Super Poly-harmonic Properties, Liouville Theorems and Classification of Nonnegative Solutions to Equations Involving Higher-order Fractional Laplacians

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Abstract. In this paper, we are concerned with equations (1.1) involving higher-order fractional Laplacians. By introducing a new approach, we prove the super poly-harmonic properties for nonnegative solutions to (1.1) (Theorem 1.1). Our theorem seems to be the first result on this problem. As a consequence, we derive many important applications of the super poly-harmonic properties. For instance, we establish Liouville theorems, integral representation formula and classification results for nonnegative solutions to fractional higher-order equations (1.1) with general nonlinearities \( f(x, u, Du, \cdots) \) including conformally invariant and odd order cases. In particular, our results completely improve the classification results for third order equations in Dai and Qin [21] by removing the assumptions on integrability. We also derive a characterization for \( \alpha \)-harmonic functions via averages in the appendix.

Keywords: Super poly-harmonic properties; Higher-order fractional Laplacians; Conformally invariant equations; Nonnegative classical solutions; Classification of solutions; Liouville theorems.

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1. Introduction

1.1. Background and setting of the problem. In this paper, we mainly consider nonnegative classical solutions to the following equations involving higher-order fractional Laplacians

\[
\begin{aligned}
&\left\{ (-\Delta)^{m+\frac{2}{\alpha}} u(x) = f(x, u, Du, \cdots), \quad x \in \mathbb{R}^n, \\
&u \in C^2_{loc}[\alpha, \alpha + \epsilon] \cap L_\alpha(\mathbb{R}^n), \quad u(x) \geq 0, \quad x \in \mathbb{R}^n,
\end{aligned}
\]

where \( n \geq 2, 1 \leq m < +\infty \) is an integer, \( 0 < \alpha < 2, \epsilon > 0 \) is arbitrarily small, \([\alpha]\) denotes the integer part of \( \alpha \), \( \{\alpha\} := \alpha - [\alpha] \), the higher-order fractional Laplacians \((-\Delta)^{m+\frac{2}{\alpha}} := (-\Delta)^m (-\Delta)^{\frac{2}{\alpha}} \) and nonlinearity \( f(x, u, Du, \cdots) \geq 0 \) is an arbitrary nonnegative function (may depend on \( x, u \) and derivatives of \( u \)) which is continuous with respect to \( x \in \mathbb{R}^n \).

For any \( u \in C^2_{loc}[\alpha, \alpha + \epsilon](\mathbb{R}^n) \cap L_\alpha(\mathbb{R}^n) \), the nonlocal operator \((-\Delta)^{\frac{2}{\alpha}} \) \((0 < \alpha < 2)\) is defined by (see [6, 12, 21, 22, 37, 41])

\[
(-\Delta)^{\frac{2}{\alpha}} u(x) = C_{n,\alpha} P.V. \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x - y|^{n+\alpha}} \, dy := C_{n} \lim_{\varepsilon \to 0} \int_{|y-x| \geq \varepsilon} \frac{u(x) - u(y)}{|x - y|^{n+\alpha}} \, dy,
\]

where the function space

\[
L_\alpha(\mathbb{R}^n) := \left\{ u : \mathbb{R}^n \to \mathbb{R} \mid \int_{\mathbb{R}^n} \frac{|u(x)|}{1 + |x|^{n+\alpha}} \, dx < \infty \right\}.
\]

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The fractional Laplacians \((-\Delta)^{\frac{\alpha}{2}}\) can also be defined equivalently (see [13]) by Caffarelli and Silvestre’s extension method (see [15]) for \(u \in C^{[\alpha],[\alpha]+\epsilon}(\mathbb{R}^n) \cap L_2(\mathbb{R}^n)\). Throughout this paper, we define \((-\Delta)^{m+\frac{\alpha}{2}} u := (-\Delta)^m (-\Delta)^{\frac{\alpha}{2}} u\) for \(u \in C^{2m+\alpha}[\alpha]+\epsilon(\mathbb{R}^n) \cap L_2(\mathbb{R}^n)\), where \((-\Delta)^{\frac{\alpha}{2}} u\) is defined by definition (1.2). Due to the nonlocal feature of \((-\Delta)^{\frac{\alpha}{2}}\), we need to assume \(u \in C^{2m+\alpha}[\alpha]+\epsilon(\mathbb{R}^n)\) with arbitrarily small \(\epsilon > 0\) (merely \(u \in C^{2m+\alpha}[\alpha]\) is not enough) to guarantee that \((-\Delta)^{\frac{\alpha}{2}} u \in C^{2m}(\mathbb{R}^n)\) (see [13, 37]), and hence \(u\) is a classical solution to equation (1.1) in the sense that \((-\Delta)^{\frac{\alpha}{2}} u\) is pointwise well-defined and continuous in the whole \(\mathbb{R}^n\).

For \(0 < \gamma < +\infty\), PDEs of the form
\[
(-\Delta)^{\frac{\alpha}{2}} u(x) = f(x, u, \cdots)
\]
have numerous important applications in conformal geometry and Sobolev inequalities, which also model many phenomena in mathematical physics, astrophysics, probability and finance (see [3, 12, 13, 15, 16, 17, 31, 39, 40] and the references therein). We say that equation (1.4) is in critical order if \(\gamma = n\), is in sub-critical order if \(0 < \gamma < n\) and is in super-critical order if \(n < \gamma < +\infty\).

1.2. Super poly-harmonic properties of nonnegative solutions. First, we will investigate the super poly-harmonic properties of nonnegative solutions to (1.1). It is well known that the super poly-harmonic properties of nonnegative solutions play a crucial role in establishing the integral representation formulae, Liouville type theorems and classification of solutions to higher order PDEs in \(\mathbb{R}^n\) or \(\mathbb{R}^n_+\) (see [2, 3, 4, 5, 11, 18, 19, 20, 21, 23, 25, 26, 31, 33, 39] and the references therein).

For integer higher-order equations (i.e., \(\alpha = 0\) in (1.1)), the super poly-harmonic properties of nonnegative solutions usually can be derived via the “spherical average, re-centers and iteration” arguments in conjunction with careful ODE analysis (we refer to [5, 31, 33, 39], see also [3, 4, 11, 20, 23, 25] and the references therein). However, for the fractional higher-order equation (1.1), so far there is no result on the super poly-harmonic properties. The reason for this is that \((-\Delta)^{\frac{\alpha}{2}}\) is nonlocal and \((-\Delta)^{\frac{\alpha}{2}} f(r)\) cannot be calculated or expanded accurately (\(0 < \alpha < 2\) and \(f(r)\) is a radially symmetric function), thus the strategy for integer higher-order equations does not work any more for equation (1.1) involving higher-order fractional Laplacians. To overcome these difficulties we need to implement new ideas and arguments. In this paper, by taking full advantage of the Poisson representation formulae for \((-\Delta)^{\frac{\alpha}{2}}\) and developing some new integral estimates on the average \(\int_{R^+} \frac{u(r)}{r^{n+\alpha}} dr\) and iteration techniques, we will introduce a new approach to overcome these difficulties and establish the super poly-harmonic properties of nonnegative classical solutions to (1.1) (see Section 2). Our theorem seems to be the first result on this problem.

**Theorem 1.1.** Assume \(n \geq 2\), \(m \geq 1\), \(0 < \alpha < 2\) and \(f \geq 0\) is continuous w.r.t. \(x \in \mathbb{R}^n\). Suppose that \(u\) is a nonnegative classical solution to (1.1). Then, we have, for every \(i = 0, 1, \cdots, m - 1\),
\[
(-\Delta)^{i+\frac{\alpha}{2}} u(x) \geq 0, \quad \forall x \in \mathbb{R}^n.
\]

Now suppose the nonlinearity \(f(x, u, Du, \cdots) \leq 0\) in (1.1) is an arbitrary function (may depend on \(x, u\) and derivatives of \(u\)) which is continuous with respect to \(x \in \mathbb{R}^n\), by using the ideas in proving the super poly-harmonic properties in Theorem 1.1 we can derive the following sub poly-harmonic properties of nonnegative classical solutions to (1.1).
Theorem 1.2. Assume $n \geq 2$, $m \geq 1$, $0 < \alpha < 2$ and $f \leq 0$ is continuous w.r.t. $x \in \mathbb{R}^n$. Suppose that $u$ is a nonnegative classical solution to \((1.1)\). Then, we have, for every $i = 0, 1, \cdots, m - 1$,
\begin{equation}
(-\Delta)^{i+\frac{\alpha}{2}} u(x) \leq 0, \quad \forall \ x \in \mathbb{R}^n.
\end{equation}

1.3. Liouville theorems, integral representation formula and classification of nonnegative solutions. In this subsection, by applying the super poly-harmonic properties in Theorem 1.1, we will derive some important results on equations involving higher-order fractional Laplacians.

(i) Liouville theorem for fractional poly-harmonic functions in $\mathbb{R}^n$.

Assume $u \geq 0$ is a nonnegative fractional poly-harmonic functions in $\mathbb{R}^n$, that is,
\begin{equation}
(-\Delta)^{m+\frac{\alpha}{2}} u(x) = 0, \quad \forall \ x \in \mathbb{R}^n,
\end{equation}
where $n \geq 2$, $1 \leq m < +\infty$ is an integer and $0 < \alpha < 2$.

As a consequence of the super poly-harmonic properties in Theorem 1.1 and the sub poly-harmonic properties in Theorem 1.2, we deduce that $(-\Delta)^{m+\frac{\alpha}{2}} u \equiv 0$ in $\mathbb{R}^n$ for every $i = 0, \cdots, m - 1$. In particular, one has $(-\Delta)^{\frac{\alpha}{2}} u \equiv 0$ in $\mathbb{R}^n$, and hence from the Liouville theorem for fractional Laplacians $(-\Delta)^{\frac{\alpha}{2}} u$ with $0 < \alpha < 2$ in [11, 41], it follows that $u \equiv C$ in $\mathbb{R}^n$ for some nonnegative constant $C \geq 0$. Therefore, we have the following Liouville theorem for fractional poly-harmonic functions in $\mathbb{R}^n$.

Theorem 1.3. Assume $n \geq 2$, $m \geq 1$ and $0 < \alpha < 2$. Suppose $u$ is a nonnegative fractional poly-harmonic functions in $\mathbb{R}^n$ satisfying \((1.7)\), then $u \equiv C \geq 0$ in $\mathbb{R}^n$.

(ii) Subcritical order cases $2m + \alpha < n$.

Equation \((1.1)\) is closely related to the following integral equation
\begin{equation}
u(x) = \int_{\mathbb{R}^n} \frac{R_{2m+\alpha,n}}{|x-y|^{n-2m-\alpha}} f(y,u(y),\cdots)dy,
\end{equation}
where the Riesz potential’s constants $R_{\gamma,n} := \frac{\Gamma(\frac{n-\gamma}{2})}{\pi^{\frac{n+\gamma}{2}} \Gamma(\frac{\gamma}{2})}$ for $0 < \gamma < n$ (see [38]).

From the super poly-harmonic properties of nonnegative solutions in Theorem 1.1 by using the methods in [6, 41], we can deduce the following equivalence between PDEs \((1.1)\) and IEs \((1.8)\).

Theorem 1.4. Assume $2m + \alpha < n$, $m \geq 1$, $0 < \alpha < 2$ and $f \geq 0$ is continuous w.r.t. $x \in \mathbb{R}^n$. Suppose that $u$ is a nonnegative classical solution to \((1.1)\), then $u$ is also a nonnegative solution to integral equation \((1.8)\), and vice versa.

Remark 1.5. Based on Theorem 1.1 the proof of Theorem 1.4 is entirely similar to [6, 41] (see also [20, 22]), so we omit the details here.

Remark 1.6. One can observe that, Theorem 1.1 and Theorem 1.4 hold for PDEs \((1.1)\) and IEs \((1.8)\) if we take the nonlinearities $f = |x|^a u^p$ ($a \geq 0$, $p > 0$), $f = |x|^a e^{nu}$ ($a \geq 0$) or $f = |x|^a u^p (1 + |\nabla u|^2)^{\frac{\alpha}{2}}$ ($a \geq 0$, $p > 0$, $\kappa > 0$) and so on. If we consider positive solution $u > 0$, then Theorem 1.1 and Theorem 1.4 are also valid for PDEs \((1.1)\) and IEs \((1.8)\) with $f = |x|^a u^{-q}$ ($a \geq 0$, $q > 0$).

