Gosper summability of rational multiples of hypergeometric terms

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ABSTRACT

By the telescoping method, Sun has recently given some hypergeometric series whose sums are related to $\pi$. We investigate these series from the point of view of Gosper’s algorithm. Given a hypergeometric term $t_k$, we consider the Gosper summability of $r(k)t_k$ for $r(k)$ being a rational function of $k$. We give an upper bound and a lower bound on the degree of the numerator of $r(k)$ such that $r(k)t_k$ is Gosper summable. We also show that the denominator of $r(k)$ can be read off from the Gosper representation of $t_{k+1}/t_k$. Based on these results, we give a systematic method to construct series whose sums can be derived from the known ones. We also illustrate the corresponding super-congruences and the $q$-analogue of the approach.

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1. Introduction

In [12], Sun derived several identities involving $\pi$ by the telescoping method. For example, from Bauer’s series [1]

$$
\sum_{k=0}^{\infty} \frac{(2k)^3}{(64)^k} = \frac{2}{\pi}
$$

and the telescoping sum

$$
\sum_{k=0}^{n} \frac{(2k)^2}{(2k-1)^2(-64)^k} = \frac{8(2n+1)}{(64)^n} \left(\frac{2n}{n}\right)^3,
$$

he deduced

$$
\sum_{k=0}^{\infty} \frac{k(4k-1)(2k)^3}{(2k-1)^2(-64)^k} = -\frac{1}{\pi}.
$$

(1)

We aim to give a systematic method to construct series like (1). This motivates us to consider the following problem: Given a hypergeometric term $t_k$, for which rational functions $r(k)$ is the product $r(k)t_k$ Gosper summable?
Recall that a hypergeometric term $t_k$ is Gosper summable [4] (in brief, summable) if there exists a hypergeometric term $z_k$ such that

$$t_k = z_{k+1} - z_k.$$  

The well-known Gosper’s algorithm [11, Chap. 5] finds $z_k$ if it exists. More precisely, suppose that

$$\frac{t_{k+1}}{t_k} = \frac{a(k) c(k+1)}{b(k) c(k)},$$

where $a(k), b(k), c(k)$ are polynomials such that

$$\gcd(a(k), b(k+h)) = 1, \quad \forall \ h \geq 0, \ h \in \mathbb{Z}.$$  

Then $z_k$ exists if and only if there exists a polynomial $x(k)$ such that

$$a(k)x(k+1) - b(k-1)x(k) = c(k). \quad (2)$$

In many cases, $t_k$ is not summable itself. However, if we multiply it with a suitable rational function $r(k)$, the product $r(k)t_k$ will be summable. Let $K$ be a field of characteristic zero. By considering the space

$$\{a(k)x(k+1) - b(k-1)x(k) \mid x(k) \in K[k]\},$$

we derive an upper bound and a lower bound on the degree of the numerator of $r(k)$ in Section 2. By the aid of Gosper’s algorithm, we also give candidates for the denominator of $r(k)$.

Based on this discussion, we are able to give a systematic method to construct series of the form (1) in Section 3. We not only recover the series given in [12], but also find some new ones. Moreover, we derive some super-congruences for the partial sums of the series and illustrate the $q$-analogue of this approach by an example.

### 2. The numerator and the denominator of $r(k)$

Let $t_k$ be a hypergeometric term over a field $K$ of characteristic zero. That is, $t_{k+1}/t_k$ is a rational function of $k$ over $K$. In this section, we will give some properties of the rational function $r(k)$ such that $r(k)t_k$ is summable.

Suppose

$$\frac{t_{k+1}}{t_k} = \frac{a(k) c(k+1)}{b(k) c(k)}, \quad (3)$$

where $a(k), b(k), c(k) \in K[k] \setminus \{0\}$ and

$$\gcd(a(k), b(k+h)) = 1, \quad \forall \ h \geq 0, \ h \in \mathbb{Z}, \quad (4)$$

we say $(a(k), b(k), c(k))$ is a Gosper representation of $t_k$. 
We first consider the case when \( r(k) \) is a polynomial. In this case, we write \( p(k) \) instead of \( r(k) \). To give an upper bound on the degree of \( p(k) \), we need the concept of degeneration introduced in [6]. Let
\[
  u(k) = a(k) - b(k - 1),
\]
and
\[
  d = \max\{\deg u(k), \deg a(k) - 1\}.
\]

The pair \((a(k), b(k))\) is said to be degenerated if
\[
  \deg u(k) = \deg a(k) - 1 \quad \text{and} \quad -lc(u(k))/lc(a(k)) \in \mathbb{N},
\]
where \( lc(f(k)) \) denotes the leading coefficient of \( f(k) \).

