Eigenvalues for a Schrödinger operator on a closed Riemannian manifold with holes

Olivier Lablée

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Abstract
In this article we consider a closed Riemannian manifold $(M,g)$ and $A$ a subset of $M$. The purpose of this article is the comparison between the eigenvalues $(\lambda_k(M))_{k \geq 1}$ of a Schrödinger operator $P := -\Delta_g + V$ on the manifold $(M,g)$ and the eigenvalues $(\lambda_k(M - A))_{k \geq 1}$ of $P$ on the manifold $(M - A, g)$ with Dirichlet boundary conditions.

1 Introduction
The behaviour of the spectrum of a Riemannian manifold $(M,g)$ under topological perturbation has been the subject of many research. The most famous exemple is the crushed ice problem [Kac], see also [Ann]. This problem consists to understand the behaviour of Laplacian eigenvalues with Dirichlet boundary on a domain with small holes. This subject was first studied by M. Kac [Kac] in 1974. Then, J. Rauch and M. Taylor [Ra-Ta] studied the case of Euclidian Laplacian in a compact set $M$ of $\mathbb{R}^n$: they showed that the spectrum of $\Delta_{\mathbb{R}^n}$ is invariant by a topological excision of a $M$ by a compact subset $A$ with a Newtonian capacity zero. Later, S. Osawa, I. Chavel and E. Feldman [Ca-Fe1], [Ca-Fe2] treated the Riemannian manifold case. They used complex probalistic techniques based on Brownian motion. In [Ge-Zh], F. Gesztesy and Z. Zhao investigate the study the case of a Schrödinger operator with Dirichlet boundary conditions $\mathbb{R}^n$, they use probabilistic tools. In 1995, in a nice article [Cou] G. Courtois studied the case of Laplace Beltrami operator on closed Riemannian manifold. He used very simple techniques of analysis. In [Be-Co] J. Bertrand and B. Colbois explained also the case of Laplace Beltrami operator on compact Riemannian manifold. In this article we focus on the the Schrödinger operator $-\Delta_g + V$ case on a closed Riemannian manifold.

Assumption. The manifold is closed (i.e. compact without boundary); the function $V$ is bounded on the manifold $M$ and $\min_M V > 0$.

In this work we show that under “little” topological excision of a part $A$ from the manifold, the spectrum of $-\Delta_g + V$ on $M - A$ is close of the spectrum on $M$. More precisely, the “good” parameter for measuring the littleness of $A$ is a type of electrostatic capacity defined by :

$$\text{cap}(A) := \inf \left\{ Q(u), u \in H^1(M), \int_M u dV_g = 0, u + e_1 \in H^1_0(M - A) \right\}$$
where \( e_1 \) denotes the first eigenfunction of the operator \(-\Delta_g + V\) on the manifold \( M \), and \( Q \) is the following quadratic form:

\[
Q(\phi) := \int_M |d\phi|^2 \, dV_g + \int_M V |\phi|^2 \, dV_g
\]

and \( H^1_0(M-A) \) is the Sobolev space defined by:

\[
H^1_0(M-A) := \{ g \in H^1(M), \, g = 0 \text{ on a open neighborhood of } A \}
\]

the closure is for the norm \( \| \cdot \|_{H^1(M)} \), \( H^1(M) \) is the usual Sobolev space on \( M \).

Indeed, more \( \text{cap}(A) \) is small, more the spectrum \(-\Delta_g + V\) on \( M - A \) is close of the spectrum on \( M \) in the following sense:

**Theorem.** Let \((M,g)\) a closed Riemannian manifold. For all integer \( k \geq 1 \), there exists a constant \( C_k \) depending on the manifold \((M,g)\) and on the potential \( V \) such that for all subset \( A \) of \( M \) we have:

\[
0 \leq \lambda_k(M-A) - \lambda_k(M) \leq C_k \sqrt{\text{cap}(A)}.
\]

The organization of this paper is the following: in the part 2 we start by recall some classicals results in spectral theory and about usual Sobolev spaces, next we define our specific Sobolev space \( H^1_0(M-A) \) and the notion of Schrödinger capacity. In particular, we explain the link between the functional Hilbert space \( H^1_0(M-A) \) and Schrödinger capacity \( \text{cap}(A) \). The last part of this paper is a detailed proof of the main theorem.

## 2 Spectral problem background

### 2.1 Schrödinger operator on a Riemannian manifold

We recall here some generality on spectral geometry. In Riemannian geometry, the **Laplace Beltrami operator** is the generalisation of Laplacian \( \Delta = \sum_{j=1}^n \partial_{x_j} \partial_{x_j} \) on \( \mathbb{R}^n \). For a \( C^2 \) real valued function \( f \) on a Riemannian manifold and for a local chart \( \phi : U \subset M \to \mathbb{R} \) of the manifold \( M \), the Laplace Beltrami operator is given by the local expression:

\[
\Delta_g f = \frac{1}{\sqrt{g}} \sum_{j,k=1}^n \frac{\partial}{\partial x_j} \left( \sqrt{g} g^{jk} \frac{\partial (f \circ \phi^{-1})}{\partial x_k} \right) \tag{2.1}
\]

where \( g = \det(g_{ij}) \) and \( g^{jk} = (g_{jk})^{-1} \).

The spectrum of this operator is a nice geometric invariant, see Berger, Gauduchon and Mazet [BGM] and [Ré-Be]. The spectrum of Laplace Beltrami operator has many applications in geometry topology, physics, etc...

