LOCAL WELL-POSEDNESS FOR
THE MAXWELL-SCHRÖDINGER EQUATION

MAKOTO NAKAMURA AND TAKESHI WADA

Abstract. Time local well-posedness for the Maxwell-Schrödinger equation in the Coulomb gauge is studied in Sobolev spaces by the contraction mapping principle. The Lorentz gauge and the temporal gauge cases are also treated by the gauge transform.

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1. Introduction

We consider the Maxwell-Schrödinger equation (MS):

\begin{align}
    i \partial_t u &= (\mathcal{H}(A) + \phi) u, \quad (1.1) \\
    -\Delta \phi - \partial_t \text{div } A &= \rho(u), \quad (1.2) \\
    (\partial_t^2 - \Delta) A + \nabla (\partial_t \phi + \text{div } A) &= J(u, A), \quad (1.3)
\end{align}

where \((u, \phi, A) : \mathbb{R}^{1+3} \to C \times \mathbb{R} \times \mathbb{R}^3, \mathcal{H}(A) = -(\nabla - iA)^2, \rho(u) = |u|^2, J(u, A) = 2 \text{Im } \bar{u}(\nabla - iA)u\). This system describes the evolution of a charged nonrelativistic quantum mechanical particle interacting with the (classical) electro-magnetic field it generates; \(u\) is the wave function of the particle and \((\phi, A)\) is the electro-magnetic potential.

The solutions of MS have some freedom coming from the gauge invariance, which is the consequence of the fact that the observables are gauge invariant but \(u, \phi, \) and \(A\) themselves are not observables. Namely, for any function \(\lambda : \mathbb{R}^{1+3} \to \mathbb{R},\) MS is invariant under the gauge transform

\[(u', \phi', A') = (\exp(i\lambda)u, \phi - \partial_t \lambda, A + \nabla \lambda).\] (1.4)

By this fact, (1.1)-(1.3) itself are not adequate to discuss well-posedness. For, the uniqueness of the solution clearly does not hold. To remove this uncertainty, we need to indicate how to choose representative elements from gauge equivalence classes. Such conditions are called gauge conditions. One of the well-known gauge condition is the Coulomb gauge

\[\text{div } A = 0.\] (1.5)

In this gauge, (1.2) and (1.3) become

\[\begin{align}
    -\Delta \phi &= \rho(u), \quad (1.6) \\
    (\partial_t^2 - \Delta) A &= PJ(u, A),
\end{align}\]
where $P = 1 - \nabla \text{div} \Delta^{-1}$ is the projection onto the solenoidal subspace. The first equation in (1.6) is easily solved by the Newtonian potential. Therefore $\text{MS}$ in the Coulomb gauge ($\text{MS-C}$) is expressed by

$$i\partial_t u = (H(A) + \phi(u))u, \quad (\partial_t^2 - \Delta)A = PJ(u, A),$$

where $\phi(u) = (-\Delta)^{-1}|u|^2$, and the Coulomb gauge condition (1.5) is required. In this gauge $\phi$ does not need the initial datum. The condition (1.5) is conserved if the initial data $A(0)$ and $\partial_t A(0)$ satisfy (1.5). Therefore we consider the time local well-posedness of $\text{MS-C}$ with initial data

$$(u(0), A(0), \partial_t A(0)) = (u_0, A_0, A_1) \in X^{s,\sigma}, \quad (1.7)$$

where $X^{s,\sigma} = \{(u_0, A_0, A_1) \in H^s \oplus H^\sigma \oplus H^{\sigma-1}; \text{div } A_0 = \text{div } A_1 = 0\}$.

Our purpose in this paper is to show the local well-posedness for $\text{MS-C}$ in the Sobolev space as wide as possible by the contraction mapping principle. The main theorem is the following.

**Theorem 1.1.** Let $s \geq 5/3$ and $\max\{4/3, s-2, (2s-1)/4\} \leq \sigma \leq \min\{s+1, (5s-2)/3\}$ with $(s, \sigma) \neq (5/2, 7/2), (7/2, 3/2)$. Then for any $(u_0, A_0, A_1) \in X^{s,\sigma}$, there exists $T > 0$ such that $\text{MS-C}$ with initial condition (1.7) has a unique solution $(u, A)$ satisfying $(u, A, \partial_t A) \in C([0, T]; X^{s,\sigma})$. Moreover if $\sigma \geq \max\{(s-1), (2s+1)/4\}$ with $(s, \sigma) \neq (5/2, 3/2)$, then the map $(u_0, A_0, A_1) \mapsto (u, A, \partial_t A)$ is continuous as a map from $X^{s,\sigma}$ to $C([0, T]; X^{s,\sigma})$.
Remark. (1) $T$ depends only on $s, \sigma$ and $\|(u_0, A_0, A_1); X^{s,\sigma}\|$. 

(2) For any $s$ and $\sigma$ satisfying the assumption above for the unique existence of the solution, the map $(u_0, A_0, A_1) \mapsto (u, A, \partial_t A)$ is continuous in weak-star sense.

Nakamitsu-Tsutsumi [10] showed the time local well-posedness in $X^{s,s}$ with $s > 5/2$. In fact, they treated the case of Lorentz gauge mentioned below, but the Coulomb gauge case can be treated analogously. Generally, the most difficult point of the treatment of MS-C is to overcome the loss of derivative which may be caused by the term $A\nabla u$ in (1.1). In [10] it is done by usual energy method. The fact that $\text{Re}\langle A\nabla u, u \rangle_{H^s} = -\int |\Omega^s u|^2 \text{div} A \, dx = 0$, where $\Omega = (1 - \Delta)^{1/2}$, is used to obtain a differential inequality $\frac{d\|u; H^\sigma\|}{dt} \leq \|A; H^\sigma\|\|\partial u\|_{\infty} + \|\partial A\|_{\infty}\|u; H^\sigma\|$. The assumption $s > 5/2$ is needed to treat $\|\partial u\|_{\infty}$ and $\|\partial A\|_{\infty}$. In order to refine the result, in the present paper we derive the estimate for $\|H u; H^{s-2}\|$ or $\|\partial u; H^{s-2}\|$ instead of $\|u; H^\sigma\|$ itself. Then the self-adjointness of $H$ in $L^2$ helps us to overcome the loss of derivative. We also remark that the energy inequality for the wave equation in $H^\sigma$ requires that the inhomogeneous term belongs to $H^{\sigma-1}$, from which the assumption $\sigma \leq s$ seems to be needed. However, we actually need a weaker condition for $\sigma$ by the use of the projection $P$ (see Lemma 4.1.1). On the other hand, Guo-Nakamitsu-Strauss [4] constructed a time global solution in $X^{1,1}$ although they did not show the uniqueness. Indeed, MS-C has the conservation laws of the charge and the energy from which we can obtain the boundedness of $\|(u, A, \partial_t A); X^{1,1}\|$. Therefore this result is obtained by the compactness method. Our result fills some part of the gap between [4] and [10] but not completely.

Next we consider the Lorentz gauge

$$\partial_t \phi + \text{div} A = 0. \tag{1.8}$$

MS in the Lorentz gauge (MS-L) is expressed as

$$i\partial_t u = (H(A) + \phi)u, \quad (\partial_t^2 - \Delta)\phi = \rho(u), \quad (\partial_t^2 - \Delta)A = J(u, A).$$

In this case, we need the initial data

$$(u(0), \phi(0), \partial_t \phi(0), A(0), \partial_t A(0)) = (u_0, \phi_0, \phi_1, A_0, A_1) \in Y^{s,\sigma}. \tag{1.9}$$

Here

$$Y^{s,\sigma} = \{(u_0, \phi_0, \phi_1, A_0, A_1) \in H^s \oplus H^\sigma \oplus H^{\sigma-1} \oplus H^\sigma \oplus H^{\sigma-1}; \text{div} A_0 + \phi_1 = \text{div} A_1 + \Delta \phi_0 + |u_0|^2 = 0\}.$$

The condition (1.8) is conserved if the initial datum belongs to $Y^{s,\sigma}$. The result for MS-L is the following.

**Theorem 1.2.** Let $s \geq 5/3$ and $\max\{4/3, s - 1\} \leq \sigma \leq \min\{s + 1, (5s - 2)/3\}$ with $(s, \sigma) \neq (5/2, 7/2)$. Then for any $(u_0, \phi_0, \phi_1, A_0, A_1) \in Y^{s,\sigma}$, there exists $T > 0$ such that
MS-L with initial condition \( (1.9) \) has a unique solution \((u, \phi, A)\) satisfying
\[
(u, \phi, \partial_t \phi, A, \partial_t A) \in C([0, T]; Y^{s, \sigma}).
\]
Moreover, if \( \sigma \geq (2s + 1)/4 \) with \( (s, \sigma) \neq (5/2, 3/2) \), then the map \((u_0, \phi_0, \phi_1, A_0, A_1) \mapsto (u, \phi, \partial_t \phi, A, \partial_t A)\) is continuous as a map from \( Y^{s, \sigma} \) to \( C([0, T]; Y^{s, \sigma}) \).

We can also treat the temporal gauge, namely
\[
\phi = 0. \tag{1.10}
\]
In this gauge MS becomes the following system, which is referred to as MS-T:
\[
i \partial_t u = \mathcal{H}(A)u, \quad (\partial_t^2 - \Delta)A + \nabla \text{div} A = J(u, A).
\]
For MS-T, we need the initial data
\[
(u(0), A(0), \partial_t A(0)) = (u_0, A_0, A_1) \in \tilde{Y}^{s, \sigma}, \tag{1.11}
\]
where \( \tilde{Y}^{s, \sigma} = \{(u_0, A_0, A_1) \in H^s \oplus H^\sigma \oplus H^{\sigma-1}; -\text{div} A_1 = |u_0|^2\} \).

**Theorem 1.3.** Let \( s \geq 5/3 \) and \( \max\{4/3, s - 1\} \leq \sigma \leq \min\{s + 1, (5s - 2)/3\} \) with \( (s, \sigma) \neq (5/2, 7/2) \). Then there exists \( T > 0 \) such that MS-T with initial condition \( (1.11) \) has a unique solution \((u, A)\) satisfying \((u, A, \partial_t A) \in C([0, T]; \tilde{Y}^{s, \sigma})\). Moreover, if \( \sigma \geq (2s + 1)/4 \) with \( (s, \sigma) \neq (5/2, 3/2) \), then the map \((u_0, A_0, A_1) \mapsto (u, A, \partial_t A)\) is continuous as a map from \( \tilde{Y}^{s, \sigma} \) to \( C([0, T]; \tilde{Y}^{s, \sigma}) \).

This paper is organized as follows. In Section 2, we introduce some elementary estimates required in this paper. In Section 3, we construct the evolution operator for the linear Schrödinger equation. In Section 4, we prepare a priori estimates for the solutions of linearized equation. In Sections 5 and 6, we prove Theorem 1.1 by the contraction mapping principle except for the continuous dependence of the solutions on the data, which is proved in Section 7. In Section 8, We prove Theorems 1.2 and 1.3.

We conclude this section by giving the notation used in this paper. \( \omega = (-\Delta)^{1/2} \) and \( \Omega = (1 - \Delta)^{1/2} \). \( L^p = L^p(\mathbb{R}^3) \) is the usual Lebesgue space and its norm is denoted by \( \| \cdot \|_p \) or \( \| \cdot ; 1/p \| \). \( p' = p/(p - 1) \) is the dual exponent of \( p \). This symbol is used only for Lebesgue exponents. \( H^{s, p} = \{ \phi \in S'(\mathbb{R}^3) ; \| \Omega^s \phi \|_p < \infty \} \) is the usual Sobolev space. For any interval \( I \subset \mathbb{R} \) and Banach space \( X \), \( L^p(I; X) \) denotes the space of \( X \)-valued strongly measurable functions on \( I \) whose \( X \)-norm belong to \( L^p(I) \). This space is often abbreviated to \( L^pX \) when we fix the time interval \( I \). \( W^{m, p}(I; X) \) denotes the space of functions in \( L^p(I; X) \) whose derivatives up to the \( (m - 1) \)-times are locally absolutely continuous and the derivatives up to the \( m \)-times belong to \( L^p(I; X) \). The inequality \( a \lesssim b \) means \( a \leq Cb \), where \( C \) is a positive constant that is not essential. \( (a) = \sqrt{1 + a^2} \). \( a \vee b \) and \( a \wedge b \) denote the maximum and the minimum of \( a \) and \( b \) respectively. We use the following unusual but convenient symbol: \( a_+ \) means \( a \wedge 0 \) if \( a \neq 0 \), whereas \( 0_+ \) means a sufficiently small positive number. Namely \( b \geq a_+ \) means \( b \geq a \wedge 0 \) if \( a \neq 0 \) and \( b > 0 \) if
$a = 0$. It is useful to express sufficient conditions for Sobolev type embeddings $H^{s,r} \hookrightarrow L^p$ by the inequality $(1/r - s/3)_+ \leq 1/p \leq 1/r$.

2. Preliminaries

**Lemma 2.1.** Let $s, s_1, s_2, s_3$ satisfy $0 \leq s \leq s_3$, $s_1 \wedge s_2 \geq (s-2) \vee 0$, and $s_1 + s_2 > 0$. Let

$$s_1 + s_2 + s_3 \wedge (3/2) \geq s + 1$$

and the inequality be strict if (1) $s_j = 3/2$ for some $1 \leq j \leq 3$ or (2) $s = s_3 < 3/2$. Then the following estimate holds:

$$\|\omega^{-2}(u_1 u_2) u_3; H^s\| \lesssim \prod_{j=1}^3 \|u_j; H^{s_j}\|. \quad (2.1)$$

**Proof.** By Leibniz’s rule the left-hand side of (2.1) is bounded by some constant times

$$\|\omega^{-2}(u_1 u_2)\|_{p_1} \|u_3; H^{s_3}\| + \|\omega^{-2+s}(u_1 u_2)\|_{p_3} \|u_3\|_{p_4} \equiv I + II \quad (2.2)$$

with $1/2 = 1/p_1 + 1/p_2 = 1/p_3 + 1/p_4$. We begin with the treatment of the first term. We choose $p_2$ as large as possible provided that $H^{s_2} \hookrightarrow H^{s_3}$ and that the operator $\omega^{-2}: L^\nu \to L^{p_1}$ ($0 < 1/\nu = 1/p_1 + 2/3 < 1$) is bounded by virtue of the Hardy-Littlewood-Sobolev inequality. Certainly if $p_1 = \infty$, we use Hölder’s inequality instead. With such a choice of $(p_1, p_2)$ we have $I \lesssim \|u_1 u_2\|_\nu \|u_3; H^{s_3}\|$. Next we apply Hölder’s inequality and Sobolev’s inequality for the first factor in order to obtain $\|u_1 u_2\|_\nu \lesssim \prod_{j=1}^3 \|u_j; H^{s_j}\|$. Then we obtain

$$I \lesssim \prod_{j=1}^3 \|u_j; H^{s_j}\|.$$

Specifically, if $s_3 = s$, we choose $p_1 = \infty$, $p_2 = 2$. Then we need $s_1 + s_2 > 1$ to obtain the estimate above. If $0 < s_3 - s < 1$, we choose $1/p_2 = 1/2 - (s_3 - s)/3$, and then we need $s_1 + s_2 + s_3 \geq s + 1$. If $s_3 - s \geq 1$, we choose $p_1 = 3 + 0$ so that $\nu = 1 + 0$, and then we need $s_1 + s_2 > 0$.

