SOME REMARKS ON THE QUALITATIVE QUESTIONS FOR
BIHARMONIC EQUATIONS

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Abstract. In this article, we obtain several interesting remarks on the qualitative questions such as stability criteria, Morse index, Picone’s identity for biharmonic equations.

1. Introduction

In the recent years there has been a good amount of interest on the existence and multiplicity of solutions to biharmonic equations. Recently, A.E. Lindsay and J. Lega [16] obtain multiple quenching solutions of a fourth order parabolic partial differential equation

\[
\begin{cases}
  u_t = -\Delta^2 u + \delta \Delta u - \lambda \frac{h(x)}{1+u} \quad &\text{in } \Omega \subset \mathbb{R}^2, \\
  u = 0 = \frac{\partial u}{\partial v} \quad &\text{on } \partial \Omega, \\
  u = 0, \quad t = 0, \quad x \in \Omega.
\end{cases}
\]

Eq. (1.1) models a microelectromechanical systems (MEMS) capacitor, where \( u(x, t) \) represents the deflection of the device and \( \delta \) represents the relative effects of tension and rigidity on the deflecting plate, \( \lambda \geq 0 \) represents the ratio of electric forces to elastic forces and \( h \) represents possible heterogeneities in the deflecting surface’s dielectric profile. For the details on this subject, we refer the reader to [19]. The steady state of Eq. (1.1) (when \( u(x, t) \) is independent of \( t \)) is:

\[
\begin{cases}
  \Delta^2 u - \delta \Delta u = -\lambda \frac{h(x)}{1+u} \quad &\text{in } \Omega, \\
  u = 0 = \frac{\partial u}{\partial v} \quad &\text{on } \partial \Omega.
\end{cases}
\]

For the existence of positive solutions to problems similar to (1.2) in \( \mathbb{R}^N \), we refer to [20] and the references therein and for the existence and bifurcation results to more general problem

\[
\begin{cases}
  \Delta^2 u - \Delta_g u = f(\lambda, x, u) \quad &\text{in } \Omega, \\
  u = 0 = \frac{\partial u}{\partial v} \quad &\text{on } \partial \Omega,
\end{cases}
\]

we refer to [14]. Equations of type (1.2) are also discussed on Riemannian manifold \((M^n, g)\), \( n \geq 5 \), see [17], where the author obtain the existence of classical solutions to

\[
\Delta^2 u - \text{div}(a(x) \nabla u) + b(x)u = f(x)|u|^{N-2}u \quad \text{in } M^n.
\]
There is also a good amount of work on the qualitative questions such as stability criteria, Picone’s identity, Morse index, Sturm comparison theorem for Laplace as well as p-Laplace equations but very little is known for biharmonic equations. Recently, there have been some investigations on the stability of solutions to p-Laplace equations/quasilinear elliptic equations, see for instance, [8, 13, 23, 24] and the references therein. We refer to [6] for the stability results to biharmonic equations and [25] for Liouville theorems for stable radial solutions for the biharmonic operators. Very recently, J. Wei and D. Ye [26] prove Liouville type results for stable solutions to biharmonic equations with sign changing nonlinearity.

Theorem 1.2. [22] Let \( v > 0 \) and \( u \geq 0 \) be differentiable. Denote

\[
L(u, v) = |\nabla u|^p + (p - 1) \frac{u^p}{v^p} |\nabla v|^p - p \frac{u^{p-1}}{v^p} |\nabla u|v^{p-2} \nabla v.
\]

\[
R(u, v) = |\nabla u|^p - \nabla \left( \frac{u^p}{v^{p-1}} \right) |\nabla v|^p v^{p-2} \nabla v.
\]

Then \( L(u, v) = R(u, v) \). Moreover, \( L(u, v) \geq 0 \) and \( L(u, v) = 0 \) a.e. in \( \Omega \) if and only if \( \nabla u \) = 0 a.e. in \( \Omega \).

Recently, the second author obtains a nonlinear analogue of (1.6) in [22] and obtained some qualitative results. The nonlinear analogue of (1.6) reads as follows:

Theorem 1.2. [22] Let \( v \) be a differentiable function in \( \Omega \) such that \( v \neq 0 \) in \( \Omega \) and \( u \) be a non-constant differentiable function in \( \Omega \). Let \( f(y) \neq 0, \forall 0 \neq y \in \mathbb{R} \) and suppose that there exists \( \alpha > 0 \) such that \( f'(y) \geq \frac{1}{\alpha}, \forall 0 \neq y \in \mathbb{R} \). Denote

\[
L(u, v) = \alpha |\nabla u|^2 - \frac{|\nabla u|^2}{f'(v)} + \left( \frac{u \sqrt{f'(v)} \nabla v}{f(v)} - \frac{\nabla u}{\sqrt{f'(v)}} \right)^2.
\]
Then $L(u, v) = R(u, v)$. Moreover, $L(u, v) \geq 0$ and $L(u, v) = 0$ in $\Omega$ if and only if $u = c_1v + c_2$ for some arbitrary constants $c_1$, $c_2$.

K. Bal [5] extended the nonlinear Picone’s identity of [22] to deal with p-Laplace equations. The extension reads as follows:

**Theorem 1.3. [5]** Let $v > 0$ and $u \geq 0$ be two non-constant differentiable functions in $\Omega$. Also assume that $f'(y) \geq (p - 1)[f(y)\frac{p}{p-2}]$ for all $y$. Define

$$L(u, v) = |\nabla u|^{p-1} \frac{p u \nabla u \nabla v |\nabla v|^{p-2} \nabla v}{f(v)} + \frac{u^p f'(v) |\nabla v|^{p-2} \nabla v}{f(v)}.$$  

$$R(u, v) = |\nabla u|^{p} - \nabla \left( \frac{u^p}{f(v)} \right) |\nabla v|^{p-2} \nabla v.$$  

Then $L(u, v) = R(u, v) \geq 0$. Moreover $L(u, v) = 0$ a.e. in $\Omega$ if and only if $\nabla (\frac{u}{v}) = 0$ a.e. in $\Omega$.

