LOCAL SOLUTIONS WITH INFINITE ENERGY OF THE MAXWELL-CHERN-SIMONS-HIGGS SYSTEM IN LORENZ GAUGE

HARTMUT PECHER
FACHBEREICH MATHEMATIK UND NATURWISSENSCHAFTEN
BERGISCHE UNIVERSITÄT WUPPERTAL
GAUSSSTR. 20
42119 WUPPERTAL
GERMANY
E-MAIL PECHER@MATH.UNI-WUPPERTAL.DE

Abstract. We consider the Maxwell-Chern-Simons-Higgs system in Lorenz gauge and use a null condition to show local well-posedness for low regularity data. This improves a recent result of J. Yuan.

1. Introduction and main results

The Lagrangian of the (2+1)-dimensional Maxwell-Chern-Simons-Higgs model which was proposed in [LLM] is given by

\[
\mathcal{L} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \frac{\kappa}{4} \epsilon^{\mu\nu\rho} F_{\mu\nu} A_\rho + D_\mu \phi \overline{D^\mu \phi} + \frac{1}{2} \partial_\mu N \partial^\mu N - \frac{1}{2} (e|\phi|^2 + \kappa N - e v^2)^2 - e^2 N^2 |\phi|^2
\]

in Minkowski space \( \mathbb{R}^{1+2} = \mathbb{R} \times \mathbb{R}^2 \) with metric \( g_{\mu\nu} = \text{diag}(1, -1, -1) \). We use the convention that repeated upper and lower indices are summed, Greek indices run over 0,1,2 and Latin indices over 1,2. Here

\[
D_\mu := \partial_\mu - ieA_\mu
\]

\[
F_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu
\]

Here \( F_{\mu\nu} : \mathbb{R}^{1+2} \to \mathbb{R} \) denotes the curvature, \( \phi : \mathbb{R}^{1+2} \to \mathbb{C} \) and \( N : \mathbb{R}^{1+2} \to \mathbb{R} \) are scalar fields, and \( A_\mu : \mathbb{R}^{1+2} \to \mathbb{R} \) are the gauge potentials. \( e \) is the charge of the electron and \( \kappa > 0 \) the Chern-Simons constant, \( v \) is a real constant. We use the notation \( \partial_0 = \partial_t \), where we write \((x^0, x^1, ..., x^n) = (t, x^1, ..., x^n)\) and also \( \partial_0 = \partial_t \). \( \epsilon^{\mu\nu\rho} \) is the totally skew-symmetric tensor with \( \epsilon^{012} = 1 \).

The corresponding Euler-Lagrange equations are given by

\[
\partial_\lambda F^{\lambda\rho} + \frac{\kappa}{2} \epsilon^{\mu\nu\rho} F_{\mu\nu} + 2eIm(\overline{\phi} D^\rho \phi) = 0
\]

\[
D_\mu D^\mu \phi + U_\phi(|\phi|^2, N) = 0
\]

\[
\partial_\mu \partial^\mu N + U_N(|\phi|^2, N) = 0
\]
where
\[ U(|\phi|^2, N) = (e|\phi|^2 + \kappa N - ev^2)\phi + e^2N^2\phi \]
\[ U_N(|\phi|^2, N) = \kappa(e|\phi|^2 + \kappa N - ev^2) + 2e^2N|\phi|^2. \]

(1) can be written as follows
\[
\begin{align*}
- \Delta A_0 + \partial_t(\partial_1 A_1 + \partial_2 A_2) + \kappa F_{12} + 2eIm(\phi \overline{D\phi}) &= 0 \\
(\partial_1^2 - \partial_2^2)A_1 - \partial_1(\partial_1 A_0 - \partial_2 A_2) - \kappa F_{02} + 2eIm(\phi \overline{D\phi}) &= 0 \\
(\partial_2^2 - \partial_1^2)A_2 - \partial_2(\partial_1 A_0 - \partial_2 A_1) + \kappa F_{01} + 2eIm(\phi \overline{D\phi}) &= 0.
\end{align*}
\]

The initial conditions are
\[ A_\nu(0) = a_\nu, \quad (\partial_1 A_\nu)(0) = a_{\nu 1}, \quad \phi(0) = \phi_0, \quad (\partial_t \phi)(0) = \phi_1 \]
\[ N(0) = N_0, \quad (\partial_1 N)(0) = N_1. \]

The Gauss law constraint (4) requires the initial data to fulfill the following condition:
\[ \Delta a_{00} - \partial_1 a_{11} - \partial_2 a_{21} - \kappa(\partial_1 a_{01} - \partial_2 a_{10}) - 2eIm(\phi_0 \overline{\phi_1}) + 2e^2a_{000}|\phi_0|^2 = 0. \]

The energy \( E(t) \) of (1), (2), (3) is (formally) conserved, where
\[
E(t) = \int \frac{1}{2} \sum_i F_{0i}(x, t) + \frac{1}{2} F_{12}(x, t)
+ \sum_\mu |D_\mu \phi(x, t)|^2 + \sum_\mu |\partial_t N(x, t)|^2 + U(|\phi|^2, N)(x, t) dx
\]
with
\[ U(|\phi|^2, N) = \frac{1}{2}(e|\phi|^2 + \kappa N - ev^2)^2 + e^2N^2|\phi|^2. \]

There are two possible natural asymptotic conditions to make the energy finite: either the "nontopological" boundary condition \((\phi, N, A) \rightarrow (0, \frac{ev^2}{\kappa}, 0)\) as \(|x| \rightarrow \infty\) or the "topological" boundary condition \((|\phi|^2, N, A) \rightarrow (e^2, 0, 0)\) as \(|x| \rightarrow \infty\).

