Anisotropic Inflation with Non-Abelian Gauge Kinetic Function

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Abstract

We study an anisotropic inflation model with a gauge kinetic function for a non-abelian gauge field. We find that, in contrast to abelian models, the anisotropy can be either a prolate or an oblate type, which could lead to a different prediction from abelian models for the statistical anisotropy in the power spectrum of cosmological fluctuations. During a reheating phase, we find chaotic behaviour of the non-abelian gauge field which is caused by the nonlinear self-coupling of the gauge field. We compute a Lyapunov exponent of the chaos which turns out to be uncorrelated with the anisotropy.
1 Introduction

In an inflationary scenario, quantum vacuum fluctuations during inflation accounts for the origin of the large scale structure of the universe. The nature of such primordial fluctuations is understood from the approximate symmetry in quasi-deSitter inflation. First of all, in order to have the inflation, we need an approximate translation invariance for an inflaton field, which forbids strong nonlinearity in the action. Hence, we have Gaussian statistics of fluctuations. Moreover, the approximate deSitter symmetry makes the power spectrum of fluctuations scale invariant and statistically isotropic. Note that the scale invariance originates from the temporal part of the deSitter symmetry and the statistical isotropy comes from the spatial part of that. Since these predictions are based on the symmetry, they are robust in the standard single field inflationary scenario.

However, the symmetry in quasi-deSitter inflation is not accurate from the point of view of precision cosmology. In fact, if the translation symmetry is exact, inflation never ends. Since the violation of the translational symmetry is characterized by the slow roll parameters, it is natural that the non-Gaussianity in a single inflaton model is of the order of the slow roll parameters. Apparently, the violation of deSitter symmetry is also characterized by slow roll parameters of the order of a few percent. Due to the violation of temporal part of deSitter symmetry, a deviation from the scale invariant spectrum can be expected to be of the order of the slow roll parameters. Actually, this deviation has been observationally confirmed. In this line of thought, it is legitimate to suspect that the spatial part of deSitter symmetry also breaks down slightly. The violation of the spatial part of deSitter symmetry would lead to an anisotropy in the cosmic expansion, namely, anisotropic inflation. As a consequence, quantum fluctuations generated during the anisotropic inflation must have the statistical anisotropy. Therefore, it is quite natural to expect the statistical anisotropy of the order of the slow roll parameter.

Historically, there have been many attempts to construct anisotropic inflationary models. However, it has been shown that these models suffer from the instability, or a fine tuning problem, or a naturalness problem. Recently, a successful anisotropic inflationary model has been proposed. More precisely, it turned out that the presence of a non-trivial gauge kinetic function in supergravity theory can accommodate an anisotropic inflation. It is well known that the supergravity is characterized by a superpotential, a Kahler potential, and a gauge kinetic function. These functions should be constrained by comparing predictions of inflation with cosmological observations. For example, the tilt of the power spectrum has given interesting information for the superpotential and the Kahler potential. The information provides a hint to the fundamental theory. Amazingly, so far, the gauge kinetic function in supergravity has been neglected in making predictions of inflation. The reason is partially due to the cosmic no-hair theorem which states that the anisotropy, curvature, and any matter will vanish once the inflation commences. It has been proved that this is merely a prejudice. The reason is simply that we do
not have a cosmological constant because of the violation of the translation symmetry for
the inflaton. In spite of the absence of the cosmic no-hair theorem, since the anisotropic
inflation is an attractor, the predictability of the model still remains [15–18]. Indeed, the
imprints of the anisotropic expansion could be seen in the CMB [19].

From the particle physics point of view, it is important to explore the role of the gauge ki-
netic function in inflation. In a previous work, we have considered an abelian gauge field [12].
However, in reality, we have non-abelian gauge fields in particle physics models. Hence,
the main purpose of this paper is to investigate a cosmological role of non-abelian gauge
fields in an inflationary scenario. There are two important differences between abelian and
non-abelian gauge fields, that is, the non-abelian gauge fields have multi-gauge-components
and nonlinear self-couplings. Thus, in this paper, we focus on consequences stemming from
these two features.

Here, we should note other works on the statistical anisotropy. The statistical anisotropy
generated by vector fields is first investigated using δN formalism in [20] where the possi-
bility that the anisotropy appears strongly only in the non-gaussianity is pointed out. The
model has been further extended in various ways [21–25]. In particular, the formalism has
been generalized to non-abelian gauge models [26–28]. As a different approach, there are
attempts to see the remnant of the universe before inflation [29–35]. This could be possible
if the duration of inflation is sufficiently short. In the non-inflationary scenario, there is
another mechanism for producing the statistical anisotropy [35].

