Fixed points for branched covering maps of the plane

by

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Abstract. A well-known result of Brouwer states that any orientation preserving homeomorphism of the plane with no fixed points has an empty non-wandering set. In particular, the existence of an invariant compact set implies the existence of a fixed point. In this paper we give sufficient conditions for degree 2 branched covering maps of the plane to have a fixed point, namely:

• A totally invariant compact subset that does not separate the critical point from its image.
• An invariant compact subset with a connected neighbourhood $B$ such that $\text{Fill}(B \cup f(B))$ does not contain the critical point nor its image.
• An invariant continuum such that the critical point and its image belong to the same connected component of its complement.

1. Introduction. The existence of periodic and fixed points for continuous maps of the plane has been extensively studied. A key theorem for the development of this area was given by Brouwer in 1912 [Brou12]:

Theorem 1.1. Let $f : \mathbb{C} \to \mathbb{C}$ be an orientation preserving homeomorphism such that $\text{Fix}(f) = \emptyset$. Then every point is wandering.

This result triggered a great amount of research, and addresses the simplest case of an open question for plane dynamics [Ste35]: Does a continuous function of the plane, taking a non-separating continuum into itself, necessarily have a fixed point?

To be precise, let us introduce the following notions:

Definition 1.2. Let $S$ be a surface, $f : S \to S$, $K$ a subset of $S$, and $f^{-1}(K) = \{ p \in S : f(p) \in K \}$. Then:

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• $K$ is an invariant set if $f(K) = K$,
• $K$ is a totally invariant set if $f^{-1}(K) = K$.

Cartwright and Littlewood [CL51] proved in 1951 that if $f$ is an orientation preserving homeomorphism of the plane with an invariant non-separating continuum, then it has a fixed point in it. A brief elegant proof of that result was given two decades later by Brown [Bro77].

Bell [Bel78] generalized this result to all homeomorphisms in 1978, and announced in 1984 that the theorem could be extended to all holomorphic maps (see also [Aki99])—note that these are a particular type of branched covering maps of the plane (see Section 2 for definitions). Kuperberg [Kup91] extended the previous theorem to orientation reversing homeomorphisms in 1991, removing the hypothesis of the continuum being non-separating.

More recently, the Cartwright–Littlewood theorem was further extended to all orientation preserving branched covering maps of the plane by Fokkink, Mayer, Oversteegen and Tymchatyn in 2008. Their proof can be found in [BF+13].

We want to find sufficient conditions for branched covering maps of the plane to have fixed (or periodic) points. In particular, we are interested in the following question: Does a branched covering map of the plane with an invariant compact subset have a fixed point?

In this paper we study the dynamics of degree 2 branched covering maps of the plane. Simple examples can be given of such maps that are periodic point free: see, for example, [BO09].

A similar problem for branched covering maps on the sphere was addressed by Iglesias, Portela, Rovella and Xavier [IP+20] in 2016; they proved the following result:

**Theorem 1.3.** Let $f$ be a branched covering map of the sphere of degree $d$, $|d| > 1$. Suppose there exists a simply connected open set $U$ whose closure is disjoint from the set of critical values, and such that $f^{-1}(U) \subset U$. Then $f^n$ has at least $d^n$ fixed points.

Note that in this case we can restrict the dynamics of $f$ to a branched covering map of the plane—with the same degree—and get a totally invariant non-separating compact set $K = \bigcap_{n \in \mathbb{N}} f^{-n}(U)$. Note that this set has infinitely many connected components.

The authors of [IP+16] looked for sufficient conditions on surface branched covering maps of degree $d$ to satisfy

$$\limsup_{n \to \infty} \frac{1}{n} \log(#\text{Fix}(f^n)) \geq \log(d).$$

Equivalently, it means there is a subsequence of iterates $n_k \to \infty$ such that $\text{#Fix}(f^{n_k})$ grows exponentially as $d^{n_k}$, which is the fastest growth to be
expected in a non-degenerate case. When that inequality is satisfied, the map is said to have the rate.

The same authors in 2016 gave sufficient conditions for covering maps of the annulus to have the rate (see [IP+16]), and as a consequence got the following results, which will be key tools in our proof.

**Theorem 1.4.** Let $f$ be a covering map of the annulus, of degree $d > 1$, such that there exists a compact totally invariant subset $K$. Then $f^n$ has at least $d^n - 1$ fixed points.

**Theorem 1.5.** Let $f$ be a covering map of the annulus, of degree $d > 1$, with an invariant essential continuum $K$. Then $f^n$ has at least $d^n - 1$ fixed points.

