Dynamic load identification of stochastic structures based on unscented transformation

Chongwen Wang and Chengbin Du*

Department of engineering mechanics, Hohai University, Nanjing, Jiangsu Province, 211100, China

*Corresponding author’s e-mail: cbdu@hhu.edu.cn

Abstract. In view of some unknown parameters in structures, which are described by the Gaussian distribution model in this paper, a new dynamic load identification method for stochastic structures is proposed based on unscented transformation (UT). Firstly, the convolution equation for solving structural response is discretized, and the sampling points, namely, sigma points of stochastic parameters are calculated according to the unscented transformation. Then, when the stochastic parameters take each sigma point, the dynamic load is calculated by the direct inversion method combined with the regularization method. Finally, the mean value and standard deviation of the identified load are obtained, and the coefficient of variation and the upper and lower bounds of the identified load are defined. The result of an example shows that compared with the Monte Carlo simulation (MCS) and perturbation method (PM), the proposed method has higher computational efficiency and accuracy.

1. Introduction
Accurate understanding of dynamic loads of structures is the premise of evaluating its state and safety. However, in many cases, due to the constraints of economic and technological conditions, the dynamic loads acted on many structures cannot be directly measured or the cost of measurement is very high, so the indirect method, namely, the identification of dynamic loads has gradually been developed.

The dynamic load identification technology was first developed in the military field in the 1970s [1]. So far, a variety of methods have been developed [2], including the direct inversion method, regularization method, Kalman filter method, neuron method, etc.. However, most of these methods are for deterministic structures. However, for more engineering structures, due to the influence of manufacturing, assembly and service life, the actual parameters are bound to have some errors compared with the design values, which makes structures uncertain. Therefore, in recent years, some methods combining the existing dynamic load identification methods with the uncertainty theory have been developed to be applied to the dynamic load identification of uncertain structures.

Han and Liu et al. [3,4] described the uncertain parameters by interval model, and expanded the dynamic load into linear functions of uncertain parameters through the first-order Taylor expansion. The upper and lower bounds of the identified load are determined by interval expansion calculation. Sun et al. [5] proposed an method for load identification of stochastic structures based on perturbation method. This method transforms the dynamic load identification problem of stochastic structures into two kinds of deterministc problems. Based on Gegenbauer polynomial expansion, Liu et al.[6] approximated stochastic parameters by stochastic variables or their derivatives with \( \lambda \) probability density function, and then the load identification problem of a stochastic structure is transformed into that of an equivalent deterministic structure.
Based on the unscented transformation, a new method for dynamic load identification of stochastic structures is proposed in this paper. Firstly, the convolution equation for solving structural response is established and is discretized according to sampling time points. Then, the sigma points of stochastic parameters are calculated according to the unscented transformation, and the dynamic load is obtained by the direct inversion method combined with the regularization method when the stochastic parameters take each sigma point. Finally, the mean and standard deviation of the identified load are obtained, and the coefficient of variation and the upper and lower bounds of the identified load are defined. In an example of a 10-DOF spring mass structure, the proposed method is compared with the Monte Carlo simulation and the perturbation method [5]. Meanwhile, the influence of stochastic parameters on the identified load is also discussed.

2. Discretization of the convolution equation for solving structural response

For a linear single-DOF damped structure, the dynamic equation can be expressed as follows:

\[ m\ddot{y}(t) + c\dot{y}(t) + ky(t) = f(t) \]  

in which, \( m \), \( c \) and \( k \) are the mass, damping and stiffness respectively, \( f(t) \) is the dynamic load, and \( y(t) \), \( \dot{y}(t) \) and \( \ddot{y}(t) \) are the displacement, velocity and acceleration respectively. \n
Supposing the initial velocity and displacement are both zero, the convolution equation for solving the response can be obtained by the Duhamel integral:

\[ y(t) = \int_0^t (1/m\omega_d) \exp[-\xi\omega(t-\tau)]\sin[\omega_d(t-\tau)]f(\tau)d\tau \]  

in which, \( \xi = c / 2km \), which is the damping ratio, \( \omega = (k/m)^{1/2} \), which is the natural frequency, and \( \omega_d = \omega(1-\xi^2)^{1/2} \), which is the damped natural frequency.