We decide to study the "nontopological" boundary condition. Replacing \( N \) by \( N - \frac{ev^2}{\kappa} \) and denoting it again by \( N \) we obtain \((\phi, N, A) \rightarrow (0, 0, 0)\) as \(|x| \rightarrow \infty\), thus leading to solutions in standard Sobolev spaces, and in (2), (3) we now have
\[
\begin{align*}
U(|\phi|^2, N) &= (e|\phi|^2 + \kappa N)\phi + e^2(N + \frac{ev^2}{\kappa})^2\phi \\
U_N(|\phi|^2, N) &= \kappa(e|\phi|^2 + \kappa N) + 2e^2(N + \frac{ev^2}{\kappa})|\phi|^2.
\end{align*}
\]

For the "topological" boundary condition the problem can also be reduced to a system for \((\phi, N, A)\) which fulfills \((\phi, N, A) \rightarrow (0, 0, 0)\) as \(|x| \rightarrow \infty\) for a modified function \(\phi\), if one makes the assumption that \(\phi \rightarrow \lambda \in \mathbb{C}\) as \(|x| \rightarrow \infty\) with \(|\lambda| = v\). In this case one simply replaces \(\phi\) by \(\phi - \lambda\). For details we refer to Yuan’s paper [X]. It is easy to see that the system in this case can be studied in the same way as in the "nontopological" case.

The equations (1), (2), (3) are invariant under the gauge transformations
\[ A_\mu \rightarrow A'_\mu = A_\mu + \partial_\mu \lambda, \quad \phi \rightarrow \phi' = \exp(ie\chi)\phi, \quad D_\mu \rightarrow D'_\mu = \partial_\mu - ieA'_\mu. \]

We consider exclusively the Lorenz gauge
\[ \partial^\mu A_\mu = 0, \]
so that we have to assume that the data fulfill
\[ \partial^\mu a_\mu = 0. \]
We want to prove local well-posedness of the Cauchy problem for (1), (2), (3) for data with minimal regularity assumptions especially for $\phi_0$ and $\phi_1$.

Chae-Chae [CC] assumed $(\phi_0, \phi_1) \in H^2 \times H^1$ and proved local and even global well-posedness using energy conservation. This was improved by J. Yuan [Y] to $(\phi_0, \phi_1), (a_{\mu 0}, a_{\mu 1}), (N_0, N_1) \in H^s \times H^{s-1}$ with $s > \frac{3}{2}$, who obtained a local solution $\phi, A_\mu, N \in C^0([0, T], H^s) \cap C^1([0, T], H^{s-1})$, which is unique in a suitable subset of $X^{s,b}$-type. Using energy conservation this solution exists globally, if $s \geq 1$.

We now further lower down the regularity of the data to $(\phi_0, \phi_1) \in H^s \times H^{s-1}, (a_{\mu 0}, a_{\mu 1}) \in H^{2s-\frac{3}{2}} \times H^{2s-\frac{3}{2}}, (N_0, N_1) \in H^{\frac{3}{2}} \times H^\frac{3}{2}$ under the condition $s > \frac{3}{2}$ . We obtain a local solution $\phi \in C^0([0, T], H^s) \cap C^1([0, T], H^{s-1}), A_\mu \in C^0([0, T], H^{2s-\frac{3}{2}}) \cap C^1([0, T], H^{2s-\frac{3}{2}}), N \in C^0([0, T], H^{\frac{3}{2}}) \cap C^1([0, T], H^\frac{3}{2})$, which is unique in a suitable subspace of $X^{s,b}$-type.

Moreover it is easy to see that as a consequence of these results unconditional uniqueness holds for the solutions of Yuan, i.e. , for $\phi, A_\mu, N \in C^0([0, T], H^s) \cap C^1([0, T], H^{s-1})$ and $s > \frac{3}{2}$ , especially global well-posedness for finite energy solutions ($s=1$).

Whereas Chae-Chae only used standard energy type estimates Yuan applied bilinear Strichartz type estimates which were given in the paper of d’Ancona, Foschi and Selberg [AFS]. We also use this type of estimates but additionally take advantage of a crucial null condition of the term $A_\mu D^\alpha \phi$ in the wave equation for $\phi$. This was detected by Klainerman-Machedon [KM] and Selberg-Tesfahun [ST1] for the Maxwell-Klein-Gordon equations and also by Selberg-Tesfahun [ST1] for the corresponding problem for the Chern-Simons-Higgs equations. When combined with the bilinear Strichartz type estimates this leads to the improved lower bound for the regularity for the problem at hand.

We denote the Fourier transform $\mathcal{F}$ with respect to space as well as to space and time by $\hat{}$. The operator $\langle \nabla \rangle^\alpha$ is defined by $\mathcal{F}(\langle \nabla \rangle^\alpha f)(\xi) = \langle \xi \rangle^\alpha \hat{f}(\xi)$, where $\langle \xi \rangle := (1 + |\xi|^2)^{\frac{1}{2}}$. Define $a+ := a + \epsilon$ for $\epsilon > 0$ sufficiently small, so that $a < a+ < a++$ and similarly $a- < a- < a$, and $a-b$ means $(a-b)-$.

Besides the Sobolev spaces $H^{b,p}$ we use the spaces $X_{s,b}^\pm$ of Bougain-Klainerman-Machedon type defined as the completion of $S(\mathbb{R}^3)$ with the norm $||u||_{X_{s,b}^{\pm}} = || \langle \xi \rangle^s (\pm |\xi|) b(\tau, \xi) \hat{u}(\tau, \xi) ||_{L^2_{\tau \xi}}$ and similarly the wave-Sobolev spaces $X_{s,b}^\pm$ with norm $||u||_{X_{s,b}^\pm} = || \langle \xi \rangle^s (\mp |\xi|) b(\tau, \xi) \hat{u}(\tau, \xi) ||_{L^2_{\tau \xi}}$. The spaces of restrictions to $[0, T] \times \mathbb{R}^2$ with induced norms are $X_{s,b}^\pm[0, T]$ and $X_{s,b}^\pm[0, T]$. Remark that $||u||_{X_{s,b}^\pm}$ for $b \leq 0$ and the reverse estimate for $b \geq 0$.