The organization of the paper is as follows. In section II, we introduce inflationary
models inspired by supergravity where the Yang-Mills field couples with an inflaton. In
section III, we first solve basic equations numerically and obtain solutions which exhibit
anisotropic expansion and chaos during reheating. Next, we present analytical formula
for the degree of the anisotropy of the cosmic expansion during inflation. We also discuss
observational implication of our finding. In section IV, we calculate a Lyapunov exponent of
the chaos during reheating and find no correlation between the anisotropy and the Lyapunov
exponent. The final section is devoted to conclusion. In the appendix A, we explain how
to reduce the degree of freedom of the non-abelian gauge fields using the symmetry in the
system.

2 Inflation model in supergravity

In this section, we present an inflationary model based on supergravity where we have
a non-trivial gauge kinetic function for a gauge field. Although the gauge group could
be general, we choose SU(2) for concreteness. Using Pauli matrices σ^a, we can define
generators of SU(2) by T^a = σ^a/2 (a = 1, 2, 3) satisfying

\[ [T^a, T^b] = i\epsilon^{abc}T^c, \quad \text{tr}(T^aT^b) = \frac{1}{2}\delta^{ab}, \]

(2.1)
where $\epsilon^{abc}$ is a Levi-Civita symbol and $\text{tr}$ denotes the trace of the matrix representation. Here, $\delta^{ab}$ is a usual Kronecker delta. The $SU(2)$ gauge field is defined as $A = A_\mu dx^\mu = A_a dx^a$. We note that the gauge field $A = A_a T^a$ has multi-gauge-component $A^a$.

The action for the gravitational field, the inflaton $\phi$, and the gauge field reads

$$S = \int d^4x \sqrt{-g} \left[ \frac{1}{2\kappa^2} R - \frac{1}{2} (\partial \phi)^2 - V(\phi) - \frac{1}{2} f^2(\phi) \text{tr}(F_{\mu \nu}^a F_{\mu \nu}^a) \right],$$

where $R$ is the scalar curvature, $g$ represents a determinant of the spacetime metric, $V(\phi)$ is a potential for the inflaton and the field strength $F_{\mu \nu}^a$ of the $SU(2)$-gauge field is defined as $F_{\mu \nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + ig_Y A_\mu^a A_\nu^a$. Here, $g_Y$ is a Yang-Mills coupling constant. The above action is invariant under the local $SU(2)$ gauge transformation,

$$A_\mu \rightarrow \gamma^{-1} A_\mu \gamma - \frac{i}{g_Y} \gamma^{-1} \partial_\mu \gamma,$$

where $\gamma \in SU(2)$. The gauge kinetic function $f(\phi)$ will be specified later. Equations of motion derived from the action (2.2) are given by

$$\frac{1}{\kappa^2} G_{\mu \nu} = \nabla_\mu \phi \nabla_\nu \phi - \frac{1}{2} g_{\mu \nu} (\nabla \phi)^2 - g_{\mu \nu} V(\phi) + 2 f^2(\phi) \text{tr}(F_{\mu \nu}^a F_{\mu \nu}^a) - \frac{1}{4} g_{\mu \nu} F^2$$

$$\nabla^2 \phi - V'(\phi) - f(\phi) f'(\phi) \text{tr}(F^2) = 0,$$

$$D_\nu [f^2(\phi) F_{\mu \nu}^a] = 0,$$

where $G_{\mu \nu}$ is the Einstein tensor, $\nabla_\mu$ represents a covariant derivative with respect to the metric $g_{\mu \nu}$ and we have defined the derivative $' \equiv d/d\phi$ and the gauge covariant derivative $D_\mu = \nabla_\mu + ig_Y [A_\mu, *]$. 

Now, let us consider a cosmological background spacetime. For simplicity, we consider the axially symmetric Bianchi type-I metric

$$ds^2 = -dt^2 + e^{2\alpha(t)} [e^{-4\sigma(t)} dx^2 + e^{2\sigma(t)} (dy^2 + dz^2)],$$

where $\alpha$ describes the average expansion and $\sigma$ characterizes the anisotropy of the expansion. The symmetry in this spacetime is characterized by Killing vectors $\partial_x$, $\partial_y$, $\partial_z$ and $\xi_\phi \equiv -z \partial_y + y \partial_z$. In particular, $\xi_\phi$ generates the rotational symmetry in $(y, z)$-plane. Imposing the symmetry on the inflaton $\phi$ and the gauge field $A$, we can reduce variables into the following form

$$\phi(x^\mu) = \phi(t), \quad A(x^\mu) = v_1(t) T^1 dx + v_2(t) T^2 dy + T^3 dz.$$

