Linearization of the box-ball system with box capacity L

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Abstract

We construct a bijection between the state of the box-ball system with box capacity \( L \) and a pair of two sequences. In time evolution, one of the sequences moves at speed 1, and the other follows the rules of the box-ball system with box capacity one, which can be linearized by the Kerov-Kirillov-Reshetikhin (KKR) bijection. Our method can be applied to a state including a negative value or a value greater than the box capacity.

Keywords: soliton cellular automata, box-ball system, Kerov-Kirillov-Reshetikhin bijection

1 Introduction

In 1990, Takahashi and Satsuma introduced a soliton cellular automaton called the box-ball system (BBS) [1]. The state of the original BBS is described with infinite boxes that can hold only one ball (that is, box capacity is one) and finite balls. It has since been studied from a variety of perspectives, including ultradiscretization of soliton equations [2, 3], crystal bases [4], and the inverse scattering method [5].

Kuniba et al. found that the time evolution of the BBS can be linearized using the Kerov-Kirillov-Reshetikhin (KKR) bijection [4, 5]. The KKR bijection was originally introduced for the analysis of solvable lattice models [6] and was later investigated in relation to Kashiwara crystals. For the BBS with box capacity one, the procedure to compose the KKR bijection was simplified with “01-arc lines” and an elementary proof was given by Kakei et al. [7]. The 01-arc lines are used to describe the time evolution of the BBS, and they are applied to BBS analysis in the context of probability theory [8].

In this paper, we give a method of linearizing the time evolution of the BBS with box capacity \( L \) by decomposing a state into two sequences: 1) a sequence that shifts to the right at speed one and 2) a binary sequence that exhibits the time evolution of the BBS with box capacity one. Our method of decomposition can be easily applied to a state which includes a negative value or a value greater than the box capacity.

We use the following notation in this paper:

- Semi-infinite integer sequence: \( \eta = (\eta_0, \eta_1, \eta_2, \ldots), \eta_j \in \mathbb{Z} \) (\( j = 0, 1, 2, \ldots \)).
- The \( j \)-th component of \( \eta \): \( (\eta)_j = \eta_j \).

2 BBS with Box Capacity \( L \) for a Sequence of Integer Values

2.1 Time evolution

In this paper, we consider the BBS with box capacity \( L \in \mathbb{Z}_{\geq 0} \) (hereinafter, we call this BBS(L)). First, we introduce the original BBS(L) for \( L + 1 \) values. The set of BBS(L) states is denoted by \( S_L \):

\[
S_L = \left\{ \eta \in [0, 1, \ldots, L]^\mathbb{Z} \left| \sum_{j=0}^{\infty} \eta_j < \infty, \eta_0 = 0, \sum_{j=0}^{i} (L - \eta_j) \geq \sum_{j=0}^{i+1} \eta_j (i = 0, 1, \ldots) \right. \right\}. \quad (2.1)
\]
The time evolution $T_L : S_L \rightarrow S_L; \eta^i \mapsto \eta^{i+1} = T_j(\eta^i)$ of the BBS($L$) can be described with a carrier that transports balls from left to right according to the following rules:

(i) The carrier starts from the leftmost site with no balls, and it runs to the right.

(ii) When the carrier passes in front of the $j$-th box, it performs the following two operations simultaneously:
- if that box contains at least one ball, the carrier picks the ball(s) up,
- if that box is not full and the carrier has at least one ball, the carrier drops off as many balls as possible into that box.

(iii) When all the balls are transported to another box, the carrier stops.

By repeating this procedure, the time evolution series of BBS($L$) can be obtained.

The time evolution $T_L$ can be rewritten as a piecewise linear equation known as the ultradiscrete Korteweg-de Vries(uKdV) equation:

$$u_0^j = 0, \quad \eta_j = \min(L - \eta_j, u_j), \quad \eta_j = \max(0, \eta_j + u_j - L),$$

where $\eta_j$ is the number of balls in the $j$-th box at time $t$, and $u^j$ is the number of balls in the carrier just before passing the $j$-th box at time $t$. Eq. (2.4) means that the total number of balls is conserved in the time evolution as $u_j + \eta_j = \eta_{j+1} + u_{j+1}$.

The BBS with finite balls has the reverse time evolution uniquely. Let $x$ be a non-negative integer such that $\eta_j = 0$ for all $j > x$. Then, the variables satisfy

$$u_i^j = 0, \quad \eta_i^j = \min(L - \eta_j, u_{j+1}^j), \quad \eta_j^i = \max(0, \eta_j^i + u_{j+1}^j - L).$$

**Example 2.1.** (Box capacity $L = 3$)

$$\eta^0 = 00023200021010000000000000\cdots$$

$$T_L(\eta^0) = \eta^1 = 00000133002101000000000\cdots$$

$$\langle T_L \rangle^2(\eta^0) = \eta^2 = 000000003312101000000000\cdots$$

$$\langle T_L \rangle^3(\eta^0) = \eta^3 = 0000000021232100000000\cdots$$

$$\langle T_L \rangle^4(\eta^0) = \eta^4 = 0000000021012320000000\cdots$$

$$\langle T_L \rangle^5(\eta^0) = \eta^5 = 000000000210103300000\cdots$$

$$\langle T_L \rangle^6(\eta^0) = \eta^6 = 000000000021010033100\cdots$$

In the argument in the next section, we allow the state value $\eta_j$ to take a negative value or a value greater than the box capacity $L$. For a state of arbitrary integer values, let $M = \max_{j \in \mathbb{Z}_0}(-\eta_j, \eta_j - L)$. Define the set of states as

$$S_{LM} = \left\{ \eta \in \mathbb{Z}_0 | \sum_{j=0}^{\infty} |\eta_j| < \infty, 0 \leq \eta_0 \leq 0, \max_{j \in \mathbb{Z}_0}(-\eta_j, \eta_j - L) = M, \sum_{j=0}^{i} (L - \eta_j) \geq \sum_{j=0}^{i+1} \eta_j (i = 0, 1, \ldots) \right\}. \quad (2.8)$$
We set variables $\tilde{\eta}'_j = \eta'_j + M, \tilde{u}'_j = u'_j + M, \tilde{L} = L + 2M[9]$. Then, the time evolution Eqs. (2.2), (2.3) and (2.4) become

\begin{align*}
\tilde{u}'_0 &= M, \quad (2.9) \\
\tilde{\eta}'_{j+1} &= \min(\tilde{L} - \tilde{\eta}'_j, \tilde{u}'_j), \quad (2.10) \\
\tilde{u}'_{j+1} &= \tilde{\eta}'_j + \tilde{u}'_j - \tilde{\eta}'_{j+1} \\
&= \tilde{\eta}'_j + \max(0, \tilde{\eta}'_j + \tilde{u}'_j - \tilde{L}). \quad (2.11)
\end{align*}

Since Eqs. (2.10) and (2.11) are equivalent to Eqs. (2.3) and (2.4) respectively, we can regard this system as a BBS($\tilde{L}$). Note that the carrier starts with $M$ balls and stops when the carrier and every box to the right of the carrier have $M$ balls. Let $T^M_L$ denote a time evolution of uKdV Eqs. (2.7) and (2.8) with the initial value $\tilde{u}'_0 = M$.

The BBS($\tilde{L}$) can be expressed in terms of the BBS(1) by thinking transformation between a binary sequence and an $L + 1$ value sequence[10]. We define two binary sequences $r_I$ and $l_I$ from a state $\eta' \in S_{L,M}$. Let $J(\eta')$ be a set of indices $j$, where $\eta'_j + u'_j \geq L$,

\begin{align*}
J(\eta') &= \{ j \in \mathbb{Z}_{\geq 0} | \eta'_j + u'_j \geq L \} \\
&= \{ j \in \mathbb{Z}_{\geq 0} | \tilde{\eta}'_j + \tilde{u}'_j \geq \tilde{L} \}.
\end{align*}

For a subset $I = \{i_0,i_1,\ldots,i_m\} \subset J(\eta')$, let $r_I(\eta_j)$ be the binary sequence of length $\tilde{L}$ as

\begin{align*}
\begin{cases}
1_{\tilde{L}-\tilde{\eta}'_j} & \text{if } j \in I, \\
0_{\tilde{L}-\tilde{\eta}'_j} & \text{if } j \notin I,
\end{cases}
\end{align*}

and let $r_I(\tilde{\eta}'_j)$ be the concatenation of $r_I(\tilde{\eta}'_j)$. Similarly, let $J'(\eta')$ be a set of indices $j$, where $\eta'_{j'} + u'_{j+1} \geq L$,

\begin{align*}
J'(\eta') &= \{ j \in \mathbb{Z}_{\geq 0} | \eta'_{j'} + u'_{j+1} \geq L \} \\
&= \{ j \in \mathbb{Z}_{\geq 0} | \tilde{\eta}'_{j'} + \tilde{u}'_{j+1} \geq \tilde{L} \}.
\end{align*}

For a subset $I' = \{i'_0,i'_1,\ldots,i'_m\} \subset J'(\eta')$, let $l_{I'}(\eta_j)$ be the binary sequence of length $\tilde{L}$ as

\begin{align*}
\begin{cases}
0_{\tilde{L}-\tilde{\eta}'_{j'}} & \text{if } j' \in I', \\
1_{\tilde{L}-\tilde{\eta}'_{j'}} & \text{if } j' \notin I',
\end{cases}
\end{align*}

and let $l_{I'}(\tilde{\eta}'_{j'})$ be the concatenation of $l_{I'}(\tilde{\eta}'_{j'})$. For $\eta' \in S_{L,M}$ and $I = \{i_0,i_1,\ldots,i_m\} \subset J(\eta')$,

\begin{align*}
T^M_L(r_I(\eta')) = l_I(T_L(\eta')). \quad (2.18)
\end{align*}

**Theorem 2.2.** For $\eta' \in S_{L,M}$ and $I = \{i_0,i_1,\ldots,i_m\} \subset J(\eta')$,

\begin{align*}
T^M_L(r_I(\eta')) = l_I(T_L(\eta')). \quad (2.18)
\end{align*}

*Proof.* From Eq. (2.4), we have $\eta'_j + u'_j = \eta'_{j+1} + u'_{j+1}$ and $J(\eta') = J'(T_L(\eta'))$. Consider the carrier passing through $\tilde{L}$ boxes from $j\tilde{L}$-th to $(j+1)\tilde{L} - 1$-th on $r_I(\eta')$.

