The star trellis decoding of Reed-Solomon codes

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Abstract

The new method for Reed-Solomon codes decoding is introduced. The method is based on the star trellis decoding of the binary image of Reed-Solomon codes.

1 The Golay code in star representation

For the Golay code $G_{Golay}$ with codewlength $n = 24$ the star trellis was proposed in [1].

The time axis consists of a number of parts $I = \bigcup_j I_j$, which are joint in one common point. In Figure 1 an example factor graph representation is depicted. This example has three equal length parts. The time axis of the parts are given by $I_j = \{0, (j - 1)n/3 + 1, \ldots, jn/3, \infty\}$, where $\infty$ denotes the junction. This junction in Figure 1 is represented by a square and the state space is denoted $S_i$.

Each part of the time axis is associated with the conventional trellis shortening with a single starting state and some end states. The star trellis consists of a union of all conventional trellis shortening in the junction $\infty$. The star trellis of the Golay code is given in Figure 2.
Figure 1: Factor Graph Representation of the Star Trellis.

Figure 2: The Star Trellis of the Golay Code.

$G_{Golay}$ may be represented by the following generator matrix:

$$G_{Golay} = \begin{bmatrix}
11110000 & 00000000 & 11110000 \\
01011010 & 00000000 & 01011010 \\
00111100 & 00000000 & 00111100 \\
00000000 & 11110000 & 11110000 \\
00000000 & 01011010 & 01011010 \\
00000000 & 00111100 & 00111100 \\
11111111 & 00000000 & 00000000 \\
00000000 & 11111111 & 00000000 \\
00000000 & 00000000 & 11111111 \\
10011010 & 10011010 & 10011010 \\
11001001 & 11001001 & 11001001 \\
01111000 & 01111000 & 01111000 \\
\end{bmatrix}.$$
This result was obtained by a permutation of the Turyn-construction \[2, 18.7.4\].

Each trellis shortening consists of \(n/3 = 8\) sections and has a single starting state and eight possible end states. The end states correspond to the three last rows of the generator matrix \(G_{\text{Golay}}\). We connect three corresponding end-states in eight special states to obtain a star trellis for the Golay code.

A valid Golay codeword is one-to-one correspondence union of three paths on trellis shortening, which starts in a single starting state and ends in one of the eight possible end states. For all eight possibilities the special linear dependencies need to be satisfied in order to obtain a valid Golay codeword.

## 2 The Vardy–Be’ery decomposition

The map of the Reed-Solomon (RS) code \(\mathcal{RS}\) into its binary image \(\text{Im}(\mathcal{RS})\) was presented in \[3\].

Let us introduce some notations for the RS code. Let \(\mathcal{RS}\) be a \((N, K, D)\) RS code of length \(N = 2^m - 1\), dimension \(K\) and minimum Hamming distance \(D = N - K + 1\) over \(GF(2^m)\). The RS code generator polynomial is \(G(x)\) with roots \(\alpha, \alpha^2, \ldots, \alpha^{D-1}\), where \(\alpha\) is a primitive element of \(GF(2^m)\).

By analogy, let us introduce some notations for the Bose-Chaudhuri-Hocquenghem (BCH) code \(\mathcal{BCH}\) with the same parity-check matrix that the \(\mathcal{RS}\) has. Let \(\mathcal{BCH}\) be a binary \((n, k, d)\) BCH code of length \(n = N\), dimension \(k \leq K\) and minimum Hamming distance \(d \geq D\). The BCH code generator polynomial is \(g(x) \in GF(2)[x]\) with roots \(\alpha, \alpha^2, \ldots, \alpha^{D-1}\) and their cyclotomic conjugates over \(GF(2)\). It is obvious that \(G(x) \mid g(x)\). The \(\mathcal{BCH}\) code has a generator matrix \(G_{\text{BCH}}\).

Let \(\{\gamma_1, \gamma_2, \ldots, \gamma_m\}\) be any basis in \(GF(2^m)\).

