q-Deformed quantum Lie algebras

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Abstract
Attention is focused on q-deformed quantum algebras with physical importance, i.e. $U_q(su_2)$, $U_q(so_4)$ and q-deformed Lorentz algebra. The main concern of this article is to assemble important ideas about these symmetry algebras in a consistent framework which shall serve as starting point for representation theoretic investigations in physics, especially quantum field theory. In each case considerations start from a realization of symmetry generators within the differential algebra. Formulae for coproducts and antipodes on symmetry generators are listed. The action of symmetry generators in terms of their Hopf structure is taken as q-analog of classical commutators and written out explicitly. Spinor and vector representations of symmetry generators are calculated. A review of the commutation relations between symmetry generators and components of a spinor or vector operator is given. Relations for the corresponding quantum Lie algebras are computed. Their Casimir operators are written down in a form similar to the undeformed case.

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1 Introduction

It is an old idea that limiting the precision of position measurements by
a fundamental length will lead to a new method for regularizing quantum
field theories [1]. It is also well-known that such a modification of classical
spacetime will in general break its Poincaré symmetry [2]. One way out
of this difficulty is to change not only spacetime, but also its underlying
symmetry.

Quantum groups can be seen as deformations of classical spacetime sym-
metries, as they describe the symmetry of their comodules, the so-called
quantum spaces. From a physical point of view the most realistic examples
for quantum groups and quantum spaces arise from q-deformation [3–9].
In our work we are interested in q-deformed versions of Minkowski space
and Euclidean spaces as well as their corresponding symmetries, given by
q-deformed Lorentz algebra and algebras of q-deformed angular momentum,
respectively [10–14]. Remarkably, Julius Wess and his coworkers were able
to show that q-deformation of spaces and symmetries can indeed lead to dis-
cretizations, as they result from the existence of a smallest distance [15,16].
This observation nourishes the hope that q-deformation might give a new
method to regularize quantum field theories [17–20].

In our previous work [21–27] attention was focused on q-deformed quan-
tum spaces of physical importance, i.e. two-dimensional Manin plane, q-
deformed Euclidean space in three or four dimensions and q-deformed Min-
kowski space. If we want to describe fields on q-deformed quantum spaces
we need to consider representations of the corresponding quantum symme-
tries, given by $U_q(su_2), U_q(so_4)$ and q-deformed Lorentz algebra. The study
of such quantum algebras has produced a number of remarkable results
during the last two decades. For a review we recommend the reader the
presentations in [28–30] and references therein. In this article we want to
adapt these general ideas to our previous considerations about q-deformed
quantum spaces. In doing so, we provide a basis for performing concrete cal-
culations, as they are necessary in formulating and evaluating field theories
on quantum spaces.

In particular, we intend to proceed as follows. In Sec. 2 we cover the
ideas our considerations about q-deformed quantum symmetries are based
on. In the subsequent sections we first recall for each quantum algebra
under consideration how its generators are realized within the correspond-
ing q-deformed differential calculus. Then we are going to present explicit
formulae for coproduct and antipode on a set of independent symmetry
generators. With this knowledge at hand we should be able to write down
explicit formulae for so-called q-commutators between symmetry generators and representation space elements. In addition to this, we are going to consider spinor and vector representations of the independent symmetry generators and give a complete review of the commutation relations between symmetry generators and components of a spinor or vector operator. Furthermore we are going to calculate the adjoint action of the independent symmetry generators on each other. In this manner, we will get relations for quantum Lie algebras. We will close our considerations by writing down q-analogs of Casimir operators. Finally, Sec. 6 shall serve as a short conclusion. For reference and for the purpose of introducing consistent and convenient notation, we provide a review of key notations and results in Appendix A.

We should also mention that most of our results were obtained by applying the computer algebra system Mathematica [31]. We are convinced that in the future this powerful tool will be inevitable in managing the extraordinary complexity of q-deformation.

2 Basic ideas on q-deformed quantum symmetries

Roughly speaking a quantum space is nothing else than an algebra generated by non-commuting coordinates $X_1, X_2, \ldots, X_n$, i.e.

$$\mathcal{A}_q = \mathbb{C}[[X_1, \ldots, X_n]]/\mathcal{I},$$

where $\mathcal{I}$ denotes the ideal generated by the relations of the non-commuting coordinates. The quantum spaces we are interested in for physical reasons are two-dimensional Manin plane, q-deformed Euclidean space in three or four dimensions as well as q-deformed Minkowski space. For their definition we refer the reader to Appendix A.

On each of these quantum spaces exist two differential calculi [32–34] with

$$\partial^i X^j = g^{ij} + k(\hat{R}^{-1})_{kl}^i X^k \partial^l, \quad k \in \mathbb{R},$$

$$\hat{\partial}^i X^j = g^{ij} + k^{-1}(\hat{R})_{kl}^i X^k \hat{\partial}^l,$$

where $\hat{R}$ and $g^{ij}$ denote respectively the $R$-matrix and the corresponding quantum metric of the underlying quantum symmetry. There is a q-deformed antisymmetriser $P_A$ that enables us to define the components of q-deformed orbital angular momentum as antisymmetrized products of coordinates and derivatives [35]. Specifically, we have

$$L^{ij} \sim (P_A)_{kl}^i X^k \partial^l \sim (P_A)_{kl}^i X^k \hat{\partial}^l.$$
It can be shown that the $L_{ij}$ together with a central generator $U$ being subject to
\[ U - 1 \sim \lambda g_{ij} \hat{X}^i \hat{\partial}^j, \quad \lambda \equiv q - q^{-1}, \] (4)
span the quantum algebras describing the underlying quantum symmetry. This way we will obtain $U_q(su_2)$, $U_q(so_4)$ as well as q-deformed Lorentz algebra in the sequel of this article.

In accordance with the classical case, a single particle wave function should be defined as element of the tensor product of a finite vector space $\mathcal{S}$ holding the spin degrees of freedom and the algebra of space functions $\mathcal{M}$, where $\mathcal{S}$ and $\mathcal{M}$ are both modules of the q-deformed symmetry algebra under consideration. In contrast to the classical situation, we now have to distinguish between right and left wave functions, i.e.

\[ \psi_R \equiv \sum_j e^j \otimes \psi^j, \quad \psi_L \equiv \sum_j \psi^j \otimes e^j, \] (5)

where $\{e^j\}$ is a basis in $\mathcal{S}$ and $\psi^j$ are elements of $\mathcal{M}$. The reason for this lies in the fact that due to the braiding between the two tensor factors we are not allowed to interchange them by a trivial flip [36].

The transformation properties of our wave functions are determined by the coproduct of the symmetry generators, since we have (we write the coproduct in the so-called Sweedler notation, i.e. $\Delta(a) = a(1) \otimes a(2)$)

\[ L_{ij} \triangleright \psi_R = \sum_k (L_{ij})^{(1)}_k \triangleright e^k \otimes (L_{ij})^{(2)}_k \triangleright \psi^k, \]
\[ L_{ij} \triangleright \psi_L = \sum_k (L_{ij})^{(1)}_k \triangleright \psi^k \otimes (L_{ij})^{(2)}_k \triangleright e^k. \] (6)

For scalar wave functions $\mathcal{S}$ has to be a one-dimensional vector space with the corresponding representation on it being the trivial one. In this manner, the symmetry transformation is reduced to the action on space functions. In Ref. [22] we derived explicit formulae for such actions. In the sequel of this article we will restrict ourselves to representations of symmetry generators on a spinorial or vectorial basis. Physical fields of higher spin should then be built up from spinor or vector fields.

It should be noted that we can combine a quantum algebra $\mathcal{H}$ with its representation space $\mathcal{A}$ to form a left cross product algebra $\mathcal{A} \ltimes \mathcal{H}$ built on $\mathcal{A} \otimes \mathcal{H}$ with product

\[ (a \otimes h)(b \otimes g) = a(h(1) \triangleright b) \otimes h(2) g, \quad a, b \in \mathcal{A}, \quad h, g \in \mathcal{H}. \] (7)

There is also a right-handed version of this notion called a right cross product algebra $\mathcal{H} \ltimes \mathcal{A}$ and built on $\mathcal{H} \otimes \mathcal{A}$ with product

\[ (h \otimes a)(g \otimes b) = hg(1) \otimes (a \triangleright g(2)) b. \] (8)
The last two identities tell us that the commutation relations between symmetry generators and representation space elements are completely determined by coproduct and action of the symmetry generators.

In this article it can be seen that there is a remarkable correspondence between q-deformed symmetry algebras and their classical counterparts. Towards this end we have to introduce the notion of a q-commutator which is nothing other than the action of a symmetry generator $L^{ij}$ on a representation space element $V$. Expressing that action in terms of the Hopf structure of $L^{ij}$ the q-brackets become

$$\begin{align*}
[L^{ij}, V]_q &\equiv L^{ij} \triangleright V = L^{ij}_{(1)} VS(L^{ij}_{(2)}), \\
[V, L^{ij}]_q &\equiv V \triangleleft L^{ij} = S^{-1}(L^{ij}_{(2)})V L^{ij}_{(1)},
\end{align*}$$

where $S$ and $S^{-1}$ denote the antipode and its inverse, respectively. From their very definition it follows that q-commutators obey the q-deformed Jacobi identities

$$\begin{align*}
[[L^{ij}, L^{kl}], V]_q &\equiv [L^{ij}_{(1)}, [L^{kl}, V]_q], \\
[[L^{ij}, L^{kl}]_q, V] &\equiv [L^{ij}_{(1)}, [L^{kl}, [S(L^{ij}_{(2)}), V]]_q].
\end{align*}$$

Now, we are able to introduce the notion of a quantum Lie algebra as it was given in [37–39]. A quantum Lie algebra can be regarded as a subspace of a q-deformed enveloping algebra $U_q(g)$ being invariant under the adjoint action of $U_q(g)$. The point now is that the $L^{ij}$ are the components of a tensor operator and this is the reason why their adjoint action on each other equals to a linear combination of the $L^{ij}$ [40], i.e. the $L^{ij}$ span a quantum Lie algebra with

$$\begin{align*}
[L^{ij}, L^{kl}]_q &= L^{ij} \triangleright L^{kl} = L^{ij} \triangleleft L^{kl} = (C^{ij})^k_m L^{mn},
\end{align*}$$

where the $C^{ij}$ are the so-called quantum structure constants. In the subsequent sections we are going to determine those constants in the case of $U_q(su_2)$, $U_q(so_4)$ and q-deformed Lorentz algebra.

Finally, let us mention that the q-brackets give another way to write down commutation relations between symmetry generators and components of a vector or spinor operator. More formally, we have

$$\begin{align*}
[L^{ij}, X^k]_q &\equiv (\tau^{ij}_L)^k_l X^l, \\
[X_k, L^{ij}]_q &\equiv X_l (\tau^{ij}_R)^l_k,
\end{align*}$$

and

$$\begin{align*}
[L^{ij}, \theta^\alpha]_q &\equiv (\sigma^{ij}_L)^\alpha_\beta \theta^\beta, \\
[\theta^\alpha, L^{ij}]_q &\equiv \theta_\beta (\sigma^{ij}_R)^\beta_\alpha,
\end{align*}$$
where $X^k$ and $\theta^\alpha$ stand respectively for components of vector and spinor operators. Notice that for $\tau_{L/R}$ and $\sigma_{L/R}$ we have to substitute the representation matrix of $L^{ij}$, as it comes out for the vector and spinor representation, respectively. It should be appreciated that the relations in (11), (12), and (13) become part of a q-deformed Super-Euclidean or Super-Poincaré algebra, if such objects exist.

### 3 Quantum Lie algebra of three-dimensional angular momentum

#### 3.1 Representation of three-dimensional angular momentum within q-deformed differential calculus

In the case of three-dimensional q-deformed Euclidean space (for its definition see Appendix A) the generators of orbital angular momentum are defined by [14]

$$L^A \equiv \Lambda^{1/2} X^C \hat{\partial}^D \epsilon_{DC}^A, \quad A \in \{+3,-\},$$

where $\epsilon_{DC}^A$ denotes a q-analog of the completely antisymmetric tensor of third rank and $\Lambda$ stands for a scaling operator subject to

$$\Lambda X^A = q^{4} X^A \Lambda, \quad \Lambda \hat{\partial}^A = q^{-4} \hat{\partial}^A \Lambda, \quad A \in \{+,3,-\}.$$

(15)

If not stated otherwise summation over all repeated indices is to be understood. Substituting for $\epsilon_{DC}^A$ their explicit form, we obtain from Eq. (14)

$$L^+ = -q^{-1} \Lambda^{1/2} X^+ \hat{\partial}^3 + q^{-3} \Lambda^{1/2} X^3 \hat{\partial}^+,$$

$$L^3 = -q^{-2} \Lambda^{1/2} X^+ \hat{\partial}^+ + q^{-2} \Lambda^{1/2} X^- \hat{\partial}^- - q^{-2} \lambda \Lambda^{1/2} X^3 \hat{\partial}^3,$$

$$L^- = -q^{-1} \Lambda^{1/2} X^3 \hat{\partial}^- + q^{-3} \Lambda^{1/2} X^- \hat{\partial}^3.$$

(16)

Using the Leibniz rules for partial derivatives in the form

$$X^A \hat{\partial}^B = g^{AB} + (\hat{R}^{-1})^{AB} CD \hat{\partial}^C X^D,$$

and taking into account the identities

$$g^{AB} \epsilon_{BA}^C = 0, \quad (\hat{R}^{-1})^{AB} CD \epsilon_{BA}^E = -q^4 \epsilon_{DC}^E,$$

the generators in Eq. (14) can alternatively be written as

$$L^A = -q^4 \Lambda^{1/2} \hat{\partial}^C X^D \epsilon_{DC}^A.$$

(19)
As already mentioned, there is a second set of derivatives $\partial A$ which can be linked to the first one via the relation [14]

$$\hat{\partial} A = \Lambda^{-1}(\partial A + q^3 \lambda X^A \partial^B \partial^C g_{BC}). \quad (20)$$

Making use of this identity together with $X^C X^D \epsilon_{DC} A = 0$, one can show that we additionally have

$$L^A = q^4 \Lambda^{-1/2} X^C \partial^D \epsilon_{DC} A = -\Lambda^{-1/2} \partial^C X^D \epsilon_{DC} A. \quad (21)$$

3.2 Hopf structure of $U_q(su_2)$ and corresponding q-commutators

As it is well-known, the generators $L^+, L^3$, and $L^-$ together with a generator $\tau^{1/2}$ can be viewed as elements of the quantum algebra $U_q(su_2)$ [14]. Its defining relations read

$$\tau^{1/2} L^\pm = q^{\mp 2} L^\pm \tau^{1/2}, \quad \tau^{1/2} L^3 = L^3 \tau^{1/2},$$

$$L^- L^+ - L^+ L^- = \tau^{-1/2} L^3,$$

$$L^\pm L^3 - L^3 L^\pm = \mp q^{\pm 1} L^\pm \tau^{-1/2}. \quad (22)$$

Furthermore, this algebra has a Hopf structure given by

$$\Delta(L^\pm) = L^\pm \otimes \tau^{-1/2} + 1 \otimes L^\pm, \quad (23)$$

$$\Delta(L^3) = L^3 \otimes \tau^{-1/2} + \tau^{1/2} \otimes L^3$$

$$+ \lambda \tau^{1/2}(qL^+ \otimes L^- + q^{-1} L^- \otimes L^+),$$

$$S(L^\pm) = q^{\mp 2} S^{-1}(L^\pm) = -L^\pm \tau^{1/2}, \quad (24)$$

$$S(L^3) = S^{-1}(L^3) = -q^{-2} L^3 + \lambda \lambda \tau^{1/2} L^+ L^-,$$

$$\varepsilon(L^A) = 0, \quad A \in \{+, 3, -\}, \quad (25)$$

where $\lambda_+ \equiv q + q^{-1}$. In addition to this, let us notice that $\tau^{1/2}$ is a grouplike generator.

With this Hopf structure at hand we are in a position to write down expressions for q-commutators. Specifically, we get from Eq. [9] for left-commutators

$$[L^\pm, V]_q = (L^\pm V - VL^\pm) \tau^{1/2}, \quad (26)$$

$$[L^3, V]_q = L^3 V \tau^{1/2} - q^{-2} \tau^{1/2} VL^3.$$
\[- \lambda \tau^{1/2}(q^{-1}L^+ V \tau^{1/2}L^- + qL^- V \tau^{1/2}L^+) \]
\[+ \lambda \lambda_V \tau^{1/2}V \tau^{1/2}L^+L^- \],

and likewise for right-commutators,
\[[V, L^+]_q = \tau^{1/2}(V L^+ - L^+ V) \]
\[[V, L^3]_q = \tau^{1/2}VL^3 - q^{-2}L^3V \tau^{1/2} \]
\[- \lambda \tau^{1/2}(q^{-1}L^+ V \tau^{1/2}L^- + qL^- V \tau^{1/2}L^+) \]
\[+ \lambda \lambda \tau^{1/2}L^+L^- \tau^{1/2} \],

where \(V\) denotes an element living in a representation space of \(U_q(su_2)\).

### 3.3 Matrix representations of \(U_q(su_2)\) and commutation relations with tensor operators

Next, we would like to turn our attention to some special representations of the symmetry generators \(L^+, L^3,\) and \(L^-\), namely spinor and vector representations [28, 41]. The finite dimensional representations of \(U_q(su_2)\) are already well-known (see for example Refs. [28] and [41]). With our conventions (see also Appendix A) the spinor representations on symmetry generators become

\[L^A \triangleright \theta^\alpha = (\sigma^A)^\alpha_\beta \theta^\beta, \quad \tau^{1/2} \triangleright \theta^\alpha = (\tau^{1/2})^\alpha_\beta \theta^\beta,\]
\[\theta^\alpha \triangleleft L^A = \theta_\beta (\sigma^A)^\beta_\alpha, \quad \theta^\alpha \triangleleft \tau^{1/2} = \theta_\beta (\tau^{1/2})^\beta_\alpha,\]

where we have introduced as some kind of q-deformed sigma matrices
\[(\sigma^+)_{\alpha \beta} = -q^{1/2} \lambda^+_{\alpha \beta} \left( \begin{array}{cc} 0 & 0 \\ 0 & 1 \end{array} \right), \quad (\sigma^-)_{\alpha \beta} = q^{-1/2} \lambda^+_{\alpha \beta} \left( \begin{array}{cc} 0 & 1 \\ 0 & 0 \end{array} \right),\]
\[(\sigma^3)_{\alpha \beta} = \lambda^+_{\alpha \beta} \left( \begin{array}{cc} -q & 0 \\ 0 & q^{-1} \end{array} \right), \quad (\tau^{1/2})_{\alpha \beta} = \left( \begin{array}{cc} q & 0 \\ 0 & q^{-1} \end{array} \right).\]

Here and in what follows, we shall take the convention that upper and lower matrix indices refer to columns and rows, respectively. That the above matrices indeed give a representation can easily be checked. Towards this end, we have to substitute in (22) the sigma matrices for the algebra generators. Then we can show by usual matrix multiplication that the algebra relations are fulfilled.

