CORRECTING PARAMETERS OF EVENTS BASED ON THE ENTROPY OF MICROLENSING ENSEMBLE

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ABSTRACT

We entertain the idea that robust theoretical expectations can become a tool in removing hidden observational or data-reduction biases. We illustrate this approach for a specific problem associated with gravitational microlensing. Using the fact that a group is more than just a collection of individuals, we derive formulae for correcting the distribution of the dimensionless impact parameters of events, \( u_{\text{min}} \). We refer to the case when undetected biases in the \( u_{\text{min}} \) distribution can be alleviated by multiplication of impact parameters of all events by a common constant factor. We show that in this case the general maximum likelihood problem of solving an infinite number of equations reduces to two constraints, and we find an analytic solution. Under the above assumptions, this solution represents a state in which the “entropy” of a microlensing ensemble is at its maximum; that is, the distribution of \( u_{\text{min}} \) resembles a specific, theoretically expected, box-like distribution to the highest possible extent. We also show that this technique does not allow one to correct the parameters of individual events on an event-by-event basis, independently from each other.

Subject headings: dark matter — gravitation — gravitational lensing — methods: statistical — stars: fundamental parameters

1. INTRODUCTION

There are two complementary processes that have led to progress in the physical sciences: (1) new hypotheses have triggered experiments that either verify or falsify these hypotheses, and (2) observations of unexpected phenomena force theorists to refine old or invent new models and mathematical descriptions. Most of the achievements in astrophysics have followed the second pattern. Here we argue that an extreme variation of the first path can be very useful in some astrophysical problems. We present the idea that robust theoretical expectations can become a tool in removing hidden observational biases. We illustrate this approach by deriving possible corrections to the distribution of impact parameters for a sample of microlensing events.

Over the last decade, the microlensing surveys have grown from a fascinating idea entertained by a group of enthusiastic theoreticians (Paczyński 1986, 1991; Griest 1991) to a reality of data sets containing tens and hundreds of real events (Alcock et al. 2000; Popowski et al. 2001; Udalski et al. 2000). The more data we gather, the more we appreciate the ability to do precise microlensing analyses. The effects of “parallax” (Gould 1992; Alcock et al. 1995), binary caustic crossing (Mao & Paczyński 1991; Afonso et al. 2000), and finite source (Gould 1994; Alcock et al. 1997) are particularly appreciated because of their ability to break degeneracies present in the simplest cases and provide useful constraints on stellar physics. On the other hand, blend fits (needed to determine how much flux of a few unresolved stars at a given location has been microlensed) are a standard tool in investigating crowding/seeing biases common to most of the events. As long as one tries to infer the geometry from the fluxes themselves, the determinations are sensitive to the sky level and other weakly controlled factors. Therefore, it is possible that even doing the best possible analysis on reported events, one still has to correct an entire data set for undetected biases.

Here we argue that it is possible to obtain a more accurate determination of the parameters of events using the information that they belong to a microlensing family. The robust prediction of microlensing is that different impact parameters are equally likely, and, as a result, the distribution of the impact parameter is boxlike, with values bracketed by 0 and \( u_{\text{min}}^* \), where \( u_{\text{min}}^* \) is the maximum value of \( u_{\text{min}} \) allowed by the minimum amplification chosen for a particular event selection (see Fig. 1 and the following sections). This robust prediction may appear to break for blended events if blending tends to populate certain \( u_{\text{min}} \) ranges at the expense of the others. Similarly, it may seem to break if observing strategy, conditions, or instrumentation preferentially select certain \( u_{\text{min}} \) ranges. In the following treatment we will assume either that such effects are insignificant or that they have been reduced to a negligible level through some correction procedures (e.g., deblending). Therefore, we concentrate on hypothetical irremovable biases that may originate at the stage of data reduction and analysis. We suggest that, when dealing with a clean (almost clean) microlensing sample, one should correct all individual \( u_{\text{min},i} \) in such a way as to achieve the highest possible agreement between the observed and theoretically predicted cumulative distribution of \( u_{\text{min}} \). The corrected \( u_{\text{min},i} \) values may be used to redetermine the duration of microlensing events. This, in turn, would lead to a new estimate of the microlensing optical depth and modification of most likely lens masses.