- Case 1: $j \notin I$ and $\tilde{\eta}'_j + \tilde{u}'_j < \tilde{L}$

  The carrier drops off $\tilde{u}'_j$ balls into the first $\tilde{u}'_j$ empty boxes and picks up $\tilde{\eta}'_j$ balls. Then, the number of balls in the boxes in this interval becomes $\tilde{u}'_j$, which is equal to $\tilde{\eta}'_{j+1} = \min(\tilde{L} - \tilde{\eta}'_j, \tilde{u}'_j)$. 

Example 2.3. (\(L = 2, \eta' = 0, 3, 2, -1, 3, 0, 0, 0, 0, \ldots\))
\[ M = \max_{j \in \mathbb{Z}_+} (-\eta', \eta' - L) = 1, \quad L = 4, \text{ and } \tilde{\eta}' = 1, 4, 3, 0, 4, 1, 1, 1, 1, \ldots. \] Using time evolution Eqs. (2.2), (2.3) and (2.4), we get
\[
\begin{align*}
\eta' &= 0, 0, 4, 6, 2, 6, 4, 2, 0, 0, \ldots, \\
\eta'^{+1} &= -1, 0, 3, -1, 2, 2, 0, 0, \ldots, \\
\tilde{u}' &= 1, 1, 5, 7, 3, 7, 5, 3, 1, 1, \ldots, \\
\tilde{\eta}' &= 1, 0, 1, 4, 0, 3, 3, 3, 1, 1, \ldots.
\end{align*}
\]
Then, \( J(\eta') = \{ j \in \mathbb{Z}_{\geq 0} \mid \tilde{n}_j + \tilde{u}_j' \geq L \} \) becomes
\[
J(\eta') = \{ 1, 2, 4, 5, 6 \}.
\]

Choosing a subset of \( J(\eta') \) as \( I = \{2, 5\} \), we obtain
\[
\begin{align*}
 r_I(\eta') &= 001111111100000111110000010010001\cdots, \\
 T_I^1(r_I(\eta')) &= 1000000000011111000001111110110100\cdots, \\
 I_I(T_L(\eta')) &= 1000000000011111000001111110110100\cdots.
\end{align*}
\]

### 2.2 Decomposition

We define the following sets of semi-infinite sequences:
\[
S^{(b)}_{LM} = \left\{ c \in S_{LM} \mid c_j + c_{j+1} \leq L \text{ for } j = 0, 1, \ldots \right\},
\]
\[
S^{(f)}_L = \{ d \in S_1 \mid \text{the length of every maximal subsequence that contains only 0s/1s is larger than } L \}. \tag{2.19}
\]

First, we decompose \( \eta \in S_{LM} \) into two sequences \( c \in S^{(b)}_{LM} \) and \( d \in S^{(f)}_L \) through the decomposition map \( \beta_L : S_{LM} \to S^{(b)}_{LM} \oplus S^{(f)}_L : \eta \mapsto (c, d) \) defined by the following procedure:

1. Let \( \tilde{\eta} = (\tilde{n}_0, \tilde{n}_1, \ldots) \) be as defined above, \( \tilde{n}_j = \eta_j + M \).
2. Define a soliton flag sequence \( \alpha = (a_0, a_1, \ldots) \in S_1 \) as
\[
\begin{align*}
a_0 &= 0, \\
a_j+1 &= \begin{cases} 
0 & \tilde{n}_j + \tilde{n}_{j+1} < L, \\
1 & \tilde{n}_j + \tilde{n}_{j+1} = L, \\
2 & \tilde{n}_j + \tilde{n}_{j+1} > L.
\end{cases}
\end{align*}
\]

We say \( j \) is in a **0-segment** if \( a_j = 0 \), and in a **1-segment** if \( a_j = 1 \).

3. Let \( I = \{ j \in \mathbb{Z}_{\geq 0} \mid a_j = 1, a_{j+1} = 0 \} \), which is the set of the right ends of contiguous **1-segments**. We define
\[
\bin(\eta) = r_I(\eta). \tag{2.22}
\]

where \( r_I \) is defined in Eqs. (2.13) and (2.14). In Example 2.4 below, we underline each group of \( L \) numbers to make it easy to see the boxes.

4. We write “\( \times \)” on the subsequence corresponding to the \( j \)-th and the \((j + 1)\)-th boxes where \( j \in I \). This means that we mark the right end of every contiguous **1-segment** and just left of there.

5. We iteratively draw 10-arc lines from 1s in the \( j \)-th box without “\( \times \)” and 0s in the \((j + 1)\)-th box. Then, the number of arc lines that connect the \( j \)-th box and the \((j + 1)\)-th box is \( \min(\tilde{n}_j, \tilde{n}_{j+1}) \).

6. Let \( \tilde{b} = (\tilde{b}_0, \tilde{b}_1, \ldots) \), where \( \tilde{b}_j \) is the number of 10-arc lines that connect 1s in the \( j \)-th box and 0s in the \((j + 1)\)-th box. We will use this sequence in the proof of Theorem 2.12. If \( a_{j+1} = 0, \tilde{n}_j + \tilde{n}_{j+1} \leq L \), then \( \min(\tilde{n}_j, \tilde{n}_{j+1}) = \tilde{n}_j \). If \( a_{j+1} = 1, \tilde{n}_j + \tilde{n}_{j+1} \geq L \), then \( \min(\tilde{n}_j, \tilde{n}_{j+1}) = \tilde{n}_{j+1} \). Thus, we get
\[
\tilde{b}_j = \begin{cases} 
\tilde{n}_j & a_j = a_{j+1} = 0, \\
L - \tilde{n}_{j+1} & a_{j+1} = a_{j+2} = 1, \\
0 & (a_j, a_{j+1}) = (1, 0) \text{ or } (a_{j+1}, a_{j+2}) = (1, 0).
\end{cases} \tag{2.23}
\]
7. Define a raised background sequence \( \tilde{b} = (\tilde{c}_0, \tilde{c}_1, \ldots) \) as a sequence obtained by skipping terms in the sequence \( b \) if "\( \times \)" is written.

8. Define a raised soliton sequence \( \tilde{d} = (\tilde{d}_0, \tilde{d}_1, \ldots) \) as a binary sequence that is obtained by eliminating 1s and 0s connected with the 10-arcs from \( \text{bin}(\tilde{\eta}) \). Let \( \tilde{s}_j \) be the number of consecutive 0s and \( \tilde{t}_j \) be the number of consecutive 1s in \( \tilde{d} \) as \( \tilde{d} = 0^5 1^5 0^5 1^5 \cdots \).

9. Let \( c = (c_0, c_1, \ldots) \in S^{(b)}_{L,M} \), where \( c_j = \tilde{c}_j - M \) (\( j = 0, 1, \ldots \)). This sequence is called the background sequence [11].

10. Let \( d = (d_0, d_1, \ldots) = 0^5 1^5 0^5 1^5 \cdots \), where \( s_1 = \tilde{s}_1 - M, s_j = \tilde{s}_j - 2M \) (\( j = 2, 3, \ldots \)), \( t_j = \tilde{t}_j - 2M \) (\( j = 1, 2, \ldots \)). We call this a soliton sequence.

**Example 2.4.** \( (L = 2, M = 1 \text{ and } L = 4.) \)

\[
\begin{align*}
\eta &= 0 \ 0 \ 0 \ 3 \ 1 \ -1 \ 2 \ 1 \ 1 \ 0 \ 0 \ \cdots \\
a &= 0 \ 0 \ 1 \ 1 \ 0 \ 0 \ 1 \ 1 \ 0 \ 0 \ \cdots \\
\tilde{\eta} &= 1 \ 1 \ 4 \ 2 \ 0 \ 3 \ 2 \ 2 \ 1 \ 1 \ \cdots \\
\text{bin}(\eta) &= 0001000111111100000000111100111100000010001 \cdots \\
\tilde{b} &= 1 \ 0 \ 0 \ 0 \ 0 \ 2 \ 0 \ 0 \ 1 \ 1 \ \cdots \\
\tilde{c} &= 1 \ 0 \ 0 \ 2 \ 1 \ 1 \ \cdots \\
c &= 0 \ -1 \ -1 \ 1 \ 0 \ 0 \ \cdots \\
\tilde{d} &= 000 \ 001 \ 111111000000001 \ 11 \ 1100 \ 000 \ \cdots \\
&= 0^5 \ 1^5 \ 0^5 \ 1^5 \ 0^5 \\
d &= 0^4 \ 1^5 \ 0^5 \ 1^3 \ 0^5 \\

\end{align*}
\]

Define the forward-shift operator \( \Lambda \) on \( S^{(b)}_{L,M} \) as

\[
(\Lambda(c))_j = \begin{cases} 
0 & j = 0, \\
(c)_{j-1} & \text{otherwise.}
\end{cases} \quad (2.24)
\]

**Theorem 2.5.** For \( \eta \in S_{L,M} \), let \( c(\eta) \) be the background sequence and \( d(\eta) \) be a soliton sequence, as defined above.

\[
\begin{align*}
\Lambda(c(\eta)) &= c(T_1(\eta)) \quad (2.25) \\
T_1(d(\eta)) &= d(T_1(\eta)) \quad (2.26)
\end{align*}
\]

The background sequence \( c \) is shifted to the right at speed one, and the soliton sequence \( d \) follows the time evolution of BBS(1). Because the time evolution of BBS(1) can be linearized [7], we can linearize the time evolution of BBS(L). This claim will be proved from Theorems 2.9 and 2.12.

