Geometry of $k$-harmonic maps and the second variational formula of the $k$-energy

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Abstract

In [6], J.Eells and L. Lemaire introduced $k$-harmonic maps, and T. Ichiyama, J. Inoguchi and H.Urakawa [1] showed the first variation formula. In this paper, we give the second variation formula of $k$-harmonic maps, and show non-existence theorem of proper $k$-harmonic maps into a Riemannian manifold of non-positive curvature ($k \geq 2$). We also study $k$-harmonic maps into the product Riemannian manifold, and describe the ordinary differential equations of $3$-harmonic curves and $4$-harmonic curves into a sphere, and show their non-trivial solutions.

Introduction

Theory of harmonic maps has been applied into various fields in differential geometry. The harmonic maps between two Riemannian manifolds are critical maps of the energy functional $E(\phi) = \frac{1}{2} \int_M \|d\phi\|^2 v_g$, for smooth maps $\phi : M \to N$.

On the other hand, in 1981, J. Eells and L. Lemaire [6] proposed the problem to consider the $k$-harmonic maps: they are critical maps of the functional $E_k(\phi) = \int_M e_k(\phi) v_g$, $(k = 1, 2, \cdots)$, where $e_k(\phi) = \frac{1}{2} \|(d + d^*)^k \phi\|^2$ for smooth maps $\phi : M \to N$. G.Y. Jiang [5] studied the first and second variation formulas of the bi-energy $E_2$, and critical maps of $E_2$ are called biharmonic maps. There have been extensive studies on biharmonic maps.

Recently, in 2009, T. Ichiyama, J. Inoguchi and H. Urakawa [1] studied the first variation formula of the $k$-energy $E_k$, whose critical maps are called $k$-harmonic maps. Harmonic maps are always $k$-harmonic maps by definition. In this paper, we study $k$-harmonic maps and show the second variational formula of $E_k$.

In [1] we introduce notation and fundamental formulas of the tension field and the $k$-stress energy tension field.

In [2] we recall $k$-harmonic maps, and state the second variation formula of the $k$-energy $E_k$ which will be proved in [3].

In [4] we show the non-existence theorem of proper $k$-harmonic maps into a Riemannian manifold of non-positive curvature $(k \geq 2)$.

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In §5, we introduce the notion of stable $k$-harmonic maps, and show the non-existence theorem of non-trivial stable $k$-harmonic maps into constant sectional curvature manifolds.

In §6, we show the reduction theorem of $k$-harmonic maps into the product spaces.

Finally, in §7, we derive the necessary and sufficient condition to be $k$-harmonic maps into a sphere, and determine the ODEs of the 3-harmonic and 4-harmonic curve equations into a sphere, and show their non-trivial solutions, respectively.

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1 Preliminaries

Let $(M, g)$ be an $m$ dimensional Riemannian manifold, $(N, h)$ an $n$ dimensional one, and $\phi : M \to N$, a smooth map. We use the following notation. The second fundamental form $B(\phi)$ of $\phi$ is a covariant differentiation $\tilde{\nabla}d\phi$ of 1-form $d\phi$, which is a section of $\otimes^2 T^*M \otimes \phi^{-1}TN$. For every $X,Y \in \Gamma(TM)$, let

$$B(X,Y) = (\tilde{\nabla}d\phi)(X,Y) = (\tilde{\nabla}_Xd\phi)(Y) = \nabla_X\phi(Y) - d\phi(\nabla_XY).$$

(1)

Here, $\nabla$, $\nabla^N$, $\tilde{\nabla}$ are the induced connections on the bundles $TM$, $TN$, $\phi^{-1}TN$, respectively.

If $M$ is compact, we consider critical maps of the energy functional

$$E(\phi) = \int_M e(\phi)v_g,$$

where $e(\phi) = \frac{1}{2}\|d\phi\|^2 = \sum_{i=1}^m \frac{1}{2}(d\phi(e_i),d\phi(e_i))$ which is called the energy density of $\phi$, and the inner product $\langle \cdot, \cdot \rangle$ is a Riemannian metric $h$. The tension field $\tau(\phi)$ of $\phi$ is defined by

$$\tau(\phi) = \sum_{i=1}^m (\tilde{\nabla}d\phi)(e_i,e_i) = \sum_{i=1}^m (\tilde{\nabla}_e_i d\phi)(e_i).$$

(3)

Then, $\phi$ is a harmonic map if $\tau(\phi) = 0$.

The curvature tensor field $\tilde{R}(\cdot,\cdot)$ of the Riemannian metric on the bundle $T^*M \otimes \phi^{-1}TN$ is defined as follows:

$$\tilde{R}(X,Y) = \tilde{\nabla}_X\tilde{\nabla}_Y - \tilde{\nabla}_Y\tilde{\nabla}_X - \tilde{\nabla}_{[X,Y]}, \quad (X,Y \in \Gamma(TM)).$$

(4)

Furthermore, we define the following: For any $Z \in \Gamma(TM)$,

$$\tilde{R}(X,Y)d\phi)(Z) = R^{\phi^{-1}TN}(X,Y)d\phi(Z) - d\phi(R^M(X,Y)Z)$$

$$= R^N(d\phi(X),d\phi(Y))d\phi(Z) - d\phi(R^M(X,Y)Z),$$

(5)
where $R^M, R^N$ and $R^{\phi-1TN}$ are the Riemannian curvature tensor fields on $TM, TN$ and $\phi^{-1}TN$, respectively. And we define

$$\Delta = \tilde{\nabla}^* \tilde{\nabla} = - \sum_{k=1}^{m} (\tilde{\nabla}_{e_k} \tilde{\nabla}_{e_k} - \tilde{\nabla}_{\nabla_{e_k} e_k}), \quad (6)$$

is the rough Laplacian.

A section of $\mathcal{O}^2 T^* M$ defined by $S_{\phi} = e(\phi)g - \phi^* h$ is called the stress-energy tensor field, and $\phi$ is said to satisfy the conservation law if $\text{div} S_{\phi} = 0$. As in [6], it holds that

$$(\text{div} S_{\phi})(X) = - \langle \tau(\phi), d\phi(X) \rangle, \quad (X \in \Gamma(TM)), \quad (7)$$

$k$-harmonic maps

J. Eells and L. Lemaire [6] proposed the notation of $k$-harmonic maps. The Euler-Lagrange equations for the $k$-harmonic maps was shown by T. Ichiyama, J. Inoguchi and H. Urakawa [1]. We first recall it briefly.

Definition 2.1 ([6]). For $k = 1, 2, \cdots$ the $k$-energy functional is defined by

$$E_k(\phi) = \frac{1}{2} \int_M \| (d + d^*)^k \phi \|^2 v_g, \quad \phi \in C^\infty(M, N).$$

Then, $\phi$ is $k$-harmonic if it is a critical point of $E_k$, i.e., for all smooth variation $\{\phi_t\}$ of $\phi$ with $\phi_0 = \phi$,

$$\frac{d}{dt} \bigg|_{t=0} E_k(\phi_t) = 0.$$

We say for a $k$-harmonic map to be proper if it is not harmonic.

Then, the first variational formula of $E_k$ can be obtained as follows. First, notice the following lemma which will be used to show Theorem 1.2.

