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ALPHA-INvariants AND PURELY LOG TERMINAL BLOW-UPS

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Abstract. We prove that the sum of the α-invariants of two different Kollár components of a Kawamata log terminal singularity is less than 1.

Let $V$ be a normal irreducible projective variety of dimension $n \geq 1$, and let $\Delta_V$ be an effective $\mathbb{Q}$-divisor on $V$. Write

$$\Delta_V = \sum_{i=1}^{r} a_i \Delta_i,$$

where each $\Delta_i$ is a prime divisor, and each $a_i$ is a positive rational number. Suppose that the log pair $(V, \Delta_V)$ has at most Kawamata log terminal singularities. Then, in particular, each $a_i$ does not exceed 1. Suppose also that the divisor $-(K_V + \Delta_V)$ is ample, so that $(V, \Delta_V)$ is a log Fano variety. Finally, suppose that $V$ is faithfully acted on by a finite group $G$ such that the divisor $\Delta_V$ is $G$-invariant. Let $\alpha_G(V, \Delta_V)$ be the real number

$$\sup \left\{ \lambda \in \mathbb{Q} \middle| \text{the pair } (V, \Delta_V + \lambda D_V) \text{ has Kawamata log terminal singularities} \right\}.$$ 

This number is known as the α-invariant of the log Fano variety $(V, \Delta_V)$, or its global log canonical threshold (see [12, Definition 3.1]). If $G$ is trivial, we put $\alpha(V, \Delta_V) = \alpha_G(V, \Delta_V)$.

Example 1. The divisor $-(K_{\mathbb{P}^1} + \Delta_{\mathbb{P}^1})$ is ample if and only if $\sum_{i=1}^{r} a_i < 2$. One has

$$\alpha(\mathbb{P}^1, \Delta_{\mathbb{P}^1}) = \frac{1 - \max(a_1, \ldots, a_r)}{2 - \sum_{i=1}^{r} a_i}.$$ 

We put $\alpha_G(V) = \alpha_G(V, \Delta_V)$ if $\Delta_V = 0$.

Example 2. A finite group $G$ acting faithfully on $\mathbb{P}^1$ is one of the following finite groups: the alternating group $\mathfrak{A}_5$, the symmetric group $\mathfrak{S}_4$, the alternating group $\mathfrak{A}_4$, a dihedral group $D_{2m}$ of order $2m$, or a cyclic group $\mu_m$ of order $m$ (including the case $m = 1$, that is, the trivial group). The number $\frac{\alpha_G(\mathbb{P}^1)}{2}$ is equal to the length of the smallest $G$-orbit in $\mathbb{P}^1$, which gives

$$\alpha_G(\mathbb{P}^1) = \begin{cases} 
6 & \text{if } G \cong \mathfrak{A}_5, \\
3 & \text{if } G \cong \mathfrak{S}_4, \\
2 & \text{if } G \cong \mathfrak{A}_4, \\
1 & \text{if } G \cong D_{2m}, \\
\frac{1}{2} & \text{if } G \cong \mu_m.
\end{cases}$$

We assume that all varieties are defined over the field $\mathbb{C}$. 

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If both $\Delta_V = 0$ and $G$ is trivial, we put $\alpha(V) = \alpha_G(V, \Delta_V)$. This is the most classical case. Namely, if $V$ is a smooth Fano variety, then by [11, Theorem A.3] the number $\alpha(V)$ coincides with the $\alpha$-invariant of $V$ defined by Tian in [15]. Its values were found or estimated in many cases. For example, in the toric case the explicit formula for $\alpha(V)$ is given by Cheltsov and Shramov in [11, Lemma 5.1]. It gives $\alpha(\mathbb{P}^n) = \frac{1}{n+1}$, which can also be verified by an easy explicit computation. The $\alpha$-invariants of smooth del Pezzo surfaces were computed in [2].

**Theorem 3.** Let $V$ be a smooth del Pezzo surface. Then one has

$$\alpha(V) = \begin{cases} 
1 & \text{if } K_V^2 = 1 \text{ and } \left| -K_V \right| \text{ contains no cuspidal curves}, \\
\frac{5}{6} & \text{if } K_V^2 = 1 \text{ and } \left| -K_V \right| \text{ contains a cuspidal curve}, \\
\frac{5}{6} & \text{if } K_V^2 = 2 \text{ and } \left| -K_V \right| \text{ contains no tacnodal curves}, \\
\frac{3}{4} & \text{if } K_V^2 = 2 \text{ and } \left| -K_V \right| \text{ contains a tacnodal curve}, \\
\frac{3}{4} & \text{if } V \text{ is a cubic in } \mathbb{P}^3 \text{ with no Eckardt points}, \\
\frac{2}{3} & \text{if either } V \text{ is a cubic in } \mathbb{P}^3 \text{ with an Eckardt point, or } K_V^2 = 4, \\
\frac{1}{2} & \text{if } V \cong \mathbb{P}^1 \times \mathbb{P}^1 \text{ or } K_V^2 \in \{5, 6\}, \\
\frac{1}{3} & \text{in the remaining cases}.
\end{cases}$$

The $\alpha$-invariants of all del Pezzo surfaces with Du Val singularities were computed in [4, 43, 38, 37, 7].

**Example 4.** Let $V$ be a singular cubic surface in $\mathbb{P}^3$ that has at most Du Val singularities. Then one has

$$\alpha(V) = \begin{cases} 
\frac{2}{3} & \text{if } V \text{ has unique singular point, and it is of type } A_1, \\
\frac{1}{3} & \text{if } V \text{ contains singular point of type } A_4, \\
\frac{1}{3} & \text{if } V \text{ has unique singular point, and it is of type } D_4, \\
\frac{1}{3} & \text{if } V \text{ contains two singular points of type } A_2, \\
\frac{1}{4} & \text{if } V \text{ contains singular point of type } A_5, \\
\frac{1}{4} & \text{if } V \text{ has unique singular point, and it is of type } D_5, \\
\frac{1}{6} & \text{if } V \text{ has unique singular point, and it is of type } E_6, \\
\frac{1}{2} & \text{in all the remaining cases}.
\end{cases}$$

The $\alpha$-invariants of many non-Gorenstein singular del Pezzo surfaces that are quasi-smooth well-formed complete intersections in weighted projective spaces were computed
The \( \alpha \)-invariants of many smooth and singular Fano threefolds were computed or estimated in \([23, 11, 3, 5, 6, 25]\). The \( \alpha \)-invariants of smooth Fano hypersurfaces were estimated in \([1, 8, 40, 10]\).