**Example 2.6.** \( L = 3 \), and \( \eta = 0, 0, 3, 1, -1, 2, 1, 1, 0, 0, \ldots \in S_{2,1} \) (as in Example 2.4)

\[
\begin{align*}
c(\eta) &= 0, -1, -1, 1, 0, 0, \ldots \\
d(\eta) &= 00001111100000011100000 \cdots \\
T_3(\eta) &= 0, 0, -1, 1, 3, 0, 1, 1, 2, 0, \ldots \\
c(T_3(\eta)) &= 0, 0, -1, -1, 1, 0, \ldots \\
d(T_3(\eta)) &= 00000000011111000001100 \cdots 
\end{align*}
\]
1. Define a soliton flag sequence $a' = (a'_{i-1}, a'_{i}, \ldots) \in S_1$ as

$$a'_{i-1} = 0,$$

$$a'_{i-1} = \begin{cases} 0 & \eta_{i+1} + \eta_{i+2} < \tilde{L}, \\ a'_{i} & \eta_{i+1} + \eta_{i+2} = \tilde{L}, \\ 1 & \eta_{i+1} + \eta_{i+2} > \tilde{L}, \end{cases}$$

$$a'_{i+1} = \begin{cases} 0 & \eta_{j} + \eta_{j+1} < L, \\ a_{j} & \eta_{j} + \eta_{j+1} = L, (j = 0, 1, 2, \ldots) \\ 1 & \eta_{j} + \eta_{j+1} > L. \end{cases}$$

We say $j$ is in a right-$0$-segment if $a'_j = 0$, and in a right-$1$-segment if $a'_j = 1$. By definition, it is clear that

$$a'_j = a'_{j-1} \quad (j = 0, 1, \ldots).$$

2. Let $I' = \{ j \in \mathbb{Z}_{\geq 0} | a'_j = 1, a'_{j-1} = 0 \}$, which is the set of the left ends of contiguous right-$1$-segments. We define

$$\tilde{\text{bin}}'(\eta) = I'_r(\tilde{\eta}),$$

where $I'_r$ is defined in Eqs. (2.16) and (2.17).

3. We write “$\times$” on the subsequence corresponding to the $j$-th and the $(j + 1)$-th boxes where $j \in I$. This means that we mark the left end of every contiguous right-$1$-segment and just right of there.

4. We iteratively draw 01-arc lines from 0s in the $(j - 1)$-th box without “$\times$” and 1s in the $j$-th box repeatedly. Then, the number of arc lines that connect the $(j - 1)$-th box and the $j$-th box is $\min(\eta_j, \tilde{L} - \eta_{j-1}).$

5. Let $\tilde{b}' = (\tilde{b}'_0, \tilde{b}'_1, \ldots)$, where $\tilde{b}'_j$ is the number of 01-arc lines that connect 0s in the $(j - 1)$-th box and 1s in the $j$-th box. If $a'_{j-1} = 0$, $\eta_{j-1} > \tilde{L}$, then $\min(\eta_j, \tilde{L} - \eta_{j-1}) = \tilde{L}$. If $a_{j-1} = 1$, $\eta_{j-1} + \tilde{L} > \tilde{L}$, then $\min(\eta_{j}, \tilde{L} - \eta_{j-1}) = \tilde{L} - \tilde{L}_{j-1}$. Thus, we get

$$\tilde{b}'_j = \begin{cases} \eta_{j} & a'_{j-1} = a'_j = 0, \\ L - \tilde{\eta}_{j-1} & a'_{j-2} = a'_{j-1} = 1, \\ 0 & (a'_{j-2}, a'_{j-1}) = (0, 1) \text{ or } (a'_{j-1}, a'_j) = (0, 1), \end{cases}$$

6. Let $\tilde{c}' = (\tilde{c}'_0, \tilde{c}'_1, \ldots)$ be a sequence obtained by skipping terms in the sequence $\tilde{b}'$ if “$\times$” is written.

7. Let $\tilde{d}' = (\tilde{d}'_0, \tilde{d}'_1, \ldots)$ be a binary sequence that is obtained by eliminating 1s and 0s connected with the 01-arcs from $\text{bin}(\tilde{\eta})$. Let $\tilde{s}'_j$ be the number of consecutive 0s and $\tilde{r}'_j$ be the number of consecutive 1s in $\tilde{d}'$ as $\tilde{d}' = 0^{\tilde{s}'_j} 1^{\tilde{r}'_j} 0^{\tilde{s}'_{j+1}} 1^{\tilde{r}'_{j+1}} \ldots$. 
8. Let \( c' = (c'_0, c'_1, \ldots) \in S_{L,M}^{(0)} \), where \( c'_j = \tilde{c}'_j - M \) \( (j = 0, 1, \ldots) \).

9. Let \( d' = (d'_0, d'_1, \ldots) = 0^5 1^5 0^5 1^5 \ldots \), where \( s_j = \tilde{s}'_j - 2M, \tilde{i}'_j = \tilde{t}'_j - 2M \) \( (j = 1, 2, \ldots) \).

**Lemma 2.7.** For \( \eta \in S_{L,M} \), the following conditions are equivalent.

(i) \((a(\eta))_j = 1 \) and \((a(\eta))_{j+1} = 0 \)

(ii) \((a'(T_k(\eta)))_j = 1 \) and \((a'(T_L(\eta)))_{j-1} = 0 \)

**Proof.** Let \( a(\eta) = (a_0, a_1, \ldots), a'(T_k(\eta)) = (a'_0, a'_1, \ldots) \). Here, we prove that (i) \( \Rightarrow \) (ii) by using Eqs. (2.3) and (2.4). The reverse (ii) \( \Rightarrow \) (i) can be proved similarly.

- **Case 1:** \( \eta'_{j-1} + \eta'_j > L, \eta'_j + \eta'_{j+1} < L \)

  \[ \eta'_{j+1} = \min(u'_j, L - \eta'_j) \]

  \[ = L - \eta'_j. \]

  Therefore, we have

  \[ \eta'_{j-1} + \eta'_{j+1} - L = \min(u'_{j-1}, L - \eta'_{j+1}) + (L - \eta'_j) - L \]

  \[ = \min(u'_{j-1} - \eta'_j, L - \eta'_{j-1} - \eta'_j) \]

  \[ < 0. \]

  By using \( u'_{j+1} = \eta'_j + \max(0, \eta'_j + u'_j - L) > u'_j \):

  \[ \eta'_{j+1} + \eta'_{j+1} - L = (L - \eta'_j) + \min(u'_{j+1}, L - \eta'_{j+1}) - L \]

  \[ = \min(u'_{j+1} - \eta'_j, L - \eta'_j - \eta'_{j+1}) \]

  \[ > 0. \]

- **Case 2:** The case \( \eta'_{k-1} + \eta'_k > L, \eta'_k + \eta'_{k+1} < L \)

  \[ \eta'_i + u'_i - L \geq \eta'_i + \eta'_{i-1} - L \geq 0 \] for \( i = k, k + 1, \ldots, j \) and

  \[ \eta'_{i+1} = \min(u'_j, L - \eta'_j) \]

  \[ = L - \eta'_j \] \( (i = k, k + 1, \ldots, j) \).

  From \( \eta'_{k} + u'_k - L \geq \eta'_{k} + \eta'_{k-1} - L \geq 0 \), we have \( u'_{k+1} > \eta'_{k} \). Similarly, we have \( u'_{j+1} > \eta'_{j} \).
Therefore, we have

\[ \eta_{k-1}^{t+1} + \eta_k^{t+1} - L = \min(u_{k-1}^{t+1}, L - \eta_{k-1}^t) + (L - \eta_k^t) - L \]
\[ = \min(u_{k-1}^{t+1} - \eta_k^t, L - \eta_k^t - \eta_k^t) \]
\[ < 0 \]

\[ \eta_{i-1}^{t+1} + \eta_i^{t+1} - L = (L - \eta_{i-1}^t) + (L - \eta_i^t) - L \]
\[ = L - \eta_{i-1}^t - \eta_i^t \]
\[ = 0 \quad (i = k + 1, \ldots, j) \]

\[ \eta_{j+1}^{t+1} + \eta_{j+1}^{t+1} - L = (L - \eta_{j+1}^t) + \min(u_{j+1}^{t+1}, L - \eta_{j+1}^t) - L \]
\[ = \min(u_{j+1}^{t+1} - \eta_{j+1}^t, L - \eta_{j+1}^t - \eta_{j+1}^t) \]
\[ > 0 \]

\[ \square \]

Lemma 2.8. For \( \eta \in S_{LM} \),

\[ T_1^M(\text{bin}(\eta)) = \text{bin}'(T_L(\eta)) \] (2.31)

Proof. Let \( I \) be the set of indices

\[ I = \{ i \in \mathbb{Z}_{\geq 0} \mid (a(\eta))_i = 1, (a(\eta))_{i+1} = 0 \} \]
\[ \subset \{ i \in \mathbb{Z}_{\geq 0} \mid \eta_i + u_i \geq L \}. \]

Then, from Lemma 2.7,

\[ I = \{ i \in \mathbb{Z}_{\geq 0} \mid (a'(T_L(\eta)))_{i-1} = 0, (a'(T_L(\eta)))_i = 1 \}. \]

From Theorem 2.2, we have \( T_1^M(\text{bin}(\eta)) = \text{bin}'(T_L(\eta)) \). \( \square \)

Theorem 2.9. For \( \eta \in S_{LM} \)

\[ \Lambda(c(\eta)) = c'(T_L(\eta)) \] (2.32)

\[ T_1^M(d(\eta)) = d'(T_L(\eta)) \] (2.33)

To prove Theorem 2.9, we introduce 10-arc lines on the binary sequence \( \eta \in S_1 \) [12] which can express the time evolution of BBS(1) according to the following rules.

i) For \( \eta \in S_1 \), connect all 10 pairs with arc lines.

ii) Neglecting the 1s and 0s which were connected already, connect all the remaining 10 pairs with arc lines.

iii) Repeat the above procedure until all the 1s are connected to 0s.

iv) \( T_1(\eta) \) is the state obtained by exchanging the 1s and 0s in every connected 10 pair.

