Fault estimation for complex networks with model uncertainty and stochastic communication protocol

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**ABSTRACT**

This paper aims to investigate the fault estimation problem for a class of complex networks with model uncertainty and stochastic communication protocol. The model uncertainty existing in the system is norm-bounded. Stochastic communication protocol is employed to cope with possible data collisions in the multiple signal transmissions. An augmented system is constructed by forming an augmented state vector consisting of system states and related faults. By designing the state estimator, the state estimation problem, in the presence of model uncertainty and random disturbance is solved. The parameters of the estimator are obtained by solving several recursive matrix equations such that an upper bound of the estimation error covariance is established and it is minimized. Finally, we give a simulation example to verify the feasibility of the proposed state estimation scheme.

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1. Introduction

Modern engineering and technology systems are moving in a direction of large scale and complexity. Once such systems fail, huge losses of people and property may be caused. Therefore, it is great significance to effectively guarantee the reliability and safety of modern complex systems. Fortunately, the emergence of fault diagnosis technology has opened up a new way to improve the reliability of complex systems (Cheng, Yang, & Jiang, 2016; Fan & Wang, 2016), and the past several decades have witnessed a rapid development of fault diagnosis technology and much literature has been published (Chiang, Russell, & Braatz, 2001; Shi, He, Wang, & Zhou, 2014; To, Paul, & Liu, 2018; Vachtsevanos, Georgoulas, & Nikolakopoulos, 2016; Yin, Zhu, & Kaynak, 2015; Zhou, Qin, He, Yan, & Deng, 2017). Fault diagnosis technology has been used in aviation (Lu, Wang, & Wang, 2017), aerospace (Wang, Liu, Qing, Liu, & He, 2017), robots (Zhou, Qian, Ma, & Dai, 2017), multi-vehicle (Shi, Zhou, Yang, & Sun, 2018) and numerous industrial processes and tremendous economic benefits have been achieved.

Complex networks consist of nodes, edges, and topological matrices. With the establishment of the small-world (Strogatz & Watts, 1998), scale-free model (Barabasi & Albert, 1999) for complex networks, complex networks model has been applied in different disciplines such as engineering disciplines (such as electric power network and transportation network), life sciences (such as neural networks and metabolic networks) and social sciences (such as social networks). Thus, the research on complex networks is great significance. It is worth noticing that, with obtaining network state information being a precondition for network dynamics, synchronization control, topology identification, containment control, and fault diagnosis, the state of complex networks plays a key role on the engineering practice. However, in actual networks, due to lots of reasons such as intricate topological relationships among network nodes, noise, communication delays and the difficulty of measurement, the information of state can not be gained. Thus, investigating the state estimation problem for complex networks is crucial. Until now, complex networks studies have mainly focussed on synchronization phenomenon and state estimation (Sasirekha & Rakkiyappan, 2017). For instance, Lv, Liang, and Cao (2011) has studied the neural network state estimation under noise disturbance. State observer gain matrices have been given in terms of the solutions of linear matrix inequalities and the prescribed $\mathcal{H}_\infty$ performance index has been guaranteed simultaneously. Liu, Lu, Lü, and Hill (2009) has designed an adaptive state observer to estimate the state of a complex dynamic network with transmission delay, and has applied to the estimation to the topology identification of the network. Till now, there are mainly three
methods to deal with the complex network filtering problem subject to external noise disturbances: final bounded state estimation or set-membership state estimation for bounded noise (Schweppe, 1968), minimum variance state estimation for random noise known probability distribution (Li, Shen, Wang, & Alsaadi, 2018; Liu, Wang, & Zhou, 2018; Sheng, Wang, Zou, & Alsaadi, 2017), and $\mathcal{H}_\infty$ state estimation for energy bounded noise (Ding, Wang, Shen, & Dong, 2015; Han, Wei, Ding, & Song, 2017; Wang, Xia, Zhou, & Duan, 2017; Yan, Zhang, Ding, Liu, & Alsaadi, 2017).

The reliability of complex network systems is very important. Besides, if a fault occurs, it may lead to a bad reaction in complex network systems and even instability. Thus, fault diagnose technology is crucial to accomplish the operation of systems (Liu, Wang, He, & Zhou, 2017). As a necessary part of the fault diagnose theory, fault estimation plays a key role in obtaining fault information, and much literature is available. Dong, Wang, Ding, and Gao (2014) has proposed the concept of stochastic faults and has developed a novel Riccati difference equation method to solve the finite-horizon $\mathcal{H}_\infty$ fault estimation problem.

