ON SMOOTH CHAS-SULLIVAN LOOP PRODUCT IN QUILLEN’S GEOMETRIC COMPLEX COBORDISM OF HILBERT MANIFOLDS

CENAP ÖZEL

ABSTRACT. In [1], by using Fredholm index we developed a version of Quillen’s geometric cobordism theory for infinite dimensional Hilbert manifolds. This cobordism theory has a graded group structure under topological union operation and has push-forward maps for complex orientable Fredholm maps. In [19], by using Quinn’s Transversality Theorem [23], it has been shown that this cobordism theory has a graded ring structure under transversal intersection operation and has pull-back maps for smooth maps. It has been shown that the Thom isomorphism in this theory was satisfied for finite dimensional vector bundles over separable Hilbert manifolds and the projection formula for Gysin maps has been proved. In [4], Chas and Sullivan described an intersection product on the homology of loop space \( LM \).

In [4], R. Cohen and J. Jones described a realization of the Chas-Sullivan loop product in terms of a ring spectrum structure on the Thom spectrum of a certain virtual bundle over the loop space. In this paper, we will extend this product on cobordism and bordism theories.

1. The Fredholm Index and Complex Cobordism of Hilbert Manifolds.

In [22], Quillen gave a geometric interpretation of cobordism groups which suggests a way of defining the cobordism of separable Hilbert manifolds equipped with suitable structure. In order that such a definition be sensible, it ought to reduce to his for finite dimensional manifolds and smooth maps of manifolds and be capable of supporting reasonable calculations for important types of infinite dimensional manifolds such as homogeneous spaces of free loop groups of finite dimensional Lie groups.

1.1. Cobordism of separable Hilbert manifolds. By a manifold, we mean a smooth manifold modelled on a separable Hilbert space; see Lang [13] for details on infinite dimensional manifolds. The facts about Fredholm map can be found in [5].

Definition 1.1. Suppose that \( f : X \to Y \) is a proper Fredholm map with even index at each point. Then \( f \) is an admissible complex orientable map if there is a smooth factorization

\[
X \xrightarrow{f} \xi \xrightarrow{g} Y,
\]

where \( q : \xi \to Y \) is a finite dimensional smooth complex vector bundle and \( \tilde{f} \) is a smooth embedding endowed with a complex structure on its normal bundle \( \nu(f) \).

A complex orientation for a Fredholm map \( f \) of odd index is defined to be one for the map \( (f, \varepsilon) : X \to Y \times \mathbb{R} \) given by \( (f, \varepsilon)(x) = (f(x), 0) \) for every \( x \in X \). At \( x \in X \), \( \text{index}(f, \varepsilon)_x = (\text{index } f_x) - 1 \). Also the finite dimensional complex vector bundle \( \xi \) in the smooth factorization will be replaced by \( \xi \times \mathbb{R} \).

Suppose that \( f \) is an admissible complex orientable map. Then since the map \( f \) is the Fredholm and \( \xi \) is a finite dimensional vector bundle, we see \( \tilde{f} \) is also a Fredholm map. By the surjectivity of \( q \),

\[
\text{index } \tilde{f} = \text{index } f - \dim \xi.
\]

Before we give a notion of equivalence of such factorizations \( \tilde{f} \) of \( f \), we want to give some definitions.

Definition 1.2. Let \( X, Y \) be the smooth separable Hilbert manifolds and \( F : X \times \mathbb{R} \to Y \) a smooth map. Then we will say that \( F \) is an isotopy if it satisfies the following conditions.

1. For every \( t \in \mathbb{R} \), the map \( F_t \) given by \( F_t(x) = F(x, t) \) is an embedding.
2. There exist numbers \( t_0 < t_1 \) such that \( F_t = F_{t_0} \) for all \( t \leq t_0 \) and \( F_t = F_{t_1} \) for all \( t \geq t_1 \).

The closed interval \( [t_0, t_1] \) is called a proper domain for the isotopy. We say that two embeddings \( f : X \to Y \) and \( g : X \to Y \) are isotopic if there exists an isotopy \( F_t : X \times \mathbb{R} \to Y \) with proper domain \( [t_0, t_1] \) such that \( f = F_{t_0} \) and \( g = F_{t_1} \).

Date: September 02, 2003.
Mathematics Subject Classification. Algebraic topology, Global Analysis.
Key words and phrases. cobordism, Fredholm map, Hilbert manifold, loop space, Pontrjagin-Thom construction, Chas-Sullivan loop product.
Proposition 1.3. (see [13]) The relation of isotopy between smooth embeddings is an equivalence relation.

