THE EXOTIC HEAT-TRACE ASYMPTOTICS OF A REGULAR-SINGULAR OPERATOR REVISITED

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Abstract. We discuss the exotic properties of the heat-trace asymptotics for a regular-singular operator with general boundary conditions at the singular end, as observed by Falomir, Muschietti, Pisani and Seeley in [FMPS03] as well as by Kirsten, Loya and Park in [KLP06]. We explain how their results alternatively follow from the general heat kernel construction by Mooers [Moo99], a natural question that has not been addressed yet, as the latter work did not elaborate explicitly on the singular structure of the heat trace expansion beyond the statement of non-polyhomogeneity of the heat kernel.

1. Introduction

In this paper we revisit the discussion of the heat kernel and the unusual properties of the heat trace expansion for a general self-adjoint realization in $L^2(0,1)$ of the regular-singular operator $(\mathbb{R}^+ = (0,\infty))$

$$\Delta_\nu = -\frac{d^2}{dx^2} + \frac{1}{x^2} \left( \nu^2 - \frac{1}{4} \right) : C^\infty_0(\mathbb{R}^+) \to C^\infty_0(\mathbb{R}^+), \quad \nu \geq 0,$$

where $C^\infty_0(\mathbb{R}^+)$ denotes the space of smooth functions with compact support in $\mathbb{R}^+$.

The heat trace expansion of $\Delta_\nu$ for $\nu > 0$ is well-understood in [KLP08] and in [Moo99], and in fact does not exhibit new phenomena, compare in particular the general discussion of Gil, Krainer and Mendoza [GKM10]. The intricate case is rather $\nu = 0$, where the relation between the explicit approaches of [FMPS03], [KLP06] and the methods of the heat kernel construction in [Moo99] is less obvious.

The regular-singular operators $\Delta_\nu$ arise naturally in the spectral geometry of spaces with isolated conical singularities, modelled by a bounded generalized cone $(\mathcal{C}(N) = \mathbb{R}^+ \times N, g = dx^2 \oplus x^2 g_N)$ over a closed Riemannian manifold $(N, g^N)$ of dimension $n$. This includes the example of a higher-dimensional disc $\mathcal{D} \subset \mathbb{R}^{n+1}$, where the euclidean metric

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takes the form \( dx^2 \oplus x^2 g_{S^1} \) with respect to standard polar coordinates. The Laplace-Beltrami operator on \( (\mathcal{C}(N) = \mathbb{R}^+ \times N, g) \) is a symmetric operator in \( L^2(\mathcal{C}(N), g) \) and takes the form
\[
\Delta_{\mathcal{C}(N)} = -\frac{\partial^2}{\partial x^2} - \frac{n}{x} \frac{\partial}{\partial x} + x^{-2} \Delta_N.
\]
Under the unitary transformation (see [BrSe87])
\[
\Phi : L^2(\mathbb{R}^+, L^2(N, g_N), dx) \to L^2(\mathcal{C}(N), g), \omega \mapsto x^{n/2} \omega,
\]
the Laplacian \( \Delta_{\mathcal{C}(N)} \) unitarily transforms to
\[
\Phi \circ \Delta_{\mathcal{C}(N)} \circ \Phi^{-1} = -\frac{\partial^2}{\partial x^2} + \frac{1}{x^2} \left( \Delta_N - \frac{1}{4} \right) =: \Delta_{\mathcal{C}(N)}^\phi.
\]
The spectral decomposition on the cross section \((N, g_N)\) decomposes \( \Delta_{\mathcal{C}(N)}^\phi \) into a direct sum of operators \( \Delta_\nu \) over the \( \nu^2 \)-eigenspaces of \( \Delta_N \).

We are interested here in the operator over the kernel of \( \Delta_N \)
\[
\Delta = -\frac{d^2}{dx^2} - \frac{1}{4x^2} : C^\infty_0(\mathbb{R}^+) \to C^\infty_0(\mathbb{R}^+).
\]
The spectral properties of \( \Delta \) for general self-adjoint boundary conditions have been subject of careful analysis by Falomir, Muschietti, Pisani and Seeley in [FMPS03] as well as by Kirsten, Loya and Park in [KLP06]. In both instances the authors uncovered new unusual phenomena in the heat trace expansion of \( \Delta \), if the boundary conditions at \( x = 0 \) do not define the Friedrichs extension. Their (and our) result reads as follows.

