On rumour propagation among sceptics

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Abstract: Junior, Machado and Zuluaga (2011) introduced a model to understand the spread of a rumour. This model consisted of individuals situated at i.i.d. points of the line \( \mathbb{N} \). An individual at the origin 0 starts a rumour and passes it to all individuals in the interval \([0, \rho_0]\), where \( \rho_0 \) is a positive random variable. An individual located in this interval receives the rumour and transmits it further. The rumour process is said to survive if all the individuals in the model receive the rumour. We study this model, when the individuals are more sceptical and they transmit only if they receive the rumour from at least two different sources.

In stochastic geometry the equivalent of this rumour process is the study of coverage of the space \( \mathbb{N}^d \) by random sets. Our study here extends the study of coverage of space and considers the case when each vertex of \( \mathbb{N}^d \) is covered by at least two distinct random sets.

1 Introduction

Let \( \{X_i : 1 \geq 1\} \) be a collection of i.i.d. Bernoulli \((p)\) random variables, i.e.,

\[
X_i = \begin{cases} 
1 & \text{with probability } p \\
0 & \text{with probability } 1 - p.
\end{cases}
\]

(1)

Also let \( \{\rho_i : 1 \geq 1\} \) be a collection of i.i.d. \( \mathbb{N} \) valued random variables, independent of the collection \( \{X_i : 1 \geq 1\} \). Let \( \rho \) denote a generic random variable with the same distribution as \( \rho_i \).

In addition, let \( \rho_0 \) an independent \( \mathbb{N} \) valued random variables, independent of the collections \( \{X_i : 1 \geq 1\} \) and \( \{\rho_i : 1 \geq 1\} \), with \( \rho_0 \) having the same distribution as \( \rho \). Taking \( X_0 \equiv 1 \), consider the region

\[
C := \bigcup_{\{i \geq 0 : X_i = 1\}} ([i, i + \rho_i]).
\]

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Junior, Machado and Zuluaga (2011) takes \( \{ i : X_i = 1 \} \) as the location of individuals on \( \mathbb{N} \) and a rumour is started at the origin which is received by individuals located in \([0, \rho_0]\). An individual at \( i \) on receiving the rumour spreads it among all individuals in the region \([i, i + \rho_i]\), and thus the rumour propagates. This model they call the \textit{firework process}. They also introduce another model of rumour propagation, called the \textit{reverse firework process}, where they study the region

\[ \tilde{C} := \bigcup_{\{i \geq 0 : X_i = 1\}} ([i - \rho_i, i]). \]

Their object of interest is under what conditions on the processes \( \{ X_i : 1 \geq 1 \} \) and \( \{ \rho_i : 1 \geq 1 \} \) are the regions \( C \) and \( \tilde{C} \) unbounded connected regions with positive probability, i.e. the rumour process ‘percolates’.

\textbf{Remark 1.1.} A simple argument using Kolmogorov’s 0-1 law yields that \( C \) (or \( \tilde{C} \)) being an unbounded connected region with positive probability is equivalent to \( C \) (or \( \tilde{C} \)) containing a region \([t, \infty)\), for some \( t \geq 1 \) with probability 1.

We study the spread of rumour among sceptics. A sceptic individual at \( i \) spreads it among all individuals in the region \([i, i + \rho_i]\) only if s/he received the rumour from at least two distinct sources, i.e.,

(i) for the firework process, there are two individuals at \( j \) and \( k \) (say) with \( j \neq k \) and \( -1 \leq j, k < i \) such that \( i \in [j, j + \rho_j] \cap [k, k + \rho_k] \); here we assume that there is an individual at location \(-1\), who spreads the rumour in the region \([-1, -1 + \rho_{-1}]\), where \( \rho_{-1} \) is a random variable independent of all other random variables and having the same distribution as \( \rho_i \).

(ii) for the reverse firework process, there are two individuals at \( j \) and \( k \) (say) with \( j \neq k \) and \( -1 \leq j, k < i \) such that \( j, k \in [i - \rho_i, i] \).

Towards this we define the regions

\[ D := \{ x \in \mathbb{R} : \text{there exist } j, k \geq -1 \text{ with } j \neq k \text{ and } X_j = X_k = 1 \text{ such that } x \in ([j, j + \rho_j] \cap [k, k + \rho_k]) \}, \]

\[ \tilde{D} := \{ x \in \mathbb{R} : x \in [i - \rho_i, i] \text{ for some } i \text{ with } X_i = 1 \text{ and there exist } -i \leq j, k < i \text{ with } j \neq k \text{ and } X_j = X_k = 1 \text{ such that } j, k \in [i - \rho_i, i] \}. \]

We look for conditions on on the processes \( \{ X_i : 1 \geq 1 \} \) and \( \{ \rho_i : 1 \geq 1 \} \) such that the regions \( D \) and \( \tilde{D} \) are unbounded connected regions with positive probability.

