Review on the Kleisli and Eilenberg-Moore 2-adjunctions

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Abstract

We collect some isomorphisms of categories and bijections of structures using the Kleisli and Eilenberg-Moore 2-adjunctions.

1 Introduction

Motivated by [2], the authors were constructing a pair of 2-adjunctions but they realized that [3] discovered them already, in a very general case, nevertheless due to the fact of different motivations, the examples they apply such 2-adjunctions to, are other. Among the examples given in this article, there are two that are of high importance. The first one is the equivalence between the monoidal structures for the category of Eilenberg-Moore algebras with the colax monad structures, and the dual case for Kleisli categories, cf [8] and [11]. The second example is the equivalence between the right strong monads and left actions of the underlying category over the Kleisli one, cf [6].

We give the structure of the article along with the list of compiled examples.

In Section 2, we give the formal 2-adjunction corresponding to the Kleisli situation. In Section 3, we give the formal 2-adjunction corresponding to the Eilenberg-Moore case.

In Section 4, we apply the case where the 2-category is 2Cat to the 2-adjunction of EM. In Section 5, we proved the theorem of I. Moerdijk on the equivalence for the monoidal structures induced on the category of algebras and colax monads.

In Section 6, we use the Kleisli 2-adjunction for the case 2Cat. In Section 7, we apply this 2-adjunction to induced a monoidal structure on the Kleisli category and relate it with lax monads.

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In Section 8, we apply the adjunction to the most known case of liftings of functors and commutative diagrams for the forgetful functor, check [1] and [10], just to mention a couple of references.

In Section 9, we relate actions of the category $C$ over its Kleisli category $C_F$ with strong monads.

In Section 10, we finalize with left and right functor algebras for a monad and relate this to certain liftings and extensions, respectively, for the underlying functors, cf. [1].

We give some remarks on notation. Suppose that we had an adjunction of the form $\mathcal{L} \dashv \mathcal{R}$, then the unit and counit for this adjunction will be denoted as $\eta^{\mathcal{RL}}$ and $\varepsilon^{\mathcal{LR}}$, respectively. We are completely aware that this notation is very cumbersome, nonetheless, it is clear and in the case of the proliferation of several adjunctions it has the property of avoiding the need for extra greek letters to denote the new units and counits. As the article develops, the reader will see the advantage in using this notation.

We will be working with monoidal categories denoted as $(\mathcal{C}, \otimes, I, a, l, r)$ and also as $(\mathcal{C}, \otimes, I)$, as a contraction, that leaves understood the natural constrain transformations. We will be working with the constant functor $\delta^I: 1 \rightarrow \mathcal{C}$, on $I$, where $1$ is the category with only one object $0$ and only one arrow $1_0$. That is to say, $\delta^I(0) = I$.

On the other hand, it is known that a category with binary products and a terminal object has a canonical (cartesian) monoidal structure. This is the case for the category $\textit{Cat}$, of small categories. The natural constraint transformations, taken on components, are functors, for example, for $\mathcal{C}, \mathcal{D}, \mathcal{E}$, $a_{\mathcal{C}, \mathcal{D}, \mathcal{E}}: (\mathcal{C} \times \mathcal{D}) \times \mathcal{E} \rightarrow \mathcal{C} \times (\mathcal{D} \times \mathcal{E})$ is the obvious functor. In order to compact the notation, we will agree that in the case that the component be the object $\mathcal{C}, \mathcal{C}, \mathcal{C}$, the associativity functor will be denoted simply as $a_{\mathcal{C}}$. In turn, the respective constraint functors will be denoted as $l_{\mathcal{C}}$ and $r_{\mathcal{C}}$.

The horizontal composition in a general 2-category $\mathcal{A}$ will be denoted as $\cdot$ or by juxtaposition, this notation will be used indistinctively. The vertical composition on 2-cells will be given the symbol $\circ$.

2 Formal Kleisli 2-Adjunction

Consider a 2-category such that $\mathcal{A}^{op}$ admits the construction of algebras. Due to this property of the 2-category $\mathcal{A}^{op}$, we will be able to construct a 2-adjunction of the form

$$
\begin{array}{c}
\text{Mnd}(\mathcal{A}^{op}) \xrightarrow{\Phi_K} \text{Adj}_R(\mathcal{A}^{op})
\end{array}
$$

If we describe the adjunction over $\mathcal{A}$ and not on the opposite one then the 2-category $\text{Mnd}(\mathcal{A}^{op})$ will be isomorphic to $\text{Mnd}^*(\mathcal{A})$ and the 2-category $\text{Adj}_R(\mathcal{A}^{op})$ will be isomorphic to $\text{Adj}_L(\mathcal{A})$. Note that in [9], the category $\mathcal{A}^{op}$ is denoted as $\mathcal{A}^*$.

The description of the 2-category $\text{Mnd}^*(\mathcal{A})$ is given as follows.

1.- The 0-cells are monads in $\mathcal{A}$, i.e. $(\mathcal{A}, f, \mu', \eta')$. The short notation $(\mathcal{A}, f)$ will be used for such a monad.
2.- The 1-cells, which we call indistinctively as morphisms of monads, are pairs of the form \((m, \pi) : (A, f) \rightarrow (B, h)\); where \(m : A \rightarrow B\) is a 1-cell in \(\mathcal{A}\) and \(\pi : mf \rightarrow hm\) is a 2-cell in \(\mathcal{A}\) such that the following diagrams commute

\[
\begin{array}{cccccc}
mf & \xrightarrow{\pi f} & hmf & \xrightarrow{h\pi} & hm \\
m\mu f & \downarrow & & \downarrow & \rho \mu m \\
mf & \xrightarrow{\pi} & hm
\end{array}
\]

3.- The 2-cells, which we call indistinctively as transformations of monads, have the form \(\vartheta : (m, \pi) \rightarrow (n, \tau) : (A, f) \rightarrow (B, h)\), such that \(\vartheta : m \rightarrow n : A \rightarrow B\) is a 2-cell in \(\mathcal{A}\) and the following diagram commutes

\[
\begin{array}{cccccc}
mf & \xrightarrow{\pi} & hm \\
\vartheta f & \downarrow \vartheta & \downarrow \vartheta h & \downarrow \vartheta h \\
nf & \xrightarrow{\tau} & hn
\end{array}
\]

This 2-cell is displayed as follows

\[
\begin{array}{cccccc}
\text{(m, \pi)} & \xrightarrow{\vartheta} & \text{(n, \tau)} \\
\text{(A, f)} & \xrightarrow{\vartheta} & \text{(B, h)}
\end{array}
\]

The structure of the 2-category \(\text{Adj}_L(\mathcal{A})\) is given as follows

1.- The 0-cells are made of adjunctions

\[
\begin{array}{cccccc}
A & \xrightarrow{r} & B
\end{array}
\]

2.- The 1-cells are pairs of the form \((j, k)\) such that the second diagram is the 2-cell mate to the first one

\[
\begin{array}{cccccc}
A & \xrightarrow{j} & A & \leftarrow & \text{A} \\
l & \downarrow & \downarrow & \rho & \downarrow \\
B & \xrightarrow{k} & B & \leftarrow & \text{B}
\end{array}
\]

The mate is described by

\[
\rho = \eta k \varepsilon \circ \eta j r
\] (2)

This morphism can be represented as
3.- The 2-cells are made of a pair of 2-cells in $A$, $(\alpha, \beta)$ as in

\[
\begin{array}{c}
A \\
\downarrow j \\
B
\end{array} \quad \begin{array}{c}
A \\
\downarrow k \\
B
\end{array} \\
\begin{array}{c}
l \dashv r \\
\downarrow \rho \\
\end{array} \quad \begin{array}{c}
l' \dashv r' \\
\downarrow \rho' \\
\end{array}
\]

such that they fulfill one of the following equivalent conditions

(i) $l\alpha = \beta l$,

(ii) $\rho' \circ \alpha r = r'\beta \circ \rho$.

\textbf{Remark 2.1} Note that the previous conditions can be seen as commutative surface diagrams.

This 2-cell can be displayed as follows

\[
\begin{array}{c}
A \\
\downarrow j \\
B
\end{array} \quad \begin{array}{c}
A \\
\downarrow k \\
B
\end{array} \\
\begin{array}{c}
l \dashv r \\
\downarrow \alpha \\
\end{array} \quad \begin{array}{c}
l' \dashv r' \\
\downarrow \beta \\
\end{array}
\]

\[
\begin{array}{c}
j \quad \rho \\
\downarrow j' \\
\end{array} \quad \begin{array}{c}
l \dashv r \\
\downarrow \rho' \\
\end{array}
\]

The $n$-cell structure described arrange itself to form a 2-category.

Before going into the details on the construction of the 2-functor $\Psi_K$, we develop some calculations. These calculations are dual to those made in [9]. Note that we are going to be switching between the 2-categories $A^{op}$ and $\text{Mnd}(A^{op})$ to $A$ and $\text{Mnd}^*(A)$, respectively.

Since the 2-category $A^{op}$ admits the construction of algebras, the functor $\text{Inc}_{A^{op}} : A^{op} \rightarrow \text{Mnd}(A^{op})$ admits a right adjoint, denoted as $\text{Alg}_{A^{op}} : \text{Mnd}(A^{op}) \rightarrow A^{op}$. These 2-functors are going to be short denoted as $I^*$ and $A^*$ respectively.

The corresponding counit, on the component $(A, f^{op})$, is $\varepsilon I^*(A, f^{op}) : \text{Inc}_{A^{op}} \text{Alg}_{A^{op}}(A, f^{op}) \rightarrow (A, f^{op})$. If we define $\text{Alg}_{A^{op}}(A, f^{op}) = A_f$ then $\varepsilon I^*(A, f^{op}) = (g_f, \iota_f) : (A, f) \rightarrow (A_f, 1_{A_f})$. This last 1-cell belongs to $\text{Mnd}^*(A)$, where $g_f : A \rightarrow A_f$ and $\iota_f : g_f v_f g_f \rightarrow g_f$. 
Following [9], for any monad \((A, f^\text{op})\) in \(\text{Mnd}(A^{\text{op}})\), there exists an adjunction in \(A\),

\[
A \quad \xleftarrow{\eta_f} \quad A_f \quad \xrightarrow{\epsilon_f} \quad A
\]
such that it generates the monad \((A, f)\), with unit \(\eta_f\) and counit \(\epsilon_f\). It can be checked that \(\nu_f = \epsilon_f \gamma_f \eta_f\).

Suppose that there is a morphism of monads \((m, f^\text{op}) : (B, h^\text{op}) \rightarrow (A, f^\text{op})\) in \(\text{Mnd}(A^{\text{op}})\), i.e. \((m, \pi) : (A, f) \rightarrow (B, h)\) in \(\text{Mnd}^*(A)\). Take the following composition of morphisms of monads

\[
(g_h, \nu_h) \cdot (m, \pi) = (g_h m \circ \nu_h \cdot g_h, \pi) : (A, f) \rightarrow (B, h, 1)\]

Since the counit is universal from \(\text{Inc}_{A^{\text{op}}}\) to \((A, f^\text{op})\), there exists a 1-cell \(\pi : A_f \rightarrow B_h\), in \(A\), such that the following diagram commute

\[
\begin{array}{ccc}
(A, f) & \xrightarrow{(g_f, \nu_f)} & (g_h m \circ \nu_h \cdot g_h, \pi) \\
(A_f, 1_A) & \xrightarrow{(m, \nu)} & (B_h, 1_B)
\end{array}
\]

In particular, \(g_h m = m \nu_f g_f\) and \(\nu_h m \circ g_h \pi = m \nu_f\). Note that the associated mate to the first equality is \(\rho = \nu_f \epsilon_f \gamma_f \circ f \circ \eta_m v_f\) and that \(\rho \circ \nu_f = \pi\).