In the most general case, one would like to use a Kolmogorov test to choose a set of \( u_{\text{min},i} \) that is consistent with

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4 The assumption of a clean microlensing sample is discussed in detail in § 5.
micro-lensing at the highest possible confidence level. The distribution function constraint coming from the Kolmogorov test is formally equivalent to an infinite number of constraints on all the moments of the distribution. This does not mean that the information contained in Kolmogorov statistic (see §4) is equivalent to the information contained in all the moments of the distribution. However, the determination of the Kolmogorov statistic may require the knowledge of all the moments of the distribution. The moments are easier to deal with in the general approach of maximum likelihood that we are going to invoke here. The high moments of the distribution (even the shape parameters of skewness and kurtosis) are very weakly constrained in the case of small number statistics. One may think that it should be advantageous to use just lower, well-constrained moments. This is not always correct. Despite the fact that higher moments are correlated with lower moments, the infinite number of very weak constraints coming from higher moments may overcome the statistical signal of the well-constrained mean and variance of the distribution. Therefore, in the general case the limited moment approach to correcting microlensing parameters may be biased in an unpredictable way. However, there is one type of correction that is completely determined just by the mean, μ, and the variance, σ², of the u_min distribution. In this case described in §3, it is mathematically more elegant and convenient to use the moment formalism as an ersatz for the complete Kolmogorov statistic. The structure of this paper is the following: In §2 we briefly review the basic principles of microlensing and the possible biases in the determination of stellar parameters. In §3 we present our new formalism pointing to its likely practical application. In §4 we discuss the question of how to obtain microlensing constraints from the traditional Kolmogorov procedure in the general case. We argue that it is not possible to improve the parameter determination of the fits to the individual highly degenerate blending events. Finally, in §5 we summarize our results.

2. GENERAL MICROLENSING

The microlensing of stellar light is produced when a massive object (e.g., a star) passes very close to the line of sight between the source of light and the observer (see, e.g., Gould 2001 for a theoretical review). The light is gravitationally deflected, and the processed image changes its surface as projected on the plane of the source. Because during microlensing the surface brightness is conserved, the observed flux from the source increases proportionally to the change in the image surface. As a result the source is magnified, and the maximum magnification A_max depends on the impact parameter u_min, which describes the level of alignment between the source, lens, and observer. The centroid of the new distribution of light in the sky changes; but because the shift is ≲100 μas for a typical event (Boden, Shao, & van Buren 1998), it cannot be observed from the ground with current instruments. Therefore, the classical signature of the simplest microlensing is a characteristic achronic light curve, which is time-symmetric with respect to the epoch with the highest flux.

Microlensing surveys typically have noisy photometry and, therefore, are forced to allow only the events with A_max exceeding 1.5 or so. The minimum recorded maximum amplification, A_max*, sets the limits on the largest recorded impact parameter u_min*:

\[ u_{\text{min}}^* = \sqrt{-2 + \frac{2A_{\text{max}}}{\sqrt{A_{\text{max}}^2 - 1}}} \]  

The stellar systems in which microlensing is observed contain enough objects on random enough orbits so that all the values of u_min in the range between 0 and u_min* are equally likely.

