The super local antimagic $\mathcal{H}$-decomposition coloring of graph

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Abstract. Let $G$ be a nontrivial, finite, connected graph with vertex set $V$ and edge set $E$. A bijection $f : V(G) \cup E(G) \rightarrow \{1, 2, 3, ... |V(G)| + |E(G)|\}$ is called a local antimagic $\mathcal{H}$-decomposition labeling for any two adjacent subgraph $\mathcal{H}_1$ and $\mathcal{H}_2$, $w(\mathcal{H}_1) \neq w(\mathcal{H}_2)$, where $w(\mathcal{H}) = \sum_{u \in V(\mathcal{H})} f(u) + \sum_{e \in E(\mathcal{H})} f(e)$. The local antimagic $\mathcal{H}$-decomposition labeling that induces a proper decomposition coloring of $G$ where the decomposition $\mathcal{H}$ is assigned by the color $w(\mathcal{H})$ is called local antimagic $\mathcal{H}$-decomposition coloring. We call labeling of local antimagic $\mathcal{H}$-decomposition coloring is super if we add vertices label start from 1 until the sum of vertices, edges label start from the sum of vertices plus 1 until the sum of vertices and edges. The minimum number of colors in super local antimagic $\mathcal{H}$-decomposition coloring is super local antimagic $\mathcal{H}$-decomposition coloring chromatic number and denoted by $\gamma_{\text{lat}}(G)$. In this paper, we used some special graph such as wheel graph ($W_n$), sun graph $Su_n$, and star graph $S_n$.

1. Introduction

A graph $G$ is nontrivial, finite, and connected graph. A graph $G$ have sets $(V(G), E(G))$ where $V(G)$ is nonempty set and $E(G)$ are possibly empty set. The elements of $V(G)$ and $E(G)$ are vertices and edges of graph $G$. The order of graph $G$ is denoted as $|V(G)|$ and the size of graph $G$ is denoted as $|E(G)|$. We can see [5] [3] for more detail about graphs. A graph coloring is a subclass of graph labeling. Labeling here means giving a certain color to the vertex to some extent. There are three kinds of graph coloring. First, vertex coloring is to give different colors at each vertex adjacent so that no two vertex are adjacent have the same color. Second, edge coloring is to give different colors at each edge adjacent so that no two edges are adjacent have the same color. Third, face coloring is to give different colors at each face adjacent so that no two face are adjacent have the same color. The applications of graph coloring is usually applied to scheduling problems, allocation lists, sudoku games, etc.

Graph labeling is a bijective function that mapping natural number to elements of graph. Based on the domain of bijective function, the labeling is respectively divided into vertex labeling, edge labeling, face labeling, and total labeling. We can find a general survey of graph labeling in [4]. The concept of antimagic labeling was introduced by Hartsfield and Ringel [6]. A covering of $G$ is a family $\mathcal{H} = \{H_1, H_2, H_3, ..., H_k\}$ of subgraphs with the property that each edge of $G$ is contained in at least one graph $H_i$ for some $i \in \{1, 2, 3, ..\}$. Let $H$ be a subgraph of $G$, if every $H_i$ is isomorphic to $H$, such a covering is called an $H$-covering of $G$. In such a case, we say that $G$ admits an $H$-covering. Furthermore, if an $H$-covering of $G$ has a property
that each edge of \( G \) is contained in exactly one graph \( H_i \) for some \( i \in \{1, 2, \ldots, k\} \), \( H \)-covering is called an \( H \)-decomposition. In such a case, \( G \) is said to be \( H \)-decomposable or \( G \) admits an \( H \)-decomposition \[7\].

Graph are colored differently whenever two vertex are adjacent. In a complete \( n \)-coloring, for each pair of different colors, there are two adjacent vertex which are assigned for these two colors. The chromatic number \( \chi(G) \) of a graph \( G \) is the minimum number \( n \) such that \( G \) has an \( n \)-coloring [1] and [2]. Arumugam et al. [2] also introduces the same concept which is the of local antimagic vertex coloring of graph \( G \).