We proceed to the treatment of the second term, which is essentially similar to that of the first term, but we have to divide the proof into cases in a different fashion. If $s = 0$, we do not need the estimate for $II$. If $0 < s \leq (s_3 \wedge 3/2) - 1$, we choose $1/p_3 = (1 + s)/3 - 0$, $1/p_4 = 1/2 - (1 + s)/3 + 0$. Then we have $\|\omega^{-2+s}(u_1 u_2)\|_{p_3} \lesssim \prod_{j=1}^2 \|u_j; H^{s_j}\|$ provided $s_1 + s_2 > 0$. Since we also have $\|u_3\|_{p_4} \lesssim \|u_3; H^{s_3}\|$, we obtain

$$II \lesssim \prod_{j=1}^3 \|u_j; H^{s_j}\|. \quad (2.3)$$

If $(s_3 \wedge 3/2) - 1 < s \leq 2$ and if $s_3 \neq 3/2$, we choose $1/p_3 = (s_3/3) \wedge (1/2)$, $1/p_4 = (1/2 - s_3/3) \vee 0$. Then we obtain (2.3) provided $s_1 + s_2 + (s_3 \wedge 3/2) \geq s + 1$. We can similarly estimate $II$ even if $s_3 = 3/2$, but the limiting case $s_1 + s_2 + s_3 = s - 1/2$ is excluded from the sufficient condition for the estimate because of the exception of Sobolev’s embedding theorem, namely $H^{3/2} \not\hookrightarrow L^\infty$. If $s > 2$, we choose $p_3 = 2, p_4 = \infty$.\[\]
Then we obtain (2.3) by virtue of Leibniz’s formula together with Sobolev’s inequality. The exceptionally prohibited case in the statement of the lemma comes from the exception of Sobolev’s embedding theorem. □

Lemma 2.2. (1) Let \( \sigma \geq s \vee (1/2) \vee (-s-1) \) and \((s, \sigma) \neq (1/2, 1/2), (-3/2, 1/2)\). Then

\[
\| (\nabla - iA)v; H^s \| \lesssim \| v; H^{s+1} \| \langle \| A; H^\sigma \| \rangle \tag{2.4}
\]

for any \( v \) and \( A \).

(2) Let \( s, s_1, s_2 \) satisfy \( 0 \leq s \leq s_1 \wedge s_2, s_1 + s_2 \geq s + 3/2, (s_1, s_2) \neq (s, 3/2), (3/2, s), \) and let \( \sigma \) satisfy \( \sigma \geq s_2 \vee (1/2), (s, \sigma) \neq (1/2, 1/2) \). Then

\[
\| w(\nabla - iA)v; H^s \| \lesssim \| w; H^{s_1} \| \langle \| A; H^\sigma \| \rangle \tag{2.5}
\]

for any \( w, A \) and \( v \).

Proof. (1) If \( s \geq 0 \), by the Leibniz formula and the Sobolev inequality, we have

\[
\| Av; H^s \| \lesssim \| A; H^\sigma \| \| v; H^{s+1} \|
\]

with \( \sigma \geq (1/2) \vee s, (s, \sigma) \neq (1/2, 1/2) \). If \( -1 \leq s < 0 \), again by the Sobolev inequality

\[
\| Av; H^s \| \lesssim \| A \|_3 \| v; 1/2 - (s + 1)/3 \| \lesssim \| A; H^\sigma \| \| v; H^{s+1} \|
\]

with \( \sigma \geq 1/2 \). If \( s < -1 \), we use duality. We have by the result for \( s \geq 0 \)

\[
\| \langle Av, \psi \rangle \| \leq \| v; H^{s+1} \| \langle A\psi; H^{-s-1} \rangle
\]

\[
\lesssim \| v; H^{s+1} \| \langle A; H^\sigma \| \psi; H^{-s} \|
\]

This estimates yields \( \| Av; H^s \| \lesssim \| A; H^\sigma \| \| v; H^{s+1} \| \). Consequently we have this inequality for all \( s, \sigma \). Using this estimate with the trivial estimate \( \| \nabla v; H^s \| \lesssim \| v; H^{s+1} \| \), we obtain (2.4).

(2) By the Leibniz formula, we have

\[
\| w(\nabla - iA)v; H^s \| \lesssim \| w\|_{r_j}\| (\nabla - iA)v; H^{s,2r_j/(r_j-2)} \| + \| w; H^{s,2r_j/(r_j-2)} \| \| (\nabla - iA)v\|_{r_j},
\]

where \( 1/r_j = (1/2 - s_j/3)^+, j = 1, 2 \). Under the assumption for \( s, s_1 \) and \( s_2 \), the right-hand side does not exceed some positive constant times \( \| w; H^s \| \| (\nabla - iA)v; H^{s_2} \| \) by virtue of the Sobolev inequality. Therefore we obtain (2.5) by (1). □

We define \( \Gamma \) by

\[
\Gamma \equiv \left\{ (s, \sigma); \begin{array}{l}
s \geq 0, \sigma \geq (3/2 - s/2) \vee (1/2) \vee (s/2 - 1/4) \vee (s - 2), \\
(s, \sigma) \neq (7/2, 3/2), (3/2, 1/2), (1/2, 1/2) 
\end{array} \right\}.
\]

Lemma 2.3. Let \( (s, \sigma) \in \Gamma \), \( \text{div} A = 0 \). Then

\[
\| (\mathcal{H}(A) + \phi(u))v; H^{s-2} \| \lesssim \| v; H^s \| \langle \| A; H^\sigma \| \vee \| u; H^{(s-1)\nu_0} \| \rangle \|^2. \tag{2.6}
\]

Moreover if \( s > 0, \sigma > (1/2) \vee (3/4 - s/2) \), then

\[
\| v; H^s \| \lesssim \| (\mathcal{H}(A) + \phi(u))v; H^{s-2} \| + \langle \| A; H^\sigma \| \vee \| u; H^{(s-1)\nu_0} \| \rangle^\alpha \| v \|_2, \tag{2.7}
\]
where \( \alpha = \alpha(s, \sigma) \) is a positive constant independent of \( v \) and \( A \).

**Proof.** First we show

\[
\|(2iA \nabla + A^2)v; H^{s-2}\| \lesssim \|v; H^s\| \|A; H^\sigma\|^2. \tag{2.8}
\]

For \( 0 \leq s \leq 2 \), by the Sobolev inequality,

\[
\|A \nabla v; H^{s-2}\| \lesssim \begin{cases} \|A \nabla v; 1/2 - (s - 2)/3\| & \text{if } 1 \leq s \leq 2, \\ \|A v; H^{s-1}\| & \text{if } 0 < s < 1 \end{cases}
\lesssim \|A\|_3\|v; H^r\|.
\]

Here we have used \( \text{div} \, A = 0 \) when \( 0 \leq s < 1 \). We also have

\[
\|A^2 v; H^{s-2}\| \lesssim \|A^2 v\|_r \lesssim \|A; (1/2 - \sigma/3)_+\|^2\|v; 1/r - 2(1/2 - \sigma/3)_+\|
\lesssim \|A; H^\sigma\|^2\|v; H^s\|,
\]

where \( 1/2 \leq 1/r \leq 1 - (1/2 - (2 - s)/3)_+ = (1/2 - s/3)_+ \leq 1/r - 2(1/2 - \sigma/3)_+ \leq 1/2 \). Such \( r \) exists if \( (s, \sigma) \in \Gamma \) with \( 0 \leq s \leq 2 \). These estimates imply (2.8) for \( 0 \leq s \leq 2 \).

For \( s > 2 \), by the Leibniz formula and the Sobolev inequality, we have

\[
\|A \nabla v; H^{s-2}\| \lesssim \|A\|_3\|v; H^r\|\|v\|_\infty + \|A\|_2^2\|v; H^{s-2,2r/(r-2)}\|
\lesssim \|A\|_3\|v; H^r\|\|v\|_\infty + \|A\|_2^2\|v; H^{s-2,2r/(r-4)}\|
\lesssim \|A; H^\sigma\|^2\|v; H^s\|,
\]

where \( \sigma \geq s - 2 + 3/r \), \( 1/r = (1/2 - \sigma/3)_+ \). Thus (2.8) has been established. We remark that actually we have

\[
\|A \nabla v; H^{s-2}\| \lesssim \|A; H^\sigma\| \|v; H^{s-\delta}\| \tag{2.9}
\]

\[
\|A^2 v; H^{s-2}\| \lesssim \|A; H^\sigma\|^2 \|v; H^{s-\delta}\|. \tag{2.10}
\]

if \( s > 0 \), \( \sigma > 1/2 \) and \( \sigma > (3/4 - s/2) \). Here \( \delta \) is a sufficiently small positive number.

These inequalities will be used to prove (2.7). The estimates (2.6) follows from (2.8) and the inequality

\[
\|\phi(u)v; H^{s-2}\| \lesssim \|u; H^{(s-1)\gamma_0}\|^2\|v; H^{(s-1)+}\|. \tag{2.11}
\]

If \( s \geq 2 \), this inequality follows from Lemma 2.1 directly. If \( s < 2 \), this follows from the duality argument such as

\[
|\langle \phi(u)v, \psi \rangle| = |\langle u, \omega^{-2}(\bar{v}\psi)u \rangle| \lesssim \|u\|_2\|u; H^{(s-1)\gamma_0}\|^2\|v; H^{(s-1)+}\|\|\psi; H^{2-s}\|.
\]

Next we show (2.7). Clearly

\[
\|v; H^s\| \lesssim \|(H(A) + \phi(u))v; H^{s-2}\| + \|(2iA \nabla + A^2 + \phi(u))v; H^{s-2}\| + \|u\|_2.
\]

We apply the interpolation inequality

\[
a\|v; H^{s-\delta}\| \lesssim a\|v\|_2^{\delta/s}\|v; H^\delta\|^{(s-\delta)/s} \lesssim \varepsilon\|v; H^s\| + C(\varepsilon)a^{\delta/\delta}\|v\|_2 \tag{2.12}
\]

where \( \alpha = \alpha(s, \sigma) \) is a positive constant independent of \( v \) and \( A \).
to (2.9)-(2.11), where $a > 0$ is a constant. Then we obtain (2.7) by taking $\varepsilon$ sufficiently small. □

**Remark.** Let the assumption for (2.7) be satisfied and let $A(t) \in C(I; H^s)$, $u(t), v(t) \in C(I; H^{s-\delta})$ for some $\delta > 0$. Then by estimates similar to (2.9)-(2.11), we have $A\nabla v, A^2 v, \phi(u)v \in C(I; H^s)$. This fact will be used later.

### 3. Estimates for solutions to linear Schrödinger equations

For our treatment of MS, we need energy estimates for linear Schrödinger equations with electro-magnetic potentials. Let $I \subset \mathbb{R}$ be a compact interval, $\phi : I \times \mathbb{R}^3 \to \mathbb{R}$, $A : I \times \mathbb{R}^3 \to \mathbb{R}^3$, and $t_0 \in I$. We consider the equation

$$i\partial_t v = \mathcal{H}(A)v + \phi v \quad (3.1)$$

with initial data

$$v(t_0) = v_0. \quad (3.2)$$

For a while we regard $u$ and $A$ as known functions, and consider the linear Cauchy problem (3.1)-(3.2).

Before we proceed to energy estimates, we clarify the concept of the solution. Let $s \geq 0$. A function $v$ is called an $H^s$-solution to (3.1) in $I$ if $v \in C(I; H^s) \cap W^{1,1}(I; H^{s-2})$ and satisfies (3.1) almost every $t \in I$. Moreover, if $v_0 \in H^s$ and $v$ satisfies (3.2), then $v$ is called an $H^s$-solution to (3.1)-(3.2).

The $L^2$-norm of $H^1$-solutions are clearly conserved. For, if $v(t)$ is an $H^1$-solution, then $\|v(t)\|_2^2$ is absolutely continuous, and (3.1) yields

$$d\|v(t)\|_2^2/dt = 2\text{Re} \langle \partial_t v, v \rangle_{H^{-1} \times H^1} = 0$$

for almost every $t \in I$. This also implies the uniqueness of the $H^1$-solution to (3.1)-(3.2).

For a while we assume the following:

**Assumption (A1).** (1) $A \in L^\infty(I; H^1) \cap W^{1,1}(I; L^3)$ with $\text{div} A = 0$;

(2) $\phi = \phi(u) = \omega^{-2}|u|^2$ with $u \in L^\infty(I; H^{3/4})$.

**Lemma 3.1.** We assume (A1). Then (3.1)-(3.2) has a unique $H^2$-solution $v$, which satisfies the estimate

$$\|\mathcal{H}(A)v(t)\|_2 + \langle l \rangle^4\|v(t)\|_2 \leq \left\{ \|\mathcal{H}(A(t_0))v_0\|_2 + \langle l \rangle^4\|v_0\|_2 \right\} \times \exp \left\{ C\|u\|_{L^2(I; H^{3/4})}^2 + C\|\partial_t A; L^1(I; L^3)\| \right\}$$

\[(3.3)\]

for any $t \in I$. Here $l = \|A; L^\infty(I; H^1)\|$. Moreover, if $u \in C(I; L^2)$ then $v \in C^1(I; L^2)$.\]
Proof. We first prove (3.3) rather formally. By direct computation, self-adjointness of $\mathcal{H}(\mathbf{A})$ and Schwarz’s inequality
\[
\frac{1}{2} \frac{d}{dt} \|\mathbf{A}v\|_2^2 = \text{Im} \langle \mathcal{H}(\mathbf{A} + \phi)v, \mathcal{H}v \rangle + 2\partial_t \mathbf{A}(\nabla - i\mathbf{A})v, \mathcal{H}v \rangle \\
\leq \left\{ \|\mathcal{H}\phi v\|_2 + 2\|\partial_t \mathbf{A}(\nabla - i\mathbf{A})v\|_2 \right\} \|\mathbf{A}v\|_2.
\]
The quantities in the brackets are estimated by Sobolev’s inequality, Lemmas 2.1 and (2.7) with $u = 0$. Indeed we have
\[
\|\mathcal{H}\phi v\|_2 \lesssim \|\phi v; H^2\| + \langle l \rangle^4 \|\phi v\|_2 \\
\lesssim \|u; H^3/4\|^2 \{ \|v; H^2\| + \langle l \rangle^4 \|v\|_2 \} \\
\lesssim \|u; H^3/4\|^2 \{ \|\mathcal{H}v\|_2 + \langle l \rangle^4 \|v\|_2 \},
\]
and
\[
\|\partial_t \mathbf{A}(\nabla - i\mathbf{A})v\|_2 \leq \|\partial_t \mathbf{A}\|_3 \|(\nabla - i\mathbf{A})v\|_6 \lesssim \|\partial_t \mathbf{A}\|_3 \{ \|\mathcal{H}v\|_2 + \langle l \rangle^4 \|v\|_2 \}.
\]
We remark that we can take $\alpha = 4$ in (2.7). In the last inequality we have used the estimate
\[
\|\mathbf{A}v\|_6 \leq \|\mathbf{A}\|_6 \|v\|_\infty \lesssim \|\mathbf{A}; H^1\|^1 \|v\|^{1/4}_2 \|H^2\|^{3/4} \lesssim \|v; H^2\| + l^4 \|v\|_2.
\]
Therefore we obtain the differential inequality
\[
\frac{d}{dt} \{ \|\mathbf{A}v\|_2 + \langle l \rangle^4 \|v\|_2 \} \lesssim \{ \|u; H^3/4\|^2 + \|\partial_t \mathbf{A}\|_3 \} \{ \|\mathcal{H}v\|_2 + \langle l \rangle^4 \|v\|_2 \},
\]
which yields (3.3) by virtue of Gronwall’s inequality. Here we have used the $L^2$-norm conservation law in the light-hand side. We next prove the existence of the solution. If $\mathbf{A}$ and $u$ are sufficiently smooth, the existence of the solution is proved by Kato’s abstract method [5, 6], or the parabolic regularization technique. Indeed, the condition $u, \mathbf{A} \in L^2(I; H^{s+1/2/4+0})$ with $s \geq 2$ will suffice to prove the $H^s$-wellposedness. To construct the solution under the assumption $\mathbf{A}_k = \eta_k * \mathbf{A}$, $u_k = \eta_k * u$, $k = 1, 2, \cdots$, and consider the problem (3.1)-(3.2) with $(u, \mathbf{A})$ replaced by $(u_k, \mathbf{A}_k)$, where $\eta_k(x) = k^3 \eta(kx)$, $\eta \in \mathcal{S}$. Let $v_k$ be the corresponding solution to the regularized equation mentioned above. Then $v_k$ satisfies the estimate (3.3). Accordingly $\sup_k \|v_k; L^{\infty}(I; H^2)\| < \infty$ by virtue of Lemma 2.2. Therefore there exists a subsequence of $\{v_k\}$ that converges to some function $v \in L^{\infty}(I; H^2)$ in $w^*$-sense. We can easily check that $v$ satisfies (3.3). The function $v$ belongs to $W^{1,1}(I; L^2)$ and satisfies (3.1) almost every $t$ since $v$ satisfies the integral version of (3.1)-(3.2), namely
\[
v(t) = v_0 - i \int_{t_0}^{t} [\mathcal{H}(\mathbf{A}) + \phi(u)] v(\tau) d\tau.
\]
For, each $v_k$ clearly satisfies (3.4) with $(u, \mathbf{A})$ replaced by $(u_k, \mathbf{A}_k)$, and $\{v_k\}$ converges to $v$ in $w^*$-sense as $k \to \infty$ along some suitable subsequence. Finally we prove the strong continuity of $v(t)$ in $H^2$. To this end we remark that $v \in C_w(I; H^2) \cap C(I; H^s)$ with
s < 2 by (3.4) and that \( A \in C(I; L^p) \) with \( 3 \leq p < 6 \) by (A1). Hence we can show that 
\( 2iA \nabla v + |A|^2 v \) is strongly continuous in \( L^2 \) and that \( \mathcal{H}(A(t))v(t) \) is weakly continuous in \( L^2 \). We use the estimate (3.3) with \( I = [t_0, t] \) and the conservation of the \( L^2 \)-norm to obtain 
\[ \limsup_{t \to t_0} \| \mathcal{H}(A(t))v(t) \|_2 \leq \| \mathcal{H}(A(t_0))v(t_0) \|_2. \]
This inequality and the weak continuity conclude the strong continuity of \( \mathcal{H}(A(t))v(t) \) in \( L^2 \), and hence \( v(t) \) is strongly continuous in \( H^2 \). The last part of the lemma is so easy that we omit the proof. □