Based on the equivalence between PDEs \((1.1)\) and IEs \((1.8)\), we will first consider the conformally invariant case $f = u^{|\frac{n+2m+n}{2m-n}}$, which is geometrically interesting.
The quantitative and qualitative properties of solutions to fractional order or higher order conformally invariant equations of the form
\begin{equation}
(\Delta)^\gamma u = u^{\frac{n+\alpha}{n-\gamma}} \quad \text{with} \quad 0 < \gamma < n
\end{equation}
have been extensively studied (see \cite{7, 12, 14, 16, 20, 21, 27, 31, 32, 33, 34, 39, 40} and the references therein). The classification results for conformally invariant equations (1.9) have important applications in many problems from conformal geometry (i.e., prescribing scalar curvature problems, variational problems involving Paneitz operators on compact Riemannian manifolds, applications in many problems from conformal geometry (i.e., prescribing scalar curvature problems, variational problems involving Paneitz operators on compact Riemannian manifolds, see \cite{7, 8, 9, 10, 14, 16, 21, 31, 32, 33, 34, 39, 40}). In \cite{14}, by developing the method of moving planes in integral forms, Chen, Li and Ou classified all positive solutions to (1.9) under additional weak integrability assumption \cite{14} (Theorem 1 in \cite{14}).

One should observe that, when \(\gamma \in (2, n)\) is an odd integer, or more general, when \(\gamma = 2m + \alpha < n\) with \(m \geq 1\) and \(0 < \alpha < 2\), classification for positive classical solutions to (1.9) is still open. In the particular case \(\gamma = 3\), by applying the harmonic asymptotic expansions for \((-\Delta)^{\frac{3}{2}} u\) (\(u\) is the Kelvin transform of \(u\)) and the method of moving planes to the third-order integral equation (1.9) directly, Dai and Qin \cite{21} derived the classification of nonnegative classical solutions to (1.9) under additional weak integrability assumption \(\int_{\mathbb{R}^n} \frac{u^{\frac{n+\alpha}{2}}}{|x|^{n-3}} \, dx < \infty\).

In this paper, by the classification of positive \(L_{loc}^{\frac{n-2m-\alpha}{2}}\) solutions to integral equation (1.8) in \cite{14} (Theorem 1 in \cite{14}) and the equivalence between PDE (1.1) and integral equation (1.8) in Theorem 1.4, we can classify all positive classical solutions to (1.1) in the conformally invariant cases \(f = u^{\frac{n+2m+\alpha}{2-2m}}\) without any assumptions on integrability or decay of \(u\).

Our classification result for (1.1) in the conformally invariant cases is as follows.

**Theorem 1.7.** Assume \(2m + \alpha < n\), \(m \geq 1\), \(0 < \alpha < 2\) and \(f = u^{\frac{n+2m+\alpha}{2-2m}}\). Suppose that \(u\) is a nonnegative classical solution of (1.1), then either \(u \equiv 0\) or \(u\) is of the following form
\[u(x) = \mu \frac{n-2m-\alpha}{2m+\alpha} Q(\mu(x - x_0)) \quad \text{for some} \quad \mu > 0 \quad \text{and} \quad x_0 \in \mathbb{R}^n,
\]
where
\[Q(x) := \left(\frac{1}{R_{2m+\alpha,n}} I(\frac{n-2m-\alpha}{2(2m+\alpha)}) \right)^{n-2m-\alpha} \left(1 + \frac{1}{|x|^2}\right)^{\frac{n-2m-\alpha}{2-2m+\alpha}}\]
with \(I(s) := \frac{\pi^{\frac{n-s}{2}}}{\Gamma\left(1 - \frac{s}{2}\right)}\) for \(0 < s < \frac{n}{2}\).

**Remark 1.8.** Theorem 1.7 follows directly from Theorem 1 in \cite{14} and Theorem 1.4 so we omit the details here. The exact constants in the expression of \(Q(x)\) are given by formula (37) in Lemma 4.1 in \cite{14}.

**Remark 1.9.** Combining Theorem 1.7 with the classification theorems in \cite{7, 12, 14, 27, 31, 39} gives us the complete classification results for conformally invariant equations (1.1) in all the cases \(0 < \gamma < n\). If we take \(\alpha = 1\), then Theorem 1.7 gives the classification results for all the odd order conformally invariant equations (1.1). In particular, Theorem 1.7 completely
improves the classification results for third order conformally invariant equations in [21] by removing the integrability assumption $\int_{\mathbb{R}^n} \frac{|u|^p}{|x|^m} \, dx < \infty$.

Next, we take $f = |x|^\alpha u^p$ ($a \geq 0$, $p > 0$) and study the Liouville property of nonnegative solutions in the subcritical cases.

For PDEs (1.1) and IEs (1.8), we say the Hardy-Hénon type nonlinearities $f = |x|^\alpha u^p$ is subcritical if $0 < p < p_c(a) := \frac{n+2m+n+2\alpha}{n-2m-\alpha}$, critical if $p = p_c(a)$ and super-critical if $p > p_c(a)$. There are also lots of literature on Liouville type theorems for fractional order or higher order Hardy-Hénon type equations in the subcritical cases, and we refer to [11, 12, 13, 14, 21, 22, 23, 24, 25, 26, 31, 35, 39, 41] and the references therein. It should be noted that, all the known results focused on the cases $m = 0$ or $\alpha = 0$, hence Liouville type theorems for general fractional higher-order cases $m \geq 1$ and $0 < \alpha < 2$ are still open. In the particular case $m = \alpha = 1$ and $f = u^p$ with $1 \leq p < \frac{n+3}{n-3}$, Dai and Qin [21] derived Liouville type theorem for nonnegative classical solutions to (1.1) under additional weak integrability assumption $\int_{\mathbb{R}^n} \frac{u^p}{|x|^{n-2}} \, dx < \infty$.

In this paper, by applying the method of scaling spheres developed recently by Dai and Qin [22] (see also [23, 24, 26]), we will establish Liouville type theorem for nonnegative solutions to IEs (1.8). Our Liouville type result for IEs (1.8) is as follows.

**Theorem 1.10.** Assume $2m + \alpha < n$, $m \geq 1$, $0 < \alpha < 2$ and $f = |x|^\alpha u^p$ with $a \geq 0$ and $0 < p < p_c(a)$. Suppose $u \in C(\mathbb{R}^n)$ is a nonnegative solution to IEs (1.8), then $u \equiv 0$ in $\mathbb{R}^n$.

**Remark 1.11.** It is clear from the proof of Theorem 1.10 that (see (4.41) in Section 3), the Liouville type results in Theorem 1.10 are also valid for $f = |x|^\alpha u^p$ ($i = 1, 2, \ldots, n$) with $a \geq 0$ and $0 < p < p_c(a)$. Theorem 1.10 can also be available for more general nonlinearities $f(x, u)$ satisfying appropriate assumptions, we leave the details to readers (we refer to [22, 23, 24, 26]).

From the equivalence between PDEs (1.1) and IEs (1.8) in Theorem 1.4 and Theorem 1.10, we derive the following Liouville type result for nonnegative classical solutions to PDEs (1.1) immediately.

**Corollary 1.12.** Assume $2m + \alpha < n$, $m \geq 1$, $0 < \alpha < 2$ and $f(x, u) = |x|^\alpha u^p$ with $a \geq 0$ and $0 < p < p_c(a)$. Suppose $u$ is a nonnegative classical solution to PDEs (1.1), then $u \equiv 0$ in $\mathbb{R}^n$.

**Remark 1.13.** If we take $\alpha = 1$, then Corollary 1.12 gives Liouville type results for all the odd order equations (1.1) with $f = |x|^\alpha u^p$ in subcritical cases $0 < p < p_c(a)$. In particular, Corollary 1.12 completely improves the Liouville theorem for third order equations (1.1) with $f = u^p$ ($1 \leq p < \frac{n+3}{n-3}$) in [21] by removing the integrability assumption $\int_{\mathbb{R}^n} \frac{u^p}{|x|^{n-2}} \, dx < \infty$ and extending $1 \leq p < \frac{n+3}{n-3}$ to the full subcritical range $0 < p < \frac{n+3}{n-3}$.

(iii) Critical and super-critical order cases: $n \leq 2m + \alpha < +\infty$.

As an immediate consequence of the super poly-harmonic properties in Theorem 1.1 by arguments developed by Chen, Dai and Qin [3], we can establish Liouville type theorem for nonnegative solutions to (1.1) with general nonlinearities $f$ in both critical and super-critical order cases. For the particular case $\alpha = 0$, Liouville type theorems for integer higher-order Hénon-Hardy type equations in $\mathbb{R}^n$ or $\mathbb{R}^n_+$ have been derived by Chen, Dai and Qin [3] and Dai and Qin [23] in both critical and super-critical order cases. Our result will extend the results in [3] to general fractional higher-order cases $0 < \alpha < 2$ and general nonlinearities $f(x, u, \cdots)$. For the critical and super-critical order cases we have the following result.
Theorem 1.14. Assume $n \geq 3$, $m \geq 1$, $0 < \alpha < 2$, $\frac{\alpha}{2} \leq m + \frac{\alpha}{2} < +\infty$, $f \geq 0$ is continuous w.r.t. $x \in \mathbb{R}^n$ and $f > 0$ at some point in $\mathbb{R}^n$ if $u > 0$ in the whole $\mathbb{R}^n$. Suppose that $u$ is a nonnegative classical solution to (1.1), then $u \equiv 0$ in $\mathbb{R}^n$.

Remark 1.15. If we take $\alpha = 1$, then Theorem 1.14 gives Liouville type results for all the critical and super-critical order equations (1.1) involving odd order Laplacians. One should observe that, if $f \geq 0$ is continuous w.r.t. $x \in \mathbb{R}^n$ and $f \geq C|x|^a u^p$ for some $a \in \mathbb{R}$, $p > 0$, $C > 0$ and some point $x \neq 0$ in $\mathbb{R}^n$, then $f$ satisfies the assumptions in Theorem 1.14 and hence Theorem 1.14 is valid for equations (1.1) with such kind of nonlinearities.

Remark 1.16. If we consider positive solution $u > 0$, suppose $f \geq 0$ is continuous w.r.t. $x \in \mathbb{R}^n$ and $f \geq C|x|^a u^{-q}$ for some $a \in \mathbb{R}$, $q > 0$, $C > 0$ and some point $x \neq 0$ in $\mathbb{R}^n$, then Theorem 1.14 implies nonexistence of positive solutions and thus extend Theorems 1.2 and 1.3 in [36] to general fractional higher-order cases $0 < \alpha < 2$ and general nonlinearities $f(x, u, \ldots)$.

This paper is organized as follows. In Section 2, we will carry out our proof of Theorem 1.1. In Section 3, we will prove Theorem 1.2. Section 4 and 5 are devoted to proving Theorems 1.10 and 1.14 respectively. In the Appendix, we establish an important characterization for $\alpha$-harmonic functions via the averages $\int_R^+ \frac{R^a}{r^{n+\alpha}} \pi(r) dr$ and deduce some important properties for $\alpha$-harmonic functions.

Throughout this paper, we will use $C$ to denote a general positive constant that may depend on $u$ and the quantities appearing in the subscript, and whose value may differ from line to line.

2. Proof of Theorem 1.1

In this section, we will carry out our proof of the super poly-harmonic properties for non-negative solutions to (1.1) (i.e., Theorem 1.1) via contradiction arguments.

Let $v_i := (-\Delta)^{i+\frac{\alpha}{2}} u$ for $i = 0, 1, \ldots, m - 1$, then it follows from equation (1.1) that

$$
\begin{align*}
(-\Delta)^{\frac{\alpha}{2}} u &= v_0 & \text{in } \mathbb{R}^n, \\
-\Delta v_0 &= v_1 & \text{in } \mathbb{R}^n, \\
& \vphantom{(-\Delta)^{\frac{\alpha}{2}}} \ldots \\
-\Delta v_{m-1} &= f \geq 0 & \text{in } \mathbb{R}^n.
\end{align*}
$$

(2.1)

Suppose that Theorem 1.1 does not hold, then there must exist a largest integer $0 \leq k \leq m - 1$ and a point $x_0 \in \mathbb{R}^n$ such that

$$
v_k(x_0) = (-\Delta)^{k+\frac{\alpha}{2}} u(x_0) < 0.
$$

(2.2)

Let

$$
\bar{g}(r) = \bar{g}(|x - x_0|) := \frac{1}{|\partial B_r(x_0)|} \int_{\partial B_r(x_0)} g(x) d\sigma
$$

(2.3)

be the spherical average of a function $g$ with respect to the center $x_0$.

