ANISOTROPIES OF COSMIC BACKGROUND RADIATION FROM A LOCAL COLLAPSE

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Abstract

We present an exact solution of the anisotropies of cosmic background radiation (CBR) from a local collapse described by a spherical over-dense region embedded in a flat universe, with the emphasis on the relationship between the dipole \((\Delta T/T)_d\) and the quadrupole \((\Delta T/T)_q\) anisotropy. This result has been used to examine the kinematic quadrupole correction \((\Delta T/T)_q = (\Delta T/T)_d^2/2\), which is usually applied to remove the contamination of the quadrupole produced by local density inhomogeneities when finding the cosmic amplitude of the quadrupole at the surface of last scattering. We have found that the quadrupole of local collapse origin cannot always be approximately described by the kinematic quadrupole. Our numerical result shows that the difference between the kinematic and local quadrupoles depends on the size and matter density in the peculiar field, and the position of the observer. For a given dipole, the local quadrupole can be different from the kinematic quadrupole by a factor as large as 3. Therefore, the kinematic quadrupole correction remains an uncertain factor in the determination of the amplitude of a cosmic quadrupole. Nevertheless, a preliminary analysis shows that this uncertainty might not dramatically change the cosmological origin of the COBE-DMR’s quadrupole, unless a huge peculiar gravitational field is assumed.

Subject headings: cosmic background radiation - cosmology: theory
1. INTRODUCTION

This paper is aimed at studying the contribution of local collapse of matter to the anisotropies of cosmic background radiation (CBR), especially the relationship between the dipole and the kinematic quadrupole.

It is generally believed that the CBR dipole anisotropy comes from a Doppler effect. If the observer moves with velocity $v$ with respect to the CBR rest frame, the special relativity (SR) Doppler effect would lead to a frequency-independent thermodynamic temperature distribution on CBR (Peebles & Wilkinson, 1968)

$$T = T_0 \left[ 1 + \frac{v}{c} \cos \Psi + \frac{1}{2} \left( \frac{v}{c} \right)^2 \cos 2\Psi + \cdots \right]$$

The terms in the right-hand side are, respectively, the monopole, the dipole and the kinematic quadrupole anisotropies.

Eq.(1) is often used 1) to determine the peculiar velocity $v$, and 2) to calculate the correction of the kinematic quadrupole. For instance, the recent COBE-DMR first year sky maps show a dipole amplitude $(\Delta T)_d = 3.365 \pm 0.027$ mK toward direction $(l^{II}, b^{II}) = (264.4^\circ \pm 0.3^\circ, 48.4^\circ \pm 0.5^\circ)$ (Smoot et al, 1992; Bennett et al, 1992; Kogut et al, 1993). If the entirely observed dipole results from our peculiar motion, the kinematic quadrupole anisotropy should be $(\Delta T)_q = (\Delta T)_d^2 / 2T \approx 2.1 \mu K$. This result has been used by the COBE-DMR team to obtain the CBR quadrupole anisotropy (Smoot et al, 1991, 1992). They removed the kinematic quadrupole term from the DMR maps in order to eliminate the influence of local density inhomogeneities on the quadrupole amplitude. Since the amplitude of the kinematic quadrupole is about 13% of the cosmic quadrupole, the difference between the quadrupoles produced by the SR kinematics and local density inhomogeneities would be one of the factors leading to uncertainty in the amplitude of the cosmological quadrupole. Therefore, it is necessary to study the conditions, under which the kinematic quadrupole correction given by eq.(1) is valid.

The initial velocity of cosmic matter underwent a decrease due to the expansion of the universe. It can then be reasonably assumed that our present peculiar velocity results totally from the infall motion toward the center of the local collapse of matter, i.e., the peculiar motion of the observer is completely given by the gravitation of local matter clustering. In linear approximation, the relationship between the matter density fluctuation $\delta$ and the peculiar velocity is (Peebles, 1980)

$$v = \frac{1}{3} (H_0 D) \Omega^{0.6} \delta$$

where $D$ is the distance to the center of the perturbation, $\Omega$, the density parameter and $H_0$, the Hubble constant. Thus, eqs.(1) and (2) imply that the CBR dipole and kinematic quadrupole anisotropies are essentially caused by the local density inhomogeneities. In other word, the CBR’s dipole and quadruple anisotropy should be calculated as an effect of a locally time-dependent density inhomogeneity.

The question is then: can the contribution of a local collapse of matter to the CBR dipole and quadrupole anisotropies be approximately described by a Doppler
effect of eq.(1), i.e. a pure SR effect? Obviously, the SR Doppler effect description is principally different from that of a local collapse. The former is a kinematic effect, of which the only parameter is the observer’s velocity \( v \). The latter is a dynamical model, and governed by the matter distribution or the initial density contrast \( \delta \), the location of the observer and the size of the local collapse.

Eqs.(1) and (2) also tell us that the dipole anisotropy depends on the first order of the density fluctuation \( \delta \) and then the kinematic quadrupole is of the second order. It is well known from general relativity that in a linear approximation the behavior of a comoving object in an expansion or collapsing metric can be equivalently described as a Doppler motion. But such an equivalence will no longer be held if the higher orders are involved. Indeed, in terms of the dipole term one can not distinguish between a SR Doppler effect and a local collapse. The velocity \( v \) determined by a dipole anisotropy is just equal to the \( v \) caused by the locally gravitational collapse. However, one can not expect that the SR kinematic quadrupole remains the same as the quadrupole caused by the local collapse.

