Abstract

We briefly review some results concerning the problem of classical singularities in general relativity, obtained with the help of the theory of differential spaces. In this theory one studies a given space in terms of functional algebras defined on it. Then we present a generalization of this method consisting in changing from functional (commutative) algebras to noncommutative algebras. By representing such an algebra as a space of operators on a Hilbert space we study the existence and properties of various kinds of singular space-times. The obtained results suggest that in the noncommutative regime, supposedly reigning in the pre-Planck era, there is no distinction between singular and non-singular states of the universe, and that classical singularities are produced in the transition process from the noncommutative geometry to the standard space-time physics.
1 Introduction

The standard method of dealing with singularities in general relativity consists in probing the incompleteness of space-time (geodesic incompleteness, Schmidt's b-incompleteness or some other suitably chosen incompleteness) with the help of various structures defined on the space-time manifold (such as the chronological or causal structures) and certain conditions superimposed on it (such as different energy conditions). In this approach singularities are treated as ideal or boundary points of space-time which can be reached only by investigating regular (i.e., nonsingular) regions of space-time (see [12, 33]). In the series of works [9, 10, 14, 15, 16, 17, 22] we have proposed an alternative method of investigating singularities in general relativity. The idea goes back to Geroch [8] who suggested that the space-time geometry, instead of being studied in terms of the usual charts and atlases on a manifold $\mathcal{M}$, can be investigated in terms of the algebra $C^\infty(\mathcal{M})$ of smooth functions on $\mathcal{M}$. Although this method is, in principle, equivalent to the traditional one, it is more algebraic and more global as far as its intuitive character is concerned and, first of all, it seems to be more open for further generalizations. The tempting idea is to consider a more general functional algebra $\tilde{\mathcal{C}}$ and treat it as - ex definitione - an algebra of smooth functions on a larger space $\tilde{\mathcal{M}}$ such that if we restrict $\tilde{\mathcal{C}}$ to $\mathcal{M}$ we obtain the standard geometry on a smooth manifold $\mathcal{M}$. One could hope that the set $\tilde{\mathcal{M}} \setminus \mathcal{M}$ contains space-time singularities which, in this case, would be directly accessible to investigations in terms of the algebra $\tilde{\mathcal{C}}$.

It turns out that such a program can indeed be implemented provided that the algebra $\tilde{\mathcal{C}}$ is subject to some further conditions which would allow one to define standard differential tools in terms of it. The pair $(\tilde{\mathcal{M}}, \tilde{\mathcal{C}})$, where $\tilde{\mathcal{C}}$ is a functional algebra satisfying suitable conditions, is usually called differential space. The theory of these spaces is a quickly developing field of research in mathematics (for a bibliographical review see [2]; in [13], to develop the above sketched program we have used the theory of differential spaces elaborated by Sikorski [29, 30, 31]).

In Section 2, to make the paper self-contained, we briefly review the main results concerning the problem of singularities obtained with the help of the theory of differential spaces and, in Section 3, we explicitly show a geometrical mechanism of the formation of singularities, and illustrate it with some examples. In this respect, the functional algebra method seems to be more
efficient than the traditional one. However, also this method trivializes if we try to apply it to the investigation of stronger type singularities. It turns out that the functional algebra method can be generalized by replacing commutative algebras by noncommutative ones. In Section 4, we effectively construct such a noncommutative algebra $A$, and in Sections 5 and 6, we show how can it be used to the study of singularities. Even the strongest singularities (which we call malicious singularities) surrender to this method. We study the existence and structure of malicious elementary quasiregular and regular singularities. The obtained results can be interpreted in the following way. The pre-Planck era is modelled by a noncommutative geometry, and in this era there is no distinction between singular and nonsingular states of the universe. Passing through the Planck threshold consists in changing from the noncommutative algebra to (a subset of) its center. In this process the standard space-time physics emerges and classical singularities are produced. Consequently, the question of the existence of singularities at the fundamental level is meaningless. It is only from the point of view of the macroscopic observer that one can ask whether the universe had the initial singularity in its finite past or will have the final singularity in its finite future.

2. Singularities in Terms of Functional Algebras

Let us consider a family $C$ of real valued functions on a set $M$ which we endow with the weakest topology $\tau_C$ in which the functions of $C$ are continuous.

A function $f$, defined on $A \subset M$, is called a local $C$-function if, for any $x \in A$, there is a neighborhood $B$ of $x$ in the topological space $(A, \tau_A)$, with $\tau_A$ the topology induced in $A$ by $\tau_C$, and a function $g \in C$ such that $g|B = f|B$. Let $C_A$ denote the set of all local $C$-functions. It can be easily seen that $C \subset C_M$; if $C = C_M$, the family $C$ is said to be closed with respect to localization.

$C$ is said to be closed with respect to superposition with smooth Euclidean functions if for any $n \in \mathbb{N}$ and each function $\omega \in C^\infty(\mathbb{R}^n)$, $f_1, ..., f_n \in C$ implies $\omega \circ (f_1, ..., f_n) \in C$. It is easy to see that this condition implies that $C$ is a linear algebra.