The gauge field $A$ is parametrized by two functions, $v_1(t)$ and $v_2(t)$. In the appendix A, we explain how to achieve the above form for the gauge field by using the symmetry. Substituting Eqs. (2.7) and (2.8) into Eqs. (2.4-2.6), we obtain basic equations for the cosmological
background spacetime. From the time-time component of Einstein equations, we obtain a constraint equation
\[ \frac{3}{\kappa^2}(-\dot{\alpha}^2 + \dot{\sigma}^2) + \frac{1}{2} \dot{\phi}^2 + V(\phi) \]
\[ + \frac{1}{6} f^2(e^{-2\alpha + 4\sigma} \dot{v}_1^2 + 2e^{-2\alpha - 2\sigma} \dot{v}_2^2 + 2g_Y^2 e^{-4\alpha + 2\sigma} v_1^2 v_2^2 + g_Y^2 e^{-4\alpha - 4\sigma} v_1^4 v_2^4) = 0, \tag{2.9} \]
where we defined a derivative \( \cdot \equiv d/dt \) with respect to the cosmic time. We also have the evolution equations
\[ \frac{2}{\kappa^2} \ddot{\alpha} + 3 \dot{\alpha} \ddot{\sigma} - \frac{\kappa^2}{3} f^2(e^{-2\alpha + 4\sigma} \dot{v}_1^2 - e^{-2\alpha - 2\sigma} \dot{v}_2^2 - g_Y^2 e^{-4\alpha + 2\sigma} v_1^2 v_2^2 + g_Y^2 e^{-4\alpha - 4\sigma} v_1^4 v_2^4) = 0. \tag{2.10} \]
The equation for the inflaton yields
\[ \ddot{\phi} + 3 \dot{\phi} \dot{\phi} + V' - f' f^2(e^{-2\alpha + 4\sigma} \dot{v}_1^2 + 2e^{-2\alpha - 2\sigma} \dot{v}_2^2 - 2g_Y^2 e^{-4\alpha + 2\sigma} v_1^2 v_2^2 - g_Y^2 e^{-4\alpha - 4\sigma} v_1^4 v_2^4) = 0 \tag{2.11} \]
From Yang-Mills equations, we have
\[ \ddot{v}_1 + 2 \frac{f'}{f} \dot{\phi} \dot{v}_1 + (\dot{\alpha} + 4 \dot{\sigma}) v_1 + 2g_Y^2 e^{-2\alpha - 2\sigma} v_1 v_2^2 = 0 \tag{2.13} \]
and
\[ \ddot{v}_2 + 2 \frac{f'}{f} \dot{\phi} \dot{v}_2 + (\dot{\alpha} - 2 \dot{\sigma}) v_2 + g_Y^2 e^{-2\alpha + 4\sigma} v_1^2 v_2 + g_Y^2 e^{-2\alpha - 2\sigma} v_3 = 0. \tag{2.14} \]
Note that, when we put \( v_2 = 0 \), the above equations reduce to those in abelian cases.

We need to specify the inflaton potential \( V(\phi) \) and the coupling function \( f(\phi) \) in order to solve the above equations. From the previous analysis in [12], we know that the condition
\[ \frac{f' V'}{f V} > 2\kappa^2, \tag{2.15} \]
is necessary for anisotropic inflation to commence. When we consider a simple chaotic inflation
\[ V(\phi) = \frac{1}{2} m^2 \phi^2, \tag{2.16} \]
the simplest choice is [36]
\[ f(\phi) = e^{c\phi^2}/2. \tag{2.17} \]
Then, the condition (2.13) for anisotropic inflation yields \( c > 1 \). It should be stressed that the anisotropic inflation occurs for a quite broad class of potential and gauge kinetic functions as long as the condition (2.15) is satisfied [37][40].

In the next section, we solve Eqs. (2.9-2.14) and study the anisotropic inflation caused by the gauge kinetic function for the \( SU(2) \) Yang-Mills field.
3 Anisotropic inflation

Because of the non-linearity of basic equations of motion, it is difficult to obtain exact solutions. Hence, we first solve equations of motion numerically and find features of anisotropic inflation. From the numerical analysis, it turns out that the nonlinearity of gauge fields can be neglected during anisotropic inflation. Thus, we can make an analytic treatment of basic equations of motion under the slow roll approximation.

