EXISTENCE OF SOLUTIONS FOR SPACE-FRACTIONAL PARABOLIC HEMIVARIATIONAL INEQUALITIES

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Abstract. This paper is devoted to the existence of solutions for space-fractional parabolic hemivariational inequalities by means of the well-known surjectivity result for multivalued \((S_+)\) type mappings.

1. Introduction. Fractional-order-in-space mathematical models, in which an integer-order differential operator is replaced by a corresponding fractional one, are becoming increasingly popular, since they provide an adequate description of many processes that exhibit anomalous diffusion. This is due to the fact that the nonlocal nature of the fractional operators enables one to capture the spatial heterogeneity that characterizes these processes. The study of problems in the framework of integro-differential equations is quite recent and has risen a great interest particularly in connection with problems involving nonlocal effects.

Nonlocal operators naturally appear in elasticity problems [29], water waves [9, 10, 31], crystal dislocation [34], thin obstacle problems [3, 32], phase transition [1, 2, 30], flames propagation [4], stratified materials [26], quasi-geostrophic flows [5, 21], among other phenomena.

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The aim of this paper is to prove the existence of at least one solution for nonlocal parabolic hemivariational inequalities as follows:

\[
\begin{aligned}
&u' + L_K u + \partial J(u) \ni f \quad \text{in } Q = \Omega \times (0, T), \\
u(x, t) = 0 & \quad \text{in } \Omega^c \times (0, T), \\
u(x, 0) = u_0(x) & \quad \text{in } \Omega.
\end{aligned}
\]  

(1)

where \( \Omega := \mathbb{R}^N \setminus \Omega, \Omega \subset \mathbb{R}^N \) is an open bounded set with Lipschitz boundary. The nonlocal operator \( L_K \) defined as follows:

\[
L_K u(x) := -\int_{\mathbb{R}^N} [u(x + y) + u(x - y) - 2u(x)]K(y) dy, \quad \forall x \in \mathbb{R}^N,
\]

(2)

where \( K : \mathbb{R}^N \setminus \{0\} \to (0, +\infty) \) is a function satisfying the assumption (A):

(i) \( \gamma K \in L^1(\mathbb{R}^N) \), where \( \gamma(x) = \min\{ |x|^2, 1 \} \);

(ii) there exists \( \lambda > 0 \) such that \( K(x) \geq \lambda |x|^{-(N+2s)} \), \( \forall x \in \mathbb{R}^N \setminus \{0\} \);

(iii) \( K(x) = K(-x), \forall x \in \mathbb{R}^N \setminus \{0\} \).

A typical example for \( K \) is given by \( K(x) = |x|^{-(N+2s)} \) for \( s \in (0, 1)(N > 2s) \). In this case \( L_K \) is the fractional Laplace operator \(-\Delta^s\), which (up to normalization factor) is defined as

\[-(\Delta)^su(x) := \int_{\mathbb{R}^N} \frac{u(x+y) + u(x-y) - 2u(x)}{|y|^{N+2s}} dy, \quad \forall x \in \mathbb{R}^N.\]

(3)

\( \partial J(\cdot) \) denotes the generalized subdifferential in the sense of Clarke (cf.\[7, 22\]), \( f : \Omega \times (0, T) \to \mathbb{R} \).

We remark that the Dirichlet datum is given in \( \Omega^c \) and not simply on \( \partial \Omega \), consistently with the non-local character of the operator \( L_K \).

Hemivariational inequalities arise in variational expressions for some mechanical problems with nonsmooth and nonconvex energy superpotentials. The derivative of hemivariational inequality is based on the mathematical notion of the generalized gradient of Clarke (cf. \[8, 11, 13, 14, 15, 16, 17, 22, 23\]).

Recently, Liu and Tan [19] obtained an existence result for nonlocal elliptic hemivariational inequalities with Dirichlet boundary condition by use of pseudomonotone theory. While Teng [33] and Xi et al. [35] established multiplicity of weak solutions to nonlocal elliptic hemivariational inequalities by using the nonsmooth critical point theory and nonsmooth version of the three-critical-points theorem under the framework of the nonsmooth functional. However, to the best of our knowledge, the mathematical literature dedicated to the existence results for nonlocal parabolic hemivariational inequalities are still untreated and this fact is the motivation of the present work.

