Turning the nuclear energy density functional method into a proper effective field theory: reflections

R. J. Furnstahl

Department of Physics, Ohio State University, Columbus, OH 43210, USA

Received: date / Revised version: date

Abstract. Nuclear energy density functionals (EDFs) have a long history of success in reproducing properties of nuclei across the table of the nuclides. They capture quantitatively the emergent features of bound nuclei, such as nuclear saturation and pairing, yet greater accuracy and improved uncertainty quantification are actively sought. Implementations of phenomenological EDFs are suggestive of effective field theory (EFT) formulations and there are hints of an underlying power counting. Multiple paths are possible in trying to turn the nuclear EDF method into a proper EFT. I comment on the current situation and speculate on how to proceed using an effective action formulation.

1 Introduction

The many levels of emergent phenomena in strong interaction physics have led to a corresponding set of models that have successfully described experiment at each energy scale (see Table 1). The tension between simplified models for these phenomena and a reductive description via an underlying theory has been bridged in almost all cases by an effective field theory (EFT). An EFT is characterized by a selection of the relevant degrees of freedom, an identification of symmetries to enforce, and a power counting that establishes an order-by-order expansion, incorporating fine-tuning as needed. One might observe that behind every successful emergent phenomenology there is an EFT (or possibly more than one) waiting to be uncovered. The insight, inspiration, and validation from a more microscopic description can be absorbed by the EFT or guide its construction through matching. In turn, the EFT provides clarity to models, computational simplification, and an opportunity for greater precision.

The one clear gap in the EFT hierarchy of Table 1 is for describing ground state properties of nuclei. Energy density functionals (EDFs) remain the phenomenological method of choice for these observables across the bulk of the table of nuclides. This is despite great progress in the reach of ab initio methods using chiral EFT Hamiltonians, for there is still much of the table uncovered and the accuracy for the ground states is not comparable although generally superior for low-lying spectroscopy. The form and organization of these EDFs are suggestive of EFTs, but there remain major holes to fill and questions to answer before a robust theory is available. My goal here is to survey where we stand and the possibilities for filling the holes. A theme I will adopt is that we should use the phenomenological success as a close, though not infallible, guide to formulating an appropriate EFT. I will touch upon many promising avenues being pursued and then outline a particular path that has not been explored in depth. I caution the reader that this is a selection and assessment based on personal prejudices, not a comprehensive review, and therefore some approaches will not be addressed and references to the literature will not be exhaustive.

1.1 Emergent features of nuclei captured by EDFs

As I intend to use the phenomenological successes of EDFs as a guide, let me briefly review some of the emergent features of nuclei manifested by empirical EDFs such as Skyrme models or Gogny models. Nuclear saturation is characterized by quantitatively precise liquid drop systematics overlaid with a regular shell structure. Every EDF that is successful in fitting nuclear masses and radii predicts almost the same nuclear matter saturation properties: very close binding energies and equilibrium densities, with greater spread in the symmetry energy and compressibility. (There are some systematic differences between nonrelativistic and covariant functionals, possibly because of the incompleteness of their respective functional forms, but these differences are small.) Bulk deformations are well described as are separation energies.

To describe pairing, modern EDFs implement some variety of the Hartree-Fock-Bogoliubov (HFB) formalism. Pairing is manifested in the form of an even-odd staggering of masses, with the magnitude reflecting the size of...
the pairing gap. The specification of the force or functional in the pairing channel is usually much less detailed than in the particle-hole channel. EDFs that are not derived from a Hamiltonian generally treat these channels independently while others such as Gogny implementations originate from an effective force that may be the same in both channels. We will return to the apparent lower resolution in the pairing channel later, as it is a relevant element to consider for a EFT implementation. Finally, mean collective properties of excitations such as giant resonances are generally well described in a small-amplitude time-dependent mean-field approach (RPA or QRPA), although not damping or fragmentation [14]. Note that these features do not exhaust the scope of EDFs (see ref. [5] for a recent overview), but are sufficient for our discussion.

The tension between underlying complexity and emergent simplicity is evident when comparing the functionals to microscopic inter-nucleon interactions based on chiral EFT. On the one hand, there is evidence that the naturalness of low-energy constants in the chiral EFT Lagrangians, when scaled according to naive dimensional analysis, is inherited by parameter values in Skyrme and covariant functionals [15,16,17] (with the caveat that while the evidence is compelling it is not absolutely convincing because of the limited orders available). Thus the EDFs in this way appear to reflect underlying chiral physics. At the same time, there is no overt support in the functionals for chiral symmetry constraints [18]. Furthermore, multiple studies over the years have shown that relatively few parameter combinations determine most of the physics [19,20,21,13], for example, from doing a singular value decomposition to uncover the dominant combinations. These combinations reflect emergent saturation properties, not the chiral Lagrangian parameters (which are themselves interpretable via resonance saturation in terms of meson exchange [22]). The limited role of pions in mean-field descriptions is perhaps clearest in covariant functionals, where they appear in a defining Lagrangian, but are suppressed because of spin and isospin averaging in the bulk. An analogous averaging argument for the dominance of SU(4)-invariant interactions was made recently by Lu et al. while considering “essential elements for nuclear binding”: these elements did not include pions [23]. Are we seeing phenomenological signals that pionic degrees of freedom are not optimal (or necessary) for nuclear EDFs?

One might question whether the nuclear properties I have identified should be called “emergent” based on how that term is used elsewhere. For my purposes it means that the phenomena reflect a complexity or collectivity not evident in the degrees of freedom (dofs) of the underlying Hamiltonian, such that a more transparent description would require different dofs. Nuclear saturation and pairing are certainly emergent from the viewpoint of QCD, where even the nucleons themselves are highly non-trivial consequences of the QCD Hamiltonian. But with respect to chiral EFT they are not obviously emergent by my definition. Rather, one might say that it is evident from the mid-range attraction and short-range repulsion built into fitting phase shifts, plus a repulsive three-body force, that there will be saturation (i.e., a liquid phase); that low-lying collective modes are inevitable when thinking of vibrating liquid drops; and that attraction near the Fermi surface always means there will be pairing. While this may all be true, the quantitative properties are very finely determined. We do know that if we fit chiral EFT parameters to few-body properties, we will get saturation, just not necessarily in the right place (e.g., see [24]). And it is a precise determination that is key to a quantitative, not just qualitative, description of nuclear properties.

### Table 1. Emergent phenomenologies and the EFTs that have supplanted them.

| Emergent phenomena | Phenomenology | Effective field theory (EFT) |
|--------------------|---------------|-----------------------------|
| nucleons as confined quarks/glue | constituent quark model | chiral quark model [6] |
| inter-nucleon forces | meson exchange models | chiral EFT with nucleons, $\Delta$s, pions [4] |
| universal large-scattering-length physics and halo nuclei | effective range expansion and cluster models | pionless EFT: nucleons only [7] or nucleons and clusters (halo) [8] |
| saturation, pairing, shell structure | energy density functionals | What goes here??? |
| low-lying excitations, superfluidity | nuclei as Fermi liquids [9] | EFT at the Fermi surface [10,11] |
| collective rotational/vibrational motions of deformed nuclei | collective models | effective theory for deformed nuclei: systematic collective dofs [12] |

### 1.2 Why should we try to do better?

The motivations for doing better than the current empirical EDFs are well aligned with what an EFT could (in principle) do for us.

- The breakdown and failure mode for EDFs is unclear. For example, should they work all the way to the dripline as currently applied or should they degrade when bulk systematics become less dominant? The internalization of the breakdown scale is a hallmark of the EFT approach.

- How do we improve EDFs in a controlled manner? Are density dependencies too simplistic? How do you know? How should we organize possible terms in the EDF? All of these related questions are addressed by EFT power counting.