The original intention of the kinematic quadrupole correction in the COBE sky maps is to remove the contamination of a quadrupole given by the local density inhomogeneity. Therefore, in principle, this correction should not be done by the kinematic quadrupole, but the local collapse of matter. We need then to conduct a quantitative comparison of the kinematic quadrupole with that of a local collapse.

The CBR anisotropies from a collapse at redshift greater than 1 have been addressed by several authors (e.g. Rees & Sciama, 1968; Dyer, 1976; Olson & Silk; 1979; Raine & Thomas, 1981; Kaiser, 1982; Occhionero, Santangelo & Vittorio, 1983; Nottale, 1984; Dyer & Ip, 1988; Arnau et al, 1993). However, the contribution from a local collapse to the CBR anisotropies, especially the relationship between the dipole and local quadrupole, has not been carefully studied thus far. We will investigate the CBR dipole and quadrupole anisotropies from a locally spherical collapse described by a Tolman-Bondi universe. This model has recently been used to analyze the CBR anisotropy produced by perturbations with sizes of 100 ~ 1000 Mpc (Fang and Wu, 1993, hereafter Paper I). The advantage of this model is that one can find all needed exact solutions, which allow us to check the availability of the kinematic quadrupole correction of eq.(1).

In Section 2, we will discuss the metric and null geodesic in a spherically symmetric collapse. Section 3 shows the linear, the high-order and the exact numerical solutions of the dipole and quadrupole anisotropies. In Section 4, a relationship between peculiar velocity and CBR anisotropies will be presented. Finally, a brief summary and discussion will be given in Section 5.

2. METRIC AND NULL GEODESIC

For simplicity, we consider a spherical density perturbation embedded in a flat universe. The universe can then be generally modeled as a Tolman-Bondi metric (Paper I):

\[
ds^2 = e^{\lambda(x,t)}dx^2 + r^2(x,t)(d\theta^2 + \sin^2\theta d\phi^2) - dt^2
\]  (3)
and
\[ e^{\lambda(x,t)} = \frac{r^2}{1 + x^2 H_i^2(1 - \frac{\overline{\rho}(x,t_i)}{\rho_{ci}})} \] (4)

where \( H_i \), \( \overline{\rho}(x, t_i) \) and \( \rho_{ci} \) are, respectively, the Hubble constant, the density and the critical density of the universe at epoch \( t_i \). We denote that \( ' = \partial/\partial x \) and \( \dot{} = \partial/\partial t \).

The dynamics of the collapse from the initial perturbation at \( x = 0 \) is described by the collapse factor \( S(x,t) \) which is defined as
\[ r = S(x,t)x, \quad S(x,t_i) = 1 \] (5)

The dynamical equation of \( S(x,t) \) is
\[ \ddot{S} - H_i^2 \frac{\overline{\rho}(x,t_i)}{\rho_{ci}} \frac{1}{S} = H_i^2(1 - \frac{\overline{\rho}(x,t_i)}{\rho_{ci}}) \] (6)

Let the initial density perturbation be \( \delta(x) \), the density distribution \( \overline{\rho}(x,t_i) \) in eqs.(4) and (6) can then be written as
\[ \overline{\rho}(x,t_i) = \rho_{ci}(1 + \delta(x)) \] (7)

If the initial density perturbation \( \delta_0 \) is assumed to be constant in the range of \( x < x_c \), one has
\[ \delta(x) = \begin{cases} \delta_0, & x \leq x_c \\ \delta_0(x_c/x)^3, & x > x_c. \end{cases} \] (8)

In this case, the dynamical equation (6) can be solved analytically (Paper I).

The 0th-component of the null geodesic in the metric of eq.(3) is
\[ \frac{dk^0}{d\sigma} = -\frac{1}{2} e^{\frac{\lambda}{2}} \left( \frac{dx}{d\sigma} \right)^2 - nr^2 \left( \frac{d\phi}{d\sigma} \right)^2 \] (9)

where \( k^0 \) is the 0th-component of a photon’s four momentum and \( \sigma \) is an affine parameter of the null geodesic. From the condition of \( ds = 0 \), one can find the energy shift of a photon, which is assumed to be emitted at \( t = t_e \) with frequency \( \nu_e \) and received at \( (x_0, t_0) \) with frequency \( \nu_0 \),
\[ \frac{\nu_e}{\nu_0} = \exp \left( \int_{t_e}^{t_0} \frac{1}{2} \lambda dt + \int_{t_e}^{t_0} \left( \frac{\dot{r}}{r} - \frac{1}{2} \lambda \right) r^2 \left( \frac{d\phi}{dt} \right)^2 dt \right) \] (10)

The trajectories of the photon can be obtained from
\[ \frac{dx}{dt} = \pm e^{-\lambda/2} \sqrt{1 - r^2 \left( \frac{d\phi}{dt} \right)^2} \] (11)
\[
\frac{d^2 \phi}{dt^2} + \left[ \frac{2}{r} \left( \frac{dr}{dt} \right) - \frac{1}{2} \lambda \right] \frac{d\phi}{dt} = \left( \frac{\dot{r}}{r} - \frac{1}{2} \lambda \right) r^2 \left( \frac{d\phi}{dt} \right)^3
\]

(12)

Considering the trajectory of a photon is approaching a straight line when \( r \) is large, one can obtain \( \frac{d\phi}{dt} \) from solving the eq.(12), and then the energy shift eq.(10) and the trajectory eq.(11) become

\[
\frac{\nu_e}{\nu_0} = e^{\int_{t_1}^{T_0} \frac{1}{2} \frac{\lambda e}{r} dT} \left( 1 - \int_{1}^{T_0} U_t e dT \right)^{1/2}
\]