For every Riemannian manifold \((M,g)\) with dimension \( n \geq 1 \) we have the “natural” Hilbert space \( L^2(M) = L^2(M, dV_g) \), \( V_g \) is the Riemannian volume form associated to the metric \( g \). For \( V \) a function from \( M \) to \( \mathbb{R} \), we define the Schrödinger operator on the manifold \((M,g)\) by the linear unbounded operator on the set of smooth compact supports real valued functions \( C^\infty_c(M) \subset L^2(M) \) by:

\[
-\Delta_g + V.
\]
2.2 Sobolev spaces

Let us denote by \( C^\infty_c(M) \) the set of smooth functions with compact support in \( M \). The set \( C^\infty_c(M) \) is also called the set of test functions in the language of distributions. Recall first that the Lebesgue space \( L^2(M) \) on the manifold \((M,g)\) is defined by:

\[
L^2(M) := \left\{ f : M \to \mathbb{R} \text{ measurable such that } \int_M |f|^2 \, dV_g < +\infty \right\}.
\]

This space is a Hilbert space for the scalar product:

\[
\langle u, v \rangle_{L^2} := \int_M uv \, dV_g.
\]

Next the Sobolev space \( H^1(M) \) is defined by:

\[
H^1(M) := \overline{C^\infty(M)}
\]

where the closure is for the norm \( \| \|_{H^1} : \| u \|_{H^1} := \sqrt{\| u \|_{L^2}^2 + \| du \|_{L^2}^2} \).

An other point of view to define the space \( H^1(M) \) is the following:

\[
H^1(M) = \left\{ u \in L^2(M); du \in L^2(M) \right\}
\]

where the derivation is the sense of distribution.

The space \( H^1(M) \) is a Hilbert space for the scalar product:

\[
\langle u, v \rangle_{H^1} := \langle u, v \rangle_{L^2} + \langle du, dv \rangle_{L^2}.
\]

For finish, the Sobolev space \( H^1_0(M,g) \) is defined by:

\[
H^1_0(M) := \overline{C^\infty_c(M)}
\]

the closure is for the norm \( \| \|_{H^1(M)} \).

So we have:

\[
C^\infty_c(M) \subset H^1_0(M) \subset H^1(M) \subset L^2(M).
\]

Recall that, for the norm \( \| \|_{L^2(M)} \) we have:

\[
\overline{C^\infty_c(M)} = L^2(M).
\]

2.3 Spectral problem

The spectral problem is the following: find all pairs \((\lambda, u)\) with \( \lambda \in \mathbb{R} \) and \( u \in L^2(M) \) such that:

\[
-\Delta_g u + Vu = \lambda u \tag{2.2}
\]

(with \( u \in L^2(M) \) in the non-compact case).
In the case of manifold with boundary, we need boundary conditions on the functions \( u \), for example the Dirichlet conditions: \( u = 0 \) on the boundary of \( M \), or Neumann conditions: \( \frac{\partial u}{\partial n} = 0 \) on the boundary of \( M \). In the case of closed manifolds (compact without boundary) we don’t have conditions.

For our context (the closed case) the natural space to look here is the Sobolev space \( H^1(M) \).

Recall here a classical theorem of spectral theory (see for example [Re-Si]):

**Theorem.** For the above problems, the operator \( -\Delta_g + V \) is self-adjoint, the spectrum of the operator \( -\Delta_g + V \) consists of a sequence of infinite increasing eigenvalues with finite multiplicity:

\[
\lambda_1(M) \leq \lambda_2(M) \leq \cdots \leq \lambda_k(M) \leq \cdots \to +\infty.
\]

Moreover, the associate eigenfunctions \( (v_k)_{k \geq 0} \) is a Hilbert basis of the space \( L^2(M) \).

**Definition.** We define the quadratic form \( Q \) with domain \( D(Q) := H^1(M) \) by:

\[
Q(\varphi) := \int_M |d\varphi|^2 dV_g + \int_M V|\varphi|^2 dV_g.
\]

Recall also (see for example [Co-Hil]) the minimax variational characterization for eigenvalues: for all \( k \geq 1 \)

\[
\lambda_k(M) = \min_{E \in H^1(M)} \max_{\|\varphi\|_E = 1} R(\varphi)
\]

where \( R(\varphi) \) is the Rayleigh quotient of the function \( \varphi \):

\[
R(\varphi) := \frac{Q(\varphi)}{\int_M \varphi^2 dV_g}.
\]

In our context, a consequence of the minimax principle is:

**Proposition.** The first eigenvalue \( \lambda_1(M) \) and \( e_1 \) the first eigenfunction of the operator \( -\Delta_g + V \) on the manifold \((M, g)\) satisfy \( \lambda_1(M) \geq \min_M V > 0 \) and \( e_1 \) does not vanish on \( M \) in \( M \).

**Proof.** It’s clear that

\[
\int_M |\partial_1|^2 dV_g + \int_M V|\partial_1|^2 dV_g \geq \min_M V \|\partial_1\|_{L^2(M)}^2
\]

and on the other hand

\[
\int_M |\partial_1|^2 dV_g + \int_M V|\partial_1|^2 dV_g = -\int_M \Delta_g \partial_1 \partial_1 dV_g + \int_M V|\partial_1|^2 dV_g
\]

\[
= \int_M ( -\Delta_g + V ) \partial_1 \partial_1 dV_g = \lambda_1(M) \|\partial_1\|_{L^2(M)}^2
\]

so \( \lambda_1(M) \geq \min_M V \). Next, suppose the function \( \partial_1 \) changes sign into \( M \), since \( \partial_1 \in H^1(M) \), the function \( f := |\partial_1| \) belongs to \( H^1(M) \) and \( |df| = |\partial_1| \) (see for example [Gi-Tr]), hence \( R(f) = R(\partial_1) \). So, the function \( f \) is a first eigenfunction of \( -\Delta_g + V \) on the manifold \( M \) which satisfies \( f \geq 0 \) on \( M \), \( f \) vanish into \( M \) and \( ( -\Delta_g + V ) f = \lambda_1(M) f \geq 0 \) on \( M \). Using the maximum principle [Pr-We], the function \( f \) can not achieved it minimum in an interior point of the manifold \( M \), hence \( f \) does not vanish on \( M \), so we obtain a contradiction. \( \square \)
3 Proof of the main theorem

3.1 Some other useful spaces

We define on the space $H^1(M)$ the $\star$-norm by:
\[
\|u\|_{\star}^2 := \int_M |u|^2 \, dV_g + \int_M V |u|^2 \, dV_g
\]
so, without difficulty we have:

**Proposition.** The application $\|\cdot\|_{\star}$ is a norm on the space $H^1(M)$; moreover, this norm is equivalent to the Sobolev norm $\|\cdot\|_{H^1(M)}$. In particular $H^1(M)$, $\|\cdot\|_{\star}$ is a Banach space.