Lemma 2.2 ([1]). For $k = 2, 3, \cdots$, the $k$-energy functional $E_k$ is given as follows:

Case 1: $k = 2l, \ l = 1, 2, \cdots$ ($k$ is even).

$$E_{2l}(\phi) = \frac{1}{2} \int_M \| \underbrace{\Delta \cdots \Delta}_{l-1} \tau(\phi) \|^2 v_g.$$

Case 2: $k = 2l + 1, \ l = 1, 2, \cdots$ ($k$ is odd).

$$E_{2l+1}(\phi) = \frac{1}{2} \int_M \| \nabla (\underbrace{\Delta \cdots \Delta}_{l-1} \tau(\phi)) \|^2 v_g.$$
Then, we have

**Theorem 2.3** (II). Let \( k = 2, 3, \ldots \). Then, we have

\[
\frac{d}{dt} \bigg|_{t=0} E_k(\phi_t) = -\int_M \langle \tau_k(\phi), V \rangle v_g,
\]

where

\[
\tau_k(\phi) := J \left( \Delta^{(k-2)} \tau(\phi) \right) = \Delta \left( \Delta^{(k-2)} \tau(\phi) \right) - \Re \left( \Delta^{(k-2)} \tau(\phi) \right),
\]

and

\[
\Delta^{(k-2)} \tau(\phi) = \underbrace{\Delta \cdots \Delta}_{k-2} \tau(\phi).
\]

As a corollary of this theorem, we have

**Corollary 2.4** (II). \( \phi : (M, g) \to (N, h) \) is a \( k \)-harmonic map if

\[
(8) \quad \tau_k(\phi) := J \left( \Delta^{(k-2)} \tau(\phi) \right) = \Delta \left( \Delta^{(k-2)} \tau(\phi) \right) - \Re \left( \Delta^{(k-2)} \tau(\phi) \right) = 0.
\]

Notice here that any harmonic map is \( k \)-harmonic.

We recall the results of Jiang [5] on the second variation formula of the \( 2 \)-energy \( E_2 \).

**Theorem 2.5** (II). Let \( \phi : M \to N \) be a 2-harmonic map from a compact Riemannian manifold \( M \) into an arbitrary Riemannian manifold \( N \), and \( \{ \phi_t \} \) an arbitrary \( C^\infty \) variation of \( \phi \) satisfying (II) (12). Then, the second variation formula of \( \frac{1}{2} E_2(\phi_t) \) is given as follows.

\[
\frac{d^2}{dt^2} \bigg|_{t=0} E_2(\phi_t) = \int_M \left( \nabla^N \nabla V + R^N(V, d\phi(e_i))d\phi(e_i), \right. \\
- \left. \nabla^N \nabla V + R^N(V, d\phi(e_i))d\phi(e_i) \right) v_g \\
+ \int_M \langle V, \nabla^N_{\tau(\phi)} R^N(V, d\phi(e_i))d\phi(e_i) \rangle v_g \\
+ \langle \nabla^N_{\tau(\phi)} R^N(V, d\phi(e_i))d\phi(e_i) \rangle v_g \\
+ R^N(V, \tau(\phi)) \tau(\phi) \\
+ 2 R^N(V, d\phi(e_k)) \nabla_{e_k} \tau(\phi) \\
+ 2 R^N(\tau(\phi), d\phi(e_i) \nabla_{e_i} V) v_g.
\]

(9) Then, we show the second variation formula of the \( k \)-energy \( E_k \).

**Theorem 2.6.** Let \( \phi : M \to N \) be a \( k \)-harmonic map from a compact Riemannian manifold \( M \) into an arbitrary Riemannian manifold \( N \), and \( \{ \phi_t \} \) an
arbitrary $C^\infty$ variation of $\phi$ satisfying (11), (12). Then, the second variation formula of $\frac{1}{2}E_k(\phi_t)$ is given as follows.

$$
\frac{d^2}{dt^2}E_k(\phi_t)\bigg|_{t=0} = \int_M \left\langle \nabla^2 \nabla V - R^N(V, d\phi(e_i))d\phi(e_i), \right.
\triangle^{k-2}\{(\nabla^2 \nabla^2 \nabla V - R^N(V, d\phi(e_i))d\phi(e_i))\} v_g \bigg\rangle
$$

(10) $$\int_M \left\langle V, \triangle^{k-2}\{(\nabla^2 \nabla^2 \nabla V - R^N(V, d\phi(e_i))V\tau(\phi) + R^N(V, d\phi(e_i))d\phi(e_i)\} + (\nabla^2 \nabla^2 \nabla V - R^N(V, d\phi(e_i))d\phi(e_i)\}
+ R^N(V, d\phi(e_i))V\tau(\phi) + 2R^N(V, d\phi(e_i))d\phi(e_i)\}
- 2R^N(V, d\phi(e_i))\nabla^2 \nabla^2 \nabla V \bigg\rangle v_g.
$$

3 Proof of the second variational formula of $k$-energy

In this section, we calculate the second variation formula of the $k$-energy $E_k$.

Assume that $\phi : M \rightarrow N$ is a smooth map, $M$ is a compact Riemannian manifold, $N$ and is a Riemannian manifold. First, let

$$
\phi_t : M \rightarrow N, \ t \in I_c = (-\epsilon, \epsilon), \epsilon > 0,
$$

be a $C^\infty$ one parameter variation of $\phi$ which yields a vector field $V \in \Gamma(\phi^{-1}TN)$ along $\phi$ in $N$ by

$$
\phi_0 = \phi, \ \frac{\partial \phi_t}{\partial t} \bigg|_{t=0} = V.
$$

The variation $\{\phi_t\}$ yields a smooth map $F : M \times I_c \rightarrow N$, which is defined by

$$
F(p, t) = \phi_t(p), \ (p \in M, \ t \in I_c).
$$

Taking the usual Euclidean metric on $I_c$, and the product Riemannian metric on $M \times I_c$, we denote by $\nabla, \nabla$, and $\nabla$, the induced Riemannian connection on $T(M \times I_c)$, $F^{-1}TN$ and $T^*(M \times I_c) \otimes F^{-1}TN$, respectively. If $\{e_i\}$ is an orthonormal frame field defined on a neighborhood $U$ of $p \in M$, $\{e_i, \frac{\partial}{\partial t}\}$ is also an orthonormal frame field on a coordinate neighborhood $U \times I_c$ in $M \times I_c$, and it holds that

$$
\nabla_{\frac{\partial}{\partial t}} e_i = 0, \nabla e_i e_j = \nabla e_i e_j = \nabla e_i \frac{\partial}{\partial t} = 0.
$$

It also holds that

$$
\frac{\partial F_t}{\partial t} = \frac{\partial F^a}{\partial t} \frac{\partial}{\partial y^a} = dF\left(\frac{\partial}{\partial t}\right), d\phi_t(e_i) = dF(e_i),
$$

where $F(\cdot)$ is a smooth map.
and

\[(\tilde{\nabla}_e_i d\phi_t)(e_j) = \nabla^N_{d\phi_t(e_i)} d\phi_t(e_j) - df_t(\nabla e_i e_j) = (\tilde{\nabla}_e_i dF)(e_j)\]