In this paper, the writer introduces a new concept in labeling and coloring study namely super local antimagic \( H \)-decomposition coloring. A bijection \( f : V(G) \cup E(G) \to \{1, 2, 3, \ldots|V(G)| + |E(G)| \} \) is called a local antimagic \( H \)-decomposition labeling for any two adjacent subgraph \( H_1 \) and \( H_2 \), \( w(H_1) \neq w(H_2) \), where \( w(H) = \sum_{u \in V(H)} f(u) + \sum_{e \in E(H)} f(e) \). The local antimagic \( H \)-decomposition labeling that induces a proper decomposition coloring of \( G \) where the decomposition \( H \) is assigned by the color \( w(H) \) is called local antimagic \( H \)-decomposition coloring. We call labeling of local antimagic \( H \)-decomposition coloring is super if we add vertices label start from 1 until the sum of vertices, edges label start from the sum of vertices plus 1 until the sum of vertices and edges. The minimum number of colors in super local antimagic \( H \)-decomposition coloring is super local antimagic \( H \)-decomposition coloring chromatic number and denoted by \( \gamma_{lat}\mathcal{H}(G) \). In this paper, we used some special graph such as wheel graph \( (W_n) \), sun graph \( Su_n \) and star graph \( S_n \).

2. The Results

We present our result super local antimagic \( H \)-decomposition coloring of some special graph. Furthermore, we have found chromatic number of super local antimagic \( H \)-decomposition in wheel graph \((W_n)\), sun graph \( Su_n \) and star graph \( S_n \). 

**Lemma 1** Let \( G \) be a connected graph \( G = \mathcal{H} \) if only if the super local antimagic \( \mathcal{H} \)-decomposition coloring of \( G \) is 1.[8]

**Lemma 2** Let \( G \) and \( \mathcal{H} \) be a connected graph \( \mathcal{H} \subset G \), the super local antimagic \( \mathcal{H} \)-decomposition coloring of \( G \) isat least 2. [8]

**Observation 1** For any \( G \) graph, \( \gamma_{lat}\mathcal{H}(G) \geq \chi_{\mathcal{H}}(G) \), where \( \chi_{\mathcal{H}}(G) \) is the chromatic number of \( \mathcal{H} \)-decomposition coloring of graph \( G \).

2.1. The Chromatic number of super local antimagic \( \mathcal{H} \)-decomposition coloring in wheel graph \((W_n)\)

Let \( n \leq 3 \) be a natural number, wheel graph \((W_n)\) is a graph with \( V(W_n) = \{y\} \cup \{x_i; 1 \leq i \leq n\} \), \( E(W_n) = \{yx_i; 1 \leq i \leq n\} \cup \{x_i x_{i+1}; 1 \leq i \leq n-1\} \cup \{x_i x_n; i = 1\} \) and \( \mathcal{H} \cong P_3 \), then we determine chromatic number of super local \( \mathcal{H} \)-decomposition antimagic coloring in wheel graph as follows.

**Theorem 1** For every natural number \( n \geq 3 \), \( \gamma_{lat}\mathcal{H}(W_n) = n \).

**Proof.** To proof chromatic number of super local antimagic \( \mathcal{H} \)-decomposition coloring of wheel graph, we will use label of elements graph to find chromatic number as \( \gamma_{lat}\mathcal{H}(W_n) \leq n \). We define vertex labeling as a \( f \) bijective function to natural number until the order of \( W_n \) is as follows:

\[
f(y) = 1; \ n \geq 3
\]

for \( n \) is odd.
\[ f(x_i) = \begin{cases} \frac{i+3}{2}, & \text{for } i \text{ is odd} \\ n + \frac{i}{2} + 1, & \text{for } i \text{ is even} \end{cases} \]

for \( n \) is even

\[ f(x_i) = \begin{cases} i + 1, & \text{for } i = 1, 2 \\ n - i + 4, & \text{for } 3 \leq i \leq n \end{cases} \]

