RESUMMING THE EFFECTIVE ACTION

Andrei Leonidov\textsuperscript{(a)} and Andrei Zelnikov\textsuperscript{(b)}

(a) Physics Department
University of Bielefeld D-4800, Bielefeld, Germany
and
Theoretical Physics Department
P.N.Lebedev Physics Institute, 117924 Leninsky pr.53, Moscow, Russia

(b) Institut fur Theoretische Physik
ETH-Honggerberg CH-8093, Zurich, Switzerland
and
P.N.Lebedev Physics Institute
117924 Leninsky pr.53, Moscow, Russia

Abstract

At the simple example of a massless scalar field propagating in the static background we study the resummed expressions for the effective action at zero and finite temperature that are free from a usual infrared sickness of the effective action induced by massless particles.
1. Introduction

In this note we discuss at simple examples the ways of dealing with the well-known problem of the infrared sickness of the one-loop effective action for the massless test particles propagating in the external field.

Let us first remind the essence of the above-mentioned problem and consider the simplest case of a scalar field \( \varphi(x) \), propagating in the background field \( \Phi(x) \). The corresponding Lagrangian has the form

\[
L[\varphi, \Phi] = \frac{1}{2} \varphi(x)(-\Box + m^2 + \Phi^2(x))\varphi(x)
\]

where \( \Box \) is a free Laplacian and the corresponding effective action for \( \Phi(x) \) induced by the test particles \( \varphi(x) \) (or using the terminology from statistical mechanics, the induced entropy of the external field) at the one loop level is

\[
W[\Phi] = -\frac{1}{2} \int_0^\infty \frac{ds}{s} e^{-m^2 s} Tr(e^{(-\Box+\Phi^2(x))s} - e^{\Box s})
\]

where we have used the proper time representation and the interesting contribution is isolated by subtracting the free propagator contribution. Certainly the above expression should be regularized. In the following we shall exploit the following variant of the \( \zeta \)-regularization:

\[
\text{LogDet}A = -\frac{d}{\delta} \left( \frac{M^{2\delta}}{\Gamma(\delta)} \int_0^\infty \frac{ds}{s^{1+\delta}} Tr e^{-A s} \right)_{\delta=0}
\]

where \( M \) is an ultraviolet cutoff. In a situation when the external field \( \Phi(x) \) is essentially inhomogeneous one is usually confined to using only a certain number of terms in the expansion of the trace of the heat kernel

\[
TrK(s) = Tr e^{(-\Box+\Phi^2(x))s}
\]

in the powers of the proper time \( s \)

\[
TrK(s) = \sum_{n=0}^\infty b_{-\omega+n}(\Phi)s^{-\omega+n}
\]

where \( 2\omega \) is a (Euclidean) space-time dimension. We see that in the massless case \( m = 0 \) starting from some term in the heat kernel expansion the corresponding effective action is term by term divergent at the upper limit of the integration over \( s \) and therefore the effective action is an operationally ill-defined quantity in the infrared domain. The same problem arises in the finite temperature calculations of the free energy of a system of massless
particles in the inhomogeneous static external field. Namely, we have for the free energy of this system at the one-loop level

\[ -\beta F = \frac{1}{2} \int_0^\infty \frac{ds}{s} \frac{\beta}{(4\pi s)^\frac{1}{2}} \sum_{-\infty}^\infty e^{-\frac{2\pi^2}{4s} Tr K_{2\omega-1}(s)} \]  

which after using the Poisson resummation formula

\[ \sum_{n=-\infty}^\infty e^{-\frac{n^2 \beta^2}{4s}} = \left( \frac{4\pi s}{\beta^2} \right)^\frac{1}{2} \sum_{n=-\infty}^\infty e^{-\frac{4\pi^2}{\beta^2 n^2}} \]  

takes the form

\[ -\beta F = \frac{1}{2} \int_0^\infty \frac{ds}{s} (1 + 2 \sum_{n=1}^\infty e^{-\frac{4\pi^2}{\beta^2 n^2}}) Tr K_{2\omega-1}(s) \]  

where \( \beta \) is an inverse temperature (let us recall that here the euclidean time is compactified on a circle of a length \( \beta \)). We see that for \( n = 0 \) (entropy) term when using a usual semiclassical expansion for the spatial heat kernel \( K_{2\omega-1}(s) \) we face the same problem of a term by term infrared divergence at the upper limit of the integration over \( s \).

This unpleasant situation (which is of course arising for any type of the massless test particle: gluons, gravitons, etc.) forces us to find a way to give an operational definition for the effective action by using such approximations for the heat kernel \( K(s) \) that the corresponding effective action would be finite. In the following we shall consider two such approximations actually corresponding to the two different types of resumming certain infinite sequences of terms in the heat kernel.