(13)

\[
\frac{dX}{dT} = \pm e^{-\lambda/2} \left( 1 - \frac{\xi}{1 - \int_{T_0}^{T} U_t e dT} \right)^{1/2}
\]

(14)

where the new time and space coordinates \((T, X)\) are defined by \( T = t/t_e \) and \( X = x/t_e \). Therefore, \( T_0 = t_0/t_e \), \( X_0 = x_0/t_e \) and \( X_c = x_c/t_e \). \( U \) and \( \xi \) in eqs.(13) and (14) are, respectively,

\[
U = 2\xi \left( \frac{\dot{\lambda}}{2} - \frac{\dot{r}}{r} \right)
\]

(15)

\[
\xi = \left( \frac{r_0}{r} \right)^2 \sin^2 \Psi e^{\int_{T_0}^{T} \frac{1}{2} \dot{\lambda} e dT}
\]

(16)

with \( r_0 = r(x_0, t_0) \), and \( \Psi \) is the incidence angle of the photon, i.e. the angle between the directions of the photon’s trajectory and the line of sight from observer to the center of the perturbation.

Since the universe is flat, the CBR anisotropy produced by the local collapse is given by

\[
\frac{\Delta T}{T} = \frac{\nu_0}{\nu_e} T_0^{2/3} - 1
\]

(17)

where \( T_0 = (1 + z_d)^{3/2} \) and \( z_d \) is the redshift at decoupling time \((t_e)\). Without a loss of generality, we will take \( t_i = t_e \), i.e., the \( \delta(x) \) is the density fluctuation at the recombination time \( t_e \).

Because we are interested in the effect of a local collapse, in the following, only the case of \( x_0 < x_c \) or \( X_0 < X_c \) will be considered. Using the expansion of \( S(X, T) \) given in the Appendix, one find the solution of \( \Delta T/T \) up to the second order of \( \delta_0 \) as

\[
\frac{\Delta T}{T} = - \int_{T_1}^{T_0} f_1 \delta dT + X_0^2 \sin^2 \Psi \int_{T_1}^{T_{e_0}} \frac{u_1}{X(0)^2} \delta dT
\]

\[
+ \frac{1}{2} \left[ \int_{T_1}^{T_0} f_1 \delta dT - X_0^2 \sin^2 \Psi \int_{T_1}^{T_{e_0}} \frac{u_1}{X(0)^2} \delta dT \right]^2 + \frac{1}{2} \int_{T_1}^{T_{e_0}} f_1 \frac{\Delta X}{X(0)} \delta dT
\]

6
\[ -5X_0^2 \sin^2 \Psi \int_1^{T_{c0}} \frac{u_1}{X(0)^2} \frac{\Delta X}{X(0)} \delta dT + \left[ 3f_1(T_{c0}) + X_0^2 \sin^2 \Psi \frac{u_1(T_{c0})}{X_c^2} \right] \delta_0 \Delta T_c \]

\[ - \int_1^{T_0} f_2 \delta^2 dT + X_0^2 \sin^2 \Psi \int_1^{T_{c0}} \frac{u_2}{X(0)^2} \delta^2 dT \]

\[ + 2\delta_0 X_0^2 \sin^2 \Psi \frac{S_1(T_0)}{T_0^{2/3}} \int_1^{T_{c0}} \frac{u_1}{X(0)^2} \delta dT - 2X_0^2 \sin^2 \Psi \int_1^{T_{c0}} \frac{S_1(T)}{T^{2/3}} \frac{u_1}{X(0)^2} \delta^2 dT \]

\[ + 2X_0^2 \sin^2 \Psi \int_1^{T_{c0}} \frac{u_1}{X(0)^2} \delta \int_0^T f_1 \delta dT + X_0^4 \sin^4 \Psi \left[ \int_1^{T_{c0}} \frac{u_1}{X(0)^2} \delta dT \right]^2 \] (18)

where \( f_i \) and \( u_i \) (\( i = 1, 2 \)) are defined in the Appendix. \( \Delta X \) is the first-order correction to the photon trajectory, and \( \Delta T_c \), the cross time correction when the photon enters into the perturbation regime. They are, respectively,

\[ \Delta X = \pm \sqrt{(X(0))^2 - X_0^2 \sin^2 \Psi} \int_1^{T_{c0}} \frac{F(X(0), T)}{T^{2/3}} dT \] (19)

\[ \Delta T_c = T_{c0}^{2/3} \int_1^{T_{c0}} \frac{F(X(0), T)}{T^{2/3}} dT \] (20)

where \( X(0) \) is the zero-order solution of the photon’s trajectory, which can be found from eq.(14) by taking \( \delta = 0 \). It is

\[ (X(0))^2 = X_0^2 \sin^2 \Psi + (3T_0^{1/3} - 3T_0^{1/3} - X_0 \cos \Psi)^2 \] (21)

Similarly, the zero-order solution to the photon cross time (\( T_c \)) is found to be

\[ T_{c0} = \left[ T_0^{1/3} - \frac{X_0 \cos \Psi}{3} - \frac{1}{3} \sqrt{X_c^2 - X_0^2 \sin^2 \Psi} \right]^3 \] (22)

The function \( F(X, T) \) in eqs.(19) and (20) is defined as

\[ F(X, T) = g_1 T^{2/3} \delta - \frac{X_0^2 \sin^2 \Psi}{X^2 - X_0^2 \sin^2 \Psi} \]

\[ \left[ \frac{S_1(T_0)}{T_0^{2/3}} \delta_0 - \frac{S_1(T)}{T^{2/3}} \delta + \int_0^T f_1 \delta dT + X_0^2 \sin^2 \Psi \int_T^{T_0} \frac{u_1}{X_c^2} \delta dT \right] \] (23)

and \( g_1 \) is also given in the Appendix.