In this paper, we show the existence of at least one solution for the nonlocal parabolic hemivariational inequalities. The basic tools used in our paper are the surjectivity result for \((S_\epsilon)\) and coercive operators, properties of the generalized subdifferential in the sense of Clarke. We believe that our result gives a natural approach to the theory of the nonlocal evolutional hemivariational inequalities. Furthermore, we emphasise that our methods in this paper are also applicable to periodic and anti-periodic nonlocal evolutional hemivariational inequalities.

2. Mathematical framework. Let \( E \) be a real reflexive Banach space densely and continuously imbedded in a real Hilbert space \( H \). Identifying \( H \) with its dual, we have \( E \subseteq H \subseteq E^\ast \), where \( E^\ast \) stands for the dual of \( E \). The norm of any Banach
space $B$ is denoted by $\| \cdot \|_B$. The duality pairing between $B$ and its dual $B^*$ is denoted by $\langle \cdot, \cdot \rangle_B$.

In the following, we always assume $s \in (0, 1)(N > 2s), \Omega \subset \mathbb{R}^N$ is an open bounded set with Lipschitz boundary. $K : \mathbb{R}^N \setminus \{0\} \to (0, +\infty)$ satisfies assumption (A).

We recall some preliminary material on function spaces and norms. Denote $Q = \mathbb{R}^{2N} \setminus \mathcal{O}$, where $\mathcal{O} = (\mathbb{R}^N \setminus \Omega) \times (\mathbb{R}^N \setminus \Omega)$. We denote

$$X = \{ u : \mathbb{R}^N \to \mathbb{R}, u|_{\Omega} \in L^2(\Omega), (u(x) - u(y))\sqrt{K(x - y)} \in L^2(\mathbb{R}^{2N} \setminus \mathcal{O}) \},$$

$$X_0 = \{ u \in X : u = 0 \text{ a.e. in } \mathbb{R}^N \setminus \Omega \},$$

where $u|_{\Omega}$ represents the restriction to $\Omega$ of function $u(x)$.

It is easy to check that $X$ and $X_0$ are norm linear spaces endowed with the norm:

$$\| u \| = \| u \|_{L^2(\Omega)} + \left( \int_\mathcal{Q} |u(x) - u(y)|^2 K(x - y) \, dxdy \right)^{1/2}. \quad (4)$$

In $X_0$, we may also use the norm

$$\| u \|_{X_0} = \left( \int_\mathcal{Q} |u(x) - u(y)|^2 K(x - y) \, dxdy \right)^{1/2}, \quad (5)$$

with the inner product for $u, v \in X_0$

$$\langle u, v \rangle_{X_0} := \int_\mathcal{Q} (u(x) - u(y))(v(x) - v(y))K(x - y) \, dxdy. \quad (6)$$

Then the two norms defined by (4) and (5) in $X_0$ are equivalent (see, [27] for details).

In the sequel, we denote by $H^s(\Omega)$ the usual fractional Sobolev space endowed with the norm (the Gagliardo norm)

$$\| u \|_{H^s(\Omega)} = \| u \|_{L^2(\Omega)} + \left( \int_{\Omega \times \Omega} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} \, dxdy \right)^{1/2}. \quad (7)$$

We remark that even in the model case in which $K(x) = |x|^{-(N+2s)}$, the norm in (4) and (7) are not the same, because $\Omega \times \Omega$ is strictly contained in $Q$ (this makes the classical fractional Sobolev space approach not sufficient for studying the problem). For further details on the fractional Sobolev spaces we refer to [25] and the references therein.

We stress that $C_0^\infty(\Omega) \subseteq X_0$ (see, e.g., [28]). So $X_0$ is non-empty and dense in $L^2(\Omega)$. We may collect the useful facts on the space $X_0$ (see [27], for more details) as follows.

