ON ENRIQUES SURFACES WITH FOUR CUSPS

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ABSTRACT. We study Enriques surfaces with four $A_2$-configurations. In particular, we construct open Enriques surfaces with fundamental groups $(\mathbb{Z}/3\mathbb{Z})^{\oplus 2} \times \mathbb{Z}/2\mathbb{Z}$ and $\mathbb{Z}/6\mathbb{Z}$, completing the picture of the $A_2$-case from [7].

1. Introduction

The main aim of this note is to study Enriques surfaces with four disjoint $A_2$-configurations, the maximum number possible. We shall make heavy use of elliptic fibrations to show that such Enriques surfaces come in exactly two families $\mathcal{F}_{3,3,3,3}, \mathcal{F}_{4,3,1}$. This observation which relies on the understanding of Picard-Lefschetz reflections on the Enriques surface and its K3-cover following [7], enables us to determine the fundamental groups of the open Enriques surfaces obtained by removing the $A_2$-configurations (often also referred to as the cusps).

Our paper draws on the classification of possible fundamental groups of open Enriques surfaces (i.e. complements of configurations of smooth rational curves) initiated in [7]. Keum and Zhang state a list of 26 possible groups and give 24 examples. Here we supplement and correct their results by adding one example and one group supported by another example. Our main result is as follows.

Theorem 1.1. Let $G \in \{ S_3 \times \mathbb{Z}/3\mathbb{Z}, (\mathbb{Z}/3\mathbb{Z})^{\oplus 2} \times \mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/6\mathbb{Z} \}$. Then there is a complex Enriques surface $S$ with a set $\mathcal{A}$ of four disjoint $A_2$-configurations such that

$$\pi_1(S \setminus \mathcal{A}) \cong G.$$ 

For a more concise statement, the reader is referred to Theorem 4.3. This completes the picture of the $A_2$-case.

Another key point of our paper is the clarification that there are indeed Enriques surfaces admitting different sets of four disjoint $A_2$-configurations which lead to each alternative of the fundamental group in Theorem 1.1. This issue will be discussed in detail in Section 5 and also supported by an explicit example, see §5.6.

While some of the constructions involved in our methods are analytic in nature, notably the notion of logarithmic transformations of elliptic surfaces, we will crucially facilitate Enriques involutions of base change type as studied systematically in [5] since this algebro-geometric technique grants us good control.

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of the curves on the surfaces and their moduli. We review this among all the prerequisites and basics necessary for the understanding of this paper in Section 2. Section 3 introduces the two families $F_{3,3,3,3}$ and $F_{4,3,1}$ and prepares for the proof of Theorem 1.1 which is given in Section 4. The paper concludes with considerations concerning the moduli of Enriques surfaces with 4 disjoint $A_2$-configurations.

Convention: In this note the base field is always $\mathbb{C}$. Root lattices $A_n, D_k, E_l$ are taken to be negative definite.

2. Preliminaries and basics

2.1. $A_2$-configurations. Let $S$ be an Enriques surface that contains four disjoint $A_2$-configurations, i.e., eight smooth rational curves $F'_1, F''_1, \ldots, F'_4, F''_4$ such that

$$F'_j F''_j = 1 \text{ for } j = 1, \ldots, 4,$$

and the rational curves in question are mutually disjoint otherwise. We say that a collection of $A_2$-configurations $F'_1, F''_1, \ldots, F'_l, F''_l$ is three-divisible if and only if one can label the rational curves in each $A_2$-configuration such that the divisor

$$(2.1) \quad \sum_{j=1}^l (F'_j - F''_j)$$

is divisible by 3 in $\text{Pic}(S)$. Equivalently, since

$$\text{Pic}(S) = H^2(S, \mathbb{Z}) = \mathbb{Z}^{10} \oplus \mathbb{Z}/2\mathbb{Z},$$

the class given by (2.1) is 3-divisible in $\text{Num}(S)$, the lattice given by divisors up to numerical equivalence. Recall that

$$\text{Num}(S) = U + E_8$$

where $U$ denotes the unimodular hyperbolic plane. One can easily check using the integrality of $\text{Num}(S)$ that a 3-divisible set on an Enriques surface consists of exactly three $A_2$-configurations (i.e., $l = 3$).

We follow the approach of [7, §3] and put $M$ (resp. $\overline{M}$) to denote the lattice spanned by $F'_1, \ldots, F''_4$ in $\text{Num}(S)$ (resp. its primitive closure).

**Lemma 2.1.** The index of $M$ inside $\overline{M}$ satisfies $[\overline{M} : M] \in \{3, 3^2\}$.

**Proof.** The lattice $M$ has the discriminant $d(M) = 3^4$, so $[\overline{M} : M] \in \{1, 3, 3^2\}$. We claim that the first case is impossible. Indeed, suppose that $M = \overline{M}$. Then $M \hookrightarrow \text{Num}(S)$ is a primitive embedding, so

$$M^\perp / M \cong (M^\perp)^\perp / M^\perp.$$  

By assumption the left-hand side is isomorphic to $(\mathbb{Z}/3\mathbb{Z})^4$ while the right-hand side comes from the rank-2 lattice $M^\perp$, thus has rank 2 and thus length at most 2, contradiction. \qed

**Corollary 2.2.** The four $A_2$-configurations $F'_1, \ldots, F''_4$ contain either one or four 3-divisible sets.
In particular, we can infer that
\[
F_1', \ldots, F_4' \text{ contain four 3-divisible sets iff } \overline{M} \text{ is unimodular.}
\]
In other words, in this case each triplet of the $A_2$-configurations in question is 3-divisible up to relabelling the rational curves.

2.2. Elliptic fibrations. We start by recalling some basic concepts and relations. Any complex Enriques surface $S$ admits an elliptic fibration
\[
\varphi : S \rightarrow \mathbb{P}^1.
\]
There are two fibres of multiplicity two; their supports are usually called half-pencils. The difference of the two half-pencils gives the canonical divisor which represents the two-torsion in $H^2(S, \mathbb{Z})$. This already shows that the fibration cannot have a section, but by [2, Prop. VIII.17.6] there always is a bisection $R$ of square $R^2 = 0$ or $-2$, i.e. a smooth irreducible curve $R$ such that $R.F = 2$ for any fibre of \(2.3\).

The moduli of Enriques surfaces can be studied through the universal cover
\[
\pi : Y \rightarrow S
\]
which is a K3 surface. By construction, this induces an elliptic fibration
\[
\bar{\varphi} : Y \rightarrow \mathbb{P}^1
\]
which fits into the commutative diagram
\[
\begin{array}{ccc}
Y & \xrightarrow{2:1} & S \\
\downarrow \varphi & & \downarrow \varphi \\
\mathbb{P}^1 & \xrightarrow{2:1} & \mathbb{P}^1
\end{array}
\]
The bottom row degree-2 morphism
\[
\mathbb{P}^1 \xrightarrow{2:1} \mathbb{P}^1
\]
ramifies exactly in the points below the multiple fibres. Moreover, the universal covering induces a primitive embedding
\[
U(2) + E_8(2) \cong \pi^* \operatorname{Num}(S) \hookrightarrow \operatorname{Pic}(Y)
\]
which lends itself to a study of K3 surfaces with the above lattice polarisation. Abstractly, a complex K3 surface admits an Enriques involution if and only if there is a primitive embedding of $U(2) + E_8(2)$ into Pic without perpendicular roots (i.e. classes of smooth rational curves). In view of this, it is evident that a bisection $R$ of square $R^2 = 0$ occurs generically, since on the contrary any $(-2)$-curve on $S$ necessarily splits into two disjoint smooth rational curves on the K3 cover $Y$; these give sections of \(2.3\), causing the Picard number to go up to 11 at least. The same generic behaviour will occur on our families $F_3, 3, 3, 3, F_4, 3, 1$ in Section 3.