A family $C$ of real valued functions on $M$ which is both closed with re-
spect to localization and closed with respect to superposition with Euclidean functions is called a differential structure on $M$, and a pair $(C, M)$, where $C$ is a differential structure on $M$, is called a differential space. Of course, every differential manifold is a differential space with $C = C^\infty(M)$ as its differential structure.

The above construction is more general and more flexible if we use a sheaf $\mathcal{C}$ of linear functional algebras on a topological space $M$ (with any topology $\tau$) instead of a single functional algebra $C$ on $M$ (with the $\tau_C$ topology). In such a case, the condition of the closeness with respect to localization is already contained in the sheaf axioms. The triple $(M, \tau, \mathcal{C})$ is called a structured space, and $\mathcal{C}$ its differential structure. The theory of these spaces was developed in [15, 16].

It is evident that any differential space can be trivially regarded as a structured space, but not vice versa. It can be shown that a structured space $(M, \tau, \mathcal{C})$ is a differential space if for every open set $U \in \tau$ and any point $x \in U$, there exists a function $\varphi$, called bump function, such that $\varphi(p) = 1$ and $\varphi|_{M \setminus U} = 0$.

A space-time with singularities (i.e., with a singular boundary) can be organized into a structured space and, in this way, singularities can be investigated with the help of the theory of structured spaces. This is done by using the method of prolongations of differential structures. Let $M$ be a space-time manifold, which is, of course, also a structured space $(M, \tau, \mathcal{C})$ such that $\tau = \tau_C$ where $\tau_C$ is the weakest topology in which functions of $\mathcal{C}$ are continuous. Let $\bar{M} = M \cup \partial M$ where $\partial M$ is a singular boundary of $M$. A sheaf $\bar{\mathcal{C}}$ on $\bar{M}$ such that $\bar{\mathcal{C}}(M) = \mathcal{C}(M)$ is said to be a prolongation of the differential structure $\mathcal{C}$ on $M$ to that of $\bar{M}$. Since $M$ is dense in $\bar{M}$ the prolongation is unique. It turns out that such a prolongation always exists although in some cases it is trivial (see below).

In the study of singularities we assume that $M$ is a space-time and $\bar{M}$ this space-time with singularities. In this way, regular singularities and elementary quasiregular singularities (in the classification of Ellis and Schmidt [7]) can be fully analyzed [14, 27]). In particular, the conic singularity in the space-time of a cosmic string has been thoroughly studied by using this method [14, 17], and the results have been found consistent with those obtained by Vickers [34, 35, 36] who used other methods. These are the easiest cases to deal with. As another extremity, the situations have been studied in which the singular boundary is not Hausdorff separated from the rest.
of space-time. It has been demonstrated that in such cases only constant functions can be prolonged from the differential structure of a given space-time to its singular boundary \[15, 16\]. As it is well known, situations of this kind occur in the closed Friedman world model and in the Schwarzschild solution when their curvature singularities are interpreted as Schmidt’s b-boundaries. In the closed Friedman world model another pathology occurs: both the initial and final singularities turn out to be the same and the only point of the b-boundary of this space-time \([1, 24]\). This situation is transparently explained in terms of structured spaces: the differential structure \(\overline{C}(\overline{M})\) of the closed Friedman space-time \(M\) together with its b-boundary \(\partial M\), \(\overline{M} = M \cup \partial M\), consists only of constant functions, \(\overline{C}(\overline{M}) \simeq \mathbb{R}\), which do not distinguish between points, i.e., there is no function \(f \in \overline{C}(\overline{M})\) such that \(f(x) \neq f(y)\) for any \(x, y \in \overline{M}, x \neq y\). However, if we do not “touch” any singularity, i.e., if we consider the differential structure \(C(M)\) of space-time \(M\) rather than \(\overline{C}(\overline{M})\) everything remains all right \([15, 16]\).

More generally, we have proved that if \(x_0\) is a b-boundary point, and if the fiber \(\pi_\rho^{-1}(x_0)\) in the (generalized) fiber bundle of linear frames over \(\overline{M} = M \cup \partial b\) degenerates to a single point, then the only global cross sections of the sheaf \(\overline{C}\) over \(\overline{M}\) are constant functions, i.e., \(\overline{C}(\overline{M}) \simeq \mathbb{R}\) (see \([10]\)). In such a case \(x_0\) is called malicious singularity. Singularities of the closed Friedman universe and of the Schwarzschild solution belong to this class of singularities.

3 Origin of Classical Singularities in Terms of Differential Spaces

In this section we shall study a “mathematical mechanism” of the appearance of classical singularities, and illustrate it with some examples.