- More accuracy is wanted, e.g., for r-process calculations; is this even possible? That is, what is the theoretical limit of accuracy? This is again an EFT feature because there is an expansion parameter that enables a quantification of missing physics [25].
There is apparent model dependence seen upon comparing different parameterizations of empirical EDFs. How should this be interpreted? A consequence is that extrapolations to the driplines, to large A, and to high density are at least to some extent uncontrollable. The completeness of an operator basis in an EFT addresses issues of model dependence.

An important contemporary theme in nuclear theory is the robust estimation of theoretical uncertainties, which ties in with some of the other issues. There have been significant advances in EDF uncertainty quantification (UQ) [5], but a proper EFT would add important prior knowledge from the expansion convergence pattern [25].

What observables can be calculated in a controlled way and how does one know how to couple to external currents (e.g., for electroweak properties)? Within formal density functional theory (DFT)\(^2\) certain quantities such as single-particle levels are not guaranteed to accurately correspond to physical quantities, although they are still extracted from EDFs. The consistent construction of a Hamiltonian and currents is another fundamental feature of EFTs. The formulation of the EDF as an EFT (such as in an effective action framework, see below) leads to a clear identification of accessible observables and a guide to extensions for other observables (e.g., for single-particle levels one can relate the full and Kohn-Sham Green’s functions [27]).

Finally, we desire connections to other nuclear EFTs in the hierarchy leading to quantum chromodynamics (QCD). The search for intersections raises many questions we will be able to address with an EFT: When is pion physics resolved? Does near-unitarity of nuclear forces matter? What is the connection to the many-body forces that play a vital role in nuclear saturation in chiral EFT?

Together these form a strong case for seeking a proper EFT formulation of EDFs. So having decided an EFT is worth pursuing, how should we proceed? We could try top-down, starting from ab initio methods, or bottom-up with a “general” functional, matching to experiment and/or ab initio calculations. A hybrid plan on the way to a proper EFT could be to extend or modify existing EDF forms in (semi-)controlled way, using the microscopic many-body theory for guidance. There has been progress in nuclear many-body theory that touches all of these paths. I will first comment on implications of this progress before turning to my current favorite bottom-up plan.

2 Progress report

About a decade ago I co-authored a summary and forward-looking review entitled “Toward ab initio density functional theory for nuclei” [28]. The focus was how to go from ab initio microscopic methods to a DFT, with some discussion of EFT for EDFs. The article described two different general approaches: through many-body perturbation theory (MBPT), possibly resummed, or through an effective action formalism. Much in the nuclear theory landscape has changed since then. Here we will consider elements of the contemporary landscape in light of the goal of a proper EFT for EDFs.

2.1 Lattice quantum chromodynamics (LQCD)

Numerical calculation on a space-time lattice is a well-established method for solving QCD in the confinement regime (functional renormalization group methods also have many promising aspects and Schwinger-Dyson methods offer good QCD-motivated phenomenology). LQCD is now capable of first-principles calculations of hadron masses and decay constants, with controlled errors at the percent level. Despite this progress, not long ago it was thought that accurately calculating the residual force between nucleons was beyond reach in principle because of signal-to-noise arguments. However, the worst of those problems have been overcome and there are active calculations of multi-hadron systems from multiple groups [29].

There are still formidable technical problems and qualitative disagreements to work out for the two major methods being employed (e.g., see [30,31,32]). But even if resolved, at this stage it appears the most likely scenario for the role of LQCD will be to match to chiral EFT to determine low-energy constants and calibrate experimentally challenging phenomena such as three-neutron forces and strangeness, rather than providing direct input to an EDF formulation. This is consistent with the general paradigm based on the renormalization group (RG) that a tower of EFTs replaces direct calculations with the underlying theory. The more reductive frameworks are relevant for proofs-of-principle and to identify missing physics, but for tractable precision calculations and cleaner physical insight, an EFT at the appropriate resolution for nuclei near their ground states is the way to go.

So what is the appropriate resolution? Given that the Fermi momentum inside of a nucleus is larger than the pion mass, one would naively imagine that the EFT must include the pion as a resolved degree of freedom, which implies chiral EFT, which we consider next.

2.2 Ab initio with chiral EFT

We have entered an era of precision calculations of nuclear structure and reactions, which seek to address the full table of nuclides and astrophysical systems such as...
neutron stars. The precision era has been enabled by advances in theoretical methods, both conceptual and algorithmic: computational capabilities; and enhanced confrontation with experiment. How should we exploit these new capabilities in the context of improved EDFs? One possible route is to use direct extensions of ab initio many-body methods in a top-down manner via MBPT [28]. Ab initio in this context means calculations of nuclei based on a free-space Hamiltonian with proton and neutron degrees of freedom that is fit to (at least) two-body and few-body data (usually scattering phase shifts for the former and bound-state properties for the latter).

This route is conceivable because the last decade has seen astounding progress in the reach of ab initio methods. Within the last half-dozen years, the upper bound for 5% data (usually scattering phase shifts for the former and nuclear spectral functions) has grown from calculations with realistic Hamiltonians (which generally require three-body forces) has grown from \( A = 14 \) to \( A \) of order 80, with calculations in selected regions reaching even higher [33]. The Hamiltonians in these calculations are generally from chiral EFT with the pionic interactions converted to potentials, for example by decoupling the pion sector via unitary transformations [7][34]. Although long recognized as important, \( \Delta \) degrees of freedom are just now being included in mainstream potentials.

A wide range of complementary computational methods are used to implement chiral EFT for nuclei [35]. Each exhibits characteristic strengths. Lattice EFT, in which spacetime is discretized as in LQCD and pion interactions are replaced by auxiliary fields, is naturally suited to described clustering. Stochastically improved wave functions with the auxiliary field diffusion Monte Carlo method can handle short-range correlations that are problematic in general. The no-core shell model (NCSM) uses large-scale diagonalization, which gives complete excitation spectra that demonstrate the emergence of collective rotational bands [29]. (There have also been many developments of both shell model EFT and EFT in an oscillator basis.) Coupled cluster methods and the in-medium renormalization group, which sum infinite-order sets of diagrams through nonlinear equations and flow equations, respectively, have favorable scaling to large nuclei and effective interactions for the phenomenological shell model emerge naturally. Self-consistent Green’s function (SCGF) methods enable conserving approximations and direct access to nuclear spectral functions.

The proliferation of these methods has been paralleled by a recent proliferation of chiral EFT Hamiltonians. These Hamiltonians have one of two classes of physics content, with or without \( \Delta s \), but otherwise only differ in the type of regularization, which is used to cut off high momenta. In principle differences in the regularization should be absorbed in the process of renormalization, but the current implementations are not strictly renormalizable in the sense of being cutoff independent at a fixed order. An immediate consequence is that there are regulator artifacts that can degrade the EFT performance. More generally problematic is that the renormalization group (RG) cannot be directly applied to these interactions within the EFT. It is also important to note that the power counting for these potentials, which determines what contributes at each EFT order, is done in free space and then the many-body Schrödinger equation is solved as exactly as possible using the resulting potential. So there is no accounting for possible altered power counting at sufficiently high density.

The paradigm for descending a tower of EFTs is to match an emergent EFT to an underlying one at a scale where degrees of freedom are eliminated. This is where naive dimensional analysis estimates of low-energy constants are expected to be good. Then one uses the renormalization group to evolve to an appropriate lower scale for phenomena under consideration. For free-space nuclear Hamiltonians, RG methods such as the similarity RG are used to integrate out high-momentum modes, which makes the interaction softer, meaning more perturbative. In general, such a softening should be associated with having a more appropriate resolution for the problem at hand because the coupling between states that appears in perturbative expansions is reduced. The practical counterbalance in the nuclear case is the growth of many-body forces, which are difficult to treat with ab initio methods; this has limited the use of the SRG and similar approaches to only moderate softening.

