DOUBLE CONSTRUCTIONS OF FROBENIUS ALGEBRAS, CONNES COCYCLES AND THEIR DUALITY

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Abstract. We construct an associative algebra with a decomposition into the direct sum of the underlying vector spaces of another associative algebra and its dual space such that both of them are subalgebras and the natural symmetric bilinear form is invariant or the natural antisymmetric bilinear form is a Connes cocycle. The former is called a double construction of Frobenius algebra and the latter is called a double construction of Connes cocycle which is interpreted in terms of dendriform algebras. Both of them are equivalent to a kind of bialgebras, namely, antisymmetric infinitesimal bialgebras and dendriform D-bialgebras respectively. In the coboundary cases, our study leads to what we call associative Yang-Baxter equation in an associative algebra and $D$-equation in a dendriform algebra respectively, which are analogues of the classical Yang-Baxter equation in a Lie algebra. We show that an antisymmetric solution of associative Yang-Baxter equation corresponds to the antisymmetric part of a certain operator called $O$-operator which gives a double construction of Frobenius algebra, whereas a symmetric solution of $D$-equation corresponds to the symmetric part of an $O$-operator which gives a double construction of Connes cocycle. By comparing antisymmetric infinitesimal bialgebras and dendriform D-bialgebras, we observe that there is a clear analogy between them. Due to the correspondences between certain symmetries and antisymmetries appearing in the analogy, we regard it as a kind of duality.

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2000 Mathematics Subject Classification. 16W30, 17A30, 17B60, 57R56, 81T45.
Key words and phrases. Associative algebra; Frobenius algebra; Connes cocycle; Yang-Baxter equation.
1. Introduction

Throughout this paper, an associative algebra is a nonunital associative algebra. There are two important (nondegenerate) bilinear forms on an associative algebra given as follows.

**Definition 1.0.1.** A bilinear form $B(\cdot, \cdot)$ on an associative algebra $A$ is invariant if

$$B(xy, z) = B(x, yz), \ \forall \ x, y, z \in A. \quad (1.0.1)$$

**Definition 1.0.2.** An antisymmetric bilinear form $\omega(\cdot, \cdot)$ on an associative algebra $A$ is a cyclic 1-cocycle in the sense of Connes if

$$\omega(xy, z) + \omega(yz, x) + \omega(zx, y) = 0, \ \forall \ x, y, z \in A. \quad (1.0.2)$$

We also call $\omega$ a Connes cocycle for abbreviation.

1.1. *Frobenius algebras.* A Frobenius algebra $(A, B)$ is an associative algebra $A$ with a non-degenerate invariant bilinear form $B(\cdot, \cdot)$. It was first studied by Frobenius ([Fro]) in 1903 and then named by Brauer and Nesbitt ([BrN]). In fact, Frobenius algebras appear in many fields in mathematics and mathematical physics, such as (modular) representations of finite groups ([Kap]), Hopf algebras ([LS]), statistical models over 2-dimensional graphs ([BFN]), Yang-Baxter equation ([St]), Poisson brackets of hydrodynamic type ([BaN]) and so on. In particular, they play a key role in the study of topological quantum field theory ([Ko], [RFFS], etc.). There are a lot of references on the study of Frobenius algebras (for example, see [Kap] or [Y] and the references therein).

A Frobenius algebra $(A, B)$ is symmetric if $B$ is symmetric. In this paper, we mainly consider a class of symmetric Frobenius algebras $(A, B)$ satisfying the following conditions:

1. $A = A_1 \oplus A_1^*$ as the direct sum of vector spaces;
2. $A_1$ and $A_1^*$ are associative subalgebras of $A$;
3. $B$ is the natural symmetric bilinear form on $A_1 \oplus A_1^*$ given by

$$B(x + a^*, y + b^*) = \langle x, b^* \rangle + \langle a^*, y \rangle, \ \forall x, y \in A_1, \ a^*, b^* \in A_1^*, \quad (1.1.1)$$
where $\langle \cdot, \cdot \rangle$ is the natural pair between the vector space $A_1$ and its dual space $A_1^*$. We call it a double construction of Frobenius algebra.

Such a double construction of Frobenius algebra is quite different from the “double extension construction” of a Lie algebra with a nondegenerate invariant bilinear form ([Kac], [MR1-2], etc.) or the “$T^*$-extension” of Frobenius algebra given by Bordemann in [Bo].

Moreover, the above double constructions of Frobenius algebras were also considered by Zhelyabin in [Z] and Aguiar in [A3] (under the name of “balanced Drinfeld double $D_b(A)$”) with different motivations and approaches respectively. They are closely related to Lie bialgebras. Lie bialgebras were introduced by Drinfeld ([D]) and they play a crucial role in symplectic geometry and quantum groups. They are equivalent to Manin triples (see [CP] and the references therein or subsection 5.2).

It is easy to show that the commutator of a Frobenius algebra from the above double construction gives a Manin triple (hence a Lie bialgebra). Furthermore, such a double construction has many properties similar to a Lie bialgebra. It is equivalent to an antisymmetric infinitesimal bialgebra (which is the same structure under the names of “associative D-algebra” in [Z] and “balanced infinitesimal bialgebra” in the sense of the opposite algebra in [A3]) and under a “coboundary” condition, it leads to an analogue of the classical Yang-Baxter equation ([Se]) in an associative algebra $A_1$

$$r_{12}r_{13} + r_{13}r_{23} - r_{23}r_{12} = 0,$$

where $r = \sum_{i} x_i \otimes y_i \in A_1 \otimes A_1$ and

$$r_{12}r_{13} = \sum_{i,j} x_i x_j \otimes y_i \otimes y_j, \quad r_{13}r_{23} = \sum_{i,j} x_i \otimes x_j \otimes y_i y_j, \quad r_{23}r_{12} = \sum_{i,j} x_j \otimes x_i y_j \otimes y_i. \quad (1.1.3)$$

In particular, an antisymmetric solution of the above equation in $A_1$ gives a double construction of Frobenius algebra $(A = A_1 \oplus A_1^*, \mathcal{B})$.

On the other hand, we introduce the new notion of antisymmetric infinitesimal bialgebra in order to express explicitly its relation with the known notion of infinitesimal bialgebra, although there are certain notions for the same or similar structures. An infinitesimal bialgebra is a triple $(A, m, \Delta)$, where $(A, m)$ is an associative algebra, $(A, \Delta)$ is a coassociative algebra and

$$\Delta(ab) = \sum_{i} ab_i \otimes b_2 + \sum_{i} a_1 \otimes a_2 b, \quad \forall a, b \in A. \quad (1.1.4)$$

It was introduced by Join and Rota ([JR]) in order to provide an algebraic framework for the calculus of divided difference. Furthermore, Aguiar studied the cases of principal derivations and gave the associative Yang-Baxter equation ([A1])

$$r_{13}r_{12} - r_{12}r_{23} + r_{23}r_{13} = 0. \quad (1.1.5)$$

Note that equation (1.1.2) is equation (1.1.5) in the opposite algebra and, when $r$ is antisymmetric, equation (1.1.5) is just equation (1.1.2) under the operation $\sigma_{13}(x \otimes y \otimes z) = z \otimes y \otimes x$. 
We would like to point out that although there have been many results on the double constructions of Frobenius algebras, there has not been a complete and explicit interpretation yet. In fact, most of these results were given in a scattered way with different motivations. For example, Zhelyabin in [Z] introduced the notion of associative D-algebra as an important step to develop a bialgebra theory of Jordan algebras (there was not an explicit study of coboundary cases for the associative algebras themselves). In [A3], Aguiar introduced the notion of balanced infinitesimal bialgebra and then studied the antisymmetric solution of equation (1.1.5) in order to compare them with Lie bialgebras and the classical Yang-Baxter equation in a Lie algebra respectively, and the balanced Drinfeld double $D_b(A)$ appears as an important consequence. We will formulate the known results by a different and systematic approach (for example, the “invariant” antisymmetry appears naturally). Moreover such an approach is useful and convenient for the whole study in this paper.

1.2. **O-operators and dendriform algebras.** When $r$ is antisymmetric, besides the standard tensor form (1.1.2) or (1.1.5), the associative Yang-Baxter equation has an equivalent operator form, that is, a special case of a certain operator called $O$-operator. An $O$-operator associated to a bimodule $(l, r, V)$ of an associative algebra $A$ is a linear map $T: V \rightarrow A$ satisfying

$$T(u) \cdot T(v) = T(l(T(u))v + r(T(v)u)), \quad \forall \ u, v \in V. \quad (1.2.1)$$

In fact, an antisymmetric solution of associative Yang-Baxter equation is an $O$-operator associated to the bimodule $(R^*, L^*)$. The notion of $O$-operator was introduced in [BGN1] (such a structure appeared independently in [U] under the name of generalized Rota-Baxter operator) which is an analogue of the $O$-operator defined by Kupershmidt as a natural generalization of the operator form of the classical Yang-Baxter equation ([Kn3] and a further study in [Bai1]). Conversely, the antisymmetric part of an $O$-operator satisfies the associative Yang-Baxter equation in a larger associative algebra.

From an $O$-operator, one can get a dendriform algebra. Dendriform algebras are equipped with an associative product which can be written as a linear combination of nonassociative compositions. They were introduced by Loday ([Lo1]) with motivation from algebraic $K$-theory and have been studied quite extensively with connections to several areas in mathematics and physics, including operads ([Lo3]), homology ([Fra1-2]), Hopf algebras ([Cha2], [H1-2], [Ron], [LR2]), Lie and Leibnitz algebras ([Fra2]), combinatorics ([LR1]), arithmetic ([Lo2]) and quantum field theory ([F1]) and so on (see [EMP] and the references therein).

Furthermore, there is a compatible dendriform algebra structure on an associative algebra $A$ if and only if there exists an invertible $O$-operator of $A$, or equivalently, there exists an invertible (usual) 1-cocycle (see equation (3.1.6)) associated to certain suitable bimodule of $A$ ([BGN2]). Thus a close relation between the associative Yang-Baxter equation (hence the antisymmetric...
infinitesimal bialgebras and the double constructions of Frobenius algebras) and dendriform algebras is obviously given (see also [A3], [E1-2]).

1.3. Connes cocycles. Note that a Connes cocycle given by equation (1.0.2) is in fact a Hochschild 2-cocycle which satisfies antisymmetry. It corresponds to the original definition of cyclic cohomology by Connes ([C]). Also note that in cyclic cohomology a cyclic n-cocycle in the sense of Connes is an $n + 1$ linear form, although a Connes cocycle was called a cyclic 2-cocycle in some references (like [A3]) from some different viewpoints. Moreover, although Connes used it in the unital framework and in the nonunital framework cyclic homology has a very different behavior, we still use the terminology “Connes cocycle” in this paper.

In this paper, we will see that, from a nondegenerate Connes cocycle on an associative algebra $A$, one can get a compatible dendriform algebra structure on $A$. Moreover, the dendriform algebra structures play a key role in the following constructions of nondegenerate Connes cocycles, which is one of the main contents in this paper. We call $(A, \omega)$ a double construction of Connes cocycle if it satisfies the following conditions:

1. $A = A_1 \oplus A_1^*$ as the direct sum of vector spaces;
2. $A$ is an associative algebra and $A_1$ and $A_1^*$ are associative subalgebras of $A$;
3. $\omega$ is the natural antisymmetric bilinear form on $A_1 \oplus A_1^*$ given by
   \[
   \omega(x + a^*, y + b^*) = -\langle x, b^* \rangle + \langle a^*, y \rangle, \quad \forall x, y \in A_1, \quad a^*, b^* \in A_1^*,
   \]

and $\omega$ is a Connes cocycle on $A$.

In this paper, the double construction of Connes cocycle is interpreted in terms of dendriform algebras. We find that such a structure is quite similar to a double construction of Frobenius algebra or a Lie bialgebra. Briefly speaking, a double construction of Connes cocycle is equivalent to a certain bialgebra structure, namely, a dendriform D-bialgebra structure. Both antisymmetric infinitesimal bialgebras and dendriform D-bialgebras have many similar properties as Lie bialgebras. In particular, there are the so-called coboundary dendriform D-bialgebras which lead to another analogue ($D$-equation in a dendriform algebra) of the classical Yang-Baxter equation. A symmetric solution of the $D$-equation corresponds to the symmetric part of an $O$-operator, which gives a double construction of Connes cocycle.

1.4. Duality between bialgebras. By comparing antisymmetric infinitesimal bialgebras and dendriform D-bialgebras, we observe that there is a clear analogy between them. Moreover, due to the correspondences between certain symmetries and antisymmetries appearing in the analogy, we regard it as a kind of duality.

There is a similar study in the version of Lie algebras ([CP], [Bai2]). In fact, there is also a double construction of a Lie algebra with a nondegenerate invariant bilinear form (Manin triple or Lie bialgebra) or with a nondegenerate 2-cocycle of Lie algebra (parakähler Lie algebra or
pre-Lie bialgebra). There are the \(\mathcal{O}\)-operators and a kind of algebras called pre-Lie algebras (Lie-admissible algebras whose left multiplication operators form a Lie algebra) which play the same roles of the \(\mathcal{O}\)-operators and dendriform algebras. And there is a similar duality between Lie bialgebras and pre-Lie bialgebras.

Moreover, due to Chapoton ([Cha1]), there is a close relationship among the Lie algebras, associative algebras, pre-Lie algebras and dendriform algebras as follows (in the sense of commutative diagram of categories).

\[
\begin{array}{ccc}
\text{dendriform algebras} & 
\rightarrow & \text{pre-Lie algebras} \\
\downarrow & & \downarrow \\
\text{associative algebras} & 
\rightarrow & \text{Lie algebras}
\end{array}
\]

We will extend the above relationship at the level of bialgebras with the dualities in a commutative diagram. In particular, the relation between antisymmetric infinitesimal bialgebras (the special case of infinitesimal Hopf algebras) and Lie bialgebras have been mentioned in [A3]. Furthermore, these types of bialgebras fit into the general framework of “generalized bialgebras” as introduced by Loday in [Lo4].

The paper is organized as follows. In section 2, we give an explicit and systematic study on the double constructions of Frobenius algebras and then get the associative Yang-Baxter equation naturally. In section 3, we introduce the close relations between \(\mathcal{O}\)-operators and dendriform algebras. In section 4, we study the double constructions of Connes cocycles in terms of dendriform algebras. In section 5, we give the clear analogy between antisymmetric infinitesimal bialgebras and dendriform D-bialgebras, which we regard it as a kind of duality. After recalling a similar duality between Lie bialgebras and pre-Lie bialgebras, we express a close relationship among associative algebras, Lie algebras, pre-Lie algebras and dendriform algebras at the level of bialgebras.

Throughout this paper, all algebras are finite-dimensional, although many results still hold in the infinite-dimensional case.

2. Double constructions of Frobenius algebras and another approach to associative Yang-Baxter equation

2.1. Bimodules and matched pairs of associative algebras.

**Definition 2.1.1.** Let \(A\) be an associative algebra and \(V\) be a vector space. Let \(l, r : A \rightarrow \mathfrak{gl}(V)\) be two linear maps. \(V\) (or the pair \((l, r)\), or \((l, r, V)\)) is called a **bimodule** of \(A\) if

\[
l(xy)v = l(x)l(y)v, \quad r(xy)v = r(y)r(x)v, \quad l(x)r(y)v = r(y)l(x)v, \quad \forall x, y \in A, v \in V.
\]

(2.1.1)

In fact, according to [Sc], \((l, r, V)\) is a bimodule of an associative algebra \(A\) if and only if the direct sum \(A \oplus V\) of vector spaces is turned into an associative algebra (the semidirect sum) by
defining multiplication in $A \oplus V$ by

$$(x_1 + v_1) \ast (x_2 + v_2) = x_1 \cdot x_2 + (l(x_1)v_2 + r(x_2)v_1), \quad \forall x_1, x_2 \in A, v_1, v_2 \in V. \quad (2.1.2)$$

We denote it by $A \ltimes_{l,r} V$ or simply $A \ltimes V$.

The following conclusion is obvious.

**Lemma 2.1.2.** Let $(l, r, V)$ be a bimodule of an associative algebra $A$.

1. Let $l^*, r^* : A \to \mathfrak{gl}(V^*)$ be the linear maps given by

$$\langle l^*(x)u^*, v \rangle = \langle l(x)v, u^* \rangle, \quad \langle r^*(x)u^*, v \rangle = \langle r(x)v, u^* \rangle, \quad \forall x \in A, u^* \in V^*, v \in V. \quad (2.1.3)$$

Then $(r^*, l^*, V^*)$ is a bimodule of $A$.

2. $(l, 0, V)$, $(0, r, V)$, $(r^*, 0, V^*)$ and $(0, l^*, V^*)$ are bimodules of $A$.

**Example 2.1.3.** Let $A$ be an associative algebra. Let $L(x)$ and $R(x)$ denote the left and right multiplication operator respectively, that is, $L(x)(y) = xy$, $R(x)(y) = yx$ for any $x, y \in A$. Let $L : A \to \mathfrak{gl}(A)$ with $x \to L(x)$ and $R : A \to \mathfrak{gl}(A)$ with $x \to R(x)$ (for every $x \in A$) be two linear maps. Then $(L, 0)$, $(0, R)$ and $(L, R)$ are bimodules of $A$. On the other hand, $(R^*, 0)$, $(0, L^*)$ and $(R^*, L^*)$ are bimodules of $A$, too.

**Theorem 2.1.4.** Let $(A, \cdot)$ and $(B, \circ)$ be two associative algebras. Suppose that there are linear maps $l_A, r_A : A \to \mathfrak{gl}(B)$ and $l_B, r_B : B \to \mathfrak{gl}(A)$ such that $(l_A, r_A)$ is a bimodule of $A$ and $(l_B, r_B)$ is a bimodule of $B$ and they satisfy the following conditions:

$$l_A(x)(a \circ b) = l_A(r_B(b)x)b + (l_A(x)a) \circ b; \quad (2.1.4)$$

$$r_A(x)(a \circ b) = r_A(l_B(b)x)a + a \circ (r_A(x)b); \quad (2.1.5)$$

$$l_B(a)(x \cdot y) = l_B(r_A(x)a)y + (l_B(a)x) \cdot y; \quad (2.1.6)$$

$$r_B(a)(x \cdot y) = r_B(l_A(y)a)x + x \cdot (r_B(a)y)); \quad (2.1.7)$$

$$l_A(l_B(a)x)b + (r_A(x)a) \circ b - r_A(r_B(b)x)a - a \circ (l_A(x)b) = 0; \quad (2.1.8)$$

$$l_B(l_A(x)a)y + (r_B(a)x) \cdot y - r_B(r_A(y)a)x - x \cdot (l_B(a)y) = 0, \quad (2.1.9)$$

for any $x, y \in A, a, b \in B$. Then there is an associative algebra structure on the direct sum $A \oplus B$ of the underlying vector spaces of $A$ and $B$ given by

$$(x+a) \ast (y+b) = (x \cdot y + l_B(a)y + r_B(b)x) + (a \circ b + l_A(x)b + r_A(y)a), \quad \forall x, y \in A, a, b \in B. \quad (2.1.10)$$

We denote this associative algebra by $A \rtimes_{l_A, r_A} B$ or simply $A \rtimes B$. On the other hand, every associative algebra with a decomposition into the direct sum of the underlying vector spaces of two subalgebras can be obtained from the above way.

**Proof.** It is straightforward. □
Definition 2.1.5. Let \((A, \cdot)\) and \((B, \circ)\) be two associative algebras. Suppose that there are linear maps \(l_A, r_A : A \to \mathfrak{gl}(B)\) and \(l_B, r_B : B \to \mathfrak{gl}(A)\) such that \((l_A, r_A)\) is a bimodule of \(A\) and \((l_B, r_B)\) is a bimodule of \(B\). If equations \((2.1.4)-(2.1.9)\) are satisfied, then \((A, B, l_A, r_A, l_B, r_B)\) is called a matched pair of associative algebras.

Remark 2.1.6. Obviously \(B\) is an ideal of \(A \bowtie B\) if and only if \(l_B = r_B = 0\). If \(B\) is a trivial (that is, all the products of \(B\) are zero) ideal, then \(A \bowtie_{0,0} B \cong A \bowtie l_A, r_A B\). Moreover, some other special cases of Theorem 2.1.4 have already been studied. For example, the case that \(A\) is left \(B\)-module and \(B\) is a right \(A\)-module was considered in [A1], that is, \(l_A = 0\) and \(r_B = 0\).

2.2. Double constructions of Frobenius algebras and antisymmetric infinitesimal bialgebras. Recall that a (symmetric) Frobenius algebra is an associative algebra \(A\) with a nondegenerate (symmetric) invariant bilinear form. Let \((A, \cdot)\) be an associative algebra. Suppose that there is an associative algebra structure “\(\circ\)” on its dual space \(A^*\). We construct an associative algebra structure on the direct sum \(A \oplus A^*\) of the underlying vector spaces of \(A\) and \(A^*\) such that \((A, \cdot)\) and \((A^*, \circ)\) are subalgebras and the symmetric bilinear form on \(A \oplus A^*\) given by equation \((1.1.1)\) is invariant. That is, \((A \oplus A^*, B)\) is a symmetric Frobenius algebra. Such a construction is called a double construction of Frobenius algebra associated to \((A, \cdot)\) and \((A^*, \circ)\) and we denote it by \((A \bowtie A^*, B)\).

Theorem 2.2.1. Let \((A, \cdot)\) be an associative algebra. Suppose that there is an associative algebra structure “\(\circ\)” on its dual space \(A^*\). Then there is a double construction of Frobenius algebra associated to \((A, \cdot)\) and \((A^*, \circ)\) if and only if \((A, A^*, R^*_A, L^*_A, R^*_B, L^*_B)\) is a matched pair of associative algebras.

Proof. If \((A, A^*, R^*_A, L^*_A, R^*_B, L^*_B)\) is a matched pair of associative algebras, then it is straightforward to show that the bilinear form \((1.1.1)\) is invariant on the associative algebra \(A \bowtie_{R^*_A, L^*_B} A^*\) given by equation \((2.1.10)\). Conversely, set

\[
x \ast a^* = l_A(x)a^* + r_A^*(a^*)x, \quad a^* \ast x = l_A^*(a^*)x + r_A(x)a^*, \quad \forall x \in A, a^* \in A^*.
\]

Then \((A, A^*, l_A, r_A, l_A^*, r_A^*)\) is a matched pair of associative algebras. Note that

\[
\langle l_A(x)a^*, y \rangle = \langle r_A(y)a^*, x \rangle \quad \text{and} \quad \langle l_A^*(b^*)x, a^* \rangle = \langle r_A^*(a^*)x, b^* \rangle = \langle a^* \circ b^*, x \rangle,
\]

where \(x, y \in A, a^*, b^* \in A^*\). Hence, \(l_A = R^*_A, r_A = L^*_A, l_A^* = R^*_B, r_A^* = L^*_B\).

