On the asymptotic normality of kernel density estimators for linear random fields

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Abstract

We establish sufficient conditions for the asymptotic normality of kernel density estimators, applied to causal linear random fields. Our conditions on the coefficients of linear random fields are weaker than known results, although our assumption on the bandwidth is not minimal. The proof is based on the \( m \)-approximation method. As a key step, we prove a central limit theorem for triangular arrays of stationary \( m \)-dependent random fields with unbounded \( m \). We also apply a moment inequality recently established for stationary random fields.

Keywords: central limit theorem, \( m \)-dependence, moment inequality.

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1 Introduction

Let \( \{X_i\}_{i \in \mathbb{Z}^d}, d \in \mathbb{N} \) be a stationary zero-mean random field, such that the marginal probability density function \( p(\cdot) \) exists. We are interested in the Parzen–Rosenblatt kernel density estimator of \( p(x) \) in the form of

\[
f_n(x) = \frac{1}{n^d b_n} \sum_{i \in [1,n]^d} K\left(\frac{x - X_i}{b_n}\right), \quad x \in \mathbb{R}.
\] (1)

Throughout this paper, we assume that the kernel \( K : \mathbb{R} \to \mathbb{R} \) is a bounded Lipschitz-continuous density function, and the bandwidth \( b_n \) satisfies

\[
b_n \to 0 \quad \text{and} \quad n^d b_n \to \infty \quad \text{as} \quad n \to \infty.
\] (2)

We also write, for \( a, b \in \mathbb{Z}, [a, b] \equiv \{a, a+1, \ldots, b\} \).

This problem was first considered by Rosenblatt [20] and Parzen [15], in the case that \( X_i \)’s are independent and identically distributed (i.i.d.) random variables: in particular, one can show the consistency

\[
\lim_{n \to \infty} f_n(x) = p(x),
\]

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and the asymptotic normality
\[(n^d b_n)^{1/2}(f_n(x) - \mathbb{E} f_n(x)) \Rightarrow \mathcal{N}(0, \sigma_x^2) \text{ as } n \to \infty,\]
where \(\sigma_x^2 = p(x) \int K^2(s) ds\). See for example Silverman [21] for more references on density estimation problems with i.i.d. data.

The case that \(X_i\)'s are dependent, however, has presented more challenges, and we focus on establishing the asymptotic normality (3) in this paper. The dependent one-dimensional case has been considered by Robinson [19], Castellana and Leadbetter [2], Bosq et al. [1], Wu and Mielniczuk [27] and Dedecker and Merlevède [8], among others. In particular, Wu and Mielniczuk [27] investigated thoroughly the case when \(\{X_i\}_{i \in \mathbb{Z}}\) is a linear process. That is,
\[X_i = \sum_{k=-\infty}^{\infty} a_k \epsilon_{i-k}, \ i \in \mathbb{Z},\]
where \(\sum_k a_k^2 < \infty\) and the innovations \(\{\epsilon_i\}_{i \in \mathbb{Z}}\) are i.i.d. random variables. Linear processes are important in the study of stationary processes, as any stationary process can be represented as linear combinations of linear processes (the so-called superlinear processes) with martingale-difference innovations (Volný et al. [24]).

The asymptotic normality of kernel density estimators for random fields has been considered by Tran [22], Hallin et al. [13], Cheng et al. [4] and El Machkouri [10, 11], among others. The extension of results in one dimension to high dimensions, however, is not trivial. As summarized in Hallin et al. [13], 'the points of \(\mathbb{Z}^d\) do not have a natural ordering. As a result, most techniques available for one-dimensional processes do not extend to random fields.' See more references in [13] on related discussions.

In particular, a notorious difficulty for kernel density estimation of random fields, is that one often needs more assumptions on the bandwidth \(b_n\) than the minimal one (2). This condition is minimal in the sense that it is the natural condition for the asymptotic normality (3) to hold when \(X_i\)'s are i.i.d. To the best of our knowledge, only the recent results by El Machkouri [10] [11] assume no other but minimal condition (2) on \(b_n\) for dependent random fields.

In this paper, we focus on the kernel density estimation for causal linear random fields \(\{X_i\}_{i \in \mathbb{Z}^d} (d \in \mathbb{N})\) in form of
\[X_i = \sum_{k \in \mathbb{Z}^d, k \geq 0} a_k \epsilon_{i-k}, \ i \in \mathbb{Z}^d,\]
where \(\sum_{i \geq 0} a_i^2 < \infty\) and \(\{\epsilon_i\}_{i \in \mathbb{Z}^d}\) are i.i.d. zero-mean random variables with finite second moments. Throughout this paper, we let ‘\(i \geq k\)’ denote ‘\(i_\tau \geq k_\tau\)’
for all \( \tau = 1, \ldots, d' \) for \( i, k \in \mathbb{Z}^d \), and write \( 0 = (0, \ldots, 0), 1 = (1, \ldots, 1) \in \mathbb{Z}^d \).

We provide new conditions on the coefficient \( \{a_i\}_{i \in \mathbb{Z}^d} \) such that the asymptotic normality (3) holds (see Theorem 1 below), and compare with results obtained by Hallin et al. [13] and El Machkouri [11]. In both cases, our conditions are weaker on the coefficients \( \{a_i\}_{i \in \mathbb{Z}^d} \). On the other hand, our condition on the bandwidth improves the one in [13], but it is still stronger than the minimal one (2) assumed in [11]. We do not compare our result with Cheng et al. [4], as there is a mistake in their proof (see Remark 6 below).

Our proof is based on the \( m \)-approximation approach. As we will see, to address this problem one has to establish an \( m \)-approximation with unbounded \( m \) \((m_n \to \infty \text{ as } n \to \infty)\). As a key step of our approach, we establish a central limit theorem for triangular arrays of stationary \( m \)-dependent random fields with unbounded \( m \) (Theorem 2). This result improves a central limit theorem established by Heinrich [14]. Our \( m \)-approximation method is also involved with certain moment inequalities for stationary random fields (Lemma 2). These moment inequalities are variations of the one established in Wang and Woodroofe [25], based on the maximal inequalities for stationary sequences \((d = 1)\) by Peligrad and Utev [16] (see also [17, 23]). In general, the \( m \)-approximation method has been successful in proving central limit theorems for random fields (see e.g. Cheng et al. [3], Wang and Woodroofe [25] and El Machkouri et al. [12]). In particular, El Machkouri [10, 11] also established \( m \)-approximations with unbounded \( m \), combined with Lindeberg’s method (see e.g. Rio [18] and Dededecker [6]), to prove asymptotic normality.

