Dirac fermions in an inhomogeneous magnetic field

Ahmed Jellal\textsuperscript{1,2,3} and Abderrahim El Mouhafid\textsuperscript{3}

\textsuperscript{1} Physics Department, College of Science, King Faisal University, PO Box 380, Alahsa 31982, Saudi Arabia
\textsuperscript{2} Saudi Center for Theoretical Physics, Dhahran, Saudi Arabia
\textsuperscript{3} Theoretical Physics Group, Faculty of Sciences, Chouaib Doukkali University, PO Box 20, 24000 El Jadida, Morocco

E-mail: jellal@pks.mpg.de, ahjellal@kfut.edu.sa and elmouhafid@gmail.com

Received 27 April 2010, in final form 12 September 2010
Published 29 November 2010
Online at stacks.iop.org/JPhysA/44/015302

Abstract

We study a confined system of Dirac fermions in the presence of an inhomogeneous magnetic field. Splitting the system into different regions, we determine their corresponding energy spectrum solutions. We underline their physical properties by considering the conservation energy where some interesting relations are obtained. These are used to discuss the reflexion and transmission coefficients for Dirac fermions and check the probability condition for different cases. We generalize the obtained results to a system with gap and make some analysis. After evaluating the current-carrying states, we analyze the Klein paradox and report interesting discussions.

PACS numbers: 73.63.−b, 73.23.−b, 11.80.−m

(Some figures in this article are in colour only in the electronic version)

1. Introduction

The Dirac formalism plies an important role not only from mathematical point of view but physical one as well. The recent observation of the anomalous quantum Hall effect in graphene \cite{1, 2} renewed the interest to this formalism. In fact, many questions, raised in graphene, found their solutions by adopting the Dirac formalism as cornerstone. Among them, we cite the confinement \cite{3} that is much needed to describe the transport properties in graphene, the subject that attracted much attention where interesting developments appeared when dealing with different issues, for instance refer to \cite{4, 5}.

On the other hand, the quantum wires (electron waveguides) with quantized conductance can be formed in graphene \cite{6}. Such electron waveguides are indispensable parts of any conceivable all-graphene device. In lithographically formed graphene ‘ribbons’, the electronic bandstructure is theoretically expected to very sensitively depend on the width and on details
of the boundary [7]. On top of that, disorder and structural inhomogeneity are substantial in real graphene [8]. For narrow graphene ribbons or electrostatically formed graphene wires [9], conventional conductance quantization thus seems unlikely [10]. This expectation is in accordance with recent experiments [11].

Magnetic barrier technology is well developed [12–14] and its application to graphene samples appears to pose no fundamental problems [6]. In fact, snake states are experimentally studied in other materials [12, 15], mainly motivated by the quest for electrical rectification. On the theory side, the confined Schrödinger fermions in the magnetic field (with $B' = 0$) are discussed [16] as well as the asymmetric cases [17]. For the Dirac–Weyl quasiparticles encountered in graphene, however, such calculations are not reported. The inhomogeneous magnetic field case in graphene is analyzed in [18]. Theoretically, the electron waveguides, in graphene created by suitable inhomogeneous magnetic fields, is considered [6]. The properties of unidirectional snake states are discussed. For a certain magnetic field profile, two spatially separated counter-propagating snake states are formed, leading to conductance quantization insensitive to backscattering by impurities or irregularities of the magnetic field.

Subsequently, the tunneling effect of two-dimensional Dirac fermions in a constant magnetic field is studied [19]. This is done by using the continuity equation at fixed points to determine the corresponding reflexion and transmission coefficients. For this, a system made of graphene, as superposition of two different regions where the second is characterized by an energy gap $t'$ is considered. In fact, concrete systems are treated to practically give two illustrations: barrier and diode. For each case, the transmission in terms of the ratio of the energy conservation and $t'$ is discussed. Moreover, the resonant tunneling by introducing a scalar Lorentz potential is analyzed where it is shown that a total transmission is possible.

Motivated by the above progress and in particular [6, 19], we deal with other features of the system considered in [6]. Such a system is composed of different regions submitted to two magnetic fields and confined to a constant potential. This allows us to treat each region separately by determining the corresponding energy spectrum solutions. We underline some physical properties of their spectrum by taking into account of the energy conservation where interesting relations are obtained. Using the continuity at different points, we explicitly evaluate the reflexion and transmission coefficients. Combining all, we show that the probability condition is well verified. As the second task, we consider the present system with energy gap $t'$ and do the same to derive its eigenspinors as well as eigenvalues. It is shown that even the reflexion and transmission coefficients take new forms in terms of gap but the probability condition still verified. Interesting limits are discussed, which concern total reflexion and transmission of the system with gap.

Finally, we treat the Klein paradox by using the current-carrying states where different limits and discussions are presented. More precisely, we evaluate the currents for each region and use their relations to the reflexion and transmission coefficients to check the probabilities. Subsequently, three different cases are considered, which correspond to week, intermediate and strong potentials. We note that two last cases are shown negative transmissions. However, by combining all coefficients we end up with a sum equal to unity.

This paper is organized as follows. In section 2, we consider a confined Dirac fermion in an inhomogeneous magnetic field (1). After getting the eigenvalues and eigenfunctions, we analyze the energy conservation that allows us to derive interesting relations between involved quantum numbers and parameters. In section 3, we study scattering between two regions to determine the reflexion and transmission coefficients, which will be used to discuss the probability conditions of the present system. We do the same job in section 4 but by considering three regions where the first is equivalent to the third. The continuity at each point leads to express the coefficients entering in the game in terms of different parameters. In
section 5, we introduce a gap like a mass term and analyze the tunneling effect of such a case. We study the Klein paradox in section 6 by involving the currents corresponding to different regions and consider three cases. Finally, we close by concluding our work.

2. Dirac fermions in an inhomogeneous magnetic field

We consider a system of massless Dirac fermions through a strip of graphene characterized by the length $d$ and width $W$ in the presence of the inhomogeneous magnetic field. More precisely, we introduce two magnetic fields $B$ and $B'$, such as

$$B(x) = \begin{cases} 
  B, & x < -d \\
  B', & |x| < d \\
  B, & x > d.
\end{cases}$$

According to configuration (1), we decompose the present system into three regions. Schematically, we end up with figure 1. Clearly, regions I and III are similar but different with respect to region II. Note that the system characterized by figure 1 has been analyzed in [6] for possible quantum wires in graphene. However, in the present work we study other features of such a system to deal with different issues, which concern the tunneling effect and Klein paradox.

2.1. Dirac Hamiltonian

Before writing down the appropriate Hamiltonian of the system (figure 1), let us derive the corresponding gauge field to configuration (1). Indeed, using the continuity of the potential to obtain

$$A_j(x) = \begin{cases} 
  A_{I}(x) = Bx + (B - B')d, & x < -d \\
  A_{II}(x) = B'x, & |x| < d \\
  A_{III}(x) = Bx - (B - B')d, & x > d,
\end{cases}$$

where $j$ is labeling regions I, II and III. It is clear that for $B = B'$ we end up with one potential and therefore three regions become similar to each other.