Proposition 2.2.2. Let \((A, \cdot)\) be an associative algebra. Suppose that there is an associative algebra structure “\(\circ\)” on its dual space \(A^*\). Then \((A, A^*, R^*_A, L^*_A, R^*_B, L^*_B)\) is a matched pair of associative algebras if and only if for any \(x \in A^*, a^*, b^* \in A^*\),

\[
R^*_A(x)(a^* \circ b^*) = R^*_A(L^*_B(a^*)x)b^* + (R^*_A(x)a^*) \circ b^*; \tag{2.2.1}
\]

\[
R^*_B(R^*_B(a^*)x)b^* + L^*_A(x)a^* \circ b^* = L^*_A(L^*_B(b^*)x)a^* + a^* \circ (R^*_A(x)b^*). \tag{2.2.2}
\]
Proof. Obviously, equation (2.2.1) is just equation (2.1.4) and equation (2.2.2) is just equation (2.1.8) in the case $l_A = R^*, r_A = L^*, l_B = l_{A^*}, r_B = r_{A^*} = L_{A^*}^*$. By equation (2.1.3), it is easy to show that in this situation,

$$\text{equation (2.1.4)} \iff \text{equation (2.1.5)} \iff \text{equation (2.1.6)} \iff \text{equation (2.1.7)};$$

$$\text{equation (2.1.8)} \iff \text{equation (2.1.9)}.$$ 

Therefore the conclusion holds. \hfill \Box

Before the next study, we give some notations as follows. Let $A$ be an associative algebra. Let $\sigma : A \otimes A \to A \otimes A$ be the exchange operator defined as

$$\sigma(x \otimes y) = y \otimes x, \ \forall x, y \in A. \quad (2.2.3)$$

There are several ways to make $A \otimes A$ into a bimodule of $A$. For example, let $id$ be the identity map on $A$. Then $(id \otimes L, R \otimes id)$ given by (for any $x, a, b \in A$)

$$(id \otimes L)(x)(a \otimes b) = (id \otimes L(x))(a \otimes b) = a \otimes xb, \ (R \otimes id)(x)(a \otimes b) = (R(x) \otimes id)(a \otimes b) = ax \otimes b, \quad (2.2.4)$$

is a bimodule of $A$. Similarly, $(L \otimes id, id \otimes R)$ is also a bimodule of $A$. In fact, equation (1.1.4) given in the introduction can be rewritten as

$$\Delta(ab) = (L(a) \otimes id)\Delta(b) + (id \otimes R(b))\Delta(a), \quad (2.2.5)$$

which gives the notion of infinitesimal bialgebra \((\text{JR})\).

For a linear map $\phi : V_1 \to V_2$, we denote the dual (linear) map by $\phi^* : V_2^* \to V_1^*$ given by

$$\langle v, \phi^*(u^*) \rangle = \langle \phi(v), u^* \rangle, \ \forall v \in V_1, u^* \in V_2. \quad (2.2.6)$$

**Theorem 2.2.3.** Let $(A, \cdot)$ be an associative algebra. Suppose there is an associative algebra structure "•" on its dual space $A^*$ given by a linear map $\Delta^* : A^* \otimes A^* \to A^*$. Then \((A, A^*, R^*, L^*, R_1^*, L_1^*)\) is a matched pair of associative algebras if and only if $\Delta : A \to A \otimes A$ satisfies the following two conditions:

$$\Delta(x \cdot y) = (id \otimes L(x))\Delta(y) + (R(y) \otimes id)\Delta(x); \quad (2.2.7)$$

$$(L(y) \otimes id - id \otimes R(y))\Delta(x) + \sigma[(L(x) \otimes id - id \otimes R(x))\Delta(y)] = 0, \ \forall x, y \in A. \quad (2.2.8)$$

Proof. Let \(\{e_1, \cdots, e_n\}\) be a basis of $A$ and \(\{e_1^*, \cdots, e_n^*\}\) be its dual basis. Set $e_i \cdot e_j = \sum_{k=1}^n c_{ij}^k e_k$ and $e_i^* \circ e_j^* = \sum_{k=1}^n f_{ij}^k e_k^*$. Therefore, we have $\Delta(e_k) = \sum_{i,j=1}^n f_{ij}^k e_i \otimes e_j$ and

$$R^*(e_i) e_j^* = \sum_{k=1}^n c_{kj}^i e_k^* \quad L^*(e_i) e_j^* = \sum_{k=1}^n c_{ik}^j e_k^* \quad R_1^*(e_i^*) e_j = \sum_{k=1}^n f_{ik}^j e_k \quad L_1^*(e_i^*) e_j = \sum_{k=1}^n f_{ik}^j e_k.$$

Hence the coefficient of $e_j \otimes e_k$ in

$$\Delta(e_i \cdot e_m) = (id \otimes L(e_i))\Delta(e_m) + (R(e_m) \otimes id)\Delta(e_i)$$
gives the following relation (for any \(i, j, k, m\))

\[
\sum_{l=1}^{n} c_{ml} f_{jl} = \sum_{l=1}^{n} (c_{ml} f_{jl} + c_{i}^{j} f_{lk})
\]

which is just the relation given by the coefficient of \(e_{m}^{*}\) in

\[
R^{*}(e_{i})(e_{j}^{*} \circ e_{k}^{*}) = R^{*}(L_{o}(e_{j}^{*})e_{i}^{*})e_{k}^{*} + (R^{*}(e_{i})e_{j}^{*}) \circ e_{k}^{*}.
\]

Similarly, equation (2.2.8) corresponds to equation (2.2.2).  \(\square\)

**Remark 2.2.4.** From the symmetry of the associative algebras \((A, \cdot)\) and \((A^{*}, \circ)\) appearing in the double construction, we also can consider the operation \(\beta : A^{*} \to A^{*} \otimes A^{*}\) such that \(\beta^{*} : A \otimes A \to A\) gives an associative algebra structure on \(A\). It is easy to show that \(\Delta\) satisfies equations (2.2.7) and (2.2.8) if and only if \(\beta\) satisfies

\[
\beta(a^{*} \circ b^{*}) = (id \otimes L_{o}(a^{*}))\beta(b^{*}) + (R_{o}(b^{*}) \otimes id)\beta(a^{*}); \quad (2.2.9)
\]

\[
(L_{o}(b^{*}) \otimes id - id \otimes R_{o}(b^{*}))\beta(a^{*}) + \sigma[(L_{o}(a^{*}) \otimes id - id \otimes R_{o}(a^{*}))\beta(b^{*})] = 0, \quad \forall a^{*}, b^{*} \in A. \quad (2.2.10)
\]

**Definition 2.2.5.** Let \(A\) be an associative algebra. An **antisymmetric infinitesimal bialgebra** structure on \(A\) is a linear map \(\Delta : A \to A \otimes A\) such that

(a) \(\Delta^{*} : A^{*} \otimes A^{*} \to A^{*}\) defines an associative algebra structure on \(A^{*}\);

(b) \(\Delta\) satisfies equations (2.2.7) and (2.2.8).

We denote it by \((A, \Delta)\) or \((A, A^{*})\).

**Corollary 2.2.6.** Let \((A, \cdot)\) and \((A^{*}, \circ)\) be two associative algebras. Then the following conditions are equivalent.

1. There is a double construction of Frobenius algebra associated to \((A, \cdot)\) and \((A^{*}, \circ)\);
2. \((A, A^{*}, R^{*}, L^{*}, R_{o}^{*}, L_{o}^{*})\) is a matched pair of associative algebras;
3. \((A, A^{*})\) is an antisymmetric infinitesimal bialgebra.

**Proof.** It follows from Theorems 2.2.1 and 2.2.3.  \(\square\)

**Remark 2.2.7.** As we have pointed out in the introduction, an antisymmetric infinitesimal bialgebra is exactly an associative D-algebra in \([Z]\) where the above equivalence between (1) and (3) was given and a balanced infinitesimal bialgebra in the sense of the opposite algebra in \([A^{3}]\) where the corresponding double construction of Frobenius algebra was called a balanced Drinfeld double as an important consequence. On the other hand, the notion of antisymmetric infinitesimal bialgebra is due to the fact that equation (2.2.7) (in the sense of the opposite algebra) corresponds to equation (2.2.5) which gives the notion of infinitesimal bialgebra and equation (2.2.8) expresses certain antisymmetry.
**Definition 2.2.8.** Let \((A, \Delta_A)\) and \((B, \Delta_B)\) be two antisymmetric infinitesimal bialgebras. A *homomorphism of antisymmetric infinitesimal bialgebras* \(\varphi : A \to B\) is a homomorphism of associative algebras such that
\[
(\varphi \otimes \varphi)\Delta_A(x) = \Delta_B(\varphi(x)), \quad \forall x \in A.
\]
(2.2.11)

An *isomorphism of antisymmetric infinitesimal bialgebras* is an invertible homomorphism of antisymmetric infinitesimal bialgebras.

**Definition 2.2.9.** Let \((A_1 \bowtie A_1^*, B_1)\) and \((A_2 \bowtie A_2^*, B_2)\) be two double constructions of Frobenius algebras. They are *isomorphic* if and only if there exists an isomorphism of associative algebras \(\varphi : A_1 \bowtie A_1^* \to A_2 \bowtie A_2^*\) such that
\[
\varphi(A_1) = A_2, \quad \varphi(A_1^* ) = A_2^*, \quad B_1(x, y) = \varphi^* B_2(x, y) = B_2(\varphi(x), \varphi(y)), \quad \forall x, y \in A_1 \bowtie A_1^*.
\]
(2.2.12)

**Proposition 2.2.10.** Two double constructions of Frobenius algebras are isomorphic if and only if their corresponding antisymmetric infinitesimal bialgebras are isomorphic.

**Proof.** Let \((A_1 \bowtie A_1^*, B_1)\) and \((A_2 \bowtie A_2^*, B_2)\) be two double constructions of Frobenius algebras. Let \(\{e_1, \cdots, e_n\}\) be a basis of \(A_1\) and \(\{e_1^*, \cdots, e_n^*\}\) be its dual basis. If \(\varphi : A_1 \bowtie A_1^* \to A_2 \bowtie A_2^*\) is an isomorphism of double constructions of Frobenius algebras, then \(\varphi|_{A_1} : A_1 \to A_2\) and \(\varphi|_{A_1^*} : A_1^* \to A_2^*\) are isomorphisms of associative algebras. Moreover, \(\varphi|_{A_1^*} = (\varphi|_{A_1})^{*^{-1}}\) since
\[
\langle \varphi|_{A_1^*}(e_i^*), \varphi(e_j) \rangle = B_2(\varphi|_{A_1^*}(e_i^*), \varphi(e_j)) = B_1(e_i^*, e_j) = \delta_{i j} = \langle e_i^*, e_j \rangle
\]
\[
= \langle \varphi^* (\varphi|_{A_1})^{*^{-1}}(e_i^*), e_j \rangle = \langle ((\varphi|_{A_1})^{*^{-1}}(e_i^*), \varphi(e_j) \rangle.
\]

Hence \((A_1, A_1^*)\) and \((A_2, A_2^*)\) are isomorphic as antisymmetric infinitesimal bialgebras. Conversely, let \(\varphi' : A_1 \to A_2\) be an isomorphism between two antisymmetric infinitesimal bialgebras \((A_1, A_1^*)\) and \((A_2, A_2^*)\). Set \(\varphi : A_1 \oplus A_1^* \to A_2 \oplus A_2^*\) be a linear map given by
\[
\varphi(x) = \varphi'(x), \varphi(a^*) = (\varphi'^*)^{-1}(a^*), \quad \forall x \in A_1, a^* \in A_1^*.
\]
Then it is easy to show that \(\varphi\) is an isomorphism of double constructions of Frobenius algebras between \((A_1 \bowtie A_1^*, B_1)\) and \((A_2 \bowtie A_2^*, B_2)\).

**Example 2.2.11.** Let \((A, \Delta)\) be an antisymmetric infinitesimal bialgebra. Then its dual \((A^*, \beta)\) given in Remark 2.2.4 is also an antisymmetric infinitesimal bialgebra.

**Example 2.2.12.** Let \(A\) be an associative algebra. If the associative algebra structure on \(A^*\) is trivial, then either \((A, 0)\) or \((A, A^*)\) is an antisymmetric infinitesimal bialgebra. Moreover, its corresponding Frobenius algebra is given by the semidirect sum \(A \ltimes_{R,L} A^*\) with the natural invariant bilinear form \(B\) given by equation (1.1.1). Dually, if \(A\) is a trivial associative algebra, then the antisymmetric infinitesimal bialgebra structures on \(A\) are in one-to-one correspondence with the associative algebra structures on \(A^*\).
Example 2.2.13. Let \((A, A^*)\) be an antisymmetric infinitesimal bialgebra. In the next subsection, we will prove that there exists a canonical antisymmetric infinitesimal bialgebra structure on the direct sum \(A \oplus A^*\) of the underlying vector spaces of \(A\) and \(A^*\).

2.3. Coboundary (principal) antisymmetric infinitesimal bialgebras. In fact, for an associative algebra \(A\), \(\Delta : A \rightarrow A \otimes A\) satisfying equation (2.2.7) is a 1-cocycle or a derivation of \(A\) associated to the bimodule \((id \otimes L, R \otimes id)\). So it is natural to consider the special case that \(\Delta\) is a 1-coboundary or a principal derivation.

Definition 2.3.1. An antisymmetric infinitesimal bialgebra \((A, \Delta)\) is called coboundary if there exists a \(r \in A \otimes A\) such that

\[
\Delta(x) = (id \otimes L(x) - R(x) \otimes id)r, \quad \forall x \in A. \tag{2.3.1}
\]

Let \(A\) be an associative algebra and \(r \in A \otimes A\). If \(\Delta : A \rightarrow A \otimes A\) is given by equation (2.3.1), then it is obvious that \(\Delta\) satisfies equation (2.2.7). Therefore, \((A, \Delta)\) is an antisymmetric infinitesimal bialgebra if and only if the following two conditions are satisfied:

1. \(\Delta^* : A^* \otimes A^* \rightarrow A^*\) defines an associative algebra structure on \(A^*\).
2. \(\Delta\) satisfies equation (2.2.8).

Lemma 2.3.2. ([A1, Proposition 5.1]) Let \(A\) be an associative algebra and \(r \in A \otimes A\). Define \(\Delta : A \rightarrow A \otimes A\) by

\[
\Delta(a) = [L(x) \otimes id - id \otimes R(x)]r, \quad \forall x \in A. \tag{2.3.2}
\]

Then \(\Delta^* : A^* \otimes A^* \rightarrow A^*\) defines an associative algebra structure on \(A^*\) if and only if

\[
(L(x) \otimes id \otimes id - id \otimes id \otimes R(x))(r_{13}r_{12} + r_{23}r_{13} - r_{12}r_{23}) = 0, \quad \forall x \in A, \tag{2.3.3}
\]

where the notations \(r_{13}r_{12}, r_{23}r_{13}, r_{12}r_{23}\) are given similarly as equation (1.1.3).

Therefore for (1), we use a similar discussion to get the following conclusion.

Proposition 2.3.3. Let \(A\) be an associative algebra and \(r \in A \otimes A\). Define \(\Delta : A \rightarrow A \otimes A\) by equation (2.3.1). Then \(\Delta^* : A^* \otimes A^* \rightarrow A^*\) defines an associative algebra structure on \(A^*\) if and only if

\[
(id \otimes id \otimes L(x) - R(x) \otimes id \otimes id)(r_{12}r_{13} + r_{13}r_{23} - r_{23}r_{12}) = 0. \quad \forall x \in A. \tag{2.3.4}
\]

Proposition 2.3.4. Let \(A\) be an associative algebra and \(r \in A \otimes A\). Define \(\Delta : A \rightarrow A \otimes A\) by equation (2.3.1). Then \(\Delta\) satisfies equation (2.2.8) if and only if \(r\) satisfies

\[
[L(x) \otimes id - id \otimes R(x)][id \otimes L(y) - R(y) \otimes id](r + \sigma(r)) = 0, \quad \forall x, y \in A. \tag{2.3.5}
\]

Proof. It is straightforward. 

Combining Proposition 2.3.3 and Proposition 2.3.4, we have the following conclusion.
Theorem 2.3.5. Let $A$ be an associative algebra and $r \in A \otimes A$. Then the linear map $\Delta$ defined by equation (2.3.1) induces an associative algebra structure on $A^*$ such that $(A, A^*)$ is an antisymmetric infinitesimal bialgebra if and only if equations (2.3.4) and (2.3.5) are satisfied.

Theorem 2.3.6. Let $(A, \Delta_A)$ be an antisymmetric infinitesimal bialgebra. Then there is a canonical antisymmetric infinitesimal bialgebra structure on the direct sum $A \oplus A^*$ of the underlying vector spaces of $A$ and $A^*$ such that both the inclusions $i_1 : A \rightarrow A \oplus A^*$ and $i_2 : A^* \rightarrow A \oplus A^*$ into the two summands are homomorphisms of antisymmetric infinitesimal bialgebras. Here the antisymmetric infinitesimal bialgebra structure on $A^*$ is $(A^*, -\beta_{A^*})$, where $\beta_{A^*} : A^* \rightarrow A^* \otimes A^*$ is given in Remark 2.2.4.

Proof. Let $r \in A \otimes A^* \subset (A \oplus A^*) \otimes (A \oplus A^*)$ correspond to the identity map $id : A \rightarrow A$. Let $\{e_1, \cdots, e_n\}$ be a basis of $A$ and $\{e_1^*, \cdots, e_n^*\}$ be its dual basis. Then $r = \sum_{i=1}^{n} e_i \otimes e_i^*$. Suppose that the associative algebra structure “$*$” on $A \oplus A^*$ is given by $\mathcal{AD}(A) = A \otimes_{R^*, L^*} A^*$. Then by Theorem 2.1.4, we have (for any $x, y \in A, a^*, b^* \in A^*$)

$$x \ast y = x \cdot y, \quad a^* \ast b^* = a^* \circ b^*, \quad x \ast a^* = R^*(x)a^* + L^*_0(a^*)x, \quad a^* \ast x = R^*_0(a^*)x + L^*_0(x)a^*.$$ 

If $r$ satisfies equations (2.3.4) and (2.3.5), then

$$\Delta_{\mathcal{AD}}(u) = (id \otimes L(u) - R(u) \otimes id)r, \quad \forall u \in \mathcal{AD}(A),$$

induces an antisymmetric infinitesimal bialgebra structure on $\mathcal{AD}(A)$.

In fact, for equation (2.3.5), we prove a little stronger conclusion (for any $\mu \in \mathcal{AD}(A)$):

$$(id \otimes L(\mu) - R(\mu) \otimes id)(r + \sigma(r)) = \sum_i (e_i \otimes \mu \ast e_i^* + e_i \otimes \mu \ast e_i - e_i \otimes \mu \ast e_i^* - e_i \otimes \mu \otimes e_i) = 0. \quad (2.3.6)$$

If $\mu = e_j$, then

$$\sum_i e_i \otimes e_j \ast e_i^* = \sum_m e_m \cdot e_j \otimes e_m + \sum_{i,m} (e_i^* \circ e_m, e_j)e_i \otimes e_m; \quad \sum_i e_i^* \otimes e_j \ast e_i = \sum_i e_i^* \otimes e_j \cdot e_i;$$

$$\sum_i e_i \ast e_j \otimes e_i^* = \sum_i e_i \cdot e_j \otimes e_i^*; \quad e_i^* \ast e_j \otimes e_i = \sum_{i,m} (e_j, e_m \circ e_i^*)e_m \otimes e_i + \sum_m e_m \otimes e_j \cdot e_m.$$ 

Hence equation (2.3.6) holds for $\mu = e_j$ by exchanging some indices. Similarly, equation (2.3.6) holds for $\mu = e_j^*$. Therefore equation (2.3.5) holds. Furthermore,

$$r_{12}r_{13} + r_{13}r_{23} - r_{23}r_{12} = \sum_{i,j} \{e_j \otimes e_i \ast e_j^* \otimes e_i^* - e_j \cdot e_i \ast e_j^* \otimes e_i^* - e_i \otimes e_j \ast e_i^* \circ e_j^*\}.$$ 

Since $e_i \ast e_j^* = \sum_m (\langle e_j^*, e_m \cdot e_i \rangle e_m + \langle e_j^* \circ e_m, e_i \rangle e_m)$, we show that $r_{12}r_{13} + r_{13}r_{23} - r_{23}r_{12} = 0$. So $\mathcal{AD}(A)$ is an antisymmetric infinitesimal bialgebra.
For \( e_i \in A \), we have
\[
\Delta_{AD}(e_i) = \sum_{m,k} \{ (e_m^*, e_k \cdot e_i) e_m \otimes e_k^* + (e_m^* \circ e_k^*, e_i) e_m \otimes e_k - (e_m^* \cdot e_k \cdot e_i) e_m \otimes e_k^* \}
\]
\[
= \sum_{m,k} (e_m^* \circ e_k^*, e_i) e_m \otimes e_k = \Delta_A(e_i).
\]
Therefore the inclusion \( i_1 : A \to A \oplus A^* \) is a homomorphism of antisymmetric infinitesimal bialgebras. Similarly, the inclusion \( i_2 : A^* \to A \oplus A^* \) is also a homomorphism of antisymmetric infinitesimal bialgebras since \( \Delta_{AD}(e_i^*) = -\beta_{A^*}(e_i^*) \), where \( \beta_{A^*} \) is given in Remark 2.2.4. □

**Definition 2.3.7.** Let \((A, A^*)\) be an antisymmetric infinitesimal bialgebra. With the antisymmetric infinitesimal bialgebra structure given in Theorem 2.3.6, \(A \oplus A^*\) is called an associative double of \(A\). We denote it by \( AD(A) \).

**Remark 2.3.8.** If we use the opposite algebra, then Theorem 2.3.6 and its proof overlap [A3, Theorem 5.9 and Proposition 5.10] partly. Moreover, the associative double \( AD(A) \) is a balanced Drinfeld double which was denoted by \( D_b(A) \) in [A3].