Consider a 2-cell of monads \(\vartheta : (m, \pi) \rightarrow (n, \tau) : (A, f) \rightarrow (B, h)\) in \(\text{Mnd}^*(A)\). Because of the construction of algebras for \(A^{\text{op}}\), the 2-adjunction \(\text{Alg}_{A^{\text{op}}} \dashv \text{Inc}_{A^{\text{op}}}\) provides an isomorphism of categories, for \((A, f^\text{op})\) in \(\text{Mnd}(A^{\text{op}})\) and \(B\) in \(A^{\text{op}}\), of the form

\[
\text{Hom}_{\text{Mnd}(A^{\text{op}})}((A, f^{\text{op}}), \text{Inc}_{A^{\text{op}}}(B)) \cong \text{Hom}_{A^{\text{op}}}(\text{Alg}_{A^{\text{op}}}(A, f^{\text{op}}), B)
\]

this translates, in the non-opposite case, into the following assignment

\[
\begin{array}{ccc}
A_f & \xrightarrow{\alpha_f} & (A, f) \\
\alpha & \circ & \alpha_g \\
\beta & \circ & \beta_f
\end{array}
\]

On the other hand, we have an equality of 2-cells

\[
\begin{array}{ccc}
A_f & \xrightarrow{\alpha_f} & (A, f) \\
\alpha & \circ & \alpha_g \\
\beta & \circ & \beta_f
\end{array}
\]

(3)
Therefore, to the 2-cell \(g_h \vartheta\) there corresponds, through the asignment \(\mathfrak{3}\), a 2-cell \(\beta \vartheta = \text{Alg}_{A^\circ \varphi}(g_h \vartheta)\), such that \(g_h \vartheta = \beta \vartheta g_f\), where \(\vartheta : m_\pi \to n_\tau\). We change, at this point, the notation as \(\beta \vartheta = \tilde{\vartheta}\).

Without any further ado, we provide the description of the 2-functor \(\Psi_K\)

1. For the monad \((A, f, \mu', \eta')\) in \(\text{Mnd}^*(A)\), \(\Psi_K(A, f) = g_f \dashv v_f\).
2. For the morphism \((m, \pi) : (A, f) \to (B, h)\), \(\Psi_K(m, \pi) = (m, m_\pi, \rho_\pi)\).
3. For the transformation \(\vartheta : (m, \pi) \to (n, \tau) : (A, f) \to (B, g)\), \(\Psi_K(\vartheta) = (\vartheta, \tilde{\vartheta})\), where \(\tilde{\vartheta}\) is given as above.

The description of the functor \(\Phi_K\) is given as follows

1. For the adjunction \(l \dashv r\), \(\Phi_K(l \dashv r) = (A, rl)\).
2. For the morphism of adjunctions \((j, k, \rho) : (l \dashv r) \to (l' \dashv r')\), \(\Phi_K(j, k, \rho) = (j_\rho, k_\rho, \pi_\rho)\). Where \(\pi_\rho = \rho l\).
3. For the transformation of adjunctions \((\alpha, \beta) : (j, k, \rho) \to (j', k', \rho') : (l \dashv r) \to (l' \dashv r')\), \(\Phi_K(\alpha, \beta) = \vartheta(\alpha, \beta)\).

Yet again, following \([9]\), it can be shown that for the adjunction \(l \dashv r\), there exists a dual comparison 1-cell \(k_{rl} : A_{rl} \to B\), such that \(l = k_{rl}g_{rl}, v_{rl} = r k_{rl}\) and \(\varepsilon^rl = k_{rl}v_{rl}\).

The unit of the 2-adjunction in \((1)\), \(\eta^{\Phi_K} : 1_{\text{Mnd}^*(A)} \to \Phi_K\Psi_K\) is defined, in the component \((A, f)\), as follows

\[
\eta^{\Phi_K}(A, f) := (1_A, 1_f) : (A, f) \to (A, f) \quad \text{in} \quad \text{Mnd}^*(A)
\]

In turn, the counit \(\varepsilon^{\Psi_K} : \Psi_K\Phi_K \to 1_{\text{Adj}_L(A)}\) is defined, in the component \(l \dashv r\), as follows

\[
\varepsilon^{\Psi_K}(l \dashv r) := (1_A, k_{rl}, 1_{v_{rl}}) : g_{rl} \dashv v_{rl} \to l \dashv r \quad \text{in} \quad \text{Adj}_L(A)
\]

**Theorem 2.1** There exists a 2-adjunction \(\Psi_K \dashv \Phi_K\).

**Proof:**

We prove only one of the triangular identities, \(i.e. \Phi_K \varepsilon^{\Psi_K} \circ \eta^{\Phi_K} \Phi_K = 1_{\Phi_K}\).
\[
\left( \Phi_K \circ \Psi \Phi_K \circ \eta \Psi \Phi_k \Phi_K \right)(l \dashv r) = \Phi_K \circ \Psi \Phi_K (l \dashv r) \cdot \eta \Psi \Phi_k \Phi_K (l \dashv r) \\
= \Phi_K (1_A, k_{rl}, 1_{vl}) \cdot \eta \Psi \Phi_k (A, rl) \\
= (1_A, 1_{vl} \eta_{rl}) \cdot (1_A, 1_{rl}) = (1_A, 1_{rl}) = 1_{(A, rl)} \\
= 1_{\Phi_K (l \dashv r)} = 1_{\Phi_K (l \dashv r)}.
\]

\[
□
\]

3 Formal Eilenberg-Moore 2-Adjunction

Consider a 2-category \(A\) which admits the construction of algebras. With this property of \(A\), we will construct a 2-adjunction of the form

\[
\text{Adj}_R (A) \xrightarrow{\Psi_E} \text{Mnd} (A) \xleftarrow{\Phi_E}.
\]

The 2-category \(\text{Adj}_R (A)\) is described as follows

1.- The 0-cells are made of adjunctions

\[
A \xrightarrow{r} B.
\]

2.- The 1-cells are pairs, of 1-cells in \(A\), \((j, k)\) such that the first diagram is the 2-cell mate to the second one

\[
\begin{array}{ccc}
A & \xrightarrow{j} & \bar{A} \\
\downarrow l & \swarrow \lambda & \downarrow \theta \\
B & \xleftarrow{k} & \bar{B}
\end{array}
\]

The mate is described by

\[
\lambda = \varepsilon_{kl} \circ \bar{\eta}_{j\theta}
\]

This morphism can be represented as

\[
\begin{array}{ccc}
A & \xrightarrow{j} & \bar{A} \\
\downarrow l & \swarrow \lambda & \downarrow \tau \\
B & \xleftarrow{k} & \bar{B}
\end{array}
\]

and denoted as \((j, k, \lambda) : l \dashv r \longrightarrow \bar{l} \dashv \bar{r} \).
3.- The 2-cells are made of a pair of 2-cells in \( A \), \((\alpha, \beta)\) as in

\[
\begin{array}{c}
\begin{tikzpicture}
\node (A) at (0,0) {$A$};
\node (B) at (0,-2) {$B$};
\node (j) at (0,1) {$j$};
\node (l) at (0,-1) {$l$};
\node (r) at (0,-0.5) {$r$};
\node (i) at (0,-0.75) {$i$};
\node (k) at (0,-1.5) {$k$};
\node (k') at (0,-1.75) {$k'$};
\node (j') at (0,0.75) {$j'$};
\node (l') at (0,1.25) {$l'$};
\node (r') at (0,0.75) {$r'$};
\node (α) at (0.5,0) {$\alpha$};
\node (β) at (0.5,-2) {$\beta$};
\draw[->] (A) to (B);
\draw[->] (B) to (A);
\draw[->] (A) to (j);
\draw[->] (j) to (B);
\draw[->] (A) to (l);
\draw[->] (l) to (B);
\draw[->] (A) to (r);
\draw[->] (r) to (B);
\draw[->] (A) to (i);
\draw[->] (i) to (B);
\draw[->] (A) to (k);
\draw[->] (k) to (B);
\draw[->] (A) to (k');
\draw[->] (k') to (B);
\end{tikzpicture}
\end{array}
\]

such that they fulfill one of the following equivalent conditions

(i) \( \lambda' \circ l \alpha = \beta l \circ \lambda \),
(ii) \( \alpha r = r \beta \).

**Remark 3.1** Note that the previous conditions can be seen as commutative surface diagrams.

This 2-cell can be displayed as follows

\[
\begin{array}{c}
\begin{tikzpicture}
\node (A) at (0,0) {$l$};
\node (B) at (0,-2) {$r$};
\node (j) at (0,1) {$j$};
\node (k) at (0,-1) {$k$};
\node (l) at (0,-0.5) {$l$};
\node (r) at (0,-0.75) {$r$};
\node (i) at (0,-0.75) {$i$};
\node (k') at (0,-1.5) {$k'$};
\node (j') at (0,0.75) {$j'$};
\node (l') at (0,1.25) {$l'$};
\node (r') at (0,0.75) {$r'$};
\node (α) at (0.5,0) {$\alpha$};
\node (β) at (0.5,-2) {$\beta$};
\draw[->] (A) to (B);
\draw[->] (B) to (A);
\draw[->] (A) to (j);
\draw[->] (j) to (B);
\draw[->] (A) to (l);
\draw[->] (l) to (B);
\draw[->] (A) to (r);
\draw[->] (r) to (B);
\draw[->] (A) to (i);
\draw[->] (i) to (B);
\draw[->] (A) to (k);
\draw[->] (k) to (B);
\draw[->] (A) to (k');
\draw[->] (k') to (B);
\end{tikzpicture}
\end{array}
\]

The described \( n \)-cell structure arrange itself to form a 2-category.

The 2-category \( \text{Mnd}(A) \) is formed as follows

1.- The 0-cells are monads in \( A \), \((A, f, \mu^f, \eta^f)\). The short notation \((A, f)\) will be used for such a monad.