**Proposition 2.1.** $X_0$ is a Hilbert space and for $p \in [1, 2^*)$, there exists a positive constant $c(p)$ such that

$$\| u \|_{L^p(\mathbb{R}^N)}^p \leq c(p) \| u \|_{X_0}^p, \quad \forall u \in X_0, \quad (8)$$

where $2^* = \frac{2N}{N-2s}$. Furthermore, the embedding is compact if $p \in [1, 2^*)$.

Next, let us recall some useful facts from the theory of nonlinear operators of monotone type. Let $L : D(L) \subseteq E \to E^*$ be a linear maximal monotone operator.

**Definition 2.2.** We say that a multivalued mapping $A$ from $D(A) \subseteq E \to 2^{E^*}$ has the $(S_\bullet)$ property with respect to $D(L)$ if the following conditions hold:

(i) The set $A u$ is nonempty, bounded, closed and convex for each $u \in D(A)$.

(ii) $A$ is finitely weakly upper-semicontinuous, i.e., for each finite dimensional subspace $S$ of $E$, $A$ is an upper-semicontinuous mapping of $D(A) \cap S$ into $2^{E^*}$ with $E^*$
given its weak topology.

(iii) For any sequence \( \{u_i\} \) in \( D(A) \cap D(L) \) converging weakly to an element \( u \) of \( D(L) \) in \( E \), \( Lu_i \) converging weakly to \( Lu \) in \( E^* \) and for any sequence \( w_i \) in \( E^* \) with \( w_j \in A(u_j) \) for each \( j \geq 1 \), the condition

\[
\lim \sup \langle w_j, u_j - u \rangle_E \leq 0
\]

implies the strong convergence of \( \{u_i\} \) to \( u \) and there exists a subsequence \( \{w_{n_j}\} \) of \( \{w_j\} \) such that \( \{w_{n_j}\} \) converges weakly to \( w \in Au \).

The main tool we use in this paper is a surjectivity result for multivalued \((S_+)\) type mappings (cf. [12, 20, 22]). For the convenience of the reader we include it here.

**Theorem 2.3.** [20] If \( E \) is a reflexive Banach space, \( L : D(L) \subseteq E \rightarrow E^* \) is a closed densely linear maximal monotone operator, and \( S : E \rightarrow 2^{E^*} \) is of class \((S_+)\) with respect to \( D(L) \), and coercive (i.e., \( \inf \{\langle w, u \rangle_E : w \in Su \} / \|u\|_E \rightarrow \infty \) as \( \|u\|_E \rightarrow \infty \)), then \( RL = E^* \) (i.e., the operator \((L+S)(\cdot)\) is surjective).

Let us recall \( h^0(u,v) \) the Clarke generalized directional derivative of a locally Lipschitz functional \( h : E \rightarrow \mathbb{R} \) at \( u \in E \) in the direction \( v \in E \), i.e.,

\[
h^0(u,v) = \lim \sup_{\lambda \rightarrow 0^+, w \rightarrow u} \frac{h(w + \lambda v) - h(w)}{\lambda}
\]

and the generalized Clarke subdifferential of \( h \) at \( u \in E \)

\[
\partial h(u) := \{u^* \in V^* \mid h^0(u,v) \geq \langle u^*, v \rangle \text{ for all } v \in E\}.
\]

The next proposition provides basic properties of the generalized directional derivative and the generalized gradient.

**Proposition 2.4.** [7, 22] If \( h : U \rightarrow \mathbb{R} \) is a locally Lipschitz function on a subset \( U \) of \( E \), then

(i) for every \( x \in U \) the gradient \( \partial h(x) \) is a nonempty, convex, and weakly* compact subset of \( E^* \) which is bounded by the Lipschitz constant \( K_\epsilon > 0 \) of \( h \) near \( u \).