3. SOLUTIONS OF \( \Delta T/T \)
3.1 First-Order Solution

The first-order solution of $\Delta T/T$ can be obtained by substituting the zero-order solution of the photon trajectory $X^{(0)}$ of eq.(21) into the first two terms of eq.(18). After a straightforward computation, the first-order solution is found to be

$$\frac{\Delta T}{T} = \delta_0 \left( \frac{X_c^2}{15} - \frac{X_0^2}{45} + \frac{2}{15} T_0^{1/3} X_0 \cos \Psi - \frac{2}{135} \frac{X_0^3}{T_0^{1/3}} + O(T_0^{-2/3}) \right)$$

(24)

The above expression is also an expansion with respect to the parameter $(1/T_0)$. The largest term is of the order of $(1/T_0)^{-1/3}$. Eq.(24) shows that in the approximation up to the first-order of $\delta_0$, $\Delta T/T$ consists mainly of two parts: a monopole term

$$\left( \frac{\Delta T}{T} \right)_0 \simeq \left( \frac{X_c^2}{15} - \frac{X_0^2}{45} - \frac{2}{135} \frac{X_0^3}{T_0^{1/3}} \right) \delta_0$$

(25)

and a dipole term

$$\left( \frac{\Delta T}{T} \right)_d \simeq \frac{2}{15} T_0^{1/3} X_0 \delta_0 \cos \Psi$$

(26)

The physical explanations of these results are simple. The monopole is given by the gravitational redshift of the local matter inhomogeneity $\delta_0$, which leads to an isotropic increase of the CBR temperature. The dipole term depends on the distance between the observer and the center of the density perturbation ($X_0$). In the case of $X_0 = 0$, i.e., the observer sits at the center of the local collapse, the dipole term will disappear. This indicates that the dipole anisotropy is indeed due to the asymmetry of the local collapse around the observer. However, it should be pointed out that the dipole anisotropy depends on the time-dependence of the local inhomogeneity. If the gravitation field around the observer is static, the asymmetry ($X_0 \neq 0$) do not produce such a dipole anisotropy.

Comparing eq.(26) with eq.(1), we have

$$v = \frac{2}{15} T_0^{1/3} X_0 \delta_0$$

(27)

This means that the CBR dipole anisotropy produced by a local collapse can be equivalently described as a SR Doppler effect if the observer is assumed to have a peculiar velocity given by eq.(27). Let’s show that eq.(27) is indeed the observer’s peculiar velocity produced by the gravitation field of the local collapse. The proper distance $D$ corresponding to $X = x/t_e$ is

$$D = \frac{2}{3} \frac{c}{H_0 \sqrt{1 + z_t}} X$$

(28)
where $z_i$ is the redshift when the perturbation occurred. We will take it to be the redshift at decoupling era, i.e. $z_i = z_d$. In the paper I, we have found that for a given $\delta_0$ the present density contrast of the local collapse should be

$$\frac{\Delta \rho}{\rho} \approx \frac{3}{5} (1 + z_d) \delta_0.$$ (29)

In a flat universe, $\rho = \rho_{cr} = 3H_0^2/8\pi G$. Using eqs.(28) and (29), eq.(27) can be rewritten as

$$v = \frac{1}{3} (H_0 D) \left( \frac{\Delta \rho}{\rho} \right) = \frac{2}{3H_0} g$$ (30)

where $g = G\Delta M/D^2$ is the observer’s acceleration raised by the extra-mass $\Delta M = (4\pi/3) D^3 \Delta \rho_0$. Therefore, $v$ in eq.(27) is the same as that in eq.(2). In a word, in the linear approximation it is reasonable to determine the observer’s peculiar velocity by the CBR dipole anisotropy and the Doppler effect formula (1).

If the CBR dipole is totally given by the local collapse, one can then find from eq.(26) that $X_0 \delta_0 = (15/2) (\Delta T/T)_{d} T_0^{-1/3} \sim 2.6 \times 10^{-4}$. This is, in fact, a constrain to the local collapse causing the dipole. For instance, if we assume the size of this local collapse is of the order of the distance to the Great Aractor, i.e. $X_0 \sim 1$, the initial density perturbation $\delta_0$ in this area should be about $\delta_0 \sim 2 \times 10^{-4}$, or from eq.(29) today’s density contrast is about $1 \times 10^{-1}$.

### 3.2 Second-Order Solutions and Quadrupole Anisotropy

In a similar way, the second-order solution of $\Delta T/T$ can be found from eq.(18) as follows

$$\frac{\Delta T}{T} = \delta_0^2 \left[ \left( \frac{3}{175} X_c^2 - \frac{11}{1575} X_b^2 \right) T_0^{2/3} + \frac{4}{175} T_0 X_0 \cos \Psi + \frac{2}{225} T_0^{2/3} X_0^2 \delta_0^2 \cos 2\Psi \right]$$ (31)

which is also written in the series of $(1/T_0)$ up to the order of $(1/T_0)^{-2/3}$. Eq.(31) indicates that, up to the $\delta_0^2$ approximation, the dominant term is still the dipole, because the amplitude of dipole is of the order of $T_0$, while the amplitudes of the monopole and quadrupole terms are only of the order of $T_0^{2/3}$.