We can draw 01 arc lines in the same fashion, and the following lemma [7] is obvious from the definition.

Lemma 2.10. The 10-arc lines for \( \eta \in S_1 \) coincide with the 01-arc lines for \( T_1(\eta) \).
(Proof of Theorem 2.9.) From Lemmas 2.7 and 2.8, the 10-arc lines for $\text{bin}(\eta)$ coincide with the 01-arc lines for $\tilde{\text{bin}}' (T_2(\eta))$. Using the 01/10-arc lines, $\min(\tilde{\eta}_j, \tilde{L} - \tilde{\eta}_{j+1})$ can be described as the number of 10-arc lines on $\tilde{\text{bin}}' (T_2(\eta))$ that connect the balls in the $j$–th box and the vacancies in the $(j + 1)$–th box. Similarly, $\min((T_L(\eta))_j, \tilde{L} - (T_L(\eta))_{j-1})$ can be explained as the number of 01-arc lines on $\text{bin}' (T_L(\eta))$ that connect the balls in the $j$–th box and the vacancies in the $(j - 1)$–th box. Thus, it follows that $\Lambda(\tilde{b}(\eta)) = \tilde{b}' (T_L(\eta))$ and $\Lambda(\tilde{c}(\eta)) = \tilde{c}' (T_L(\eta))$.

When we obtain the raised soliton sequences $\tilde{d}(\eta), \tilde{d}' (T_L(\eta))$ deleting 1s, 0s and 10/01-arc lines that correspond to background materials, other 10-arc lines on $\text{bin}(\eta)$ and 01-arc lines on $\text{bin}' (T_L(\eta))$ do not change. Thus, we obtain $T_1^M (d(\eta)) = d'(T_L(\eta))$, and $T_1 (d(\eta)) = d'(T_L(\eta))$.

**Example 2.11.** $L = 2$. $\eta = 0, 0, 3, -1, 1, 3, 0, 1, 1, 2, 0, \ldots$ (as in Examples 2.4 and 2.6.)

\[
\begin{align*}
T_2(\eta) & = 0 \ 0 \ -1 \ 1 \ 3 \ 0 \ 1 \ 1 \ 2 \ 0 \ldots \\
a'(T_2(\eta)) & = 0 \ 0 \ 0 \ 1 \ 1 \ 0 \ 0 \ 1 \ 1 \ 0 \ldots \\
T_2(\eta) & = 1 \ 1 \ 0 \ 2 \ 4 \ 1 \ 2 \ 2 \ 3 \ 1 \ldots \\
\text{bin}' (T_2(\eta)) & = 0^{10}100100111111100100\ldots \\
\tilde{b}' (T_2(\eta)) & = 1 \ 1 \ 0 \ 0 \ 0 \ 0 \ 2 \ 0 \ 0 \ 1 \ldots \\
\tilde{c}' (T_2(\eta)) & = 1 \ 1 \ 0 \ 0 \ 2 \ 1 \ldots \\
\tilde{c}' (T_2(\eta)) & = 0 \ 0 \ -1 \ -1 \ 1 \ 0 \ldots \\
\tilde{d}' (T_2(\eta)) & = 000 \ 000 \ 0000111111111001111100\ldots \\
& = 0^{10}1^{15}0^{1}1^{15}0^{1}0^{1,0^{1}} \\
d & = 0^{10}1^{15}0^{1}1^{15}0^{1}0^{1} \text{ or }
\end{align*}
\]

**Theorem 2.12.**

\[
\begin{align*}
e(\eta) &= e'(\eta) \quad (2.34) \\
d(\eta) &= d'(\eta) \quad (2.35)
\end{align*}
\]

**Proof.** For the soliton flag sequence $a$, define indices $i_1, i_2, \ldots, i_N, k_1, k_2, \ldots, k_N \in \mathbb{Z}_{\geq 0}$ as

\[
\begin{cases}
\{ j \in \mathbb{Z}_{\geq 0} \mid a_j = 0, a_{j+1} = 1 \} = \{ i_1 < i_2 < \ldots < i_N \} \\
\{ j \in \mathbb{Z}_{\geq 0} \mid a_j = 1, a_{j+1} = 0 \} = \{ k_1 < k_2 < \ldots < k_N \}.
\end{cases}
\]

These indices satisfy the interlacing condition $0 < i_1 < k_1 < i_2 < k_2 < \cdots < i_N < k_N$. Similarly, for the soliton flag sequence $a'$, define indices $i'_1, i'_2, \ldots, i'_{N'}, k'_1, k'_2, \ldots, k'_{N'} \in \mathbb{Z}_{\geq 0}$ as

\[
\begin{cases}
\{ j \in \mathbb{Z}_{\geq 0} \mid a'_{j-1} = 0, a'_j = 1 \} = \{ i'_1 < i'_2 < \ldots < i'_{N'} \} \quad (2.38) \\
\{ j \in \mathbb{Z}_{\geq 0} \mid a'_{j-1} = 1, a'_j = 0 \} = \{ k'_1 < k'_2 < \ldots < k'_{N'} \} \quad (2.39).
\end{cases}
\]

From Eq. (2.28), we have $i'_m = i_m, k'_m = k_m (m = 1, 2, \ldots, N)$.

Letting $k_0 = k_0' = -1$, we have

\[
\tilde{b}'_j = \begin{cases}
\tilde{\eta}_j & \text{if } j = k_m - 1, k_m, \\
\tilde{L} - \tilde{\eta}_{j+1} & \text{if } i_m \leq j \leq k_{m-2}, (m = 0, 1, 2, \ldots) \\
0 & \text{if } m \neq 0, 1, 2, \ldots.
\end{cases}
\]

from Eq. (2.23), and

\[
\tilde{b}_j = \begin{cases}
\tilde{\eta}_j & \text{if } j = i'_m, i'_m + 1. \\
\tilde{L} - \tilde{\eta}_{j+1} & \text{if } i'_m + 2 \leq j \leq k'_m, (m = 0, 1, 2, \ldots) \\
0 & \text{if } j = i'_m, i'_m + 1.
\end{cases}
\]

from Eq. (2.30). We obtain $\tilde{e}$ from $\tilde{b}$ by skipping the terms $j = k_m - 1, k_m$, and $\tilde{c}'$ from $\tilde{b}'$ by skipping the terms $j = i'_m, i'_m + 1$. Thus, we obtain $\tilde{e} = \tilde{c}'$ and $e = c'$. 

Letting $\tilde{\eta}_{-1} = L$, we can write the soliton sequence $\tilde{d}$ as $\tilde{d} = 0^{i_1}1^{i_2}0^{i_3}1^{i_4} \cdots 1^{i_N}0^\infty$, where

$$
\tilde{s}_m = \sum_{j=k_m}^{i_m} (L - \tilde{\eta}_j) - \sum_{j=k_{m+1}}^{i_{m-1}} \tilde{b}_j, 
$$

(2.42)

and $d'$ as $d' = 0^{i_1'}1^{i_2'}0^{i_3'}1^{i_4'} \cdots 1^{i_N'}0^\infty$, where

$$
\hat{s}'_1 = \sum_{j=0}^{i_0} (L - \tilde{\eta}_j) - \sum_{j=0}^{i_{N-1}} \tilde{b}_j' + M, 
$$

(2.44)

$$
\hat{s}'_m = \sum_{j=k_m}^{i_m} (L - \tilde{\eta}_j) - \sum_{j=k_{m+1}}^{i_{m-1}} \tilde{b}_j' \quad (m = 2, 3, \ldots, N),
$$

(2.45)

and

$$
\hat{t}'_m = \sum_{j=k_m}^{i_m} \tilde{\eta}_j - \sum_{j=k_{m+1}}^{i_{m-1}} \tilde{b}_j' \quad (m = 1, 2, \ldots, N). 
$$

(2.46)

Using the Eqs. (2.40), (2.41) and $i'_m = i_m, k'_m = k_m$, we have

$$
\hat{s}_1 = \hat{s}'_1 - M = L + \sum_{j=1}^{i_0} (L - \tilde{\eta}_j - \tilde{\eta}_{j+1}),
$$

(2.47)

$$
\hat{s}_m = \hat{s}'_m = L + \sum_{j=k_m}^{i_m} (L - \tilde{\eta}_j - \tilde{\eta}_{j+1}) \quad (m = 2, 3, \ldots, N),
$$

(2.48)

and

$$
\hat{t}_m = \hat{t}'_m = L + \sum_{j=1}^{k_m} \tilde{\eta}_j + \tilde{\eta}_{j+1} - L \quad (m = 1, 2, \ldots, N). 
$$

(2.49)

Therefore, we get $\tilde{d}(\eta) = \tilde{d}'(\eta)$. Since $L - \tilde{\eta}_j - \tilde{\eta}_{j+1} = L - \eta_j - \eta_{j+1}$, the soliton sequences $d = 0^{i_1}1^{i_2}0^{i_3}1^{i_4} \cdots 1^{i_N}0^\infty$ and $d' = 0^{i_1'}1^{i_2'}0^{i_3'}1^{i_4'} \cdots 1^{i_N'}0^\infty$ can be written as

$$
s_m = s'_m = L + \sum_{j=k_m}^{i_m} (L - \eta_j - \eta_{j+1}),
$$

(2.50)

$$
t_m = t'_m = L + \sum_{j=1}^{k_m} (\eta_j + \eta_{j+1} - L) \quad (m = 1, 2, \ldots, N). 
$$

(2.51)

□

A binary sequence $d \in S_1$ is associated with a rigged configuration by the KKR bijection[6], and the time evolution of BBS with box capacity 1 can be linearized. We briefly review the definition of a rigged configuration briefly[4]. Consider a partition $\mu = (\mu_1, \ldots, \mu_m)$. Define $m_j$ as the number of rows in $\mu$ whose lengths are $j (j = 1, \ldots, \mu_1)$. A rigged configuration is a set $(\mu, J)$, where $J = (J_1, J_2, \ldots, J_m), J_k = (J_{k,1}, J_{k,2}, \ldots, J_{k,m_k}) \in (\mathbb{Z})^{m_k}$. $J_{k,1}, J_{k,2}, \ldots, J_{k,m_k}$ are the riggings corresponding to the rows of length $k$.