On the other hand, we can consider the Jacobian fibration of \(2.3\). This will be a rational elliptic surface
\[
X \rightarrow \mathbb{P}^1
\]
with section and is thus governed by means of explicit classifications, e.g. using the theory of Mordell-Weil lattices in [15]. Naturally $S$ and $X$ share the same singular fibers, except that on $S$, certain smooth or semi-stable fibers (Kodaira type $I_n, n \geq 1$) may come with multiplicity two. The Enriques surface $S$ can be recovered from $X$ through a logarithmic transform which depends on the choice of non-trivial 2-torsion points in two distinct smooth or semi-stable fibers of (2.7) (see e.g. [3, § 1.6]). Intrinsically this leads to another K3 surface in terms of the jacobian elliptic fibration arising from (2.7) through the quadratic base change (2.6) ramified in the two distinct fibers where the logarithmic transform changed the multiplicities of fibers. It is clear from the construction, that at the same time this K3 surface features as the Jacobian of (2.4). That is, we get another commutative diagram

\[
\begin{array}{ccc}
\text{Jac}(Y) & \xrightarrow{2:1} & X_{3,3,3,3} \\
\downarrow & & \downarrow \\
\mathbb{P}^1 & \xrightarrow{2:1} & \mathbb{P}^1
\end{array}
\]

Recall that the depicted elliptic fibrations on $Y$ and $\text{Jac}(Y)$ share the same configurations of singular fibers and the same Picard numbers.

For some purposes, the above construction has the drawback of being analytical in nature. This can be circumvented in the special situation where the elliptic fibration (2.4) is already jacobian, i.e. admits a section. For instance, this occurs in the presence of a bisection $R$ of (2.3) with square $R^2 = -2$ as indicated above. A more general framework for this to occur was introduced in terms of involutions of base change type in [5]. Here one considers the quadratic twist $X'$ of $X$ which acquires $I_n^*$ fibres ($n \geq 0$) at the two ramification points of (2.6). In consequence, the quadratic base change (2.6) applied to either $X$ and $X'$ gives the same K3 surface $Y$.

For any section on $X'$, the pull-back to $Y$ is anti-invariant with respect to the involution $\iota$ on $Y$ induced by the deck transformation of (2.6) (s.t. $Y/\iota = X$). It follows that $\iota$ composed with translation by the anti-invariant section defines another non-symplectic involution on $Y$. This has fixed points, necessarily in the ramified fibers, if and only if the section meets the identity components of the two twisted fibres on $X'$. Otherwise, for instance if the section is two-torsion, we obtain an Enriques involution on $Y$ we will refer to as an involution of base change type.

2.3. Picard-Lefschetz reflections. Recall that by Kodaira’s work [9], the irreducible components of a singular fibre of an elliptic fibration correspond to an extended Dynkin diagram; a Dynkin diagram, or equivalently root lattice, can be obtained from the singular fiber by omitting any simple component. Given $A_2$-configurations, it is thus natural to ask whether these correspond to rational curves supported on the fibres of an elliptic fibration on $S$. While this may not be true in general, we can weaken the limitations by considering the question up to automorphisms of $H^2(S, \mathbb{Z})$. This will allow us to reduce the problem of 3-divisible sets of $A_2$-configurations to the study of certain elliptic fibrations on Enriques surfaces. To this end, recall that each smooth rational
curve $E$ in $S$ (more generally each $(-2)$-class in $\text{H}^2(S, \mathbb{Z})$) defines a Picard-Lefschetz reflection

$$s_E : \text{H}^2(S, \mathbb{Z}) \ni D \mapsto D + (D.E)E \in \text{H}^2(S, \mathbb{Z}).$$

In the sequel we will crucially use the following corrected version of [7, Claim 3.5.1] (which did not include the configurations (2.10) neither the degenerate case of (2.11)):

**Lemma 2.3.** There exists an elliptic pencil $|E|$ on $S$ and smooth rational curves $E_1, \ldots, E_k \subset S$ such that the image of each curve $F'_j, F''_j$, where $j = 1, \ldots, 4$, under the map

$$(2.9) \quad p_S := (s_{E_1} \circ \ldots \circ s_{E_k})$$

is the class of a smooth rational curve which is an irreducible component of a member of the pencil $|E|$. Moreover, the elliptic fibration given by $|E|$ is either of the type

$$(2.10) \quad I_3^4, I_3^3 \oplus 2I_3, I_3^2 \oplus (2I_3)^2$$

or of the type

$$(2.11) \quad IV^* \oplus I_3 \oplus I_1, IV^* \oplus 2I_3 \oplus I_1, IV^* \oplus 2I_3 \oplus 2I_1, IV^* \oplus IV.$$

**Proof.** The existence of rational curves $E_1, \ldots, E_k$ such that all $p_S(F'_j), p_S(F''_j)$ are components of members of an elliptic pencil $|E|$ is shown in the proof of [7, Claim 3.5.1].

As explained in 2.2, the jacobian fibration of $\pi_{|E|}$ is a rational elliptic surface $X$. With 4 disjoint $A_2$-configurations in the fibers, $X$ is automatically extremal by the Shioda-Tate formula, i.e. $X$ has finite Mordell-Weil group. Going through the classification in [11], one finds that $X$ may have the following configurations:

$$I_3^4, IV^* \oplus I_3 \oplus I_1, IV^* \oplus IV.$$

The possible configurations (2.10) and (2.11) follow immediately. □

**Remark 2.4.** On the extremal rational elliptic surfaces, the orthogonal $A_2$-configurations gives rise to 3-torsion sections by way of 3-divisibility. Essentially, this holds because $\text{H}^2(X, \mathbb{Z})$ is unimodular. Since the same applies to $\text{Num}(S)$, we will be able to establish the same results on $S$, even though there is no section, see Lemma 3.5.

In the last part of this section we study Picard-Lefschetz reflections on the K3-cover $Y$ of $S$. Let $\pi : Y \rightarrow S$ be the K3-cover with induced elliptic fibration (2.3) and let $\psi \curvearrowleft Y$ be the Enriques involution. Given a smooth rational curve $E$ in $S$, the preimage $\pi^{-1}(E)$ consists of two disjoint smooth rational curves $E^+, E^-$. We maintain the notation of Lemma 2.3 and define

$$(2.12) \quad p_Y := (s_{E_1^+} \circ s_{E_1^-} \circ \ldots \circ s_{E_k^+} \circ s_{E_k^-}).$$

This reflection is independent of the order of $E_i^+, E_i^-$ as we shall exploit below. Let $D \in \text{Pic}(S)$. Observe that $(D.E_1) = (\pi^*D.E_1^+) = (\pi^*D.E_1^-)$. In particular,
we have
\[(s_{E_1^+} \circ s_{E_1^-})(\pi^* D) = \pi^* D + (\pi^* D.E_1^+)E_1^+ + (\pi^* D.E_1^-)E_1^- \]
\[= \pi^* (D + (D.E_1)E_1) = \pi^*(s_{E_1}(D)).\]
This yields the equality
\[p_Y \circ \pi^* = \pi^* \circ p_S.\]
Similarly, one can show that
\[p_Y \circ \psi^* = \psi^* \circ p_Y.\]
Moreover, one has the equality
\[\pi_*(p_Y(E^+)) = \pi_*(p_Y(E^-)) = p_S(E).\]
The latter implies that \(p_Y(F_j^{\pm})\), \(p_Y(F_j^\mu\pm)\) are represented, up to sign, by smooth rational curves contained in singular fibers of the elliptic fibration \([2,4]\) induced by \(\lfloor \pi^* E \rfloor\); here we use that a nodal class on a K3 surface is represented by a unique effective divisor by Riemann-Roch. In particular, \(Y\) inherits 8 orthogonal \(A_2\)-configurations from \(S\). We label the curves \(F_j^{\pm}, F_j^\mu\pm, p_SF_j^{\pm}, p_SF_j^\mu\pm\) in such a way that
\[p_Y(F_j^{\pm}) = p_S F_j^{\pm}\text{ and }p_Y(F_j^\mu\pm) = p_S F_j^\mu\pm \text{ for } j = 1, \ldots, 4.\]

3. Two families of Enriques surfaces with four cusps

In this section we construct families of Enriques surfaces with four disjoint \(A_2\)-configurations supported on the fibers of an elliptic fibration (following Lemma [2,3]) and study 3-divisible sets on them.