(ii) for each \( y \in E \), there exists \( z_x \in \partial h(x) \) such that

\[
h^0(x;y) = \max \{\langle z, y \rangle_E \mid z \in \partial h(x)\} = \langle z_x, y \rangle_E.
\]

(iii) the graph of the generalized gradient \( \partial h \) is closed in \( E \times (w^* - E^*) \) topology, i.e., if \( \{x_k\} \subseteq U \) and \( \{\zeta_k\} \subseteq E^* \) are sequences such that \( \zeta_k \in \partial h(x_k) \) and \( x_k \rightarrow x \) in \( E, \zeta_k \rightarrow \zeta \) weakly* in \( E^* \) then \( \zeta \in \partial h(x) \) where, recall, \( w^* - E^* \) denotes the space \( E^* \) equipped with weak* topology.

(vi) The multifunction \( U \ni x \rightarrow \partial h(x) \subseteq E^* \) is upper semicontinuous from \( U \) into \( w^* - E^* \).

Let \( Q = \Omega \times (0,T) \). We assume that \( j : Q \times \mathbb{R} \rightarrow \mathbb{R}, j(\cdot,0) \in L^1(Q) \) satisfying the assumption (H):

(i) \( j(\cdot,s) : Q \rightarrow \mathbb{R} \) is measurable for all \( s \in \mathbb{R} \);

(ii) \( j(x, \cdot, t) : \mathbb{R} \rightarrow \mathbb{R} \) is locally Lipschitz for a.e. \( (x, t) \in Q \);

(iii) there exist \( p \geq 1, c > 0 \) and \( b \in L_{loc}^{\frac{N}{p-1}}(Q) \) such that

\[
|z| \leq b(x, t) + c|s|^{p-1}, \forall (x, t) \in Q, \forall z \in \partial_s j(x, t, s).
\]

Define the integral functional

\[
J(v) := \int_Q j(x, t, v(x, t)) \, dx \, dt \text{ for all } v \in L^p(Q).
\]
Proposition 2.5. [13] Under the assumption (H), the functional \( J \) in (9) is locally Lipschitz and the following inequalities hold:

\[
J^0(u, v) \leq c_1(1 + \|u\|^\frac{p-1}{p})\|v\|_{L^p} \forall u, v \in L^p(Q)
\]

and

\[
\|w\|_{L^{p'}} \leq c_1(1 + \|u\|^\frac{p-1}{p}) \forall w \in \partial(J|_{L^p})(u), u \in L^p(Q),
\]

where \( \frac{1}{p} + \frac{1}{p'} = 1 \) and \( c_1 \) is a positive constant.

Remark 1. Identifying \( L^2(\Omega) \) with its dual, we have \( X_0 \subseteq L^2(\Omega) \subseteq (X_0)^* \), where \( (X_0)^* \) stands for the dual of \( X_0 \). Let \( \mathcal{X} = L^2([0, T], X_0), L^2(Q) = L^2([0, T], L^2(\Omega)) \), \( \mathcal{X}^* = L^2([0, T], (X_0)^*) \) and \( \mathcal{W} = \{ u \in \mathcal{X}; \frac{\partial u}{\partial t} \in \mathcal{X}^* \} \). Then \( \mathcal{W} \) with the norm \( \|u\|_W = \|u\|_X + \|\frac{\partial u}{\partial t}\|_{\mathcal{X}^*} \) is a Banach space. Furthermore, the imbedding \( \mathcal{W} \subseteq L^2(Q) \) is compact and the embedding \( \mathcal{W} \subseteq C([0, T], L^2(\Omega)) \) is continuous (for more details, see [36]).

3. Main results.

Definition 3.1. \( u \in \mathcal{W} \) is said to be a weak solution to (1) with \( f \in \mathcal{X}^* \) and \( u_0 \in L^2(\Omega) \) if there exists \( w \in \partial J(u) \) such that

\[
\int_{\Omega} \frac{\partial u}{\partial t}v(x, t) \, dx \, dt + \int_{\Omega} v(x, t)\mathcal{L}Ku(x, t) \, dx \, dt \\
+ \int_{\Omega} w(x, t)v(x, t) \, dx \, dt \\
= \int_{\Omega} f(x, t)v(x, t) \, dx \, dt, \forall v \in \mathcal{X}.
\]

We define a linear functional \( \mathcal{K} : \mathcal{X} \rightarrow \mathcal{X}^* \) by \( \mathcal{K}u \in \mathcal{X}^* \) such that \( (\mathcal{K}u)(v) =: \int_{\Omega} \mathcal{L}Ku(x, t) \, dx \, dt, \forall v \in \mathcal{X} \), which is well-defined by Proposition 3.2 below.

