Classification of Topological Excitations in Quadratic Bosonic Systems

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We investigate the classification of topological excitations in quadratic bosonic systems with an excitation band gap. Time-reversal, charge-conjugation, and parity symmetries in bosonic systems are introduced to realize a ten-fold symmetry classification. We find a specific decomposition of the quadratic bosonic Hamiltonian and use it to prove that each quadratic bosonic system is homotopic to a direct sum of two single-particle subsystems. The classification table of topological excitations is thus derived via inheriting from that of Atland-Zirnbauer classes and unique topological phases of bosons are predicted. Finally, concrete bosonic models are proposed to demonstrate the peculiarity of bosonic topological excitations.

Introduction. Searching for topological phases of a many-body system with specific symmetries becomes an important issue in both condensed matter and cold atom physics. As a milestone work, gapped free-fermion systems including the Chern insulator [1], topological insulator [2–5] and topological superconductor [6–9] are categorized into ten Atland-Zirnbauer (AZ) symmetry classes according to the time-reversal, charge-conjugation, and chiral symmetries [10, 11]. The relevant topological phases are classified by a periodic table of K-theory [12–15]. Recently, the concept of topological phases has also been extended to dynamical [16–19] and open quantum-mechanical systems [20–24]. The classifications of Floquet topological insulator [25] and non-Hermitian systems [26, 27] are also established based on similar classification principles.

Parallel to the fermionic insulator, topological phases also emerge from excitations of a bosonic system. Topological excitations (TEs) of bosons not only are attainable from the simulation of single-particle topological bands [28–39] but also are discovered in peculiar bosonic systems without fermionic analogy. It is reported that the 2D bosonic Bogoliubov-de Gennes (BdG) model containing two-boson annihilation/creation interactions [40–42] is capable of hosting excitation bands with non-vanishing Chern numbers, which is realizable in magnonic crystals [43–46], nonlinear bosonic systems [47], photonic crystals [48], and ultracold bosonic atoms in optical lattices [49–52]. The bosonic excitation modes are obtained by pseudo-unitary diagonalization due to the bosonic commutation relation [53, 54]. Thus, the topological obstruction, i.e., the energy gap, is defined in an exotic way. Moreover, the stability of bosonic Hamiltonians requires the semi-positive definiteness that brings a limitation on the symmetries. These facts suggest that the bosonic system can potentially generate peculiar TEs beyond the common AZ symmetry and topological classification.

Then a problem naturally arise: It remains unclear if the bosonic TEs exist in other dimension and symmetry class. Hence it is of great necessity to achieve the classifications of the symmetries and TEs in bosonic systems.

In this Letter, we focus on excitations of a quadratic bosonic system (QBS) with an excitation band gap and systematically classify their topological phases according to symmetries. We firstly inherit the AZ classification scheme to introduce the time reversal, charge conjugation, and their composite interpreted as parity and generate a ten-fold symmetry classification for the QBS. We then explore the topological structure of the bosonic Hamiltonian via a specific decomposition which reveals that each QBS is homotopic to a direct sum of two single-particle subsystems. Therefore, the classification table of bosonic TEs is derived via the periodic table of AZ classes. We further apply these results to predict unique topological phases of bosons and construct concrete bosonic models without fermionic single-particle counterpart. Our work opens a route of exploring the topological phases and effects of bosons.

Model for QBS. We consider a quadratic Hamiltonian \( \mathcal{H} = \sum_k \phi_+^{\dagger}(k) H(k) \phi(k) \) composed by bosonic field operators \( \phi^{\dagger}(k) = (a_1^{\dagger}(k) \ b_1(−k) ) \) and an Hermitian matrix \( H(k) \) which is continuous with regard to wave vector \( k \). Here, \( a_1^{\dagger} = (a_1^{\dagger} \ldots a_N^{\dagger}) \) and \( b_1 = (b_1 \ldots b_N) \) are bosonic creation and annihilation operators, respectively. The field operators obey bosonic commutation relations

\[
[\phi_i(k), \phi_j^\dagger(\bar{k})] = \tau_{ij}, \quad \tau = \begin{pmatrix} \mathbb{I}_N & \mathbb{I}_N \cr \mathbb{I}_N & -\mathbb{I}_N \end{pmatrix},
\]

where \( \mathbb{I}_N \) denotes \( N \times N \) identity matrix. To stabilize the system, \( H(k) \) is required to be semi-positive definite. This general QBS has included the single-particle system when \( b(k) \) vanishes [29, 35, 36, 38, 39, 55] and the widely studied bosonic BdG system when \( b_i(k) = a_i(k) \) (\( N = N' \)) [43, 47–52].