**Theorem 2.13.** (Theorem 3 and Theorem 12 in Kakei et al. [7])

Let $(\mu, J)$ be the rigged configuration associated with $d \in S_1$, and $(\mu, \overline{J})$ be the rigged configuration associated with $T_1(d)$. Then,

$$
\overline{\mu}_i = \mu_i \quad (i = 1, \ldots, m)
$$

(2.52)

and

$$
\overline{J}_{k,l} = J_{k,l} + k \quad (k = 1, \ldots, \mu_1, l = 1, \ldots, m_k).
$$

(2.53)
Example 2.14. \((L = 2)\)

\[
\eta = 0, 0, 3, 1, -1, 2, 1, 1, 0, 0, \ldots
\]
\[
c(\eta) = 0, -1, -1, 1, 0, 0, \ldots
\]
\[
d(\eta) = 0001111100000111000000 \cdots
\]
\[
T_2(\eta) = 0, 0, -1, 1, 3, 0, 1, 1, 2, 0, \ldots
\]
\[
c(T_2(\eta)) = 0, 0, -1, -1, 1, 0, \ldots
\]
\[
d(T_2(\eta)) = 000000001111000111000000 \cdots
\]

\[
\begin{array}{c}
\eta \\
T_2 \\
\downarrow
\end{array}
\quad
\begin{array}{c}
T_2(\eta) \\
\downarrow
\end{array}
\quad
\begin{array}{c}
\eta \\
T_2
\end{array}
\]

\[
\begin{array}{|c|c|}
\hline
j & c(j) \\
\hline
0 &  \ldots \\
1 & 1 \\
2 & 0 \\
3 & 0 \\
4 & 1 \\
5 & 1 \\
6 & 0 \\
7 & -1 \\
8 & 4 \\
\hline
\end{array}
\]

3 \ reconstruction of BBS sequence \(\eta\) and bijectivity of \(\beta_L\)

In this section, we prove the bijectivity of the map \(\beta_L\). First, we define the reconstruction map \(\beta_L^{-1} : S_{LM}^b \oplus S_L^f \rightarrow S_{LM}; (c, d) \mapsto \eta\) as follows.

Let \(M = \max_{j \in \mathbb{Z}_0} (-c_j, c_j - L)\), \(\tilde{L} = L + 2M\). Let \(\tilde{c} = (\tilde{c}_0, \tilde{c}_1, \ldots)\) where \(\tilde{c}_j = c_j + M\), and \(\tilde{c}_-1 = 0\). When \(d\) is represented as \(d = 0^t \cdot 1^{s_1} \cdots 1^{s_0} \cdots \), let \(\tilde{d} = 0^t + M 1^{s_1 + 2M} 0^{s_2} + 2M 1^{s_3} + 2M \ldots 1^{s_0 + 2M} 0^{s_0}\).

Here, \(k^{(j)}\) denotes the position of the left end of the subsequence in \(d\) that corresponds to the \(j\)-th box (\(k^{(0)} = 0\)). Let \(\tilde{c}^{(j)}\) be the sequence that is obtained by inserting 0s into \(\tilde{c}\) up to the \(j\)-th box (\(\tilde{c}^{(0)} = \tilde{c}\)). Repeat the following procedure for \(j = 0, 1, 2, \ldots\) in this order, and stop if \(\tilde{c}^{(m)} = M\) for all \(l > j\) and \(\tilde{d}_m = 0\) for all \(m > k^{(j)}\).

1. For the binary sequence \(d\), let \(X_j = i - k^{(j)}\) where \(i\) is the minimal integer that satisfies \(i \geq k^{(j)}\), \(\tilde{d}_{-1} = 1\), and \(\tilde{d}_i = 0\). If there is no such integer \(i\), let \(X_j = +\infty\).

2. • Case 1: \(X_j < 2\tilde{L} - \tilde{c}_j^{(j+1)} - \tilde{c}_j^{(j)}\)

Let \(\tilde{c}^{(j+2)}\) be the sequence obtained by inserting 0 between \(\tilde{c}_j^{(j)}\) and \(\tilde{c}_j^{(j+1)}\). The length of the subsequence of \(\tilde{d}\) that corresponds to the \(j\)-th box is \(\tilde{L} - \tilde{c}_j^{(j+1)} - \tilde{c}_j^{(j+1)}\) and let \(\tilde{k}^{(j+1)}\) be

\[
k^{(j+1)} = k^{(j)} + (\tilde{L} - \tilde{c}_j^{(j+1)})
\]

Underline the subsequence of \(\tilde{d}\), from \(\tilde{d}_{k^{(j)}}\) to \(\tilde{d}_{k^{(j+1)}}\).

• Case 2: \(X \geq 2\tilde{L} - \tilde{c}_j^{(j+1)} - \tilde{c}_j^{(j)}\)

Let \(\tilde{c}^{(j+1)} = \tilde{c}^{(j)}\). Then, the length of the subsequence of \(\tilde{d}\) that corresponds to the \(j\)-th box is \(\tilde{L} - \tilde{c}_j^{(j+1)} - \tilde{c}_j^{(j+1)}\), and let \(\tilde{k}^{(j+1)}\) be

\[
k^{(j+1)} = k^{(j)} + (\tilde{L} - \tilde{c}_j^{(j+1)} - \tilde{c}_j^{(j+1)})
\]

Underline the subsequence of \(\tilde{d}\), from \(\tilde{d}_{k^{(j)}}\) to \(\tilde{d}_{k^{(j+1)}}\).
After stopping this procedure, let \( \bar{\eta}_i \) be the sum of \( \bar{c}^{(j)} \) and the number of 1s on the \( j \)-th underline in \( \bar{d} \) if \( i = 0, 1, \ldots, j-1 \) and be zero if \( i \geq j \).

**Example 3.1.** Consider \( L = 2, \ c = 0, -1, -1, 1, 0, \ldots, \ d = 00001111110000001110000 \cdots \). Then, \( M = -1, \ L = 4, \ \bar{c} = 1, 0, 0, 2, 1, \ldots, \bar{d} = 00000111111100000001111000000 \cdots \).

- \( j = 0 \) \( k^{(0)} = 0, i = 12, \) and \( X_0 = 12. \)
  \[ 2L - \bar{c}^{(0)} - \bar{c}^{(0)} = 7 \leq 12 \]
  \[ \bar{c}^{(1)} = 1, 0, 0, 2, 1, \ldots, L - \bar{c}^{(1)} - \bar{c}^{(1)} = 3. \]
  \[ \bar{d} = 000011111100000001111100000 \cdots \]

- \( j = 1 \) \( k^{(1)} = 3, i = 12, \) and \( X_1 = 9. \)
  \[ 2L - \bar{c}^{(1)} - \bar{c}^{(1)} = 7 \leq 9 \]
  \[ \bar{c}^{(2)} = 1, 0, 0, 2, 1, \ldots, L - \bar{c}^{(2)} - \bar{c}^{(2)} = 3. \]
  \[ \bar{d} = 000011111100000001111100000 \cdots \]

- \( j = 2 \) \( k^{(2)} = 6, i = 12, \) and \( X_2 = 6. \)
  \[ 2L - \bar{c}^{(2)} - \bar{c}^{(2)} = 8 \geq 6 \]
  \[ \bar{c}^{(3)} = 1, 0, 0, 0, 2, 1, \ldots, L - \bar{c}^{(3)} - \bar{c}^{(3)} = 4. \]
  \[ \bar{d} = 000011111100000001111100000 \cdots \]

- \( j = 3 \) \( k^{(3)} = 10, i = 12, \) and \( X_3 = 2. \)
  \[ 2L - \bar{c}^{(3)} - \bar{c}^{(3)} = 8 \geq 2 \]
  \[ \bar{c}^{(4)} = 1, 0, 0, 0, 0, 2, 1, \ldots, L - \bar{c}^{(4)} - \bar{c}^{(4)} = 4. \]
  \[ \bar{d} = 000011111100000001111100000 \cdots \]

- \( j = 4 \) \( k^{(4)} = 14, i = 24, \) and \( X_4 = 10. \)
  \[ 2L - \bar{c}^{(4)} - \bar{c}^{(4)} = 8 \leq 10 \]
  \[ \bar{c}^{(5)} = 1, 0, 0, 0, 0, 2, 1, \ldots, L - \bar{c}^{(5)} - \bar{c}^{(5)} = 4. \]
  \[ \bar{d} = 000011111100000001111100000 \cdots \]

