SOFFER’S INEQUALITY *

Gary R. Goldstein  
Department of Physics  
Tufts University  
Medford, Massachusetts 02155

R. L. Jaffe and Xiangdong Ji  
Center for Theoretical Physics  
Laboratory for Nuclear Science  
and Department of Physics  
Massachusetts Institute of Technology  
Cambridge, Massachusetts 02139

(MIT-CTP-2402  HEP-PH /9501297  Submitted to: Physical Review D1  January 1995)

Abstract

Various issues surrounding a recently proposed inequality among twist-two quark distributions in the nucleon are discussed. We provide a rigorous derivation of the inequality in QCD, including radiative corrections and scale dependence. We also give a more heuristic, but more physical derivation, from which we show that a similar inequality does not exist among twist-three quark distributions. We demonstrate that the inequality does not constrain the nucleon’s tensor charge. Finally we explore physical mechanisms for saturating the inequality, arguing it is unlikely to occur in Nature.

*This work is supported in part by funds provided by the U.S. Department of Energy (D.O.E.) under cooperative agreement #DF-FC02-94ER40818 and #DE-FG02-92ER40702.
I. INTRODUCTION

In a recent letter [1], Soffer has proposed a new inequality among the nucleon’s twist-two quark distributions, $f_1, g_1,$ and $h_1$ [2,3]:

$$f_1 + g_1 \geq 2|h_1|.$$  \hfill (1)

$f_1$ is the well-known spin average quark distribution which measures the probability to find a quark in a nucleon independent of its spin orientation. $g_1$ measures the polarization asymmetry in a longitudinally polarized nucleon — the probability to find a quark polarized along the nucleon’s spin minus the probability to find a quark polarized against the nucleon’s spin. $h_1$, which is less familiar, measures the polarization asymmetry in a transversely polarized nucleon. $f_1$ and $g_1$ have been measured in many deep inelastic scattering experiments. $h_1$ decouples from lepton scattering and has not yet been measured. Proposals to measure $h_1$ at HERA and RHIC have generated efforts to characterize $h_1$, hence the interest in this inequality [4,5].

Soffer derives Eq. (1) by analogy between quark-nucleon scattering and nucleon-nucleon scattering, where helicity amplitudes analogous to $f_1, g_1,$ and $h_1$ obey inequalities derived many years ago [6]. There are potential problems with this analogy. The intermediate states in quark-hadron scattering, which are treated as on-shell physical states in Soffer’s derivation, are, in fact, colored and gauge dependent. The distribution functions $f_1, g_1,$ and $h_1$ are, in fact, integrals of quark-hadron forward scattering amplitudes over transverse momentum with cutoffs at $k_\perp \approx \sqrt{Q^2}$. In QCD, the definitions of quark distributions such as $f_1, g_1,$ and $h_1$ are scale and renormalization scheme dependent. Any relations among them must be accompanied by a precise description of the procedure with which they are extracted from experimental data. In contrast, the well-known inequalities and positivity constraints among distribution functions such as $f_1 \geq |g_1|$ are general properties of lepton-hadron scattering, derived without reference to quarks, color and QCD.

In this Paper we consider Soffer’s inequality in the context of QCD. We find that Eq. (1) can be derived in a “parton model approximation” to QCD, but that radiative corrections modify Eq. (1) in a significant way. Each term in Eq. (1) is multiplied by a power series in $\alpha_s(Q^2)/\pi$. So the inequality as presented by Soffer is of limited practical use — it is strictly valid only at asymptotic $Q^2$ where $\alpha_s \to 0$ and the distribution functions vanish for all $x > 0$. Thus the inequality has a similar status in QCD as the Callan–Gross relation [7] — a parton model result which is invalidated by QCD radiative corrections. One should remember, however, that the Callan–Gross relation is a very useful, although approximate tool in deep-inelastic phenomenology. A one-loop calculation of the radiative corrections to Eq. (1), which we have not attempted, would yield an improved result which would be useful at experimentally accessible $Q^2$.

In §II we study Soffer’s inequality from the consideration of current–hadron scattering amplitudes. This treatment has the same level of rigor as the derivation of standard deep-inelastic inequalities such as $f_1 \geq |g_1|$, and demonstrates the presence of radiative corrections in QCD. In §III we present a second derivation closer in spirit to Soffer’s earlier analysis to nucleon-nucleon scattering. This derivation is heuristic. In particular, it ignores QCD radiative corrections. However, it enables us to make contact with standard operator definitions of the distributions $f_1, g_1,$ and $h_1$. It is then straightforward to generalize the
analysis to twist-three (corrections of $O(1/\sqrt{Q^2})$). In his paper Soffer suggested that there would be a twist-three generalization of his inequality [1]. Although there is a natural correspondence between the three twist-two distributions, $f_1$, $g_1$, and $h_1$, on the one hand, and the three twist-three distributions, $e$, $g_T$, and $h_L$, on the other [2], we find that there is no such inequality at twist three. Also in his paper, Soffer claims that the inequality places a constraint on the nucleon's “tensor charge,” the lowest moment of $h_1$. Using the formalism of §III we show that this result is invalidated by the presence of antiquarks in the nucleon wavefunction and that there is no way to define the notion of a “valence quark” to give a useful result.

Soffer noted that his inequality appeared to be saturated for single quarks in simple quark models such as the non-relativistic quark model and the bag model [1,2]. In §IV we demonstrate that this feature is not preserved by even the simplest quark model wavefunctions. For example, the inequality is saturated for down-quarks in the quark model proton, but not for up-quarks. Also, saturation is not preserved by evolution. We comment on the possibility of using saturation (e.g. for down-quarks in the proton) as “boundary data” [8,9].