At last, we point out that when the asymptotic normality (3) holds, the random variables are often said to have \emph{weak dependence}, in the sense that they behave asymptotically as i.i.d. random variables. On the other hand, when the dependence is strong enough, the normalization for obtaining limiting distributions is of different order from \( n^{d'b_n} \) in (3), and the asymptotic limit may be no longer Gaussian (see e.g. Cs"orgo and Mielniczuk [5] for one-dimensional case). These two regimes are sometimes referred to as \emph{short-range dependence} and \emph{long-range dependence}, respectively. For linear processes, Wu and Mielniczuk [27] addressed both short-range and long-range dependence cases. For the linear random fields, however, to the best of our knowledge, the long-range dependence case remains open. It seems that the \( m \)-approximation method is limited to the short-range dependence case. Therefore, the long-range dependence case is beyond the scope of this paper.

The paper is organized as follows. Our assumptions and main results are presented in Section 2. Examples and comparison with other results are provided in Section 3. Section 4 is devoted to the central limit theorem for triangular arrays of \( m \)-dependent random fields. Section 5 establishes
asymptotic normality by \(m\)-approximation. Auxiliary proofs are given in Section 6.

2 Assumptions and the main result

We first introduce our conditions. For each \(m \in \mathbb{N}, i \in \mathbb{Z}^d\), write

\[
X_{i,m} = \sum_{k \in [0,m-1]^d} a_k \epsilon_{i-k} \quad \text{and} \quad \tilde{X}_{i,m} = X_i - X_{i,m}.
\]  

Let \(p, p_m\) and \(\tilde{p}_m\) denote the probability density function of \(X_0, X_{0,m}\) and \(\tilde{X}_{0,m}\), respectively. Let \(p_i\) and \(p_{i,m}\) denote the joint density functions of \((X_0, X_i)\) and \((X_{0,m}, X_{i,m})\), respectively. Our first condition is on the regularity of the density functions. Define the supremum \(\overline{p} = \sup_x p(x), \overline{p}_i = \sup_{x,y} p_i(x,y)\) and similarly \(p_m\) and \(\overline{p}_{i,m}\).

**Condition A.**

(i) The density functions \(p\) and \(\{p_m\}_{m \in \mathbb{N}}\) exist. They are \(c_0\)-Lipschitz continuous with certain constant \(c_0 < \infty\), independent of \(m\) (i.e., \(\max(|p(x) - p(y)|, |p_m(x) - p_m(y)|) \leq c_0 |x - y|\)). Furthermore,

\[
\overline{p} < \infty \quad \text{and} \quad \sup_m \overline{p}_m < \infty.
\]  

(ii) The density functions \(p_i\) and \(p_{i,m}\) exist for all \(i \neq 0, m \in \mathbb{N}\). Furthermore,

\[
\sup_i \overline{p}_i < \infty \quad \text{and} \quad \sup_m \sup_i \overline{p}_{i,m} < \infty.
\]  

Condition A can be satisfied, for example, by simply assuming that the probability density function \(p_\epsilon\) of \(\epsilon_0\) exists and is Lipschitz. This was assumed also in Wu and Mielniczuk [27].

**Lemma 1.** If \(p_\epsilon\) exists and is Lipschitz, then Condition A holds.

The proof is deferred to Section 6.

Our second condition is on the decay of coefficients and bandwidth \(b_n\).

Define

\[
A_k = \left( \sum_{i \geq k} a_i^2 \right)^{1/2}, k \in \mathbb{Z}^d \quad \text{and} \quad B_m = \left( \sum_{i \in [0,\infty]^d} a_i^2 \right)^{1/2}, m \in \mathbb{N},
\]

with \(|i|_\infty = \max_{\tau=1,\ldots,d} |i_\tau|\). Write

\[
\Delta_n = \sum_{k \in [1,n]^d} \frac{A_{k-1}}{\prod_{\tau=1}^d k_{\tau}^{1/2}},
\]  

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Condition B. There exist a sequence of integers \( \{m_n\}_{n \in \mathbb{N}} \) such that \( m_n \to \infty \) as \( n \to \infty \), and the following limits hold:

\[
\lim_{n \to \infty} B_{m_n} / b_n = 0, \quad (8)
\]

\[
\lim_{n \to \infty} m_n^d b_n = 0, \quad (9)
\]

\[
\lim_{n \to \infty} m_n^d \log d_n / n^{d/2} b_n = 0. \quad (10)
\]

Theorem 1. If Conditions 1 and 2 hold and \( \mathbb{E}(|\epsilon_0|^\alpha) < \infty \) for some \( \alpha > 2 \), then the asymptotic normality (3) holds.

We will prove Theorem 1 in Section 5. We conclude this section with a few remarks.

Remark 1. We briefly comment on each condition in Condition B.

(i) Condition (8) is slightly weaker than

\[
\Delta_\infty \equiv \sum_{k \in [1, \infty]^d} A_{k-1} / \prod_{j=1}^d k_j^{1/2} < \infty.
\]

It was shown in [25], Corollary 1 that, the above condition implies the asymptotic normality of \( \sum_{i \in [1, n]^d} [f(X_i) - \mathbb{E}f(X_0)] / n^{d/2} \) for Lipschitz continuous function \( f \) such that \( \mathbb{E}f^2(X_0) < \infty \).

(ii) Condition (9) implies that

\[
\lim_{n \to \infty} \mathbb{E} |\tilde{X}_{0,m_n}| / b_n = 0. \quad (12)
\]

Indeed, Wu [26], Lemma 4 showed that for i.i.d. zero-mean random variables \( \{\epsilon_i\}_{i \in \mathbb{Z}} \) with \( \mathbb{E}(|\epsilon_0|^{2p}) < \infty, p > 0 \),

\[
\mathbb{E}\left( \left| \sum_i a_i \epsilon_i \right|^{2p} \right) \leq C \left( \sum_i a_i^2 \right)^p. \quad (13)
\]

Intuitively, \( \tilde{X}_{0,m_n} \) can be viewed as the remainder of \( X_0 \) after the \( m_n \)-truncation. Condition (12) tells that \( m_n \) needs to tend to infinity fast enough, so that the central limit theorem holds.

