RISK MEASURE OPTIMIZATION: PERCEIVED RISK AND OVERCONFIDENCE OF STRUCTURED PRODUCT INVESTORS

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ABSTRACT. In financial optimization, it is important to quantify the risk of structured financial products. This paper quantifies the risk of structured financial products by perceived risk measures based on a standard measure of risk, and then we construct the risk perception and decision-making models of individual investors considering structured products. Moreover, based on bullish and bearish binary structured products, we introduce the psychological bias of overconfidence to explore how this bias affects investors' perceived risk. This study finds that overconfident investors believe in private signals and underestimate the variance of noise in private signals, which affects their expectation of the underlying asset price of structured financial products. Furthermore, overconfidence bias leads investors to overestimate the probability of obtaining a better return. With the increase in overconfidence, the overestimation of the probability is intensified, which eventually leads to lower perceived risk.

1. Introduction. Structured financial products, originated in the 1980s, have experienced explosive growth in European markets in the 1990s. They have become the most innovative financial products over the past 20 years. Das [14] defines such products as a new type of financial instrument resulting from a combination of fixed income securities and derivatives, which can be simply expressed as “bonds + options”. Compared with the traditional financial products, the return of the structured financial products is linked to the underlying asset price, exhibiting a “high-risk, high-yield” feature.

Structured financial products are flexible and diverse. They have diverse maturity, underlying assets, and payoff formulas. The diversity not only helps individual investors gain exposure to specific markets but also reduces the risk of individual

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investors investing directly in the capital market. However, there is a large risk associated with the prosperity of structured financial products. They seldom reach a high return under their best-case scenario—the headline rate—and their disclosures are strategically and intentionally selected. In terms of the external environment, the needs for meaningful regulations and supervisions are serious. In terms of the product itself, structured products are designed to be more complex, which results in it being difficult for investors to judge the price trends of underlying assets. The issuers even have the motive of designing increasingly complex products to induce psychological biases of investors for selfish profits.

Since structured products have such a huge risk, why do investors still choose to invest in such products? The studies on structured financial products are more focused on empirical evidence that psychological biases may affect investors’ decision-making so that investors have irrational investment behaviors (i.e., behavioral bias). However, they do not explore or provide a detailed explanation for this mechanism [5, 8, 33]. Furthermore, Ofir and Wiener [32], Rieger and Hens [36], Hens and Rieger [19], and other studies have specified that the related psychological biases include loss aversion, disposal effect, herding, and possibility misinterpretation. Because investors’ decision-making process or preference is significantly affected by perceived risk, overconfidence, as the most widely recognized psychological bias, will inevitably affect the perceived risk of structured product investors and thereby affect their decision-making. Thus, it is of theoretical and practical significance to study the impact of overconfidence on the perceived risk of structured financial product investors.

In the field of behavioral finance, several studies have verified that overconfidence affects individual financial decision-making, and this psychological bias can even be widely found in professional financial practitioners [17, 21]. Because models based on a completely rational human hypothesis cannot describe the phenomenon of excessive transaction and predictable return in the real security market, introducing psychological bias, such as overconfidence, becomes an effective way to explain financial market anomalies [18, 6, 1]. In the classical models established by Odean [31], Daniel et al. [13], and Hirshleifer and Luo [20], investors underestimate the precision of private signals to describe the overconfidence. We introduce this method into our research on the perceived risk of structured financial product investors. The “private information” in this paper refers to the information about the underlying asset price at maturity that the investors obtain through their own efforts. It can be either disclosed or inside information. Even if the information is disclosed, investors still need to work hard to obtain it. Then, the investors will overestimate its accuracy.

To explore how overconfidence affects the perceived risk of structured product investors, we must identify the forming process of perceived risk, that is, to determine the risk measure that can more truly describe the forming mechanism of investors’ risk perception.

Recently, studies on perceived risk have been more focused on the field of online consumer behavior, considering perceived risk as a multidimensional quantity, using a structural equation model to explore how perceived risk affects consumer online purchase attitude and purchase intention, etc [2, 12, 16, 26, 30].

For most of the decisions under uncertainty, the objects can be interpreted explicitly as lotteries with numerical payoffs and probabilities. Coombs and Huang [11] find that any two-outcome lottery can be obtained by three transformations of
the original lottery, different transformations have different effects on perceived risk, and their empirical results support a distributive model of the transformation functions. However, Barron [3] demonstrates that this distributive model is inconsistent with reality. Pollatsek and Tversky [35] deduce a real-valued risk measure based on seven axioms indicating that individuals' perceived risk is a linear combination of the mean and variance of a lottery. The axiomatization of risk judgment proposed by Pollatsek and Tversky is of great significance to the research of perceived risk. However, Coombs and Bowen [10] find that the continuity axiom is not consistent with the empirical results. Furthermore, Luce [27, 28] explores a new measure of risk with the perspective of lottery transformations affecting perceived risk. Based on two categories of assumptions, he produces four perceived risk measure models.

Weber [39] proposes that Luce’s first set of assumptions is doubtful, and the second class of assumptions is not consistent with their empirical results. Moreover, Keller et al. [25] argue that the three models of Luce’s risk functions are not good measures to characterize perceived risk. Luce and Weber [29] combine the empirical results of Weber [40] to improve and revise the risk measure proposed by Luce [27, 28], and they propose a conjoint expected risk (CER) model. However, the mixture space axioms and conjoint structure axioms of the CER model are also questionable [41].

Because the above perceived risk measure models are imperfect, Jia and Dyer [22] propose a standard measure of risk based on the converted expected utility of normalized lotteries with zero-expected values that can reflect the “pure” risks of the original lotteries. Additionally, the three major assumptions (axioms) under the standard measure of risk and the risk-value model have been confirmed by empirical tests on individuals’ risk judgement [7].

The standard measure of risk has the following excellent properties to describe the risk of lottery form decisions:

First, the standard measure of risk is derived from the same axioms of the expected utility model; thus, it is compatible with the expected utility model explicitly or implicitly. If and only if the risk independence condition is satisfied, we can express the expected utility function as a risk-value trade-off. The trade-off framework can be included in many proposed optimization models characterizing preference, such as Bell’s disappointment model [4, 24]. When investors decide whether to invest in structured products, they actually go through an optimization process based on a risk-value trade-off. Second, the standard measure of risk depends on the individual’s utility function, which can reflect the different preference and psychological characteristics of different individuals. Different utility functions can embody the general standard measure of risk into some common risk measures proposed in financial studies, such as absolute deviation, variance, semi-variance, skewness, etc. Third, the standard measure of risk has no restriction on the distribution of the outcomes of the lottery and the form of the utility function. We can calculate the real-value risk of the lottery under certain risk attitudes when given a specific utility function and a specific distribution of a lottery. Fourth, there are many desirable properties of the standard measure of risk that are in line with the conception of risk. The perceived risk models based on the standard measure of risk are consistent with the previous empirical research results and can reasonably explain the related finance paradox [23].