Let \((M, C)\) be a differential space (in the sense of Sikorski), and \(\rho \subset M \times M\) an equivalence relation in \(M\). The family

\[\overline{C} := C/\rho = \{\bar{f} : M/\rho \to \mathbb{R} : \bar{f} \circ \pi_\rho \in C\},\]

where \(\pi_\rho : M \to M/\rho := \overline{M}\) is the canonical projection, is the largest differential structure on \(M/\rho\) such that \(\pi_\rho\) is smooth (in the sense of the theory of differential spaces). When going from the set \(M\) to the set \(\overline{M}\) some
elements of $M$ are glued together forming various kinds of singularities. We shall say that a function passes through a singularity if the singularity is in its domain. It is obvious that the differential structure $\bar{C}$ is the maximal set of functions passing through singularities in the quotient space $\bar{M} = M/\rho$. Let us define another linear algebra of functions $C^\rho := \{ f \in \bar{C} : \forall x,y \in M \, x\rho y \Rightarrow f(x) = f(y) \}$. Any function belonging to $C^\rho$ is called $\rho$-consistent. There exists an isomorphism of linear algebras $\Phi : \bar{C} \to C^\rho$ given by $\Phi(f) = f \circ \pi_\rho$ for $f \in \bar{C}$ (all these facts are proven in [22]). As we can see, there is an isomorphism between functions passing through singularities and $\rho$-consistent functions.

In many cases considered in general relativity, the equivalence relation $\rho$ is defined by the action of a group $\Gamma$ on $M$: $M \times \Gamma \to M$. In such a case, for $x,y \in M$,

$$x\rho y \Leftrightarrow \exists g \in \Gamma \text{ such that } y = xg.$$  

The family $C_\Gamma = \{ f \in C : \forall x \in M, g \in \Gamma, f(x) = f(xg) \}$ of functions belonging to $C$, which are constant on the orbits of $\Gamma$, is called the family of $\Gamma$-invariant functions. It is clearly a linear algebra.

To summarize the above analysis we can say that singularities are formed from the initially smooth space-time modelled by a differential space $(M, C)$, where $M$ is a smooth manifold and $C = C^\infty(M)$, in the process of forming the quotient space $\bar{M} = M/\rho$. The space-time with singularities is now modelled by the quotient differential space $(M/\rho, C/\rho)$ such that $C/\rho$ is isomorphic with $C^\rho$. This schema can be visualized in the form of the following diagram

$$(M, C) \Rightarrow (M/\rho, C/\rho) \Leftrightarrow (M, C^\rho).$$

We shall illustrate this schema with some simple examples.

**Example 1: Regular singularity.** Let $M = \mathbb{R}^2$ and $\Gamma = \text{O}(2)$. The differential space without singularities is $(\mathbb{R}^2, C^\infty(\mathbb{R}^2))$. Let us define the equivalence relation $\rho$ in the following way: $p\rho q, p,q \in M$, iff there exists $g \in \Gamma$ such that $q = pg$. When we go to the quotient differential space $(M/\rho, C/\rho)$ singularities are formed. Equivalence classes of $\rho$ have the form of the concentric circles with the degenerate circle, the singularity, at the origin. The singularity is a fixed point of the action of $\Gamma$, i. e., the isotropy subgroup $\Gamma_{(0,0)}$ of the point $(0,0)$ coincides with the entire group $\Gamma$. The
isotropy group of other points of $M/\rho$ is trivial, i.e., $\Gamma_{(x,y)} = \{I\}$ for all $(x, y) \neq (0, 0)$. Now, we define the linear algebra of $\Gamma$-invariant functions

$$C_{\Gamma} = \{ f \in C^\infty(\mathbb{R}^2) : f(x, y) = \omega(x^2 + y^2), (x, y) \in \mathbb{R}^2, \omega \in C^\infty(\mathbb{R}^2) \}.$$ 

The isomorphism $\Phi : C/\rho \to C_{\Gamma}$ is given by

$$\Phi[(x, y)] = x^2 + y^2.$$ 

The points along the concentric circles are now identified and the quotient space becomes a half-line $\mathbb{R}_+$. We obtain the quotient differential space $(\mathbb{R}_+, C^\infty(\mathbb{R}_+))$ which originates from cutting off parts of the original differential space $(\mathbb{R}^2, C^\infty(\mathbb{R}^2))$, and consequently the singularities which are formed in this construction are regular singularities.

**Example 2: Quasiregular singularity.** Let $M = \mathbb{R}^2$ and $\Gamma = \{O_{2\pi/3}, O_{4\pi/3}, O_0\}$, the rotation group by the angles $2\pi/3, 4\pi/3, 0$, respectively. The set of fixed points of the action of $\Gamma$ consists of one point $(0, 0)$, and each orbit, for nonsingular points, consists of three points. The isotropy subgroups are

$$\Gamma_{(x,y)} = \begin{cases} \Gamma & \text{for } (x, y) = (0, 0) \\ I & \text{for } (x, y) \neq (0, 0) \end{cases}$$

After making suitable identifications we obtain a cone. For details see [17].