(iii) Conditions (10) and (11) are useful when we apply a central limit theorem for \( m \)-dependent random variables with unbounded \( m \) in Proposition 1 below.

Throughout this paper, let \( C \) denote constants that do not depend on \( i, k, m, n, x, y \). The value of \( C \) may change from line to line.
3 Examples and discussions

Theorem 1, and particularly Condition B, is not convenient to apply for concrete models. Instead, we provide a corollary for practical reason. Write

\[ A[n] = \max\{A_{n,1,...,1}, A_{1,n,1,...,1}, \ldots, A_{1,...,1,n}\} \].

**Corollary 1.** Suppose \( A[n] \leq c_1 n^{-\beta} \) and \( \beta > 0 \), and \( b_n = c_2 n^{-\gamma} \). Then a sufficient condition such that Condition B holds is

\[ \gamma < \frac{d\beta}{d + \beta} \quad \text{and} \quad \beta > d. \]  

Consequently, if \( \mathbb{E}(|\epsilon_0|^{\alpha}) < \infty \) for some \( \alpha > 2 \), and Condition A and (14) hold, then the asymptotic normality (3) follows.

**Proof.** Assume that \( m_n \) takes the form of \( \lfloor n^{\delta} \rfloor \). Observe that \( B_{m_n} \) is of the same order of \( A[m_n] \) as \( n \to \infty \). Then, the limit conditions (9), (10) and (11) are implied by

\[ \lim_{n \to \infty} n^{-\beta\delta+\gamma} + n^{d\delta-\gamma} + n^{\delta-1} = 0, \]

which is equivalent to \( \gamma/\beta < \delta < \min\{\gamma/d, 1 - \gamma/d\} \). Since \( \beta > d \) implies that \( \Delta_\infty < \infty \), the desired result follows.

**Remark 2.** Under the assumptions of Corollary 1, Condition (14) is very close to necessary for Condition B to holds. Indeed, if \( A[n] = l(n)n^{-\beta} \) with \( \lim_{n \to \infty} l(n) = c_2 > 0 \), then the same argument above yields that Condition B is equivalent to (14).

Below, we provide examples of coefficients so that Condition B holds. We assume that \( b_n = n^{-\gamma} \) for some \( \gamma \in (0, d) \).

**Example 1.** We compare our conditions and the ones by Hallin et al. [13]. They considered the case that \( |a_i| \leq C|i|^{\alpha}, i \geq 0 \). Then, they require

\[ q > \max(d + 3, 2d + 1/2) \quad \text{and} \quad \lim_{n \to \infty} n^{d} b_n^{(2q-1+6d)/(2q-1-4d)} = \infty. \]  

Our condition (14) imposes weaker assumption in this case (with \( b_n = n^{-\gamma} \)). First, observe that

\[ A_{n,1,...,1}^2 \leq B_n^2 \leq C \sum_{i=n}^{\infty} i^{d-1} i^{-2q} \leq C n^{d-2q}. \]

We can apply Corollary 1 with \( \beta = q - d/2 \). Then, (14) becomes

\[ q > \frac{3d}{2} \quad \text{and} \quad \gamma < \frac{d q - d/2}{q + d/2}. \]  

Thus, to establish the asymptotic normality (3), our condition (16) is less restrictive than (15) on both \( q \) and \( \gamma \).
Example 2. We compare our conditions and the ones by El Machkouri [11]. Note that his results apply to general stationary random fields and the linear random fields are a specific case. In particular, he showed that for causal linear random fields, if
\[ \sum_{i \in \mathbb{Z}^d} |i|^q |a_i| < \infty \]  
with \( q = 5d/2 \), then the asymptotic normality follows.

In this case, our condition on the coefficients is weaker, requiring only \( q > d \). Indeed, suppose (17) holds with some \( q > 0 \). Then, to apply Corollary [1] it suffices to observe
\[
A_{n,1}^2 = \sum_{i_1=0}^{\infty} \sum_{i_2, \ldots, i_d \in \mathbb{N}} |a_i|^2 \leq C n^{-2q} \sum_{i_1=0}^{\infty} \sum_{i_2, \ldots, i_d \in \mathbb{N}} |i|^2 |a_i|^2 < C n^{-2q},
\]
and take \( \beta = q \).

At the same time, our result requires \( \gamma < dq/(q + d) \) for the bandwidth, in addition to the minimal one (2) assumed in [11]. Recall also that we assume \( \mathbb{E}(|\epsilon_0|^{\alpha}) < \infty \) for some \( \alpha > 2 \), while El Machkouri’s result needs only finite-second-moment assumption on \( \epsilon_0 \).

Remark 3. Finally, we compare our result to Wu and Mielniczuk [27]. In the one-dimensional case, to have asymptotic normality they assume only finite variance of \( \epsilon_0 \) and weaker assumption on the coefficient:
\[ \sum_{i=0}^{\infty} |a_i| < \infty. \]  
(18)

This is weaker than our condition in one dimension (with \( q > d = 1 \) in [17]).

Wu and Mielniczuk followed a martingale approximation approach. It remains an open question that in high dimension, whether the condition \( q > d \) in (17) can be improved to match (18) in dimension one.

4 A central limit theorem for \( m \)-dependent random fields

In this section, we prove a central limit theorem for stationary triangular arrays of \( m \)-dependent random fields. Throughout this section, let \( \{Y_{n,i} : i \in \mathbb{N}^d\}_{n \in \mathbb{N}} \) denote stationary zero-mean triangular arrays. That is, for each \( n \), \( \{Y_{n,i} : i \in \mathbb{N}^d\} \) is stationary and \( Y_{n,i} \) has zero mean. Furthermore, we assume that \( \{Y_{n,i} : i \in \mathbb{N}^d\} \) is \( m \)-dependent in the sense that \( Y_{n,i} \) and \( Y_{n,j} \) are independent if \( |i - j|_\infty \geq m \). We provide conditions such that
\[ \frac{S_n(Y)}{n^{d/2}} = \frac{\sum_{i \in [1,n]^d} Y_{n,i}}{n^{d/2}} \Rightarrow \mathcal{N}(0, \sigma^2) \quad \text{as} \quad n \to \infty. \]  
(19)
A key condition is the following:

\[
\left\| \sum_{i \in \mathbb{N}^d, 1 \leq i \leq j} Y_{n,i} \right\|_2 \leq C(j_1 \cdots j_d)^{1/2} \text{ for all } n \in \mathbb{N}, j \in \mathbb{N}^d. \tag{20}
\]

**Remark 4.** Inequality (20) has been established, under various conditions on the dependence of stationary random fields, by Dedecker [7], Wang and Woodroofe [25], and El Machkouri et al. [12], among others.