In this way, we may say that \( u \in \mathcal{W} \) is a weak solution to problem (1) with \( f \in \mathcal{X}^* \) and \( u_0 \in L^2(\Omega) \) if there exists \( w \in \partial J(u) \) such that

\[
\left\{ \begin{array}{l}
\frac{\partial u}{\partial t} + \mathcal{K}u + w = f \\
u(x, 0) = u_0(x) \text{ in } \Omega.
\end{array} \right.
\]

Proposition 3.2. \( \mathcal{K} : \mathcal{X} \rightarrow \mathcal{X}^* \) is a linear bounded strongly monotone operator.

Proof. By means of Assumption A, we have for all \( u, v \in \mathcal{X} \),

\[
(\mathcal{K}u)(v) =: \int_{\Omega} \mathcal{L}Ku(x, t) \, dx \, dt \\
= -\int_0^T dt \int_{\Omega} v(x, t) \, dx \int_{\Omega} [u(x + y, t) + u(x - y, t) - 2u(x, t)]K(y) \, dy \\
= -\int_{\Omega} \int_{\Omega} u(x + y, t) - u(x, t)[v(x, t)K(y) \, dy \, dx \\
- \int_{\Omega} \int_{\Omega} [u(x, t) - u(x - y, t)]v(x, t)K(y) \, dy \, dx \\
= -\int_{\Omega} \int_{\Omega} [u(y, t) - u(x, t)]v(x, t)K(x - y) \, dy \, dx \\
- \int_{\Omega} \int_{\Omega} [u(y, t) - u(x, t)]v(x, t)K(y - x) \, dy \, dx
\]
which implies \( K : X \to X^* \) is a linear bounded strongly monotone operator and
\[
\| K u \|_{X^*} \leq \| u \|_X.
\]

The proof is complete.

\[\square\]

Remark 2. By (13), we also may say that \( u \in W \) is said to be a weak solution to (1) with \( f \in X^* \) and \( u_0 \in L^2(\Omega) \) if there exists \( w \in \partial J(u) \) such that
\[
\int_0^T \int_{\mathbb{R}^n} \frac{\partial u}{\partial t} v(x,t) \, dx \, dt + \int_0^T \int_{\mathbb{R}^n} w(x,t) v(x,t) \, dx \, dt + \int_0^T \int_{\mathbb{R}^n} \sum_{i=1}^n [(u(x,t) - u(y,t))(v(x,t) - v(y,t)) - K(x-y) \, dy dx \, dt \]
\[
= \int_0^T \int_{\mathbb{R}^n} f(x,t) v(x,t) \, dx \, dt, \quad \forall v \in X.
\]

Firstly, we are going to discuss the problem (1) with zero-initial condition, i.e.,
\[
\left\{ \begin{array}{l}
\frac{\partial u}{\partial t} + Ku + w = f \\
u(x,0) = 0 \text{ in } \Omega.
\end{array} \right.
\]

For this purpose, we define
\[
Lu = \frac{\partial u}{\partial t}, \quad D(L) = \{ u \in W : u(0) = 0 \}.
\]

Here \( \frac{\partial u}{\partial t} \) stands for the generalized derivative of \( u \), i.e.,
\[
\int_0^T \int_{\mathbb{R}^n} \frac{\partial u}{\partial t} (t) \phi(t) \, dx \, dt = - \int_0^T \int_{\mathbb{R}^n} u(t) \frac{\partial \phi}{\partial t} (t) \, dx \, dt, \quad \forall \phi \in C_0^\infty([0,T]).
\]

Then, \( L : D(L) \subseteq X \to X^* \) defined by (15) is a closed densely linear maximal monotone operator (cf.[36]).