First, we will show that $0 \leq k \leq m - 1$ is even. Suppose not, assume $k$ is an odd integer. From (2.1) and the well-known property $\Delta u = \Delta \bar{u}$, we get

$$
\bar{v}_k(r) \leq \bar{v}_k(0) := -c_0 < 0, \quad \forall \ r > 0.
$$

(2.4)

It follows immediately that

$$
\bar{v}_{k-1}(r) \geq \bar{v}_{k-1}(0) + \frac{c_0}{2n} r^2, \quad \forall \ r > 0,
$$

(2.5)
and
\[ (2.6) \quad \overline{v}_{k-2}(r) \leq \overline{v}_{k-2}(0) - \frac{r^2}{2n} \overline{v}_{k-1}(0) - \frac{c_0}{8n(n+2)} r^4, \quad \forall \ r > 0. \]

Repeating the above argument, we get
\[ (2.7) \quad \overline{v}_{0}(r) \geq \overline{v}_{0}(0) + c_1 r^2 + c_2 r^4 + \cdots + c_k r^{2k}, \quad \forall \ r > 0, \]
where \( c_k > 0 \). From (2.7), we infer that there exists a \( r_0 \) large enough, such that
\[ (2.8) \quad \overline{v}_{0}(r) \geq \frac{1}{2} c_k r^{2k}, \quad \forall \ r > r_0. \]

From the first equation in (2.1), we conclude that, for arbitrary \( R > 0 \),
\[ (2.9) \quad u(x) = \int_{B_R(x_0)} G_{R}^{\alpha}(x, y)v_0(y)dy + \int_{|y-x_0| > R} P_{R}^{\alpha}(x, y)u(y)dy, \quad \forall \ x \in B_R(x_0), \]
where the Green’s function for \((-\Delta)^{\frac{\alpha}{2}}\) with \( 0 < \alpha < 2 \) on \( B_R(x_0) \) is given by
\[ (2.10) \quad G_{R}^{\alpha}(x, y) := \frac{C_{n, \alpha}}{|x-y|^{n-\alpha}} \int_{t_R}^{s_R} \frac{b_\alpha^{-1}}{(1+b)^{\frac{\alpha}{2}}} db \quad \text{if } x, y \in B_R(x_0) \]
with \( s_R = \frac{|x-y|^2}{R^2} \), \( t_R = \left(1 - \frac{|x-y|^2}{R^2}\right) \left(1 - \frac{|y-x_0|^2}{R^2}\right) \), and \( G_{R}^{\alpha}(x, y) = 0 \) if \( x \) or \( y \in \mathbb{R}^n \setminus B_R(x_0) \) (see [28]), and the Poisson kernel \( P_{R}^{\alpha}(x, y) \) for \((-\Delta)^{\frac{\alpha}{2}}\) in \( B_R(x_0) \) is defined by \( P_{R}^{\alpha}(x, y) := 0 \) for \(|y-x_0| < R\) and
\[ (2.11) \quad P_{R}^{\alpha}(x, y) := \frac{\Gamma\left(\frac{\alpha}{2}\right)}{\pi^{\frac{n}{2}+1}} \sin \frac{\pi \alpha}{2} \left(\frac{R^2 - |x-x_0|^2}{|y-x_0|^2 - R^2}\right)^{\frac{\alpha}{2}} \frac{1}{|x-y|^n} \]
for \(|y-x_0| > R\) (see [13]). Therefore, we have
\[ (2.12) \quad +\infty > u(x_0) = \int_{B_R(x_0)} \frac{C_{n, \alpha}}{|y-x_0|^{n-\alpha}} \left(\int_{0}^{\frac{R^2}{|y-x_0|^2} - R^2} \frac{b_\alpha^{-1}}{(1+b)^{\frac{\alpha}{2}}} db \right) v_0(y)dy + \]
\[ + C_{n, \alpha} \int_{|y-x_0| > R} \frac{R^0}{|y-x_0|^n} \frac{u(y)}{|y-x_0|^n} dy \]
\[ = C_{n, \alpha} \int_{0}^{R} r^{\alpha-1} \left(\int_{0}^{\frac{R^2}{r^2} - R^2} \frac{b_\alpha^{-1}}{(1+b)^{\frac{\alpha}{2}}} db \right) \overline{v}_{0}(r)dr + C_{n, \alpha} \int_{R}^{+\infty} \frac{R^0}{r^{\alpha-1}} \overline{u}(r)dr. \]

Observe that, if \( 0 < r \leq \frac{R}{2} \), then \( 3 \leq \frac{R^2}{r^2} - 1 < +\infty \), and hence
\[ (2.13) \quad \int_{0}^{3} \frac{b_\alpha^{-1}}{(1+b)^{\frac{\alpha}{2}}} db \leq \int_{0}^{R^2} \frac{b_\alpha^{-1}}{(1+b)^{\frac{\alpha}{2}}} db \leq \int_{0}^{+\infty} \frac{b_\alpha^{-1}}{(1+b)^{\frac{\alpha}{2}}} db. \]

As a consequence of (2.7), (2.8), (2.12) and (2.13), we deduce that
\[ (2.14) \quad u(x_0) \geq C_{n, \alpha} \int_{r_0}^{R} r^{\alpha-1} \overline{v}_{0}(r)dr - \tilde{C}_{n, \alpha} \int_{0}^{r_0} r^{\alpha-1} |\overline{v}_{0}(r)|dr \]
\[ \geq C \int_{r_0}^{R} r^{2k+\alpha-1}dr - \tilde{C} \geq CR^{2k+\alpha} - \tilde{C}. \]
for any $R > 2r_0$. By letting $R \to +\infty$ in (2.14), we get immediately a contradiction. Therefore, $k$ must be even.

Next, we will show that $k = 0$. Suppose on contrary that $2 \leq k \leq m - 1$ is even, through similar procedure as in deriving (2.7), we obtain

$$\tag{2.15} \overline{v}_0(r) \leq \overline{v}_0(0) - c_1 r^2 - c_2 r^4 - \cdots - c_k r^{2k}, \quad \forall \ r > 0,$$

where $c_k > 0$. Thus there exists a $r_1 > 0$ large enough such that

$$\tag{2.16} \overline{v}_0(r) \leq -\frac{1}{2} c_k r^{2k}, \quad \forall \ r > r_1.$$

Observe that, if $\frac{R}{2} < r < R$, then $0 < \frac{R^2}{r^2} - 1 < 3$, and hence

$$\tag{2.17} \int_0^{\frac{R^2}{r^2} - 1} \frac{b_2^{-1}}{(1 + b)^2} db \geq \int_0^{\frac{R^2}{r^2} - 1} \frac{b_2^{-1}}{2n} db \geq C_{n, \alpha} \left( \frac{R^2}{r^2} - 1 \right)^{\frac{2}{\alpha}}.$$

It follows from (2.12), (2.13), (2.15), (2.16) and (2.17) that, for any $R > 2r_1$,

$$\tag{2.18} \int_R^{+\infty} \frac{R^\alpha \bar{u}(r)}{r (r^2 - R^2)^{\frac{2}{\alpha}}} dr \geq -C_{n, \alpha} \int_0^R r^{\alpha - 1} \left( \int_0^{\frac{R^2}{r^2} - 1} \frac{b_2^{-1}}{(1 + b)^2} db \right) \overline{v}_0(r) dr$$

$$\geq C \int_{r_1}^R r^{2k+\alpha - 1} dr - \tilde{C} \int_0^{r_1} r^{\alpha - 1} |\overline{v}_0(r)| dr + C \int_R^{+\infty} r^{2k+\alpha - 1} \left( \frac{R^2}{r^2} - 1 \right)^{\frac{2}{\alpha}} dr$$

$$\geq CR^{2k+\alpha} - \tilde{C}.$$

Thus there exists a $r_2 > 2r_1$ large enough such that

$$\tag{2.19} \int_R^{+\infty} \frac{R^\alpha \bar{u}(r)}{r (r^2 - R^2)^{\frac{2}{\alpha}}} dr \geq CR^{2k+\alpha}, \quad \forall \ R > r_2.$$

Since $u \in \mathcal{L}_\alpha(\mathbb{R}^n)$, we have

$$\tag{2.20} \int_{|x-x_0| > 1} \frac{u(x)}{|x-x_0|^{n+\alpha}} dx = C \int_1^{+\infty} \frac{\overline{u}(r)}{r^{1+\alpha}} dr < +\infty,$$

and hence, for any $\delta > 0$,

$$\tag{2.21} \int_1^{+\infty} \frac{1}{R^{1+\alpha+\delta}} \int_R^{+\infty} \frac{R^\alpha \bar{u}(r)}{r (r^2 - R^2)^{\frac{2}{\alpha}}} dr dR = \int_1^{+\infty} \frac{\overline{u}(r)}{r} \int_1^r \frac{1}{R^{1+\delta}(r^2 - R^2)^{\frac{2}{\alpha}}} dR dr$$

$$\leq C \int_1^{+\infty} \frac{\bar{u}(r)}{r^{1+\alpha}} \int_1^{+\infty} \frac{1}{R^{1+\delta}} dR dr + C \int_1^{+\infty} \frac{\bar{u}(r)}{r^{2+\delta}} \int_1^{+\infty} \frac{1}{R^{2+\delta}(r^2 - R^2)^{\frac{2}{\alpha}}} dR dr$$

$$\leq C \delta \int_1^{+\infty} \frac{\bar{u}(r)}{r^{1+\alpha}} dr + C \int_1^{+\infty} \frac{\bar{u}(r)}{r^{1+\alpha}} dr < +\infty,$$

which is a contradiction with (2.19) and thus $k = 0$.

Since $k = 0$, we deduce that

$$\tag{2.22} \overline{v}(r) \leq \overline{v}_0(0) := -c_0 < 0, \quad \forall \ r > 0.$$
Thus (2.12), (2.13), (2.17) and (2.22) yield that, for any \( R > 0 \),

\[
\int_R^{+\infty} \frac{R^\alpha u(r)}{r^{(r^2 - R^2)^{\frac{1}{2}}}dr} \geq C \int_0^R r^{\alpha - 1} dr + C \int_0^{R^{\frac{1}{2}}} r^{\alpha - 1} \left( \frac{R^2}{r^2} - 1 \right)^{\frac{\alpha}{2}} dr
\]

\[
\geq CR^\alpha + C \int_0^R R^{\frac{\alpha}{2} - 1} (R - r)^{\frac{\alpha}{2}} dr \geq CR^\alpha.
\]

Since \( u \in \mathcal{L}_\alpha(\mathbb{R}^n) \), we have

\[
\int_{|x-x_0| > N} \frac{u(x)}{|x-x_0|^{n+\alpha}} dx = C \int_N^{+\infty} \frac{\overline{u}(r)}{r^{1+\alpha}} dr = o_N(1)
\]

as \( N \to +\infty \), and hence

\[
\int_2^{2R} \frac{R^\alpha u(r)}{r^{(r^2 - R^2)^{\frac{1}{2}}}dr} dr \leq CR^\alpha \int_2^{2R} \frac{\overline{u}(r)}{r^{1+\alpha}} dr = o_2(1)R^\alpha
\]

as \( R \to +\infty \). We can choose \( R_0 > 0 \) sufficiently large such that, \( o_2(1) < \frac{C}{2} \) for any \( R > R_0 \) with the same constant \( C \) as in the RHS of (2.23). Consequently, it follows from (2.23) and (2.25) that

\[
\int_R^{2R} \frac{R^\alpha u(r)}{r^{(r^2 - R^2)^{\frac{1}{2}}}dr} dr > CR^\alpha, \quad \forall R > R_0.
\]

By (2.20), we arrive at

\[
\int_1^{+\infty} \frac{1}{R^{1+\alpha}} \int_R^{2R} \frac{R^\alpha u(r)}{r^{(r^2 - R^2)^{\frac{1}{2}}}dr} dR = \int_1^{+\infty} \overline{u}(r) \int_r^{+\infty} \frac{1}{R^{(r^2 - R^2)^{\frac{1}{2}}}dR} dr 
\]

\[
\leq C \int_1^{+\infty} \overline{u}(r) \int_r^{+\infty} \frac{1}{(r-R)^{\frac{1}{2}}} dR dr \leq C \int_1^{+\infty} \overline{u}(r) dr < +\infty,
\]

which is a contradiction with (2.26). Therefore, the super poly-harmonic properties in Theorem 1.1 holds and hence Theorem 1.1 is proved.

3. Proof of Theorem 1.2

In this section, we show sub poly-harmonic properties for nonnegative classical solutions to equations (1.1) with \( f \leq 0 \), i.e. Theorem 1.2.

Suppose that \( u \) is a nonnegative classical solution to (1.1). Let \( u_i(x) := (-\Delta)^{\frac{i}{2}} u(x) \) and \( u_i(x) := (-\Delta)^{i-1} u_1(x) \) for \( i = 2, \cdots, m \). Then, from equations (1.1), we have

\[
\begin{cases}
(-\Delta)^{\frac{i}{2}} u(x) = u_1(x) & \text{in } \mathbb{R}^n, \\
-\Delta u_1(x) = u_2(x) & \text{in } \mathbb{R}^n, \\
\cdots & \\
-\Delta u_m(x) = f \leq 0 & \text{in } \mathbb{R}^n.
\end{cases}
\]

Our aim is to prove that \( u_i \leq 0 \) in \( \mathbb{R}^n \) for every \( i = 1, \cdots, m \).

