Revisiting Shor’s quantum algorithm for computing general discrete logarithms

Martin Ekerå

1KTH Royal Institute of Technology, Stockholm, Sweden
2Swedish NCSA, Swedish Armed Forces, Stockholm, Sweden

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Abstract

We heuristically show that Shor’s algorithm for computing general discrete logarithms achieves an expected success probability of approximately 60% to 82% in a single run when modified to enable efficient implementation with the semi-classical Fourier transform.

By slightly increasing the number of group operations that are evaluated quantumly and performing a single limited search in the classical post-processing, or by performing two limited searches in the post-processing, we show how the algorithm can be further modified to achieve a success probability that heuristically exceeds 99% in a single run. We provide concrete heuristic estimates of the success probability of the modified algorithm, as a function of the group order \( r \), the size of the search space in the classical post-processing, and the additional number of group operations evaluated quantumly. In the limit as \( r \to \infty \), we heuristically show that the success probability tends to one.

In analogy with our earlier works, we show how the modified quantum algorithm may be heuristically simulated classically when the logarithm \( d \) and \( r \) are both known. Furthermore, we heuristically show how slightly better tradeoffs may be achieved, compared to our earlier works, if \( r \) is known when computing \( d \). We generalize our heuristic to cover some of our earlier works, and compare it to the non-heuristic analyses in those works.

1 Introduction

In 1994, in a seminal work with profound cryptologic implications, Shor [29, 30] introduced polynomial-time quantum algorithms for factoring integers — via a reduction to order finding in cyclic subgroups of \( \mathbb{Z}_N^* \), for \( N \) the integer to be factored — and for computing discrete logarithms in \( \mathbb{F}_p^* \). The algorithms may be generalized to compute orders and discrete logarithms in any cyclic group for which the group operation may be implemented efficiently quantumly.

Shor’s algorithms are important, in that the security of virtually all currently widely deployed asymmetric cryptosystems is underpinned by either of the two aforementioned problems: The security of RSA [27] is based on the intractability of the factoring problem. Schemes such as Diffie–Hellman [5] and DSA [22] rely on the intractability of the discrete logarithm problem.

In this paper, we analyze the success probability of Shor’s algorithm for computing discrete logarithms, when modified to enable efficient implementation with the semi-classical quantum Fourier transform [12] and control qubit recycling [19, 24]. We heuristically demonstrate that the modified algorithm achieves a success probability that exceeds 99% in a single run, if the number of group operations that are evaluated quantumly is slightly increased compared to Shor’s original algorithm and a limited search is performed in the classical post-processing, or if two limited searches are performed in the classical post-processing.

Furthermore, compared to our earlier result in [9], we heuristically show how slightly better tradeoffs may be achieved — between the number of runs, and the number of group operations that are evaluated quantumly in each run — when the group order is known. We compare our proposed approach to making tradeoffs to a related approach proposed by Kaliski [14].
The results in this paper may be used to develop estimates of the overall complexity of attacking schemes that rely on the computational intractability of the discrete logarithm problem, and to enable fair comparisons of quantum algorithms for computing discrete logarithms.

1.1 Preliminaries

As in [7–9], we let \( g \) be a generator of a finite cyclic group \( \langle g \rangle \) under \( \circ \) of order \( r \), and
\[
x = [d]g = g \circ \cdots \circ g, \text{ } d \text{ times}
\]

Given \( x \) and \( g \), the discrete logarithm problem is to compute the logarithm \( d = \log_g(x) \). In the general discrete logarithm problem \( d \in \{ 0, r \} \cap \mathbb{Z} \), whereas \( d \) is smaller than \( r \) by some order of magnitude in the short discrete logarithm problem. The order \( r \) may either be known or unknown when \( d \) is to be computed: Both cases are interesting and merit study. In cryptologic schemes that are based on the computational intractability of the discrete logarithm problem, the group \( \langle g \rangle \) is typically a subgroup of \( \mathbb{F}_p^\ast \) or an elliptic curve group.

Given \( g \), the order-finding problem is the problem of computing \( r \). In what follows below, we survey quantum algorithms for both the order-finding and discrete logarithm problems.

1.1.1 Notation

Throughout this paper, \( u \mod n \) and \( \{u\}_n \) denotes \( u \) reduced modulo \( n \) constrained to the intervals \([0, n)\) and \([-n/2, n/2)\), respectively. Furthermore, \([u]\), \([u]_1\) and \([u]\) denotes \( u \) rounded up, down and to the closest integer. Ties are broken by requiring that \([u] = u - \{u\}_1\).

The Euclidean norms of \( u \in \mathbb{C} \), and of the vector \( \mathbf{u} \), are denoted \(|u|\) and \(|\mathbf{u}|\), respectively. In heuristic arguments, \( u \sim v \) and \( u \simeq v \) denote that \( u \) and \( v \) are of similar size, and heuristically equal, respectively. Furthermore, \( u \ll v \) denotes that \( u \) is much smaller than \( v \), and \( \text{len}(u) \) is the bit length \( \lceil \log_2(u) \rceil + 1 \) of the non-negative integer \( u \).

1.2 Earlier works

Shor’s order-finding algorithm works by inducing a periodicity in \( r \), extracting information on the period using the quantum Fourier transform (QFT), and classically post-processing the output by forming a quotient and expanding it in a continued fraction to recover \( r \).

Specifically, a control register of \( 2m \) qubits is first initialized to a uniform superposition of all values \( a \in \{ 0, 2^m \} \cap \mathbb{Z} \), for \( m \) an upper bound on the bit length of \( r \). Then \([a]g\) is computed to a second register. When the QFT is applied to the control register, this maps
\[
\frac{1}{2^m} \sum_{a=0}^{2^m-1} [a,g] \xrightarrow{\text{QFT}} \frac{1}{2^{2m}} \sum_{a=0}^{2^m-1} \sum_{j=0}^{2^m-1} e^{2\pi i aj/2^{2m}} |j, [a]g\rangle.
\]

In practice, the generalized exponentiation \([a]g\) is computed using the generalized square-and-multiply algorithm: Let \( a = a_0 + 2a_1 + 2^2a_2 + \ldots \) for \( a_i \) selected uniformly at random from \{0, 1\} for \( i \in [0, 2m] \cap \mathbb{Z} \). Then \([a]g = [a_0 + 2a_1 + 2^2a_2 + \ldots] g = [a_0]g \circ [2a_1]g \circ [2^2a_2]g \circ \ldots \) so if the second register is initialized to the identity \([0]g\), we may classically pre-compute \([2^i]g\) and operate with this element on the second register conditioned on \( a_i \) for \( i \in [0, 2m] \cap \mathbb{Z} \) to perform the exponentiation. A total of \( 2m \) group operations are then evaluated quantumly.

When each \( a_i \) is selected uniformly at random from \{0, 1\}, as is the case above, the QFT may be implemented in an interleaved fashion using the semi-classical QFT [12] with control qubit recycling [19,24]. A single control qubit, and a few standard quantum operations, then

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3Shor [30] originally introduced the order-finding algorithm to factor integers \( N \), and therefore picked the control register length \( l \) such that \( N^2 \leq 2^l < 2N^2 \) as a function of \( N > r \) to ensure that \( 2^l > r^2 \). Since we consider the order-finding problem in arbitrary cyclic groups, we instead pick \( l = 2m \) for \( m \) such that \( r < 2^m \).
suffice to implement the first control register and the QFT. The quantum cost of the algorithm is then dominated by the cost of performing the $2m$ group operations.

If $[1]$ is observed, the frequency $j$ and $y = \lfloor e \rfloor g$ for $e \in [0, r) \cap \mathbb{Z}$, are yielded with probability

$$
\frac{1}{2^{4m}} \left| \sum_{t=0}^{M} e^{2\pi i (e + tr) / 2^{2m}} \right|^2 = \frac{1}{2^{4m}} \left| \sum_{t=0}^{M} e^{2\pi i jr / 2^{2m}} \right|^2 = \frac{1}{2^{4m}} \left[ \sum_{e=1}^{2m} e^{2\pi i e / 2^{2m}} \right]^2
$$

where $M = \lfloor (2^{2m} - e - 1) / r \rfloor$. Constructive interference is expected to arise when $r j / 2^{2m} \approx z$, for $z$ an integer, as all unit vectors in the above sum then point in approximately the direction of the real axis. To recover $r$, Shor expands $j / 2^{2m}$ in a continued fraction to attempt to find the unknown quotient $z / r \approx j / 2^{2m}$. Shor uses that $r < 2^m$ to stop the expansion short.

If $r$ is not prime, it may be that $z$ and $r$ have a common factor $\tau = \gcd(z, r)$, in which case $z / \tau$ and $r / \tau$ will be found instead of $z$ and $r$. To address this problem, Shor proposes to search for $\tau$ in $\{1, 2, \ldots, B_r\}$, for $B_r$ some bound, to attempt to find the order $r$. This significantly increases the success probability. Shor furthermore proposes to consider not only $j$, but also $j \pm 1, \ldots, j \pm B_j$, for $B_j$ some bound, to further increase the success probability.

In [11], Ekerå describes how the two aforementioned searches may be performed efficiently; both when using continued fractions-based post-processing, as originally proposed by Shor, and when using lattice-based post-processing. Furthermore, Ekerå gives lower bounds on the probability of successfully recovering $r$ in a single run of the quantum part of the order-finding algorithm, as a function of the search space in the classical post-processing part of the algorithm, and of the length of the control register in the quantum part of the algorithm:

Already for moderate $r$, a high success probability that well exceeds 99% can be guaranteed without increasing the length of the control register compared to Shor. Asymptotically, in the limit as $r$ and hence $m \to \infty$, the search space can be bounded as a function of $m$ so that the success probability tends to one without compromising the polynomial runtime.

By slightly decreasing the length of the control register, work in the classical post-processing part of the algorithm may be traded for work in the quantum part [11 see Lem. 4.3]: The steep reduction in the success probability that arises when the control register length is decreased may be compensated for to some extent by searching an increased space in the post-processing, and vice versa. Typically, one would strive to keep the control register as short as possible.

For earlier works that seek to increase the success probability of Shor’s order-finding algorithm by increasing the control register length, see e.g. [7] p. 11 and App. C], [21] the paragraph containing Eq. (5.44) on p. 227, or the analysis in [2]. See also the survey in [11 Sect. 1.5].

1.2.1 Tradeoffs in order finding

In [28], Seifert introduces the idea of making tradeoffs across runs in Shor’s order-finding algorithm — the basic idea being to perform $n \geq s$ runs with a $(1 + 1/s)n$-bit exponent, and to then simultaneously solve the outputs of these $n$ runs for $r$ in a classical post-processing step by generalizing Shor’s continued fractions-based post-processing to higher dimensions.

Seifert’s original motivation for introducing this variation of Shor’s algorithm was to reduce the number of control qubits required to implement the algorithm. In a more modern interpretation, where the semi-classical QFT is used and the control qubits are recycled, the effect is rather to reduce the circuit depth, and hence the number of group operations that need to be evaluated in each run of the algorithm, at the expense of having to run it multiple times.

In [9] App. A], Ekerå describes how the quantum part of Shor’s order-finding algorithm, and of Seifert’s variation thereof, may be simulated classically for known $r$. Ekerå furthermore uses the simulator to estimate the number of runs $n$ with tradeoff factor $s \geq 1$ that are needed to solve for $r$ with at least 99% success probability when using lattice-based post-processing.

1.2.2 Factoring general integers via order finding

Shor originally introduced the order-finding algorithm for the purpose of factoring general integers $N$ via a classical reduction that follows from the work of Miller [16]:

}\]
Specifically, to factor an odd integer \( N \) that is not a perfect prime power\(^2\), Shor first selects an integer \( g \in (1, N) \cap \mathbb{Z} \). In the unlikely event that \( g \) is a non-trivial divisor of \( N \) there is no need to proceed — it suffices to return \( g \). Otherwise, \( g \) is perceived as a generator of a subgroup of \( \mathbb{Z}_N^* \) and its order \( r \) computed quantumly with the order-finding algorithm.

If \( r \) is even and \( g^{r/2} \not\equiv -1 \pmod{N} \), Shor then uses that
\[
g^r - 1 = (g^{r/2} - 1)(g^{r/2} + 1) \equiv 0 \pmod{N}
\]
to split \( N \) into two non-trivial factors by computing \( \gcd((g^{r/2} \mod N) \pm 1, N) \) in a classical post-processing step. If \( r \) is odd, or if \( g^{r/2} \equiv -1 \pmod{N} \) causing only trivial factors to be found, Shor originally proposed to simply re-run the algorithm for a new \( g \). Shor gives a proof that the probability of having to perform a re-run for either of these two reasons is at most 1/2.

It is possible to do better, however: Ekerå shows in \([10]\) how any integer \( N \) may be completely factored with very high probability of success, that asymptotically tends to one, given the order \( r \) of a single element \( g \) selected uniformly at random from \( \mathbb{Z}_N^* \).

By combining \([10]\) with \([11]\), Ekerå obtains a lower bound \([11]\) Sect. 4.2.1] on the probability of completely factoring any integer \( N \) successfully in a single run of Shor’s order-finding algorithm. The probability is large already for moderate \( N \), and it tends to one asymptotically.

For earlier proposals for how the success probability of the classical post-processing step may be increased, the reader is referred to the surveys in \([10]\) Sect. 2 and \([11]\) Sects. 1.4–1.5.

### 1.2.3 Breaking RSA

If the integer \( N \) is known to be on a special form, this fact may be leveraged: Integers on the form \( N = pq \), for \( p, q \) distinct random primes of similar length in bits, are of particular importance in cryptology, as the problem of factoring such integers — commonly referred to as RSA integers — underpins the security of the widely deployed RSA cryptosystem \([27]\).

Ekerå and Hästad \([7,8]\) describe how RSA integers may be factored by computing a short discrete logarithm on the quantum computer, rather than by using order finding as originally proposed by Shor. Ekerå and Hästad’s quantum algorithm has a lower overall quantum cost than Shor’s order-finding algorithm, in that it requires approximately \( 3m/2 \) group operations to be evaluated in each run when not making tradeoffs \([8]\) App. A.2.1], compared to \( 2m \) group operations in each run of Shor’s algorithm. Furthermore, the probability of Ekerå and Hästad’s algorithm successfully finding the factors of the RSA integer in a single run of the quantum algorithm exceeds 99% when not making tradeoffs \([8]\) App. A.2.1]

If tradeoffs are made in Ekerå–Hästad’s algorithm, in analogy with Seifert’s \([28]\) idea for making tradeoffs in Shor’s order-finding algorithm, the number of group operations that need to be evaluated quantumly in each run can be further reduced to \((1/2 + 1/s)m\), for \( s \geq 1 \) the tradeoff factor, at the expense of performing \( n \geq s \) runs. This allows the requirements on the quantum computer to be further reduced, beyond what is possible with Seifert’s tradeoffs.

Ekerå has estimated the number of runs required for a success probability that exceeds 99% to be achieved when making tradeoffs for different tradeoff factors \([8]\) App. A.2].

### 1.2.4 Computing discrete logarithms

In analogy with the order-finding algorithm, Shor’s algorithm for computing general discrete logarithms works by inducing a periodicity in the logarithm \( d \), extracting the period using the quantum Fourier transform (QFT), and classically post-processing the output to recover \( d \).

