Limit constructions over Riemann surfaces and their parameter spaces, and the commensurability group actions

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1 Introduction

This paper is a sequel to the work we had started, with Dennis Sullivan, in our earlier publication [BNS]. In that work the universal commensurability mapping class group, $MC_\infty(X)$, was introduced, and it was shown that this group acts by biholomorphic modular transformations on the universal direct limit, $T_\infty(X)$, of Teichmüller spaces of compact surfaces. This space $T_\infty(X)$ was named the universal commensurability Teichmüller space.

Let $X$ be a compact connected oriented surface of genus at least two. We recall that the elements of the universal commensurability mapping class group, $MC_\infty(X)$, arise from closed circuits, starting and terminating at $X$, in the graph of all topological coverings of $X$ by other compact connected oriented topological surfaces. The edges of the circuit represent covering morphisms, and the vertices represent the corresponding covering surfaces. The group $MC_\infty(X)$ is naturally isomorphic with the group of virtual automorphisms of the fundamental group $\pi_1(X)$ [BN1]. A virtual automorphism is an isomorphism between two finite index subgroups of $\pi_1(X)$; two such isomorphisms are identified if they agree on some finite index subgroup.

The Teichmüller space for $X$, denoted $T(X)$, is the space of all conformal structures on $X$ quotiented by the group of diffeomorphisms of $X$ path connected to the identity diffeomorphism. Any unramified covering $p : Y \to X$ induces an embedding $T(p) : T(X) \to T(Y)$, defined by sending a complex structure on $X$ to its pull back on $Y$ using $p$. The complex analytic “ind-space”, $T_\infty(X)$, is the direct limit of the finite dimensional Teichmüller spaces of connected coverings of $X$, the connecting maps of the direct system being the maps $T(p)$. (This inductive system of Teichmüller spaces is built over the directed set of all unramified covering surfaces of $X$. The precise definitions are in the pointed category; see [BNS] and Section 2 below.)
As stated earlier, there is a natural action of $MC_\infty(X)$ on $T_\infty(X)$. For exact definitions we refer to section III.1 below. Let $CM_\infty(X)$ denote the image of $MC_\infty(X)$, arising via this action, in the holomorphic automorphism group of $T_\infty(X)$. The group $CM_\infty(X)$ was called the universal commensurability modular group in [BNS].

We prove in Theorem 3.14 of this paper that the action of $MC_\infty(X)$ on $T_\infty(X)$ is effective. In other words, the projection of $MC_\infty(X)$ onto $CM_\infty(X)$ is an isomorphism.

As noted in [BNS], the direct limit space $T_\infty(X)$ is the universal parameter space of compact Riemann surfaces, and it can be interpreted as the space of transversely locally constant complex structures on the universal hyperbolic solenoid $H_\infty(X) := \tilde{X} \times_{\pi_1(X)} \hat{\pi}_1(X)$.

Here $\tilde{X}$ is the universal cover of $X$ and $\hat{\pi}_1(X)$ is the profinite completion of $\pi_1(X)$. The transverse direction mentioned above refers to the fiber direction for the natural projection of $H_\infty(X)$ onto $X$.

In this article, to any compact connected Riemann surface $X$, we associate a subgroup of $MC_\infty(X)$, which we denote by $\text{ComAut}(X)$, that may be called the commensurability automorphism group of $X$. The members of $\text{ComAut}(X)$ arise from closed circuits, again starting and ending at $X$, whose edges represent holomorphic coverings amongst compact connected Riemann surfaces. Indeed, this group $\text{ComAut}(X)$, turns out to be precisely the stabilizer, of the (arbitrary) point $[X] \in T_\infty(X)$, for the action of the universal commensurability modular group $CM_\infty(X)$.

As we mentioned earlier, each point of $T_\infty(X)$ represents a complex structure on the universal hyperbolic solenoid, $H_\infty(X)$. The base leaf in $H_\infty(X)$ is the path connected subset $\tilde{X} \times_{\pi_1(X)} \hat{\pi}_1(X)$. We show that $\text{ComAut}(X)$ acts by holomorphic automorphisms on this complex analytic solenoid, preserving the base leaf. In fact, we demonstrate that $\text{ComAut}(X)$ is the full group of base leaf preserving holomorphic automorphisms of $H_\infty(X)$.

The study of the isotropy group associated to any point of $T_\infty(X)$ makes direct connection with the well-known theory of commensurators of the corresponding uniformizing Fuchsian groups. The commensurator, denoted by $\text{Comm}(G)$, of a Fuchsian group, $G \subset PSL(2,\mathbb{R})$, is the group consisting of those Möbius transformations in $PSL(2,\mathbb{R})$ that conjugate $G$ onto a group that is “commensurable” (≡ finite-index comparable) with $G$ itself. In other words, $g \in \text{Comm}(G)$ if and only if $G \cap gGg^{-1}$ is of finite index in both $G$ and $gGg^{-1}$.

Let $X = \Delta/G$, where $\Delta$ is the unit disk, and $G$ is a torsion free co-compact Fuchsian group. We will demonstrate in Section 4 that $\text{ComAut}(X)$ is canonically isomorphic to $\text{Comm}(G)$.

If the genus of $X$ is at least three, then the subgroup of $MC_\infty(X)$ that fixes the stra-
tum $T(X) \subset T_\infty(X))$ pointwise, is shown to be precisely a copy of $\pi_1(X)$, [Proposition 4.4]. The group $\pi_1(X)$ is realized as a subgroup of the group of virtual automorphisms of $\pi_1(X)$ by the inner conjugation action.

Now, the commensurator of the Fuchsian group, $G$, associated to a generic compact Riemann surface (of genus at least three), is known to be simply $G$ itself [Gr1], [Sun]. On the other hand, it is a deep result following from the work of G.A. Margulis [Mar], that $\text{Comm}(G)$ (for $G$ any finite co-volume Fuchsian group) is dense in $\text{PSL}(2, \mathbb{R})$ if and only if the group $G$ was an arithmetic subgroup. We explain these matters in Section 4.

The countable family of co-compact arithmetic Fuchsian groups play a central rôle in one result of this paper, which we would like to highlight. In Theorem 5.1 we assert that the biholomorphic action of $\text{ComAut}(X)$ on the complex solenoid $H_\infty(X)$ turns out to be ergodic precisely when the Fuchsian group uniformizing $X$ is arithmetic. The proof of this theorem utilizes strongly the result of Margulis quoted above. Here the ergodicity is with respect to a natural measure that exists on each of these complex analytic solenoids $H_\infty(X)$. In fact, the product measure on $\tilde{X} \times \tilde{\pi}_1(X)$, arising from the Poincaré measure on $\tilde{X}$ and the Haar measure on $\tilde{\pi}_1(X)$, is actually invariant under the action of $\pi_1(X)$. Consequently it induces the relevant natural measure on $H_\infty(X)$. (There are also elegant alternative ways to construct this measure; see Section 5.)

Another aspect regarding the applications of the group $MC_\infty(X)$, as well as of its isotropy subgroups, arises in lifting these actions to vector bundles over $T_\infty(X)$. In fact, the space $T_\infty(X)$ supports certain natural holomorphic vector bundles, where each fiber can be interpreted as the space of holomorphic $i$-forms on the corresponding complex solenoid. The action of the modular group $MC_\infty(X)$ does in fact lift canonically to these bundles, and the action on the relevant fiber of the isotropy subgroup, $\text{ComAut}(X)$, is studied. The very basic question, asking whether or not the action of the commensurability automorphism group is effective on the corresponding infinite dimensional fiber, is settled in the affirmative [Theorem 6.5]. It is also shown, [section VI.3], that the action of the commensurability modular group preserves a natural Hermitian structure on the bundles.

Some of the results presented here were announced in [BN2].

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2 The universal limit objects $H_\infty$ and $\mathcal{T}_\infty$

The Teichmüller space $\mathcal{T}(X)$ parametrizes isotopy classes of complex structures on any compact, connected, oriented topological surface, $X$. We recall that if $\text{Conf}(X)$ is the space of all complex structures over $X$ compatible with the orientation, and $\text{Diff}_0(X)$ is the group of all diffeomorphisms of $X$ homotopic to the identity map, then $\mathcal{T}(X) = \text{Conf}(X)/\text{Diff}_0(X)$.

Fix a compact connected oriented surface $X$. Consider any orientation preserving unbranched covering over $X$:

\begin{equation}
(2.1) \quad p : Y \longrightarrow X,
\end{equation}

where $Y$ is allowed to be an arbitrary compact connected oriented surface. Associated to $p$ is a proper injective holomorphic immersion

\begin{equation}
(2.2) \quad \mathcal{T}(p) : \mathcal{T}(X) \longrightarrow \mathcal{T}(Y),
\end{equation}

which is defined by mapping (the isotopy class of) any complex structure on $X$ to its pull back by $p$. It is easy to check that the injective map

\[ p^* : \text{Conf}(X) \longrightarrow \text{Conf}(Y), \]

obtained using pull back of complex structures by $p$, actually descends to a map $\mathcal{T}(p)$ between the Teichmüller spaces. The map $\mathcal{T}(p)$ respects the Teichmüller metrics; the Teichmüller metric determines the quasiconformal-distortion. The association of $\mathcal{T}(p)$ to $p$ is a contravariant functor from the category of surfaces and covering maps to the category of complex manifolds and holomorphic maps.

We will now recall some basic constructions from [BNS] and [BN1]. For our present purposes we first need to carefully explain the various related directed sets over which our infinite limit constructions will proceed.

II.1. Directed sets and the solenoid $H_\infty$: Henceforth assume that $X$ has genus greater than one, and also fix a base point $x \in X$. Fix a pointed universal covering of $X$:

\begin{equation}
(2.3) \quad u : (\tilde{X}, \star) \longrightarrow (X, x),
\end{equation}

and canonically identify $G := \pi_1(X, x)$ as the group of deck transformations of the covering map $u$. Note that any two choices of the pointed universal covering are canonically isomorphic.

Let $\mathcal{I}(X)$ denote the directed set consisting of all arbitrary unbranched pointed coverings $p : (Y, y) \longrightarrow (X, x)$ over $X$. Note that if $p$ is a (unpointed) covering of
degree \( N \), as in (2.1), then there are \( N \) distinct members of \( \mathcal{I}(X) \) corresponding to it, each being a copy of \( p \) but with the \( N \) distinct choices of a base point \( y \in p^{-1}(x) \) on \( Y \). The partial order in \( \mathcal{I}(X) \) is determined in the obvious way by base-point preserving factoring of covering maps. More precisely, given another pointed covering \( q : (Z, z) \rightarrow (X, x) \) in \( \mathcal{I}(X) \), we say \( q \geq p \) if and only if there is a pointed covering map

\[
r : (Z, z) \rightarrow (Y, y)
\]

such that \( p \circ r = q \). It is important to note that a factoring map, when exists, is uniquely determined because we work in the pointed category.

Let \( \text{Sub}(G) \) denote the directed set of all finite index subgroups in \( G \), ordered by reverse inclusion. In other words, for two subgroups \( G_1, G_2 \subset G \), we say \( G_1 \geq G_2 \) if and only if \( G_1 \subseteq G_2 \).

There are order-preserving canonical maps each way:

\[
A : \mathcal{I}(X) \rightarrow \text{Sub}(G) \quad \text{and} \quad B : \text{Sub}(G) \rightarrow \mathcal{I}(X)
\]

The map \( A \) associates to any \( p \in \mathcal{I}(X) \), as above, the image of the monomorphism \( p_* : \pi_1(Y, y) \rightarrow \pi_1(Y, y) \). The latter map, \( B \), sends the subgroup \( H \in \text{Sub}(G) \) to the pointed covering \( (\tilde{X}/H, \ast) \rightarrow (X, x) \), with \( \ast \) denoting, of course, the \( H \)-orbit of the base point in the universal cover \( \tilde{X} \). These covers arising from quotienting \( \tilde{X} \) by subgroups of \( G \), provide canonical models, up to isomorphism, for arbitrary members of \( \mathcal{I}(X) \). Notice that the composition \( A \circ B \) is the identity map on \( \text{Sub}(G) \). Consequently, \( A \) is surjective, and \( B \) maps \( \text{Sub}(G) \) injectively onto a cofinal subset in \( \mathcal{I}(X) \). As mentioned, this cofinal subset contains a representative for every isomorphism class of pointed covering of \( X \).

It is also convenient to introduce the directed cofinal subset, \( \mathcal{I}_{\text{gal}}(X) \) comprising only the normal (Galois) coverings in \( \mathcal{I}(X) \). The corresponding cofinal subset in \( \text{Sub}(G) \) is denoted \( \text{Sub}_{\text{nor}}(G) \), and it consists of all the normal subgroups in \( G \) of finite index.

**Remark 2.5 :** For the construction of the projective and inductive limit objects, \( H_{\infty} \) and \( \mathcal{T}_{\infty} \) mentioned in the Introduction, we note that we can work with any of the directed sets \( (\mathcal{I}(X), \mathcal{I}_{\text{gal}}(X), \text{Sub}(G), \text{or} \text{Sub}_{\text{nor}}(G)) \) as introduced above; by utilizing the relationships described above, it follows that the actual limit objects will not be affected by which directed set we happen to use. It is, however, a rather remarkable thing, that to define the commensurability groups of automorphisms on these very limit objects (see Section 3 below), one is forced to work with the sets like \( \mathcal{I}(X) \), or, equivalently, with the actual monomorphisms between surface groups; in other words, working with just their image subgroups in \( G \) does not suffice.

Denote by \( H_\infty(X) \) the inverse limit, \( \varprojlim X_\alpha \), where \( \alpha \) runs through the index set \( \mathcal{I}(X) \), and \( X_\alpha \) being the covering surface (the domain of the map \( \alpha \)). Introduced in
the space \( H_{\infty}(X) \) is known as the *universal hyperbolic solenoid*. The “universality” of this object resides in the evident but crucial fact that these spaces \( H_{\infty}(X) \), as well as their Teichmüller spaces \( \mathcal{T}(H_{\infty}(X)) \), do not really depend on the choice of the base surface \( X \). If we were to start with a pointed surface \( X' \) of different genus (both genera being greater than one), we could pass to a common covering surface of \( X \) and \( X' \) (always available), and hence the limit spaces we construct would be isomorphic. More precisely, there is a natural isomorphism between \( H_{\infty}(X) \) and \( H_{\infty}(X') \) whenever we fix a surface \((Y, y)\) together with pointed covering maps of it onto \( X \) and \( X' \).

We are therefore justified in suppressing \( X \) in our notation and referring to \( H_{\infty}(X) \) as simply \( H_{\infty} \). For each surface \( X \) there is a natural projection

\[
(2.6) \quad p_{\infty} : H_{\infty}(X) \rightarrow X
\]

induced by coordinate projection from \( \prod X_{\alpha} \) onto \( X \). Each fiber of \( p_{\infty} \) is a perfect, compact and totally disconnected space — homeomorphic to the Cantor set. The space \( H_{\infty}(X) \) itself is compact and connected, but not path connected. The path components of \( H_{\infty}(X) \) are christened “leaves”. Each leaf, equipped with the “leaf-topology” (which is strictly finer than the subspace topology it inherits from \( H_{\infty}(X) \)), is a simply connected two-manifold; when restricted to any leaf, the map \( p_{\infty} \) is an universal covering projection on \( X \). There are uncountably many leaves in \( H_{\infty}(X) \), and each is dense in \( H_{\infty}(X) \). The base point of \( \tilde{X} \) determines a base point in \( H_{\infty}(X) \). The base leaf is, by definition, the one containing the base point.