First, we will prove \( u_m \leq 0 \) by contradiction arguments. If not, then there exists \( x_0 \in \mathbb{R}^n \) such that \( u_m(x_0) > 0 \). By taking spherical average w.r.t. center \( x_0 \) to all equations except the
first equation in (3.1), we have
\[
\begin{align*}
\begin{cases}
(-\Delta)^{\frac{\alpha}{2}} u(x) = u_1(x) & \text{in } \mathbb{R}^n, \\
-\Delta u_1(r) = \bar{u}_2(r), & \forall r \geq 0, \\
\cdots \\
-\Delta \bar{u}_m(r) = \bar{f}(r) \leq 0, & \forall r \geq 0.
\end{cases}
\end{align*}
\tag{3.2}
\]

From last equation of (3.2), one has
\[
-\frac{1}{r^{n-1}} \left(r^{n-1} \bar{u}_m'(r)\right)' \leq 0, \quad \forall r \geq 0.
\tag{3.3}
\]

Integrating both sides of (3.3) twice gives
\[
\bar{u}_m(r) \geq \bar{u}_m(0) = u_m(x_0) =: c_0 > 0,
\tag{3.4}
\]
for any \( r \geq 0 \). Then from the last but one equation of (3.2) we derive
\[
-\frac{1}{r^{n-1}} \left(r^{n-1} \bar{u}_{m-1}'(r)\right)' \geq c_0, \quad \forall r \geq 0.
\tag{3.5}
\]

Again, by integrating both sides of (3.5) twice, we arrive at
\[
\bar{u}_{m-1}(r) \leq \bar{u}_{m-1}(0) - c_1 r^2, \quad \forall r \geq 0,
\tag{3.6}
\]
where \( c_1 = \frac{c_0}{2n} > 0 \). Continuing this way, we finally obtain that
\[
(-1)^{m-1} \bar{u}_1(r) \geq a_{m-1} r^{2(m-1)} + \cdots + a_0, \quad \forall r \geq 0,
\tag{3.7}
\]
where \( a_{m-1} > 0 \). Hence, we have
\[
(-1)^{m-1} \bar{u}_1(r) \geq C r^{2(m-1)},
\tag{3.8}
\]
for any \( r > R_0 \) with \( R_0 \) sufficiently large. One should observe that it is very difficult to take spherical average to the first equation in (3.2), since the fractional Laplacian \((-\Delta)^{\frac{\alpha}{2}}\) is a nonlocal operator. Instead of taking spherical average, we will apply the Green-Poisson representation formulae for \((-\Delta)^{\frac{\alpha}{2}}\) in the first equation in (3.2) to overcome this difficulty.

From the first equation in (3.2), we have
\[
\begin{align*}
\int_{B_R(x_0)} G_R^\alpha(x, y) u_1(y) dy + \int_{|y-x_0| > R} P_R^\alpha(x, y) u(y) dy, \quad \forall x \in B_R(x_0),
\end{align*}
\tag{3.9}
\]
where \( G_R^\alpha \) is the Green’s function for \((-\Delta)^{\frac{\alpha}{2}}\) with \( 0 < \alpha < 2 \) on \( B_R(x_0) \) and \( P_R^\alpha(x, y) \) is the Poisson kernel for \((-\Delta)^{\frac{\alpha}{2}}\) in \( B_R(x_0) \). Taking \( x = x_0 \) in (3.9) gives
\[
\begin{align*}
u(x_0) &= \int_{B_R(x_0)} C_{n,\alpha} \left( \int_0^{\frac{R^2}{|y-x_0|^2}} \frac{b_{\frac{\alpha}{2}}^{-\frac{1}{2}}}{(1+b)^{\frac{\alpha}{2}}} db \right) u_1(y) dy \\
&\quad + C_{n,\alpha}' \int_{|y-x_0| > R} \frac{R^\alpha}{(|y-x_0|^2 - R^2)^{\frac{\alpha}{2}}} u(y) dy \\
&\quad = \int_{0}^{R} \bar{C}_{n,\alpha} \left( \int_0^{\frac{R^2}{r^2}} \frac{b_{\frac{\alpha}{2}}^{-\frac{1}{2}}}{(1+b)^{\frac{\alpha}{2}}} db \right) \bar{u}_1(r) dr + \bar{C}_{n,\alpha} \int_{R}^{\infty} \frac{R^\alpha \bar{u}(r)}{r(r^2 - R^2)^{\frac{\alpha}{2}}} dr.
\end{align*}
\tag{3.10}
\]
One can easily observe that for \(0 < r \leq \frac{R}{2}\), \(R^2 - 1 \geq 3\) and hence \(\int_0^{\frac{R^2 - 1}{2}} \frac{b_2^{\alpha - 1}}{(1 + b)^{\frac{\alpha}{2}}} db \geq 0\). For \(\frac{R}{2} < r < R\), one has \(0 < \frac{R^2}{r^2} - 1 < 3\), thus \(\int_0^{\frac{R^2}{r^2}} \frac{b_2^{\alpha - 1}}{(1 + b)^{\frac{\alpha}{2}}} db > \int_0^{\frac{R^2}{r^2}} \frac{b_2^{\alpha - 1}}{2^{\alpha/2}} db =: C_2\left(\frac{R^2 - r^2}{r^2}\right)^{\frac{\alpha}{2}}\). Thus, we conclude that

\[
\int_0^{\frac{R^2 - 1}{2}} \frac{b_2^{\alpha - 1}}{(1 + b)^{\frac{\alpha}{2}}} db \geq C_1 \chi_{0 < r \leq \frac{R}{2}} + C_2 \chi_{\frac{R}{2} < r < R}\left(\frac{R^2 - r^2}{r^2}\right)^{\frac{\alpha}{2}}
\]

for any \(0 < r < R\). Then, from (3.8), (3.10) and (3.11), we derive that, for \(R > 2R_0\),

\[
(-1)^m \int_0^{R} \frac{R^2 \bar{u}(r)}{r(r^2 - R^2)^{\frac{\alpha}{2}}} dr = C \int_0^{R} \frac{1}{r^{1-\alpha}} \left(\int_0^{\frac{R^2}{r^2} - 1} \frac{b_2^{\alpha - 1}}{(1 + b)^{\frac{\alpha}{2}}} db\right) \left(-1\right)^{m-1} \bar{u}(r) dr + (-1)^m u(x_0) \\
\geq C \int_0^{R} \frac{1}{r^{1-\alpha}} \left[ C_1 \chi_{0 < r \leq \frac{R}{2}} + C_2 \chi_{\frac{R}{2} < r < R}\left(\frac{R^2 - r^2}{r^2}\right)^{\frac{\alpha}{2}}\right] r^{2(m-1)} dr \\
+ C \int_0^{R} \frac{1}{r^{1-\alpha}} \left(\int_0^{\frac{R^2}{r^2}} \frac{b_2^{\alpha - 1}}{(1 + b)^{\frac{\alpha}{2}}} db\right) \left(-1\right)^{m-1} \bar{u}(r) dr + (-1)^m u(x_0) \\
\geq CR^{2m-2+\alpha} - C_0 + (-1)^m u(x_0).
\]

It is easy to see that if \(m\) is odd, (3.12) implies that \(\int_0^{R} \frac{R^2 \bar{u}(r)}{r(r^2 - R^2)^{\frac{\alpha}{2}}} dr < 0\) for \(R\) sufficiently large, which contradicts with \(u \geq 0\). Therefore, we only need to consider the case that \(m\) is even. In such cases, (3.12) implies

\[
\int_0^{+\infty} \frac{R^2 \bar{u}(r)}{r(r^2 - R^2)^{\frac{\alpha}{2}}} dr \geq CR^{2m-2+\alpha}
\]

for \(R\) sufficiently large. Since \(u \in L_\alpha(\mathbb{R}^n)\), we have

\[
\int_{|x - x_0| > 1} \frac{u(x)}{|x - x_0|^{n+\alpha}} dx = C \int_1^{+\infty} \bar{u}(r) \frac{1}{r^{1+\alpha}} dr < +\infty.
\]

Then, by (3.14) and the fact that \(2m - 2 \geq 0\), for \(R\) sufficiently large, we have

\[
\int_0^{2R} \frac{R^2 \bar{u}(r)}{r(r^2 - R^2)^{\frac{\alpha}{2}}} dr \geq CR^{2m-2+\alpha} - \int_0^{+\infty} \frac{R^2 \bar{u}(r)}{r(r^2 - R^2)^{\frac{\alpha}{2}}} dr \\
\geq CR^{2m-2+\alpha} - o_R(1) R^\alpha \\
\geq CR^{2m-2+\alpha}.
\]
On the one hand, by (3.14), we have

\[
\int_1^{+\infty} \frac{1}{R^{1+\alpha}} \int_R^{2R} \frac{\alpha \overline{u}(r)}{r(r^2 - R^2)^{\frac{\alpha}{2}}} dr dR = \int_1^{+\infty} \frac{\bar{u}(r)}{r} \int_{\frac{r}{2}}^{r} \frac{1}{R(r^2 - R^2)^{\frac{\alpha}{2}}} dR dr \\
\leq C \int_1^{+\infty} \frac{\bar{u}(r)}{r^{1+\alpha}} dr < +\infty.
\]

On the other hand, by (3.15), we derive

\[
\int_1^{+\infty} \frac{1}{R^{1+\alpha}} \int_R^{2R} \frac{\alpha \overline{u}(r)}{r(r^2 - R^2)^{\frac{\alpha}{2}}} dr dR \geq \int_1^{+\infty} \frac{1}{R} dR = +\infty,
\]

where \(N\) is sufficiently large such that (3.15) holds for any \(R > N\). Combining (3.16) with (3.17), we get a contradiction. Hence, we must have \(u_m \leq 0\) in \(\mathbb{R}^n\). One should observe that, in the proof of \(u_m \leq 0\), we have mainly used the property \(-\Delta u_m \leq 0\). Therefore, through a similar argument as above, one can prove that \(u_{m-1} \leq 0\). Continuing this way, we obtain that \(u_i = (-\Delta)^{i-1+\frac{n}{2}} u \leq 0\) for every \(i = 1, 2, \ldots, m\). This completes the proof of Theorem 1.2.

4. Proof of Theorem 1.10

In this section, we will prove Theorem 1.10 by way of contradiction and the method of scaling spheres developed by Dai and Qin [22] (see also [23, 24, 26]). For more related literature on the method of moving planes (spheres), we refer to [2, 3, 4, 6, 7, 8, 9, 10, 12, 14, 17, 18, 19, 21, 25, 27, 30, 31, 32, 34, 39, 40] and the references therein.

Now suppose, on the contrary, that \(u \geq 0\) satisfies integral equations (1.8) but \(u\) is not identically zero, then there exists a point \(\bar{x} \in \mathbb{R}^n\) such that \(u(\bar{x}) > 0\). It follows from (1.8) immediately that

\[
u(x) > 0, \quad \forall \ x \in \mathbb{R}^n,
\]

i.e., \(u\) is actually a positive solution in \(\mathbb{R}^n\). Moreover, there exists a constant \(C > 0\), such that the solution \(u\) satisfies the following lower bound:

\[
u(x) \geq \frac{C}{|x|^{n-2m-\alpha}} \quad \text{for} \ |x| \geq 1.
\]

Indeed, since \(u > 0\) satisfies the integral equation (1.8), we can infer that

\[
\begin{align*}
u(x) & \geq C_{n,m,\alpha} \int_{|y| \leq \frac{1}{2}} \frac{|y|^a}{|x-y|^{n-2m-\alpha}} u^p(y) dy \\
& \geq \frac{C}{|x|^{n-2m-\alpha}} \int_{|y| \leq \frac{1}{2}} |y|^a u^p(y) dy =: \frac{C}{|x|^{n-2m-\alpha}}
\end{align*}
\]

for all \(|x| \geq 1\).

Next, we will apply the method of scaling spheres to show the following lower bound estimates for positive solution \(u\), which contradict with the integral equations (1.8) for \(0 < p < \frac{n+2m+\alpha+2a}{n-2m-\alpha}\).

