ALGEBRAIC K-THEORY, K-REGULARITY, AND T-DUALITY OF
\(O_\infty\)-STABLE \(C^*\)-ALGEBRAS

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ABSTRACT. We develop an algebraic formalism for topological \(T\)-duality. Our main result shows that topological \(T\)-duality induces a specific isomorphism between noncommutative motives that implements the well-known isomorphism between twisted K-theories (up to a shift). In order to establish this result we model topological K-theory via algebraic K-theory. We construct an \(E_\infty\)-operad starting from any strongly self-absorbing \(C^*\)-algebra \(D\) and show that the algebraic K-theory of such a \(C^*\)-algebra is an algebra over this operad; moreover, the algebraic K-theory of any \(D\)-stable \(C^*\)-algebra is a module over it. Along the way we obtain a highly structured spectra valued functorial model for the topological K-theory of \(C^*\)-algebras. We also show that \(O_\infty\)-stable \(C^*\)-algebras are K-regular. We conclude with an explicit description of the algebraic K-theory of \(ax + b\)-semigroup \(C^*\)-algebras coming from number theory and that of \(O_\infty\)-stabilized noncommutative tori.

Introduction

An interesting physical duality that has received a lot of attention in the mathematical literature is \(T\)-duality. One aspect of this theory is the Bunke–Schick topological \(T\)-duality. It is insensitive to subtle geometric structures but its mathematical underpinnings are very well understood \(\cite{6,5,7}\). The main objective of this article is to develop an algebraic formalism for topological \(T\)-duality relating it to the theory of noncommutative motives \(\cite{30,53,37}\). Along the way we obtain several interesting applications to algebraic K-theory and K-regularity of \(C^*\)-algebras. The novelty of our approach lies in the use of the Cuntz algebra \(O_\infty\).

Let us briefly describe our main results. To any \(C^*\)-algebra \(A\) we functorially associate its noncommutative motive \(\text{HPf}_{dg}(A)\). As a mathematical object \(\text{HPf}_{dg}(A)\) is a differential graded category that is defined purely algebraically. Then we show that if \(A\) and \(A'\) are KK-equivalent separable \(C^*\)-algebras, then \(\text{HPf}_{dg}(A\hat{\otimes}O_\infty)\) and \(\text{HPf}_{dg}(A'\hat{\otimes}O_\infty)\) are isomorphic objects in the category of noncommutative motives (cf. Theorem \(\ref{thm:main}\)) and Corollary \(\ref{cor:main}\). We also show that the nonconnective K-theory of the noncommutative motive \(\text{HPf}_{dg}(A\otimes O_\infty)\) is isomorphic to the topological K-theory of \(A\) (cf. Theorem \(\ref{thm:comparison}\)). It is known that under favourable circumstances topological \(T\)-duality can be expressed as a KK-equivalence between two separable \(C^*\)-algebras \(\cite{2,3}\). Thus our results show that in such cases one actually has an isomorphism of noncommutative motives that implements the well-known isomorphism (up to a shift) between the twisted K-theories. Since noncommutative motives

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1

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constitute the universal cohomology theory of noncommutative spaces, our results demonstrate that topological $T$-duality implements an isomorphism of universal cohomology theories. The treatment here is completely algebraic; we model topological K-theory via algebraic K-theory refining our earlier approach in [35].

Let us now explain the significance of $O_\infty$ in this context that made no appearance in [35]. It is a prominent example of a strongly self-absorbing $C^\ast$-algebra [51] with very interesting structural properties. Using a result of Cortiñas–Phillips [11] (see also [27]) one can deduce that nonconnective algebraic K-theory agrees with topological K-theory for $O_\infty$-stable $C^\ast$-algebras. Exploiting this result we obtain a new functorial highly structured spectra valued model for the topological K-theory of $C^\ast$-algebras (cf. Theorem 2.4 and Remark 2.5). This result could have been deduced from the Karoubi conjecture without invoking $O_\infty$. However, we produce more algebraic structure on K-theory using our formalism. Indeed we construct for every strongly self-absorbing $C^\ast$-algebra $D$ an $E_\infty$-operad that we call the strongly self-absorbing $D$-operad (cf. Definition 3.2 and Proposition 3.4). Then we show that the connective algebraic K-theory spectrum of $D$ is an algebra over the $D$-operad and the connective algebraic K-theory spectrum of any unital $D$-stable $C^\ast$-algebra is a module over this algebra (cf. Theorem 3.8). These operadic structures up to coherent homotopy can be further rectified to strict algebraic structures in spectra (see, for instance, Theorem 1.4 of [22]). To the best of the author’s knowledge this is the first appearance of such highly structured commutative ring spectra as K-theory spectra of noncommutative $C^\ast$-algebras.

Using the same circle of ideas we show that $A\hat{\otimes}O_\infty$ is K-regular for any $C^\ast$-algebra $A$ (cf. Theorem 4.1), supporting a conjecture of Rosenberg [47]. We are also able to carry out an explicit computation of the algebraic K-theory of $ax+b$-semigroup $C^\ast$-algebras associated to number rings [15, 32] (cf. Theorem 5.1). Noncommutative tori constitute arguably the most widely studied class of noncommutative spaces. Geometric invariants of them were studied extensively by Connes and Rieffel (see, for instance, [8, 9, 44]). We show in the sequel that the algebraic K-theory of noncommutative tori are explicitly computable after $O_\infty$-stabilization (cf. Theorem 5.2). Using some powerful results of Rieffel [44], one also obtains a clear understanding of the elements in the algebraic K-theory groups (in low degrees).

**Remark.** Some of the arguments below exploit a cute trick (cf. Lemma 1.4). The range of applicability of this trick is much broader than the case explored here (see, for instance, Proposition 1.1.2 of [15]). The author is grateful to D. Enders for pointing out that Proposition 2.2, that uses this trick, can be generalized to all $C^\ast$-algebras of the form $A\hat{\otimes}B$ with $B$ properly infinite. We encourage the readers to consult [11] for an even more general result.

**Notations and Conventions:** In the sequel we denote the category of all $C^\ast$-algebras by $C^\ast$ and $\hat{\otimes}$ stands for the maximal $C^\ast$-tensor product. We denote by $K(-)$ [resp. $K_n(-)$] the nonconnective algebraic K-theory spectrum [resp. algebraic K-theory group] functor on $C^\ast$. Finally, we are going to denote by $hSp$ the triangulated stable homotopy category. Unless otherwise stated, all spaces are assumed to be Hausdorff.

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1. Topological $\mathbb{T}$-duality and noncommutative motives

Let us recall very briefly axiomatic topological $\mathbb{T}$-duality from [6]. Let $B$ be a topological base space. Consider the category of pairs $(E, h)$, where $\pi : E \to B$ is a principal $S^1$-bundle over $B$ and $h \in H^3(E, \mathbb{Z})$. Two such pairs $(E_1, h_1)$ and $(E_2, h_2)$ are isomorphic if there is an isomorphism $F : E_1 \to E_2$ of principal bundles such that $F^\ast h_2 = h_1$. Two pairs $(E_1, h_1)$ and $(E_2, h_2)$ are said to be $\mathbb{T}$-dual if there is a Thom class $\mathbf{Th}$ for $S(V)$ such that $h_1 = i_1^\ast \mathbf{Th}$ and $h_2 = i_2^\ast \mathbf{Th}$. Here $S(V)$ is the sphere bundle of $V := E_1 \times_{S^1} \mathbb{C} \oplus E_2 \times_{S^1} \mathbb{C}$ and $i_k : E_k \to S(V)$ are the canonical maps for $k = 1, 2$. This definition implies the following correspondence picture: Let $\pi_k : E_k \to B$ with $k = 1, 2$ be two principal $S^1$-bundles and $(E_1, h_1)$ and $(E_2, h_2)$ be $\mathbb{T}$-dual pairs. Then there is a commutative diagram

\[
\begin{array}{ccc}
E_1 \times_B E_2 & \xrightarrow{pr_1} & E_1 \\
\downarrow{\pi_1} & & \downarrow{\pi} & \downarrow{\pi_2} \\
B, & \xrightarrow{q} & E_2 \\
\end{array}
\]

such that $pr_1^\ast (h_1) = pr_2^\ast (h_2)$. This basic correspondence picture relates topological $\mathbb{T}$-duality to cohomological quantization (see, for instance, [41, 49]).