The \( \alpha \)-invariant plays an important role in Kähler geometry. If \( V \) is a smooth Fano variety, then \( V \) admits a \( G \)-invariant Kähler–Einstein metric provided that
\[
\alpha_G(V) > \frac{\dim(V)}{\dim(V) + 1}.
\]
This was proved by Tian in \([45]\). In \([19]\), this result was improved by Fujita. He proved that \( V \) admits a Kähler–Einstein metric if it is smooth and \( \alpha(V) \geq \frac{\dim(V)}{\dim(V) + 1} \). In particular, all smooth hypersurfaces in \( \mathbb{P}^d \) of degree \( d \) are Kähler–Einstein, because their \( \alpha \)-invariants are at least \( \frac{d - 1}{d} \) by \([1, 8]\).

The \( \alpha \)-invariant also plays an important role in birational geometry. It was first observed by Park in \([35]\), where he proved the following

**Theorem 5 (\([4\), Theorem 5.7]).** Let \( X \) be a variety with at most terminal \( \mathbb{Q} \)-factorial singularities. Suppose that there is a flat morphism \( \phi: X \to Z \) such that \( Z \) is a curve, and \( -K_X \) is \( \phi \)-ample. Let \( P \) be a point in \( Z \), and let \( E_X \) be a scheme fiber of \( \phi \) over \( P \). Suppose that \( E_X \) is irreducible, reduced, normal, and has at most Kawamata log terminal singularities, so that \( E_X \) is a Fano variety by the adjunction formula. Suppose also that there is a commutative diagram

\[
\begin{array}{ccc}
X & \xrightarrow{\rho} & Y \\
\phi \downarrow & & \psi \\
Z \\
\end{array}
\]

such that \( Y \) is a variety with at most terminal \( \mathbb{Q} \)-factorial singularities, \( \psi \) is a flat morphism, the divisor \( -K_Y \) is \( \psi \)-ample, and \( \rho \) is a birational map that induces an isomorphism
\[
X \setminus \text{Supp}(E_X) \cong Y \setminus \text{Supp}(E_Y),
\]
where \( E_Y \) is a scheme fiber of \( \psi \) over \( P \). Suppose, in addition, that \( E_Y \) is irreducible. Then \( \rho \) is an isomorphism provided that \( \alpha(E_X) \geq 1 \). Moreover, if \( E_Y \) is reduced, normal and has at most Kawamata log terminal singularities, then \( \rho \) is an isomorphism provided that \( \alpha(E_X) + \alpha(E_Y) > 1 \).

Theorem 5 gives a necessary condition in terms of \( \alpha \)-invariants for the existence of a non-biregular fiberwise birational transformation of a Mori fibre space over a curve. It follows from \([29\), Theorem 1.1] that this condition is not a sufficient condition. Nevertheless, the bound is sharp (one can find many examples in \([35, 36]\)).
**Example 6.** Let $S$ be a $\mathbb{P}^1$-bundle over a curve. Then we have an elementary transformation to another $\mathbb{P}^1$-bundle over the same curve. Note that the $\alpha(\mathbb{P}^1) = \frac{1}{2}$ by Example 2.

**Example 7** ([13, Example 5.8]). Let $S$ be a smooth cubic surface in $\mathbb{P}^3$ with an Eckardt point $O$. Denote by $L_1, L_2, L_3$ the lines in $S$ passing through $O$. Put $X = S \times \mathbb{A}^1$, and let $\phi$ be the natural projection $X \to \mathbb{A}^1$. Let us identify $S$ with a fiber of $\phi$. Then there is a commutative diagram

\[
\begin{array}{ccc}
U & \xrightarrow{\psi} & \overline{U} \\
\downarrow{\alpha} & & \downarrow{\beta} \\
 X & \xrightarrow{\rho} & Y \\
\downarrow{\phi} & & \downarrow{\psi} \\
 \mathbb{A}^1 & & \mathbb{A}^1
\end{array}
\]

where $\alpha$ is the blow up of the point $O$, the map $\psi$ is the anti-flip along the proper transforms of the curves $L_1, L_2, L_3$, and $\beta$ is the contraction of the proper transform of the surface $S$. The scheme fiber of $\psi$ over the point $\phi(S)$ is a cubic surface in $\mathbb{P}^3$ that has one singular point of type $\mathbb{D}_4$. Its $\alpha$-invariant is $\frac{1}{3}$ by Example 4. On the other hand, we have $\alpha(S) = \frac{2}{3}$ by Theorem 3.

**Example 8** ([35, Example 5.3]). Let $X$ and $Y$ be subvarieties in $\mathbb{A}^1 \times \mathbb{P}^3$ given by equations

\[
x^3 + y^2z + z^2w + t^{12}w^3 = 0 \quad \text{and} \quad x^3 + y^2z + z^2w + w^3 = 0,
\]

respectively, where $t$ is a coordinate on $\mathbb{A}^1$, and $(x : y : z : w)$ are homogeneous coordinates on $\mathbb{P}^3$. Then the projections $\phi : X \to \mathbb{A}^1$ and $\psi : Y \to \mathbb{A}^1$ are birational to cubic surfaces, and the map

\[
(t, x, y, z, w) \mapsto (t, t^2x, t^3y, z, t^6w)
\]

gives a non-biregular birational fiberwise map $\rho : X \dashrightarrow Y$ between them. The fiber of $\phi$ over the point $t = 0$ is a cubic surface that has one Du Val singular point of type $\mathbb{E}_6$, so that its $\alpha$-invariant is $\frac{1}{6}$ by Example 4 and the scheme fiber of $\psi$ over the point $t = 0$ is a smooth cubic surface with an Eckardt point, so that its $\alpha$-invariant is $\frac{2}{3}$ by Theorem 3.

