NON-TOPOLOGICAL SOLUTIONS IN A GENERALIZED
CHERN-SIMONS MODEL ON TORUS

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Abstract. We consider a quasi-linear elliptic equation with Dirac source terms arising in a generalized self-dual Chern-Simons-Higgs gauge theory. In this paper, we study doubly periodic vortices with arbitrary vortex configuration. First of all, we show that under doubly periodic condition, there are only two types of solutions, topological and non-topological solutions as the coupling parameter goes to zero. Moreover, we succeed to construct non-topological solution with \( k \) bubbles where \( k \in \mathbb{N} \) is any given number. We believe that it is the first result for the existence of non-topological doubly periodic solution of the quasi-linear elliptic equation arising in a generalized self-dual Chern-Simons-Higgs gauge theory. To find a suitable approximate solution, it is important to understand the structure of quasi-linear elliptic equation.

1. Introduction

In this paper, we study a generalized self-dual Chern-Simons-Higgs gauge theory introduced by Burzlaff, Chakrabarti, Tchrakian in [1]. The Lagrangian density of the model in \((2 + 1)\) dimensions is

\[
\mathcal{L} = \sqrt{2\varepsilon}\epsilon^{\mu\nu}\left[A_\alpha - 2i\left(1 - \frac{|\phi|^2}{2}\right)|D_\mu\phi|^2\right]F_{\mu\nu} + 2(1 - |\phi|^2)|D_\mu\phi|^2 - V,
\]

where \( A = (A_0, A_1, A_2) \) is a 3-vector gauge field, \( F_{\alpha\beta} = \frac{\partial}{\partial x_\alpha}A_\beta - \frac{\partial}{\partial x_\beta}A_\alpha \) is the corresponding curvature, \( \phi = \phi_1 + i\phi_2 \) is a complex scalar field called the Higgs field, \( D_j = \frac{\partial}{\partial x_j} - iA_j, j = 0, 1, 2 \) is the gauge covariant derivative associated with \( A, \alpha, \beta, \mu, \nu = 0, 1, 2, \varepsilon > 0 \) is a constant referred to as the Chern-Simons coupling parameter, \( \epsilon^{\alpha\beta\gamma} \) is the Levi-Civita totally skew-symmetric tensor with \( \epsilon^{012} = 1 \), \( V \) is the Higgs potential function. The corresponding Bogomol’nyi equations for unknowns \( \phi, A \) defined on \( \mathbb{R}^2 \) are

\[
\begin{align*}
D_1\phi & = iD_2\phi, \\
(1 - |\phi|^2)F_{12} & = i(D_1\phi D_2\phi - D_1\phi D_2\phi) + \frac{1}{2\varepsilon}(|\phi|^2(1 - |\phi|^2)^2).
\end{align*}
\]
In view of Jaffe-Taubes’ argument in [14], we introduce unknown \( v \) defined by
\[
\phi(z) = \exp \left( \frac{v(x)}{2} + i \sum_{j=1}^{N} \arg(z - p_j) \right), \quad z = x_1 + ix_2 \in \mathbb{C},
\]
where \( \{p_j\}_{j=1}^{N} \) are the zeros of \( \phi(z) \), allowing their multiplicities. Then we obtain the following reduced equation:
\[
(1 - e^v) \Delta v - e^{v(x)}|\nabla v|^2 + \frac{1}{\varepsilon^2} e^{v(x)} (1 - e^{v(x)})^2 = 4\pi \sum_{j=1}^{N} \delta_{p_j}. \tag{1.1}
\]

Here \( p_j \) is called a vortex point. The equation (1.1) can be considered in \( \mathbb{R}^2 \) or a two dimensional flat torus \( \Omega \) due to the theory suggested by ’t Hooft in [20].

We fix \( \varepsilon > 0 \) for a while. In \( \mathbb{R}^2 \), a solution \( v(x) \) is called a topological solution if \( \lim_{|x| \to +\infty} v(x) = 0 \), and is called a non-topological solution if \( \lim_{|x| \to +\infty} v(x) = -\infty \). Yang in [24] found topological multi-vortex solutions of (1.1) by using the variational structure of the elliptic problem to produce an iteration scheme that yields the desired solution. After then, Chae and Imanuvilov in [4] constructed a non-topological multi-vortex solution \( v(x) \) of (1.1) satisfying \( v(x) = -(2N + 4 + \sigma) \ln |x| + O(1) \) as \( |x| \to +\infty \) for some \( \sigma > 0 \). To obtain the non-topological solution of (1.1), the authors in [4] observed that (1.1) is a perturbation of the Liouville equation and applied the arguments developed in [3]. In [3], Chae and Imanuvilov showed the existence of non-topological multi-vortex solutions of the relativistic Chern-Simons-Higgs model (see (1.6) below), using the implicit function theorem argument with Lyapunov-Schmidt reduction method.

Now we consider the equation (1.1) on flat two torus \( \Omega \), where \( \varepsilon \) goes to 0. Since
\[
(1 - e^v) \Delta v - e^{v(x)}|\nabla v|^2 = \text{div}( (1 - e^v) \nabla v),
\]
as any solution \( v(x) \) to (1.1) satisfies
\[
\int_{\Omega} \frac{1}{\varepsilon^2} e^{v(x)} (1 - e^{v(x)})^2 \, dx = 4\pi N. \tag{1.2}
\]

Moreover, from the maximum principle (see also [11, Lemma 3.1]), we note that any solution \( v(x) \) to (1.1) satisfies
\[
v(x) \leq 0 \quad \text{on} \quad \Omega. \tag{1.3}
\]

For the well known Chern-Simons-Higgs equation with \( \varepsilon \to 0 \) (see (1.6) below), the corresponding properties (1.2) and (1.3) were important to classify the solutions according to their asymptotic behavior as \( \varepsilon \to 0 \) (see [9, 8]). So it is natural that we expect that the solutions to (1.1) can also be classified according to the asymptotic behavior. Now we have the following theorem:
**Theorem 1.1.** For any given vortex configuration \( \{p_j\} \), let \( v_\varepsilon \) be a sequence of solutions of (1.1). Then, up to subsequence, one of the following holds true:

(i) \( v_\varepsilon \to 0 \) a.e. as \( \varepsilon \to 0 \). Moreover, \( v_\varepsilon \to 0 \) in \( L^p(\Omega) \) for any \( p > 1 \) (topological type);
(ii) \( v_\varepsilon \to -\infty \) a.e. as \( \varepsilon \to 0 \) (non-topological type).

Recently, Han in [11, Theorem 3.1] proved the existence of critical value of the coupling parameter \( \varepsilon_c = \varepsilon_c(p_1, ..., p_N) > 0 \) such that there is a solution to (1.1) on \( \Omega \) if and only if \( 0 < \varepsilon < \varepsilon_c \). He obtained a maximal solution \( v_{\varepsilon,M} \) to (1.1) by using a super-sub solutions method (see [11, Theorem 2.1]). Here the maximal solution means that \( v_{\varepsilon,M} \geq v_\varepsilon \) on \( \Omega \) for any solution \( v_\varepsilon \) to (1.1). In [11, Lemma 3.5], he also showed that the maximum solutions \( v_{\varepsilon,M} \) of (1.1) are a monotone family in the sense that \( v_{\varepsilon_1,M} > v_{\varepsilon_2,M} \) whenever \( 0 < \varepsilon_1 < \varepsilon_2 < \varepsilon_c \). Therefore, in view of Theorem 1.1, the maximal solution obtained in [11] is a topological solution.

At this point, one might ask the existence of non-topological solution to (1.1) on \( \Omega \). In this paper, we obtain the affirmative answer for this question by constructing a bubbling non-topological solution \( v_\varepsilon \) to (1.1) on \( \Omega \) satisfying

\[
\lim_{\varepsilon \to 0} \sup_{\Omega} v_\varepsilon = -\infty, \quad \frac{e^{v_\varepsilon}}{\int_{\Omega} e^{v_\varepsilon} \, dx} \to \frac{1}{k} \sum_{i=1}^{k} \delta_{q_i}, \quad q_i \in \Omega \setminus \bigcup_{i=1}^{2k} \{p_i\}, \quad (1.4)
\]

in the sense of measure as \( \varepsilon \to 0 \).