Investors faced with structured financial products will simplify them as a lottery. Different trigger points change the probabilities of outcomes, and different final gains
produce different payoffs. The measure of perceived risk and the risk-value trade-off framework based on the standard measure of risk can consider the risk derived both from the price volatility of the underlying asset and the complicated structure of the structured financial product. Thus, they can more rationally characterize the risk judgment and decision-making process of structured product investors with different utility functions. Recently, scholars have introduced new risk models of lotteries [9, 38, 34]. However, these methods require further empirical tests.

In financial optimization, it is important to quantify the risk of structured financial products, and investors prefer portfolios with low perceived risk to maximize their utility. In this paper, we propose a model to explore the effect of overconfidence on the perceived risk of structured financial product investors. The innovation is reflected in at least two aspects. First, we regard structured products as lotteries with different return scenarios and corresponding probabilities of occurrence, and we use perceived risk measures based on a standard measure of risk to characterize the risk perception. Second, we introduce the psychological bias of overconfidence into the process. Based on the assumption that investors deduce the conditional distribution of the asset price at maturity based on the Bayes theorem, the subjective expectation is proven to be a normal distribution. Overconfidence distorts the probability estimate of investors by leading them to underestimate the noise variance and then affecting the mean and variance of their expected distribution.

The next sections of this paper are organized as follows. Section 2 summarizes the related concepts and measures, including the standard measure of risk, relative measure of risk, measure of perceived risk, and risk-value trade-off framework, to analyze the forming mechanism of the perceived risk of structured financial product investors. Section 3 explores how the psychological bias of overconfidence affects investors’ expectation when investors optimize their portfolios. Section 4 further explores how overconfidence affects the perceived risk of structured product investors by influencing their expectation. Finally, Section 5 summarizes the paper.

2. Measures of perceived risk. Jia et al. [23] propose a perceived risk model based on a standard measure of risk and introduce the perceived risk model into a risk-value framework of preference. The standard measure of risk can be replaced by a relative measure of risk.

2.1. The standard measure of risk. Let \( P \) be a convex set of all simple probability distributions or lotteries \( \{X, Y, Z, \cdots\} \) on a nonempty set of outcomes. Define a subset of \( P \), a set of normalized probability distributions, as follows:

\[
P^0 = \{X'|X' = X - \bar{X}, X \in P\},
\]

where \( \bar{X} \) is the mean of \( X \). We call \( P^0 \) the “risk set” of the original probability distributions and call its element \( X' \) the “standard risk” of a lottery \( X \). The standard risk \( X' \) is a normalized lottery and reflects the “pure” risk of the original lottery \( X \). For any degenerate lottery, \( X' = 0 \). Therefore, lottery \( X \) can be expressed as \( (\bar{X}, X') \), which indicates a lottery can be determined by its mean and pure risk factor.

Assume the existence of expected utility axioms and a consistent risk-preference condition. Then, there exists a measure of risk \( R: P^0 \rightarrow \mathbb{R} \) (where \( \mathbb{R} \) is a set of real numbers), such that for any \( X', Y' \in P^0 \), \( X' \succeq_R Y' \) if and only if \( R(X') > R(Y') \), where \( R(X') \) is defined as follows:

\[
R(X') = -E[u(X - \bar{X})],
\]

(2)
where $u$ is a von Neumann-Morgenstern utility function, and $>_R$ denotes a binary risk relation on $P^0$. The specific form of the standard measure of risk depends on the individual utility function. According to the specific individual utility function, we can derive the corresponding standard measure of risk.

2.2. The relative measure of risk. The relative measure of risk is defined as follows [15]:

$$ R(X^*) = -E[u(X^*)] = -E[u(X/\bar{X})]. \quad (3) $$

In the standard measure of risk (2), $X' = X - \bar{X}$ is a standard risk variable with an expected value of 0. In the relative measure of risk (3), $X^* = X/\bar{X}$ is a relative risk variable with an expected value of 1. $X^* = X/\bar{X}$ interprets risk in terms of a standardized “rate of return”, a concept with powerful implications for finance. In the relative measure of risk (3), the lottery $X$ must be nonnegative (while degenerate lotteries are required to be positive).

The standard measure of risk and the relative measure of risk models are both reasonable. Different people think of risk in different ways. Some investors are more concerned with the absolute measure, while others are more concerned with the relative measure.

2.3. The measure of perceived risk. For lotteries with an expected value of 0, the standard measure of risk can reasonably describe perceived risk, although for general lotteries, this measure of risk cannot. Studies have shown that adding a positive constant to all outcomes can decrease the perceived risk, implying that perceived risk is associated with the means of lotteries. However, the standard measure of risk depends only on the “pure” risk factor and has nothing to do with the mean [25].

Jia et al. [23] propose a measure of perceived risk based on a two-dimensional structure of the standard risk of lottery and its mean.

Let $>_\tilde{R}$ be a strict binary risk relation, $\sim_{\tilde{R}}$ an indifference risk relation on $P$, and $>_R$ be a strict binary risk relation on $P^0$. To obtain a decomposed form of the perceived risk model, the following assumptions concerning the risk judgment are proposed:

**Assumption 1.** For $X', Y' \in P^0$, if there exists a $w^0 \in \mathbb{R}$ for which $(w^0, X') >_{\tilde{R}} (w^0, Y')$, then $(w, X') >_{\tilde{R}} (w, Y')$ for all $w \in \mathbb{R}$.

**Assumption 2.** For $X', Y' \in P^0$, $(0, X') >_{\tilde{R}} (0, Y')$ if and only if $X' >_R Y'$.

**Assumption 3.** For $(\bar{X}, X') \in P$, then $(\bar{X}, X') >_{\tilde{R}} (\bar{X} + \Delta, X')$ for any constant $\Delta > 0$.

**Assumption 4.** For any $w \in \mathbb{R}$, $(w, 0) \sim_{\tilde{R}} (0, 0)$.

**Theorem 2.1.** (Jia et al., 1999) If and only if the above four assumptions are satisfied, the perceived risk model can be represented in the following two-attribute form:

$$ R_p(\bar{X}, X') = \varphi(\bar{X})[R(X') - R(0)], \quad (4) $$

where $R_p(\bar{X}, X')$ is the perceived risk of lottery investors. $\varphi(\bar{X}) > 0$ is a decreasing function, indicating that when the pure risk factor is unchanged, increasing the mean proportionally reduces perceived risk.

We can consider appropriate forms for $\varphi(\bar{X})$ and $R(X')$ based on different theories or experimental results to construct many different models to characterize the perceived risks of different people.
2.4. **Risk-value trade-off framework.** Assume the existence of the expected utility model. Then,

\[
E[u(X)] = u(\bar{X}) - \varphi(\bar{X})[R(X') - R(0)],
\]

if and only if the risk independence condition holds, where \( \varphi(x) > 0 \).