In all, the prospects are good for ab initio methods improving their coverage of the table of nuclides, with the advantage of consistent currents. However, it is uncertain at best whether they will be able to challenge the accuracy of the phenomenological EDFs for masses. Calculations at the percent level are feasible, but for \(^{208}\text{Pb}\), for example, that would mean an error of order 15 MeV in the binding energy, while the best EDFs are at the 1 MeV level or below. This may be an insurmountable barrier, but time will tell. It is noteworthy that the best reproduction of nuclear masses and radii is achieved when the chiral EFT low-energy constants are determined by fits that include nuclei well above \( A = 4 \). In essence, they must be fine-tuned to saturation properties, as is done with EDFs.

An alternative to directly solving for nuclear properties using chiral EFT and ab initio methods is to use them to inform nuclear EDFs (see also sect. 2.5). Duguet and collaborators [37][38][39][40] have developed a strategy to use MBPT and coupled cluster theory as frameworks for constructing novel parameterizations of nuclear EDFs that overcome problematic issues for EDFs. A key element is a proper treatment of symmetry breaking and restoration (see sect. 3.5). The goal is “to guide the construction of safe, explicitly correlated and systematically improvable parameterizations.” We will return to the critical issue of broken symmetries later.

2.3 Using phase space estimates for power counting

For any quantum field theory, a regularization scheme and scale must be introduced to define the theory. To regulate divergent loops, chiral EFT introduces cutoff regulators, which become an intrinsic part of the nuclear potentials. With the Weinberg power counting scheme, which is used
in essentially all microscopic calculations to date, iteration of the potential beyond leading order generates divergences without all the corresponding counterterms. As a result, the EFT is no longer strictly renormalizable and, as already noted, there are significant regulator artifacts. Until recently, almost all calculations used potentials with similar non-local regulators.

However, there are now local regulators for use with quantum Monte Carlo calculations [11,12] and semi-local regulators designed for improved power-counting behavior in free space [13,14]. Recent work has studied the impact of different regulators for finite density calculations [45]. The strategy was to use uniform matter as a testbed and work in MBPT [35]. A Monte-Carlo sampling method was developed to probe the interaction phase space in energy integrals using a variety of regulators, which revealed some striking differences between different regulators. (The observed impact of the regulators led to revisiting the implementation of the density matrix expansion, which had stalled in previous attempts that did not include regulators; see the next section.) A subsequent phase-space-based effort with the same sampling approach was aimed at developing and validating a many-body power counting for softened potentials in uniform matter [47]. This power counting completely alters the conventional hole-line-expansion power counting developed for hard potentials. At densities near nuclear matter saturation, Pauli blocking and decoupling of high-momentum modes combine to dramatically change the counting one expects in free-space (e.g., fine-tuning in the S waves goes away).

The diagnostic sampling tools to analyze many-body contributions showed that quantitative estimates are feasible in the particle-particle channel [47]. This approach is being extended to the two-body particle-hole channels, where a power counting prescription is less clear, and to more completely examine the role of three- and higher-body forces. Such empirical studies using the sampling technique, if applied to the effective action formalism advocated in sect. 15, can give us direct insight and validation into how power counting should work for a nuclear EDF.

2.4 Generating an EDF with pions using the DME

The perturbative in-medium results from low-momentum (e.g., RG-evolved) potentials could be taken to suggest that pion as dofs are still appropriate at nuclear matter densities but that an alternative EFT power counting from chiral EFT is needed. Kaiser and collaborators have proposed a perturbative chiral theory approach to nuclear matter and then to finite nuclei through an EDF functional [48,49]. They consider Lagrangians both for nucleons and pions and for nucleons, pions, and ∆'s, and fit parameters to nuclear saturation properties. They construct a loop expansion for the nuclear matter energy per particle, which leads to an energy expansion of the form

\[ E(k_f) = \sum_{n=2}^{\infty} k^n_f f_n(k_f/m_\pi, \Delta/m_\pi) , \]

\[ \Delta \equiv M_\Delta - MN \approx 300 \text{ MeV} , \]

where each \( f_n \) is determined from a finite number of in-medium Feynman diagrams, which incorporate the long-distance physics. All powers of \( k_f/m_\pi \) and \( \Delta/m_\pi \) are kept in the \( f_n \)'s because these ratios are not small quantities [49].

A semi-quantitative description of nuclear matter is found even with just the lowest two terms without \( \Delta \)'s and adding \( \Delta \)'s brings uniform improvement (e.g., in the neutron matter equation of state). By applying the density matrix expansion (DME) in momentum space to this expansion, they derive a Skyrme-like EDF for nuclei [49]. The DME provides a general way to map nonlocal functionals into local ones by converting one-body density matrices (OBDMs) into local densities. In particular, the nonlocality in the OBDMs arising from finite-range potentials is factorized into products of local densities multiplied by density-dependent couplings. The resulting EDF has qualitatively correct features but the accuracy is not competitive to phenomenological EDFs.

A more phenomenological approach is to apply the DME to long-range pion terms following the Weinberg power-counting expansion for the potential, then to merge the resulting functional with a conventional Skyrme functional and re-optimize. The idea is to inject novel density dependence from the long-range pion physics while adjusting the Skyrme parameters to absorb short-distance physics. Table 2 (adapted from ref. [50]) shows intriguing results from the application of the coordinate-space, regulated DME formulation by Dydak et al. [51] to NN and NNN diagrams. In particular, with the same fitting protocol and the same free parameters, the r.m.s. residual for binding energies improved significantly compared to the UNEDF-2 reference when chiral NN contributions are added, particularly with an explicit \( \Delta \) (e.g., to 1.41 MeV for NLO\( \Delta \) and 1.26 MeV for N2LO\( \Delta \) compared to 1.98 MeV for UNEDF-2). But after including three-body contributions, the result does not improve or even becomes significantly worse (compare N2LO\( \Delta+3N \) to N2LO\( \Delta \)). Whether the superior performance of the functional was due to including correct density dependence from pion exchange or to a better optimization from the new terms is still under investigation.

The plot of binding energy residuals in fig. 1 for the most favorable case from ref. [50] shows definite improvement when compared to the UNEDF-2 standard, but also manifests similar systematic deviations from zero. At least some of these deviations are attributable to “beyond mean field” physics, meaning that one needs to go beyond the standard Hartree-Fock-Bogoliubov (HFB) energy density functional (EDF), such as doing symmetry restoration and incorporating coupling to vibrational modes. An example of how such corrections reduce the pattern of residuals seen in fig. 1 is given in ref. [52] (see also [53]). The bottom
Table 2. Root mean square (r.m.s.) deviations between experimental and theoretical binding energies in MeV [50]. The ordering is according to free-space Weinberg power counting, with and without three-body forces (3N) and with and without $\Delta$s.

| EDF            | r.m.s. residual (MeV) |
|----------------|-----------------------|
| UNEDF-2        | 1.98                  |
| LO             | 1.99                  |
| NLO            | 2.02                  |
| N2LO           | 1.57                  |
| N2LO+3N        | 1.58                  |
| NLO-$\Delta$   | 1.41                  |
| NLO+$\Delta$+3N| 1.46                  |
| N2LO-$\Delta$  | 1.26                  |
| N2LO+$\Delta$+3N| 1.72                 |

Subsequently there have been interesting recent developments by others based on pionless EFT but going beyond the low-density perturbative expansion and by connecting to EDFs beyond the mean field level. These include constraining terms in the functional by requiring renormalizability [62,63] or going beyond a perturbative expansion by considering approximate self-energies [64]. A particularly compelling idea is to push the expansion of nuclei about the unitarity limit that dictates the universal physics manifested in cold atoms and dilute neutron gases [65,66,67]. See refs. [68,69] for a review of these ideas. At this stage these are all worthwhile avenues to pursue; none of them is conclusive as yet.