II. DERIVATION OF THE INEQUALITY FROM CURRENT-HADRON AMPLITUDES

It is useful to review the textbook derivation of the inequalities or “positivity constraints” on the familiar structure functions of deep inelastic lepton scattering, $f_1$, $f_2$, $g_1$, and $g_2$ [10]. They follow from demanding that cross sections for forward, vector current-hadron scattering are positive definite. These cross sections are proportional to

$$W(\epsilon) = \frac{1}{4\pi} \sum X (2\pi)^4 \delta^4(P + q - P_X) \|\langle X|J \cdot \epsilon|P,S\rangle\|^2,$$

$$= \epsilon^{\mu\nu} W_{\mu\nu}(q, P, S) \epsilon^\nu,$$  \hspace{1cm} (2)

which is manifestly positive definite for any $\epsilon$. $P^\mu$ and $S^\mu$ are the momentum and spin of the target ($P^2 = -S^2 = M^2$, $P \cdot S = 0$), and $\epsilon^\mu$ is the polarization vector of the (virtual) photon. $J_\mu$ is the electromagnetic current operator, which in QCD would be $\sum_a e_a \bar{\psi}^a \gamma_\mu \psi^a$, where $a$ is a flavor label. For simplicity we consider a single quark flavor with unit charge. Hence the relations we derive will be valid for each flavor separately. $W_{\mu\nu}$ is the usual current-current correlation function of deep inelastic scattering,

$$W_{\mu\nu}(q, P, S) = \frac{1}{4\pi} \int d^4\xi e^{i q \cdot \xi} \langle P, S| [J_\mu(\xi), J_\nu(0)] |P, S\rangle,$$

$$= -g_{\mu\nu} f_1(q^2, \nu) + \frac{1}{\nu} P_\mu P_\nu f_2(q^2, \nu) + \text{gauge terms} + \frac{i}{\nu} \epsilon_{\mu\nu\rho\lambda} q^\rho S^\lambda g_1(q^2, \nu) + \frac{i}{\nu^2} \epsilon_{\mu\nu\rho\lambda} q^\rho \left(\nu S^\lambda - q \cdot SP^\lambda\right) g_2(q^2, \nu)$$  \hspace{1cm} (3)

where $q^2 < 0$, and $\nu = P \cdot q > 0$. Substituting this expansion back into Eq. (2) and taking the Bjorken scaling limit yields $f_1 + g_1 \geq 0$ or $f_1 - g_1 \geq 0$ for transverse photons and definite nucleon helicity states, hence $f_1 \geq |g_1|$. The current $\bar{\psi}^a \gamma_\mu \psi^a$ creates and annihilates antiquarks as well as quarks so the structure functions all receive both quark and antiquark contributions. In the Bjorken limit, $\lim_{Bj}$
\( Q^2 = -q^2, \nu \to \infty, x \equiv -q^2/2\nu \) fixed) of QCD, \( f_1 \) and \( g_1 \) reduce to quark distribution functions which we label with the flavor \( a \) or \( \bar{a} \) of quark or antiquark,

\[
\lim_{Bj} f_1(q^2, \nu) = f^a_1(x, \ln Q^2) + f^\bar{a}_1(x, \ln Q^2),
\]

\[
\lim_{Bj} g_1(q^2, \nu) = g^a_1(x, \ln Q^2) + g^\bar{a}_1(x, \ln Q^2),
\]

and the positivity constraints will apply to such combinations. The physical meaning of the inequality can be seen from the fact that the combination \( f_1 + g_1 \) in parton model is simply the probability to find a quark or antiquark with spin parallel to the target nucleon,

\[
\lim_{Bj} [f_1(q^2, \nu) + g_1(q^2, \nu)] = q^a(\ln Q^2) + \bar{q}^\bar{a}(\ln Q^2).
\]

and conversely for \( f_1 - g_1 \). The \( \ln Q^2 \) dependence comes from the evolution of the distributions under scale transformation. Note that these distributions have been defined in terms of deep-inelastic vector-current structure functions. Quark distributions are in general process-dependent and relations among quarks distributions extracted from different experiments can be calculated in QCD perturbation theory \[\[\].

Of course the quark and antiquark distributions \( f_1^a, g_1^a \) and \( f_1^{\bar{a}}, g_1^{\bar{a}} \) are separately constrained. We must understand how this comes about in order to obtain the strongest possible bounds that include the transverse structure function, \( h_1 \). We would like to replace \( J_\mu \) by a current which couples only to quarks. The chiral currents \( J^\pm_\mu = \frac{1}{2}(V_\mu \pm A_\mu) \), which are given by \( \frac{1}{2}\bar{\psi}\gamma_\mu(1 \pm \gamma_5)\psi \) in QCD, are candidates. \( J^-_\mu \), for example, couples to left-handed quarks and right-handed antiquarks. If we choose the polarization vector, \( \epsilon^\mu \), judiciously, we can select left-handed quanta, thereby decoupling the antiquarks. To be specific, we choose the momentum \( \vec{q} \) to be in the positive \( \hat{e}_3 \) direction, \( q^\mu = (q^0, 0, 0, q^3) \), and \( \vec{P} \) to be in the \( -\hat{e}_3 \) direction. If we employ the V–A current, negative helicity for the target nucleon, and \( \epsilon^-_\mu = \frac{1}{\sqrt{2}}(0, 1, -i, 0) \), then the current selects left-handed quarks and right-handed antiquarks in the left-handed target: \( q^\mu + \bar{q}^\bar{a} \). The right-handed antiquarks decouple from the product \( J^- \cdot \epsilon^- \) because they have \( J_z = -\frac{1}{2} \) and cannot absorb \( \Delta J_z = -1 \). It is quite easy to see that \( f_1^a + g_1^a \geq 0 \) results. Analogous choices yield constraints on \( f_1^a - g_1^a \) and on antiquark distributions.