Definition 1.4. Two factorizations $f : X \xrightarrow{F} \xi \xrightarrow{g} Y$ and $f : X \xrightarrow{F'} \xi' \xrightarrow{g'} Y$ are equivalent if $\xi$ and $\xi'$ can be embedded as subvector bundles of a vector bundle $\xi'' \to Y$ such that $\tilde{f}$ and $\tilde{f}'$ are isotopic in $\xi''$ and this isotopy is compatible with the complex structure on the normal bundle. That is, there is an isotopy $F$ such that for all $t \in [t_0, t_1]$, $F_t : X \to \xi''$ is endowed with a complex structure on its normal bundle which matches that of $f$ and $f'$ in $\xi''$ at $t_0$ and $t_1$ respectively.

By Proposition 1.3 we have Proposition 1.5. The relation of equivalence of admissible complex orientability of proper Fredholm maps between separable Hilbert manifolds is an equivalence relation.

This generalizes Quillen’s notion of complex orientability for maps of finite dimensional manifolds. We can also define a notion of cobordism of admissible complex orientable maps between separable Hilbert manifolds. First we recall some ideas on the transversality.

Definition 1.6. Let $f_1 : M_1 \to N, f_2 : M_2 \to N$ be smooth maps between Hilbert manifolds. Then $f_1$ and $f_2$ are transverse at $y \in N$ if

$$df_1(T_x, M_1) + df_2(T_{x_2}, M_2) = T_y N$$

whenever $f_1(x_1) = f_2(x_2) = y$. The maps $f_1$ and $f_2$ are said to be transverse if they are transverse at every point of $N$.

Lemma 1.7. Smooth maps $f_i : M_i \to N$ ($i = 1, 2$) are transverse if and only if $f_1 \times f_2 : M_1 \times M_2 \to N \times N$ is transverse to the diagonal map $\Delta : N \to N \times N$.

Definition 1.8. Let $f_1 : M_1 \to N, f_2 : M_2 \to N$ be transverse smooth maps between smooth Hilbert manifolds. The topological pullback

$$M_1 \coprod_N M_2 = \{(x_1, x_2) \in M_1 \times M_2 : f_1(x_1) = f_2(x_2)\}$$

is a submanifold of $M_1 \times M_2$ and the diagram

$$\begin{array}{ccc}
M_1 \coprod_N M_2 & \xrightarrow{f_2^*(f_1)} & M_2 \\
\downarrow f_1^*(f_2) & & \downarrow f_2 \\
M_1 & \xrightarrow{f_1} & N
\end{array}$$

is commutative, where the map $f_i^*(f_j)$ is pull-back of $f_j$ by $f_i$.

Definition 1.9. Let $f_i : X_i \to Y$ ($i = 0, 1$) be admissible complex oriented maps. Then $f_0$ is cobordant to $f_1$ if there is an admissible complex orientable map $h : W \to Y \times \mathbb{R}$ such that the maps $\varepsilon_i : Y \to Y \times \mathbb{R}$ given by $\varepsilon_i(y) = (y, i)$ for $i = 0, 1$, are transverse to $h$ and the pull-back map $\varepsilon_i^* h$ is equivalent to $f_i$. The cobordism class of $f : X \to Y$ will be denoted by $[X, f]$.

Proposition 1.10. If $f : X \to Y$ is an admissible complex orientable map and $g : Z \to Y$ a smooth map transverse to $f$, then the pull-back map

$$g^*(f) : Z \coprod_Y X \to Z$$

is an admissible complex orientable map with finite dimensional pull-back vector bundle

$$g^*(\xi) = Z \coprod_Y \xi = \{(z, v) \in Z \times \xi : g(z) = q(v)\}$$

in the factorization of $g^*(f)$, where $q : \xi \to Y$ is the finite-dimensional complex vector bundle in the factorization of $f$ as in Definition 1.4.

The next result was proved in [17] by essentially the same argument as in the finite dimensional situation using the Implicit Function Theorem [13].

Theorem 1.11. Cobordism is an equivalence relation.

Definition 1.12. For a separable Hilbert manifold $Y$, $\mathcal{U}_d(Y)$ is the set of cobordism classes of the admissible complex orientable proper Fredholm maps of index $-d$. 

2
My next result is the following.

**Theorem 1.13.** If \( f : X \to Y \) is an admissible complex orientable Fredholm map of index \( d_1 \) and \( g : Y \to Z \) is an admissible complex orientable Fredholm map of index \( d_2 \), then \( g \circ f : X \to Z \) is an admissible complex orientable Fredholm map with index \( d_1 + d_2 \).

Let \( g : Y \to Z \) be an admissible complex orientable Fredholm map of index \( r \). By Theorem 1.13 we have push-forward, or Gysin map

\[ g_* : \mathcal{U}^d(Y) \to \mathcal{U}^{d+r}(Z) \]

given by \( g_*([X,f]) = ([X, g \circ f]) \).

We show in [14] that it is well-defined. If \( g' : Y \to Z \) is a second map cobordant to \( g \) then \( g'_* = g_* \); in particular, if \( g \) and \( g' \) are homotopic through proper Fredholm maps they induce the same Gysin maps. Clearly, we have \((h \circ g)_* = h_* g_* \) for admissible complex orientable Fredholm maps \( h, g \) and \( \text{Id}_* = \text{Id} \).