For the construction of bubbling solution solution \( v_\varepsilon \) to (1.1) on \( \Omega \) satisfying (1.4), we assume that \( N = 2k \in 2\mathbb{N} \). We note that since the equation (1.1) is quasi-linear, it is not easy to deal it directly. As in [11, 23, 24], we introduce a new dependent variable \( u \) defined by

\[
u = F(v) := 1 + v - e^v.
\]

We have that \( F'(v) = 1 - e^v \) and \( F''(v) = -e^v \), which implies \( F \) is strictly increasing and invertible over \((-\infty, 0)\). Let \( G \) be the inverse function of \( F \) over \((-\infty, 0]\). Then we see that \( G(u) = v = G(1 + v - e^v) \). Let \( u_\varepsilon = F(v_\varepsilon) = 1 + v_\varepsilon - e^{v_\varepsilon} \). Then \( v_\varepsilon \) satisfies (1.1) if and only if \( u_\varepsilon \) satisfies

\[
\Delta u_\varepsilon + \frac{1}{\varepsilon^2} e^{G(u_\varepsilon(x))} (1 - e^{G(u_\varepsilon(x))})^2 = 4\pi \sum_{i=1}^{2k} \delta_{p_i} \quad (1.5)
\]
We remark that if \( \lim_{\varepsilon \to 0} \sup_{\Omega} u_\varepsilon = -\infty \), then the equation (1.5) would be a perturbation of bubbling solutions \( W_\varepsilon \) of the following Chern-Simons-Higgs equation:

\[
\Delta W_\varepsilon + \frac{1}{\varepsilon^2} e^{W_\varepsilon(x)} \left( 1 - e^{W_\varepsilon(x)} \right) = 4\pi \sum_{i=1}^{2k} \delta_{p_i} \text{ on } \Omega.
\] (1.6)

The relativistic Chern-Simons-Higgs model has been proposed in [12] and independently in [13] to describe vortices in high temperature superconductivity. The above equation was derived from the Euler-Lagrange equations of the CSH model via a vortex ansatz, see [12, 13, 22, 25]. The equation (1.6) has been extensively studied not only in a flat torus \( \Omega \) but also in the whole \( \mathbb{R}^2 \). We refer the readers to [2, 3, 5, 6, 7, 8, 15, 16, 17, 18, 19, 21, 22] and references therein. Among them, in a recent paper [16], Lin and Yan succeeded to construct bubbling non-topological solutions to (1.6) on \( \Omega \). Compared to (1.6), our equation has a difficulty caused by the nonlinear terms including implicit function \( G \).

Therefore, to choose a suitable approximate solution, we should investigate the behavior of the function \( G \) near \(-\infty\) and carry out the analysis carefully. To state our result exactly, we introduce the following notations:

Let \( G \) be the Green function satisfying

\[-\Delta_x G(x, y) = \delta_y - \frac{1}{|\Omega|} \quad \text{for } x, y \in \Omega, \quad \text{and} \quad \int_{\Omega} G(x, y) dx = 0.\]

We let \( \gamma(x, y) = G(x, y) + \frac{1}{2\pi} \ln |x - y| \) be the regular part of the Green function \( G(x, y) \), and

\[u_0(x) \equiv -4\pi \sum_{i=1}^{N} G(x, p_i).\]

Then \( u_0 \) satisfies the following problem:

\[
\begin{cases}
\Delta u_0 = -\frac{4\pi N}{|\Omega|} + 4\pi \sum_{i=1}^{N} \delta_{p_i}, \\
\int_{\Omega} u_0 dx = 0.
\end{cases}
\]

We remind that \( N = 2k \). We denote \( \Omega^{(k)} := \{ (x_1, \ldots, x_k) \mid x_i \in \Omega \setminus \bigcup_{i=1}^{2k} \{p_i\} \} \) for \( 1 \leq i \leq k, x_i \neq x_j \text{ if } i \neq j \). Let \( \mathbf{q} = (q_1, \ldots, q_k) \in \Omega^{(k)} \) be the critical point of the following function:

\[G^*(\mathbf{q}) := \sum_{i=1}^{k} u_0(q_i) + 8\pi \sum_{i \neq j} G(q_i, q_j).\]

We define

\[D(\mathbf{q}) := \lim_{r \to 0} \left( \sum_{i=1}^{k} \rho_i \left( \int_{\Omega \setminus B_r(q_i)} \frac{e^{f_{q_i} - 1}}{|y - q_i|^4} dy - \int_{\mathbb{R}^2 \setminus \Omega_i} \frac{1}{|y - q_i|^4} dy \right) \right),\]
where $\Omega_i$ is any open set satisfying $\Omega_i \cap \Omega_j = \emptyset$ if $i \neq j$, $\cup_{i=1}^{k} \Omega_i = \bar{\Omega}$, $B_{\delta_i}(x_i) \subset \subset \Omega_i$, $i = 1, \ldots, k$,

$$f_{q_i}(y) := 8\pi \left( \gamma(y, q_i) - \gamma(q_i, q_i) + \sum_{j \neq i} (G(y, q_j) - G(q_i, q_j)) \right) + u_0(y) - u_0(q_i),$$

and

$$\rho_i = e^{8\pi \left( \gamma(q_i, q_i) + \sum_{j \neq i} G(q_i, q_j) \right) + u_0(q_i)}.$$

At this point, we introduce our main result.

**Theorem 1.2.** Let $q = (q_1, \ldots, q_k) \in \Omega^{(k)}$ be a non-degenerate critical point of $G^*(q)$. Suppose that $D(q) < 0$. Then for $\varepsilon > 0$ small, there exists a non-topological solution $v_\varepsilon$ to (1.1) such that

$$\lim_{\varepsilon \to 0} \sup_{\Omega} v_\varepsilon = -\infty, \quad \frac{e^{v_\varepsilon}}{\int_{\Omega} e^{v_\varepsilon} dx} \to \frac{1}{k} \sum_{i=1}^{k} \delta_{q_i} \text{ in the sense of measure as } \varepsilon \to 0.$$

To the best of our knowledge, Theorem 1.2 is the first result for the existence of non-topological solution solutions to (1.1) on $\Omega$. We remark that in our paper, a limiting equation for (1.1) is Liouville equation since $\lim_{\varepsilon \to 0} \sup_{\Omega} v_\varepsilon = -\infty$. It would be an interesting problem to find other types of non-topological solution solution to (3.1), for example, satisfying $\sup_{\Omega} v_\varepsilon \geq -c_0 > -\infty$ for some constant $c_0 > 0$.

The organization of this paper is as follows. In Section 2, we prove Theorem 1.1. In Section 3, to prove Theorem 1.2 we present some preliminaries results and discuss about the invertibility of a linearized operator. Moreover, we find a suitable approximate solution and complete the proof of Theorem 1.2.

## 2. PROOF OF THEOREM 1.1

**Proof of Theorem 1.1.**

Our arguments will be based on [9, Theorem 3.1]. We consider the equation (1.5), which is equivalent to (1.1). Let $\{u_\varepsilon\}$ be a sequence of solutions of (1.5). Let $d_\varepsilon = \frac{1}{|\Omega|} \int_{\Omega} u_\varepsilon dx$ and $u_\varepsilon = w_\varepsilon + u_0 + d_\varepsilon$. Then $w_\varepsilon$ satisfies

$$\Delta w_\varepsilon + \frac{1}{\varepsilon^2} e^{G(u_\varepsilon)} (1 - e^{G(u_\varepsilon)})^2 = \frac{4\pi N}{|\Omega|} \text{ on } \Omega,$$

and $\int_{\Omega} w_\varepsilon dx = 0.$
We claim that there exist $C_q > 0$ such that $\|\nabla w_\varepsilon\|_{L^q(\Omega)} \leq C_q$ for any $q \in (1,2)$. Let $q' = \frac{q}{q-1} > 2$. Then

$$\|\nabla w_\varepsilon\|_{L^q(\Omega)} \leq \sup \left\{ \left| \int_\Omega \nabla w_\varepsilon \nabla \phi dx \right| \mid \phi \in W^{1,q'}(\Omega), \int_\Omega \phi dx = 0, \|\phi\|_{W^{1,q'}(\Omega)} = 1 \right\}. \tag{2.2}$$

By lemma 7.16 in [10], if $\int_\Omega \phi dx = 0$, then there exist $c, C > 0$ such that

$$|\phi(x)| \leq c \int_\Omega \frac{|\nabla \phi|}{|x-y|} dy \leq C \|\nabla \phi\|_{L^{q'}(\Omega)} \quad \text{for } x \in \Omega. \tag{2.3}$$

Thus in view of (2.1), (2.3), and (1.2), we see that there exists constant $C > 0$, independent of $\phi$ satisfying $\int_\Omega \phi dx = 0$ and $\|\phi\|_{W^{1,q'}(\Omega)} = 1$, such that

$$\left| \int_\Omega \nabla w_\varepsilon \nabla \phi dx \right| = \left| \int_\Omega \Delta w_\varepsilon \phi dx \right| \leq \|\phi\|_{L^\infty(\Omega)} \left| \int_\Omega \frac{1}{\varepsilon^2} e^{G(u_\varepsilon)} (1 - e^{G(u_\varepsilon)})^2 dx + 4\pi N \right| \leq C. \tag{2.4}$$

Now using (2.2), we complete the proof of our claim.