**Corollary 2.3.9.** Let \((A, A^*)\) be an antisymmetric infinitesimal bialgebra. Then the associative double \( AD(A) \) of \(A\) is an antisymmetric infinitesimal bialgebra and it is a symmetric Frobenius algebra with the bilinear form given by equation (1.1.1).

2.4. **Associative Yang-Baxter equation and its properties.**

**Corollary 2.4.1.** Let \(A\) be an associative algebra and \( r \in A \otimes A \). Suppose that \( r \) is antisymmetric. Then the map \( \Delta \) defined by equation (2.3.1) induces an associative algebra structure on \( A^* \) such that \((A, A^*)\) is an antisymmetric infinitesimal bialgebra if
\[
r_{12}r_{13} + r_{13}r_{23} - r_{23}r_{12} = 0. \tag{2.4.1}
\]

**Definition 2.4.2.** Let \(A\) be an associative algebra and \( r \in A \otimes A \). Equation (2.4.1) is called associative Yang-Baxter equation in \(A\).

**Remark 2.4.3.** In [A1] and [A3], the associative Yang-Baxter equation is given as
\[
r_{13}r_{12} + r_{23}r_{13} - r_{12}r_{23} = 0. \tag{2.4.2}
\]

Note that equation (2.4.1) is equation (2.4.2) in the opposite algebra. Moreover, if \( r \) satisfies
\[
(L(x) \otimes id \otimes id - id \otimes id \otimes R(x))(r_{12} + r_{21}) = 0,
\]
then ([A3, Lemma 3.4])
\[
\sigma_{13}(r_{12}r_{13} + r_{13}r_{23} - r_{23}r_{12}) = r_{13}r_{12} + r_{23}r_{13} - r_{12}r_{23}, \tag{2.4.3}
\]
where the linear map \( \sigma_{13} : A \otimes A \otimes A \to A \otimes A \otimes A \) is given by \( \sigma_{13}(x \otimes y \otimes z) = z \otimes y \otimes x \) for any \( x, y, z \in A \). In particular, when \( r \) is antisymmetric, the above two associative Yang-Baxter equations are equivalent.
In order to be self-contained, in the following we give some properties of associative Yang-Baxter equation from the point of view of Frobenius algebras, although some of them have already been given in [A3]. Let $A$ be a vector space. For any $r \in A \otimes A$, $r$ can be regarded as a map from $A^*$ to $A$ in the following way:

$$\langle u^* \otimes v^*, r \rangle = \langle u^*, r(v^*) \rangle, \quad \forall \ u^*, v^* \in A^*. \quad (2.4.4)$$

**Proposition 2.4.4.** Let $(A, \cdot)$ be an associative algebra and $r \in A \otimes A$ be an antisymmetric solution of associative Yang-Baxter equation in $A$. Then the associative algebra structure on the associative double $AD(A)$ is given from the products in $A$ as follows.

(a) $a^* \circ b^* = a^* \circ b^* = R^*(r(a^*))b^* + L^*(r(b^*))a^*$, for any $a^*, b^* \in A^*$; \hspace{1cm} (2.4.5)

(b) $x \circ a^* = x \cdot r(a^*) - r(R^*(x)a^*) + R^*(x)a^*$, for any $x \in A$, $a^* \in A^*$; \hspace{1cm} (2.4.6)

(c) $a^* \circ x = r(a^*) \cdot x - r(L^*(x)a^*) + L^*(x)a^*$, for any $x \in A$, $a^* \in A^*$. \hspace{1cm} (2.4.7)

**Proof.** Let $\{e_1, \ldots, e_n\}$ be a basis of $A$ and $\{e_1^*, \ldots, e_n^*\}$ be its dual basis. Suppose that $e_i \circ e_j = \sum_k c_{ij}^k e_k$ and $r = \sum_{ij} a_{ij} e_i \otimes e_j$, where $a_{ij} = -a_{ji}$. Then for any $i$, we have

$$\Delta(e_i) = \sum_{\alpha, \beta, l} a_{\alpha \beta} (e_\alpha \otimes e_l - e_\alpha \otimes e_l) = \sum_{\alpha, \beta} (a_{\alpha \beta} c_{il}^\beta e_\alpha \otimes e_\beta).$$

Therefore we show that (for any $i, j$)

$$e_i^* \circ e_j^* = \sum_{t, t} (a_{tt} c_{il}^t - a_{ij} c_{il}^t) e_t^* = \sum_{t, t} (a_{tt} \langle e_t \cdot e_i, e_j^* \rangle - a_{ij} \langle e_l \cdot e_i, e_j^* \rangle) e_t^*$$

$$= \sum_t ((\langle e_t \cdot r(e_i^*) \rangle, e_j^*) + \langle r(e_j^*) \cdot e_l, e_i^* \rangle) e_t^* = R^*(r(e_i^*))e_j^* + L^*(r(e_j^*))e_i^*.$$

Similarly, equations (2.4.6) and (2.4.7) hold. \qed

**Theorem 2.4.5.** ([A3, Proposition 2.1]) Let $A$ be an associative algebra and $r \in A \otimes A$. Suppose that $r$ is antisymmetric and nondegenerate. Then $r$ is a solution of associative Yang-Baxter equation in $A$ if and only if the inverse of the isomorphism $A^* \rightarrow A$ induced by $r$, regarded as a bilinear form $\omega$ on $A$ (that is, $\omega(x, y) = \langle r^{-1} x, y \rangle$ for any $x, y \in A$), is a Connes cocycle.

**Corollary 2.4.6.** Let $(A, \cdot)$ be an associative algebra and $r \in A \otimes A$ be a nondegenerate antisymmetric solution of associative Yang-Baxter equation in $A$. Suppose the associative algebra structure “$\circ$” on $A^*$ is induced by $r$ from equation (2.4.5). Then we have

$$a^* \circ b^* = r^{-1}(r(a^*) \cdot r(b^*)), \quad \forall a^*, b^* \in A^*. \quad (2.4.8)$$

Therefore $r : A^* \rightarrow A$ is an isomorphism of associative algebras.

**Proof.** Set $\omega(x, y) = \langle r^{-1}(x), y \rangle$ for any $x, y \in A$. Then $\omega$ is a Connes cocycle of $A$. Hence

$$\langle a^* \circ b^*, x \rangle = \langle r(b^*) \cdot x, a^* \rangle + \langle x \cdot r(a^*), b^* \rangle = \omega(r(a^*), r(b^*) \cdot x) + \omega(r(b^*), x \cdot r(a^*))$$

$$= -\omega(x, r(a^*) \cdot r(b^*)) = \langle r^{-1}(r(a^*) \cdot r(b^*)), x \rangle, \quad \forall a^*, b^* \in A^*, x \in A.$$
So equation (2.4.8) holds. Therefore $r$ is an isomorphism of associative algebras. \hfill \Box

Next we turn to the general antisymmetric solutions of associative Yang-Baxter equation.

**Theorem 2.4.7.** Let $(A, \cdot)$ be an associative algebra and $r \in A \otimes A$ be antisymmetric. Then $r$ is a solution of associative Yang-Baxter equation in $A$ if and only if $r$ satisfies

$$r(a^* \cdot b^*) = r(R^* (r(a^*)) b^* + L^* (r(b^*)) a^*), \quad \forall a^*, b^* \in A^*.$$ (2.4.9)

**Proof.** Let $\{e_1, \ldots, e_n\}$ be a basis of $A$ and $\{e_1^*, \ldots, e_n^*\}$ be its dual basis. Suppose that $e_i \cdot e_j = \sum_k c_{ij}^k e_k$ and $r = \sum_{i,j} a_{ij} e_i \otimes e_j$, $a_{ij} = -a_{ji}$. Hence $r(e_i^*) = \sum_k a_{ki} e_k$. Then $r$ is a solution of associative Yang-Baxter equation in $A$ if and only if (for any $i, j, k$)

$$\sum_{m,l} \{ c_{kl}^m a_{ik} a_{jl} - c_{il}^k a_{jm} a_{km} - c_{jk}^l a_{im} a_{ik} \} = 0.$$  

The left-hand side of the above equation is just the coefficient of $e_m$ in

$$r(e_i^*) \cdot r(e_j^*) - r(R^* (r(e_i^*)) e_j^*) + L^* (r(e_j^*)) e_i^*. $$

Therefore the conclusion follows. \hfill \Box

Combining Proposition 2.4.4 and Theorem 2.4.7, we have the following conclusion which extends Corollary 2.4.6.

**Corollary 2.4.8.** Let $(A, \cdot)$ be an associative algebra and $r \in A \otimes A$ be an antisymmetric solution of associative Yang-Baxter equation in $A$. Suppose the associative algebra structure “$\circ$” on $A^*$ is induced by $r$ from equation (2.4.5). Then we have

$$r(a^* \circ b^*) = r(a^*) \cdot r(b^*), \quad \forall a^*, b^* \in A^*.$$ (2.4.10)

Therefore $r : A^* \to A$ is an homomorphism of associative algebras.

Recall that two Frobenius algebras $(A_1, \mathcal{B}_1)$ and $(A_2, \mathcal{B}_2)$ are isomorphic if and only if there exists an isomorphism of associative algebras $\varphi : A_1 \to A_2$ such that

$$\mathcal{B}_1(x, y) = \varphi^* \mathcal{B}_2(x, y) = \mathcal{B}_2(\varphi(x), \varphi(y)), \quad \forall x, y \in A_1.$$ (2.4.11)

**Theorem 2.4.9.** Let $(A, \cdot)$ be an associative algebra. Then as Frobenius algebras, the Frobenius algebra $(A \bowtie_{R^* L^*} A^*, \mathcal{B})$ given by an antisymmetric solution $r$ of associative Yang-Baxter equation in $A$ is isomorphic to the Frobenius algebra $(A \ltimes_{R^* L^*} A^*, \mathcal{B})$, where $\mathcal{B}$ is given by equation (1.1.1). However, in general, they are not isomorphic as the double constructions of Frobenius algebras (or equivalently, as antisymmetric infinitesimal bialgebras).

**Proof.** Let $r$ be an antisymmetric solution of associative Yang-Baxter equation in $A$. Define a linear map $\varphi : A \ltimes_{R^* L^*} A^* \to A \bowtie_{R^* L^*} A^*$ satisfying

$$\varphi(x) = x, \quad \varphi(a^*) = -r(a^*) + a^*, \quad \forall x \in A, a^* \in A^*.$$
It is straightforward to show that \( \varphi \) is an isomorphism of associative algebras. Moreover,
\[
\varphi^* B(x + a^* y + b^*) = \langle a^*, -r(b^*) + y \rangle + \langle x - r(a^*), b^* \rangle = \langle a^*, y \rangle + \langle x, b^* \rangle = B(x + a^*, y + b^*).
\]
Therefore \( \varphi \) is an isomorphism of Frobenius algebras. However in general, as antisymmetric infinitesimal bialgebras, they are not isomorphic. In fact, if \( \psi \) is an isomorphism of antisymmetric infinitesimal bialgebras between \( A \rtimes_{R^*, L^*} A^* \) and \( A 
abla_{R^*, L^*} A^* \), then for any \( u^*, v^* \in A^* \), there exist \( a^*, b^* \in A^* \) such that \( \psi(a^*) = u^*, \psi(b^*) = v^* \). However, \( \psi(a^* \circ b^*) = 0 \) and \( \psi(a^*) \ast \psi(b^*) = u^* \ast v^* = R^*(r(a^*))b^* + L^*(r(b^*))a^* \) is not zero in general, which is a contradiction.

**Corollary 2.4.10.** Let \( (A, \cdot) \) be an associative algebra. Then as Frobenius algebras, the Frobenius algebras \( (A 
abla_{R^*, L^*} A^*, B) \) given by all antisymmetric solutions of associative Yang-Baxter equation in \( A \) are isomorphic to the Frobenius algebra \( (A \rtimes_{R^*, L^*} A^*, B) \) given by the zero solution.

### 2.5. Associative Yang-Baxter equation and \( O \)-operators.

**Definition 2.5.1.** Let \( (A, \cdot) \) be an associative algebra and \( (l, r, V) \) be a bimodule. A linear map \( T : V \to A \) is called an \( O \)-operator associated to \( (l, r, V) \) if \( T \) satisfies
\[
T(u) \cdot T(v) = T(l(T(u))v + r(T(v))u), \quad \forall u, v \in V.
\]

**Example 2.5.2.** Let \( (A, \cdot) \) be an associative algebra. Then the identity map \( id \) is an \( O \)-operator associated to the bimodule \( (L, 0) \) or \( (0, R) \).

**Example 2.5.3.** Let \( (A, \cdot) \) be an associative algebra. A linear map \( R : A \to A \) is called a Rota-Baxter operator on \( A \) of weight zero (cf. \cite{Bax}, \cite{Rot}) if \( R \) satisfies
\[
R(x) \cdot R(y) = R(R(x) \cdot y + x \cdot R(y)), \quad \forall x, y \in A.
\]
In fact, a Rota-Baxter operator on \( A \) is just an \( O \)-operator associated to the bimodule \( (L, R) \).

**Example 2.5.4.** Let \( (A, \cdot) \) be an associative algebra and \( r \in A \otimes A \) be antisymmetric. Then \( r \) is a solution of associative Yang-Baxter equation in \( A \) if and only if \( r \) is an \( O \)-operator associated to the bimodule \( (R^*, L^*) \).

**Theorem 2.5.5.** (\cite{BGN1}) Let \( (A, \cdot) \) be an associative algebra and \( (l, r, V) \) be a bimodule. Let \( (r^*, l^*, V^*) \) be the bimodule of \( A \) given by Lemma 2.1.2. Let \( T : V \to A \) be a linear map which is identified as an element in \( (A \rtimes_{r^*, l^*} V^*) \otimes (A \rtimes_{r^*, l^*} V^*) \). Then \( r = T - \sigma(T) \) is an antisymmetric solution of the associative Yang-Baxter equation in \( A \rtimes_{r^*, l^*} V^* \) if and only if \( T \) is an \( O \)-operator associated to the bimodule \( (l, r, V) \).

**Corollary 2.5.6.** (cf. Corollary 3.1.5) Let \( (A, \cdot) \) be an associative algebra. Then
\[
r = \sum_{i=1}^{n} (e_i \otimes e_i^* - e_i^* \otimes e_i)
\]
is a solution of the associative Yang-Baxter equation in $A \ltimes_{R^*,0} A^*$ or $A \ltimes_{0,L^*} A^*$, where $\{e_1, \cdots, e_n\}$ is a basis of $A$ and $\{e_1^*, \cdots, e_n^*\}$ is its dual basis. Moreover, there is a natural Connes cocycle $\omega$ on $A \ltimes_{R^*,0} A^*$ or $A \ltimes_{0,L^*}$ induced by $r^{-1} : A \oplus A^* \rightarrow (A \oplus A^*)^*$, which is given by equation (1.4.1).

Proof. Note that $id$ is an $O$-operator associated to the bimodule $(L, 0, A)$ or $(0, R, A)$. Then the conclusion follows from Theorems 2.5.5 and 2.4.5. \qed

3. Dendriform algebras

3.1. $O$-operators and dendriform algebras.

There are close relations between $O$-operators and a class of algebras, namely, dendriform algebras, which are given in [BGN2]. In order to be self-contained, we list them in this subsection.

Definition 3.1.1. ([Lo1]) Let $A$ be a vector space over a field $F$ with two bilinear products denoted by $<$ and $>$. $(A, <, >)$ is called a dendriform algebra if for any $x, y, z \in A$,

\[(x \prec y) \prec z = x \prec (y \prec z), \quad (x \succ y) \prec z = x \succ (y \succ z), \quad x \succ (y \succ z) = (x \ast y) \succ z, \quad (3.1.1)\]

where $x \ast y = x \prec y + x \succ y$.

Let $(A, <, >)$ be a dendriform algebra. For any $x \in A$, let $L_>(x), R_>(x)$ and $L_<(x), R_<(x)$ denote the left and right multiplication operators of $(A, <)$ and $(A, >)$ respectively, that is,

\[L_>(x)(y) = x \succ y, \quad R_>(x)y = y \succ x, \quad L_<(x)y = x < y, \quad R_<(x)(y) = y < x, \quad \forall \; x, y \in A.\]

Moreover, let $L_>, R_>, L_<, R_<_A \rightarrow gl(A)$ be four linear maps with $x \rightarrow L_>(x), x \rightarrow R_>(x), x \rightarrow L_<(x)$ and $x \rightarrow R_<(x)$ respectively. It is known that the product given by ([Lo1])

\[x \ast y = x < y + x > y, \quad \forall x, y \in A, \quad (3.1.2)\]

defines an associative algebra. We call $(A, \ast)$ the associated associative algebra of $(A, >, <)$ and $(A, >, <)$ is called a compatible dendriform algebra structure on the associative algebra $(A, \ast)$. Moreover, $(L_>, R_>)$ is a bimodule of the associated associative algebra $(A, \ast)$.

Theorem 3.1.2. ([BGN2]) Let $A$ be an associative algebra and $(l, r, V)$ be a bimodule. Let $T : V \rightarrow A$ be an $O$-operator associated to $(l, r, V)$. Then there exists a dendriform algebra structure on $V$ given by

\[u > v = l(T(u))v, \quad u \prec v = r(T(v))u, \quad \forall \; u, v \in V. \quad (3.1.3)\]

So there is an associated associative algebra structure on $V$ given by equation (3.1.2) and $T$ is a homomorphism of associative algebras. Moreover, $T(V) = \{T(v) | v \in V\} \subset A$ is an associative subalgebra of $A$ and there is an induced dendriform algebra structure on $T(V)$ given by

\[T(u) \succ T(v) = T(u > v), \quad T(u) \prec T(v) = T(u \prec v), \quad \forall \; u, v \in V. \quad (3.1.4)\]
Its corresponding associated associative algebra structure on $T(V)$ given by equation (3.1.2) is just the associative subalgebra structure of $A$ and $T$ is a homomorphism of dendriform algebras.

**Corollary 3.1.3. ([BGN2])** Let $(A, *)$ be an associative algebra. There is a compatible dendriform algebra structure on $A$ if and only if there exists an invertible $O$-operator of $(A, *)$.

In fact, if $T$ is an invertible $O$-operator associated to a bimodule $(l, r, V)$, then the compatible dendriform algebra structure on $A$ is given by

$$x \triangleright y = T(l(x)T^{-1}(y)), \quad x \triangleleft y = T(r(y)T^{-1}(x)), \quad \forall x, y \in A. \quad (3.1.5)$$

Conversely, let $(A, \triangleright, \triangleleft)$ be a dendriform algebra and $(A, *)$ be the associated associative algebra. Then the identity map $id$ is an $O$-operator associated to the bimodule $(L_\triangleright, R_\triangleleft)$ of $(A, *)$.

**Remark 3.1.4.** If $T$ is an invertible $O$-operator associated to a bimodule $(l, r, V)$, then the linear map $f = T^{-1}: A \to V$ satisfies

$$f(x * y) = l(x)f(y) + r(y)f(x), \quad \forall x, y \in A. \quad (3.1.6)$$

Such a linear map is a 1-cocycle of $(A, *)$ associated to the bimodule $(l, r, V)$.

**Corollary 3.1.5. ([BGN2])** Let $(A, \triangleright, \triangleleft)$ be a dendriform algebra. Then

$$r = \sum_{i=1}^{n} (e_i \otimes e_i^* - e_i^* \otimes e_i) \quad (3.1.7)$$

is a solution of the associative Yang-Baxter equation in $A \ltimes_{R_\triangleright, L_\triangleleft} A^*$, where $\{e_1, \ldots, e_n\}$ is a basis of $A$ and $\{e_1^*, \ldots, e_n^*\}$ is its dual basis. Moreover there is a natural Connes cocycle $\omega$ on $A \ltimes_{R_\triangleright, L_\triangleleft} A^*$ induced by $r^{-1}: A \oplus A^* \to (A \oplus A^*)^*$, which is given by equation (1.4.1).

**Remark 3.1.6.** It is easy to see that Corollary 2.5.6 is just a special case of the above conclusion, that is, the former corresponds to the trivial dendriform algebra structure on an associative algebra $(A, \cdot)$ given by $\triangleright = \triangleleft = \cdot = 0$ or $\triangleright = 0, \triangleleft = \cdot$.

### 3.2. Bimodules and matched pairs of dendriform algebras.

**Definition 3.2.1. ([A4])** Let $(A, \triangleright, \triangleleft)$ be a dendriform algebra and $V$ be a vector space. Let $l_\triangleright, r_\triangleright, l_\triangleleft, r_\triangleleft : A \to \mathfrak{gl}(V)$ be four linear maps. $V$ (or $(l_\triangleright, r_\triangleright, l_\triangleleft, r_\triangleleft)$, or $(l_\triangleright, r_\triangleright, l_\triangleleft, r_\triangleleft, V)$) is called a bimodule of $A$ if the following equations hold (for any $x, y \in A$).

$$l_\triangleright(x \triangleleft y) = l_\triangleright(x)l_\triangleright(y); \quad r_\triangleright(x)l_\triangleright(y) = l_\triangleright(y)r_\triangleright(x); \quad r_\triangleright(x)r_\triangleright(y) = r_\triangleright(y * x); \quad (3.2.1)$$

$$l_\triangleleft(x \triangleright y) = l_\triangleleft(x)l_\triangleleft(y); \quad r_\triangleleft(x)l_\triangleleft(y) = l_\triangleleft(y)r_\triangleleft(x); \quad r_\triangleleft(x)r_\triangleleft(y) = r_\triangleleft(y < x); \quad (3.2.2)$$

$$l_\triangleright(x * y) = l_\triangleright(x)l_\triangleright(y); \quad r_\triangleright(x)l_\triangleright(y) = l_\triangleright(y)r_\triangleright(x); \quad r_\triangleright(x)r_\triangleright(y) = r_\triangleright(y > x), \quad (3.2.3)$$

where $x * y = x \triangleright y + x < y, l_\ast = l_\triangleright + l_\triangleleft, r_\ast = r_\triangleright + r_\triangleleft.$
By a direct computation or according to \( \text{Sc}\), \((l_>, r_>, l_\prec, r_\prec, V)\) is a bimodule of a dendriform algebra \((A, >, <)\) if and only if there exists a dendriform algebra structure on the direct sum \(A \oplus V\) of the underlying vector spaces of \(A\) and \(V\) given by \((\forall x, y \in A, u, v \in V)\)

\[
(x + u) > (y + v) = x > y + l_\prec(x)v + r_\prec(y)u, \quad (x + u) < (y + v) = x < y + l_\prec(x)v + r_\prec(y)u.
\]

We denote it by \(A \ltimes_{l_>, r_>, l_\prec, r_\prec} V\).