\textbf{Remark 1.2.} As in the case of rumour propagation in \( C \) and \( \tilde{C} \), a simple argument using Kolmogorov’s 0-1 law yields that \( D \) (or \( \tilde{D} \)) being an unbounded connected region with positive probability is equivalent to \( D \) (or \( \tilde{D} \)) containing a region \([t, \infty)\), for some \( t \geq 1 \) with probability 1.

Thus we define
Definition 1.3. The firework process (respectively, reverse firework process,) percolates among sceptics if, with probability 1, $D$ (respectively, $\tilde{D}$,) contains a region $[t, \infty)$, for some $t \geq 1$.

Remark 1.4. Using this equivalent definition obtained by the tail event properties allows us to study the model without considering the influence of the two initiators $X_{-1}$ and $X_0$.

Proposition 1.5. Let

$$l := \liminf_{j \to \infty} j \mathbb{P}(\rho \geq j) > 1 \quad \text{and} \quad L := \limsup_{j \to \infty} j \mathbb{P}(\rho \geq j) < \infty.$$  

We have that both the firework process and the reverse firework process percolate if $p > 1/l$ and neither of them percolates if $p < 1/L$.

We note here that this is the same condition that Junior, Machado and Zuluaga (2011) obtain for the percolation among ‘non-sceptical’ individuals. Indeed, the above proposition goes through among more radical sceptics too, i.e. if individuals need to receive the rumour from $k \geq 1$ distinct sources before they transmit the rumour.

As noted in the review article Junior, Machado and Ravishankar (2016), the above model is related to the study of coverage processes in stochastic geometry. Athreya, Roy and Sarkar (2004) introduce a notion of ‘eventual coverage’ which, for 1-dimension, is identical to the equivalent formulation of the percolation of rumour process as given in Remark 1.1. We state the model in brief here and present the results obtained.

Let $\{X_i : i \in \mathbb{N}^d\}$ be a collection of i.i.d. Bernoulli ($p$) random variables and $\{\rho_i : i \in \mathbb{N}^d\}$ a collection of i.i.d. $\mathbb{N}$ valued random variables, independent of the collection $\{X_i : i \in \mathbb{N}^d\}$. Let $\rho$ denote a generic random variable with the same distribution as $\rho_i$.

Let

$$C := \bigcup_{i=1}^{n} (i + [0, \rho_i])^d$$

denote the covered region of $\mathbb{N}^d$.

Athreya, Roy and Sarkar (2004) define a notion of eventual coverage as follows: $\mathbb{N}^d$ is eventually covered if there exists a $t \in \mathbb{N}^d$ such that $t + \mathbb{N}^d \subseteq C$. For our purposes we say $\mathbb{N}^d$ is eventually doubly covered if there exists a $t \in \mathbb{N}^d$ such that $t + \mathbb{N}^d \subseteq D$, where

$$D := \{x \in \mathbb{R}^d : \text{there exist } i, j \in \mathbb{N}^d \text{ with } i \neq j \text{ and } X_i = X_j = 1 \text{ such that } x \in (i + [0, \rho_i])^d \cap (j + [0, \rho_j])^d\}.$$  

We have

Proposition 1.6. (i) For $d = 1$, let

$$l := \liminf_{j \to \infty} j \mathbb{P}(\rho \geq j) > 1 \quad \text{and} \quad L := \limsup_{j \to \infty} j \mathbb{P}(\rho \geq j) < \infty.$$  

We have

$$\mathbb{P}_p(\mathbb{N} \text{ is eventually doubly covered}) = \begin{cases} 1 & \text{if } p > 1/l \\ 0 & \text{if } p < 1/L. \end{cases}$$
(ii) For $d \geq 2$, we have

$$\mathbb{P}_p(\mathbb{N}^d \text{ is eventually doubly covered}) = \begin{cases} 1 & \text{if } \lim \inf_{j \to \infty} j \mathbb{P}(\rho \geq j) > 0 \\ 0 & \text{if } \lim_{j \to \infty} j \mathbb{P}(\rho \geq j) = 0. \end{cases}$$

In stochastic geometry the notion of coverage of space has received widespread attention. In particular Hall (1988) and Chiu, et al (2013) provide a review of the topics studied. Our endeavour in this paper may be viewed as an effort to introduce a notion of ‘reinforced coverage’.