There are several effects that complicate this simple picture. First of all, there are binary lenses, which produce lines of formally infinite magnification (caustics) and light curves with a zoo of shapes (Griest & Hu 1992). Here we mean both stellar binaries as well as stars with planetary companions. Second, the sources are not pointlike, and as a result the magnification pattern differs from the basic case. Third, the constraints coming from the survey instruments, sampling, and site weather introduce a whole class of biases. Most of them can be summarized in two categories: blending and efficiency. Blending is mostly about atmospheric seeing. All microlensing surveys have to deal with seeing of about 1" or more. The seeing disk of that size typically covers more than one star in the sky, especially when one wants to account for the faint end of the luminosity function. However, in almost all cases there is only one star (or stellar system like a binary) that is microlensed. One has to make “blend fits” that determine what fraction of the light observed in the seeing disk has actually been microlensed. Efficiency is about the duration of the survey, the frequency of sampling the light curves, and the photometric response of the system. The events shorter than an average interval between observations are likely to go unnoticed as are the events that last much longer than the whole survey. The magnitude-limited character of the survey will not substantially bias the detection of bright sources but will allow only highly magnified faint sources. These effects leave their mark on the u_min distribution and are likely to show preference for specific u_min ranges. The qualitative character of such preferences can be revealed only through a complete analysis conducted by a given survey. Once necessary adjustments are applied, the u_min distribution should be approximately boxlike. In an ideal case such distribution would not require any further correction. The more realistic case is presented in §3.
3. BASIC FORMALISM

As mentioned in the previous section all values of \( u_{\text{min}} \) in the range between 0 and \( u_{\text{min}}^* \) are theoretically equally likely, and therefore \( u_{\text{min}} \) should have a boxlike distribution:

\[
f(u_{\text{min}}) = \begin{cases} \frac{1}{u_{\text{min}}^*} & \text{if } 0 \leq u_{\text{min}} \leq u_{\text{min}}^* \ , \\ 0 & \text{if } u_{\text{min}} > u_{\text{min}}^* . \end{cases}
\]

Therefore, the moments of the distribution should be given by

\[
m_n = \frac{\int_0^\infty x^n f(u_{\text{min}}) \, dx}{\int_0^\infty f(u_{\text{min}}) \, dx} = \frac{\int_0^{u_{\text{min}}^*} x^n (1/u_{\text{min}}^*) \, dx}{\int_0^{u_{\text{min}}^*} (1/u_{\text{min}}^*) \, dx} = (u_{\text{min}}^*)^n \ . \quad (3)
\]

As a result,

\[
E[\mu(u_{\text{min}})] \equiv m_1 = \frac{1}{2} u_{\text{min}}^* , \quad (4)
\]

\[
E[\sigma^2(u_{\text{min}})] \equiv m_2 - m_1^2 = \frac{1}{12} (u_{\text{min}}^*)^2 , \quad (5)
\]

where \( E \) stands for the expectation value. Now the question is how one can use this information to make a better estimate of the microlensing parameters. Here we are going to call on a thermodynamic analogy, which we think corresponds closely to the currently considered case. Imagine a huge reservoir of particles that are characterized by a certain temperature and, therefore, have a certain distribution of kinetic energies. To measure some properties of these particles, one could investigate average properties of any subvolume of this huge reservoir because such a subvolume is representative of the entire volume. In practice, one would rather attach to the reservoir a small chamber separated from the main volume by a partition. If the partition is semi-permeable, the number of particles in the small chamber will increase until the experiment is finished and the particle properties are measured. As long as the number of particles in the small chamber is much smaller than the number of particles in the huge reservoir, the chamber will be filled with particles that originate from the reservoir but are, on average, more energetic. This is because the faster particles are more likely to hit the partition than one would infer from their number density. However, if one knows the original velocity distribution, one can make an adjustment for this effect. After this correction, the small chamber collection should be equivalent to a randomly chosen subvolume mentioned above and, therefore, representative of the huge reservoir. The most likely state of such a subvolume is the one that maximizes entropy. The state that maximizes entropy is the one in which the distributions of energies of particles in the reservoir and the corrected energies of the particles in the chamber are identical.