**Theorem 1.1.** The self-adjoint boundary conditions of \( \Delta \) at \( x = 0 \) are parametrized by \( \theta \in [0, \pi) \), where \( \theta = \pi/2 \) corresponds to the Friedrichs extension. We write \( \Delta(\theta) \) for the corresponding self-adjoint realization. With
\[
\kappa_\theta := \gamma - \log 2 + \tan \theta,
\]
where \( \gamma \) denotes the Euler-Mascheroni constant, we find for \( \theta \neq \pi/2 \)
\[
\text{Tr}_1 e^{-t \Delta(\theta)} := \int_0^1 e^{-t \Delta(\theta)}(x, x) \, dx \sim \text{Tr}_1 e^{-t \Delta(\pi/2)} + \sum_{j=0}^{\infty} a_j t^j,
\]
\[
+ \frac{1}{\pi} \text{Im} \left( \int_1^\infty \frac{e^{-ty}}{y} (\log y + i\pi + 2\kappa_\theta)^{-1} dy \right), \quad t \to 0.
\]
Both discussions of [FMPS03] and [KLP06] have been performed independently from the earlier work by Mooers [Moo99], who in particular constructed the heat kernel of \( \Delta \) for a general self-adjoint realization and observed its non-polyhomogeneity in case of a non-Friedrichs extension at \( x = 0 \).
However, [Moo99] did not elaborate further on the particular structure of the non-polyhomogeneous heat kernel and its heat trace behaviour, leading to the natural question of whether her analysis can be reconciled with the explicit results in [FMPS03] and [KLP06]. Hereby, certain inconsistencies in [Moo99] become apparent, see Remark 4.2.

In this note we address this issue and show how the heat trace expansion results in [KLP06] follow straightforwardly from the heat kernel construction of Mooers in [Moo99]. At various points we chose to provide arguments, alternative to [Moo99].

This paper is organized as follows. We first classify the self-adjoint realizations for $\Delta$ in §2, and study the solution to the signaling problem in §3. The solution to the signaling problem is the central ingredient in the construction of the heat kernel for general self-adjoint boundary conditions, which is explained in §4 and is basically a revision of [Moo99]. In §5 we establish well-definition of the various expressions in the general heat kernel formula. In the final §6 we derive the heat trace expansion directly from the heat kernel structure in §4, thus reproving Theorem 1.1 by a different method.

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2. Self-adjoint boundary conditions

Consider the differential operator
\[ \Delta = -\frac{d^2}{dx^2} - \frac{1}{4x^2} : C_0^\infty(\mathbb{R}^+) \to C_0^\infty(\mathbb{R}^+) . \]

acting on \( C_0^\infty(\mathbb{R}^+) \subset L^2(\mathbb{R}^+) \). We always denote by \( C_0^\infty(I) \) the space of smooth functions with compact support in \( I \subset \mathbb{R} \). We define the minimal closed extension \( \Delta_{\text{min}} \) to be the graph closure of \( \Delta \), which is a densely defined and symmetric. The maximal closed extension is \( \Delta_{\text{max}} := (\Delta_{\text{min}})^\ast \).

We then have the following the well-known characterization of the maximal domain \( D(\Delta_{\text{max}}) \), written out in various sources, including [BrSe87], [Moo99], [FMPS03], [KLP06], [LeVe11] and [Ver09].

**Proposition 2.1.** Let \( f \in D(\Delta_{\text{max}}) \). Then there exist constants \( c_+(f), c_-(f) \in \mathbb{C} \) depending only on \( f \), and a continuously differentiable \( \tilde{f} \in D(\Delta_{\text{min}}) \), with \( \tilde{f}(x) = O(x^{3/2} \log(x)) \) and \( \tilde{f}'(x) = O(x^{1/2} \log(x)) \), as \( x \to 0 \), such that

\[
    f(x) = c_+(f)\sqrt{x} + c_-(f)\sqrt{x} \log(x) + \tilde{f}(x).
\]

Moreover, for any \( f, g \in D(\Delta_{\text{max}}) \) the following Green’s identity holds

\[
    \langle \Delta f, g \rangle - \langle f, \Delta g \rangle = c_-(f)c_+(g) - c_+(f)c_-(g).
\]

An extension \( D(\Delta_{\text{min}}) \subseteq D \subseteq D(\Delta_{\text{max}}) \) is self-adjoint, if

\[
    D = \{ f \in D(\Delta_{\text{max}}) \mid \forall g \in D : \langle \Delta f, g \rangle_{L^2} = \langle f, \Delta g \rangle_{L^2} \}.
\]

Consequently we obtain a self-adjoint extension of \( \Delta \) by choosing the following boundary operators on \( f \in D(\Delta_{\text{max}}) \)

\[
    B_\theta(f) := \cos \theta \cdot c_+(f) + \sin \theta \cdot c_-(f), \quad \theta \in [0, \pi).
\]

We define then

\[
    D(\Delta(\theta)) := \{ f \in D(\Delta_{\text{max}}) \mid B_\theta(f) = 0 \}.
\]

The extensions \( \Delta(\theta), \theta \in [0, \pi) \) define self-adjoint realizations of \( \Delta \) by Proposition 2.1, and in fact classify completely all self-adjoint boundary conditions at \( x = 0 \).