There are also several interesting articles dealing with Picone’s identity in different contexts. We just name a few articles, for instance, for a Picon’s type identity to half-linear elliptic operators with higher order half linear differentiable operators, we refer to [13] and the references therein, for Picone identities to half-linear elliptic operators with $p(x)$-Laplacians, we refer to [28] and for Picone-type identity to pseudo p-Laplacian with variable power, we refer to [7]. In [9], D.R.Dunninger established a Picone identity for a class of fourth order elliptic differential inequalities. This identity says that if $u$, $v$, $a\Delta u$, $A\Delta v$ are twice continuously differentiable functions with $v(x) \neq 0$ and $a$ and $A$ are positive weights, then

$$\text{div} \left[ u \nabla (a\Delta u) - a\Delta u \nabla u - \frac{u}{v} \nabla (A\Delta v) + A\Delta v \nabla \left( \frac{u^2}{v} \right) \right]$$

$$= - \frac{u^2}{v} \Delta (A\Delta v) + u\Delta (a\Delta u) + (A - a)(\Delta u)^2$$

$$- A \left( \Delta u - \frac{u}{v} \Delta v \right)^2 + A \frac{2\Delta v}{v} \left( \nabla u - \frac{u}{v} \nabla v \right)^2.$$  

(1.9)

In this context, there is a natural question. Can we establish a nonlinear analogue of (1.9)? More precisely, the aim of this article is twofold. Firstly, we establish a nonlinear analogue of Picone’s identity which could deal with biharmonic equations and secondly using the similar techniques, we consider the stability of a positive weak solution $u \in H_0^2(\Omega) \cap L^\infty(\Omega)$ of (1.5) in any arbitrary smooth bounded domain for sign changing nonlinearity. In this paper, we assume that $\Omega \subset \mathbb{R}^N$ is an open, smooth and bounded subset and $a \in L^\infty(\Omega)$ are such that (1.5) has a positive weak solution $u \in H_0^2(\Omega) \cap L^\infty(\Omega)$.

We make the following hypothesis on the nonlinearity $f_1$:

(H1) Let $f_1 \in C(\overline{\Omega} \times \mathbb{R}, \mathbb{R})$ and $C^1$ in the $y$ variable and satisfies

$$\frac{\partial f_1(x, y)}{\partial y} \geq \frac{f_1(x, y)}{y}, \quad \forall \ 0 < y \in \mathbb{R}, \ \forall \ x \in \overline{\Omega}.$$  

The plan of this paper is as follows. Section 2 deals with the nonlinear analogue of Picone’s identity which could deal with biharmonic equations. In Section 3, we give several applications of Picone’s identity to biharmonic equations. In Section 4, we establish a stability theorem of a positive weak solution to (1.5).
2. Nonlinear analogue of Picone’s identity

In this section, we establish a nonlinear analogue of Picone’s identity. The next lemma can be obtained from \[1.9\] with some assumptions. Since the proof is short and interesting so we write it independently here with useful insights.

**Lemma 2.1.** (Picone’s identity) Let \(u\) and \(v\) be twice continuously differentiable functions in \(\Omega\) such that \(v > 0\), \(-\Delta v > 0\) in \(\Omega\). Denote

\[
L(u, v) = \left(\Delta u - \frac{u}{v} \Delta v\right)^2 - \frac{2\Delta v}{v} \left(\nabla u - \frac{u}{v} \nabla v\right)^2.
\]

\[
R(u, v) = |\Delta u|^2 - \Delta \left(\frac{u^2}{v}\right) \Delta v.
\]

Then (i) \(L(u, v) = R(u, v)\) (ii) \(L(u, v) \geq 0\) and (iii) \(L(u, v) = 0\) in \(\Omega\) if and only if \(u = \alpha v\) for some \(\alpha \in \mathbb{R}\).

**Proof.** Let us expand \(R(u, v)\):

\[
R(u, v) = |\Delta u|^2 - \Delta \left(\frac{u^2}{v}\right) \Delta v
\]

\[
= |\Delta u|^2 + \frac{u^2}{v^2} |\Delta v|^2 - \frac{2u}{v} \Delta u \Delta v - \frac{2\Delta v}{v} |\nabla u|^2 \Delta v + \frac{4u}{v^2} \nabla u \nabla v \Delta v - \frac{2u^2}{v^3} |\nabla v|^2 \Delta v
\]

\[
= \left(\Delta u - \frac{u}{v} \Delta v\right)^2 - \frac{2\Delta v}{v} \left(\nabla u - \frac{u}{v} \nabla v\right)^2
\]

\[
= L(u, v),
\]

which proves the first part. Now using the fact that \(v > 0\), \(-\Delta v > 0\) in \(\Omega\), one can see that \(L(u, v) \geq 0\) and therefore (ii) is proved. Now \(L(u, v) = 0\) in \(\Omega\) implies that

\[
0 = \left(\Delta u - \frac{u}{v} \Delta v\right)^2 - \frac{2\Delta v}{v} \left(\nabla u - \frac{u}{v} \nabla v\right)^2, \text{ i.e.,}
\]

\[
0 \leq -\frac{2\Delta v}{v} \left(\nabla u - \frac{u}{v} \nabla v\right)^2 = -\left(\Delta u - \frac{u}{v} \Delta v\right)^2 \leq 0,
\]

which implies that there exists some \(\alpha \in \mathbb{R}\) such that \(u = \alpha v\). Conversely, when \(u = \alpha v\), one can see easily that \(L(u, v) = 0\), and therefore (iii) is proved. \(\Box\)

**Remark 2.2.** We note that the above lemma also holds if we replace \(v > 0\) and \(-\Delta v > 0\) in \(\Omega\) by \(v < 0\) and \(-\Delta v < 0\) in \(\Omega\), respectively.

In the next proposition, we establish a nonlinear analogue of Picone’s identity for biharmonic equations.

**Proposition 2.3.** (Nonlinear analogue of Picone’s identity) Let \(u\) and \(v\) be twice continuously differentiable functions in \(\Omega\) such that \(v > 0\), \(-\Delta v > 0\) in \(\Omega\). Let \(f: \mathbb{R} \to (0, \infty)\) be a \(C^2\) function such that \(f''(y) \leq 0\), \(f'(y) \geq 1\), \(\forall 0 \neq y \in \mathbb{R}\). Denote

\[
L(u, v) = |\Delta u|^2 - \left|\frac{\Delta u}{f'(v)}\right|^2 + \left(\frac{\Delta u}{f'(v)} - \frac{u}{f(v)} \sqrt{f'(v)} \Delta v\right)^2
\]

\[
- \frac{2\Delta v}{f(v)} \left(\nabla u - \frac{uf'(v)}{f(v)} \nabla v\right)^2 + \frac{u^2 f''(v)}{f(v)} |\nabla v|^2 \Delta v.
\]

\[
R(u, v) = |\Delta u|^2 - \Delta \left(\frac{u^2}{f(v)}\right) \Delta v.
\]
Then (i) \( L(u, v) = R(u, v) \) (ii) \( L(u, v) \geq 0 \) and (iii) \( L(u, v) = 0 \) in \( \Omega \) if and only if \( u = cv + d \) for some \( c, d \in \mathbb{R} \).

