On a conjecture by Boyd

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Abstract

The aim of this note is to prove the Mahler measure identity

\[ m(x + x^{-1} + y + y^{-1} + 5) = 6m(x + x^{-1} + y + y^{-1} + 1) \]

which was conjectured by Boyd. The proof is achieved by proving relationships between regulators of both curves.

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1 Introduction

Boyd [3] studied the Mahler measure of families of polynomials. In particular, he considered the two-variable family

\[ P_k(x, y) = x + \frac{1}{x} + y + \frac{1}{y} + k. \]

The zeros of \( P_k(x, y) \) correspond, generically to a curve of genus 1. Let \( E_k \) denote the elliptic curve corresponding to the algebraic closure of \( P_k(x, y) = 0 \).

Recall that the (logarithmic) Mahler measure of a non-zero Laurent polynomial, \( P(x_1, \ldots, x_n) \), with complex coefficients is defined as

\[ m(P) = \int_0^1 \cdots \int_0^1 \log |P(e^{2\pi i t_1}, \ldots, e^{2\pi i t_n})| \ dt_1 \cdots t_n. \]

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Let us denote \( m(k) := m(P_k) \). Boyd computed \( m(k) \) for \( k \) a positive integer less than or equal to 100 (it is easy to see that the Mahler measure does not depend on the sign of \( k \) for this family). He found that

\[
m(k) = r_k L'(E_k, 0),
\]

where \( r_k \) is a rational number and the question mark stands for an equality that has only been established numerically (typically to at least 50 decimal places).

The case with \( k = 1 \) (resulting in \( r_k = 1 \)) was considered in detail by Deninger [5], who found an explanation for such a formula by relating it to evaluations of regulators in the context of the Bloch–Beilinson conjectures. Rodriguez-Villegas [8] also considered this family in the context of the Bloch-Beilinson conjectures, including more general cases where \( k^2 \in \mathbb{Q} \). He was able to prove identities for the cases where the Bloch–Beilinson conjectures are known to be true, such as when \( E_k \) has complex multiplication.

When the curves \( E_{k_1} \) and \( E_{k_2} \) are isogenous, their \( L \)-functions coincide. One can then compare the values in equation (1) and conjecture identities of the form \( r_{k_2} m(k_1) = r_{k_1} m(k_2) \). For example,

**Theorem 1**

\[
m(8) = 4m(2), \quad m(5) = 6m(1).
\]

The first identity was proved in [7]. In this note, we prove the second one.

## 2 Functional Identities

Functional identities for \( m(k) \) have been studied by Kurokawa and Ochiai in [6], and by Rogers and the author in [7]. The simplest ones are given as follows:

**Theorem 2** We have the following functional equations for \( m(k) \):

- [6]: For \( h \in \mathbb{R}\setminus\{0\} \):

\[
m(4h^2) + m\left(\frac{4}{h^2}\right) = 2m\left(2\left(h + \frac{1}{h}\right)\right).
\]
• [7]: If $h \neq 0$, and $|h| < 1$:

$$m \left( 2 \left( h + \frac{1}{h} \right) \right) + m \left( 2 \left( ih + \frac{1}{ih} \right) \right) = m \left( \frac{4}{h^2} \right). \quad (5)$$

If we set $h = \frac{1}{\sqrt{2}}$ in both identities, we obtain

$$m \left( 2 \right) + m \left( 8 \right) = 2m \left( 3\sqrt{2} \right),$$
$$m \left( 3\sqrt{2} \right) + m \left( i\sqrt{2} \right) = m \left( 8 \right).$$

Similarly, if we set $h = \frac{1}{\pi}$, we obtain

$$m \left( 1 \right) + m \left( 16 \right) = 2m \left( 5 \right),$$
$$m \left( 5 \right) + m \left( -3i \right) = m \left( 16 \right).$$

Thus, in order to prove (2) and (3), we need to find one additional equation for each of the above linear systems.

### 3 The relationship with the regulator

In this section, we sometimes write $x_k$ and $y_k$ for $x$ and $y$, so we can distinguish them when we look at different curves.

After the works of Deninger [5] and Rodriguez-Villegas [8], we write

$$m(k) = \frac{1}{2\pi} r_k(\{x_k, y_k\}),$$

where $r_k$ is a period of the regulator in the symbol $\{x_k, y_k\} \in K_2(E_k)$. For our purposes, we can reduce to $K_2(\mathbb{C}(E_k))$, so that $x_k, y_k$ are elements of $\mathbb{C}(E_k)$. See [5] and [8] for general details, and [7] for the specific treatment of this particular example.

In our context, it is enough to take into account that

$$r_k(\{x_k, y_k\}) = \alpha D_k((x_k) \diamond (y_k)),$$

where $\alpha$ is a constant independent of $k$ and $D_k$ is the elliptic dilogarithm in $E_k$ constructed by Bloch (see [2]).
We will briefly explain the meaning of \((x) \diamond (y)\). Let \(E\) be an elliptic curve with \(x, y \in \mathbb{C}(E)\). Consider the divisors
\[
(x) = \sum a_S(S), \quad (y) = \sum b_T(T).
\]
Now define
\[
(x) \diamond (y) = \sum a_S b_T(S - T).
\]
This is an element in
\[
\mathbb{Z}[E(\mathbb{C})] = \mathbb{Z}[E(\mathbb{C})] / \sim,
\]
where the equivalence relation stands for \((-T) \sim -(T)\).

