Letter

Disordered contacts can localize helical edge electrons

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Abstract

It is well known that quantum spin Hall (QSH) edge modes being helical are immune to backscattering due to non-magnetic disorder within the sample. Thus, QSH edge modes are non-localized and show a vanishing Hall resistance along with quantized 2-terminal, longitudinal and non-local resistances even in presence of sample disorder. However, this is not the case for contact disorder. This paper shows that when all contacts are disordered in a N-terminal QSH sample, then transport via these helical QSH edge modes can have a significant localization correction. All the resistances in a N-terminal QSH sample deviate from their values derived while neglecting the phase acquired at disordered contacts, and this deviation is called the quantum localization correction. This correction term increases with the increase of disorderedness of contacts but decreases with the increase in number of contacts in a N terminal sample. The presence of inelastic scattering, however, can completely destroy the quantum localization correction.

Keywords: helical, edge mode, localization

(Some figures may appear in colour only in the online journal)

1. Introduction

The QSH effect observed in a 2D topological insulator is known for transport via dissipation-less helical 1D edge modes. These 1D helical edge modes are robust to sample disorder and are observed in systems like HgTe/CdTe heterostructures at low temperatures, due to bulk spin orbit effects and in absence of a magnetic field [1–3]. QSH edge modes are helical, i.e. at the upper edge a spin-up electron moves in one direction while spin-down electron moves in opposite direction while at the lower edge the directions are reversed, see figure 1. Thus, QSH systems are invariant under time reversal symmetry. Due to the topological nature of these edge modes, the Hall resistance vanishes, while the 2-terminal, longitudinal and non-local resistances are quantized at $\frac{e^2}{2h}$, $\frac{e^2}{2h}$ and $\frac{1}{6} \frac{e^2}{h}$ respectively in a six terminal ideal QSH sample (without any disordered contacts). The Hall, longitudinal, 2-terminal and non-local conductances/resistances are determined by resorting to the Landauer–Buttiker (L–B) theory [5, 6]. In this formalism, for a QSH device with N contacts, the current at contact i at zero temperature is [5–7]:

$$I_i = \frac{e^2}{h} \sum_{\sigma, \sigma'} \sum_{j=1}^{N} \left[ T_{ij}^{\sigma\sigma'} V_j - T_{ij}^{\sigma'\sigma} V_i \right],$$

with $T_{ij}^{\sigma\sigma'} = \text{Tr}[s_{ij}^{\sigma\sigma'} s_{ij}^{\sigma\sigma'}]$, (1)

where $T_{ij}^{\sigma\sigma'}$ is the transmission probability for an electronic edge mode from contact j to contact i with initial spin $\sigma'$ to final spin $\sigma$, $V_i$ being the voltage bias applied at contact i, while $s_{ij}^{\sigma\sigma'}$ are the elements of the scattering matrix $S$ of the N-terminal sample.
2. Motivation

In quantum diffusive transport regime, localization of electronic states is well known [4, 5], the resistance of a sample increases exponentially with sample length ($l$) for $l > \xi$ ($\xi$ being localization length) [5]. This is known as strong or Anderson localization [12, 13, 20]. On the other hand, when the sample length $l \leq \xi$, the system shows a unique property: the resistance increases from the Ohmic result by universal factor $h/2e^2$. This increase by the universal factor $h/2e^2$ is called as weak localization correction. The QSH edge modes, as shown in figure 1, are immune to backscattering, e.g. if there is disorder in the sample (see, figure 1), edge modes will move around the disorder without their transmission probabilities getting affected due to topological protection. In this work we however predict that, if a contact is disordered, i.e. can reflect edge modes partially then a ‘quantum’ localization correction can arise for edge modes too but only when all contacts are disordered. What happens is backscattering of the electrons within the sample takes place when all contacts are disordered and thus multiple paths are generated from one contact to another. As a result, the transmission probabilities and resistances become dependent on the disordered-ness of contacts [8, 14, 15]. However, it should noted that this quantum localization observed for QSH edge modes is different from the weak localization correction seen in context of quantum diffusive transport. In quantum diffusive transport regime, the weak localization correction is universal ($h/2e^2$), while in our case, the correction due to localization as will be discussed in more detail in sections 3 and 4, depends on the strength of disorder at contacts and on the number of contacts. Further, this quantum localization correction is present only when all contacts are disordered, see figure 2(a). In figure 2(a), $a_i^\sigma$ and $b_i^\sigma$ refer to the incoming and outgoing edge states respectively from sample to contact $i$ with $\sigma$ being the spin index for that edge state. In figure 2(a), we see that a spin up electron in the $a_1^\uparrow$ edge state at contact 1 can either transmit into the sample with probability $T_1$ or reflect back again to contact 1 with $R_1$. After entering the sample, this edge state electron can reach contact 3 via reflection at contact 2 with probability $R_2$ and then transmit to contact 3 with probability $T_3$. Thus the transmission probability for a spin-up edge electron from contact 1 to 3 is $T_1 R_2 T_3$. This is one among the infinite number of paths possible. For example, it can also reach contact 3 by taking second path after reflecting at contacts 3, 4, 1, 2 and then finally transmitting into contact 3 with transmission probability $T_1 T_3 R_2 R_3 R_4$. Thus, summing all paths from contact 1 to 3, we get the net transmission probability for the spin up edge state-$T_{31}^{\uparrow} = T_1 R_2 T_3 / (1 - R_1 R_2 R_3 R_4)$. However, by taking recourse to scattering amplitudes instead of probabilities we get the transmission amplitude from contact 1 to 3 as

$$T_{31}^{\uparrow} = -t_1 t_2 t_3 e^{i(\phi - \phi_3)} / (1 - r_1 r_2 r_3 r_4 e^{i\phi})$$