3.1 Numerical analysis

Let us solve the Eqs.(2.10-2.14) numerically. In our numerical calculations, we set parameters as

$$\kappa = 1, \quad c = 2, \quad g_Y = 0.01, \quad m = 10^{-5},$$

and initial conditions as

$$\phi = 12, \quad \dot{\phi} = 0, \quad v_1 = 0, \quad \dot{v}_1 = 2.47 \times 10^{-75}, \quad v_2 = 0, \quad \alpha = \sigma = \dot{\sigma} = 0.$$  (3.2)

The initial value for $\dot{\alpha}$ is determined by the constraint equation (2.9). We use these parameters for all numerical calculations in this paper since the qualitative result does not change even if we change these parameters. However, the initial condition for $\dot{v}_2$, which is not included in the above set (3.2), changes the qualitative behaviour of the inflation. Thus, in our numerical calculations, we vary the value of $\dot{v}_2$ and study features of anisotropic inflation for various $\dot{v}_2$’s.

Figure 1: The phase flow in $\dot{\phi}$-$\phi$ space for $\dot{v}_2/\dot{v}_1 = 0.5$. We can see two phases of inflations which correspond to isotropic and anisotropic inflations.

In Fig.1, we show a trajectory with the initial condition $\dot{v}_2/\dot{v}_1 = 0.5$ in $\dot{\phi}$-$\phi$ plane. Taking look at Fig.1, we see that the behaviour is similar to that in an anisotropic inflation.
in the case of the $U(1)$-gauge field $[12]$. There are two phases of inflations, isotropic and anisotropic inflations. Since we have started with negligible vector fields, the trajectory goes into a conventional isotropic inflation. During this stage, the energy density of the vector fields rapidly increases and the inflation soon becomes anisotropic in the second slow roll stage. There, the increase of the energy density of the vector fields saturates due to the backreaction of the vector field. Therefore, the anisotropic inflation is an attractor solution. After a sufficient e-folding, the inflation ends with reheating.

![Figure 2: Time evolution of anisotropy $\Sigma/H$ is plotted against e-holding number $\alpha$. These curves correspond to $\dot{v}_2/\dot{v}_1 = 0.5, 0.75, 1.33$ and 2.0 from top to bottom. We see that the anisotropy can be either positive or negative depending on the ratio $\dot{v}_2/\dot{v}_1$. We also see the rapid oscillation of the anisotropy during the reheating.](image)

In Fig.2, we show time evolution of anisotropy, $\Sigma/H = \dot{\sigma}/\dot{\alpha}$, for $\dot{v}_2/\dot{v}_1 = 0.5, 0.75, 1.33$ and 2.0. It is remarkable that the anisotropy can be either positive or negative depending on the initial ratio $\dot{v}_2/\dot{v}_1$. On the other hand, for the anisotropic inflation by a $U(1)$-gauge field, the anisotropy did not depend on the initial condition for the gauge field and it was always positive. In $[18]$, it was shown that the statistical anisotropy generated by anisotropic inflation with a $U(1)$-gauge field has an opposite sign to the one claimed by the analysis of WMAP data in the CMB. The negative anisotropy, if true, may suggests that the anisotropic inflation assisted by non-abelian gauge field generates the statistical anisotropy. We should note that the massive vector fields could also produce the negative anisotropy $[40]$. We note that the initial condition dependence does not mean the loss of predictability because the initial condition dependence comes into only in the degree of the anisotropy and hence it can be absorbed by rescaling the model parameter. Indeed, the consistency relations among observables found in $[18]$ are independent on the initial conditions.

In the reheating stage, we see the rapid oscillation of the anisotropy in Fig.2, which implies a rapid oscillation of the gauge field. In Fig.4, we show the dynamics of the gauge
Figure 3: The dynamics of $v_1(t)$ and $v_2(t)$ with the initial condition $\dot{v}_2/\dot{v}_1 = 0.5$. The dashed and solid curves correspond to $v_1(t)$ and $v_2(t)$. The horizontal axis is the cosmological time $t$ in the unit of $\kappa = 1$. We can see the chaotic behaviour of the gauge field after inflation.

field, $v_1(t)$ and $v_2(t)$ for $\dot{v}_2/\dot{v}_1 = 0.5$. We can see the chaotic behaviour of the gauge field during the reheating. We discuss the Lyapnov exponent which characterize the chaos in section 4.

3.2 Slow roll approximation

In the previous subsection, we found the slow roll inflation with the anisotropy of the expansion of the universe. Now, we solve basic equations (2.9-2.14) using the slow roll approximation.