\( u(\bar{X}) \) is a subjective measure of the expected value of a lottery, namely, the “value measure”. Because \( R(X') - R(0) \) is simply transformed by \( R(X') \), \( R(X') - R(0) \) is also a standard measure of risk. Therefore, we call model (5) the “standard risk-value model”.

\( \varphi(\bar{X})[R(X') - R(0)] \) is a measure of perceived risk for general lotteries with different means. \( \varphi(\bar{X}) \) is a trade-off factor reflecting the risk effect on preference or the relative weight of risk and value.

Expected utility is a measure of individual preference. Jia and Dyer [22] and Jia et al. [24, 23] argue that individual preference depends on the trade-off between perceived value and perceived risk. The basic form of preference can be represented as follows:

\[
f(\bar{X}, X') = V(\bar{X}) - \varphi(\bar{X})[R(X') - R(0)].
\]

\( f(\bar{X}, X') \) is the preference, reflecting how likely an individual would choose the lottery. A larger preference value corresponds to a greater selection tendency.

\( V(\bar{X}) \) is a subjective value measure that is an increasing function of the mean \( \bar{X} \).

\( \varphi(\bar{X}) > 0 \) is a trade-off factor that depends on the mean \( \bar{X} \).

\( R(X') \) is a standard measure of risk that characterizes the risk derived from the pure risk of lotteries.

In the risk-value model (6), we can select the utility function, the value function, and the trade-off factor independently according to different theories or empirical results. This makes the risk-value model very flexible for modeling individual risk choices.

3. **The effect of overconfidence on investors’ expectation.** In this section, we analyze the impact of overconfidence on the expectation of structured financial product investors, including the distribution of the price at maturity of the underlying asset and the probability of obtaining a better return.

3.1. **Structured financial products embedded with binary options.** Structured financial products embedded with binary options exhibit payoff scenarios like a lottery with two outcomes. The return rates at maturity in the two scenarios are defined by a pre-specified formula that does not change with the underlying asset price. However, the return rate that investors can actually obtain depends on the price at maturity of the underlying asset. If the price satisfies a condition (such as above or below the trigger price), the investor receives a fixed return rate; otherwise, the investor obtains another return rate.

Assume that each investor has a given amount of principal \( M \). Two types of structured financial products embedded with binary options—bullish and bearish structured products—are defined as follows:

**Bullish products:** If the price at maturity \( \theta \) (or price on the observation day) of the underlying asset is greater than the trigger point \( A \), the investor earns a return rate of \( r_1 \); otherwise, the investor earns a return rate of \( r_2 \), where \( r_1 > r_2 \).
Bearish products: If the price at maturity $\theta$ (or price on the observation day) of the underlying asset is less than the trigger point $A$, the investor earns a return rate of $r_1$; otherwise, the investor earns a return rate of $r_2$, where $r_1 > r_2$.

The probability density of the asset price at maturity can be obtained by a Monte Carlo simulation of the underlying asset price path. Then, the structured products embedded with binary options can be transformed into the following lottery form:

$$P = \left\{ \begin{array}{ll}
P(\theta > A) = \int_A^{+\infty} f(\theta) \, d\theta, & \text{if bullish}, \\
P(\theta < A) = \int_{-\infty}^{A} f(\theta) \, d\theta, & \text{if bearish}. \\
\end{array} \right. \quad (7)$$

3.2. The effect of overconfidence on investors’ expectation of price distribution. The classic behavioral asset pricing model, the DHS model [13], argues that overconfidence exists widely in human cognition. Overconfident investors ignore public signals and attach greater psychological weight to private signals acquired through their own efforts. When overconfident investors receive a private signal, they will overestimate its accuracy, which in turn affects the investors’ judgment on the underlying asset price at maturity.

Assume that overconfident investors are risk neutral, $\theta$ is the underlying asset expiry price subject to the public signal, and $\theta \sim N(\mu, \sigma^2_\theta)$. $\varepsilon$ is the noise in the private signal $\theta + \varepsilon$, and $\varepsilon \sim N(0, \sigma^2_\varepsilon)$. $\varepsilon$ and $\theta$ are independent random variables.

Overconfident investors overestimate the accuracy of private signals. They will underestimate the variance of $\varepsilon$ to be $\sigma^2_\varepsilon \phi$. That is, there is a $\phi$, $0 \leq \phi < 1$, for which $\sigma^2_\varepsilon = \phi \sigma^2_\varepsilon$. $\phi$ is an overconfidence coefficient. Smaller overconfidence coefficient values correspond to increased investor overconfidence.

After receiving a private signal, the risk-neutral investors will reason the conditional distribution $\theta|\theta + \varepsilon$ of the asset price at maturity based on their private signal and the Bayes theorem.

**Proposition 1.** If $\theta \sim N(\mu, \sigma^2_\theta)$ and $\varepsilon \sim N(0, \sigma^2_\varepsilon)$, then $\theta|\theta + \varepsilon$ is normally distributed, where $\varepsilon$ and $\theta$ are independent random variables.

**Proof.** It can be proven that

$$F(x, y) = P \{ \theta \leq x, \theta + \varepsilon \leq y \}$$

$$= \int \int_{\varepsilon \leq \varepsilon + \varepsilon \leq y} f(\theta, \varepsilon) \, d\theta \, d\varepsilon$$

$$= \int_{-\infty}^{x} d\theta \int_{-\infty}^{y-\theta} f(\theta, \varepsilon) \, d\varepsilon$$
\[ E[\theta] = \frac{\int_{-\infty}^{\infty} f_0(\theta) f_\varepsilon(\varepsilon) \, d\varepsilon}{\int_{-\infty}^{\infty} f_\varepsilon(\varepsilon) \, d\varepsilon} \]

\[ = \int_{-\infty}^{x} f_0(\theta) d\theta \int_{-\infty}^{y-\theta} f_\varepsilon(\varepsilon) \, d\varepsilon \]

\[ = \int_{-\infty}^{x} f_0(\theta) \left[ \int_{-\infty}^{y-\theta} f_\varepsilon(\varepsilon) \, d\varepsilon \right] d\theta, \]

and

\[ \frac{\partial F}{\partial y} = \int_{-\infty}^{x} f_0(\theta) \left[ \frac{\partial}{\partial y} \int_{-\infty}^{y-\theta} f_\varepsilon(\varepsilon) \, d\varepsilon \right] d\theta \]

\[ = \int_{-\infty}^{x} f_0(\theta) f_\varepsilon(y - \theta) \, d\theta. \]