By virtue of the lemma above, we can define the evolution operator for (3.1). Under the assumption (A1), we define a two-parameter family of operators \( \{U(t, \tau)\}_{t, \tau \in I} \) by the relation
\[ U(t, \tau)v(\tau) = v(t). \]
Namely, we arbitrarily give the initial data at the time \( \tau \), say \( v(\tau) \), and solve (3.1) up to the time \( t \); then we define the image of \( v(\tau) \) by \( v(t) \). In what follows we omit the lower indices \( u, A \) unless it causes any confusion. Clearly this family of operators is well-defined, and has the group property:
\[ U(t, \tau)U(\tau, \tau') = U(t, \tau'), \]
\[ U(t, t) = 1 \] (3.6)
for \( t, \tau \in I \). On account of Lemma 3.1 together with Lemma 2.3, \( U(t, \tau) \) are uniformly bounded operators on \( H^2 \) with the estimate
\[ K_2 \equiv \sup_{t, \tau \in I} \| U(t, \tau); H^2 \to H^2 \| \]
\[ \lesssim \{1 + \| A; L^\infty(I; H^1)\|^4 \exp \{ C\| u; L^2(I; H^{3/4})\|^2 + C\| \partial_t A; L^1(I; L^3)\| \}. \] (3.7)
This family is strongly continuous in \( H^2 \). Namely, for any \( \psi \in H^2 \), the function
\[ (t, \tau) \in I \times I \mapsto U(t, \tau)\psi \]
is strongly continuous in \( H^2 \). Indeed, \( U(t, \tau)\psi \) is strongly continuous in \( t \). Combining this fact with (3.6)-(3.7), we obtain the strong continuity as a two-variable function.

**Lemma 3.2.** Under the assumption (A1) \( \{U(t, \tau)\} \) defined by (3.5) can be uniquely extended to a strongly continuous two-parameter family of operators on \( H^s \), \( 0 \leq s \leq 2 \), with the estimate
\[ K_s \equiv \sup_{t, \tau \in I} \| U(t, \tau); H^s \to H^s \| \leq K_2^{s/2}. \] (3.8)
Especially, \( \{U(t, \tau)\} \) is a unitary group on \( L^2 \) and
\[ U(t, \tau)^* = U(\tau, t). \] (3.9)
Moreover, for any \( v_0 \in H^s \), \( U(t, t_0)v_0 \) is a unique \( H^s \)-solution to (3.1)-(3.2).
Proof. \( \{U(t,\tau)\} \) can be extended as a family of unitary operators in \( L^2 \) on account of the \( L^2 \)-norm conservation law and the fact that each \( U(t,\tau) \) is a bijection on \( H^2 \). Therefore (3.8) is proved by interpolation. Therefore the first part of the lemma has been proved except strong continuity of \( U \); this is a consequence of the continuity of the \( H^s \)-solution, which is proved below. The relation (3.3) follows from the unitarity and the group property. The latter part is proved by approximation. Let \( \{v_{0j}\}_{j=1}^\infty \subset H^2 \) be a sequence converging to \( v_0 \) in \( H^s \). Then \( v_j(t) = U(t,t_0)v_{0j} \) are \( H^2 \)-solutions and the sequence \( \{v_j\} \subset C(I;H^2) \) strongly converges to \( v(t) \equiv U(t,t_0)v_0 \) in \( L^\infty(I;H^s) \) by virtue of the estimate (3.8). Hence \( v \in C(I;H^s) \). Moreover, \( v \) satisfies (3.3) since each \( v_j \) satisfies this equation with \( v_0 \) replaced by \( v_{0j} \), and since \( v_j \to v \) strongly in \( C(I;H^s) \). This fact implies that \( v \in W^{1,1}(I;H^{s-2}) \). Therefore \( v \) is an \( H^s \)-solution. Finally we show uniqueness, which has yet to be proved in the case \( s < 1 \). Let \( v(t) \) be an \( H^s \)-solution, and \( \psi \in H^2 \) be an arbitrary function. Then 

\[
\frac{d}{dt} \langle v(t), U(t,\tau)\psi \rangle = \langle -i(\mathcal{H} + \phi) v, U(t,\tau)\psi \rangle + \langle v(t), -i(\mathcal{H} + \phi) U(t,\tau)\psi \rangle = 0.
\]

Therefore \( \langle v_0, U(t_0,\tau)\psi \rangle = \langle v(t), U(t_0,\tau)\psi \rangle = \langle v(\tau), U(\tau,\psi) \rangle = \langle v(\tau), \psi \rangle \). This means \( v(\tau) = U(\tau,t_0)v_0 \) for any \( \tau \in I \). Therefore the uniqueness has been proved. \( \square \)

Corollary 3.1. Under the assumption \([A1]\) \( \{U(t,\tau)\} \) defined by (3.3) can be extended uniquely to a strongly continuous family on \( H^{-s}, 0 < s \leq 2, \) and 

\[
\|U(t,\tau); H^{-s} \to H^{-s}\| \leq K_s.
\]

Proof. The corollary immediately follows from Lemma 3.2 by duality. \( \square \)

Next we consider the inhomogeneous problem.

Lemma 3.3. We assume \([A1]\) Let \( f \in L^1(I;H^{-2}) \), and \( v \in C(I;L^2) \cap W^{1,1}(I;H^{-2}) \) be an \( L^2 \)-solution to

\[
i\partial_t v = \mathcal{H}(A)v + \phi(u)v + f.
\]

(3.10)

Then for any \( t_0 \in I \),

\[
v(t) = U(t,t_0)v(t_0) - i \int_{t_0}^t U(t,\tau)f(\tau)d\tau.
\]

(3.11)

Here \( \{U(t,\tau)\} \) is the evolution operator for (3.1).

Proof. We take \( \psi \in H^2 \) arbitrarily. Then \( \langle v(\tau), U(\tau,t)\psi \rangle \) is absolutely continuous with respect to \( \tau \) and for almost every \( \tau \in I \)

\[
\frac{d}{d\tau} \langle v(\tau), U(\tau,t)\psi \rangle = \langle -if(\tau), U(\tau,t)\psi \rangle = -i\langle U(t,\tau)f(\tau), \psi \rangle.
\]

Integrating this formula with respect to \( \tau \) on \( [t_0,t] \), we obtain

\[
\langle v(t), \psi \rangle = \langle U(t,t_0)v(t_0), \psi \rangle - i \int_{t_0}^t \langle U(t,\tau)f(\tau), \psi \rangle d\tau.
\]
This means (3.11). □

We proceed to the case $s > 2$.

**Lemma 3.4.** Let $s > 2$, $\sigma \geq \max\{s - 2, (2s - 1)/4, 1\}$ and $(s, \sigma) \neq (7/2, 3/2)$. Let $A \in \bigcap_{j=0}^{2} W^{j, \infty}(I; H^{\sigma-j})$ with $\text{div } A = 0$ and $\partial_t A \in L^1(I; L^3)$. Let $u \in \bigcap_{j=0}^{2} W^{j, \infty}(I; H^{s-2j})$. Let $v_0 \in H^s$. Then the $H^2$-solution $v$ to (3.1)–(3.2) actually belongs to $C^\infty(I; H^s)$ and satisfies

\[
\|v; L^\infty(I; H^s)\| \lesssim K_{s-4}(M^2_\sigma \vee R^1_{s-1})\alpha \|v_0; H^s\| \times \exp\left(CK_{s-4}(M^2_\sigma \vee R^1_{s-1})^2 \int_I \langle \|\partial_t A\|_3 \rangle dt \right). \tag{3.12}
\]

Here $M^k_\sigma = \max_{0 \leq j \leq k} \|\partial_t^j A; L^\infty(I; H^{\sigma-j})\|$, $R^k = \max_{0 \leq j \leq k} \|\partial_t^j u; L^\infty(I; H^{s-2j})\|$ and $\alpha$ is some positive constant. Moreover if $v = u$ and $s \geq 5/2$, we have

\[
\|u; L^\infty(I; H^s)\| \lesssim K_{s-4}(M^2_\sigma \vee R^1_{s-1})^\alpha \|u_0; H^s\| \times \exp\left(CK_{s-4}(M^2_\sigma \vee R^1_{s-1})^2 \int_I \langle \|\partial_t A\|_3 \rangle dt \right). \tag{3.13}
\]

**Remark.** For $-2 \leq s \leq 2$, $K_s$ is defined and estimated as in Lemma 3.2 and Corollary 3.1. Therefore once we have obtained (3.12), this estimate ensures that $\{U(t, \tau)\}$ is a family of operators on $H^s$ and gives an upper bound of $K_s$ for $2 < s \leq 6$. Repeating this process, we can inductively estimate $K_s$ for all $s > 2$.

**Proof.** In the following proof, the exponent $\alpha$ may be different line to line; precisely we have to replace $\alpha$ by the greatest one that has ever appeared, but for simplicity we omit this process and use the same letter $\alpha$. We estimate $\|\partial_t^2 u; H^{s-4}\|$ instead of $\|u; H^s\|$ since they are expected to be equivalent. To this end, we differentiate (3.1) in $t$ twice. By simple calculation, we obtain

\[
i\partial_t^2 v = (\mathcal{H} + \phi)\partial_t v + (2i\partial_t A(\nabla - iA) + \partial_t \phi)v, \tag{3.14}
\]

\[
i\partial_t^3 v = (\mathcal{H} + \phi)\partial_t^2 v + 4i\partial_t A(\nabla - iA)\partial_t v
\]

\[
+ 2\partial_t \partial_t A(\nabla - iA)v + 2(\partial_t A)^2 v + \partial_t^2 \phi v
\]

\[
\equiv (\mathcal{H} + \phi)\partial_t^2 v + F_1 + \cdots + F_5. \tag{3.15}
\]

We put $F \equiv \sum_{j=1}^{5} F_j$. By virtue of Lemma 3.3 we convert (3.15) to the integral form. Precisely we need $s \geq 4$ to apply the lemma, but we have the expression below for $s > 2$ by regularizing technique:

\[
\partial_t^2 v(t) = U(t, t_0)\partial_t^2 v(t_0) - i \int_{t_0}^{t} U(t, \tau)F(\tau)d\tau. \tag{3.16}
\]

Therefore

\[
\|\partial_t^2 v(t); H^{s-4}\| \leq K_{s-4}\{\|\partial_t^2 v(t_0); H^{s-4}\| + \int_{t_0}^{t} \|F(\tau); H^{s-4}\|d\tau\}. \tag{3.17}
\]
We estimate the right-hand side. First, we prove that the following equivalence holds for the solution $v$:

$$
\|v; H^s\| + <M_1^1 \lor R_{s-1}^1>^\alpha \|v\|_2 \simeq \|\partial_t^2 v; H^{s-4}\| + <M_1^1 \lor R_{s-1}^1>^\alpha \|v\|_2. \quad (3.18)
$$

By the use of (2.7) twice and equations (3.1), (3.14), we obtain for $s > 2$

$$
\|v; H^s\| \lesssim \|\partial_t^2 v; H^{s-4}\| + \|\partial_t A(\nabla - iA)v; H^{s-4}\| + \|\partial_t \phi v; H^{s-4}\|
$$

$$
+ <M_0^0 \lor R_{s-1}^0>^\alpha \|v\|_2 + <M_0^0 \lor R_{s-1}^0>^\alpha \|v\|_2
$$

$$
\equiv \|\partial_t^2 v; H^{s-4}\| + F_6 + \cdots + F_9.
$$

We begin with the estimate of $F_6$. If $s \geq 4$, by the Leibniz formula and (2.4) we have

$$
\|\partial_t A(\nabla - iA)v; H^{s-4}\|
$$

$$
\lesssim \|\partial_t A; H^{s-4,6}\|\|\nabla - iA\|_3 + \|\partial_t A\|_6\|\nabla - iA\|_3\| A\|_6
$$

$$
\lesssim \|\partial_t A; H^{s-3}\|\|\nabla - iA\| v; H^{s-3}\|
$$

$$
\lesssim \|\partial_t A; H^{s-3}\|\|v; H^{s-2}\|\| A; H^s\|). \quad (3.19)
$$

If $2 < s < 4$, we choose $p$ so that $(1/2 - (s - 2)/3) \leq 1/p \leq 1/2 - (1/2 - (4 - s)/3)$. Then by the Sobolev inequality we have the continuous embeddings $H^{s-2} \hookrightarrow L^p$ and $L^q \hookrightarrow H^{s-4}$, where $1/q = 1/2 + 1/p$. Using these embeddings together with (2.4) we obtain

$$
\|\partial_t A(\nabla - iA)v; H^{s-4}\| \lesssim \|\partial_t A(\nabla - iA)v\|_q \lesssim \|\partial_t A\|_2\| (\nabla - iA) v\|_p
$$

$$
\lesssim \|\partial_t A\|_2\| (\nabla - iA) v; H^{s-2}\| \lesssim \|\partial_t A\|_2\|v; H^{s-1}\|\| A; H^s\|).
$$

Therefore we get

$$
F_6 \lesssim <M_1^1>^2\|v; H^{s-1}\|.
$$

Next we derive the estimate for $F_7$. We have

$$
F_7 \lesssim <R_{s-1}^1>^2\|v; H^{s-2}\|.
$$

If $s \geq 3$, this is proved by Lemma 2.1 as

$$
F_7 \lesssim \|\partial_t \phi u; H^{s-3}\| \lesssim \|\partial_t u; H^{s-3}\|\|u; H^{s-1}\|\|v; H^{s-2}\|.
$$

If $2 < s < 3$, we use the duality estimate as follows, from which we obtain the desired result:

$$
|\langle \omega^{-2} \partial_t u \bar{u}, \psi \rangle| = |\langle \partial_t u, \omega^{-2}(\bar{\psi} u) \rangle| \leq \|\partial_t u; H^{s-3}\|\|\omega^{-2}(\bar{\psi} u); H^{3-s}\|
$$

$$
\lesssim \|\partial_t u; H^{s-3}\|\|v; H^{s-2}\|\|\psi; H^{4-s}\|\|u; H^{s-1}\|.
$$