2.- The 1-cells are morphims of monads \((p, \varphi) : (A, f) \rightarrow (B, h)\) which consist of a 1-cell \( p : A \rightarrow B \) and a 2-cell \( \varphi : hp \rightarrow pf \) such that the following diagrams commutes

\[
\begin{array}{c}
\begin{tikzpicture}
\node (A) at (0,0) {$hp$};
\node (B) at (1,0) {$hf$};
\node (C) at (2,0) {$pf$};
\draw[->] (A) to node[auto] {$h\varphi$} (B);
\draw[->] (B) to node[auto] {$\varphi f$} (C);
\draw[->] (A) to node[auto] {$h\mu^f$} (B);
\draw[->] (B) to node[auto] {$p\mu^f$} (C);
\end{tikzpicture}
\end{array}
\]

3.- The 2-cells or transformations of monads \( \theta : (p, \varphi) \rightarrow (q, \psi) : (A, f) \rightarrow (B, h) \), consists of a 2-cell \( \theta : p \rightarrow q \) in \( A \) and fulfills the commutativity of the following diagram

\[
\begin{array}{c}
\begin{tikzpicture}
\node (A) at (0,0) {$hp$};
\node (B) at (1,0) {$hf$};
\node (C) at (2,0) {$pf$};
\draw[->] (A) to node[auto] {$\varphi$} (C);
\draw[->] (A) to node[auto] {$h\theta$} (B);
\draw[->] (B) to node[auto] {$\theta f$} (C);
\end{tikzpicture}
\end{array}
\]
This 2-cell is displayed as follows

\[
\begin{array}{ccc}
(A, f) & \theta & (B, h) \\
\downarrow & & \downarrow \\
(q, \psi) & & (p, \varphi)
\end{array}
\]

The description of 2-functor \( \Phi_E \) is given as follows

1. On 0-cells, \( \Phi_E(l \dashv r) = (A, rl, r\varepsilon_l, \eta) \), i.e. the induced monad by an adjunction.
2. On 1-cells, \((j, k, \lambda) : (A, rl) \to (A, r)\), \( \Phi_E(j, k, \lambda) = (j, r\lambda) : (A, rl) \to (A, r) \).
3. On 2-cells, \((\alpha, \beta) : (j, k, \lambda) \to (j', k', \lambda'), \Phi_E(\alpha, \beta) = \alpha : (j, r\lambda) \to (j', r\lambda') \).

Before the description of the 2-functor \( \Psi_E \), we realize some calculations.

Since the 2-category \( \mathcal{A} \) admits the construction of algebras, the 2-functor \( \text{Inc}_\mathcal{A} : \mathcal{A} \to \text{Mnd}(\mathcal{A}) \) admits a right adjoint, denoted as \( \text{Alg}_\mathcal{A} : \text{Mnd}(\mathcal{A}) \to \mathcal{A} \).

The corresponding counit, on the component \((A, f)\), is \( \varepsilon^I_A(A, f) : \text{Inc}_\mathcal{A}(A, f) \to (A, f) \).

If we define \( \text{Alg}_\mathcal{A}(A, f) = A' \) then \( \varepsilon^I_A(A, f) : (u', \chi') : (A', 1_{A'}) \to (A, f) \), where \( u' : A' \to A \) and \( \chi' : u'u'du' \to u' \).

In Theorem 2, at \( \text{[9]} \), the author proved that if \( \mathcal{A} \) admits the construction of algebras then for any monad \((A, f)\) in \( \text{Mnd}(\mathcal{A}) \), there exists an adjunction in \( \mathcal{A} \)

\[
A \xrightarrow{u'} A', \quad d' \xleftarrow{\lambda'} A
\]

such that it generates the monad \((A, f)\), with unit \( \eta' \) and counit \( \varepsilon^I d'u' \). It can be checked that \( \chi' = u' \varepsilon d'u' \).

Suppose there is a morphism of monads \((p, \varphi) : (A, f) \to (B, h)\). Take the composition of morphisms of monads \((p, \varphi) \cdot (u', \chi') = (pu', p\chi' \circ \varphi u') : \text{Inc}_\mathcal{A}(A') = (A', 1_{A'}) \to (B, h) \).

The previous counit, \( \varepsilon^I_A \), is universal from the functor \( \text{Inc}_\mathcal{A} \), in particular, for the 1-cell \((pu', p\chi' \circ \varphi u') : \text{Inc}_\mathcal{A}(A') \to (A, f) \) exists a unique 1-cell in \( \mathcal{A} \) of the form \( p^\varphi : A' \to \text{Alg}_\mathcal{A}(B, h) = B^h \) such that the following diagram commutes

\[
\begin{array}{ccc}
\text{Inc}_\mathcal{A}(A') & \xrightarrow{\text{Inc}_\mathcal{A}(p^\varphi)} & \text{Inc}_\mathcal{A}(B^h) \\
(pu', p\chi' \circ \varphi u') \downarrow & & (u^h, \chi^h) \\
(A, f) & \xleftarrow{(u^h, \chi^h)} & (A, f)
\end{array}
\]

In particular, \( pu' = u^h p^\varphi \) and \( p\chi' \circ \varphi u' = \chi^h p^\varphi \). Observe that the associated mate, to the first equality, is \( \lambda = \varepsilon^h d'p^\varphi \circ d^h p\eta' \) and that \( u^h \lambda = \varphi \).
Consider a 2-cell of monads, \( \theta : (p, \varphi) \to (q, \psi) : (A, f) \to (B, h) \). Because of the construction of algebras for \( \mathcal{A} \), the 2-adjunction provides an isomorphism of categories, for \( A \) in \( \mathcal{A} \) and \( (X, f) \) in \( \mathbf{Mnd}(\mathcal{A}) \),

\[
\text{Hom}_\mathcal{A}(A, \text{Alg}_\mathcal{A}(X, f)) \cong \text{Hom}_{\mathbf{Mnd}(\mathcal{A})}(\text{Inc}_\mathcal{A}(A), (X, f))
\]

given by the following assignment

\[
A \xrightarrow{\alpha} X' \quad \mapsto \quad (A, 1_A) \xrightarrow{u'\alpha} (X, f)
\]

(5)

cf. [9]. On the other hand, we have an equality of 2-cells

\[
(A, f) \xrightarrow{\theta u'} (B, h) = (A, f) \xrightarrow{\eta \Psi \Phi^E} (B, h)
\]

Therefore, to the 2-cell \( \theta u' \) there corresponds, through the assignment (5), a 2-cell \( \text{Alg}_\mathcal{A}(\theta u') \eta^A(A') := \beta^\theta \), where \( \beta^\theta : p'' \to q'' \) and such that \( u''\beta^\theta = \theta u' \). We change the notation as follows \( \beta^\theta = \tilde{\theta} \).

With these calculations at hand, we define the 2-functor \( \Psi^E \).

1.- On 0-cells, \((A, f) \), \( \Psi^E(A, f) = d' \Rightarrow u' \).

2.- On 1-cells, \((p, \varphi) : (A, f) \to (B, h) \), \( \Psi^E(p, \varphi) = (p, p'') : d' \Rightarrow u' \to d'' \Rightarrow u'' \).

3.- On 2-cells, \( \theta : (p, \varphi) \to (q, \psi) : (A, f) \to (B, h) \), \( \Psi^E(\theta) = (\theta, \tilde{\theta}) : (p, p'') \to (q, q'') : d' \Rightarrow u' \to d'' \Rightarrow u'' \).

The unit and the counit for this 2-adjunction are given as follows. The component of the unit, at \( l \Rightarrow r \), is \( \eta^\Psi\Phi^E(l \Rightarrow r) : l \Rightarrow r \to \Psi^E \Phi^E(l \Rightarrow r) \), where \( \Phi^E \Phi^E(l \Rightarrow r) = d'' \Rightarrow u'' \). In [9], Theorem 3, the author proved the existence of a comparison 1-cell \( k^{rl} : B \to A^{rl} \), such that \( u^{rl} k^{rl} = r \) and \( d^{rl} = k^{rl} l \). Therefore, we can make the following definition \( \eta^\Psi\Phi^E(l \Rightarrow r) = (1_A, k^{rl}, 1_{d^{rl}}) : l \Rightarrow r \to d^{rl} \Rightarrow u^{rl} \).

In turn, the component of the counit, at \((A, f)\), is \( \varepsilon^\Phi\Psi^E(A, f) : \Phi^E \Psi^E(A, f) \to (A, f) \), where \( \Phi^E \Psi^E(A, f) = (A, f) \). In this case, the counit is defined as \( \varepsilon^\Phi\Psi^E(A, f) = (1_A, 1_f) : (A, f) \to (A, f) \).

**Theorem 3.1** There exists a 2-adjunction \( \Phi^E \dashv \Psi^E \).

**Proof:**

---

10
We prove only one of the triangular identities and the other one is left to the reader. Using the definition of the unit and counit for this 2-adjunction, the triangular identity \( \varepsilon \Phi \Psi \eta \Phi \varepsilon = 1 \Phi \varepsilon \) is proved as indicated.

\[
(\varepsilon \Phi \Psi \eta \Phi \varepsilon)(l \mapsto r) = \varepsilon \Phi \Psi \eta \Phi \varepsilon \Phi \eta \Phi \varepsilon (l \mapsto r) = \varepsilon \Phi \Psi \eta \Phi \varepsilon \Phi \eta \Phi \varepsilon (l \mapsto r) = (1_A, 1_{\Phi \varepsilon}) \cdot (1_A, u^{lr} 1_{d^r})
\]

\[
= (1_A, 1_{\Phi \varepsilon}) = 1_{\Phi \varepsilon} (l \mapsto r) = 1_{\Phi \varepsilon} (l \mapsto r).
\]

\[\square\]

## 4 Eilenberg-Moore 2-Adjunction

In this section, we apply the results of the Section 3 to the 2-category \( \mathcal{2} \text{Cat} \), the 2-category of small categories and functors, because the 2-category \( \mathcal{2} \text{Cat} \) admits the construction of algebras. The 2-adjunction given in that section gives a usual adjunction,

\[
\text{Adj}_R(\mathcal{2} \text{Cat}) \xrightarrow{\Psi_E} \text{Mnd}(\mathcal{2} \text{Cat}) \xleftarrow{\Phi_E} \text{Mnd}(\mathcal{2} \text{Cat})
\]

Since the complete description, for a general \( A \), has been given above, we only give some remarks on the derived properties for this particular 2-category.

The description of the 2-functor \( \Psi_E \), for this particular 2-category, is given by the following entries

1.- On 0-cells, \( \Psi_E(\mathcal{C}, F) = D^F \rightleftharpoons U^F \), i.e. the Eilenberg-Moore adjunction.

2.- On 1-cells, \( (P, \varphi) : (\mathcal{C}, F) \rightarrow (\mathcal{D}, H) \), \( \Psi_E(P, \varphi) = (P, P^\varphi, \lambda^\varphi) \). The action of the functor \( P^\varphi : \mathcal{C}^F \rightarrow \mathcal{D}^H \) is the following

(i) On objects, \( (M, \chi_M) \) in \( \mathcal{C}^F \), \( P^\varphi(M, \chi_M) = (PM, P\chi_M \cdot \varphi_M) \).

(ii) On morphisms, \( p \), \( P^\varphi(p) = Pp \).

