Charmonium Spectrum from Quenched QCD with Overlap Fermions

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Abstract

We present the first study of the charmonium spectrum using overlap fermions, on quenched configurations. Simulations are performed on $16^3 \times 72$ lattices, with Wilson gauge action at $\beta = 6.3345$. We demonstrate that we have discretization errors under control at about 5%. We obtain 88(4) MeV for hyperfine splitting using $r_0$ scale, and 121(6) MeV using the $(1\bar{P} − 1\bar{S})$ scale. This paper raises the possibility that the discrepancy between the lattice results and the experimental value for charmonium hyperfine splitting can be resolved using overlap fermions to simulate the charm quark on lattice.

1 Introduction

Over the last few years, numerical simulations of chiral fermions have matured. The stage of testing has passed for simulating valence chiral fermions, and physically relevant results have been reported in lattice simulations. All the studies so far have concentrated on simulating light quarks. This is natural, as chiral symmetry plays an important role for small quark masses. However, the use of overlap fermions to simulate heavy as well as light quarks has been suggested in \cite{1}. In this paper we want to make the point that overlap fermions can also alleviate some problems related to simulating heavy quarks. Here we present the first quantitative study of a heavy quark system using overlap fermions. This opens the door for the simulation of experimentally more interesting heavy–light systems. Using the unequal mass Gell-Mann-Oakes-Renner relation as the renormalization condition, the renormalization factor in the heavy-light current can be determined non-perturbatively to a high precision for overlap fermions. \cite{1} This is important for computation of heavy-light decay constants.

We demonstrate the value of overlap fermions to simulate heavy quarks using hyperfine splitting in the charmonium system. It is known that with staggered quarks, there is an ambiguity about Nambu-Goldstone (NG) and non-NG modes for the $\eta_c$, resulting in widely different estimates of hyperfine splitting – 51(6) MeV (non-NG) and 404(4) MeV (NG) \cite{2}. NRQCD converges only slowly for charm \cite{3}. Including $O(v^6)$ terms changed the result from 96(2) MeV to 55(5) MeV. Wilson fermions have $O(a)$ errors. Hyperfine splitting is very sensitive to the coefficient of the correction term, $c_{SW}$. There are many studies \cite{2, 3, 4} using Wilson

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type valence quarks, including some with non-perturbative $c_{SW}$, and with continuum extrapolation. The quenched clover estimate of hyperfine splitting has stabilized around 70–75 MeV using $r_0$ scale [5, 6, 7], and a higher number of about 85 MeV using $(1P−1S)$ scale [5]. Results from a 2+1 dynamical simulation using tree-level $c_{SW}$ still fall short of the experimental value by about 20% [8].

Although costly to simulate, overlap fermions [9] have the following desirable features:

- Exact chiral symmetry on the lattice.
- No additive quark mass renormalization.
- No flavor symmetry breaking.
- No $O(a)$ error.
- The $O(m^2a^2)$ and $O(\Lambda_{QCD}ma^2)$ errors are also small, from dispersion relation and renormalization constants.

The first two features are especially significant for light quarks. Many exciting results at low quark masses have been reported using overlap fermions [10]. The last three features are more important for computing charmonium hyperfine splitting using overlap fermions. The last feature, demonstrated in [11], is an unexpected bonus in this regard. The key observation is that the discretization errors are only about 5% all the way up to $ma \approx 0.5$. In Fig. 1 we reproduce a plot of the speed of light, obtained from the pseudoscalar meson dispersion relation, as a function of $ma$ from Ref. [1]. This is obtained using a $16^3 \times 28$ lattice at a spacing of 0.20 fm. It is harder to study the dispersion relation on the configurations we use for this paper, because on the small volume lattice box we use, one unit of momentum corresponds to about 1.6 GeV. This is a huge amount of momentum, and as a result, the data is noisier. The effective energies for 0,1 and 2 units of lattice momentum are shown in Fig. 2. There is no clean plateau already for 2 units of momentum. This results in a large error bar for the energy corresponding to that momentum. Fig. 2 corresponds to $ma = 0.35$. For smaller values of $ma$, the data is even more noisy, and it is hard to obtain the speed of light reliably for smaller masses. However, it is expected that the deviation of speed of light from 1 is larger for higher values of $ma$. Fig. 3 shows percent deviation of the speed of light from unity, obtained from a fit to the dispersion relation as a function of quark mass using the equation

$$ (E(p)a)^2 = c^2(pa)^2 + (E(0)a)^2. $$

(1)

It is clear from this figure that we have discretization errors under control at about the 5–7% level near the charm mass, which is near $ma \approx 0.35$.