**Theorem 2.** Suppose that there exists a constant \(C\) such that (20) holds. If there exists a sequence \(\{l_n\}_{n \in \mathbb{N}} \subset \mathbb{N}, m_n/l_n \to 0\) and \(l_n/n \to 0\) as \(n \to \infty\), such that

\[
\lim_{n \to \infty} \frac{1}{l_n^d} \mathbb{E}\left[ \left( \sum_{k \in [1,l_n]^d} Y_{n,k} \right)^2 \right] = \sigma^2, \tag{21}
\]

\[
\lim_{n \to \infty} \frac{1}{l_n^d} \mathbb{E}\left[ \left( \sum_{k \in [1,l_n]^d} Y_{n,k} \right)^2 \mathbf{1}\left\{ \sum_{k \in [1,l_n]^d} |Y_{n,k}| > n^{d/2} \epsilon \right\} \right] = 0, \tag{22}
\]

for all \(\epsilon > 0\), then (19) holds.

**Proof.** Consider partial sums over big blocks of size \(l_n^d\), denoted by

\(\eta_{n,k} = \sum_{i \in [1,l_n]^d} Y_{n,i+k(l_n+m_n)}, k \in \mathbb{N}^d\).

In this way, for each \(n \in \mathbb{N}\), \(\{\eta_{n,k}\}_{k \in \mathbb{N}^d}\) are i.i.d., as we separate neighboring blocks by distance \(m_n\), and \(\{Y_{n,i}\}_{i \in \mathbb{Z}^d}\) are \(m_n\)-dependent. Set

\(S_n(\eta) = \sum_{k \in [0,\lfloor n/(l_n+m_n)\rfloor-1]^d} \eta_{n,k}, n \in \mathbb{N}\).

Then, (20) implies that

\[
\left\| \frac{S_n(Y)}{n^{d/2}} - \frac{S_n(\eta)}{n^{d/2}} \right\|_2 \to 0 \text{ as } n \to \infty.
\]

To see this, for the sake of simplicity, we consider the case \(n/(l_n+m_n) = \lfloor n/(l_n+m_n) \rfloor\). Indeed, by the triangular inequality, the left-hand side above can be bounded by sums in form of \(\| \sum_{i \in B} Y_{n,i} \|_2/n^{d/2}\), where \(B\) can be a rectangle of size \(n^{d-r}m_n^r\) with \(r \in \{1, \ldots, d-1\}\). Focusing on the dominant term with \(r = 1\), we then bound the left-hand side above by \(C(n/(l_n+m_n))^{1/2}(n^{d-1}m_n)^{1/2}/n^{d/2} = Cn^{1/2}/(l_n+m_n)^{1/2} \to \infty \text{ as } n \to \infty\).

As a consequence, it suffices to show \(S_n(\eta)/n^{d/2} \Rightarrow \mathcal{N}(0, \sigma^2)\). This, under conditions (21) and (22), follows from the standard central limit theorem for triangular arrays of independent random variables (see e.g. [9], Chapter 2, Theorem 4.5). \(\square\)
Remark 5. Central limit theorems for $m_n$-dependent random fields has been considered by Heinrich [14]. His result has been recently applied, with $m_n = m$ fixed, by El Machkouri et al. [12] to establish a central limit theorem for stationary random fields.

Our application requires us to take $m_n \to \infty$ (see Remark 6 below). In this case our condition in Theorem 2 is weaker than Heinrich's. In particular, he assumed

$$
\lim_{n \to \infty} \frac{m_n^{2d}}{n^d} \sum_{i \in [1,n]^d} \mathbb{E} \left( Y_{n,i}^2 1_{\{|Y_{n,i}| > \epsilon n^{d/2} m_n^{-2d}\}} \right) = 0, \text{ for all } \epsilon > 0.
$$

This is stronger than (22).

5 Asymptotic normality by $m$-approximation

In this section, we prove Theorem 1 by an $m$-approximation argument. Fix $x \in \mathbb{R}$ and write

$$
Z_{n,i} = \frac{1}{\sqrt{b_n}} K \left( \frac{x - X_i}{b_n} \right) \text{ and } \zeta_{n,i} = \frac{1}{\sqrt{b_n}} K \left( \frac{x - X_{i,m_n}}{b_n} \right), \quad i \in \mathbb{Z}^d.
$$

In this way, $\{\zeta_{n,i} : i \in \mathbb{Z}^d\}$ are $m_n$-dependent. We will use $\{\zeta_{n,i} : i \in \mathbb{Z}^d\}$$_{n \in \mathbb{N}}$ to approximate $\{Z_{n,i} : i \in \mathbb{Z}^d\}$$_{n \in \mathbb{N}}$. We also write $\overline{Z}_{n,i} = Z_{n,i} - \mathbb{E}Z_{n,i}$ and $\overline{\zeta}_{n,i} = \zeta_{n,i} - \mathbb{E}\zeta_{n,i}$. Setting

$$
S_n(\zeta) = \sum_{i \in [1,n]^d} \overline{\zeta}_{n,i} \quad \text{and} \quad S_n(\overline{Z} - \overline{\zeta}) = \sum_{i \in [1,n]^d} (\overline{Z}_{n,i} - \overline{\zeta}_{n,i}),
$$

we decompose

$$
(n^d b_n)^{1/2} (f_n(x) - \mathbb{E}f_n(x)) = \frac{S_n(\zeta)}{n^{d/2}} + \frac{S_n(\overline{Z} - \overline{\zeta})}{n^{d/2}}.
$$

To prove Theorem 1 it suffices to establish the following two results.

