OPTIMAL CONTROL FOR THE THIN-FILM EQUATION:
CONVERGENCE OF A MULTI-PARAMETER APPROACH TO TRACK
STATE CONSTRAINTS AVOIDING DEGENERACIES

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Abstract. We consider an optimal control problem subject to the thin-film equation which is deduced from the Navier–Stokes equation. The PDE constraint lacks well-posedness for general right-hand sides due to possible degeneracies; state constraints are used to circumvent this problematic issue and ensure well-posedness, and the rigorous derivation of necessary optimality conditions for the optimal control problem. A multi-parameter regularization addressing both, the possibly degenerate term in the equation and the state constraint, is considered, and convergence is shown for vanishing regularization parameters by decoupling both effects. The fully regularized optimal control problem allows for practical simulations which are provided, including the control of a dewetting scenario, evidence of the need of the state constraint, and proper scalings of involved regularization and numerical parameters.

1. Introduction

Let $\Omega = (a, b) \subset \mathbb{R}$, $0 < T < \infty$, $g_0 : (0, \infty) \to \mathbb{R}$ be a given function, $u \in L^2(0, T; \Omega)$ and $y_0 \geq C_0 > 0$ be smooth enough. The one-dimensional thin-film equation (weak slip) reads as follows: Find $y : \Omega_T \to \mathbb{R}$ such that

$$y_t = -(f(y)y_{xxx})_x - (g_0(y)y_x)_x + u_x,$$

(1.1)

together with the initial condition $y(0, \cdot) = y_0$ and boundary conditions $y_x = y_{xxx} = 0$ in $a, b$.

The variable $y$ describes the height of a thin fluid film on some (flat) surface. Here and below, we assume $f(y) = \lambda |y|^3$ for some $\lambda > 0$. The potential function $g_0 : \mathbb{R} \to \mathbb{R}$ will be specified in the following. The potential function $g_0$ models forces which are present in the evolution of the film. Roughly speaking, $g_0$ has to be in such a way that the solution of (1.1) for $u \equiv 0$ is strictly positive provided initial data $y_0$ are strictly positive. Relevant potentials $g_0$ are $g_0 \equiv 0$ (cf. [7]), $g_0 \equiv -1$ (cf. [8]), or $g_0(y) = -y^{-\kappa}$ for some $\kappa > 0$ (cf. [4, 9]). Typically, the potential $g_0$ becomes singular for $y \to 0$, resulting in strong forces in the equation (1.1). A survey addressing general issues of the equation (1.1) is given in [9] and the references therein.

In this work, we study the following constrained optimization problem related to (1.1).

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Problem 1.1. Let $\tilde{y} \in L^2(\Omega_T)$ be a given, $\alpha > 0$, and $C_0 > 0$. Find a minimum $(y^*, u^*)$ of

$$J(y, u) := \frac{1}{2} \int_0^T \int_\Omega |y - \tilde{y}|^2 \, dx \, dt + \frac{\alpha}{2} \int_0^T \int_\Omega |u|^2 \, dx \, dt$$

subject to (1.1) and $y \geq C_0$ in $\Omega_T$.

The aim of the problem is to control the evolution height $y$ of a fluid film which is driven by an external control $u_x : \Omega_T \to \mathbb{R}$. The governing equation (1.1) is in divergence form, avoiding evaporation or wetting effects. The first term on the right-hand side of (1.1) models the dynamics of the fluid film coming from the Navier–Stokes equations. The term $g_0$ models intrinsic forces such as gravity, van der Waals forces between molecules, etc. Some further properties of $g_0$ are detailed in Hypothesis 3.3 and Hypothesis 3.8. The last contribution on the right-hand side $u_x$ is also in divergence form and models external forces. It is known that in the absence of external forces and for some potential functions $g_0$, the solution $y$ of (1.1) converges to its spatial mean value in the limit $t \to \infty$ as long as a global solution can be provided; cf. [8]. In the context of the optimization, we want to force the optimal solution not to form a flat profile in time, i.e., not to be a flat profile for big values of $t$, but to approximate the spatial shape $\tilde{y}$.

We refer the reader to [9, 10] for the equation (1.1), and to [22] for the derivation of the equation and corresponding models (e.g., the strong slip case). The fundamental work for the equation with $g_0, u \equiv 0$ is [7]. Since our goal is to show existence and derive optimality conditions for Problem 1.1, we need to recapitulate and modify the proofs given in [7] for (1.1) with $g_0, u \neq 0$. Typically, a solution of the leading equation (1.1) is endowed with an energy equation and an entropy inequality, from which we may deduce non-negativity of solutions. It is due to the presence of the $u$-term, that an entropy inequality is not clear to hold any more; see Figures 1 and 2, where the solution to a given non-trivial right-hand side $u$ is displayed. For a general given target profile $\tilde{y}$, the external control $u$ is not expected to have a sign, and $u$ should force the state $y$ to take almost zero values if $\tilde{y}$ is of this kind. This is the reason why we do not expect that in such a case an entropy inequality holds for equation (1.1).

A possible application of the above optimal control Problem 1.1 is in the fabrication of electronic chips, where thin layers of different material are deposited on a Si wafer. For an efficient electronical circuit, each layer has to constitute a specific profile, which defines where there is material and where no material is allowed. The problem is to find external forces such that the solution of (1.1) is near the desired profile $\tilde{y}$. Typically, the initial condition in this application is constant and the goal is to form the profile by so-called dewetting; see [6] and subsection 7.7 below. This goal can either be accomplished by background engineering knowledge or by solving Problem 1.1.

Equation (1.1) is derived in, e.g., [5, 22]: We consider the fluid to be thin, i.e., $\tau := \text{height/length} \ll 1$. A nondimensional transformation from the classical Navier–Stokes equation which is based on the small ratio $\tau$ and a Taylor expansion of the terms, together with the assumption of a low Reynolds number (cf. [22 p. 361]) leads to an asymptotic expansion in $\tau$. Neglecting higher order terms of $\tau$, and the proper use of Navier-slip boundary conditions then leads to (1.1).
It is important to note that through the transformation process, a conservative force on the right-hand side of the Navier–Stokes equation transforms into a potential function $g_0$ in (1.1). Hence, a control problem for the Navier–Stokes equation where a distributed conservative force is to be found (cf. [1]) transforms “naturally” into an optimal control problem of the thin-film equation, where a potential function is to be found: Instead of searching a $L^2$ control function $u$, one would like to find a potential function like $g_0$ in (1.1) for minimizing the functional $J$. However, we do not know how to accomplish this goal since nothing about the potential is known. Up to our knowledge there is no work in the context of optimal control where a potential as control variable is to be found. The coupling of such a control and state variables is much stronger as in the case where they are only coupled via the right-hand side of a partial differential equation or similar. In particular, it is not clear if this coupling allows to derive continuous solution operators to deduce boundedness of the state variable to prove existence of optimal controls. Also, it seems that there is no literature on deriving necessary optimality conditions for such a situation. Only in cases where a specific algebraic form of the potential is given (i.e., if the potential is polynomial or a sum of given potentials), there is hope that the corresponding optimal control problem is well-posed—and in these cases the space of controls becomes finite-dimensional since only a few real numbers are to be found. The authors are not aware of any literature dealing with optimal control problems, where potential functions are the subject of interest.

The problem in solving Problem 1.1 is that the nonlinear term of the equation (1.1) may degenerate, and therefore the equation could not be well-posed. To avoid this deficiency, we can only take into account those exterior forces $u$, where the corresponding solution $y$ exists. Unfortunately, we cannot give a good characterization to it, and we do not know topological properties of this set. There are a few recent articles dealing with degenerate optimal control problems for different equations than (1.1); see e.g., [13, 14, 15].

There are two possible ways to overcome this problematic issue:

1. Restrict to a rich enough class of external forces $u : \Omega_T \to \mathbb{R}$ such that solvability of (1.1) is ensured and solutions $y$ are strictly positive. From an optimization viewpoint this strategy is convenient since only control constraints appear. Unfortunately, there is no such result for equation (1.1), and the possibility of too severely restricted controls sets in order to ensure well-posedness of (1.1) has to be encountered.

2. Force the solution to be strictly positive by state constraints as indicated in Problem 1.1. In this case, we only aim for strict positivity of a solution $y$ of (1.1), but have no further restriction regarding controls $u$, i.e., solutions $y$ near an almost degenerate target function $\tilde{y}$ are possible and can be reached by the optimization procedure. As a drawback, we have to overcome several mathematical difficulties.

We refer the reader to [14] where the authors compare both strategies for a different equation, and conclude that the second scenario is more suitable in order to cope with possible degeneracies arising in the governing equation since the set of external forces in the first scenario may not be rich enough, and therefore possible target profiles $\tilde{y}$ may not be reached.

We are able to show existence for the optimal control Problem 1.1. With the help of an abstract result for state constrained optimization problems from [3], we are able to derive necessary optimality conditions for Problem 1.1. In order to overcome technical difficulties arising later in the convergence proofs, we want to make the optimal control problem at this
point compatible and need to aim for right-hand sides \( u \) from \( L^2(H^1_0) \). This can be accomplished in two different ways: The first possibility is to consider a different cost functional \( J \) including the \( L^2(H^1) \)-norm of the control \( u \), whereby the issue is solved directly. However, we encounter difficulties later in the convergence analysis for the optimal control problem in gaining uniform bounds: The derivative of the new cost functional would include a second derivative in space of \( u \), which is not known to be uniformly bounded. Instead of this idea, we consider \( (\Delta^{-1}(u_x))_x \) as external control in (1.1) instead of \( u_x \), where \( \Delta^{-1} \) is the inverse Poisson operator with homogeneous Dirichlet boundary conditions. As a result, the driving term in (1.1) takes values in \( L^2(L^2) \) rather than in \( L^2(H^{-1}) \). We emphasize that this is only needed in order to pass to proper limiting functions in section 5. Later in section 6 we consider a finite dimensional version of the optimal control Problem 1.1, where no convergence analysis will be done, hence there is no need to modify the right-hand side of (1.1), and we will use there \( u_x \) as external control.

The optimality conditions (4.13) involve non-regular Lagrange multipliers in the dual space of \( L^2(H^4) \cap H^1(L^2) \), where it is not clear how to handle them in a numerical simulation: Typical strategies for solving such problems use some sort of relaxation of the state constraint \( y \geq C_0 \) such as penalty approximation [11], the Moreau-Yosida approximation [20], or mixed control-state constraints (Lavrentiev regularization) [24]. The problem in our case is that the state constraint is not additional, but essential in order to ensure well-posedness of the equation (1.1).

In order to circumvent the problematic issue of possibly loosing the well-posedness property of the state equation in the context of relaxation methods, our strategy is as follows: First, we regularize the state equation (1.1) by adding \( \varepsilon y_{xxxx} \), which introduces a regularization to the equation and ensures well-posedness for general exterior forces \( u \). We consider the optimal control problem subject to the regularized equation (3.1) and state constraints (see Problem 5.1). Similarly to the original Problem 1.1 we show existence of an optimum, and derive necessary optimality conditions. We show that the sequence of solutions of the optimal control problem is uniformly bounded with respect to \( \varepsilon \) which allows to construct limiting functions and Lagrange multipliers. In order to show the bounds, it is crucial that we modified the external control term in (1.1) to \( (\Delta^{-1}(u_x))_x \), which helps us at this particular point to bound all corresponding Lagrange multipliers in their particular spaces. We are able to show that these derived limiting functions and Lagrange multipliers solve the necessary optimality conditions (4.13) of the original Problem 4.1. Here, the additional term plays an important role.

Finally, we consider the optimal control problem subject to the regularized equation without state constraints, but with a modified cost functional \( J_\gamma \) which additionally contains a penalization term to account for the state constraint with a parameter \( \gamma \geq 0 \); see Problem 6.1. Since the equality constraint is well-posed for every \( \varepsilon > 0 \), we may use a standard numerical approach to solve the corresponding optimality conditions (6.4) in section 7. We can show that the sequence of minimizers of the fully regularized Problem 6.1 converges to functions solving the intermediate optimization Problem 5.1 for \( \gamma \to 0 \). We use here the penalty approach because of its simple implementation and flexibility. However, the drawback is that the condition number of the underlying problem grows for \( \gamma \to 0 \). This leads to ill-posed problems
on the level of numerical linear algebra, which can also be observed in the numerical experiments in section 7. An alternative way would be to use more sophisticated regularization approaches as mentioned above, where some further details are provided in section 6.

