On expansions for the Black-Scholes prices and hedge parameters

Jean-Philippe Aguilar
BRED Banque Populaire, Modeling Department, 18 quai de la Râpée, Paris - 75012
jean-philippe.aguilar@bred.fr

Abstract
We derive new formulas for the price of the European call and put options in the Black-Scholes model, under the form of uniformly convergent series generalizing previously known approximations. We also provide precise boundaries for the convergence speed and apply the results to the calculation of hedge parameters (Greeks).

Key words— Option pricing, Black-Scholes formula, Hedge parameters, Risk sensitivities, Series expansion, Mellin transform, Multidimensional complex analysis

I. Introduction

Several ways exist to derive the celebrated Black-Scholes formula [4] (eight different derivations are described in [2], and even ten in the textbook [18]), the most famous of which being the resolution of the Black-Scholes partial differential equation (PDE). This PDE is classically obtained by a hedging argument combined with elementary stochastic calculus (which is the original derivation used by Black and Scholes in their seminal paper [4]) but also as a limiting case of the binomial model [5] or by using the Capital Asset Pricing Model (CAPM) (see [2, 13] for instance). For non-PDE approaches, let us mention probabilistic derivations such as the martingale technique, which, although losing the direct connection with the hedging argument, provides a precise interpretation of the Black-Scholes formula in terms of risk-neutral probabilities [10, 12], or purely economic approaches like the "representative investor" as introduced by Rubinstein [14].

In this paper, we would like to document a new derivation, based on a representation for the option price as a complex integral over a domain of \( \mathbb{C}^2 \) (obtained via suitable transforms of the Green function for the Black-Scholes PDE). This approach is fruitful because, evaluating this complex integral by means of multidimensional residues, we obtain series expansions for the price of the call and put options, which turn out to be simple and fast convergent; these expansions recover existing approximations, that were known in some very specific market configurations (when the asset is at-the-money forward, it is easy to expand the Black-Scholes formula as a power series of the variance [3]). Moreover, they are unconditionally and uniformly convergent, which would not be the case with a naive Taylor expansion: as already noticed by Estrella in [7], the Black-Scholes formula mixes two components of strongly different natures (the logarithmic function, possessing an expansion converging very fast but only for a small range of arguments, and the normal distribution, whose expansion converges on the whole real axis but with fewer level of accuracy) resulting in situations where, for a plausible range of parameter values, the Taylor series for the Black-Scholes price diverges.

The paper is organized as follow. In section 2, we first write down the option price as a double complex integral, and evaluate it by help of residue theory in \( \mathbb{C}^2 \). This will allow us to...
expression the call option price under the form of a simple series (19), which will be refined into another series (42) exhibiting more clearly its financial meaning; last for this section, we prove the exponential convergence to the option price. In section 3, we apply (42) to the specific at-the-money configuration, and we compute the series expansions for the hedge parameters (Delta, Rho, Vega, Theta). After the conclusive section and for reader’s convenience, we have equipped the paper with an appendix summing up (without proof) the main concept of multidimensional complex analysis used in the present article.

**Notations**

In all of the following, $S$ will denote the market (spot) price of an underlying asset, $r$ the annual risk-free interest rate and $\sigma$ the market volatility. Call and (resp. put) options are assumed to be Europeans with maturity $T$, strike $K$ and payoff $[S - K]^+$ (resp. $[K - S]^+$), and their prices will be denoted by $C$ (resp. $P$). We introduce the notations for the reversed time $\tau$, the variance $z$, the forward strike price $F$ and the log-forward moneyness $k$:

$$
\tau := T - t \quad z := \sigma \sqrt{\tau} \quad F := Ke^{-rt} \quad k := \log \frac{S}{F} = \log \frac{S}{K} + rt
$$

Within this system of notations, the Black-Scholes formula [4, 17] reads:

$$
C = SN\left(\frac{k}{z} + \frac{z}{2}\right) - FN\left(\frac{k}{z} - \frac{z}{2}\right) = F\left[e^{kN}\left(\frac{k}{z} + \frac{z}{2}\right) - N\left(\frac{k}{z} - \frac{z}{2}\right)\right]
$$

As far as multidimensional complex analysis will be concerned, we will denote the vectors in $\mathbb{C}^2$ by $u = [u_1, u_2]$, $u_1, u_2 \in \mathbb{C}$; we will also adopt the standard wedge notation for differential forms, which has the properties [9]:

$$
du_1 \wedge du_2 = -du_2 \wedge du_1 \quad \text{and} \quad du_1 \wedge du_1 = 0
$$

and we will denote $du := du_1 \wedge du_2$.

**II. Pricing formulas**

Our starting point is the Black-Scholes PDE [4]:

$$
\begin{cases}
\frac{\partial C}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + rS \frac{\partial C}{\partial S} - rC = 0 & t \in [0, T] \\
C = [S - K]^+ & t = T
\end{cases}
$$

It is known (see the classic textbook by Wilmott [17] for instance) that, with the change of variables

$$
\begin{cases}
x := \log S + (r - \frac{\sigma^2}{2}) \tau \\
\tau := T - t \\
C := e^{-rt}W
\end{cases}
$$

then the Black-Scholes PDE [4] resumes to the diffusion (or heat) equation

$$
\frac{\partial W}{\partial \tau} - \frac{\sigma^2}{2} \frac{\partial^2 W}{\partial x^2} = 0
$$
whose fundamental solution (i.e., Green function) is the heat kernel:

\[ g(x, K, r, \sigma, \tau) := \frac{1}{\sigma \sqrt{2\pi \tau}} e^{-\frac{x^2}{2\sigma^2\tau}} \]  

(7)

In this new set of variables the terminal condition becomes an initial condition \( W(\tau = 0) = [e^x - K]^+ \) and therefore, by the method of Green functions, we know that we can express \( W \) as

\[ W = \int_{-\infty}^{+\infty} [e^x + y - K]^+ g(y, K, r, \sigma, \tau) \, dy \]  

(8)

Turning back to the initial variables:

\[ C = e^{-\sigma r} \int_{-\infty}^{+\infty} [Se^{\sigma r - \frac{z^2}{2\tau}} + y - 1] + \frac{1}{\sigma \sqrt{2\pi \tau}} e^{-\frac{y^2}{2\sigma^2\tau}} \, dy \]  

(9)

I. The call price as a complex integral

Let us express the call option price (9) under the form of an integral over a domain of \( C^2 \).