Specifically, two control registers, each of length \( m \) qubits for \( m \) the bit length of \( r \), are first initialized to a uniform superposition of all \( a, b \in [0, r) \cap \mathbb{Z}_r \). Then \( |a \rangle g \otimes [-b] x \) is computed to a third register. When QFTs of size \( 2^m \) are applied to the two control registers, this maps
\[
\frac{1}{r} \sum_{a=0}^{r-1} \sum_{b=0}^{r-1} |a, b, [a] g \otimes [-b] x \rangle \xrightarrow{\text{QFT}}
\]
\(^2\)These requirements do not imply a loss of generality: Even integers and perfects powers are trivial to factor.
\(^3\)Shor \([20]\) originally described the algorithm for the full multiplicative group \( \mathbb{Z}_p^* \) and so set \( r = p - 1 \).
\[
\frac{1}{2^{m_r}} \sum_{a=0}^{r-1} \sum_{b=0}^{r-1} \sum_{j=0}^{2^m-1} \sum_{k=0}^{2^m-1} e^{2\pi i (aj + bk) / 2^m} |j, k, [a - bd] g\rangle
\]  
(2)

after which \(j, k\) and \(y = [v] g\) for \(v \in [0, r) \cap \mathbb{Z}\) are observed. The frequencies \(j\) and \(k\) are then post-processed using a classical algorithm to find \(d\). This algorithm requires \(r\) to be known.

As for the order-finding algorithm, the quantum cost of the above algorithm (2) is dominated by the \(2m\) group operations. However, the implementation is complicated by the fact that the individual qubits \(a_i\) and \(b_i\) in the two control registers are not selected uniformly at random from \(\{0, 1\}\), as required by the semi-classical QFT with control qubit recycling. In this paper, we therefore analyze a modified version of Shor’s algorithm for computing discrete logarithms, in which the superpositions in the control registers are uniform over \([0, 2^m) \cap \mathbb{Z}\).

1.2.5 On the discrete logarithm problem and our motivation

The discrete logarithm problem underpins the security of virtually all widely deployed asymmetric schemes. Notable examples include RSA [27], Diffie–Hellman [5], DSA [22, 23] and virtually all forms of elliptic curve cryptography (ECC) [15, 17]. Despite this, Shor’s algorithm for computing discrete logarithms has seemingly received less attention in the literature than Shor’s order-finding and factoring algorithms: In particular, there is seemingly no tight analysis of the success probability of Shor algorithm for computing discrete logarithms, when modified as described above to facilitate efficient implementation. A lower bound that slightly exceeds 65% is given in Mosca’s PhD thesis [18, Cor. 19 on p. 58], for a slightly modified version of Shor’s algorithm that admits efficient implementation (see Sect. 4.6.1 for further details).

In this work, we aim to develop a tighter estimate, in light of the following:

Ekerå and Håstad have developed derivatives of Shor’s algorithm for solving cryptologically relevant instances of the discrete logarithm problem. It is known that a success probability that exceeds 99% may be achieved in a single run of Ekerå’s algorithm for computing short discrete logarithms [6, 8], in Ekerå–Håstad’s algorithm for computing short discrete logarithms and factoring RSA integers with tradeoffs [7, 8], and in Ekerå’s algorithm for computing general discrete logarithms with tradeoffs [9]. All of these algorithms may be simulated classically for known \(d\) and \(r\), enabling the success probability of the classical post-processing to be verified.

The latter of the three aforementioned algorithms computes general discrete logarithms, as does Shor’s algorithm for the discrete logarithm problem. Unlike Shor’s algorithm, Ekerå’s algorithm does however not require the group order to be known. If the order is unknown, it may be computed along with the discrete logarithm in the classical post-processing at no additional quantum cost. When computing discrete logarithms without making tradeoffs, Ekerå’s algorithm requires approximately \(3m\) group operations to be evaluated quantumly, compared to \(2m\) operations in Shor’s algorithm. This is explained by the fact that Ekerå’s algorithm computes both \(d\) and \(r\), whereas Shor’s algorithm computes only \(d\) given \(r\).

If \(r\) is unknown, it is hence more efficient to run Ekerå’s algorithm, compared to first running Shor’s order-finding algorithm to find \(r\), and then running Shor’s algorithm for the discrete logarithm problem to find \(d\) given \(r\), in terms of the overall number of group operations that are evaluated quantumly. If \(r\) is known — as is very often the case in practice — and tradeoffs are not made, then Shor’s algorithm is expected to outperform Ekerå’s algorithm.

However, this is only true if one assumes Shor’s algorithm to have a success probability that is on par with that of Ekerå’s algorithm: Without a tight analysis of the success probability of Shor’s algorithm, making a proper comparison is difficult. This fact served as one of our original motivations for writing down the analysis presented in this work.

It should be noted that Proos and Zalka [26, the short note in App. A.2] discuss the success probability of Shor’s algorithm. They state that it should be possible to achieve a success probability that tends to one, either by performing a limited classical search, or by evaluating additional group operations quantumly, making reference to the situation being similar to that

\footnote{E.g. via the reduction to a short discrete logarithm problem.}

\footnote{But not in terms of the number of group operations per run, and hence not in terms of the circuit depth.}
in Shor’s order-finding algorithm. However, no formal analysis is provided: In particular, no concrete estimate of the success probability, as a function of the size of the search space, or of the number of additional group operations, is given.

1.3 Our contributions

In this work, we join an additional piece to the puzzle laid out in the previous sections: We heuristically show that Shor’s algorithm for computing general discrete logarithms achieves an expected success probability of approximately 60% to 82% in a single run when modified to enable the use of the semi-classical QFT \[12\] with control qubit recycling \[19,24\].

By slightly increasing the number of group operations that are evaluated quantumly and performing a single limited search in the classical post-processing, or by performing two limited searches in the classical post-processing, we show how the algorithm can be further modified to achieve a success probability that heuristically exceeds 99% in a single run.

We provide concrete heuristic estimates of the success probability of the modified algorithm, as a function of the order \(r\), the size of the search space in the classical post-processing, and the additional number of group operations evaluated quantumly. This enables fair comparisons to be made between Shor’s algorithm for computing discrete logarithms and other algorithms. In the limit as \(r \to \infty\), we heuristically show that the success probability tends to one.

In analogy with our earlier works \[8,9\], we show how the modified quantum algorithm may be heuristically simulated classically when the logarithm \(d\) and \(r\) are both known. Furthermore, we heuristically show how slightly better tradeoffs may be achieved, compared to our earlier work \[9\], if \(r\) is known when computing \(d\). Finally, we generalize our heuristic to cover some of our earlier works \[7–9\], and compare it to the non-heuristic analyses in those works.

1.4 Assumptions and reductions

When computing discrete logarithms in groups of known order \(r\), as is the primary focus in this paper, we may without loss of generality assume \(r\) to have no small prime factors. To see why this is, note that we may efficiently find all small factors of \(r\) classically, and use Pohlig–Hellman decomposition \[25\] to reduce the problem in the order \(r\) group to problems in one or more small, and one large, subgroup. We may then solve the problems in the small-order subgroups classically, leaving only the problem in the large-order subgroup with no small factors to be solved quantumly. For this reason, \(r\) is typically prime in cryptologic schemes that are based on the computational intractability of the discrete logarithm problem.

Furthermore, when computing general discrete logarithms in groups of known order \(r\), we may without loss of generality assume the logarithm \(d\) to be selected uniformly at random from \([0, r) \cap \mathbb{Z}\). To see why this is, note that the discrete logarithm problem may be randomized:

Given \(g\) and \(x = [d'] g\) for \(d' \in [0, r) \cap \mathbb{Z}\), we may pick an offset \(t\) uniformly at random from \([0, r) \cap \mathbb{Z}\), compute \(x = x' \circ [t] g = [d' + t] g\), compute the randomized discrete logarithm \(d = d' + t = \log_g x\) quantumly, and then compute \(d' \equiv d - t \pmod{r}\) classically.

1.5 Overview

The remainder of this paper is organized as follows:

In Sect. \[2\] we describe the quantum algorithm that upon input of \(g\) and \(x = [d] g\) computes an integer pair \((j, k)\). In Sect. \[3\] we heuristically derive a closed-form expression for the probability of observing \((j, k)\). In Sect. \[4\] we introduce the notion of \(B_{\eta} B_{\Delta}\)-good pairs \((j, k)\), and heuristically estimate the probability of the algorithm yielding such a pair as a function of \(B_{\eta}, B_{\Delta}\) and other relevant parameters. Furthermore, we discuss the soundness of the heuristic, and compare our heuristic estimates to other estimates in the literature.

In Sect. \[5\] we explain how the quantum algorithm may be simulated classically for known \(d\) and \(r\). In Sect. \[6\] we describe how \(B_{\eta} B_{\Delta}\)-good pairs \((j, k)\) may be post-processed classically to recover the logarithm \(d\) given \(r\), both when making and when not making tradeoffs. We discuss generalizations in Sect. \[7\] and summarize and conclude the paper in Sect. \[8\].
2 The quantum algorithm

Given \( g \) and \( x = [d] g \), the quantum algorithm that we analyze henceforth initializes two control registers, of length \( m + \zeta \) and \( \ell \leq m + \zeta \) bits, respectively, to a uniform superposition of all \( a \in [0, 2^{m+\zeta}) \cap \mathbb{Z} \) and \( b \in [0, 2^\ell) \cap \mathbb{Z} \), respectively, and computes \([a \ g \odot [-b] \ x \) to a third register.

Applying QFTs of size \( 2^{m+\zeta} \) and \( 2^\ell \), respectively, to the two control registers then maps

\[
\frac{1}{\sqrt{2^{m+\zeta}\ell}} \sum_{a=0}^{2^{m+\zeta}-1} \sum_{b=0}^{2^\ell -1} \{a, b, [a \ g \odot [-b] \ x}\ \xrightarrow{\text{QFT}} \frac{1}{\sqrt{2^{m+\zeta}\ell}} \sum_{a=0}^{2^{m+\zeta}-1} \sum_{b=0}^{2^\ell -1} \sum_{j=0}^{2^\ell -1} \sum_{k=0}^{2^{m+\zeta} - 1} e^{2\pi i (aj + 2^{m+\zeta} - \ell bk)/2^{m+\zeta}} \langle j, k, [a - bd] g \rangle .
\]

Compared to Shor’s original algorithm \([2]\), that induces a uniform superposition of \( r \) values in the first two registers before applying the QFTs, the above algorithm \([3]\) induces a uniform superposition of \( 2^{m+\zeta} \) and \( 2^\ell \) values, respectively. This enables it to directly employ the semi-classical QFT \([12]\) with control qubit recycling \([19, 24]\) by interleaving the computational steps.

To compute general discrete logarithms \( d \) in groups of known order \( r \), as is our primary focus, we let \( m \) be the bit length of \( r \), \( \ell \sim m/s \) for \( s \) a positive integer that we refer to as the tradeoff factor, and \( \zeta \) be a small non-negative integer that we refer to as the padding length.

2.1 Notes on generalizations

Although not the primary focus of this paper, it is interesting to note that \( m \), \( \zeta \) and \( \ell \) may be selected to make the above algorithm \([3]\) equivalent to Ekerå’s derivative \([9]\) for computing general discrete logarithms \( d \), and optionally the group order \( r \), in groups of unknown order, and to Ekerå–Håstad’s derivative \([7, 8]\) for computing short discrete logarithms \( d \) in groups of unknown order. In this sense, the above algorithm \([3]\), and — as we shall soon see in what follows — our heuristic analysis of the probability distribution it induces, also captures our previous works \([7, 9]\). We consider such generalizations in Sect. \([7]\).

3 Analysis of the probability distribution

When the superposition \([3]\) is observed, it collapses to \([j, k]\) and \([e \ g\) with probability

\[
\frac{1}{2^{2(m+\zeta+\ell)}} \sum_{(a, b)} \exp \left[ \frac{2\pi i}{2^{m+\zeta}} (aj + 2^{m+\zeta} - \ell bk) \right] ^2
\]

where \((a, b)\) runs over all \( a \in [0, 2^{m+\zeta}) \cap \mathbb{Z} \) and \( b \in [0, 2^\ell) \cap \mathbb{Z} \) such that \( e \equiv a - bd \pmod r \).

3.1 Arguments and angles

In analogy with the analyses in our earlier works \([7, 9]\), let

\[
\alpha_r = \{rj\}_{2^{m+\zeta}}, \quad \alpha_d = \{dj + 2^{m+\zeta} - \ell k\}_{2^{m+\zeta}}, \quad \theta_r = \frac{2\pi \alpha_r}{2^{m+\zeta}}, \quad \theta_d = \frac{2\pi \alpha_d}{2^{m+\zeta}};
\]

where we recall that \([u]_n\) denotes \( u \) reduced modulo \( n \) constrained to \([-n/2, n/2] \).

3.2 Simplifying the expressions

Substituting \( a \) for \( e + bd + n_r r \) in \([4]\) where \( a \in [0, 2^{m+\zeta}) \cap \mathbb{Z} \) yields

\[
\frac{1}{2^{2(m+\zeta+\ell)}} \sum_{b=0}^{2^\ell -1} \sum_{n_r = -[(e+bd)/r]}^{[(2^{m+\zeta}-(e+bd))/r]-1} \exp \left[ \frac{2\pi i}{2^{m+\zeta}} ((e + bd + n_r r)j + 2^{m+\zeta} - \ell bk) \right] ^2 =
\]
where we have introduced \( \delta_b = (e + bd)/r + [-e + bd]/r = (e + bd)/r \mod 1 \in \mathbb{R} \).

In particular, in the last step, we used that \([-t] = -[t]\) for any \(t \in \mathbb{R}\) to show that

\[
\left[ \frac{2^{m+\varsigma}}{r} - (e + bd)/r \right] - [-e + bd]/r = \left[ \frac{2^{m+\varsigma}}{r} - \left( \frac{(e + bd)/r + \delta_b}{} \right) \right] - [-e + bd]/r = \left[ \frac{2^{m+\varsigma}}{r} - \delta_b \right] + [-e + bd]/r - [-e + bd]/r = \left[ \frac{2^{m+\varsigma}}{r} - \delta_b \right]
\]

when simplifying the interval for the inner sum. To simplify the exponential function, we used that \([-e + bd]/r = (e + bd)/r + [-e + bd]/r - (e + bd)/r = \delta_b - (e + bd)/r\), and adjusted the global phase to drop the \(e/r\) term. Furthermore, we used at the outset that

\[
0 \leq a = e + bd + n_r r < 2^{m+\varsigma} \Rightarrow -(e + bd)/r \leq n_r < (2^{m+\varsigma} - (e + bd))/r,
\]

which in turn implies that \([-e + bd]/r \leq n_r < [2^{m+\varsigma} - (e + bd))/r]\).

### 3.3 Expanding the inner sum in a Fourier series heuristically

By letting \(\lambda(\delta_b)\) denote the inner sum in (5), and expanding \(\lambda(\delta_b)\) in a Fourier series, we have

\[
\lambda(\delta_b) = \sum_{n_r = 0}^{2^{m+\varsigma}/r - \delta_b} e^{i\theta_r (n_r + \delta_b)} \simeq \sum_{\eta = -\infty}^{\infty} \hat{\lambda}_\eta e^{2\pi i \eta \delta_b} \quad \text{where} \quad \hat{\lambda}_\eta = \int_0^1 \lambda(\delta) e^{-2\pi i \eta \delta} \, d\delta,
\]

and where we recall that \(\delta_b = (e + bd)/r \mod 1\), so \(\lambda(\delta_b)\) is defined for \(\delta_b \in [0, 1]\). This explains why we pick the integration interval \([0, 1]\) in the above expansion.