**3. The signaling solution**

The fundamental component in the heat kernel construction of Mooers in [Moo99] is the *signaling solution* \( F(h)(\cdot, t) \in D(\Delta_{\text{max}}), t \in \mathbb{R}^+_0 \), with \( \mathbb{R}^+_0 = [0, \infty) \), defined for any given \( h \in L^1_{\text{loc}}(\mathbb{R}^+_0) \cap C^\infty(\mathbb{R}^+) \), as a solution to the so-called signaling problem.
\[(\partial_t + \Delta)F(h)(x, t) = 0, \quad F(h)(\cdot, 0) \equiv 0, \quad c_-(F(h)(\cdot, t)) = h(t), \quad t > 0.\] 

(3.1)

Note that there does not exist \(\theta \in [0, \pi)\), such that 
\[F(h)(\cdot, t) \in \mathcal{D}(\Delta(\theta)) \quad \text{for} \quad t \in \mathbb{R}_0^+,\]

since by uniqueness of solutions to the heat equation, 
\[F(h)(\cdot, 0) \equiv 0 \quad \text{then implies} \quad F(h) \equiv 0.\]

The signaling solution is in fact unique, since for 
\[c_-(F(h)(\cdot, t)) = 0, \quad F(h)(\cdot, t) \in \mathcal{D}(\Delta(\pi/2)) \quad \text{and hence} \quad F(h)(\cdot, 0) \equiv 0 \quad \text{then implies} \quad F(h) \equiv 0.\]

Let the differential expression \(\Delta\) act on \(C_0^\infty(\mathbb{R}^+)\) and consider the heat kernel 
\[E_+(x, \bar{x}, t)\] of the Friedrichs extension of \(\Delta\) in \(L^2(\mathbb{R}^+)\). We put 
\[N'E_+(x, t) = \left. \frac{\sqrt{x\bar{x}}}{2t} \exp\left( -\frac{x^2 + \bar{x}^2}{4t} \right) \right|_{\bar{x}} = 0.\]

By \([\text{Les97}]\) and the asymptotics of the modified Bessel functions, see \([\text{AbSt92}]\)
\[I_{\nu}(z) \sim \frac{z^\nu}{2^\nu \Gamma(\nu + 1)}, \quad \text{as} \quad z \to 0,\]

we have
\[E_+(x, \bar{x}, t) = \frac{\sqrt{x\bar{x}}}{2t} I_0\left( \frac{x\bar{x}}{2t} \right) \exp\left( -\frac{x^2 + \bar{x}^2}{4t} \right),\]
\[N'E_+(x, t) = \frac{\sqrt{x}}{2t} \exp\left( -\frac{x^2}{4t} \right).\] 

(3.2)

**Lemma 3.1.** \([\text{Moo99, Proposition 4.2}]\)

For any \(h \in L^1_{\text{loc}}(\mathbb{R}_0^+) \cap C^\infty(\mathbb{R}^+)\) the signaling solution is given by

\[F(h)(x, t) := -\int_0^t h(t - \tau)N'E_+(x, \tau)d\tau.\]

**Proof.** \([\text{Moo99, Proposition 4.2}]\) establishes the statement by deriving the formula for \(F(h)\) conceptually. Here, for purposes of brevity, we simply show by a direct computation that the solution \(F(h)\) above indeed provides the signaling solution. With \(N'E_+\) solving the heat equation, clearly \((\partial_t + \Delta)F(h)(x, t) = 0\).

By the explicit formulas (3.2) we find for any fixed \(t > 0\), 
\(F(h)(\cdot, t) \in L^2(\mathbb{R}^+)\). Similarly, differentiating \(F(h)(x, t)\) explicitly in the first argument, we find \(\Delta F(h)(\cdot, t) \in L^2(\mathbb{R}^+)\) and consequently, 
\(F(h)(\cdot, t) \in \mathcal{D}(\Delta_{\text{max}})\).

It remains to identify \(c_-(F(h)(\cdot, t))\). We write \(h(t - \tau) = h(t) - \tau \mathcal{H}(t - \tau)\), where \(\mathcal{H} \in C^\infty(\mathbb{R}^+)\) by smoothness of \(h\). Then, substituting \(T = \).
in the integrals below, we find

\[ F(h)(x, t) = - \int_0^t h(t - \tau) N' E_+(x, \tau) d\tau \]

\[ = - \int_0^\infty h \left( t - \frac{x^2}{T} \right) \frac{\sqrt{x}}{2T} \exp \left( -\frac{T}{4} \right) dT \]

\[ = -h(t) \sqrt{x} \int_0^\infty \frac{1}{2T} \exp \left( -\frac{T}{4} \right) dT \]

\[ + (\sqrt{x})^2 \int_0^\infty \mathcal{H} \left( t - \frac{x^2}{T} \right) \frac{1}{2T^2} \exp \left( -\frac{T}{4} \right) dT \]

\[ =: I_1 + I_2. \]

For the first integral and \( x^2 < t \) (note \( t > 0 \) is fixed) we obtain

\[ I_1 = -h(t) \sqrt{x} \int_0^1 \sum_{k=0}^\infty \frac{(-1)^k T^{k-1}}{2k!} dT - h(t) \sqrt{x} \int_1^\infty \frac{1}{2T} \exp \left( -\frac{T}{4} \right) dT \]

\[ = h(t) \sqrt{x} \log(x) + O(\sqrt{x}), \ x \to 0. \]

For the second integral \( I_2 \) we obtain similarly \( I_2 = O(\sqrt{x}) \) as \( x \to 0 \), and hence indeed find \( c_-(F(h)(\cdot, t)) = h(t) \).