There are several related directions motivated by EFT principles being pursued by Dobaczewski and collaborators. The first was to extend existing Skyrme functionals with additional terms in a controlled, order-by-order gradient expansion [70]. However this does not overcome the evident limitations of the Skyrme functional form. Next there is the formulation in terms of pseudopotentials, which specify the EDF upon folding with an uncorrelated Slater determinant, which is found self-consistently. Thus the full functional is given within the Hartree-Fock (or HFB) approximation. The secret of the EFT approach based on local Lagrangians is due to the expectation (not generally proved) that the most general Lagrangian consistent with the symmetries of the system leads to the most general S-matrix. Thus a complete operator basis in the Lagrangian ensures a model-independent formulation because everything possible can be accommodated. The question is: can the pseudopotential formulation provide the analogous general blueprint for a nuclear energy functional? If so, we would have a framework for a proper EFT for EDFs.

In refs. [71,72,73], pseudopotentials, both zero-range and finite-range, are derived including all possible terms in a derivative expansion, following a well-defined power counting that uses an identified in-medium scale. An interesting argument is made that the resolution scale of chiral EFT is much higher than is needed. In particular, rather than using momentum $k \leq 2m_\pi$ or $k_F$, one should consider $\delta k$ to dissociate a nucleon:

$$\delta E_{\text{kin}} = \frac{\hbar^2 k_\delta}{M} \approx 0.25 \hbar c \delta k \approx 8 \text{ MeV}$$

which implies $\delta k \approx 32 \text{ MeV}/hc$. Nuclear excitations and shell-effects at the 1 MeV energy implies $\delta k \approx 4 \text{ MeV}/hc$ and below. From this perspective, the pion is a high-energy dof. It will be important to trace how this argument plays out in diagrammatic constructions.
In the pseudopotential approach, coupling constants are fit to data (with constraints). By deriving the EDF in Hartree-Fock form, self-interaction pathologies that plague beyond-mean-field calculations are avoided. The resulting functionals pass tests for scale independence, convergence, and naturalness. More recent pseudopotentials generate a spuriosity-free nonlocal EDF that can describe pairing without density-dependent terms [74]. Finally, an approach to connect ab initio calculations to a pseudopotential-based EDF is laid out in ref. [75]. Here one derives the couplings not from experiment but from theoretical calculations of finite systems using a constrained variation method, which is one way to define a density functional theory.

Finally, we consider the work by Saperstein and collaborators [76], which is rooted in Landau’s Fermi liquid theory as extended to finite nuclei by Migdal [9,77]. The Fermi liquid approach has led to the very successful Fayans functionals, which have significant phenomenological components [13]. But the underlying formalism is in the spirit of EFTs. Indeed, Landau theory has been formalized by Polchinski [10] and Shankar [11] as an expansion about the Fermi surface. If this can be extended to include the bulk properties, building on the self-consistent theory of finite fermi systems in [76], it would provide insight or even a direct path to a proper nuclear EDF.

2.6 So what do we conclude?

The many approaches touched on in this section give multiple avenues for making progress toward improved EDFs. I’m afraid it is not at all clear to me which should be favored and I do not propose to choose a winner. But we have accumulated many clues about how we might choose dofs and formulate an EFT for nuclear EDFs, so we will proceed with one particular path not yet followed.

3 Effective action as a framework for proper EFT for EDFs

Here I will lay out my own motivation and plan for making progress on a proper EFT for nuclear EDFs, based on a bottom-up EFT for DFT using an effective action formulation with auxiliary fields and an explicit treatment of zero modes.

3.1 Addressing questions about EFT for DFT

Any path toward a proper effective theory for nuclear EDFs must confront some pertinent questions about the general features of an EFT. Here are my prejudices at present for the answers based on insights from the explorations outlined in the previous section.

- What are the optimal dofs? The first point is whether the dofs of chiral EFT are appropriate. My take is that the resolution of chiral EFT with nucleons and pions is too high for an efficient EFT, based on the smaller excitations scales at finite density and the success of the phenomenological EDFs. A pionless theory expanded about free space does not naturally include the liquid drop systematics. But at finite density one could identify the dofs as quasineutron densities and the associated mean fields. For low-lying collective degrees of freedom, time-dependent mean fields seem to be the natural choice. A challenge to address is that quasiparticles are usually only considered as meaningful near the Fermi surface, but we want a description for the bulk as well (as suggested in [76]).

- Power counting: what is the expansion parameter for an EDF? First, could it be the counting of chiral EFT? The fact that saturation in chiral EFT as currently formulated is driven by the repulsive three-body force, which does not appear until third order (N^3LO), suggests this is not a counting we want for heavy nuclei (so this reinforces an alternative choice of dofs). Indeed, we expect a proper EFT would produce saturation in leading order [68]. The work by Dyhdalo et al. on medium-power counting with RG-softerned interactions indicates how Pauli-blocking modifies the power counting [17]. Unfortunately, the actual expansion parameter for a phase-space-based approach is not obvious as yet. When identified, it should inform about the breakdown scale and therefore the range of applicability.

- How should the EDF as EFT be formulated? One path is to use MBPT [28]; I believe this is worth pursuing. However, as I will review in sect. 3.3 effective actions are the natural and appropriate formalism for a field theoretic version of density functional theory (DFT) [78]. This still leaves open multiple options for the implementation based on different types of effective action.

- How can we implement or expand about liquid drop physics? The phenomenological EDFs are conventionally thought of as mean-field approximations, for which the liquid drop physics and also shell structure are natural consequences. In an effective action formalism treated in a loop expansion, the leading piece is a saddlepoint contribution that defines mean fields and should improve with nuclear size as EDFs do.

- What role should the renormalization group play? The renormalization group (RG) should be an integral part of the ultimate EFT-for-EDFs formulation. For now we can highlight several areas where RG technology and insight is relevant. The first is the softening of interactions from decoupling high-momentum dofs. As discussed in ref. [28], this is a natural prelude to nuclear EDFs that clearly affects the power counting of MBPT [17]. The second is the use of RG to scale toward the Fermi surface in the EFT version of Landau theory [10]; is something analogous helpful for EDFs (i.e., a different type of scaling than with respect to the full momenta of nucleons in the medium)? The third is a top-down approach to ab initio but orbit-free DFT (i.e., not Kohn-Sham) proposed by Schwenk and Polonyi that uses a clever RG evolution [79]. The
basic idea is to introduce an effective action for a nucleus with a low-momentum interaction included with a multiplicative factor $\lambda$ and a confining background potential (e.g., a harmonic oscillator trap) with a factor $(1 - \lambda)$. As $\lambda$ flows from 0 to 1, the background potential is turned off and the interactions, with associated many-body correlations, are turned on. This evolution is dictated by an RG equation in $\lambda$. The first test implementations were by Braun and collaborators [80,81] and there are recent related developments by others [82,83,84]. The practicality of this approach for nuclei will have to be demonstrated but at the least it should give valuable insight about the structure of a low-resolution functional and how to treat self-bound systems in DFT.

- **How might we reconcile the different EDF representations via EFT?** One element is in the range of choices in implementing an EFT, such as the use of dimer fields in free-space pionless EFT as an alternative to only nucleon fields. Thus we can use auxiliary fields or expand in gradients with point couplings (contact terms). We also expect freedom from the choices how to regularize and renormalize, in analogy to the scale and scheme dependence of interactions in chiral EFT. Finally, there is also the change in representation possible from field redefinitions (or unitary transformations).

- **How should we deal with symmetry breaking that is so integral to the standard EDF approach?** In the context of effective actions, the symmetry breaking is precipitated by approximations stemming from a saddlepoint evaluation. A consequence is zero modes when one expands about the saddlepoint — part of what is called going “beyond mean field” in the EDF literature. The zero modes must be dealt with as a priority for the EFT program to succeed; my preferred plan is to adapt methods used for gauge theories to handle collective coordinates, as has been done in other contexts, although this is not yet shown to be a viable approach for DFT.