2 Proof of Proposition 1.5

Let $S$ be the event that "the firework process percolates" and $\tilde{S}$ be the event that "the reverse firework process percolates". Observe that,

$$S = \bigcap_{n=1}^{\infty} \bigcup_{i=1}^{\infty} \bigcup_{j=1, j \neq i}^{\infty} ([\rho_{n-i} \geq i + 1] \cap [\rho_{n-j} \geq j + 1]),$$

and

$$\tilde{S} = \bigcap_{n=1}^{\infty} \bigcup_{i=1}^{\infty} \bigcup_{j=1, j \neq i}^{\infty} ([\rho_{n+i} \geq i] \cap [\rho_{n+j} \geq j]).$$

We will show that if $p > 1/l$, then we have $\mathbb{P}(S) = 1$, $\mathbb{P}(\tilde{S}) = 1$ and if $p < 1/L$, then $\mathbb{P}(S) = 0$, $\mathbb{P}(\tilde{S}) = 0$. We prove the proposition for the firework process, a similar proof works for the reverse firework process. Using De Morgan’s law, we have

$$S^c = \bigcup_{n=1}^{\infty} \bigcap_{i=1}^{\infty} \bigcap_{j=1, j \neq i}^{\infty} ([\rho_{n-i} \leq i] \cup [\rho_{n-j} \leq j]).$$
Since \( p > 1/l \), there exists \( \eta > 1 \) such that \( l > \eta/p \). Also let \( i_0 \) be such that \( i \mathbb{P}(\rho \geq i) > \frac{\eta}{p} \), for all \( i \geq i_0 \). Therefore, for all \( i, j \geq i_0 \) and each fixed \( n \),

\[
\mathbb{P} \left( \bigcap_{i=i_0}^{\infty} \bigcap_{j=i+1}^{\infty} \left( \rho_{n-i} \leq i \right) \cup \left( \rho_{n-j} \leq j \right) \right)
\]

\[
= \mathbb{P} \left( \left\{ \bigcap_{i=i_0}^{\infty} \bigcap_{j=i+1}^{\infty} \left( \rho_{n-j} \leq j \right) \right\} \cup \left\{ \bigcap_{j=i_0}^{\infty} \bigcap_{i=j+1}^{\infty} \left( \rho_{n-i} \leq i \right) \right\} \right)
\]

\[
= \mathbb{P} \left( \bigcap_{i=i_0}^{\infty} \bigcap_{j=i+1}^{\infty} \left( \rho_{n-j} \leq j \right) \right) + \mathbb{P} \left( \bigcap_{j=i_0}^{\infty} \bigcap_{i=j+1}^{\infty} \left( \rho_{n-i} \leq i \right) \right)
\]

\[
\leq \mathbb{P} \left( \bigcap_{j=i_0+1}^{\infty} \left( \rho_{n-j} \leq j \right) \right) + \mathbb{P} \left( \bigcap_{i=i_0+1}^{\infty} \left( \rho_{n-i} \leq i \right) \right)
\]

\[
= \prod_{j=i_0+1}^{\infty} \left[ 1 - \mathbb{P}(\rho \geq j + 1) \right] + \prod_{i=i_0+1}^{\infty} \left[ 1 - \mathbb{P}(\rho \geq i + 1) \right]
\]

\[
\leq \prod_{j=i_0+1}^{\infty} \left[ 1 - \frac{\eta}{p(j+1)} \right] + \prod_{i=i_0+1}^{\infty} \left[ 1 - \frac{\eta}{p(i+1)} \right]
\]

\[
\leq \prod_{j=i_0+1}^{\infty} \left[ 1 - \frac{1}{j+1} \right] + \prod_{i=i_0+1}^{\infty} \left[ 1 - \frac{1}{i+1} \right].
\]

But, \( \prod_{j=i_0+1}^{\infty} \left[ 1 - \frac{1}{j+1} \right] = 0 \) as well as \( \prod_{i=i_0+1}^{\infty} \left[ 1 - \frac{1}{i+1} \right] = 0 \). Therefore, \( \mathbb{P}(S^c) = 0 \).

Finally

\[
\mathbb{P}(\text{percolation among sceptic individuals}) \leq \mathbb{P}(\text{percolation among ‘non sceptical’ individuals}) = 0
\]

for \( p < 1/L \), where the equality above follows from Theorem 2.1 of Junior, Machado and Ravishankar (2016).