**Proof.** Let us expand \( R(u, v) \):

\[
R(u, v) = |\Delta u|^2 - \Delta \left( \frac{u^2}{f(v)} \right) \Delta v
\]

\[
= |\Delta u|^2 - \frac{|\Delta u|^2}{f'(v)} + \left( \frac{|\Delta u|^2}{f'(v)} + \frac{u^2 f'(v)}{f^2(v)} |\Delta v|^2 - \frac{2u \Delta u \Delta v}{f(v)} \right)
\]

\[
- \frac{2 \Delta v}{f(v)} \left( \nabla u \cdot \nabla v \right)^2 + \frac{u^2 f''(v)}{f^2(v)} |\nabla v|^2 \Delta v
\]

\[
= |\Delta u|^2 - \frac{|\Delta u|^2}{f'(v)} + \left( \frac{\Delta u}{\sqrt{f'(v)}} - \frac{u}{f(v)} \sqrt{f'(v)} \Delta v \right)^2
\]

\[
- \frac{2 \Delta v}{f(v)} \left( \nabla u - \frac{uf'(v)}{f(v)} \nabla v \right)^2 + \frac{u^2 f''(v)}{f^2(v)} |\nabla v|^2 \Delta v
\]

\[
= L(u, v),
\]

which proves the first part. Now using the fact that \(-\Delta v > 0\), \(f'(y) \geq 1\), and \(f''(y) \leq 0\), \(\forall y \in \mathbb{R}\), we get \(L(u, v) \geq 0\) and therefore (ii) is proved. Now \(L(u, v) = 0\) in \(\Omega\) implies that

\[
(2.1) \quad |\Delta u|^2 - \frac{|\Delta u|^2}{f'(v)} = 0 \text{ and } \nabla u - \frac{uf'(v)}{f(v)} \nabla v = 0.
\]

This gives \(f'(v) = 1\) or \(f(v) = v + c_1\) where \(c_1\) is a constant, which yields

\[(\nabla u)(v + c_2) - u \nabla (v + c_2) = 0 \text{ or } \nabla \left( \frac{u}{v + c_2} \right) = 0 \text{ i.e., } u = cv + d\]

for some constants \(c\) and \(d\). Conversely, let us assume (2.1) holds. We need to show that \(L(u, v) = 0\). From (2.1), we get that \(f'(v) = 1\) and therefore \(f''(v) = 0\). Now it remains to show that

\[
\left( \frac{\Delta u}{\sqrt{f'(v)}} - \frac{u}{f(v)} \sqrt{f'(v)} \Delta v \right) = 0 \text{ i.e., } f(v) \Delta u = uf'(v) \Delta v.
\]

From (2.1), we get

\[
0 = f(v) \nabla u - uf'(v) \nabla v
\]

\[
0 = f(v) \Delta v + f'(v) \nabla u \nabla v - f'(v) \nabla u \nabla v - uf''(v) |\nabla v|^2 - uf'(v) \Delta v
\]

\[
f(v) \Delta u = uf'(v) \Delta v,
\]

which completes the proof. \(\square\)

### 3. Applications

This section deals with the applications of Lemma 2.1 and Proposition 2.3. In next theorem, we obtain a Hardy-Rellich type inequality.

**Theorem 3.1.** Assume that there is a \(C^2\) function \(v\) satisfying

\[
(3.1) \quad \Delta^2 v \geq \lambda gf(v), \quad v > 0, -\Delta v > 0 \text{ in } \Omega,
\]
for some $\lambda > 0$ and a nonnegative continuous function $g$ on $\Omega$ and $f$ satisfies the conditions of Proposition 2.3. Then for any $u \in C^\infty_c(\Omega)$,

\begin{equation}
\int_\Omega |\Delta u|^2 dx \geq \lambda \int_\Omega g|u|^2 dx.
\end{equation}

Proof. Take $\phi \in C^\infty_c(\Omega)$, by Proposition 2.3, we have
\begin{align*}
0 & \leq \int_\Omega L(\phi,v)dx = \int_\Omega R(\phi,v)dx \\
& = \int_\Omega |\Delta \phi|^2 dx - \int_\Omega \Delta \left( \frac{\phi^2}{f(v)} \right) \Delta v dx \\
& = \int_\Omega |\Delta \phi|^2 dx - \int_\Omega (\Delta^2 v) \frac{\phi^2}{f(v)} dx, \text{ on integration,} \\
& \leq \int_\Omega |\Delta \phi|^2 dx - \lambda \int_\Omega \phi^2 g dx \quad \text{by (3.1)}.
\end{align*}

Letting $\phi \to u$, we have
\begin{equation}
\int_\Omega |\Delta u|^2 dx \geq \lambda \int_\Omega g|u|^2 dx.
\end{equation}

The next lemma deals with a necessary condition for the nonnegative solutions of biharmonic equations.

Lemma 3.2. Let $u \in H^2(\Omega) \cap H_0^1(\Omega)$ be a nonnegative weak solution (not identically zero) of
\begin{equation}
\Delta^2 u = a(x)u \text{ in } \Omega, \quad u = \Delta u = 0 \text{ on } \partial \Omega,
\end{equation}
where $0 \leq a \in L^\infty(\Omega)$, then $-\Delta u > 0$ in $\Omega$.