We emphasize that it is necessary to study the intermediate optimization Problem 5.1 since it is not possible to simultaneously let both regularization parameters tend to zero. It is important that the parameter $\gamma > 0$ dealing with the regularization of the state constraint is the first which tends to zero: Here, we benefit from the well-posedness of the involved equality constraint for every $\varepsilon > 0$ to construct a solution of the intermediate optimization Problem 5.1. Vice versa, a direct approximation with the penalty method and $\gamma > 0$ only (or any other relaxation method) could lead to an optimal control problem with possibly non-invertable state equation due to the non-feasibility of the iterates—which was the issue why we introduced the state constraint in the first place.

To sum up, our main result is to show the existence of a solution of Problem 1.1 and to construct it by a multi-parameter regularization of the optimal control problem, where the limit of a corresponding subsequence solves the original Problem 1.1.

The paper is organized as follows:

- In section 3, we study a regularization of the state equation (3.1) and derive results for it, with either the regularization parameter $\varepsilon > 0$ or state constraints being present. In the context of this section, we consider rather general potential functions $g_0$ pertaining to improved regularity results for solutions of (4.1). A few results are only shown for $g_0 \equiv 0$ or $g_0 \equiv -1$.
- In section 4, we study a modification (i.e., Problem 4.1) of Problem 1.1 which uses the external control $(\Delta^{-1}u_x)_x$ with a non-regularized equation and the state constraint.
- In section 5, we study an optimization problem (i.e., Problem 5.1) which is connected to the one in the previous section, but with a regularization term scaled by $\varepsilon > 0$ in the equation to make it well-posed. This problem can be understood as an intermediate one between the original Problem 4.1 in section 4 and Problem 6.1 in section 6 which is suitable as starting point for numerical studies. We show solvability and construct a sequence of minimizers which converges to a function (up to a subsequence) which solves the optimality conditions from section 4 for $\varepsilon \to 0$.
- In section 6, we study the penalty approximation with parameter $\gamma > 0$ of Problem 5.1 introduced in the previous section, which allows to get rid of the non-regular Lagrange multiplier associated to the state constraint. We will show that the sequence of solutions converges to a minimum of Problem 5.1 from section 5 for $\gamma \to 0$.
- In section 7, we present computational studies, using a mixed first order finite element method. We detail how to implement the optimal control problem and compare different parameters. In particular, we show that a goal like optimal dewetting mentioned above can be accomplished; see Figure 2, we show that the state constraints are necessary to provide solutions being positive; we give hints concerning how other parameters like $\alpha, \varepsilon$ should be chosen in order to perform reasonable experiments.

We do not include convergence studies for the involved parameters, hence the change of $u_x$ to $(\Delta^{-1}(u_x))_x$ in (1.1) is unnecessary, and this replacement may be considered as a scaling of the control for a fixed spatial discretization parameter $h > 0$. 
Therefore, we do not include this modification into our computational experiments in section 7.

2. Preliminaries

We write $\|\cdot\|$ for the $L^2(\Omega)$ or $L^2(\Omega_T)$-norm, when it is clear if we only integrate in space or both, in space and time. Let $W^{k,p}$ and $H^k := W^{k,2}$ denote standard Sobolev spaces. By

$$W^{k,p}(W^{m,q}) := W^{k,p}(0,T;W^{m,q})$$

we refer the reader to standard Bochner spaces. The space $C$ denotes the space of continuous functions, while $C^{0,\alpha}$ denotes corresponding Hölder spaces.

The dual pairing of $X$ and its dual space $X^*$ is written as $\langle \cdot, \cdot \rangle_{X,X^*}$. For the scalar products in $L^2$ and $L^2(L^2)$, respectively, of $f$ and $g$, we write $(f,g)$ in cases where no confusion arises; otherwise, we add the corresponding space as index to the scalar product.

We use $C$ as a generic nonnegative constant; to indicate dependencies, we write $C(\cdot)$.

3. The regularized state equation

In this section, we will show properties of solutions of a regularization of the equation (1.1). At the end of this section, we discuss aspects of its solvability, which relies on the proven results and is crucially depending on the chosen potential function $g_0$.

**Problem 3.1.** Let $\varepsilon > 0$. Find $y : \Omega_T \to \mathbb{R}$ such that

$$y_t = -([f(y) + \varepsilon]y_{xxx} + (g_0(y)y_x)_x + u_x),$$

(3.1)

together with initial condition $y(t,0) = y_0$ and boundary conditions $y_x = y_{xxx} = 0$ in $a, b$, where $f(y) = \lambda|y|^3$ for a given $\lambda > 0$. We will also use the abbreviation $f_\varepsilon(y) := f(y) + \varepsilon$.

3.1. Regularity and properties of solutions.

**Lemma 3.2.** Let $\varepsilon > 0$, $u \in L^2(\Omega_T)$ and let $y$ be a solution of (3.1). Then, the mass is conserved, i.e.,

$$\int_\Omega y(t,.)\,dx = \int_\Omega y_0\,dx \quad \forall 0 \leq t \leq T.$$

(3.2)

**Proof.** Integrate (1.1) over $\Omega$ and use the divergence theorem to proof (3.2). $\Box$

The following hypothesis gathers minimum requirements concerning the potential function $g_0$ to prove some regularity properties of possible solutions $y$ of (3.1).

**Hypothesis 3.3.** Assume that one of the following hypothesis is true.

(A1) The potential $g_0$ is continuous and uniformly bounded in $L^2(0,T;L^\infty([C_0,\infty)))$.

(A2) The potential $g_0$ is smooth, uniformly bounded in $L^2(0,T;L^2([C_0,\infty)))$, and $g_0 \leq 0$ as well as $g_0'' \geq 0$ in $\Omega_T$.

In both cases, a term like $\|g_0\|_X$ (where $X$ is one of the spaces above) means that the range of $g_0$ in the particular domain is bounded for all possible arguments in $g_0$. 

Lemma 3.4. Let $\varepsilon > 0$, let Hypothesis 3.3 be true and let $y : \Omega_T \to \mathbb{R}$ be a solution of (3.1) with $y \geq C_0$ a.e. Then there exists a constant $C > 0$ such that the following energy inequality holds
\begin{equation}
\|y_t\|^2_{L^\infty=L^2} + (\lambda C_0^3 + \varepsilon)\|y_{x\!xx}\|^2_{L^2} \leq C \left(T, C_0, \|y_0\|_{H^1}, \|u\|_{L^2} \right).
\end{equation}
In particular, $y$ is Hölder continuous in space, i.e., there exists a constant $H_{\text{space}} > 0$ such that
\begin{equation}
|y(t, x_1) - y(t, x_2)| \leq H_{\text{space}}|x_1 - x_2|^{1/2} \quad \forall 0 \leq t \leq T, \; x_1, x_2 \in \Omega.
\end{equation}

Proof. We multiply (3.1) with $-y_{x\!xx}$, integrate over $\Omega$, and arrive for almost all $t \in [0, T]$ at
\begin{equation}
\frac{1}{2} \frac{d}{dt}\|y_x\|^2 + \int_\Omega f_\varepsilon(y) y_{x\!xx}^2 dx = -\int_\Omega g_0(y) y_{x\!xx}^2 dx - \int_\Omega u_x y_{x\!xx} dx =: I + II.
\end{equation}
We estimate now the terms $I$ and $II$, depending on whether $g_0$ satisfies (A1) or (A2).

If (A1) is true, we estimate $I$ as follows
\begin{equation}
I \leq \|g_0\|_{L^\infty=L^\infty} \|y_x\| \|y_{x\!xx}\| \leq \sigma \|y_{x\!xx}\|^2 + C(\sigma) \|y_0\|^2_{L^\infty} \|y_x\|^2,
\end{equation}
where $\sigma > 0$. In the case of (A2), we calculate
\begin{equation}
I = \int_\Omega g_0(y) y_{x\!xx}^2 dx + \int_\Omega g_0'(y) y_{x\!xx}^2 y_x dx = \int_\Omega g_0(y) y_{x\!xx}^2 dx - \frac{1}{3} \int_\Omega g_0''(y) y_x^4 dx \leq 0,
\end{equation}
since by integration by parts, there holds
\begin{equation}
\int_\Omega g_0'(y) y_{x\!xx}^2 y_x dx = -\int_\Omega g_0''(y) y_x^4 dx - 2 \int_\Omega g_0'(y) y_{x\!xx} y_x^2 dx.
\end{equation}
The term $II$ can be estimated by
\begin{equation}
II = \int_\Omega u y_{x\!xx} dx \leq \sigma \|y_{x\!xx}\|^2 + C(\sigma) \|u\|^2.
\end{equation}
Putting things together, using that $f_\varepsilon(y) \geq \lambda C_0^3 + \varepsilon$, and Gronwall’s inequality, we have proven the lemma. The Hölder continuity follows by one-dimensional Sobolev embeddings. \hfill \Box

Lemma 3.5. Let $\varepsilon > 0$, let Hypothesis 3.3 be true, $u \in L^2(\Omega_T)$, and let $y : \Omega_T \to \mathbb{R}$ be a solution of (3.1) with $y \geq C_0$ a.e. Then there exists a constant $H_{\text{time}} \equiv H_{\text{time}}(T, C_0, y_0, u) > 0$ such that
\begin{equation}
|y(t_2, x) - y(t_1, x)| \leq H_{\text{time}} |t_2 - t_1|^{1/2} \quad \forall 0 \leq t_1, t_2 \leq T, \; x \in \Omega.
\end{equation}

Proof. The proof uses arguments similar (for $g_0 \equiv 0$) to those given in [7] Lemma 2.1.

Step 1: Assume the statement is not correct. Then for every $M > 0$ there exist $x_0 \in \Omega$ and $0 \leq t_1, t_2 \leq T$ such that
\begin{equation}
|y(t_2, x_0) - y(t_1, x_0)| > M|t_2 - t_1|^\beta
\end{equation}
for $\beta = \frac{1}{2}$. Without restriction let us assume that $t_1 < t_2$ and $y(t_2) > y(t_1)$. Then (3.5) reads as
\begin{equation}
y(t_2, x_0) - y(t_1, x_0) > M(t_2 - t_1)^\beta.
\end{equation}
In the proof, we will show that $M$ can be uniformly bounded with respect to $x_0, t_1$ and $t_2$, which contradicts (3.6).
We construct an appropriate test function of the equation (3.1). Let

$$
\xi(x) := \xi_0 \left( \frac{x - x_0}{\frac{M^2}{16H^2_{space}}(t_2 - t_1)^{2\beta}} \right),
$$

where $M$ is from (3.6), $H_{space}$ from Lemma 3.4. The function $\xi_0 \in C^\infty_0$ has the properties $\xi_0(x) = \xi_0(-x)$, $\xi_0(x) \equiv 1$ for $0 \leq x < \frac{L}{2}$ for some $L > 0$ $(L$ will be chosen later and will only depend on $H_{space} > 0$ from Lemma 3.4 and on $\Omega$), $\xi_0(x) \equiv 0$ for $x \geq 1$ and $\xi_0'(x) \leq 0$ for $x \geq 0$. In particular, we have

$$
\xi(x) = \begin{cases} 
0, & |x - x_0| \geq \frac{M^2}{16H^2_{space}}(t_2 - t_1)^{2\beta}, \\
1, & |x - x_0| \leq \frac{1}{2L} \frac{M^2}{16H^2_{space}}(t_2 - t_1)^{2\beta}.
\end{cases}
$$

We define the function $\theta_\delta$ by

$$
\theta_\delta(t) := \int_{-\infty}^{t} \theta_\delta'(s) \, ds,
$$

where

$$
\theta_\delta'(t) = \begin{cases} 
\frac{1}{\delta}, & |t - t_2| < \delta, \\
-\frac{1}{\delta}, & |t - t_1| < \delta, \\
0, & \text{else}
\end{cases}
$$

for $0 < \delta < \min\{\frac{1}{2}(t_2 - t_1), t_1, T - t_2\}$ small enough.