We aim to investigate the excitation bands on top of a bosonic ground state, which are solved via a linear transformation \( \phi(k) = V(k) \psi(k) \). To satisfy the bosonic commutation relation Eq. (1), the transformation matrix needs to obey \( V^\dagger \tau V = \tau \) and forms a pseudo-unitary group \( \mathbb{U}(N, N') \). As a mathematical theorem [54], each positive definite Hermitian matrix \( H \) is pseudo-unitarily congruent to a positive diagonal matrix \( \Lambda \), i.e.,

\[
V^\dagger HV = \Lambda, \quad V \in \mathbb{U}(N, N').
\]
Then the positive definite Hamiltonian takes a decoupled form $H = \sum_k \psi_k (\Lambda(k) \psi(k))$ with excitation modes $\psi(k)$ and energy spectra $\Lambda(k)$. The Hamiltonian with zero excitation modes can be regarded as a limit case and suits the same treatment. To generate a topological obstruction, we assume an excitation band gap such that several energy bands are always lower than the others. It leads to the appearance of bosonic TE(s) which cannot be continuously mapped to a trivial flat-band model when the gap keeps open and the symmetries keep invariant.

**Symmetry classification.** For the symmetry classification of the QBS, we inherit the AZ classification scheme to introduce time-reversal $T$, charge-conjugation $C$, and their composite $P = T \cdot C$ symmetries. The time-reversal operator is anti-unitary, i.e., $T i T^{-1} = -i$. The charge-conjugation operator is only adaptive to the case of $N = N'$, which is unitary and capable of reversing the sign of the charge $Q = \sum_{i,k} (a_{i,k}^+ a_{i,k} - b_{i,k}^+ b_{i,k})$, i.e., $CQC^{-1} = -Q$. Their physical properties are summarized in Tab. I. In the first four columns, we specify the actions of $O = T, C, P$ upon the field operator, i.e., $O \phi(k) O^{-1}$, and derive the symmetric constraint of an $O$-symmetric Hamiltonian ($OHO^{-1} = H$) which takes the form of $OH(k)O^{-1} = H(\pm k)$. Here we redefine the symmetry operators $O = T, C, P$ acting on $H(k)$ instead of $H$, in which $T, C$ are anti-unitary and $P$ becomes unitary. We also find the structure of $O, i.e., O\tau O^{-1} = \pm \tau$, which results from the $\tau\tau$-action upon $H(k)$, i.e., $[O \phi(k) O^{-1}, O \phi(k) O^{-1}] = \tau_{ij}$. Furthermore, we assume $T, C, P$ as involutive operators (twice action making any system return to itself) and obtain the fifth column by applying Schur’s lemma. The representation matrices of $U_{T,C,P}$ that satisfy the above properties are provided in the last column.

Comparing to the AZ classification [10, 15], we find three different features: (1) the symmetric constraints for $T$ and $C$ take the same form; (2) $T$ and $C$ are identified by their relations to $\tau$; (3) $P$ should be named parity according to the symmetric constraint, in contrast to the fermionic case where $T \cdot C$ is interpreted as chirality. This is because the fermionic anti-commutation relation is replaced by the bosonic commutation relation.

Based on the presence or absence of these three symmetries, the symmetry classification of the QBS is listed in the first four columns of Tab. II. This result resembles the AZ ten-fold symmetry classification for fermions. Nevertheless, the symmetry classes correspond to repeated classifying spaces as revealed later. Thus, different labels compared to Cartan’s are used.

**Classification of TEs.** As a key technique in topological analysis, we introduce a specific decomposition of the elementary-excitation Hamiltonian. According to the definition $V^\dagger \tau V = \tau$, each $V \in U(N, N')$ can be factorized as

$$V \equiv \begin{pmatrix} v_a & v_x \\ v_y & v_b \end{pmatrix} = \exp \left( \begin{pmatrix} w \dagger & \cdot \\ \cdot & \cdot \end{pmatrix} \right) \cdot \begin{pmatrix} u_a & u_x \\ u_y & u_b \end{pmatrix},$$

where $u_{a,b} = (v_{a,b} v_{a,b}^\dagger)^{-\frac{1}{2}} v_{a,b}$ are unitary matrices and

$$W = \begin{pmatrix} w \dagger & \cdot \\ \cdot & w \end{pmatrix} = \tanh^{-1} \left( \begin{pmatrix} v_y v_a & -v_x v_b^{-1} \end{pmatrix} \right)$$

is an Hermitian matrix. The detail is shown in Appendix A. By reversing the pseudo-unitary diagonalization process of Eq. (2), we can recast $H(k)$ as

$$H(k) = e^{-W} H_0 e^{-W}, \quad H_0(k) = \begin{pmatrix} h_0 & 0 \\ 0 & h'_0 \end{pmatrix},$$

where $h_0(k) = u_a E u_a^\dagger$ and $h'_0(k) = u_b E' u_b^\dagger$ serve as two single-particle Hamiltonians with the spectra given by $\Lambda(k) = E(k) \oplus E'(k)$. It implies that a QBS is decomposed into two effective single-particle subsystems $h_0(k), h'_0(k)$ with a coupling generator $w(k)$, and its excitation spectra are identical to those of the subsystems. Conversely, components $W(k)$ and $H_0(k)$ can also be expressed in term of $H(k)$. We note $\tau H \tau = e^W H_0 e^W = e^{2W} H e^{2W}$ and find the unique solution

$$e^{2W} = H^{-\frac{1}{2}} \left( H^{\frac{1}{2}} \tau H \tau H^{\frac{1}{2}} \right)^{-\frac{1}{2}} H^{-\frac{1}{2}}, \quad H_0 = e^W H e^W.$$

From these explicit expressions, we know that $W(k)$ and $H_0(k)$ are continuous with regard to $k$ and obey the same symmetric constraints as the original Hamiltonian $H(k)$.

