Cosmological Constant
or
Cosmological Potential*

P. Fiziev

Theoretical Physics Department
Sofia University
fiziev@phys.uni-sofia.bg

March 24, 2022

*talk given at Conference ”PROBLEMS OF VACUUM ENERGY” Copenhagen, 23–26 August, 2000
Introduction:
At present the original general relativity (GR) is:
A SUCCESSFUL THEORY of gravity in description of gravitational phenomena at:
• laboratory and earth surface scales,
• solar system and star systems scales.
QUITE GOOD in description of these phenomena:
• at galaxies scales, and
• at the scales of the whole Universe.
PROBLEMATIC in description of:
• rotation of galaxies and their motion in galactic clasters
• initial singularity problem, early Universe and inflation,
• recently discovered accelerated expansion of the Universe,
• vacuum energy problem.
The most promising modern theories like supergravity and (super)string theories incorporate naturally GR but at present:

- are not developed enough to allow real experimental test,
- introduce large number of new fields without real physical basis.

Therefore it seems meaningful to look for some **minimal extension of GR** which:

- is compatible with known gravitational experiments,
- promises to overcome at least some of the problems,
- may be considered as a part of more general modern theories.

Such **minimal** extension must include one new scalar field degree of freedom. Its contribution in the action of the theory may be described in different (sometimes equivalent) ways, being not fixed a priori.
Here we outline the general properties of such model with one additional scalar field $\Phi$ which differs from known inflationary models, being conformal equivalent in scalar-tensor sector to some quintessence models. We call

**MINIMAL DILATONIC GRAVITY (MDG)**

the scalar-tensor model of gravity with action

$$A_{G,\Lambda} = -\frac{c}{2\kappa} \int d^4x \sqrt{|g|} \Phi \left( R + 2\Lambda \Pi(\Phi) \right), \quad (1)$$

- Branse-Dicke parameter $\omega(\Phi) \equiv 0$ (i.e., **without standard kinetic term for $\Phi$!**),
- **cosmological constant** $\Lambda$ and
- **dimensionless cosmological factor** $\Pi(\Phi)$.

The matter action $A_M$ and matter equations of motion will have usual GR form.
Equations for metric $g_{\alpha\beta}$ and dilaton field $\Phi$:

$$\Phi \left( G_{\alpha\beta} - \Lambda \Pi(\Phi) g_{\alpha\beta} \right) - (\nabla_\alpha \nabla_\beta - g_{\alpha\beta} \Box) \Phi = \frac{\kappa}{c^2} T_{\alpha\beta},$$

$$\Box \Phi + \Lambda \frac{dV}{d\Phi}(\Phi) = \frac{\kappa}{3c^2} T.$$  \hspace{1cm} (2)

yield usual energy-momentum conservation law:

$$\nabla_\alpha T^{\alpha}_\beta = 0.$$  \hspace{1cm}

In addition we have the important relation:

$$R + 2\Lambda \left( \Phi \frac{d\Pi}{d\Phi}(\Phi) + \Pi(\Phi) \right) = 0.$$  \hspace{1cm} (3)

In (2) the quantity:

$$\frac{dV}{d\Phi} = \frac{2}{3} \Phi \left( \Phi \frac{d\Pi}{d\Phi} - \Pi \right)$$

introduces dilatonic potential $V(\Phi)$.

It is convenient to introduce, too, a cosmological potential:

$$U(\Phi) = \Phi \Pi(\Phi).$$
Investigation of MDG was started by O’Hanlon (PRL, 1972) in connection with Fujii’s theory of massive dilaton, but without any relation with cosmological constant problem.

