \( A^\alpha \) is the gauge field and its associated terms are defined to be absorbed into their corresponding hadron functions

\[
\omega_\alpha (-gA^\alpha) \phi_\pi(u) = \omega_\alpha \int \frac{d\lambda}{2\pi} \int \frac{d\eta}{2\pi} e^{i\lambda u} e^{in(u' - u)} \langle \pi| \bar{q}(\frac{n}{E}\lambda)(-gA^\alpha(\frac{n}{E}n)q(0)|0 \rangle
\]

\[
= \omega_\alpha \phi_\pi^{\alpha'}(u, u').
\]

The total effect, when one \( \not p \) factor can contact with the partons of the \( \pi \) meson, is

\[
\text{Tr} \left[ H_\mu^{(1)}(\xi, \pi) \phi_B(\xi) \phi_\pi(u) \right] = \text{Tr} \left[ \left( (i\gamma_\alpha) \frac{i}{2n \cdot l_q + i\epsilon} H_\mu^{(1)}(\xi, u) + H_\mu^{(1)}(\xi, u) \frac{-i}{2n \cdot l_q - i\epsilon} \right) \omega_\alpha \phi_B(\xi) \phi_\pi^{\alpha'}(u, u') \right],
\]

where

\[
\phi_\pi^{\alpha'}(u, u') = \int \frac{d\lambda}{2\pi} \int \frac{d\eta}{2\pi} e^{i\lambda u} e^{in(u' - u)} \langle \pi| \bar{q}(\frac{n}{E}\lambda)iD^\alpha'(\frac{n}{E}n)q(0)|0 \rangle,
\]

where \( iD^\alpha' = i\partial^\alpha' - gA^\alpha' \). Since \( n \cdot l_{q(\hat{q})} \) are of order \( E \) for collinear \( l_{q(\hat{q})} \), the related contributions are suppressed by one additional \( E^{-1} \)order. We call these terms as abnormal terms. The other terms are identified as normal terms. The normal terms will be kept in the reduced hard scattering functions, while the abnormal terms are dropped.
In the form factors, the $\xi$ is of order $O(\Lambda/E)$ due to the twist-2 $B$ meson distribution amplitude, $\phi_B^{\text{tw}}(\xi)$. The expansion into short distance and long parts of the quark propagator can not help to separate different twist contributions from the $B$ meson. In this work, we only consider the contributions from the two parton Fock state $|b\bar{q}\rangle$ of the $B$ meson. The contributions from the sub-leading twist state of $B$ meson are left to other places. The equation of motion for the $b$ quark, $(P_b - m_b)b = 0$, is the only condition. This fact reflects in the derivation of the reduced hard scattering functions. (Refer to following text and Appendix A.)

Considering all possibilities of the applications of equations of motion of quarks, the spin structures of the hard scattering functions are strongly restricted. It is useful to take the diagram in Fig.1(a) as an example to explain this operation. After the color index factorization, the amplitude for Fig.1(a) is proportional to

$$\int d\xi \int du \text{Tr}[H^{(1),\mu,(1a)}(\xi, u)\phi_B(\xi)\phi_\pi(u)].$$

(58)

The spin structure of $H^{(1),\mu,(1a)}(\xi, u)$ has the expression

$$[H^{(1),\mu,(1a)}(\xi, u)]_{ij,kl} \propto [\gamma^\alpha(\hat{k} - \kappa)\gamma^\mu]_{ij}[\gamma_\alpha]_{kl},$$

(59)

where $\hat{k} = u \hat{p}_\pi$ and $\kappa = \vec{p} n \cdot P_b - \bar{u} \vec{p}_\pi$. By using

$$g_{\alpha\beta} = \bar{n}^{\alpha} n^{\beta} + n^{\alpha} \bar{n}^{\beta} + d_{\perp}^\alpha \delta_{\beta},$$

$$\gamma_{\alpha} = \bar{n}^{\alpha} \bar{\gamma} + n^{\alpha} \gamma + \gamma_\alpha^\perp,$$

(60)

(61)

where $\gamma_\perp = d_{\perp}^{\alpha\beta} \gamma_{\beta}$ and $g_{\alpha} = d_{\perp}^\alpha = -2$. The spin structure of $H^{(1),\mu,(1a)}(\xi, u)$ is then expanded into a series in terms of $\bar{\gamma}, \bar{\gamma}, \gamma_\perp$. Each term in the expansion series is then examined to determine whether it is of the considered twist or of higher twist according to the equations of motion for the quarks in the $\pi$ meson and the $B$ meson. The abnormal terms are subtracted from the expression. We note that this analysis is automatically fulfilled for leading twist hard scattering functions, because the leading twist spin structure of the $\pi$ meson can eliminate the possible $\bar{n}$ factor by $\bar{p}_\pi \bar{n} \propto \bar{n} \bar{p} = 0$. For the sub-leading twist amplitudes, this procedure of subtraction of possible abnormal terms is necessary. For example, the spin structures of the twist-3 PS or PT DA are proportional to $\gamma_5$ or $\epsilon_{\mu\nu\alpha\beta}\sigma^{\mu\nu}\bar{n}^\alpha n^{\beta}$. These spin factors can not eliminate the abnormal terms by the traditional method. In the following, we assume that the hard scattering functions are determined according to the above analysis.
and the resultant hard scattering functions are called reduced hard scattering functions. The relevant reduced hard scattering functions are given in Appendix A.

The spin index factorization is performed in the following way,

\[ \text{Tr}_s[H^{(1),\mu}(\xi, u)\phi_B(\xi)\phi_\pi(u)] = [H^{(1),\mu}(\xi, u)]_{ij,kl}[\phi_B(\xi)]_{ik}[\phi_\pi(u)]_{jl}, \] (62)

\[ \text{Tr}_s[H^{(1),\mu}_\eta(\xi, u, u')\phi_B(\xi)\phi_\pi(u, u') = [H^{(1),\mu}_\eta(\xi, u, u')]_{ij,kl}[\phi_B(\xi)]_{ik}[\phi_\pi(u, u')]_{jl}, \] (63)

where the meson functions are expanded as

\[ [\phi_B(\xi)]_{ik} = \frac{1}{4}(\gamma^\rho\gamma_5)_{ik}\text{Tr}[\phi_B(\xi)\gamma^\rho\gamma_5] + \frac{1}{4}(\gamma_5)_{ik}\text{Tr}[\phi_B(\xi)\gamma_5] + \cdots, \] (64)

\[ [\phi_\pi(u)]_{jl} = \frac{1}{4}(\gamma_5)_{jl}\text{Tr}[\phi_\pi(u)\gamma_5] + \frac{1}{4}(\gamma^\rho)_{jl}\text{Tr}[\phi_\pi(u)\gamma^\rho\gamma_5] + \frac{1}{8}(\sigma_{\rho\lambda}\gamma_5)_{jl}\text{Tr}[\phi_\pi(u)\sigma^\rho\lambda\gamma_5] + \cdots, \] (65)

\[ [\phi_\pi^{'\nu}(u, u')]_{jl} = \frac{1}{8}(\sigma_{\rho\lambda}\gamma_5)_{jl}\text{Tr}[\phi_\pi^{\prime\nu}(u, u')\sigma^\rho\lambda\gamma_5] + \cdots. \] (66)

The dots denote those terms are not of our interesting. Each coefficient in the above expansion is attributed by a DA according to the following definitions

\[ \text{Tr}[\phi_B(\xi)\gamma^\rho\gamma_5] = if_B[P_B,\rho B(\xi) + E(n_\rho - \bar{n}_\rho)\bar{\phi}_B] + \cdots, \] (67)

\[ \text{Tr}[\phi_B(\xi)\gamma_5] = if_Bm_B\phi_B(\xi) + \cdots, \] (68)

\[ \text{Tr}[\phi_\pi(u)\gamma^\rho\gamma_5] = -if_\pi p_{\pi,\rho}P(\xi) + \cdots, \] (69)

\[ \text{Tr}[\phi_\pi(\xi)\gamma_5] = -if_\pi \mu_\chi \phi_\chi(\xi) + \cdots, \] (70)

\[ \text{Tr}[\phi_\pi(u)\sigma^\rho\lambda\gamma_5] = -f_\pi \mu_\chi [\bar{n}_\rho, n_\lambda] \phi_\chi(\xi) + \cdots, \] (71)

\[ \text{Tr}[\phi_\pi^{\prime\nu}(u, u')\sigma^\rho\lambda\gamma_5] = -f_\pi \mu_\chi (p_\pi^{\prime\nu}d^\rho - p_\pi^{\prime\nu}d^\rho')d^\rho'\phi_\pi^{3p}(u, u') + \cdots. \] (72)

In these definitions for the DAs, only relevant ones are shown and the others are left into the dots terms. Note that the $\phi_\pi^{\sigma}(u)$ are defined as the the energetic limits of the PS and PT DAs of the $\pi$ meson, respectively. The spin projector of $\phi_\pi^{\sigma}(u)$ is the leading part of the full spin projector $[p_\pi^{\sigma}, z^\chi]$ of the PT DA under the energetic limit. The $\phi_\pi^{c}(u)$ corresponds to the $d\phi_\pi^{c}(u)/du$ of the usually used PT DA, $\phi_\pi^{c}(u)$. The factor $\mu_\chi = m_\pi^2/(m_q + m_{\bar{q}})$ with $m_\pi$ the pion mass and $m_{q(\bar{q})}$ the current quark(anti-quark) masses. The Dirac matrices in the expansion series are absorbed by the hard scattering functions. The spin index factorization is completed.

