We present a data-driven framework for incorporating side information in dynamic optimization under uncertainty. Specifically, our approach uses predictive machine learning methods (such as $k$-nearest neighbors, kernel regression, and random forests) to weight the relative importance of various data-driven uncertainty sets in a robust optimization formulation. Through a novel measure concentration result for local machine learning methods, we prove that the proposed framework is asymptotically optimal for stochastic dynamic optimization with covariates. We also describe a general-purpose approximation for the proposed framework, based on overlapping linear decision rules, which is computationally tractable and produces high-quality solutions for dynamic problems with many stages. Across a variety of examples in shipment planning, inventory management, and finance, our method achieves improvements of up to 15% over alternatives and requires less than one minute of computation time on problems with twelve stages.

**Key words**: Distributionally robust optimization; machine learning; dynamic optimization.

**History**: This paper was first submitted in May 2019.

1. Introduction

Dynamic decision making under uncertainty forms the foundation for numerous fundamental problems in operations research and management science. In these problems, a decision maker attempts to minimize an uncertain objective over time, as information incrementally becomes available. For example, consider a retailer with the goal of managing the inventory of a new short life cycle product. Each week, the retailer must decide an ordering quantity to replenish its inventory. Future demand for the product is unknown, but the retailer can base its ordering decisions on the remaining inventory level, which depends on the realized demands in previous weeks. A risk-averse investor faces a similar problem when constructing and adjusting a portfolio of assets in order to achieve a desirable risk-return tradeoff over a horizon of many months. Additional examples abound in energy planning, airline routing, and ride sharing, as well as in many other areas.

To make high quality decisions in dynamic environments, the decision maker must accurately model future uncertainty. Often, practitioners have access to side information or auxiliary covariates, which can help predict that uncertainty. For a retailer, although the future demand for a newly introduced clothing item is unknown, data on the brand, style, and color of the item, as well
as data on market trends and social media, can help predict it. For a risk-averse investor, while the returns of the assets in future stages are uncertain, recent asset returns and prices of relevant options can provide crucial insight into upcoming volatility. Consequently, organizations across many industries are continuing to prioritize the use of predictive analytics in order to leverage vast quantities of data to understand future uncertainty and make better operational decisions.

A recent body of work has aimed to leverage predictive analytics in decision making under uncertainty. For example, [Hannah et al. (2010), Ban and Rudin (2018), Bertsimas and Kallus (2014)] and [Ho and Hanasusanto (2019)] investigate prescriptive approaches, based on sample average approximation, that use local machine learning to assign weights to the historical data based on covariates. [Bertsimas and Van Parys (2017)] propose adding robustness to those weights to achieve optimal asymptotic budget guarantees. [Elmachtoub and Grigas (2017)] develop an approach for linear optimization problems in which a machine learning model is trained to minimize the decision cost. All of these approaches are specialized for single-stage or two-stage optimization problems, and do not readily generalize to problems with many stages. For a class of dynamic inventory problems, [Ban et al. (2018)] propose a data-driven approach by fitting the stochastic process and covariates to a parametric regression model, which is asymptotically optimal when the model is correctly specified. [Bertsimas and McCord (2019)] propose a different approach based on dynamic programming that uses nonparametric machine learning methods to handle auxiliary covariates. However, these dynamic approaches require scenario tree enumeration and suffer from the curse of dimensionality. To the best of our knowledge, no previous work leverages machine learning in a computationally tractable, data-driven framework for decision making in dynamic environments with covariates.

Recently, [Bertsimas et al. (2018a)] developed a data-driven approach for dynamic optimization under uncertainty that they call sample robust optimization (SRO). Their SRO framework solves a robust optimization problem in which an uncertainty set is constructed around each historical sample path. They show this data-driven framework enjoys nonparametric out-of-sample performance guarantees for a class of dynamic linear optimization problems without covariates and show that this framework can be approximated using decision rule techniques from robust optimization.

### 1.1. Contributions

In this paper, we present a new framework for leveraging side information in dynamic optimization. Specifically, we propose combining local machine learning methods with the sample robust optimization framework. Through a new measure concentration result, we show that the proposed sample robust optimization with covariates framework is asymptotically optimal, providing the
assurance that the resulting decisions are nearly optimal in the presence of big data. We also
demonstrate the tractability of the approach via an approximation algorithm based on overlapping
linear decision rules. To the best of our knowledge, our method is the first nonparametric
approach for tractably solving dynamic optimization problems with covariates, offering practition-
ers a general-purpose tool for better decision making with predictive analytics. We summarize our
main contributions as follows:

- We present a general-purpose framework for leveraging machine learning in data-driven
dynamic optimization with covariates. Our approach extends the sample robust optimization
framework by assigning weights to the uncertainty sets based on covariates. The weights
are computed using machine learning methods such as $k$-nearest neighbor regression, kernel
regression, and random forest regression.

- We provide theoretical justification for the proposed framework in the big data setting. First,
we develop a new measure concentration result for local machine learning methods (Theo-
rem 2), which shows that the weighted empirical distribution produced by local predictors
converges quickly to the true conditional distribution. To the best of our knowledge, such a
result for local machine learning is the first of its kind. We use Theorem 2 to establish that
the proposed framework is asymptotically optimal for dynamic optimization with covariates
without any parametric assumptions (Theorem 1).

- To find high quality solutions for problems with many stages in practical computation times,
we present an approximation scheme based on overlapping linear decision rules. Specifically,
we propose using separate linear decision rules for each uncertainty set to approximate the
costs incurred in each stage. We show that the approximation is computationally tractable,
both with respect to the number of stages and size of the historical dataset.

- By using all available data, we show that our method produces decisions that achieve improved
out-of-sample performance. Specifically, in a variety of examples (shipment planning, inventory
management, and finance), across a variety of time horizons, our proposed method outperforms
alternatives, in a statistically significant manner, achieving up to 15% improvement in average
out-of-sample cost. Moreover, our algorithm is practical and scalable, requiring less than one
minute on examples with up to twelve stages.

The paper is organized as follows. Section 2 introduces the problem setting and notation. Section 3
proposes the new framework for incorporating machine learning into dynamic optimization. Sec-
tion 4 develops theoretical guarantees on the proposed framework. Section 5 presents the general
multi-policy approximation scheme for dynamic optimization with covariates. Section 6 presents
a detailed investigation and computational simulations of the proposed methodology in shipment planning, inventory management, and finance. We conclude in Section 7.

1.2. Comparison to Related Work

This paper follows a recent body of literature on data-driven optimization under uncertainty in operations research and management science. Much of this work has focused on the paradigm of distributionally robust optimization, in which the optimal solution is that which performs best in expectation over a worst-case probability distribution from an ambiguity set. Motivated by probabilistic guarantees, distributionally robust optimization has found particular applicability in data-driven settings in which the ambiguity set is constructed using historical data, such as Delage and Ye (2010), Xu et al. (2012), Esfahani and Kuhn (2018), Van Parys et al. (2017). In particular, the final steps in our convergence result (Section 4.4) draw heavily from similar techniques from Esfahani and Kuhn (2018) and Bertsimas et al. (2018a). In contrast to previous work, this paper develops a new measure concentration result for the weighted empirical distribution (Section 4.3) which enables machine learning and covariates to be incorporated into sample robust optimization and Wasserstein-based distributionally robust optimization for the first time.

Several recent papers have focused on tractable approximations of two- and multi-stage distributionally and sample robust optimization. Many approaches are based around policy approximation schemes, including lifted linear decision rules (Bertsimas et al. 2018b), K-adaptivity (Hanasusanto et al. 2016), and finite adaptability (Bertsimas et al. 2018a). Alternative approaches include tractable approximations of copositive formulations (Hanasusanto and Kuhn 2018). Closest related to the approximation scheme in this paper are Chen et al. (2019) and Bertsimas et al. (2019), which address two-stage problems via overlapping decision rules. Chen et al. (2019) propose a scenario-wise modeling approach that leads to novel approximations of various distributionally robust applications, including two-stage distributionally robust optimization using Wasserstein ambiguity sets and expectations of piecewise convex objective functions in single-stage problems. Independently, Bertsimas et al. (2019) investigate a multi-policy approximation of two-stage sample robust optimization by optimizing a separate linear decision rule for each uncertainty set and prove that this approximation gap converges to zero as the amount of data goes to infinity. In Section 5 of this paper, we show how to extend similar techniques to dynamic problems with many stages for the first time.

As discussed previously, the methodology in this paper also follows recent work on incorporating covariates in optimization under uncertainty using local predictive methods (such as k-nearest neighbor regression, kernel regression, and random forests). In particular, the asymptotic optimality
justification of Bertsimas and Kallus (2014) in single-stage settings relies on the strong universal consistency for local predictive models (e.g., Walk (2010)). Our proof of asymptotic optimality instead relies on convergence guarantees rooted in distributionally robust optimization. The reason we use a different approach is that the arguments for the convergence for local predictive models from Bertsimas and Kallus (2014) require finite dimensional decision variables. In contrast, the convergence guarantees in this paper apply for dynamic optimization over general spaces of policies.

2. Problem Setting

We consider finite-horizon discrete-time stochastic dynamic optimization problems. The uncertain quantities observed in each stage are denoted by random variables $\xi_1 \in \Xi_1 \subseteq \mathbb{R}^{d_1}, \ldots, \xi_T \in \Xi_T \subseteq \mathbb{R}^{d_T}$. The decisions made in each stage are denoted by $x_1 \in X_1 \subseteq \mathbb{R}^{d_1}, \ldots, x_T \in X_T \subseteq \mathbb{R}^{d_T}$. Given realizations of the uncertain quantities and decisions, we incur a cost of

$$c(\xi_1, \ldots, \xi_T, x_1, \ldots, x_T) \in \mathbb{R}. $$

A decision rule $\pi = (\pi_1, \ldots, \pi_T)$ is a collection of measurable functions $\pi_t : \Xi_1 \times \cdots \times \Xi_{t-1} \to X_t$ which specify what decision to make in stage $t$ based on the information observed up to that point. Given realizations of the uncertain quantities and choice of decision rules, the resulting cost is

$$c^\pi(\xi_1, \ldots, \xi_T) := c(\xi_1, \ldots, \xi_T, \pi_1, \ldots, \pi_T(\xi_1, \ldots, \xi_{T-1})).$$

Before selecting the decision rules, we observe auxiliary covariates $\gamma \in \Gamma \subseteq \mathbb{R}^{d_\gamma}$. For example, in the aforementioned fashion setting, the auxiliary covariates may information on the brand, style, and color of a new clothing item and the remaining uncertainties representing the demand for the product in each week of the lifecycle. Given a realization of the covariates $\gamma = \bar{\gamma}$, our goal is to find decision rules which minimize the conditional expected cost:

$$v^*(\bar{\gamma}) := \min_{\pi \in \Pi} \mathbb{E} \left[ c^\pi(\xi_1, \ldots, \xi_T) \mid \gamma = \bar{\gamma} \right]. \quad (1)$$

We refer to $\Pi$ as dynamic optimization with covariates. The optimization takes place over a collection $\Pi$ which is any subset of the space of all non-anticipative decision rules.

In this paper, we assume that the joint distribution of the covariates and uncertain quantities $(\gamma, \xi_1, \ldots, \xi_T)$ is unknown, and our knowledge consists of historical data of the form

$$(\gamma^1, \xi^1_1, \ldots, \xi^1_T), \ldots, (\gamma^N, \xi^N_1, \ldots, \xi^N_T),$$

where each of these tuples consists of a realization of the auxiliary covariates and the following realization of the random variables over the stages. For example, in the aforementioned fashion
setting, each tuple corresponds to the covariates of a past fashion item as well as its demand over its lifecycle. We will not assume any parametric structure on the relationship between the covariates and future uncertainty.

The goal of this paper is a general-purpose, computationally tractable, data-driven approach for approximately solving dynamic optimization with covariates. In the following sections, we propose and analyze a new framework which leverages nonparametric machine learning, trained from historical data, to predict future uncertainty from covariates in a way that leads to near-optimal decision rules to \( (1) \).

### 2.1. Notation

The joint probability distribution of the covariates \( \gamma \) and uncertain quantities \( \xi = (\xi_1, \ldots, \xi_T) \) is denoted by \( \mathbb{P} \). For the purpose of proving theorems, we assume throughout this paper that the historical data are independent and identically distributed (i.i.d.) samples from this distribution \( \mathbb{P} \). In other words, we assume that the historical data satisfies

\[
((\gamma^1, \xi^1), \ldots, (\gamma^N, \xi^N)) \sim \mathbb{P}^N,
\]

where \( \mathbb{P}^N := \mathbb{P} \times \cdots \times \mathbb{P} \) is the product measure. The set of all probability distributions supported on \( \Xi := \Xi_1 \times \cdots \times \Xi_T \subseteq \mathbb{R}^{dT} \) is denoted by \( \mathcal{P}(\Xi) \). For each of the covariates \( \gamma \in \Gamma \), we assume that its conditional probability distribution satisfies \( \mathbb{P}_\gamma \in \mathcal{P}(\Xi) \), where \( \mathbb{P}_\gamma(\cdot) \) is shorthand for \( \mathbb{P}(\cdot \mid \gamma = \gamma) \). We sometimes use subscript notation for expectations to specify the underlying probability distribution; for example, the following two expressions are equivalent:

\[
\mathbb{E}_{\xi \sim \mathbb{P}_\gamma} [f(\xi_1, \ldots, \xi_T)] \equiv \mathbb{E}[f(\xi_1, \ldots, \xi_T) \mid \gamma = \gamma].
\]

Finally, we say that the cost function resulting from a policy \( \pi \) is upper semicontinuous if

\[
\limsup_{\zeta \to \bar{\zeta}} c^\pi(\zeta_1, \ldots, \zeta_T) \leq c^\pi(\bar{\zeta}_1, \ldots, \bar{\zeta}_T)
\]

for all \( \zeta \in \Xi \).

