Analysis of data sets of stochastic systems

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This paper deals with the analysis of data sets of stochastic systems which can be described by a Langevin equation. By the method presented in this paper drift and diffusion terms of the corresponding Fokker-Planck equation can be extracted from the noisy data sets, and deterministic laws and fluctuating forces of the dynamics can be identified. The method is validated by the application to simulated one- and two-dimensional noisy data sets.

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I. INTRODUCTION

In biological, economical, physical or technical systems noisy data sets occur very frequently. For describing and/or influencing these complex systems, it is necessary to know the deterministic laws and the strength of the fluctuations controlling the dynamics. Especially in the dynamics of order parameters close to instability points fluctuations play an important role [1], [2], [3]. This paper discusses a general method which allows one to determine drift and diffusion coefficients of the Fokker-Planck equation for stationary continuous complex systems with Markov properties. So a complete description of the stochastic process is found. This problem has also been addressed by [4] and [5], but in a different way.

II. BASIC RELATIONS

A. Langevin equation and Markovian property

The dynamics of a continuous Markovian system is governed by a Langevin equation [6] for a set of \( n \) variables \( q_k(t) \), \( k = 1, \ldots n \) [6]:

\[
\frac{d q_i(t)}{d t} = h_i\{q_k(t), t\} + \sum_j g_{ij}\{q_k(t), t\} \Gamma_j(t) \tag{1}
\]

\( k = 1, \ldots n \)

The fluctuating Langevin forces \( \Gamma_j(t) \) are considered to be \( \delta \)-correlated noise functions with vanishing mean as expressed in equation (2) and (3).

\[
< \Gamma_i(t) > = 0 \quad \forall i \tag{2}
\]

\[
< \Gamma_i(t) \Gamma_j(t') > = Q \cdot \delta_{ij} \delta(t-t') \quad \forall i, j \tag{3}
\]

The condition of Markovian property demands that the dynamics of the process depends only on the present state of the system and not on its history. With definition (4) for conditional probability density distributions \( p \), the condition for the validity of Markovian property can be expressed by equation (5)

\[
p(q_n, t_n | q_{n-1}, t_{n-1}; \ldots; q_1, t_1) = p(q_n, t_n | q_{n-1}, t_{n-1}), \tag{4}
\]

where \( t_{n-1} > t_{n-2} > \ldots > t_1 \),

B. Fokker-Planck equation

Usually a formal general solution of the stochastic differential Langevin equation (1) cannot be given. Therefore a Fokker-Planck equation (6) is set up by which the probability density distribution \( w(q_k(t), t) \) of the stochastic variables can be calculated [10]:

\[
\frac{\partial}{\partial t} w(q_k(t), t) = - \sum_{i=1}^{n} \frac{\partial}{\partial q_i} \left( D^{(1)}_i \{q_k(t), t\} w(q_k(t), t) \right) + \frac{1}{2} \sum_{i,j=1}^{n} \frac{\partial^2}{\partial q_i \partial q_j} \left( D^{(2)}_{ij} \{q_k(t), t\} w(q_k(t), t) \right) \tag{6}
\]

\( w(q_k(t), t) \) is the probability to find the system in \( q_i \ldots q_1 \) \( dq_i \) at time \( t_i \) for all \( i \).

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\( w(q_k(t), t) \) is the probability to find the system in \( q_i \ldots q_1 \) \( dq_i \) at time \( t_i \) for all \( i \).
The coefficients $D_{ij}^{(1)}$ are called drift coefficients, the terms $D_{ij}^{(2)}$ diffusion coefficients, they are defined in equation (9) and (10) [8].

$$D_{ij}^{(1)}(\{q(t)\}, t) = \lim_{\tau \to 0} \frac{1}{\tau} \langle \tilde{q}_j(t + \tau) - q_i \rangle_{\{\tilde{q}_k(t)\} = \{q_k\}}$$

$$D_{ij}^{(2)}(\{q_k(t)\}, t) = \lim_{\tau \to 0} \frac{1}{\tau} \langle (\tilde{q}_j(t + \tau) - q_j) \rangle_{\{q_k(t)\} = \{q_k\}}$$

where $\{\tilde{q}_k(t + \tau)\}$ with $\tau > 0$ is a solution of (11) which at time $t$ has the sharp value $\{\tilde{q}_k(t)\} = \{q_k\}$.