In the systems made of graphene, the two Fermi points, each with a two-fold band degeneracy, can be described by a low-energy continuum approximation with a four-component envelope wavefunction whose components are labeled by a Fermi-point pseudospin $\pm 1$ and a sublattice forming an honeycomb. Specifically, the Hamiltonian for the one-pseudospin component for the present system can be written as

$$H_j = \mu_F \vec{\sigma} \cdot \vec{\pi}_j + V_j(x),$$

where the components of the conjugate momentum $\vec{\pi}_j = \vec{p} + e \vec{A}_j$ are given by

$$\pi_{x,j} = p_x, \quad \pi_{y,j} = p_y + \frac{e}{c} A_j(x).$$
and $V_j(x)$ is the potential barrier that has a rectangular shape, which is infinite along the $y$-axis and has the form

$$V_j(x) = \begin{cases} V_0, & -d < x < d \\ 0, & \text{otherwise}, \end{cases}$$

where $V_0 > 0$. Substituting all into (3) we get

$$H_j = v_F \left( \begin{array}{cc} \frac{V_j(x)}{v_F} & p_x - ip_y - i\xi A_j(x) \\ p_x + ip_y + i\xi A_j(x) & \frac{V_j(x)}{v_F} \end{array} \right).$$

(6)

At this stage, it is convenient to introduce the annihilation and creation operators. They can be defined as

$$a_j = ip_x + p_y + \frac{e}{c} A_j(x), \quad a_j^\dagger = -ip_x + p_y + \frac{e}{c} A_j(x)$$

(7)

which obey the canonical commutation relations

$$[a_j, a_j^\dagger] = \begin{cases} \frac{2\hbar^2}{I_2}, & j = I \\ \frac{2\hbar^2}{I_2|B|}, & j = II \\ \frac{2\hbar^2}{I_3}, & j = III \end{cases}$$

where the magnetic lengths $I_B = \sqrt{\frac{hc}{eB}}$ and $I_{|B|} = \sqrt{\frac{hc}{e|B|}}$ are corresponding to the magnetic fields $B$ and $B'$, respectively. The Hamiltonian (6) can be written in terms of $a_j$ and $a_j^\dagger$ as

$$H_j = i\hbar\omega_c \left( \begin{array}{cc} 0 & -a_j \\ a_j^\dagger & \frac{V_j(x)}{i\hbar} \end{array} \right)$$

(9)

which is encoding all regions. This will be used to study each region separately and derive the corresponding energy spectrum solutions.

### 2.2. Energy spectrum solutions

We determine the eigenvalues and eigenspinors of the Hamiltonian $H_j$. Indeed, the Dirac Hamiltonian describing region I is obtained from (9) as

$$H_I = i\hbar\omega_c \left( \begin{array}{cc} 0 & -a_I \\ a_I^\dagger & \frac{V_I(x)}{i\hbar} \end{array} \right).$$

(10)

The operators $a_I$ and $a_I^\dagger$ can be rescaled to define others, such as

$$b_I = \frac{I_B}{\sqrt{2\hbar}} a_I, \quad b_I^\dagger = \frac{I_B}{\sqrt{2\hbar}} a_I^\dagger,$$

(11)

which verify

$$[b_I, b_I^\dagger] = \mathbb{I}.$$

(12)

Using these we write $H_I$ as

$$H_I = i\hbar\omega_c \left( \begin{array}{cc} 0 & -b_I \\ b_I^\dagger & 0 \end{array} \right).$$

(13)

where we have set $\omega_c = \sqrt{2\frac{e}{I_B}}$ as cyclotron frequency.
To get the energy spectrum solutions of (13), we solve the eigenvalue equation for a given spinor $\psi = (\psi_1 \psi_2)$ of $H_I$. This is

$$H_I \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = E_I \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix},$$

which is equivalent to

$$i\hbar \omega_c \begin{pmatrix} 0 & -b_I \\ b_I^\dagger & 0 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = E_I \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix},$$

and leads to two relations between spinor components

$$-i\hbar \omega_c b_I \psi_2 = E_I \psi_1$$

(16)

$$i\hbar \omega_c b_I^\dagger \psi_1 = E_I \psi_2.$$

(17)

Now substitute (16) into (17) to obtain a differential equation of second order for $\psi_2$:

$$\hbar^2 \omega_c^2 b_I^\dagger b_I \psi_2 = E_I^2 \psi_2.$$

(18)

It is clear that $\psi_2$ is an eigenstate of the number operator $\hat{n} = b_I^\dagger b_I$ and therefore we identify $\psi_2$ to the eigenstates of the harmonic oscillator $|n\rangle$, namely

$$\psi_2 \sim |n\rangle$$

(19)

and its eigenvalues read

$$E_{I,n} = \hbar \omega_c \sqrt{|n|},$$

(20)

where $n$ is obviously the eigenvalues of $\hat{n}$, with $n = 0, \pm 1, \pm 2, \ldots$, and $s = \text{sgn}(n)$. Note that $n = 0$ corresponds to the lowest Landau level, i.e. zero mode energy.

Use (16), (19) and (20) to get the first component as

$$\psi_1 = -\frac{i\hbar \omega_c}{E_I} b_I |n\rangle = -i s |n - 1\rangle$$

(21)

which gives the eigenspinors

$$\phi_{I,n} \sim \begin{pmatrix} -s \sqrt{n-1} \\ |n\rangle \end{pmatrix}.$$

(22)

In terms of the parabolic cylinder functions $D_n(x + x_{01})$ [20], the eigenspinors in the plane $(x, y)$ are

$$\phi_{I,n,k}(x, y) = \frac{1}{\sqrt{2}} \begin{pmatrix} -s i D_{|n|-1}(x + x_{01}) \\ D_n(x + x_{01}) \end{pmatrix} e^{iky},$$

(23)

where $x_{01} = k_i l_B^2 + (1 - |B'|)d$ and $D_n(x)$ are related to Hermite polynomials via

$$D_n(x) = (l_B \sqrt{\pi n} 2^n)^{-1/2} \exp \left( -\frac{x^2}{2} \right) H_n(x).$$

(24)

As far as region II is concerned, we use the mapping $n \rightarrow m$ and $s \rightarrow s'$ in (23) to obtain the eigenspinors $\phi_{II,m,k}(x, y)$ in terms of $x_{02} = q_i l_B^2 |B'|$ and the corresponding eigenvalues

$$E_{II,m} = s' \hbar \omega_c' \sqrt{|m|} + V_0$$

(25)

where $s' = \text{sgn}(m)$ and $\omega_c' = \sqrt{2 \xi_{x'}}$ is the cyclotron frequency associated with the magnetic field $B'$. 


Finally, for region III the eigenvalues and the eigenspinors are similar to those of region I except that the correspondence

\[ x_{01} \rightarrow x_{03} = k_y l_B^2 - \left( 1 - \frac{|B'|}{B} \right) d \]  

must be taken into account in (23). Note that for \( B' = -B \) (with \( B' < 0 \)) the eigenspinors for three regions can be expressed with the same position \( x_0 = k_y l_B^2 \).

### 2.3. Illustrations

To give some illustrations, we focus on the eigenfunctions of four lowest states in region II, which are summarized in table 1. Different plots are given in figure 2, which show that the \( m \)th eigenfunction has \( m \) nodes, namely there are \( m \) values of \( x \) for which \( \phi_{II,m}(x) = 0 \).

On the other hand, the wavefunctions have observable properties. Indeed, if the position coordinate is changed from \( x \) to \( -x \), the eigenfunction has a definite symmetry

\[
\begin{align*}
\phi_{II,m}(-x) &= \pm \phi_{II,m}(x) & \text{if } m \text{ is even} \\
\phi_{II,m}(-x) &= -\phi_{II,m}(x) & \text{if } m \text{ is odd},
\end{align*}
\]
Figure 3. The position probability densities $|\phi_{m}(x)|^2$ corresponding to table 1.

