OPTIMALITY CONDITIONS OF FENCHEL-LAGRANGE DUALITY AND FARKAS-TYPE RESULTS FOR COMPOSITE DC INFINITE PROGRAMS

GANG LI
Department of Mathematics, Zhejiang Sci-Tech University
Hangzhou 310018, China

YINGHONG XU
Department of Mathematics, Zhejiang Sci-Tech University
Hangzhou 310018, China

ZHENHUA QIN
Department of Information Technology
Zhejiang Institute of Mechanical & Electrical Engineering
Hangzhou 310053, China

(Communicated by Jinyan Fan)

Abstract. This paper is concerned with a DC composite programs with infinite DC inequalities constraints. Without any topological assumptions and generalized increasing property, we first construct some new regularity conditions by virtue of the epigraph technique. Then we give some complete characterizations of the (stable) Fenchel-Lagrange duality and the (stable) Farkas-type assertions. As applications, corresponding assertions for the DC programs with infinite inequality constraints and the conic programs with DC composite function are also given.

1. Introduction. Let $X$ be a real locally convex Hausdorff topological vector space, $C$ be a nonempty convex subset of $X$ and $T$ be a nonempty (possibly infinite) index set. We discuss the following DC infinite programs:

$$(P_0) \inf_{x \in C} \{ f(x) - g(x) \} \text{ s.t. } h_t(x) - \varphi_t(x) \leq 0, t \in T,$$

where $f, g, h_t, \varphi_t : X \to \overline{\mathbb{R}} := \mathbb{R} \cup \{+\infty\}$ are proper, convex functions.

We say that $(P_0)$ is the semi-infinite programming (cf. [13, 24]) if $X$ is finite dimensional. In the case when $g = \varphi_t = 0$, then $(P_0)$ reduces to the infinite

2020 Mathematics Subject Classification. Primary: 90C26, 90C30; Secondary: 90C46.

Key words and phrases. DC programs, stable Farkas-type results, conical programming, stable strong duality.

*Corresponding author.

This work was supported by the Natural Science Foundation of China (11401533), the Zhejiang Provincial Natural Science Foundation of China (LY18A010030), the Scientific Research Fund of Zhejiang Provincial Education Department (19060042-F) and the Science Foundation of Zhejiang Sci-Tech University (19062150-Y).
programs:

\[(P_1) \quad \inf \{ f(x) \} \quad \text{s.t.} \quad x \in C, h_t(x) \leq 0, t \in T.\]

If \( \varphi_t = 0 \), then the problem \((P_0)\) turns out to be the DC infinite programs:

\[(P_2) \quad \inf \{ f(x) - g(x) \} \quad \text{s.t.} \quad x \in C, h_t(x) \leq 0, t \in T.\]

The above DC optimization problems have attracted extensive attention. As pointed out in [6], DC optimization problems play a significant role in optimization theory and applications, and various dual schemes are studied for different kinds of DC programs; see [3–7,9,10,12,16–22,26] and the references therein.

Another related interesting problem is the so-called composed convex optimization problem. Since many convex optimization problems generated from practical fields (such as transports and location or finance and economics) relate to composite convex functions, researches of these problems have been aroused wide interests; see [1,2,15,27] and the references therein.

In recent years, duality of DC composite optimization problems is of great interest and numerous major results have been obtained. Here, we just refer to the works of [11, 23, 25]. In [11], Fang et al. considered a DC programs with two composite functions, and presented some new regularity conditions to completely characterize the weak (zero/strong/total) dualities. For a DC optimization with a composite function, Sun and Fu [23] gave some generalized subdifferential conditions which characterize the global optimal solution under a closed qualification condition. In [25], by using the standard convexification technique, Sun et al. introduced a Fenchel-Lagrange dual problem and obtained a closedness qualification condition for the strong duality and extended Farkas lemma.

Enlightened by the discussions above, we proceed to consider problem \((P_0)\) by \( f := f_1 \circ f_2 \) and \( g := g_1 \circ g_2 \), that is, the DC problem defined by

\[(P) \quad \inf \{ (f_1 \circ f_2)(x) - (g_1 \circ g_2)(x) \} \quad \text{s.t.} \quad x \in C, h_t(x) - \varphi_t(x) \leq 0, t \in T,\]

where \( Y \) and \( Z \) are real locally convex Hausdorff topological vector spaces. \( K \subseteq Y \) and \( S \subseteq Z \) are two closed convex cones. \( f_1 : Y \to \mathbb{R}, g_1 : Z \to \mathbb{R} \) are proper, convex functions, \( f_2 : X \to Y^* \) is a proper, \( K \)-convex function, \( g_2 : X \to Z^* \) is a proper, \( S \)-convex function, and \( h_t, \varphi_t : X \to \mathbb{R} \) are proper, convex functions. More details will be given in the section that follows.

Under the additional assumption that \( f_1 \) is \( K \)-increasing, \( g_1 \) is \( S \)-increasing and \( \varphi_t(x) = 0 \), Fang and Gong [8] obtained some new regularity conditions in \( X^* \times \{0\} \times \{0\} \times \mathbb{R} \) by using the epigraph technique, and gave the complete characterizations of the dualities and the Farkas lemma. By virtue of epigraph of the conjugate functions and convexification approach, this paper aims to present some new constraint qualifications in \( X^* \times \mathbb{R} \), which provide complete characterization of the weak (strong/stable) Fenchel-Lagrange duality and the (stable) Farkas-type assertions. Similar to [11, 22], in this paper there is no any topological assumption imposed on \( C \) and the functions \( f_1, f_2, g_1, g_2, h_t, \varphi_t \), that is, \( C \) may not necessarily be a closed set, \( f_1, f_2, g_1, g_2, h_t \) and \( \varphi_t \) may not necessarily be lower semicontinuous. Moreover, \( f_1 \) is not necessarily \( K \)-increasing and \( g_1 \) is not necessarily \( S \)-increasing as shown in Example 2. Results obtained in the paper are shown to extend and improve the associated results in [8,22].
The organization of the present paper is as follows. In the next section, we recall necessary notations and give preliminary results. Subsequently, the Fenchel-Lagrange dual problem associated with \((P)\) is constructed, and two characteristic sets involved are given in Section 3. The weak (strong/stable) duality statements and the Farkas-type assertions are established in Section 4. Moreover, the corresponding results for the DC programs with infinite inequality constraints and the conic programs with DC composite function are obtained in Section 5.

2. Notations and preliminaries. In the section, we present the corresponding notations used in the sequel (cf. [3, 28]). Throughout this paper, we assume that \(X, Y\) and \(Z\) are real locally convex Hausdorff topological vector spaces, \(X^*, Y^*\) and \(Z^*\) are the dual spaces associated with \(X, Y\) and \(Z\), and their weak*-topologies are \(w^*(X^*, X), w^*(Y^*, Y)\) and \(w^*(Z^*, Z)\). We always use the notation \(\langle x^*, x \rangle\) to denote the value of \(x^* \in X^*\) at \(x \in X\), that is, \(\langle x^*, x \rangle := x^*(x)\). For a set \(D \subseteq X\), \(\text{cl}D\) and cone \(D\) denote the closure and convex cone hull of \(D\). By convention, cone \(D = \{0\}\) if \(D = \emptyset\). For a set \(W \subseteq X^*\), we denote the weak*-closure of \(W\) by \(\text{cl}W\).

The indicator function \(\delta_D : X \to \mathbb{R}\) of the nonempty set \(D\) is defined by

\[
\delta_D(x) := \begin{cases} 
0, & \text{if } x \in D, \\
+\infty, & \text{otherwise}.
\end{cases}
\]

For \(\lambda_t \in \mathbb{R}, t \in T\), we use \(\mathbb{R}^T\) to denote the space of \(\lambda = (\lambda_t)_{t \in T}\). The collection of \(\lambda = (\lambda_t)_{t \in T}\) with only finitely many \(\lambda_t \neq 0\) is denoted by \(\mathbb{R}^T(+)\) and the nonnegative cone in \(\mathbb{R}^T(+)\), that is,

\[
\mathbb{R}^T(+) := \{\lambda \in \mathbb{R}^T : \lambda_t \geq 0 \text{ for all } t \in T\}.
\]

Let supp \(\lambda := \{t \in T : \lambda_t \neq 0\}\). Then, one has

\[
\langle \lambda, u \rangle := \sum_{t \in T} \lambda_t u_t = \sum_{t \in \text{supp} \lambda} \lambda_t u_t,
\]

where \(u \in \mathbb{R}^T, \lambda \in \mathbb{R}^T(+)\).

The dual cone \(K^+\) of a closed convex cone \(K \subseteq Y\) is defined by

\[
K^+ := \{y^* \in Y^* : \langle y^*, y \rangle \geq 0 \text{ for all } y \in K\}.
\]

By \(\leq_K\), we denote the partial ordering relationship on \(Y\) induced by \(K\), i.e.,

\[
y_1 \leq_K y_2 \iff y_2 - y_1 \in K \text{ for all } y_1, y_2 \in Y.
\]

Moreover, we use \(\infty_Y\) to denote a greatest element with respect to the partial order \(\leq_K\). Let \(Y^* := Y \cup \{\infty_Y\}\). Then we have that \(y \leq_K \infty_Y\) for each \(y \in Y^*\).

For a proper function \(f : X \to \overline{\mathbb{R}}\), let dom \(f\) and epi \(f\) denote the effective domain and the epigraph of \(f\), that is, dom \(f := \{x \in X : f(x) < +\infty\}\) and epi \(f := \{(x, r) \in X \times \mathbb{R} : f(x) \leq r\}\). Then \(f\) is said to be proper, if dom \(f \neq \emptyset\).