The derivation we have just outlined would be quite complicated for non-asymptotic \( q^2 \) and \( \nu \). The introduction of chiral currents and polarized targets requires all the machinery developed for neutrino scattering from polarized targets \[\[\]. Such an analysis would lead to a very general constraint, valid independent of QCD and the Bjorken limit. However, it is only useful in the Bjorken limit where only the familiar twist-two invariant structure functions \( f_1 \) or \( g_1 \) survive. The same remark will apply in the case of Soffer’s inequality to which we now turn.

The quark currents \( \bar{\psi}\gamma_\mu\gamma_5\psi \) and \( \bar{\psi}\gamma_\mu\gamma_5\gamma_3\psi \) preserve quark chirality. So does the leading term in the product of two such currents at short distances. The distribution function \( h_1 \), in contrast, couples quarks of opposite chirality \[\[\] and therefore does not appear in any of these relations. This suggests that constraints involving \( h_1 \) might be obtained by considering the interference between the V–A current and a current of opposite chirality. This is in fact the case. So, in addition to the V–A current, \( J^-_\mu \), we introduce a hypothetical current, \( J \), which is composed of scalar and pseudoscalar currents, along with tensor and pseudo-tensor currents
\[ \mathcal{J} \equiv (S + P - T^+ - T_s^-)/2\sqrt{2} \, . \] (6)

This ungainly choice has been engineered to select out the distribution functions of interest. Unlike the vector and axial currents which are defined by symmetries, these currents cannot be defined independent of quarks and QCD. For example, different constraints on distribution functions would be obtained from \( S = \tilde{\psi}\psi \) or \( S = \tilde{\psi}\gamma_5\psi \). We define the currents as follows: \( S(\xi) = Z_S \tilde{\psi}(\xi)\psi(\xi), \) \( P(\xi) = Z_P \tilde{\psi}(\xi)\gamma_5\psi(\xi), \) \( T^{\mu\nu}(\xi) = Z_T \tilde{\psi}(\xi) [\gamma^\mu, \gamma^\nu] \psi(\xi), \) and \( T_s^{\mu\nu}(\xi) = Z_T \tilde{\psi}(\xi) [\gamma^\mu, \gamma^s] \gamma^5 \psi(\xi). \) Because these currents are not constrained by Ward-identities, they are non-trivially renormalized in QCD. As a consequence in addition to the ambiguities already mentioned, they are regularization and renormalization scheme dependent. However, for any choice of scheme, the derivation of the inequality remains the same, and, of course, the physical implications of the inequality are scheme independent. For simplicity, however, we choose dimensional regularization and (modified) minimal subtraction. The renormalization scale in currents is set at the virtual-boson mass, \( Q^2 \). The tensor and pseudo-tensor currents combine with the scalar and pseudo-scalar currents to project the “good” light-cone components of the right-handed chiral fermions (as will be discussed in the next section) from the field \( \psi \). When positive helicity is chosen for the nucleon, the right-handed quark field will remain, rather than the left-handed anti-quark.

The desired inequality follows from consideration of a judiciously chosen fictitious “cross section.” Consider the quantity,

\[
\mathcal{W}(q, P) = \frac{1}{4\pi} \sum_X (2\pi)^4 \delta^4(P + q - P_X) \| \langle X | J_\perp \cdot \epsilon_- | P_- \rangle \| \pm \langle X | \mathcal{J} | P_+ \rangle \|^2,
\]

\[
= \frac{1}{4\pi} \int d^4\xi e^{iq\xi} \left[ \langle P, - | \left[ J_\perp^\dagger(\xi) \cdot \epsilon_-^*, J_\perp(0) \cdot \epsilon_- \right] | P, - \rangle + \langle P, + | \left[ \mathcal{J}^\dagger(\xi), \mathcal{J}(0) \right] | P, + \rangle \right]
\]

\[
\pm \frac{1}{2\pi} \text{Re} \int d^4\xi e^{iq\xi} \left[ \langle P, + | \left[ \mathcal{J}^\dagger(\xi), \mathcal{J}(0) \cdot \epsilon_- \right] | P, - \rangle \right],
\]

(7)

which is manifestly positive. \( \mathcal{W} \) involves three terms. Referring back to Eq. (3) it is clear that the \( J_\perp^\dagger \cdot \epsilon_-^* \otimes J_\perp \cdot \epsilon_- \) term will reduce to \( f_1^a + g_1^a \) in the Bjorken limit. Likewise, it is clear from general considerations that the \( \mathcal{J}^\dagger \otimes \mathcal{J} \) term will also involve \( f_1^a \) and \( g_1^a \) in the Bjorken limit. However, since \( \mathcal{J}^\dagger \otimes \mathcal{J} \) suffers different radiative corrections than \( J_\perp^\dagger \cdot \epsilon_-^* \otimes J_\perp \cdot \epsilon_- \), \( f_1 \) and \( g_1 \) will be multiplied by a series in \( a_s(Q^2)/\pi \). The interference term, \( \mathcal{J}^\dagger \otimes J \cdot \epsilon_- \), is chiral-odd and can only involve \( h_1^a \) in the Bjorken limit. Combined with the other two terms, we obtain

\[
\lim_{\delta \to 0} \mathcal{W} = R_f(a_s(Q^2)) f_1^a(x, \ln Q^2) + R_g(a_s(Q^2)) g_1^a(x, \ln Q^2) \pm 2R_h(a_s(Q^2)) h_1^a(x, \ln Q^2) \, . \]

(8)