### 3. Sample Robust Optimization with Covariates

In this section, we present our approach for incorporating machine learning in dynamic optimization. We first review sample robust optimization, and then we introduce our new sample robust optimization with covariates framework.
3.1. Preliminary: sample robust optimization

Consider a stochastic dynamic optimization problem of the form (1) in which there are no auxiliary covariates. The underlying joint distribution of the random variables \( \xi \equiv (\xi_1, \ldots, \xi_T) \) is unknown, but we have data consisting of sample paths, \( \xi^i \equiv (\xi^i_1, \ldots, \xi^i_T), \ldots, \xi^N \equiv (\xi^N_1, \ldots, \xi^N_T) \). For this setting, sample robust optimization can be used to find approximate solutions in stochastic dynamic optimization. To apply the framework, one constructs an uncertainty set around each sample path in the training data and then chooses the decision rules that optimize the average of the worst-case realizations of the cost. Formally, this framework results in the following robust optimization problem:

\[
\min_{\pi \in \Pi} \sum_{i=1}^{N} \frac{1}{N} \sup_{\zeta \in \mathcal{U}_i} c^{\pi}(\zeta_1, \ldots, \zeta_T),
\]

where \( \mathcal{U}_i \subseteq \Xi \) is an uncertainty set around \( \xi^i \). Intuitively speaking, (2) chooses the decision rules by averaging over the historical sample paths which are adversarially perturbed. Under mild probabilistic assumptions on the underlying joint distribution and appropriately constructed uncertainty sets, Bertsimas et al. (2018a) show that sample robust optimization converges asymptotically to the underlying stochastic problem and that (2) is amenable to approximations similar to dynamic robust optimization.

3.2. Incorporating covariates into sample robust optimization

We now present our new framework, based on sample robust optimization, for solving dynamic optimization with covariates. In the proposed framework, we first train a machine learning algorithm on the historical data to predict future uncertainty \( (\xi_1, \ldots, \xi_T) \) as a function of the covariates. From the trained learner, we obtain weight functions \( w^i_N(\bar{\gamma}) \), for \( i = 1, \ldots, N \), each of which captures the relevance of the \( i \)th training sample to the new covariates, \( \bar{\gamma} \). We incorporate the weights into sample robust optimization by multiplying the cost associated with each training example by the corresponding weight function. The resulting sample robust optimization with covariates framework is as follows:

\[
\hat{v}^N(\bar{\gamma}):= \min_{\pi \in \Pi} \sum_{i=1}^{N} w^i_N(\bar{\gamma}) \sup_{\zeta \in \mathcal{U}_i} c^{\pi}(\zeta_1, \ldots, \zeta_T),
\]

where the uncertainty sets are defined

\[
\mathcal{U}_i := \{ \zeta \in \Xi : \|\zeta - \xi^i\| \leq \epsilon_N \},
\]

and \( \| \cdot \| \) is some \( \ell_p \) norm with \( p \geq 1 \).

The above framework provides the flexibility for the practitioner to construct weights from a variety of machine learning algorithms. We focus in this paper on weight functions which come from
nonparametric machine learning methods. Examples of viable predictive models include k-nearest neighbors (kNN), kernel regression, classification and regression trees (CART), and random forests (RF). We describe these four classes of weight functions.

**Definition 1.** The k-nearest neighbor weight functions are given by:

\[
 w_{N,kNN}(\gamma) := \begin{cases} 
 \frac{1}{k_N}, & \text{if } \gamma^i \text{ is a } k_N\text{-nearest neighbor of } \hat{\gamma}, \\
 0, & \text{otherwise}. 
\end{cases}
\]

Formally, \( \gamma^i \) is a \( k_N \)-nearest neighbor of \( \hat{\gamma} \) if \( |\{ j \in \{1, \ldots, N\} \setminus i : \| \gamma^j - \hat{\gamma} \| < \| \gamma^i - \hat{\gamma} \| \}| < k_N \). For more technical details, we refer the reader to Biau and Devroye (2015).

**Definition 2.** The kernel regression weight functions are given by:

\[
 w_{N, KR}(\gamma) := \frac{K(\| \gamma - \hat{\gamma} \| / h_N)}{\sum_{j=1}^{N} K(\| \gamma - \hat{\gamma} \| / h_N)},
\]

where \( K(\cdot) \) is the kernel function and \( h_N \) is the bandwidth parameter. Examples of kernel functions include the Gaussian kernel, \( K(u) = \frac{1}{\sqrt{2\pi}}e^{-u^2/2} \), the triangular kernel, \( K(u) = (1 - u)1\{u \leq 1\} \), and the Epanechnikov kernel, \( K(u) = \frac{3}{4}(1 - u^2)1\{u \leq 1\} \). For more information on kernel regression, see Friedman et al. (2001, Chapter 6).

The next two types of weight functions we present are based on classification and regression trees (Breiman et al. 1984) and random forests (Breiman 2001). We refer the reader to Bertsimas and Kallus (2014) for technical implementation details.

**Definition 3.** The classification and regression tree weight functions are given by:

\[
 w_{N, CART}(\gamma) := \begin{cases} 
 \frac{1}{|l_N(\gamma)|}, & i \in l_N(\gamma), \\
 0, & \text{otherwise}, 
\end{cases}
\]

where \( l_N(\gamma) \) is the set of indices \( i \) such that \( \gamma^i \) is contained in the same leaf of the tree as \( \hat{\gamma} \).

**Definition 4.** The random forest weight functions are given by:

\[
 w_{N, RF}(\gamma) := \frac{1}{B} \sum_{b=1}^{B} w_{N, CART}(\gamma),
\]

where \( B \) is the number of trees in the ensemble, and \( w_{N, CART}(\gamma) \) refers to the weight function of the \( b \)th tree in the ensemble.

All of the above weight functions come from nonparametric machine learning methods. They are highly effective as predictive methods because they can learn complex relationships between the covariates and the response variable without requiring the practitioner to state an explicit
parametric form. Similarly, as we prove in Section 4, solutions to (3) with these weight functions are asymptotically optimal for (1) without any parametric restrictions on the relationship between $\gamma$ and $\xi$. In other words, incorporating covariates into sample robust optimization via (3) leads to better decisions asymptotically, even without specific knowledge of how the covariates affect the uncertainty.

4. Asymptotic Optimality

In this section, we establish asymptotic optimality guarantees for sample robust optimization with auxiliary covariates. We prove that, under mild conditions, (3) converges to (1) as the number of training samples goes to infinity. Thus, as the amount of data grows, sample robust optimization with covariates becomes an optimal approximation of the underlying stochastic dynamic optimization problem. Crucially, our convergence guarantee does not require parametric restrictions on the space of decision rules (e.g., linearity) or parametric restrictions on the joint distribution of the covariates and uncertain quantities. These theoretical results are consistent with empirical experiments in Section 6.

4.1. Main result

We begin by presenting our main result. The proof of the result depends on some technical assumptions and concepts from distributionally robust optimization. For simplicity, we defer the statement and discussion of technical assumptions regarding the underlying probability distribution and cost until Sections 4.3 and 4.4, and first discuss what is needed to apply the method in practice. The practitioner needs to select a weight function, parameters associated with that weight function, and the radius, $\epsilon_N$, of the uncertainty sets. While these may be selected by cross validation, we show that the method will in general converge if the parameters are selected to satisfy the following:

**Assumption 1.** The weight functions and uncertainty set radius satisfy one of the following:

1. \{w^i_N(\cdot)\} are $k$-nearest neighbor weight functions with $k_N = \min(\lceil k_3 N^{\delta} \rceil, N - 1)$ for constants $k_3 > 0$ and $\delta \in (\frac{1}{2}, 1)$, and $\epsilon_N = \frac{k_1}{N^p}$ for constants $k_1 > 0$ and $0 < p < \min\left(\frac{1-\delta}{d_\gamma}, \frac{2-\delta}{d_\xi} + \frac{1}{2}\right)$.

2. \{w^i_N(\cdot)\} are kernel regression weight functions with the Gaussian, triangular, or Epanechnikov kernel function and $h_N = k_4 N^{-\delta} \gamma$ for constants $k_4 > 0$ and $\delta \in \left(0, \frac{1}{2d_\gamma}\right)$, and $\epsilon_N = \frac{k_1}{N^p}$ for constants $k_1 > 0$ and $0 < p < \min\left(\delta, \frac{1-\delta d_\gamma}{2+d_\xi}\right)$.

Given Assumption 1, our main result is the following.
Theorem 1. Suppose the weight function and uncertainty sets satisfy Assumption 1, the joint probability distribution of $(\gamma, \xi)$ satisfies Assumptions 2-4 from Section 4.3, and the cost function satisfies Assumption 5 from Section 4.4. Then, for every $\gamma \in \Gamma$,

$$\lim_{N \to \infty} \hat{v}_N^N(\gamma) = v^*(\gamma), \quad \mathbb{P}^\infty\text{-almost surely.}$$

The theorem says that objective value of (3) converge almost surely to the optimal value of the full-information problem, (1), as $N$ goes to infinity. The assumptions of the theorem require that the joint distribution and the feasible decision rules are well behaved. We will discuss these technical assumptions in more detail in the following sections.

In order to prove the asymptotic optimality of sample robust optimization with covariates, we view (3) through the more general lens of Wasserstein-based distributionally robust optimization. We first review some properties of the Wasserstein metric and then prove a key intermediary result, from which our main result follows.

4.2. Review of the Wasserstein metric

The Wasserstein metric provides a distance function between probability distributions. In particular, given two probability distributions $Q, Q' \in \mathcal{P}(\Xi)$, the type-1 Wasserstein distance is defined as the optimal objective value of a minimization problem:

$$d_1(Q, Q') := \inf \left\{ \mathbb{E}_{(\xi, \xi') \sim \Pi} \| \xi - \xi' \| : \Pi \text{ is a joint distribution of } \xi \text{ and } \xi' \text{ with marginals } Q \text{ and } Q', \text{ respectively} \right\}.$$ 

The Wasserstein metric is particularly appealing because a distribution with finite support can have a finite distance to a continuous distribution. This allows us to construct a Wasserstein ball around an empirical distribution that includes continuous distributions, which cannot be done with other popular measures such as the Kullback-Leibler divergence [Kullback and Leibler 1951]. We remark that the 1-Wasserstein metric satisfies the axioms of a metric, including the triangle inequality [Clement and Desch 2008]:

$$d_1(Q_1, Q_2) \leq d_1(Q_1, Q_3) + d_1(Q_3, Q_2), \quad \forall Q_1, Q_2, Q_3 \in \mathcal{P}(\Xi).$$

Important to this paper, the 1-Wasserstein metric admits a dual form, as shown by Kantorovich and Rubinstein [1958],

$$d_1(Q, Q') = \sup_{\text{Lip}(h) \leq 1} \| \mathbb{E}_{\xi \sim Q} [h(\xi)] - \mathbb{E}_{\xi' \sim Q'} [h(\xi)] \|,$$

where the supremum is taken over all 1-Lipschitz functions. Note that the absolute value is optional in the dual form of the metric, and the space of Lipschitz functions can be restricted to those which
satisfy \( h(0) = 0 \) without loss of generality. Finally, we remark that Fournier and Guillin (2015) prove under a light-tailed assumption that the 1-Wasserstein distance between the empirical distribution and its underlying distribution concentrates around zero with high probability. Theorem 2 in the following section extends this concentration result to the setting with auxiliary covariates.

### 4.3. Concentration of the weighted empirical measure

Given a local predictive method, let the corresponding weighted empirical measure be defined as

\[
\hat{P}_N^\gamma := \sum_{i=1}^{N} w_i^\gamma \xi_i,
\]

where \( \delta_\xi \) denotes the Dirac probability distribution which places point mass at \( \xi \). In this section, we prove under mild assumptions that the weighted empirical measure \( \hat{P}_N^\gamma \) concentrates quickly to \( P_\gamma \) with respect to the 1-Wasserstein metric. We introduce the following assumptions on the underlying joint probability distribution:

**Assumption 2 (Conditional Subgaussianity).** There exists a parameter \( \sigma > 0 \) such that

\[
P\left( \|\xi\| - E[\|\xi\| \mid \gamma = \gamma] > t \mid \gamma = \gamma \right) \leq \exp\left( -\frac{t^2}{2\sigma^2} \right) \quad \forall t > 0, \gamma \in \Gamma.
\]

**Assumption 3 (Lipschitz Continuity).** There exists \( 0 < L < \infty \) such that

\[
d_1(P_\gamma, P_{\gamma'}) \leq L \|\gamma - \gamma'\|, \quad \forall \gamma, \gamma' \in \Gamma.
\]

**Assumption 4 (Smoothness of Auxiliary Covariates).** The set \( \Gamma \) is compact, and there exists \( g > 0 \) such that

\[
P(\|\gamma - \gamma\| \leq \epsilon) \geq g e^{d_1\epsilon}, \quad \forall \epsilon > 0, \gamma \in \Gamma.
\]

With these assumptions, we are ready to prove the concentration result, which is proved using a novel technique that relies on the dual form of the Wasserstein metric and a discrete approximation of the space of 1-Lipschitz functions.