- \( j = 5 \) \( k^{(5)} = 18, i = 24, \) and \( X_5 = 6. \)
  \[ 2L - \bar{c}^{(5)} - \bar{c}^{(5)} = 6 \leq 6 \]
  \[ \bar{c}^{(6)} = 1, 0, 0, 0, 0, 2, 1, \ldots, L - \bar{c}^{(6)} - \bar{c}^{(6)} = 2. \]
  \[ \bar{d} = 000011111100000001111100000 \cdots \]

- \( j = 6 \) \( k^{(6)} = 20, i = 24, \) and \( X_6 = 4. \)
  \[ 2L - \bar{c}^{(6)} - \bar{c}^{(6)} = 5 \geq 4 \]
  \[ \bar{c}^{(7)} = 1, 0, 0, 0, 0, 2, 0, 1, \ldots, L - \bar{c}^{(7)} - \bar{c}^{(7)} = 2. \]
  \[ \bar{d} = 000011111100000001111100000 \cdots \]

- \( j = 7 \) \( k^{(7)} = 22, i = 24, \) and \( X_7 = 2. \)
  \[ 2L - \bar{c}^{(7)} - \bar{c}^{(7)} = 7 \geq 2 \]
  \[ \bar{c}^{(8)} = 1, 0, 0, 0, 0, 2, 0, 0, 1, \ldots, L - \bar{c}^{(8)} - \bar{c}^{(8)} = 4. \]
  \[ \bar{d} = 000011111100000001111100000 \cdots \]
Then, we obtain $\tilde{\eta} = 1, 1, 4, 2, 0, 3, 2, 2, 1, \ldots$ and $\eta = 0, 0, 3, 1, -1, 2, 1, 1, 0, \ldots$

**Theorem 3.2.** (Injectivity of $\beta_L$)  
For $\eta \in S_{LM}$, $(\beta_L^{-1} \circ \beta_L)(\eta) = \eta$.

**Proof.** If we have the background sequence $\tilde{b} = (\tilde{b}_0, \tilde{b}_1, \ldots) \in S^b_{LM}$ and a soliton sequence $\tilde{d}$, we can get a binary subsequence that corresponds to the $j$-th box by dividing $\tilde{d}$ every $\tilde{L} - \tilde{b}_{j-1} - \tilde{b}_j$. Further, $\tilde{\eta}_j$ is the sum of $\tilde{b}_j$ and the number of 1s in the $j$-th subsequence. Let $e$ and $d$ respectively be the raised background sequence and soliton sequence constructed from $\eta$. We will prove that $X_j < 2\tilde{L} - \tilde{e}_j - \tilde{c}_j$ if and only if $(a_i, a_{i+1}) = (1, 0)$ or $(a_{i+1}, a_{i+2}) = (1, 0)$.

- Case 1: $a_{i+2k-1} = 1, a_{i+2k} = 0$ (k = 1, 2, ..., n) and $(a_{i+2n+1}, a_{i+2n+2}) \neq (1, 0)$.

- Case 2: $\tilde{b}_i = \tilde{c}_i = \tilde{L} - \tilde{\eta}_i$, $a_{i+1} = \cdots = a_k = 1$, $a_{k+1} = 0$. (We need not insert zeros.)

$$X_i = (L - \tilde{\eta}_i - \tilde{c}_i) + \tilde{\eta}_i + \tilde{\eta}_{i+1}$$

$$= \tilde{L} - \tilde{c}_i + \tilde{\eta}_{i+1}$$

From $a_{i+2k-1} = 1, a_{i+2k} = 0$ (k = 1, 2, ..., n), we get $\tilde{\eta}_{i+2k-2} + \tilde{\eta}_{i+2k-1} > \tilde{L}$, $\tilde{\eta}_{i+2k-1} + \tilde{\eta}_{i+2k} < \tilde{L}$ and $\tilde{b}_{i+2} = \tilde{c}_i \leq \tilde{\eta}_{i+2}$.

Therefore

$$X_i < (L - \tilde{c}_i + \tilde{\eta}_{i+1}) + \sum_{k=1}^{n}(L - \tilde{\eta}_{i+2k-2} - \tilde{\eta}_{i+2k})$$

$$+ \sum_{k=2}^{n}(\tilde{\eta}_{i+2k-2} + \tilde{\eta}_{i+2k-2} - L)$$

$$= 2\tilde{L} - \tilde{c}_i + \tilde{\eta}_{i+2n}$$

$$\leq 2\tilde{L} - \tilde{c}_i - \tilde{c}_i$$

$$X_{i+1} = X_i - (L - \tilde{c}_{i+1})$$

$$< \tilde{L} - \tilde{c}_i$$

$$\leq 2\tilde{L} - \tilde{c}_i - \tilde{c}_{i+1}.$$
For $d(3)$ if $\hat{\eta} = \tilde{a} \tilde{a} \cdots \tilde{a} \hat{\eta} \cdots \hat{\eta}_{k-1} \hat{\eta}_k \hat{\eta}_{k+1}$
\begin{align*}
\text{bin}(q) : & \quad 0^{L-\hat{\eta}} 1^\hat{\eta} 0^{L-\hat{\eta}+1} 1^{\hat{\eta}+1} \cdots 0^{L-\hat{\eta}+1} 1^\hat{\eta} 0^{L-\hat{\eta}+1} 1^{\hat{\eta}+1} \\
\hat{b} : & \quad \hat{c}_i = L - \hat{\eta}_{i+1} \hat{c}_{i+1} = L - \hat{\eta}_{i+2} \cdots \hat{c}_{k-2} = L - \hat{\eta}_{k-1} 0 0 \hat{c}_{k-1} \\
\hat{c} : & \quad \hat{c}_i = \hat{c}_i \hat{c}_{i+1} = \hat{c}_{i+1} \cdots \hat{c}_{k-2} = \hat{c}_{k-2} \hat{c}_{k-1} \hat{c}_k \\
\hat{d} : & \quad 0^{L-\hat{\eta}_{i-1}} 1^{\hat{\eta}+L} 1^{\hat{\eta}+\hat{\eta}+L} \cdots 1^{\hat{\eta}+\hat{\eta}+L} 1^{\hat{\eta}+\hat{\eta}+L} 0 0^{L-\hat{\eta}+1} 1^{\hat{\eta}+1} \\
\end{align*}

From $a_j + 1 = 1$, we get $\hat{\eta}_j + \hat{\eta}_{j+1} - \hat{L} \geq 0$ for $j = i + 1, \ldots, k - 1$, and from $\hat{b}_i = \hat{c}_i = L - \hat{\eta}_{i+1}$, we get $\hat{\eta}_{i+1} = L - \hat{c}_i$,
\[ X_i \geq \hat{L} + (\hat{L} - \hat{c}_i) - \hat{c}_{i-1} = 2L - \hat{c}_i - \hat{c}_{i-1} \]

- Case 3: $\hat{b}_i = \hat{c}_i = \hat{\eta}_i$, $\hat{\eta}_i = \cdots = \hat{\eta}_0 = 0$, $\hat{\eta}_{i+1} = \cdots = \hat{\eta}_k = 1$. (We need not insert zeros.)
\begin{align*}
\hat{\eta} : & \quad \hat{\eta}_i \hat{\eta}_{i+1} \cdots \hat{\eta}_l \hat{\eta}_{l+1} \cdots \hat{\eta}_{k-1} \hat{\eta}_k \hat{\eta}_{k+1} \\
a : & \quad 0 0 \cdots 0 1 \cdots 1 1 0 \\
\text{bin}(q) : & \quad 0^{L-\hat{\eta}} 1^{\hat{\eta}} 0^{L-\hat{\eta}+1} 1^{\hat{\eta}+1} \cdots 0^{L-\hat{\eta}+1} 1^{\hat{\eta}} 0^{L-\hat{\eta}+1} 1^{\hat{\eta}+1} \\
\hat{b} : & \quad \hat{c}_i = \hat{\eta}_i \hat{c}_{i+1} = \hat{\eta}_{i+1} \cdots \hat{c}_l = L - \hat{\eta}_{l+1} \hat{c}_{l+1} = L - \hat{\eta}_{l+2} \cdots 0 0 \hat{c}_{l+1} \\
\hat{c} : & \quad \hat{c}_i = \hat{c}_i \hat{c}_{i+1} \cdots \hat{c}_l = \hat{c}_l \hat{c}_{l+1} \cdots \hat{c}_{l+1} = \hat{c}_{l+1} \\
\hat{d} : & \quad 0^{L-\hat{\eta}_{i-1}} 1^{\hat{\eta}+L} 0^{L-\hat{\eta}-1} 1^{\hat{\eta}+L} \cdots 1^{\hat{\eta}+L} 1^{\hat{\eta}} 0^{L-\hat{\eta}} 0^{L-\hat{\eta}+1} 1^{\hat{\eta}+1} \\
\end{align*}

\[ X_i = (\hat{L} - \hat{c}_i - \hat{c}_{i-1}) + \sum_{j=i}^{l-1} (\hat{L} - \hat{\eta}_j - \hat{\eta}_{j+1}) \]
\[ + \sum_{j=i}^{k-2} (\hat{\eta}_j + \hat{\eta}_{j+1} - \hat{L}) + \hat{\eta}_{k-1} + \hat{\eta}_k \]
\[ = 2L - \hat{\eta}_i - \hat{c}_{i-1} + \sum_{j=i}^{l-1} (\hat{L} - \hat{\eta}_j - \hat{\eta}_{j+1}) \]
\[ + \sum_{j=i}^{k-1} (\hat{\eta}_j + \hat{\eta}_{j+1} - \hat{L}) \]

For $j = i, \ldots, l - 1$, from $a_j + 1 = 0$, we get $\hat{L} - \hat{\eta}_j - \hat{\eta}_{j+1} \geq 0$, and for $j = l, \ldots, k - 1$, from $a_j + 1 = 1$, we get $\hat{\eta}_j + \hat{\eta}_{j+1} - \hat{L} \geq 0$, and $\hat{\eta}_i = \hat{c}_i$,
\[ X_i \geq 2L - \hat{c}_{i-1} - \hat{c}_i. \]

\[ \square \]

**Theorem 3.3.** (Surjectivity of $\beta_{L})$

For $e \in S_{L \cdot M}^{(b)}$, $d \in S_{L}^{(c)}$, $(\beta_{L} \circ \beta_{L}^{-1})(e, d) = (e, d)$.