The above results enable us to write down commutation relations between symmetry generators and components of a spinor operator. Such a
spinor operator with components $\theta^\alpha$, $\alpha = 1, 2$, is completely determined by its transformation properties

$$[L^A, \theta^\alpha]_q = L^A \triangleright \theta^\alpha, \quad [\theta^\alpha, L^A]_q = \theta^\alpha \triangleleft L^A. \quad (30)$$

Inserting the results of (28) and (29) into the relations of (30) and then rearranging, it follows that

$$L^+ \theta^1 = \theta^1 L^+ - q^{1/2} \lambda_+^{-1/2} \theta^2 \tau^{-1/2} - q^{-1/2} \lambda_+^{-1} \theta^1 \tau^{-1/2}, \quad (31)$$

$$L^+ \theta^2 = \theta^2 L^+, \quad L^3 \theta^1 = q^{1/2} L^3 - q^{-1/2} \lambda_+^{-1/2} \theta^2 L^+ - q^{-1/2} \lambda_+^{-1} \theta^1 \tau^{-1/2}, \quad L^3 \theta^2 = q^{-1/2} L^3 + q^{1/2} \lambda_+^{-1/2} \theta^1 L^+ + q^{-1/2} \lambda_+^{-1} \theta^2 \tau^{-1/2}, \quad L^- \theta^1 = \theta^1 L^-, \quad L^- \theta^2 = \theta^2 L^- + q^{-1/2} \lambda_+^{-1/2} \theta^1 \tau^{-1/2},$$

and likewise,

$$\theta_1 L^+ = L^+ \theta_1, \quad (32)$$

$$\theta_2 L^+ = L^+ \theta_2 - q^{1/2} \lambda_+^{-1/2} \tau^{-1/2} \theta_1, \quad \theta_1 L^3 = q L^3 \theta_1 + q^{-1/2} \lambda_+^{-1/2} L^+ \theta_2 + q^{-1} \lambda_+^{-1} \tau^{-1/2} \theta_1, \quad \theta_2 L^3 = q^{-1} L^3 \theta_2 - q^{1/2} \lambda_+^{-1/2} L^- \theta_1 + q^{-1} \lambda_+^{-1} \tau^{-1/2} \theta_2, \quad \theta_1 L^- = L^- \theta_1 + q^{-1/2} \lambda_+^{-1/2} \tau^{-1/2} \theta_2, \quad \theta_2 L^- = L^- \theta_2.$$

Next, we turn to the vector representations of $U_q(su_2)$. They take the form

$$L^A \triangleright X^B = (\tau^A)^B_C X^C, \quad X_B \triangleleft L^A = X_C (\tau^A)^C_B, \quad (33)$$

with

$$(\tau^+)^B_C = \begin{pmatrix} 0 & -q & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}, \quad (\tau^-)^B_C = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & q^{-1} & 0 \end{pmatrix}, \quad (\tau^3)^B_C = \begin{pmatrix} q^{-1} & 0 & 0 \\ 0 & -\lambda & 0 \\ 0 & 0 & -q \end{pmatrix}, \quad (\tau^{1/2})^B_C = \begin{pmatrix} q^{-2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & q^2 \end{pmatrix}, \quad (34)$$

where rows and columns are arranged in the order $+, 3, -$ (from left to right and top to bottom). Their property of being a representation can
be verified in very much the same way as was done for the spinor representation. Finally, let us note that the above representation matrices and the q-deformed $\varepsilon$-tensor are linked via

$$(\tau^A)^B_C = q^2 \varepsilon^{AB} C, \quad A \in \{+, 3, -\}. \quad (35)$$

With the same reasonings already applied to the case of spinor representations we can derive commutation relations between symmetry generators and the components of a vector operator. In complete accordance to the spinor case, their general form is given by

$$[L^A, X^B]_q = L^A \triangleright X^B, \quad [X_B, L^A]_q = X_B \triangleleft L^A, \quad (36)$$

from which we find by specifying q-commutator and vector representation

$$L^\pm X^\pm = X^\pm L^\pm, \quad (37)$$

and

$$X^\pm L^\pm = L^\pm X^\pm, \quad (38)$$

This way, we regain the commutation relations already presented in [14].

### 3.4 Quantum Lie algebra of $U_q(su_2)$ and its Casimir operator

Recalling that $L^A, A \in \{+, 3, -\}$, is a vector operator, the identities in (36) also apply to the case with $X^A$ being replaced by $L^A$. In doing so, we are led to the relations for the quantum Lie algebra of $U_q(su_2)$, i.e.

$$[L^A, L^B]_q = q^2 \varepsilon^{AB} C L^C, \quad (39)$$
or more concretely,
\begin{align}
[L^A, L^A]_q &= 0, \quad A \in \{+, -\}, \\
[L^3, L^3]_q &= -\lambda L^3, \\
[L^\pm, L^3]_q &= \mp q^{\pm 1} L^\pm, \\
[L^\pm, L^\mp]_q &= \mp L^3. 
\end{align}
(40)

These relations are equivalent to those in (22), as can be seen in a straightforward manner by writing out the q-commutators explicitly.

Instead of the vector operator $L^A$, one can just as well use a complete antisymmetric tensor operator of second rank given by

\[ M^{AB} \equiv \varepsilon^{ABC} L^C. \]  
(41)

Its antisymmetry requires the following identities to hold, which can easily be read off from its very definition:
\begin{align}
M^{++} &= M^{--} = 0, \quad M^{\pm 3} = -q^{\pm 2} M^{3\pm}, \\
M^{-+} &= -M^{+-}, \quad M^{33} = \lambda M^{+-}. 
\end{align}
(42)

Consequently, we have three independent components, for which we can choose
\begin{align}
M^{3\pm} &= -q^{\pm 1} L^\pm, \quad M^{+-} = -q^{-2} L^3. 
\end{align}
(43)

Last but not least we want to deal with the Casimir operator of our quantum Lie algebra. Let us recall that a Casimir operator $C$ has to be subject to

\[ [L^A, C]_q = [C, L^A]_q = 0 \quad \text{for all} \quad A \in \{+, 3, -\}. \]  
(44)

By a direct calculation making use of
\begin{align}
[L^A, UV]_q &= [L^A_{(1)}, U]_q [L^A_{(2)}, V]_q, \\
[UV, L^A]_q &= [U, L^A_{(2)}]_q [V, L^A_{(1)}]_q, 
\end{align}
(45)

it can be seen that a solution to (44) is given by the expression

\[ L^2 \equiv g_{AB} L^A L^B = -q L^+ L^- + L^3 L^3 - q^{-1} L^- L^+, \]  
(46)

which, in fact, is a q-analog for the Casimir operator of classical angular momentum algebra. Equivalently, the Casimir operator for $U_q(su_2)$ can also be written in terms of the $M^i$’s. In this manner, it becomes
\[ M^2 \equiv g_{AB} g_{CD} M^{AC} M^{BD}. \]  
(47)
\[ q^{-1} M^{+3} M^{3-} - M^{+-} M^{-+} + q^{-3} M^{3-} M^{+3}, \]

which agrees with \( L^2 \) up to a constant factor.

Before closing this section, we wish to specify our Casimir operator for the different representations addressed so far. Substituting for the symmetry generators their representations we finally obtain

a) (operator representation)
\[
L^2 = -(X \circ X)(\partial \circ \partial) + q^2 (X \circ \partial)(X \circ \partial) + q^{-2} X \circ \partial,
\]
with \( U \circ V \equiv g_{AB} U^A V^B \),

b) (spinor representation)
\[
L^2 = q^{-2} \lambda_+^{-2} [[3]]_{q^2} \mathbb{1}_{2 \times 2},
\]

c) (vector representation)
\[
L^2 = q^{-2} [[2]]_{q^4} \mathbb{1}_{3 \times 3},
\]

where the antisymmetric q-numbers are defined by
\[
[[n]]_q \equiv \frac{1 - q^{an}}{1 - q^a}, \quad n \in \mathbb{N}, \quad a \in \mathbb{C}.
\]

4 Quantum Lie algebra of four-dimensional angular momentum

As next example we would like to consider the q-deformed algebra of four-dimensional angular momentum. This case can be treated in very much the same way as the three-dimensional one. Thus, we restrict ourselves to stating the results, only.

4.1 Representation of four-dimensional angular momentum within q-deformed differential calculus

First of all, let us start with the defining representation. Within the differential calculus the generators of q-deformed four-dimensional angular momentum are represented by (see for example Ref. [42])
\[
L^{ij} \equiv -q^{-2} \lambda_+ \Lambda^{1/2} (P_A)^{ij}_{kl} X^k \partial^l = \lambda_+ \Lambda^{1/2} (P_A)^{ij}_{kl} \partial^k X^l,
\]
or

\[ L^{ij} \equiv -\lambda + \Lambda^{-1/2}(P_A)^{ij}_{kl} X^k \partial^l = q^{-2} \lambda + \Lambda^{-1/2}(P_A)^{ij}_{kl} \partial^k X^l, \]  

(53)

where \( P_A \) is a q-analog of an antisymmetrizer and \( \Lambda \) denotes a scaling operator subject to

\[ \Lambda X^i = q^2 X^i \Lambda, \quad \Lambda \partial^i = q^{-2} \partial^i \Lambda. \]  

(54)

Eq. (53) shows us that the \( L^{ij} \) are components of an antisymmetric tensor operator. More specifically, they satisfy

\[
\begin{align*}
L^{i1} &= 0, \quad i = 1, \ldots, 4, \\
L^{j1} &= -qL^{ij}, \quad L^{4j} = -qL^{j4}, \quad j = 2, 3, \\
L^{41} &= -L^{14}, \quad L^{32} = -L^{23} + \lambda L^{14}.
\end{align*}
\]

Thus, we have in complete analogy to the classical case only six independent generators, for which we can choose the set of \( L^{ij} \) with \( i < j \). Taking the explicit form of \( P_A \) into consideration, we get for them more explicitly

\[
\begin{align*}
L^{1i} &= -\Lambda^{-1/2}(q^{-1} X^i \hat{\partial}^i - X^i \hat{\partial}^1), \quad i = 1, 2, \\
L^{i4} &= -\Lambda^{-1/2}(q^{-1} X^i \hat{\partial}^4 - X^i \hat{\partial}^3), \\
L^{14} &= 2\lambda_+^1 \Lambda^{-1/2}(X^1 \hat{\partial}^4 - X^4 \hat{\partial}^1) \\
&\quad - \lambda \lambda_+^1 \Lambda^{-1/2}(X^2 \hat{\partial}^3 + X^3 \hat{\partial}^2), \\
L^{23} &= (q^2 + q^{-2}) \lambda_+^1 \Lambda^{-1/2}X^2 \hat{\partial}^3 \\
&\quad - 2\lambda_+^1 \Lambda^{-1/2}X^3 \hat{\partial}^2 + \lambda \lambda_+^1 \Lambda^{-1/2}(X^1 \hat{\partial}^4 - X^4 \hat{\partial}^1).
\end{align*}
\]

Together with two grouplike braiding operators \( K_i, i = 1, 2 \), the generators \( L^{kl}, k < l \), span the Hopf algebra \( U_{q}(so_4) \). Applying the commutation relations for coordinates, partial derivatives and the two braiding operators (see for example Appendix A and Ref. [22]) leaves us with the nontrivial relations

\[
\begin{align*}
L^{14}L^{ij} - L^{ij}L^{14} &= -K_i L^{12}, \quad (i, j) \in \{(2, 2), (1, 3)\}, \\
L^{14}L^{14} - L^{14}L^{14} &= -q^{-2} K_k L^{14}, \quad (k, l) \in \{(1, 2), (2, 3)\}, \\
L^{23}L^{12} - L^{12}L^{23} &= -qK_2 L^{12}, \quad L^{23}L^{13} - L^{13}L^{23} = q^{-1}K_1 L^{13}, \\
L^{24}L^{23} - L^{23}L^{24} &= q^{-3}K_1 L^{24}, \quad L^{34}L^{23} - L^{23}L^{34} = -q^{-1}K_2 L^{34}, \\
K_1 L^{13} &= q^{-2}L^{13} K_1, \quad K_2 L^{12} = q^{-2}L^{12} K_2, \\
K_1 L^{24} &= q^2L^{24} K_1, \quad K_2 L^{34} = q^2L^{34} K_2.
\end{align*}
\]
4.2 Hopf structure of $U_q(so_4)$ and corresponding q-commutators

As in the three-dimensional case we need the Hopf structure to proceed further. For the independent generators this reads as (see also Ref. [42])

$$\Delta(L^{ij}) = L^{ij} \otimes K_i + 1 \otimes L^{12}, \quad (i, j) \in \{(2, 2), (1, 3)\},$$

$$\Delta(L^{14}) = L^{14} \otimes K_k + 1 \otimes L^{14}, \quad (k, l) \in \{(1, 2), (2, 3)\},$$

$$\Delta(L^{14}) = q\lambda_+^{-1}L^{14} \otimes K_1 + q^{-1}\lambda_+^{-1}L^{14} \otimes K_2$$
$$+ q\lambda_+^{-1}K_1^{-1} \otimes L^{14} + q^{-1}\lambda_+^{-1}K_2^{-1} \otimes L^{14}$$
$$- \lambda_+^{-1}L^{23} \otimes K_1 + \lambda_+^{-1}L^{23} \otimes K_2$$
$$- \lambda_+^{-1}K_1^{-1} \otimes L^{23} + \lambda_+^{-1}K_2^{-1} \otimes L^{23}$$
$$- q\lambda_+^{-1}L^{24}K_1^{-1} \otimes L^{13} - q\lambda_+^{-1}K_1^{-1}L^{13} \otimes L^{24}$$
$$- q\lambda_+^{-1}L^{34}K_2^{-1} \otimes L^{12} - q\lambda_+^{-1}K_2^{-1}L^{12} \otimes L^{34},$$

$$\Delta(L^{23}) = -\lambda_+^{-1}K_1^{-1} \otimes L^{14} + \lambda_+^{-1}K_2^{-1} \otimes L^{14}$$
$$- \lambda_+^{-1}L^{14} \otimes K_1 + \lambda_+^{-1}L^{14} \otimes K_2$$
$$+ q^{-1}\lambda_+^{-1}L^{23} \otimes K_1 + q^{-1}\lambda_+^{-1}L^{23} \otimes K_2$$
$$+ q^{-1}\lambda_+^{-1}K_1^{-1} \otimes L^{23} + q^{-1}\lambda_+^{-1}K_2^{-1} \otimes L^{23}$$
$$+ \lambda_+^{-1}L^{24}K_1^{-1} \otimes L^{13} + \lambda_+^{-1}K_1^{-1}L^{13} \otimes L^{24}$$
$$- q^2\lambda_+^{-1}L^{34}K_2^{-1} \otimes L^{12} - q^2\lambda_+^{-1}K_2^{-1}L^{12} \otimes L^{34},$$

$$S(L^{ij}) = q^{-2}S^{-1}(L^{ij}) = -L^{ij}K_1^{-1}, \quad (i, j) \in \{(2, 2), (1, 3)\},$$

$$S(L^{14}) = q^2S^{-1}(L^{14}) = -L^{14}K_k^{-1}, \quad (k, l) \in \{(1, 2), (2, 3)\},$$

$$S(L^{14}) = S^{-1}(L^{14}) = L^{14} - q^{-1}\lambda^{-1}((K_1 + K_2) - (K_1^{-1} + K_2^{-1})),$$

$$S(L^{23}) = S^{-1}(L^{23}) = L^{23} + q^{-1}\lambda^{-1}(q^{-1}(K_1 - K_1^{-1}) - q(K_2 - K_2^{-1})),$$

$$\varepsilon(L^{mn}) = 0.$$

Again, this Hopf structure enables us to introduce q-commutators in complete analogy to the three-dimensional case. Explicitly written out we obtain for the q-commutator with an element living in a representation space of $U_q(so_4)$

$$[L^{ij}, V]_q = (L^{ij}V - VL^{ij})K_1^{-1}, \quad (i, j) \in \{(2, 2), (1, 3)\},$$

$$[L^{14}, V]_q = (L^{14}V - VL^{14})K_k^{-1}, \quad (k, l) \in \{(1, 2), (2, 3)\},$$

$$[L^{14}, V]_q = -q^{-1}\lambda^{-1}(K_1^{-1}VK_1 + K_2^{-1}VK_2)$$
\[
\begin{align*}
&+ q^{-1}\lambda^{-1}(K_1^{-1}V K_1^{-1} + K_2^{-1}V K_2^{-1}) \\
&- \lambda^{-1}_+(K_2^{-1}V L^{23} + L^{23}V K_2^{-1} - K_1^{-1}V L^{23} + L^{23}V K_1^{-1}) \\
&+ q^{-1}\lambda^{-1}_+(K_2^{-1}V L^{14} + L^{14}V K_2^{-1}) \\
&+ q\lambda^{-1}_+(K_1^{-1}V L^{14} + L^{14}V K_1^{-1}) \\
&+ q\lambda\lambda^{-1}_+(K_1^{-1}L^{13}VL^{24}K_1^{-1} + K_2^{-1}L^{12}VL^{34}K_2^{-1}) \\
&+ q\lambda\lambda^{-1}_+(L^{24}K_1^{-1}VL^{13}K_1^{-1} + L^{34}K_2^{-1}VL^{12}K_2^{-1}),
\end{align*}
\]