The lower semicontinuous hull \(\text{cl} f : X \to \overline{\mathbb{R}}\) of \(f\) is defined by

\[
\text{epi}(\text{cl} f) = \text{cl}(\text{epi} f).
\]

The classical conjugate function \(f^* : X^* \to \overline{\mathbb{R}}\) of \(f\) is defined by

\[
f^*(x^*) := \sup_{x \in X} \{\langle x^*, x \rangle - f(x)\}.
\]
Then by [28, Theorem 2.3.1 (iv)], one has \( f^* = (\text{cl}f)^* \). Moreover, if \( \text{cl}f \) is proper convex, [28, Theorem 2.3.4] ensures that

\[
f^{**} = \text{cl}f.
\]  

(1)

For two proper convex functions \( f, h : X \to \mathbb{R} \) satisfying dom \( f \cap \text{dom} h \neq \emptyset \), the following inclusion holds:

\[
\text{epi} f^* + \text{epi} h^* \subseteq \text{epi}(f + h)^*.
\]

(2)

Let \( x^* \in X^* \) and \( r \in \mathbb{R} \). Then, we have

\[
\text{epi}(f + x^* + r)^* = \text{epi} f^* + (x^*, -r).
\]

(3)

Furthermore, let \( f \Box g : X \to \mathbb{R} \cup \{\pm\infty\} \) denote the infimal convolution of \( f \) and \( h \), i.e.,

\[
(f \Box g)(x) := \inf_{z \in X} \{ f(z) + g(x - z) \}
\]

for each \( x \in X \).

For a function \( f_1 : Y \to \mathbb{R} \), \( f_1 \) is said to be \( K \)-increasing if for any \( y_1, y_2 \in Y \) satisfying \( y_1 \leq_K y_2 \), one has \( f_1(y_1) \leq f_1(y_2) \).

For a function \( f_2 : X \to Y^* \), we use \( \text{dom} f_2 \) and \( \text{epi}_K f_2 \) to denote the effective domain and the \( K \)-epigraph of \( f_2 \), that is, \( \text{dom} f_2 := \{ x \in X : f_2(x) \in Y \} \) and \( \text{epi}_K f_2 := \{ (x, y) \in X \times Y : y \in f_2(x) + K \} \). Then, \( f_2 \) is said to be proper if \( \text{dom} f_2 \neq \emptyset \), and \( f_2 \) is called \( K \)-epi-closed if \( \text{epi}_K f_2 \) is closed. Moreover, \( f_2 \) is called \( K \)-convex if the following generalized inequality holds:

\[
f_2(tx_1 + (1 - t)x_2) \leq_K t f_2(x_1) + (1 - t)f_2(x_2), \quad \text{for any } x_1, x_2 \in X, t \in [0, 1].
\]

For \( g^* \in \text{dom} f_1^* \), \( (g^*, f_2) : X \to \mathbb{R} \) denotes the value of \( g^* \) at \( f_2(x) \), that is,

\[
(g^*, f_2)(x) := \begin{cases} 
(g^*, f_2(x)), & \text{if } x \in \text{dom} f_2, \\
+\infty, & \text{otherwise}.
\end{cases}
\]

Now, recall the following lemmas which will be used in the rest of the paper.

**Lemma 2.1** ([10]). Suppose that \( f, h : X \to \mathbb{R} \) are proper, convex functions such that \( \text{dom} f \cap \text{dom} h \) is nonempty, either \( f \) or \( h \) is continuous at some \( x_0 \in \text{dom} f \cap \text{dom} h \). Then, the relation \( \text{epi}(f^* \Box h^*) = \text{epi}(f + h)^* = \text{epi} f^* + \text{epi} h^* \) holds.

**Lemma 2.2** ([28]). Assume that \( \{p_i\}_{i \in I} \) is a family of functions, where \( I \) is an index set. Then, we have

(i) \( \text{epi}(\sup_{i \in I} p_i) = \bigcap_{i \in I} \text{epi} p_i \).

(ii) \( (\inf_{i \in I} p_i)^* = \sup_{i \in I} p_i^* \); consequently, \( \text{epi}(\inf_{i \in I} p_i)^* = \bigcap_{i \in I} \text{epi} p_i^* \).

3. **Fenchel-Lagrange dual problems.** Let \( \infty_Y \) and \( \infty_Z \) denote the greatest elements with respect to \( \leq_K \) and \( \leq_S \) (the partial orders imposed on \( Y \) and \( Z \)). Denote \( Y^* := Y \cup \{\infty_Y\} \) and \( Z^* := Z \cup \{\infty_Z\} \). The solution set \( A \) of the constraint system \( \{ x \in C : h_1(x) - \varphi_1(x) \leq 0, t \in T \} \) is defined by

\[
A := \{ x \in C : h_1(x) - \varphi_1(x) \leq 0, \forall t \in T \}.
\]

(4)

We assume that \( h_1, \varphi_1 : X \to \mathbb{R} \), \( f_1 : Y \to \mathbb{R} \) and \( g_1 : Z \to \mathbb{R} \) are proper, convex functions, \( f_2 : X \to Y^* \) is a proper, \( K \)-convex function, \( g_2 : X \to Z^* \) is a proper, \( S \)-convex function satisfying \( \text{dom}(f_1 \circ f_2) \cap A \cap \text{dom} (g_1 \circ g_2) \neq \emptyset \), where \( f_1(\infty_Y) = +\infty \) and \( g_1(\infty_Z) = +\infty \).

Following [28, Page 39], in this paper we adopt the following operations:

\[
(+\infty) - (+\infty) = (+\infty) + (-\infty) = +\infty, \quad 0 \cdot (+\infty) = +\infty, \quad 0 \cdot (-\infty) = 0.
\]
By the similar approach in [8], we construct the Fenchel-Lagrange dual problem associated with \((P)\). In the case when \(f_1\) is \(K\)-increasing, \(g_1\) is \(S\)-increasing and lower semicontinuous, \(g_2\) is \(S\)-epi-closed and \(\varphi_t, t \in T\) are lower semicontinuous, we have \(\varphi_t = \varphi_t^*(\text{cf.}(1))\). Then \((P)\) can be reformulated as follows:
\[
(P) \quad \inf_{\mu \in H^*} \inf_{x \in \mathbb{R}^T} \left\{ g_1^*(\alpha) + (\alpha g_2)^*(\beta) + (f_1 \circ f_2)(x) - \langle \beta, x \rangle \right\}
\text{s.t.} \quad h_t(x) + \varphi_t^*(\gamma_t) - \langle \gamma_t, x \rangle \leq 0, \ t \in T, x \in C,
\]
where \(\mu := (\alpha, \beta, \gamma)\) with \(\gamma = (\gamma_t)_{t \in T}\) and
\[
H^* := \text{dom } g_1^* \times \text{dom } g_2^* \times \prod_{t \in T} \text{dom } \varphi_t^*.
\tag{5}
\]
Note that for each \(\mu \in H^*\), the subproblem
\[
(P^\mu) \quad \inf_{x \in \mathbb{R}^T} \left\{ g_1^*(\alpha) + (\alpha g_2)^*(\beta) + (f_1 \circ f_2)(x) - \langle \beta, x \rangle \right\}
\text{s.t.} \quad h_t(x) + \varphi_t^*(\gamma_t) - \langle \gamma_t, x \rangle \leq 0, \ t \in T, x \in C,
\]
is convex. Therefore, we have the corresponding Lagrange dual problem:
\[
(D^\mu) \quad \sup_{\lambda \in \mathbb{R}^T_+} \inf_{x \in \mathbb{R}^T} \left\{ g_1^*(\alpha) + (\alpha g_2)^*(\beta) + \sum_{t \in \text{supp } \lambda} \lambda_t \left( h_t(x) + \varphi_t^*(\gamma_t) - \langle \gamma_t, x \rangle \right) \right\}
\]
i.e.,
\[
(D^\mu) \quad \sup_{\lambda \in \mathbb{R}^T_+} \inf_{x \in \mathbb{R}^T} \left\{ g_1^*(\alpha) + (\alpha g_2)^*(\beta) + \sum_{t \in \text{supp } \lambda} \lambda_t \varphi_t^*(\gamma_t) + (f_1 \circ f_2)(x) + \sum_{t \in \text{supp } \lambda} \lambda_t h_t(x) - \langle \beta + \sum_{t \in \text{supp } \lambda} \lambda_t \gamma_t, x \rangle \right\}.
\tag{6}
\]
Furthermore, we can reformulate the corresponding Fenchel-Lagrange dual problem of \((P^\mu)\), that is
\[
(D^\mu) \quad \sup_{w \in W^*} \Phi(\mu, w, \lambda),
\]
where \(w := (u, v, s)\) with \(s = (s_t)_{t \in T}\),
\[
W^* := \text{dom } f_1^* \times \text{dom } f_2^* \times \prod_{t \in T} \text{dom } h_t^*,
\tag{6}
\]
and \(\Phi : H^* \times W^* \times \mathbb{R}^T \rightarrow \mathbb{R}\) is expressed as follows,
\[
\Phi(\mu, w, \lambda) := g_1^*(\alpha) + (\alpha g_2)^*(\beta) + \sum_{t \in \text{supp } \lambda} \lambda_t \left( \varphi_t^*(\gamma_t) - h_t^*(s_t) \right) - f_1^*(u) - (uf_2)^*(v) - \delta_C^*(\beta + \sum_{t \in \text{supp } \lambda} \lambda_t \gamma_t - s_t) - v.
\tag{7}
\]
Thus, motivated by this reformulation above, the dual problem associated with \((P)\) can be defined by
\[
(D) \quad \inf_{\mu \in H^*} \sup_{w \in W^*} \Phi(\mu, w, \lambda).
\tag{8}
\]
Take \(p \in X^*\). For the corresponding linearly perturbed optimization problem
\[
(P^p) \quad \inf_{x \in A} \left\{ (f_1 \circ f_2)(x) - (g_1 \circ g_2)(x) - (p, x) \right\},
\tag{9}
\]
we can also get the corresponding dual problem

\[
(D^p) \quad \inf_{\mu \in H^*} \sup_{w \in W^*, \lambda \in R^T_+} \Phi^p(\mu, w, \lambda),
\]

where

\[
\Phi^p(\mu, w, \lambda) := g_1^*(\alpha) + (\alpha g_2)^*(\beta) + \sum_{t \in \text{supp } \lambda} \lambda_t (\varphi_t^*(\gamma_t) - h_t^*(s_t)) - f_2^*(u) - (uf_2)^*(v) - \delta_C^*(\beta + p + \sum_{t \in \text{supp } \lambda} \lambda_t (\gamma_t - s_t) - v).
\]

As usual, we use \(v(P), v(P^p), v(D), v(D^p)\) and \(v(D^p)\) to denote the optimal values of \((P), (P^p), (D), (D^p)\) and \((D^p)\).

