Organization of cooperation in fractal structures

Dan Peng¹ and Ming Li²,*

¹Department of Network and New Media, Anhui University, Hefei 230601, P. R. China
²Department of Thermal Science and Energy Engineering, University of Science and Technology of China, Hefei 230026, P. R. China
(Dated: October 15, 2020)

It is known that the small-world structure constitutes sufficient conditions to sustain cooperation and thus enhances cooperation. On the contrary, the fractal/large-world structure, of which the average distance between nodes scales in a power-law manner with the system size, is thought of as suppressing the emergence of the cooperation. In this paper we show that the fractal structure does not always play a negative role in the organization of cooperation. Compared to regular networks, the fractal structure might even facilitate the emergence of cooperation. This mainly depends on the existence of locally compact clusters. The sparse connections between these clusters construct an asymmetric barrier that the defection strategy is almost impossible to cross, but the cooperation strategy has a not too small chance. Indeed, the network needs not to be a standard fractal, as long as such structures exist. In turn, when this typical structure is absent, the fractal structure will also suppress the emergence of the cooperation, such as the fractal configuration obtained by diluting a random tree-like network. Our findings also clarify some contradictions in the previous studies, and suggest that both removing and inserting links from/into a regular network can enhance cooperation.

I. INTRODUCTION

In the study of complex systems, a long-standing question is how the collective behavior of interacting individuals is influenced by the topology of those interactions. Inspired by network science, the interaction topologies extracted from real networked systems have been widely studied in the last two decades [1, 2]. Two typical examples are the structures with scale-free property and small-world effect.

Mathematically, the scale-free property translates into a power-law degree distribution of a network, where a node’s degree is the number of nodes that are adjacent to the node. Obviously, this formula allows the existence of very high degree nodes, called hubs of networks, rather than the comparable degrees in Erdős-Rényi (ER) networks or regular networks [3]. As a result, the dynamics on such a network bears a strong heterogeneity induced by the existence of hubs [2]. Specifically, the change of a hub’s state can affect a large number of nodes, while the influences of periphery nodes are limited to their few neighbors.

It is conceivable that the average distance between nodes is also an important characteristic of a topology structure [3]. A short average distance can facilitate communications between nodes, while a long one lengthens the response time for communications. If the average distance \( l \) grows proportionally to the logarithm of the network size \( N \), i.e., \( l \sim \ln N \), it just refers to the small-world effect [4]. In general, the small-world effect can be constructed by inserting long-range connections into a regular network [4]. In contrast, there is another typical topology called fractal [5], showing a long average distance in a power-law manner with the system size. Due to the long distance, the fractal structure is usually considered to be not favorable for the spreading process, as contrasted with the small-world networks that can reduce the threshold for an epidemic outburst [6, 7], yield the most effective information spreading [8], and promote cooperation [9–11].

In addition to the long average distance, the fractal exhibits similar patterns at increasingly smaller scales, called self-similarity [5]. This pattern generally renders a structure composed of many locally compact clusters with some sparse connections between them. That is which distinguishes the dynamics on fractal structures from that of regular structures, which also have long average distances but uniform connections. For instance, in an evolutionary game model, Wang et al. found that the structure formed at the percolation threshold [12, 13], which presents a fractal structure [14], can optimally promote cooperation (compared to that of far way from the percolation threshold, which can be seen as more like a regular network). This is mainly because the organization and the stability of cooperator clusters are highly dependent on the local connections of individuals [15]. However, some works also showed that the long distance between nodes generally reduces cooperation [9–11]. Besides, Tang et al. also found that an ER network requires a relatively dense connectivity to achieve an optimal cooperation [16], however, for average degree 1, of which ER network shows a fractal structure, no cooperators can survive.

In this paper we employ the prisoner’s dilemma game (PDG) to clarify these seemingly paradoxical phenomena. Our results will show that these phenomena are mainly ascribed to the existence of compact clusters with sparse connections between them, which can often be found in the fractal structure. In the followings, we will first introduce the model used in this study, and then the results on two different fractal structures will be discussed.