Since

\[ f(x, y) = \frac{\partial^2 F}{\partial y \partial x} = f_\theta(x) f_\varepsilon(y - x), \]

we have the following:

\[ f_{\theta|\theta+\varepsilon}(x|y) = \frac{f_\theta(x) f_\varepsilon(y - x)}{f_{\theta+\varepsilon}(y)} \]

\[ = \frac{1}{\sqrt{2\pi} \sigma_z} e^{-\frac{(x-\mu)^2}{2\sigma_z^2}} \frac{1}{\sqrt{2\pi} \sigma_\varepsilon} e^{-\frac{(y-x)^2}{2\sigma_\varepsilon^2}} \]

\[ = \frac{1}{\sqrt{2\pi}} \frac{\sqrt{\sigma_\theta^2 + \phi \sigma_\varepsilon^2}}{\sqrt{\phi \sigma_\theta \sigma_\varepsilon}} e^{-\frac{1}{2} \left( \frac{(x-\mu)^2}{\sigma_\theta^2} + \frac{(y-x)^2}{\sigma_\varepsilon^2} - \frac{(y-x)^2}{\sigma_\theta^2 + \phi \sigma_\varepsilon^2} \right)}. \]

Thus,

\[ f_{\theta|\theta+\varepsilon}(x|y) = \frac{1}{\sqrt{2\pi}} \frac{\sqrt{\sigma_\theta^2 + \phi \sigma_\varepsilon^2}}{\sqrt{\phi \sigma_\theta \sigma_\varepsilon}} e^{-\frac{1}{2} \left( \frac{x^2 + \phi \varepsilon^2}{\sigma_\theta^2 + \phi \sigma_\varepsilon^2} \right) \left( x - \frac{\phi \mu \sigma_\varepsilon^2}{\sigma_\theta^2 + \phi \sigma_\varepsilon^2} \right)^2}. \] (8)

Proposition 1 is proved. \( \square \)

Proposition 1 implies that when the actual price at maturity of the underlying asset is subject to a normal distribution, the expectation of the structured product investor based on Bayesian law after receiving a private signal is also subject to a normal distribution.

**Proposition 2.** If \( \theta + \varepsilon > \mu \), then \( E[\theta|\theta+\varepsilon] > \mu \) and \( \frac{\partial E[\theta|\theta+\varepsilon]}{\partial \sigma_\theta} < 0 \); if \( \theta + \varepsilon < \mu \), then \( E[\theta|\theta+\varepsilon] < \mu \) and \( \frac{\partial E[\theta|\theta+\varepsilon]}{\partial \sigma_\theta} > 0 \). \( Var[\theta|\theta+\varepsilon] - \sigma_\theta^2 < 0 \) and \( \frac{\partial Var[\theta|\theta+\varepsilon]}{\partial \sigma_\theta} > 0 \) always hold.

**Proof.** Formula (8) implies that the conditional distribution \( \theta|\theta+\varepsilon \) is subject to a normal distribution with mean \( E[\theta|\theta+\varepsilon] \) and variance \( Var[\theta|\theta+\varepsilon] \):

\[ E[\theta|\theta+\varepsilon] = \frac{(\theta + \varepsilon) \sigma_\theta^2 + \mu \phi \sigma_\varepsilon^2}{\sigma_\theta^2 + \phi \sigma_\varepsilon^2}, \] (9)

\[ Var[\theta|\theta+\varepsilon] = \frac{\phi \sigma_\varepsilon^2 \sigma_\theta^2}{\sigma_\theta^2 + \phi \sigma_\varepsilon^2}. \] (10)

If \( \theta + \varepsilon > \mu \), the expected price at maturity will be \( E[\theta|\theta+\varepsilon] > \mu \). If \( \theta + \varepsilon < \mu \), the expected price at maturity will be \( E[\theta|\theta+\varepsilon] < \mu \).
Formula (10) minus \( \sigma^2_\theta \) results in the following:

\[
\frac{\phi \sigma^2_\theta \sigma^2_\varepsilon}{\sigma^2_\theta + \phi \sigma^2_\varepsilon} - \sigma^2_\theta = \frac{-\sigma^4_\theta}{\sigma^2_\theta + \phi \sigma^2_\varepsilon} < 0. \tag{11}
\]

Thus, \( \text{Var}[\theta | \theta + \varepsilon] - \sigma^2_\theta < 0 \) always holds.

Find the partial derivative of (9) with respect to \( \phi \):

\[
\frac{\partial E[\theta | \theta + \varepsilon]}{\partial \phi} = \frac{[\mu - (\theta + \varepsilon)] \sigma^2_\theta \sigma^2_\varepsilon}{(\sigma^2_\theta + \phi \sigma^2_\varepsilon)^2}. \tag{12}
\]

For \( \theta + \varepsilon > \mu \), \( \frac{\partial E[\theta | \theta + \varepsilon]}{\partial \phi} < 0 \). For \( \theta + \varepsilon < \mu \), \( \frac{\partial E[\theta | \theta + \varepsilon]}{\partial \phi} > 0 \).

Find the partial derivative of (10) with respect to \( \phi \):

\[
\frac{\partial \text{Var}[\theta | \theta + \varepsilon]}{\partial \phi} = \frac{\sigma^4_\theta \sigma^2_\varepsilon}{(\sigma^2_\theta + \phi \sigma^2_\varepsilon)^2} > 0. \tag{13}
\]

Formula (13) means that \( \frac{\partial \text{Var}[\theta | \theta + \varepsilon]}{\partial \phi} > 0 \) always holds. Proposition 2 is proved.

The financial meaning of Proposition 2 is as follows. When the private signal received by the investors indicates that the price at maturity of the underlying asset is greater than the expected price of the actual distribution, overconfidence will lead investors to overestimate the expected price at maturity and underestimate the variance. Moreover, the overestimation of the expected price at maturity and the underestimation of the variance will be intensified with increased overconfidence.

When the private signal received by the investors indicates that the price at maturity of the underlying asset is less than the expected price of the actual distribution, overconfidence will lead investors to underestimate the expected price at maturity and the variance. Moreover, the underestimation of the expected price at maturity and the variance will be intensified with increased overconfidence.

Take the case \( \theta + \varepsilon > \mu \) as an example. As shown in Figure 2, after the overconfident investors receive the private signal and reason based on Bayesian statistics, they will overestimate the expected maturity price and underestimate the variance. Moreover, there will be a further overestimation of the mean and a further underestimation of the variance as the degree of overconfidence increases.

**Figure 2.** Expected price distribution of overconfident investors when \( \theta + \varepsilon > \mu \)
3.3. The effect of overconfidence on investors’ expectation of the probability of obtaining a better return. In this subsection, we take investors who buy bullish structured products as an example to analyze how overconfidence affects investors’ estimation of the probability of obtaining a better return.

If the maturity price in the private signal is greater than the trigger point, namely, \( \theta + \varepsilon > A \), investors tend to buy bullish products. Three cases result: \( \mu < A < \theta + \varepsilon \), \( A < \mu < \theta + \varepsilon \), and \( A < \theta + \varepsilon < \mu \).

