Robust parsimonious search with scale-free stochastic resetting

Łukasz Kuśmierz and Taro Toyoizumi

Laboratory for Neural Computation and Adaptation, RIKEN Center for Brain Science, 2-1 Hirosawa, Wako, Saitama 351-0198, Japan

A new model of search based on stochastic resetting is introduced, wherein rate of resets depends explicitly on time elapsed since the beginning of the process. It is shown that rate inversely proportional to time leads to paradoxical diffusion which mixes self-similarity and linear growth of the mean square displacement with non-locality and non-Gaussian propagator. It is argued that such resetting protocol offers a general and efficient search-boosting method that does not need to be optimized with respect to the scale of the underlying search problem (e.g., distance to the goal) and is not very sensitive to other search parameters. Additionally, subdiffusive and superdiffusive regimes are demonstrated with different rate functions.

Any search is intrinsically governed by randomness: the need to perform it underscores the ignorance of the searching agent and its inability to predict the outcome. How can the efficiency of a search strategy be boosted without acquiring any information beforehand? It is often the case that the longer the search lasts, the longer the expected remaining time to find the target. In these cases restarting the process anew may be beneficial as it lowers the expected completion time. More recently, Evans and Majumdar introduced a model with time-homogeneous stochastic resetting (i.e., with exponentially distributed waiting times between the resets) in tandem with diffusion [9]. They showed that one-dimensional [9] and multidimensional diffusion with resets exhibit finite mean first hitting times (MFHTs) which can be optimized with respect to resetting rate \( r \). Moreover, resets lead to non-equilibrium steady-states with a combination of local and non-local currents [9,11,12]. The simplicity and nontrivial behavior of these systems has sparked the interest of the physics community, with a considerable amount of research focusing on modifying the model to include nontrivial boundary conditions [13–15], external potentials [16], anomalous transport [11,17,21], drift [22,23], and non-exponential distributions of waiting times between resets [12,23,24]. Resets were also studied in the context of stochastic energetics [25], enzymatic reactions [24], fluctuating interfaces [27], and power-law distributions in non-equilibrium systems [28].

Non-exponential waiting times between the resets can be generated by time-dependent resetting rates [23,24]. In models introduced in these previous works, rate depends on the time elapsed since the last reset. Here we introduce a new model with resetting rate \( r(t) \) that depends on the time elapsed since the start of the process. As we will show, this modification dramatically affects the behavior of the process. If \( r(t) \) decays with time, the process does not converge to any stationary distribution with variance growing indefinitely. Moreover, a properly chosen \( r(t) \) provides an efficient and robust mechanism to boost the efficiency of a search.

We start from one-dimensional diffusion as the underlying search process, which will help to build the intuition. At the end we present a more general framework. Since the process of diffusion with memoryless resetting is Markovian, it is fully characterized by the initial position distribution and the propagator, \( \rho(x,t|x_0,t_0) \), i.e., the probability density function of \( x \) of a particle that at time \( t_0 \) is at \( x_0 \). The free propagator solves the following partial differential equation in the absence of absorbing or reflecting boundaries

\[
\partial_t \rho(x,t|x_0,t_1) = D \partial_x^2 \rho(x,t|x_0,t_1) - r(t)\rho(x,t|x_0,t_1) + r(t)\delta(x-x_0) \tag{1}
\]

where \( \partial_t \) denotes the partial derivative with respect to \( t \) and \( D \) is the diffusion coefficient. This equation describes the probability density function of a particle which starts at time \( t = t_0 \) starts its movement from \( x = x_0 \) and diffuses till the resetting event, which brings the particle back to the initial position \( x_0 \) [29]. It then continues to diffuse until the next reset. Such process has been analyzed extensively in the case of a constant rate \( r(t) = r \), with the corresponding resets described by the homogeneous Poisson point process. It has been shown that this process attains a non-equilibrium stationary state and that there exists an optimal rate \( r^* \propto x_0^{-2} \) at which the mean first hitting time (MFHT) to a single target at the origin is minimized.