Here the \( R_f \) and \( R_g \) factors take into account the radiative corrections mentioned above. The \( R_h \) factor arises because the definition of \( h_1 \) is process dependent. If we chose to define \( h_1 \) through our fictitious process then \( R_h = 1 \) by definition. On the other hand, if \( h_1 \) is defined through a physical process such as Drell-Yan \( \mu \)-pair production with transversely polarized beams \( [3] \), then \( R_h = 1 + O(a_s) \). Another subtlety in this calculation is that the vector-scalar interference terms have the (nucleon) helicity structure \( \langle P \pm | \ldots | P\mp \rangle \), which does not correspond to an expectation value in a state of definite spin. However the helicity structure required can be extracted by combining expectation values in states with \( \hat{S} = \hat{\epsilon}_1 \) and \( \hat{S} = \hat{\epsilon}_2 \). Radiative corrections aside, the result is straightforwardly obtained by
calculating the current correlation functions at tree-level in the Bjorken limit, and using the standard definitions of the distribution functions \( f_1^a, g_1^a, \) and \( h_1^a \).

Since \( W \) is manifestly positive, eq (3) is the desired inequality. Of course \( W \) is positive for all \( q^2 \) and \( \nu \). So (1) implies a constraint among the many invariant structure functions that occur in the decomposition of \( W \) at sub-asymptotic \( q^2 \) and \( \nu \). There is no point, however, in displaying this inequality explicitly, since nearly all the novel structure functions, such as those involved in the invariant decomposition of \( T_{\mu\nu} \otimes J^\pm_1 \), are not directly measurable.

This derivation shows that Soffer’s inequality holds independently for each quark and antiquark flavor. Also, it is clear that careful attention must be given to the specific “process”, in which the quark distributions can be defined unambiguously. The “natural” choice would be to define \( f_1 \) and \( g_1 \) in vector-current deep-inelastic scattering, and \( h_1 \) in polarized Drell-Yan. It is clear that Soffer’s identity is a parton model approximation (no radiative corrections) to a more useful identity which can be obtained by computing the factors \( R_f, R_g, \) and \( R_h \) at least through (lowest non-trivial) order \( \alpha_s/\pi \).

Armed with this rigorous, if rather unphysical, derivation, we turn to examine the inequality from the more familiar viewpoint of the quark parton model and its coordinate space equivalent, the light-cone expansion.

### III. DERIVATION OF THE INEQUALITY FROM QUARK HADRON AMPLITUDES

We begin with a simple, heuristic “parton model” derivation of the inequality postponing any complexity. Next we introduce the bilocal light-cone correlation functions which allow us to give a more convincing derivation and study twist-three distribution functions. Only QCD radiative corrections will be left out at this stage. The derivation of the previous section shows how their effects can be included.

In the most elementary parton model, deep inelastic processes are summarized by the “handbag” diagram of Fig. 1a. At the bottom of this diagram is the imaginary part of a quark-nucleon scattering amplitude. We focus on this amplitude. Since the quark (nucleon) begins and ends with the same momentum, \( k (P) \), the amplitude describes forward scattering. Since the quark is initially removed from the nucleon and then replaced, the diagram actually corresponds to a \( u \)-channel discontinuity of forward quark-nucleon scattering, as shown in Fig. 1b. We label the \( u \)-channel discontinuities \( A_{Hh,H'h'} \), where \( H \) and \( H' \) are the initial and final nucleon helicities and \( h \) and \( h' \) are the outgoing and incoming quark helicities respectively. For spin-1/2 quarks and nucleons parity and time-reversal invariance reduce the number of independent helicity amplitudes to three. Three convenient choices shown in Fig. 2, are \( A_{++,++}, A_{+-,-+}, \) and \( A_{++,-} \) respectively. Amplitudes that fail to satisfy conservation of angular momentum along the collision axis, \( H + h' = H' + h \), vanish. Other helicity amplitudes are either related to these by parity, \( A_{Hh,H'h'} = A_{-H,-h,-H'--h'} \), or time reversal, \( A_{Hh,H'h'} = A_{H'h',Hh} \). It is easy to show that the three twist-two structure

---

1 The propagator on the quark leg is not truncated.
functions, \( f_1, g_1 \) and \( h_1 \) are (suitably normalized) linear combinations of \( A_{++,+}, A_{+-,-} \), and \( A_{+-,0} \): \( f_1 = A_{++,+} + A_{+-,-}, g_1 = A_{++,+} - A_{+-,-}, \) and \( h_1 = A_{++,0} \).

To obtain the Soffer’s inequality it is necessary to consider the quark-hadron amplitudes which are related to the \( \{A\} \) by unitarity. Define four amplitudes \( a_{Hh} \) by

\[
a_{Hh}(X) = \langle X | \phi_h | PH \rangle ,
\]

where \( \phi \) is the quark field, and \( X \) is an arbitrary final state. Unitarity requires that the \( \{A\} \) are proportional to products of the form \( \sum_X a_{H'h'}^\ast(X) a_{Hh}(X) \), so

\[
f_1 \propto \sum_X a_{++}(X)^* a_{++}(X) + a_{-+}(X)^* a_{+-}(X) ,
\]

\[
g_1 \propto \sum_X a_{++}(X)^* a_{++}(X) - a_{-+}(X)^* a_{+-}(X) ,
\]

\[
h_1 \propto \sum_X a_{++}(X)^* a_{-+}(X) ,
\]

The desired inequality follows from the observation that

\[
\sum_X \|a_{++}(X) \pm a_{-+}(X)\|^2 \geq 0 ,
\]

and that \( A_{++,+} = A_{--,-} \) and \( A_{++,-} = A_{--,++} \) by parity.