where $t_i$ and $r_i$ are the scattering amplitudes and the reflection coefficients respectively.
and $r_i$ are the transmission and reflection amplitudes at contact $i$ with $\phi_i$ being the phase acquired by the electron at contact $i$ and $\phi = \sum \phi_i$. This scattering amplitude will lead to the transmission probability from contact 1 to 3 for spin up edge state $T_{13}^{\uparrow\uparrow} = \vert t_{13}^{\uparrow\uparrow} \vert^2 = T_1 R_2 T_3 / (1 + R_1 R_2 R_3 - 2 \sqrt{R_1 R_3} R_i \cos(\phi)$. This is different to what was derived earlier for $T_{31}^{\uparrow\uparrow}$. Similarly, rest of the transmission probabilities can be calculated by considering transmission probabilities or via following scattering amplitudes, and these too will be different for each case. Thus, when an infinite number of paths exist from one contact to another then a difference between the average resistances derived from scattering amplitudes $\langle R_X^{\text{Amp}} \rangle$ (wherein $X = H, L, 2T, NL$ denotes hall, longitudinal, two-terminal and non-local) and resistances derived from probabilities $R_X$, i.e. $\langle R_X^{\text{Amp}} \rangle \neq R_X$ is seen. This situation changes, if however at least one of the contacts is not disordered, see figure 2(b) (wherein contact 4 is not disordered), in this case there are a finite number of paths from one contact to another. This can be seen as follows: in figure 2(b), a spin up edge state from contact 1 can reach contact 3 by following only one path via reflection at contact 2 with probability $T_{31}^{\uparrow\uparrow} = T_1 R_2 T_3$. There is no second path to reach contact 3, since contact 4 is not disordered, this edge state can not reflect from contact 4. Further, the scattering amplitude from contact 1 to 3 is $t_{13}^{\uparrow\uparrow} = - t_1 r_2 t_3 e^{i(\phi - \phi_a)}$, which gives the transmission probability $T_{31}^{\uparrow\uparrow} = \vert t_{13}^{\uparrow\uparrow} \vert^2 = T_1 R_2 T_3$. Thus the calculation using scattering probabilities and that with scattering amplitudes yield identical results for the case when less than $N$ contacts are disordered. This results in $\langle R_X^{\text{Amp}} \rangle = R_X$ for the case when less than $N$ contacts are disordered and thus quantum localization correction vanishes. Similar, to what is described here for QSH system, was also shown recently for quantum Hall (QH) system in [9]. This is the main motivation of our work, can we see a similar quantum localization correction for QSH samples? Since QSH edge modes are helical (spin polarized) rather than chiral (spin unpolarized) as in QH sample, it will be interesting to see the effect of spin polarized and helical edge modes on the quantum localization correction. Further, to compare the characteristics of this quantum localization correction in various resistances for both QH and QSH systems is another motivation of this paper. We elaborate on this in sections 3–5 for four, six and $N$-terminal QSH samples respectively. The topic of research undertaken in this paper is both timely as well as novel. Since understanding why in QSH experiments the robust quantized conductance is absent is a hotly debated topic of research. Reasons for the less than robust quantization of spin Hall conduction have ranged from magnetic impurities to inelastic scattering as well as to hyperfine interaction which will break time reversal symmetry and therefore induce backscattering of edge modes [10]. In this manuscript, we show that even when there is no inelastic scattering or in absence of magnetic impurities or even for no hyperfine interaction [11] there still can be loss in quantized conductance which we call a quantum localization correction due to disordered contacts alone. Further all the proposals to explain the loss of quantization of helical conduction in QSH samples rely on some kind of inelastic scattering which is dealt with via many body interactions. Our paper is novel in that we via a single particle theory explain the loss of quantized conductance which we dub the quantum localization correction to helical edge transport.

The organization of this paper is as follows: in section 3, we deal with a 4-terminal QSH sample with all disordered contacts and derive an expression for the quantum localization correction, while in sections 4 and 5 we discuss the six and $N$-terminal QSH samples. Next in section 6, we study the impact of inelastic scattering on this quantum localization correction. We conclude with a table summarizing the main results of our paper and compare it with results derived in [9].

3. Four terminal system with all disordered contacts

A 4-terminal QSH sample is shown in figure 3(a) with all disordered contacts. The strength of disorder at contact $i$ is defined by $D_i$ and it is related to the reflection ($R_i$) and transmission probabilities ($T_i$) of an edge state at contact $i$ by the relation $D_i = R_i = 1 - T_i$. Contacts 1, 3 are current probes while contacts 2, 4 are voltage probes, such that $I_2 = I_4 = 0$. For calculating the current at each of these contacts, we need to derive the edge state transmission probability $T_{ij}^{\sigma\sigma}$ between these contacts. Since all contacts are disordered, we need to consider the scattering amplitudes to calculate the transmission probabilities $T_{ij}^{\sigma\sigma}$, from equation (1). First we write down the scattering matrix $S_j$ at each contact $j$ separately relating incoming edge modes $(a_j^{\uparrow\sigma}, a_j^{\downarrow\sigma})$ to outgoing edge modes $(b_j^{\uparrow\sigma}, b_j^{\downarrow\sigma}, b_j^{\nu\sigma}, b_j^{\nu\sigma})$ at that particular contact $j$ and then deduce the full scattering matrix $S$ of the system out of the contact scattering matrices $S_j$, see [18]. The scattering matrix $S_j$ is defined as follows