The final result is written as

\[ M^\mu = M^{tw2,\mu} + M^{B,\mu} + M^{tw3,ps,\mu} + M^{tw3,pt,\mu} + M^{tw3,3p,\mu} + \cdots, \] (73)
where the expansion terms in the right hand side of Eq. (73) are defined as

\[
M^{tw2,\mu} = \frac{1}{16} f_B f_\pi \int d \xi \phi_B(\xi) \int du \phi^P_\xi(u) \text{Tr} \left[ \gamma_5 \not{p} H^{(1),\mu}(\xi, u)(P_B + m_B)\gamma_5 \right],
\]

\[
M^{R,\mu} = \frac{1}{16} f_B f_\pi E \int d \xi \phi_B(\xi) \int du \phi_\pi^P(u) \text{Tr} \left[ \gamma_5 \not{p} H^{(1),\mu}(\xi, u)(\not{p} - \not{q})\gamma_5 \right],
\]

\[
M^{tw3,ps,\mu} = \frac{1}{16} f_B f_\pi \mu \int d \xi \phi_B(\xi) \int du \phi_\pi^\sigma(u) \text{Tr} \left[ \gamma_5 H^{(1),\mu}(\xi, u)(P_B + m_B)\gamma_5 \right],
\]

\[
M^{tw3,pt,\mu} = -\frac{i}{32} f_B f_\pi \mu \int d \xi \phi_B(\xi) \int du \phi_\pi^\sigma(u) \text{Tr} \left[ \epsilon_\perp \cdot \sigma H^{(1),\mu}(\xi, u)(P_B + m_B)\gamma_5 \right] \delta(1 - u_q - u_q - u_g) \text{Tr} \left[ \sigma_{\rho\lambda} \gamma_5 H^{(1),\mu}(\xi, u_q, u_q, u_g)(P_B + m_B)\gamma_5 \right] \omega_\eta \Gamma^{\rho\lambda\eta\nu}(p_\pi),
\]

where \( \Gamma^{\rho\lambda\eta\nu}(p_\pi) = (p_\pi^\rho d^{\eta\nu} - p_\pi^\eta d^{\rho\nu})p_\pi^\lambda \) and \( \epsilon_\perp \cdot \sigma = \epsilon_{\perp,\alpha\beta} \sigma^{\alpha\beta} = \epsilon_{\alpha\beta\eta\gamma} \sigma^{\alpha\beta} n^\eta n^\gamma \) with \( \sigma^{\alpha\beta} = i[\gamma^\alpha, \gamma^\beta]/2 \). \( M^{tw2,\mu} \) has been calculated in previous section. The other four twist-3 terms, \( M^{tw3,R,ps,pt,3p,\mu} \), are calculated in next Section. In the above expression, we have used a transformation of the integrations over \( u \) and \( u' \) into the integrations over \( u_q, u_q, \) and \( u_g \) to make the expression more explicitly. The \( u_q, u_q, \) and \( u_g \) are defined as the fractions of the quark, anti-quark, and gluon partons of the \( \pi \) meson.

The retain of \( \xi \) in the internal \( b \) quark propagator can partially resolve the end-point divergences at twist-3, which are linear divergences due to the constant nature of the \( \pi \) meson’s twist-3 distribution amplitude (the pseudo-scalar one, \( \phi^P_\pi(u) \)). The PS DA \( \phi^P_\pi(u) = 1 \) for the \( \pi \) meson is determined by the equation of motion for the quark of the \( \pi \) meson in the soft pion state, where the energy of the pion, \( E_\pi \), the pion mass, \( m_\pi \), or the scale of the parton transverse momentum, \( l_\perp \), are of the same order, \( E_\pi \simeq m_\pi \simeq l_\perp \simeq O(\Lambda) \). The final state \( \pi \) meson in the semi-leptonic \( B^0 \to \pi^- l^+ \nu_l \) process can carry an energetic momentum, where \( E_\pi \gg m_\pi \). The equation of motion for \( \phi^P_\pi(u) \) can be further approximated to a reduced equation of motion [41]. According to this reduced equation of motion, the \( \pi \) meson’s PS DA and the PT DA, \( \phi^\sigma_\pi(u) \), become equal. In this way, \( \phi^P_\pi(u) \) and \( \phi^\sigma_\pi(u) \) can be modeled similar to the twist-2 \( \pi \) meson’s distribution amplitude, \( \phi^P_\pi(u) \), to take their asymptotic forms, \( \phi^\sigma_\pi(u) = \phi^P_\pi(u) = 6u(1 - u) \). The constant feature of \( \phi^\sigma_\pi(u) \) as found in literature is disappeared in the \( \pi \) meson’s energetic state.

In the \( \pi \) meson’s soft pion state, different twist order contributions ordered by \( O((\Lambda/E_\pi)^n) \) become equally important under the limit \( E_\pi \simeq \Lambda \). On the other hand, in the \( \pi \) meson’s energetic state [41], different twist order contributions ordered by \( O((\Lambda/E_\pi)^n) \) are restric-
tively ordered under the limit $E_\pi \gg \Lambda$. This explains why $\phi_\pi^p(u)$ can have different forms in the energetic and chiral limits. Once the non-constant $\phi_\pi^p(u)$ are substituted into the relevant expressions for the amplitude under the $\xi^R$, the linear divergences automatically disappear and the related end-point divergent problem is resolved. There are no similar end-point divergences for the twist-3 three parton amplitude $M^{tuc,3p,\mu}$ since the three parton DA $\phi_\pi^{3p}(u_q, u_d, u_g)$ is not a constant. As a result, we obtain a factorization formula for the $B \rightarrow \pi$ form factors up-to twist-3 order.
Figure 3: Feynman diagrams for the hard scattering functions $H^\mu_{\nu\eta}$ of the $C^f$ amplitudes for the $\bar{B} \to \pi l\bar{\nu}_l$ decays. The external meson states $|\bar{B}\rangle$ and $\langle \pi |$ are not shown. The cross vertex denotes the vector operator $\gamma^\mu$. The dot vertex denotes the subscript $\eta$ in $H^\mu_{\nu\eta}$. Only contributions from diagrams (1-4) are of twist-3 order. The contributions from diagram (5-36) are of higher than twist-3 order.

The calculations of three parton contributions are straightforward by using the reduced hard scattering functions given in Appendix A. The important steps in the calculations are described below. There are totally 36 Feynman diagrams as depicted in Fig. 3. Only
4 Feynman diagrams (Fig. 3(1)-(4)) are needed at twist-3 order. The rest 32 Feynman diagrams (Fig. 3(5)-(36)) are of higher twist. Under covariant gauge, the contraction of the spin projector \( \Gamma_{\rho\lambda\eta\nu}(p_\pi) = (p_\rho d_\lambda - p_\lambda d_\rho)(p_\nu) \) with those contributions from Fig. 3(1)-(4) contained in \( \text{Tr} \left[ \sigma_\rho \gamma_5 H^{(1)}_{\mu\nu}(\xi, u_q, u_\bar{q}, u_g)(P_B + m_B)\gamma_5 \right] \) results in Eq. (91).

IV. TWIST-3 AMPLITUDES AND FORM FACTORS

In this Section, we present the twist-3 amplitudes \( M^{B,\mu}, M^{\text{tw}_3,ps,\mu}, M^{\text{tw}_3,pt,\mu}, \) and \( M^{\text{tw}_3,3p,\mu} \) calculated by the collinear expansion introduced in last Section. The twist-3 form factors \( f_{1,2}^{\text{tw}_3} \) are extracted from each twist-3 amplitude. The twist-3 form factors \( F_{+,0}^{\text{tw}_3} \) are determined from the twist-3 form factors \( f_{1,2}^{\text{tw}_3} \). The calculations of \( M^{\text{tw}_3,ps,\mu}, M^{B,\mu}, M^{\text{tw}_3,pt,\mu}, \) and \( M^{\text{tw}_3,3p,\mu} \) are straightforward according to Eqs. (76, 77, 79). The reduced two parton hard scattering functions \( H^{(1),\mu}(\xi, u) \) and the reduced three parton hard scattering functions \( H^{(1),\mu}_{3\nu}(\xi, u) \) are given in Appendix A. By substituting these reduced hard scattering functions into Eqs. (76, 77, 79), one can obtain the following twist-3 contributions.

A. \( \bar{\phi}_B \) contributions

The \( \bar{\phi}_B \) contributions are written as

\[
M^{B,\mu} = \frac{\pi \alpha_s C_F}{N_c} f_\pi f_B \int_0^1 d\xi \bar{\phi}_B(\xi) \int_0^1 du \phi^B_\pi(u) H^{B,\mu}(\xi, u),
\]

(79)

where

\[
H^{B,\mu} = \frac{1}{\eta \xi \bar{u} E^2} P_\mu^B + \frac{(1 + \eta) \eta \bar{u} - \eta - (1 - \eta) \xi}{\eta^2 \xi \bar{u} (\xi - \eta \bar{u}) E^2} p_\mu^B.
\]

Similar to the twist-2 case, the form factors are defined as

\[
M^{B,\mu} = f_1^B(q^2) P_\mu^B + f_2^B(q^2) p_\mu^B,
\]

(80)

where \( f_{1,2}^B \) are given by

\[
f_1^B(q^2) = \frac{\pi \alpha_s C_F}{E^2 N_c} f_\pi f_B \int d\xi \bar{\phi}_B(\xi) \int du \phi^B_\pi(u) \left[ \frac{1}{\eta \xi \bar{u}} \right],
\]

(81)

\[
f_2^B(q^2) = \frac{\pi \alpha_s C_F}{E^2 N_c} f_\pi f_B \int d\xi \bar{\phi}_B(\xi) \int du \phi^B_\pi(u) \left[ \frac{(1 + \eta) \eta \bar{u} - \eta - (1 - \eta) \xi}{\eta^2 \xi \bar{u} (\xi - \eta \bar{u})} \right].
\]

(82)
The form factors $F^B_{\pi,0}(q^2)$ are obtained by $f_{1,2}^{tw3,B}$ as

$$F^B_+(q^2) = \frac{\pi \alpha_s C_F}{2E^2 N_c} f_{\pi} f_B \int d\xi \phi_B(\xi) \int du \phi_\pi^P(u) \left[ \frac{\eta \bar{u} - \eta - (1 - 2\eta)\xi}{\eta^2 \xi \bar{u} (\xi - \eta \bar{u})} \right],$$

$$F^B_0(q^2) = \frac{\pi \alpha_s C_F}{2E^2 N_c} f_{\pi} f_B \int d\xi \bar{\phi}_B(\xi) \int du \phi_\pi^P(u) \left[ \frac{\xi - (1 - 2\eta)\eta \bar{u} - \eta}{\eta \xi \bar{u} (\xi - \eta \bar{u})} \right].$$