Our first step in improving this admittedly heuristic derivation is to clarify the relationship between the helicity amplitudes \( \{A\} \) and \( \{a\} \) and the operator expressions which define the distribution functions \( f_1, g_1, \) and \( h_1 \) in QCD. First we will derive Eqs. (10) from standard definitions of \( f_1, g_1, \) and \( h_1 \). Then it will be straightforward to show that the inequality does not generalize to twist-three. Also it will be clear that the tensor charge is not constrained by Eq. (4).

In QCD parton distributions are defined by the light-cone Fourier transformation of forward matrix elements of operator products. The quark distributions of interest to us are related to matrix elements of bilinear quark operators,

\[
\int \frac{d\lambda}{4\pi} e^{i \lambda x} \langle PS | \bar{\psi}(0) \gamma_{\mu} \psi(\lambda n) | PS \rangle = f_1(x) p_{\mu} + M^2 f_1(x) n_{\mu} \quad (12)
\]

\[
\int \frac{d\lambda}{4\pi} e^{i \lambda x} \langle PS | \bar{\psi}(0) \gamma_{\mu} \gamma_5 \psi(\lambda n) | PS \rangle = g_1(x) p_{\mu} S \cdot n + [g_1(x) + g_2(x)] S_{\perp \mu} + M^2 g_3(x) n \cdot S n_{\mu} \quad (13)
\]

\[
\int \frac{d\lambda}{4\pi} e^{i \lambda x} \langle PS | \bar{\psi}(0) \psi(\lambda n) | PS \rangle = Me(x) \quad (14)
\]

\[
\int \frac{d\lambda}{4\pi} e^{i \lambda x} \langle PS | \bar{\psi}(0) \sigma_{\mu \nu} i \gamma_5 \psi(\lambda n) | PS \rangle = h_1(x) (S_{\perp \mu} p_{\nu} - S_{\perp \nu} p_{\mu}) / M + [h_2(x) + h_1(x)/2] M(p_{\mu} n_{\nu} - p_{\nu} n_{\mu}) S \cdot n + h_3(x) M S_{\perp \mu} n_{\nu} - S_{\perp \nu} n_{\mu} \quad (15)
\]

where \( n \) and \( p \) are null vectors of mass dimension \(-1\) and \(1\), respectively \((n^2 = p^2 = 0, n^+ p^- = 0, n \cdot p = 1)\). \( P \) and \( S \) may be decomposed in terms of \( n \) and \( p \), \( P = p + \frac{M^2}{2} n, S_{\mu} = S \cdot n p_{\mu} + S \cdot p n_{\mu} + S_{\perp \mu} \). For a target moving in the \( \hat{e}_3 \) direction, \( p = \frac{1}{\sqrt{2}} (\Lambda, 0, 0, \Lambda), n = \)
\[ \frac{1}{2} (1, 0, 0, -\frac{1}{2}). \] In Eqs. (12) the four-component Dirac field for the quark. The flavor label on \( \psi \) and the corresponding distribution functions has been suppressed.

Eqs. (12) are written in \( n \cdot A = 0 \) gauge. In any other gauge a Wilson link would be required between \( \psi \) and \( \bar{\psi} \) to maintain gauge invariance. Gluon radiative corrections, which generate a renormalization point dependence for these operators and an associated \( q^2 \) dependence for the distribution functions, have been suppressed in Eqs. (12).

The leading twist contributions to Eqs. (12) are the distributions functions \( f_1, g_1 \), and \( h_1 \) respectively. They may be projected out by contracting the equations with \( n^\mu, n^\mu, \) and \( n^\mu S^{+\nu} \) respectively. In every case the projection operator \( P^+ \equiv \gamma^0 (\gamma^0 + \gamma^3)/2 = (1 + \alpha_3)/2 \) emerges from the Dirac algebra. \( P^+ \) projects the four component Dirac spinor \( \psi \) onto the two dimensional subspace of “good” light-cone components which are canonically independent fields (13). Likewise, \( P^- \equiv (\gamma^0 - \gamma^3)/2 = (1 - \alpha_3)/2 \) projects on the two dimensional subspace of “bad” light-cone components which are interaction dependent fields and should not enter at leading twist (3). Much of our analysis is simplified by choosing a representation for the Dirac matrices tailored to the light-cone (13),

\[
\gamma^0 = \rho_1 \sigma_3, \quad \gamma^1 = i \sigma_1, \quad \gamma^2 = i \sigma_2, \quad \gamma^3 = -i \rho_2 \sigma_3, \quad \gamma_5 = i \gamma^0 \gamma^1 \gamma^2 \gamma^3 = \rho_3 \sigma_3 .
\] (16)

Where \( \{ \rho_k \} \) and \( \{ \sigma_k \} \) are 2 \times 2 Pauli matrices. This is to be contrasted to the familiar Dirac-Pauli representation, \( \{ \rho_3, i \rho_2 \sigma_1, i \rho_2 \sigma_2, i \rho_2 \sigma_3, \rho_1 \} \) which is convenient for many other purposes. In the light-cone representation \( P^\pm, \gamma_5 \) and \( \mathbf{\sigma} \cdot \mathbf{\hat{e}}_3 \) are all diagonal,

\[
P^+ = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad P^- = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad \gamma_5 = \begin{pmatrix} \sigma_3 & 0 \\ 0 & -\sigma_3 \end{pmatrix}, \quad \mathbf{\sigma} \cdot \mathbf{\hat{e}}_3 = \begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix} .
\] (17)

where \( 1 \) and \( 0 \) are the \( 2 \times 2 \) identity and null matrices respectively. In this basis \( P^+ \) and \( P^- \) project onto the upper and lower two components of the Dirac spinor respectively,

\[
\phi \equiv P^+ \psi = \begin{pmatrix} \phi_+ \\ \phi_- \end{pmatrix} , \quad \chi \equiv P^- \psi = \begin{pmatrix} \chi_+ \\ \chi_- \end{pmatrix} .
\] (18)