The quadrupole anisotropy of the local collapse now is

$$\left( \frac{\Delta T}{T} \right)_q = \frac{2}{225} T_0^{2/3} X_0^2 \delta_0^2 \cos 2\Psi$$ (32)

Comparing this amplitude with that in eq.(26), one find

$$\left( \frac{\Delta T}{T} \right)_q = \frac{1}{2} \left( \frac{\Delta T}{T} \right)_{d}^2$$ (33)
This is just the SR relationship between the anisotropies of the dipole and kinematic quadrupole. Therefore, the SR kinematic quadrupole correction is available till the approximation of eq.(32) is correct. However, when the terms of the order of \( T_0^{1/3} \), \( T_0^0 \) are taken into account, the dipole-quadrupole relation of eq(33) will no longer hold true. For instance, up to the order of \( \delta_0^2 \) and \( T_1/3_0 \), eq.(33) should be corrected as

\[
\left( \frac{\Delta T}{T_0} \right)_q = \frac{1}{2} \left( \frac{\Delta T}{T} \right)^2_d + T_0^{1/3} X_0^2 \Delta q \delta_0^2
\]

where the factor \( \Delta q \) is given by

\[
\Delta q = \frac{19X_c}{3780} - \frac{1}{X_0} \left[ \frac{X_0}{140X_c} + \frac{229X_0^3}{61440X_c^3} + \frac{261X_0^5}{81920X_c^5} + \frac{3X_0^7}{4096X_c^7} \right]
+ X_0 \left[ \frac{41X_0}{9800X_c} - \frac{1333X_0^3}{2064384X_c^3} + \frac{467X_0^5}{5734400X_c^5} + \frac{3833X_0^7}{11468800X_c^7} + O\left( \frac{X_0^9}{X_c^9} \right) \right]
\]

Eq.(35) shows that the term of the SR kinematic quadrupole does not always dominate the quadrupole anisotropy shown in eq.(34). Therefore, the SR kinematic quadrupole may not always be a good approximation of the quadrupole produced by a local collapse. In principle, the kinematic quadrupole correction of eq.(33) should also be replaced by the local quadrupole correction of eq.(34).

Let’s define a ratio between the local quadrupole to the SR kinematic quadrupole as

\[
q = \frac{(\Delta T/T)_q}{(1/2)(\Delta T/T)_d}
\]

Figure 1 plots the dependence of \( q \) on \( X_c \) and \( X_0 \) as given by eq.(35). One can see that \( q \) does not sensitively depend on the observer distance but the whole size of the local inhomogeneity. The difference between local and kinematic quadrupole, i.e. that \( q \) significantly deviates from 1, mainly occurs in two cases: 1) very small clustering with a scale less than a few Mpc, and 2) very large clustering with a scale greater than \( 10^3 \) Mpc.

### 3.3 Numerical Solution

In order to accurately compare the kinematic quadrupole with the local collapse model, we made the numerical solutions of the dipole and quadrupole amplitude of the local collapse model. These solutions depend on three parameters: 1) the initial density fluctuation \( \delta_0 \), 2) the size of local collapse \( X_c \), and 3) the distance of the observer to the center of the local collapse \( X_0 \).

We still lack the detailed knowledge of the local collapse. It is generally believed that the bulk motion of horizon sized volume is negligible. Therefore, the Local Group’s peculiar velocity, which may govern the dipole term, was probably induced
by the local inhomogeneities in the density field with size comparable with horizon at the time of decoupling. We have shown in Paper I that the COBE-DMR result of CBR anisotropy on an angular scale of $10^\circ$ implies the existence of collapses on scale as large as about 1000 $h_{50}^{-1}$ Mpc with an initial density fluctuation $\delta_0 \sim (2.8 - 6.9) \times 10^{-6}$, or its present density enhancement is $(1.7 - 4.2) \times 10^{-3}$. On the other hand, the distance of our Local Group to the center of the collapse should at least be greater than the distance to the Great Attractor, which is estimated to be 80 $h_{50}^{-1}$ Mpc (Lynden-Bell et al, 1988), i.e. $X_0 = 0.7$. Therefore, it would be valuable to consider the following two cases: $X_c = 1.4$ ($\sim 150 h_{50}^{-1}$ Mpc) and 10 ($\sim 1000 h_{50}^{-1}$ Mpc), i.e., the lower value of $X_c$ is about 2 times of the distance to the Great Attractor and the higher value of $X_c$ is about the size of horizon.

The numerical results are listed in Table 1, in which $[\text{exact}]$ indicates the exact solution of the dipole amplitude, and $[\delta_0]$ and $[\delta_0^2]$ denote, respectively, the solutions up to the first- and second-order of $\delta_0$. The density fluctuations at the decoupling epoch ($z_d = 10^3$) are taken to be $10^{-3}$, $10^{-4}$ and $10^{-5}$, respectively. The corresponding values of the SR kinematic quadrupole $(1/2)(\Delta T/T)_q^2 = (1/2)[\text{exact}]^2$ have also been listed for comparisons. The term of $(\Delta T/T)_q$ is the exact solution of the quadrupole amplitude.

One can find from Table 1 that: a) For the dipole anisotropy the linear approximation is good if the initial density fluctuation is less than $10^{-4}$. b) The dipole is nearly independent of the size of the local collapse. c) The local quadrupole sensitively depends on the size of the perturbation. d) The kinematic quadrupole is always larger than the exact solution of the local quadrupole. e) The difference between the kinematic and exact quadrupoles does not vanish with the decrease of density perturbation $\delta_0$. In the case of $X_c = 10$ the ratio $1/q$ can be as large as 3.