\[
[L^{23}, V]_q = q\lambda^{-1}_+(K_2^{-1}V L^{23} + L^{23}V K_2^{-1}) \\
+ q^{-1}\lambda^{-1}_+(K_1^{-1}V L^{23} + L^{23}V K_1^{-1}) \\
+ \lambda^{-1}_+(L^{14}V K_2^{-1} + K_2^{-1}V L^{14} - L^{14}V K_1^{-1} - K_1^{-1}V L^{14}) \\
+ \lambda^{-1}(K_2^{-1}V K_2^{-1} - K_2^{-1}V K_2) \\
+ q^{-1}\lambda^{-1}(K_1^{-1}V K_1 - K_1^{-1}V K_1^{-1}) \\
+ q^{2}\lambda\lambda^{-1}_+(K_2^{-1}L^{12}VL^{34}K_2^{-1} + L^{34}K_2^{-1}VL^{12}K_2^{-1}) \\
- \lambda\lambda^{-1}_+(K_1^{-1}L^{13}VL^{24}K_1^{-1} - L^{24}K_1^{-1}VL^{13}K_1^{-1}),
\]

and likewise for their right versions,

\[
\begin{align*}
[V, L^{1j}]_q &= K_i^{-1}(VL^{1j} - L^{1j}V), \quad (i, j) \in \{(2, 2), (1, 3)\}, \\
[V, L^{14}]_q &= K_i^{-1}(VL^{14} - L^{14}V), \quad (k, l) \in \{(1, 2), (2, 3)\}, \\
[V, L^{14}]_q &= q\lambda^+_+(L^{14}V K_1^{-1} + K_1^{-1}VL^{14}) \\
+ q^{-1}\lambda^+_+(L^{14}V K_2^{-1} + K_2^{-1}VL^{14}) \\
+ \lambda^+_+(L^{23}V K_2^{-1} + K_2^{-1}VL^{23} - L^{23}V K_1^{-1} - K_1^{-1}VL^{23}) \\
+ q^{-1}\lambda^{-1}(K_1^{-1}V K_1^{-1} - K_1^{-1}V K_1^{-1}) \\
+ q^{-1}\lambda^{-1}(K_2^{-1}V K_1^{-1} - K_2^{-1}V K_1^{-1}) \\
+ q\lambda\lambda^{-1}_+(K_1^{-1}L^{13}VL^{24}K_1^{-1} + K_1^{-1}L^{24}K_1^{-1}L^{13}) \\
+ q\lambda\lambda^{-1}_+(K_2^{-1}L^{34}VL^{24}K_2^{-1} + L^{24}K_2^{-1}L^{34}K_2^{-1}),
\end{align*}
\]

\[
[V, L^{23}]_q = q\lambda^+_+(L^{23}V K_2^{-1} + K_2^{-1}VL^{23}) \\
+ q^{-1}\lambda^+_+(L^{23}V K_1^{-1} + K_1^{-1}VL^{23}) \\
+ \lambda^+_+(L^{14}V K_2^{-1} + K_2^{-1}VL^{14} - L^{14}V K_1^{-1} - K_1^{-1}VL^{14}) \\
+ \lambda^{-1}(K_2^{-1}V K_2^{-1} - K_2^{-1}V K_2^{-1}) \\
+ q^{-2}\lambda^{-1}(K_1^{-1}V K_1^{-1} - K_1^{-1}V K_1^{-1}) \\
+ \lambda\lambda^{-1}_+(q^2 K_2^{-1}L^{34}VK_2^{-1}L^{12} + q^2 K_2^{-1}L^{12}VL^{34}K_2^{-1}) \\
- \lambda\lambda^{-1}_+(K_1^{-1}L^{13}VL^{24}K_1^{-1} - K_1^{-1}L^{24}VK_1^{-1}L^{13}).
\]
4.3 Matrix representations of $U_q(so_4)$ and commutation relations with tensor operators

Let us now go on to the spinor and vector representations of the $L^{ij}$. As in the classical case, we can distinguish two types of spinor representations, i.e. $(1/2,0)$ and $(0,1/2)$ (see for example Ref. [42]). Explicitly, we have

$$L^{ij} \triangleright \theta^\alpha = (\sigma^{ij})^{\alpha}_{\beta} \theta^\beta,$$
$$\theta_\alpha \triangleleft L^{ij} = \theta_\beta (\sigma^{ij})^\beta_{\alpha},$$
with

$$(\sigma^{13})^{\alpha}_{\beta} = \begin{pmatrix} 0 & -\frac{1}{q^2} \\ 0 & 0 \end{pmatrix}, \quad (\sigma^{24})^{\alpha}_{\beta} = \begin{pmatrix} 0 & 0 \\ -q^{-1} & 0 \end{pmatrix},$$

$$(\sigma^{14})^{\alpha}_{\beta} = -q(\sigma^{23})^{\alpha}_{\beta} = q^{-1} \lambda_+^{-1} \begin{pmatrix} -q & 0 \\ 0 & q^{-1} \end{pmatrix},$$
$$(\sigma^{34})^{\alpha}_{\beta} = (\sigma^{12})^{\alpha}_{\beta} = 0.$$

The matrices with tilde one gets most easily from the identities

$$\tilde{\sigma}^{13} = (\sigma^{12})^{\alpha}_{\beta}, \quad \tilde{\sigma}^{12} = (\sigma^{13})^{\alpha}_{\beta},$$
$$\tilde{\sigma}^{14} = (\sigma^{14})^{\alpha}_{\beta}, \quad \tilde{\sigma}^{23} = (\sigma^{23})^{\alpha}_{\beta},$$
$$\tilde{\sigma}^{34} = (\sigma^{34})^{\alpha}_{\beta}, \quad \tilde{\sigma}^{24} = (\sigma^{24})^{\alpha}_{\beta}.$$

Notice that the tilde on the spinor components shall remind us of the fact that they transformed differently. That these matrices determine a representation of $U_q(so_4)$ can again be proven by inserting them together with

$$(K_1)^{\alpha}_{\beta} = \begin{pmatrix} q & 0 \\ 0 & q^{-1} \end{pmatrix}, \quad (\tilde{K}_2)^{\alpha}_{\beta} = \begin{pmatrix} q^{-1} & 0 \\ 0 & q \end{pmatrix},$$

$$\tilde{K}_1 = K_2 = 1,$$

into relations (57) - (59).

From the spinor representation of $U_q(so_4)$ we can - as usual - compute commutation relations between the $L^{ij}$ and the components of a spinor operator. For this to achieve we apply the identities

$$[L^{ij}, \theta^\alpha]_q = (\sigma^{ij})^{\alpha}_{\beta} \theta^\beta, \quad [\theta^\alpha, L^{ij}]_q = \theta_\beta (\sigma^{ij})^\beta_{\alpha}.$$

If we write out the q-commutators and substitute the representation matrices without tilde for the $L^{ij}$ we get the commutation relations

$$L^{13} \theta^2 = \theta^2 L^{13} - q^{-2} \theta^1 K_1,$$
\[ L^{24} \theta^1 = \theta^1 L^{24} - q^{-1} \theta^2 K_1, \]
\[ L^{14} \theta^1 = (q^2 + q^{-1}) \lambda_+^{-1} \theta^1 L^{14} - (q - 1) \lambda_+^{-1} \theta^1 L^{23} + q \lambda \lambda_+^{-1} \theta^2 L^{13} - \lambda_+^{-1} \theta^1 K_1, \]
\[ L^{14} \theta^2 = (q^{-1} + 1) \lambda_+^{-1} \theta^2 L^{14} + (q^{-1} - 1) \lambda_+^{-1} \theta^2 L^{23} + \lambda \lambda_+^{-1} \theta^1 L^{24} + q^{-2} \lambda_+^{-1} \theta^2 K_1, \]
\[ L^{23} \theta^1 = (q + 1) \lambda_+^{-1} \theta^1 L^{23} - (q - 1) \lambda_+^{-1} \theta^1 L^{14} - \lambda_+^{-1} \theta^2 L^{13} + q^{-1} \lambda_+^{-1} \theta^1 K_1, \]
\[ L^{23} \theta^2 = (q^2 + q) \lambda_+^{-1} \theta^2 L^{23} + (1 - q^{-1}) \lambda_+^{-1} \theta^2 L^{14} - q^{-1} \lambda_+^{-1} \theta^1 L^{24} - q^{-3} \lambda_+^{-1} \theta^2 K_1, \]

and
\[ \theta_1 L^{13} = L^{13} \theta_1 - q^{-2} K_1 \theta_2, \]
\[ \theta_2 L^{24} = L^{24} \theta_2 - q^{-1} K_1 \theta_1, \]
\[ \theta_1 L^{14} = (q^2 + q^{-1}) \lambda_+^{-1} L^{14} \theta_1 - (q - 1) \lambda_+^{-1} L^{23} \theta_1 + \lambda \lambda_+^{-1} L^{24} \theta_2 - \lambda_+^{-1} K_1 \theta_1, \]
\[ \theta_2 L^{14} = (q^{-1} + 1) \lambda_+^{-1} L^{14} \theta_2 + (1 - q^{-1}) \lambda_+^{-1} L^{23} \theta_2 + q \lambda \lambda_+^{-1} L^{13} \theta_1 + q^{-2} \lambda_+^{-1} K_1 \theta_2, \]
\[ \theta_1 L^{23} = (q + 1) \lambda_+^{-1} L^{23} \theta_1 - (q - 1) \lambda_+^{-1} L^{14} \theta_1 - q^{-1} \lambda_+^{-1} L^{24} \theta_2 + q^{-1} \lambda_+^{-1} K_1 \theta_1, \]
\[ \theta_2 L^{23} = (q + q^{-2}) \lambda_+^{-1} L^{23} \theta_2 + (1 - q^{-1}) \lambda_+^{-1} L^{14} \theta_2 + \lambda_+^{-1} L^{13} \theta_1 - q^{-3} \lambda_+^{-1} K_1 \theta_2. \]

The remaining relations are trivial, i.e. these relations take the form \( L^{ij} \theta^k = \theta^k L^{ij} \). Repeating the same steps as before for spinors (and the representation matrices) with tilde we arrive at the following nontrivial relations:

\[ L^{12} \tilde{\theta}^2 = \tilde{\theta}^2 L^{12} - q^{-2} \tilde{\theta}^1 K_2, \]
\[ L^{34} \tilde{\theta}^1 = \tilde{\theta}^1 L^{34} - q^{-1} \tilde{\theta}^2 K_2, \]
\[ L^{14} \tilde{\theta}^1 = (q + 1) \lambda_+^{-1} \tilde{\theta}^1 L^{14} + (q - 1) \lambda_+^{-1} \tilde{\theta}^1 L^{23} + q \lambda \lambda_+^{-1} \tilde{\theta}^2 L^{12} - \lambda_+^{-1} \tilde{\theta}^1 K_2, \]
\[ L^{14} \tilde{\theta}^2 = (q^{-2} + q) \lambda_+^{-1} \tilde{\theta}^2 L^{12} + (q^{-1} - 1) \lambda_+^{-1} \tilde{\theta}^2 L^{23} + \lambda \lambda_+^{-1} \tilde{\theta}^1 L^{34} + q^{-2} \lambda_+^{-1} \tilde{\theta}^2 K_2, \]
\[ L^{23} \tilde{\theta}^1 = (q^{-1} + q^2) \lambda_+^{-1} \tilde{\theta}^1 L^{23} + (q - 1) \lambda_+^{-1} \tilde{\theta}^1 L^{14} \]
\[ + q^2 \lambda \lambda^{-1} \tilde{\theta}^2 L^{12} - q \lambda^{-1} K_2, \]
\[ L^{23} \tilde{\theta}^2 = (q^{-1} + 1) \lambda \lambda^{-1} \tilde{\theta}^2 L^{23} + (q^{-1} - 1) \lambda^{-1} \tilde{\theta}^2 L^{14} \]
\[ + q \lambda \lambda^{-1} \tilde{\theta} L^{34} + q^{-1} \lambda \lambda^{-1} \tilde{\theta}^2 K_2, \]

and

\[ \tilde{\theta}_1 L^{12} = L^{12} \tilde{\theta}_1 - q^{-2} K_2 \tilde{\theta}_2, \quad (73) \]
\[ \tilde{\theta}_2 L^{34} = L^{34} \tilde{\theta}_2 - q^{-1} K_2 \tilde{\theta}_1, \]
\[ \tilde{\theta}_1 L^{14} = (q + 1) \lambda^{-1} L^{14} \tilde{\theta}_1 + (q - 1) \lambda^{-1} L^{23} \tilde{\theta}_1 \]
\[ + \lambda \lambda^{-1} L^{34} \tilde{\theta}_2 - \lambda^{-1} K_2 \tilde{\theta}_1, \]
\[ \tilde{\theta}_2 L^{14} = (q + q^{-2}) \lambda^{-1} L^{14} \tilde{\theta}_2 + (q^{-1} - 1) \lambda^{-1} L^{23} \tilde{\theta}_2 \]
\[ + q \lambda \lambda^{-1} L^{12} \tilde{\theta}_1 + q^{-2} \lambda^{-1} K_2 \tilde{\theta}_2, \]
\[ \tilde{\theta}_1 L^{23} = (q^2 + q^{-1}) \lambda^{-1} L^{23} \tilde{\theta}_1 + (q - 1) \lambda^{-1} L^{14} \tilde{\theta}_1 \]
\[ + q \lambda \lambda^{-1} L^{34} \tilde{\theta}_2 - q \lambda^{-1} K_2 \tilde{\theta}_1, \]
\[ \tilde{\theta}_2 L^{23} = (1 + q^{-1}) \lambda^{-1} L^{23} \tilde{\theta}_2 + (q^{-1} - 1) \lambda^{-1} L^{14} \tilde{\theta}_2 \]
\[ + q^2 \lambda \lambda^{-1} L^{12} \tilde{\theta}_1 + q^{-1} \lambda^{-1} K_2 \tilde{\theta}_2. \]

These considerations carry over to the vector representations of the independent generators of \( U_q(so_4) \). Its right and left versions are given by

\[ L^{ij} \triangleright X^k = (\tau^{ij})^k_m X^m, \quad X_k \triangleleft L^{ij} = X_m ((\tau^{ij})^m_k), \quad (74) \]

with

\[ (\tau^{14})^k_m = q^{-1} \lambda \lambda^{-1} \left( \begin{array}{cccc} -q & 0 & 0 & 0 \\ 0 & -\lambda & 0 & 0 \\ 0 & 0 & -\lambda & 0 \\ 0 & 0 & 0 & 2q^{-1} \end{array} \right), \quad (75) \]
\[ (\tau^{12})^k_m = q^{-2} \left( \begin{array}{cccc} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right), \quad (\tau^{13})^k_m = q^{-2} \left( \begin{array}{cccc} 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array} \right), \]
\[ (\tau^{24})^k_m = q^{-1} \left( \begin{array}{cccc} 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array} \right), \quad (\tau^{34})^k_m = q^{-1} \left( \begin{array}{cccc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{array} \right). \]

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\[(\tau^{23})^m = q^{-1}\lambda_+^{-1} \begin{pmatrix}
-q\lambda & 0 & 0 & 0 \\
0 & -(q^2 + q^{-2}) & 0 & 0 \\
0 & 0 & 2 & 0 \\
0 & 0 & 0 & q^{-1}\lambda
\end{pmatrix}.
\]

If we want to check that the above matrices give a representation of \(U_q(so_4)\)
we additionally need the matrices corresponding to the braiding operators, i.e.
\[(K_1)^k_m = \begin{pmatrix}
q^{-1} & 0 & 0 & 0 \\
0 & q & 0 & 0 \\
0 & 0 & q^{-1} & 0 \\
0 & 0 & 0 & q
\end{pmatrix}, \quad (K_2)^k_m = \begin{pmatrix}
q^{-1} & 0 & 0 & 0 \\
0 & q^{-1} & 0 & 0 \\
0 & 0 & q & 0 \\
0 & 0 & 0 & q
\end{pmatrix}.
\] (76)

We are again in a position to write down commutation relations between
the independent \(L^{ij}\) and the components of a vector operator. Starting from
\[[L^{ij}, X^k]_q = (\tau^{ij})^k_l X^l, \quad \text{(77)}\]
and proceeding in very much the same way as in the spinor case we obtain
the relations
\[
\begin{align*}
L^{ij}X^1 &= X^1L^{ij}, \\
L^{ij}X^j &= X^jL^{ij}, \\
L^{ik}X^i &= X^iL^{ik} - q^{-2}X^1K_i, \\
L^{1m}X^4 &= X^4L^{1m} + q^{-2}X^mK_n, \\
L^{m'4}X^1 &= X^1L^{m'4} - q^{-1}X^{m'}K_n, \\
L^{4j}X^j &= X^jL^{4j}, \\
L^{4k}X^k &= X^kL^{4k} + q^{-1}X^4K_i, \\
L^{44}X^4 &= X^4L^{44},
\end{align*}
\] (79)

and
\[
\begin{align*}
L^{14}X^1 &= qX^1L^{14} - \lambda_+^{-1}X^1(K_1 + K_2) \\
&\quad + q\lambda_+^{-1}X^2L^{13} + X^3L^{12}, \\
L^{14}X^2 &= 2\lambda_+^{-1}X^2L^{14} + \lambda_+^{-1}X^2(q^2K_1 - K_2) + \lambda\lambda_+^{-1}X^2L^{23} \\
&\quad + \lambda\lambda_+^{-1}(X^1L^{24} - qX^4L^{12}),
\end{align*}
\] (80)
\[ L^{14}X^3 = q^{-2}(q^4 + 1)\lambda_+^{-1}X^3L^{14} + \lambda_+^{-1}X^3(q^{-2}K_2 - K_1) - \lambda\lambda_+^{-1}X^3L^{23} + \lambda\lambda_+^{-1}(X^1L^{34} - qX^4L^{13}), \]
\[ L^{14}X^4 = q^{-1}X^4L^{14} + q^{-2}\lambda_+^{-1}X^4(K_1 + K_2) - \lambda\lambda_+^{-1}(X^2L^{34} + X^3L^{24}), \]
\[ L^{23}X^1 = qX^1L^{23} + \lambda_+^{-1}X^1(q^{-1}K_1 - qK_2) - \lambda\lambda_+^{-1}(X^2L^{13} - q^2X^3L^{12}), \]
\[ L^{23}X^2 = q^{-2}(q^4 + 1)X^2L^{23} - \lambda_+^{-1}X^2(q^{-3}K_1 + qK_2) - q^2\lambda_+^{-1}X^4L^{12} - \lambda\lambda_+^{-1}(q^{-1}X^1L^{24} + X^2L^{14}), \]
\[ L^{23}X^3 = 2\lambda_+^{-1}X^3L^{23} + q^{-1}\lambda_+^{-1}X^3(K_1 + K_2) - \lambda\lambda_+^{-1}X^3L^{14} + \lambda\lambda_+^{-1}(qX^1L^{34} + X^4L^{13}), \]
\[ L^{23}X^4 = q^{-1}X^4L^{23} - q^{-1}\lambda_+^{-1}X^4(q^{-2}K_1 - K_2) - \lambda\lambda_+^{-1}(qX^2L^{34} - q^{-1}X^3L^{24}), \]

where

\[ j = 2, 3, \quad (k, l, i) \in \{(2, 3, 2), (3, 2, 1)\}, \]
\[ (m, n) \in \{(2, 2), (3, 1)\}, \quad m' \equiv 5 - m'. \]