The first method corresponds to a summation of some sequence of terms of all orders in the external field \( \Phi(x) \) using the analogy between the trace of the heat kernel and the partition function, for which the corresponding resummation has been discussed in the literature\[1\]. We find that for the simple external scalar field configurations the infrared problem is cured and it is possible to get an infrared-finite answer for the effective action (entropy) already for the simplest of previously discussed resummation for the partition function.

The second method corresponds to summing all derivatives for the terms having some definite power in the external field. This corresponds to a calculation of a nonlocal effective action [2-5]. In this method it was proved both at zero [2-4] and finite temperature [5] that this procedure allows to get an infrared-finite answer for the effective action (free energy). Below we illustrate this situation - again using the simplest localized scalar field configurations. Here a (small) new step is using the explicitely nonlocal formulas for the effective action.

Let us notice, that formally it is possible to exponentiate the nonlocal terms in the effective action too (at least for scalar [6] and electromagnetic [7] interactions). However it
is not at all clear whether this method could be used for actual computations. We hope to return to the analysis of this question elsewhere.

Let us also mention that in estimating the effective action one can use the method of term-by-term infrared regularization of the effective action using the series (5) and optimizing with respect to a cutoff [8], but in this case it seems to be difficult to trace the interrelations between the relevant scales to motivate the expansion parameter.

2. Resumming the potential

The origin of the above-described infrared catastrophe is clearly in using the expansion of the trace of the heat kernel in the powers of the proper time. Thus an obvious idea is to use an expression for the

trace of the heat kernel which is (at least approximately) exponential in the external field. In fact the expressions of such type have already been discussed in the statistical physics context [1], where the analogue of the trace of the heat kernel is a partition function, the inverse temperature being the analogue of the proper time. As our main purpose here is to illustrate the corresponding possibilities at the simple solvable examples, we shall use just the original version of the formula for the partition function as calculated by Goldberger and Adams [1]. In our notation the expression for the trace of the heat kernel for the massless scalar particle propagating in the external field $\Phi(x)$ reads

\[
Tr (K(s) - K_0(s)) = \frac{1}{(4\pi s)^\omega} \int d^2\omega x (e^{-\Phi^2(x)s}(1 - s^2(\frac{1}{6}\Phi^2(x) - \frac{1}{12}(\nabla\Phi^2(x))^2 + ...)) - 1)
\]  

(9)

In the finite temperature case (as seen from Eq.(8)) this trace should be taken over 2-$\omega - 1$-dimensional space (for the static external field configurations).

Let us now consider a localized spherically symmetric static external field configuration in 3 dimensions

\[
\Phi(r) = \frac{\Phi_0}{(1 + r/a)^2}
\]  

(10)

and first calculate the effective action in the zero temperature case. Substituting the external field configuration (10) into the expression for the heat kernel Eq.(9) we get in the leading approximation for the heat kernel ($\omega = 2$):

\[
Tr (K(s) - K_0(s)) = \frac{\pi^2\Phi_0^2 a^3}{(4\pi \sigma)^2}(\sigma^\frac{3}{4}\gamma(-\frac{1}{4}, \sigma) - 2\sigma^\frac{1}{4}\gamma(-\frac{1}{2}, \sigma) + \sigma^\frac{3}{4}\gamma(-\frac{3}{4}, \sigma) + \frac{4}{3})
\]  

(11)

where $\tau$ is an euclidean time, $\gamma(\alpha, \sigma)$ is an incomplete gamma-function and we have introduced the dimensionless variable $\sigma = \Phi_0^2 s$. The terms originating from the derivative
corrections in Eq.(9) can also be calculated, and finally one gets for the effective action

\[ W[\Phi_0, a, M] = -\frac{1}{64\pi^2} \tau (\frac{4}{3} \pi a^3) \Phi_0^4 \frac{1}{35} \log \frac{M^2}{\Phi_0^2} (1 + \frac{70}{63} \frac{1}{\Phi_0^2 a^2}) + \ldots \]  

(12)

We can conclude that in the simplest approximation for the exponentiated heat kernel the basic difference from the familiar constant external field case (the first term in the above expression would just correspond to a usual effective potential) is the appearance of the effective volume \( a^3 \) of the localized configuration instead of the total spatial volume \( V(3) \). The logarithmic factor still depends only on the external field amplitude and taking into account the terms with derivatives results in the expansion in the inverse powers of \( \Phi_0^2 a^2 \).

Let us now calculate the free energy of a massless scalar field propagating in the static background field configuration Eq.10. Combining the Eqs. 8 and 11 we get:

\[ -\frac{F}{T} = \frac{(\Phi_0 a)^3}{90} - \frac{2}{3} (\Phi_0 a)^3 \sum_{n=1}^{\infty} (1 + \mu n^2)^{\frac{3}{2}} f(\mu, n) \]  

(13)

where \( \mu = \frac{4\pi^2 T^2}{\Phi_0^2} \) and

\[ f(\mu, n^2) = 2F_1(1; -\frac{3}{2}; \frac{3}{4}; \frac{1}{1 + \mu n^2}) - 2F_1(1; -\frac{1}{2}; \frac{1}{2}; \frac{1}{1 + \mu n^2}) + \frac{1}{3} 2F_1(1; -\frac{3}{2}; \frac{1}{4}; \frac{1}{1 + \mu n^2}) - \frac{1}{3} (\frac{\mu n^2}{1 + \mu n^2})^{\frac{3}{2}} \]  

(14)

where \( 2F_1(a; b; c; x) \) is a hypergeometric function and for simplicity we took into account only the leading exponential term (the derivative corrections can be derived in a completely analogous way). From this expression one can work out the low- and high-temperature expansions in a standard way.