2 Simulation Details

Our simulations are performed on $16^3 \times 72$ isotropic lattices. We present results on 100 configurations. The Wilson gauge action is used at $\beta = 6.3345$. We use a multi-mass inverter to obtain propagators for 26 masses ranging from 0.020–0.85 in lattice units. Only five of these masses in the range 0.25–0.50 are used for this study.

Since overlap simulations are computationally expensive, it is important to choose the required residues carefully – blindly requiring extremely precise inversions is not the optimal use of computing resources. For overlap simulations, there are three relevant numbers: residue for eigenvectors projected out to reduce the condition number of the matrix to be inverted in the inner loop, residue for inner loop which computes the overlap operator, and residue for the outer loop which actually computes the quark propagators. For the lattices we use, we only need to project out about 15 eigenvectors, so we simply demand a very small residue, $10^{-10}$ for this step. Unlike this step, however, the inner and outer loop residues demanded affect the computational cost substantially. To determine what residue is good enough, we repeat quark propagator inversion for one spin, one color and one configuration, and compare the “pseudoscalar” two-point function for various quark masses. This is not a physical quantity since no trace over spin and color is performed, and no configuration average is taken – we are simply studying precision issues here. Comparing results for inner loop residue of $10^{-6}$ with those from inner loop residue of $10^{-7}$, we find no change for small quark masses.
Figure 1: This is a plot of the speed of light, $c$, obtained from the dispersion relation. It can be seen that the discretization errors are only a few percent till $ma \approx 0.5$. This data comes from a $16^3 \times 28$ lattice at a spacing of 0.20fm.

Figure 2: Effective energies for pseudoscalar mesons, for 0, 1 and 2 units of lattice momentum, from the $16^3 \times 72$ lattice, at $ma = 0.350$. The effective energy for 2 units of momentum is very noisy, as explained in text.
However for heavy quarks, the two-point function falls through many orders of magnitude, and becomes very small at the center of the lattice. To get this precisely, we find we need a small inner loop residue – $10^{-6}$ is not sufficient. In Fig. 4 we show the effect of inner loop residue on “pseudoscalar” propagators for heavy quarks. The curves are slightly shifted for clarity. For $ma = 0.450$, even an inner loop residue of $10^{-6}$ appears to be good enough. However, for a larger $ma = 0.630$, this residue is not good enough for $t > 30$. For our production runs, we choose an inner loop residue of $10^{-8}$ and outer loop residue of $10^{-5}$. We have tested outer loop residue of $10^{-7}$, two orders of magnitude better. This affects results at less than half percent level, so we deem outer loop residue of $10^{-5}$ to be sufficient. This residue of $10^{-5}$ is demanded for the lightest quark mass. Near the charm mass, the residue obtained through the multi-mass inversion algorithm is $≈ 2 \times 10^{-9}$.

3 Analysis

In this paper, we study five charmonium states shown in Table 1 – $\eta_c(1S_0)$, $J/\Psi(3S_1)$, $h_c(1P_1)$, $\chi^0_c(3P_0)$ and $\chi^1_c(3P_1)$. For the $P$ states, there are two possible operators – one (denoted by $\Gamma$) simply using appropriate $\gamma$ matrices and the other (denoted by $\Delta$) using a derivative as well as $\gamma$ matrices. We always use a $\Gamma$ operator for the source, because using a $\Delta$ operator for source would require additional inversions. (It is for this reason we do not study $\chi_2^c$. This state has no $\Gamma$ operator.) Using a $\Delta$ sink does not cost additional inversions. Thus for our $P$ state analysis, we have three possibilities – $\Gamma$, $\Delta$ or $\Gamma \Delta$. The last one is our notation for a simultaneous fit to both $\Gamma$ and $\Delta$ sink correlators.

The effective mass plots for the pseudoscalar and the vector states are shown in Fig. 5. The lower half of this figure shows the effective hyperfine splitting from the ratio of vector to pseudoscalar correlators. These show a long plateau to justify a single exponential fit. For the $P$ states, the effective masses are shown in Fig. 6. These have much larger error bars, but they are still flat. The data gets noisy beyond $t = 30$ and precision problems cannot be excluded for channels other than the pseudoscalar meson. We do not use time-slices beyond 30 in our fits.

We use two ways to set the scale – from $r_0$ (using 0.5 fm) and from the $(1\bar{P} - 1\bar{S})$ splitting in the charmonium system. The singlet $P$ mass $m_{h_c}$ is used for $\bar{P}$, and $(3m_{J/\Psi} + m_{\eta_c})/4$ for $\bar{S}$ mass. The $(1\bar{P} - 1\bar{S})$ scale analysis has three sub-cases, depending on which of $\Gamma$, $\Delta$ or $\Gamma \Delta$ fit is used for $h_c$. 