2. **Notation and preliminaries.** Throughout this paper, a surface $S$ will be a two-dimensional orientable topological manifold. We will say that $S$ is respectively a plane, an annulus or a sphere if it is homeomorphic to $\mathbb{C}$, $\mathbb{C}\setminus\{0\}$ or $S^2$. A set $U \subset S$ will be a disc if it is homeomorphic to $D = \{z \in \mathbb{C} : z < 1\}$. To lighten notation, we will define $m_k : \mathbb{C} \to \mathbb{C}$ as the map such that $m_k(z) = z^k$.

All maps considered in this paper are continuous.

**Definition 2.1.** Given $p, p' \in S$, a path or curve from $p$ to $p'$ is a function $\gamma : [0, 1] \to S$ such that $\gamma(0) = p$, $\gamma(1) = p'$. The path is simple when $\gamma$ is injective, and closed when $p = p'$. We will say that $\gamma$ is a segment, a line, or a circle if its image is homeomorphic to $[0, 1], \mathbb{R}, S^1$ respectively.

**Definition 2.2.** Given a surface $S$, an oriented topological foliation $\mathcal{F}$ is a partition of $S$ into one-dimensional manifolds such that for each $p \in S$, there exists a neighbourhood $U_p$ and a homeomorphism $h : U_p \to (-1, 1) \times (-1, 1)$ which preserves orientation and sends $\mathcal{F}$ into the foliation by vertical lines, oriented from bottom to top.

**Definition 2.3.** Let $U \subset \mathbb{C}$, $c \in U$, and $f : U \to \mathbb{C}$. We say that $f$ is geometrically conjugate to $m_k$ if there exist foliations $\mathcal{F}$ in $U \setminus \{c\}$ and $\mathcal{F}'$ in $f(U) \setminus \{f(c)\}$, and homeomorphisms $\phi : U \to \mathbb{D}$ and $\phi' : f(U) \to \mathbb{D}$ such that both $\mathcal{F}$ and $\mathcal{F}'$ are mapped into the radial foliation in $\mathbb{D}$ and the following diagram commutes:

$$
\begin{array}{ccc}
U & \xrightarrow{f} & f(U) \\
\downarrow{\phi} & & \downarrow{\phi'} \\
\mathbb{D} & \xrightarrow{m_k} & \mathbb{D}
\end{array}
$$

**Definition 2.4.** A branched covering map $f : S \to S$ (or simply a branched covering) is a map that is a local homeomorphism at each point
\( p \in S \), except for finitely many critical points, each having a neighbourhood on which \( f \) is geometrically conjugate to \( m_k \) with some \( k \in \mathbb{Z}^+ \).

In that context, the degree of a critical point will be \( k \). We define \( \text{Crit}(f) \) as the set of critical points of \( f \). Each point which is not the image of a critical point has the same number \( d \) of preimages, from which we deduce that the degree of \( f \) is \( d \) if \( f \) is orientation preserving, and \(-d\) if \( f \) is orientation reversing. If \( \text{Crit}(f) = \emptyset \), we say \( f \) is a covering map.

**Definition 2.5.** Let \( K \) be a compact subset of the plane. Then the filling of \( K \) (notation: \( \text{Fill}(K) \)) is the union of \( K \) and the bounded connected components of its complement.

**Definition 2.6.** Let \( B \) be a bounded open subset of the plane. We define \( \text{Fill}(B) = \text{Fill}(\overline{B}) \).

**Definition 2.7.** Let \( B \) be a bounded open subset of the annulus \( A \), and suppose that \( \overline{B} \) is contained in a disc of \( A \). The set \( \text{Fill}(B) \) is defined as follows. Take the universal covering \( \tilde{A} \) of \( A \), which is a plane, the covering projection \( \tilde{\pi} \), and a connected component \( \tilde{B} \) of \( \tilde{\pi}^{-1}(B) \), which is homeomorphic to \( B \). Then \( \text{Fill}(B) = \tilde{\pi}(\text{Fill}(\tilde{B})) \).

**3. Goals and sketch of the proofs.** The goal of this paper is to prove the following three results:

**Theorem 3.1.** Let \( f \) be a degree 2 branched covering map of the plane, and \( c \) the unique critical point of \( f \). Suppose there exists a compact set \( K \) with \( f^{-1}(K) = K \) such that \( c \) and \( f(c) \) belong to the same connected component of \( \mathbb{C} \setminus K \). Then \( \text{Fix}(f) \neq \emptyset \).

**Theorem 3.2.** Let \( f \) be a degree 2 branched covering map of the plane, and \( c \) the unique critical point of \( f \). Suppose that there is a compact invariant set \( K \) (not necessarily totally invariant), and that there exists a connected neighbourhood \( U \) of \( K \) such that \( \text{Fill}(U \cup f(U)) \) contains neither \( c \) nor \( f(c) \). Then \( \text{Fix}(f) \neq \emptyset \).