(16) \[(\tilde{\nabla}_e_k \tilde{\nabla}_e_i d\phi_t)(e_j) = \nabla^N_{d\phi_t(e_k)} ((\tilde{\nabla}_e_i d\phi_t)(e_j)) - (\tilde{\nabla}_e_i d\phi_t)(\nabla e_k e_j) = (\tilde{\nabla}_e_k \tilde{\nabla}_e_i dF)(e_j),\]

and so on. Then,

**Lemma 3.1** \([\text{[3]}]\). We have

\[
\nabla_{e_k}((\tilde{\nabla}_e_i dF)(e_i))
\]

(17) \[= (\tilde{\nabla}_e_i \tilde{\nabla}_e_i dF) \left( \frac{\partial}{\partial t} \right) - (\tilde{\nabla}_e_i \nabla e_i dF) \left( \frac{\partial}{\partial t} \right)

- R^N (dF(e_i), dF) \left( \frac{\partial}{\partial t} \right) dF(e_i),\]

\[
\tilde{\nabla}_{e_k} \nabla e_k ((\tilde{\nabla}_e_i dF)(e_i))
\]

(18) \[= \nabla_{e_k} \nabla_{e_k} \left[ (\tilde{\nabla}_e_i \tilde{\nabla}_e_i dF) \left( \frac{\partial}{\partial t} \right) - (\tilde{\nabla}_e_i \nabla e_i dF) \left( \frac{\partial}{\partial t} \right)

- R^N (dF(e_i), dF) \left( \frac{\partial}{\partial t} \right) dF(e_i) \right]

- R^N (dF(e_k), dF) \left( \frac{\partial}{\partial t} \right) \nabla e_k ((\tilde{\nabla}_e_i dF)(e_i)),\]

and

\[
\nabla_{e_k} \nabla_{e_k e_k} ((\tilde{\nabla}_e_i dF)(e_i))
\]

(19) \[= \nabla_{e_k e_k} \left[ (\tilde{\nabla}_e_i \tilde{\nabla}_e_i dF) \left( \frac{\partial}{\partial t} \right) - (\tilde{\nabla}_e_i \nabla e_i dF) \left( \frac{\partial}{\partial t} \right)

- R^N (dF(e_i), dF) \left( \frac{\partial}{\partial t} \right) dF(e_i) \right]

- R^N (dF(\nabla e_k e_k), dF) \left( \frac{\partial}{\partial t} \right) ((\tilde{\nabla}_e_i dF)(e_i)).\]

By using Lemma 3.1, we prove Theorem 2.6.

**Proof of Theorem 2.6**

As in \([\text{[1]}]\), we have

\[
\frac{d}{dt} E_k(\phi_t) = \int_M (dF(\frac{\partial}{\partial t}) (\nabla e_k \nabla e_k - \nabla_{e_k e_k e_k} (\Delta^{k-2} ((\tilde{\nabla}_e j dF)(e_j)))) v_g

+ \int_M (R^N (dF(\frac{\partial}{\partial t}), dF(e_i))) dF(e_i), \Delta^{k-2} ((\tilde{\nabla}_e j dF)(e_j))) v_g.
\]

(20)
Therefore, we have

\[
\frac{d^2}{dt^2} E_k(f_i) = \int_M (\nabla_{\dot{e}_j} dF(\frac{\partial}{\partial t}), -\Delta^{k-2}((\nabla_{\dot{e}_j} dF)(e_i))) \\
+ R^N(\Delta^{k-2}((\nabla_{\dot{e}_j} dF)(e_j)), dF(e_j)) e_g \\
+ \int_M (dF(\frac{\partial}{\partial t}), \nabla_{\dot{e}_j} dF, -\Delta^{k-2}((\nabla_{\dot{e}_j} dF)(e_j))) \\
+ R^N(\Delta^{k-2}((\nabla_{\dot{e}_j} dF)(e_j)), dF(e_j)) e_g. 
\]

(21)

Then, we calculate \( \nabla_{\dot{e}_j} \left[ R^N(\Delta^{k-2}((\nabla_{\dot{e}_j} dF)(e_j)), dF(e_i)) dF(e_i) \right] \).

Using (17) and second Bianchi’s identity, we have

\[
\nabla_{\dot{e}_j} \left[ R^N(\Delta^{k-2}((\nabla_{\dot{e}_j} dF)(e_j)), dF(e_i)) dF(e_i) \right] \\
= (\nabla_{\dot{e}_j} R^N(\Delta^{k-2}((\nabla_{\dot{e}_j} dF)(e_j))) (dF(\frac{\partial}{\partial t}), dF(e_i)) dF(e_i) \\
+ (\nabla_{\dot{e}_j} R^N(\Delta^{k-2}((\nabla_{\dot{e}_j} dF)(e_j))), dF(\frac{\partial}{\partial t})) dF(e_i) \\
+ R^N(\Delta^{k-2}((\nabla_{\dot{e}_j} dF)(e_j)), \nabla_{\dot{e}_j} dF(\frac{\partial}{\partial t})) dF(e_i) \\
+ R^N(\Delta^{k-2}((\nabla_{\dot{e}_j} dF)(e_j)), dF(e_i)) dF(e_i) \\
+ R^N(\Delta^{k-2}((\nabla_{\dot{e}_j} dF)(e_j)), (\nabla_{\dot{e}_j} dF)(\frac{\partial}{\partial t})) dF(e_i) \\
+ R^N(\Delta^{k-2}((\nabla_{\dot{e}_j} dF)(e_j)), dF(e_i)) (\nabla_{\dot{e}_j} dF)(\frac{\partial}{\partial t}).
\]

(22)

Using (17), (18), (19) and (22), we have
\[ \frac{d^2}{dt^2} E_k(f_t) = \int_M \left( \nabla \cdot dF \left( \frac{\partial}{\partial t} \right) \right) - \Delta(\Delta^{k-2}(\tilde{\nabla}_e dF(e_j))) \\
+ R^N(\Delta^{k-2}(\tilde{\nabla}_e dF(e_j)), dF(e_j))v_g dF(e_i) \\
+ \int_M \left\langle dF \left( \frac{\partial}{\partial t} \right), \right. \\
\Delta^{k-2} \left\{ \nabla_{e_1} \nabla_{e_k} \left[ (\tilde{\nabla}_e \tilde{\nabla}_e dF) \left( \frac{\partial}{\partial t} \right) - (\tilde{\nabla}_{e_1} e_k dF) \left( \frac{\partial}{\partial t} \right) \right. \\
- R^N(dF(e_j), dF(e_i)) \right] \\
- \nabla_e \left[ R^N(dF(e_k), dF(e_i)) \right] (\tilde{\nabla}_e dF(e_j)) \\
- \nabla_{e_1} e_k \left[ (\tilde{\nabla}_e \tilde{\nabla}_e dF) \left( \frac{\partial}{\partial t} \right) - (\tilde{\nabla}_{e_1} e_k dF) \left( \frac{\partial}{\partial t} \right) \right. \\
- R^N(dF(e_j), dF(e_i)) \right] \\
\left. \rightangle v_g \right. \\
\right. \\
(23) \\
\right. \\
Putting \ t = 0 \ in \ (23), \ the \ first \ term \ of \ the \ RHS \ of \ (23) \ vanishes \ since \ \phi \ is \ k-harmonic.\]
Therefore, we have 
\[
\frac{d^2}{dt^2} E_k(\phi_t) \bigg|_{t=0} = \int_M \left\langle -\nabla^2 \nabla V - R^N(d\phi_e, V)d\phi(e_i), \nabla^2 \left\{ (-\nabla^2 \nabla V - R^N(d\phi(e_j), V)d\phi(e_j)) \right\} \right\rangle v_g 
\]
(24) 

By the Bianchi’s second identity, 
\[
- (\nabla^N \triangle^k - 2 \tau(\phi) R^N)(V, d\phi(e_i)) - (\nabla^N d\phi(e_i) R^N)(\nabla^k - 2 \tau(\phi), V)d\phi(e_i) 
\]
and Bianchi’s first identity 
\[
- R^N(\nabla^k - 2 \tau(\phi), d\phi(e_i)) + R^N(\nabla^k - 2 \tau(\phi), d\phi(e_i)) V, 
\]
we have Theorem 2.6. 