The formula for assessing whether investors overestimate or underestimate the probability of obtaining a better return is as follows:

\[
\int_b^{+\infty} f(x) \, dx - \int_c^{+\infty} f(x) \, dx = \int_b^c f(x) \, dx, \tag{14}
\]

where \( \int_b^{+\infty} f(x) \, dx \) is the subjective probability and \( \int_c^{+\infty} f(x) \, dx \) is the objective probability. When \( \int c b f(x) \, dx > 0 \), namely, \( c > b \), investors will overestimate the probability of obtaining a better return. When \( \int c b f(x) \, dx < 0 \), namely, \( c < b \), investors will underestimate the probability.

In (14), \( f(x) \) is the probability density of the standard normal distribution, and \( b \) and \( c \) are normalized trigger points relative to the subjective expected distribution and the actual distribution of the maturity price of underlying asset, respectively:

\[
b = \frac{A - \frac{(\theta + \varepsilon)\sigma_\theta^2 + \mu\phi\sigma_e^2}{\sqrt{\sigma_\theta^2 + \phi\sigma_e^2}}}{\sqrt{\sigma_\theta^2 + \phi\sigma_e^2}}, \tag{15}
\]

\[
c = \frac{A - \mu}{\sigma_\theta}. \tag{16}
\]

From (16) minus (15), we have the following:

\[
c - b = \frac{(A - \mu)\sqrt{\sigma_\theta^2}}{\sqrt{\sigma_\theta^2 + \phi\sigma_e^2}} - A + \frac{(\theta + \varepsilon)\sigma_\theta^2 + \mu\phi\sigma_e^2}{\sqrt{\sigma_\theta^2 + \phi\sigma_e^2}}. \tag{17}
\]

When \( c = b \), we can obtain

\[
A = \frac{(\theta + \varepsilon)\sigma_\theta^2 + \mu\phi\sigma_e^2 - \mu\sigma_e\sqrt{\theta} \sqrt{\sigma_\theta^2 + \phi\sigma_e^2}}{(\sigma_\theta^2 + \phi\sigma_e^2) - \sigma_e \sqrt{\theta} \sqrt{\sigma_\theta^2 + \phi\sigma_e^2}} = a(\phi). \tag{18}
\]

The subjective probability of the investor is equal to the objective probability, and \( a(\phi) \) is called the perceived equilibrium point.

**Proposition 3.** If \( A < a \), then \( c > b \); if \( A > a \), then \( c < b \). If \( \theta + \varepsilon > \mu \), then \( \frac{\partial a}{\partial \phi} > 0 \); if \( \theta + \varepsilon < \mu \), then \( \frac{\partial a}{\partial \phi} < 0 \).

**Proof.** Find the partial derivative of (17) with respect to \( A \):

\[
\frac{\partial(c - b)}{\partial A} = \frac{\sqrt{\phi\sigma_e} - \sqrt{\sigma_\theta^2 + \phi\sigma_e^2}}{\sqrt{\phi\sigma_\theta\sigma_e}}. \]

Since \( \sqrt{\phi\sigma_e} - \sqrt{\sigma_\theta^2 + \phi\sigma_e^2} < 0 \), \( \frac{\partial(c-b)}{\partial A} < 0 \); moreover, \( c - b \) is strictly monotonically decreasing with \( A \).

Thus, if \( A < a \), then \( c > b \); if \( A > a \), then \( c < b \).
Find the partial derivative of (18) with respect to $\phi$:

$$\frac{\partial a}{\partial \phi} = \frac{\left[(\theta + \varepsilon) - \mu\right] \sigma_x \sigma_\varepsilon \left(\sqrt{\sigma_x^2 + \phi \sigma_\varepsilon^2} - \sqrt{\sigma_x^2 + \phi \sigma_\varepsilon^2}ight)^2}{2 \sqrt{\sigma_x^2 + \phi \sigma_\varepsilon^2} \left[\left(\sigma_x^2 + \phi \sigma_\varepsilon^2\right) - \sqrt{\sigma_x^2 + \phi \sigma_\varepsilon^2} \sqrt{\sigma_x^2 + \phi \sigma_\varepsilon^2}\right]^2}. \quad (19)$$

According to (19), when $\theta + \varepsilon > \mu$, $\frac{\partial a}{\partial \phi} > 0$ holds, and the perceived equilibrium point $a$ decreases with decreasing $\phi$. When $\theta + \varepsilon < \mu$, $\frac{\partial a}{\partial \phi} < 0$ holds and the perceived equilibrium point $a$ increases with decreasing $\phi$. The proposition is proved.

Proposition 3 means that when the trigger point designed in the bullish structured product is less than the investor’s perceived equilibrium point, the investor will overestimate the probability of obtaining a better return. When the trigger point designed in the bullish structured product is greater than the investor’s perceived equilibrium point, the investor will underestimate the probability. For bearish conditions, the opposite holds.

The limits of (9) and (18) can be obtained as follows:

$$\lim_{\phi \to 0} E[\theta|\theta + \varepsilon] = \theta + \varepsilon, \quad (20)$$

$$\lim_{\phi \to 0} a = \theta + \varepsilon. \quad (21)$$

According to the analysis of limits (20) and (21), when $\theta + \varepsilon > \mu$, $E[\theta|\theta + \varepsilon] \leq \theta + \varepsilon \leq a$ holds; moreover, $E[\theta|\theta + \varepsilon]$ and $a$ tend to be $\theta + \varepsilon$ from two sides with $\phi$ varying from 1 to 0. When $\theta + \varepsilon < \mu$, $a \leq \theta + \varepsilon \leq E[\theta|\theta + \varepsilon]$ holds; moreover, $a$ and $E[\theta|\theta + \varepsilon]$ tend to be $\theta + \varepsilon$ from two sides with $\phi$ varying from 1 to 0.

The perceived equilibrium point $a$, the mean $E[\theta|\theta + \varepsilon]$, and the variance $\text{Var}[\theta|\theta + \varepsilon]$ are all functions of the overconfidence coefficient $\phi$, which means that the degree of overconfidence can affect the investor’s subjectively estimated probability. This effect includes two levels: the absolute level and the relative level. Absolute level refers to how the subjectively estimated probability of obtaining a better return varies with overconfidence. The relative level corresponds to how overconfidence affects the investor’s overestimation or underestimation of the probability compared with the actual objective probability.

To analyze the effect, $\frac{\partial b}{\partial \phi}$ must be derived.

$$b = \frac{\left[A - (\theta + \varepsilon)\sigma_x^2 + \mu \phi \sigma_\varepsilon^2\right]}{\sigma_x^2 + \phi \sigma_\varepsilon^2} \frac{\sqrt{\sigma_x^2 + \phi \sigma_\varepsilon^2}}{\sqrt{\sigma_x^2 + \phi \sigma_\varepsilon^2}} \quad (22)$$

$$b = \frac{\left[A - (\theta + \varepsilon)\sigma_x^2 + \mu \phi \sigma_\varepsilon^2\right]}{\sqrt{\sigma_x^2 + \phi \sigma_\varepsilon^2}} \left(\frac{\sigma_x^2 + \phi \sigma_\varepsilon^2}{\sqrt{\sigma_x^2 + \phi \sigma_\varepsilon^2}}\right).$$

It can be proven that $\frac{\partial b}{\partial \phi} > 0 \iff [2(\theta + \varepsilon) - A - \mu \phi \sigma_\varepsilon^2 - [A - (\theta + \varepsilon)]\sigma_x^2 > 0$.