In our special scalar-tensor model of gravity the cosmological factor $\Pi(\Phi)$ (or the cosmological potential $U(\Phi)$) is the only unknown function which has to be chosen:

- to comply with all gravitational experiments and observations
- to solve the following inverse cosmological problem:

Determination of the factor $\Pi(\Phi)$ which yields given time evolution of the scale parameter $A(t)$ in Robertson-Walker (RW) model of Universe.
The action (1) of MDG is a Helmholz action of nonlinear gravity with lagrangian

\[ L_{NLG} \sim \sqrt{|g|} f(R), \]

or a (4d) low energy limit of superstringy action (for metric and dilaton only) in some **new frame**. It follows from stringy lagrangian

\[ L_{String} \sim \sqrt{|g|} e^{-2\phi} \left( R + 4(\partial \phi)^2 + V_{SUSY}(\phi) \right) \]

after conformal transformation (in \( D \)-dimensions):

\[ g_{\mu\nu} \rightarrow e^{\pm 4\phi/\sqrt{(D-2)(D-1)}} g_{\mu\nu}, \]

\[ \phi \rightarrow \Phi = \exp \left( -2\phi \left( 1 \pm \sqrt{D-2}/(D-1) \right) \right). \]

The potential \( V_{SUSY}(\phi) \) may originate from SUSY breaking due to gaugino condensation, or may appear in more complicated way. At present its form is not known exactly.

The action (1) appears, too, in a new model of gravity with torsion and unusual local conformal symmetry after its breaking in metric-dilaton sector only (PF: gr-qc/9809001).
An **essential new element** of our MDG (PF: gr-qc/9911037) is the **nonzero cosmological constant** $\Lambda$.

Nevertheless at present still exist doubts in astrophysical data:

$$\Omega_{\Lambda} = .65 \pm .13, \quad H_0 = (65 \pm 5) \, km \, s^{-1} \, Mps^{-1}$$

which determine

$$\Lambda^{obs} = 3\Omega_{\Lambda} H_0^2 c^{-2} = (.98 \pm .34) \times 10^{-56} cm^{-2}$$

we accept this observed value of cosmological constant **as a basic quantity** which **defines natural units** for all other cosmological quantities, namely:

- **cosmological length**:

$$A_c = 1/\sqrt{\Lambda^{obs}} = (1.02 \pm .18) \times 10^{28} cm,$$

- **cosmological time**:

$$T_c := A_c/c = (3.4 \pm .6) \times 10^{17} s = (10.8 \pm 1.9) Gyr,$$
• cosmological energy density:
\[ \varepsilon_c := \frac{\Lambda c^2}{\kappa} = (1.16 \pm .41) \times 10^{-7} \text{ g cm}^{-1} \text{ s}^{-2}, \]

• cosmological energy:
\[ E_c := 3A_c^3 \varepsilon_c = 3\Lambda^{-1/2}c^2\kappa^{-1} = (3.7 \pm 0.7) \times 10^{77} \text{ erg}, \]

• cosmological momentum:
\[ P_c := 3c/(\kappa \sqrt{\Lambda^{obs}}) = (1.2 \pm 0.2) \times 10^{67} \text{ g cm s}^{-1} \]

• and cosmological unit for action:
\[ A_c := 3c/(\kappa \Lambda^{obs}) = (1.2 \pm 0.4) \times 10^{122} \hbar, \]

\(\kappa\) being Einstein constant.

Further we use dimensionless variables like:
\[ \tau := t/T_c, \ a := A/A_c, \ h := H T_c \ (H := A^{-1}dA/dt \text{ being Hubble parameter}), \ \varepsilon_c = \varepsilon_c/|\varepsilon_c| = \pm 1, \]
\[ \varepsilon := \varepsilon/|\varepsilon_c|\text{-matter energy density, etc.} \]
Solar System and Earth-Surface Gravitational Experiments

Properties of cosmological factor $\Pi(\Phi)$ derived from known gravitational experiments:

1. The MDG with $\Lambda = 0$ contradicts to solar system gravitational experiments. The cosmological term $\Lambda\Pi(\Phi)$ is needed in action (1) to overcome this problem.

2. In contrast to O’Hanlon’s model we wish MDG to reproduce GR with $\Lambda \neq 0$ for some $\Phi = \Phi = const \neq 0$, i.e., we consider a MDG solution which coincides with original de Sitter solution.