**Definition 3.1.** We say that

(i) the weak duality between \((P)\) and \((D)\) holds, if and only if \(v(P) \geq v(D)\);
(ii) the strong duality between \((P)\) and \((D)\) holds, if and only if \(v(P) = v(D)\) and for each \(\mu \in H^*\) satisfying \(v(D^\mu) = v(D)\), the dual problem \((D^p)\) has an optimal solution;
(iii) the stable weak (strong) duality between \((P)\) and \((D)\) holds, if and only if for each \(p \in X^*\), the weak (strong) duality between \((P^p)\) and \((D^p)\) holds.

**Remark 1.** Different from convex case, in general the weak duality between \((P)\) and \((D)\) does not necessarily hold, which will be shown in Example 1.

In order to give complete characterization of the dualities and the Farkas-type results, we will present some new qualifications. We first denote the characteristic sets in \(X^* \times \mathbb{R}\) as follows:

\[
\mathcal{K} := \text{epi}(f_1 \circ f_2 - g_1 \circ g_2 + \delta_A)^*.
\]

and

\[
\mathcal{J} := \bigcap_{\mu \in H^*} \left( \bigcup_{u \in \text{dom } f_1^*} \left( \text{epi}(uf_2)^* + (0, f_1^*(u)) \right) + \text{epi} \delta_C^* \right.

\left. + \text{cone} \left( \bigcup_{t \in T} \left( \text{epi} h_t^* - (\gamma_t, \varphi_t^*(\gamma_t)) \right) \right) - \left( \beta, g_1^*(\alpha) + (\alpha g_2)^*(\beta) \right) \right).
\]

**Remark 2.** (a) If \(f_2\) is the identity operator on \(X\), i.e., \(f_2 = \text{Id}_X\), we have that

\[
(uf_2)^*(v) = \begin{cases} 0, & \text{if } v = u, \\ +\infty, & \text{otherwise}, \end{cases}
\]

where \(u \in \text{dom } f_1^*\). This follows that \(\text{epi}(uf_2)^* = \{u\} \times [0, +\infty)\). Hence,

\[
\bigcup_{u \in \text{dom } f_1^*} \left( \text{epi}(uf_2)^* + (0, f_1^*(u)) \right) = \text{epi } f_1^*.
\]

(b) It is note that, the inclusions between \(\mathcal{J}\) and \(\mathcal{K}\) do not necessarily hold in generally.

**Example 1.** Let \(X = Y = Z = C := \mathbb{R}\), \(K = S = T := [0, +\infty)\), \(f_1 := \delta_{[0, +\infty)}\), \(f_2 := g_2 := \text{Id}_\mathbb{R}\), \(h_t(x) := -tx\), \(\varphi_t := \delta_{[-\infty, 0]}\) and

\[
g_1(x) := \begin{cases} 0, & \text{if } x > 0, \\ 1, & \text{if } x = 0, \\ +\infty, & \text{otherwise}. \end{cases}
\]
Then, \( X^* = Y^* = Z^* = \mathbb{R}, K^+ = S^+ = [0, +\infty), f_2 \) is a proper, \( K \)-convex function, \( g_2 \) is a proper, \( S \)-convex function and \( f_1, g_1, h_t, \varphi_t \) are proper, convex functions.

By simple calculation, \( \mathcal{A} = \{ x \in \mathbb{R} : h_t(x) - \varphi_t(x) \leq 0 \} = \{ 0 \}, \)

\[
(f_1 \circ f_2 - g_1 \circ g_2 + \delta_\mathcal{A})(x) = \begin{cases} 
-1, & \text{if } x = 0, \\
+\infty, & \text{otherwise},
\end{cases}
\]  

and \((f_1 \circ f_2 - g_1 \circ g_2 + \delta_\mathcal{A})^*(x^*) = 1. \) Hence

\[
\mathcal{K} = \mathbb{R} \times [1, +\infty).
\]  

By (12), for each \( u \in \text{dom } f_1^*, \alpha \in \text{dom } g_1^*, \)

\[
(u f_2)^*(v) = \begin{cases} 
0, & \text{if } v = u, \\
+\infty, & \text{otherwise},
\end{cases}
\]  

\[
(\alpha g_2)^*(\beta) = \begin{cases} 
0, & \text{if } \beta = \alpha, \\
+\infty, & \text{otherwise}.
\end{cases}
\]  

By (13), we have that

\[
\bigcup_{u \in \text{dom } f_1^*} \left( \text{epi}(u f_2)^* + (0, f_1^*(u)) \right) = \text{epi } f_1^*.
\]  

Moreover, since

\[
f_1^* = g_1^* = \delta_{(-\infty, 0]}, \varphi_t^* = \delta_{[0, +\infty)}, h_t^* = \delta_{[-t]}, \delta_C^* = \delta_0,
\]  

it follows that

\[
\text{epi } f_1^* = (-\infty, 0] \times [0, +\infty), \text{epi } \delta_C^* = \{ 0 \} \times [0, +\infty), \text{epi } h_t^* = \{-t\} \times [0, +\infty),
\]

and for each \( \gamma_t \geq 0, \)

\[
\text{cone} \left( \bigcup_{t \in T} \left( \text{epi } h_t^* - (\gamma_t, \varphi_t^*(\gamma_t)) \right) \right) = (-\infty, 0] \times [0, +\infty).
\]

This combined with (17) implies that

\[
\mathcal{J} = \bigcap_{\mu \in H^*} \left( \bigcup_{u \in \text{dom } f_1^*} \left( \text{epi}(u f_2)^* + (0, f_1^*(u)) \right) + \text{epi } \delta_C^* 
+ \text{cone} \left( \bigcup_{t \in T} \left( \text{epi } h_t^* - (\gamma_t, \varphi_t^*(\gamma_t)) \right) \right) - (\beta, g_1^*(\alpha) + (\alpha g_2)^*(\beta)) \right)
= \bigcap_{\beta \leq 0} ((-\infty, 0] \times [0, +\infty) + \{ 0 \} \times [0, +\infty) + (-\infty, 0] \times [0, +\infty) - (\beta, 0))
= (-\infty, 0] \times [0, +\infty),
\]

which, together with (15), implies that \( \mathcal{J} \not\subseteq \mathcal{K} \) and \( \mathcal{K} \not\subseteq \mathcal{J} \). Furthermore, by (14), (16) and (18), we have

\[
v(P) = -1 < v(D) = 0.
\]

Therefore, the weak duality fails to hold.

Finally, in the section, we suppose that \( g_1 \circ g_2 \) is convex, \( \text{cl}(g_1 \circ g_2) \) and \( \text{cl } \varphi_t \) are proper for all \( t \in T \). Let \( \tilde{f} := f_1 \circ f_2 - \text{cl}(g_1 \circ g_2), \tilde{h}_t := h_t - \text{cl } \varphi_t \). Then \( \tilde{f} \) and \( \tilde{h}_t \) are proper.

\[
(f_1 \circ f_2)(x) - (g_1 \circ g_2)(x) \leq \tilde{f}(x), \ h_t(x) - \varphi_t(x) \leq \tilde{h}_t(x).
\]  

The following lemma shows the relationship between \( \mathcal{J} \) and \( \text{epi } \tilde{f}^* + \text{epi } \delta_C^* + \text{cone} \left( \bigcup_{t \in T} \text{epi } \tilde{h}_t^* \right). \)
Lemma 3.2. Suppose that \((g_1 \circ g_2)^*(\beta) = \min_{\alpha \in \text{dom} g_1} \{ g_1^*(\alpha) + (\alpha g_2)^*(\beta) \}\) for all \(\beta \in \text{dom}(g_1 \circ g_2)^*\), then the following inclusion holds:

\[
\mathcal{J} \subseteq \text{epi} \tilde{f}^* + \text{epi} \delta_C^* + \text{cone} \left( \bigcup_{t \in T} \text{epi} \tilde{h}_t^* \right). \tag{20}
\]

Proof. Since \(\text{cl}(g_1 \circ g_2)\) and \(\text{cl} \varphi_t, t \in T\) are proper convex, and lower semicontinuous, it follows from (1) that

\[
\text{cl}(g_1 \circ g_2) = (g_1 \circ g_2)^{**}, \text{cl} \varphi_t = \varphi_t^*, t \in T.
\]

By Lemma 2.2 (ii), we have

\[
\tilde{f}^* = \left( \inf (f_1 \circ f_2 - \beta + (g_1 \circ g_2)^*(\beta)) \right)^* = \sup \left( f_1 \circ f_2 - \beta + (g_1 \circ g_2)^*(\beta) \right)^*.
\]