3.1. First family of Enriques surfaces. Let \(X_{3,3,3,3}\) be the extremal rational elliptic surface with four singular fibers of the type \(I_3\). Locating them at the third roots of \((-1)\) and at \(\infty\), the surface is given by the Hesse pencil
\[X_{3,3,3,3} : x^3 + y^3 + z^3 + 3\lambda xyz = 0.\]
Here the 3-torsion section alluded to in Remark [2,4] enter as the base points of the cubic pencil. An Enriques surface is obtained from \(X_{3,3,3,3}\) by applying logarithmic transformations of order 2 to the elliptic fibers over two distinct points \(P_1, P_2 \in \mathbb{P}^1\). As explained in [2,2] this depends on the choice of 2-torsion points in the fibers of \(X_{3,3,3,3}\) over \(P_1, P_2\). However, this subtility will not be relevant for our purposes (e.g. the moduli of the covering K3 surface do not depend on the choice of 2-torsion points, see Proposition [5,11]). Therefore we will allow ourselves to abuse notation and denote the resulting Enriques surface simply by \(S_{P_1, P_2}\). We obtain a 2-dimensional family
\[\mathcal{F}_{3,3,3,3} := \{S_{P_1, P_2} : P_1 \neq P_2\}\]
of Enriques surfaces parametrized by pairs of 2-torsion points in distinct fibers of the fibration on \(X_{3,3,3,3}\).

Now, let \(S = S_{P_1, P_2} \in \mathcal{F}_{3,3,3,3}\) be an Enriques surface with K3-cover \(Y\) and elliptic fibrations \(\varphi, \tilde{\varphi}\) in the notation of [2,2] We continue by establishing some information about \(Y\) with the help of \(\text{Jac}(Y)\). As soon as \(P_1, P_2\) do not hit \(\infty\)
and third roots of \((-1)\), we obtain eight fibers of the type \(I_3\) on \(Y\) and \(\text{Jac}(Y)\), so the Picard number \(\rho(Y) = \rho(\text{Jac}(Y))\) is at least 18 by the Shioda-Tate formula.

**Lemma 3.1.** If \(\rho(Y) = 18\), then \(\text{NS}(Y)\) has discriminant \(d(\text{NS}(Y)) = -324\).

**Proof.** By assumption, \(\text{Jac}(Y)\) has finite Mordell-Weil group. The configuration of singular fibers only accommodates 3-torsion, so we infer

\[
\text{MW}(\text{Jac}(Y)) \cong (\mathbb{Z}/3\mathbb{Z})^2
\]

by pull-back from \(X_{3,3,3,3}\). Hence \(d(\text{NS}(\text{Jac}(Y))) = -81\). By the existence of a bisection on \(Y\) (induced from \(S\), see 2.2), we infer from \(\text{Lemma 2.1}\) that

\[
\text{either } d(\text{NS}(Y)) = -81 \text{ or } d(\text{NS}(Y)) = -324
\]

as soon as \(\rho(Y) = 18\). (Here the former equality holds iff \(\tilde{\phi}\) admits a section i.e. iff \(Y = \text{Jac}(Y)\).) Lemma 3.1 now results immediately from the following proposition. \(\Box\)

**Proposition 3.2.** Let \(Y\) be the K3-cover of an Enriques surface. Then

\[
2^{20-\rho(Y)} | d(\text{NS}(Y)).
\]

In particular, if \(d(\text{NS}(Y))\) is odd, then \(\rho(Y) = 20\).

**Proof.** We shall use the primitive embedding

\[
L := U(2) + E_8(2) \cong \pi^* \text{Num}(S) \rightarrow \text{NS}(Y).
\]

We follow the notation of \(\text{[13, \S 5]}\) and denote the discriminant group of \(L\) by

\[
A_L := L^\vee/L;
\]

likewise for other primitive sublattices of \(\text{NS}(Y)\) such as \(L_L\). Define the finite abelian group

\[
H := \text{NS}(Y)/(L \oplus L_L).
\]

Obviously we have the inclusion

\[
H \subset A_L \oplus A_L L_L.
\]

Let \(p_L\) (resp. \(p_{L_L}\)) be the projection from \(A_L \oplus A_L L_L\) onto the first (resp. the second) summand. By \(\text{[13, p. 111]}\) either projection is an embedding. The first embedding implies

\[
H \cong (\mathbb{Z}/2\mathbb{Z})^l,
\]

while the second shows \(l \leq \rho - 10\) since the length of \(A_L L_L\) is bounded by the rank of \(L_L\). We obtain

\[
d(\text{NS}(Y)) = d(L \oplus L_L)/|H|^2 = 2^{10-l} \cdot (d(L_L)/2^l).
\]

Note that the right-most term in brackets is an integer since \(|H| = |p_{L_L}(H)|\) divides \(|A_L L_L| = d(L_L)|. Hence we infer that \(2^{20-\rho} | d(\text{NS}(Y))\) as claimed. \(\Box\)

**Remark 3.3.** A detailed analysis using the 2-length of the groups involved allows one to strengthen the above line of arguments to prove that the K3 cover \(Y\) of an Enriques surface has \(A_{\text{NS}(Y)}\) of 2-length at least \(20 - \rho(Y)\).
3.2. 3-divisible sets. We shall now investigate the 3-divisible sets among the
4 $A_2$-configurations given by fibers of an Enriques surface $S \in \mathcal{F}_{3,3,3,3}$. Our
main results will be formulated in Lemma 3.5 and Lemma 3.6.

Let $G$ be a 2-section of the elliptic fibration $\varphi$ and let $F_j, F'_j, F''_j$, where
$j = 1, \ldots, 4$, be the components of the $I_3$-fibers of $\varphi$. In order to streamline our
notation we label the components of the singular fibers in the following way relative to $G$:

**Notation 3.4.** If $G$ meets only one component of an $I_3$-fiber we denote this
component by $F_j$. Otherwise, $F'_j, F''_j$ stand for the components of the $I_3$-fiber
that meet the 2-section $G$ (i.e. we have $G.F_j = 0$ then).

In particular, if $(F_j + F'_j + F''_j)$ happens to be a half-pencil of the fibration
in question, we assume that $G.F_j = 1$. After those preparations we can study
3-divisible sets in the fibers of the elliptic fibration $\varphi$ on $S$ and $\tilde{\varphi}$ on $Y$.

**Lemma 3.5.** Let $S \in \mathcal{F}_{3,3,3,3}$. The $A_2$-configurations
\begin{equation}
F'_1, F''_1, \ldots, F'_4, F''_4
\end{equation}
contain four 3-divisible sets.

**Proof.** By (2.2) it suffices to prove that $\overline{M}$ is unimodular, the primitive closure
of the lattice $M$ spanned in $\text{Num}(S)$ by the curves (3.2). Equivalently, $M^\perp = \overline{M}^\perp$ is unimodular. To see this, define an auxiliary divisor class
\begin{equation}
D := G + \sum_{\{j: G.F_j = 0\}} (F'_j + F''_j) \in M^\perp.
\end{equation}

Let $B$ denote a half-pencil of the fibration $\varphi$. By construction, $B \in M^\perp$, and $B, D$ span the hyperbolic plane $U$ since $D.B = G.B = 1$ and $B^2 = 0$. Thus $\overline{M}$
is unimodular, and the proof of Lemma 3.5 is completed by (2.2). $\square$

We shall now eliminate all but one 3-divisible classes by considering a different
configuration of 4 $A_2$’s on $S \in \mathcal{F}_{3,3,3,3}$. Recall that $F'_j, F''_j$ stand for the
$(-2)$-curves on the K3-cover $\pi : Y \to S$ that lie over the smooth rational
curve $F_j$, and likewise for $F'_j, F''_j$. A discussion of properties of 3-divisible
sets of $A_2$-configurations on K3 surfaces can be found in [1]. We call a set
of $A_2$-configurations on a K3-surface a trivial 3-divisible set iff it is a linear
combination of rational curves with coefficients in $3\mathbb{Z}$. In particular, by [1,
Lemma 1], a (non-trivial) 3-divisible set of $A_2$-configurations on a K3 surface
consists always of six or nine such configurations.

**Lemma 3.6.** Let $S \in \mathcal{F}_{3,3,3,3}$. Then
(a) The four $A_2$ configurations
\begin{equation}
F'_1, F''_1, \ldots, F'_4, F''_4
\end{equation}
on the Enriques surface $S$ contain exactly one 3-divisible set.
(b) The eight $A_2$ configurations
\begin{equation}
F''_1, F''_1, F''_1, F''_1, F'_3, F'_3, F''_4, F''_4
\end{equation}
on the K3-cover $Y$ contain exactly one 3-divisible set.
Proof. (a): By \((3.5)\) we are to show that \((3.3)\) does not contain four 3-divisible sets. Suppose to the contrary. Then each triplet of \(A_2\)-configurations in \((3.3)\) is 3-divisible. In particular, we have

\[
\sum_{j=2}^{3} (\lambda_j^1 F_j^1 + \lambda_j^2 F_j^2 + \lambda_j^3 F_j^3) = 3\mathcal{L},
\]

where \(\{\lambda_j^1, \lambda_j^2\} = \{1, -1\}\).