The $\alpha$-invariant also plays an important role in singularity theory. Let $U \ni P$ be a germ of a Kawamata log terminal singularity. Then it follows from [47, Lemma 1] that there is a birational morphism $\phi : X \to U$ such that its exceptional locus consists of a single prime divisor $E_X$ such that $\phi(E_X) = P$, the log pair $(X, E_X)$ has purely log terminal singularities, and the divisor $-(K_X + E_X)$ is $\phi$-ample. Then

\[
-(K_X + E_X) \sim_{\mathbb{Q}} -\delta_X E_X
\]

for some positive rational number $\delta_X$. Recall from [39, Definition 2.1] that the birational morphism $\phi : X \to U$ is a purely log terminal blow-up of the singularity $U \ni P$.

By [26] Theorem 7.5, the divisor $E_X$ is a normal variety that has rational singularities. Moreover, it can be naturally equipped with a structure of a log Fano variety. Let $R_1, \ldots, R_s$ be all the irreducible components of the locus $\text{Sing}(X)$ of codimension 2 that are contained in $E_X$. Put

\[
\text{Diff}_{E_X}(0) = \sum_{i=1}^{s} \frac{m_i - 1}{m_i} R_i,
\]
where $m_i$ is the smallest positive integer such that the divisor $m_i E_X$ is Cartier in a general point of $R_i$. Then $\text{Diff}_{E_X}(0)$ is usually called the different of the pair $(X, E_X)$. One has

$$-\delta_X E_X \Big|_{E_X} \sim_Q - \left( K_X + E_X \right) \Big|_{E_X} \sim_Q - (K_{E_X} + \text{Diff}_{E_X}(0)).$$

Furthermore, the singularities of the log pair $(E_X, \text{Diff}_{E_X}(0))$ are Kawamata log terminal by Adjunction, see [31, 3.2] or [27, 17.6]. This means that $(E_X, \text{Diff}_{E_X}(0))$ is a log Fano variety with Kawamata log terminal singularities, because $-E_X$ is $\phi$-ample.

**Definition 9** (cf. [31 Definition 1.1]). The log Fano variety $(E_X, \text{Diff}_{E_X}(0))$ is a Kollár component of $U \ni P$.

Let us show how to compute $\alpha(E_X, \text{Diff}_{E_X}(0))$ in three simple cases.

**Example 10.** Let $U \ni P$ be a germ of a Du Val singularity, and $f: W \to U$ be the minimal resolution of this singularity. Then the exceptional curves of $f$ are smooth rational curves whose self-intersections are $-2$, and their dual graph is of type $A_m, D_m, E_6, E_7, \text{ or } E_8$. Let $E_W$ be one of the exceptional curves that is chosen as follows. If $U \ni P$ is not a singularity of type $A_m$, let $E_W$ be the only $f$-exceptional curve that intersects three other $f$-exceptional curves, i.e., $E_W$ is the “fork” of the dual graph. If $U \ni P$ is a singularity of type $A_m$, choose $E_W$ to be the $k$-th curve in the dual graph. In this case, we may assume that $k \leq \frac{m+1}{2}$. In all cases, there exists a commutative diagram

\[
\begin{array}{ccc}
W & \xrightarrow{h} & Y \\
\downarrow{f} & & \downarrow{g} \\
U & \xleftarrow{g} & Y \\
\end{array}
\]

where $h$ is the contraction of all $f$-exceptional curves except $E_W$, and $g$ is the contraction of the proper transform of $E_W$ on the surface $Y$. Denote the $g$-exceptional curve by $E_Y$. Then $Y$ has at most Du Val singularities of type $A$, the curve $E_Y$ is smooth, and it contains all singular points of the surface $Y$, if any. One can check that the log pair $(Y, E_Y)$ has purely log terminal singularities, see [28, Theorem 4.15(3)]. Also, the divisor $-(K_Y + E_Y)$ is $g$-ample. Thus, the curve $E_Y$ is a Kollár component of the singularity $U \ni P$. Moreover, if $U \ni P$ is a singularity of type $A_m$, then

$$\alpha(E_Y, \text{Diff}_{E_Y}(0)) = \frac{k}{m+1} \leq \frac{1}{2}.$$ 

Indeed, if $U \ni P$ is a singularity of type $A_1$, then $h$ is an isomorphism and $Y$ is smooth, so that $\text{Diff}_{E_Y}(0) = 0$, which gives $\alpha(E_Y, \text{Diff}_{E_Y}(0)) = \frac{1}{2}$. Similarly, if $U \ni P$ is a singularity of type $A_m, m \geq 2$, and $k = 1$, then $Y$ has a singular point $P_1$ that is a Du Val singular point of type $A_{m-1}$. In this case, we have

$$\text{Diff}_{E_Y}(0) = \frac{m-1}{m} P_1,$$

which gives $\alpha(E_Y, \text{Diff}_{E_Y}(0)) = \frac{1}{m+1}$. Finally, if $U \ni P$ is a singularity of type $A_m, m \geq 3$, and $2 \leq k \leq \frac{m+1}{2}$, then $Y$ has two singular points $P_1$ and $P_2$, which are Du Val singular points of type $A_{k-1}$ and $A_{m-k}$. In this case, we have

$$\text{Diff}_{E_Y}(0) = \frac{k-1}{k} P_1 + \frac{m-k}{m-k+1} P_2,$$
so that \( \alpha(E_Y, \text{Diff}_{E_Y}(0)) = \frac{k}{m+1} \). Likewise, if \( U \ni P \) is a singularity of type \( \mathbb{D}_m \) with \( m \geq 4 \), then \( \alpha(E_Y, \text{Diff}_{E_Y}(0)) = 1 \). Indeed, in this case \( Y \) has three singular points \( P_1, P_2 \) and \( P_3 \) such that \( P_1 \) and \( P_2 \) are Du Val singular points of type \( \mathbb{A}_1 \), and \( P_3 \) is a Du Val singular point of type \( \mathbb{A}_{m-3} \), so that