$$S_j = \begin{pmatrix} r_j e^{i\phi_j^{\uparrow\sigma}} & 0 & t_j e^{i\phi_j^{\downarrow\sigma}} & 0 \\ 0 & r_j e^{i\phi_j^{\uparrow\sigma}} & 0 & t_j e^{i\phi_j^{\downarrow\sigma}} \\ t_j e^{i\phi_j^{\downarrow\sigma}} & 0 & r_j e^{i\phi_j^{\uparrow\sigma}} & 0 \\ 0 & t_j e^{i\phi_j^{\downarrow\sigma}} & 0 & r_j e^{i\phi_j^{\uparrow\sigma}} \end{pmatrix}.$$ (2)

where $r_j$ and $t_j$ are the reflection and transmission amplitudes respectively at contact $j$. $\phi_j^{\sigma\sigma}$ and $\phi_j^{\nu\sigma}$ are the reflection and transmission phase acquired by the spin $\sigma = \uparrow / \downarrow$ edge electron via scattering at the disordered contact $j$. Unitarity of the scattering matrix $S_j$ dictates $S_j^{\dagger} S_j = S_j S_j^{\dagger} = I$, which implies $\phi_j^{\sigma\sigma} = \phi_j^{\nu\sigma} - \frac{\pi}{2} = \phi_j$ (dropping the spin index $\sigma$ from the phase as dissorter is spin independent). Thus the scattering matrix $S_j$ reduces to
Each element of the full scattering matrix $S$ can be calculated from these $S_j$ matrices in the following manner: an electron in $a_1$ edge state can scatter into $b_1$ edge state directly with amplitude $r_1 e^{i\phi_1}$, but then, it can also follow a different path via scattering at contacts 2, 3, 4 and reach $b_1$ edge state with amplitude: $i/2 r_1 r_2 r_3 r_4 e^{i\phi_1} (1/2)$ and so on. Summing over all these paths we get the $S_j$th element of full scattering matrix $S$ of the system, which is: $\sum_j S_j = (r_1 - r_2 r_3 r_4 e^{i\phi}) e^{i\phi_1}/(1 - r_1 r_2 r_3 r_4 e^{i\phi})$, with $\phi = \phi_1 + \phi_2 + \phi_3 + \phi_4$. Similarly, rest of the elements of the $S$ matrix can be derived. The scattering matrix for 4-terminal QSH sample in figure 3(a) is thus:
where $a = 1 - r_1 r_2 r_3 r_4 e^{i\phi}$. This full scattering matrix $S$ relates the incoming edge modes to the outgoing edge modes (see, figure 3(a)) of the system via the relation $(b_1', b_2', b_3', b_4')^T = S(a_1', a_2', a_3', a_4')^T$. Current conservation is guaranteed by the unitarity of the scattering matrix $S$. The conductance matrix $G$ of the system deduced from the full scattering matrix $S$, following from equation (1), is

$$
G = \frac{e^2}{h} \frac{1}{a'} \begin{pmatrix}
2(1 - R_1 R_2 R_3) & -T_1 T_2 (R_2 R_4 + 1) & -T_1 T_3 (R_2 + R_4) & -T_1 T_4 (1 + R_2 R_4)
-T_1 T_2 (1 + R_1 R_3) & 2(1 - R_1 R_2 R_3) & -T_2 T_3 (R_1 R_4 + 1) & -T_2 T_4 (R_1 + R_3)
-T_1 T_3 (R_2 + R_4) & -T_2 T_3 (1 + R_1 R_4) & 2(1 - R_1 R_2 R_3) & -T_3 T_4 (R_1 R_2 + 1)
-T_1 T_4 (1 + R_3 R_2) & -T_2 T_4 (R_1 + R_3) & -T_3 T_4 (1 + R_1 R_2) & 2(1 - R_1 R_2 R_3) & T_4
\end{pmatrix},
$$

where $a' = (1 + R_1 R_2 R_3 R_4 - 2 \sqrt{R_1 R_2 R_3 R_4} \cos \phi)$. Conductance matrix $G$ connects currents and voltages at each contact via the relation $(I_1, I_2, I_3, I_4)^T = G(V_1, V_2, V_3, V_4)^T$. Since currents through voltage probes 2 and 4 are zero, so $I_2 = I_4 = 0$, and choosing reference potential $V_3 = 0$ we get voltages $V_2$ and $V_4$ in terms of $V_1$. Hall resistance $R^\text{Hall}_{12} = R_{13,24} = \frac{(V_2 - V_4)}{I_1}$, 2-terminal resistance is $R^\text{2T}_{12} = R_{13,13} = \frac{(V_1 - V_3)}{I_1}$, and non-local resistance is $R^\text{NL}_{13,43} = \frac{(V_4 - V_3)}{I_1}$ (to calculate the non-local resistance contacts 1, 2 are used as current probes while contacts 3, 4 as voltage probes, see [19]). Here, we consider $D_1 = D_2 = D_3 = D_4 = D_1$ for Hall resistance since for equally disordered contacts it vanishes. For 2-terminal and non-local, we consider $D_2 = D_3 = D$. Thus,

$$
R^\text{Hall}_{12} = \frac{h}{2e^2} (D_1 - D_2)^2 (1 + D_1 D_2),
$$

$$
R^\text{2T}_{12} = \frac{h}{2e^2} (1 + D_1 D_2)^2,
$$

and $R^\text{NL}_{13,43} = \frac{h}{2e^2} (1 + D_1 D_2)^2 (1 + D_1 D_2)$. (6)

The mean Hall, 2-terminal and non-local resistances obtained by averaging over the phase 'φ' acquired by the electronic edge modes due to multiple scattering at disordered contacts is thus-

$$
(R^\text{Hall}_{12})^\text{amp} = \frac{h}{2e^2} (D_1 - D_2)^2 (1 + D_1^2 D_2^2),
$$

$$
(R^\text{2T}_{12})^\text{amp} = \frac{h}{2e^2} (1 + D_1^2),
$$

and $R^\text{NL}_{13,43} = \frac{h}{2e^2} (1 + D_1^2) (1 + D_1^2)$. (7)