In view of Poincaré inequality, we also have $\|w_\varepsilon\|_{L^q(\Omega)} \leq c \|\nabla w_\varepsilon\|_{L^q(\Omega)}$. Then there exist $w \in W^{1,q}(\Omega)$ and $p > 1$ such that, as $\varepsilon \to 0$,

$$w_\varepsilon \rightharpoonup w \quad \text{weakly in } W^{1,q}(\Omega), \quad w_\varepsilon \to w \quad \text{strongly in } L^p(\Omega), \quad w_\varepsilon \to w \quad \text{a.e.} \tag{2.5}$$

Since $v_\varepsilon \leq 0$ on $\Omega$, we see that $u_\varepsilon \leq 0$ and $0 \leq e^{d_\varepsilon} \leq 1$. Then there exists $A \geq 0$ such that $\limsup_{\varepsilon \to 0} e^{d_\varepsilon} = A$. If $A \equiv 0$, that is, $\lim_{\varepsilon \to 0} d_\varepsilon = -\infty$, then by using (2.5), we get that $u_\varepsilon = w_\varepsilon + d_\varepsilon + u_0 \to -\infty$ a.e. in $\Omega$.

If $A > 0$, then by using Fatou’s lemma and (2.5), we see that

$$4\pi N \varepsilon^2 = \int_\Omega e^{G(u_\varepsilon)} (1 - e^{G(u_\varepsilon)})^2 dx \geq \int_\Omega e^{G(w + u_0 + \ln A)} (1 - e^{G(w + u_0 + \ln A)})^2 dx,$$

which implies that $G(w + u_0 + \ln A) = -\infty$ or $G(w + u_0 + \ln A) = 0$ a.e. in $\Omega$. Since $G$ is strictly increasing on $(-\infty,0)$ and $G(0) = 0$, we see that $w + u_0 + \ln A = -\infty$ or $w + u_0 + \ln A = 0$ a.e. in $\Omega$. By $A > 0$ and $w, u_0 \in L^p(\Omega)$, we have $w + u_0 + \ln A = 0$ a.e. in $\Omega$. From $\int_\Omega w + u_0 dx = 0$, we see that $A \equiv 1$, and $w + u_0 = 0$ a.e. in $\Omega$. By using (2.5), we get that $u_\varepsilon = w_\varepsilon + d_\varepsilon + u_0 \to w + \ln A + u_0 = 0$ a.e. in $\Omega$ and $u_\varepsilon \to 0$ in $L^p(\Omega)$ for any $p > 1$ (since $q \in (1,2)$ in (2.2) can be arbitrary). Now we complete the proof of Theorem 1.1. □
3. Existence of bubbling non-topological solution solution

In this section, we want to construct a bubbling non-topological solution solution $v_\epsilon$ to (1.1) satisfying $\lim_{\epsilon \to 0} \sup_\Omega v_\epsilon = -\infty$, and

$$\frac{e^{v_\epsilon}}{\int_\Omega e^{v_\epsilon} \, dx} \to \frac{1}{k} \sum_{i=1}^k \delta_{\rho_i} \quad \text{in the sense of measure as } \epsilon \to 0,$$

where $q = (q_1, \ldots, q_k) \in \Omega^{(k)}$ is a non-degenerate critical point of $G^*(q)$ and $D(q) < 0$. Without loss of generality, from now on, we assume that $|\Omega| = 1$.

We note that $v_\epsilon$ satisfies (1.1) if and only if $u_\epsilon = F(v_\epsilon) = 1 + v_\epsilon - e^{v_\epsilon}$ satisfies

$$\Delta u_\epsilon + \frac{1}{\epsilon^2} e^{G(u_\epsilon)} (1 - e^{G(u_\epsilon)})^2 = 4\pi \sum_{i=1}^{2k} \delta_{\rho_i} \quad \text{(3.1)}$$

As we mentioned in the introduction, if $\lim_{\epsilon \to 0} \sup_\Omega u_\epsilon = -\infty$, then $u_\epsilon$ would be related to the following Chern-Simons-Higgs equation:

$$\Delta W_\epsilon + \frac{1}{\epsilon^2} e^{W_\epsilon} (1 - e^{W_\epsilon}) = 4\pi \sum_{j=1}^{2k} \delta_{\rho_j}.$$

In [16], bubbling solutions for the above Chern-Simons-Higgs equation have been constructed as following:

$$W_\epsilon(y) \simeq u_0(y) + w^*_{x,\mu}(y) - \int_\Omega w^*_{x,\mu}(z) \, dz + c(w_{x,\mu}),$$

where $x = (x_1, \ldots, x_k)$, $x_i \in \Omega$, $\mu \in [\frac{\beta_0}{\sqrt{\epsilon}}, \frac{\beta_1}{\sqrt{\epsilon}}]$ for some $0 < \beta_0 \ll 1$, $\beta_1 \gg 1$,

$$\rho_i := e^{8\pi \gamma(z_i, x_i) + 8\pi \sum_{j \neq i} G(z_i, x_i) + u_0(x_i)},$$

$$(\mu_1, \ldots, \mu_k) := (\mu, \sqrt{\frac{\rho_1}{\rho_2}}, \ldots, \sqrt{\frac{\rho_1}{\rho_k}} \mu),$$

and $d > 0$ is a fixed small constant, $d_i^2 := d - 1/\mu_i^2$, $u_{x_i,\mu_i}(y) := \ln \frac{8\mu_i^2}{(1 + \mu_i^2 |y-x_i|^2)^2}$,

$$w^*_{x,\mu}(y) := \sum_{i=1}^k w_{x_i,\mu_i}(y),$$

$$w^*_{x_i,\mu_i}(y) := \begin{cases} u_{x_i,\mu_i}(y) + 8\pi \gamma(y, x_i) \left(1 - \frac{1}{d_i^2}\right), & y \in B_{d_i}(x_i), \\ u_{0,\mu_i}(d_i) + 8\pi \left(G(y, x_i) - \frac{1}{2\pi} \ln \frac{1}{d_i} \right) \left(1 - \frac{1}{d_i^2}\right), & y \in \Omega \setminus B_{d_i}(x_i), \end{cases}$$

$$w_{x,\mu}(y) := u_{x,\mu}(y) - \int_\Omega w^*_{x,\mu}(z) \, dz,$$

$$c(w_{x,\mu}) := \ln \frac{16k\pi^2}{\int_\Omega e^{\alpha_0 + w_{x,\mu}} \, dy \left(1 + \sqrt{1 - 32k\pi^2 \int_\Omega e^{\alpha_0 + w_{x,\mu}} \, dy} \right)^2}.$$
We note that \( u_{x_i,\mu} \) satisfies
\[
\begin{aligned}
    -\Delta u_{x_i,\mu} (y) & = e^{u_{x_i,\mu}(y)} \quad \text{in } \mathbb{R}^2, \\
\int_{\mathbb{R}^2} e^{u_{x_i,\mu}(y)} dy & = 8\pi.
\end{aligned}
\]
We denote
\[
\tilde{W}_{x,\mu}(y) := w^*_{x,\mu}(y) - \int_{\Omega} w^*_{x,\mu}(z) dz + c(w_{x,\mu}).
\]
We want to find solution \( u_\varepsilon \) to (3.1) in the following form:
\[
u
\begin{aligned}
u
u
u
u
u
\end{aligned}
\]
where \( \eta_{x,\mu} \) is a perturbation term. To find \( \eta_{x,\mu} \) which makes that \( u_\varepsilon \) in the form (3.2) is a solution to (3.1), we consider the following linearized operator
\[
L_{x,\mu}(\eta_{x,\mu}) := (\Delta + h_\mu(y)) \eta \quad \text{with } h_\mu(y) := \sum_{i=1}^{k} 1_{B_{\delta_i}(x_i)} e^{u_{x_i,\mu}(y)}.
\]
We see that \( u_\varepsilon \) is a solution to (3.1) if \( \eta_{x,\mu} \) satisfies
\[
L_{x,\mu} \eta_{x,\mu} = g_{x,\mu}(\eta_{x,\mu}),
\]
where
\[
g_{x,\mu}(\eta_{x,\mu}) := h_\mu(y) \eta_{x,\mu} + 8k\pi - \Delta \tilde{W}_{x,\mu} - \frac{1}{\varepsilon^2} e^{G(1+u_0 + \tilde{W}_{x,\mu} + \eta_{x,\mu})}(1 - e^{G(1+u_0 + \tilde{W}_{x,\mu} + \eta_{x,\mu})})^2.
\]
To show the invertibility of the linear operator \( L_{x,\mu} \), we need to introduce suitable function spaces. For fixed a small constant \( \alpha \in (0,\frac{1}{2}) \), we define
\[
\rho(y) = (1 + |y|)^{1+\frac{\alpha}{2}} , \quad \tilde{\rho}(y) = \frac{1}{(1 + |y|) (\ln(2 + |y|))^{1+\frac{\alpha}{2}}}.
\]
Let \( \Omega' := \bigcup_{i=1}^{k} B_{\delta_i}(x_i) \) and \( \tilde{\xi}_i(y) := \xi(x_i + \mu^{-1}y) \). We say that \( \xi \in X_{\alpha,x,\mu} \) if
\[
\|
\begin{aligned}
\end{aligned}
\]
and \( \xi \in Y_{\alpha,x,\mu} \) if
\[
\|
\begin{aligned}
\end{aligned}
\]
Let \( \chi_i(|y|) \) be a smooth function satisfying \( \chi_i = 1 \) in \( B_i(0) \), \( \chi_i = 0 \) in \( \mathbb{R}^2 \setminus B_{2i}(0) \), and \( 0 \leq \chi_i \leq 1 \). We use the following notations