In this Letter we limit our attention to the special case of

\[
r(t) = \frac{\alpha}{t} \tag{2}
\]

The only parameter \( \alpha \) is dimensionless—this form of resetting does not introduce any timescale. For this reason, we refer to (2) as a scale-free resetting. Resets described by (2) are generated by the inhomogeneous Poisson point process with an average intensity (expected number of events) within a time period \([t_0,t_1]\) given by

\[
N(t_0,t_1) = \alpha \ln \frac{t_1}{t_0}. \tag{3}
\]

As evident from (1), the process (1) with (2) preserves the self-similarity of the pure Brownian motion, i.e., the re-scaling \( x \to cx \) and \( t \to c^2t \) does not change any observables (see Fig. 1). We can therefore expect that the mean square displacement (MSD) scales linearly with time. Indeed, straightforward calculations lead to

\[
\langle (x(t) - x_0)^2 \rangle = \int_{-\infty}^{\infty} dxx^2 \rho(x,t|x_0,0) = \frac{2Dt}{1+\alpha}. \tag{4}
\]
Although the self-similarity and associated linear time-dependence of the MSD bear a strong resemblance to free diffusion, resets vastly modify the dynamics of the system which can be seen in other statistics. In particular, the propagator is non-Gaussian and has a cusp at $x = x_0$. Its Fourier transform reads

$$\rho(k, t|x_0, 0) = \alpha e^{i k x_0} t^{-\alpha} e^{-Dk^2 t} \int_0^t d\tau \tau^{-\alpha - 1} e^{Dk^2 \tau}$$  \hspace{1cm} (5)$$

and takes a particularly simple form for $\alpha = 1$

$$\rho(k, t|x_0, 0) = e^{i k x_0} \frac{1 - e^{-Dk^2 t}}{Dk^2 t}.$$  \hspace{1cm} (6)$$

Another striking difference with respect to free diffusion can be observed in the statistics of the first hitting times, a problem which we will focus on in the following. Let us introduce a function $T_\alpha(x_0)$, which denotes the MFHT to a target at the origin as a function of the initial position and the dimensionless parameter $\alpha$. Due to the self-similarity we can expect that $T_\alpha(x_0) \propto x_0^2$ if the MFHT is finite. In order to obtain the statistics of the first hitting times we write down and solve the following partial differential equation with a perfectly absorbing sink at $x = 0$

$$\partial_t f(x, t|x_0) = D \partial^2_x f(x, t|x_0) - \frac{\alpha}{t} f(x, t|x_0) + \frac{\alpha}{t} S_\alpha(t) \delta(x - x_0) - p_\alpha(t) \delta(x),$$  \hspace{1cm} (7)$$

where $p_\alpha(t)$ denotes the probability distribution function of the first hitting times, $S_\alpha(t)$ the corresponding survival times $S_\alpha(t) = \int_0^\infty d\tau p_\alpha(\tau)$, and $f(x, t|x_0)$ the unnormalized density function with $f(x, t = 0|x_0) = \delta(x - x_0)$, $f(x = 0, t|x_0) = 0$, and $\int_0^\infty dxf(x, t|x_0) = S_\alpha(t)$. Notice that the functions $S_\alpha(t)$ and $p_\alpha(t)$ depend on $x_0$, which we do not write explicitly for brevity. We rewrite (7) in the Fourier-Laplace space (which we denote by changing the arguments $x \rightarrow k$ and $t \rightarrow s$)

$$\left(s + Dk^2\right) \partial_s f(k, s|x_0) + (1 - \alpha) f(k, s|x_0) = h(k, s),$$  \hspace{1cm} (8)$$

with

$$h(k, s) \equiv -\alpha e^{i k x_0} S_\alpha(s) - \partial_s p_\alpha(s).$$  \hspace{1cm} (9)$$

in an infinitesimal time. This set of equations simplifies significantly for $\alpha = 1$. In this case the formal solution of (8) reads

$$f(k, s|x_0) = \int_0^s du \frac{h(k, u)}{u + Dk^2}.$$  \hspace{1cm} (10)$$

From the absorbing boundary condition we have

$$f(x = 0, s|x_0) = \frac{1}{2} \int_{-\infty}^{\infty} dk f(k, s|x_0) = 0,$$  \hspace{1cm} (11)$$

therefore we integrate (10) with respect to $k$ and arrive at the differential equation

$$\partial_s p_1(s) + \exp(-\sqrt{s \tau_{\text{diff}}}) S_1(s) = 0,$$  \hspace{1cm} (12)$$

where $\tau_{\text{diff}} \equiv x_0^2/D$. Eq. (12) can be solved by employing the identity

$$S_\alpha(s) = \frac{1 - p_\alpha(s)}{s}$$  \hspace{1cm} (13)$$

and its solution takes a simple form

$$S_1(s) = s^{-1} \exp\left(-\int_s^{\infty} du u^{-1} e^{-\sqrt{\tau_{\text{diff}}}}\right).$$  \hspace{1cm} (14)$$