**Proposition 1.** Let \( P \subset C^2 \) be the polyhedra \( P := \{ f \subset C^2 \, , \, Re(2t_1 + t_2) > 2 \, , \, Re(t_2) < 1 \} \). Then, for any \( c \in P \),

\[ C = F \int_{\mathbb{C}+i\mathbb{R}^2} (-1)^{-t_2} 2^{1-t_1} \frac{\Gamma(t_2)\Gamma(1 - t_2)\Gamma(-2 + 2t_1 + t_2)}{\Gamma(t_1 + \frac{1}{2})} \frac{\Gamma(t_1)}{2^{2t_1 - 2t_2}} \frac{1}{\left(2\pi\right)^2} dt_1 \]  

(10)

**Proof.** With our notations (1), we can re-write the call price (9) as:

\[ C = F \frac{e^{-\frac{z^2}{2\tau}}}{\sqrt{2\pi \tau}} \int_{\frac{z}{2} - k}^{\frac{z}{2} + \infty} (e^{k - \frac{z^2}{2\tau}} - 1) \frac{1}{z} e^{-\frac{y^2}{2\tau}} \, dy \]  

(11)

Let us introduce a Mellin-Barnes representation for the heat kernel-term in (11) (see (76) in Appendix, and [8, 6] or any monograph on integral transforms):

\[ \frac{1}{z} e^{-\frac{z^2}{2\tau}} = \frac{1}{z} \int_{c_1 - i\infty}^{c_1 + i\infty} \Gamma(t_1) \left( \frac{y^2}{2\tau} \right)^{-t_1} \frac{dt_1}{2\pi i} \]  

(12)

We thus have:

\[ C = \frac{K e^{-\sigma r}}{\sqrt{2\pi}} \int_{c_1 - i\infty}^{c_1 + i\infty} 2^{t_1} \Gamma(t_1) \int_{\frac{z}{2} - k}^{\frac{z}{2} + \infty} (e^{k - \frac{z^2}{2\tau}} - 1) y^{2t_1} \, dy \frac{z^{2t_1 - 1}}{2\pi i} \]  

(13)

Integrating by parts in the \( y \)-integral yields:

\[ C = \frac{F}{\sqrt{2\pi}} \int_{c_1 - i\infty}^{c_1 + i\infty} 2^{t_1} \frac{\Gamma(t_1)}{2t_1 - 1} \int_{\frac{z}{2} - k}^{\frac{z}{2} + \infty} e^{k - \frac{z^2}{2\tau}} y^{1 - 2t_1} \, dy \frac{z^{2t_1 - 1}}{2\pi i} \]  

(14)
II Residue summation

Let us introduce another Mellin-Barnes representation for the remaining exponential term (again, see (76) in Appendix):

\[ e^{k - \frac{z^2}{2} + y} = \int_{c_2 - i\infty}^{c_2 + i\infty} (-1)^{-t_2} \Gamma(t_2) \left( k - \frac{z^2}{2} + y \right)^{-t_2} \frac{dt_2}{2\pi i} \quad (c_2 > 0) \]  (15)

Therefore the call price is:

\[ C = \frac{F}{\sqrt{2\pi}} \times \int_{c_1 + i\infty}^{c_2 + i\infty} \int_{c_1 - i\infty}^{c_2 - i\infty} (-1)^{-t_2} \frac{2t_1}{2t_1 - 1} \Gamma(t_1) \Gamma(t_2) \int_{\frac{z^2}{2} - k}^{\infty} y^{1 - 2t_1} \left( k - \frac{z^2}{2} + y \right)^{-t_2} dy z^{2t_1 - 1} \frac{dt_1}{2\pi i} \wedge \frac{dt_2}{2\pi i} \]  (16)

The \( y \)-integral is a particular case of Bêta-integral [11] and equals:

\[ \int_{\frac{z^2}{2} - k}^{\infty} y^{1 - 2t_1} \left( k - \frac{z^2}{2} + y \right)^{-t_2} dy = \left( \frac{z^2}{2} - k \right)^{2 - 2t_1 - t_2} \frac{\Gamma(1 - t_2) \Gamma(-2 + 2t_1 + t_2)}{\Gamma(2t_1 - 1)} \]  (17)

and converges on the conditions \( \text{Re}(t_2) < 1 \) and \( \text{Re}(2t_1 + t_2) > 2 \); plugging into (16), using the Gamma function functional relation \( (2t_1 - 1)\Gamma(2t_1 - 1) = \Gamma(2t_1) \) and the Legendre duplication formula for the Gamma function [11]:

\[ \frac{\Gamma(t_1)}{\Gamma(2t_1)} = 2^{1 - 2t_1} \sqrt{\pi} \frac{1}{\Gamma(t_1 + \frac{1}{2})} \]  (18)

we obtain the integral [10].

II. Residue summation

Now we compute the double integral [10] by means of summation of multidimensional residues.