The graded cobordism set \( \mathcal{U}^*(Y) \) of the separable Hilbert manifold \( Y \) has a group structure given as follows. Let \([X_1, f_1] \) and \([X_2, f_2] \) be cobordism classes. Then \([X_1, f_1] + [X_2, f_2] \) is the class of the map \( f_1 \sqcup f_2 : X_1 \sqcup X_2 \to Y \), where \( X_1 \sqcup X_2 \) is the topological sum (disjoint union) of \( X_1 \) and \( X_2 \). We show in [14] that this sum is well-defined. As usual, the class of the empty set \( \emptyset \) is the zero element of the cobordism set and the negative of \([X, f] \) is itself with the opposite orientation on the normal bundle of the embedding \( f \). Then we have

**Theorem 1.14.** The graded cobordism set \( \mathcal{U}^*(Y) \) of the admissible complex orientable maps of \( Y \) is a graded abelian group.

Now we define relative cobordism.

**Definition 1.15.** If \( A \) is a finite dimensional submanifold of \( Y \), the relative cobordism set \( \mathcal{U}^*(Y, A) \) is the set of the admissible complex orientable maps of \( Y \) whose images lie in \( Y - A \).

More generally,

**Theorem 1.16.** Let \( A \) be a finite dimensional submanifold of \( Y \). Then the relative cobordism set \( \mathcal{U}^*(Y, A) \) is a graded abelian group and there is a homomorphism \( \kappa^* : \mathcal{U}^*(Y, A) \to \mathcal{U}^*(Y) \) by \( \kappa^*[M \xrightarrow{h} Y] = [M \xrightarrow{h} Y] \) with \( h(M) \subseteq Y - A \).

If our cobordism functor \( \mathcal{U}^*(\cdot) \) of admissible complex orientable Fredholm maps is restricted to finite dimensional Hilbert manifolds, it agrees Quillen’s complex cobordism functor \( MU^*(\cdot) \).

**Theorem 1.17.** For finite dimensional separable Hilbert manifolds \( A \subseteq Y \), there is a natural isomorphism

\[ \mathcal{U}^*(Y, A) \cong MU^*(Y, A). \]

1.2. **Transversal approximations, contravariance and cup products.** We would like to define a product structure on the graded cobordism group \( \mathcal{U}^*(Y) \). Given cobordism classes \([X_1, f_1] \in \mathcal{U}^{d_1}(Y_1)\) and \([X_2, f_2] \in \mathcal{U}^{d_2}(Y_2)\), their external product is

\[ [X_1, f_1] \times [X_2, f_2] = [X_1 \times X_2, f_1 \times f_2] \in \mathcal{U}^{d_1 + d_2}(Y_1 \times Y_2). \]

Although there is the external product in the category of cobordism of separable Hilbert manifolds, we can not necessarily define an internal product on \( \mathcal{U}^*(Y) \) unless \( Y \) is a finite dimensional manifold. However, if admissible complex orientable Fredholm map \( f_1 \times f_2 : X_1 \times X_2 \to Y \times Y \) is transverse to the diagonal imbedding \( \Delta : Y \to Y \times Y \), then we do have an internal (cup) product

\[ [X_1, f_1] \cup [X_2, f_2] = \Delta^*[X_1 \times X_2, f_1 \times f_2]. \]

If \( Y \) is finite dimensional, then by Haefliger and Thom’s Transversality Theorem in [23], every complex orientable map to \( Y \) has a transverse approximation, hence the cup product \( \cup \) induces a graded ring structure on \( \mathcal{U}^*(Y) \). The unit element \( 1 \) is represented by the identity map \( Y \to Y \) with index \( 0 \). However F. Quinn [23] proved the generalization of Thom’s Transversality Theorem for separable Hilbert manifolds using smooth transversal approximations of Sard functions in fine topology.

By Quin’s Transversality Theorem, a smooth map (even continuous map) \( g : Z \to Y \) can be deformed to a smooth map \( g' : Z \to Y \) by a small correction until it is transverse to an admissible complex orientable map \( f : X \to Y \). It is obvious that they are homotopic each other. By definition of Cobordism and Proposition 1.10, the cobordism functor is contravariant for any smooth map between separable Hilbert manifolds.
Theorem 1.18. Let \( f : X \to Y \) be an admissible complex oriented map and let \( g : Z \to Y \) be a smooth (may be continuous) map. Then the cobordism class of the pull-back \( Z \coprod X \to Z \) depends only on the cobordism class of \( f \), hence there is a map \( g^* : \mathcal{U}^d(Y) \to \mathcal{U}^d(Z) \) given by

\[
g^*[X, f] = g''*[X, f] = [Z \coprod X, g''(f)],
\]

where \( g' \) is a smooth \( \varepsilon \)-approximation of \( g \) which is transverse to \( f \). Moreover, \( g^* \) depends only on the homotopy class of \( g \).