The MFHT can be calculated as $\lim_{s \rightarrow 0} S_1(s)$ and reads

$$T_1(x_0) = e^{\gamma x_0^2 / D} \approx 3.17 x_0^2 / D,$$  \hspace{1cm} (15)$$

where $\gamma \approx 0.5772$ stands for the Euler-Mascheroni constant. As it turns out, the mean time to reach a single target is finite and, as expected from the dimensionality analysis, scales quadratically with the initial distance to the target. Although we were unable to obtain analytical results for general $\alpha \neq 1$, our numerical simulations show (cf. Fig. 2) that the MFHT is finite for any $\alpha$ and attains its minimum value of $T_\alpha(x_0) / \tau_{\text{diff}} \approx 1.97$ at $\alpha^* \approx 3.5$. This result is quite remarkable: the scale-free resetting provides a technique for an efficient search without employing any knowledge about the underlying process. In order to illustrate this it is instructive to compare diffusion with scale-free resetting to diffusion with constant-rate resetting. The optimal MFHT of the latter is given by $T_{\text{const}}(x_0, r^\ast) \approx 1.54 x_0^2 / D$ with $r^\ast \propto x_0^2$.
Although $\mathbb{T}_{\text{const}}(x_0, r') < \mathbb{T}_r(x_0)$, the essential difference lies in the fact that, while the optimal scale-free search parameter $\alpha^*$ does not involve any knowledge about the distance to the target, the optimal constant rate $r'$ depends strongly on $x_0$, see Fig. 2. Such sensitive strategy has to be adapted to the specific search scenario. However, this may be hard, or even impossible, if the target’s position is unknown. Consider a scenario of a single, immobile target placed at a random position and assume the location of the target does not change between the resets, but is drawn independently for separate trials [18,52,53]. This is equivalent to the resetting (and initial) position being drawn from a distribution $\rho_X(x_0)$, i.e., the MFHT in this case can be calculated as $(\mathbb{T}(x_0))_{\alpha^*-\rho_X}$. Distribution $\rho_X$ may represent the real variability of the environment or ignorance of the searching agent. As an important special case, let us take the Laplace distribution $\rho_X(x_0) = \exp(-|x_0|/\lambda)/(2\lambda)$, which maximizes entropy for a given average distance to the target $\lambda = \langle |x_0| \rangle$. In this case, the MFHT of diffusion with constant resetting rate $r$ reads

$$\mathbb{T}_{\text{const}}(x_0, r) = \frac{\lambda^2}{D}\frac{1}{\sqrt{r^2 - r^2}}$$

with the minimum value of $4\lambda^2/D$ at $1/r^* = 4\lambda^2$. Since the variance of the Laplace distribution is equal to $2\lambda^2$, the MFHT of diffusion with scale-free resetting is given by

$$\langle \mathbb{T}(x_0) \rangle_{\alpha^*-\rho_X} = f(\alpha)\frac{2\lambda^2}{D},$$

where $f(\alpha) = \mathbb{T}_a(x_0)/\tau_{\text{diff}}$ is the same prefactor as in the case of constant $x_0$ (Fig. 2) with the minimum $f(\alpha^*) \approx 1.97$ at $\alpha^* \approx 3.5$—in this case a properly chosen $\alpha$ leads to a (slightly) better efficiency than any constant resetting rate. More importantly, in that case the optimal choice of $\alpha$ does not depend on $\lambda$, to which the optimal constant rate is quite sensitive. Indeed, if the chosen constant rate is larger than $4\lambda^2 = A^{-2}$, the MFHT diverges, see (16). More broadly, the MFHT of constant resetting search diverges for any distribution of $x_0$ that has tails heavier than exponential, in particular for any power-law distribution. In contrast, the MFHT of diffusion with scale-free resetting remains finite for any distribution of $x_0$ with finite variance and the choice of the optimal parameter $\alpha$ does not depend on any aspect of the distribution of $x_0$.