Then from action (1) we obtain normalization condition for cosmological factor

$$\Pi(\Phi) = 1$$

and Einstein constant

$$\kappa = \bar{\kappa}/\Phi.$$
In vacuum, far from matter MDG have to allow weak field approximation \( |\zeta| \ll 1 \):

\[
\Phi = \bar{\Phi}(1 + \zeta).
\]

Then the linearized dilaton equation (2):

\[
\Box \zeta + \zeta/l_\Phi^2 = \frac{\kappa}{3c^2} T
\]

gives

\[
\frac{d\Pi}{d\Phi}(\bar{\Phi}) = 1/\bar{\Phi}.
\]

Taylor series expansion of the function \( \frac{dV}{d\Phi}(\Phi) \) around the value \( \bar{\Phi} \) gives relation

\[
\frac{d^2\Pi}{d\Phi^2}(\bar{\Phi}) = \frac{3}{2} p^{-2} \bar{\Phi}^{-2}.
\]

Then

\[
\Pi = 1 + \zeta + \frac{3}{4p^2} \zeta^2 + O(\zeta^3),
\]

\[
p = \frac{l_\Phi}{A_c}
\]

being dimensionless Compton length of dilaton in cosmological units.
4. Point particles of masses $m_a$ as source of metric and dilaton fields give in Newtonian approximation gravitational potential $\varphi(r)$ and dilaton field $\Phi(r)$:

$$\varphi(r)/c^2 = -\frac{G}{c^2} \sum_a \frac{m_a}{|r-r_a|} \left(1 + \alpha(p) e^{-|r-r_a|/l\Phi}\right)$$
$$- \frac{1}{6} p^2 \sum_a \frac{m_a}{M} (|r-r_a|/l\Phi)^2, \quad (4)$$

$$\Phi(r)/\bar{\Phi} = 1 + \frac{2}{3} \frac{G}{c^2 (1 - \frac{4}{3} p^2)} \sum_a \frac{m_a}{|r-r_a|} e^{-|r-r_a|/l\Phi}, \quad (5)$$

$G = \frac{\kappa c^2}{8\pi} (1 - \frac{4}{3} p^2)$ is Newton constant, $M = \sum_a m_a$. The term

$$- \frac{1}{6} p^2 \sum_a \frac{m_a}{M} (|r-r_a|/l\Phi)^2 = -\frac{1}{6} \Lambda |r - \sum_a \frac{m_a}{M} r_a|^2 + \text{const}$$

in $\varphi$ is known from GR with $\Lambda \neq 0$. It represents an universal anty-gravitational interaction of test mass $m$ with any mass $m_a$ via repellent elastic force

$$F_\Lambda a = \frac{1}{3} \Lambda m c^2 \frac{m_a}{M} (r - r_a). \quad (6)$$
For **solar system distances** $l \leq 1000\text{AU}$ neglecting the $\Lambda$ term (of order $\leq 10^{-24}$) we compare the gravitational potential $\varphi$ with specific MDG coefficient

$$\alpha(p) = \frac{1 + 4p^2}{3 - 4p^2}$$

with Cavendish type experiments and obtain an experimental constraint $l_\Phi \leq 1.6\text{[mm]}$, or

$$p < 2 \times 10^{-29}.$$ 

Hence, in the solar system the factor $e^{-l/l_\Phi}$ has a fantastic small values ($< \exp(-10^{13})$ for the Earth-Sun distances $l$, or $< \exp(-3 \times 10^{10})$ for the Earth-Moon distances $l$) and there is no hope to find some differences between MDG and GR in this domain.

The corresponding constraint

$$m_\Phi c^2 \geq 10^{-4}\text{[eV]}$$

does not exclude a small value (with respect to the elementary particles scales) for the rest energy of hypothetical $\Phi$-particle.
5. The parameterized-post-Newtonian (PPN) solution of equation (2) is complicated, but because of the constraint \( p < 10^{-28} \) we may use with great precision Helbig’s PPN formalism (for \( \alpha = \frac{1}{3} \)). Because of the condition \( \omega \equiv 0 \) we obtain much more definite predictions then usual general relations between \( \alpha \) and the length \( l_\Phi \):

- **Nordtvedt Effect:**

In MDG a body with significant gravitational self-energy \( E_G = \sum_{b \neq c} G \frac{m_b m_c}{|r_b - r_c|} \) will not move along geodesics due to additional universal anty-gravitational force:

\[
F_N = -\frac{2}{3} E_G \nabla \Phi. \tag{7}
\]

For usual bodies it is too small even at distances \( |r - r_a| \leq l_\Phi \), because of the small factor \( E_G \). Hence, in MDG we have no strict strong equivalence principle nevertheless the week equivalence principle is not violated.
The experimental data for Nordtvedt effect, caused by the Sun, are formulated as a constraint $\eta = 0 \pm 0.0015$ on the parameter $\eta$ which in MDG becomes a function of the distance $l$ to the source: $\eta(l) = -\frac{1}{2}(1 + l/l_\Phi)e^{-l/l_\Phi}$. This gives constraint $l_\Phi \leq 2 \times 10^{10}[m]$.