- **Can we implement an EFT for EDFs without losing the favorable computational scaling of current nuclear EDFs?** This is a pertinent question, but one that can’t be answered at present. If the computational cost to evaluate the functional does increase significantly, there is the option to use statistical emulators (e.g., Gaussian processes).

- **What makes it possible to have a simple form for the phenomenological EFTs?** In Coulomb DFT this is related to the robustness of local density approximations (LDA). This realized in nuclear EDFs through a gradient expansion of densities. In field theoretic terms, the question becomes whether low-order operator product expansions can be used for the action.

It should be clear that these are provisional answers and may not be exclusionary, e.g., there may be more than one reasonable choice of degrees of freedom.

### 3.2 Schematic look at effective actions

To implement a bottom-up EFT for EDFs, we need an appropriate formalism. I believe the natural choice is an effective action formalism [78,85]. Motivation will be given in sect. 3.3, but first we give a schematic overview of some of the key features, neglecting at first the issue of zero modes. More detail can be found in ref. [28] and references cited there.

If we focus only on ground states, we can build on the intuition physicists have for ordinary thermodynamics with $N$ particles as temperature $T \rightarrow 0$. The thermodynamic potential is derived from the grand canonical partition function, with the chemical potential $\mu$ acting as a source to change $N = \langle \hat{N} \rangle$,

$$\Omega(\mu) = -kT \ln Z(\mu) \quad \text{and} \quad N = - \left( \frac{\partial \Omega}{\partial \mu} \right)_{TV}. \quad (4)$$

Because $\Omega$ is convex, $N$ is a monotonically increasing function of $\mu$ and we can invert to find $\mu(N)$ and apply a Legendre transform to obtain

$$F(N) = \Omega(\mu(N)) + \mu(N) N. \quad (5)$$

This is our (free) energy function of the particle number, which is analogous to the DFT energy functional of the density $\rho$.

If we generalize to a spatially dependent chemical potential $J(x)$, then

$$Z(\mu) \rightarrow Z[J(x)]$$

$$\mu N = \mu \int \bar{\psi} \psi \rightarrow \int J(x) \bar{\psi} \psi(x) . \quad (7)$$

Continuing the analogy, we can do a functional Legendre transform from $\ln Z[J(x)]$ to $I[\rho(x)]$, where $\rho = \langle \bar{\psi} \psi \rangle$, and we have DFT with $I$ simply proportional to the energy functional.

The functional $I$ is one type of effective action [85]. An effective action is generically the Legendre transform of a generating functional with an external source (or sources). For DFT, we use a source to adjust the density (cf. using an external applied magnetic field to adjust the magnetization in a spin system). Consider first the simplest case of a single external source $J(x)$ coupled to the density operator $\bar{\rho}(x) \equiv \bar{\psi}(x)\psi(x)$ in the partition function

$$Z[J] = e^{-W[J]} \sim \text{Tr} e^{-\beta(\hat{H} + J \bar{\rho})} \quad (8)$$

$$\sim \int D[\bar{\psi}] D[\psi] e^{-\int [L + J \bar{\psi} \psi]}, \quad (9)$$

for which we can construct a (Euclidean) path integral representation with Lagrangian $L$ [87]. (Note: because our treatment is schematic, for convenience we neglect normalization factors and take the inverse temperature $\beta$ and the

---

3 Because $\nu_{\text{ext}}$ is typically given rather than eliminated, for a closer analogy we would also define $\Omega_{\nu_{\text{ext}}}(\hat{N}) = F(\hat{N}) - \mu \hat{N}$. This depends explicitly on both $N$ and $\mu$. This gives the grand potential when minimized with respect to $N$ [86].
volume $\Omega$ equal to unity.) The static density $\rho(x)$ in the presence of $J(x)$ is

$$\rho(x) \equiv \langle \hat{\rho}(x) \rangle_J = \frac{\delta W[J]}{\delta J(x)} , \tag{10}$$

which we invert to find $J[\rho]$ and then Legendre transform from $J$ to $\rho$:

$$\Gamma[\rho] = -W[J] + \int dx J(x) \rho(x) , \tag{11}$$

with

$$J(x) = \frac{\delta \Gamma[\rho]}{\delta \rho(x)} \to \frac{\delta \Gamma[\rho]}{\delta \rho(x)} \bigg|_{\rho_{gs}(x)} = 0 . \tag{12}$$

For static $\rho(x)$, $\Gamma[\rho]$ is proportional to the conventional Hohenberg-Kohn energy functional, which by eq. $12$ is extremized at the ground state density $\rho_{gs}(x)$ (note that thermodynamic arguments establish that it is a minimum $[89]$).

Consider the partition function in the zero-temperature limit of a Hamiltonian with time-independent source $J(x)$ $[90]$:

$$\hat{H}(J) = \hat{H} + \int J \psi^\dagger \psi . \tag{13}$$

If the ground state is isolated (and bounded from below),

$$e^{-\beta \hat{H}} = e^{-\beta E_0} \left( |0\rangle \langle 0| + O(e^{-\beta(E_1-E_0)}) \right) . \tag{14}$$

As $\beta \to \infty$, $Z[J]$ yields the ground state of $\hat{H}(J)$ with energy $E_0(J)$:

$$E_0(J) = \lim_{\beta \to \infty} -\frac{1}{\beta} \log Z[J] = \frac{1}{\beta} W[J] . \tag{15}$$

Substitute and separate out the pieces:

$$E_0(J) = \langle \hat{H}(J) \rangle_J = \langle \hat{H} \rangle_J + \int J \langle \psi^\dagger \psi \rangle_J \tag{16}$$

$$= \langle \hat{H} \rangle_J + \int J \rho(J) . \tag{17}$$

Rearranging, the expectation value of $\hat{H}$ in the ground state generated by $J[\rho]$: $[4]$:

$$\langle \hat{H} \rangle_J = E_0(J) - \int J \rho = \frac{1}{\beta} \Gamma[\rho] . \tag{18}$$

Now put it all together:

$$\frac{1}{\beta} \Gamma[\rho] = \langle \hat{H} \rangle_J \to E_0 \tag{19}$$

$$J(x) = -\frac{\delta \Gamma[\rho]}{\delta \rho(x)} \to \frac{\delta \Gamma[\rho]}{\delta \rho(x)} \bigg|_{\rho_{gs}(x)} = 0 . \tag{20}$$

So for static $\rho(x)$, $\Gamma[\rho]$ is indeed proportional to the DFT energy density functional. Furthermore, the true ground state (with $J = 0$) is a variational minimum $[4]$ so additional sources should be better than just one source coupled to the density (these sources will couple to additional densities such as the kinetic energy density in a Skyrme EDF). The universal dependence on a non-zero external potential $v$ follows directly in this formalism:

$$\Gamma_v[\rho] = W_v[J] - \int J \rho \tag{21}$$

$$= W_{v=0}[J + v] - \int [(J + v) - v] \rho \tag{22}$$

$$= \Gamma_{v=0}[\rho] + \int v \rho . \tag{23}$$

Thus allowing for non-zero $v_{ext}$ is a trivial modification to $\Gamma[\rho]$. For the nuclear application, however, it is the lack of an external potential that complicates the formulation.

To summarize, conventional microscopic DFT follows naturally from calculating the response of a many-body system to external sources, as in Green’s function methods, only with local, static sources that couple to densities rather than fundamental fields. (Time-dependent sources can be used for certain excited states.) We can consider the zero temperature limit of the partition function $Z$ for the (finite) system of interest in the presence of external sources coupled to various quantities of interest (such as the fermion density). We derive energy functionals of these quantities by Legendre transformations with respect to the sources $[91]$. These sources probe, in a variational sense, configurations near the ground state.

The work by Lieb $[92]$ on the Hohenberg-Kohn theorem $[23]$ establishes that the critical issue for DFT is the existence of the Legendre transform $F[\rho]$ of the ground state energy as a functional $E[v]$ of the potential. The details involve sophisticated mathematics ($e.g.$, convex-functional analysis) that is not readily accessible; I recommend ref. $[91]$ by Kutzelnigg as a gateway to the mathematically rigorous literature behind DFT in terms of Legendre transformations $[7]$.