According to (3) and Lemma 2.2 (i), we get that

\[
\text{epi} \tilde{f}^* = \bigcap_{\beta \in \text{dom}(g_1 \circ g_2)^*} \text{epi} \left( f_1 \circ f_2 - \beta + (g_1 \circ g_2)^*(\beta) \right)^* = \bigcap_{\beta \in \text{dom}(g_1 \circ g_2)^*} \left( \text{epi}(f_1 \circ f_2)^* - (\beta, (g_1 \circ g_2)^*(\beta)) \right). \tag{21}
\]

In the same way, we have

\[
\text{epi} \tilde{h}_t^* = \bigcap_{\gamma \in \text{dom} \varphi_t^*} \text{epi} \left( h_t - \gamma_t + \varphi_t^*(\gamma_t) \right)^* = \bigcap_{\gamma \in \text{dom} \varphi_t^*} \left( \text{epi} h_t^* - (\gamma_t, \varphi_t^*(\gamma_t)) \right). \tag{22}
\]

For any \(\gamma = (\gamma_t)_{t \in T} \in \Pi_{t \in T} \text{dom} \varphi_t^*\), one has

\[
\bigcap_{\gamma \in \Pi_{t \in T} \text{dom} \varphi_t^*} \text{cone} \left( \bigcup_{t \in T} \text{epi} h_t^* - (\gamma_t, \varphi_t^*(\gamma_t)) \right) \subseteq \text{cone} \left( \bigcup_{t \in T} \text{epi} h_t^* - (\gamma_t, \varphi_t^*(\gamma_t)) \right). \tag{23}
\]

Note that \(\gamma = (\gamma_t)_{t \in T} \in \Pi_{t \in T} \text{dom} \varphi_t^*\) is arbitrary. Hence,

\[
\bigcap_{\gamma \in \Pi_{t \in T} \text{dom} \varphi_t^*} \text{cone} \left( \bigcup_{t \in T} \text{epi} h_t^* - (\gamma_t, \varphi_t^*(\gamma_t)) \right) \subseteq \text{cone} \left( \bigcup_{t \in T} \text{epi} h_t^* \right), \tag{23}
\]

where the equality follows from (22).

We claim that

\[
\bigcup_{u \in \text{dom} f_1^*} \left( \text{epi}(u f_2)^* + (0, f_1^*(u)) \right) \subseteq \text{epi}(f_1 \circ f_2)^* \tag{24}
\]
Remark 3. Suppose that \((x^*, r) \in \bigcup_{u \in \text{dom} f_1^*} \left( \text{epi}(u f_2)^* + (0, f_1^*(u)) \right) \). Then, there exists \(u \in \text{dom} f_1^* \) satisfying \((x^*, r) = (x_1^*, r_1) + (0, r_2)\), where \((x_1^*, r_1) \in \text{epi}(u f_2)^*\), \(r_2 = f_1^*(u)\). We have that
\[
x^* = x_1^*, \quad r = r_1 + r_2.
\]
Thus,
\[
(x_1^*, r_1) \in \text{epi}(u f_2)^* \quad \Rightarrow \quad r_1 \geq \langle x_1^*, x \rangle - (u f_2)(x), \quad \forall x \in X,
\]
\[
r_2 = f_1^*(u) \quad \Rightarrow \quad r_2 \geq \langle u, y \rangle - f_1(y), \quad \forall y \in Y.
\]
This follows that
\[
r = r_1 + r_2 \geq \langle x_1^*, x \rangle - (u f_2)(x) + \langle u, y \rangle - f_1(y).
\]
By taking \(y = f_2(x)\), (26) can be simplified as follows
\[
r \geq \langle x^*, x \rangle - (f_1 \circ f_2)(x),
\]
which implies that \((x^*, r) \in \text{epi}(f_1 \circ f_2)^*\). So, (24) holds.

To prove (25), for each \((\alpha, \beta) \in \text{dom} g_1^* \times \text{dom} g_2^*\), let \(r_3 = (g_1^*(\alpha) + (\alpha g_2)^*(\beta))\). Since \((g_1 \circ g_2)^*(\beta) = \min_{\alpha \in \text{dom} g_1^*} \{ g_1^*(\alpha) + (\alpha g_2)^*(\beta) \} \) for all \(\beta \in \text{dom}(g_1 \circ g_2)^*\), one has that
\[
r_3 \geq (g_1 \circ g_2)^*(\beta).
\]
Hence,
\[
\bigcap_{\alpha \in \text{dom} g_1^*} \left( \text{epi}(f_1 \circ f_2)^* - (\beta, g_1^*(\alpha) + (\alpha g_2)^*(\beta)) \right)
= \bigcap_{\alpha \in \text{dom} g_1^*} \left( \text{epi}(f_1 \circ f_2)^* - (0, r_3 - (g_1 \circ g_2)^*(\beta)) - (\beta, (g_1 \circ g_2)^*(\beta)) \right)
= \bigcap_{\beta \in \text{dom}(g_1 \circ g_2)^*} \left( \text{epi}(f_1 \circ f_2)^* - (\beta, (g_1 \circ g_2)^*(\beta)) \right),
\]
which together with (21) implies that (25) holds. By (23), (24) and (25), we have that (20) holds, and this lemma is proved.

**Remark 3.** Suppose that \((g_1 \circ g_2)^*(\beta) = \min_{\alpha \in \text{dom} g_1^*} \{ g_1^*(\alpha) + (\alpha g_2)^*(\beta) \} \) for all \(\beta \in \text{dom}(g_1 \circ g_2)^*\), the equation (3.5) in [10] and Lemma 3.2 ensure that \(J \subseteq \text{epi}(\tilde{f} + \delta_{\mathcal{A}^d})^*\) holds, where \(\mathcal{A}^d := \{ x \in C : \tilde{h}_t(x) \leq 0, t \in T \}\). By (4) and (19), we have \(\mathcal{A}^d \subseteq \mathcal{A}\). Moreover, if \(g_1 \circ g_2\) and \(\varphi_t\) are lower semicontinuous, one has that \(\tilde{f} = f_1 \circ f_2 - g_1 \circ g_2\) and \(\mathcal{A}^d = \mathcal{A}\). This follows that \(J \subseteq K\).

4. **Stable strong Fenchel–Lagrange duality.** Throughout this section, the notations \(f_1, f_2, g_1, g_2, h_t, \varphi_t, C, \mathcal{A}, \mathcal{K}\) and \(J\) are as explained at Section 3. The following lemma shows the relationship between \(\mathcal{K}, J, v(P^p)\) and \(v(D^p)\), which plays a key tool in the proofs of duality results and Farkas-type assertions.

**Lemma 4.1.** Let \(p \in X^*, r \in \mathbb{R}\). Then, the following assertions are true.

(i) \((p, r) \in \mathcal{K}\) if and only if \(v(P^p) \geq -r\).
(ii) \((p,r) \in \mathcal{J}\) if and only if \(v(D^p) \geq -r\) and for any \(\mu \in H^*\), there exists 
\((w,\lambda) \in W^* \times \mathbb{R}_+^{(T)}\) satisfying 
\[
\Phi^p(\mu, w, \lambda) \geq -r, 
\]
where \(\mu = (\alpha, \beta, \gamma)\) with \(\gamma = (\gamma_t)_{t \in T}\), \(w = (u, v, s)\) with \(s = (s_t)_{t \in T}, H^*, W^*\) 
and \(\Phi^p\) are defined in (5), (6) and (11), respectively.

**Proof.** (i) By the expression of problem \((P^p)\) (cf. (9)), one has 
\[
v(P^p) = -(f_1 \circ f_2 - g_1 \circ g_2 + \delta_\lambda)^*(p).
\]

Hence, the result holds naturally.

(ii) \(\Rightarrow\) Let \((p, r) \in \mathcal{J}\). Then, for each \(\mu = (\alpha, \beta, \gamma) \in H^*\) with \(\gamma = (\gamma_t^*)_{t \in T}\), 
there exists \(u \in \text{dom} f_1^*\) such that 
\[
(p, r) \in \text{epi}(uf_2)^* + (0, f_1^*(u)) + \text{epi} \delta_C^* + \text{cone} \left( \bigcup_{t \in T} \left( \text{epi} h_t^* - (\gamma_t^*, \varphi_t^*(\gamma_t^*)) \right) \right) - (\beta, g_1^*(\alpha) + (\alpha g_2)^*(\beta)).
\]

Then, there exist \(\lambda \in \mathbb{R}_+^{(T)}\), \((v, r_1) \in \text{epi}(uf_2)^*\), \((x_2^*, r_2) \in \text{epi} \delta_C^*, (s_t, \beta_t) \in \text{epi} h_t^*\) 
with \(t \in \text{supp} \lambda\) such that 
\[
(p, r) = (v, r_1) + (0, f_1^*(u)) + (x_2^*, r_2) + \sum_{t \in \text{supp} \lambda} \lambda_t \left( (s_t, \beta_t) - (\gamma_t^*, \varphi_t^*(\gamma_t^*)) \right) - (\beta, g_1^*(\alpha) + (\alpha g_2)^*(\beta)),
\]
which shows that 
\[
p = v + x_2^* + \sum_{t \in \text{supp} \lambda} \lambda_t (s_t - \gamma_t) - \beta, \tag{28}
\]

and 
\[
r + g_1^*(\alpha) + (\alpha g_2)^*(\beta) + \sum_{t \in \text{supp} \lambda} \lambda_t \varphi_t^*(\gamma_t) = r_1 + f_1^*(u) + r_2 + \sum_{t \in \text{supp} \lambda} \lambda_t \beta_t. \tag{29}
\]

Since \((uf_2)^*(v) \leq r_1, \delta_C^*(x_2^*) \leq r_2, h_t^*(s_t) \leq \beta_t\), it follows from (28) and (29) that 
\[
r + g_1^*(\alpha) + (\alpha g_2)^*(\beta) + \sum_{t \in \text{supp} \lambda} \lambda_t \varphi_t^*(\gamma_t) 
\geq (uf_2)^*(v) + f_1^*(u) + \delta_C^*(x_2^*) + \sum_{t \in \text{supp} \lambda} \lambda_t h_t^*(s_t)
\]
\[
=(uf_2)^*(v) + f_1^*(u) + \delta_C^* \left( \beta + p - v + \sum_{t \in \text{supp} \lambda} \lambda_t (\gamma_t - s_t) \right) + \sum_{t \in \text{supp} \lambda} \lambda_t h_t^*(s_t).
\]

Then 
\[
\begin{align*}
g_1^*(\alpha) + (\alpha g_2)^*(\beta) - f_1^*(u) - (uf_2)^*(v) 
&+ \sum_{t \in \text{supp} \lambda} \lambda_t (\varphi_t^*(\gamma_t) - h_t^*(s_t)) - \delta_C^* \left( \beta + p + \sum_{t \in \text{supp} \lambda} \lambda_t (\gamma_t - s_t) - v \right) 
\geq -r.
\end{align*}
\]

Hence, \(v(D^p) \geq -r\) and for each \(\mu \in H^*\), there exists \((w, \lambda) \in W^* \times \mathbb{R}_+^{(T)}\) satisfying 
(27).
Theorem 4.2. (stable) weak duality.