In the inflationary phase, the scalar field takes the value $\kappa \phi \sim 10$. Then, the gauge kinetic function has a very large value, $f(\phi) \sim e^{100}$. As one can see from the action (2.2), the $g_Y/f(\phi)$ can be regarded as an effective gauge coupling. During the inflation, the effective gauge coupling becomes very small $g_Y/f(\phi) \sim e^{-100}$. Therefore, we can neglect the gauge coupling during inflation. Then, we can integrate Eqs.(2.13) and (2.14) as

$$\dot{v}_1 = f^{-2}(\phi)e^{-\alpha - 4\sigma}p_1, \quad \dot{v}_2 = f^{-2}(\phi)e^{-\alpha + 2\sigma}p_2,$$  

(3.3)

where $p_1$ and $p_2$ are constants of integration. Since the energy of the gauge field should be subdominant during inflation, we can ignore $\sigma$ in Eqs.(2.9-2.10). Thus, using the slow roll conditions, $\dot{\phi}^2 \ll V$ and $\ddot{\phi} \ll V'$, these equation can be written as

$$\dot{\alpha}^2 = \frac{\kappa^2}{6}[m^2 \phi^2 + e^{-\kappa^2 \phi^2 - 4\alpha}(p_1^2 + 2p_2^2)] \quad (3.4)$$

$$3\dot{\phi} - m^2 \phi - e^{\kappa^2 \phi^2 - 4\alpha}(p_1^2 + 2p_2^2) = 0,$$  

(3.5)
When the effect of the vector field is comparable with that of the inflaton in (3.5), namely, when
\[ m^2 \sim c \kappa^2 e^{-c \kappa^2 \phi^2 - 4 \alpha (p_1^2 + 2p_2^2)} \],
we find
\[ e^{-c \kappa^2 \phi^2 - 4 \alpha (p_1^2 + 2p_2^2)}/(m^2 \phi^2) \sim 1/(c \kappa^2 \phi^2) \sim 10^{-2}. \]
Thus, we can neglect the second term in the right hand side of Eq. (3.4). Then, from Eqs. (3.4) and (3.5), we find
\[ \frac{d}{d\alpha} \phi = -\frac{2}{\kappa^2} + \frac{2c}{m^2} e^{-c \kappa^2 \phi^2 - 4 \alpha (p_1^2 + 2p_2^2)}, \]
Integrating the above equation, we obtain
\[ e^{-c \kappa^2 \phi^2 - 4 \alpha (p_1^2 + 2p_2^2)} = \frac{m^2 (c-1)}{c^2 \kappa^2 (p_1^2 + 2p_2^2)} \]
From Eq. (2.11), we obtain
\[ 3 \ddot{\alpha} \dot{\sigma} = \left( \kappa^2 / 3 \right) e^{-c \kappa^2 \phi^2 - 4 \alpha (p_1^2 - p_2^2)}. \]
Therefore, the anisotropy can be evaluated as
\[ \frac{\Sigma}{H} = \frac{\kappa^2}{9 \dot{\alpha}^2} e^{-c \kappa^2 \phi^2 - 4 \alpha (p_1^2 - p_2^2)} = \frac{2(c-1)(p_1^2 - p_2^2)}{3c^2 (p_1^2 + 2p_2^2)} \frac{1}{\kappa^2 \phi^2}. \]
To obtain the last expression, we have used Eqs. (3.4) and (3.7). From (2.3) and (2.11), we have
\[ \ddot{\alpha} = -(\kappa^2 / 2) \dot{\phi}^2 - (\kappa^2 / 3) e^{-c \kappa^2 \phi^2 - 4 \alpha (p_1^2 + 2p_2^2)}. \]
Thus, the slow roll parameter is given by
\[ \epsilon = -\frac{\ddot{\alpha}}{\dot{\alpha}^2} = \frac{2}{c \kappa^2 \phi^2} \]
Therefore, the anisotropy can be written as
\[ \frac{\Sigma}{H} = \frac{(c-1)(p_1^2 - p_2^2)}{3c (p_1^2 + 2p_2^2)} \epsilon. \]
We should notice that the abelian result can be recovered if we put \( p_2 = 0 \). From the expression (3.10), we obtain an inequality as
\[ \frac{\Sigma}{H} \leq \frac{c-1}{3c} \epsilon. \]
It implies that \( \Sigma/H \) is suppressed by the slow roll parameter. The anisotropy can be either positive or negative depending on the ratio \( p_2/p_1 \). In particular, when \( p_1 = p_2 \), we have no anisotropy. Except for this accidental case, we have the anisotropy proportional to the slow roll parameter. In particular, for the cases \( p_2 > p_1 \), we have a negative anisotropy \( \Sigma/H \) which never occurs in abelian models. This analytical result explains the numerical result shown in Fig.4.