**Theorem 3.3.** Let \( f \) be a degree 2 branched covering map of the plane, and \( c \) the unique critical point of \( f \). Suppose there is an invariant continuum \( K \) that does not separate \( c \) from \( f(c) \). Then \( \text{Fix}(f) \neq \emptyset \). In addition, if \( c \) and \( f(c) \) belong to the unbounded connected component of \( \mathbb{C} \setminus K \), then \( f \) has a fixed point in \( \text{Fill}(K) \).

In order to prove these, we will dedicate the next section to build our main tool: a perturbation \( h \) of the map \( f \) with good properties, given below.

**Proposition 3.4.** Let \( f \) be a degree 2 branched covering map of the plane \( \mathbb{C} \), with a compact invariant set \( K \), and \( c \) the unique critical point of \( f \). Suppose there exists a path from \( f(c) \) to \( c \) which is contained in \( \mathbb{C} \setminus K \).
and contains no fixed point of $f$. Then there exists an open neighbourhood $A$ of $K$ and an orientation preserving homeomorphism $h : \mathbb{C} \to \mathbb{C}$ with $h|_A = \text{Id}$ and $\text{Fix}(h \circ f) \setminus \text{Fix}(f) = \{c\}$.

Let us outline the roadmap for the proof of this proposition.

Given that $c$ is a degree 2 critical point, we know that there exists a neighbourhood of $c$ such that $f$ is locally geometrically conjugate to $m_2 : \mathbb{C} \to \mathbb{C}$ (see Section 2 for definitions). In Lemma 4.1 we build a neighbourhood $U$ of $c$ with that property and such that $U \cap f(U)$ is nonempty but is also contained in $\partial U$. We use Lemma 4.2 to control the shape of $U$, letting us assume that $U = \mathbb{D}, c = 0$.

In Lemma 4.3 we then construct the domain $V$ of the perturbation $h$, which will be a neighbourhood of a path from $c$ to $f(c)$ which is contained in $U \cup f(U)$ except for one point (remember that $U \cap f(U) = \emptyset$). We then change coordinates one more time, so that the path becomes a horizontal line with $c$ on the left of $f(c)$, and the whole neighbourhood $V$ becomes a rectangle, in such a way that we can control the dynamics of $f|_V$.

The last part of the proof is to define the perturbation on the rectangle (it is the identity map on its boundary), which is heuristically the composition of a vertical contraction towards the path and a translation to the left on each horizontal line.

4. The good perturbation. The only condition we will require of $A$ is that itself and its iterated images are far from the critical point and its image; more precisely, if we define $A_2 := A \cup f(A) \cup f^2(A)$, then we need that

- $d(c, A_2) > 0, d(f(c), A_2) > 0$;
- both $c$ and $f(c)$ belong to the same connected component of $\mathbb{C} \setminus A_2$.

It is important to note that, as $f$ is continuous and $K$ is invariant, we can always find a neighbourhood $A$ as described (simply start by taking $A_2$ satisfying those two conditions, then take $A$ small enough so that its first two iterates are contained in $A_2$). For ease of notation, set $A_1 := A \cup f(A)$.

**Lemma 4.1.** There exists a disc $U$ containing $c$ such that

- $d(U, A_1) > 0$,
- $f|_U$ is geometrically conjugate to $m_2$,
- $\text{Fix}(f) \cap f(U) = \emptyset$,
- $\overline{U} \cap \overline{f(U)} \neq \emptyset$,
- $U \cap f(U) \subset \partial U$.

**Proof.** Since $f(c)$ is not a fixed point (its only preimage is $c$) and $d(c, A_2) > 0$, we may take a disc $U_0$ satisfying the first three properties: take a sufficiently small neighbourhood $W$ of $f(c)$ and define $U_0 = f^{-1}(W)$. Take
a path $\gamma : [0,1] \to \mathbb{C}$ from $f(c)$ to $c$ avoiding the fixed points of $f$ such that $d(\gamma, A_2) > 0$.