Table 1. Normalized eigenfunctions for four lowest states of a one-dimensional potential energy field.

| Number | Energy eigenvalue | Energy eigenfunction |
|--------|------------------|---------------------|
| $m = 0$ | $E_{0,0} = V_0$ | $\phi_{0,0}(x) = \left(\frac{1}{\sqrt{\pi|B'|}}\right)^{1/2} e^{-\frac{(x+x_0)^2}{2|B'|^2}}$ |
| $m = 1$ | $E_{1,1} = \hbar \omega' + V_0$ | $\phi_{1,1}(x) = \left(\frac{1}{\sqrt{\pi|\sigma'|\sqrt{\pi|B'|}}}\right)^{1/2} \left[2 \left(\frac{x+x_0}{\sqrt{\pi|B'|}}\right)^2 - 2\right] e^{-\frac{(x+x_0)^2}{2|B'|^2}}$ |
| $m = 2$ | $E_{2,2} = \sqrt{2}\hbar \omega' + V_0$ | $\phi_{2,2}(x) = \left(\frac{1}{\sqrt{\pi|\sigma'|\sqrt{\pi|B'|}}}\right)^{1/2} \left[4 \left(\frac{x+x_0}{\sqrt{\pi|B'|}}\right)^3 - 12 \left(\frac{x+x_0}{\sqrt{\pi|B'|}}\right)\right] e^{-\frac{(x+x_0)^2}{2|B'|^2}}$ |
| $m = 3$ | $E_{3,3} = \sqrt{3}\hbar \omega' + V_0$ | $\phi_{3,3}(x) = \left(\frac{1}{\sqrt{\pi|\sigma'|\sqrt{\pi|B'|}}}\right)^{1/2} \left[8 \left(\frac{x+x_0}{\sqrt{\pi|B'|}}\right)^4 - 12 \left(\frac{x+x_0}{\sqrt{\pi|B'|}}\right)^2\right] e^{-\frac{(x+x_0)^2}{2|B'|^2}}$ |

which is nothing but the parity symmetry. Furthermore, the position probability density of fermion is given by

$$|\phi_{m}(x, y)|^2 = |\phi_{m}(x)|^2.$$  

(28)

By plotting this for some specific values of $m$, we deduce an interesting conclusion. From figure 3, we note that the fermion can have any location between $x = -\infty$ and $x = +\infty$, in marked contrast with a classical fermion, which is confined to the region $-A < x < +A$, where $A$ is the amplitude of the oscillation.
Figure 4. Variation of the ratio \( \frac{E}{V_0} \) in terms of \( \frac{|B'|}{B} |m| \) for \( E > 0 \) and \( \frac{B}{B'} |n| \) for \( E < 0 \).

Table 2. Positive and negative energies and their corresponding quantum number configurations.

| Energy | Quantum Numbers |
|--------|-----------------|
| \( E > 0 \) \( (s = s' = +1) \) | \( |n| > \frac{|B'|}{B} |m| \) |
| \( E < 0 \) \( (s = s' = -1) \) | \( |n| < \frac{|B'|}{B} |m| \) |

2.4. Energy conservation

In the interface between regions, there is conservation of the tangent components of the wave vector, i.e. \( k_y = q_y \), and conservation of the energy. This is

\[
E_I = E_{\parallel} = E,
\]

which leads to the constraint

\[
\frac{|n|}{|m|} = \frac{|B'|}{B} \frac{E^2}{(E - V_0)^2}.
\]

Since \( n \) and \( m \) are integer values, the r.h.s term must be a fractional number, which can be written as

\[
|n| = K |m|, \quad K \in \mathbb{Q}^+.
\]

This relation is very important because without such set one cannot talk about the tunneling effect in the present case. We will clarify this statement from the next section and exactly when we begin by calculating different quantities in order to check the probability condition.

Now let us return to (20) and (25) to write the ratio as

\[
\frac{E}{V_0} = \frac{\sqrt{|n|}}{\sqrt{|m|} - \sqrt{\frac{|B'|}{B} |m|}}.
\]

Recall that in region II we have \( V_0 > 0 \), which implies that the energy \( E \) can be either positive or negative. Therefore, we should distinguish between two situations as listed below.

These energies can be plotted to explicitly illustrate their behavior in terms of different quantities entering in the game, which are given in figure 4.
J. Phys. A: Math. Theor. 44 (2011) 015302  A Jellal and A El Mouhafid

3. Two regions in an inhomogeneous magnetic field

To treat a concrete example of the present system, we consider a barrier submitted to an inhomogeneous magnetic field. This barrier can be seen as the superposition of two regions separated by an interface localized at a fixed point. We study the tunneling effect by evaluating the reflection and transmission coefficients at interface, which in our case corresponds to the point zero. The coefficients will be used to show that the probability condition is exactly one and emphasis what makes difference with respect to without confinement case \[19\]. To perform this task, we follow the same lines as has been done in \[19\] and distinguish between propagation with positive and negative incidences. In both cases we deal with propagation from region I to II, II to III and vice versa.

3.1. Propagation with positive incidence

To proceed, let us first define the eigenspinors for three regions in the positive and negative direction of the variable \(x\). In region I, we write

\[
\phi_{I,+} = \frac{1}{\sqrt{2}} \left( \begin{array}{c} -s_i D_{\mu[-1]}(x + x_{01}) \\ D_{\nu}(x + x_{01}) \end{array} \right) e^{ik_x y},
\]

\[
\phi_{I,-} = \frac{1}{\sqrt{2}} \left( \begin{array}{c} -s_i D_{\mu[-1]}(-x - x_{01}) \\ D_{\nu}(-x - x_{01}) \end{array} \right) e^{ik_x y}.
\]

Similarly, in region II we have

\[
\phi_{II,+} = \frac{1}{\sqrt{2}} \left( \begin{array}{c} -s_i D_{\mu[-1]}(x + x_{02}) \\ D_{\nu}(x + x_{02}) \end{array} \right) e^{ik_x y},
\]

\[
\phi_{II,-} = \frac{1}{\sqrt{2}} \left( \begin{array}{c} -s_i D_{\mu[-1]}(-x - x_{02}) \\ D_{\nu}(-x - x_{02}) \end{array} \right) e^{ik_x y}.
\]

as well as in region III

\[
\phi_{III,+} = \frac{1}{\sqrt{2}} \left( \begin{array}{c} -s_i D_{\mu[-1]}(x + x_{03}) \\ D_{\nu}(x + x_{03}) \end{array} \right) e^{ik_x y},
\]

\[
\phi_{III,-} = \frac{1}{\sqrt{2}} \left( \begin{array}{c} -s_i D_{\mu[-1]}(-x - x_{03}) \\ D_{\nu}(-x - x_{03}) \end{array} \right) e^{ik_x y},
\]

where \((\pm)\) refer to the positive and negative propagations, respectively.

We start by analyzing the case of propagation from region I to region II. Indeed, at the interface \(x = 0\) (for all \(y\)), the continuity of the system gives

\[
\phi_{I,+} + r_{nm}^+ \phi_{I,-} = t_{nm}^+ \phi_{II,+},
\]

where \(r_{nm}^+\) and \(t_{nm}^+\) are the reflection and transmission coefficients, respectively, in positive propagation. From (33) and (34), we obtain

\[
\left( \begin{array}{c} -s_i D_{\mu[-1]}(x_{01}) \\ D_{\nu}(x_{01}) \end{array} \right) + r_{nm}^+ \left( \begin{array}{c} -s_i D_{\mu[-1]}(-x_{01}) \\ D_{\nu}(-x_{01}) \end{array} \right) = t_{nm}^+ \left( \begin{array}{c} -s_i D_{\mu[-1]}(x_{02}) \\ D_{\nu}(x_{02}) \end{array} \right).
\]

They can be solved to get the coefficients in terms of some constants, which are magnetic field dependent. They are

\[
r_{nm}^+(x_{01}, x_{02}) = (-1)^{|\mu|} \frac{sA_1 - s'B_1}{sA_1 + s'B_1},
\]

\[
t_{nm}^+(x_{01}, x_{02}) = \frac{2sC_1}{sA_1 + s'B_1}.
\]
where we have set
\[ A_I = A_{nm}(x_{01}, x_{02}) = D_{|n|-1}(x_{01})D_{|m|}(x_{02}), \]
\[ B_I = B_{nm}(x_{01}, x_{02}) = D_{|m|-1}(x_{02})D_{|n|}(x_{01}), \]
\[ C_I = C_n(x_{01}) = D_{|n|-1}(x_{01})D_{|n|}(x_{01}). \]