It should be noted that the naive expansion of \( F(h) \) does not lead to a closed expression of \( c_+(F(h)) \), which requires rather integral transform arguments and is the fundamental aspect of the next section.

4. Heat kernel for general boundary conditions

We can now discuss the heat kernel for a self-adjoint operator

\[ \mathcal{D}(\Delta(\theta)) = \{ f \in \mathcal{D}(\Delta_{\text{max}}) \mid B_\theta(f) := \cos \theta c_+(f) + \sin \theta c_-(f) = 0 \}, \]

with \( \theta \in [0, \pi) \). By the Duhamel principle, the heat trace expansion of any self-adjoint extension of \( \Delta \) in \( L^2(0, 1) \) with boundary conditions \( B_\theta \) at \( x = 0 \) and separated boundary conditions at \( x = 1 \), differs from the heat trace expansion of \( \Delta(\theta) \) (trace integral taken over \([0, 1]\)) by the classical trace contribution from the regular boundary \( x = 1 \) and \( O(t^\infty), t \to 0, \) cf. for instance Lesch [Les97, Theorem 1.4.11].

The heat kernel \( E_+ \) for the Friedrichs extension \( \Delta(\pi/2) \) is well-known with the explicit expression written out in (3.2). Fix any \( \theta \in [0, \pi] \setminus \{\pi/2\} \). We do not distinguish notationally between the pseudodifferential operators and their Schwartz kernels. Consider \( \phi \in C_0^\infty(\mathbb{R}_0^+) \), vanishing to infinite order as \( x \to 0 \), and put \( u = E_+ \phi \).
We seek to correct \(u\) by \(F(h)\) for an appropriate \(h \in L^1_{\text{loc}}(\mathbb{R}^+_0) \cap C^\infty(\mathbb{R}^+)\), to satisfy the boundary conditions \(B_\theta(u + F(h)) = 0\). Note

\[
\omega := u + F(h) \in \mathcal{D}(\Delta(\theta))
\]

\[
\Leftrightarrow c_+(u) + c_+(F(h)) = -\tan(\theta) c_-(F(h)) \quad (4.1)
\]

\[
\Leftrightarrow N'E_+ \phi + G * t h(t) = -\tan(\theta) h(t),
\]

where \(G\) is the convolution kernel mapping \(h \equiv c_-(F(h))\) to \(c_+(F(h))\). We explify \(G(t)\) using the Laplace transform. For any \(g \in L^1_{\text{loc}}(\mathbb{R}^+_0) \cap C^\infty(\mathbb{R}^+)\), not growing exponentially as \(t \to \infty\), the Laplace transform is defined as follows

\[
(\mathcal{L} g)(\zeta) = \int_{\mathbb{R}^+} g(t) \exp(-\zeta t) dt, \quad \text{Re}(\zeta) > 0. \quad (4.2)
\]

The inverse Laplace transform is given for any any \(\delta > 0\) and analytic \(L(\zeta)\), integrable over \(\text{Re}(\zeta) = \delta\), by

\[
(\mathcal{L}^{-1} L)(t) = \frac{1}{2\pi i} \int_{\delta+i\mathbb{R}} e^{\zeta t} L(\zeta) d\zeta. \quad (4.3)
\]

Lemma 4.1.

\[
\mathcal{L} G(\zeta) = \log \sqrt{\zeta} + \gamma - \log 2. \quad (4.4)
\]

Proof. We compute

\[
(\mathcal{L} F(h))(x, \zeta) = -(\mathcal{L} h)(\zeta) \cdot (\mathcal{L} N'E_+)(x, \zeta)
\]

\[
= -\sqrt{x} K_0(x \sqrt{\zeta})(\mathcal{L} h)(\zeta), \quad \text{Re}(\zeta) > 0
\]

where \(K_0\) is the modified Bessel function of second kind, and in the definition of \(\sqrt{\zeta}\) we fix the branch of logarithm in \(\mathbb{C} \setminus \mathbb{R}^-\). We assume that \(h(t)\) is not of exponential growth as \(t \to \infty\), so that \((\mathcal{L} h)(\zeta)\) is well-defined for \(\text{Re}(\zeta) > 0\). The Bessel function \(K_0(z)\) admits an asymptotic expansion, see [AbSt92]

\[
K_0(z) \sim -\log(z) + (\log 2 - \gamma) + \tilde{K}_0(z), \quad \tilde{K}_0(z) = O(z), \quad z \to 0, \quad (4.5)
\]

where \(\gamma \in \mathbb{R}\) is the Euler constant. Consequently

\[
c_+(\mathcal{L} F(h))(\cdot, \zeta) = -(\mathcal{L} h)(\zeta)(\log 2 - \log \sqrt{\zeta} - \gamma),
\]

\[
c_-(\mathcal{L} F(h))(\cdot, \zeta) = (\mathcal{L} h)(\zeta). \quad (4.6)
\]