The topological feature of bosonic excitations is fully encoded in the homotopic property of the elementary-excitation Hamiltonian. We define the homotopy as follows: If $H(k)$ can be continuously mapped to $H'(k)$ without breaking the symmetry and closing the excitation gap, we say that $H(k)$ and $H'(k)$ are homotopic, denoted as $H(k) \approx H'(k)$. Based on Eq. (4), we construct a continuous series of Hamiltonian,

$$H_\epsilon(k) = e^{-\epsilon W(k)} H_0(k) e^{-\epsilon W(k)}, \quad \epsilon \in [0, 1],$$

which evidently shares the same symmetries and energy spectra of $H(k)$. Therefore, we immediately achieve a homotopic relation

$$H(k) = H_1(k) \approx H_0(k) = h_0(k) \oplus h'_0(k).$$

This means that a QBS is topologically equivalent to two gapped fermion-like subsystems which belong to AZ classes. As an intuitive comprehension, each $V = e^W (u_a \oplus u_b)$ can be continuously mapped to $u_a \oplus u_b$ through linearly decreasing $W$ to zero. This mapping forms a deformation retraction of $U(N, N')$ onto $U(N) \times U(N')$ which implies that the two Lie groups are homotopy equivalent. Then Hamiltonian $H(k)$ diagonalized by $V(k)$ can be retracted to $H_0(k)$ diagonalized
are essentially the Chern numbers of the bands below class AII (\(T\)) with periodicity 2, 8, 8, respectively. In the last eight columns, entries like 

\[ (\mathcal{C}P) \]

are attributed to class A (\(U\)). This is a natural extension of the topological classification for the AZ classes, with the details stated in Appendix B. Eventually, the classification table of bosonic TEs is derived from that of the AZ classes [12–15] and presented in Tab. II.

**Topological invariants.** With the TEs classified, we still need characteristic numbers to distinguish different topological phases in each symmetry class. Inherited from the fermionic case, the \(\mathbb{Z}\)-type invariants in Tab. II are essentially the Chern numbers of the bands below the gap (for even \(d\)), and the \(\mathbb{Z}_2\)-type invariants are interpreted as the Chern-Simons invariants for odd \(d\) or the Fu-Kane invariants for even \(d\) [15]. We completely list the characteristic numbers in Tab. III and provide their formulas in Appendix C.

We further predict unique topological phases of bosonic excitations from Tab. II, including (1) asymmetric system in class C for \(d = 2, 4\) and (2) \(T\)-invariant system in class H for \(d = 2, 3\). Their topological structures are characterized by a pair of Chern numbers and a pair of \(\mathbb{Z}_2\) indices, respectively, which double the results of their fermionic counterparts. When \(d = 1\), all the symmetry classes are trivial because the disappearance of chirality invalidates the winding number that labels the topological phase. Besides, topological phases may remain unchanged during the retraction.

Next, we need to figure out the intrinsic structure and interrelation between two subsystems. When there is only \(T\) symmetry or no symmetry, two subsystems are fully independent. Thus, the topological classification of \(H\) is given by that of \(h_0(k) \oplus h_0'(k)\). When \(H\) is \(C\)-invariant or \(P\)-invariant, there are constraints among two subsystems as given by

\[
u_c^{-1} h_0^*(k) u_c = h_0'(k) , u_p^{-1} h_0(k) u_p = h_0'(k) .
\]

This implies that the topological classification of \(H\) is fully determined by \(h_0(k)\). In other word, the \(C\) and \(P\) symmetries in the QBS establishes the relation between \(h_0(k)\) and \(h_0'(k)\) rather than confine their intrinsic structures. Therefore, subsystems \(h_0(k)\) and \(h_0'(k)\) only contain the same \(T\) symmetry as the original system \(H\), attributed to class A (\(T = 0\)), class AI (\(T^2 = +1\)), or class AII (\(T^2 = -1\)) in the AZ classification [10].

Now we are able to finish the classification of bosonic TEs. We regard a class of TEs which are homotopic up to flat bands as one topological phase and aim to count out all the possible topological phases in each symmetry class. This is a natural extension of the topological classification for the AZ classes, with the details stated in Appendix B. Eventually, the classification table of bosonic TEs is derived from that of the AZ classes [12–15] and presented in Tab. II.

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arise in every symmetry class for \( d = 4 \) as predicted in Tab. II, which are possibly implemented in artificial dimensions.

**Interaction-driven TEs.** The technique of Hamiltonian decomposition allows us to construct peculiar bosonic models without fermionic counterpart. The typical feature of a QBS is the two-boson annihilation/creation interactions \( ba / a \dagger b \dagger \) which may arise from the two-photon squeezing in a photonic crystal or the atomic interaction of ultracold atoms. It is possible to impose two-boson interactions on the trivial single-particle parts to make the complete Hamiltonian topological, generating interaction-driven TEs.

