SIGMA FUNCTIONS FOR A SPACE CURVE OF TYPE $(3, 4, 5)$

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Abstract. In this article, a generalized Kleinian sigma function for an affine $(3, 4, 5)$ space curve of genus two was constructed as the simplest example of the sigma function for an affine space curve, and in terms of the sigma function, the Jacobi inversion formulæ for the curve are obtained. An interesting relation between a space curve with a semigroup generated by $(6, 13, 14, 15, 16)$ and Norton number associated with Monster group is also mentioned in the Appendix by the second author.

1. Introduction

Recently the Kleinian sigma function for hyperelliptic curves, a natural generalization of the Weierstrass sigma function, is re-evaluated because in terms of the sigma functions, it is more convenient to investigate the properties of the abelian functions and their interesting properties are revealed naturally [2, 6, 17]. Further Enolskii, Elbeek, and Leykin [5] discovered a construction which generalizes the Kleinian sigma function associated with hyperelliptic curves to one for an affine $(r, s)$ plane curve, where $r$ and $s$ ($r < s$) are coprime positive integers $g = (r - 1)(s - 1)/2$. There they have constructed also the fundamental differential of the second kind over an affine $(r, s)$ plane curve and using it, obtained the Legendre relation as the symplectic structure over the curve. Using the Legendre relation, they defined the generalized Kleinian sigma function over the image of the abelian map $C^g$. They have found also the natural Jacobi inversion formulæ in terms of their sigma function. We call this construction EEL construction in this article. Using the EEL construction, we have some interesting results [20, 21].

In this article, we consider a generalized Kleinian sigma function for an affine $(3, 4, 5)$ space curve of genus two, which is the simplest affine space curve. Our purpose of this article is to show that the sigma function is also defined for an affine space curve as we can do for plane curves.

Following the EEL-construction, we define the fundamental differential of the second kind over it and obtain the Legendre relation as the symplectic structure over it.
With the abelian map to $\mathbb{C}^2$, we show that the symplectic structure determines the sigma function. Further using the sigma function, we obtain the Jacobi inversion formulae for the curve and the Jacobian following the previous works [20, 21]. It means that the generalization of the sigma functions for the affine plane curves to ones for the space curves is basically possible and is useful. Recently, Korotkin with Shramchenko defined a sigma function for a compact Riemann surface [15] but it is not directly associated with an algebraic curve. Further A yano introduced sigma functions for space curves of special class [1], which are called telescop ic curves, but the class does not include this (3,4,5) curve.

In Remark 21, we also show a problem of a space curve associated with the semi-group generated by $(6, 13, 14, 15, 16)$ with an Appendix by Komeda. The semi-group might be related to Norton number associated with the Monster group, the simple largest sporadic finite group [22].

2. Preliminary

2.1. Numerical Semigroup

Here we give a short overview of recent study of the numerical semigroups as sub-semigroups of non-negative integers $\mathbb{N}_0$ related to algebraic curves. We call an additive semigroup in $\mathbb{N}_0$ numerical semigroup if its complement in $\mathbb{N}_0$ is finite. For a numerical semigroup $H := H(M)$ generated by $M$, the number of elements of $L(H) := \mathbb{N}_0 \setminus H$ is called genus and $L(H)$ is called gap sequence. For example, we have semigroups $H_2, H_4, H_{12}$ generated by $M_2 := \langle 3, 4, 5 \rangle$, $M_4 := \langle 3, 7, 8 \rangle$, $M_{12} := \langle 6, 13, 14, 15, 16 \rangle$ respectively whose genera are $g(H_g)$ for $g = 2, 4, 12$ due to $L(H_2) = \{1, 2\}$, $L(H_4) = \{1, 2, 4, 5\}$, $L(H_{12}) = \{1, 2, 3, 4, 5, 7, 8, 9, 10, 11, 17, 23\}$.

For a complete non-singular irreducible curve $C$ of genus $g$ over an algebraically closed field $k$ of characteristic $0$, the field of its rational functions $k(C)$, and a point $P \in C$, we define

$$H(P) := \{n \in \mathbb{N}_0 : \text{there exists } f \in k(C) \text{ such that } (f)_{\infty} = nP \}$$

and refer further on as the Weierstrass semigroup of the point $P$. If $L(H(P)) := \mathbb{N}_0 \setminus H(P) \text{ differs from the set } \{1, 2, \cdots, g\}$, we call $P$ Weierstrass point of $C$.

A numerical semigroup $H$ is said to be Weierstrass if there exists a pointed algebraic curve $(C, P)$ such that $H = H(P)$. Hurwitz considered whether every numerical semigroup $H$ is Weierstrass. This was a long-standing problem but Buchweitz finally showed that every $H$ is not Weierstrass. His first counterexample is the semigroup $H_B$ generated by $13, 14, 15, 16, 17, 18, 20, 22$ and $23$.

2.2. Weierstrass Point

A Weierstrass point is a point $P$ on a curve $C$ such that the Weierstrass semigroup $H(P)$ of $P$ is a numerical semigroup. The Weierstrass semigroup of a point $P$ is defined as the set of all integers $n$ for which there exists a rational function $f$ on $C$ such that $(f)_{\infty} = nP$. A Weierstrass point is a point where the Weierstrass semigroup is a numerical semigroup. A numerical semigroup is a sub-semigroup of the non-negative integers that is not finite.

2.3. Weierstrass Semigroup

The Weierstrass semigroup of a point $P$ on a curve $C$ is defined as the set of all integers $n$ for which there exists a rational function $f$ on $C$ such that $(f)_{\infty} = nP$. The Weierstrass semigroup is a numerical semigroup if and only if it is a sub-semigroup of the non-negative integers.

2.4. Weierstrass Property

A curve $C$ is said to have the Weierstrass property if for every point $P$ on $C$, the Weierstrass semigroup $H(P)$ is a numerical semigroup.

2.5. Weierstrass Theorem

The Weierstrass theorem states that a non-singular curve of genus $g$ has a point $P$ with Weierstrass property if and only if $g \geq 2$.

2.6. Weierstrass Formulas

The Weierstrass formulas are the formulas that relate the Weierstrass semigroups and the gap sequences of a curve $C$ and a point $P$.

2.7. Weierstrass Gap Sequence

The gap sequence of a point $P$ on a curve $C$ is the set of all integers $n$ for which there exists a rational function $f$ on $C$ such that $(f)_{\infty} = nP$. The gap sequence is a numerical sequence if and only if it is a sub-sequence of the non-negative integers.