Taking the inverse Laplace transform, we obtain

\[
F(h)(x, t) = \sqrt{x} \log(x) \mathcal{L}^{-1}(c_-(\mathcal{L} F(h))) + \sqrt{x} \mathcal{L}^{-1}(c_+(\mathcal{L} F(h)))
\]

\[
+ \sqrt{x} \mathcal{L}^{-1}((\mathcal{L} h)\tilde{K}_0(x \sqrt{\gamma})),
\]
where each $L^{-1}(c_\pm(LF(h)))$ exists and $L^{-1}((Lh)\tilde{K}_0(x\sqrt{\cdot})) = O(x^{3/2})$, uniformly as $x \to 0$. Consequently, indeed

$$L(c_\pm(F(h))) = c_\pm(LF(h)).$$

This yields the following explicit expression for the Laplace transform of $G$

$$(\mathcal{L}G)(\zeta) = \log \sqrt{\zeta} + \gamma - \log 2.$$  \hfill (4.7)

As already noted at the end of §3, the explicit expression for $c_+(F(h)) = G \ast_t h(t)$, obtained in Lemma 4.1 by means of integral transforms, cannot be obtained by naive expansion of $F(h)(x, t)$ in $x$ directly.

**Remark 4.2.** Similar argument may be performed using Fourier transformation instead of the Laplace transform. The Fourier transform has been used in [Moo99, Lemma 4.4], where however the factor $(\gamma - \log 2)$ from (4.7) is incorrectly missing. More precisely, the scaling argument outlined in [Moo99] determines the Fourier and the Laplace transform of $G$ to be given by $\log \sqrt{\zeta}$ only up to an additive constant, which cannot be specified by her method.

Returning back to (4.1) and taking the Laplace transform of its third identity, we obtain

$$h(t) = -L^{-1}((\mathcal{L}G) + \tan(\theta))^{-1} \ast_t N'E_+\phi =: -K_\theta \ast_t N'E_+\phi,$$

$$u + F(h) = (E_+ + N'E_+ \ast_t K_\theta \ast_t N'E_+) \phi \in \mathcal{D}(\Delta(\theta)), \hfill (4.8)$$

where the inverse Laplace transform in the formula for $K_\theta$ is given explicitly as follows. For any $\delta > 0$ and $\kappa_\theta := \gamma - \log 2 + \tan \theta$ we have a priori

$$K_\theta(t) = \frac{1}{2\pi i} \int_{\frac{1}{2} + i\mathbb{R}} \frac{e^{\kappa \zeta}}{(\log \sqrt{\zeta + \kappa_\theta})} d\zeta = \frac{1}{2\pi i} \int_{i\mathbb{R}} \frac{e^{\kappa \zeta}}{(\log \sqrt{\zeta + \kappa_\theta})} d\zeta$$

$$- \lim_{R \to \infty} \frac{1}{2\pi i} \int_0^\delta e^{itR} e^{xt} \left(\log \sqrt{R + i \arctan(x/\delta)/2 + \kappa_\theta}\right)^{-1} dx$$

$$+ \lim_{R \to \infty} \frac{1}{2\pi i} \int_0^\delta e^{-itR} e^{xt} \left(\log \sqrt{R - i \arctan(x/\delta)/2 + \kappa_\theta}\right)^{-1} dx.$$

The latter two integrals behave as $O((\log R)^{-1})$, as $R \to \infty$, and hence vanish in the limit. Consequently

$$K_\theta(t) = \frac{1}{2\pi i} \int_{i\mathbb{R}} e^{\kappa \zeta} \left(\log \sqrt{\zeta + \gamma - \log 2 + \tan(\theta)}\right)^{-1} d\zeta. \hfill (4.9)$$
Well-definement of \(K_\theta *_t N'E_+ \phi\), and regularity of \(h, F(h)\) is not obvious and requires a detailed analysis of \(K_\theta\), which is the content of §5 below. Then, in view of (4.8) we have proved the following

**Theorem 4.3.** The heat kernel \(E_\theta\) of \(\Delta(\theta), \theta \in (0, \pi)\) is given by

\[
E_\theta = E_+ + N'E_+ *_t K_\theta *_t N'E_+.
\]

5. **Analysis of the convolution kernel \(K_\theta\)**

The first key step here is a specific integral representation of \(K_\theta(t)\).