Proof. Let $-\Delta u = v$. Then writing (3.3) into system form, we get
\begin{equation}
\begin{cases}
-\Delta u = v \text{ in } \Omega, \\
-\Delta v = a(x)u \text{ in } \Omega, \\
u = 0 = v \text{ on } \partial \Omega,
\end{cases}
\end{equation}

Since $a(x) \geq 0$ in $\Omega$, so by maximum principle, we get $v \geq 0$. By strong maximum principle, either $v > 0$ or $v \equiv 0$ in $\Omega$. If $v \equiv 0$, then we have
\[-\Delta u = 0 \text{ in } \Omega; \quad v = 0 \text{ on } \partial \Omega.\]

Again by maximum principle, we get $u \equiv 0$, which is a contradiction and therefore $v > 0$ in $\Omega$ and hence
\[-\Delta u > 0 \text{ in } \Omega.\]

Next, we consider the following singular system of fourth order elliptic equations:
\begin{equation}
\begin{cases}
\Delta^2 u = f(v) \text{ in } \Omega, \\
\Delta^2 v = \left(\frac{f(v)}{u}\right)^2 \text{ in } \Omega, \\
u > 0, v > 0 \text{ in } \Omega, \\
u = \Delta u = 0 = v = \Delta v \text{ on } \partial \Omega,
\end{cases}
\end{equation}
where $f$ is as defined in Proposition 2.3. In the next theorem, we show a linear relationship between the components $u$ and $v$, where $(u, v)$ is a solution of (3.5).

**Theorem 3.3.** Let $(u, v)$ be a weak solution of (3.5) and $f$ satisfy the conditions of Proposition 2.3. Then $u = c_1 v + c_2$, where $c_1, c_2$ are constants.

**Proof.** Let $(u, v)$ be a weak solution of (3.5). Then

\begin{align}
\int_{\Omega} \Delta u \Delta \phi_1 dx &= \int_{\Omega} f(v) \phi_1 dx, \tag{3.6} \\
\int_{\Omega} \Delta v \Delta \phi_2 dx &= \int_{\Omega} \frac{f^2(v)}{u} \phi_2 dx \tag{3.7}
\end{align}

hold for any $\phi_1, \phi_2 \in H^2(\Omega) \cap H^1_0(\Omega)$. Now choosing $\phi_1 = u$ and $\phi_2 = \frac{u^2}{f(v)}$ in (3.6) and (3.7), respectively, we obtain

\[ \int_{\Omega} |\Delta u|^2 dx = \int_{\Omega} f(v) u dx = \int_{\Omega} \Delta v \Delta \left( \frac{u^2}{f(v)} \right) dx. \]

Hence we have

\[ \int_{\Omega} R(u, v) dx = \int_{\Omega} \left[ |\Delta u|^2 - \Delta v \Delta \left( \frac{u^2}{f(v)} \right) \right] dx = 0. \]

By positivity of $R(u, v)$, we get $R(u, v) = 0$ and by Lemma 3.2, we have $-\Delta u > 0$, $-\Delta v > 0$ in $\Omega$.

Now an application of Proposition 2.3 yields that $u = c_1 v + c_2$ for some constants $c_1$ and $c_2$. \qed

Let us consider the following eigenvalue problem

\[ \Delta^2 u = \lambda a(x)u \quad \text{in} \quad \Omega, \quad u = \Delta u = 0 \quad \text{on} \quad \partial \Omega, \quad \text{where} \quad \Omega \subset \mathbb{R}^N \quad \text{is an open, bounded subset} \quad \text{and} \quad 0 \leq a \in L^\infty(\Omega). \]

We recall that a value $\lambda \in \mathbb{R}$ is an eigenvalue of (3.8) if and only if there exists $u \in H^2(\Omega) \cap H^1_0(\Omega)/\{0\}$ such that

\[ \int_{\Omega} \Delta u \Delta \phi dx = \int_{\Omega} a(x) u \phi dx, \quad \forall \ \phi \in H^2(\Omega) \cap H^1_0(\Omega) \]

and $u$ is called an eigenfunction associated with $\lambda$. The least positive eigenvalue of (3.8) is defined as

\[ \lambda_1 = \inf \left\{ \int_{\Omega} |\Delta u|^2 dx : \ u \in H^2(\Omega) \cap H^1_0(\Omega) \quad \text{and} \quad \int_{\Omega} a(x)|u|^2 dx = 1 \right\}. \]

**Lemma 3.4.** $\lambda_1$ is attained.

**Proof.** For showing the above infimum is attained, let us introduce the functionals

\[ J, \quad G : H^2(\Omega) \cap H^1_0(\Omega) \rightarrow \mathbb{R} \quad \text{defined by} \]

\[ J(u) = \frac{1}{2} \int_{\Omega} |\Delta u|^2 dx, \quad G(u) = \frac{1}{2} \int_{\Omega} a(x)|u|^2 dx, \quad u \in H^2(\Omega) \cap H^1_0(\Omega). \]

It is easy to see that $J$ and $G$ are $C^1$ functionals. By definition, $\lambda \in \mathbb{R}$ is an eigenvalue of (3.8) if and only if there exists $u \in H^2(\Omega) \cap H^1_0(\Omega)/\{0\}$ such that

\[ J'(u) = \lambda G'(u). \]
Let us define
\[ M = \left\{ u \in H^2(\Omega) \cap H_0^1(\Omega) \mid \frac{1}{2} \int_{\Omega} a(x)|u|^2 \, dx = 1 \right\}. \]
Since \( a \geq 0 \) so \( M \neq \emptyset \) and \( M \) is a \( C^1 \) manifold in \( H^2(\Omega) \cap H_0^1(\Omega) \). It is also easy to see that \( J \) is coercive and (sequentially) weakly lower semicontinuous on \( M \) and \( M \) is a weakly closed subset of \( H^2(\Omega) \cap H_0^1(\Omega) \). Now by an application of Theorem 1.2 [21], \( J \) is bounded from below on \( M \) and attains its infimum in \( M \). Also by Lagrange’s multiplier rule
\[ J'(u) = \lambda_1 G'(u) \]
and therefore \( \lambda_1 \) is attained. \( \square \)

In the next lemma, we show that the first eigenfunction \( u \) corresponding to the first eigenvalue \( \lambda_1 \) of (3.8) is of one sign. We use the following theorem.

**Theorem 3.5.** [12] (Dual cone decomposition theorem) Let \( H \) be a Hilbert space with scalar product \( (\cdot, \cdot)_H \). Let \( K \subset H \) be a closed, convex nonempty cone. Let \( K^* \) be its dual cone, namely
\[ K^* = \{ w \in H \mid (w, v)_H \leq 0, \quad \forall v \in K \}. \]
Then for any \( u \in H \), there exists a unique \( (u_1, u_2) \in K \times K^* \) such that
\[ u = u_1 + u_2, \quad (u_1, u_2)_H = 0. \]
In particular,
\[ ||u||_H^2 = ||u_1||_H^2 + ||u_2||_H^2. \]
Moreover, if we decompose arbitrary \( u, v \in H \) according to (3.10), i.e.,
\[ ||u - v||_H^2 \geq ||u_1 - v_1||_H^2 + ||u_2 - v_2||_H^2. \]
In particular, the projection onto \( K \) is Lipschitz continuous.