We consider the function $\phi(t, x) := \xi(x)\theta_\delta(t)$, multiply (3.1) with $\phi$, integrate over $\Omega_T$ and get

$$
\int_0^T \int_\Omega y \phi_t \, dx \, dt = -\int_0^T \int_\Omega f_\varepsilon(y) y_{xx} \phi_x \, dx \, dt - \int_0^T \int_\Omega g_0(y) y_x \phi_x \, dx \, dt + \int_0^T \int_\Omega u \phi \, dx \, dt. \quad (3.7)
$$

**Step 2:** We derive a lower bound for the left-hand-side of (3.7). By the construction of $\theta_\delta$, its time derivative approximates like a Dirac function evaluated at $t_1$ and $t_2$, respectively. More precisely, we have for $\delta \to 0$

$$
\int_0^T \int_\Omega y(t, x) \xi(x) \theta_\delta'(t) \, dx \, dt \to \int_\Omega \xi(x) [y(t_2, x) - y(t_1, x)] \, dx. \quad (3.8)
$$

We consider points $x$ such that

$$
|x - x_0| \leq \frac{M^2}{16H^2_{space}}(t_2 - t_1)^{2\beta} \quad (3.9)
$$

since outside this ball, the corresponding integral in (3.7) vanishes. For such $x$, there holds by (3.6) and Lemma 3.4

$$
y(t_2, x) - y(t_1, x) = [y(t_2, x) - y(t_2, x_0)] + [y(t_2, x_0) - y(t_1, x_0)] + [y(t_1, x_0) - y(t_1, x)]
\geq -2H_{space}|x - x_0|^{\frac{1}{2}} + M(t_2 - t_1)^{\beta} \geq \frac{M}{2}(t_2 - t_1)^{\beta},
$$
where we also used (3.9). For $L = L(\Omega, H) > 0$ appropriate, we have $\{\xi = 1\} \subset \Omega$. We may estimate the term in (3.8) from below as follows.

$$\int_\Omega \xi(x) [y(t_2, x) - y(t_1, x)] \, dx \geq \frac{M}{2} (t_2 - t_1)^\beta \frac{M^2}{2L \|H\|_{H^2}} (t_2 - t_1)^{3\beta} = CM^3(t_2 - t_1)^{3\beta}. \quad (3.10)$$

**Step 3:** We derive an upper bound for the right-hand side of (3.7). The first term can be estimated as follows

$$\int_0^T \int_\Omega f_\varepsilon(y) y_{xxx} \phi_x \, dx \, dt$$

$$\leq \|f_\varepsilon(y)\|_{L^\infty(\Omega_T)} \|y_{xxx}\|_{L^2(L^2)} \left( \int_0^T \int_\Omega [\xi'(x)]^2 [\theta_\delta(t)]^2 \, dx \, dt \right)^{\frac{1}{2}}$$

$$\leq \|f_\varepsilon(y)\|_{L^\infty(\Omega_T)} \|y_{xxx}\|_{L^2(L^2)} \left( \int_\Omega [\xi'(x)]^2 \, dx \right)^{\frac{1}{2}} \left( \int_0^T [\theta_\delta(t)]^2 \, dt \right)^{\frac{1}{2}}$$

$$\leq C(H_{\text{space}}) \frac{1}{16 \|H\|_{H^2}} (t_2 - t_1)^{2\beta} \|\xi_0\|_{L^\infty(\Omega)} \frac{M}{4 \|H\|_{H^2}} (t_2 - t_1)^{2\beta} (t_2 - t_1 + 2\delta)^{\frac{1}{2}},$$

where we used that the first two norms are uniformly bounded via Lemma 3.4 by $C(H_{\text{space}})$. The factor $\frac{M}{4 \|H\|_{H^2}} (t_2 - t_1)^{2\beta}$ is the integral of 1 over $\text{supp} \xi$, while $(t_2 - t_1 + 2\delta)^{\frac{1}{2}}$ is the Lebesgue measure of the support of $\theta_\delta$, where we use that $\theta_\delta$ is uniformly bounded by 2 (We highlight the affiliation of each term in the last estimate). We emphasize that the constant $C$ does depends on $H_{\text{space}}$ from Lemma 3.4 (i.e., on $T, C_0, y_0,$ and $u$), but it does not depend on $\varepsilon, M$ or $\delta$.

We estimate the remaining terms in (3.7),

$$\int_0^T \int_\Omega g_0(y) y_x \phi_x \, dx \, dt \leq \|g_0(y)\|_{L^\infty(C_0, \infty)} \|y_x\| \|\phi_x\|$$

$$\leq C(H_{\text{space}}) \frac{1}{M} (t_2 - t_1)^{-\beta} (t_2 - t_1 + 2\delta)^{\frac{1}{2}},$$

where we used the same calculation as above for $\phi_x$ as well as Lemma 3.4 and we considered (A1) from Hypothesis 3.3. In case of (A2) from Hypothesis 3.3 we estimate as follows.

$$\int_0^T \int_\Omega g_0(y) y_x \phi_x \, dx \, dt \leq \|g_0(y)\|_{L^2(C_0, \infty)} \|y_x\|_{L^\infty(\Omega_T)} \|\phi_x\|$$

$$\leq C(H_{\text{space}}) \frac{1}{M} (t_2 - t_1)^{-\beta} (t_2 - t_1 + 2\delta)^{\frac{1}{2}},$$
where we used Lemma 3.4 and Sobolev embeddings. The last term is easy to estimate,
\[ \int_0^T \int_\Omega u \phi_x \, dx \, dt \leq \|u\|_2 \|\phi_x\| \leq C(H_{\text{space}}) \frac{1}{M} (t_2 - t_1)^{1/2} (t_2 - t_1 + 2 \delta)^{1/2}. \]

**Step 4:** For \( \delta \to 0 \), we get at the end
\[ M^{3/2} (t_2 - t_1)^{3/2} \leq C \frac{1}{M} (t_2 - t_1)^{1/2}, \]
where the constant \( C \) is independent of \( x_0, t_1, t_2 \) and \( M \). This leads to \( M \leq \sqrt{C} \), which contradicts (3.6), and the lemma follows.

\[ \square \]

The following lemma does not hold for each potential function \( g_0 \) which satisfies Hypothesis 3.3 since derivatives of \( g_0 \) come into play. We restrict ourselves to the two main examples.

**Lemma 3.6.** Let \( \varepsilon > 0 \), let \( g_0 \equiv 0 \) or \( g_0 \equiv -1 \). Moreover, let \( u \in L^q(H^1) \), and let \( y : \Omega_T \to \mathbb{R} \) be a solution of (3.1) with \( y \geq C_0 \) a.e. Then, for every \( \sigma > 0 \), there holds
\[ \|y_{xx}\|_{L^\infty(L^2)} + (C_0^3 + \varepsilon) \|y_{xxxx}\|_{L^2(L^2)} \leq \sigma \|y_{xxxx}\|_{L^2(L^2)} C(\sigma)(\|u_x\|_{L^2(L^2)} + 1), \]
where \( C(\sigma) \) denotes a positive constant depending on \( \sigma > 0 \).

**Proof.** We rewrite the main part of the equation (3.1) in non-divergence form,
\[ ([f(y) + \varepsilon]y_{xxx}) = [f(y)]y_{xxx} + [f(y) + \varepsilon]y_{xxxx}, \quad (g_0(y)y_x)_x = g_0(y)y_{xx}, \]
where we already know that \([f(y)]_x = 3y^2y_x\). We multiply (3.1) with \( y_{xxxx} \), integrate over \( \Omega \) and arrive for \( \sigma > 0 \) at
\[ \frac{1}{2} \frac{d}{dt} \|y_{xx}\|^2 + \int_\Omega [f(y) + \varepsilon]y_{xxx}^2 \, dx \leq - \int_\Omega [f(y)]y_{xx}y_{xxxx} \, dx \]
\[ =: I_1 + I_2 + I_3 \]

We calculate for \( \sigma > 0 \)
\[ I_1 \geq (\lambda C_0^3 + \varepsilon) \|y_{xxxx}\|^2, \]
\[ I_2 \leq C \|y_x\|_{L^\infty(\Omega)^2} \|y_{xx}\| \|y_{xxx}\| \|y_{xxxx}\|, \]
\[ I_3 \leq \sigma \|y_{xxxx}\|^2 + C(\sigma) \|g_0(y)\|_{L^\infty(\Omega)} \|y_{xx}\|^2, \]
\[ I_4 \leq C \|y_x\|^{1/2} \|y_{xx}\|^{1/2} \leq C \|y_x\| + C \|y_{xx}\|, \]
\[ I_5 \leq \sigma \|y_{xxxx}\| + C(\sigma) \|y_x\|, \]
where we used that \( \Omega \subset \mathbb{R} \) (for \( I_4 \)) and we used \[ \text{Theorem 5.2(1)} \] (for \( I_5 \)). With the estimates of \( I_4 \) and \( I_5 \), we arrive at
\[ I_2 + I_3 = \sigma C_1 \|y_{xxxx}\|^2 + \sigma \|y_{xxxx}\|^2 \]
\[ \text{(3.13)} \]
\[ + C(\sigma)\|y\|_{L^\infty(\Omega)}^2 (\|y_x\| + \|y\|)^2 + C(\sigma)\|g_0(y)\|_{L^\infty(\Omega)}^2 \|y_{xx}\|_2^2, \]

where \( C_1 \) depends on \( T, C_0, y_0, u \) and comes from Lemma 3.4 but is independent of \( \sigma \). The constant \( C(\sigma) \) is also justified from Lemma 3.4 and depends on \( T, C_0, y_0, u, \) and on \( \sigma \).

We absorb the first two terms (with a leading \( \sigma > 0 \)) in the first row of (3.13) into the lower bound of \( I_1 \). Since the remaining two terms in the last row of (3.13) which are led by \( C(\sigma) \) are integrable in time by (3.3), we deduce (3.11) with Gronwall’s lemma. \[ \square \]

3.2. Existence. For every \( \varepsilon > 0 \), the regularized equation (3.1) has at least one weak solution.

Lemma 3.7. Let \( \varepsilon > 0 \), let Hypothesis 3.3 be true, and let \( u \in L^2(\Omega_T) \). Then (3.1) has at least one solution \( y \in L^2(H^3) \cap H^1(H^{-1}) \).

Proof. This follows from standard parabolic theory since the leading part of the equation is uniformly parabolic. \[ \square \]

At least for \( g_0 \equiv 0 \) or \( g_0 \equiv -1 \), it is possible to show uniqueness of solutions. We note that the subsequent analysis does not require uniqueness of (3.1), hence we do not go into further detail.

In general, there is no solution of (1.1) for \( \varepsilon = 0 \), \( g_0 \), and \( u \) arbitrary. For the next section, we have to restrict ourselves to special cases of \( g_0 \) which provides a solution for at least one right-hand side \( u \) for \( \varepsilon = 0 \) in order to have a non-empty feasible set for the optimization problem.

Hypothesis 3.8. We assume that the potential function \( g_0 \) is in such a way that there exists at least one function \( u \in L^2(H_1^3) \) such that there exists a global solution \( y \) of (3.1) with \( y \geq 2C_0 \) in \( \Omega_T \) for some constant \( C_0 > 0 \).

Remark 3.9. Hypothesis 3.8 is valid for, e.g., \( g_0 \equiv 0 \) (see [7]), or \( g_0 \equiv -1 \) (see [8]). In both cases, Hypothesis 3.8 holds for \( u \equiv 0 \).

Even for \( g_0 \equiv 0 \), equation (1.1) might be too degenerate to have a solution for a general right-hand side \( u \in L^2(H^1) \). There are two ways to construct a solution of (1.1) by a sequence \( (y_\varepsilon) \) solving (3.1) for a sequence \( \varepsilon \to 0 \): Either, we restrict ourselves to more regular right-hand sides \( u \in L^2(H^2) \) which allows uniform estimates as in Lemma 3.6 with respect to \( \varepsilon > 0 \). Another possibility, which we will use in the optimization problem is the following: If all iterates \( y_\varepsilon \) have a pointwise lower bound which is uniformly bounded away from zero with respect to \( \varepsilon > 0 \), then it is also possible to pass to the limit, even without the use of more regular right-hand sides \( u \). The uniform lower bound is obtained, e.g., when the sequence \( (y_\varepsilon) \) ensembles from solutions of an optimization problem with suitable state constraints. The following two lemmas reflect both situations separately.

Lemma 3.10. Let \( g_0 \equiv 0 \) or \( g_0 \equiv -1 \), \( u \in L^2(H^2) \) and let \( y_\varepsilon \) be the solution of (3.1). Then, there exist a function \( y \in H^1(L^2) \cap L^2(H^4) \) and a subsequence (still denoted by \( \varepsilon \)) such that \( y_\varepsilon \to y \) uniformly in \( \Omega_T \). The limit function \( y \) solves (1.1).