\[
\text{Diff}_{E_Y}(0) = \frac{1}{2} P_1 + \frac{1}{2} P_2 + \frac{m-3}{m-2} P_3,
\]

which easily gives \( \alpha(E_Y, \text{Diff}_{E_Y}(0)) = 1 \). If \( U \ni P \) is a singularity of type \( \mathbb{E}_m \), then \( Y \) has three Du Val singular points \( P_1, P_2, \) and \( P_3 \) such that \( P_1 \) and \( P_2 \) are Du Val singular points of type \( \mathbb{A}_1 \), and \( P_3 \) is a Du Val singular point of type \( \mathbb{A}_{m-3} \), so that

\[
\text{Diff}_{E_Y}(0) = \frac{1}{2} P_1 + \frac{1}{3} P_2 + \frac{m-4}{m-3} P_3.
\]

Example 11. Let \( U \ni P \) be a germ of a Du Val singularity of type \( \mathbb{A}_m \), and let \( f: W \to U \) be the minimal resolution of this singularity. Let \( Q \) be a point on one of the two exceptional curves that correspond to “tails” of the dual graph such that \( Q \) is not contained in any other exceptional curve. Let \( \xi: \mathring{W} \to W \) be the blow up at \( Q \), and \( \zeta \) be the contraction of the proper transforms of all the \( f \)-exceptional curves. Thus, there exists a commutative diagram

\[ W \xrightarrow{\xi} \mathring{W} \xrightarrow{\zeta} Y \xrightarrow{g} U. \]

Denote the \( g \)-exceptional curve by \( E_Y \). Then \( Y \) has a unique singular point \( O \), the dual graph of the exceptional curves of its minimal resolution \( \zeta: \mathring{W} \to Y \) is a chain, the self-intersection numbers of the exceptional curves of \( \zeta \) are \(-3, -2, \ldots, -2\), and the proper transform of \( E_Y \) intersects only the “tail” component of this chain. The curve \( E_Y \) is smooth, and it contains the singular point \( O \). By [28, Theorem 4.15(3)] the log pair \( (Y, E_Y) \) has purely log terminal singularities. Also, the divisor \(- (K_Y + E_Y)\) is \( g \)-ample. Thus, the curve \( E_Y \) is a Kollár component of the singularity \( U \ni P \). Moreover, we have

\[
\alpha(E_Y, \text{Diff}_{E_Y}(0)) = 2 \text{ if } m = 6, \quad 3 \text{ if } m = 7, \quad 6 \text{ if } m = 8.
\]

Example 12. Let \( U \ni P \) be a germ of a Du Val singularity of type \( \mathbb{A}_m \), \( m \geq 2 \), and let \( f: W \to U \) be the minimal resolution of this singularity. Let \( Q \) be the intersection point of the \( k \)-th and \( (k+1) \)-th exceptional curves of \( f \), where \( 1 \leq k \leq \frac{m}{2} \). Let \( \xi: \mathring{W} \to W \) be the
blow up at $Q$, and $\zeta$ be the contraction of the proper transforms of all the $f$-exceptional curves. As in Example 11 there is a commutative diagram

\[
\begin{array}{ccc}
\hat{W} & \xrightarrow{\zeta} & Y \\
\xi \circ f & \downarrow & \downarrow g \\
U. & \rightarrow & \rightarrow Y
\end{array}
\]

Denote the $g$-exceptional curve by $E_Y$. Then $Y$ has two singular points $P_1$ and $P_2$, the dual graphs of the exceptional curves of the minimal resolution of singularities $\hat{\zeta}: \hat{W} \to Y$ are chains such that the self-intersection numbers of the exceptional curves are $-3, -2, \ldots, -2$, and the proper transform of $E_Y$ intersects only the “tail” components of these chains. The curve $E_Y$ is smooth, and it contains both the points $P_1$ and $P_2$. By [28, Theorem 4.15(3)] the log pair $(Y, E_Y)$ has purely log terminal singularities. Also, the divisor $-(K_Y + E_Y)$ is $g$-ample. Thus, the curve $E_Y$ is a Kollár component of the singularity $U \ni P$. As in Example 11 one can check that each $P_i$ is a cyclic quotient singularity of the surface $Y$, which is a quotient of $\mathbb{C}^2$ by the cyclic group $\mu_{2n_i+1}$, where $n_1 = k$ and $n_2 = m - k$. This implies

\[
\text{Diff}_{E_Y}(0) = \frac{2k}{2k+1}P_1 + \frac{2(m - k)}{2(m - k) + 1}P_2.
\]

Therefore,

\[
\alpha(E_Y, \text{Diff}_{E_Y}(0)) = \frac{2k + 1}{2m + 2} \leq \frac{1}{2}.
\]

In particular, we see that $\alpha(E_Y, \text{Diff}_{E_Y}(0)) = \frac{1}{2}$ if and only if $m$ is even, and $Q$ is the “central point” of the configuration of the $f$-exceptional curves.

It is easy to see from [28, Theorem 4.15] that if $U \ni P$ is a Du Val singularity of type $D$ or $E$, and the exceptional curve $E_W$ in Example 10 is not chosen to be the “fork” of the dual graph, then the corresponding curve $E_Y$ is not a Kollár component. This is not a coincidence: we will see later that in these cases the singularity $U \ni P$ has a unique Kollár component, which is described in Example 10. This is not true in general, i.e., a Kollár component of a singularity $U \ni P$ may not be unique, as one can see from Examples 10, 11 and 12. Nevertheless, Li and Xu established in [31, Theorem B] the following:

**Theorem 13.** A K-semistable Kollár component of $U \ni P$ is unique if it exists.

The K-semistable Kollár components of two-dimensional Du Val singularities are described in our Examples 10 and 12. They are precisely the Kollár components whose $\alpha$-invariants are at least $\frac{1}{2}$ (cf. [32, Example 4.7]).