**Theorem 1.** Let \( Z := \frac{z}{\sqrt{2}} \) (normalized variance); then the Black-Scholes price of the European call is:

\[ C = \frac{F}{\sqrt{2\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n}{n! \Gamma(1 + \frac{m-n}{2})} \left( \frac{z^2}{2} - k \right)^n Z^{m-n} \]  (19)

**Proof.** Let \( \omega \) be the complex differential 2-form

\[ \omega := (-1)^{-t_2} 2^{\frac{1}{2} - t_1} \frac{\Gamma(t_2) \Gamma(1 - t_2) \Gamma(-2 + 2t_1 + t_2)}{\Gamma(t_1 + \frac{1}{2})} \left( \frac{z^2}{2} - k \right)^{2 - 2t_1 - t_2} \frac{dt_1}{2\pi i} \wedge \frac{dt_2}{2\pi i} \]  (20)

so that we can write the call price [10] under the form:

\[ C = F \int_{c_1 + i\mathbb{R}^2} \omega \quad (c \in P) \]  (21)

This complex integral can be performed by means of summation of \( C^2 \)-residues, by virtue of a multidimensional analogue to the residue theorem valid for this specific class of integrals (see [92].
II Residue summation

Figure 1: The divisors $D_1$ (oblique lines) are induced by the $\Gamma(-2 + 2t_1 + t_2)$ term, and $D_2$ (horizontal lines) by the $\Gamma(t_2)$ term. The intersection set $D_1 \cap D_2$ (the dots), located in the compatible green cone $\Pi$, gives birth to residues whose sum in the whole cone equals the integral (21).

and references in Appendix). Indeed, the characteristic quantity (see definition (84) in appendix) associated to the differential form (20) is:

$$\Delta = \begin{bmatrix} 2 - 1 \\ 1 - 1 + 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

and the admissible half-plane is

$$\Pi_\Delta := \{ t \in \mathbb{C}^2, \Re(\Delta \cdot t) < \Delta \cdot c \}$$

in the sense of the euclidean scalar product. This half-plane is therefore located under the line

$$t_2 = -t_1 + c_1 + c_2$$

In this half-plane, the cone $\Pi$ as shown of fig. 1 and defined by

$$\Pi := \{ t \in \mathbb{C}^2, \Re(t_2) \leq 0, \Re(2t_1 + t_2) \leq 2 \}$$

contains and is compatible with the two family of divisors

$$\begin{cases} D_1 = \{ t \in \mathbb{C}^2, -2 + 2t_1 + t_2 = -n_1, \ n_1 \in \mathbb{N} \} \\ D_2 = \{ t \in \mathbb{C}^2, t_2 = -n_2, \ n_2 \in \mathbb{N} \} \end{cases}$$

induced by $\Gamma(-2 + 2t_1 + t_2)$ and $\Gamma(t_2)$ respectively. This configuration is shown on fig. 1. To compute the residues associated to every element of the singular set $D_1 \cap D_2$, we change the variables:

$$\begin{cases} u_1 := -2 + 2t_1 + t_2 \\ u_2 := t_2 \end{cases} \quad \rightarrow \quad \begin{cases} t_1 = \frac{1}{2}(2 + u_1 - u_2) \\ t_2 = u_2 \end{cases} \quad dt_1 \wedge dt_2 = \frac{1}{2} du_1 \wedge du_2$$
so that in this new configuration $\omega$ reads

$$
\omega = \frac{1}{2} (-1)^{-u_2} 2^{\frac{u_2-1}{2}} \frac{\Gamma(u_2) \Gamma(1-u_2) \Gamma(u_1)}{\Gamma(1 + \frac{u_1-u_2+1}{2})} \left( \frac{z^2}{2} - k \right)^{-u_1} z^{u_1-u_2+1} \frac{du_1}{2i\pi} \wedge \frac{du_2}{2i\pi} 
$$

(28)

With this new variables, the divisors $D_1$ and $D_2$ are induced by the $\Gamma(u_1)$ and $\Gamma(u_2)$ functions in $(u_1, u_2) = (-n, -m)$, $n, m \in \mathbb{N}$. From the singular behavior of the Gamma function around a singularity (74), we can write

$$
\omega \sim (-n, -m)
$$

$$
\frac{1}{2} (-1)^{-u_2} \frac{(-1)^{n+m}}{n!m!} 2^{\frac{u_2-1}{2}} \frac{\Gamma(1-u_2)}{\Gamma(1 + \frac{u_1-u_2+1}{2})} \left( \frac{z^2}{2} - k \right)^{-u_1} z^{u_1-u_2+1} \frac{du_1}{2i\pi(u_1+n)} \wedge \frac{du_2}{2i\pi(u_2+m)}
$$

(29)

and therefore the residues are, by the Cauchy formula:

$$
\text{Res}(u_1 = -n, u_2 = -m) = (-1)^n 2^{\frac{u_2-1}{2}} \frac{1}{n! \Gamma(1 + \frac{m-n}{2})} \left( \frac{z^2}{2} - k \right)^n z^{m-n+1}
$$

(30)

By virtue of the residue theorem (92), the sum of the residues in the whole cone equals the integral (27):

$$
V = 2 \sum_{n=0}^{\infty} (-1)^n 2^{\frac{u_2-1}{2}} \frac{1}{n! \Gamma(1 + \frac{m-n}{2})} \left( \frac{z^2}{2} - k \right)^n z^{m-n+1}
$$

(31)

We can further simplify by changing the index $m \rightarrow m + 1$ and introducing $Z := \frac{z}{\sqrt{2}} = \frac{\sqrt{\pi}}{\sqrt{2}}$, and we finally obtain the series (19).