**Proposition 3.2.2.** Let \((l_>, r_>, l_\prec, r_\prec, V)\) be a bimodule of a dendriform algebra \((A, >, <)\). Let \((A, *)\) be the associated associative algebra. Then we have the following results.

1. Both \((l_>, r_\prec, V)\) and \((l_\prec + l_>, r_\prec + r_\prec, V)\) are bimodules of \((A, *)\).
2. For any bimodule \((l, r, V)\) of \((A, *)\), \((l, 0, 0, r, V)\) is a bimodule of \((A, >, <)\).
3. Both \((l_\prec + l_>, 0, 0, r_\prec + r_\prec, V)\) and \((l_\prec, 0, 0, r_\prec, V)\) are bimodules of \((A, >, <)\).
4. The dendriform algebras \(A \ltimes_{l_>, r_>, l_\prec, r_\prec} V\) and \(A \ltimes_{l_\prec + l_>, 0, 0, r_\prec + r_\prec} V\) have the same associated associative algebra \(A \ltimes_{l_\prec, 0, 0, l_\prec} V^*\).
5. Let \(l_*^+, r_*^+, l_*^-, r_*^- : A \to \mathfrak{gl}(V^*)\) be the linear maps given by

\[
\langle l_*^+(x)a^*, y \rangle = \langle l_\prec(x)y, a^* \rangle, \quad \langle r_*^+(x)a^*, y \rangle = \langle r_\prec(x)y, a^* \rangle, \quad (3.2.5)
\]

\[
\langle l_*^-(x)a^*, y \rangle = \langle l_\prec(x)y, a^* \rangle, \quad \langle r_*^-(x)a^*, y \rangle = \langle r_\prec(x)y, a^* \rangle.
\]

(3.2.6)

Then \((l_*^+, r_*^+, l_*^-, r_*^-, V^* )\) is a bimodule of \((A, >, <)\).

6. Both \((r_*^+, l_*^+, 0, 0, l_*^-, V^*)\) and \((r_*^-, l_*^-, 0, 0, l_*^+, V^*)\) are bimodules of \((A, >, <)\).

7. Both \((r_*^+, l_*^+, l_*^+, 0, V^*)\) and \((r_*^-, l_*^-, l_*^-, 0, V^*)\) are bimodules of \((A, *)\).

8. The dendriform algebras \(A \ltimes_{r_*^+, l_*^+, l_*^+, l_*^-, V^*} V\) and \(A \ltimes_{r_*^-, l_*^-, l_*^-, l_*^+, V^*} V\) have the same associative algebra \(A \ltimes_{r_*^-, l_*^-, l_*^-, l_*^+, V^*} V^*\).

**Proof.** It is straightforward. \(\square\)

**Example 3.2.3.** Let \((A, >, <)\) be a dendriform algebra. Then

\[
(l_\prec, R_\prec, L_\prec, R_\prec, A), \quad (L_\succ, 0, 0, R_\succ, A) \quad \text{and} \quad (L_\succ + L_\prec, 0, 0, R_\succ + R_\prec, A)
\]

are bimodules of \((A, <, >)\). On the other hand,

\[
(R_*^+, R_*^+, L_*^+, L_*^+, 0, 0, L_*^+, 0, L_*^+, A^*) \quad \text{and} \quad (R_*^+, R_*^+, 0, 0, L_*^+, L_*^+, A^*)
\]

are bimodules of \((A, >, <)\), too. There are two compatible dendriform algebra structures

\[
A \ltimes_{R_*^+, R_*^+, R_*^+, L_*^+, L_*^+, L_*^+, A^*} \quad \text{and} \quad A \ltimes_{R_*^+, 0, 0, L_*^+, A^*}
\]

on the same associative algebra \(A \ltimes_{R_*^+, L_*^+, A^*}\).

**Theorem 3.2.4.** Let \((A, >_A, <_A)\) and \((B, >_B, <_B)\) be two dendriform algebras. Suppose that there are linear maps \(l_{>_A}, r_{>_A}, l_{<_A}, r_{<_A} : A \to \mathfrak{gl}(B)\) and \(l_{>_B}, r_{>_B}, l_{<_B}, r_{<_B} : B \to \mathfrak{gl}(A)\) such
that \((l_{\succ A}, r_{\succ A}, l_{\prec A}, r_{\prec A})\) is a bimodule of \(A\) and \((l_{\succ B}, r_{\succ B}, l_{\prec B}, r_{\prec B})\) is a bimodule of \(B\) and they satisfy the following 18 equations:

\[
\begin{align*}
r_{\prec A}(x)(a \prec_B b) &= a \prec_B (r_A(x)b) + r_{\prec A}(l_B(b)x)a; \\
l_{\prec A}(l_{\prec B}(x)a)b + (r_{\prec A}(x)a) &\prec_B b = a \prec_B (l_A(x)b) + r_{\prec A}(r_B(b)x)a; \\
l_{\prec A}(x)(a \ast_B b) &= (l_{\prec A}(x)a) \prec_B b + l_{\prec A}(r_{\prec B}(a)x)b; \\
r_{\prec A}(x)(a \succ_B b) &= r_{\prec A}(l_{\prec B}(b)x)a + a \succ_B (r_{\prec A}(x)b); \\
l_{\prec A}(l_{\succ B}(a)x)b + (r_{\prec A}(x)a) &\prec_B b = a \succ_B (l_{\prec A}(x)b) + r_{\prec A}(r_{\prec B}(b)x)a; \\
l_{\prec A}(x)(a \prec_B b) &= (l_{\prec A}(x)a) \prec_B b + l_{\prec A}(r_{\prec B}(a)x)b; \\
r_{\prec A}(x)(a \ast_B b) &= a \succ_B (r_{\prec A}(x)b) + r_{\prec A}(l_{\prec B}(b)x)a; \\
A \succ_B (l_{\prec A}(x)b) + r_{\prec A}(r_{\prec B}(b)x)a = l_{\prec A}(l_B(a)x)b + (r_A(x)a) \succ_B b; \\
l_{\prec A}(x)(a \succ_B b) &= (l_A(x)a) \succ_B b + l_{\prec A}(r_B(a)x)b; \\
r_{\prec B}(x)(x \prec_A y) &= x \prec_A (r_B(a)y) + r_{\prec B}(l_A(y)a)x; \\
l_{\prec B}(l_{\prec A}(x)a)y + (r_{\prec B}(a)x) &\prec_A y = x \prec_A (l_B(a)y) + r_{\prec B}(r_A(y)a)x; \\
l_{\prec B}(a)(x \ast_A y) &= (l_{\prec B}(a)x) \prec_A y + l_{\prec B}(r_{\prec A}(x)a)y; \\
r_{\prec B}(a)(x \succ_A y) &= r_{\prec B}(l_{\prec A}(y)a)x + x \succ_A (r_{\prec B}(a)y); \\
l_{\prec B}(l_{\succ A}(x)a)y + (r_{\prec B}(a)x) &\prec_A y = x \succ_A (l_{\prec B}(a)y) + r_{\prec B}(r_{\prec A}(y)a)x; \\
l_{\prec B}(a)(x \prec_A y) &= (l_{\prec B}(a)x) \prec_A y + l_{\prec B}(r_{\prec A}(x)a)y; \\
r_{\prec B}(a)(x \ast_A y) &= x \succ_A (r_{\prec B}(a)y) + r_{\prec B}(l_{\prec A}(y)a)x; \\
x \succ_A (l_{\prec B}(a)y) + r_{\prec B}(r_{\prec A}(y)a)x = l_{\prec B}(l_A(x)a)y + (r_B(a)x) \succ_A y; \\
l_{\prec B}(a)(x \succ_A y) &= (l_B(a)x) \succ_A y + l_{\prec B}(r_A(x)a)y; \\
for any \(x, y \in A, a, b \in B\) and \(l_A = l_{\succ A} + l_{\prec A}, r_A = r_{\succ A} + r_{\prec A}, l_B = l_{\succ B} + l_{\prec B}, r_B = r_{\succ B} + r_{\prec B}\). Then there is a dendriform algebra structure on the direct sum \(A \oplus B\) of the underlying vector spaces of \(A\) and \(B\) given by

\[
\begin{align*}
(x + a) \succ (y + b) &= (x \succ_A y + r_{\prec B}(b)x + l_{\succ B}(a)y) + (l_{\succ A}(x)b + r_{\succ A}(y)a + a \prec_B b), \\
(x + a) \prec (y + b) &= (x \prec_A y + r_{\prec B}(b)x + l_{\prec B}(a)y) + (l_{\prec A}(x)b + r_{\prec A}(y)a + a \prec_B b),
\end{align*}
\]
for any \(x, y \in A, a, b \in B\). We denote this dendriform algebra by \(A \gg_{l_{\succ A} \prec_A l_{\prec A}, r_{\succ A} \prec_B} B\) or simply \(A \gg B\). On the other hand, every dendriform algebra which is the direct sum of the underlying vector spaces of two subalgebras can be obtained from the above way.

Proof. It is straightforward.
Definition 3.2.5. Let \((A, \succ_A, \prec_A)\) and \((B, \succ_B, \prec_B)\) be two dendriform algebras. Suppose that there are linear maps \(l_{\succ_A}, r_{\succ_A}, l_{\prec_A}, r_{\prec_A} : A \to \mathfrak{gl}(B)\) and \(l_{\succ_B}, r_{\succ_B}, l_{\prec_B}, r_{\prec_B} : B \to \mathfrak{gl}(A)\) such that \((l_{\succ_A}, r_{\succ_A}, l_{\prec_A}, r_{\prec_A})\) is a bimodule of \(A\) and \((l_{\succ_B}, r_{\succ_B}, l_{\prec_B}, r_{\prec_B})\) is a bimodule of \(B\). If equations (3.2.7)-(3.2.24) are satisfied, then \((A, B, l_{\succ_A}, r_{\succ_A}, l_{\prec_A}, l_{\succ_B}, r_{\succ_B}, l_{\prec_B}, r_{\prec_B})\) is called a matched pair of dendriform algebras.

Remark 3.2.6. Obviously \(B\) is an ideal of \(A \bowtie B\) if and only if \(l_{\succ_B} = r_{\succ_B} = l_{\prec_B} = r_{\prec_B} = 0\). If \(B\) is a trivial ideal, then \(A \bowtie_{0,0,0} B \simeq A \bowtie_{l_{\succ_A}, r_{\succ_A}, l_{\prec_A}, r_{\prec_A}} B\).

Corollary 3.2.7. Let \((A, B, l_{\succ_A}, r_{\succ_A}, l_{\prec_A}, r_{\prec_A}, l_{\succ_B}, r_{\succ_B}, l_{\prec_B}, r_{\prec_B})\) be a matched pair of dendriform algebras. Then \((A, B, l_{\succ_A} + l_{\prec_A}, r_{\succ_A} + r_{\prec_A}, l_{\succ_B} + l_{\prec_B}, r_{\succ_B} + r_{\prec_B})\) is a matched pair of the associated associative algebras \((A, *_A)\) and \((B, *_B)\).

Proof. In fact, the associated associative algebra \((A \bowtie B, *)\) is exactly the associative algebra obtained from the matched pair \((A, B, l_A, r_A, l_B, r_B)\) of associative algebras:

\[(x + a) * (y + b) = x *_A y + l_B(a)y + r_B(b)x + a *_B b + l_A(x)b + r_A(y)a, \quad \forall x, y, a, b \in B,\]

where \(l_A = l_{\succ_A} + l_{\prec_A}, r_A = r_{\succ_A} + r_{\prec_A}, l_B = l_{\succ_B} + l_{\prec_B}, r_B = r_{\succ_B} + r_{\prec_B}.\)

4. Double constructions of Connes cocycles and an analogue of the classical Yang-Baxter equation

4.1. Connes cocycles and dendriform algebras.

Theorem 4.1.1. Let \((A, *)\) be an associative algebra and \(\omega\) be a nondegenerate Connes cocycle. Then there exists a compatible dendriform algebra structure \(\succ, \prec\) on \(A\) given by

\[\omega(x \succ y, z) = \omega(y, z * x), \quad \omega(x \prec y, z) = \omega(x, y * z), \quad \forall x, y, z \in A.\] (4.1.1)

Proof. Define a linear map \(T : A \to A^*\) by \(T(x, y) = \omega(x, y), \quad \forall x, y \in A\). Then \(T\) is invertible and \(T^{-1}\) is an \(O\)-operator of the associative algebra \((A, *)\) associated to the bimodule \((R^*_a, L^*_a)\).

By Corollary 3.1.3, there is a compatible dendriform algebra structure \(\succ, \prec\) on \((A, *)\) given by

\[x \succ y = T^{-1}R^*_a(x)T(y), \quad x \prec y = T^{-1}L^*_a(y)T(x), \quad \forall x, y \in A,\]

which gives exactly equation (4.1.1). \(\square\)

Next, we turn to the double construction of Connes cocycles. Let \((A, *_A)\) be an associative algebra and suppose that there is a associative algebra structure \(*_{A^*}\) on its dual space \(A^*\). We construct an associative algebra structure on the direct sum \(A \oplus A^*\) of the underlying vector spaces of \(A\) and \(A^*\) such that both \(A\) and \(A^*\) are subalgebras and the antisymmetric bilinear form on \(A \oplus A^*\) given by equation (1.4.1) is a Connes cocycle on \(A \oplus A^*\). Such a construction is called a double construction of Connes cocycle associated to \((A, *_A)\) and \((A^*, *_{A^*})\) and we denote it by \((T(A) = A \bowtie A^*, \omega)\).
Corollary 4.1.2. Let \((T(A) = A \bowtie A^*, \omega)\) be a double construction of Connes cocycle. Then there exists a compatible dendriform algebra structure \(\succ, \prec\) on \(T(A)\) defined by equation (4.1.1). Moreover, \(A\) and \(A^*\) are dendriform subalgebras with this product.

Proof. The first half follows from Theorem 4.1.1. Let \(x, y \in A\). Set \(x \succ y = a + b^*\), where \(a \in A\), \(b^* \in A^*\). Since \(A\) is an associative subalgebra of \(T(A)\) and \(\omega(A, A) = \omega(A^*, A^*) = 0\), we have

\[
\omega(b^*, A^*) = \omega(b^*, A) = \omega(x \succ y, A) = \omega(y, A \ast x) = 0.
\]

Therefore \(b^* = 0\) due to the nondependence of \(\omega\). Hence \(x \succ y = a \in A\). Similarly, \(x \prec y \in A\). Thus \(A\) is a dendriform subalgebra of \(T(A)\) with the product \(\succ, \prec\). By symmetry of \(A\) and \(A^*\), \(A^*\) is also a dendriform subalgebra. \(\square\)

Definition 4.1.3. Let \((T(A_1) = A_1 \bowtie A_1^*, \omega_1)\) and \((T(A_2) = A_2 \bowtie A_2^*, \omega_2)\) be two double constructions of Connes cocycles. They are isomorphic if there exists an isomorphism of associative algebras \(\varphi : T(A_1) \to T(A_2)\) satisfying the following conditions:

\[
\varphi(A_1) = A_2, \quad \varphi(A_1^*) = A_2^*, \quad \omega_1(x, y) = \varphi^* \omega_2(x, y) = \omega_2(\varphi(x), \varphi(y)), \forall x, y \in A_1. \tag{4.1.3}
\]

Proposition 4.1.4. Two double constructions of Connes cocycles \((T(A_1) = A_1 \bowtie A_1^*, \omega_1)\) and \((T(A_2) = A_2 \bowtie A_2^*, \omega_2)\) are isomorphic if and only if there exists a dendriform algebra isomorphism \(\varphi : T(A_1) \to T(A_2)\) satisfying equation (4.1.3), where the dendriform algebra structures on \(T(A_1)\) and \(T(A_2)\) are given by equation (4.1.1) respectively.

Proof. It is straightforward. \(\square\)

Theorem 4.1.5. Let \((A, \succ_A, \prec_A)\) be a dendriform algebra and \((A, \ast_A)\) be the associated associative algebra. Suppose that there is a dendriform algebra structure \(\succ_{A^*}, \prec_{A^*}\) on its dual space \(A^*\) and \((A^*, \ast_{A^*})\) is the associated associative algebra. Then there exists a double construction of Connes cocycle associated to \((A, \ast_A)\) and \((A, \ast_{A^*})\) if and only if \((A, A^*, R^*_{\prec_A}, L^*_{\succ_A}, R^*_{\prec_{A^*}}, L^*_{\succ_{A^*}})\) is a matched pair of the associative algebras. Moreover, every double construction of Connes cocycle can be obtained from the above way.

Proof. The conclusion can be obtained by a similar proof as of Theorem 2.2.1. \(\square\)

Corollary 4.1.6. Let \((A, \succ, \prec)\) be a dendriform algebra and \((R^*_{\prec}, L^*_{\succ})\) be the bimodule of the associated associative algebra \((A, \ast)\). Then \((T(A) = A \ltimes R^*_{\prec}, L^*_{\succ} A^*, \omega)\) is a double construction of Connes cocycle. Conversely, let \((T(A) = A \ltimes A^*, \omega)\) be a double construction of Connes cocycle. If \(A^*\) is an ideal of \(T(A)\), then \(A^*\) is a trivial associative algebra and hence \(T(A)\) is isomorphic to the semidirect \(A \ltimes_{T(A), R_{T(A)}} A^*.\) Furthermore, this double construction of Connes cocycle is isomorphic to the double construction of Connes cocycle \((T(A) = A \ltimes R^*_{\prec}, L^*_{\succ} A^*, \omega)\) which the dendriform algebra structure on \(A\) is given by \(\omega\) from equation (4.1.1).
Proof. By Remark 2.1.6, \((A, A^*, R^*_{<A}, L^*_{<A}, 0, 0)\) with the associative algebra structure on \(A^*\) being trivial is always a matched pair of associative algebras, the first half follows immediately. Conversely, if \(A^*\) is an ideal, then for any \(a^*, b^* \in A^*\), we have
\[
T(A) \ast a^*, \ b^* \ast T(A) \in A^* \implies \omega(a^* \ast b^*, T(A)) = -\omega(T(A) \ast a^*, b^*) - \omega(b^* \ast T(A), a^*) = 0.
\]
Thus \(a^* \ast b^* = 0\). Then \(T(A)\) is isomorphic to \(A \ltimes_{L_T(A), R_{T(A)}} A^*\). By Remark 2.1.6 again, we show that \((T(A) = A \bowtie A^*, \omega)\) is isomorphic to the double construction of Connes cocycle \((T(A) = A \ltimes_{R^*_{<A}, L^*_{<A}} A^*, \omega)\).

**Theorem 4.1.7.** Let \((A, \succsim_A, \prec_A)\) be a dendriform algebra and \((A, \ast_A)\) be the associated associative algebra. Suppose that there is a dendriform algebra structure \(\succsim_A, \prec_A\) on its dual space \(A^*\) and \((A^*, \ast_A)\) is the associated associative algebra. Then \((A, A^*, R^*_{<A}, L^*_{<A}, R^*_{>A}, L^*_{>A})\) is a matched pair of associative algebras if and only if
\[
(A, A^*, R^*_{>A} - L^*_{<A}, -R^*_{>A}, R^*_{<A} - L^*_{<A}, -R^*_{>A}, R^*_{>A} + L^*_{<A}, -R^*_{>A}, L^*_{>A} + L^*_{<A})
\]
is a matched pair of dendriform algebras.

**Proof.** The “if” part follows from Corollary 3.2.7. We need to prove the “only if” part. If \((A, A^*, R^*_{<A}, L^*_{<A}, R^*_{>A}, L^*_{>A})\) is a matched pair of associative algebras, then \((A \bowtie_{R^*_{<A}, L^*_{<A}} A^*, \omega)\) is a double construction of Connes cocycle. Hence there exists a compatible dendriform algebra structure on \(A \bowtie_{R^*_{<A}, L^*_{<A}} A^*\) given by equation (4.1.1). By a simple and direct computation, we show that \(A\) and \(A^*\) are its subalgebras and the other products are given by
\[
x \succ a^* = (R^*_{>A} + R^*_{<A})(x)a^* - L^*_{<A}(a^*)x, \quad x \prec a^* = -R^*_{>A}(x)a^* + (L^*_{<A} + L^*_{>A})(a^*)x,
\]
\[
a^* \succ x = (R^*_{>A} + R^*_{<A})(a^*)x - L^*_{<A}(a^*)x, \quad a^* \prec x = -R^*_{>A}(a^*)x + (L^*_{<A} + L^*_{>A})(x)a^*,
\]
for any \(x \in A, a^* \in A^*\). Therefore
\[
(A, A^*, R^*_{>A} + R^*_{<A} - L^*_{<A} - R^*_{>A}, R^*_{<A} + R^*_{>A} - L^*_{<A} - R^*_{>A}, R^*_{>A} + L^*_{<A} + L^*_{>A})
\]
is a matched pair of dendriform algebras.

