Harmonic Analysis and Localization Technique

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Abstract

We study relationships between different formulations of the local principle. Also we establish a connection among the local principle and the non-commutative Fourier transform approach to the investigation of convolution operator algebras.

1 Introduction

This paper is devoted to the local principle, which is a canonical method for the study of operator algebras. Theorem 2.5 establish the coincidence of the easy to use local principle based on the existence of a central commutative subalgebra [1] and the general local principle constructed on a set of ideals [2]. The second question of this paper is the correspondence between the local principle and the non-commutative Fourier transform. If we have a group $G$ with the group operation $*$ and the Haar (= invariant) measure $dg$, it seems quite natural to introduce a group algebra $\mathcal{G}$ associated with the group $G$.

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Now the noncommutative Fourier transformation established a direct connection between representations of the group algebra $\mathfrak{G}$ and representations of the group $G$ itself (see [3, 4]).

On the other hand, the convolution algebra $\mathfrak{G}$ has a regular representation in the space of bounded operators $\mathcal{B}(L_2(G))$. This representation is introduced as the integral of the shift operator $\pi_r(g)$

$$[\pi_r(g)f](h) = f(h \ast g)$$

giving rise to the regular representation of the group $G$ on the space $L_2(G)$:

$$\pi_r : \mathfrak{G} \rightarrow \mathcal{B}(L_2(G)) : k(g) \mapsto K = \int_G k(g)\pi_r(g) \, dg, \quad (1.1)$$

Therefore the local principle from the operator theory can be applied to the operator algebra $\mathfrak{G}$. The two ways lead to the same answer, so a relation between the noncommutative Fourier transformation and the local principle should exist. Note, that harmonic analysis is described in terms of the Plancherel measure and sectional representations are constructed in field $*$-topology. The correspondence between these two different mathematical objects is clarified in Theorem 3.2. An application of the achieved result to the support of the Plancherel measure is given in Corollary 3.3. All proofs are easy and they are skipped.

# 2 A System of Ideals and the Algebra Center

To apply the local principle from [2] one need to find an appropriate system of ideals. If the algebra under consideration has a non-trivial abelian subalgebra then the answer is given by the following

**Theorem 2.1** [2, Proposition 4.5] If $\mathfrak{R}$ is a $C^*$-algebra, $\mathcal{Z}$ is an abelian $C^*$-subalgebra contained in $\mathfrak{R}$ having maximal ideal space $M_{\mathcal{Z}}$, and for $x$ in $M_{\mathcal{Z}}$, 

$I_x$ is a closed ideal in $\mathfrak{A}$ generated by the maximal ideal $\{Z \in \mathcal{Z} : \hat{Z}(m) = 0\}$, then $\bigcap_{x \in M_Z} I_x = \{0\}$. In particular, if $\Phi_x$ is *-homomorphism from $\mathfrak{A}$ to $\mathfrak{A}/I_x$ then $\sum_{x \in M_Z} \bigoplus \Phi_x$ is *-isomorphism of $\mathfrak{A}$ into $\sum_{x \in M_Z} \bigoplus \mathfrak{A}/I_x$. Moreover, $T$ is invertible in $\mathfrak{A}$ if and only if $\Phi_x(T)$ is invertible in $\mathfrak{A}/I_x$ for $x$ in $M_Z$.

In the case of a trivial (or unsuitable) center of the algebra $\mathfrak{A}$ the following trick is commonly used: one can introduce a larger algebra $\tilde{\mathfrak{A}} \supset \mathfrak{A}$ with appropriate center $\hat{Z}$ and use Theorem 2.1 for the algebra $\tilde{\mathfrak{A}}$. Then the description of the algebra $\mathfrak{A}$ can be obtained as a subalgebra of the algebra $\sum_{x \in M_Z} \bigoplus \tilde{\mathfrak{A}}/I_x$.

**Example 2.2** The algebra $\mathfrak{B}$ of bisingular operators $[6]$ has only operators of multiplication by a constant in its center. Nevertheless, one can introduce the largest algebra $\tilde{\mathfrak{A}}$ of operators having the form $a(x)F^{-1}b(\xi)F$ with $a(x) \in \mathbb{R}^2$ and $b(\xi)$ be a homogeneous degree zero function whose restriction on $S^1$ is piecewise continuous. This algebra contains the bisingular operators and have a central commutative (up to compact operators) subalgebra of Calderon-Mikhlin-Zigmund operators with continuous symbols. The application of Douglas’ technique allows to obtain the full description of the algebra of symbols of the algebra $\mathfrak{A}$ (and the algebra $\mathfrak{B}$ correspondingly).