However, this attractive picture of DFT in field theoretic terms is not sufficient for the nuclear case, because nuclei are self-bound systems. The problems are clear even in our schematic treatment because we need the source to couple only to internal dofs. For now we will assume this will all work out and return to discuss a solution in sect. $3.3$.5.

### 3.3 Motivation for effective action EFT for EDFs

Here I enumerate the advantages of the effective actions in path integral form for implementing an EFT for nuclear EDFs $[25]$. Note that because there will be an underlying

$4$ A Minkowski-space formulation of the effective action with time-dependent sources leads naturally to an RPA-like generalization of DFT that can be used to calculate properties of collective excitations.

$5$ The functionals will change with resolution or field redefinitions; only stationary points are observables. This can be seen from eq. $13$, where $\Gamma[\rho]$ is not the expectation value of $\hat{H}$ in an eigenstate unless $J = J[\rho_{gs}]$.

$6$ For the Minkowski-space version of this discussion, see ref. $[90]$.

$7$ There are important formal details $[94]$, such as that we need $E[v]$ to be concave to carry out the transform.
EFT Hamiltonian, these EDFs will be generalizations of the functionals of DFT.

- Effective actions are the natural theoretical framework for Legendre transforms [95,90,89], which is the underlying basis for DFT. Note that these aspects tend to be hidden when using wave function methods.
- The path integral construction of DFT is transparent, such as the role and usefulness of additional densities/sources.
- Path integral effective actions are particularly suited for symmetry breaking, such as encountered with pairing. The renormalization issues in pairing are manifest rather than hidden.
- Connections to the RG, and therefore to EFT and power counting, can be more accessible.
- The path integral formulation puts the DFT construction in a broader perspective, which can suggest connections and generalizations not apparent otherwise. For example, there are alternative effective actions using auxiliary fields or with a two-particle-irreducible nature. The former will be considered in sect. 3.4. The latter may be related to more general EDF constructions proposed in ref. [90].
- The quantization of gauge theories was greatly facilitated by Faddeev-Popov and BRST methods using path integrals; analogous techniques offer alternative possibilities for implementing collective coordinates to robustly address the issue of symmetries broken at the mean-field level [97] (see sect. 3.5).
- The path integral formulation can suggest different types of nonperturbative approximations, such as $1/N$ expansions, that can organize for extensions beyond the mean-field level.

In summary, effective actions are a natural framework to implement Legendre transformations, motivate approximations not obvious in MBPT, and allow us to consider generalizations of conventional DFT. One limitation of DFT is the exclusive role of local potentials (sources) and densities, where locality is in reference to coordinate space. Kutzelnigg points out that this is in contrast to many-body methods that introduce a finite basis in which operators are expanded, for which local operators have no privileged place. In this sense, density matrix functional theory, as proposed for nuclei in ref. [90], seems more natural [91]. By looking at effective actions as a broader context, the limitations and problems of local sources are apparent, but also the opportunities for generalizations.

3.4 Auxiliary fields

I propose a bottom-up construction, building on the insights from the phase-space approach, the successful phenomenology of nuclear EDFs, and the DFT framework. In particular, I would like to build in that the pairing and particle-hole channels seem to have different associated scales and that collective modes are dominant low-energy dofs beyond the mean fields. To accomplish both, we can look to textbook treatment for analogous condensed matter systems, which introduce auxiliary bosonic fields for fermion bilinears via Hubbard-Stratonovich (HS) transformations [78,91,90,48].

To my knowledge, this path for nuclei has not yet been explored. In previous work with my collaborators, we studied effective actions of composite operators, because that closely paralleled the form of Skyrme functions. This required use of the so-called “inversion method” to carry out the Legendre transformation. It seemed well adapted to EFT because the inversion was carried out order-by-order in an expansion. Pairing was included by coupling sources not only to the proton and neutron densities but to the anomalous density [61,100]. This meant, however, that the particle-hole and pairing channels were treated on the same footing. It also meant that collective excitations would not be naturally included in a loop expansion but would have to be built up from coherent nucleon pairs at different orders.

In the auxiliary field (HS) method [95,101], one couples operators such as $\psi^\dagger \psi$ to an auxiliary field $\phi$, and eliminate all or part of $(\psi^\dagger \psi)^2$. The Legendre transform follows from adding a source term $J\phi$ and performing a loop expansion about the expectation value $\langle \phi \rangle$. The inversion is direct [95], in contrast to the case of composite operators where a perturbative inversion method is needed. We integrate out the fermion fields, whence a saddlepoint expansion gives a mean-field approximation in leading order (with freedom to choose how this is organized, e.g., Hartree, HF, HFB [77]) and then the next order has collective contributions (corresponding to the RPA). In practice for nuclei we evaluate the fermion determinant using solutions to Schrödinger equations for single-particle quasi-nucleon wave functions in the mean fields; this generates the Kohn-Sham system. We can use the freedom of the expansion to require the density be unchanged at each order.

In an EFT approach that restricts momenta, one would directly introduce a complete (but not redundant) set of bosonic operators coupled to quadratic fermion bilinears. The “trick” in this case is to introduce fields for all of the channels, not just one (they are indicated schematically for two-body interactions in fig. 2). If one summed contributions from all momenta to all orders, this would lead to double counting. But for small momenta near the Fermi surface, the different channels are effectively independent (these are different small momenta in the different channels), so one can and should include both [95,90]. This would then be consistent with nuclear EDF phenomenology.

The first steps for testing whether this formulation is viable is to apply it to a uniform system (an effective action treatment with one auxiliary field was explored in [95]) and then to extend to a system in a trap. It is possible that the subtle renormalization issues arising with pairing when coupling to composite densities [61] will be easier to deal with in terms of the auxiliary fields. We also note that the proposed approach is close to phenomenological covariant DFT phenomenology with meson fields.
Ordinary nuclei are self-bound, which presents conceptual issues about whether Kohn-Sham DFT is well defined and practical problems on how to deal with the consequences of symmetry breaking by the Kohn-Sham potentials, which will not have all of the symmetries of the Hamiltonian [103]. These broken symmetries include the $U(1)$ phase symmetry for fermion number and translational and rotational invariance.

In conventional discussions of EDFs the issue of symmetry breaking plays an important role; indeed a distinction between DFT and the EDF method is made on this basis [26]. The dilemma with properly treating symmetries in the nuclear many-body problem is that one wants simple wave functions (e.g., Slater determinants) but this misses correlations from symmetries (e.g., plane waves won’t describe clustering into nuclei) [104]. In developing nuclear DFT or some variation of it within a field theory (effective action) formalism and as an EFT, the symmetry breaking manifests as zero modes. In particular these arise when one does a saddlepoint or stationary phase expansion of the path integral for the nuclear ground state, which in turn leads one to pick out a mean-field reference state. The quantum corrections will naively be found to be infinite because there are fluctuations possible in flat (symmetry-wise) directions. Mathematically, one must calculate a determinant when evaluating quadratic fluctuations and zero eigenvalues (hence “zero modes”) will cause divergences.

The textbook discussions of how to restore mean-field broken symmetries tend to follow one of these two related lines of discussion:

- States connected by a unitary transformation $U(\alpha)$ corresponding to a broken symmetry are degenerate:
  $$|\phi \alpha\rangle = U(\alpha)|\phi\rangle$$

with $|\phi\rangle$ a “deformed” state, implies
  $$\langle \phi \alpha | H_N | \phi \alpha \rangle = \langle \phi | H_N | \phi \rangle .$$

The degeneracy can be removed by diagonalizing in the subspace spanned by the degenerate states. The group parameter $\alpha$ for continuous groups can be considered a collective coordinate, which specifies the orientation in gauge space of the deformed state $|\phi\rangle$. A general strategy is to transform from 3A particle coordinates into collective and internal coordinates [104].