On the other hand, considering propagation from region II to region I, the continuity at point zero reads
\[ \phi_{II, +} + r_{mn}^+ \phi_{II, -} = t_{mn}^+ \phi_{I, +}, \] (40)
which implies
\[ \left( s' D_{|m|-1}(x_{02}) \right) + r_{mn}^+ \left( s' D_{|n|-1}(-x_{02}) \right) = t_{mn}^+ \left( s D_{|n|-1}(x_{01}) \right). \] (41)
These lead to the solution
\[ r_{mn}^+ (x_{01}, x_{02}) = (-1)^{|m|} \frac{s'B_I - sA_I}{sA_I + s'B_I}, \] (42)
\[ t_{mn}^+ (x_{01}, x_{02}) = \frac{2s'F_I}{sA_I + s'B_I}, \] (43)
where \( F_I = F_m(x_{02}) = D_{|m|-1}(x_{02})D_{|m|}(x_{02}). \) (44)

In a similar way, we show that the reflection and transmission coefficients corresponding to propagation from II to III are given by
\[ r_{mn}^+ (x_{02}, x_{03}) = (-1)^{|m|} \frac{s'B_{II} - sA_{II}}{sA_{II} + s'B_{II}}, \] (45)
\[ t_{mn}^+ (x_{02}, x_{03}) = \frac{2s'C_{II}}{sA_{II} + s'B_{II}}, \] (46)
where the constants read
\[ A_{II} = A_{nm}(x_{02}, x_{03}), \quad B_{II} = B_{nm}(x_{02}, x_{03}). \] (47)

As far as the propagation from III to II is concerned, we find
\[ r_{nm}^+ (x_{02}, x_{03}) = (-1)^{|m|} \frac{sA_{II} - s'B_{II}}{sA_{II} + s'B_{II}}, \] (48)
\[ t_{nm}^+ (x_{02}, x_{03}) = \frac{2sC_{II}}{sA_{II} + s'B_{II}}, \] (49)
where \( C_{II} = C_n(x_{03}). \) (50)

This summarizes our analysis for propagation with positive incidence, which together will be used to discuss different issues and before doing so, we need to analyze negative incidence.
3.2. Propagation with negative incidence

We determine the reflection and transmission coefficients for the propagation with negative incidence, which will be denoted as \( r_{mn/sl} \) and \( t_{mn/sl} \), for the cases of the propagations from I to II, II to III and vice versa. To prove this, we use the same analysis as before but one should take into account the negative sign of variable.

In performing our task, for region I we write the corresponding eigenspinors as

\[
\phi_{l,+} = \frac{1}{\sqrt{2}} \left( \begin{array}{c} -sD_{[m]}(x-x_0) \\ D_{[n]}(x-x_0) \end{array} \right) e^{i\delta}, \quad \phi_{l,-} = \frac{1}{\sqrt{2}} \left( \begin{array}{c} -sD_{[m]}(x+x_0) \\ D_{[n]}(x+x_0) \end{array} \right) e^{i\delta}.
\]

For region II, we have

\[
\phi_{l,+} = \frac{1}{\sqrt{2}} \left( \begin{array}{c} -s'D_{[m]}(x-x_0) \\ D_{[n]}(x-x_0) \end{array} \right) e^{i\delta}, \quad \phi_{l,-} = \frac{1}{\sqrt{2}} \left( \begin{array}{c} -s'D_{[m]}(x+x_0) \\ D_{[n]}(x+x_0) \end{array} \right) e^{i\delta}.
\]

In region III, we write

\[
\phi_{l,+} = \frac{1}{\sqrt{2}} \left( \begin{array}{c} -sD_{[m]}(x-x_0) \\ D_{[n]}(x-x_0) \end{array} \right) e^{i\delta}, \quad \phi_{l,-} = \frac{1}{\sqrt{2}} \left( \begin{array}{c} -sD_{[m]}(x+x_0) \\ D_{[n]}(x+x_0) \end{array} \right) e^{i\delta}.
\]

Now let us treat each case to evaluate reflexion and transmission coefficients. Consider the propagation: I \( \rightarrow \) II to obtain

\[
\phi_{l,+} + r_{mn} \phi_{l,-} = t_{mn} \phi_{l,+}
\]

at point zero. Inserting (51) and (52) into (54), one gets

\[
r_{mn}(x_0, x_0) = (-1)^{|m|} \frac{sA_1 - s'B_1}{sA_1 + s'B_1}, \quad t_{mn}(x_0, x_0) = (-1)^{|m|+|n|} \frac{2sC_1}{sA_1 + s'B_1}.
\]

For II \( \rightarrow \) I a similar relation to (54) reads

\[
\phi_{l,+} + r_{lm} \phi_{l,-} = t_{lm} \phi_{l,+}.
\]

The solutions are given by

\[
r_{lm}(x_0, x_0) = (-1)^{|m|} \frac{s'B_1 - sA_1}{sA_1 + s'B_1}, \quad t_{lm}(x_0, x_0) = (-1)^{|m|+|n|} \frac{2sF_1}{sA_1 + s'B_1}.
\]

After applying the same technique as before, we show that the propagation: II \( \rightarrow \) III gives

\[
r_{mn}(x_0, x_0) = (-1)^{|m|} \frac{s'B_{02} - sA_{02}}{sA_{02} + s'B_{02}}.
\]
\[
T_{nn}^{\pm}(x_02, x_03) = (-1)^{|n|+|m|} \frac{2sF_I}{sA_{II} + s'B_{II}}
\]

and III \(\rightarrow\) II leads to
\[
T_{nm}^{\pm}(x_02, x_03) = (-1)^{|n|} \frac{sA_{II} - s'B_{II}}{sA_{II} + s'B_{II}}
\]
\[
T_{mn}^{\pm}(x_02, x_03) = (-1)^{|n|+|m|} \frac{2sC_{II}}{sA_{II} + s'B_{II}}.
\]

By inspecting the forms of different coefficients obtained so far, one can establish a symmetry between them. Indeed, we show the relation
\[
\frac{t_{ij}^{\pm}(x_01, x_02)}{t_{ji}^{\pm}(x_01, x_02)} = (-1)^{|n|+|m|},
\]
where the pair of index is chosen to be \((i \neq j)\) \(\in \{n, m\}\). Note that the same relations are also valid for the couple \((x_02, x_03)\).

3.3. Reflexion and transmission amplitudes

Now let us collect the products of our results by checking their importance. In fact, we discuss the reflexion and transmission amplitudes between regions to emphasize the influence of each parameter on them. For propagation between I and II, we define the reflexion and transmission amplitudes as
\[
\rho(x_01, x_02) = r_{ij}^{\pm}(x_01, x_02)r_{ji}^{\mp}(x_01, x_02), \quad \tau(x_01, x_02) = t_{ij}^{\pm}(x_01, x_02)t_{ji}^{\pm}(x_01, x_02).
\]

After replacing, we end up with
\[
\rho(x_01, x_02) = \frac{[sA_I - s'B_I]^2}{[sA_I + s'B_I]^2},
\]
\[
\tau(x_01, x_02) = \frac{4ss'C_{II}F_I}{[sA_I + s'B_I]^2} = \frac{4ss'A_B}{[sA_I + s'B_I]^2},
\]
where the relation \(C_{II}F_I = A_B\) is satisfied. A straightforward calculation shows that the probability sums to unity, namely
\[
\rho(x_01, x_02) + \tau(x_01, x_02) = 1.
\]

To stress how the amplitudes behave in terms of different parameters, we give figure 5. It is clear that \(\rho(x_01, x_02)\) and \(\tau(x_01, x_02)\) change with respect to variation in the magnetic field \(B'\) for different values of \(B, m, d\) and \(k_y\).

It is worthwhile to invert the situation by varying \(B\) for two values of \(B'\) a given configuration of the parameters \((m, d, k_y)\). This is summarized as follows (figure 6).