### 3 Proof of Proposition 1.6

As mentioned earlier (i) of Proposition 1.6 is just Proposition 1.5 rephrased. Thus we need to prove (ii).

First note that from Proposition 3.2(a) of Athreya, Roy and Sarkar (2004), we know that if \( \lim_{j \to \infty} j \mathbb{P}(\rho \geq j) = 0 \) then \( \mathbb{P}_p(\mathbb{N}^d \text{ is eventually covered}) = 0 \), and so

\[
\mathbb{P}_p(\mathbb{N}^d \text{ is eventually doubly covered}) = 0.
\]
We prove Proposition [1.6] (ii) for the case $d = 2$; the proof carries through in a straightforward fashion for higher dimensions. Fix $0 < p < 1$ and we assume that \( \liminf_{j \to \infty} j \mathbb{P}(\rho \geq j) > 0. \)

For \((i, j) \in \mathbb{N}^2\), define

\[
B_{i,j} := \{(i, j) \not\in D\}.
\]

If we show that, for some $N \geq 1$,

\[
\sum_{i,j \geq N} P_p(B_{i,j}) < \infty,
\]

then Borel-Cantelli lemma guarantees that, with probability 1, there exists $N_0 \geq 1$ such that \((i, j) \in D\) for all $i, j \geq N_0$, i.e. we have eventual coverage.

For \(u := (i - l_1, j - l_2) \in \mathbb{N}^2\) with $1 \leq l_1 \leq i$, $1 \leq l_2 \leq j$, we observe that \((i, j) \in u + [0, \rho_u]^2\) if and only if $\rho_u \geq \max\{l_1, l_2\}$. Also note that for $i \geq j$ and $0 \leq t \leq j - 1$ there are exactly $2t + 1$ vertices \((i - l_1, j - l_2)\) with $\max\{l_1, l_2\} = t$ (viz. \(\{(i - l_1, j - l_2) : l_1 = t \text{ and } 0 \leq l_2 \leq t \text{ or } l_2 = t \text{ and } 0 \leq l_1 \leq t\}\)), while for $j \leq t \leq i$ there are exactly $j$ vertices \((i - l_1, j - l_2)\) with $\max\{l_1, l_2\} = t$ (viz. \(\{(i - l_1, j - l_2) : l_1 = t \text{ and } 0 \leq l_2 \leq j - 1\}\))

Thus, for $A_{i,j} := \{(i, j) \not\in C\}$, and, for $i \geq j$,

\[
P_p(B_{i,j}) = \mathbb{P}_p(A_{i,j}) + \prod_{k=1}^{i-j} [1 - pG(k + j - 1)]^{2t} \left\{ \sum_{t=0}^{j-1} pG(t) [1 - pG(t)]^{2t} \prod_{l \neq t, l=0}^{j-1} [1 - pG(l)]^{2t+1} \right\} + \prod_{t=0}^{j-1} [1 - pG(t)]^{2t+1} \times \left\{ \sum_{k=1}^{i-j} pG(k + j - 1) [1 - pG(k + j - 1)]^{2t-1} \prod_{l \neq k, l=1}^{j-1} [1 - pG(l + j - 1)]^{2t-1} \right\}.
\]

(2)

Noting that for $i \geq j$

\[
P_p(A_{i,j}) = \prod_{l=0}^{j-1} [1 - pG(l)]^{2t+1} \prod_{k=1}^{i-j} [1 - pG(k + j - 1)]^{j}
\]

(the last product is taken to be 1 if $i = j$), (2) simplifies to

\[
P_p(B_{i,j}) = \mathbb{P}_p(A_{i,j}) + p \mathbb{P}_p(A_{i,j}) \sum_{t=0}^{i-1} \frac{G(t)}{1 - pG(t)}.
\]

Before we proceed we fix some quantities. Since $\liminf_{j \to \infty} j \mathbb{P}(\rho \geq j) > 0$ we may choose $\eta > 0$ such that $\liminf_{j \to \infty} j \mathbb{P}(\rho \geq j) > \eta$. Also, for this $\eta$ and our fixed $p \in (0, 1)$ let $a$ be
such $0 < e^{-pq} < a < 1$. Now we choose $N \geq 1$ such that for all $j \geq N$ the following hold:

(i) $f \mathbb{P}(\rho \geq j) > \eta$

(ii) $(1 - p\eta j^{-1})^j < a$

(iii) $pj \eta > 1$. \hspace{1cm} (4)

Note that (ii) above guarantees that for all $j \geq N$, we have $(1 - pG(j))^j < a$.