II. COOPERATION EVOLUTION

Initially, cooperators (C) and defectors (D) are randomly and uniformly distributed over the network. In accordance with the standard definition of the PDG, we set the payoff of
To understand how that works, we next check the behavior of an individual encountering with a D (adjacent to) a D is 0, in contrast with encountering with a C, a C and a D respectively gain payoff 1 and $b$ ($b > 1$). To represent the cooperation evolution, at each time step a randomly chosen individual $x$ tries to enforce its strategy (C or D) on a randomly chosen neighbor $y$ with the probability

$$w_{x,y} = \frac{1}{1 + e^{-(p_x - p_y)/\kappa}},$$

where $p_x$ ($p_y$) is the total payoff of individual $x$ ($y$) gaining from all its neighbors, and $\kappa$ determines the level of uncertainty by strategy adoptions. Without loss of generality we set $\kappa = 0.1$ in this paper. Note that when an individual’s strategy changed, all the payoffs gaining from this individual need to be recalculated. For convenience, we rescale the time step by the system size $N$, thus one unit contains $N$ time steps, called a Monte Carlo step.

### III. HIERARCHICAL LATTICE

To study the effects of the fractal structure on the evolution of cooperation, we first consider a hierarchical lattice, which can demonstrate both fractal and small-world characteristics by applying different connection probability $p$ [17]. This network model adopts a growth mechanism from a single link, called generation $n = 0$. The generation $n+1$ is constructed by expanding each normal link (those are not long-range links) of generation $n$ to be a diamond-type lattice, and the original link is preserved as a long-range link with probability $p$, see Fig.1 (a). Note that the long-range links do not participate in this growth process.

For $p = 1$, the network exhibits the typical small-world property with a high clustering coefficient. Due to the absence of long-range links, the case $p = 0$ is a large-world lattice exhibiting an average distance $l \sim N^{1/2}$ [17]. As shown in Fig.1 (b), a typical local structure of this network is that two large degree nodes are bridged by two degree-2 nodes, meanwhile, there exists a link (long-range link) between the two nodes with probability $p$. As will be discussed later, this typical structure plays a key role in the evolution of cooperation.

It is widely acknowledged that the small-world effect can significantly enhance the cooperation [9, 10, 18]. To demonstrate this, in Fig.2 (a) we show the cooperator density $f_C$ in hierarchical networks under three relatively large $b$, for which the regular lattice, such as the square lattice (average degree 4) and the honeycomb lattice (average degree 3) [19], gives the full D state [20]. We can find that due to the small-world effect, a large $p$ always corresponds to a high cooperator density $f_C$. However, compared to regular lattices, which gives the full D state for parameters used in Fig.2 (a) [20], the fractal structure ($p \to 0$) is also in favour of the emergence of the cooperation [20]. This suggests that the hierarchical lattice with large $p$ (small-world effect) does not have a distinctly unique advantage in the emergence of cooperation. Without the long range link, the typical structure shown in Fig.1 (b) can also constitutes sufficient conditions to sustain cooperation, and even induces a nearly full C state (see the case $b = 1.05$ in Fig.2 (a)). To understand how that works, we next check the spreading of the two strategies on such a structure.

Specially, assuming that the two compact clusters at the two ends of the degree-2 node are a C cluster and a D cluster, respectively (see Fig.1 (b)). When the long-range link is absent, the growths of the two clusters have to go through the degree-2 node. If the degree-2 node is a D, it can gain a payoff $b$ from the C neighbor in the C cluster. However, this payoff is much smaller than the payoff of its C neighbor, labeled as $m$ for convenience, since it can gain mutual benefit from the C cluster. From Eq.(1), we can know that the probability that D can invade into the C cluster through the degree-2 node is $w_{b,m} = [1 + e^{(m-b)/\kappa}]^{-1}$, which is very small for large $m$. This means that the C cluster in such a structure is much more stable than that of a regular lattice.