### 4.2. Dendriform D-bialgebras

**Theorem 4.2.1.** Let \((A, \succsim_A, \prec_A)\) be a dendriform algebra whose products are given by two linear maps \(\beta^*_>, \beta^-_A : A \otimes A \to A\). Suppose that there is a dendriform algebra structure \(\succsim_{A^*}, \prec_{A^*}\) on its dual space \(A^*\) given by two linear maps \(\Delta^<_A, \Delta^>_A : A^* \otimes A^* \to A^*\). Then \((A, A^*, R^*_{<A}, L^*_{<A}, R^*_{>A}, L^*_{>A})\) is a matched pair of associative algebras if and only if the following equations hold (for any \(x, y \in A\) and \(a^*, b^* \in A^*\)):
\[
\begin{align*}
1. \quad \Delta_<(x \ast_A y) &= (id \otimes L_{<A}(x))\Delta_<(y) + (R_A(y) \otimes id)\Delta_<(x); \quad (4.2.1) \\
2. \quad \Delta_>(x \ast_A y) &= (id \otimes L_A(x))\Delta_>(y) + (R_{<A}(y) \otimes id)\Delta_>(x); \quad (4.2.2) \\
3. \quad \beta_<(a^* \ast_A b^*) &= (id \otimes L_{<A}(a^*))\beta_<(b^*) + (R_A(b^*) \otimes id)\beta_<(a^*); \quad (4.2.3)
\end{align*}
\]
(4) \( \beta_- (a^* \cdot_A b^*) = (id \otimes L_{\Delta, A} (a^*)) \beta_- (b^*) + (R_{\Delta, A} (b^*) \otimes id) \beta_- (a^*) \);  
(4.2.4)

(5) \( (L_A(x) \otimes id - id \otimes R_{\Delta, A} (x)) \Delta_\prec (y) + \sigma [(L_{\Delta, A} (y) \otimes -id \otimes R_A (y)) \Delta_\succ (x)] = 0; \)  
(4.2.5)

(6) \( (L_{A^*} (a^*) \otimes id - id \otimes R_{\Delta, A^*} (a^*)) \beta_- (b^*) + \sigma [(L_{\Delta, A^*} (b^*) \otimes -id \otimes R_{A^*} (b^*)) \beta_- (a^*)] = 0; \)  
where \( L_A = L_{\Delta, A} + L_{\Delta, A}; \ R_A = R_{\Delta, A} + R_{\Delta, A}; \ L_{A^*} = L_{\Delta, A^*} + L_{\Delta, A^*}; \ R_{A^*} = R_{\Delta, A^*} + R_{\Delta, A^*}. \)

Proof. Let \{\( e_1, \ldots, e_n \)\} be a basis of \( A \) and \{\( e_1^*, \ldots, e_n^* \)\} be its dual basis. Set

\[
e_i \succ_A e_j = \sum_{k=1}^n a_{ij}^k e_k, \quad e_i \prec_A e_j = \sum_{k=1}^n b_{ij}^k e_k, \quad e_i^* \succ_{A^*} e_j^* = \sum_{k=1}^n c_{ij}^k e_k, \quad e_i^* \prec_{A^*} e_j^* = \sum_{k=1}^n d_{ij}^k e_k.
\]

Therefore the coefficient of \( e_i^* \) in

\[
R_{\Delta, A}^* (e_i) (e_j^* \cdot_A e_k^*) = R_{\Delta, A}^* (L_{\Delta, A^*}^* (e_i) e_k^* + R_{\Delta, A}^* (e_i) e_j^* \cdot_A e_k^*)
\]
gives the following relation (for any \( i, j, k, l \))

\[
\sum_{m=1}^n b_{ij}^m c_{mk}^n + b_{ij}^m d_{mk}^n = \sum_{m=1}^n [c_{jm}^m b_{mk}^n + b_{jm}^m c_{mk}^n]
\]

which is precisely the relation given by the coefficient of \( e_i^* \otimes e_k^* \) in

\[
\beta_- (e_j^* \cdot_A e_k^*) = (R_{A^*} (e_j^*) \otimes id) \beta_- (e_k^*) + (id \otimes L_{\Delta, A^*} (e_j^*)) \beta_- (e_k^*).
\]

So equation (2.1.4) in the case \( L_A = R_{\Delta, A}^*, r_A = L_{\Delta, A}^*, \ l_B = l_{A^*} = R_{\Delta, A}^*, r_B = r_{A^*} = L_{\Delta, A}^* \) is equation (4.2.3). Similarly, in this situation, we have the following correspondences:

- equation (2.1.5) \( \iff \) equation (4.2.4);
- equation (2.1.6) \( \iff \) equation (4.2.1);
- equation (2.1.7) \( \iff \) equation (4.2.2);
- equation (2.1.8) \( \iff \) equation (4.2.6);
- equation (2.1.9) \( \iff \) equation (4.2.5).

Therefore the conclusion holds due to Theorem 2.1.4. \( \square \)

Definition 4.2.2. Let \( A \) be a vector space. A dendriform \( D \)-bialgebra structure on \( A \) is a set of linear maps \( (\Delta_\prec, \Delta_\succ, \beta_- , \beta_+) \) such that \( \Delta_\prec, \Delta_\succ : A \rightarrow A \otimes A, \beta_- , \beta_+ : A^* \rightarrow A^* \otimes A^* \) and

(a) \( (\Delta_\prec, \Delta_\succ) : A^* \otimes A \rightarrow A^* \) defines a dendriform algebra structure \( (\prec, \succ) \) on \( A; \)

(b) \( (\beta_- , \beta_+) : A \otimes A \rightarrow A \) defines a dendriform algebra structure \( (\prec, \succ) \) on \( A; \)

(c) Equations (4.2.1-4.2.6) are satisfied.

We also denote it by \((A, A^*, \Delta_\prec, \Delta_\succ, \beta_- , \beta_+)\) or simply \((A, A^*)\).

Remark 4.2.3. In fact, there have already been the notions of dendriform bialgebra ([LR1-2], [Ron], [A4]) and bidendriform bialgebras ([F2]) which are the special dendriform bialgebras. We use the terminology “D-bialgebra” in order to express its relation with the double construction. All of these bialgebras are dendriform algebras equipped with coassociative cooperations verifying some (different) compatibility relations. We would like to point out that the dendriform D-bialgebras are quite different from the other types of bialgebras. For example, one of the differences is that the term \( a \otimes b \) appears in both \( \Delta_\prec (a \cdot b) \) and \( \Delta_\succ (a \cdot b) \) in a bidendriform bialgebra, whereas it does not appear in a dendriform D-bialgebra.
Theorem 4.2.4. Let \((A, \prec_A, \succ_A)\) and \((A^*, \prec_{A^*}, \succ_{A^*})\) be two dendriform algebras. Let \((A, \ast_A)\) and \((A^*, \ast_{A^*})\) be the associated associative algebras respectively. Then the following conditions are equivalent.

1. There is a double construction of Connes cocycle associated to \((A, \ast_A)\) and \((A, \ast_{A^*})\).
2. \((A, A^*, R^*_{\prec_A}, L^*_{\prec_A}, R^*_{\prec_{A^*}}, L^*_{\prec_{A^*}})\) is a matched pair of the associative algebras.
3. \((A, A^*, R^*_{\prec_A} + R^*_{\prec_{A^*}}, -L^*_{\prec_A}, -R^*_{\prec_{A^*}}, -R^*_{\prec_{A^*}}, -R^*_{\prec_{A^*}}, -L^*_{\prec_{A^*}}, -L^*_{\prec_{A^*}}, -L^*_{\prec_{A^*}})\) is a matched pair of dendriform algebras.
4. \((A, A^*)\) is a dendriform D-bialgebra.

Proof. It follows from Theorems 4.1.5, 4.1.7 and 4.2.1. □

Definition 4.2.5. Let \((A, A^*, \Delta_\prec, \Delta_\succ, \beta_\prec, \beta_\succ)\) and \((B, B^*, \Delta_\prec, \Delta_\succ, \beta_\prec, \beta_\succ)\) be two dendriform D-bialgebras. A homomorphism of dendriform D-bialgebras \(\varphi : A \to B\) is a homomorphism of dendriform algebras such that \(\varphi^* : B^* \to A^*\) is also a homomorphism of dendriform algebras, that is, \(\varphi\) satisfies

\[
(\varphi \otimes \varphi)\Delta_\prec (x) = \Delta_\prec (\varphi(x)), \quad (\varphi \otimes \varphi)\Delta_\succ (x) = \Delta_\succ (\varphi(x)),
\]

\[
(\varphi^* \otimes \varphi^*)\beta_\prec (a^*) = \beta_\prec (\varphi^*(a^*)), \quad (\varphi^* \otimes \varphi^*)\beta_\succ (a^*) = \beta_\succ (\varphi^*(a^*)),
\]

for any \(x \in A, a^* \in B^*\). An isomorphism of dendriform D-bialgebras is an invertible homomorphism of dendriform D-bialgebras.

Proposition 4.2.6. Two double constructions of Connes cocycles are isomorphic if and only if their corresponding dendriform D-bialgebras are isomorphic.

Proof. It follows from a similar proof as of Proposition 2.2.10. □

Example 4.2.7. Let \((A, A^*, \Delta_\prec, \Delta_\succ, \beta_\prec, \beta_\succ)\) be a dendriform D-bialgebra. Then its dual \((A^*, A, \beta_\prec, \beta_\succ, \Delta_\prec, \Delta_\succ)\) is also a dendriform D-bialgebra.

Example 4.2.8. Let \((A, \prec_A, \succ_A)\) be a dendriform algebra. If the dendriform algebra structure on \(A^*\) is trivial, then \((A, A^*, 0, 0, \beta_\prec, \beta_\succ)\) is a dendriform D-bialgebra. And its corresponding dendriform algebra is \(A \ltimes R^*_{\prec_A}, L^*_{\prec_A} - L^*_{\prec_A}, -R^*_{\prec_A}, L^*_{\prec_A} + L^*_{\prec_A} A^*\). Moreover, its corresponding double construction of Connes cocycle is just the semidirect sum \(A \ltimes R^*_{\prec_A} L^*_{\prec_A} A^*\) with the bilinear form \(\omega\) given by equation (1.4.1). Dually, if \(A\) is a trivial dendriform algebra, then the dendriform D-bialgebra structures on \(A\) are in one-to-one correspondence with the dendriform algebra structures on \(A^*\).

Example 4.2.9. Let \((A, A^*)\) be a dendriform D-bialgebra. In the next subsection, we will prove that there exists a canonical dendriform D-bialgebra structure on the direct sum \(A \oplus A^*\) of the underlying vector spaces of \(A\) and \(A^*\).
4.3. Coboundary dendriform D-bialgebras. By Theorem 4.2.1, we have shown that both \( \Delta_\succ \) and \( \Delta_\prec \) (\( \beta_\succ \) and \( \beta_\prec \) respectively) are the 1-cocycles of the associated associative algebra \((A, \ast_A)((A^*, \ast_A)\) respectively). So it is natural to consider the special case that they are 1-coboundaries or principal derivations, as we have done in subsection 2.3.

Let \((\succ, \prec, \ast)\) be a dendriform algebra and \(r_\succ, r_\prec \in A \otimes A\). Set

\[
\Delta_\succ(x) = (id \otimes L(x) - R_\succ(x) \otimes id)r_\succ; \quad (4.3.1)
\]

\[
\Delta_\prec(x) = (id \otimes L_\prec(x) - R_\prec(x) \otimes id)r_\prec, \quad (4.3.2)
\]

for any \(x \in A\). It is obvious that \(\Delta_\succ\) satisfies equation (4.2.1) and \(\Delta_\prec\) satisfies equation (4.2.2).

Moreover, by equation (4.2.5), we show that

\[
(L(x) \otimes id - id \otimes R_\prec(x))(id \otimes L_\prec(y) - R(y) \otimes id)(r_\prec + \sigma(r_\prec)) = 0, \ \forall x, y \in A. \quad (4.3.3)
\]

Therefore \((A, \Delta_\succ, \Delta_\prec, \beta_\succ, \beta_\prec)\) is a dendriform D-bialgebra if and only if the following conditions are satisfied:

1. \(\Delta_\succ^*, \Delta_\prec^* : A^* \otimes A^* \to A^*\) defines a dendriform algebra structure on \(A^*\).
2. \(\beta_\succ, \beta_\prec\) satisfy equations (4.2.3-4.2.4) and (4.2.6), where the dendriform algebra structure on \(A^*\) is given by (1).

**Proposition 4.3.1.** Let \((\succ, \prec, \ast)\) be a dendriform algebra whose products are given by two linear maps \(\beta_\succ^*, \beta_\prec^* : A \otimes A \to A\) and \(r_\succ, r_\prec \in A \otimes A\). Suppose there exists a dendriform algebra structure “\(\succ_A, \prec_A\)” on \(A^*\) given by \(\Delta_\succ^*, \Delta_\prec^* : A^* \otimes A^* \to A^*\), where \(\Delta_\succ\) and \(\Delta_\prec\) are two linear maps given by equations (4.3.1) and (4.3.2) respectively. Then

1. Equation (4.2.3) holds if and only if \(r_\succ, r_\prec\) satisfy

\[
[R_\prec(x) \otimes L_\prec(y) - id \otimes L_\prec(y \prec x) - R_\prec(y \succ x) \otimes id](r_\succ + r_\prec) = 0, \ \forall x, y \in A. \quad (4.3.4)
\]

2. Equation (4.2.4) holds if and only if \(r_\succ, r_\prec\) satisfy equation (4.3.4).

3. Equation (4.2.6) holds if and only if \(r_\succ, r_\prec\) satisfy (for any \(x, y \in A\))

\[
[L_\prec(x) \otimes id - id \otimes R_\prec(x)][-id \otimes L_\succ(y) + R_\succ(y) \otimes id](r_\prec + r_\succ) + [L_\succ(x) \otimes id - id \otimes R_\succ(x)][R_\prec(y) \otimes id(r_\prec + \sigma(r_\prec)) - id \otimes L_\prec(y)(\sigma(r_\prec) + r_\prec)] = 0. \quad (4.3.5)
\]

**Proof.** Let \(\{e_1, \ldots, e_n\}\) be a basis of \(A\) and \(\{e_1^*, \ldots, e_n^*\}\) be its dual basis. Set

\[
r_\prec = \sum_{i,j} a_{ij} e_i \otimes e_j, \quad r_\succ = \sum_{i,j} b_{ij} e_i \otimes e_j.
\]

\[
e_i \succ e_j = \sum_{k=1}^n a_{ij}^k c_k, \quad e_i \prec e_j = \sum_{k=1}^n b_{ij}^k e_k, \quad e_i^* \succ e_j^* = \sum_{k=1}^n c_{ij}^k e_k^*, \quad e_i^* \prec e_j^* = \sum_{k=1}^n d_{ij}^k e_k^*.
\]

By equations (4.3.1) and (4.3.2), we have (for any \(i, k, l\))

\[
c_{kl}^i = \sum_{m=1}^n |b_{km}(a_{lm}^i + b_{lm}^k) - b_{ml}^i b_{km}^k|, \quad d_{kl}^i = \sum_{m=1}^n |a_{km} a_{lm}^i - a_{ml}^i (a_{mi}^k + b_{mi}^k)|. \quad (4.3.6)
\]
(1) Equation (4.2.3) holds (taking $a^* = e^*_i, b^* = e^*_j$) if and only if (for any $i, j, m, t$)
\[
\sum_{k=1}^n (e^k_{ij} + d^k_{ij}) b^k_{ml} = \sum_{k=1}^n [b^j_{mk} c^i_{ik} + b^k_{kt}(e^m_{kj} + d^m_{kj})].
\]
Substituting equation (4.3.6) into the above equation and after rearranging the terms suitably, we have
\[
(F1) + (F2) + (F3) + (F4) + (F5) + (F6) = 0,
\]
where
\[
(F1) = \sum_{k,l} (a_{kl} + b_{kl})(a^i_{ml} b^j_{kt}); \quad (F2) = \sum_{k,l} (b_{kl} b^j_{kl} b^i_{ml} - b_{lk} b^i_{ln} b^j_{mk});
\]
\[
(F3) = \sum_{k,l} (a_{il} + b_{il})(-a^j_{kl} b^k_{ml}); \quad (F4) = \sum_{k,l} b_{il} [b^j_{mk} (a^k_{il} + t^k_{il}) - b^k_{ml} b^j_{kl}];
\]
\[
(F5) = \sum_{k,l} (a_{ij} + b_{ij})(b^j_{ml} b^i_{lk} - b^i_{lm} b^j_{lk}); \quad (F6) = \sum_{k,l} a_{ij} (a^j_{ik} b^k_{ml} - a^k_{im} b^j_{lk}).
\]
(F1) is the coefficient of $e_i \otimes e_j$ in $[R_<(e_t) \otimes L_>(e_m)](r_\succ + r_\prec)$;
(F2) = 0 by interchanging the indices $k$ and $l$;
(F3) is the coefficient of $e_i \otimes e_j$ in $-[id \otimes L_>(e_m < e_t)](r_\succ + r_\prec)$;
(F4) = 0 since the term in the bracket is the coefficient of $e_j$ in
\[
e_m < (e_t \succ e_t \prec e_t) - (e_m \prec e_n) < e_t = 0;
\]
(F5) is the coefficient of $e_i \otimes e_j$ in $-[R_<(e_m > e_t) \otimes id](r_\succ + r_\prec)$;
(F6) = 0 since the term in the bracket is the coefficient of $e_i$ in
\[
e_i \succ (e_m \prec e_t) - (e_t \succ e_m) \prec e_t = 0.
\]
Therefore we have
\[
[R_<(e_t) \otimes L_>(e_m) - id \otimes L_>(e_m < e_t) - R_<(e_m > e_t) \otimes id](r_\succ + r_\prec) = 0.
\]

(2) Similarly, we show that equation (4.2.4) holds if and only if $r_\succ, r_\prec$ satisfy equation (4.3.4). In fact, comparing with the proof in (1), the difference appears in $(F2)'$, $(F4)'$ and $(F6)'$, where
\[
(F2)' = \sum_{k,l} (a^i_{mk} a^j_{lt} - a^i_{kt} a^j_{ml}) = 0 \quad \text{by interchanging the indices } k \text{ and } l;
\]
\[
(F4)' = \sum_{k,l} b_{il} (a^k_{mt} b^i_{lk} - a^k_{ml} b^i_{lk}) = 0 \quad \text{since the term in the bracket is the coefficient of } e_j \text{ in}
\]
\[
(e_m \succ e_t) < e_t - e_m \succ (e_t \prec e_t) = 0;
\]
\[
(F6)' = \sum_{k,l} a_{ij} (a^k_{lj} b^k_{ml} + b^k_{lj}) - a^k_{im} a^j_{kt} = 0 \quad \text{since the term in the bracket is the coefficient of } e_i \text{ in}
\]
\[-e_t > (e_m > e_t) \prec (e_t < e_m) > e_t = 0.
\]

(3) Equation (4.2.6) holds (taking $a^* = e^*_i, b^* = e^*_j$) if and only if (for any $i, j, m, t$)
\[
\sum_{l=1}^n [(e^m_{il} + d^m_{il}) b^j_{lt} - b^j_{ml} d^i_{lt} + a^i_{lm} c^j_{jl} - a^i_{tl} (e^m_{lj} + d^m_{ij})] = 0.
\]
Substituting equation (4.3.6) into the above equation and after rearranging the terms suitably, we have

\[(F1) + (F2) + (F3) + (F4) + (F5) + (F6) + (F7) + (F8) + (F9) + (F10) = 0,
\]

where

\[
(F1) = \sum_{k,l}(a_{kl} + b_{kl})(-b_{it}^j b_{km}^i) \implies -R_<(e_m) \otimes R_>(e_t)(r_+ + r_-);
\]

\[
(F2) = \sum_{k,l}(a_{lk} + b_{lk})(-a_{jk}^l b_{it}^j) \implies -L_>(e_t) \otimes L_<(e_m)(r_+ + r_-);
\]

\[
(F3) = \sum_{k,l}(a_{kl} + b_{kl})(-a_{km}^l b_{it}^j) \implies -R_<(e_m) \otimes R_>(e_t)(\sigma(r_+) + r_-);
\]

\[
(F4) = \sum_{k,l}(a_{lk} + b_{lk})(-a_{mk}^l b_{it}^j) \implies -L_>(e_t) \otimes L_<(e_m)(r_+ + \sigma(r_-));
\]

\[
(F5) = \sum_{k,l}(a_{ik} + b_{ik})a_{mk}^l b_{it}^j \implies id \otimes R_<(e_t)L_<(e_m)(r_+ + r_-);
\]

\[
(F6) = \sum_{k,l}a_{kl}(a_{ik}^l + b_{ik}^l) b_{mk}^i \implies id \otimes R_<(e_t)L_<(e_m)(\sigma(r_-));
\]

\[
(F7) = \sum_{k,l}b_{ik} b_{lt}^i b_{lm}^j \implies id \otimes R_<(e_t)L_<(e_m)(r_-);
\]

\[
(F8) = \sum_{k,l}(a_{kj} + b_{kj})a_{it}^l b_{km}^j \implies L_>(e_t)R_<(e_m) \otimes id(r_+ + r_-);
\]

\[
(F9) = \sum_{k,l}a_{jk} a_{mk}^l b_{it}^j \implies L_>(e_t)R_<(e_m) \otimes id(r_-);
\]

\[
(F10) = \sum_{k,l}b_{jk} a_{lm}^i (a_{ik}^l + b_{ik}^l) \implies L_>(e_t)R_<(e_m) \otimes id(\sigma(r_-)).
\]

Therefore equation (4.3.5) holds.

\[\square\]

By the definition of a dendriform algebra, we have the following conclusion (cf. [F2]).

**Lemma 4.3.2.** Let \(A\) be a vector space and \(\Delta_>, \Delta_\prec : A \otimes A \to A\) be two linear maps. Then \(\Delta_\prec, \Delta_\succ : A^* \otimes A^* \to A^*\) define a dendriform algebra structure on \(A^*\) if and only if the following conditions are satisfied:

\[(1) \ (\Delta_\prec \otimes id) \Delta_\succ = (id \otimes (\Delta_\succ + \Delta_\prec)) \Delta_\prec; \quad (4.3.7)\]

\[(2) \ (id \otimes \Delta_\prec) \Delta_\succ = (\Delta_\prec \otimes id) \Delta_\prec; \quad (4.3.8)\]

\[(3) \ (id \otimes \Delta_\succ) \Delta_\prec = ((\Delta_\succ + \Delta_\prec) \otimes id) \Delta_\succ. \quad (4.3.9)\]

**Proposition 4.3.3.** Let \((A, >, \prec)\) be a dendriform algebra and \(r_+, r_- \in A \otimes A\). Define \(\Delta_>, \Delta_\prec : A \to A \otimes A\) by equations (4.3.1-4.3.2). Then \(\Delta_\prec, \Delta_\succ : A^* \otimes A^* \to A^\ast\) define a dendriform algebra
structure on $A^*$ if and only if the following equations are satisfied (for any $x \in A$)

$$(R(x) \otimes id \otimes id)[(r_{\prec,12} * r_{\prec,13} + r_{\prec,13} < r_{\prec,23} - r_{\prec,23} \succ r_{\prec,12}) + r_{\prec,13} \succ (r_{\prec,23} + r_{\succ,23}) - (r_{\prec,23} + r_{\succ,23}) < r_{\prec,12}] + (id \otimes L_\prec(x) \otimes id)r_{\prec,12} + (id \otimes id \otimes L_\prec(x))(-r_{\prec,12} * r_{\prec,13} - r_{\prec,13} < r_{\prec,23} + r_{\succ,23} \succ r_{\prec,12}) - [(id \otimes id \otimes L_\prec(x))r_{\prec,13}] > (r_{\prec,23} + r_{\prec,23}) = 0; \tag{4.3.10}$$

$$(R_\succ(x) \otimes id \otimes id)(r_{\prec,23} * r_{\succ,12} - r_{\succ,12} < r_{\succ,13} - r_{\prec,13} \succ r_{\prec,23}) - (id \otimes id \otimes L_\succ(x))(r_{\prec,23} * r_{\succ,12} - r_{\succ,12} < r_{\succ,13} - r_{\prec,13} \succ r_{\prec,23}) = 0; \tag{4.3.11}$$

$$(R_\succ(x) \otimes id \otimes id)(-r_{\prec,13} * r_{\succ,23} + r_{\succ,23} \succ r_{\succ,12} - r_{\prec,12} \succ r_{\prec,13}) - (r_{\succ,12} + r_{\prec,12}) < [(R_\succ(x) \otimes id \otimes 1)r_{\succ,13}] + [(id \otimes R_\prec(x) \otimes id)r_{\prec,23}] > (r_{\succ,12} + r_{\prec,12}) + (id \otimes id \otimes L(x))[r_{\prec,13} * r_{\succ,23} - r_{\succ,23} < r_{\succ,12} + r_{\prec,12} \succ r_{\prec,13} + (r_{\succ,12} + r_{\prec,12})] = 0. \tag{4.3.12}$$

The operation between two $r$s is given in an obvious and similar way as equation (1.1.3).