(iii) The natural transformation \( \lambda^\varphi \) is the mate of the identity \( U^H P^\varphi = PU^F \). Using (4), we get the component of \( \lambda^\varphi \) at \( A \), in \( \mathcal{C} \),

\[
\lambda^\varphi A = (\varepsilon \Phi \Psi \eta \Phi \varepsilon \Phi \eta \Phi \varepsilon)(l \mapsto r) = (1_A, 1_{\Phi \varepsilon}) \cdot (1_A, u^{lr} 1_{d^r})
\]

\[
= (1_A, 1_{\Phi \varepsilon}) = 1_{\Phi \varepsilon} (l \mapsto r) = 1_{\Phi \varepsilon} (l \mapsto r).
\]

3.- On 2-cells, \( \theta : (P, \varphi) \rightarrow (Q, \psi) \), we have

\[
\Psi_E(\theta) = (\alpha^\theta, \beta^\theta) = (\theta, \tilde{\theta})
\]

The induced natural transformation \( \tilde{\theta} : P^\varphi \rightarrow Q^\psi : \mathcal{C}^F \rightarrow \mathcal{D}^H \) is defined through its components as

\[
\tilde{\theta}(M, \chi_M) = \theta M
\]

It is clear that this definition is equivalent to the condition \( \theta U^F = U^H \tilde{\theta} \).
Since we have a 2-adjunction, the following isomorphism of categories takes place, natural for all $L ⊣ R$ and $(Χ, H)$:

$$\text{Hom}_{\text{Adj}_{\mathcal{S} \text{Cat}}}(L ⊣ R, \Psi_E(Χ, H)) \cong \text{Hom}_{\text{Mnd}(\mathcal{S} \text{Cat})}(Φ_E(L ⊣ R), (Χ, H))$$ \hfill (6)

5 \hspace{1em} \textbf{Monoidal Liftings (Eilenberg-Moore Type)}

5.1 \hspace{1em} \textbf{Colax Monads}

In this section, we give the definition of a colax monad.

\textbf{Definition 5.1} \hspace{1em} A \textit{colax monad $((F, ξ, γ), μ^F, η^F)$ over the monoidal category $(C, ⊗, I)$ consists of the following}

1. $(F, μ^F, η^F)$ is a monad on $C$.

2. $(F, ξ, γ) : (C, ⊗, I) → (C, ⊗, I)$ is a colax monoidal functor. That is to say, the natural transformations $ξ : F ⊗ → ⊗ (F × F)$ and $γ : F · δ_I → δ_I$ fulfills the commutativity on the following diagrams

$$\begin{align*}
F((A ⊗ B) ⊗ C) & \xrightarrow{ξ_{A,B,C}} F(A ⊗ B) ⊗ FC \\
& \xrightarrow{ξ_{A,B} ⊗ FC} (FA ⊗ FB) ⊗ FC
\end{align*}$$ \hfill (7)

$$\begin{align*}
F(A ⊗ (B ⊗ C)) & \xrightarrow{ξ_{A,B,C}} FA ⊗ F(B ⊗ C) \\
& \xrightarrow{FA ⊗ ξ_{B,C}} FA ⊗ (FB ⊗ FC)
\end{align*}$$

$$\begin{align*}
F(I ⊗ A) & \xrightarrow{ξ_{I,A}} FI ⊗ FA \\
& \xrightarrow{γ ⊗ FA} I ⊗ FA
\end{align*}$$ \hfill (8)

$$\begin{align*}
FA ⊗ I & \xrightarrow{FA ⊗ γ} FA ⊗ FI \\
& \xrightarrow{ξ_{A,I}} F(A ⊗ I)
\end{align*}$$

3. $μ^F : (F, ξ, γ) · (F, ξ, γ) → (F, ξ, γ)$ and $η^F : (1_C, 1_⊗, 1_δ_I) → (F, ξ, γ)$ are colax natural transformations, i.e. apart from the fact that they are natural transformations, they fulfill additionally the following commutative diagrams

$$\begin{align*}
FF ⊗ & \xrightarrow{Fξ} F ⊗ (F × F) \\
& \xrightarrow{ξ(F × F)} ⊗(FF × FF)
\end{align*}$$

$$\begin{align*}
FFδ_I & \xrightarrow{Fγ} Fδ_I \\
& \xrightarrow{γ} δ_I
\end{align*}$$
Since the natural transformation $\gamma$ has only one component, at 0, then this natural transformation and its component will be denoted indistinctly as $\gamma$.

Using the isomorphism (6), the following bijection can be obtained, cf. [8]

**Theorem 5.1** There is bijective correspondence between the following structures

1.- Colax monads $\left((F,\xi,\gamma),\mu^F,\eta^F\right)$, for the monoidal structure $\left(C,\otimes, I,a,l,r\right)$.

2.- Morphisms and natural transformations of monads of the form

\[
(\otimes,\xi) : (C \times C, F \times F) \to (C, F),
\]

\[
(\delta_I,\gamma) : (1,1_1) \to (C, F)
\]

\[
a : (\otimes \cdot (\otimes \times C),\otimes(\xi \times F) \circ \xi(\otimes \times C)) \to (\otimes \cdot (C \times \otimes) \cdot a_C,\otimes(F \times \xi)a_C \circ \xi(C \times \otimes)a_C)
\]

\[
: ((C \times C) \times C, (F \times F) \times F) \to (C, F),
\]

\[
l : (\otimes \cdot (\delta_I \times C) \cdot l_C^{-1},\otimes(\gamma \times F)l_C^{-1} \circ \xi(\delta_I \times C)l_C^{-1}) \to (1_C,1_F) : (C, F) \to (C, F),
\]

\[
r : (\otimes \cdot (C \times \delta_I) \cdot r_C^{-1},\otimes(F \times \gamma)r_C^{-1} \circ \xi(C \times \delta_I)r_C^{-1}) \to (1_C,1_F) : (C, F) \to (C, F).
\]

3.- Monoidal structures for the Eilenberg-Moore category, $\left(C^P,\hat{\otimes},\hat{I},\hat{a},\hat{l},\hat{r}\right)$ such that the following diagram of arrows and surfaces commutes

\[
(a) \hspace{10cm} (b)
\]

\[
\begin{array}{ccc}
C \times C & \xrightarrow{\otimes} & C \\
U^{F \times U^P} \downarrow & & \downarrow U^P \\
C^P \times C^P & \xrightarrow{\otimes} & C^P
\end{array}
\]

\[
\begin{array}{ccc}
1 \hspace{10cm} \delta_I \\
\downarrow U^1 \hspace{10cm} \downarrow U^P \\
1 \hspace{10cm} \delta_I
\end{array}
\]
Proof:

1 ⇒ 2)

Consider a colax monad \(((F, \xi, \gamma), \mu^F, \eta^F)\), for the monoidal structure \((C, \otimes, I)\). In particular, the multiplication and the unit of the monad are colax natural transformations and the first diagrams in (9) and (10) commute. Therefore, we have a monad morphism \((\otimes, \xi) : (C \times C, F \times F) \to (C, F)\).

Likewise, the commutativity of the second diagrams in (9) and (10) implies that \((\delta_I, \gamma) : (1, 1_I) \to (C, F)\) is a morphism of monads. Note that the requirement \((\delta_I, \gamma)\) is a monad morphism is equivalent to the statement \((I, \gamma)\) is an Eilenberg-Moore algebra.

Since \((\otimes, \xi)\) is a morphism of monads then the following morphisms are also morphisms of monads \((\otimes(\otimes \times C), \otimes(\xi \times F) \circ \xi(\otimes \times C))\) and \((\otimes(\otimes \times a_C, \otimes(F \times \xi) \circ \xi(\otimes \times a_C))\) from \(((C \times C) \times C, (F \times F) \times F)\) to \((C, F)\) and due to the commutativity of the diagram [7], the following is a 2-cell in \textbf{Mnd}

\[
\begin{array}{c}
\begin{array}{c}
\otimes(\otimes(\otimes \times C), \otimes(\xi \times F) \circ \xi(\otimes \times C)) \\
((C \times C) \times C, (F \times F) \times F) \\
\otimes(\otimes(\otimes \times a_C, \otimes(F \times \xi) \circ \xi(\otimes \times a_C)) \end{array} \\
\overset{a}{\longrightarrow} \\
\begin{array}{c}
(C, F) \\
((C \times C) \times C, (F \times F) \times F) \\
\otimes(\otimes(\otimes \times a_C, \otimes(F \times \xi) \circ \xi(\otimes \times a_C)) \end{array}
\end{array}
\]

Likewise, because \((\otimes, \xi)\) and \((\delta_I, \gamma)\) are monad morphisms, \((\otimes(\delta_I \times C) \cdot l_C^{-1}, \otimes(\gamma \times F) \cdot l_C^{-1} \circ \xi(\delta_I \times C) \cdot l_C^{-1})\) is also a monad morphism. Using the commutativity of the first diagram in (5), we can consider the monad 2-cell

\[
\begin{array}{c}
\begin{array}{c}
\otimes(\delta_I \times C) \cdot l_C^{-1}, \otimes(\gamma \times F) \cdot l_C^{-1} \circ \xi(\delta_I \times C) \cdot l_C^{-1) \\
(C, F) \\
\otimes(\delta_I \times C) \cdot l_C^{-1}, \otimes(\gamma \times F) \cdot l_C^{-1} \circ \xi(\delta_I \times C) \cdot l_C^{-1) \end{array} \\
\overset{t}{\longrightarrow} \\
\begin{array}{c}
(C, F) \\
((C \times C) \times C, (F \times F) \times F) \\
\otimes(\otimes(\otimes \times a_C, \otimes(F \times \xi) \circ \xi(\otimes \times a_C)) \end{array}
\end{array}
\]
In a similar way, the following is a monad transformation, \( r : (\otimes \cdot (C \times \delta I) \cdot r_C^{-1}, \otimes (F \times \gamma) r_C^{-1} \circ \xi(C \times \delta I) r_C^{-1}) \rightarrow (1_C, 1_F) : (C, F) \rightarrow (C, F) \).

2 \Rightarrow 1)

Note that the aforementioned claims can be reverted.

2 \Rightarrow 3)

Take the monad morphism \((\otimes, \xi) : (C \times C, F \times F) \rightarrow (C, F)\). In order to use the isomorphism (6), we make \( L \dashv R = D F \times D F \dashv U F \times U F \) and \((X, H, \mu^H, \eta^H) = (C, F, \mu^F, \eta^F)\). Therefore, to this monad morphism corresponds a morphism of adjunctions of the form \((\otimes, \otimes \xi) : D F \times D F \dashv U F \times U F \rightarrow D F \dashv U F\) such that a diagram like (11a) commutes. According to the definition of \(\Psi^E\), the functor \(\otimes \xi\) acts as follows

\[
\otimes \xi((M, \chi_M), (N, \chi_N)) = (\otimes (M, N), \otimes (\chi_M, \chi_N) \cdot \xi_{M,N})
\]

The previous action is defined at the beginning of the proof of Theorem 7.1, [8].

\[
\otimes \xi(p, q) = \otimes(p, q)
\]

We change the notation from \(\otimes \xi\) to \(\hat{\otimes}\).

If in the isomorphism (6), we make \( L \dashv R = 1_1 \dashv 1_1 \) and \((X, H, \mu^H, \eta^H) = (C, F, \mu^F, \eta^F)\). The monad morphism \((\delta I, \gamma)\) has an associated morphism of adjunctions of the form \((\delta I, \delta I \gamma) : (1_1 \dashv 1_1) \rightarrow D F \dashv U F\) such that a diagram like (11b) commutes. According to the definition of \(\Psi^E\), the functor \(\delta I \gamma\) acts as follows

\[
\delta I \gamma(0, 1_0) = (\delta I(0), \delta I(1_0) \cdot \gamma) = (I, \gamma).
\]

On morphisms,

\[
\delta I \gamma(1_0) = \delta I(1_0) = 1_I = 1_{(I, \gamma)}.
\]

If we make the following definition \(\hat{I} = (I, \gamma)\), then \(\delta I \gamma := \hat{\delta I}\). The algebra \((I, \gamma)\) is the unit of the monoidal structure on \(C^F\).