We now formulate our main results. One easily checks that a solution of (1), (2), (3) (with (9), (10)) under the Lorenz condition

$$\partial^\mu A_\mu = 0$$

(12)

also fulfills the following system

$$(\square + 1) A_0 = -\kappa F_{12} - 2 \epsilon \text{Im}(\phi D_0 \overline{\phi}) + A_0$$

(13)

$$(\square + 1) A_i = -\kappa e^{ij} F_{0j} - 2 \epsilon \text{Im}(\phi D_i \overline{\phi}) + A_i$$

(14)

$$(\square + 1) \phi = 2 \epsilon e^{ab} A_0 \partial_a \phi - 2 \epsilon e^{ij} A_j \overline{\phi} - \epsilon^2 A^2 A_0 \phi + \epsilon^2 A^2 \phi - U_\infty(\phi^2, N) + \phi$$

(15)

$$\square + 1) N = -U_N(|\phi|^2, N) + N.$$ (16)

Here we replaced $\square$ by $\square + 1$ by adding a linear terms on both sides of the equations in order to avoid the operator $(-\Delta)^{-\frac{3}{2}}$, which is unpleasant especially in two dimensions.
Defining

\[ A_{\mu,\pm} = \frac{1}{2}(A_\mu \pm i^{-1}(\nabla)^{-1}\partial_t A_\mu) \quad \iff A_\mu = A_{\mu,+} + A_{\mu,-}, \quad \partial_t A_\mu = i(\nabla)(A_{\mu,+} - A_{\mu,-}) \]

\[ \phi_{\pm} = \frac{1}{2}(\phi \pm i^{-1}(\nabla)^{-1}\partial_t \phi) \quad \iff \phi = \phi_+ + \phi_-, \quad \partial_t \phi = i(\nabla)(\phi_+ - \phi_-) \]

\[ N_{\pm} = \frac{1}{2}(N \pm i^{-1}(\nabla)^{-1}\partial_t N) \quad \iff N = N_+ + N_-, \quad \partial_t N = i(\nabla)(N_+ - N_-) \]

we obtain the equivalent system

\[ (i\partial_t \pm (\nabla))A_{0,\pm} = \pm 2^{-1}(\nabla)^{-1}(\text{R.H.S. of } (13)) \quad (17) \]
\[ (i\partial_t \pm (\nabla))A_{1,\pm} = \pm 2^{-1}(\nabla)^{-1}(\text{R.H.S. of } (14)) \quad (18) \]
\[ (i\partial_t \pm (\nabla))\phi_{\pm} = \pm 2^{-1}(\nabla)^{-1}(\text{R.H.S. of } (15)) \quad (19) \]
\[ (i\partial_t \pm (\nabla))N_{\pm} = \pm 2^{-1}(\nabla)^{-1}(\text{R.H.S. of } (16)) \quad (20) \]

We obtain the following result:

**Theorem 1.1.** Assume \(1 > s > \frac{1}{2}\) and

\[ \phi_0 \in H^s, \quad \phi_1 \in H^{s+1}, \quad a_{\mu,0} \in H^{2s-\frac{4}{5}-}, \quad a_{\mu,1} \in H^{2s-\frac{4}{5}+} \quad (\mu = 0, 1, 2), \]

\[ n_0 \in H^\frac{4}{5}, \quad n_1 \in H^{-\frac{4}{5}}. \]

There exists \(T > 0\) such that the system (13), (14), (15), (16) with (17), (18) and Cauchy conditions

\[ A_\mu(0) = a_{\mu,0}, \quad \partial_t A_\mu(0) = a_{\mu,1}, \quad \phi_0 = \phi_1, \quad \partial_t \phi(0) = \phi_1, \quad N(0) = N_0, \quad \partial_t N(0) = N_1 \]

has a unique local solution

\[ \phi \in X^{\frac{4}{5}+}_+[0,T] + X^{-\frac{4}{5}+}_+[0,T] \]
\[ A_\mu \in X^{2s-\frac{4}{5}-2s-\frac{4}{5}+}_+[0,T] + X^{2s-\frac{4}{5}-2s-\frac{4}{5}+}_-[0,T] \quad \text{(for } \frac{1}{2} < s \leq \frac{5}{8}) \]
\[ A_\mu \in X^{2s-\frac{4}{5}-1+}_+[0,T] + X^{2s-\frac{4}{5}-1-}_-[0,T] \quad \text{(for } \frac{5}{8} < s \leq 1) \]
\[ N \in X^{\frac{4}{5}+}_+[0,T] + X^{-\frac{4}{5}+}_+[0,T]. \]

It has the properties

\[ \phi \in C^0([0,T], H^s(\mathbb{R}^2)) \cap C^1([0,T], H^{s-1}(\mathbb{R}^2)) \]
\[ A_\mu \in C^0([0,T], H^{2s-\frac{4}{5}-}(\mathbb{R}^2)) \cap C^1([0,T], H^{2s-\frac{4}{5}+}(\mathbb{R}^2)) \]
\[ N \in C^0([0,T], H^\frac{4}{5}(\mathbb{R}^2)) \cap C^1([0,T], H^{-\frac{4}{5}}(\mathbb{R}^2)). \]

This result is proven in section 2.