Let turn back the interior(cup) products in \( \mathcal{U}^* \). Given cobordism classes \([X, f_1] \in \mathcal{U}^d_1(Y_1)\) and \([X, f_2] \in \mathcal{U}^d_2(Y_2)\), their external product is

\[
[X, f_1] \times [X, f_2] = [X \times X, f_1 \times f_2] \in \mathcal{U}^{d_1 + d_2}(Y_1 \times Y_2).
\]

If admissible complex orientable Fredholm map \( f_1 \times f_2 \) is transverse to the diagonal imbedding \( \Delta : Y \to Y \times Y \), then we do have an internal (cup) product

\[
[X, f_1] \cup [X, f_2] = \Delta^*[X \times X, f_1 \times f_2].
\]

If the diagonal imbedding \( \Delta : Y \to Y \times Y \) is not transverse to smooth proper Fredholm map \( f_1 \times f_2 : X_1 \times X_2 \to \), by Quim’s transversality Theorem, we can find a smooth \( \varepsilon \)-approximation \( \Delta' \) of \( \Delta \) which is transverse to \( f_1 \times f_2 \). Then

Theorem 1.19. If \([X, f_1] \in \mathcal{U}^d(Y_1)\) and \([X, f_2] \in \mathcal{U}^d_2(Y_2)\), internal(cup) product

\[
[X, f_1] \cup [X, f_2] = \Delta^*[X \times X, f_1 \times f_2] = \Delta'^*[X \times X, f_1 \times f_2] \in \mathcal{U}^{d_1 + d_2}(Y)
\]

where \( \Delta' \) is a smooth \( \varepsilon \)-approximation of \( \Delta \) which is transverse to \( f_1 \times f_2 \).

The cup product is well-defined and associative.

Then, \( \mathcal{U}^*(\cdot) \) is a multiplicative contravariant functor for smooth functions on the separable Hilbert manifolds.

We define the Euler class of a finite dimensional complex vector bundle on a separable Hilbert manifold. Note that Theorem 1.18 implies that this Euler class is a well-defined invariant of the bundle \( \pi \).

Definition 1.20. Let \( \pi : \xi \to B \) be a finite dimensional complex vector bundle of dimension \( d \) on a separable Hilbert manifold \( B \) with zero-section \( i : B \to \xi \). The \( \mathcal{U} \)-theory Euler class of \( \xi \) is the element

\[
\chi(\pi) = i^*i_*(1) \in \mathcal{U}^{2d}(B).
\]

Let \( \pi : \xi \to X \) be a finite dimensional complex vector bundle of dimension \( d \) on a separable Hilbert manifold \( X \) with zero-section \( i : X \to \xi \).

Now we need a useful lemma from [29].

Lemma 1.21. A smooth split submanifold of a smooth separable Hilbert manifold has a smooth tubular neighborhood.

The map \( i \) is proper so that we have the Gysin map

\[
i_* : \mathcal{U}^j(X) \to \mathcal{U}^{j+2d}(\xi, \xi - U)
\]

where \( U \) is a smooth neighborhood of the zero section.

The map \( \pi \) is not proper. However if \( U \) is contained in a tube \( U^r \) of finite radius \( r \), then \( \pi|_{\bar{U}} \) is proper and we can define

\[
\pi_* : \mathcal{U}^{j+2d}(\xi, \xi - U) \to \mathcal{U}^j(X).
\]

Since \( \pi \) is proper we have \( \pi_*i_* = \text{Id} \). The composite map \( \pi^* \) is homotopic to \( \text{Id}_\xi \). If \( U = U^c \) is itself a tube, the homotopy moves on \( U \) and we have Thom isomorphism

\[
\mathcal{U}^{j+2d}(\xi, \xi - U) \cong \mathcal{U}^j(X).
\]
Let $M_d$ be a closed oriented $d$-dimensional smooth manifold, and let $LM = C^\infty(S^1, M)$ be the space of smooth loops in $M$. In [3], Chas and Sullivan described an intersection product on the homology $H_*(LM)$, having total degree $-d$, \[ o : H_q(LM) \otimes H_r(LM) \to H_{q+r-d}(LM). \]

In [3], R. Cohen and J. Jones described a realization of the Chas-Sullivan loop product in terms of a ring spectrum structure on the Thom spectrum of a certain virtual bundle over the loop space. We want to extend this product to the $U^*$-theory.

Let $M_d$ be a closed complex $d$-dimensional smooth manifold, and let $LM = C^\infty(S^1, M)$ be the space of smooth loops in $M$. Let consider the standard parameterization of the circle by the unit interval, \[ \exp : [0, 1] \to S^1 \text{ defined by } \exp(t) = e^{2\pi i t}. \] With respect to this parameterization we can regard a loop $\gamma \in LM$ as a map $\gamma : [0, 1] \to M$ with $\gamma(0) = \gamma(1)$. Let consider the evaluation map $ev : LM \to M$ by $\gamma \to \gamma(1)$.