**Theorem 2.** Suppose the weight function and uncertainty sets satisfy Assumption 2 and the joint probability distribution of \( (\gamma, \xi) \) satisfies Assumptions 3, 4. Then, for every \( \gamma \in \Gamma \),

\[
P^\infty \left( \{d_1(P_{\gamma}, \hat{P}_N^\gamma) > \epsilon_N \} \ i.o. \right) = 0.
\]
Proof. Without loss of generality, we assume throughout the proof that all norms \( \| \cdot \| \) refer to the \( \ell_\infty \) norm. Fix any \( \gamma \in \Gamma \). It follows from Assumption 1 that

\[
\{ w^i_N(\gamma) \} \text{ are not functions of } \xi^1, \ldots, \xi^N; \quad (4)
\]

\[
\sum_{i=1}^N w^i_N(\gamma) = 1 \text{ and } w^1_N(\gamma), \ldots, w^N_N(\gamma) \geq 0, \quad \forall N \in \mathbb{N}; \quad (5)
\]

\[
\epsilon_N = \frac{k_1}{N^p}, \quad \forall N \in \mathbb{N}, \quad (6)
\]

for constants \( k_1, p > 0 \). Moreover, Assumption 1 also implies that there exists constants \( k_2 > 0 \) and \( \eta > p(2 + d_\epsilon) \) such that

\[
\lim_{N \to \infty} \frac{1}{\epsilon_N} \sum_{i=1}^N w^i_N(\gamma) \| \gamma^i - \gamma \| = 0, \quad \mathbb{P}^\infty\text{-almost surely; (7)}
\]

\[
\mathbb{E}_{\gamma,\gamma^i} \left[ \exp \left( \frac{-\theta}{\sum_{i=1}^N w^i_N(\gamma)} \right) \right] \leq \exp(-k_2 \theta N^\eta), \quad \forall \theta \in (0, 1), N \in \mathbb{N}. \quad (8)
\]

The proof of the above statements under Assumption 1 is found in Appendix EC.1. Now, choose any fixed \( q \in (0, \eta/(2 + d_\epsilon) - p) \), and let

\[
b_N := N^q, \quad B_N := \left\{ \xi \in \mathbb{R}^{d_\xi} : \| \xi \| \leq b_N \right\}, \quad I_N := 1 \left\{ \xi^1, \ldots, \xi^N \in B_N \right\}.
\]

Finally, we define the following intermediary probability distributions:

\[
\hat{Q}^N_N := \sum_{i=1}^N w^i_N(\gamma) \mathbb{P}_{\gamma^i}, \quad \hat{Q}^N_{\gamma|B_N} := \sum_{i=1}^N w^i_N(\gamma) \mathbb{P}_{\gamma^i|B_N},
\]

where \( \mathbb{P}_{\gamma|B_N}(\cdot) \) is shorthand for \( \mathbb{P}(\cdot | \gamma = \gamma^i, \xi \in B_N) \).

Applying the triangle inequality for the 1-Wasserstein metric and the union bound,

\[
\mathbb{P}^\infty \left( \left\{ d_1(\mathbb{P}_\gamma, \hat{P}^N_N) > \epsilon_N \right\} \text{ i.o.} \right) \leq \mathbb{P}^\infty \left( \left\{ d_1(\mathbb{P}_\gamma, \hat{Q}^N_N) > \frac{\epsilon_N}{3} \right\} \text{ i.o.} \right) + \mathbb{P}^\infty \left( \left\{ d_1(\hat{Q}^N_N, \hat{Q}^N_{\gamma|B_N}) > \frac{\epsilon_N}{3} \right\} \text{ i.o.} \right) + \mathbb{P}^\infty \left( \left\{ d_1(\hat{Q}^N_{\gamma|B_N}, \hat{P}^N_N) > \frac{\epsilon_N}{3} \right\} \text{ i.o.} \right).
\]

We now proceed to bound each of the above terms.

\footnote{To see why this is without loss of generality, consider any other \( \ell_p \) norm where \( p \geq 1 \). In this case,

\( \| \xi - \xi' \|_p \leq d_1^{1/p}_\xi \| \xi - \xi' \|_\infty \).

By the definition of the 1-Wasserstein metric, this implies

\( d_1^{1/p}_\xi(\mathbb{P}_\gamma, \hat{P}^N_N) \leq d_1^{1/p}(\mathbb{P}_\gamma, \hat{P}^N_N), \)

where \( d_1^{1/p} \) refers to the 1-Wasserstein metric with the \( \ell_p \) norm. If \( \epsilon_N \) satisfies Assumption 1, \( \epsilon_N/d_1^{1/p}_\xi \) also satisfies Assumption 1 so the result for all other choices of \( \ell_p \) norms follows from the result with the \( \ell_\infty \) norm.}
**Term 1:** $d_1(\mathbb{P}_\gamma, \hat{\mathbb{P}}^N_\gamma)$: By the dual form of the 1-Wasserstein metric,

$$d_1(\mathbb{P}_\gamma, \hat{\mathbb{P}}^N_\gamma) = \sup_{\text{Lip}(h) \leq 1} \left| \mathbb{E}[h(\xi) | \gamma = \tilde{\gamma}] - \sum_{i=1}^{N} w^i_N(\tilde{\gamma}) \mathbb{E}[h(\xi) | \gamma = \gamma^i] \right|,$$

where the supremum is taken over all 1-Lipschitz functions. By [5] and Jensen’s inequality, we can upper bound this by

$$d_1(\mathbb{P}_\gamma, \hat{\mathbb{P}}^N_\gamma) \leq \sum_{i=1}^{N} w^i_N(\tilde{\gamma}) \left( \sup_{\text{Lip}(h) \leq 1} |\mathbb{E}[h(\xi) | \gamma = \tilde{\gamma}] - \mathbb{E}[h(\xi) | \gamma = \gamma^i]| \right)$$

$$= \sum_{i=1}^{N} w^i_N(\tilde{\gamma}) d_1(\mathbb{P}_\gamma, \mathbb{P}_\gamma^i)$$

$$\leq L \sum_{i=1}^{N} w^i_N(\tilde{\gamma}) \|\tilde{\gamma} - \gamma^i\|,$$

where the final inequality follows from Assumption [3]. Therefore, it follows from (7) that

$$\mathbb{P}^\infty \left( \left\{ d_1(\mathbb{P}_\gamma, \hat{\mathbb{P}}^N_\gamma) > \frac{\epsilon N}{3} \right\} \right. \left. \text{i.o.} \right) = 0. \quad (9)$$

**Term 2:** $d_1(\hat{\mathbb{Q}}^N_\gamma, \hat{\mathbb{Q}}^N_{\gamma | B_N})$: Consider any Lipschitz function $\text{Lip}(h) \leq 1$ for which $h(0) = 0$, and let $N \in \mathbb{N}$ satisfy $b_N \geq \sigma + \sup_{\gamma \in \Gamma} \mathbb{E}[\|\xi\| | \gamma = \tilde{\gamma}]$ (which is finite because of Assumption [4]). Then, for all $N \geq \hat{N}$, and all $\gamma' \in \Gamma$,

$$\mathbb{E}[h(\xi) | \gamma = \tilde{\gamma}] - \mathbb{E}[h(\xi) | \gamma = \gamma', \xi \in B_N]$$

$$= \mathbb{E}[h(\xi) \mathbb{1}_{\{\xi \not\in B_N\}} | \gamma = \gamma'] + \mathbb{E}[h(\xi) \mathbb{1}_{\{\xi \in B_N\}} | \gamma = \gamma', \xi \in B_N] - \mathbb{E}[h(\xi) | \gamma = \gamma', \xi \in B_N]$$

$$= \mathbb{E}[h(\xi) \mathbb{1}_{\{\xi \not\in B_N\}} | \gamma = \gamma'] + \mathbb{E}[h(\xi) | \gamma = \gamma', \xi \in B_N] \mathbb{P}(\xi \in B_N | \gamma = \gamma') - \mathbb{E}[h(\xi) | \gamma = \gamma', \xi \in B_N]$$

$$= \mathbb{E}[h(\xi) \mathbb{1}_{\{\xi \not\in B_N\}} | \gamma = \gamma'] - \mathbb{E}[h(\xi) | \gamma = \gamma', \xi \in B_N] \mathbb{P}(\xi \not\in B_N | \gamma = \gamma')$$

$$\leq \mathbb{E}[\|\xi\| \mathbb{1}_{\{\xi \not\in B_N\}} | \gamma = \gamma'] + b_N \mathbb{P}(\xi \not\in B_N | \gamma = \gamma')$$

$$= \int_{b_N}^{\infty} \mathbb{P}(\|\xi\| > t | \gamma = \gamma') \, dt + b_N \mathbb{P}(\|\xi\| \geq b_N | \gamma = \gamma')$$

$$\leq (\sigma + b_N) \exp \left( -\frac{1}{2\sigma^2} \left( b_N - \sup_{\gamma' \in \Gamma} \mathbb{E}[\|\xi\| | \gamma = \gamma'] \right)^2 \right).$$

The first inequality follows because $|h(\xi)| \leq b_N$ for all $\xi \in B_N$ and $|h(\xi)| \leq \|\xi\|$ otherwise. For the second inequality, we used the Gaussian tail inequality $\int_x^{\infty} e^{-t^2/2} \, dt \leq e^{-x^2/2}$ for $x \geq 1$ (Vershynin 2018) along with Assumption [2]. Because this bound holds uniformly over all $h$, and all $\gamma' \in \Gamma$, it follows that

$$d_1(\hat{\mathbb{Q}}^N_\gamma, \hat{\mathbb{Q}}^N_{\gamma | B_N}) = \sup_{\text{Lip}(h) \leq 1, h(0) = 0} \left| \sum_{i=1}^{N} w^i_N(\gamma) \left( \mathbb{E}[h(\xi) | \gamma = \gamma^i] - \mathbb{E}[h(\xi) | \gamma = \gamma', \xi \in B_N] \right) \right|$$
\[
\leq \sum_{i=1}^{N} w_N(\gamma) \sup_{\text{Lip}(h) \leq 1, h(0) = 0} \left| \mathbb{E}[h(\xi) \mid \gamma = \gamma_i] - \mathbb{E}[h(\xi) \mid \gamma = \gamma, \xi \in B_N] \right|
\]
\[
\leq \sup_{\gamma' \in \Gamma} \sup_{\text{Lip}(h) \leq 1, h(0) = 0} \left| \mathbb{E}[h(\xi) \mid \gamma = \gamma'] - \mathbb{E}[h(\xi) \mid \gamma = \gamma', \xi \in B_N] \right|
\]
\[
\leq (\sigma + b_N) \exp \left( -\frac{1}{2\sigma^2} \left( b_N - \sup_{\gamma' \in \Gamma} \mathbb{E}[\|\xi\| \mid \gamma = \gamma'] \right)^2 \right),
\]
for all \(N \geq \bar{N}\). It is easy to see that the right hand side above divided by \(\epsilon_N / 3\) goes to 0 as \(N\) goes to infinity, so
\[
\mathbb{P}^\infty \left( \left\{ d_1(\hat{\mathcal{Q}}_{\gamma_i}^N, \hat{\mathcal{Q}}_{\gamma_i}^N) > \frac{\epsilon_N}{3} \right\} \right) = 0.
\]

**Term 3:** \(d_1(\hat{\mathcal{Q}}_{\gamma_i}^N, \hat{\mathcal{Q}}_{\gamma_i}^N)\): By the law of total probability,
\[
\mathbb{P}^N \left( d_1(\hat{\mathcal{Q}}_{\gamma_i}^N, \hat{\mathcal{Q}}_{\gamma_i}^N) > \frac{\epsilon_N}{3} \right) \leq \mathbb{P}^N(I_N = 0) + \mathbb{P}^N \left( d_1(\hat{\mathcal{Q}}_{\gamma_i}^N, \hat{\mathcal{Q}}_{\gamma_i}^N) > \frac{\epsilon_N}{3} \mid I_N = 1 \right).
\]

We now show that each of the above terms have finite summations. First,
\[
\sum_{N=1}^{\infty} \mathbb{P}^N(I_N = 0) \leq \sum_{N=1}^{\infty} N \sup_{\gamma' \in \Gamma} \mathbb{P}(\xi \notin B_N \mid \gamma = \gamma') \leq \sum_{N=1}^{\infty} N \sup_{\gamma' \in \Gamma} \exp \left( -\frac{(b_N - \mathbb{E}[\|\xi\| \mid \gamma = \gamma')]^2}{2\sigma^2} \right) < \infty.
\]
The first inequality follows from the union bound, the second inequality follows from Assumption 2 and the final inequality follows because \(\sup_{\gamma' \in \Gamma} \mathbb{E}[\|\xi\| \mid \gamma = \gamma'] < \infty\) and the definition of \(b_N\).