Supposing \( H(t) = (1/m\omega_d) \exp[-\xi\omega(t)]\sin(\omega_d t) \), which is the impulse response function, equation (2) can be discretized according to the sampling time points:

\[
\begin{bmatrix}
  y(t_1) \\
  y(t_2) \\
  \vdots \\
  y(t_Q)
\end{bmatrix} =
\begin{bmatrix}
  H(t_1) & 0 & \cdots & 0 \\
  H(t_2) & H(t_1) & \cdots & 0 \\
  \vdots & \vdots & \ddots & \vdots \\
  H(t_Q) & H(t_{Q-1}) & \cdots & H(t_1)
\end{bmatrix}
\begin{bmatrix}
  f(t_0) \\
  f(t_1) \\
  \vdots \\
  f(t_{Q-1})
\end{bmatrix}
\]  

in which, \( \Delta t \) is the sampling time interval, \( Q \) is the number of sampling time points, \( t_k = k\Delta t \), which is the \( k \)-th sampling time, and \( y(t_k) \), \( f(t_k) \) and \( H(t_k) \) are the values of the response, dynamic load and impulse response function at \( t_k \) respectively.

For a multi-input and multi-output linear structure, the response of each DOF is the linear superposition of the response caused by each load, which can be expressed as follows:

\[
\begin{bmatrix}
  y_1 \\
  y_2 \\
  \vdots \\
  y_M
\end{bmatrix} =
\begin{bmatrix}
  H_{11} & H_{12} & \cdots & H_{1N} \\
  H_{21} & H_{22} & \cdots & H_{2N} \\
  \vdots & \vdots & \ddots & \vdots \\
  H_{M1} & H_{M2} & \cdots & H_{MN}
\end{bmatrix}
\begin{bmatrix}
  f_1 \\
  f_2 \\
  \vdots \\
  f_N
\end{bmatrix}
\]  

in which, \( M \) is the number of measured points, \( N \) is the number of dynamic loads, \( y_i \) (\( i = 1, 2, \cdots, M \)) is the response vector of the \( i \)-th measured point, \( f_j \) (\( j = 1, 2, \cdots, N \)) is the \( j \)-th load vector and \( H_{ij} \) (\( i = 1, 2, \cdots, M \), \( j = 1, 2, \cdots, N \)) is the corresponding impulse response matrix.
3. Unscented transformation
Supposing the mean vector and variance matrix of an $n$-dimensional stochastic vector $\mathbf{x}$ are respectively $\bar{\mathbf{x}}$ and $\mathbf{P}_x$, $\mathbf{y}$ is obtained by the nonlinear transformation $f$ of $\mathbf{x}$, namely:

$$\mathbf{y} = f(\mathbf{x})$$

(5)

Firstly, $2n+1$ sigma points of $\mathbf{x}$ are constructed:

$$\mathbf{x}_0 = \bar{\mathbf{x}}$$

(6)

$$\mathbf{x}_i = \bar{\mathbf{x}} + [\sqrt{(n+\lambda)}\mathbf{P}_x]_i, \quad i = 1, 2, \ldots, n$$

(7)

$$\mathbf{x}_i = \bar{\mathbf{x}} - [\sqrt{(n+\lambda)}\mathbf{P}_x]_i, \quad i = n+1, n+2, \ldots, 2n$$

(8)

in which, $[\sqrt{(n+\lambda)}\mathbf{P}_x]_i$ represents the $i$-th row or $i$-th column of $\sqrt{(n+\lambda)}\mathbf{P}_x$, $\lambda = \alpha^2(n+\kappa) - n$, $\alpha$ is usually a smaller positive number ($10^{-4} < \alpha \leq 1$), and $\kappa$ usually takes $3-n$ or 0.