An upper bound on the degree of \( p(k) \) can be given in terms of \( d \).

**Theorem 2.1:** Let \( t_k \) be a hypergeometric term with a Gosper representation \((a(k), b(k), c(k))\). Then there exists a non-zero polynomial \( p(k) \) such that \( p(k)t_k \) is summable. Moreover, an upper bound on the degree of \( p(k) \) is given by
\[
  B = \begin{cases} 
  d + 1, & \text{if } (a(k), b(k)) \text{ is degenerated or } \deg u(k) < \deg a(k) - 1, \\
  d, & \text{otherwise.}
\end{cases}
\]

**Proof:** Let
\[
  S_{a,b} = \{a(k)x(k + 1) - b(k - 1)x(k) \mid x(k) \in K[k]\}.
\]
By Theorem 2.3 of [6], the dimension of the quotient space \( K[k]/S_{a,b} \) is bounded by \( B \). Therefore, the \( B + 1 \) vectors
\[
  \overline{c(k)}, \overline{k \cdot c(k)}, \ldots, \overline{k^B \cdot c(k)}
\]
are linearly dependent in \( K[k]/S_{a,b} \). Hence there exists a non-zero polynomial \( p(k) \) of degree less than or equal to \( B \) such that \( c(k)p(k) \in S_{a,b} \), implying that \( p(k)t_k \) is summable.

As a corollary, we have

**Corollary 2.2:** Let \( t_k \) be a hypergeometric term and \( q(k) \) be a non-zero polynomial. Suppose \((a(k), b(k), c(k))\) is a Gosper representation of \( t_k/q(k) \) and \( B \) is given by (7). Then there exists a non-zero polynomial \( p(k) \) of degree no more than \( B \) such that \( p(k)t_k/q(k) \) is summable.

**Remark 2.1:** When \( a(k), b(k) \) are shift-free, i.e. \( \gcd(a(k), b(k + h)) = 1 \) for any \( h \in \mathbb{Z} \), Chen, Huang, Kauers and Li [3] showed that one may take \( d \) as the upper bound.

The following examples show that when \( a(k), b(k) \) are not shift-free, we need to take \( d + 1 \) as the upper bound.
Example 2.1: Let
\[ t_k = \frac{k}{(k+1)^2(k+2)}. \]
A Gosper representation of \( t_k \) is given by
\[ \frac{a(k)}{b(k)} \cdot \frac{c(k+1)}{c(k)} = \frac{(k+1)^2}{(k+2)(k+3)} \cdot \frac{k+1}{k}. \]
One sees that \((a(k), b(k))\) is degenerated and \( d = 1 \). But there are no polynomials \( p(k) \) of degree less than or equal 1 such that \( p(k)t_k \) is summable.

Let
\[ t_k = \frac{1}{(k+1)^2}. \]
A Gosper representation of \( t_k \) is given by
\[ \frac{a(k)}{b(k)} \cdot \frac{c(k+1)}{c(k)} = \frac{(k+1)^2}{(k+2)^2} \cdot \frac{1}{1}. \]
One sees that \( \deg u(k) < \deg a(k) - 1 \) and \( d = 1 \). But there are no polynomials \( p(k) \) of degree less than or equal 1 such that \( p(k)t_k \) is summable.

Of course, it is possible that there exists a polynomial \( p(k) \) of degree less than \( B \) such that \( p(k)t_k \) is summable.

Example 2.2: Let
\[ t_k = \frac{(2k)^4}{(2k-1)^4256^k}. \]
A Gosper representation of \( t_k \) is given by
\[ \frac{a(k)}{b(k)} \cdot \frac{c(k+1)}{c(k)} = \frac{(2k-1)^4}{16(k+1)^4} \cdot \frac{1}{1}. \]
We find that \( d = 3 \) and \( B = 4 \), while \((4k-1)t_k \) is summable, although the degree of \( p(k) \) is much smaller than the upper bound.