From the proof of Proposition 3.2(b) of Athreya, Roy and Sarkar (2004) we know that, for $j \geq N$,

$$\sum_{i=1}^{\infty} \mathbb{P}_p(A_{i,j}) < \infty.$$ 

Thus if we show that, for $j \geq N$,

$$\sum_{i=1}^{\infty} \mathbb{P}_p(A_{i,j}) \sum_{t=0}^{i-1} \frac{G(t)}{1 - pG(t)} < \infty \hspace{1cm} (5)$$

then we will have

$$\sum_{i=1}^{\infty} \mathbb{P}_p(B_{i,j}) < \infty \text{ whenever } j \geq N. \hspace{1cm} (6)$$

Note that,

$$\sum_{i=1}^{\infty} \mathbb{P}_p(A_{i,j}) \sum_{t=0}^{i-1} \frac{G(t)}{1 - pG(t)} = \sum_{t=0}^{j-1} \frac{G(t)}{1 - pG(t)} + \sum_{k=1}^{i-j} \frac{G(k + j - 1)}{1 - pG(k + j - 1)}.$$ 

Now, for fixed $j \geq N$,

$$\sum_{i=j}^{\infty} \mathbb{P}_p(A_{i,j}) \sum_{k=1}^{i-j} \frac{G(k + j - 1)}{1 - pG(k + j - 1)}$$

$$= \sum_{i=j}^{j-1} \prod_{l=0}^{i-1} [1 - pG(t)]^{2i+1} \prod_{i=1}^{i-j} [1 - pG(l+j-1)]^j \sum_{k=1}^{i-j} \frac{G(k + j - 1)}{1 - pG(k + j - 1)}$$

$$= \prod_{t=0}^{j-1} [1 - pG(t)]^{2i+1} \sum_{k=1}^{i-j} \prod_{i=1}^{i-j} [1 - pG(t)]^j \sum_{k=j}^{i-1} \prod_{h \neq k, h=j}^{i-1} [1 - pG(h)]$$

$$= \prod_{t=0}^{j-1} (1 - pG(t))^{2i+1} \sum_{k=1}^{i-j} \prod_{l=1}^{i-j} [1 - pG(l)]^{j-1} G(k) \prod_{h \neq k, h=j}^{i-1} [1 - pG(h)] \text{ for } j \geq N. \hspace{1cm} (7)$$

Taking

$$e_k := \sum_{i=k+1}^{\infty} \prod_{l=j}^{i-1} [1 - pG(l)]^{j-1} G(k) \prod_{h \neq k, h=j}^{i-1} [1 - pG(h)]$$
as the inner sum in (7), we have
\[ e_k = G(k) \prod_{l=j}^{k} [1 - pG(l)]^{j-1} \prod_{h=j}^{k-1} [1 - pG(h)] \]
\[ \times \left( 1 + \sum_{i=k+1}^{\infty} \prod_{l=k+1}^{i} [1 - pG(l)]^{j-1} \prod_{h=k+1}^{i} [1 - pG(h)] \right) \]
\[ = G(k) \prod_{l=j}^{k} [1 - pG(l)]^{j-1} \prod_{h=j}^{k-1} [1 - pG(h)] \times \left( 1 + [1 - pG(k)]^j \right) \]
\[ \times \left( 1 + \sum_{i=k+1}^{\infty} \prod_{l=k+1}^{i} [1 - pG(l)]^{j-1} \prod_{h=k+1}^{i} [1 - pG(h)] \right). \] \( (8) \)

Similarly, for \( e_{k+1} \), as in the term in the first equality above, we have
\[ e_{k+1} = G(k+1) \prod_{l=j}^{k+1} [1 - pG(l)]^{j-1} \prod_{h=j}^{k} [1 - pG(h)] \]
\[ \times \left( 1 + \sum_{i=k+2}^{\infty} \prod_{l=k+2}^{i} [1 - pG(l)]^{j-1} \prod_{h=k+2}^{i} [1 - pG(h)] \right). \] \( (9) \)

Taking
\[ C(k, j) := 1 + \sum_{i=k+2}^{\infty} \prod_{l=k+2}^{i} [1 - pG(l)]^{j-1} \prod_{h=k+2}^{i} [1 - pG(h)] \]
we see that
\[ \frac{e_{k+1}}{e_k} = C(k, j) \frac{G(k+1) [1 - pG(k+1)]^{j-1} [1 - pG(k)]}{G(k) 1 + C(k, j) [1 - pG(k+1)]^j}. \] \( (10) \)