Proof. We just only prove (i). Same argument of (i) guarantees that (ii) holds.

\[ r + g_1^*(\alpha) + (\alpha g_2)^*(\beta) + \sum_{t \in \text{supp} \lambda} \lambda_t (\varphi_t^*(\gamma_t) - h_t^*(s_t)) - f_1^*(u) - (u f_2)^*(v) \geq \delta_C^*(\beta + p + \sum_{t \in \text{supp} \lambda} \lambda_t (\gamma_t - s_t) - v), \]

which implies

\[ (\beta + p + \sum_{t \in \text{supp} \lambda} \lambda_t (\gamma_t - s_t) - v, r + g_1^*(\alpha) + (\alpha g_2)^*(\beta) - f_1^*(u) - (u f_2)^*(v) + \sum_{t \in \text{supp} \lambda} \lambda_t (\varphi_t^*(\gamma_t) - h_t^*(s_t))) \in \text{epi} \delta_C^*. \]  

Furthermore, we can get that

\[ p = v + \sum_{t \in \text{supp} \lambda} \lambda_t (s_t - \gamma_t) - \beta + (\beta + p + \sum_{t \in \text{supp} \lambda} \lambda_t (\gamma_t - s_t) - v) \]  

and

\[ r = f_1^*(u) + (u f_2)^*(v) + \sum_{t \in \text{supp} \lambda} \lambda_t (h_t^*(s_t) - \varphi_t^*(\gamma_t)) - g_1^*(\alpha) - (\alpha g_2)^*(\beta) + (r + g_1^*(\alpha) + (\alpha g_2)^*(\beta) - f_1^*(u) - (u f_2)^*(v) + \sum_{t \in \text{supp} \lambda} \lambda_t (\varphi_t^*(\gamma_t) - h_t^*(s_t))). \]

By (30), (31) and (32), we get

\[ (p, r) \in \bigcup_{u \in \text{dom} f_1^*} \left( \text{epi}(u f_2)^* + (0, f_1^*(u)) \right) + \text{epi} \delta_C^* \]

\[ + \text{cone} \left( \bigcup_{t \in T} \left( \text{epi} h_t^* - (\gamma_t, \varphi_t^*(\gamma_t)) \right) \right) - (\beta, g_1^*(\alpha) + (\alpha g_2)^*(\beta)). \]

Noting that \( \mu \in H^* \) is arbitrary, \((p, r) \in J\), and this completes the proof. \( \square \)

Based on the inclusions is arbitrary, \((p, r) \in J\), and this completes the proof.

Theorem 4.2. The following assertions hold.

(i) The stable weak duality between \((P)\) and \((D)\) holds if and only if \(J \subseteq K\).

(ii) The weak duality between \((P)\) and \((D)\) holds if and only if \(J \cap (\{0\} \times \mathbb{R}) \subseteq K \cap (\{0\} \times \mathbb{R})\).

Proof. We just only prove (i). Same argument of (i) guarantees that (ii) holds.

\[ \Rightarrow \] Assume that the stable weak duality holds, i.e., \(v(P^p) \geq v(D^p), p \in X^*\). Let \((p, r) \in J\). Then Lemma 4.1(ii) ensures \(v(D^p) \geq -r\). By the stable weak duality, one has that \(v(P^p) \geq -r\). By Lemma 4.1(i), \((p, r) \in K\). This follows that \(J \subseteq K\).

\[ \Leftarrow \] Suppose \(J \subseteq K\). If \(v(P^p) = +\infty\), the assertion holds naturally. So we assume that \(v(P^p) < +\infty\). To show \(v(P^p) \geq v(D^p)\), we prove by contradiction, assuming that \(v(P^p) < v(D^p)\). Consequently, there is a real number \(r \in \mathbb{R}\) satisfying...
\( v(P^p) < -r < v(D^p) \). It follows from the definition of \( v(D^p) \) (cf. (10) and (11)) that for each \( \mu \in H^* \), there is a pair of \((w, \lambda) \in W^* \times \mathbb{R}_+^{(T)} \) satisfying (27). By Lemma 4.1(ii), \((p, r) \in \mathcal{J} \). The hypothesis \( \mathcal{J} \subseteq \mathcal{K} \) guarantees \((p, r) \in \mathcal{K} \). By Lemma 4.1(i), one has that \( v(P^p) \geq -r \), which yields a contradiction with \( v(P^p) < -r \). Hence, \( v(P^p) \geq v(D^p) \) and this theorem is proved. \( \square \)

The following theorem characterizes completely the (stable) strong duality.

**Theorem 4.3.** The following statements hold.

(i) The stable strong duality between \((P)\) and \((D)\) holds if and only if \( \mathcal{J} = \mathcal{K} \).

(ii) The strong duality between \((P)\) and \((D)\) holds if and only if \( \mathcal{J} \cap \{(0) \times \mathbb{R} \} = \mathcal{K} \cap \{(0) \times \mathbb{R} \} \).

**Proof.** We only prove (i). Same argument of (i) guarantees that (ii) holds.

\[ \Rightarrow \] Assume that the stable strong duality holds, that is \( v(P^p) = v(D^p), p \in X^* \).

By Theorem 4.2(i) and the stable strong duality, one has that

\[ \mathcal{J} \subseteq \mathcal{K}. \] (33)

Thus, we only need to show that \( \mathcal{K} \subseteq \mathcal{J} \). Let \((p, r) \in \mathcal{K} \). Lemma 4.1 (i) shows that \( v(P^p) \geq -r \). By the stable strong duality, one has \( v(D^p) \geq -r \) and there exists \((w, \lambda) \in W^* \times \mathbb{R}_+^{(T)} \) satisfying (27) for each \( \mu \in H^* \). This combined with Lemma 4.1(ii) shows that \((p, r) \in \mathcal{J} \). Consequently, \( \mathcal{K} \subseteq \mathcal{J} \). By (33), we have \( \mathcal{J} = \mathcal{K} \).

\[ \Leftarrow \] Suppose that \( \mathcal{J} = \mathcal{K} \) holds. This implies that \( \mathcal{J} \subseteq \mathcal{K} \). By Theorem 4.2(i), the stable weak duality holds. If \( v(P^p) = -\infty \), the assertion holds naturally. We just prove \( v(P^p) \leq v(D^p) \). Let \( r = v(P^p) \in \mathbb{R} \). Then, \((p, -r) \in \mathcal{K} \) thanks to Lemma 4.1(i). This follows from the hypothesis that \((p, -r) \in \mathcal{J} \). By Lemma 4.1(ii), one sees that \( v(D^p) \geq r \), and for any \( \mu \in H^* \), there exists a pair of \((w, \lambda) \in W^* \times \mathbb{R}_+^{(T)} \) satisfying

\[ \Phi^p(\mu, w, \lambda) \geq r, \] (34)

which, combined with the weak duality, shows that \( v(D^p) = v(P^p) = r \) and for any \( \mu \in H^* \), there exists \((w, \lambda) \in W^* \times \mathbb{R}_+^{(T)} \) satisfying (34). Consequently, the stable strong duality holds, and the theorem is proved. \( \square \)

The following theorem provide complete characterizations of the corresponding stable Farkas-type results.

**Theorem 4.4.** \( \mathcal{J} = \mathcal{K} \) holds if and only if for each \( a \in \mathbb{R} \) and \( p \in X^* \), the following assertions are equivalent:

(i) \( x \in A \implies (f_1 \circ f_2)(x) - (g_1 \circ g_2)(x) \geq \langle p, x \rangle + a \).

(ii) For each \( \mu \in H^* \), there exists \((w, \lambda) \in W^* \times \mathbb{R}_+^{(T)} \) such that

\[ \Phi^p(\mu, w, \lambda) \geq a, \] (35)

where \( \mu = (\alpha, \beta, (\gamma_t)_{t \in T}), w = (u, v, (s_t)_{t \in T}), H^*, W^* \) and \( \Phi^p \) are defined in (5),(6) and (11), respectively.

**Proof.** \( \Rightarrow \) Suppose that \( \mathcal{J} = \mathcal{K} \) holds. Then, Theorem 4.3(i) guarantees that the stable strong duality holds. If (i) holds, then \( v(P^p) \geq a \). The stable strong duality implies that \( v(P^p) = v(D^p) \geq a \). Hence, (ii) holds. Conversely, if (ii) holds, (35) shows that

\[ \inf_{\mu \in H^*} \sup_{w \in W^*, \lambda \in \mathbb{R}_+^{(T)}} \Phi^p(\mu, w, \lambda) \geq a. \]
which shows that \( v(D^p) \geq a \). Therefore \( v(P^p) \geq a \) thanks to the stable strong duality. Consequently, (i) holds.