In [6] Bunke–Schick showed that the association $B \mapsto \{\text{isom. classes of pairs over } B\}$ as a functor on topological spaces is representable. The representing space $E$ supports a universal pair and any pair on $B$ can be obtained up to isomorphism via a pullback along some map $B \to E$ (defined uniquely up to homotopy). Using the explicit construction of the universal object and the $\mathbb{T}$-dual of the universal pair the authors were able to prove the existence and uniqueness of $\mathbb{T}$-duality for $S^1$-bundles. One of the salient features of $\mathbb{T}$-duality is the following: If $(E_1, h_1)$ and $(E_2, h_2)$ are $\mathbb{T}$-dual pairs, then there is an isomorphism of twisted $K$-theories:

\[
K^{\text{od}}(E_1, h_1) \simeq K^{\text{ev}}(E_2, h_2) \quad \text{and} \quad K^{\text{ev}}(E_1, h_1) \simeq K^{\text{od}}(E_2, h_2).
\]

The theory of topological $\mathbb{T}$-duality is not limited to $S^1$-bundles. However, for more general $(\prod_{i=1}^n S^1)$-bundles with $n > 1$ the theory becomes quite subtle [5, 7] and sometimes necessitates the use of $C^\ast$-algebras [38]. Moreover, $C^\ast$-algebras appear quite naturally in the context of twisted $K$-theory [16]. Thus it seems natural to study $\mathbb{T}$-duality via $C^\ast$-algebras from the outset. The readers may refer to [48] for a survey on the interactions between $C^\ast$-algebras, $K$-theory, noncommutative geometry, and $\mathbb{T}$-duality. Some recent results indicate that $\mathbb{T}$-duality can even be related to Langlands duality [21, 4].

We denote the category of separable $C^\ast$-algebras by $\mathcal{SC}^\ast$ and the bivariant $K$-theory category by $\mathbf{KK}$. There is a canonical functor $\iota : \mathcal{SC}^\ast \to \mathbf{KK}$, which is identity on objects and admits a universal characterization [23, 14]. Recall from [28] that there is a category of noncommutative motives $\mathbf{Hm}_0$, whose objects are $k$-linear DG categories ($k = \mathbb{C}$ for our purposes). The theory of noncommutative motives is an active area of research with interesting applications to $K$-theory as well as a wide variety of other mathematics [30, 53, 37]. Building upon an earlier work of Quillen [42] the author constructed a functorial passage
Theorem 1.1. There is a dashed functor below making the following diagram of categories commute (up to a natural isomorphism):

\[
\begin{array}{ccc}
\text{SC}^* & \xrightarrow{A \mapsto A \hat{} \otimes K} & \text{SC}^* \\
\downarrow \iota & & \downarrow \text{HPf}_{\text{dg}} \\
\text{KK} & \xrightarrow{} & \text{Hmo}_0.
\end{array}
\]

Theorem 1.2. For any \( A \in \text{SC}^* \) the homotopy groups of the nonconnective K-theory spectrum of \( \text{HPf}_{\text{dg}}(A \hat{} \otimes K) \) are naturally isomorphic to the topological K-theory groups of \( A \).

Remark 1.3. In [35] the author phrased the results in terms of \( \text{NCC}_{\text{dg}} \), which was called the category of noncommutative DG correspondences. The category \( \text{NCC}_{\text{dg}} \) is equivalent to \( \text{Hmo}_0 \).

Moreover, in Theorem 3.7 of [35] actually the connective version of Theorem 1.2 was proven. However, the extension to the nonconnective version is straightforward.

A crucial insight of Rosenberg in [46] is that certain bundles of compact operators \( \mathbb{K} \) on locally compact spaces can be used to model twisted K-theory. More precisely, given any pair \((E, h)\) with \( E \) locally compact one can construct a noncommutative stable \( C^* \)-algebra \( \text{CT}(E, h) \), whose topological K-theory is the twisted K-theory of the pair \((E, h)\). This formalism extends to certain infinite dimensional spaces through the use of \( \sigma \)-\( C^* \)-algebras [34]. In [2, 3] the authors extended the formalism of \( T \)-duality to \( C^* \)-algebras and showed that under favourable circumstances if \( B \) and \( B' \) are \( T \)-dual \( C^* \)-algebras, then there is an invertible element in \( KK_0(B, \Sigma B') \) that implements the twisted K-theory isomorphism (as in [2]). The Connes–Skandalis picture of KK-theory [10] is pertinent to their construction. Thanks to Theorem 1.1 we conclude that if two stable \( C^* \)-algebras \( B \) and \( B' \) are \( T \)-dual, such that there is an invertible element \( \alpha \in KK_0(B, \Sigma B') \), then their noncommutative motives \( \text{HPf}_{\text{dg}}(B) \) and \( \text{HPf}_{\text{dg}}(B') \) are isomorphic in \( \text{Hmo}_0 \). Furthermore, Theorem 1.2 asserts that the invertible element \( \alpha \) implements the twisted K-theory isomorphism.

Recall that the Cuntz algebra \( \mathcal{O}_\infty \) is the universal unital \( C^* \)-algebra generated by a set of isometries \( \{s_i \mid i \in \mathbb{N}\} \) with mutually orthogonal range projections \( s_is_i^* \) [13]. Observe that \( \mathcal{O}_\infty \) is a unital \( C^* \)-algebra, so that \( \mathcal{O}_\infty \)-stabilization preserves unitality (unlike \( \mathbb{K} \)-stabilization). The following Lemma is crucial and it exploits the fact that \( \mathcal{O}_\infty \) is purely infinite.

Lemma 1.4. There is a commutative diagram in \( C^* \)

\[
\begin{array}{ccc}
\mathcal{O}_\infty & \xrightarrow{\iota} & \mathcal{O}_\infty \\
\downarrow \theta & \xrightarrow{\kappa} & \mathcal{O}_\infty \hat{} \otimes \mathbb{K} \\
\mathcal{O}_\infty \hat{} \otimes \mathbb{K}, & & \\
\end{array}
\]

where the top horizontal arrow \( \iota : \mathcal{O}_\infty \to \mathcal{O}_\infty \) is an inner endomorphism.

Proof. Observe that the subset \( \{s_is_j^* \mid i, j \in \mathbb{N}\} \subset \mathcal{O}_\infty \) generates a copy of the compact operators \( \mathbb{K} \) inside \( \mathcal{O}_\infty \). Consider the \( * \)-homomorphism \( \kappa : \mathcal{O}_\infty \hat{} \otimes \mathbb{K} \to \mathcal{O}_\infty \), which is defined as \( a \otimes e_{ij} \mapsto s_is_j^* \). Due to the simplicity of all the \( C^* \)-algebras in sight, \( \kappa \) is injective. Let
\[ \theta : O_\infty \to O_\infty \hat{\otimes} K \] be simply the corner embedding, sending \( a \mapsto a \otimes e_{11} \). The composite \( \iota = \kappa \theta \) is given by \( \iota(a) = s_1 as_1^* \). This \(*\)-homomorphism is manifestly inner. \( \square \)

Recall that a functor \( F : \mathcal{S}^* \to \text{Hmo}_0 \) is called \textit{split exact} if it sends a split exact sequence in \( \mathcal{S}^* \) to a direct sum diagram in the additive category \( \text{Hmo}_0 \). It follows from Lemma 3.1 of [35] that the functor \( \text{HPf}_{\hat{O}}(-) \) is split exact.