**Example 3: Malicious singularity.** We recall that a singularity is called malicious if the fiber over it in the generalized frame bundle over the space-time with such a singularity degenerates to a single point. Such singularities occur in the closed Friedman world model and in the Schwarzschild solution. In [16] we have shown that the only functions that pass through the singularity are constant functions, i.e., in such a case $C_\rho \simeq \mathbb{R}$. Since constant functions do not distinguish points, the differential space $(M/\rho, C/\rho)$ modelling space-time with a malicious singularity is diffeomorphic with the space $\{\text{point}\}, \mathbb{R})$. This also explains why the b-boundary of a space-time with malicious singularities consists of a single point. For instance, such a situation occurs for the closed Friedman model in which the initial and final singularities are the same and the unique point of the b-boundary [1, 24]. For this model $C_\rho \simeq \mathbb{R}$, and consequently only zero vector fields can be prolonged to the b-completion of the corresponding space-time. Therefore, any curve joining the initial and final singularities must have the zero “bundle length”. (For details see [13, 14].)
The analysis carried out in the present section can also be done in terms of structured spaces. The method of using sheaves of functional algebras rather than algebras is even more flexible since it does not presuppose a priori fixed topology. Nevertheless, the main difficulty remains: although the source of difficulties with malicious singularities is beautifully explained, their structure is not analyzable with the help of this method.

4 Desingularization Procedure

To gain more insight into what happens in malicious singularities we have proposed to replace commutative functional algebras, regarded as differential structures, by noncommutative ones [18, 19]. The idea is to generalize a space-time $M$ with a singular boundary $\partial M$, $\bar{M} = M \cup \partial M$, to a noncommutative space in the sense of Connes [4]. Such a space is essentially nonlocal, and when the above generalization is done we loose information on single points, but we gain the information about states, and both “singular” and “nonsingular” states are on equal footing. In this sense, the proposed construction can be regarded as a desingularization procedure. In doing it we closely follow the method described by Connes [4, p.99] and elaborated by him for other purposes.

Let $M$ be a space-time and $\overline{OM}$ the Cauchy completed total space of the orthonormal frame bundle over $M$ with the structural group $\Gamma = \text{SO}(3,1)$. Then $\bar{M} = \overline{OM}/\Gamma$ is the b-completion of the space-time $M$, and $\partial_b M = \bar{M} \setminus M$ is its b-boundary. The Cartesian product $G = \bar{M} \times \Gamma$ has the structure of a groupoid [in this case, it can be called a (generalized) transformation groupoid]. The elements of $G$ are pairs $\gamma = (p, g)$ where $p \in \overline{OM}$ and $g \in \Gamma$, and evidently two such pairs $\gamma_1 = (p_1, g_1)$ and $\gamma_2 = (p_2, g_2)$ can be composed if $p_2 = p_1 g_1$. If we represent $\gamma = (p, g)$ as an arrow beginning at $p$ and ending at $pg$, then two arrows $\gamma_1$ and $\gamma_2$ can be composed if the beginning of $\gamma_2$ coincides with the end of $\gamma_1$. Let us notice that the “frame” $p_0$ belonging to the “singular fibre”, i.e., $p_0 \in \pi^{-1}(x_0)$ where $x_0 \in \partial_b M$, is not an ordinary frame but rather the limit of Cauchy sequences of orthonormal frames (see [18]). From Schmidt’s construction it follows that such limits always exist.

Let us define two functions: $\text{beg}(\gamma) = p$ and $\text{end}(\gamma) = pg$ for $\gamma = (p, g) \in \overline{OM}$. 
G. It is immediate to see that the sets of all arrows that begin at \( p \in \overline{OM} \)

\[ G^p := \{ \gamma \in G : \text{beg}(\gamma) = p \} = \{(p, g) : g \in \Gamma \} \]

and the set of all arrows that end at \( q \in \overline{OM} \)

\[ G_q := \{ \gamma \in G : \text{end}(\gamma) = q \} = \{(qg^{-1}, q) : g \in \Gamma \} \]

can be given the structure of the group manifold \( \Gamma = SO(3,1) \) even if \( p \) or \( q \) belong to the fiber over a malicious singularity. In this way, also malicious singularities can be represented by well behaved structures. This is another reason for the name “desingularization” procedure.