For any element \(\alpha^j = \sum_{i=1}^{m} a_i \gamma_i \in GF(2^m)\) let us introduce its binary image \(\text{Im}(\alpha^j) = (a_1, a_2, \ldots, a_m)\); and the binary image of 0 is \(\text{Im}(0) = (0, 0, \ldots, 0)\).

Without loss of generality, we shall use a standard basis \(\{\alpha^0, \alpha^1, \alpha^2, \ldots, \alpha^{m-1}\}\).

For any codeword of the \(\mathcal{RS}\) code we have
\[
\gamma_i c = (\gamma_i c_0, \gamma_i c_1, \ldots, \gamma_i c_{n-1}) \in \mathcal{RS}, \ i \in [1, m],
\]

\[
\begin{bmatrix}
\gamma_1 c_0 & \gamma_1 c_1 & \ldots & \gamma_1 c_{n-1} \\
\gamma_2 c_0 & \gamma_2 c_1 & \ldots & \gamma_2 c_{n-1} \\
\vdots & \vdots & \ddots & \vdots \\
\gamma_m c_0 & \gamma_m c_1 & \ldots & \gamma_m c_{n-1}
\end{bmatrix}
\in \mathcal{RS},
\]
\[
\begin{bmatrix}
\text{Im}(\gamma_1 c_0) & \text{Im}(\gamma_1 c_1) & \ldots & \text{Im}(\gamma_1 c_{n-1}) \\
\text{Im}(\gamma_2 c_0) & \text{Im}(\gamma_2 c_1) & \ldots & \text{Im}(\gamma_2 c_{n-1}) \\
\vdots & \vdots & \ddots & \vdots \\
\text{Im}(\gamma_m c_0) & \text{Im}(\gamma_m c_1) & \ldots & \text{Im}(\gamma_m c_{n-1})
\end{bmatrix} \in \text{Im}(\mathcal{RS}).
\]

Any codeword of the \(\mathcal{BCH}\) code is also a codeword of the \(\mathcal{RS}\) code. We have \(b = (b_0, b_1, \ldots, b_{n-1}) \in \mathcal{BCH}\), \(b_i \in \text{GF}(2)\), \(b = (b_0, b_1, \ldots, b_{n-1}) \in \mathcal{RS}\), \(\gamma_i b = (\gamma_i b_0, \gamma_i b_1, \ldots, \gamma_i b_{n-1}) \in \mathcal{RS}\), \(i \in [1, m]\).

We use the standard basis \(\gamma_i = \alpha^{i-1}\), \(i \in [1, m]\), and obtain

\[
\text{Im}(\gamma_{i+1} b_j) = \text{Im}(\alpha^i b_j) = (0 \ldots 1 \ i \ i+1 \ldots m-1),
\]

\[
\gamma_{i+1} b = (\alpha^i b_0, \alpha^i b_1, \ldots, \alpha^i b_{n-1}) \in \mathcal{RS}, \ i \in [0, m-1],
\]

\[
I_b = \begin{bmatrix}
0 b_0 \ldots 0 & b_1 \ldots 0 & \ldots & b_{n-1} \ldots 0 \\
0 b_0 \ldots 0 & 0 b_1 \ldots 0 & \ldots & 0 b_{n-1} \ldots 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 0 \ldots 0 & 0 0 \ldots b_1 & \ldots & 0 0 \ldots b_{n-1}
\end{bmatrix} \in \text{Im}(\mathcal{RS}).
\]

Let us introduce a permutation for the columns of the matrix \(I_b\):

\[
\text{Per}\left((0, 0), (0, 1), \ldots, (0, m-1) \mid (1, 0), (1, 1), \ldots, (1, m-1) \mid \ldots \right)
\]

\[
(n-1, 0), (n-1, 1), \ldots, (n-1, m-1) =
\]

\[
((0, 0), (1, 0), \ldots, (n-1, 0) \mid (0, 1), (1, 1), \ldots, (n-1, 1) \mid \ldots \mid (0, m-1),
\]

\[
(1, m-1), \ldots, (n-1, m-1)\].
\]