Proof. In the case \( u \in L^2(H^2) \), the proof of Lemma 3.4 can be modified such that \( y_{\varepsilon,x} \in L^\infty(L^2) \) without the need of \( y_\varepsilon \geq C_0 \) (we have to perform integration by parts on the term
\(-(u_x, y_{xx}) = (u_{xx}, y_x) \leq \|u_{xx}\|^2 + \|y_x\|^2\) in (3.4), which can then be treated by Gronwall’s lemma. In the case of \(g_0 \equiv -1\), the first term on the right-hand side of (3.4) is just \(-\|y_{xx}\|^2\) after integration by parts., i.e., we have \(y_\varepsilon \in C^{(0, \frac{1}{2})}\) bounded uniformly with respect to \(\varepsilon > 0\). Together with Lemma 3.5, we can deduce that the sequence \((y_\varepsilon)\) is bounded uniformly in \(C^{0, \frac{1}{2}}(C^{0, \frac{1}{2}})\), i.e., \(y_\varepsilon\) is equicontinuous and uniformly bounded and there exist a subsequence and \(y\) such that \(y_\varepsilon \to y\) uniformly.

The fact that \(y\) solves (1.1) follows from [7, Theorem 3.1].

**Lemma 3.11.** Let \(g_0 \equiv 0\) or \(g_0 \equiv -1\), \(u \in L^2(H^1)\) and let \(y_\varepsilon\) be the solution of (3.1) with \(y_\varepsilon \geq C_0\) independent of \(\varepsilon > 0\). Then, there exist a \(y \in H^1(L^2) \cap L^2(H^1)\) and a subsequence (still denoted by \(\varepsilon\)) such that \(y_\varepsilon \to y\) uniformly in \(\Omega_T\). The limiting function \(y\) solves (1.1).

**Proof.** If \(y_\varepsilon \geq C_0\) uniformly in \(\varepsilon > 0\), then it is possible to absorb all the terms to the second term on the left-hand side of (3.4) in the proof of Lemma 3.4, i.e., we get uniform (with respect of \(\varepsilon > 0\)) bounds for \(y_\varepsilon\) in the \(L^2(H^2) \cap H^1(\varepsilon-1)\) norm. We follow the proof of Lemma 3.6 to show that \((y_\varepsilon)\) is uniformly bounded in \(L^2(H^4) \cap H^1(L^2)\). By the uniform bounds, there exists a limiting function \(y \in L^2(H^4) \cap H^1(L^2)\) such that \(y_\varepsilon \to y\) weakly in \(L^2(H^4) \cap H^1(L^2)\) (up to a subsequence). It remains to show that \(y\) solves (1.1). We can now either use the second part of the proof of Lemma 3.10 to conclude, or we verify it by hand: For the linear terms, this is clear. For the nonlinear terms, we calculate for \(\varphi \in C^\infty(\Omega_T)\) and the subsequence mentioned

\[ (f(y_\varepsilon)y_{\varepsilon,xxx} - f(y)y_{xxx}\varphi_x) = ([f(y_\varepsilon) - f(y)]y_{\varepsilon,xxx}\varphi_x) + (f(y)[y_{\varepsilon,xxx} - y_{xxx}]\varphi_x) \to 0. \]

For the second nonlinear term, we calculate

\[ (g_\varepsilon(y_\varepsilon)y_{\varepsilon,x} - g_\varepsilon(y)y_{xx}\varphi_x) = ([g_\varepsilon(y_\varepsilon) - g_\varepsilon(y)]y_{\varepsilon,x}\varphi_x) + (g_\varepsilon(y)[y_{\varepsilon,x} - y_{xx}]\varphi_x). \]

This concludes the proof.

In this part of the section, we discussed different cases for which solvability of (1.1) and (3.1) may be established.

1. For \(\varepsilon > 0\), \(g_0\) satisfying Hypothesis 3.3 and an arbitrary \(u \in L^2(\Omega_T)\), a solution \(y_\varepsilon\) of (3.1) exists; see Lemma 3.7.
2. For \(\varepsilon = 0\), \(g_0\) satisfying Hypothesis 3.8 \(u \equiv 0\), and state constraints (i.e., \(y \geq C_0\)) being absent, equation (1.1) is solvable; cf. Hypothesis 3.8 and Remark 3.9.
3. For \(\varepsilon > 0\), \(g_0 \equiv 0\), and regular right-hand sides \(u \in L^2(H^4)\), the sequence of solutions \((y_\varepsilon)\) of (3.1) is uniformly bounded, and the obtained limit \(y\) for \(\varepsilon \to 0\) solves (1.1); see Lemma 3.10.
4. For \(\varepsilon > 0\), \(g_0 \equiv 0\) or \(g_0 \equiv -1\), the sequence of solutions \((y_\varepsilon)\) of (3.1) converges to a solution \(y\) of (1.1), if all \(y_\varepsilon\) are uniformly (with respect to \(\varepsilon\)) bounded away from zero, i.e., there exists a constant \(C_0\) (independent of \(\varepsilon > 0\)) such that \(y_\varepsilon \geq C_0\); see Lemma 3.11.

4. Analysis of the Optimization Problem without Regularization

In this section, we want to show solvability for the original optimization Problem 1.1 and derive necessary optimality conditions. This seems to be possible for a general potential \(g_0\).
which satisfies Hypothesis 3.3 and Hypothesis 3.8. The analysis in the following sections relies on Lemmas 3.10 and 3.11 which is only shown for $g_0 \equiv 0$ and $g_0 \equiv -1$ (hence also for $g_0 \equiv c$ for $c < 0$). In order to keep arguments and calculations as easy as possible, we set $g_0 \equiv 0$ from now on, but the results are also valid (at least) for $g_0 \equiv -1$.

In order to use the Lagrange multiplier theorem, we have to ensure a certain regularity for the optimal solution of the optimization problem stated below. Since the control $u$ in (1.1) is only in $L^2(\Omega_T)$ (due to the structure in the cost functional $J$), the desired regularity of the corresponding optimal state may not be reached. This regularity is a crucial property needed in the next section. To overcome this issue, we restrict proper controls in (1.1) to those of the form $(\Delta^{-1}u_x)_x$ instead of $u_x$; see (4.2). The choice of this particular term is not immediate, but crucial for the rest of the paper: First, in order to exclude a trivial optimization problem, we want the mass to be conserved (otherwise an optimal control would lead to local evaporation or wetting effects), hence the modified term needs to be in divergence form. To determine the involved amount of derivatives in the modified term, a deep look into the proof of Theorem 5.4 is needed in order to uniformly bound all emerging terms there.

As we already discussed in the introduction of this paper, an alternative is the use of an additional term $\|u_x\|^2$ in the functional, which also ensures the desired regularity. However, this would lead to second spatial derivatives of $u$ in the optimality condition (4.13g). This equation is later used in order to show uniform bounds of all involved functions, but since we do not know if second derivatives of $u$ are uniformly bounded, this alternative approach does not seem promising.

We now state the modified form of Problem 1.1.

**Problem 4.1.** Let $\alpha > 0$, $\tilde{y} \in L^2(\Omega_T)$. Minimize

$$J(y, u) := \frac{1}{2} \int_0^T \int_{\Omega} |y - \tilde{y}|^2 \, dx \, dt + \frac{\alpha}{2} \int_0^T \int_{\Omega} |u|^2 \, dx \, dt$$

subject to $y \geq C_0$ and

$$y_t = -(f(y)y_{xxx})_x + q_x, \quad y_x = y_{xxx} = 0$$

in $a, b$.

**Theorem 4.2.** Problem 4.1 has at least one solution.

**Proof.**

**Step 1:** By Hypothesis 3.8, there exist at least $q \in L^2(H^1)$ and $u \in L^2(\Omega_T)$ such that all side constraints (i.e., the equation (4.1), (4.2), and $y \equiv y(q(u)) \geq C_0$ in $\Omega_T$) are satisfied. Therefore, we have

$$\inf J(y, u) =: J^* > -\infty,$$

where the infimum is taken over all feasible pairs $(y, u)$.

**Step 2:** By the first step, there exists a sequence $\{ (y_i, q_i) \}$ fulfilling (4.1), (4.2), and $y_i \geq C_0$ with $J(y_i, u_i) \uparrow J^*$. Therefore, $u_i$ is bounded in $L^2(\Omega_T)$ and there exists a $u \in L^2(\Omega_T)$ such that $u_i \rightharpoonup u$ weakly in $L^2(\Omega_T)$ (up to subsequences).
By the weak lower semicontinuity of the functional Step 3: By the weak lower semicontinuity of the functional $J$, $(y, u)$ is a minimum of Problem [4.1]

By the nonlinearity of the leading equation [4.1], it is clear that a minimum need not be unique. In the remainder of this section, we will derive necessary optimality conditions for a minimum obtained by Theorem 4.2. A problem here is that a classical Slater type result (i.e., the feasible set must have nonempty interior) cannot be used, since the Slater type condition of a state constraint $y \geq C_0$ requires pointwise information (i.e., in $C(\Omega_T)$; but this is not available since we need to get it on the set of solutions which has no interior; cf. [3]).

The key step to derive this is the following abstract result about optimal control problems with state constraints, which is obtained in [3].

**Lemma 4.3.** Let $X, V, W$ be Banach spaces, $U$ be a separable Banach space, let $J : X \times U \to \mathbb{R}$, $G : X \times U \to V$, $H : X \to W$ be mappings, and $C \subseteq W$ be a set.

Let $x, \tilde{u} \in X \times U$ be a minimum of the optimal control problem

$$J(x, \tilde{u}) = \min_{(x, u) \in S} J(x, u)$$

with

$$S := \{(x, u) \in X \times U : G(x, u) = 0, H(x) \in C\}$$

and let the following assumptions be true.

(1) $G : X \times U \to V$ is Frechet differentiable at $(\tilde{x}, \tilde{u})$,

(2) $H : X \to W$ is Frechet differentiable at $\tilde{x}$,

(3) $\emptyset \neq C \subseteq W$ is a convex subset with nonempty interior (measured in the topology of $W$),

(4) $G_u^t(\tilde{x}, \tilde{u}) : X \to V$ is surjective.

Then there exist $(p, \mu, \zeta) \in V^* \times W^* \times \mathbb{R}$ such that

$$\zeta \langle J'_x(\tilde{x}, \tilde{u}), x \rangle_{X, X^*} + \langle p, G'_x(\tilde{x}, \tilde{u}) x \rangle_{V, V^*} + \langle \mu, H'(\tilde{x}) x \rangle_{W, W^*} = 0 \quad \forall x \in X, \quad (4.3a)$$

$$\zeta \langle J'_u(\tilde{x}, \tilde{u}), u \rangle_{U, U^*} + \langle p, G'_u(\tilde{x}, \tilde{u}) u \rangle_{V, V^*} = 0 \quad \forall u \in U, \quad (4.3b)$$

$$\zeta \geq 0, \quad (4.3c)$$

$$\langle \mu, w - H(\tilde{x}) \rangle_{W, W^*} \leq 0 \quad \forall w \in C \quad (4.3d)$$

and if $\zeta = 0$ then $\langle \mu, w \rangle_{W, W^*} \neq 0$ for some $w \in C$.

If we additionally assume that there exists $(\bar{x}, \bar{u}) \in X \times U$ such that

$$G'_x(\tilde{x}, \tilde{u}) \bar{x} + G'_u(\tilde{x}, \tilde{u}) (\bar{u} - \tilde{u}) = 0, \quad (4.4a)$$

$$H(\tilde{x}) + H'(\tilde{x}) \bar{x} \in \text{int } C, \quad (4.4b)$$
then we can take $\zeta = 1$.

We now apply this general result to our setup in Problem 4.1. We define the spaces
$$X := X_y \times X_q, \quad X_y := L^2(H^1) \cap H^1(L^2), \quad X_q := L^2(H^1_0),$$
as well as the spaces $U := L^2(L^2)$, $V := L^2(L^2) \times L^2(H^{-1})$, $W := X_y$, and the set $C := \{v \in W : v \geq C_0 \text{ in } \Omega_T\}$. Since $W \subset C(\Omega_T)$ by Sobolev embeddings, the set $C$ is well-defined.

The function $G$ is given by
$$G((y, q), u) := \left( y_t + (f(y)y_{xx})_x - q_x, -q_{xx} - u_x \right),$$
while $H$ is given by $H(y, q) := y$. We omit initial conditions and boundary conditions in $G$, which may be treated by standard methods; see, e.g., [17, Section 2.6].

**Lemma 4.4.**

1. The function $G : X \times U \to V$ is well-defined.
2. The function $H : X \to W$ is well-defined.
3. The set $C$ is convex with nonempty interior (measured in the topology of $W$).