**B. Twist-3 two parton pseudo-scalar and pseudo-tensor contributions**

The pseudo-scalar contributions and pseudo-tensor contributions are found identical as

$$M^{tw3,ps,\mu} = M^{tw3,pt,\mu} = f_{\pi} f_B \frac{\pi \alpha_s C_F}{2N_c} r_\chi \int d\xi \phi_B(\xi) \int du \phi_\pi^P(u) H_{tw3,2p,\mu}(\xi, u),$$

where

$$H_{tw3,2p,\mu} = -\frac{1}{\eta^2 \bar{u} E^2} P^\mu_B - \frac{(1 + \eta)\eta \bar{u} - \xi}{\eta^2 \bar{u} (\xi - \eta \bar{u}) E^2} P^\mu_\pi.$$

The form factors are defined as

$$M^{tw3,ps(pt),\mu} = f_1^{tw3,ps(pt)}(q^2) P^\mu_B + f_2^{tw3,ps(pt)}(q^2) P^\mu_\pi,$$

where the form factors $f_1^{tw3,ps(pt)}$ are given by $(2E^2 = m_B^2$ and $r_\chi = 2\mu_\chi/m_B)$

$$f_1^{tw3,ps(pt)}(q^2) = -\frac{\pi \alpha_s C_F}{2E^2 N_c} f_{\pi} f_B r_\chi \int d\xi \phi_B(\xi) \int du \phi_\pi^P(u) \left[ \frac{1}{\eta^2 \bar{u} \xi} \right],$$

$$f_2^{tw3,ps(pt)}(q^2) = -\frac{\pi \alpha_s C_F}{2E^2 N_c} f_{\pi} f_B r_\chi \int d\xi \phi_B(\xi) \int du \phi_\pi^P(u) \left[ \frac{(1 + \eta)\eta \bar{u} - \xi}{\eta^2 \bar{u} (\xi - \eta \bar{u})} \right].$$

The form factors $F^B_{+,-,tw3,ps}(q^2)$ are obtained by $f_1^{tw3,ps}$ as

$$F^B_{+,tw3,ps}(q^2) = -\frac{\pi \alpha_s C_F}{4E^2 N_c} f_{\pi} f_B r_\chi \int d\xi \phi_B(\xi) \int du \phi_\pi^P(u) \left[ \frac{\eta \bar{u} - (1 - \eta)\xi}{\eta^2 \xi \bar{u} (\xi - \eta \bar{u})} \right],$$

$$F^B_{-,tw3,ps}(q^2) = \frac{\pi \alpha_s C_F}{4E^2 N_c} f_{\pi} f_B r_\chi \int d\xi \phi_B(\xi) \int du \phi_\pi^P(u) \left[ \frac{(1 - 2\eta)\eta \bar{u} - (1 - \eta)\xi}{\eta^2 \xi \bar{u} (\xi - \eta \bar{u})} \right].$$

We found that the twist-3 two pseudo-tensor amplitude $M^{tw3,pt,\mu}$ is equal to the twist-3 two pseudo-scalar amplitude $M^{tw3,ps,\mu}$. The corresponding form factors defined by these two amplitudes are identical. This is consistent with the reduced equations of motion $\phi_\pi^P(u) = \phi_\pi^P(u)$. 

25
C. Twist-3 three parton contributions

The twist-3 three parton contributions are written as

\[ M_{tw3p,\mu}^{tw3p,\mu} = \frac{\pi \alpha_s C_F}{N_c} f_{\pi} f_{BR} \pi \int d\xi \phi_B(\xi) \int du_q \int du_{q'} \int du_g \times \delta(1 - u_q - u_{q'} - u_g) \phi_3^p(u_q, u_{q'}, u_g) H_{3p,\mu}(\xi, u_q, u_{q'}, u_g), \]  

(90)

where

\[ H_{3p,\mu}(\xi, u_q, u_{q'}, u_g) = \frac{1}{u_q u_{q'} u_g} \times \left[ \frac{1}{\eta \xi u_g E^2} P_B + \frac{(1 + \eta u_g) \eta u_{q'} - \xi}{\eta^3 \xi (\xi - \eta u_q) E^2 P_B^\mu} \right]. \]  

(91)

The twist-3 three parton form factors are defined as

\[ M_{tw3p,\mu}^{tw3p,\mu} = f_{1}^{tw3p,\mu}(q^2) P_B^\mu + f_{2}^{tw3p,\mu}(q^2) P_B^\mu, \]

where the form factors \( f_{1,2}^{tw3p} \) are given by \((2E^2 = m_B^2 \text{ and } r_B = 2\mu_B/m_B)\)

\[ f_{1}^{tw3p,\mu}(q^2) = \frac{\pi \alpha_s C_F}{E^2} \frac{1}{N_c} f_{\pi} f_{BR} \pi \int d\xi \phi_B(\xi) \int du_q \int du_{q'} \int du_g \delta(1 - u_q - u_{q'} - u_g) \phi_3^p(u_q, u_{q'}, u_g) \frac{1}{u_q u_{q'} u_g} \frac{1}{\eta \xi u_g E^2} P_B + \frac{(1 + \eta u_g) \eta u_{q'} - \xi}{\eta^3 \xi (\xi - \eta u_q) E^2 P_B^\mu}, \]  

(92)

\[ f_{2}^{tw3p,\mu}(q^2) = \frac{\pi \alpha_s C_F}{E^2} \frac{1}{N_c} f_{\pi} f_{BR} \pi \int d\xi \phi_B(\xi) \int du_q \int du_{q'} \int du_g \delta(1 - u_q - u_{q'} - u_g) \phi_3^p(u_q, u_{q'}, u_g) \frac{1}{u_q u_{q'} u_g} \frac{(1 + \eta u_g) \eta u_{q'} - \xi}{\eta^3 \xi u_g (\xi - \eta u_q)} \frac{1}{P_B}. \]  

(93)

The form factors \( F_{+,0}^{B\pi,tw3p}(q^2) \) are obtained by Eq.(34)

\[ F_{+,0}^{B\pi,tw3p}(q^2) = \frac{\pi \alpha_s C_F}{2E^2} \frac{1}{N_c} f_{\pi} f_{BR} \pi \int d\xi \phi_B(\xi) \int du_q \int du_{q'} \int du_g \delta(1 - u_q - u_{q'} - u_g) \phi_3^p(u_q, u_{q'}, u_g) \frac{1}{u_q u_{q'} u_g \eta} \frac{(1 - \eta u_g) \eta u_{q'} - \eta \xi}{\eta^3 \xi u_g (\xi - \eta u_q)}, \]  

(94)

\[ F_{0}^{B\pi,tw3p}(q^2) = \frac{\pi \alpha_s C_F}{2E^2} \frac{1}{N_c} f_{\pi} f_{BR} \pi \int d\xi \phi_B(\xi) \int du_q \int du_{q'} \int du_g \delta(1 - u_q - u_{q'} - u_g) \phi_3^p(u_q, u_{q'}, u_g) \frac{1}{u_q u_{q'} u_g} \frac{\eta \xi - (1 - \eta (2 - u_g)) \eta u_{q'}}{\eta^2 \xi u_g (\xi - \eta u_q)} \frac{1}{P_B^\mu}. \]  

(95)
The resultant form factors are

\[ F^{B\pi}(q^2) = F^{tw2}(q^2) + F^{B}(q^2) + F^{tw3,ps}(q^2) + F^{tw3,pt}(q^2) + F^{tw3,3p}(q^2) \]

\[ = \frac{\pi \alpha_s(t)}{2E^2} C_F f_B \int d\xi \phi_B(\xi) \int du \left[ \frac{\phi_P^P(u)(\eta - \xi + \eta \bar{u}) + \phi_P^P(u)(1 - \eta)\bar{u}}{\eta^2 \xi \bar{u}(\xi - \eta \bar{u})} \right] \]

\[ + \frac{\pi \alpha_s(t)}{2E^2} C_F f_B \int d\xi \phi_B(\xi) \int du \phi_P^P(u) \left[ \frac{\eta - \eta - (1 - 2\eta)\bar{u}}{\eta^2 \xi \bar{u}(\xi - \eta \bar{u})} \right] \]

\[ + \frac{\pi \alpha_s(t)}{2E^2} C_F f_B r_{\chi} \int d\xi \phi_B(\xi) \int du \int du \int du \int du \delta(1 - u - u - u) \]

\[ \times \frac{\phi_P^P(u, u, u, u)}{\eta^2 \xi \bar{u}(\xi - \eta \bar{u})} \left[ \frac{(1 - \eta u)(\eta u - \eta \bar{u})}{\eta^2 \xi \bar{u}(\xi - \eta \bar{u})} \right] \].

The form factor \( F^{B\pi}(q^2) \) is analytic in the whole range of \( q^2 \), \( 0 \leq q^2 \leq q_{\max}^2 = 26.42 \) GeV\(^2\) for \( m_B = 5.28 \) GeV and \( m_\pi = 0.14 \) GeV. Under the maximal recoil limit, \( \eta_{\max} = 1 \), the form factor \( F^{B\pi}_+(q^2) \) becomes

\[ \lim_{q^2 \rightarrow 0} F^{B\pi}_+(q^2) = \frac{\pi \alpha_s(t)}{2E^2} C_F f_B \int d\xi \phi_B(\xi) \int du \left[ \frac{\phi_P^P(u)(\bar{u} - \xi)}{\eta \xi \bar{u}(\xi - \eta \bar{u})} \right] \]

\[ + \frac{\pi \alpha_s(t)}{2E^2} C_F f_B \int d\xi \phi_B(\xi) \int du \phi_P^P(u) \left[ \frac{\bar{u} - \xi}{\eta \xi \bar{u}(\xi - \eta \bar{u})} \right] \]

\[ + \frac{\pi \alpha_s(t)}{2E^2} C_F f_B r_{\chi} \int d\xi \phi_B(\xi) \int du \int du \int du \int du \delta(1 - u - u - u) \]

\[ \times \frac{\phi_P^P(u, u, u, u)}{\eta^2 \xi \bar{u}(\xi - \eta \bar{u})} \left[ \frac{1}{\xi (\xi - 1)} \right] \].