Proof. We need to prove that equations (4.3.7-4.3.9) are equivalent to equations (4.3.10-4.3.12) respectively. Here we only give an explicit proof that equation (4.3.10) holds if and only if equation (4.3.7) holds since the proof of the other two equations is similar. Let $x \in A$. After rearranging the terms suitably, we divide equation (4.3.7) into three parts:

$$(\Delta_\prec \otimes id)\Delta_\prec(x) - (id \otimes (\Delta_\prec + \Delta_\succ))\Delta_\prec(x) = (F1) + (F2) + (F3),$$

where

$$(F1) = \sum_{i,j} \{ (a_i \succ x + a_i < x) \otimes [a_j \otimes b_i \succ b_j - (a_j \succ b_i + a_j < b_i) \otimes b_j + c_j \otimes (b_i \succ d_j) + b_i \otimes d_j - c_j \otimes b_i \otimes d_j] + [a_j \succ (a_i \succ x + a_i < x) + a_j < (a_i \succ x + a_i < x)] \otimes b_j \otimes b_i \};$$

$$(F2) = \sum_{i,j} \{ a_i \otimes [a_j \succ (x \succ b_i) + a_j \otimes (x \succ b_i)] \otimes b_j + a_i \otimes c_j \otimes (x \succ b_i) \otimes d_j - a_j \otimes (a_i \succ x + a_i < x) \succ b_j \otimes b_i \};$$

$$(F3) = \sum_{i,j} \{ [a_i \otimes (a_i \succ b_j) - (a_j \succ a_i + a_j \succ a_i) \otimes b_j] \otimes (x \succ b_i) - a_i \otimes a_j \otimes [(x \succ b_i) \succ b_j] - a_i \otimes c_j \otimes [(x \succ b_i) \succ d_j + (x \succ b_i) \succ d_j] \}. $$
On the other hand,

\begin{align*}
(F1a) &= (R(x) \otimes id \otimes id)(r_{\prec,12} * r_{\prec,13}) = \sum_{i,j}[(a_i * a_j) * x \otimes b_i \otimes b_j] \\
&= \sum_{i,j}[(a_i > x + a_i < x) + a_j < (a_i > x + a_i < x)] \otimes b_j \otimes b_i]; \\
(F1b) &= (R(x) \otimes id \otimes id)(r_{\prec,13} < r_{\succ,23}) = \sum_{i,j}[(a_i * x) \otimes c_j \otimes (b_i < d_j)] \\
&= \sum_{i,j}[(a_i > x + a_i < x) \otimes c_j \otimes (b_i < d_j)]; \\
(F1c) &= (R(x) \otimes id \otimes id)(-r_{\prec,23} > r_{\prec,12}) = \sum_{i,j}[-(a_i * x) \otimes (a_j > b_i) \otimes b_j] \\
&= \sum_{i,j}[-(a_i > x + a_i < x) \otimes (a_j > b_i) \otimes b_j]; \\
(F1d) &= (R(x) \otimes id \otimes id)[r_{\succ,13} > (r_{\prec,23} + r_{\succ,23})] \\
&= \sum_{i,j}[(a_i > x + a_i < x) \otimes (a_j \otimes (b_i > b_j) + c_j \otimes (b_i > d_j))]; \\
(F1e) &= (R(x) \otimes id \otimes id)[-r_{\prec,23} + r_{\succ,12}] \\
&= -\sum_{i,j}[(a_i > x + a_i < x) \otimes [(a_j < b_i) \otimes b_j + (c_j < b_i) \otimes d_j]]; \\
(F2') &= (r_{\prec,23} + r_{\succ,23}) < [id \otimes L_{\prec}(x) \otimes id) r_{\prec,12}] \\
&= \sum_{i,j}a_i \otimes [a_j < (x > b_i) \otimes b_j + c_j < (x > b_i)] \otimes d_j] \\
(F3a) &= (id \otimes id \otimes L_{\succ}(x))(-r_{\prec,12} * r_{\prec,13}) = \sum_{i,j}-a_i * a_j \otimes b_i \otimes (x > b_j) \\
&= \sum_{i,j}[-(a_i > a_j + a_i < a_j) \otimes b_i \otimes (x > b_j)]; \\
(F3b) &= (id \otimes id \otimes L_{\succ}(x))(-r_{\prec,13} < r_{\succ,23}) = \sum_{i,j}[-a_i \otimes c_j \otimes x > (b_i < d_j)]; \\
(F3c) &= (id \otimes id \otimes L_{\prec}(x))(r_{\succ,13} > r_{\prec,12}) = \sum_{i,j}[a_i \otimes (a_j > b_i) \otimes (x > b_j)]; \\
(F3d) &= -[(id \otimes id \otimes L_{\succ}(x))r_{13}] > (r_{\succ,23} + r_{\prec,23}) \\
&= -\sum_{i,j}a_i \otimes [a_j \otimes (x > b_i) \otimes b_j + c_j \otimes (x > b_i) \otimes d_j].
\end{align*}

It is obvious that

\begin{align*}
(F1) &= (F1a) + (F1b) + (F1c) + (F1d) + (F1e), \\
(F2) &= (F2'), \quad (F3) = (F3a) + (F3b) + (F3c) + (F3d).
\end{align*}

Therefore equation (4.3.10) holds if only if equation (4.3.7) holds. \qed

Combining Propositions 4.3.1 and 4.3.3, we obtain the following conclusion.
Therefore equation (4.3.4) holds automatically. By a similar proof as of Theorem 2.3.6, we show that the dendriform D-bialgebra structure $DD$ can induce a dendriform D-bialgebra structure on $A$ such that $(A, A^*)$ is a dendriform D-bialgebra if and only if $r_\succ$ and $r_\prec$ satisfy equations (4.3.3-4.3.5) and (4.3.10-4.3.12).

**Definition 4.3.5.** A dendriform D-bialgebra $(A, A^*)$ is called *coboundary* if its structure is given by $r_\succ, r_\prec \in A \otimes A$ through Theorem 4.3.4.

**Theorem 4.3.6.** Let $(A, A^*, \Delta_\succ, \Delta_\prec, \beta_\succ, \beta_\prec)$ be a dendriform D-bialgebra. Then there is a canonical dendriform bialgebra structure on the direct sum $A \oplus A^*$ of the underlying vector spaces of $A$ and $A^*$ such that both the inclusions $i_1 : A \to A \oplus A^*$ and $i_2 : A^* \to A \oplus A^*$ into the two summands are homomorphisms of dendriform D-bialgebras, where the dendriform D-bialgebra structure on $A^*$ is given in Example 4.2.7.

**Proof.** Let $r = \sum e_i \otimes e_i^* \in A \otimes A^* \subset (A \oplus A^*) \otimes (A \oplus A^*)$ which corresponds to the identity map $id : A \to A$, where $\{e_1, \cdots, e_n\}$ is a basis of $A$ and $\{e_1^*, \cdots, e_n^*\}$ is its dual basis. Suppose that the dendriform D-bialgebra structure “$\succ$, $\prec”$ on $A \oplus A^*$ is given by

$$DD(A) = A \oplus R_{\succ A^*}^{\succ} + R_{\prec A^*}^{\prec} - L_{\prec A^*}^{\prec} - R_{\prec A^*}^{\succ} - L_{\succ A^*}^{\succ} + L_{\prec A^*}^{\prec} + L_{\succ A^*}^{\prec}. A^*.$$ 

Then we have (for any $x, y \in A, a, b \in A^*$)

$$x \succ y = x \succ_A y, \quad x \prec a = R_A^\prec(a)x - L_A^\prec(a)x, \quad x \succ a = R_A^\succ(a)x - L_A^\prec(a)x,$$

$$a \prec x = -R_{A^*}^\succ(a)x + L_{A^*}^\succ(a)x, \quad a \succ b = a \succ_{A^*} b, \quad a \prec b = a \prec_{A^*} b.$$

If $r_\succ = r$ and $r_\prec = -r$ satisfies equations (4.3.3)-(4.3.5) and (4.3.10)-(4.3.12), then

$$\Delta_{DD, \succ}(u) = (id \otimes L(u) - R_\prec(u) \otimes id)(r_\succ), \quad \Delta_{DD, \prec}(u) = (id \otimes L_\prec(u) - R(u) \otimes id)(r_\prec), \quad \forall u \in DD(A),$$

can induce a dendriform D-bialgebra structure on $DD(A)$.

In fact, we have

$$r_\prec + r_\succ = 0, \quad r_\succ + \sigma(r_\succ) = \sum_i (-e_i \otimes e_i^* + e_i^* \otimes e_i)$$

Therefore equation (4.3.4) holds automatically. By a similar proof as of Theorem 2.3.6, we show that equations (4.3.3) and (4.3.5) hold and

$$r_{12} \cdot r_{13} - r_{13} \cdot r_{23} < r_{23} - r_{23} \succ r_{12} = -r_{23} \cdot r_{12} + r_{12} \prec r_{13} + r_{13} \succ r_{23}$$

$$r_{13} \cdot r_{23} - r_{23} \cdot r_{12} < r_{12} + r_{12} \succ r_{13} \succ r_{13} = 0.$$
So equations (4.3.10)-(4.3.12) are satisfied. Hence $\mathcal{D}\mathcal{D}(A)$ is a dendriform $D$-bialgebra. Furthermore, for $e_k \in A$, we have
\[
\Delta_{\mathcal{D}\mathcal{D},\prec}(e_k) = \sum_i [e_i \otimes e_k * e_i^\ast - (e_i * e_k) \otimes e_i^\ast] = \sum_i (e_k, e_i^\ast \triangleright e_i^\ast) e_i \otimes e_j = \Delta_\prec(e_k);
\]
\[
\Delta_{\mathcal{D}\mathcal{D},\succ}(e_k) = \sum_i [-e_i \otimes e_k \triangleright e_i^\ast + (e_i * e_k) \otimes e_i^\ast] = \sum_i (e_k, e_i^\ast \triangleright e_i^\ast) e_i \otimes e_j = \Delta_\succ(e_k).
\]

Therefore the inclusion $i_1 : A \rightarrow A \oplus A^*$ is a homomorphism of dendriform $D$-bialgebras. Similarly, the inclusion $i_2 : A^* \rightarrow A \oplus A^*$ is also a homomorphism of dendriform $D$-bialgebras, where the dendriform $D$-bialgebra structure on $A^*$ is given in Example 4.2.7. □

**Definition 4.3.7.** Let $(A, A^*)$ be a dendriform $D$-bialgebra. With the dendriform $D$-bialgebra structure given in Theorem 4.3.6, $A \oplus A^*$ is called a *dendriform double* of $A$. We denote it by $\mathcal{D}\mathcal{D}(A)$.

**Corollary 4.3.8.** Let $(A, A^*)$ be a dendriform $D$-bialgebra. Then the dendriform double $\mathcal{D}\mathcal{D}(A)$ of $A$ is a dendriform $D$-bialgebra and the bilinear form $\omega$ given by equation (1.4.1) is a Connes cocycle.

At the end of this subsection, we would like to point out that, unlike the symmetry of 1-cocycles of $A$ and $A^*$ appearing in the definition of a dendriform $D$-bialgebra $(A, A^*)$, it is not necessary that $\beta$ is also a 1-coboundary of $A^*$ for a coboundary dendriform $D$-bialgebra $(A, A^*, \Delta_\succ, \Delta_\prec, \beta_\succ, \beta_\prec)$, where $\Delta_\succ, \Delta_\prec$ are given by equations (4.3.1-4.3.2).

### 4.4. $D$-equation and its properties.

In this subsection, we consider some simple and special cases to satisfy the equations (4.3.3-4.3.5) and (4.3.10-4.3.12).

At first, due to equation (4.3.3), we consider the condition
\[
r_\prec = r, \ r_\succ = -\sigma(r), \ r \in A \otimes A. \quad (4.4.1)
\]

**Corollary 4.4.1.** Let $(A, \succ, \prec)$ be a dendriform algebra and $r = \sum_i a_i \otimes b_i \in A \otimes A$. Then the maps $\Delta_\succ, \Delta_\prec$ defined by equations (4.3.1) and (4.3.2) with $r_\succ, r_\prec$ satisfying equation (4.4.1) induce a dendriform algebra structure on $A^*$ such that $(A, A^*)$ is a dendriform $D$-bialgebra if and only if $r$ satisfies the following equations
\[
[P(x \succ y) - (id \otimes L_\succ(x))P(y)](r - \sigma(r)) = 0; \quad (4.4.2)
\]
\[
\sigma(P(x))P(y)(r - \sigma(r)) = 0; \quad (4.4.3)
\]
\[
(R(x) \otimes id \otimes id - id \otimes id \otimes L_\succ(x))[(r_{12} * r_{13} - r_{13} \prec r_{32} - r_{23} \succ r_{12})
+ \sum_i (a_i * x) \otimes P(b_i)(r - \sigma(r)) - a_i \otimes [P(x \succ b_i)(r - \sigma(r))] = 0; \quad (4.4.4)
\]
\[
(R_\prec(x) \otimes id \otimes id - id \otimes id \otimes L_\prec(x))(-r_{23} * r_{21} + r_{21} \prec r_{13} + r_{31} \succ r_{23}) = 0; \quad (4.4.5)
\]
\[
(R_\prec(u) \otimes id \otimes id - id \otimes id \otimes L(u))(-r_{31} * r_{32} + r_{32} \prec r_{21} + r_{12} \succ r_{31})
\]
\[ + \sum_i [P(b_i)(r - \sigma(r)) \otimes x \ast a_i - P(b_i \prec x)(r - \sigma(r)) \otimes a_i] = 0, \quad (4.4.6) \]

where \( x, y \in A, P(x) = \text{id} \otimes L_\prec(x) - R_\prec(x) \otimes \text{id}. \)

**Remark 4.4.2.** Let \( \sigma_{123}, \sigma_{132} : A \otimes A \otimes A \to A \otimes A \otimes A \) be two linear maps given by
\[
\sigma_{123}(x \otimes y \otimes z) = z \otimes x \otimes y, \quad \sigma_{132}(x \otimes y \otimes z) = y \otimes z \otimes x. \quad \forall x, y, z \in A. \quad (4.4.7)
\]

Then we have
\[
(r_{23} \ast r_{21} \prec r_{13} \prec r_{13} \succ r_{23}) = \sigma_{123}(r_{12} \ast r_{13} \prec r_{13} \prec r_{23} \succ r_{12});
\]
\[
(r_{31} \ast r_{32} \prec r_{21} \prec r_{12} \succ r_{31}) = \sigma_{132}(r_{12} \ast r_{13} \prec r_{13} \prec r_{23} \succ r_{12}).
\]

**Remark 4.4.3.** We also can consider the case that \( r_\succ + r_\prec = 0 \) as we have done in the proof of Theorem 4.3.6. Obviously, if in addition, \( r_\prec = r \) is symmetric, then this case is as the same as the case satisfying equation (4.4.1).

The simplest way to satisfy equations (4.4.2-4.4.6) is to assume that \( r \) is symmetric and
\[
r_{12} \ast r_{13} = r_{13} \prec r_{23} \prec r_{23} \succ r_{12}. \quad (4.4.8)
\]

**Corollary 4.4.4.** Let \((A, \succ, \prec)\) be a dendriform algebra and \( r \in A \otimes A \). Suppose \( r \) is symmetric and \( r \) satisfies equation (4.4.8). Then the maps \( \Delta_\succ, \Delta_\prec \) defined by equations (4.3.1) and (4.3.2) with \( r_\succ = -r, r_\prec = r \) induce a dendriform algebra structure on \( A^* \) such that \((A, A^*)\) is a dendriform \( D \)-bialgebra.

**Definition 4.4.5.** Let \((A, \succ, \prec)\) be a dendriform algebra and \( r \in A \otimes A \). Equation (4.4.8) is called \( D \)-equation in \( A \).

By Remark 4.4.2, when \( r \) is symmetric, the equivalent forms of \( D \)-equation are given as
\[
r_{23} \ast r_{12} = r_{12} \prec r_{13} \prec r_{23}; \quad \text{or} \quad r_{13} \ast r_{23} = r_{23} \prec r_{12} \prec r_{13}. \quad (4.4.9)
\]

By a similar proof as of Proposition 2.4.4, we have the following conclusion.

**Proposition 4.4.6.** Let \((A, \succ, \prec)\) be a dendriform algebra and \( r \in A \otimes A \) be a symmetric solution of \( D \)-equation in \( A \). Then the dendriform algebra structure and its associated associative algebra structure on the dendriform double \( \mathcal{D} \mathcal{D}(A) \) is given from the products in \( A \) as follows (for any \( x \in A, a^*, b^* \in A^* \)).

\[
(a) \quad a^* \prec b^* = -R_\succ^*(r(a^*))b^* + L^*(r(b^*))a^*, \quad a^* \succ b^* = R^*(r(a^*))b^* - L^*_\prec(r(b^*))a^*; \quad (4.4.10)
\]
\[
(b) \quad a^* \ast b^* = a^* \succ b^* + a^* \prec b^* = R_\succ^*(r(a^*))b^* + L^*_\prec(r(b^*))a^*; \quad (4.4.11)
\]
\[
(c) \quad x \prec a^* = x \prec r(a^*) - r(R^*(x)a^*) + R^*(x)a^* = 0, \quad (4.4.12)
\]
\[
(d) \quad x \ast a^* = x \ast r(a^*) - r(R^*_\prec(x)a^*) + R^*_\prec(x)a^*; \quad (4.4.13)
\]
Theorem 4.4.7. Let $(A, \succ, \prec)$ be a dendriform algebra and $r \in A \otimes A$. Suppose that $r$ is symmetric and nondegenerate. Then $r$ is a solution of $D$-equation in $A$ if and only if the inverse of the isomorphism $A^* \to A$ induced by $r$, regarded as a bilinear form $B$ on $A$ (that is, $B(x, y) = \langle r^{-1}x, y \rangle$ for any $x, y \in A$) satisfies

$$B(x \cdot y, z) = B(y, z \cdot x) + B(x, y \cdot z), \quad \forall x, y, z \in A.$$  \hfill (4.4.16)

Proof. Let $r = \sum_i a_i \otimes b_i$. Since $r$ is symmetric, $r(v^*) = \sum_i \langle v^*, a_i \rangle b_i$ for any $v^* \in A^*$. Since $r$ is nondegenerate, for any $x, y, z \in A$, there exist $u^*, v^*, w^* \in A^*$ such that $x = r(u^*), y = r(v^*), z = r(w^*)$. Therefore

$$B(x \cdot y, z) = \langle r(u^*) \cdot r(v^*), w^* \rangle = \sum_{i,j} \langle u^*, b_i \rangle \langle v^*, b_j \rangle \langle w^*, a_i \cdot a_j \rangle = \langle w^* \otimes u^* \otimes v^*, r_{12} \cdot r_{13} \rangle;$$

$$B(y, z \cdot x) = \langle r^*(v^*) \cdot r^*(u^*), w^* \rangle = \sum_{i,j} \langle u^*, b_i \rangle \langle v^*, b_j \rangle \langle w^*, a_i \cdot a_j \rangle = \langle w^* \otimes u^* \otimes v^*, r_{13} \cdot r_{23} \rangle;$$

$$B(x, y \cdot z) = \langle r(v^*) \cdot r(w^*), u^* \rangle = \sum_{i,j} \langle v^*, b_i \rangle \langle w^*, b_j \rangle \langle u^*, a_i \cdot a_j \rangle = \langle w^* \otimes u^* \otimes v^*, r_{23} \cdot r_{12} \rangle.$$ 

Therefore $B$ satisfies equation (4.4.16) if and only if $r$ is a solution of $D$-equation in $A$. \hfill \square

Definition 4.4.8. Let $(A, \succ, \prec)$ be a dendriform algebra. A bilinear form $B$ on $A$ is called a 2-cocycle if $B$ satisfies equation (4.4.16).

Remark 4.4.9. Let $B$ be 2-cocycle on a dendriform algebra $(A, \succ, \prec)$. Then it is easy to show that $\omega(x, y) = B(x, y) - B(y, x)$ (for any $x, y \in A$) is a Connes cocycle of the associated associative algebra $(A, \cdot)$. On the other hand, $B$ satisfies

$$B(x \cdot y, z) - B(x, y \cdot z) = B(y \cdot x, z) - B(y, x \cdot z), \quad \forall x, y, z \in A,$$  \hfill (4.4.17)

where $x \cdot y = x \succ y - y \prec x$ for any $x, y \in A$. Furthermore, $(A, \cdot)$ is a pre-Lie algebra (see subsections 5.2 and 5.3) and a bilinear form on a pre-Lie algebra $A$ satisfying equation (4.4.17) is called a 2-cocycle on $A$ (Ku2). Moreover, a pre-Lie algebra $A$ over the real number field $\mathbb{R}$ is called Hessian if there exists a symmetric and positive definite 2-cocycle on $A$. In geometry, a Hessian manifold $M$ is a flat affine manifold provided with a Hessian metric $g$, that is, $g$ is a Remanning metric such that for any each point $p \in M$ there exists a $C^\infty$-function $\phi$ defined on a neighborhood of $p$ such that $g_{ij} = \frac{\partial^2 \phi}{\partial x_i \partial x_j}$. A Hessian pre-Lie algebra corresponds to an affine Lie group $G$ with a $G$-invariant Hessian metric (Shl). Therefore a symmetric and positive definite 2-cocycle on a real dendriform algebra can give a Hessian structure.
Corollary 4.4.10. Let \((A, \rhd_A, \prec_A)\) be a dendriform algebra and \(r \in A \otimes A\) be a nondegenerate symmetric solution of \(D\)-equation in \(A\). Suppose the dendriform algebra structure \(\rhd_{A^*}, \prec_{A^*}\) on \(A^*\) is induced by \(r\) through Proposition 4.4.6. Then we have
\[
a^* \rhd_{A^*} b^* = r^{-1}(r(a^*) \rhd_A r(b^*)), a^* \prec_{A^*} b^* = r^{-1}(r(a^*) \prec_A r(b^*)), \quad \forall a^*, b^* \in A^*. \tag{4.4.18}
\]
Therefore \(r : A^* \to A\) is an isomorphism of dendriform algebras.