Let $\iota : M \to \mathbb{C}^{N+d}$ be a fixed smooth imbedding of $M$ into codimension $N$ Unitary space. Let $\iota^* \nu^N \to M$ be the $2N$-dimensional normal bundle. Let $Th(\iota^* \nu^N)$ be the Thom space of this bundle. We know that $Th(\iota^* \nu^N)$ is Spanier-Whitehead dual to $M_+$ where $M_+$ denotes $M$ with a disjoint basepoint. Let $M^{-TM}$ be the spectrum given by desuspending the Thom space, \[ M^{-TM} = \sum_{-2(N+d)} \Th(\iota^* \nu^N). \]

We have the following spectra maps \[ S^0 \to M_+ \wedge M^{-TM} \quad \text{and} \quad M_+ \wedge M^{-TM} \to S^0, \] where $M^{-TM}$ is $S$-dual of $M_+$. These maps induce an equivalence with the function spectrum $M^{-TM} \simeq \Map(M_+, S^0)$. Since $U^*(X) \simeq MU^*(X)$ for finite dimensional manifolds $X$ and the contravariant cobordism theory $MU^*$ is dual to the covariant bordism theory $MU_*$, we have the following isomorphisms \[ U^q(M_+) \cong U_{-q}(M^{-TM}) \]
\[ U^{-q}(M^{-TM}) \cong U_q(M_+) \]
for all $q \in \mathbb{Z}$. These duality isomorphisms are induced by the compositions \[ U^{-q}(M^{-TM}) \xrightarrow{\tau} U^{-q+2d}(M_+) \xrightarrow{\rho} U_q(M_+) \]
where $\tau$ is the Thom isomorphism, and $\rho$ is the Poincar’e duality isomorphism for compact manifolds.

By duality, the diagonal map $\Delta : M \to M \times M$ induces a map of spectra \[ \Delta^* : M^{-TM} \wedge M^{-TM} \to M^{-TM} \]
that makes $M^{-TM}$ into a ring spectrum with unit $S^0 \to M^{-TM}$.

Let $Th(\iota^* \nu^N)$ be the Thom space of the pull back bundle $\iota^* \nu^N \to LM$ where $LM = C^\infty(S^1, M)$ is smooth manifold over separable Hilbert space $\mathbb{H}$. Let define the spectrum \[ LM^{-TM} = \sum_{-2(N+d)} \Th(\iota^* \nu^N). \]

The representing dual manifold $X$ of the spectrum $LM^{-TM}$ is also a smooth separable Hilbert manifold, e.g. $LM^{-TM} = [X, f] \in U^{2d}(LM)$.

Now we will give the main theorem of this work.

**Theorem 2.1.** The spectrum $LM^{-TM}$ is a homotopy commutative ring spectrum with unit, whose multiplication $\mu : LM^{-TM} \wedge LM^{-TM} \to LM^{-TM}$ satisfies the following properties.

1. The evaluation map $ev : LM^{-TM} \to M^{-TM}$ is a map of ring spectra.

2. There is a map of ring spectra $\rho : LM^{-TM} \to \bigoplus_{\infty} \Omega M$ where the target is the suspension spectrum of the based loop space with a disjoint basepoint. Its ring structure is induced by the usual product on the based loop space. In bordism the map $\rho_*$ is given by the composition \[ \rho_* : MU_q(LM^{-TM}) \xrightarrow{\tau} MU_{q+2d}(LM) \xrightarrow{\delta} MU_q(\Omega M) \]
where \( \tau \) is the Thom isomorphism and \( i \) takes a bordism class with dimension \((q + 2d)\) and by intersecting with the based loop \( \Omega M \) as a codimension \(2d\), e.g., \( i = i^* : MU_*(LM) \to MU_{-2d}(\Omega M) \) is an induced homomorphism from the embedding \( i : \Omega M \to LM \).

3. The ring structure is compatible with the Chas-Sullivan loop product in the sense that the following diagrams commute.

\[
\begin{align*}
U^q(LM^{-TM}) \times U^r(LM^{-TM}) & \xrightarrow{\text{ext}} U^{q+r}(LM^{-TM} \wedge LM^{-TM}) \xrightarrow{\Delta^*} U^{q+r}(LM^{-TM}) \\
\cong & (u_* \downarrow) \quad \text{and} \quad \cong (u_* \downarrow)
\end{align*}
\]

and

\[
\begin{align*}
U_q(LM^{-TM}) \times U_r(LM^{-TM}) & \xrightarrow{\text{ext}} U_{q+r}(LM^{-TM} \wedge LM^{-TM}) \xrightarrow{\mu_*} U_{q+r}(LM^{-TM}) \\
\cong & (u_* \downarrow) \quad \text{and} \quad \cong (u_* \downarrow)
\end{align*}
\]

where \( \text{ext} \) is the external product, \( u_* \) is the Thom isomorphism, \( \Delta : LM^{-TM} \times LM^{-TM} \to LM^{-TM} \) is the diagonal map which is adjoint to the multiplication map

\[
\mu : LM^{-TM} \wedge LM^{-TM} \to LM^{-TM}
\]

and \( \circ \) is the Chas-Sullivan loop product in cobordism.