Second, for each \(l \in \mathbb{N}\), we define several quantities. Let \(\mathcal{P}_l\) be the partitioning of \(B_N = [-b_N, b_N]^d\) into \(2^{ld}\) translations of \((-b_N 2^{-l}, b_N 2^{-l})^d\). Let \(\mathcal{H}_l\) be the set of piecewise constant functions which are constant on each region of the partition \(\mathcal{P}_l\), taking values on \(\{kbN 2^{-l} : k \in \{0, \pm 1, \pm 2, \pm 3, \ldots, \pm 2^l\}\}\). Note that \(|\mathcal{H}_l| = (2^{l+1} + 1)^{2^{ld}}\). Then, we observe that for all Lipschitz functions \(\text{Lip}(h) \leq 1\) which satisfy \(h(0) = 0\), there exists a \(\hat{h} \in \mathcal{H}_l\) such that
\[
\sup_{\zeta \in B_N} |h(\zeta) - \hat{h}(\zeta)| \leq b_N 2^{-l+1}.
\]
Indeed, within each region of the partition, \(h\) can vary by no more than \(b_N 2^{-l+1}\). The possible function values for \(\hat{h}\) are separated by \(b_N 2^{-l}\). Because \(h\) is bounded by \(\pm b_N\), this implies the existence of \(\hat{h} \in \mathcal{H}_l\) such that \(\hat{h}\) has a value within \(b_N 2^{-l+1}\) of \(h\) everywhere within that region. The identical reasoning holds for all other regions of the partition.

Therefore, for every \(l \in \mathbb{N}\),
\[
\mathbb{P}^N \left( d_1(\hat{\mathcal{Q}}_{\gamma_i}^N, \hat{\mathcal{Q}}_{\gamma_i}^N) > \frac{\epsilon_N}{3} \mid I_N = 1 \right)
\]
\[
= \mathbb{P}^N \left( \sup_{\text{Lip}(h) \leq 1} \sum_{i=1}^{N} w_N(\gamma) (h(\xi_i) - \mathbb{E}[h(\xi) \mid \gamma = \gamma', \xi \in B_N]) > \frac{\epsilon_N}{3} \right)
\]
\[
= \mathbb{P}^N \left( \sup_{\text{Lip}(h) \leq 1} \sum_{i=1}^{N} w_N(\gamma) (h(\xi_i) - \mathbb{E}[h(\xi) \mid \gamma = \gamma', \xi \in B_N]) > \frac{\epsilon_N}{3} \right).
\]
\[
\begin{align*}
&\leq \mathbb{P}^N \left( \sup_{h \in \mathcal{H}} \sum_{i=1}^{N} w_N^i (\gamma) \left( \hat{h}(\xi^i) - \mathbb{E} [\hat{h}(\xi) | \gamma = \gamma^i, \xi \in B_N] \right) > \frac{\epsilon_N}{3} - 2 \cdot b_N 2^{-l+1} \bigg| I_N = 1 \right) \\
&\leq |\mathcal{H}| \sup_{h \in \mathcal{H}} \mathbb{P}^N \left( \sum_{i=1}^{N} w_N^i (\gamma) \left( \hat{h}(\xi^i) - \mathbb{E} [\hat{h}(\xi) | \gamma = \gamma^i, \xi \in B_N] \right) > \frac{\epsilon_N}{3} - b_N 2^{-l+2} \bigg| I_N = 1 \right),
\end{align*}
\]

where the final inequality follows from the union bound. We choose \( l = \left[ 2 + \log_2 \frac{\delta b_N}{\epsilon_N} \right] \), in which case

\[
\frac{\epsilon_N}{3} - b_N 2^{-l+2} \geq \frac{\epsilon_N}{6}.
\]

Furthermore, for all sufficiently large \( N \),

\[
|\mathcal{H}| = (2^{l+1} + 1)^{4d\xi} \leq \left( \frac{96b_N}{\epsilon_N} \right)^{24d\xi (b_N/\epsilon_N)^{d\xi}} = \exp \left( 24d\xi \left( \frac{b_N}{\epsilon_N} \right)^{d\xi} \log \frac{96b_N}{\epsilon_N} \right).
\]

Applying Hoeffding’s inequality, and noting \(|\hat{h}(\xi^i)|\) is bounded by \( b_N \) when \( \xi^i \in B_N \), we have the following for all \( \hat{h} \in \mathcal{H} \):

\[
\begin{align*}
&\mathbb{P}^N \left( \sum_{i=1}^{N} w_N^i (\gamma) \left( \hat{h}(\xi^i) - \mathbb{E} [\hat{h}(\xi) | \xi \in B_N, \gamma = \gamma^i] \right) > \frac{\epsilon_N}{6} \bigg| I_N = 1 \right) \\
&= \mathbb{E} \left[ \mathbb{P}^N \left( \sum_{i=1}^{N} w_N^i (\gamma) \left( \hat{h}(\xi^i) - \mathbb{E} [\hat{h}(\xi) | \xi \in B_N, \gamma = \gamma^i] \right) > \frac{\epsilon_N}{6} \bigg| I_N = 1, \gamma^1, \ldots, \gamma^N \right) \bigg| I_N = 1 \right] \\
&\leq \mathbb{E} \left[ \exp \left( -\frac{\epsilon_N^2}{72 \sum_{i=1}^{N} (w_N^i(\gamma))^2 b_N^2} \right) \bigg| I_N = 1 \right] \\
&= \mathbb{E} \left[ \exp \left( -\frac{\epsilon_N^2}{72 \sum_{i=1}^{N} (w_N^i(\gamma))^2 b_N^2} \right) I_N \right] \left( \frac{1}{\mathbb{P}^N (I_N = 1)} \right) \\
&\leq 2 \mathbb{E} \left[ \exp \left( -\frac{\epsilon_N^2}{72 \sum_{i=1}^{N} (w_N^i(\gamma))^2 b_N^2} \right) \right] \\
&\leq 2 \exp \left( -\frac{k_2N^2\epsilon_N^2}{72b_N^2} \right),
\end{align*}
\]

for \( N \) sufficiently large that \( \mathbb{P}(I_N = 1) \geq 1/2 \) and \( \epsilon_N^2/72b_N^2 < 1 \). Note that (8) was used for the final inequality. Combining these results, we have

\[
\begin{align*}
&\mathbb{P}^N \left( d_1 (\mathbb{P}_N^N, \mathcal{Q}_N^N_{\gamma|B_N}) > \frac{\epsilon_N}{3} \bigg| I_N = 1 \right) \\
&\leq 2 \exp \left( 24d\xi \left( \frac{b_N}{\epsilon_N} \right)^{d\xi} \log \frac{96b_N}{\epsilon_N} - \frac{k_2\epsilon_N^2N^u}{72N^2b_N^2} \right),
\end{align*}
\]

for \( N \) sufficiently large. For some constants \( c_1, c_2 > 0 \), and sufficiently large \( N \), this is upper bounded by

\[
2 \exp \left( -c_1N^{\eta-2(p+q)} + c_2N^{d\xi(p+q)} \log N \right).
\]

Since \( 0 < d\xi (p+q) < \eta - 2(p+q) \), we can conduct a limit comparison test with \( 1/N^2 \) to see that this term has a finite sum over \( N \), which completes the proof. \( \square \)
4.4. Proof of main result

Theorem 2 provides the key ingredient for the proof of the main consistency result. We state one final assumption, which requires that the objective function of (1) is upper semicontinuous and bounded by linear functions of the uncertainty.

Assumption 5. For all \( \pi \in \Pi \), \( c^\pi(\zeta_1, \ldots, \zeta_T) \) is upper semicontinuous in \( \zeta \) and \( |c(\zeta, x)| \leq C(1 + \|\zeta\|) \) for all \( \zeta \in \Xi \) and some \( C > 0 \).

Under this assumption, the proof of Theorem 1 follows from Theorem 2 via arguments similar to those used by Esfahani and Kuhn (2018) and Bertsimas et al. (2018a). We state it fully in Appendix EC.2.

5. Tractable Approximations

In the previous sections, we presented the new framework of sample robust optimization with covariates and established its asymptotic optimality without any significant structural restrictions on the space of decision rules. In this section, we focus on tractable methods for approximately solving the robust optimization problems that result from this proposed framework. Specifically, we develop a formulation which uses auxiliary decision rules to approximate the cost function. In combination with linear decision rules, this approach enables us to find high-quality decisions for real-world problems with more than ten stages in less than one minute, as we demonstrate in Section 6.

We focus in this section on dynamic optimization problems with cost functions of the form

\[
c(\xi_1, \ldots, \xi_T, x_1, \ldots, x_T) = \sum_{t=1}^T \left( f_t^\top x_t + g_t^\top \xi_t + \min_{y_t \in \mathbb{R}^d_y} \left\{ h_t^\top y_t : \sum_{s=1}^t A_{t,s} x_s + \sum_{s=1}^t B_{t,s} \xi_s + C_t y_t \leq d_t \right\} \right). \tag{10}
\]

Such cost functions appear frequently in applications such as inventory management and supply chain networks. Unfortunately, it is well known that these cost functions are convex in the uncertainty \( \xi_1, \ldots, \xi_T \). Thus, even evaluating the worst-case cost over a convex uncertainty set is computationally demanding in general, as it requires the maximization of a convex function.

As an intermediary step towards developing an approximation scheme for (3) with the above cost function, we consider the following optimization problem:

\[
\hat{v}^N(\gamma) := \min_{\pi \in \Pi, y \in \mathbb{R}_y} \sum_{i=1}^N w_i^N(\gamma) \sup_{\zeta \in \mathcal{U}_y} \sum_{t=1}^T \left( f_t^\top \pi_t(\zeta_1, \ldots, \zeta_{t-1}) + g_t^\top \xi_t + h_t^\top y_t(\zeta_1, \ldots, \zeta_t) \right) \tag{11}
\]

subject to

\[
\sum_{s=1}^t A_{t,s} \pi_s(\zeta_1, \ldots, \zeta_{s-1}) + \sum_{s=1}^t B_{t,s} \xi_s + C_t y_t(\zeta_1, \ldots, \zeta_t) \leq d_t
\]

\( \forall \zeta \in \mathcal{U}_y \), \( i \in \{1, \ldots, N\} \), \( t \in \{1, \ldots, T\} \).
where \( \mathcal{R}_t \) is the set of all functions \( y: \Xi_1 \times \cdots \times \Xi_t \rightarrow \mathbb{R}^{d_y} \). In this problem, we have introduced auxiliary decision rules which capture the minimization portion of (10) in each stage. We refer to (11) as a multi-policy approach, as it involves different auxiliary decision rules for each uncertainty set. The following theorem shows that (11) is equivalent to (3).

**Theorem 3.** For cost functions of the form (10), \( \hat{v}^N(\bar{\gamma}) = \bar{v}^N(\bar{\gamma}) \).

**Proof.** See Appendix EC.3

We observe that (11) involves optimizing over decision rules, and thus is computationally challenging to solve in general. Nonetheless, we can obtain a tractable approximation of (11) by further restricting the space of primary and auxiliary decision rules. For instance, we can restrict all primary and auxiliary decision rules as linear decision rules of the form

\[
\pi_t(\zeta_1, \ldots, \zeta_{t-1}) = x_{t,0} + \sum_{s=1}^{t-1} X_{t,s} \zeta_s, \quad y^i_t(\zeta_1, \ldots, \zeta_t) = y^i_{t,0} + \sum_{s=1}^{1} Y^i_{t,s} \zeta_s.
\]

One can alternatively elect to use a richer class of decision rules, such as lifted linear decision rules (Chen and Zhang 2009, Georghiou et al. 2015). In all cases, feasible approximations that restrict the space of decision rules of (11) provide an upper bound on the cost \( \hat{v}^N(\bar{\gamma}) \) and produce decision rules that are feasible for (11).

The key benefit of the multi-policy approximation scheme is that it offers many degrees of freedom in approximating the nonlinear cost function. Specifically, in (11), a separate auxiliary decision rule \( y^i_t \) captures the value of the cost function for each uncertainty set in each stage. We approximate each \( y^i_t \) with a linear decision rule, which only needs to be locally accurate, i.e., accurate for realizations in the corresponding uncertainty set. As a result, (11) with linear decision rules results in significantly tighter approximations of (3) compared to using a single linear decision rule, \( y_t \), for all uncertainty sets in each stage. Moreover, these additional degrees of freedom come with only a mild increase in computation cost, and we substantiate these claims via computational experiments in Section 6.2. In Appendix EC.4, we provide the reformulation of the multi-policy approximation scheme with linear decision rules into a deterministic optimization problem using standard techniques from robust optimization.

### 6. Computational Experiments

We perform computational experiments to assess the out-of-sample performance and computational tractability of the proposed methodologies across several applications. These examples are two-stage shipment planning (Section 6.1), dynamic inventory management (Section 6.2), and portfolio optimization (Section 6.3).
We compare several methods using different machine learning models. These methods include the proposed sample robust optimization with covariates, sample average approximation (SAA), the predictions to prescriptions (PtP) approach of Bertsimas and Kallus (2014), and sample robust optimization without covariates (SRO). In Table 1, we show that each of the above methods are particular instances of (3) from Section 3. The methods in the left column ignore covariates by assigning equal weights to each uncertainty set, and the methods in the right column incorporate covariates by choosing the weights based on predictive machine learning. The methods in the top row do not incorporate any robustness, and the methods in the bottom row incorporate robustness via a positive $\epsilon_N$ in the uncertainty sets. In addition, for the dynamic inventory management example, we also implement and compare to the residual tree algorithm described in Ban et al. (2018). In each experiment, the relevant methods are applied to the same training datasets, and their solutions are evaluated against a common testing dataset. Further details are provided in each of the following sections.