**Theorem 1.5.** If \( A \) and \( A' \) are isomorphic in \( \text{KK} \), then the noncommutative motives of \( A \hat{\otimes} O_\infty \) and \( A' \hat{\otimes} O_\infty \) are isomorphic in \( \text{Hmo}_0 \).

**Proof.** Let us first assume that \( A, A' \) are unital and let \( \alpha \in \text{KK}_0(A, A') \) be any invertible element. Consider the commutative diagram that is obtained by applying \( A \hat{\otimes} - \) to the commutative diagram [3]

\[
\begin{array}{ccc}
A \hat{\otimes} O_\infty & \overset{\text{id}_{A \hat{\otimes} \kappa}}{\longrightarrow} & A \hat{\otimes} O_\infty \\
\downarrow{R := \text{id}_{A \hat{\otimes} \theta}} & & \downarrow{S := \text{id}_{A \hat{\otimes} \kappa}} \\
A \hat{\otimes} O_\infty \hat{\otimes} K.
\end{array}
\]

Now from Theorem [1.1] one obtains a diagram in \( \text{Hmo}_0 \)

\[
\begin{array}{ccc}
\text{HPf}_{\hat{O}}(A \hat{\otimes} O_\infty) & \overset{\text{HPf}_{\hat{O}}(R)}{\longrightarrow} & \text{HPf}_{\hat{O}}(A \hat{\otimes} O_\infty \hat{\otimes} K) \\
\downarrow{\beta := \text{HPf}_{\hat{O}}(\alpha \hat{\otimes} \text{id}_{O_\infty} \hat{\otimes} \text{id}_K)} & & \downarrow{\text{HPf}_{\hat{O}}(S')} \\
\text{HPf}_{\hat{O}}(A' \hat{\otimes} O_\infty) & \overset{\text{HPf}_{\hat{O}}(R')}{\longrightarrow} & \text{HPf}_{\hat{O}}(A' \hat{\otimes} O_\infty \hat{\otimes} K).
\end{array}
\]

where \( R' \) and \( S' \) are defined in the obvious manner (replace \( A \) by \( A' \) in diagram [1]. Since \( \alpha \) is invertible, so are \( \alpha \hat{\otimes} \text{id}_{O_\infty} \) and \( \alpha \hat{\otimes} \text{id}_{O_\infty} \hat{\otimes} \text{id}_K \). Therefore, the middle vertical arrow \( \beta \) is an isomorphism. Observe that \( S \circ R \) is an inner endomorphism in \( \mathcal{S}^* \) of the form \( x \mapsto (1_A \otimes s_1)x(1_A \otimes s_1)^* \) (and so is \( S' \circ R' \) similarly). It is known that if \( F \) is a matrix stable functor on \( \mathcal{C}^* \) (resp. \( \mathcal{S}^* \)) and \( f \) is an inner endomorphism in \( \mathcal{C}^* \) (resp. \( \mathcal{S}^* \)), then \( F(f) \) is the identity map (see, for instance, Proposition 3.16. of [17]). It was shown in Lemma 2.3 of [33] that the functor \( \text{HPf}_{\hat{O}}(-) \) is matrix stable on \( \mathcal{S}^* \), whence we get

\[ \text{HPf}_{\hat{O}}(S) \circ \text{HPf}_{\hat{O}}(R) = \text{id}_{\text{HPf}_{\hat{O}}(A \hat{\otimes} O_\infty)} \quad \text{and} \quad \text{HPf}_{\hat{O}}(S') \circ \text{HPf}_{\hat{O}}(R') = \text{id}_{\text{HPf}_{\hat{O}}(A' \hat{\otimes} O_\infty)}. \]

Thus the maps \( \text{HPf}_{\hat{O}}(R) \) and \( \text{HPf}_{\hat{O}}(R') \) possess left inverses. An inspection of diagram [3] reveals that they also possess right inverses. The composite \(*\)-homomorphism \( K \overset{i}{\to} O_\infty \overset{\theta}{\to} O_\infty \hat{\otimes} K \) defines an invertible element \( \theta \circ i = \gamma \in \text{KK}_0(K, O_\infty \hat{\otimes} K). \) Consequently, \( \text{id}_A \hat{\otimes} \gamma \in \text{KK}_0(A \hat{\otimes} K, A \hat{\otimes} O_\infty \hat{\otimes} K) \) is an invertible element. By Theorem [1.1] \( \text{id}_A \hat{\otimes} \gamma \hat{\otimes} \text{id}_K = (\text{id}_A \hat{\otimes} \theta \hat{\otimes} \text{id}_K) \circ (\text{id}_A \hat{\otimes} i \hat{\otimes} \text{id}_K) \) induces an isomorphism

\[ \text{HPf}_{\hat{O}}(A \hat{\otimes} K \hat{\otimes} K) \xrightarrow{\sim} \text{HPf}_{\hat{O}}(A \hat{\otimes} O_\infty \hat{\otimes} K \hat{\otimes} K). \]
Let us set $I = \text{id}_{A \hat{\otimes} i}$, so that $\text{HPf}_{dg}(R \hat{\otimes} \text{id}_K) \circ \text{HPf}_{dg}(I \hat{\otimes} \text{id}_K)$ is the above isomorphism. Now consider the following commutative diagram

$$
\begin{array}{ccc}
A \hat{\otimes} K & \longrightarrow & A \hat{\otimes} K \hat{\otimes} K \\
\downarrow I & & \downarrow I \hat{\otimes} \text{id}_K \\
A \hat{\otimes} \mathcal{O}_\infty & \longrightarrow & A \hat{\otimes} \mathcal{O}_\infty \hat{\otimes} K \\
\downarrow R & & \downarrow R \hat{\otimes} \text{id}_K \\
A \hat{\otimes} \mathcal{O}_\infty \hat{\otimes} K & \longrightarrow & A \hat{\otimes} \mathcal{O}_\infty \hat{\otimes} K \hat{\otimes} K.
\end{array}
$$

Here all the horizontal arrows are corner embeddings. Now the top and the bottom horizontal arrows are homotopic to isomorphisms. Since $\text{HPf}_{dg}(\_)$ is homotopy invariant on stable $C^*$-algebras, it sends the top and the bottom horizontal arrows to isomorphisms. We already know that it sends $(R \hat{\otimes} \text{id}_K) \circ (I \hat{\otimes} \text{id}_K)$ to an isomorphism. It follows that $\text{HPf}_{dg}(R)$ has a right inverse. Similarly, one can prove that $\text{HPf}_{dg}(R')$ has a right inverse. Now using split exactness of $\text{HPf}_{dg}(\_)$ one can extend the result to nonunital $C^*$-algebras.

**Corollary 1.6.** The functor $\text{HPf}_{dg}(\_ \hat{\otimes} \mathcal{O}_\infty)$ is $C^*$-stable and it factors through $\mathcal{K}K$.

**Proof.** For any separable $C^*$-algebra $A$ the corner embedding $A \rightarrow A \hat{\otimes} K$ is $\mathcal{K}K$-invertible whence $\text{HPf}_{dg}(\_ \hat{\otimes} \mathcal{O}_\infty)$ is $C^*$-stable. It follows from Lemma 3.1 of [35] that the functor $\text{HPf}_{dg}(\_ \hat{\otimes} \mathcal{O}_\infty)$ is split exact. The second assertion now is a consequence of the universal characterization of $\mathcal{K}K$. \qed

Now we prove the $\mathcal{O}_\infty$-analogue of Theorem 1.2.

**Theorem 1.7.** For any $A \in \mathcal{S} C^*$ the homotopy groups of the nonconnective K-theory spectrum of $\text{HPf}_{dg}(A \hat{\otimes} \mathcal{O}_\infty)$ are naturally isomorphic to the topological K-theory groups of $A$.