**Lemma 3.6.** The eigenfunction \( u \) corresponding to the first eigenvalue \( \lambda_1 \) of (3.8) is of one sign.

**Proof.** Using Theorem 3.5 and classical maximum principle for \( -\Delta \), Ferrero et al. [10] obtain the positivity of the minimizers of the problem
\[ S_p = \min_{w \in X \setminus \{0\}} \frac{||\Delta w||_2^2}{||w||_p^2}, \quad 1 \leq p < \frac{2n}{n-4}, \]
where \( X = H^2(B) \cap H_0^1(B) \), \( B \) denotes the unit ball in \( \mathbb{R}^n \). The same proof works for eigenfunction \( u \) corresponding to the first eigenvalue \( \lambda_1 \) of (3.8) in \( \Omega \). For this, we refer to [10] and omit the details. \( \square \)

Next we show the strict monotonicity of the principle eigenvalue \( \lambda_1 \).

**Theorem 3.7.** Suppose \( \Omega_1 \subset \Omega_2 \) and \( \Omega_1 \neq \Omega_2 \). Then \( \lambda_1(\Omega_1) > \lambda_1(\Omega_2) \), if both exist.

**Proof.** Let \( u_i \) be a positive eigenfunction associated with \( \lambda_1(\Omega_i) \), \( i = 1, 2 \). For \( \phi \in C_c^\infty(\Omega_1) \),
\[ 0 \leq \int_{\Omega_1} L(\phi, u_2) \, dx = \int_{\Omega_1} R(\phi, u_2) \, dx \]
\[ = \int_{\Omega_1} \left( |\Delta \phi|^2 - \Delta(\frac{\phi^2}{u_2}) \Delta u_2 \right) \, dx \]
\[
\int_{\Omega_1} |\Delta \phi|^2 \, dx - \int_{\Omega_1} \frac{\phi^2}{u_2^2} \Delta^2 u_2 \, dx
\]
(3.11) \[
= \int_{\Omega_1} |\Delta \phi|^2 \, dx - \lambda_1(\Omega_2) \int_{\Omega_1} a(x) \phi^2 \, dx
\]
Letting \( \phi \to u_1 \) in (3.11), we obtain
\[
0 \leq \int_{\Omega_1} L(u_1, u_2) \, dx = (\lambda_1(\Omega_1) - \lambda_1(\Omega_2)) \int_{\Omega_1} a(x) u_1^2 \, dx.
\]
This gives \( \lambda_1(\Omega_1) - \lambda_1(\Omega_2) \geq 0 \). Now if \( \lambda_1(\Omega_1) - \lambda_1(\Omega_2) = 0 \) then \( L(u_1, u_2) = 0 \) and an application of Lemma 2.1 implies that \( u_1 = cu_2 \), which is not possible as \( \Omega_1 \subset \Omega_2 \) and \( \Omega_1 \neq \Omega_2 \). This completes the proof. \( \square \)

In the next theorem, using Picone's identity (Lemma 2.1), we show that \( \lambda_1 \) is simple, i.e., the eigenfunctions associated to it are a constant multiple of each other.

**Theorem 3.8.** \( \lambda_1 \) is simple.

**Proof.** Let \( u \) and \( v \) be two eigenfunctions associated with \( \lambda_1 \). From Lemma 3.6 without any loss of generality, we can assume that \( u \) and \( v \) are positive in \( \Omega \). Now by Lemma 3.2 we have
\[
-\Delta u > 0, \quad -\Delta v > 0 \quad \text{in} \quad \Omega.
\]
Let \( \epsilon > 0 \). From Lemma 2.1, we have
\[
0 \leq \int_{\Omega} L(u, v + \epsilon) \, dx
= \int_{\Omega} R(u, v + \epsilon) \, dx
= \int_{\Omega} \left[ |\Delta u|^2 - \Delta \left( \frac{u^2}{v + \epsilon} \right) \Delta v \right] \, dx
= \lambda_1 \int_{\Omega} a(x) u^2 \, dx - \int_{\Omega} \Delta \left( \frac{u^2}{v + \epsilon} \right) \Delta v \, dx.
\]
(3.12) The function \( \phi = \frac{u^2}{v + \epsilon} \) and is admissible in the weak formulation of
\[
\Delta^2 v = \lambda_1 a(x) v, \quad \text{i.e.,}
\]
\[
\int_{\Omega} \Delta v \Delta \left( \frac{u^2}{v + \epsilon} \right) \, dx = \lambda_1 \int_{\Omega} a(x) v \left( \frac{u^2}{v + \epsilon} \right) \, dx.
\]
(3.13) From (3.12) and (3.13), we get
\[
0 \leq \int_{\Omega} L(u, v + \epsilon) \, dx
= \lambda_1 \int_{\Omega} a(x) \left[ u^2 - v \left( \frac{u^2}{v + \epsilon} \right) \right] \, dx.
\]
Letting \( \epsilon \to 0 \), in the above inequality, we get
\[
L(u, v) = 0
\]
and again by an application of Lemma 2.1, there exists \( \alpha \in \mathbb{R} \) such that
\[
u = \alpha v,
\]
which proves the simplicity of \( \lambda_1 \). \( \square \)
Next, we show the sign changing nature of any eigenfunction $v$ associated to a positive eigenvalue $0 < \lambda \neq \lambda_1$.

**Proposition 3.9.** Any eigenfunction $v$ associated to a positive eigenvalue $0 < \lambda \neq \lambda_1$ changes sign.