In the case of microlensing, the reservoir is filled with all possible events. The small chamber corresponds to a microlensing survey, which gathers the data. The factors biasing the distribution of event parameters are both detection efficiencies of the survey as well as blending of the events. Once corrected for the known biases, the distribution of \( u_{\text{min}} \) of events in the survey should closely resemble the distribution in the reservoir. And it does unless there are hidden biases associated with data reduction and analysis. We will define the state of maximum microlensing entropy as the one in which the \( u_{\text{min}} \) distribution resembles the expected distribution to the highest possible extent. Because the reservoir contains a tremendous number of possible microlensing events, its distribution of \( u_{\text{min}} \) is to a very high accuracy boxlike. We want to modify the \( u_{\text{min}} \) distribution of the survey to make it look like the expected boxlike distribution.

For simplicity, we will present here only the conditions imposed on the mean and the variance of the distribution. We define

\[
u_{\text{min}}^\text{new} = g_i(P)u_{\text{min},i} , \quad (6)
\]

where \( g_i(P) \) is a function of different parameters designated by the capital \( P \) (in what follows we will use just \( g_i \) to avoid clutter). The two most extreme cases are that all \( g_i(P) \) are equal to 1, and so the entire transformation is in effect an identity, and that all \( g_i(P) \) are different and unrelated to each other. For the treatment with only the first two moments taken into account (flawed in the general case as described in §1), the likelihood of a certain configuration of \( u_{\text{min}} \) values (if no other information is available) is

\[
L(u_{\text{min},i}, i \in \Gamma, N) = \exp \left( -\frac{\chi^2_\mu}{2} \right) \exp \left( -\frac{\chi^2_{\sigma^2}}{2} \right) . \quad (7)
\]

The maximization of this likelihood is equivalent to the minimization of

\[
-2 \ln L = \chi^2_\mu + \chi^2_{\sigma^2} , \quad (8)
\]

where

\[
\chi^2_\mu = \left[ \frac{(1/N) \sum_{i=1}^N g_i u_{\text{min},i} - (1/2) u_{\text{min}}^*]^2}{(1/N) \{1/(N-1) \sum_{i=1}^N \left[ g_i u_{\text{min},i} - (1/N) \sum_{i=1}^N g_i u_{\text{min},i} \right]^2 \}} \right] , \quad (9)
\]

and

\[
\chi^2_{\sigma^2} = \left[ \left\{ 1/(N-1) \sum_{i=1}^N \left[ g_i u_{\text{min},i} \right]^2 \right\} - (1/12) (u_{\text{min}}^*)^2 \right]^2 \left\{ [K(u_{\text{min}}, g) - 1]/(N-1) \right\} \left\{ 1/(N-1) \right\} \left\{ 1/N \right\} \left\{ \sum_{i=1}^N \left[ g_i u_{\text{min},i} - (1/N) \sum_{i=1}^N g_i u_{\text{min},i} \right]^2 \right\}^2 . \quad (10)
\]

In the general case, the kurtosis \( K(u_{\text{min}}, g) \) is a function of both the original distribution and the modifying functions and so it enters the minimization in an active way making all the expressions very complicated. Here we are going to consider a special case where

\[
\chi_{\frac{0}{1}} = \alpha = \text{const} . \quad (11)
\]
Then the equations (9) and (10) are simplified to
\[
\chi_{\mu}^2 = \frac{[\alpha - (1/2)\mu_{\min}^*]^2}{\alpha^2 \sigma^2}, \quad (12)
\]
\[
\chi_{\sigma^2}^2 = \frac{[\alpha^2 \sigma^2 - (1/12)(\mu_{\min}^*)^2]^2}{\{K(\mu_{\min}) - 1/N\} \alpha^2 \sigma^4}, \quad (13)
\]
where \(\mu\) and \(\sigma^2\) are the mean and dispersion of the original distribution of \(\mu_{\min}\), respectively.