The statistical anisotropy induced by an anisotropic inflation can be characterized by the direction dependent power spectrum
\[ P(k) = P_0(k) \left[ 1 + g_s \left( k \cdot n \right)^2 \right], \]

\[ \frac{\Sigma}{H} \]

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where $P_0(k)$ is the isotropic part of the power spectrum, $k$ is a wavenumber vector of fluctuations and $n$ is a specific direction, in our case, this is the $x$-direction. Here, the number $g_*$ is a magnitude of the statistical anisotropy. According to the recent analysis of the CMB data [41], $g_*$ takes a positive value. The prediction of anisotropic inflation based on the abelian gauge field models was inconsistent with this result [15–18]. In our non-abelian models, however, the anisotropy can give rise to a positive $g_*$ since the anisotropy of the expansion can take any signature depending on the ratio $p_2/p_1$. Thus, if the CMB data shows the statistical anisotropy with positive $g_*$, it may imply the anisotropic inflation with a gauge kinetic function for a non-abelian gauge field. Of course, we should keep it in mind that the data analysis may contain systematic errors.

4 Chaos during reheating

In this section, we study the chaos during reheating in Fig.3. The chaos in Yang-Mills cosmology has been studied previously [12–14]. In those analysis, the existence of coherent non-abelian gauge fields is assumed. Here, the initial coherent non-abelian gauge field is provided by the anisotropic inflation.

As we explained at the beginning of subsection 3.2, the gauge coupling is extremely weak during the inflation. Hence, we can neglect the non-linearity of the gauge field. During this phase, the coherent gauge field is produced due to the rapid variation of the gauge kinetic function. While, at the end of the inflation, the gauge kinetic function becomes $f(\phi) \sim 1$ and effective gauge coupling $g_Y/f(\phi)$ becomes of the order of $g_Y$. The chaotic behaviour after the inflation occurs due to this relatively large effective gauge coupling. Therefore, without the time varying gauge kinetic function, this chaotic behavior of the gauge field never happens.

From the numerical analysis, the pattern of oscillation in Fig.3 seems to change depending on the ratio $\dot{v}_2/\dot{v}_1$. The anisotropy $\Sigma/(\epsilon H)$ also depends on the same ratio. Hence, we might have some relation between the anisotropy and the chaos. So, we need to check if the relation between anisotropy and chaotic behaviour exists. We consider linear perturbations of Eqs.(2.9-2.14) by the substitution $\alpha \rightarrow \alpha + \delta\alpha$, $\sigma \rightarrow \sigma + \delta\sigma$, $\phi \rightarrow \phi + \delta\phi$, $v_1 \rightarrow v_1 + \delta v_1$ and $v_2 \rightarrow v_2 + \delta v_2$. Since the chaotic system is sensitive to initial conditions, the perturbation would grow at late time. We define “Lyapunov exponent” $\lambda(t)$ as $\delta v_2 \propto e^{\lambda(t)t}$. The $\lambda(t)$ represents the growing rate of the perturbation. In Fig.4, we depict the $\lambda(t)$ for $\Sigma/(\epsilon H) = 6.20 \times 10^{-2}$, $2.26 \times 10^{-3}$ and $-7.10 \times 10^{-2}$, respectively. We cannot find any correlation between $\lambda(t)$ and $\Sigma/(\epsilon H)$. It indicates that Lyapunov exponent does not depend on the anisotropy in the anisotropic inflation. Although there is a gauge ambiguity

1 Mathematically, the Lyapunov exponent $\lambda$ is defined by $\lambda = \lim_{t \to \infty} \ln(|w(t)|/|w(0)|)/t$ where $w(t) = (\delta\alpha, \cdots, \delta v_2, \delta\alpha, \cdots, \delta v_2)$. Since it is difficult to solve Eqs.(2.9-2.14) for sufficiently long time in our numerical calculations, we define the $\lambda(t)$ in this way.
in defining the Lyapunov exponent, our conclusion itself does not depend on the choice of the gauge.