Case I. $\mu < A < \theta + \varepsilon$

In case I, both the maturity price in the private signal and the trigger price of the structured product are greater than the mean of the actual price distribution of the underlying asset. The expected price distribution of the investors is shown in Figure 3.

Absolute level: Subjective probability increases monotonically with increasing overconfidence.
Relative level: Investors will always overestimate the probability, and the overestimation is intensified with increasing overconfidence.

In fact, since
\[ \mu \phi \sigma_z^2 - \mu \sqrt{\phi \sigma_z} \sqrt{\sigma_0^2 + \phi \sigma_z^2} < 0, \]
we have
\[ a = \frac{(\theta + \varepsilon) \sigma_0^2 + \mu \phi \sigma_z^2 - \mu \sigma_z \sqrt{\phi \sigma_z} \sqrt{\sigma_0^2 + \phi \sigma_z^2}}{(\sigma_0^2 + \phi \sigma_z^2) - \sigma_z \sqrt{\phi \sigma_z} \sqrt{\sigma_0^2 + \phi \sigma_z^2}} > A. \] (23)

Formula (23) shows that regardless of the degree of overconfidence, overconfident bullish investors will always overestimate the probability of obtaining a better return.

Since
\[ 2(\theta + \varepsilon) - A - \mu > 0 \quad \text{and} \quad A - (\theta + \varepsilon) < 0, \]
we have
\[ [2(\theta + \varepsilon) - A - \mu] \phi \sigma_z^2 - [A - (\theta + \varepsilon)] \sigma_0^2 > 0. \]

Thus, \( \frac{\partial b}{\partial \phi} > 0 \), and \( b \) decreases monotonically with decreasing \( \phi \).

**Case II.** \( A < \mu < \theta + \varepsilon \)

In case II, the maturity price in the private signal is greater than the mean of the actual price distribution of the underlying asset, and the trigger price of the structured product is less than the mean of actual price distribution. The expected price distribution of the investors is shown in Figure 4.

Absolute level: Subjective probability increases monotonically with increasing overconfidence.

Relative level: Investors will always overestimate the probability, and the overestimation is intensified with increasing overconfidence.

In fact, since
\[ \mu \phi \sigma_z^2 - \mu \sqrt{\phi \sigma_z} \sqrt{\sigma_0^2 + \phi \sigma_z^2} < 0, \]
we have
\[ a = \frac{(\theta + \varepsilon) \sigma_0^2 + \mu \phi \sigma_z^2 - \mu \sigma_z \sqrt{\phi \sigma_z} \sqrt{\sigma_0^2 + \phi \sigma_z^2}}{(\sigma_0^2 + \phi \sigma_z^2) - \sigma_z \sqrt{\phi \sigma_z} \sqrt{\sigma_0^2 + \phi \sigma_z^2}} > \theta + \varepsilon > A. \] (24)
Formula (24) shows that regardless of the degree of overconfidence, overconfident bullish investors will always overestimate the probability of obtaining a better return.

Since

\[ 2(\theta + \varepsilon) - A - \mu > 0 \quad \text{and} \quad A - (\theta + \varepsilon) < 0, \]

we have

\[ [2(\theta + \varepsilon) - A - \mu]\phi \sigma_{\varepsilon}^2 - [A - (\theta + \varepsilon)]\sigma_{\varepsilon}^2 > 0. \]

Thus, \( \frac{\partial b}{\partial \phi} > 0 \), and \( b \) decreases monotonically with decreasing \( \phi \).

**Case III.** \( A < \theta + \varepsilon < \mu \)

In case III, both the maturity price in the private signal and the trigger price of the structured product are less than the mean of the actual price distribution of the underlying asset. The expected price distribution of the investors is shown in Figure 5.

It can be proven that \( \frac{\partial b}{\partial \phi} > 0 \Leftrightarrow [(\theta + \varepsilon) - A](\sigma_{\theta}^2 + \phi \sigma_{\varepsilon}^2) > [\mu - (\theta + \varepsilon)]\phi \sigma_{\varepsilon}^2. \)

Since

\[ (\theta + \varepsilon) - A > 0 \quad \text{and} \quad \mu - (\theta + \varepsilon) > 0, \]

it can be proven that \( \frac{\partial b}{\partial \phi} > 0 \Leftrightarrow \frac{(\theta + \varepsilon) - A}{\mu - (\theta + \varepsilon)} > \frac{\phi \sigma_{\varepsilon}^2}{\sigma_{\theta}^2 + \phi \sigma_{\varepsilon}^2}. \)

Since,

\[ \frac{\phi \sigma_{\varepsilon}^2}{\sigma_{\theta}^2 + \phi \sigma_{\varepsilon}^2} = \frac{\sigma_{\varepsilon}^2}{\sigma_{\theta}^2 + \sigma_{\varepsilon}^2}, \]

\[ \frac{\sigma_{\theta}^2 + \phi \sigma_{\varepsilon}^2}{\sigma_{\theta}^2 + \sigma_{\varepsilon}^2} = 1. \]
\[
\lim_{\phi \to 0} \frac{\phi \sigma^2_z}{\sigma^2_\theta + \phi \sigma^2_z} = 0,
\]

the term \(\frac{\phi \sigma^2_z}{\sigma^2_\theta + \phi \sigma^2_z}\) decreases monotonically from \(\frac{\sigma^2_\theta + \sigma^2_z}{\sigma^2_\theta + \phi \sigma^2_z}\) to 0 as \(\phi\) decreases from 1 to 0. When \(\phi = 1\), we have

\[a(1) = a(\phi = 1) = \frac{(\theta + \epsilon)\sigma^2_\theta + \mu \sigma^2_z - \mu \sigma_z \sqrt{\sigma^2_\theta + \sigma^2_z}}{(\sigma^2_\theta + \sigma^2_z) - \sigma_z \sqrt{\sigma^2_\theta + \sigma^2_z}}.
\]

When case III is satisfied, \(a\) increases monotonically from \(a(1)\) to \(\theta + \epsilon\) as \(\phi\) decreases from 1 to 0.

Then, case III can be classified based on the following conditions:

(A) \(A < a(1) (b_{\phi=1} < c)\)

This subcase means investors will overestimate the probability of obtaining a better return when \(\phi = 1\).

**Subcase III(1).** \(0 < \frac{(\theta + \epsilon) - A}{\mu - (\theta + \epsilon)} < \frac{\sigma^2_\theta + \sigma^2_z}{\sigma^2_\theta + \sigma^2_z}\)

When subcase III(1) is satisfied, \(\frac{\partial b}{\partial \phi}\) is initially less than 0 and then greater than 0 as \(\phi\) decreases from 1 to 0, which means that \(b\) first increases and then decreases.