**Proof.** Assume by contradiction that $v \geq 0$, the case $v \leq 0$ can be dealt similarly. By Lemma 3.2, $v > 0$ in $\Omega$. Let $\phi > 0$ be an eigenfunction associated with $\lambda_1 > 0$. For any $\epsilon > 0$, we apply Lemma 2.1 to the pair $\phi, v + \epsilon$ and get

\[
0 \leq \int_{\Omega} \left[ |\Delta \phi|^2 - \Delta \left( \frac{\phi^2}{v + \epsilon} \right) \Delta v \right] dx
\]

\[
= \int_{\Omega} \left[ \lambda_1 a(x)\phi^2 - \Delta \left( \frac{\phi^2}{v + \epsilon} \right) \Delta v \right] dx.
\]

(3.14)

For every $\phi \in H^2(\Omega) \cap H_0^1(\Omega)$, $\frac{\phi^2}{v + \epsilon} \in H^2(\Omega) \cap H_0^1(\Omega)$ and is admissible in the weak formulation of

$\Delta^2 v = \lambda a(x)v$ in $\Omega$; $v = \Delta v = 0$ on $\partial \Omega$.

This implies that

\[
\int_{\Omega} \Delta v \Delta \left( \frac{\phi^2}{v + \epsilon} \right) dx = \lambda \int_{\Omega} a(x)v \frac{\phi^2}{v + \epsilon} dx.
\]

From (3.14) and (3.15), we get

\[
0 \leq \int_{\Omega} \left[ \lambda_1 a(x)\phi^2 - \lambda a(x)v \frac{\phi^2}{v + \epsilon} \right] dx.
\]

Letting $\epsilon \to 0$ in the above inequality, we get

\[
0 \leq (\lambda_1 - \lambda) \int_{\Omega} a(x)\phi^2 dx,
\]

which is a contradiction, because $\int_{\Omega} a(x)\phi^2 dx > 0$ and hence $v$ must change sign. \hfill \square

For the application of Lemma 2.1 on Morse index, let us consider the following boundary value problem

\[
\Delta^2 u = a(x)G(u) \text{ in } \Omega; \quad u = \Delta u = 0 \text{ on } \partial \Omega.
\]

(3.16)

For the existence of positive solution to the equations similar to (3.16), we refer the reader to [11]. By the standard elliptic regularity theory, $u \in C^4(\Omega) \cap C^5(\Omega)$. We shall assume that there exists a positive $C^4$ solution $u$ of the boundary value problem (3.16). For the solution $u \in C^4(\Omega)$, the Morse index is defined via the eigenvalue problem for the linearization at $u$:

**Definition 3.10.** Morse index: The Morse index of a solution $u$ of (3.16) is the number of negative eigenvalues of the linearized operator

\[
\Delta^2 - a(x)G'(u)
\]

(3.17)
acting on $H^2(\Omega) \cap H^1_0(\Omega)$, i.e., the number of eigenvalues $\lambda$ such that $\lambda < 0$, and the boundary value problem

$$\Delta^2 w - a(x)G'(u)w = \lambda w \text{ in } \Omega; \quad w = 0 = \Delta w \text{ on } \partial \Omega$$

has a nontrivial solution $w$ in $H^2(\Omega) \cap H^1_0(\Omega)$.

The next theorem gives an application of Lemma 2.1.

**Theorem 3.11.** Let us consider (3.16). Let $a \in C^\alpha(\Omega)$, $0 < \alpha < 1$ and $G \in C^1(\mathbb{R}, \mathbb{R})$ be such that

$$G(v) \geq G'(0) \geq 0, \quad \forall \ 0 < v \in \mathbb{R}.$$ 

Then the trivial solution of (3.16) has Morse index 0.

**Proof.** Let $v$ be a positive weak solution of (3.16). Then

$$\int_\Omega \Delta v \Delta \psi dx = \int_\Omega a(x)G(v)\psi dx, \quad \forall \psi \in H^2(\Omega) \cap H^1_0(\Omega).$$

For any $w \in C_c^\infty(\Omega)$, let us take $w^2 v$ as a test function in (3.19) and obtain

$$\int_\Omega \Delta v \Delta \left( \frac{w^2}{v} \right) dx = \int_\Omega a(x)G(v) \frac{w}{v} w^2 dx.$$ 

Since $v$ is a positive solution of (3.16), so using the fact that $G(v) \geq 0$ and in view of Lemma 3.2, one can see that

$$-\Delta v > 0.$$ 

Now an application of Lemma 2.1 for $u = w$ yields that

$$\int_\Omega |\Delta w|^2 dx \geq \int_\Omega \Delta v \Delta \left( \frac{w^2}{v} \right) dx = \int_\Omega a(x)G(v) \frac{w}{v} w^2 dx \geq \int_\Omega a(x)G'(0)w^2 dx.$$ 

Consider the eigenvalue problem associated with the linearization for (3.16) at 0, which is

$$\Delta^2 w - a(x)G'(0)w = \lambda w \text{ in } \Omega; \quad w = 0 = \Delta w \text{ on } \partial \Omega.$$ 

By the variational characterization of the eigenvalue in (3.22), from (3.21), one can see that $\lambda \geq 0$, which proves the claim. □

4. **Stability of positive solutions**

In this section, we consider the stability of a positive solution to (1.5). The functional associated with (1.5) is

$$E: H^2_0(\Omega) \rightarrow \mathbb{R}$$
defined by

\[ E(u) = \frac{1}{2} \int_{\Omega} |\Delta u|^2 dx + \delta \int_{\Omega} |\nabla u|^2 dx - \frac{1}{2} \int_{\Omega} a(x)u^2 dx + \int_{\Omega} F_1(x, u)dx, \]

where

\[ F_1(x, s) = \int_0^s f_1(x, t)dt. \]

The weak formulation of (1.5) is the following:

\[ (4.1) \int_{\Omega} \Delta u \Delta \phi dx + \delta \int_{\Omega} \nabla u \cdot \nabla \phi dx = \int_{\Omega} a(x)u \phi dx - \int_{\Omega} f_1(x, u) \phi dx, \quad \forall \phi C^2_c(\Omega). \]

By the classical elliptic regularity theory, \( u \in C^3(\Omega) \), see Theorem 2.20 [12] when \( \delta = 0 \) in (1.5) and in fact the same proof works in case \( \delta \neq 0 \). Therefore, we assume that the solution of (1.5) belongs to \( C^3(\Omega) \).