\section{4 \textit{k}-harmonic maps into Riemannian manifold of non positive sectional curvature}

Jiang [5] showed the following proposition.

\textbf{Proposition 4.1} [5]. Assume that $M$ is compact and $N$ is non positive curvature, i.e., Riemannian curvature of $N$ $\text{Riem}^N \leq 0$. Then, every 2-harmonic map $\phi : M \to N$ is harmonic.

In this section, we generalize this proposition to every $k$-harmonic map. Namely, we have

\textbf{Theorem 4.2}. Assume that $M$ is compact and $N$ is non positive curvature, $\text{Riem}^N \leq 0$. Then, every $k$-harmonic map $\phi : M \to N$ is harmonic.

To prove this theorem, we show Theorem 4.5. First, we show the following two lemmas.
Lemma 4.3. Let $l = 1, 2, \ldots$. If $\nabla_{e_i} \Delta^{(l-1)} \tau(\phi) = 0$ ($i = 1, \ldots, m$), then $\Delta^{l} \tau(\phi) = 0$.

Proof. Indeed, we can define a global vector field $X_\phi \in \Gamma(TM)$ defined by

$$X_\phi = \sum_{j=1}^{m} \langle -\nabla_{e_j} \nabla^{(l-1)} \tau(\phi), \nabla^{l} \tau(\phi) \rangle e_j.$$  \hfill (27)

Then, the divergence of $X_\phi$ is given as

$$\text{div}(X_\phi) = \langle \nabla^{l} \tau(\phi), \nabla^{l} \tau(\phi) \rangle + \sum_{j=1}^{m} \langle -\nabla_{e_j} \nabla^{(l-1)} \tau(\phi), \nabla_{e_j} \nabla^{l} \tau(\phi) \rangle = \langle \nabla^{l} \tau(\phi), \nabla^{l} \tau(\phi) \rangle,$$

by the assumption. Integrating this over $M$, we have

$$0 = \int_{M} \text{div}(X_\phi) v_g = \int_{M} \langle \nabla^{l} \tau(\phi), \nabla^{l} \tau(\phi) \rangle v_g,$$

which implies $\Delta^{l} \tau(\phi) = 0$. \hfill \Box

Lemma 4.4. Let $l = 1, 2, \ldots$. If $\nabla^{l} \tau(\phi) = 0$, then

$$\nabla_{e_i} \nabla^{(l-1)} \tau(\phi) = 0, \quad (i = 1, \ldots, m).$$

Proof. Indeed, by computing the Laplacian of the $2l$-energy density $e_{2l}(\phi)$, we have

$$\Delta e_{2l}(\phi) = \sum_{i=1}^{m} \left\langle \nabla_{e_i} \nabla^{(l-1)} \tau(\phi), \nabla_{e_i} \nabla^{(l-1)} \tau(\phi) \right\rangle$$

$$= \sum_{i=1}^{m} \left\langle \nabla \nabla(\Delta^{(l-1)} \tau(\phi)), \Delta^{(l-1)} \tau(\phi) \right\rangle \geq 0.$$ \hfill (28)

By Green’s theorem $\int_{M} \Delta e_{2l}(\phi) v_g = 0$, and (28), we have $\Delta e_{2l}(\phi) = 0$. Again, by (28), we have

$$\nabla_{e_i} \nabla^{(l-1)} \tau(\phi) = 0, \quad (i = 1, \ldots, m, \quad l = 1, 2, \ldots).$$ \hfill \Box

Lemma 4.5. Let $l = 1, 2, \ldots$. If $\nabla^{l} \tau(\phi) = 0$ or $\nabla_{e_i} \nabla^{(l-1)} \tau(\phi) = 0$, ($i = 1, 2, \ldots, m$), then every $k$-harmonic map $\phi : M \to N$ from a compact Riemannian manifold $M$ into another Riemannian manifold $N$ is a harmonic map.
Proof. By using Lemma 4.3 and 4.4, we have Lemma 4.5. By using Lemma 4.5, we show Theorem 4.2.

Proof of Theorem 4.2

By computing the Laplacian of the $2(k - 1)$-energy density $e_{2(k-1)}(\phi)$, we have

\[
\triangle e_{2(k-1)}(\phi) = \left( \nabla_{\phi} \Delta_{(k-2)}^{(k-2)} \tau(\phi), \nabla_{\phi} \Delta_{(k-2)}^{(k-2)} \tau(\phi) \right) - \left( \nabla_{\phi} \nabla(\Delta_{(k-2)}^{(k-2)} \tau(\phi)), \Delta_{(k-2)}^{(k-2)} \tau(\phi) \right).
\]

By using (30)

\[
\tau_k(\phi) = \Delta_{(k-1)}^{(k-1)} \tau(\phi) - \mathcal{R}(\Delta_{(k-2)}^{(k-2)} \tau(\phi)) = 0.
\]

Then, we have

\[
-\left( \nabla_{\phi} \Delta_{(k-2)}^{(k-2)} \tau(\phi), \nabla_{\phi} \Delta_{(k-2)}^{(k-2)} \tau(\phi) \right) = - \left( \mathcal{R}(\Delta_{(k-2)}^{(k-2)} \tau(\phi)), \Delta_{(k-2)}^{(k-2)} \tau(\phi) \right) \geq 0,
\]

due to $\text{Riem}^N \leq 0$. By Green’s theorem,

\[
0 = \int_M \Delta e_{2(k-1)}(\phi) v_g = \int_M \left( \nabla_{\phi} \Delta_{(k-2)}^{(k-2)} \tau(\phi), \nabla_{\phi} \Delta_{(k-2)}^{(k-2)} \tau(\phi) \right) v_g
\]

\[
- \int_M \left( \mathcal{R}(\Delta_{(k-2)}^{(k-2)} \tau(\phi)), \Delta_{(k-2)}^{(k-2)} \tau(\phi) \right) v_g.
\]

Then, the both terms of the RHS of (32) are non-negative, so that we have

\[
0 = \Delta e_{2(k-1)}(\phi) = \left( \nabla_{\phi} \Delta_{(k-2)}^{(k-2)} \tau(\phi), \nabla_{\phi} \Delta_{(k-2)}^{(k-2)} \tau(\phi) \right)
\]

\[
- \left( \mathcal{R}(\Delta_{(k-2)}^{(k-2)} \tau(\phi)), \Delta_{(k-2)}^{(k-2)} \tau(\phi) \right).
\]

Then, since the both terms of (33) are non-negative again, we have

\[
\nabla_{\phi} \Delta_{(k-2)}^{(k-2)} \tau(\phi) = 0, \quad (i = 1, \cdots, m).
\]

By using Lemma 4.5, we have $\tau(\phi) = 0$. □

5 Stable $k$-harmonic maps

In this section, we generalize the results of Jiang [5] on stable 2-harmonic maps to stable $k$-harmonic maps.