**Proposition 5.1.** Let \(\kappa(\theta) := \gamma - \log 2 + \tan(\theta)\). Then there exists a bounded \(K_\theta^1 \in C^\infty(\mathbb{R}^+_0)\), such that

\[
K_\theta(t) = \frac{(-1)}{\pi} \text{Im} \left\{ \int_1^\infty e^{-ty} \left( \log \sqrt{y} + i\pi/2 + \kappa(\theta) \right)^{-1} dy \right\} + K_\theta^1(t)
\]

\[= 2 \int_1^\infty e^{-ty} \left( (\log y + 2\kappa(\theta))^2 + \pi^2 \right)^{-1} dy + K_\theta^1(t).\]

**Proof.** We compute from (4.9)

\[
K_\theta(t) = \frac{1}{2\pi i} \int e^{\zeta} \left( \log \sqrt{\zeta + \kappa(\theta)} \right)^{-1} d\zeta
\]

\[= \frac{1}{2\pi i} \int_{i[-1,1]} e^{\zeta} \left( \log \sqrt{\zeta + \kappa(\theta)} \right)^{-1} d\zeta
\]

\[+ \frac{1}{2\pi} \int_1^\infty e^{itx} \left( \log \sqrt{x + i\pi/4 + \kappa(\theta)} \right)^{-1} dx
\]

\[+ \frac{1}{2\pi} \int_1^\infty e^{-itx} \left( \log \sqrt{\zeta - i\pi/4 + \kappa(\theta)} \right)^{-1} dx
\]

\[=: \tilde{K}_\theta^1(t) + K_\theta^2(t) + K_\theta^3(t).\]

The first summand \(\tilde{K}_\theta^1 \in C^\infty(\mathbb{R}^+_0)\) is smooth and bounded. For the summand \(K_\theta^2\) we deform the integration contour to \(i[1, \infty)\). Let the contour \(\gamma_R := \{R \exp(i\phi) \mid \phi \in [0, \pi/2]\}\) be oriented counterclockwise.
Figure 1. The integration contour $\gamma_R$.

We use the $O$-notation for the asymptotics as $R \to \infty$. Then, substituting $x = R \exp(i\phi)$, we find for some $C > 0$

$$
\int_{\gamma_R} | e^{itx} \left( \log \sqrt{x} + i\pi/4 + \kappa(\theta) \right)^{-1} | \, dx \\
= R \int_0^{\pi/2} \left| \frac{\exp(-tR \sin \phi + i(\phi + tR \cos \phi))}{\log \sqrt{R} + i(\phi/2 + \pi/4) + \kappa(\theta)} \right| \, d\phi \\
= R \int_0^{\pi/4} \left| \frac{\exp(-tR \sin \phi + i(\phi + tR \cos \phi))}{\log \sqrt{R} + i(\phi/2 + \pi/4) + \kappa(\theta)} \right| \, d\phi \\
\leq \frac{CR}{\log R} \int_0^{\pi/4} \exp(-tR \sin \phi) \cos \phi \, d\phi + O(R^{-\infty}) \\
= \frac{CR}{\log R} \int_0^{\sin \pi/4} \exp(-tRy) \, dy + O(R^{-\infty}) = O\left(\frac{1}{\log R}\right),
$$

where in the fourth line we have used the fact that $\cos \phi$ is bounded from below for $\phi \in [0, \pi/4]$, and in the final line we substituted $y = \sin \phi$. Consequently, we may indeed deform the integration contour of $K_\theta^2$ to $i[1, \infty)$ and find

$$
K_\theta^2(t) = \frac{i}{2\pi} \int_1^\infty e^{-ty} \left( \log \sqrt{y} + i\pi/2 + \kappa(\theta) \right)^{-1} \, dy \\
+ \int_{\gamma_1} e^{itx} \left( \log \sqrt{x} + i\pi/4 + \kappa(\theta) \right)^{-1} \, dx,
$$

where the second summand is clearly smooth and bounded on $\mathbb{R}_0^+$, since $\text{Im}(x) \geq 0$ for $x \in \gamma_1$. We denote the second summand in the expression for $K_\theta^2$ by $\tilde{K}_\theta^2 \in C^\infty(\mathbb{R}_0^+)$. By a similar exercise we deform
the integration contour of $K_3^3$ to $(-i[1, \infty))$ and find

$$K_3^3(t) = \frac{-i}{2\pi} \int_{-i}^{i} e^{-ty} (\log \sqrt{y} - i\pi/2 + \kappa(\theta))^{-1} \, dy$$

$$+ \int_{\gamma_1} e^{-itz} (\log \sqrt{x} - i\pi/4 + \kappa(\theta))^{-1} \, dx,$$