\[
Y_{x, \mu, 0} := -\frac{1}{\mu_1} + \sum_{i=1}^k \sqrt{\rho_i \mu_i \left( 1 + \mu^2_i |y - x_i|^2 \right)} \chi_i(y - x_i),
\]

\[
Y_{x, \mu, i, j} := \chi_i(y - x_i) \frac{\mu^2_i (y_j - x_{ij})}{1 + \mu^2_i |y - x_i|^2}, \quad i = 1, \ldots, k, \ j = 1, 2,
\]

where \( x_i = (x_{i1}, x_{i2}) \) and \( y = (y_1, y_2) \). The estimations for \( Y_{x, \mu, 0}, Y_{x, \mu, i, j} \) has been known:

**Lemma 3.1.** \([16]\)**

\[
L_{x, \mu} Y_{x, \mu, 0} = O(\mu^{-3}), \quad L_{x, \mu} Y_{x, \mu, i, j} = O(1), \quad i = 1, \ldots, k, \ j = 1, 2.
\]

**Proof.** See the estimation (3.8) in \([16]\). \(\square\)

From Lemma 3.1, we see that \( Y_{x, \mu, 0}, Y_{x, \mu, i, j} \) are the approximate kernels for \( L_{x, \mu} \).

Let

\[
Z_{x, \mu, 0} = -\Delta Y_{x, \mu, 0} + h_\mu(y) Y_{x, \mu, 0},
\]

and

\[
Z_{x, \mu, i, j} = -\Delta Y_{x, \mu, i, j} + h_\mu(y) Y_{x, \mu, i, j}, \quad i = 1, \ldots, k, \ j = 1, 2.
\]

We define two subspace of \( X_{\alpha, x, \mu}, Y_{\alpha, x, \mu} \) as

\[
E_{x, \mu} := \{ \xi \in X_{\alpha, x, \mu} \mid \int_\Omega Z_{x, \mu, 0} \xi dx = \int_\Omega Z_{x, \mu, i, j} \xi dx = 0, \ i = 1, \ldots, k, \ j = 1, 2 \},
\]

\[
F_{x, \mu} := \{ \xi \in Y_{\alpha, x, \mu} \mid \int_\Omega Y_{x, \mu, 0} \xi dx = \int_\Omega Y_{x, \mu, i, j} \xi dx = 0, \ i = 1, \ldots, k, \ j = 1, 2 \}.
\]

and projection operator \( Q_{x, \mu} : Y_{\alpha, x, \mu} \to F_{x, \mu} \) by

\[
Q_{x, \mu} \xi = \xi - c_0 Z_{x, \mu, 0} - \sum_{j=1}^2 \sum_{i=1}^k c_{ij} Z_{x, \mu, i, j},
\]

where \( c_0, c_{ij} \) are chosen so that \( Q_{x, \mu} \xi \in F_{x, \mu} \). For the projection operator \( Q_{x, \mu} \), we have the following result.

**Lemma 3.2.** \([16] \text{Lemma 3.1} \)** There is a constant \( C > 0 \), independent of \( x \) and \( \mu \), such that

\[
\| Q_{x, \mu} u \|_{Y_{\alpha, x, \mu}} \leq C \| u \|_{Y_{\alpha, x, \mu}}.
\]

The following lemma will be useful for our arguments.
Lemma 3.3. $\frac{1}{\epsilon^2} e^{\tilde{W}_{\mathbf{x}, \mu}} = O(\sum_{i=1}^{k} e^{u_{\mathbf{x}, \mu_i}} 1_{B_{d_i}(x_i)} + O(\epsilon)(1 - \sum_{i=1}^{k} 1_{B_{d_i}(x_i)}))$.

Proof. On $B_{d_i}(x_i)$, we see that

$$\frac{1}{\epsilon^2} e^{\tilde{W}_{\mathbf{x}, \mu_i}} = \frac{1}{\epsilon^2} e^{w^*_{\mathbf{x}, \mu_i}(y) + \sum_{j \neq i} u_{0, \mu_i}(d_j) + \Gamma_i - \int_{\Omega} w^*_{\mathbf{x}, \mu}(z) dz + c(w_{\mathbf{x}, \mu})},$$

where $\Gamma_i := 8\pi \left[ \gamma(y, x_i) \left(1 - \frac{1}{d\mu_i} \right) + \sum_{j \neq i} (G_{\mathbf{x}, \mu}(y, x_j) + \frac{1}{2\pi} \ln d_j) \left(1 - \frac{1}{d\mu_j} \right) \right]$. In the proof of [16 Proposition 2.1], the following estimations were obtained (see the estimations (2.13) and (2.22) in [16]):

$$- \int_{\Omega} w^*_{\mathbf{x}, \mu_i}(y) dy = 2 \ln \mu_i + O(1), \quad c(w_{\mathbf{x}, \mu}) = -6 \ln \mu + O(1).$$

Moreover, $u_{0, \mu_i}(d_j) = O(\ln \frac{1}{\mu_i})$ on $\Omega \setminus [B_{d_j}(x_j)]$ and $\mu_i = O(\frac{1}{\sqrt{\epsilon}})$ for all $i = 1, \ldots, k$. Thus we get that

$$\frac{1}{\epsilon^2} e^{\tilde{W}_{\mathbf{x}, \mu}} = \frac{1}{\epsilon^2} e^{u_{\mathbf{x}, \mu_i}(y) O(\mu_i^{-2(k-1)+2k-6}) = O(e^{u_{\mathbf{x}, \mu_i}(y)}) \text{ on } B_{d_i}(x_i).$$

Similarly we get that

$$\frac{1}{\epsilon^2} e^{W_{\mathbf{x}, \mu}} = O(\epsilon) \text{ on } \Omega \setminus [\cup_i B_{d_i}(x_i)].$$

\[\square\]

The following invertibility result for the operator $Q_{\mathbf{x}, \mu} L_{\mathbf{x}, \mu}$ obtained in [16] is essential for our arguments:

Theorem 3.4. [16 Theorem A.2] The operator $Q_{\mathbf{x}, \mu} L_{\mathbf{x}, \mu}$ is an isomorphism from $E_{\mathbf{x}, \mu}$ to $F_{\mathbf{x}, \mu}$. Moreover, if $w \in E_{\mathbf{x}, \mu}$ and $h \in F_{\mathbf{x}, \mu}$ satisfy

$$Q_{\mathbf{x}, \mu} L_{\mathbf{x}, \mu} w = h,$$

then there is a constant $C > 0$, independent of $\mathbf{x}$ and $\mu$, such that

$$\|w\|_{L^\infty(\mathbb{R}^2)} + \|w\|_{\bar{X}_{\alpha, \mathbf{x}, \mu}} \leq C \ln \mu \|h\|_{\bar{Y}_{\alpha, \mathbf{x}, \mu}}.$$

We define $\tilde{g}_{\mathbf{x}, \mu}(\eta)$ as

$$\tilde{g}_{\mathbf{x}, \mu}(\eta) := h_{\mu}(y) \eta - \frac{1}{\epsilon^2} e^{u_{0} + \tilde{W}_{\mathbf{x}, \mu} + \eta} \left(1 - e^{u_{0} + \tilde{W}_{\mathbf{x}, \mu} + \eta} \right) - \Delta \tilde{W}_{\mathbf{x}, \mu} + 8k\pi.$$