Note that Du Val singularities are two-dimensional rational quasi-homogeneous isolated hypersurface singularities. The K-semistable Kollár components of many three-dimensional rational quasi-homogeneous isolated hypersurface singularities have been described in [9, 15]. Similarly, the K-semistable Kollár components of many four-dimensional rational quasi-homogeneous isolated hypersurface singularities have been described in [23].

The purpose of this paper is to prove the following analogue of Theorem 5.
**Theorem 14.** Suppose that there is a commutative diagram

\[
\begin{array}{ccc}
X & \xrightarrow{\rho} & Y \\
\phi \downarrow & & \psi \downarrow \\
U & \xrightarrow{} & \end{array}
\]

where \(\psi\) is a birational morphism such that its exceptional locus consists of a single prime divisor \(E_Y\) with \(\psi(E_Y) = P\), the log pair \((Y, E_Y)\) has purely log terminal singularities, and the divisor \(-(K_Y + E_Y)\) is \(\psi\)-ample. Suppose also that

\[
\alpha(E_X, \text{Diff}_{E_X}(0)) + \alpha(E_Y, \text{Diff}_{E_Y}(0)) \geq 1.
\]

Then \(\rho\) is an isomorphism.

Before proving this result, let us consider its applications. Suppose that

\[
\alpha(E_X, \text{Diff}_{E_X}(0)) \geq \dim(U) - 1.
\]

By Theorem 14, this inequality implies that the \(\alpha\)-invariant of another Kollár component of the singularity \(U \ni P\), if any, must be less than \(\frac{1}{\dim(U)}\), so that it should be K-unstable. Of course, this also follows from Theorem 13, because the inequality (15) implies that the log Fano variety \((E_X, \text{Diff}_{E_X}(0))\) is K-semistable.

Theorem 14 also implies

**Corollary 16.** If \(\alpha(E_X, \text{Diff}_{E_X}(0)) \geq 1\), then the Kollár component of \(U \ni P\) is unique.

This corollary is well known: it follows from [39, Theorem 4.3] and [30, Theorem 2.1]. Recall from [39] Definition 4.1 that the singularity \(U \ni P\) is said to be weakly exceptional if it has a unique purely log terminal blow-up. This is equivalent to the condition that there is a Kollár component \(E_X\) of \(U \ni P\) such that \(\alpha(E_X, \text{Diff}_{E_X}(0)) \geq 1\), see [39, Theorem 4.3], [30, Theorem 2.1]. It follows from Example 10 that Du Val singularities of types \(\mathbb{D}\) and \(\mathbb{E}\) are weakly exceptional. On the other hand, Du Val singularities of type \(\mathbb{A}\) are not weakly exceptional, since each of them admits several Kollár components (see Examples 10, 11, and 12), and thus has several purely log terminal blow-ups.

**Remark 17.** Du Val singularities are special examples of two-dimensional quotient singularities. Note that quotient singularities are always Kawamata log terminal. For each of them, it is easy to describe one Kollár component. Let \(\hat{G}\) be a finite subgroup in \(\text{GL}_{n+1}(\mathbb{C})\). Suppose that \(U \ni P\) is a quotient singularity \(\mathbb{C}^{n+1}/\hat{G}\). By the Chevalley–Shephard–Todd theorem, we may assume that the group \(\hat{G}\) does not contain quasi-reflections (cf. [13, Remark 1.16]). Let \(\eta: \mathbb{C}^{n+1} \to U\) be the quotient map. Then there is a commutative diagram

\[
\begin{array}{ccc}
W & \xrightarrow{\omega} & Y \\
\pi \downarrow & & \psi \downarrow \\
\mathbb{C}^{n+1} & \xrightarrow{\eta} & U
\end{array}
\]

where \(\pi\) is the blow up at the origin, the morphism \(\omega\) is the quotient map that is induced by the action of \(\hat{G}\) lifted to the variety \(W\), and \(\psi\) is a birational morphism. Denote by \(\tilde{E}\) the exceptional divisor of \(\pi\), and denote by \(E_Y\) the exceptional divisor of \(\psi\). Then \(\tilde{E} \cong \mathbb{P}^n\), and \(E_Y\) is naturally isomorphic to the quotient \(\mathbb{P}^n/G\), where \(G\) is the image of the group.
\[ \hat{G} \] in \( \text{PGL}_{n+1}(\mathbb{C}) \). Moreover, the log pair \((Y, E_Y)\) has purely log terminal singularities, and the divisor \(-(K_Y + E_Y)\) is \(\psi\)-ample. Thus, the log Fano variety \((E_Y, \text{Diff}_{E_Y}(0))\) is a Kollár component of the singularity \(U \ni P\). Also, it follows from [31, Example 7.1(1)] and [31, Theorem 1.2] that \(E_Y\) is K-semistable. Furthermore, one has
\[ \alpha(E_Y, \text{Diff}_{E_Y}(0)) = \alpha_G(\mathbb{P}^n), \]
see [12, Proof of Theorem 3.16]. Thus, if \(\alpha_G(\mathbb{P}^n) \geq 1\), then this Kollár component is unique by Corollary [16]. One can find many subgroups \(G \subset \text{PGL}_{n+1}(\mathbb{C})\) with \(\alpha_G(\mathbb{P}^n) \geq 1\) in [33, 12, 13, 41, 14, 42, 16]. Note also that one always has \(\alpha_G(\mathbb{P}^n) \leq 1184036\) by [46].

In the remaining part of the paper, we prove Theorem [13]. Let us use its assumptions and notations. We have to show that \(\rho\) is an isomorphism. Suppose that this is not the case. Let us seek for a contradiction.

We may assume that \(U\) is affine. There exists a commutative diagram

\[
\begin{array}{ccc}
W & \xrightarrow{f} & X \\
\downarrow{g} & & \downarrow{\rho} \\
Y & \xleftarrow{\phi} & U \\
\end{array}
\]

such that \(W\) is a smooth variety, and \(f\) and \(g\) are birational morphisms. Denote by \(E_X^W\) and \(E_Y^W\) the proper transforms of the divisors \(E_X\) and \(E_Y\) on the variety \(W\), respectively. Then \(E_X^W\) is \(g\)-exceptional, and \(E_Y^W\) is \(f\)-exceptional. We may assume that \(E_X^W\), \(E_Y^W\) and the remaining exceptional divisors of \(f\) and \(g\) form a divisor with simple normal crossings.