**Remark:** The system (13)-(16) is not scaling invariant, but ignoring lower order terms we can write it schematically as

\[ \square A = \phi \nabla^2 \phi + A \phi^2 \]
\[ \square \phi = A \nabla^2 \phi + A^2 \phi + N \phi^2 + \phi^3 \]
\[ \square N = N \phi^2. \]

This system is invariant under the scaling

\[ A(x,t) \mapsto \lambda A(\lambda x, \lambda^2 t), \quad \phi(x,t) \mapsto \lambda^2 \phi(\lambda x, \lambda t), \quad N(x,t) \mapsto \lambda N(\lambda x, \lambda t). \]

Thus the critical data space with respect to scaling for \(A(0), \phi(0), N(0)\) in dimension 2 is \(L^2\), so that there remains a gap between this space and our minimal assumptions in Theorem 1.1, namely \(\phi(0) \in H^{\frac{4}{5}+}, A(0) \in H^{\frac{4}{5}+}, N(0) \in H^{\frac{4}{5}}\).
In section 3 we show the following theorem and its corollary as a consequence of Theorem 1.1

**Theorem 1.2.** Assume $1 > s > \frac{4}{3}$. Moreover assume that the data satisfy the assumptions of Theorem 1.1 and also

\[
\Delta a_{00} - \partial_1 a_{11} - \partial_2 a_{21} - \kappa (\partial_1 a_{20} - \partial_2 a_{10}) - 2\epsilon \text{Im}(\phi_0 \phi_1) + 2\epsilon^2 a_{00} |\phi_0|^2 = 0
\]

and

\[
\partial^\mu a_\mu = 0.
\]

The solution of Theorem 1.1 is the unique solution of the Cauchy problem for the system

\[
\partial_\lambda F^{\lambda \rho} + \frac{\kappa}{2} e^{\mu \rho} F_{\mu \nu} + 2\epsilon \text{Im}(\phi D^{\nu} \phi) = 0
\]

\[
D_\mu D^\mu \phi + U^S(|\phi|^2, N) = 0
\]

\[
\partial_\mu \partial^\nu N + U_N(|\phi|^2, N) = 0,
\]

where

\[
U^S(|\phi|^2, N) = (\epsilon|\phi|^2 + \kappa N) \phi + \epsilon^2 (N + \frac{\epsilon^2}{\kappa})^2 \phi
\]

\[
U_N(|\phi|^2, N) = \kappa (\epsilon|\phi|^2 + \kappa N) + 2\epsilon^2 (N + \frac{\epsilon^2}{\kappa}) |\phi|^2
\]

with initial conditions

\[
A_\mu(0) = a_{\mu 0}, \quad \partial_\mu A_\mu(0) = a_{\mu 1}, \quad \phi_0 = \phi_0, \quad \phi_1(0) = \phi_1, \quad N(0) = N_0, \quad \partial_\lambda N(0) = N_1,
\]

which fulfills the Lorenz condition $\partial^\mu A_\mu = 0$.

**Corollary 1.1.** Let $s > \frac{4}{3}$, $T > 0$ and $\phi_0, a_{\mu 0}, n_0 \in H^s$, $a_{\mu 1}, n_1 \in H^{s-1}$ satisfying (21), (23). Then the solution of (23), (24), (25) with initial conditions (28) under the Lorenz condition $\partial^\mu A_\mu = 0$ is (unconditionally) unique in the space $\phi, A_\mu, N \in C^0([0, T], H^s) \cap C^1([0, T], H^{s-1})$. Combined with the existence result of Yuan [Y] we obtain local well-posedness and in energy space and above ($s \geq 1$) global well-posedness.

Fundamental for the proof of our theorem are the following bilinear estimates in wave-Sobolev spaces which were proven by d’Ancona, Foschi and Selberg in the two-dimensional case $n = 2$ in [AFS] in a more general form which include many limit cases which we do not need.

**Theorem 1.3.** Let $n = 2$. The estimate

\[
\|uv\|_{X^{s_0, b_0}} \lesssim \|u\|_{X^{s_1, b_1}} \|v\|_{X^{s_2, b_2}}
\]

holds, provided the following conditions are satisfied:

\[
b_0 + b_1 + b_2 > \frac{1}{2}, \quad b_0 + b_1 > 0, \quad b_0 + b_2 > 0, \quad b_1 + b_2 > 0,
\]

\[
s_0 + s_1 + s_2 > \frac{3}{2} - (b_0 + b_1 + b_2)
\]

\[
s_0 + s_1 + s_2 > 1 - \min(b_0 + b_1, b_0 + b_2, b_1 + b_2)
\]

\[
s_0 + s_1 + s_2 > \frac{1}{2} - \min(b_0, b_1, b_2)
\]

\[
s_0 + s_1 + s_2 > \frac{3}{4}
\]
In the case $s > 0$, which easily follows by Sobolev’s embedding for $s > 1$, we have to show
\[\|A_{0,\pm}\|_{X^s_{1,0}[0,T]} \lesssim \|A_{0,\pm}(0)\|_{\dot{H}^s} + T^\epsilon \|\nabla\|^{-1}(R.H.S. of (13))\|_{X^s_{1,0-1+}[0,T]}\]
\[\lesssim \|A_{0,\pm}(0)\|_{\dot{H}^s} + T^\epsilon \|R.H.S. of (13)\|_{X^s_{1,0-1+}[0,T]},\]
which holds for $l \in \mathbb{R}$, $0 < T \leq 1$, $\frac{1}{2} < b \leq 1$, $0 < \epsilon \leq 1 - b$.

The linear terms are easily treated and therefore omitted here.