The function $\tilde{g}_{\mathbf{x}, \mu}(\eta)$ was introduced in [16], and the following estimations were obtained:
Lemma 3.5. \cite[Proposition 3.2]{16} There is an \(\varepsilon_0 > 0\), such that for each \(\varepsilon \in (0, \varepsilon_0]\), \(x\) which is closed to \(q\) with \(|DG^* (x)| \leq \frac{C}{\mu}\), and \(\mu \in \left[\frac{\varepsilon_0}{\sqrt{\varepsilon}}, \frac{1}{\sqrt{\varepsilon}}\right]\), if \(\eta, \eta' \in E_{x, \mu}\) satisfies \(\|\tilde{\eta}\|_{L^\infty (\Omega)} + \|\tilde{\eta}\|_{X_{\alpha, x, \mu}} \leq \frac{1}{\mu}\) where \(\tilde{\eta} = \eta, \eta'\), then we have

\[
\|\tilde{g}_{x,\mu} (\eta)\|_{Y_{\alpha, x, \mu}} \leq \frac{C}{\mu^{\frac{3}{2} - \frac{\alpha}{2}}},
\]

and

\[
\|\tilde{g}_{x,\mu} (\eta) - \tilde{g}_{x,\mu} (\eta')\|_{Y_{\alpha, x, \mu}} \leq \frac{C}{\mu} \|\eta - \eta'\|_{L^\infty (\Omega)},
\]

where \(C > 0\) is a constant, independent of \(x, \mu, \eta, \eta'\).

Proof. See \cite[(3.16)]{16} for the estimation (3.4) and \cite[(3.21),(3.22)]{16} for the estimation (3.5). \(\square\)

We remind that \(q\) is a non-degenerate critical point of \(G^* (q)\) with \(D (q) < 0\). Now we have the following proposition.

Proposition 3.6. There is an \(\varepsilon_0 > 0\), such that for each \(\varepsilon \in (0, \varepsilon_0]\), \(x\) which is closed to \(q\) with \(|DG^* (x)| \leq \frac{C}{\mu}\), and \(\mu \in \left[\frac{\varepsilon_0}{\sqrt{\varepsilon}}, \frac{1}{\sqrt{\varepsilon}}\right]\), there exists \(\eta_{x, \mu} \in E_{x, \mu}\) satisfying

\[
Q_{x, \mu} (L_{x, \mu} (\eta_{x, \mu})) = Q (g_{x, \mu} (\eta_{x, \mu}))
\]

Moreover,

\[
\|\eta_{x, \mu}\|_{L^\infty} + \|\eta_{x, \mu}\|_{X_{\alpha, x, \mu}} \leq \frac{C \ln \mu}{\mu^{\frac{3}{2} - \frac{\alpha}{2}}},
\]

where \(C > 0\) is independent of \(\varepsilon > 0\). Here \(\alpha \in (0, \frac{1}{2})\) is the same constant as in \(X_{\alpha, x, \mu}\) and \(Y_{\alpha, x, \mu}\).

Proof. Define

\[
S_{x, \mu} := \left\{ \eta \in E_{x, \mu} \mid \|\eta\|_{L^\infty (\Omega)} + \|\eta\|_{X_{\alpha, x, \mu}} \leq \frac{1}{\mu} \right\}.
\]

We denote \(\|\eta\|_{S_{x, \mu}} := \|\eta\|_{L^\infty (\Omega)} + \|\eta\|_{X_{\alpha, x, \mu}}\) as the norm in \(S_{x, \mu}\). We consider the following mapping

\[
B_{x, \mu} : \eta \mapsto (Q_{x, \mu} L_{x, \mu})^{-1} [Q_{x, \mu} g_{x, \mu} (\eta_{x, \mu})].
\]

Step 1. First, we claim that \(B_{x, \mu}\) maps \(S_{x, \mu}\) to \(S_{x, \mu}\).

In view of Theorem 3.3 and Lemma 3.2 we have for some constant \(C > 0\), independent of \(\varepsilon > 0\),

\[
\|B_{x, \mu} (\eta)\|_{S_{x, \mu}} \leq C \ln \mu \|g_{x, \mu} (\eta)\|_{Y_{\alpha, x, \mu}}.
\]
From the definition of $\tilde{g}_{x,\mu}(\eta)$ and $G(1 + s - \varepsilon^s) = s$, we see that

$$g_{x,\mu}(\eta) - \tilde{g}_{x,\mu}(\eta)$$

$$= \frac{1}{\varepsilon^2} e^{u_0 + W_{x,\mu} + \eta} (1 - e^{u_0 + W_{x,\mu} + \eta}) - \frac{1}{\varepsilon^2} e^{G(1+u_0 + \bar{W}_{x,\mu} + \eta)} (1 - e^{G(1+u_0 + \bar{W}_{x,\mu} + \eta)})^2$$

$$= \frac{1}{\varepsilon^2} e^{u_0 + W_{x,\mu} + \eta} (1 - e^{u_0 + W_{x,\mu} + \eta}) - \frac{1}{\varepsilon^2} e^{u_0 + W_{x,\mu} + \eta} (1 - e^{u_0 + W_{x,\mu} + \eta})^2$$

$$+ \frac{1}{\varepsilon^2} e^{G(1+u_0 + \bar{W}_{x,\mu} + \eta - e^{u_0 + \bar{W}_{x,\mu} + \eta})(1 - e^{G(1+u_0 + \bar{W}_{x,\mu} + \eta - e^{u_0 + \bar{W}_{x,\mu} + \eta})})^2}$$

$$- \frac{1}{\varepsilon^2} e^{G(1+u_0 + \bar{W}_{x,\mu} + \eta - e^{u_0 + \bar{W}_{x,\mu} + \eta})(1 - e^{G(1+u_0 + \bar{W}_{x,\mu} + \eta - e^{u_0 + \bar{W}_{x,\mu} + \eta})})^2}.$$
and thus

\[ g_{x,\mu}(\eta) - \tilde{g}_{x,\mu}(\eta) = \frac{2\varepsilon^{3u_0 + 3\tilde{W}_{x,\mu} + 3\eta}}{\varepsilon^2} \]

\[ + \left\{ e^{G(1 + u_0 + \tilde{W}_{x,\mu} + \eta - \theta e^{u_0 + \tilde{W}_{x,\mu} + \eta})} - e^{G(1 + u_0 + \tilde{W}_{x,\mu} + \eta - \theta e^{u_0 + \tilde{W}_{x,\mu} + \eta})} \right\} \]

\[ \times \left( 1 - 6e^{G(1 + u_0 + \tilde{W}_{x,\mu} + \eta - \theta e^{u_0 + \tilde{W}_{x,\mu} + \eta})}(\theta - 1) \varepsilon^{3u_0 + 2\tilde{W}_{x,\mu} + 2\eta} \right) \]

\[ + \left( 1 - 6e^{G(1 + u_0 + \tilde{W}_{x,\mu} + \eta - \theta e^{u_0 + \tilde{W}_{x,\mu} + \eta})}(\theta - 1) \varepsilon^{3u_0 + 2\tilde{W}_{x,\mu} + 2\eta} \right) \]

\[ = \frac{2\varepsilon^{3u_0 + 3\tilde{W}_{x,\mu} + 3\eta}}{\varepsilon^2} \]

\[ + \left( 1 - 6e^{G(1 + u_0 + \tilde{W}_{x,\mu} + \eta - \theta e^{u_0 + \tilde{W}_{x,\mu} + \eta})}(\theta - 1) \varepsilon^{3u_0 + 3\tilde{W}_{x,\mu} + 3\eta} \right) \]

By Lemma 3.3 and \( G(-\infty) = -\infty \), we see that as \( \varepsilon \to 0 \), \( e^{\tilde{W}_{x,\mu}} = O(\varepsilon) \) and \( u_0 + \tilde{W}_{x,\mu} + \eta \to -\infty \) uniformly on \( \Omega \), which implies that

\[ g_{x,\mu}(\eta) - \tilde{g}_{x,\mu}(\eta) = O\left(\varepsilon^3\right) \text{ on } \Omega \setminus \{(\cup_{k=1}^k B_d(x_i))\} \]  

(3.7)

In view of Lemma 3.3, we see that \( e^{\tilde{W}_{x,\mu}} = O(\varepsilon^3) \) on \( \Omega \setminus \{(\cup_{k=1}^k B_d(x_i))\} \), and thus

\[ g_{x,\mu}(\eta) - \tilde{g}_{x,\mu}(\eta) = O\left(\varepsilon^7\right) \text{ on } \Omega \setminus \{(\cup_{k=1}^k B_d(x_i))\} \]  

(3.8)