The right version of (77), i.e.
\[ [X_k, L^{ij}]_q = X_i (t^{ij})^l_k, \]
\[ X_1L^{1m} = L^{1m}X_1 - q^{-2}K_nX_3, \]
\[ X_\ell L^{1k} = L^{1k}X_\ell + q^{-2}K_iX_4, \]
\[ X_j^l L^{ij} = L^{ij}X_j^l, \]
\[ X_4 L^{ij} = L^{ij}X_4, \]
\[ X_1 L^{j4} = L^{j4}X_1, \]
\[ X_k^\ell L^{i4} = L^{i4}X_2 - q^{-1}K_iX_1, \]
\[ X_j^\ell L^{j4} = L^{j4}X_j^\ell, \]
\[ X_4 L^{m4} = L^{m4}X_4 + q^{-1}K_nX_m, \]
\[ 20 \]
and

\[ X_1 L^{14} = q L^{14} X_1 - \lambda_+^{-1}(K_1 + K_2) X_1 \]
\[ + \lambda \lambda_+^{-1}(L^{24} X_2 + L^{34} X_3), \]
\[ X_2 L^{14} = 2 \lambda_+^{-1} L^{14} X_2 + \lambda_+^{-1}(q^{-2} K_1 - K_2) X_2 \]
\[ + \lambda \lambda_+^{-1} L^{23} X_2 - \lambda \lambda_+^{-1}(L^{34} X_4 + q L^{13} X_1), \]
\[ X_3 L^{14} = q^{-2}(q^4 + 1) \lambda_+^{-1} L^{14} X_3 + \lambda_+^{-1}(q^{-2} K_2 - K_1) X_3 \]
\[ - \lambda \lambda_+^{-1} L^{23} X_3 + \lambda \lambda_+^{-1}(q L^{12} X_1 - L^{24} X_4), \]
\[ X_4 L^{14} = q^{-1} L^{14} X_4 + q^{-2} \lambda_+^{-1}(K_1 + K_2) X_4 \]
\[ - q \lambda \lambda_+^{-1}(L^{12} X_2 + L^{13} X_3), \]

\[ X_1 L^{23} = q L^{23} X_1 + q^{-1} \lambda_+^{-1}(K_1 - q^2 K_2) X_1 \]
\[ + \lambda \lambda_+^{-1}(q L^{34} X_3 - q^{-1} L^{24} X_2), \]
\[ X_2 L^{23} = q^{-2}(q^4 + 1)L^{23} X_2 - \lambda_+^{-1}(q^{-3} K_1 + q K_2) X_2 \]
\[ + \lambda \lambda_+^{-1} L^{14} X_2 - \lambda \lambda_+^{-1}(L^{13} X_1 + q L^{34} X_4), \]
\[ X_3 L^{23} = 2 \lambda_+^{-1} L^{23} X_3 + q^{-1} \lambda_+^{-1}(K_1 + K_2) X_3 \]
\[ - \lambda \lambda_+^{-1} L^{14} X_3 + \lambda \lambda_+^{-1}(q^2 L^{12} X_1 + q^{-1} L^{24} X_4), \]
\[ X_4 L^{23} = q^{-1} L^{23} X_4 + q^{-1} \lambda_+^{-1}(K_2 - q^{-2} K_1) X_4 \]
\[ + \lambda \lambda_+^{-1}(L^{13} X_3 - q^2 L^{12} X_2). \]

4.4 Quantum Lie algebra of \( U_q(s_0) \) and its Casimir operators

The \( L^{ij} \) can act on themselves through the adjoint representation, which can be obtained via the following simple reasoning. We calculate the action of the \( L^{ij} \) on a product of antisymmetrised vector coordinates (for their definition see Ref. [26]), i.e.

\[ L^{ij} \triangleright \xi^k \xi^l = \left( (L^{ij})_{(1)} \triangleright \xi^k \right) \left( (L^{ij})_{(2)} \triangleright \xi^l \right) = \sum_{mn} (a^{ij,kl})_{mn} \xi^m \xi^n. \] (88)

As already mentioned, the \( L^{ij} \) are components of an antisymmetric tensor operator. Hence, they have to act on themselves in the same way as they act on a product of antisymmetrised coordinates. In this sense we read off from (88) for their quantum Lie algebra

\[ [L^{ij}, L^{kl}]_q = L^{ij} \triangleright L^{kl} = \sum_{mn} (a^{ij,kl})_{mn} L^{mn}. \] (89)
Explicitly, the non vanishing $q$-commutators for the independent $L^{ij}$ become

\begin{align}
[L^{12}, L^{23}]_q &= -q^{-2}[L^{23}, L^{12}]_q = q^{-1}L^{12}, \\
[L^{12}, L^{34}]_q &= -[L^{34}, L^{12}]_q = -q^{-2}L^{23} - q^{-3}L^{14}, \\
[L^{13}, L^{23}]_q &= -q^{-2}[L^{23}, L^{13}]_q = -q^{-3}L^{13}, \\
[L^{13}, L^{24}]_q &= -[L^{24}, L^{13}]_q = q^{-2}L^{23} - q^{-1}L^{14}, \\
[L^{14}, L^{1i}]_q &= -q^2[L^{1i}, L^{14}]_q = -L_i^{1i}, \quad i = 2, 3, \\
[L^{14}, L^{14}]_q &= -q^{-1}\lambda L^{14}, \\
[L^{14}, L^{23}]_q &= [L^{23}, L^{14}]_q = -q^{-1}\lambda L^{23}, \\
[L^{14}, L^{i4}]_q &= -q^{-2}[L^{i4}, L^{14}]_q = q^{-2}L^{i4}, \quad i = 2, 3, \\
[L^{23}, L^{23}]_q &= -q^{-1}\lambda L^{14} - q^{-1}\lambda^2 L^{23}, \\
[L^{23}, L^{24}]_q &= -q^{-2}[L^{24}, L^{23}]_q = -q^{-3}L^{24}, \\
[L^{23}, L^{34}]_q &= -q^{-2}[L^{34}, L^{23}]_q = q^{-1}L^{34}.
\end{align}

Let us remark that these identities can be checked by means of the spinor and vector representations of the $L^{ij}$. Towards this end, one has to write out the $q$-commutators using (63) or (64). Then we replace the generators by their representation matrices and apply usual matrix multiplication. Proceeding in this manner will again show us the validity of the above identities.

At last, let us consider the Casimir operators, which can be introduced as in the three-dimensional case. The first one is given by the expression

\begin{align}
C_1 = g_{ik}g_{jm}L^{ij}L^{km} &= 2L^{23}L^{23} + \lambda_+(L^{12}L^{34} + L^{13}L^{24}) \\
&= +q^2\lambda_+(L^{24}L^{13} + L^{34}L^{12}) + (q^2 + q^{-2})L^{14}L^{14} \\
&\quad - \lambda(L^{14}L^{23} + L^{23}L^{14}),
\end{align}

and the second one by

\begin{align}
C_2 = \varepsilon_{ijkl}L^{ij}L^{kl} &= q^2\lambda_+^2(L^{14}L^{23} + L^{23}L^{14}) + q^2\lambda_+^2(L^{12}L^{34} - L^{13}L^{24}) \\
&\quad + q^4\lambda_+^2(L^{34}L^{12} - L^{24}L^{13}) - q^2\lambda_+^2L^{14}L^{14},
\end{align}

where $g_{ik}$ and $\varepsilon_{ijkl}$ denote respectively quantum metric and $q$-deformed epsilon-tensor (for its definition see Ref. [43]) corresponding to $U_q(so_4)$. Again, we would like to specify the Casimir operators for the different representations. In doing so, we get the following:
5 Quantum Lie algebra of Lorentz transformations

In this section we deal with the quantum Lie algebra of Lorentz transformations. Everything so far applies to this case, which from a physical point of view is the most interesting one we consider in this article.

5.1 Representation of Lorentz generators within q-deformed differential calculus

In complete analogy to the previous section we start with a realization of the $V^{\mu\nu}$ given by [14]

$$V^{\mu\nu} \equiv \Lambda^{1/2}(P_A)_{\mu\nu}^{\rho\sigma} X^\rho \partial^\sigma = -q^{-2} \Lambda^{1/2}(P_A)_{\mu\nu}^{\rho\sigma} \partial^\rho X^\sigma,$$

or equivalently

$$V^{\mu\nu} \equiv q^{-2} \Lambda^{-1/2}(P_A)^{\mu\nu}_{\rho\sigma} X^\rho \partial^\sigma = -\Lambda^{-1/2}(P_A)^{\mu\nu}_{\rho\sigma} \partial^\rho X^\sigma,$$

where $\Lambda$ is a scaling operator being subject to

$$\Lambda X^\mu = q^{-2} X^\mu \Lambda, \quad \Lambda \partial^\mu = q^2 \partial^\mu \Lambda,$$

and $P_A$ denotes the antisymmetrizer for q-deformed Lorentz symmetry. More explicitly, we have for example

$$V^{+3} = 2q^2 \lambda^+ \partial^3 X^+ \partial^3 - 2\lambda^2 X^3 \partial^+$$

$$+ 2q^{-1} \lambda X \partial^+,$$

$$C_1 = 2q^{-2}(X \circ X)(\partial \circ \partial) + 2q^2(X \circ \partial)(X \circ \partial)$$

$$+ 2q^{-1} \lambda X \circ \partial,$$

$$C_2 = 0,$$

with $U \circ V \equiv g_{AB}U^A V^B$.
\[ V^{+0} = 2\lambda_+^{-2}X^+\hat{\partial}^0 - q^{-2}(q^4 + 1)\lambda_+^{-2}X^0\hat{\partial}^+ + \lambda\lambda_+^{-2}(qX^+\hat{\partial}^3 - q^{-1}X^3\hat{\partial}^+), \]
\[ V^{++} = 2\lambda_+^{-2}(\lambda^2 + 1)X^+\hat{\partial}^- - 2\lambda_+^{-2}X^-\hat{\partial}^+ - \lambda\lambda_+^{-2}(X^3\hat{\partial}^0 + X^0\hat{\partial}^3), \]
\[ V^{30} = 2\lambda_+^{-2}X^3\hat{\partial}^0 - q^{-2}(q^4 + 1)\lambda_+^{-2}X^0\hat{\partial}^3 + \lambda\lambda_+^{-2}(\lambda^2 + 1)(X^+\hat{\partial}^0 - X^-\hat{\partial}^+), \]
\[ V^{3-} = 2q^2\lambda_+^{-2}X^3\hat{\partial}^- - 2\lambda_+^{-2}X^-\hat{\partial}^3 - q\lambda\lambda_+^{-2}(X^0\hat{\partial}^- + X^-\hat{\partial}^0), \]
\[ V^{0-} = 2\lambda_+^{-2}X^0\hat{\partial}^- - q^{-2}(q^4 + 1)\lambda_+^{-2}X^-\hat{\partial}^0 - \lambda\lambda_+^{-2}(qX^3\hat{\partial}^0 - q^{-1}X^-\hat{\partial}^3). \]

The antisymmetry of the \( V^{\mu\nu} \) implies that we have six independent components for which we can choose
\[ V^{+3}, V^{+0}, V^{+-}, V^{30}, V^{3-}, V^{0-}, \] (102)
since the remaining ones are related to them by the relations
\[ V^{++} = V^{00} = V^{--} = 0, \]
\[ V^{33} = \lambda V^{++}, \ V^{--} = -V^{+-}, \]
\[ V^{\pm3} = -q^{\pm2}V^{3\pm}, \]
\[ V^{\pm0} = \mp q^{\pm1}\lambda V^{3\pm} - V^{0\pm}, \]
\[ V^{03} = \lambda V^{+-} - V^{30}. \] (103)

In Ref. [14] it was shown that the \( V^{\mu\nu} \) together with two additional generators, denoted in the following by \( U^1, U^2 \), span q-deformed Lorentz algebra. However, in what follows it is convenient to introduce another set of generators which are related to the \( V^{\mu\nu} \) by
\[ R^+ = q^{-1}\lambda_+^{-1}(V^{+0} - V^{+3}), \]
\[ R^3 = q^{-1}\lambda_+^{-1}(V^{30} - qV^{+-}), \]
\[ R^- = q^{-1}\lambda_+^{-1}(q^{-2}V^{3-} + V^{0-}), \]
\[ S^+ = q^{-1}\lambda_+^{-1}(V^{+0} + q^{-2}V^{+3}), \]
\[ S^3 = q^{-1}\lambda_+^{-1}(V^{30} + q^{-1}V^{+-}), \] (105)
$S^- = q^{-1} \lambda_+^{-1} (V^3 - V^0).$

In terms of the generators $R^A$ and $S^A$ the relations of q-deformed Lorentz algebra take the rather compact form

\begin{align}
\epsilon_{CB}^A R^B R^C &= q^{-4} \lambda_+^{-1} U^1 R^A, \\
\epsilon_{CB}^A S^B S^C &= -q^{-4} \lambda_+^{-1} U^2 S^A, \\
R^A S^B &= q^2 (\hat{R}_{SO_q(3)})^{AB}_{CD} S^C R^D,
\end{align}

while $U^1$ and $U^2$ are both central in the algebra. $\epsilon_{CB}^A$ in (106) denotes the epsilon tensor from Sec. 3 and $\hat{R}_{SO_q(3)}$ stands for the vector representation of the universal R-matrix of $SO_q(3)$. For more details we refer the reader to [14].

### 5.2 Hopf structure of q-deformed Lorentz algebra and corresponding q-commutators

To write down q-commutators with Lorentz generators $V^{\mu\nu}$ we need to know their Hopf structure. From Ref. [44], however, we know the coproduct for the generators $R^A$ and $S^A$, $A \in \{+, 3, -\}$. By solving (104) and (105) for the independent generators given in (102) and making use of the algebra homomorphism property of the coproduct we are able to find expressions for the coproducts of the $V^{\mu\nu}$.

For compactness, we introduce

\begin{align}
\rho &\equiv q^2 \lambda_+ R^3 + U^1, \\
\sigma &\equiv q^2 \lambda_+ S^3 - U^2,
\end{align}

and the generators of an $U_q(su_2)$-subalgebra, given by [44]

\begin{align}
L^A &\equiv -q^2 \lambda_+ (U^1 S^A - U^2 R^A + q^2 \lambda_+ \epsilon_{CB}^A R^B S^C), \\
\tau^{-1/2} &\equiv U^1 U^2 - q^4 \lambda_+ g_{AB} R^A S^B + \lambda L^3,
\end{align}

where $g_{AB}$ is the quantum metric of Sec. 3. In terms of the new quantities the wanted coproducts read as

\begin{align}
\Delta(V^{+3}) &= -q^2 \sigma \otimes R^+ + q^2 R^+ \otimes \rho, \\
&\quad - q^2 \tau^{1/2} R^+ \otimes \tau^{1/2} \rho + q^2 \tau^{1/2} \sigma \otimes \tau^{1/2} R^+ \\
&\quad - q^3 \lambda_+^{-1} \tau^{1/2} \sigma L^+ \otimes \sigma - q^3 \lambda_+^{-1} \tau^{1/2} \sigma \otimes \tau^{1/2} \sigma L^+ \\
&\quad + q^7 \lambda_+^2 \tau^{1/2} R^+ \otimes \tau^{1/2} S^- L^+ + q^9 \lambda_+^2 \tau^{1/2} R^+ L^+ \otimes S^-, \\
\Delta(V^{+0}) &= \sigma \otimes R^+ - R^+ \otimes \rho
\end{align}
\[ - q^2 \tau^{1/2} R^+ \otimes \tau^{1/2} \rho + q^2 \tau^{1/2} \sigma \otimes \tau^{1/2} R^+ \\
- q^3 \lambda^1 \tau^{1/2} \sigma L^+ \otimes \sigma - q^3 \lambda^1 \tau^{1/2} \sigma \otimes \tau^{1/2} \sigma L^+ \\
+ q^7 \lambda^2 \tau^{1/2} R^+ \otimes \tau^{1/2} S^- L^+ + q^9 \lambda^2 \tau^{1/2} R^+ \otimes S^- , \\
\Delta(V^+) = qR^3 \otimes \rho - q \tau^{1/2} \sigma \otimes R^3 \\
+ q^2 S^3 \otimes \sigma - q \tau^{1/2} \rho \otimes S^3 \\
- q^4 \lambda \lambda \lambda S^- \otimes R^+ - q^2 \lambda \lambda \lambda R^+ \otimes S^- \\
+ q^4 \lambda \lambda \tau^{1/2} R^+ \otimes R^- + q^3 \lambda \tau^{1/2} \sigma L^- \otimes R^+ \\
+ q^2 \lambda \lambda \tau^{1/2} S^- \otimes S^+ + q^7 \lambda \tau^{1/2} \rho L^+ \otimes S^- , \\
\Delta(V^{30}) = - R^3 \otimes \rho + \tau^{1/2} \sigma \otimes R^3 \\
+ q^2 S^3 \otimes \sigma - q \tau^{1/2} \rho \otimes S^3 \\
+ q^3 \lambda \lambda \lambda S^- \otimes R^+ - q^3 \lambda \lambda \lambda R^+ \otimes S^- \\
- q^3 \lambda \lambda \lambda \tau^{1/2} R^+ \otimes R^- - q^2 \lambda \tau^{1/2} \sigma L^- \otimes R^+ \\
+ q^3 \lambda \lambda \lambda \tau^{1/2} S^- \otimes S^+ + q^6 \lambda \tau^{1/2} \rho L^+ \otimes S^- , \\
\Delta(V^3^-) = q^2 S^- \otimes \sigma - R^3 \otimes S^- \\
- q^2 \tau^{1/2} S^- \otimes \tau^{1/2} \sigma + q^2 \tau^{1/2} \rho \otimes \tau^{1/2} S^- \\
- q^3 \lambda^1 \tau^{1/2} \rho L^- \otimes \rho - q^3 \lambda^1 \tau^{1/2} \rho \otimes \tau^{1/2} \rho L^- \\
+ q^5 \lambda^2 \lambda^2 \tau^{1/2} S^- L^- \otimes R^+ + q^7 \lambda^2 \lambda^2 \tau^{1/2} S^- \otimes \tau^{1/2} R^+ L^- , \\
\Delta(V^{0^-}) = - S^- \otimes \sigma + \rho \otimes S^- \\
- q^2 \tau^{1/2} S^- \otimes \tau^{1/2} \sigma + q^2 \tau^{1/2} \rho \otimes \tau^{1/2} S^- \\
- q^3 \lambda^1 \tau^{1/2} \rho L^- \otimes \rho - q^3 \lambda^1 \tau^{1/2} \rho \otimes \tau^{1/2} \rho L^- \\
+ q^5 \lambda^2 \lambda^2 \tau^{1/2} S^- L^- \otimes R^+ + q^7 \lambda^2 \lambda^2 \tau^{1/2} S^- \otimes \tau^{1/2} R^+ L^- . \\
\]

