NUMBER IDENTITIES AND INTEGER PARTITIONS

CRAIG CULBERT

Abstract. Using a specific form of the triple product identity, polygonal number identities are stated. Further number identities are examined that can be considered identities related to modular sets of numbers. The identities can be used to give results on integer partitions with parts from numbers in modular arithmetic progression. This includes recursive formulas for the number of partitions using these modular parts. The triple product identity can derive further recursive formulas. Additionally, there is a recursive formula for the related sum of divisors function. The specific triple product identity provides a framework to examine all the identities and can be used to define related theta functions.

1. Number identities

Euler’s classic pentagonal number identity, Theorem 353 of [8] or Corollary 1.7 of [2], has exponents that are the general pentagonal numbers indexed by the integers. There are identities that involve triangular numbers and square numbers that can be found in Section 19.9 of [8]. Considering other polygonal numbers, Theorem 355 of [8] is an identity where the exponents are the general heptagonal numbers. Additionally, two general polygonal number identities attributed to Berger can be found in [5] or [4]. All these identities can be derived from Jacobi’s triple product identity: Theorem 352 of [8] or Theorem 2.8 of [2]. This form of the triple product identity will be referred to as the traditional form of the identity.

In Section 57 of [16], Sylvester references that Jacobi took the traditional form of the triple product identity and through substitution made the form more convenient for the study of number identities. This will be the strategy employed in this article, but rather than use this form of the identity, an alternative form of the triple product identity is used. Denote the integers by $\mathbb{Z}$, the positive integers by $\mathbb{I}$ and the natural numbers including 0 by $\mathbb{N}$. The variables $q$ and $z$ are complex numbers and consider the following form of the triple product identity.

---

Date: May 23, 2018.

2010 Mathematics Subject Classification. 05A17, 11P81.

Key words and phrases. Integer partitions, number identities, polygonal numbers, recursive formulas, theta functions.
**Theorem 1.1.** For $|q| < 1$ and $z \neq 0$,

$$
\prod_{m \in I} (1 - q^m)(1 + q^{m-1}) = \sum_{j \in \mathbb{Z}} q^{\frac{j^2}{2}} z^j
$$

(1.1)

The identity is well known and can be proven by modifying the proof of Theorem 2.8 of [2]. The identity can be found in a slightly different form on page 285 of [8] and as Theorem 11 from Chapter 8 of [1]. Following Jacobi, in the triple product identity replace $q$ by $q^k$ and $z$ by $\pm q^\ell$. More generally, $z$ can be replaced by $q^\ell z$.

As the eventual goal is to relate the identities to integer partitions with positive integer parts, consider $k$ a positive integer and $\ell$ a natural number restricted to $0 \leq \ell \leq \frac{k}{2}$. Each of the following infinite products is same for the pairs $(k, \ell)$ and $(k, k-\ell)$. The equivalent series have the same terms except they are indexed distinctly. Consequently, it is possible to consider the restriction on $\ell$ of $0 \leq \ell \leq \frac{k}{2}$.

$$
\prod_{m \in I} (1 - q^{km})(1 + q^{km-\ell})(1 + q^{k(m-1)+\ell}) = \sum_{j \in \mathbb{Z}} q^{k(j-1)+\ell j}
$$

(1.2)

$$
\prod_{m \in I} (1 - q^{km})(1 - q^{km-\ell})(1 - q^{k(m-1)+\ell}) = \sum_{j \in \mathbb{Z}} (-1)^j q^{k(j-1)+\ell j}
$$

(1.3)

For $\gamma = \pm 1$, the products can be written using $q$-Pochhammer symbols as:

$$(q^k; q^k)_{\infty} (-\gamma q^{k-\ell}; q^k)_{\infty} (-\gamma q^{\ell}; q^k)_{\infty} = (q^k, -\gamma q^{k-\ell}, -\gamma q^{\ell}; q^k)_{\infty}$$

The two identities are similar to the special identities of section 19.9 of [8]. The second identity can be related naturally to integer partitions and is similar to Corollary 2.9 of [2] where the $k$ is replaced by $2k + 1$. It is possible now to consider polygonal number identities for the general polygonal numbers, those indexed by $\mathbb{Z}$.