Formula (19) is a new representation for the Black-Scholes price of the European call option, and can be very easily implemented so as to make precise calculations; one may note that this expansion is surprisingly simple and compact, when compared to the Black-Scholes formula (2).

Let us also remark that, when the asset is at-the-money forward (that is, when $S = F$ and therefore when $k = 0$), we get the power series

$$
C = \frac{S}{2} \sum_{n=0}^{\infty} (-1)^n \frac{1}{n! \Gamma(1 + \frac{m-n}{2})} Z^{m+n}
$$

(32)

However, despite its simplicity, formula (19) does not exhibit clearly the ordering in powers of $Z$ and the simplifications that can be operated; for instance in the power series (32) all even powers of $Z$ vanish (as expected from the Taylor series for the normal distribution). Let us therefore refine (19) into an expansion with an even more tractable structure.

**Corollary 1.** Let the $\varphi_j$ be the real functions defined by

$$
\varphi_J(x) = \sum_{n=0}^{J-1} (-1)^n \frac{1}{n! \Gamma(1 + \frac{1}{2} - n)} (1 - x)^n \quad J = 1, 2, \ldots
$$

(33)

Then the Black-Scholes price of the European call is

$$
C = \frac{F}{2} \sum_{j=1}^{\infty} Z^j \varphi_j \left( \frac{k}{Z^2} \right)
$$

(34)
**II Residue summation**

**Proof.** Let \( J := m + n \) in the series (19), \( J \geq 1 \). As \( m \geq 1 \), this implies \( J - n \geq 1 \), that is, the upper bound of the \( n \) summation within this new configuration is \( n \leq J - 1 \). This means that we are performing the summation along oblique lines instead of horizontal or vertical ones (see Fig 2); we can therefore introduce the \( J \)-th partial sum (recall that now \( m - n = J - 2n \))

\[
C_J := \frac{F}{2} \sum_{n=0}^{J-1} \frac{(-1)^n}{n!\Gamma(1 + \frac{1}{2} - n)} (Z^2 - k)^n Z^{J-2n}
\]

and summing over all the oblique lines \( J \geq 1 \) yields formula (34)

\[
\frac{Z^{J}}{2} \sum_{n=0}^{J-1} \frac{(-1)^n}{n!\Gamma(1 + \frac{1}{2} - n)} \left( 1 - \frac{k}{Z^2} \right)^n
\]

(Fig. 2: The summation (19) can be performed on oblique lines \((J = 1, 2, \ldots)\) instead of vertical lines \(m = 1, 2, \ldots\) or horizontal lines \(n = 0, 1, \ldots\).)

**Proposition 2.** For any integer \( j \geq 1 \),

\[
Z^{2j} \varphi_{2j} \left( \frac{k}{Z^2} \right) = \frac{k^j}{j!}
\]

**Proof.** By definition of \( \varphi_{2j} \), we have

\[
\varphi_{2j} \left( \frac{k}{Z^2} \right) = \sum_{n=0}^{2j-1} \frac{(-1)^n}{n!\Gamma(j + 1 - n)} \left( 1 - \frac{k}{Z^2} \right)^n
\]

As the Gamma function in the denominator is infinite for negative integers, all terms after \( n = j \) vanish and therefore the sum can be written as:

\[
\varphi_{2j} \left( \frac{k}{Z^2} \right) = \sum_{n=0}^{j} \frac{(-1)^n}{n!\Gamma(j + 1 - n)} \left( 1 - \frac{k}{Z^2} \right)^n
\]

\[
= \sum_{n=0}^{j} \frac{1}{n!(j-n)!} \left( \frac{k}{Z^2} \right)^n
\]

\[
= \frac{1}{j!} \frac{k^j}{Z^{2j}}
\]

in virtue of the binomial theorem.
III Speed of convergence

Theorem 2. The Black-Scholes price of the European call option is:

\[ C = \frac{1}{2} (S - F) + \frac{F}{2} \sum_{j=0}^{n} \sum_{i=0}^{2j} \frac{(-1)^{i} n! \Gamma(j + j - n)}{n! \Gamma(j + i + j - n)} \left(1 - \frac{k}{Z^2}\right)^n \] (42)

Proof. Let us write

\[ C = \frac{F}{2} \left[ \sum_{i=1}^{\infty} Z^{2j} \varphi_{2j} \left(\frac{k}{Z^2}\right) + \sum_{j=0}^{\infty} Z^{2j+1} \varphi_{2j+1} \left(\frac{k}{Z^2}\right) \right] \] (43)

\[ C = \frac{F}{2} \left[ (e^k - 1) + \sum_{j=0}^{\infty} Z^{2j+1} \varphi_{2j+1} \left(\frac{k}{Z^2}\right) \right] \] (44)

where we have used proposition 2 to perform the first sum. Recalling that \( e^k = \frac{S}{F} \) and simplifying yields eq. (42).

Formula (42) is less compact than formula (19), but is more suitable for practical applications. Notably, it immediately follows from the call-put parity \( C = P = S - F \) that the Black-Scholes put is given by:

\[ P = \frac{1}{2} (F - S) + \frac{F}{2} \sum_{j=0}^{n} \sum_{i=0}^{2j} \frac{(-1)^{i} n! \Gamma(j + j - n)}{n! \Gamma(j + i + j - n)} \left(1 - \frac{k}{Z^2}\right)^n \] (45)

In (46) we write down the expansion (42) up to order \( Z^5 \); note that the value of gamma functions at half-integers are actually all known analytically, because \( \Gamma(1/2) = \sqrt{\pi} \) and \( \Gamma(z + 1) = z\Gamma(z) \).