2.8. Weierstrass Genus

The genus of a curve $C$ is the number of elements in the gap sequence of any point $P$ on $C$. The genus of a curve is a non-negative integer.

2.9. Weierstrass Semigroup Generating Set

The Weierstrass semigroup generating set of a point $P$ on a curve $C$ is the set of all integers $n$ for which there exists a rational function $f$ on $C$ such that $(f)_{\infty} = nP$. The Weierstrass semigroup generating set is a numerical semigroup if and only if it is a sub-semigroup of the non-negative integers.

2.10. Weierstrass Semigroup Extension

The Weierstrass semigroup extension of a point $P$ on a curve $C$ is the set of all integers $n$ for which there exists a rational function $f$ on $C$ such that $(f)_{\infty} = nP$. The Weierstrass semigroup extension is a numerical semigroup if and only if it is a sub-semigroup of the non-negative integers.

2.11. Weierstrass Semigroup Property

A curve $C$ is said to have the Weierstrass property if for every point $P$ on $C$, the Weierstrass semigroup $H(P)$ is a numerical semigroup.

2.12. Weierstrass Semigroup Formulas

The Weierstrass semigroup formulas are the formulas that relate the Weierstrass semigroups and the gap sequences of a curve $C$ and a point $P$. The Weierstrass semigroup formulas are the formulas that relate the Weierstrass semigroups and the gap sequences of a curve $C$ and a point $P$. The Weierstrass semigroup formulas are the formulas that relate the Weierstrass semigroups and the gap sequences of a curve $C$ and a point $P$.
Sigma Functions for a Space Curve of Type \((3, 4, 5)\) whose genus is 16. Thus in general, it is not so trivial whether a given semigroup is Weierstrass or not. Komeda has been investigated this problem with Ohbuchi and Kim [10–12, 14].

2.2. Commutative Algebra

Here we review a normal ring and normalization in commutative ring [16]. We assume that every ring is a commutative ring with unit.

\[ B \text{ is a ring and } A \text{ is a subring of } B. \text{ If } B \text{ is integral over } A \text{ if } b \text{ satisfies a monic polynomial over } A, \text{ i.e., there exist } n \text{ and } \{a_i\}_{i=1,\ldots,n} \in A \text{ such that } b^n + a_1b^{n-1} + \cdots + a_n = 0. \]

We say that \( B \) is integral over \( A \), or \( B \) is an integral ring over \( A \), or \( B \) is an integral extension of \( A \) if every element \( b \) of \( B \) is integral over \( A \).

An integral closure in \( B \) over \( A \) is defined by \( \tilde{A} := \{b \in B ; b \text{ is integral over } A\} \). If \( A = \tilde{A} \), \( A \) is integral closed in \( B \).

Definition 1. Let \( A \) is a ring and \( Q(A) \) is a quotient ring of \( A \). We assume that \( A \) is an integral domain. \( A \) is normal if \( A \) is integral closed in \( Q(A) \), i.e., for \( \tilde{A} := \{q \in Q(A) ; \text{there exist } n \text{ and } a_j \in A \text{ such that } q^n + a_1q^{n-1} + \cdots + a_n = 0\} \), \( A = \tilde{A} \).

We define the minimum extension \( \hat{A} \) of \( A \) in \( Q(A) \) so that \( \hat{A} \) is integral closed in \( Q(A) \). We say that \( \hat{A} \) is normalization of \( A \) or the normalized ring of \( A \).

Through the correspondence between an algebraic variety and a commutative ring, we have the well-known normalization theorem [9, p.5, p.68]

**Theorem 2.** For any irreducible algebraic curve \( X \subset \mathbb{C}P^2 \), there exists a compact Riemann surface \( \tilde{X} \) and a holomorphic mapping \( s : \tilde{X} \to \mathbb{C}P^2 \) such that \( s(\tilde{X}) = X \) and \( s \) is injective on the inverse image of the set of smooth points of \( X \). Further the Riemann surface is unique up to its isomorphism; if there are two Riemann surfaces \( \tilde{X} \) and \( \tilde{X}' \) given by normalizations of \( X \), there is a biholomorphic from \( \tilde{X} \) to \( \tilde{X}' \).

As illustrations of Theorem 2, we give three examples.

**Example 3.** \((x^3 - y^2)t, R := \mathbb{C}[X, Y]/(X^3 - Y^2)\) is not normal because \( \frac{d}{dt} \in R \setminus \tilde{R} \subset Q(R) \) due to \((\frac{d}{dt})^2 - X = 0\). Since \( R \approx \mathbb{C}[t^2, t^3] \), the normalized ring is \( \tilde{R} = \mathbb{C}[t] \).
Example 4. \((y^3 = x^3 - 1\) and \(w^3 = z - z^5)\). Following Theorem 2, we consider the covering of a curve of \(f(x, y) := y^3 - x^3 + 1\). Let us consider a homogeneous polynomial \(F(X, Y, Z) := Y^3Z^2 - X^3 + Z^6 \in \mathbb{C}[X, Y, Z]\). Around \(Z \neq 0\), we have \(F(X, Y, Z) = Z^2\left(\frac{1}{X^2} - \frac{X}{Y^2} + 1\right)\) and thus by regarding that \(x = X/Z\) and \(y = Y/Z\), we have \(F(X, Y, Z) = Z^2f(X/Z, Y/Z)\). \(R_0 := \mathbb{C}[x, y]/(f(x, y))\) is a normal ring. On the other hand, around \(Z = 0\) and \(X \neq 0\), we have \(F(X, Y, Z) = X^3\left(\frac{1}{Y^2} - \frac{X}{Z^3} + 1\right)\), and then we obtain a polynomial, \(g(w, z) = w^3z^4 - 1 + z^5\) by regarding \(w = Y/X\) and \(z = Z/X\). However \(R_\infty := \mathbb{C}[w, z]/(g(w, z))\) is not a normal ring. As a vector space, \(R_\infty\) is \(\mathbb{C}^1 + \mathbb{C}x + \mathbb{C}y + \mathbb{C}xz + \mathbb{C}x^2z + \cdots + \mathbb{C}z^3 + \mathbb{C}z^4 + \mathbb{C}z^5\). We show that \(q \in Q(R_\infty)\) \(R_\infty\) exists such that \(q^a + a_1q^{a-1} + \cdots + a_n = 0\) for certain \(a_i \in R_\infty\). Noting \(1 - x^2\) \(g(w, z) = \frac{w^3z^4}{1 - z^2} + 1 + z^2 + z^3 = 0 \in Q(R_\infty)\) we consider \(q := \frac{w^3z^4}{1 - z^2} + \frac{1}{1 - z} \in Q(R_\infty)\), which is integral over \(R_\infty\). By normalizing, we define \(w := wz/y^3, R_\infty := \mathbb{C}[w, z]/(g(w, z))\) is a normal ring, where \(\hat{g}(w, z) := w^3 - z^4 + z^5\). The minimal condition is obvious.