2. Summing the derivatives

Let us now turn to the second possibility of constructing an infrared-safe approximation for the calculation of the effective action for massless test particles. This method corresponds to a summation of all derivative terms for a given power of the external field. The resulting effective action is therefore an essentially nonlocal object. In a pioneering paper [3] Barvinsky and Vilkovisky have shown that this procedure provides infrared-convergent integrals for the effective action in all orders in the external field. Later this technique was generalized to a finite-temperature case [5]. In [2-4,5] all the formfactors were written as infinite series in the free laplacian. This is quite suitable for the formal purposes, but for physical applications it is desirable to have the expressions which are explicitly nonlocal in the external fields,
thus facilitating the analysis of the impact of the scales (correlation lengths) set up by the external field. Below we shall analyse the explicitly nonlocal effective action for the same case of a massless scalar field propagating in an localized external field configuration.

For the trace of the heat kernel we have a general expansion

\[ \text{Tr}K(s) = \sum_{n=0}^{\infty} \text{Tr}K_n(s) \]  

where

\[ \text{Tr}K_n(s) = \frac{s^n}{n} \int_{\alpha_i \geq 0} d^n \alpha \delta \left( 1 - \sum_{i=1}^{n} \alpha_i \right) \text{Tr} \left[ V e^{s \alpha_1} \ldots V e^{s \alpha_n} \right] \]  

and for the external field potential \( V \) we shall take an \( O(3) \)-symmetric external field configuration

\[ V = \Phi_0^2 e^{-\frac{r^2}{2a^2}} \]  

where the choice of the configuration to consider is dictated by a computational simplicity. For the effective action at zero temperature we get in the third order in the external field perturbation:

\[ W = \frac{1}{64\pi^2} a C (\frac{\tau}{4\Phi_0^2 a^2})^2 \text{Log} \left( \frac{1}{M^2 a^2} \right) - \frac{c}{192(2\pi)^2} \left( \frac{\tau}{a} (\frac{\Phi_0^2 a^2}{2})^3 \right) \]  

where the constant \( c = 1.30348 \) was obtained by numerical integration. We see that the basic difference of this answer from that obtained in the first section is that the charge renormalization logarithm is now saturated by the slope of the field, and not by its amplitude.

For the free energy one gets in the same limit

\[ -\beta F = \frac{\pi}{16} (\Phi_0 a)^4 \]
\[ + \frac{\pi^2}{4} (\Phi_0 a)^4 (aT)^2 \sum_{n=1}^{\infty} n^2 \int_0^1 d\alpha \alpha^{-\frac{3}{2}} (1 - \alpha)^{-\frac{1}{2}} \Psi \left( \frac{3}{2}, 2, \frac{4\pi^2 a^2 T^2 n^2}{\alpha(1 - \alpha)} \right) \]  

where \( \Psi(a, c; x) \) is a confluent hypergeometric function. This expression can serve as a starting point for constructing the low- and high- temperature expansions by standard methods.

4. Acknowledgments

A.L is grateful to Prof. H. Satz for kind hospitality at the University of Bielefeld where this paper was finished. His work was partially supported by the Russian Fund for Fundamental Research, Grant 93-02-3815. A.Z. is grateful to Prof. C. Schmid for kind hospitality at the Institut for Theoretical Physics, ETH-Honggerberg (Zurich). His work was partially supported by Soros Fund and American Astronomical Society.
References

1. M.L. Goldberger, E.N. Adams. *Journ. Chem. Phys.* 20 (1952), 240;
2. A.O. Barvinsky, G.A. Vilkovisky *Nucl. Phys* B282 (1987), 163;
3. A.O. Barvinsky, G.A. Vilkovisky *Nucl. Phys* B333 (1990), 471;
4. A.O. Barvinsky, G.A. Vilkovisky *Nucl. Phys* B333 (1990), 512;
5. A.V. Leonidov, A.I. Zelnikov *Phys. Lett* B276 (1992), 122;
6. Y. Fujiwara, T.A. Osborn and S.F.J. Wilk *Phys. Rev.* A25 (1982), 14;
7. A.O. Barvinsky, T.A. Osborn Manitoba University preprint *MANI-92-01, May 1992.*
8. D.I. Diakonov, V.Yu. Petrov and A.V. Yung *Phys. Lett.* B130 (1983), 385; *Sov. J. Nucl. Phys.* 39 (1984), 150.