We now consider the right hand side of (13):
\[-2\epsilon i m(\phi D_0 \phi) = -2\epsilon i m(\phi + \phi_+)(-i)(\nabla)(\phi_+ - \phi_-) = -2\epsilon^2 A_0|\phi|^2.\]

In the case $\frac{1}{2} < s < \frac{4}{5}$, we need the estimate
\[\|\phi_{\pm 1}(\nabla)\phi_{\pm 2}\|_{X^{2s-\frac{2}{5},-\frac{2}{5}}_{1,1}} \lesssim \|\phi_{\pm 1}\|_{X^{s,\frac{1}{5}}_{1,1}} \|\nabla\phi_{\pm 2}\|_{X^{s-1,\frac{1}{5}}_{1,1}},\]
where here and in the following $\pm$ and $\pm$ denote independent signs. This follows from
\[\|uv\|_{X^{2s-\frac{2}{5},-\frac{2}{5}}_{1,0}} \lesssim \|u\|_{X^{s,\frac{1}{5}}_{1,1}} \|v\|_{X^{s-1,\frac{1}{5}}_{1,1}},\]
which follows similarly from Theorem 1.3 with the same choice of the parameters as before except $b_0 = 0$. This implies $s_0 + b_0 = 2s$.

The cubic term $2\epsilon^2 A_0|\phi|^2$ is handled as follows. In the case $\frac{1}{2} < s \leq \frac{6}{7}$, we obtain
\[\|uvw\|_{X^{2s-\frac{2}{7},-\frac{2}{7}}_{1,0}} \lesssim \|u\|_{X^{2s-\frac{2}{7},-\frac{2}{7}}_{1,0}} \|vw\|_{X^{0,0}}\]
and in the case $\frac{5}{8} < s < 1$
\[\|uvw\|_{X^{2s-\frac{5}{8},-\frac{5}{8}}_{1,0}} \lesssim \|u\|_{X^{2s-\frac{5}{8},-\frac{5}{8}}_{1,0}} \|vw\|_{X^{0,0}},\]
by Theorem 1.3. Combining this with the estimate
\[\|vw\|_{X^{0,0}} \lesssim \|v\|_{X^{1,\frac{1}{5}}_{1,1}} \|w\|_{X^{s,\frac{1}{5}}_{1,1}}\]
which easily follows by Sobolev’s embedding for $s > \frac{1}{2}$, we obtain the desired estimate.

The right hand side of (13) can be handled in the same way.

It remains to consider the right hand side of (15). We start with the most interesting quadratic term, where the null conditions come into play, namely
2\epsilon A_j \partial^\mu \phi. Defining the modified Riesz transforms \( R_j := \langle \nabla \rangle^{-1} \partial_j \) and splitting \( A_j \) into divergence-free and curl-free parts and a smooth remainder we obtain
\[
A_j = A_j^{df} + A_j^{cf} + \langle \nabla \rangle^{-2} A_j ,
\]
where
\[
A_1^{df} = R_2 (R_1 A_2 - R_2 A_1), \quad A_2^{df} = R_1 (R_2 A_1 - R_1 A_2)
\]
\[
A_1^{cf} = -R_1 (R_1 A_1 + R_2 A_2), \quad A_2^{cf} = -R_2 (R_1 A_1 + R_2 A_2).
\]
Now we have
\[
A_\mu \partial^\mu \phi = (A_0 \partial_\mu \phi - A_j^{df} \partial^j \phi) - A_j^{df} \partial^j \phi + \langle \nabla \rangle^{-2} A_j \partial^j \phi .
\]
(29)
The first two terms on the right hand side are of null form type, where for the first term we have to use the Lorenz condition (these arguments go back to Selberg-Tesfahun).

We calculate
\[
A_j^{df} \partial^j \phi = R_2 (R_1 A_2 - R_2 A_1) R_1 \langle \nabla \rangle \phi - R_1 (R_1 A_2 - R_2 A_1) R_2 \langle \nabla \rangle \phi
\]
\[
= \sum_{\pm_1, \pm_2} Q_{\pm,1, \pm_2}^{12} ((R_1, \pm_1 A_2, \pm_2 - 1, R_2 A_1, \pm_1 1), \langle \nabla \rangle \phi \pm_2) ,
\]
where
\[
Q_{\pm_1, \pm_2}^{12}(u, v) := R_2 (\pm_1 u) R_1 (\pm_2 v) - R_1 (\pm_1 u) R_2 (\pm_2 v)
\]
is the standard null form with Fourier symbol
\[
\sigma_1 (\pm_1 \xi, \pm_2 \eta) := \frac{(\pm_1 \xi_2)(\pm_2 \eta_1) - (\pm_1 \xi_1)(\pm_2 \eta_2)}{\langle \xi \rangle \langle \eta \rangle} ,
\]
which can be estimated by
\[
|\sigma_1 (\pm_1 \xi, \pm_2 \eta)| \lesssim \frac{||\xi||\eta||}{\langle \xi \rangle \langle \eta \rangle} \Theta (\pm_1 \xi, \pm_2 \eta) \leq \Theta (\pm_1 \xi, \pm_2 \eta) ,
\]
(30)
where \( \Theta (\xi, \eta) \) denotes the angle between two vectors \( \xi, \eta \in \mathbb{R}^2 \).