Let us denote by $C^\infty_0(M-A)$ the set of smooth functions with compact support on $M-A$. For a compact subset $A$ of the manifold $M$ the usual Sobolev space $H^1_0(M-A)$ is defined by the closure of $C^\infty_0(M-A)$ for the norm $\|\cdot\|_{H^1(M)}$:
\[
H^1_0(M-A) := \overline{C^\infty_0(M-A)}.
\]

What happens when the set $A$ is not compact? For example if $A$ is a dense and countable subset of points of the manifold $M$, the space of test functions $C^\infty_0(M-A)$ is reduced to $\{0\}$. Therefore we cannot define the space $H^1_0(M-A)$. In this case, we propose a definition of $H^1_0(M-A)$ for any subset $A$ of $M$.

**Definition.** We define the Sobolev spaces $H^1_0(M-A)$ and $H^1_0(M-A)$ by:
\[
H^1_0(M-A) := \left\{ g \in H^1(M), g = 0 \text{ on a open neighborhood of } A \right\};
\]
\[
H^1_0(M-A) := \overline{H^1_0(M-A)}
\]
where the closure is for the norm $\|\cdot\|_{H^1(M)}$.

We have the:

**Proposition.** If the set $A$ is compact, the previous definition of the space $H^1_0(M-A)$ coincides with the usual ones.

**Proof.** Let $f \in H^1_0(M-A) := \overline{H^1_0(M-A)}$, then by definition: for all $\varepsilon \geq 0$ there exists $g \in H^1_0(M-A)$ such that $\|f-g\|_{H^1(M)} \leq \varepsilon$. So, we will show that we can write $g$ as a limit of sequence from the space $C^\infty_0(M-A)$ and conclude.

Since $g \in H^1_0(M-A)$ there exists an open set $U \supset A$ such that $g|_U = 0$. Consider two open sets $U_1$ and $U_2$ of the manifold $M$ such that:
\[
A \subset U_1, \quad M-U \subset U_2, \quad U_1 \cap U_2 = \emptyset;
\]
and consider also a function $\varphi \in \mathcal{D}(M)$ such that:
\[
\varphi|_{U_1} = 0, \quad \varphi|_{U_2} = 1.
\]

Of course, the function $\varphi$ belongs to the space $C^\infty_0(M-A)$. Next, since $g \in H^1_0(M-A) \subset H^1(M)$ and as the set of smooth functions $C^\infty(M)$ is dense in...
$H^1(M)$: there exists a sequence $(g_n)_n$ in $C_c^\infty(M)$ such that $\lim_{n \to +\infty} g_n = g$ for the norm $\| \cdot \|_{H^1(M)}$. Therefore we claim that $\lim_{n \to +\infty} \varphi g_n = g$ for the norm $\| \cdot \|_{H^1(M)}$. Indeed, start by, for all integer $n$:

$$\| \varphi g_n - g \|_{L^2(U)}^2 \leq \| g_n - g \|_{H^1}^2 + \| \varphi g_n - g \|_{H^1(U)}^2.$$ 

As a consequence, we have for all integer $n$:

$$\| \varphi g_n - g \|_{H^1(U)}^2 = \| \varphi g_n \|_{H^1(U)}^2$$

$$= \int_U |\varphi g_n|^2 \, dV_g + \int_U |d\varphi g_n + \varphi dg_n|^2 \, dV_g$$

$$\leq \int_U |\varphi g_n|^2 \, dV_g + \int_U |d\varphi|_g^2 \, dV_g + \int_U |\varphi|_g^2 \, dV_g + 2 \int_U |d\varphi g_n| \, dV_g$$

$$\leq \| \varphi \|_{L^2(U)}^2 \| g_n \|_{L^2(U)}^2 + \| d\varphi \|_{L^2(U)}^2 \| g_n \|_{L^2(U)}^2 + 2 \| d\varphi \|_{L^\infty(M)} \| g_n \|_{L^2(U)} \| d\varphi \|_{L^2(U)}$$

$$\leq \| \varphi \|_{L^2(U)}^2 \| g_n \|_{L^2(U)}^2 + \| d\varphi \|_{L^\infty(M)} \| g_n \|_{L^2(U)} \| d\varphi \|_{L^2(U)}$$

Next, we observe that, for all integer $n$:

$$\| \varphi g_n - g \|_{L^2(U)}^2 \leq \| g_n \|_{L^2(U)}^2 \left( 2 \| \varphi \|_{L^\infty}^2 + \| d\varphi \|_{L^\infty}^2 + 2 \| d\varphi \|_{L^\infty} \| \varphi \|_{L^\infty} \right).$$

As a consequence, we have for all integer $n$:

$$\| \varphi g_n - g \|_{L^2(U)}^2 \leq \| g_n - g \|_{H^1(M-U)}^2 + \| g_n \|_{H^1(U)}^2 \left( 2 \| \varphi \|_{L^\infty}^2 + \| d\varphi \|_{L^\infty}^2 + 2 \| d\varphi \|_{L^\infty} \| \varphi \|_{L^\infty} \right).$$

Now, it suffices to note that $\| g_n \|_{H^1(U)}^2 = \| g_n - g \|_{H^1(M)}^2 \leq \| g_n - g \|_{H^1(M)}^2$ (since $g = 0$ on the open set $U$) and we finally have:

$$\| \varphi g_n - g \|_{H^1(M)}^2 \leq$$

$$\| g_n - g \|_{H^1(M)}^2 \left( 1 + 2 \| \varphi \|_{L^\infty}^2 + \| d\varphi \|_{L^\infty}^2 + 2 \| d\varphi \|_{L^\infty} \| \varphi \|_{L^\infty} \right).$$

The sequence $(\varphi g_n)_n$ belong to $C_c^\infty(M - A)^N$, and since $\lim_{n \to +\infty} g_n = g$ for the norm $\| \cdot \|_{H^1(M)}$ the previous inequality implies $\lim_{n \to +\infty} \varphi g_n = g$ for the norm $\| \cdot \|_{H^1(M)}$.

