The Analyses of Node Swapping Networks by New Graph Index

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Abstract

We have proposed two new dynamic networks where two nodes are swapped each other, and showed that the both networks behave as a small world like in the average path length but can not make any effective discussions on the clustering coefficient because of the topological invariant properties of the networks. In this article we introduce a new index, "hamming coefficient" or "multiplicity", that can act well for these dynamic networks. The hamming coefficient or multiplicity is shown essentially to behave as the clustering coefficient in the small world network proposed by Watts and Strogatz\[4\]. By evaluating the new index, we uncover another properties of the two networks.

key words: Small world network, Scale free network, node swapping network, clustering coefficient, hamming distance, multiplicity

1 Introduction

In a social network, we may need to consider the possibility that people is transferred to another place. Then the physical (direct) relations among them are often lost by the movement. In terms of a network theory, this means that some nodes break the present connections with neighboring nodes, move and there build new connections with nodes. For simplicity, we here consider only that two nodes exchange the place each other on the network. Such exchange is assumed to be constantly carried out. Some properties such as the diameter, the average path length, the propagation when one virus is placed on the network, have been studied by the author\[10\] where it has been pointed out that the swapping networks look like a little small world property but have rather intermediate properties between the small world network (SW-NET) introduced by Watts and Strogatz\[4, 5\] and regular lattices. In this article, we study the dynamic networks in more details.

The clustering coefficient is used in usual network analyses. There are, however, two difficult points in estimating it in node swapping networks (NSN). First is that general dynamic networks such as swapping networks has a time dependent clustering coefficient, unlike static networks such as SW-NET or the preferencial scale free model (SF-NET) introduced by Barabasi and Albert\[1, 2, 3\], where they are usually analyzed after networks are completed. Second is that the topology of the NSN is invariant under the swapping of nodes, because swapped nodes inherit the all links connected with old nodes after swapping, and so the clustering coefficient is trivially constant. Thus we need to introduce a sort of new index corresponding to the clustering coefficient. In any dynamic networks, it would be crucial that some propagation process is considered. We consider the situation that a test virus is randomly placed on a node on NSN. The virus propagates a next one connected with the first target node. We compare the similarity of the friendships of two nodes, the first node \(i\) and propagated node \(j\), that is to say, estimate the hamming distance between the two nodes with adjacent vectors, \(v_i\) and \(v_j\) that are the \(i\)-th row component and \(j\)-th row component of the adjacent matrix corresponding to the network, respectively. The estimated quantity is related only to the simirality of two connected nodes. The clustering coefficient are related to the similarity of all nodes connected with a target
node. So the new index can be interpreted to be a sort of the shortening of the clustering coefficient. A virus propagates from the target node to connected nodes one after another and we estimate the index every time the virus infects some connected node. We consider the average hamming distance of them during propagation till all nodes are infected. We compare the average hamming distance with the clustering coefficient of SW-NET. As result we turn out that they show the similar behaviours as the rewiring probability increases. So this suggests that we can use this new index instead of the clustering coefficient in dynamic networks. By evaluating the new index, we can observe that the NSN certainly behaves as SW-NET in the average hamming distance, unlike in an average path length. Thus we conclude that the NSN is not so small world like as SW-NET but looks like SW-NET with respect to this new index corresponding to the usual clustering coefficient.

This article is planned as follows. First section is devoted to the Introduction, and we give a brief review of the swapping network together with the constructive definition that was given in the previous article[10]. In section 3 we give the definition of the new index, average hamming distance and multiplicity, and compare it with the clustering coefficient of the well known SW-NET. After that, we evaluate the indices for NSN and the preferential NSN [10] to study the properties of the network in more details in the section 4. In the last section, 5, we give concluding remarks.

2 Node swapping network and average path length between nodes

In this section we introduce the NSN by presenting a constructive definition, and review some properties of the network that have been discussed in [10].