Note that this step is heuristic since the sum does not converge absolutely. Note furthermore that \(\lambda(\delta_b)\) has two jump discontinuities when perceived as a cyclic function on \([0, 1]\): One when the function cycles, and one when the number of elements in the sum decreases by one.

Substituting (6) back into (5) yields

\[
\left[ \frac{2^{m+\varsigma}}{r} - \delta_b \right] - 1 \end{equation}

\[
\sum_{n_r = 0}^{2^{m+\varsigma}/r - \delta_b} e^{i\theta_r (n_r + \delta_b)} \right] \right| \right|^2
\]
where we have adjusted the global phase to drop the imaginary factor.

\[
\phi_\eta = \left\{ \theta_d - \frac{d}{r} (\theta_r - 2\pi \eta) \right\} \mod 2\pi
\]

is close to zero. When \( \phi_\eta \approx 0 \), the primary contribution is yielded by

\[
\hat{\lambda}_\eta = \int_0^1 \lambda(\delta) e^{-2\pi i n \delta} \, d\delta = \int_0^1 \sum_{n_r = 0}^{2^{m + 1} / r - 1} e^{i \theta_r (n_r + \delta) - 2\pi i n \delta} \, d\delta
\]

\[
= \int_0^\beta e^{i (\theta_r - 2\pi \eta) \delta} \sum_{n_r = 0}^{2^{m + 1} / r - 1} e^{i \theta_r n_r} \, d\delta + \int_\beta^1 e^{i (\theta_r - 2\pi \eta) \delta} \sum_{n_r = 0}^{2^{m + 1} / r - 2} e^{i \theta_r n_r} \, d\delta
\]

\[
= \frac{e^{i (\theta_r - 2\pi \eta) \beta} - 1}{\theta_r - 2\pi \eta} - 1 + i e^{i (\theta_r - 2\pi \eta) \beta} e^{i (2^{m + 1} / r - 1) - 1} - 1
\]

\[
= \frac{1 - e^{i (\theta_r - 2\pi \eta) (2^{m + 1} / r - 1)}}{\theta_r - 2\pi \eta} - 1 = \frac{1 - e^{i (\theta_r - 2\pi \eta) 2^{m + 1} / r}}{\theta_r - 2\pi \eta} - 1
\]

where \( \beta = 2^{m + 1} / r \mod 1 \). Plugging the expression for \( \hat{\lambda}_\eta \) in (9) back into (7) yields

\[
\sum_{\eta = -\infty}^{\infty} \hat{\lambda}_\eta e^{2\pi i \eta r} \sum_{b = 0}^{2^{m - 1}} e^{i b \phi_\eta} \]

\[
= \frac{1}{2^{2(m + c + r)}} \sum_{\eta = -\infty}^{\infty} \frac{1 - e^{i (\theta_r - 2\pi \eta) 2^{m + 1} / r}}{\theta_r - 2\pi \eta} e^{2\pi i \eta r} \sum_{b = 0}^{2^{m - 1}} e^{i b \phi_\eta}
\]

where we have adjusted the global phase to drop the imaginary factor.

### 3.4 Deriving closed-form expressions heuristically

For each \( \eta \), we expect constructive interference to arise for pairs \( (j, k) \) such that \( \phi_\eta \approx 0 \). For \( \theta_d \) such that \( \phi_\eta \approx 0 \) as a function of \( \theta_d \) and \( \eta \), we may therefore heuristically approximate the expression \( P(\theta_d, \theta_r) \) for the probability of observing an integer pair \( (j, k) \) with angle
pair \((\theta_d, \theta_r)\) summed over all \(c \in [0, r) \cap \mathbb{Z}\), as given by

\[
P(\theta_d, \theta_r) \simeq \frac{1}{2^{2(m+\ell+\ell)}} \sum_{e = 0}^{r-1} \sum_{\eta = -\infty}^{\infty} \frac{1 - e^{i((\theta_r - 2\pi\eta)2^{m+\ell}/r)}}{\theta_r - 2\pi\eta} e^{2\pi i\eta/r} \sum_{b = 0}^{2^f-1} e^{ib\phi_\eta},
\]

by the sum

\[
\sum_{\eta = -B}^{B} P_\eta(\theta_d, \theta_r)
\]

for \(B\) a non-negative integer, and

\[
P_\eta(\theta_d, \theta_r) = \frac{1}{2^{2(m+\ell+\ell)}} \left| \sum_{e = 0}^{r-1} \frac{1 - e^{i((\theta_r - 2\pi\eta)2^{m+\ell}/r)}}{\theta_r - 2\pi\eta} e^{2\pi i\eta/r} \sum_{b = 0}^{2^f-1} e^{ib\phi_\eta} \right|^2
\]

\[
= \frac{r}{2^{2(m+\ell)}} \left| \frac{1 - e^{i((\theta_r - 2\pi\eta)2^{m+\ell}/r)}}{\theta_r - 2\pi\eta} \sum_{b = 0}^{2^f-1} e^{ib\phi_\eta} \right|^2
\]

\[
= \frac{r}{2^{2(m+\ell)}} \left| \frac{1 - e^{i((\theta_r - 2\pi\eta)2^{m+\ell}/r)}}{\theta_r - 2\pi\eta} \right|^2 \left| \frac{1}{2^f} \sum_{b = 0}^{2^f-1} e^{ib\phi_\eta} \right|^2,
\]

where \(P_\eta\) yields the contribution from the main term \(\lambda_\eta\) associated with \(\phi_\eta\).

We expect this to be a good heuristic whenever the peaks associated with different \(\eta\) do not interfere with each other: That is to say, when \(d/r\) has no good rational approximation \(p/q\) with denominator \(q \ll r\). It is reasonable to assume that this assumption holds since \(d\) may be randomized, and since \(r\) is typically prime in cryptologic schemes that are based on the intractability of the discrete logarithm problem, see Sect. 1.4 for further details.

The two functions \(f_\eta(\theta_r)\) and \(h(\phi_\eta)\) introduced above may be placed on closed form, as

\[
f_\eta(\theta_r) = \frac{r}{2^{2(m+\ell)}} \frac{2(1 - \cos((\theta_r - 2\pi\eta)2^{m+\ell}/r))}{(\theta_r - 2\pi\eta)^2} \quad \text{for } \theta_r \neq 2\pi\eta \quad \text{and} \quad f_0(0) = \frac{1}{r},
\]

where we use that \(|\theta_r| \leq \pi\) implies that \(\theta_r - 2\pi\eta = 0\) if and only if \(\eta = 0\), and as

\[
h(\phi_\eta) = \frac{1}{2^{2\ell}} \frac{\cos(\phi_\eta 2^\ell) - 1}{\cos(\phi_\eta) - 1} \quad \text{for } \phi_\eta \neq 0 \quad \text{and} \quad h(0) = 1.
\]

4 Capturing the probability distribution

In what follows below, we describe how the probability distribution induced by the quantum algorithm may be heuristically captured by summing up the parts of the distribution where the probability mass is expected to be concentrated due to constructive interference arising.

4.1 The notion of \(B_\eta-B_\Delta\)-good pairs \((j, k)\)

For each \(\eta\) and \(j\), there is an optimal value \(k_{\eta,0}\) of \(k\) that minimizes \(|\phi_\eta|\), as given by

\[
k_{\eta,0} = \left[ \frac{1}{2^{m+\ell-t}} \left( -dj + \frac{d}{r} \left( \{rj\}2^{m+\ell} - 2^{m+\ell}\eta \right) \right) \mod 2^\ell \right]
\]

\[
= \left( \frac{1}{2^{m+\ell-t}} \left( -dj + \frac{d}{r} \left( \{rj\}2^{m+\ell} - 2^{m+\ell}\eta \right) \right) - \delta_\eta \right) + 2^\ell \nu_\eta
\]
for some $\delta_\eta \in [-\frac{1}{2}, \frac{1}{2})$ and some integer $\nu_\eta$. More specifically,
\[
\delta_\eta = \left\{ \frac{1}{2m^{+\varsigma} - \ell} \left( -dj + \frac{d}{r} \{rj\} 2^{m^{+}\varsigma} - 2^{m^{+}\varsigma} \nu_\eta \right) \right\}_1.
\] (16)

For $k = k_{\eta,\Delta} = (k_{\eta,0} + \Delta) \mod 2^\ell$, for $\Delta \in [2^{\ell-1}, 2^{\ell-1}) \cap \mathbb{Z}$, it then holds that
\[
\phi_\eta = \left\{ \frac{\theta_d - d}{r} \left( \theta_r - 2\pi \eta \right) \right\} = \frac{2\pi}{2^{m^{+}\varsigma}} \left\{ \frac{\alpha_d - d}{r} \left( \alpha_r - 2^{m^{+}\varsigma} \nu_\eta \right) \right\}_{2^{m^{+}\varsigma}},
\] (17)

which is close to zero for small $\Delta$, leading to constructive interference arising.

It is convenient therefore to introduce the notion of a $B_{\eta}$-$B_{\Delta}$-good pair $(j,k)$:

**Definition 1.** A pair $(j,k)$ is said to be $B_{\eta}$-$B_{\Delta}$-good if

\[
\phi_\eta = \frac{2\pi}{2^{m^{+}\varsigma}} \left\{ dj + 2^{m^{+}\varsigma} t_k - \frac{d}{r} \{rj\} 2^{m^{+}\varsigma} - 2^{m^{+}\varsigma} \nu_\eta \right\}_{2^{m^{+}\varsigma}} = \frac{2\pi}{2^{\ell}} \{\Delta - \delta_\eta\}_{2^{\ell}}
\]

for $\eta, \Delta$ integers such that $|\eta| \leq B_{\eta}$ and $|\Delta| \leq B_{\Delta} < 2^{\ell-1}$, and some $\delta_\eta \in [-\frac{1}{2}, \frac{1}{2})$.

### 4.2 The distribution of arguments $\alpha_r$

Before proceeding to capture the distribution, we need to understand how the arguments $\alpha_r$ are distributed on $[-2^{m^{+}\varsigma-1}, 2^{m^{+}\varsigma-1}) \cap \mathbb{Z}$ as a function of $r$. In analogy with [9] App. A.3:

**Claim 1.** There are $2^{2\nu}\nu_r$ distinct $j \in \{0, 2^{m^{+}\varsigma}\} \cap \mathbb{Z}$ that yield a given argument $\alpha_r = \{rj\} 2^{m^{+}\varsigma}$ for $2^{\nu}\nu_r$ the greatest power of two to divide both $r$ and $2^{m^{+}\varsigma}$. It follows that there are $2^{m^{+}\varsigma-\nu_r}$ distinct arguments $\alpha_r$, that are all multiples of $2^{\nu}\nu_r$, and that all occur with multiplicity $2^{\nu}\nu_r$.

**Proof.** It is easy to see that the frequencies $j + 2^{m^{+}\varsigma-\nu_r} t_r \mod 2^{m^{+}\varsigma}$ for $t_r \in \{0, 2^{\nu}\nu_r\} \cap \mathbb{Z}$ yield the same argument $\alpha_r = \{rj\} 2^{m^{+}\varsigma}$, and that all arguments must be multiples of $2^{\nu}\nu_r$, since the modulus when computing $\alpha_r$ is $2^{m^{+}\varsigma}$, and so the claim follows.

In what follows, we shall assume that $r$ is not a multiple of $2^{m^{+}\varsigma}$, in which case $2^{\nu}\nu_r$ is the greatest power of two to divide $r$. This slightly simplifies the analysis. Note that the situation where $r$ is a multiple of $2^{m^{+}\varsigma}$ cannot arise when computing general discrete logarithms in groups of known order $r$, since $m$ is then the bit length of $r$. When computing short discrete logarithms, it can in theory arise, but not if $r$ is prime as is typically the case, see Sect. 4.4.

A notable special case is Ekerå–Hastad’s algorithm [7, 8] for factoring $N = pq$, where $p, q$ are two distinct random primes of similar bit length, by means of computing a short discrete logarithm in a cyclic $r$-order subgroup of $\mathbb{Z}_N^*$. In this algorithm, $r$ must divide $\operatorname{lcm}(p - 1, q - 1)$, so if $2^{m^{+}\varsigma}$ divides $r$ then $2^{m^{+}\varsigma}$ must divide either $p - 1$ or $q - 1$, and furthermore $2^m$ is of the same size as $p + q$ or greater, so the factors $p, q$ may then be deduced classically.

### 4.3 Probability of observing a $B_{\eta}$-$B_{\Delta}$-good pair

By the heuristic in (11), the probability of observing a $B_{\eta}$-$B_{\Delta}$-good pair is

\[
\sum_{\eta = -B_{\eta}}^{B_{\eta}} \sum_{\alpha_r' = -2^{m^{+}\varsigma-\nu_r}}^{2^{m^{+}\varsigma-\nu_r}} 2^{\nu_r} f_{\eta} \left( \frac{2\pi}{2^{m^{+}\varsigma}} 2^{\nu_r} \alpha_r' \right) \sum_{\Delta = -B_{\Delta}}^{B_{\Delta}} h \left( \frac{2\pi}{2^{\ell}} (\Delta - \delta_\eta) \right),
\]

where we sum over all arguments $\alpha_r$ with multiplicity $2^{\nu}\nu_r$ by introducing $\alpha_r'$, see Sect. 4.2, and where $\delta_\eta$ is a function of $\alpha_r = \{rj\} 2^{m^{+}\varsigma}$ and hence of $\alpha_r'$, see Sect. 4.1. Furthermore, as $h(\phi_\eta)$ is $2\pi$-periodic, we have that $h(\phi_\eta) = h(2\pi(\Delta - \delta_\eta)/2^{\ell}) = h(2\pi(\Delta - \delta_\eta)/2^{\ell})$.  

11
4.4 Asymptotic soundness of the heuristic

By Thm. 1 below, the heuristic expression (18) for the probability of observing a \(B_\eta-B_\Delta\)-good pair tends to one asymptotically, when summed over all \(\Delta\), and \(B_\eta\) and/or \(\varsigma\) tend to infinity:

**Theorem 1.** It holds that

\[
\lim_{B_\eta \to \infty \text{ and/or } \varsigma \to \infty} \sum_{\eta = -B_\eta}^{B_\eta} \sum_{\alpha' = -2^{m+\kappa - 1}}^{2^{m+\kappa - 1}} 2^{\kappa r} f_\eta \left( \frac{2\pi}{2m+\varsigma} 2^{\kappa r} \alpha' \right) \sum_{\Delta = -B_\Delta}^{B_\Delta} h \left( \frac{2\pi}{2r} (\Delta - \delta_\eta) \right) = 1.
\]

**Proof.** The theorem follows by combining Lem. C.1 and Lem. C.3 in App. C. ■

4.5 Lower-bounded probability of observing a \(B_\eta-B_\Delta\)-good pair

By Thm. 2 below, the heuristic expression (18) for the probability of observing a \(B_\eta-B_\Delta\)-good pair quickly tends to one as \(B_\eta\) and \(B_\Delta\) and/or \(\varsigma\) tend to infinity:

**Theorem 2.** The probability of observing a \(B_\eta-B_\Delta\)-good pair, as heuristically given by

\[
\sum_{\eta = -B_\eta}^{B_\eta} \sum_{\alpha' = -2^{m+\kappa - 1}}^{2^{m+\kappa - 1}} 2^{\kappa r} f_\eta \left( \frac{2\pi}{2m+\varsigma} 2^{\kappa r} \alpha' \right) \sum_{\Delta = -B_\Delta}^{B_\Delta} h \left( \frac{2\pi}{2r} (\Delta - \delta_\eta) \right) \geq \max \left( 0, 1 - \frac{1}{2m+\varsigma} \frac{1}{2} \frac{1}{B_\eta} \right),
\]

for \(B_\eta\) and \(\varsigma\) non-negative integers, for \(B_\Delta \in \{0, 2^{\ell-1}\} \cap \mathbb{Z}\), and for

\[
\epsilon_{B_\eta} = \epsilon(B_\eta + 1/2) \quad \text{and} \quad \epsilon_{B_\Delta} = \epsilon(2^{m+\kappa - \varsigma}(B_\eta + 1/2)) \quad \text{for} \quad \epsilon(x) = \frac{1}{2x^2} + \frac{1}{6x^2}.
\]

**Proof.** The theorem follows by combining Lem. C.2 and Lem. C.4 in App. C. ■

The lower bound (19) in Thm. 2 on the heuristic expression for the probability of observing a \(B_\eta-B_\Delta\)-good pair is tabulated in Tabs. 1-2 in App. A.1 for \(\varsigma = 0\) as a function of \(B_\eta\) and \(B_\Delta\), and for \(B_\eta = 0\) as a function of \(\varsigma\) and \(B_\Delta\), respectively. As may be seen in Tabs. 1-2, it is necessary to select \(B_\eta\) and \(B_\Delta\) and/or \(\varsigma\), greater than zero for the probability to exceed 99%.