In turn, if the degree-2 node is a C, it gains payoff 1 from its C neighbor in the C cluster, while its D neighbor in the D cluster gains payoff $b$ from it. Since $b > 1$, the degree-2 node will change its strategy to be a D with a relatively large probability,
then this reduces to the former case. Nevertheless, since the value \( b \) of interest in the study is often not much larger than 1, the update rule Eq. (1) also allows a not too small chance \( w_{1,b} = [1 + e^{(b-1)/s}]^{-1} \) that the degree-2 node can enforce its strategy \( C \) to its \( D \) neighbor in the \( D \) cluster.

In conclusion, the probability \( w_{1,b} \) that \( C \) can invade into the \( D \) cluster could be much larger than the probability \( w_{b,m} \) that \( D \) can invade into the \( C \) cluster. Taking \( b = 1.05 \) and \( m = 3 \) as an example, we have \( w_{1,b} \approx 0.38 \) and \( w_{b,m} \approx 3.4 \times 10^{-9} \). This means that the structure shown in Fig.1 (b) (without the long-range link) constructs an asymmetric barrier for the growth of \( C \) and \( D \) clusters, for which \( D \) strategy is almost impossible to cross, while \( C \) strategy has a not too small chance. In other words, the \( C \) cluster is more stable than the \( D \) cluster. This explains why for small \( p \) the hierarchical lattice can also show a very high cooperation level. Moreover, a very large \( b \) will break the asymmetry, and this structure would no longer facilitate cooperation, see the cases \( b = 1.5 \) and \( b = 1.75 \) of Fig.2 (a).

In fact, this asymmetric barrier comes from that an individual has a small chance to enforce its strategy to the one with a larger payoff. If such a strategy update is not allowed, namely, individuals can only enforce their strategies to the ones with payoffs smaller than theirs, this structure will no longer facilitate the emergence of the cooperation. The corresponding results are shown in Fig.2 (b). This further confirms the above analysis, and is also consistent with the result found in Ref.[11].

As a second consequence, the fractal structure can also stretch the convergence time of the system as shown in Fig.3. This is mainly due to two reasons. First, the long average distance obviously extends the spreading process. Second, due to the sparse connections between the compact clusters of fractal structures, a successful invasion of \( D \) cluster might need a larger number of attempts and thus lead to a very large convergence time. An immediate consequence of this is that the long-term growth of the cooperators density \( f_C \) is more like a step function, where a jump corresponds to a complete conversion of a \( D \) cluster, see Fig.3.

Generally speaking, the spreading of strategies is cramped by the sparse connections between different clusters, but not completely blocked, which increases the convergence time and ultimately favours the emergence of cooperation. Note that for small \( p \), the convergence time is so large that there is no uniform criterion for the convergence of the system. For some cases, the convergence even cannot be achieved in an acceptable time, see case \( p = 0 \) in Fig.3. From this perspective, we suspect that for small \( b \), such as the case \( b = 1.05 \) shown in Fig.2 (a), a large enough system will also give a full \( C \) state for small \( p \). However, due to the high time complexity, we have no firm numerical evidence to support this.

It is also worth pointing out that the compact cluster mentioned here is not equivalent to the community structure, although a large convergence time was also found on the network with community structures [21]. In general, the community is a macrostructure scaling in the order of the system size, while the compact cluster mentioned here is a microstructure containing only a small amount of nodes. In a community the clusters of different strategies could coexist, while there is often only one strategy in a small compact cluster.

\section*{IV. NETWORKS NEAR THE PERCOLATION THRESHOLD}

Excluding the small-word effect, hubs also exist in the hierarchical lattice, which has also been proved to be extremely important in the evolution of the cooperation [9, 10, 16, 18]. In this section we will show that the above findings do not depend on the existence of hubs.