Now the kurtosis \(K\) is only a function of the initial distribution of \(\mu_{\min}\), because the higher standardized moments are invariable against multiplication of the entire distribution by a constant. Therefore, in minimization \(K\) can be treated as a constant of the sought value of \(\alpha\). More importantly, all the standardized moments higher than variance remain constant under the multiplication of the whole distribution by a constant. In this case, the infinite number of conditions on all the moments of the distribution is equivalent to two conditions on the mean and variance of the distribution. We start with separate minimizations of equations (12) and (13) to earn some intuitive understanding of the desired corrections to the observed set of \(\mu_{\min}\) values. From equation (12),
\[
\frac{\partial \chi_{\mu}^2}{\partial \alpha} = \frac{N \mu_{\min}^* \{\alpha - (1/2)\mu_{\min}^*\}}{\alpha^3 \sigma^2} = 0 \iff \alpha = \frac{(1/2)\mu_{\min}^*}{\mu}. \quad (14)
\]
Therefore,
\[
\alpha = \frac{(1/2)\mu_{\min}^*}{\mu}. \quad (15)
\]
From equation (13),
\[
\frac{\partial \chi_{\sigma^2}^2}{\partial \alpha} = \frac{\{N/[3(K-1)]\} (\mu_{\min}^*)^2 \{\alpha^2 \sigma^2 - (1/12)(\mu_{\min}^*)^2\}}{\alpha^2 \sigma^4} = 0 \iff \alpha^2 = \frac{(1/12)(\mu_{\min}^*)^2}{\sigma^2}. \quad (16)
\]
Therefore,
\[
\alpha = \frac{(1/12)(\mu_{\min}^*)^2}{\sigma^2}. \quad (17)
\]
Solutions (15) and (17) are what one would expect. They force either the mean or variance of the modified distribution to be equal to the theoretically expected values reported in equations (4) and (5). Note that the requirement that the \(\alpha\) returned by equations (15) and (17) be identical to each other is equivalent to \(\mu^2 = 3\sigma^2\) (every boxlike distribution with a support in the range from 0 to a constant meets this condition). Now we combine equations (14) and (16) to find the value of \(\alpha\) that is optimum from the point of view of both mean and variance of the distribution:
\[
\frac{\partial \chi_{\mu}^2}{\partial \alpha} + \frac{\partial \chi_{\sigma^2}^2}{\partial \alpha} = 0 \iff \alpha^3 - \frac{2 - 3(K - 1) \mu_{\min}^*}{3(K - 1)} \alpha^2 - \frac{1}{36(K - 1)} \frac{(\mu_{\min}^*)^3}{\mu \sigma^2} = 0. \quad (18)
\]
We define
\[
B \equiv \frac{2 - 3(K - 1) \mu_{\min}^*}{3(K - 1)} \mu, \quad D \equiv \frac{1}{36(K - 1)} \frac{(\mu_{\min}^*)^3}{\mu \sigma^2}. \quad (19)
\]
We introduce a new variable \(y = \alpha + (B/3)\). Then equation (18) takes the form of
\[
y^3 - \frac{B^2}{3} y + \left(\frac{2B^3}{27} + D\right) = 0. \quad (20)
\]
From the theory of solving third-order equations, one knows that equation (20) has two complex and only one real solution if
\[
\Delta \equiv D \left(\frac{B^3}{27} + \frac{D}{4}\right) > 0. \quad (21)
\]
If \(K > 1\), then we see from equation (19) that \(D < 0\). Therefore, condition (21) is equivalent to
\[
\left(\frac{B^3}{27} + \frac{D}{4}\right) < 0. \quad (22)
\]
Condition (22) is true, independent of the values of \(\mu\) and \(\sigma^2\), if
\[
B \leq 0 \iff K \geq \frac{5}{3}. \quad (23)
\]
Condition (23) is likely to be the case most of the time, and the only real root of equation (18) will be the most probable solution for \(\alpha\). This solution can be written as
\[
\alpha = \sqrt[3]{\frac{B^3}{27} + \frac{D}{2}} - \sqrt{\Delta + \sqrt{\Delta^2}} \pm \frac{B}{2} + \frac{\sqrt{\Delta} - B}{3}. \quad (24)
\]
In the case when all the roots of equation (18) are real, one has to choose the one that minimizes the combined \(\chi^2\) of equation (8), instead of just taking solution (24). We do not discuss such complications in any detail because the procedure is mathematically straightforward and completely standard.