By the second variation formula, Jiang [5] defined the notion of stable 2-harmonic maps as follows.
Definition 5.1 ([5]). Let \( \phi : M \to N \) be a 2-harmonic map of a compact Riemannian manifold \( M \) into a Riemannian manifold \( N \). Then, \( \phi \) is stable if the second variation of 2-energy is non-negative for every variation \( \{ \phi_t \} \) along \( \phi \).

Notice that by definition of the 2-energy, any harmonic maps are stable 2-harmonic maps. This also can be seen as follows: since \( \tau(\phi) = 0 \), for a vector field \( V \) of any variation \( \{ \phi_t \} \), it holds that

\[
\left. \frac{d^2}{dt^2} E_2(\phi_t) \right|_{t=0} = \int_M \| - \nabla^* \nabla V + R^N(V, d\phi(e_i))d\phi(e_i) \|^2 v_g \geq 0.
\]

Theorem 5.2 ([5]). Assume that \( M \) is a compact Riemannian manifold, and \( N \) is a Riemannian manifold with a non-negative constant sectional curvature \( K \geq 0 \). Then, there is no non-trivial stable 2-harmonic map satisfying the conservation law.

By the second variation formula (cf. Theorem 2.6), we can introduce the notion of stable \( k \)-harmonic maps.

Definition 5.3. Let \( \phi : M \to N \) be any \( k \)-harmonic map of a compact Riemannian manifold \( M \) into a Riemannian manifold \( N \). Then, \( \phi \) is stable if the second variation of \( k \)-energy is non-negative for every variation \( \{ \phi_t \} \) of \( \phi \), i.e., (35) in Theorem 2.6 is non-negative for every vector field \( V \) along \( \phi \).

We have immediately

**Proposition 5.4.** All harmonic maps \( \phi : M \to N \) are stable 2\( l \)-harmonic maps \( (l = 1, 2, \cdots) \).

Proof. Let \( k = 2l \) \((l = 1, 2, \cdots)\). Since \( \tau(\phi) = 0 \), by Theorem 2.6 we have

\[
\left. \frac{d^2}{dt^2} E_k(\phi_t) \right|_{t=0} = \int_M \| \Delta^{l-1} (\nabla^* \nabla V - R^N(V, d\phi(e_i))d\phi(e_i)) \|^2 v_g \geq 0.
\]

Furthermore, one can consider a stable \( k \)-harmonic map into a Riemannian manifold \((N, h)\) of constant sectional curvature. Then, we have

**Theorem 5.5.** Assume that \( M \) is a compact Riemannian manifold, and \( N \) is a Riemannian manifold of non-negative constant sectional curvature \( K \geq 0 \). Then, there are no stable proper \( k \)-harmonic maps satisfying the conservation law, the \( k \)-conservation law and the \( 2(k - 1) \)-conservation law.
Proof. Since $N$ has constant curvature, $\nabla^N R^N = 0$, so that \eqref{eq:10} becomes

\begin{equation}
\left. \frac{d^2}{dt^2} E_k(\phi_t) \right|_{t=0} = \int_M \left\langle \nabla^N \nabla V - R^N(V, d\phi(e_i))d\phi(e_i), \Delta^{k-2} (\nabla^N \nabla V - R^N(V, d\phi(e_i))d\phi(e_i)) \right\rangle v_g \\
- \int_M \left\langle V, \Delta^{k-2} \left\{ R^N(\nabla(\tau(\phi), V)\nabla(\tau(\phi)) \right. \right.
\left. \left. + R^N(d\phi(e_k), \nabla_{e_k} V)\nabla(\tau(\phi)) + 2R^N(d\phi(e_k), V)\nabla_{e_k} \tau(\phi) \right) \right. \right.
\left. \left. + R^N(\nabla_{e_k} V, d\phi(e_i))\Delta^{k-2} \tau(\phi) \right) \right. \right.
\left. \left. - 2R^N(\Delta^{k-2} \tau(\phi), d\phi(e_i))\nabla_{e_k} V \right\rangle v_g. \right)
\end{equation}

Especially, if we take $V = \Delta^{k-2} \tau(\phi)$, then the first term of the RHS of \eqref{eq:36} must vanish. So we have

\begin{equation}
\left. \frac{d^2}{dt^2} E_k(\phi_t) \right|_{t=0} = - \int_M \left\langle \Delta^{2(k-2)} \tau(\phi), R^N(\nabla(\tau(\phi), \Delta^{k-2} \tau(\phi))\nabla(\tau(\phi)) \right. \right.
\left. \left. + R^N(d\phi(e_k), \nabla_{e_k} \Delta^{k-2} \tau(\phi))\nabla(\tau(\phi)) + 2R^N(d\phi(e_k), \nabla_{e_k} \tau(\phi))\nabla_{e_k} \tau(\phi) \right) \right. \right.
\left. \left. + R^N(\nabla_{e_k} V, d\phi(e_i))\Delta^{k-2} \tau(\phi) \right) \right. \right.
\left. \left. - 2R^N(\Delta^{k-2} \tau(\phi), d\phi(e_i))\nabla_{e_k} \Delta^{k-2} \tau(\phi) \right\rangle v_g. \right)
\end{equation}
Here, we have
\[
\langle d\phi(e_i), \nabla_{e_i} \tau(\phi) \rangle = e_i \langle d\phi(e_i), \tau(\phi) \rangle - \langle \nabla_{e_i} d\phi(e_i), \Delta^{k-2} \tau(\phi) \rangle
\]
(38)
\[
= -\langle \tau(\phi), \Delta^{k-2} \tau(\phi) \rangle - \langle d\phi(\nabla_{e_i} e_i), \Delta^{k-2} \tau(\phi) \rangle
\]
\[
= -\langle \tau(\phi), \Delta^{k-2} \tau(\phi) \rangle.
\]
By the assumptions, we have that
\[
\begin{align*}
\langle \tau(\phi), d\phi(X) \rangle &= 0, \\
\langle d\phi(X), \Delta^{k-2} \tau(\phi) \rangle &= 0, \\
\langle d\phi(X), \Delta^{2(k-2)} \tau(\phi) \rangle &= 0.
\end{align*}
\]
(39)
for all $X \in \Gamma(TM)$. And we have $\langle d\phi(e_k), \nabla_{e_k} \tau(\phi) \rangle = -\|\tau(\phi)\|^2$. Thus, we have
\[
\frac{d^2}{dt^2} E_k(\phi_t) \bigg|_{t=0} = -K \int_M \langle \tau(\phi), \Delta^{2(k-1)} \tau(\phi) \rangle \Delta^{k-2} \tau(\phi) \tau(\phi) \\
+ \|\tau(\phi)\|^2 \langle \Delta^{k-2} \tau(\phi), \Delta^{2(k-2)} \tau(\phi) \rangle \\
+ 2\|\tau(\phi)\|^2 \|\Delta^{k-2} \tau(\phi)\| v_g \leq 0.
\]
(40)
Now, we divide the situation into two cases.