Absolute level: Subjective probability first decreases and then increases with increasing overconfidence.

Relative level: Investors’ overestimation of the probability will increase after decreasing as overconfidence increases.

However, when \(A < a(1)\), \(\frac{(\theta + \epsilon) - A}{\mu - (\theta + \epsilon)} > \frac{\sigma^2_\theta + \sigma^2_z}{\sigma^2_\theta + \sigma^2_z}\) always holds. Thus, subcase III(1) does not exist.

(B) \(A > a(1) (b_{\phi=1} > c)\)

This subcase means that investors will underestimate the probability of obtaining a better return when \(\phi = 1\).

**Subcase III(2).** \(\frac{(\theta + \epsilon) - A}{\mu - (\theta + \epsilon)} > \frac{\sigma^2_\theta + \sigma^2_z}{\sigma^2_\theta + \sigma^2_z}\)

When subcase III(2) is satisfied, \(\frac{\partial b}{\partial \phi}\) > 0 holds, which means that \(b\) decreases monotonically as \(\phi\) decreases from 1 to 0.

Absolute level: Subjective probability increases monotonically with increasing overconfidence.

Relative level: Investors will always overestimate the probability, and the overestimation is intensified with increasing overconfidence.

(B) \(A > a(1) (b_{\phi=1} > c)\)

This subcase means that investors will underestimate the probability of obtaining a better return when \(\phi = 1\).

**Subcase III(3).** \(0 < \frac{(\theta + \epsilon) - A}{\mu - (\theta + \epsilon)} < \frac{\sigma^2_\theta + \sigma^2_z}{\sigma^2_\theta + \sigma^2_z}\)

When subcase III(3) is satisfied, \(\frac{\partial b}{\partial \phi}\) is initially less than 0 and then greater than 0 as \(\phi\) decreases from 1 to 0, which means that \(b\) first increases and then decreases.

Absolute level: Subjective probability first decreases and then increases with increasing overconfidence.

Relative level: Investors’ underestimation of the probability will increase and then decrease. Investors will overestimate the probability, and the overestimation is intensified with increasing overconfidence.

**Subcase III(4).** \(\frac{(\theta + \epsilon) - A}{\mu - (\theta + \epsilon)} > \frac{\sigma^2_\theta + \sigma^2_z}{\sigma^2_\theta + \sigma^2_z}\)

When subcase III(4) is satisfied, \(\frac{\partial b}{\partial \phi}\) > 0 holds, which means that \(b\) decreases monotonically as \(\phi\) decreases from 1 to 0.
Absolute level: Subjective probability increases monotonically with increasing overconfidence.

Relative level: Investors’ underestimation of the probability will weaken. Then, investors will overestimate the probability, and the overestimation is intensified with increasing overconfidence.

Based on the above analyses of case I, case II, and case III, we conclude the following.

**Conclusion 1.** Overconfidence distorts investors’ estimate of the probability of obtaining a better return. With increasing overconfidence, the subjective probability increases monotonically (case I, case II, and case III(2)(4)). Otherwise, after overconfidence reaches a certain level, the subjective probability increases monotonically with increasing overconfidence (case III(3)). Moreover, overconfidence can lead the investor to overestimate the probability, and the overestimation is intensified with increasing overconfidence.

For bullish investors, \( \theta + \varepsilon > A \) exists, and it can be shown that \( \lim_{\phi \to 0} E[\theta + \varepsilon] = \theta + \varepsilon \), \( \lim_{\phi \to 0} \text{Var}[\theta + \varepsilon] = 0 \), and \( \lim_{\phi \to 0} b = -\infty \); thus, the subjective probability of absolute overconfident investors is 1, \( \phi \to 0 \), namely, \( P(\theta + \varepsilon > A) = 1 \).

If the maturity price in the private signal is less than the trigger point, namely, \( \theta + \varepsilon < A \), investors tend to buy bearish products. The impact of overconfidence on the subjectively estimated probability of bearish investors is similar to that of bullish investors; thus, we do not need to repeat the analysis.

4. **The effect of overconfidence on perceived risk.** In this section, we further explore how the psychological bias of overconfidence affects investors’ perceived risk by influencing their expectations.

According to the assumptions regarding the properties of the structured products in section 3.1, the mean of the lottery can be calculated as follows:

\[
\bar{X} = Mr_1 P + Mr_2 (1 - P) = M[P(r_1 - r_2) + r_2].
\] (25)

According to the corresponding risk measures in the section 2, we can obtain the pure risk factor:

\[
X' = \begin{cases} 
Mr_1 - M[P(r_1 - r_2) + r_2] = M(r_1 - r_2)(1 - P) \\
Mr_2 - M[P(r_1 - r_2) + r_2] = -M(r_1 - r_2)P
\end{cases}.
\] (26)

Because the utility functions of investors in the market are very different, this paper uses the most classic utility function proposed by Tversky and Kahneman [37] to analyze perceived risk. Assume the following utility function:

\[
u(x) = \begin{cases} 
x, x \geq 0 \\
\lambda x, x < 0
\end{cases},
\] (27)

where \( \lambda > 1 \) is the loss aversion coefficient. A greater \( \lambda \) means investors have a greater degree of loss aversion. Investors are more sensitive to loss because a unit of loss has a greater impact on utility than a unit of gain.

The standard measure of risk \( R(X') \) is

\[
R(X') = -E[u(X')]
= \lambda M(r_1 - r_2)(1 - P)P - M(r_1 - r_2)P(1 - P)
= M(r_1 - r_2)P(1 - P)(\lambda - 1).
\] (28)
The perceived risk based on the standard measure of risk is

\[ \varphi(\bar{X})[R(X') - R(0)] = \varphi(\bar{X})[M(r_1 - r_2)P(1 - P)(\lambda - 1) - R(0)], \]  

where \( \varphi(\bar{X}) \) is greater than 0 and a decreasing function of the mean of lottery, and \( R(0) \) is a constant. 

The result of (29) shows that the perceived risk is an increasing function of the loss aversion coefficient, namely, the perceived risk increases with increasing loss aversion. To explore the effect of overconfidence on perceived risk, we must determine the properties of investors’ perceived risk towards binary structured products.

By determining the partial derivative of (29) with respect to \( P \), we obtain the following:

\[
\frac{d\varphi(\bar{X})[R(X') - R(0)]}{dP} = M(r_1 - r_2)(\lambda - 1)[\frac{d\varphi(\bar{X})}{d\bar{X}} M(r_1 - r_2)P(1 - P) + \varphi(\bar{X})(1 - 2P)]. \tag{30}
\]

Formula (30) is a quadratic function of the subjective probability \( P \) with a symmetric axis greater than \( \frac{1}{2} \). Additionally, the function has two zero points. It is easy to demonstrate that one zero point is between 0 and \( \frac{1}{2} \) and the other one is greater than 1. Thus, the perceived risk is monotonically increasing and then monotonically decreasing with the subjective probability of obtaining a better return regardless of the parameter values. Moreover, the perceived risk has a maximum point that is between 0 and \( \frac{1}{2} \). This property is consistent with intuition. Investors’ perceived risks are affected by two aspects of a structured product: the mean and the standard risk.