We are now ready to define the noncommutative involutive algebra \( A = (C_\infty^c(\Gamma, C), \ast, \ast) \) of compactly supported complex valued functions on the groupoid \( G \). The multiplication in this algebra is defined to be the convolution

\[(s \ast t)(\gamma) = \int_{G_p} s(\gamma_1) t(\gamma_1^{-1}\gamma) d\gamma_1 \]

for every \( s, t \in A \), where \( \gamma = \gamma_1 \circ \gamma_2 \), \( G_p \) is the fiber over \( p \in \overline{OM} \), and the integral is taken over the Haar measure on \( G_p \). The involution is defined in the following way

\[ s^\ast(\gamma) = \overline{s(\gamma^{-1})} \]

where \( \gamma^{-1} \) is the “reversed arrow”. Geometry based on the algebra \( A \) should be regarded as a noncommutative version of the space-time geometry with singularities. Indeed, when \( A \) is restricted to the nondegenerate part of \( G \) (to the nondegenerate fibres of \( G \)) it is strongly Morita equivalent to the algebra \( C^\infty(M) \) of smooth functions on the space-time manifold \( M \) (strong Morita equivalence can be regarded as a noncommutative counterpart of isomorphism, see \[25, p.179\]). The states of the algebra \( A \), i. e., positive and suitably normed linear functionals on \( A \), represent states in the physical sense. The algebra \( A \) is ‘desingularized”, i. e., it does not distinguish between “singular” and “nonsingular states” \[13\].

It is the standard thing to represent a noncommutative algebra in a Hilbert space \( \mathcal{H} \) and to study it in terms of the algebra of bounded operators on \( \mathcal{H} \). Following Connes \[4, p.102\], we define, for every \( q \in \overline{OM} \), the following representation of the algebra \( A \)

\[ \pi_q : A \rightarrow \mathcal{B}(\mathcal{H}) \]
by
\[(\pi_q(s))(\xi) = (s_q \ast \xi),\]
for every \(s \in \mathcal{A}\), where \(\xi \in L^2(G_q) := \mathcal{H}\). This representation is involutive and nondegenerate. The completion of \(\mathcal{A}\) with respect to the norm
\[\|s\| = \sup_{q \in \mathcal{OM}} \|\pi_q(s)\|\]
is a \(C^*\)-algebra. As we shall see in the following, singularities (even malicious ones) can be studied in terms of these representations.

To make contact with the analysis carried out in the preceding Section in terms of commutative differential structures on space-time it is important to answer the question when the commutative algebra \(\mathcal{A}\) reproduces the commutative differential structure on a manifold. Let us consider two fibres \(G_p\) and \(G_q\), \(p, q \in \mathcal{OM}\), of the groupoid \(G\). They are said to be equivalent if there is \(g \in \Gamma\) such that \(q = pg\). A function of \(\mathcal{A}\) which is constant on the equivalence classes of equivalent fibres is said to be projectible. The set of all such functions, denoted by \(\mathcal{A}_{\text{proj}}\), forms a subalgebra of \(\mathcal{A}\). It can be easily seen that if \(f, g \in \mathcal{A}_{\text{proj}}\), then their convolution \(f \ast g\) becomes the usual (commutative) multiplication, and \(\mathcal{A}_{\text{proj}} \subset Z(\mathcal{A})\) where \(Z(\mathcal{A})\) is the center of \(\mathcal{A}\). One can readily show that \(\mathcal{A}_{\text{proj}}\) is isomorphic to an algebra of complex valued functions on \(\bar{M}\). It is evident that if there are no singularities then \(\mathcal{A}_{\text{proj}}\) is isomorphic with \(C^\infty(M)\). If \(\mathcal{A}_H\) denotes all Hermitian elements of \(\mathcal{A}\), then \(\mathcal{A}_{\text{proj}} \cap \mathcal{A}_H\) is isomorphic with a family of real valued functions on \(M\).

5 Nonlocal Geometry of Space-Time with Singularities

As it is well known (see, for instance, [25, p.11]), every commutative \(C^*\)-algebra \(C\) corresponds to a Hausdorff topological space \(M\) in the sense that \(C\) is isometrically \(*\)-isomorphic to the algebra of (complex valued) functions on \(M\). Indeed, let us consider the space \(\hat{C}\) of characters of the algebra \(C\), i. e. the space of functionals \(\phi : C \to \mathbb{C}\) such that \(\phi(fh) = \phi(f)\phi(h)\), and equip it with the topology of pointwise convergence. If \(f \in C\), then the mapping \(\hat{f} : \hat{C} \to \mathbb{C}\) defined by \(\hat{f}(\phi) = \phi(f)\), for all \(\phi \in \hat{C}\), is a continuous complex valued function on \(\hat{C}\) (called the Gel’fand transform of \(f\)). The
The Gel’fand-Neimark theorem states that all continuous functions on \( \hat{C} \) are of this form (for some \( f \in C \)). Consequently, one can regard elements of \( C \) as complex valued functions on \( \hat{C} \). Equivalently, one can define \( \hat{C} \) as the set of maximal ideals of the algebra \( C \) with the Jacobson topology \([25, \text{p.12}]\). Every such ideal is the set of functions vanishing at some point \( x \in M \).