So we have shown that every function $f \in H^1_0(M - A) := \overline{H^1_0(M - A)}$ is a limit (for the norm $\| \cdot \|_{H^1(M)}$) of a sequence of $C_c^\infty(M - A)$. Conversely, since $C_c^\infty(M - A) \subset H^1_0(M - A)$ we get:

$$H^1_0(M - A) := \overline{C_c^\infty(M - A)} \subset H^1_0(M - A) := \overline{H^1_0(M - A)}.$$
Let us also denote the spaces $H^1_1(M)$ and $S_A(M)$ by:

$$H^1_1(M) := \left\{ f \in H^1(M), \int_M f \, d\nu_S = 0 \right\};$$

and

$$S_A(M) := \left\{ u \in H^1_1(M), u - e_1 \in H^1_0(M - A) \right\}.$$

In the definition of the space $H^1_1(M)$ the condition $\int_M f \, d\nu_S = 0$ is analog to a boundary condition. We observe that the space $H^1_1(M)$ is a Hilbert space for the norm:

$$\|u\|_* := \int_M |du|^2 \, d\nu_S + \int_M V |u|^2 \, d\nu_S;$$

and $S_A(M)$ is just an affine closed subset of $H^1(M)$.

### 3.2 Schrödinger capacity

Next, we introduce the Schrödinger capacity of the set $A$;

**Definition.** Let us consider the Schrödinger capacity $\text{cap}(A)$ of the set $A$ defined by

$$\text{cap}(A) := \inf \left\{ \int_M |du|^2 \, d\nu_S + \int_M V |u|^2 \, d\nu_S, u \in S_A(M) \right\}. \quad (3.1)$$

Let us remark that: there exists an unique function $u_A \in S_A(M)$ such that

$$\text{cap}(A) = d_*(0, S_A(M)) := \inf \{ \|u\|_*, u \in S_A(M) \} = \|u_A\|_*.$$

In the following lemma we give the relationships between the capacity $\text{cap}(A)$, the functions $u_A$, $e_1$ and the Sobolev spaces $H^1_0(M - A)$, $H^1(M)$.

**Lemma.** For all subset $A$ of the manifold $M$, the following properties are equivalent:

(i) $\text{cap}(A) = 0$;

(ii) $u_A = 0$;

(iii) $e_1 \in H^1_0(M - A)$;

(iv) $H^1_1(M - A) = H^1(M)$.

**Proof.** It is clear from the formula (3.1) that (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii). Next, suppose the property (iii) holds: so there exists a sequence $(v_n)_n \in H^1_0(M - A)^N$ such that $\lim_{n \to +\infty} v_n = e_1$ for the norm $\| \cdot \|_{H^1(M)}$. So, for all smooth function $\varphi \in C^\infty(M)$ we have $\lim_{n \to +\infty} (\varphi v_n)/e_1 = \varphi$ for the norm $\| \cdot \|_{H^1(M)}$, indeed for all integer $n$:

$$\left\| \frac{\varphi v_n}{e_1} - \varphi \right\|_{H^1(M)}^2 = \int_M \left| \frac{\varphi v_n}{e_1} - \varphi \right|^2 \, d\nu_S + \int_M \left| d \left( \frac{\varphi v_n}{e_1} \right) - d\varphi \right|^2 \, d\nu_S.$$
First, we have for all integer $n$:

$$\int_M \frac{|\varphi v_n|}{e_1} - \varphi \|^2 d\mathcal{V}_g = \int_M \frac{1}{|e_1|} \varphi (v_n - e_1)^2 d\mathcal{V}_g$$

$$\leq \left\| \frac{1}{e_1} \right\|_\infty \| \varphi \|_\infty \| v_n - e_1 \|^2_{L^2(M)}$$

so, since $\lim_{n \to +\infty} v_n = e_1$ for the norm $\| \cdot \|_{H^1(M)}$ we have

$$\lim_{n \to +\infty} \int_M \frac{|\varphi v_n|}{e_1} - \varphi \|^2 d\mathcal{V}_g = 0.$$

On the other hand, for all integer $n$:

$$\int_M \left( \frac{|\varphi v_n|}{e_1} - \varphi \right) - d\varphi \|^2 d\mathcal{V}_g = \int_M \left( \frac{d(\varphi v_n) e_1 - \varphi v_n d e_1}{e_1^2} - d\varphi \right) d\mathcal{V}_g$$

$$= \int_M \left( \frac{1}{e_1^2} \right) \left| d(\varphi) v_n e_1 + \varphi d(v_n) e_1 - \varphi v_n d(e_1) - d(\varphi) e_1 \right|^2 d\mathcal{V}_g$$

$$\leq \left\| \frac{1}{e_1} \right\|_\infty \left\| d\varphi v_n e_1 - d\varphi e_1 \right\|_{L^2(M)}^2 + \left\| \varphi d v_n e_1 - \varphi v_n d e_1 \right\|_{L^2(M)}^2$$

$$\leq \left\| \frac{1}{e_1} \right\|_\infty \left[ \left\| d\varphi \right\|_\infty \| e_1 \|_\infty \| v_n - e_1 \|_{L^2(M)} + \left\| \varphi \right\|_\infty \| e_1 \|_\infty \| v_n - e_1 \|_{L^2(M)} \right]^2$$

$$\leq \left\| \frac{1}{e_1} \right\|_\infty \left[ \left\| d\varphi \right\|_\infty \| e_1 \|_\infty \| v_n - e_1 \|_{L^2(M)} + \left\| \varphi \right\|_\infty \| d e_1 \|_\infty \| v_n - e_1 \|_{L^2(M)} \right]^2$$

so, since $\lim_{n \to +\infty} v_n = e_1$ for the norm $\| \cdot \|_{H^1(M)}$ we have

$$\lim_{n \to +\infty} \int_M \left( \frac{|\varphi v_n|}{e_1} - \varphi \right) - d\varphi \|^2 d\mathcal{V}_g = 0.$$

Therefore, for all function $\varphi \in C^\infty(M)$ we have $\lim_{n \to +\infty} \frac{\varphi v_n}{e_1} = \varphi$ for the norm $\| \cdot \|_{H^1(M)}$.