The estimate for $F_8$ is easy. Indeed we obtain by (2.4)

$$
F_8 \lesssim \|v; H^2\|\| A; H^s\| \lor \|u\|_2^\alpha \lesssim <M_1^1 \lor R_{s-1}^0>^\alpha \|v; H^2\|.
$$
Using these estimates with (2.12), we obtain for any $\varepsilon > 0$
\[
\sum_{j=6}^{9} F_j \leq C(\varepsilon)\|v\|_2\langle M_1^1 \vee R_{s-1}^1 \rangle^\alpha + \varepsilon\|v; H^s\|,
\] (3.20)
where $C(\varepsilon)$ is a positive constant. Therefore we have proved
\[
\|v; H^s\| \lesssim \|\partial_t^2 v; H^{s-4}\| + \langle M_1^1 \vee R_{s-1}^1 \rangle^\alpha \|v\|_2.
\]
The opposite inequality in (3.18) is similarly proved. Applying this inequality to (3.17) together with the $L^2$-norm conservation law, we obtain the following intermediate estimate:
\[
\|v(t); H^s\| \lesssim K_{s-4}\langle M_1^1 \vee R_{s-1}^1 \rangle^\alpha \|v_0; H^s\| + \int_0^t \|F(\tau); H^{s-4}\|d\tau.
\]
The next step is the estimate of $F = \sum_{j=1}^5 F_j$ in $H^{s-4}$. We begin with the estimate of $F_1$. If $s \geq 4$, by the estimate (3.19) with $v$ replaced by $\partial_t v$, we have
\[
\|F_1; H^{s-4}\| \lesssim \|\partial_t A; H^{s-3}\|\langle\|A; H^\sigma\||\partial_t v; H^{s-2}\|.
\]
If $3 \leq s < 4$, we choose $1/p = 1/2 + (4-s)/3$, $1/q = 1/2 - (s-3)/3$. Then by the Sobolev inequality and (2.4)
\[
\|F_1; H^{s-4}\| \lesssim \|\partial_t A; H^{s-3}\|\langle\|A; H^\sigma\||\partial_t v; H^{s-2}\|.
\]
For $2 < s < 3$, we use $\text{div } A = 0$ and (2.4) to have
\[
\|F_1; H^{s-4}\| = \|\langle \nabla - iA\rangle(\partial_t A) \partial_t v; H^{s-4}\| \lesssim \|\partial_t A\|_3\langle\|A; H^1\||\partial_t v; H^{s-2}\|.
\]
Here we have used the Sobolev inequality twice as in the previous case. In any cases we have by (2.6)
\[
\|F_1; H^{s-4}\| \lesssim \langle M_1^0 \vee R_{s-1}^0 \rangle^4 \|\partial_t A\|_3\|v; H^s\|.
\]
By (2.6) and the estimate for $F_7$ with $v$ replaced by $\partial_t v$, we have
\[
\|F_2; H^{s-4}\| \lesssim \langle M_1^0 \vee R_{s-1}^1 \rangle^4 \|v; H^s\|.
\]
The estimate of $F_3$. If $s \geq 4$, by the Leibniz formula and (2.4) we have
\[
\|F_3; H^{s-4}\| \lesssim \|\partial_t^2 A; H^{s-4}\|\langle\|\nabla - iA\|\infty + \|\partial_t^2 A; 1/2 - \varepsilon\|\langle\|\nabla - iA\|v; H^{s-4,\varepsilon-1}\|\|\partial_t^2 A; H^{s-4}\|\langle\|A; H^\sigma\||\|v; H^{s-1}\|,
\]
where $\varepsilon$ is a sufficiently small number, and the second term in the right-hand side of the first inequality is removed if $s = 4$. If $2 < s < 4$, we use duality. By (2.5), we have
\[
\langle\langle F_3, \psi\rangle\rangle = \langle\langle \partial_t^2 A, \psi(\nabla + iA)\bar{v}\rangle\rangle \leq \|\partial_t^2 A; H^{\sigma - 2}\|\|\psi; H^{4-s}\| \|v; H^{\sigma + 1}\|\langle\|A; H^\sigma\||\|v; H^{s-1}\|,
with \(1 \vee (2s - 1)/4 \vee (s - 2) \leq \sigma_0 \leq 2, (s, \sigma_0) \neq (7/2, 3/2), (5/2, 1)\). Taking \(\sigma_0\) so that \(\sigma_0 \leq \sigma \wedge (s - 1)\), we obtain
\[
\| F_3; H^{s-4} \| \lesssim \| \partial_t^2 A; H^{s-2} \| \| \langle \| A; H^{s} \| \rangle \| v; H^{s} \|.
\]
Therefore we have
\[
F_3 \lesssim \langle M^2_\sigma \rangle^2 \| v; H^{s} \|.
\]
The estimate of \(F_4\). If \(s \geq 4\), we have by the Leibniz rule and the Sobolev inequality
\[
\| F_4; H^{s-4} \| \lesssim \| \partial_t A; H^{s-4,6} \| \| \partial_t A \|_3 \| v \|_\infty + \| \partial_t A \|_6^2 \| v; H^{s-4,6} \|
\lesssim \| \partial_t A; H^{s-1} \| \| v; H^{s} \|.
\]
If \(s < 4\), we have by the Sobolev inequality
\[
\| F_4; H^{s-4} \| \lesssim \left\{ \begin{array}{ll}
\| \partial_t A \|_3 \| \partial_t A \|_2 \| v \|_\infty & \text{if } s \leq 3,
\| \partial_t A \|_3 \| \partial_t A \|_1/2 - (s - 3)/3 \| v \|_\infty & \text{if } s > 3
\end{array} \right.
\lesssim \| \partial_t A; L^3 \| \| \partial_t A; H^{s-1} \| \| v; H^{s} \|.
\]
Therefore we have
\[
\| F_4; H^{s-4} \| \lesssim \langle \| \partial_t A; L^3 \| \rangle \langle M^0_\sigma \rangle^2 \| v; H^{s} \|.
\]
The estimate for \(F_5\). If \(s \geq 4\), by Lemma \(2.1\) we have
\[
\| \omega^{-2}(\partial_t^2 \bar{u})v; H^{s-4} \| \lesssim \| \partial_t^2 u; H^{s-4} \| \| u; H^{s-2} \| \| v; H^{s-4} \|,
\]
\[
\| \omega^{-2}(\partial_t u \partial_t \bar{u})v; H^{s-4} \| \lesssim \| \partial_t u; H^{s-3} \| \| v; H^{s-2} \|,
\]
which lead
\[
\| F_5; H^{s-4} \| \lesssim \langle R^2_\delta \rangle^2 \| v; H^{s-2} \|.
\]
If \(2 < s < 4\), it is sufficient to show
\[
\| \omega^{-2}(\bar{u} \partial_t^2 u)v, \psi \rangle = \| \langle \partial_t^2 u, \omega^{-2} (\bar{u} \psi) \rangle u \| \lesssim \| \partial_t^2 u; H^{s-4} \| \| v; H^{s-1} \| \| \psi; H^{4-s} \| \| u; H^{s} \|,
\]
\[
\| \omega^{-2}(\partial_t u \partial_t \bar{u})v; H^{s-4} \| \lesssim \| \partial_t u; H^{s-2} \| \| v; H^{s-1} \|,
\]
where we have used Lemma \(2.1\) and \(L^2 \hookrightarrow H^{s-4}\) at the second inequality. Therefore we obtain
\[
\| F_5; H^{s-4} \| \lesssim \langle R^2_\delta \rangle^2 \| v; H^{s-1} \|.
\]
Collecting all the estimates, we obtain
\[
\| v(t); H^{s} \| \lesssim K_{s-4} \{ \| v_0; H^{s} \| \langle M^1_\sigma \vee R^1_{s-1} \rangle^\alpha \}
+ \int_{t_0}^t \| v(\tau); H^{s} \| \langle \| \partial_t A \|_3 \rangle \langle M^2_\sigma \vee R^2_{s-1} \rangle^4 d\tau \},
\]
which leads \(3.12\) by the Gronwall inequality.
To prove (3.13), we only have to modify the estimate of $F_5$ for $5/2 \leq s < 4$. Indeed, if $s \geq 5/2$ we have
\[
\| \omega^{-2}(\bar{u}\partial_t^2 u) \|_{H^s} \lesssim \| \partial_t^2 u \|_{H^{s-4}} \| u \|_{H^{s-1}}^2
\]
\[
\lesssim \langle \| A \| H^\alpha \rangle \vee \| u \|_{H^{s-1}}^\alpha \| u \|_{H^s}
\]
and
\[
\| \omega^{-2}(\partial_t u \partial_t \bar{u}) \|_{H^s} \lesssim \| \partial_t u \|_{H^{s-2}} \| \partial_t \bar{u} \|_{H^{s-3}} \| u \|_{H^{s-1}}
\]
\[
\lesssim \langle \| A \| H^\alpha \rangle \vee \| u \|_{H^{s-1}}^\alpha \| u \|_{H^s}
\]
by virtue of the duality argument. □

**Corollary 3.2.** Let $s, \sigma, u, A$ and $v_0$ satisfy the assumption of Lemma 3.4. Then for the solution $v$ to (3.1)-(3.2) we have the following.

1. The estimate
\[
\max_{j=1,2} \| \partial_j^2 v \|_{L^\infty(I; H^{s-2j})} \lesssim \langle M_\sigma^1 \vee R_{s-1}^1 \rangle^\alpha \| v \|_{L^\infty(I; H^s)}
\]
holds. Here $M_\sigma^1, R_{s-1}^1$ are defined as in Lemma 3.4 and $\alpha$ is some positive number.

2. If $A \in C(I; H^\sigma)$, then $v \in \bigcap_{j=0}^2 C(I; H^{s-2j})$. Especially, $\{U(t, \tau)\}$ is a strongly-continuous family on $H^s$.

**Proof.** (1) is the consequence of (2.6) and (3.13). We prove (2) for $2 < s \leq 6$. $F = \sum_{j=1}^5 F_j$ in (3.15) belongs to $L^1(I; H^{s-4})$. Therefore by virtue of Lemma 3.1 together with Lebesgue’s convergence theorem, the right-hand side of (3.16), and hence $\partial_t^2 v$ belong to $C(I; H^{s-4})$. To prove $\partial_t v \in C(I; H^{s-2})$, it suffices to show $\Delta \partial_t v \in C(I; H^{s-4})$ since we have $\partial_t v \in C(I; L^2)$. If we recall the remark for Lemma 2.3 and use the estimates for $F_6, F_7$ in the proof of Lemma 3.3, we can show that all the terms except $\Delta \partial_t v$ in (3.14) belong to $C(I; H^{s-4})$. Therefore $\Delta \partial_t v \in C(I; H^{s-4})$. Analogously we can prove $v \in C(I; H^s)$ by (3.1). General case is proved by induction. □

### 4. Linearized equation for MS-C

To solve MS-C, we consider the linearized equation below:
\[
i \partial_t v = (\mathcal{H}(A) + \phi(u))v,
\]
\[
v(0) = u_0,
\]
\[
(\partial_t^2 - \Delta + 1)B = PJ(u, A) + A,
\]
\[
B(0) = A_0, \partial_t B(0) = A_1.
\]
We always assume $\text{div } A = 0$ and $(u_0, A_0, A_1) \in X^{s,\sigma}$. In later sections we often use the equations with $(u, A, v, B)$ replaced by $(u', A', v', B')$ and $(u_0, A_0, A_1)$ by $(u_0', A_0', A_1')$. We refer to such equations as (1.1)', (1.2)', and we often abbreviate $\mathcal{H}(A'), \phi(u'), J(u', A')$ to $\mathcal{H}', \phi', J'$. If we define the map
\[
\Phi : (u, A) \mapsto (v, B),
\]
the fixed points of $\Phi$ solve \textbf{MS-C}. To prove the unique existence of the solution, we show that $\Phi$ is a contraction map in some appropriate function space. In this section we prove a priori estimates for (4.1)-(4.2) which yields that $\Phi$ is a map from some function space to itself. We treat (4.1) by the linear estimates discussed in Section 3, and (4.2) mainly by the Strichartz estimate for the Klein-Gordon equation stated in Lemma 4.1. We also need Lemma 4.2 to treat the nonlinear term in (4.2).

The Klein-Gordon equation
\begin{align*}
(\partial_t^2 - \Delta + 1)A &= f, \\
A(0) &= A_0, \quad \partial_t A(0) = A_1
\end{align*}
is solved as
\begin{align*}
A(t) &= \cos t\Omega A_0 + \frac{\sin t\Omega}{\Omega} A_1 + \int_0^t \frac{\sin(t - \tau)\Omega}{\Omega} f(\tau) d\tau. \tag{4.3}
\end{align*}
We call a pair $(q, r)$ admissible if $2 \leq r < \infty$, $1/r + 1/q = 1/2$. We put $\beta(r) \equiv 1 - 2/r = 2/q$. With this notation, we have the following.

**Lemma 4.1.** Let $(q_j, r_j)$, $j = 0, 1$, be any admissible pairs, $I \subset \mathbb{R}$ be an interval containing 0. Let $(A_0, A_1) \in H^\sigma \oplus H^{\sigma - 1}$ with $\sigma \in \mathbb{R}$, and $f \in L_t^{q_0}(I; H^{\sigma - 1 + \beta(r_0), r_0'})$. Then $A$ in (4.3) belongs to $C(I; H^\sigma) \cap C^1(I; H^{\sigma - 1})$ and satisfies
\begin{align*}
&\|A; L_t^{q_0}(I; H^{\sigma - \beta(r_0), r_0})\| + \|\partial_t A; L_t^{q_0}(I; H^{\sigma - \beta(r_0) - 1, r_0'})\| \\
&\lesssim \|(A_0, A_1); H^\sigma \oplus H^{\sigma - 1}\| + \|f; L_t^{q_0}(I; H^{\sigma - 1 + \beta(r_1), r_1'})\|.
\end{align*}

**Proof.** See for example \cite{1,2,3,11}.

**Lemma 4.2.** (1) Let $s, \sigma, p$ satisfy $s > 1$, $\sigma \geq 0$, max\{3/(s - 1), 2\} $\leq p < \infty$, $(s, p) \neq (5/2, 2)$. Then
\begin{align*}
\|P(u \nabla v); H^{\sigma, \sigma'}\| &\lesssim \|u; H^\sigma\| \|v; H^s\| + \|u; H^s\| \|v; H^\sigma\|.
\end{align*}

(2) Moreover if $1 \leq \sigma \leq s$ or $\sigma \leq s - 1$, then
\begin{align*}
\|P(u \nabla v); H^{\sigma, \sigma'}\| &\lesssim \|u; H^\sigma\| \|v; H^s\|.
\end{align*}

**Proof.** We rewrite and estimate the left-hand side as
\begin{align*}
\|P(u \nabla v); H^{\sigma, \sigma'}\| &= \|P([\Omega^\sigma, u] \nabla v - \nabla u \Omega^\sigma v)\|_{p'} \lesssim \|\,[\Omega^\sigma, u] \nabla v - \nabla u \Omega^\sigma v\|_{p'},
\end{align*}
where we have used the property $P(u \nabla w) = -P(\nabla uw)$ and the fact that $P$ is a bounded operator on $L^{p'}$. By the Kato-Ponce commutator estimate (see Appendix in \cite{8}, Lemma 2.10 in \cite{6}), we have
\begin{align*}
\|\,[\Omega^\sigma, u] \nabla v - \nabla u \Omega^\sigma v\|_{p'} &\lesssim \Omega^\sigma u_2 \|\nabla v\|_{2p/(p-2)} + \|\nabla u\|_{p_1} \|\Omega^\sigma v\|_{p_2}
\end{align*}
with $1/p' = 1/p_1 + 1/p_2$, $p' \leq p_2 < \infty$. By putting $p_2 = 2$ and using the embedding $H^{\sigma - 1} \hookrightarrow L^{2p/(p-2)}$, we obtain the required estimate (1). The proof for (2) when $\sigma \leq s - 1$
follows from the direct application of the Leibniz formula to \( \|u \nabla v; H^{\sigma', \nu'} \| \) and the Sobolev inequality. For the proof of (2) when \( 1 \leq \sigma \leq s \), we may assume \( s - 1 < \sigma \leq s \). Putting \( 1/p_2 = 1/2 - (s - \sigma)/3 \), we obtain \( \| \nabla u \|_{p_1} \| \Omega^2 v \|_{p_2} \lesssim \| u; H^{\sigma} \| \| v; H^{s} \| \) again by the Sobolev inequality. \( \Box \)

**Lemma 4.3.** Let \( s, \sigma \) satisfy \( s > 1 \) and \( 1 \leq \sigma \leq \min\{ (5s - 2)/3, s + 1 \} \) with \( (s, \sigma) \neq (5/2, 7/2) \). Let \( A, B \) satisfy (1.2). Then for \( I = [0, T] \) with \( 0 < T \leq 1 \) and for any admissible pair \( (q, r) \) the following estimate holds.

\[
\max_{j=0,1} \| \partial_t^j B; L^\infty(I; H^{\sigma - j}) \cap L^q(I; H^{\sigma - (r - j)/r}) \| \\
\lesssim \| (A_0, A_1); H^\sigma \oplus H^{\sigma - 1} \| + T^{1/2} \{ \| u; L^\infty(I; H^s) \| \lor \| A; L^\infty(I; H^\sigma) \| \}^3. \quad (4.4)
\]

\( B \in \bigcap_{j=0}^1 \bigcap_{s} \bigcup_{\sigma} \mathcal{C}(I; H^{\sigma - j}) \) if the right-hand side is finite. Moreover we have

\[
\| \partial_t^2 B; L^\infty(I; H^{\sigma - 2}) \| \\
\lesssim \| (A_0, A_1); H^\sigma \oplus H^{\sigma - 1} \| + \{ \| u; L^\infty(I; H^s) \| \lor \| A; L^\infty(I; H^\sigma) \| \}^3. \quad (4.5)
\]