2.1 Review of node swapping network (NSN)

As explained in the previous section, we consider that nodes swap each other on a regular network. This network may be seemed to look like a small world network. However, it is necessarily not the case. By the movement, the nodes and the edges accompanied with them are entirely cut, and the nodes are connected with new edges each other at the new position. Notice that the network topology is apparently invariant under the procedure. In small world networks the static properties are only pursued but the dynamic properties such as NSN are rather important.

The algorithm for formulating the NSN is as follows;
1. Prepare a regular (typically one dimensional) network with a periodic boundary condition such as a ring.
2. Randomly choose two nodes on its network and swap them. This procedure is repeated \( Q \) times.
3. Evaluate correct quantities of the network.
4. 1~3, which is one round, is repeated \( M \) times.

In such a way, the network is dynamically analyzed as edges are cut and pasted to new nodes.

We analyze some properties of the network by doing computer simulations. First of all we discuss the diameter \( D \) and the average distance \( L \) between any pairs of nodes of the NSN, which have been given in [10].

The diameter of usual random networks behaves as \( \frac{\log n}{\log <k>} \), where \( <k> \) is the average degree of nodes and \( n \) is the size of network, that is, the number of nodes. We have conveniently introduced a handy network in [10] with the same properties essentially as random networks, instead of usual random networks. This new network has been called "random graph with fixed degree", RNFD, where the degrees of all nodes are contrived to be a constant number \( k \).
Fig. 1 shows the size $n$ vs. $D$ of the NSN constructed from degree $k = 4$ regular lattice and RNFD with $k = 4$, respectively\[10\]. The points and curved lines in the figures show simulation data and its approximate curves, respectively. This shows that $n$ dependence of $D$ in the NSN is exponential, while that in RAFD is logarithmic such as random networks. Their essential properties are independent of $Q$ in NSN or $k$ in RNFD. Since the behaviour is linear, $D = \frac{n^2}{k}$, in one dimensional regular lattice with the periodic boundary condition, it turns out that NSN is a network intermediate between regular lattices and random networks or SW-NET. The existence of $D$ also means that NSN is an overall connected network.

To clear the point we study average distance $L$ between any pairs of nodes. (Notice that the behaviour of $L$ is not necessarily equal to that of $D$ in dynamic networks, because the network in calculating the distances from a target node to nearby nodes is not the same as that in calculating the distance from the target node to faraway one. Thus $L$ is the average over different networks. $D$ is the step number from a target one to the most faraway node.) Fig.2 shows $L$-$n$ curves of RNFD with $k = 4$ and NSN with $Q = 10$ and $k = 4$. Essentially $L$’s have the same property as $D$. The reason will be that the number of steps needed for the complete estimation of $D$ is nearly equal to that of $L$. The properties are also independent of $Q$ or $k$. Since a regular lattice shows linear dependence in $L$-$n$ relation such as $D$-$n$, the NSN is not only so small world and but also so large world after all. Fig. 3 refers to theoretical $L$-$n$ curves of SW-NET\[7\] and SF-NET\[8, 9\] that are given by

$$L(n) = \begin{cases} \frac{\log(4np)}{\log k} & \text{for } 2np \gg 1 \text{ and SW-NET}, \\ \frac{\log n}{\log \log n} & \text{for SF-NET}, \end{cases}$$

and their numerically approximated curves. $p$ is the rewiring probability in SW-NET, taken $p = 0.05$ in Fig.3. In SF-NET, the logarithmic function phenomenologically fits almost perfectly. Though it is also possible that both of NSN and SF-NET can be approximated by exponential functions, they are very different from each other in the absolute value of the index. This property is essentially invariant under changing $Q$ value. As we increase $Q = 1, 5, 10, ...$, the index of the exponential decreases to $s = 0.83, 0.62, 0.58, ...$ in NSN. As for SF-NET $s = 0.07$, different from those of NEN in order, and it seems not to be able to overcome the difference (we should interpret that the excessively small $s$ means that it is rather the logarithmic function).

Thus SF-NET and NSN are essentially thought to be different networks in terms of the average path length. In summary we conclude that the relation

$$RNFD \sim SF - NET < SW - NET < NSN < Regular \ lattice$$

appllys in $L$.