Since \(G.(\lambda_j^1 F_j^1 + \lambda_j^2 F_j^2) \in 3\mathbb{Z}\) for \(j = 2, 3\), we obtain \(G.(\lambda_4 F_4 + \lambda_4^3 F_4^3) \in 3\mathbb{Z}\).

If \(G\) meets only the curve \(F_4\) in the fiber \((F_4 + F_4^1 + F_4^2)\) (resp. \(2(F_4 + F_4^1 + F_4^2)\) iff we deal with a half-pencil) we have \(G.F_4 \in \{2, 1\}\) and \(G.F_4^3 = 0\), so \(\lambda_4 \in 3\mathbb{Z}\). Contradiction.

Otherwise, \(G\) meets the fiber \((F_4 + F_4^1 + F_4^3)\) in two different points, i.e. \(G.F_4 = G.F_4^3 = 1\) and \(G.F_4 = 0\), which yields \(\lambda_4 \in 3\mathbb{Z}\). Again we arrive at a contradiction, which implies by symmetry and Lemma \(2.1\) that

\[
(3.5) \quad F_1^1, F_1^3, F_2^2, F_2^3, F_3^2, F_3^3 \text{ form the unique 3-divisible set in } (3.3).
\]

(b): Since the pull-back of a (non-trivial) 3-divisible divisor under \(\pi\) is (non-trivial) 3-divisible, \((3.5)\) implies that the six \(A_2\)-configurations

\[
(3.6) \quad F_{1+}, F_{1-}, F_{1+}, F_{2+}, F_{2-}, F_{2+}, F_{3+}, F_{3-}, F_{3+}
\]

are 3-divisible on the K3-cover \(Y\). To show that they form the unique 3-divisible configuration in \((3.3)\), assume that the \(A_2\)-configuration \(F_{1+}, F_{1+}\) is contained in another non-trivial 3-divisible set on \(Y\).

Suppose that the curves \(F_{1-}, F_{1-}\) are not contained in the 3-divisible divisor in question. Since \(\pi\) is unramified, push-forward yields a non-trivial 3-divisible set of three \(A_2\)-configurations in \((3.3)\) that contains \(F_4, F_4^3\). The latter is impossible by \((3.5)\).

Thus we can assume that the curves \(F_{1-}, F_{1-}, F_{1+}, F_{1+}\) are contained in the support of the 3-divisible divisor in question. From the properties of the push-forward \(\pi_*\) and \((3.5)\), we infer the existence of \(\lambda_1^+, \lambda_1^- \in \{1, -1\}\), such that one has

\[
\sum_{j=1}^{3} (\lambda_j^1 F_j^1 + \lambda_j^2 F_j^2 + \lambda_j^3 F_j^3 + \lambda_j^4 F_j^4 + \lambda_j^5 F_j^5 + \lambda_j^6 F_j^6) + (F_{1-} - F_{1-}) = 3\tilde{\mathcal{L}}
\]

for a divisor \(\tilde{\mathcal{L}}\) on \(Y\). By Lemma \(3.5\) each triplet of \(A_2\)-configurations in \((3.2)\) is 3-divisible, so we can assume that for \(j = 2, 3, 4\) there exist \(\mu_j^1, \mu_j^3\), such that

\[
\sum_{j=2}^{4} (\mu_j^1 F_j^1 + \mu_j^2 F_j^2) + \mu_j^3 (F_j^3 + F_j^4) = 3\tilde{\mathcal{L}} \text{ and } \{\mu_j^1, \mu_j^3\} = \{1, -1\}
\]

for some \(\tilde{\mathcal{L}} \in \text{Pic}(Y)\). After exchanging components, if necessary, we can assume that \((\mu_j^1, \mu_j^3) = (1, -1)\). By adding the previous two equalities and subtracting three copies of the fiber of the elliptic fibration \(\tilde{\varphi}\), we derive a 3-divisible divisor

\[
D = (F_{1+}^1 - F_{1+}^1 - F_{1+}^1)
\]

with \(\text{supp}(D)\) contained in the union of the curves \(F_j^{1\pm}, F_j^{3\pm}\) for \(j = 1, 2, 3\). We continue to establish a contradiction.
Recall (see e.g. [16 § 5]) that each non-trivial 3-divisible set on \( Y \) corresponds to a line \( \mathbb{F}_3v \), where \( v \) is a non-zero vector in the kernel of the \( \mathbb{F}_3 \)-linear map

\[
\mathbb{F}_3^6 \ni (\lambda_1^+, \ldots, \lambda_4^-) \mapsto \sum_{j=1}^4 \lambda_j^+ (F_j^+ - F_j'^+) + \sum_{j=1}^4 \lambda_j^- (F_j^- - F_j''^-) \in \text{Pic}(Y) \otimes \mathbb{F}_3.
\]

Thus the kernel in question is a ternary \([8, d, 6]\)-code (i.e. a \( d \)-dimensional subspace of \( \mathbb{F}_3^6 \), such that all its non-zero vectors have exactly 6 non-zero coordinates). By the Griesmer bound (see e.g. [23, Thm (5.2.6)]), we have \( d \leq 2 \) and \( F_j^\pm, F_j'^\pm \), where \( j = 1, \ldots, 4 \), contain at most four sets of 3-divisible \( \mathbb{A}_2 \)-configurations. On the other hand we obtain four non-trivial \( \psi^* \)-invariant 3-divisible sets by pulling-back the 3-divisible sets from \( S \) (see Lemma [5.5]). Observe that the 3-divisible set given by \([3,7] \) is not \( \psi^* \)-invariant. Contradiction.

\[ \square \]

3.3. Second family of Enriques surfaces. In the following paragraphs, we work out Enriques surfaces with elliptic fibrations of the types \([2,11] \). To this end, we consider another extremal rational elliptic surface, or in fact two of them. Consider the rational elliptic surface \( X \) given in Weierstrass form

\[
X : \quad y^2 + cxy + ty = x^3, \quad c \in \mathbb{C}.
\]

This has a fibre of Kodaira type \( IV^* \) at \( \infty \) and a 3-torsion section at \((0,0)\). The fibre type at \( t = 0 \) depends on \( c \) as follows.

If \( c \neq 0 \), then we can rescale \( x, y \) to reach the normalisation \( c = 1 \). We denote the resulting elliptic surface with singular fibers of types \( IV^*, I_3, I_1 \) by \( X_{4,3,1} \).

If \( c = 0 \), then the singular fibre at \( t = 0 \) degenerates to Kodaira type \( IV \) as the fibration becomes isotrivial \((j = 0)\). This is the special case in \([2,11] \) omitted in \([7] \). For our purposes, it is not necessary to pay special attention to this isolated surface for the following reason: the Enriques surfaces arising from this rational elliptic surfaces via logarithmic transform have only one-dimensional moduli because we can still move around one fiber. In terms of the elliptic K3 surfaces arising via the quadratic base \([2,6] \), this corresponds to a 1-dimensional subfamily of the 2-dimensional family arising from \( X_{4,3,1} \) (see Remark 5.7). Thus we will restrict to the study of \( X_{4,3,1} \) in what follows.

For two distinct points \( P_1, P_2 \in \mathbb{P}^1 \setminus \{\infty\} \), we let \( S'_{P_1,P_2} \) denote the Enriques surface obtained by applying logarithmic transformations of order 2 to the fibers of \( X_{4,3,1} \) over \( P_1, P_2 \). Suppressing the choice of 2-torsion points in the fibers for simplicity as before, we obtain another 2-parameter family

\[ \mathcal{F}_{4,3,1} := \{ S'_{P_1,P_2} : P_1, P_2 \neq \infty \text{ and } P_1 \neq P_2 \} \]

of Enriques surfaces. On \( S' = S'_{P_1,P_2} \in \mathcal{F}_{4,3,1} \), we put

\[ 3F_0 + \sum_{j=1}^3 (F_j' + 2F_j'') \]

(resp. \( F_4 + F_4' + F_4'' \)) to denote the \( IV^* \)-fiber (resp. the \( I_3 \)-fiber) of the induced elliptic fibration \( \varphi \). It is immediate that, up to the choice of the curve \( F_4 \), the rational curves

\[ F_1', F_1'', F_4', F_4'' \]
form the only set of four disjoint $A_2$-configurations contained in the singular fibers of the fibration $\varphi$.