- In finite systems, broken symmetries arise only as a result of approximations. This usually happens with variational calculations over trial wavefunctions that are too restricted; a mean-field approximation is an example. The symmetry can be restored by using a linear superposition of degenerate states:
  $$|\psi\rangle = \int d\alpha f(\alpha)|\phi \alpha\rangle ,$$

when minimized with respect to the $f(\alpha)$’s projects states of good symmetry [103]. (Because minimizing with respect to $|\phi\rangle$ and with respect to $f(\alpha)$ do not commute, there are two types of projection. It is most accurate to project first and then find the best deformed state corresponding to a given quantum number.) For example, particle number projection for EDF’s is described in refs. [105, 106].

When implemented, these approaches are considered to be beyond EDF, where there are only densities and not a wavefunction. From a different perspective, the restoration of broken symmetries of GCM-type configuration mixings can be considered as a “multi-reference” extension of the usual “single-reference” EDF implementation (see refs. [107, 108, 109]).

For nuclear DFT, the conceptual question was highlighted by Engel [110], who pointed out that the ground state of a self-bound system, with a plane wave describing the center of mass, has a density distributed uniformly over space. Clearly this is not the density one wants to find from DFT, so there is a question of how to proceed. There are two separate considerations: i) Does Kohn-Sham DFT exist in a useful form for self-bound systems? ii) If so, how does one formulate and implement it? Engel and other authors have addressed this issue some time ago [110, 111, 112, 113, 114, 115, 116, 26, 17], with a consensus that HK existence proofs for DFT are still well founded, but for internal densities (e.g., meaning independent of the center-of-mass motion when considering broken translational symmetry).

Wave function methods have several related strategies for dealing with the “center of mass” (COM) problem:

1. Isolate the “internal” degrees of freedom, typically by introducing Jacobi coordinates. Then the observables are by construction independent of the COM. This gets

In other contexts, such densities are called “intrinsic”, but this has a different meaning in the context of symmetry breaking, so “internal” is typically used instead.
increasingly cumbersome with greater numbers of particles.

2. Work in a harmonic-oscillator Slater determinant basis, for which the COM decouples, and introduce a potential for the COM that allows its contribution to be subtracted.

3. Work with the internal Hamiltonian \( (i.e., \) subtract the COM kinetic energy \( T_{CM} \) \) so that the COM part factorizes and does not contribute to observables to good approximation (see in particular ref. \[118\] for coupled cluster calculations).

Versions of the first two possibilities are in fact among the ideas considered for DFT in refs. \[110,111,112,113,114,115\].

For the effective action approach, the issue of broken symmetry was first addressed long ago in the study of solitons \[119,85\], where it also arises as the problem of dealing with zero modes when calculating quantum fluctuations. Methods found in the literature include those similar to the textbook treatments of mean-field broken symmetry cited above. A subsequent field-theoretic functional approach used a Fadeev-Popov construction to introduce collective coordinates with ghost degrees of freedom (recall that these ghost fields are spin-zero but anticommuting) \[119\]. This works, but is cumbersome, particularly if the symmetry is non-Abelian. For quantizing gauge theories, a more effective approach uses BRST symmetry.

BRST can be applied to the collective coordinate methods by building on the equivalence of a theory with constraints and a gauge theory. It is then just another way to deal with the original constrained system, by imposing an eigenvalue condition on the states of the theory to eliminate spurious dofs \[120\]. There exists a substantial literature, primarily from Bes and collaborators (see ref. \[97,121\] and references therein), on how to apply the BRST methods to translational, rotational, and pairing dofs. They have demonstrated both variational and perturbative approximations, mostly using algebraic methods.

The procedure starts with adding fields for the collective degrees of freedom and Lagrange multipliers, so there is an overcomplete Hilbert space. But rather than project out physical dofs, one doubles down and adds still more fields: ghost fields and their conjugate momenta. This might seem to complicate things, but in fact it simplifies them because there is a supersymmetry between field variables and ghosts, called BRST symmetry \[95\]. As in other contexts, the ghost fields serve to cancel spurious contributions, as enforced by the BRST symmetry. Associated with the symmetry is a charge \( \hat{Q} \).

For a gauge theory, one can ensure physical results by working with gauge-invariant operators in the Lagrangian or Hamiltonian. With the BRST, the focus is instead on BRST-invariant operators, \( i.e., \) to exploit BRST symmetry rather than the original gauge invariance. The trick of the BRST technique is to replace the notion of a gauge transformation that shifts operators by c-number functions with a BRST transformation that mixes operators with different statistics. Gauge fixing is accomplished by adding a BRST-invariant function to the Hamiltonian that is not invariant under an ordinary gauge transformation and then work with \( \hat{H}_{BRST} \). Physical states have \( Q = 0 \) and matrix elements of \( \hat{H}_{BRST} \) in these states will give the physical internal energy.

What this means for the broken-symmetry application is that \( \hat{H}_{BRST} \) commutes with the generator of collective transformations, so it displays unbroken collective symmetry, and states are labeled by the corresponding quantum numbers. But \( \hat{H}_{BRST} \) does not commute with the symmetry generator because of the gauge-fixing function (they differ by a so-called null function), so there are no zero modes, no infrared divergences. In this way there is a unique ground state and projection is automatically achieved. The BRST approach has commonality with the features sought for nuclear DFT in refs. \[110,112,115\], such as maintaining the full set of orbitals.

Furthermore, it is ok if due to approximations the state is not exactly a \( Q = 0 \) state, because approximately broken supersymmetry does not lead to zero modes. So in practice states will be strictly states of the collective generator but approximate eigenstates of \( \hat{Q} \). This doesn’t disturb the favorable features and a better approximation means better projection. Of course, it remains to be seen if suitable approximations are going to be feasible.

We can now imagine how eq. \[13\] would be modified in the BRST framework. By replacing \( \hat{H} \) by \( \hat{H}_{BRST} \) and then coupling external sources to a BRST-invariant field combination to probe near the ground state, one stays within the sector of Hilbert space with internal (physical) dofs. Thus the conditions to carry out the Legendre transformation and have a variational state can be fulfilled. The details have not been worked out before for this case and there are both subtleties and unexplored freedom, so there is still much to be done.

4 Outlook

This is an exciting time in low-energy nuclear theory, with steadily improving calculations of structure and reactions. There are multiple opportunities for improving the performance energy of the density functional method; I believe the best overall strategy at this stage is to pursue all promising avenues. People are focusing on the relevant questions and developing impressive technology that opens doors to new approaches.

The progress toward formulating a proper effective field theory to supplant nuclear EDFs has been less obvious. I have proposed a path to follow toward an effective action formulation, but now is just the beginning of the journey. As indicated, the most critical step in this program is implementing the BRST approach. There have been several applications to simple quantum mechanical models to illustrate how it works, although in an operator formalism. There are also illuminating toy models for path-integral effective actions (even as zero-dimensional field theories). The plan is to start by merging the path-integral toys with the toy models in refs. \[122,120\] for proofs-of-principle and
then graduate to simple models like those considered in refs. [10, 12] before finally considering realistic nuclei.

I acknowledge many illuminating discussions over many years with my colleagues on effective field theory and energy density functionals that have contributed to my reflections here. However, all misunderstandings, misstatements, and misinterpretations are my own. Supported in part by the US National Science Foundation under Grant No. PHY–1614460 and the NUCLEI SciDAC Collaboration under US Department of Energy MSU subcontract RC107839-OSU.