Let $\hat{\gamma}$ be the interior of the path, $\hat{\gamma} = \gamma \setminus \{c, f(c)\}$. Take $\varepsilon > 0$ such that $B(f(c), \varepsilon) \subset f(U_0)$, and $X^\varepsilon = \{z \in \mathbb{C} : d(z, \gamma) < \varepsilon\}$. Note that $\delta := \inf \{d(f(p), p) : p \in \gamma\} > 0$, because $\gamma$ is compact and passes through no fixed points of $f$. Then there exists a neighbourhood $V^\varepsilon$ of $\hat{\gamma}$, homeomorphic to $(0,1) \times (-\varepsilon, \varepsilon)$ by means of a homeomorphism $h^\varepsilon$, such that

- $d(V^\varepsilon, A_2) > 0$,
- $V^\varepsilon \subset X^\varepsilon$,
- $h^\varepsilon(t,0) = \gamma(t)$.
- $\text{Fix}(f) \cap V^\varepsilon = \emptyset$.

Note that $V^\varepsilon$ is a disc in $\mathbb{C}$, and can be seen as a fiber bundle with base $\hat{\gamma}$ (see Figure 1). We then define $V^\varepsilon_t := h^\varepsilon^{-1}((0,t) \times (-\varepsilon, \varepsilon))$ and $\tilde{V}_t := V^\varepsilon_t \cup f(U_0)$. Set

$$t_0 = \sup \{t \in [0,1] : \tilde{V}_t \cap f^{-1}(\tilde{V}_t) = \emptyset\}.$$  

Then we may define $U := f^{-1}(\tilde{V}_{t_0})$, and note that the desired intersection is nonempty and is included in the boundary, so $U$ satisfies the last two properties of the lemma. From the fact that $\text{Fix}(f) \cap V^\varepsilon = \emptyset$ and the construction of $U_0$, we find that $U$ satisfies the third property. Moreover, from $d(V^\varepsilon, A_2) > 0$ and the fact that $K$ is an invariant set, we infer that $U$ also satisfies the first property. Finally, since $f(c)$ is the image of the unique critical point, any preimage of a neighbourhood of $f(c)$ (such as $U$) will satisfy the second property: simply take any radial foliation in $f(U_0) \setminus \{f(c)\}$, and its preimage will be a foliation in $U_0 \setminus \{c\}$ as in the definition. ■

**Lemma 4.2.** Modulo a change of coordinates, we may assume that $c = 0$, $U = \mathbb{D}$, and that the radial foliation in $\mathbb{D} \setminus \{0\}$ is mapped 2 : 1 by $f$ into a foliation by lines in $f(U) \setminus f(0)$.
Proof. Since $\partial U$ is a simple closed curve by construction, the Jordan–Schoenflies theorem gives a homeomorphism $g : \mathbb{C} \to \mathbb{C}$ such that $g(U) = \mathbb{D}$, $g(\overline{U}) = \overline{\mathbb{D}}$ (we may assume $g(c) = 0$). Then $f' := g \circ f \circ g^{-1}$ is dynamically conjugate to $f$, so $f'|_\mathbb{D} : \mathbb{D} \to f'(\mathbb{D})$ is geometrically conjugate to $m_2$, and we obtain a foliation by lines $\mathcal{F}$ in $\mathbb{D} \setminus \{0\}$ as at the end of Section 2. Finally, we may conjugate $\mathcal{F}$ to the radial foliation: since each leaf $\phi \in \mathcal{F}$ gets out of $\mathbb{D}$, simply take the point of the leaf which is in the boundary of $\mathbb{D}$, and define the image of the leaf as the ray which goes through that point.

We will now define the domain of the perturbation, that is, $V \subset \mathbb{C}$ such that $h|_{\mathbb{C}\setminus V} = \text{Id}$. We start by taking a neighbourhood $U_0 \subset U$ of $c$ satisfying the first two properties of Lemma 4.1.

**Lemma 4.3.** There exists a disc $V$ such that

- $V \cap A_1 = \emptyset$,
- $V \cap U$ and $V \cap f(U)$ are discs,
- $U_0 \cup f(U_0) \subset V$,
- $f(V \cap U) \cap V = f(U_0)$,
- $f(V \setminus (U \cup f(U))) \cap V = \emptyset$.

**Proof.** Let $z \in \partial U \cap \partial f(U)$ (exists by Lemma 4.1). Then take $\hat{z} \in \partial U$ such that $f(\hat{z}) = z$. Let $\gamma$ and $\hat{\gamma}$ be the rays from $z$ to 0 and from 0 to $\hat{z}$ respectively, parametrized by arc length. We define $\gamma' := f(\hat{\gamma}) \cdot \gamma$.

For simplicity, take neighbourhoods $V^\varepsilon$ of the interior of $\gamma'$ as we did with $\gamma$ in Lemma 4.1 having local product structure, and with $V^\varepsilon \subset X^\varepsilon$ (see Figure 2). Note that there exists $\varepsilon_1$ such that $V^{\varepsilon_1} \cap A_1 = \emptyset$, because $d(\gamma', A_1) > 0$.