**Proposition 1.** Under Condition A and (8), (10), (11) of Condition B,

$$
\frac{S_n(\zeta)}{n^{d/2}} \Rightarrow \mathcal{N}(0, \sigma_x^2).
$$

**Proposition 2.** Under Condition A and (8), (9) of Condition B,

$$
\frac{S_n(\overline{Z} - \overline{\zeta})}{n^{d/2}} \xrightarrow{p} 0.
$$

To prove the above two propositions, a key step is to establish the following moment inequalities.
Lemma 2. There exists a constant $C > 0$, such that for all $n \in \mathbb{N}$,
\[
\|S_n(Z - \zeta)\|_2 \leq Cn^{d/2}\left(\|Z_{n,0} - \zeta_{n,0}\|_2 + b_n^{1/2}\Delta_n\right).
\] (27)

In addition, if $\mathbb{E}(|\varepsilon_0|^\alpha) < \infty$ for some $\alpha \geq 2$, then
\[
\left\| \sum_{i \in \mathbb{N}^d, 1 \leq i \leq j} \zeta_{n,i} \right\|_\alpha \leq C(j_1 \cdots j_d)^{1/2}\left(\|\zeta_{n,0}\|_\alpha + b_n^{1/2}\Delta_n\right), \text{ for all } j \in \mathbb{N}^d.
\] (28)

These inequalities are consequences of the moment inequality recently established in Wang and Woodroofe [25]. The proof is deferred to Section 6.

**Proof of Proposition 1.** Observe that $S_n(Z)/n^{d/2}$ is a partial sum of $m_n$-dependent random fields and we apply Theorem 2. Observe that since $\|\zeta_{n,0}\|_\alpha \rightarrow \sigma_x$ as $n \rightarrow \infty$, (28) with $\alpha = 2$ and assumption (8) entail (20).

Thus, to prove (25), it suffices to show, for $l_n = m_n \log n$,
\[
\lim_{n \rightarrow \infty} \frac{1}{l_n^2} \mathbb{E}\left(\sum_{i \in [1,l_n]^d} \zeta_{n,i}^2\right) = \sigma_x^2,
\] (29)

and, writing $\xi_n = \sum_{i \in [1,l_n]^d} \zeta_{n,i}$,
\[
\lim_{n \rightarrow \infty} \frac{1}{l_n^d} \mathbb{E}\left(\xi_n^2 1_{\{\|\xi_n\|_2 > n^{d/2}\epsilon\}}\right) = 0, \text{ for all } \epsilon > 0. \] (30)

By standard calculation, under (7) of Condition A for all $n \in \mathbb{N}$ and $i \neq 0$,
\[
\|\mathbb{E}(\zeta_{n,0}, \zeta_{n,i})\|_\alpha \leq C\|\zeta_{n,0}\|_\alpha b_n \leq Cb_n.
\]

Therefore,
\[
\left| \frac{1}{l_n} \mathbb{E}\left(\sum_{i \in [1,l_n]^d} \zeta_{n,i}^2\right) - \mathbb{E}\zeta_{n,0}^2\right| \leq 2 \sum_{i \in [-m_n, m_n]^d} |\mathbb{E}(\zeta_{n,0}, \zeta_{n,i})| 1_{\{i \neq 0\}} \leq Cm_n^d b_n.
\]

Thus, assumption (10) entails (29). To prove (30), observe that
\[
\mathbb{E}(\xi_n^2 1_{\{\|\xi_n\|_2 > n^{d/2}\epsilon\}}) \leq \|\xi_n\|_\alpha^2 \mathbb{P}(\|\xi_n\|_2 > n^{d/2}\epsilon)^{(\alpha - 2)/\alpha} \leq \|\xi_n\|_\alpha^2 \left(\frac{\|\xi_n\|_\alpha^2}{n^{d/2}\epsilon^2}\right)^{(\alpha - 2)/\alpha}.
\]

This time, (28) and (3) yield $\|\xi_n\|_2 \leq Cn^{d/2}$. For $\alpha > 2$, observe that, since $K$ is bounded,
\[
\|\zeta_{n,0}\|_\alpha = (\mathbb{E}|\zeta_{n,0}|^\alpha)^{1/\alpha} \leq \left(\frac{C}{b_n^{(\alpha - 2)/2}}\right)\|\zeta_{n,0}\|_2^{1/\alpha} \leq Cb_n^{(\alpha - 2)/(2\alpha)}.
\]
So, \( \|\xi_n\|_2^2 \leq C n^{d(b-\alpha)/2} \). To sum up, we have obtained that
\[
\frac{1}{n} \mathbb{E}(\xi_n^2 \mathbb{1}_{\{\|\xi_n\| > n^{d/2}\epsilon\}}) \leq C \left( \frac{d}{n^d b_n} \right)^{(\alpha-2)/\alpha}.
\]
Now, (11) entails (30). \( \square \)

**Proof of Proposition 2.** To obtain the desired result, it suffices to combine (27), assumptions (8) and (9) and Lemma 3 below.

**Lemma 3.** Under the assumption of Condition A, there exists a constant \( C \), such that for all \( n \in \mathbb{N} \),
\[
\|\sqrt{n} \xi_{n,0} - \sqrt{n} Z_{n,0}\|_2 \leq C \left[ \left( \frac{B_{m_n}}{b_n} \right)^{1/2} + b_n^{1/2} \right]. \tag{31}
\]
The proof is deferred to Section 6.

**Remark 6.** Cheng et al. [4] also considered the asymptotic normality of kernel density estimators for linear random fields. Their approach combines an \( m \)-approximation with a martingale approximation by defining an appropriate filtration in \( \mathbb{Z}^2 \). However, there is a mistake in Lemma 2 therein. In our notation, they claimed that, instead of (31), there exists a constant \( C \), such that (in the case \( d = 2 \))
\[
\|\xi_{n,0} - Z_{n,0}\|_2 \leq C b_n \text{ with } m_n \equiv m. \tag{32}
\]
To see that (32) is not true, observe that
\[
\|\xi_{n,0} - Z_{n,0}\|_2^2 = \mathbb{E} \xi_{n,0}^2 + \mathbb{E} Z_{n,0}^2 - 2 \mathbb{E}(\xi_{n,0} Z_{n,0}).
\]
By standard calculations, \( \lim_{n \to \infty} \mathbb{E} \xi_{n,0}^2 = p(x) \int K(s)^2 ds \), and if \( m_n \equiv m \), then \( \lim_{n \to \infty} \mathbb{E} Z_{n,0}^2 = m(x) \int K(s)^2 ds \) and \( \lim_{n \to \infty} \mathbb{E}(\xi_{n,0} Z_{n,0}) = 0 \). Therefore, the left-hand side of (32) has a strictly positive limit as \( n \to \infty \) (unless \( p(x) = m(x) = 0 \)), thus a contradiction. Their approach might still work by adapting an \( m \)-approximation with unbounded \( m \), although it is not clear to us what conditions it would lead to.