In some cases, the upper bound \( B \) is also a lower bound.

Theorem 2.3: Let \( t_k \) be a hypergeometric term with a Gosper representation \((a(k), b(k), c(k))\) and let \( u(k) \) be given by (5). Suppose that
\[ \deg u(k) = \max\{\deg a(k), \deg b(k)\} \quad \text{and} \quad c(k) = 1. \quad (8) \]
If \( p(k)t_k \) is summable, then
\[ \deg p(k) \geq \max\{\deg a(k), \deg b(k)\}. \]
**Proof:** Notice that
\[ \deg u(k) = \max\{\deg a(k), \deg b(k)\} \]
implies that for any polynomial \( x(k) \),
\[ \deg (a(k)x(k + 1) - b(k - 1)x(k)) = \max\{\deg a(k), \deg b(k)\} + \deg x(k), \]
while \( p(k)t_k \) is summable if and only if there exists \( x(k) \) such that
\[ a(k)x(k + 1) - b(k - 1)x(k) = p(k). \]
Therefore,
\[ \deg p(k) \geq \max\{\deg a(k), \deg b(k)\} = \deg u(k). \]

Notice that
\[ \deg u(k) < \deg a(k) \implies \deg a(k) = \deg b(k). \]
Hence the condition \( \deg u(k) = \max\{\deg a(k), \deg b(k)\} \) implies that \( B = d \) in Theorem 2.1.

**Example 2.3:** Let
\[ t_k = \frac{(2k)^2}{64^k}. \]
A Gosper representation of \( t_k \) is given by
\[ \frac{a(k)}{b(k)} \cdot \frac{c(k + 1)}{c(k)} = \frac{(2k + 1)^2}{16(k + 1)^2} \cdot \frac{1}{1}. \]
We find that \( B = d = 2 \) and the quadratic polynomial \( p(k) = 1 + 4k - 12k^2 \) satisfies the condition that \( p(k)t_k \) is summable.

Example 2.3 shows that the condition \( \deg u(k) = \max\{\deg a(k), \deg b(k)\} \) is necessary. The following example indicates that the condition \( c(k) \neq 1 \) is also necessary.

**Example 2.4:** Let
\[ t_k = (3k + 2)\binom{2k}{k}. \]
A Gosper representation of \( t_k \) is given by
\[ \frac{a(k)}{b(k)} \cdot \frac{c(k + 1)}{c(k)} = \frac{2(2k + 1)}{(k + 1)} \cdot \frac{3k + 5}{3k + 2}. \]
It is easy to check that \( B = d = 1 \), but \( t_k \) itself is summable.

Now we consider the denominator of \( r(k) \) such that \( r(k)t_k \) is summable.
Theorem 2.4: Let \( t_k \) be a hypergeometric term with a Gosper representation \( (a(k), b(k), c(k)) \) and let \( p(k), q(k) \) be two polynomials such that \( p(k)t_k/q(k) \) is summable. Suppose that
\[
gcd(q(k), a(k-1-h)) = gcd(q(k), b(k+h)) = 1, \quad \forall h \in \mathbb{N} \tag{9}
\]
and
\[
gcd(q(k), c(k)) = 1. \tag{10}
\]
Suppose further that
\[
gcd(q(k), q(k+1+h)) = 1, \quad \forall h \in \mathbb{N}. \tag{11}
\]
Then we have \( q(k) \mid p(k) \).

Proof: Let \( \hat{t}_k = p(k)t_k/q(k) \). We have
\[
\frac{\hat{t}_{k+1}}{\hat{t}_k} = \frac{a(k)q(k)}{b(k)q(k+1)} \cdot \frac{c(k+1)p(k+1)}{c(k)p(k)}.
\]
By (9) and (11), we have
\[
gcd(a(k)q(k), b(k+h)q(k+1+h)) = gcd(a(k), b(k+h)q(k+1+h))
\]
\[
= gcd(a(k), b(k+h)) = 1, \quad \forall h \in \mathbb{N}.
\]
Therefore, \( (a(k)q(k), b(k)q(k+1), c(k)p(k)) \) is a Gosper representation of \( p(k)t_k/q(k) \).
Hence \( \hat{t}_k \) is summable if and only if there exists a polynomial \( x(k) \) such that
\[
q(k)(a(k)x(k+1) - b(k-1)x(k)) = c(k)p(k).
\]
By (10), we derive that \( q(k) \mid p(k) \). \( \blacksquare \)

The following example shows that condition (11) is necessary.