Observe that \(E_X^W \neq E_Y^W\). Indeed, if \(E_X^W = E_Y^W\), then \(\rho\) is small, which is impossible, because \(-E_X\) is \(\phi\)-ample, and \(-E_Y\) is \(\psi\)-ample (see [17, Proposition 2.7]). Let \(F_1, \ldots, F_m\) be the prime divisors on \(W\) that are contracted by both \(f\) and \(g\). Then
\[
K_W + E_X^W + aE_Y^W + \sum_{i=1}^{m} a_iF_i \sim_Q f^*(K_X + E_X)
\]
for some rational numbers \(a, a_1, \ldots, a_m\). Since the log pair \((X, E_X)\) has purely log terminal singularities, all numbers \(a, a_1, \ldots, a_m\) are strictly less than 1. Also, we have
\[
E_X^W \sim_Q f^*(E_X) - bE_Y^W - \sum_{i=1}^{m} b_iF_i,
\]
where \(b, b_1, \ldots, b_m\) are non-negative rational numbers. Then \(b > 0\), since \(f(E_Y^W) \subset E_X\).

Fix an integer \(n \gg 0\). Put \(M_X = \lvert -nE_X \rvert\). Then \(M_X\) does not have base points. Denote its proper transforms on \(Y\) and \(W\) by \(M_X^W\) and \(M_X^W\), respectively. Then
\[
M_X^W \sim_Q f^*(nE_X) \sim_Q -nE_X^W - nbE_Y^W - \sum_{i=1}^{m} nb_iF_i,
\]
which implies that \(M_X^W \sim_Q -nbE_Y\). On the other hand, we have \(-(K_Y + E_Y) \sim_Q -\delta_Y E_Y\) for some positive rational number \(\delta_Y\). Put \(\epsilon_X = \frac{2\delta_Y}{nb}\). Then \(\epsilon_X M_X^Y \sim_Q -(K_Y + E_Y)\), so
Now take any non-negative rational numbers $K$. Since $a < \alpha$, there exist rational numbers $\alpha, \alpha$ such that

$$\epsilon_Y M_X^W \sim_{Q} -(K + E_X),$$

so that

$$K_W + E_Y^W + \epsilon_X M_X^W + a E_Y^W + \sum_{i=1}^{m} \alpha_i F_i \sim_{Q} g^* \left( K_Y + E_Y + \epsilon_X M_X^Y \right) \sim_{Q} 0$$

for some rational numbers $\alpha, \alpha_1, \ldots, \alpha_m$. Similarly, let $M_Y$ be the base point free linear system $|-nE_Y|$. Denote by $M_X^Y$ and $M_Y^W$ its proper transforms on $X$ and $W$, respectively. Then there is a positive rational number $\epsilon_Y$ such that $\epsilon_Y M_X^W \sim_{Q} -(K + E_X)$, so that

$$K_W + E_Y^W + \epsilon_Y M_Y^W + \beta E_Y^W + \sum_{i=1}^{m} \beta_i F_i \sim_{Q} f^* \left( K_X + E_X + \epsilon_Y M_Y^X \right) \sim_{Q} 0$$

for some rational numbers $\beta, \beta_1, \ldots, \beta_m$.

**Lemma 18.** One has $\alpha > 1$ and $\beta > 1$. In particular, the singularities of the log pairs $(Y, E_Y + \epsilon_X M_X^X)$ and $(X, E_X + \epsilon_Y M_Y^Y)$ are not log canonical.

**Proof.** It is enough to show that $\alpha > 1$. We have

$$E_Y^W + \epsilon_X M_X^W + a E_X^W + \sum_{i=1}^{m} \alpha_i F_i \sim_{Q} 0 \sim_{Q} E_X^W + a E_Y^W + \sum_{i=1}^{m} \alpha_i F_i - f^* \left( K_X + E_X \right).$$

This gives

$$\epsilon_X M_X^W \sim_{Q} (1 - \alpha) E_X^W + (a - 1) E_Y^W + \sum_{i=1}^{m} (a_i - \alpha_i) F_i - f^* \left( K_X + E_X \right).$$

(19)  

It implies that

$$\epsilon_X M_X \sim_{Q} -(K + E_X) - (\alpha - 1) E_X.$$  

Recall that $-(K + E_X) \sim_{Q} -\delta_X E_X$. We then obtain

$$\epsilon_X M_X \sim_{Q} -\left( K_X + E_X \right) - (\alpha - 1) E_X \sim_{Q} -t_X \left( K_X + E_X \right),$$

where $t_X = 1 + \frac{1}{\delta_X} > 1$. On the other hand, from (19) we obtain

$$(1 - \alpha) E_X^W + \sum_{i=1}^{m} (a_i - \alpha_i) F_i \sim_{Q} (1 - a) E_Y^W + (1 - t_X) f^* \left( K_X + E_X \right).$$

Since $a < 1$, Negativity Lemma (see [28] Lemma 3.39) implies $\alpha > 1$. \qed

As in the proof of Lemma 18, put $t_Y = 1 + \frac{1}{\delta_Y} > 1$. Then

$$\epsilon_Y M_Y \sim_{Q} -t_Y \left( K_Y + E_Y \right).$$

Now take any non-negative rational numbers $\lambda$ and $\mu$ such that $\lambda + \mu \geq 1$. One has

$$K_X + E_X + \lambda \epsilon_Y M_Y^X + \mu \epsilon_X M_X \sim_{Q} -(\lambda + \mu t_X - 1) \left( K_X + E_X \right),$$

so that $K_X + E_X + \lambda \epsilon_Y M_Y^X + \mu \epsilon_X M_X$ is $\phi$-ample. Similarly, we see that

$$K_Y + E_Y + \lambda \epsilon_Y M_Y + \mu \epsilon_X M_X^Y \sim_{Q} -(\lambda t_Y + \mu - 1) \left( K_Y + E_Y \right),$$

so that $K_Y + E_Y + \lambda \epsilon_Y M_Y + \mu \epsilon_X M_X^Y$ is $\psi$-ample.