Example 5. (A space curve \(y^3 = x^3(x^2 - 1)\) and \(w^3 = x(x^2 - 1)^2\). Let us consider a polynomial \(f(x, y) := y^3 - x^3(x^2 - 1)\) and show that \(R_0 := \mathbb{C}[x, y]/(f(x, y))\) is a normal ring. As a vector space, \(R_0\) is \(\mathbb{C}^1 + \mathbb{C}x + \mathbb{C}x^2 + \cdots + \mathbb{C}y + \mathbb{C}yx + \mathbb{C}y^2 + \cdots + \mathbb{C}y^2x + \mathbb{C}y^3x + \mathbb{C}y^3x^2 + \cdots\). We show that \(w \in Q(R_0)\) \(R_0\) exists such that \(u^a + a_1u^{a-1} + \cdots + a_n = 0\) for certain \(a_i\)’s of \(R_0\). In other words, noting that \(y \sim \sqrt{x(x^2 - 1)}\) and \(y^2 \sim x(x^2 - 1)\), one of \(w\) is \(w := \sqrt{x}x^2 - 1\) because \(w^3 = \sqrt{x}x^2 - 1\) \(w^3 = x(x^2 - 1)^3\) or \(w^3 = x(x^2 - 1)^2 = 0 \in R_0\). Let \(g(x, w) = x(x^2 - 1)^2\).

Noting the relations that \(w = \frac{\sqrt{x}x^2 - 1}{x}, y = \frac{\sqrt{x}x^2 - 1}{x}, \) and \(wy = (x^2 - 1)x, \) we have \(R_0 := \mathbb{C}[x, y, w]/(f_1(x, y, z), f_2(x, y, z)f_3(x, y, z))\), as the normalized ring of \(R_0\), where \(f_1(x, y, z) = y^2 - wx, f_2(x, y, w) = wy - (x^2 - 1)x, \) and \(f_3(x, y, w) = w^3 - y(x^2 - 1)\). The minimal condition is also obvious. This example corresponds to the special case of the affine (3, 4, 5) space curve in this article. Due to Theorem 2, the corresponding Riemann surface uniquely exists up to an isomorphism.

3. A Curve (3, 4, 5)

Since \(H_2\) generated by (3, 4, 5) is Weierstrass and is the simplest semigroup whose cardinality of the generators is greater than two, we consider a curve \(C(H_2)\) explicitly in order to construct the sigma functions for \(C(H_2)\) following the EEL construction.
As a normalization of these singular curves, we have the commutative ring
\[ f_{3,12}(x, y) := y_1^2 - k_2(x), \quad f_{4,15}(x, y) := y_2^2 - k_3(x) \]
where \( k_2(x) := k_2(x)k_1(x)^2, k_3(x) := k_2(x)^2k_1(x), k_2(x) := (x - b_1)(x - b_2) = x^2 + \lambda_0^{(2)} x + \lambda_2^{(2)}, \) and \( k_1(x) := (x - b_0) = x + \lambda_1^{(1)} \) for finite \( b_0 \in \mathbb{C} \) \((\alpha = 1, 2, 3)\) which is distinct from each other. Let us consider commutative rings \( R_3 := \mathbb{C}[x, y]/(f_{3,12}(x, y)) \) and \( R_4 := \mathbb{C}[x, y]/(f_{4,15}(x, y)) \) related to \( X_3 \) and \( X_4 \) respectively. These genera of the semigroups associated with their Weierstrass non-gap sequences at \( \infty \)-points are three and four respectively, though the geometric genera are not. Following the normalization in Section 2, we normalize \( R_3 \) and \( R_4 \). Since in terms of the language of the commutative algebra \([16]\), \( y_1^2 \) is integral over \( R_3 \) in \( Q(R_3) \) and \( y_2^2 \) is integral over \( R_4 \) in \( Q(R_4) \). \( R_3 \) and \( R_4 \) are not normal rings. Thus we will normalise them in \( \mathbb{C}[x, y_1, y_2]\) in the meaning of the commutative algebra \([16]\) (see Example 5 in \S2.2).

For the zeroes of \( f_{3,12}(x, y_1) \) and \( f_{4,15}(x, y_4) \), we could have the relations,
\[ y_1 y_2 = k_2(x)k_1(x), \quad y_1 = \frac{y_1^2}{(x - b_0)}, \quad y_2 = \frac{y_1^2}{(x - b_1)(x - b_2)} \quad (2) \]
Here for the primitive root \( \zeta_3 \) \((\zeta_3^3 = 1, \zeta_3 \neq 1)\), \( \zeta_3 \) acts on \( X_3 \) and \( X_4 \) respectively.

The first relation is chosen in the possibilities \( y_1 y_2 = \zeta_3^k k_3(x)k_1(x) \) \( i = 0, 1, 2 \). As a normalization of these singular curves, we have the commutative ring,
\[ R_3 := \mathbb{C}[x, y_1, y_2]/(f_3, f_4, f_{10}) \]
and \( X_3 := \text{Spec } R_3 \). Here we define \( f_3, f_4, f_{10} \in \mathbb{C}[x, y_1, y_2] \) by
\[ f_3 = y_1^2 - y_1 k_2(x), \quad f_4 = y_1 y_2 - k_2(x)k_1(x), \quad f_{10} = y_1^2 - y_2 k_3(x) \]
which are also regarded as the \( 2 \times 2 \) minors of \( \begin{vmatrix} k_2(x) & y_1 & y_2 \\ y_1 & y_1 & k_1(x) \end{vmatrix} \). Here \( \zeta_3 \) acts on \( X_3 \) by \( \zeta_3 \) \((x, y_1, y_2) \) \( = (x, \zeta_3 y_1, \zeta_3^2 y_2) \).