One observes that the mean Hall, 2-terminal and non-local resistances lose their quantization due to disordered contacts. To calculate the quantum localization correction, we need to calculate the resistances using probabilities ignoring the phases acquired by edge modes at disordered contacts. The conductance matrix $G$ is then

$$
G = \frac{e^2}{h} \frac{1}{a''} \begin{pmatrix}
(1 - R_1 R_2 R_3) & -T_1 T_2 R_2 R_4 & -T_1 T_3 R_2 + R_4 & -T_1 T_4 (1 + R_2 R_4)
-T_1 T_2 (1 + R_1 R_3) & 2(1 - R_1 R_2 R_3) & -T_2 T_3 R_1 R_4 + 1 & -T_2 T_4 (R_1 + R_3)
-T_1 T_3 R_2 + R_4 & -T_2 T_3 (1 + R_1 R_4) & 2(1 - R_1 R_2 R_3) & -T_3 T_4 (R_1 R_2 + 1)
-T_1 T_4 (1 + R_3 R_2) & -T_2 T_4 (R_1 + R_3) & -T_3 T_4 (1 + R_1 R_2) & 2(1 - R_1 R_2 R_3) & T_4
\end{pmatrix},
$$

where $a'' = (1 - R_1 R_2 R_3 R_4)$. As before, the current through voltage probes 2, 4 is zero, and the reference potential $V_3 = 0$. Thus, the potentials $V_2$ and $V_4$ are derived in terms of $V_1$. The Hall resistance $R^\text{Hall}_{12}$, 2-terminal resistance $R^\text{2T}_{12}$, and nonlocal resistance $R^\text{NL}_{13,43}$ calculated via probabilities are then-
\[
R_H = \frac{h}{e^2} \frac{(D_a - D_l)^2}{4(1 + D_l)(1 + D_a)(1 - D_l D_a)}, \quad R_{2T} = \frac{h}{e^2} \frac{1 + D_l^2}{1 + D_l},
\]
and \[R_{NL} = \frac{h}{e^2} \frac{1 - D_l}{1 + D_l}.
\]

The quantum localization correction is defined as the difference in the resistances calculated from amplitudes and that from probabilities, is then \[R_X^Q = (R_X^{\text{amp}}) - R_X\], with \(X = H, 2T, NL\).

\[
R_H^Q = \frac{h}{2e^2} \frac{D_l^2 D_m^2 (D_l - D_m)^2}{2(1 + D_l)(1 + D_m)(1 - D_l D_m)}, \quad R_{2T}^Q = \frac{h}{2e^2} \frac{2D^4}{(1 - D^2)^2}, \quad R_{NL}^Q = \frac{h}{2e^2} \frac{2(1 + D)^2(1 + D^2).}{(1 + D^2).}
\]

It should be noted here, if \(D_a = D_l\), then the quantum localization correction in case of Hall resistances \(R_H^Q\) vanishes for equally disordered contacts. However, quantum localization correction does not vanish for 2-terminal and non-local resistances for equally disordered contacts. For 2-terminal and non-local resistances the quantum localization correction increases as disorder increases. Here, we also see from equation (10) that if at least one of the contacts is not disordered, i.e. \(D_i = 0\) for either contacts \(i = 1, 2, 3, 4\) then quantum localization correction vanishes for 2-terminal and non-local resistances too (the factor \(D^4\) in the numerator in the expression of 2-terminal and non-local resistances of equation (10) comes from the product of \(D_1, D_2, D_3 \) and \(D_4\), i.e. \(D^4 = D_1 D_2 D_3 D_4\), when all contacts are equally disordered). Thus the 2-terminal and non-local resistances calculated via probabilities and via amplitudes are identical for the case when one contact is not disordered. This condition holds for any number of contacts as shown in the following sections.

4. Six terminal QSH system with all disordered contacts

Figure 3(b) shows a 6-terminal QSH sample with all disordered contacts. Contacts 1, 4 are used as current probes while 2, 3, 5, 6 are used as voltage probes, such that current through these contacts is zero, i.e. \(I_2 = I_3 = I_5 = I_6 = 0\). The scattering matrix of the system shown in figure 3(b) is

\[
S = \frac{1}{b} \begin{pmatrix}
-\rho_{1a} e^{\phi_1} & 0 & -\rho_{1b} e^{\phi_1} & 0 & -\rho_{1c} e^{\phi_1} & 0 & -\rho_{1d} e^{\phi_1} & 0 & -\rho_{1e} e^{\phi_1} & 0 & -\rho_{1f} e^{\phi_1} & 0 \\
0 & -\rho_{2a} e^{\phi_2} & 0 & -\rho_{2b} e^{\phi_2} & 0 & -\rho_{2c} e^{\phi_2} & 0 & -\rho_{2d} e^{\phi_2} & 0 & -\rho_{2e} e^{\phi_2} & 0 & -\rho_{2f} e^{\phi_2} \\
-\rho_{3a} e^{\phi_3} & 0 & -\rho_{3b} e^{\phi_3} & 0 & -\rho_{3c} e^{\phi_3} & 0 & -\rho_{3d} e^{\phi_3} & 0 & -\rho_{3e} e^{\phi_3} & 0 & -\rho_{3f} e^{\phi_3} & 0 \\
0 & -\rho_{4a} e^{\phi_4} & 0 & -\rho_{4b} e^{\phi_4} & 0 & -\rho_{4c} e^{\phi_4} & 0 & -\rho_{4d} e^{\phi_4} & 0 & -\rho_{4e} e^{\phi_4} & 0 & -\rho_{4f} e^{\phi_4} \\
-\rho_{5a} e^{\phi_5} & 0 & -\rho_{5b} e^{\phi_5} & 0 & -\rho_{5c} e^{\phi_5} & 0 & -\rho_{5d} e^{\phi_5} & 0 & -\rho_{5e} e^{\phi_5} & 0 & -\rho_{5f} e^{\phi_5} & 0 \\
0 & -\rho_{6a} e^{\phi_6} & 0 & -\rho_{6b} e^{\phi_6} & 0 & -\rho_{6c} e^{\phi_6} & 0 & -\rho_{6d} e^{\phi_6} & 0 & -\rho_{6e} e^{\phi_6} & 0 & -\rho_{6f} e^{\phi_6} \\
\end{pmatrix},
\]