As an example, we consider a 2D bosonic BdG Hamiltonian

\[
H_{\text{BdG}} (k) = \begin{pmatrix}
    a(k) & h_x(k) \\
    h_x^*(k) & -h_y(k)
\end{pmatrix}.
\]

This model possesses \( C \) symmetry with \( C^2 = +\mathbb{I} \) and is attributed to class CI. According to Tabs. II and III, its topological phases are classified by \( Z \) and characterized by a Chern number in even dimensions. (Classifying the pseudo-Hermitian Hamiltonian \( \tau H_{\text{BdG}} (k) \) with a line gap gives the same result and provides a verification [27].) Here we propose the trivial block matrices as follows,

\[
h_a(k) = \sigma^1 \sin k_1 + \sigma^2 \sin k_2 - \mu \mathbb{I}, \quad (\mu < 0), \quad (9)
\]

\[
h_x(k) = \sigma^3 (m + \cos k_1 + \cos k_2) + \xi \mathbb{I}, \quad (m, \xi \in \mathbb{C}), \quad (10)
\]

where \( \sigma^1, \sigma^2, \sigma^3 \) are Pauli matrices and \( k_{1,2} \in [0, 2\pi] \) form a 2-torus. To achieve the topological phase diagram, we consider a homotopic Hamiltonian \( H(k; \Omega) = H_{\text{BdG}}(k) + \Omega \mathbb{I} \approx H_{\text{BdG}}(k) \) for \( \Omega \to +\infty \) (see Appendix D) and apply perturbation theory to reduce Eq. (5) to \( h_0(k; \Omega) = h_a(k) - \frac{1}{2\Omega} h_x h_x^\dagger (k) + \Omega \mathbb{I} \). After calculations, we obtain the Chern number of \( h_0(k; \Omega) \approx h_0(k) \) as follows [57]

\[
\text{CN} = \begin{cases}
    \text{sgn} (\text{Re} \xi^* m), & 0 < |\text{Re} \xi^* m| < 2 |\text{Re} \xi| \\
    2 |\text{Re} \xi| & |\text{Re} \xi^* m| \geq 2 |\text{Re} \xi|
\end{cases} \quad (11)
\]

To demonstrate a TE, we solve the effective single-particle Hamiltonian \( h_0(k) = \lambda \mathbb{I} + \mathbf{l} \cdot \sigma \) \((\lambda > 0 \) and \( \mathbf{l} \in \mathbb{R}^3 \)) of this model by Eq. (5). The spin texture \( \mathbf{l} / |\mathbf{l}| \) and excitation spectra \( E(k) = \lambda \pm |\mathbf{l}| \) for parameters \( \mu = -5, \xi = 2 \) and \( m = 1 \) are presented in Fig. 1 (a) and (c), respectively. We can see that \( E(k) \) in Fig. 1 (a) opens a gap in the presence of the two-boson interactions. And the skyrmion in Fig. 1 (c) reflects the non-triviality of the interaction-driven TE. We also present the Berry curvature \( \mathcal{F} = -i \int \gamma (k) d\mathbf{k}_1 \wedge d\mathbf{k}_2 \) (see Appendix C) in Fig. 1 (d) and gain the Chern number \( \text{CN} = \frac{1}{4\pi} \int \mathcal{F} = 1 \) which coincides with the analytical result of Eq. (11). Finally, we consider the lattice model of \( H_{\text{BdG}}(k) \) and adopt 1D open and 1D periodical boundary condition. Two edge modes crossing the excitation gap are observed in the excitation spectra as shown in Fig. 1 (b).

We also propose a \( Z_2 \)-type TE in 2D and 3D based on the similar construction. We still choose the BdG Hamiltonian with block matrices \( h_a(k) = \sum_{j=1}^3 \gamma_j \sin k_j - \mu \mathbb{I} \) and \( h_x(k) = \gamma^0 (m + \sum_{j=1}^d \cos k_j) + \xi \mathbb{I} \) \((m < 0, m, \xi \in \mathbb{R})\). Here \( \gamma^0, \gamma^j \) are the Clifford generators and \( k_j \in [0, 2\pi] \) form a \( d \)-torus \((d = 2, 3)\). This model possesses all the \( T, C, P \) symmetries and belongs to class III whose TEs are classified by \( Z_2 \). Similar to the above analysis, we find \( H_{\text{BdG}}(k) + \Omega \mathbb{I} \approx H_{\text{BdG}}(k) \) (see Appendix D) and \( h_a(k) - \frac{1}{2\Omega} h_x h_x^\dagger (k) + \Omega \mathbb{I} \approx h_0(k) \) which serves as a \( T \)-invariant topological insulator [58]. The non-trivial phase is given by \( d - 2 < |m| < d \) and the trivial one is corresponding to \( |m| > d \) or \( |m| < d - 2 \) if the gapped condition \( \xi \neq 0 \) has not been satisfied.

**Conclusions.** We have classified the symmetries and TEs in the QBS with an excitation band gap. Three basic symmetries in QBS are introduced and the ten-fold symmetry classification is realized. A specific decomposi-
tion of the elementary-excitation Hamiltonian is applied to reveal its algebraic and topological structures. Then the classification table of TEs is derived based on that of AZ classes. Unique topological phases of bosons are discussed and concrete bosonic models without fermionic counterpart are constructed. Our work provides a framework to explore richer topological physics of bosons.