The nodes of a complex network are distributed at different locations in space. Therefore, distributed information transmission is very important in complex networks. In most available literature, all sensors are assumed to be simultaneously accessed the communications networks to send or receive signals. However, this assumption is usually not very practical in engineering because the network of the real system is inevitably affected by the limited bandwidth, and multiple transmissions may cause data collisions in the case of multiple visits at the same time. One of the effective ways to prevent data collisions is to arrange signal transmission according to certain protocols. So far, communication protocols have been introduced and used to determine which sensors gain to access to the communication network. The commonly used communication protocols in the industry include the Round-Robin protocol (Ugrinovskii & Fridman, 2014), the Try-Once-discard protocol (Walsh, Ye, & Bushnell, 2006), and the stochastic communication protocol (Zhang, Yu, & Feng, 2011). Therefore, it is very important to consider the impact of stochastic communication protocols on complex networks. Nevertheless, the problem of fault estimation for complex networks with model uncertainties and stochastic protocols has not yet been solved. Therefore, the joint estimation of states and faults uses robust Kalman filtering (Hu, Wang, Gao, & Stergioulas, 2012) to recursion the state estimation.

Based on the above discussion, the key issues we need to study are as follows: (1) how to establish performance indicators to evaluate the estimation effect and establish a sufficient (minute) condition so that the estimation error meets a given performance index; (2) how to establish a recursive algorithm to solve the problem of state estimation for time-varying complex networks; (3) how to handle the influence of model uncertainty and stochastic protocols on the estimation problem. In this paper, what we mainly need to do is to solve the problem of fault estimation for a complex network with model uncertainty.

Based on the above summary, we aim to solve the problem of recursive fault estimation for discrete time-varying coupled complex network arrays with model uncertainties and stochastic communication protocol. Summarize this article as follows: (1) the fault estimation problem is, first, investigated for complex networks with model uncertainty and stochastic protocol; (2) based on the solution of the Riccati-like difference equation, an explicit form of the fault estimation parameter is given.

The other parts are arranged as follows. In the second section, a linear complex network system with model uncertainties and stochastic communication protocols is introduced, and an estimator is formulated based on the system equations. In the third section, the estimation error covariance is gained based on the designed estimator, and the Riccati difference equation is used to obtain the upper bound of the estimated error covariance. In addition, the required observer gain matrix is obtained by solving the minimization problem of a class of Riccati difference equation constraints. In the fourth quarter, some simulations are carried out to verify the validity of the theory. Finally, the conclusions are drawn in section 5.

**Notations:** In this article, the notations used are all standard. $\mathbb{R}^n$ and $\mathbb{R}^{n \times m}$ represent a set of $n$-dimensional Euclidean space and all $n \times m$ real matrices. $\mathbb{N}$ represents a set of integers. If $S \geq T$ ($S > T$), then $S$ and $T$ are real symmetric matrices. And if $B$ is a matrix, $B^T$ indicates the transposition of $B$. $\text{Tr}(B)$ represents the trajectory of $B$. $0$ is the zero matrix. In order not to cause confusion, the $n$-dimensional unit matrix is denoted by $\text{in}_{l_n}$ or denoted by a simple $I$. $\text{diag}(\cdots)$ is the diagonal matrix of the piece. $\mathbb{E}(x)$ and $\mathbb{E}(x \mid y)$ represent the expected value of the random variable $x$ and the expected value of $y$. For a given vector $x$, $\|x\|$ denotes the norm of $x$ in Euclid. $\otimes$ is the Kronecker product defined as

$$
A \otimes B = \begin{bmatrix}
    a_{1,1}B & \cdots & a_{1,n}B \\
    \vdots & \ddots & \vdots \\
    a_{m,1}B & \cdots & a_{m,n}B
\end{bmatrix}
$$

$\otimes$ is the Hadamard product of matrices.
2. Problem statement and preliminaries

Consider the following complex network consisting of $N$ coupled nodes:

$$
x_{i,k+1} = A_{i,k}x_{i,k} + (L_{i,k} + \Delta L_{i,k})u_{i,k} + \sum_{j=1}^{N} w_{ij} \Gamma x_{j,k} + B_{i,k}w_{i,k} + F_{i,k}f_{i,k},
$$

where $x_{i,k} \in \mathbb{R}^n$ is the state vector of the $i$th node; $y_{i,k} \in \mathbb{R}^m$ is the measurement output vector of the $i$th node; $u_{i,k} \in \mathbb{R}^p$ is the control input vector of the $i$th node; $v_{i,k} \in \mathbb{R}^l$ denotes the measurement noise with zero-mean and covariance $S_{i,k} > 0$; $w_{i,k} \in \mathbb{R}^{l'w}$ denotes the process noise with zero-mean and covariance $R_{i,k} > 0$. $f_{i,k}$ denotes the fault vector, which includes actuator fault and plant fault; the initial state of the system $x_{i,0}$ has the mean $\bar{x}_{i,0}$ and $\mathbb{E}[x_{i,0}x_{i,0}^T] = \Omega_{i,0}$, $A_{i,k}$, $B_{i,k}$, $G_{i,k}$, $L_{i,k}$, $F_{i,k}$ and $Q_{i,k}$ denote known time-varying matrices with appropriate dimensions; $\Delta L_{i,k}$ is model uncertainty satisfying $\|\Delta L_{i,k}\| \leq \delta_{i,k}$. The $i$th node, $\phi_k = i$, is allowed to get access to central node, whose probability distribution is $P(\phi_k = i) = \lambda_{i,k}, i \in \{1, 2, \ldots, N\}$, $\sum_i^{N} \lambda_{i,k} = 1$. $\xi_{i,k}$ represents a stochastic variable whose probability distribution is