Multiplication of the entire \(\mu_{\min}\) distribution by \(\alpha\) is likely to change the number of events in the range \((0, \mu_{\min})\). The described procedure should be repeated on new sets as many times as necessary for the results to converge. Therefore, the result of the analysis can be given as \(\alpha = \prod_{i=1}^{R} \alpha_i\), where \(R\) is the number of needed repetitions and \(\alpha_i\) are corrections of individual iterations.

In Figure 2 we present the application of equation (24) to an artificial set of 17 \(\mu_{\min}\) values. The dashed line is a theoretically expected distribution for the magnification threshold of \(4_{\text{max}} = 1.34\) or \(\mu_{\min} = 1.0\). The thick solid line is an original, uncorrected cumulative distribution of \(\mu_{\min}\) values. The thick solid line is this distribution multiplied by \(\alpha = 0.9\), as obtained from equation (24). The improvement is easily visible and can be quantified by the reduction in maximum vertical distance between the two distributions \(D_N\) (formally introduced in the next section).

4. THE MORE GENERAL TREATMENT AND THE IMPOSSIBILITY OF CONSTRAINING INDIVIDUAL BLENDED EVENTS

We will now consider the most general case as described by equations (9), (10), and the infinite number of other equations for the higher moments. Because all the moments now contain the statistical information, the only solution is to turn to the original Kolmogorov treatment.

5 The kurtosis of the boxlike distribution is \(K = 1.8\), so most of the deviations from this distribution will have \(K \geq 5/3\).
The Kolmogorov test verifies the hypothesis that a continuous variable $X$ could have been drawn from a cumulative distribution function $F_0(x)$. The test statistic is

$$D_N = \sup_X |F_0(x) - S_N(x)|,$$

where $S_N(x)$ is the observed cumulative distribution function based on the ordered sample of $N$ measurements. If $d_N(1 - \beta)$ designates quantiles of the statistic $D_N$, then $P[D_N \geq d_N(1 - \beta)] = \beta$. The higher $\beta$, the less sensitive the test is.

Suppose that we construct all the possible sets of corrected $u_{\text{min}}$ values from the interval 0 to $u^*$. Then for each set we can find a $D_N$ statistic and a corresponding significance level $\beta$. In a formal Kolmogorov-type analysis, it is inappropriate to conclude that the relative probabilities of different sets scale as $P(\text{set1})/P(\text{set2}) = \beta(\text{set2})/\beta(\text{set1})$. However, even though we cannot assign relative probabilities of certain corrected configurations, we can select the set of $u_{\text{min}}$ values with the smallest $\beta$ as the preferred solution (for the cumulative distribution only).

The main problem with this approach is that from the point of view of the cumulative distribution only, there will be a number of optimum solutions, all of which, except one, will mix the original order of $u_{\text{min}}$ values. To avoid total chaos one can consider a simplified situation when the following condition is met:

**Condition A.**—The order of $u_{\text{min}}$ to be corrected is going to be conserved, i.e., the event with the $i$th lowest $u_{\text{min}}$ will remain the event with the $i$th lowest $u_{\text{min}}$ after the correction has been performed.

Imagine that we operate in this idealized situation and select $N$ microlensing events and order them according to increasing $u_{\text{min}}$. One will reach the highest possible agreement if one sets

$$u_{\text{min},i} = \frac{i - 0.5}{N + 1} u^*.$$

If correcting factors $g_i$ can be represented as a function of $u_{\text{min},i}$, that is, $g_i = g(u_{\text{min},i})$, then as long as $\partial g / \partial u_{\text{min}} > -1/u_{\text{min}}^2$ in the range $(0, u^*)$, condition A does not have to be imposed but is naturally met. Condition A makes the most sense if one deals with a separated set of events. A “separated set” here is a well-defined concept described as