Thus, the Mahler measure depends just on \(D_k\) and \((x_k) \diamond (y_k)\). For example, if the elliptic curves are isomorphic, \(D_k\) does not change and the Mahler measure only depends on \((x_k) \diamond (y_k)\). This idea was discovered by Rodriguez-Villegas [9], and also used by Bertin [1]. We applied this idea again in [7], to isogenous elliptic curves, in order to prove identities like (5).

A Weierstrass model for \(E_k\) is given by
\[
Y^2 = X \left( X^2 + \left( \frac{k^2}{4} - 2 \right) X + 1 \right),
\]
where
\[
x = \frac{kX - 2Y}{2X(X - 1)}, \quad y = \frac{kX + 2Y}{2X(X - 1)}.
\]
It is not hard to see that \(E_k(\mathbb{Q}(k))_{\text{tor}} \cong \mathbb{Z}/4\mathbb{Z}\). To fix notation, we will denote a generator by
\[
P = \left( 1, \frac{k}{2} \right).
\]
Then we have \(2P = (0, 0)\). Eventually, we will perform computations in the curve with parameter \(k = h + \frac{1}{h}\). In this curve, we will denote
\[
Q = \left( -\frac{1}{h^2}, 0 \right),
\]
which is a point of order 2. Notice that \(P + Q = (-1, h - \frac{1}{h})\) and \(2P + Q = (-h^2, 0)\).

In [7] we prove
\[
(x) \diamond (y) = 8(P).
\]
Consider the isomorphism
\[ \phi : E_2(h+\frac{1}{h}) \to E_2(ih+\frac{1}{ih}), \quad (X,Y) \to (-X,iY), \]
which relates two of the curves in equation (5). We use this isomorphism to pull the rational functions \( x, y \in \mathbb{C}(E_2(ih+\frac{1}{ih})) \) back to \( \mathbb{C}(E_2(h+\frac{1}{h})) \):
\[ r_2(ih+\frac{1}{ih})(\{x,y\}) = r_2(h+\frac{1}{h})(\{x \circ \phi, y \circ \phi\}). \]

On the other hand, it is easy to see that
\[ (x \circ \phi) \odot (y \circ \phi) = 8(P + Q). \]

4 Relationships between divisors

From the previous section, the problem reduces to finding relations between \((P)\) and \((P + Q)\) in \( \mathbb{Z}[E_2(h+\frac{1}{h})](\mathbb{C}) \). In order to do that, we will look for elements that are trivial in \( K_2(\mathbb{C}(E_2(h+\frac{1}{h}))) \). In other words, we will find combinations of Steinberg symbols \( \{g, 1 - g\} \) with \( g \in \mathbb{C}(E_2(h+\frac{1}{h})) \), such that the corresponding combination \( (g) \odot (1 - g) \) yields a linear combination of \((P)\) and \((P + Q)\). Since \( \{g, 1 - g\} \) is trivial in \( K \)-theory, we conclude that \( (g) \odot (1 - g) \sim 0 \), yielding a linear combination involving \((P)\) and \((P + Q)\).

Consider the function
\[ f = \frac{Y}{2h} + \left( \frac{1}{2} - \frac{1}{2h^2} \right) X. \]
We have
\[ 1 - f = 1 - \frac{Y}{2h} - \left( \frac{1}{2} - \frac{1}{2h^2} \right) X. \]
Then
\[ (f) = (2P) + 2(P + Q) - 3O, \quad (1 - f) = (P) + (A) + (B) - 3O, \]
where
\[ A = \left( \frac{-3 + \sqrt{9 - 16h^2}}{2}, \frac{7h}{2} - \frac{3}{2h} - \left( h - \frac{1}{h} \right) \frac{\sqrt{9 - 16h^2}}{2} \right). \]
\[ B = \left( \frac{-3 - \sqrt{9 - 16h^2}}{2}, \frac{7h}{2} - \frac{3}{2h} + \left( h - \frac{1}{h} \right) \frac{\sqrt{9 - 16h^2}}{2} \right). \]

In particular, for \( h = \frac{1}{\sqrt{2}} \), we get

\[ A = 3P + Q, \quad B = Q, \]

implying

\[ (f) \circ (1 - f) = 6(P) - 10(P + Q) \sim 0 \]

yielding the expected relation.