Proof. The conclusion can be obtained by a similar proof as of Corollary 2.4.6. \(\Box\)

Theorem 4.4.11. Let \((A, \rhd, \prec)\) be a dendriform algebra and \(r \in A \otimes A\) be symmetric. Then \(r\) is a solution of \(D\)-equation in \(A\) if and only if \(r\) satisfies
\[
r(a^*) \rhd r(b^*) = r(R^*_\prec(r(a^*))b^* + L^*_\prec(r(b^*))a^*), \quad \forall a^*, b^* \in A^*. \tag{4.4.19}
\]
Proof. The conclusion can be obtained by a similar proof as of Theorem 2.4.7. \(\Box\)

Combining Theorem 4.4.11 and Theorem 3.1.2, we have the following conclusion.

Corollary 4.4.12. Let \((A, \rhd, \prec)\) be a dendriform algebra and \(r \in A \otimes A\) be symmetric. Then \(r\) is a solution of \(D\)-equation in \(A\) if and only if \(r\) is an \(\mathcal{O}\)-operator of the associated associative algebra \((A, \ast)\) associated to \((R^*_\prec, L^*_\prec)\). Therefore there is a dendriform algebra structure on \(A^*\) given by
\[
a^* \rhd b^* = R^*_\prec(r(a^*))b^*, \quad a^* \prec b^* = L^*_\prec(r(b^*))a^*, \quad \forall a^*, b^* \in A^*. \tag{4.4.20}
\]
It has the same associated associative algebra of the dendriform algebra on \(A^*\) given by equation (4.4.11), which is induced by \(r\) in the sense of coboundary dendriform \(D\)-bialgebras. If \(r\) is nondegenerate, then there is a new compatible dendriform algebra structure on \(A\) given by
\[
x \rhd' y = r(R^*_\prec(x)r^{-1}y), \quad x \prec' y = r(L^*_\prec(y)r^{-1}x), \quad \forall x, y \in A, \tag{4.4.21}
\]
which is just the dendriform algebra structure given by
\[
\mathcal{B}(x \rhd' y, z) = \mathcal{B}(y, z \ast x), \quad \mathcal{B}(x \prec' y, z) = \mathcal{B}(x, y \ast z), \quad \forall x, y, z \in A, \tag{4.4.22}
\]
where \(\mathcal{B}\) is the symmetric 2-cocycle on \(A\) induced by \(r^{-1}\).

Theorem 4.4.13. Let \((A, \ast)\) be an associative algebra and \((l, r, V)\) be a bimodule. Let \((r^*, l^*, V^*)\) be the bimodule of \(A\) given by Lemma 2.1.2. Suppose that \(T : V \to A\) is an \(\mathcal{O}\)-operator associated to \((l, r, V)\). Then \(r = T + \sigma(T)\) is a symmetric solution of the \(D\)-equation in \(T(V) \ltimes_{r^*, 0, 0, l^*} V^*\), where \(T(V) \subset A\) is a dendriform algebra given by equation (3.1.4) and \((r^*, 0, 0, l^*)\) is a bimodule since its associated associative algebra \(T(V)\) is an associative subalgebra of \(A\), and \(T\) can be identified as an element in \(T(V) \otimes V^* \subset (T(V) \ltimes_{r^*, 0, 0, l^*} V^*) \otimes (T(V) \ltimes_{r^*, 0, 0, l^*} V^*)\).
Proof. Let \( \{e_1, \ldots, e_n\} \) be a basis of \( A \). Let \( \{v_1, \ldots, v_m\} \) be a basis of \( V \) and \( \{v_1^*, \ldots, v_m^*\} \) be its dual basis. Set \( T(v_i) = \sum_{k=1}^{n} a_{ik} e_k, i = 1, \ldots, m \). Then

\[
T = \sum_{i=1}^{m} T(v_i) \otimes v_i^* = \sum_{i=1}^{m} \sum_{k=1}^{n} a_{ik} e_k \otimes v_i^* \in T(V) \otimes V^* \subset (T(V) \rtimes_r \mathbb{L}^* \otimes V^*) \otimes (T(V) \rtimes_r \mathbb{L}^* \otimes V^*).
\]

Therefore we have

\[
\sum_{i,j=1}^{m} r^*(T(v_i))v_j^* \otimes v_i^* \otimes T(v_j) = \sum_{i,j=1}^{m} v_j^* \otimes v_i^* \otimes T(r(T(v_i))v_j);
\]

\[
\sum_{i,j=1}^{m} l^*(T(v_j))v_i^* \otimes T(v_i) \otimes v_j^* = \sum_{i,j=1}^{m} v_i^* \otimes T(l(T(v_j))v_i) \otimes v_j^*;
\]

\[
\sum_{i,j=1}^{m} T(v_i) \otimes v_j^* \otimes l^*(T(v_j))v_i^* = \sum_{i,j=1}^{m} T(l(T(v_j))v_i) \otimes v_j^* \otimes v_i^*;
\]

\[
\sum_{i,j=1}^{m} T(v_j) \otimes r^*(T(v_i))v_j^* \otimes v_i^* = \sum_{i,j=1}^{m} T(r(T(v_i))v_j) \otimes v_j^* \otimes v_i^*.
\]

Since \( T \) is an \( \mathcal{O} \)-operator of \( A \) associated to \( (l, r, V) \) and

\[
T(u) \succ T(v) = T(l(T(u))v), \quad T(u) \prec T(v) = T(r(T(v))u), \quad \forall u, v \in V,
\]

we show that \( r \) is a symmetric solution of the \( D \)-equation in \( T(V) \rtimes_r \mathbb{L}^* \otimes V^* \). \( \square \)

Remark 4.4.14. Roughly speaking, a symmetric solution of \( D \)-equation corresponds to the symmetric part of an \( \mathcal{O} \)-operator, whereas an antisymmetric solution of associative Yang-Baxter equation corresponds to the antisymmetric part of an \( \mathcal{O} \)-operator.

Corollary 4.4.15. Let \( (A, \succ, \prec) \) be a dendriform algebra. Then

\[
r = \sum_{i=1}^{n} (e_i \otimes e_i^* + e_i^* \otimes e_i)
\]

is a symmetric solution of the \( D \)-equation in \( A \rtimes_{R_{>0},0,L_{>0}^*} A^* \), where \( \{e_1, \ldots, e_n\} \) is a basis of \( A \) and \( \{e_1^*, \ldots, e_n^*\} \) is its dual basis. Moreover, \( r \) is nondegenerate and the induced 2-cocycle \( \mathcal{B} \) on \( A \rtimes_{R_{>0},0,L_{>0}^*} A^* \) is given by equation (1.1.1).

Proof. Let \( V = A, l = L_{>0}, r = R_{>0} \) and \( T = id \) in Theorem 4.4.13. Then the conclusion follows immediately. \( \square \)
Remark 4.4.16. Comparing with Theorem 4.3.6, we show that (the non-symmetric) $T = \sum_{i=1}^{n} e_i \otimes e_i^{*}$ induces a dendriform D-bialgebra structure on $A \ltimes_{L^*, -L^*, -L^*, L^*} A^*$, whereas the above (symmetric) $r = T + \sigma(T)$ induces a dendriform D-bialgebra structure on $A \ltimes_{R^*, 0, L^*, 0, L^*} A^*$.

Recall that two Connes cocycles $(A_1, \omega_1)$ and $(A_2, \omega_2)$ are isomorphic if and only if there exists an isomorphism of associative algebras $\varphi : A_1 \to A_2$ such that
\[
\omega_1(x, y) = \varphi^{*}\omega_2(x, y) = \omega_2(\varphi(x), \varphi(y)), \quad \forall x, y \in A_1.
\]

By a similar proof as of Theorem 2.4.9, we have the following conclusion.

Theorem 4.4.17. Let $(A, >, \prec)$ be a dendriform algebra. Then as Connes cocycles of associative algebras, the double construction of Connes cocycle (or the dendriform D-bialgebra) $(T(A) = A \ltimes A^*, \omega)$ given by a symmetric solution $r$ of $D$-equation in $A$ is isomorphic to the double construction of Connes cocycle (or the dendriform D-bialgebra) $(T(A) = A \ltimes_{L^*, L^*} A^*, \omega)$, where $\omega$ is given by equation (1.4.1). However, in general, they are not isomorphic as double constructions of Connes cocycles (or dendriform D-bialgebras).

Corollary 4.4.18. Let $(A, >, \prec)$ be a dendriform algebra. Then as Connes cocycles of associative algebras, the double constructions of Connes cocycles given by all symmetric solutions of $D$-equation in $A$ are isomorphic to the double construction of Connes cocycle $(T(A) = A \ltimes_{R^*, L^*} A^*, \omega)$ given by the zero solution.

5. Comparison (duality) between bialgebra structures

5.1. Comparison (duality) between antisymmetric infinitesimal bialgebras and dendriform D-bialgebras.

The results in the previous sections allow us to compare antisymmetric infinitesimal bialgebras and dendriform D-bialgebras in terms of the following properties: 1-cocycles of associative algebras, matched pairs of associative algebras, associative algebra structures on the direct sum of the associative algebras in the matched pairs, bilinear forms on the direct sum of the associative algebras in the matched pairs, double structures on the direct sum of the associative algebras in the matched pairs, algebraic equations associated to coboundary cases, nondegenerate solutions, $\mathcal{O}$-operators of associative algebras and constructions from dendriform algebras. We list the them in Table 1. From this table, we observe that there is a clear analogy between them and in particular, double constructions of Frobenius algebras correspond to double constructions of Connes cocycles in this sense. Moreover, due to the correspondences between certain symmetries and antisymmetries appearing in the Table 1, we regard it as a kind of duality.

Next we consider the case that a dendriform D-bialgebra is also an antisymmetric infinitesimal bialgebra.
Table 1. Comparison between antisymmetric infinitesimal bialgebras and dendriform D-bialgebras

| Algebras | Antisymmetric infinitesimal bialgebras | Dendriform D-bialgebras |
|----------|----------------------------------------|-------------------------|
| 1-cocycles of associative algebras | \((\text{id} \otimes L, R \otimes \text{id})\) | \((\text{id} \otimes L_\prec, R \otimes \text{id}), (\text{id} \otimes L, R_\prec \otimes \text{id})\) |
| Matched pairs of associative algebras | \((A, A^*, R_\prec^*, L_\prec_A^*, R_A^*, L_A^*)\) | \((A, A^*, R_\prec^*_A, R_A^*_L, L_A^*_R, R_A^*_L, L_A^*_R)\) |
| Associative algebra structures on the direct sum of the associative algebras in the matched pairs | double constructions of Frobenius algebras | double constructions of Connes cocycles |
| Bilinear forms on the direct sum of the associative algebras in the matched pairs | symmetric | antisymmetric |
| Double structures on the direct sum of the associative algebras in the matched pairs | associative doubles | dendriform doubles |
| Algebraic equations associated to coboundary cases | antisymmetric solutions | symmetric solutions |
| Nondegenerate solutions | Connes cocycles of associative algebras | 2-cocycles of dendriform algebras |
| \(\mathcal{O}\)-operators of associative algebras | associated to \((R^*_\prec, L^*_A)\) | associated to \((R^*_\prec, L^*_A)\) |
| Constructions from dendriform algebras | induced bilinear forms | induced bilinear forms |
| \(r = \sum_{i=1}^{n} (e_i \otimes e_i^* - e_i^* \otimes e_i)\) | \(r = \sum_{i=1}^{n} (e_i \otimes e_i^* + e_i^* \otimes e_i)\) |

**Theorem 5.1.1.** Let \((A, A^*, \Delta_\prec, \Delta_A, \beta_\prec, \beta_A)\) be a dendriform D-bialgebra. Then \((A, A^*)\) is an antisymmetric infinitesimal bialgebra if and only if the following two equations hold:

\[
\langle L_{\prec, A}^*(b^*)y, L_{\prec, A}^*(x)a^* \rangle = \langle R_{\prec, A}^*(a^*)x, R_{\prec, A}^*(y)b^* \rangle; \tag{5.1.1}
\]

\[
\langle L_{\prec, A}^*(b^*)y, R_{\prec, A}^*(x)a^* \rangle + \langle L_{\prec, A}^*(a^*)x, R_{\prec, A}^*(y)b^* \rangle = \langle R_{\prec, A}^*(a^*)x, L_{\prec, A}^*(y)a^* \rangle + \langle R_{\prec, A}^*(a^*)y, L_{\prec, A}^*(x)b^* \rangle, \tag{5.1.2}
\]

for any \(x, y \in A^*, a^*, b^* \in A^*\).

**Proof.** The conclusion can be obtained by a similar proof as of Proposition 2.2.2. \(\square\)

**Corollary 5.1.2.** Let \((A, >, \prec)\) be a dendriform algebra and \(r \in A \otimes A\) be a symmetric solution of \(D\)-equation in \(A\). Suppose the dendriform algebra structure on \(A^*\) is induced by \(r\) from equation (4.4.11). Then \((A, A^*)\) is an antisymmetric infinitesimal bialgebra if and only if the following
two equations hold:
\[ \langle y <_A (x >_A r(a^*)) - y *_A r(R_{<_A}^r(x)a^*), b^* \rangle = \langle r(L_{<_A}^r(y)b^*) *_A x - (r(b^*)) <_A y >, a^* \rangle; \quad (5.1.3) \]
\[ \langle y <_A (r(a^*)) <_A x \rangle - (y >_A r(a^*)) >_A x + r(R_{<_A}^r(y)a^*) *_A x - y *_A r(L_{<_A}^r(x)a^*), b^* \rangle \]
\[ = \langle -x <_A (r(b^*)) <_A y \rangle + (x >_A r(b^*)) >_A y - y *_A r(R_{<_A}^r(x)a^*) *_A y + x *_A r(L_{<_A}^r(x)a^*), a^* \rangle, \quad (5.1.4) \]
for any \( x, y \in A \) and \( a^* \in A^* \).

**Corollary 5.1.3.** Let \( (A, A^*, \Delta_>, \Delta_<, \beta_<, \beta_> \) be a dendriform \( D \)-bialgebra. If equations (5.1.1-5.1.2) are satisfied, then there are two associative algebra structures \( A \bowtie^{R_{<_A}^r, L_{<_A}^r} \ A^* \) and \( A \bowtie^{R_{<_A}^r, L_{<_A}^r} \ A^* \) on the direct sum \( A \oplus A^* \) of the underlying vector spaces of \( A \) and \( A^* \) such that both \( A \) and \( A^* \) are associative subalgebras and the bilinear form given by equation (1.4.1) is a Connes cocycle on \( A \bowtie^{R_{<_A}^r, L_{<_A}^r} \ A^* \) and the bilinear form given by equation (1.1.1) is invariant on \( A \bowtie^{R_{<_A}^r, L_{<_A}^r} \ A^* \). Moreover, These two associative algebras are not isomorphic in general.

**Example 5.1.4.** Let \( (A, *_A) \) be an associative algebra and \( \omega \) be a Connes cocycle on \( (A, *_A) \). Then there is an antisymmetric infinitesimal bialgebra whose associative algebra structure on \( A^* \) is given by a nondegenerate solution \( r \) of associative Yang-Baxter equation as follows.

\[ \Delta(x) = (id \otimes L(x) - R(x) \otimes id) r, \quad \forall x \in A, \quad (5.1.5) \]
where \( r : A^* \rightarrow A \) is given by \( \omega(x, y) = \langle r^{-1}(x), y \rangle \). On the other hand, there exists a compatible dendriform algebra structure \( \succ_A, \prec_A \) on \( A \) given by equation (4.1.1), that is,

\[ \omega(x \succ_A y, z) = \omega(y, z *_A x), \quad \omega(x \prec_A y, z) = \omega(x, y *_A z), \quad \forall x, y, z \in A. \quad (5.1.6) \]
Moreover, there exists a compatible dendriform algebra structure on the associative algebra \( A^* \) given by

\[ a^* \succ_A b^* = r^{-1}(r(a^*) \succ_A r(b^*)), \quad a^* \prec_A b^* = r^{-1}(r(a^*) \prec_A r(b^*)), \quad \forall a^*, b^* \in A. \quad (5.1.7) \]
Furthermore, it is easy to show that

\[ L_{<_A}^r(x)a^* = r^{-1}(r(a^*) \succ_A x), \quad R_{<_A}^r(x)a^* = -r^{-1}(x \prec_A r(a^*)), \quad L_{<_A}^r(x)a^* = -r^{-1}(r(a^*) \succ_A x), \]
\[ R_{<_A}^r(x)a^* = r^{-1}(x *_A r(a^*)), \quad L_{<_A}^r(a^*)x = x *_A r(a^*), \quad R_{<_A}^r(a^*)x = -r(a^*) \prec_A x, \]
\[ L_{<_A}^r(a^*)x = -x \succ_A r(a^*), \quad R_{<_A}^r(a^*)x = r(a^*) *_A x, \quad \forall x \in A, a^* \in A^*. \quad (5.1.8) \]
Therefore according to Theorem 4.2.4, \( (A, A^*) \) (as dendriform algebras) is a dendriform D-bialgebra if and only if \( (A, A^*, R_{<_A}^r, L_{<_A}^r, R_{<_A}^r, L_{<_A}^r) \) a matched pair of associative algebras, if and only if \( A \) is 2-step nilpotent, that is, \( x *_A y *_A z = 0 \) for any \( x, y, z \in A \). In this case, by equation (5.1.6), we show that it is equivalent to

\[ x \succ_A (y \succ_A z) = x \prec_A (y \prec_A z) = x \succ_A (y \prec_A z) = x \succ_A (y \prec_A z) = 0, \quad \forall x, y, z \in A. \quad (5.1.9) \]
Therefore, under such conditions, equations (5.1.1-5.1.2) hold naturally.
5.2. Duality in the version of Lie algebras: Lie bialgebras and pre-Lie bialgebras.

There is a similar duality in the version of Lie algebras which was given in [Bai2]. In order to be self-contained, we give a brief introduction in this subsection. We would like to point out that, although we give the Lie bialgebras and pre-Lie bialgebras as the similar structures of antisymmetric infinitesimal bialgebras and dendriform D-bialgebras here, in fact, it is the Manin triples (Lie bialgebras) that have been first studied and then motivate us to study the other structures.

There are two kinds of important (nondegenerate) bilinear forms on Lie algebras as follows. A bilinear form \( B( , ) \) on a Lie algebra \( A \) is invariant if
\[
B([x, y], z) = B(x, [y, z]), \quad \forall \ x, y \in A.
\] (5.2.1)

A 2-cocycle (symplectic form) on a Lie algebra \( A \) is an antisymmetric bilinear form \( \omega \) satisfying
\[
\omega([x, y], z) + \omega([y, z], x) + \omega([z, x], y) = 0, \quad \forall \ x, y, z \in A.
\] (5.2.2)

Moreover, the algebras play a similar role of dendriform algebras in the double constructions of Frobenius algebras and Connes cocycles are pre-Lie algebras. In fact, pre-Lie algebras (or under other names like left-symmetric algebras, quasi-associative algebras, Vinery algebras and so on) are a class of natural algebraic systems appearing in many fields in mathematics and mathematical physics (see a survey article [Bu] and the references therein).

**Definition 5.2.1.** Let \( A \) be a vector space over a field \( F \) with a bilinear product \( (x, y) \rightarrow xy \). \( A \) is called a pre-Lie algebra if
\[
(xy)z - x(yz) = (yx)z - y(xz), \quad \forall x, y, z \in A.
\] (5.2.3)

Let \( A \) be a pre-Lie algebra. For any \( x, y \in A \), let \( L(x) \) and \( R(x) \) denote the left and right multiplication operator respectively, that is, \( L(x)(y) = xy \), \( R(x)(y) = yx \). Let \( L : A \rightarrow gl(A) \) with \( x \rightarrow L(x) \) and \( R : A \rightarrow gl(A) \) with \( x \rightarrow R(x) \) (for every \( x \in A \)) be two linear maps. For a Lie algebra \( G \), we let \( \text{ad}(x) \) denote the adjoint operator, that is, \( \text{ad}(x)y = [x, y] \), and \( \text{ad} : G \rightarrow gl(G) \) with \( x \rightarrow \text{ad}(x) \) be a linear map.

**Proposition 5.2.2.** Let \( A \) be a pre-Lie algebra.

1. The commutator
\[
[x, y] = xy - yx, \quad \forall x, y \in A,
\] (5.2.4)
defines a Lie algebra \( G(A) \), which is called the sub-adjacent Lie algebra of \( A \) and \( A \) is also called a compatible pre-Lie algebra structure on the Lie algebra \( G(A) \).

2. The map \( L : A \rightarrow gl(A) \) gives a representation of the Lie algebra \( G(A) \).

**Proposition 5.2.3.** ([Chu]) Let \( G \) be a Lie algebra and \( \omega \) be a nondegenerate 2-cocycle on \( G \) (such a Lie algebra called a symplectic Lie algebra). Then there exists a compatible pre-Lie
algebra structure on $G$ defined by
\[
\omega(x * y, z) = -\omega(y, [x, z]), \quad \forall x, y, z \in G.
\] (5.2.5)

Next we give the “double constructions” of Lie algebras with nondegenerate invariant bilinear forms or nondegenerate 2-cocycles. In fact, both of them have their own (independent) interests in many fields.