An alternative construction of \( H_{\infty}(X) \) that we will be using repeatedly is as follows.

First let us recall the definition of the profinite completion of any group \( G \). For us \( G \) will be \( \pi_1(X) \).

The *profinite completion* of \( G \) is the projective limit

\[
(2.7) \quad \hat{G} = \varprojlim (G/H),
\]

where the limit is taken over all \( H \in \text{Sub}_{\text{nor}}(G) \). For \( G \) a surface group, note that each \( G/H \) can be identified with the deck transformation group of the finite Galois cover corresponding to \( H \). This group \( \hat{G} \), with the discrete topology being assigned on each of the finite groups \( G/H \), is homeomorphic to the Cantor set. There is a natural homomorphism from \( G \) into \( \hat{G} \) induced by the projections of \( G \) onto \( G/H \). Since \( G \) is residually finite, one sees that this homomorphism of \( G \) into \( \hat{G} \) is injective.

We will also require another useful description of the profinite completion group \( \hat{G} \). Consider the Cartesian product

\[
\prod_{H \in \text{Sub}_{\text{nor}}(G)} G/H,
\]
which, using Tychonoff’s theorem, is a compact topological space. There is a natural homomorphism of $G$ into it; the injectivity of this homomorphism is equivalent to the assertion that $G$ is residually finite. The closure of $G$ in this product space is the profinite completion $\hat{G}$.

Denoting by $G$ the fundamental group, $\pi_1(X, x)$, of the pointed surface $(X, x)$, the universal cover, $u : \tilde{X} \to X$, has the structure of a principal $G$-bundle over $X$. It is not difficult to see that the solenoid $H_\infty(X)$ can also be defined as the principal $\hat{G}$-bundle

\begin{equation}
 p_\infty : H_\infty(X) \to X,
\end{equation}

obtained by extending the structure group of the principal $G$-bundle, defined by the universal covering, using the natural inclusion homomorphism of $G$ into its profinite completion $\hat{G}$. The typical fiber of the projection $p_\infty$ is the Cantor group $\hat{G}$.

In other words, the solenoid $H_\infty(X)$ is identified with the quotient of the product $\tilde{X} \times \hat{G}$ by a natural properly discontinuous and free $G$-action:

\begin{equation}
 H_\infty(X) \equiv \tilde{X} \times_G \hat{G}
\end{equation}

Here $G$ acts on $\tilde{X}$ by the deck transformations, and it acts by left translations on $\hat{G}$.

One further notes that the projection:

\begin{equation}
 \Pi_G : \tilde{X} \times \hat{G} \to \tilde{X} \times_G \hat{G}
\end{equation}

enjoys the property that its restriction to any slice of the form $\tilde{X} \times \{\hat{\gamma}\}$, for arbitrarily fixed member $\hat{\gamma} \in \hat{G}$, is a homeomorphism onto a path connected component (i.e., a “leaf”) of $\tilde{X} \times_G \hat{G}$. It will be useful to remark that a point $(z, \hat{\gamma}) \in \tilde{X} \times \hat{G}$ maps by $\Pi_G$ into the base leaf, if and only if there exists a fixed $g \in G$ such that, for every $H \in \text{Sub}_{\text{nor}}(G)$ the coset of $G/H$ listed as the $H$-coordinate in $\hat{\gamma}$ is $gH$.

We leave it to the reader to check these elementary matters, as well as to trace through the canonical identifications between the various descriptions of the solenoid that we have offered above.

\textbf{II.2. The Teichmüller functor and $T_\infty$} : We now apply the “Teichmüller functor” to the tower of coverings parametrized by $I(X)$; that produces an inductive system of Teichmüller spaces, and we set:

\begin{equation}
 T_\infty \equiv T_\infty(X) = \varinjlim T(X_\alpha)
\end{equation}

This is the direct limit of finite dimensional Teichmüller spaces, where $\alpha$ runs through the same directed set $I(X)$. The connecting morphisms in the direct system are the immersions defined in (2.2), and the limit object is called the \textit{universal commensurability Teichmüller space}. 

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This space $\mathcal{T}_\infty$ is an universal parameter space for compact Riemann surfaces of genus at least two. It is a metric space with a well-defined Teichmüller metric. Indeed, $\mathcal{T}_\infty(X)$ also carries a natural Weil-Petersson Kähler structure obtained from scaling the Weil-Petersson pairing on each finite dimensional stratum $\mathcal{T}(X_\alpha)$. (See [BNS], [BN1], for details.) We will now interpret $\mathcal{T}_\infty(X)$ as the space of a certain class of complex structures on the universal hyperbolic solenoid.

Local (topological) charts for the solenoid, $H_\infty(X)$, are “lamination charts”, namely homeomorphisms of open subsets of the solenoid to $\{\text{disc}\} \times \{\text{transversal}\}$. Now, by definition, a complex structure on the solenoid is the assignment of a complex structure on each leaf, such that the structure changes continuously in the transverse (Cantor fiber) direction. Details may be found in [S]. When the solenoid is equipped with a complex structure we say that we have a complex analytic solenoid, alternatively called a Riemann surface lamination. The space of leaf-preserving isotopy classes of complex structures on $H_\infty$ constitutes a complex Banach manifold — the Teichmüller space of the hyperbolic solenoid — and it is denoted by $\mathcal{T}(H_\infty) = \mathcal{T}(H_\infty(X))$ ([S], [NS]). In fact, this Banach manifold contains $\mathcal{T}_\infty(X)$, as we explain next, and is actually the Teichmüller metric completion of the direct limit object $\mathcal{T}_\infty(X)$. (Note [NS].)

Each point of $\mathcal{T}_\infty(X)$ corresponds to a well-defined Riemann surface lamination. Indeed, fix a complex structure on $X_\alpha$, with $\alpha : X_\alpha \rightarrow X$ being any member of $\mathcal{I}(X)$. Then for every $\beta \in \mathcal{I}(X)$, with $\beta \geq \alpha$, we obtain a complex structure on $X_\beta$ by pulling back the complex structure on $X_\alpha$, using the pointed (factorizing) covering map $\sigma : X_\beta \rightarrow X_\alpha$. (The factorization $\beta = \alpha \circ \sigma$, in the pointed category, uniquely determines $\sigma$.) It can be now verified that there is a unique complex structure on $H_\infty(X)$ enjoying the property that the natural projection of $H_\infty(X)$ onto any $X_\beta$, where $\beta \geq \alpha$, is holomorphic with respect to the complex structure on $X_\beta$ just constructed.

The complex structures so obtained on $H_\infty(X)$ are more than just continuous in the transversal direction. They are, in fact, precisely those complex structures that are transversely locally constant. This demonstrates that $\mathcal{T}_\infty(X)$ is naturally a subset of $\mathcal{T}(H_\infty(X))$.

3 The commensurability modular action on $\mathcal{T}_\infty$

The group of topological self correspondences of $X$, which arise from undirected cycles of finite pointed coverings starting at and returning to $X$, gave rise to the universal commensurability mapping class group $MCinX$. This group acts by holomorphic automorphisms on $\mathcal{T}_\infty(X)$ — and that is one of the main themes of this paper.

III.1. The $H_\infty$ and $\mathcal{T}_\infty$ functors on finite covers: We proceed to recall in
some detail a chief construction introduced in [BNS], and followed up in [BN1], [BN2], [BN3]. Indeed, a quite remarkable yet evident fact about the construction of the genus-independent limit object $H_\infty$, is that every member of $\mathcal{I}(X)$, namely every pointed covering, $p : Y \to X$, induces a natural, homeomorphism, mapping the base leaf to the base leaf, between the two copies of the universal solenoid obtained from the two different choices of base surface:

$$(3.1) \quad H_\infty(p) : H_\infty(X) \to H_\infty(Y).$$

In fact, any compatible string of points, one from each covering surface, representing a point of $H_\infty(X)$, becomes just such a compatible string representing an element of $H_\infty(Y)$ — simply by discarding the coordinates in all the strata that did not factor through $p$.

As for $T_\infty$, the same cover $p$ also induces a bijective identification between the two corresponding models of $T_\infty$ built with these two different choices of base surface:

$$(3.2) \quad T_\infty(p) : T_\infty(Y) \to T_\infty(X)$$

The mapping above corresponds to the obvious order preserving map of directed sets, $\mathcal{I}(Y)$ to $\mathcal{I}(X)$, defined by $\theta \mapsto p \circ \theta$. The image of $\mathcal{I}(Y)$ is cofinal in $\mathcal{I}(X)$. That induces a natural morphism between the direct systems that define $T_\infty(Y)$ and $T_\infty(X)$, respectively; the limit map is the desired $T_\infty(p)$. That the map $T_\infty(p)$ is invertible follows simply because the pointed coverings with target $Y$ are cofinal with those having target $X$.

The bijection $T_\infty(p)$ is easily seen to be a Teichmüller metric preserving biholomorphism.

Since both the maps $H_\infty(p)$ as well as $T_\infty(p)$ are invertible, it immediately follows that every (undirected) cycle of pointed coverings starting and ending at $X$ produces:

(i) a self-homeomorphism, which preserves the base leaf (as a set), of $H_\infty(X)$ on itself;

(ii) a biholomorphic automorphism of $T_\infty(X)$.

The above observation, [BNS, Section 5], leads one to define the universal commensurability mapping class group of $X$, denoted $MC_\infty(X)$, as the group of equivalence classes of such undirected cycles of topological coverings starting and terminating at $X$. Notice that $MC_\infty(X)$ is a purely topological construct, whose definition has nothing to do with the theory of Teichmüller spaces. The equivalence relation among undirected polygons of pointed coverings is obtained, as explained both in [BNS] and in [BN1], by replacing any edge (i.e., a covering) by any factorization of it, thus allowing us to reroute through fiber product diagrams.
Indeed, by repeatedly using appropriate fiber product diagrams, we know from the papers cited above that any cycle (with arbitrarily many edges) is equivalent to just a two-edge cycle. Thus every element of $MC_\infty(X)$ arises from a finite topological “self correspondence” (two-arrow diagram) on $X$.

Fix any such self correspondence given by an arbitrary pair of pointed, orientation preserving, topological coverings of $X$, say:

\[(3.3)\quad p : Y \to X \quad \text{and} \quad q : Y \to X\]

We have the following induced \textit{automorphism}:

\[(3.4)\quad A_{(p,q)} = T_\infty(q) \circ T_\infty(p)^{-1}\]

of $T_\infty(X)$. The set of all automorphisms of $T_\infty(X)$ arising this way constitute a group of biholomorphic automorphisms of $T_\infty(X)$, which is called the \textit{universal commensurability modular group} $CM_\infty(X)$, acting on $T_\infty(X)$ as well as on its Banach completion $\mathcal{T}(H_\infty(X))$.

In Theorem 3.14 below we will prove that this natural map from $MC_\infty(X)$ to $CM_\infty(X)$ is an isomorphism of groups.

**III.2. The virtual automorphism group of $\pi_1(X)$:** The group of \textit{virtual automorphisms} of any group $G$, $\text{Vaut}(G)$, comprises equivalence classes of isomorphisms between arbitrary finite index subgroups of $G$. To be explicit, an element of $\text{Vaut}(G)$ is represented by an isomorphism $a : G_1 \to G_2$, where $G_1$ and $G_2$ are finite index subgroups of $G$; another such isomorphism $b : G_3 \to G_4$ is identified with $a$ if and only if there is a subgroup $G' \subset G_1 \cap G_3$ of finite index in $G$, such that $a$ and $b$ coincide on $G'$.

For us $G$ will always be the fundamental group, $\pi_1(X, x)$, of a closed oriented surface.

Let $\text{Vaut}^+(G) \subset \text{Vaut}(G)$ denote the subgroup of index two that consists of the orientation preserving elements. Given a virtual automorphism of the surface group $G$, it is possible to check whether it is in $\text{Vaut}^+(G)$ by looking at the action on the second (group) cohomology level. We will be dealing only with the subgroup $\text{Vaut}^+(G)$.

We recall a proposition from [BNI]. For any pointed covering $p : (Y, y) \to (X, x)$, the induced monomorphism $\pi_1(Y, y) \to \pi_1(X, x)$ will be denoted by $\pi_1(p)$.

**Proposition 3.5.** [BNI, Proposition 2.10] The group $\text{Vaut}^+(\pi_1(X))$, is naturally isomorphic to $MC_\infty(X)$. The element of $MC_\infty(X)$ determined by the pair of covers $(p, q)$ as in (3.3), corresponds to the virtual automorphism represented by the isomorphism: $\pi_1(q) \circ \pi_1(p)^{-1} : H \to K$, where $H = \text{Image}(\pi_1(p))$, $K = \text{Image}(\pi_1(q))$. 

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Let Möbius($S^1$) denote the group of (orientation preserving) diffeomorphisms of $S^1$ defined by the Möbius transformations that map $\Delta$ on itself. Recall that for any Fuchsian group $F$ (of the first kind), the Teichmüller space $\mathcal{T}(F)$ consists of equivalence classes of all monomorphisms $\alpha : F \to \operatorname{Möbius}(S^1)$ with discrete image. Two monomorphisms, say $\alpha$ and $\beta$, are in the same equivalence class if $\beta = \operatorname{Conj}(A) \circ \alpha$, where $\operatorname{Conj}(A)$ denotes the inner automorphism of Möbius($S^1$) achieved by an arbitrary Möbius transformation $A$ in that group. In fact, the monomorphism $\alpha : F \to \operatorname{Möbius}(S^1)$, representing any point of $\mathcal{T}(F)$ can be chosen to be the unique “Fricke-normalized” one in its class — see [Ab], [N], and III.5 below.

We will need to describe explicitly the action of $MC_\infty$ on $\mathcal{T}_\infty$, when we model $\mathcal{T}_\infty$ via any co-compact, torsion free, Fuchsian group $\Gamma$. Namely:

\[(3.6) \quad \mathcal{T}_\infty(\Gamma) = \varinjlim \mathcal{T}(H)\]

the inductive limit being over the directed set $\operatorname{Sub}(\Gamma)$. The connecting maps, $\mathcal{T}(H_1) \to \mathcal{T}(H_2)$, whenever $H_2 \subset H_1(\subset \Gamma)$, are obvious.