Now, \( \frac{G(k+1)}{G(k)} \leq 1 \) for all \( k \geq 0 \), and, for fixed \( j \),

(i) for \( k \) large enough, \( 0.9 < [1 - pG(k+1)]^{j-1} [1 - pG(k+1)] \leq 1 \);

(ii) for \( j \geq N \), from equation (3.5) of Athreya, Roy and Sarkar (2004), we have
\[ \sum_{i=k+2}^{\infty} \prod_{l=k+2}^{i} [1 - pG(l)]^{j-1} < \infty \] and hence
\[ \sum_{i=k+2}^{\infty} \prod_{l=k+2}^{i} [1 - pG(l)]^{j-1} \prod_{h=k+2}^{i} [1 - pG(h)] < \infty; \]
which ensures that, for \( k \) large enough, \( 1 \leq C(k, j) < 1.1 \).

Thus, for \( j \geq N \) and \( k \) large enough, we have \( \frac{e_{k+1}}{e_k} < \frac{1}{1.1} \), and so, by ratio test, \( \sum_{k=1}^{\infty} e_k < \infty \). This shows, from (7), that (8) and thereby (6) hold.
Now we show that, for \( N \) as above, \( \sum_{i,j \geq N} \mathbb{P}_p(B_{i,j}) < \infty \). Towards this, we first observe that, for \( i, j \geq 1 \), by symmetry we have \( \mathbb{P}_p(B_{i,j}) = \mathbb{P}_p(B_{j,i}) \), thus we need to show
\[
\sum_{i,j \geq N} \mathbb{P}_p(B_{i,j}) = 2 \sum_{i=N}^{\infty} \sum_{j=N}^{i-1} \mathbb{P}_p(B_{i,j}) + \sum_{i=N}^{\infty} \mathbb{P}_p(B_{i,i}) < \infty. \tag{11}
\]

We will show separately that
\[
\sum_{i=N}^{\infty} \sum_{j=N}^{i-1} \mathbb{P}_p(B_{i,j}) < \infty \quad \text{and} \quad \sum_{i=N}^{\infty} \mathbb{P}_p(B_{i,i}) < \infty. \tag{12}
\]

First, for any \( i, j \geq 1 \),
\[
\mathbb{P}_p(B_{i,j}) = \left[ 1 + p \sum_{t=0}^{i-1} \frac{G(t)}{1 - pG(t)} \right] \mathbb{P}_p(A_{i,j}).
\]

From the proof of Proposition 3.2(b) of Athreya, Roy and Sarkar (2004) we know that,
\[
\sum_{i=N}^{\infty} \sum_{j=N}^{i-1} \mathbb{P}_p(A_{i,j}) < \infty \quad \text{and} \quad \sum_{i=N}^{\infty} \mathbb{P}_p(A_{i,i}) < \infty,
\]

thus we need to show that
\[
\sum_{i=N}^{\infty} \sum_{j=N}^{i-1} \sum_{t=0}^{i-1} \frac{G(t)}{1 - pG(t)} \mathbb{P}_p(A_{i,j}) < \infty \quad \text{and} \quad \sum_{i=N}^{\infty} \sum_{t=0}^{i-1} \frac{G(t)}{1 - pG(t)} \mathbb{P}_p(A_{i,i}) < \infty. \tag{13}
\]

For the first sum, interchanging the order of summation we have
\[
\sum_{i=N}^{\infty} \sum_{j=N}^{i-1} \sum_{t=0}^{i-1} \frac{G(t)}{1 - pG(t)} \mathbb{P}_p(A_{i,j}) = \sum_{i=N}^{\infty} \sum_{t=0}^{i-1} \frac{G(t)}{1 - pG(t)} \sum_{j=N+1}^{\infty} \sum_{i=\max(t,j)+1}^{\infty} \mathbb{P}_p(A_{i,j}).
\]

Also breaking up the inner two sums according to the values taken by \( i \), writing out the expression for \( \mathbb{P}_p(A_{i,j}) \) and collecting terms together, we have
\[
\sum_{j=N+1}^{\infty} \sum_{i=\max(t+1,j)+1}^{\infty} \mathbb{P}_p(A_{i,j})
= \sum_{m=1}^{\infty} \sum_{r=\max(t-N+1,m+1)}^{\infty} \mathbb{P}_p(A_{N+r,N+m})
= \sum_{m=1}^{t-N} \sum_{r=t+1-N}^{\infty} \prod_{l=0}^{N+m-1} [1 - pG(l)]^{2l+1} \prod_{k=1}^{r-m} [1 - pG(k + N + m - 1)]^{N+m}
+ \sum_{m=t-N+1}^{\infty} \sum_{r=m+1}^{\infty} \prod_{l=0}^{N+m-1} [1 - pG(l)]^{2l+1} \prod_{k=1}^{r-m} [1 - pG(k + N + m - 1)]^{N+m}. \tag{14}
\]