**Proof.** By the above Corollary the nonconnective K-theory spectra of $\text{HPf}_{dg}(A \hat{\otimes} \mathcal{O}_\infty)$ and $\text{HPf}_{dg}(A \hat{\otimes} \mathcal{O}_\infty \hat{\otimes} K)$ are weakly equivalent. By Theorem 1.1 the homotopy groups of the nonconnective K-theory spectrum of $\text{HPf}_{dg}(A \hat{\otimes} \mathcal{O}_\infty \hat{\otimes} K)$ are isomorphic to the topological K-theory groups of $A \hat{\otimes} \mathcal{O}_\infty$, which are in turn isomorphic to those of $A$. \qed

**Remark 1.8.** Since noncommutative motives constitute the universal additive invariant [52], an isomorphism of therein is the most fundamental (co)homological isomorphism. It certainly explains the isomorphism between twisted K-theories (up to a shift) under $\mathbb{T}$-duality.

**Remark 1.9.** The above results provide a connection between the Dixmier–Douady theory via $\mathcal{O}_\infty \hat{\otimes} K$-bundles due to Dadarlat–Pennig [18] and noncommutative motives.

### 2. The Generalized Homology Theory $K(\_ \hat{\otimes} \mathcal{O}_\infty)$

A functor $F : \mathcal{C}^* \longrightarrow \mathcal{hSp}$ is called *homotopy invariant* if it sends the evaluation at $t$ map $\text{ev}_t : A[0,1] \rightarrow A$ to an isomorphism in $\mathcal{hSp}$ for all $A \in \mathcal{C}^*$. Such a functor is called *excisive* if for any short exact sequence $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ in $\mathcal{C}^*$ the induced diagram $F(A) \rightarrow F(B) \rightarrow F(C) \rightarrow \Sigma F(A)$ is an exact triangle in $\mathcal{hSp}$. A homotopy invariant excisive functor $F : \mathcal{C}^* \longrightarrow \mathcal{hSp}$ is called an $\mathcal{hSp}$-valued *generalized homology theory* on $\mathcal{C}^*$.

It is known that the algebraic K-theory functor acquires special properties after compact stabilization. We are going to show that the same is true after $\mathcal{O}_\infty$-stabilization.
Proposition 2.1. The functor \( K(\hat\otimes \mathcal{O}_\infty) : \mathcal{C}^* \to hSp \) is an excisive functor.

Proof. It follows from the Suslin–Wodzicki Theorem [50, 51] that the functor \( K \) is excisive on \( \mathcal{C}^* \). Since maximal \( C^* \)-tensor product is exact, the functor \(-\hat\otimes \mathcal{O}_\infty \) preserves exactness in \( \mathcal{C}^* \) whence \( K(-\hat\otimes \mathcal{O}_\infty) \) is excisive. □

Thanks to the Karoubi conjecture, which is now a Theorem [50, 51], we know that the nonconnective algebraic K-theory of a stable \( C^* \)-algebra is isomorphic to its topological K-theory. In fact, there is a canonical comparison map of spectra that induces the isomorphisms \( c_n(A) : K_n(A) \to K_n^{\text{top}}(A) \) for all \( n \in \mathbb{Z} \) when \( A \) is stable [26] (see also [17]). The comparison map \( c_0(A) : K_0(A) \to K_0^{\text{top}}(A) \) is always an isomorphism.

Proposition 2.2 (Cortiñas–Phillips). For any \( C^* \)-algebra \( A \) the comparison map
\[
\hat c_n(A \hat\otimes \mathcal{O}_\infty) : K_n(A \hat\otimes \mathcal{O}_\infty) \to K_n^{\text{top}}(A \hat\otimes \mathcal{O}_\infty)
\]
is an isomorphism for all \( n \in \mathbb{Z} \).

Proof. Let us first assume that \( A \) is a unital \( C^* \)-algebra. After applying \( A \hat\otimes - \) to the commutative diagram in Lemma [12] we obtain
\[
\begin{array}{ccc}
A \hat\otimes \mathcal{O}_\infty & \xrightarrow{id \hat\otimes \iota} & A \hat\otimes \mathcal{O}_\infty \\
R := id \hat\otimes \theta & & S := id \hat\otimes \kappa \\
A \hat\otimes \mathcal{O}_\infty \hat\otimes \mathcal{K} & & \\
\end{array}
\]

where the top horizontal arrow is an inner endomorphism. Now applying the functors \( K_n(-) \), \( K_n^{\text{top}}(-) \) and using the naturality of \( c_n \), we get a commutative diagram
\[
\begin{align*}
K_n(A \hat\otimes \mathcal{O}_\infty) & \xrightarrow{K_n(R)} K_n(A \hat\otimes \mathcal{O}_\infty \hat\otimes \mathcal{K}) \xrightarrow{K_n(S)} K_n(A \hat\otimes \mathcal{O}_\infty) \\
K_n^{\text{top}}(A \hat\otimes \mathcal{O}_\infty) & \xrightarrow{K_n^{\text{top}}(R)} K_n^{\text{top}}(A \hat\otimes \mathcal{O}_\infty \hat\otimes \mathcal{K}) \xrightarrow{K_n^{\text{top}}(S)} K_n^{\text{top}}(A \hat\otimes \mathcal{O}_\infty).
\end{align*}
\]

Since \( S \circ R \) is the inner endomorphism \( id \hat\otimes : A \hat\otimes \mathcal{O}_\infty \to A \hat\otimes \mathcal{O}_\infty \), we conclude that \( K_n(S) \circ K_n(R) \) is the identity map due to the matrix stability of algebraic K-theory on the category of unital \( C^* \)-algebras. Moreover, \( K_n^{\text{top}}(S) \circ K_n^{\text{top}}(R) \) is also the identity map due to the matrix stability of \( K_n^{\text{top}}(-) \). The assertion for unital \( A \) now follows by a simple diagram chase. Indeed, it is easily seen that \( K_n(R) \) must be injective and \( K_n^{\text{top}}(S) \) must be surjective. Since \( A \hat\otimes \mathcal{O}_\infty \hat\otimes \mathcal{K} \) is stable, we conclude that \( c_n(A \hat\otimes \mathcal{O}_\infty \hat\otimes \mathcal{K}) \) is an isomorphism. Thus \( c_n(A \hat\otimes \mathcal{O}_\infty \hat\otimes \mathcal{K}) \circ K_n(R) \) is injective whence so is \( c_n(A \hat\otimes \mathcal{O}_\infty) \) (the left vertical one). Similarly, \( K_n^{\text{top}}(S) \circ c_n(A \hat\otimes \mathcal{O}_\infty \hat\otimes \mathcal{K}) \) is surjective whence so is \( c_n(A \hat\otimes \mathcal{O}_\infty) \) (the right vertical one).

The proof for nonunital \( A \) follows by a simple excision argument (see Proposition [21]). □

Remark 2.3. The above Proposition is a special case of a result of Cortiñas–Phillips. They proved that the comparison map \( \hat c_n(A) : K_n(A) \to K_n^{\text{top}}(A) \) is an isomorphism in a very general situation [11]. Notice that the \( C^* \)-algebra \( A \hat\otimes \mathcal{O}_\infty \) is purely infinite for any separable \( C^* \)-algebra \( A \) [29]. We have decided to include our simple proof as it involves only elementary (homological) algebra and hence comprehensible to non-experts on \( C^* \)-algebras. The above argument is based on the ideas of Karoubi–Wodzicki [27] with some simplifications that exploit the special properties of \( \mathcal{O}_\infty \).

\[\text{Page 7}\]
Theorem 2.4. The functor $K(- \otimes O_\infty) : C^* \to hSp$ is a model for topological K-theory.