Case 1) $k = 2l$ ($l = 1, 2, \cdots$). In this case, we have
\[
0 \leq \frac{d^2}{dt^2} E_{2l}(\phi_t) \bigg|_{t=0} = -K \int_M \|\Delta^{2l-1} \tau(\phi)\|^2 \|\Delta^{-1} \tau(\phi)\|^2 \\
+ \|\tau(\phi)\|^2 \|\Delta^{2l-2} \tau(\phi)\|^2 \\
+ 2\|\Delta^{-1} \tau(\phi)\|^2 \|\Delta^{2l-1} \tau(\phi)\|^2 v_g \leq 0.
\]
(41)
By using Lemma[45] we obtain $\tau(\phi) = 0$.

Case 2) $k = 2l + 1$ ($l = 1, 2, \cdots$). In this case, we have
\[
0 \leq \frac{d^2}{dt^2} E_{2l+1}(\phi_t) \bigg|_{t=0} = -K \int_M \|\nabla_{e_i} \Delta^{2l-1} \tau(\phi)\|^2 \|\nabla_{e_i} \Delta^{-1} \tau(\phi)\|^2 \\
+ \|\tau(\phi)\|^2 \|\nabla_{e_i} \Delta^{2l-2} \tau(\phi)\|^2 \\
+ 2\|\nabla_{e_i} \Delta^{-1} \tau(\phi)\|^2 \|\Delta^{2l-1} \tau(\phi)\|^2 v_g \leq 0.
\]
(42)
By using Lemma[45] we obtain $\tau(\phi) = 0$.

So, we have Theorem[55].

6 The $k$-harmonic maps into the product spaces

In this section, we describe the necessary and sufficient condition to be $k$-harmonic maps into the product spaces. First, let us recall the result of Y.-L. Ou[3].
Theorem 6.1 ([4]). Let \( \varphi : (M, g) \to (N_1, h_1) \) and \( \psi : (M, g) \to (N_2, h_2) \) be two maps. Then, the map \( \phi : (M, g) \to (N_1 \times N_2, h_1 \times h_2) \) with \( \phi(x) = (\varphi(x), \psi(x)) \) is 2-harmonic if and only if the both map \( \varphi \) or \( \psi \) are 2-harmonic. Furthermore, if one of \( \varphi \) or \( \psi \) is 2-harmonic and the other is a proper 2-harmonic map, then \( \phi \) is a proper 2-harmonic map.

We generalize Theorem 6.1 for \( k \)-harmonic maps. Namely, we have the following theorem which is useful to construct examples the \( k \)-harmonic maps.

Theorem 6.2. Let \( \varphi : (M, g) \to (N_1, h_1) \) and \( \psi : (M, g) \to (N_2, h_2) \) be two maps. Then, the map \( \phi : (M, g) \to (N_1 \times N_2, h_1 \times h_2) \) with \( \phi(x) = (\varphi(x), \psi(x)) \) is \( k \)-harmonic if and only if the both map \( \varphi \) or \( \psi \) are \( k \)-harmonic. Furthermore, if one of \( \varphi \) or \( \psi \) is harmonic and the other is a proper \( k \)-harmonic map, then \( \phi \) is a proper \( k \)-harmonic map.

Proof. As in [4], \( \tau(\phi) = \tau(\varphi) + \tau(\psi) \). We only notice that \( \nabla_{\phi} \nabla_{\phi} \tau(\phi) = \nabla_{\phi} \nabla_{\phi} \tau(\varphi) + \nabla_{\phi} \nabla_{\phi} \tau(\psi) \).

Similary, \( \nabla_{\phi}^{(k-2)} \tau(\phi) = \nabla_{\phi}^{(k-2)} \tau(\varphi) + \nabla_{\phi}^{(k-2)} \tau(\psi) \).

We use the property of the curvature of the product manifold to have

\[
R^{N_1 \times N_2}(d\phi(e_i), \nabla_{\phi}^{(k-2)} \tau(\phi))d\phi(e_i)
= R^{N_1}(d\varphi(e_i), \nabla_{\phi}^{(k-2)} \tau(\varphi))d\varphi(e_i) + R^{N_2}(d\psi(e_i), \nabla_{\phi}^{(k-2)} \tau(\psi))d\psi(e_i).
\]

Therefore, we have

\[
\tau_k(\phi) = \nabla_{\phi}^{(k-1)} \tau(\phi) + R^{N_1 \times N_2}(d\phi(e_i), \nabla_{\phi}^{(k-2)} \tau(\phi))d\phi(e_i)
= \nabla_{\phi}^{(k-1)} \tau(\varphi) + R^{N_1}(d\varphi(e_i), \nabla_{\phi}^{(k-2)} \tau(\varphi))d\varphi(e_i)
+ \nabla_{\phi}^{(k-1)} \tau(\psi) + R^{N_2}(d\psi(e_i), \nabla_{\phi}^{(k-2)} \tau(\psi))d\psi(e_i)
= \tau_k(\varphi) + \tau_k(\psi).
\]

The following corollary generalizes Corollary 3.4 in [4].

Corollary 6.3. Let \( \psi : (M, g) \to (N, h) \) be a smooth map. Then, the graph \( \phi : (M, g) \to (M \times N, g \times h) \) with \( \phi(x) = (x, \psi(x)) \) is a \( k \)-harmonic map if and only if the map \( \psi : (M, g) \to (N, h) \) is a \( k \)-harmonic map. Furthermore, if \( \psi \) is proper \( k \)-harmonic, then so is the graph.

Proof. This follows from Theorem 6.2 with \( \varphi : (M, g) \to (N, h) \) being identity map which is harmonic.
Example 6.4. Let $\phi : \mathbb{R} \to \mathbb{R} \times S^n$ be a smooth curve parametrized by the arc length in the product space $\mathbb{R} \times S^n$ with the standard product metric defined by

$$\phi(x) = (x, \cos(\sqrt{2}x)c_1 + \sin(\sqrt{2}x)c_2 + c_4),$$

where $c_1$, $c_2$ and $c_4$ are constant vectors in $\mathbb{R}^{n+1}$ orthogonal to each other with $|c_1|^2 = |c_2|^2 = |c_4|^2 = \frac{1}{2}$ as in Example 6.4. Then, $\phi : \mathbb{R} \to \mathbb{R} \times S^n$ is $k$-harmonic for $k = 2, 3$ and 4.

7 Determination of $k$-harmonic curves into a sphere

We determine that the ODEs of the 3-harmonic, and 4-harmonic curve equations into a sphere, respectively.

Theorem 7.1. Let $\gamma : I \to S^n \subset \mathbb{R}^{n+1}$ be a smooth curve defined on an interval of $\mathbb{R}$ parametrized by arc length. Then, $\gamma$ is $k$-harmonic curve if and only if

$$\left((\nabla_{\gamma'} \nabla_{\gamma'})^{(k-1)}(\nabla_{\gamma'} \gamma') + (\nabla_{\gamma'} \nabla_{\gamma'})^{(k-2)}(\nabla_{\gamma'} \gamma')ight) - g \left((\nabla_{\gamma'} \nabla_{\gamma'})^{(k-2)}(\nabla_{\gamma'} \gamma'), \gamma'\right) \gamma' = 0,$$

where $g$ is the standard Riemannian metric on $S^n$ of constant sectional curvature 1, we denote by $\gamma'$ the differential of $\gamma$ with respect to the arc length.