On the other hand, for \( h = \frac{1}{2} \), our function \( f \) becomes

\[ f = Y - \frac{3}{2}X. \]

In this case, \( A \) and \( B \) are given by:

\[ A = \left( \frac{-3 - \sqrt{5}}{2}, -\frac{5 - 3\sqrt{5}}{4} \right), \quad B = \left( \frac{-3 + \sqrt{5}}{2}, -\frac{5 + 3\sqrt{5}}{4} \right). \]

In particular, we have the relations

\[ 2A = 2B = P, \quad B - A = 2P, \quad A + B = -P. \]

We obtain

\[ (f) \circ (1 - f) = (P) + (2P - A) + (2P - B) - 3(2P) + 2(Q) + 2(P + Q - A) + 2(P + Q - B) - 6(P + Q) - 3(-P) - 3(-A) - 3(-B) + 9O \]

\[ = 2(Q + A) + 2(Q + B) - 6(P + Q) + 4(P) + 2(A) + 2(B). \]

We need further relations among the divisors \((A),(B)\). Thus we consider the following function

\[ g = \frac{\sqrt{5} - 1}{10}Y + \frac{3 + \sqrt{5}}{20}(X + 4), \]

\[ 1 - g = 1 - \frac{\sqrt{5} - 1}{10}Y - \frac{3 + \sqrt{5}}{20}(X + 4). \]
We have
\((g) = (Q) + (A) + (−Q − A) − 3O, \quad (1 − g) = (−P) + 2(B) − 3O.\)

The diamond operation yields a new relation:
\[
(g) ∘ (1 − g) = (Q + P) + 2(Q − B) − 3(Q) + (A + P) + 2(A − B) − 3(A) \]
\[
+ (−Q − A + P) + 2(−Q − A − B) − 3(−Q − A) − 3(P) − 6(−B) + 9O \]
\[
= 3(Q + P) − 2(Q + B) − 3(A) + 4(Q + A) − 3(P) + 5(B).
\]

In order to get more relations, we apply the Galois conjugate,
\[
(g^σ) ∘ (1 − g^σ) = 3(Q + P) − 2(Q + A) − 3(B) + 4(Q + B) − 3(P) + 5(A).
\]

The last two equations yield
\[
(g) ∘ (1 − g) + (g^σ) ∘ (1 − g^σ) = 6(Q + P) + 2(Q + A) + 2(Q + B) + 2(A) + 2(B) − 6(P).
\]

Finally, we obtain
\[
(f) ∘ (1 − f) − (g) ∘ (1 − g) − (g^σ) ∘ (1 − g^σ) = −12(Q + P) + 10(P) \sim 0.
\]

5 Conclusion of the proof
Given a relationship of the form
\[
a(P) \sim b(P + Q),
\]
we get
\[
ar_2(\frac{h}{h + 1}) \left\{ x_2(\frac{h}{h + 1}), y_2(\frac{h}{h + 1}) \right\} = \left\{ x_2(\frac{ih}{h + 1}), y_2(\frac{ih}{h + 1}) \right\},
\]
and
\[
am \left( 2 \left( h + \frac{1}{h} \right) \right) = bm \left( 2 \left( ih + \frac{1}{ih} \right) \right).
\]
Thus, for \( h = \frac{1}{\sqrt{2}}, \) we recover
\[
m(8) = \frac{8}{5} m \left( 3\sqrt{2} \right) = \frac{8}{3} m \left( i\sqrt{2} \right) = 4m(2).
\]
For $h = \frac{1}{2}$, we conclude

$$m(16) = \frac{11}{6} m(5) = \frac{11}{5} m(-3i) = 11m(1).$$

$$m(5) = 6m(1).$$

□

Questions that remain open are how to predict identities such as (2) and (3) and, more precisely, to list all such identities.

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References

[1] M. J. Bertin, Mesure de Mahler d’une famille de polynômes. J. Reine Angew. Math. 569 (2004), 175–188.

[2] S. J. Bloch, Higher regulators, algebraic $K$-theory, and zeta functions of elliptic curves. CRM Monograph Series, 11. American Mathematical Society, Providence, RI, 2000. x+97 pp.

[3] D. W. Boyd, Mahler’s measure and special values of L-functions, Experiment. Math. 7 (1998), 37-82.

[4] J. W. S. Cassels, Lectures on elliptic curves. London Mathematical Society Student Texts, 24. Cambridge University Press, Cambridge, 1991. vi+137 pp.
[5] C. Deninger, Deligne periods of mixed motives, $K$-theory and the entropy of certain $\mathbb{Z}^n$-actions, *J. Amer. Math. Soc.* 10 (1997), no. 2, 259–281.

[6] N. Kurokawa and H. Ochiai, Mahler measures via crystalization, *Commentarii Mathematici Universitatis Sancti Pauli*, 54 (2005), 121-137.

[7] M. N. Lalín, M. D. Rogers, Functional equations for Mahler measures of genus-one curves, *Algebra Number Theory* 1 (2007), no. 1, 87–117.

[8] F. Rodriguez-Villegas, Modular Mahler measures I, Topics in number theory (University Park, PA, 1997), 17–48, Math. Appl., 467, Kluwer Acad. Publ., Dordrecht, 1999.

[9] F. Rodriguez-Villegas, Identities between Mahler measures, *Number theory for the millennium, III (Urbana, IL, 2000)*, 223–229, A K Peters, Natick, MA, 2002.