References

1. H. Georgi, Ann. Rev. Nucl. Part. Sci. 43, 209 (1993)
2. C.P. Burgess, Ann. Rev. Nucl. Part. Sci. 57, 329 (2007), hep-th/0701053
3. R.J. Furnstahl, G. Rupak, T. Schfer, Ann. Rev. Nucl. Part. Sci. 58, 1 (2008), 0801.0729
4. M. Bender, P.H. Heenen, P.G. Reinhard, Rev. Mod. Phys. 75, 121 (2003)
5. N. Schunck, ed., Energy Density Functional Methods for Atomic Nuclei, 2053-2563 (IOP Publishing, 2019)
6. A. Manohar, H. Georgi, Nucl. Phys. B 234, 189 (1984)
7. E. Epelbaum, H.W. Hammer, U.G. Meissner, Rev. Mod. Phys. 81, 1773 (2009), 0811.1338
8. H.W. Hammer, C. Ji, D.R. Phillips, J. Phys. 44, 103002 (2017), 1702.08605
9. A. Migdal, Theory of Finite Fermi Systems: And Applications to Atomic Nuclei, Interscience monographs and texts in physics and astronomy, v. 19 (Interscience Publishers, 1967)
10. J. Polchinski (1992), hep-th/9210046
11. R. Shankar, Rev. Mod. Phys. 66, 129 (1994)
12. T. Papenbrock, H. Weidemann, Phys. Rev. C 80, 014334 (2014), 1307.1181
13. A. Bulgac, M.M. Forbes, S. Jin, R. Navarro Perez, N. Schunck, Phys. Rev. C 97, 044313 (2018), 1708.08771
14. M. Harakeh, A. Woude, Giant Resonances: Fundamental High-frequency Modes of Nuclear Excitation, Oxford science publications (Oxford University Press, 2001)
15. J.L. Friar, D.G. Mada11, B.W. Lynn, Phys. Rev. C 53, 3085 (1996), nucl-th/9512011
16. J.J. Rusnak, R.J. Furnstahl, Nucl. Phys. A 627, 945 (1997), nucl-th/9708040
17. M. Kortelainen, R.J. Furnstahl, W. Nazarewicz, M.V. Stoitsov, Phys. Rev. C 82, 011304 (2010), 1005.2552
18. R.J. Furnstahl, A. Schwenk, J. Phys. G 37, 064004 (2010), 1001.0327
19. D. Vautherin, D.M. Brink, Phys. Rev. C 5, 626 (1972)
20. R.J. Furnstahl, B.D. Serot, Nucl. Phys. A 671, 447 (2000), nucl-th/9911019
21. G.F. Bertsch, B. Sabbe, M. Ussnakki, Phys. Rev. C 71, 054311 (2005), nucl-th/0412091
22. E. Epelbaum, U.G. Meissner, W. Glöckle, C. Elster, Phys. Rev. C 65, 044001 (2002), nucl-th/0106007
23. B.N. Lu, N. Li, S. Elhayisari, D. Lee, E. Epelbaum, U.G. Meiner (2018), 1812.10928
24. C. Drischler, K. Hebeler, A. Schwenk, Phys. Rev. Lett. 122, 042501 (2019), 1710.08220
25. J.A. Melendez, R.J. Furnstahl, D.R. Phillips, M.T. Pratola, S. Wesolowski (2019), 1904.10581
26. T. Duguet, J. Sadoudi, J. Phys. G 37, 064009 (2010), 1001.0673
27. A. Bhattacharyya, R.J. Furnstahl, Phys. Lett. B 607, 259 (2005), nucl-th/0410105
28. J.E. Drut, R.J. Furnstahl, L. Platter, Prog. Part. Nucl. Phys. 64, 120 (2010), 0906.1463
29. Z. Davoudi (NPLQCD), Light Nuclei from Lattice QCD: Spectrum, Structure and Reactions, in 22nd International Conference on Few-Body Problems in Physics (FB22) Caen, France, July 9-13, 2018 (2019), 1902.04959
30. T. Iritani, S. Aoki, T. Doi, T. Hatsuda, Y. Ikeda, T. Inoue, N. Ishii, H. Nemura, K. Sasaki, Phys. Rev. D 96, 034521 (2017), 1703.07210
31. S.R. Beane et al. (2017), 1705.09239
32. T. Iritani, S. Aoki, T. Doi, T. Hatsuda, Y. Ikeda, T. Inoue, N. Ishii, H. Nemura, K. Sasaki (HAL QCD), JHEP 03, 007 (2019), 1812.08539
33. H. Hergert, J. Yao, T.D. Morris, N.M. Parzuchowski, S.K. Bogner, J. Engel, J. Phys. Conf. Ser. 1041, 012007 (2018), 1805.09221
34. R. Machleidt, D. Entem, Phys. Rept. 503, 1 (2011), 1105.2919
35. M. Hjorth-Jensen, M.P. Lombardo, U. van Kolck, Lect. Notes Phys. 936, 1 (2017)
36. P. Maris, M.A. Caprio, J.P. Vary, Phys. Rev. C 91, 014310 (2015), [Erratum: Phys. Rev. C 99, 029902 (2019)], 1409.0881
37. T. Duguet, J. Phys. G 42, 025107 (2015), 1406.7183
38. T. Duguet, A. Signoracci, J. Phys. G44, 015103 (2017), [Erratum: J. Phys.G44,no.4,049601(2017)], 1512.02878
39. T. Duguet, M. Bender, J.P. Ebran, T. Lesinski, V. Som, Eur. Phys. J. A51, 162 (2015), 1502.03672
40. J. Ripoche, D. Lacroix, D. Gambaccatura, J.P. Ebran, T. Duguet, Phys. Rev. C95, 014326 (2017), 1610.04063
41. A. Gezerlis, I. Tews, E. Epelbaum, S. Gandolfi, K. Hebeler et al., Phys. Rev. Lett. 111, 032503 (2013), 1303.6243
42. M. Piarulli, L. Girlanda, R. Schiavilla, R. Navarro Prez, J.E. Amaro, E. Ruiz Arriola, Phys. Rev. C 93, 054304 (2016), 1512.0142
43. E. Epelbaum, H. Krebs, U.G. Meiner, Eur. Phys. J. A51, 53 (2015), 1412.0142
44. P. Reinert, H. Krebs, E. Epelbaum, Eur. Phys. J. A 54, 86 (2018), 1711.08821
45. A. Dyhdalo, R.J. Furnstahl, K. Hebeler, I. Tews, Phys. Rev. C 94, 034001 (2016), 1602.08038
46. K. Hebeler, S. Bogner, R. Furnstahl, A. Nogga, A. Schwenk, Phys. Rev. C 83, 031301 (2011), 1012.3831
47. A. Dyhdalo, S.K. Bogner, R.J. Furnstahl, Phys. Rev. C 96, 054005 (2017), 1707.07199
48. N. Kaiser, W. Weise, Nucl. Phys. A 836, 256 (2010), 0912.3207
49. J.W. Holt, N. Kaiser, W. Weise, Prog. Part. Nucl. Phys. 73, 35 (2013), 1304.6350
50. R. Navarro Prez, N. Schunck, A. Dyhdalo, R.J. Furnstahl, S.K. Bogner, Phys. Rev. C 97, 054304 (2018), 1803.08615
51. A. Dyhdalo, S.K. Bogner, R.J. Furnstahl, Phys. Rev. C 95, 054314 (2017), 1611.03849
52. S. Goriely, S. Hilaire, M. Girod, S. Pru, Eur. Phys. J. A 52, 202 (2016)
118. G. Hagen, T. Papenbrock, D.J. Dean, Phys. Rev. Lett. 103, 062503 (2009), 0905.3167
119. R. Rajaraman, Solitons and Instantons (North Holland, New York, 1982)
120. D. Nemeschansky, C.R. Preitschopf, M. Weinstein, Annals Phys. 183, 226 (1988)
121. D.R. Bes, Phys. Scripta 91, 063010 (2016)
122. D. R. Bes, O. Civitarese, American Journal of Physics - AMER J PHYS 70 (2002)
123. M.G. Bertolli, T. Papenbrock, Phys. Rev. C 78, 064310 (2008), 0805.2856