$$
P(\xi_{i,k} = 1) = P(\phi_k = i) = \lambda_{i,k},
$$

$$
P(\xi_{i,k} = 0) = 1 - \lambda_{i,k},
$$

where $\xi_{i,k} = 1$ stands for the $i$th node being transmitted to the central node; $\Gamma = \text{diag}((\gamma_1, \gamma_2, \ldots, \gamma_N)) \succeq 0$ denotes the inner coupling matrix which links the $j$th node if $\gamma_j \neq 0$; $W = (w_{ij}) \in \mathbb{R}^{N \times N}$ denotes the coupled configuration matrix of the complex network (1), where $w_{ij}$ satisfies $w_{ij} \geq 0 (i \neq j)$.

In the industrial process, step faults and ramp faults are two kinds of common faults. For step faults and ramp faults, it is obvious that $\Delta^2 f_{i,k} = 0$ holds.

Define the following augmented state vector:

$$
\check{x}_{i,k} = \begin{bmatrix} x_{i,k}^T & f_{i,k}^T & \Delta^T f_{i,k} \end{bmatrix}^T \in \mathbb{R}^{\bar{n}}
$$

where $\bar{n} = n + 2l'$.

Then, we can obtain the following augmented system:

$$
\check{x}_{i,k+1} = \begin{bmatrix} A_{i,k} & 0 & 0 \\ L_{i,k} & 0 & 0 \\ 0 & 0 & I_{l'} \end{bmatrix} \check{x}_{i,k} + \sum_{j=1}^{N} w_{ij} \Gamma \check{x}_{j,k} + B_{i,k}w_{i,k} + F_{i,k}f_{i,k},
$$

$$
\check{y}_{i,k} = \check{G}_{i,k} \check{x}_{i,k} + v_{i,k},
$$

$$
\check{y}_{i,k} = \xi_{i,k} \check{G}_{i,k} \check{x}_{i,k} + \xi_{i,k} v_{i,k}.
$$

Next, the augmented system (4) is shown as follows:

where

$$
\check{A}_{i,k} = \begin{bmatrix} A_{i,k} & F_{i,k} & 0 \\ L_{i,k} & 0 & 0 \\ 0 & 0 & I_{l'} \end{bmatrix} \in \mathbb{R}^{\bar{n} \times \bar{n}},
$$

$$
\check{L}_{i,k} = \begin{bmatrix} L_{i,k} \\ 0 \\ 0 \end{bmatrix} \in \mathbb{R}^{n \times p},
$$

$$
\check{B}_{i,k} = \begin{bmatrix} B_{i,k} \\ 0 \\ 0 \end{bmatrix} \in \mathbb{R}^{n \times l'},
$$

Denoting

$$
\check{A}_k = \text{diag}(\check{A}_{1,k}, \check{A}_{2,k}, \ldots, \check{A}_{N,k}),
$$

$$
\check{B}_k = \text{diag}(\check{B}_{1,k}, \check{B}_{2,k}, \ldots, \check{B}_{N,k}),
$$

$$
\check{G}_k = \text{diag}(\check{G}_{1,k}, \check{G}_{2,k}, \ldots, \check{G}_{N,k}),
$$

$$
\check{L}_k = \text{diag}(\check{L}_{1,k}, \check{L}_{2,k}, \ldots, \check{L}_{N,k}),
$$

$$
\Delta \check{L}_k = \text{diag}(\Delta \check{L}_{1,k}, \Delta \check{L}_{2,k}, \ldots, \Delta \check{L}_{N,k}),
$$

$$
Q_k = \text{diag}(Q_{1,k}, Q_{2,k}, \ldots, Q_{N,k}),
$$

$$
\Phi_k = \text{diag}(\Phi_{1,k}, \Phi_{2,k}, \ldots, \Phi_{N,k}).
$$

Then, construct the following Kalman-type state estimator for (5):

$$
\hat{\check{x}}_{k+1 | k} = [\check{A}_k + (\check{L}_k + \Delta \check{L}_k)Q_k \check{G}_k + W \Omega \check{\Gamma}] \check{x}_k + \check{B}_k \omega_k + (\check{L}_k + \Delta \check{L}_k)Q_k v_k,
$$

$$
\hat{\check{y}}_k = \Phi_k \check{G}_k \check{x}_k + \Phi_k v_k.
$$

where '⊙' represents the Kronecker product.