**Proof.** The proof was done in [4] by essentially the same argument for homology but their proof had a fundamental mistake. In this proof we will sort out this mistake and we will do the modification for cobordism and bordism theories.

Let \( \Delta : M \to M \times M \) be the diagonal embedding of closed oriented manifold \( M \). The normal bundle is isomorphic to the tangent bundle, \( \nu_\Delta \cong TM \) so that the Pontrjagin-Thom map is a complex orientable map \( \tau : M \times M \to M^{TM} \) with index zero. So we have Gysin map in cobordism,

\[
MU_*(M \times M) \xrightarrow{\tau} MU_*(M^{TM}) \xrightarrow{\cong} MU_{s-2d}(M)
\]

which is the transversal intersection product.

Here we will apply the Pontrjagin-Thom construction to the diagonal embedding \( \Delta : M \to M \times M \) using the canonical bundle \(-TM \times -TM\) over \( M \times M \). We get a map of Thom spectra

\[
\tau : (M \times M)^{-TM \times -TM} \to M^{TM \oplus \Delta^*(-TM \times -TM)}
\]

or,

\[
\tau : M^{-TM} \wedge M^{-TM} \to M^{-TM}.
\]

The details about the Pontrjagin-Thom construction can be found in [4].

To construct the ring spectrum product

\[
\mu : LM^{-TM} \wedge LM^{-TM} \to LM^{-TM},
\]

they pull back the structure \( \tau \) over the loop space \( LM \).

For this, they define \( LM \times_M LM \) which is fiber product in the following diagram

\[
\begin{array}{ccc}
LM \times_M LM & \xrightarrow{\Delta} & LM \times LM \\
\downarrow \text{ev} & & \downarrow \text{ev} \\
M & \xrightarrow{\Delta} & M \times M.
\end{array}
\]

They note that \( LM \times_M LM \) is a codimension 2d submanifold of the infinite dimensional manifold \( LM \times LM \) and it is equal to

\[\{(\alpha, \beta) \in LM \times LM : \alpha(0) = \beta(0)\}\].

Since \( \text{ev} : LM \to M \) is a submersion, the fiber product corresponds to transversal intersection of maps, so \( \Delta \) is pull back of the diagonal map \( \Delta \) under the submersion map \( \text{ev} \times \text{ev} : LM \times LM \to M \times M \). The induced map \( \Delta \) is a Fredholm map with index 2d and consequently \( LM \times_M LM \) is a codimension 2d smooth submanifold of the infinite dimensional manifold \( LM \times LM \).
Also they note that there is a natural map \( \gamma : LM \times_M LM \to LM \) defined by first applying \( \alpha \) and then \( \beta \). That is,
\[
\gamma : LM \times_M LM \to LM \quad \text{by} \quad \gamma((\alpha, \beta)) = \alpha \ast \beta,
\]
where
\[
(\alpha \ast \beta)(t) = \begin{cases} 
\alpha(2t) & \text{if } 0 \leq t \leq \frac{1}{2} \\
\beta(2t - 1) & \text{if } \frac{1}{2} \leq t \leq 1.
\end{cases}
\]
But the image of \( \gamma \) is not smooth loop as they have described it above. However there is a standard way to modify the definition of \( \gamma \) so that the target is smooth. The resolution is in the parametrization of the loop. With continuous loops one just defines:
\[
(\alpha \ast \beta)(t) = \begin{cases} 
\alpha(2t) & \text{if } 0 \leq t \leq \frac{1}{2} \\
\beta(2t - 1) & \text{if } \frac{1}{2} \leq t \leq 1.
\end{cases}
\]
but this may be not be smooth at \( \frac{1}{2} \) or at 1. To modify this we need a bijective smooth function from \([0, 1]\) to \([0, 1]\) which has all derivatives zero at 0 and 1 since we will use this to reparameterize the two loops. Then we can patch them together without losing smoothness.