Example 2.5: Let
\[
t_k = \frac{16^k}{\binom{2k}{k}^2}.
\]
A Gosper representation of \( t_k \) is given by
\[
\frac{a(k)}{b(k)} \cdot \frac{c(k+1)}{c(k)} = \frac{4(k+1)^2}{(2k+1)^2} \cdot \frac{1}{1}.
\]
Let
\[
r(k) = \frac{p(k)}{q(k)} = \frac{1 - 40k - 32k^2}{(4k+1)(4k+5)}.
\]
It is easy to check that \( r(k)t_k \) is summable and \( q(k) \) satisfies (9) and (10).

Let \( r(k) \) be a rational function such that \( r(k)t_k \) is summable. Write \( r(k) = p(k)/q(k) \) in reduced form, i.e. \( gcd(p(k), q(k)) = 1 \), and assume that \( q(k) \) satisfies (11). Then Theorem 2.4 tells us that \( q(k) \) contains a factor of \( c(k), a(k-1-h) \), or \( b(k+h) \) for some \( h \in \mathbb{N} \).
3. Constructing new hypergeometric series

In this section, we will use the results from Section 2 to give a systematic method of constructing series similar to (1).

Given a hypergeometric series whose sum is known

\[ \sum_{k=0}^{\infty} t_k = C. \]

We first compute a Gosper representation \((a(k), b(k), c(k))\) of \(t_k\) and make \(c(k)\) to be 1 by setting \(\hat{t}_k = t_k/c(k)\). Then we try to find rational functions \(r(k) = p(k)/q(k)\) such that \((c(k) + r(k))\hat{t}_k\) is summable. Theorem 2.4 tells us that a good choice for \(q(k)\) is a factor of \(a(k - 1)b(k)\). Given \(q(k)\), we can search for \(p(k)\) by applying the Extended Zeilberger’s algorithm [2] to

\[ t_k, \hat{t}_k/q(k), \hat{k}\hat{t}_k/q(k), \ldots, k^m\hat{t}_k/q(k). \]

In fact, the algorithm finds \(a_0, a_1, \ldots, a_m\) and \(g(k)\) such that

\[ t_k + (a_0 + a_1k + \cdots + a_mk^m)\hat{t}_k/q(k) = g(k + 1) - g(k). \]

One can also use the reduction based creative telescoping proposed by Chen, Huang, Kauers and Li [3] to find the polynomial \(p(k)\). Notice that Corollary 2.2 ensures the existence of non-trivial \(a_i's\) and Theorem 2.3 gives a lower bound for the degree \(m\). Let

\[ p(k) = a_0 + a_1k + \cdots + a_mk^m. \]

We finally derive a new hypergeometric series

\[ \sum_{k=0}^{\infty} \frac{p(k)}{q(k)c(k)} t_k = -C + \lim_{k \to \infty} g(k) - g(0). \]

Note that when \(\deg p(k) - \deg q(k) < \deg c(k)\), the new series converges faster than the original one.

We will illustrate the method by the following example.

Example 3.1: Consider Bauer’s series

\[ \sum_{k=0}^{\infty} \frac{(4k + 1)(2k)^3}{(-64)^k} = \frac{2}{\pi}. \]

Let

\[ t_k = \frac{(4k + 1)(2k)^3}{(-64)^k}. \]

A Gosper representation of \(t_k\) is given by

\[ \frac{a(k)}{b(k)} \cdot \frac{c(k + 1)}{c(k)} = \frac{-(2k + 1)^3}{8(k + 1)^3} \cdot \frac{4k + 5}{4k + 1}. \]
So we set
\[ \hat{t}_k = \frac{t_k}{4k + 1} = \frac{\left(\frac{2k}{k}\right)^3}{(-64)^k}. \]

We first take \( q(k) \) to be the factor \((2k - 1)^2\) of \( a(k - 1) \). In this case, we find that
\[ t_k + \frac{(8k^2 - 2k)}{(2k - 1)^2} t_k = g(k + 1) - g(k), \]
with
\[ g(k) = -\frac{8k^3}{(2k - 1)^2} \left(\frac{2k}{k}\right)^3. \] \hspace{1cm} \text{(12)}

Noting that \( \lim_{k \to \infty} g(k) = g(0) = 0 \), we thus derive from Bauer’s series
\[ \sum_{k=0}^{\infty} \frac{(4k - 1)k}{(2k - 1)^2} \left(\frac{2k}{k}\right)^3 (-64)^k = -\frac{1}{\pi}, \]
which is exactly (1).