Figure 1: Diameters of the NEN with $Q = 10$ for average of 50 times (left) and RNFD with $k = 4$ for average of 100 times (right). Approximate formula of them are $D = 0.4725n^{0.619}$ and $D = 1.7507\log n - 1.9778$, respectively.
Figure 2: Average distances between two nodes for average of 100 times: The left is an average $L$ of RNFD with $k = 4$. The right is that of NEN with $Q = 10$ and $k = 4$. The approximate formula of them are $L = 0.7861 \times \log_e n - 0.2182$ and $L = 0.6509 \times n^{0.579}$, respectively.

Figure 3: Average distances between two nodes: The left is an average $L$ of SW-NET with $k = 4$ and $p = 0.05$, and the right is that of SF-NET. The approximate formula of them are $L = 2.5 \log_e n - 4.0236$ and $L = 0.2421 \log_e n + 1.9031$ or $L = 2.1888 \times n^{0.0707}$, respectively.

3 Hamming coefficient and clustering coefficient

The clustering coefficient and the degree distribution have no significance in the NSN, because the network topology in NSN is apparently invariant temporally so that they take the same values as those of the original regular lattice. As for this, we may have to introduce a sort of new kind of index to investigate NSN in more details.

The most effective way would be to explore the propagation of a test virus on dynamic networks. We adopt the idea, basically. Instead of exploring the similarities of friendship among all nodes connected with a target node such as the clustering coefficient, we estimate the similarity of friendship between a node $i$ connected with a target node $j$ and the target node. We measure it by calculating the hamming distance between adjacent vectors $v_i$ and $v_j$ where the adjacent vector $v_i(v_j)$ is the $i$-th ($j$-th) row vector in the adjacent matrix of the network. Then node $i$ is chosen at random among the connected nodes with $j$, which reflects the situation that a virus randomly infects some node connected with the target node $j$. In place of the usual clustering coefficient, we evaluate the averaged hamming distance $D_H$ during the time all nodes will be infected. More exactly, we introduce the multiplicity $M$ as

$$M = 1 - \frac{D_H}{D_n}, \quad D_H = v_i \bullet v_j \quad \text{(where } \bullet \text{ means the Boolean inner product)}$$

(2)
in order to measure a similarity of two nodes, while the hamming distance itself means the difference of friendship between two connected nodes. We take $D_n = 2k$ as the normalization factor (The reason will be given later).

Next we compare the multiplicity to the usual clustering coefficient in well known networks such as the SW-NET. In Fig. 4 the two indices in the SW-NET with degree $k = 12$ and $n = 500$ are given. The fact that both act in a similar way suggests that the multiplicity can play the same role as the clustering coefficient. Of course both indices are originally different ones and so it is not necessary that they take a same value or behave in same way exactly. The multiplicity is only a substitute for the clustering coefficient. However, it can play an important role in dynamic networks such as NSN as discussed in the next section.

Here we have a little theoretical discussion on $D_H$ to speculate the value of $D_n$. In regular lattice, we can analytically estimate $D_H$:

$$D_H = \frac{\delta}{2} \sum_{i=1}^{\delta} 2^i = \delta + 1$$  \hspace{1cm} (3)

where $k = 2\delta$. This corresponds to the limit of $p \to 0$ in SW-NET. The fact that $D_H = \delta + 1 = 7$ for $\delta = 6$ agrees with Fig.5 where $D_H = 7.16$.