**Proof.**

1. This follows from Lemma 3.6.
2. Clear by definition.
3. Clearly, the set $C$ is convex, since it is the intersection of two convex sets. We note that the set $\hat{C} := \{v \in C(\Omega_T) : v \geq C_0 \text{ in } \Omega_T\}$ has nonempty interior (e.g., $\hat{v} \equiv 2C_0$ is an interior point), i.e., there exist a point $\hat{v} \in \hat{C}$ and $r > 0$ such that $B_r(\hat{v}) \subset \hat{C}$. Without loss of generality, we can assume that $\hat{v} \in W$ due to the density of $W \subset C(\Omega_T)$. Since the embedding mapping $\text{id} : W \to C(\Omega_T)$ is continuous by Sobolev embeddings, the preimage $\text{id}^{-1}(B_r(\hat{v})) \subset C$ is open, hence there exists an open neighborhood of $\text{id}^{-1}(\hat{v})$, which means that $C$ has nonempty interior in the topology of $W$.

\[\square\]

**Remark 4.5.** As of this place, it seems non straight-forward to use $W = L^2(H^1) \cap H^1(L^2)$ and $C = \{v \in W : v \geq C_0 \text{ in } \Omega_T\}$, instead of simply using $W = C(\Omega_T)$ and $C$ accordingly.

This particular choice will be evident in the proof of Theorem 5.4, where we need to bound the Lagrange multipliers $\mu_\varepsilon$ associated to the state constraint $y \geq C_0$ uniformly with respect to $\varepsilon > 0$, i.e., we need to bound some dual pairings $\langle \mu_\varepsilon, \varphi \rangle$ for all $\varphi$ with $\|\varphi\|_W \leq 1$. If we choose $W = C(\Omega_T)$, we would only know $\sup_{(t, x) \in \Omega_T} |\varphi(t, x)| \leq 1$, which is not enough to bound all emerging terms. However, the choice $W = L^2(H^1) \cap H^1(L^2)$ allows to bound all those terms and thus to prove Theorem 5.4.

We now check that the remaining assumptions in Lemma 4.3 are valid. In order to write down (4.3), we have to show that $G'_x(\bar{x}, \bar{u}) : X \to V$ is surjective, which is done in the following.

**Lemma 4.6.**
The function \( G_1 \), which is defined as
\[
G_1((y, q), u) := y_t + (f(y)y_{xxx})_x - q_x,
\]
has the following Frechet derivative
\[
\begin{align*}
\langle G'_{1,y}((\bar{y}, \bar{q}), \bar{u}), \delta y \rangle &= (\delta y)_t + ((f'(\bar{y}), \delta y)y_{xxx})_x + (f(\bar{y})(\delta y)_{xxx})_x \quad \forall \delta y \in X_y, \\
\langle G'_{1,q}((\bar{y}, \bar{q}), \bar{u}), \delta q \rangle &= - (\delta q)_x \quad \forall \delta q \in X_q, \\
\langle G'_{1,u}((\bar{y}, \bar{q}), \bar{u}), \delta u \rangle &= 0 \quad \forall \delta u \in U.
\end{align*}
\]

The function \( G_2 \), which is defined as
\[
G_2((y, q), u) := -q_{xx} - u_x,
\]
has the following Frechet derivative
\[
\begin{align*}
\langle G'_{2,y}((\bar{y}, \bar{q}), \bar{u}), \delta y \rangle &= 0 \quad \forall \delta y \in X_y, \\
\langle G'_{2,q}((\bar{y}, \bar{q}), \bar{u}), \delta q \rangle &= - (\delta q)_{xx} \quad \forall \delta q \in X_q, \\
\langle G'_{2,u}((\bar{y}, \bar{q}), \bar{u}), \delta u \rangle &= - (\delta u)_x \quad \forall \delta u \in U.
\end{align*}
\]

\begin{proof}
The function \( G \) is smooth and the derivation of it is a straightforward calculation. \( \square \)
\end{proof}

Lemma 4.7.

1. For every \( \Phi \in L^2(L^2) \) and every \( w \in L^2(H^1_0) \), there exists a \( v \in L^2(H^1) \cap H^1(L^2) \) such that
\[
\langle G'_{1,y}((\bar{y}, \bar{q}), \bar{u}), v \rangle + \langle G'_{1,q}((\bar{y}, \bar{q}), \bar{u}), w \rangle = \Phi
\]
(4.5)
together with the initial conditions \( v(0, .) = 0 \) and \( w(0, .) = 0 \) as well as the boundary conditions \( v_x = v_{xxx} = 0 \) and \( w = 0 \) in \( a, b \).
2. For every \( \Psi \in L^2(H^{-1}) \) and every \( v \in L^2(H^4) \cap H^1(L^2) \), there exists a \( w \in L^2(H^1_0) \) such that
\[
\langle G'_{2,y}((\bar{y}, \bar{q}), \bar{u}), v \rangle + \langle G'_{2,q}((\bar{y}, \bar{q}), \bar{u}), w \rangle = \Psi
\]
(4.6)
together with the initial conditions \( v(0, .) = 0 \) and \( w(0, .) = 0 \) as well as the boundary conditions \( v_x = v_{xxx} = 0 \) and \( w = 0 \) in \( a, b \).

\begin{proof}
\begin{enumerate}
1. Inserting the derivative of \( G_1 \) with respect to \( y \) by Lemma 4.6, equation (4.5) reads as
\[
v_t + (f(\bar{y})v_{xxx})_x + \text{lower order terms} = \Phi.
\]
(4.7)
For a test function \( \varphi \in X_y \), we write
\[
\langle (f(\bar{y})v_{xxx})_x, \varphi \rangle = - \langle f(\bar{y})v_{xxx}, \varphi_x \rangle = \langle f(\bar{y})v_{xx}, \varphi_x \rangle + \langle f'(\bar{y})\bar{y}_x v_{xx}, \varphi_x \rangle.
\]
(4.8)
Since \( f'(\bar{y}) = 3\bar{y}^{-1}f(\bar{y}) \), on nothing that \( \bar{y} \geq C_0 > 0 \), we can estimate the last term in (4.8) as follows
\[
\langle f'(\bar{y})\bar{y}_x v_{xx}, \varphi_x \rangle \leq \sigma \| f(\bar{y})v_{xx} \|^2 + C(\sigma) \| \bar{y}\bar{y}_x \varphi_x \|^2
\]
with \( \sigma > 0 \). The remaining term in (4.8) is either uniformly \( H^2 \)-coercive (since \( \bar{y} \geq C_0 \)) or is of lower order. Therefore, there exists a solution \( v \in L^2(H^2) \cap H^1(H^{-1}) \) of (4.7).

As in the proof of Lemma 3.6, we can write

\[
(f(\bar{y})v_{xxx})_x = f(\bar{y})v_{xxxx} + f'(\bar{y})\bar{y}_x v_{xxx},
\]

i.e., the leading part of the equation (4.7) is uniformly elliptic since \( \bar{y} \geq C_0 \). Similar to the proof in Lemma 3.6, it is possible to multiply the equation with \( v_{xxx} \) and to absorb the lower order terms into the leading term in (4.9). Therefore, it is possible to show that the solution \( v \) is as regular as claimed.

The choice of \( w \) is arbitrary, since it is of lower order and the related can be put into the right hand side \( \Phi \) in the first place.

(2) By Lemma 4.6, equation (4.6) reads as the Poisson equation, which is surjective on the corresponding spaces.

We will now show that the regular point conditions (4.4a) and (4.4b) from Lemma 4.3 are fulfilled. For this goal, it is important to make use of the surjectivity of the derivative of \( G \).

**Lemma 4.8.** There exists \( (\bar{x}, \bar{u}) \in X \times U \) such that (4.4a) and (4.4b) are fulfilled.

**Proof.**

**Step 1:** First, we note that \( \text{int} C - H(\bar{x}) = \{ f \in C(\Omega_T) : f > C_0 - \bar{y} \} \). Since \( H'(\bar{x})x = y \), we have to choose \( y \in X_y \) such that \( y > C_0 - \bar{y} \) in \( \Omega_T \) to meet (4.4b), which is always possible (e.g., we can choose \( y = 2C_0 \)).

**Step 2:** Now, we take a look at the first component of the equation (4.4a)

\[
G'_x(\bar{x}, \bar{u})x + G'_u(\bar{x}, \bar{u})(u - \bar{u}) = 0,
\]

which can be written as

\[
\langle G'_{1,y}((\bar{y}, \bar{q}), \bar{u}), y \rangle = q_x
\]

due to Lemma 4.6. By Lemma 4.7, the left-hand side of (4.10) is surjective, i.e., there exists a \( q \in X_q \) such that (4.10) holds.

**Step 3:** Finally, we take a look at the second component of (4.4a), which reads as

\[
-q_{xx} = u_x - \bar{u}_x =: \tilde{u}_x.
\]

This equation is the Poisson equation, which is known to be surjective on the corresponding spaces, i.e., there exists \( \tilde{u}_x \in L^2(H^{-1}) \) such that (4.11) holds. Since \( \tilde{u}_x \) is known and we do not have additional constraints on \( u \), there exists a \( u_x \in L^2(H^{-1}) \) such that (4.11) holds. But this implies the existence of \( \bar{u} \in L^2(L^2) = U \) such that (4.11) holds.

To summarize, we have constructed \( (\bar{y}, q, \bar{u}) \in X \times U \) such that both conditions (4.4a) and (4.4b) hold.

**Remark 4.9.** For a leading equation of second order (instead of the fourth order equation, which we have here), the proof of Lemma 4.8 would work in a much more general setting: In
(4.4b), we have to show that there exists a \( y \in X_y \) such that \( \bar{y} + y > C_0 \) and \( y \) is a solution of the linearized equation (4.4a). Since \( u \) can be chosen arbitrarily, (4.4a) reads as

\[
G_y'(\bar{x}, \bar{u})\bar{z} = \Phi,
\]

where \( \Phi \) can have an arbitrary sign (There are no additional constraints on \( u \)). If \( G \) contains an parabolic equation of second order, equation (4.12) would read as an linear parabolic equation of second order. There holds a maximum principle for such equations, i.e., if \( \Phi \) has a certain sign, we can guarantee that \( x \) has also a sign making it easier to show (4.4b), where this information is useful.

**Theorem 4.10.** Let \((y,q,u)\) be a solution of Problem 4.1. Then, there exist \( z \in L^2(L^2) \), \( \eta \in L^2(H^1_0) \), and \( \mu \in (L^2(H^1) \cap H^1(L^2))^* \) such that the following optimality conditions are fulfilled.

\[
y_t = -(\langle f(y) \rangle y_{xxx})_x + q_x, \quad (4.13a)
\]

\[
-q_{xx} = u_x, \quad (4.13b)
\]

\[
y \geq C_0, \quad (4.13c)
\]

\[
0 \geq (w - y, \mu) \quad \forall X_y \ni w \geq C_0, \quad (4.13d)
\]

\[
0 = \langle y - \bar{y}, \varphi \rangle + \langle z, \varphi_t + (f'(y)_y_{xxx} \varphi)_x \rangle + \langle z, (\langle f(y) \rangle \varphi_{xxx})_x \rangle + \langle \varphi, \mu \rangle \quad \forall \varphi \in X_y, \quad (4.13e)
\]

\[
0 = z_x - \eta_{xx}, \quad (4.13f)
\]

\[
0 = \alpha u + \eta_x, \quad (4.13g)
\]

together with initial conditions \( y(0,.) = y_0, \) \( z(T,.) = 0; \) boundary conditions \( y_x = y_{xxx} = z_x = z_{xxx} = 0 \) in \( a,b; \) as well as \( q = \eta = 0 \) in \( a,b. \)

**Proof.** We use Lemma 4.3, the conditions there are fulfilled by Lemma 4.4, Lemma 4.6, Lemma 4.7, and Lemma 4.8. □

## 5. Optimization with Regularization in the Equation

In this section, we consider a modification of Problem 4.1 where the state equation is regularized; the functional remains the same. After having shown solvability and having derived corresponding optimality conditions in Theorem 5.2 and Theorem 5.3 respectively, we will show that solutions of this problem converge to objects which solve (4.13), i.e., we show that solutions of the modified problem converge to those of the original problem in a certain sense.