**Theorem 4.1.** Assume \(m \geq 1, 0 < \alpha < 2, 2m + \alpha < n, 0 \leq a < +\infty, 0 < p < \frac{n+2m+\alpha+2a}{n-2m-\alpha}\). Suppose \(u \in C(\mathbb{R}^n)\) is a positive solution to (1.8), then it satisfies the following lower bound...
estimates: for $|x| \geq 1$,
\begin{equation}
(4.4) \quad u(x) \geq C_\kappa |x|^\kappa \quad \forall \kappa < \frac{2m + \alpha + a}{1 - p}, \quad \text{if} \quad 0 < p < 1;
\end{equation}
\begin{equation}
(4.5) \quad u(x) \geq C_\kappa |x|^\kappa \quad \forall \kappa < +\infty, \quad \text{if} \quad 1 \leq p < \frac{n + 2m + \alpha + 2a}{n - 2m - \alpha}.
\end{equation}

**Proof.** Given any $\lambda > 0$, we first define the Kelvin transform of a function $u : \mathbb{R}^n \to \mathbb{R}$ centered at 0 by
\begin{equation}
(4.6) \quad u_\lambda(x) = \left(\frac{\lambda}{|x|}\right)^{n-2m-\alpha} u\left(\frac{\lambda^2 x}{|x|^2}\right)
\end{equation}
for arbitrary $x \in \mathbb{R}^n \setminus \{0\}$. It’s obvious that the Kelvin transform $u_\lambda$ may have singularity at 0 and $\lim_{|x| \to \infty} |x|^{n-2m-\alpha} u_\lambda(x) = \lambda^{n-2m-\alpha} u(0) > 0$. By (4.6), one can infer from the regularity assumptions on $u$ that $u_\lambda \in C(\mathbb{R}^n \setminus \{0\})$.

Next, we will carry out the process of scaling spheres with respect to the origin 0 in $\mathbb{R}^n$.

To this end, let $\lambda > 0$ be an arbitrary positive real number and let
\begin{equation}
(4.7) \quad \omega^\lambda(x) := u_\lambda(x) - u(x)
\end{equation}
for any $x \in B_\lambda(0) \setminus \{0\}$. We will first show that, for $\lambda > 0$ sufficiently small,
\begin{equation}
(4.8) \quad \omega^\lambda(x) \geq 0, \quad \forall \ x \in B_\lambda(0) \setminus \{0\}.
\end{equation}

Then, we start dilating the sphere $S_\lambda$ from a place near the origin 0 outward as long as (4.8) holds, until its limiting position $\lambda = +\infty$ and derive lower bound estimates on $u$. Therefore, the scaling sphere process can be divided into two steps.

**Step 1. Start dilating the sphere from near $\lambda = 0$.** Define
\begin{equation}
(4.9) \quad B^-_\lambda := \{x \in B_\lambda(0) \setminus \{0\} | \omega^\lambda(x) < 0\}.
\end{equation}

We will show that, for $\lambda > 0$ sufficiently small,
\begin{equation}
(4.10) \quad B^-_\lambda = \emptyset.
\end{equation}

Since $u \in C(\mathbb{R}^n)$ is a positive solution to integral equations (4.8), through direct calculations, we get
\begin{equation}
(4.11) \quad u(x) = C \int_{B_\lambda(0)} \frac{|y|^\alpha}{|x - y|^{n-2m-\alpha}} u^p(y) dy + C \int_{B_\lambda(0)} \frac{|y|^\alpha}{|y|^{n-2m-\alpha}} \left(\frac{\lambda}{|y|}\right)^\tau u^p_\lambda(y) dy
\end{equation}
for any $x \in \mathbb{R}^n$, where $\tau := n + 2m + \alpha + 2a - p(n - 2m - \alpha) > 0$. Direct calculations deduce that $u_\lambda$ satisfies the following integral equation
\begin{equation}
(4.12) \quad u_\lambda(x) = C \int_{\mathbb{R}^n} \frac{|y|^\alpha}{|x - y|^{n-2m-\alpha}} \left(\frac{\lambda}{|y|}\right)^\tau u^p_\lambda(y) dy
\end{equation}
for any $x \in \mathbb{R}^n \setminus \{0\}$, and hence, it follows immediately that
\begin{equation}
(4.13) \quad u_\lambda(x) = C \int_{B_\lambda(0)} \frac{|y|^\alpha}{|x - y|^{n-2m-\alpha}} u^p(y) dy + C \int_{B_\lambda(0)} \frac{|y|^\alpha}{|y|^{n-2m-\alpha}} \left(\frac{\lambda}{|y|}\right)^\tau u^p_\lambda(y) dy.
\end{equation}
From the integral equations (4.11) and (4.13), one can derive that, for any $x \in B_\lambda^-$,

\begin{equation}
0 > \omega^\lambda(x) = u_\lambda(x) - u(x) = C \int_{B_\lambda(0)} \left( \frac{|y|^a}{|x-y|^{n-2m-\alpha}} - \frac{|y|^a}{|\lambda x - \frac{\lambda}{|y|} y|^{n-2m-\alpha}} \right) \left( \left( \frac{\lambda}{|y|} \right)^\tau u_\lambda^p(y) - u^p(y) \right) dy
\end{equation}

\begin{eqnarray*}
& > & C \int_{B_\lambda^-} \left( \frac{|y|^a}{|x-y|^{n-2m-\alpha}} - \frac{|y|^a}{|\lambda x - \frac{\lambda}{|y|} y|^{n-2m-\alpha}} \right) \max \left\{ u^{p-1}(y), u_\lambda^{p-1}(y) \right\} \omega^\lambda(y) dy \\
& \geq & C \int_{B_\lambda^-} \frac{|y|^a}{|x-y|^{n-2m-\alpha}} \max \left\{ u^{p-1}(y), u_\lambda^{p-1}(y) \right\} \omega^\lambda(y) dy.
\end{eqnarray*}

By Hardy-Littlewood-Sobolev inequality and (4.14), we have, for any $\frac{n}{n-2m-\alpha} < q < \infty$,

\begin{equation}
\|\omega^\lambda\|_{L^q(B_\lambda^-)} \leq C \left\| |x|^a \max \{ u^{p-1}, u_\lambda^{p-1} \} \omega^\lambda \right\|_{L^{\frac{nq}{n-2m-\alpha q}}(B_\lambda^-)} \leq C \left\| |x|^a \max \{ u^{p-1}, u_\lambda^{p-1} \} \right\|_{L^{\frac{n}{2m+\alpha}}(B_\lambda^-)} \right\| \omega^\lambda \|_{L^q(B_\lambda^-)}.
\end{equation}

Since (4.3) implies that

\begin{equation}
\inf_{x \in B_\lambda(0) \setminus \{0\}} u_\lambda(x) \geq C
\end{equation}

for any $\lambda \leq 1$, there exists a $\epsilon_0 > 0$ small enough, such that

\begin{equation}
C \left\| |x|^a \max \{ u^{p-1}, u_\lambda^{p-1} \} \right\|_{L^{\frac{n}{2m+\alpha}}(B_\lambda^-)} \leq \frac{1}{2}
\end{equation}

for all $0 < \lambda \leq \epsilon_0$. Thus (4.15) implies

\begin{equation}
\|\omega^\lambda\|_{L^q(B_\lambda^-)} = 0, \quad \forall 0 < \lambda \leq \epsilon_0,
\end{equation}

which means $B_\lambda^- = \emptyset$. Consequently for all $0 < \lambda \leq \epsilon_0$,

\begin{equation}
\omega^\lambda(x) \geq 0, \quad \forall x \in B_\lambda(0) \setminus \{0\},
\end{equation}

which completes Step 1.

**Step 2. Dilate the sphere $S_\lambda$ outward until $\lambda = +\infty$ to derive lower bound estimates on $u$.**

Step 1 provides us a start point to dilate the sphere $S_\lambda$ from place near $\lambda = 0$. Now we dilate the sphere $S_\lambda$ outward as long as (4.8) holds. Let

\begin{equation}
\lambda_0 := \sup\{ \lambda > 0 \mid \omega^\mu \geq 0 \text{ in } B_\mu(0) \setminus \{0\}, \forall 0 < \mu \leq \lambda \} \in (0, +\infty],
\end{equation}

and hence, one has

\begin{equation}
\omega^{\lambda_0}(x) \geq 0, \quad \forall x \in B_{\lambda_0}(0) \setminus \{0\}.
\end{equation}

In what follows, we will prove $\lambda_0 = +\infty$ by contradiction arguments.

Suppose on contrary that $0 < \lambda_0 < +\infty$. In order to get a contradiction, we will first show that

\begin{equation}
\omega^{\lambda_0}(x) > 0, \quad \forall x \in B_{\lambda_0}(0) \setminus \{0\}.
\end{equation}

Then, we will obtain a contradiction with (4.20) via showing that the sphere $S_\lambda$ can be dilated outward a little bit further. More precisely, there exists a $\varepsilon > 0$ small enough such that $\omega^\lambda \geq 0$ in $B_\lambda(0) \setminus \{0\}$ for all $\lambda \in [\lambda_0, \lambda_0 + \varepsilon]$. 
Now we start to prove (4.22). Indeed, if we suppose that

$$\omega^{\lambda_0}(x) \equiv 0, \quad \forall x \in B_{\lambda_0}(0) \setminus \{0\},$$

then by the second equality in (4.14) and (4.23), we arrive at

$$0 = \omega^{\lambda_0}(x) = u_{\lambda_0}(x) - u(x)$$

then by the second equality in (4.14) and (4.23), we arrive at

$$0 = \omega^{\lambda_0}(x) = u_{\lambda_0}(x) - u(x)$$

for any $x \in B_{\lambda_0}(0) \setminus \{0\}$, which is absurd. Thus there exists a point $x^0 \in B_{\lambda_0}(0) \setminus \{0\}$ such that $\omega^{\lambda_0}(x^0) > 0$, which implies that by continuity, there exists a small $\delta > 0$ and a constant $c_0 > 0$ such that

$$B_{\delta}(x^0) \subset B_{\lambda_0}(0) \setminus \{0\} \quad \text{and} \quad \omega^{\lambda_0}(x) \geq c_0 > 0, \quad \forall x \in B_{\delta}(x^0).$$

From (4.25) and the integral equations (4.11) and (4.13), one can derive that, for any $x \in B_{\lambda_0}(0) \setminus \{0\}$,

$$\omega^{\lambda_0}(x) = u_{\lambda_0}(x) - u(x)$$

and thus arrive at (4.22). Furthermore, (4.26) also implies that there exists a $0 < \eta < \lambda_0$

small enough such that, for any $x \in B_{\eta}(0) \setminus \{0\}$,

$$\omega^{\lambda_0}(x) > c_4 + C \int_{B_{\eta}(x^0)} c_2 c_2^{-1} c_0 dy =: \tilde{c}_0 > 0.$$
By (4.14), one can easily verify that inequality as (4.15) (with the same constant $C$) also holds for any $\lambda \in [\lambda_0, \lambda_0 + r_0]$, that is, for any $\frac{n}{n - 2m - \alpha} < q < \infty$,

\[
\|\omega^\lambda\|_{L^q(B_0)} \leq C \|x|^a \max \{u^{p-1}, u_{\lambda}^{p-1}\} \|\omega^\lambda\|_{L^q(B_0)}.
\]

From (4.22) and (4.27), we can infer that there exists a $\lambda < \varepsilon$ such that, for any $\lambda \in [\lambda_0, \lambda_0 + \varepsilon]$,\n
\[
\|\omega^\lambda\|_{L^q(B_0)} \leq C \|x|^a \max \{u^{p-1}, u_{\lambda}^{p-1}\} \|\omega^\lambda\|_{L^q(B_0)}.
\]

By (4.31), one can easily verify that inequality as (4.15) (with the same constant $C$) also holds for any $\lambda \in [\lambda_0, \lambda_0 + \varepsilon]$,\n
\[
\omega^\lambda(x) \geq m_0 > 0, \quad \forall x \in B_{\lambda_0 - r_0}(0) \setminus \{0\}.
\]

In order to prove (4.32), one should observe that (4.31) is equivalent to\n
\[
|x|^{n-2m-\alpha} u(x) - \lambda_0^{n-2m-\alpha} u(\lambda_0 x) \geq m_0 \lambda_0^{n-2m-\alpha}, \quad \forall |x| \geq \frac{\lambda_0^2}{\lambda_0 - r_0}.
\]

Since $u$ is uniformly continuous on arbitrary compact set $K \subset \mathbb{R}^n$ (say, $K = B_{\lambda_0}(0)$), we can deduce from (4.31) that, there exists a $0 < \varepsilon_1 < r_0$ sufficiently small, such that, for any $\lambda \in [\lambda_0, \lambda_0 + \varepsilon_1]$,

\[
\|\omega^\lambda\|_{L^q(B_0)} \leq C \|x|^a \max \{u^{p-1}, u_{\lambda}^{p-1}\} \|\omega^\lambda\|_{L^q(B_0)}.
\]

From (4.30) and (4.32), we have proved (4.32).