Let \( X \) be the Riemann surface which is naturally obtained as an extension of \( X_2 \) as mentioned in Theorem 2, i.e., \( X = X_2 \cup \{ \infty \} \) as a set. It is noted that when \( x \) diverges, \( y_1 \) and \( y_2 \) also diverge vise versa. Thus the infinity point \( \infty \) uniquely exists in \( X \). \( G_m \) acts on \( R \) by setting \( g_m^a x, g_m^a y_1, \) for \( x, y_1, g_m \in G_m \) and \( a = 4, 5 \). By Nagata’s Jacobi-method \([16]\), it can be proved that \( X \) is non-singular.
Though they do not explicitly appear, we may also implicitly consider parametrizations of \( y_4 \) and \( y_5 \) by \( y_4 = w_2 w_1^2 \) and \( y_5 = w_2^2 w_1 \), where \( w_1 = k_1 \) and \( w_2 = k_2 \).

When we consider \( \bar{R} := \mathbb{C}[x, w_1, w_2]/(w_1^3 - k_1(x), w_2^2 - k_2(x)) \), it is related to a natural covering of \( X \).

### 3.1. The Weierstrass Gap and Holomorphic One Forms

The Weierstrass gap sequences at \( \infty \) are given in Table 1. For the local parameter \( t_\infty \) at \( \infty \), we have

\[
x = \frac{1}{t_\infty}, \quad y_4 = \frac{1}{t_\infty^2}(1 + d_2(t_\infty)), \quad y_5 = \frac{1}{t_\infty}(1 + d_2(t_\infty)).
\]

Here for a given local parameter \( t \) at some \( P \) of \( X \), the series of \( t \), whose orders of zero at \( P \) are greater than \( t \) or equal to \( t \), is denoted by \( d_2(t') \). \( H(\infty) \) in (1) is \( H(3, 4, 5) \) as Pinkham considered \((3, 4, 5)\) curve as the simplest example of the numerical semigroup \( H(3, 4, 5) \) [23, Section 14]. Its monomial curve is defined by, \( Z_4^4 = Z_5^2 Z_3, Z_4 Z_5 = Z_4^3, Z_3^2 = Z_4^2 Z_3 \), or the 2 \times 2 minor of \( [Z_4 \ Z_5 \ Z_3] \).

\( Z_4 \) and \( Z_5 \) correspond to \( \frac{1}{x^4} \) and \( \frac{1}{y_5} \) respectively and these relations correspond to (2).

| \( x_4 \) | 1 | - | - | - | - | - | - | - | - | - |
| \( y_4 \) | \( x^2 \) | \( x^2 y \) | \( x^3 \) | \( x^3 y \) | \( x^4 \) | \( x^4 y \) | \( x^5 \) | \( x^5 y \) | \( x^6 \) | \( x^6 y \) |
| \( x_5 \) | 1 | - | - | - | - | - | - | - | - | - |
| \( y_5 \) | \( y^2 \) | \( y^2 x \) | \( y^3 \) | \( y^3 x \) | \( y^4 \) | \( y^4 x \) | \( y^5 \) | \( y^5 x \) | \( y^6 \) | \( y^6 x \) |

There we define \( \phi_{i_1}^{(g)} \) as a non-gap monomial in \( \bar{R}_g \) for \( g = 2, 3, 4 \) and e.g., \( \phi_{1}^{(2)} = 1 \), \( \phi_{1}^{(2)} = x \), \( \phi_{1}^{(2)} = y_4 \), \( \phi_{1}^{(2)} = y_5 \), \( \phi_{1}^{(2)} = x^2 \), \( \phi_{1}^{(2)} = x y_4 \), \( \phi_{1}^{(2)} = x y_5 \). We introduce the weight \( N^{(g)}(\alpha) \) by letting \( N^{(g)}(\alpha) := -\text{wt}(\phi_{i_1}^{(g)}) \), where \( \text{wt}() \) is the degree of divisor at \( \infty \) of each curve \( \bar{X} \). It is noted that \( H_2 \) is identical to \( \{ N^{(2)}(\alpha) ; n = 0, 1, 2, \ldots \} \). For later convenience, we also introduce \( \phi_{i_1} \in \bar{R} \) (\( i = 1, 2, 3, \ldots \)) by \( \phi_{i_1} := y_4 \), \( \phi_{i_1} := y_5 \), \( \phi_{i_1} := x y_4 \), \( \phi_{i_1} := x y_5 \), for \( i > 3 \), \( \phi_{i_1} := \left\{ \begin{array}{ll} x^{[i-4]/3}y_4 & i \equiv 1 \text{ mod } 3, \\
x^{[i+1]/3}y_4 & i \equiv 2 \text{ mod } 3, \\
x^{i/3}y_5 & i \equiv 0 \text{ mod } 3. \end{array} \right.$
We also define the weight $N_{H^1}(n)$ by $N_{H^1}(n) := -\text{wt}((f_{H^1}))$. $N_{H^1}(0) = 4$, $N_{H^1}(1) = 5$, and $N_{H^1}(n) = n + 5$ for $n \geq 2$. By letting
\[
\Lambda^{(2)} := N_{H^1}(2) - N_{H^1}(i - 1) + i - 3, \\
\Lambda^{(3)} := N^{(2)}(g) - N^{(2)}(i - 1) - g + i - 1, \quad (g = 3, 4)
\]
the corresponding Young diagrams, $\Lambda \equiv \Lambda^{(2)} := (A_1, A_2) = (1, 1)$, $\Lambda^{(3)} := (A^{(3)}_1, A^{(3)}_2, A^{(3)}_3) = (3, 1, 1)$ and $\Lambda^{(4)} := (A^{(4)}_1, A^{(4)}_2, A^{(4)}_3, A^{(4)}_4) = (4, 2, 1, 1)$ are given respectively as

The Young diagram $\Lambda$ is not symmetric, whereas $^{\dagger}\Lambda^{(3)} = \Lambda^{(3)}$ and $^{\dagger}\Lambda^{(4)} = \Lambda^{(4)}$.

Then the following propositions are obvious

**Proposition 6.** The $b$-small bases of the holomorphic one forms over $X$ are expressed by $\nu_i = \frac{dx}{3y} \text{ and } \nu_j = \frac{dy}{3x}$, or $\nu_i = \frac{\phi d\nu}{3y} \text{ and } \nu_j = \frac{\phi d\nu}{3x}$, $i = 1, 2$.

We note their divisors and linear equivalences for $B_a := (b_a, 0, 0)$, $a = 0, 1, 2$.