Proof. It follows from Proposition 2.2 that the natural comparison map of spectra is a weak equivalence. Since the canonical $*$-homomorphism $A \to A \otimes O_\infty$ sending $a \mapsto a \otimes 1_{O_\infty}$ is a KK-equivalence, we have a diagram of weak equivalences of spectra (see Section 8.3 of [17])

$$K(A \otimes O_\infty) \xrightarrow{\sim} K^{\text{top}}(A \otimes O_\infty) \xleftarrow{\sim} K^{\text{top}}(A),$$

where $K^{\text{top}}$ denotes the topological K-theory spectrum. Thus for every $A \in C^*$ there is an isomorphism $K(A \otimes O_\infty) \cong K^{\text{top}}(A)$ in $hSp$. □

Remark 2.5. Applying the nonconnective Waldhausen K-theory functor $[55, 1]$ to $A \otimes O_\infty$ for any unital $C^*$-algebra $A$ one actually obtains a highly structured spectrum model for the topological K-theory of $A$ [25]. Using the Green–Julg–Rosenberg Theorem one can also get such a spectrum model for $G$-equivariant topological K-theory for a finite group $G$. Indeed, one can simply apply the nonconnective Waldhausen K-theory functor to $(A \times G) \otimes O_\infty$ for any unital $G$-$C^*$-algebra $A$. This construction would be different from the one in [21].

3. STRONGLY SELF-ABSORBING OPERADS

Operadic structures have pervaded many areas of mathematics and physics with wide ranging applications. From the viewpoint of topology the operadic machinery can be effectively used to recognise (infinite) loop spaces. An operad in a symmetric monoidal category $(C, \otimes, 1_C)$ consists of a collection of objects $\{C(j)\}_{j \geq 0}$ with each $C(j)$ carrying a right action of the permutation group $\Sigma_j$, a unit map $\eta : 1_C \to C(1)$, and product or composition maps

$$\gamma = \gamma_{j_1, \ldots, j_k} : C(k) \otimes C(j_1) \otimes \cdots \otimes C(j_k) \to C(j)$$

for $k \geq 1$ and $j_s \geq 0$ for all $s = 1, \ldots, k$ subject to $j = \sum_s j_s$. These data should be inter-compatible in a specific manner, i.e., satisfy certain associativity, unitality, and equivariance axioms (see, for instance, [31]). Neglecting the actions of the permutation groups and the corresponding equivariance conditions one arrives at the notion of a nonsymmetric operad. A space is tacitly assumed to be compactly generated and weakly Hausdorff. Observe that such spaces constitute a symmetric monoidal category under cartesian product with pt as a unit object and hence one may consider operads in spaces.

In this section we shall work with a specific model for spectra, namely, symmetric spectra $Sp^E$ [25]. Several choices for the algebraic K-theory functor land inside the category of symmetric spectra in simplicial sets. By applying the geometric realisation functor one can pass to the category of symmetric spectra in spaces. One may choose one’s favourite strictly functorial model for algebraic K-theory that takes values in $Sp^E$ (and not in the stable homotopy category $hSp$). The category of symmetric spectra $Sp^E$ has an associative and commutative smash product $\wedge$ and it is tensored over based spaces (see [25]).

Recall that a unital separable $C^*$-algebras $D$ ($D \neq \mathbb{C}$) is called strongly self-absorbing if there is an isomorphism $D \to D \otimes D$ that is approximately unitarily equivalent to the first factor embedding $D \to D \hat{\otimes} D$ sending $d \mapsto d \otimes 1_D$ [54]. Such $C^*$-algebras turn out to be simple and nuclear. The Cuntz algebra $O_\infty$ is a prominent example of such a $C^*$-algebra. For any strongly self-absorbing $C^*$-algebra $D$ we set $D(j) = \text{Hom}_1(D \hat{\otimes} j, D)$, i.e., the space of unital full $*$-homomorphisms $D \hat{\otimes} j \to D$ with the point-norm topology. Since $D$ is a separable $C^*$-algebra, it follows from Lemma 22 of [10] that each $D(j)$ is a metrizable topological space. Hence they are all compactly generated and Hausdorff spaces.
**Lemma 3.1.** The collection \( \{D(j)\}_{j \geq 0} \) can be promoted to an operad in spaces.

**Proof.** Let us define \( \gamma \) and \( \eta \) as follows:

\[
\gamma : D(k) \times D(j_1) \times \cdots \times D(j_k) \to D(j) = D(j_1 + \cdots + j_k) \\
(\alpha, \beta_1, \ldots, \beta_k) \mapsto \alpha \circ (\beta_1, \ldots, \beta_k)
\]

and \( \eta : pt \to D(1) \) sends the unique element in \( pt \) to \( id : D \to D \). If we let the permutation group \( \Sigma_j \) to act on \( D(j) = \text{Hom}_1(D^\otimes j, D) \) by permuting the tensor factors of \( D^\otimes j \), then it can be verified that the data satisfy the associativity, unitality, and equivariance axioms. \( \square \)

Thanks to the above Lemma we introduce the following operad:

**Definition 3.2.** For any strongly self-absorbing \( C^* \)-algebra \( D \) we call the operad that the collection \( \{D(j)\}_{j \geq 0} \) defines as the **strongly self-absorbing \( D \)-operad**.

**Remark 3.3.** For \( j = 0 \) we get \( D(0) = pt \), i.e., a singleton set containing the unique unital inclusion \( C \hookrightarrow D \). Hence \( D \) is a reduced operad.

**Proposition 3.4.** Every strongly self-absorbing \( D \)-operad is an \( E_\infty \)-operad.

**Proof.** We need to show that each \( D(j) \) for \( j \geq 0 \) is contractible and the action of \( \Sigma_j \) on each \( D(j) \) is free. The contractibility of each \( D(j) \) follows from Theorem 2.3 of [18] and the fact that \( D \cong D^\otimes j \). In order to see the freeness of the \( \Sigma_j \)-action on \( D(j) \) we check the stabilizers. For any \( f \in D(j) \) suppose \( f_\sigma = f \). Owing to the simplicity of \( D^\otimes n \) such an \( f \) must be a monomorphism whence \( \sigma \) is the trivial permutation. \( \square \)

**Remark 3.5.** The underlying nonsymmetric operad of every strongly self-absorbing \( D \)-operad is an \( A_\infty \)-operad.

Let \( K^c \) denote a strictly functorial model for K-theory that takes values in \( Sp^F \), whose homotopy groups are the connective algebraic K-theory groups (see, for instance, [39] [22]). This specific choice \( K^c \) takes a small permutative category as input. For a unital ring \( A \) let \( P(A) \) denote the category, whose objects are pairs \((A^n, i)\). Here \( n \in \mathbb{N} \) and \( i : A^n \to A^n \) is an idempotent left \( A \)-module endomorphism. A morphism from \((A^n, i)\) to \((A^m, j)\) is a left \( A \)-module isomorphism from \( \text{Im}(i) \) to \( \text{Im}(j) \). The category \( P(A) \) becomes a permutative category under direct sum of modules and idempotents. The K-theory of this permutative category is isomorphic to Quillen’s algebraic K-theory of the ring \( A \) (see Examples in page 171 of [22]).

**Proposition 3.6.** There is a natural map \( \kappa = \kappa_{A,B} : K^c(A) \wedge K^c(B) \to K^c(A \hat{\otimes} B) \) for any unital \( A, B \in C^* \). In particular, there is a natural map \( \kappa : K^c(D)^\otimes j \to K^c(D^\otimes j) \) for all \( j \geq 0 \).