It is known that the configuration obtained at the percolation threshold also demonstrates a fractal structure [14], see Fig.4 as an example. It thus provides a framework to adjust...
a network between a regular structure and a fractal structure. Next, we employ the bond percolation model to further explore the effects of the fractal structure generated by diluting a regular network.

Put simply, the percolation configuration can be realized by randomly removing a fraction \(1 - q\) of links of a network, where \(q\) is often called occupied probability. If there exists a giant cluster of the order of system size, we call the system percolates. The percolation threshold is just the critical point \(q_c\), above which the system percolates. For bond percolation on square lattice discussed here, the percolation threshold has the exact solution \(q_c = 1/2\) [14].

As shown in Fig.4, the configuration around the percolation threshold can also present a typical structure as that shown in Fig.1 (b) but without hubs, i.e., locally compact clusters with some sparse connections between them. The simulation results shown in Fig.5 indicate that the configuration around the percolation threshold can also enhance cooperation as the fractal structure generated by hierarchical lattices.

Specifically, compared with regular networks \((q \to 1)\), the configuration around the percolation threshold can facilitate the emergence of cooperation, and the cooperation level is optimized near the percolation threshold, see Fig.5 (a). If individuals cannot enforce their strategies to the ones with a larger payoff, this optimization disappears, and the cooperators density \(f_c\) decreases monotonically with the increasing of \(q\), see Fig.5 (b). However, whatever the strategy update rule, Fig.5 suggests that removing some of links from the regular network can enhance the cooperation, i.e., showing a larger cooperators density \(f_c\) than that of \(q = 1\). Note that this makes sense only above or near the percolation threshold. When \(q < q_c\), the network falls to pieces [22], and the system state is almost entirely determined by the initial distribution of the two strategies.

The main difference between the two fractal structures shown in Figs.1 and 4 is that the former has hubs, i.e., the nodes with very large degrees. This suggests that the found manner of the fractal structure does not rely on the existence of hubs. From the fact that the cooperation is significantly enhanced for a wide range of \(q\) (optimized near the percolation threshold), we can further know that a standard fractal structure (exactly at the percolation threshold) is not needed to exhibit this phenomenon. This can be also seen from the results in the last section, which shows an high cooperator density for a wide range of \(p\).

However, our finding does not mean that all the fractal structures have such a property. A typical example is the ER network at the percolation threshold, i.e., the one with average degree 1. For this case, the connections of the giant cluster is so sparse that the locally compact clusters are absent. Therefore, this structure cannot constitute sufficient conditions to sustain cooperation. This is consistent with the results of Ref.[16], which suggested that the density of cooperators peaks at a large average degree depending on \(b\).

V. CONCLUSION

In this paper we discussed the organization of cooperation in the fractal structure. As we know, in the spatial evolutionary game, cooperators can survive only by forming clusters, while the fractal structure naturally provides many locally compact connections for the forming of cooperators clusters. We found that when \(b\) is not too large the sparse connections between these clusters construct many asymmetric barriers for the growth of C and D clusters, for which D strategy is almost impossible to cross, but C strategy has a not too small chance. This suggests that the network with a long average distance could also show a very high cooperation level. We also pointed out that the scale of these compact clusters is much smaller than community. The community is a macrostructure, in which C and D can coexist. Here, the small compact clusters are often fully occupied by either C or D. However, due to sparse connections between clusters/communities, both of them lead to a long convergence time.

In summary, we found that the fractal structure does not always play a negative role in the organization of cooperation. Compared with regular networks, the fractal structure can also facilitate the emergence of cooperation, suggesting that both removing and inserting links from/into a regular network can enhance cooperation. This means that a global scale communication channel is not strictly necessary to improve cooperation. By optimizing local connections, the cooperation can also be enhanced. This has important implications for our understanding of the emergence and organization of cooperation in different scales.