Let $\pi : Y' \to S'$ be the K3-cover and let $\tilde{\varphi}$ be the fibration induced by $\varphi$ on $Y'$. The number of 3-divisible sets on $S'$ (resp. $Y'$) supported on the components of fibers of $\varphi$ (resp. $\tilde{\varphi}$) can be found using [7, Lemma 3.5 (1)].

**Lemma 3.7.** Let $S' \in F_{4,3,1}$. Then the four $A_2$ configurations

$$F'_1, F''_1, \ldots, F'_4, F''_4$$

contain exactly one 3-divisible set, whereas the eight $A_2$-configurations $(F'_1, F''_1, \ldots, F'_4, F''_4)$ on the K3-cover contain four 3-divisible sets.

**Proof.** By [7, Lemma 3.5 (1)] the set $F'_1, F''_1, \ldots, F'_4, F''_4$ is not 3-divisible, whereas the six $A_2$-configurations $F'^{+}_1, F''^{+}_1, F'^{-}_1, F''^{-}_1, \ldots, F'^{+}_4, F''^{+}_4$ form a 3-divisible set on $Y'$ (pushing down to a trivially 3-divisible divisor on $S'$). The former assertion rules out the second possibility of (2.2).

On the other hand, the curves $F'_1, F''_1, \ldots, F'_4, F''_4$ contain at least one 3-divisible set by (2.2). As the pullback under $\pi$ we obtain another 3-divisible set on $Y'$, so the K3-cover contains exactly four 3-divisible sets (see the proof of Lemma 3.6). $\square$

3.4. **Explicit examples supporting each case.**

**Example 3.8.** Let $S'$ be the Enriques surface with finite automorphism group $S_4 \times \mathbb{Z}/2\mathbb{Z}$ considered in [10, Example V] arising from the Kummer surface of $E^2$ for the elliptic curve with zero $j$-invariant. By [10, Table 2 on p. 132] we have

$$S' \in F_{4,3,1} \setminus F_{3,3,3,3}.$$ 

In fact, by [10, Remark (4.29)] no surface in $F_{3,3,3,3}$ has finite automorphism group, so we can find no example of a surface from $F_{3,3,3,3}$ in [10]. Therefore, we will use the construction of Enriques involution of base change type (as reviewed in (2.2)) to obtain an explicit example of such an Enriques surface.

**Example 3.9.** Let $Y$ denote the singular K3 surface with transcendental lattice

$$(3.8) \quad T(Y) = \begin{pmatrix} 6 & 3 \\ 3 & 12 \end{pmatrix}.$$ 

In what follows, we will sketch in a rather conceptual way that $Y$ admits an Enriques involution of base change type whose quotient surface is in $F_{3,3,3,3}$.

We start from elliptic curves $E$ parametrised by the $j$-invariant. To $E^2$, we can associate the Kummer surface $\text{Kum}(E^2)$, but also by way of what is called a Shioda-Inose structure nowadays (cf. [19]) a K3 surface which recovers the transcendental lattice of $E^2$. We thus obtain a one-dimensional family of K3 surfaces $\mathcal{Y'}$ with generic transcendental lattice

$$T(\mathcal{Y'}) = U + (2).$$ 

By Nishiyama’s method [14], $\mathcal{Y'}$ comes with an elliptic fibration with a fiber of Kodaira type $I_{18}$ and generically $\text{MW}(\mathcal{Y'}) \cong \mathbb{Z}/3\mathbb{Z}$. Quotienting out by
translation by the 3-torsion sections, we obtain another family of K3 surfaces \( \mathcal{Y} \), generically with one fiber of type \( I_6 \), 6 fibers of type \( I_3 \) and
\[
MW(\mathcal{Y}) \cong (\mathbb{Z}/3\mathbb{Z})^2.
\]
It follows that \( \mathcal{Y} \) arises from \( X_{3,3,3,3} \) by the one-dimensional family of base changes \((2.6)\) ramified at a given singular fiber, say \( t = \infty \). That is, there is an involution \( \iota \cap \mathcal{Y} \) such that \( \mathcal{Y}/\iota = X_{3,3,3,3} \). For discriminant reasons, the transcendental lattice is scaled by a factor of 3
\[
(3.9) \quad T(\mathcal{Y}) \cong T(\mathcal{Y}')/(3)
\]
where this equality not only holds generically, but also on the level of single members of the families (cf. [17, Lemma 8]).

We continue by specialising to a member \( Y \in \mathcal{Y} \) in order to endow \( Y \) with a section which combines with \( \iota \) to an Enriques involution of base change type. To this end, choose \( E \) to be the elliptic curve with CM by \( \mathbb{Z}[\omega] \) for \( \omega = (1+\sqrt{-7})/2 \) (j-invariant \( -15 \)). By [20],
\[
T(E)^2 \cong \begin{pmatrix} 2 & 1 \\ 1 & 4 \end{pmatrix}
\]
which exactly gives rise to \((3.3)\) by \((3.9)\). Inside the family \( \mathcal{Y} \), this can only be accounted for by a section \( Q \) of height \( 7/6 \). It is induced from a section \( Q' \) of height \( 7/12 \) on the quadratic twist \( X_{3,3,3,3}' \) of \( X_{3,3,3,3} \). Here \( X_{3,3,3,3}' \) has singular fibers of types \( I_0^* \), 3 times \( I_3 \) and \( I_0^* \); the given height can only be attained if \( Q' \) is perpendicular to \( O' \) and intersects nontrivially \( I_0^* \) (far simple component), one \( I_3 \) and \( I_0^* \). In consequence, the pull-back \( Q \) on \( Y \) is disjoint from \( O \) and anti-invariant for \( \iota \). Hence
\[
j := \iota \circ \text{(translation by } Q)\]
defines an Enriques involution on \( Y \) such that \( Y/j \in \mathcal{F}_{3,3,3,3} \) as claimed. All of this can be made explicit without much difficulty. For instance, spelling out the conditions for \( Q' \) on \( X_{3,3,3,3}' \), one finds that the second ramification point of the quadratic base change \((2.6)\) is located at \( t = 5/4 \). We leave the details to the reader.

4. The fundamental groups of open Enriques surfaces

Let \( S \) be an Enriques surface that contains four disjoint A_2-configurations \( F'_1, F''_1, \ldots, F'_4, F''_4 \) and let \( \pi : Y \to S \) be the K3-cover. As in the preceding sections, the \((-2)\)-curves in \( \pi^{-1}(F'_j) \) are denoted by \( F'^+_j, F'^-_j \). Moreover, we put
\[
\mathcal{A} := \{ F'_1, F''_1, \ldots, F'_4, F''_4 \} \quad \text{and} \quad \mathcal{A}^\pm := \{ (F'_j)^+, (F'_j)^-, (F''_j)^+, (F''_j)^- \}.
\]
Given the pair \((S, \mathcal{A})\), we follow [17] and define the fundamental group of the open Enriques surface \( S^o = S \setminus \mathcal{A} \):
\[
\pi_1(S, \mathcal{A}) := \pi_1(S^o).
\]
To deal with Enriques surfaces with four A_2-configurations in more generality we introduce the following notation:
Notation 4.1. We say that the pair \((S, \mathcal{A})\) belongs to \(\mathcal{F}_{4,3,1}\) (resp. \(\mathcal{F}_{3,3,3,3}\)) iff there exists a composition of Picard-Lefschetz reflections \(2.9\) such that all curves \(p_S(F'_j), p_S(F''_j)\), where \(j = 1, \ldots, 4\), are components of members of an elliptic pencil that has fibers of the types \(2.11\) (resp. \(2.10\)). To simplify our notation we write

\[ (S, \mathcal{A}) \in \mathcal{F}_{4,3,1} \text{ (resp. } (S, \mathcal{A}) \in \mathcal{F}_{3,3,3,3} \text{)} \]

when the above condition is satisfied. Then \(p_S(\mathcal{A})\) stands for the set of the four \(A_2\)-configurations \(p_S(F'_1), \ldots, p_S(F''_4)\) for a fixed composition \(p_S\).

Recall that \(p_S\) induces the map \(p_Y\) (see \(2.12\)). In the sequel we maintain the notation \(2.13\) and use \(p_Y(\mathcal{A}^\pm)\) to denote the set of the eight \(A_2\)-configurations on the K3-cover (again supported on the fibers of an elliptic fibration).