**Proof.** For $e \in S_{L \cdot M}^{(b)}$ and $d \in S_{L}^{(c)}$, let $\eta = \beta_{L}^{-1}(e, d)$, and let $a$ be the soliton flag sequence calculated from $\eta$. Let $Y_j$ be the number of continuous 0s in soliton sequence $d$ from $d_{k^{(j)}}$ (If $d_{k^{(j)}} = 1$, let $Y_j = 0$). We will prove the following four statements:

1. If $\hat{L} \leq X_j < 2\hat{L} - \hat{c}_{j-1} - \hat{c}_j$, then $a_{j+1} = 1$ and $a_{j+2} = 0$.
2. If $X_j < \hat{L}$, then $a_j = 1$ and $a_{j+1} = 0$.
3. If $X_j \geq 2\hat{L} - \hat{c}_{j-1} - \hat{c}_j$ and $Y_j < \hat{L} - \hat{c}_{j-1} - \hat{c}_j$, then $a_{j+1} = a_{j+2} = 1$. 

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(4) If \( X_j \geq 2\tilde{L} - \tilde{c}_{j-1}^{(j)} - \tilde{c}_{j}^{(j)} \) and \( Y_j \geq \tilde{L} - \tilde{c}_{j-1}^{(j)} - \tilde{c}_{j}^{(j)} \), then \( a_j = a_{j+1} = 0 \).

In this proof, let \( D_k \) be the subsequence of \( d \) corresponding to the \( k \)-th box.

(1) \(( \tilde{L} \leq X_j < 2\tilde{L} - \tilde{c}_{j-1}^{(j)} - \tilde{c}_{j}^{(j)} )\)

\[
\tilde{D} = \begin{cases} 
0 \ldots 0 1 \ldots 1 & X_j - (L - \tilde{c}_{j-1}^{(j)}) \\
1 \ldots 1 0 \ldots 0 & \tilde{L} - \tilde{c}_{j}^{(j)} \\
0 \ldots 0 1 \ldots 1 & B
\end{cases}
\]

\[\tilde{\eta} = A + \tilde{c}_{j}^{(j+1)} + \tilde{c}_{j}^{(j+2)} \]

From the algorithm of reconstruction, we have \( \tilde{c}_{j}^{(j+1)} = 0 \). Since \( X_j \geq \tilde{L} - \tilde{c}_{j-1}^{(j)} \), \( X_{j+1} \) becomes

\[
X_{j+1} = X_j - \left( L - \tilde{c}_{j}^{(j)} \right)
\]

\[
< (2\tilde{L} - \tilde{c}_{j-1}^{(j)} - \tilde{c}_{j}^{(j)}) - (L - \tilde{c}_{j}^{(j)})
\]

\[
= L - \tilde{c}_{j}^{(j)}
\]

\[
\leq L - \tilde{c}_{j}^{(j)} + \tilde{c}_{j}^{(j)} - \tilde{c}_{j}^{(j+1)} + (L - \tilde{c}_{j}^{(j+1)})
\]

\[
= 2\tilde{L} - \tilde{c}_{j}^{(j+1)} - \tilde{c}_{j}^{(j+2)},
\]

and we get \( \tilde{c}_{j+1}^{(j+2)} = 0 \). First, we prove \( a_{j+1} = 1 \). Let \( A \) denote the number of 1s in \( D_j \).

(i) If \( A < \tilde{L} - \tilde{c}_{j-1}^{(j)} \). Since the number of continuous 1s in \( d \) is larger than \( \tilde{L} \), we get \( A + X_j - (\tilde{L} - \tilde{c}_{j-1}^{(j)}) > \tilde{L} \).

Then,

\[
\tilde{\eta} + \tilde{\eta}_{j+1} - \tilde{L} = A + X_j - L + \tilde{c}_{j-1}^{(j)} - \tilde{L}
\]

\[
> 0,
\]

and we get \( a_{j+1} = 1 \).

(ii) If \( A = \tilde{L} - \tilde{c}_{j-1}^{(j)} \). Let \( i \) denote the maximum index less than \( j \) such that the leftend of contiguous 1s in \( d \) is at \( D_i \). From (i), \( a_{i+1} = 1 \). For \( k = i + 2, \ldots, j + 1 \),

\[
\tilde{\eta}_{k-1} + \tilde{\eta}_k - \tilde{L} = (L - \tilde{c}_{k-2}^{(j)}) + (L - \tilde{c}_{k-1}^{(j)}) - \tilde{L}
\]

\[
= L - \tilde{c}_{k-2}^{(j)} - \tilde{c}_{k-1}^{(j)}
\]

\[
\geq 0,
\]

and we get \( a_k = 1 \).

Next, we prove \( a_{j+2} = 0 \). Let \( B \) denote the number of 1s in \( D_{j+2} \).

(i) If \( B > 0 \): Since the number of continuous 0s in \( d \) is larger than \( \tilde{L} \), we get

\[
(\tilde{L} - (X_j - \tilde{L} + \tilde{c}_{j-1}^{(j)})) + (\tilde{L} - \tilde{c}_{j}^{(j)} - B)
\]

\[
= 2\tilde{L} - X_j - B - \tilde{c}_{j-1}^{(j)} - \tilde{c}_{j}^{(j)}
\]

\[
> 0.
\]
Here, using $\tilde{\mathcal{C}}_{j+2}^{(j+3)} = \tilde{\mathcal{C}}_{j}^{(j)}$ or 0, we get

$$\tilde{\eta}_{j+1} + \tilde{\eta}_{j+2} - \tilde{L} = (X_j - \tilde{L} + \tilde{\mathcal{C}}_{j-1}^{(j)}) + (B + \tilde{\mathcal{C}}_{j+2}^{(j+3)}) - \tilde{L}$$

$$= -2\tilde{L} + X_j + B + \tilde{\mathcal{C}}_{j-1}^{(j)} + \tilde{\mathcal{C}}_{j+2}^{(j+3)}$$

$$< 0,$$

and $a_{j+2} = 0$.

(ii) If $B = 0$:

$$\tilde{\eta}_{j+1} + \tilde{\eta}_{j+2} - \tilde{L} = (X_j - \tilde{L} + \tilde{\mathcal{C}}_{j-1}^{(j)}) + \tilde{\mathcal{C}}_{j+2}^{(j+3)} - \tilde{L}$$

$$< (2\tilde{L} - \tilde{\mathcal{C}}_{j-1}^{(j)} - \tilde{\mathcal{C}}_{j}^{(j)}) + \tilde{\mathcal{C}}_{j-1}^{(j+1)} + \tilde{\mathcal{C}}_{j+2}^{(j+3)} - 2\tilde{L}$$

$$= \tilde{\mathcal{C}}_{j+2}^{(j+3)} - \tilde{\mathcal{C}}_{j}^{(j)}$$

$$\leq 0$$

and $a_{j+2} = 0$.

(2) $(X_j < \tilde{L})$

| $\tilde{\mathcal{C}}_{j}^{(j)}$ | $\tilde{\mathcal{C}}_{j-1}^{(j)}$ | $\tilde{\mathcal{C}}_{j+1}^{(j+1)}$ | $\tilde{\mathcal{C}}_{j}^{(j)}$ |
|----------------|----------------|----------------|----------------|
| $\tilde{\mathcal{C}}_{j+1}^{(j+1)}$ | $\tilde{\mathcal{C}}_{j-1}^{(j)}$ | $\tilde{\mathcal{C}}_{j+1}^{(j+1)}$ |

| $\tilde{d}$: | $\begin{array}{} 0 \cdots 0 \cdot 1 \cdots 1 \end{array}$ | $\begin{array}{} 1 \cdots 1 \cdot 0 \cdots 0 \end{array}$ |
|----------------|----------------|----------------|
| $\tilde{\mathcal{C}}_{j}^{(j)}$ | $\begin{array}{} A \end{array}$ | $\begin{array}{} X_j \end{array}$ |
| $\tilde{\mathcal{C}}_{j+2}^{(j+3)}$ | $\begin{array}{} L - \tilde{\mathcal{C}}_{j-1}^{(j)} \end{array}$ | $\begin{array}{} L - \tilde{X}_j \end{array}$ |

If we assume that $X_{j-1} \geq 2\tilde{L} - \tilde{\mathcal{C}}_{j-2}^{(j-1)} - \tilde{\mathcal{C}}_{j-1}^{(j-1)}$ and $\tilde{\mathcal{C}}_{j-1}^{(j)} = \tilde{\mathcal{C}}_{j-1}^{(j-1)}$, then

$$X_{j-1} = (\tilde{L} - \tilde{\mathcal{C}}_{j-2}^{(j-1)} - \tilde{\mathcal{C}}_{j-1}^{(j-1)}) + X_j$$

$$< \tilde{L} - \tilde{\mathcal{C}}_{j-2}^{(j-1)} - \tilde{\mathcal{C}}_{j-1}^{(j-1)} + L$$

$$= 2\tilde{L} - \tilde{\mathcal{C}}_{j-2}^{(j-1)} - \tilde{\mathcal{C}}_{j-1}^{(j-1)}.$$

Therefore, we get $X_{j-1} < 2\tilde{L} - \tilde{\mathcal{C}}_{j-2}^{(j-1)} - \tilde{\mathcal{C}}_{j-1}^{(j-1)}$ and from Case (1) in Theorem 3.3, $a_j = 1$ and $a_{j+1} = 0$.