where \(b = 1 - \rho e^{i\phi}\) with \(\phi_{ij,m} = \phi_i + \phi_j + ... + \phi_m\). For simplicity, we consider all contacts to be equally disordered. \(r\) and \(t\) denote reflection and transmission amplitudes at contact \(i\). Scattering matrix \(S\) of the 6-terminal QSH sample relates the incoming spin-polarized edge states to the outgoing states (see, figure 3(b)) of the system via the relation:

\[
(b_1', b_2', b_3', b_4', b_5', b_6')^T = S(a_1', a_2', a_3', a_4', a_5', a_6')^T. \quad \text{Conductance matrix } G \text{ of the sample deduced from scattering matrix } S \text{ of equation (11), and using equation (1)}, \text{ is}
\]

\[
G = \frac{2e^2}{h} \begin{pmatrix}
2(1 - R_3^5)T & -T^2 R^2 & -T^2 R & -T^2 R & -T^2 R & -T^2 R \\
-T^2 R & 2(1 - R_3^5)T & -T^2 R & -T^2 R & -T^2 R & -T^2 R \\
-T^2 R & -T^2 R & 2(1 - R_3^5)T & -T^2 R & -T^2 R & -T^2 R \\
-T^2 R & -T^2 R & -T^2 R & 2(1 - R_3^5)T & -T^2 R & -T^2 R \\
-T^2 R & -T^2 R & -T^2 R & -T^2 R & 2(1 - R_3^5)T & -T^2 R \\
-T^2 R & -T^2 R & -T^2 R & -T^2 R & -T^2 R & 2(1 - R_3^5)T \\
\end{pmatrix}.
\]
where \( b' = (1 + R^6 - 2R^3 \cos \phi) \). Since current through voltage probes 2, 3, 5 and 6 is zero, so \( I_2 = I_3 = I_5 = I_6 = 0 \), and choosing reference potential \( V_4 = 0 \) we get potentials \( V_2, V_3, V_5 \) and \( V_6 \) in terms of \( V_1 \). Thus, the Hall resistance \( R_{HL}^{\text{amp}} = R_{14,26} = \frac{(V_2 - V_6)}{h} \), 2-terminal resistance \( R_{2F}^{\text{amp}} = R_{14,14} = \frac{(V_2 - V_4)}{h} \), longitudinal resistance \( R_{L}^{\text{amp}} = R_{12,54} = \frac{(V_2 - V_4)}{h} \) and non-local resistance \( R_{NL}^{\text{amp}} \) (to calculate non-local resistance, as before, contacts 1, 2 are used as current probes while contacts 3, 4, 5, 6 are voltage probes are)

\[
\begin{align*}
R_{HL}^{\text{amp}} &= \frac{e^2}{h} (D_n - D_l) * e, \\
&= 0, \quad \text{(when } D_l = D_n), \\
R_{2F}^{\text{amp}} &= \frac{h}{e^2} (3 - D^2)(1 + D^6 - 2D^3 \cos \phi), \\
&= 0, \quad \text{(when } D_l = D_n), \\
R_{L}^{\text{amp}} &= \frac{h}{e^2} (1 + D^6 - 2D^3 \cos \phi), \\
&= 0, \quad \text{(when } D_l = D_n), \\
R_{NL}^{\text{amp}} &= \frac{h}{e^2} (1 + D^6 - 2D^3 \cos \phi).
\end{align*}
\]

where,

\[
F = \frac{(1 + D_l(2 - D_n) - 2D_n(1 + D_l^2) - 2 \sqrt{D_l^4 D_n^2 \cos \phi})}{6(1 + D_l)(1 + D_n)(1 - D_l D_n)^2(1 + D_l D_n)(1 + D_l D_n)}.
\]

All contacts are considered to be equally disordered, i.e. \( D_i = D \) (for \( i = 1 \)–6). To calculate the Hall resistance only we have considered \( D_1 = D_2 = D_3 = D_n \) and \( D_4 = D_5 = D_6 = D_l \), otherwise for equally disordered contacts the Hall resistance is always zero. After averaging over phase shift \( \phi \) we get,

\[
\begin{align*}
R_{HL}^{\text{amp}} &= \frac{h}{e^2} (D_n - D_l) * e', \\
&= 0, \quad \text{(when } D_l = D_n), \\
R_{2F}^{\text{amp}} &= \frac{h}{e^2} (3 - D^2)(1 + D^6), \\
R_{L}^{\text{amp}} &= \frac{h}{e^2} (1 + D^6), \\
R_{NL}^{\text{amp}} &= \frac{3h}{e^2} (1 + D^6),
\end{align*}
\]

where,

\[
F' = \frac{(1 + D_l(2 - D_n) - 2D_n(1 + D_l^2))^2}{6(1 + D_l)(1 + D_n)(1 - D_l D_n)^2(1 + D_l D_n)(1 + D_l D_n)}.
\]