The flexibility and robustness of the proposed resetting mechanism does not come without a price, as it brings about large fluctuations of the first hitting times. To show this we rewrite the survival probability for $\alpha = 1$ [14] as a function of the upper incomplete gamma function

$$S_1(s) = s^{-1} \exp\left(-\Gamma(0, \sqrt{\tau_{\text{diff}} s})\right),$$

which admits the following expansion around $s = 0$

$$S_1(s) = \tau_{\text{diff}} \exp\left(2 + 2 \sum_{k=1}^{\infty} \frac{(\sqrt{\tau_{\text{diff}} s})^k}{k \cdot k!}\right).$$

From [19] we infer that $p_1(t) \sim 1/t^{1+\nu}$ with $\nu = 3/2$ for $t \gg \tau_{\text{diff}}$, i.e., the probability distribution function of the first hitting times has a fat, power-law tail and its variance diverges. This suggests that resetting protocols entail natural trade-off between insensitivity of the mean search times to the details of the underlying search process and the variability of the search times. Indeed, optimal deterministic and periodic (“sharp”) resetting shows shorter MFHT and lower relative fluctuations than stochastic resetting in a known search scenario [54]. However, sharp resets are at the same time very sensitive to the details of the search process. In the case of diffusion they lead to divergent MFHTs for any Laplace distribution of $x_0$ [52].

Is the robustness of the scale-free resetting protocol limited to diffusive search? We now present more a general framework and use it to illustrate that this is not the case, i.e., we prove that the search time of a process with scale-free resetting scales linearly with the scale of the underlying search problem. We build upon the general renewal approach introduced recently by Pal and Reuveni [54]. Instead of assuming that the underlying search process is described by a simple, one-dimensional diffusion, we consider here any search process, by abstracting out most of its details and considering only its completion time statistics. Let us introduce positive random variables: completion time of a search process without resets $T_0$, waiting time for the next reset $R(t)$, and remaining completion time of a process with scale-free resets $T_a(t, g)$, with the underlying resets-free random completion time being equal to $gT_0$ with $g > 0$ representing the scale of the search problem at hand. For example, in the case of diffusive search we could write $g = \tau_{\text{diff}} = \lambda^2/D$. Since the scale-free resetting protocol is non-stationary, $R$ and $T_a$ depend on the elapsed time $t$. The renewal structure of the process allows us to write the equation

$$T_a(t, g) = T_{R(t) \leq T_0}(R(t) + T_a(t + R(t), g)) + I_{R(t) > g T_0} g T_0,$$

where $I_{R(t) \leq T_0}$ is the indicator function that $R(t) \leq T_0$. The MFHT of the process can then be expressed as

$$\mathbb{T}(x_0, t, g) = \mathbb{T}_{\text{const}}(x_0, r) + \mathbb{T}_r(x_0)\frac{1}{\sqrt{r^2 - r^2}}.$$
where $T'_\alpha$ is an independent realization of the random variable $T_\alpha$ and the indicator function $I_{a\leq b}$ is 1 if $a < b$ and $I_{a \geq b} = 0$ if $a \geq b$. The distribution of $R(t)$ can be calculated as

$$
\psi_R(\tau|t) = \frac{d}{dt} \left( \int_0^t \epsilon(u) \exp \left( - \int_0^\tau \mu(u) \right) \right) = \frac{\alpha t^{\alpha-1}}{(t + \tau)^{\alpha+1}},
$$

which directly shows that the scale-character of resets brings about the identity

$$
R(t) \sim g R(t/g),
$$

where $\sim$ denotes an equality of the distributions of the random variables. Using (22) we re-write (20) in the form

$$
T_\alpha(tg, g) = I_{R(t) \leq T_0} \left( R(t) + T_\alpha(g + gR(t), g) \right) + I_{R(t) > T_0} T_0,
$$

from which we see that

$$
T_\alpha(tg, g) \sim g T_\alpha(t, 1).
$$

Since we are interested in the process starting at $t = 0$ we take the limit of $t \to 0^+$ and obtain $T_\alpha(0^+, g) \sim g T_\alpha(0^+, 1)$, i.e., the completion times of the process under scale-free restarts scale linearly with the magnitude of the completion times of the underlying resets-free process $g$. As a corollary, the mean value, if exists, is proportional to $g$

$$
\bar{T}_\alpha(0^+, g) = g \bar{T}_\alpha(0^+, 1).
$$

which explains why in the case of diffusion $\bar{T}_\alpha \propto x_0^2$ and shows that in general scale-free restart of the form (22) yield the MFHT that is proportional to the scale of the underlying search problem. Due to this feature, the optimal parameter $\alpha$ is insensitive to the distribution of $g$ and, thus, we say that the protocol is parsimonious.