We choose the bases $\nu_i \equiv \nu_j \equiv \nu_k \equiv \nu_l \equiv (dx/y^2) = 2(3\infty - B_1 - B_2)$ and $(\nu_i') \equiv (\nu_j') \equiv (\nu_k') \equiv (\nu_l') \equiv B_1 + B_2 \sim (dx/y^2) = 2(2\infty - B_0) = 2(\infty + (\infty - B_0))$.

**Proposition 7.** $\sum a_i \hat{\nu}_i$ belongs to $H^1(X \setminus \infty, O_X)$, where $\hat{\nu}_i := \frac{\alpha_{1i} dx}{3y g_{1i}}$ and the order of the singularity of $(\hat{\nu}_i)$ at $\infty$ is given by $N_{H^1}(n) - 5$.

**Lemma 8.** $a_0 \frac{dx}{3y g_0} \equiv a_1 \frac{xdx}{9y g_1} \equiv a_2 \frac{x^2 dx}{9y g_2}$ is not holomorphic one form over $X$ if $a_i$ does not vanish.

**Proof:** For $n < 3$, every $\sum_{i=1}^n a_i \frac{dx}{3y g_i}$ has singularities at points in $X \setminus \infty$. ■

We choose the bases $\alpha_i, \beta_j$ $(1 \leq i, j \leq 2)$ of $H_1(X, \mathbb{Z})$ such that their intersection numbers are $\alpha_i \cdot \alpha_j = \beta_i \cdot \beta_j = 0$ and $\alpha_i \cdot \beta_j = \delta_{ij}$, and we denote the period matrices by $[\omega^i \omega^j] = \frac{1}{2} \left[ \left\langle \nu^i_j, \nu^k_l \right\rangle \right]_{i,j=1,2}$. Let $\Pi_2$ be a lattice generated by $\omega^i$ and $\omega^j$. For a point $P \in X$, the abelian map $\tilde{u}_{\alpha} : X \to \mathbb{C}^2$ is defined by $\tilde{u}_{\alpha}(P) = \int_{\infty}^{P} \nu^j_i \in \mathbb{C}^2$
and for a point \((P_1, \cdots, P_k) \in S^k X\), i.e., the \(k\)-th symmetric product of \(X\), the shifted abelian map \(\hat{u} : S^k X \to C^2\) by

\[\hat{u}(P_1, \cdots, P_k) := \hat{u}_0(P_1, \cdots, P_k) + \hat{u}_0(B_0)\]

where \(\hat{u}_0(P_1, \cdots, P_k) := \sum_{i=1}^{k} \hat{u}_0(P_i)\). Then we define the Jacobian \(J_2\) and its subvariety \(W_k\), \(k = 0, 1, 2\) by

\[\kappa : C^2 \to J_2 = C^2 / \Pi_2 = W_2, \quad W_k := \kappa \hat{u}(S^k X)\]

respectively. Further the singular locus of \(S^2 X\) is denoted by \(S^2 X\) as in [20].

For a point \((P_1, P_2) \in S^2 X\) around the infinity point, by letting their local parameters \(t_{\infty,1}\) and \(t_{\infty,2}\), \(u \equiv (u_1, u_2) := \hat{u}_0(P_1, P_2)\) is given by

\[u_1 = \frac{1}{2} (t_{\infty,1}^2 + t_{\infty,2}^2)(1 + d_{\infty}(t_{\infty,1}, t_{\infty,2})), \quad u_2 = (t_{\infty,1} + t_{\infty,2})(1 + d_{\infty}(t_{\infty,1}, t_{\infty,2})), \quad d_{\infty}(t_1, t_2) \text{ is a natural extension of } d_\infty(t)\]

3.2. Differentials of the Second and the Third Kinds

Following the EEL-construction [5] for an \((n, s)\) curve, we give here an algebraic representation of a differential form which is equal to the fundamental normalized differential of the second kind in [7, Corollary 2.6], up to a tensor of holomorphic one forms

**Definition 9.** A two-form \(\Omega(P_1, P_2)\) on \(X \times X\) is called a fundamental differential of the second kind if it is symmetric, \(\Omega(P_1, P_2) = \Omega(P_2, P_1)\), it has its only pole (of second order) along the diagonal of \(X \times X\), and in the vicinity of each point \((P_1, P_2)\) is expanded in power series as

\[\Omega(P_1, P_2) = \left(\frac{1}{(t_{P_1} - t_{P_2})^2 + d_{\infty}(1)}\right) dt_{P_1} \otimes dt_{P_2}, \quad \text{as } P_1 \to P_2\]

where \(t_P\) is a local coordinate at a point \(P \in X\).

Here we use the convention that for \(P_0 \in X\), \(P_0\) is represented by \((x_0, y_{4,a}, y_{5,a})\) or \((x_{P_0}, y_{4,P_0}, y_{5,P_0})\) and for \(P \in X\), \(P\) is expressed by \((x, y_4, y_5)\). Then the following propositions holds.

**Proposition 10.** By letting

\[\Sigma(P, Q) := \frac{y_{4,Q} y_{4,P} + y_{4,P} y_{4,Q} + y_{4,Q} y_{4,P}}{(x_P - x_Q)^2 y_{4,P} y_{4,Q}}\]

\(\Sigma(P, Q)\) has the properties

\[\Sigma(P, Q)\]
Proposition 11.

1) \( \Sigma(P, Q) \) as a function of \( P \) is singular at \( Q = (x_Q, y_Q, \nu_Q) \) and \( \infty \), and vanishes at \( \xi(Q) = (x_Q, \xi_1(Q), \xi_2(Q), \ldots, \xi_l(Q)), l = 1, 2, \ldots \).

2) \( \Sigma(P, Q) \) as a function of \( Q \) is singular at \( P = \infty \).

Proof: Direct computations provide the results.

\[ \text{Proposition 11. There exist differentials } \nu_i^{(k)} = \nu_i^{(k)}(x, y, \nu), j = 1, 2 \text{ of the second kind such that they have their only pole at } \infty \text{ and satisfy the relation} \]

\[ d_0\Sigma(P, Q) - d_0\Sigma(Q, P) = \sum_{i=1}^2 \left( \nu_i^{(k)}(Q) \otimes \nu_i^{(k)}(P) - \nu_i^{(k)}(P) \otimes \nu_i^{(k)}(Q) \right) \]

where \( d_0\Sigma(P, Q) := dx_P \otimes dx_Q \frac{\partial}{\partial x_Q} \frac{y_4(P) + y_4(Q) + y_4(x, y, \nu, P)}{(x - x_Q)^3 y_4(P, Q)} \).