Next, by density of $C^\infty(M)$ in $H^1(M)$: for all function $f \in H^1(M)$ we have $\lim_{n \to +\infty} f v_n = f$. Since the sequence $\left( \frac{f v_n}{e_1} \right)_n \in \mathcal{H}_0^1(M - A)^N$ we get finally that $f$ belongs to space $H^1_0(M - A)$. Finally, it is easy to see that $(iv) \Rightarrow (iii)$. \(\Box\)

An obvious consequence of this lemma is the following result:

**Proposition.** The spectrum of $-\Delta_g + V$ on the manifold $(M, g)$ and on the manifold $(M - A, g)$ are equal if and only if $\text{cap}(A) = 0$. 

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3.3 The Poincaré inequality

Now, let introduce the Poincaré inequality:

**Theorem.** If \( \lambda_1(M) \) denotes the first eigenvalue of the operator \(-\Delta_g + V\) on the manifold \((M,g)\), the following inequality

\[
\|u_A\|^2_{L^2(M)} \leq \frac{\text{cap}(A)}{\lambda_1(M)}
\]

holds for all subset \( A \) of \( M \).

**Proof.** The case \( \text{cap}(A) = 0 \) is an obvious consequence of the lemma in section 3.2. Suppose here that \( \text{cap}(A) > 0 \), then \( \|u_A\|^2_{L^2(M)} > 0 \). The first eigenvalue \( \lambda_1(M) \) of the operator \(-\Delta_g + V\) on the manifold \((M,g)\) is given by:

\[
\lambda_1(M) = \min_{E \subset H^1_0(M)} \max_{\phi \in E \quad \text{dim}(E) = 1 \quad \phi \neq 0} \frac{\int_M |d\phi|^2 + V|\phi|^2 \, dV_g}{\int_M |\phi|^2 \, dV_g}
\]

\[
= \min_{\phi \in H^1_0(M) \quad \phi \neq 0} \frac{\int_M |d\phi|^2 + V|\phi|^2 \, dV_g}{\int_M |\phi|^2 \, dV_g}
\]

Since \( u_A \) belongs to the space \( H^1_0(M) \) we get \( \lambda_1(M) \leq \frac{\text{cap}(A)}{\|u_A\|^2_{L^2(M)}} \).

3.4 The main theorem

Recall our main result:

**Theorem.** Let \((M,g)\) a compact Riemannian manifold. For all integer \( k \geq 1 \), there exists a constant \( C_k \) depending on the manifold of \((M,g)\) and the potential \( V \) such that for all subset \( A \) of \( M \) we have:

\[
0 \leq \lambda_k(M - A) - \lambda_k(M) \leq C_k \sqrt{\text{cap}(A)}.
\]

**Remark.** We can easily adapt the proof for a compact Riemannian manifold with boundary.

**Proof.** Let us denote by \((e_k)_{k \geq 1} \) an orthonormal basis of the space \( L^2(M) \) with eigenfunctions of the operator \(-\Delta_g + V\) on the manifold \((M,g)\). For all integer \( k \geq 1 \), we consider the sets

\[
F_k := \text{span}\{e_1,e_2,\ldots,e_k\}
\]

and

\[
E_k := \left\{ f \left(1 - \frac{u_A}{e_1}\right), \ f \in F_k \right\}.
\]

First, observe that \( E_k \subset H^1_0(M - A) \). For all \( j \in \{1,\ldots,k\} \) we introduce also the functions \( \phi_j := e_j \left(1 - \frac{u_A}{e_1}\right) \in E_k \).
hence by Poincaré inequality we have
\[ \lambda \leq B \]
where (and for the same reasons as in the study of \( \lambda \) and the eigenvalue \( B \).

Therefore, there exists \( \epsilon_k \in [0,1] \) (depends on the constant \( B \)) such that for all \( A \subset M \) we have:
\[ \cap(A) \leq \epsilon_k \Rightarrow \dim(E_k) = k \quad \forall j \in \{1, \ldots, k\}, \quad \bigg| \| \phi_j \|^2_{L^2(M)} - 1 \bigg| \leq D_k \sqrt{\cap(A)} \]

where (and for the same reasons as in the study of \( B \)) for all integer \( k \), the constant \( D_k \) depends only on \( M \) and \( V \), but \( D_k = D_k(M, V) \).

**Step 2:** Let a function \( \phi = f \left( 1 - \frac{d}{c} \right) \in E_k \), with \( f \in F \). Without loss generality we can assume that \( \| f \|^2_{L^2(M)} = 1 \), indeed : we have \( R(\phi) = R \left( \frac{\phi}{\| f \|^2_{L^2(M)}} \right) \) and in our context we interest in the Galois quotient of \( \phi \) (see the end of the final step of the proof).

Set \( v_A := \frac{\| f \|^2_{L^2(M)}}{c} \), we have:
\[ \int_M |d\phi|^2 dV_A = \int_M |df - d(fv_A)|^2 dV_A \]
\[ = \int_M |df|^2 dV_A + \int_M |dfv_A + f dv A|^2 dV_A - 2 \int_M df (fv_A) dV_A \]
\[ = \int_M |df|^2 dV_A + \int_M |dfv_A|^2 dV_A + \int_M |fv_A|^2 dV_A \]
Recall we have \( dv_A = \frac{du_A - u da}{e^i} \), and:

\[
\int_M |\phi|^2 dV_g = \int_M V |f|^2 dV_g - 2 \int_M V |f|^2 v_A dV_g + \int_M V |a|^2 dV_g
\]

hence

\[
\int_M |\phi|^2 dV_g + \int_M V |\phi|^2 dV_g = \int_M |f|^2 dV_g + \int_M V |f|^2 dV_g + \int_M |df|^2 dV_g
\]

\[
= \int_M |df|^2 dV_g + \int_M V |f|^2 dV_g
\]

\[
+ \int_M |df|^2 dV_g + \int_M V |f|^2 dV_g - 2 \left( \int_M |df|^2 v_A dV_g + \int_M V |f|^2 v_A dV_g \right)
\]

\[
- 2 \int_M dV_A f (1 - v_A) dV_g.
\]