**Problem 5.1.** Let \( \varepsilon > 0. \) Suppose \( \bar{y} \in L^2(\Omega_T) \). Minimize

\[
J(y, u) := \frac{1}{2} \int_0^T \int_\Omega |y - \bar{y}|^2 \, dx \, dt + \frac{\alpha}{2} \int_0^T \int_\Omega |u|^2 \, dx \, dt
\]

subject to \( y \geq C_0, \) (3.1), and (4.2), together with the initial condition \( y(0) = y_0. \)

**Theorem 5.2.** Problem 5.1 has at least one solution.

**Proof.** The proof uses the same argument as the proof of Theorem 4.2. □
Theorem 5.3. Let \((y,q,u)\) be a minimum of Problem 5.1. Then, there exist Lagrange multipliers \(z \in L^2(L^2), \eta \in L^2(H^1_0)\), and \(\mu \in (L^2(H^4) \cap H^1(L^2))^*\) such that the following equations are fulfilled.

\[
\begin{align*}
y_t &= -((f(y) + \varepsilon y_{xxx})_x + q_x), & (5.1a) \\
-\eta_{xx} &= u_x, & (5.1b) \\
y &\geq C_0 & (5.1c) \\
0 &\geq \langle w - y, \mu \rangle \quad \forall X_y \ni w \geq C_0, & (5.1d) \\
0 &= \langle y - \bar{y}, \varphi \rangle + \langle z, \varphi_t + (f'(y)y_{xxx}\varphi)_x \rangle + \langle z, ([f(y) + \varepsilon ]\varphi_{xxx})_x \rangle + \langle \varphi, \mu \rangle \quad \forall \varphi \in X_y, & (5.1e) \\
0 &= z_x - \eta_{xx}, & (5.1f) \\
0 &= \alpha u + \eta_x & (5.1g)
\end{align*}
\]

together with initial conditions \(y(0,.) = y_0, z(T,.) = 0\); boundary conditions \(y_x = y_{xxx} = z_x = z_{xxx} = 0\) in \(a,b\); as well as \(q = \eta = 0\) in \(a,b\).

Proof. This is a direct consequence of Lemma 4.3. The details are similar to the proof of Theorem 4.10. \qed

Theorem 5.4. Let \(\{(y_\varepsilon, q_\varepsilon, u_\varepsilon)\}\) be a sequence of solutions of Problem 5.1. Then, there exists \((y^*, q^*, u^*) \in (H^1(L^2) \cap L^2(H^4)) \times L^2(H^1_0) \times L^2(L^2)\) such that \((y_\varepsilon, q_\varepsilon, u_\varepsilon) \rightharpoonup (y^*, q^*, u^*)\) weakly in \((H^1(L^2) \cap L^2(H^4)) \times L^2(H^1_0) \times L^2(L^2)\) (up to a subsequence). The limit functions \((y^*, q^*, u^*)\) are a solution of (4.13).

Proof.

Step 1: First we prove that \((u_\varepsilon)\) is uniformly bounded in \(L^2(L^2)\): To do so, we want to find a function \(\bar{u}\) and a corresponding solution \((\bar{y}_\varepsilon, \bar{q}_\varepsilon)\), which is feasible for every \(\varepsilon > 0\) small enough, i.e., which is solving (3.1) and (4.2), together with \(\bar{y}_\varepsilon \geq C_0\). For \(u \equiv 0\) (hence \(q \equiv 0\) and \(\varepsilon = 0\), there exists a solution \(\bar{y}\) of (1.1) (See Hypothesis 3.8), which satisfies \(\bar{y} \geq 2C_0\). Let \(y^{(0)}_\varepsilon\) be the solution of (3.1) for \(q = 0\) (hence \(u = 0\)). Then there exists \(y : \Omega_T \to \mathbb{R}\) such that \(y^{(0)}_\varepsilon \rightharpoonup y\) uniformly for \(\varepsilon \to 0\), cf. Lemma 3.10. Hence there exists an \(\varepsilon_0 > 0\) such that \(y^{(0)}_\varepsilon \geq C_0\) for every \(0 < \varepsilon \leq \varepsilon_0\).

Since \(\bar{y}_\varepsilon\) is uniformly bounded (with respect to \(\varepsilon > 0\)) in \(L^2(H^4) \cap H^1(\Omega_T)\) by a constant depending on the fixed norm of \(u \equiv 0\), we may deduce that the solution \((y_\varepsilon, q_\varepsilon, u_\varepsilon)\) of Problem 5.1 satisfies \(J(y_\varepsilon, u_\varepsilon) \leq J(y^{(0)}_\varepsilon, 0) < \infty\), i.e., by construction of the functional \(J\), the sequence \((u_\varepsilon)\) is bounded uniformly in \(L^2(L^2)\). Hence there exists a \(u^* \in L^2(L^2)\) such that \(u_\varepsilon \rightharpoonup u^*\) weakly in \(L^2(L^2)\).

Step 2: By standard estimates for the Poisson equation, there exists a constant \(C > 0\) such that for the solution \(q_\varepsilon\) of (4.2) holds

\[
\|q_\varepsilon\|_{L^2(H^1_0)} \leq C\|u_\varepsilon\|_{L^2(L^2)},
\]

i.e., there exists an \(q^* \in L^2(H^1_0)\) such that \(q \rightharpoonup q^*\) weakly in \(L^2(H^1_0)\) thanks to the boundedness of \((u_\varepsilon)\) from the last step.

Step 3: By Lemma 3.6, the solution \(y_\varepsilon\) of (3.1) is uniformly bounded (with respect to \(\varepsilon > 0\)) in \(L^2(H^4) \cap H^1(L^2)\), i.e., there exists \(y^* \in L^2(H^4) \cap H^1(L^2)\) such that \(y_\varepsilon \to y^*\) weakly
in $L^2(H^4) \cap H^1(L^2)$. Since all $y_\varepsilon \geq C_0$, we have $y^* \geq C_0$ and $y^*$ solves (4.1) by Lemma 3.11.

**Step 4:** We have shown so far that there exists $(y^*, q^*, u^*)$ in the given spaces and they solve (4.13a), (4.13b) and (4.13c). It remains to show that the Lagrange multipliers $(\mu_\varepsilon, z_\varepsilon, \eta_\varepsilon)$ are uniformly bounded (with respect to $\varepsilon$) and that their limits solve the remaining equations in (4.13).

**Step 5:** Since $(u_\varepsilon)$ is bounded in $L^2(L^2)$, we have $(\eta_\varepsilon)$ bounded in $L^2(H_1^1)$ by (5.1g). By (5.1f), $(z_{\varepsilon}, x)$ is bounded uniformly in $L^2(H^{-1})$. We will now consider (5.1e) and may show that $\mu_\varepsilon$ is uniformly bounded in $(L^2(H^4) \cap H^1(L^2))^*$, i.e., we have to show that

$$\|\mu_\varepsilon\|_{(L^2(H^4) \cap H^1(L^2))^*} = \sup_{\psi \in L^2(H^4) \cap H^1(L^2)} |\langle \mu_\varepsilon, \psi \rangle|$$

is bounded independently from $\varepsilon > 0$. Since $\mu_\varepsilon$ is on the right-hand side of (5.1e), we can represent $\mu_\varepsilon$ by means of $y_\varepsilon$ and $z_\varepsilon$, i.e., we have

$$|\langle \mu_\varepsilon, \psi \rangle| \leq |\langle \mu_\varepsilon - \tilde{y}, \psi \rangle| + |\langle z_\varepsilon, \psi \rangle| + |\langle z_{\varepsilon, x}, f(\varepsilon, y_\varepsilon, x, x, x) \psi \rangle|$$

$$+ |\langle z_{\varepsilon, x}, f(\varepsilon, y_\varepsilon) + \varepsilon, \psi_{xx} \rangle| =: I_1 + I_2 + I_3 + I_4.$$

We estimate those terms as follows and use the bounds from the first steps (and Sobolev embeddings),

$$I_1 \leq \|y_\varepsilon - \tilde{y}\| \|\psi\| \leq C,$$

$$I_2 = |\langle z_{\varepsilon, x}, \psi \rangle| \leq \|z_{\varepsilon, x}\| \|\psi\| \leq C,$$

$$I_3 \leq \|z_{\varepsilon, x}\| \|f(\varepsilon, y_\varepsilon, x, x, x) \| \leq C,$$

$$I_4 \leq \|z_{\varepsilon, x}\| \|f(\varepsilon) + \varepsilon \| \|\psi_{xx}\| \leq C,$$

where we used that $\|\psi\|_{L^2(H^4) \cap H^1(L^2)} \leq 1$.

Adding up, we arrive at $\sup_{\varepsilon > 0} \|\mu_\varepsilon\|_{(L^2(H^4) \cap H^1(L^2))^*} \leq C$, i.e., $\{\mu_\varepsilon\}$ is uniformly bounded with respect to $\varepsilon > 0$.

**Step 6:** By the bounds from the previous step, there exist $\eta^* \in L^2(H_1^1)$, $z^*_x \in L^2(H^{-1})$, and $\mu^* \in (L^2(H^4) \cap H^1(L^2))^*$ such that $\eta_\varepsilon \rightharpoonup \eta^*$ weakly in $L^2(H_1^1)$, $z_{\varepsilon, x} \rightharpoonup z^*_x$ weakly in $L^2(H^{-1})$, and $\mu_\varepsilon \rightharpoonup \mu^*$ weakly in $(L^2(H^4) \cap H^1(L^2))^*$.

**Step 7:** With the bounds and the convergence from the last step, it is possible to show that by taking the limit in (5.1d), (5.1e), (5.1f), and (5.1g), respectively, that $(\eta^*, z^*, \mu^*)$ solve (4.13d), (4.13e), (4.13f) and (4.13g), respectively. This concludes the proof.

\[\square\]

### 6. Penalty approximation

In this section, we investigate a penalty approximation of Problem 5.1. The main idea is to add an additional non-negative term to the functional, which increases in value in cases where the state constraint $y \geq C_0$ does not apply. This additional term allows us to get rid of the state constraint, hence we can get rid of the non-regular Lagrange multiplier $\mu$ in the optimality system (5.1). On the opposite, a drawback is that in general this generates non-feasible solutions (with respect to the constraint $y \geq C_0$). When considering a well-posed equation,
this is not that crucial, but in our case the original equation (4.1) may degenerate, and non-feasible solutions might even not exist. That is the reason for introducing the intermediate Problem 5.1 with the regularized equation (3.1).

We now introduce a penalty approximation of Problem 5.1, and prove the existence of a corresponding minimum, as well as convergence of minimizers to a minimum of Problem 5.1 for a fixed $\epsilon > 0$. Then we derive optimality conditions, which are the starting point for numerical studies in section 7. For more details to the penalty approximation we refer the reader to [11, Section 1.10] and [23, Section 3].

**Problem 6.1.** Let $\epsilon, \gamma > 0$. We define the functional

$$J_\gamma(y, u) := J(y, u) + \frac{1}{2\gamma} \int_0^T \int_\Omega \left( |(C_0 - y)^+|^2 + \overline{\chi} + \gamma(C_0 - y) \right)^2 \, dx \, dt.$$  \hfill (6.1)

Find $(y_\gamma, q_\gamma, u_\gamma)$ as the minimum of $J_\gamma$ subject to (3.1) and (4.2).

**Remark 6.2.** As mentioned in the introduction, the penalty method is not the only method for the regularization of the state constraint in Problem 5.1. We add a few words about two other prominent methods.

(1) In the Moreau-Yosida approximation [20], we consider the function

$$J_\gamma(y, u) := J(y, u) + \frac{1}{2\gamma} \int_0^T \int_\Omega \left( \overline{\chi} + \gamma(C_0 - y) \right)^+ \, dx \, dt,$$

where $\overline{\chi} \in L^2(L^2)$ is given and $\gamma > 0$ should tend to zero. Note that the scaling is different from that in (6.1). For the Moreau-Yosida based approximation, a class of effective solvers are available, cf. [19]. However, it is not clear if the convergence result in [20] also holds for the nonlinear equation in (1.1).

(2) In the Lavrentiev approximation [24], the state constraint $y \geq C_0$ is replaced by a mixed control-state constraint $\gamma u + y \geq C_0$ for some $\gamma > 0$, which should tend to zero. It is standard to show existence of an optimum subject to the mixed constraints instead of pure state constraints. The necessary optimality conditions now involve a Lagrange multiplier $\mu_\gamma \in L^2(\Omega_T)$ with $\mu_\gamma \geq 0$ and the complementary condition [24, Theorem 3.3]

$$\langle \mu_\gamma, C_0 - \gamma \bar{u} - \bar{y} \rangle = 0,$$  \hfill (6.2)

which is then solved together with the remaining part of the optimality condition with a semi-smooth Newton method. As in the case of the Moreau-Yosida approximation, the authors are not aware if the Lavrentiev converges in the case of the governing equation (1.1), and thus is left open at this place.