**Proof.** By Lemma 4.1 the left-hand side of (4.4) is estimated by

\[
\| (A_0, A_1); H^\sigma \oplus H^{\sigma - 1} \| + \| A; L^1 H^{\sigma - 1} \| \\
+ \| P(\tilde{u} \nabla u); L^{q'} H^{\sigma + \beta(r_1 - 1, r_1')} \| + \| A|u|^2; L^1 H^{\sigma - 1} \|,
\]

where \( (q_1, r_1) \) is an admissible pair. Under the assumption of the lemma, there exists an exponent \( r_1 \) such that \( \max\{2, 3/(s - 1)\} \leq r_1 < \infty \) and \( 0 \leq \sigma + \beta(r_1) - 1 \leq s \). By (1) of Lemma 4.2 we have

\[
\| P(\tilde{u} \nabla u); H^{\sigma + \beta(r_1 - 1, r_1')} \| \lesssim \| u; H^{\sigma + \beta(r_1 - 1)} \| \| u; H^s \| \lesssim \| u; H^s \|^2
\]

for this \( r_1 \). On the other hand we have

\[
\| A|u|^2; H^{\sigma - 1} \| \lesssim \| A; H^{\sigma - 1, 6} \| \| u \|_6^2 + \| A|u|; H^{\sigma - 1, p_1} \| \| u \|_{p_2}
\lesssim \| A; H^\sigma \| \| u; H^s \|^2,
\]

where \( 1/\nu = 1/2 - 1/p_1 - 1/p_2 \). We choose \( p_1 = p_2 = 6 \) if \( \sigma \leq s \), \( 1/p_1 = 1/2 - (s + 1 - \sigma)/3 \), \( 1/p_2 = (1/2 - s/3) \) if \( s < \sigma \leq s + 1 \) so that \( H^s \hookrightarrow H^{\sigma - 1, p_1}; L^{p_2} \). With such a choice \( H^\sigma \hookrightarrow L^\nu \) under the assumption of the lemma. With these estimates and the Hölder inequality for the time variable, we obtain (4.4). Finally we prove (4.5). By (4.2) we have

\[
\| \partial_t^2 B; L^\infty H^{\sigma - 2} \| \leq \| B; L^\infty H^\sigma \| + \| A + P J; L^\infty H^{\sigma - 2} \|
\]

We have \( \| P J; L^\infty H^{\sigma - 2} \| \lesssim \{ \| u; L^\infty(I; H^s) \| \lor \| A; L^\infty(I; H^\sigma) \| \}^3 \) similarly as above, since \( H^{\sigma + \beta(r_1 - 1, r_1')} \hookrightarrow H^{\sigma - 2} \). Therefore we obtain the required result. \( \Box \)
Now we define the function spaces where we consider the map $\Phi$. We put for $s \leq 2$
\[
Z_{s,\sigma} = \left\{ (u, A) \in L^\infty(I; H^s \oplus H^\sigma); \|u; L^\infty(I; H^s)\| \leq l_S, \quad \right.
\]
\[
 \text{div} A = 0, A \in W^{1,0}(I; L^3), \|A; L^\infty(I; H^\sigma)\| \vee \|\partial_t A; L^6(I; L^3)\| \leq l_M \right\}.
\] (4.6)

Here $I = [0, T]$. For $s > 2$, we put
\[
Z_{s,\sigma} = Z_{s,\sigma}, \cap \tilde{Z}_{s,\sigma}
\] (4.7)

with
\[
\tilde{Z}_{s,\sigma} = \left\{ (u, A) \in \bigcap_{j=0}^{2} W^{j,\infty}(I; H^{s-2j} \oplus H^{\sigma-j}); \max_{0 \leq j \leq 2} \|\partial_t^j u; L^\infty(I; H^{s-2j})\| \leq l_S^s, \quad \right.
\]
\[
 \max_{j=0,1} \|\partial_t^j A; L^\infty(I; H^{\sigma-j})\| \leq l_M^\sigma, \|\partial_t^2 A; L^\infty(I; H^{\sigma-2})\| \leq \tilde{l}_M^\sigma \right\}.
\]

Here $s_*(s-1) \vee 2$ and $\sigma_* \leq \sigma$ is a number such that $(s_*,\sigma_*)$ satisfies the assumption of Proposition 4.1 below with $(s,\sigma)$ replaced by $(s_*,\sigma_*)$.

**Proposition 4.1.** Let $s \geq 6/5$, $\max\{4/3, s-2, (2s-1)/4\} \leq \sigma \leq \min\{s+1, (5s-2)/3\}$ and $(s,\sigma) \neq (5/2, 7/2), (7/2, 3/2)$. Let the map $\Phi$ be defined by (4.1), (4.2).

1. If $s \leq 2$, there exist $l_S, l_M, T$ so that $\Phi$ is a map from $Z_{s,\sigma}$ to itself.

2. Let $s > 2$ and let $\Phi$ map $Z_{s,\sigma}$ to itself. Then there exist $L^s_S, L^\sigma_M, \tilde{l}_M^\sigma, T$ so that $\Phi$ is a map from $Z_{s,\sigma}$ to itself.

**Remark.** If $\Phi(Z_{s,\sigma}) \subset Z_{s,\sigma}$, then the same inclusion holds even if we replace $T$ by a smaller one.

**Proof.** (1) Let $(v, B) = \Phi(u, A)$. By Lemmas 3.2 and 4.3 we have
\[
\|v; L^\infty(I; H^s)\| \leq C\langle l_M \rangle^{2s} \exp(CT_S^2 + CT_S^5/l_M)\|u_0; H^s\|
\]

and
\[
\|B; L^\infty(I; H^\sigma)\| \vee \|\partial_t B; L^6(I; L^3)\| \leq C\|\langle A_0, A_1 \rangle; H^\sigma \oplus H^{\sigma-1}\| + CT^{1/2}\langle l_S \vee l_M \rangle^3.
\]

We choose $l_S, l_M$ and $T$ as follows. First, we choose $l_M$ so that $C\|\langle A_0, A_1 \rangle; H^\sigma \oplus H^{\sigma-1}\| \leq l_M/2$. Next we choose $l_S$ so that $C\langle l_M \rangle^{2s}\|u_0; H^s\| \leq l_S/2$. Finally we choose $T$ so that $\exp(CT_S^2 + CT_S^5/l_M) \leq 2$ and that $CT^{1/2}\langle l_S \vee l_M \rangle^3 \leq l_M/2$. Then we have
\[
\|v; L^\infty(I; H^s)\| \leq l_S \text{ and } \|B; L^\infty(I; H^\sigma)\| \vee \|\partial_t B; L^6(I; L^3)\| \leq l_M. \quad \therefore \Phi(Z_{s,\sigma}) \subset Z_{s,\sigma}.
\]

(2) By Lemma 3.3 together with Corollary 3.2 and Lemma 4.3 we have
\[
\max_{0 \leq j \leq 2} \|\partial_t^j v; L^\infty(I; H^{s-2j})\| \leq CK_{s-1}l_M^\sigma \langle l_M \vee \tilde{l}_M^\sigma \rangle\alpha
\]
\[
\times \exp(CK_{s-1}l_M^\sigma \langle l_M \vee \tilde{l}_M^\sigma \rangle T^{5/6}\langle l_M \rangle)\|u_0; H^s\|,
\]
The proof is similar to that of the previous proposition, and left to the reader. □

We also need the following space:

\[ \tilde{Z}_{s,\sigma} = \{(u, A) \in Z_{s,\sigma} \cap W^{1,\infty}(I; H^{s-2} \oplus H^{\sigma-1}); \partial_t A \in C(I; L^2) \} \]

\[ \| \partial_t u; L^{\infty}(I; H^{s-2}) \| \leq l_s, \| \partial_t A; L^{\infty}(I; H^{\sigma-1}) \| \leq l_M \}. \quad (4.8) \]

This space is mainly used to discuss the uniqueness for \( s < 7/4 \).

**Proposition 4.2.** Let \( 6/5 \leq s \leq 2 \) and \( 4/3 \leq \sigma \leq (5s - 2)/3 \). Then there exist \( l_s, l_M \) and \( T \) so that \( \Phi \) is a map from \( \tilde{Z}_{s,\sigma} \) to itself.

The proof is similar to that of the previous proposition, and left to the reader.

## 5. The contraction argument for \( s \geq 7/4 \)

We give the proof of Theorem [14] in the case of \( s \geq 7/4 \). We consider the metric

\[ d(u, A, u', A') \equiv \| u - u'; L^{\infty}(I; L^2) \| \vee \| A - A'; L^{\infty}(I; H^{1/2} \cap L^4(I; L^4)) \|. \quad (5.1) \]

We prepare the following proposition on this metric.

**Proposition 5.1.** Let \( I = [0, T] \) with \( 0 < T \leq 1 \). Let \( (u, A), (u', A') \in Z_{7/4,4/3} \). Let \( (v, B) \) and \( (v', B') \) be the solutions to \( (1.1) \sim (1.2) \) and \( (1.1)' \sim (1.2)' \) respectively. Then the estimate

\[ d(v, B, v', B') \lesssim \|(u_0 - u'_0, A_0 - A'_0, A_1 - A'_1); X^{0.1/2}\| \]

\[ + T^{1/2}(\| v'; L^{\infty}(I; H^{7/4}) \| \vee l_s \vee l_M)^2 d(u, A, u', A') \]

holds.

**Proof.** We write the difference of the equations for \( v \) and \( v' \) as

\[ i \partial_t (v - v') = (\mathcal{H} + \phi)(v - v') + (\mathcal{H} + \phi - \mathcal{H}' - \phi') v'. \quad (5.2) \]

We regard \( (\mathcal{H} + \phi - \mathcal{H}' - \phi') v' \) as the inhomogeneous term and convert this equation to the integral form by virtue of Lemma [33]. Since \( U_{u,A}(t, \tau) \) is unitary on \( L^2 \), we have

\[ \|(v - v')(t)\|_2 \leq \| u_0 - u'_0 \|_2 + \|(\mathcal{H} + \phi - \mathcal{H}' - \phi') v'; L^1 L^2 \|. \quad (5.3) \]

For the second term of the right-hand side we use the identity

\[ (\mathcal{H} + \phi - \mathcal{H}' - \phi') v' = 2i(A - A')(\nabla - i(A + A'))/2 v' + \omega^{-2} Re(u - u')(u - u') v' \]
and estimate it in $L^2$ by the inequalities
\[
\| (A - A')(\nabla - i(A + A')/2)v' \|_2 \lesssim \| A - A' \|_4 \| (\nabla - i(A + A')/2)v' ; H^{3/4} \| \\
\lesssim \| A - A' \|_4 \| (A + A'; H^1) \| \| v' ; H^{7/4} \|,
\]
\[
\| \omega^{-2}(\text{Re}(u - u')(u + u'))v' \|_2 \lesssim \| u - u' \|_2 \| u + u' ; H^1 \| \| v' ; H^1 \|,
\]
where we have used (2.4) and Lemma 2.4. Therefore, by the Hölder inequality for the time variable we have
\[
\| v - v' ; L^\infty L^2 \| \lesssim \| u_0 - u'_0 \|_2 + T^{3/4} \| (v') ; L^\infty H^{7/4} \| \cap l_S \cap l_M \|^2 d(u, A, u', A').
\] (5.4)

On the other hand, by Lemma 4.1 we have
\[
\| B - B' ; L^\infty H^{1/2} \cap L^1 L^1 \| \lesssim \| (A_0 - A'_0, A_1 - A'_1) ; H^{1/2} \oplus H^{-1/2} \|
\]
\[
+ \| A - A' ; L^1 H^{1/2} \| + \| P(J - J') ; L^{4/3} L^{4/3} \|.
\]

For the last term we have the identity
\[
P(J - J') = 2P \text{Im} \{ (u - u')(\nabla - iA)u - i\bar{u}'(A - A')u - (u - u')(\nabla - i\bar{A'})u' \} \tag{5.5}
\]
since $P\{u'(\nabla - iA)(u - u')\} = -P\{(u - u')(\nabla - iA)u\}$. We apply the Hölder and the Sobolev inequalities to (5.5) together with (2.4) and obtain
\[
\| P(J - J') ; L^{4/3} \| \lesssim \langle l_S \cap l_M \rangle (\| u - u' \|_2 \| A - A' \|_4).
\]

Therefore the Hölder inequality for the time variable yields
\[
\| A - A' ; L^1 H^{-1/2} \| + \| P(J - J') ; L^{4/3} L^{4/3} \| \lesssim T^{1/2} \langle l_S \cap l_M \rangle d(u, A, u', A').
\]

Thus we obtain the required estimate. □

**Proof of Theorem 1.1.** Part 1. We prove the unique existence of the solution for $s \geq 7/4$. The case $s < 7/4$ and the continuous dependence on the data will be proved in later sections. We consider the complete metric space $(Z_{s,\sigma}, d)$ and the map $\Phi$ (see Proposition 4.1 and (5.1) for the definition). By Proposition 4.1 we can choose $l_S, l_M, \ldots, L^s_S, L^s_M, L^\sigma_M, T$ so that $\Phi$ is a map from $(Z_{s,\sigma}, d)$ to itself. On the other hand by Proposition 5.1
\[
d(\Phi(u, A), \Phi(u', A')) \leq CT^{1/2} \langle l_S \cap l_M \rangle d(u, A, u', A').
\]

If we take suitable $T$, then $\Phi$ becomes a contraction mapping on $(Z_{s,\sigma}, d)$. This yields the unique existence of the fixed point $(u, A)$. This is the unique solution stated in the theorem. Precisely the following is yet to be checked. First, we check $(u, A, \partial_t A) \in C(I; X^{s,\sigma})$. Indeed we have $(A, \partial_t A) \in C(I; H^\sigma \oplus H^{\sigma-1})$ by Lemma 4.3. On the other hand by virtue of Lemma 5.2 and Corollary 5.2 we have $u \in C(I; H^\sigma) \cap C^1(I; H^{\sigma-2})$ since $v = u$ is a solution to (4.1). Next, we have to check the uniqueness; we have used slightly different spaces from $C(I; X^{s,\sigma})$. To this end we recall Lemma 4.3. From this lemma, we automatically have $A \in L^4 L^4$ if $(u, A)$ is a solution in the required class. Therefore the contraction argument above implies the uniqueness. □
6. The contraction argument for $5/3 \leq s < 7/4$

In this section we consider the following more complicated metric to refine the result on unique existence of the solution. We put
\[
\tilde{d}(u, A, u', A') = \max_{j=0,1} \| \partial_{ij}^j (u - u'); L^\infty H^{s-1-2j} \| \\
\vee \| A - A'; L^q H^{2-s,r} \cap L^2 L^\infty \cap L^\infty H^1 \| \vee \| \partial_t (A - A'); L^\infty L^2 \|. 
\] (6.1)
Here $(q, r) = (6/(2s-1), 3/(2-s))$. The space $L^q H^{2-s,r}$ is removed if $s = 2$.

**Proposition 6.1.** Let $5/3 \leq s \leq 2$, $I = [0, T]$ with $0 < T \leq 1$. Let $(u, A), (u', A') \in \hat{Z}_{s,4/3}$ and let $(v, B)$ and $(v', B')$ be the solutions to (4.1)-(4.2) and (4.1)-(4.2)', respectively. Then the estimate
\[
\tilde{d}(v, B, v', B') \leq C(L)(\|(u_0 - u_0', A_0 - A'_0, A_1 - A'_1)\|_s H^{s-1,1+1/r} \| + C(L)\|(u - u')(0); H^{s-3}\| \vee \|(A - A')(0)\|_2\|v'; L^\infty H^s \| \\
+ C(L)T^{1/2}\{(\|v'; L^\infty H^s\| \vee \|\partial_t v'; L^\infty H^{s-2}\|)\tilde{d}(u, A, u', A') + \|v - v'; L^\infty H^{s-1}\| \}
\]
holds, where $L = l_S \vee l_M$. If $s = 2$, the space $X^{s-1,1+1/r}$ in the estimate above is replaced by $X^{1,1+\delta}$ for sufficiently small $\delta > 0$.

To prove this proposition, we need the following lemma which allows us to exchange $\|v - v'; L^\infty H^{s-1}\|$ and $\|\partial_t (v - v'); L^\infty H^{s-3}\|$. We state the lemma in general form although we use only (6.3) to prove Proposition 6.1. This is because we need (6.2) to prove the continuous dependence of the solution on the data in Section 7.