### 6 Proofs

**Proof of Lemma 1.** (i) The existence and Lipschitz continuity of \( p \) and \( p_m \) have been proved by Wu and Mielniczuk [27], Lemma 1. To prove (ii), observe that
\[
|p_m(y) - p(y)| \leq \int |p_m(y) - p_m(y - x)| \tilde{p}_m(x) dx \\
\leq C \int |x| \tilde{p}_m(x) dx = C \mathbb{E} |\tilde{X}_{0,m}|. \tag{33}
\]
This entails that $p_m(x) \to p(x)$ uniformly for $x \in \mathbb{R}$ as $m \to \infty$. Therefore, (6) holds.

(ii) Fix $i \in \mathbb{Z}^d \setminus \{0\}$ and let $F_i$ denote the joint distribution function of $(X_0, X_i)$. For the sake of simplicity, we prove the case of $a_0 = 1$. Write $R = X_0 - \epsilon_0$ and $R_i = X_i - \epsilon_i - a_i \epsilon_0$. Now, $R$ and $R_i$ are dependent random variables. First, we show that

$$p_i(x, y) \equiv \frac{\partial^2}{\partial x \partial y} F_i(x, y) = \mathbb{E}[p_\epsilon(x - R)p_\epsilon(y - R_i - a_i x)]. \quad (34)$$

Indeed,

$$F_i(x, y) = \mathbb{P}(X_0 \leq x, X_i \leq y) = \mathbb{P}(\epsilon_0 + R \leq x, \epsilon_i + a_i \epsilon_0 + R_i \leq y) = \mathbb{E} \Phi_i(x - R, y - R_i), \quad (35)$$

with, letting $F_\epsilon$ denote the cumulative distribution function of $\epsilon_0$,

$$\Phi_i(x, y) = \int_{-\infty}^x F_\epsilon(y - a_i x')F_\epsilon(dx').$$

Differentiating (35) yields (34) (see e.g. [9], Appendix A.9 on the validation of exchange of differentiation and expectation).

Next, we prove (7) by establishing the following two steps:

$$\lim_{|i| \to \infty} \sup_{x, y} |p_i(x, y) - p(x)p(y - a_i x)| = 0, \quad (36)$$

and

$$\lim_{m \to \infty} \sup_{x, y, i} |p_i(x, y) - p_{i,m}(x, y)| = 0. \quad (37)$$

Then, (36) implies the first part of (7), and the two limits imply the second part.

To prove (36), set

$$\tilde{D}_i = \mathbb{E}(R_i \mid \sigma(\epsilon_k : k \leq 0)) \quad \text{and} \quad D_i = R_i - \tilde{D}_i, i \in \mathbb{Z}^d.$$ 

By definition, $D_i$ and $R$ are independent. Introducing an intermediate term $\mathbb{E}[p_\epsilon(x - R)p_\epsilon(y - D_i - a_i x)] = p(x)\mathbb{E}p_\epsilon(y - D_i - a_i x)$, we then bound $|p_i(x, y) - p(x)p(y - a_i x)| \leq \Psi_1 + \Psi_2$ with, under the assumption that $p_\epsilon$ is bounded and Lipschitz,

$$\Psi_1 = |p_i(x, y) - \mathbb{E}[p_\epsilon(x - R)p_\epsilon(y - D_i - a_i x)]| \leq \mathbb{E}|p_\epsilon(x - R)| |R_i - D_i| \leq C \mathbb{E} |\tilde{D}_i|,$$
and

\[ \Psi_2 = |p(x)p(y - a_i x) - E[p_e(x - R) p_e(y - D_i - a_i x)]| \]
\[ \leq p(x)E |p_e(y - a_i x - R_i + a_i \epsilon_0) - p_e(y - D_i - a_i x)| \]
\[ \leq C(E|D_i| + |a_i|). \]

By (13), \(|p_i(x, y) - p(x)p(y - a_i x)| \rightarrow 0\) as \(|i|_\infty \rightarrow \infty.\)

To prove (37), define \(R_m = X_{0,m} - \epsilon_0\) and \(R_{i,m} = X_{i,m} - \epsilon_i - a_i \epsilon_0 1_{\{|i|_\infty < m\}}.\) Then, similarly as (54), one has

\[ p_{i,m}(x,y) = E[p_e(x - R_m)p_e(y - a_i x 1_{\{|i|_\infty < m\}} - R_{i,m})]. \]

Introducing an intermediate term \(E[p_e(x - R)p_e(y - a_i x 1_{\{|i|_\infty < m\}} - R_{i,m})],\)
we obtain that

\[ |p_{i,m}(x,y) - p_i(x,y)| \]
\[ \leq E[p_e(x - R)(|a_i| 1_{\{|i|_\infty \geq m\}} + |R_i - R_{i,m}|)] + C E|R - R_m| \]
\[ \leq C(|x|p(x)|a_i| 1_{\{|i|_\infty \geq m\}} + |R - R_m| + |R_i - R_{i,m}|). \]

Clearly, since \(X_0\) has finite second moment and \(p\) is bounded and Lipschitz, \(\sup_x |x|p(x) < \infty.\) The summability assumption on \(a_i\) implies that \(\lim_{m \rightarrow \infty} \sup_{|i|_\infty \geq m} |a_i| = 0,\) and \(\sup_i (|R - R_m| + |R_i - R_{i,m}|) \rightarrow 0\) as \(m \rightarrow \infty\) (recall (13)). Therefore, we have thus proved (37).

\[ \square \]