We define edge labeling as a bijective function to the order of \( W_n \) plus one until the order and edges of \( W_n \) is as follows:

for \( n \) is odd

\[ g(x_iy) = \begin{cases} n + \frac{i+3}{2}, & \text{for } i \text{ is odd} \\ \frac{3n+i+3}{2}, & \text{for } i \text{ is even} \end{cases} \]

\[ g(x_ix_{i+1}) = \begin{cases} \frac{5n-i+2}{2}, & \text{for } i \text{ is odd} \\ \frac{3n-i-2}{2}, & \text{for } i \text{ is even} \end{cases} \]

for \( n \) is even

\[ g(x_iy) = \begin{cases} n + i + 1, & \text{for } i = 1, 2 \\ 2n - i + 4, & \text{for } 3 \leq i \leq n \end{cases} \]

\[ g(x_ix_{i+1}) = \begin{cases} 3n - i + 2, & \text{for } i = 1, 2 \\ 2n + i - 1, & \text{for } 3 \leq i \leq n \end{cases} \]

\[ g(x_nx_i) = 3n - 1; n \geq 4 \]

it is easy to get a super local antimagic \( \mathcal{H} \)-decomposition coloring of wheel graph and the decomposition weight are follows:

for \( n \) is odd

\[ w(\mathcal{H}_i) = \begin{cases} \frac{9n+2i+15}{2}, & \text{for } i \leq i \leq n - 1 \\ \frac{9n+i+5}{2}, & \text{for } i = n \end{cases} \]

for \( n \) is even

\[ w(\mathcal{H}_i) = \begin{cases} 6n - 2i + 11, & \text{for } 3 \leq i \leq n - 1 \\ 4n + 10, & \text{for } i = n \\ 4n - i + 10, & \text{for } i = 1 \\ 5n + 4i, & \text{for } i = 2 \end{cases} \]

Based on observation 1, we get \( \gamma_{\text{lat}}(W_n) \geq n \) and \( \gamma_{\text{lat}}(W_n) \leq n \) as lower bound and upper bound of super local antimagic \( \mathcal{H} \)-decomposition coloring of wheel graph \( (W_n) \). To prove \( \gamma_{\text{lat}}(W_n) \geq n \), assume it \( \gamma_{\text{lat}}(W_n) = n - 1 \), if \( \gamma_{\text{lat}}(W_n) = n - 1 \) then there is any adjacent subgraph have the same color \( c(\mathcal{H}_i) = c(\mathcal{H}_j) \) for \( 1 \leq i \leq n, 1 \leq j \leq n \) where \( \mathcal{H}_i = yx_ix_{i+1}, 1 \leq i \leq n - 1 \) so that contradiction with observation 1 then \( \gamma_{\text{lat}}(W_n) \geq n \).

We get the chromatic number of super local antimagic \( \mathcal{H} \)-decomposition coloring in wheel graph is \( \gamma_{\text{lat}}(W_n) = n \) \( \square \)
Figure 1. super local antimagic $H$-decomposition coloring in wheel graph $W_{11}$ and $W_{12}$

2.2. The Chromatic number of super local antimagic $H$-decomposition coloring in sun graph $(Sn_m)$

Let $m \leq 3$ be a natural number, sun graph $(Sn_m)$ is a graph with $V(Sn_m) = \{x_i; 1 \leq i \leq m\} \cup \{y_i; 1 \leq i \leq m\}$, $E(Sn_m) = \{y_i x_i; 1 \leq i \leq m\} \cup \{y_i y_{i+1}; 1 \leq i \leq m-1\} \cup \{y_i y_m; i = 1\}$ and $H \cong P_3$, then we determine chromatic number of super local antimagic $H$-decomposition coloring in wheel graph as follows.

**Theorem 2** For every natural number $m \geq 3$

\[ \gamma_{latH}(Sn_m) = \begin{cases} 
2, & \text{for } m \text{ is even} \\
3, & \text{for } m \text{ is odd}
\end{cases} \]

**Proof.** To proof chromatic number of super local antimagic $H$-decomposition coloring of sun graph, we divided the proof into two cases.