For a product with two scenarios, when the subjective probability of better return is less than \( \frac{1}{2} \), the standard risk increases with the probability; otherwise, the standard risk decreases with the probability. When the probability of obtaining a better return is equal to \( \frac{1}{2} \), uncertainty is at a maximum. Because perceived risk is the combination of the two effects, it will always show a right-skewed hump-shaped function, and the maximum point depends on the relative magnitude of the two effects.

To illustrate the effect of psychological bias on perceived risk clearly, we set the parameters as follows to draw a general figure of perceived risk: \( \varphi(\bar{X}) = \frac{10}{[\ln(\bar{Y})]^2} \), \( M = 100 \), \( r_1 = 5\% \), \( r_2 = 3\% \), and \( \lambda = 2.25 \).

The subjectively estimated probability of investors is \( P_a = \int_0^{+\infty} f(x) \, dx \). When \( \phi \to 1 \), the probability is denoted by \( P_s1 \); when \( \phi \to 0 \), the probability is \( P_s0 \).

**Case I. \( \mu < A < \theta + \varepsilon \)**

When case I is satisfied, \( P_{s1} < P_{s0} \) and \( \frac{\partial b}{\partial \varepsilon} > 0 \) hold, which means that \( b \) decreases monotonically with decreasing \( \phi \) (\( \phi: 1 \to 0 \)), and the subjective probability of bullish investors increases monotonically with increasing overconfidence.

When \( \phi = 1 \), \( b = \frac{(A-y)\sigma_y^2+(A-\mu)\sigma_x^2}{\sigma_x\sigma_y} \sqrt{\sigma_y^2+\sigma_x^2} \) (where \( y = \theta + \varepsilon \)). Then, it can be proven that \( b > 0 \leftrightarrow (A-y)\sigma_y^2+(A-\mu)\sigma_x^2 > 0 \).

With only \( \mu < A < \theta + \varepsilon \), we cannot determine whether \( (A-y)\sigma_y^2+(A-\mu)\sigma_x^2 \) is positive or negative, namely, the sign of \( b \). Thus, when case I is satisfied, the relative position of \( P_{s1} \) and the maximum point of the perceived risk cannot be determined.
The general change in perceived risk is shown in Figure 6 for increasing overconfidence (decreasing $\phi$).

Figure 6. Perceived risk of overconfident bullish investors of type I

When $P_{s1}$ is less than the maximum point, perceived risk first increases and then decreases with increasing overconfidence; after overconfidence reaches a certain level, perceived risk decreases monotonically with increasing overconfidence. When $P_{s1}$ is greater than the maximum point, perceived risk decreases monotonically with increasing overconfidence.

Case II & Case III(2)(4). When Case II & Case III(2)(4) are satisfied, $\frac{1}{2} < P_{s1} < P_{s0} = 1$ and $\frac{\partial b}{\partial \phi} > 0$ hold, which means that $b$ decreases monotonically with decreasing $\phi$, ($\phi$: 1 $\rightarrow$ 0), and the subjective probability of bullish investors increases monotonically with increasing overconfidence.

Figure 7. Perceived risk of overconfident bullish investors of type II or type III(2)(4)

As shown in Figure 7, perceived risk decreases monotonically with increasing overconfidence (decreasing $\phi$).

Case III(3). When case III(3) is satisfied, $\frac{1}{2} < P_{inflexion} < P_{s1} < P_{s0} = 1$ holds. $\frac{\partial b}{\partial \phi}$ is first less than 0 and then greater than 0 as $\phi$ decreases from 1 to 0, which means that $b$ first increases and then decreases. The subjective probability of structured product investors first decreases and then increases with increasing overconfidence.

As shown in Figure 8, perceived risk first increases and then decreases with increasing overconfidence (decreasing $\phi$). After overconfidence reaches a certain level, the perceived risk decreases monotonically with increasing overconfidence.
Figure 8. Perceived risk of overconfident bullish investors of type III(3)

$P_{inflexion}$ is a subjective probability corresponding to the maximum point of $b$. In case III, $P_{inflexion} > \frac{1}{2}$ always holds.

Based on the above analysis of different cases, we can obtain the following conclusion.

**Conclusion 2.** The psychological bias of overconfidence can lead investors to perceive lower risk. The perceived risk decreases with increasing overconfidence (case II and case III(2)(4)). Otherwise, after overconfidence reaches a certain level, the perceived risk decreases with increasing overconfidence (case I and case III(3)).

**5. Conclusion.** For individual investors, it is important to quantify and optimize their investment portfolio risks. Particularly for structured financial product investors, the payoff formulas of these products may make it difficult to judge their returns at maturity. This paper uses measures of perceived risk based on a standard measure of risk to characterize how the perceived risk of structured product investors forms. Structured financial products are actually lotteries with different return scenarios and corresponding probabilities.

Considering a psychological bias, overconfident investors ignore public information and give higher psychological weight to private information. They underestimate the variance of the noise in private information and follow Bayesian law to predict the distribution of the expiring price of the underlying asset. Investors of structured products predict the maturity price of the underlying asset based on information, although this information does not have a substantial impact on the underlying asset price. According to the conditional distribution properties, it can be proven that the expectation of investors based on their private information will obey a normal distribution. Additionally, the subjective expectation is influenced by both the private signal and the degree of overconfidence. We can use the mean and variance to analyze how overconfidence affects investors’ expectations because a subjective expectation follows a normal distribution, which in turn makes the mechanism for how overconfidence affects the probability estimate clearer. The results suggest that overconfidence affects structured product investors’ perceived risks by influencing their expectations of maturity prices and probabilities of obtaining a better return. Overconfidence can lead investors to overestimate the probability, and the overestimation is intensified with increasing overconfidence. Simultaneously, overconfidence can lead the investors’ subjectively estimated probability to increase, which finally contributes to a lower perceived risk. Thus, the psychological bias plays an important role in investors’ risk-value optimization process.
This paper provides a new perspective for the study of probability distortion. The distorted judgment of the probability of structured financial product investors is related to the private information they receive, the level of overconfidence, and the trigger point designed in structured financial product, and it is also related to the relatively “good” and “bad” scenarios. Investors tend to overestimate the probability of obtaining a better payoff, which is more consistent with reality. Moreover, micro-level decision-making models can provide a theoretical basis for the computational financial experiments for the macro-market. That is, heterogeneous agents have different utility functions, so they will choose different structured products to maximize their returns according to different perceived risk functions, preference functions, and private signals. Thus, this study helps to build primary and secondary analog markets of structured financial products for further research.

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