By (4.32), we know that for any $\lambda \in [\lambda_0, \lambda_0 + \varepsilon_1]$,

\[
B_\lambda \subset A_{\lambda_0 + r_0, 2r_0},
\]

and hence, estimates (4.28) and (4.30) yields\n
\[
\|\omega^\lambda\|_{L^q(B_\lambda)} = 0.
\]

Therefore, for any $\lambda \in [\lambda_0, \lambda_0 + \varepsilon_1]$, we deduce from (4.36) that, $B_\lambda \subset \emptyset$, that is,

\[
\omega^\lambda(x) \geq 0, \quad \forall x \in B_\lambda(0) \setminus \{0\},
\]

which contradicts with the definition (4.20) of $\lambda_0$. Thus we must have $\lambda_0 = +\infty$, that is,

\[
u(x) \geq \left(\frac{\lambda}{|x|}\right)^{n-2m-\alpha} u \left(\frac{\lambda^2 x}{|x|^2}\right), \quad \forall |x| \geq \lambda, \quad \forall 0 < \lambda < +\infty.
\]

For arbitrary $|x| \geq 1$, let $\lambda := \sqrt{|x|}$, then (4.38) yields that\n
\[
u(x) \geq \left(\frac{1}{|x|^{\frac{n-2m-\alpha}{2}}} u \left(\frac{x}{|x|}\right)\right),
\]

and hence, we arrive at the following lower bound estimate:

\[
u(x) \geq \left(\min_{x \in S_1} u(x)\right) \left(\frac{1}{|x|^{\frac{n-2m-\alpha}{2}}} := \frac{C_0}{|x|^{\frac{n-2m-\alpha}{2}}}, \quad \forall |x| \geq 1.
\]

The lower bound estimate (4.40) can be improved remarkably for $0 < p < \frac{n+2m+\alpha+2s}{n-2m-\alpha}$ using the “Bootstrap” iteration technique and the integral equations (1.8).
In fact, let \( \mu_0 := \frac{n-2m-\alpha}{2} \), we infer from the integral equations (1.8) and (4.40) that, for \(|x| \geq 1\),

\[
(4.41) \quad u(x) \geq C \int_{2|x| \leq |y| \leq 4|x|} \frac{1}{|x-y|^{n-2m-\alpha} |y|^{p\mu_0-a}} dy \\
\geq \frac{C}{|x|^{n-2m-\alpha}} \int_{2|x| \leq |y| \leq 4|x|} \frac{1}{|y|^{p\mu_0-a}} dy \\
\geq \frac{C}{|x|^{n-2m-\alpha}} \int_{2|x|}^{|x|} r^{n-1-p\mu_0+a} dr \\
\geq \frac{C_1}{|x|^{p\mu_0-(a+2m+\alpha)}}.
\]

Let \( \mu_1 := p\mu_0 - (a + 2m + \alpha) \). Due to \( 0 < p < \frac{n+2m+\alpha+2a}{n-2m-\alpha} \), our important observation is

\[
(4.42) \quad \mu_1 := p\mu_0 - (a + 2m + \alpha) < \mu_0.
\]

Thus we have obtained a better lower bound estimate than (4.40) after one iteration, that is,

\[
(4.43) \quad u(x) \geq \frac{C_1}{|x|^{\mu_1}}, \quad \forall \ |x| \geq 1.
\]

For \( k = 0, 1, 2, \ldots \), define

\[
(4.44) \quad \mu_{k+1} := p\mu_k - (a + 2m + \alpha).
\]

Since \( 0 < p < \frac{n+2m+\alpha+2a}{n-2m-\alpha} \), it is easy to see that the sequence \( \{\mu_k\} \) is monotone decreasing with respect to \( k \). Repeating the above iteration process involving the integral equation (1.8), we have the following lower bound estimates for every \( k = 0, 1, 2, \ldots \),

\[
(4.45) \quad u(x) \geq \frac{C_k}{|x|^{\mu_k}}, \quad \forall \ |x| \geq 1.
\]

Now Theorem 4.1 follows easily from the obvious properties that as \( k \to +\infty \),

\[
(4.46) \quad \mu_k \to -\frac{a + 2m + \alpha}{1-p} \quad \text{if} \ 0 < p < 1; \quad \mu_k \to -\infty \quad \text{if} \ 1 \leq p < \frac{n+2m+\alpha+2a}{n-2m-\alpha}.
\]

This finishes our proof of Theorem 4.1 \( \square \)

We have proved that a nontrivial nonnegative solution \( u \) to integral equations (1.8) is actually a positive solution. For \( 0 < p < \frac{n+2m+\alpha+2a}{n-2m-\alpha} \), the lower bound estimates in Theorem 1.1 contradicts with the following integrability

\[
(4.47) \quad C \int_{\mathbb{R}^n} \frac{u^p(x)}{|x|^{n-2m-\alpha-a}} dx = u(0) < +\infty
\]

indicated by integral equations (1.8). Therefore, \( u \equiv 0 \) in \( \mathbb{R}^n \), that is, the unique nonnegative solution to IEs (1.8) is \( u \equiv 0 \) in \( \mathbb{R}^n \). The proof of Theorem 1.10 is therefore completed.
5. Proof of Theorem \[1.14\]

In this section, using Theorem 1.1 and the arguments from Chen, Dai and Qin [3], we will prove the Liouville properties in Theorem 1.14 in both critical order cases \( m + \frac{\alpha}{2} = \frac{n}{2} \) and super-critical order cases \( m + \frac{\alpha}{2} > \frac{n}{2} \).

We will prove Theorem 1.14 by using contradiction arguments. Suppose on the contrary that \( u \geq 0 \) satisfies equation (1.1) but \( u \) is not identically zero, then there exists a point \( \bar{x} \in \mathbb{R}^n \) such that \( u(\bar{x}) > 0 \). By Theorem 1.1 we can deduce from \( (-\Delta)^{\frac{\alpha}{2}} u \geq 0, \ u \geq 0, u(\bar{x}) > 0 \) that

\[
(5.1) \quad u(x) > 0, \quad \forall \ x \in \mathbb{R}^n.
\]

Suppose not, then there exists a point \( \bar{x} \in \mathbb{R}^n \) such that \( u(\bar{x}) = 0 \), and hence we have

\[
(5.2) \quad (-\Delta)^{\frac{\alpha}{2}} u(\bar{x}) = C_{n, \alpha} \text{P.V.} \int_{\mathbb{R}^n} \frac{-u(y)}{||x - y||^{n+\alpha}} dy < 0,
\]

which is absurd. Moreover, by maximum principle and induction, we can also infer further from \((-\Delta)^{i+}\frac{\alpha}{2} u \geq 0 \ (i = 0, \ldots, m - 1), \ u > 0\) the assumptions on \( f \) and equation (1.1) that

\[
(5.3) \quad (-\Delta)^{i+}\frac{\alpha}{2} u(x) > 0, \quad \forall \ i = 0, \ldots, m - 1, \ \forall \ x \in \mathbb{R}^n.
\]

Since \( m + \frac{\alpha}{2} \geq \frac{n}{2} \), it follows immediately that either \( m = \frac{n-1}{2} \) with \( n \) odd or \( m \geq \lceil \frac{n}{2} \rceil \), where \( \lceil x \rceil \) denotes the least integer not less than \( x \).

In the following, we will try to obtain contradictions by discussing the two different cases \( m = \frac{n-1}{2} \) with \( n \) odd and \( m \geq \lceil \frac{n}{2} \rceil \) separately.

**Case i):** \( m = \frac{n-1}{2} \) and \( n \) is odd. Since \( m + \frac{\alpha}{2} \geq \frac{n}{2} \), we have \( 1 \leq \alpha < 2 \). Now we will first show that \((-\Delta)^{m-1+}\frac{\alpha}{2} u \) satisfies the following integral equation

\[
(5.4) \quad (-\Delta)^{m-1+}\frac{\alpha}{2} u(x) = \int_{\mathbb{R}^n} \frac{R_{2,n}}{|x - y|^{n-2}} f(y, u(y), \ldots) dy, \quad \forall \ x \in \mathbb{R}^n,
\]

where the Riesz potential’s constants \( R_{\alpha,n} := \frac{\Gamma(\frac{n-\alpha}{2})}{\pi^{\frac{n}{2}} \Gamma(\frac{\alpha}{2})} \) for \( 0 < \alpha < n \).

To this end, for arbitrary \( R > 0 \), let \( f_1(u)(x) := f(x, u(x), \ldots) \) and

\[
(5.5) \quad v_1^R(x) := \int_{B_R(0)} G^2_R(x, y) f_1(u(y)) dy,
\]

where the Green’s function for \(-\Delta\) on \( B_R(0) \) is given by

\[
(5.6) \quad G^2_R(x, y) = R_{2,n} \left[ \frac{1}{|x - y|^{n-2}} - \frac{1}{(|x| \cdot |x - y|^{n-2} - \frac{y}{R})^{n-2}} \right], \quad \text{if} \ x, y \in B_R(0),
\]

and \( G^2_R(x, y) = 0 \) if \( x \) or \( y \in \mathbb{R}^n \setminus B_R(0) \). Then, we can derive that \( v_1^R \in C^2(\mathbb{R}^n) \) and satisfies

\[
(5.7) \quad \begin{cases}
-\Delta v_1^R(x) = f(x, u(x), \ldots), & x \in B_R(0), \\
v_1^R(x) = 0, & x \in \mathbb{R}^n \setminus B_R(0).
\end{cases}
\]

Let \( w_1^R(x) := (-\Delta)^{m-1+}\frac{\alpha}{2} u(x) - v_1^R(x) \). By Theorem 1.1, 1.11 and (5.7), we have \( w_1^R \in C^2(\mathbb{R}^n) \) and satisfies

\[
(5.8) \quad \begin{cases}
-\Delta w_1^R(x) = 0, & x \in B_R(0), \\
w_1^R(x) > 0, & x \in \mathbb{R}^n \setminus B_R(0).
\end{cases}
\]
By maximum principle, we deduce that for any $R > 0$,
(5.9) \[ w_1^R(x) = (-\Delta)^{m-1+\frac{\alpha}{2}}u(x) = v_1^R(x) > 0, \quad \forall \, x \in \mathbb{R}^n. \]

Now, for each fixed $x \in \mathbb{R}^n$, letting $R \to \infty$ in (5.9), we have
(5.10) \[ (-\Delta)^{m-1+\frac{\alpha}{2}}u(x) \geq \int_{\mathbb{R}^n} \frac{R_{2,n}}{|x - y|^{n-2}}f_1(y)dy =: v_1(x) > 0. \]

Take $x = 0$ in (5.10), we get
(5.11) \[ \int_{\mathbb{R}^n} \frac{f(y, u(y), \cdots)}{|y|^{n-2}}dy < +\infty. \]

One can easily observe that $v_1 \in C^2(\mathbb{R}^n)$ is a solution of
(5.12) \[ -\Delta v_1(x) = f(x, u(x), \cdots), \quad x \in \mathbb{R}^n. \]

Define $w_1(x) := (-\Delta)^{m-1+\frac{\alpha}{2}}u(x) - v_1(x)$. Then, by (1.1), (5.10) and (5.12), we have $w_1 \in C^2(\mathbb{R}^n)$ and satisfies
(5.13) \[ \begin{cases} -\Delta w_1(x) = 0, & x \in \mathbb{R}^n, \\ w_1(x) \geq 0, & x \in \mathbb{R}^n. \end{cases} \]

From Liouville theorem for harmonic functions, we can deduce that
(5.14) \[ w_1(x) = (-\Delta)^{m-1+\frac{\alpha}{2}}u(x) - v_1(x) \equiv C_1 \geq 0. \]

Therefore, we have
(5.15) \[ (-\Delta)^{m-1+\frac{\alpha}{2}}u(x) = \int_{\mathbb{R}^n} \frac{R_{2,n}}{|x - y|^{n-2}}f(y, u(y), \cdots)dy + C_1 =: f_2(u(x)) > C_1 \geq 0. \]

Next, for arbitrary $R > 0$, let
(5.16) \[ v_2^R(x) := \int_{B_R(0)} G_R^2(x, y) f_2(u(y))dy. \]

Then, we can get
(5.17) \[ \begin{cases} -\Delta v_2^R(x) = f_2(u(x)), & x \in B_R(0), \\ v_2^R(x) = 0, & x \in \mathbb{R}^n \setminus B_R(0). \end{cases} \]

Let $w_2^R(x) := (-\Delta)^{m-2+\frac{\alpha}{2}}u(x) - v_2^R(x)$. By Theorem 1.1 (5.15) and (5.17), we have
(5.18) \[ \begin{cases} -\Delta w_2^R(x) = 0, & x \in B_R(0), \\ w_2^R(x) > 0, & x \in \mathbb{R}^n \setminus B_R(0). \end{cases} \]

By maximum principle, we deduce that for any $R > 0$,
(5.19) \[ w_2^R(x) = (-\Delta)^{m-2+\frac{\alpha}{2}}u(x) - v_2^R(x) > 0, \quad \forall \, x \in \mathbb{R}^n. \]