**Proof.** We may apply the external tensor product in algebraic K-theory to get the first map.

\[
K^c(A) \wedge K^c(B) \to K^c(A \otimes Z B) \to K^c(A \hat{\otimes} B).
\]

Now one may use induction on \( j \) to prove the second assertion noticing that the cases \( j = 0, 1 \) are trivial. Naturality follows from that of the external pairing. \( \square \)

**Lemma 3.7.** The above maps \( \kappa : K^c(A) \wedge K^c(B) \to K^c(A \hat{\otimes} B) \) can be arranged to be associative and symmetric, i.e., for any unital \( A, B, C \in C^* \).
(1) the following diagram can be made to commute:

\[
\begin{array}{ccc}
(K^c(A) \land K^c(B)) \land K^c(C) & \xrightarrow{\kappa \land \text{id}} & K^c(A) \land (K^c(B) \land K^c(C)) \\
\downarrow \kappa \land \text{id} & & \downarrow \text{id} \land \kappa \\
K^c(A \hat{\otimes} B) \land K^c(C) & \xrightarrow{\kappa} & K^c(A) \land K^c(B \hat{\otimes} C) \\
\end{array}
\]

where the horizontal maps are furnished by the associativity of $\land$ and $\hat{\otimes}$, and

(2) the following diagram can be made to commute:

\[
\begin{array}{ccc}
K^c(A) \land K^c(B) & \xrightarrow{\kappa} & K^c(A \hat{\otimes} B) \\
\downarrow & & \downarrow \\
K^c(B) \land K^c(A) & \xrightarrow{\kappa} & K^c(B \hat{\otimes} A),
\end{array}
\]

where the vertical maps $K^c(A) \land K^c(B) \to K^c(B) \land K^c(A)$ and $K^c(A \hat{\otimes} B) \to K^c(B \hat{\otimes} A)$ are furnished by the symmetries of $\land$ and $\hat{\otimes}$ respectively.

Proof. Let $\otimes$ stand for $\otimes_Z$. The following diagram can be made to commute using the techniques of [39]:

\[
\begin{array}{ccc}
(K^c(A) \land K^c(B)) \land K^c(C) & \xrightarrow{\kappa} & K^c(A) \land (K^c(B) \land K^c(C)) \\
\downarrow & & \downarrow \\
K^c(A \otimes B) \land K^c(C) & \xrightarrow{\kappa} & K^c(A) \land K^c(B \otimes C) \\
\downarrow & & \downarrow \\
K^c((A \otimes B) \otimes C) & \xrightarrow{\kappa} & K^c(A \otimes (B \otimes C)).
\end{array}
\]

Now using the strict functoriality of the construction $K^c(-)$ and the canonical homomorphisms (? $\otimes$ -) $\to$ (? $\hat{\otimes}$-) we get

\[
\begin{array}{ccc}
(K^c(A) \land K^c(B)) \land K^c(C) & \xrightarrow{\kappa} & K^c(A) \land (K^c(B) \land K^c(C)) \\
\downarrow & & \downarrow \\
K^c(A \hat{\otimes} B) \land K^c(C) & \xrightarrow{\kappa} & K^c(A) \land K^c(B \hat{\otimes} C) \\
\downarrow & & \downarrow \\
K^c((A \hat{\otimes} B) \otimes C) & \xrightarrow{\kappa} & K^c(A \otimes (B \hat{\otimes} C)) \\
\downarrow & & \downarrow \\
K^c((A \hat{\otimes} B) \hat{\otimes} C) & \xrightarrow{\kappa} & K^c(A \hat{\otimes} (B \hat{\otimes} C)).
\end{array}
\]
Observe that the associator \((A \otimes B) \otimes C \cong A \otimes (B \otimes C)\) is induced by the associativity isomorphism \((A \otimes B) \otimes C \cong A \otimes (B \otimes C)\). Similar arguments can be used to prove \(\Box\).

Let \(C\) be any operad in spaces. A symmetric spectrum \(X\) is said to be an algebra over \(C\) if there are maps of spectra \(\theta : C(j)_+ \wedge X^j \to X\) for all \(j \geq 0\), which are associative, unital, and equivariant in a suitable sense \([31]\). Moreover, a symmetric spectrum \(M\) is said to be a module over \(X\) if there are maps of spectra \(\lambda : C(j)_+ \wedge X^{j-1} \wedge M \to M\) for all \(j \geq 1\) that are again associative, unital, and equivariant as explained in \([31]\).

**Theorem 3.8.** Let \(\mathcal{D}\) be any strongly self-absorbing \(C^*\)-algebra. The spectrum \(K^c(\mathcal{D})\) is an algebra over the strongly self-absorbing \(\mathcal{D}\)-operad. Moreover, for any unital \(C^*\)-algebra \(A\), the spectrum \(K^c(A \otimes \mathcal{D})\) is a module over \(K^c(\mathcal{D})\).

**Proof.** We define the maps for \(j \geq 0\)
\[
\theta : \mathcal{D}(j)_+ \wedge K^c(\mathcal{D})^j \to K^c(\mathcal{D})
\]
\[
(f, (x_1, \cdots, x_j)) \mapsto f_*(\kappa(x_1, \cdots, x_j)).
\]
Here \(f_* : K^c(\mathcal{D} \hat{\otimes} j) \to K^c(\mathcal{D})\) is the map induced by \(f \in \mathcal{D}(j)\) and \(\kappa : K^c(\mathcal{D})^j \to K^c(\mathcal{D} \hat{\otimes} j)\) is the canonical map from Proposition 3.6. Similarly, we define the maps for all \(j \geq 1\)
\[
\lambda : \mathcal{D}(j)_+ \wedge K^c(\mathcal{D})^{j-1} \wedge K^c(\mathcal{D} \hat{\otimes} A) \to K^c(\mathcal{D} \hat{\otimes} A)
\]
\[
(f, (x_1, \cdots, x_j), y) \mapsto (f \otimes \text{id})_*(\kappa'(x_1, \cdots, x_j, y)).
\]
Here \(\kappa' : K^c(\mathcal{D})^{j-1} \wedge K^c(\mathcal{D} \hat{\otimes} A) \to K^c(\mathcal{D} \hat{\otimes} A) \to K^c(\mathcal{D} \hat{\otimes} \mathcal{D} \hat{\otimes} A)\) is induced by the composition of the canonical maps furnished by Proposition 3.6 and \((f \otimes \text{id})_* : K^c(\mathcal{D} \hat{\otimes} A) \to K^c(\mathcal{D} \hat{\otimes} \mathcal{D} \hat{\otimes} A)\) is the map induced by \(f \otimes \text{id} : \mathcal{D} \hat{\otimes} A \to \mathcal{D} \hat{\otimes} A\). The axiom for unitality says that the following diagrams commute:

\[
\begin{array}{ccc}
S^0 \wedge K^c(\mathcal{D}) & \cong & K^c(\mathcal{D}) \\
\downarrow \eta \wedge \text{id} & & \downarrow \theta \\
\mathcal{D}(1)_+ \wedge K^c(\mathcal{D}) & \cong & K^c(\mathcal{D} \hat{\otimes} A) \\
\end{array}
\]

This condition is clear from the fact that \(\eta\) maps the non-basepoint in \(S^0\) to \(\text{id} : \mathcal{D} \to \mathcal{D}\). Now using Lemma 3.7 and the strict functoriality of \(K^c(-)\) one can check the required associativity and equivariance conditions. \(\Box\)

**Remark 3.9.** It follows from Proposition 3.4 that \(K^c(\mathcal{D})\) is an \(E_\infty\)-ring in symmetric spectra.

Using Theorem 1.4 of \([22]\) one can rectify this \(E_\infty\)-ring structure on \(K^c(\mathcal{D})\) (resp. the module structure on \(K^c(\mathcal{D} \hat{\otimes} A)\)) to a strictly commutative symmetric ring spectrum structure (resp. to a strict module structure over it).