5 Conclusion

We have studied the anisotropic inflation model inspired by supergravity where the gauge kinetic function for the $SU(2)$ Yang-Mills field is non-trivial. We found that the anisotropy of expansion rate can take either positive or negative value depending on the initial ratio $\dot{v}_2/\dot{v}_1$. Namely, the shape of the comoving volume becomes either prolate or oblate depending on the initial configurations of the gauge field. This new feature can be attributed to the multi-component nature of non-abelian gauge fields. In principle, this could occur even in multi-abelian models. However, it would be difficult to organize the multi-abelian fields so that the same result as the non-abelian models can be obtained. On the other hand, the gauge structure of the non-abelian gauge field self-organizes the configuration. In spite of the above initial configuration dependence, the anisotropic inflation is still an attractor in the sense that the details of initial conditions are irrelevant except for one relevant parameter which controls the anisotropy of the universe. In addition to the anisotropic expansion, we found the chaotic behaviour of the gauge field during reheating. This is due to the nonlinear self-coupling of non-abelian gauge fields. We calculated the Lyapunov exponent of the chaos and found that the Lyapunov exponent does not correlate with the anisotropy of the inflation. This indicates the universality of the chaotic behaviour of Yang-Mills field in an anisotropic inflationary scenario with a gauge kinetic function for a non-abelian gauge field.
It is remarkable that the anisotropy $\Sigma/H$ can be negative in our inflation model. In [18], it was shown that statistical anisotropy generated by anisotropic inflation by $U(1)$-gauge field takes an opposite sign to the one of observationally favored value. While, the negative anisotropy would lead to the observationally favored signature. It may indicate that the anisotropic inflation with a gauge kinetic function for a non-abelian gauge field generates the statistical anisotropy in the present CMB. Probably, it would be too early to conclude something, at least, we should wait for more precise CMB data provided by PLANCK [13]. It is also intriguing to observe that the new relevant parameter $\dot{v}_2/\dot{v}_1$ helps to weaken the constraint on the model parameter $c$ found in [18]. This is because we have two parameters $\dot{v}_2/\dot{v}_1$ and $c$ to control the anisotropy.

In this paper, we have assumed axial symmetry for simplicity. It is a straightforward exercise to consider more general configurations. As a future work, we can consider non-gaussianity in an anisotropic inflation along the previous paper [20]. From this point of view, it is interesting to extend anisotropic inflationary scenario to a non-slow roll type inflationary scenario such as the DBI inflation model [16]. It is also interesting to study implications of chaos in the early universe. For example, gravitational waves may be generated at a reheating phase. Because of the chaotic behaviour of the gauge field, we may be able to find relic of the chaos during the reheating.

Acknowledgements
JS would like to thank Sugumi Kanno and Masa-aki Watanabe for previous collaboration and useful comments. We are grateful to Kazuya Koyama and David Wands for the hospitality during our stay at ICG, the University of Portsmouth, where most of this work has been done. KM is supported by a grant for research abroad by the JSPS (Japan). JS is supported by the Grant-in-Aid for Scientific Research (A) (No.21244033), the Grant-in-Aid for Scientific Research Fund of the Ministry of Education, Science and Culture of Japan No.22540274, the Grant-in-Aid for Scientific Research (A) (No. 22244030), the Grant-in-Aid for Scientific Research on Innovative Area No.21111006, JSPS under the Japan-Russia Research Cooperative Program and the Grant-in-Aid for the Global COE Program “The Next Generation of Physics, Spun from Universality and Emergence”.

A Axi-symmetric gauge fields

A.1 Symmetry constraints

Using symmetry, we can make a reduction of variables [14]. In the present system, there are spacetime isometry and gauge symmetry. Using these symmetry, we will make the variables as simple as possible.

Imposing the translation invariance along $\partial_x$, $\partial_y$ and $\partial_z$, we can put the gauge field as $A = A_t(t)dt + A_i(t)dx^i$. Furthermore, using the local $SU(2)$ gauge freedom, we can fix the
time component of the gauge field as $A_t(t) = 0$. Then, the gauge field can be written as

$$A^a = \beta^a(t) dx + \gamma^a(t) dy + \delta^a(t) dz . \quad (A.1)$$

The residual global gauge transformation is given by

$$\delta_g A^a = i[A, u]^a = \epsilon^{abc} u^b A^c$$

$$= \epsilon^{abc} u^b (\beta^c(t) dx + \gamma^c(t) dy + \delta^c(t) dz)$$

$$= (\vec{u} \times \vec{\beta})^a dx + (\vec{u} \times \vec{\gamma})^a dy + (\vec{u} \times \vec{\delta})^a dz . \quad (A.2)$$

where $u^a$ are constants and we used a vector notation like as $\vec{u} = (u^1, u^2, u^3)$, $\vec{\beta} = (\beta^1, \beta^2, \beta^3)$, etc. Without loss of generality, choosing an appropriate basis in Lie algebra, we can put $\vec{u} = (u^1, 0, 0) = u^1 \vec{e}_1$.