The linearized operator \( L_u \) associated with (1.5) at a given solution \( u \) is defined by following duality:

\[ L_u : v \in H^2_0(\Omega) \rightarrow L_u(v) \in (H^2_0(\Omega))', \]

where

\[ L_u(v) : \psi \in H^2_0(\Omega) \rightarrow L_u(v, \psi) \]

and

\[ L_u(v, \psi) = \int_{\Omega} \Delta v \Delta \psi dx + \delta \int_{\Omega} \nabla v \cdot \nabla \psi dx - \int_{\Omega} a(x)v \psi dx + \int_{\Omega} \frac{\partial f_1(x, u)}{\partial u} v \psi dx. \]

It is easy to see that \( L_u \) is well-defined and the first eigenvalue of \( L_u \) is given by

\[ (4.2) \lambda_1 = \inf_{v \in H^2_0(\Omega), v \neq 0} \frac{L_u(v, v)}{\int_{\Omega} v^2 dx}. \]

We say that the solution \( u \) of (1.5) is stable if

\[ (4.3) \int_{\Omega} |\Delta v|^2 dx + \delta \int_{\Omega} |\nabla v|^2 dx - \int_{\Omega} a(x)v^2 dx + \int_{\Omega} \frac{\partial f_1(x, u)}{\partial u} v^2 dx \geq 0 \]

for every \( v \in C^2_c(\Omega) \), see [27] for the definition of stability of solutions to biharmonic problems. Actually, (4.3) implies that the principal eigenvalue of the linearized equation associated with (1.5) is nonnegative and hence the solution \( u \) of (1.5) is stable.

Next, we state and prove the stability theorem.

**Theorem 4.1.** Let \( u \in H^2_0(\Omega) \cap L^\infty(\Omega) \) be a positive solution to (1.5) in \( \Omega \). Let (H1) hold. Then \( u \) is stable.

**Proof.** Since \( u \in H^2_0(\Omega) \cap L^\infty(\Omega) \) be a positive solution of (1.5) so by the classical elliptic regularity theory, \( u \in C^2(\Omega) \). Now for any \( v \in C^2_c(\Omega) \), we choose

\[ \phi = \frac{v^2}{u} \]

as a test function in (4.1). Since

\[ \nabla \phi = \frac{2uv\nabla v - u^2\nabla u}{u^2}, \]

and

\[ \Delta \phi = \frac{2u^3|\nabla v|^2 - 4uvu^2\nabla u \cdot \nabla v + 2u^2u^2|\nabla u|^2 + 2vu^3\Delta v - v^2u^2\Delta u}{u^4}, \]

we have

\[ \int_{\Omega} (\Delta \phi)^2 dx + \delta \int_{\Omega} (\nabla \phi)^2 dx - \int_{\Omega} a(x)\phi^2 dx + \int_{\Omega} \frac{\partial f_1(x, u)}{\partial u} \phi^2 dx \geq 0. \]

Therefore, the solution \( u \) of (1.5) is stable.
so from (4.1), we get

\[
\int_{\Omega} \Delta u \left[ \frac{2u^3|\nabla v|^2 - 4uv |\nabla u \cdot \nabla v|}{u^4} + 2v^2 |\nabla u|^2 + 2uv^3 \Delta v - v^2 u^2 \Delta u \right] dx
\]

\[
+ \delta \int_{\Omega} \nabla u \left[ \frac{2uv \nabla v - v^2 \nabla u}{u^2} \right] dx = \int_{\Omega} a(x)v^2 dx - \int_{\Omega} \frac{f_1(x, u)}{u} v^2 dx.
\]

This yields that

\[
\int_{\Omega} -\frac{4v}{u^2} \Delta u \nabla u \cdot \nabla v dx + \int_{\Omega} \frac{2v}{u} \Delta u \Delta v dx + \int_{\Omega} \frac{2}{u} |\nabla v|^2 \Delta u dx - \int_{\Omega} \frac{v^2}{u^2} |\Delta u|^2 dx
\]

\[
+ \int_{\Omega} \frac{2v^2}{u^3} |\nabla u|^2 dx + \delta \int_{\Omega} \nabla u \left[ \frac{2v \nabla v - v^2}{u^2} \nabla u \right] dx - \int_{\Omega} a(x)v^2 dx
\]

\[
+ \int_{\Omega} f_1(x, u) \frac{v^2}{u} dx + \int_{\Omega} |\Delta v|^2 dx - \int_{\Omega} |\Delta v|^2 dx + \frac{\delta}{\int_{\Omega} |\nabla v|^2 dx} - \frac{\delta}{\int_{\Omega} |\nabla v|^2 dx} = 0.
\]

Retaining underlined terms on left hand side we get

\[
\int_{\Omega} |\Delta v|^2 dx + \delta \int_{\Omega} |\nabla v|^2 dx - \int_{\Omega} a(x)v^2 dx + \int_{\Omega} f_1(x, u) \frac{v^2}{u} dx
\]

\[
= \int_{\Omega} \left[ |\Delta v|^2 + \frac{4v}{u^2} \Delta u \nabla u \cdot \nabla v - \frac{2v}{u^2} \Delta u \Delta v
\]

\[
- \frac{2}{u} |\nabla v|^2 \Delta u + \frac{v^2}{u^2} |\Delta u|^2 - \frac{2v^2}{u^3} |\nabla u|^2 \Delta u - \frac{2v}{u} \nabla u \nabla v + \delta \frac{v^2}{u^2} |\nabla u|^2 \right] dx
\]

\[
= \int_{\Omega} \left[ (\Delta v - \frac{v}{u} \Delta u)^2 + \delta \left( \nabla v - \frac{v}{u} \nabla u \right)^2 + \frac{4v}{u^2} \Delta u \nabla u \nabla v - \frac{2v}{u} |\nabla v|^2 \Delta u - \frac{2v^2}{u^3} |\nabla u|^2 \Delta u \right] dx.
\]

This implies that

\[
\int_{\Omega} |\Delta v|^2 dx + \delta \int_{\Omega} |\nabla v|^2 dx - \int_{\Omega} a(x)v^2 dx + \int_{\Omega} f_1(x, u) \frac{v^2}{u} dx
\]

\[
\geq \int_{\Omega} \left[ \delta \left( \nabla v - \frac{v}{u} \nabla u \right)^2 + \frac{4v}{u^2} \Delta u \nabla u \nabla v - \frac{2v}{u^2} |\nabla v|^2 \Delta u - \frac{2v^2}{u^3} |\nabla u|^2 \Delta u \right] dx
\]

\[
= \int_{\Omega} \left[ \delta \left( \nabla v - \frac{v}{u} \nabla u \right)^2 + \frac{2}{u} \Delta u \left( |\nabla v|^2 + \frac{v^2}{u^2} |\nabla u|^2 - \frac{2v \nabla u \nabla v}{u} \right) \right] dx
\]

\[
= \int_{\Omega} \left( \delta - \frac{2}{u} \Delta u \right) \left( \nabla v - \frac{v}{u} \nabla u \right)^2 dx
\]

\[\geq 0.\]