As we explained around Lemma 2.3, the authors of [7] claim that after applying an appropriate composition of Picard-Lefschetz reflections \(p_S\), the four \(A_2\)-configurations on the Enriques surface \(S\) become components of singular fibers of the fibration of type \(2.11\). The latter implies the erroneous claim that \(\mathcal{A}\) never contains four 3-divisible sets (7, Lemma 3.5 (2))) and the fundamental group \(\pi_1(S^0)\) of the open Enriques surface is either \(\mathbb{Z}/6\mathbb{Z}\) or \(S_3 \times \mathbb{Z}/3\mathbb{Z}\) (see [7] Lemma 3.6 (3)). Here we correct these claims.

Lemma 4.2. Let \(S\) be an Enriques surface with four \(A_2\)-configurations \(\mathcal{A}\). Then

\[ \pi_1(S^0) \in \{S_3 \times \mathbb{Z}/3\mathbb{Z}, \mathbb{Z}/6\mathbb{Z}, (\mathbb{Z}/3\mathbb{Z})^\oplus 2 \times \mathbb{Z}/2\mathbb{Z}\}. \]

Moreover, one has the following characterizations:

(a) Both \(\mathcal{A}\) and \(\mathcal{A}^\pm\) contain exactly one 3-divisible set iff \(\pi_1(S^0) = \mathbb{Z}/6\mathbb{Z}\).

(b) \(\mathcal{A}\) contains exactly one 3-divisible set and \(\mathcal{A}^\pm\) contains four 3-divisible sets iff

\[ \pi_1(S^0) = S_3 \times \mathbb{Z}/3\mathbb{Z}. \]

(c) \(\mathcal{A}\) contains four 3-divisible sets iff \(\pi_1(S^0) = (\mathbb{Z}/3\mathbb{Z})^\oplus 2 \times \mathbb{Z}/2\mathbb{Z}\).

Proof. The proof follows almost verbatim the first part of the proof of [7] Lemma 3.6 (3), but there is one addition to be made: Lemma 3.5 shows that one cannot use [7] Lemma 3.5 (2) to rule out the existence of \(S\) with \(\pi_1(S^0) = (\mathbb{Z}/3\mathbb{Z})^\oplus 2 \times \mathbb{Z}/2\mathbb{Z}\).

With these preparations we can prove the following precise version of Theorem 4.3.

Theorem 4.3. Let \(S\) be an Enriques surface with a set of four mutually disjoint \(A_2\)-configurations \(\mathcal{A}\). Then we have

\[ (S, \mathcal{A}) \in \mathcal{F}_{4,3,1} \cup \mathcal{F}_{3,3,3,3}. \]

More precisely,

(a) if \((S, \mathcal{A}) \in \mathcal{F}_{4,3,1}\), then \(\pi_1(S^0) = S_3 \times \mathbb{Z}/3\mathbb{Z}\);

(b) if \(S \in \mathcal{F}_{3,3,3,3}\), then there exist \(\mathcal{A}\) and \(\mathcal{A}'\) such that

\[ \pi_1(S, \mathcal{A}) = (\mathbb{Z}/3\mathbb{Z})^\oplus 2 \times \mathbb{Z}/2\mathbb{Z} \text{ and } \pi_1(S, \mathcal{A}') = \mathbb{Z}/6\mathbb{Z}. \]

In particular, all groups given in Lemma 4.2 are realized by Enriques surfaces.
Proof. By every elliptic fibration \( \varphi \) of the type listed in (2.10) (resp. (2.11)) on an Enriques surface can be obtained by performing a logarithmic transformation on \( X_{3,3,3,3} \) (resp. \( X_{4,3,1} \)). Thus Lemma 2.3 yields (4.2).

For the remaining parts of Theorem 4.3 we shall replace \( A \) by \( p_S(A) \). Observe that by definition of the map \( p_S \), the set \( A \) contains exactly one (resp. four) 3-divisible set(s) iff \( p_S(A) \) contains exactly one (resp. four) 3-divisible set(s). The analogous claims hold for \( p_Y(A^\pm) \). Now it is easy to deduce the assertions of Theorem 4.3 from our previous results in this paper:

The claim (a) follows from Lemma 3.7 and Lemma 4.2 (b).

As for (b), the existence of \( A \) (resp. \( A' \)) results immediately from Lemma 3.5 and Lemma 4.2 (c) (resp. Lemma 3.6 and Lemma 4.2 (a)). \( \square \)

Finally, we use the jacobian fibration to verify that surfaces in \( F_{4,3,1} \cap F_{3,3,3,3} \) have some special properties; notably the families \( F_{4,3,1}, F_{3,3,3,3} \) only overlap on proper subfamilies:

**Proposition 4.4.** Let \( S \) be an Enriques surface and let \( Y \) be the K3-cover of \( S \). If \( S \in F_{4,3,1} \cap F_{3,3,3,3} \), then \( \rho(Y) \geq 19 \).

**Proof.** We compare the discriminants of the K3-covers. For \( S \in F_{3,3,3,3} \) with K3-cover \( Y \) of Picard number \( \rho(Y) = 18 \), Lemma 3.1 gives \( d(\text{NS}(Y)) = -324 \).

A completely analogous argument applies to \( S' \in F_{4,3,1} \) with K3-cover \( Y' \) such that \( \rho(Y') = 18 \). We find that \( d(\text{NS}(Y')) = -36 \). This implies that \( S' \notin F_{3,3,3,3} \) and vice versa for \( S \). \( \square \)

In the next section, we shall take this result as a starting point to take a closer look at the moduli of our families \( F_{3,3,3,3} \) and \( F_{4,3,1} \).

5. **Moduli**

5.1. **Algebro-geometric construction.** We start by giving an algebro-geometric description of the family \( F_{4,3,1} \). As opposed to the analytic construction of logarithmic transformations, it will be based on Enriques involutions of base change type as outlined in [2,2]. Our starting point is another extremal rational elliptic surface \( X_{6,3,2,1} \), this time with \( \text{MW}(X_{6,3,2,1}) \cong \mathbb{Z}/6\mathbb{Z} \). As a cubic pencil, it can be given by

\[
X_{6,3,2,1} : (x + y)(y + z)(z + x) + \lambda xyz = 0.
\]

More precisely, \( X_{6,3,2,1} \) is the relatively minimal resolution of the above cubic pencil model in \( \mathbb{P}^2 \times \mathbb{P}^1 \), obtained by blowing up the three double base points at \([1, 0, 0], [0, 1, 0], [0, 0, 1]\). The blow-up results in a fiber of Kodaira type \( I_5 \) at \( \infty \); the other singular fibers are \( I_3 \) at \( t = 0, I_2 \) at \( t = 1 \) and \( I_1 \) at \( t = -8 \). The other three base points of the cubic pencil are actually points of inflection. Fixing one of them as zero \( O \) for the group law, say \([1, -1, 0]\), we find that \( P = [0, 0, 1] \) has order 2 inside \( \text{MW}(X_{6,3,2,1}) \). Thus it lends itself to (the classical case of) the construction of an Enriques involution of base change type. To this end, consider a quadratic base change (2.6) that does not ramify at the \( I_3 \) and \( I_1 \) fibers. Denote the pull-back surface by \( Y_{6,3,2,1} \); this is an elliptic K3 surface, generically with all singular fibers of \( X_{6,3,2,1} \) duplicated. The deck
transformation \( \iota \) enables us to define a fixed point-free involution \( \psi \) on \( Y_{6,3,2,1} \) by

\[
\psi = \iota \circ \text{(fibrewise translation by } P).\]

The quotient surface will be an Enriques surface \( S = S_{6,3,2,1} \) with elliptic fibration \( \pi_0 \) with the same singular fibers as \( X_{6,3,2,1} \) (generically reduced); here \( O \) and \( P \) map to a smooth rational bisection \( R \).

**Lemma 5.1.** \( S \) contains 4 perpendicular \( A_2 \) configurations.

**Proof.** Consider the singular fibres of \( S \) together with the bisection \( R \). The following figure depicts how they intersect and indicates the 4 \( A_2 \) configurations.