\( \diamond \) Study of \( A(f) := \int_M |f|^2 dV_g + \int_M V |f|^2 dV_g \geq 0 \) : since \( f \in F_k \) we can write \( f = \sum_{i=1}^k x_i e_i \) where \( (x_i)_{1 \leq i \leq k} \in \mathbb{R}^k \) and with \( \sum_{i=1}^k x_i^2 = 1 \) (since \( \|f\|_{L^2(M)} = 1 \)), thus we get

\[
A(f) = \left( \sum_{i=1}^k x_i d e_i, \sum_{i=1}^k x_i d e_i \right)_{L^2(M)} + \left( \sqrt{\sum_{i=1}^k x_i e_i}, \sqrt{\sum_{i=1}^k x_i e_i} \right)_{L^2(M)}
\]

\[
= \sum_{i,j} x_i x_j \left( \langle d e_i, d e_i \rangle_{L^2(M)} + \int_M V e_i e_i dV_g \right)
\]

\[
= \sum_{i,j} x_i x_j \left( \langle - \Delta e_i, V e_i \rangle_{L^2(M)} + \int_M V e_i e_i dV_g \right)
\]

\[
= \sum_{i,j} x_i x_j \lambda_i(M) \langle e_i, e_i \rangle_{L^2(M)} = \sum_{i=1}^k x_i^2 \lambda_i(M) \leq \lambda_k(M).
\]

Hence, for all integer \( k \) and for all function \( f \in F_k \) such that \( \|f\|_{L^2(M)} = 1 \) we have
\[ 0 \leq A(f) \leq \lambda_k(M). \quad (3.3) \]

\[ \text{\# Study of } B(f) := \int_M \| df \|_{L^2}^2 dV_g : \text{ here } v_A = \frac{du}{du} \text{ and } dv_A = \frac{dv du}{\epsilon_1}, \]
so we get \[ B \leq \| df \|_{L^\infty}^2 \| v_A \|_{L^2(M)}^2 \]
and, with the Poincaré inequality:
\[ \| v_A \|_{L^2(M)}^2 \leq \left\| \frac{1}{\epsilon_1} \right\|_{\infty}^2 \| u_A \|_{L^2(M)}^2 \leq \left\| \frac{1}{\epsilon_1} \right\|_{\infty}^2 \frac{\text{cap}(A)}{\lambda_1(M)} \]

hence, for all integer \( k \), and for all function \( f \in F_k \) such that \( \| f \|_{L^2(M)} = 1 \) we have
\[ 0 \leq B(f) \leq E_k \text{cap}(A) \quad (3.4) \]
where \( E_k = E_k(e_1, \lambda_1(M)) > 0 \), moreover since the eigenfunction \( e_1 \) and the eigenvalue \( \lambda_1(M) \) depends only on \((M,g)\) and \( V \), for all integer \( k \) the constant \( E_k \) depends only on \((M,g)\) and \( V \), ie : \( E_k = E_k(M,V) \).

\[ \text{\# Study of } C(f) : \text{ here } C(f) \text{ is equal to } \int_M | f dv_A |^2 dV_g + \int_M V | v_A f |^2 dV_g. \]

Let us observe first \( C_1(f) : \)
\[ C_1(f) \leq \| f \|_{L^\infty}^2 \| dv_A \|_{L^2(M)}^2 \]
and
\[ \| dv_A \|_{L^2(M)}^2 = \int_M \left| \frac{du_A e_1 - u_A d e_1}{\epsilon_1^2} \right|^2 dV_g \]
\[ \leq \left\| \frac{1}{\epsilon_1} \right\|_{\infty}^2 \int_M | u_A e_1 - u_A d e_1 |^2 dV_g \]
\[ \leq \left\| \frac{1}{\epsilon_1} \right\|_{\infty}^2 \left( \int_M | u_A e_1 |^2 dV_g + 2 \int_M | u_A d e_1 e_1 u_A | dV_g + \frac{1}{\epsilon_1} \right) \]
\[ \leq \left\| \frac{1}{\epsilon_1} \right\|_{\infty}^2 \left( \| u_A \|^2_{L^2(M)} \| e_1 \|_{L^\infty}^2 + 2 \| d e_1 \|_{L^\infty} \| e_1 \|_{L^\infty} \| u_A \|_{L^2(M)} \| u_A \|_{L^2(M)} + \| d e_1 \|_{L^\infty}^2 \| u_A \|_{L^2(M)}^2 \right). \]

Next we have also:
\[ C_2(f) = \int_M V | v_A f |^2 dV_g \leq \| f \|_{L^\infty}^2 \int_M V | v_A |^2 dV_g \]
\[ \leq \| f \|_{L^\infty}^2 \left\| \frac{1}{\epsilon_1} \right\|_{\infty}^2 \int_M V | u_A |^2 dV_g. \]

Hence we get:
\[ C(f) \leq \| f \|_{L^\infty}^2 \left\| \frac{1}{\epsilon_1} \right\|_{\infty}^2 \left( \| u_A \|^2_{L^2(M)} \| e_1 \|_{L^\infty}^2 \right) + 2 \| d e_1 \|_{L^\infty} \| e_1 \|_{L^\infty} \| u_A \|_{L^2(M)} \| u_A \|_{L^2(M)} + \| d e_1 \|_{L^\infty}^2 \| u_A \|_{L^2(M)}^2 \]
\begin{align*}
&+ \|f\|_\infty^2 \left[ \frac{1}{e_1} \int_M V |u_A|^2 \, dV_g \right] \\
&\leq \|f\|_\infty^2 \left[ \frac{1}{e_1} \int_M (\|du_A\|_{L^2(M)}^2 + 2\|de_1\| \|e_1\| \|du_A\|_{L^2(M)} \|u_A\|_{L^2(M)} + \|de_1\|_{L^2(M)}^2) \|u_A\|_{L^2(M)}^2 \right] \\
&\leq \|f\|_\infty^2 \left[ \frac{1}{e_1} \int_M (\|du_A\|_{L^2(M)}^2 + \|V\|_{\infty} \|u_A\|_{L^2(M)}^2) \right] \\
&\leq \max \left( \|df\|_\infty \frac{1}{e_1} \int_M \frac{|u_A|}{e_1} \, dV_g, \left\| \frac{Vf}{e_1} \right\|_\infty \int_M |u_A| \, dV_g \right) \\
&\leq \max \left( \|df\|_\infty \frac{1}{e_1} \int_M \frac{|Vf^2|}{e_1} \, dV_g \right) \sqrt{\text{Vol}(M)} \|u_A\|_{L^2(M)} \\
&\leq \max \left( \|df\|_\infty \frac{1}{e_1} \int_M \frac{|Vf^2|}{e_1} \, dV_g \right) \sqrt{\text{Vol}(M)} \frac{\text{cap}(A)}{\lambda_1(M)}
\end{align*}