The dependence of strong coupling constant \( \alpha_s(t) \) on \( t \propto \sqrt{\eta} \) restrict the effective range of \( q^2 \) to be \( 0 \leq q^2 \leq 16 \) GeV\(^2\), in which the values of \( \alpha_s \) are varying within 0.33 ~ 0.48.
V. NUMERICAL CALCULATIONS

We apply previous results for numerical calculations of the form factor $F_{B^+\pi}(q^2)\rangle$ for $\bar{B}^0 \rightarrow \pi^- l^+ \bar{\nu}_l$ decays. The shape parameter $\omega_B$ of the $B$ meson DA and the value of $|V_{ub}|$ will be determined. The $q^2$ shape of the form factor $F_{B^+\pi}(q^2)$ is given according to the value of $\omega_B$.

A. analysis method

In order to explore the full range $q^2$ shape of $F_{+}\langle q^2\rangle$ based on our calculations, we employ a statistic analysis method. We assume that the form factor $F_{+}\langle q^2\rangle$ can smoothly vary with $q^2$ from small $q^2$ region to large $q^2$ region. We first collect 17 data points of $F_{+}\langle q^2\rangle$ at $0 \leq q^2 \leq 16 \text{ GeV}^2$ and $\Delta q^2 = 1\text{ GeV}^2$. We denote these 17 data points as the $C_f$ set of data points. Based on the $C_f$ set of data points, we derive a fit formula for $F_{+}\langle q^2\rangle$ by a minimum $\chi^2$ fit according to a $\chi^2_{FF}$ function. The fit form factor $F_{+}\langle q^2\rangle^\text{fit}$ is a function of $q^2$ with 9 parameters, $a_i$, $i = 0, 1, \cdots , 8$, as

$$F_{+}\langle x\rangle^\text{fit} = \sum_{i=0}^{8} a_i x^i. \quad (99)$$

The $\chi^2_{FF}$ function is defined as

$$\chi^2_{FF} = \sum_{i}^{N} \frac{(F_{+}\langle q_i^2\rangle^\text{fit} - F_{+}\langle q_i^2\rangle^\text{data})^2}{\sigma_i^2}, \quad (100)$$

where $i$ denotes the $i$-th bin and $N$ is the total bin number of data points. $q_i^2$ and $\sigma_i$ are the reference $q^2$ value and the uncertainty for the $i$-th bin of the used data set. $F_{+}\langle q_i^2\rangle^\text{fit}$ is the value of $F_{+}\langle x\rangle^\text{fit}$ at $x = q_i^2$. $F_{+}\langle q_i^2\rangle^\text{data}$ is the $i$-bin data. A 8% error is taken for the input parameters ($\alpha_s$, $f_\pi$, $f_B$, $r_\chi$). See more detailed explanations about this 8% error in next Subsection.

The correctness of the $q^2$ shape of $F_{+}\langle q^2\rangle$ for $q^2 > 16 \text{ GeV}^2$ is estimated by comparing the predictions of $F_{+}\langle q^2\rangle^\text{fit}$ for $q^2 > 16 \text{ GeV}^2$ to the LQCD calculations by the FNAL collaboration \cite{16, 51, 52} and by the HPQCD collaboration \cite{13}, respectively.

The fit form factors is used to make a prediction for the branching ratio denoted as $Br_{SL}^\text{th}$ for $\bar{B} \rightarrow \pi l\bar{\nu}_l$. The $\omega_B$ and $|V_{ub}|$ are determined by comparing the predictions and experimental data, $Br_{SL}^\text{exp}$. This is equivalent to a fit for the value of $|V_{ub}F_{+}(0)|$. Because the $q^2$ shape of $F_{+}\langle q^2\rangle$ is fixed by Eq. (96), theoretically, the normalization point $|V_{ub}F_{+}(0)|$
can be determined by the fitting method. The $|F_+(0)|$ and $|V_{ub}|$ are determined separately. Our best fit to the central value of $Br_{SL}^{exp,full} = (1.36 \pm 0.09) \times 10^{-4}$ (PDG08) \[45\] gives

$$|V_{ub}F_+(0)| = (6.471 \pm 0.196_{\text{exp}} \pm 0.523_{\text{th}}) \times 10^{-4},$$

$$|F_+(0)| = 0.164^{+0.010}_{-0.009}|_{\alpha_s \pm 0.008}|_{f_B}$$

$$= 0.164 \pm 0.013,$$

$$|V_{ub}| = (3.946^{+0.128}_{-0.133_{\text{exp}} - 0.282_{\text{th}}}) \times 10^{-3},$$

$$\omega_B = 0.315 \text{GeV}.$$  

The errors are experimental (exp) and theoretical (th). The theoretical errors are only considering those uncertainties from $\alpha_s$ and $f_B$. $|F_+(0)|$ is calculated according to the fit form factor and the fitted value $\omega_B$. $|V_{ub}|$ is determined by a combination of $|V_{ub}F_+(0)|$ and $|F_+(0)|$. The central value of $|V_{ub}F_+(0)|$ is determined by a best fit to the central value of the $Br_{SL}^{exp}$. The determinations of the fit form factor and corresponding $|V_{ub}|$ will be respectively described in detail latter.

The $C^f$ form factor $F^C_+(q^2)$ is perturbatively meaningful for $q^2 \leq 16 \text{ GeV}^2$. We perform an similar analysis as above by using the partial branching fraction $Br_{SL}^{exp,q^2<16\text{GeV}^2} = (0.93 \pm 0.06) \times 10^{-4}$.  

\[1\] The best fit result is given by

$$|V_{ub}F_+(0)| = (7.34 \pm 0.233_{\text{exp}} \pm 0.597_{\text{th}}) \times 10^{-4},$$

$$|F_+(0)| = 0.188^{+0.011}_{-0.010}|_{\alpha_s \pm 0.009}|_{f_B}$$

$$= 0.188 \pm 0.014_{\text{th}},$$

$$|V_{ub}| = (3.904^{+0.124}_{-0.128_{\text{exp}} + 0.356_{\text{th}}}) \times 10^{-3},$$

$$\omega_B = 0.281 \text{GeV},$$

The errors are the same as explained previous. We note that our best fit value $|V_{ub}F_+(0)| = 6.5 \times 10^{-4}$ is lower than the value \[23\], which was obtained by a fitting to the branching fraction and $q^2$ spectrum data by BaBar collaboration \[53\],

$$|V_{ub}F_{\pi}^{B}(0)| = (9.1 \pm 0.3 \pm 0.6) \times 10^{-4},$$

\[1\] The value is quoted from the ICHEP08 averages given in the online update at [http://www.slac.stanford.edu/xorg/hfag](http://www.slac.stanford.edu/xorg/hfag) by the HFAG collaboration. Note that the full branching fraction $Br_{SL}^{exp,full} = (1.34 \pm 0.08) \times 10^{-4}$ has been updated by HFAG at 2008.
where the first error is from the uncertainty on $Br(\bar{B} \to \pi l \bar{\nu}_l)$, and the second from the parametrization of shape of the form factor versus $q^2$. The $|V_{ub}F_+(0)|$ in Eqs. (101) and (102) is consistent with the result $|V_{ub}F_+(0)| = (7.6 \pm 1.9) \times 10^{-4}$ obtained with SCET and QCD factorization [36].

The analysis procedure is the following. We first choose a reference value of $\omega_B$ to prepare the $C_f$ set of data point for $F_+$. Using this $C_f$ set of data, the fit form factor is determined according to the $\chi^2_{FF}$ function. The fit form factor is then used to make a prediction for the branching fraction for $\bar{B} \to \pi l \bar{\nu}_l$ denoted as $Br^{fit}_{SL}$ by varying the value of $|V_{ub}|$. A best fit for the predicted branching fraction and experimental one is performed according to the $\chi^2_{Br}$ function

$$\chi^2_{Br} = \frac{(Br^{fit}_{SL} - Br^{exp}_{SL})^2}{\sigma^2}$$

where $\sigma$ is the experimental error calculated by adding the systematic and statistical errors in quadrature. Once a best fit is found, $|V_{ub}|$ is determined. Otherwise, another $\omega_B$ is chosen and the whole procedure is run from start, again.

**B. form factors**

The values of $\alpha_s(t)$ at different values of $q^2$ are calculated by the program [54] and given in Table I. The program [54] has considered experimental and theoretical uncertainties. The scale $t$ is defined as the scale of the internal virtual gluon, which are assumed to carry a hard-collinear energy. $t = \sqrt{\eta \Lambda_h m_B}$ with $\Lambda_h = 0.5$ GeV and $m_B = 5.28$ GeV.