The above construction does not work for noncommutative algebras. In general, such algebras have no maximal ideals, and the structure which is the closest one to the concept of point is the primitive ideal of a noncommutative algebra \( \mathcal{A} \), i.e., the kernel of an irreducible \( * \)-representation of \( \mathcal{A} \). If \( \mathcal{A} \) is a \( C^* \)-algebra, \( \pi \) a representation of \( \mathcal{A} \) in a Hilbert space \( \mathcal{H} \), and \( \xi \in \mathcal{H} \), then \( f \mapsto (\pi(f)\xi, \xi) \) is a positive form on \( \mathcal{A} \). If additionally \( \| f \| = 1 \), this form is called a state. There exists a correspondence between (equivalence classes) of representations of \( \mathcal{A} \) in a Hilbert space and states on \( \mathcal{A} \). If the representation \( \pi \) is nonzero and irreducible, the corresponding state is the pure state (for details see \([26, \text{pp.140-149}]\)).

The algebra \( \mathcal{A} = C_c^\infty(G, C) \) is noncommutative, and consequently nonlocal in the sense that it has no maximal ideals. One can regard it, in analogy with the commutative case, as describing a certain space, usually called a noncommutative space. We shall also speak of the associated space with the algebra \( \mathcal{A} \). Such a space can be identified with the set \( \text{Prim} \mathcal{A} \) of all primitive ideals of the algebra \( \mathcal{A} \). Since this algebra encodes information on the structure of space-time with singularities we could say that the space-time with singularities is a “pointless” space: the information on individual points of space-time has been lost, but the information about the “structure of singularities” has been gained. We could say that both “singular” and “nonsingular” states (modelled by the states on the algebra \( \mathcal{A} \) in the mathematical sense) are on equal footing.

In the next section, we shall give a series of theorems characterizing the existence of singularities of various types in terms of the noncommutative algebra \( \mathcal{A} = C_c^\infty(G, C) \).

### 6 The Existence of Singularities

**Lemma 1.** For every \( q \in \partial M \), if \( s \in \mathcal{A}_{\text{proj}} \), then 

\[
\pi_q(s)(\xi) = k(q)T(\xi)
\]
where \( k(q) \in \mathbb{C} \) is the value of the constant function \( s \) on the fiber \( \pi_{\overline{OM}}^{-1}(q) \), and \( T(\xi) = \int_{G_q} \xi(\gamma^{-1}_1) \).

**Proof.** Let \( s \in \mathcal{A}_{proj} \). Representation (1) gives

\[
\pi_q(s)(\xi) = s_q \ast \xi = kT(\xi)(\gamma)
\]

for \( \xi \in L^2(G_q) \). \( \square \)

The function \( s \) from the above proposition is “constant on a given fibre”, but need not be constant on different fibres unless a malicious singularity is present.

**Theorem 1.** \( \mathcal{A}_{proj} \simeq \mathbb{C} \) if and only if the space-time associated with the algebra \( \mathcal{A} \) contains at least one malicious singularity. In such a case

\[
\pi_q(s)(\xi) = kT(\xi)
\]

where \( k \in \mathbb{C} \) is a value of a constant function \( s \in \mathcal{A}_{proj} \) on any fibre \( \pi_{\overline{OM}}^{-1}(q) \), \( q \in \overline{OM} \), and \( T(\xi) = \int_{G_q} \xi(\gamma^{-1}_1) \).

**Proof.** The first part of the theorem follows from the previous analysis in terms of functional algebras and the fact that \( \mathcal{A}_{proj} \) is isomorphic with the algebra of complex valued functions on \( \overline{M} \). If we replace \( \mathcal{A}_{proj} \) with \( \mathcal{A}_{proj} \cap \mathcal{A}_H \) we should replace \( \mathbb{C} \) with \( \mathbb{R} \) (for details see [15].

The proof of the second part of the theorem is the same as that of lemma 1; one should only notice that since \( \mathcal{A}_{proj} \simeq \mathbb{C} \), \( k \) is constant on all fibres. \( \square \)

The above results can be rephrased in terms of the total representation of the algebra \( \mathcal{A} \) on a Hilbert space \( L^2(G) := \bigoplus_{q \in \overline{OM}} L^2(G_q) \). This representation

\[
\pi : \mathcal{A} \to \bigoplus_{p \in \overline{OM}} \pi_q(s)
\]

is defined by

\[
\pi(s) = \bigoplus_{q \in \overline{OM}} \pi_q(s).
\]

From lemma 1 it follows that

\[
\pi(s)(\xi) = \bigoplus_{q \in \overline{OM}} k(q)T(\xi),
\]

and the second part of theorem 1 asserts that

\[
\pi(s)(\xi) = k \bigoplus_{q \in \overline{OM}} T(\xi).
\]
Let us notice that each \( s \in A_{\text{proj}} \) defines the function \( \tilde{s} : OM \to C \) by \( \tilde{s} = s(q, e) \) where \( e \) is the neutral element of \( \Gamma \); \( s \) is then a function which, for a given \( q \in OM \), assumes the value \( \tilde{s}(q) \) equal to the value of the constant function \( s \in A_{\text{proj}} \) on the fiber \( G_q \).