Then, the one-step prediction of $\hat{\check{x}}_k$ with initial condition $\hat{\check{x}}_0 | 0 = \check{x}_0 = [\check{x}_{i,0}^T \check{x}_{i,0}^T \ldots \check{x}_{N,0}^T]^T$; $\check{x}_{i+1 | k}$ represents the one-step prediction of $\check{x}_k$; $\check{y}_{i+1 | k}$ is the actual measurement output at time step $k+1$; $\Phi_k = \text{diag}(\mu_{1,k}, \mu_{2,k}, \ldots, \mu_{N,k})$ denotes the mathematical expectation of random matrix $\Lambda_k$ and $H_{k+1}$ is the estimator parameter to be designed.
Set the one-step prediction error as $e_{k+1|k} = \hat{x}_{k+1} - \hat{x}_{k+1|k}$ and the estimation error as $e_{k-1|k+1} = \hat{x}_{k+1} - \hat{x}_{k+1|k+1}$. So

$$e_{k+1|k} = [(\hat{A}_k + \hat{L}_k Q_k \hat{G}_k + W \otimes \hat{\Gamma}) e_{k|k} + (\hat{L}_k + \Delta \hat{L}_k)] \times Q_k v_k + \Delta L_k Q_k \hat{G}_k \hat{x}_k + \hat{B}_k w_k,$$  

(8)

Denoting $\hat{I} = \text{diag}(I, I, \ldots, I)$, $H_{k-1} = \text{diag}(H_{k-1,1}, H_{k-1,2}, \ldots, H_{k-1,k-1})$ and according to (5) and (7), we have

$$e_{k+1|k} = e_{k+1|k} - H_{k-1} [\hat{G}_{k-1} \hat{G}_{k-1}] e_{k+1|k} - H_{k-1} [\Phi_{k-1} + 1] e_{k+1|k} \hat{x}_{k+1}.$$

(9)

The aim of this paper can be boiled down to two aspects. One is to design an estimator in the form of (6)–(7) to ensure that an upper bound of the estimation error covariance matrix exists. Therefore, we need to find a positive definite matrix $\Pi_{k+1|k+1}$ satisfying

$$\mathbb{E}[e_{k+1|k}^T e_{k+1|k+1}] \leq \Pi_{k+1|k+1}$$

(10)

The other is to minimize $\Pi_{k+1|k+1}$ by designing appropriate estimator parameters at each time.

Next, the following lemmas are given and applied to the proof of the main results.

**Lemma 2.1 (Horn & Johnson, 1991):** Let $P = [p_{ij}]_{n \times n}$ be a real-valued matrix and $Q = \text{diag}(q_1, q_2, \ldots, q_n)$ be a diagonal stochastic matrix. Then

$$\mathbb{E}[QPO^T] = \begin{bmatrix} \mathbb{E}[q_1^2] & \mathbb{E}[q_1 q_2] & \cdots & \mathbb{E}[q_1 q_n] \\ \mathbb{E}[q_2 q_1] & \mathbb{E}[q_2^2] & \cdots & \mathbb{E}[q_2 q_n] \\ \vdots & \vdots & \ddots & \vdots \\ \mathbb{E}[q_n q_1] & \mathbb{E}[q_n q_2] & \cdots & \mathbb{E}[q_n^2] \end{bmatrix} \circ P,$$

where “$\circ$” denotes the Hadamard product.

**Lemma 2.2 (Boyd, Ghaoul, Feron, & Balakrishnan, 1994):** For given any two vectors $x, y \in \mathbb{R}^n$ with the same dimension, the following inequality holds:

$$xy^T + yx^T \leq \varepsilon xx^T + \varepsilon^{-1} yy^T.$$

(11)

where $\varepsilon > 0$ is an any scalar.

**Lemma 2.3:** For given matrices $A, B, Y$ and $P$ with suitable dimensions, then the following relationships hold:

$$\frac{\partial \text{tr}(AYB)}{\partial Y} = A^T B^T,$$

$$\frac{\partial \text{tr}(AY^T B)}{\partial Y} = BA,$$

$$\frac{\partial \text{tr}(AYBP(AYB)^T)}{\partial Y} = 2A^T AYBPB^T.$$

### 3. Main results

In this chapter, we shall first calculate the one-step prediction error covariance matrix and the estimation error covariance matrix based on (8)–(9). Then, their upper bound will be determined and minimized by selecting appropriate estimator parameters.