6.1. Shipment planning

We first consider a two-stage shipment planning problem in which a decision maker seeks to satisfy demand in several locations from several production facilities while minimizing production and transportation costs. Our problem setting closely follows Bertsimas and Kallus (2014), in which the decision maker has access to auxiliary covariates (promotions, social media, market trends), which may be predictive of future sales in each retail location.

**Problem Description.** The decision maker first decides the quantity of inventory $x_f \geq 0$ to produce in each of the production facilities $f \in F := \{1, \ldots, |F|\}$, at a cost of $p_1$ per unit. The demands $\xi_\ell \geq 0$ in each location $\ell \in L := \{1, \ldots, |L|\}$ are then observed. The decision maker fulfills these demands by shipping $s_{f\ell} \geq 0$ units from facility $f \in F$ to location $\ell \in L$ at a per-unit cost of $c_{f\ell} > 0$. Additionally, after observing demand, the decision maker has the opportunity to produce additional units $y_f \geq 0$ in each facility at a cost of $p_2 > p_1$ per unit. The fulfillment of each unit of
demand generates $r > 0$ in revenue. Given the above notation and dynamics, the cost incurred by the decision maker is

$$c(\xi, x) = \sum_{f \in \mathcal{F}} p_1 x_f - \sum_{\ell \in \mathcal{L}} r \xi_\ell + \minimize_{s \in \mathbb{R}^{|\mathcal{L} \times \mathcal{F}|}, y \in \mathbb{R}^{|\mathcal{F}|}} \sum_{f \in \mathcal{F}} p_2 y_f + \sum_{f \in \mathcal{F}} \sum_{\ell \in \mathcal{L}} c_{f\ell} s_{f\ell}$$

subject to

$$\sum_{f \in \mathcal{F}} s_{f\ell} \geq \xi_\ell \quad \forall \ell \in \mathcal{L}$$

$$\sum_{\ell \in \mathcal{L}} s_{f\ell} \leq x_f + y_f \quad \forall f \in \mathcal{F}.$$ 

Experiments. We perform computational experiments using the same parameters and data generation procedure as Bertsimas and Kallus (2014). Specifically, we consider an instance with $|\mathcal{F}| = 4$, $|\mathcal{L}| = 12$, $p_1 = 5$, $p_2 = 100$, and $r = 90$. The network topology, transportation costs, and the joint distribution of the covariates $\gamma \in \mathbb{R}^3$ and demands $\xi \in \mathbb{R}^{12}$ are the same as Bertsimas and Kallus (2014), with the exception that we generate the covariates as i.i.d. samples as opposed to an ARMA process (but with the same marginal distribution).

In our experiments, we compare sample robust optimization with covariates, sample average approximation, sample robust optimization, and predictions to prescriptions. For the robust approaches (bottom row of Table 1), we construct the uncertainty sets from Section 3 using the $\ell_1$ norm and $\Xi = \mathbb{R}^{12}_+$, solve these problems using the multi-policy approximation with linear decision rules described in Section 5, and consider uncertainty sets with radius $\epsilon \in \{100, 500\}$. For the approaches using covariates (right column of Table 1), we used the $k_N$-nearest neighbors with parameter $k_N = \frac{2N}{5}$. All solutions were evaluated on a test set of size 100 and the results were averaged over 100 independent training sets.

Results. In Figure 1, we present the average out-of-sample profits of the various methods. The results show that the best out-of-sample average profit is attained when using the proposed sample robust optimization with covariates. Interestingly, we observe no discernible differences between sample average approximation and sample robust optimization in Figure 1 suggesting the value gained by incorporating covariates in this example. Compared to the approach of Bertsimas and Kallus (2014), sample robust optimization with covariates achieves a better out-of-sample average performance for each choice of $\epsilon$. Table 2 shows that these differences are statistically significant. This example demonstrates that, in addition to enjoying asymptotic optimality guarantees, sample robust optimization with covariates provides meaningful value across various values of $N$.

6.2. Dynamic inventory management

We next consider a dynamic inventory control problem over the first $T = 12$ weeks of a new product. In each week, a retailer observes demand for the product and can replenish inventory
via procurement orders to different suppliers with lead times. Our problem setting closely follows [Ban et al. (2018)], motivated by the fashion industry in which retailers have access to auxiliary covariates on the new product (color, brand) which are predictive of how demand unfolds over time.

**Problem Description.** In each stage $t \in \{1, \ldots, T\}$, the retailer procures inventory from multiple suppliers to satisfy demand for a single product. The demands for the product across stages are denoted by $\xi_1, \ldots, \xi_T \geq 0$. In each stage $t$, and before the demand $\xi_t$ is observed, the retailer places procurement orders at various suppliers indexed by $\mathcal{J} = \{1, \ldots, |\mathcal{J}|\}$. Each supplier $j \in \mathcal{J}$ has per-unit order cost of $c_{tj} \geq 0$ and a lead time of $\ell_j$ stages. At the end of each stage, the firm incurs a

---

**Figure 1** Out-of-sample profit for the shipment planning example.

**Table 2** Statistical significance for shipment planning problem.

| $N$  | $\epsilon = 100$ | $\epsilon = 500$ |
|------|-----------------|-----------------|
| 50   | $4.6 \times 10^{-13}$ | $5.3 \times 10^{-16}$ |
| 75   | $1.3 \times 10^{-14}$ | $6.4 \times 10^{-12}$ |
| 100  | $1.2 \times 10^{-13}$ | $1.1 \times 10^{-7}$ |
| 125  | $2.6 \times 10^{-15}$ | $1.5 \times 10^{-11}$ |
| 150  | $3.4 \times 10^{-12}$ | $1.2 \times 10^{-6}$ |
| 200  | $1.4 \times 10^{-12}$ | $1.0 \times 10^{-8}$ |
| 250  | $3.4 \times 10^{-10}$ | $1.0 \times 10^{-4}$ |
| 300  | $1.8 \times 10^{-6}$ | $5.2 \times 10^{-4}$ |

The $p$-values from the Wilcoxon signed rank test for comparison with the predictive to prescriptive analytics method (PtP-$k$NN) and sample robust optimization with covariates (SRO-$k$NN). After adjusting for multiple hypothesis testing, all results are significant at the $\alpha = 0.05$ significance level because all $p$-values are less than $\frac{0.05}{|\mathcal{J}|} \approx 3.1 \times 10^{-3}$. 

---
per-unit holding cost of $h_t$ and a backorder cost of $b_t$. Inventory is fully backlogged and the firm starts with zero initial inventory. The cost incurred by the firm over the time horizon is captured by

$$c(\xi_1, \ldots, \xi_T, x_1, \ldots, x_T) = \sum_{t=1}^{T} \sum_{j \in J} c_{tj} x_{tj} + \minimize_{y_t \in \mathbb{R}} y_t$$

subject to

$$y_t \geq h_t \left( \sum_{j \in J} \sum_{s=1}^{t-\ell_j} x_{sj} - \sum_{s=1}^{t} \xi_s \right)$$

$$y_t \geq -b_t \left( \sum_{j \in J} \sum_{s=1}^{t-\ell_j} x_{sj} - \sum_{s=1}^{t} \xi_s \right).$$

**Experiments.** The parameters of the procurement problem were chosen based on Ban et al. (2018). Specifically, we consider the case of two suppliers where $c_{t1} = 1.0$, $c_{t2} = 0.5$, $h_t = 0.25$, and $b_t = 11$ for each stage. The first supplier has no lead time and the second supplier has a lead time of one stage. We generate training and test data from the same distribution as the shipment planning problem in Section 6.1. In this case, the demands produced by this process are interpreted as the demands over the $T = 12$ stages. We perform computational experiments comparing the proposed sample robust optimization with covariates and the residual tree algorithm proposed by Ban et al. (2018). In particular, we compare sample robust optimization with covariates with the multi-policy approximation as well as without the multi-policy approximation (in which we use a single auxiliary linear decision rule for $y_t$ for all uncertainty sets in each stage). The uncertainty sets from Section 3 are defined with the $\ell_2$ norm and $\Xi = \mathbb{R}_+^{12}$. The out-of-sample cost resulting from the decision rules were averaged over 100 training sets of size $N = 40$ and 100 testing points, and sample robust optimization with covariates used $k$-nearest neighbors with varying choices of $k$ and radius $\epsilon \geq 0$ of the uncertainty sets.

**Results.** In Table 3, we show the average out-of-sample cost resulting from sample robust optimization with covariates using linear decision rules, with and without the multi-policy approximation from Section 5. In both settings, we used $k$-nearest neighbors as the machine learning method and evaluated the out-of-sample performance by applying the linear decision rules for the ordering quantities. The results of these computational experiments in Table 3 demonstrate that significant improvements in average out-of-sample performance are found when combining the multi-policy approximation with covariates via $k$-nearest neighbors. We show in Table 4 that these results are statistically significant. For comparison, we also implemented the residual tree algorithm from Ban et al. (2018). When using their algorithm with a binning of $B = 2$ in each stage, their approach resulted in an average out-of-sample cost of 27142. We were unable to run with a binning of $B = 3$. 


in each stage due to time limitations of $10^3$ seconds, as the size of the resulting linear optimization problem scales on the order $O(B^T)$. Such results are consistent with the estimations of computation times presented in [Ban et al. 2018 Section 6.3]. The running times of the various methods are displayed in Table 5.

### 6.3. Portfolio optimization

Finally, we consider a single-stage portfolio optimization problem in which we wish to find an allocation of a fixed budget to $n$ assets. Our goal is to simultaneously maximize the expected return while minimizing the the conditional value at risk (cVaR) of the portfolio. Before selecting our portfolio, we observe auxiliary covariates which include general market indicators such as index performance as well as macroeconomic numbers released by the US Bureau of Labor Statistics.
Problem Description. We denote the portfolio allocation among the assets by \( x \in X := \{ x \in \mathbb{R}^n_+ : \sum_{j=1}^n x_j = 1 \} \), and the returns of the assets by the random variables \( \xi \in \mathbb{R}^n \). The conditional value at risk at the \( \alpha \in (0,1) \) level measures the expected loss of the portfolio, conditional on losses being above the \( 1 - \alpha \) quantile of the loss distribution. Rockafellar and Uryasev (2000) showed that the cVaR of a portfolio can be computed as the optimal objective value of a convex minimization problem. Therefore, our portfolio optimization problem can be expressed as a convex optimization problem with an auxiliary decision variable, \( \beta \in \mathbb{R} \). Thus, given an observation \( \bar{\gamma} \) of the auxiliary covariates, our goal is to solve

\[
\min_{x \in X, \beta \in \mathbb{R}} \mathbb{E} \left[ \beta + \frac{1}{\alpha} \max(0, -x^T\xi - \beta) - \lambda x^T\xi \middle| \bar{\gamma} = \gamma \right],
\]

where \( \lambda \in \mathbb{R}_+ \) is a trade-off parameter that balances the risk and return objectives.

Experiment. Our experiments are based on a similar setting from Bertsimas and Van Parys (2017, Section 5.2). Specifically, we perform computational experiments on an instance with parameters \( \alpha = 0.05 \) and \( \lambda = 1 \), and the joint distribution of the covariates and asset returns are chosen the same as Bertsimas and Van Parys (2017, Section 5.2). In our experiments, we compare sample robust optimization with covariates, sample average approximation, sample robust optimization, and predictions to prescriptions. For the robust approaches (bottom row of Table 1), we construct the uncertainty sets from Section 3 using the \( \ell_1 \) norm. For each training sample size, we compute the out-of-sample objective on a test set of size 1000, and we average the results over 100 instances of training data.

In order to select \( \epsilon_N \) and other tuning parameters associated with the machine learning weight functions, we first split the data into a training and validation set. We then train the weight
functions using the training set, compute decisions for each of the instances in the validation set, and compute the out-of-sample cost on the validation set. We repeat this for a variety of parameter values and select the combination that achieves the best cost on the validation set.

Following a similar reformulation approach as Esfahani and Kuhn (2018), we solve the robust approaches exactly by observing that

$$\min_{x \in X, \beta \in \mathbb{R}^N} \sum_{i=1}^N w_i \gamma \sup_{\zeta \in U_i} \left\{ \beta + \frac{1}{\alpha} \max\{0, -x^\top \zeta - \beta\} - \lambda x^\top \zeta \right\}$$

$$= \min_{x \in X, \beta \in \mathbb{R}^N} \sum_{i=1}^N w_i \gamma \sup_{\zeta \in U_i} \left\{ \max\left\{ \beta - \lambda x^\top \zeta, \left(\frac{1}{\alpha} + \lambda\right) x^\top \zeta \right\} \right\}$$

$$= \min_{x \in X, \beta \in \mathbb{R}, v \in \mathbb{R}^N} \sum_{i=1}^N w_i \gamma v_i$$

subject to

$$v_i \geq \beta - \lambda x^\top \zeta$$

$$v_i \geq \left(\frac{1}{\alpha} + \lambda\right) x^\top \zeta$$

$$\forall \zeta \in U_i, i \in \{1, \ldots, N\}.$$

The final expression can be reformulated as a deterministic optimization problem by reformulating the robust constraints.