Then we take \( q(k) = (k + 1)^2 \) which is a factor of \( b(k) \). This time we find that
\[ t_k - \frac{1}{4} \frac{(4k + 3)(2k + 1)}{(k + 1)^2} t_k = g(k + 1) - g(k), \]
with
\[ g(k) = -2k \left(\frac{2k}{k}\right)^3. \] \hspace{1cm} \text{(13)}

We thus derive from Bauer’s series
\[ \sum_{k=0}^{\infty} \frac{(4k + 3)(2k + 1)}{(k + 1)^2} \left(\frac{2k}{k}\right)^3 (-64)^k = \frac{8}{\pi}. \] \hspace{1cm} \text{(14)}

Finally we take \( q(k) = (k + 1)(2k - 1) \). In this case, we discover the identity
\[ \sum_{k=0}^{\infty} \frac{4k + 1}{(k + 1)(2k - 1)} \left(\frac{2k}{k}\right)^3 (-64)^k = -\frac{4}{\pi}. \] \hspace{1cm} \text{(15)}

Similarly, when \( q(k) \) is of degree 3, we get two more identities
\[ \sum_{k=0}^{\infty} \frac{4k - 1}{(2k - 1)^3} \left(\frac{2k}{k}\right)^3 = \frac{2}{\pi}, \]
\[ \sum_{k=0}^{\infty} \frac{4k + 3}{(k + 1)^3} \left(\frac{2k}{k}\right)^3 = 8 - \frac{16}{\pi}, \]
where the constant 8 comes from \( g(0) \).
Here is another example, in which the new series may not start from the first term of the original one.

**Example 3.2:** In 1993, Zeilberger [13] used the WZ method to show that

\[ \sum_{k=1}^{\infty} \frac{21k - 8}{k^3 \left( \begin{array}{c} 2k \\ k \end{array} \right)^3} = \frac{\pi^2}{6}. \]

In this case, we have

\[ \frac{a(k)}{b(k)} \cdot c(k + 1) = \frac{k^3}{8(2k + 1)^3} \cdot \frac{21k + 13}{21k - 8}, \]

and

\[ \hat{t}_k = \frac{1}{k^3 \left( \begin{array}{c} 2k \\ k \end{array} \right)^3}. \]

Taking \( q(k) = (2k + 1)^2 \), we derive

\[ \sum_{k=1}^{\infty} \frac{28k^2 + 31k + 8}{(2k + 1)^2 k^3 \left( \begin{array}{c} 2k \\ k \end{array} \right)^3} = \frac{\pi^2 - 8}{2}. \] (16)

which is found by Sun [11]. If we take \( q(k) = (k - 1)^2 \), the series should start from \( k = 2 \). We thus derive

\[ \sum_{k=2}^{\infty} \frac{7k^2 - 8k + 2}{(k - 1)^2 k^3 \left( \begin{array}{c} 2k \\ k \end{array} \right)^3} = \frac{10 - \pi^2}{16}. \] (17)

By considering the congruent properties of the partial sums of the series and the boundary values, we are able to derive some super-congruences.