III Speed of convergence

Proposition 3. Let \( \alpha := \max \left(1, \left|1 - \frac{k}{Z^2}\right|\right) \). Then the generic term of the series (42) is bounded (uniformly in \( n \)) by

\[ \left| Z^{2j+1} \frac{(-1)^{i} n! \Gamma(j + j - n)}{n! \Gamma(j + i + j - n)} \left(1 - \frac{k}{Z^2}\right)^n \right| \leq Z^2 (aZ)^{2j} \left(\frac{\alpha}{Z^2}\right)^{2j} \] (47)

Proof. Let \( R_{j,n} := \frac{(-1)^{i} n! \Gamma(j + j - n)}{n! \Gamma(j + i + j - n)} \). We have:

\[ \left| Z^{2j+1} \frac{(-1)^{i} n! \Gamma(j + j - n)}{n! \Gamma(j + i + j - n)} \left(1 - \frac{k}{Z^2}\right)^n \right| \leq Z^{2j+1} \left|R_{j,n}\right| \alpha^{2j} \] (48)
Derivating with respect to $n$, it is easy to see that $R_{j,n}$ is maximal on the line $n = \lfloor j \rfloor + 1$:

$$|R_{j,n}| \leq |R_{j,\lfloor j \rfloor + 1}| = \frac{1}{(\lfloor j \rfloor + 1)! \Gamma\left(\frac{1}{2} + j - \lfloor j \rfloor\right)}$$ (49)

As $x \to \Gamma\left(\frac{1}{2} + x\right)$ grows for $x > 0$, we have

$$\Gamma\left(\frac{1}{2} + j - \lfloor j \rfloor\right) \geq \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$ (50)

which terminates the proof.

Therefore, if we wish to attain a precision of $\epsilon$ in the series representation (42), we just need to find the integer $j_e$ such as

$$\frac{Z}{\sqrt{\pi} (\alpha Z)^{2j_e} (\lfloor j_e \rfloor + 1)!} < \epsilon$$ (51)

On each $j$-line, there are $(2j + 1)$ terms to sum and therefore, the total number of terms it suffices to consider to attain the desired precision is:

$$\sum_{j=0}^{j_e} (2j + 1) = (j_e + 1)^2$$ (52)

In table 1, we define $M_j$ to be the l.h.s. of (51) and we fix a typical set of market parameters ($S = 4200$, $K = 4000$, $\sigma = 20\%$, $r = 1\%$ and $\tau = 1Y$). We compute the values of $M_j$ for $j \geq 2$ and deduce which precision is attained and after how many terms.

| $j$ | $M_j$ | Attained precision ($\epsilon$) | Total number of terms $(j_e + 1)^2$ |
|-----|-------|-------------------------------|-----------------------------------|
| 2   | 0.0002258 | $10^{-8}$                     | 9                                 |
| 3   | 0.0000170 | $10^{-3}$                     | 16                                |
| 4   | 4.27 × $10^{-7}$ | $10^{-6}$ | 25                                |
| 5   | 3.21 × $10^{-8}$ | $10^{-7}$ | 36                                |
| 6   | 6.03 × $10^{-10}$ | $10^{-9}$ | 49                                |
| 7   | 4.54 × $10^{-11}$ | $10^{-10}$ | 64                                |

Table 1: Table containing the numerical values for the $(j,n)$-term in the series (42) for the option price ($S = 3800$, $K = 4000$, $r = 1\%$, $\sigma = 20\%$, $\tau = 1Y$). Only 16 terms are needed to attain a precision of $10^{-3}$, and 64 for a $10^{-10}$ precision.

In table 2, we plot the first terms of the series representation (42) under the form of lower triangular $(j,n)$-matrix (the first entry of the matrix is the $\frac{1}{2}(S - F)$ term) for the same set of parameters.

| $(j,n)$ terms | 0 | 1 | 2  | 3  | 4  | 5  | 6  | 7 |
|--------------|---|---|----|----|----|----|----|---|
| 0            | 119.900 | 315.978 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1            | 4.213 | 12.257 | 5.943 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 2            | 0.034 | 0.163 | 0.238 | 0.077 | -0.019 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 3            | 0.000 | 0.001 | 0.003 | 0.003 | 0.001 | -0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |

Table 2: Table containing the numerical values for the $(j,n)$-term in the series (42) for the option price ($S = 4200$, $K = 4000$, $r = 1\%$, $\sigma = 20\%$, $\tau = 1Y$). The call price converges to a precision of $10^{-5}$ after summing only very few terms of the series.
III.  APPROXIMATIONS AND HEDGE PARAMETERS

I. At-the-money price

The asset is said to be at the money forward if \( S = F \) (and therefore \( k = 0 \)). In this case, the series (42) becomes

\[
C_{\text{ATMF}} = \frac{S}{2} \sum_{j \geq 0} \frac{(-1)^n}{n! \Gamma\left(\frac{3}{2} + j - n\right)} Z^{2j + 1}
\]

This series is now a series of positive powers of \( Z \). It starts for \( n = 0, m = 1 \) and goes as follows:

\[
C_{\text{ATMF}} = \frac{S}{2} \frac{1}{\Gamma\left(\frac{3}{2}\right)} Z + O(Z^3)
\]

Recalling that \( Z = \frac{\sigma \sqrt{\tau}}{\sqrt{2}} \) and that \( \Gamma\left(\frac{3}{2}\right) = \frac{\sqrt{\pi}}{2} \) \[\text{(1)}\], we get

\[
C_{\text{ATMF}} = \frac{S}{\sqrt{2\pi}} \sigma \sqrt{\tau} + O((\sigma \sqrt{\tau})^3)
\]

As:

\[
\frac{1}{\sqrt{2\pi}} \simeq 0.399 \simeq 0.4
\]

we thus recover the well-known Brenner-Subrahmanyam approximation \[\text{(3)}\]:

\[
C_{\text{ATMF}} = 0.4 S \sigma \sqrt{\tau}
\]

II. Greeks

The boundary (47) shows that the series (42) converges normally to the call price \( C \), and therefore the greeks can be easily obtained by term by term differentiation of this series.