**Lemma 3.1:** One-step prediction error covariance $Z_{k+1|k}$ and the estimation error covariance $Z_{k+1|k+1} = \mathbb{E}[e_{k+1|k}^T e_{k+1|k+1}]$ are determined by

$$Z_{k+1|k} = [\hat{A}_k + \hat{L}_k Q_k \hat{G}_k + W \otimes \hat{\Gamma}] Z_{k+1|k} [\hat{A}_k + \hat{L}_k Q_k \hat{G}_k + W \otimes \hat{\Gamma}]+ \Delta \hat{L}_k \hat{G}_k \hat{G}_k^T \hat{L}_k + \hat{B}_k \hat{S}_k \hat{B}_k^T + \Delta \hat{L}_k Q_k \hat{G}_k \hat{G}_k^T \hat{L}_k \otimes \hat{\Gamma} E[e_{k+1|k}] \hat{x}_{k+1} \hat{x}_{k+1}^T + [\hat{A}_k + \hat{L}_k Q_k \hat{G}_k + W \otimes \hat{\Gamma}] E[e_{k+1|k}] \hat{x}_{k+1} \hat{x}_{k+1}^T$$

(12)

and

$$Z_{k+1|k+1} = [\hat{I} - H_{k-1} \hat{F}_{k-1} \hat{G}_{k-1}] Z_{k+1|k} [\hat{I} - H_{k-1} \hat{F}_{k-1}] + \Delta \hat{L}_k \hat{G}_k \hat{G}_k^T \hat{L}_k + \hat{B}_k \hat{S}_k \hat{B}_k^T + \Delta \hat{L}_k Q_k \hat{G}_k \hat{G}_k^T \hat{L}_k \otimes \hat{\Gamma} E[e_{k+1|k}] \hat{x}_{k+1} \hat{x}_{k+1}^T$$

(13)

where

$$\Omega_k = \mathbb{E}[\hat{x}_k \hat{x}_k^T],$$

$$R_k = \text{diag}(R_{1,k}, R_{2,k}, \ldots, R_{N,k}),$$

$$S_k = \text{diag}(S_{1,k}, S_{2,k}, \ldots, S_{N,k}),$$

$$\Phi_{k+1} = \text{diag}(\delta_{1,k}^2, \delta_{2,k}^2, \ldots, \delta_{N,k}^2).$$

**Proof:** According to (8), it is obvious that the formula (12) holds.

From (9), we have

$$Z_{k+1|k+1} = [\hat{I} - H_{k-1} \hat{F}_{k-1} \hat{G}_{k-1}] Z_{k+1|k} [\hat{I} - H_{k-1} \hat{F}_{k-1}] + \Delta \hat{L}_k \hat{G}_k \hat{G}_k^T \hat{L}_k + \hat{B}_k \hat{S}_k \hat{B}_k^T + \Delta \hat{L}_k Q_k \hat{G}_k \hat{G}_k^T \hat{L}_k \otimes \hat{\Gamma} E[e_{k+1|k}] \hat{x}_{k+1} \hat{x}_{k+1}^T$$

(14)
Then, denote $\tilde{\Phi}_{k+1} = \Phi_{k+1} - \tilde{\Phi}_{k+1}$. Following Lemma 2.1 and noticing the statistical properties of stochastic communication protocol $\xi_{i,k}$ one can that

$$
\mathbb{E}\{\tilde{\Phi}_{k+1}\tilde{G}_{k+1}{\tilde{\Phi}_{k+1}}^T\tilde{G}_{k+1}^T\tilde{\Phi}_{k+1}\} = \\
\mathbb{E}\{\tilde{\Phi}_{k+1}\tilde{G}_{k+1}{\tilde{\Phi}_{k+1}}^T\tilde{G}_{k+1}^T\tilde{\Phi}_{k+1}\} = \\
\mathbb{E}\{\tilde{\Phi}_{k+1}\tilde{G}_{k+1}\} = \tilde{\Phi}_{k+1} \circ (\tilde{G}_{k+1}\mathbb{E}[\tilde{G}_{k+1}\tilde{G}_{k+1}^T])
$$

(15)

holds. Substituting (15) into (14), one can easily see (13) is ensured, which completes the proof of this lemma.

**Theorem 3.1:** Let $\varepsilon_1$, $\varepsilon_2$ and $\varepsilon_3$ be a positive scalar. For joint estimation system (5), the estimation error is bounded if the solution of the following two Riccati-like difference equations exist under the initial condition $Z_{0|0} = \Pi_{0|0}$:

$$
\Pi_{k+1|k} = (1 + \varepsilon_1)(\tilde{A}_k + \tilde{L}_kQ_k\tilde{G}_k + W \otimes \tilde{\Gamma})\Pi_{k|k}\tilde{A}_k + \tilde{L}_k \times \tilde{G}_k + \tilde{Q}_k\tilde{G}_k + W \otimes \tilde{\Gamma})^T + (1 + \varepsilon_3)\tilde{L}_k\tilde{Q}_k\tilde{R}_k\tilde{G}_k^T\tilde{Q}_k + \\
\times \delta_{k,2}^1W_k\epsilon_{j,2}^1 + \tilde{B}_k\tilde{S}_k\tilde{B}_k^T + (1 + \varepsilon_1)^{-1} \times \delta_{k,2}^1W_k\epsilon_{j,2}^1
$$