Fix an element $[\lambda] \in \operatorname{Vaut}^+(\Gamma)$ represented by the isomorphism $\lambda : H \to K$, as in the setting above. The present aim is to describe the automorphism:

$[\lambda]_* : \mathcal{T}_\infty(\Gamma) \to \mathcal{T}_\infty(\Gamma)$

Now, any isomorphism $\lambda$ of a Fuchsian group $H$ onto a Fuchsian group $K$ determines the following natural “allowable isomorphism” between their Teichmüller spaces:

\[(3.7) \quad \mathcal{T}(\lambda) : \mathcal{T}(K) \to \mathcal{T}(H)\]

defined by precomposition of monomorphisms by $\lambda$. See [N, Section 2.3.12] for the details.

Evidently, $\lambda$ induces an order-preserving isomorphism between the directed sets $\operatorname{Sub}(H)$ and $\operatorname{Sub}(K)$. The definitions of $\mathcal{T}_\infty(H)$ and $\mathcal{T}_\infty(K)$ as inductive limits proceed over these two directed sets, respectively.

Moreover, if $\mathcal{T}(Z)$ is the Teichmüller space of any such group $Z$, where $Z \in \operatorname{Sub}(H)$, then there is the corresponding allowable isomorphism, induced by $\lambda$, between the following two Teichmüller spaces:

\[(3.8) \quad \tau^Z_\lambda : \mathcal{T}(Z) \to \mathcal{T}(\lambda(Z))\]

The collection of all these allowable isomorphisms, (as $Z$ runs through $\operatorname{Sub}(H)$), defines a morphism of direct systems, thus resulting in a map, say $\mathcal{T}_\infty(\lambda)$, mapping isomorphically $\mathcal{T}_\infty(H)$ onto $\mathcal{T}_\infty(K)$.
But for any finite index subgroup $G \subset \Gamma$, the cofinality of $\text{Sub}(G)$ in $\text{Sub}(\Gamma)$ certainly gives us an isomorphism of the corresponding limit Teichmüller spaces:

$$I_{G \subset \Gamma} : \mathcal{T}_\infty(G) \rightarrow \mathcal{T}_\infty(\Gamma)$$

It follows by tracing through the definitions, that the assigned $[\lambda] \in \text{Vaut}^+(\Gamma)$ acts on $\mathcal{T}_\infty(\Gamma)$ by the commensurability modular automorphism:

$$(3.9) \quad [\lambda]_* = I_{K \subset \Gamma} \circ \mathcal{T}_\infty(\lambda) \circ I_{H \subset \Gamma}^{-1}$$

### III.3. Representation of $\text{Vaut}^+(\pi_1(X))$ within $\text{Homeo}_{q.s.}(S^1)$

We will utilize the result, [BN1], that $\text{Vaut}(\pi_1(X))$ allows certain natural representations in the homeomorphism group of the unit circle $S^1$, by the theory of boundary homeomorphisms. The general theory of quasisymmetric boundary homeomorphisms that arise in Teichmüller theory can be found, for example, in [N, Chapter 2].

Let us fix any Riemann surface structure on $X$. Then the universal covering $\tilde{X}$ can be conformally identified as the unit disc $\Delta \subset \mathbb{C}$, with the base point being mapped to $0 \in \Delta$; the cover transformations group, $G$, then becomes a co-compact torsion free Fuchsian group, say $\Gamma$:

$$\Gamma \subset \text{Möbius}(S^1) \equiv \text{PSU}(1,1) \equiv \text{Aut}(\Delta)$$

By $\text{Möbius}(S^1)$ we simply mean the restrictions of the holomorphic automorphisms of the unit disc ($\text{PSU}(1,1)$) to the boundary circle.

Let $[\rho] \in \text{Vaut}^+(\Gamma)$ be represented by the group isomorphism $\rho : H \rightarrow K$, where $H$ and $K$ are Fuchsian subgroups of finite index within $\Gamma$. A description of the boundary homeomorphism associated to this virtual automorphism is as follows: Consider the natural map, $\sigma_\rho$, that $\rho$ defines from the orbit of the origin ($= 0 \in \Delta$) under $H$ to the orbit of 0 under $K$. In other words, the map

$$\sigma_\rho : H(0) \rightarrow K(0)$$

is defined by $h(0) \mapsto \rho(h)(0)$. But each orbit under these co-compact Fuchsian groups $H$ and $K$ accumulates everywhere on the boundary $S^1$. Therefore, it follows that the map $\sigma_\rho$ extends by continuity to define a homeomorphism of $S^1$. That homeomorphism is quasisymmetric, and it is the one that we naturally associated to the element $[\rho]$ of $\text{Vaut}^+(\Gamma)$ — see [BN1].

Thus, we have a faithful representation

$$\Sigma : \text{Vaut}^+(\pi_1(X)) \rightarrow \text{Homeo}_{q.s.}(S^1).$$
Here \( \text{Homeo}_{q.s.}(S^1) \) denotes the group of orientation preserving quasisymmetric homeomorphisms of the circle. The image of \( \Sigma \) is exactly the \textit{group of virtual normalizers of} \( \Gamma \) \textit{amongst quasisymmetric homeomorphisms}. By this we mean

\[
V_{\text{norm}}_{q.s.}(\Gamma) = \{ f \in \text{Homeo}_{q.s.}(S^1) : f \text{ conjugates some finite index subgroup of } \Gamma \text{ to another such subgroup of } \Gamma \}
\]

See \([BN1]\) for details.

\textit{Remark :} This faithful copy, (3.10), of \( MC_\infty(X) \) demonstrates that the normalizers in \( \text{Homeo}_{q.s.}(S^1) \) of every finite index subgroup of \( \Gamma \) sit naturally embedded in \( V_{\text{norm}}_{q.s.}(\Gamma) \cong MC_\infty(X) \). Any such normalizer, say \( N_{q.s.}(H) \), for \( H \in \text{Sub}(\Gamma) \), is precisely the “extended modular group” for the Fuchsian group \( H \), as defined by Bers \([B2]\). As \( H \) ranges over all the finite index subgroups of \( \Gamma \), these extended modular groups sweep through the “mapping class like” \(([BN1], [Od]\) elements of \( MC_\infty(X) \).

\[ III.4. \] \( MC_\infty \) as subgroup of Bers’ universal modular group : The representation of \( \text{Vaut}^+(\pi_1(X)) \) above allows us to consider the action of \( MC_\infty \) on \( T_\infty \) via the usual type of right translations by quasisymmetric homeomorphisms, as is standard for the classical action of the universal modular group on the universal Teichmüller space.

Recall that the \textit{Universal Teichmüller space} of Ahlfors-Bers, \( T(\Delta) \), is the homogeneous space of right cosets (i.e., Möbius\((S^1)\) acts by post-composition):

\[
T(\Delta) := \text{Möbius}(S^1)\backslash\text{Homeo}_{q.s.}(S^1)
\]

The coset of \( \phi \in \text{Homeo}_{q.s.}(S^1) \), viz. \([\text{Möbius}(S^1)]\phi \), will be denoted by \([\phi]\). There is a natural base point \([Id]\) \( \in T(\Delta) \) given by the coset of the identity homeomorphism \( Id : S^1 \to S^1 \). The Teichmüller space of an arbitrary Fuchsian group \( G \) embeds naturally in \( T(\Delta) \) as the cosets of those quasisymmetric homeomorphisms that are compatible with \( G \). Compatibility \((B2)\) of \( \phi \) with \( G \) means that \( \phi G\phi^{-1} \subset \text{Möbius}(S^1) \).

Since \( T(\Delta) \) is a homogeneous space for \( \text{Homeo}_{q.s.}(S^1) \), the group \( \text{Homeo}_{q.s.}(S^1) \) acts (in fact, by biholomorphic automorphisms) on this complex Banach manifold, \( T(\Delta) \). The action is by right translation (i.e., by precomposition). In other words, each \( f \in \text{Homeo}_{q.s.}(S^1) \) induces the automorphism:

\[
f_* : T(\Delta) \longrightarrow T(\Delta) ; \quad f_*([\phi]) = [\phi \circ f]
\]

This action on \( T(\Delta) \) is classically called the universal modular group action (see \([B2], [B3]\), or [N, Chapter 2]). Let us note here that \textit{every} non-trivial element of \( \text{Homeo}_{q.s.}(S^1) \), including all the non-identity elements of the conformal group \( \text{Möbius}(S^1) \), acts \textit{non-trivially} on the homogeneous space \( T(\Delta) \). Of course, the set of universal modular transformations that keep the base point fixed are precisely those that arise from \( \text{Möbius}(S^1) \).
Having fixed the Fuchsian group $\Gamma$ uniformizing the reference compact Riemann surface $X$, we see that a copy of the universal commensurability Teichmüller space, $T_\infty(X)$, appears embedded in $T(\Delta)$ as follows:

\begin{equation}
T_\infty(X) \cong T_\infty(\Gamma) = \{[\phi] \in T(\Delta) : \phi \in \text{Homeo}_{q.s.}(S^1) \text{ is compatible} \text{ with some finite index subgroup of } \Gamma\}
\end{equation}

Indeed, one notes that $T_\infty(\Gamma)$ is precisely the union, in $T(\Delta)$, of the Teichmüller spaces of all the finite index subgroups of $\Gamma$. The reader will observe that (3.13) is simply (3.6) — but now embedded within the ambient space $T(\Delta)$.

This embedded copy of $T_\infty$ (see [NS]) was called the “$\Gamma$-tagged” copy. In connection with the discussion (see II.2 above) of the full space of complex structures on the solenoid, it is relevant to point out that the topological closure in $T(\Delta)$ of any such $\Gamma$-tagged copy of $T_\infty$ is a model of $T(H_\infty)$.

Finally then we will need the important fact, (we refer again to [BN1]), that the action of $MC_\infty(X)$ on $T_\infty(X)$ coincides with the action, by right translations, of the subgroup of the universal modular group corresponding to $V_{\text{norm}}q.s.(\Gamma) \subset \text{Homeo}_{q.s.}(S^1)$. In fact, the universal modular transformations of $T(\Delta)$ induced by the members of $V_{\text{norm}}q.s.(\Gamma)$ preserve the subset $T_\infty(\Gamma)$, and, under the canonical identification of $T_\infty(X)$ with $T_\infty(\Gamma)$, these transformations on $T_\infty(\Gamma)$ correspond to the universal commensurability modular transformations acting on $T_\infty(X)$.

**III.5. $MC_\infty$ acts effectively on $T_\infty(X)$:** We are ready to prove a very basic fact that will be important in what follows:

**Theorem 3.14.** $\text{Vaut}^+(\pi_1(X))$ acts effectively on $T_\infty(X)$. In other words, the natural homomorphism $MC_\infty(X) \to CM_\infty(X)$ has trivial kernel.

To prove the theorem we need the following lemma:

**Lemma 3.15.** Assume that the genus of $X$ is at least three. Suppose that $p, q : (Y, y) \to (X, x)$ are any two pointed unbranched finite coverings such that the induced monomorphisms $\pi_1(p)$ and $\pi_1(q)$ are unequal, and further that they remain unequal even after any inner conjugation in either the domain or the target group is applied. Then the corresponding induced embeddings $T(p)$ and $T(q)$, (of $T(X)$ into $T(Y)$), must be unequal.

**Proof of Lemma 3.15 :** Let $X$ have genus $g, g \geq 3$. Fix any complex structure on $X$, and let $X = \Delta/\Gamma$ where $\Gamma$ is the uniformizing Fuchsian group (isomorphic to $\pi_1(X, x)$). Let $H$ and $K$ denote the images of the monomorphisms $\pi_1(p)$ and $\pi_1(q)$, respectively. Denote by $\lambda$:

\begin{equation}
\lambda := \pi_1(q) \circ \pi_1(p)^{-1} : H \to K
\end{equation}
the non-trivial isomorphism of $H$ onto $K$ that is given to us. By assumption it represents some non-identity element of $\text{Vaut}^+(\Gamma)$.

By Nielsen’s theorem, there exists a based diffeomorphism, say $\Theta$, of $\Delta/H$ onto $\Delta/K$ whose action on $\pi_1$ is given by $\lambda$. Lift this diffeomorphism to the universal covering, $\Delta$, and consider the homeomorphism, say $\theta : S^1 \rightarrow S^1$ defined by the boundary action of the lift. The map $\theta$ is exactly the quasisymmetric homeomorphism that we associated — see III.3 — as the boundary homeomorphism that corresponds to the given element $[\lambda] \in \text{Vaut}^+(\Gamma)$. In fact, the isomorphism $\lambda$ is realized as conjugation by $\theta$ of members of $H$:

\begin{equation}
\lambda(h) = \theta \circ h \circ \theta^{-1}, \quad h \in H.
\end{equation}

Note that if we extend $\theta$ as a quasiconformal homeomorphism of $\Delta$ by using the conformally natural Douady-Earle extension operator, then the extension (we will still call it $\theta$) will satisfy the above equation not only on the boundary circle but throughout the unit disc.

We recall that there is a unique “Fricke-normalized” monomorphism $\alpha : F \rightarrow \text{M"obius}(S^1)$, representing any point $[\alpha]$ of the Teichmüller space $\mathcal{T}(F)$, (see section III.2 above), once we have chosen standard generators $\{A_1, B_1, A_2, B_2, \ldots, A_g, B_g\}$ for the genus $g$ Fuchsian group $F$. More precisely, we want to normalize the positions of three of the four fixed points (on the unit circle) for the hyperbolic Möbius transformations that get assigned by the monomorphism to $B_g$ and $A_g$. For exact details see [N, Section 2.5.2] or [AH, Chapter II]. Such a normalization eliminates the 3-parameter Möbius ambiguity in identifying Teichmüller equivalent monomorphisms.

Now, since $H$ and $K$ are subgroups of $\Gamma$, we have natural embeddings between the Teichmüller spaces given by restricting the monomorphisms of $\Gamma$ into $\text{M"obius}(S^1)$ to the respective subgroups:

\begin{equation}
E_H : \mathcal{T}(\Gamma) \rightarrow \mathcal{T}(H) \quad \text{and} \quad E_K : \mathcal{T}(\Gamma) \rightarrow \mathcal{T}(K).
\end{equation}

Let us identify the surface $Y$ with $\Delta/H$; then the embeddings between Teichmüller spaces $\mathcal{T}(\Gamma) \rightarrow \mathcal{T}(Y)$ that we desire to compare are given by:

\begin{equation}
\mathcal{T}(p) = E_H : \mathcal{T}(\Gamma) \rightarrow \mathcal{T}(H), \quad \mathcal{T}(q) = \mathcal{T}(\lambda) \circ E_K : \mathcal{T}(\Gamma) \rightarrow \mathcal{T}(H)
\end{equation}

where $\mathcal{T}(\lambda)$ is the allowable isomorphism between Teichmüller spaces described in (3.7). The lemma will be demonstrated by proving that the two mappings above are unequal.

By assumption, the map $\lambda$ which acts on $H$ as follows : $h \mapsto \theta \circ h \circ \theta^{-1}$, is not the identity map on $H$. Therefore, let $h_1$ be a primitive element of $H$ which is not equal to $\lambda(h_1)$. Let us choose a set of $2g$ generators $\{A_1, B_1, A_2, B_2, \ldots, A_g, B_g\}$ for $\Gamma$ such that $A_1 = h_1$, satisfying the standard single relation $\Pi[A_j, B_j] = 1$. 