![Fig. 2. The domain of our perturbation is the union of the three darker coloured sets (red, blue and gray in the pdf file).](image-url)
We recall Lemma 4.1 and use Fix$(f) \cap \overline{f(U)} = \emptyset$ to conclude that $f(z) \neq z$. Then $z$ is the only point of $\gamma'$ which is outside the discs $U$ and $f(U)$, so there exists $\varepsilon_2$ such that $f(V^{\varepsilon_2} \setminus (U \cup f(U))) \cap V^{\varepsilon_2} = \emptyset$.

Moreover, $f(\gamma) \cap f(\hat{\gamma}) = f(c)$, so there also exists $\varepsilon_3$ such that $(f(V^{\varepsilon_3} \cap U) \cap V) \subset f(U_0)$. Finally, taking $\varepsilon_0 := \min\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$, we conclude that $V = U_0 \cup V^{\varepsilon_0} \cup f(U_0)$ satisfies the desired properties. 

**Remark 4.4.** With a new change of coordinates, we may assume that

- $V$ is the rectangle $(0,10) \times (-1,1)$,
- $U_0 = (0,2) \times (-1,1)$, $f(U_0) = (8,10) \times (-1,1)$,
- $c = (1,0)$, $f(c) = (9,0)$,
- $\gamma'$ is the segment between $c$ and $f(c)$,
- if $(x,y) \in U \cap V$ and $0 < x' < x$, then $(x',y) \in U \cap V$,
- if $(x,y) \in f(U) \cap V$ and $x < x' < 10$, then $(x',y) \in f(U) \cap V$.

Note that the last two properties state that there is a well-defined notion of left and right inside the rectangle, namely, $U \cap V$ is to the left of $f(U) \cap V$.

**Remark 4.5.** We may also require that in these coordinates, $f|_{U_0}$ sends the radial foliation to the radial foliation in $f(U_0)$, sending the ray with angle $\theta$ to the ray with angle $2\theta$. Moreover, since our change of coordinates is by conjugating a geometrical model by a radial map affine on each ray that sends a disc onto a square, we may assume that in the new coordinates, the map $f|_{U_0}$ is affine on each ray.

**Remark 4.6.** The expansion (or contraction) rate of $f|_{U_0}$ on each ray is uniformly bounded by the quotient between the longest and the shortest ray, which is $\sqrt{2}$.

We now proceed to build the perturbation $h$.

**Proof of Proposition 3.4** We begin by taking $h_1$, a perturbation to be extended as the identity outside $f(U_0)$, so we will define it on the square $(8,10) \times (-1,1)$. Heuristically, we want strong contraction towards the horizontal $y = 0$, but we need to adjust it so it becomes the identity on the boundary, so we will define a piecewise affine map. We impose symmetry with respect to the line $x = 9$, and make it preserve verticals, that is, $h_1(x,y) = (x,h_12(x,y))$, where

$$h_12(x,y) = \begin{cases} 
\frac{y}{4} & \text{if } \frac{81}{10} \leq x \leq \frac{99}{10}, |y| \leq \frac{1}{2}, \\
\frac{(7y-3)}{4} & \text{if } \frac{81}{10} \leq x \leq \frac{99}{10}, |y| > \frac{1}{2}, \\
(10x-99)y + (100-10x)y/4 & \text{if } x > \frac{99}{10}, |y| \leq \frac{1}{2}, \\
(10x-99)y + (100-10x)(7y-3)/4 & \text{if } x > \frac{99}{10}, |y| > \frac{1}{2}, \\
(81-10x)y + (10x-80)y/4 & \text{if } x < \frac{81}{10}, |y| \leq \frac{1}{2}, \\
(81-10x)y + (10x-80)(7y-3)/4 & \text{if } x < \frac{81}{10}, |y| > \frac{1}{2}.
\end{cases}$$
Note that we may extend $h_1$ as the identity to the boundary of $f(U_0)$. Next we define $h_2$ supported in $V$, preserving horizontals and sending $f(c)$ into $c$, in a similar fashion to how we defined $h_1$. Let $h_2(x, y) = (h_{21}(x, y), y)$, with

$$h_{21}(x, y) = \begin{cases} |y|x + (1 - |y|)x/9 & \text{if } x \leq 9, \\ |y|x + (1 - |y|)(9x - 80) & \text{if } x > 9. \end{cases}$$

Note that $h_2$ may also be extended as the identity to the boundary of $V$.