Next the Lorenz condition gives
\[
R_1 A_1 + R_2 A_2 = \langle \nabla \rangle^{-1} (\partial_1 A_1 + \partial_2 A_2) = \langle \nabla \rangle^{-1} \partial_1 A_0 ,
\]
so that
\[
A_j^{cf} = -R_j (R_1 A_1 + R_2 A_2) = -\langle \nabla \rangle^{-1} R_j \partial_1 A_0 = -i R_j (A_{0,+} - A_{0,-}) .
\]

Thus
\[
A_0 \partial_\mu \phi - A_j^{df} \partial^j \phi = (A_{0,+} + A_{0,-}) i (\langle \nabla \rangle (\phi_+ - \phi_-) - i R_j (A_{0,+} - A_{0,-}) \partial^j (\phi_+ + \phi_-)
\]
\[
= i \sum_{\pm_1, \pm_2} (A_0, \pm_1 \langle \nabla \rangle (\pm_2 \phi_\pm_2) - R_j (\pm_1 A_0, \pm_1 \partial^j \phi_\pm_2)
\]
\[
= i \sum_{\pm_1, \pm_2} (A_0, \pm_1 \langle \nabla \rangle (\phi_\pm_2) - R_j (\pm_1 A_0, \pm_1 \partial^j \phi_\pm_2) )
\]
\[
= i \sum_{\pm_1, \pm_2} \tilde{Q}_{\pm_1, \pm_2}^1 (A_0, \pm_1, \langle \nabla \rangle \phi_\pm_2) ,
\]
where
\[
\tilde{Q}_{\pm_1, \pm_2}^1 (u, v) := u(\pm_2 v) - (\pm_1 R_j u) R^j v
\]
has Fourier symbol
\[
\sigma_2 (\pm_1 \xi, \pm_2 \eta) := (\pm_2 v) - \frac{(\pm_1 \xi) \cdot \eta}{\langle \xi \rangle \langle \eta \rangle} = \frac{(\pm_2 \langle \xi \rangle \langle \eta \rangle) - (\pm_1 \xi) \cdot \eta}{\langle \xi \rangle \langle \eta \rangle} .
\]
Lemma 2.1.

\( |\pm_2 \left< \xi \right| \langle \eta \rangle - \left< \pm_1 \xi \cdot \eta \right> | = \left< \eta \right| \left< \xi \right| - \frac{(\pm_1 \xi) \cdot (\pm_2 \eta)}{\langle \eta \rangle} \right| \\
\leq |\eta| \left| \left( \frac{\langle \pm_1 \xi \rangle \cdot \langle \pm_2 \eta \rangle}{\langle \eta \rangle} \right) \right| + \left| (\pm_1 \xi) \cdot (\pm_2 \eta) \left( \frac{1}{|\eta|} - \frac{1}{\langle \eta \rangle} \right) \right| \right| \\
\leq |\eta| \left| (|\xi| - |\eta|) + |\xi| \left( 1 - \frac{(\pm_1 \xi) \cdot (\pm_2 \eta)}{|\eta| |\xi|} \right) \right| + \left| (\pm_1 \xi) (\pm_2 \eta) \left( \frac{1}{|\eta|} - \frac{1}{\langle \eta \rangle} \right) \right| \right| \\
\lesssim |\eta| \left( |\xi| \Theta(|\pm_1 \xi|, |\pm_2 \eta|) + \frac{|\xi|}{\langle \eta \rangle} \left( |\eta| - |\eta| \right) \right) \right| \\
\lesssim \left< \eta \right| \Theta(|\pm_1 \xi|, |\pm_2 \eta|) + \frac{1}{\langle \xi \rangle} + \frac{1}{\langle \eta \rangle} \right|,
\right.

(31)

Our aim is to prove the following estimate in the case \( \frac{1}{2} < s \leq \frac{5}{8} \):

\[ \| A_{\mu \pm 1} \partial_j \phi_{\pm 2} \|_{L_2^{x \pm 1 \cdots \frac{1}{2} \cdots \frac{1}{2} \cdots 1 \pm 1}} \lesssim \sum_{\mu} \| A_{\mu \pm 1} \|_{L_2^{x \pm 1 \cdots \frac{1}{2} \cdots \frac{1}{2} \cdots 1 \pm 1}} \| (\nabla) \phi_{\pm 1} \|_{L_2^{x \pm 1 \frac{1}{2}}} \]

and in the case \( \frac{5}{8} < s < 1 \):

\[ \| A_{\mu \pm 1} \partial_j \phi_{\pm 2} \|_{L_2^{x \pm 1 \cdots \frac{1}{2} \cdots \frac{1}{2} \cdots 1 \pm 1}} \lesssim \sum_{\mu} \| A_{\mu \pm 1} \|_{L_2^{x \pm 1 \cdots \frac{1}{2} \cdots \frac{1}{2} \cdots 1 \pm 1}} \| (\nabla) \phi_{\pm 1} \|_{L_2^{x \pm 1 \frac{1}{2}}} \]

We first estimate the last term on the right hand side of (29) in the whole range \( \frac{1}{2} < s < 1 \). We have the sufficient estimate

\[ \| (\nabla)^{-2} A_{\mu \pm 1} \partial_j \phi_{\pm 2} \|_{L_2^{x \pm 1 \cdots \frac{1}{2} \cdots \frac{1}{2} \cdots 1 \pm 1}} \lesssim \| A_{\mu \pm 1} \|_{L_2^{x \pm 1 \cdots \frac{1}{2} \cdots \frac{1}{2} \cdots 1 \pm 1}} \| (\nabla) \phi_{\pm 1} \|_{L_2^{x \pm 1 \frac{1}{2}}} \]

which follows from

\[ \| uv \|_{L_2^{x \pm 1 \cdots \frac{1}{2} \cdots \frac{1}{2} \cdots 1 \pm 1}} \lesssim \| u \|_{L_2^{x \pm 1 \cdots \frac{1}{2} \cdots \frac{1}{2} \cdots 1 \pm 1}} \| v \|_{L_2^{x \pm 1 \frac{1}{2}}} \]

by an application of Sobolev’s embedding using \( (1 - s) + (2s + \frac{5}{4}) + (s - 1) > 1 \).