Proof. $\gamma$ is a $k$-harmonic curve if and only if

$$\tau_k(\gamma) := \Delta^{(k-2)}(\gamma) - R(\Delta^{(k-2)} \tau(\gamma)) = 0.$$

Now, we have $\tau(\gamma) = \nabla_{\gamma'} \gamma'$, $R(V) = V - g(V, \gamma') \gamma'$, and $\Delta \gamma = -\nabla_{\gamma'} \nabla_{\gamma'} V$, respectively for all $V \in \Gamma(\gamma^{-1} TS^n)$. By

$$\Delta^{(k-2)} V = (-1)^{(k-2)}(\nabla_{\gamma'} \nabla_{\gamma'})^{(k-2)} V,$$

we have Theorem 7.1. 

Proposition 7.2. Let $\gamma : I \to S^n \subset \mathbb{R}^{n+1}$ be a smooth curve parametrized by arc length. Then $\gamma$ is 3-harmonic curve if and only if

$$\gamma^{(6)} + 2\gamma^{(4)} + (2 - g_{22})\gamma'' - 4g_{23}\gamma' + (2 - 3g_{22} - 9g_{24} - 8g_{33})\gamma = 0,$$

where $g_{ij} = g_0(\gamma^{(i)}, \gamma^{(j)}), (i, j = 0, 1, \ldots)$, and $g_0$ is the standard metric on the Euclidean space $\mathbb{R}^{n+1}$.

Proof. $\nabla^0_{\gamma'} \gamma' = \sigma(\gamma', \gamma') + \nabla_{\gamma'} \gamma'$,

which yields that $\nabla_{\gamma'} \gamma' = \nabla^0_{\gamma'} \gamma' + g(\gamma', \gamma') \gamma$.

Therefore, we have $\nabla_{\gamma'} \gamma' = \gamma'' + \gamma$. Similarly,

$$(\nabla_{\gamma'} \nabla_{\gamma'}) \gamma' = \gamma^{(4)} + \gamma'' + (g_{13} + 1)\gamma.$$
\[
(\nabla_{\gamma'} \nabla_{\gamma'})^2 (\nabla_{\gamma'} \gamma') = \gamma^{(6)} + \gamma^{(4)} + (g_{13} + 1)\gamma'' + (2g_{23} + 3g_{14})\gamma' + (1 + g_{33} + 3g_{24} + 3g_{15} + g_{22} + 3g_{13})\gamma.
\]

(46)

Here, we used \(g_{13} = -g_{22}, g_{14} = -3g_{23}, g_{15} = -3g_{33} - 4g_{24}\). So we have Proposition 7.2.

Proposition 7.3. Let \(\gamma : I \rightarrow S^n \subset \mathbb{R}^{n+1}\) be a smooth curve parametrized by the arc length. Then, \(\gamma\) is 4-harmonic curve if and only if

\[
\begin{align*}
\gamma^{(8)} &+ 2\gamma^{(6)} + (2 - g_{22})\gamma^{(4)} - 11g_{23}\gamma^{(3)} \\
&+ (-24g_{33} - 25g_{24} + 2 - 3g_{22})\gamma'' \\
&+ (19g_{34} + 9g_{16} - 13g_{23})\gamma' \\
&+ (5g_{44} + 11g_{35} + 10g_{26} + 5g_{17} - 40g_{34} - 43g_{24} + g_{22}^2 - 5g_{22} + 2)\gamma &= 0,
\end{align*}
\]

(47)

where, \(g_{ij} = g_{0}(\gamma^{(i)}, \gamma^{(j)}), (i, j = 0, 1, \ldots)\).

Proof. We calculate \((\nabla_{\gamma'} \nabla_{\gamma'})^3 (\nabla_{\gamma'} \gamma')\) as follows.

\[
\begin{align*}
(\nabla_{\gamma'} \nabla_{\gamma'})^3 (\nabla_{\gamma'} \gamma') &= \gamma^{(8)} + \gamma^{(6)} + (g_{13} + 1)\gamma^{(4)} + (4g_{23} + 5g_{14})\gamma^{(3)} \\
&+ (6g_{33} + 15g_{24} + 2g_{22} + 3g_{13})\gamma'' \\
&+ (19g_{34} + 20g_{25} + 9g_{16} + 12g_{23} + 10g_{14})\gamma' \\
&+ (5g_{44} + 11g_{35} + 10g_{26} + 5g_{17} + 10g_{33} + 22g_{24} + 14g_{15} + g_{13}^2 + 4g_{13} + 1 + g_{22})\gamma.
\end{align*}
\]

By using (46), we have Proposition 7.3.

We can derive the ODE to be 3-harmonic or 4-harmonic in terms of the Frenet-frame. Indeed, the Frenet-frame is given as follows:

\[
\begin{align*}
\gamma' &= T, \quad \nabla_N T = \kappa N, \quad \nabla_T N = -\kappa T, \\
\langle T, N \rangle &= 0, \quad \langle T, T \rangle = 1, \quad \langle N, N \rangle = 1,
\end{align*}
\]

(48)

where \(\kappa\) is the geodesic curvature and \(\langle \cdot, \cdot \rangle = g\) the standard Riemannian metric on \(S^2\). Then, we have the following.

Proposition 7.4. Let \(\gamma : I \rightarrow (S^2, \langle \cdot, \cdot \rangle)\) be a smooth curve parametrized by the arc length. Then, \(\gamma\) is a 3-harmonic curve if and only if

\[
\begin{align*}
\kappa^{(4)} - 12(\kappa')^2 - 10\kappa^2\kappa'' + \kappa^5 - 3\kappa(\kappa')^2 + \kappa'' - \kappa &= 0, \\
\kappa\kappa^{(3)} - 2\kappa^3\kappa' + 2\kappa'\kappa'' &= 0,
\end{align*}
\]

where \(\kappa\) is the geodesic curvature of \(\gamma\).
Proof. We calculate \((\nabla^N_N \nabla^N_N)^2 \tau(\gamma)\) as follows.

\[
(\nabla^N_N \nabla^N_N)^2 \tau(\gamma) = (\kappa(4))^2 - 12(\kappa')^2 - 10\kappa'' + \kappa^3 - 3\kappa(\kappa')^2)N
+ (-5\kappa^3 + 10\kappa - 10\kappa'')T.
\]

Therefore, \(\gamma\) is 3-harmonic if and only if

\[
\begin{align*}
(\kappa(4))^2 - 12(\kappa')^2 - 10\kappa'' + \kappa^3 - 3\kappa(\kappa')^2)N \\
+ (-5\kappa^3 + 10\kappa - 10\kappa'')T &= 0.
\end{align*}
\]

So we have Proposition 7.5. \(\square\)

Proposition 7.5. Let \(\gamma : I \to (S^2, \langle \cdot, \cdot \rangle)\) be a smooth curve parametrized by the arc length. Then, \(\gamma\) is 4-harmonic curve if and only if

\[
\begin{align*}
\kappa(6) - 24(\kappa'')^2 - 24\kappa'\kappa''' - 45(\kappa')^2\kappa'' - 46\kappa(\kappa'')^2 - 81\kappa'\kappa(3) - 21\kappa^2(4) \\
+ 93\kappa^3(4) + 35\kappa^4\kappa'' + 12\kappa^2(\kappa')^2 - \kappa^7 \\
+ (\kappa(4))^2 - 12(\kappa')^2 - 10\kappa'' + \kappa^3 - 3\kappa(\kappa')^2)N \\
- 7\kappa^3(4) + 48\kappa'\kappa'' + 162\kappa^2(\kappa'') + 35\kappa^5(\kappa')^2 - 21\kappa^5 + 69\kappa(\kappa')^3 \\
- 21\kappa'\kappa(4) + 12(\kappa')^3 - 35\kappa''\kappa(3)) = 0,
\end{align*}
\]

where \(\kappa\) is the geodesic curvature of \(\gamma\).