4. DIPOLE AS THE FUNCTIONS OF DENSITY CONTRAST AND PECULIAR VELOCITY

In this section, we intend to represent the CBR’s dipole anisotropy by the observable parameters such as the present density contrast and peculiar velocity. Up to the second order of $\delta_0$ the dipole anisotropy is (see eqs.(24) and (31))

$$
\left( \frac{\Delta T}{T} \right)_d = \left( \frac{2}{15} T_0^{1/3} \delta_0 + \frac{4}{175} T_0 \delta_0^2 + \ldots \right) X_0 \cos \Psi
$$

First, the present density contrast $\Delta \rho/\rho$ as a function of initial fluctuation $\delta_0$ is given by

$$
\frac{\Delta \rho}{\rho} = \left( \frac{S_0}{S(X,T_0)} \right)^3 (1 + \delta_0) - 1
$$

Using eq.(A1–A5), one can find the expression of $\Delta \rho/\rho$ expanded as a series of $\delta_0$

$$
\frac{\Delta \rho}{\rho} = \delta_0 \left[ \frac{3}{5} T_0^{2/3} + \frac{2}{5} T_0^{-1} \right]
$$
\[+ \delta_0^2 \left[ \frac{51}{175} T_0^{4/3} - \frac{2}{5} T_0^{2/3} + \frac{4}{25} T_0^{-1/3} - \frac{6}{35} T_0^{-1} + \frac{3}{25} T_0^{-2} \right] \]

\[+ \delta_0^3 \left[ \frac{341}{2625} T_0^2 - \frac{68}{175} T_0^{4/3} + \frac{1}{3} T_0^{2/3} + \frac{34}{875} T_0^{1/3} - \frac{92}{525} T_0^{-1/3} + \frac{2}{21} T_0^{-1} \right. \]

\[+ \frac{14}{375} T_0^{-4/3} - \frac{18}{175} T_0^{-2} + \frac{4}{125} T_0^{-3} \right] + O(\delta_0^4) \quad (39)\]

Reversing this expansion and retaining the first three terms, we have

\[\delta_0 \approx \frac{1}{1 + z_d} \left( \frac{\Delta \rho}{\rho} \right) \left[ \frac{5}{3} - \frac{85}{63} \left( \frac{\Delta \rho}{\rho} \right) + \frac{14075}{11907} \left( \frac{\Delta \rho}{\rho} \right)^2 + \ldots \right] \quad (40)\]

Substituting this result into eq. (37), one finds the expression of dipole up to the second-order of the present density contract \(\Delta \rho/\rho\),

\[\left( \frac{\Delta T}{T} \right)_d = \frac{1}{3} \left( \frac{H_0 D}{c} \right) \left( \frac{\Delta \rho}{\rho} \right) \left( 1 - \frac{11}{21} \frac{\Delta \rho}{\rho} \right). \quad (41)\]

Therefore, in linear approximation, the dipole anisotropy would be overestimated from the measurement of the local density fluctuation.

Second, the peculiar velocity inside the perturbation regime can be derived from the continuity equation (Peebles, 1980)

\[\frac{\partial \rho}{\partial T} + \frac{1}{S_0} \nabla \cdot (1 + \Delta \rho/\rho) \mathbf{v} = 0. \quad (42)\]

It is

\[v = \frac{1}{2} \left( H_0 D \right) T_0^{1/3} \left( \frac{\partial \rho}{\partial T} \right) \frac{S_0}{1 + \Delta \rho/\rho} \quad (43)\]

Using eqs. (38), (39) and (40), the peculiar velocity can be expanded by \(\Delta \rho/\rho\) as

\[v \approx (H_0 D) \left[ \frac{1}{3} \left( \frac{\Delta \rho}{\rho} \right) - \frac{4}{63} \left( \frac{\Delta \rho}{\rho} \right)^2 + \frac{328}{11907} \left( \frac{\Delta \rho}{\rho} \right)^3 + \cdots \right] \quad (44)\]

This result is consistent with the recent work by Gramann (1993), who showed that the peculiar velocity would be overestimated by the linear approximation. Substituting this expression into eq. (37), we have the solution of the dipole anisotropy as a function of peculiar velocity

\[\left( \frac{\Delta T}{T} \right)_d = \frac{c}{v} \left[ 1 + \frac{25}{63} \left( \frac{c}{DH_0} \right) \left( \frac{v}{c} \right) + \ldots \right] \quad (45)\]
or

\[ \frac{v}{c} = \left( \frac{\Delta T}{T} \right)_d / \left( 1 + \frac{25}{198} \frac{\Delta \rho}{\rho} \right) \]  (46)

Therefore, the SR relation eq.(1) can be used to determine peculiar velocity from the CBR measurement only if the local density contrast \( \Delta \rho / \rho \) is very small. Considering the fact that the present density contrast \( \Delta \rho / \rho \) is of the order of 1 on the scale of superclusters, the higher order correction for the peculiar field may not be negligible.

The peculiar velocity divergence in our model is simply

\[ \nabla \cdot v = -\dot{S}_0 \left( \frac{\Delta \rho}{\rho} \right) \left[ 1 - \frac{4}{21} \left( \frac{\Delta \rho}{\rho} \right) + \frac{328}{3969} \left( \frac{\Delta \rho}{\rho} \right)^2 + ... \right] \]  (47)

or

\[ \nabla \cdot v = -\frac{\dot{S}_0 (\Delta \rho / \rho)}{1 + 0.190 (\Delta \rho / \rho) - 0.046 (\Delta \rho / \rho)^2 + ...} \]  (48)

A similar result has been empirically found recently by Nusser and Dekel (1991, 1993).