Then, substituting $\mathbf{x}_i$ ($i = 0, 1, \ldots, 2n$) into equation (5) to obtain $\mathbf{y}_i$ ($i = 0, 1, \ldots, 2n$), the mean vector $\bar{\mathbf{y}}$ and variance matrix $\mathbf{P}_y$ of $\mathbf{y}$ can be obtained by weighted calculation:

$$\bar{\mathbf{y}} = \sum_{i=0}^{2n} W_i^{(m)} \mathbf{y}_i$$

(9)

$$\mathbf{P}_y = \sum_{i=0}^{2n} W_i^{(c)} (\mathbf{y}_i - \bar{\mathbf{y}})(\mathbf{y}_i - \bar{\mathbf{y}})^T$$

(10)

in which, the superscript $'T'$ represents matrix transpose, $W_i^{(m)}$ and $W_i^{(c)}$ are both the weighted coefficients, and

$$W_0^{(m)} = \lambda / (\lambda + n)$$

(11)

$$W_i^{(m)} = 1 / [2(\lambda + n)] \quad i = 1, 2, \ldots, 2n$$

(12)

$$W_0^{(c)} = \lambda / (\lambda + n) + (1 - \alpha^2 + \beta)$$

(13)

$$W_i^{(c)} = 1 / [2(\lambda + n)] \quad i = 1, 2, \ldots, 2n$$

(14)

in which, for the Gaussian distribution, the optimal value of $\beta$ is 2, but in the case of a single stochastic variable, the optical value of $\beta$ is 0.

4. Identification of dynamic loads of stochastic structures based on unscented transformation
Supposing the structure contains $n$ stochastic parameters, represented by $\mathbf{p} = [p_1, p_2, \ldots, p_n]^T$, equation (4) can be simplified into the following form:

$$\mathbf{y} = \mathbf{H}(\mathbf{p})f$$

(15)

Firstly, the sigma point $\mathbf{p}_i$ ($i = 0, 1, \ldots, 2n$) of $\mathbf{p}$ is constructed according to the unscented transformation, and accordingly, the impulse response matrix $\mathbf{H}(\mathbf{p}_i)$ ($i = 0, 1, \ldots, 2n$) is calculated; then, we can get:

$$\mathbf{y} = \mathbf{H}(\mathbf{p}_i)f_i = \mathbf{H}_i f_i \quad i = 0, 1, \ldots, 2n$$

(16)

Because the measured response always contains some noise and matrix inversion is usually accompanied by the ill-posed problem, the Tikhonov regularization method [7] is used here to suppress the influence of noise on the identification load:

$$\min J(f_i, \lambda) = \|\mathbf{H}_i f_i - \mathbf{y}\|^2 + \lambda \|f_i\|^2 \quad i = 0, 1, \ldots, 2n$$

(17)

in which, $\lambda > 0$, which is the regularization parameter and can be determined by general cross-validation (GCV) criterion[8] or L-curve criterion[9]; and $\|\|^{\prime}$ represents the 2-norm of a vector.
Then, \( f_i \) can be obtained by the least square method:

\[
f_i = (H_i^T H_i + \lambda E)^{-1} H_i^T y \quad i = 0, 1, \cdots, 2n
\]

where \( E \) is an identity matrix.

Finally, the mean vector \( \bar{f} \) and variance matrix \( P_f \) of the identified load \( f \) can be obtained:

\[
\bar{f} = \sum_{i=0}^{2n} W_i^{(m)} f_i
\]

\[
P_f = \sum_{i=0}^{2n} W_i^{(m)} (f_i - \bar{f}) (f_i - \bar{f})^T
\]

5. Evaluation of the identified load

The standard deviation vector \( S_f = [S_1 S_2 \cdots S_Q]^T \) of the identified load can also be obtained:

\[
S_i = \sqrt{(P_f)_{ii}} \quad i = 1, 2, \cdots, Q
\]

Generally, for a stochastic variable \( x \) with the mean value \( \bar{x} \) and standard deviation of \( S_x \), the following equation is usually adopted to define the coefficient of variation \( CV_x \):

\[
CV_x = \frac{S_x}{\bar{x}} \times 100\% 
\]