The possible studies in future include the implementation of the predicted TEs in realistic systems, the extension of the classification table by considering lattice symmetries, and a systematical investigation on the bulk-edge correspondence in the bosonic case.

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[1] F. D. M. Haldane, Phys. Rev. Lett. 61, 2015 (1988).
[2] C. L. Kane and E. J. Mele, Phys. Rev. Lett. 95, 146802 (2005).
[3] C. L. Kane and E. J. Mele, Phys. Rev. Lett. 95, 226801 (2005).
[4] B. A. Bernevig, T. L. Hughes, and S.-C. Zhang, Science 314, 1757 (2006).
[5] L. Fu and C. L. Kane, Phys. Rev. B 76, 045302 (2007).
[6] N. Read and D. Green, Phys. Rev. B 61, 10267 (2000).
[7] A. Y. Kitaev, Phys.-Usp. 44, 131 (2001).
[8] V. Mourik, K. Zuo, S. M. Frolov, S. R. Plissard, E. P. A. M. Bakkers, and L. P. Kouwenhoven, Science 336, 1003 (2012).
[9] M. Sato and Y. Ando, Rep. Prog. Phys. 80, 076501 (2017).
[10] A. Altland and M. R. Zirnbauer, Phys. Rev. B 55, 1142 (1997).
[11] M. R. Zirnbauer, ArXiv e-prints (2010), arXiv:1001.0722 [math-ph].
[12] M. Stone, C.-K. Chiu, and A. Roy, J. Phys. A: Math. Theor. 44, 045001 (2011).
[13] A. Kitaev, AIP Conf. Proc. 1134, 22 (2009).
[14] Y. X. Zhao and Z. D. Wang, Phys. Rev. Lett. 110, 240404 (2013).
[15] C.-K. Chiu, J. C. Y. Teo, A. P. Schnyder, and S. Ryu, Rev. Mod. Phys. 88, 035005 (2016).
[16] P.-Y. Chang, Phys. Rev. B 97, 224304 (2018).
[17] C. Yang, L. Li, and S. Chen, Phys. Rev. B 97, 060304 (2018).
[18] Z. Gong and M. Ueda, Phys. Rev. Lett. 121, 250601 (2018).
[19] X. Qiu, T.-S. Deng, G.-C. Guo, and W. Yi, Phys. Rev. A 98, 021601 (2018).
[20] H. Shen, B. Zhen, and L. Fu, Phys. Rev. Lett. 120, 146402 (2018).
[21] K. Kawabata, K. Shiozaki, and M. Ueda, Phys. Rev. B 98, 165148 (2018).
[22] S. Yao and Z. Wang, Phys. Rev. Lett. 121, 086803 (2018).
[23] S. Yao, F. Song, and Z. Wang, Phys. Rev. Lett. 121, 136802 (2018).
[24] K. Kawabata, S. Higashikawa, Z. Gong, Y. Ashida, and M. Ueda, Nat. Commun. 10, 297 (2019).
[25] R. Roy and F. Harper, Phys. Rev. B 96, 155118 (2017).
[26] Z. Gong, Y. Ashida, K. Kawabata, K. Takasan, S. Higashikawa, and M. Ueda, Phys. Rev. X 8, 031079 (2018).
[27] K. Kawabata, K. Shiozaki, M. Ueda, and M. Sato, arXiv e-prints, arXiv:1812.09133 (2018), arXiv:1812.09133 [cond-mat.mes-hall].
[28] F. D. M. Haldane and S. Raghu, Phys. Rev. Lett. 100, 013904 (2008).
[29] Z. Wang, Y. D. Chong, J. D. Joannopoulos, and M. Soljačić, Phys. Rev. Lett. 100, 013905 (2008).
[30] Y. D. Chong, X.-G. Wen, and M. Soljačić, Phys. Rev. B 77, 235125 (2008).
[31] M. Hafezi, E. A. Demler, M. D. Lukin, and J. M. Taylor, Nat. Phys. 7, 907 (2011).
[32] M. Hafezi, S. Mittal, J. Fan, A. Migdall, and J. M. Taylor, Nat. Photonics 7, 1001 (2013).
[33] Y. Poo, R.-x. Wu, Z. Lin, Y. Yang, and C. T. Chan, Phys. Rev. Lett. 106, 093903 (2011).
[34] Y. Poo, C. He, C. Xiao, M.-H. Lu, R.-X. Wu, and Y.-F. Chen, Sci. Rep. 6, 20380 (2016).
[35] S. A. Skirlo, L. Lu, Y. Igarashi, Q. Yan, J. Joannopoulos, and M. Soljačić, Phys. Rev. Lett. 113, 133904 (2014).
[36] S. A. Skirlo, L. Lu, Y. Igarashi, Q. Yan, J. Joannopoulos, and M. Soljačić, Phys. Rev. Lett. 115, 253901 (2015).
[37] Z. Wang, Y. Chong, J. D. Joannopoulos, and M. Soljačić, Nature 461, 772 (2009).
[38] M. Aidelsburger, M. Atala, M. Lohse, J. T. Barreiro, B. Paredes, and I. Bloch, Phys. Rev. Lett. 111, 185301 (2013).
[39] M. Aidelsburger, M. Lohse, C. Schweizer, M. Atala, J. Barreiro, S. Nascimbène, N. Cooper, I. Bloch, and N. Goldman, Nat. Phys. 11, 162 (2014).