\( \phi_\pm \) are the “good” light-cone components of the quark field, which are independent canonical variables in the light-cone formulation. \( \chi_\pm \) are the “bad” light-cone components which may be regarded as composite fields built from quarks (the “good” light-cone components) and transverse gluons. The \( \pm \) labels on \( \phi \) and \( \chi \) refer to the eigenvalue of \( \sigma_3 \) which is proportional to helicity, \( \mathbf{\sigma} \cdot \mathbf{\hat{P}} \), for quarks moving in the \( \mathbf{\hat{e}}_3 \) direction, \( \{ \mathbf{\sigma} \cdot \mathbf{\hat{P}} \phi_\pm = \pm \phi_\pm, \mathbf{\sigma} \cdot \mathbf{\hat{P}} \chi_\pm = \pm \chi_\pm \} \), not to chirality. From Eqs. (17) and (18) it is clear that helicity and chirality are the same for \( \phi \), but opposite for \( \chi \). This is easy to understand when one recognizes that the bad light-cone components are actually composites of the canonically independent operators \( \phi_\pm \) and \( \mathbf{\hat{A}}_\perp \).

The positive helicity component of \( \chi_+ \) involves a transverse gluon (with positive helicity) and a good light-cone component of the quark field, \( \phi_- \) (with negative helicity and therefore negative chirality).

It is now straightforward to project \( f_1, g_1 \), and \( h_1 \) out of Eqs. (12) and rewrite the result in terms of \( \phi_\pm \),

\[
f_1(x) = \int \frac{d\lambda}{4\pi P} e^{ix\lambda} \langle P + |\phi_+(0)\phi_+(\lambda n) + \phi_-(0)\phi_-(\lambda n)|P+ \rangle ,
\]

\[
g_1(x) = \int \frac{d\lambda}{4\pi P} e^{ix\lambda} \langle P + |\phi_+(0)\phi_+(\lambda n) - \phi_-(0)\phi_-(\lambda n)|P+ \rangle ,
\]

\[
h_1(x) = \int \frac{d\lambda}{8\pi P} e^{ix\lambda} \{ \langle P + |\phi_+(0)\phi_-(\lambda n)|P- \rangle + \langle P - |\phi_-(0)\phi_+(\lambda n)|P+ \rangle \} ,
\] (19)
If we insert a complete set of intermediate states between $\phi^\dagger$ and $\phi$, translate the fields and carry out the $\lambda$ integration, we obtain,

$$f_1(x) = \frac{1}{2P} \sum_x \delta(x - 1 + n \cdot P_X) \{|a_{++}(X)|^2 + |a_{--}(X)|^2\},$$

$$g_1(x) = \frac{1}{2P} \sum_x \delta(x - 1 + n \cdot P_X) \{|a_{++}(X)|^2 - |a_{--}(X)|^2\},$$

$$h_1(x) = \frac{1}{2P} \sum_x \delta(x - 1 + n \cdot P_X) a_{++}(X)^* a_{--}(X). \quad (20)$$

This reproduces Eq. (10) and shows that the “generic” quark fields which appear there should be identified with the chiral components of the “good” light-cone components of the quark field. This derivation illustrates the questionable procedure required to obtain Soffer’s inequality using traditional parton-model/light-cone methods: the states in $|X\rangle$ are colored; and the bilocal operators in Eq. (13) do not actually exist since each term in their Taylor expansion about $\lambda = 0$ is renormalized differently by radiative corrections. However the result is correct (modulo the important radiative corrections discussed in §II) and the derivation is considerably more “physical” than the more rigorous one presented in the previous section.

The light-cone formalism defined in this section allows us to examine the possible extension of Soffer’s identity to the twist-three distributions, $e$, $g_T$, and $h_L$. $e$ is defined in Eq. (14), and the others are defined by, $g_T = g_1 + g_2$ and $h_L = \frac{1}{2}h_1 + h_2$. Examination of Eqs. (12, 15) shows that $e(q^2, \nu)$ is spin-independent and chiral-odd. $h_L$ and $g_T$ are spin-dependent and chiral-odd and chiral-even respectively. $h_L$ is associated with longitudinally polarized targets and $g_T$ with transversely polarized targets. In summary, the spin attributes of $\{e, h_L, g_T\}$ correspond to $\{f_1, g_1, h_1\}$ respectively. The astute reader will note that this correspondence appears to be inconsistent with the chirality assignments of the distribution functions. For example, $f_1$ is spin average, and therefore diagonal in helicity — $f_1 \propto \phi^\dagger_+ \phi_+ + \phi^\dagger_- \phi_-$. Clearly $f_1$ preserves quark chirality — i.e. it is chiral-even. $e$ on the other hand is claimed to be chiral-odd, even though it, like $f_1$, averages over helicity. The resolution of this apparent contradiction comes from the classification of $e$ with respect to the light-cone projection operators $P_\pm$. It is easy to see that $e \propto \chi^\dagger_+ \phi_+ + \chi^\dagger_- \phi_- + \text{h.c.}$. A glance at the chirality assignments of $\chi_\pm$ confirms that $e$ flips chirality — i.e. it is chiral-odd. An analogous analysis applies to $h_L$ and $g_T$.