For writing down q-commutators we also need the following antipodes and their inverses:

\[ S(R^+) = q^2 S^{-1} (R^+) = - q^2 \tau^{1/2} R^+ , \]
\[ S(R^3) = S^{-1} (R^3) = - q^{-2} \lambda^{-1} \lambda^{-1} (U^1 + \tau^{1/2}) \sigma , \]
\[ S(R^-) = q^{-2} S^{-1} (R^-) = - S^- - q^{-1} \lambda^{-1} \tau^{1/2} L^- \sigma , \]
\[ S(S^+) = q^2 S^{-1} (S^+) = - R^+ - q^3 \lambda^1 \tau^{1/2} L^+ \rho , \]
\[ S(S^3) = S^{-1} (S^3) = q^{-2} \lambda^{-1} \lambda^{-1} (U^2 - \tau^{1/2} \rho) , \]
\[ S(S^-) = q^{-2} S^{-1} (S^-) = - q^{-2} \tau^{1/2} S^- , \]
After these preparations we are ready to write down q-commutators. For their left versions we have found

\[
\begin{align*}
S(U^i) &= S^{-1}(U^i) = U^i, \quad i = 1, 2, \\
S(\rho) &= S^{-1}(\rho) = -\tau^{1/2}\sigma, \\
S(\sigma) &= S^{-1}(\sigma) = -\tau^{1/2}\rho.
\end{align*}
\]

\[
[\mathcal{V}^3, \mathcal{V}]_q = q^3\lambda_+^{-1}\tau^{1/2}\sigma(L^+\mathcal{V} - \mathcal{V}L^+)\tau^{1/2}\rho \\
+ q^7\lambda_+^2\tau^{1/2}R^+(\mathcal{V}L^+ - L^+\mathcal{V})\tau^{1/2}\mathcal{S}^- \\
+ q^2(\sigma\mathcal{V}\tau^{1/2}R^+ - \tau^{1/2}\sigma\mathcal{V}R^+) \\
+ q^2(\tau^{1/2}(R^+\mathcal{V}\sigma - \mathcal{V}R^+\tau^{1/2}\sigma)),
\]

\[
[\mathcal{V}^0, \mathcal{V}]_q = q^3\lambda_+^{-1}\tau^{1/2}\sigma(L^+\mathcal{V} - \mathcal{V}L^+)\tau^{1/2}\rho \\
+ q^7\lambda_+^2\tau^{1/2}R^+(\mathcal{V}L^+ - L^+\mathcal{V})\tau^{1/2}\mathcal{S}^- \\
- q^2(\sigma\mathcal{V}\tau^{1/2}R^+ + \tau^{1/2}\sigma\mathcal{V}R^+) \\
+ q^2(\tau^{1/2}R^+\mathcal{V}\sigma + \mathcal{V}R^+\tau^{1/2}\sigma),
\]

\[
[\mathcal{V}^-, \mathcal{V}]_q = q(R^3\mathcal{V}\tau^{1/2}\sigma + S^3\mathcal{V}\tau^{1/2}\rho) \\
+ q^{-1}\lambda_1^{-1}\lambda_+^{-1}\tau^{1/2}(\sigma\mathcal{V}(U^1 + \tau^{1/2}\sigma) - \rho\mathcal{V}(U^2 - \tau^{1/2}\rho)) \\
- q^3\lambda\tau^{1/2}(\rho L^+\mathcal{V}\tau^{1/2}\mathcal{S}^- + q^2S^-\mathcal{V}\tau^{1/2}L^+\rho) \\
- q^3\lambda\tau^{1/2}(q^2\sigma L^-\mathcal{V}\tau^{1/2}R^+ + \mathcal{V}R^+\tau^{1/2}\mathcal{S}^-\sigma) \\
+ q^4\lambda_1(\lambda^2S^-\mathcal{V}\tau^{1/2}R^+ - \tau^{1/2}R^+\mathcal{V}S^-) \\
+ \lambda_1(\mathcal{V}R^+\tau^{1/2}\mathcal{S}^- - q^2\tau^{1/2}\mathcal{S}^-\mathcal{V}R^+),
\]

\[
[V^{30}, V]_q = R^3\mathcal{V}\tau^{1/2}\sigma - q^2S^3\mathcal{V}\tau^{1/2}\rho \\
- q^{-2}\lambda_1^{-1}\lambda_+^{-1}\tau^{1/2}(\sigma\mathcal{V}(U^1 + \tau^{1/2}\sigma)) + q^2\rho\mathcal{V}(U^2 - \tau^{1/2}\rho) \\
- q^4\lambda\tau^{1/2}(\rho L^+\mathcal{V}\tau^{1/2}\mathcal{S}^- + q^2S^-\mathcal{V}\tau^{1/2}L^+\rho) \\
+ q^2\lambda\tau^{1/2}(q^2\sigma L^-\mathcal{V}\tau^{1/2}R^+ + \mathcal{V}R^+\tau^{1/2}\mathcal{S}^-\sigma) \\
+ q^3\lambda_1(\lambda^2S^-\mathcal{V}\tau^{1/2}R^+ - \tau^{1/2}R^+\mathcal{V}S^-) \\
+ q\lambda_1(\mathcal{V}R^+\tau^{1/2}\mathcal{S}^- - q^2\tau^{1/2}\mathcal{S}^-\mathcal{V}R^+),
\]

\[
[V^{3-}, V]_q = \rho\mathcal{V}\tau^{1/2}\mathcal{S}^- - q^2\tau^{1/2}\rho\mathcal{V}\mathcal{S}^- \\
+ q^2(\tau^{1/2}\mathcal{S}^-\rho - \mathcal{S}^-\mathcal{V}\tau^{1/2}\rho).
\]
and likewise for the corresponding right versions,

$$ [V^0, V]_q = -q^{-2} \rho V \tau^{1/2} S^- - q^2 \tau^{1/2} \rho V S^- + q^2 \tau^{1/2} S^- V \rho + S^\tau \tau^{1/2} \rho $$

$$ - q^3 \lambda^- \tau^{1/2} \rho (L_V - VL^-) \tau^{1/2} R^+, $$

$$ V^0, V = q^2 \sigma \tau \tau^{1/2} R^+ - q^2 \sigma \tau^{1/2} V R^+ $$

and likewise for the corresponding right versions,

$$ [V, V^+^0]_q = q^2 \sigma \tau \tau^{1/2} R^+ - q^2 \sigma \tau^{1/2} V R^+ $$

$$ - R^+ \tau^{1/2} \sigma R^+ \tau^{1/2} \sigma $$

$$ - q^3 \lambda^- (\rho V \tau^{1/2} \sigma L^+ - L^+ \rho \tau^{1/2} \sigma) $$

$$ + q^2 \lambda^2 \lambda^2 \tau^{1/2} (L^+ S^\tau \tau^{1/2} R^+ - q^2 S^\tau \tau^{1/2} R^+ L^+), $$

$$ [V, V^{+3}]_q = q^2 \sigma \tau \tau^{1/2} R^+ - q^2 \sigma \tau^{1/2} V R^+ $$

$$ - R^+ \tau^{1/2} \sigma R^+ \tau^{1/2} \sigma $$

$$ - q^3 \lambda^- (\rho V \tau^{1/2} \sigma L^+ - L^+ \rho \tau^{1/2} \sigma) $$

$$ + q^2 \lambda^2 \lambda^2 \tau^{1/2} (L^+ S^\tau \tau^{1/2} R^+ - q^2 S^\tau \tau^{1/2} R^+ L^+), $$

$$ [V, V^{+3}]_q = -q^2 (\rho \tau^{1/2} V S^3 + \sigma \tau^{1/2} V R^3) $$

$$ + q^{-1} \lambda^{-1} \lambda^1 \left((-U^2 + \rho \tau^{1/2}) V \tau^{1/2} \rho + (U^1 + \sigma \tau^{1/2}) V \tau^{1/2} \sigma \right) $$

$$ - q \lambda (L^+ \rho \tau^{1/2} V \tau^{1/2} S^- + q^6 \tau^{1/2} \sigma V \tau^{1/2} \rho L^+) $$

$$ - q \lambda (q^4 \tau^{1/2} L^- \sigma V \tau^{1/2} R^+ + R^+ \tau^{1/2} \sigma V \tau^{1/2} L^-) $$

$$ + \lambda \lambda^+ (q^4 S^\tau \tau^{1/2} V R^+ - R^+ \tau \tau^{1/2} S^-) $$

$$ + q^2 \lambda \lambda^+ (R^+ \tau \tau^{1/2} V S^- - q^4 S^\tau \tau^{1/2} R^+), $$

$$ [V, V^{+3}]_q = \sigma \tau \tau^{1/2} R^3 - q^2 \rho \tau^{1/2} V S^3 $$

$$ - q^{-2} \lambda^{-1} \lambda^1 \left((-U^2 + \sigma \tau^{1/2}) V \tau^{1/2} \sigma + q^2 (U^2 - \rho \tau^{1/2}) V \tau^{1/2} \rho \right) $$

$$ + \lambda (R^+ \tau \tau^{1/2} V \tau^{1/2} \sigma L^- + q^4 \tau^{1/2} L^- \sigma V \tau^{1/2} R^+) $$

$$ - q^2 \lambda (L^+ \rho \tau^{1/2} V \tau^{1/2} S^- - q^6 \tau^{1/2} V \tau^{1/2} \rho L^+) $$

$$ - q \lambda \lambda^+ (R^+ V \tau \tau^{1/2} S^- + R^+ \tau \tau^{1/2} V S^-) $$

$$ + q^5 \lambda \lambda^+ (S^\tau \tau^{1/2} R^+ + S^- \tau^{1/2} V R^+), $$

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\[ [V, V^{3-}] = - q^2 \rho \tau^{1/2} V S^- + q^2 \rho V \tau^{1/2} S^- \]
\[ - q^4 S^- \tau^{1/2} \rho + q^4 S^- \tau^{1/2} \rho \]
\[ + q^3 \lambda_1^{-1} (\sigma \tau^{1/2} V \tau^{1/2} \rho L^- - \tau^{1/2} L^- \sigma V \tau^{1/2} \rho) \]
\[ + q^3 \lambda_2^2 \lambda_3^2 (q^2 \tau^{1/2} L^- R^+ V \tau^{1/2} S^- - R^+ \tau^{1/2} V \tau^{1/2} S^- L^-) \],

\[ [V, V^{0-}] = \rho \tau^{1/2} V S^- + q^2 \rho V \tau^{1/2} S^- \]
\[ - q^4 S^- \tau^{1/2} \rho - q^4 S^- \tau^{1/2} \rho \]
\[ + q^3 \lambda_1^{-1} (\sigma \tau^{1/2} V \tau^{1/2} \rho L^- - \tau^{1/2} L^- \sigma V \tau^{1/2} \rho) \]
\[ - q^3 \lambda_2^2 \lambda_3^2 (q^2 \tau^{1/2} L^- R^+ V \tau^{1/2} S^- - R^+ \tau^{1/2} V \tau^{1/2} S^- L^-) \].

5.3 Matrix representations of q-deformed Lorentz algebra and commutation relations with tensor operators

Let us now consider spin representations of q-deformed Lorentz algebra. From a physical point of view q-deformed analogs of the two spinor representations \((1/2, 0)\) and \((0, 1/2)\) as well as the vector representation \((1/2, 1/2)\) are the most interesting cases [10, 12, 45, 47]. For a systematic treatment of finite dimensional representations of q-Lorentz algebra we refer for example to [48]. As in the previous sections, we begin with spinor representations, for which we have

\[ V^{\mu\nu} \triangleright \theta^\alpha = (\sigma^{\mu\nu})^\alpha_\beta \theta^\beta, \quad V^{\mu\nu} \triangleright \bar{\theta}^\alpha = (\bar{\sigma}^{\mu\nu})^\alpha_\beta \bar{\theta}^\beta \]
\[ \theta_\alpha \triangleleft V^{\mu\nu} = \theta_\beta (\sigma^{\mu\nu})^\beta_\alpha, \quad \bar{\theta}_\alpha \triangleleft V^{\mu\nu} = \bar{\theta}_\beta (\bar{\sigma}^{\mu\nu})^\beta_\alpha \],

and the spin matrices are given by

\[ (\sigma^{+3})^\alpha_\beta = q^2 (\sigma^{+0})^\alpha_\beta = -q^{1/2} \lambda_+^{-3/2} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \]
\[ (\sigma^{+0})^\alpha_\beta = -q (\sigma^{00})^\alpha_\beta = q^{-1} \lambda_+^{-2} \begin{pmatrix} -q & 0 \\ 0 & q^{-1} \end{pmatrix}, \]
\[ (\sigma^{-0})^\alpha_\beta = (\sigma^{-0})^\alpha_\beta = q^{-1/2} \lambda_+^{-3/2} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \]

in connection with

\[ (\bar{\sigma}^{+3})^\alpha_\beta = (\sigma^{+3})^\alpha_\beta, \quad (\bar{\sigma}^{+0})^\alpha_\beta = -q^2 (\sigma^{+0})^\alpha_\beta \].
\[(\tilde{\sigma}^+)^\alpha_\beta = (\sigma^+)^\alpha_\beta, \quad (\tilde{\sigma}_3)^\alpha_\beta = (\sigma^3)^\alpha_\beta, \quad (\tilde{\sigma}^0)^\alpha_\beta = -q^2(\sigma^0)^\alpha_\beta, \quad (\tilde{\sigma}^-)^\alpha_\beta = -q^2(\sigma^-)^\alpha_\beta.\]

The spinor representations of the generators \(U^1\), \(U^2\), and \(\sigma\) are given by diagonal matrices which fulfill the identities

\[-q^2(q^2 + q^{-2})^{-1}\lambda_+(U^1)^\alpha_\beta = -(U^2)^\alpha_\beta = (\sigma)^\alpha_\beta = \delta^\alpha_\beta. \quad (129)\]

Furthermore, for \(\rho\) and the generators of the \(U_q(su_2)\)-subalgebra we have

\[(L^+)^\alpha_\beta = -q^{1/2}\lambda_+^{-1/2} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad (L^-)^\alpha_\beta = -q^{-1/2}\lambda_+^{-1/2} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad (\tau^{-1/2})^\alpha_\beta = -(\rho)^\alpha_\beta = \left( \begin{array}{cc} q^{-1} & 0 \\ 0 & q \end{array} \right). \quad (130)\]

What we have done so far, enables us to write out the commutation relations between Lorentz generators and components of a spinor operator. In general, they are equivalent to

\[
[V^{\mu\nu}, \theta^{\alpha}]_q = (\sigma^{\mu\nu})^\alpha_\beta \theta^{\beta}, \quad [V^{\mu\nu}, \bar{\theta}^{\alpha}]_q = (\bar{\sigma}^{\mu\nu})^\alpha_\beta \bar{\theta}^{\beta}, \quad [\theta^{\alpha}, V^{\mu\nu}]_q = \theta^{\beta} (\sigma^{\mu\nu})^\beta_\alpha, \quad [\bar{\theta}^{\alpha}, V^{\mu\nu}]_q = \bar{\theta}^{\beta} (\bar{\sigma}^{\mu\nu})^\beta_\alpha. \quad (131)
\]

Unfortunately, it is rather difficult to derive from (131) the explicit form of the commutation relations between Lorentz generators and spinors. Thus, we proceed differently and apply the identities

\[
V^{\mu\nu}a = ((V^{\mu\nu})_{(1)} \triangleright a)(V^{\mu\nu})_{(2)},
\]

\[
aV^{\mu\nu} = (V^{\mu\nu})_{(2)}(a \triangleleft (V^{\mu\nu})_{(1)}),
\]

which result from Eqs. (7) and (8). This way the first relation in (132) implies

\[
V^{+3}\theta^1 = (q + 1)\lambda_+^{-1}\theta^1V^{+3} + (q^2 - q)\lambda_+^{-1}\theta^1V^{+0} \quad (133)
\]

\[
- q^{1/2}\lambda_+^{-1/2}\theta^2(V^{+-} + qV^{30}) + q^{1/2}\lambda_+^{-3/2}\theta^2\rho,
\]

\[
V^{+3}\theta^2 = (q + q^{-2})\lambda_+^{-1}\theta^2V^{+3} + (1 - q)\lambda_+^{-1}\theta^2V^{+0},
\]

\[
V^{+0}\theta^1 = (q^2 + q^{-1})\lambda_+^{-1}\theta^1V^{+0} + (1 - q^{-1})\lambda_+^{-1}\theta^1V^{+3} \quad (134)
\]

\[
- q^{1/2}\lambda_+^{-1/2}\theta^2(V^{+-} + qV^{30}) - q^{-3/2}\lambda_+^{-3/2}\theta^2\rho,
\]

\[
V^{+0}\theta^2 = (1 + q^{-1})\lambda_+^{-1}\theta^2V^{+0} + (q^{-2} - q^{-1})\lambda_+^{-1}\theta^2V^{+3},
\]