On \( B_d(x_i) \), from Lemma 3.3, we see that \( e^{\tilde{W}_{x,\mu}} = O\left(\frac{\varepsilon}{(1 + \mu_i^2 \|y - x_i^2\|^2)^\alpha}\right) = O\left(\frac{\varepsilon}{(1 + \mu_i^2 \|y - x_i^2\|^2)^\alpha}\right) \) and

\[ g_{x,\mu}(\eta) - \tilde{g}_{x,\mu}(\eta) = O\left(\frac{\varepsilon}{(1 + \mu_i^2 \|y - x_i^2\|^2)^\alpha}\right), \]

which implies that

\[ \frac{1}{\mu_i^2} \left\{ g_{x,\mu}(\eta) - \tilde{g}_{x,\mu}(\eta) \right\}(x_i + \mu_i^{-1} y)(1 + |y|)^{1 + \frac{2}{\alpha}} \right\} L^2(B_{d,\mu_i}(0)) \]

\[ = \left\| O(\varepsilon^2)(1 + |y|)^{1 + \frac{2}{\alpha}} \right\|_{L^2(B_{d,\mu_i}(0))} = O(\varepsilon^2). \]  

(3.9)

From the above arguments (3.8), (3.9), we have

\[ \| g_{x,\mu}(\eta) - \tilde{g}_{x,\mu}(\eta) \|_{\mathcal{V}_{\eta,\gamma,\mu}} \leq \frac{C}{\mu_i^2}. \]  

(3.10)
Combining Lemma 3.5 and the estimation (3.10) together, we obtain

\[ \| g_{x,\mu}(\eta) \|_{Y_{\alpha,x,\mu}} \leq \frac{C}{\mu^{2-\frac{\alpha}{2}}}. \]  

(3.11)

By (3.11), we see that for large \( \mu > 0 \) (i.e. for small \( \varepsilon > 0 \)), \( B_{x,\mu} \) maps \( S_{x,\mu} \) to \( S_{x,\mu} \).

Step 2. Now we claim that \( B_{x,\mu} \) is a contraction map.

In view of Theorem 3.4 and Lemma 3.2, there is a constant \( C > 0 \), independent of \( \varepsilon > 0 \), satisfying for any \( \eta, \eta' \in S_{x,\mu} \),

\[ \| B_{x,\mu}(\eta) - B_{x,\mu}(\eta') \|_{S_{x,\mu}} \leq C \ln \mu \| g_{x,\mu}(\eta) - g_{x,\mu}(\eta') \|_{Y_{\alpha,x,\mu}}. \]  

(3.12)

To estimate \( \| g_{x,\mu}(\eta) - g_{x,\mu}(\eta') \|_{Y_{\alpha,x,\mu}} \), we observe that

\[ \| g_{x,\mu}(\eta) - g_{x,\mu}(\eta') \|_{Y_{\alpha,x,\mu}} \leq \| g_{x,\mu}(\eta) - \tilde{g}_{x,\mu}(\eta) - (g_{x,\mu}(\eta') - \tilde{g}_{x,\mu}(\eta')) \|_{Y_{\alpha,x,\mu}} \]

\[ + \| \tilde{g}_{x,\mu}(\eta) - \tilde{g}_{x,\mu}(\eta') \|_{Y_{\alpha,x,\mu}}. \]

We see that

\[
\begin{align*}
g_{x,\mu}(\eta) - \tilde{g}_{x,\mu}(\eta) - (g_{x,\mu}(\eta') - \tilde{g}_{x,\mu}(\eta')) & = \frac{1}{\varepsilon^2} e^{2u_0 + 2\tilde{W}_{x,\mu} + \eta} (1 - e^{u_0 + \tilde{W}_{x,\mu} + \eta}) - \frac{1}{\varepsilon^2} e^{2u_0 + 2\tilde{W}_{x,\mu} + 2\eta'} (1 - e^{u_0 + \tilde{W}_{x,\mu} + \eta'}) \\
& + \frac{1}{\varepsilon^2} e^{G(1 + u_0 + \tilde{W}_{x,\mu} + \eta - e^{u_0 + \tilde{W}_{x,\mu} + \eta})} (1 - e^{G(1 + u_0 + \tilde{W}_{x,\mu} + \eta - e^{u_0 + \tilde{W}_{x,\mu} + \eta})})^2 \\
& - \frac{1}{\varepsilon^2} e^{G(1 + u_0 + \tilde{W}_{x,\mu} + \eta' - e^{u_0 + \tilde{W}_{x,\mu} + \eta'})} (1 - e^{G(1 + u_0 + \tilde{W}_{x,\mu} + \eta' - e^{u_0 + \tilde{W}_{x,\mu} + \eta'})})^2 \\
& - \frac{1}{\varepsilon^2} e^{G(1 + u_0 + \tilde{W}_{x,\mu} + \eta)} (1 - e^{G(1 + u_0 + \tilde{W}_{x,\mu} + \eta)})^2 + \frac{1}{\varepsilon^2} e^{G(1 + u_0 + \tilde{W}_{x,\mu} + \eta')} (1 - e^{G(1 + u_0 + \tilde{W}_{x,\mu} + \eta')})^2.
\end{align*}
\]
Then for some numbers $\xi_0, \xi_1, \xi_2, \xi_3$ between $\eta$ and $\eta'$, and some $\theta \in (0, 1)$, $\theta' \in (\theta, 1)$, we have

$$g_{x,\mu}(\eta) - \tilde{g}_{x,\mu}(\eta) - (g_{x,\mu}(\eta') - \tilde{g}_{x,\mu}(\eta'))$$

$$= \frac{1}{\varepsilon^2} e^{2u_0+2W_{x,\mu}+2\xi_0} (2 - 3 e^{u_0+W_{x,\mu}+\xi_1}) (\eta - \eta')$$

$$+ \frac{e^{G(1+u_0+W_{x,\mu}+\xi_1)}}{\varepsilon^2} (1 - 3 e^{G(1+u_0+W_{x,\mu}+\xi_1)-e^{u_0+W_{x,\mu}+\xi_1}}) (1 - e^{u_0+W_{x,\mu}+\xi_1}) (\eta - \eta')$$

$$+ \frac{1}{\varepsilon^2} e^{G(1+u_0+W_{x,\mu}+\xi_2)} (1 - 3 e^{G(1+u_0+W_{x,\mu}+\xi_2)} (\eta - \eta')$$

$$= \frac{1}{\varepsilon^2} e^{2u_0+2W_{x,\mu}+2\xi_0} (2 - 3 e^{u_0+W_{x,\mu}+\xi_1}) (\eta - \eta')$$

$$+ \left\{ \frac{e^{G(1+u_0+W_{x,\mu}+\xi_1)-e^{u_0+W_{x,\mu}+\xi_1}}}{\varepsilon^2} (1 - 3 e^{G(1+u_0+W_{x,\mu}+\xi_1)-e^{u_0+W_{x,\mu}+\xi_1}})$$

$$- \frac{1}{\varepsilon^2} e^{2u_0+2W_{x,\mu}+2\xi_1} (1 - 3 e^{G(u_0+W_{x,\mu}+\xi_1-e^{u_0+W_{x,\mu}+\xi_1})} (\eta - \eta')$$

$$= O\left(\varepsilon^{2u_0+2W_{x,\mu}}\right) (\eta - \eta')$$

$$+ \left\{ \frac{1}{\varepsilon^2} e^{G(1+u_0+W_{x,\mu}+\xi_3-\theta e^{u_0+W_{x,\mu}+\xi_1})} (\eta - \eta') (\xi_1 - e^{u_0+W_{x,\mu}+\xi_1})$$

$$\times \left\{ \frac{e^{u_0+W_{x,\mu}+\xi_1}}{\varepsilon^2} (1 - e^{G(1+u_0+W_{x,\mu}+\xi_3-\theta e^{u_0+W_{x,\mu}+\xi_1})} + e^{u_0+W_{x,\mu}+\xi_3}$$

$$= O\left(\varepsilon^{2u_0+2W_{x,\mu}}\right) (\eta - \eta') + O\left(\varepsilon^{2u_0+2W_{x,\mu}}\right) (\eta - \eta')^2.$$
From Lemma 3.5 and the above estimations (3.13)-(3.14), we have
\[ \|g_{x,\mu}(\eta) - g_{x,\mu}(\eta')\|_{Y_{\alpha,x,\mu}} \leq (\|\eta\|_{L^\infty(\Omega)} + \|\eta'\|_{L^\infty(\Omega)} + O(\varepsilon^{1/2}))\|\eta - \eta'\|_{L^\infty(\Omega)}. \] (3.15)

In view of the estimations (3.12)-(3.15), we obtain that \( B_{x,\mu} \) is a contraction map on \( S_{x,\mu} \).