The classic polygonal numbers, those related to polygonal figures, are indexed by $I$ and can be found on page 1 of [4]. Consider $k$ a positive integer and let $\ell$ equal 1, then the following are the general polygonal number identities of Berger where $k + 2$ is the number of sides of the polygon.

**Proposition 1.2.** For $|q| < 1$,

$$
\prod_{m \in I} (1 - q^{km})(1 + q^{km-1})(1 + q^{k(m-1)+1}) = \sum_{j \in \mathbb{Z}} q^{k(j-1)+j}
$$

$$
\prod_{m \in I} (1 - q^{km})(1 - q^{km-1})(1 - q^{k(m-1)+1}) = \sum_{j \in \mathbb{Z}} (-1)^j q^{k(j-1)+j}
$$
These identities are found on page 31 of [5] including a special general hexagonal identity, or in [4]. Adding or subtracting a pair of these identities gives further identities indexed by either even or odd integers. In order to consider what classes of numbers are present in the more general identities, specific sets of natural numbers are defined.

Let $k, \ell \in \mathbb{N}$, $k > 0$ with $0 \leq \ell \leq k$. Define $m_{k,\ell}(i) = k(i - 1) + \ell$ for $i \in \mathbb{I}$. These numbers constitute a modular arithmetic progression or sequence and $m_{k,\ell}(i)$ is the $i$-th number in the sequence. These numbers can be considered gnomons as defined on page 1 of [5]. The sum of these gnomons leads to specific classes of numbers, natural numbers greater than or equal to zero.

Consider the sum of these numbers in the progression, for $j \in \mathbb{I}$.

$$\sum_{i=1}^{j} m_{k,\ell}(i) = \sum_{i=1}^{j} k(i - 1) + \ell = \frac{k}{2}j(j - 1) + \ell j = M_{k,\ell}(j)$$

The following is also true:

$$\sum_{i=1}^{j} m_{k,k-\ell}(i) = \frac{k}{2}j(j + 1) - \ell j$$

These resulting sums form a class of numbers that can be considered figurate in a very general sense. The set of these numbers $M_{k,\ell}(\mathbb{I}) = \{M_{k,\ell}(j) \mid j \in \mathbb{I}\}$ are the modular figurate numbers for parameters $k$ and $\ell$. $M_{k,\ell}(j)$ is the $j$-th number. When indexed by the integers $\mathbb{Z}$, these are the general modular figurate numbers. Then $M_{k,\ell}(-j) = M_{k,k-\ell}(j)$, $M_{k,\ell}(\mathbb{Z}) = M_{k,k-\ell}(\mathbb{Z})$, and for a positive integer $c$, $M_{c,k,\ell}(j) = c \cdot M_{k,\ell}(j)$. All of the polygonal numbers and the general polygonal numbers can be considered modular figurate numbers for $\ell = 1$. Following the argument of Berger for the polygonal numbers in [4], for $i \neq j, M_{k,\ell}(i) \neq M_{k,\ell}(j)$ unless $\ell = 0$ with $i = 1 - j$, $\ell = k$ with $i = -1 - j$, or $\ell = \frac{k}{2}$ with $i = -j$. These number forms are also referenced in [11].

The modular figurate numbers are not the figurate numbers as defined on page 7 of [5]. Instead, basic trapezoidal figures are related to the form of these numbers, but the numbers are not all trapezoidal numbers. The modular figurate numbers with the above restriction on $\ell$ can be considered generalized trapezoidal numbers, the $k$-trapezoidal numbers defined in [9] without the restriction on $\ell$.