**Proof of Lemma 2.** We only prove (27). The proof of (28) is similar. By [27], Corollary 2, there exists a constant \(C,\) such that

\[ \frac{\|S_n(Z - \bar{\zeta})\|_2}{n} \leq C \sum_{k \in [1,n]^d} \frac{\|E(Z_{n,k} | F_1)\|_2}{\prod_{\tau=1}^d k_{\tau}^{1/2}}, \] (38)

where \(F_1 = \sigma(\epsilon_k : k \in \mathbb{Z}^d, k \leq 1).\) By the definition of \(\bar{\zeta}_{n,i,}\) (38) equals (up to the multiplicative constant \(C),\)

\[ \sum_{k \in [1,n]^d} \frac{\|E(Z_{n,k} | F_1)\|_2}{\prod_{\tau=1}^d k_{\tau}^{1/2}} + \sum_{k \in [1,m_n]^d} \frac{\|E(Z_{n,k} - \bar{\zeta}_{n,k} | F_1)\|_2}{\prod_{\tau=1}^d k_{\tau}^{1/2}} \]
\[ \leq \|Z_{n,0} - \bar{\zeta}_{n,0}\|_2 + \sum_{k \in [1,n]^d} \|E(Z_{n,k} | F_1)\|_2 \frac{\prod_{\tau=1}^d k_{\tau}^{1/2}}{\prod_{\tau=1}^d k_{\tau}^{1/2}} + \sum_{k \in [1,m_n]^d} \|E(\zeta_{n,k} | F_1)\|_2 \frac{\prod_{\tau=1}^d k_{\tau}^{1/2}}{\prod_{\tau=1}^d k_{\tau}^{1/2}} \]
\[ \leq C \left(\|Z_{n,0} - \bar{\zeta}_{n,0}\|_2 + b_n^{1/2} \sum_{k \in [1,n]^d} \frac{A_{k-1}}{\prod_{\tau=1}^d k_{\tau}^{1/2}} \right), \]

where the last inequality follows from Lemma 3 below. \(\square\)
Lemma 4. Suppose that in addition to Condition\textsuperscript{A}, $\mathbb{E}(|\xi_0|^{\alpha}) < \infty$ for some $\alpha \geq 2$. For all $k \in \mathbb{N}^d, k \neq 1$,

\[
\begin{align*}
\|\mathbb{E}(\mathbb{Z}_{n,k} | \mathcal{F}_1)\|_{\alpha} &\leq Cb_n^{1/2}A_{k-1}, \quad \text{(39)} \\
\|\mathbb{E}(\zeta_{n,k} | \mathcal{F}_1)\|_{\alpha} &\leq Cb_n^{1/2}A_{k-1}. \quad \text{(40)}
\end{align*}
\]

Proof of Lemma 4. First, we control $\|\mathbb{E}(\mathbb{Z}_{n,k} | \mathcal{F}_1)\|_{\alpha}$. For each $k \in \mathbb{Z}^d$, introduce the notation

\[
\Gamma(k) := \{i \in \mathbb{Z}^d : i \leq k\},
\]

and write

\[
X_k = \sum_{i \in \Gamma(k)} a_{k-i} c_i = \left( \sum_{i \in \Gamma(1)} + \sum_{i \in \Gamma(k) \setminus \Gamma(1)} \right) a_{k-i} c_i =: D_k + T_k.
\]

For the sake of simplicity, write $D \equiv D_k, T \equiv T_k$, and, given a random variable $Y$, let $\mathbb{E}Y(\cdot) := \mathbb{E}(\cdot | Y)$ denote the conditional expectation given the $\sigma$-algebra generated by $Y$. Since $k \geq 1, k \neq 1$, $T_k$ is a non-degenerate random variable. Then,

\[
\mathbb{E}(\mathbb{Z}_{n,k} | \mathcal{F}_1) = \frac{1}{\sqrt{b_n}} \left[ \mathbb{E}D K\left(\frac{x - D - T}{b_n}\right) - \mathbb{E}K\left(\frac{x - D - T}{b_n}\right) \right].
\]

Let $\tilde{D}$ be a copy of $D$, independent of $D$ and $T$. Then, the above identity becomes, letting $p_T$ denote the density of $T$,

\[
\frac{1}{\sqrt{b_n}} \mathbb{E}D \mathbb{E}_{D,\tilde{D}} \left[ K\left(\frac{x - D - T}{b_n}\right) - K\left(\frac{x - \tilde{D} - T}{b_n}\right) \right]
\]

\[
= b_n^{1/2} \mathbb{E}D \int K(t) \left[ p_T(x - b_n t - D) - p_T(x - b_n t - \tilde{D}) \right] dt.
\]

Since $p_T$ is Lipschitz, the absolute value of the above term is bounded by $C \int |K(s)| ds b_n^{1/2} \mathbb{E}_D |D - \tilde{D}|$, almost surely. (Here $p_T$ depends on $k, n$, but one can show that the Lipschitz constant can be chosen independently from $k, n$. See e.g.\textsuperscript{26}, Lemma 1.) To sum up, we have

\[
\|\mathbb{E}(\mathbb{Z}_{n,k} | \mathcal{F}_1)\|_{\alpha} \leq Cb_n^{1/2} \left\| \mathbb{E}_D |D - \tilde{D}| \right\|_{\alpha} \leq Cb_n^{1/2} \|D\|_{\alpha} \leq Cb_n^{1/2} A_{k-1},
\]

where the last inequality follows from \textsuperscript{13}. We have thus proved \textsuperscript{39}. To prove \textsuperscript{10}, a similar argument yields $\|\mathbb{E}(\zeta_{n,k} | \mathcal{F}_1)\|_{\alpha} \leq Cb_n^{1/2} A_{k,m_n}$ with $A_{k,m_n} = (\sum_{i \in [0,m_n-1]} a_i)^{1/2} \leq A_{k-1}$. \hfill $\square$
Proof of Lemma 3. For random variables $Z_n, \tilde{Z}_n, \zeta_n, \tilde{\zeta}_n$, we replace the index ‘$n, 0$’ by ‘$n$’ for the sake of simplicity. First observe that

$$(\mathbb{E}Z_n)^2 + (\mathbb{E}\zeta_n)^2 \leq C(p^2 b_n + p_m^2 b_n) \leq C b_n,$$

where the last step we applied (7). Then,

$$|\mathbb{E}[(\zeta_n^2 - Z_n^2)]| \leq \left| \int K^2(y)[p_m(x - b_n y) - p(x - b_n y)]dy \right| + C b_n$$

$$\leq \sup_y |p_m(y) - p(y)| \int K^2(s)ds + C b_n.$$  \hspace{1cm} (42)

where the last inequality follows from (33). Next, write

$$\|\zeta_n - Z_n\|_2^2 = \mathbb{E}Z_n^2 - \mathbb{E}\zeta_n^2 + 2(\mathbb{E}\zeta_n - \mathbb{E}(Z_n \tilde{\zeta}_n)).$$  \hspace{1cm} (43)