The claimed estimate for the first two terms on the right hand side of (29) reduces to

\[ \int \frac{\widehat{u}(|\xi|, \tau)}{( - \tau \pm_1 |\xi|)^{2s - \frac{5}{4}} (\xi)^{2s - \frac{5}{4}}} \frac{\widehat{v}(\eta, \lambda)}{( - \lambda \pm_2 |\eta|)^{2s - \frac{5}{4}} (\eta)^{2s - \frac{5}{4}}} \left| (\lambda + \tau) - (\xi + \eta) \right|^{s - 1} d\xi d\lambda d\eta \]

\[ \lesssim \| u \|_{L_2^{x \pm 1 \frac{1}{2}}} \| v \|_{L_2^{x \pm 1 \frac{1}{2}}} \]

for \( j = 1, 2 \), where we may assume that the Fourier transforms are nonnegative.

In order to estimate the symbols using \( \Theta \) and \( \Theta_4 \) we use the following

\[ \Theta(|\pm_1 \xi|, |\pm_2 \eta|) \lesssim \left| ( - \tau \pm_1 |\xi|) + ( - \lambda \pm_2 |\eta|) \right|^{\frac{s}{2}} + \left| (\lambda + \tau) - (\xi + \eta) \right|^{1 - s} \left( \min(|\xi|, |\eta|) \right)^{\frac{s}{2} - s} \]

\[ \forall \xi, \eta \in \mathbb{R}^2, \lambda, \tau \in \mathbb{R}. \]
Thus \( \text{(41)} \) reduces to the following estimates

\[
\left\| u_{\nu \nu} \right\|_{X^s} \lesssim \left\| u \right\|_{X^{2s-\frac{1}{4}}} \left\| v \right\|_{X^{s-\frac{1}{4}}} < s < 1
\]

where the last two estimates take account of the last two terms in \( \text{(31)} \) and suffice for the whole range \( \frac{1}{2} < s < 1 \). In the case \( \frac{3}{2} < s < 1 \) the terms of the form \( \left\| u \right\|_{X^{2s-\frac{1}{4}}} \) have to be replaced by \( \left\| u \right\|_{X^{2s-\frac{1}{4}}} \) by \( \left\| u \right\|_{X^{2s-\frac{1}{4}}} \). All these estimates follow from Theorem 1.3 \( \text{(22)} \) and \( \text{(43)} \) are easy consequences of the Sobolev embedding theorem using \( 1 - s + 2s + \frac{3}{4} - s - 1 > \frac{3}{2} > 1 \).

We consider only \( \text{(36)} \). The parameters in Theorem 1.3 are:

Concerning \( \xi^2 \phi \) we show

\[
\left\| u_{\nu \nu} \right\|_{X^s} \lesssim \left\| u \right\|_{X^{2s-\frac{1}{4}}} \left\| v \right\|_{X^{s-\frac{1}{4}}}
\]

by two applications of Theorem 1.2

For the terms \( \phi^2 \phi \) and \( N^2 \phi \) we obtain similarly

\[
\left\| u_{\nu \nu} \right\|_{X^s} \lesssim \left\| u \right\|_{X^{2s-\frac{1}{4}}} \left\| v \right\|_{X^{s-\frac{1}{4}}}
\]

The term \( N \phi \) can be handled even more easily. Finally, we have to consider the terms on the right hand side of \( \text{(16)} \). The term \( N^2 \phi \) (and similarly \( \phi^2 \)) is treated by Bounded in Theorem 1.3 and Sobolev as follows:

\[
\left\| u_{\nu \nu} \right\|_{X^s} \lesssim \left\| u \right\|_{X^{2s-\frac{1}{4}}} \left\| v \right\|_{X^{s-\frac{1}{4}}}
\]

The proof of Theorem 1.3 is now complete.

3. Proof of Theorem 1.2 and Corollary 1.3

Proof of Theorem 1.2. We first prove the following

\[
W := \partial_t A_0 - \partial_1 A_1 - \partial_2 A_2, \quad V := \Delta A_0 - \kappa F_{12} - 2\varepsilon Im(\bar{\phi}D^\alpha\phi) - \partial_t (\Phi A_1 + \partial_2 A_2). \]

With this notation we know that \( W(0) = V(0) = 0 \) and want to show \( W(t) = V(t) = 0 \) \( \forall t \in [0, T] \). We claim that \( (W, V) \) is a solution of the system

\[
\partial_t W = V, \quad \partial_t V = \Delta W + 2\varepsilon |\phi|^2 W. \]

(44)
We easily calculate using (13)
\[ \partial_t W = \partial_t^2 A_0 - \partial_t (\partial_t A_1 + \partial_t A_2) \]
\[ = \Delta A_0 - \kappa F_{12} - 2\epsilon \text{Im}(\phi \bar{D}^2 \phi) - \partial_t (\partial_t A_1 + \partial_t A_2) = V . \]

Moreover, by (14)
\[ \Delta W = \partial_t A_0 - \partial_t \Delta A_1 - \partial_t \delta A_2 \]
\[ = \partial_t \Delta A_0 - \partial_t (\kappa F_{02} + 2\epsilon \text{Im}(\phi \bar{D}^1 \phi) - \partial_t^2 A_1 \]
\[ - \partial_t (-\kappa F_{01} + 2\epsilon \text{Im}(\phi \bar{D}^2 \phi)) - \partial_t^2 \delta A_2 \]
\[ = \partial_t \Delta A_0 - \partial_t (\kappa F_{12}) - 2\epsilon (\partial_t \text{Im}(\phi \bar{D}^1 \phi) + \partial_t \text{Im}(\phi \bar{D}^2 \phi)) - \partial_t^2 (\partial_t A_1 + \partial_t A_2) . \]