**Case 1.** For $m$ is even, we will use label of elements graph to find chromatic number as $\gamma_{latH}(Sn_m) \leq 2$. We define vertex labeling as a $f$ bijective function to natural number until the order of $Sn_m$ is as follows:

\[ f(x_i) = \begin{cases} 
\frac{i+1}{2}, & \text{for } i \text{ is odd} \\
\frac{m+i}{2}, & \text{for } i \text{ is even}
\end{cases} \]

\[ f(y_i) = 2n - i + 1, 1 \leq i \leq n \]

We define edge labeling as a $g$ bijective function to the order of $Sn_m$ plus one until the order and edges of $Sn_m$ is as follows:

\[ g(x_i y_i) = \begin{cases} 
2m + i + 1, & \text{for } 1 \leq i \leq m - 1 \\
2m + 1, & \text{for } i = m
\end{cases} \]
\[ g(y_iy_{i+1}) = \begin{cases} \frac{6m+i+1}{2}, & \text{for } i \text{ is odd} \\ \frac{7m+i-1}{2}, & \text{for } i \text{ is even} \end{cases} \]

it is easy to get a super local antimagic $H$-docomposition coloring of sun graph and the decomposition weight are follows:

\[ w(H_i) = \begin{cases} \frac{18m+6}{2}, & \text{for } i \text{ is even} \\ \frac{20m+4}{2}, & \text{for } i \text{ is odd} \end{cases} \]

Based on observation 1 we get $\gamma_{lat}(Sn_m) \geq 2$ and $\gamma_{lat}(Sn_m) \leq 2$ as lower bound and upper bound. Based on lemma 2, $G \cong Sn_m$ such that $\gamma_{lat}(Sn_m) \geq 2$ however we can attain the best lower bound. We get $\gamma_{lat}(Sn_m) = 2$ for $n$ is even.

**Case 2.** For $m$ is odd, we will use the label of elements graph to find chromatic number as $\gamma_{lat}(Sn_m) \leq 3$. We define vertex labeling as a $f$ bijective function to natural number until the order of $Sn_m$ is follows:

\[ f(x_i) = \begin{cases} \frac{i+2}{2m+i+2}, & \text{for } i \text{ is odd} \\ \frac{3m-i+2}{2m+i+2}, & \text{for } i \text{ is even} \end{cases} \]

\[ f(y_i) = \begin{cases} \frac{3m-i+2}{2m+i+2}, & \text{for } i \text{ is odd} \\ \frac{3m+i+1}{2}, & \text{for } i \text{ is even} \end{cases} \]

We define edge labeling as a $g$ bijective function to the order of $Sn_m$ plus one until the order and edges of $Sn_m$ is as follows:

\[ g(x_iy_i) = \begin{cases} \frac{5m+i}{2m-i+1}, & \text{for } i \text{ is odd} \\ \frac{5m+i}{2}, & \text{for } i \text{ is even} \end{cases} \]

\[ g(y_iy_{i+1}) = \begin{cases} \frac{4m-i+1}{3m+i+2}, & \text{for } i \text{ is odd} \\ \frac{4m+i+1}{3m+i+2}, & \text{for } i \text{ is even} \end{cases} \]

it is easy to get a super local antimagic $H$-docomposition coloring of sun graph and the decomposition weight are follows:

\[ w(H_i) = \begin{cases} \frac{19m+7}{2}, & \text{for } i \text{ is odd} \\ \frac{19m+5}{2}, & \text{for } i \text{ is even} \\ \frac{18m+6}{2}, & \text{for } i = m \end{cases} \]