[\( \Rightarrow \)] Assume that (i) and (ii) are equivalent. Then by Lemma 4.1, one has \( J = \mathcal{K} \). This completes the proof.

By the similar discussion, we get the corresponding Farkas-type results.

**Theorem 4.5.** \( J \cap \{(0) \times \mathbb{R}\} = \mathcal{K} \cap \{(0) \times \mathbb{R}\} \) holds if and only if for each \( a \in \mathbb{R} \), the following assertions are equivalent:

(i) \( x \in A \implies (f_1 \circ t_2)(x) - (g_1 \circ t_2)(x) \geq a \).

(ii) For each \( \mu \in H^* \), there exists \( (w, \lambda) \in W^* \times \mathbb{R}^+_+ \) such that \( \Phi(\mu, w, \lambda) \geq a \), where \( \Phi \) is defined by (7).

5. **Special cases.**

5.1. **The case when** \( f_2 = g_2 = \text{Id}_X \), \( \varphi_t = 0 \). Let \( X = Y = Z \), \( f_2 = g_2 = \text{Id}_X \) and \( \varphi_t = 0 \), \( t \in T \). Then problem \((P)\) is simplified into the following DC programming problem:

\[
(P_1) \quad \underset{x \in A_1}{\inf} \{ f_1(x) - g_1(x) \}
\]

where \( A_1 := \{ x \in C : h_t(x) \leq 0, t \in T \} \), By (12), the dual problem \((D)\) is turned into

\[
(D_1) \quad \underset{\alpha \in H^*_1}{\inf} \sup_{w \in W^*_1, \lambda \in \mathbb{R}^+_+^{(T)}} \Phi_1(\alpha, w, \lambda),
\]

where \( w := (u, s), s = (s_t)_{t \in T}, H^*_1 := \text{dom} g_1^*, W^*_1 := \text{dom} f_1 \times \prod_{t \in T} \text{dom} h_t^* \) and

\[
\Phi_1(\alpha, w, \lambda) := g_1^*(\alpha) - f_1^*(u) - \sum_{t \in \supp \lambda} \lambda_t h_t^*(s_t) - \delta_C^*(\alpha - \sum_{t \in \supp \lambda} \lambda_t s_t - u).
\]

Take \( p \in X^* \). Consider the DC optimization problem with a linear perturbation:

\[
(P^p) \quad \underset{x \in A_1}{\inf} \{ f_1(x) - g_1(x) - \langle p, x \rangle \}.
\]

Similarly, we can get the associated dual problem

\[
(D^p) \quad \underset{\alpha \in H^*_1}{\inf} \sup_{w \in W^*_1, \lambda \in \mathbb{R}^+_+^{(T)}} \Phi^p_1(\alpha, w, \lambda),
\]

where

\[
\Phi^p_1(\alpha, w, \lambda) := g_1^*(\alpha) - f_1^*(u) - \sum_{t \in \supp \lambda} \lambda_t h_t^*(s_t) - \delta_C^*(\alpha + p - \sum_{t \in \supp \lambda} \lambda_t s_t - u).
\]

We use \( \mathcal{K}_1 \) to denote the characteristic set of \((P_1)\), that is

\[
\mathcal{K}_1 := \text{epi}(f_1 - g_1 + \delta_{A_1})^*.
\]

By (12) and (13), the characteristic set \( J \) becomes

\[
J := \bigcap_{\alpha \in H^*_1} \left( \text{epi} f_1^* + \text{epi} \delta_C^* + \text{cone} \left( \bigcup_{t \in T} \text{epi} h_t^* \right) - \langle \alpha, g_1^*(\alpha) \rangle \right).
\]

By the similar discussions of Section 4, we also obtain the corresponding theorems directly.

**Theorem 5.1.** The following statements hold.

(i) The stable weak duality between \((P_1)\) and \((D_1)\) holds if and only if \( J_1 \subseteq \mathcal{K}_1 \).
Theorem 5.2. The following statements hold.

(i) The stable strong duality between \((P_1)\) and \((D_1)\) holds if and only if \(\mathcal{J}_1 = \mathcal{K}_1\).

(ii) The strong duality between \((P_1)\) and \((D_1)\) holds if and only if \(\mathcal{J}_1 \cap \{0\} \times \mathbb{R} \subseteq \mathcal{K}_1 \cap \{0\} \times \mathbb{R}\).

Theorem 5.3. \(\mathcal{J}_1 = \mathcal{K}_1\) holds if and only if for each \(a \in \mathbb{R}\) and \(p \in X^*\), the following assertions are equivalent:

(i) \(x \in \mathcal{A}_1 \implies f_1(x) - g_1(x) \geq \langle p, x \rangle + a\).

(ii) For each \(\alpha \in H_1^*\), there exists \((w, \lambda) \in W_1^* \times \mathbb{R}_+^{(T)}\) such that \(\Phi_1^p(\alpha, w, \lambda) \geq a\).

Theorem 5.4. \(\mathcal{J}_1 \cap \{0\} \times \mathbb{R} = \mathcal{K}_1 \cap \{0\} \times \mathbb{R}\) holds if and only if for each \(a \in \mathbb{R}\), the following assertions are equivalent:

(i) \(x \in \mathcal{A}_1 \implies f_1(x) - g_1(x) \geq a\).

(ii) For each \(\alpha \in H_1^*\), there exists \((w, \lambda) \in W_1^* \times \mathbb{R}_+^{(T)}\) such that \(\Phi_1(\alpha, w, \lambda) \geq a\).

Note that Theorem 5.1-5.4 also appeared in [22].

5.2. The case when \(T = S^+, h_t = th, \varphi_t = 0\). Let \(h_t = th, \varphi_t = 0, t \in T = S^+\).

Then problem \((P)\) is simplified into the following conical composite DC optimization problem:

\[
(P_2) \quad \inf_{x \in C} \{ (f_1 \circ f_2)(x) - (g_1 \circ g_2)(x) \} \quad \text{s.t.} \quad x \in C, h(x) \in -S.
\]

For \(p \in X^*\), problem \((P^p)\) is reduced to the conic composite DC optimization problem with a linear perturbation:

\[
(P^p_2) \quad \inf_{x \in C} \{ (f_1 \circ f_2)(x) - (g_1 \circ g_2)(x) - \langle p, x \rangle \} \quad \text{s.t.} \quad x \in C, h(x) \in -S
\]

where \(h : X \to \mathbb{Z}^*\) is a proper, \(S\)-convex function.

The corresponding dual problem of \((P_2)\) becomes

\[
(D_2) \quad \inf_{\mu \in H_2^*} \sup_{w \in W_2^*} \Phi_2(\mu, w, \lambda),
\]

and the dual problem of \((P^p_2)\) becomes

\[
(D^p_2) \quad \inf_{\mu \in H_2^*} \sup_{w \in W_2^*} \Phi^p_2(\mu, w, \lambda),
\]

where

\[
\Phi_2(\mu, w, \lambda) := g_1^*(\alpha) + (\alpha g_2)^*(\beta) - f_1^*(u) - (uf_2)^*(v)
\]

\[-(\lambda h)^*(s) - \delta_C^*(\beta - s - v),
\]

\[
\Phi^p_2(\mu, w, \lambda) := g_1^*(\alpha) + (\alpha g_2)^*(\beta) - f_1^*(u) - (uf_2)^*(v)
\]

\[-(\lambda h)^*(s) - \delta_C^*(\beta + p - s - v),
\]

with \(\mu := (\alpha, \beta), w := (u, v, s), H_2^* := \text{dom } g_1^* \times \text{dom } g_2^*\) and \(W_2^* := \text{dom } f_1^* \times \text{dom } f_2^* \times \text{dom } h^*\).

As usual, the solution set and the characteristic set of \((P_2)\) is denoted by \(\mathcal{A}_2\) and \(\mathcal{K}_2\),

\[
\mathcal{A}_2 := \{ x \in C : h(x) \in -S \}, \quad \mathcal{K}_2 := \text{epi}(f_1 \circ f_2 - g_1 \circ g_2 + \delta_{A_2})^*.
\]
In fact, $\bigcup_{\lambda \in S^+} \text{epi}(\lambda h)^*$ is a closed convex cone (cf. [14]). This follows that,

$$\text{cone} \left( \bigcup_{\lambda \in S^+} \text{epi}(\lambda h)^* \right) = \bigcup_{\lambda \in S^+} \text{epi}(\lambda h)^*.$$ 

Then the characteristic set $J$ becomes

$$J := \bigcap_{\mu \in H_2^*} \left( \bigcup_{u \in \text{dom} f_1^*} \left( \text{epi}(\mu u f_2)^* + (0, f_1^*(u)) \right) + \text{epi} \delta_C^* \right)$$

$$+ \bigcup_{\lambda \in S^+} \text{epi}(\lambda h)^* - \left( \beta, g_1^*(\alpha) + (\alpha g_2)^*(\beta) \right).$$

By the similar discussions of Section 4, we obtain the corresponding results directly.

**Theorem 5.5.** The following assertions hold.

1. The stable weak duality between $(P_2)$ and $(D_2)$ holds if and only if $J_2 \subseteq K_2$.
2. The weak duality between $(P_2)$ and $(D_2)$ holds if and only if $J_2 \cap \{0\} \times \mathbb{R} \subseteq K_2 \cap \{0\} \times \mathbb{R}$.