[40] P. Ring and P. Schuck, The Nuclear Many-Body Problem (Springer, New York, 1980).
[41] R. Rossignoli and A. M. Kowalski, Phys. Rev. A 72, 032101 (2005).
[42] J. P. Blaizot and G. Ripka, Quantum Theory of Finite Systems (MIT Press, Cambridge, MA, 1986).
[43] R. Shindou, R. Matsumoto, S. Murakami, and J.-i. Ohe, Phys. Rev. B 87, 174427 (2013).
[44] R. Chisnell, J. S. Helton, D. E. Freedman, D. K. Singh, R. I. Bewley, D. G. Nocera, and Y. S. Lee, Phys. Rev. Lett. 115, 147201 (2015).
[45] A. Roldán-Molina, A. S. Nascimbène, N. Cooper, I. Bloch, and J. A. Clerk, Nat. Phys. 11, 10779 (2016).
[46] G. Engelhardt and T. Brandes, Phys. Rev. A 91, 053621 (2015).
[47] Z.-F. Xu, L. You, A. Hemmerich, and W. V. Liu, Phys. Rev. Lett. 117, 085301 (2016).
[48] M. Di Liberto, A. Hemmerich, and C. Morais Smith, Phys. Rev. Lett. 117, 163001 (2016).
[49] G.-Q. Luo, A. Hemmerich, and Z.-F. Xu, Phys. Rev. A 98, 053617 (2018).
[53] J. H. P. Colpa, Physica A: Statistical Mechanics and its Applications 134, 417 (1986).
[54] R. Simon, S. Chaturvedi, and V. Srinivasan, J. Math. Phys. 40, 3632 (1999).
[55] Y. Yan and Q. Zhou, Phys. Rev. Lett. 120, 235302 (2018).
[56] A deformation retraction $F_\epsilon : G \to G$ satisfies that (1) $F_0 : G \to G$ is an identity mapping; (2) $F_1 : G \to G_1$ is a retraction mapping; (3) $F_\epsilon : G_1 \to G_1$ is identity mapping for each $\epsilon$.
[57] X.-L. Qi, Y.-S. Wu, and S.-C. Zhang, Phys. Rev. B 74, 045125 (2006).
[58] B. A. Bernevig, Topological Insulators and Topological Superconductors (Princeton University Press, 2013).
From the definition \( V^\dagger r V = \tau \), each \( V = \begin{pmatrix} v_a & v_x \\ v_y & v_b \end{pmatrix} \in U(N, N') \) satisfies
\[
\begin{aligned}
v_a^\dagger v_a - v_y^\dagger v_y = 1, & \quad v_b^\dagger v_b - v_x^\dagger v_x = 1, \\
v_a^\dagger v_x - v_y^\dagger v_b = 0. & \quad (12)
\end{aligned}
\]
From \( \det v_a^\dagger v_a \geq 1 \) and \( \det v_b^\dagger v_b \geq 1 \) one knows \( \det v_{a,b} \neq 0 \) such that \( v_a \) and \( v_b \) are invertible matrices. After denoting \( r = v_x v_b^{-1} = (v_y v_a^{-1})^\dagger \), one can reduce the above equations to
\[
\begin{aligned}
\mathbb{I} - rr^\dagger = (v_a v_a^\dagger)^{-1}, & \quad \mathbb{I} - r^\dagger r = (v_b v_b^\dagger)^{-1}, \\
& \quad (13)
\end{aligned}
\]
and further recast the pseudo-unitary matrix as
\[
V = \begin{pmatrix} \mathbb{I} & r \\ r^\dagger & \mathbb{I} \end{pmatrix} \begin{pmatrix} v_a & v_b \\ v_y & v_x \end{pmatrix} = \begin{pmatrix} \mathbb{I} & r \\ r^\dagger & \mathbb{I} \end{pmatrix} \begin{pmatrix} (\mathbb{I} - rr^\dagger)^{-\frac{1}{2}} u_a \\ (\mathbb{I} - r^\dagger r)^{-\frac{1}{2}} u_b \end{pmatrix},
\]
(14)
Here \( u_{a,b} = (v_{a,b} v_{a,b}^\dagger)^{-\frac{1}{2}} v_{a,b} \) are unitary matrices. Then we make substitution \( \begin{pmatrix} r^\dagger \\ r \end{pmatrix} = \tanh W \) to simplify the above expression as
\[
V = (\mathbb{I} + \tanh W) (1 - \tanh^2 W)^{-\frac{1}{2}} \begin{pmatrix} u_a \\ u_b \end{pmatrix} = e^W \begin{pmatrix} u_a \\ u_b \end{pmatrix},
\]
(15)
Since \( \tanh x \) is an odd function, \( W = \tanh^{-1} \begin{pmatrix} r^\dagger \\ r \end{pmatrix} \) as a power series of \( \begin{pmatrix} r^\dagger \\ r \end{pmatrix} \) only contains odd-power terms, taking the form of \( W = \begin{pmatrix} w \\ w^\dagger \end{pmatrix} \). Therefore, a pseudo-unitary matrix \( V \) is composed by an \( N \times N' \) matrix \( w \) and two unitary matrices \( u_{a,b} \), i.e., \( U(N, N') = \mathbb{C}^{N,N'} \times U(N) \times U(N') \).