where $\gamma_1 = \{ \tau \mid \tau \in \gamma_1 \}$ is oriented clockwise; again the second summand is clearly smooth and bounded on $\mathbb{R}_0^+$, since $\text{Im}(x) \leq 0$ for $x \in \gamma_1$. We denote the second summand in the expression for $K_3^3$ by $\widetilde{K}_3^3 \in C^\infty(\mathbb{R}_0^+)$. In total we now obtain using $i(z - \overline{z}) = -2\text{Im}(z)$ and setting $K_1^1 \theta := \widetilde{K}_1^3 + \widetilde{K}_2^3 + \widetilde{K}_3^3$

$$K_\theta(t) = \frac{(-1)}{\pi} \text{Im} \left\{ \int_{1}^{\infty} e^{-ty} (\log \sqrt{y} + i\pi/2 + \kappa(\theta))^{-1} \, dy \right\} + K_1^1 \theta(t)$$

$$= 2 \int_{1}^{\infty} e^{-ty} ((\log y + 2\kappa(\theta))^2 + \pi^2)^{-1} \, dy + K_1^1 \theta(t).$$

(5.1)

As a consequence of Proposition 5.1, we find that $K_\theta \in L^1_{\text{loc}}(\mathbb{R}_0^+) \cap C^\infty(\mathbb{R}_0^+)$ is bounded as $t \to \infty$. Integrability at $t = 0$ follows using Fubini theorem. Indeed we find for any $T > 0$, using (5.1)

$$\int_{1}^{\infty} \int_{0}^{T} e^{-ty} (\log y + 2\kappa(\theta))^2 + \pi^2)^{-1} \, dy$$

$$= \int_{1}^{\infty} \left( 1 - e^{-Ty} \right) ((\log y + 2\kappa(\theta))^2 + \pi^2)^{-1} \, dy < \infty.$$

On the other hand, also $N' E_+ \phi \in L^1_{\text{loc}}(\mathbb{R}_0^+) \cap C^\infty(\mathbb{R}_0^+)$, bounded as $t \to \infty$. Indeed, substituting $X = x/\sqrt{t}$ we find for $\text{supp}(\phi) \subset [0, 1)$

$$N' E_+ \phi(t) = \int_{0}^{1} \sqrt{X} \exp \left( -\frac{x^2}{4t} \right) \phi(x) \, dx$$

$$= t^{-1/4} \int_{0}^{1/\sqrt{t}} \sqrt{X} \exp \left( -\frac{X^2}{4} \right) \phi(X\sqrt{t}) \, dX,$$

which shows integrability at $t = 0$. Consequently, the convolution $h = K_\theta *_t N' E_+ \phi$ indeed exists and $h \in C^\infty(\mathbb{R}_0^+)$ is continuous at $t = 0$ and not growing exponentially as $t \to \infty$. In particular, the Laplace transform $\mathcal{L}(h)(\zeta)$ is well-defined for $\text{Re}(\zeta) > 0$. Similarly, for each fixed $x \in \mathbb{R}_0^+$, the signaling solution $F(h)(x, \cdot) \in C^\infty(\mathbb{R}_0^+)$ is continuous at $t = 0$ and its Laplace transform $\mathcal{L} F(h)(\zeta)$ is also well-defined for $\text{Re}(\zeta) > 0$. 
6. Unusual heat trace expansions

In this final section we study the asymptotic expansion of $\text{Tr}_1(E_\theta), \theta \in [0, \pi) \setminus \{\pi/2\}$ and establish Theorem 1.1. Discussion of the kernel $N' E_+ *_t K_\theta *_t N' E_+$ is central and consists of two steps. We begin with an explicit evaluation of asymptotics of $\text{Tr}_1 N' E_+ *_t N' E_+$ as $t \to 0$.

**Proposition 6.1.**

$$\text{Tr}_1 N' E_+ *_t N' E_+ \equiv \int_0^1 (N' E_+ *_t N' E_+)(x, t) \, dx = \frac{1}{2} + O(t^\infty), \; t \to 0.$$  

**Proof.** We first substitute $y = x^2$

$$\int_0^1 (N' E_+ *_t N' E_+)(x, t) \, dx = \int_0^1 \int_0^t N' E_+(x, t-\tau) N' E_+(x, \tau) \, d\tau \, dx$$

$$= \int_0^1 \int_0^t \frac{x}{4\tau(t-\tau)} \exp \left( -x^2 \left( \frac{1}{4(t-\tau)} - \frac{1}{4\tau} \right) \right) \, d\tau \, dx$$

$$= \int_0^1 \int_0^t \frac{1}{8\tau(t-\tau)} \exp \left( -y \left( \frac{1}{4(t-\tau)} - \frac{1}{4\tau} \right) \right) \, d\tau \, dy.$$

By Fubini theorem we may interchange the integration orders and find after integration in $y$

$$\int_0^1 \int_0^t \frac{1}{8\tau(t-\tau)} \exp \left( -y \left( \frac{1}{4(t-\tau)} - \frac{1}{4\tau} \right) \right) \, d\tau \, dy$$

$$= \frac{1}{2\tau} \int_0^t \left( 1 - \exp \left( -\frac{t}{4\tau(t-\tau)} \right) \right) \, d\tau = \frac{1}{2} + O(t^\infty), \; t \to 0,$$

where in the last step we have estimated for $\tau \in [0, t]$

$$\exp \left( -\frac{t}{4\tau(t-\tau)} \right) \leq \exp \left( -\frac{1}{4t} \right).$$

\[ \square \]

We can now discuss the trace of full kernel $N' E_+ *_t K_\theta *_t N' E_+$.