**Lemma 20.** At least one of the log pairs $(X, E_X + \lambda \epsilon_Y M_Y^X)$ and $(Y, E_Y + \mu \epsilon_X M_X^Y)$ is not log canonical.
Proof. Suppose that both \((X, E_X + \lambda \epsilon_Y \mathcal{M}^X_Y)\) and \((Y, E_Y + \mu \epsilon_X \mathcal{M}^Y_X)\) are log canonical. Then the log pairs \((X, E_X + \lambda \epsilon_Y \mathcal{M}^X_Y + \mu \epsilon_X \mathcal{M}^X_X)\) and \((Y, E_Y + \lambda \epsilon_Y \mathcal{M}^Y_Y + \mu \epsilon_X \mathcal{M}^Y_X)\) are also log canonical. On the other hand, we have

\[
K_W + E_X^W + \lambda \epsilon_Y \mathcal{M}^X_Y + \mu \epsilon_X \mathcal{M}^X_X + cE_X^W + \sum_{i=1}^{m} c_i F_i \sim_{Q} f^* \left( K_X + E_X + \lambda \epsilon_Y \mathcal{M}^X_Y + \mu \epsilon_X \mathcal{M}^X_X \right)
\]

for some rational numbers \(c, c_1, \ldots, c_m\) that do not exceed 1. Similarly, we have

\[
K_W + E_Y^W + \lambda \epsilon_Y \mathcal{M}^Y_Y + \mu \epsilon_X \mathcal{M}^Y_X + dE_Y^W + \sum_{i=1}^{m} d_i F_i \sim_{Q} g^* \left( K_Y + E_Y + \lambda \epsilon_Y \mathcal{M}^Y_Y + \mu \epsilon_X \mathcal{M}^Y_X \right),
\]

where \(d, d_1, \ldots, d_m\) are rational numbers that do not exceed 1. Denote by \(D_W\) the boundary \(\lambda \epsilon_Y \mathcal{M}^W_Y + \mu \epsilon_X \mathcal{M}^W_X + E_X^W + E_Y^W + \sum_{i=1}^{m} F_i\). Then

\[
K_W + D_W \sim_{Q} f^* \left( K_X + E_X + \lambda \epsilon_Y \mathcal{M}^X_Y + \mu \epsilon_X \mathcal{M}^X_X \right) + (1-c)E_Y^W + \sum_{i=1}^{m} (1-c_i) F_i \sim_{Q} g^* \left( K_Y + E_Y + \lambda \epsilon_Y \mathcal{M}^Y_Y + \mu \epsilon_X \mathcal{M}^Y_X \right) + (1-d)E_X^W + \sum_{i=1}^{m} (1-d_i) F_i.
\]

Moreover, the log pair \((W, D_W)\) is log canonical, since \(W\) is smooth, the linear systems \(\mathcal{M}^W_Y\) and \(\mathcal{M}^W_X\) are free from base points, and the divisors \(E_X^W, E_Y^W, F_1, \ldots, F_m\) form a simple normal crossing divisor. Since \(K_X + E_X + \lambda \epsilon_Y \mathcal{M}^X_Y + \mu \epsilon_X \mathcal{M}^X_X\) is \(\psi\)-ample, it follows from [28, Corollary 3.53] that the log pair \((X, E_X + \lambda \epsilon_Y \mathcal{M}^X_Y + \mu \epsilon_X \mathcal{M}^X_X)\) is the canonical model of the log pair \((W, D_W)\). Similarly, the log pair \((Y, E_Y + \lambda \epsilon_Y \mathcal{M}^Y_Y + \mu \epsilon_X \mathcal{M}^Y_X)\) is also the canonical model of the log pair \((W, D_W)\), because \(K_Y + E_Y + \lambda \epsilon_Y \mathcal{M}^Y_Y + \mu \epsilon_X \mathcal{M}^Y_X\) is \(\psi\)-ample. Since the canonical model is unique by [28, Theorem 3.52], we see that \(\rho\) is an isomorphism. Since \(\rho\) is not an isomorphism by assumption, we obtain a contradiction. This completes the proof of the lemma.

Let \(\lambda = \alpha(E_X, \text{Diff}_{E_X}(0))\) and \(\mu = \alpha(E_Y, \text{Diff}_{E_Y}(0))\). We may assume that the log pair \((X, E_X + \lambda \epsilon_Y \mathcal{M}^X_Y)\) is not log canonical. Then \((E_X, \text{Diff}_{E_X}(0) + \lambda \epsilon_Y \mathcal{M}^X_Y|_{E_X})\) is not log canonical by Inversion of adjunction, see [27, 17.6]. On the other hand, we have

\[
\epsilon_Y \mathcal{M}^X_Y \bigg|_{E_X} \sim_{Q} \left( K_X + E_X \right) \bigg|_{E_X} \sim_{Q} \left( K_{E_X} + \text{Diff}_{E_X}(0) \right).
\]

This is impossible by the definition of the \(\alpha\)-invariant \(\alpha(E_X, \text{Diff}_{E_X}(0))\).