Suppose that we have a natural transformation of the form \(a : (\otimes \cdot (\otimes \times C), \otimes (\xi \times F) \circ \xi (\otimes \times C)) \rightarrow (\otimes \cdot (C \times \otimes) \cdot a_C, \otimes (F \times \xi) a_C \circ \xi (C \times \otimes) a_C) : ((C \times C) \times C, (F \times F) \times F) \rightarrow (C, F)\) then we can make
In order to reduce expressions, we used and will be using the following notation

\[ \cdot \xi^2 := \otimes (\xi \times F) \circ \xi (\otimes \times C), \]
\[ \xi^2 := \otimes (F \times \xi) a_C \circ \xi (C \times \otimes) a_C, \]
\[ (\cdot)^3 := (\cdot \times \cdot) \times \cdot. \]

It can be prove that \([\otimes \cdot (\otimes \times C)] \xi^2 = \hat{\otimes} \cdot (\hat{\otimes} \times C^F). \) On objects and morphisms

\[
[\otimes \cdot (\otimes \times C)] \xi^2 \left(\left(\langle M, \chi_M \rangle, \langle N, \chi_N \rangle, \langle M', \chi_{M'} \rangle\right)\right) \\
= \left[\langle \otimes (\otimes \times C) \rangle \left(\langle M, \chi_M \rangle, \langle M', \chi_{M'} \rangle\right\rangle, \langle \otimes (\otimes \times C) \rangle \left(\langle \chi_M \times \chi_N \rangle, \langle \chi_M \times \chi_{M'} \rangle\right\rangle \right]\right) \cdot (\xi_{M,N} \otimes F M') \cdot (\xi_{M,N} \otimes M') \\
= \left[\langle M \otimes N \rangle \otimes M', \langle (\chi_M \otimes \chi_N) \otimes (\chi_M \otimes \chi_{M'})\rangle \right] \cdot (\xi_{M,N} \otimes F M') \cdot (\xi_{M,N} \otimes M') \\
= \left[\langle M \otimes N \rangle \otimes M', \langle (\chi_M \otimes \chi_N) \cdot (\xi_{M,N} \otimes (\chi_M \otimes \chi_{M'}). \rangle \cdot (\xi_{M,N} \otimes \xi_{M', M'})\rangle \right] \\
= \hat{\otimes} \left(\langle M \otimes N \rangle, \langle (\chi_M \otimes \chi_N), (\xi_{M,N} \otimes (\chi_M \otimes \chi_{M'}). \rangle \right) \cdot (\xi_{M,N} \otimes \xi_{M', M'})\right) \\
= \hat{\otimes} \cdot (\hat{\otimes} \times C^F) \left(\langle (\chi_M, \chi_N), (\xi_{M,N} \otimes (\chi_M \otimes \chi_{M'}). \rangle \right) \cdot (\xi_{M,N} \otimes \xi_{M', M'})\right)
\]

\[
[\otimes \cdot (\otimes \times C)] \xi^2((p, q), (p', q')) = \otimes (\otimes \times C)((p, q), (p', q')) \\
= (p \otimes q) \otimes (p' \otimes q')
\]

In the same way, we can check that \([\otimes \cdot (C \times \otimes) \cdot a_C] \xi^2 = \hat{\otimes} \cdot (\hat{\otimes} \times C^F) \cdot a_{C^F}. \) We change the notation \(\beta^a\) for \(\hat{a}\) and we get a natural transformation \(\hat{a} : \hat{\otimes}(\hat{\otimes} \times C^F) \longrightarrow \hat{\otimes}(\hat{\otimes} \times C^F) \cdot a_{C^F} : (C^F \times C^F) \rightarrow C^F. \) Using the definition of the functor \(\Psi_F\) on the 2-cell \(a\), we get the component at \(\langle (\langle M, \chi_M \rangle, (\langle N, \chi_N \rangle, (\langle M', \chi_{M'} \rangle)\rangle\rangle\) \(\hat{a}((\langle M, \chi_M \rangle, (\langle N, \chi_N \rangle, (\langle M', \chi_{M'} \rangle)) = a(M, N, M')\)
Suppose we have a 2-cell in $\text{Mnd}$ of the form $l : (\otimes : (\delta_I \times C) \cdot l_C^{-1}, \otimes(\gamma \times F)l_C^{-1} \circ \xi(\delta_I \times C), l_C^{-1}) \longrightarrow (1_C, 1_F) : (C, F) \longrightarrow (C, F)$. If in the isomorphism (6), we make $L \dashv R = D^p \dashv U^p$ and $(X, H, \mu^H, \eta^H) = (C, F, \mu^F, \eta^F)$, it can be obtained a 2-cell in the 2-category $\text{Adj}_{\text{K}}(2\text{Cat})$ of the form $(l, \beta^l) : (\otimes : (\delta_I \times C) \cdot l_C^{-1}, 1_C) \longrightarrow ([\otimes : (\delta_I \times C) \cdot l_C^{-1}] \gamma_{\otimes}, [1_C]^{1_F}) : D^p \dashv U^p \longrightarrow D^p \dashv U^p$. Where we used the notation $\gamma \circ \xi = (\otimes(\gamma \times F)l_C^{-1} \circ \xi(\delta_I \times C), l_C^{-1})$. We change the notation from $\beta^l$ to $\hat{l}$.

In the same way as before, it can be proved that $[\otimes : (\delta_I \times C) \cdot l_C^{-1}] \gamma_{\otimes} = \hat{\otimes}(\delta_I \times C)l_C^{-1}$ and $[1_C]^{1_F} = 1_{C^F}$. Therefore, we obtain a natural transformation $\hat{l} : \otimes(\delta_I \times C)l_C^{-1} : (1_C, 1_F) : (C, F) \longrightarrow (C, F)$ there corresponds a natural transformation $\hat{r} : \otimes(C^F \times \delta_I)l_C^{-1} : 1_{C^F} : C^F \longrightarrow C^F$. The component of this natural transformation, at $(M, \chi_M)$, is

$$\hat{r}(M, \chi_M) = r_M \quad (12)$$

Since the natural transformations $a, l$ and $r$ fulfill the coherence conditions for a monoidal struture and $U^p$ is faithful then $\hat{a}, \hat{l}$ and $\hat{r}$ fulfill the pentagon and the triangle coherence conditions. Therefore, $(C^F, \otimes, \hat{I}, \hat{a}, \hat{l}, \hat{r})$ is a monoidal structure over $C^F$.

$3 \Rightarrow 2)$

Note that the aforementioned statements can be reverted. For example, take the morphism of adjunctions $(a, \hat{a}) : (\otimes : (\otimes \times C, \otimes(\otimes \times C^F)) \longrightarrow (\otimes : (C \times \otimes) \cdot a_C, \otimes(\otimes \times C^F) : (U^p \times U^p) \times U^p \longrightarrow (D^p \times D^p) \times D^p \longrightarrow U^p \dashv U^p)$. The image of this 2-cell, under $\Phi_{\text{K}}$, is $a : \otimes(\otimes \times C, \otimes(\otimes \times C), \otimes(\otimes \times C^F) : (U^p \times U^p) \times U^p \longrightarrow (D^p \times D^p) \times D^p \longrightarrow U^p \dashv U^p) \longrightarrow (C, F)$, i.e.

$$a : (\otimes(\otimes \times C), \otimes(\otimes \times C) \cdot a_C, \otimes(F \times \varphi_\otimes) \cdot a_C : (C^3, F^3) \longrightarrow (C, F)$$

Everytime we used the isomorphism (4), the monad $(C, F, \mu^F, \eta^F)$ was always taken fixed, therefore the implication $2 \Rightarrow 3$ is natural in the monad $(C, F, \mu^F, \eta^F)$.

The authors did not check for the naturality of the implication $1 \Rightarrow 2$, but the reader can do it.

### 6 Kleisli 2-Adjunction

Based on either [2] or [3], the following 2-adjunction takes place

$$\text{Mnd}^\circ(2\text{Cat}) \xrightarrow{\Phi_{\text{K}}} \text{Adj}_{\text{K}}(2\text{Cat}) \xleftarrow{\Psi_{\text{K}}}$$
which can also be deduced from the general 2-adjunction given by \( \text{(1)}. \) In this sense, we provide only a few remarks on the structure for the several objects that build this 2-adjunction.

The description of 2-functor, \( \Psi_K, \) is given completely in order to provide the necessary notation. The structure of such 2-functor goes as follows

1.- On 0-cells, \( \Psi_K(C, F) = G_F \sqcup V_F, \) i.e. the Kleisli adjunction.

2.- On 1-cells, \( (P, \pi) : (C, F) \to (D, H), \) \( \Psi_K(P, \pi) = (P, P_\pi, \rho_\pi). \) In the definition of the functor \( P_\pi : \mathcal{C}_F \to \mathcal{D}_H, \) we use the notation \( (\cdot)^\sharp \) given for a morphism in \( \mathcal{C}_F \) and \( (\cdot)^\flat \) for a morphism in \( \mathcal{D}_H. \) This notation is used in \([5]\) and \([10]\).

(i) On objects, \( X \) in \( \mathcal{C}_F, \) \( P_\pi X = PX. \)

(ii) On morphisms, \( x^\sharp : X \to Y \) in \( \mathcal{C}_F, \) \( P_\pi x^\sharp = (\pi C x^\sharp \cdot P x)^\flat, \) where \( C x^\sharp \) is the notation for the codomain of the morphism \( x^\sharp \) as in \( \mathcal{C}_F, \) which in this case is \( Y. \)

(iii) In order to define \( \rho_\pi \) we have to prove that the following equality of functors takes place, \( G_HP = P_\pi G_F. \)

On objects and morphisms \( f : A \to B \) in \( \mathcal{C}, \)

\[
G_HPA = PA = P_\pi A = P_\pi G_F A,
\]

\[
G_HPf = (HP f \cdot \eta^H PA)^\flat = (HP f \cdot \pi A \cdot P \eta^F A)^\flat
\]

\[
= (\pi B \cdot P \eta^F A)^\flat = P_\pi (P f \cdot \eta^F A)^\sharp = P_\pi G_F f
\]

where the second equality takes place because of the unitality condition on \( \pi \) and the third one is due to the naturality on \( \pi. \)

Using \( \text{(2)}, \) we get the mate for this identity

\[
\rho_\pi = V_\pi P_\pi \varepsilon^{FU} \circ \eta^H PV_F,
\]

whose component, at \( X \) in \( \mathcal{C}_F, \) is \( \rho_\pi X = \mu^H PX \cdot H \pi X \cdot \eta^H PF X = \pi X. \)

3.- On 2-cells, \( \vartheta : (P, \pi) \to (Q, \tau), \) we have

\[
\Psi_K(\vartheta) = (\alpha_\vartheta, \beta_\vartheta)
\]

where \( \alpha_\vartheta := \vartheta \) and we rename \( \beta_\vartheta \) as \( \tilde{\vartheta}. \) The induced natural transformation \( \tilde{\vartheta} : P_\pi \to Q_\pi : \mathcal{C}_F \to \mathcal{D}_H \) is defined through its components as

\[
\tilde{\vartheta} X = (\eta^H QX \cdot \vartheta X)^\flat
\]

It is clear that this definition is equivalent to the condition \( G_H \vartheta = \vartheta G_F. \)

Since we have a 2-adjunction, the following isomorphism of categories takes place, natural in \( (X, H) \) and \( L \sqcup R \)

\[
\text{Hom}_{\text{Mnd}^*_{(2\text{Cat})}}((X, H), \Phi_K(L \sqcup R)) \cong \text{Hom}_{\text{Adj}_{2\text{Cat}}}(\Psi_K(X, H), L \sqcup R)
\]  \( \text{(14)} \)
7 Monoidal Extensions (Kleisli Type)

7.1 Lax Monads

Dual to colax monads, we give the definition of a lax monad.