As far as the propagation between II and III is concerned, we use the same definition as above to write the amplitudes
\[
\rho(x_02, x_03) = r_{ij}^{\pm}(x_02, x_03)r_{ji}^{\mp}(x_02, x_03), \quad \tau(x_02, x_03) = t_{ij}^{\pm}(x_02, x_03)t_{ji}^{\pm}(x_02, x_03),
\]
which give
\[
\rho(x_02, x_03) = \frac{[sA_{II} - s'B_{II}]^2}{[sA_{II} + s'B_{II}]^2}.
\]
Figure 5. Reflection $\rho(x_{01}, x_{02})$ (red line) and transmission $\tau(x_{01}, x_{02})$ (green line) coefficients between regions I and II for a magnetic barrier of width 2D at various magnetic field $B'$ for two cases: (a) $B = 10$ and (b) $B = 15$ and for different values of $(m, d, k_y)$.

\[
\tau(x_{02}, x_{03}) = \frac{4ss'C_{II}F_1}{[sA_{II} + ss'B_{II}]^2} = \frac{4ss'A_{II}B_{II}}{[sA_{II} + ss'B_{II}]^2},
\]

(71)

where $C_{II}F_1 = A_{II}B_{II}$. Using these we verify

\[
\rho(x_{02}, x_{03}) + \tau(x_{02}, x_{03}) = 1.
\]

(72)

As before we illustrate this result by making different plots, which are in figures 7 and 8.

These are among the interesting results derived so far. In fact, it tells us that the transmission of barrier in inhomogeneous magnetic fields cannot be greater than 1, which is analogous to what was obtained in the one field case, without confinement [19].
Let us present some discussions and derive interesting results which have applications in physics areas. Indeed, similar relations to what obtained above exist also for the photon optics cases. For instance, one can write (68) and (72) when a light beam is reflected and refracted in a diopter between regions I–II and II–III. The first case is characterized by the configuration
Figure 8. Reflexion $\rho(x_{01}, x_{02})$ (red line) and transmission $\tau(x_{02}, x_{03})$ (green line) coefficients between region II and III for a magnetic barrier of width $2d$ at various magnetic field $B$ for two cases: (a) $B' = 10$ and (b) $B' = 15$ and for different values of $(m, d, k_y)$.

**For $n = 0$**

$$\begin{cases}
\rho(x_{01}, x_{02}) = 1, & \tau(x_{01}, x_{02}) = 0 \\
\rho(x_{02}, x_{03}) = 1, & \tau(x_{02}, x_{03}) = 0.
\end{cases}$$

This is an expected result since the transmission $\tau = 1 - \rho$ must be zero in the case where the wave in the $n$-region enters in the $m$-region and vice versa. In fact, the interface between tree regions behaves like a mirror where the reflexion is total. The second case is described by

**For $s = s'$ and $n = \pm m$**

$$\begin{cases}
\rho(x_{01}, x_{02}) = 0, & \tau(x_{01}, x_{02}) = 1 \\
\rho(x_{02}, x_{03}) = 0, & \tau(x_{02}, x_{03}) = 1,
\end{cases}$$

which means that the interface between regions I–II and II–III behaves like a non-reflective dioptr, namely there is a total transmission. Note that these two cases have interesting interpretation in optics physics.

### 4. Tree regions in inhomogeneous magnetic fields

We study another case of a physical system composed of a region indexed by the quantum number $m$ of length $2w$ separating two others indexed by the same quantum number $n$. This will allow us to see how the above results will be changed to the present case and underline what makes difference with respect to the former analysis.

#### 4.1. Reflexion and transmission coefficients

The present situation is quiet different from the former one. We use the above tool to write the continuity equation at the points $x = -d$ and $x = d$. Then, we derive the quantities needed to discuss the reflexion and transmission coefficients as well as the corresponding probabilities. This will be done by treating propagation with positive and negative incidences.
We study the positive incidence by considering the geometry that corresponds to the first interface, which is given by figure (9). Following this process we get the relation

\[ \phi_{I,+}(-d) + r^+ \phi_{I,-}(-d) = \alpha \phi_{II,+}(-d) + \beta \phi_{II,-}(-d), \]  

(73)

where \( r^+ \) is the reflection coefficient for positive incidence, which will be determined together with the parameters \( \alpha \) and \( \beta \). Equation (73) gives

\[ sD_{[m-1]}(d_1) + r^+ sD_{[m-1]}(-d_1) = \alpha sD_{[m]}(d_2) + \beta sD_{[m]}(-d_2), \]

(74)

\[ D_{[m]}(d_1) + r^+ D_{[m]}(-d_1) = \alpha D_{[m]}(d_2) + \beta D_{[m]}(-d_2), \]

(75)

with the constants \( d_1 = x_{01} - d \) and \( d_2 = x_{02} - d \). These can be solved for \( \alpha \) and \( \beta \) to obtain

\[ \alpha = \frac{sA_{nm}(d_1, d_2) + s'B_{nm}(d_1, d_2) + r^+(-1)^n [s'B_{nm}(d_1, d_2) - sA_{nm}(d_1, d_2)]}{2s'F_m(d_2)} \]

(76)

\[ \beta = \frac{s'B_{nm}(d_1, d_2) - sA_{nm}(d_1, d_2) + r^+(-1)^n [sA_{nm}(d_1, d_2) + s'B_{nm}(d_1, d_2)]}{2(-1)^m s'F_m(d_2)} \]

(77)

In terms of the reflection and transmission coefficients, we have

\[ \alpha = \frac{1}{t_{mn}^+(d_1, d_2)} - r^+ \left[ \frac{r_{nm}^+(d_1, d_2)}{t_{mn}^+(d_1, d_2)} \right] \]

(78)

\[ \beta = (-1)^{|n|+|m|} \left[ \frac{r^+}{t_{mn}^+(d_1, d_2)} - \frac{r_{nm}^-(d_1, d_2)}{t_{mn}^-(d_1, d_2)} \right]. \]

(79)

To accomplish such analysis we consider the second interface as shown in figure (10). At the point \( x = d \), we have

\[ \alpha \phi_{II,+}(d) + \beta \phi_{III,-}(d) = t^+ \phi_{II,+}(d), \]

(80)

where \( t^+ \) is the transmission coefficient for positive incidence. After replacing, we end up with a system of equations, such as

\[ \alpha s'D_{[m-1]}(d_2) + \beta s'D_{[m-1]}(-d_2) = t^+ s'D_{[m-1]}(d_3) \]

(81)
After calculation, we find

\[ \alpha d D_{m|n}(d'_2) + \beta D_{m|n}(-d'_3) = t^* D_{m|n}(d_3), \]  

with \( d'_2 = d + x_{02} \) and \( d'_3 = d + x_{03} \). The solution reads

\[ \alpha = t^* \frac{s A_{mn}(d_3, d'_2) + s' B_{mn}(d_3, d'_2)}{2s' F_m(d'_2)} \]

\[ \beta = t^* \frac{(-1)^{|m|} s' B_{mn}(d_3, d'_2) - s A_{mn}(d_3, d'_2)}{2s' F_m(d'_2)} \]

To determine the coefficients for positive incidence we simply use \(78\)–\(79\) and \(83\)–\(84\). Combing all we obtain

\[ r^+ = r^+_{n,m}(d_1, d_2) - r^-_{n,m}(d'_2, d'_3) \]

\[ r^- = r^-_{n,m}(d_1, d_2) - r^-_{n,m}(d'_2, d'_3) \]

\[ t^+ = t^+_m(d'_2, d'_3) \left[ 1 - r^-_{n,m}(d_1, d_2) r^-_{n,m}(d'_2, d'_3) \right] \]

\[ t^- = t^-_{n,m}(d'_2, d'_3) \left[ 1 - r^-_{n,m}(d_1, d_2) r^-_{n,m}(d'_2, d'_3) \right] \]

Now let see how the above results will be written by considering the negative incidence case. Indeed, applying the same machinery as before we get

\[ r^- = r^-_{n,m}(d_1, d_2) - r^-_{n,m}(d'_2, d'_3) \]

\[ t^- = t^-_{n,m}(d'_2, d'_3) \left[ 1 - r^-_{n,m}(d_1, d_2) r^-_{n,m}(d'_2, d'_3) \right] \]

Having obtained the above results, we analyze the corresponding probability and give comments. This issue and related matter will be considered in the following subsection.