Therefore, to get a solution of the problem (1) with zero-initial condition means to solve the following abstract equation:
\[
Lu + Ku + w = f, \text{ where } w \in \partial J(u).
\]

Proposition 3.3. Under the assumption (H), \( K + \partial J : X \to 2^{X^*} \) has \((S_+)\) property with respect to \( D(L) \). Furthermore, if \( 1 \leq p < 2 \) or \( p = 2 \) with \( c_1 c(2) < 1 \), where \( c(2) \) and \( c_1 \) are the constants in (8) and (10), respectively, then \( K + \partial J : X \to 2^{X^*} \) is coercive.
**Proof.** We shall firstly show that $Ku + \partial J(u)$ has the $(S_+)$ property with respect to $D(L)$. By Proposition 2.4, we observe that $\partial J$ is nonempty, convex, weak-compact subset of $\mathcal{X}^*$. Then for each $u \in \mathcal{X}$, $Ku + \partial J(u)$ is nonempty, bounded, closed and convex subset of $\mathcal{X}^*$. Moreover, $Ku + \partial J(u)$ is upper semicontinuous from $\mathcal{X}$ to $\mathcal{X}^*$.

For any sequence $\{u_k\}$ in $D(L)$ converging weakly to an element $u$ of $D(L)$ in $\mathcal{X}$, $Lu_k$ converging weakly to $Lu$ in $\mathcal{X}^*$ and for any sequence $w_k$ in $\mathcal{X}^*$ with $w_k \in \partial J(u_k)$ for each $k \geq 1$, if the following condition holds

$$\limsup_{k \to \infty} \langle Ku_k + w_k, u_k - u \rangle_{\mathcal{X}} \leq 0, \quad (16)$$

we have to show that the strong convergence of $\{u_k\}$ to $u$ and there exists a subsequence $\{w_{n_k}\}$ of $\{w_k\}$ such that $\{w_{n_k}\}$ converges weakly to $w \in \partial J(u)$, which means that $K + \partial J$ has the $(S_+)$ property with respect to $D(L)$.

By Proposition 2.1, we have $X_0 \subseteq L^2(\Omega) \subseteq (X_0)^*$ and the embedding $X_0 \hookrightarrow L^2(\Omega)$ is compact. Therefore, by Remark 1, we have that the imbedding $\mathcal{W} \hookrightarrow L^2(Q)$ is compact. Hence,

$$u_k \to u \text{ strongly in } L^2(Q).$$

Applying Theorem 2.2 in [6], we have

$$\partial(J|_{\mathcal{X}}(u)) \subseteq \partial(J|_{L^2(Q)})(u), \forall u \in \mathcal{X}.$$ 

So we get

$$\langle w_k, u_k - u \rangle_{\mathcal{X}} = \langle w_k, u_k - u \rangle_{L^2(Q)}. \quad (18)$$

On the other hand, we see that $\{w_k\}$ is bounded in $L^2(Q)$ from (11). Furthermore, $\{w_k\}$ is bounded in $\mathcal{X}^*$. So we may assume that

$$w_k \to w \text{ weakly in } \mathcal{X}^*. \quad (19)$$

and

$$\lim_{k \to \infty} \langle w_k, u_k - u \rangle_{\mathcal{X}} = \lim_{k \to \infty} \langle w_k, u_k - u \rangle_{L^p(Q)} = 0. \quad (20)$$

Then, from (17) we have

$$\limsup_{k \to \infty} \langle Ku_k, u_k - u \rangle_{\mathcal{X}} \leq 0. \quad (21)$$

Therefore, we obtain from $u_k \to u$ weakly in $\mathcal{X}$

$$\limsup_{k \to \infty} \langle Ku_k - Ku, u_k - u \rangle_{\mathcal{X}} \leq 0.$$ 