Definition 7.1 A lax monad \(((F,ζ,ω),μ^F,η^F)\) over a monoidal category \((C,⊗,I,a,l,r)\) consists of the following

1. \((F,μ^F,η^F)\) is a monad on \(C\).

2. \((F,ζ,ω) : (C,⊗,I) \to (C,⊗,I)\) is a lax monoidal functor. This means that the natural transformations \(ζ : ⊗ · (F × F) \to F · ⊗\) and \(ω : δ_I \to F · δ_I\), fulfills the commutativity on the following diagrams

   \[
   (FA ⊗ FB) ⊗ FC \xrightarrow{ζ_{A,B,C}} F(A ⊗ B) ⊗ FC \xrightarrow{ζ_{A⊗B,C}} F((A ⊗ B) ⊗ C)
   \]

3. \(μ^F : (F,ζ,ω) · (F,ζ,ω) \to (F,ζ,ω)\) and \(η^F : (1_C,1_⊗,1_δ) \to (F,ζ,ω)\) are lax natural transformations, the adjective lax adds, to the naturality, the following commutative diagrams
Note 7.2 Necessarily $\omega(0) = \eta^I$.

Since the natural transformation $\omega$ has only one component then this natural transformation and its component will be denoted indistinctly with $\omega$, for example $\omega = \omega(0) = \eta^I$.

We are going to make use of the isomorphism (14). The result we want to obtain using this isomorphism is the following.

Theorem 7.3 There is a bijective correspondence between the following structures

1.- Colax monads $((F, \zeta, \omega), \mu^F, \eta^F)$, for the monoidal structure $(C, \otimes, I, a, l, r)$.

2.- Morphisms and transformations of monads of the form

$$(\otimes, \zeta) : (C \times C, F \times F) \to (C, F),$$

$$(\delta_l, \omega) : (1, 1) \to (C, F)$$

$a : (\otimes \cdot (\otimes \times C), \zeta(\otimes \times C) \circ \otimes(\zeta \times F)) \to (\otimes \cdot (\otimes \times C) \cdot a_C, \zeta(\otimes \times C) a_C \circ \otimes(F \times \zeta a_C)$$

$$(\delta_l \circ C, F \times F \times F) \to (C, F),$$

$l : (\otimes \cdot (\delta_l \times C) \cdot l_c, \zeta(\delta_l \times C) l_c^{-1} \circ \otimes(\omega \times F) l_c^{-1}) \to (1_c, 1) : (C, F) \to (C, F),$$

$r : (\otimes \cdot (\delta_l \times C) \cdot r_c^{-1}, \zeta(\delta_l \times C) r_c^{-1} \circ \otimes(F \times \omega) r_c^{-1}) \to (1_c, 1) : (C, F) \to (C, F).$$

3.- Monoidal structures for the Kleisli category $(C_F, \tilde{\otimes}, \tilde{I})$ such that the following diagrams of arrows and surfaces commute

\[ (a) \]

\[ (b) \]

\[ (19) \]
Proof:

1 ⇒ 2)

Consider a lax monad \(((F, \zeta, \omega), \mu^F, \eta^F)\) for the monoidal category \((C, \otimes, I)\). In particular, \(\mu^F\) and \(\eta^F\) are natural lax monoidal transformations. Therefore, the commutativity of the first diagram in (17) and the first one in (18) is equivalent to the condition that the following be a monad morphism \((\otimes, \zeta) : (C \times C, F \times F) \to (C, F)\).

The commutativity condition on the second diagrams in (17) and (18) is equivalent to the condition for the following be a monad morphism \((\delta_I, \omega) : (1, 1) \to (C, F)\).

Since \((\otimes, \zeta)\) is a morphism of monads so are \((\otimes \cdot (\otimes \times C), \zeta(\otimes \times C) \circ \otimes(\zeta \times F))\) and \((\otimes \cdot (C \times \otimes) \cdot ac, \zeta(C \times ac) \circ \otimes(F \times \zeta)ac)\). Yet again, since \(((F, \zeta), \mu^F, \eta^F)\) is a lax monad over the monoidal category \((C, \otimes, I, a, l, r)\), then a commutative diagram like (15) takes place. Therefore the following is a 2-cell in \(\text{Mnd}^\bullet(2\text{Cat})\).

\[
\begin{array}{ccc}
((C \times C) \times C, (F \times F) \times F) & \xrightarrow{a} & (C, F) \\
(\otimes \cdot (C \times \otimes) \cdot ac, \zeta(C \times ac) \circ \otimes(F \times \zeta)ac) & & \\
\end{array}
\]

Since \((\otimes, \zeta)\) and \((\delta_I, \omega)\) are monad morphisms so is \((\otimes \cdot (\delta_I \times C) \cdot l^{-1}_C, \zeta(\delta_I \times C)l^{-1}_C \circ \otimes(\omega \times F)l^{-1}_C)\) and taking into account the commutativity of the diagram (16a), we can state that the following is a 2-cell in \(\text{Mnd}^\bullet(2\text{Cat})\)

\[
\begin{array}{ccc}
(C, F) & \xrightarrow{t} & (C, F) \\
(1_C, 1_F) & & \\
\end{array}
\]

In the very same way, the following is a 2-cell of monads, \(r : (\otimes \cdot (C \times \delta_I) \cdot r^{-1}_C, \zeta(C \times \delta_I)r^{-1}_C \circ \otimes(F \times \omega)r^{-1}_C)\)

2 ⇒ 1) The previous assertions can be reverted.

2 ⇒ 3)

Suppose we have a monad morphism \((\otimes, \zeta)\). Use the isomorphism (14), with \((\mathcal{D}, H, \mu^H, \eta^H) = (C \times C, F \times F, \mu^F \times \mu^F, \eta^F \times \eta^F)\) and \(L \dashv R = G_F \dashv V_F\) to get an associated morphism of adjunctions \((\otimes, \otimes) : G_F \times G_F \dashv V_F \times V_F \to G_F \dashv V_F\), such that a diagram like (19a) commutes. According to the definition of \(\Psi_K\), the functor \(\otimes \zeta\) acts as follows. On objects,

\[\otimes\zeta(X, Y) = \otimes(X, Y) = X \otimes Y,\]

and on morphisms,
\[\otimes_\zeta(x^\sharp,y^\sharp) = (\zeta_{C^\sharp},C_{y^\sharp} \cdot (x \otimes y))^\sharp\]

where \(C^\sharp\) is codomain of the morphism \(x^\sharp\) for example. We rename \(\otimes\) as \(\tilde{\otimes}\).

For the monad morphism, \((\delta_I,\omega) : (1,1_1) \to (\mathcal{C},F)\), use the mentioned isomorphism with \((D,H,\mu^H,\eta^H) = (1,1_1,1_{11},1_{11})\), \textit{i.e.} the trivial monad on the category \(1\), and \(L \dashv R = G_F \dashv V_F\). Therefore, there exists an adjunction morphism \((\delta_I,\tilde{\delta}_I)_\omega : G_{1_1} \dashv V_{1_1} \to G_F \dashv V_F\). According to the 2-functor \(\Psi_K\), the functor \([\delta_I]_\omega : 1 \to \mathcal{C}_F\), acts in the following way

\[\tilde{\delta}_I(0) = \omega(0) \cdot \delta_I(1) = (\eta^F I)^\sharp\]

That is to say \([\delta_I]_\omega = \delta_I : 1_{1_1} \to \mathcal{C}_F\), where \(\tilde{I} = I\).

Suppose that we have the following 2-cell in \(\text{Mnd}\),

\[\begin{align*}
\otimes\zeta^2 &= \zeta(\otimes \times C) \circ \otimes (\zeta \times F), \\
\zeta^2 &= \zeta(C \times \otimes) a_C \circ \otimes (F \times \zeta) a_C, \\
(\cdot)^3 &= (\cdot \times \cdot) \times \cdot.
\end{align*}\]

According to the isomorphism of categories given by \([\square]\), to the previous 2-cell in \(\text{Mnd}^*(\text{2Cat})\) corresponds a 2-cell, \((\alpha_a, \beta_a)\) in \(\text{Adj}_{\square}(\text{2Cat})\), where \(\alpha_a = a\) and we rename \(\beta_a = \tilde{a}\) and such that
It can be shown that

\[ ([\otimes \cdot (\otimes \times C)]_{\zeta^2} = \tilde{\otimes} \cdot (\tilde{\otimes} \times C_F) \]

\[ ([\otimes \cdot (C \times \otimes) \cdot a_C]_{\zeta^2} = \tilde{\otimes} \cdot (C_F \times \tilde{\otimes}) \cdot a_{C_F} \]

Therefore, we have a natural transformation \( \tilde{a} : \tilde{\otimes} \cdot (\tilde{\otimes} \times C_F) \rightarrow \tilde{\otimes} \cdot (C_F \times \tilde{\otimes}) \cdot a_{C_F} \) that will be part of a monoidal structure on \( C_F \). According to the 2-functor \( \Psi_K \), the component of \( \tilde{a} \) at \((X, Y, Z)\) is

\[ \tilde{a}_{X, Y, Z} = (\eta^F (X \otimes (Y \otimes Z))) \cdot a_{X, Y, Z} \]

Suppose that we have a 2-cell in \( \text{Mnd}^* (2\text{Cat}) \) of the form \( l : (\otimes \cdot (\delta_I \times C) \cdot l_C^{-1}, \zeta (\delta_I \times C) l_C^{-1} \circ \otimes (\omega \times F)l_C^{-1}) \rightarrow (1_{C}, 1_{F}) : (C, F) \rightarrow (C, F) \).

Therefore, we obtain a natural transformation \( \tilde{l} : \tilde{\otimes} \cdot (\delta_I \times C_F) \cdot l_C^{-1} \rightarrow 1_{C_F} \). Using the definition of the functor \( \Psi_K \) on the 2-cell \( l \), the component of \( \tilde{l} \), on the object \( X \) in \( C_F \), is

\[ \tilde{l}_X = (\eta^F X \cdot lX)^{\tilde{z}} \quad (20) \]

Similarly, for the monad morphism \( r : (\otimes \cdot (C \times \delta_I) \cdot r_C^{-1}, \zeta (C \times \delta_I) r_C^{-1} \circ \otimes (F \times \omega) r_C^{-1}) \rightarrow (1_{C}, 1_{F}) : (C, F) \rightarrow (C, F) \), we obtain a natural transformation \( \tilde{r} : \tilde{\otimes} \cdot (\delta_I \times C_F) \cdot r_{C_F}^{-1} \rightarrow 1_{C_F} : C_F \rightarrow C_F \).