### 5. DISCUSSION AND CONCLUSIONS

It is usually believed that in linear approximation the CBR anisotropies consist primarily of three parts, namely, 1) a Doppler effect from the motion of the observer with respect to the CBR rest frame, 2) a Sachs-Wolfe effect at the surface of last scattering and 3) a time-dependent potential effect along the photon path (Martínez-González, Sanz & Silk, 1990). In this description, one cannot, in fact, distinguish between the effects of a time-dependent potential and a Doppler motion. The contribution of a local collapse of matter to CBR anisotropy can totally be contained in the Doppler effect of the observer’s peculiar velocity. One can then use the linear relationship between the peculiar velocity \( v \) and local density contrast \( \Delta \rho / \rho \) to calculate the CBR dipole anisotropy \( (\Delta T/T)_d \) [eq.(1)].

However, in terms of the CBR quadrupole anisotropy, the equivalence between the Doppler effect and the local gravitational field will no longer exists. This is because kinematic quadrupole is essentially non-linear. As a result, the quadrupole anisotropy produced by a local inhomogeneity is not generally equal to the SR kinematic quadrupole. Especially, if the peculiar field is comparable with the horizon size, the relationship between the dipole and the quadrupole terms will be substantially different from that given by the SR Doppler effect. We have found that, based on a simply spherical density perturbation model, the local quadrupole can be different from the kinematic quadrupole by a factor as large as 3.

The purpose of the SR kinematic quadrupole correction used in the data reduction of COBE observation is to remove the local quadrupole amplitude from the CBR sky temperature maps, and then the remaining maps will contain only the component of
the cosmic quadrupole, i.e. the component given by the density fluctuation on the last scattering surface. The kinematic quadrupole correction is, in fact, the correction of local quadrupole. Yet, as we have shown, the local quadrupole can not be simply replaced by the SR kinematic quadrupole. Therefore, the original purpose of the kinematic quadrupole correction may not be completely achieved by removing the quadrupole given by eq.(1).

The quadrupole amplitude of the local density inhomogeneities depends on the matter distribution, the density contrast, the size of the local gravitational field and the position of the observer. All these parameters are poorly known at the moment. Therefore, it seems to be very hard to precisely identify the local quadrupole. A possible way to obtain the local quadrupole would be to analyze the temperature map to see if they contain a component with polar axis paralleled to the direction of dipole. This is because the axes of both the dipole and local quadrupole should point out to the center of the local inhomogeneity. Principally, without a precise local quadrupole correction, the observed quadrupole should be regarded as a sum of cosmological and local components. Thus, the local quadrupole correction leads to an uncertain factor in finding the cosmological quadrupole term from observed CBR maps (e.g., the COBE measurement).

In spite of the fact that the local quadrupole may be remarkably different from the kinematic quadrupole, our simple model shows that for a given dipole amplitude the amplitude of the local quadrupole is always less than the corresponding kinematic quadrupole, if the local peculiar field is of the order of 100 Mpc in size. This means that the kinematic quadrupole seems to be an upper limit to the local quadrupole. Because the presently observed quadrupole amplitude by COBE is about 8 times larger than this upper limit, one may concludes that the uncertainty of the local quadrupole would not lead to an uncertainty greater than 13% of the observed cosmic quadrupole.

Certainly, this conclusion is model-dependent. The present model is actually a toy one, which has assumed the simplest matter distribution – a constant density contrast. It may largely deviate from the real local matter distribution. For instance, the peculiar velocity distribution derived from this toy model [eq.(44)] shows a linear increase with the distance from the perturbation center, which does not fit with the observations around the Great Attractor (Faber & Burstein, 1989). Moreover, if the initial density perturbation is assumed to be highly non-spherical, the local quadrupole may have an amplitude greater than $(\Delta T/T)_c^2$. Therefore, the loophole hidden in the kinematic quadrupole correction has not been totally closed yet. One needs to simulate this effect by using a more realistic model of local collapse of matter, e.g., the models given by N-body simulations. The study of the interaction between the local matter distribution and the CBR quadrupole would be of great significance for a better understanding of both the initial perturbation and the local collapse of matter.

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For the initial perturbation eq.(8), the collapse factor $S(x, t)$ can be expanded in the series of $\delta_0$ as

$$S(X, T) = S_0(T) + S_1(T)\delta_0 + S_2(T)\delta_0^2 + S_3(T)\delta_0^3 + ...$$  \hfill (A1)

where the coefficients $S_i (i = 0, 1, 2, \cdots)$ can be determined by eq.(6) to be

$$S_0(T) = T^{2/3}$$  \hfill (A2)

$$S_1(T) = -\frac{2}{15T^{1/3}} + \frac{T^{2/3}}{3} - \frac{T^{4/3}}{5}$$  \hfill (A3)

$$S_2(T) = -\frac{1}{225T^{4/3}} + \frac{4}{315T^{1/3}} + \frac{4T^{1/3}}{75} - \frac{T^{2/3}}{9} + \frac{T^{4/3}}{15} - \frac{3T^2}{175}$$  \hfill (A4)

$$S_3(T) = -\frac{4}{10125T^{7/3}} + \frac{11}{4725T^{4/3}} - \frac{2}{1125T^{2/3}} + \frac{2}{945T^{1/3}} - \frac{64T^{1/3}}{1575}$$

$$+ \frac{5T^{2/3}}{81} + \frac{6T}{875} - \frac{2T^{4/3}}{45} + \frac{3T^2}{175} - \frac{23T^{8/3}}{7875}$$  \hfill (A5)