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Now let the Fricke-normalized monomorphism $\sigma : \Gamma \rightarrow \text{Möbius}(S^1)$ represent an arbitrary point of $\mathcal{T}(\Gamma)$. To obtain a contradiction, we may assume that for every such $\sigma$ we have:

$$\sigma(\theta \circ h \circ \theta^{-1}) = \sigma(h)$$

for all $h \in H$. But, in order to produce a normalized monomorphism $\sigma$, we can (essentially arbitrarily) assign hyperbolic elements of $\text{Möbius}(S^1)$ to the first $2g - 3$ generators of $\Gamma$, and then fill in for the last three generators judiciously in order to maintain the relation in the group, and to keep valid the Fricke normalization. (Vide [N, p. 134 ff].) Since $\Gamma$ is a surface group of genus at least three, (so that it has six or more generators in its standard presentation), it is easy to produce some (in fact, infinitely many) normalized monomorphisms $\sigma$ so that $\sigma(h_1) \neq \sigma(\lambda(h_1))$. This completes the proof of the lemma.

**Remark:** We avoid genus 2, because if we take $p$ and $q$ to be the hyperelliptic involution and the identity homeomorphism, respectively, on $Y = X =$ a surface of genus two, then, in fact, $\mathcal{T}(p) = \mathcal{T}(q)$. This is the well-known non-effectiveness of the action of the mapping class group in genus 2 (vide [N, Section 2.3.7]). But, in the context of compact hyperbolic Riemann surfaces, that is the one and only case when a nontrivial mapping class group element induces the identity on the Teichmüller space.

**Proof of Theorem 3.14:** First let us note the crucial fact that the limit constructions we are pursuing are independent of the genus of the base surface. In fact, if $\alpha : X_\alpha \rightarrow X$ is a covering in $\mathcal{I}(X)$, then $\alpha$ sets up a natural isomorphism between the pairs:

$$(3.20) \quad (\mathcal{T}_\infty(X), MC_\infty(X)) \quad \text{and} \quad (\mathcal{T}_\infty(X_\alpha), MC_\infty(X_\alpha))$$

Therefore, to understand the action of the universal commensurability mapping class group, we may, and therefore do, take $X$ to be of genus greater than or equal to three.

In view of the description of $MC_\infty(X)$ as the group $\text{Vaut}^+ (\Gamma)$ given in Proposition 3.5, a copy of the group $\Gamma$ itself sits embedded inside $MC_\infty(X)$. Indeed, each element of $\Gamma$ determines a virtual automorphism of $\Gamma$ by inner conjugation. Let us first take care of these elements of $\text{Vaut}^+ (\Gamma)$, which play a rather special rôle.

Given any non-identity element $\gamma \in \Gamma$, we utilize the residual finiteness of the surface group $\Gamma$ to find a finite index subgroup $H \subset \Gamma$ so that $\gamma$ is not in $H$. Then in the direct limit construction of $\mathcal{T}_\infty(\Gamma)$, it follows easily that the automorphism of $\mathcal{T}_\infty(\Gamma)$ arising from $\gamma$ will already act nontrivially on the stratum $\mathcal{T}(H)$.

It is also clear by a similar argument that every non-identity mapping class like element of $MC_\infty(X)$, (see the Remark following (3.10), and [BN1]), will act non-trivially on $\mathcal{T}_\infty(X)$. We have already disposed of members of $\Gamma$ itself, therefore, let the element under scrutiny be given by $\sigma \in \text{Vnorm}_{q.s.}(\Gamma) \setminus \Gamma$. By assumption $\sigma$ is
mapping class like, — one therefore sees easily that it must preserve some appropriate stratum \( \mathcal{T}(X) \) (as a set), and will act as the standard modular transformation on that Teichmüller space. But the classical genus \( g \) mapping class group, say \( MC_g \), is known to act effectively, (see [B2], [N, Chapter 2.3]), on the genus \( g \) Teichmüller space \( \mathcal{T}_g \), for every \( g \geq 3 \). That takes care of \( \sigma \). Actually, by essentially the above argument, we can see that every member of \( V_{\text{norm}}(\Gamma) \cap \text{Möbius}(S^1) \) acts non-neutrally on \( \mathcal{T}_\infty(X) \).

We now come to the interesting case when the element of \( MC_\infty(X) \) being investigated is not of the above types. Take therefore a nontrivial element of \( MC_\infty(X) \) determined by a self correspondence \((p, q)\), namely by the two coverings \( p \) and \( q \) from \((Y, *)\) onto \((X, *)\), as in (3.3). The condition on the element of \( MC_\infty(X) \) so determined implies that the hypothesis of Lemma 3.15 may be assumed satisfied.

Let \( t \in \mathcal{T}_\infty(X) \) be a point that is represented as a Riemann surface \( X_\mu \), (\( \mu \) being a complex structure on \( X \)), in the base stratum \( \mathcal{T}(X) \). Remember that in the direct limit construction of \( \mathcal{T}_\infty(X) \) (over the directed set \( I(X) \)), there are different strata, each corresponding to a copy of \( \mathcal{T}(Y) \), but tagged by every distinct choice of finite pointed covering map \( Y \to X \). Let us agree to denote the stratum \( \mathcal{T}(Z) \) corresponding to any such pointed covering \( r : Z \to X \), by \( \mathcal{T}(Z)_r \).

Thus the point \( t = [X_\mu] \) may be represented in the stratum \( \mathcal{T}(Y)_p \) as \( Y_{p, \mu} \in \mathcal{T}(Y)_p \), (and, of course, also as \( Y_{q, \mu} \in \mathcal{T}(Y)_q \)). Here, in self-evident notation, we are writing \( p^* \mu \) for the complex structure on \( Y \) obtained as the pull back of \( \mu \) via \( p \).

Now from the work in sections III.1 and III.2 we note that the automorphism \( A_{(p, q)} \) of \( \mathcal{T}_\infty(X) \) determined by the self correspondence \((p, q)\) on \( X \) acts as follows: take any point \( t = [Y_{q, \mu} \in \mathcal{T}(Y)_q] \); then

(3.21) \[
A_{(p, q)}(t) = [Y_{p, \mu} \in \mathcal{T}(Y)_q]
\]

is valid. The point of the equation (3.21) is that we have arranged both the element \( t \), as well as its image under the \( MC_\infty(X) \)-automorphism, \( A_{(p, q)} \), to be represented in one and the same stratum. We deduce immediately that if \( t = A_{(p, q)}(t) \), for each \( t \) coming from the base stratum \( \mathcal{T}(X)(\subset \mathcal{T}_\infty(X)) \), then the mappings \( \mathcal{T}(p) \) and \( \mathcal{T}(q) \) coincide (as embeddings of \( \mathcal{T}(X) \) into \( \mathcal{T}(Y) \)). Thus Lemma 3.15 is contradicted, and we are through. Notice that we have actually proved that each \( A_{(p, q)} \) of this type is already non-trivial when restricted to the base stratum. \[\square\]

Remark 3.22 : In the context of the model \( \mathcal{T}_\infty(\Gamma) \) of \( \mathcal{T}_\infty(X) \), which we exhibited as an embedded complex analytic “ind-space” within the Ahlfors-Bers universal Teichmüller space \( \mathcal{T}(\Delta) \), the result of Theorem 3.14 asserts that each one of the universal modular transformations of \( \mathcal{T}(\Delta) \) arising from any non-trivial member of \( V_{\text{norm}}(\Gamma) \cap \text{Möbius}(S^1) \) must move nontrivially points of \( \mathcal{T}_\infty(\Gamma) \). A little reflection shows that although it is easy to create some arbitrary quasisymmetric homeomorphism \( \phi \), such that \([\phi]\) is actually
moved (by any given universal modular transformation), it is not quite trivial to produce such a $\phi$ with the extra property that it is compatible with the Fuchsian group $\Gamma$ [in the sense that it conjugates some finite index subgroup of $\Gamma$ to again such a subgroup].

4 Isotropy subgroups of $MC_\infty$

Fix an arbitrary point $t \in T_\infty(X)$. We want to study the stabilizer subgroup at this point of the universal commensurability modular action.

Utilizing again the observation, (3.20), above regarding the natural isomorphism of pairs, it is evident that we lose no generality by assuming that the point $t$ is already represented in the base stratum $T(X)$.

Therefore, in this section, $X = X_\mu$ will be a Riemann surface, with the complex structure $\mu$. The universal covering $\tilde{X}$ is then identified biholomorphically with $\Delta$, and we let $G$ denote the Fuchsian group uniformizing $X$.

IV.1. Commensurable subgroups of $\text{Möbius}(S^1)$ : Two subgroups, say $H$ and $K$, of Aut($\Delta$) $\equiv$ $\text{Möbius}(S^1)$ are called commensurable if $H \cap K$ is of finite index in both $H$ and $K$. We define the commensurability automorphism group for the Riemann surface $X = \Delta/G$, denoted as ComAut($X$) $\equiv$ ComAut($\Delta/G$) by setting :

$$\text{(4.1)} \quad \text{ComAut}(X) \equiv \{ g \in \text{Aut}(\Delta) | \ gGg^{-1} \text{ and } G \text{ are commensurable} \}$$

This group, which is the commensurator of the Fuchsian group $G$, will be identified by us as arising from the finite holomorphic self correspondences of the Riemann surface $X$. Namely, the members of ComAut($X$) appear from undirected cycles of holomorphic covering maps that start and end at $X$.

IV.2. Isotropy in $MC_\infty$ and commensurators : Let the point of $T_\infty(X)$ represented by the Riemann surface $X = X_\mu$ be denoted by $[X]$.

**Theorem 4.2.** (a) The subgroup ComAut($X$) of Aut($\Delta$) is the virtual normalizer of $G$ among the Möbius transformations of $\Delta$. Namely :

$$\text{ComAut}(X) = \text{Vnorm}_{\text{Aut}(\Delta)}(G) = \text{Aut}(\Delta) \cap \text{Vnorm}_{q.s.}(G)$$

(b) The group ComAut($X$) is naturally isomorphic to the isotropy subgroup at $[X]$ for the action of $MC_\infty(X)$ on $T_\infty(X)$.

**Proof.** Part (a) : Suppose $\gamma \in \text{ComAut}(X)$. Let $C_\gamma : \text{Möbius}(S^1) \to \text{Möbius}(S^1)$ denote the inner conjugation in Möbius($S^1$) given by $\phi$, i.e., $C_\gamma(A) = \gamma \circ A \circ \gamma^{-1}$. Set
\[ H = G \cap C_\gamma^{-1}(G) \text{ and } K = G \cap C_\gamma(G). \] It is evident that \( \gamma \) will conjugate \( H \) onto \( K \), and it is easily proved that both \( H \) and \( K \) are finite index subgroups of \( G \). This shows that \( \text{ComAut}(X) \) lies in the virtual normalizer of \( G \) in the group \( \text{Möbius}(S^1) \).

Conversely, if a finite index subgroup, \( H \subset G \) is carried, under conjugation by some Möbius transformation \( \rho \), to another finite index subgroup \( K \subset G \), then it is an easy exercise to demonstrate that \( \rho G \rho^{-1} \) and \( G \) are commensurable. \( \square \)

Part (b): To prove part (b) we choose to model the universal commensurability Teichmüller space, \( T_\infty(X) \), as the subset \( T_\infty(G) \) of the universal Teichmüller space \( T(\Delta) \), (we explained this in section III.4 above). Now, the identifying isomorphism \( T_\infty(X) \to T_\infty(G) \) maps the point \([X] \in T_\infty(X)\) to the base point \([1] \in T_\infty(G)\). Thus the problem of identifying the stabilizer of the point \([X]\) in the commensurability modular action \( CM_\infty(X) \) is equivalent to the problem of identifying the subgroup of \( \text{Vnorm}_{q,s}(G) \) that fixes, (in its action by universal modular transformation on \( T(\Delta) \)), the base point \([1]\). But, as explained in III.4, the only universal modular transformations that keep \([1]\) fixed arise from Möbius(\( S^1 \)). Consequently, the stabilizer subgroup in \( CM_\infty(X) \) of the point \([X]\) is canonically identified with the intersection of \( \text{Vnorm}_{q,s}(G) \) with Möbius(\( S^1 \)). By part (a) above, this intersection is exactly the commensurator of the Fuchsian group \( G \). The proof of the theorem is finished. \( \square \)

\textbf{ComAut}(X) and holomorphic circuits of coverings :} We will now delineate the crucial point that we have mentioned already in the Introduction; namely, that the isotropy subgroup \( \text{ComAut}(X) \) arises from undirected cycles of holomorphic covers that start and end at \( X \).

We retain the notations of part (a) of the proof of Theorem 4.2. Thus choose any Möbius transformation \( \gamma : \Delta \to \Delta \) that is a member of \( \text{ComAut}(\Delta/G) \). We know that there exist two finite index subgroups \( H \) and \( K \) in \( G \) such that the conjugation map \( C_\gamma \), carries \( H \) isomorphically onto \( K \). It follows that \( \gamma \) descends to a biholomorphic isomorphism, say

\[ \gamma_* : Y \to Z \]

between the compact Riemann surfaces \( Y = \Delta/H \) and \( Z = \Delta/K \).

Let \( \alpha : Y \to X \) and \( \beta : Z \to X \) denote the holomorphic finite covers corresponding to the group inclusions \( H \subset G \) and \( K \subset G \). Then the chosen element \( \gamma \) of \( \text{ComAut}(X) \) corresponds to the circuit of holomorphic covering morphisms given by \( \alpha \) (with arrow reversed), followed by \( \gamma_* \), followed by \( \beta \).

Thus, \( \gamma \) in \( \text{ComAut}(X) \) is represented by the holomorphic two-arrow diagram arising from the two holomorphic coverings \( p = \alpha \) and \( q = \beta \circ \gamma_* \) from the Riemann surface \( Y \) onto the Riemann surface \( X \). This should be carefully compared with (3.3) of Section 3.
It is interesting to note from the above, that the element of the stabilizer subgroup in $MC_{\infty}(X)$, arising from any such finite circuit of holomorphic and unramified coverings, is well-defined without reference to base points on the Riemann surfaces involved. A little reflection shows that this phenomenon is due to the well-known *rigidity* of holomorphic covering maps between compact hyperbolic Riemann surfaces.