(3) $(X_j \geq 2\tilde{L} - \tilde{\mathcal{C}}_{j-1}^{(j)} - \tilde{\mathcal{C}}_{j}^{(j)}$ and $Y_j < \tilde{L} - \tilde{\mathcal{C}}_{j-1}^{(j)} - \tilde{\mathcal{C}}_{j}^{(j)}$)

| $\tilde{\mathcal{C}}_{j}^{(j)}$ | $\tilde{\mathcal{C}}_{j+1}^{(j+1)}$ | $\tilde{\mathcal{C}}_{j+2}^{(j+2)}$ | $\tilde{\mathcal{C}}_{j+3}^{(j+3)}$ |
|----------------|----------------|----------------|----------------|
| $\tilde{\mathcal{C}}_{j-1}^{(j)}$ | $\tilde{\mathcal{C}}_{j+1}^{(j+1)}$ | $\tilde{\mathcal{C}}_{j+2}^{(j+2)}$ |
| $\tilde{\mathcal{C}}_{j-2}^{(j)}$ | $\tilde{\mathcal{C}}_{j+1}^{(j+1)}$ | $\tilde{\mathcal{C}}_{j+2}^{(j+2)}$ |
| $\tilde{\mathcal{C}}_{j-3}^{(j-1)}$ | $\tilde{\mathcal{C}}_{j+1}^{(j+1)}$ | $\tilde{\mathcal{C}}_{j+2}^{(j+2)}$ |

| $\tilde{d}$: | $\begin{array}{} 0 \cdots 0 \cdot 1 \cdots 1 \end{array}$ | $\begin{array}{} 1 \cdots 1 \cdot 0 \cdots 0 \end{array}$ |
|----------------|----------------|----------------|
| $\tilde{\mathcal{C}}_{j}^{(j)}$ | $\begin{array}{} Y_j \end{array}$ | $\begin{array}{} B \end{array}$ |
| $\tilde{\mathcal{C}}_{j+2}^{(j+3)}$ | $\begin{array}{} L - \tilde{\mathcal{C}}_{j-1}^{(j)} - \tilde{\mathcal{C}}_{j}^{(j)} \end{array}$ | $\begin{array}{} L - \tilde{\mathcal{C}}_{j-1}^{(j)} - \tilde{\mathcal{C}}_{j}^{(j)} \end{array}$ |
| $\tilde{\mathcal{C}}_{j+3}^{(j+4)}$ | $\begin{array}{} L - \tilde{\mathcal{C}}_{j-1}^{(j)} - \tilde{\mathcal{C}}_{j}^{(j)} \end{array}$ | $\begin{array}{} L - \tilde{\mathcal{C}}_{j-1}^{(j)} - \tilde{\mathcal{C}}_{j}^{(j)} \end{array}$ |

| $\tilde{\eta}$: | $\tilde{\eta}_j = \tilde{L} - \tilde{\mathcal{C}}_{j-1}^{(j)} - Y_j$ | $\tilde{\eta}_{j+1} = \tilde{L} - \tilde{\mathcal{C}}_{j}^{(j)}$ |
|----------------|----------------|----------------|
| $\tilde{\mathcal{C}}_{j+2}^{(j+3)}$ | $\tilde{\eta}_{j+2} = B + \tilde{\mathcal{C}}_{j+2}^{(j+3)}$ | $\begin{array}{} B \end{array}$ |
First,
\[
\tilde{\eta}_j + \tilde{\eta}_{j+1} - L = (L - \tilde{c}^{(j)}_{j-1} - Y_j) + (\tilde{\eta}_{j+1} = L - \tilde{c}^{(j)} - L)
\]
\[
= \tilde{L} - \tilde{c}^{(j)}_{j-1} - \tilde{c}^{(j)} - Y_j
\]
\[
> 0,
\]
and we get \(a_{j+1} = 1\). Next, we prove \(a_{j+2} = 1\). Let \(B\) be the number of 1s in \(D_{j+2}\).

(i) If \(B < L - \tilde{c}^{(j+2)}_{j+1} - \tilde{c}^{(j+3)}\) 

\[
B = X_j - (\tilde{L} - \tilde{c}^{(j)}_{j-1} - \tilde{c}^{(j)}_j) - (L - \tilde{c}^{(j)}_{j} - \tilde{c}^{(j+2)}_{j+1})
\]
\[
= \{X_j - (2\tilde{L} - \tilde{c}^{(j)}_{j-1} - \tilde{c}^{(j)}_j) + \tilde{c}^{(j)}_j + \tilde{c}^{(j+2)}
\]
\[
\geq \tilde{c}^{(j)}_j + \tilde{c}^{(j+2)}_j,
\]

we obtain
\[
\tilde{\eta}_{j+1} + \tilde{\eta}_{j+2} - L = B - \tilde{c}^{(j)}_j - \tilde{c}^{(j+3)}_{j+2}
\]
\[
\geq \tilde{c}^{(j+2)}_{j} + \tilde{c}^{(j+3)}_{j+2}
\]
\[
\geq 0,
\]

and then, \(a_{j+2} = 1\).

(ii) If \(B = L - \tilde{c}^{(j+2)}_{j+1} - \tilde{c}^{(j+3)}\):

\[
\tilde{\eta}_{j+1} + \tilde{\eta}_{j+2} - L = (\tilde{L} - \tilde{c}^{(j)}_j) + (L - \tilde{c}^{(j+2)}_{j+1}) - \tilde{L}
\]
\[
= \tilde{L} - \tilde{c}^{(j+2)} - \tilde{c}^{(j+3)}_{j+1}
\]
\[
\geq 0
\]

and \(a_{j+2} = 1\).

(4) \((X_j \geq 2L - \tilde{c}^{(j)}_{j-1} - \tilde{c}^{(j)}_j)\) and \(Y_j \geq \tilde{L} - \tilde{c}^{(j)}_{j-1} - \tilde{c}^{(j)}_j\)

\[
\tilde{c}^{(j)}_{j-1}
\]
\[
\tilde{c}^{(j+1)}_{j-1}
\]
\[
\tilde{c}^{(j+2)}_{j-1}
\]
\[
\tilde{d}: \hspace{0.5cm} \begin{array}{c}
A
\end{array}
\]
\[
\tilde{\eta}_j = \tilde{c}^{(j)}_j
\]
\[
\tilde{\eta}_{j+1} = B + \tilde{c}^{(j+2)}_{j+1}
\]

First, we prove \(a_j = 0\).

(i) If \(X_{j-1} \leq \tilde{L} - \tilde{c}^{(j)}_{j-2} - \tilde{c}^{(j)}_{j-1}\): Since \(X_{j-1} \leq \tilde{L}\) and Case (2) in Theorem 3.3, we get \(a_{j-1} = 1, a_j = 0\).

(ii) If \(X_{j-1} > \tilde{L} - \tilde{c}^{(j)}_{j-2} - \tilde{c}^{(j)}_{j-1}\): Let \(i\) denote the maximum index less than \(j\) such that the leftend of contiguous 0s in \(d\) is at \(D_i\). From (i), \(a_i = 1, a_{i+1} = 0\). For \(k = i + 2, \ldots, j\),

\[
\tilde{\eta}_{k-1} + \tilde{\eta}_k - \tilde{L} = \tilde{c}^{(j)}_{k-1} + \tilde{c}^{(j)}_k - \tilde{L}
\]
\[
\leq 0,
\]

and we get \(a_k = 0\).
Next, we prove $a_{j+1} = 0$. Let $B$ the number of 1s in $D_{j+1}$.

(i) If $B > 0$: Since $Y_j$ is the number of continuous 0s in soliton sequence $d$ from $d_{k_0}$, we have

$$Y_j + B = (L - \tilde{\zeta}_{j-1}^{(j)} - \tilde{\varepsilon}_j^{(j)}) + (L - \tilde{\varepsilon}_j^{(j)} - \tilde{\varepsilon}_{j+1}^{(j+2)}).$$

Using this, we obtain

$$\tilde{\eta}_j + \tilde{\eta}_{j+1} - L = B + \tilde{\zeta}_j^{(j)} + \tilde{\varepsilon}_{j+1}^{(j+2)} - L = L - \tilde{\varepsilon}_{j-1}^{(j)} - \tilde{\varepsilon}_j^{(j)} - Y_j \leq 0.$$

(ii) If $B = 0$:

$$\tilde{\eta}_j + \tilde{\eta}_{j+1} - L = \tilde{\varepsilon}_j^{(j)} + \tilde{\varepsilon}_{j+1}^{(j+2)} - L \leq 0,$$

and we obtain $a_j = a_{j-1} = 0$. \hfill $\square$

From Theorems 3.2 and 3.3, the bijectivity of the map $\beta_L$ is proved, and we have the following theorem.

**Theorem 3.4.** A BBS state with box capacity $L$ can be decomposed into a soliton sequence and a background sequence.

$$S_{L,M} = S_{L_M}^{(b)} \otimes S_{L}^{(f)}. \quad (3.1)$$

4 Conclusion

We proposed a method to linearize the time-evolution of BBS($L$) by decomposing a state into a sequence that shifts to the right at speed 1 and a binary sequence that exhibits the time evolution of BBS(1). For a state including a negative value or a value greater than the box capacity, this method is applicable with a simple variable transformation.

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