Let us define $h := h_2 \circ h_1$. It only remains to prove

**Lemma 4.7.** Fix$(h \circ f) \setminus \text{Fix}(f) = \{c\}$.

**Proof.** Let $w \in \text{Fix}(h \circ f) \setminus \text{Fix}(f)$. We start by proving $w \in U_0$.

Since $h$ is supported in $V$, we need $f(w) \in V$ (otherwise, we would get $h \circ f(w) = f(w)$). Furthermore, as $h$ sends $V$ into itself, we also need $w \in V$ (so that we can get $h \circ f(w) = w$). By the last property in Lemma 4.3 we have $f(V \setminus (U \cup f(U))) \cap V = \emptyset$, so $w \in U$ or $w \in f(U)$.

If $w \in f(U)$, then $f(w) \notin f(U)$ because we defined $U$ as the preimage of a set which we then renamed as $f(U)$ (see Lemma 4.1). Thus we deduce $h \circ f(w) = h_2 \circ f(w)$. Moreover, by the last property of Remark 4.4 and recalling that $h_2$ preserves horizontals and sends points to the left, we conclude that $w \notin \text{Fix}(h_2 \circ f)$. On the other hand, if $w \in U$, we use $f(V \cap U) \cap V = f(U_0)$ (Lemma 4.3), and recall the construction of $U_0$ in Lemma 4.1 as preimage of a set which is then renamed as $f(U_0)$, to conclude that $w \in U_0$.

Let $w = (\hat{x}, \hat{y})$. Note that necessarily $w \in h \circ f(U_0)$. Since $h_1$ is a bijection of $f(U_0)$, we obtain $h \circ f(U_0) = h_2 \circ f(U_0)$, and therefore

$$w \in (U_0 \cap (h_2 \circ f)(U_0)) = U_1.$$

In order to understand the shape of that intersection, it is enough to find $\partial U_0 \cap \partial (h \circ f(U_0))$. These points have $x$-coordinate $2$, and their preimages have $x$-coordinate $8$. Using the explicit form of $h_2$ we find that these two points are $(2, 5/32)$ and $(2, -5/32)$. We may then conclude that $|y| \leq 5/32 < \sqrt{2}/8$. On the other hand, any $w \in h \circ f(U_0)$ must have $x$-coordinate at least $8/9$: simply check this fact for points in $U_0 \cap \partial (h \circ f(U_0))$. The worst case is with $y = 0$, which gives us the lower bound $x = 8/9).$ We then get $w \in U_1 \subset [8/9, 2] \times [-\sqrt{2}/8, \sqrt{2}/8]$. Define

$$W_1 := \{(x, y) \in f(U_0) : x \leq 81/10\}, \quad W_2 := \{(x, y) \in f(U_0) : x \geq 99/10\}.$$

Observe the following:

- $f(U_1) \subset [8, 10] \times [-1/2, 1/2]$: this is because the map $\theta \mapsto 2\theta$ in the circle multiplies the $y$-coordinate by at most $2$. As $U_0$ and $f(U_0)$ are squares, we use Remark 4.6 to conclude that $f$ multiplies the $y$-coordinate by at most $2\sqrt{2}$. 

\[ f(U_1) \cap W_1 = \emptyset: \] let \( p \in U_1 \). If its angular coordinate in \( U_0 \) is no more than \( \pi/4 \), then by Remark 4.3 we get \( f(p) \notin W_1 \). Otherwise, \( d(p, c) \leq \sqrt{2} \cdot (\sqrt{2}/8) = 1/4 \), and by Remark 4.6 we get \( d(f(p), f(c)) \leq \sqrt{2}/4 \), and conclude that \( f(p) \notin W_1 \).

- \( h_2(W_2) \cap U_0 = \emptyset \): substituting \( x = 99/10 \) in \( h_{21} \), we get \( h_{21}(x, y) \geq 91/10 \), which proves this step.

Since outside the vertical stripes \( W_1, W_2 \), the map \( h_2 \) divides the height by 4 in \( f(U_1) \), we obtain \( w \in U_2 := h \circ f(U_1) \subset [0,2] \times [-1/8,1/8] \), so \( w \in U_2 \).

Proceeding inductively, we define \( U_n := h \circ f(U_{n-1}) \), then observe that \( U_n \subset [0,2] \times [-\sqrt{2^n}/2^{n+2}, \sqrt{2^n}/2^{n+2}] \), and use the fact that \( w \in \bigcap_{n \geq 0} U_n \) to deduce \( \dot{y} = 0 \). Furthermore, \( \dot{x} \geq 1 \) (since \( f(\rho, \theta) = (\rho, 2\theta) \)), so the candidates \( w \) are in the interval \( I \) with ends \( c \) and \( (2,0) \). It remains to observe that \( f|f(x,0) = (9x-8,0) \) has \( c = (1,0) \) as its unique fixed point, which concludes the proof.