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For the second set of spinors we get from (132)

\[
V^+\theta^1 = q^{-1}(2q^2 + \lambda_+)\lambda_+^{-2}\theta^1 V^+ + q^{-1}(q - 1)^2\lambda_+^{-2}\theta^1 V^{30}
\]

\[
+ q^{1/2}(\lambda_+ - 1)\lambda_+^{-3/2}\theta^2 V^{0-} - q^{-3/2}(q^2\lambda_+ + 1)\lambda_+^{-3/2}\theta^2 V^{3-} + \lambda_+^{-2}\theta^1 U^1,
\]

\[
V^+\theta^2 = q^{-1}(2q^2 + \lambda_+)\lambda_+^{-2}\theta^2 V^+ + q^{-1}(q - 1)^2\lambda_+^{-2}\theta^2 V^{30}
\]

\[
+ q^{-1/2}\lambda_+^{-3/2}\theta^1(V^+ - V^{30}) - q^{-2}\lambda_+^{-2}\theta^2 U^1,
\]

\[
V^{30}\theta^1 = q^{-1}(q^2\lambda_+ + 2)\lambda_+^{-2}\theta^1 V^{30} + q^{-1}(q - 1)^2\lambda_+^{-2}\theta^1 V^+ + q^{-1/2}(1 + q^2\lambda_+)\lambda_+^{-3/2}\theta^2 V^{0-}
\]

\[
+ q^{-5/2}(1 - q^4\lambda_+)\lambda_+^{-3/2}\theta^2 V^{3-} - q^{-1}\lambda_+^{-2}\theta^1 U^1,
\]

\[
V^{30}\theta^2 = q^{-1}(q^2\lambda_+ + 2)\lambda_+^{-2}\theta^2 V^{30} + q^{-1}(q - 1)^2\lambda_+^{-2}\theta^2 V^+ + q^{-3/2}\lambda_+^{-3/2}\theta^1(V^+ - V^{30}) + q^{-3}\lambda_+^{-2}\theta^2 U^1,
\]

\[
V^3\theta^1 = (q^{-1} + 1)\lambda_+^{-1}\theta^1 V^{3-} + (q - 1)\lambda_+^{-1}\theta^2 V^{0-}
\]

\[
- q^{-1/2}\lambda_+^{-3/2}\theta^1 \rho,
\]

\[
V^3\theta^2 = (q^2 + q^{-1})\lambda_+^{-1}\theta^2 V^{3-} - (q^2 - q)\lambda_+^{-1}\theta^1 V^{0-}
\]

\[
- q^{-1/2}\lambda_+^{-3/2}\theta^1 \rho,
\]

\[
V^0\theta^1 = (q + q^{-2})\lambda_+^{-1}\theta^1 V^{0-} + (q^{-1} - q^{-2})\lambda_+^{-1}\theta^1 V^{3-},
\]

\[
V^0\theta^2 = (q + 1)\lambda_+^{-1}\theta^2 V^{0-} + (q^{-1} - 1)\lambda_+^{-1}\theta^2 V^{3-}
\]

\[
- q^{-1/2}\lambda_+^{-3/2}\theta^1 \rho.
\]

For the second set of spinors we get from (132)

\[
V^+\tilde{\theta}^1 = (q + q^{-1})\lambda_+^{-1}\tilde{\theta}^1 V^+ + (q - 1)\lambda_+^{-1}\tilde{\theta}^1 V^{0-}
\]

\[
- q^{1/2}\lambda_+^{-3/2}\tilde{\theta}^2 \sigma,
\]

\[
V^+\tilde{\theta}^2 = (q^2 + q^{-1})\lambda_+^{-1}\tilde{\theta}^2 V^+ + (q^2 - q)\lambda_+^{-1}\tilde{\theta}^2 V^{0-},
\]

\[
V^0\tilde{\theta}^1 = (q + q^{-2})\lambda_+^{-1}\tilde{\theta}^1 V^{0-} + (q^{-1} - q^{-2})\lambda_+^{-1}\tilde{\theta}^1 V^{3-}
\]

\[
- q^{1/2}\lambda_+^{-3/2}\tilde{\theta}^2 \sigma,
\]

\[
V^0\tilde{\theta}^2 = (q + 1)\lambda_+^{-1}\tilde{\theta}^2 V^{0-} + (1 - q^{-1})\lambda_+^{-1}\tilde{\theta}^2 V^{3-},
\]

\[
V^+\tilde{\theta}^1 = q^{-1}(2 + q^2\lambda_+)\lambda_+^{-2}\tilde{\theta}^1 V^+ - q^{-1}(q - 1)^2\lambda_+^{-2}\tilde{\theta}^1 V^{30}
\]

\[
+ q^{1/2}\lambda_+^{-3/2}\tilde{\theta}^2 (V^{0-} - V^{3-}) + \lambda_+^{-2}\tilde{\theta}^1 U^2,
\]

\[
V^+\tilde{\theta}^2 = q^{-1}(2 + q^2\lambda_+)\lambda_+^{-2}\tilde{\theta}^2 V^+ + q^{-1}(q - 1)^2\lambda_+^{-2}\tilde{\theta}^2 V^{30}
\]

\[
+ q^{1/2}\lambda_+^{-3/2}\tilde{\theta}^1 (V^+ - V^{30}) + q^{-2}\lambda_+^{-2}\tilde{\theta}^2 U^1,
\]
\[- \frac{1}{2} q \lambda (\lambda_+ + 1) \lambda_+^{-3/2} \theta^1 V^0 \]
\[+ \frac{5}{2} q \lambda (q^2 \lambda_+ + 1) \lambda_+^{-3/2} \theta^1 V^+ - \frac{1}{2} \lambda_+^{-2} \theta^2 U^1, \]

\[V^{30} \theta^1 = q^{-1}(\lambda_+ + 2q^2) \lambda_+^{-2} \theta^1 V^{30} - q^{-1}(q - 1)^2 \lambda_+^{-2} \theta^1 V^+ - \frac{3}{2} \lambda_+^{3/2} \theta^2 (V^0 - V^3) + q \lambda_+^{-2} \theta^1 U^2, \]

\[V^{30} \theta^2 = q^{-1}(\lambda_+ + 2q^2) \lambda_+^{-2} \theta^2 V^{30} - q^{-1}(q - 1)^2 \lambda_+^{-2} \theta^2 V^+ \]
\[- \frac{3}{2} \lambda_+^{3/2} \theta^1 V^+ + q^{-3/2} \lambda (\lambda_+ + 1) \lambda_+^{-3/2} \theta^1 V^+ \]
\[+ q^{-3/2} (q_+ + 2q^2) \lambda_+^{-3/2} \theta^1 V^+ - q^{-1} \lambda_+^{-3/2} \theta^2 U^2, \]

\[V^3 \theta^1 = (q + 1) \lambda_+^{-1} \theta^1 V^3 - (q^2 - q) \lambda_+^{-1} \theta^1 V^0, \]
\[V^3 \theta^2 = (q + q^{-2}) \lambda_+^{-1} \theta^2 V^3 - (q - 1) \lambda_+^{-1} \theta^2 V^0 + \frac{1}{2} \lambda_+^{3/2} \theta^1 (qV^+ - V^{30}) + q^{-1} \lambda_+^{-3/2} \theta^1 \sigma, \]

\[V^0 \theta^1 = (q^2 + q^{-1}) \lambda_+^{-1} \theta^1 V^0 - (1 - q^{-1}) \lambda_+ \theta^1 V^0 \]
\[+ q^{-1/2} \lambda_+^{-1/2} \theta^1 (qV^+ - V^{30}) - q^{-5/2} \lambda_+^{-3/2} \theta^1 \sigma, \]

The right versions of the above relations read

\[\theta_1 V^+ = (q + 1) \lambda_+^{-1} V^+ \theta_1 + (q^2 - q) \lambda_+^{-1} V^0 \theta_1, \]
\[\theta_2 V^+ = (q + q^{-2}) \lambda_+^{-1} V^+ \theta_2 - (q - 1) \lambda_+^{-1} V^0 \theta_2 \]
\[- \frac{1}{2} \lambda_+^{-1/2} (V^+ - qV^{30}) \theta_1 + q^{-1} \lambda_+^{-3/2} \theta_1 \theta_1, \]
\[\theta_1 V^0 = (q^2 + q) \lambda_+^{-1} V^0 \theta_1 + (1 - q^{-1}) \lambda_+^{-1} V^3 \theta_1, \]
\[\theta_2 V^0 = (1 + q^{-1}) \lambda_+^{-1} V^0 \theta_2 - (q^{-1} - q^{-2}) \lambda_+^{-1} V^3 \theta_2 \]
\[- \frac{1}{2} \lambda_+^{-1/2} (V^+ - qV^{30}) \theta_1 + q^{-1} \lambda_+^{-3/2} \theta_1 \theta_1, \]

\[\theta_1 V^+ = q^{-1}(q^4 + 1 + \lambda_+) \lambda_+^{-2} V^+ \theta_1 \]
\[- q^{-2}(q^4 + 1 + q^2 \lambda_+) \lambda_+^{-2} V^3 \theta_1 \]
\[+ q^{-1} \lambda_+^{-3/2} (V^+ - q^{-1} V^0) \theta_2 \]
\[+ q^{-1} \lambda_+^{-2} \theta_1 + q^{-2} \lambda_+^{-2} U^1 \theta_1, \]
\[\theta_2 V^+ = q^{-1}(q^4 + 1 + \lambda_+) \lambda_+^{-2} V^+ \theta_2 \]
\[- q^{-2}(q^4 + 1 - q^2 \lambda_+) \lambda_+^{-2} V^3 \theta_2 \]
\[ \theta_1 V^{30} = q^{-3}(q^4 + 1 + q^4 \lambda_+) \lambda_+^{-2} V^{30} \theta_1 \\
- q^{-2}(q^4 + 1 - q^2 \lambda_+) \lambda_+^{-2} V^{+} \theta_1 \\
+ q^{-3/2} \lambda_+^{-3/2} (V^{+} + V^{0}) \theta_2 \\
- q^{-2} \lambda_+^{-2} \rho \theta_1 - q^{-3} \lambda_+^{-2} U^1 \theta_1 \]
\[ \theta_2 V^{30} = q^{-3}(q^4 + 1 + q^4 \lambda_+) \lambda_+^{-2} V^{30} \theta_2 \\
- q^{-2}(q^4 + 1 - q^2 \lambda_+) \lambda_+^{-2} V^{+} \theta_2 \\
+ q^{-1/2}(q^2 \lambda_+ + 1) \lambda_+^{-3/2} V^{0-} \theta_1 \\
+ q^{-5/2}(1 - q^4 \lambda_+) \lambda_+^{-3/2} V^{3-} \theta_1 \\
- q^{-2} \lambda_+^{-2} \rho \theta_2 + q^{-1} \lambda_+^{-2} U^1 \theta_2 \]

\[ \theta_1 V^{3-} = (1 + q^{-1}) \lambda_+^{-1} V^{3-} \theta_1 + (q - 1) \lambda_+^{-1} V^{0-} \theta_1 \\
- q^{-1/2} \lambda_+^{-3/2} \rho \theta_2 \]
\[ \theta_2 V^{3-} = q^{-1}(q^3 + 1) \lambda_+^{-1} V^{3-} \theta_2 - (q^2 - q) \lambda_+^{-1} V^{0-} \theta_2 \]
\[ \theta_1 V^{0-} = q^{-2}(q^3 + 1) \lambda_+^{-1} V^{0-} \theta_1 + q^{-2}(q - 1) \lambda_+^{-1} V^{3-} \theta_1 \\
- q^{-1/2} \lambda_+^{-3/2} \rho \theta_2, \]
\[ \theta_2 V^{0-} = (q + 1) \lambda_+^{-1} V^{0-} \theta_2 - (1 - q^{-1}) \lambda_+^{-1} V^{3-} \theta_2 \]

and similar for spinors with bar,

\[ \bar{\theta}_1 V^{+3} = (1 + q^{-1}) \lambda_+^{-1} V^{+3} \bar{\theta}_1 + (q - 1) \lambda_+^{-1} V^{+0} \bar{\theta}_1 \]
\[ \bar{\theta}_2 V^{+3} = q^{-1}(q^3 + 1) \lambda_+^{-1} V^{+3} \bar{\theta}_2 - (q^2 - q) \lambda_+^{-1} V^{+0} \bar{\theta}_2 \\
- q^{1/2} \lambda_+^{-3/2} \sigma \bar{\theta}_1 \]
\[ \bar{\theta}_1 V^{+0} = q^{-2}(q^3 + 1) \lambda_+^{-1} V^{+0} \bar{\theta}_1 + (q^{-1} - q^{-2}) \lambda_+^{-1} V^{+3} \bar{\theta}_1 \]
\[ \bar{\theta}_2 V^{+0} = (q^{-1} - 1) \lambda_+^{-1} V^{+3} \bar{\theta}_2 + (q + 1) \lambda_+^{-1} V^{+0} \bar{\theta}_2 \\
- q^{1/2} \lambda_+^{-3/2} \sigma \bar{\theta}_1 \]
\[ \bar{\theta}_1 V^{+-} = q^{-3}(q^4 + 1 + q^4 \lambda_+) \lambda_+^{-2} V^{+-} \bar{\theta}_1 \]

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\[ + q^{-1}(q - 1)^2(\lambda_+ + 1)\lambda_+^{-2}V^{30}\tilde{\theta}_1 \\
+ (q^2\lambda_+ + 1)q^{-5/2}\lambda_+^{-3/2}V^{+3}\bar{\theta}_2 \\
+ q^{-1/2}(1 - \lambda_+)\lambda_+^{-3/2}V^{+0}\bar{\theta}_2 \\
- q^{-1}\lambda_+^2\sigma\bar{\theta}_1 + q^{-2}\lambda_+^{-2}U^2\bar{\theta}_1, \]
\[
\bar{\theta}_2V^{+-} = q^{-3}(q^4 + 1 + q^4\lambda_+)\lambda_+^{-2}V^{+-}\tilde{\theta}_2 \\
+ q^{-1}(q - 1)^2(\lambda_+ + 1)\lambda_+^{-2}V^{30}\bar{\theta}_2 \\
+ q^{1/2}\lambda_+^{-3/2}(V^{0-} - V^{3-})\bar{\theta}_1 \\
- q^{-1}\lambda_+^2\sigma\bar{\theta}_2 - \lambda_+^{-2}U^2\bar{\theta}_2, \]
\[
\tilde{\theta}_1V^{30} = q^{-1}(q^4 + 1 + \lambda_+)\lambda_+^{-2}V^{30}\tilde{\theta}_1 \\
+ q^{-1}(q - 1)^2(\lambda_+ + 1)\lambda_+^{-2}V^{+-}\tilde{\theta}_1 \\
+ q^{-3/2}(1 - \lambda_+)\lambda_+^{-3/2}V^{+-}\bar{\theta}_2 \\
+ q^{-3/2}(q^4 + \lambda_+)\lambda_+^{-3/2}V^{+0}\bar{\theta}_2 \\
- \lambda_+^2\sigma\tilde{\theta}_1 + q^{-1}\lambda_+^{-2}U^2\bar{\theta}_1, \]
\[
\bar{\theta}_2V^{30} = q^{-1}(q^4 + 1 + \lambda_+)\lambda_+^{-2}V^{30}\bar{\theta}_2 \\
+ q^{-1}(q - 1)^2(\lambda_+ + 1)\lambda_+^{-2}V^{+-}\bar{\theta}_2 \\
+ q^{3/2}\lambda_+^{-3/2}(V^{0-} - V^{3-})\tilde{\theta}_1 \\
- \lambda_+^2\sigma\bar{\theta}_2 - q\lambda_+^{-2}U^2\bar{\theta}_2, \]
\[
\tilde{\theta}_1V^{3-} = (q + 1)\lambda_+^{-1}V^{3-}\tilde{\theta}_1 + (q^2 - q)\lambda_+^{-1}V^{0-}\tilde{\theta}_1 \\
+ q^{1/2}\lambda_+^{-1/2}(qV^{+-} - V^{30})\bar{\theta}_2 + q^{-1/2}\lambda_+^{-3/2}\sigma\bar{\theta}_2, \]
\[
\bar{\theta}_2V^{3-} = q^{-2}(q^3 + 1)\lambda_+^{-1}V^{3-}\bar{\theta}_2 - (q - 1)\lambda_+^{-1}V^{0-}\bar{\theta}_2, \]
\[
\tilde{\theta}_1V^{0-} = q^{-1}(q^3 + 1)\lambda_+^{-1}V^{0-}\tilde{\theta}_1 + (1 - q^{-1})\lambda_+^{-1}V^{3-}\tilde{\theta}_1 \\
+ q^{1/2}\lambda_+^{-1/2}(qV^{+-} - V^{30})\bar{\theta}_2 - q^{-5/2}\lambda_+^{-3/2}\sigma\bar{\theta}_2, \]
\[
\bar{\theta}_2V^{0-} = (q + q^{-1})\lambda_+^{-1}V^{0-}\bar{\theta}_2 - (q^{-1} - q^{-2})\lambda_+^{-1}V^{3-}\bar{\theta}_2. \]

Next, we turn to the vector representations for the \(V^{\mu\nu}\), which are seen to be
\[
V^{\mu\nu} \triangleright X^\rho = (\tau^{\mu\nu})^\rho_\sigma X^\sigma, \quad X_\rho \triangleleft V^{\mu\nu} = X_\sigma(\tau^{\mu\nu})^\sigma_\rho; \tag{158}
\]
with

$$(\tau^3)^{\rho \sigma} = \lambda_+^{-2} \begin{pmatrix} 0 & -2q & -\lambda & 0 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & -q\lambda \\ 0 & 0 & 0 & 0 \end{pmatrix},$$ (159)

$$(\tau^0)^{\rho \sigma} = \lambda_+^{-2} \begin{pmatrix} 0 & -\lambda & 2q^{-1} & 0 \\ 0 & 0 & 0 & -q^{-1}\lambda \\ 0 & 0 & 0 & -(q^2 + q^{-2}) \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$(\tau^+)^{\rho \sigma} = \lambda_+^{-2} \begin{pmatrix} 2q^{-2} & 0 & 0 & 0 \\ 0 & -2q^{-1}\lambda & -q^{-1}\lambda & 0 \\ 0 & q^{-1}\lambda & 0 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix},$$

$$(\tau^-)^{\rho \sigma} = \lambda_+^{-2} \begin{pmatrix} \lambda q^{-2} & 0 & 0 & 0 \\ 0 & -q^{-1}\lambda^2 & 2q & 0 \\ 0 & q^{-1}(q^2 + q^{-2}) & 0 & 0 \\ 0 & 0 & 0 & -\lambda \end{pmatrix},$$

$$(\tau^{30})^{\rho \sigma} = \lambda_+^{-2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ -q^{-1}\lambda & 0 & 0 & 0 \\ 0 & 2q^{-1} & -\lambda & 0 \end{pmatrix},$$

$$(\tau^{3-})^{\rho \sigma} = \lambda_+^{-2} \begin{pmatrix} q^{-1}\lambda & 0 & 0 & 0 \\ 2q^{-2} & 0 & 0 & 0 \\ 0 & q^{-2}\lambda & -q^{-1}(q^2 + q^{-2}) & 0 \\ 0 & 0 & 0 & -q^{-1}\lambda_+ \end{pmatrix}.$$}