**Theorem 6.3.** There exists at least a solution $(y_\gamma, q_\gamma, u_\gamma)$ of Problem 6.1.

**Proof.** Similar to the proof of Theorem 4.2 and Theorem 5.2. \hfill $\Box$

**Theorem 6.4.** Let $\epsilon > 0$, and let $(y_\gamma, q_\gamma, u_\gamma)$ be a sequence of solutions of Problem 6.1. Then, there exist $y^* \in L^2(H^1) \cap H^1(L^2)$, $q \in L^2(H_0^1)$, and $u^* \in L^2(L^2)$ such that $y_\gamma \rightharpoonup y^*$ weakly in $L^2(H^1) \cap H^1(L^2)$, $q_\gamma \rightharpoonup q^*$ weakly in $L^2(H_0^1)$, and $u_\gamma \rightharpoonup u^*$ weakly in $L^2(L^2)$ for $\gamma \to 0$. Moreover, $(y^*, q^*, u^*)$ is a solution of Problem 5.1.
Step 2: We first show that the functional is uniformly bounded (with respect to \( \gamma > 0 \)): Let \((\bar{y}, \bar{q}, \bar{u})\) be the solution of Problem 5.1, i.e., \((\bar{y}, \bar{q}, \bar{u})\) solve (3.1) and (4.2), \(\bar{y} \geq C_0\) and \(J(\bar{y}, \bar{u})\) is minimal for all such \((y, q, u)\). Since \(\bar{y} \geq C_0\), we have \(J_\gamma(y, u) = J(\bar{y}, \bar{u})\) independent of \(\gamma > 0\).

By the minimizing property of \((y_\gamma, q_\gamma, u_\gamma)\), there holds
\[ J_\gamma(y_\gamma, u_\gamma) \leq J_\gamma(\bar{y}, \bar{u}) = J(\bar{y}, \bar{u}) < \infty. \]

Hence, \(J_\gamma(y_\gamma, u_\gamma)\) is uniformly bounded with respect to \(\gamma > 0\).

Step 3: We want to get weak limit functions: \(\gamma > 0\), it remains to show that \(\gamma > 0\) bounded in \(L^2(H_0^1)\). By the a-priori estimates from Lemma 3.6, \(y_\gamma\) is uniformly (with respect to \(\gamma > 0\)) bounded in the \(L^2(H^1) \cap H^1(L^2)\)-norm. Therefore, there exists \((y^*, q^*, u^*) \in (L^2(H^1) \cap H^1(L^2)) \times L^2(H_0^1) \times L^2(L^2)\) such that \((y_\gamma, q_\gamma, u_\gamma) \rightharpoonup (y^*, q^*, u^*)\) weakly in the corresponding spaces.

Step 4: Finally, we show that \((y^*, q^*, u^*)\) are feasible for Problem 5.1. It is easy to verify that \((y^*, q^*, u^*)\) solves (3.1) and (4.2) like it was done, e.g., in the proof of Theorem 4.2. It remains to show that \(\gamma \geq C_0\). Since \(J_\gamma(y_\gamma, u_\gamma) \leq C\) uniformly in \(\gamma > 0\), we know that for \(\gamma \to 0\),
\[ \int_0^T \int_\Omega \left| (C_0 - y_\gamma) \right|^2 dt \, dx \to 0, \]

i.e., we have \((C_0 - y_\gamma)^+ \to 0\) a.e. in \(\Omega_T\), which means \(y^* \geq C_0\).

Proof. Let \((y_\gamma, q_\gamma, u_\gamma)\) be the solution of Problem 6.1.

As in the last sections, we can now derive an analogon to (4.13) and (5.1), respectively.

**Theorem 6.5.** Let \((y, q, u)\) be a minimum of Problem 6.1. Then, there exist Lagrange multiplier \(z \in L^2(L^2)\) and \(\eta \in L^2(H_0^1)\) such that the following equations are fulfilled.

\[
\begin{align*}
  y_t &= -\left( [f(y) + \varepsilon]y_{xxx} \right)_x + q_x, \quad (6.4a) \\
  -q_{xx} &= u_x, \quad (6.4b) \\
  0 &= \langle y - \bar{y}, \varphi \rangle + \langle z, \varphi_t + (f'(y)y_{xxx}\varphi)_x \rangle + \langle z, (|f(y) + \varepsilon|\varphi_{xxx})_x \rangle. \quad (6.4c)
\end{align*}
\]
$$+ \frac{1}{\gamma} \langle \varphi, (C_0 - y)^+ \mu \rangle \quad \forall \varphi \in X_y,$$

$$0 = z_x - \eta_{xx}, \quad (6.4d)$$

$$0 = \alpha u + \eta_x, \quad (6.4e)$$

Together with initial conditions $y(0,.) = y_0$, $z(T,.) = 0$; boundary conditions $y_x = y_{xxx} = z_x = z_{xxx} = 0$ in $a, b$; as well as $q = \eta = 0$ in $a, b$.

7. Computational studies

In order to study numerical experiments for the optimal control Problem 6.1, we first have to discretize the optimization problem to obtain a finite dimensional problem: We use the “first discretize, then optimize” ansatz, which has several advantages such as that the system of necessary optimality conditions is well-posed, and the adjoint equation inherits a discretization from the discretization of the state equation. As in the last sections, we also consider here $g_0 \equiv 0$.

7.1. Discretization of the equation. We use the following space-time discretization scheme for (3.1), which was originally suggested for (1.1) in [4].

Let $h_{\text{space}} = b - a$ and $x_i := a + ih$ for $i = 0, \ldots, N_{\text{space}}$ denote the set of spatial nodes. Define the standard finite element space $V_h$, containing piecewise linear functions, via

$$V_h := \{ v_h \in C([a,b]) : v_h \big|_{[x_i,x_{i+1}]} \in P_1 \},$$

cf. [10]. The function $P_h : L^2 \to V_h$ denotes the projection onto $V_h$ with respect to the $L^2$ scalar product.

Let $k_{\text{time}} = T$, and let $t_n := nk$ for $n = 0, \ldots, N_{\text{time}}$ denote the nodal points of a time grid which covers $[0,T]$.

We will use the following notation for discrete functions: The notation $\{V^n\} \subseteq X_h$ describes a family of finite element functions evaluated at subsequent times $t_n$, while $V : \Omega_T \to \mathbb{R}$ stands for the piecewise affine, globally continuous time interpolant of $\{V^n\}$. Sometimes, we also write $V(t = t_n)$ instead of $V^n$.

The discrete version of (3.1) (for $g_0 \equiv 0$) reads as follows.

**Problem 7.1.** Let $Y_0 := P_h y_0 \in V_h$. Set $Y^0 := Y_0$, find $P^0 \in V_h$ such that

$$(Y^0_x, \Phi_x) - (P^0, \Phi) = 0 \quad \forall \Phi \in V_h.$$  

Then for $n = 1, \ldots, N_{\text{time}} - 1$ find $Y^{n+1} \in V_h$, $P^{n+1} \in V_h$ and $P^{n+1} \in V_h$, such that

$$\frac{1}{k} (Y^{n+1} - Y^n, \Phi) + (f_x(Y^{n+1}) P^{n+1}_x, \Phi_x) = (U_x(t_{n+1}), \Phi) \quad \forall \Phi \in V_h,$$

$$\quad (Y^{n+1}_x, \Phi_x) - (P^{n+1}, \Phi) = 0 \quad \forall \Phi \in V_h.$$  

(7.1)

**Remark 7.2.** Note that in this whole section, we will not include the term regularization $\langle \Delta^{-1} u_x \rangle_x$, which we introduced in section 4 in order to cope with spatial regularity and convergence issues, but use instead the term $u_x$. In all our experiments, we do not study the effects in the discretization parameter $h$ dealing with spatial resolution, i.e., $h > 0$ is kept
fixed. On a finite dimensional level, the problem with the easier term $u_x$ (from Problem 1.1) is related to the properly scaled Problem 4.1.

The coupled system (7.1) is solved by Newton’s method with exact derivatives, and all terms (which are polynomials of higher order) are assembled exactly using an accurate quadrature rule.

Lemma 3.7 motivates solvability of (7.1) for $\varepsilon > 0$. However, for small $\varepsilon > 0$, the system matrix has a high condition number in the presence of related large values of the approximation of $U_x(t_n)$ and small values of $\{Y^n\}$ due to the algebraic form of $f_\varepsilon$. We encountered this problem in the form of a singular system matrix on the level of numerical linear algebra. Smaller values of $k$, bigger values of $\varepsilon$ and—in the context of optimal control—state constraints help to overcome this issue.

For all experiments in this section, we choose $\lambda = 1.0$ and Newton’s method as nonlinear algebraic solver stops if the difference of two consecutive iterations is less than $10^{-10}$, or if the maximum number of iterations exceeds 1 000. However, except for those experiments with singular system matrices, the observed number of iterates was well below (average 2–5/max. 30 iterations; highly depending on the specific experiment).

### 7.2. Simulations of the equation

For the first experiment, we take $[a, b] = [0, 5]$, $T = 1.0$, $N_{\text{space}} = 8$, $N_{\text{time}} = 5 000$, and $\varepsilon = 0$; we take a fixed right-hand side $U$ and solve (7.1). The output is displayed in Figure 1. In this experiment we see that the solution takes negative values for a general function $u$, which motivates that the state constraint in the optimization Problem 1.1 is really needed. For comparison, we included corresponding simulations for $U \equiv 0$, where we know from Hypothesis 3.8 that the solution stays positive.

In order to exclude effects concerning spatial discretization, we change data to $N_{\text{space}} = 30$, $\varepsilon = 0.03$ and repeat the experiment for a given right-hand side (actually the same like above, but the magnitude is decreased by a factor of 0.7); see Figure 2. The change in the right-hand side and the small, but positive value of $\varepsilon$ have to be done since the system matrix was singular otherwise. For comparison, we included corresponding simulations for $U \equiv 0$ and $\varepsilon = 0.03$ as well as for $\varepsilon = 0$, which both stay positive for the whole time. We see that small values of $\varepsilon$ do have only a small effect on the solution being negative if at all.

### 7.3. Discretization of the optimal control problem

We use a “first discretize, then optimize” (cf. [21]) approach to state the following discrete version of Problem 6.1.

**Problem 7.3.** Let $\varepsilon > 0$, $\gamma \geq 0$, and let $t_k$ like above. Define $J_{\gamma, \text{disc}} : V_h^{N_{\text{time}}+1} \times V_h^{N_{\text{time}}+1} \to \mathbb{R}$ via

$$J_{\gamma, \text{disc}}(Y, U) := \frac{k}{2} \sum_{n=0}^{N_{\text{time}}} \|Y^n - \hat{Y}^n\|^2 + \frac{\alpha k}{2} \sum_{n=0}^{N_{\text{time}}} \|U^n\|^2 + \frac{k}{2\gamma} \sum_{n=0}^{N_{\text{time}}} \|(C_0 - Y^n)^+\|^2,$$

where the last term is ignored if we set $\gamma = 0$. If $\hat{Y}^n \notin V_h$, we instead insert the interpolation of it into $J_{\gamma, \text{disc}}$.

Find $(Y, U)$ as the minimum of $J_{\gamma, \text{disc}}$ subject to (7.1).

**Theorem 7.4.** Let $\varepsilon > 0$ and $\gamma \geq 0$. Then there exists a solution of Problem 7.3.
Figure 1. Solution $Y$ at different times for a given right-hand side $U \neq 0$ (---) and $U \equiv 0$ (---).

Figure 2. Solution $Y$ at different times for a given right-hand side $U \neq 0$ and $\varepsilon = 0.03$ (---), for $U \equiv 0$ and $\varepsilon = 0.03$ (---), and for $U \equiv 0$ and $\varepsilon = 0$ (---).