Step 3. In view of Step 1, Step 2, and contraction mapping theorem, there exists a unique solution \( \eta_{x,\mu} \in S_{x,\mu} \) of (3.6). Moreover, from Theorem 3.4, Lemma 3.2, and (3.11), we obtain that
\[ \|\eta_{x,\mu}\|_{L^\infty} + \|\eta_{x,\mu}\|_{X_{\alpha,x,\mu}} \leq C \ln \frac{\mu}{\mu^2 - \alpha^2}, \]
where \( C > 0 \) is independent of \( \varepsilon > 0 \). Now we complete the proof of Proposition 3.6. □

By Proposition 3.6, we get that for any \( \mu \in \left[ \beta_0 \sqrt{\varepsilon}, \beta_1 \sqrt{\varepsilon} \right] \), and any \( x \) close to \( q \), where \( q \) is a non-degenerate critical point of \( G^*(q) \) with \( D(q) < 0 \), there is \( \eta_{x,\mu} \in S_{x,\mu} \) such that
\[ \Delta(\tilde{W}_{x,\mu} + \eta_{x,\mu}) = -\frac{1}{\varepsilon^2} e^{G(u_0 + \tilde{W}_{x,\mu} + \eta_{x,\mu})} (1 - e^{G(u_0 + \tilde{W}_{x,\mu} + \eta_{x,\mu})})^2 + 8k\pi + c_0 Z_{x,\mu,0} + \sum_{i=1}^{k} \sum_{j=1}^{2} c_{ij} Z_{x,\mu,j}, \] (3.16)
where \( c_0, c_{ij} \) are constants satisfying
\[ L_{x,\mu}(\eta_{x,\mu}) - g_{x,\mu}(\eta_{x,\mu}) - c_0 Z_{x,\mu,0} - \sum_{i=1}^{k} \sum_{j=1}^{2} c_{ij} Z_{x,\mu,j} \in F_{x,\mu}. \]

In the following, we will choose \( x, \mu \) suitably (depending on \( \varepsilon \)) such that the corresponding \( c_0, c_{ij} \) are zero and hence the solution \( \eta_{x,\mu} \) is exactly the solution to (3.3) which implies that \( u_\varepsilon = 1 + u_0 + \tilde{W}_{x,\mu} + \eta_{x,\mu} \) is a solution to (3.1). It is standard to prove the following lemma.

**Lemma 3.7.** If \[
\int_{\Omega} \left[ \Delta \eta_{x,\mu} + \frac{1}{\varepsilon^2} e^{G(u_0 + \tilde{W}_{x,\mu} + \eta_{x,\mu})} (1 - e^{G(u_0 + \tilde{W}_{x,\mu} + \eta_{x,\mu})})^2 + \Delta \tilde{W}_{x,\mu} - 8k\pi \right] Y_{x,\mu,0} \, dx = 0,
\]
and
\[
\int_{\Omega} \left[ \Delta \eta_{x,\mu} + \frac{1}{\varepsilon^2} e^{G(u_0 + \tilde{W}_{x,\mu} + \eta_{x,\mu})} (1 - e^{G(u_0 + \tilde{W}_{x,\mu} + \eta_{x,\mu})})^2 + \Delta \tilde{W}_{x,\mu} - 8k\pi \right] Y_{x,\mu,i,j} \, dx = 0,
\]
then \( c_0 = c_{ij} = 0 \) for \( i = 1, \ldots, k \) and \( j = 1, 2 \).

Let \( x_i = (x_{i1}, x_{i2}) \). By using the proof of [16, Theorem 1.2], we get the following result.
Proposition 3.8. We have

\[
\int_{\Omega} \left[ \Delta \eta_{k,\mu} + \frac{1}{\varepsilon^2} e^{G(u_0 + \bar{W}_{x,\mu} + \eta_{x,\mu})} (1 - e^{G(u_0 + \bar{W}_{x,\mu} + \eta_{x,\mu})})^2 + \Delta \bar{W}_{x,\mu} - 8k\pi \right] Y_{x,\mu,ij} dx
\]

\[
= A_0 \frac{\partial G^*(x)}{\partial x_{ij}} + O \left( \frac{\ln \mu}{\mu^{2 - \frac{1}{2}}} \right) \quad \text{for} \; j = 1, 2,
\]

and

\[
\int_{\Omega} \left[ \Delta \eta_{k,\mu} + \frac{1}{\varepsilon^2} e^{G(u_0 + \bar{W}_{x,\mu} + \eta_{x,\mu})} (1 - e^{G(u_0 + \bar{W}_{x,\mu} + \eta_{x,\mu})})^2 + \Delta \bar{W}_{x,\mu} - 8k\pi \right] Y_{x,\mu,0} dx
\]

\[
= \frac{8}{\rho_1 \mu^3} \left( \sum_{i=1}^{k} \rho_i \left( \int_{\Omega \setminus B_{d_i}(x_i)} \frac{e^{f_{x,i}} - 1}{|y - x_i|^2} - \int_{\mathbb{R}^2 \setminus \Omega_i} \frac{1}{|y - x_i|^4} \right) \right) + B_0 \varepsilon^2 \mu + \frac{1}{\mu^3} O (|DG^*(x)|^2 \ln \mu + \delta^2) + O \left( \frac{1}{\mu^5} \right),
\]

where \( A_0, B_0 > 0 \) are constants, \( \delta > 0 \) is any small constant, \( \Omega_1, \ldots, \Omega_k \) are any open set with \( \Omega_i \cap \Omega_j = \emptyset \) for \( i \neq j \), \( \cup_{i=1}^{k} \Omega_i = \Omega \), \( B_{d_i}(x_i) \subset \subset \Omega_i \), \( i = 1, \ldots, k \), and \( f_{x,i}(y) = 8\pi \left( \gamma(y, x_i) - \gamma(x_i, x_i) + \sum_{j \neq i} (G(y, x_j) - G(x_i, x_j)) \right) + u_0(y) - u_0(x_i) \).

Proof. In \([16]\), if \( \eta_{x,\mu} \in X_{\alpha,\Omega,\mu} \) satisfies \( \|\eta_{x,\mu}\|_{L^\infty(\Omega)} + \|\eta_{x,\mu}\|_{X_{\alpha,\Omega,\mu}} \leq \frac{C \ln \mu}{\mu^{2 - \frac{1}{2}}} \), then the followings hold:

\[
\int_{\Omega} \left( \Delta \left( \bar{W}_{x,\mu} + \eta_{x,\mu} \right) + \frac{1}{\varepsilon^2} e^{u_0 + \bar{W}_{x,\mu} + \eta_{x,\mu}} (1 - e^{u_0 + \bar{W}_{x,\mu} + \eta_{x,\mu}}) - 8k\pi \right) Y_{x,\mu,ij} dx
\]

\[
= A_0 \frac{\partial G^*(x)}{\partial x_{ij}} + O \left( \frac{\ln \mu}{\mu^{2 - \frac{1}{2}}} \right) \quad \text{for} \; j = 1, 2,
\]

and

\[
\int_{\Omega} \left( \Delta \left( \bar{W}_{x,\mu} + \eta_{x,\mu} \right) + \frac{1}{\varepsilon^2} e^{u_0 + \bar{W}_{x,\mu} + \eta_{x,\mu}} (1 - e^{u_0 + \bar{W}_{x,\mu} + \eta_{x,\mu}}) - 8k\pi \right) Y_{x,\mu,0} dx
\]

\[
= \frac{8}{\rho_1 \mu^3} \left( \sum_{i=1}^{k} \rho_i \left( \int_{\Omega \setminus B_{d_i}(x_i)} \frac{e^{f_{x,i}} - 1}{|y - x_i|^2} - \int_{\mathbb{R}^2 \setminus \Omega_i} \frac{1}{|y - x_i|^4} \right) \right) + B_0 \varepsilon^2 \mu + \frac{1}{\mu^3} O (|DG^*(x)|^2 \ln \mu + \delta^2) + O \left( \frac{1}{\mu^5} \right).
\]

Indeed, the estimation (3.19) was obtained in \([16\ (4.26)]\) and the estimation (3.20) was obtained in \([16\ (4.28)]\).

Comparing to our integral (3.17)-(3.19) and (3.18)-(3.20), the differences are the following
In view of (3.19)-(3.24), we complete the proof of Proposition 3.8. □

From Lemma 3.3, we see that

From (3.7), we remind that

Similarly, we also see that from (3.21)-(3.22),

Then from (3.21)-(3.22), we see that

From Lemma 3.3, we see that

Then from (3.21)-(3.22), we see that

Similarly, we also see that from (3.21)-(3.22),

In view of (3.19)-(3.21), we complete the proof of Proposition 3.8.