**Theorem 2.** In the space-time associated with the algebra \( A \) there is no singularity if and only if \( A_{\text{proj}} \simeq C^\infty(M, C) \)

**Proof.** Let us assume that in the space-time associated with \( A \) there is no singularity, and let \( s \in A_{\text{proj}} \). The function \( \tilde{s} : OM \to C \) is constant on fibres of the frame bundle \( \pi_M : OM \to M \), and consequently it defines the smooth complex valued function \( f \) on \( M \) by \( f(x) = \tilde{s}(q) \), for \( x \in M \), where \( q \) is any element of the fibre \( \pi^{-1}_M(x) \). Since \( \pi_M(q) = x \) one has \( f(\pi_M(q)) = \tilde{s} \), i.e., \( f \circ \pi_M = \tilde{s} \). The smoothness of \( f \) follows from the fact that \( \pi_M : OM \to M \) is a principal, locally trivial fibre bundle. It is also clear that each function of \( A_{\text{proj}} \) can be obtained by lifting a function \( f \in C^\infty(M, C) \).

Now, let us assume that \( A_{\text{proj}} \simeq C^\infty(M, C) \). From the construction of the algebra \( A \) one has that \( \Gamma_0 \)-invariant functions lift to all functions of \( A_{\text{proj}} \). Hence \( \tilde{M} = M \) which means that there are no singularities. □

Let \( \mathcal{A}_{\Gamma_0} \) be the family of \( \Gamma_0 \)-invariant functions belonging to \( A \), i.e., the family of functions which are constant on the orbits of the action of \( \Gamma_0 \).

**Theorem 3.** In the space-time associated with the algebra \( A \) there is an elementary quasiregular singularity (but there are no stronger singularities) if and only if there exists a discrete group \( \Gamma_0 \) of isometries of \( M \) such that \( A_{\text{proj}} \simeq C^\infty(M)_{\Gamma_0} \).

**Proof.** Elementary quasiregular singularities are those which are produced in the procedure of making the quotient of space-time by a finite subgroup \( \Gamma_0 \) of its isometries \( \mathcal{M} \). Only \( \Gamma_0 \)-invariant functions pass through such singularities \([7, 27]\). \( \Gamma_0 \)-invariant functions lift to all functions of \( A_{\text{proj}} \). □

Let us notice that, for malicious singularities, \( A_{\text{proj}} \) consists only of constant functions; for elementary quasiregular singularities \( A_{\text{proj}} \) consists of \( \Gamma_0 \)-invariant functions; if there are no singularities, \( A_{\text{proj}} \) consists of functions isomorphic with all smooth functions on space-time. If in the given space-time there are singularities of various kinds, the strongest singularity determines the structure of the algebra \( A \). In agreement with the non-local character of the noncommutative algebra \( A \), the above theorems convey the information about the structure of singular space-times (space-times with
the structure of singularities themselves.

**Theorem 4.** In the space-time associated with the algebra \( \mathcal{A} \) there is a regular singularity (but there are no stronger singularities) if and only if the groupoid \( G = \overline{OM} \times \Gamma \) is a subspace of a “larger” groupoid \( \bar{G} = \bar{E} \times \Gamma \), where \( \overline{OM} \) is a subspace of constant dimension (in the sense of Sikorski) of the space \( \bar{E} \). In such a case \( A_{\text{proj}} \) is a localization of \( \hat{A}_{\text{proj}} \) where \( \hat{A}_{\text{proj}} \) is the subalgebra of projectible functions on \( G \), i.e., \( A_{\text{proj}} = (\hat{A}_{\text{proj}})_G \).

**Proof.** We remind that a differential space \((M,C)\) is of constant dimension (in the sense of Sikorski) if (1) \( \dim T_x M = n \) for every \( x \in M \); (2) the module of vector fields \( \mathcal{X}(M) \) is locally free of rank \( n \). In agreement with the definition of regular singularities such singularities originate if we cut off a space-time \( M \) from the larger space-time \( \tilde{M} \) so as not to alter its dimension (for the construction see [14]). This implies that also \( \overline{OM} \times \Gamma \) is a subspace of constant dimension of \( \bar{E} \times \Gamma \).

The second part of the theorem is a consequence of the following implication

\[
(C^\infty(M) = C^\infty(\tilde{M})_M) \Rightarrow (A_{\text{proj}} = (\hat{A}_{\text{proj}})_G)
\]

where \( C^\infty(\tilde{M})_M \) denotes the localization of \( C^\infty(\tilde{M}) \) to \( M \) (see, beginning of Section 2) \( \square \)

It can be seen from the above that regular singularities are very mild (they can hardly be called singularities): they do not change the family \( A_{\text{proj}} \) but only narrow its domain.