### 4.2. Probability

To characterize the behavior of the present system, we study the incident beam. This can be achieved by calculating the probability of reflexing and transmitting beam. Indeed, let us adopt the definition

\[ R = r^+ r^-, \quad T = t^+ t^- \]

After calculation, we find

\[ R = \frac{\rho(d_1, d_2) + \rho(d'_2, d'_3) - 2r^-_{n,m}(d_1, d_2) r^-_{n,m}(d'_2, d'_3)}{1 + \rho(d_1, d_2) \rho(d'_2, d'_3) - 2r^-_{n,m}(d_1, d_2) r^-_{n,m}(d'_2, d'_3)} \]
\[ T = \frac{1 + \rho(d_1, d_2)\rho(d'_2, d_3) - \rho(d_1, d_2) - \rho(d'_2, d_3)}{1 + \rho(d_1, d_2)\rho(d'_2, d_3) - 2r_{n,m}(d_1, d_2)r_{n,m}(d'_2, d_3)} . \] (91)

Combining all to end up with probability
\[ R + T = 1. \] (92)

From this, we summarize the following conclusions:

- the probabilities of reflection and transmission sum to unity, as must be the case, since they are the only possible outcomes for a fermion incident on the barrier;
- Equations (90) and (91) yield that under resonance conditions: \(|n| = |m|, s = s'\)
  
  the barrier becomes transparent, i.e. \(T = 1\);
- more significantly, however, the barrier always remains perfectly transparent for \(|n| = |m|\);
- \(T = 1\) is the feature unique to massless Dirac fermions;
- \(T = 1\) is directly related to the Klein paradox in quantum electrodynamics.

5. Introducing gap

In this study, we consider the confined system in the inhomogeneous magnetic field given by configuration (1) but in the presence of an energy gap \(t'\) in the region II. We will show how the above results will be generalized to the gap case.

5.1. Hamiltonian formalism

As far as regions I and III are concerned, the eigenvalues and the eigenfunctions are those given before for case of without \(t'\). However, in region II the Dirac Hamiltonian can be written as

\[ H_{II} = H_{II}^R = v_F \vec{\sigma} \vec{\pi} + V_0 + t' \sigma_z. \] (93)

Clearly, the mass term \(t' \sigma_z\) makes difference with respect to the former analysis. In fact, it will play a crucial role and lead to discover interesting results. In terms of matrix, \(H_{II}\) takes the form

\[ H_{II} = v_F \begin{pmatrix} 0 & p_x - i p_y - i \varepsilon A_2(x) \\ p_x + i p_y + i \varepsilon A_2(x) & 0 \end{pmatrix} + \begin{pmatrix} V_0 + t' & 0 \\ 0 & V_0 - t' \end{pmatrix}. \] (94)

For the next purpose, we determine the energy spectrum solutions of (94). In doing so, let us fix \(\phi_{II} = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}\) as a spinor of \(H_{II}\) in the presence of an energy gap \(t'\), such as

\[ H_{II} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = E_{II} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \] (95)

which implies two relations:

\[ -i \hbar \omega_2 D_x \psi_2 = (E_{II} - V_0 - t') \psi_1 \] (96)

\[ i \hbar \omega_1 D_y \psi_1 = (E_{II} - V_0 + t') \psi_2. \] (97)

They show

\[ \hbar^2 \omega_2^2 D_y^2 \psi_2 = [(E_{II} + V_0)^2 - t'^2] \psi_2. \] (98)
Its solution gives the second spinor component as
\[
\psi_2(x, y) = D_{|m|}(x + x_{02}) e^{i q y}, \quad m \in \mathbb{Z}.
\] (99)

From (98), it is easy to obtain the eigenvalues (figure 11)
\[
E_{II,m} = s' \sqrt{h^2 \omega^2 |m| + t'^2} + V_0.
\] (100)

We note that in the presence of the energy gap, there is a gap separating the conduction and valence bands, which is missing in the case where \( t' = 0 \) (figure 12(b)).
To complete the derivation of eigenspinors, we determine the first component $\psi_1$. Then, from (96) and (97) we obtain

$$\psi_1(x, y) = -i\hbar \omega' \sqrt{|m|} D_{|m|}^{-1}(x + x_{02}) e^{i q_y y}. \tag{101}$$

After normalization the eigenspinors read

$$\phi_{II,m,qy}(x, y) = \frac{1}{\sqrt{2}} \left( -a_m D_{|m|}^{-1}(x + x_{02}) \right) e^{i q_y y} \tag{102}$$

where the constants are given by

$$a_m = s' \sqrt{E_{II} - V_0 + t' s}, \quad b_m = \frac{E_{II} - V_0 - t' s}{E_{II}}. \tag{103}$$

As concerning regions I and III, the corresponding eigenspinors $\phi_{I,n,ky}(x, y, x_{01})$ and $\phi_{III,n,ky}(x, y, x_{03})$ can be written in compact form as

$$\phi_{I,n,ky}(x, y, x_{0}) = \left( -a_m D_{|m|}^{-1}(x + x_{02}) \right) e^{i q_y y} \tag{104}$$

such that $x = -d$ and $x = d$ give $x_0 = x_{01}$ and $x_0 = x_{03}$, respectively.

On the other hand, the energy conservation between regions I and II gives

$$E_I = E_{II} = E. \tag{105}$$

After replacing, we show that the quantum numbers $n$ and $m$ verify the relation

$$\frac{|n|}{|m|} = \frac{|B'|}{B} \frac{E^2}{(E - V_0)^2 - t'^2}. \tag{106}$$

We have some remarks in order. In region III we have $V_0 = 0$ and $t' = 0$; thus (106) reduces to

$$\frac{|n|}{|m|} = \frac{|B'|}{B}. \tag{107}$$

However, in region II there are two cases

$$(E - V_0)^2 > t'^2 \implies |m| = +m \tag{108}$$

$$\quad (E - V_0)^2 < t'^2 \implies |m| = -m.$$

Finally, the analogue of (32) is now given by

$$E = \frac{\sqrt{|m|}}{\sqrt{|n|}} \frac{s}{s' \sqrt{|B'|}} \left| \frac{a_m}{b_m} \right| + \frac{t'}{s' \sqrt{|B'|}}. \tag{109}$$

5.2. Reflexion and transmission coefficients in the presence of $t'$

We will see how the results obtained before can be generalized to the present case. To proceed, we consider two (barrier) and three regions (diode). For barrier, we show that the reflexion and transmission coefficients are

$$\rho'(x_0, x_{02}) = \frac{[sb_m A_1 - a_m B_1]^2}{[sb_m A_1 + a_m B_1]^2} \tag{110}$$

$$\tau'(x_0, x_{02}) = \frac{4sb_m a_m C F}{[sb_m A_1 + a_m B_1]^2} \tag{111}$$
Figure 13. Reflexion $\rho'(x_0, x_{02})$ (red line) and transmission $\tau'(x_0, x_{02})$ (green line) coefficients for a magnetic barrier of width $2d$ at various energy $E$ for two cases: (a) $V_0 = 15, t' = 5$ and (b) $V_0 = 30, t' = 10$.