$$\bigwedge_{i \in 1,N} \delta u_{\text{min},i} \leq \min \left( (u_{\text{min},(i+1)} - u_{\text{min},i}), (u_{\text{min},i} - u_{\text{min},(i-1)}) \right),$$

where $\delta u_{\text{min},i}$ is the formal error in $u_{\text{min},i}$. At first glance, one may think that because of the requirement of small formal errors, the inequality (27) automatically guarantees that all $u_{\text{min},i}$ were determined correctly. However, that may not be the case, when unrecognized biases are present. In a separated set case, the constraints from the fits to individual events are crucial in establishing the order of $u_{\text{min}}$ values, and the Kolmogorov test is used to stretch or squeeze the entire distribution. The prescription described here and in § 3 can be implemented in an iterative program, which can converge on the solution using both the cumulative distribution information as well as other constraints on individual $u_{\text{min},i}$, with the relative weights selected by the investigator.

It is important to note that one is not allowed to draw the same statistical conclusions by fixing the $u_{\text{min}}$ positions of some events and “filling the gaps” with the others. To give an example, one may hope that the information contained in the microlensing character of the ensemble can help to deblend the events with highly degenerate fits. Imagine that we have $(N - 1)$ events with nicely determined parameters (for which we would be happy to put $g_i \equiv 1$) and one blending event with a very uncertain blending fraction (let us say $k$th on the ordered list). This corresponds to a situation when only one $g_i \neq 1 = \alpha$. In this case, the blindly applied Kolmogorov analysis is going to return quite a strong preference for a certain value of $u_{\text{min},k}$ of this particular event. However, the actual probability that the considered event will be in this narrow preferred range is low, and all the values of $u_{\text{min}}$ between 0 and $u^*$ are almost equally likely. This is because the right treatment here is to consider just a conditional probability of a certain $u_{\text{min}}$ value given the set of $(N - 1)$ other $u_{\text{min}}$ values. Because the reservoir of possible microlensing events is huge, this conditional probability is extremely close to the flat distribution expected from the infinite set. Therefore, the rule is to either adjust the parameters of all events applying the Kolmogorov approach (and extra constraints) to the whole set of $u_{\text{min}}$ values or leave it untouched. All the naive “partial” procedures involving the Kolmogorov test will produce spurious results.

### 5. SUMMARY

We have investigated the question of how the information that events constitute a microlensing family can help to
reduce systematic errors in parameter determination. We have argued that the Kolmogorov test for the agreement between the observed and the theoretical cumulative distributions of the impact parameter $u_{\text{min}}$ can put useful additional constraints on microlensing parameters. However, there is no natural way to incorporate the classical Kolmogorov approach into the convenient maximum likelihood analysis. Fortunately, the constraints on the cumulative distribution can be replaced with the constraints on the infinite number of the moments of the distribution. We explain why taking into account only a few lower moments of the distribution is flawed, and, therefore eliminates the usefulness of such limited moment approach. However, we show that when the whole distribution of $u_{\text{min}}$ values is multiplied by the same factor $\alpha$, all the information about the probabilities of different $\alpha$-values is stored in the mean and variance of the original distribution and the selection criteria of events. Assuming that no other statistical information on events is available (or relevant at this point), we solve for $\alpha$. Our results are given in equation (24) of § 3.