(16)

and

$$
\Pi_{k+1|k+1} = (1 - \varepsilon_2)(\tilde{I} - H_{k+1}\tilde{\Phi}_{k+1}\tilde{G}_{k+1}) \times \Pi_{k+1|k+1} \times \Pi_{k+1|k} \times \Pi_{k+1|k} \times \Pi_{k+1|k} \times \Pi_{k+1|k} \times \Pi_{k+1|k}
$$

(17)

where

$$
H_{k+1} = \text{diag}(H_{1,k+1}, H_{2,k+1}, \ldots, H_{n,k+1}),
$$

$$
r_{1,k} = \lambda_{\text{max}}(Z_{k|k}),
$$

$$
r_{2,k} = \lambda_{\text{max}}(Q_k\tilde{G}_k\tilde{G}_k^TQ_k^T),
$$

$$
r_{3,k} = \lambda_{\text{max}}(Q_k\tilde{G}_k\tilde{G}_k^TQ_k^T)
$$

$$
\Theta_{1,j} = \begin{bmatrix} 0, 0, \ldots, 0, n_{k} \\ 0, 0, \ldots, 0, n_{k} \\ \vdots \\ 0, 0, \ldots, 0, n_{k} \end{bmatrix},
$$

$$
\Theta_{2,j} = \begin{bmatrix} 0, 0, \ldots, 0, n_{k} \\ 0, 0, \ldots, 0, n_{k} \\ \vdots \\ 0, 0, \ldots, 0, n_{k} \end{bmatrix},
$$

$$
M_{k+1} = (1 - \varepsilon_2)^{-1}\tilde{\Phi}_{k+1} \circ (\tilde{G}_{k+1}\tilde{G}_{k+1}^T) + \Phi_{k+1}\tilde{R}_{k+1}\tilde{G}_{k+1}^T + \tilde{\Phi}_{k+1}\tilde{G}_{k+1}\tilde{G}_{k+1}^T
$$

(18)

**Proof:** This theorem is proved by mathematical induction. According to the initial condition $Z_{0|0} = \Pi_{0|0}$, assuming $Z_{k|k} = \Pi_{k|k}$ it needs to be proven that $Z_{k+1|k+1} \leq \Pi_{k+1|k+1}$ holds.

First, according to Lemma 2.2, we have

$$
Z_{k+1|k} \leq (1 + \varepsilon_1)(\tilde{A}_k + \tilde{L}_kQ_k\tilde{G}_k + W \otimes \tilde{\Gamma})\Pi_{k|k}\tilde{A}_k + \tilde{L}_k \times \tilde{G}_k + \tilde{Q}_k\tilde{G}_k + W \otimes \tilde{\Gamma})^T + (1 + \varepsilon_3)\tilde{L}_k\tilde{Q}_k\tilde{R}_k\tilde{G}_k^T + \\
\times \delta_{k,2}^1W_k\epsilon_{j,2}^1 + \tilde{B}_k\tilde{S}_k\tilde{B}_k^T + (1 + \varepsilon_1)^{-1} \times \delta_{k,2}^1W_k\epsilon_{j,2}^1
$$

(20)

where $\varepsilon_1$, $\varepsilon_2$ and $\varepsilon_3$ are arbitrary scalars. Noticing (13) and Lemma 2.2, we have

$$
\Omega_{k+1} = [\tilde{A}_k + \tilde{L}_k + \tilde{L}_k\tilde{Q}_k\tilde{G}_k + W \otimes \tilde{\Gamma}]\Omega_k[\tilde{A}_k + \tilde{L}_k + \tilde{L}_k\tilde{Q}_k\tilde{G}_k + W \otimes \tilde{\Gamma}]^T + (1 + \varepsilon_4)\tilde{L}_k\tilde{Q}_k\tilde{R}_k\tilde{G}_k^T + \\
\times (\tilde{L}_k + \tilde{L}_k)\tilde{Q}_k\tilde{R}_k\tilde{G}_k^T + \tilde{B}_k\tilde{S}_k\tilde{B}_k^T
$$

(21)

where $\varepsilon_4$ are arbitrary scalars. Then, we have

$$
Z_{k+1|k+1} \leq (1 + \varepsilon_2)(\tilde{I} - H_{k+1}\tilde{\Phi}_{k+1}\tilde{G}_{k+1})Z_{k+1|k}(\tilde{I} - H_{k+1} \times \tilde{\Phi}_{k+1}\tilde{G}_{k+1})^T + (1 - \varepsilon_2)^{-1}H_{k+1}\tilde{\Phi}_{k+1} \circ (\tilde{G}_{k+1} \times \Omega_{k+1}\tilde{G}_{k+1}^T)H_{k+1} + H_{k+1}\tilde{\Phi}_{k+1}\tilde{R}_{k+1}\tilde{G}_{k+1}^T
$$