The quantum localization correction is the difference between the resistances calculated using probabilities, i.e. neglecting the phase acquired by the edge electrons and the resistance determined from scattering amplitudes, equation (13). The conductance matrix \( G \) derived from scattering probabilities is

\[
G = \frac{2e^2}{h b} \begin{pmatrix}
2(1 - R^6) & -T^2 R & -T^2 R & -T^2 R & -T^2 R & -T^2 R \\
-T^2 R & 2(1 - R^6) & -T^2 R & -T^2 R & -T^2 R & -T^2 R \\
-T^2 R & -T^2 R & 2(1 - R^6) & -T^2 R & -T^2 R & -T^2 R \\
-T^2 R & -T^2 R & -T^2 R & 2(1 - R^6) & -T^2 R & -T^2 R \\
-T^2 R & -T^2 R & -T^2 R & -T^2 R & 2(1 - R^6) & -T^2 R \\
-T^2 R & -T^2 R & -T^2 R & -T^2 R & -T^2 R & 2(1 - R^6)
\end{pmatrix},
\]

where \( b' = (1 - R^6) \). As before, current through voltage probes 2, 3, 5, 6 is zero, and choosing reference potential \( V_4 = 0 \) we get potentials \( V_2 \) and \( V_4 \) in terms of \( V_1 \). Thus, Hall resistance \( R_{HL} \), 2-terminal resistance \( R_{2F} \), longitudinal resistance \( R_L \), and non-local resistance \( R_{NL} \) calculated via probabilities are then

\[
\begin{align*}
R_{HL} &= \frac{(D_n - D_l)(1 + 2D_n - D_l(2 + D_n))}{6(1 + D_l)(1 + D_n)(1 - D_l D_n)}, \\
&= 0, \quad \text{when } D_l = D_n, \\
R_{2F} &= \frac{h}{2e^2} (3 - D(2 - 3D)), \\
and \quad R_L &= \frac{3h}{2e^2} (1 - D) \\
\end{align*}
\]

The quantum localization corrections to the above calculated Hall, longitudinal, 2-terminal and non-local resistances in the 6-terminal QSH sample thus are \( R_{HL}^{\text{QL}} = (R_{HL}^{\text{amp}}) - R_X \), with \( X = H, 2T, NL, L \).

\[
\begin{align*}
R_{HL}^{\text{QL}} &= \frac{(D_n - D_l)(1 - D + D_n - 2D_l^3 D_n^3 - 2D_l^3 D_n^3)}{3(1 + D_l)(1 + D_n)(1 - D_l D_n)^2(1 + D_l D_n)(1 + D_l D_n)}, \\
&= 0, \quad \text{when } D_l = D_n, \\
R_{2F}^{\text{QL}} &= \frac{h}{2e^2} (3 - 2D(2 - 3D)), \\
and \quad R_L^{\text{QL}} &= \frac{3h}{2e^2} (1 - D)^2(1 + D^2 + D^3).
\end{align*}
\]

From equation (17) we see that the quantum localization correction for Hall resistance in a six terminal QSH sample can be positive as well as negative depending on the strength of disorder at different contacts while for four terminal QSH sample it is always positive, see equation (10). This negative correction term does not imply anti-localization of the helical electrons, rather it comes from the fact that the Hall resistance for QSH sample itself can be negative. However, the absolute value of resistances calculated via amplitudes is always greater than the absolute value of the resistances derived via probabilities, i.e. \( |(R_{HL}^{\text{amp}})| > |R_{HL}| \). This negative quantum localization correction for Hall resistance is unique to QSH samples only and not present for QH samples, see [9]. From equation (17) it can also be noted that for equally disordered contacts the quantum localization correction for Hall resistance vanishes for QSH samples while for QH samples it is finite, see [9]. The quantum localization correction to the 2-terminal, longitudinal and non-local resistances increases with increasing disorder while the same for Hall resistance increases with the increase of the difference between the disorder strength of upper (\( D_n \)) and lower (\( D_l \)) contacts.
calculated (a) via scattering amplitudes and (b) via probabilities for an N terminal QSH sample with lower edge contacts disordered with strength $D_1$ and upper edge contacts disordered with strength $D_n$. (c) the quantum localization correction to the longitudinal resistance.

$D_{	ext{u}}=0.5, D_{	ext{D}}=0.2, D_{	ext{L}}=0.5$
$D_{	ext{u}}=0.2, D_{	ext{D}}=0.2, D_{	ext{L}}=0.2$
$D_{	ext{u}}=0.1, D_{	ext{D}}=0.1$

$\text{Figure 5.}$ Longitudinal resistance in units of $\frac{e^2}{h}$ calculated (a) via scattering amplitudes, and (b) via probabilities for an N-terminal QSH sample with all contacts equally disordered, and (c) the quantum localization correction to the longitudinal resistance.

5. N terminal system with all contacts disordered

An N-terminal QSH sample is shown in figure 3(c) with all contacts equally disordered, i.e. $D_1 = D_2 = ... = D_N = D$. Contacts 1 and $k$ are current probes and contacts $2, 3, ... k-1, k+1, ... N$ are voltage probes, thus current through these contacts, i.e. $I_2 = I_3 = ... = I_{k-1} = I_{k+1} = ... = I_{N} = 0$. The scattering matrix for the N-terminal QSH sample in figure 3(c) is

$$S = \frac{1}{c} \begin{pmatrix} (r - p^{N-1}) e^{\phi_{b_i}} & 0 & -2 p^{N-k} e^{\phi_{b_i}} & 0 & ... & 0 & 0 \\ 0 & (r - p^{N-1}) e^{\phi_{a_i}} & 0 & ... & -2 p^{N-k} e^{\phi_{a_i}} & ... & 0 \\ -2 p^{N-k} e^{\phi_{b_i}} & 0 & (r - p^{N-1}) e^{\phi_{a_i}} & 0 & ... & 0 & 0 \\ 0 & -2 p^{N-k} e^{\phi_{b_i}} & 0 & (r - p^{N-1}) e^{\phi_{a_i}} & 0 & ... & 0 \\ ... & ... & ... & ... & ... & ... & ... \\ -2 p^{N-k} e^{\phi_{b_i}} & 0 & ... & ... & 0 & (r - p^{N-1}) e^{\phi_{a_i}} & 0 \\ 0 & -2 p^{N-k} e^{\phi_{b_i}} & 0 & ... & ... & 0 & (r - p^{N-1}) e^{\phi_{a_i}} \end{pmatrix}$$