**IV.3. The commensurator of a generic Fuchsian group** : We have now described the isotropy subgroup of the commensurability modular action at every point $[Y] \in T_\infty(X)$, where $Y$ is any compact hyperbolic Riemann surface, as precisely the commensurator of the Fuchsian group, $G$, uniformizing $Y$. Note therefore that this isotropy is always *infinite*, — it always contains a copy of the fundamental group $\pi_1(Y)$. In fact, $G$ is contained in its normalizer, $N(G)$ (in the Möbius group), while $N(G)$ in its turn is contained in the virtual normalizer of $G$ — namely:

$$G \subset N(G) \subset \text{Comm}(G)$$

Clearly, $\text{Comm}(G)$ contains the normalizer subgroup in the Möbius group, say $N(H)$, for any finite index subgroup $H \subset G$. The union of these $N(H)$, over all subgroups $H \in \text{Sub}(G)$, constitute the *mapping class like* members of $\text{Comm}(G) = \text{ComAut}(Y)$

Now, $G$ is of course normal in $N(G)$, and it is well-known (as well as rather easy to see), that the quotient $N(G)/G = \text{HolAut}(X)$ is the group of usual holomorphic automorphisms of $X$. This quotient is always a finite group (for any compact hyperbolic Riemann surface) ($\text{order}(N(G)/G) \leq 84(g-1)$). Indeed, $\text{HolAut}(X)$ is non-trivial only on some union of lower dimensional subvarieties in each moduli space $M_g$, for $g \geq 3$.

The *isotropy subgroup* at any $[X] \in T_g$, of the action of the classical Teichmüller modular group $MC_g$ on the $(3g-3)$ dimensional Teichmüller space $T_g$, is identifiable as the group $\text{HolAut}(X)$. See [B1] and [N], and references therein, for details.

In our infinite limit situation we are noting therefore the following interesting parallel with the above classical theory. Indeed, we are asserting that the *isotropy subgroup* at each $[Y] \in T_\infty$, of the action of the *universal commensurability modular group* $MC_\infty$, is canonically identified with the new group $\text{ComAut}(Y)$. This group, as we said, is always infinite, and, as we saw in Section 3, the action of $MC_\infty$ is always effective. We note that $G$ need not be normal in $\text{ComAut}(\Delta/G) = \text{Comm}(G)$ — so that a quotient group (as in the case of $N(G)/G$) cannot be defined in general.

**Generically** $\text{ComAut}(\Delta/G) = \text{Comm}(G) = G$ : By [Gr1], a co-compact torsion free Fuchsian group representing a compact Riemann surface from the moduli space $M_g$, $g \geq 3$, is actually *maximal* amongst discrete subgroups of Möbius($S^1$), provided we discard the groups that lie on certain lower dimensional subvarieties in $M_g$. See also the interesting discussion of this point in [Sun].
For $g = 2$ remember that every member of $\mathcal{M}_2$ is hyperelliptic, so that the holomorphic automorphism group contains at least a $\mathbb{Z}_2$. Generically again, the group $\text{HolAut}(X)$ is just $\mathbb{Z}_2$ in this genus. It follows that, on an open dense subset of points of $\mathcal{M}_2$ the commensurator of the corresponding Fuchsian group, $G$, is simply a degree two extension of $G$.

**IV.4. Compact Riemann surfaces possessing large $\text{ComAut}(X)$:** At the other end of the spectrum from the generic considerations explained just above, we want to now explore the possibility of creating interesting elements of the commensurability automorphism group of certain special Riemann surfaces. We first explain a method we have devised of finding non-trivial elements in $\text{ComAut}(X) \setminus G$ for certain Riemann surfaces $X = \Delta/G$, that allow large automorphism groups. The method is to utilize certain finite quotients of $X$.

Let us point out first the evident, yet important, fact that the two commensurability automorphism groups $\text{ComAut}(X)$ and $\text{ComAut}(Y)$ are isomorphic whenever there is a holomorphic unramified finite covering map from $X$ onto $Y$, (or vice versa). That is evident since the commensurator of a Fuchsian group is completely insensitive to either extending or contracting the group, up to finite index.

Suppose therefore that we start with some compact Riemann surface, $X = \Delta/G$, of genus $g \geq 2$, with $\text{HolAut}(X)$ being its (finite) group of holomorphic automorphisms. Suppose that $\text{HolAut}(X)$ contains a subgroup, $P$, such that:

(i) every non-identity member of $P$ acts fixed point freely on $X$;

(ii) the subgroup $P$ is *not* a normal subgroup of $\text{HolAut}(X)$.

By condition (ii) there exists $\alpha \in \text{HolAut}(X)$ such that the subgroup of $\text{HolAut}(X)$ given by $Q := \alpha P \alpha^{-1}$ is *not* equal to $P$, and of course, every $q \in Q$ also acts fixed point freely on $X$ (since the members of $P$ acted in that fashion).

Consider then the two quotient Riemann surfaces: $Y := X/P$ with the finite unbranched (normal) holomorphic covering projection: $f_P : X \to Y$, and, correspondingly, $Z := X/Q$ with holomorphic covering projection: $f_Q : X \to Z$.

But, since $\alpha$ conjugates $P$ onto $Q$, it descends to a biholomorphic isomorphism $\alpha_* : Y \to Z$.

We thus have a diagram of unramified holomorphic finite coverings:

$$
\begin{array}{ccc}
X & \xrightarrow{f_P} & X \\
\downarrow & & \downarrow \\
Y & \xrightarrow{\alpha_*} & Z
\end{array}
$$

If we put the conjugating automorphism $\alpha$ itself as the map on the horizontal top arrow, then this diagram will commute and it therefore produces no interesting element of the
commensurability automorphism groups (of any of the Riemann surfaces in sight). But, and here is a crucial point, if we use the identity map as the top horizontal arrow, then the diagram will not commute, — and we have thus found, by circuiting through this diagram, an interesting element of $\text{ComAut}(Y) \cong \text{ComAut}(Z)$ (and therefore also of $\text{ComAut}(X)$). We note that the description of this commensurability automorphism necessitates the utilization of non-trivial holomorphic coverings (i.e., of covering degree at least two).

Of course, we must find a supply of Riemann surfaces $X$ allowing enough holomorphic automorphisms so as to be able to carry through the above construction. But here is a well-known result: Given any finite group $F$, there exists some compact hyperbolic Riemann surface, $X$, with $\text{HolAut}(X) \cong F$, [Gr2]. (The proof of this actually uses the theory of Teichmüller spaces.)

We also refer the reader to the article [K], and the references quoted therein, for some explicit examples of compact Riemann surfaces $X$ which have large automorphism groups, so that the above construction can be carried through explicitly and easily in many instances.

**Arithmetic Fuchsian groups**: A more deep way to pinpoint Fuchsian groups with large commensurators is to involve number theory. In the following sections we will need to utilize heavily Margulis’ well known result [Ma], regarding the fact that the commensurator of $G$ actually becomes a dense subset of $\text{Möbius}(S^1) \cong PSL(2, \mathbb{R})$ precisely when the Fuchsian group $G$ is an arithmetic subgroup of the ambient Lie group $PSL(2, \mathbb{R})$. That situation happens for only countably many compact Riemann surfaces in each genus. Consequently there are only countably many hyperbolic compact Riemann surfaces with arithmetic Fuchsian groups, even when counting over all the genera greater than one.

*Definition of arithmeticity for Fuchsian lattices*: It may be convenient to recall here the definition of when a finite co-volume Fuchsian group $G$ in $PSL(2, \mathbb{R})$ is called arithmetic. The requirement is that, (after conjugating $G$ in $PSL(2, \mathbb{R})$, if necessary), $G$ is commensurable with the group of matrices whose entries are from the integers of some (arbitrary) number field. Of course, the standard example is the subgroup $PSL(2, \mathbb{Z})$. (This example is neither co-compact, nor torsion-free.) Arithmetic Fuchsian groups will be at the very center of our work in Section 5.

**IV.5. Subgroup of $MC_\infty(X)$ acting trivially on the base stratum**: The following result is obtained by considering an appropriate intersection of the isotropy subgroups we described.

**Proposition 4.4.** Let $X$ have genus at least three. The Fuchsian group $G$, considered as a subgroup of $\text{Vaut}^+(G) = MC_\infty(X)$ using inner conjugation, coincides with the
subgroup of \(MC∞(X)\) that fixes pointwise the stratum \(T(X)\) of the inductive limit space \(T∞(X)\).

Proof: A member of \(\text{Vaut}^+(G)\), considered as a quasisymmetric homeomorphism \(f \in \text{Vnorm}_{q,a}(G)\), will act as the identity on the base stratum if and only if, for every quasisymmetric homeomorphism \(\phi\) that is compatible with \(G\), it is true that:

\[
(4.5) \quad \phi \circ f \circ \phi^{-1} \text{ is in } \text{Möbius}(S^1)
\]

Condition (4.5) is checked by tracing through the various canonical identifications that we have explained amongst the models for the action of \(MC∞(X)\) on \(T∞(X)\).

Consequently, for \(f \in G\), it follows, from the definition of \(\phi\) being compatible with \(G\), that (4.5) is satisfied. This part is true even if \(X\) has genus two.

Conversely, the set of transformations of \(MC∞(X)\) holding \(T(X)\) pointwise fixed must, of course, lie in \(\text{ComAut}(X)\). But note that we are free to choose any co-compact torsion free Fuchsian group \(G\), since the base surface \(X\) is at our disposal to fix. If we choose any \(G\) so that \(\text{Comm}(G) = G\), we are through. But with genus \(X\) greater than two, as we said, such a choice of \(G\) is in fact generic. In other words, an open dense set in the moduli space \(\mathcal{M}_g (g \geq 3)\) corresponds to Fuchsian groups whose commutators are no larger than themselves. That completes the proof. \(\square\)

IV.6. Biholomorphic identification of solenoids: Let \(G \subset \text{Aut}(\Delta)\) be, as before, the torsion-free co-compact Fuchsian group under study, and \(X = \Delta/G\).

Let \(H \subset G\) be any subgroup of finite index. The inclusion homomorphism, \(i\), of \(H\) into \(G\) induces an injective homomorphism between the profinite completions:

\[
(4.6) \quad \hat{i} : \hat{H} \longrightarrow \hat{G}
\]

Now, the map

\[
(4.7) \quad \text{Id} \times \hat{i} : \Delta \times \hat{H} \longrightarrow \Delta \times \hat{G}
\]

induces a natural map

\[
(4.8) \quad Q_H : \Delta \times_H \hat{H} \longrightarrow \Delta \times_G \hat{G}
\]

between the above two copies of the universal solenoid. (Recall the discussion in section II.1.) The action of \(G\) on \(\Delta\) in (4.8) is the tautological action of \(\text{Aut}(\Delta)\) on \(\Delta\).

Note that both \(\Delta \times_H \hat{H}\) and \(\Delta \times_G \hat{G}\) carry complex structures. The following lemma says that \(Q_H\) is a biholomorphism with respect to these complex structures.

\textbf{Lemma 4.9.} The map \(Q_H\) is a base leaf preserving biholomorphic homeomorphism.
**Proof:** The continuity of the map \( Id \times \hat{i} \) defined in (4.7) implies that the map \( Q_H \) is continuous.

Since the subset \( \Delta \times_H H \) (respectively, \( \Delta \times_G G \)) is the base leaf in \( \Delta \times_H \hat{H} \) (respectively, in \( \Delta \times_G \hat{G} \)), it is immediate that the map \( Q_H \) sends the base leaf into the base leaf.

The chief issue is to show that \( Q_H \) is a bijection. Let

\[
(4.10) \quad \tilde{i} : H \backslash \hat{H} \rightarrow G \backslash \hat{G}
\]

be the map induced by \( \hat{i} \) (see (4.6)) between the coset spaces. Note that, from the remarks following equation (2.10), it follows that these two coset spaces are precisely the parameter spaces of leaves of the respective associated solenoids. The lemma will be proved by showing that \( \tilde{i} \) is a bijection.

Let us denote the image of the injection \( \hat{i} \) also as \( \hat{H} \). Since \( \hat{H} \) is an open subgroup of \( \hat{G} \), and \( G \) is a dense subgroup of \( \hat{G} \), we have \( G \cdot \hat{H} = \hat{G} \). Therefore, to prove that \( \tilde{i} \) is a bijection it suffices to show that \( \hat{H} \cap G = H \). But the projection \( G \rightarrow G/H \) extends to a continuous map from \( \hat{G} \) to \( G/H \). Since the inverse image of the identity coset contains \( \hat{H} \), we deduce \( \hat{H} \cap G = H \).

Consider next the natural projection of \( \Delta \times_G \hat{G} \), (respectively, \( \Delta \times_H \hat{H} \)), onto \( G \backslash \hat{G} \), (respectively, \( H \backslash \hat{H} \)). These projections fit into the following commutative diagram :

\[
(4.11) \quad \begin{array}{ccc}
\Delta \times_H \hat{H} & \xrightarrow{Q_H} & \Delta \times_G \hat{G} \\
\downarrow & & \downarrow \\
H \backslash \hat{H} & \xrightarrow{\tilde{i}} & G \backslash \hat{G}
\end{array}
\]

But every fiber of the two vertical projections may be identified with \( \Delta \) (after choosing an element to represent the corresponding coset). Since \( \tilde{i} \) is bijective, it follows immediately that \( Q_H \) is also a bijection.

We discuss now the holomorphy of \( Q_H \). Consider the laminated surfaces \( \Delta \times \hat{H} \) and \( \Delta \times \hat{G} \). The complex structure of \( \Delta \) induces a natural complex structure on each of them which is actually constant in the transverse direction. (Recall the definition of a complex structure on a laminated surface given in Section II.2.) The map \( Id \times \hat{i} \) is evidently holomorphic from \( \Delta \times \hat{H} \) to \( \Delta \times \hat{G} \), with the above complex structures.

The complex structure on \( \Delta \times_H \hat{H} \) (respectively, \( \Delta \times_G \hat{G} \)) is induced by descending the complex structure on \( \Delta \times \hat{H} \) (respectively, \( \Delta \times \hat{G} \)) using the complex structure preserving action of \( H \) (respectively, \( G \)). It is easy to see that this descended complex structure coincides with complex structure on a solenoid constructed in Section II.2 from a point of \( T_\infty(X) \).

Since the map \( Q_H \) is obtained by descending \( Id \times \hat{i} \) using the action of \( G \), the
holomorphicity of the map $Id \times \hat{i}$ immediately implies the holomorphicity of $Q_H$. This completes the proof of the lemma. □

For an unramified pointed covering $p : Y \to X$, if we set $H = \pi_1(Y) \subset \pi_1(X)$, then the homeomorphism $Q_H$ obtained in (4.8) can be seen to coincide with the inverse of the homeomorphism $H_\infty(p) : H_\infty(X) \to H_\infty(Y)$ that was constructed in (3.1). Now, the fact that $H_\infty(p)$ is a biholomorphic homeomorphism, when $X$ and $Y$ are Riemann surfaces with $p$ being a holomorphic covering space, follows directly from the very definitions of $H_\infty(X)$ and $H_\infty(Y)$ as inverse limits over towers of Riemann surfaces. However, for our work in Section 5 below, the above construction and analysis of $Q_H$ will be very useful.

IV.7. Holomorphic action of $\text{ComAut}(X)$ on $H_\infty(X)$ : We would like to bring out now a point that is crucial to our work. As we explained with the equations (3.1) and (3.2), the commensurability mapping class group, $MC_\infty$, acts by self-bijections of the appropriate type on the (genus-independent) limit objects like $H_\infty$ and $T_\infty$, — simply because both of those constructions proceed over the tower $I(X)$ of topological finite covers of the base surface. By the same token then, the isotropy group $\text{ComAut}(X)$, which arises for us from the circuits of holomorphic finite covers over $X$, will operate as automorphisms (for the same purely set-theoretic reasons) on any limit object that is created over the directed tower, say $I_{hol}(X)$, comprising only the holomorphic coverings of the given Riemann surface $X$. A first application of this principle is seen in the Proposition 4.12 below, which describes the base leaf preserving holomorphic automorphisms of any complex analytic solenoid. Further applications are manifest in the work of Section 6 below.