9
To simplify the expressions we take $\sigma_t := \frac{G(t)}{1 - pG(t)}$, $s_m := \prod_{l=0}^{N+m-1} [1 - pG(l)]^{2l+1}$ and $g(k, m) := [1 - pG(k + N + m - 1)]^{N+m}$. Using this notation, from the previous two equations we have

$$\sum_{i=N}^{\infty} \sum_{j=N}^{\infty} \sum_{l=0}^{\infty} \frac{G(t)}{1 - pG(t)} \mathbb{P}_l(A_{i,j}) = \sum_{t=1}^{\infty} \sigma_t \sum_{m=1}^{t-N} \sum_{r=t+1-N}^{\infty} s_m \prod_{k=1}^{r-m} g(k, m)$$

$$+ \sum_{t=1}^{\infty} \sigma_t \sum_{m=t-N+1}^{\infty} \sum_{r=m+1}^{\infty} s_m \prod_{k=1}^{r-m} g(k, m). \quad (14)$$

We start with the first term on the right in the above equation. Reordering the sums, we have

$$\sum_{t=1}^{\infty} \sigma_t \sum_{m=1}^{t-N} \sum_{r=t+1-N}^{\infty} s_m \prod_{k=1}^{r-m} g(k, m) = \sum_{m=1}^{\infty} s_m \sum_{t=m+N}^{\infty} \sigma_t \sum_{r=t+1-N}^{\infty} \prod_{k=1}^{r-m} g(k, m). \quad (15)$$

Let

$$\alpha_m := s_m \sum_{t=m+N}^{\infty} \sigma_t \sum_{r=t+1-N}^{\infty} \prod_{k=1}^{r-m} g(k, m)$$

denote the summand. Observe that

$$\frac{\alpha_{m+1}}{\alpha_m} = \frac{\sum_{t=m+1+N}^{\infty} \sum_{r=t+1-N}^{\infty} \sigma_t \sum_{k=1}^{r-m-1} g(k, m+1)}{\sum_{t=m+N}^{\infty} \sum_{r=t+1-N}^{\infty} \prod_{k=1}^{r-m} g(k, m)}$$

$$= \frac{[1 - pG(N + m)]^{2(N+m)+1} \sum_{t=m+1+N}^{\infty} \sum_{r=t+1-N}^{\infty} \prod_{k=1}^{r-m} g(k, m+1)}{\sum_{t=m+N}^{\infty} \sum_{r=t+1-N}^{\infty} \prod_{k=1}^{r-m} g(k, m)}$$

$$= \frac{[1 - pG(N + m)]^{N+m}}{[1 - pG(N+m)]^{N+m+1}} \sum_{t=m+1+N}^{\infty} \prod_{k=1}^{r-m+1} g(k, m) + \sum_{t=m+1+N}^{\infty} \sigma_t \sum_{r=t+1-N}^{\infty} \prod_{k=1}^{r-m-1} g(k, m+1) \times \frac{s_{m+1} \prod_{k=1}^{r-m} g(k, m+1)}{s_m \prod_{k=1}^{r-m} g(k, m)}.$$
then $\frac{a_{m+1}}{a_m} < a$; and so an application of the ratio test will yield that the sum in (15) is finite.

To show $\sum_{t=m+1+N}^{\infty} \sigma_t \sum_{r=t+1-N}^{\infty} \prod_{k=1}^{r-m-1} g(k, m+1) < \infty$, we again apply a ratio test. Let $\tau_t := \sigma_t \sum_{r=t+1-N}^{\infty} \prod_{k=1}^{r-m-1} g(k, m+1) < \infty$. Then,

$$\frac{\tau_{t+1}}{\tau_t} = \frac{G(t+1)}{G(t)} [1 - pG(t)]$$

$$\times \frac{[1 - pG(t + 1)]^{N+m}}{1 + [1 - pG(t + 1)]^{N+m+1}} \left(1 + \sum_{r=t+2-N-m}^{\infty} \prod_{k=t+2-N-m}^{r} g(k, m+1) \right).$$