Completion of the proof of Theorem 1.2 In view of Proposition 3.6, we can find \( \eta_{x, \mu} \) satisfying (3.16). To complete the proof of Theorem 1.2, we need to find \((x, \mu)\)
suitably depending on $\varepsilon > 0$ such that the corresponding $c_0, c_{ij}$ are zero in (3.16). By using Proposition 3.8, we see that the conditions in Lemma 3.7 are equivalent to

$$DG^*(x) = O\left(\frac{\ln \mu}{\mu^2} \right),$$

(3.25)

and

$$\frac{8}{\rho_1 \mu^3} \left( \sum_{i=1}^k \rho_i \left( \int_{\Omega_1 \setminus B_{\delta}(x_i)} e^{f_{x_i}} - \frac{1}{|y - x_i|^4} - \int_{\mathbb{R}^2 \setminus \Omega_1} \frac{1}{|y - x_i|^4} \right) \right) + B_0 \varepsilon^2 \mu$$

$$= \frac{1}{\mu^3} O \left( |DG^*(x)|^2 \ln \mu + \delta^2 \right) + O \left( \frac{1}{\mu^5} \right).$$

(3.26)

Since $D(q) < 0$, we can find a small $\delta > 0$, such that for $x$ close to $q$, we have

$$\sum_{i=1}^k \rho_i \left( \int_{\Omega_1 \setminus B_{\delta}(x_i)} e^{f_{x_i}} - \frac{1}{|y - x_i|^4} - \int_{\mathbb{R}^2 \setminus \Omega_1} \frac{1}{|y - x_i|^4} \right) + O(\delta^2) < 0.$$

Then we obtain a solution $(x, \mu) = (x(\varepsilon), \mu(\varepsilon))$ of (3.25)-(3.26) satisfying

$$|DG^*(x(\varepsilon))| = O(\varepsilon^{1-\frac{2}{k}} \ln \varepsilon), \quad \mu(\varepsilon) \in \left( \frac{\beta_0}{\sqrt{\varepsilon}}, \frac{\beta_1}{\sqrt{\varepsilon}} \right),$$

which implies the existence of a solution $u_\varepsilon$ to (3.1). In view of $e^{W_{x,\mu}} = O(\varepsilon)$ on $\Omega$, $u_\varepsilon = F(v_\varepsilon)$, and

$$u_\varepsilon(y) = 1 + u_0 + \tilde{W}_{x,\mu} + \eta_{x,\mu},$$

we obtain that $\lim_{\varepsilon \to 0} \sup_{\Omega} v_\varepsilon = -\infty$. Moreover, we remind that from (16),

$$\int_{B_{\delta}(x_i)} e^{w_{x,\mu} + u_0} dx = \frac{8^{k-1} \rho_1}{\Pi_{i=2}^k \mu_i^2} \left( 8\pi + O\left( \frac{\ln \mu}{\mu^2} \right) \right),$$

(3.27)

$$\int_{\Omega} e^{w_{x,\mu} + u_0} dx = \frac{8^{k-1} \rho_1}{\Pi_{i=2}^k \mu_i^2} \left( 8k\pi + O\left( \frac{\ln \mu}{\mu^2} \right) \right),$$

(3.28)

and

$$w_{x,\mu}^*(x) = \sum_{i=1}^k w_{x_i,\mu_i}^*(x) = -2k \ln \mu + O(1) \quad \text{on} \quad \Omega \setminus \bigcup_{i=1}^k B_\delta(x_i) \quad \text{for any} \quad \delta > 0. \quad (3.29)$$

Indeed, the estimation (3.27) was obtained in (16) (2.9), the estimation (3.28) was obtained in (16) (2.10), and the estimation (3.29) was obtained in (16) (2.12). Then we obtain that

$$\frac{e^{G(1+u_0+\tilde{W}_{x,\mu}+\eta_{x,\mu})}}{\int_{\Omega} e^{G(1+u_0+\tilde{W}_{x,\mu}+\eta_{x,\mu})} dx} = \frac{e^{G(u_\varepsilon)}}{\int_{\Omega} e^{G(u_\varepsilon)} dx} \to \frac{1}{k} \sum_{i=1}^k \delta_{q_i},$$

in the sense of measure as $\varepsilon \to 0$. At this point, we complete the proof of Theorem 1.2. \hfill \Box
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References

[1] Burzlaff, J., Chakrabarti, A., Tchrakian, D. H.: Generalized self-dual Chern-Simons vortices. Phys. Lett. B 293, 127-131 (1992).
[2] Caffarelli, L. A., Yang, Y.S.: Vortex condensation in the Chern-Simons Higgs model: an existence theorem. Comm. Math. Phys. 168, 321-336 (1995).
[3] Chae, D., Imanuvilov, O. Y.: The existence of non-topological multivortex solutions in the relativistic self-dual Chern-Simons theory, Comm. Math. Phys. 215, 119-142 (2000).
[4] Chae, D., Imanuvilov, O. Y.: Non-topological solutions in the generalized self-dual Chern-Simons-Higgs theory. Calc. Var. Partial Differential Equations 16, 47-61 (2003).
[5] Chan, H., Fu, C.-C., Lin, C.-S.: Non-topological multivortex solutions to the self-dual Chern-Simons-Higgs equation. Comm. Math. Phys. 231, 189-221 (2002).
[6] Choe, K.: Uniqueness of the topological multivortex solution in the self-dual Chern-Simons theory. J. Math. Phys. 46, 012305, 22 pp. (2005).
[7] Choe, K.: Asymptotic behavior of condensate solutions in the Chern-Simons-Higgs theory. J. Math. Phys. 48, 103501, 17 pp. (2007).
[8] Choe, K., Kim, N.: Blow-up solutions of the self-dual Chern-Simons-Higgs vortex equation. Ann. Inst. H. Poincaré Anal. Non Linéaire 25, 313-338 (2008).
[9] Ding, W., Jost, J., Li, J., Peng, X., Wang, G.: Self duality equations for Ginzburg-Landau and Seiberg-Witten type functionals with 6th order potentials. Comm. Math. Phys. 217 383-407 (2001).
[10] Gilbarg, D., Trudinger, N.S.: Elliptic Partial Differential Equations of Second Order. vol. 224, second ed., Springer, Berlin, (1983).
[11] Han, X.: Existence of doubly periodic vortices in a generalized Chern-Simons model. Nonlinear Anal. Real World Appl. 16, 90-102 (2014).
[12] Hong, J., Kim, Y., Pac, P.Y.: Multi-vortex solutions of the Abelian Chern-Simons-Higgs theory, Phys. Rev. Lett. 64, 2230-2233 (1990).
[13] Jackiw, R., Weinberg, E. J.: Self-dual Chern-Simons vortices, Phys. Rev. Lett. 64, 2234-2237 (1990).
[14] Jaffe, A., Taubes, C.: Vortices and Monopoles, Birkhäuser, Boston (1980).
[15] Lin, C.-S., Yan, S.: Bubbling solutions for relativistic abelian Chern-Simons model on a torus. Comm. Math. Phys. 297, 733-758 (2010).
[16] Lin, C.-S., Yan, S.: Existence of Bubbling solutions for Chern-Simons model on a torus. Arch. Ration. Mech. Anal. 207, 353-392 (2013).
[17] Nolasco, M., Tarantello, G.: On a sharp Sobolev-type inequality on two dimensional compact manifolds. Arch. Ration. Mech. Anal. 145, 161-195 (1998).

[18] Nolasco, M., Tarantello, G.: Double vortex condensates in the Chern-Simons-Higgs theory. Calc. Var. Partial Differential Equations 9, 31-94 (1999).

[19] Spruck, J., Yang, Y.: Topological solutions in the self-dual Chern-Simons theory: existence and approximation. Ann. Inst. H. Poincare Anal. Non Lineaire 12, 75-97 (1995).

[20] 't Hooft, G.: A property of electric and magnetic flux in nonabelian gauge theories. Nucl. Phys. B153 141-160 (1979).

[21] Tarantello, G.: Multiple condensate solutions for the Chern-Simons-Higgs theory. J. Math. Phys. 37, 3769-3796 (1996).

[22] Tarantello, G.: Selfdual Gauge Field Vortices. An analytical approach. Progress in Nonlinear Differential Equations and their Applications. Birkhauser Boston, Inc., Boston (2008).

[23] Tchrakian, D. H., Yang, Y.: The existence of generalised self-dual Chern-Simons vortices. Lett. Math. Phys. 36, 403-413 (1996).

[24] Yang, Y.: Chern-Simons solitons and a nonlinear elliptic equation, Helv. Phys. Acta 71 (5), 573-585 (1998).

[25] Yang, Y.: Solitons in Field Theory and Nonlinear Analysis, Springer Monographs in Mathematics, Springer-Verlag, New York (2001).

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