(18)

where $c = 1 - r^{N} e^{\phi}$ and $\phi_{b_{i,k}} = \phi_{b_i} + \phi_{b_j} + ... + \phi_{b_k}$. The scattering matrix connects the incoming edge states to the outgoing edge states via the relation $(b_{i1}, b_{i2}, ..., b_{in}, b_{0}, b_{+}) = S(a_{i1}, a_{i2}, ..., a_{in}, a_{0}, a_{-})$. The conductance matrix $G$ of the N-terminal QSH sample derived from the scattering matrix $S$, following equation (1), is thus-

$$G = \frac{1}{c^2} \begin{pmatrix} 2T(1 - R^{N-1}) & ... & -T^2(R^{N-k} + R^{-k}) & ... & -T^2(1 + R^{N-2}) \\ ... & ... & ... & ... & ... \\ -T^2(R^{N-k} + R^{-k}) & ... & 2T(1 - R^{N-1}) & ... & -T^2(R^{k-1} + R^{N-k-1}) \\ ... & ... & ... & ... & ... \\ -T^2(R^{N-2} + 1) & ... & -T^2(R^{N-k-1} + R^{-k-1}) & ... & 2T(1 - R^{N-1}) \end{pmatrix}.$$  

(19)
where \( c' = 1 + R^N - 2R^{N/2}\cos \phi \). Since currents through voltage probes 2, 3, ..., \( k-1, k+1, ..., N \) is zero, so \( I_2 = I_3 = ... = I_{k-1} = I_{k+1} = I_N = 0 \), and choosing reference potential \( V_i = 0 \) we get potentials \( V_2, V_3, V_{k-1}, V_{k+1} \) and \( V_N \) in terms of \( V_1 \). So, Hall resistance \( R_{hh}^{amp} = R_{1k,2N} = \frac{(V_2 - V_N)}{I} \), 2-terminal resistance \( R_{2T}^{amp} = R_{1k,1k} = \frac{(V_2 - V_3)}{I} \), longitudinal resistance \( R_{ampL}^{L} = R_{12,23} = \frac{(V_2 - V_3)}{I} \) and non-local resistance \( R_{NL}^{amp} = R_{12,3,k+1} = \frac{(V_2 - V_3)}{I} \). To calculate non-local resistance we consider contacts 1, 2 as current probes and contacts 3, 4, ..., \( k-1, k, k+1, ..., N \) as voltage probe. As the expressions for these resistances are large, we analyze them via plots, see figures 4–7. The average resistances for \( N \)-terminal case are found by averaging over the phases. Thus \( \langle R_{amp} \rangle = \frac{1}{2\pi} \int_{0}^{2\pi} R_{1k,1k} \, d\phi \). To calculate the quantum localization correction, we need to calculate the conductance using probabilities ignoring the phase acquired by the edge electrons. The conductance matrix \( G \) derived via transmission probabilities is then

![Figure 6](image1.png)

**Figure 6.** 2-terminal resistance in units of \( \frac{h}{e^2} \) calculated (a) via scattering amplitudes, (b) via probabilities for a \( N \)-terminal QSH sample with all contacts equally disordered, and (c) the quantum localization correction to the 2T resistance.

![Figure 7](image2.png)

**Figure 7.** Non-local resistance in units of \( \frac{h}{e^2} \) calculated (a) via scattering amplitudes, (b) via probabilities for a \( N \)-terminal QSH sample with all contacts equally disordered, and (c) the quantum localization correction to the non-local resistance.

![Figure 8](image3.png)

**Figure 8.** 6 terminal QSH bar with all disordered contacts and inelastic scattering.

| Table 1. Comparison of the quantum localization correction in 6-terminal QH [9] and QSH sample with equally disordered contacts. |
|---|---|---|
| Quantum Hall | Quantum spin Hall |
| \( R_{QH}^{QL} \) | \( \frac{h}{2\pi} \frac{2D}{1-D} \) | 0 |
| \( R_{QH}^{NL} \) | 0 | \( \frac{h}{2\pi} \frac{2D}{(1+D)(1+D^3+D^2)} \) |
| \( R_{QH}^{LT} \) | \( \frac{h}{2\pi} \frac{2D/(1+D)}{(1-D)(1+D^3+D^2)} \) | \( \frac{h}{2\pi} \frac{D(3-2D+3D^2)}{(1-D^2)(1+D^2+D^2)} \) |
| \( R_{NL}^{NL} \) | 0 | \( \frac{h}{2\pi} \frac{D}{(1+D)(1+D^3+D^2)} \) |