**Results.** In Figure 2, we show the average out-of-sample objective values using the various methods. Consistent with the computational results of Esfahani and Kuhn (2018) and Bertsimas and McCord.
Van Parys (2017), the results underscore the importance of robustness in preventing overfitting and achieving good out-of-sample performance in the small data regime. Indeed, we observe that the sample average approximation, which ignores the auxiliary data, outperforms PtP-$k$NN and PtP-CART when the amount of training data is limited. We believe this is due to the fact the latter methods both throw out training examples, so the methods overfit when the training data is limited, leading to poor out-of-sample performance. In contrast, our methods (SRO-$k$NN and SRO-CART) typically achieve the strongest out-of-sample performance, even though the amount of training data is limited.

7. Conclusion

In this paper, we introduced sample robust optimization with covariates, a new framework for solving dynamic optimization problems with side information. Through three computational examples, we demonstrated that our method achieves significantly better out-of-sample performance than scenario-based alternatives. We complemented these empirical observations with theoretical analysis, showing our nonparametric method is asymptotically optimal via a new concentration measure result for local learning methods. Finally, we showed our approach inherits the tractability of robust optimization, scaling to problems with many stages via the multi-policy approximation scheme.

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**Electronic Companion**

**EC.1. Properties of Weight Functions**

In this section, we show that the $k$-nearest neighbor and kernel regression weight functions satisfy several guarantees. These results are used in the proof of Theorem 2 found in Section 4.3. The main result of this section is the following. For convenience, the equations below are numbered the same as in the proof of Theorem 2.

**Theorem EC.1.** If Assumptions 1 and 4 hold, then

$$\{w_N^i(\bar{\gamma})\} \text{ are not functions of } \xi^1, \ldots, \xi^N;$$

$$\sum_{i=1}^N w_N^i(\gamma) = 1 \text{ and } w_N^1(\bar{\gamma}), \ldots, w_N^N(\bar{\gamma}) \geq 0, \quad \forall N \in \mathbb{N}. \quad (5)$$

Moreover, there exists constants $k_2 > 0$ and $\eta > p(2 + d_\xi)$ such that

$$\lim_{N \to \infty} \frac{1}{\epsilon_N} \sum_{i=1}^N w_N^i(\gamma) \|\gamma^i - \bar{\gamma}\| = 0, \quad \mathbb{P}^\infty-a\text{almost surely}; \quad (7)$$

$$\mathbb{E}_{\mathbb{P}^N} \left[ \exp \left( -\frac{\theta}{\sum_{i=1}^N w_N^i(\gamma)^2} \right) \right] \leq \exp(-k_2 \theta N^\eta), \quad \forall \theta \in (0, 1), N \in \mathbb{N}. \quad (8)$$

**Proof.** We observe that (4) and (5) follow directly from the definitions of the weight functions. The proofs of (7) and (8) are split into two parts, one for the $k$-nearest neighbor weights and one for kernel regression weights.

**k-Nearest Neighbors:** For the proof of (7), we note

$$\sum_{i=1}^N w_N^i(\gamma) \|\gamma^i - \bar{\gamma}\| \leq \|\gamma(k_N)(\bar{\gamma}) - \bar{\gamma}\|,$$

where $\gamma(k_N)(\bar{\gamma})$ denotes the $k_N$th nearest neighbor of $\bar{\gamma}$ out of $\gamma^1, \ldots, \gamma^N$. Therefore, for any $\lambda > 0$,

$$\mathbb{P}^N \left( \sum_{i=1}^N w_N^i(\gamma) \|\gamma^i - \bar{\gamma}\| > \lambda \epsilon_N \right) \leq \mathbb{P}^N \left( \|\gamma(k_N)(\bar{\gamma}) - \bar{\gamma}\| > \lambda \epsilon_N \right) \leq \mathbb{P}^N \left( \left| \left\{ i : \|\gamma^i - \bar{\gamma}\| \leq \lambda \epsilon_N \right\} \right| \leq k_N - 1 \right).$$

By Assumption 4, this probability is upper bounded by $\mathbb{P}(\beta \leq k - 1)$, where $\beta \sim \text{Binom}(N, g(\lambda \epsilon_N)^{d_\gamma})$. By Hoeffding’s inequality,

$$\mathbb{P}^N \left( \sum_{i=1}^N w_N^i(\gamma) \|\gamma^i - \bar{\gamma}\| > \lambda \epsilon_N \right) \leq \exp \left( \frac{-2(N g(\lambda k_1/N^p)^{d_\gamma} - k_N + 1)^2}{N} \right).$$
for \( k_N \leq Ng(\lambda k_1/N^p)^{d_\gamma} + 1 \). We note that this condition on \( k_N \) is satisfied for \( N \) sufficiently large because \( \delta + pd_\gamma < 1 \) by Assumption \([1]\) Because the right hand side in the above inequality has a finite sum over \( N \), (7) follows by the Borel Cantelli lemma.

For the proof of (8), it follows from Assumption \([1]\) that

\[
\sum_{i=1}^{N} w_i^i(\bar{\gamma})^2 \leq k_3 N^{1-2\delta}
\]
deterministically (for all sufficiently large \( N \) such that \( [k_3 N^{\delta}] \leq N - 1 \) and \( p(2 + d_\varepsilon) > 2\delta - 1 \). Thus, (8) follows with \( \eta = 2\delta - 1 \).

**Kernel regression:** Assumption \([1]\) stipulates that the kernel function \( K(\cdot) \) is Gaussian, triangular, or Epanechnikov, which are defined in Section \([3]\) It is easy to verify that these kernel functions satisfy the following:

1. \( K \) is nonnegative, finite valued, and monotonically decreasing (for nonnegative inputs).
2. \( u^\alpha K(u) \to 0 \) as \( u \to \infty \) for any \( \alpha \in \mathbb{R} \).
3. \( \exists u^* > 0 \) such that \( K(u^*) > 0 \).

For the proof of (7), define \( q > 0 \) such that \( p < q < \delta \). Letting \( D \) be the diameter of \( \Gamma \) and \( g_N(\bar{\gamma}) = \sum_{i=1}^{N} K(\|\gamma_i - \bar{\gamma}\|/h_N) \), we have

\[
\sum_{i=1}^{N} w_i^i(\bar{\gamma})\|\gamma_i - \bar{\gamma}\|
= \sum_{i=1}^{N} w_i^i(\bar{\gamma}) \mathbb{1}\{\|\gamma_i - \bar{\gamma}\| \leq N^{-q}\}\|\gamma_i - \bar{\gamma}\| + \frac{1}{g_N(\bar{\gamma})} \sum_{i=1}^{N} K\left(\frac{\|\gamma_i - \bar{\gamma}\|}{h_N}\right) \mathbb{1}\{\|\gamma_i - \bar{\gamma}\| > N^{-q}\}\|\gamma_i - \bar{\gamma}\| \
\leq N^{-q} + \frac{ND K(N^{-q}/h_N)}{g_N(\bar{\gamma})},
\]

where the inequality follows from the monotonicity of \( K \). By construction, \( N^{-q}/\varepsilon_N \to 0 \), so we just need to handle the second term. We note, for any \( \lambda > 0 \),

\[
\mathbb{P}^N\left(\frac{ND K(N^{-q}/h_N)}{g_N(\bar{\gamma})} > \lambda \varepsilon_N\right) \leq \mathbb{P}^N\left(\sum_{i=1}^{N} Z_i^N K(u^*) < \frac{ND K(N^{-q}/h_N)}{\lambda \varepsilon_N}\right),
\]

where \( Z_i^N = \mathbb{1}\{\|\gamma_i - \bar{\gamma}\| \leq u^* h_N\} \). To achieve this inequality, we lower bounded each term in \( g_N(\bar{\gamma}) \) by \( K(u^*) \) or 0, because of the monotonicity of \( K \). By Hoeffding’s inequality,

\[
\mathbb{P}^N\left(\sum_{i=1}^{N} Z_i^N K(u^*) < \frac{ND K(N^{-q}/h_N)}{\lambda \varepsilon_N}\right) \leq \exp\left( -\frac{2 \left( N\mathbb{E}Z_i^N - \frac{ND}{\lambda \varepsilon_N K(u^*)} K(N^{-q}/h_N) \right)^2}{N} \right) \\
\leq \exp\left( -\frac{2 \left( Ng(u^* h_N)^{d_\gamma} - \frac{ND}{\lambda \varepsilon_N K(u^*)} K(N^{-q}/h_N) \right)^2}{N} \right) \\
= \exp\left( -\left( k_3 N^{1/2 - \delta d_\gamma} - k_6 N^{1/2 + p} K(k_4 N^{-q + \delta}) \right)^2 \right),
\]
for some constants $k_5, k_6 > 0$ that do not depend on $N$. We used Assumption 4 for the second inequality. Because $\delta > q$, the second kernel property implies $N^{1/2 + p}K(k_4 N^{-q+\delta})$ goes to 0 as $N$ goes to infinity, so that term is irrelevant. Because $1/2 - \delta d_\gamma > 0$ by Assumption 1, the right hand side of the inequality has a finite sum over $N$, and thus follows from the Borel Cantelli lemma.

For the proof of (8), define $\nu^N = \begin{pmatrix} K(\|\gamma - \bar{\gamma}\|/h_N) \\ \vdots \\ K(\|\gamma - \bar{\gamma}\|/h_N) \end{pmatrix}$.

We note that

$$\sum_{i=1}^N w_N^i(\gamma)^2 = \|\nu^N\|_2^2 \leq \|\nu^N\|_1 \leq \frac{K(0)}{K(u*)} \sum_{i=1}^N Z_i^N,$$

where $Z_i^N$ is defined above. The first inequality follows from Holder's inequality, and the second inequality follows from the monotonicity of $K$. Next, we define $\bar{Z}_i^N$ to be a Bernoulli random variable with parameter $g(u^* h_N)^{d_\gamma}$ for each $i$. For any $\theta \in (0, 1)$,

$$\mathbb{E}_{\nu^N} \left[ \exp \left( -\frac{\theta}{\sum_{i=1}^N w_N^i(\gamma)^2} \right) \right] \leq \mathbb{E}_{\nu^N} \left[ \exp \left( -\frac{\theta K(u*) \sum_{i=1}^N \bar{Z}_i^N}{K(0)} \right) \right] = \left( 1 - g(u^* h_N)^{d_\gamma} + g(u^* h_N)^{d_\gamma} \exp(-\theta K(u*)/K(0)) \right)^N \leq \exp \left( -N g(u^* h_N)^{d_\gamma} (1 - \exp(-\theta K(u*)/K(0))) \right) \leq \exp \left( -N g(u^* h_N)^{d_\gamma} \frac{\theta K(u*)}{2K(0)} \right) = \exp \left( -\frac{\theta K(u*)g(k_4 u^*)^{d_\gamma} N^{1-\delta d_\gamma}}{2K(0)} \right).$$

The first inequality follows because $g(u^* h_N)^{d_\gamma}$ is an upper bound on $\mathbb{P}(\|\gamma - \bar{\gamma}\| \leq u^* h_N)$ by Assumption 4. The first equality follows from the definition of the moment generating function for a binomial random variable. The next line follows from the inequality $e^x \geq 1 + x$ and the following from the inequality $1 - e^{-x} \geq x/2$ for $0 \leq x \leq 1$. Because $1 - \delta d_\gamma > p(2 + d_\xi)$, this completes the proof of (8) with $\eta = 1 - \delta d_\gamma$, and $k_2 = K(u^*)g(k_4 u^*)^{d_\gamma}/2K(0)$. □

**EC.2. Proof of Theorem 1**

In this section, we present our proof of Theorem 1. First, we must introduce some necessary terminology. To connect Theorem 2 to sample robust optimization, we consider the $\infty$-Wasserstein metric, which is given by:

$$d_\infty(Q, Q') = \inf \left\{ \Pi-ess \sup_{\xi, \xi'} \|\xi - \xi\| : \Pi \text{ is a joint distribution of } \xi \text{ and } \xi', \text{ with marginals } Q \text{ and } Q', \text{ respectively} \right\},$$
where the essential supremum of the joint distribution is defined as
\[
\Pi\text{-ess sup}_{\Xi \times \Xi} \|\xi - \xi'\| = \inf \left\{ M : \Pi \left( \|\xi - \xi'\| > M \right) = 0 \right\}.
\]

We make use of the following result from Bertsimas et al. (2018a):

**Lemma EC.1.** For any measurable \( f : \Xi \to \mathbb{R} \),
\[
\sum_{i=1}^{N} w_i^N(\gamma) \sup_{\zeta \in u_i^N} f(\zeta) = \sup_{Q \in \mathcal{P}(\Xi): d_1(\hat{Q}, Q) \leq \epsilon_N} \mathbb{E}_{\xi \sim Q}[f(\xi)].
\]

The proof of Lemma [EC.1] follows identical reasoning as in Bertsimas et al. (2018a) and is thus omitted.