**Example 3.3:** For primes \( p \geq 3 \), Bauer’s series has a nice \( p \)-adic analogue [10]

\[ \sum_{k=0}^{p-1} \frac{(4k + 1) \left( \begin{array}{c} 2k \\ k \end{array} \right)^3}{(-64)^k} \equiv p \left( \frac{-1}{p} \right) \pmod{p^3}. \]

By Example 3.1, we have

\[ \frac{(4k + 1) \left( \begin{array}{c} 2k \\ k \end{array} \right)^3}{(-64)^k} + \frac{(8k^2 - 2k) \left( \begin{array}{c} 2k \\ k \end{array} \right)^3}{(2k - 1)^2 (-64)^k} = g(k + 1) - g(k), \]

where

\[ g(k) = -\frac{8k^3 \left( \begin{array}{c} 2k \\ k \end{array} \right)^3}{(2k - 1)^2 (-64)^k}. \]

It is known [9] that for prime \( p > 3 \),

\[ \left( \frac{p - 1}{p - 1} \right) \equiv (-1) \frac{\sum_{i=0}^{p-1} 2^{2p-2}}{p^3} \pmod{p^3}. \]
Therefore,
\[ g \left( \frac{p + 1}{2} \right) = (-64)^{-(p-1)/2} p \left( \frac{p - 1}{p - 1/2} \right)^3 \equiv p 8^{p-1} \pmod{p^4}. \] (18)

Summing Equation (18) on \( k \) from 0 to \((p - 1)/2\), we derive that
\[ \sum_{k=0}^{p-1} \frac{(8k^2 - 2k) (2k)^3}{(2k - 1)^2 (-64)^k} \equiv -p \left( \frac{-1}{p} \right) + p 8^{p-1} \pmod{p^3}. \] (19)

Note that when \( p = 3 \), the above congruence also holds.

In a similar way, we derive that
\[ \sum_{k=0}^{p-1} \frac{(4k + 3)(2k + 1)}{4(k + 1)^2} \left( \frac{2k}{64} \right)^3 \equiv p \left( \frac{-1}{p} \right) \pmod{p^3}, \] (20)

and
\[ \sum_{k=0}^{p-1} \frac{4k + 1}{2(k + 1)(2k - 1)} \left( \frac{2k}{64} \right)^3 \equiv p^2 - p \left( \frac{-1}{p} \right) \pmod{p^3}. \] (21)

We remark that all the discussion works smoothly for the \( q \)-analogue \([8]\) of hypergeometric series. We will give an example of the \( q \)-case to conclude the paper.

**Example 3.4:** A \( q \)-analogue of Bauer’s identity is given by \([5, Eq.(1.5)]\)
\[ \sum_{k=0}^{\infty} \frac{(-1)^k (1 - q^{4k+1}) q^{k^2}}{1 - q} \frac{(q; q^2)_k^3}{(q^2; q^2)_k^3} = \frac{(q; q^2)_\infty (q^3; q^2)_\infty}{(q^2; q^2)_\infty}. \]

Let
\[ t_k = \frac{(-1)^k (1 - q^{4k+1}) q^{k^2}}{1 - q} \frac{(q^2; q^2)_k^3}{(q^2; q^2)_k^3}. \]

A \( q \)-Gosper representation of \( t_k \) is given by
\[ \frac{a(k)}{b(k)} \cdot \frac{c(k + 1)}{c(k)} = -q^{2k+1} (1 - q^{2k+1})^3 \cdot \frac{1 - q^{4k+5}}{1 - q^{4k+1}}, \]
so that
\[ \hat{t}_k = \frac{(-1)^k q^{k^2}}{1 - q} \frac{(q^2; q^2)_k^3}{(q^2; q^2)_k^3}. \]

Taking \( q(k) = (1 - q^{2k-1})^2 \), we derive
\[ \sum_{k=0}^{\infty} \frac{(-1)^k (1 - q^{2k})(1 - q^{4k-1}) q^{k^2 - 1}}{(1 - q^{2k-1})^2} \frac{(q^2; q^2)_k^3}{(q^2; q^2)_k^3} = -\frac{(q; q^2)_\infty (q^3; q^2)_\infty}{(q^2; q^2)_\infty}, \] (22)
which is a \( q \)-analogue of (1) and Equation (1.9) of [7].
Taking \( q(k) = (1 - q^{2k-1})(1 - q^{2k+2}) \), we derive

\[
\sum_{k=0}^{\infty} \frac{(-1)^k q^{k^2+2k-1}(1 - q^{4k+1})}{(1 - q^{2k-1})(1 - q^{2k+2})} \frac{(q^3; q^2)_k^3}{(q^2; q^2)_k^3} = -\frac{(q^3; q^2)_\infty^2}{(q^2; q^2)_\infty^2},
\]

which is a \( q \)-analogue of (15).

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