In [11] an application of these equations (7) and (8) together with the definitions of conditional probability density distributions (9) to the statistical properties of a turbulent cascade is discussed.

The relations between the sets of coefficients of the Langevin equation and the Fokker-Planck equation in consideration of the Stratonovich definitions are [8]

$$D_{ij}^{(1)}(\{q_k(t)\}, t) = h_i(\{q_k(t)\}, t)$$

$$+ \frac{Q}{2} \sum_{i,j} g_{ij}(\{q_k(t)\}, t) \frac{\partial}{\partial q_i} g_{ij}(\{q_k(t)\}, t)$$

$$D_{ij}^{(2)}(\{q_k(t)\}, t)$$

$$= Q \cdot \sum_k g_{it}(\{q_k(t)\}, t) g_{jt}(\{q_k(t)\}, t)$$

$$k = 1, \ldots n$$

III. ANALYSIS OF STOCHASTIC DATA SETS

A continuous Markovian system in the presence of white noise is governed by a Langevin equation (1) respectively by the corresponding Fokker-Planck equation (6). A complete analysis of such a complex system, therefore, should yield the quantities $h_i$ and $g_{ij}$ of the Langevin equation (1) respectively drift and diffusion coefficients $D_{ij}^{(1)}$ and $D_{ij}^{(2)}$ of the Fokker-Planck equation (6). Both sets of terms are according to (7) and (8) correlated with each other.

For the regarded class of stationary continuous Markovian systems with white dynamical noise, where the validity of the Markovian property may have been achieved by the introduction of delay coordinates, it is always possible to determine drift and diffusion terms directly from the data sets by using the equations (6), (7) and (8). Because of the stationarity, $D_{ij}^{(1)}$ and $D_{ij}^{(2)}$ have no explicit time dependence. The needed conditional probability density distribution can be determined numerically from the data set according to (4) by calculating histograms.

This general algorithm has apparently not been recognized in literature up to now. The numerical method is completely general, no ansatz for the coefficients is needed. If analytical formulas for the evolution equations of a process are needed, the gained numerical results may be approximated by analytical functions.
IV. APPLICATIONS OF THE PRESENTED ALGORITHM

In the following the discussed algorithm will be applied to various one- and two-dimensional synthetically determined data sets.

A. First example

The first example deals with the case of one-dimensional systems, following the Langevin equation (11)

\[ \dot{q}(t) = \varepsilon q(t) - q(t)^3 + \Gamma(t) \]  

which is valid for systems exhibiting noisy pitchfork bifurcations. Figure 1 presents a part of the calculated noisy time series.

![Figure 1: Variable q versus time t. The time series is calculated according to the Langevin equation (11)].

\[ D^{(1)}(q) = 0.1q - q^3 \]

\[ D^{(2)}(q) = 0.0025 \]

![Figure 2: Drift and diffusion coefficient \(D^{(1)}\) and \(D^{(2)}\) versus variable q. According to the discussed algorithm the conditional probability distribution of the noisy time series shown in figure 1 has been determined and drift and diffusion terms have been calculated. The dots exhibit the numerically determined values, the lined curves show the theoretical functions for the coefficients according to equation (9) and (10)].

B. Second example

The second example is concerned with the case of one-dimensional systems following the Langevin equation (12)

\[ \dot{\phi}(t) = \omega + \sin(\phi) + \Gamma(t) \]  

This type of equation describes the dynamics of a phase difference \(\phi = \phi_1 - \phi_2\), where \(\phi_1, \phi_2\) are the phases of two coupled nonlinear oscillators. For two different sets of parameters the noisy time series and calculated drift coefficients are shown in fig. 3 and 4. Because of the singularities of the time series there is only a small probability for the corresponding variable values. These rapid changes of the phase are responsible for the strong noise of the drift coefficient.
FIG. 3: Phase Difference $\phi$ versus time $t$ and drift coefficient $D^{(1)}$ versus $\phi$: The noisy time series belongs to the Langevin equation $\dot{\phi}(t) = 0.2 + \sin(\phi(t)) + 0.6 \cdot F(t)$ where $F(t)$ is a Gaussian distributed fluctuating force. The corresponding drift coefficient $D^{(1)}$ has been calculated and is plotted (dots) together with the theoretical function (lined curve).

Since the Langevin equation (12) is valid for a wide class of biological systems applications of the algorithm for many different fields are expected.