Figure 6. 2-terminal resistance in units of \( \frac{h}{e^2} \) calculated (a) via scattering amplitudes, (b) via probabilities for a \( N \)-terminal QSH sample with all contacts equally disordered, and (c) the quantum localization correction to the 2T resistance.
where \( c'' = (1 - R_N) \). Setting the current, as before, through voltage probes \( 2, 3, ..., k - 1, k + 1, ..., N \) to zero, and choosing reference potential \( V_k = 0 \) we get potentials \( V_1, V_2, V_{k+1} \) in terms of \( V_1 \). Similarly, we need to calculate the Hall resistance \( R_H \), 2-terminal resistance \( R_{2T} \), and nonlocal resistance \( R_{NL} \) via probabilities from the conductance matrix as in equation (20). As these expressions are large, we analyze them in figures 4–7. The quantum localization correction, as defined before, is \( R^{QL}_X = (R^{\text{Amp}}_X) - R_X \) with \( X = H, 2T, L, NL \). One can get a closed form expression for a general \( N \) (with \( N = \text{even} \))-terminal system as well by looking at the 6, 8, 10... terminal resistances. This is written below for the quantum localization correction, resistance derived via probabilities and that derived from amplitudes in case of longitudinal and non-local resistances:

\[
G = \frac{1}{e^2} \begin{pmatrix}
2T(1 - R_N^{-1}) & -T^2(R_N^{-k} + R^{k-2}) & \cdots & -T^2(1 + R_N^{-k-2}) \\
\vdots & \vdots & \ddots & \vdots \\
-T^2(R^{k-2} + R_N^{-k}) & 2T(1 - R_N^{-1}) & \cdots & -T^2(1 + R_N^{-k-1}) \\
\vdots & \vdots & \ddots & \vdots \\
-T^2(R_N^{-2} + 1) & -T^2(R_N^{-k-1} + R^{k-1}) & \cdots & 2T(1 - R_N^{-1})
\end{pmatrix},
\]

(20)

In figures 6(a) and (b) we see that the 2-terminal resistance for QSH case increases with number of contacts (unlike the QH case), which implies the 2-terminal resistance increases as a function of the length of the sample. This is similar to what is observed for Ohmic behavior. In figure 6(c) we see that the quantum localization correction is very small for \( D < 1/2 \), only for \( D > 1/2 \) it becomes substantial. In figures 4–7 we see that for large number of terminals the quantum localization correction disappears. Quantum localization correction is substantial only for strong disorder and few terminals.

6. Effect of inelastic scattering on quantum localization correction

A 6-terminal QSH sample with all disordered contacts and with inelastic scattering is shown in figure 8. When the length between the disordered contacts is larger than the phase coherence length for electronic edge modes, inelastic scattering occurs. In presence of inelastic scattering spin up edge electrons coming out of contact 1 equilibrate with other spin up and down electrons at equilibrating potential \( V'_1 \) and lose their phase acquired via scattering at the contacts via equilibration of their energy. Similarly spin down electrons coming out of contact 1 lose their phase at equilibrating potential \( V'_2 \) via equilibration of their energies with other spin up and down electrons. Thus, there is no possibility for an electron in a edge state to get back to the same contact after emerging out of it at that energy and with an unique phase. Thus, there is no difference between resistances calculated via probabilities and that via amplitudes. This implies absence of quantum localization correction in presence of inelastic scattering. Using probabilities the resistances have already been derived, see [16, 17], as:

\[
R_H = 0, \quad R_{2T} = \frac{h}{e^2} \frac{(3 - D)}{(1 - D^2)} \quad R_L = 3R_{NL} = \frac{h}{e^2} \frac{1}{(1 + D)},
\]

(22)

with \( R_X = \langle R^{\text{Amp}}_X \rangle \), \( X = H, 2T, L, NL \). Here, we have only concentrated on the six terminal QSH system, as in 4- and \( N \)-terminal QSH sample we obtain exactly similar results wherein inelastic scattering completely kills the quantum localization correction.

7. Conclusion

We see that resistances are affected by the quantum localization correction but only when all contacts are disordered. The
quantum localization correction for the resistances for both QH (see [9]) and QSH six terminal samples are summarized and compared in table 1. From table 1, we see that for equally disordered contacts in QH sample only 2-terminal and Hall resistances are affected by the quantum localization correction, while in QSH sample the 2-terminal, longitudinal and non-local resistances are affected by the same correction. Quantum localization correction term arises in a QSH or QH sample due to multiple paths available edge mode electrons due to the fact that all contacts are disordered as explained in section 2. However, summing the multiple paths available for helical edge modes in QSH samples and chiral edge modes in QH sample leads to a difference in the quantum localization correction. A remark on the table- the vanishing quantum localization correction does not mean \( \langle R_{Amp}^{LNL} \rangle = \langle R_{LNL} \rangle \) for a QH sample or \( \langle R_{Amp}^{H} \rangle = \langle R_{H} \rangle \) for a QSH sample but rather because \( \langle R_{Amp}^{LNL} \rangle = \langle R_{LNL} \rangle = 0 \) for QH sample and same for Hall resistance in QSH sample. This suggests that the quantum localization correction term is finite only when resistances calculated via scattering amplitudes or probabilities are themselves finite.

In QSH samples we even see a negative localization correction, which is not due to the anti localization of the states, but rather due to the fact that the Hall resistance in a QSH system can itself turn negative. In presence of inelastic scattering this quantum localization term vanishes for both QH and QSH cases. In this letter, we have assumed disorder only at the contacts, there is no disorder within the sample. Generally, edge modes in QH/QSH samples suffer some amount of scattering at contacts. The presence of disorder within the sample wont affect the results of our letter, since it is well known that QH and QSH edge modes are robust to sample disorder. Disorder at contacts works as a barrier to edge mode transport, edge modes can partially transmit into the contacts through the barrier with probability \( T \) or can be partially reflected with probability \( R \). In case one has completely clean contacts, one can design sample contacts to partially reflect edge modes at contacts by directly doping non-magnetic impurities or via creating an electrostatic barrier at the contacts. In [21, 22], the authors have studied sample disorder in QH systems via doping impurities within the sample. Similarly, impurities can be doped into contacts in a QSH sample thus realizing our setups and verifying the quantum localization correction.

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