The chromatic number of super local antimagic $H$-docomposition coloring in sun graph is at least 2, because sun graph has at least two adjacent subgraph. For $m = 3$, $c(H_1) \neq c(H_2), c(H_2) \neq c(H_3)$ and $c(H_3) \neq c(H_1)$. We can see that every subgraph of $Sn_3$ must have 3 different colors. Based on observation 1, we get $\gamma_{lat}(Sn_m) \geq 3$ and $\gamma_{lat}(Sn_m) \leq 3$ as a lower bound and upper bound of super local antimagic $H$-docomposition coloring of sun graph ($Sn_m$). To proof $\gamma_{lat}(Sn_m) \geq 3$, assume it $\gamma_{lat}(Sn_m) = 2$ then there is any adjacent subgraph have the same color $c(H_i) = c(H_j)$ where $H_i = x_iy_iy_{i+1}, 1 \leq i \leq n, 1 \leq j \leq n$, so that contradiction with observation 1. We get $\gamma_{lat}(Sn_m) = 3$ for $m$ is odd. \( \square \)
Figure 2. super local antimagic $\mathcal{H}$-decomposition coloring in sun graph $S_{n_9}$ and $S_{n_{10}}$

2.3. The Chromatic number of super local antimagic $\mathcal{H}$-decomposition coloring in star graph ($S_n$)
Let $n \leq 3$ be a natural number, star graph $(S_n)$ is a graph with $V(S_n) = \{x_i; 1 \leq i \leq n\} \cup \{A\}, E(S_n) = \{x_iA; 1 \leq i \leq n\}$ and $\mathcal{H} \cong P_3$, then we determine chromatic number of super local antimagic $\mathcal{H}$-decomposition coloring in star graph as follows.

**Theorem 3** For every natural number $n \geq 3$ where $n$ is odd

**Proof.** To proof chromatic number of super local antimagic $\mathcal{H}$-decomposition coloring of star graph, we will use label of elements graph to find chromatic number as $\gamma_{lat}(S_n) \leq n$. We define vertex labeling as a $f$ bijective function to natural number until the order of $S_n$ is as follows:

$$f(x_i) = \begin{cases} 
\frac{i+1}{2}, & \text{for } i \text{ is odd} \\
\frac{n+i+2}{2}, & \text{for } i \text{ is even}
\end{cases}$$

$$f(y_i) = \frac{n+2}{2}, n \geq 4$$

We define edge labeling as a $g$ bijective function to the order of $S_n$ plus one until the order and edges of $S_n$ is as follows:

$$f(x_iy_i) = \begin{cases} 
\frac{2n+i+3}{4n-2i+1}, & \text{for } i \text{ is odd} \\
\frac{2n+i+3}{4n-2i+1}, & \text{for } i \text{ is even}
\end{cases}$$

it is easy to get a super local antimagic $\mathcal{H}$-decomposition coloring of star graph and the decomposition weight are follows:

$$w(\mathcal{H}_i) = 4n + 2i + 5, 1 \leq i \leq n$$
Based on observation 1, we get $\gamma_{lat}(S_n) \geq n$ and $\gamma_{lat}(S_n) \leq n$ as lower bound and upper bound of super local antimagic $H$-decomposition coloring of star graph $(S_n)$. To proof $\gamma_{lat}(S_n) \geq n$, assume it $\gamma_{lat}(S_n) = n - 1$, if $\gamma_{lat}(S_n) = n - 1$ then there is any adjacent subgraph have the same color $c(H_i) = c(H_j)$ $1 \leq i \leq n, 1 \leq j \leq n$ where $H_i = x_i x_{i+1}$ $1 \leq i \leq n - 1$ so that contradiction with observation 1 then $\gamma_{lat}(S_n) \geq n$. We get the chromatic number of super local antimagic $H$-decomposition coloring in star graph is $\gamma_{lat}(S_n) = n$ where $n$ is odd.  

3. Conclusion
In this section, we have given the result of super local antimagic $H$-decomposition coloring of some graph. We determine the super local antimagic $H$-decomposition coloring chromatic number of some graph such as wheel graph, sun graph and star graph. We provide an open problem for further researcher as follows.

Open problem 1. Determine of super local antimagic $H$-decomposition coloring of some classes graphs $G$.

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