II.1 Delta

By definition of \( k \) we have \( \frac{\partial k}{\partial S} = \frac{1}{S} \) and therefore, by differentiating (42) with respect to \( S \) and rearranging the terms:

\[
\frac{\partial C}{\partial S} = \frac{1}{2} - \frac{1}{2} \frac{F}{S} \sum_{j \geq 1} \frac{(-1)^n}{n! \Gamma\left(\frac{1}{2} + j - n\right)} Z^{2j - 1} \left(1 - \frac{k}{Z^2}\right)^n
\]

In the ATM-forward configuration \( (S = F, k = 0) \) we are left with:

\[
\frac{\partial C}{\partial S} = \frac{1}{2} - \frac{1}{2} \sum_{j \geq 1} \frac{(-1)^{n+1}}{n! \Gamma\left(\frac{1}{2} + j - n\right)} Z^{2j - 1}
\]

\[
= \frac{1}{2} + \frac{1}{2\sqrt{\pi}} Z + O(Z^3)
\]
II.2 Rho

By definition of $F$ and $k$ we have $\frac{\partial F}{\partial r} = -\tau F$ and $\frac{\partial k}{\partial r} = \tau$ and therefore, by differentiating \[42\] with respect to $r$:

$$\frac{\partial C}{\partial r} = \frac{1}{2} \tau F \left[ 1 - \sum_{j \geq 0 \atop n \leq 2j} Z^{2j-1}(Z^2 - k + n) \frac{(-1)^n}{n!\Gamma(\frac{3}{2} + j - n)} \left( 1 - \frac{k}{Z^2} \right)^{n-1} \right]$$ \[(61)\]

In the ATM-forward configuration ($S = F, k = 0$) we are left with:

$$\frac{\partial C}{\partial r} = \frac{1}{2} \tau F \left[ 1 - \sum_{j \geq 0 \atop n \leq 2j} Z^{2j-1}(Z^2 + n) \frac{(-1)^n}{n!\Gamma(\frac{3}{2} + j - n)} \right]$$ \[(62)\]

$$= \frac{1}{2} \tau F \left[ 1 - \frac{1}{\sqrt{\pi}} Z + O(Z^2) \right]$$ \[(63)\]

II.3 Vega

By definition of $Z$ we have $\frac{\partial Z}{\partial \sigma} = \sqrt{\frac{\tau}{2}}$ and therefore, by differentiating \[42\] with respect to $\sigma$:

$$\frac{\partial C}{\partial \sigma} = \frac{F}{2} \sqrt{\frac{\tau}{2}} \sum_{j \geq 0 \atop n \leq 2j} Z^{2j-1}(Z^2 - k + n) \frac{(-1)^n}{n!\Gamma(\frac{3}{2} + j - n)} P_{j,n} \left( 1 - \frac{k}{Z^2} \right)^{n-1}$$ \[(64)\]

where

$$P_{j,n} = (Z^2 - k)(1 + 2j) + 2nk$$ \[(65)\]

In the ATM-forward configuration ($S = F, k = 0$) then $P_{j,n} = (1 + 2j)Z^2$ and we are left with:

$$\frac{\partial C}{\partial \sigma} = \frac{S}{\sqrt{\frac{\tau}{2}}} \sum_{j \geq 0 \atop n \leq 2j} (1 + 2j) Z^{2j} \frac{(-1)^n}{n!\Gamma(\frac{3}{2} + j - n)}$$ \[(66)\]

$$= \frac{S}{\sqrt{\frac{\tau}{2}}} \left[ \frac{1}{\sqrt{\pi}} - \frac{1}{4\sqrt{\pi}} Z^2 + O(Z^4) \right]$$ \[(67)\]

II.4 Theta

By definition of $Z, F$ and $k$ we have $\frac{\partial Z}{\partial \tau} = \frac{\sigma}{2\sqrt{\tau}}$, $\frac{\partial F}{\partial \tau} = -rF$, $\frac{\partial k}{\partial \tau} = r$ and therefore, by differentiating \[42\] with respect to $\tau$:

$$\frac{\partial C}{\partial \tau} = \frac{1}{2} rF + \sigma^2 F \sum_{j \geq 0 \atop n \leq 2j} Z^{2j-3} \frac{(-1)^n}{n!\Gamma(\frac{3}{2} + j - n)} Q_{j,n} \left( 1 - \frac{k}{Z^2} \right)^{n-1}$$ \[(68)\]

where

$$Q_{j,n} = \frac{1}{8} \left[ (1 + 2j - 2r\tau)(Z^2 - k) + 2n(k - r\tau) \right]$$ \[(69)\]
IV CONCLUDING REMARKS

In the ATM-forward configuration \((S = F, k = 0)\) then \(Q_{j,n} = \frac{1}{8} \left[(1 + 2j - 2r\tau)Z^2 - 2nr\tau\right]\) and we are left with:

\[
\frac{\partial C}{\partial \tau} = \frac{1}{2} rF + \sigma^2 F \sum_{j \geq 0} \sum_{n \leq 2j} Z^{2j-3} \frac{(-1)^n((1 + 2j - 2r\tau)Z^2 - 2nr\tau)}{8n!\Gamma(\frac{3}{2} + j - n)}
\]