The differentials \( \{\nu_i^{(k)}, \nu_i^{(k)}\} \) are determined modulo the \( \mathbb{C} \)-linear space spanned by \( \{\nu_1^{(k)}, \nu_2^{(k)}\} \) and we fix

\[ \{\nu_i^{(k)}, \nu_i^{(k)}\} = \left\{ \frac{\left(2x + \lambda_i^{(2)}\right)dx}{3y_4}, -\frac{xdx}{3y_4} \right\} \]

as their representative.

Proof: \( \frac{\partial}{\partial x_Q} \frac{y_4(P) + y_4(Q) + y_4(x, y, \nu, P)}{(x - x_Q)^3 y_4(P, Q)} \) is equal to

\[ \frac{1}{(x - x_Q)^3 y_4(P, Q)} \left[ 3\left(\frac{y_4(P) + y_4(Q) + y_4(x, y, \nu, P)}{x - x_Q}\right) y_4(P, Q) \right. \]

\[ \left. + \left(\frac{y_4(P)}{y_4(Q)}(2k_2 \cdot k_2' + k_2 \cdot k_2') + y_4(P) \frac{y_4(P)}{y_4(Q)}(2k_2 \cdot k_2' + k_2 \cdot k_2') \right) \right] \]

Here \( k_{a, P} = k_{a}(x_P) \) and \( k_{a, P}' = \frac{dk_{a}(x_P)}{dx_P} \). We have

\[ \frac{\partial}{\partial x_Q} \frac{y_4(P) + y_4(Q) + y_4(x, y, \nu, P)}{(x - x_Q)^3 y_4(P, Q)} = \frac{\partial}{\partial x_P} \frac{y_4(P) + y_4(Q) + y_4(x, y, \nu, P)}{(x - x_P)^3 y_4(P, Q)} \]

\[ = \frac{1}{(x - x_Q)^3 y_4(P, Q)} \left( B_2(P, Q) - B_2(Q, P) \right) \]

where \( B_2(P, Q) = y_4(P, Q) \left(2x_Q + \lambda_i^{(2)} - x_P\right) \). Then we obtain the statements.
Corollary 12. 1) The one form, \( \Omega_{P_1}^{P_2}(P) := \Sigma(P, P_1) - \Sigma(P, P_2) \), is a differential of the third kind, whose only (first-order) poles are \( P = P_1 \) and \( P = P_2 \), and residues +1 and −1 respectively.

2) \( \Omega(P_1, P_2) \) is defined by
\[
\Omega(P_1, P_2) = \frac{F(P_1, P_2)dx_1 \otimes dx_2}{(x_{P_1} - x_{P_2})} + \sum_{i=1}^{2} \nu_i^I(P_1) \otimes \nu_i^J(P_2)
\]

where \( F \) is an element of \( R \otimes R \).

Proof: Direct computations give the claims.

Lemma 13. We have
\[
\lim_{P_1 \to \infty} \frac{F(P_1, P_2)}{\phi_{P_1}(P_1)(x_{P_1} - x_{P_2})^2} = \phi_{P_2}(P_2) = x_{P_2}y_4, P_2.
\]

Proof: \( B_2 \) which appears in the proof of Proposition 11 ensures the result.

For later convenience we introduce the quantity
\[
\Omega_{P_1, P_2}^{Q_1, Q_2} = \int_{P_1}^{P_2} \int_{Q_1}^{Q_2} \Omega(P, Q),
\]

\[
\Omega_{Q_1, Q_2}^{P_1, P_2} = \int_{P_1}^{P_2} (\Sigma(P, Q_1) - \Sigma(P, Q_2)) + \sum_{i=1}^{2} \int_{Q_1}^{Q_2} \nu_i^I(P) \int_{P_1}^{P_2} \nu_i^J(P). \tag{6}
\]

4. The sigma Function for \((3, 4, 5)\) Curve

4.1. Generalized Legendre Relation

Corresponding to the complete integral of the first kind, we define the complete integral of the second kind
\[
\left[ \nu^I \nu^J \right] := \frac{1}{2} \left[ \int_{Q_1}^{Q_2} \nu^I \int_{P_1}^{P_2} \nu^J \right]_{i, j = 1, 2}.
\]

Let \( \tau_{Q_1, Q_2} \) be the normalized differential of the third kind such that \( \tau_{Q_1, Q_2} \) has residues +1 and −1 at \( Q_1 \) and \( Q_2 \) respectively, is regular everywhere else, and is normalized \( \int_{P_1}^{P_2} \tau_{PQ} = 0 \) for \( i = 1, 2 \) [7, p.4]. The following Lemma corresponding to Corollary 2.6 (ii) in [7] holds


Lemma 14. By letting $\gamma = \omega^{-1} \eta'$, we have

$$\Omega_{P_1 Q_1, P_2 Q_2} = \int_{P_2}^{P_1} \tau_{Q_1 Q_2} + \sum_{i,j=1}^{3} \gamma_{ij} \int_{Q_1}^{Q_2} \nu_i \nu_j.$$

Proof: The same as [20, I: Lemma 4.1].

The following Proposition provides a symplectic structure in the Jacobian $\mathcal{J}_2$, known as the generalized Legendre relation [3, 4, 20]

Proposition 15. $M \begin{bmatrix} -1 \\ 1 \end{bmatrix} M = 2\pi \sqrt{-1} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ for $M := \begin{bmatrix} 2\omega' & 2\omega'' \\ 2\eta' & 2\eta'' \end{bmatrix}$.

Proof: The same as [20, I: Proposition 4.2].

4.2. The $\sigma$ Function

Due to the Riemann relations [7], $\text{Im}(\omega^{-1}\omega')$ is positive definite. Theorem 1.1 in [7] gives $\delta := \begin{bmatrix} \delta'' \\ \delta' \end{bmatrix} \in \frac{Z}{2} \leq$ the theta characteristic which is equal to the Riemann constant $\xi_R$ and the period matrix $2[\omega' \omega'']$. We note that $\xi_R = \hat{u}(P_R)$ for a point $P_R \in X$ satisfying $2P_R + 2B_0 - 4\infty \sim 0$. We define an entire function of (a column-vector) $u = (u_1, u_2) \in \mathbb{C}^2$

$$\sigma(u) = ce^{-\frac{1}{2} \omega \omega^{-1} \omega''} \sum_{n \in \mathbb{Z}^2} \tau \psi^{-1} \begin{bmatrix} \eta(u + \delta') \omega^{-1} \omega''(u + \delta') + \eta(u + \delta') \omega^{-1} u + \delta' \end{bmatrix}$$

where $c$ is a certain constant as in (7).