It follows from Thm. 2 that asymptotically the sum over \(h\) is \(1 - O(B_\eta^{-1})\), whereas the sum over \(f_\eta\) is \(1 - O(2^{\varsigma})\) and \(1 - O(B_\eta^{-1})\), respectively. By parameterizing \(B_\Delta\), \(B_\eta\) and/or \(\varsigma\), to grow slowly as functions of \(m\), it follows that we may make the probability tend to one asymptotically whilst retaining the polynomial runtime of the algorithm:

**Corollary 2.1.** For \(m\) the bit length of \(r\), \(B_\eta = \omega_m(1)\), and \(B_\Delta = \omega_m(1) < 2^{\ell-1}\), the probability of observing a \(B_\eta-B_\Delta\)-good pair, as heuristically given by

\[
\sum_{\eta = -B_\eta}^{B_\eta} \sum_{\alpha' = -2^{m+\kappa - 1}}^{2^{m+\kappa - 1}} 2^{\kappa r} f_\eta \left( \frac{2\pi}{2m+\varsigma} 2^{\kappa r} \alpha' \right) \sum_{\Delta = -B_\Delta}^{B_\Delta} h \left( \frac{2\pi}{2r} (\Delta - \delta_\eta) \right),
\]

tends to one in the limit as \(m \to \infty\).

**Proof.** The corollary follows by taking the limit of the lower bound in Thm. 2. ■

Recall that the two heuristic assumptions up to this point are that it is safe to expand \(\lambda(\delta_\eta)\) in a Fourier series in Sect. 3.3 and that \(d/r\) has no good rational approximation \(p/q\) with denominator \(q \ll r\) so that \(p/(\theta_d, \theta_r)\) can be approximated by a sum over \(F_\eta(\theta_d, \theta_r)\) in Sect. 3.4.
4.6 Expected probability of observing a $B_\eta$-$B_\Delta$-good pair

The lower bound on the heuristic in Thm. 2 tends to one asymptotically, but it does not provide a good estimate for the expected probability of obtaining a $B_\eta$-$B_\Delta$-good pair for small values of $B_\Delta$, and $B_\eta$ and/or $\varsigma$. In this section, we heuristically derive such an estimate.

Specifically, substituting (17) into (11), heuristically replacing $\delta_\eta$ with a stochastic variable $\delta$ uniformly distributed on $[-\frac{r}{2}, \frac{r}{2}]$, and computing the expectation value, yields

$$\sum_{\eta = -B_\eta}^{B_\eta} f_\eta(\theta_r) \int_{-1/2}^{1/2} h \left( \frac{2\pi}{2r}(\Delta - \delta) \right) \, d\delta. \tag{20}$$

Integrating (20) over all $\theta_r = 2\pi \alpha_r/2^{m+\varsigma}$ with multiplicity, see Sect. 4.2, and then summing over all integers $\Delta$ such that $|\Delta| \leq B_\Delta$ where $B_\Delta < 2^{\ell-1}$, yields

$$\sum_{\eta = -B_\eta}^{B_\eta} 2^\epsilon r \int_{-2^{m+\varsigma}-\kappa_r-1}^{2m+\varsigma-1} f_\eta \left( \frac{2\pi}{2^{m+\varsigma}} 2^\epsilon \alpha_r' \right) \, d\alpha_r' \sum_{\Delta = -B_\Delta}^{B_\Delta} \int_{-1/2}^{1/2} h \left( \frac{2\pi}{2r}(\Delta - \delta) \right) \, d\delta$$

$$= \sum_{\eta = -B_\eta}^{B_\eta} 2^\epsilon r \int_{-2^{m+\varsigma}-\kappa_r-1}^{2m+\varsigma-1} f_\eta \left( \frac{2\pi}{2^{m+\varsigma}} 2^\epsilon \alpha_r' \right) \, d\alpha_r' \int_{-B_\Delta-1/2}^{B_\Delta+1/2} h \left( \frac{2\pi}{2r} v \right) \, dv. \tag{21}$$

The expected probability of observing a $B_\eta$-$B_\Delta$-good pair, as heuristically given by (21), is tabulated in Tabs. 3-4 in App. A.2, for $\varsigma \leq 0$ as a function of $B_\eta$ and $B_\Delta$, and for $B_\eta = 0$ as a function of $\varsigma$ and $B_\Delta$, respectively. As may be seen in Tabs. 3-4, it is necessary to select $B_\Delta$, and $B_\eta$ and/or $\varsigma$, greater than zero for the probability to exceed 99%.

The choice of $m$ and $s$ is of little consequence, for as long as $m$ is sufficiently large, and for as long as $s$ is not very large in relation to $m$. The size of $r$ in relation to $2^m$ is important, however. In Tab. 4, the order $r = 2^m - 1$ is maximal. If $r$ is instead close to the lower end of the interval, say $r = 2^{m-1} + 1$, then essentially all rows in the table are shifted upwards one step (as halving $r$ is equivalent to adding a padding bit) yielding higher probability estimates.

In practice, the order $r$ is often a random prime on $(2^{m-1}, 2^m)$ in cryptologic schemes that are based on the computational intractability of the discrete logarithm problem. A notable exception are the NIST P-192, P-224, P-256, P-384 and P-521 elliptic curve groups [3 Sect. 3.2.1] that have orders very close to $2^m$. This is a side effect of the base field moduli having been selected to enable fast arithmetic. Some other curves also have this property.

4.6.1 On the relation to Mosca’s lower bound

Mosca gives a lower bound of $((r - 1)/r)(8/\pi^2)^2 \approx (8/\pi^2)^2 \approx 0.6570$ on the success probability in his PhD thesis [18 Cor. 19 on p. 58]. Specifically, Mosca’s bound is for Shor’s algorithm for computing discrete logarithms, as modified in Sect. 2 to enable qubit recycling.

Mosca does not parameterize the algorithm in $\varsigma$ and $\ell$. Furthermore, Mosca does not consider making tradeoffs, or searching in the classical post-processing. For $r \in (2^{m-1}, 2^m) \cap \mathbb{Z}$, we may fix $B_\eta = 0$, $\varsigma = 2$ and $\ell = m + \varsigma$ to make our algorithm equivalent to the algorithm analyzed by Mosca. For this choice of parameters, Mosca’s lower bound is consistent with our heuristic: Specifically, by (21) our heuristic yields a higher expected success probability of

$$2^\epsilon r \int_{-2^{m+\varsigma}-\kappa_r-1}^{2m+\varsigma-1} f_0 \left( \frac{2\pi}{2^{m+\varsigma}} 2^\epsilon \alpha_r' \right) \, d\alpha_r' \int_{-2}^2 h \left( \frac{2\pi}{2r} v \right) \, dv \approx 0.9024$$

for maximal $r = 2^m - 1$, where we have evaluated the expression numerically for $m = 128$ with Mathematica. The same result is obtained for greater $m$. By comparison, our lower bound on the heuristic (19) in Thm. 2 yields $\approx 0.4770$ for $B_\Delta = 1$ when rounding down.

Note that it is required that $t = |r(\Delta - \delta_\eta)/2^{m+\varsigma}| = 0$ and $\eta = 0$ to solve without searching as in Shor’s original post-processing algorithm, see Sect. 6.1. In the limit as $m \to \infty$, it holds that $t = 0$ when $v = \Delta - \delta_\eta \in [-2, 2)$, giving rise to the above integration limits in $v$, and to the choice of $B_\Delta = 1$. We consider only the contribution from $f_0$ since $\eta = 0$. 

13
5 Simulating the quantum algorithm

We now have the necessary framework in place to heuristically simulate the quantum algorithm classically for a known problem instance given by \(d\) and \(r\). This in the sense that we may heuristically sample the probability distribution induced by the quantum algorithm to generate outputs \((j, k)\) that are heuristically representative of outputs that would be produced by the quantum algorithm if executed on a quantum computer.

Note explicitly that the simulator requires \(d\) and \(r\) to be known: It cannot be used to simulate problem instances given on the form of two group elements \(g\) and \(x = [d]g\).

5.1 Sampling the probability distribution

For all integers \(\eta\) such that \(|\eta| \leq B_\eta\), we first construct a high-resolution histogram in \(\alpha_r\) by piecewise numerical integration of \(f_\eta(\theta_r)\) as defined in \(\text{(13)}\) over small intervals in \(\alpha_r\) (where we recall that \(\theta_r = 2\pi\alpha_r/2^{m+\varsigma}\), see Sect. 3.1 and that \(\alpha_r\) is distributed as explained in Sect. 4.2).

To sample \(\eta\) and \(j\), we first sample \(\eta\) and an argument \(\alpha_r\) from the resulting histograms, and then sample \(j\) from the set of all integers \(j\) that yield \(\alpha_r\), as given by

\[
j = \left( \frac{\alpha_r}{2^{m+\varsigma}} \left( \frac{r}{2^{\kappa_r}} \right)^{-1} + 2^{m+\varsigma} \right) \mod 2^{m+\varsigma}
\]

as \(t_r\) runs through all integers on \([0, 2^{\kappa_r}) \cap \mathbb{Z}\). This sampling procedure is analogous to that in \([\text{9}]\) Lem. A.4 in App. A.3]. Note that all arguments \(\alpha_r\) are multiples of \(2^{\kappa_r}\), see Sect. 4.2 and that this fact must be accounted for when sampling \(\alpha_r\) from the histogram.

Given \(\eta\) and \(j\), we let \(k_{\eta,0}\) be the value of \(k \in [0, 2^\ell) \cap \mathbb{Z}\) that minimizes \(|\phi_\eta|\), where we recall that the expression for \(k_{\eta,0}\) is given in \(\text{(15)}\), and that

\[
\phi_\eta = \left\{ \theta_d - d \left( \theta_r - 2\pi \eta \right) \right\} \mod 2^{m+\varsigma} = \frac{2\pi}{2^{m+\varsigma}} \left\{ \frac{\alpha_d}{2} \left( \alpha_r - 2^{m+\varsigma} \right) \right\} \mod 2^{m+\varsigma}.
\]

For all integers \(\Delta\) such that \(|\Delta| \leq B_\Delta\) where \(B_\Delta < 2^{\ell-1}\), we then compute the probability

\[
h \left( \frac{2\pi}{2^{m+\varsigma}} \left\{ dj + 2^{m+\varsigma} \right\} \right) \mod 2^{m+\varsigma} \]  

of observing \(k = k_{\eta,\Delta} = (k_{\eta,0} + \Delta) \mod 2^\ell\), for \(h\) as in \(\text{(14)}\), to construct a histogram in \(\Delta\).

We sample \(\Delta\), and hence \(k = k_{\eta,\Delta}\), from the resulting histogram, and return \((j, k)\) as the output from the simulator. If either \(\eta\) and \(\alpha_r\), or \(\Delta\), fall outside the respective ranges of the histograms when sampled, we instead return a sampling error. By increasing \(B_\Delta\) and \(B_\eta\) and/or \(\varsigma\), we may suppress the probability of such errors occurring in practice.

Note that it is permissible to first sample \(\eta\) and \(j\) from the distribution induced by \(f_\eta\) as above, and to then sample \(k\) given \(\eta\) and \(j\) via the distribution induced by \(h\): This is equivalent to first computing and observing \(j\), and to then computing and observing \(k\) given \(j\).

5.1.1 Implementation remarks

We have implemented the simulator for the purpose of heuristically evaluating the efficiency of the two classical post-processing algorithms described in Sect. 6.

5.2 On the relation to our earlier works

The simulator described in this section is similar to the simulators previously developed for Shor’s and Seifert’s order-finding algorithms \([\text{9}]\) App. A.1, for Ekerä’s algorithm \([\text{9}]\) for computing general discrete logarithms with tradeoffs in groups of unknown order, and for Ekerä’s and Ekerä–Håstad’s algorithms \([\text{6}, \text{8}]\) for computing short discrete logarithms.
6 Classical post-processing

When not making tradeoffs (i.e., when $s = 1$), we may individually post-process pairs $(j, k)$ yielded by the quantum algorithm in the manner originally proposed by Shor [29].

Shor’s original post-processing algorithm recovers $d$ given $r$ from any pair $(j, k)$ for which the offset $\Delta = \{k - \kappa_\eta\delta\}_t$ is close to zero when $\eta = 0$. Specifically, the algorithm requires $t = \lceil r(\Delta - \delta_\eta)/2^\ell \rceil = 0$ to be successful. In Sect. 6.1 below, we describe a slightly modified version of Shor’s post-processing algorithm. The modified algorithm performs two searches, over $t$ and $\eta$ respectively, to recover $d$ from any $B_\eta B_\Delta$-good pair $(j, k)$.

To handle tradeoff factors $s \geq 1$, we adapt the lattice-based post-processing algorithms from our earlier works [6–9], so as to recover $d$ given $r$ from a set of $n$ pairs $\{(j_1, k_1), \ldots, (j_n, k_n)\}$ resulting from $n \geq s$ independent runs of the quantum algorithm. This lattice-based post-processing algorithm is described in Sect. 6.2.

6.1 Solving individual runs by modifying Shor’s post-processing

Assume that the quantum algorithm yields a $B_\eta B_\Delta$-good pair $(j, k)$.