In addition to the presented applications to search problems, a generalization of the proposed resetting mechanism can be applied to model anomalous transport phenomena. In particular, the MSD of (1) with $r(t) = \alpha/t^\mu$ is given by the formula

$$
\left\langle (x(t) - x_0)^2 \right\rangle = 2D \exp \left( -\frac{\alpha}{1 - \mu} t^{1-\mu} \right) \int_0^t dt' \exp \left( \frac{\alpha}{1 - \mu} t'^{1-\mu} \right),
$$

which for $\mu < 1$ exhibits a smooth transition between diffusive behavior with the MSD $\approx 2Dt$ for $t \ll \tau_\alpha$ and subdiffusive behavior with the MSD $\approx 2Dp^\mu/\alpha$ for $t \gg \tau_\alpha$, where $\tau_\alpha = \alpha^{-1/(1-\mu)}$ is the timescale introduced by the power-law resetting. Such subdiffusive search may also be efficient in search problems, although it introduces a timescale and its scaling with $g$ is supralinear. It is therefore not parsimonious, as the optimal $\alpha$ depends on $g$. In the limit of $\mu \to 0$ the standard constant-resetting case is recovered and one can expect that the fluctuations of the first hitting times are smaller for lower values of $\mu$. In the opposite case of $\mu > 1$ the long-term behavior is diffusive, whereas at short times an apparent superdiffusivity with the MSD $\approx 2D t^\mu/\alpha$ is observed. By assuming $r(t) \propto (\log t)^{-1}$ one can also model an ultra-slow diffusion with the MSD $\propto \log t$, a behavior previously uncovered in the strongly non-Markovian random walks with preferential relocations to places visited in the past [33]. One could involve heavy-tails jump distributions leading to a competition between superdiffusivity of the Lévy flights and subdiffusive tendency from resets, similar to the competition observed in the CTRW scheme [34, 38]. Many different variants of a combination of anomalous transport with constant-rate stochastic resetting were previously studied [11, 17, 21] and were shown to exhibit non-trivial features, including first and second order transitions of the optimal search [17, 18, 24, 39] and a non-monotonic behavior of the MSD [21]. For this reason, we expect that a combination of scale-free resets with independent constant-rate resets should also lead to thought-provoking phenomena. The rich family of models with absolute-time-dependent stochastic resetting and the associated trade-off between maximizing robustness and minimizing variability will be the subject of a separate study.

We envision multiple applications of scale-free resetting, especially in optimization problems. In particular, it would be beneficial to study its relation to simulated annealing [40], and its applicability in evolutionary algorithms [3] and deep learning via gradient methods [7, 41]. The robustness of the scale-free resetting protocol should offer additional benefits in the non-stationary setup of curriculum learning [42] and could potentially explain why aging [43, 44] may be useful in learning. Notice that the practical optimization problems are high-dimensional and thus the general renewal framework (20) should prove useful in the construction and analysis of the practical algorithms. Another interesting avenue of possible applications are models of evidence accumulation and decision making, as recently it was shown that stationary resets can modify splitting probabilities [45]. Restarts in this context may be interpreted as useful forgetting.

A number of open problems are left for future studies. In order to assess the efficiency of the proposed scheme, one could compare it to diffusion in time-dependent, scale-free potentials—such comparison in the case of static potentials and resets seems to favor the latter [31, 32]. Moreover, it is important to find conditions under which scale-free resets can lower the expected completion time, similar to the simple criterion in the case of constant-rate resets [26, 40, 47]. A related question is how the optimal $\alpha$ depends on the details of a search problem at hand. Another important issue is the divergence of (22) at $t = 0$ which is infeasible and in practice a short-time cut-off has to be introduced. The optimal cut-off should depend on a cost associated with resets.