**Remark 3.10.** Note that \(O_\infty\) is a purely infinite \(C^*\)-algebra. In fact, Proposition 2.2 above is applicable to all strongly self-absorbing \(C^*\)-algebras that are purely infinite. Thus for such a \(C^*\)-algebra \(\mathcal{D}\) we conclude that \(K^c(\mathcal{D})\) is a commutative symmetric ring spectrum model for (connective) topological K-theory and for a unital \(C^*\)-algebra \(A\) the symmetric spectrum \(K^c(\mathcal{D} \hat{\otimes} A)\) is a module over it. Similar results were obtained by different methods in \([19]\).
4. **K-regularity of $O_\infty$-stable $C^*$-algebras**

Let $F$ be any functor on $C^*$. A $C^*$-algebra $A$ is called $F$-regular if the canonical inclusion $A \rightarrow A[t_1, \ldots, t_n]$ induces an isomorphism $F(A) \sim F(A[t_1, \ldots, t_n])$ for all $n \in \mathbb{N}$. This map has a one-sided inverse induced by the evaluation map $ev_0$. Rosenberg conjectured that any $C^*$-algebra $A$ is $K_0$-regular. Using the techniques developed to prove the Karoubi conjectures [24], it is shown in Theorem 3.4 of [17] that the conjecture is true if $A$ is stable. In fact, the Theorem in [17] asserts that a stable $C^*$-algebra is $K_m$-regular for all $m \in \mathbb{Z}$. A $C^*$-algebra is called $K$-regular if it is $K_m$-regular for all $m \in \mathbb{Z}$.

**Theorem 4.1.** The $C^*$-algebras $A \hat{\otimes} O_\infty$ are K-regular for all $A \in C^*$.

**Proof.** For the sake of better readability let us set $A_\infty := A \hat{\otimes} O_\infty$ and $B^n := B[t_1, \ldots, t_n]$ for any $A, B \in C^*$. Using excision we may assume that $A$ is unital. Arguing as in the proof of Proposition [2.2] we obtain a commutative diagram

$$
\begin{array}{ccc}
K_m(A_\infty) & \rightarrow & K_m((A_\infty)\hat{\otimes} K) \\
\downarrow & & \downarrow \\
K_m((A_\infty)^n) & \rightarrow & K_m((A_\infty)\hat{\otimes} (K)^n)
\end{array}
$$

Due to the stability of $A_\infty \hat{\otimes} K$ the middle vertical arrow is an isomorphism. Moreover, the compositions of the top and the bottom horizontal arrows are again isomorphisms due to the matrix stability of the functor $K_m(-)$ for unital algebras. Observe that the composite $*$-homomorphisms $A_\infty \rightarrow A_\infty \hat{\otimes} K \rightarrow A_\infty$ and $(A_\infty)^n \rightarrow (A_\infty)\hat{\otimes} (K)^n \rightarrow (A_\infty)^n$ are still inner. Now a similar diagram chase as before enables one to conclude that the left vertical arrow must be an isomorphism.

**Remark 4.2.** Purely infinite simple $C^*$-algebras like $O_\infty$ can be regarded as maximally noncommutative. Rather surprisingly, one needs fairly sophisticated techniques to establish the K-regularity of commutative $C^*$-algebras (see [47, 12]).

**Remark 4.3.** Using Proposition [2.2] and Theorem 4.1 the reduction principle for assembly maps (see Theorem 1.1 of [36]) can be generalized to include coefficients in $O_\infty$, i.e., for a countable, discrete, and torsion free group $G$, if the Baum–Connes assembly map with complex coefficients is injective (resp. split injective), then the Farrell–Jones assembly map in algebraic K-theory with coefficients in $O_\infty$ is also injective (resp. split injective).

5. **Algebraic K-theory of certain $O_\infty$-stable $C^*$-algebras**

We now explicitly compute the algebraic K-theory groups of certain $O_\infty$-stable $C^*$-algebras. It must be noted that complete calculation of the algebraic K-theory groups of an arbitrary ring is an extremely difficult task in general.

5.1. **Semigroup $C^*$-algebras coming from number theory.** A recent result of Li asserts that for a countable integral domain $R$ with vanishing Jacobson radical (which is, in addition, not a field) the left regular $ax + b$-semigroup $C^*$-algebra $C^*_\Lambda(R \rtimes R^\times)$ is $O_\infty$-absorbing, i.e., $C^*_\Lambda(R \rtimes R^\times) \hat{\otimes} O_\infty \cong C^*_\Lambda(R \rtimes R^\times)$ (see Theorem 1.3 of [33]).
Now we focus on the main object of our interest, namely, the left regular $ax + b$-semigroup $C^*$-algebra of the ring of integers $R$ of a number field $K$. It is shown in [16] that

$$K^\top_*(C^*_\lambda(R \rtimes R^\times)) \cong \bigoplus_{[X] \in G \setminus \mathcal{I}} K^\top_*(C^*_\lambda(G_X)),$$

where $\mathcal{I}$ is the set of fractional ideal of $R$, $G = K \rtimes K^\times$, and $G_X$ is the stabilizer of $X$ under the $G$-action on $\mathcal{I}$. The orbit space $G \setminus \mathcal{I}$ can be identified with the ideal class group of $K$.

As a consequence of Proposition 2.2 we obtain

**Theorem 5.1.** The algebraic K-theory of the $ax + b$-semigroup $C^*$-algebra of the ring of integers $R$ of a number field $K$ is 2-periodic and explicitly given by

$$K_*(C^*_\lambda(R \rtimes R^\times)) \cong \bigoplus_{[X] \in G \setminus \mathcal{I}} K^\top_*(C^*_\lambda(G_X)).$$

5.2. $O_\infty$-stabilized noncommutative tori. We recall some basic material before stating our result. A good reference for generalities on noncommutative tori is Rieffel’s survey [44].

For any real-valued skew bilinear form $\theta$ on $\mathbb{Z}^n$ ($n \geq 2$) the $C^*$-algebra of the noncommutative $n$-torus $A^n_\theta$ can be defined as the universal $C^*$-algebra generated by unitaries $U_x \in \mathbb{Z}^n$ subject to the relation

$$U_x U_y = \exp(\pi i \theta(x, y)) U_{x+y} \quad \forall x, y \in \mathbb{Z}^n.$$

Using the Pimsner–Voiculescu exact sequence one can compute the $K^\top$-theory of $A^n_\theta$ as an abelian group, namely,

$$(9) \quad K^\top_0(A^n_\theta) \simeq \mathbb{Z}^{2^{n-1}} \quad \text{and} \quad K^\top_1(A^n_\theta) \simeq \mathbb{Z}^{2^{n-1}}.$$

**Theorem 5.2.** The algebraic K-theory of the $O_\infty$-stabilized noncommutative $n$-torus $A^n_\theta$ is 2-periodic and explicitly given by

$$K_0(A^n_\theta \hat{\otimes} O_\infty) \simeq \mathbb{Z}^{2^{n-1}} \quad \text{and} \quad K_1(A^n_\theta \hat{\otimes} O_\infty) \simeq \mathbb{Z}^{2^{n-1}}.$$

**Proof.** By Proposition 2.2 one has an isomorphism $K_*(A^n_\theta \hat{\otimes} O_\infty) \cong K^\top_*(A^n_\theta \hat{\otimes} O_\infty)$. Using the Künneth Theorem one now computes that $K^\top_*(A^n_\theta \hat{\otimes} O_\infty) \cong K^\top_*(A^n_\theta)$. Observe that $K^\top_*(O_\infty) \simeq \mathbb{Z}$ and $K^\top_1(O_\infty) \simeq 0$ and all $C^*$-algebras in sight belong to the UCT-class. Now use Equation (9). \qed

We just determined the isomorphism type of the algebraic K-theory groups of $A^n_\theta \hat{\otimes} O_\infty$. One can also describe the elements in these groups using Rieffel’s results.