Next, we state a result from Bertsimas et al. (2018a) (their Theorem EC.1), which bounds the difference in worst case objective values between 1-Wasserstein and \( \infty \)-Wasserstein distributionally robust optimization problems. We note that Bertsimas et al. (2018a) proved the following result for the case that \( Q' \) is the unweighted empirical measure, but their proof carries through for the case here in which \( Q' \) is a weighted empirical measure.

**Lemma EC.2.** Let \( Z \subseteq \mathbb{R}^d, f : Z \to \mathbb{R} \) be measurable, and \( \zeta_1, \ldots, \zeta_N \in Z \). Suppose that
\[
Q' = \sum_{i=1}^{N} w_i' \delta_{\zeta_i}
\]
for given weights \( w^1, \ldots, w^N \geq 0 \) that sum to one. If \( \theta_2 \geq 2\theta_1 \geq 0 \), then
\[
\sup_{Q \in \mathcal{P}(Z): d_1(Q', Q) \leq \theta_1} \mathbb{E}_{\xi \sim Q}[f(\xi)] \leq \sup_{Q \in \mathcal{P}(Z): d_\infty(Q', Q) \leq \theta_2} \mathbb{E}_{\xi \sim Q}[f(\xi)] + \frac{4\theta_1}{\theta_2} \sup_{\zeta \in Z} |f(\zeta)|.
\]

We now restate and prove the main result, which combines the new measure concentration result from this paper with similar proof techniques as Bertsimas et al. (2018a) and Esfahani and Kuhn (2018).

**Theorem 1.** Suppose the weight function and uncertainty sets satisfy Assumption 1, the joint probability distribution of \((\gamma, \xi)\) satisfies Assumptions 2-4 from Section 4.3, and the cost function satisfies Assumption 5 from Section 4.4. Then, for every \( \bar{\gamma} \in \Gamma \),
\[
\lim_{N \to \infty} \hat{v}^N(\bar{\gamma}) = v^*(\bar{\gamma}), \quad \mathbb{P}^\infty\text{-almost surely}.
\]

**Proof.** We break the limit into upper and lower parts. The proof of the lower part follows from an argument similar to that used by Bertsimas et al. (2018a). The proof of the upper part follows from the argument used by Esfahani and Kuhn (2018).
Lower bound. We first show
\[
\liminf_{N \to \infty} \hat{v}^N(\gamma) \geq v^*(\gamma), \quad P^\infty\text{-almost surely.} \tag{EC.1}
\]
To begin, we define
\[
D_N := \{ \zeta : \| \zeta \| \leq \log N \},
\]
and let \( P_{\gamma | D_N}(\cdot) \) be shorthand for \( P(\cdot | \gamma = \hat{\gamma}, \xi \in D_N) \). Then, applying Assumption 2
\[
P^N \left( \bigcup_{i=1}^N \mathcal{U}_i \not\subseteq D_N \right) \leq P \left( \max_{i \leq N} \| \xi \| + \epsilon_N > \log N \right)
\leq N \mathbb{P}(\| \xi \| > \log N - \epsilon_N)
= NE \left[ \mathbb{P}(\| \xi \| - E[\| \xi \| | \gamma] > \log N - \epsilon_N - E[\| \xi \| | \gamma] | \gamma] \right]
\leq N \mathbb{E} \left[ P \left( \| \xi \| - E[\| \xi \| | \gamma] > \log N - \epsilon_N - \sup_{\gamma' \in \Gamma} E[\| \xi \| | \gamma = \gamma'] | \gamma \right) \right]
\leq N \mathbb{E} \left[ 2 \exp \left( - \frac{(\log N - \epsilon_N - \sup_{\gamma' \in \Gamma} E[\| \xi \| | \gamma = \gamma'])^2}{2\sigma^2} \right) \right]
= 2 \exp \left( \log N - \frac{(\log N - \epsilon_N - \sup_{\gamma' \in \Gamma} E[\| \xi \| | \gamma = \gamma'])^2}{2\sigma^2} \right), \tag{EC.2}
\]
which has a finite sum over \( N \in \mathbb{N} \). Therefore, by the Borel-Cantelli lemma, there exists \( N_0 \in \mathbb{N} \), \( P^\infty\)\text{-almost surely, such that
\[
\bigcup_{i=1}^N \mathcal{U}_i \subseteq D_N \quad \forall N \geq N_0.
\]
We now choose any \( r > 0 \) such that \( \epsilon_N N^{-r} \) satisfies Assumption 1 and define \( N_1 := \max\{N_0, 2^{1/r}\} \). Then, the following holds for all \( N \geq N_1 \) and \( \pi \in \Pi \):
\[
\sup_{Q \in \mathcal{P}(D_N \cap \Xi) : d_1(Q, \bar{P}_N) \leq \frac{1}{N^r}} \mathbb{E}_{\xi \sim Q}[\pi(\xi_1, \ldots, \xi_T)]
\leq \sup_{Q \in \mathcal{P}(D_N \cap \Xi) : d_1(Q, \bar{P}_N) \leq \frac{1}{N^r}} \mathbb{E}_{\xi \sim Q}[\pi(\xi_1, \ldots, \xi_T)] + \frac{4}{N^r} \sup_{\zeta \in D_N \cap \Xi} \sup_{\xi} |\pi(\zeta, \xi_1, \ldots, \xi_T)|
= \sum_{i=1}^N w_i^N(\hat{\gamma}) \sup_{\zeta \in D_N} \pi(\zeta, \xi_1, \ldots, \xi_T) + \frac{4}{N^r} \sup_{\zeta \in D_N \cap \Xi} \sup_{\xi} |\pi(\zeta, \xi_1, \ldots, \xi_T)|
\leq \sum_{i=1}^N w_i^N(\hat{\gamma}) \sup_{\zeta \in \hat{U}_N} \pi(\zeta, \xi_1, \ldots, \xi_T) + \frac{4C}{N^r}(1 + \log N). \tag{EC.3}
\]
Indeed, the first supremum satisfies the conditions of Lemma EC.2 since \( N \geq N_0 \) and \( N \geq 2^{1/r} \), and the equality follows from Lemma EC.1 since \( N \geq N_0 \). The final inequality follows from Assumption 5 and the construction of \( D_N \). We observe that the second term on (EC.3) converges to zero as \( N \to \infty \). Next, we observe that
\[
\mathbb{E}[\pi(\xi_1, \ldots, \xi_T) | \gamma = \hat{\gamma}] \triangleq \mathbb{E}_{\xi \sim \bar{P}_{\hat{\gamma}}}[\pi(\xi_1, \ldots, \xi_T)]
= \mathbb{E}_{\xi \sim \bar{P}_{\hat{\gamma}}}[\pi(\xi_1, \ldots, \xi_T) \mathbb{1}\{\xi \not\in D_N\}] + \mathbb{E}_{\xi \sim \bar{P}_{\hat{\gamma}}}[\pi(\xi_1, \ldots, \xi_T) \mathbb{1}\{\xi \in D_N\}].
\]
We handle the first term with the Cauchy-Schwartz inequality,
\[
\mathbb{E}_{\xi \sim \mathcal{P}_N} [c^\pi(\xi_1, \ldots, \xi_T) \mathbb{1} \{ \xi \notin D_N \}] \leq \sqrt{\mathbb{E}_{\xi \sim \mathcal{P}_N} [c^\pi(\xi_1, \ldots, \xi_T)^2] \mathbb{P}_N(\xi \notin D_N)}.
\]

By Assumptions 2 and 5, the above bound is finite and converges to zero as \( N \to \infty \) uniformly over \( \pi \in \Pi \). We handle the second term by the new concentration measure from this paper. Specifically, it follows from Theorem 2 that there exists an \( N_2 \geq N_1 \), \( \mathbb{P}^\infty \)-almost surely, such that
\[
d_1(\mathbb{P}_N, \hat{\mathbb{P}}^N) \leq \frac{\epsilon_N}{N^r} \quad \forall N \geq N_2.
\]

Therefore, for all \( N \geq N_2 \) and decision rules \( \pi \in \Pi \):
\[
\mathbb{E}_{\xi \sim \mathcal{P}_N} [c^\pi(\xi_1, \ldots, \xi_T) \mathbb{1} \{ \xi \in D_N \}] = \mathbb{E}_{\xi \sim \mathcal{P}_N} \left[ \left( c^\pi(\xi_1, \ldots, \xi_T) - \inf_{\xi \in D_N \cap \Xi} c^\pi(\xi_1, \ldots, \xi_T) \right) \mathbb{1} \{ \xi \in D_N \} \right] + \mathbb{P}_N(\xi \notin D_N) \inf_{\xi \in D_N \cap \Xi} c^\pi(\xi_1, \ldots, \xi_T)
\]
\[
\leq \sup_{Q \in \mathcal{P}(\Xi) : d_1(Q, \hat{Q}_N^N) \leq \frac{\epsilon_N}{N}} \mathbb{E}_{\xi \sim Q} \left[ \left( c^\pi(\xi_1, \ldots, \xi_T) - \inf_{\xi \in D_N \cap \Xi} c^\pi(\xi_1, \ldots, \xi_T) \right) \mathbb{1} \{ \xi \in D_N \} \right] + \alpha_N
\]
\[
= \sup_{Q \in \mathcal{P}(\Xi \cap D_N) : d_1(Q, \hat{Q}_N^N) \leq \frac{\epsilon_N}{N}} \mathbb{E}_{\xi \sim Q} \left[ c^\pi(\xi_1, \ldots, \xi_T) - \inf_{\xi \in D_N \cap \Xi} c^\pi(\xi_1, \ldots, \xi_T) \right] + \alpha_N
\]
\[
= \sup_{Q \in \mathcal{P}(\Xi \cap D_N) : d_1(Q, \hat{Q}_N^N) \leq \frac{\epsilon_N}{N}} \mathbb{E}_{\xi \sim Q} [c^\pi(\xi_1, \ldots, \xi_T)] - \mathbb{P}_N(\xi \notin D_N) \inf_{\xi \in D_N \cap \Xi} c^\pi(\xi_1, \ldots, \xi_T).
\]

Indeed, the inequality follows because \( N \geq N_2 \). It follows from Assumption 5 and (EC.2) that the second term in the final equality converges to zero as \( N \to \infty \) uniformly over \( \pi \in \Pi \). Combining the above, we conclude that
\[
\lim \inf_{N \to \infty} \hat{v}^N(\gamma) = \lim \inf_{N \to \infty} \inf_{\pi \in \Pi} \sum_{i=1}^N w_i^N(\gamma) \sup_{\xi \in \mathcal{U}_N} c^\pi(\xi_1, \ldots, \xi_T) \geq \inf_{\pi \in \Pi} \mathbb{E}_{\xi \sim \mathcal{P}_N} [c^\pi(\xi_1, \ldots, \xi_T) | \gamma = \gamma] = v^*(\gamma),
\]
where the inequality holds \( \mathbb{P}^\infty \)-almost surely. This completes the proof of (EC.1).

**Upper bound.** We now prove that
\[
\lim \sup_{N \to \infty} \hat{v}^N(\gamma) \leq v^*(\gamma), \quad \mathbb{P}^\infty \text{-almost surely.} \tag{EC.4}
\]

Indeed, for any arbitrary \( \delta > 0 \), let \( x_\delta \in \mathcal{X} \) be a \( \delta \)-optimal solution for \((1)\). By Esfahani and Kuhn (2018 Lemma A.1) and Assumption 5 there exists a non-increasing sequence of functions \( f^j(\zeta_1, \ldots, \zeta_T), j \in \mathbb{N}, \) such that
\[
\lim_{j \to \infty} f^j(\zeta_1, \ldots, \zeta_T) = c^{x_\delta}(\zeta_1, \ldots, \zeta_T), \quad \forall \zeta \in \Xi
\]
and $f^j$ is $L_j$-Lipschitz continuous. Furthermore, for each $N \in \mathbb{N}$, choose any probability distribution $\hat{Q}^N \in \mathcal{P}(\Xi)$ such that $d_1(\hat{Q}^N, \hat{P}^N) \leq \epsilon_N$ and

$$\sup_{Q \in \mathcal{P}(\Xi): d_1(Q, \hat{P}^N) \leq \epsilon_N} E_{\xi \sim Q}[c^{x_1}(\xi_1, \ldots, \xi_T)] \leq E_{\xi \sim \hat{Q}^N}[c^{x_1}(\xi_1, \ldots, \xi_T)] + \delta.$$  