Theorem 7.5. Let $(Y, U) \in V_{h_{\text{time}}}^{N_{\text{time}}+1} \times V_{h_{\text{time}}}^{N_{\text{time}}+1}$ be a minimum of Problem 7.3. Then, there exist Lagrange multipliers $Z \in V_{h_{\text{time}}}^{N_{\text{time}}+1}$ and $S \in V_{h_{\text{time}}}^{N_{\text{time}}+1}$, such that for all $n = 1, \ldots, N_{\text{time}} - 1$ the following equations are fulfilled:

\[ \frac{1}{k}(Y^{n+1} - Y^n, \Phi) + (f_\varepsilon(Y^{n+1}) P_x^{n+1}, \Phi_x) = (U_x(t_{n+1}), \Phi) \quad \forall \Phi \in V_h, \quad (7.2a) \]

\[ (Y_x^{n+1}, \Phi_x) - (P^{n+1}, \Phi) = 0 \quad \forall \Phi \in V_h, \quad (7.2b) \]

\[ \frac{1}{k}(\Phi, Z^n) + (f'(Y^{n+1}) \Phi P_x^{n+1}, Z_x^n) + (\Phi_x, S_x^n) = \frac{1}{k}(\Phi, Z^{n+1}) + (\Phi, \bar{Y}^{n+1} - Y^{n+1}) \]

\[ + \frac{1}{\gamma} \left( \Phi, (C_0 - Y^{n+1})^+ \right) \quad \forall \Phi \in V_h, \quad (7.2c) \]

\[ (f_\varepsilon(Y^{n+1}) \Phi_x, Z^n) - (\Phi, S^n) = 0 \quad \forall \Phi \in V_h, \quad (7.2d) \]
\[ \alpha(U, \Phi) + (Z_x, \Phi) = 0 \quad \forall \Phi \in V_h, \]  
\[ \text{(7.2c)} \]

together with initial conditions \( Y^0 = Y_0, Z^{N_{\text{time}}} = 0 \). Conditions \( \text{(7.2b)}, \text{(7.2d)}, \) and \( \text{(7.2c)} \) are also valid for \( n = 0 \).

By the uniqueness of solutions for the continuous equation \( (3.1) \) (which is valid at least for \( g_0 \equiv 0 \) as well for the discrete version of it, \( (7.1) \) (which can be shown for \( k > 0 \) is small enough), the operator \( U \mapsto Y(U) \) is well-defined. Therefore, we can use a steepest descent algorithm in order to solve Problem \( 7.3 \) numerically instead of addressing directly \( (7.2) \) with, e.g., a SQP-algorithm, which suffers from a huge system matrix for the present evolutionary problem.

We write \( Y(U) \) for the solution of \( (7.1) \) for a given \( U \) and can restate Problem \( 7.3 \) by minimizing the functional
\[ \bar{J}(U) := J_{\text{disc}}(Y(U), U) \]
without any constraints. From \( \text{(7.2c)} \), we know that the gradient of \( \bar{J} \) is given by the finite element projection of \( \alpha U + Z_x \), which we use as search direction for the steepest descent method, in combination with an Armijo step size rule, which is very flexible and ensures by its selection of valid step sizes a monotone decrease of \( \bar{J}(U_r) \) as \( r \to \infty \). For general details regarding our used method and a recent overview about the theoretical background we refer the reader to \[18, 21\].

The corresponding algorithm reads as follows.

**Algorithm 7.6.** Set \( U_0 \equiv 0 \) and fix \( \sigma_*> 0, 0 < \beta < 1, \delta_{\text{tol}} > 0 \). Compute \((Y_1, P_1)\) from solving \( (7.1) \), then compute \((Z_1, S_1)\) from solving \( \text{(7.2c)} \) and \( \text{(7.2d)} \). Repeat for \( r \geq 0 \):

1. Evaluate \( \nabla \bar{J}(U_r) = \alpha U_r + (Z_r)_x \) and evaluate \( \bar{J}(U_r) \).
2. Repeat for \( s \geq 0 \):
   a. Define \( U^{(s)}_{r+1} := U_r - \beta^s \nabla \bar{J}(U_r) \).
   b. Compute \((Y^{(s)}_{r+1}, P^{(s)}_{r+1})\) from solving \( (7.1) \) for \( U^{(s)}_{r+1} \) as right-hand side.
   c. STOP, if
   \[ \bar{J}(U^{(s)}_{r+1}) - \bar{J}(U_r) \leq -\sigma_\varepsilon \beta^s \|
   \nabla \bar{J}(U_r) \|^2, \]
   and set \( U_{r+1} := U^{(s)}_{r+1} \).
3. Compute \((Z_{r+1}, S_{r+1})\) from solving \( \text{(7.2c)} \) and \( \text{(7.2d)} \).
4. STOP, if \( \|
   \nabla \bar{J}(U_{r+1}) \|^2 \leq \delta_{\text{tol}} \) and set \( U_{\text{opt}} = U_{r+1}, \ Y_{\text{opt}} = Y_{r+1} \).

In all the studies below, we set \( \sigma_* := 10^{-5} \) and \( \beta := 0.15 \). The stopping condition is set to be \( \delta_{\text{tol}} := 5 \cdot 10^{-5} \), which is obtained after 700 up to 50,000 iterations. The number of iterations highly depends on the given data (i.e., on \( Y_0, Y \), and on \( \alpha, \varepsilon, \gamma > 0 \)). A typical evaluation with respect to the number of iterations of the functional \( \bar{J} \) and the gradient \( \nabla \bar{J} \) with respect to the number of iterations is shown in Figure 3. For the majority of the steps in this example, the biggest step size was considered as being suitable, i.e., \( (7.3) \) was fulfilled for \( s = 0 \). In cases where more nested iterations in Algorithm 7.6 are needed, the values of \( \bar{J} \) and \( \|
   \nabla \bar{J} \|^2 \) decrease more slowly with respect to the number of iterations.

In Figure 3 we plotted two different scenarios depending on \( \gamma > 0 \): In both scenarios, \( r \mapsto \bar{J}(U_r) \) is monotonously decreasing, thanks to the definition of step size. This is also
the case for \( r \mapsto \|\nabla \tilde{J}(U_r)\|^2 \) if \( \gamma \equiv 0 \). For \( \gamma > 0 \), the function \( \tilde{J} \) is more complicated, and so is the norm of its gradient; see Figure 3b. The time dynamics of the optimal solutions of this particular experiment is displayed in Figure 8 and the experiment is explained in subsection 7.6.

It is clear that the choice of \( \beta \) is one of the most crucial parameters for the performance of the algorithm. Smaller values of \( \beta \) rule out bigger step sizes, leading to more iterations. Bigger values of \( \beta \) allow for potentially bigger step sizes, which could also lead to a longer runtime since it takes more nested iterations to obtain a valid step size in the spirit of (7.3). In the context of an optimal control problem with a degenerate equation such as in the present case, it is important not to choose \( \beta \) too big: Then, the new potential control \( U_{\text{new}} = U_{\text{old}} - \beta \nabla \tilde{J}(U_{\text{old}}) \) can be too “destructive”, in a sense that the corresponding system matrix for the potential new state \( Y(U_{\text{new}}) \) is (close to) singular and hence would affect the linear algebra solver. With our choice of \( \beta \), we observed a reasonable behavior of Algorithm 7.6.

We note that the performance of Algorithm 7.6 can be improved in the following way: First solve Problem 7.3 with coarse discretization parameters \( h, k > 0 \). Then, transfer the solution \( U_{\text{opt}} \) to \( U_0 \), and solve Problem 7.3 with the finer discretization with the different start value for \( U_0 \). Clearly, the number of iterations can be rapidly decreased in this way.

![Figure 3.](image)

(A) Evolution of \( \tilde{J}(U_r) \) with respect to the number of iterations of Algorithm 7.6

(B) Evolution of \( \|\nabla \tilde{J}(U_r)\|^2 \) with respect to the number of iterations of Algorithm 7.6

7.4. Comparison of the parameter \( \varepsilon \). In the next experiment we take \( [a, b] = [0, 5] \), \( T = 1.0 \), \( N_{\text{space}} = 30 \), \( N_{\text{time}} = 5000 \), and \( \alpha = 10^{-6} \); and we solve (7.1) for \( U \equiv 0 \) to study the dependencies on \( \varepsilon > 0 \); see Figure 4. The bigger the value of \( \varepsilon \), the more dissipative is the evolution, and the solution becomes almost flat after a short time. In contrast to this, for a small value of \( \varepsilon \), the solution needs longer to approach a flat profile.

For a large value of \( \varepsilon \), the solution is slightly negative in some regions; see Figures 4c, 4e, and 4f. This is due to the fact that there is no maximum principle for the biharmonic problem, which would force the solution to stay positive. This effect vanishes for decreasing values of \( \varepsilon \).
We repeat the above experiment with the same parameters in the context of optimal control Problem 7.3 for $\gamma \equiv 0$; see Figure 5. In contrast to the previous experiment from Figure 4, there is not such a big difference between the computed evolution of the optimal states, depending on the value of $\varepsilon$. This is due to the fact that the optimal state $Y = Y(\varepsilon)$ belongs to different optimal controls $U = U(\varepsilon)$ which force the solution to obtain the given target profile $\tilde{Y}$. The experiment which is shown in Figure 5 demonstrates that relevant controls are active since the dynamics of the solutions completely differs from the case without control which was shown in Figure 4.

7.5. Comparison of the parameter $\alpha$. In this experiment, we take $\varepsilon = 0.05$, $\gamma \equiv 0$, $N_{\text{space}} = 54$, $N_{\text{time}} = 5000$, and compare different values of $\alpha > 0$; see Figure 6 (state) and Figure 7 (control). Here, $\tilde{Y}$ is constant in time. We can see that a small value of $\alpha$ allows for bigger controls; see Figure 7. The optimal state $Y$ (with small $\alpha$) almost agrees with the target state $\tilde{Y}$ after a very short time, while the optimal state $Y$ (with bigger $\alpha$) needs more time for that. The snapshot in Figure 6e shows the first time when the optimal state $Y$ coincides with the target state $\tilde{Y}$ for all values of $\alpha$.

We note that the optimal controls displayed in Figure 7 are typical for many experiments: The control acts near the spatial boundary, i.e., it could be worth to consider Problem 1.1 with boundary control instead of a distributed control. Also, the amplitude of the controls decreases in time, which is typical for parabolic optimal control problems with a constant
target profile $\tilde{y}$: A large control in the beginning of the experiment enforces the solution to be near the target profile $\tilde{y}$, which decreases immediately the tracking term $\|y - \tilde{y}\|$ in the functional, while a large control near $t = T$ has almost no impact on the optimal state $y$, but increases the cost term $\|u\|^2$ in the functional.

7.6. **Comparison of the parameter $\gamma$.** In this experiment, we take $C_0 = 0.01$, $\alpha = 10^{-6}$, $\varepsilon = 0.1$, $N_{\text{space}} = 42$, $N_{\text{time}} = 5000$ and simulate different values of $\gamma > 0$; see Figure 8. Here, $\tilde{Y}$ is constant in time and the profile is given in the figure. We can see that even for a moderate choice of $\gamma > 0$, this parameter has a significant effect on the simulation: If this penalization term is missing, the solution ceases to be positive, while the solution is positive (except for some single points) over the whole simulation if the penalization is active. As we have noted before, the condition number is increasing for decreasing values of $\gamma$. This leads to a longer runtime for smaller $\gamma$; sometimes it also happens that matrices are identified as singular by the linear algebra solver due to this increasing condition number.

Vice versa, for bigger values of $\gamma$, the state condition is not resolved properly, i.e., system matrices can also become singular in this case. This leads to the conclusion that – as long as $\varepsilon > 0$ is kept small, and as long as no sophisticated linear algebra solvers are used – there is only a small range for $\gamma > 0$ where simulations are likely to terminate in a reasonable amount of time.
Figure 6. Target $\tilde{Y}$ (---) and optimal states $Y$ for $\alpha = 10^{-2}$ (---) and $\alpha = 10^{-10}$ (---) at different times.

Figure 7. Control $U$ for $\alpha = 10^{-2}$ (---) and $\alpha = 10^{-10}$ (---) at different times. Note that the different plots are scaled by different factors.

Also, the more complex structure of the functional $\tilde{J}$ for $\gamma > 0$ leads to an increase of the needed amount of iterations in the steepest descent algorithm. The corresponding evolution of $\tilde{J}$ and $\|\nabla \tilde{J}\|^2$ are displayed for both values of $\gamma$ in Figure 3.

7.7. Dewetting application. In the last experiment, we simulate the solution of a simplified version of the problem arising in [6]: Given a constant initial value $Y_0$, a profile $\tilde{Y}$ should be accomplished, where there is a bigger region (nearly) without any fluid; this evolution is referred to as dewetting procedure. We set $[a, b] = [0, 5]$, $T = 1.0$, $N_{\text{space}} = 54$, $N_{\text{time}} = 25000$, $\alpha = 10^{-6}$, $C_0 = 0.01$, and $\gamma = 0.01$; see Figure 9.
Figure 8. Target $\tilde{Y}$ and optimal states $Y$ for $\gamma \equiv 0$ and $\gamma = 0.02$ at different times.

Figure 9. Target $\tilde{Y}$ and optimal states $Y$ at different times.

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