(22)

Thus, we can get $Z_{k+1|k+1} \leq \Pi_{k+1|k+1}$. Next, we are in the position to minimize the upper bound $\Pi_{k+1|k+1}$ by selecting appropriate the estimator parameter $H_{k+1}$ in (19). It is worth nothing that $\Pi_{k+1|k+1}$
in (17) can be written in the following form

\[
\frac{d}{dt} \Pi_{k+1|k+1} = (1 - \varepsilon_2) \Pi_{k+1|k} - H_{k+1} \Phi_{k+1} \tilde{G}_{k+1} \Pi_{k+1|k+1} - H_{k+1} M_{k+1} H_{k+1}^T
\]

By substituting \( H_{k+1} = \sum_{i=1}^{N} (\Theta_{1i}^T H_{i,k+1} \Theta_{2i}) \) into (23), we get the new form of the trace \( \Pi_{k+1|k+1} \) as follows:

\[
\text{tr}(\Pi_{k+1|k+1})
\]

\[
= \text{tr}\left[ (1 - \varepsilon_2) \Pi_{k+1|k} - (1 - \varepsilon_2) \sum_{i=1}^{N} (\Theta_{1i}^T H_{i,k+1} \Theta_{2i}) \right]
\]

\[
\times \Phi_{k+1} \tilde{G}_{k+1} \Pi_{k+1|k} - (1 - \varepsilon_2) \sum_{i=1}^{N} (\Theta_{1i}^T H_{i,k+1} \Theta_{2i}) \Phi_{k+1} \tilde{G}_{k+1} 
\]

\[
\times \sum_{i=1}^{N} (\Theta_{1i}^T H_{i,k+1} \Theta_{2i}) + (1 - \varepsilon_2) \sum_{i=1}^{N} (\Theta_{1i}^T H_{i,k+1} \Theta_{2i}) 
\]

\[
\times M_{k+1} (\Theta_{1i}^T H_{i,k+1} \Theta_{2i})^T, \tag{24}
\]

Through the value of Lemma 2.3, we can obtain

\[
\frac{d}{dt} \text{tr}(\Pi_{k+1|k+1}) = 0 - 2(1 - \varepsilon_2) \Theta_{1i}^T H_{i,k+1} \Theta_{2i} + 2 \Theta_{1i}^T H_{i,k+1} \Theta_{2i} M_{k+1} \Theta_{2i}^T = 0. \tag{25}
\]

Here \( M_{k+1} > 0 \), then \( (\Theta_{2i}^T M_{k+1} \Theta_{2i})^T \) is invertible. Next, through the collation operation, the estimator parameters are given by (25) and \( \Theta_{1i}^T H_{i,k+1} = I_{n \times n} \), \( H_{i,k+1} = (1 - \varepsilon_2) \Theta_{1i}^T H_{i,k+1} \Theta_{2i} M_{k+1} \Theta_{2i}^T \).

The proof of the theorem has been completed. \( \blacksquare \)

4. Numerical example

In this section, we use a numerical simulation to prove the validity of the proposed theorem. Consider the complex network (1) with the following parameters:

\[
W = \begin{bmatrix}
-0.2 & 0.1 & 0.1 \\
0.1 & -0.2 & 0.1 \\
0.1 & 0.1 & -0.2
\end{bmatrix}, \quad \Gamma = \begin{bmatrix}
0.5 & 0 \\
0 & 0.5
\end{bmatrix},
\]

\[
A_1(k) = \begin{bmatrix}
1.1 \sin(k) & 0 \\
0 & 0.98
\end{bmatrix},
\]

\[
A_2(k) = \begin{bmatrix}
0.85 \cos(0.9k) & 0 \\
0 & 0.99
\end{bmatrix},
\]

\[
A_3(k) = \begin{bmatrix}
0.98 & 0 \\
0 & 0.98 \sin(0.2k)
\end{bmatrix}, \quad B_1(k) = \begin{bmatrix}
0.15 \\
0.1
\end{bmatrix}, \quad B_2(k) = \begin{bmatrix}
0.6 \\
0.1
\end{bmatrix}
\]

\[
G_1(k) = \begin{bmatrix}
0.2 \sin(0.3k) & 0.9 \\
0.5 & 0.85
\end{bmatrix}, \quad G_2(k) = \begin{bmatrix}
0.1 & 0 \\
0.15 & 0.1
\end{bmatrix}, \quad L_1(k) = \begin{bmatrix}
0.1 \\
0.1
\end{bmatrix}, \quad L_2(k) = \begin{bmatrix}
0.2 \\
0.15
\end{bmatrix}, \quad Q = \begin{bmatrix}
0.1 & 0 & 0 \\
0.1 & 0.1 & 0 \\
0 & 0 & 0.1
\end{bmatrix}
\]