B. CLASSIFICATION PRINCIPLE

We present the classification principle of AZ classes for free-fermion systems and extend it to our quadratic bosonic systems. For the fermionic AZ classes, one first introduces the trivial Hamiltonian
\[
\sigma = \begin{pmatrix} \mathbb{I}_r \\ -\mathbb{I}_r \end{pmatrix},
\]
(16)
which corresponds to a flat-band system with equal conduction and valence bands. Then one applies K-theory to define the stable equivalence of single-particle Hamiltonians \( h(k) \) and \( h'(k) \), i.e.,
\[
h(k) \oplus \sigma \approx h'(k) \oplus \sigma',
\]
(17)
denoting as \( h(k) \sim h'(k) \). Here \( \sigma \) and \( \sigma' \) are independent trivial Hamiltonians with unlimited matrix sizes. The stable equivalence studies the homotopy of Hamiltonians whose intrinsic space is enlarged to \( \infty \)-dimension by attaching flat bands. It is a looser condition than the original homotopy such that the classification becomes much easier. People usually regard the stably equivalent class \([h(k)]\) as a topological phase and achieve the classification by counting out all the \([h(k)]\) for given symmetries and \(k\)-space dimension \([13, 15]\). The classification table is derived from the Bott periodicity theorem \([12]\). We present the results of class A, AI, and AII in Tab. IV, where the \(k\)-space is chosen as sphere \( S^d \). If the \(k\)-space becomes the usual torus \( T^d \) in band theory, the final result is the present answer plus some weak topological invariants \([13]\).
Table IV. Topological classification of class A, AI, and AII. The $k$-space is chosen as sphere $S^2$. In the second column, symbol $0$ refers to absence of symmetry and $\pm$ refers to the presence of symmetry with $T^2 = \pm \mathbb{I}$.

| AZ class | $T$ | Classifying space $d = 0$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---------|-----|------------------------|---|---|---|---|---|---|---|
| A       | 0   | $\mathbb{C}_0 = \mathbb{Z} \times BU$ | $\mathbb{Z}$ | 0 | $\mathbb{Z}$ | 0 | 0 | 0 | 0 |
| AII     | -   | $\mathbb{R}_4 = \mathbb{Z} \times BS\mathbb{P}$ | $\mathbb{Z}$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | 0 | 0 | 0 | 0 |

As for the quadratic bosonic system, we naturally extend the trivial Hamiltonian $O$ as follows,

$$O = \begin{pmatrix} o_{\tau=+1} & 0 \\ 0 & o_{\tau=-1} \end{pmatrix}, \quad o = \begin{pmatrix} E_{\text{up low}} \end{pmatrix},$$

in which the block matrix $o_{\tau=\pm 1}$ acting on the $\tau = \pm 1$ eigenspace corresponds to a flat-band subsystem with equal upper and lower excitation bands (the excitation bands above and below the gap). Hence, attaching it on $H(k)$ does not change the sign difference of $\tau$ nor the difference of upper/lower band numbers. Then we call $H(k)$ stably equivalent to $H'(k)$ if

$$H(k) \oplus o \approx H'(k) \oplus O',$$  \hspace{1cm} (19)

denoting as $H(k) \sim H'(k)$. We regard the stable equivalent class $[H(k)]$ as a topological phase and achieve the classification by counting out all the $[H(k)]$. Since the homotopic property of $H(k)$ is determined by $h_0(k)$ or $h_0(k) \oplus h'_0(k)$, the classification of $[H(k)]$ reduces to the classification of $[h_0(k)]$ or $[h_0(k) \oplus h'_0(k)]$. Here the stable equivalence for the subsystem is reduced from Eq. (19), given by

$$h_0(k) \oplus o \approx h''_0(k) \oplus o''.$$  \hspace{1cm} (20)

We know that $h_0(k)$ is attributed to class A, AI, or AII, and $o$ is homotopic to the fermionic trivial Hamiltonian $\sigma$. Therefore, the classification of $[h_0(k)]$ is directly given by the periodic table of AZ classes Tab. IV. For independent $h_0(k)$ and $h'_0(k)$, the classification of $[h_0(k) \oplus h'_0(k)] = [h_0(k)] \oplus [h'_0(k)]$ simply gets doubled.