It should now be clear that an identity analogous to Eq. (11) cannot be obtained at twist-three. The reason is that an object of the form $\langle \Psi | \chi^\dagger_\pm \phi_{\pm} | \Psi \rangle$ could only arise by starting with positive definite structure such as $\| \langle X | \chi_\pm | \Psi \rangle + \langle X | \phi_{\pm} | \Psi \rangle\|^2$. This object would generate twist-three distributions in the interference, but twist-two, and more problematically, twist-four distributions such as $\langle \Psi | \chi^\dagger_\pm \chi_{\pm} | \Psi \rangle$ would be unavoidable. The conclusion then is that any positivity constraint involving the twist-three distributions $e$, $g_T$, and $h_L$ would inextricably include twist-four distributions which are very difficult to measure. Hence Soffer’s speculation is incorrect.

\[\text{For further discussion of } \{e, h_L, g_T\}, \text{ see } \text{[3].}\]
Finally we consider the relationship imposed on the lowest moment of \( h_1 \) by the inequality, Eq. (1). The nucleon’s tensor charge, \( \delta q^a(Q^2) \) is defined by analogy to the axial charge, \( \Delta q^a \) \[2\],

\[
S^k \delta q^a(Q^2) \equiv \frac{1}{2} \langle PS|\bar{q}\sigma^{ak}i\gamma_5\frac{\lambda^a}{2}q|Q^2|PS \rangle = S^k \int_{-1}^{1} dx [h_1^a(x, \ln Q^2) - h_1^a(x, \ln Q^2)] ,
\]

\[
S^k \Delta q^a \equiv \frac{1}{2} \langle PS|\bar{q}\gamma^k\gamma_5\frac{\lambda^a}{2}q|PS \rangle = S^k \int_{-1}^{1} dx [g_1^a(x, \ln Q^2) + g_1^a(x, \ln Q^2)] .
\]

In contrast to the nucleon’s axial charge which figures in beta-decay, the tensor charge does not appear in weak matrix elements and has not been measured. Note that \( \delta q^a \) is renormalization point dependent, whereas \( \Delta q^a \) is not (because the axial current in QCD is conserved apart from quark mass terms). Note also that \( h_1^a(g_1^a) \) enters Eq. (10) with a minus (plus) sign reflecting the fact that the operator \( \bar{q}\sigma^{ik}\gamma_5q \) (\( \bar{q}\gamma^{ik}\gamma_5q \)) is odd (even) under charge conjugation. There is no way to combine Eqs. (21) with the inequalities \( f_1^a + g_1^a \geq 2| h_1^a | \), and \( f_1^a + g_1^a \geq 2| h_1^a | \) to obtain any useful information about \( \delta q^a \) without further assumptions. Soffer \[1\] suggests that his inequality applies to the valence quark distributions in the nucleon, however the only circumstances in which we find a useful bound is if we assume that the nucleon contains no antiquarks at all \( (f_1^a = g_1^a = h_1^a = 0) \), which is known to be false.

IV. SATURATION OF SOFFER’S INEQUALITY

There are some special circumstances for which Soffer’s inequality is saturated, i.e. \( 2|h_1^a(x)| = f_1^a(x) + g_1^a(x) \). It is useful to consider such cases in order to develop some intuition about the distribution of spin within the nucleon and to speculate on how saturation may be used to estimate \( h_1(x) \) in regions of experimental interest. The most trivial case is a model in which all the spin and flavor information of the proton is carried by a single quark, either in a non-relativistic quark model (NRQM) or the bag model. In the NRQM, if two quarks are always in a spin and flavor scalar configuration, then the third quark will yield \( h_1^a(x) = f_1^a(x) = g_1^a(x) \) — a consequence of the rather trivial Dirac structure of non-relativistic spinors. The bag model is less trivial due to the lower component \( p \)-wave contribution. Nonetheless, the saturation remains valid. In more realistic case of an SU(6) wave function, the saturation only holds for the d-quark, as we will demonstrate below.

The possibility of saturation is related to a possible symmetry between the amplitudes \( a_{++}(X) \) and \( a_{--}(X) \) defined in Eq. (3). In particular, if \( a_{++}(X) = a_{--}(X) \) for all states \( X \) contributing to the sums which define \( f_1 \), etc. in Eq. (10), then the inequality is saturated with the + sign for the absolute value. To relate \( a_{++} \) to \( a_{--} \) consider the unitary operator, \( U \) defined as the product of parity, \( \Pi \), and a rotation by 180° about an axis perpendicular to \( \hat{P} \), \( U \equiv \Pi R_2(\pi) \). Here we have chosen \( \hat{P} \) to define the \( \hat{e}_3 \)-axis and rotated (by \( \pi \)) about the \( \hat{e}_2 \)-axis. It is easy to see that \( U \) transforms \( |P^+ \rangle \) into \( |P^- \rangle \) up to a phase. Likewise, \( U \) transforms \( \phi_+ \) into \( \phi_- \) up to a phase [Note that \( \Pi \psi(0)\Pi = \gamma^0\psi(0) = \rho_1\sigma_3\psi(0) \) and \( R_2(\pi)\psi(0)R_2(\pi) = -i\rho_3\sigma_2\psi(0) \), so \( U \psi(0)U = -i\rho_3\sigma_1\psi(0) \).] Applying this transformation to \( a_{++}(X) \) we obtain,

\[
a_{++}(X) = \text{phase} \times a_{--}(UX).
\]

(22)
So the saturation of the identity resolves down to the question of whether \( X \) is an eigenstate of the operator \( U \). In simple valence quark models, the state \( X \) consists merely of the two spectator quarks left behind when the operator \( \phi_+ \) annihilates one quark in the target state \( |P+\rangle \).