Thus,

\[
\text{Per}(I_b) = \begin{bmatrix}
0 b_0 b_1 \ldots b_{n-1} & 0 0 \ldots 0 & \ldots & 0 0 \ldots 0 \\
0 0 \ldots 0 & b_0 b_1 \ldots b_{n-1} & \ldots & 0 0 \ldots 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 0 \ldots 0 & 0 0 \ldots b_1 & \ldots & b_0 b_1 \ldots b_{n-1}
\end{bmatrix} \in \text{Per}(\text{Im}(\mathcal{RS})).
\]

It is correct for any codeword \(b \in \mathcal{BCH}\). Hence the generator matrix of the permutation for the binary image RS code may be represented as

\[
G_{\text{Per}(\text{Im}(\mathcal{RS}))} = \begin{bmatrix}
G_{\mathcal{BCH}} & 0 & \ldots & 0 \\
0 & G_{\mathcal{BCH}} & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & G_{\mathcal{BCH}}
\end{bmatrix},
\]

\(\text{Per}(\text{Im}(\mathcal{RS}))\).

\]
where the submatrix “glue vectors” is a $m(K - k) \times Nm$ matrix.

We consider the RS $(7,5,3)$ code. The permutation of columns of the generator matrix for the binary image RS code is the generator matrix for the binary $(21,15,3)$ code [4]:

$$G_{\text{Per}(\text{Im}(\text{RS}))} = \begin{bmatrix}
1101000 & 0000000 & 0000000 \\
0110100 & 0000000 & 0000000 \\
0011010 & 0000000 & 0000000 \\
0001101 & 0000000 & 0000000 \\
0000000 & 1101000 & 0000000 \\
0000000 & 0110100 & 0000000 \\
0000000 & 0011010 & 0000000 \\
0000000 & 0001101 & 0000000 \\
0000000 & 0000000 & 1101000 \\
0000000 & 0000000 & 0110100 \\
0000000 & 0000000 & 0011010 \\
0000000 & 0000000 & 0001101 \\
1000000 & 0000100 & 0010000 \\
0100000 & 0000010 & 0001000 \\
0010000 & 0000001 & 0000100
\end{bmatrix}.$$

3 Decoding method

The star trellis can be constructed for any Reed-Solomon code. The star trellis consists of $m$ parts. Each part is a conventional trellis shortening with a single starting state and $2^{m(K - k)}$ end states. The end states are defined by “glue vectors” in the generator matrix of the permutation for the binary image RS code.

The decoding method has two stages. The first stage is the soft-decision decoding for $m$ trellis shortening. The result of this stage is a list of code-words. The cardinality of the list is not more than $2^{m(K - k)}$. On the second stage the nearest codeword form the list to the received vector is chosen.

The simulation for the RS $(7,5,3)$ code is executed. Bit Error Rate (BER) and Codeword Error Rate (CER) performance dependence on signal-to-noise ratio (SNR) for additive white Gaussian noise (AWGN) channel is given in Table 1. By classic decoding method for RS code we understand decoding of both errors and erasures (see, for example, Forney or Chase algorithms [5, 6]) with the Berlekamp-Massey algorithm for the key equation solving.
Table 1

| SNR | BER New method | BER Classic method | CER New method | CER Classic method |
|-----|----------------|-------------------|---------------|------------------|
| 1   | 0.0460         | 0.0640            | 0.20          | 0.45             |
| 2   | 0.0173         | 0.0433            | 0.10          | 0.36             |
| 3   | 0.0153         | 0.0227            | 0.09          | 0.17             |
| 4   | 0.0047         | 0.0080            | 0.03          | 0.07             |
| 5   | 0.0000         | 0.0040            | 0.00          | 0.03             |

Simulation results indicate that the new decoding method can achieve up to 2–3 dB of coding gain on AWGN channel in comparison to classic decoding method.

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