The proof of the coherence conditions are left to the reader.
In summary, \((C_F, \tilde{\otimes}, \tilde{I}, \tilde{a}, \tilde{l}, \tilde{r})\) has a monoidal structure on \(C_F\).

3 \Rightarrow 2)

Using the isomorphism, given by \([14]\), we get the return of the proof. For example, the image, under \(\Phi_K\), for the 2-cell of adjunctions \((a, \tilde{a}) : (\otimes \cdot (\otimes \times C), \otimes \cdot (\otimes \times c_c) \cdot a_c) \to (\otimes \cdot (\otimes \times C_F), \otimes \cdot (C_F \times \otimes) \cdot a_{C_F}) : (G_F \times G_F) \times G_F \to (V_F \times V_F) \times V_F\) is

\[
\Phi_K((a, \tilde{a})) = a : (\otimes, \pi_\otimes)(\otimes \times C, \pi_\otimes \times C) \to (\otimes, \pi_\otimes)(C \times \otimes, \pi_{C \times \otimes})(a_c, \pi_{a_c}) : (C^3, F^3) \to (C, F)
\]

\[
= a : (\otimes, \pi_\otimes)(\otimes \times C, \otimes \times F) \to (\otimes, \pi_\otimes)(C \times \otimes, F \times \pi_\otimes)(a_c, 1_{F \times (P \times F) \cdot a_c}) : (C^3, F^3) \to (C, F)
\]

Note that \(a_c\) is a morphism of adjunctions. \(\Box\)

8 Liftings to the Eilenberg-Moore algebras & Extensions to the Kleisli Categories

This is probably the more explored section of all this article, a few examples of the detailed proofs for the following statements are found in \([1]\) and \([10]\). In this section, we treated these statements only as direct consequences of the isomorphisms of categories given by \([6]\) and \([14]\).

**Theorem 8.1** There is a bijective correspondence, natural in \((C, F, \mu^F, \eta^F)\) and \((D, H, \mu^H, \eta^H)\), between the following structures

1.- Liftings to the Eilenberg-Moore algebras, for the functor \(P : C \to D\). That is to say, the following diagram commutes

\[
\begin{array}{ccc}
C^P & \xrightarrow{Q} & D^H \\
\downarrow{U^F} & & \downarrow{U^H} \\
C & \xrightarrow{P} & D
\end{array}
\]

2.- Morphisms of monads \((P, \varphi) : (C, F) \to (D, H)\). That is to say, a natural transformation \(\varphi : HP \to PF\), such that the following diagrams commute

\[
\begin{array}{ccc}
HHP & \xrightarrow{H\varphi} & HPF \\
\downarrow{\mu^H_P} & & \downarrow{\varphi^F} \\
HP & \xrightarrow{\varphi} & PF
\end{array}
\]

\[
\begin{array}{ccc}
HHP & \xrightarrow{H\varphi} & HPF \\
\downarrow{\mu^H_P} & & \downarrow{\varphi^F} \\
HP & \xrightarrow{\varphi} & PF
\end{array}
\]
Theorem 8.2 There exists a bijective correspondence, natural in \((C, F, \mu^F, \eta^F)\) and \((D, H, \mu^H, \eta^H)\), between the following structures

1.- Extensions to the Kleisli categories, for the functor \(P : C \rightarrow D\). That is to say, the following diagram commutes

\[
\begin{array}{c}
C \xrightarrow{P} D \\
\downarrow \quad \downarrow \\
C_F \quad D_H \\
\end{array}
\]

\[
\begin{array}{c}
G_F \quad G_H \\
\end{array}
\]

2.- Morphisms of monads \((P, \varphi) : (C, F) \rightarrow (D, H)\). That is to say, a natural transformation \(\varphi : P \rightarrow H\)

\[
\begin{array}{c}
P \quad P \quad H \quad H \quad P \\
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
P \quad \varphi \\
\end{array}
\]

\[
\begin{array}{c}
P \mu^F \\
\end{array}
\]

\[
\begin{array}{c}
P \mu^H \\
\end{array}
\]

\[
\begin{array}{c}
\varphi \\
\end{array}
\]

9 Actions on the Kleisli Category

9.1 Categorical Actions

In this section we give the definition of a categorical action.

Definition 9.1 Let \((C, \otimes, I)\) be a monoidal category. A left \(C\)-action on the category \(B\) is a functor \(\otimes : C \times B \rightarrow B\) together with natural transformations \(\nu : \otimes(\otimes \times B) \rightarrow \otimes(\otimes \times)\alpha_s : (C \times C) \times B \rightarrow B\) and \(j : \otimes(\delta_1 \times B)\iota^{-1} \rightarrow 1_B : B \rightarrow B\) such that they fulfill the following commutative diagrams, for objects \(C, C', C''\) in \(C\) and \(B\) in \(B\),

\[
\begin{array}{c}
[(C \otimes C') \otimes C''] \otimes B \\
\downarrow \quad \downarrow \\
C \otimes [(C' \otimes C'') \otimes B] \\
\end{array}
\]

\[
\begin{array}{c}
\otimes_{C, C', C''} B \\
\end{array}
\]

\[
\begin{array}{c}
(C \otimes C') \otimes (C'' \otimes B) \\
\downarrow \\
(C \otimes C') \otimes (C'' \otimes B) \\
\end{array}
\]

\[
\begin{array}{c}
\nu_{C, C', C''} B \\
\end{array}
\]

\[
\begin{array}{c}
\nu_{C, C', C''} B \\
\end{array}
\]

\[
\begin{array}{c}
C \otimes [(C' \otimes C'') \otimes B] \\
\downarrow \\
C \otimes [(C' \otimes C'') \otimes B] \\
\end{array}
\]

\[
\begin{array}{c}
\nu_{C, C', C''} B \\
\end{array}
\]

\[
\begin{array}{c}
C \otimes [(C' \otimes C'') \otimes B] \\
\downarrow \\
C \otimes [(C' \otimes C'') \otimes B] \\
\end{array}
\]

\[
\begin{array}{c}
\nu_{C, C', C''} B \\
\end{array}
\]

\[
\begin{array}{c}
C \otimes [(C' \otimes C'') \otimes B] \\
\downarrow \\
C \otimes [(C' \otimes C'') \otimes B] \\
\end{array}
\]

\[
\begin{array}{c}
\nu_{C, C', C''} B \\
\end{array}
\]

\[
\begin{array}{c}
C \otimes [(C' \otimes C'') \otimes B] \\
\downarrow \\
C \otimes [(C' \otimes C'') \otimes B] \\
\end{array}
\]

\[
\begin{array}{c}
\nu_{C, C', C''} B \\
\end{array}
\]

\[
\begin{array}{c}
C \otimes [(C' \otimes C'') \otimes B] \\
\downarrow \\
C \otimes [(C' \otimes C'') \otimes B] \\
\end{array}
\]
9.2 Strong Monads

In this section we give the definition of a **strong monad**.

**Definition 9.1** A right strong monad $((F, \sigma^r), \mu^F, \eta^F)$, on the monoidal category $(C, \otimes, I)$, is a usual monad $(F, \mu^F, \eta^F)$, on $C$, with a natural transformation $\sigma^r : A \otimes FB \to F(A \otimes B)$ such that the following diagrams commute

\[
\begin{align*}
(a) & \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \q
2.- Morphisms and transformations of monads of the form

\[ (\otimes, \sigma^r) : (C \times C, C \times F) \rightarrow (C, F) \]
\[ a : (\otimes \cdot (\otimes \times C), \sigma^r(\otimes \times C)) \rightarrow (\otimes \cdot (C \times \otimes) \cdot a_C, \sigma^r(C \times \otimes) a_C \circ (\otimes \times \sigma^r) a_C) \]
\[ l : (\otimes \cdot (\delta I \times C) \cdot l_C^{-1}, \sigma^r(\delta I \times C) l_C^{-1}) \rightarrow (1_C, 1_F) : (C, F) \rightarrow (C, F) \]

3.- Left actions on the Kleisli category, \( C_F, \boxtimes : C \times C_F \rightarrow C_F \) such that the following diagrams of morphisms and surfaces commute

\[ \begin{array}{ccc}
C \times C & \overset{\otimes}{\rightarrow} & C \\
\downarrow & & \downarrow \\
C \times G_F & \underset{G_F}{\rightleftharpoons} & C_F \\
\downarrow & & \downarrow \\
C \times C_F & \underset{\boxtimes}{\rightarrow} & C_F
\end{array} \] (23)

\[ \begin{array}{ccc}
C \times C & \overset{\otimes \cdot (\otimes \times C)}{\rightarrow} & C \\
\downarrow & & \downarrow \\
C^2 \times C & \underset{\boxtimes \cdot (\otimes \times C) - a_C}{\rightarrow} & C_F \\
\downarrow & & \downarrow \\
C^2 \times G_F & \underset{G_F}{\rightleftharpoons} & C_F
\end{array} \quad \begin{array}{ccc}
C \times C & \overset{\otimes \cdot (\delta I \times C) \cdot l_C^{-1}}{\rightarrow} & C \\
\downarrow & & \downarrow \\
C \overset{l_C}{\rightarrow} & C_F \\
\downarrow & & \downarrow \\
C_F & \underset{\boxtimes \cdot (\delta I \times C_F) \cdot l_C^{-1}}{\rightarrow} & C_F
\end{array} \] (24)

**Proof:**

1 \( \Rightarrow \) 2)

Take a right strong monad \((F, \sigma^r)\) over the monoidal category \((C, \otimes, I)\). Due to this fact, the following is a morphism of monads \((\otimes, \sigma^r) : (C \times C, C \times F) \rightarrow (C, F)\), and so are the following, \((\otimes \cdot (\otimes \times C), \sigma^r(\otimes \times C))\) and \((\otimes \cdot (C \times \otimes) \cdot a_C, \sigma^r(C \times \otimes) a_C \circ (\otimes \times \sigma^r) a_C )\).

The commutativity of the diagram \([22a]\) implies that the following is a transformation of monads

\[ ((C \times C) \times C, (C \times C) \times F) \overset{a}{\rightarrow} (C, F) \]
\[ (\otimes \cdot (C \times \otimes) \cdot a_C, \sigma^r(C \times \otimes) a_C \circ (\otimes \times \sigma^r) a_C ) \]
By the same reason as before, the following is a morphism of monads \((\otimes \cdot (\delta_I \times C))_{C}^{-1}, \sigma^r(\delta_I \times C)_{C}^{-1}\). Furthermore, due to the commutativity of (12), we have a 2-cell of monads \(l : (\otimes \cdot (\delta_I \times C))_{C}^{-1}, \sigma^r(\delta_I \times C)_{C}^{-1}) \rightarrow (1_C, 1_F) : (C, F) \rightarrow (C, F)\) as required.