Similarly, the coefficient of variation \( CV_f \) of the identified load can also be defined:

\[
CV_f = \frac{\| S_f \|}{\| \bar{f} \|} \times 100\% 
\]

In addition, as we all know, for a Gaussian distribution, 99.7% of the values of a stochastic variable are distributed within 3 standard deviations to the mean value. Therefore, to more intuitively reflect the influence of stochastic parameters on the identified load, the upper and lower bounds of the identified load are respectively defined as:

\[
f_{up} = \bar{f} + 3S_f
\]

\[
f_{low} = \bar{f} - 3S_f
\]

6. An example

In this section, a 10-DOF damped spring mass structure is used as the calculation example, as shown in figure 1.

![Figure 1. A 10-DOF spring mass structure with damping.](image_url)

The stiffness design values \( k_1 = k_2 = \cdots k_{11} = 10^5 \text{N/m} \), and the mass design values \( m_1 = m_2 = \cdots m_{10} = 10\text{kg} \). The damping is assumed to be proportional, and the damping ratios of the first two orders are both 0.03, and then it can be obtained that the mass matrix correlation coefficient of the damping matrix is 0.36s\(^{-1}\), and the stiffness matrix correlation coefficient of the damping matrix is 2.24\(\times\)10\(^{-4}\).s.

In addition, a sinusoidal load as shown in figure 2 is acted at \( m_3 \) in the horizontal direction and the responses at \( m_1 \) and \( m_6 \) in the horizontal direction are used to identify the dynamic load. Meanwhile, 5% Gaussian white noise is added into the response.
6.1. $k_4$ and $k_7$ are stochastic and independent of each other

In this section, $\beta$ takes 2, and because the number of stochastic parameters is 2, $\kappa$ takes 1. Supposing the coefficients of variation $CV_k$ of $k_4$ and $k_7$ are both 10%, the errors of the standard deviation of the identified load between the proposed method and Monte Carlo simulation under different $\alpha$ are listed in Table 1.

Table 1. Errors (%) of the standard deviation of the identified load between the proposed method and Monte Carlo simulation under different $\alpha$.

| $\alpha$ | 0.1  | 0.2  | 0.3  | 0.4  | 0.5  | 0.6  | 0.7  | 0.8  | 0.9  | 1.0  |
|----------|------|------|------|------|------|------|------|------|------|------|
| $CV_k$ 10% | 2.61 | 2.53 | 2.40 | 2.22 | 1.98 | 1.70 | 1.35 | 0.96 | 0.51 | 0.11 |

Moreover, the method proposed in literature [5] based on the perturbation method is also used to calculate the example, and when $CV_k$ takes 10%, the error of the standard deviation of the identified load between the perturbation method and Monte Carlo simulation is 3.59%. Compared with the results in Table 1, the errors of the proposed method in this paper are all clearly smaller, especially when $\alpha$ takes a larger value, which shows that the proposed method has higher accuracy.

6.2. All stiffness are stochastic

In this section, two cases will be discussed: all stiffness are completely correlated and all stiffness are independent of each other.

For the case where all stiffness are completely correlated, when $CV_k$ takes 10%, the upper and lower bounds of the identified load are shown in Figure 2.(a), and the coefficient of variation of the identified load is 10.37%. For the case where all stiffness are independent of each other, when $CV_k$ takes 10%, the upper and lower bounds of the identified load are shown in Figure 2.(b), and the coefficient of variation of the identified load is 3.95%.

![Figure 2](image)

Figure 2. The upper and lower bounds of the identified load.

It is very clear that when all the stiffness are completely correlated, it has a great influence on the identified load. However, when all stiffness are independent of each other, the influence on the identified load is far less than the former case.

7. Conclusions

In this paper, a new method for dynamic load identification of stochastic structures is proposed based on the unscented transformation. Through the selection of sigma points, this method transforms the stochastic problem into several deterministic problems and is far superior to the Monte Carlo simulation in computational efficiency. Moreover, the result of the example shows that the accuracy of
the proposed method is better than the perturbation method. The discussion on the stochastic parameters on the identified load can also provide some guidance for engineering.

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