Similarly, one can find the expansions of $r$, $\lambda$, $\xi$ and $U$ with respect to $\delta_0$ by their definitions (Paper I)

$$\dot{\lambda} = \frac{2\dot{r}'}{r'}$$  \hfill (A6)

and

$$\frac{1}{2}\dot{\lambda} = \frac{x\dot{S}' + \dot{S}}{xS' + S}$$  \hfill (A7)

as well as eqs.(15) and (16). Let

$$\frac{\dot{M}_e}{2} = \frac{2}{3T} + f_1\delta_0 + f_2\delta_0^2 + ...$$  \hfill (A8)

$$\dot{\lambda} - \frac{\dot{r}}{r} = u_1\delta_0 + u_2\delta_0^2 + ...$$  \hfill (A9)

$$e^{-\lambda/2} = \frac{1}{T^{2/3}} + g_1\delta_0 + g_2\delta_0^2 + ...$$  \hfill (A10)
where the coefficients $f_i = f_i(T)$, $u_i = u_i(T)$ and $g_i = g_i(X,T)$ $(i = 1, 2, 3, \ldots)$ are found to be

1. for $X < X_c$,

   $f_1 = \frac{2}{15T^2} - \frac{2}{15T^{1/3}}$  
   \hspace{1cm} (A11)

   $f_2 = \frac{2}{75T^3} - \frac{2}{35T^2} - \frac{2}{225T^{4/3}} + \frac{4}{45T^{1/3}} - \frac{26T^{1/3}}{525}$  
   \hspace{1cm} (A12)

   $u_i = 0, \quad i = 1, 2, 3, \ldots$  
   \hspace{1cm} (A13)

   $g_1 = \frac{1}{5} + \frac{2}{15T^{5/3}} - \frac{1}{3T^{2/3}} - \frac{2X^2}{9T^{2/3}}$  
   \hspace{1cm} (A14)

   $g_2 = -\frac{1}{5} + \frac{1}{45T^{8/3}} - \frac{32}{315T^{5/3}} + \frac{2}{9T^{2/3}} + \frac{2T^{2/3}}{35}$
   \hspace{1cm}  
   \hspace{1cm} \hspace{1cm} - \frac{2X^2}{45} - \frac{4X^2}{135T^{5/3}} + \frac{2X^4}{27T^{2/3}} - \frac{2X^4}{81T^{2/3}}$  
   \hspace{1cm} (A15)

2. for $X > X_c$,

   $f_1 = -\frac{4}{15T^2} + \frac{4}{15T^{1/3}}$  
   \hspace{1cm} (A16)

   $f_2 = \frac{2}{75T^3} - \frac{4}{35T^2} + \frac{28}{225T^{4/3}} - \frac{2}{45T^{1/3}} + \frac{4T^{1/3}}{525}$  
   \hspace{1cm} (A17)

   $u_1 = -\frac{2}{5T^2} + \frac{2}{5T^{1/3}}$  
   \hspace{1cm} (A18)

   $u_2 = -\frac{2}{35T^2} + \frac{2}{15T^{4/3}} - \frac{2}{15T^{1/3}} + \frac{2T^{1/3}}{35}$  
   \hspace{1cm} (A19)

   $g_1 = -\frac{2}{5} - \frac{4}{15T^{5/3}} + \frac{2}{3T^{2/3}} - \frac{2X^2}{9T^{2/3}}$  
   \hspace{1cm} (A20)

   $g_2 = -\frac{1}{5} + \frac{11}{225T^{8/3}} - \frac{92}{315T^{5/3}} + \frac{12}{25T} - \frac{1}{9T^{2/3}}$
   \hspace{1cm}  
   \hspace{1cm} \hspace{1cm} + \frac{13T^{2/3}}{175} + \frac{4X^2}{45} + \frac{8X^2}{135T^{5/3}} - \frac{4X^2}{27T^{2/3}} - \frac{2X^4}{81T^{2/3}}$  
   \hspace{1cm} (A21)
Table 1: Comparison of the SR Kinematic Quadrupole with the Exact Solutions

| size   | $\delta_0$ | $(\Delta T/T)_d$ | $|(\Delta T/T)_d|^2/2 \times \delta_0^{-2}$ | $(\Delta T/T)_q$ | $\times \delta_0^{-2}$ |
|--------|------------|------------------|---------------------------------------------|------------------|-----------------------|
|        | $|\delta_0|/\delta_0$ | $|\delta_0 + \delta_2^2|/\delta_0$ | $|\text{exact}|/\delta_0$ |                  |                      |
| $X_c = 1.4$ | $10^{-3}$ | 2.951 | 3.457 | 3.638 | 6.6 | 6.0 |
|         | $10^{-4}$ | 2.951 | 3.002 | 3.003 | 4.5 | 4.1 |
|         | $10^{-5}$ | 2.951 | 2.957 | 2.957 | 4.4 | 4.1 |
| $X_c = 10$  | $10^{-3}$ | 2.951 | 3.457 | 3.632 | 6.6 | 2.6 |
|         | $10^{-4}$ | 2.951 | 3.002 | 3.000 | 4.5 | 1.4 |
|         | $10^{-5}$ | 2.951 | 2.957 | 2.953 | 4.4 | 1.4 |
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Figure Captions

Figure 1. The ratio $q$ of the local quadrupole $(\Delta T/T)_q$ to the SR kinematic quadrupole $(1/2)(\Delta T/T)^2$. Only the terms with orders greater than $T_0^{1/3}$ in eqs.(34) are considered. This plot illustrates the dependence of the ratio $q$ on the size of the inhomogeneity and the position of the observer. For the collapse with intermediate scale of about $10^2$ Mpc, we have $q \approx 1$. For both the very small and very large scale collapses, the local quadrupole will significantly be different from the kinematic quadrupole.