**Remark 4.8.** As \( V \cap A_1 = \emptyset \), the dynamics in \( A \) remains unchanged, that is, \( h \circ f|A = f|A \).

We now give one last remark, which will be used in the proof of Theorem 3.2.

**Remark 4.9.** Let \( B \subset \mathbb{C} \) be an open set, and set \( C = \text{Fill}(B \cup f(B)) \). Suppose that, at the beginning of the construction of the perturbation in Lemma 4.1, the path \( \gamma \) is contained in \( \mathbb{C} \setminus C \). Then we can build the domain \( V \) of the perturbation \( h \) (as in Lemma 4.3) so that

- \( h(C) = \text{Fill}(B \cup (h \circ f(B))) \),
- \( c \notin h(C) \).

For the first claim, it is enough to prove \( V \cap B = \emptyset \). Recall Lemma 4.1 where the set \( U \) is built as the preimage of the union of a neighbourhood \( W \) of \( f(c) \), and a sufficiently small neighbourhood of \( \gamma \). Then, by Lemma 4.3 the set \( V \) is contained in \( U \cup f(U) \), except for a neighbourhood of one point \( z \), whose size may be controlled. Therefore, it is enough to take \( \tilde{V}_{t_0} \) (as in Lemma 4.1) that does not intersect \( C \). It is easy to do this as \( d(\gamma, C) > 0 \), because \( C \) is a compact set.

For the second claim, observe that \( h \) is a homeomorphism of the plane which preserves orientation. As \( f(c) \notin C \), we get \( c = h(f(c)) \notin h(C) \).

**5. Proof of the main results.** This section is devoted to the proofs of the three theorems stated in Section 3.

**Proof of Theorem 3.1.** The idea is to take a perturbation \( h \) as in Proposition 3.4. We start by taking for \( A \) any sufficiently small neighbourhood of \( K \),
because we only need the map $h$ not to modify the dynamics in $K$. Then the critical point becomes fixed, and we do not generate any other fixed point in this process. Since $h \circ f$ has degree 2, we have $(h \circ f)^{-1}(c) = c$, so we can puncture the plane at the critical point, and restrict the dynamics to a covering map $g$ of the open annulus $\mathbb{C} \setminus \{c\}$.

The set $K$ is then totally invariant for $g$, so we are under the hypothesis of Theorem 1.4 which lets us infer that $\text{Fix}(g) \neq \emptyset$. We then conclude that $\{c\}$ is a proper subset of $\text{Fix}(h \circ f)$, which finishes the proof. ■

In order to prove Theorem 3.2 we will use the following result, which can also be found in [LeC07].

**Lemma 5.1.** Let $f$ be an orientation preserving homeomorphism of the plane, with a compact invariant set $K$. Let $B$ be a connected neighbourhood of $K$. Then $f$ has a fixed point in the compact set $C = \text{Fill}(B \cup f(B))$.

**Proof.** We know that $f$ has at least one fixed point, by Theorem 1.1. Suppose that there is no fixed point in $C = \text{Fill}(B \cup f(B))$. Take $S = \mathbb{C} \setminus \text{Fix}(f)$. Since $\text{Fix}(f)$ is a (closed) invariant set, we may restrict $f$ to $S$. We may also assume that $S$ is connected (otherwise, $f$ preserves the connected component of $S$ which contains $K$).

We then consider the universal covering $\tilde{S}$ of $S$, which is a plane. Since $C$ is contained in a disc of $S$ (otherwise we take a slightly smaller $B$), every connected component of $\tilde{\pi}^{-1}(C)$ is homeomorphic to $C$. For the same reason, for any lift $\tilde{f} : \tilde{S} \rightarrow \tilde{S}$ of $f$, and any connected component $\tilde{B}$ of $\tilde{\pi}^{-1}(B)$, the set $\tilde{f}(\tilde{B})$ intersects exactly one connected component of $\tilde{\pi}^{-1}(C)$.

Therefore, if $\tilde{K}$ and $\tilde{B}$ are the lifts of $K$ and $B$ respectively contained in $\tilde{C}$, we may take a lift $\tilde{f}_0$ of $f$ such that $\tilde{f}_0(\tilde{B})$ only intersects $\tilde{C}$, thus having $\tilde{f}_0(\tilde{K}) = \tilde{K}$. Again by Theorem 1.1, $\tilde{f}_0$ has a fixed point, which is a contradiction. ■

**Lemma 5.2.** Let $f$ be a covering of the annulus of degree $d > 0$, $K$ a compact invariant set of $f$, $B$ a connected neighbourhood of $K$, and suppose that $B \cup f(B)$ is inessential (contained in a disc). Then $f$ has a fixed point in $C = \text{Fill}(B \cup f(B))$.