Figure 3: Percent deviation of the speed of light from unity, as a function of $ma$. This serves as an estimate of the percent discretization error. Near our charm mass, the discretization errors are about 5%.
Figure 4: Effect of inner loop precision on pseudoscalar propagators for heavy quarks. We study output of one spin and one color for a single configuration for this illustration. Curves are slightly shifted horizontally for clarity.

We present the $r_0$ results first. The lattice spacing for the $\beta$ we use is 0.0561 fm [11]. The experimental $m_{J/\psi}$ is used to set $m_c$ (in lattice units). Interpolation for $m_{J/\psi}$ as a function of $ma$ is shown in Fig. 7. A straight line fit is used. Interpolation for the hyperfine splitting is shown in Fig. 5. Knowing the charm mass and the scale, the hyperfine splitting in MeV can be determined. Our result for the hyperfine splitting using $r_0$ scale is 88(4) MeV. This is considerably higher than the quenched results from Wilson-type fermions. The spectrum obtained using $r_0$ scale is shown in Fig 9. The corresponding results can be found in Table 2.

The $(1\bar{P} - 1\bar{S})$ scale has the advantage that it is set within the charmonium system, using masses of physical particles, so it is expected to be more relevant for this system, and it is model independent. However, we have large errors on the $P$ states. Consequently, the scale set from $(1\bar{P} - 1\bar{S})$ splitting itself will have about 12% error, which is not included in the direct statistical errors on various masses quoted below. The interpolation for the $\Gamma\Delta$ fit for $m_{h_c}$ is shown in Fig. 7, along with the interpolations for $m_{J/\psi}$ and $m_{\eta_c}$. We also show $m_{h_c}$ obtained using $\Gamma$ and $\Delta$ fits on the same plot. It is clear from this plot that $m_{h_c}$ obtained from the three fits completely agree within error bars. However, the slight difference in $m_{h_c}$ in the three cases changes the scale, the charm mass and the hyperfine splitting values considerably.

In the case of the spin splitting scale, the determination of $a$ and $m_c a$ is entangled. The procedure we follow to disentangle these is as follows. As shown in Fig. 3 all hadron masses in lattice units are fitted to a straight line, $m_h a = A_h ma + B_h$. Lattice spacing $a$ and bare charm quark mass $m_c a$ are two unknowns; $m_{J/\psi}$ and $m(1\bar{P} - 1\bar{S})$ in physical units are the two inputs. We solve for $a$ and $m_c a$ to obtain values shown in Table 2. The charm masses obtained are indicated in Fig. 7. We would like to point out that while $m_c a$ in lattice units differs considerably in the three sub-cases of $(1\bar{P} - 1\bar{S})$ analysis, values for $m_c$ in GeV, tabulated in Table 2 cluster much tighter.

The value we obtain for the hyperfine splitting in MeV is extremely sensitive to the value used for the lattice spacing $a$. For a slightly smaller $a$, the hyperfine splitting in lattice units is considerably larger, since it falls rapidly with increasing $a$, as seen in Fig. 3. Converting this to physical units further increases the value. As a result, our results from the three sub-cases of $(1\bar{P} - 1\bar{S})$ analysis look quite different – 113(5) MeV using $\Gamma$, 121(6) MeV using $\Gamma\Delta$ and 144(9) MeV using $\Delta$. We would like to emphasize here that the
Table 1: Charmonium states. For the $P$ states, there are two possible interpolating fields, denoted by $\Gamma$ and $\Delta$. Experimental masses in GeV are shown.

Errors quoted are only direct statistical errors, and the errors on $a$ are large enough to bring these results in statistical agreement with each other.

Fig. 9 shows the charmonium spectrum obtained from both $r_0$ and $1\bar{P} - 1\bar{S}$ analysis. Agreement with the experimental values is much better for the $1\bar{P} - 1\bar{S}$ scale. The agreement with experimental numbers for all the particles studied is very reasonable, indicating that the discretization errors must indeed be small for overlap fermions. This is because the different mass differences are supposed to measure differently defined quark masses $M_1^2$, $M_2^2$, $M_E^2$, etc. [13]. The inequality of these quark masses implies discretization errors. If all the mass differences come out right, it would imply that $M_1 \approx M_2 \approx M_E$, and small discretization errors.