First consider, for definiteness, the \( d \) down quark distribution in a simple constituent quark model of the proton. The two spectator \( u \)–quarks must be in a \( J = 1 \) state on account of Fermi statistics. Thus the angular momentum structure of the wavefunction is,

\[
|\hat{P} = \hat{e}_3 + \rangle = \sqrt{\frac{7}{3}} \{uu\}^{J=1,J_3=1} d^\dagger - \sqrt{\frac{1}{3}} \{uu\}^{J=1,J_3=0} d^\dagger.
\]  

(23)

Only the second term contributes to \( a_{++} \), leaving the spectator state \( |X\rangle = -\sqrt{\frac{1}{3}} \{uu\}^{J=1,J_3=0} \), which clearly is an eigenstate of \( U \). A careful accounting of all the phases yields

\[
a_{++}(X) = -\eta_P \eta_u^3 a_{--}(X)
\]  

(24)

where \( \eta_P \) (\( \eta_u \)) is the intrinsic parity of the nucleon (quark) and the negative sign arises from the conventional Condon and Shortley phases in the Clebsch-Gordon series. Since the relative parity of the quark and nucleon is positive, the factor \(-\eta_P \eta_u^3\) is minus one, and the inequality is saturated with the absolute value of \( h_d^4 \). The structure functions are in the ratios \((f^u_1 : g^u_1 : h^u_4) = (1 : -\frac{1}{3} : -\frac{1}{3})\). However, due to the effects of \( p \)-wave, the saturation does not occur for \( d \) quark in the bag model.

For the \( u \) up quark distribution in the proton the situation is different. The spectator \( u \) and \( d \) quarks are in a mixed spin state, \( J = 1 \) and 0. Annihilating a \( u \)-quark with positive helicity in Eq. (23) leaves the spectator state \( |X\rangle = \sqrt{\frac{1}{11}} \{ud\}^{J=1,J_3=0} + 3 \{ud\}^{J=0} \). Annihilating a \( u \)-quark with negative helicity in the proton with negative helicity leaves the state \( |X\rangle = \sqrt{\frac{2}{10}} \{ud\}^{J=1,J_3=0} - 3 \{ud\}^{J=0} \). The relative sign change for the \( J = 0 \) and \( J = 1 \) parts means that \( a_{++}(X) \) is not a simple multiple of \( a_{--}(X) \) — there is no analog of Eq. (24) and hence, no saturation. In fact \((f^u_1 : g^u_1 : h^u_4) = (2 : \frac{4}{3} : \frac{4}{3})\) for the NRQM.

We see that the saturation of Soffer’s inequality for the \( d \)-quark follows from the particularly simple spin structure of the nucleon in quark models. It is easy to construct a more elaborate model in which even that saturation fails. For example, suppose we introduce a component into the nucleon wavefunction in which the spectators are coupled to total angular momentum \( J = 0 \), say, \(|\{uug\}^{J=0} d^\dagger\rangle\), where \( g \) is a gluon. Then the state \( X \) is a superposition of components, one with \( J = J_3 = 0 \) and the other with \( J = 1, J_3 = 0 \). These two components transform with opposite sign under \( U \) and thereby ruin Eq. (24).

Finally we consider the relationship between QCD evolution and saturation of the inequality. Since \( f_1, g_1 \) and \( h_1 \) evolve differently with \( Q^2 \), saturation is incompatible with evolution. We can understand this in light of the discussion of the previous paragraph — evolution mixes gluons (and \( q\bar{q} \) pairs) into the nucleon wavefunction destroying the simple structure responsible for saturation. Quark model relationships, like saturation of the inequality for \( d \)-quarks in the proton, should be interpreted as “boundary data” for QCD evolution [3], valid at some low scale \( \mu_0^2 \). The implications for experiments carried out at much larger scales must be obtained by evolution from \( \mu_0^2 \) to the experimental scale, \( Q^2 \). In the case of saturation, some remnant of a prediction for the down quark contribution to \( h_1 \)
in the proton might be obtained when good data on the $d$-quark contributions to $f_1$ and $g_1$ become available.

V. ACKNOWLEDGEMENTS

We thank Jacques Soffer for discussions and a prepublication copy of Ref. [1].
REFERENCES

[1] J. Soffer, Marseille Preprint CPT-94/P.3059 (September 1994).
[2] R. L. Jaffe and X. Ji, *Phys. Rev. Lett.* **67** (1991) 552 .
[3] R. L. Jaffe and X. Ji, *Nucl. Phys.* B375 (1992) 527.
[4] G. Bunce, *et. al.*, *Particle World* **3** (1991) 1.
[5] The HERMES collaboration, a proposal to HERA.
[6] C. Bourrely, E. Leader, and J. Soffer, *Phys. Reports* **59** (1980) 95.
[7] C. Callan and D. J. Gross, *Phys. Rev. Lett.* **22** (1969) 156.
[8] G. Parisi and R. Petronzio, *Phys. Lett.* **62B** (1976) 331.
[9] R. L. Jaffe and G. G. Ross, *Phys. Lett.* **93B** (1980) 313 .
[10] B. L. Ioffe, V. A. Khoze, and L. N. Lipatov, *Hard Processes*, North-Holland, Amsterdam, 1984.
[11] G. G. Altarelli, R. K. Ellis, and G. Martinelli, *Nucl. Phys.* **B143** (1978) 521.
[12] X. Ji, *Nucl. Phys.* **B402** (1993) 217.
[13] J. Kogut and D. Soper, *Phys. Rev. D* **1** (1970) 2901 .
FIGURES

FIG. 1. a). The hand-bag diagram for deep-inelastic scattering. b) Quark-nucleon scattering amplitudes in $s$ and $u$ channels. The momentum and helicity labels are shown explicitly.

FIG. 2. Three independent helicity amplitudes in $u$-channel.