**Lemma 6.1.** Let $s > 1/2, \sigma \geq \max\{1, (2s-1)/4, s-2\}$ with $(s, \sigma) \neq (7/2, 3/2)$. Let $v$ and $v'$ be the solutions to (1.1) and (1.1)', respectively. Then the following inequality holds:
\[
\|v - v'; H^s\| \\
\lesssim \|\partial_t (v - v'); H^{s-2}\| + C(l)\{\|v - v\|_2 + \|v'; H^s\|(\|u - u'; H^{s-2}\| + \|A - A'; H^\sigma\|)\} \\
\lesssim C(l)\{\|v - v'; H^s\| + \|v'; H^s\|(\|u - u'; H^{s-2}\| + \|A - A'; H^\sigma\|)\}. 
\] (6.2)
Here $l = \|u; H^s\| \vee \|u'; H^s\| \vee \|A; H^\sigma\| \vee \|A'; H^\sigma\|$. Moreover, the following inequality holds for $3/2 < s \leq 2$:
\[
\|v - v'; H^{s-1}\| \\
\lesssim \|\partial_t (v - v'); H^{s-3}\| + C(l)\{\|v - v\|_2 + \|v'; H^s\|(\|u - u'; H^{s-3}\| + \|A - A'; H^{s-3}\|)\} \\
\lesssim C(l)\{\|v - v'; H^{s-1}\| + \|v'; H^s\|(\|u - u'; H^{s-3}\| + \|A - A'; H^{s-3}\| + \|A - A\|_2)\}. 
\] (6.3)

**Proof.** In the beginning, we prove the first inequality of (6.2). Applying (2.7) to (5.2), we obtain
\[
\|\partial_t (v - v'); H^{s-2}\| \lesssim \|v - v'; H^s\| + C(l)\|v - v\|_2 + \|(\mathcal{H} + \phi - \mathcal{H}' - \phi')v'; H^{s-2}\|.
\]
For the estimate of $((\mathcal{H} + \phi - \mathcal{H}' - \phi'))v'$, we need the following inequalities:

\[
\|(A - A')(\nabla - i(A + A')/2)v'; H^{s-2}\| \lesssim \|A - A'; H^s\| \|\|A + A'; H^s\||v'; H^s\|, \tag{6.4}
\]

\[
\|(\phi - \phi')v'; H^{s-2}\| \lesssim \|u - u'; H^{s-2}\| \|u + u'; H^s\| \|v'; H^s\|. \tag{6.5}
\]

We can show (6.4) by the same method in the proof of Lemma 2.3. To prove (6.5) for $s \geq 2$, we use Lemma 2.1. If $1/2 < s < 2$, we also use duality. Therefore the first inequality of (6.2) has been established. The second inequality of (6.2) can be proved analogously. To prove (6.3), we use similar estimates with $s$ replaced by $s - 1$, but we also need to use the estimate

\[
\|(A - A')(\nabla - i(A + A')/2)v'; H^{s-3}\| \lesssim \|A - A'; 2\|1\|A + A'; H^1\||v'; H^s\|
\]

instead of (6.4). This inequality is proved by the embedding $L^{6/(9-2s)} \hookrightarrow H^{s-3}$ and (2.3). \hfill \Box

**Proof of Proposition 6.1** Taking the difference of (3.14) and the corresponding equation for $v'$, we have

\[
i\partial_t^2(v - v') = (\mathcal{H} + \phi)\partial_t(v - v') + (\mathcal{H} + \phi - \mathcal{H}' - \phi')\partial_t v'
+ (2i\partial_t A(\nabla - iA) + \partial_t\phi)(v - v')
+ (2i\partial_t A(\nabla - iA) + \partial_t\phi - 2i\partial_t A'(\nabla - iA') - \partial_t\phi')v'
\equiv (\mathcal{H} + \phi)\partial_t(v - v') + \sum_{j=1}^3 f_j. \tag{6.6}
\]

We convert this equation to the integral form by Lemma 3.3 regarding $f \equiv \sum_{j=1}^3 f_j$ as the inhomogeneous term, and take the $H^{s-3}$-norm of the both-sides. Then we obtain

\[
\|\partial_t(v - v')(t); H^{s-3}\| \leq K_{s-3}\{\|\partial_t(v - v')(0); H^{s-3}\| + \int_0^t \|f(\tau); H^{s-3}\| d\tau\}. \tag{6.7}
\]

$K_{s-3} \leq C(L)$ clearly follows from Lemma 3.2. We begin with the estimate of the inhomogeneous term. We have

\[
\|f_1; H^{s-3}\| \lesssim \langle L\rangle\|\partial_t v'; H^{s-2}\|\|A - A'; H^{s-2, r} \cap L^{\infty}\| \|u - u'; H^{s-1}\|.
\]

If $s = 2$, we do not need $H^{2, s, r}$. We can show this inequality by the duality argument such as

\[
|\langle(A - A')(\nabla - i(A + A')/2)\partial_t v', \psi\rangle|
\leq \|\partial_t v'; H^{s-2}\|\|A - A')(\nabla - i(A + A')/2)\psi; H^{2-s}\|
\lesssim \|\partial_t v'; H^{s-2}\|\|A - A'; H^{2-s, r} \cap L^{\infty}\|(\nabla - i(A + A')/2)\psi; H^{2-s}\|
\lesssim \|\partial_t v'; H^{s-2}\|\|A - A'; H^{2-s, r} \cap L^{\infty}\|\|A + A'; H^1\|\|\psi; H^{3-s}\|
\]
and
\[ \|((\phi - \phi')\partial_t v'; \psi)\| \leq \|\partial_t v'; H^{s-2}\||((\phi - \phi')\psi; H^{2-s}) \]
\[ \lesssim \|\partial_t v'; H^{s-2}\||u - u'; H^{s-1}\|u + u'||_2\psi; H^{3-s}\| , \]
where we have used (2.4) and Lemma 2.1. We next estimate \( f_2 \). We have
\[ \|f_2; H^{s-3}\| \lesssim \|v - v'; H^{s-1}\||\|\partial_t A\|_3\langle L\rangle^2 \]
again by the duality argument as follows:
\[ |\langle \partial_t A(\nabla - iA)(v - v'), \psi\rangle| \leq \|\partial_t A(v - v'); H^{s-2}\||\langle \nabla - iA\rangle\psi; H^{2-s}\| \]
\[ \lesssim \|\partial_t A\|_3\|v - v'; H^{s-1}\||\|\partial_t A; H^{3-s}\|\langle \|A\|; H^1\rangle\rangle , \]
\[ |\langle \omega^{-2}(\partial_t u\bar{u})(v - v'), \psi\rangle| \leq \|\partial_t u; H^{s-2}\||\langle \omega^{-2}(v - v')u\rangle; H^{2-s}\| \]
\[ \lesssim \|\partial_t u; H^{s-2}\||v - v'; H^{s-1}\|\|\partial_t A; H^{3-s}\|\|u; H^1\| . \]

For \( f_3 \), we have
\[ \|f_3; H^{s-3}\| \lesssim \langle L\rangle\|v'; H^s\|\max\{\|\partial_t^j(A - A')\|; H^{1-j}\| + \|\partial_t^j(u - u'); H^{s-1-2j}\|\} \]
by the following estimates:
\[ \|\partial_t A(A - A')v'; H^{s-3}\| \lesssim \|\partial_t A\|_2\|A - A'\|_6\|v'; H^{s-1}\| , \]
\[ \|\partial_t(A - A')(\nabla - iA')v'; H^{s-3}\| \lesssim \|\partial_t(A - A')\|_2\|\langle \nabla - iA'\rangle v'; H^{s-3/2}\| \]
\[ \lesssim \|\partial_t(A - A')\|_2\|v'; H^{s-1/2}\|\langle \|A'\|; H^1\rangle\rangle , \]
\[ |\langle \omega^{-2}(\partial_t(u - u')\bar{u})v', \psi\rangle| \leq \|\partial_t(u - u'); H^{s-3}\||\langle \omega^{-2}(\psi\bar{u})u\rangle; H^{3-s}\| \]
\[ \lesssim \|\partial_t(u - u'); H^{s-3}\||\|v'; H^s\|\|\psi; H^{3-s}\|\|u; H^s\| , \]
\[ |\langle \omega^{-2}(\partial_t u'\bar{u} - u\bar{u'})v', \psi\rangle| \leq \|\partial_t u'; H^{s-2}\||\langle \omega^{-2}(\bar{u}\psi)(u - u')\rangle; H^{2-s}\| \]
\[ \lesssim \|\partial_t u'; H^{s-2}\||\|v'; H^s\|\|\psi; H^{3-s}\|\|u - u'; H^{s-1}\| . \]

Therefore we obtain
\[ \|F; L^1 H^{s-3}\| \leq C(L)T^{1/2}(\|v'; L^\infty H^s\| + \|\partial_t v'; L^\infty H^{s-2}\|)\tilde{d}(u, A, u', A') \]
\[ + C(L)T^{5/6}\|v - v'; L^\infty H^{s-1}\| . \]

The estimate for \( \|\partial_t(u - u'); L^\infty H^{s-3}\| \) is completed if we apply (6.3) to the right-hand side of (6.1). We also need the estimate for \( \|u - u'; L^\infty H^{s-1}\| \). By virtue of (6.3), we only have to estimate \( \|u - u'; L^\infty H^{s-3}\|, \|A - A'; L^\infty L^2\| \) and \( \|v - v'; L^\infty L^2\| \). To estimate the first two norms, we use the trivial inequality \( \|u - u'; H^{s-3}\| \leq \|(u - u')(0); H^{s-3}\| + \|\partial_t(u - u'); L^1 H^{s-3}\| \) and the corresponding one for \( A - A' \). The norm \( \|v - v'; L^\infty L^2\| \) is estimated by using (5.3), but the estimate of the second term of the right-hand side is
slightly different. In this case we use the inequality
\[
\| (\mathcal{H} + \phi - \mathcal{H}' - \phi')v' \|_2 \lesssim \| A - A' \|_\infty \| v' ; H^1 \| \langle \| A + A' ; H^1 \| \rangle \\
+ \| u - u' ; H^{1/2} \| \| u + u' ; H^1 \| \| v' \|_2
\]
obtained by (2.4) and Lemma 2.1. By the Hölder inequality for the time variable we have
\[
\| v - v' ; L^\infty L^2 \| \lesssim \| u_0 - u_0' \|_2 + T^{1/2} \tilde{d}(u, A, u', A') \| v' ; H^1 \| \langle L \rangle.
\]
Collecting the estimates above, we obtain
\[
\| v - v' ; L^\infty H^{s-1} \|
\leq C(L) \| u_0 - u_0' ; H^{s-1} \| + \langle \| (u - u')(0) ; H^{s-3} \| \vee \| (A - A')(0) \|_2 \| v' ; L^\infty H^s \|
+ T^{1/2} \{ \| v' ; L^\infty H^s \| \vee \| \partial_t v' ; L^\infty H^{s-2} \| \tilde{d}(u, A, u', A') + \| v - v' ; L^\infty H^{s-1} \| \}.
\]
The estimate for the Schrödinger part has been completed. We proceed to the Maxwell part. First we consider the case $s < 2$; later we mention how to modify the proof when $s = 2$. We begin with the estimate in $L^q H^{2-s,r}$ with $s < 2$. We put $(\tilde{q}, \tilde{r}) \equiv (6/(4s-5), 3/(4-2s))$, which is an admissible pair. By Lemma 4.1 we have
\[
\| B - B' ; L^q H^{2-s,r} \| \lesssim \| (A_0 - A'_0, A_1 - A'_1) ; H^{\sigma_1} \oplus H^{\sigma_1-1} \|
+ T \| A - A' ; L^\infty H^1 \| + \| P(J - J') ; L^{\tilde{q}} H^{\sigma_1-1+\beta(\tilde{r}), \tilde{r}} \|,
\]
where $\sigma_1 = 1 + 1/r$ and $(\tilde{q}, \tilde{r})$ is an admissible pair. We should estimate the last term in the right-hand side. We decompose $P(J - J')$ as in (5.5). Since $s - 1 < \sigma_1 < s$, we can choose $\tilde{r}$ so that $\sigma_1 - 1 + \beta(\tilde{r}) = s - 1$. With this choice, we have
\[
\| (u - u')(\nabla - iA)u ; H^{s-1, \tilde{r}} \| \lesssim \| u - u' ; H^{s-1} \| \| (\nabla - iA)u ; H^{s-1} \|
\lesssim \| u - u' ; H^{s-1} \| \| u ; H^s \| \langle \| A ; H^q \| \rangle
\]
if $1/\tilde{r} \leq (s - 1)/3$. This inequality holds provided $s \geq 5/3$. We also have
\[
\| u'(A - A')u ; H^{s-1, \tilde{r}} \| \lesssim \| u' ; H^{s-1,3} \| \| A - A' \|_6 \| u ; 1/2 - 1/\tilde{r} \|
+ \| u' \|_\infty \| A - A' ; H^{s-1} \| \| u ; 1/2 - 1/\tilde{r} \|
\lesssim \| u' ; H^s \| \| u ; H^s \| \| A - A' ; H^1 \|.
\]
Using these estimates together with the Hölder inequality for the time variable, we have
\[
\| B - B' ; L^q H^{2-s,r} \| \lesssim \| (A_0 - A'_0, A_1 - A'_1) ; H^{1+1/r} \oplus H^{1/r} \|
+ T^{1/2} C(L) \tilde{d}(u, A, u', A').
\]
We can estimate $\| B - B' ; L^\infty H^1 \| \vee \| \partial_t (B - B') ; L^\infty L^2 \|$ in the same way. Indeed these norms are estimated by the right-hand side of (6.10) with $H^{1+1/r} \oplus H^{1/r}$ replaced by $H^1 \oplus L^2$. Next we mention the estimate in $L^2 H^{\infty}$; the exponent $(2, \infty)$ is the prohibited endpoint. However, for any admissible pair $(q_0, r_0)$ and any $\varepsilon > 0$, we have
\[
\| B - B' ; L^2 H^{\infty} \| \lesssim \| B - B' ; L^{q_0} H^{3/r_0 + \varepsilon, r_0} \|.
\]
Here we have used the Sobolev inequality for the spatial variable and the Hölder inequality for the time variable together with $0 < T \leq 1$. Therefore we have by Lemma 4.1
\[
\| B - B' ; L^2 L^\infty \| \lesssim \|( A_0 - A'_0 , A_1 - A'_1 ) ; H^{\tilde{s}_1} + H^{\tilde{s}_1 - 1} \| \\
+ \| P( J - J') ; L^{\tilde{g}_0} H^{\tilde{s}_1 - 1 + \beta(\tilde{r}_0)} \tilde{r}_0 \| + T \| A - A' ; L^\infty H^1 \|, 
\] (6.11)
where $\tilde{s}_1 = 1 + \varepsilon + 1/r_0$ and $(\tilde{g}_0, \tilde{r}_0)$ is an admissible pair. If $8/5 < s \leq 2$, we can take $\varepsilon > 0$ and $r_0, \tilde{r}_0 \in [2, \infty)$ satisfying
\[
0 < 1/r_0 \leq \min\{1/r - \varepsilon, (5s - 8)/3 - \varepsilon\}, \quad \tilde{s}_1 - 1 + \beta(\tilde{r}_0) = s - 1.
\]
With such a choice, we have $H^{1+1/r} \oplus H^{1/r} \hookrightarrow H^{1+\varepsilon + 1/r_0} \oplus H^{\varepsilon + 1/r_0}$ and
\[
\| P( J - J') ; L^{\tilde{g}_0} H^{s - 1, \tilde{r}_0} \| \lesssim T^{1/2} C(L) \tilde{d}(u, A, u', A'),
\]
since the estimates (6.8) and (6.9) hold with $\tilde{r}$ replaced by $\tilde{r}_0$ in the same manner. Thus we have obtained the required result for the Maxwell part when $s < 2$. For $s = 2$, we do not need $L^g H^{2-s,r}$. The estimate in $L^2 L^\infty$ is almost same. We only have to replace the condition $1/r_0 \leq 1/r - \varepsilon$ by $1/r_0 \leq \delta - \varepsilon$. The norms $\| B - B' ; L^\infty H^1 \|$ and $\| \partial_t( B - B') ; L^\infty L^2 \|$ are also bounded by the right-hand side of (6.11) for the sake of Lemma 4.1. Collecting the above estimates, we obtain the required result. \(\square\)

**Proof of Theorem 1.1 Part 2.** Here we treat the case $5/3 \leq s < 7/4$. We put
\[
\mathcal{B} \equiv \{(u, A) \in \mathcal{Z}_{s,\sigma}; (u(0), A(0), \partial_t A(0)) = (u_0, A_0, A_1)\}
\]
and consider the complete metric space $(\mathcal{B}, \tilde{d})$. By Propositions 4.2 and 6.1, $\Phi$ is a map from $\mathcal{B}$ to itself and
\[
\tilde{d}(v, B, v', B') \leq C( l_M, l_S ) T^{1/2} (\tilde{d}(u, A, u', A') + \| v - v' ; L^\infty H^{s-1} \|)
\]
with $(v, B) = \Phi(u, A)$ and $(v', B') = \Phi(u', A')$. Therefore $\Phi$ is a contraction mapping if we take sufficiently small $T$. \(\square\)

**Remark.** This proof is still valid if $7/4 \leq s \leq 2$. Therefore the solution obtained in Section 5 actually belongs to $\mathcal{Z}_{s,\sigma}$ for some $l_S, l_M$.