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References

[1] I. Cheltsov, *Log canonical thresholds on hypersurfaces*, Sb. Math. **192** (2001), 1241–1257.
[2] I. Cheltsov, *Log canonical thresholds of del Pezzo surfaces*, Geom. Funct. Anal. **18** (2008), 1118–1144.
[3] I. Cheltsov, *Fano varieties with many selfmaps*, Adv. Math. **217** (2008), 97–124.
[4] I. Cheltsov, *On singular cubic surfaces*, Asian J. Math. **19** (2009), 191–214.
[5] I. Cheltsov, *Log canonical thresholds of three-dimensional Fano hypersurfaces*, Izv. Math. **73** (2009), 727–795.
[6] I. Cheltsov, *Extremal metrics on two Fano varieties*, Sb. Math. **200** (2009), 95–132.
[7] I. Cheltsov, D. Kosta, *Computing $\alpha$-invariants of singular del Pezzo surfaces*, J. Geom. Anal. **24** (2014), 798–842.
[8] I. Cheltsov, J. Park, *Global log-canonical thresholds and generalized Eckardt points*, Sb. Math. **193** (2002), 779–789.
[9] I. Cheltsov, J. Park, C. Shramov, *Exceptional del Pezzo hypersurfaces*, J. Geom. Anal. **20** (2010), 787–816.
[10] I. Cheltsov, J. Park, J. Won, *Log canonical thresholds of certain Fano 3-folds*, Math. Z. **276** (2014), 51–79.
[11] I. Cheltsov, C. Shramov, *Log canonical thresholds of smooth Fano threefolds*, Russian Math. Surveys **63** (2008), 73–180.
[12] I. Cheltsov, C. Shramov, *On exceptional quotient singularities*, Geom. Topol. **15** (2011), 1843–1882.
[13] I. Cheltsov, C. Shramov, *Six-dimensional exceptional quotient singularities*, Math. Res. Lett. **18** (2011), 1121–1139.
[14] I. Cheltsov, C. Shramov, *Sporadic simple groups and quotient singularities*, Izv. Math. **77** (2013), 846–854.
[15] I. Cheltsov, C. Shramov, *Del Pezzo zoo*, Exp. Math. **22** (2013), 313–326.
[16] I. Cheltsov, C. Shramov, *Weakly-exceptional singularities in higher dimensions*, J. Reine Angew. Math. **689** (2014), 201–241.
[17] A. Corti, *Factoring birational maps of threefolds after Sarkisov*, J. Algebraic Geom. **4** (1995), 223–254.
[18] A. Corti, *Del Pezzo surfaces over Dedekind schemes*, Ann. of Math. **144** (1996), 641–683.
[19] K. Fujita, *K-stability of Fano manifolds with not small alpha invariants*, to appear in J. Inst. Math. Jussieu.
[20] K. Fujita, *Uniform K-stability and plt blowups of log Fano pairs*, [arXiv:1701.00203] (2017).
[21] K. Fujita, *Openness results for uniform K-stability*, to appear in arXiv.
[22] K. Fujita, Y. Odaka, *On the K-stability of Fano varieties and anticanonical divisors*, to appear in Tohoku Math. J.
[23] J. Johnson, J. Kollár, *Fano hypersurfaces in weighted projective 4-spaces*, Exp. Math. **10** (2001), 151–158.
[24] I. Kim, J. Park, *Log canonical thresholds of complete intersection log del Pezzo surfaces*, Proc. Edinb. Math. Soc. **58** (2015), 445–483.
[25] I. Kim, T. Okada, J. Won, *Alpha invariants of birationally rigid Fano threefolds*, to appear in Int. Math. Res. Not.
[26] J. Kollár, *Singularities of pairs*, Proc. Sympos. Pure Math. **62** (1997), 221–287.
[27] J. Kollár et al., *Flips and abundance for algebraic threefolds*, Societe Mathematique de France, Astérisque **211**, 1992.
[28] J. Kollár, S. Mori, *Birational geometry of algebraic varieties*, Cambridge University Press (1998).
[29] I. Krylov, *Rationally connected non-Fano type varieties*, [arXiv:1406.3752] (2014).
[30] S. Kudryavtsev, *Pure log terminal blow-Ups*, Math. Notes **69**:6, 814–819 (2001).
[31] C. Li, C. Xu, *Stability of valuations and Kollár components*, [arXiv:1604.05398] (2016).
[32] Y. Liu, *The volume of singular Kähler–Einstein Fano varieties*, [arXiv:1605.01034] (2016).
[33] D. Markushevich, Yu. Prokhorov, *Exceptional quotient singularities*, Amer. J. Math. **121** (1999), 1179–1189.
[34] Y. Odaka, Y. Sano, *Alpha invariants and K-stability of Q-Fano varieties*, Adv. Math. **217** (2008), 97–124.
[35] J. Park, *Birational maps of del Pezzo fibrations*, J. Reine Angew. Math. **538** (2001), 213–221.
[36] J. Park, *A note on del Pezzo fibrations of degree 1*, Comm. Algebra 31 (2003), 5755–5768.
[37] J. Park, J. Won, *Log canonical thresholds on Gorenstein canonical del Pezzo surfaces*, Proc. Edinb. Math. Soc. 54 (2011), 187–219.
[38] J. Park, J. Won, *Log canonical thresholds on del Pezzo surfaces of degree $\geq 2$*, Nagoya Math. J. 200 (2010), 1–26.
[39] Yu. Prokhorov, *Blow-ups of canonical singularities*, Algebra (Moscow, 1998), de Gruyter, Berlin (2000), 301–317.
[40] A. Pukhlikov, *Birational geometry of Fano direct products*, Izv. Math. 69 (2005), 1225–1255.
[41] D. Sakovics, *Weakly-exceptional quotient singularities*, Cent. Eur. J. Math. 10 (2012), 885–902.
[42] D. Sakovics, *Five-dimensional weakly exceptional quotient singularities*, Proc. Edinb. Math. Soc. 57 (2014), 269–279.
[43] Y. Shi, *On the $\alpha$-invariants of cubic surfaces with Eckardt points*, Adv. Math. 225 (2010), 1285–1307.
[44] V. Shokurov, *Three-fold log flips*, Russian Acad. Sci. Izv. Math. 40 (1993), 95–202.
[45] G. Tian, *On Kähler–Einstein metrics on certain Kähler manifolds with $c_1(M) > 0$*, Invent. Math. 89 (1987), 225–246.
[46] P. Tiep, *The $\alpha$-invariant and Thompson’s conjecture*, Forum Math. Pi 4 (2016), e5, 28 pp.
[47] C. Xu, *Finiteness of algebraic fundamental groups*, Compos. Math. 150 (2014), 409–414.

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