(70)

\[
= \frac{1}{2} rF + \sigma^2 F \left[ \frac{1 - r\tau}{4\sqrt{\pi}} \frac{1}{Z} + O(Z) \right]
\]

(71)

IV. CONCLUDING REMARKS

In this paper, we have provided two new formulas for the European options in the Black-Scholes model, under the form of rapidly converging series whose terms are straightforward to calculate: formulas (19) and (42). Both are simple, but (42) is more tractable for practical applications thanks to his factorized structure in terms of odd powers of the variance; as this series is uniformly convergent, we were also able to differentiate it term by term, and obtain series expansions for the hedge parameters. Besides their novelty and own mathematical interest, these series can be very easily used in practice and allow to control the calculations to an arbitrary order of precision.
A. APPENDIX: MELLIN TRANSFORMS AND RESIDUES

We briefly present here some of the concepts used in the paper. The theory of the one-dimensional Mellin transform is explained in full detail in [8]. An introduction to multidimensional complex analysis can be found in the classic textbook [9], and applications to the specific case of Mellin-Barnes integrals is developed in [15, 16].

I. One-dimensional Mellin transforms

1. The Mellin transform of a locally continuous function $f$ defined on $\mathbb{R}^+$ is the function $f^*$ defined by

$$f^*(s) := \int_0^\infty f(x) x^{s-1} \, dx$$

(72)

The region of convergence $\{\alpha < \Re(s) < \beta\}$ into which the integral (72) converges is often called the fundamental strip of the transform, and sometimes denoted $<\alpha, \beta>$.

2. The Mellin transform of the exponential function is, by definition, the Euler Gamma function:

$$\Gamma(s) = \int_0^\infty e^{-x} x^{s-1} \, dx$$

(73)

with strip of convergence $\{\Re(s) > 0\}$. Outside of this strip, it can be analytically continued, except at every negative $s = -n$ integer where it admits the singular behavior

$$\Gamma(s) \sim_{s \to -n} (\frac{-1}{n!})^n \frac{1}{s+n} \quad n \in \mathbb{N}$$

(74)

3. The inversion of the Mellin transform is performed via an integral along any vertical line in the strip of convergence:

$$f(x) = \int_{c-i\infty}^{c+i\infty} f^*(s) x^{-s} \, \frac{ds}{2i\pi} \quad c \in (\alpha, \beta)$$

(75)

and notably for the exponential function one gets the so-called Cahen-Mellin integral:

$$e^{-x} = \int_{c-i\infty}^{c+i\infty} \Gamma(s) x^{-s} \, \frac{ds}{2i\pi} \quad c > 0$$

(76)

4. When $f^*(s)$ is a ratio of products of Gamma functions of linear arguments:

$$f^*(s) = \frac{\Gamma(a_1s + b_1) \ldots \Gamma(a_n s + b_n)}{\Gamma(c_1 s + d_1) \ldots \Gamma(c_m s + d_m)}$$

(77)

then one speaks of a Mellin-Barnes integral, whose characteristic quantity is defined to be

$$\Delta = \sum_{k=1}^{n} a_k - \sum_{j=1}^{m} c_j$$

(78)
\( \Delta \) governs the behavior of \( f^*(s) \) when \(|s| \to \infty \) and thus the possibility of computing (75) by summing the residues of the analytic continuation of \( f^*(s) \) right or left of the convergence strip:

\[
\begin{cases}
\Delta < 0 & f(x) = - \sum_{\text{Re}(s_N) > \beta} \text{Res}_{S_N} f^*(s)x^{-s} \\
\Delta > 0 & f(x) = \sum_{\text{Re}(s_N) < \alpha} \text{Res}_{S_N} f^*(s)x^{-s}
\end{cases}
\]  

(79)

For instance, in the case of the Cahen-Mellin integral one has \( \Delta = 1 \) and therefore:

\[
e^{-x} = \sum_{\text{Re}(s_n) < 0} \text{Res}_{s_n} \Gamma(s)x^{-s} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^n
\]

(80)
as expected from the usual Taylor series of the exponential function.

II. Multidimensional Mellin transforms

1. Let the \( \alpha_k, \xi_j \) be vectors in \( \mathbb{C}^2 \), and the \( b_k, d_j \) be complex numbers. Let \( t := \begin{bmatrix} t_1 \\ t_2 \end{bmatrix} \) and \( \xi := \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} \) in \( \mathbb{C}^2 \) and "." represent the euclidean scalar product. We speak of a Mellin-Barnes integral in \( \mathbb{C}^2 \) when one deals with an integral of the type

\[
\int_{\xi + \mathbb{R}^2} \omega
\]

(81)

where \( \omega \) is a complex differential 2-form who reads

\[
\omega = \frac{\Gamma(a_1 t_1 + b_1) \cdots \Gamma(a_n t_n + b_n)}{\Gamma(\xi_1 t_1 + d_1) \cdots \Gamma(\xi_m t_m + d_m)} x^{-t_1} y^{-t_2} \frac{dt_1}{2i\pi} \wedge \frac{dt_2}{2i\pi} x, y \in \mathbb{R}
\]

(82)
The singular sets induced by the singularities of the Gamma functions

\[
D_k := \{ t \in \mathbb{C}^2, a_k t_k + b_k = -n_k, n_k \in \mathbb{N} \} \quad k = 0 \ldots n
\]

(83)
are called the \textit{divisors} of \( \omega \). The \textit{characteristic vector} of \( \omega \) is defined to be

\[
\Delta = \sum_{k=1}^{n} a_k - \sum_{j=1}^{m} \xi_j
\]