The return of the implication is immediate. For example, a monad transformation \(a\) as indicated implies that a diagram like (22a) commutes.

\[2 \Rightarrow 3)\]

In the isomorphism (14), make \((X, H) = (C \times C, C \times F)\) and \(L \downarrow R = G_F \downarrow V_F\). Therefore, exists a bijection between morphisms of monads of the form \((\otimes, \varphi)\) and morphisms of adjunctions \((\otimes, \varphi)\), where the corresponding induced functor is denoted as \(\otimes = \otimes_{c} \varphi\). This pair of functors make a diagram like (23) commute. In particular, the action of the second functor, on morphisms, is \(\mathbb{K}(f, x^2) = (\varphi_{A', Y} \cdot (f \otimes x))^2\), where \(x^2 : X \rightarrow Y\) and \(f : A \rightarrow A'\).

Yet again, use the isomorphism (14) with \((X, H) = ((C \times C) \times C, (C \times C) \times F)\) and \(L \downarrow R = G_F \downarrow V_F\). Therefore, there exists a bijection between the transformations of monads of the form \(a\) and transformations of adjunctions \((a, \tilde{a})\). Where the second natural transformation has the form \(\tilde{a} : [\otimes \cdot (\otimes \times C)]_{\varphi(\otimes \times C)} \rightarrow [\otimes \cdot (\otimes \times C) \cdot a_{C}])_{\varphi^2} : (C \times C) \times C_F \rightarrow C_F\). Note that we use the following short notation \(\varphi^2 = \varphi(C \times \otimes) a_{C} \circ (\otimes \circ (C \times \varphi) a_{C}\).

We prove only that \([\otimes \cdot (\otimes \times C)]_{\varphi(\otimes \times C)} = \mathbb{K}(\otimes \times C)\). Since for objects there is nothing to prove, let \(((f, g), x^2) : ((A, B), X) \rightarrow ((A', B'), Y)\) be a morphin in \((C \times C) \times C_F\), therefore

\[
[\otimes \cdot (\otimes \times C)]_{\varphi(\otimes \times C)}((f, g), x^2) = (\varphi(\otimes \times C)((A', B'), Y) \cdot (\otimes \times C)((f, g), x))^2
= (\varphi(\otimes \times C)((A', B'), Y) \cdot (f \otimes g) \otimes X)^2 = \mathbb{K}(f \otimes g, x^2)
= \mathbb{K}(\otimes \times C)((f, g), x^2)
\]

At this moment, we change the notation to \(\tilde{a} = \nu\). Therefore, the referred natural transformation can be written as \(\nu : \mathbb{K}(\otimes \times C_F) \rightarrow \mathbb{K}(\otimes \times C) a_{C}, : (C \times C) \times C_F \rightarrow C_F\), according to the requirement. Note that the notation \(a_{C}\) stands for the object \(a_{C, C, C_F}\). The component of the natural transformation \(\nu\), on the object \(((A, A'), X)\), is

\[\nu_{A, A', X} = (\eta^r(A \otimes (A' \otimes X)) \cdot a_{A, A', X})^2\]

according to the equation (13).

The same procedure can be applied to the natural transformation \(l\), in order to get a natural transformation \(j : \mathbb{K}(\delta_I \times C_F)_{C_F}^{-1} \rightarrow 1_{C_F} : C_F \rightarrow C_F\), whose component, on the object \(X\), is \(j_X = (\eta^r X \cdot l_X)^2\).
The reader is invited to realize the remain calculations in order to prove that the structure \((C_F, \boxtimes, \nu, j)\) is that of an left \(C\)-action on \(C_F\).

3 \Rightarrow 2)

Since we have the isomorphism, the return is already given, nonetheless we comment some part of the procedure.

Under the isomorphism, a commutative diagram like (23) give rise to a morphism of monads of the form \((\otimes, \varphi^{\boxtimes}) : (C \times C, C \times F) \rightarrow (C, F)\) where the commutative diagrams that fulfill this morphism are (21) and (22).

If we have a commutative surface like (24a), we obtain, through the isomorphism, a transformation of monads of the form \(a : (\otimes \cdot (\otimes \times C), \varphi^{\boxtimes}(\otimes \times C)) \rightarrow (\otimes \cdot (C \times \otimes) \cdot a_C, \varphi^{\boxtimes}(C \times \otimes) a_C \circ \otimes (C \times \varphi^{\boxtimes}) a_C)\).

\[\square\]

We state the dual theorem

**Theorem 9.4** There exists a bijection between the following structures

1.- Left strong monads \(((F, \sigma^t), \mu^F, \eta^F)\) on the monoidal category \((C, \otimes, I, a, r, l)\).

2.- Morphisms and transformations of monads of the form

\[
\begin{align*}
(\otimes, \varphi) : (C \times C, F \times C) & \rightarrow (C, F) \\
a : (\otimes \cdot (\otimes \times C), \varphi(\otimes \times C) \circ \otimes (\varphi \times C)) & \rightarrow (\otimes \cdot (C \times \otimes) \cdot a_C, \varphi(C \times \otimes) a_C) \\
r : ((C \times C) \times C, (F \times C) \times C) & \rightarrow (C, F) \\
r : (\otimes \cdot (C \times \delta I) \cdot r^{-1}_{C}, \varphi(C \times \delta I) r^{-1}_{C}) & \rightarrow (1_C, 1_F) : (C, F) \rightarrow (C, F)
\end{align*}
\]

3.- Right actions on the Kleisli category, \(C_F, \boxtimes : C_F \times C \rightarrow C_F\) such that the following diagrams of morphisms and surfaces commute

\[\begin{array}{ccc}
C \times C & \overset{\otimes}{\longrightarrow} & C \\
\downarrow{G_F \times C} & & \downarrow{G_F} \\
C_F \times C & \overset{\otimes}{\longrightarrow} & C_F
\end{array}\]
We left to the reader the writing of dual statements, i.e. the ones that corresponds to the Eilenberg-Moore category, where the direction of the natural transformations are inverted, for example \( \hat{\sigma}_{A,B} : F(A \otimes B) \to A \otimes FB \).

### 10  Functor Algebras

Check Proposition II.1.1 in [4] and [7] for this section. we define the category of H-left functor algebras for a given monad \((D, H, \mu^H, \eta^H)\).

**Definition 10.1** The category of left H-functor algebras, for the pair \((C, D)\), denoted as \(H\, \mathcal{F}\) or \(H\, \mathcal{M}\) is defined as follows. The objects are given by \((J, \lambda_J)\), where \(J : C \to D\) is a functor and \(\lambda_J : HJ \to J\) is a natural transformation such that the following diagrams commute

\[
\begin{align*}
\begin{array}{c}
HJ & \xrightarrow{\mu^H J} & HJ \\
\lambda_J & \downarrow & \downarrow \lambda_J \\
HJ & \xrightarrow{H\lambda_J} & J
\end{array}
\end{align*}
\]

\[\tag{25}\]

A morphism of functor algebras, is a natural transformation, \(\theta : (J, \lambda_J) \to (K, \lambda_K)\), \(\theta : J \to K\) such that the following diagram commute

\[
\begin{align*}
\begin{array}{c}
HJ & \xrightarrow{H\theta} & HK \\
\lambda_J & \downarrow & \downarrow \lambda_K \\
J & \xrightarrow{\theta} & K
\end{array}
\end{align*}
\]

\[\tag{26}\]

We realize that the diagrams given by \((25)\), for a left H-functor algebra, account for a monad morphism of the form \((J, \lambda_J) : (C, 1_C) \to (D, H)\). In the same way, the commutative diagram for
morphism of left $H$-functor algebras, as in (26), account for a monad transformation $\theta : (J, \lambda_j) \to (K, \lambda_K) : (C, 1_C) \to (D, H)$.

Using the isomorphism for the Eilenberg-Moore 2-adjunction, given by (6), the category $\mathcal{F}_H$ is isomorphic to the following category, named possibly as category of liftings to $D^H$, for the pair $(C, D)$. The objects of such category are functor pairs $(J, \hat{J})$ such that they complete to an adjunction morphism, in $\text{Adj}_{(2\text{Cat})}$, of the form $(J, \hat{J}) : 1_C \dashv 1_C \to D^H \dashv U^H$. That is to say, the following diagram commutes

\[
\begin{array}{ccc}
C & \xrightarrow{J} & D \\
\downarrow{1_C} & & \downarrow{U^H} \\
C & \xrightarrow{j} & D^H
\end{array}
\]

i.e.

\[
\begin{array}{ccc}
C & \xrightarrow{J} & D \\
\downarrow{j} & & \downarrow{U^H} \\
D^H & \xrightarrow{\hat{J}} & C
\end{array}
\]

The morphisms of such category are the usual morphisms of adjunctions $(\alpha, \beta) : (J, \hat{J}) \to (K, \hat{K}) : 1_C \dashv 1_C \to D^H \dashv U^H$. We proved then the following theorem.

**Theorem 10.1** There exists an isomorphism, natural on $C$ and $(D, H)$, between the following categories

1.- The category of left $H$-functor algebras $\mathcal{F}_H$.

2.- The category of liftings to $D^H$, for the pair $(C, D)$.

\[\square\]

Dually, we have the category of right $H$-functor algebras, for the monad $(D, H, \mu^H, \eta^H)$, denoted as $\mathcal{F}_H$ or $\mathcal{M}_H$. The objects are pairs $(J, \rho_j)$, where the natural transformation $\rho_j : JH \to J$ is such that it fulfills diagrams dual to those in (25). In the same (dual) way as before, this category is the same as the category $\text{Hom}_{\text{Mnd}^*(2\text{Cat})}(\mathcal{D}, (C, 1_C))$. Therefore using the isomorphism (14), the previous category is isomorphic to the category named as extensions from $D^H$, for the pair $(D, C)$. The objects of this category are pairs of functors $(J, \hat{J})$ such that they complete to an adjunction morphism $(J, \hat{J}) : G_H \dashv V_H \to 1_C \dashv 1_C$ in $\text{Adj}_{(2\text{Cat})}$. In particular, the following diagram commutes

\[
\begin{array}{ccc}
D & \xrightarrow{J} & C \\
\downarrow{G_H} & & \downarrow{j} \\
D^H & \xrightarrow{\hat{J}} & C
\end{array}
\]

We also proved the following theorem

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Theorem 10.2 There exists an isomorphism, natural on \((\mathcal{D}, H)\) and \(\mathcal{C}\), between the following categories

1. The category of right \(H\)-functor algebras \(\mathcal{F}_H\).
2. The category of extensions from \(\mathcal{D}_H\), for the pair \((\mathcal{D}, \mathcal{C})\).

\(\Box\)

11 Conclusions and Future Work

This review has the objective to show how several situations for the theory of monads are connected in a very simple way, through a 2-adjunction. Any person who has taught a course on monads would agree that this structure, of a 2-adjunction, can be used as an educational purpose in the sense that a simple structure can account for several situations and which can spare the, otherwise cumbersome, details of the proofs.

For future work we have a few recommendations. The reader may find interesting to extent the part of strong monads and actions over the Kleisli categories to strong symmetrical monads and use the actions for the Eilenberg-Moore case. It would be interesting also to contextualize the case of the monoidal liftings and monoidal extensions according to the formal theory of monoidal monads, and the *standard* objects, given in [11].

The reader may want to find more situations in the monad theory that can use the isomorphism provided by this pair of 2-adjunctions, the authors will certainly pursue this issue.
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