- **Time Delay of Electromagnetic Waves**

The action of electromagnetic field does not depend on the field $\Phi$. Therefore influence of $\Phi$ on the electromagnetic waves in vacuum is possible only via influence of $\Phi$ on the space-time metric. The solar system measurements of the time delay of the electromagnetic pulses give the value $\gamma = 1 \pm 0.001$ of this post Newtonian parameter. In MDG this yields the relation $(1 \pm 0.001)g(l_{AU}) = 1$ and gives once more the constraint $l_\Phi \leq 2 \times 10^{10}[m]$. Here $g(l) := 1 + \frac{1}{3}(1 + l/l_\Phi)e^{-l/l_\Phi}$. 
• **Perihelion Shift**

For the perihelion shift of a planet orbiting around the Sun (with mass $M_\odot$) in MDG we have:

$$\delta \varphi = \frac{k(l_p)}{g(l_p)} \delta \varphi_{GR}.$$  

Here $l_p$ is the semimajor axis of the orbit of planet and

$$k(l_p) \approx 1 + \frac{1}{18} \left( 4 + \frac{l_p^2}{l_\Phi^2 GM_\odot^2} \right) e^{-l_p/l_\Phi} - \frac{1}{27} e^{-2l_p/l_\Phi}$$

is obtained neglecting its eccentricity. The observed value of perihelion shift of Mercury gives the constraint $l_\Phi \leq 10^9 [m]$.

*Conclusion:*

In presence of dilaton field $\Phi$ are impossible essential deviations from GR in solar system.

Observable deviations from Newton law of gravity may not be expected at distances greater then few mm.
Vacuum Energy and True Vacuum Solution in MDG

Total (true) tensor of energy momentum is:

\[ TT_{\mu\nu} := T_{\mu\nu} + \langle \rho_0 \rangle c^2 g_{\mu\nu}, \quad (8) \]

\( \langle \rho_0 \rangle \) being the averaged energy density of zero quantum fluctuations. For true vacuum solution of MDG:

\[ \Phi = \Phi_0 = \text{const}, \quad g_{\mu\nu} = \eta_{\mu\nu} \quad (9) \]

from dynamical equations (2) we obtain:

\[ \Phi_0 \frac{d\Pi}{d\Pi} (\Phi_0) + \Pi(\Phi_0) = 0 \quad (10) \]

\[ T^0_{\mu\nu} = -\frac{c^2}{\kappa} \Lambda U_0 g_{\mu\nu} = TT^0_{\mu\nu} - \langle \rho_0 \rangle c^2 g_{\mu\nu}, \quad (11) \]

where \( U_0 = \Phi_0 \Pi(\Phi_0) = \Phi_0 \Pi_0 \). But for true vacuum solution we must have

\[ TT^0_{\mu\nu} \equiv 0. \quad (12) \]

This way we obtain

\[ \langle \rho_0 \rangle = \frac{1}{\kappa} \Lambda U_0 = \frac{1}{\kappa} \Lambda \Pi_0 \quad (13) \]
Hence in MDG:

True Vacuum  \Rightarrow  Minkowski Space-Time:

\[ TT_{\mu\nu} \equiv 0. \]

Physical Vacuum  \Rightarrow  de Sitter Space-Time:

\[ TT_{\mu\nu} = \langle \rho_0 \rangle c^2 g_{\mu\nu}. \]

- a physically sound picture!