For the arguments $\alpha_d, \alpha_r$ and angle $\phi_\eta$ associated with this pair, it then holds that
\[
\frac{2^{m+c}}{2\pi} \phi_\eta = \left\{ \frac{\alpha_d}{r} \left( \alpha_r - 2^{m+c} \eta \right) \right\}_{2^{m+c}} = 2^{m+c-\ell} (\Delta - \delta_\eta)
\]
\[
\Rightarrow \left\{ dj + 2^{m+c-\ell} k \right\}_{2^{m+c}} - \frac{d}{r} \left\{ r j \right\}_{2^{m+c}} = 2^{m+c-\ell} (\Delta - \delta_\eta)
\]
\[
\Rightarrow d j + 2^{m+c-\ell} k - \frac{d}{r} \left\{ r j \right\}_{2^{m+c}} = 2^{m+c-\ell} (\Delta - \delta_\eta)
\]
\[
\Rightarrow r d j + 2^{m+c-\ell} r k - \{ r j \}_{2^{m+c}} + 2^{m+c} n_d = 2^{m+c-\ell} r (\Delta - \delta_\eta)
\]
\[
\Rightarrow d (r j - \{ r j \}_{2^{m+c}}) + 2^{m+c-\ell} r k + 2^{m+c} n_d = 2^{m+c-\ell} r (\Delta - \delta_\eta)
\]
\[
\Rightarrow d j + 2^{m+c-\ell} r k + 2^{m+c} n_d + n_d r = \frac{r (\Delta - \delta_\eta)}{2^\ell}
\]
\[
\Rightarrow d(z + \eta) + \left\lceil \frac{r k}{2^\ell} \right\rceil + n_d r = \left\lfloor \frac{r (\Delta - \delta_\eta)}{2^\ell} \right\rfloor
\]
\[
\Rightarrow d(z + \eta) + \left\lceil \frac{r k}{2^\ell} \right\rceil \equiv t \pmod{r}
\]

which implies that we may solve for $d$ if $z + \eta$ is invertible modulo $r$. Specifically
\[
d \equiv \left( t - \left\lceil \frac{r k}{2^\ell} \right\rceil \right) (z + \eta)^{-1} \pmod{r} \tag{23}
\]

where we need to search at most $2 \cdot |t| + 1 \leq 2 \cdot \lceil r(B_\Delta + 1/2)/2^\ell \rceil + 1$ values of $t$ for each of the at most $2B_\eta + 1$ values of $\eta$ to recover $d$ from $(j, k)$ given $r$. Suppose that $\ell = m + \nu_\ell$ for some fixed $\nu_\ell \in \mathbb{Z}$ such that $|\nu_\ell|$ is small. Then we need to search at most
\[
2 \cdot \lceil (B_\Delta + 1/2)/2^{m+\nu_\ell} \rceil + 1 \leq 2 \cdot \lceil (\Delta + 1/2)/2^{\nu_\ell} \rceil + 1
\]

values of $t$ for each of the $2B_\eta + 1$ values of $\eta$, which is feasible if $B_\eta$ and $B_\Delta$ are sufficiently small. A minimum requirement for the search to be efficient is that $B_\eta, B_\Delta \in O(\text{poly}(m))$.

If $\nu_\ell$ is negative, work in the quantum algorithm is traded for work in the classical post-processing, and vice versa. A reasonable choice is to fix $\nu_\ell = 0$, or to fix some negative $\nu_\ell$.

\footnote{When modified to account for the uniform initialization of the two control registers, and to account for the registers being of length $m + c$ and $\ell$ qubits, respectively, but not to account for searching.}
6.1.1 On the probability of $z + \eta$ being invertible modulo $r$

Even though we search over $\eta$ in the post-processing, we require $z + \eta$ to be invertible modulo $r$
for the correct $\eta$ only. When $r$ is prime, the inverse of $z + \eta$ modulo $r$ exists if and only if
$z + \eta \not\equiv 0 \pmod{r}$, allowing us to bound the probability of $z + \eta$ not being invertible as follows:

**Lemma 1.** The probability that $z + \eta \equiv 0 \pmod{r}$ for some fixed $\eta \in \mathbb{Z}$ is at most $2^{\ell+5-m}$,
for $m$ the bit length of $r$ and $\ell \geq 0$. For fixed $\ell \geq 0$, the upper bound on the probability tends
to zero at an exponential rate as $r \rightarrow \infty$ and hence $m \rightarrow \infty$.

**Proof.** The lemma follows immediately from Lem. C.5 in App. C.3, since $z \equiv -\eta \pmod{r}$ with
probability at most $2^{\ell+5-m}$ by said lemma.

To generalize the above, regardless of whether $r$ is prime or composite, the modular inverse
exists if $\gcd(z + \eta, r) \neq 1$. If we assume that $r$ has no small prime factors, we may obtain a
bound on the probability of $z + \eta$ not being invertible modulo $r$ as follows:

**Lemma 2.** The probability that $\gcd(z + \eta, r) \neq 1$ for some fixed $\eta \in \mathbb{Z}$ is at most

$$\frac{2^\ell 5 \log_2(r)}{B_r \log_2(B_r)}$$

if $r$ has no prime factor less than $B_r$, and for $m$ the bit length of $r$ and $\ell \geq 0$. For fixed $\ell \geq 0$
and $B_r = \Omega(m)$, the upper bound on the probability tends to zero as $r \rightarrow \infty$ and hence $m \rightarrow \infty$.

**Proof.** The proof follows by combining Lem. C.5 with Claim C.4 both in App. C.3.

For $\phi$ Euler’s totient function, there are $r - \phi(r)$ values of $z' = \pmod{z}$ mod $r$ such that $z' + \eta$ is
not coprime to $r$, and hence $r - \phi(r)$ values of $z'$ such that $\gcd(z' + \eta, r) \neq 1$, or equivalently
such that $\gcd(z + \eta, r) \neq 1$. By Lem. C.5 in App. C.3 the probability of observing $(j, k)$ with $j$
yielding one of these $r - \phi(r)$ values of $z'$ is at most

$$\frac{2^\ell 5 \log_2(r)}{B_r \log_2(B_r)}$$

where we have used Claim C.4 in App. C.3 to bound $r - \phi(r)$, and so the lemma follows.

Recall that $r$ may be assumed to have no small prime factors without loss of generality,
see Sect. 1.4 since we can efficiently find all factors less than $B_r$ when $B_r = O(\text{poly}(m))$ and
perform Pohlig–Hellman decomposition. For this reason, $r$ is typically prime in cryptographic
schemes that are based on the computational intractability of the discrete logarithm problem.

Note that the resulting post-processing algorithm is identical to that originally described
by Shor [29], except for the searches over $t$, $\eta$ and the partial factorization of $r$.

6.1.2 On the asymptotic success probability

As $r \rightarrow \infty$, the probability of recovering $d$ given $r$ in a single run of the quantum algorithm
heuristically tends to one, if we first perform Pohlig–Hellman decomposition for all factors of $r$
less than $B_r = \Omega(m)$, pick $\ell \sim m$, and perform two searches in the post-processing:

**Theorem 3.** The probability of recovering $d$ given $r$ in a single run of the quantum algorithm
in Sect. 3 with $m$ the bit length of $r$, $\ell \geq 0$ and $\ell = m + \nu_t$ for some fixed $\nu_t \in \mathbb{Z}$ such that $|\nu_t|$
is small, heuristically tends to one in the limit as $r \rightarrow \infty$, when

i) $r$ has no prime factors $< B_r = \Omega(m)$,

ii) using the post-processing in Sect. 6.1 and searching all $\eta, t \in \mathbb{Z}$ such that $|\eta| \leq B_\eta$ and
$|t| \leq \lfloor (B_\Delta + 1/2)/2^{\nu_t} \rfloor$, where $B_\eta = \omega_m(1)$ and $B_\Delta = \omega_m(1) < 2^{\ell-1}$, and

iii) using the heuristic [11] to bound the probability of the run yielding a $B_\eta$-$B_\Delta$-good pair.

16
Proof. By Cor. 2.1, the probability as given by the heuristic (11) of observing a $B_\eta$-$B_\Delta$-good pair $(j, k)$ tends to one in the limit in the theorem, and for such a pair the classical post-processing algorithm in Sect. 6.1 recovers $d$ given $r$ when searching all $\eta, t \in \mathbb{Z}$ such that $|\eta| \leq B_\eta$ and $|t| \leq \lfloor (B_\Delta + 1/2)/2^\nu \rfloor$ except if gcd$(z + \eta, r) \neq 1$ for the correct $\eta$.

By Lem. 2, the probability that gcd$(z + \eta, r) \neq 1$ for the correct $\eta$ tends to zero in the limit as $r$ and hence $m \to \infty$ when $B_r = \Omega(m)$, and so the theorem follows. ■

6.1.3 On simulating the complete algorithm

We have implemented the post-processing algorithm and tested it using the simulator in Sect. 5.

6.1.4 On the success probability of Shor’s original algorithm

Recall that it is required that $t = \lfloor r(\Delta - \delta_0)/2^{m+c} \rfloor = 0$ and $\eta = 0$ to solve without searching as in Shor’s original post-processing algorithm, see Sect. 6.1.

For $\varsigma = 0$ and $\ell = m$, as in Shor’s original algorithm\(^5\) and maximal $r = 2^m - 1$, we have that $t = 0$ when $\Delta - \delta_0 \in [-\frac{1}{2}, \frac{1}{2})$ in the limit as $m \to \infty$. This implies that Shor’s original post-processing algorithm succeeds if $(j, k)$ is a $B_\eta$-$B_\Delta$-good pair for $B_\eta = B_\Delta = 0$. Such a pair is heuristically yielded with expected probability approximately 0.5986 by Tab. 4.\(^6\)

At the other extreme, for $r = 2^{m-1} + 1$ (where we add one to avoid making $r$ a perfect power of two), we have that $t = 0$ when $v = \Delta - \delta_0 \in [-1, 1)$, yielding the expected probability

$$2^{2r} \int_{-2^{m+c}-\nu+\varepsilon}^{2^{m+c}+\nu-\varepsilon} f_0 \left( \frac{2\pi}{2^{m+c}} 2^{2r} a_r \right) d\alpha_r \int_{-1}^{1} h \left( \frac{2\pi}{2^{m+c}} v \right) dv \approx 0.8151$$

by the heuristic (21), where we have evaluated the expression numerically for $m = 128$ with Mathematica.\(^7\) The same result is obtained for greater $m$.

We conclude from the above that Shor’s original algorithm, modified to allow the semiclассical QFT to be used with control qubit recycling, achieves an expected success probability of approximately 60% to 82% according to our heuristic.

6.2 Solving using lattice-based post-processing

Assume that $s \geq 1$, and that $n \geq s$ independent runs of the quantum algorithm are performed with $\ell \sim m/s$, yielding pairs $(j_i, k_i)$ for $i \in [1, n] \cap \mathbb{Z}$ with associated arguments

$$\alpha_{r,i} = \{r j_i\}_{2^{m+c}} \quad \text{and} \quad \alpha_{d,i} = \{d j_i + 2^{m+c-\ell} k_i\}_{2^{m+c}}.$$

Furthermore, let $\eta_i$ for $i \in [1, n] \cap \mathbb{Z}$ be integers that minimize $|\phi_{\eta,i}|$, where

$$\phi_{\eta,i} = \frac{2\pi}{2^{m+c}} \left\{ \alpha_{d,i} - \frac{d}{r} \left( \alpha_{r,i} - 2^{m+c+\varsigma} \eta_i \right) \right\}_{2^{m+c}}. \quad (24)$$

If $\varsigma$ is sufficiently large, then with high probability $\eta_i = 0$ for all $i \in [1, n] \cap \mathbb{Z}$. Otherwise, the $\eta_i$ are still typically small, enabling the correct $\eta_i$ to be found exhaustively if $n$ is small.

\(^5\)Modified to induce a uniform superposition over $[0, 2^m) \cap \mathbb{Z}$ instead of $[0, r) \cap \mathbb{Z}$ for $r$ the bit length of $r$.

\(^6\)In the initial pre-print of this paper, for $r = 2^m - 1 + 1$, we accidentally took $v = \Delta - \delta_0 \in [-\frac{1}{2}, \frac{1}{2})$ as opposed to $v \in [-1, 1)$, leading us to estimate the success probability to approximately 70% as opposed to 82%.
Let $L$ be the lattice given by integer combinations of the rows of
\[
A = \begin{bmatrix}
j_1 + \frac{(2^{m+\varsigma} \eta_1 - \alpha_{r,1})}{r} & \cdots & j_n + \frac{(2^{m+\varsigma} \eta_n - \alpha_{r,n})}{r} & 2^{\varsigma - t} \\
\vdots & \ddots & \vdots & \vdots \\
0 & \cdots & 2^{m+\varsigma} & 0 \\
\end{bmatrix}
\begin{bmatrix}
\tilde{a}_0 \\
\tilde{a}_1 \\
\vdots \\
\tilde{a}_n \\
\end{bmatrix}
\] (25)
for $2^{\varsigma - t}$ a scaling factor elaborated on below, and enumerate the vectors in $L$ closest to
\[
\vec{v} = (-2^{m+\varsigma - t} k_1, \ldots, -2^{m+\varsigma - t} k_n, 0) \in \mathbb{Z}^{n+1}
\]
with the aim of finding
\[
\vec{u} = d \cdot \tilde{a}_0 + \sum_{i=1}^{n} t_i \cdot \tilde{a}_i \in L \quad \text{for some} \quad t_1, \ldots, t_n \in \mathbb{Z}
\]
where $\tilde{a}_i$ for $i \in [0, n] \cap \mathbb{Z}$ are the rows of $A$, see the definition in \[20\].

The vector $\vec{u}$ yields $d$, as its last component is $2^{\varsigma - t} d$. Furthermore, $\vec{u}$ is within distance
\[
|\vec{u} - \vec{v}| = |(\Delta_1, \ldots, \Delta_n, 2^{\varsigma - t} d)|
\]
of $\vec{v}$, where, for some appropriately selected $t_i \in \mathbb{Z}$ for $i \in [1, n] \cap \mathbb{Z}$, each
\[
\Delta_i = d j_i + 2^{m+\varsigma - t} k_i - \frac{d}{r} (\alpha_{r,i} - 2^{m+\varsigma} \eta_i) + 2^{m+\varsigma} t_i
\]
so we expect $|\Delta_i| \sim 2^{m+\varsigma - t}$ for $i \in [1, n] \cap \mathbb{Z}$. Furthermore $2^{\varsigma - t} d \in [0, 2^{m+\varsigma - t})$. This implies
\[
|\vec{u} - \vec{v}| = |(\Delta_1, \ldots, \Delta_n, 2^{\varsigma - t} d)| \sim \sqrt{n + 1} \cdot 2^{m+\varsigma - t} \sim \sqrt{n + 1} \cdot 2^{m(1-1/s)+\varsigma}.
\]

Note that we select the scaling factor $2^{\varsigma - t}$ such that $2^{\varsigma - t} d \sim |\Delta_i|$, to avoid having the last component of $\vec{u}$ dominate $|\vec{u} - \vec{v}|$, whilst maximizing $\det L$ to facilitate the enumeration of $L$.

### 6.2.1 On the expected number of runs required to solve

As the determinant
\[
\det L = 2^{(m+\varsigma)n+\varsigma - t} \sim 2^{m(n-1/s)+\varsigma(n+1)}
\]
the shortest non-zero vector in $L$ is expected, by the Gaussian heuristic, to be of norm
\[
\lambda_1 = \sqrt{\frac{n + 1}{2\pi e}} \cdot n + \sqrt{\det L} \sim \sqrt{\frac{n + 1}{2\pi e}} \cdot 2^{m(n-1/s)+(n+1)+\varsigma}
\]
if $L$ behaves like a random lattice. If we are to quickly find $\vec{u}$ when enumerating a ball in $L$ centered on $\vec{v}$, we need $\lambda_1 \sim |\vec{u} - \vec{v}|$. Simplifying this relation yields
\[
\sqrt{\frac{n + 1}{2\pi e}} \cdot 2^{m(n-1/s)+(n+1)+\varsigma} \sim \sqrt{n + 1} \cdot 2^{m(1-1/s)+\varsigma} \sim |\vec{u} - \vec{v}| \\
\Rightarrow m(n - 1/s)/(n + 1) \sim \log_2(\sqrt{2\pi e}) \sim m(1 - 1/s)
\]
\[
\Rightarrow m(n - 1/s) \sim \log_2(\sqrt{2\pi e})(n + 1) + m(1 - 1/s)(n + 1)
\]
\[
\Rightarrow n - 1/s \sim \log_2(\sqrt{2\pi e})(n + 1)/m + (1 - 1/s)(n + 1)
\]
\[
\Rightarrow n - 1/s \sim \log_2(\sqrt{2\pi e})(n + 1)/m + 1 - n/s - 1/s
\]
\[
\Rightarrow n/s \sim \log_2(\sqrt{2\pi e})(n + 1)/m + 1
\]
\[
\Rightarrow n \sim \log_2(\sqrt{2\pi e})(n + 1)s/m + s
\]

Hence, as $m$ grows large for fixed $s$, we expect to perform roughly $n \sim s$ runs to efficiently solve for $d$ given $r$ without having to enumerate very many vectors in $L$. 