At first, recall that $(G, H, \rho, \mu)$ is a matched pair of Lie algebras if $G$ and $H$ are Lie algebras and $\rho : G \to \text{gl}(H)$ and $\mu : H \to \text{gl}(G)$ are representations satisfying
\[
\rho(x)[a, b] - [\rho(x)a, b] - [a, \rho(x)b] + \rho(\mu(x)a)b - \rho(\mu(b)x)a = 0;
\] (5.2.6)
\[
\mu(a)[x, y] - [\mu(a)x, y] - [x, \mu(a)y] + \mu(\rho(x)a)y - \mu(\rho(y)a)x = 0,
\] (5.2.7)
for any $x, y \in G$ and $a, b \in H$. In this case, there exists a Lie algebra structure on the direct sum $G \oplus H$ of the underlying vector spaces of $G$ and $H$ given by
\[
[x + a, y + b] = [x, y] + \mu(a)y - \mu(b)x + [a, b] + \rho(x)b - \rho(y)a, \quad \forall x, y \in G, a, b \in H.
\] (5.2.8)
We denote it by $G \bowtie H$ or simply $G \bowtie H$. Moreover, every Lie algebra which is the direct sum of the underlying vector spaces of two subalgebras can be obtained from a matched pair of Lie algebras as above.

**Definition 5.2.4.** Let $G$ be a Lie algebra. Suppose that there is a Lie algebra structure on the direct sum of the underlying vector spaces of $G$ and its dual space $G^*$ such that $G$ and $G^*$ are Lie subalgebras.

(a) If the natural symmetric bilinear form on $G \oplus G^*$ given by equation (1.1.1) is invariant, then $(G \bowtie G^*, G, G^*)$ is called a (standard) Manin triple.

(b) If the natural antisymmetric bilinear form on $G \oplus G^*$ given by equation (1.4.1) is a 2-cocycle, then it is called a phase space of the Lie algebra $G$ ([Ku1]). $(G \bowtie G^*, G, G^*)$ is also called a parakähler structure on the Lie algebra $G \bowtie G^*$ ([Kan]).

For a Lie algebra $G$ and a representation $(\rho, V)$ of $G$, recall that a 1-cocycle $T$ associated to $\rho$ (denoted by $(\rho, T)$) is a linear map from $G$ to $V$ satisfying
\[
T([x, y]) = \rho(x)T(y) - \rho(y)T(x), \quad \forall x, y \in G.
\] (5.2.9)

**Definition 5.2.5.** (a) Let $G$ be a Lie algebra. A Lie bialgebra structure on $G$ is an antisymmetric linear map $\delta : G \to G \otimes G$ such that $\delta^* : G^* \otimes G^* \to G^*$ is a Lie bracket on $G^*$ and $\delta$ is a 1-cocycle of $G$ associated to $\text{ad} \otimes \text{id} + \text{id} \otimes \text{ad}$ with values in $G \otimes G$. We denote it by $(G, G^*)$ or $(G, \delta)$.

(b) Let $A$ be a vector space. A pre-Lie bialgebra structure on $A$ is a pair of linear maps $(\Delta, \beta)$ such that $\Delta : A \to A \otimes A, \beta : A^* \to A^* \otimes A^*$ and

1. $\Delta^* : A^* \otimes A^* \to A^*$ defines a pre-Lie algebra structure on $A^*$;
2. $\beta^* : A \otimes A \to A$ defines a pre-Lie algebra structure on $A$;
(3) $\Delta$ is a 1-cocycle of $\mathcal{G}(A)$ associated to $L \otimes id + id \otimes ad$ with values in $A \otimes A$;
(4) $\beta$ is a 1-cocycle of $\mathcal{G}(A^*)$ associated to $L \otimes id + id \otimes ad$ with values in $A^* \otimes A^*$.

We denote it by $(A, A^*, \Delta, \beta)$ or simply $(A, A^*)$.

**Theorem 5.2.6.** (a) Let $(\mathcal{G}, \{ , \}^\mathcal{G})$ and $(\mathcal{G}^*, \{ , \}^\mathcal{G}^*)$ be two Lie algebras. Then the following conditions are equivalent:

1. $(\mathcal{G} \bowtie \mathcal{G}^*, \mathcal{G}, \mathcal{G}^*)$ is a standard Manin triple with the bilinear form (1.1.1);
2. $(\mathcal{G}, \mathcal{G}^*, ad^\mathcal{G}_G, ad^\mathcal{G}^*_G)$ is a matched pair of Lie algebras;
3. $(\mathcal{G}, \mathcal{G}^*)$ is a Lie bialgebra.

(b) Let $(A, \cdot)$ and $(A^*, \circ)$ be two pre-Lie algebras. Then the following conditions are equivalent:

1. $(\mathcal{G}(A) \bowtie \mathcal{G}(A^*), \mathcal{G}(A), \mathcal{G}(A^*))$ is a parakähler Lie algebra with the bilinear form (1.4.1).
2. $(\mathcal{G}(A), \mathcal{G}(A^*), L^*, L^*_0)$ is a matched pair of Lie algebras;
3. $(A, A^*)$ is a pre-Lie bialgebra.

In fact, a Lie bialgebra is the Lie algebra $\mathcal{G}$ of a Poisson-Lie group $G$ equipped with additional structures induced from the Poisson structure on $G$ and a Poisson-Lie group is a Lie group with a Poisson structure compatible with the group operation in a certain sense. Poisson-Lie groups play an important role in symplectic geometry and quantum group theory (cf. [D] and the references therein). On the other hand, in geometry, a parakähler manifold is a symplectic manifold with a pair of transversal Lagrangian foliations ([Li]). A parakähler Lie algebra $\mathcal{G}$ is the Lie algebra of a Lie group $G$ with a $G$-invariant parakähler structure ([Kan]).

We have already obtained many properties of Lie bialgebras and pre-Lie algebras which are similar to our study in the previous sections. We put them in the Appendix and we compare pre-Lie bialgebras and Lie bialgebras in terms of their certain properties in Table 2. From Table 2, we observe that there is also a clear analogy between them and in particular, due to the correspondences between certain symmetries and antisymmetries appearing in the analogy, we can regard it as a kind of duality again which is similar to the duality appearing in the Table 1.

**5.3. Relationships among four bialgebras.**

**Proposition 5.3.1.** ([Cha1], [A2]) Let $(A, >, <)$ be a dendriform algebra. Then there is a pre-Lie algebra structure on $(A, \cdot)$ given by

$$x \cdot y = x > y - y < x, \quad \forall x, y \in A.$$  \hspace{1cm} (5.3.1)

**Corollary 5.3.2.** Let $(A, >, <)$ be a dendriform algebra. Then the sub-adjacent Lie algebra of the pre-Lie algebra $(A, \cdot)$ given by equation (5.3.1) is as the same as the commutator Lie algebra of the associated associative algebra $(A, *)$, that is,

$$[x, y] = x * y - y * x = x \cdot y - y \cdot x = x > y + x < y - y > x - y < x, \quad \forall x, y \in A.$$  \hspace{1cm} (5.3.2)
Table 2. Comparison between Lie bialgebras and pre-Lie bialgebras

| Algebras                              | Lie bialgebras          | Pre-Lie bialgebras      |
|---------------------------------------|-------------------------|-------------------------|
| Corresponding Lie groups              | Poisson-Lie groups      | parakähler Lie groups   |
| 1-cocycles of Lie algebras            | \( id \otimes ad + ad \otimes id \) | \( L \otimes id + id \otimes ad \) |
| Matched pairs of Lie algebras         | \((G, G^*, ad_G^{\ast}, ad_{G^*})\) | \((G(A), G(A^*), L_A^\ast, L_{A^*}^\ast)\) |
| Lie algebra structures on the direct sum of the Lie algebras in the matched pairs | symmetric | antisymmetric |
| Bilinear forms on the direct sum of the Lie algebras in the matched pairs | \( \langle x + a^\ast, y + b^\ast \rangle \) | \( \langle x + a^\ast, y + b^\ast \rangle \) |
|                                       | \( = \langle x, b^\ast \rangle + \langle a^\ast, y \rangle \) | \( = -\langle x, b^\ast \rangle + \langle a^\ast, y \rangle \) |
| Double structures on the direct sum of the Lie algebras in the matched pairs | Drinfeld doubles | symplectic doubles |
| Algebraic equations associated to coboundary cases | classical Yang-Baxter equations in Lie algebras | \( S \)-equations in pre-Lie algebras |
| Nondegenerate solutions               | 2-cocycles of Lie algebras | 2-cocycles of pre-Lie algebras |
|                                       | symplectic structures   | Hessian structures      |
| \( \mathcal{O} \)-operators of Lie algebras | associated to \( ad^* \) | associated to \( L^* \) |
|                                       | antisymmetric parts     | symmetric parts         |
| Constructions from pre-Lie algebras    | \( r = \sum_{i=1}^n (e_i \otimes e_i^* - e_i^* \otimes e_i) \) | \( r = \sum_{i=1}^n (e_i \otimes e_i^* + e_i^* \otimes e_i) \) |
|                                       | induced bilinear forms  | induced bilinear forms  |
|                                       | \( \langle x + a^\ast, y + b^\ast \rangle \) | \( \langle x + a^\ast, y + b^\ast \rangle \) |
|                                       | \( = -\langle x, b^\ast \rangle + \langle a^\ast, y \rangle \) | \( = (x, b^\ast) + (a^\ast, y) \) |

Therefore, as Chapoton pointed out in [Cha1] (also see [A2], [A4], [EMP]), there is the following commutative diagram of categories.

\[
\begin{array}{c}
\text{dendriform algebras} \\
\downarrow \\
\text{associative algebras}
\end{array} \quad \longrightarrow \quad \begin{array}{c}
\text{pre-Lie algebras} \\
\downarrow \\
\text{Lie algebras}
\end{array}
\]

In this diagram, the left vertical arrow is given by equation (3.1.2), the top horizontal arrow is given by equation (5.3.1), the bottom arrow is given by equation (5.2.4) since an associative algebra is a special pre-Lie algebra and the right vertical arrow is given by equation (5.2.4).

Obviously, if a symmetric or antisymmetric bilinear form on an associative algebra is invariant or a Connes cocycle respectively, then it is also invariant or a 2-cocycle on the commutator Lie algebra respectively.

**Theorem 5.3.3.** (1) A double construction of Frobenius algebra gives a standard Manin triple (on the commutator Lie algebra) naturally.
(2) A double construction of Connes cocycles gives a parakähler Lie algebra (on the commutator Lie algebra) naturally.

**Corollary 5.3.4.** (1) Any antisymmetric infinitesimal bialgebra is a Lie bialgebra (in the sense of its commutator Lie algebra).

(2) Any dendriform D-bialgebra is a pre-Lie bialgebra (in the sense of equation (5.3.1)).

**Corollary 5.3.5.** We have the following relationship among the antisymmetric infinitesimal bialgebras, dendriform algebras, Lie bialgebras and pre-Lie bialgebras.

\[
\begin{array}{ccc}
\text{dendriform D-bialgebras} & \hookrightarrow & \text{pre-Lie bialgebras} \\
\downarrow \text{dual} & & \downarrow \text{dual} \\
\text{antisymmetric infinitesimal bialgebras} & \hookrightarrow & \text{Lie bialgebras}
\end{array}
\]

The $\downarrow$ means that the duality given in subsections 5.1 and 5.2 and the $\hookrightarrow$ means the inclusion in the sense of Corollary 5.3.4.

**Remark 5.3.6.** The conclusion (1) in Corollary 5.3.4 and the relation given by the bottom $\hookrightarrow$ in the above diagram were also pointed out in [A3].

**Corollary 5.3.7.** Let \((A, A^*, \Delta_>, \Delta_<, \beta_>, \beta_<)\) be a dendriform D-bialgebra. If equations (5.1.1-5.1.2) hold, then \((A, A^*)\) is an antisymmetric infinitesimal bialgebra. \((A, A^*)\) is also a pre-Lie bialgebra in the sense of equation (5.3.1). Furthermore, as the commutator Lie algebras, \((G(A), G(A)^*)\) is a Lie bialgebra. Therefore, there is an associative algebra structure and a Lie algebra structure on the direct sum \(A \oplus A^*\) of the underlying space of \(A\) and \(A^*\) such that the natural symmetric bilinear form given by equation (1.1.1) is invariant on both of them and the natural antisymmetric bilinear form given by equation (1.4.1) is a Connes cocycle on the associative algebra and a 2-cocycle on the Lie algebra. Moreover, under such a condition, there is the following commutative diagram in an obvious way.

\[
\begin{array}{ccc}
\text{dendriform D-bialgebras} & \hookrightarrow & \text{pre-Lie bialgebras} \\
\downarrow & & \downarrow \\
\text{antisymmetric infinitesimal bialgebras} & \hookrightarrow & \text{Lie bialgebras}
\end{array}
\]

**Acknowledgments**

The author thanks Professors M. Aguiar, A. Connes, L. Guo and J.-L. Loday for important suggestion. This work was supported in part by NSFC (10621101, 10920161), NKBRC (2006CB805905) and SRFDP (200800550015).

**Appendix: Some properties of Lie bialgebras and pre-Lie bialgebras**

In this appendix, we list some properties of Lie bialgebras and pre-Lie bialgebras. Most of the results can be found in [Bai2] and the references therein.
Proposition A1. (a) Let \((G, G^*)\) be a Lie bialgebra. Then there is a canonical Lie bialgebra structure on \(G \oplus G^*\) such that the inclusions \(i_1 : G \to G \oplus G^*\) and \(i_2 : G^* \to G \oplus G^*\) into the two summands are homomorphisms of Lie bialgebras, where the Lie bialgebra structure on \(G^*\) is given by \(-\delta_{G^*}\). Such a structure is called a classical (Drinfeld) double of \(G\).

(b) Let \((A, A^*, \Delta, \beta)\) be a pre-Lie bialgebra. Then there is a canonical pre-Lie bialgebra structure on \(A \oplus A^*\) such that both the inclusions \(i_1 : A \to A \oplus A^*\) and \(i_2 : A^* \to A \oplus A^*\) into the two summands are homomorphisms of pre-Lie bialgebras. Such a structure is called a symplectic double of \(A\).

Definition A2. (a) A Lie bialgebra \((G, \delta)\) is called coboundary if \(\delta\) is a 1-coboundary of \(G\) associated to \(\text{ad} \otimes id + id \otimes \text{ad}\), that is, there exists a \(r \in G \otimes G\) such that
\[
\delta(x) = (\text{ad}(x) \otimes id + id \otimes \text{ad}(x))r, \quad \forall x \in G. \tag{A1}
\]

(b) A pre-Lie bialgebra \((A, A^*, \Delta, \beta)\) is called coboundary if \(\Delta\) is a 1-coboundary of \(G(A)\) associated to \(L \otimes id + id \otimes \text{ad}\), that is, there exists a \(r \in A \otimes A\) such that
\[
\Delta(x) = (L(x) \otimes id + id \otimes \text{ad}x)r, \quad \forall x \in A. \tag{A2}
\]

Theorem A3. (a) Let \(G\) be a Lie algebra and \(r \in G \otimes G\). Then the map \(\delta : G \to G \otimes G\) defined by equation (A1) induces a Lie bialgebra structure on \(G\) if and only if the following two conditions are satisfied (for any \(x \in G\)):

1. \((\text{ad}(x) \otimes id + id \otimes \text{ad}(x))(r + \sigma(r)) = 0;
2. \((\text{ad}(x) \otimes id \otimes id + id \otimes \text{ad}(x) \otimes id + id \otimes id \otimes \text{ad}(x))(\sigma_1, \sigma_2) + [\sigma_1, \sigma_2] = 0.

(b) Let \(A\) be a pre-Lie algebra and \(r \in A \otimes A\). Then the map \(\Delta\) defined by equation (A2) induces a pre-Lie algebra structure on \(A^*\) such that \((A, A^*)\) is a pre-Lie bialgebra if and only if the following two conditions are satisfied (for any \(x, y \in A\)):

1. \([P(x \cdot y) - P(x)P(y)](r - \sigma(r)) = 0;
2. \(Q(x)[r, r] = 0,
where \(Q(x) = L(x) \otimes id \otimes id + id \otimes L(x) \otimes id + id \otimes id \otimes \text{ad}x, \quad P(x) = L(x) \otimes id + id \otimes L(x) \) and
\[
[r, r] = r_{12} \cdot r_{23} - r_{23} \cdot r_{12} + [r_{23}, r_{12}] - [r_{12}, r_{23}]. \tag{A3}
\]

Corollary A4. (a) Let \(G\) be a Lie algebra and \(r \in G \otimes G\). If \(r\) is antisymmetric and \(r\) satisfies
\[
[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0, \tag{A4}
\]
then the map \(\delta : G \to G \otimes G\) defined by equation (A1) induces a Lie bialgebra structure on \(G\).

(b) Let \(A\) be a pre-Lie algebra and \(r \in A \otimes A\). Suppose that \(r\) is symmetric. Then the map \(\Delta\) defined by equation (A2) induces a pre-Lie algebra structure on \(A^*\) such that \((A, A^*)\) is a pre-Lie bialgebra if
\[
-r_{12} \cdot r_{13} + r_{12} \cdot r_{23} + [r_{13}, r_{23}] = 0. \tag{A5}
\]
Definition A5.  
(a) Let $G$ be a Lie algebra and $r \in G \otimes G$. Equation (A4) is called classical Yang-Baxter equation in $G$. 
(b) Let $A$ be a pre-Lie algebra and $r \in A \otimes A$. Equation (A5) is called $S$-equation in $A$. 

Let $G$ be a Lie algebra and $\rho : G \to \text{gl}(V)$ be its representation. Recall that a linear map $T : V \to G$ is called an $O$-operator of $G$ associated to $\rho$ if $T$ satisfies

$$[T(u), T(v)] = T(\rho(T(u))v - \rho(T(v))u), \forall u, v \in V.$$  \hspace{1cm} (A6)

Proposition A6.  
(a) Let $G$ be a Lie algebra and $r \in G \otimes G$.

(1) Suppose $r$ is antisymmetric and nondegenerate. Then $r$ is a solution of classical Yang-Baxter equation in $G$ if and only if the isomorphism $G^* \to G$ induced by $r$, regarded as a bilinear form on $G$ is a 2-cocycle on $G$.

(2) Suppose $r$ is antisymmetric. Then $r$ is a solution of classical Yang-Baxter equation in $G$ if and only if $r$ is an $O$-operator of $G$ associated to $\text{ad}^*$, that is, $r$ satisfies

$$[r(a^*), r(b^*)] = r(\text{ad}^*(r(a^*))b^* - \text{ad}^*(r(b^*))a^*), \forall a^*, b^* \in G^*.$$ \hspace{1cm} (A7)

(b) Let $A$ be a pre-Lie algebra and $r \in A \otimes A$.

(1) Suppose that $r$ is symmetric and nondegenerate. Then $r$ is a solution of $S$-equation in $A$ if and only if the inverse of the isomorphism $A^* \to A$ induced by $r$, regarded as a bilinear form $B$ on $A$ is a 2-cocycle on $A$ (see equation (4.4.17)).

(2) Suppose that $r$ is symmetric. Then $r$ is a solution of $S$-equation in $A$ if and only if $r$ is an $O$-operator of $G(A)$ associated to $L^*$, that is, $r$ satisfies

$$[r(a^*), r(b^*)] = r(L^*(r(a^*))b^* - L^*(r(b^*))a^*), \forall a^*, b^* \in A^*.$$ \hspace{1cm} (A8)

Lemma A7.  Let $G$ be a Lie algebra and $\rho : G \to \text{gl}(V)$ be a representation. Let $T : V \to G$ be an $O$-operator associated to $\rho$. Then the product

$$u \circ v = \rho(T(u))v, \forall u, v \in V$$ \hspace{1cm} (A9)

defines a pre-Lie algebra structure on $V$. Therefore $V$ is a Lie algebra as the sub-adjacent Lie algebra of this pre-Lie algebra and $T$ is a homomorphism of Lie algebras. Furthermore, $T(V) = \{T(v) | v \in V\} \subset G$ is a Lie subalgebra of $G$ and there is an induced pre-Lie algebra structure on $T(V)$ given by

$$T(u) \cdot T(v) = T(u \circ v) = T(\rho(T(u))v), \forall u, v \in V.$$ \hspace{1cm} (A10)

Moreover, its sub-adjacent Lie algebra structure is just the Lie subalgebra structure of $G$ and $T$ is a homomorphism of pre-Lie algebras.
Proposition A8. Let $G$ be a Lie algebra and $\rho : G \to gl(V)$ be a representation. Let $\rho^* : G \to gl(V^*)$ be the dual representation of $\rho$.

(a) A linear map $T : V \to G$ is an $O$-operator of $G$ associated to $\rho$ if and only if $r = T - \sigma(T)$ is an antisymmetric solution of the classical Yang-Baxter equation in $G \ltimes_{\rho^*} V^*$.

(b) Let $T : V \to G$ be an $O$-operator associated to $\rho$. Then $r = T + \sigma(T)$ is a symmetric solution of the $S$-equation in $T(V) \ltimes_{\rho^*,0} V^*$, where $T(V) \subset G$ is a pre-Lie algebra given by equation (A10) and $(\rho^*,0)$ is a bimodule since its sub-adjacent Lie algebra $\mathcal{G}(T(V))$ is a Lie subalgebra of $G$, and $T$ can be identified as an element in $T(V) \otimes V^* \subset (T(V) \ltimes_{\rho^*,0} V^*) \otimes (T(V) \ltimes_{\rho^*,0} V^*)$.

Proposition A9. Let $(A,\cdot)$ be a pre-Lie algebra. Let $\{e_1, \cdots, e_n\}$ be a basis of $A$ and $\{e_1^*, \cdots, e_n^*\}$ be its dual basis.

(a) $r$ given by equation (2.5.3) is an antisymmetric solution of the classical Yang-Baxter equation in $G(A) \ltimes_{L^*} G(A)^*$. Moreover, $r$ is nondegenerate and the induced 2-cocycle $B$ of $G(A) \ltimes_{L^*} G(A)^*$ is given by equation (1.4.1).

(b) $r$ given by equation (4.4.23) is a symmetric solution of the $S$-equation in $A \ltimes_{L^*,0} A^*$. Moreover, $r$ is nondegenerate and the induced 2-cocycle $B$ of $A \ltimes_{L^*,0} A^*$ is given by equation (1.1.1).

Theorem A10 Let $(A, A^*, \Delta, \beta)$ be a pre-Lie bialgebra. Then $(\mathcal{G}(A), \mathcal{G}(A^*))$ is a Lie bialgebra if and only if

\[
\langle R^*(x)a^*, R_0^*(b^*)y \rangle + \langle R^*(x)b^*, R_0^*(a^*)y \rangle = \langle R^*(y)b^*, R_0^*(a^*)x \rangle + \langle R^*(y)a^*, R_0^*(b^*)x \rangle,
\]

for any $x, y \in A^*, a^*, b^* \in A^*$.

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