Clearly, what makes difference with respect to the former results is the appearance of the constant parameters $a_m$ and $b_m$. These coefficients can be used to verify the probability condition

$$\rho'(x_0, x_{02}) + \tau'(x_0, x_{02}) = 1.$$  \hspace{1cm} (112)

At $(V_0 = 15, t' = 5)$ and $(V_0 = 30, t' = 10)$ the transmission and reflexion profile (figure 13) between regions I-II and II-II show

- $-t' < E < t'$: $\tau'(x_0, x_{02}) \to 1$ and $\rho'(x_0, x_{02}) \to 0$,
- $t' < E < V_0 - t'$: $\tau'(x_0, x_{02})$ decreases and $\rho'(x_0, x_{02})$ increases,
- $V_0 - t' < E < V_0 + t'$: there is no transmission and non-reflexion (not allowed states) and
- $E > V_0 + t'$: $\tau'(x_0, x_{02}) \to 1$ and $\rho'(x_0, x_{02}) \to 0$.

As far as three regions are concerned, one can inspire from the case without gap to obtain the reflection and transmission amplitudes, such as

$$R' = \frac{\rho'(d_1, d_2) + \rho'(d_3, d'_2) - 2r_{n,m}^+(d_1, d_2)r_{n,m}^+(d_3, d'_2)}{1 + \rho'(d_1, d_2)\rho'(d_3, d'_2) - 2r_{n,m}^+(d_1, d_2)r_{n,m}^+(d_3, d'_2)}$$ \hspace{1cm} (113)

$$T' = \frac{1 + \rho'(d_1, d_2)\rho'(d'_2, d_3) - \rho'(d_1, d_2) - \rho'(d'_2, d_3)}{1 + \rho'(d_1, d_2)\rho'(d'_2, d_3) - 2r_{n,m}^+(d_1, d_2)r_{n,m}^+(d'_2, d_3)}.$$ \hspace{1cm} (114)

After a straightforward calculation, we find

$$R' + T' = 1.$$ \hspace{1cm} (115)

Note that, for $B = B'$ we discover the results obtained in [19], which shows that our results are more general.

6. Klein paradox

We complete this work by analyzing the Klein paradox for the present system. This can be done by introducing other considerations based on the current-carrying states and study different limiting cases.
6.1. Propagation from region I to region II: \((x = -d)\)

We consider the scattering of a Dirac fermion of energy \(E\) from an electrostatic step-function potential as shown in figure \ref{fig:potential}. This problem is an archetype problem in nonrelativistic quantum mechanics. For relativistic quantum mechanics, we will find that the solution leads to a paradox (Klein paradox) when the potential is strong.

According to the previous analysis for two regions, it is easy to note that the solution of the Dirac equation in regions I and II are given by

\[
\begin{align*}
\phi_{x<-d} &= \frac{1}{\sqrt{2}} \left( -siD|_{n}|^{-1}(x + x_{01}) \right) + \frac{1}{\sqrt{2}} R \left( -siD|_{n}|^{-1}(-x - x_{01}) \right) \\
\phi_{x>-d} &= \frac{1}{\sqrt{2}} T \left( -amD|_{m}|^{-1}(x + x_{02}) \right) 
\end{align*}
\]

(116)

(117)

where \(R\) and \(T\) are reflected and transmitted coefficients, respectively. Imposing the boundary condition that \(\phi\) be continuous at \((x = -d)\) gives the relation

\[
\left( sD|_{n}|^{-1}(-d + x_{01}) \right) + R \left( sD|_{n}|^{-1}(d - x_{01}) \right) = T \left( amD|_{m}|^{-1}(-d + x_{02}) \right). 
\]

(118)

Solving for \(R\) and \(T\) we obtain

\[
R = \frac{sb_{m}u_{1}v_{3} - amv_{1}u_{3}}{amv_{2}u_{3} - sb_{m}u_{2}v_{3}} 
\]

(119)

\[
T = \frac{s(u_{1}v_{2} - v_{1}u_{2})}{amv_{2}u_{3} - sb_{m}u_{2}v_{3}}, 
\]

(120)

where we use the notation

\[
\begin{align*}
D|_{n}|^{-1}(-d + x_{01}), \quad u_{2} &= D|_{m}|^{-1}(d - x_{01}), \quad u_{3} = D|_{m}|^{-1}(-d + x_{02}) \\
v_{1} &= D|_{n}|(-d + x_{01}), \quad v_{2} = D|_{n}|(d - x_{01}), \quad v_{3} = D|_{n}|(-d + x_{02}) 
\end{align*}
\]

(121)

To proceed further, we introduce the current-carrying states. This is based on the current associated with the Dirac equation, which is

\[
J = ev \sum_{i} \phi^{+} \sigma_{i} \phi, 
\]

(122)

where \(i = x, y\). As an immediate application, the incident current is given by

\[
J_{1} = ev(\phi_{1}^{+} \sigma_{3} \phi_{1} + \phi_{1}^{+} \sigma_{3} \phi_{1}) = seuv_{1}v_{1}. 
\]

(123)
This can be used to evaluate the final currents to the left and right of the potential boundary, which read

\[
J_{x<-d} = sev(u_1 + Ru_2)(v_1 + Rv_2) \tag{124}
\]

\[
J_{x>-d} = \frac{ev}{2}|T|^2 u_3 v_3 \left[ i(a_m^* b_m - a_m b_m^*) + (a_m^* b_m + a_m b_m^*) \right]. \tag{125}
\]

Equation (125) can be split into three parts

\[
J_{x>-d} = \begin{cases} 
-sev|T|^2 a_m b_m u_3 v_3, & V_0 > E + t' \\
-iev|T|^2 a_m b_m u_3 v_3, & E - t' < V_0 < E + t' \\
ev|T|^2 a_m b_m u_3 v_3, & V_0 < E - t',
\end{cases} \tag{126}
\]

with the condition \( a_m b_m > 0 \).

Recall that the reflection and transmission amplitudes are related to the currents through

\[
R = \frac{J_1 - J_{x<-d}}{J_1}, \quad T = \frac{J_{x>-d}}{J_1}. \tag{127}
\]

After replacing, we end up with

\[
R = \frac{u_1 v_1 (a_m v_2 u_3 - sb_m u_2 v_3)^2 - a_m b_m u_3 v_3 (u_1 v_2 - v_1 u_2)^2}{u_1 v_1 (a_m v_2 u_3 - sb_m u_2 v_3)^2} \tag{128}
\]

\[
T = \frac{u_3 v_3 [i(a_m^* b_m - a_m b_m^*) + (a_m^* b_m + a_m b_m^*)] (u_1 v_2 - v_1 u_2)^2}{2u_1 v_1 (a_m v_2 u_3 - sb_m u_2 v_3)^2}. \tag{129}
\]

These show that the probability is

\[
R + T = 1. \tag{130}
\]

We can further inspect (126) to derive other results. This can be achieved by considering three interesting cases.

6.2. Limiting cases

In region II there are three distinct cases, depending on the strength of the potential. This is shown in figure 15.

Let us analyze each case separately and underline its physical properties. Indeed, in the weak potential that corresponds to \( E - V_0 > t' \), we have a restriction on the quantum numbers and parameter constants, such as

\[
|m| = +m, \quad a_m^\dagger = a_m, \quad b_m^\dagger = b_m.
\]

In such a case, the reflection and transmission amplitudes are given by

\[
R = \frac{u_1 v_1 (a_m v_2 u_3 - sb_m u_2 v_3)^2 - a_m b_m u_3 v_3 (u_1 v_2 - v_1 u_2)^2}{u_1 v_1 (a_m v_2 u_3 - sb_m u_2 v_3)^2} \tag{131}
\]

\[
T = \frac{a_m b_m u_3 v_3 (u_1 v_2 - v_1 u_2)^2}{u_1 v_1 (a_m v_2 u_3 - sb_m u_2 v_3)^2}. \tag{132}
\]

They verify the probability condition (130). Thus, the incident beam is partly reflected and partly transmitted. This is similar to the result obtained in nonrelativistic quantum mechanics. The last expression shows that the total probability is conserved.
As far as the intermediate potential is concerned, i.e. $|E - V_0| < t'$, different quantities reduce as

$$|m| = -m, \quad a_m^\dagger = -a_m, \quad b_m^\dagger = b_m.$$ 

The corresponding amplitudes are

$$R = \frac{u_1 v_1 (a_m^2 v_2^2 u_2^2 - b_m^2 u_2^2 v_2^2)^2 + i a_m b_m u_3 v_3 (u_1 v_2 - v_1 u_2)^2}{u_1 v_1 (a_m^2 v_2^2 u_2^2 - b_m^2 u_2^2 v_2^2)^2}$$

$$T = -\frac{i a_m b_m u_3 v_3 (u_1 v_2 - v_1 u_2)^2}{u_1 v_1 (a_m^2 v_2^2 u_2^2 + b_m^2 u_2^2 v_2^2)},$$

where the probabilities sum to unity, as must be the case, since reflection and transmission are the only possible outcomes for a fermion incident on the barrier.