![A2-configurations on S_{6,3,2,1}](image)

**Figure 1.** \( A_2 \)-configurations on \( S_{6,3,2,1} \)

By Theorem 4.3, we conclude that \( S \in \mathcal{F}_{3,3,3,3} \) or \( S \in \mathcal{F}_{4,3,1} \). For discriminant reasons (compare the proof of Proposition 4.4), the second alternative should hold. Here we will give a purely geometric argument:

**Lemma 5.2.** \( S \in \mathcal{F}_{4,3,1} \).

**Proof.** It suffices to identify a divisor of Kodaira type \( IV^* \) on \( S \) with orthogonal \( A_2 \). Then its linear system will induce an elliptic fibration \( \pi \) on \( S \) with singular fibres of types \( IV^* \) and \( I_3 \) or \( IV \); thus \( S \in \mathcal{F}_{4,3,1} \). This is easily achieved: simply connect three \( A_2 \)'s in Figure 1 through one of the remaining components of the original \( I_6 \) fibre.

**Remark 5.3.** A bisection for the fibration \( \pi \) can be given quite easily: take a half-pencil \( B \) of the fibration \( \pi_0 \). Out of the curves depicted in Figure 1, \( B \) only meets \( R \) with multiplicity one. Hence on the fibration \( \pi \), \( B \) meets the \( IV^* \) fibre only in the double component \( R \), so \( B \) indeed is a bisection. In consequence, the fiber of type \( I_3 \) or \( IV \) is met twice in the component not visible in Figure 1.

5.2. **K3-cover for \( \mathcal{F}_{4,3,1} \).** Overall, there are 6 configurations how a bisection may intersect the two reducible fibers of a given Enriques surface \( S \in \mathcal{F}_{4,3,1} \). For 3 of them, including the one sketched in Remark 5.3, we can conversely derive the configuration of rational curves on \( S_{6,3,2,1} \) originating from the Enriques involution of base change type. Here we detail on one example:
Example 5.4. For the above configuration of a bisection $P$ meeting the fibres of type $IV^*$ and $I_3/IV$ on $S$, one finds that $R$ automatically is a half-pencil of the elliptic fibration $\pi|_{2R}$ since it is met by some (nodal) curve (the double component of the $IV^*$ fibre) with multiplicity 1. This fibration has singular fibers accounting for the root lattices $A_5 + A_1$ and $A_2$ obtained from the extended Dynkin diagrams $\tilde{E}_6, A_2$ by omitting the curve met by $R$. With a nodal bisection, we necessarily end up generically on a quotient of $X_{6,3,2,1}$ by an Enriques involution of base change type.

For the three remaining configurations, this does not seem to be possible. However, the next proposition and its corollary show that they are still covered by $Y_{6,3,2,1}$. In other words, we expect that they generically arise from $Y_{6,3,2,1}$ by another kind of Enriques involution.

**Proposition 5.5.** Let $S \in \mathcal{F}_{4,3,1}$ such that the K3 cover $Y$ has $\rho(Y) = 18$. Then

$$\text{NS}(Y) \cong U(2) + A_2 + E_6 + E_8.$$  

**Proof.** The elliptic fibration on $Y$ induced from $S$ comes automatically with a bisection $R$. Since $\rho(Y) = 18$, we can assume that $R^2 = 0$. The key step in proving the proposition is the observation that we can modify $R$ to a divisor $D$ by adding fiber components as correction terms such that $D$ is perpendicular to $2A_2$ and $2E_6$ configurations on $Y$ (in the fibers of $\pi$). For fibers of type $I_3, IV$, this has been exhibited in the proof of Lemma 3.5. For $IV^*$ fibers, it is a similar exercise. For instance, if $R$ meets a double component, then simply subtract the adjacent simple component.

Crucially, we now use that the singular fibers come in pairs which are met by $R$ in exactly the same way (there cannot be non-reduced singular fibers since $\rho(Y) = 18$). In consequence, the correction terms for $D$ also come in pairs, so

$$D^2 \equiv R^2 \equiv 0 \mod 4.$$  

Hence $D$ and the general fiber $F$ span the lattice $U(2)$, and we obtain a finite index sublattice

$$U(2) + A_2^2 + E_6^2 \subset \text{NS}(Y).$$  

To compute $\text{NS}(Y)$, it remains to take the 3-divisible class in $A_2^2 + E_6^2$ into account. From the lattice viewpoint, this behaves exactly like the 3-torsion section on $\text{Jac}(Y)$. Thus it is an easy computation to verify that the overlattice is as claimed. \hfill \Box

**Corollary 5.6.** Any Enriques surface $S \in \mathcal{F}_{4,3,1}$ is covered by a K3 surface $Y_{6,3,2,1}$.

**Proof.** The Néron-Severi lattice of the covering K3 surface admits a unique embedding into the K3 lattice $U^3 + E_8^3$ up to isometries by [13, Thm. 1.14.4]. Hence the K3 surfaces with this lattice polarisation form an irreducible two-dimensional family, and the corollary ends up being a consequence of Lemma 5.2 in the reverse direction. \hfill \Box
Remark 5.7. We can use the above description to check the claim from §3 about the locus inside $\mathcal{F}_{4,3,1}$ where the singular fiber types degenerate from $(I_3 + I_1)$ to $IV$. For this, we normalise the base change (2.6) to take the shape

$$t \mapsto 1 - \frac{(t - 1)(t - \lambda)}{t},$$

so that $S_{6,3,2,1}$ has fibers of type $I_6$ at $0$, $\infty$ and $I_2$ at $1$, $\lambda$. Then we extract the elliptic fibration with 2 fibers of types $IV^*$ and 2 perpendicular $A_2$ inducing $X_{4,3,1}$. This turns to be isotrivial (with zero j-invariant) exactly for $\lambda = 1 - 3/\gamma$.

5.3. Comparison with $Y_{6,3,2,1}$. As a sanity check, we will compute $\text{NS}(Y_{6,3,2,1})$ at a very general moduli point directly. Incidentally, this will allow us to draw interesting consequences, see Theorem 5.8.

Consider a K3 surface $Y_{6,3,2,1}$ with $\rho(Y_{6,3,2,1}) = 18$. In order to compute $\text{NS}(Y_{6,3,2,1})$ directly, we will identify two perpendicular divisors $D_1, D_2$ of Kodaira type $II^*$ among the plentitude of $(-2)$-curves visible in the elliptic fibration $\pi_0$ as fiber components and torsion sections. To define $D_1$, connect the zero section $O$ in three directions: by a component of either $I_2$ fiber, two components of an $I_3$ and a chain $\Theta_0, \ldots, \Theta_4$ of five components of an $I_6$. Similarly, the divisor $D_2$ comprises the 6-torsion section disjoint from $D_1$ (i.e. meeting the remaining fibre component $\Theta_5$ of the chosen $I_6$ fiber) and fibre components of the other $I_2, I_3$ and $I_6$ fibres.

This approach has several advantages. First it reveals that $Y_{6,3,2,1}$ admits an elliptic fibration $\pi_{|D_1}$ with two fibers of type $II^*$. This comes with multisections of degree 6, given for instance by $\Theta_5$. In consequence, the Jacobian has

$$\text{NS}(\text{Jac}(Y_{6,3,2,1}, \pi_{|D_1})) \cong U + E_8^2.$$ 

With this Néron-Severi lattice, $\text{Jac}(Y_{6,3,2,1}, \pi_{|D_1})$ is sandwiched by the Kummer surface of two elliptic curves by [22]. All of this occurs in the framework of Shioda-Inose structures and shows the following result:

Theorem 5.8. The Hodge structure of $Y_{6,3,2,1}$ is governed by a product of two elliptic curves.

Remark 5.9. Theorem [5.8] provides a conceptual way to exhibit explicit K3 surfaces $Y_{6,3,2,1}$ with $\rho(Y_{6,3,2,1}) = 18$, paral-lelling [18, §4.7]. Using the involution of base change type from [5.1] we obtain explicit very general members of the family $\mathcal{F}_{4,3,1}$, as opposed to the extraordinary Example 3.8.

As a second application, we return to the computation of $\text{NS}(Y_{6,3,2,1})$. Consider the orthogonal projection inside $\text{NS}(Y_{6,3,2,1})$ with respect to the sublattice $E_8^2$ specified above. The multisection $C$ is taken to a divisor $C'$ of square $C'^2 = 48$. It follows that $C'$ and a fiber of $\pi_{|D_1}$ generate the lattice $U(6)$. Thus we obtain

$$\text{NS}(Y_{6,3,2,1}) \cong U(6) + E_8^2.$$ 

(Here equality holds since the discriminants match by [18, (22)].) One easily checks that the discriminant forms of the Néron-Severi lattices in Proposition 5.5 and in (5.2) agree. By [13, Cor. 1.13.3], this suffices to prove that the lattices are isometric as required.
Proposition 5.10. If \( Y_{6,3,2,1} \) has \( \rho(Y_{6,3,2,1}) = 18 \), then \( T(Y_{6,3,2,1}) \cong U + U(6) \).