Now, for each fixed $x \in \mathbb{R}^n$, letting $R \to \infty$ in (5.19), we have
(5.20) \[ (-\Delta)^{m-2+\frac{\alpha}{2}}u(x) \geq \int_{\mathbb{R}^n} \frac{R_{2,n}}{|x - y|^{n-2}}f_2(u(y))dy =: v_2(x) > 0. \]

Take $x = 0$ in (5.20), we get
(5.21) \[ \int_{\mathbb{R}^n} \frac{C_1}{|y|^{n-2}}dy \leq \int_{\mathbb{R}^n} \frac{f_2(u(y))}{|y|^{n-2}}dy < +\infty, \]
it follows easily that \( C_1 = 0 \), and hence we have proved (5.4), that is,
\[ (-\Delta)^{m-1+\frac{2}{\alpha}} u(x) = f_2(u)(x) = \int_{\mathbb{R}^n} \frac{R_{2,n}}{|x-y|^{n-2}} f(y, u(y), \ldots) dy. \]

One can easily observe that \( v_2 \) is a solution of
\[ -\Delta v_2(x) = f_2(u)(x), \quad x \in \mathbb{R}^n. \]
Define \( w_2(x) := (-\Delta)^{m-2+\frac{2}{\alpha}} u(x) - v_2(x) \), then it satisfies
\[ \begin{cases} 
-\Delta w_2(x) = 0, \quad x \in \mathbb{R}^n, \\
w_2(x) \geq 0, \quad x \in \mathbb{R}^n.
\end{cases} \]

From Liouville theorem for harmonic functions, we can deduce that
\[ w_2(x) = (-\Delta)^{m-2+\frac{2}{\alpha}} u(x) - v_2(x) \equiv C_2 \geq 0. \]
Therefore, we have proved that
\[ (-\Delta)^{m-2+\frac{2}{\alpha}} u(x) = \int_{\mathbb{R}^n} \frac{R_{2,n}}{|x-y|^{n-2}} f_2(u)(y)dy + C_2 =: f_3(u)(x) \geq C_2 \geq 0. \]
By the same methods as above, we can prove that \( C_2 = 0 \), and hence
\[ (-\Delta)^{m-2+\frac{2}{\alpha}} u(x) = f_3(u)(x) = \int_{\mathbb{R}^n} \frac{R_{2,n}}{|x-y|^{n-2}} f_2(u)(y)dy. \]
Repeating the above argument, defining
\[ f_{k+1}(u)(x) := \int_{\mathbb{R}^n} \frac{R_{2,n}}{|x-y|^{n-2}} f_k(u)(y)dy \]
for \( k = 1, 2, \ldots, m \), then by Theorem 1.1 and induction, we have
\[ (-\Delta)^{m-k+\frac{2}{\alpha}} u(x) = f_{k+1}(u)(x) = \int_{\mathbb{R}^n} \frac{R_{2,n}}{|x-y|^{n-2}} f_k(u)(y)dy \]
for \( k = 1, 2, \ldots, m - 1 \), and
\[ (-\Delta)^{\frac{2}{\alpha}} u(x) = \int_{\mathbb{R}^n} \frac{R_{2,n}}{|x-y|^{n-2}} f_m(u)(y)dy + C_m = f_{m+1}(u)(x) + C_m > C_m \geq 0. \]
For arbitrary \( R > 0 \), let
\[ v_{m+1}^R(x) := \int_{B_R(0)} G_R^\alpha(x, y) (f_{m+1}(u)(y) + C_m) dy, \]
where the Green’s function for \((-\Delta)^{\frac{2}{\alpha}}\) with \( 0 < \alpha < 2 \) on \( B_R(0) \) is given by
\[ G_R^\alpha(x, y) := \frac{C_{n,\alpha}}{|x-y|^{n-\alpha}} \int_{s_R}^{t_R} \frac{b^{\frac{\alpha}{2}-1}}{(1 + b)^{\frac{\alpha}{2}}} db \quad \text{if } x, y \in B_R(0) \]
with \( s_R = \frac{|x-y|^2}{R^2}, t_R = \left(1 - \frac{|x|^2}{R^2}\right) \left(1 - \frac{|y|^2}{R^2}\right) \), and \( G_R^\alpha(x, y) = 0 \) if \( x \) or \( y \) \( \not\in \mathbb{R}^n \setminus B_R(0) \) (see [28]). Then, we can get
\[ \begin{cases} 
(-\Delta)^{\frac{2}{\alpha}} v_{m+1}^R(x) = f_{m+1}(u)(x) + C_m, \quad x \in B_R(0), \\
v_{m+1}^R(x) = 0, \quad x \in \mathbb{R}^n \setminus B_R(0).
\end{cases} \]
Let $w^R_{m+1}(x) := u(x) - v^R_{m+1}(x)$. By Theorem 5.30 and (5.33), we have

\[
\begin{cases}
(-\Delta)^{\frac{\alpha}{2}} w^R_{m+1}(x) = 0, & x \in B_R(0), \\
w^R_{m+1}(x) > 0, & x \in \mathbb{R}^n \setminus B_R(0).
\end{cases}
\]

(5.34)

Now we need the following maximum principle for fractional Laplacians $(-\Delta)^{\frac{\alpha}{2}}$, which can been found in [12, 37].

**Lemma 5.1.** Let $\Omega$ be a bounded domain in $\mathbb{R}^n$ and $0 < \alpha < 2$. Assume that $u \in L_\alpha \cap C^{1,1}_\text{loc}(\Omega)$ and is l.s.c. on $\overline{\Omega}$. If $(-\Delta)^{\frac{\alpha}{2}} u \geq 0$ in $\Omega$ and $u \geq 0$ in $\mathbb{R}^n \setminus \Omega$, then $u \geq 0$ in $\mathbb{R}^n$. Moreover, if $u = 0$ at some point in $\Omega$, then $u = 0$ a.e. in $\mathbb{R}^n$. These conclusions also hold for unbounded domain $\Omega$ if we assume further that

\[
limit_{|x| \to \infty} u(x) \geq 0.
\]

By Lemma 5.1 we can deduce immediately from (5.34) that for any $R > 0$,

\[
w^R_{m+1}(x) = u(x) - v^R_{m+1}(x) > 0, \quad \forall \ x \in \mathbb{R}^n.
\]

(5.35)

Now, for each fixed $x \in \mathbb{R}^n$, letting $R \to \infty$ in (5.35), we have

\[
u(x) \geq \int_{\mathbb{R}^n} \frac{R_{\alpha,n}}{|x - y|^{n-\alpha}} (f_{m+1}(u)(y) + C_m) dy > 0.
\]

(5.36)

Take $x = 0$ in (5.36), we get

\[
\int_{\mathbb{R}^n} \frac{C_m}{|y|^{n-\alpha}} dy \leq \int_{\mathbb{R}^n} \frac{f_{m+1}(u)(y) + C_m}{|y|^{n-\alpha}} dy < +\infty,
\]

(5.37)

it follows easily that $C_m = 0$, and hence we have

\[
(-\Delta)^{\frac{\alpha}{2}} u(x) = f_{m+1}(u)(x) = \int_{\mathbb{R}^n} \frac{R_{2,n}}{|x - y|^{n-\alpha}} f_{m}(u)(y) dy,
\]

(5.38)

and

\[
u(x) \geq \int_{\mathbb{R}^n} \frac{R_{\alpha,n}}{|x - y|^{n-\alpha}} f_{m+1}(u)(y) dy.
\]

(5.39)

In particular, it follows from (5.29), (5.38) and (5.39) that

\[
+\infty > (-\Delta)^{m-k+\frac{\alpha}{2}} u(0) \geq \int_{\mathbb{R}^n} \frac{R_{2,n}}{|y|^{k-1|n-2|}} f_k(u)(y) dy
\]

\[
\geq \int_{\mathbb{R}^n} \frac{R_{2,n}}{|y|^{k-1|n-2|}} \int_{\mathbb{R}^n} \frac{R_{2,n}}{|y^{k-|n-2|}|} \cdots \int_{\mathbb{R}^n} \frac{R_{2,n}}{|y^{2-1|n-2|}|} f(y^1, u(y^1), \cdots) dy^1 dy^2 \cdots dy^k
\]

for $k = 1, 2, \cdots, m$, and

\[
+\infty > u(0) \geq \int_{\mathbb{R}^n} \frac{R_{\alpha,n}}{|y|^{n-\alpha}} f_{m+1}(u)(y) dy
\]

\[
\geq \int_{\mathbb{R}^n} \frac{R_{\alpha,n}}{|y|^{m+1-\alpha}} \cdots \int_{\mathbb{R}^n} \frac{R_{\alpha,n}}{|y^{2-1|n-2|}|} f(y^1, \cdots) dy^1 \cdots dy^{m+1}.
\]

(5.41)

From the properties of Riesz potential, for any $\alpha_1, \alpha_2 \in (0, n)$ such that $\alpha_1 + \alpha_2 \in (0, n)$, one has (see [38])

\[
\int_{\mathbb{R}^n} \frac{R_{\alpha_1,n}}{|x - y|^{n-\alpha_1}} \frac{R_{\alpha_2,n}}{|y - z|^{n-\alpha_2}} dy = \frac{R_{\alpha_1+\alpha_2,n}}{|x - z|^{n-(\alpha_1+\alpha_2)}}.
\]

(5.42)
By applying (5.42) and direct calculations, we obtain that
\begin{equation}
(5.43) \quad \int_{\mathbb{R}^n} \frac{R_{2,n}}{|y^{n+1} - y^m|^{n-2}} \cdots \int_{\mathbb{R}^n} \frac{R_{2,n}}{|y^3 - y^2|^{n-2}} \cdot \frac{R_{2,n}}{|y^2 - y^1|^{n-2}} \, dy^2 \cdots \, dy^m
= \frac{R_{2m,n}}{|y^{m+1} - y^1|^{n-2m}}.
\end{equation}

Now, note that \( m = \frac{n-1}{2} \) and \( n \) is odd, we can deduce from (5.41), (5.43) and Fubini’s theorem that
\begin{equation}
(5.44) \quad +\infty > u(0) \geq \int_{\mathbb{R}^n} \frac{R_{2,n}}{|y^{m+1} - y^1|^{n-\alpha}} \left( \int_{\mathbb{R}^n} \frac{R_{n-\alpha,n}}{|y^{m+1} - y^1|} f(y^1, u(y^1), \cdots) \, dy^1 \right) \, dy^{m+1}
= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \frac{1}{|y^{1}|^{n-\alpha}} \left( \int_{\mathbb{R}^n} \frac{1}{|y - z|} f(z, u(z), \cdots) \, dz \right) \, dy.
\end{equation}

We will get a contradiction from (5.44). Indeed, if we assume that \( u \) is not identically zero, then by (5.1), \( u > 0 \) in \( \mathbb{R}^n \). By the assumptions on \( f \), we have, there exists a point \( x_0 \in \mathbb{R}^n \) such that \( f > 0 \) at \( x_0 \). Hence by the integrability (5.11), we have
\begin{equation}
(5.45) \quad 0 < C_0 := \int_{\mathbb{R}^n} \frac{f(z, u(z), \cdots)}{|z|^{n-2}} \, dz < +\infty.
\end{equation}

For any given \( |y| \geq 3 \), if \( |z| \geq (\ln |y|)^{\frac{1}{n-2}} \), then one has immediately
\begin{equation}
(5.46) \quad |y - z| \leq |y| + |z| \leq \left( |y| (\ln |y|)^{\frac{1}{n-2}} + 1 \right) |z| \leq 2 |y| (\ln |y|)^{\frac{1}{n-2}} |z|.
\end{equation}
Thus it follows from (5.45) and (5.46) that, there exists a \( R_0 \geq 3 \) sufficiently large such that, for any \( |y| \geq R_0 \), we have
\begin{equation}
(5.47) \quad \int_{\mathbb{R}^n} \frac{f(z, u(z), \cdots)}{|y - z|} \, dz \geq \frac{1}{2 |y| \ln |y|} \int_{|z| \geq (\ln |y|)^{\frac{1}{n-2}}} \frac{f(z, u(z), \cdots)}{|z|^{n-2}} \, dz \geq \frac{1}{4 |y| \ln |y|} \int_{\mathbb{R}^n} \frac{f(z, u(z), \cdots)}{|z|^{n-2}} \, dz \geq \frac{C_0}{4 |y| \ln |y|}.
\end{equation}

As a consequence, we can finally deduce from (5.44), (5.47) and \( 1 \leq \alpha < 2 \) that
\begin{equation}
(5.48) \quad +\infty > u(0) \geq \frac{C_0}{4(2\pi)^n} \int_{|y| \geq R_0} \frac{1}{|y|^{n-\alpha + 1} \ln |y|} \, dy = +\infty,
\end{equation}
which is a contradiction. Therefore \( u \equiv 0 \) in \( \mathbb{R}^n \). This proves Theorem 1.14 in Case i): \( m = \frac{n-1}{2} \) and \( n \) is odd.