**Theorem 5.6.** The following statements hold.

1. The stable strong duality between $(P_2)$ and $(D_2)$ holds if and only if $J_2 = K_2$.
2. The strong duality between $(P_2)$ and $(D_2)$ holds if and only if $J_2 \cap \{0\} \times \mathbb{R} = K_2 \cap \{0\} \times \mathbb{R}$.

**Theorem 5.7.** $J_2 = K_2$ holds if and only if for each $a \in \mathbb{R}$ and $p \in X^*$, the following assertions are equivalent:

1. $x \in A_2 \Rightarrow (f_1 \circ f_2)(x) - (g_1 \circ g_2)(x) \geq \langle p, x \rangle + a$.
2. For each $\mu \in H_2^*$, there exists $(w, \lambda) \in W_2^* \times S^+$ such that $\Phi_2(\mu, w, \lambda) \geq a$.

**Theorem 5.8.** $J_2 \cap \{0\} \times \mathbb{R} = K_2 \cap \{0\} \times \mathbb{R}$ holds if and only if for each $a \in \mathbb{R}$, the following assertions are equivalent:

1. $x \in A_2 \Rightarrow (f_1 \circ f_2)(x) - (g_1 \circ g_2)(x) \geq a$.
2. For each $\mu \in H_2^*$, there exists $(w, \lambda) \in W_2^* \times S^+$ such that $\Phi_2(\mu, w, \lambda) \geq a$.

**Remark 4.** (a) Under the assumption that $f_1$ is $K$-increasing and $g_1$ is $S$-increasing,

$$f_1 \circ f_2 - g_1 \circ g_2 + \delta_{A_2}$$

the authors in [8] introduced the dual problems of $(P_2)$ and $(P_2^p)$

$$\begin{align*}
(D_S) & \quad \inf_{\alpha \in \text{dom} g_1^*} \sup_{\beta \in \text{dom} g_2^*} \left\{ g_1^*(\alpha) - f_1^*(u) - (uf_2 + \delta_C + \lambda h)^*(\beta) + (\alpha g_2)^*(\beta) \right\}, \\
(D_{sp}) & \quad \inf_{\alpha \in \text{dom} g_1^*} \sup_{\beta \in \text{dom} g_2^*} \left\{ g_1^*(\alpha) - f_1^*(u) - (uf_2 + \delta_C + \lambda h)^*(p + \beta) + (\alpha g_2)^*(\beta) \right\},
\end{align*}$$

and the characteristic sets $K_2'$ and $T$ defined as follows

$$K_2' := \{(x^*, 0, 0, r) : (x^*, r) \in \text{epi}(f_1 \circ f_2 - g_1 \circ g_2 + \delta_{A_2})^*\},$$

$$T := \bigcap_{\alpha \in \text{dom} g_1^*} \left( \bigcup_{\beta \in \text{dom} g_2^*} \left( \left\{ (x^*, -u, 0, r) : (x^*, r) \in \text{epi}(uf_2 + \delta_C + \lambda h)^* \right\} + \left\{ (0, y^*, 0, r) : (y^*, r) \in \text{epi} f_1^* \right\} \right) - (\beta, 0, 0, g_1^*(\alpha) + (\alpha g_2)^*(\beta)) \right).$$
Proposition 1. Suppose that \( P \) and \( D \) hold if and only if \( T \subseteq K_2' \). Proposition 1 will give the relationships between the conditions.

(b) It is noted that the assumption (36) is not essential in our paper as shown in Example 2.

The relationship between the conditions above is characterized in the following proposition.

**Proposition 1.** Suppose that \( \text{dom} f_2 \cap \text{dom} h \cap C \neq \emptyset \), either \( f_2 \) or \( h \) is continuous at some \( x_0 \in \text{dom} f_2 \cap \text{dom} h \cap C \). Then the following statements hold:

(i) For \((x^*, r) \in X^* \times \mathbb{R}, \)

\[
(x^*, 0, 0, r) \in T \Leftrightarrow (x^*, r) \in J_2.
\]

(ii) \( T = K_2' \Leftrightarrow J_2 = K_2 \).

**Proof.** It results from the definitions of \( T \) and \( K_2' \) that (i) implies (ii). Here we just only prove (i).

\[ \Rightarrow \] Take \((x^*, 0, 0, r) \in T \). For each \( \alpha \in \text{dom} g_1^*, \beta \in \text{dom} g_2^*, \) there exist \( u \in \text{dom} f_1^*, \lambda \in S^+ \) satisfying

\[
(x^*, 0, 0, r) = (x_1^*, -u, 0, r_1) + (0, u, 0, r_2) - (\beta, 0, 0, g_1^*(\alpha) + (\alpha g_2)^*(\beta)),
\]

where \((x_1^*, r_1) \in \text{epi}(uf_2 + \delta_C + \lambda h)^*, (u, r_2) \in \text{epi} f_1^* \). This implies that

\[
(x^*, r) = (x_1^*, r_1) + (0, r_2) - (\beta, g_1^*(\alpha) + (\alpha g_2)^*(\beta)).
\]

Lemma 2.1 guarantees that

\[
\text{epi}(uf_2 + \delta_C + \lambda h)^* = \text{epi}(uf_2)^* + \text{epi} \delta_C + \text{epi}(\lambda h)^*.
\]

So there exist \((x_{11}^*, r_{11}) \in \text{epi}(uf_2)^*, (x_{12}^*, r_{12}) \in \text{epi} \delta_C, (x_{13}^*, r_{13}) \in \text{epi}(\lambda h)^* \) satisfying the following equation:

\[
(x_1^*, r_1) = (x_{11}^*, r_{11}) + (x_{12}^*, r_{12}) + (x_{13}^*, r_{13}).
\]

Furthermore \((u, r_2) \in \text{epi} f_1^* \) implies that \( r_2 \geq f_1^*(u) \). So we have \( r_2 = f_1^*(u) + \gamma \), where \( \gamma \geq 0 \).

\[
(x_{11}^*, r_{11}) + (0, r_2) = (x_{11}^*, r_{11}) + (0, f_1^*(u)) + (0, \gamma) = (x_{11}^*, r_{11} + \gamma) + (0, f_1^*(u))
\]

which implies that

\[
(x_{11}^*, r_{11}) + (0, r_2) \in \text{epi}(uf_2)^* + (0, f_1^*(u)).
\]

By (38), (39) and (40), we see that

\[
(x^*, r) = (x_{11}^*, r_{11}) + (0, r_2) + (x_{12}^*, r_{12}) + (x_{13}^*, r_{13}) - (\beta, g_1^*(\alpha) + (\alpha g_2)^*(\beta)) \in \bigcup_{u \in \text{dom} f_1^*} \left( \text{epi}(uf_2)^* + (0, f_1^*(u)) \right) + \text{epi} \delta_C
\]

\[
+ \bigcup_{\lambda \in S^+} \text{epi}(\lambda h)^* - (\beta, g_1^*(\alpha) + (\alpha g_2)^*(\beta)) \bigg).
\]

Therefore, by the arbitrariness of \( \alpha \in \text{dom} g_1^*, \beta \in \text{dom} g_2^* \), we have that \((x^*, r) \in J_2 \).
[\Leftarrow] Let \((x^*, r) \in J_2\). By (2),
\[
\bigcup_{u \in \text{dom} f_2^*} \left( \text{epi}(uf_2)^* + (0, f_1^*(u)) \right) + \text{epi} \delta_C^* + \bigcup_{\lambda \in S^+} \text{epi}(\lambda h)^* \\
\subseteq \bigcup_{u \in \text{dom} f_2^*} \left( \text{epi}(uf_2 + \delta_C + \lambda h)^* + (0, f_1^*(u)) \right).
\]
This follows that, for each \((\alpha, \beta) \in H_2^*\), there exist \(u \in \text{dom} f_2^*\) and \(\lambda \in S^+\) satisfying
\[
(x^*, r) = (x^*_1, r_1) + (0, f_1^*(u)) - (\beta, g_1^*(\alpha) + (\alpha g_2)^*(\beta)),
\]
where \((x^*_1, r_1) \in \text{epi}(uf_2 + \delta_C + \lambda h)^*\). Therefore, by (41), we have that,
\[
(x^*, 0, 0, r) = (x^*_1, -u, 0, r_1) + (0, u, f_1^*(u)) - (\beta, 0, 0, g_1^*(\alpha) + (\alpha g_2)^*(\beta)),
\]
which implies that \((x^*, 0, 0, r) \in T\). Consequently, (37) holds. \(\square\)

The following example implies that Theorem 5.6(ii) still holds even without the assumption that \(f_1\) is \(K\)-increasing and \(g_1\) is \(S\)-increasing.