[1] Y. Huang, C. Kintala, N. Kolettis, and N. D. Fulton, Software
rejuvenation: Analysis, module and applications, in fcs, p. 0381, IEEE, 1995.
[2] T. Dohi, K. Gõseva-Pop Stojanova, and K. Trivedi, Comput. J. 44, 473 (2001).
[3] A. P. Van Moorsel and K. Wolter, IEEE Trans. Softw. Eng., 547 (2006).
[4] H. Okamura and T. Dohi, Journal of Japan Industrial Management Association 66, 416 (2016).
[5] A. S. Fukunaga, Restart scheduling for genetic algorithms, in International Conference on Parallel Problem Solving from Nature, pp. 357–366, Springer, 1998.
[6] T. Jansen, On the analysis of dynamic restart strategies for evolutionary algorithms, in International conference on parallel problem solving from nature, pp. 33–43, Springer, 2002.
[7] M. A. Luersen and R. Le Riche, Comput. Struct. 82, 2251 (2004).
[8] I. Loschilov and F. Hutter, arXiv:1608.03983 (2016).
[9] M. R. Evans and S. N. Majumdar, Phys. Rev. Lett. 106, 160601 (2011).
[10] M. R. Evans and S. N. Majumdar, J. Phys. A: Math. Theor. 47, 285001 (2014).
[11] V. Méndez and D. Campos, Phys. Rev. E 93, 022106 (2016).
[12] S. Eule and J. J. Metzger, New J. Phys. 18, 033006 (2016).
[13] C. Christou and A. Schadschneider, J. Phys. A: Math. Theor. 48, 285003 (2015).
[14] A. Chatterjee, C. Christou, and A. Schadschneider, Phys. Rev. E 97, 062106 (2018).
[15] A. Pal and V. Prasad, arXiv:1812.03009 (2018).
[16] A. Pal, Phys. Rev. E 91, 012113 (2015).
[17] L. Kušmierz, S. N. Majumdar, S. Sarhapanot, and G. Schehr, Phys. Rev. Lett. 113, 220602 (2014).
[18] L. Kušmierz and E. Gudowska-Nowak, Phys. Rev. E 92, 052127 (2015).
[19] V. Shkilev, Phys. Rev. E 96, 012126 (2017).
[20] M. Montero, A. Mas-Guiu-Dellosas, and J. Villarroel, Eur. Phys. J. 90, 176 (2017).
[21] L. Kušmierz and E. Gudowska-Nowak, arXiv:1812.08489 (2018).
[22] S. Ray, D. Mondal, and S. Reuveni, arXiv:1811.08239 (2018).
[23] A. Nagar and S. Gupta, Phys. Rev. E 93, 060102 (2016).
[24] A. Pal, A. Kundu, and M. R. Evans, J. Phys. A: Math. Theor. 49, 225001 (2016).
[25] J. Fuchs, S. Goldt, and U. Seifert, EPL 113, 60009 (2016).
[26] T. Rotbart, S. Reuveni, and M. Urbakh, Phys. Rev. E 92, 060101 (2015).
[27] S. Gupta, S. N. Majumdar, and G. Schehr, Phys. Rev. Lett. 112, 220601 (2014).
[28] S. C. Manrubia and D. H. Zanette, Phys. Rev. E 59, 4945 (1999).
[29] We are assuming that the initial position corresponds to the resetting position. However, due to the singular behavior of the function $r(t)$ at $t = 0$, the choice of the initial position is inconsequential: If it does not match the resetting position, the probability mass is transferred to the resetting position in an infinitesimal time.
[30] R. Mannella, Phys. Lett. A 254, 257 (1999).
[31] M. R. Evans, S. N. Majumdar, and K. Mallick, J. Phys. A: Math. Theor. 46, 185001 (2013).
[32] L. Kušmierz, M. Bier, and E. Gudowska-Nowak, J. Phys. A: Math. Theor. 50, 185003 (2017).
[33] R. G. Pińský, arXiv:1803.10463 (2018).
[34] A. Pal and S. Reuveni, Phys. Rev. Lett. 118, 030603 (2017).
[35] D. Boyer and C. Solís-Salas, Phys. Rev. Lett. 112, 240601 (2014).
[36] E. W. Montroll and G. H. Weiss, J. Math. Phys. 6, 167 (1965).
[37] R. Metzler and J. Klafter, Phys. Rep. 339, 1 (2000).
[38] M. Magdziarz and A. Weron, Phys. Rev. E 75, 056702 (2007).
[39] D. Campos and V. Méndez, Phys. Rev. E 92, 062115 (2015).
[40] S. Kirkpatrick, C. D. Gelatt, and M. P. Vecchi, Science 220, 671 (1983).
[41] Y. LeCun, Y. Bengio, and G. Hinton, Nature 521, 436 (2015).
[42] Y. Bengio, J. Louradour, R. Collobert, and J. Weston, Curriculum learning, in Proceedings of the 26th annual international conference on machine learning, pp. 41–48, ACM, 2009.
[43] J. Villain, J. Phys. C: Solid State Phys. 10, 1717 (1977).
[44] R. Metzler, J.-H. Jeon, A. G. Cherstvy, and E. Barkai, Phys. Chem. Chem. Phys. 16, 24128 (2014).
[45] S. Belan, Phys. Rev. Lett. 120, 080601 (2018).
[46] S. Reuveni, M. Urbakh, and J. Klafter, Proc. Natl. Acad. Sci. U.S.A., 201318122 (2014).
[47] S. Reuveni, Phys. Rev. Lett. 116, 170601 (2016).