The real word looks like de Sitter Universe created by zero quantum vacuum fluctuations and perturbed by other matter and radiation fields.
For $<\rho_0>$ calculated using Plank length as a quantum cuttoff the observed value of $\Lambda$ gives:

$$U(\Phi_0)/U(\bar{\Phi}) = \kappa <\rho> /\Lambda \approx 10^{122}$$

and causes the famous cosmological constant problem in standard theory. In different articles this number varies from $10^{118}$ to $10^{123}$.

We see that:

- It is obviously close in order to the ratio of cosmological action $A_c$ and Planck constant $\hbar$:
  $$U(\Phi_0)/U(\bar{\Phi}) \approx A_c/\hbar.$$  

- In MDG there is no crisis caused by this big number, because it gives ratio of the values of cosmological potential for different solutions: $\Phi_0$ and $\bar{\Phi}$, i.e. in different universes.
If we calculate the values

\[ A^0_{G,\Lambda} = -\Lambda \frac{c}{\kappa} U_0 \text{ Voll} \]

and

\[ \bar{A}_{G,\Lambda} = \Lambda \frac{c}{\kappa} \bar{U} \text{ Voll} \]

of the very action (1) and introduce corresponding specific actions \( \alpha_0 = -\Lambda \frac{c}{\kappa} U_0 \) and \( \bar{\alpha} = \Lambda \frac{c}{\kappa} \bar{U} \), i.e., actions per unit volume, we can rewrite the above observed result in a form:

\[ \bar{\alpha} \approx -\alpha_0 \times \bar{\eta}/A_c = |\alpha_0| \times 10^{-122}. \]

One can hope that such new and quite radically changed formulation of the cosmological constant problem in MDG will bring us to its resolution. For example it’s easy to think that this results is determined by evolution of the Universe.
Indeed, consider the simplest model of Universe build of Bohr hydrogen atoms in ground state only, i.e. let’s just for simplicity describe the whole content of the Universe with such **effective Bohr hydrogen** (EBH) atoms.

Then for the time of the existence of Universe $T_U \sim 4 \times 10^{17} sec$ one EBH atom with Bohr angular velocity $\omega_B = me^4\hbar^{-3} \sim 4 \times 10^{16} sec^{-1}$ accumulates classical action

$$A_{EBH} = \frac{3}{2} \omega_B T_U \hbar \sim 2.4 \times 10^{34} \hbar.$$ 

Hence, to explain the present day action of the Universe $\sim A_c$ the number of EBH in it must be

$$N_{EBH} \sim 5 \times 10^{87}$$

which seems to be quite reasonable number from physical point of view, taking into account that the observed number of barions in the observable Universe is

$$N_{\text{barions}} \sim 10^{78}.$$
This means that in our approach we have disposable some 9 orders of magnitude to solve cosmological constant problem taking into account the contribution to the action of Universe of all other constituents of matter and radiation (quarks, leptons, gamma quanta, etc) during the evolution of Universe from Big Bang to present day state.

The main conclusion of this qualitative consideration is that actually in MDG the observed nonzero value of cosmological constant $\Lambda^{obs} \neq 0$ restricts the number of degrees of freedom in the observable Universe and forbids existence of much more levels of matter structure below the quark level.
Application of MDG in Cosmology

Consider RW adiabatic homogeneous isotropic Universe with

\[ ds_{RW}^2 = c^2 dt^2 - A^2 dl_k^2, \]

\[ t = T_c \tau, \quad A(t) = A_c a(\tau) \]

and dimensionless

\[ dl_k^2 = \frac{dl^2}{1 - kl^2} + l^2 (d\theta^2 + \sin^2 \theta) d\varphi^2 \]

\((k = -1, 0, 1)\) in presence of matter with energy-density \( \varepsilon = \varepsilon_c \varepsilon(a)/\bar{\Phi} \) and pressure \( p = \varepsilon_c p\varepsilon(a)/\bar{\Phi} \).