On the other hand, in random lattice, we can estimate $D_H$ as an expectation value of the probability that $i$-th element in an $n$ bit string, whose component randomly takes 0 or 1, is different from $i$-th one in another random $n$ bit string. So we obtain

$$D_H = \left[ 1 - \left( \frac{2\delta}{n} \right)^2 + \left( \frac{n-2\delta}{n} \right)^2 \right] \times n = 4\delta \frac{(n-\delta)}{n}$$ \hspace{1cm} (4)

where the inner parts of $\{ \}$ is the sum of two probabilities that both $i$-th elements are 0 and that they are 1 together. Lastly $n$ is multiplied to take an average for $n$ bits. Simply we can also evaluate it as the expectation value of the probability that $i$-th elements of two random $n$-bits strings are different each other;

$$D_H = 2 \times n \times \frac{n-2\delta}{n} \times \frac{2\delta}{n} = 4\delta \frac{(n-\delta)}{n}$$ \hspace{1cm} (5)

where $n$ is multiplied to take the average for $n$ bits as before and 2 is multiplied due to the permutation symmetry of two $n$-bit strings. More elaborate derivation will be given in Appendix. This happens at large $p$ for SW-NET and so $D_H = 23.8$ in the present case with $\delta = 6$ and $n = 500$, which agrees well with Fig.5 where $D_H = 22.7$.

Anyway $\frac{4\delta(n-\delta)}{n}$ is the maximal value of $D_H$. For $n >> \delta$, $\frac{4\delta(n-\delta)}{n} \sim 4\delta$. Thus the normalization factor $D_n = 2k = 4\delta$ is taken in the equation (2). This ensures $0 \leq M \leq 1$.
4 Multiplicity of simple node swapping network and preferential node swapping network

In this section, we estimate the multiplicities of NSN and their variation, which will be defined in 4.2, to analyse network properties in more details.

4.1 Hamming coefficient of simple node swapping network

In this section we evaluate new index, the hamming distance, for NSN. For it, we need a little extension of the index so that it adapts in dynamic networks. In dynamic networks, we define the incoming edges as the edges that a target node \( i \) leaves, and the outgoing edges as those that the node \( h \) infected from the node \( i \) leaves. When no swapping happens, the index can be obtained by calculating \( v_i \cdot v_h \). However, notice that when the infected node \( h \) is swapped, outgoing edges are different from those without swapping because of rewiring effect. When the infected node \( h \) is swapped, the outgoing edges are those that the infected node gets at the new position. Then the hamming distance turns to the Boolean inner product between the target node \( i \) and the node \( j \) that is swapped with the node \( h \) connected with \( i \), that is \( v_i \cdot v_j \), ultimately. Thus we evaluate it for NSN and the preferential NSN, which will be explained in the successive subsection.

4.2 Multiplicities of simple and preferential node swapping network

First of all we explain a variation of the NSN. There are a little similarity between the NSN and SF-NET apparently as suggested before. We pursue this point still more. Scale free property usually appears from both of the evolution and the preferential attachment. We apply the idea of the preferential attachment to this dynamic NSN. We assume that the nodes which has been transferred once are also transferred with high probability after that. At \( m \) round and \( q \) times, the probability \( p_i(t) \) that a node \( i \) is chosen as a swapping node is assumed that

\[
p_i(t) = \begin{cases} \frac{1 + p_i(t-1)N(t-1)}{N(t-1) + 2} & \text{when the node } i \text{ was chosen as exchange node at time } t-1, \\ p_i(t-1) & \text{others,} \end{cases}
\]

where

\[
N(t) = n + 2t, \quad t = mQ + q, \quad p_i(0) = \frac{1}{N(0)} = \frac{1}{n} \text{ for all } i.
\]
This reflects the fact that while active people often transfer, others trend to stay in one place. We call this type of networks Preferential Node Swapping Network (PNSN). On the other hand, NSN introduced in the previous subsection is called simple NSN when we need to distinguish them. The results of computer simulation of $L$ and $D$ on PNSN are just similar to those of the NEN [10].

Fig. 6 shows the multiplicity of the simple NSN and PNSN with $k = 4$, $n = 500$ and $Q = 5$. More simulations will prove that changing $Q$ does not have any crucial effects in the multiplicity. We can observe that the behaviour of NSN is the almost same as that of simple NSN. So it seems that there is not preferential effect in NSN, even when swapping increases in a number of times, which corresponds to large $p$.

From Fig.4 and Fig.6 where the multiplicity showly drops off in the similar manner as SW-NET, we can observe that the behaviour of (P)NSN looks like that of SW-NET in $M$. This means that NSN is definitely different from SF-NET.