**C. CHARACTERISTIC NUMBERS**

We provide the computation formulas of the Chern number, Chern-Simons invariant, and Fu-Kane invariant for the single-particle Hamiltonians $h_0(k)$ and $h'_0(k)$ [15]. Firstly, the Chern number is defined by

$$\text{CN} = \int_{\text{BZ}} \text{det} \left( I + \frac{i}{2\pi} F \right),$$

with Berry curvature 2-form $F = dA + A \wedge A$. Here $A$ denotes the Berry connection form of the bands below the gap, and BZ refers to the Brillouin zone, namely, the $k$-space. It is a topological invariant that measures the twisting of the energy bands (vector bundle). Secondly, the Chern-Simons invariant for $d = 2m - 1$ is a geometrical invariant defined by

$$\text{CS} = \exp \left( 2\pi i \int_{\text{BZ}} \text{cs}_m \right),$$

in which the Chern-Simons form reads

$$\text{cs}_m = \frac{1}{(m-1)!} \left( \frac{i}{2\pi} \right)^m \int_0^1 \text{d}t \text{Tr} \left( A \wedge F_t^{m-1} \right),$$

with $F_t = tdA + t^2A^2$. With the existence of time-reversal symmetry, CS takes discrete values $\pm 1$ for $d = 3, 7$ and then keeps invariant under the continuous deformation of $A(k)$, which becomes a topological invariant. Thirdly, the Fu-Kane invariant for $d = 2m$ is defined by
\[ \text{FK} = \int_{\text{BZ}/2} \frac{1}{m!} \text{Tr} \left( \frac{i}{2\pi} \mathcal{F} \right)^m - \int_{\partial\text{BZ}/2} \text{cs}_m, \]  

(24)

in which \( \text{BZ}/2 \) refers to a half of Brillouin zone. It also takes discrete values \( \pm 1 \) for \( d = 2, 6 \) with the existence of the \( T \) symmetry and thus becomes a topological invariant.

Finally, we provide the expressions of the Berry connection. Without losing generality, we suppose the first \( n \) elements of \( E = u_a^\dagger h_0 u_a \) lower than the energy gap. Then the Berry connection form of \( h_0(k) \) is given by

\[ \mathcal{A} = \Psi^\dagger u_a^\dagger d u_a \Psi = \left( I_n 0 \right). \]  

(25)

By using the factorization of pseudo-unitary matrix \( V = \exp \left( \tau \mathcal{W} \right) \cdot \left( u_a u_b \right) \), we can recast the Berry connection as

\[ \mathcal{A} = \left( \Psi^\dagger, 0 \right) V^\dagger \tau d V \left( \Psi 0 \right). \]  

(26)

Similarly, we suppose that the first \( n' \) elements of \( E' = u_b^\dagger h_0' u_b \) are lower than the energy gap. Then the Berry connection of \( h_0'(k) \) is given by

\[ \mathcal{A}' = \Psi'^\dagger u_b^\dagger d u_b \Psi' = - \left( 0, \Psi'^\dagger \right) V^\dagger \tau d V \left( 0 \right), \]  

(27)

with \( \Psi'^\dagger = \left( I_{n'} 0 \right) \).

**D. SKILL OF HOMOTOPY**

We prove \( H(k; \Omega) = H_{\text{BdG}}(k) + \Omega I \approx H_{\text{BdG}}(k) \) which is used to analyze our models. Since \( H(k; \Omega) \) obviously keeps the symmetries of \( H_{\text{BdG}}(k) \), we just need to prove that the gap keeps open while \( \Omega \) increases from 0 to \( +\infty \).

For the first model in class C, the additional inversion symmetry \( U_I^{-1} H_{\text{BdG}}(-k; \Omega) U_I = H_{\text{BdG}}(k; \Omega) \) with \( U_I = \gamma_2 \otimes \sigma^3 \) results in \( \sigma^3 h_0(-k; \Omega) \sigma^3 = h_0(k; \Omega) \) such that the spectra obey \( E(k; \Omega) = E(-k; \Omega) \). Due to the \( C \) symmetry, we also find \( E'(k; \Omega) = E^*(k; \Omega) = E(k; \Omega) \). Therefore, the gapless condition of \( H_{\text{BdG}}(k; \Omega) \) simply reads \( H_0(k; \Omega) = \lambda I \). According to the Hamiltonian decomposition, it is equivalent to \( (\tau H)^2 = \lambda^2 I \). By substituting the expression of \( H(k; \Omega) \) into it, we reduce the gapless condition to \( \sin k_j = 0 \) and \( \xi^* \left( m \cos k_1 + \cos k_2 \right) + \text{c.c.} = 0 \) which are independent on \( \Omega \). As a result, \( H(k; \Omega) \) is gapped as long as \( H_{\text{BdG}}(k) \) is gapped. Therefore, \( H(k; \Omega) \) is a homotopy for \( \Omega \in [0, +\infty) \).

For the second model in class III, the \( P \) symmetry guarantees \( E(k) = E'(k) \) and the Kramer degeneracy from \( T^2 = -I \) symmetry reduces the spectra to two separate bands. Then the gapless condition is also given by \( H_0(k; \Omega) = \lambda I \) which is equivalent to \( (\tau H)^2 = \lambda^2 I \). We still find that the gapped condition is independent on \( \Omega \). Therefore, \( H(k; \Omega) \) is a homotopy for \( \Omega \in [0, +\infty) \).