Hence, for all integer \( k \), and for all function \( f \in F_k \) such that \( \|f\|_{L^2(M)} = 1 \):

\begin{align}
|D(f)| \leq G_k \sqrt{\text{cap}(A)} \tag{3.6}
\end{align}

where (and for the same reasons as in the study of \( F \), see the constant \( F_k \)) for all integer \( k \), the constant \( G_k \) depends only on \( M \) and \( V \), ie \( G_k = G_k (M, V) \).

\( \diamond \) Study of \( |E(f)| \) : recall that \( E(f) = \int_M df \nu A f (1 - \nu_A) \, dV_g \), hence
\[ |E(f)| \leq \int_M |df| d\nu_A |f| d\nu_\delta + \int_M |df| \|f\|_\delta d\nu_\delta. \]

For the first term \( \int_M |df| d\nu_A |f| d\nu_\delta \) we have:

\[ \int_M |df| d\nu_A |f| d\nu_\delta \leq \|f\|_\delta \|df\|_\delta \sqrt{\text{Vol}(M)} \|d\nu_A\|_{L^2(M)}; \]

we have see in the study of \( C(f) \) that

\[ \|d\nu_A\|_{L^2}^2 \]

\[ \leq \left\| \frac{1}{\| e_1 \|_\infty} \right\|^2 \left( \|du_A\|_{L^2(M)}^2 \|e_1\|_\infty^2 + 2 \|de_1\|_\infty \|e_1\|_\infty \|du_A\|_{L^2(M)} \|u_A\|_{L^2(M)} + \|de_1\|_\infty \|u_A\|_{L^2(M)}^2 \right) \]

so with \( K := \|f\|_\delta \|df\|_\delta \sqrt{\text{Vol}(M)} \frac{1}{\| e_1 \|_\infty} \) we get

\[ \int_M |df| d\nu_A |f| d\nu_\delta \leq K \sqrt{\|du_A\|_{L^2(M)}^2 \|e_1\|_\infty^2 + 2 \|de_1\|_\infty \|e_1\|_\infty \sqrt{\text{cap}(A)} \sqrt{\text{cap}(A)} \frac{1}{\| e_1 \|_\infty} \lambda_1(M) + \|de_1\|_\infty^2 \text{cap}(A) \frac{1}{\| e_1 \|_\infty} \lambda_1(M)} \]

\[ \leq H_k \sqrt{\text{cap}(A)} \]

where (same reasons as above), for all integer \( k \), the constant \( H_k \) depends only on \( M \) and \( V \), i.e. \( H_k = H_k(M, V) \).

Next, for the second term \( \int_M |df| d\nu_A |f| d\nu_\delta \) we have:

\[ \int_M |df| d\nu_A |f| d\nu_\delta \leq \|df\|_\delta \|f\|_\delta \|d\nu_A\|_{L^2(M)} \|\nu_A\|_{L^2(M)} \]

\[ \leq \|df\|_\delta \|f\|_\delta \|d\nu_A\|_{L^2(M)} \frac{1}{\| e_1 \|_\infty} \|u_A\|_{L^2(M)} \]

\[ \leq \|df\|_\delta \|f\|_\delta \frac{1}{\| e_1 \|_\infty} \sqrt{\text{cap}(A)} \lambda_1(M) H_k \sqrt{\text{cap}(A)} \]

\[ \leq H_k' \text{cap}(A). \]

where (same reasons as above), for all integer \( k \), the constant \( H_k' \) depends only on \( M \) and \( V \), i.e. \( H_k' = H_k'(M, V) \).

So, for all integer \( k \):

\[ |E(f)| \leq H_k''(M, V). \]

(3.7)

where \( H_k'' := H_k''(M, V) \).

Finally, with the study of \( A(f), B(f), C(f), |D(f)| \) and \( |E(f)| \), for all integer \( k \),
for any function $\phi = f \left(1 - \frac{\|A\|^{2}}{\|A\|^{2}_{i}}\right) \in E_k$, with $f \in F_k$ such that $\|f\|_{L^{2}(M)} = 1$ we get:

$$\int_{M} |d\phi|^{2} dV_{S} + \int_{M} V |\phi|^{2} dV_{S} \leq \lambda_{k}(M) + I_{k}\left(\sqrt{\text{cap}(A)} + \text{cap}(A)\right) \tag{3.8}$$

where, for all integer $k$, the constant $I_{k}$ depends only on $M$ and $V$, i.e.: $I_{k} = I_{k}(M, V)$.

**Step 3:** Now we claim that: for all $A \subset M$ such that $\text{cap}(A) \leq \varepsilon_{k}$ and for any function $\phi \in E_{k}$ we have:

$$\|\phi\|_{L^{2}(M)}^{2} \geq 1 - I'_{k,M} \sqrt{\text{cap}(A)} \tag{3.9}$$

where, for all integer $k$, the constant $I'_{k,M}$ depend only on $M$ and $V$, i.e.: $I'_{k,M} = I'_{k,M}(M, V)$.