The uncertainties on the $F_+$ could be (1) phenomenological models for meson DAs, (2) input parameters ($\alpha_s$, $f_\pi$, $f_B$, $r_\chi^$, $\omega_B$, $\eta_3$, and $\omega$). The uncertainty on phenomenological models is fixed by taking models determined from other processes. The twist-2 and twist-3 meson DAs takes the following models

$$\phi^P_\pi(u) = \phi^P_\pi(u) = \phi^0_\pi(u) = 6u(1-u),$$

$$\phi^{3P}_\pi(u_q, u_q, u_g) = 360\eta_3 u_q u_g u_g^2(1 + \frac{\omega}{2}(7u_g - 3)).$$

where $\eta_3 = 0.015$ and $\omega = -3.0$ are used. The chiral factor is set $r_\chi^ = 0.76$ GeV. The total uncertainties on the input parameters are assumed constant of 8%. The recent LQCD calculations for $f_B$ are $f_B = 204^{+37}_{-28}$ MeV (CP-PACS) [55], $f_B = 190 \pm 14$ MeV (JL-QCD)
Table I: The values of \( \alpha_s \) at \( q^2 \), \( 0 \leq q^2 \leq 16 \text{GeV}^2 \), are calculated by the program \[54\].

| \( q^2(\text{GeV}^2) \) | 0  | 2  | 4  | 6  | 8  | 10 | 12 | 14 | 16 |
|------------------|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| \( \alpha_s(t) \) | 0.333 | 0.340 | 0.348 | 0.357 | 0.368 | 0.400 | 0.421 | 0.447 | 0.484 |
| \( \delta \alpha_s(t) \) | +0.020 | +0.020 | +0.022 | +0.023 | +0.025 | +0.032 | +0.036 | +0.043 | +0.053 |

For example, \( f_B = 206 \pm 10 \text{MeV} \) (ALPHA) \[54\], \( f_B = 216 \pm 11 \text{MeV} \) (HPQCD) \[58, 59\], \( f_B = 214 \pm 9 \text{MeV} \) (Guo-Weng) \[60\]. Except of the CP-PACS result with 16\% errors, the other calculations contain about 5\% uncertainties. In our calculations, we choose \( f_B = 200 \text{MeV} \) as a reference value and associate a 5\% error on it. The chiral enhanced contributions are about 20\% in our calculations. We choose \( r_\pi = 2m_\pi^2/(m_B(m_u + m_d)) = 0.76 \text{GeV} \). This corresponds to use \( (m_u + m_d) = 9.8 \text{MeV} \) \[45\], \( m_\pi = 140 \text{MeV} \), and \( m_B = 5.28 \text{GeV} \). The pion decay constant \( f_\pi^+ = (130.4 \pm 0.04 \pm 0.02) \text{MeV} \) and \( f_\pi^0 = (130 \pm 5) \text{MeV} \) \[45\]. The errors on \( f_\pi^- \) can be neglected. The uncertainties on \( \alpha_s \) are 5 - 10\%. There are about 10\% errors associated with \( r_\chi \). Because the chiral contributions with \( r_\chi \) are about 20\% of the leading twist (twist-2) contributions, the errors on \( r_\chi \) are about 0.2 \times 0.1 \approx 2\% and can be neglected safely. The constant 8\% errors on \( F_+(q^2) \) is assumed according to the errors on \( \alpha_s \) and \( f_B \). Instead of using \( \bar{\Lambda} = m_B - m_b \) and \( \lambda_B \) for the value of \( \omega_B \), we take \( \omega_B \) as a free parameter and propose to determine \( \omega_B \) and \( |V_{ub}| \) by the world branching fraction \( Br_{SL}^{\text{exp}} \) for \( \bar{B} \rightarrow \pi l\bar{\nu}_l \)

\[
Br_{SL}^{\text{exp}}(\bar{B}^0 \rightarrow \pi^- l^+ \nu_l) = (1.36 \pm 0.06_{\text{stat}} \pm 0.07_{\text{sys}}) \times 10^{-4}
\]

(104)

for \( 0 \leq q^2 \leq q^2_{\text{max}} \), where \( q^2_{\text{max}} = 26.42 \text{GeV}^2 \).

17 bins of \( C_f^\tau \) data points of \( F_+^{B\pi}(q^2) \) with \( \Delta q^2 = 1\text{GeV}^2 \) are prepared by taking a test value of \( \omega_B \). In our present case, we choose \( \omega_B = 0.46 \text{GeV} \) as the first value. The \( F_+^{\text{fit}}(q^2) \) is then used to make a \( \chi^2 \) fit to these 17 bins of data. The best fit is determined by requiring the \( \chi^2_{FF} \) function to be minimal. The fit form factor is then substituted in Eq. (109) to make a prediction for the branching fraction, \( Br_{SL}^{\text{th}} \), by varying the value of \( |V_{ub}| \) in \( 2.8 \times 10^{-3} \leq |V_{ub}| \leq 4.5 \times 10^{-3} \). The best fit is found by using the \( \chi^2_{Br} \) function to calculate the minimal \( \chi^2 \) value. Our best fit gives

\[
\omega_B = 0.315 \pm 0.000 \text{GeV} \, , \, |V_{ub}| = 3.946^{+0.006}_{-0.002} \times 10^{-3}
\]

(105)

under \( \chi^2_{Br} \leq 0.001 \). The errors on the fit value of \( \omega_B \) and \( |V_{ub}| \) are statistic. The fit form
factor is assumed to have 9 parameters

\[ F_{\text{fit}}^+(x) = \sum_{i=0}^{i=8} a_i x^i. \]  

(106)

The fitting of \( F_{\text{fit}}^+(x) \) to the 17 data points is performed by using different numbers of parameters, 1 to 9. The minimal \( \chi^2 \) value of the \( \chi^2_{FF} \) function is required to derive a best fit. The \( \chi^2_{FF}/N \) is equal to 0.05/(17 - 5) by assuming an 8% constant errors on the \( C_f^f \) data set. The values of parameters \( a_i, i = 0 \cdot \cdot \cdot 8 \), are given in Table II. The errors on \( a_i, i = 0 \cdot \cdot \cdot 8 \), are statistic due to the best fit of \( \omega_B \). We denote the fit form factor as \( F_{Cf}^+(q^2) \).

By \( F_{Cf}^+(q^2) \), we compare our calculations with the same form factor calculated by the other approaches, including the \( k_f^T \) method [31], the LCSR method [61], the LQCD-HPQCD method [13], the LQCD-Fermilab/MILC/2005 (FNAL05) method [16, 51] and the LQCD-Fermilab/MILC/2008 (FNAL08) method [52]. The comparisons are shown in Fig. 4. The twist-2, \( F_{B\pi,tw2}^+ \), and twist-3, \( F_{B\pi,tw3}^+ \), components of the form factor \( F_{B\pi}^+ \) are calculated at different \( q^2 \), \( 0 \leq q^2 \leq 16 \text{ GeV}^2 \) in Table. III. We note that \( F_{Cf}^+(0) = 0.164 \) is obtained. This is close to the finding by the SCET analysis for \( B \to \pi\pi \) decays [36], that \( F_{SCET}^{Cf}(0) = 0.170 \).

We note that \( F_{Cf}^+(0) = 0.164 \) is smaller than \( F_{t}(0) = 0.25 \sim 0.30 \) derived by other theories (such as, LCSR, LQCD, \( k_f^T \)), and experiments. As was firstly found by Bauer et al [36], there is an inconsistency between the small \( F_{SCET,Cf}^{SCET}(0) = 0.17 \) implied in \( B \to \pi\pi \) decays and the large \( F_{t}(0) = 0.25 \sim 0.30 \) derived by other theories and experiment. For example, \( F_{B\pi}^+(0) = 0.28 \) was used in the QCD factorization for \( B \to \pi\pi \) decays and the
Table II: The values of parameters of $F_{\pi}^{fit}(q^2)$ and the best fit value of $\omega_B$ in unit of GeV. The best fit means $\chi^2_{BR} < 0.001$. The errors on $\omega_B$ and $a_i$, $i = 0, \cdots, 8$, are statistic.

| Method               | $\chi^2_N$ | $\omega_B \times 10^1$ | $a_0 \times 10^1$ | $a_1 \times 10^3$ | $a_2 \times 10^3$ | $a_3 \times 10^4$ | $a_4 \times 10^5$ | $a_5$ | $a_6$ | $a_7$ | $a_8$ |
|----------------------|-------------|------------------------|-------------------|-------------------|-------------------|-------------------|-------------------|-------|-------|-------|-------|
| full                 | $0.05_{12}$ | $3.151^{+3}_{-3}$      | $1.636^{+2}_{-2}$ | $8.403^{+12}_{-10}$ | $1.903^{+2}_{-2}$ | $-1.873^{+2}_{-2}$ | $1.407^{+2}_{-2}$ | $0$    | $0$    | $0$    | $0$    |
| partial $q^2 \leq 16$ GeV^2 | $0.00_{12}$ | $2.810^{+5}_{-5}$      | $1.881^{+4}_{-4}$ | $9.773^{+23}_{-22}$ | $2.185^{+5}_{-5}$ | $-2.143^{+5}_{-5}$ | $1.625^{+3}_{-4}$ | $0$    | $0$    | $0$    | $0$    |

Figure 4: Comparisons between $C_f$(solid line), $k_f^T$(dash-dot-dash line), LCSR(dot line), LQCD(FNAL05) (data point), and LQCD-HPQCD (data point) form factors.

QCD factorization predictions for the branching ratios of $B \to \pi \pi$ decays were inconsistent with the experimental data. On the other hand, if $F_{\pi}^{SCET}(0) = 0.17$, then the QCD factorization based on SCET can explain the experimental data consistently. According to our calculations, the small $F_{\pi}^{C_f}(0) = 0.164 \pm 0.013$ can also explain the semi-leptonic decay $\bar{B} \to \pi l \bar{\nu}_l$ with a value of $|V_{ub}|$ consistent with the world averaged value. Recently, a general parameterization approach [14, 15, 16, 17, 18] has been used to study the $q^2$ shape of $F_+(q^2)$. Among of different parameterization approaches [14, 15, 16, 17, 18], Arneson et al [16] found that it is possible to consistently explain the branching ratio and BaBar differential rate data by using the SCET value $F_{\pi}^{SCET}(0) = 0.17$. The other parameterization approaches [15, 17, 18] employed the LCSR value $F_{\pi}(0) = 0.26$ as an input. Whether the value of $F_{\pi}(0)$ is small or large is a controversial topic, it needs more theoretical and experimental works to clarify this problem.
Table III: The values of different contributions of $|F^B_{\pi}(q^2)|$ at $q^2$, $0 \leq q^2 \leq 16 \text{GeV}^2$. $\omega_B = 0.315 \text{GeV}$ is used.

| $q^2(\text{GeV}^2)$ | 0  | 2  | 4  | 6  | 8  | 10 | 12 | 14 | 16 |
|---------------------|----|----|----|----|----|----|----|----|----|
| $|F^B_{\pi,\text{tw2}}(q^2)| \times 10^3$ | 175 | 205 | 240 | 286 | 351 | 444 | 582 | 787 | 1100 |
| $|F^B_{\pi,B}(q^2)| \times 10^4$ | 129 | 152 | 180 | 218 | 270 | 347 | 463 | 640 | 919 |
| $|F^B_{\pi,\text{tw3,ps+pt}}(q^2)| \times 10^4$ | 196 | 248 | 318 | 417 | 568 | 810 | 1213 | 1911 | 3202 |
| $|F^B_{\pi,\text{tw3,3p}}(q^2)| \times 10^4$ | 106 | 127 | 154 | 193 | 255 | 360 | 547 | 900 | 1615 |
| total $\times 10^3$ | 162 | 188 | 219 | 259 | 314 | 393 | 508 | 679 | 943 |