In the follow-up, given the corrected $u_{\text{min}}$ values, one may keep them fixed and re-determine the other parameters. Such an approach is applicable if there is a systematic homogeneous bias of all $u_{\text{min}}$ values. The fully consistent treatment should proceed through the simultaneous determination of the multiplying factor $\alpha$ and the parameters of individual events. A useful consistency check would be to select different $u_{\text{min}}$ values and make sure that $\alpha$ does not strongly depend on this choice.\footnote{Here we mean $u_{\text{min}}^* \equiv u_{\text{min}}$ values that do not differ too much from the original one — $\alpha$ is a local correction and as such is expected to vary if $u_{\text{min}}^*$ changes substantially.}

One may ask when it is justified to apply an $\alpha$-type correction to the data. This is a serious concern because it is normal that every actual realization of the $u_{\text{min}}$ distribution will be affected by finite number statistics and, as a result, $\alpha$ is expected to be $\neq 1$ even for unbiased data, which need no correction. However, in this case, correction by $\alpha$ will typically be small and will not significantly change the properties of the distribution. Consequently, it seems safe to apply the $\alpha$-type correction even if the original distribution is consistent with the expected one.

Finally, we should examine the assumption that the microlensing sample under analysis is clean. In the traditional approach of massive microlensing searches directed at random events, it is very hard to construct a completely clean microlensing sample. In defense of the $\alpha$-type correction, we should notice that sensitivity to contamination is common to all determinations associated with microlensing. For example: (1) few long-duration variables mistakenly classified as microlenses can change the microlensing optical depth by a factor of 2, (2) few very short stellar flares mistakenly classified as microlenses can entirely change the slope and cutoff of the mass function of lenses derived from the duration distribution of events. Similarly our method may bias the final $u_{\text{min}}$ distribution if some candidate events are not microlenses. Therefore, a particular survey should apply this method to their data only if the bias caused by sample contamination is smaller than the advantage gained due to the described adjustment. A potential improvement can be judged using a learning set of highly constrained events.\footnote{Such a learning set can also answer the question of what type of systematic errors are best removed with an $\alpha$-type correction. However, in general, the understanding of the nature of systematic errors is not required to achieve improvement in parameter determination.}

However, the $u_{\text{min}}$ test is often used as evidence for the microlensing character of the events. This makes its status somewhat different from the optical depth or duration distribution of the events. As a result, what we need in an ideal case is a way to separate the “convoluted” nature of the $u_{\text{min}}$ test (which intrinsically allows one to check both the clean character of the sample and the correctness of the $u_{\text{min}}$ values) into two independent issues. If we could select a clean sample, then we could limit the role of the $u_{\text{min}}$ test to a single task of parameter control. In this way the $u_{\text{min}}$ distribution could become an active, unbiased tool in parameter determination, as we have presented it above. Here we give two examples of realistic situations, in which the clean character of the sample can be established with a very high probability.

1. Events on demand from high proper motion lenses.—In the current microlensing searches there is no expectation of the location of the next event. The next event happens somewhere on one of $\sim 10^7$ monitored stars. In this scenario and with typical optical depths on the order of $10^{-2}$ to $10^{-4}$, it is expected that most varying objects will not be microlensing events. Moreover, there will be enough variables detected per each microlensing event to produce cases that look very much like microlensing and can contaminate the sample. Fortunately, observations do not have to follow this pattern. Some high proper motion stars toward dense stellar fields (e.g., the Galactic bulge) will within $\sim 10$ yr become sufficiently aligned with the known bulge sources to produce microlensing events (Salim & Gould 2000; Drake et al. 2002). As the occurrence of such events can be predicted based on known stellar motions, one can check the sky patch under investigation in advance for possible nonmicrolensing contaminants. More importantly, if the event happens when scheduled, the probability that it is not microlensing is negligible.

2. Astrometrically constrained events.—Single variable stars (including cataclysmic ones) are the main contaminants of microlensing samples. With high accuracy astrometric observations of candidate events carried out by the Space Interferometry Mission, or a similar astrometric satellite, it is easy to eliminate such contaminants: the centroids of varying flux due to variables do not move, whereas the centroids of light from microlensing images follow a well-defined path (e.g., Boden et al. 1998). A detection of the centroid shift of varying flux (even the one insufficient to constrain microlensing parameters from astrometry alone) will be a very good indication of the microlensing character of the event.

Both types of samples described above should be clean enough for the purpose of the nonrestrictive application of $\alpha$-type correction, which may be based on light curves alone.

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