Now, we consider the rotational symmetry. The rotational transformation is generated by $L_\phi dx = 0$, $L_\phi dy = -dz$ and $L_\phi dz = dy$. Under the infinitesimal rotational transformation, the gauge field transforms as

$$L_\phi A^a = -\gamma^a(t) dz + \delta^a(t) dy . \quad (A.3)$$

For the rotational invariance, the above rotational transformation must be absorbed by the gauge transformation $[A.2]$, namely,

$$L_\phi A^a = \delta_g A^a . \quad (A.4)$$

Since the gauge field has to be the same after making a round, $A = \exp(2\pi L_\phi) A$ holds. Thus, using Eq.$(A.4)$ and $[L_\phi, \delta_g] = 0$, we derive a relation

$$A = \exp(2\pi L_\phi) A = \exp(2\pi \delta_g) A . \quad (A.5)$$

Therefore, $\exp(2\pi \delta_g)$ must be an identical transformation. This implies $u^1 = n \in \mathbb{Z}$. Then, substituting Eqs.$(A.2)$ and $(A.3)$ into Eq.$(A.4)$, we obtain

$$n \vec{e}_1 \times \vec{\beta} = 0 , \quad (A.6)$$

$$n \vec{e}_1 \times \vec{\gamma} = \vec{\delta} , \quad (A.7)$$

$$n \vec{e}_1 \times \vec{\delta} = -\vec{\gamma} . \quad (A.8)$$

For $n = 0$, we find

$$\vec{\beta} = (\beta^1(t), \beta^2(t), \beta^3(t)) , \quad \vec{\gamma} = \vec{\delta} = 0 . \quad (A.9)$$

For $n = 1$, from Eq.$(A.6)$, we obtain $\vec{\beta} \parallel \vec{e}_1$. From Eqs.$(A.7)$ and $(A.8)$, we find $\vec{e}_1 \perp \vec{\gamma} \perp \vec{\delta} \perp \vec{e}_1$. Furthermore, taking the absolute value in Eq.$(A.7)$, we have $|\vec{\gamma}| = |\vec{\delta}|$. Thus, the vectors $\vec{\beta}$, $\vec{\gamma}$ and $\vec{\delta}$ can be written as

$$\vec{\beta} = (\beta^1(t), 0, 0) , \quad \vec{\gamma} = (0, \gamma^2(t), \gamma^3(t)) , \quad \vec{\delta} = (0, -\gamma^3(t), \gamma^2(t)) . \quad (A.10)$$

For $n > 1$, Eqs.$(A.7)$ and $(A.8)$ cannot be satisfied unless $\vec{\gamma} = \vec{\delta} = 0$. Thus, we obtain

$$\vec{\beta} = (\beta^1(t), 0, 0) , \quad \vec{\gamma} = \vec{\delta} = 0 . \quad (A.11)$$
A.2 Yang-Mills constraints

In the previous subsection, we classified the gauge field into the three types (A.9), (A.10) and (A.11). Here, we impose the Yang-Mills constraints on these expressions.

First, we consider the case (A.9). Substituting Eq.(A.9) into a time component of Eq.(2.6), we obtain
\[ \dot{\beta}_1 \beta_2 - \dot{\beta}_2 \beta_1 = \dot{\beta}_2 \beta_3 - \dot{\beta}_3 \beta_2 = \dot{\beta}_3 \beta_1 - \dot{\beta}_1 \beta_3 = 0 . \] (A.12)

Therefore, we have \( \vec{\beta} = \beta_1(t) \vec{c} \), where \( \vec{c} \) is a constant vector. Thus, choosing an appropriate basis in Lie algebra, we can put
\[ \vec{\beta} = (\beta_1(t), 0, 0) , \quad \vec{\gamma} = \vec{\delta} = 0 . \] (A.13)

Next, we consider the case (A.10). Then, from a time component of Eq.(2.6), we find
\[ \dot{\gamma}_2 \gamma_3 - \dot{\gamma}_3 \gamma_2 = 0 . \] (A.14)

Thus, we get \( \gamma^2 = c \gamma^3 \), where \( c \) is a constant. Choosing an appropriate basis in the subspace of Lie algebra spanned by \( T^2 \) and \( T^3 \), the vectors \( \vec{\beta}, \vec{\gamma} \) and \( \vec{\delta} \) can be written as
\[ \vec{\beta} = (\beta^1(t), 0, 0) , \quad \vec{\gamma} = (0, \gamma^2(t), 0) , \quad \vec{\delta} = (0, 0, \gamma^2(t)) . \] (A.15)

For the case (A.11), constraint equations in Eq.(2.6) are trivially satisfied and we have the same expression as Eq.(A.13). Therefore, Eq.(A.13) is the most general expression for an axially symmetric gauge field.

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