The fit result $\omega_B = 0.315 \text{GeV}$ implies $\bar{\Lambda} = 0.471 \text{ GeV}$ or $\lambda_B = 0.315 \text{GeV}$. We note that $\lambda_B = 0.315 \text{GeV}$ is lower than $0.460 \pm 0.110 \text{ GeV}$ by Braun et al. [62], $0.454 \text{ GeV}$ by Grozin and Neubert [46], $0.479 \pm 0.089 \text{ GeV}$ by Lee and Neubert [47], $0.600 \text{ GeV}$ by Ball [63]. $m_b = m_B - \bar{\Lambda} = (5.28 - 0.471) \text{ GeV} = 4.57 \text{ GeV}$, is consistent with the $m_b$ determined by different schemes, $m_b^{1S} = 4.701 \pm 0.030 \text{ GeV}$ (1S scheme) [64], $m_b^{SF} = 4.630 \pm 0.060 \text{ GeV}$ (shape function scheme) [64], and $m_b^{\text{kin}} = 4.613 \pm 0.035 \text{ GeV}$ (kinetic scheme) [64], but is larger than $m_b^{MS} = 4.164 \pm 0.025 \text{ GeV}$ (MS bar scheme) [65]. At the chiral point, $E_\pi \simeq m_\pi$ and $q^2(E_\pi) = 26.42 \text{ GeV}^2$, the chiral perturbation theory predicts that

$$F^{\text{Chiral}}_+(q^2) = \frac{g f_B m_B}{2 f_\pi (E_\pi + m_{B^*} - m_B)},$$

where $g$ is the $B^*B\pi$ coupling. The fit form factor $F^{Cf}_+(q^2)$ predicts $F^{Cf}_+(26.42) = 5.12$, which implies $g = 0.234 \pm 0.019$ with a theoretical error by using $m_{B^*} - m_B = 45.75 \pm 0.35 \text{ MeV}$. The extracted $g = 0.234 \pm 0.019$ is consistent with $g = 0.22 \pm 0.07$ determined by the FNAL collaboration [52], and $0 < g < 0.45$ by the HPQCD collaboration [13], but is lower than the usually employed value $g = 0.51$ proposed by Stewart et al [16].

C. determination of $V_{ub}$

The differential decay rate for $B^0 \rightarrow \pi^- l^+ \bar{\nu}_l$, under the approximation that the lepton masses are vanishing, is given by

$$\left( \frac{dT}{dq^2} \right)_{\text{th}} = \frac{G_F^2 |V_{ub}|^2}{24 \pi^3} |p_\pi|^3 |F^B_{\pi}(q^2)|^2$$

(109)
where $p_\pi$ is the momentum of pion in the $B$ meson rest frame. The branching ratio $Br(B^0 \to \pi^- l^+ \bar{\nu}_l)$ is expressed in terms of $(d\Gamma/d\eta)_\text{th}$

$$Br(B^0 \to \pi^- l^+ \bar{\nu}_l) = \tau_{B^0} \int_{0}^{q^2_{\text{max}}} dq^2 \frac{d\Gamma}{dq^2}_\text{th}.$$  \hspace{1cm} (110)

where $\tau_{B^0} = (1.530 \pm 0.009)\text{ps}$ is the life time of $\bar{B}^0$ meson [45].

To determine $|V_{ub}|$, we employ

$$R(q^2_{\text{max}}) \equiv |V_{ub}|^{-2} \int_{0}^{q^2_{\text{max}}} dq^2 \left( \frac{d\Gamma}{dq^2}_\text{th} \right).$$ \hspace{1cm} (111)

The result is

$$R(q^2_{\text{max}}) = (5.709 \pm 0.913_{\text{th}})\text{ps}^{-1}. \hspace{1cm} (112)$$

By substituting $R(q^2_{\text{max}})$ into Eq. (110), we can determine

$$|V_{ub}^{C\!F}| = (3.946 \pm 0.117_{\text{exp}} \pm 0.312_{\text{th}}) \times 10^{-3}. \hspace{1cm} (113)$$

The first error on $|V_{ub}|$ is from branching ratio and the second error is from $F_+(0)$. The theoretical uncertainty is 8% in the same level with that of the inclusive method. For comparison, the theoretical errors are 15% in the LCSR method, 10 – 14% in the LQCD [45], and 12% in the parameterization approach (PA) [14, 15, 17, 18]. The $|V_{ub}|$ in Eq. (113) is consistent with the inclusive value $|V_{ub}^{\text{incl}}| = (4.12 \pm 0.15_{\text{exp}} \pm 0.40_{\text{th}}) \times 10^{-3}$ from $B \to X_u l\bar{\nu}_l$ [45] and the world averaged $|V_{ub}^{\text{avg}}| = (3.95 \pm 0.35) \times 10^{-3}$ [45]. We note that the world averaged exclusive value is $|V_{ub}^{\text{excl}}| = (3.5 \pm 0.6) \times 10^{-3}$ [45]. The discrepancy between the values of $|V_{ub}|$ determined from exclusive and inclusive methods has raised a lot of discussions in literature [23]. However, we note that a smaller inclusive value $|V_{ub}^{\text{incl08}}| = (3.98 \pm 0.15 \pm 0.30) \times 10^{-3}$ with experimental and theoretical errors was obtained by Neubert [66]. This shows that there may not exist such a discrepancy. If we only employ the partial branching ratio for $q^2 < 16 \text{ GeV}^2$, a smaller $|V_{ub}|$ will be obtained as given in Eq. (102). The analysis method is similar to the above and skipped here.

### D. $q^2$ shape

BaBar has measured the differential rate for $\bar{B} \to \pi l\bar{\nu}_l$ versus $q^2$ with good accuracy [21, 53]. The $q^2$ shape of the differential rate relates to the $q^2$ shape of $F^B_+(q^2)$. It is
interesting to compare our prediction for the spectrum with the data. Based on $F_{CF}^+$, the $q^2$ spectrum $\Delta Br/Br$ is calculated as shown in Fig. 5. We observe that the predicted shape of $q^2$ spectrum ($C^f q^2$ shape) is inconsistent with the experimental $q^2$ spectrum. The $\chi^2$ is 86.6 for 12 degrees of freedom by assuming that the 12 bins of experimental data are completely uncorrelated. Parameterization methods [14, 15, 17, 18] have been widely used to determine the $q^2$ shape of $F_+(q^2)$ according to the $q^2$ shape data and theoretical inputs from LCSR and LQCD. It is an important task to compare $F_{CF}^+(q^2)$ with these parameterization form factors to investigate their differences. This is left to other places. In summary, our calculations for the $q^2$ spectrum of the differential rate for $\bar{B} \rightarrow \pi l \bar{\nu}_l$ decays can not accommodate with the $q^2$-spectrum of $\bar{B} \rightarrow \pi l \bar{\nu}_l$. The difference can be understood by the scaling behavior of $F_{CF}^+(q^2)$ and that of $(d\Gamma/dq^2)_\text{th}$, $F_{CF}^+(q^2) \propto \eta^{-1-0.1x}$ and $(\Gamma^{-1} d\Gamma/dq^2)_\text{th} \propto \eta^{1-0.2x}$ for $0 \leq q^2 \leq 10 \text{ GeV}^2$ and $x = q^2/(1\text{GeV}^2)$, and $F_{CF}^+(q^2) \propto \eta^{-2}$ and $(\Gamma^{-1} d\Gamma/dq^2)_\text{th} \propto \eta^{-1}$ for $q^2 \geq 10 \text{ GeV}^2$. We note that the form factor implied in BaBar $q^2$ data can have a scaling $F_{PA}^+(q^2) \propto \eta^{-1.5} (x + \eta^y)^{1/2}$ with $x = -0.897$ and $y = 0.0281$. This gives $\Gamma^{-1} d\Gamma/dq^2 \propto x + \eta^y$. The residual scaling factor $\eta^y$ in $\Gamma^{-1} d\Gamma/dq^2$ has small effects for large $\eta$ (small $q^2$) and is important for small $\eta$ (large $q^2$).
VI. COMPARISONS WITH OTHER APPROACHES

Many progresses in theories for $B \to \pi$ form factors were obtained in past years. Partial $O(1/m_B)$ corrections have been calculated in the $k_T$ factorization approach [31, 35] and LCSR [5, 9, 44]. The unquenched quark effects [10, 11, 12, 13, 52] were included in the lattice QCD method. An all order proof of the factorization of the form factors was given for leading twist and twist-3 two parton contributions in the collinear factorization [49] and in the soft collinear effective theory (SCET) [50], respectively. Different strategies for a parametrization of form factors have been developed [14, 15, 16, 17, 18, 23]. We compare our result with various theoretical methods.

A. Comparisons with $k_T^f$

It is the end-point divergent problem that the $B \to \pi$ form factors contain logarithmic end-point divergences at twist-2 and linear end-point divergences at twist-3. To solve the divergent problem, the general wisdom is to use $k_T$-regularization for logarithmic end-point divergences. For linear end-point divergences, a threshold re-summation is needed, too. Twist-3 contributions dominate over the twist-2 ones in the $k_T$ factorization approach. This is known as a power expansion problem in $k_T^f$. One possible solution to this problem is to employ twist-3 wave functions with better end-point behaviors. However, this would introduce additional uncertainties.

For comparison, let’s compare the values of $F^{B\pi, tw2}(0)$ calculated under the $k_T^R$ and $\xi^R$. Under $k_T^R$, the result is $F^{B\pi, tw2}(0) = 0.120$ [31], where $f_\pi = 0.13$GeV, $f_B = 0.19$GeV, and $\omega_B = 0.40$ GeV are used. Under $\xi^R$, $F^{B\pi, tw2}(0) = 0.124$ by using the same input parameters. This shows that $\xi^R$ is as effective as $k_T^R$ at leading twist order.