For a given $u \in \mathbb{C}^2$, we introduce $u'$ and $u''$ in $\mathbb{R}^2$ so that $u = 2\omega' u' + 2\omega'' u''$.

Proposition 16. For $u, v \in \mathbb{C}^2$, $\ell \in \mathbb{R}^2 \setminus 2[\omega' \omega'']$, and introducing $L(u, v) := 2u(\eta' \omega' + \eta'' \omega'')$, $\chi(\ell) := \exp(\pi \sqrt{-1}(2\ell \omega' \omega' - 2\ell \delta'' + \ell \delta'))$, we have the translational relation

$$\sigma(u + \ell) = \sigma(u) \exp(L(u + \frac{1}{2} \ell, \ell)) \chi(\ell).$$

Proof: The same as [20, I: Proposition 4.3].

The vanishing locus of $\sigma$ is simply given by $\Theta^A := (\mathcal{W}^A \cup [-1]\mathcal{W}^A) = \mathcal{W}^A$. 


4.3. The Riemann Fundamental Relation

As in [20, I: Proposition 4.4], we have the Riemann fundamental relation

**Proposition 17.** For \((P, Q, P_i, P_i') \in X^2 \times (S^2(X) \setminus S^2(X)) \times (S^2(X) \setminus S^2(X))\)

\[
\exp \left( \sum_{i,j=1}^2 \Omega_{P_i, P_i'} \right) = \frac{\sigma \left( \sigma(P) - \tilde{u}(P, P_2) \sigma(P, P_2) \right)}{\sigma \left( \sigma(Q) - \tilde{u}(P_1, P_2) \sigma(P, P_2) \right)}
\]

Using the differential identity:

\[
\sum_{i,j=1}^2 \phi_{M+1,i} (P_i') \phi_{M,j-1} (P_i') \frac{\partial^2}{\partial u_i \partial u_j}
\]

\[
= y_{P} y_{P_i} y_{P_i'} y_{P_i'} \frac{\partial^2}{\partial u_i \partial u_j}
\]

taking logarithm of both sides of the relation and differentiating them along \(P_i = P\) and \(P_i' = P_2\), we have the differential expressions of the relation, as mentioned in [20, I: Proposition 4.5]

**Proposition 18.** For \((P, P_i, P_i') \in X \times S^2(X) \setminus S^2(X)\) and \(u := \tilde{u}(P, P_2)\), the equality

\[
\sum_{i,j=1}^2 \psi_{i,j} (u) \phi_{M+1,i} (P) \phi_{M,j-1} (P) = \frac{F(P, P_2)}{(x-x_2)^2}
\]

holds for every \(a = 1, 2\), where we set

\[
\psi_{i,j}(u) := -\frac{\sigma(u) \sigma_j(u) - \sigma(u) \sigma_i(u)}{\sigma(u)^2} = -\frac{\partial^2}{\partial u_i \partial u_j} \log \sigma(u).
\]

4.4. Jacobi Inversion Formulae

As in [20], we introduce meromorphic functions on the curve \(X\)

**Definition 19.** For \(P, P_1, \ldots, P_n \in (X \setminus \infty) \times S^n(X \setminus \infty), n = 1, 2\), we define

\[
\mu_1(P; P_i) := y_1 \frac{y_1}{y_4}, \\
\mu_2(P; P_i, P_i') := xy_1 \frac{y_1 y_2 y_4 - y_1 y_2 y_4 - y_1 y_2 y_4}{y_4 y_1 y_4 - y_4 y_1 y_4 - y_4 y_1 y_4}
\]

We note that when \(n = 1\) is characterized by the condition on the polynomial \(\mu_n = \sum_{a_i \in \mathbb{C}} a_i \phi_{M_1}(P_i)\), \(a_i \in \mathbb{C}\) and \(a_n = 1\), which has a zero at each point \(P_i\) and...
Theorem 20. 1) For \((P_1, P_2) \in X \times \left( S^2(X) \setminus S^2(X) \right)\), we have

\[ \mu_2(P; P_1, P_2) = xy - \varphi_2(\hat{u}(P_1, P_2))y_4 + \varphi_2(\hat{u}(P_1, P_2))y_5. \]

2) For \((P, P_1) \in X \times \left( X \right)\) and \(u = \hat{u}(P_1) \in \kappa^{-1}(\mathcal{W})\)

\[ \mu_1(P; P_1) = y_5 - \frac{\sigma_1(u)}{\sigma_2(u)}y_4, \quad \text{and} \quad \frac{\sigma_1(u)}{\sigma_2(u)} = \frac{y_5}{y_4}. \]

Proof: 1) is the same as [20, I: Proposition 4.6]. As in [20, I: Theorem 5.1], by considering \( \lim_{P_2 \to \infty} \varphi_2(\hat{u}(P_1, P_2)) \), we have the second result.

Following the statement by Buchstaber, Leykin and Enolski, Nakayashiki showed that the leading of the sigma function for \((r, s)\) curve is expressed by Schur function [21]. Noting (3) and degrees of \(u\), the above Jacobi inversion formulae gives

\[ \sigma(u) = \frac{1}{2} y_2^{u^2} - u_1 + \sum_{|\alpha| \geq 2} a_\alpha u^\alpha \]

where \(a_\alpha \in \mathbb{Q}[b_1, \ldots, b_5], \alpha = (\alpha_1, \alpha_2), |\alpha| = \alpha_1 + \alpha_2 \text{ and } u^\alpha = u_1^{\alpha_1} u_2^{\alpha_2}. \) The prefactor \(c\) is determined by this relation. Since for a Young diagram \(\Lambda\), \(S_\Lambda\) and \(s_\Lambda\) are the Schur functions defined by

\[ S_\Lambda(T_1, T_2) = t_1 t_2 - \frac{1}{2} T_1^2 - T_2 \]

where \(T_1 := t_1 + t_2\) and \(T_2 := \frac{1}{2}(t_1^2 + t_2^2)\), we have

\[ \sigma(u) = S_\Lambda(u_1, u_2) + \sum_{|\alpha| \geq 2} a_\alpha u^\alpha. \]
Remark 21. We showed that the EEL construction works well even for a space curve, and the sigma function associated with the curve is naturally defined. Since this construction is very natural, this study sheds a new light on the way to construction of the sigma functions for space curves. We conjectured that the EEL construction could be applied to every space curve if it is Weierstrass.