To write this special parametrization, we define a map \( \varphi : [0, 1] \to [0, 1] \) by
\[
\varphi(t) = \frac{1}{c} \int_0^t \exp\left(-\frac{s}{(c - s)^2}ight) \, ds 
\]
where
\[
c = \int_0^1 \exp\left(-\frac{s}{(c - s)^2}\right) \, ds.
\]
It is a smooth bijective function from \([0, 1]\) to \([0, 1]\) which has all derivatives zero at 0 and 1. Now we can define a map
\[
\Phi : LM \to LM 
\]
by
\[
\Phi(\alpha)(t) = \alpha(\varphi(t)) = \alpha \circ \varphi(t).
\]
It is a smooth map and let \( L_{res}M = \Phi(LM) \). \( L_{res}M \) is a smooth submanifold of \( LM \) and it is smooth homotopic retraction of \( LM \). Similarly \( L_{res}M \times_M L_{res}M \) can be constructed. It is also smooth homotopic retraction of \( LM \times_M LM \). Then we can define a new version of the map \( \gamma : LM \times_M LM \to LM \) as the composition of the following maps
\[
\gamma : LM \times_M LM \xrightarrow{\Phi \times \Phi} L_{res}M \times_M L_{res}M \xrightarrow{\alpha} L_{res}M \subseteq LM.
\]
The defined new version of \( \gamma \) is smooth. In the proof of [1], we can use \( L_{res}M \times_M L_{res}M \) instead of \( LM \times_M LM \) because it is smooth homotopic retraction of \( LM \times LM \).

If we restrict to the product of the based loop spaces, \( \Omega_{res}M \times \Omega_{res}M \subseteq L_{res}M \times_M L_{res}M \), then \( \gamma \) is just the \( H \)-space product on the based loop space, \( \Omega_{res}M \times \Omega_{res}M \to \Omega_{res}M \).

The embedding \( \Xi : L_{res}M \times_M L_{res}M \to L_{res}M \times L_{res}M \) has a tubular neighborhood \( \nu(\Xi) \) defined to be the inverse image of the tubular neighborhood of the diagonal map \( \Delta : M \to M \times M : \)
\[
\nu(\Xi) = ev^{-1}(\nu(\Delta)).
\]
Hence there is a Pontjagin-Thom construction
\[
\tau : L_{res}M \times L_{res}M \to (L_{res}M \times_M L_{res}M)^{ev^*(TM)}.
\]
The map \( \tau \) is a smooth Fredholm map. By the Pontrjagin-Thom construction, we have the following commutative diagram
\[
\begin{array}{ccc}
L_{res}M \times L_{res}M & \xrightarrow{\tau} & (L_{res}M \times_M L_{res}M)^{TM} \\
\downarrow ev & & \downarrow ev \\
M \times M & \xrightarrow{\tau} & M^{TM}.
\end{array}
\]
In bordism, we have
\[
\iota : U_s(L_{res} \times L_{res}M) \to U_{s-2d}(L_{res} \times_M L_{res}M)
\]
where \( \iota \) takes a bordism class with dimension \( n \) and intersects with the submanifold \( L_{res} \times_M L_{res}M \) as a codimension \( 2d \), i.e. pull backs by the inclusion \( L_{res} \times_M L_{res}M \to L_{res} \times L_{res}M \).

By the following commutative diagram
\[
\begin{array}{ccc}
L_{res}M \times_M L_{res}M & \xrightarrow{\gamma} & L_{res}M \\
\downarrow ev & & \downarrow ev \\
M & \xrightarrow{=} & M,
\end{array}
\]
we have an induced map of bundles $\gamma : \text{ev}^*(TM) \to \text{ev}^*(TM)$, hence we have a map of spectra

$$(L_{\text{res}}M \times_M L_{\text{res}}M)^{TM} \xrightarrow{\gamma} (L_{\text{res}}M)^{TM}.$$ 

Then we will get the following composition

$$\tilde{\mu} : L_{\text{res}}M \times L_{\text{res}}M \xrightarrow{\gamma} (L_{\text{res}}M \times_M L_{\text{res}}M)^{TM} \xrightarrow{\gamma} LM^{TM}.$$ 

In bordism, the homomorphism

$$U_*(LM \times LM) \xrightarrow{\tilde{\mu}} U_*(LM^{TM}) \xrightarrow{\gamma} U_{* -2d}(LM)$$

takes a bordism class in $LM \times LM$, intersects in with the codimension $d$ submanifold $L_{\text{res}}M \times_M L_{\text{res}}M$, maps it via $\gamma$ to $LM$. This is the definition of Chas-Sullivan product $U_*(LM)$.

Using the diagonal embedding $LM \to LM \times LM$, we can perform the Pontrjagin-Thom construction when we pull back the virtual bundle $-TM \times -TM$ over $LM \times LM$. Then we obtain

$$\tau : LM^{-TM} \wedge LM^{-TM} \to (L_{\text{res}}M \times_M L_{\text{res}}M)^{TM \oplus -2TM} = (L_{\text{res}}M \times_M L_{\text{res}}M)^{-TM}.$$ 

Then we can define the ring structure on the Thom spectrum to be the composition

$$\mu : LM^{-TM} \wedge LM^{-TM} \xrightarrow{\gamma} (L_{\text{res}}M \times_M L_{\text{res}}M)^{-TM} \xrightarrow{\gamma} LM^{-TM}.$$ 

In [4], They show that $\tilde{\mu}$ is associative.