18
6.2.2 On simulating the complete algorithm

We have implemented the lattice-based post-processing algorithm and tested it using the simulator described in Sect. 6.4. Unsurprisingly, running the post-processing algorithm with respect to heuristically simulated quantum algorithm outputs recovers $d$, provided (i) that no sampling errors occur, (ii) that $n$ is selected equal to or slightly greater than $s$, so that it is feasible to enumerate the lattice, and (iii) that the guess for $\{\eta_1, \ldots, \eta_n\}$ is correct.

As $B_\Delta$, and $B_\eta$ and/or $\varsigma$, in the sampling procedure increase, the probability of sampling errors occurring tends to zero, see Tab. 4 and Thm. 2 in Sect. 4.3. Furthermore, the probability that $\eta_1 = \ldots = \eta_n = 0$ tends to one as $\varsigma$ increases, making it easier to guess $\{\eta_1, \ldots, \eta_n\}$.

6.2.3 On the relation to our earlier works

Compared to [9], where $n \sim s$ runs of a quantum algorithm that performs $\sim (1 + 2/s)m$ group operations quantumly are required to solve for $d$ and $r$, the algorithm in this work performs only $\sim (1 + 1/s)m$ group operations quantumly when solving for $d$ given $r$ in $n \sim s$ runs.

This is a slight improvement, entirely explained by the fact that the order is required to be known in this algorithm, whereas the algorithm in [9] computes both $d$ and $r$.

6.2.4 On the relation to Kaliski’s work

Kaliski [14] has modified Shor’s quantum algorithm for computing general discrete logarithms to produce an oracle for the most significant half-bit of the logarithm: For $d$ the logarithm and $r$ the group order, the oracle estimates whether $d/r \geq 1/2$ with a non-negligible advantage.

The oracle is used by Kaliski to instantiate the Blum–Micali [1] reduction, as generalized by Kaliski [13, Ch. 5], acknowledging ideas from Goldreich. This enables general discrete logarithms to be computed in groups of known prime order by repeatedly querying the oracle.

Specifically, to compute $d$ given $x = [d]_g$ in a group of $m$-bit order $r$, the reduction queries the oracle $2m c(m)$ times in total: For each $i \in [0, m) \cap \mathbb{Z}$, it makes $2c(m)$ queries for elements on the form $[2^i] x \odot [t]_g$, where $t$ is selected uniformly at random from $[0, r)$ in each query. This allows $d$ to be classically reconstructed with a success probability that depends on $c(m)$. In [13, Figs. 5.2–5.3] Kaliski lets $c(m) = \left[4\sqrt{m} \epsilon^{-1}\right] \left[\epsilon^{-1}\right]$ for $\epsilon \in (0, 1/2)$ the average advantage of the oracle. By [13, Thm. 5.6] the success probability is then at least $1/2$.

As for the oracle, we can consider an optimized version of Kaliski’s quantum algorithm [14, Sect. 6.3], in which the first stage computes and reads out $j$, after which the second stage performs a single controlled group operation by the element $[z^{-1} \mod r] x$ for $z = \lceil rj/2^m + \varsigma \rceil$. A phase-shift gate and a Hadamard gate are then applied to the control qubit, after which it is read out. The result is correlated with $d/r$, producing the required half-bit oracle.

In summary $m + 1$ group operations are evaluated quantumly in each of the $2m e(m) = 2m \left[4\sqrt{m} \epsilon^{-1}\right] \left[\epsilon^{-1}\right]$ runs, where $\epsilon = \pi^{-1}$ for the oracle [14, Sect. 4]. In practice, it is usually possible to solve in a much smaller, but still substantial, number of runs.

By comparison, our algorithm for computing general discrete logarithms in groups of known order is parameterized under a tradeoff factor $s$ that controls the extent to which we make tradeoffs. This is in analogy with our earlier works [7,9] on computing short and general discrete logarithms in groups of unknown order, and with Seifert’s work on order finding [28].

Specifically, our quantum algorithm evaluates $\sim (1 + 1/s)m$ group operations quantumly in each run, and requires $s$ or slightly more than $s$ runs, as each run yields $\sim m/s$ bits of information on $d$. Expressed in this language of tradeoffs, Kaliski may hence be said to make maximal tradeoffs for $s = m$, whereas we make variable tradeoffs for $s$ a parameter.

Besides $s$, our algorithm takes two additional parameters, in the form of the padding length $\varsigma$ and the bound on the search space when enumerating the lattice. For small to moderate $s$, and for appropriate choices of $\varsigma$ and the bound on the search space, we expect to find good
overall tradeoffs between the requirements on the quantum computer, the requirements on the classical computer performing the post-processing, and the number of runs required.

Note that as we post-process the outputs from all \( n \) runs simultaneously using lattice-based techniques, there is a limit to how large we can grow \( s \): The dimension of the lattice \( L \) is \( n + 1 \) where \( n \geq s \). Hence, the fact that we must be able to compute a sufficiently good reduced basis for \( L \) restricts \( s \). Kaliski’s algorithm does not suffer from this restriction.

If maximal tradeoffs are sought, Kaliski’s algorithm provides the necessary means to achieve such tradeoffs. In practice, one would however arguably seek to make a good overall tradeoff, rather than a maximal tradeoff, in which case the algorithm proposed in this work is useful. It may be possible to extend Kaliski’s approach to cover a range of different tradeoffs.

7 Generalizing the analysis

Essentially, \( \varsigma \) represents the number of bits of information that we learn on \( r \) in each run of the algorithm, whereas \( \ell \) represents the number of bits of information that we learn on \( d \).

We have assumed up until this point that \( r \) is known, and sought to compute \( d \) given \( r \), as in Shor’s original algorithm. Therefore, we have picked \( \varsigma \) small, \( \ell \sim m/s \) and \( m = \text{len}(r) \).

As previously hinted at, it is however possible to relax these requirements, and to more freely select both \( m \), \( \varsigma \) and \( \ell \), without voiding the analysis in this paper.

7.1 Computing general \( d \) and optionally \( r \) when \( r \) is unknown

If \( r \) is unknown, we may select \( m = \varsigma = \ell = \text{len}(r) \) to compute both \( d \) and \( r \) in a single run, at the expense of having a \( 3m \)-bit exponent in this single run. In fact, we may do tradeoffs, and compute both \( d \) and \( r \) in \( s \) runs by picking \( m = \text{len}(r) \) and \( \varsigma = \ell \sim m/s \). This lands us in the variation of Shor’s algorithm described and analyzed in [9], extending the heuristic analysis of the probability distribution in this paper to said algorithm.

The analysis in [9] is based upon several approximation steps, where the error introduced in each approximation step is upper-bounded. The heuristic analysis in this paper complements it by giving information also for parameterizations in \( m \), \( \varsigma \) and \( \ell \) that are beyond the reach of the analysis in [9] due to the error bounds on the approximation errors growing too large.

For some parameterizations that are within reach of the analysis in [9], we have verified numerically that the analysis in [9] agrees with the heuristic analysis in this paper, see App. B.1.1.

7.2 Computing short \( d \) when \( r \) is unknown

If \( r \) is unknown, and \( d \) is short, we may select \( m = \varsigma = \ell = \text{len}(d) \) to compute \( d \) in a single run without computing \( r \). In fact, we may do tradeoffs, and select \( m = \text{len}(d) \) and \( \varsigma = \ell \sim m/s \), to compute \( d \) in \( s \) runs without computing \( r \). This lands us in the algorithm for computing short discrete logarithms in groups of unknown order described and analyzed in [7][8], extending the heuristic analysis in this paper to said algorithm.

The analyses in [7][8] are exact, but they rely on \( d \sim \sqrt{r} \) or smaller when \( s = 1 \). Specifically, they require that \( r \geq 2^{3m+1} + (2^r - 1)d \), which implies \( \log_2(d) \sim \log_2(r)/(1 + 1/s) \) or smaller. By comparison, the heuristic analysis in this paper covers the case where \( d \sim \sqrt{r} \) or greater. We have verified numerically that it agrees with the analysis in [8] when \( d \sim \sqrt{r} \), see App. B.1.2.

To see why we need \( d \sim \sqrt{r} \) or greater, note that the heuristic assumes \( b_0 = (c + bd)/r \mod 1 \) to be approximately uniformly distributed on \([0, 1)\). This implies that \( bd \) must run up to \( r \).

Since \( b \in [0, 2^r) \cap \mathbb{Z} \), this in turn implies \( \log_2(d) + \ell \sim (1 + 1/s) \log_2(r) \sim \log_2(r) \) or greater, so we have \( \log_2(d) \sim \log_2(r)/(1 + 1/s) \geq \log_2(r)/2 \) or greater, or equivalently \( d \sim \sqrt{r} \) or greater.

7.3 Future work

Unlike our earlier works [6][9], in which we provided exact analyses, or approximations with associated error bounds, the analysis presented in this paper is heuristic. We are currently exploring various options for bounding the approximation error in the analysis.
8 Summary and conclusion

We have heuristically shown that Shor’s algorithm for computing general discrete logarithms achieves an expected success probability of approximately 60% to 82% in a single run when modified to enable the use of the semi-classical QFT [12] with control qubit recycling [19,24].

By slightly increasing the number of group operations that are evaluated quantumly and performing a single limited search in the classical post-processing, or by performing two limited searches in the classical post-processing, we have shown how the algorithm can be further modified to achieve a success probability that heuristically exceeds 99% in a single run.

We have provided concrete heuristic estimates of the success probability of the modified algorithm, as a function of the order $r$, the size of the search space in the classical post-processing, and the additional number of group operations evaluated quantumly. This enables fair comparisons between Shor’s algorithm for computing discrete logarithms and other algorithms. In the limit as $r \to \infty$, we have heuristically shown that the success probability tends to one.

In analogy with our earlier works [8,9], we have shown how the modified quantum algorithm may be heuristically simulated classically when the logarithm $d$ and $r$ are both known. Furthermore, we have heuristically shown how slightly better tradeoffs may be achieved, compared to our earlier work [9], if $r$ is known when computing $d$. Finally, we have generalized our heuristic to cover some of our earlier works [7,9], and compared it to the non-heuristic analyses in those works as a means of verification. For further details, see App. B.

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A Tabulated lower bounds and expectation values

This appendix contains tabulated heuristically derived lower bounds and expectation values.

A.1 Heuristically derived lower-bounded probabilities

Tabs. 1–2 below were created by evaluating (19) in Thm. 2 numerically.

| $B_\eta$ | 0   | 10  | 100 | 250 | 500 | 1000 | 2500 | 5000 | 10000 |
|---------|-----|-----|-----|-----|-----|------|------|------|-------|
| 0       | 0.5650 | 0.5917 | 0.5935 | 0.5941 | 0.5944 | 0.5945 | 0.5946 | 0.5946 |
| 1       | 0.9037 | 0.9757 | 0.9787 | 0.9797 | 0.9802 | 0.9805 | 0.9806 | 0.9806 |
| 2       | 0.9481 | 0.9929 | 0.9959 | 0.9969 | 0.9974 | 0.9977 | 0.9978 | 0.9979 |
| 3       | 0.9492 | 0.9941 | 0.9971 | 0.9981 | 0.9986 | 0.9989 | 0.9990 | 0.9991 |
| 4       | 0.9496 | 0.9945 | 0.9975 | 0.9985 | 0.9990 | 0.9993 | 0.9994 | 0.9995 |
| 5       | 0.9498 | 0.9947 | 0.9977 | 0.9987 | 0.9992 | 0.9995 | 0.9996 | 0.9997 |
| 6       | 0.9499 | 0.9949 | 0.9979 | 0.9989 | 0.9994 | 0.9997 | 0.9998 | 0.9998 |
| 7       | 0.9500 | 0.9949 | 0.9979 | 0.9989 | 0.9994 | 0.9997 | 0.9998 | 0.9999 |
| 8       | 0.9500 | 0.9949 | 0.9979 | 0.9989 | 0.9994 | 0.9997 | 0.9998 | 0.9999 |
| 9       | 0.9500 | 0.9949 | 0.9979 | 0.9989 | 0.9994 | 0.9997 | 0.9998 | 0.9999 |

Table 1: The probability of observing a $B_\eta$-$B_\Delta$-good pair, as given by the lower bound on the heuristic in Thm. 2 for $\eta = 0$ and as $m \to \infty$ in the worst case where $r = 2^m - 1$. This table was created by evaluating the bound using Mathematica and rounding down.

| $B_\Delta$ | 0   | 10  | 100 | 250 | 500 | 1000 | 2500 | 5000 | 10000 |
|---------|-----|-----|-----|-----|-----|------|------|------|-------|
| 0       | 0.5650 | 0.5917 | 0.5935 | 0.5941 | 0.5944 | 0.5945 | 0.5946 | 0.5946 |
| 1       | 0.9037 | 0.9757 | 0.9787 | 0.9797 | 0.9802 | 0.9805 | 0.9806 | 0.9806 |
| 2       | 0.9481 | 0.9929 | 0.9959 | 0.9969 | 0.9974 | 0.9977 | 0.9978 | 0.9979 |
| 3       | 0.9492 | 0.9941 | 0.9971 | 0.9981 | 0.9986 | 0.9989 | 0.9990 | 0.9991 |
| 4       | 0.9496 | 0.9945 | 0.9975 | 0.9985 | 0.9990 | 0.9993 | 0.9994 | 0.9995 |
| 5       | 0.9498 | 0.9947 | 0.9977 | 0.9987 | 0.9992 | 0.9995 | 0.9996 | 0.9997 |
| 6       | 0.9499 | 0.9949 | 0.9979 | 0.9989 | 0.9994 | 0.9997 | 0.9998 | 0.9998 |
| 7       | 0.9500 | 0.9949 | 0.9979 | 0.9989 | 0.9994 | 0.9997 | 0.9998 | 0.9999 |
| 8       | 0.9500 | 0.9949 | 0.9979 | 0.9989 | 0.9994 | 0.9997 | 0.9998 | 0.9999 |

Table 2: The probability of observing a $B_\eta$-$B_\Delta$-good pair, as given by the lower bound on the heuristic in Thm. 2 for $\eta = 0$ and as $m \to \infty$ in the worst case where $r = 2^m - 1$. This table was created by evaluating the bound using Mathematica and rounding down.
A.2 Heuristically derived expected probabilities

Tabs. 3–4 below were created by evaluating (21) numerically.