Basic dynamical equations of MDG for RW Universe are:

\[ \frac{1}{a} \frac{da}{d\tau} + \frac{1}{a^2} \left( \frac{da}{d\tau} \right)^2 + \frac{k}{a^2} = \frac{1}{3} \left( \Phi \frac{dn}{d\Phi}(\Phi) + \Pi(\Phi) \right), \]

\[ \frac{1}{a} \frac{da}{d\tau} \frac{d\Phi}{d\tau} + \Phi \left( \frac{1}{a^2} \left( \frac{da}{d\tau} \right)^2 + \frac{k}{a^2} \right) = \frac{1}{3} \left( \Phi \Pi(\Phi) + \varepsilon(a) \right) \quad (14) \]

The use of Hubble parameter \( h(a) = a^{-1} \frac{da}{d\tau}(\tau(a)) \)
\((\tau(a) \text{ – inverse function of } a(\tau))\), new variable
\[ \lambda = \ln a \text{ and prime for differentiation with respect to } \lambda \text{ gives the system for } \Phi(\lambda) \text{ and } h^2(\lambda): \]
\[
\frac{1}{2}(h^2)' + 2h^2 + ke^{-2\lambda} = \frac{1}{3}(\Phi \frac{d\Pi}{d\Phi}(\Phi) + \Pi(\Phi)), \]
\[
h^2\Phi' + (h^2 + ke^{-2\lambda}) \Phi = \frac{1}{3}(\Phi \Pi(\Phi) + \epsilon(e^\lambda)).
\]
and relation
\[
\tau(a) = \int_{a_{in}}^{a} da/(a h(a)) + \tau_{in}.
\]
Excluding cosmological factor \( \Pi(\Phi) \) we have:

\[
\Phi'' + \left(\frac{h'}{h} - 1\right) \Phi' + 2 \left(\frac{h'}{h} - k h^{-2} e^{-2\lambda}\right) \Phi = \frac{1}{3h^2} \epsilon'. \tag{15}
\]

In terms of the function \( \psi(a) = \sqrt{|h(a)|/a} \Phi(a) \) it reads:

\[
\psi'' + n^2 \psi = \delta, \tag{16}
\]
where we introduce new functions

\[
-n^2 = \frac{1}{2} \frac{h''}{h} - \frac{1}{4} \left(\frac{h'}{h}\right)^2 - \frac{5}{2} \frac{h'}{h} + \frac{1}{4} + \frac{2k}{h^2} e^{-2\lambda},
\]
\[
\delta = \frac{1}{3} \sqrt{a/|h|^3} \frac{d\epsilon}{da}. \tag{17}
\]
Now we are ready to consider

**The inverse cosmological problem**: to find a cosmological factor $\Pi(\Phi)$ (or potentials $V(\Phi)$, or $U(\Phi)$) which yield given evolution of the Universe, determined by function $a(\tau)$.

*A remarkable property of MDG:*

An unique solution of this problem exist for almost any three times differentiable function $a(\tau)$. 
Indeed: for given $a(\tau)$ construct a function $h(\lambda)$ and find the general solution $\Phi(\lambda, C_1, C_2)$ of the linear second order differential equation (15). The two constants $C_{1,2}$ have to be determined from the additional conditions

$$\Pi(\bar{\Phi}) = 1, \quad \frac{d\Pi}{d\Phi}(\bar{\Phi}) = \bar{\Phi}^{-1}, \quad \frac{d^2\Pi}{d\Phi^2}(\bar{\Phi}) = \frac{3}{2}p^{-2}\bar{\Phi}^{-2}.$$ 

—self-consistence conditions at point $\bar{\lambda}$ which is real solution of the algebraic equation

$$r(\bar{\lambda}) = -4,$$

$$r(\lambda) = -6 \left(\frac{1}{2}h^2\right)' + 2h^2 + ke^{-2\lambda}$$

being dimensionless scalar curvature: $r = R/\Lambda$. Then:

$$\bar{\Phi} = -4\bar{\epsilon}\left(1 + \frac{4}{3}p^2\right) / \left(\bar{j}_{00}'(1 + \frac{4}{3}p^2) + 4p^2\bar{h}^2\bar{r}'\right),$$

$$\bar{\Phi}'/\bar{\Phi} = -\frac{1}{3}p^2\bar{r}' / \left(1 + \frac{4}{3}p^2\right). \quad (18)$$