For the last term on the right-hand side of (43), observe that $\mathbb{E}(Z_n \tilde{\zeta}_n) = \mathbb{E}(Z_n \zeta_n) - \mathbb{E}Z_n \mathbb{E}\zeta_n = \mathbb{E}(Z_n \zeta_n) + O(\mathbb{E}\zeta_n^2)$, for some $b_n$. We claim that $\mathbb{E}(Z_n \zeta_n)$ is very close to $\mathbb{E}\zeta_n^2$, under our restriction on the choice of $m_n$. Indeed,

$$|\mathbb{E}(Z_n \zeta_n) - \mathbb{E}\zeta_n^2| = \left| \int K^2(x)p_m(x - b_n y)dy \right|,$$  \hspace{1cm} (44)

and,

$$\mathbb{E}(Z_n \zeta_n) = \int \int \frac{1}{b_n} K\left(\frac{x - y - z}{b_n}\right)K\left(\frac{x - y}{b_n}\right)p_m(y)\tilde{p}_m(z)dydz$$

$$= \int K(y)\mathbb{E}K\left(y - \tilde{X}_{0,m_n}\right)p_m(x - b_n y)dy.$$  \hspace{1cm} (45)

Therefore, (44) can be bounded by, since $K$ is Lipschitz,

$$\int |K(y)|\left|K\left(y - \tilde{X}_{0,m_n}\right) - K(y)\right|p_m(x - b_n y)dy$$

$$\leq \frac{\mathbb{E}|\tilde{X}_{0,m_n}|}{b_n} \int |K(y)|p_m(x - b_n y)dy,$$

and $\mathbb{E}|\tilde{X}_{0,m_n}| \leq C B_{m_n}$ by (13). To sum up, we have thus shown that (recall that $b_n \downarrow 0$, whence $B_{m_n}$ is dominated by $B_{m_n}/b_n$), under (6),

$$\|\zeta_n - Z_n\|_2^2 \leq C \left(\frac{B_{m_n}}{b_n} + b_n\right).$$

$\square$
References

[1] D. Bosq, F. Merlevède, and M. Peligrad. Asymptotic normality for density kernel estimators in discrete and continuous time. *J. Multivariate Anal.*, 68(1):78–95, 1999.

[2] J. V. Castellana and M. R. Leadbetter. On smoothed probability density estimation for stationary processes. *Stochastic Process. Appl.*, 21(2):179–193, 1986.

[3] T.-L. Cheng and H.-C. Ho. Central limit theorems for instantaneous filters of linear random fields on $\mathbb{Z}^2$. In *Random walk, sequential analysis and related topics*, pages 71–84. World Sci. Publ., Hackensack, NJ, 2006.

[4] T.-L. Cheng, H.-C. Ho, and X. Lu. A note on asymptotic normality of kernel estimation for linear random fields on $\mathbb{Z}^2$. *J. Theoret. Probab.*, 21(2):267–286, 2008.

[5] S. Csörgő and J. Mielniczuk. Density estimation under long-range dependence. *Ann. Statist.*, 23(3):990–999, 1995.

[6] J. Dedecker. A central limit theorem for stationary random fields. *Probab. Theory Related Fields*, 110(3):397–426, 1998.

[7] J. Dedecker. Exponential inequalities and functional central limit theorems for a random fields. *ESAIM Probab. Statist.*, 5:77–104 (electronic), 2001.

[8] J. Dedecker and F. Merlevède. Necessary and sufficient conditions for the conditional central limit theorem. *Ann. Probab.*, 30(3):1044–1081, 2002.

[9] R. Durrett. *Probability: theory and examples*. Duxbury Press, Belmont, CA, second edition, 1996.

[10] M. El Machkouri. Asymptotic normality of the parzen-rosenblatt density estimator for strongly mixing random fields. *Stat. Inference Stoch. Process.*, 14(1):73–84, 2011.

[11] M. El Machkouri. Kernel density estimation for stationary random fields. preprint, available at [http://arxiv.org/abs/1109.2694](http://arxiv.org/abs/1109.2694), 2011.

[12] M. El Machkouri, D. Volný, and W. B. Wu. A central limit theorem for stationary random fields. Submitted, available at [http://arxiv.org/abs/1109.0838](http://arxiv.org/abs/1109.0838), 2011.

[13] M. Hallin, Z. Lu, and L. T. Tran. Density estimation for spatial linear processes. *Bernoulli*, 7(4):657–668, 2001.
[14] L. Heinrich. Asymptotic behaviour of an empirical nearest-neighbour distance function for stationary Poisson cluster processes. *Math. Nachr.*, 136:131–148, 1988.

[15] E. Parzen. On estimation of a probability density function and mode. *Ann. Math. Statist.*, 33:1065–1076, 1962.

[16] M. Peligrad and S. Utev. A new maximal inequality and invariance principle for stationary sequences. *Ann. Probab.*, 33(2):798–815, 2005.

[17] M. Peligrad, S. Utev, and W. B. Wu. A maximal $\ell_p$-inequality for stationary sequences and its applications. *Proc. Amer. Math. Soc.*, 135(2):541–550 (electronic), 2007.

[18] E. Rio. About the Lindeberg method for strongly mixing sequences. *ESAIM Probab. Statist.*, 1:35–61 (electronic), 1995/97.

[19] P. M. Robinson. Nonparametric estimators for time series. *J. Time Ser. Anal.*, 4(3):185–207, 1983.

[20] M. Rosenblatt. Remarks on some nonparametric estimates of a density function. *Ann. Math. Statist.*, 27:832–837, 1956.

[21] B. W. Silverman. *Density estimation for statistics and data analysis*. Monographs on Statistics and Applied Probability. Chapman & Hall, London, 1986.

[22] L. T. Tran. Kernel density estimation on random fields. *J. Multivariate Anal.*, 34(1):37–53, 1990.

[23] D. Volný. A nonadapted version of the invariance principle of Peligrad and Utev. *C. R. Math. Acad. Sci. Paris*, 345(3):167–169, 2007.

[24] D. Volný, M. Woodroofe, and O. Zhao. Central limit theorems for superlinear processes. *Stoch. Dyn.*, 11(1):71–80, 2011.

[25] Y. Wang and M. Woodroofe. A new condition on invariance principles for stationary random fields. Submitted, available at [http://arxiv.org/abs/1101.5195](http://arxiv.org/abs/1101.5195), 2011.

[26] W. B. Wu. Central limit theorems for functionals of linear processes and their applications. *Statist. Sinica*, 12(2):635–649, 2002.

[27] W. B. Wu and J. Mielenz. Kernel density estimation for linear processes. *Ann. Statist.*, 30(5):1441–1459, 2002.