Figure 6: Multiplicity of the simple NSN and PNSN with $k = 4$, $n = 500$ and $Q = 5$.

5 Concluding Remarks

We introduced a new index for dynamic networks to analyze them, especially NSN or PSNS, in more details. It has been shown that this index, multiplicity, can stand in for the usual clustering coefficient in the SW-NET. Using this fact, we analyse the simple NSN and the PNSN. These behaviours look like SW-NET in the point of view of the new index, $M$. Considering the results of analyses of the diameters and the average path length given in [10], we entirely obtain three main conclusions. One is that NSN is not so small world as SW-NET and SF-NET, but a little more small world than regular lattice networks, and thus NSN is something between regural networks and SW-NET. Second one is that (P)NSN shows simiar behabiours as SW-NET in the multiplicity. Third is that there is not any crucial preferential effects in NSN.

They are summarized with other well-known networks in Table 1 where the multiplicity in exchange for $C$ is shown for (P)NSN. The properties of $L$ and $C$ of all networks that have already known currently [11] are included in the Table 1 except for (P)NSN. For example, the properties of complete graphs are essentially the same as those of SW-NET with respect to $L$ and $C$, and so on. By contrast, (P)NSN are different from every one of them that have already known. Moreover as $p \to$ large, $L$ increases and the $M$ corresponding to $C$ decreases in (P)NSN. By taking large $p$, a network with large $L$ and small $M(C)$ may be constructed, which has quire novel property. To study some dynamics of NSN with these properties will be next intersting works[12].
Table 1: Comparison of various networks with (P)NSN.

| Networks  | Randm | SF-NET | SW-NET (0 < p < 1) | (P)NSN | Regular Lattice |
|-----------|-------|--------|--------------------|--------|-----------------|
| L         | small | small  | small              | middle | large           |
| C(M)      | small | small  | large              | large  | large           |

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Appendix

Analytic derivation of hamming coefficient in a random lattice

We consider a network that the number of nodes is $n$ and the degree of nodes is $2\delta = k$. The hamming coefficient $D_H$ in a random lattice can be derived as follows;

$$D_H = \sum_{m=0}^{n-2\delta} \frac{2m(2\delta)^m \times (n-2\delta)_m}{A}, \quad (8)$$

where \( \binom{n}{m} \) shows the combination

$$\binom{n}{m} = \frac{n!}{m!(n-m)!}, \quad (9)$$

and $A$ is a normalization factor;

$$A = \sum_{m=0}^{n-2\delta} \left( \frac{2\delta}{m} \right) \times \binom{n-2\delta}{m} = \binom{n}{n-2\delta} = \binom{n}{2\delta}. \quad (10)$$

Then we lead to the following equation that is exactly same as equation (4) or (5);

$$D_H = \frac{1}{A} \sum_{m=0}^{n-2\delta} 2m \left( \frac{2\delta}{m} \right) \times \frac{(n-2\delta)!}{m!(n-2\delta-m)!} \quad (11)$$

$$= \frac{1}{A} \sum_{m=0}^{n-2\delta} 2 \left( \frac{2\delta}{m} \right) \times \frac{(n-2\delta)!}{(m-1)!(n-2\delta-m)!} \quad (12)$$

$$= \frac{1}{A} \sum_{m=0}^{n-2\delta} 2 \left( \frac{2\delta}{m} \right) \times \binom{n-2\delta}{m-1} \quad (13)$$

$$= \frac{2(n-2\delta)}{A} \binom{n-1}{n-2\delta} \quad (14)$$

$$= 2(n-2\delta) \frac{(n-2\delta)!(2\delta)!}{n!(n-2\delta)!(2\delta-1)!} \quad (15)$$

$$= \frac{4\delta(n-2\delta)}{n}, \quad (16)$$

where we used the following forumul;

$$\sum_{m=0}^{q} \binom{X}{m} \times \binom{Y}{p-m} = \binom{X+Y}{q}. \quad (17)$$
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