**Remark 5.3.** It follows from [13] that for irrational $\theta$ the projections in $A^n_\theta$ generate all of $K_0(A^n_\theta \hat{\otimes} O_\infty)$ and

$$K_1(A^n_\theta \hat{\otimes} O_\infty) \cong K^\top_1(A^n_\theta \hat{\otimes} O_\infty) \cong K^\top_1(A^n_\theta) \cong U A^n_\theta / U^0 A^n_\theta.$$

Here $UA^n_\theta$ denotes the group of unitary elements in $A^n_\theta$ and $U^0 A^n_\theta$ denotes the connected component of the identity element of $UA^n_\theta$. Thus one obtains a good description of the elements of the algebraic K-theory groups in low degrees in terms of projections and unitaries.
References

[1] A. J. Blumberg, D. Gepner, and G. Tabuada. A universal characterization of higher algebraic K-theory. *Geom. Topol.*, 17(2):733–838, 2013.

[2] J. Brodzki, V. Mathai, J. Rosenberg, and R. J. Szabo. D-branes, RR-fields and duality on noncommutative manifolds. *Comm. Math. Phys.*, 277(3):643–706, 2008.

[3] J. Brodzki, V. Mathai, J. Rosenberg, and R. J. Szabo. Non-commutative correspondences, duality and D-branes in bivariant K-theory. *Adv. Theor. Math. Phys.*, 13(2):497–552, 2009.

[4] U. Bunke and T. Nikolaus. T-Duality via Gerby Geometry and Reductions. arXiv:1305.6050.

[5] U. Bunke, P. Rumpf, and T. Schick. The topology of T-duality for Tⁿ-bundles. *Rev. Math. Phys.*, 18(10):1103–1154, 2006.

[6] U. Bunke and T. Schick. On the topology of T-duality. *Rev. Math. Phys.*, 17(1):77–112, 2005.

[7] U. Bunke, T. Schick, M. Spitzweck, and A. Thom. Duality for topological abelian group stacks and T-duality. In *K-theory and noncommutative geometry*, EMS Ser. Congr. Rep., pages 227–347. Eur. Math. Soc., Zürich, 2008.

[8] A. Connes. *Noncommutative geometry*. Academic Press Inc., San Diego, CA, 1994.

[9] A. Connes and M. A. Rieffel. Yang-Mills for noncommutative two-tori. In *Operator algebras and mathematical physics (Iowa City, Iowa, 1985)*, volume 62 of *Contemp. Math.*, pages 237–266. Amer. Math. Soc., Providence, RI, 1987.

[10] A. Connes and G. Skandalis. The longitudinal index theorem for foliations. *Publ. Res. Inst. Math. Sci.*, 20(6):1139–1183, 1984.

[11] G. Cortiñas and N. C. Phillips. Algebraic K-theory and properly infinite C∗-algebras. arXiv:1402.3197.

[12] G. Cortiñas and A. Thom. Algebraic geometry of topological spaces I. *Acta Math.*, 209(1):83–131, 2012.

[13] J. Cuntz. Simple C∗-algebras generated by isometries. *Comm. Math. Phys.*, 57(2):173–185, 1977.

[14] J. Cuntz. A new look at KK-theory. *K-Theory*, 1(1):31–51, 1987.

[15] J. Cuntz, R. Meyer, and J. M. Rosenberg. Topological and bivariant K-theory, volume 36 of *Oberwolfach Seminars*. Birkhäuser Verlag, Basel, 2007.

[16] M. Dadarlat and U. Pennig. A Dixmier–Douady theory for strongly self-absorbing C∗-algebras. arXiv:1306.2583.

[17] C. Daenzer and E. Van Erp. T-Duality for Langlands Dual Groups. arXiv:1211.0763.

[18] M. Dadarlat and U. Pennig. Unit spectra of K-theory from strongly self-absorbing C∗-algebras. arXiv:1306.2583.

[19] B. Keller. On differential graded categories. In *International Congress of Mathematicians. Vol. II*, pages 151–190. Eur. Math. Soc., Zürich, 2006.

[20] E. Kirchberg and M. Rørdam. Non-simple purely infinite C∗-algebras. *Amer. J. Math.*, 122(3):637–666, 2000.
M. Kontsevich. XI Solomon Lefschetz Memorial Lecture series: Hodge structures in non-commutative geometry. In Non-commutative geometry in mathematics and physics, volume 462 of Contemp. Math., pages 1–21. Amer. Math. Soc., Providence, RI, 2008. Notes by Ernesto Lupercio.

I. Krží and J. P. May. Operads, algebras, modules and motives. Astérisque, (233):iv+145pp, 1995.

X. Li. Semigroup C*-algebras and amenability of semigroups. to appear in JFA, arXiv:1105.5539.

X. Li. Semigroup C*-algebras of ax+b-semigroups. arXiv:1306.5553.

S. Mahanta. Twisted K-theory, K-homology and bivariant Chern–Connes type character of some infinite dimensional spaces. to appear in Kyoto J. Math.; arXiv:1104.4835.

S. Mahanta. Higher nonunital Quillen K′-theory, KK-dualities and applications to topological T-dualities. J. Geom. Phys., 61(5):875–889, 2011.

V. Mathai and J. Rosenberg. On mysteriously missing T-duals, H-flux and the T-duality group. In Differential geometry and physics, volume 10 of Nankai Tracts Math., pages 350–358. World Sci. Publ., Hackensack, NJ, 2006.

J. P. May. Pairings of categories and spectra. J. Pure Appl. Algebra, 19:299–346, 1980.

R. Meyer. Categorical aspects of bivariant K-theory. In K-theory and noncommutative geometry, EMS Ser. Congr. Rep., pages 1–39. Eur. Math. Soc., Zürich, 2008.

J. Nuiten. Cohomological quantization of local prequantum boundary field theory. freely available at http://ncatlab.org/schreiber/show/master+thesis+Nuiten.

D. Quillen. K₀ for nonunital rings and Morita invariance. J. Reine Angew. Math., 472:197–217, 1996.

M. A. Rieffel. Projective modules over higher-dimensional noncommutative tori. Canad. J. Math., 40(2):257–338, 1988.

M. A. Rieffel. Noncommutative tori—a case study of noncommutative differentiable manifolds. In Geometric and topological invariants of elliptic operators (Brunswick, ME, 1988), volume 105 of Contemp. Math., pages 191–211. Amer. Math. Soc., Providence, RI, 1990.

M. Rørdam. Classification of nuclear, simple C*-algebras. In Classification of nuclear C*-algebras. Entropy in operator algebras, volume 126 of Encyclopaedia Math. Sci., pages 1–145. Springer, Berlin, 2002.

J. Rosenberg. Continuous-trace algebras from the bundle theoretic point of view. J. Austral. Math. Soc. Ser. A, 47(3):368–381, 1989.

J. Rosenberg. Comparison between algebraic and topological K-theory for Banach algebras and C*-algebras. In Handbook of K-theory. Vol. 1, 2, pages 843–874. Springer, Berlin, 2005.

J. Rosenberg. Topology, C*-algebras, and string duality, volume 111 of CBMS Regional Conference Series in Mathematics. Published for the Conference Board of the Mathematical Sciences, Washington, DC, 2009.

U. Schreiber. Quantization via Linear homotopy types. arXiv:1402.7041.

A. A. Suslin and M. Wodzicki. Excision in algebraic K-theory and Karoubi’s conjecture. Proc. Nat. Acad. Sci. U.S.A., 87(24):9582–9584, 1990.

A. A. Suslin and M. Wodzicki. Excision in algebraic K-theory. Ann. of Math. (2), 136(1):51–122, 1992.

F. Waldhausen. Algebraic K-theory of spaces. In Algebraic and geometric topology (New Brunswick, N.J., 1983), volume 1126 of Lecture Notes in Math., pages 318–419. Springer, Berlin, 1985.

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15