The large value $F_+(0)$ is related to the power expansion method used in $k_T^f$. In $k_T^f$, only $O(1/m_B)$ contributions from two parton Fock state of the pion were calculated. In [31], $\phi_p^\pi(u) = 1$ for the PS DA and $\phi_\sigma(u) = 6u(1 - u)$ for the PT DA of the pion were used. The collinear expansion scheme proposed by Beneke and Neubert (the BN scheme) [67] was employed. At $q^2 = 0$, the leading twist contributions lead to $F^{tw2}_+(0) = 0.120$ and the subleading twist contributions result in $F^{tw3}_+(0) = 0.177$. It is seen that $F^{tw3}_+(0)/F^{tw2}_+(0) = 1.5$. In comparison, the $\xi^R$ calculations give $F^{tw3}_+^{\xi^R}(0)/F^{tw2}_+^{\xi^R}(0) = 0.18$. This shows that the
twist-3 contributions in $k_T^f$ dominate over the twist-2 contributions. On the other hand, the twist-3 contributions in $C^f$ are power suppressed as expected. The authors in [31] proposed a new counting rule for the linear divergence associated with the twist-3 terms to explain this phenomenology. However, if higher order power corrections contain similar or even worse end point divergences, the power expansion would break down. This is the power expansion problem in $k_T^f$ [35]. Hwang et al [35] pointed out that intrinsic parton transverse momenta are effective for the power expansion problem. The contributions related to the intrinsic transverse momenta are unknown, in principle. They proposed to use the model

$$\Psi(x, \vec{b}) = \int_{|\vec{k}_T^f|<1/b} d^2\vec{k}_T \exp(-i\vec{k}_T \cdot \vec{b})\Psi(x, \vec{k}_T)$$

(114)

for all the wave functions (including twist-2 and twist-4 $B$ meson WFs, and twist-2 and twist-3 two parton pion WFs) used in calculations. $\vec{b}$ are the conjugate coordinates to the intrinsic transverse momenta $\vec{k}_T^f$. Because the $\Psi(x, \vec{b})$ provides a better end-point suppression than the constant model for the twist-3 pseudoscalar pion WF, the twist-3 contributions become smaller than the leading twist ones, $F_{+}^{tw3}(0)/F_{+}^{tw2}(0) = 1.5 \rightarrow 0.7$. However, it is still unclear how higher order power contributions, such as twist-3 three parton and twist-4 contributions, can be included in the $k_T^f$ approach. In addition, the introduction of intrinsic transverse momentum would arise a double counting problem when higher Fock state contributions are considered, such as three parton Fock state contributions, or four parton Fock state contributions. This is also unclear in $k_T^f$. We argue that the power expansion problem in $k_T^f$ needs more efforts to clarify. Before this problem has been solved, the large value $F_{+}^{k_T^f}(0) = 0.297$ in $k_T^f$ could be over-estimated.

B. Comparisons with SCET

In SCET, the form factor $F_{+}^{B\pi}$ contain a factorizable and a nonfactorizable parts at leading twist order, where the factorizable part is expressed in terms of a nonperturbative form factor, $\zeta_{+}^{B\pi}$ [50]. By a fitting to the experimental data for the branching ratios of $B \rightarrow \pi\pi$ decays, $F_{+}^{B\pi,SCET}(0) = 0.170$ was found [50]. It is noted that our predicted value $F_{+}^{B\pi}(0) = 0.164$ is close to the SCET result. This insures that the form factor $F_{+}^{B\pi}(0)$ may not be as large as found from other approaches. The values derived by LSCR and LQCD and $k_T^f$ are about $0.26 - 0.30$. The consistency between $F_{+}^{C_f}(0)$ and $F_{+}^{SCET}(0)$ may not be an
accident. In the energetic limit, \( E \gg m_\pi \), the \( q(0)\bar{q}(z) \) in the matrix element \( \langle 0 | q(0)\bar{q}(z) | \pi \rangle \) for defining the DA, becomes \( q(0)\bar{q}(\lambda n/E) + \cdots \), where \( q(\lambda n/E) \) is similar to the collinear quark field \( q_c(\lambda n/E) \) defined in the SCET. The collinear factorization based on full QCD with energetic limit for light meson is likely equivalent to the SCET \[49\]. However, it needs further works to show the equivalence between \( C^f \) and SCET \( I \) for \( F_{\pi}^{B\pi}(q^2) \).

The end-point divergences are also found in the SCET approach. The zero bin subtraction regularization method was proposed for dealing with soft and end-point divergences \[37\]. Since the related physics about the subtracted quantities have not been given in the SCET language, there still remained some uncertainties in this method.

### C. Comparisons with LCSR

The LCSR has been widely used for calculations of the form factors \[3, 4, 5, 6, 7, 8, 9, 44, 68, 69\]. The LCSR method calculates \( f_B F_{\pi}^{B\pi}(q^2) \) by matching the hadronic part to the partonic part of the correlator under parton-hadron duality. Let’s first compare the LCSR form factor \( F_{\pi}^{LCSR}(0) \) and the \( C^f \) form factor \( F_{\pi}^{C^f}(0) \) at twist-2 order. Numerically, \( |F_{\pi}^{LCSR,\text{tw}2}(0)| = 0.164 \sim 0.170 \) \[5, 9, 44\] and \( |F_{\pi}^{C^f,\text{tw}2}(0)| = 0.175 \) (\( \omega_B = 0.315 \) GeV) show that the LCSR and the \( C^f \) calculations are consistent at twist-2 order. At twist-3 level, we find that the ratio of twist-3 contributions to the twist-2 contributions is equal to \( |F_{\pi}^{LCSR,\text{tw}3}(0)|/|F_{\pi}^{LCSR,\text{tw}2}(0)| \sim 80\% - 100\% \) \[5, 9\]. Similarly, the ratio in the \( C^f \) approach gives \( |F_{\pi}^{C^f,\text{tw}3}(0)|/|F_{\pi}^{C^f,\text{tw}2}(0)| \sim 17\% \) (\( \omega_B = 0.315 \) GeV). We observe that twist-3 contributions in the LCSR form factor are more significant than those in the \( C^f \) form factor. This also explain why the difference between \( |F_{\pi}^{C^f}(0)| = |F_{\pi}^{C^f,\text{tw}2}(0) + F_{\pi}^{C^f,\text{tw}3}(0)| = 0.164 \) (\( \omega_B = 0.315 \) GeV) and \( |F_{\pi}^{LCSR}(0)| = |F_{\pi}^{LCSR,\text{tw}2}(0) + F_{\pi}^{LCSR,\text{tw}3}(0)| = 0.27 \) (at \( O(\alpha_s) \)) is so large. Note that there are partial cancellations in the sum of the twist-2 and twist-3 contributions of the form factors. It was also noticed that the soft contributions from end point region in the LCSR analysis are important \[3, 31\]. Because the twist-3 contributions have a more sensitive dependence on the end point behaviors of the pion DAs than the twist-2 ones, this may explain why the twist-3 contributions are so significant in the LCSR form factor. In LCSR calculation, the constant model for the PS DA of pion, \( \phi_p(u) = 1 \), is used. It is expected \[3, 31\] that if the end point contributions can be properly dealt with by appropriate method, such as Sudakov factors, the soft contributions can be reduced. We
argue that the twist-3 contributions in the LCSR form factor could be overestimated. A better power expansion method may solve this problem.

D. Comparisons with BN collinear expansion

An expansion scheme developed by Beneke and Neubert (BN) [67] is widely used in literature. The BN scheme is constructed for calculations of twist-3 two parton contributions from a final state pseudo-scalar meson. The BN scheme cannot avoid end-point divergences in the hard spectator and annihilation contributions. There exists an ambiguity that the momentum and coordinate representations for the amplitudes are used in the calculations. Besides, the BN scheme is also used in the $k_T$ factorization [31, 35]. However, the distinguishing differences between the collinear factorization and the $k_T$ factorization implies that these calculations require some cares. The reason is that the transverse parton momenta are assumed of order $O(\Lambda_{QCD})$ in the collinear factorization while they are not limited in the $k_T$ factorization. More detailed comparisons between the BN scheme and the collinear expansion refer to [41].

E. Comparisons with LQCD

The lattice QCD approach can only calculate the form factors at large $q^2$ due to limits on the inverse space length of the $\pi$ meson energy. On the other hand, PQCD approach is applicable for small $q^2$, where the virtual radiations are perturbative. Due to $\xi_R$, the $C^f$ approach developed in this paper for the form factors has extended the application range from low $q^2$ to moderate $q^2$. These two approaches are complementary and can be combined to derive form factors of full range $q^2$. The related works are left in our future publications. The comparisons between the extrapolation of $F_C^{C^f}(q^2)$ to large $q^2$ region with the LQCD calculations by FNAL [10, 11, 12, 52] and HPQCD [13] collaborations show that the $C^f$ prediction is close to the FNAL calculations (with a $\chi^2/N = 54.8/15$) but has a large deviation with the HPQCD calculations (with a $\chi^2/N = 271.1/7$). The $C^f$ prediction is close to the FNAL calculations than the HPQCD calculations.
VII. DISCUSSION AND CONCLUSIONS

In this paper, we have proposed a $\xi$-regularization for the logarithmic end-point divergences in $B \rightarrow \pi$ form factors. It can effectively resolve the end-point divergent problem. The linear end-point divergences are solved by an simultaneous use of the $\xi$-regularization and non-constant twist-3 distribution amplitudes. The $\xi$-regularization and the existence of a non-constant pseudo-scalar DA are followed by the collinear expansion. The factorizability of $B \rightarrow \pi$ form factors is shown up-to $O(\alpha_s/m_B)$ by including complete twist-3 contributions from the $\pi$ meson and partial $O(1/m_B^2)$ from the $B$ meson. The calculations are given explicitly. The form factors are calculated and applied to determine the CKM matrix element $|V_{ub}|$ according to the world average branching fraction. Eq. (101) is our main result. The result is also applied to make a prediction for the $q^2$ spectrum of $\bar{B} \rightarrow \pi l \bar{\nu}_l$. The predicted differential rate is inconsistent with the BaBar spectrum data. This discrepancy deserves further studies. Generalization of our result to other form factors are straightforward. We leave these interesting tasks to other places.