Proof. This follows directly from [5,2] using [13, Prop. 1.6.1 & Cor. 1.13.3]. □

5.4. K3-cover for \( F_{3,3,3,3} \). We can carry out similar calculations for the K3 cover \( Y' \) of an Enriques surface \( S' \in F_{3,3,3,3} \). Here we only sketch the results. Generically, \( Y' \) comes equipped with an elliptic fibration with 8 fibers of type \( I_3 \) and an irreducible bisection \( R' \) such that \( R'^2 = 0 \). Thus the argumentation from the proof of Lemma 3.5 applies to modify \( R' \) to a divisor \( D \) perpendicular to 8 \( A_2 \) configurations (supported on the fibres). Generically, we obtain the finite index sublattice

\[ U(2) + A_2^8 \leftrightarrow \text{NS}(Y') \]

which leads to the following analogue of Proposition 5.5

Proposition 5.11. Let \( S' \in F_{3,3,3,3} \) such that the K3 cover \( Y' \) has \( \rho(Y') = 18 \). Then

\[ \text{NS}(Y') \cong U(2) + A_2^8 + E_6^2. \]

As before, it follows that the K3 covers of all Enriques surfaces in \( F_{3,3,3,3} \) form an irreducible two-dimensional family. Using the discriminant form, we can compute the transcendental lattice of a very general K3 cover:

Proposition 5.12. Let \( S' \in F_{3,3,3,3} \) be an Enriques surfaces such that its K3-cover \( Y' \) has \( \rho(Y') = 18 \). Then

\[ T(Y') \cong U(3) + U(6). \]

Proof. The discriminant group \( A_{\text{NS}} \) of \( \text{NS}(Y') \) has 3-length 4 by Proposition 5.11. Since this length equals the rank of \( T(Y') \), we deduce that \( T(Y') \) is 3-divisible as an integral even lattice, i.e.

\[ T(Y') = M(3) \quad \text{for some even lattice } M. \]

By Lemma 3.1, \( M \) has discriminant 4. From Proposition 5.11 we infer the equality of discriminant forms

\[ q_M = -q_{U(2)} = q_{U(2)}. \]

Hence \( M \cong U + U(2) \) by [13, Prop. 1.6.1 & Cor. 1.13.3]. □

5.5. Overlap of \( F_{4,3,1} \) and \( F_{3,3,3,3} \). Recall from Proposition 4.4 that the two families of Enriques surfaces \( F_{4,3,1} \) and \( F_{3,3,3,3} \) only intersect on one-dimensional subfamilies. Here we shall give a lattice theoretic characterisation of two infinite series of subfamilies and work out the first case explicitly.

In essence, computing the one-dimensional subfamilies of overlap amounts to calculating even lattices \( T \) of signature \( (2, 1) \) admitting primitive embeddings into both generic transcendental lattices from Propositions 5.10, 5.12. Then one can enhance the Néron-Severi lattices by a primitive vector perpendicular to \( T \) using the gluing data encoded in the discriminant form (see [4, §3], e.g.). There are two obvious kinds of candidates for \( T \) with \( N \in \mathbb{Z}_{>0} \):
U(3) + ⟨12N⟩ \implies \begin{cases} U(3) + U(2) \cong U + U(6) \\ U(3) + U(6) \end{cases}

U(6) + ⟨6N⟩ \implies \begin{cases} U(6) + U \\ U(6) + U(3) \end{cases}

Remark 5.13. We point out that (5.3) includes families where the Jacobians of the K3 covers $Y_{3,3,3,3}$ and $Y_{4,3,1}$ overlap. In fact, this happens with transcendental lattices $U(3) + (6M)$, and one can show as in [5, Prop. 4.2] that a K3 surface with this transcendental lattice admits an Enriques involution if and only if $M$ is even. Moreover, the involution turns out to be of base change type, so we can, at least in principle, give a very explicit description of these surfaces.

5.6. Explicit component of $F_{3,3,3,3} \cap F_{4,3,1}$. We conclude this paper by working out the first case of (5.3) explicitly. That is, we aim for K3 surfaces with transcendental lattice

$$T = U(3) + ⟨12⟩.$$ (5.5)

By Remark 5.13 this could be done purely on the level of jacobians of $Y_{3,3,3,3}$ or $Y_{4,3,1}$, but here we shall rather continue to work with $Y_{6,3,2,1}$. The lattice enhancement raises the rank of the Néron-Severi lattice by one while the discriminant changes from $-36$ to $108$. By the theory of Mordell-Weil lattices [21], this can only be achieved by adding a section $Q$ of height $3$. Up to adding a torsion section, we may assume $Q$ to be induced by the quadratic twist $X'$ of $X_{6,3,2,1}$ corresponding to the quadratic base change (2.6). I.e. $Q$ comes from a section $Q'$ of height $3/2$ on $X'$. Note that $X'$ inherits the $2$-torsion section from $X_{6,3,2,1}$. At the same time, this will ease the explicit computations and limit the possible configurations for $Q'$. Indeed, using the height formula from [21] it is easy to see that there are only two possible cases for $Q'$ up to adding the two-torsion section:

- either $Q'$ meets exactly one $I_0^*$ fiber (at a component not met by the 2-torsion section) and the $I_0$ fiber (at the component met by the 2-torsion section) non-trivially,
- or it intersects non-trivially exactly one $I_0^*$ fiber (at a component not met by the 2-torsion section), the $I_0$ fibre (at a component adjacent to the zero component), and the $I_3$ fiber.

We can compute the Néron-Severi lattice and the transcendental lattice of the resulting covering K3 surfaces by the same means as in [5,3] simply compute the rank 3 orthogonal complement of $E_8^2$ inside NS. We obtain $T = U + (108)$ for the second case and the desired transcendental lattice from (5.5) for the first.

We continue to work out the first case in more detail. Let us assume that the quadratic base change (2.6) ramifies at $a, b \in \mathbb{P}^1$. For ease of computations, we shall use an extended Weierstrass form of $X'$ which locates the 2-torsion section at $(0, 0)$:

$$X': \quad y^2 = x(x^2 + (t - a)(t - b)(t^2/4 + t - 2)x + (t - a)^2(t - b)^2(1 - t))$$
Then we can implement the section $Q'$ to have the $x$-coordinate $c(t-a)$. Solving for this to give a square upon substituting into the extended Weierstrass form leads to

$$a = -\frac{1}{3} \frac{(b+2)^2}{b-4}, \quad c = (b+8)(b-1)^2/27.$$  

Thus we obtain explicitly a one-dimensional family of K3 surfaces with transcendental lattice (5.5). Unless the base change degenerates or the ramification points hit the fibers of type $I_1$ or $I_3$, i.e. for $b \not\in \{-8, -2, 0, 1, 10\}$, the resulting K3 surface $Y$ possesses the Enriques involution $\psi$ of base change type constructed in 5.1. By Lemma 5.2, the quotient surface $S$ lies in $\mathcal{F}_{4,3,1}$. We can also verify geometrically that $S \in \mathcal{F}_{3,3,3,3}$. To this end we use that the induced section $Q$ of height 3 on $Y$ meets only the two $I_6$ fibers non-trivially – in the same component as the 2-torsion section $P$, i.e. opposite the zero component – and it meets the zero section $O$ in the ramified fiber above $t = b$. Hence $Q, O$ and the identity component of either of the $I_2$ fibers form a triangle, i.e. they give a divisor $D$ of Kodaira type $I_3$. Perpendicular, we find

- another $I_3$ formed by the sections $P, (P-R)$ and the non-identity component of the other $I_2$ fiber;
- 6 $A_2$’s contributed from the $I_6$ and $I_3$ fibres;
- 4 sections of the induced elliptic fibration $\pi|_D$ given by the remaining components of the $I_6$ fibers.

We conclude that $\pi|_D$ is a jacobian elliptic fibration with 8 fibers of type $I_3$. Hence it comes from $X_{3,3,3,3}$ by some quadratic base change. Finally, one directly verifies that the above rational curves on $Y$ are interchanged by the Enriques involution $\psi$. Therefore, $\pi|_D$ induces an elliptic fibration with 4 fibers of type $I_3$ on $S = Y/\psi$. That is, $S \in \mathcal{F}_{3,3,3,3}$ as claimed.

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