For any $j \in \mathbb{N}$,

$$\limsup_{N \to \infty} \hat{v}^N(\gamma) \leq \limsup_{N \to \infty} \sup_{Q \in \mathcal{P}(\Xi): d_1(Q, \hat{P}^N) \leq \epsilon_N} E_{\xi \sim Q}[c^{x_1}(\xi_1, \ldots, \xi_T)] \leq \limsup_{N \to \infty} \sup_{Q \in \mathcal{P}(\Xi): d_1(Q, \hat{P}^N) \leq \epsilon_N} E_{\xi \sim Q}[c^{x_1}(\xi_1, \ldots, \xi_T)] \leq \limsup_{N \to \infty} E_{\xi \sim \hat{Q}^N}[f^1(\xi_1, \ldots, \xi_T)] + \delta$$

$$\leq \limsup_{N \to \infty} E_{\xi \sim \hat{Q}^N}[f^1(\xi_1, \ldots, \xi_T)] + L_j d_1(\hat{P}_\gamma, \hat{Q}^N) + \delta$$

$$\leq \limsup_{N \to \infty} E_{\xi \sim \hat{Q}^N}[f^1(\xi_1, \ldots, \xi_T)] + L_j (d_1(\hat{P}_\gamma, \hat{P}^N) + d_1(\hat{Q}^N, \hat{P}_\gamma)) + \delta$$

$$\leq \limsup_{N \to \infty} E_{\xi \sim \hat{P}_\gamma}[f^1(\xi_1, \ldots, \xi_T)] + L_j (d_1(\hat{P}_\gamma, \hat{P}^N) + \epsilon_N) + \delta$$

$$= E_{\hat{P}_\gamma}[f^1(\xi_1, \ldots, \xi_T)] + \delta, \quad \mathbb{P}^\infty\text{-almost surely},$$

where we have used the fact $d_1(\hat{P}, Q) \leq d_\infty(\hat{P}, Q)$ for the second inequality, the dual form of the 1-Wasserstein metric for the fifth inequality (because $f^j$ is $L_j$-Lipschitz), and Theorem 2 for the equality. Taking the limit as $j \to \infty$, and applying the monotone convergence theorem (which is allowed because $E_{\xi \sim \hat{P}_\gamma}[f^1(\xi_1, \ldots, \xi_T)] \leq L_1 \mathbb{E}_{\xi \sim \hat{P}_\gamma}\|\xi\| + |f^1(0)| < \infty$ by Assumption 4), gives

$$\limsup_{N \to \infty} \hat{v}^N(\gamma) \leq E_{\xi \sim \hat{P}_\gamma}[c^{x_1}(\xi_1, \ldots, \xi_T)] + \delta \leq v^*(\gamma) + 2\delta, \quad \mathbb{P}^\infty\text{-almost surely.}$$

Since $\delta > 0$ was chosen arbitrarily, the proof of (EC.4) is complete. 

**EC.3. Proof of Theorem 3**

In this section, we present our proof of Theorem 3 from Section 5. We restate the theorem here for convenience.

**Theorem 3**. For cost functions of the form (10), $\hat{v}^N(\gamma) = \hat{v}^N(\gamma)$.

**Proof.** We first show that $\hat{v}^N(\gamma) \geq \hat{v}^N(\gamma)$. Indeed, consider any primary decision rule $\hat{\pi}$ and auxiliary decision rules $\hat{y}_1, \ldots, \hat{y}_T$ for each $i \in \{1, \ldots, N\}$ which are optimal for (11). Then, it follows from feasibility to (11) that

$$h_i^j \hat{y}_i^j(\xi_1, \ldots, \xi_T) \geq \min_{y_i \in \mathbb{R}^{d_i}} \left\{ h_i^j y_i : \sum_{s=1}^t A_{t,s} \hat{\pi}_s(\xi_1, \ldots, \xi_{s-1}) + \sum_{s=1}^t B_{t,s} \xi_s + C_i y_i \leq d_i \right\}$$

If no optimal solution exists, then we may choose any $\eta$-optimal solution.
for each \( i \in \{1, \ldots, N \} \), \( \zeta \in U_i \), and \( t \in \{1, \ldots, T \} \). Thus,

\[
\hat{v}^N(\bar{\gamma}) = \min_{\pi \in \Pi} \sum_{i=1}^{N} w_i^N(\bar{\gamma}) c^\pi(\zeta_1, \ldots, \zeta_T) \\
\leq \sum_{i=1}^{N} w_i^N(\bar{\gamma}) \sup_{\zeta \in U_i} \sum_{t=1}^{T} (f_i^T \bar{\pi}_t(\zeta_1, \ldots, \zeta_{t-1}) + g_i^T \zeta_t + h_i^T \bar{y}_i(\zeta_1, \ldots, \zeta_t)) \\
\leq \sum_{i=1}^{N} w_i^N(\bar{\gamma}) \sup_{\zeta \in U_i} \sum_{t=1}^{T} (f_i^T \bar{\pi}_t(\zeta_1, \ldots, \zeta_{t-1}) + g_i^T \zeta_t + h_i^T \bar{y}_i(\zeta_1, \ldots, \zeta_t)) = \hat{v}^N(\bar{\gamma}).
\]

The other side of the inequality follows from similar reasoning. Indeed, let \( \bar{\pi} \) be an optimal solution to (3).

EC.4. Tractable Reformulation of the Multi-Policy Approximation

For completeness, we now show how to reformulate the multi-policy approximation scheme with linear decision rules from Section 5 into a deterministic optimization problem using standard techniques from robust optimization.

We begin by transforming (11) with linear decision rules into a more compact representation. First, we combine the primary linear decision rules across stages as

\[
x_0 = \begin{bmatrix} x_{1,0} \\ \vdots \\ x_{T,0} \end{bmatrix} \in \mathbb{R}^{d_x}, \quad X = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ X_{2,1} & 0 & 0 & \cdots & 0 & 0 & 0 \\ X_{3,1} & X_{3,2} & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ X_{T-2,1} & X_{T-2,2} & X_{T-2,3} & \cdots & 0 & 0 & 0 \\ X_{T-1,1} & X_{T-1,2} & X_{T-1,3} & \cdots & X_{T-1,T-2} & 0 & 0 \\ X_{T,1} & X_{T,2} & X_{T,3} & \cdots & X_{T,T-2} & X_{T,T-1} & 0 \end{bmatrix} \in \mathbb{R}^{d_x \times d_x}.
\]
We note that the zero entries in the above matrix are necessary to ensure that the linear decision rules are non-anticipative. Similarly, for each $i \in \{1, \ldots, N\}$, we represent the auxiliary linear decision rules as

$$\begin{aligned}
y_0^i &= \begin{bmatrix} y_{1,0}^i \\
\vdots \\
y_{T,0}^i \end{bmatrix} \in \mathbb{R}^{d_y}, \\
Y^i &= \begin{bmatrix} Y_{1,1}^i & \cdots & 0 & 0 \\
Y_{2,1}^i & \cdots & 0 & 0 \\
\vdots & \ddots & \vdots & \vdots \\
Y_{T-1,1}^i & \cdots & Y_{T-1,T-1}^i & 0 \\
Y_{T,1}^i & \cdots & Y_{T,T-1}^i & Y_{T,T}^i \end{bmatrix} \in \mathbb{R}^{d_y \times d_{\epsilon}}.
\end{aligned}$$

We now combine the problem parameters. Let $d = (d_1, \ldots, d_T) \in \mathbb{R}^m$ and

$$\begin{aligned}
f &= \begin{bmatrix} f_1 \\
\vdots \\
f_T \end{bmatrix} \in \mathbb{R}^{d_x}, \\
A &= \begin{bmatrix} A_{1,1} & \cdots & 0 & 0 \\
A_{2,1} & \cdots & 0 & 0 \\
\vdots & \ddots & \ddots & \vdots \\
A_{T-1,1} & \cdots & A_{T-1,T-1} & 0 \\
A_{T,1} & \cdots & A_{T,t-1} & A_{T,T} \end{bmatrix} \in \mathbb{R}^{d_x \times d_x}, \\
g &= \begin{bmatrix} g_1 \\
\vdots \\
g_T \end{bmatrix} \in \mathbb{R}^{d_{\epsilon}}, \\
B &= \begin{bmatrix} B_{1,1} & \cdots & 0 & 0 \\
B_{2,1} & \cdots & 0 & 0 \\
\vdots & \ddots & \ddots & \vdots \\
B_{T-1,1} & \cdots & B_{T-1,T-1} & 0 \\
B_{T,1} & \cdots & B_{T,t-1} & B_{T,T} \end{bmatrix} \in \mathbb{R}^{d_x \times d_x}, \\
h &= \begin{bmatrix} h_1 \\
\vdots \\
h_T \end{bmatrix} \in \mathbb{R}^{d_y}, \\
C &= \begin{bmatrix} C_{1,1} & \cdots & 0 & 0 \\
0 & \cdots & C_{T-1,T-1} & 0 \\
0 & \cdots & 0 & C_{T,T} \end{bmatrix} \in \mathbb{R}^{d_y \times d_{\epsilon}}.
\end{aligned}$$

Therefore, using the above compact notation, we can rewrite the multi-policy approximation with linear decision rules as

$$\begin{aligned}
\text{minimize} & \quad \sum_{i=1}^N w_N^i(\gamma) \sup_{\zeta \in U_N^i} \{ f^i(x_0 + X\zeta) + g^i\zeta + h^i(y_0^i + Y^i\zeta) \} \\
\text{subject to} & \quad A(x_0 + X\zeta) + B\zeta + C(y_0^i + Y^i\zeta) \leq d \\
& \quad x_0 + X\zeta \in \mathcal{X} \\
& \quad \forall \zeta \in U_N^i, \ i \in \{1, \ldots, N\},
\end{aligned} \tag{EC.5}$$

where $\mathcal{X} := \mathcal{X}_1 \times \cdots \times \mathcal{X}_T$ and the matrices $X$ and $Y$ are non-anticipative. Note that the linear decision rules in the above optimization problem are represented using $O(d_{\epsilon} \max\{d_x, Nd_y\})$ decision variables, where $d_x := d_x^1 + \cdots + d_x^T$ and $d_y := d_y^1 + \cdots + d_y^T$. Thus, the complexity of representing the primary and auxiliary linear decision rules scales efficiently both in the size of the dataset and the number of stages. For simplicity, we present the reformulation for the case in which there are no constraints on the decision variables and nonnegativity constraints on the random variables.
THEOREM EC.2. Suppose $\Xi = \mathbb{R}^d_+$ and $\mathcal{X} = \mathbb{R}^{d_z}$. Then, (EC.5) is equivalent to

$$\begin{align*}
\min_{x_0 \in \mathbb{R}^{d_z}, x \in \mathbb{R}^{d_z \times d_z}, x' \in \mathbb{R}^{d_z \times d_z}, y' \in \mathbb{R}^{d_z}, y' \in \mathbb{R}^{d_z \times d_z}, A' \in \mathbb{R}^{m \times d_z}, s \in \mathbb{R}^d}\sum_{i=1}^N w_i (x_0 + X\xi^i) + g^i \xi^i + h^i (y_0^i + Y^i \xi^i) + (s^i)^\top \xi^i + \epsilon_N \|X^i f + g + (Y^i)^\top h + s^i\|_r
\end{align*}$$

subject to

$$\begin{align*}
A (x_0 + X\xi^i) + B\xi^i + C (y_0^i + Y^i \xi^i) + \Lambda^i \xi^i + \epsilon_N \|AX + B + CY^i + \Lambda^i\|_r \leq d
\end{align*}$$

$\forall i \in \{1, \ldots, N\}$.

where $\|Z\|_r := (\|z_1\|_r, \ldots, \|z_r\|_r) \in \mathbb{R}^r$ for any matrix $Z \in \mathbb{R}^{r \times n}$.

Proof. For any $c \in \mathbb{R}^d$ and $\xi \in \Xi$, it follows directly from strong duality for conic optimization that

$$\max_{\xi \geq 0} \{c^\top \xi : \|\xi - \xi\| \leq \epsilon\} = \min_{\lambda \geq 0} \{(c + \lambda)^\top \xi + \epsilon \|c + \lambda\|_r\}.$$

We use this result to reformulate the objective and constraints of (EC.5). First, let the $j$-th rows of $A, B, C$ and the $j$-th element of $d$ be denoted by $a_j \in \mathbb{R}^{d_z}, b_j \in \mathbb{R}^d, c_j \in \mathbb{R}^{d_z},$ and $d_j \in \mathbb{R}$. Then, each robust constraint has the form

$$a_j^i (x_0 + X\xi^i) + b_j^i \xi^i + c_j^i (y_0^i + Y^i \xi^i) \leq d_j \quad \forall \xi \in U_N^i.$$

Rearranging terms,

$$(a_j^i X + b_j^i + c_j^i Y^i) \xi \leq d_j - a_j^i x_0 - c_j^i y_0^i \quad \forall \xi \in U_N^i,$$

which applying duality becomes

$$\exists \lambda_j^i \geq 0 : (X^i a_j + b_j + (Y^i)^\top c_j + \lambda_j^i)^\top \xi^i + \epsilon_N \|X^i a_j + b_j + (Y^i)^\top c_j + \lambda_j^i\|_r \leq d_j - a_j^i x_0 - c_j^i y_0^i.$$

Rearranging terms, the robust constraints for each $i \in \{1, \ldots, N\}$ are satisfied if and only if

$$\exists \lambda_j^i \geq 0 : A (x_0 + X\xi^i) + B\xi^i + C (y_0^i + Y^i \xi^i) + \Lambda^i \xi^i + \epsilon_N \|AX + B + CY^i + \Lambda^i\|_r \leq d,$$

where the dual norm for a matrix is applied separately for each row. Similarly, the objective function takes the form

$$\begin{align*}
\sum_{i=1}^N w_i (x_0 + X\xi^i) + g^i \xi^i + h^i (y_0^i + Y^i \xi^i)
\end{align*}$$

Combining the reformulations above, we obtain the desired reformulation. □