### 7. Continuous dependence on the data

In this section we prove the continuous dependence of the solution on the data, which is usually the most delicate part of the theory of well-posedness. Our method is essentially based on [7]. For a while, we assume the following:

**Assumption (A2).** (1) $(s, \sigma)$ satisfies $5/3 \leq s < \infty$, $(s, \sigma) \neq (5/2, 7/2), (7/2, 3/2)$ and
\[
\max\{4/3, s - 2, (2s - 1)/4\} \leq \sigma \leq \min\{s + 1, (5s - 2)/3\};
\]
(2) $I = [0,T]$ with $0 < T \leq 1$;
Lemma 7.1. We assume \([A2]\) with \(s \leq 2\). Then the estimate

\[
D \leq C(L) \{ ||(u_0 - u'_0, A_0 - A'_0, A_1 - A'_1); X^{s, \sigma} || + T^{1/2}(E + D) + ||u - u'; L^\infty L^2|| \}
\]

holds, where \(D = D^{s, \sigma}(u, A, u', A')\), \(E = E^s(u, A, u', A')\).

Proof. The required result is obtained analogously to the proof of Proposition 6.1. Therefore we again begin with the expression (6.6) with \(u = v, u' = v'\). In the following, we can remove \(H^{2-s,r}\) and \(L^q H^{2-s,r}\) if \(s = 2\). First, we estimate \(f = \sum_{j=1}^3 f_j\) in the inhomogeneous term. We have

\[
||f_1; H^{s-2}|| \leq C(L) ||\partial_t u'; H^{s-1}|| ||A - A'|; H^{2-s, r} \cap L^\infty || + C(L) ||u - u'; H^s||
\]

by virtue of

\[
|\langle (A - A') (\nabla - i(A + A')/2) \partial_t u', \psi \rangle| \\
\leq ||(\nabla - i(A + A')/2) \partial_t u'; H^{s-2}|| ||(A - A') \psi; H^{2-s}|| \\
\lesssim ||\partial_t u'; H^{s-1}|| ||(A + A'); H^1|| ||A - A'|; H^{2-s, r} \cap L^\infty || ||\psi; H^{2-s}||, \\
|\langle (\phi - \phi') \partial_t u', \psi \rangle| \lesssim ||u - u'; H^s|| ||(||u; H^s|| + ||u'; H^s||)|| \partial_t u'; H^{s-2}|| ||\psi; H^{2-s}||.
\]

We have

\[
||f_2; H^{s-2}|| \leq C(L) ||\partial_t A|| ||u - u'; H^s||
\]
by
\[ \| \partial_t A (\nabla - iA)(u - u'); H^{s-2} \| \lesssim \| \partial_t A \|_3 \| (\nabla - iA)(u - u'); H^{s-1} \| \lesssim \| \partial_t A \|_3 \| u - u'; H^s \| \| A; H^1 \| , \]
\[ | \langle \omega^{-2} (\partial_t u)(u - u'), \psi \rangle | \lesssim \| \partial_t u; H^{s-2} \| \| u - u'; H^s \| \| \psi; H^{2-s} \| \| u; H^s \|. \]
We have
\[ \| f_3; H^{s-2} \| \leq C(L) \{ \| A - A' \| H^1 \| + \| \partial_t (A - A') \|_3 + \max_{j=0,1} \| \partial_t^j (u - u'); H^{s-2j} \| \} \]
by
\[ \| \partial_t (A - A')(\nabla - iA)u'; H^{s-2} \| \lesssim \| \partial_t (A - A') \|_2 \| A - A'; H^1 \| \| u'; H^s \| , \]
\[ \| \partial_t (A - A')(\nabla - iA)u'; H^{s-2} \| \lesssim \| \partial_t (A - A') \|_3 \| u'; H^s \| \| A; H^1 \| , \]
\[ \| \partial_t (A - A')(\nabla - iA)u'; H^{s-2} \| \lesssim \| \partial_t (A - A') \|_3 \| u'; H^s \| \| A; H^1 \| , \]
(7.1)
\[ | \langle \omega^{-2} (\partial_t (u - u')(u - u'))(u - u'), \psi \rangle | \lesssim \| \partial_t (u - u'); H^{s-2} \| \| u; H^s \| \| \psi; H^{2-s} \| \| u'; H^s \| \| \psi; H^{2-s} \|. \]
Therefore by these inequalities together with the Hölder inequality for the time variable,
\[ \| \partial_t (u - u'); H^{s-2} \| \lesssim C(L) \| \partial_t (u - u')(0); H^{s-2} \| + T^{1/2} C(L) (E + D). \]
We obtain the estimate for \( \| \partial_t (u - u'); L^\infty H^{s-2} \| \) by using (6.2) to the first term of the right-hand side. Next we estimate \( \| u - u'; L^\infty H^s \| \). By the interpolation inequality to (6.2), we have
\[ \| u - u'; H^s \| \lesssim \| \partial_t (u - u'); H^{s-2} \| + C(L) \| u - u' \|_2 + C(L) \| A - A'; H^\sigma \|. \]
Therefore
\[ \max_{j=0,1} \| \partial_t^j (u - u'); L^\infty H^{s-2} \| \leq C(L) \{ \| (u_0 - u_0, A_0 - A_0'); H^s \oplus H^\sigma \| + T^{1/2} (E + D) \]
\[ + \| A - A'; L^\infty H^\sigma \| + \| u - u'; L^\infty L^2 \| \}. \]
(7.2)
On the other hand, by the analogous argument in the proof of Lemma 4.3, we have
\[ \max_{j=0,1} \| \partial_t^j (A - A'); L^\infty H^{s-j} \cap L^q H^{s-j-\beta(r),r} \| \]
\[ \lesssim \| (A_0 - A_0', A_1 - A_1'); H^\sigma \oplus H^{\sigma-1} \|
\[ + C(L) T^{1/2} (\| A - A'; L^\infty H^\sigma \| \vee \| u - u'; L^\infty H^\sigma \|) \]
(7.3)
Thus we have obtained the desired result. □

If \( s > 2 \), instead of \( D^{s,\sigma} \) and \( E^s \) we need
\[ \tilde{D}^{s,\sigma}(u, A, u', A') \equiv D(u, A, u', A') \vee \| \partial_t^2 (A - A') \| L^\infty H^{s-2} \| \vee \| \partial_t^2 (u - u'); L^\infty H^{s-4} \|, \]
\[ \tilde{E}^s(u, A, u', A') \equiv \| \partial_t^2 u'; H^{s-3} \| (\| A - A'; L^q H^{s-3};r \cap L^2 L^\infty \| \vee \| u - u'; L^\infty H^{s-1} \|), \]
where \( 1/r = 1/2 - (1/2 - |s - 3|/3)_+ \). We choose \( q \) so that \( (q, r) \) is an admissible pair. The space \( L^q H^{s-3};r \) in \( \tilde{E}^s \) is removed if \( s = 3 \).
Lemma 7.2. Let \( s > 2 \). Then the estimate
\[
\tilde{D} \lesssim C(L)\{\| (u_0 - u_0', A_0 - A_0', A_1 - A_1') \|; X^{s, \sigma} + T^{1/2}(\tilde{E} + \tilde{D}) + \| u - u' \|; L^\infty L^2 \}
\]
holds, where \( \tilde{D} = \tilde{D}^{s, \sigma}(u, A, u', A') \) and \( \tilde{E} = \tilde{E}^{s}(u, A, u', A') \).

Proof. By taking the difference of (3.15) with \( v = u \) and the corresponding equation for \( u' \), we have
\[
i\partial_t^2 (u - u') = (\mathcal{H} + \phi)\partial_t^2 (u - u') + (\mathcal{H} + \phi - \mathcal{H}' - \phi')\partial_t^2 u'
\]
\[
+ 2(2i\partial_t A(\nabla - iA) + \partial_t \phi)\partial_t (u - u')
\]
\[
+ 2(2i\partial_t A(\nabla - iA) + \partial_t \phi - 2i\partial_t A'(\nabla - iA') - \partial_t \phi')\partial_t u'
\]
\[
+ (2i\partial_t^2 A(\nabla - iA) + 2(\partial_t A)^2 + \partial_t^2 \phi)(u - u')
\]
\[
+ (2i\partial_t^2 A(\nabla - iA) + 2(\partial_t A)^2 + \partial_t^2 \phi - 2i\partial_t^2 A'(\nabla - iA') - 2(\partial_t A')^2 - \partial_t^2 \phi')u'
\]
\[
\equiv (\mathcal{H} + \phi)\partial_t^2 (u - u') + \sum_{j=1}^5 G_j.
\]
Regarding \( G \equiv \sum_{j=1}^5 G_j \) as the inhomogeneous term, we convert this equation to the integral form by virtue of Lemma 3.3, and take the \( H^{s-4} \)-norm of the both-sides. Then we obtain
\[
\| \partial_t^2 (u - u'); H^{s-4} \| \lesssim K_{s-4}\{\| \partial_t^2 (u - u')(0); H^{s-4} \| + \int_0^t \| G(\tau); H^{s-4} \| d\tau \}.
\]
We begin with the estimate of the inhomogeneous term. For \( G_1 \), we have
\[
\| G_1; H^{s-4} \| \leq C(L)\| \partial_t^2 u'; H^{s-3} \| (\| A - A' \|; H^{s-3}; \cap L^\infty \| \lor \| u - u'; H^{s-1} \|),
\]
where \( H^{s-3}; \) is removed if \( s = 3 \). To prove this, we should estimate
\[
G_{1,1} = (A - A')(\nabla - i(A + A')/2)\partial_t^2 u' \quad \text{and} \quad G_{1,2} = (\phi - \phi')\partial_t^2 u'.
\]
If \( s > 3 \), we can rewrite \( G_{1,1} = (\nabla - i(A + A')/2)(A - A')\partial_t^2 u' \) since \( \text{div}(A - A') = 0 \). After that, we use (2.4) and the Leibniz formula to estimate \( G_{1,1} \). If \( 2 < s \leq 3 \), we also use duality. In both cases we have
\[
G_{1,1} \lesssim C(L)\| \partial_t^2 u'; H^{s-3} \| (\| A - A' \|; H^{s-3}; \cap L^\infty \|.
\]
We have \( G_{1,2} \lesssim C(L)\| u - u'; H^{s-1} \|. \) This is obtained by Lemma 2.1 together with duality if \( s < 3 \). For the estimates of \( G_j \), \( 2 \leq j \leq 5 \), we can use the estimates for \( F_j \), \( 1 \leq j \leq 5 \), in the proof of Lemma 3.4. Indeed we have
\[
\sum_{j=2}^5 \| G_j; H^{s-4} \| \leq C(L)\| \partial_t A \|_3 \max_{j=0,1}\{\| \partial_t^j (A - A'); H^{\sigma-j} \| \lor \| \partial_t^j (u - u'); H^{s-2j} \|\}
\]
\[
+ C(L)\| \partial_t (A - A') \|_3 + C(L)\| \partial_t^2 (A - A'); H^{\sigma-2} \|.
\]
Therefore we obtain
\[
\| G; L^1 H^{s-4} \| \leq T^{1/2}C(L)\{ \tilde{D}^{s, \sigma} + \tilde{E}^{s} \}.
\]
To complete the estimate for the Schrödinger equation, we need the following estimate:
\[
\max_{j=0,1} \| \partial_t^j (u - u') \| H^{s-2j} \leq \| \partial_t^2 (u - u') \| H^{s-4} + C(L) \| u - u' \|_2 + C(L) \max_{j=0,1} \| \partial_t^j (A - A') \| H^{\sigma-j} \leq C(L) \| u - u' \| H^s + \max_{j=0,1} \| \partial_t^j (A - A') \| H^{\sigma-j} \\}
(7.5)
To this end we recall (6.6). Applying (2.7) and Lemma 6.1 to this equation, we obtain
\[
\max_{j=0,1} \| \partial_t^j (u - u') \| H^{s-2j} \leq \| \partial_t^2 (u - u') \| H^{s-4} + C(L) \| u - u' \|_2 + C(L) \| A - A' \| H^{\sigma} + \sum_{j=1}^3 \| f_j \| H^{s-4} \|
\]
where \( f_j, j = 1, 2, 3 \), are defined in (6.6) with \( v, v' \) replaced by \( u, u' \) respectively. The estimate for \( f_1 \) is obtained by (2.8) and Lemma 2.1. The estimate for \( f_2 \) and \( f_3 \) are obtained similarly to the estimate for \( F_6, F_7 \) in the proof of Lemma 3.2. Indeed we have
\[
\sum_{j=1}^3 \| f_j \| H^{s-4} \| \leq C(L) \max_{j=0,1} \| \partial_t^j (A - A') \| H^{\sigma-j} \| \vee \| \partial_t^j (u - u') \| H^{s-1-2j} \|
\]  \( (7.6) \)
The right-hand side of (7.6) does not exceed
\[
C(L) \max_{j=0,1} \| \partial_t^j (A - A') \| H^{\sigma-j} \| \vee C(L) \| u - u' \| H^{s-1} \|
\]
again by Lemma 6.1. Therefore using the interpolation inequality to \( \| u - u' \| H^{s-1} \| \), we obtain the first inequality of (7.5). Similarly we can obtain the second. Thus we have the following estimate for the Schrödinger part:
\[
\max_{j=0,1,2} \| \partial_t^j (u - u') \| H^{s-2j} \leq C(L) \| (u_0 - u_0', A_0 - A_0', A_1 - A_1') ; X^s, \sigma \| + T^{1/2} (\bar{E} + \bar{D}) + \max_{j=0,1} \| \partial_t^j (A - A') \| L^\infty H^{\sigma-j} \| + \| u - u' \| L^\infty L^2 \|
\]
The Maxwell part is easy to treat. Indeed, the estimate (7.3) is still valid for \( s > 2 \); we also have \( \| \partial_t^2 (A - A') \| L^\infty H^{\sigma-2} \| \leq C(L) D^s, \sigma \) similarly as in Lemma 4.3. Collecting the estimates both for the Schrödinger equation and the Maxwell, we obtain the desired result. □

**Lemma 7.3.** Let assume \([A2]\) Let \( N(A, A') \) be defined by the following condition:

1. If \( 5/3 \leq s < 2 \), let \( 1/r = (2 - s)/3, \sigma_1 = 1 + 1/r, \)

\[
N(A, A') \equiv \| A - A' \| L^\infty H^{\sigma_1} \cap L^a H^{2-s,r} \cap L^2 L^\infty \|
\]
(2.7)
2. If \( s > 2 \) and \( s \neq 3 \), let \( 1/r = 1/2 - (1/2 - |s - 3|/3), \sigma_1 = |s - 3| + \beta(r), \)

\[
N(A, A') \equiv \| A - A' \| L^\infty H^{\sigma_1} \cap L^a H^{2-s,r} \cap L^2 L^\infty \|
\]
(2.8)
3. If \( s = 2 \) or \( s = 3 \), let \( \sigma_1 = 1 + \varepsilon \) for sufficiently small \( \varepsilon > 0, \)

\[
N(A, A') \equiv \| A - A' \| L^\infty H^{\sigma_1} \cap L^2 L^\infty \|
\]
In any case we choose $q$ so that $(q, r)$ is an admissible pair. Then the following estimate holds:

$$
N(A, A') \lesssim \|(A_0 - A'_0, A_1 - A'_1); H^{\sigma_1} \oplus H^{\sigma_1 - 1}\| \\
+ C(L)T^{1/2} (\|u - u'; L^\infty H^{s_1} - 1\| \vee \|A - A'; L^\infty H^{\sigma_1}\|).
$$

Proof. Since $0 < T \leq 1$, there exists an admissible pair $(q_0, r_0)$ such that $L^{q_0} H^{3/r_0} + 0, r_0 \to L^2 L^\infty$ and that $1 + 1/r_0 < \sigma_1$. Therefore by Lemma 7.3 we have

$$
N(A, A') \lesssim \|(A_0 - A'_0, A_1 - A'_1); H^{\sigma_1} \oplus H^{\sigma_1 - 1}\| \\
+ \|P(J - J'); L^2 H^{\sigma_1 - 1 + \beta(\tilde{r})} - T\|A - A'; L^\infty H^{\sigma_1 - 1}||,
$$

where $(\tilde{q}, \tilde{r})$ is an admissible pair. We estimate the middle term of the right-hand side as in the proof of Proposition 6.1. We choose $\tilde{r}$ so that $1/2 - 1/\tilde{r} \geq (1/2 - (s - 1)/3)_+$ and that $\sigma_1 - 1 + \beta(\tilde{r}) \leq s - 1$. Then (6.8) and (6.9) still hold valid with $\|A - A'; H^1\|$ in (6.9) replaced by $\|A - A'; H^{\sigma_1}\|$. Therefore we can obtain the desired result. \qed

**Lemma 7.4.** Let assume (A2) with $2 < s \leq 3$. Let $s_1 = s - 1$, and $\sigma_1$ be defined in Lemma 7.3. Then the estimate

$$
D \leq C(L)(\|(u_0 - u_0', A_0 - A'_0, A_1 - A'_1); X^{s_1, \sigma_1}\| + T^{1/2}D) \tag{7.7}
$$

holds, where $D \equiv D^{s_1, \sigma_1}(u, A, u', A')$.