Proof. We calculate \((\nabla^N_N \nabla^N_N)^3 \tau(\gamma)\) as follows.

\[
(\nabla^N_N \nabla^N_N)^3 \tau(\gamma) = (\kappa(6) - 24(\kappa'')^2 - 24\kappa'\kappa''' - 45(\kappa')^2\kappa'' - 46\kappa(\kappa'')^2 - 81\kappa'\kappa(3) - 21\kappa^2(4) \\
- 81\kappa\kappa\kappa(3) - 21\kappa^2(4) + 93\kappa^3(4) + 35\kappa^4\kappa'' + 12\kappa^2(\kappa')^2 - \kappa^7)N \\
+ (-7\kappa^3(4) + 48\kappa'\kappa'' + 162\kappa^2(\kappa'') + 35\kappa^5(\kappa')^2 - 21\kappa^5 + 69\kappa(\kappa')^3 \\
- 21\kappa'\kappa(4) + 12(\kappa')^3 - 35\kappa''\kappa(3))T.
\]

Therefore, using (119), \(\gamma\) is 4-harmonic if and only if

\[
\begin{align*}
\kappa(6) - 24(\kappa'')^2 - 24\kappa'\kappa''' - 45(\kappa')^2\kappa'' - 46\kappa(\kappa'')^2 \\
- 81\kappa\kappa\kappa(3) - 21\kappa^2(4) + 93\kappa^3(4) + 35\kappa^4\kappa'' + 12\kappa^2(\kappa')^2 - \kappa^7 \\
+ (\kappa(4))^2 - 12(\kappa')^2 - 10\kappa'' + \kappa^3 - 3\kappa(\kappa')^2)N \\
+ (-7\kappa^3(4) + 48\kappa'\kappa'' + 162\kappa^2(\kappa'') + 35\kappa^5(\kappa')^2 - 21\kappa^5 + 69\kappa(\kappa')^3 \\
- 21\kappa'\kappa(4) + 12(\kappa')^3 - 35\kappa''\kappa(3))T &= 0.
\end{align*}
\]

So we have Proposition 7.5. \(\square\)

We show the following Theorem 7.6.

Theorem 7.6. Every non-harmonic \(k\)-harmonic curve into \((S^2, \langle \cdot, \cdot \rangle)\) whose geodesic curvature is constant is 2-harmonic. Namely, \(\gamma : I \to (S^2, \langle \cdot, \cdot \rangle)\) a smooth curve parametrized by the arc length. If the geodesic curvature \(\kappa\) of \(\gamma\) is constant, every \(k\)-harmonic is 2-harmonic.
Proof. γ is 2-harmonic curve if and only if
\[ (\nabla^N_N\nabla^N_N)\tau(\gamma) + \tau(\gamma) - \langle \tau(\gamma), \gamma' \rangle \gamma' = 0. \]
Now we have \( \tau(\gamma) = \kappa N \) and \( (\nabla^N_N\nabla^N_N)\tau(\gamma) = (\kappa'' - \kappa^2)N - 3\kappa'\kappa T. \) Therefore, γ is 2-harmonic if and only if \( \kappa^2 = 1. \) Next, we consider \( k \)-harmonic curve. We set geodesic curvature \( \kappa \) of γ is constant. Then, we have
\[ (\nabla^N_N\nabla^N_N)^k\tau(\gamma) = (-1)^{k-2}\kappa^{2(k-2)+1}N. \]
So, γ is \( k \)-harmonic curve if and only if
\[ 0 = (-1)^{k-2}\kappa^{2(k-2)+1}(\kappa^2 - 1)N. \]
So, we have Theorem 7.6.

Example 7.7. R. Caddeo, S. Montaldo and C. Oniciuc gave following two curves are propre 2-harmonic curves \( \gamma : I \to S^n \subset \mathbb{R}^{n+1} \) (cf. [3]).

(53) \[ \gamma(t) = \cos(\sqrt{2}t)c_1 + \sin(\sqrt{2}t)c_2 + c_4, \]
where \( c_1, \ c_2 \) and \( c_4 \) are constant vectors orthogonal to each other with \( |c_1|^2 = |c_2|^2 = |c_4|^2 = \frac{1}{2}. \)

(54) \[ \gamma(t) = \cos(at)c_1 + \sin(at)c_2 + \cos(bt)c_3 + \sin(bt)c_4, \]
where \( c_1, \ c_2, \ c_3 \) and \( c_4 \) are constant vectors orthogonal to each other with \( |c_1|^2 = |c_2|^2 = |c_3|^2 = |c_4|^2 = \frac{1}{2}, \) and \( a^2 + b^2 = 2, \ a^2 \neq b^2. \)

Proposition 7.8. The curves (53) and (54) in Example 7.7 are also 3-harmonic and 4-harmonic.

Proof. We can show this proposition by a direct computation. The proof is omitted.

Remark 7.9. Notice that a 2-harmonic map implies not always to be 3-harmonic or 4-harmonic. Proposition 7.8 is non-trivial.

References

[1] T. Ichiyama, J. Inoguchi and H. Urakawa, Bi-harmonic map and bi-Yang-Mills fields, Note di Matematica, 28 (2009), 233-275.

[2] H. Urakawa, Calculus of variation and harmonic maps, Transl. Math. Monograph. 132, Amer. Math. Soc.

[3] R. Caddeo, S. Montaldo and C. Oniciuc, Biharmonic submanifolds in spheres, Israel Journal of Mathmatics, 130 (2002), 109-123.

[4] Y.-L. Ou, Some constructions of biharmonic maps and Chen’s conjectur e on biharmonic hypersurfaces, arXiv:0912.1141v1 [math.DG] 6 Dec 2009.
[5] G. Y. Jiang, 2-harmonic maps and their first and second variational formulas, Chinese Ann. Math., 7A (1986), 388-402; the English translation, Note di Matematica, 28, (2009), 209-232.

[6] J. Eells and L. Lemaire, Selected topics in harmonic maps, CBMS, 50, Amer. Math. Soc, 1983.

[7] P. Petersen, Riemannian Geometry, Springer Science 2006.

[8] M. Spivak, A Comprehensive Introduction to Differential Geometry, I - IV, Wilmington: Publish or Perish, 1979.