Notice that rows and columns are labeled in the order $+, 3, 0, -$. For $U^1$, $U^2$, $\sigma$ and $\rho$ we obtain the matrices

$$(U^1)^{\rho \sigma} = (U^1)^{\rho \sigma} = -(q^2 + q^{-2})\lambda_+^{-1} \delta^{\rho \sigma},$$ (160)

$$(\rho)^{\rho \sigma} = \lambda_+^{-1} \begin{pmatrix} -q\lambda_+ & 0 & 0 & 0 \\ 0 & -2 & q\lambda & 0 \\ 0 & q^{-1}\lambda & -(q^2 + q^{-2}) & 0 \\ 0 & 0 & 0 & -q^{-1}\lambda_+ \end{pmatrix},$$

$$(\sigma)^{\rho \sigma} = \lambda_+^{-1} \begin{pmatrix} q\lambda_+ & 0 & 0 & 0 \\ 0 & 2 & q^{-1}\lambda & 0 \\ 0 & q\lambda & (q^2 + q^{-2}) & 0 \\ 0 & 0 & 0 & q^{-1}\lambda_+ \end{pmatrix}.$$
and likewise for the generators of the \( U_q(su_2) \)-subalgebra,

\[
(L^+)_{\sigma} = \begin{pmatrix} 0 & -q & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad (L^-)_{\sigma} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & q^{-1} & 0 & 0 \end{pmatrix}, \quad (\tau^{-1/2})_{\sigma} = \begin{pmatrix} q^2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & q^{-2} \end{pmatrix}.
\]

We can now proceed to write down commutation relations between the \( V^{\mu\nu} \) and the components of a vector operator. Let us recall that we mean by a vector operator a set of objects \( X^\rho \) which transform under Lorentz transformations according to

\[
[V^{\mu\nu}, X^\rho]_q = (\tau^{\mu\nu})_{\rho} X^\sigma, \quad [X^\rho, V^{\mu\nu}]_q = X^\sigma (\tau^{\mu\nu})^\sigma_{\rho}.
\]

These relations are equivalent to those in (132) if we substitute the vector components for the general element \( a \). By taking the coproduct of the \( V^{\mu\nu} \) together with the vector representations of the Lorentz generators we can again compute from (132) the explicit form for the commutation relations between the \( V^{\mu\nu} \) and the \( X^\rho \). Unfortunately, the results are rather lengthy:

\[
V^{+3} X^+ = q^{-2} (q^4 + 1) \lambda_+^{-1} X^+ V^{+3} - q \lambda \lambda_+^{-1} X^+ V^{+0},
\]

\[
V^{+3} X^3 = 2 \lambda_+^{-1} X^3 V^{+3} - q \lambda X^+ V^{++} + q \lambda_+^{-1} X^0 V^{+3}
- q \lambda \lambda_+^{-1} X^+ V^{30} + q \lambda_+^{-2} X^+ (U^1 + U^2),
\]

\[
V^{+3} X^0 = q^{-2} (q^4 + 1) \lambda_+^{-1} X^0 V^{+3} + q^{-1} \lambda \lambda_+^{-1} X^3 V^{+3}
- q^2 \lambda \lambda_+^{-1} X^+ V^{++} + \lambda_+^{-2} X^+ (q U^1 - q^{-1} U^2),
\]

\[
V^{+3} X^- = 2 \lambda_+^{-1} X^- V^{+3} - 2 q \lambda_+^{-1} X^3 V^{++} + q \lambda \lambda_+^{-1} X^0 V^{+3}
+ q \lambda \lambda_+^{-1} X^+ V^{+0} - q^2 \lambda \lambda_+^{-1} X^3 V^{30}
- q^2 \lambda \lambda_+^{-1} X^+ V^{00} + q^2 \lambda \lambda_+^{-1} X^+ V^{33}
+ \lambda_+^{-2} X^3 (U^1 + U^2) - \lambda_+^{-2} X^0 (U^1 - q U^2),
\]

\[
V^{+0} X^+ = 2 \lambda_+^{-1} X^+ V^{+0} - q^{-1} \lambda \lambda_+^{-1} X^+ V^{+3},
\]

\[
V^{+0} X^3 = 2 \lambda_+^{-1} X^3 V^{+0} - \lambda \lambda_+^{-1} X^+ V^{+3} - 2 q \lambda \lambda_+^{-1} X^+ V^{30}
+ q \lambda \lambda_+^{-1} X^0 V^{+0} - \lambda_+^{-2} X^+ (q^{-1} U^1 - q U^2),
\]

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\begin{align*}
V^0 X^0 &= q^{-2} \lambda_+^{-1} (q^2 + 1) X^0 V^0 + q \lambda \lambda_+^{-1} X^0 V^3 \\
&\quad + q^{-1} \lambda \lambda_+^{-1} X^1 V^0 + q^{-1} \lambda_+^{-2} X^+ (U^1 + U^2), \\
V^+ X^- &= q^{-2} \lambda_+^{-1} (q^2 + 1) X^- V^0 - q \lambda X^3 V^3 + \lambda \lambda_+^{-1} X^0 V^3 \\
&\quad + q^{-1} \lambda \lambda_+^{-1} X^- V^3 - q \lambda \lambda_+^{-1} X_3 V^3 - q^2 \lambda^2 \lambda_+^{-1} X^+ V^0 - \\
&\quad + q^2 \lambda^2 \lambda_+^{-1} X^+ V^3 - \lambda_+^{-2} X^0 (q^{-2} U^1 + q^2 U^2) \\
&\quad - \lambda_+^{-2} X^3 (q^{-2} U^1 - U^2),
\end{align*}

\begin{align*}
V^+ X^+ &= 2 \lambda_+^{-1} X^+ V^+ + q^{-1} \lambda X^3 V^3 - \lambda \lambda_+^{-1} X^3 V^0 \\
&\quad \quad - q^{-2} \lambda \lambda_+^{-1} X^0 V^3 - q^{-2} \lambda_+^{-2} X^+ (U^1 + U^2),
\end{align*}

\begin{align*}
V^+ X^3 &= (2 - \lambda^2) \lambda_+^{-1} X^3 V^+ - \lambda X^+ V^3 - q \lambda \lambda_+^{-1} X^+ V^0 \\
&\quad + q^{-1} \lambda \lambda_+^{-1} X^- V^3 - q^{-1} \lambda \lambda_+^{-1} X^- V^0 + \lambda^2 \lambda_+^{-1} X^+ V^3 \\
&\quad + q^{-2} \lambda_+^{-2} X^0 U^1 - U^2) + q^{-1} \lambda_+^{-2} X^3 (U^1 - U^2),
\end{align*}

\begin{align*}
V^0 X^0 &= q^{-2} (q^4 + 1) \lambda_+^{-1} X^0 V^0 - \lambda \lambda_+^{-1} X^3 V^3 \\
&\quad + 2 q^{-1} \lambda \lambda_+^{-1} X^0 V^0 + q^{-1} \lambda_+^{-2} X^+ (q^{-2} U^1 - U^2),
\end{align*}

\begin{align*}
V^0 X^3 &= (2 - \lambda^2) \lambda_+^{-1} X^3 V^3 + \lambda \lambda_+^{-1} X^+ V^0 + q^{-1} \lambda X^+ V^0 \\
&\quad + q^{-2} \lambda \lambda_+^{-1} (1 - q^2 \lambda_+) X^+ V^3 - q^{-2} \lambda \lambda_+^{-1} X^- V^3 \\
&\quad + \lambda^2 \lambda_+^{-1} X^0 V^0 - \lambda_+^{-2} X^3 (q^{-2} U^1 - U^2) \\
&\quad - \lambda_+^{-2} X^0 (q^{-3} U^1 + q U^2),
\end{align*}

\begin{align*}
V^3 X^0 &= q^{-2} (q^4 + 1) \lambda_+^{-1} X^0 V^0 + \lambda \lambda_+^{-1} X^+ V^0 \\
&\quad + q^{-2} \lambda^2 \lambda_+^{-1} X^+ V^3 - \lambda^2 \lambda_+^{-1} X^3 V^3 - q^{-1} \lambda_+^{-2} X^3 (U^1 + U^2),
\end{align*}

\begin{align*}
V^3 X^- &= 2 \lambda_+^{-1} X^- V^3 - \lambda \lambda_+^{-1} X^0 V^0 \\
&\quad + 2 q \lambda \lambda_+^{-1} (q + 1) X^3 V^3 - q \lambda \lambda_+^{-1} X^0 V^0 \\
&\quad + 2 q \lambda \lambda_+^{-1} X^3 V^3 - \lambda^2 \lambda_+^{-1} X^3 V^3 - \lambda_+^{-2} X^3 (q^{-1} U^1 - U^2),
\end{align*}

\begin{align*}
V^3 X^+ &= q^{-2} (q^4 + 1) \lambda_+^{-1} X^+ V^3 - q \lambda \lambda_+^{-1} X^0 V^0 \\
&\quad - q \lambda \lambda_+^{-1} X^+ V^0 - 2 q \lambda \lambda_+^{-1} X^3 V^3 - \lambda^2 \lambda_+^{-1} X^- (V^3 - V^0) \\
&\quad - \lambda \lambda_+^{-1} X^3 V^0 - \lambda_+^{-2} X^3 (U^1 + U^2) - \lambda_+^{-2} X^3 (q^{-2} U^1 - U^2),
\end{align*}
\( V^3 - X^3 = 2\lambda^+_1 X^3 V^3 - q\lambda X - V^+- - q^{-1}\lambda\lambda^+_1 X^0 V^3 \)
\( - q\lambda\lambda^+_1 X - V^30 - q^{-1}\lambda^+_2 X^{-}(U^1 + U^2), \)
\( V^3 - X^0 = q^{-2}(q^4 + 1)\lambda^+_1 X^0 V^3 - - q\lambda\lambda^+_1 X^3 V^3 \)
\( + \lambda\lambda^+_1 X^0 - V^+- + \lambda^+_2 X^{-}(qU^1 - q^{-1}U^2), \)
\( V^3 - X^- = 2\lambda^+_1 X^0 V^- - q\lambda\lambda^+_1 X^- V^0, \)
\( \)
\( V^0 - X^+ = 2\lambda^+_1 X^+ V^0 - + \lambda\lambda^+_1(q + 1) X^3 V^+ - - 2\lambda^+_1 X^3 V^30 \)  \((168)\)
\( - q^{-1}\lambda\lambda^+_1 X^+ V^3 - + \lambda\lambda^+_1 X^0 V^3 \)
\( + \lambda^2\lambda^+_1 X^-(V^+ - V^0) - \lambda^2\lambda^+_1 X^0 V^- \)
\( - \lambda^2 X^3(U^1 - q^{-2}U^2) - q^{-2}\lambda^+_2 X^{-}(U^1 + U^2), \)
\( V^0 - X^0 = 2\lambda^+_1 X^3 V^0 - - \lambda X^{-} V^30 - q^{-2}\lambda\lambda^+_1 X^- (1 - q^3\lambda_+) V^+- \)
\( - q^{-1}\lambda\lambda^+_1 X^0 V^0 - - q^{-1}\lambda^+_2 X^{-}(U^1 - q^{-2}U^2), \)
\( V^0 - X^0 = q^{-2}(q^4 + 1)\lambda^+_1 X^0 V^0 - - q\lambda\lambda^+_1 X^3 V^0 \)
\( - q^{-1}\lambda\lambda^+_1 X^0 - V^30 + q^{-1}\lambda^2\lambda^+_1 X^- V^- \)
\( + \lambda^+_2 X^{-}(qU^1 - q^{-3}U^2), \)
\( V^0 - X^- = q^{-2}(q^4 + 1)\lambda^+_1 X^- V^0 - - q^{-1}\lambda\lambda^+_1 X^- V^3. \)

If we want to commute the \( V^{\mu\nu} \) to the left of a vector operator, we have to apply instead
\( X^+ V^3 = q^{-2}(q^4 + 1)\lambda^+_1 V^+ X^3 - - q\lambda V^+ - X_3 \)  \((169)\)
\( - q\lambda\lambda^+_1 V^30 X_3 - q\lambda\lambda^+_1 V^+ X_+ - q^2\lambda\lambda^+_1 V^+- X_0 \)
\( + q^2\lambda^2\lambda^+_1 (V^3 - V^0) X_+ + q\lambda^+ (U^1 + U^2) X_3 \)
\( + \lambda^+_2 (qU^1 - q^{-1}U^2) X_0, \)
\( X^+ V^3 = 2\lambda^+_1 V^+ X_3 - 2q\lambda\lambda^+_1 V^+ X_- + q^{-1}\lambda\lambda^+_1 V^+ X_0 \)
\( - q^2\lambda\lambda^+_1 V^30 X_- + \lambda^+ (U^1 + U^2) X_-, \)
\( X^+ V^+ = q^{-2}(q^4 + 1)\lambda^+_1 V^+ X_0 + q\lambda\lambda^+_1 V^+ X_3 \)
\( + q\lambda\lambda^+_1 V^+ - X_- - \lambda^+ (U^1 - q^{-2}U^2) X_-, \)
\( X^+ V_+ = 2\lambda^+_1 V^+ X_- + q\lambda\lambda^+_1 V^+ X_-, \)
\( X^+ V^0 = 2\lambda^+_1 V^+0 X_+ - \lambda\lambda^+_1 V^+ - X_3 - q^{-1}\lambda\lambda^+_1 V^+3 X_+ \)  \((170)\)
\( - 2q\lambda\lambda^+_1 V^+3 X_3 - q\lambda\lambda^+_1 V^+0 X_0 + q^2\lambda^2\lambda^+_2 (V^3 - V^0) X_- \)
\( - \lambda^+_2 (q^{-1}U^1 - qU^2) X_3 - q^{-1}\lambda^+_2 (U^1 + U^2) X_0, \)

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\( X_3 V^+ = 2\lambda_+^{-1}V^+ X_3 - q\lambda V X_+ + q^{-1}\lambda \lambda_+^{-1}V^+ X_0 \\
- q\lambda \lambda_+^{-1}V^+ X_+ - \lambda_+^{-2}(q^{-2}U^1 - U^2)X_+ , \)
\( X_0 V^+ = q^{-2}(q^4 + 1)\lambda_+^{-1}V^+ X_0 + \lambda \lambda_+^{-1}V^3 X_+ \\
+ q\lambda \lambda_+^{-1}V^+ X_3 + \lambda_+^{-2}(q^{-2}U^1 + q^2U^2)X_+ , \)
\( X_- V^+ = q^{-2}(q^4 + 1)\lambda_+^{-1}V^+ X_- + q^{-1}\lambda \lambda_+^{-1}V^3 X_- , \)
\( X_+ V^+ = 2\lambda_+^{-1}V^+ X_+ - \lambda V^3 X_3 - q^{-1}\lambda \lambda_+^{-1}V^3 X_0 \\
+ q\lambda \lambda_+^{-1}V^0 X_3 - q^{-2}\lambda_+^{-2}(U^1 + U^2)X_+ , \)
\( X_3 V^+ = (2 - \lambda^2)\lambda_+^{-1}V^+ X_3 + q^{-1}\lambda V^3 X_+ - \lambda \lambda_+^{-1}V^0 X_+ \\
+ \lambda \lambda_+^{-1}V^0 X_3 - 2\lambda \lambda_+^{-1}V^3 X_- - \lambda^2 \lambda_+^{-1}V^+ X_0 \\
+ q^{-1}\lambda \lambda_+^{-1}(U^1 + U^2)X_3 + \lambda_+^{-2}(U^1 - q^{-2}U^2)X_0 , \)
\( X_0 V^+ = q^{-2}(q^4 + 1)\lambda_+^{-1}V^+ X_0 - q^{-2}\lambda \lambda_+^{-1}V^3 X_+ \\
+ \lambda \lambda_+^{-1}V^3 X_3 + \lambda^2 \lambda_+^{-1}V^+ X_3 + \lambda_+^{-2}(q^{-2}U^1 - U^2)X_3 , \)
\( X_- V^+ = 2\lambda_+^{-1}V^+ X_- + 2q^{-1}\lambda \lambda_+^{-1}V^3 X_3 + q^{-1}\lambda \lambda_+^{-1}V^3 X_0 \\
- q^{-1}\lambda \lambda_+^{-1}V^0 X_3 + \lambda_+^{-2}(U^1 + U^2)X_+ , \)
\( X_+ V^3 = 2\lambda_+^{-1}V^3 X_+ + q^{-2}\lambda \lambda_+^{-1}(1 - q^3\lambda_+)V^3 X_3 \\
+ q\lambda V^0 X_3 + \lambda_+^{-1}V^0 X_0 - q^{-1}\lambda^2 \lambda_+^{-1}V^3 X_0 \\
+ q^{-1}\lambda_+^{-2}(q^{-2}U^1 - U^2)X_+ , \)
\( X_3 V^3 = (2 - \lambda^2)\lambda_+^{-1}V^3 X_3 - (q + \lambda)\lambda \lambda_+^{-1}V^3 X_- \\
- q^{-1}\lambda \lambda_+^{-1}V^+ X_+ + 2\lambda \lambda_+^{-1}(q^{-2}U^1 + q^2U^2)X_+ \\
- \lambda^2 \lambda_+^{-1}V^3 X_0 - \lambda \lambda_+^{-1}X_0 - \lambda_+^{-2}(q^{-2}U^1 - U^2)X_3 , \)
\( X_0 V^3 = q^{-2}(q^4 + 1)\lambda \lambda_+^{-1}V^3 X_0 - q^{-1}\lambda \lambda_+^{-1}V^0 X_+ - q\lambda \lambda_+^{-1}V^0 X_- \\
+ \lambda^2 \lambda_+^{-1}V^3 X_3 + \lambda^2 \lambda_+^{-1}V^3 X_- - \lambda_+^{-2}(q^{-3}U^1 + qU^2)X_3 , \)
\( X_- V^3 = 2\lambda_+^{-1}V^3 X_- + q^{-1}\lambda V^0 X_0 - q^{-2}\lambda \lambda_+^{-1}V^3 X_3 \\
+ \lambda \lambda_+^{-1}V^0 X_0 - \lambda_+^{-2}(q^{-1}U^1 - qU^2)X_+ , \)
\( X_+ V^3 = q^{-2}(q^4 + 1)\lambda \lambda_+^{-1}V^3 X_+ - q\lambda \lambda_+^{-1}V^0 X_+ , \)
\( X_3 V^3 = 2\lambda_+^{-1}V^3 X_3 + 2q\lambda \lambda_+^{-1}V^+ X_+ - \lambda \lambda_+^{-1}V^3 X_+ \\
- q\lambda \lambda_+^{-1}V^3 X_- - \lambda_+^{-2}(U^1 + U^2)X_+ , \)