**Theorem 6.2.**

$$\text{Tr}_1 (N' E_+ *_t K_\theta *_t N' E_+) \sim \sum_{j=0}^\infty b_j t^j$$

$$+ \frac{1}{\pi} \text{Im} \left( \int_1^\infty \frac{e^{-ty}}{y} (\log \sqrt{y} + i\pi + 2\kappa(\theta)) \, dy, \; t \to 0. \right)$$
Proof. Using Proposition 5.1 we find

\[
\text{Tr}_1 (N' E_+ *_t K_\theta *_t N' E_+) \\
= 2 \int_0^1 \int_0^t \int_1^\infty \frac{e^{-(t-\tau)y}}{((\log y + 2\kappa(\theta))^2 + \pi^2)} (N' E_+ *_t N' E_+) (x, \tau) \, dy \, d\tau \, dx \\
+ \int_0^1 \int_0^t K_1^1 (t-\tau) (N' E_+ *_t N' E_+) (x, \tau) \, d\tau \, dx =: T_1 + T_2.
\]

We want to apply Fubini theorem and interchange the integration orders. For this we note using Proposition 6.1

\[
\int_1^\infty \int_0^t \int_0^1 \frac{e^{-(t-\tau)y}}{((\log y + 2\kappa(\theta))^2 + \pi^2)} (N' E_+ *_t N' E_+) (x, \tau) \, dx \, d\tau \, dy \\
= \int_1^\infty \int_0^t \frac{e^{-(t-\tau)y}}{((\log y + 2\kappa(\theta))^2 + \pi^2)} \left( \frac{1}{2} + O(\tau^\infty) \right) \, d\tau \, dy \\
\leq \left( \frac{1}{2} + O(t^\infty) \right) \int_1^\infty \frac{e^{-ty} - 1}{y ((\log y + 2\kappa(\theta))^2 + \pi^2)} dy < \infty,
\]
due to \( \log(y)^2 \) behaviour in the denominator of the integrand. Moreover

\[
\int_0^t \int_0^1 K_1^1 (t-\tau) (N' E_+ *_t N' E_+) (x, \tau) \, dx \, d\tau \\
= \int_0^t K_1^1 (t-\tau) \left( \frac{1}{2} + O(\tau^\infty) \right) \, d\tau \\
\leq \left( \frac{1}{2} + O(t^\infty) \right) \int_0^t K_1^1 (t-\tau) \, d\tau < \infty,
\]
due to smoothness of \( K_1^1 \). Consequently we may indeed interchange the integration orders and obtain using Proposition 6.1 and the integral representation of \( K_\theta(t) \) in Proposition 5.1

\[
T_1 = \frac{(-1)}{\pi} \int_1^\infty \int_0^t \text{Im} \left\{ \frac{e^{-(t-\tau)y}}{(\log \sqrt{y} + i\pi/2 + \kappa(\theta))^{-1}} (N' E_+ *_t N' E_+) (x, \tau) \right\} \, dy \, d\tau \\
\times \int_0^1 (N' E_+ *_t N' E_+) (x, \tau) \, dx \, d\tau \, dy \\
= \frac{1}{\pi} \int_1^\infty \text{Im} \left\{ \frac{(e^{-ty} - 1)}{y (\log \sqrt{y} + i\pi + 2\kappa(\theta))} \right\} \, dy + O(t^\infty),
\]
as $t \to 0$. For $T_2$ we obtain
\[
T_2 = \int_0^t K_1^1(t - \tau) \int_0^1 (N'E_+ *_t N'E_+)(x, \tau) \, dx \, d\tau
= \int_0^t K_1^1(t - \tau) \left( \frac{1}{2} + O(\tau^\infty) \right) \, d\tau \sim \sum_{j=0}^\infty c_j t^j, \tag{6.2}
\]
since $K_1^1 \in C^\infty(\mathbb{R}_0^+)$. The expansions (6.1) and (6.2) together yield a full asymptotic expansion of $\text{Tr}_1 (N'E_+ *_t K_\theta *_t N'E_+)$
\[
\text{Tr}_1 (N'E_+ *_t K_\theta *_t N'E_+) \sim \frac{1}{\pi} \lim_{t \to 0} \left( \int_1^\infty e^{-ty} \left( \log \sqrt{y} + i\pi + 2\kappa(\theta) \right) \right) \, dy
+ \sum_{j=0}^\infty b_j t^j, \quad t \to 0.
\]
Together with Theorem 4.3 we obtain the statement of Theorem 1.1.

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