By the proof of Proposition 3.2,

$$\limsup_{k \to \infty} \|u_n - u\|_{\mathcal{X}}^2 = \limsup_{k \to \infty} \langle Ku_k - Ku, u_k - u \rangle_{\mathcal{X}} = 0,$$

which implies that

$$u_k \to u \text{ strongly in } \mathcal{X}. \quad (22)$$

$$Ku_k \to Ku \text{ strongly in } \mathcal{X}^*. \quad (23)$$

Then, by Proposition 2.4

$$Ku + w \in Ku + \partial J(u),$$

So, $K + \partial J : \mathcal{X} \to \mathcal{X}^*$ has $(S_+)$ property with respect $D(L)$. 


In the following, we show that $K + \partial J : X \to X^*$ is coercive. By Proposition 2.1 and 2.5, we have from (11) and (18)

$$\inf\{\langle Ku + w, u \rangle_X \mid w \in \partial J(u)\}$$

$$= \langle Ku, u \rangle_X + \inf\{\langle w, u \rangle_{L^p(Q)} \mid w \in \partial J(u)\}$$

$$\geq \|u\|_X^2 - \sup\{\|w\|_{L^p(Q)} \mid w \in \partial J(u)\}\|u\|_{L^p(Q)}$$

$$\geq \|u\|_X^2 - c_1\|u\|_{L^p(Q)} - c_1\|u\|_{L^p(Q)}^p$$

$$\geq \|u\|_X^2 - c_1c(p)^{1/p}\|u\|_X - c_1c(p)\|u\|_X^p.$$
Proof. Let \( u_0 \in L^2(\Omega) \). Since \( X_0 \) is dense in \( L^2(\Omega) \), we can find a sequence \( \{u_{0n}\} \subset X_0 \) such that \( u_{0n} \) converges to \( u_0 \) in \( L^2(\Omega) \). For each \( n \geq 1 \), by Corollary 2, there exists \( u_n \in W \) and \( w_n \in \partial J(u_n) \) such that

\[
\begin{align*}
\frac{\partial u_n}{\partial t} + Ku_n + w_n &= f \quad \text{in } Q = \Omega \times (0, T), \\
u_n(x, 0) &= u_{0n}(x) \quad \text{in } \Omega.
\end{align*}
\]

(24)

Therefore, we have

\[
\frac{1}{2}\|u_n(T)\|_{L^2(\Omega)}^2 - \frac{1}{2}\|u_n(0)\|_{L^2(\Omega)}^2 + \langle Ku_n, u_n \rangle_X + \langle w_n, u_n \rangle_X = \langle f, u_n \rangle_X.
\]

By virtue of the proof of Proposition 2.5 and Proposition 3.2, we obtain

\[
\frac{1}{2}\|u_n(T)\|_{L^2(\Omega)}^2 + \|u_n\|^2_X - c_1(1 + \|u_n\|_{L^2(\Omega)})\|u_n\|_{L^2(\Omega)} \\
\leq \frac{1}{2}\|u_n(0)\|_{L^2(\Omega)}^2 + \|f\|\|x^*\|u_n\|_X.
\]

Therefore, we get

\[
\frac{1}{2}\|u_n(T)\|_{L^2(\Omega)}^2 + \|u_n\|^2_X - c_1c(2)\|u_n\|^2_X \\
\leq \frac{1}{2}\|u_n(0)\|_{L^2(\Omega)}^2 + \|f\|\|x^*\|u_n\|_X + c_1c(2)^{1/2}\|u_n\|_X.
\]

Since \( c_1c(2) < 1 \), by use of the Young inequality, we get

\[
\|u_n(T)\|_{L^2(\Omega)}^2 + C_1\|u_n\|^2_X \leq C_2 + \sup_n\{\|u_n(0)\|_{L^2(\Omega)}^2\},
\]

(25)

where \( C_1 \) and \( C_2 \) are positive constants.