39
\[X_0 V^{3-} = q^{-2}(q^4 + 1)\lambda_+^{-1}V^{3-}X_0 - q\lambda\lambda_+^{-1}V^{+-}X_+ + q^2U^1U^2X_+,
\]
\[X_- V^{3-} = 2\lambda_+^{-1}V^{3-}X_- + q\lambda V^{+-}X_3 + \lambda\lambda_+^{-1}V^{+-}X_0 + \lambda^2\lambda_+^{-1}(V^{3-} - V^{+-})X_+ + q\lambda\lambda_+^{-1}V^{0-}X_- - q\lambda\lambda_+^{-1}V^{30}X_3 + \lambda_+^{-2}(qU^1 - q^{-1}U^2)X_0 - q^{-1}\lambda_+^{-2}(U^1 + U^2)X_3,\]

\[X_+ V^{0-} = 2\lambda_+^{-1}V^{0-}X_+ - q^{-1}\lambda\lambda_+^{-2}V^{3-}X_+ \quad (174),\]
\[X_3 V^{0-} = 2\lambda_+^{-1}V^{0-}X_3 + \lambda(q^2 + \lambda)\lambda_+^{-1}V^{+-}X_+ - 2\lambda\lambda_+^{-1}V^{30}X_+ - q\lambda\lambda_+^{-1}V^{0-}X_0 - \lambda^{-2}(U^1 - q^{-2}U^2)X_+,
\]
\[X_0 V^{0-} = q^{-2}(q^4 + 1)\lambda_+^{-1}V^{0-}X_0 + \lambda\lambda_+^{-1}V^{30}X_+ - q^{-1}\lambda\lambda_+^{-1}V^{0-}X_3 - \lambda^2\lambda_+^{-1}V^{+-}X_+ - q^{-2}\lambda^{-2}(U^1 + U^2)X_+,
\]
\[X_- V^{0-} = q^{-2}(q^4 + 1)\lambda_+^{-1}V^{0-}X_- - \lambda V^{30}X_3 + q^{-1}\lambda\lambda_+^{-1}V^{3-}X_- - q^{-1}\lambda_+^{-2}X_0 + q^{-3}\lambda\lambda_+^{-1}(-q + q^{-2}\lambda_+)V^{+-}X_3 - q^{-1}\lambda_+^{-2}(U^1 - q^{-1}U^2)X_3 + \lambda_+^{-2}(qU^1 + q^{-2}U^2)X_0.
\]

### 5.4 Quantum Lie algebra of q-deformed Lorentz algebra and its Casimir operators

Last but not least we would like to present the quantum Lie algebra of q-deformed Lorentz algebra. For this to achieve, we calculate the adjoint action of the independent \(V^{\mu\nu}\) on themselves in a fashion as was done in the previous section and set the results equal to the q-commutators of the corresponding Lorentz generators. In the cases where the adjoint actions do not vanish, we obtain

\[ [V^{+3}, V^{+-}]_q = -q^2[V^{+3}, V^{+3}]_q = -\lambda_+^{-1}V^{+3}, \quad (175)\]
\[ [V^{+3}, V^{30}]_q = -q^2[V^{30}, V^{+3}]_q = -q\lambda_+^{-1}V^{+0}, \]
\[ [V^{+3}, V^{3-}]_q = -[V^{3-}, V^{+3}]_q = -q\lambda_+^{-1}V^{-}, \]
\[ [V^{+3}, V^{0-}]_q = -[V^{0-}, V^{+3}]_q = \lambda_+^{-1}(V^{30} - \lambda V^{+-}), \]
\[ [V^{+0}, V^{+3}]_q = -q^2[V^{+3}, V^{+0}]_q = -\lambda_+^{-1}V^{+0}, \quad (176)\]
\[ [V^{+0}, V^{30}]_q = -q^2[V^{30}, V^{+0}]_q = -\lambda_+^{-1}(q^{-1}V^{+3} + \lambda V^{+0}), \]
\[ [V^{+0}, V^{3-}]_q = -[V^{3-}, V^{+0}]_q = -\lambda_+^{-1}V^{30}, \]
\[ [V^{+0}, V^{0-}]_q = -[V^{0-}, V^{+0}]_q = q^{-1}\lambda_+^{-1}V^{+-}, \]

40
the representation matrices for the Lorentz generators. This can again be

\[ [V^{+-}, V^{+-}]_q = -q^{-1} \lambda \lambda^\pm_1 V^{+-}, \]
\[ [V^{+-}, V^{30}]_q = [V^{30}, V^{+-}]_q = -q^{-1} \lambda \lambda^+_1 V^{30}, \]
\[ [V^{+-}, V^{3-}]_q = -q^2 [V^{3-}, V^{+-}]_q = -\lambda^+_1 V^{3-}, \]
\[ [V^{+-}, V^{0-}]_q = -q^2 [V^{0-}, V^{+-}]_q = -\lambda^+_1 V^{0-}, \]
\[ [V^{30}, V^{30}]_q = -q^{-1} \lambda \lambda^+_1 (V^{+-} + \lambda V^{30}), \]
\[ [V^{30}, V^{3-}]_q = -q^2 [V^{3-}, V^{30}]_q = \lambda^+_1 (qV^{0-} - \lambda V^{3-}) , \]
\[ [V^{30}, V^{0-}]_q = -q^2 [V^{0-}, V^{30}]_q = q^{-1} \lambda^+_1 V^{3-}. \]

Of course, the quantum Lie algebra relations are consistent with spinor and vector representation of q-deformed Lorentz algebra. This can again be checked in a familiar way by writing out q-commutators and substituting the representation matrices for the Lorentz generators.

As Casimirs of this quantum Lie algebra we have found the two operators

\[ C^1 = \eta_{\mu\nu} \eta_{\rho\sigma} V^{\mu\rho} V^{\nu\sigma} = 2V^{30} V^{30} + (q^2 + q^{-2}) V^{+-} V^{+-} \]
\[ + 2(qV^{0+} V^{0-} - q^{-1} V^{3+} V^{3-} - q^{-3} V^{3-} V^{3+}) \]
\[ - \lambda (V^{+3} V^{0-} + V^{+0} V^{3-} + V^{+-} V^{30} + V^{30} V^{+-}) \]
\[ - q^{-2} \lambda (V^{3-} V^{0+} + V^{0-} V^{3+}), \]
\[ C^2 = \varepsilon_{\mu\nu\rho\sigma} V^{\mu\nu} V^{\rho\sigma} = (3 + q^{-4} - 2q^{-3}) (V^{+3} V^{0-} - V^{+0} V^{3-}) \]
\[ + (3 + q^{-4} - 2q^{-3}) (V^{+-} V^{30} + V^{30} V^{+-}) \]
\[ + q^{-6} (3q^4 + 1 - 2q) (V^{0+} V^{+3} - V^{3-} V^{0+}) \]
\[ - q^{-2} \lambda (2q^2 + 2q \lambda + \lambda \lambda^+_1) V^{+-} V^{+-}, \]

with \( \eta_{\mu\nu} \) and \( \varepsilon_{\mu\nu\rho\sigma} \) being q-analogs of Minkowski metric and corresponding epsilon tensor (see also Appendix A). Specifying the two Casimirs for the different representations finally yields the results:

a) (operator representation)

\[ C^1 = 2q \lambda \lambda^2 (X \circ X)(\partial \circ \partial) + 2q^{-2} \lambda \lambda^2 (X \circ \partial)(X \circ \partial) \]
\[ + 2q \lambda^{-1} X \circ \partial, \]
\[ C^2 = 0, \]

b) (spinor representation)

\[ C^1 = [[3]] q^{-4} \lambda \lambda^2 \mathbb{I}_{2 \times 2}, \]
\[ C^2 = [[3]]_{q^4}(3 + q^4 + 2q^3\lambda)I_{2\times 2}, \]

c) (vector representation)
\[ C^1 = 2[[3]]_{q^{-4}}\lambda^{-2}I_{3\times 3}, \quad (183) \]
\[ C^2 = 0. \]

6 Conclusion

Let us end with a few comments on our results. We dealt with quantum algebras describing q-deformed versions of physical symmetries. This way we considered \( U_q(su_2) \), \( U_q(so_4) \), and q-deformed Lorentz algebra. It was our aim to provide a consistent framework of basic definitions and relations which allow for representation theoretic investigations in physics.

In each case the starting point of our reasonings was the realization of symmetry generators within q-deformed differential calculus. We listed the relations of the corresponding symmetry algebras and presented expressions for coproducts and antipodes on symmetry generators. We realized that the Hopf structure of the symmetry generators allows us to define q-analogs of classical commutators. Furthermore, we concerned ourselves with q-deformed versions of such finite dimensional representations that play an important role in physics, i.e. spinor and vector representation. With the help of these representations we were able to write down q-deformed commutation relations between symmetry generators and components of a spinor or vector operator. Moreover, we calculated the adjoint action of the symmetry generators on each other and obtained relations for quantum Lie algebras this way. Finally, we gave expressions for the corresponding Casimir operators and specified them for the different representations.

Our reasonings were in complete analogy to the classical situation, but compared with the undeformed case our results are modified by terms proportional to \( \lambda = q - q^{-1} \). Hence, we regain in the classical limit as \( q \to 1 \) the familiar expressions. This observation nourishes the hope that a field theory based on quantum group symmetries can be developed along the same lines as its undeformed counterpart.

In this sense we also hope that implementing our results on a computer algebra system like Mathematica will prove useful in a systematical search for new q-identities. Furthermore, such an undertaking can be helpful to make it obvious that everything presented in this article works fine.
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A q-Deformed quantum spaces

The aim of this appendix is the following. For quantum spaces of physical importance we list their defining commutation relations as well as the nonvanishing elements of their quantum metric and q-deformed epsilon tensor.

The coordinates $\theta^\alpha$, $\alpha = 1, 2$, of two-dimensional antisymmetrised quantum plane fulfill the relation [8, 49]

$$\theta^1 \theta^2 = -q^{-1} \theta^2 \theta^1, \quad q \in \mathbb{R}, \quad q > 1,$$

(184)

whereas the spinor metric is given by a matrix $\varepsilon^{\alpha\beta}$ with nonvanishing elements

$$\varepsilon^{12} = q^{-1/2}, \quad \varepsilon^{21} = -q^{1/2}.$$ 

(185)

Furthermore, we can raise and lower indices as usual, i.e.

$$\theta^\alpha = \varepsilon^{\alpha\beta} \theta_\beta, \quad \theta_\alpha = \varepsilon_{\alpha\beta} \theta^\beta,$$

(186)

where $\varepsilon_{\alpha\beta}$ denotes the inverse of $\varepsilon^{\alpha\beta}$.

In the case of three-dimensional q-deformed Euclidean space the commutation relations between its coordinates $X^A$, $A \in \{+, 3, -\}$, read

$$X^3 X^\pm = q^{\pm 2} X^\pm X^3,$$

$$X^- X^+ = X^+ X^- + \lambda X^3 X^3.$$

(187)

The nonvanishing elements of the quantum metric are

$$g^{+-} = -q, \quad g^{33} = 1, \quad g^{-+} = -q^{-1}.$$ 

(188)

Now, the covariant coordinates are given by

$$X_A = g_{AB} X^B.$$ 

(189)
with \( g_{AB} \) being the inverse of \( g^{AB} \). The nonvanishing components of the three-dimensional q-deformed epsilon tensor take the form

\[
\begin{align*}
\varepsilon_{-3+} &= 1, & \varepsilon_{3-} &= -q^{-2}, \\
\varepsilon_{-+3} &= -q^{-2}, & \varepsilon_{+3-} &= q^{-2}, \\
\varepsilon_{3+} &= q^{-2}, & \varepsilon_{+3-} &= -q^{-4}, \\
\varepsilon_{333} &= -q^{-2}\lambda.
\end{align*}
\]

(190)

In Sec. 3 we need especially

\[
\epsilon_{ABC} \equiv g^{CD} \varepsilon_{ABD}, \quad \epsilon^{ABC} \equiv g^{AD} g^{BE} \varepsilon_{DEC}.
\]

(191)

Next we come to four-dimensional Euclidean space. For its coordinates \( X^i, i = 1, \ldots, 4 \), we have the relations

\[
\begin{align*}
X^1 X^j &= q X^j X^1, & j &= 1, 2, \\
X^j X^4 &= q X^4 X^j, \\
X^2 X^3 &= X^3 X^2, \\
X^4 X^1 &= X^1 X^4 + \lambda X^2 X^3.
\end{align*}
\]

(192)

The metric has the nonvanishing components

\[
g^{14} = q^{-1}, \quad g^{23} = g^{32} = 1, \quad g^{41} = q.
\]

(193)

Its inverse denoted by \( g_{ij} \) can again be used to introduce covariant coordinates, i.e.

\[
X_i = g_{ij} X^j.
\]

(194)

For the epsilon tensor of q-deformed Euclidean space with four dimensions one can find as components

\[
\begin{align*}
\varepsilon_{1234} &= q^4, & \varepsilon_{1432} &= -q^2, & \varepsilon_{2413} &= -q^2, \\
\varepsilon_{2134} &= -q^3, & \varepsilon_{4132} &= q^2, & \varepsilon_{4213} &= q, \\
\varepsilon_{1324} &= -q^4, & \varepsilon_{3412} &= q^2, & \varepsilon_{2341} &= -q^2, \\
\varepsilon_{3124} &= q^3, & \varepsilon_{4312} &= -q, & \varepsilon_{3241} &= q^2, \\
\varepsilon_{2314} &= q^2, & \varepsilon_{1243} &= -q^3, & \varepsilon_{2413} &= q, \\
\varepsilon_{3214} &= -q^2, & \varepsilon_{2143} &= q^2, & \varepsilon_{4231} &= -1, \\
\varepsilon_{1342} &= q^3, & \varepsilon_{1423} &= q^2, & \varepsilon_{3421} &= -q, \\
\varepsilon_{3142} &= -q^2, & \varepsilon_{4123} &= -q^2, & \varepsilon_{4321} &= 1.
\end{align*}
\]

(195)
In addition to this there are the non-classical components
\[ \varepsilon_{3232} = -\varepsilon_{2323} = q^2 \lambda. \] (196)

Now, we come to q-deformed Minkowski space [12, 13, 45] (for other deformations of spacetime and their related symmetries we refer to [50–55]). Its coordinates are subjected to the relations
\[ X^\mu X^0 = X^0 X^\mu, \quad \mu \in \{0, +, -, 3\}, \] (197)
\[ X^3 X^\pm - q^{\pm2} X^\pm X^3 = -q\lambda X^0 X^\pm, \]
\[ X^- X^+ - X^+ X^- = \lambda (X^3 X^3 - X^0 X^3), \]
and its metric is given by
\[ \eta^{00} = -1, \quad \eta^{33} = 1, \quad \eta^{+-} = -q, \quad \eta^{-+} = -q^{-1}. \] (198)

As usual, the metric can be used to raise and lower indices. In analogy to the classical case the q-deformed epsilon tensor has as nonvanishing components
\[ \varepsilon_{+0-3} = -q^{-4}, \quad \varepsilon_{+3-0} = q^{-2}, \quad \varepsilon_{3-+0} = q^{-2}, \] (199)
\[ \varepsilon_{3+0-} = q^{-2}, \quad \varepsilon_{-+03} = -q^{-2}, \quad \varepsilon_{-3+0} = -1, \]
\[ \varepsilon_{+03-} = q^{-4}, \quad \varepsilon_{0+-3} = -q^{-2}, \quad \varepsilon_{30-+} = q^{-2}, \]
\[ \varepsilon_{0+3-} = -q^{-4}, \quad \varepsilon_{-0+3} = q^{-2}, \quad \varepsilon_{03-+} = -q^{-2}, \]
\[ \varepsilon_{30+-} = -q^{-2}, \quad \varepsilon_{+-30} = q^{-4}, \quad \varepsilon_{3-0+} = -q^{-2}, \]
\[ \varepsilon_{03+-} = q^{-2}, \quad \varepsilon_{3+-0} = -q^{-2}, \quad \varepsilon_{-30+} = 1, \]
\[ \varepsilon_{+0-3} = -q^{-2}, \quad \varepsilon_{+-30} = -q^{-2}, \quad \varepsilon_{0-3+} = 1, \]
\[ \varepsilon_{00-3} = q^{-2}, \quad \varepsilon_{-+-30} = q^{-2}, \quad \varepsilon_{-03+} = -1, \]
and
\[ \varepsilon_{0-0+} = q^{-1} \lambda, \quad \varepsilon_{-0+0} = -q^{-1} \lambda, \] (200)
\[ \varepsilon_{0033} = -q^{-2} \lambda, \quad \varepsilon_{3330} = q^{-2} \lambda, \]
\[ \varepsilon_{3033} = +q^{-2} \lambda, \quad \varepsilon_{3030} = -q^{-2} \lambda, \]
\[ \varepsilon_{3303} = -q^{-2} \lambda, \quad \varepsilon_{+-00} = -q^{-3} \lambda, \]
\[ \varepsilon_{0030} = q^{-2} \lambda, \quad \varepsilon_{0+0-} = q^{-3} \lambda. \]
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