Already in the topological category, every element of $\text{Vaut}^+(G)$ acts on the solenoid $H_\infty(X) = \Delta \times_G \hat{G}$, by a self homeomorphism that preserves the base leaf. (See the note (i) following equations (3.1) and (3.2).)

**Proposition 4.12.** Let $X$ be any compact pointed hyperbolic Riemann surface. The full group of holomorphic self-homeomorphisms that preserve the base leaf of the corresponding complex analytic solenoid, $H_\infty(X)$, coincides with $\text{ComAut}(X)$.

**Proof :** Let $G$ be the Fuchsian group uniformizing $X$. Take any Möbius transformation $\gamma : \Delta \to \Delta$ that is a member of $\text{ComAut}(\Delta/G) \equiv \text{ComAut}(X)$. We must first show how $\gamma$ induces the desired kind of biholomorphic automorphism of $H_\infty(X)$.

Maintain the notations as in the proof of part (a) of Theorem 4.2, and note the remarks following the proof of that Theorem. There are two finite index subgroups, $H$ and $K$, in $G$ such that the conjugation by $\gamma$ carries $H$ isomorphically onto $K$. As in (4.3) consider the biholomorphism $\gamma_* : \Delta/H \to \Delta/K$. 

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Applying the $H_\infty$ functor, defined in (3.1), to $\gamma_\ast$, we obtain a complex analytic isomorphism:

\[(4.13) \quad H_\infty(\gamma_\ast) : H_\infty(Z) \longrightarrow H_\infty(Y)\]

Now, as explained in section II.1, we may always identify the solenoids by their group theoretic models:

\[(4.14) \quad H_\infty(X) \equiv \Delta \times_G \hat{G}, \quad H_\infty(Y) \equiv \Delta \times_H \hat{H} \quad \text{and} \quad H_\infty(Z) \equiv \Delta \times_K \hat{K}\]

Therefore apply the natural biholomorphic homeomorphisms:

\[
Q_H : \Delta \times_H \hat{H} \longrightarrow \Delta \times_G \hat{G} \quad \text{and} \quad Q_K : \Delta \times_K \hat{K} \longrightarrow \Delta \times_G \hat{G},
\]

as defined in (4.8), between the above solenoids. We thus obtain the natural biholomorphic self homeomorphism:

\[(4.15) \quad H_\infty(\gamma) = Q_K \circ H_\infty(\gamma_\ast)^{-1} \circ Q_H^{-1}\]

This map, $H_\infty(\gamma)$, is the holomorphic automorphism of $H_\infty(X)$ that corresponds to the chosen $\gamma \in \text{ComAut}(X)$. Since each factor in (4.15) is holomorphic and carries base leaf to base leaf, it is true that $H_\infty(\gamma)$ is a holomorphic automorphism preserving the base leaf of $H_\infty(X)$. That is as desired. Note that this part of the proposition holds for general torsion-free Fuchsian groups $G$. Co-compactness plays no rôle in showing that $\text{ComAut}(X)$ acts biholomorphically on the inverse limit complex solenoid.

To complete the proof of the proposition we must prove that every base leaf preserving holomorphic automorphism of $H_\infty(\Delta/G)$ comes from $\text{ComAut}(\Delta/G)$, when $G$ uniformizes a compact Riemann surface. The proof of this will be given at the end of this section. We will need a couple of lemmata to lead up to that proof.

It is crucial at this stage to point out the purely topological version of the above fact that $\text{ComAut}(X)$ acts by base leaf preserving automorphisms of $H_\infty(X)$.

**Self homeomorphisms of $H_\infty(X)$ preserving the base leaf**: Let $G$ be any discrete group, acting as a group of self homeomorphisms, on a connected and simply connected space $\tilde{X}$, the action being properly discontinuous and fixed point free. Let the quotient space be denoted $X$, $(\pi_1(X) \cong G)$, and $u : \tilde{X} \rightarrow X$ be the universal covering projection that so transpires.

Assume that $G$ is residually finite. Setting $\hat{G}$ to be the profinite completion of $G$, we can create just as in Section II.1, — see (2.8) and (2.9) — the inverse limit solenoid built with base $X$, namely $H_\infty(X) \equiv \tilde{X} \times_G \hat{G}$. The base leaf is the projection by $P_G$ (2.10) of the slice $\tilde{X} \times 1$. The base leaf is thus canonically identified with $\tilde{X}$.
**Lemma 4.17**: Let $\phi : \widetilde{X} \rightarrow \widetilde{X}$ be any homeomorphism that virtually normalizes $G$. Namely, there exist two finite index subgroups $H$ and $K$ of $G$ such that $\phi H \phi^{-1} = K$. Then there is a unique self homeomorphism

$$\Phi : \widetilde{X} \times_G \tilde{G} \rightarrow \widetilde{X} \times_G \tilde{G}$$

which preserves the base leaf, and so that the restriction of $\Phi$ to the base leaf coincides with the given homeomorphism $\phi$.

**Proof of Lemma 4.17**: The uniqueness of the extension of $\phi$ from the base leaf to the entire solenoid is automatic because the residual finiteness of $G$ implies that the base leaf is a dense subset of $\widetilde{X} \times_G \tilde{G}$.

As regards the existence, we will exhibit a formula for $\Phi$. Construct the solenoids $\widetilde{X} \times_H \tilde{H}$ and $\widetilde{X} \times_K \tilde{K}$ determined, respectively, by the two subgroups $H$ and $K$ of $G$. Define a map $\Sigma : \widetilde{X} \times \tilde{H} \rightarrow \widetilde{X} \times \tilde{K}$ by

$$\Sigma((z, \langle \gamma_F \rangle)) = (\phi(z), \langle \phi \gamma_F \phi^{-1} \rangle)$$

where $F$ runs through all finite index normal subgroups in $H$, and, of course, the $\gamma_F$ is a compatible string of cosets from the quotient groups $H/F$.

It is easily checked that $\Sigma$ descends to a homeomorphism $\Psi : \widetilde{X} \times_H \tilde{H} \rightarrow \widetilde{X} \times_K \tilde{K}$ mapping base leaf to base leaf. Therefore,

$$\Phi = Q_K \circ \Psi \circ Q_H^{-1}$$

(Compare with (4.15).) This defines the required self homeomorphism of $\widetilde{X} \times_G \tilde{G}$ with all the properties we want.

The following topological lemma, which provides a suitable converse to Lemma 4.17, will be needed. For this Lemma 4.20 we are strongly indebted to C. Odden’s thesis [Od]. The logical organization of this paper, is, however, actually independent of Lemma 4.20, as well as of the remainder of the present Section 4.

**Lemma 4.20**: In the topological set up as above, suppose moreover that the group of deck transformations $G$ (on $\widetilde{X}$) is a finitely generated group. Let

$$\Phi : \widetilde{X} \times_G \tilde{G} \rightarrow \widetilde{X} \times_G \tilde{G}$$

be any homeomorphism mapping the base leaf on itself. Assume further that $\Phi$ is actually uniformly continuous (in a natural uniform structure on the solenoid), and that $X$ has positive injectivity radius. (To be explained in a moment — see below.)

Then the restriction of $\Phi$ to the base leaf, say $\phi : \widetilde{X} \rightarrow \widetilde{X}$, must virtually normalize the group $G$. 

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Proof of 4.20: As we said, up to simple modifications, Lemma 4.20 can be found in [Oď]. For the purpose of being reasonably self-contained we outline the ideas.

To explain the uniform structure on \( \tilde{X} \times \tilde{G} \) it is easiest to work with certain metric structures on the relevant spaces. Assume that \( \tilde{X} \) carries a metric, say \( \rho \), for which the action of \( G \) is by isometries. (We could be somewhat more general, because only the uniform structure is what is actually needed. For instance, quasi-isometric action of \( G \) would suffice.) Now \( X \) has on it an induced metric (we still call that \( \rho \)).

The metric topology on \( \hat{G} : G \) is any finitely generated, residually finite group. There is a nice way to express the profinite topology via a metric. Define \( A_n \) to be the intersection of all subgroups of index \( n \) or less in \( G \). As \( G \) is assumed finitely generated, each \( A_n \) must have finite index in \( G \). Note that \( \bigcap A_n = \{1\} \), by the residual finiteness of \( G \). Odden uses this telescoping collection of subgroups to define a metric on \( G \): Let

\[
\text{ord}(g) = \max\{n : g \text{ is an element of } A_n\},
\]

(and \( \text{ord}(1) = \infty \)). Then set

\[
(4.21) \quad d(g, h) = \exp(-\text{ord}(g^{-1}h))
\]

One can verify that \( d \) is a metric, and that the completion of \( G \) with respect to \( d \) is canonically \( \hat{G} \).

Combining the \( G \)-invariant metric \( \rho \) on \( \tilde{X} \), and the profinite completion metric \( d \) above on \( \hat{G} \), one can get the obvious metric, say \( \sigma \), on \( \tilde{X} \times \tilde{G} \) that induces the inverse-limit topology.

The uniform continuity of \( \Phi \) is assumed to be with respect to this metric \( \sigma \).

Intersecting an \( \epsilon \)-ball of the \( \sigma \)-metric with the base leaf: What does a small ball in \( \tilde{X} \times \tilde{G} \) look like? If \( \epsilon \) is smaller than the injectivity radius of the quotient \( X \), then an \( \epsilon \) ball has the structure of the product of a small ball in \( \tilde{X} \) with the profinite completion of some member of the descending chain of subgroups of \( G \) described above. In effect, there exists \( A = A_n \), (for some \( n \geq 1 \)), such that the \( \epsilon \) ball in \( \tilde{X} \times \tilde{G} \) is an \( \epsilon \) ball in \( \tilde{X} \) times \( \hat{A} \).

The intersection of the base leaf and such an epsilon ball (of the \( \tilde{X} \times \tilde{G} \) metric) is an \( A \)-invariant collection of disjoint balls on the base leaf \( \tilde{X} \).

This method of choosing subgroups \( A = A_n \) in \( G \), associated to a given size of metric-ball in the solenoid, is going to provide one with the desired finite index subgroups \( H \) and \( K \) in \( G \) that need to be exhibited as getting mapped to each other by \( \phi \)-conjugation.

By the assumed uniform continuity, for each positive \( \epsilon \) there exists \( \delta > 0 \) such that \( \sigma(x, y) < \delta \) implies \( \sigma(\Phi(x), \Phi(y)) < \epsilon \). We take \( \epsilon \) itself to be smaller than half the...
injectivity radius of $X$. Find the corresponding $\delta$ (and cut it down to be smaller than $\epsilon$).

Associated to the $\epsilon$-ball and the $\delta$-ball in the $\sigma$-metric we get two corresponding finite index subgroups $K$ and $H$, say, within $G$, as explained. Now, it follows rather straightforwardly that the action of $\Phi$ on the base leaf will conjugate $H$ into a finite index subgroup of $K$. That is what was wanted. $\square$

Now we are in a position to complete the proof of Proposition 4.12.

Completion of Proof of Proposition 4.12: In our situation, $X = \Delta/G$ is compact, therefore so is $\Delta \times G\hat{G}$. The Poincaré metric on $\Delta$ plays the rôle of $\rho$. Any homeomorphism of a compact metric space is automatically uniformly continuous.

Let $\Phi$ be any holomorphic automorphism of $\Delta \times G\hat{G}$ that preserves the base leaf. It is holomorphic on the base leaf, which is canonically the unit disc, $\Delta$. Thus $\Phi|_{\Delta}$ is necessarily a Möbius transformation, and, by Lemma 4.20, it must virtually normalize $G$. Thus, by Proposition 4.2(a), we deduce that $\Phi$ is an element of $\text{ComAut}(\Delta/G)$. $\square$

In the next section we will further investigate the action of $\text{ComAut}(X)$ on $H_{\infty}(X)$.

5 Ergodic action if and only if arithmetic Fuchsian

Let $X$ be a compact connected Riemann surface. Let $G$ be a co-compact Fuchsian group $G$ acting freely on the universal cover $\Delta$, with $X = \Delta/G$.

V.1. The measure on $H_{\infty}(X)$: Consider the product measure on $\Delta \times \hat{G}$, where $\Delta$ is equipped with the volume form given by the Poincaré metric, and $\hat{G}$ is equipped with the Haar measure. For any open set $U \subset \Delta \times \hat{G}$ over which the quotient map

$$q : \Delta \times \hat{G} \rightarrow \Delta \times G\hat{G}$$

is injective, define the measure of $q(U)$ to be the product measure of $U$. The action of $G$ on $\Delta$ preserves the volume form on $\Delta$ induced by the Poincaré metric. The left action of $G$ on its profinite completion $\hat{G}$ preserves the Haar measure on $\hat{G}$. Therefore, the measure on $q(U)$ does not depend on the choice of the open set $U$. It follows that there is a unique Borel measure on $\Delta \times G\hat{G}$ whose restriction to any such open subset $q(U)$ coincides with the measure of $U$ in $\Delta \times \hat{G}$.

Let $\mu_{\infty}$ denote the Borel measure on $H_{\infty}(X) = \Delta \times G\hat{G}$ just constructed.

The action of the group $\text{ComAut}(X)$ on $H_{\infty}(X)$ preserves the measure $\mu_{\infty}$. To see this we first observe that for a finite index subgroup $H \subset G$, the natural inclusion $\hat{i} : \hat{H} \rightarrow \hat{G}$ is compatible with the Haar measures on $\hat{H}$ and $\hat{G}$ respectively, in the
sense that the image of \( \hat{i} \) is an open subgroup of \( \hat{G} \), and for any measurable subset \( U \subset \hat{H} \), the measure \( \mu \) is \( \#(G/H) \)-times the measure of \( \hat{i}(U) \), where \( \#(G/H) \) is the cardinality of \( G/H \). From this it follows immediately that the homeomorphism \( f_H \) in Lemma 4.9 is actually measure preserving. Now, from the definition of the action of \( \text{ComAut}(X) \) on \( H_\infty(X) \) it follows immediately that it preserves the measure \( \mu_\infty \).

We will describe another construction of a measure on \( H_\infty(X) \).

Consider the Poincaré measure, \( \mu_X \), on \( X \). For any unramified covering \( p : Y \longrightarrow X \) of degree \( d \), consider the measure \( \mu_Y/d \) on \( Y \), where \( \mu_Y \) is the Poincaré measure on \( Y \). For any measurable set \( U \subset X \), its measure \( \mu_X(U) \) clearly coincides with \( \mu_Y(p^{-1}(U))/d \). This compatibility condition of measures ensure that the inverse limit \( H_\infty(X) \) is equipped with a measure. This is a particular application of the Kolmogorov’s construction of measure on an inverse limit.