### 7 Perspectives

Although the analysis carried out in the present paper dealt with classical singularities, i.e., without taking into account quantum gravity effects, after making this analysis the question immediately arises: How would quantum gravity phenomena affect the obtained results? Usually, one considers two, mutually excluding themselves, possibilities: either the theory of quantum gravity, when finally discovered, will remove singularities from our picture of the universe, or singularities will remain unaffected by this theory. The first possibility has, in the last years, become a sort of common wisdom. It seems, however, that the results of the present paper open another way of looking at this problem.
As we have seen, the structure of space-time with singularities is encoded in the noncommutative algebra $\mathcal{A} = C_c^\infty(G, \mathbb{C})$. This algebra is nonlocal, in the sense that it contains no information about points and their neighborhoods. Consequently, singularities cannot be regarded as points in space-time (or in some other space). This conclusion remains in agreement with the standard understanding of singularities which are usually treated not as points of space-time, but rather as its ideal points or points of its singular boundary. As we have seen, one can meaningfully speak of (pure) states of the algebra $\mathcal{A}$, but there is no distinction between its singular and non-singular states. This corresponds to the fact that the physical system (the very early universe), modelled by the algebra $\mathcal{A}$, can occupy various states, none of which is more singular than the others. In other words, in this mathematical context, the question on the existence or non-existence of singularities does not arise.

We could speculate that this mathematical formalism is not only an artificial tool to deal with classical singularities, but it also somehow reflects physical regularities of the quantum gravity era. The fact that the states on the algebra $\mathcal{A}$ are represented as algebras of bounded operators in a Hilbert space (which is typically a quantum structure) can be viewed as a hint that the algebra $\mathcal{A}$ is indeed somehow related to quantum phenomena. In fact, there are several attempts to create a quantum gravity theory based on noncommutative geometry (see, for instance, [3, 5, 11, 21, 32]), but the above proposal is independent of the particulars of any of them.

We shall only assume that the algebra $\mathcal{A} = C_c^\infty(G, \mathbb{C})$ contains some information about the pre-Planck era of the universe, and we shall explore some possibilities hidden in this assumption. In spite of the fact that geometry determined by $\mathcal{A}$ is nonlocal, one can meaningfully speak of certain physical properties of the system modelled by it. As we have seen in Section 3, the algebra $\mathcal{A}$ can be completed to the $C^*$-algebra. This is important since $C^*$-algebras, in the noncommutative setting, generalize the standard concept of topology, and the generalization is so powerful that even non-Hausdorff cases can be dealt with by using this method (see [4, p.79]). This could provide a mathematical basis for some speculations about a “topological foam” supposedly reigning in the quantum gravity regime (see, for instance, [23]).

Moreover, with every $C^*$-algebra $\mathcal{B}$, represented as an algebra of operators on a Hilbert space, another algebra, called a von Neumann algebra, can be associated; it is defined as the commutant of its commutant, i. e., $(\mathcal{B}')' = \mathcal{B}$.
where $B'$ is the commutant of $B$. In our case, such a von Neumann algebra can be obtained as $\mathcal{R} := (\pi_q(\mathcal{A}))'$. Given this algebra one can define the one-parameter group of automorphisms of $\mathcal{R}$, $\alpha_t^\omega \in \text{Aut}\mathcal{R}$ for every $t \in \mathbb{R}$, called modular group, which depends on the state $\omega$ on $\mathcal{R}$ modulo inner automorphisms (see [6]). This means that von Neumann algebras are “dynamical objects” in the sense that they encode some (abstract) dynamical properties of a physical system modelled by a given noncommutative algebra. This is a remarkable circumstance. In spite of the fact that the algebra $\mathcal{A}$ is nonlocal (and consequently in the physical system modelled by it there is no space and no time in the usual meaning of these terms), one can define a noncommutative counterpart of dynamics with the modular group $\alpha_t^\omega$ playing the role of “generalized time” [6, 20].

The transition from the noncommutative geometry to the usual space-time geometry can be thought of as a kind of “phase transition” which happens when the universe passes through the Planck threshold. Mathematically, this corresponds to the transition from the noncommutative algebra $\mathcal{A}$ to its subalgebra $\mathcal{A}_{proj}$ as visualized in the following schema

$$(\mathcal{A} = C_c^\infty(G, C)) \Rightarrow (\mathcal{A}_{proj} \simeq \bar{C}_c^\infty(\bar{M})).$$

As we have seen in Section 3, $\mathcal{A}_{proj}$ is isomorphic with the algebra of smooth (in the sense of differential space theory) functions on the space-time $\bar{M}$ with its singular boundary $\partial M$, $\bar{M} = M \cup \partial M$. In this way, after passing through the Planck threshold both space-time (in the usual sense) and classical singularities emerge. We are entitled to say that classical singularities are produced in the process of the formation of macroscopic physics. Of course, the same can be said about final singularities, for instance in the closed Friedman world model or in the gravitational collapse of a massive object. On the fundamental level, beyond the Planck scale, there is no distinction between singular and nonsingular states. Only from the point of view of the macroscopic observer one can say that the universe had an initial singularity in its finite past, and possibly will have a final singularity in its finite future.

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