Indeed: let $\phi \in E_{k}$, we have seen below in step 1 that:

$$\text{cap}(A) \leq \varepsilon_{k} \Rightarrow \text{dim}(E_{k}) = k \text{ and } \forall j \in \{1, ..., k\}, \|\phi_{j}\|_{L^{2}(M)}^{2} - 1 \leq D_{k} \sqrt{\text{cap}(A)}$$

therefore, since $\phi \in E_{k}$, we can write $\phi = (1 - v_{A}) f$ with $f = \sum_{i=1}^{k} \alpha_{i} e_{i}$ where $(\alpha_{i})_{1 \leq i \leq k} \in \mathbb{R}^{k}$. As in the step two we can assume that $\|f\|_{L^{2}(M)} = 1$, hence we have $\sum_{i=1}^{k} \alpha_{i}^{2} = 1$. Next, compute $\|\phi\|_{L^{2}(M)}^{2}$:

$$\|\phi\|_{L^{2}(M)}^{2} = \left\|\sum_{i=1}^{k} (1 - v_{A}) \alpha_{i} e_{i}\right\|_{L^{2}(M)}^{2} = \left\|\sum_{i=1}^{k} \alpha_{i} \phi_{i}\right\|_{L^{2}(M)}^{2}$$

$$= \sum_{i=1}^{k} \alpha_{i}^{2} \|\phi_{i}\|_{L^{2}(M)}^{2} + \sum_{i,j \neq j} \alpha_{i} \alpha_{j} \langle \phi_{i}, \phi_{j}\rangle_{L^{2}(M)}.$$ And since

$$\sum_{i=1}^{k} \alpha_{i}^{2} \|\phi_{i}\|_{L^{2}(M)}^{2} = \sum_{i=1}^{k} \alpha_{i}^{2} \left[1 - 2 \int_{M} e_{i}^{2} v_{A} dV_{S} + \int_{M} e_{i}^{2} v_{A}^{2} dV_{S}\right]$$

$$= 1 - \sum_{i=1}^{k} \alpha_{i}^{2} \left[2 \int_{M} e_{i}^{2} v_{A} dV_{S} - \int_{M} e_{i}^{2} v_{A}^{2} dV_{S}\right]$$

$$= 1 - \sum_{i=1}^{k} \alpha_{i}^{2} \int_{M} e_{i}^{2} \left(2 v_{A} - v_{A}^{2}\right) dV_{S};$$

hence

$$\|\phi\|_{L^{2}(M)}^{2} = 1 - \sum_{i=1}^{k} \alpha_{i}^{2} \int_{M} e_{i}^{2} \left(2 v_{A} - v_{A}^{2}\right) dV_{S} + \sum_{i,j \neq j} \alpha_{i} \alpha_{j} \langle \phi_{i}, \phi_{j}\rangle_{L^{2}(M)}$$
we have seen in step 1 that, for \( \text{cap}(A) \) small enough:

\[
\left| \langle \phi_i, \phi_j \rangle_{L^2(M)} - \delta_{i,j} \right| \leq B_k \left( \sqrt{\text{cap}(A)} + \text{cap}(A) \right)
\]

hence, since all the \( (a_i)_{1 \leq i \leq k} \) are bounded in \( \mathbb{R} \), and for \( \text{cap}(A) \) small enough, we can find a constant \( B'_{k,M} \) which depends only on \( M \) and \( V \), ie \( B'_{k,M} = B'_k (M, V) \) such that, for \( \text{cap}(A) \) small enough:

\[
\left| \sum_{i,j \neq j} a_i a_j \langle \phi_i, \phi_j \rangle_{L^2(M)} \right| \leq B'_k \sqrt{\text{cap}(A)}
\]

and finally, in the same spirit as in the estimations in section 2, there exists a constant \( B''_{k,M} \) which depends only on \( M \) and \( V \), ie \( B''_{k,M} = B''_k (M, V) \) such that, for \( \text{cap}(A) \) small enough:

\[
\left| \sum_{i=1}^k a_i^2 \int_M \sqrt{2v_i - \sqrt{\lambda_i}} \, dV_S \right| \leq B''_k \sqrt{\text{cap}(A)}
\]

so finally we obtain:

\[
\| \phi \|_{L^2(M)}^2 \geq 1 - B''_k \sqrt{\text{cap}(A)}
\]

where the constant \( B''_k \) depend only on \( M \) and \( V \), ie \( B''_k := B''_k (M, V) \).

**Final step**: As a consequence from step 2 and 3, for all function \( \phi \in E_k \) we get:

\[
\frac{\int_M |d\phi|^2 \, dV_S + \int_M V |\phi|^2 \, dV_S}{\int_M \phi^2 \, dV_S} \leq \frac{\lambda_k(M) + L_k \left( \text{cap}(A) + \sqrt{\text{cap}(A)} \right)}{1 - B''_k \sqrt{\text{cap}(A)}}
\]

hence for \( \text{cap}(A) \) small enough (ie : \( \text{cap}(A) \leq \varepsilon_k \)) we have:

\[
\frac{\int_M |d\phi|^2 \, dV_S + \int_M V |\phi|^2 \, dV_S}{\int_M \phi^2 \, dV_S} \leq \lambda_k(M) + L_k \sqrt{\text{cap}(A)}
\]

where \( L_k := L_k (M, V) \). Next, since for all \( k \geq 1 \)

\[
\lambda_k(M - A) = \min_{E \subset H^1_0(M - A) \dim(E) = k} \max_{\varepsilon \neq 0} \frac{\int_M |d\phi|^2 \, dV_S + \int_M V |\phi|^2 \, dV_S}{\int_M \phi^2 \, dV_S}
\]

and since \( \phi \in H^1_0(M - A) \), we get for all \( k \geq 1 \)

\[
\lambda_k(M - A) \leq \frac{\int_M |d\phi|^2 \, dV_S + \int_M V |\phi|^2 \, dV_S}{\int_M \phi^2 \, dV_S} \leq \lambda_k(M) + C_k \sqrt{\text{cap}(A)}.
\]

And the statement of the theorem is established.
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Olivier Lablée

Université Grenoble 1-CNRS
Institut Fourier
UFR de Mathématiques
UMR 5582