(84)
and the \textit{admissible half-plane}:

\[
\Pi_\Delta := \{ t \in \mathbb{C}^2, \Delta t < \Delta \xi \}
\]

(85)

2. Let the \( \rho_k \) in \( \mathbb{R} \), the \( h_k : \mathbb{C} \to \mathbb{C} \) be linear applications and \( \Pi_k \) be a subset of \( \mathbb{C}^2 \) of the type

\[
\Pi_k := \{ t \in \mathbb{C}^2, \text{Re}(h_k(t_k)) < \rho_k \}
\]

(86)
A \textit{cone} in \( \mathbb{C}^2 \) is a cartesian product

\[
\Pi = \Pi_1 \times \Pi_2
\]

(87)
where \( \Pi_1 \) and \( \Pi_2 \) are of the type \( \text{(86)} \). Its \textit{faces} \( \varphi_k \) are

\[
\varphi_k := \partial \Pi_k \quad k = 1, 2
\]

(88)
and its distinguished boundary, or vertex is
\[ \partial_0 \Pi := \varphi_1 \cap \varphi_2 \]  
(89)

3. Let \( 1 < n_0 < n \). We group the divisors \( D = \bigcup_{k=0}^{n} D_k \) of the complex differential form \( \omega \) into two sub-families
\[ D_1 := \bigcup_{k=1}^{n_0} D_k \quad \text{and} \quad D_2 := \bigcup_{k=n_0+1}^{n} D_k \quad \text{such that} \quad D = D_1 \cup D_2 \]  
(90)

We say that a cone \( \Pi \subset \mathbb{C}^2 \) is compatible with the divisors family \( D \) if:
- Its distinguished boundary is \( \partial \)
- Every divisor \( D_1 \) and \( D_2 \) intersect at most one of his faces:
\[ D_k \cap \varphi_k = \emptyset \quad k = 1, 2 \]  
(91)

4. Residue theorem for multidimensional Mellin-Barnes integral [15][16]: If \( \Delta \neq 0 \) and if \( \Pi \subset \Pi_\Delta \) is a compatible cone located into the admissible half-plane, then
\[ \int_{\mathbb{C} + i \mathbb{R}^2} \omega = \sum_{\xi \in \Pi \cap (D_1 \cap D_2)} \text{Res}_\xi \omega \]  
(92)

and the series converges absolutely. The residues are to be understood as the "natural" generalization of the Cauchy residue, that is:
\[ \text{Res}_0 \left[ f(t_1, t_2) \frac{dt_1^{n_1}}{2i\pi t_1^{n_1}} \wedge \frac{dt_2^{n_2}}{2i\pi t_2^{n_2}} \right] = \frac{1}{(n_1-1)!(n_2-1)!} \frac{\partial^{n_1+n_2-2}}{\partial t_1^{n_1-1}\partial t_2^{n_2-1} f(t_1, t_2)|_{t_1=t_2=0}} \]  
(93)

REFERENCES

[1] Abramowitz, M. and Stegun, I., Handbook of mathematical functions, 1972, Dover Publications

[2] Andreasen, J., Jensen, B., Poulsen, R., Eight valuation methods in Financial Mathematics: The Black-Scholes formula as an example, 1998, Mathematical Scientist, 23(1), 18 – 40

[3] Brenner, M. and Subrahmanyam, M. G., A simple approach to option valuation and hedging in the Black-Scholes model, 1994, Financial Analysts Journal, 50(2), 25-28

[4] Black, F. and Scholes, M., The pricing of options and corporate liabilities, 1973, Journal of Political Economy, 81, 637

[5] Cox, J. C., Ross, S. A., Rubinstein, M., Option pricing: a simplified approach, 1979, Journal of Financial Economics, 7, 229-263

[6] Erdélyi, A. F., Magnus, W., Tricomi, G., Table of integral transforms, 1954, McGraw & Hill

[7] Estrella, A., Taylor, Black and Scholes: series approximations and risk management pitfalls, 1995, Federal Reserve Bank of New-York, Research paper #9501
[8] Flajolet, P., Gourdon, X., Dumas, P., Mellin transform and asymptotics: Harmonic sums, Theoretical Computer Science, 1995, **144**, 3–58

[9] Griffiths, P. and Harris, J., Principles of Algebraic Geometry, 1978, Wiley & sons

[10] Hull, J., Options, futures and other derivatives, 2008, Prentice Hall

[11] Mainardi, F, Pagnini, G. and Saxena, R., Fox H-functions in fractional diffusions, Journal of computational and applied mathematics, 2005, **178**, 321 – 331

[12] Nylsen, L. T., Understanding \( N(d_1) \) and \( N(d_2) \): risk adjusted probabilities in the Black-Scholes model, 2012, **14**, 95-106

[13] Rouah, F., Four derivations of the Black-Scholes PDE, available online at www.frouah.com

[14] Rubinstein, M., The valuation of uncertain income streams and the pricing of options, 1976, Bell journal of Economics, 7, 407 – 425

[15] Passare, M., Tsikh, A., Zhdanov, O., A multidimensional Jordan residue lemma with an application to Mellin-Barnes integrals, 1994, Aspects of Mathematics, **E26**, 233 – 241

[16] Passare, M., Tsikh, A., Zhdanov, O., Multiple Mellin-Barnes integrals as periods of Calabi-Yau manifolds, 1997, Theor. Math. Phys., **109**, 1544 – 1555

[17] Willmott, P., Paul Willmott on Quantitative Finance, 2006, Wiley & Sons

[18] Willmott, P., Frequently asked questions in Quantitative Finance, 2009, Wiley & Sons