It is remarkable that this trick may be applied to the general local principle. The following Lemma settles the particular case of regular base $B$ in the field *-bundle topology.

**Lemma 2.3** Let a $C^*$-algebra $\mathfrak{A}$ have a system $B$ of primitive ideals with property $\bigcap B = 0$ and $B$ be compact in the field *-topology. Then there is a $C^*$-algebra $\tilde{\mathfrak{A}}$ with non-trivial center $\hat{Z}$ and an embedding of the algebra $\mathfrak{A}$ on $\tilde{\mathfrak{A}}$ such that the application of the general local principle by the systems
B and the method from Theorem 2.1 to the subalgebra \( \tilde{Z} \) give the same result.

Remark 2.4 We come back to the bisingular operator algebra from Example 2.2 again. It is easy to verify it this case that the largest algebra of sections cannot be realized as an operator algebra on the initial Hilbert space \( L_2(\mathbb{R}^2) \).

Theorem 2.5 Let a \( C^* \)-algebra \( \mathfrak{A} \) have a system \( B \) of ideals such that \( \bigcap B = 0 \). Then there is a \( C^* \)-algebra \( \tilde{\mathfrak{A}} \) with non-trivial center \( \tilde{Z} \) and an embedding of the algebra \( \mathfrak{A} \) on \( \tilde{\mathfrak{A}} \) such that the application of the canonical shelf construction by the systems \( B \) and the method from Theorem 2.1 to the subalgebra \( \tilde{Z} \) give the same result.

3 The General Local Principle and the Support of the Plancherel Measure

Operator algebras in analysis usually have rich groups of symmetries, which are a symmetry group of differential equations define the analyticity property. Thus they can be treated as convolutions [5]. We would like to apply the localization technique in such an environment. Let \( B \) be a family of primitive ideals in the group \( C^* \)-algebra \( \mathfrak{G} \) with the only property:

\[ \bigcap B = 0 \quad (3.1) \]

(for example, all ideals generated by all maximal ideals of a central commutative subalgebra of our algebra, see Section 2). Then the following quotient mapping for any \( J_b \)

\[ \pi_t : (\text{convolution with kernel } k) \mapsto (\text{convolution with kernel } k)/J_b \quad (3.2) \]
generates a representation of the group algebra. Condition (3.1) ensures that all irreducible representations of the group algebra may be obtained as (sub)representations of (3.2). However, the careful extracting of irreducible representations may be a rather difficult problem [4].

The following Lemma plays a fundamental role in the establishment of a correspondence between the local principle and the Fourier transform.

**Lemma 3.1** Let $G$ be an unimodular type 1 exponential group, let $	ilde{G}$ be the support of the Plancherel measure $d\mu$ in $\hat{G}$. Then the family of two-sided ideals

$$J(\pi_i) = \{ K \mid K \text{ is a convolution with kernel } k \text{ such, that } \pi_i(k) = 0 \}, \pi_i \in \tilde{G}$$

satisfies the condition (3.1).

Lemma 3.1 suggests that one may use this family $J(\pi_i)$ of ideals for localization according to the general local principle. It obviously follows from the definition of the family $J(\pi_i)$ of ideals that the rules non-commutative Fourier transform and local technique define just the same representations of our group algebra. Thus in the mentioned case the noncommutative Fourier transformation and the local principle give, in fact, the same description of the group convolution algebra. This may be summarized as follows

**Theorem 3.2** Under the assumptions and notations of Lemma 3.1 we have:

All representations of the group algebra $\mathcal{G}$ given by the Fourier transform

$$\pi_i : \text{(convolution with kernel } k \in L_1(G)) \mapsto \pi_i(k)$$

are contained (as subrepresentation, possibly) within the representations

$$\pi_i : \text{(convolution with kernel } k) \mapsto \text{(convolution with kernel } k)/J(\pi_i),$$

where

$$J(\pi_i) = \{ K \mid K \text{ is a convolution with kernel } k \text{ such, that } \pi_i(k) = 0 \}, \pi_i \in \tilde{G}.$$
The support of the Plancherel measure $\tilde{G}$ is a dense subset of $\text{Prim}'\,\mathcal{C}$ in field $\ast$-topology.

After the remarks made the proof is not necessary. The next result shows how the described correspondence can be applied.

**Corollary 3.3** Under the assumption of Theorem 3.2, the Plancherel measure of $\hat{G}$ is supported on those representations of $G$ which do not contain the center of $G$ in their kernels.

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