\[ B_{\Delta} \]

\begin{tabular}{cccccccccccc}
0 & 0.5986 & 0.7204 & 0.7421 & 0.7662 & 0.7734 & 0.7735 & 0.7736 & 0.7737 \\
1 & 0.7204 & 0.8669 & 0.8931 & 0.9221 & 0.9307 & 0.9309 & 0.9310 & 0.9311 \\
2 & 0.7421 & 0.8931 & 0.9200 & 0.9499 & 0.9582 & 0.9588 & 0.9590 & 0.9591 \\
10 & 0.7662 & 0.9221 & 0.9499 & 0.9808 & 0.9893 & 0.9899 & 0.9901 & 0.9902 & 0.9903 \\
50 & 0.7729 & 0.9302 & 0.9582 & 0.9983 & 0.9988 & 0.9989 & 0.9989 & 0.9990 \\
100 & 0.7734 & 0.9307 & 0.9588 & 0.9989 & 0.9990 & 0.9994 & 0.9995 & 0.9996 \\
250 & 0.7735 & 0.9309 & 0.9590 & 0.9991 & 0.9994 & 0.9996 & 0.9997 & 0.9998 \\
500 & 0.7736 & 0.9310 & 0.9591 & 0.9992 & 0.9995 & 0.9997 & 0.9998 & 0.9999 \\
1000 & 0.7737 & 0.9311 & 0.9591 & 0.9993 & 0.9996 & 0.9998 & 0.9999 & 0.9999 \\
2500 & & & & & & & & & \\
\end{tabular}

Table 3: The probability of observing a \( B_\eta-B_\Delta \)-good pair, as given by the heuristically derived expectation value in (21), for \( \zeta = 0 \), \( m = 128 \), \( s = 1 \) and \( r = 2^m - 1 \). The same table is obtained for greater \( m \). The choice of \( s \) has no effect as long as \( s \) is not selected very large in relation to \( m \). Smaller \( r \in [2^{m-1}, 2^m) \cap \mathbb{Z} \) yield slightly larger probabilities. This table was created by evaluating (21) using Mathematica and rounding to the closest decimal. The symmetry is explained by \( r \) being close to \( 2^m \). For random \( r \), see Tab. 5.

\[ B_{\eta} \]

\begin{tabular}{cccccccccccc}
0 & 0.5986 & 0.7204 & 0.7421 & 0.7662 & 0.7729 & 0.7734 & 0.7735 & 0.7736 & 0.7737 \\
1 & 0.6985 & 0.8406 & 0.8659 & 0.8941 & 0.9019 & 0.9025 & 0.9026 & 0.9027 & 0.9028 \\
2 & 0.7350 & 0.8845 & 0.9111 & 0.9408 & 0.9490 & 0.9496 & 0.9497 & 0.9498 & 0.9499 \\
3 & 0.7542 & 0.9076 & 0.9349 & 0.9653 & 0.9738 & 0.9744 & 0.9746 & 0.9747 & 0.9747 \\
4 & 0.7639 & 0.9193 & 0.9470 & 0.9778 & 0.9863 & 0.9869 & 0.9871 & 0.9872 & 0.9873 \\
5 & 0.7688 & 0.9272 & 0.9531 & 0.9841 & 0.9927 & 0.9933 & 0.9935 & 0.9936 & 0.9936 \\
6 & 0.7712 & 0.9281 & 0.9561 & 0.9872 & 0.9958 & 0.9964 & 0.9966 & 0.9967 & 0.9968 \\
7 & 0.7725 & 0.9296 & 0.9576 & 0.9888 & 0.9974 & 0.9980 & 0.9982 & 0.9983 & 0.9984 \\
8 & 0.7731 & 0.9304 & 0.9584 & 0.9896 & 0.9982 & 0.9988 & 0.9990 & 0.9991 & 0.9992 \\
9 & 0.7734 & 0.9307 & 0.9588 & 0.9900 & 0.9986 & 0.9992 & 0.9994 & 0.9995 & 0.9996 \\
10 & 0.7735 & 0.9309 & 0.9590 & 0.9901 & 0.9988 & 0.9994 & 0.9996 & 0.9997 & 0.9998 \\
11 & 0.7736 & 0.9310 & 0.9591 & 0.9902 & 0.9989 & 0.9995 & 0.9997 & 0.9998 & 0.9999 \\
\end{tabular}

Table 4: The probability of observing a \( B_\eta-B_\Delta \)-good pair, as given by the heuristically derived expectation value in (21), for \( \zeta = 0 \), \( m = 128 \), \( s = 1 \) and \( r = 2^m - 1 \). The same table is obtained for greater \( m \). The choice of \( s \) has no effect as long as \( s \) is not selected very large in relation to \( m \). Smaller \( r \in [2^{m-1}, 2^m) \cap \mathbb{Z} \) yield slightly larger probabilities. This table was created by evaluating (21) using Mathematica and rounding to the closest decimal.
A.2.1 Supplementary table for a random choice of $r$

The choice of $r = 2^m - 1$ in Tab. 3 induces symmetry. To show that this is only an artefact of the choice of $r$, we also include Tab. 5 below that features a random choice of $r$.

| $B_\eta$ | 0  | 1  | 2  | 10 | 100 | 250 | 500 | 1000 | 2500 |
|----------|----|----|----|----|-----|-----|-----|------|------|
| 00       | 0.6841 | 0.8232 | 0.8480 | 0.8756 | 0.8833 | 0.8838 | 0.8840 | 0.8841 | 0.8841 |
| 1        | 0.7356 | 0.8852 | 0.9119 | 0.9416 | 0.9504 | 0.9505 | 0.9506 | 0.9507 |       |
| 2        | 0.7528 | 0.9059 | 0.9332 | 0.9635 | 0.9720 | 0.9725 | 0.9727 | 0.9729 | 0.9729 |
| 10       | 0.7685 | 0.9248 | 0.9527 | 0.9837 | 0.9923 | 0.9931 | 0.9931 | 0.9932 | 0.9932 |
| B_\Delta | 1000  | 0.7732 | 0.9304 | 0.9585 | 0.9897 | 0.9983 | 0.9991 | 0.9992 | 0.9993 |
|          | 250   | 0.7735 | 0.9308 | 0.9589 | 0.9901 | 0.9987 | 0.9993 | 0.9995 | 0.9997 |
|          | 500   | 0.7736 | 0.9310 | 0.9590 | 0.9902 | 0.9989 | 0.9995 | 0.9997 | 0.9998 |
|          | 1000  | 0.7736 | 0.9310 | 0.9591 | 0.9903 | 0.9989 | 0.9995 | 0.9997 | 0.9998 |
|          | 2500  | 0.7737 | 0.9310 | 0.9591 | 0.9903 | 0.9990 | 0.9996 | 0.9998 | 0.9999 |

Table 5: The probability of observing a $B_\eta$-$B_\Delta$-good pair, as given by the heuristically derived expectation value in (21), for $\varsigma = 0$, $m = 128$, $s = 1$ and a random choice of the order $r = 2^{341725620093007559271185988522878687746} \in [2^m - 1, 2^m) \cap \mathbb{Z}$. This table was created by evaluating (21) using Mathematica and rounding to the closest decimal.

B Numerical verification of the heuristic

We have verified the heuristic by performing various numerical checks. For instance, we have compared the probability yielded by the heuristic — for small $d$ and $r$, small $\varsigma$, small $s$ and $\ell \sim m/s$, and $B_\eta$-$B_\Delta$-good pairs $(j, k)$ for small $B_\eta$ and $B_\Delta$ — to the exact probability.

To take a concrete example: For $r = 915725$, $m = \ell = 20 = \text{len}(r)$, $\varsigma = 0$ and $d = 33979$, the probability of observing the pair $(j, k) = (965620, 199053)$ is

$$5.0487069192234575 \ldots \cdot 10^{-7},$$

as may be seen by summing the exact expression (5) over all $e \in [0, r) \cap \mathbb{Z}$.

The above probability was computed by a C program, with GMP, MPFR and MPI bindings, executing on 2560 virtual cores on the Dardel HPE Cray EX supercomputer at PDC at KTH.

By comparison, the heuristic expression (11) for $B_\eta = 10^3$ yields

$$5.0487069192200045 \ldots \cdot 10^{-7}.$$ 

Further increasing $B_\eta$ brings the probability yielded by the heuristic closer to the exact probability. Already for $B_\eta = 0$, the heuristic probability is close to the exact probability, since $(j, k)$ is a $B_\eta$-$B_\Delta$-good pair for $B_\eta = 0$ so the main contribution is yielded by $P_0$.

B.1 Numerical verification of the generalized heuristic

As explained in Sect. 7, the heuristic may be generalized to cover the algorithm in [7, 8] for computing short discrete logarithms $d$ in groups of unknown order, for which an exact closed-form expression for the probability of observing $(j, k)$ is available in [8, see $P$ on p. 2319].

Furthermore, it may be generalized to cover the algorithm in [9] for computing general discrete logarithms $d$, and optionally the order $r$, in groups of unknown order, for which an error-bounded closed-form approximation of the probability of observing $(j, k)$ is available in [9, see Thm. 3.22 on p. 371]. Both generalizations afford us opportunities to verify the heuristic for large cryptologically relevant problem instances. In the examples that follow, we pick $r$ of length 384 bits so as not to take up more than a single page.
B.1.1 General logarithms $d$ in groups of unknown order $r$

To take a concrete example, for $m = \varsigma = \ell = 384 = \text{len}(r)$,

\begin{align*}
    r &= 25469347609100372151973013820857767605031701469848799638627 \ldots \\
    &\quad 8633851214232502234886887993997116313367577810182904847102, \\
    d &= 950434975865484592034772884294848917302089403216408023623 \ldots \\
    &\quad 2390736525390273947891696095958059582489836868063545405, \\
\end{align*}

the good pair $(j, k)$ such that

\begin{align*}
    j &= 1075976306421619097001461383216750778715221797934040685563 \ldots \\
    &\quad 9910860103384940995832295315073640894204239215200760732709, \\
    k &= 1556535049017203654832951746130547334190277155748367378067 \ldots \\
    &\quad 8079753570284374152093305399973215729104403615842686810400, \\
\end{align*}

is yielded with probability

\[ 2.60840352292438536283651363 \ldots \cdot 10^{-116} \pm 5.546622 \ldots \cdot 10^{-173} \]

by the error-bounded approximation in [9, see Thm. 3.22 on p. 371 with $\sigma = 193$].

By comparison, the heuristic expression (11) for $B_0 = 0$ yields

\[ 2.60840352292438536283651363 \ldots \cdot 10^{-116}. \]

Even if many decimals are included, the heuristic and error-bounded approximation agree, as may be seen above. This is a testament to our claim that increasing $\varsigma$ stabilizes the heuristic: For large $\varsigma$, there is essentially only one peak, so it suffices to pick $B_0 = 0$.

B.1.2 Short logarithms $d$ in groups of unknown order $r$

To take a concrete example, for $m = \varsigma = \ell = 191 = \text{len}(d)$,

\begin{align*}
    d &= 3080942812686441322301364810855243932009764789700004903359, \\
    r &= 231443226061912499159923072754713729596656240520088946784 \ldots \\
    &\quad 9458865436744593996092228512856184729219291525242175740299, \\
\end{align*}

the good pair $(j, k)$ such that

\begin{align*}
    j &= 5265474986182253380969381593632747354375313412620034347729 \ldots \\
    &\quad 10390057107842143190790454793362862605243983913213352850, \\
    k &= 831703848061277749653510823317960446727318170582492589329, \\
\end{align*}

is yielded with probability

\[ 6.7696364116116706 \ldots \cdot 10^{-116} \]

by the exact expression in [8, see P on p. 2319].

By comparison, the heuristic expression (11) for $B_0 = 10^3$ yields

\[ 6.7696364113721214 \ldots \cdot 10^{-116}. \]

Further increasing $B_0$ brings the probability yielded by the heuristic closer to the exact probability. In this example $d \sim \sqrt{r}$. Increasing $d$ in relation to $r$ precludes the use of the exact expression in [8 see P on p. 2319] since it assumes that $r \geq 2^{m+\varsigma} + (2^\ell - 1)d$. Decreasing $d$ in relation to $r$ causes the heuristic to fail, for the reasons explained in Sect. 7.2.
C Supporting lemmas and claims

This appendix contains proofs of supporting lemmas and claims.

C.1 The \( h \)-function

Lemma C.1. For any real \( \delta \), it holds that

\[
\sum_{\Delta = -2^{\ell-1}}^{2^{\ell-1}-1} h \left( \frac{2\pi}{2^\ell} (\Delta - \delta) \right) = 1.
\]

Proof. This lemma is a special case of \([9\text{, Lem. C.1 in App. C.2}]): Specifically, it holds that

\[
h \left( \frac{2\pi}{2^\ell} (\Delta - \delta) \right) = \frac{1}{2^{2\ell}} \left( \sum_{b=0}^{2^{\ell-1}-1} e^{2\pi i (\Delta - \delta) b/2^\ell} \right)^2
\]

\[
= \frac{1}{2^{2\ell}} \left( \sum_{b=0}^{2^{\ell-1}-1} e^{2\pi i (\Delta - \delta) b/2^\ell} \right) \left( \sum_{b=0}^{2^{\ell-1}-1} e^{-2\pi i (\Delta - \delta) b/2^\ell} \right)
\]

\[
= \frac{1}{2^{2\ell}} \left( 2^{\ell} + \sum_{b=1}^{2^{\ell-1}-1} (2^{\ell} - b) \left( e^{2\pi i (\Delta - \delta) b/2^\ell} + e^{-2\pi i (\Delta - \delta) b/2^\ell} \right) \right)
\]

from which it follows that

\[
\sum_{\Delta = -2^{\ell-1}}^{2^{\ell-1}-1} h \left( \frac{2\pi}{2^\ell} (\Delta - \delta) \right) = 1
\]

\[
1 + \sum_{b=1}^{2^{\ell-1}-1} (2^{\ell} - b) \sum_{\Delta = -2^{\ell-1}}^{2^{\ell-1}-1} \left( e^{2\pi i (\Delta - \delta) b/2^\ell} + e^{-2\pi i (\Delta - \delta) b/2^\ell} \right) = 1
\]

and so the lemma follows. \( \blacksquare \)

C.1.1 Bounding the tails

Lemma C.2. For any \( \delta_\eta \in [-\frac{1}{2}, \frac{1}{2}] \) and \( B_\Delta \in [0, 2^{\ell-1}) \cap \mathbb{Z} \), it holds that

\[
\sum_{\Delta = -B_\Delta}^{B_\Delta} h \left( \frac{2\pi}{2^\ell} (\Delta - \delta_\eta) \right) > 1 - \frac{1}{2} \frac{1}{B_\Delta + 1/2} \left( 1 + \frac{1}{2(B_\Delta + 1/2) + \frac{1}{6(B_\Delta + 1/2)^2}} \right).
\]

Proof. Suppose that \( \Delta - \delta_\eta \neq 0 \). It then holds that

\[
h \left( \frac{2\pi}{2^\ell} (\Delta - \delta_\eta) \right) = \frac{1}{2^{2\ell}} \left| \sum_{b=0}^{2^{\ell-1}-1} e^{2\pi i (\Delta - \delta_\eta) b/2^\ell} \right|^2
\]

\[
= \frac{1}{2^{2\ell}} \left| \frac{1 - \cos(2\pi (\Delta - \delta_\eta))}{1 - \cos(2\pi (\Delta - \delta_\eta)/2^\ell)} \right|^2
\]

\[
\leq \frac{1}{2^{2\ell}} \frac{2}{2(2\pi (\Delta - \delta_\eta)/2^\ell)^2/\pi^2} = \frac{1}{2^2} \frac{1}{(\Delta - \delta_\eta)^2}
\]

28
where we have used Claim C.1 to bound the denominator. It follows that
\[
\sum_{\Delta = B_{\Delta} + 1}^{2^f - 1} h \left( \frac{2\pi}{2^f} (\Delta - \delta_n) \right) \leq \frac{1}{2^f} \sum_{\Delta = B_{\Delta} + 1}^{2^f - 1} \frac{1}{(\Delta - \delta_n)^2} < \frac{1}{2^f} \sum_{\Delta = B_{\Delta} + 1}^{\infty} \frac{1}{(\Delta - 1/2)^2} = \frac{1}{2^f} \psi'(B_{\Delta} + 1/2)
\]
for \(\psi\) the trigamma function, and analogously that
\[
\sum_{\Delta = -2^f - 1}^{-B_{\Delta} - 1} h \left( \frac{2\pi}{2^f} (\Delta - \delta_n) \right) \leq \frac{1}{2^f} \sum_{\Delta = -2^f - 1}^{-B_{\Delta} - 1} \frac{1}{(\Delta - \delta_n)^2} < \frac{1}{2^f} \sum_{\Delta = -B_{\Delta} + 1}^{\infty} \frac{1}{(\Delta - 1/2)^2} = \frac{1}{2^f} \psi'(B_{\Delta} + 1/2),
\]
which implies that
\[
\sum_{\Delta = -B_{\Delta}}^{B_{\Delta}} h \left( \frac{2\pi}{2^f} (\Delta - \delta_n) \right) = 1 - \sum_{\Delta = -2^f - 1}^{-B_{\Delta} - 1} h \left( \frac{2\pi}{2^f} (\Delta - \delta_n) \right) - \sum_{\Delta = B_{\Delta} + 1}^{2^f - 1} h \left( \frac{2\pi}{2^f} (\Delta - \delta_n) \right) > 1 - \frac{1}{2} \psi'(B_{\Delta} + 1/2) > 1 - \frac{1}{2} \frac{1}{B_{\Delta} + 1/2} \left( 1 + \frac{1}{2(B_{\Delta} + 1/2)} + \frac{1}{6(B_{\Delta} + 1/2)^2} \right)
\]
where we have used Lem. C.1 and Claim C.2 to bound \(\psi\), and so the lemma follows.