As an interesting example of a space curve, we will give a comment on a problem as follows, for which we started to study sigma functions for affine space curves. McKay considers a relation between dispersionless KP hierarchy and the replicable functions in order to obtain a further profound interpretation of the moonshine phenomena of Monster group [22]. He conjectured that it might be related to the quantised elista [18, 19]. By studying a relation between a replicable function and an algebraic curve associated with elista, Matsutani found that a semigroup \( H_{12} \) generated by \( M_{12} := \langle 6, 13, 14, 15, 16 \rangle \) has gap sequence, \( L(H_{12}) = \{1, 2, 3, 4, 5, 7, 8, 9, 10, 11, 17, 23\} \), which is identical to the Norton number, \( N_{12} := \{1, 2, 3, 4, 5, 7, 8, 9, 11, 17, 19, 23\} \) by exchanging 10 and 19. The Norton number plays the essential role in the moonshine phenomena for the Monster group [22]. The replicable function is given as an element of \( \mathbb{Q}[a_1, a_2, a_3, a_4, a_5, a_7, a_8, a_9, a_{11}, a_{17}, a_{19}, a_{23}]\). The replicable function is a generalization of the elliptic \( J \)-function, which causes the moonshine phenomena of the Monster group.

After then, Komeda proved that \( H_{12} \) is the Weierstrass semigroup and gave the fundamental relations Propositions 23 as given in Appendix, which is reported more precisely in [13]. Then we applied the EEL-construction to the curve and obtain a sigma function for a Jacobi variety \( J_{12} \) for \( C(H_{12}) \) [13]. Since the Jacobi variety \( J_{12} \) is given as 12-dimensional complex torus whose real dimension is 24, it might remind us of Witten conjecture associated with Monster group problem [8]; Witten conjectured that a 24 dimensional manifold exists such that the Monster group acts on it via Weierstrass sigma function.

Appendix (by J. Komeda). Weierstrass Properties of \( \langle 6, 13, 14, 15, 16 \rangle \)

The proofs of these propositions are given in the article [13]. Here we show only the sketch of the first one because the second one is not difficult.

Proposition 22. The numerical semigroup \( \langle 6, 13, 14, 15, 16 \rangle \) is Weierstrass.

Proof: Let \( (C, P) \) be a pointed curve with \( H(P) = \langle 3, 7, 8 \rangle \). Then

\[
2 = h^0(4P) = 4 + 1 - 4 + h^0(K - 4P) = 1 + h^0(K - 4P)
\]
which implies that \( K - 4P \sim P_1 + P_2 \) for some points \( P_1 \) and \( P_2 \in C \). Here \( K \) is a canonical divisor on \( C \). Moreover,

\[
2 = h^0(5P) = 5 + 1 - 4 + h^0(K - 5P) = 2 + h^0(K - 5P)
\]

which implies that \( h^0(K - 5P) = 0 \). Hence, we get \( P_i \neq P \) for \( i = 1, 2 \). Thus, \( K \sim 4P + P_1 + P_2 + P_i \) with \( P_i \neq P \) for \( i = 1, 2 \). We set \( D = 7P - P_1 - P_2 \). Then \( \deg(2D - P) = 9 = 2 \times 4 + 1 \), which implies that the complete linear system \( [2D - P] \) is very ample, hence base-point free. Therefore, \( 2D \sim P + Q_1 + \ldots + Q_6 \) (= a reduced divisor). Let \( \mathcal{L} \) be the invertible sheaf \( \mathcal{O}_C(-D) \) on \( C \) and \( \phi \) an isomorphism \( \mathcal{L}^{\otimes 2} \cong \mathcal{O}_C(-P - Q_1 - \ldots - Q_6) \subset \mathcal{O}_C \). Then the vector bundle \( \mathcal{O}_C \oplus \mathcal{L} \) has an \( \mathcal{O}_C \)-algebra structure through \( \phi \). The canonical morphism \( \pi : \tilde{C} = \text{Spec}(\mathcal{O}_C \oplus \mathcal{L}) \to C \), is a double covering. Its branch locus of \( \pi \) is \( \{ P, Q_1, \ldots, Q_6 \} \).

Let \( \tilde{P} \) be the ramification point of \( \pi \) over \( P \). Then it can be showed that \( H(\tilde{P}) = (6, 13, 14, 15, 16) \) using the formula, \( h^0(2n\tilde{P}) = h^0(nP) + h^0(nP - D) \) for any non-negative integer \( n \).

By considering \( h^0(2n\tilde{P}) \) for \( n = 3, 4, 5, 6, 7, 8, 9 \), we show \( H(\tilde{P}) = (6, 13, 14, 15, 16) \).

**Proposition 23.** Let \( B_{12} \) a monomial ring which is given by \( k[\sigma^a_{a \in M_{12}}] \) for the numerical semigroup \( H_{12} \). For a \( k \)-algebra homomorphism

\[
\varphi_{12} : k[Z] := k[Z_6, Z_{13}, Z_{14}, Z_{15}, Z_{16}] \to k[\sigma^a_{a \in M_{12}}]
\]

where \( Z_a \) is the weight of \( a = 6, 13, 14, 15, 16 \), the kernel of \( \varphi_{12} \) is generated by the following relations \( f_{12}^{(b)} \) \((b = 1, \ldots, 9)\)

\[
\begin{align*}
&f_{12}^{(1)} = Z_6 - Z_6^2, & f_{12}^{(2)} = Z_{13} - Z_6^2 Z_{15}, & f_{12}^{(3)} = Z_{14} - Z_6^2 Z_{15}, \\
&f_{12}^{(4)} = Z_6^2 - Z_6^3 Z_{14}, & f_{12}^{(5)} = Z_{13} - Z_6^3 Z_{15}, & f_{12}^{(6)} = Z_{14} - Z_6^3 Z_{15}, \\
&f_{12}^{(7)} = Z_6 Z_{16} - Z_6^2, & f_{12}^{(8)} = Z_{15} - Z_6 Z_{16}, & f_{12}^{(9)} = Z_6^2 - Z_6^3 Z_{13}, \\
&f_{12}^{(10)} = Z_6 Z_{16} - Z_6^2, & f_{12}^{(11)} = Z_{15} - Z_6 Z_{16}, & f_{12}^{(12)} = Z_6^2 - Z_6^3 Z_{13}.
\end{align*}
\]

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