By \( u_{0n} \to u_0 \) in \( L^2(\Omega) \), we have that \( \sup_n\{\|u_{0n}\|^2_{L^2(\Omega)}\} < +\infty \). Therefore \( \{u_n\} \) is a bounded sequence in \( \mathcal{X} \) by (25). From (H), we infer that the boundedness of \( \{u_n\} \) in \( \mathcal{X} \) implies the boundedness of \( \{\|w_n\|_{\mathcal{X}^*}\} \). So in virtue of (24), we get

\[
\|\partial u_n \|_{\mathcal{X}^*} \leq \|Ku_n\|_{\mathcal{X}^*} + \|w_n\|_{\mathcal{X}^*} + \|f\|_{\mathcal{X}^*} \leq \text{Const.},
\]

which implies that there exists a subsequence, again denoted by \( \{u_n\} \) such that

\[
u_n \to u \quad \text{weakly in } \mathcal{X}.
\]

(26)

\[
\frac{\partial u_n}{\partial t} \to \frac{\partial u}{\partial t} \quad \text{weakly in } \mathcal{X}^*.
\]

(27)

In the following we shall show that \( u \) is a weak solution of the problem (1). Now we prove that

\[
u(0) = u_0.
\]

(28)

Using Proposition 23.23 in [36] and (24)-(27), we have \( u_n, u \in C([0, T], L^2(\Omega)) \) and

\[
u_n(t) - u(t) = \int_0^t (\frac{\partial u_n}{\partial s} - \frac{\partial u}{\partial s}) ds + u_{0n} - u(0)
\]

For any \( v \in X_0 \), by \( 0 < T < +\infty \), we may say that \( v \in \mathcal{X} \). Since \( X_0 \subseteq L^2(\Omega) \subseteq (X_0)^* \), we obtain \( \mathcal{X} \subseteq \mathcal{X}^* \). Therefore, we have

\[
\langle u_n - v, u_n \rangle_X = \langle \int_0^T (\frac{\partial u_n}{\partial s} - \frac{\partial u}{\partial s}) ds, v \rangle_X + T\langle u_{0n} - u(0), v \rangle_{X_0}.
\]

Letting \( n \to \infty \), it following from (24)-(27) that \( \langle u_{0n} - u(0), v \rangle_{X_0} \to 0 \) as \( n \to \infty \) for any \( v \in X_0 \), which implies \( u_{0n} \to u(0) \) weakly in \( (X_0)^* \). But \( u_{0n} \to u_0 \) strongly in \( L^2(\Omega) \). Therefore \( u(0) = u_0 \). Now from (24), we have

\[
\langle Ku_n + w_n, u_n - u \rangle_X = \langle f, u_n - u \rangle_X - \langle \frac{\partial u_n}{\partial t}, u_n - u \rangle_X,
\]
which implies

$$\langle K u_n + w_n, u_n - u \rangle_X = \langle f, u_n - u \rangle_X - \langle \frac{\partial u_n}{\partial t}, u_n - u \rangle_X - \langle \frac{\partial u}{\partial t}, u_n - u \rangle_X$$

By (26), we obtain

$$\lim \sup \langle K u_n + w_n, u_n - u \rangle_X \leq \lim \sup \{ \frac{1}{2} \| u_0 - u_0 \|^2_{L^2(\Omega)} \} \leq 0.$$  

Using a procedure similar to that given in the proof of Proposition 3.2, we obtain $u_n \to u$ strongly in $X$. By Proposition 2.4 and Proposition 3.2, we may assume that

$$K u_n \to K u \text{ strongly in } X^*,$$  \hspace{1cm} (29)
$$w_n \to w \in \partial J(u) \text{ weakly in } X^*,$$  \hspace{1cm} (30)

In virtue of (24)-(30), letting $n \to \infty$ in (24), we obtain

$$\begin{cases}
\frac{\partial u}{\partial t} + K u + w = f, \\
u(x, 0) = u_0(x) \text{ in } \Omega.
\end{cases}$$  \hspace{1cm} (31)

So $u$ is a weak solution of the problem (1) under the assumption $u_0 \in L^2(\Omega)$. The proof is complete. 

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