Now we calculate
\[ \partial_t \text{Im}(\phi \bar{D}^1 \phi) + \partial_t \text{Im}(\phi \bar{D}^2 \phi) \]
\[ = 1 \text{Im}(\phi (\partial_t D^1 \phi + \partial_t D^2 \phi)) + \text{Im}(\partial_t \text{Im}(\phi \bar{D}^1 \phi + \partial_t \text{Im}(\phi \bar{D}^2 \phi)) \]
\[ + \text{Im}(D_2 \phi \bar{D}_2 \phi + \text{Im}(\phi \bar{D}^1 \phi + \partial_t \phi \bar{D}^2 \phi) + \text{Im}(D_2 \phi \bar{D}^2 \phi + \text{Im}(\phi \bar{D}^1 \phi + \partial_t \phi \bar{D}^2 \phi)) \]
\[ = \text{Im}(\phi (D_1 \phi + D_2 \phi)) \]
and
\[ D_2^2 \phi - D_1^2 \phi - D_2^2 \phi \]
\[ = ((\partial_2^2 - \partial_1^2 - \partial_2^2)\phi - 2\epsilon \partial_0 \phi + 2i\epsilon A_j \partial_j \phi + \epsilon^2 A_j^2 A_j \phi - \epsilon^2 A_0^2 \phi) \]
\[ + i\epsilon (\partial_1 A_1 + \partial_2 A_2 - \partial_0 A_0) \phi \]
\[ = i\epsilon (\partial_1 A_1 + \partial_2 A_2 - \partial_0 A_0) \phi - U_{\text{eq}}(\phi^2, N) = -i\epsilon W \phi - U_{\text{eq}}(\phi^2, N) , \]
where we used (13) in the last line. This implies
\[ \partial_t \text{Im}(\phi \bar{D}^1 \phi) + \partial_t \text{Im}(\phi \bar{D}^2 \phi) = \text{Im}(\phi \bar{D}^2 \phi) + \epsilon \text{Im}(\phi W \bar{\phi}) - \text{Im}(\phi U_{\text{eq}}(\phi^2, N)) \]
\[ = \text{Im}(\phi \bar{D}^2 \phi) + \epsilon W |\phi|^2 \]
using \( \text{Im}(\phi U(|\phi|^2, N)) = 0 . \) Furthermore
\[ \text{Im}(\phi \bar{D}_0 \phi) = \text{Im}(\phi (\partial_0 - i\epsilon A_0) \bar{D}_0 \phi) \]
\[ = \partial_t \text{Im}(\phi \bar{D}_0 \phi) - \text{Im}(\partial_t \phi \bar{D}_0 \phi) + \epsilon \text{Im}(\phi A_0 \bar{D}_0 \phi) - \epsilon^2 \text{Im}(\phi A_0 \bar{D}_0 \phi) \]
\[ = \partial_t \text{Im}(\phi D_0 \phi) - \text{Im}(\partial_t \phi \bar{D}_0 \phi) + \epsilon \text{Im}(\phi (\bar{D}_0 \phi) - \epsilon \text{Im}(\phi (-i) A_0 \bar{D}_0 \phi) \]
\[ = \partial_t \text{Im}(\phi D_0 \phi) . \]
Collecting all these calculations we finally arrive at
\[ \Delta W = \partial_t V - 2\epsilon |\phi|^2 W , \]
so that (14) is proven and
\[ (\partial_t^2 - \Delta) W = 2\epsilon |\phi|^2 W . \]

We remind that \( W(0) = 0 , \ (\partial_t W)(0) = V(0) = 0 . \) For sufficiently regular \( \phi \) this Cauchy problem for a linear wave equation is uniquely solvable by \( W(t) \equiv 0 . \) Thus by (14) also \( V(t) \equiv 0 , \) especially the Lorenz condition is satisfied. But the solutions obtained in Theorem 1.1 are continuously depending on the data and also persistence of higher regularity holds. Thus by regularization of the data we may assume here \( \phi, \partial_t \phi \in C^\infty_0([0, T] \times \mathbb{R}^2) \) and \( A_\mu, \partial A_\mu \in C^\infty((0, T] \times \mathbb{R}^2) , \) which justifies our calculations.
Under the Lorenz condition $W \equiv 0$ the equations (13) and (14) exactly reduce to (1), (5) and (6), which means that (1) is fulfilled. Under the Lorenz condition we also have that (2) and (3) are equivalent to (15) and (16), respectively.

Summarizing we have shown that the solution of Theorem 1.1 is a solution of the Cauchy problem for (1), (2), (3) satisfying the Lorenz condition. As remarked earlier already the reverse is also true, so that uniqueness holds.

Proof of Corollary 1.1. It suffices to show that any solution $\phi, A_\mu, N \in C^0([0,T], H^s) \cap C^1([0,T], H^{s-1})$ belongs to $\phi, A_\mu, N \in X_{1,2}^{1/2,1}([0,T], \mathbb{H}^{-1/2})$, where uniqueness holds by Theorem 1.2. This follows if the right hand sides of (17), (18), (19) and (20) belong to $L^2([0,T], H^{1/2})$. The quadratic terms are all treated similarly, e.g.

$$\langle \nabla \rangle^{-1} (A_\mu \partial_\mu \phi)$$

is estimated by the Sobolev multiplication rule as follows:

$$\|A_\mu \partial_\mu \phi\|_{L^2_t([0,T], H^{1/2})} \lesssim T^{1/2} \|A_\mu\|_{L^\infty_t([0,T], H)} \|\partial_\mu \phi\|_{L^\infty_t([0,T], H^{1/2})} < \infty.$$

The cubic terms are even easier to handle, e.g.

$$\|A_\mu^2 \phi\|_{L^2_t([0,T], H^{1/2})} \lesssim T^{1/2} \|A_\mu\|^2_{L^\infty_t([0,T], L^4)} \|\phi\|_{L^\infty_t([0,T], H)} < \infty.$$

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