In bordism, by Thom isomorphism $\mu_*$ induces the same homomorphism as $\tilde{\mu}_*$, so we have the following diagram commutes.

$$U_{q -4d}(LM^{-TM} \wedge LM^{-TM}) \xrightarrow{\mu_*} U_{q -4d}(LM^{-TM})$$

where $\circ : U_q(LM \times LM) \to U_{q -2d}(LM)$ is the Chas-Sullivan product. In complex cobordism, we define the Chas-Sullivan product by the following commutative diagram

$$U^{q -4d}(LM^{-TM} \wedge LM^{-TM}) \xrightarrow{\Delta_*} U^{q -4d}(LM^{-TM})$$

In [4], They show that $\rho : LM^{-TM} \to \sum_{\infty} (\Omega M))$ is a map of ring spectra. In bordism the map $\rho_*$ is given by the composition

$$\rho_* : MU_q(LM^{-TM}) \xrightarrow{\gamma} MU_{q +2d}(LM) \xrightarrow{\gamma} MU_q(\Omega M)$$

where $\tau$ is the Thom isomorphism and $\tau$ takes a bordism class with dimension $(q +2d)$ and by intersecting with the based loop $\Omega M$ as a codimension $2d$ e.g. $\iota = \iota^* : MU_*(LM) \to MU_{* -2d}(\Omega M)$ is an induced homomorphism from the embedding $i : \Omega M \to LM$.

□

References

[1] A. J. Baker & C. Ozel, Complex cobordism of Hilbert manifolds with some applications to flag varieties, Contemporary Mathematics 254 (2000), 1-19.
[2] R. Bonic & J. Frampton, Smooth functions on Banach manifolds, J. Math. Mech. 15 (1966), 877-898.
[3] M. Chas & D. Sullivan, String topology, preprint: math. GT/9911159, 1999.
[4] R. L. Cohen & J.D.S. Jones, A homotopy theoretic realization of string topology, preprint: math. GT/0107187, 2001.
[5] J. B. Conway, A Course in Functional Analysis, Springer-Verlag (1984).
[6] A. Dold, Geometric cobordism and the fixed point transfer, Lecture Notes in Math. Springer-Verlag 1976
[7] A. Dold, Partitions of unity in the theory of fibrations, Ann. of Math. 78 (1963), 223-255.
[8] E. Dyer, Cohomology Theories, Benjamin (1969).
[9] J. Eells & K. D. Elworthy, On the differential topology of Hilbert manifolds, Global Analysis: Proc. Symp. Pure Math. 15 (1970), 41-44.
[10] J. Eells & J. McAlpin, An approximate Morse-Sard theorem, J. Math. Mech. 17 (1968), 1055-1064.
[11] K. Jänich, Topology, Springer-Verlag (1981).
[12] H. N. Kuiper, The homotopy type of the unitary group of Hilbert space, Topology 3 (1965), 19-30.
[13] S. Lang, Differential Manifolds, Springer-Verlag (1985).
[14] P. W. Michor, Manifolds of Differentiable Mappings, Shiva Publishing Limited (1980).
[15] J. W. Milnor & J. D. Stasheff, Characteristic Classes, Princeton University Press (1974).
[16] J. J. Morava, Fredholm maps and Gysin homomorphisms, Global Analysis: Proc. Symp. Pure Math. 15 (1970), 135-156.

[17] C. Ozel, On the Complex Cobordism of Flag Varieties Associated to Loop Groups, PhD Thesis, University of Glasgow (1998).

[18] C. Ozel, On the cohomology ring of the infinite flag manifold \( LG/T \), Turkish Journal of Mathematics 22 (1998), 415-448.

[19] C. Ozel, On Fredholm index, transversal approximations and Quillen’s geometric complex cobordism of Hilbert manifolds with some applications to flag varieties of loop groups, submitted.

[20] R. S. Palais, Lusternik-Schnirelman theory on Banach manifolds, Topology 5 (1966), 115-132.

[21] A. Pressley & G. Segal, Loop Groups, Oxford University Press (1986).

[22] D. G. Quillen, Elementary proofs of some results of cobordism theory using Steenrod operations, Adv. in Math. 7 (1971), 29-56.

[23] F. Quinn, Transversal approximation on Banach manifolds, Proc. Symp. Pure Math. 15 (1970), 213-222.

[24] R. Stong, Notes on the Coborsism Theory, Princeton University Press (1968).

[25] R. Thom, Quelques propriétés des variétés différentiables, Comm. Math. Helv. 28 (1954), 17-86.

[26] A. J. Tromba, Some theorems on Fredholm maps, Proc. of the Amer. Math. Soc. 34 (1972), 578-585.

[27] E. Zeidler, Applied Functional Analysis, Main Principles and Their Applications, Springer-Verlag (1995).

AIBU GOLKOY KAMPUSU, BOLU 14280, TURKEY.

E-mail address: cenap@ibu.edu.tr