Here $j_{00} = G_{00}/\Lambda = 3 \left(h^2 + ke^{-2\lambda}\right)$ is dimensionless 00-component of Einstein tensor. Hence, the values of all ”bar” quantities (including $\bar{\kappa}$
in action (1) may be determined from time evolution \( a(\tau) \) of the Universe via the solution \( \bar{\lambda} = \ln \bar{a} \) of the equation (3). In their turn \( \bar{\Phi} \) and \( \bar{\Phi}' \) determine the values of constants \( C_{1,2} \) and an unique solution \( \Phi(\lambda) \) of the equation (15):

\[
\Phi(\lambda) = C_1 \Phi_1(\lambda) + C_2 \Phi_2(\lambda) + \Phi_\epsilon(\lambda)
\]

where \( \Phi_1(\lambda) \) and \( \Phi_2(\lambda) \) are a fundamental system of solutions of the homogeneous equation associated with non-homogeneous one (15). Then

\[
\Phi_\epsilon = \frac{\bar{a}}{(3\bar{h}\Delta)} \left( \Phi_2 \int_\lambda^\bar{\lambda} d\epsilon \frac{\Phi_1}{ah} - \Phi_1 \int_\lambda^\bar{\lambda} d\epsilon \frac{\Phi_2}{ah} \right).
\]

where \( \Delta(\lambda) = \Phi_1 \Phi'_2 - \Phi_2 \Phi'_1 \). The cosmological factor \( \Pi \) and the potential \( V \) as functions of the variable \( \lambda \) are determined by equations

\[
\Pi(\lambda) = j_{00} + 3h^2 \Phi'/\Phi - \epsilon/\Phi,
\]

\[
V(\lambda) = \frac{2}{3} \int \Phi \left( \Phi \Pi' - \Phi' \Pi \right) d\lambda \quad (19)
\]

which define functions \( \Pi(\Phi) \) and \( V(\Phi) \) implicitly, too.
Simple exactly soluble examples:

1. **Evolution law** \( a(\tau) = (\omega \tau)^{1/\nu} \), (\( \omega \) is free parameter) gives

\[
h(\lambda) = \frac{\omega}{\nu} e^{-\nu \lambda},
- n^2(\lambda) = \frac{1+10\nu+\nu^2}{4} + 2k \frac{\nu^2}{\omega^2} e^{2(\nu-1)\lambda},
\]
and the equation \( \ddot{a}^2 + \frac{3(\nu-2)\omega^2}{2\nu^2} \dot{a}^2(1-\nu) = k \) for \( \ddot{a} \).

i) For \( \nu \geq 2 \) we have real solution \( \ddot{a} \) only if \( k = +1 \):

\[
\Phi_1(a) = a^{\frac{\nu+1}{2}} I_{\mu}(ba^{\nu-1}), \quad \Phi_2(a) = a^{\frac{\nu+1}{2}} K_{\mu}(ba^{\nu-1}),
\]

\( \mu := \sqrt{\frac{1+10\nu+\nu^2}{2(\nu-1)}} \) being the order of Bessel functions \( I_{\mu}, J_{\mu}, K_{\mu}, Y_{\mu} \) and \( b := \sqrt{\frac{2\nu^2}{(\nu-1)^2\omega^2}} \). For GR-like law \( a \sim \sqrt{\tau} (\nu = 2) \) in MDG we obtain positive value \( k = +1 \) for three-space curvature, \( \ddot{\lambda} = 0, \mu = \frac{5}{2} \) and Bessel functions are reduced to elementary functions.

ii) When \( \nu < 2 \) all values \( k = -1, 0, +1 \) are admissible:
- for \( k = +1 \) the solutions \( \Phi_{1,2} \) are the same as above;

- for \( k = 0 \) we have

\[
\Phi_{1,2} = a^{\frac{\nu + 1}{2}} \pm \mu (\nu - 1);
\]

- for \( k = -1 \) the solutions are:

\[
\Phi_1 = a^{\frac{\nu + 1}{2}} J_\mu (ba^{\nu - 1}), \quad \Phi_2 = a^{\frac{\nu + 1}{2}} Y_\mu (ba^{\nu - 1}).
\]