ACKNOWLEDGMENTS

The research of D.P. was supported by the Doctoral Scientific Research Foundation of Anhui University (Grant No. Y040418184).
[1] R. Albert and A.-L. Barabási, Statistical mechanics of complex networks, *Rev. Mod. Phys.* **74**, 47 (2002).
[2] S. N. Dorogovtsev, A. V. Goltsev, and J. F. F. Mendes, Critical phenomena in complex networks, *Rev. Mod. Phys.* **80**, 1275 (2008).
[3] A.-L. Barabási, *Network science* (Cambridge university press, 2016).
[4] D. J. Watts and S. H. Strogatz, Collective dynamics of ‘small-world’ networks, *Nature* **393**, 440 (1998).
[5] A. Bunde and S. Havlin, *Fractals and disordered systems* (Springer Science & Business Media, 2012).
[6] C. Moore and M. E. J. Newman, Epidemics and percolation in small-world networks, *Phys. Rev. E* **61**, 5678 (2000).
[7] F. C. Santos, J. F. Rodrigues, and J. M. Pacheco, Epidemic spreading and cooperation dynamics on homogeneous small-world networks, *Phys. Rev. E* **72**, 056128 (2005).
[8] L. Lü, D.-B. Chen, and T. Zhou, The small world yields the most effective information spreading, *New J. Phys.* **13**, 123005 (2011).
[9] F. Santos, J. Rodrigues, and J. Pacheco, Graph topology plays a determinant role in the evolution of cooperation, *Proceedings of the Royal Society B: Biological Sciences* **273**, 51 (2005).
[10] F. C. Santos and J. M. Pacheco, Scale-free networks provide a unifying framework for the emergence of cooperation, *Phys. Rev. Lett.* **95**, 098104 (2005).
[11] C. K. Yun, N. Masuda, and B. Kahng, Diversity and critical behavior in prisoner’s dilemma game, *Phys. Rev. E* **83**, 057102 (2011).
[12] Z. Wang, A. Szolnoki, and M. Perc, Percolation threshold determines the optimal population density for public cooperation, *Phys. Rev. E* **85**, 037101 (2012).
[13] Z. Wang, A. Szolnoki, and M. Perc, If players are sparse social dilemmas are too: Importance of percolation for evolution of cooperation, *Sci. Rep.* **2**, 369 (2012).
[14] D. Stauffer and A. Aharony, *Introduction to percolation theory*, 2nd ed. (Taylor & Francis, London, 1991).
[15] M. Perc, J. J. Jordan, D. G. Rand, Z. Wang, S. Boccaletti, and A. Szolnoki, Statistical physics of human cooperation, *Phys. Rep.* **687**, 1 (2017).
[16] C.-L. Tang, W.-X. Wang, X. Wu, and B.-H. Wang, Effects of average degree on cooperation in networked evolutionary game, *The European Physical Journal B* **53**, 411 (2006).
[17] M. Hinczewski and A. Nihat Berker, Inverted Berezinskii-Kosterlitz-Thouless singularity and high-temperature algebraic order in an Ising model on a scale-free hierarchical-lattice small-world network, *Phys. Rev. E* **73**, 066126 (2006).
[18] J. Gómez-Gardeñes, M. Campillo, L. M. Floría, and Y. Moreno, Dynamical organization of cooperation in complex topologies, *Phys. Rev. Lett.* **98**, 108103 (2007).
[19] The hierarchical lattice has average degree $3 + p$ in the infinite lattice limit [17].
[20] G. Szabó, J. Vukov, and A. Szolnoki, Phase diagrams for an evolutionary prisoner’s dilemma game on two-dimensional lattices, *Phys. Rev. E* **72**, 047107 (2005).
[21] D. A. Gianetto and B. Heydari, Network modularity is essential for evolution of cooperation under uncertainty, *Sci. Rep.* **5**, 9340 (2015).
[22] Strictly speaking, below the percolation threshold, all the clusters are very small and scales in the order of $\ln N$. However, for a finite system, a cluster of order $N$ might also exist in the area that not far away from the percolation threshold.