Let \( \nu_\infty \) denote this measure on \( H_\infty(X) \). The action of the group \( \text{ComAut}(X) \) on \( H_\infty(X) \) preserves \( \nu_\infty \). Indeed, the homeomorphism \( f_H \) in Lemma 4.9 is compatible with this measure in the sense described earlier.

The measure \( \nu_\infty \) on \( H_\infty(X) \) is evidently absolutely continuous with respect to the measure \( \mu_\infty \) constructed earlier. Indeed, this is an immediate consequence of the fact that the Haar measure on the profinite completion \( \hat{G} \) can be obtained using the Kolmogorov’s inverse limit construction on the inverse limit of finite quotients \( G/H \), where \( H \) is a normal subgroup of \( G \) of finite index, and the measure on \( G/H \) being the Haar probability measure, i.e., the counting measure divided by the cardinality.

Actually the two measures on \( H_\infty(X) \) constructed above are constant multiples of each other. But for our purposes it is sufficient to know that they are absolutely continuous with respect to each other. A discussion on this measure on \( H_\infty(X) \) can also be found in Section 9 of [NS].

**V.2. The ergodicity theorem :** Using the work of Margulis that we quoted in section IV.4, we prove in the following theorem that the question of the arithmeticity of the Fuchsian group for \( X \) is equivalent to the question of whether or not \( \text{ComAut}(X) \) acts ergodically on the finite measure space \((H_\infty(X), \mu_\infty)\). [Note that since the two measures constructed on \( H_\infty(X) \), namely \( \mu_\infty \) and \( \nu_\infty \), are absolutely continuous with respect to each other, the action is ergodic with respect to \( \mu_\infty \) if and only if it is ergodic with respect to \( \nu_\infty \).]

**Theorem 5.1.** The Fuchsian group \( G \subset \text{Aut}(\Delta) \) is arithmetic if and only if the action of \( \text{ComAut}(X) \) on \( H_\infty(X) \) is ergodic. In fact, arithmeticity is also equivalent to each orbit being dense.

**Proof.** If \( G \) is not arithmetic, then by a result of Margulis, \( \text{ComAut}(X) \) is a finite extension of \( G \) [Zi, Proposition 6.2.3]. Conversely, if \( G \) is arithmetic, \( \text{ComAut}(X) \) is
dense in $\text{Aut}(\Delta)$ [\text{[6.2]}]. Since the base leaf is dense in $H_\infty(X)$, the orbits of the action of $\text{ComAut}(X)$ on $H_\infty(X)$ are dense if and only if the group $G$ arithmetic.

If $G$ is not arithmetic, then take two nonempty disjoint open subsets, say $U_1$ and $U_2$, of the compact Riemann surface $\Delta/\text{ComAut}(X)$. The inverse images of both $U_1$ and $U_2$ for the natural projection of $H_\infty(X) = \Delta \times \hat{G}$ onto $\Delta/\text{ComAut}(X)$ have positive measure. Hence the action of $\text{ComAut}(X)$ on $H_\infty(X)$ cannot be be ergodic in this case.

Now consider any locally integrable function $f$ on $H_\infty(X)$ which is invariant under the action of $\text{ComAut}(X)$. Let $\tilde{f}$ be the function on $\Delta \times \hat{G}$ obtained by pulling back $f$ using the natural projection $q$ of $\Delta \times \hat{G}$ onto $\Delta \times \hat{G}$. Since the measure $\mu_\infty$ on $\Delta \times \hat{G}$ is constructed from the projection $q$ by using the $G$-invariance property of the product measure on $\Delta \times \hat{G}$, the function $\tilde{f}$ is locally integrable with respect to the product measure.

The function $\tilde{f}$ is invariant (in the sense of equality almost everywhere) firstly, under the action of the deck transformations (action of $G$) of the covering $q$ and, secondly, under the action of $\text{ComAut}(X)$. These two invariance conditions combine together to imply that for each $g \in G$, the equality

$$\tilde{f}(x, hg) = \tilde{f}(x, h)$$

is valid for almost every $x \in \Delta$, $h \in \hat{G}$. Since $G$ is dense in $\hat{G}$, by the continuity of the associated action of $\hat{G}$ on the space of all locally integrable functions over $\Delta \times \hat{G}$, we get that $\tilde{f}(x, h)$ is constant almost everywhere in $h$; say $\tilde{f}(x, h) = \hat{f}(x)$, where $\hat{f}$ is a locally integrable function defined almost everywhere on $X$.

Since $\tilde{f}$ is invariant under the action of $\text{ComAut}(X)$ on $\Delta \times \hat{G}$, the function $\hat{f}$ must be invariant under the action of $\text{ComAut}(X)$ on $\Delta$.

Assume now that $G$ is arithmetic. Therefore, $\text{ComAut}(X)$ is dense in $\text{Aut}(\Delta)$. Using the continuity of the action of $\text{Aut}(\Delta)$ on the space of all locally integrable functions on $\Delta$, and the transitivity of the tautological action of $\text{Aut}(\Delta)$, we conclude, by an argument as above, that $\hat{f}$ must be constant almost everywhere. This completes the proof of the theorem. \hfill \square

6 Lift of the commensurability modular action on vector bundles

VI.1. Construction of natural inductive limit vector bundles over $T_\infty$ : Let $Y$ be any compact connected oriented smooth surface of negative Euler characteristic.
There is a *universal family of Riemann surfaces*:

\[(6.1) \quad f : \mathcal{Y} \longrightarrow \mathcal{T}(Y)\]

over the Teichmüller space \(\mathcal{T}(Y)\). In other words, \(f\) is a Kodaira-Spencer family, namely a holomorphic proper submersion with connected fibers, and for any point \(t \in \mathcal{T}(Y)\), the fiber \(f^{-1}(t)\) is biholomorphic to the Riemann surface represented by the point \(t\).

Let us briefly recall the construction. (Consult, for different points of view, [Gro], [B2], and [N, Chapter 5].) Let \(\text{Conf}(Y)\) denote the space of all smooth complex structures on \(Y\), compatible with the orientation. There is a tautological complex structure on \(\text{Conf}(Y) \times Y\). The group \(\text{Diff}_0(Y)\), consisting of all diffeomorphisms of \(Y\) homotopic to the identity map, acts naturally on both \(\text{Conf}(Y)\) and \(Y\). The diagonal action preserves the complex structure on \(\text{Conf}(Y) \times Y\). Consequently, the complex structure on \(\text{Conf}(Y) \times Y\) descends to a complex structure over the quotient space \((\text{Conf}(Y) \times Y) / \text{Diff}_0(Y)\). This quotient complex manifold is the universal Riemann surface \(\mathcal{Y}\). The projection \(f\) in (6.1) is obtained from the natural projection of \(\text{Conf}(Y)\) onto \(\text{Conf}(Y) / \text{Diff}_0(Y) = \mathcal{T}(Y)\).

The relative holomorphic cotangent bundle on \(\mathcal{Y}\) will be denoted by \(K_f\). In other words, \(K_f\) fits in the following exact sequence of vector bundles over \(\mathcal{Y}\):

\[0 \longrightarrow f^*\Omega^1_{\mathcal{T}(Y)} \longrightarrow \Omega^1_{\mathcal{Y}} \longrightarrow K_f \longrightarrow 0\]

For any integer \(i \geq 0\), let

\[\mathcal{V}^i(Y) := f_*K_f^{\otimes i}\]

be the holomorphic vector bundle on \(\mathcal{T}(Y)\) given by the direct image of the \(i\)-th tensor power, \(K_f^{\otimes i}\), of \(K_f\). The fiber of of the vector bundle \(\mathcal{V}^i(Y)\) over a point of \(\mathcal{T}(Y)\) represented by a Riemann surface \(Y'\) is \(H^0(Y', K_{Y'}^{\otimes i})\).

Given any holomorphic covering \(p : Y' \longrightarrow Z'\), the homomorphism

\[(6.2) \quad (dp)^* : H^0(Z', K_{Z'}^{\otimes i}) \longrightarrow H^0(Y', K_{Y'}^{\otimes i})\]

obtained using the co-differential of \(p\), \((dp)^* : p^*K_{Z'} \longrightarrow K_{Y'}\), is injective.

Given any unramified (topological) covering \(p : Y \rightarrow Z\) between compact connected oriented surfaces, the “fiberwise” construction in (6.2) gives us a bundle homomorphism

\[(6.3) \quad \hat{p}^* : \mathcal{V}^i(Z) \longrightarrow \mathcal{T}(p)^*\mathcal{V}^i(Y)\]

of holomorphic vector bundles over \(\mathcal{T}(Z)\). Here \(\mathcal{T}(p)\) is the basic embedding of Teichmüller spaces as in (2.2). In other words, there is a natural morphism of holomorphic vector bundles, \(\mathcal{V}^i(Z) \longrightarrow \mathcal{V}^i(Y)\) commuting with the embedding \(\mathcal{T}(p)\) of base spaces.
If \( q : W \to Y \) is another unramified covering, with \( W \) compact and connected, then consider the homomorphisms \( \hat{q}^* \) and \( \hat{p} \circ \hat{q}^* \) of holomorphic vector bundles over \( \mathcal{T}(Y) \) and \( \mathcal{T}(Z) \) respectively, as in (6.3). The following is a commutative diagram of homomorphisms of the relevant vector bundles over \( \mathcal{T}(Z) \):

\[
\begin{array}{ccc}
V^i(Z) & \xrightarrow{\hat{p}^*} & V^i(Z) \\
\downarrow \hat{p}^* & & \downarrow \hat{p} \circ \hat{q}^* \\
\mathcal{T}(p)^*V^i(Y) & \xrightarrow{\mathcal{T}(p)^*\hat{q}^*} & \mathcal{T}(p \circ q)^*V^i(W)
\end{array}
\]

It follows that the holomorphic vector bundles \( V^i(X_\alpha) \) over \( \mathcal{T}(X_\alpha) \), (where \( \alpha : X_\alpha \to X \) is any member of \( \mathcal{I}(X) \)), constitute an inductive system using the homomorphisms \( \hat{\alpha}^* \) constructed in (6.3). That these connecting homomorphisms do fit into an inductive system is ensured by the commutativity of the diagram in (6.4).

Therefore, we have a holomorphic vector bundle over \( \mathcal{T}_\infty(X) \) by passing to the inductive limit in this inductive system:

\[
V^i_\infty(X) := \lim_{\longrightarrow} V^i(X_\alpha)
\]

We may denote this holomorphic vector bundle by \( V^i_\infty \), suppressing in the notation the base surface \( X \). That is because, as in the work of previous sections, this construction over \( X \) produces a bundle over \( \mathcal{T}_\infty(X) \) which is holomorphically isomorphic to the corresponding construction \( V^i_\infty(Y) \) over \( \mathcal{T}_\infty(Y) \), whenever any unramified pointed topological covering \( p : Y \to X \) (member of \( \mathcal{I}(X) \)), is specified. This natural bundle isomorphism determined by \( p \)

\[
V^i_\infty(p) : V^i_\infty(Y) \to V^i_\infty(X)
\]

is constructed exactly as in the discussion of \( \mathcal{T}_\infty(p) \) (see equation (3.2)). It covers the biholomorphic identification \( \mathcal{T}_\infty(p) : \mathcal{T}_\infty(Y) \to \mathcal{T}_\infty(X) \) between the two base spaces.

The fiber of \( V^i_\infty(X) \) over any \( [Z] \in \mathcal{T}_\infty(X) \) is simply the direct limit of spaces of \( i \)-forms: \( \lim_{\longrightarrow} H^0(Z_\beta, K^{(\beta)}_{Z_\beta}) \), the index \( \beta \) running through all finite unramified holomorphic coverings \( Z_\beta \) of the Riemann surface \( Z \), with each \( Z_\beta \) a connected Riemann surface. The above direct limit vector space can be interpreted as the space of those holomorphic \( i \)-forms on the complex analytic solenoid, \( H_\infty(Z) \), which are complex analytic on the leaves and locally constant in the transverse (Cantor) direction.

**VI.2. Lifting the action of \( MC_\infty \) and allied matters** : We will now investigate the compatibility of the vector bundle \( V^i_\infty(X) \) with the action of \( MC_\infty(X) \) on \( \mathcal{T}_\infty(X) \).

**Theorem 6.7.** (a) The commensurability modular action of \( MC_\infty(X) \) on \( \mathcal{T}_\infty(X) \) lifts to \( V^i_\infty \), for every \( i \geq 0 \).
(b) Take any \( i \geq 1 \) and any point \([Z] \in \mathcal{T}_\infty(X)\). The isotropy group at \([Z]\), namely \(\text{ComAut}(Z)\), for the action of \(MC_\infty(X)\) on \(\mathcal{T}_\infty(X)\), acts effectively on the fiber of \(V^i_\infty\) over \([Z]\).

**Proof:** Part (a): That the action of \(MC_\infty(X)\) on \(\mathcal{T}_\infty(X)\) lifts to \(V^i_\infty\) is rather straightforward. Suppose that \(g \in MC_\infty(X)\) is represented by the two-arrow diagram arising from a pair of pointed unramified topological coverings:

\[
p_j : (Y, y) \rightarrow (X, x),
\]

where \(j = 1, 2\), — as in (3.3).

Then, by (6.6) above, we have two induced isomorphisms between the inductive limit bundles of \(i\)-forms over \(\mathcal{T}_\infty(X)\) and \(\mathcal{T}_\infty(Y)\), respectively. Clearly then, the commensurability modular transformation \(g\) on \(\mathcal{T}_\infty(X)\) lifts to the holomorphic bundle automorphism:

\[
(6.8) \quad V^i_\infty(p_2) \circ V^i_\infty(p_1)^{-1}
\]

Compare this with the definition of \(A_{(p_1,p_2)}\) provided in equation (3.4).

It is also worthwhile to explicitly describe the lifted action of \(g\). For this purpose, take any complex structure \(J\) on \(X\). The action of \(g\) on \(\mathcal{T}_\infty(X)\) sends the point representing the Riemann surface \((Y, p^*_1 J)\) to \((Y, p^*_2 J)\). Let \(\overline{X}, \overline{Y}_1\) and \(\overline{Y}_2\) denote the Riemann surfaces defined by the complex structures \(J, p^*_1 J\) and \(p^*_2 J\) respectively.

Let the action of \(g\) on \(V^i_\infty\) be such that it sends the subspace \((dp_1)^*_i H^0(\overline{X}, K_{\overline{X}}^{\otimes i})\) of \(H^0(\overline{Y}_1, K_{\overline{Y}_1}^{\otimes i})\) to the subspace \((dp_2)^*_i H^0(\overline{X}, K_{\overline{X}}^{\otimes i})\) of \(H^0(\overline{Y}_2, K_{\overline{Y}_2}^{\otimes i})\); the homomorphism \((dp_j)^*_i\) is defined in (6.2). The resulting isomorphism

\[
(dp_1^*)_i H^0(\overline{X}, K_{\overline{X}}^{\otimes i}) \rightarrow (dp_2^*)_i H^0(\overline{X}, K_{\overline{X}}^{\otimes i})
\]

is the identity automorphism of \(H^0(\overline{X}, K_{\overline{X}}^{\otimes i})\), after invoking the natural identification of \((dp_j)^*_i H^0(\overline{X}, K_{\overline{X}}^{\otimes i})\), \(j = 1, 2\), with \(H^0(\overline{X}, K_{\overline{X}}^{\otimes i})\).