Case ii): \( m \geq \left\lceil \frac{n}{2} \right\rceil \). Let
\begin{equation}
(5.49) \quad f_{k+1}(u)(x) := \int_{\mathbb{R}^n} \frac{R_{2,n}}{|x - y|^{n-2}} f_k(u)(y) \, dy
\end{equation}
for \( k = 1, 2, \cdots, \left\lceil \frac{n}{2} \right\rceil \), by a quite similar way as in the proof for Case i), we can infer from Theorem 1.14 and induction that
\begin{equation}
(5.50) \quad (\Delta)^{m-k+\frac{\alpha}{2}} u(x) = f_{k+1}(u)(x) = \int_{\mathbb{R}^n} \frac{R_{2,n}}{|x - y|^{n-2}} f_k(u)(y) \, dy.
\end{equation}
for $k = 1, 2, \ldots, \left[ \frac{n}{2} \right] - 1$, and

$$(-\Delta)^{m-[\frac{n}{2}]+\frac{\tau}{2}} u(x) \geq f_{[\frac{n}{2}]+1}(u)(x) = \int_{\mathbb{R}^n} \frac{R_{2,n}}{|x-y|^{n-2}} f_{[\frac{n}{2}]}(u)(y) dy.$$  \hspace{1cm} (5.51)

In particular, it follows from (5.50) and (5.51) that

$$+\infty > (-\Delta)^{m-k+\frac{\tau}{2}} u(0) = \int_{\mathbb{R}^n} \frac{R_{2,n}}{|y|^{n-2}} f_k(u)(y) dy \geq \int_{\mathbb{R}^n} \frac{R_{2,n}}{|y|^{n-2}} f_{[\frac{n}{2}]}(u)(y) dy \geq \int_{\mathbb{R}^n} \frac{R_{2,n}}{|y|^{\frac{n}{2}}|y|^{n-2}}$$

$$\cdot \left( \int_{\mathbb{R}^n} \frac{R_{2,n}}{|y^\frac{n}{2} - y^{\frac{n}{2}}|^{n-2}} \cdots \int_{\mathbb{R}^n} \frac{R_{2,n}}{|y^2 - y^1|^{n-2}} f(y^1, u(y^1), \ldots) dy^1 dy^2 \cdots dy^k \right) dy^{[\frac{n}{2}]}.$$  \hspace{1cm} (5.52)

By applying the formula (5.42) and direct calculations, we obtain that

$$\int_{\mathbb{R}^n} \frac{R_{2,n}}{|y^\frac{n}{2} - y^\frac{n}{2}|^{n-2}} \cdots \int_{\mathbb{R}^n} \frac{R_{2,n}}{|y^3 - y^2|^{n-2}} \cdots \int_{\mathbb{R}^n} \frac{R_{2,n}}{|y^2 - y^1|^{n-2}} dy^2 \cdots dy^{[\frac{n}{2}]}$$

$$= \frac{R_{2,n}}{|y^\frac{n}{2} - y^1|^{n-2} |y^\frac{n}{2} - y^{\frac{n}{2}}|^{n-2}}.$$  \hspace{1cm} (5.53)

Now, we can deduce from (5.53), (5.54) and Fubini’s theorem that

$$+\infty > (-\Delta)^{m-[\frac{n}{2}]+\frac{\tau}{2}} u(0)$$

$$\geq \int_{\mathbb{R}^n} \frac{R_{2,n}}{|y^\frac{n}{2}|^{n-2}} \left( \int_{\mathbb{R}^n} \frac{R_{2[n\frac{n}{2}]-2,n}}{|y^\frac{n}{2} - y^1|^{n-2}|y^\frac{n}{2} - y^{\frac{n}{2}}|^{n-2}+2} f(y^1, u(y^1), \ldots) dy^1 \right) dy^{[\frac{n}{2}]}$$

$$= C_n \int_{\mathbb{R}^n} \frac{1}{|y|^{n-2}} \left( \int_{\mathbb{R}^n} \frac{1}{|y-z|^{n-2}} f(z, u(z), \ldots) dz \right) dy.$$  \hspace{1cm} (5.54)

We will get a contradiction from (5.55). To do this, let $\tau(n) := n - 2[\frac{n}{2}] + 2 \in \{1, 2\}$, then it follows from (5.45) and (5.46) that, there exists a $R_0 \geq 3$ sufficiently large such that, for any $|y| \geq R_0$,

$$\int_{\mathbb{R}^n} \frac{1}{|y-z|^{\tau(n)} f(z, u(z), \ldots) dz} \geq \frac{1}{2^{\tau(n)}|y|^{\tau(n)} \ln |y|} \int_{|z| \geq (\ln |y|)^{-\frac{1}{\tau(n)}}} \frac{1}{|z|^{n-2}} f(z, u(z), \ldots) dz \geq \frac{1}{2^{\tau(n)+1}|y|^{\tau(n)} \ln |y|}.$$

Therefore, we can finally deduce from (5.53) and (5.56) that

$$+\infty > (-\Delta)^{m-[\frac{n}{2}]+\frac{\tau}{2}} u(0) \geq \frac{C_0 C_n}{2^{\tau(n)+1}} \int_{|y| \geq R_0} \frac{1}{|y|^{n-2+\tau(n)} \ln |y|} dy = +\infty,$$

which is a contradiction again. Therefore, $u \equiv 0$ in $\mathbb{R}^n$ in Case ii): $m \geq \left[ \frac{n}{2} \right]$. This concludes our proof of Theorem 1.14.
6. Appendix: A characterization for \( \alpha \)-harmonic functions via averages

One can observe from the proof of Theorem 1.1 and 1.2 that, the average \( \int_{R}^{+\infty} \frac{R^{\alpha}}{r(r^{2} - R^{2})^{\frac{\alpha}{2}}} \overline{\mu}(r) dr \) plays an basic and important role for the nonlocal fractional Laplacians \((-\Delta)^{\alpha}\) \((0 < \alpha < 2)\), which is similar as \( \overline{\mu}(r) \) for the Laplacian \(-\Delta\). In this appendix, we will characterize the \( \alpha \)-harmonic functions by using the averages \( \int_{R}^{+\infty} \frac{R^{\alpha}}{r(r^{2} - R^{2})^{\frac{\alpha}{2}}} \overline{\mu}(r) dr \) and deduce some important properties for \( \alpha \)-harmonic functions.

Let \( \Omega \subseteq \mathbb{R}^{n} \) be a (bounded or unbounded) domain. We have the following characterization for \( \alpha \)-harmonic functions in \( \Omega \).

Theorem 6.1. Assume \( 0 < \alpha < 2 \). Let \( u \in C^{[\alpha], \{\alpha\}+\epsilon}(\Omega) \cap L_{\alpha}(\mathbb{R}^{n}) \) (with \( \epsilon > 0 \) arbitrarily small) satisfy \((-\Delta)^{\frac{\alpha}{2}}u \geq 0 \) \((\leq 0)\) in \( \Omega \), then for any ball \( B = B_{R}(y) \subset \Omega \), we have

\[
(6.1) \quad u(y) \geq (\leq) C_{n,\alpha} \int_{R}^{+\infty} \frac{R^{\alpha}}{r(r^{2} - R^{2})^{\frac{\alpha}{2}}} \overline{\mu}(r) dr,
\]

where \( C_{n,\alpha} := \frac{\Gamma\left(\frac{n+\alpha}{2}\right)}{\pi^{\frac{n}{2}}} \sin \frac{\alpha\pi}{2} \) and \( \overline{\mu}(r) \) denotes the spherical average of \( u \) w.r.t. \( y \). Furthermore, \((-\Delta)^{\frac{\alpha}{2}}u = 0 \) in \( \Omega \) if and only if

\[
(6.2) \quad u(y) = C_{n,\alpha} \int_{R}^{+\infty} \frac{R^{\alpha}}{r(r^{2} - R^{2})^{\frac{\alpha}{2}}} \overline{\mu}(r) dr
\]

for any ball \( B = B_{R}(y) \subset \Omega \).

Proof. Suppose \((-\Delta)^{\frac{\alpha}{2}}u \geq 0 \) in \( \Omega \), then it follows from Green-Poisson integral representation formula that, for any ball \( B = B_{R}(y) \subset \Omega \),

\[
(6.3) \quad u(y) = \int_{B_{R}(y)} C_{\alpha}^{\alpha}(y, z)(-\Delta)^{\frac{\alpha}{2}}u(z)dz + \int_{|z-y|>R} P_{\alpha}^{\alpha}(y, z)u(z)dz
\]

\[
= \int_{B_{R}(y)} \frac{\tilde{C}_{n,\alpha}}{|z-y|^{n-\alpha}} \left( \int_{0}^{\frac{|z-y|^{2}}{r^{2}}} \frac{b_{\frac{\alpha}{2}-1}}{(1+b)^{\frac{\alpha}{2}}} \right) (-\Delta)^{\frac{\alpha}{2}}u(z)dz
\]

\[
+ C_{n,\alpha} \int_{|z-y|>R} \frac{R^{\alpha}}{|z-y|^{n+\alpha}} u(z) \frac{dz}{|z-y|^{n}}
\]

\[
\geq C_{n,\alpha} \int_{R}^{+\infty} \frac{R^{\alpha}}{r(r^{2} - R^{2})^{\frac{\alpha}{2}}} \overline{\mu}(r) dr.
\]

If \((-\Delta)^{\frac{\alpha}{2}}u \leq 0 \) in \( \Omega \), then (6.1) can be derived in entirely similar way.

Now assume that \( u \) satisfies the average property (6.2) for any ball \( B = B_{R}(y) \subset \Omega \), we will show that \( u \) is \( \alpha \)-harmonic in \( \Omega \). To this end, for any ball \( B = B_{R}(y) \subset \Omega \), let us define

\[
(6.4) \quad h(x) := C_{n,\alpha} \int_{|z-y|>R} \left( \frac{R^{2} - |x-y|^{2}}{|z-y|^{2} - R^{2}} \right)^{\frac{\alpha}{2}} \frac{u(z)}{|z-y|^{n}} dz, \quad \forall \ x \in B_{R}(y),
\]

and \( h(x) := u(x) \) if \( |x-y| \geq R \). Then it follows that \((-\Delta)^{\frac{\alpha}{2}}h = 0 \) in \( B_{R}(y) \) and hence satisfies the average property (6.2) for any ball \( B \subset \subset B_{R}(y) \). Define \( w := u - h \), then \( w(x) = 0 \) if \( |x-y| \geq R \) and \( w \) satisfies the average property (6.2) for any ball \( B \subset \subset B_{R}(y) \). Our aim is to show that \( w = 0 \) in \( B_{R}(y) \).
Indeed, suppose there exists a point \( \bar{x} \in B_R(y) \) such that \( w(\bar{x}) = M := \max_{x \in B_R(y)} w(x) > 0 \), then the average property (6.2) implies that, for any ball \( B_{\bar{R}}(\bar{x}) \subset \subset B_R(y) \),

\[
0 = w(\bar{x}) - M = C_{n, \alpha} \int_{\bar{R}}^{+\infty} \frac{\bar{R}^\alpha}{r(\bar{r}^2 - \bar{R}^2)^{\frac{\alpha}{2}}} w - M(r) dr < 0,
\]

where \( w - M(r) \) denotes the spherical average of \( w - M \) w.r.t. \( \bar{x} \). This is absurd, thus \( w \leq 0 \) in \( B_R(y) \). Similarly, we can show that \( m := \min_{x \in B_R(y)} w(x) = 0 \). Therefore, \( w = u - h = 0 \) in \( B_R(y) \) and hence \( (-\Delta)^{\frac{\alpha}{2}} u = 0 \) in \( B_R(y) \). Since \( B = B_R(y) \subset \subset \Omega \) is arbitrary, we deduce that \( (-\Delta)^{\frac{\alpha}{2}} u = 0 \) in \( \Omega \). This completes our proof of Theorem 6.1.

As an immediate consequence of Theorem 6.1 we have the following Theorem.

**Theorem 6.2.** Suppose \( \{u_n\}_{n \geq 1} \) is a sequence of \( \alpha \)-harmonic functions in \( \Omega \) and \( u_n \rightharpoonup u \) in \( \mathbb{R}^n \), then \( u \) is \( \alpha \)-harmonic in \( \Omega \).

**Remark 6.3.** The Harnack inequalities (see [29]) and Liouville theorems for \( \alpha \)-harmonic functions (see [1][41]), and Maximal principles for fractional Laplacians (-\( \Delta \))\(^{\frac{\alpha}{2}}\) (see [12][37]) can also be deduced directly from Theorem 6.1. We omit the proofs here and leave the details to readers.

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