Proof. First, we prove the following inequality:

$$
D^{s_1, \sigma_1} \lesssim \|(u_0 - u_0', A_0 - A'_0, A_1 - A'_1); X^{s_1, \sigma_1}\| \\
+ C(L)T^{1/2}(E^{s_1} + D^{s_1, \sigma_1}) + \|u - u'; L^\infty L^2\|. \tag{7.8}
$$

Here $E^{s_1} = \|\partial_t u'; L^\infty H^{s_1 - 1}||\|A - A'; L^{q_1} H^{2-s_1, r_1} \cap L^2 L^\infty\| \vee \|u - u'; L^\infty H^{s_1 - 1}||$ and $(q_1, r_1) = (6/(2s - 3), 3/(3 - s))$. This is the same inequality as the assertion of Lemma 7.1. However, we have assumed $s \geq 5/3$, $\sigma \geq 4/3$ in Lemma 7.1 $(s_1, \sigma_1)$ does not satisfy these conditions. Therefore we have to modify the proof. In the proof of (7.8), we should distinguish the assumption for $(s, \sigma)$ and that for $(s_1, \sigma_1)$. We do not need $s_1 \geq 5/3$; we need $s \geq 5/3$ only to ensure the unique existence of the solution, and $s_1 > 1$ is sufficient to obtain (7.8). In the proof of Lemma 7.1, the assumption $\sigma \geq 4/3$ is used to ensure the boundedness of $\|\partial_t A; L^6 L^3\|$ and $\|\partial_t (A - A); L^6 L^3\|$ by virtue of the Strichartz estimate. The former norm is still bounded because even in the present case $\sigma$ itself satisfies this condition. The latter norm cannot be controlled if $\sigma_1 < 4/3$, but it appears only once in the estimate of $F_3$. Therefore if we replace (7.1) by the following inequality, we obtain

$$
\|\partial_t(A - A')(\nabla - iA)u'; H^{s_1-2}\| \lesssim \|\partial_t(A - A')\|_2\|u'; H^s\|\|(A; H^1)\|.
$$
The estimate for the Maxwell part is similar to that in the proof of Proposition 6.1. Therefore (7.8) has been established. On the other hand, we have

\[ E^{s_1} \leq C(L)N(A, A') \vee \|u - u'; L^\infty H^{s-1}\| \]
\[ \leq C(L)\|A_0 - A'_0, A_1 - A'_1; H^{s_{\sigma_1}} \oplus H^{s_{\sigma_1}-1}\| + C(L)D^{s_{\sigma_1}, \sigma_1}, \]
\[ \|u - u'; L^\infty L^2\| \leq \|u_0 - u'_0\|_2 + C(L)T^{1/2}d(u, A, u', A') \]

by Lemma 7.3 and by Proposition 5.1 respectively. We obtain the desired result by collecting these estimates and using the inequality \(\|A - A'; L^4 L^4\| \leq N(A, A')\), which is obtained by interpolation. \(\Box\)

**Lemma 7.5.** Let assume (A2) with \(s > 3\). Let \(s_1 = s - 1\) and \((s_1, \sigma_*)\) satisfy \(\sigma_* < \sigma\) and (A2)(1) with \((s, \sigma)\) replaced by \((s_1, \sigma_*)\). Then the estimate

\[ \tilde{D} \leq C(L)(\|u_0 - u'_0, A_0 - A'_0, A_1 - A'_1; X^{s_1, \sigma_*}\| + T^{1/2} \tilde{D}) \]

holds, where \(\tilde{D} \equiv \tilde{D}^{s_1, \sigma_*}(u, A, u', A')\).

**Proof.** By Lemma 6.2 we have

\[ \tilde{D}^{s_1, \sigma_*} \leq C(L)\{\|u_0 - u'_0, A_0 - A'_0, A_1 - A'_1; X^{s_1, \sigma_*}\| \]
\[ + T^{1/2}(\tilde{D}^{s_1, \sigma_*} + \tilde{E}^{s_1}) + \|u - u'; L^\infty L^2\|\}. \]

We have \(\sigma_* > \sigma_1\) for \(s > 3\), where \(\sigma_1\) is defined in Lemma 7.3. Therefore we can prove the lemma similarly to Lemma 7.4. \(\Box\)

**Proof of Theorem 6.1.** Part 3. Here we prove the continuous dependence of the solution on the data. Let \((u, A)\) and \((u', A')\) be the solutions of MS-C with data \((u_0, A_0, A_1)\) and \((u'_0, A'_0, A'_1)\) respectively. We consider the case that \((u'_0, A'_0, A'_1) \to (u_0, A_0, A_1)\) in \(X^{s, \sigma}\).

Therefore we may assume that \((u'_0, A'_0, A'_1)\) is bounded in \(X^{s, \sigma}\), accordingly Assumption (A2) is satisfied for some \(L > 0\). We put \(\varepsilon = \sigma - \sigma_1\) if \(s \leq 3\), \(\varepsilon = \sigma - \sigma_*\) if \(s > 3\). \(\sigma_1\) and \(\sigma_*\) are given in Lemmas 7.3 and 7.5. In both cases \(\varepsilon > 0\). In the following, we take \(T > 0\) sufficiently small. We summarize the estimates used in the proof. By Lemma 3.4 and Corollary 3.2 we have

\[ \max_{j=0,1,2} \|\partial_j u; L^\infty H^{s+1-2j}\| \leq C(L)\|u_0; H^{s+1}\| \]

(7.9)

and the corresponding estimate for \(u'\). Since \(T > 0\) is sufficiently small, we have by Proposition 6.1 and Lemmas 7.3, 7.5

\[ \|u - u'; L^\infty H^{s-1}\| \vee N(A, A') \leq C(L)(\|u_0 - u'_0, A_0 - A'_0, A_1 - A'_1; X^{s-1, \sigma - \varepsilon}\|) \]

(7.10)
By Lemmas 7.1 and 7.2 together with (7.9) and (7.10), we also have
\[ D^{s,\sigma}(u, A, u', A') \leq C(L) \{ \| (u_0 - u_0', A_0 - A_0', A_1 - A_1') \|_{H^{s+\sigma}} + E^s(u, A, u', A') + \| u - u' \|_{L^\infty L^2} \} \leq C(L) \{ \| (u_0 - u_0', A_0 - A_0', A_1 - A_1') \|_{H^{s+\sigma}} + \langle \| u_0' \|_{H^{s+1}} \rangle \| (u_0 - u_0', A_0 - A_0', A_1 - A_1') \|_{X^{s-1,\sigma-\varepsilon}} \} \],

where \( D, E \) are replaced by \( \tilde{D}, \tilde{E} \) if \( s > 2 \). Let \( \eta \) be a rapidly decaying function such that \( \eta \) is radial and \( (\mathcal{F}\eta)(\xi) = 1 \) for \( |\xi| \leq 1 \). Here \( \mathcal{F} \) is the Fourier transform. We put \( \eta_\delta(x) = \delta^{-3}\eta(\delta^{-1}x) \) for \( \delta > 0 \). Then for \( \theta \) with \( 0 < \theta < \infty \), \( \eta_\delta \) has the properties
\[ \| \eta_\delta \ast w; H^{s+\theta} \| \lesssim \delta^{-\theta} \| w; H^s \| \text{ for any } \delta > 0, \| \eta_\delta \ast w - w; H^{s-\theta} \| = o(\delta^\theta) \text{ as } \delta \to 0. \] (7.12)

Now we put
\[ u_0^\delta \equiv \eta_\delta \ast u_0, \quad A_0^\delta \equiv \eta_\delta \ast A_0, \quad A_1^\delta \equiv \eta_\delta \ast A_1, \]
\[ u_0'^\delta \equiv \eta_\delta \ast u_0', \quad A_0'^\delta \equiv \eta_\delta \ast A_0', \quad A_1'^\delta \equiv \eta_\delta \ast A_1', \]
and let \((u^\delta, A^\delta)\) and \((u'^\delta, A'^\delta)\) be the solutions for the data \((u_0^\delta, A_0^\delta, A_1^\delta)\) and \((u_0'^\delta, A_0'^\delta, A_1'^\delta)\), respectively. For fixed \( \delta \), we have \( D^{s,\sigma}(u^\delta, A^\delta, u'^\delta, A'^\delta) \to 0 \) as \((u_0', A_0', A_1') \to (u_0, A_0, A_1)\) by virtue of (7.11). Therefore
\[ \lim_{\delta \to 0} \sup_{\text{data}} D^{s,\sigma}(u, A, u', A') \leq D^{s,\sigma}(u, A, u', A') + \lim_{\delta \to 0} \sup_{\text{data}} D^{s,\sigma}(u', A', u'^\delta, A'^\delta). \]

Here \( \lim \sup_{\text{data}} \) is the abbreviation of \( \lim \sup_{(u_0', A_0', A_1') \to (u_0, A_0, A_1)} \). The right-hand side is bounded by
\[ C(L) \{ \| (u_0 - u_0^\delta, A_0 - A_0^\delta, A_1 - A_1^\delta) \|_{X^{s,\sigma}} + \langle \| u_0^\delta \|_{H^{s+1}} \rangle \| (u_0 - u_0^\delta, A_0 - A_0^\delta, A_1 - A_1^\delta) \|_{X^{s-1,\sigma-\varepsilon}} \} \].

By (7.12), this tends to 0 as \( \delta \to 0 \). Therefore \( \lim \sup_{\text{data}} D^{s,\sigma}(u, A, u', A') = 0 \). Repeating the process above finite times, we can show the result on any compact interval where the unique existence of the solution is established. \( \square \)

8. The Lorentz gauge and the temporal gauge

We prove only Theorem 1.2, we can prove Theorem 1.3 analogously. In this section, we indicate by the superscript L the Lorentz gauge and by C the Coulomb gauge, respectively. To begin with, we heuristically explain how we construct the solution to MS-L. For any solution \((u^L, \phi^L, A^L)\) to MS-L, there exists a solution to MS-C which is gauge equivalent to \((u^L, \phi^L, A^L)\). Indeed, let us put \( \lambda = \Delta^{-1} \text{div} A^L \), \( A^C = PA^L \), \( \phi^C = (-\Delta)^{-1} \rho(u^L) \) and \( u^C = e^{-i\lambda}u^L \). Then \( A^L = A^C + \nabla \lambda \) by definition, and \( \phi^L = \phi^C - \partial_t \lambda \) by the Lorentz gauge condition and by the equation for \( \phi^L \). Therefore \((u^L, \phi^L, A^L)\) and \((u^C, \phi^C, A^C)\) are connected by the relation (1.4). Clearly \( A^C \) satisfies the Coulomb gauge condition, and
(u^C, A^C) satisfies MS-C since MS is gauge invariant. Moreover, \( \lambda \) must satisfy the wave equation

\[ (\partial_t^2 - \Delta)\lambda = \partial_t \phi^C \]  

(8.1)

with data \( \lambda_j = \Delta^{-1} \text{div} A_j \), \( j = 0, 1 \). The subscripts 0 and 1 indicate the initial datum for \( \lambda \) itself and its time derivative, respectively. Therefore we can solve the Cauchy problem for MS-L as follows. First we solve MS-C with data \((u_0^C, A_0^C, A_1^C) = (e^{-i\lambda_0} u^L, PA_0^L, P A_1^L)\). Next we solve (8.1). Then we construct the solution \((u^L, \phi^L, A^L)\) to MS-L by the gauge transform.

**Proof of Theorem 1.2.** We define \((u_0^C, A_0^C, A_1^C)\) as above. Clearly \((A_0^C, A_1^C) \in H^s \oplus H^{s-1}\) by the boundedness of \( P \) on \( H^s \). Moreover if \( \sigma \geq s - 1 \), \( u^C \in H^s \) since \( \lambda_0 \in \dot{H}^1 \cap \dot{H}^{\sigma+1} \). Therefore \((u_0^C, A_0^C, A_1^C) \in X^{s,\sigma}\). By Theorem 1.1, there exists an interval \( I = [0, T] \) such that MS-C with data \((u_0^C, A_0^C, A_1^C)\) has a unique solution \((u^C, A^C)\) with \((u^C, A^C, \partial_t A^C) \in C(I; X^{s,\sigma})\). The function \( \lambda \) is obtained by the propagator \( K(t) = \sin t \omega / \omega \) and its time derivative \( \dot{K}(t) = \cos t \omega \) as

\[
\lambda = \dot{K}(t) \lambda_0 + K(t) \lambda_1 + \int_0^t K(t - \tau) \partial_\tau (-\Delta)^{-1} |u^C|^2(\tau) d\tau \\
= \dot{K}(t) \lambda_0 + K(t)(\lambda_1 + \Delta^{-1}|u_0|^2) + \int_0^t \dot{K}(t - \tau)(-\Delta)^{-1} |u^C|^2(\tau) d\tau.
\]

Here we have used the integration by parts. By this expression, we find that \( \partial_t^j \lambda \in C^j(I; \dot{H}^1 \cap \dot{H}^{\sigma+1-j}) \) for \( j = 0, 1 \). We define \((u^L, \phi^L, A^L)\) as above. Clearly this satisfies MS-L, the initial condition and the Lorentz gauge condition. We can check that \((u^L, A^L) \in C^j(I; H^2 \cap H^{s-j}) \) for \( j = 0, 1 \). Moreover \( \phi^L \in C^j(I; H^{s-j}) \) for \( j = 0, 1 \). Indeed \( \phi^L \) satisfies

\[
\phi^L = \phi^C - \partial_t \lambda = \dot{K}(t) \phi_0^L + K(t) \phi_1^L + \int_0^t K(t - \tau)|u^L|^2(\tau) d\tau \tag{8.2}
\]

by virtue of the condition \( \phi_1^L + \text{div} A_1^L = \Delta \phi_0^L + |u_0^L|^2 + \text{div} A_1^L = 0 \). The right-hand side of (8.2) belongs to the desired space under the assumption for \((s, \sigma)\). Therefore \((u^L, \phi^L, \partial_t \phi^L, A^L, \partial_t A^L) \in C([0, T]; Y^{s,\sigma})\). The uniqueness and the continuous dependence on the data of solutions follow from the well-posedness for MS-C and (8.1). \( \square \)

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Makoto NAKAMURA
Graduate School of Information Sciences
Tohoku University
Sendai 980-8579, Japan
E-mail: m-nakamu@math.is.tohoku.ac.jp

Takeshi WADA
Department of Mathematics
Osaka University
Osaka 560-0043, Japan
E-mail: wada@math.sci.osaka-u.ac.jp