The twist-3 contributions calculated in the $k_f^T$ and LCSR approaches have been compared to our calculations. We argue that the twist-3 contributions in these two methods could be overestimated. If the twist-3 contributions in these two methods can be reduced by an appropriate power expansion method, it is possible to have a consistent result with our calculations. However, before the method has been derived under the $k_f^T$ or LCSR formalisms, the $k_f^T$ and LCSR form factors are larger than the $C_f$ form factor.

One important application of the present work is to improve the QCD factorization at leading order in $1/m_b$ expansion. The QCD factorization at leading order in $1/m_b$ expansion demonstrates that the matrix element $\langle P_1 P_2 | Q_i | B \rangle$ for $B \rightarrow PP$ decays can be expressed by the factorization formula

$$\langle P_1 P_2 | Q_i | B \rangle = F^{BP_1} \int d\xi \int du_1 \text{Tr}[T^I_i(\xi, u_1)\phi_B(\xi)\phi_{P_2}(u_1)] \tag{115}$$

$$+ \int d\xi \int du_1 \int du_2 \text{Tr}[T^{II}_i(\xi, u_1, u_2)\phi_B(\xi)\phi_{P_2}(u_1)\phi_{P_2}(u_2)],$$

where the $T^{I,II}_i$ are the type-I and type-II hard scattering kernels for non-factorizable radiative corrections. Due to the end-point divergences in the form factor $F^{BP_1}$ and in twist-3 contributions, the factorization formula is only valid at leading twist (twist-2). The form factor $F^{BP_1}$ is isolated from the factorization formula and identified to be a physical form.
factor, which can only be determined by experiments. According to the results derived in this paper and those obtained in [41], we propose a generalized QCD factorization formula, which is valid at twist-3 order. The generalized QCD factorization formula for $B \to PP$ decays with $PP$ two light pseudo-scalar mesons is given (under two parton approximation)

$$\langle P_1 P_2 |Q_i| B \rangle = f_{P_2} \int d\xi \int du_1 \text{Tr}[T^0_i(\xi, u_1) \phi_B(\xi) \phi_{P_1}(u_1)] + (P_1 \leftrightarrow P_2)$$

$$+ \int d\xi \int du_1 \int du_2 \text{Tr}[T^I_i(\xi, u_1, u_2) \phi_B(\xi) \phi_{P_1}(u_1) \phi_{P_2}(u_2)]$$

$$+ \int d\xi \int du_1 \int du_2 \text{Tr}[T^{II}_i(\xi, u_1, u_2) \phi_B(\xi) \phi_{P_1}(u_1) \phi_{P_2}(u_2)],$$

where $T^0_i$ is the hard scattering kernel for factorizable radiative corrections, $T^I_i$ is the hard scattering kernel for type-I non-factorizable radiative corrections, and $T^{II}_i$ is hard scattering kernel for type-II non-factorizable radiative corrections. The $\xi$-regularization is used to regularize the end-point divergences in form factors, the twist-3 annihilation contributions, and twist-3 hard spectator contributions in charmless $B \to PP$ decays [41]. The leading order is $O(\alpha_s)$. The improvement in this generalized QCD factorization is that the form factors are calculable and the precision order is generalized to include $O(1/m_B)$ corrections under two parton approximation. The collinear expansion is a necessary tool to derive correct twist-3 contributions in the above factorization formula. The application and the formal proof of the generalized QCD factorization formula will be given elsewhere.

The generalized QCD factorization formula is valid at twist-3 under two parton approximation. The factorization formula of the same twist order have also been shown to exist in the $k_T$ factorization and in the SCET, respectively. A consistent picture from three different approaches is obtained that the factorization theorem for $B \to PP$ decays is valid up-to twist-3 order under two parton approximation.

Appendix A: REDUCED HARD SCATTERING FUNCTIONS

The reduced hard scattering function $H^{(1)}_{\mu}(\xi, u)$ is derived according to the rules given in Section II. According to the Feynman diagrams depicted in Fig. 1 (a) and (b), the reduced hard scattering function $H^{(1)}_{\mu}(\xi, u)$ is $H^{(1)}_{\mu}(\xi, u) = H^{(1), (a)}_{\mu}(\xi, u) + H^{(1), (b)}_{\mu}(\xi, u)$ where
\[ H^{(1),a}_\mu(\xi, u) \]_{ij;kl} = \frac{C_F \pi \alpha_s}{N_c \eta^2 E^4 \xi \bar{u}} \left[ \gamma_\alpha \not\! \bar{\theta} \right]_{ij} [\gamma_\beta]_{kl} d_{\perp}^{\alpha \beta} \left( P_{B,\mu} - \frac{1}{\eta} p_{\pi,\mu} \right), \quad (A1) \\
\[ H^{(1),b}_\mu(\xi, u) \]_{ij;kl} = -\frac{C_F \pi \alpha_s}{N_c \eta^2 E^4 \xi \bar{u}(\xi - \eta u)} \left( 2 \bar{\xi} [\not\! \bar{\theta}]_{ij} [\not\! \bar{\theta}]_{kl} - \eta \bar{u} \left[ \gamma_\alpha \not\! \bar{\theta} \right]_{ij} [\gamma_\beta]_{kl} d_{\perp}^{\alpha \beta} \right) p_{\pi,\mu}. \quad (A2) \\

The reduced hard scattering function \( H^{(1)}_{\mu\nu}(\xi, u_q, u_q, u_g) \) are derived according to the Feynman diagrams depicted in Fig. 3(1) - (4). The contributions from the other diagrams depicted in Fig. 3(5) - (36) are of higher twist order than twist-3. They are neglected in this work. The expression for \( H^{(1)}_{\mu\nu}(\xi, u_q, u_q, u_g) \) is also written as \( H^{(1)}_{\mu\nu}(\xi, u_q, u_q, u_g) = H^{(1),a}_{\mu\nu}(\xi, u_q, u_q, u_g) + H^{(1),b}_{\mu\nu}(\xi, u_q, u_q, u_g) \) where

\[ H^{(1),a}_{\mu\nu}(\xi, u_q, u_q, u_g) \]_{ij;kl} = \frac{C_F \pi \alpha_s}{N_c \eta^2 E^4 \xi u_q u_q u_g \xi \bar{u}} \left[ \gamma_\nu \gamma_\alpha \not\! \bar{\theta} \right]_{ij} [\gamma_\beta]_{kl} d_{\perp}^{\alpha \beta} \left( P_{B,\mu} - \frac{1}{\eta} p_{\pi,\mu} \right), \quad (A3) \\
\[ H^{(1),b}_{\mu\nu}(\xi, u_q, u_q, u_g) \]_{ij;kl} = -\frac{C_F \pi \alpha_s}{N_c \eta^2 E^4 u_q u_q u_g \xi \bar{u}(\xi - \eta u_q)} \times \left( 2 \bar{\xi} [\gamma_\nu \gamma_\alpha \not\! \bar{\theta}]_{ij} [\not\! \bar{\theta}]_{kl} - \eta u_q \left[ \gamma_\nu \gamma_\alpha \not\! \bar{\theta} \right]_{ij} \left[ \gamma_\beta \right]_{kl} d_{\perp}^{\alpha \beta} \right) p_{\pi,\mu}. \quad (A4) \\

Appendix B: B MESON DISTRIBUTION AMPLITUDES

When subleading order effects in \( 1/m_B \) expansion are considered, the \( B \) meson DAs \( \phi_B^{tw}(\xi) \) and \( \bar{\phi}_B(\xi) \) are defined as \[ 46 \]

\[ \langle 0| \bar{q}_j(\frac{\lambda}{E} \bar{n}) b_i(0) |B \rangle = \frac{if_B}{4N_c} \int_0^1 d\xi e^{i\lambda \xi} \left[ \phi_B^{tw}(\xi) + E(\not\! \bar{\theta} - \not\! \theta) \bar{\phi}_B(\xi) \right] \gamma_5 \]_{ij} + \cdots ,

where \( \phi_B^{tw}(\xi) \) and \( \bar{\phi}_B(\xi) \) correspond to the \( \phi_B^+(\xi) \) and \( \phi_B^-(\xi) - \bar{\phi}_B(\xi) \) defined in the literature. The models for \( \phi_B^+(\xi) \) and \( \phi_B^-(\xi) \) are assumed to be \[ 46 \]

\[ \phi_B^+(\xi) = \frac{m_B^2}{\omega_B} \xi \exp\left(-\frac{m_B}{\omega_B} \xi \right), \quad \phi_B^-(\xi) = \frac{m_B}{\omega_B} \exp\left(-\frac{m_B}{\omega_B} \xi \right). \quad (B2) \]

Their first two inverse moments should satisfy the conditions

\[ \int_0^1 d\xi \phi_B^{(+,--)}(\xi) = 1, \quad (B4) \]
\[ \int_0^1 d\xi \frac{\phi_B^{(++,--)}(\xi)}{\xi} = \frac{m_B}{\lambda_B} \leq \frac{3m_B}{2} \Lambda, \quad (B5) \]

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where $\bar{\Lambda} = m_B - m_b$. In this work, we employ the following models for $\phi^{tw(1)}_B$ and $\bar{\phi}_B$:

\[
\phi^{tw(1)}_B(\xi) = \phi^+_B(\xi) = \frac{m_B^2}{\omega_B^2} \xi \exp\left(-\frac{m_B}{\omega_B} \xi\right),
\]
\[
\bar{\phi}_B(\xi) = \phi^+_B(\xi) - \phi^-_B(\xi) = \frac{m_B^2}{\omega_B^2} \left( \xi - \frac{\omega_B}{m_B} \right) \exp\left(-\frac{m_B}{\omega_B} \xi \right) \theta(\xi - \frac{\omega_B}{m_B}),
\]

where $\theta$ is a step function to insure that $\bar{\phi}_B(\xi)$ is positive.

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