**Example 2.** Let \(X = Y = Z = C := \mathbb{R}, S = K := (-\infty, 0], f_2 = g_2 = h := \text{Id}_X, g_1 := 2x\) and
\[
f_1(x) := \begin{cases} 
 x^2, & \text{if } x \geq 0, \\
 +\infty, & \text{otherwise}.
\end{cases}
\]
Obviously, \(f_1\) is not \(K\)-increasing, \(g_1\) is not \(S\)-increasing, \(f_1, g_1, h\) are proper, convex functions, \(f_2\) is a proper, \(K\)-convex function, and \(g_2\) is a proper, \(S\)-convex function. Then we have \(K^+ = S^+ = (-\infty, 0], A_2 = [0, +\infty), \)
\[
(f_1 \circ f_2 - g_1 \circ g_2 + \delta_{A_2})(x) = \begin{cases} 
 x^2 - 2x & \text{if } x \geq 0, \\
 +\infty, & \text{otherwise},
\end{cases}
\]
and
\[
(f_1 \circ f_2 - g_1 \circ g_2 + \delta_{A_2})^*(x^*) = \begin{cases} 
 0 & \text{if } x^* < -2, \\
 \frac{(x^* + 2)^2}{4} & \text{if } x^* \geq -2.
\end{cases}
\]
Thus,
\[
v(P_2) = \inf_{x \in A_2} \left\{ (f_1 \circ f_2)(x) - (g_1 \circ g_2)(x) \right\} = \inf_{x \geq 0} \{ x^2 - 2x \} = -1, \quad (42)
\]
and
\[
K_2 \cap \{(0) \times \mathbb{R}\} = \{0\} \times [1, +\infty). \quad (43)
\]
For each \(\alpha \in \text{dom} g_1^*, u \in \text{dom} f_2^*, \lambda \in S^+, (12) \) and (13) imply that
\[
(\alpha g_2)^*(\beta) = \begin{cases} 
 0, & \text{if } \beta = \alpha, \\
 +\infty, & \text{otherwise},
\end{cases} \quad (uf_2)^*(v) = \begin{cases} 
 0, & \text{if } v = u, \\
 +\infty, & \text{otherwise},
\end{cases}
\]
\[
(\lambda h)^*(s) = \begin{cases} 
 0, & \text{if } s = \lambda, \\
 +\infty, & \text{otherwise},
\end{cases} \quad \bigcup_{u \in \text{dom} f_2^*} \left( \text{epi}(uf_2)^* + (0, f_1^*(u)) \right) = \text{epi} f_1^*.
\]
Moreover, since \(g_1^* = \delta_{\{2\}}, h^* = \delta_{\{1\}}, \delta_C^* = \delta_{\{0\}}\) and for any \(u \in \mathbb{R}, \)
\[
f_1^*(u) = \begin{cases} 
 0 & \text{if } u < 0, \\
 \frac{u^2}{4} & \text{if } u \geq 0,
\end{cases}
\]
it follows that
\[\text{epi} f_1^* = (-\infty, 0] \times [0, +\infty) \cup \{(u, r) : u \geq 0, r \geq \frac{u^2}{4}\}, \text{epi} \delta_C^* = \{0\} \times [0, +\infty),\]
\[\bigcup_{\lambda \in S^+} \text{epi}(\lambda h)^* = \bigcup_{\lambda \leq 0} (\{\lambda\} \times [0, +\infty)) = (-\infty, 0] \times [0, +\infty).\]
Thus,
\[v(D_2) = \inf_{\mu \in H_2^*} \max_{w \in W_2^*} \{g_1^*(\alpha) + (\alpha g_2)^*(\beta) - f_1^*(u) - (uf_2)^*(v) - (\lambda h)^*(s) - \delta_C^*(\beta - s - v)\} = -1\]
and
\[\mathcal{J}_2 \cap \{(0) \times \mathbb{R}\} = \bigcap_{\mu \in H_2^*} \left( \left( \bigcup_{u \in \text{dom} f_1^*} (\text{epi}(uf_2)^* + (0, f_1^*(u))) \right) + \text{epi} \delta_C^* \right) + \bigcup_{\lambda \in S^+} \text{epi}(\lambda h)^* - (\beta, g_1^*(\alpha) + (\alpha g_2)^*(\beta)) \cap \{(0) \times \mathbb{R}\}
= \left( (-\infty, 0] \times [0, +\infty) \cup \{(u, r) : u \geq 0, r \geq \frac{u^2}{4}\} \right) + \{0\} \times [0, +\infty)
+ (-\infty, 0] \times [0, +\infty) - (2, 0) \right) \cap \{(0) \times \mathbb{R}\}
= \{0\} \times [1, +\infty).\]
This together with (42) and (43) implies that the strong duality holds and \(\mathcal{J}_2 \cap \{(0) \times \mathbb{R}\} = K_2 \cap \{(0) \times \mathbb{R}\}.\) Therefore Theorem 5.6(ii) holds.

6. Conclusion. In the paper, we discuss a DC composite programs with infinite DC inequalities. Without the assumption that the functions involved are lower semicontinuous, \(f_1\) is \(K\)-increasing, \(g_1\) is \(S\)-increasing and \(C\) is closed, we present the complete characterization of the weak (strong/stable) Fenchel-Lagrange duality and Farkas-type assertions by introducing two characteristic sets. We also show that the results obtained improve and extend some corresponding results in recent literature.

REFERENCES

[1] R. I. Boţ, S.-M. Grad and G. Wanka, A new constraint qualification for the formula of the subdifferential of composed convex functions in infinite dimensional spaces, Mathematische Nachrichten, 281 (2008), 1088–1107.
[2] R. I. Boţ, S.-M. Grad and G. Wanka, Generalized Moreau-Rockafellar results for composed convex functions, Optimization, 58 (2009), 917–933.
[3] R. I. Boţ, S.-M. Grad and G. Wanka, On strong and total Lagrange duality for convex optimization problems, Journal of Mathematical Analysis and Applications, 337 (2008), 1315–1325.
[4] R. I. Boţ, I. B. Hodrea and G. Wanka, Some new Farkas-type results for inequality system with DC functions, Journal of Global Optimization, 39 (2007), 595–608.
[5] N. Dinh, M. A. Goberna, M. A. López and T. Q. Song, New Farkas-type constraint qualifications in convex infinite programming, ESAIM: Control Optimisation and Calculus of Variations, 13 (2007), 580–597.
[6] N. Dinh, B. S. Mordukhovich and T. T. A. Nghia, Qualification and optimality conditions for DC programs with infinite constraints, Acta Mathematica Vietnamica, 34 (2009), 125–155.
[7] D. H. Fang, Some relationships among the constraint qualifications for Lagrangian dualities in DC infinite optimization problems, Journal of Inequalities and Applications, 2015 (2015), 41–55.
[8] D. H. Fang and X. Gong, Extended Farkas lemma and strong duality for composite optimization problems with DC functions, *Optimization*, 66 (2017), 179–196.

[9] D. H. Fang, G. M. Lee, C. Li and J. C. Yao, Extended Farkas’s lemma and strong Lagrange dualities for DC infinite programming, *Journal of Nonlinear and Convex Analysis*, 14 (2013), 747–767.

[10] D. H. Fang, C. Li and K. F. Ng, Constraint qualifications for extended Farkas’ lemmas and Lagrangian dualities in convex infinite programming, *SIAM Journal on Optimization*, 20 (2009), 1311–1332.

[11] D. H. Fang, M. D. Wang and X. P. Zhao, The strong duality for DC optimization problems with composite convex functions, *Journal of Nonlinear Convex Analysis*, 16 (2015), 1337–1352.

[12] M. A. Goberna, V. Jeyakumar and M. A. López, Necessary and sufficient conditions for solvability of systems of infinite convex inequalities, *Nonlinear Analysis*, 68 (2008), 1184–1194.

[13] M. A. Goberna and M. A. López, *Linear Semi-Infinite Optimization*, Wiley, Chichester, 1998.

[14] V. Jeyakumar, A. Rubinov, B. M. Glover and Y. Ishizuka, Inequality systems and global optimization, *Journal of Mathematical Analysis and Applications*, 202 (1996), 900–919.

[15] C. Li, D. H. Fang, G. López and M. A. López, Stable and total Fenchel duality for convex optimization problems in locally convex spaces, *SIAM Journal on Optimization*, 20 (2009), 1032–1051.

[16] C. Li and K. F. Ng, On constraint qualification for infinite system of convex inequalities in a Banach space, *SIAM Journal on Optimization*, 15 (2005), 488–512.

[17] C. Li, K. F. Ng and T. K. Pong, Constrict qualifications for convex inequality systems with applications in constrained optimization, *SIAM Journal on Optimization*, 19 (2008), 163–187.

[18] G. Li, X. Q. Yang and Y. Y. Zhou, Stable strong and total parametrized dualities for DC optimization problems in locally convex spaces, *Journal of Industrial and Management Optimization*, 9 (2013), 671–687.

[19] G. Li, L. P. Zhang and Z. Liu, The stable duality of DC programs for composite convex functions, *Journal of Industrial and Management Optimization*, 13 (2017), 63–79.

[20] W. Li and C. Nahak and I. Singer, Constrict qualifications for semi-infinite systems of convex inequalities, *SIAM Journal on Optimization*, 11 (2000), 31–52.

[21] J.-E. Martinez-Legaz and M. Volle, Duality in D.C. programming: The case of several D.C. constraints, *Journal of Mathematical Analysis and Applications*, 237 (1999), 657–671.

[22] X.-K. Sun, Regularity conditions characterizing Fenchel-Lagrange duality and Farkas-type results for DC infinite programming, *Journal of Mathematical Analysis and Applications*, 414 (2014), 590–611.

[23] X.-K. Sun and H.-Y. Fu, A note on optimality conditions for DC programs involving composite functions, *Abstract and Applied Analysis*, 2014 (2014), 203467, 6 pp.

[24] X.-K. Sun, H.-Y. Fu and J. Zeng, Robust approximate optimality conditions for uncertain nonsmooth optimization with infinite number of constraints, *Mathematics*, 7 (2019).

[25] X.-K. Sun, X.-L. Guo and Y. Zhang, Fenchel-Lagrange duality for DC programs with composite functions, *Journal of Nonlinear Convex Analysis*, 16 (2015), 1607–1618.

[26] X.-K. Sun, X.-J. Long and J. Zeng, Constraint qualifications characterizing Fenchel duality in composed convex optimization, *Journal of Nonlinear Convex Analysis*, 17 (2016), 325–347.

[27] C. Zălinescu, *Convex Analysis in General Vector Space*, World Scientific Publishing, Singapore, 2002.

Received May 2020; revised December 2020.

E-mail address: ligang@zstu.edu.cn
E-mail address: xyh7913@126.com
E-mail address: qinzhenhua@zime.edu.cn