C.1.2 Supporting claims

Claim C.1. For any \(\varphi \in [-\pi, \pi]\), it holds that
\[
\frac{2\varphi^2}{\pi^2} \leq 1 - \cos(\varphi) \leq \frac{\varphi^2}{2}.
\]

Proof. This claim is from [11]. See [11] Claim 2.4 in App. D.1 for the proof.

Claim C.2. For \(\psi'(x)\) the trigamma function and any real \(x > 0\), it holds that
\[
\psi'(x) < \frac{1}{x^2} + \frac{1}{2x^2} + \frac{1}{6x^3}.
\]

Proof. This claim is from [11] via [20]. See [11] Claim 3.2 in App. D.2 for the proof.

C.2 The \(f_\eta\)-function

Lemma C.3. It holds that
\[
\lim_{\substack{B_{\Delta} \to \infty \\ \eta \to \infty}} \sum_{\Delta = -B_{\Delta}}^{B_{\Delta}} \sum_{\eta = -B_{\eta}}^{B_{\eta}} 2^{m + x - \eta - 1} \psi \left( \frac{2\pi}{2^{m + x}} 2^{2n} \alpha' \right) = 1.
\]

Proof. Suppose that \(2^{n} \alpha' \neq 2^{m + x} \eta \neq 0\). It then holds that
\[
f_\eta \left( \frac{2\pi}{2^{m + x}} 2^{2n} \alpha' \right) = \frac{r}{2^{2(m + x)}} \frac{2(1 - \cos(2\pi(2^{n} \alpha' - 2^{m + x} \eta)/r))}{(2\pi(2^{n} \alpha'/2^{m + x} - \eta))^2} = \frac{1}{r} \frac{2(1 - \cos(2\pi(2^{n} \alpha' - 2^{m + x} \eta)/r))}{(2\pi(2^{n} \alpha'/2^{m + x} - \eta)(2^{m + x}/r))^2}
\]
and furthermore, since $\alpha'_r \in [-2^{m+\kappa_r-1}, 2^{m+\kappa_r-1}] \cap \mathbb{Z}$ and $\eta \in \mathbb{Z}$, and

$$f_0(0) = \lim_{\alpha'_r \to 0} r \frac{1}{r} \sin(2\pi \pi \alpha'_r / r^2).$$

we have that (26) also holds in the limit as $2^{\kappa_r} \alpha'_r - 2^{m+\kappa_r} \eta \to 0$. It follows that

$$\lim_{B_n \to \infty \text{ and/or } \zeta \to \infty} \sum_{B_n} \sum_{\eta = -B_n \alpha'_r = -2^{m+\kappa_r-1}}^{2^{m+\kappa_r-1}} \frac{2^{\kappa_r} f_0 \left( \frac{2\pi}{2^{m+\kappa_r}} 2^{\kappa_r} \alpha'_r \right)}{r^2} \left( \frac{2\pi}{2^{m+\kappa_r}} 2^{\kappa_r} \alpha'_r \right) \sin^2 \left( \frac{\pi (2^{\kappa_r} \alpha'_r - 2^{m+\kappa_r} \eta) / r}{(2^{\kappa_r} \pi \alpha'_r / r^2)^2} \right)$$

where we have introduced $r' = r / 2^{\kappa_r}$, and used Claim C.3 and so the lemma follows. 

### C.2.1 Supporting claims

**Claim C.3.** For any positive integer $r'$, it holds that

$$\sum_{u = -\infty}^{\infty} \frac{\sin^2(\pi u / r')}{(\pi u / r')^2} = r'.$$

**Proof.** By Parseval’s theorem, it holds that

$$\sum_{u = -\infty}^{\infty} \frac{1}{2\pi} \int_{-\pi}^{\pi} |\hat{\xi}_u|^2 \, dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} |\hat{\xi}(x)|^2 \, dx$$

for $\xi(x) : \mathbb{R} \to \mathbb{C}$ a $2\pi$-periodic function, and $\hat{\xi}_u$ its Fourier coefficients such that

$$\xi(x) = \sum_{u = -\infty}^{\infty} \hat{\xi}_u e^{iux} \text{ where } \hat{\xi}_u = \frac{1}{2\pi} \int_{-\pi}^{\pi} \xi(x) e^{-iux} \, dx.$$  

For

$$\xi(x) = \begin{cases} r' & x \in [-\pi/r', \pi/r') \\ 0 & \text{otherwise} \end{cases}$$

it then holds that

$$\hat{\xi}_u = \frac{1}{2\pi} \int_{-\pi}^{\pi} \xi(x) e^{-iux} \, dx = \frac{1}{2\pi} \int_{-\pi}^{\pi/r'} r' e^{-iux} \, dx = r' \left[ \frac{e^{-iux}}{\pi} \right]_{-\pi/r'}^{\pi/r'} = \frac{\sin(\pi u / r')}{\pi u / r'}$$

when $u \neq 0$, and furthermore

$$\hat{\xi}_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} \xi(x) \, dx = \frac{1}{2\pi} \int_{-\pi/r'}^{\pi/r'} r' \, dx = 1 = \lim_{u \to 0} \frac{\sin(\pi u / r')}{\pi u / r'}.$$
It follows that
\[
\sum_{u = -\infty}^{\infty} \frac{\sin^2(\pi u/r')}{(\pi u/r')^2} = \sum_{u = -\infty}^{\infty} |\xi_u|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |\xi(x)|^2 \, dx = \frac{1}{2\pi} \int_{-\pi/r'}^{\pi/r'} (r')^2 \, dx = r',
\]
and so the claim follows. 

**C.2.2 Bounding the tails**

**Lemma C.4.** It holds that
\[
\sum_{\eta = -B_n}^{B_n} 2^{m+\kappa_r-1} - 1 2^{\kappa_r} f_\eta \left( \frac{2\pi}{2^{m+\kappa_r}} 2^{\kappa_r} \alpha_r' \right) > 1 - \frac{2}{\pi^2} \frac{r}{2^m} \frac{1}{2^{\kappa_r}(B_n + 1/2)} \left( 1 + \epsilon B_n \right)
\]
for \( B_n \) and \( \zeta \) non-negative integers, and for
\[
\epsilon B_n = \epsilon(2^{m+\kappa_r}(B_n + 1/2)) \quad \text{for} \quad \epsilon(x) = \frac{1}{2x} + \frac{1}{6x^2}.
\]

**Proof.** It holds that
\[
T_+ = \sum_{\eta = -B_n}^{B_n} \alpha_r' = -2^{m+\kappa_r-1}\end{align}
\[
= \frac{2^{\kappa_r}}{r} \sum_{\eta = -2^{m+\kappa_r-1}}^{B_n} 2^{\kappa_r} \sin^2 \left( \frac{\pi(2^{\kappa_r} \alpha_r' - 2^{m+\kappa_r} \eta)/r}{\pi(2^{\kappa_r} \alpha_r' - 2^{m+\kappa_r} \eta)/r} \right) 2^{\kappa_r} f_\eta \left( \frac{2\pi}{2^{m+\kappa_r}} 2^{\kappa_r} \alpha_r' \right)
\]
\[
= \frac{2^{\kappa_r}}{r} \sum_{u = 2^{m+\kappa_r}(B_n + 1/2)}^{B_n} \sin^2 \left( \frac{2^{\kappa_r} \pi u/r}{2^{\kappa_r} \pi u/r} \right) \leq \frac{2^{\kappa_r}}{r} \frac{r^2}{(2^{\kappa_r} \pi)^2} \sum_{u = 2^{m+\kappa_r}(B_n + 1/2)}^{B_n} \frac{1}{u^2}
\]
\[
= \frac{r}{2^{\kappa_r} \pi^2} \sum_{u = 2^{m+\kappa_r}(B_n + 1/2)}^{B_n} \frac{1}{u^2} = \frac{r}{2^{\kappa_r} \pi^2} \psi' \left(2^{m+\kappa_r}(B_n + 1/2)\right)
\]
for \( \psi' \) the trigamma function, and where we have used \([26]\). Analogously, it holds that
\[
T_- = \sum_{\eta = B_n}^{B_n} \alpha_r' = -2^{m+\kappa_r-1}\end{align}
\[
\leq \frac{2^{\kappa_r}}{r} \sum_{u = -2^{m+\kappa_r}(B_n + 1/2)}^{B_n} \sin^2 \left( \frac{2^{\kappa_r} \pi u/r}{2^{\kappa_r} \pi u/r} \right) \leq \frac{2^{\kappa_r}}{r} \frac{r^2}{(2^{\kappa_r} \pi)^2} \sum_{u = -2^{m+\kappa_r}(B_n + 1/2)}^{B_n} \frac{1}{u^2}
\]
\[
= \frac{r}{2^{\kappa_r} \pi^2} \sum_{u = -2^{m+\kappa_r}(B_n + 1/2)}^{B_n} \frac{1}{u^2} = \frac{r}{2^{\kappa_r} \pi^2} \psi' \left(2^{m+\kappa_r}(B_n + 1/2)\right).
\]
It follows from Lem.\([\text{C.3}]\) that
\[
1 - T_+ - T_- = \sum_{\eta = -B_n}^{B_n} 2^{m+\kappa_r-1} - 1 2^{\kappa_r} f_\eta \left( \frac{2\pi}{2^{m+\kappa_r}} 2^{\kappa_r} \alpha_r' \right)
\]
\[
= 1 - \frac{2r}{2^{\kappa_r} \pi^2} \psi' \left(2^{m+\kappa_r}(B_n + 1/2)\right)
\]
\[
> 1 - \frac{2r}{2^{\kappa_r} \pi^2} \left( \frac{1}{2^{m+\kappa_r}(B_n + 1/2)} + \frac{1}{2^2 \cdot (2^{m+\kappa_r}(B_n + 1/2))^2} + \frac{1}{6 \cdot (2^{m+\kappa_r}(B_n + 1/2))^3} \right)
\]
\[ = 1 - \frac{2^m}{2^{2m}} \frac{1}{2^k (B;q + 1/2)} (1 + \epsilon_{B_n}) \]

where we have used Claim C.2 to bound \( \psi' \), and so the lemma follows. \( \blacksquare \)

C.3 Invertibility in the classical post-processing

**Lemma C.5.** The probability is at most \( 2^c + 5 - m \) of observing \((j, k)\) yielding a given \( j' \) in a single run of the quantum algorithm in Sect. 2, for \( m \) the bit length of \( r \) and \( c \geq 0 \).

**Proof.** There are at most \( M = \lceil 2^m + c/r \rceil \leq 2^m + c/r + 1 \) values of \( j \) that yield a given \( j' \).

The quantum algorithm in Sect. 2 may be implemented in an interleaved manner, where \( j \) is first computed and then \( k \). The first step, in which \( j \) is computed, is the quantum part of Shor’s order-finding algorithm. By the analysis in [11, Eq. (2)] of Shor’s order-finding algorithm, the probability of observing a given \( j \) in the first part is

\[
\left| \frac{rj}{2^m + c} \right| \mod r \in [0, r) \cap \mathbb{Z}
\]

since the sum attains its maximum when \( \theta_r = 0 \), and since

\[
\left( \frac{m+c}{1} \right) \leq \left( \frac{m+c-1}{1} \right) \leq \left( \frac{m+c-1}{1} \right) < 2^{m+c}/r.
\]

Since the upper bound \( p \) on the probability of observing a frequency \( j \) that yields a given \( j' \) does not depend on \( j \), it follows that the probability of observing one of the at most \( M \) frequencies \( j \) that yield a given \( j' \) is at most

\[
pM < \frac{r}{2^{2(m+c)}} (2^m + c/r + 1)^3 < \frac{r}{2^{2(m+c)}} (2^m + c+1/r)^3 = \frac{2^m + c+1}{r^2} \leq \frac{2^m + c+1}{2^{2(m-1)}} = 2^c + 5 - m
\]

where we have used that \( r \in \lceil 2^{m-1}, 2^m \rceil \cap \mathbb{Z} \) since \( m \) is the bit length of \( r \), which implies that \( 2^m + c/r > 1 \) since \( c \geq 0 \), and hence that \( 2^m + c / r + 1 < 2^{m+c+1}/r \), and so the lemma follows. \( \blacksquare \)

Note that Lem. C.3, although sufficient for our needs, is coarse and overestimates the probability. In practice, as explained in [11, Thm. 3.5], the probability associated with each \( z' \) is \( \sim 1/r \), i.e. the probability mass is equidistributed amongst the \( r \) peaks in the probability distribution, when \( c \) is sufficiently large.

**Claim C.4.** Suppose that no prime factor less than \( B_r \) divides \( r \). It then holds that

\[
r - \phi(r) \leq \frac{r \log_2(r)}{B_r \log_2(B_r)}
\]

for \( \phi \) Euler’s totient function.

**Proof.** It holds that \( r - \phi(r) \) is the number of integers \( u \in [0, r) \cap \mathbb{Z} \) not coprime to \( r \).

There are at most \( \log_2(r) / \log_2(B_r) \) distinct prime factors that divide \( r \). As \( u \) runs over all \( r \) integers in \( [0, r) \cap \mathbb{Z} \), the probability that each distinct prime factor \( q \) that divides \( r \) also divides \( u \) is at most \( 1/q \leq 1/B_r \). It follows that at most \( r \log_2(r) / (B_r \log_2(B_r)) \) values of \( u \) are divisible by at least one prime factor that also divides \( r \), and so the claim follows. \( \blacksquare \)