Note that $\sum_{r=t+2-N-m}^{\infty} \prod_{k=t+2-N-m}^{r} g(k, m+1) < \infty$ (from Athreya, Roy and Sarkar (2004)) and so we may obtain a $t_0$ such that, for all $t \geq t_0$,

(i) $1 \leq 1 + \sum_{r=t+2-N-m}^{\infty} \prod_{k=t+2-N-m}^{r} g(k, m+1) < 1.1$ and

(ii) $[1 - pG(t + 1)]^{N+m+1} \left(1 + \sum_{r=t+2-N-m}^{\infty} \prod_{k=t+2-N-m}^{r} g(k, m+1) \right) > 0.9.$

This choice of $t_0$ ensures that for all $t \geq t_0$ we have $\frac{\tau_{t+1}}{\tau_t} < 1.1/1.9$, and hence

$$\sum_{t=m+1+N}^{\infty} \sigma_t \sum_{r=t+1-N}^{\infty} \prod_{k=1}^{r-m-1} g(k, m+1) < \infty.$$ 

Thus the first term on the right of (14) is finite.

A similar calculation and a use of ratio test shows that the second term on the right of (14) is finite, thereby showing that the sum in (13) is finite.

Reordering the second sum in (11) and using the notation we introduced earlier, we have

$$\sum_{t=N}^{\infty} \sum_{i=0}^{t-1} \frac{G(t)}{1 - pG(t)} \mathbb{P}_p(A_{t,i}) = \sum_{t=0}^{N-1} \sum_{i=N}^{\infty} \sigma_t \mathbb{P}_p(A_{t,i}) + \sum_{t=N}^{\infty} \sum_{i=0}^{t-1} \sigma_t \mathbb{P}_p(A_{t,i}). \quad (16)$$

From the end of Section 3.1 of Athreya, Roy and Sarkar (2004) we know that with $N$ as above, $\sum_{i=N}^{\infty} \sigma_t \mathbb{P}_p(A_{t,i}) < \infty$ and hence $\sum_{t=0}^{N-1} \sum_{i=N}^{\infty} \sigma_t \mathbb{P}_p(A_{t,i}) < \infty$.

Expanding the term $\mathbb{P}_p(A_{t,i})$, we have

$$\sum_{t=N}^{\infty} \sum_{i=0}^{t-1} \sigma_t \mathbb{P}_p(A_{t,i}) = \sum_{t=N}^{\infty} \sum_{i=0}^{t-1} \sigma_t \prod_{l=0}^{i-1} [1 - pG(t)]^{2l+1} \quad (17)$$
Taking \( a_t := \sigma \sum_{i=t+1}^{\infty} \prod_{l=0}^{i-1} [1 - pG(l)]^{2l+1} \) and \( l_t := \sum_{r=t+2}^{\infty} \prod_{l=t+2}^{r} [1 - pG(l)]^{2l+1} \), we see that

\[
a_t := \sigma \prod_{l=1}^{t} [1 - pG(l)]^{2l+1} \left\{ 1 + [1 - pG(t+1)]^{2t+3} (1 + l_t) \right\}
\]

from which we have

\[
\frac{a_{t+1}}{a_t} = \frac{G(t+1)}{G(t)} \frac{[1 - pG(t)] [1 - pG(t+1)]^{2t+3} (1 + l_t)}{[1 - pG(t+1)]^{2t+2} (1 + l_t)}
\]

\[
\leq \frac{G(t+1)}{G(t)} \frac{[1 - pG(t+1)]^{2t+2} (1 + l_t)}{[1 - pG(t+1)]^{2t+3} (1 + l_t)}
\]

Also, with \( a \) as in (4),

\[
\prod_{r=t+2}^{t+1} [1 - pG(l)]^{2l+1} = [1 - pG(t+1)]^{2r+3}
\]

\[
< \left[ 1 - p \frac{r+1}{r+1} \right]^{2r+3}
\]

\[
< a \text{ for } r \geq N,
\]

so, by the ratio test \( l_t < \infty \) for \( t \geq N \). Now, since \( l_t \to 0 \) as \( t \to \infty \), from the remark following (4) we may obtain a \( t_0 \) such that for \( t \geq t_0 \), we have \([1 - pG(t+1)]^{2t+2} (1 + l_t) < a < 1 \). Also \( \frac{G(t+1)}{G(t)} < 1 \) for all \( t \), thus

\[
\frac{a_{t+1}}{a_t} < a \text{ for all } t \geq t_0,
\]

and by ratio test we have that the sum in (16) is finite.

This completes the proof of Proposition 1.6 (ii).

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