In the special case of linear evolution \( a \sim \tau \) (\( \nu = 1 \)) \(-n^2(\lambda) = 3 + \frac{2k}{\omega^2}\),

\[
\Phi_{1,2}(a) = a^{1 \pm \sqrt{-n^2}}
\]

and the root \( \bar{\lambda} = \frac{1}{2} \ln \frac{3}{2}(k + \omega^2) \) is real for all values of \( \omega^2 > 0 \), if \( k = 0, +1 \). For \( k = -1 \) the root \( \bar{\lambda} \) will be real if \( |\omega| > 1 \).
2. Evolution law $a(\tau) = e^{\omega \tau}$ gives

$$h(\lambda) = \omega, \quad -n^2(\lambda) = \frac{1}{4} + \frac{2k}{\omega^2} e^{-2\lambda}$$

and the equation $\frac{2}{3} \bar{a}^2 (1 - 3\omega^2) = k$ with root $\bar{a} = \sqrt{\frac{3}{2|1-3\omega^2|}}$. Now we have the following solutions:

i) $|\omega| < \frac{\sqrt{3}}{3}$: $\Phi_1 = a \cosh\left(\frac{\sqrt{2}}{|\omega a|}\right)$, $\Phi_2 = a \sinh\left(\frac{\sqrt{2}}{|\omega a|}\right)$, $k = +1$;

ii) $|\omega| > \frac{\sqrt{3}}{3}$: $\Phi_1 = a \cos\left(\frac{\sqrt{2}}{|\omega a|}\right)$, $\Phi_2 = a \sin\left(\frac{\sqrt{2}}{|\omega a|}\right)$, $k = -1$.

Conditions $r(\bar{\lambda}) = -4$ and (18) exclude exact exponential expansion of spatially flat Universe ($k = 0$). For inflationary scenario in this case one may use a scale factor $a(\tau) = e^{\omega \tau} + \text{const}$ which turns to be possible if $\text{const} \neq 0$. 
Specific properties of MDG:

1) If \( n > 0 \) dilatonic field \( \Phi(a) \) oscillates; if \( n < 0 \) such oscillations do not exist. Dilatonic field \( \Phi \) may change its sign, i.e. phase transitions of the Universe from gravity (\( \Phi > 0 \)) to anty-gravity (\( \Phi < 0 \)) and vice-versa are possible in general for width class of cosmological potentials.

2) In spirit of Max principle Newton constant depends on presence of matter: \( G \sim 1/\bar{\Phi} \sim 1/\bar{\epsilon} \).

3) For simple functions \( a(\tau) \) the cosmological factor \( \Pi(\Phi) \) and potentials \( V(\Phi) \) and \( U(\Phi) \) may show unexpected catastrophic behavior:

\[ \sim (\Delta \Phi)^{3/2} \]

(\( \Delta \Phi = \Phi - \Phi(\lambda^*) \)) in vicinity of the critical points \( \lambda^* \): \( \Phi'(\lambda^*) = 0 \) of the projection of analytical curve \{\( \Pi(\lambda), \Phi(\lambda), \lambda \)\} on the plain \{\( \Pi, \Phi \)\}. 
Scale factors $a(\tau)$ exist yielding an **everywhere analytical cosmological factor** $\Pi(\Phi)$ and potentials $V(\Phi)$ and $U(\Phi)$, too.

4) Clearly one can construct **MDG model of Universe without initial singularities**: $a(\tau_0) = 0$ (typical for GR) and with any desired kind of inflation.

5) Because the dilaton field $\Phi$ is quite massive, in it will be stored significant amount of energy. An interesting open question is: **may the field $\Phi$ play the role of dark matter** in the Universe?

A very important problem is to **reconstruct the cosmological factor** $\Pi(\Phi)$ of real Universe using proper experimental data and astrophysical observations.

Maybe the best way to study SUSY breaking and the corresponding potential $V_{SUSY}(\phi)$ is to look at the sky and to try to reconstruct the **real time evolution** of the Universe.
GENERAL CONCLUSIONS:

• MDG is a rich model which offers new curious possibilities and deserves further careful investigation.

• Instead of Cosmological Constant we have to prefer a Cosmological Potential, because it yields much richer theory and may be more suitable for description of the real world.
Acknowledgments:

The author is deeply indebted to NORDITA and personally to Professor Holger Bech Nielsen, Dr. Kimmo Kainulainen and Ellen Pedersen for the help and support which make possible his participation in the conference.