Since \(u\) satisfies

\[\delta u \geq 2\Delta u \text{ in } \Omega\]

so this implies that

\[
\int_{\Omega} |\Delta v|^2 dx + \delta \int_{\Omega} |\nabla v|^2 dx - \int_{\Omega} a(x)v^2 dx + \int_{\Omega} \frac{f_1(x, u)}{u} v^2 dx \geq 0
\]
and using the hypothesis (H1), we obtain
\[
(4.4) \quad \int_{\Omega} |\Delta v|^2 \, dx + \delta \int_{\Omega} |\nabla v|^2 \, dx - \int_{\Omega} a(x)v^2 \, dx + \int_{\Omega} \frac{\partial f_1(x,u)}{\partial u} v^2 \, dx \geq 0
\]
and therefore \(u\) is stable. This completes the proof of this theorem. \(\square\)

REFERENCES

[1] W. Allegretto, Positive solutions and spectral properties of weakly coupled elliptic systems, J. Math. Anal. Appl. 120 (1986), no. 2, 723–729.
[2] W. Allegretto, On the principal eigenvalues of indefinite elliptic problems, Math. Z. 195 (1987), no. 1, 29–35.
[3] W. Allegretto, Sturmian theorems for second order systems. Proc. Amer. Math. Soc. 94 (1985), no. 2, 291–296.
[4] W. Allegretto and Y.X.Huang, A Picone’s identity for the p-Laplacian and applications, Nonlinear Anal., 32(7) (1998), pp. 819–830.
[5] K.Bal, Generalized Picone’s identity and its applications, Electron. J. Diff. Equations., no. 243 (2013), pp. 1–6.
[6] E. Berchio, A. Farina, A. Ferrero, F. Gazzola, Existence and stability of entire solutions to a semilinear fourth order elliptic problem, J. Diff. Equations 252 (2012), 2596–2616.
[7] G. Bognár, O. Došlý, Picone-type identity for pseudo p-Laplacian with variable power, Electron. J. Diff. Equations 2012, No. 174, 1–8.
[8] D. Castorina, P. Esposito and B. Sciunzi, Low dimensional instability for semilinear and quasilinear problems in \(\mathbb{R}^N\), Commun. Pure Appl. Anal. 8 (2009), No. 6, 1779–1793.
[9] D.R.Dunninger, A Picone integral identity for a class of fourth order elliptic differential inequalities, Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. 50(8) (1971), pp. 630–641.
[10] A. Ferrero, F. Gazzola and T. Weth, Positivity, symmetry and uniqueness for minimizers of second-order Sobolev inequalities, Ann. Mat. Pura Appl., 186 (2007), no. 4, 565–578.
[11] J.V.A. Goncalves, E. D. Silva, M. L. Silva, On positive solutions for a fourth order asymptotically linear elliptic equation under Navier boundary conditions, J. Math. Anal. Appl. 384(2011) 387-399.
[12] F. Gazzola, H. Grunau and G. Sweers, Polyharmonic boundary value problems, A monograph on positivity preserving and nonlinear higher order elliptic equations in bounded domain, Springer, 1991.
[13] J. Jaroš, The higher-order Picone identity and comparison of half-linear differential equations of even order, Nonlinear Anal., 74(18) (2011), pp. 7513–7518.
[14] D.A. Kandilakis, M. Magiropoulos and N.Zographopoulos, Existence and bifurcation results for fourth-order elliptic equations involving two critical Sobolev exponents, Glim. Math. J. 51, no. 1 (2009), 127–141.
[15] J. Karátson, P. L. Simon, On the linearized stability of positive solutions of quasilinear problems with p-convex or p-concave nonlinearity, Nonlinear Anal., 47 (2001), pp. 4513–4520.
[16] A.E.Lindsay and L.Viga, Multiple quenching solutions of a fourth order parabolic pde with a singular nonlinearity modeling a MEMS capacitor, SIAM J. Appl. Math., Vol. 72, No. 3 (2012), 935–958.
[17] Y.Maliki, On a Q-curvature equation on complete Riemannian manifolds, Adv. Nonlinear Stud. 10 (2010), no. 1 (2010), 195–217.
[18] A. Manes, A.M. Micheletti, Un’estensione della teoria variazionale classica degli autovalori per operatori ellittici del secondo ordine. Bolletino U.M.I., 7, 1973, 285–301.
[19] J. A. Pelesko and D. H. Bernstein, Modeling MEMS and NEMS, Chapman & Hall CRC Press, Boca Raton, FL, 2002.
[20] T.Sato, T.Watanabe, Singular positive solutions for a fourth order elliptic problem in \(\mathbb{R}^N\), Commun. Pure Appl. Anal. 10, No. 1 (2011), 245–268.
[21] M. Struwe, Variational Methods: Applications to Nonlinear Partial Differential Equations and Hamiltonian Systems, Fourth Edition, Springer, 2007.
[22] J.Tyagi, A nonlinear Picone’s identity and applications, Applied Mathematics Letters, 26 (2013), 624–626.
[23] J. Tyagi, Stability of positive solutions to p&2-Laplace type equations, Diff. Equ.Appl. 5 (2015), 549–559.
[24] J. Tyagi, A note on the stability of solutions to quasilinear elliptic equations, Advances in Cal. Var., 6, Issue 4 (2013), 483–492.

[25] G. Warnault, Liouville theorems for stable radial solutions for the biharmonic operator, Asymptot. Anal. 69 (2010) 87–98.

[26] J. Wei, X. Xu, Classification of solutions of higher order conformally invariant equations, Math. Ann. 313 (1999), 207–228.

[27] J. Wei, D. Ye, Liouville theorems for stable solutions of biharmonic problem, Math. Ann. 356 (2013), No. 4, 1599–1612.

[28] N. Yoshida, Picone identities for half-linear elliptic operators with p(x)-Laplacians and applications to Sturmian comparison theory, Nonlinear Anal. 74 (2011), no. 16, 5631–5642.

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