C. Third example

After these two examples of one-dimensional systems the case of two variables will be discussed. In the following examples the presented method will be applied on vectors with two components. The third example is based on the differential equation system of a Hopf bifurcation:

$$\frac{d}{dt} q_1 = \epsilon q_1 - \gamma q_2 + (q_1^2 + q_2^2)(\mu q_1 - \omega q_2)$$  \hspace{1cm} (13)

$$\frac{d}{dt} q_2 = \gamma q_1 - \epsilon q_2 + (q_1^2 + q_2^2)(\omega q_1 + \mu q_2)$$  \hspace{1cm} (14)

Fig. 5 shows the phase diagram of the $q_1$ and $q_2$. The parameters
are chosen in a way that the behaviour of the variables becomes supercritical, i.e. the focus (0,0) is instable and the trajectory moves towards a stable limit cycle [12]. This system of differential equations is very important for describing time-spatial signals whose dynamics are determined by two order parameters.

\[ \begin{align*}
\dot{q}_1 &= 0.05q_1 - q_2 + (q_1^2 + q_2^2)(-5q_1 - 7.5q_2) \\
\dot{q}_2 &= q_1 - 0.05q_2 + (q_1^2 + q_2^2)(7.5q_1 - 5q_2)
\end{align*} \]

A Gaussian distributed white noise weighted by a factor 0.2 has been added to the system described by equations (13) and (14). Afterwards the presented method has been applied. Fig. 6 shows the two numerically determined drift coefficients \( D^{(1)}_{q_1}(q) \) and \( D^{(1)}_{q_2}(q) \). The lined surface belongs to the calculated data, the dashed surfaces are plotted according to the expected surfaces:

\[ \begin{align*}
D^{(1)}_{q_1}(q) &= 0.05q_1 - q_2 + (q_1^2 + q_2^2)(-5q_1 - 7.5q_2) \\
D^{(1)}_{q_2}(q) &= q_1 - 0.05q_2 + (q_1^2 + q_2^2)(7.5q_1 - 5q_2)
\end{align*} \]

**D. Fourth example**

The fourth example deals with the case of a co-dimension II system. The dynamics of such a system is determined by the differential equations (16) and (17):

\[ \begin{align*}
\frac{d}{dt} q_1 &= q_2 + F_1(t) \\
\frac{d}{dt} q_2 &= \epsilon q_1 + \gamma q_2 + \mu q_1^3 + \omega q_1^2 q_2 + F_2(t)
\end{align*} \]

The parameters have been chosen as

\[ \begin{align*}
\epsilon &= 0.01, \quad \gamma = 0.03, \quad \mu = -1, \quad \omega = -1.
\end{align*} \]

The fluctuation force \( F(t) \) has been taken as a Gaussian distributed noise weighted by a factor 0.05.

Fig. 7 shows the phase diagram for this system without a fluctuating force \( F(t) \). The trajectory is repelled from the instable focus and moves towards a limit cycle.
FIG. 7: Variable $q_2$ versus variable $q_1$: The phase diagram shows the dynamics of a co-dimension II instability:
\[
\begin{align*}
\dot{q}_1 &= q_2 \\
\dot{q}_2 &= 0.02q_1 + 0.03q_2 - q_1^3 - q_1^2q_2
\end{align*}
\]

Fig. 8 illustrates the drift terms $D^{(1)}_{q_1}(q)$ and $D^{(1)}_{q_2}(q)$, calculated by the presented method, together with the expected terms for the noisy system (16)-(18). In Fig. 9 one of the calculated diffusion coefficients, $D^{(2)}_{q_1q_2}(q)$, and its theoretical function can be seen.

V. SUMMARY AND OUTLOOK

By the discussed method noisy data sets that obey a Fokker-Planck equation can be analysed and described. By a numerical determination of the conditional probability distributions drift and diffusion coefficients can be calculated. Consequently the deterministic laws and the weight of the fluctuating forces underlying the dynamics of the system can be extracted.

The algorithm has been illustrated on various one- and two-dimensional systems, whose noisy time series have been simulated and afterwards analysed. A good conformance between determined and expected drift and diffusion coefficients can be seen.

In biological, physical, technical and economical systems noisy data sets referring to a Fokker-Planck equation occur very frequently. The presented method offers the possibility of analysing these systems and describing them in a mathematical way without the need of any form of ansatz or assumption. Therefore, there seems to be a broad field of a possible use of this algorithm.

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