**Proof.** Consider the universal covering of the annulus, which is the plane. As in Lemma 5.1, any lift $\tilde{C}$ of $C$ is homeomorphic to $C$ (in particular it is bounded), and therefore if $\tilde{B}$ is a lift of $B$ contained in $\tilde{C}$, we may take a lift $\tilde{f}_0$ of $f$ such that $\tilde{f}_0(\tilde{B})$ is contained in $\tilde{C}$. Then, if $\tilde{K}$ is the lift of $K$ which is contained in $\tilde{C}$, we get $\tilde{f}_0(\tilde{K}) = \tilde{K}$. Note that $\tilde{B} \cup \tilde{f}_0(\tilde{B})$ is homeomorphic to $B \cup f(B)$.

As $\tilde{f}_0$ is an orientation preserving homeomorphism of the plane, we apply Lemma 5.1 to conclude that $\tilde{f}_0$ has a fixed point in $\tilde{C}$, so $f$ has a fixed point in $\tilde{\pi}(\tilde{C}) = C$. ■
Proof of Theorem 3.2. Suppose \( \text{Fix}(f) = \emptyset \). Since neither \( c \) nor \( f(c) \) belong to \( C = \text{Fill}(B \cup f(B)) \), we may build a perturbation in the same fashion as in Proposition 3.4 without altering the dynamics in a small neighbourhood \( A \) of \( K \). We now use Remark 4.9 and note that we may build a perturbation \( h \) such that \( c \not\in h(C) \).

We know that \( h \circ f \) has no new fixed points other than \( c \). As \( c \) is totally invariant, we can restrict the dynamics of \( h \circ f \) to the annulus \( S = \mathbb{C} \setminus \{c\} \), and get a covering of the annulus of degree \( d = 2 \). We apply Lemma 5.2 to deduce that \( h \circ f \) has a fixed point in \( h(C) \), which is then fixed by \( f \). 

Remark 5.3. If \( K \) is a non-separating continuum not containing \( c \) or \( f(c) \), the hypotheses of Theorem 3.2 are immediately verified. Since \( K \) is compact and \( f \) is continuous, we may control the size of \( \text{Fill}(B \cup f(B)) \) and conclude that the fixed point must be found in \( K \), thus giving an elementary proof—for degree 2—of a result mentioned in Section 1 and given in \([BF+13]\): a positively oriented branched covering map of the plane with an invariant non-separating continuum \( K \) has a fixed point in \( K \).

We now give the proof of the last of our three results.

Proof of Theorem 3.3. We divide the proof into two cases:

Case 1: The point \( c \) belongs to the unbounded component of \( \mathbb{C} \setminus K \). Since \( f \) is proper, the bounded components of \( \mathbb{C} \setminus K \) are mapped into other bounded components, and therefore \( \text{Fill}(K) \) is an invariant non-separating continuum. By the result stated in Remark 5.3, \( f \) has a fixed point in \( \text{Fill}(K) \).

Case 2: The point \( c \) belongs to a bounded component of \( \mathbb{C} \setminus K \). Suppose \( f \) has no fixed points. Since \( c \) and \( f(c) \) are in the same bounded connected component of \( \mathbb{C} \setminus K \), we are under the hypothesis of Proposition 3.4 and we may build the perturbation, again puncture the plane and take the annulus \( A = \mathbb{C} \setminus c \). Note that \( K \) is essential in \( A \), so by Theorem 1.5, \( h \circ f \rvert_A \) has a fixed point, which is also fixed by \( f \).

Remark 5.4. As in Theorem 3.2 we may apply this result when \( K \) is an invariant non-separating continuum. In Theorem 3.3 we actually do not need \( K \) to be non-separating, but we do need to impose that \( c \) and \( f(c) \) belong to the same component of \( \mathbb{C} \setminus K \).

6. Room for improvement. The result proved in this paper leads naturally to other questions:

- Can we build the perturbation so that it does no alter the set of periodic points of \( f \)? In that case, we will have proved that under the hypothesis of Theorem 3.1, \( f \) has the rate.
• Is it possible to improve the result of Theorem 1.4 and find the rate in $K$? Recalling that the perturbation we built does not alter the dynamics in that set, this improvement would ensure that under the hypothesis of Theorem 3.1 $f$ has the rate.

• Can we adjust the techniques to the case where the degree of $f$ is larger, and get similar results? (This is quite audacious a priori, given that the amount of critical points may grow, resulting in cases with more complicated dynamics).

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