In the strong potential case, i.e. $|E - V_0| > t'$, we have

$$|m| = -m, \quad a_m^\dagger = -a_m, \quad b_m^\dagger = -b_m,$$

which is showing

$$R = \frac{u_1 v_1 (a_m b_m u_3 v_3 (u_1 v_2 - v_1 u_2)^2 + a_m v_2 u_3 - s b_m u_2 v_3)^2}{u_1 v_1 (a_m v_2 u_3 + b_m u_2 v_3)^2} > 1$$

$$T = -\frac{a_m b_m u_3 v_3 (u_1 v_2 - v_1 u_2)^2}{u_1 v_1 (a_m v_2 u_3 + b_m u_2 v_3)^2} < 0.$$ 

The probability is still conserved, but only at the cost of a negative transmission amplitude and a reflection amplitude which exceeds unity. The strong potential appears to give rise to a paradox. There is no paradox if we consider that in the strong potential case the potential is strong enough to create particle–antiparticle pairs. The antiparticles are attracted by the potential and create a negative charged current moving to the right. This is the origin of the negative transmission amplitude, i.e. (136). The particles, on the other hand, are reflected from the barrier and combined with the incident particle beam (which is completely reflected) leading to a positively charged current, moving to the left and with magnitude greater than that of the incident beam.
7. Conclusion

We considered a system composed of different regions of Dirac fermions in the presence of an inhomogeneous magnetic field and confining potential $V(x)$ in one direction. The energy spectrum solutions are obtained in terms of different parameters and quantum numbers for each region. To underline some physical properties of the obtained solutions, we analyzed the energy conservation. This allowed us to establish interesting relations and therefore solve some issues related to reflection and transmission of the system.

More precisely, by considering our system as as barrier, we derived interesting results. In fact, using the continuity equation at different points we explicitly determined the reflection and transmission coefficients. These are used to define the corresponding amplitudes and therefore to show that their probabilities sum to unity. Different cases are treated, which concerned total reflecting and transmitting beams where they are interpreted as mirror or dioptr systems.

Subsequently, we focused on three regions of two fixed points $d$ and $-d$. Writing the continuity at each point, we derived different quantities those are needed to characterize the beam of the present system. Indeed, we reached the conclusions that the probabilities of reflection and transmission sum to unity, as must be the case, since they are the only possible outcomes for a fermion incident on the barrier. Furthermore, (90) and (91) yielded that under the resonance conditions: $|n| = |m|$ and $s = s'$, the barrier becomes transparent, i.e. $T = 1$. More significantly, however, the barrier always remains perfectly transparent for $|n| = |m|$. The latter is the feature unique to massless Dirac fermions and directly related to the Klein paradox in quantum electrodynamics.

Another interesting case is analyzed, which concerned introducing a gap. After getting the energy spectrum solutions, we discussed different issues and among them the energy conservation. This allowed us to generalize the former analysis to the gap case. As interesting results, we showed that the probabilities of reflecting and transmitting amplitudes sum to unity as well. Requiring that $B = B'$, we recovered the result obtained in [19].

Finally, we discussed the Klein paradox by involving the current-carrying states for different regions. Using their relations to the reflexion and transmission amplitudes, we checked the probability by evaluating different quantities. Moreover, we treated three different limiting cases, which concern week, intermediate and strong potentials. For two last cases, the transmission amplitude is obtained with a negative sign; however, when it is added to the reflexion coefficient it gives a sum equal to unity.

Acknowledgment

The authors are grateful to Dr El Bouazzaoui Choubabi for fruitful discussions about the tunneling effect in graphene.

References

[1] Novoselov K S, Geim A K, Morozov S V, Jiang D, Katsnelson M I, Grigorieva I V, Dubonos S V and Firsov A A 2005 Nature **438** 197
[2] Zhang Y, Tan Y W, Störmer H L and Kim P 2005 Nature **438** 201
[3] Berger C et al 2006 Science **312** 1191
[4] Peres N M, Neto A H Castro and Guinea F 2006 Phys. Rev. B **73** 241403
[5] Dayi Ö F and Jellal A 2009 A novel approach to confined dirac fermions in graphene arXiv:0909.1448
[6] Ghosh T K, De Martino A, Häusler W, Dell’Anna L and Egger R R 2008 Phys. Rev. B **77** 081404(R) (arXiv:0708.1876)
[7] Nakada K, Fujita M, Dresselhaus G and Dresselhaus M S 1996 Phys. Rev. B **54** 17954

25
McCam E and Falko V I 2004 J. Phys.: Condens. Matt. 16 2371
Peres N M R, Guinea F and Neto A H C 2006 Phys. Rev. B 73 125411
Peres N M R, Guinea F and Neto A H C 2006 Phys. Rev. B 73 241403(R)
Brey L and Fertig H A 2006 Phys. Rev. B 73 235411
Abanin D A, Lee P A and Levitov L S 2006 Phys. Rev. Lett. 96 176803

[8] Meyer J C, Geim A K, Katsnelson M I, Novoselov K S, Booth T J and Roth S 2007 Nature 446 60
[9] Pereira J M, Mlinar V, Peeters F M and Vasilopoulos P 2006 Phys. Rev. B 74 045424
[10] Peres N M R, Neto A H C and Guinea F 2006 Phys. Rev. B 73 195411
Katsnelson M I 2007 Eur. Phys. J. B 57 225

[11] Han M Y, Ozyilmaz B, Zhang Y and Kim P 2007 Phys. Rev. Lett. 98 206805
[12] Ye P D et al 1995 Phys. Rev. Lett. 74 3013
[13] For recent work, see and references therein Cerchez M, Hugger S, Heinzel T and Schulz N 2007 Phys. Rev. B 75 035341
[14] For a review, see Lee S J, Souma S, Ihm G and Chang K J 2004 Phys. Rep. 394 1
[15] Lawton D, Nogaret A, Makarenko M V, Kibis O V, Bending S J and Henini M 2002 Physica E 13 699
Hara M, Endo A, Katsumoto S and Iye Y 2004 Phys. Rev. B 69 153304
[16] Peeters F M and Matulis A 1993 Phys. Rev. B 48 15166
[17] Muller J E 1992 Phys. Rev. Lett. 68 385
Badalyan S M and Peeters F M 2001 Phys. Rev. B 64 155303
Malkova N, Gomez I and Domning-Adame F 2001 Phys. Rev. B 63 035317
Reijniers J, Matulis A, Chang K, Peeters F M and Vasilopoulos P 2002 Europhys. Lett. 59 749
Lee H-W and Novikov D S 2003 Phys. Rev. B 68 155402
[18] Martino A De, DellAnna L and Egger R 2007 Phys. Rev. Lett. 98 066802
[19] Choubabi E B, Elbouziani M and Jellal A 2010 Int. J. Geom. Meth. Mod. Phys. 7 909
[20] Gradshteyn I S and Ryzhik I M 1980 Table of Integrals, Series and Products (New York: Academic)