\[
f_1(k) = \begin{bmatrix}
0 \\
0
\end{bmatrix}, \quad k < 25 \\
\begin{bmatrix}
0.5 \\
0.65
\end{bmatrix}, \quad k \geq 25
\]

\[
f_2(k) = \begin{bmatrix}
0 \\
0
\end{bmatrix}, \quad k < 30 \\
\begin{bmatrix}
0.67 \\
0.68
\end{bmatrix}, \quad k \geq 30
\]

\[
f_3(k) = \begin{bmatrix}
0.7 \\
0.85
\end{bmatrix}, \quad k < 35 \quad \text{and} \quad f_3(k) = l, \quad k \geq 35
\]

\( \omega_{ij} \text{ and } v_{ij} \) are white noises. The norm-bounded uncertainty matrices are assumed to be:

\[
\Delta L_1 = \begin{bmatrix}
0.3 \sin(k) & 0 \\
0 & \cos(0.1k)
\end{bmatrix}, \quad \Delta L_2 = \begin{bmatrix}
0.1 \\
0
\end{bmatrix}, \quad \Delta L_3 = \begin{bmatrix}
0 \\
0
\end{bmatrix}
\]

The corresponding stochastic variables \( \xi_i(k), i = 1, 2, 3 \) are with the following probability distribution:

\[
\Pr(\xi_i(k) = 0) = \frac{1}{3}, \quad \Pr(\xi_i(k) = 1) = \frac{1}{3}, \quad \Pr(\xi_i(k) = 2) = \frac{1}{3}
\]

Figure 1. The state evolution \( x_{1,1}(k) \) and its estimate \( \hat{x}_{1,1}(k) \).
Figure 2. The state evolution $x_{1,2}(k)$ and its estimate $\hat{x}_{1,2}(k)$.

Figure 3. The state evolution $x_{2,1}(k)$ and its estimate $\hat{x}_{2,1}(k)$.

Figure 4. The state evolution $x_{2,2}(k)$ and its estimate $\hat{x}_{2,2}(k)$.

Figure 5. The state evolution $x_{3,1}(k)$ and its estimate $\hat{x}_{3,1}(k)$.

Figure 6. The state evolution $x_{3,2}(k)$ and its estimate $\hat{x}_{3,2}(k)$.

Figure 7. The fault evolution $f_1(k)$ and its estimate $\hat{f}_1(k)$. 
Therefore, the expectation and variance of $\xi_i(k), i = 1, 2, 3$ can be calculated by $\mu_{1,k} = \frac{1}{3}$ and $\sigma_{1,k}^2 = \frac{2}{9}$, $\mu_{2,k} = \frac{1}{3}$ and $\sigma_{2,k}^2 = \frac{2}{9}$, $\mu_{3,k} = \frac{1}{3}$ and $\sigma_{3,k}^2 = \frac{2}{9}$. The other parameters are chosen as $\varepsilon_1 = 0.3, \varepsilon_2 = 0.6, \varepsilon_3 = 0.15, \varepsilon_4 = 0.1$. The state initial condition: $\bar{x}_{1,0} = [0.5 0.15 0 0.1]^T, \bar{x}_{2,0} = [0.9 - 0.75 0.2]^T, \bar{x}_{3,0} = [0.8 0.4 0.3]^T, \hat{x}_{1,0} = [0 0 0 0]^T, \hat{x}_{2,0} = [0 0 0 0]^T, \hat{x}_{3,0} = [0 0 0 0]^T$.

By using the Matlab Toolbox, the fault estimation problem with model uncertainty and stochastic communication protocol can be solved by Theorem 3.1. The simulation results are shown in Figures 1–9. The states of the complex network (1) and their estimations are depicted in Figures 1–6, where $x_{i,k}(i = 1, 2, 3)$ and $\hat{x}_{i,k}(i = 1, 2, 3)$ denote, respectively, the system state of $i$th node and its estimation. It can be seen that, the proposed filter could estimate the states well with model uncertainty and stochastic communication protocol. The evolutions of fault are depicted in Figures 7–9. According to the simulation results, it can be concluded that confirms the practicability and usefulness of the state estimation scheme designed in this article.

5. Conclusion

In this paper, the fault estimation problem for a class of complex networks with model uncertainty and stochastic communication protocol has been investigated. The stochastic communication, which is modelled as a series of Bernoulli random variables, has been employed to schedule the data transmission between concerned complex networks and remote filter. By augmenting the state, a recursive state estimator has been constructed such that it can simultaneously estimate states and faults. Corresponding observer parameters have been obtained in terms of the solutions of Riccati-like equations. Finally, through a simulation example, the effectiveness of the proposed designation method of minimum variance state estimator have been indicated.

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