Take any covering \(\alpha : X_\alpha \rightarrow X\), representing a point \(\alpha\) in \(\mathcal{I}(X)\). Let

\[
q_j : Y_\alpha \rightarrow X_\alpha,
\]

where \(j = 1, 2\), be the pull back of the covering \(p_j\) by \(\alpha\). Choose a complex structure \(J_\alpha\) on \(X_\alpha\). The Riemann surfaces \((X_\alpha, J_\alpha)\), will be denoted by \(\overline{X}_\alpha\). The Riemann surface \((Y_\alpha, q^*_j J_\alpha), (j = 1, 2)\), will be denoted by \(\overline{Y}_{j,\alpha}\).

The action of \(g\) on \(\mathcal{T}_\infty(X)\) sends the point of \(\mathcal{T}_\infty(X)\) represented by \(\overline{Y}_{1,\alpha}\) to the point represented by \(\overline{Y}_{2,\alpha}\).

Let us denote the fiber of the vector bundle \(V^i_\infty\) over the point of \(\mathcal{T}_\infty(X)\) represented by \(\overline{X}_\alpha\), namely \(V^1_\infty|_{\overline{X}_\alpha}\), by \((V^i_\infty)_{\overline{X}_\alpha}\).
Define the action of $g$ on $\mathcal{V}_\infty$ to be such that it sends the subspace 

$$(dq_1)_*^i H^0(X_\alpha, K_{X_\alpha}^{\otimes i}) \subset H^0(Y_{1,\alpha}, K_{Y_{1,\alpha}}^{\otimes i}) \subset (\mathcal{V}_\infty)_X$$

to the subspace 

$$(dq_2)_*^i H^0(X_\alpha, K_{X_\alpha}^{\otimes i}) \subset H^0(Y_{2,\alpha}, K_{Y_{2,\alpha}}^{\otimes i}) \subset (\mathcal{V}_\infty)_X$$

The resulting isomorphism between $(dq_1)_*^i H^0(X_\alpha, K_{X_\alpha}^{\otimes i})$ and $(dq_2)_*^i H^0(X_\alpha, K_{X_\alpha}^{\otimes i})$ is the identity automorphism of $H^0(X_\alpha, K_{X_\alpha}^{\otimes i})$, after invoking the natural identification of $(dq_j)_*^i H^0(X_\alpha, K_{X_\alpha}^{\otimes i})$, $j = 1, 2$, with $H^0(X_\alpha, K_{X_\alpha}^{\otimes i})$.

The commutativity of diagram (6.4) ensures that the above conditions on the action of $g$ are compatible. Therefore, we have demonstrated the natural lift of the action of the element $g \in MC_\infty(X)$ to the vector bundle $\mathcal{V}_\infty(X)$ over $T_\infty(X)$. That completes part (a).

Proof for part (b): To prove the effectivity of the action of the isotropy subgroup, we first consider the case $i = 1$.

Let $Z = \Delta/G$, where $G$ is a torsion free co-compact Fuchsian group. From Theorem 4.2 we know that the isotropy group at $Z$, ComAut($Z$), is exactly the commensurator $\text{Comm}(G)$.

Let $N(H) \subset \text{Comm}(G)$ denote the normalizer of $H$ in $\text{Möbius}(S^1)$, where $H$ is any finite index subgroup of $G$. (Recall section IV.3.) We will start by proving that these subgroups $N(H)$ within $\text{Comm}(G)$ act faithfully on the fiber of $\mathcal{V}_\infty^1$ over the point $[Z] \in T_\infty(X)$.

First take a non-identity element $g \in G$. Since $G$ is a residually finite group, there exists a finite index normal subgroup $H$ of $G$ such that $g$ does not belong to $H$. Let $p : Y \rightarrow Z$ be the unramified Galois covering defined by the above subgroup $H$. The Galois group $G/H$ acts effectively by deck transformations on $Y$. Thus the projection of $g$ in the quotient group $G/H$ produces a nontrivial holomorphic automorphism on the Riemann surface $Y$.

Now, it is well known that action on the space of holomorphic Abelian differentials $(H^0(Y, \Omega_Y^1))$ of any nontrivial automorphism of any Riemann surface $Y$, (of genus at least one), can never be trivial; see, for instance, [1]. Therefore, the Galois action of $g$ on $Y$ gives a nontrivial action on $H^0(Y, \Omega_Y^1)$ — implying that the action of $g$ on the fiber $\mathcal{V}_\infty^1|_Z$ is certainly nontrivial.

But every element of $N(H) \setminus G$ represents a non-trivial holomorphic automorphism of the appropriate covering surface of $Z$. Therefore, by the same token, we see that every non-identity element of every normalizer subgroup, $N(H)$, acts non-trivially on the fiber, as desired.
For the remaining case, we need to consider the “non mapping class like” elements $g \in \text{Comm}(G)$ (note section IV.3). Therefore we assume that $g$ is not a member of any of the normalizers $N(H)$. As we know from Section 4, (vide the end of section IV.2), each element $g$ is represented by a pair of holomorphic coverings from some connected Riemann surface $Y$ onto the given $Z$:

$$p_j : Y \rightarrow Z,$$

$j = 1, 2$. In order that $g$ not arise as a member of some normalizer (since we have already disposed of that), one can assume that $p_1 \circ h \neq p_2$, for any automorphism $h \in \text{HolAut}(Y)$.

Choose a point $z \in Z$, and also two points $y_j \in p^{-1}_j(z)$, $j = 1, 2$, satisfying the following condition:

1. $y_1 \neq y_2$, if $Y$ is not hyperelliptic;
2. $y_1 \neq y_2$ and also $y_1 \neq \sigma(y_2)$, if $Y$ is hyperelliptic and $\sigma$ is the hyperelliptic involution thereon.

The existence of such $z$, $y_1$ and $y_2$ is ensured by the assumption spelled out regarding $p_1$ and $p_2$.

First case: Assume $Y$ is not hyperelliptic Therefore, the holomorphic cotangent bundle $K_Y$ over $Y$ is very ample. In particular, there is a 1-form $\omega \in H^0(Y, \Omega^1_Y)$ such that $\omega(y_1) = 0$ and $\omega(y_2) \neq 0$. Therefore, the action of $g$ on $\mathcal{V}^1_{\infty}|[Z]$ does not take the line generated by $\omega$ to itself. Effectivity is established in this case.

Remaining case: Assume $Y$ is hyperelliptic If $Y$ is hyperelliptic then $K_Y$ is no longer very ample. But $K_Y$ is still base point free, and the image of corresponding map $Y \rightarrow \mathbb{P}H^0(Y, \Omega^1_Y)^*$ is $\mathbb{C}\mathbb{P}^1$, with the map itself being identifiable as the projection of $Y$ onto its own quotient by the hyperelliptic involution. Therefore, the existence of a 1-form $\omega$, with $\omega(y_1) = 0$ and $\omega(y_2) \neq 0$, is again assured. This completes the proof of effectivity of $\text{ComAut}(Z)$ on the fiber for the case of the bundle of 1-forms.

If $i \geq 2$, then the proof is identical. In fact, it is actually simpler. As is well-known, the line bundle $K_Y^\otimes i$ is very ample if $i \geq 2$ for every Riemann surface $Y$ with genus at least two. Therefore, the hyperelliptic case need not be considered separately any more. This completes the proof of the theorem.

Remarks: The above proof shows that the action of the isotropy group for $[Z]$ on the projective space $\mathbb{P}(\mathcal{V}^1_{\infty}|[Z])$ is also effective.

In the case of the bundle of $i$-forms with $i \geq 2$ we could utilize Poincaré theta series, for the relevant Fuchsian group and its subgroups, to also provide another proof of the effectivity of the action of the commensurability automorphism group on the fiber.
VI.3. Petersson hermitian structure on the bundles over $\mathcal{T}_\infty$: Let $Y$ be a connected Riemann surface of genus at least two. The Poincaré metric, $\omega$, on $Y$ induces a Hermitian metric, $h$, on any $K_Y^{\otimes i}$. For any two sections $s$ and $t$ of $H^0(Y, K_Y^{\otimes i})$, the pairing
\[ \int_Y \langle s, t \rangle_h \cdot \bar{\omega}, \]
where $\bar{\omega}$ is the Kähler form for $\omega$, defines a Hermitian inner product on the vector space $H^0(Y, K_Y^{\otimes i})$. This inner product is usually called the $L^2$-inner product; it coincides with the classical Petersson pairing of holomorphic $i$-forms on the Riemann surface.

For any covering $\alpha : X_\alpha \rightarrow X$, representing a point of $\mathcal{I}(X)$, consider the inner product on $H^0(X_\alpha, K_{X_\alpha}^{\otimes i})$ defined by
\[ \langle s, t \rangle := \frac{\int_Y \langle s, t \rangle_h \cdot \bar{\omega}}{d}, \]
where $d$ is the degree of the covering $\alpha$. This normalized $L^2$-inner product has the property that if $p : X_\beta \rightarrow X_\alpha$ is a covering map, where $\beta = p \circ \alpha \in \mathcal{I}(X)$, then the natural inclusion homomorphism
\[ (dp)_i^* : H^0(X_\alpha, K_{X_\alpha}^{\otimes i}) \rightarrow H^0(X_\beta, K_{X_\beta}^{\otimes i}), \]
defined in (6.2), actually preserves the normalized $L^2$-inner product. Therefore, the limit vector bundle $\mathcal{V}_\infty^i$ is equipped with a natural Hermitian metric. The restriction of this metric to any subspace of the type $H^0(X_\alpha, K_{X_\alpha}^{\otimes i})$ of a fiber coincides with the normalized $L^2$-inner product.

In section VI.2 above, we saw that the commensurability modular action on the base $\mathcal{T}_\infty(X)$ lifts to holomorphic vector bundle automorphisms on $\mathcal{V}_\infty^i$. The simple observation that $(dp)_i^*$ preserves the normalized $L^2$-inner product, immediately implies that the lift of each $\gamma \in CM_\infty(X)$ preserves the natural Hermitian structure of $\mathcal{V}_\infty^i(X)$. In fact, each of the bundle isomorphisms $\mathcal{V}_\infty^i(p)$ (of the type in (6.6)) is an isometric isomorphism, and the assertion follows.

VI.4. Projective limit construction of an $i$-forms vector bundle: There is a “dual” construction to the one exhibited in section VI.1. Let $p : Y \rightarrow X$ be an unramified covering map of degree $d$ between compact connected Riemann surfaces. The inverse of differential of the map $p$, namely
\[ (dp)^{-1} : K_Y \rightarrow p^* K_X, \]
induces the isomorphism $((dp)^{-1})^{\otimes i} : K_Y^{\otimes i} \rightarrow p^* K_X^{\otimes i}$. Now taking the direct image of $((dp)^{-1})^{\otimes i}$ we have
\[ (((dp)^{-1})^{\otimes i})_* : p_* K_Y^{\otimes i} \rightarrow p_* p^* K_X^{\otimes i} = K_X^{\otimes i} \otimes p_* \mathcal{O}_Y \]
The last equality is the well-known projection formula.

There is an obvious homomorphism \( p_* \mathcal{O}_Y \to \mathcal{O}_X \). Using this, we obtain

\[
\overline{p} : p_* \mathcal{K}_Y^{\otimes i} \to \mathcal{K}_X^{\otimes i}.
\]

Now, since \( H^0(Y, \mathcal{K}_Y^{\otimes i}) = H^0(X, p_* \mathcal{K}_Y^{\otimes i}) \), the above homomorphism \( \overline{p} \) induces a homomorphism

\[
\overline{p}_i : H^0(Y, \mathcal{K}_Y^{\otimes i}) \to H^0(X, \mathcal{K}_X^{\otimes i}).
\]

It is easy to see that the above homomorphism \( \overline{p}_i \) is the dual of the natural homomorphism

\[
p^* : H^1(X, T_X^{\otimes (i-1)}) \to H^1(Y, T_Y^{\otimes (i-1)})
\]

after invoking the Serre duality for both \( \mathcal{K}_X^{\otimes i} \) and \( \mathcal{K}_Y^{\otimes i} \).

For any \( \omega \in H^0(X, \mathcal{K}_X^{\otimes i}) \), it is evident that

\[
(6.10) \quad \overline{p}_i p^* \omega = d\omega.
\]

Let us denote \( \overline{p}_i/d \) by \( p_i \). If \( q : Z \to Y \) is another such covering, then evidently \( (p \circ q)_i = p_i \circ q_i \).

This compatibility condition implies that to any Riemann surface \( X \) we can associate the projective limit of spaces of \( i \)-forms of covering Riemann surfaces:

\[
\lim_{\leftarrow} H^0(X_\alpha, \mathcal{K}_X^{\otimes i}),
\]

with \( \alpha \) running through the directed set \( \mathcal{I}(X) \).

The construction of this projective limit, as the fiber over \([X]\), gives us a new holomorphic vector bundle

\[
\mathcal{V}^{\otimes i} \to \mathcal{T}_\infty(X)
\]

Furthermore, the identity \( p_* p^* \omega = \omega \) (deduced from (6.10)), implies that there is a natural injective homomorphism of vector bundles

\[
(6.11) \quad f_i : \mathcal{V}^i \to \mathcal{V}^{\otimes i}
\]

In other words, for set theoretic reasons, the inductive limit \( i \)-forms bundle injects into the newly constructed projective limit \( i \)-forms bundle. It is easy to see that the action of \( MC_\infty(X) \) on \( \mathcal{T}_\infty(X) \) lifts to \( \mathcal{V}^{\otimes i} \). Also, the two constructions are compatible, as one may check, for essentially set-theoretic reasons. In particular, the inclusion \( f_i \) in (6.11) commutes with the actions of \( MC_\infty(X) \) on these bundles.

We put down these observations in the form of the following Proposition.

**Proposition 6.12.** For any \( i \geq 0 \), the action of \( MC_\infty(X) \) on \( \mathcal{T}_\infty(X) \) lifts to \( \mathcal{V}^{\otimes i} \). The inclusion map \( f_i \) commutes with the actions of \( MC_\infty(X) \). The isotropy subgroup
at any point of $\mathcal{T}_\infty(X)$, for the action of $MC_\infty(X)$, acts faithfully on the corresponding fiber of $V_{\infty,i}$.

The last assertion regarding effectivity is clearly a consequence of Theorem 6.7 part (b).

Remark : The problem of extension of these bundles to the completion of $\mathcal{T}_\infty(X)$, namely to bundles over $\mathcal{T}(H_\infty(X))$, and the question of computing the curvature forms of these bundles as forms on the base space $\mathcal{T}_\infty(X)$, are topics to which we hope to return at a later date.

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