Finally we summarize the results in Table 2. The errors quoted are only statistical; the error on $a$ is not included. All masses are in GeV. Our value for the hyperfine splitting using $1\bar{P} - 1\bar{S}$ scale, and simultaneous fits to $\Gamma$ and $\Delta$ correlators actually agrees with experiment. This is fortuitous, because the contribution from dynamical fermions is not included, and may be significant. However, there is no real contradiction here, because we have substantial statistical and systematic errors, as detailed below:

|  | $2s+1L_J$ | $J^{PC}$ | field $\Gamma$ | field $\Delta$ | mass (GeV) |
|---|---|---|---|---|---|
| $\eta_c$ | $^1S_0$ | $0^{--}$ | $\bar{\psi}\gamma_5\psi$ | $-$ | 2.979 |
| $J/\psi$ | $^3S_1$ | $1^{--}$ | $\bar{\psi}\gamma_5\psi$ | $-$ | 3.097 |
| $h_c$ | $^1P_1$ | $1^{--}$ | $\bar{\psi}\sigma_{ji}\psi$ | $\bar{\psi}\gamma_5\Delta_i\psi$ | 3.526 |
| $\chi_{c0}$ | $^3P_0$ | $0^{++}$ | $\bar{\psi}\psi$ | $\bar{\psi}\Sigma_i\Delta_i\psi$ | 3.417 |
| $\chi_{c1}$ | $^3P_1$ | $1^{++}$ | $\bar{\psi}\gamma_5\gamma_5\psi$ | $\bar{\psi}(\gamma_i\Delta_j - \gamma_j\Delta_i)\psi$ | 3.511 |

Table 2: Charmonium spectrum (GeV). Only direct statistical errors are included; the statistical error on the lattice spacing $a$ and systematic errors are not included in this table.

1. Direct statistical errors: These are quoted in Table 2

2. Statistical error on $a$: In the $(1\bar{P} - 1\bar{S})$ scale, this is primarily due to the error on $h_c$ mass, which is about 53 MeV. This is about 12% of the physical $(1\bar{P} - 1\bar{S})$ mass difference of 458 MeV. Note, this error is absent when the scale is set using $r_0$. On the other hand, $r_0$ is a model dependent scale, and it can have comparable errors. It has been pointed out that 0.45 fm may be a better value to use for $r_0$ than 0.50 fm [13]. Using this value brings our $r_0$ results closer to the $(1\bar{P} - 1\bar{S})$ results.
3. Discretization errors: As explained in Section 1, this is estimated at about 5%, from the dispersion relation.

4. Finite volume errors: Our simulations are performed on a box size of only 0.8 fm, hence it is not inconceivable that the $P$ states have some finite volume errors. However, even this small box should be large enough for the $S$ state particles $J/\Psi$ and $\eta_c$.

5. Quenched approximation: Dynamical fermions are expected to increase the value of hyperfine splitting. This increase is about 20 MeV for the Wilson-type fermions [8]. On the other hand, a study with NRQCD [15] does not find significant contribution due to dynamical fermions.

6. Exclusion of OZI-suppressed diagrams: While a contribution of about 20 MeV cannot be ruled out, the contribution due to these appears to be small in the charm quark region [16]. Lattice calculations with smaller statistical and systematic errors are needed to settle this issue.

4 Summary

We have presented the first study of the charmonium spectrum using overlap fermions. We get a better agreement with the experimental spectrum using $1\bar{P} - 1\bar{S}$ scale rather than the $r_0$ scale. Our value for hyperfine splitting is 121(6) MeV and 88(4) MeV using $1\bar{P} - 1\bar{S}$ and $r_0$ scale respectively. This is considerably higher than the quenched clover results. This conclusion cannot be escaped even if it is argued that our $P$ state results are affected by finite volume errors. Unquenched overlap results with more statistics and somewhat larger box size may very well settle the charmonium hyperfine splitting issue.

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Figure 6: Effective masses for the P states. The filled circles correspond to $\Gamma$ operator and the open circles to $\Delta$ operator. The plots for different P states are shifted along the y-axis. These effective masses are rather noisy, and we use conservative error bars on our fitted results. Again, the plots are for $m_{a} = 0.35$.

Figure 7: We fit the meson masses linearly in quark mass. Fits are shown for $\eta_{c}$, $J/\psi$ and $h_{c}$ masses. All masses are in lattice units. $h_{c}$ masses obtained using $\Gamma$ and $\Delta$ operators are also shown, but the fit line is shown only for the $\Gamma\Delta$ fit.
Figure 8: Hyperfine splitting as a function of quark mass, with interpolation shown at $m_c a$.

Figure 9: Charmonium spectrum in physical units. Results from both $r_0$ and $1P - 1S$ scales are shown. Note, for the latter scale, a linear combination of $h_c$ and $\eta_c$ masses, along with the $J/\psi$ mass, is used for input.
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