Having defined this class of numbers, when (1.2) and (1.3) are restricted to positive integers $k$ and natural numbers $\ell$ with $0 \leq \ell \leq k$, these are identities of the modular figurate numbers. Returning to the general setting of the triple product identity, (1.1)
can also be obtained by first considering the following using Gaussian coefficients \([ ]_q\).

\[
\prod_{m=1}^{n} (1 + q^m z^{-1})(1 + q^{m-1}z) = \sum_{j=-n}^{n} \left[ \frac{2n}{n+j} \right] q^{j^2 + j} z^j \tag{1.4}
\]

The result can be proven using properties of the Gaussian coefficients and induction. It is a modification of a result of Hermite found on page 49 of \([2]\). It is also possible to substitute in this equation \(q^k\) and \(\pm q^\ell\) to find a form with values \(k\) and \(\ell\). Of course many of these identities are known and the traditional triple product identity could be used to derive all the earlier identities. The use of the present form of the triple product identity, \((1.1)\), is to give a specific symmetric framework to the study of the identities.

2. Integer partitions

In order to use \((1.3)\) to study integer partitions with distinct parts, the boundary conditions are first examined for both identities and then excluded. Assuming \(k\) is a positive integer with \(0 \leq \ell \leq k\), one boundary case is when \(\ell = 0\) or by symmetry \(\ell = k\). The other case further assumes \(k\) is an even positive integer and \(\ell = \frac{k}{2}\). As the forms of these boundary cases are related to the triangular and square numbers, their distinct properties can be derived from the identities found in Section 19.9 of \([8]\) and Corollary 2.10 of \([2]\).

For the case \(\ell = 0\), \((1.3)\) is identically zero, but it is still possible to combine the two identities to get specialized identities. It is also possible to substitute \(q^k\) into a triangular number identity, (2.2.13) of \([2]\), to get a further identity for this boundary case.

For the second boundary case when \(k\) is an even positive integer and \(\ell = \frac{k}{2}\), a substitution of \(q^\frac{k}{2}\) into a square number identity, (2.2.12) of \([2]\), gives a specialized identity for this case. Another similar identity is the following.

\[
\prod_{m \in \mathbb{I}} \frac{(1 - q^{km})(1 + q^{km-\frac{k}{2}})}{(1 + q^{km})(1 - q^{km-\frac{k}{2}})} = \sum_{j \in \mathbb{Z}} q^{\frac{k}{2}j^2} \tag{1.5}
\]

In this case, where \(k\) is a even positive integer, there is some flexibility in combining \((1.2)\) and \((1.3)\) in order to get identities indexed by odd or even integers, and to get identities indexed by natural numbers or positive integers. One additional identity is needed.

\[
\prod_{m \in \mathbb{I}} \left(1 + q^{km}\right)\left(1 + q^{km-\frac{k}{2}}\right)\left(1 - q^{km-\frac{k}{2}}\right) = 1 \tag{1.6}
\]

Excluding the boundary conditions, the earlier identities give results about integer partitions with distinct parts. Let \(k\) and \(\ell\) be positive integers with \(k \geq 3\) and
0 < ℓ < k, ℓ \neq \frac{k}{2}. Consider sets of numbers from the modular arithmetic progression.

\[ J_{k,\ell,s} = \{ m_{k,\ell}(i) \mid i \in \mathbb{I}, 1 \leq i \leq s \} \cup \{ m_{k,k-\ell}(i) \mid i \in \mathbb{I}, 1 \leq i \leq s \} \]

\[ J_{k,\ell} = \{ m_{k,\ell}(i) \mid i \in \mathbb{I} \} \cup \{ m_{k,k-\ell}(i) \mid i \in \mathbb{I} \} \]

Then \( J_{k,\ell} \) is the set of all positive integers congruent to either ±\( \ell \) modulo \( k \). It is also true that \( J_{k,\ell,s} = J_{k,k-\ell,s} \) and \( J_{k,\ell} = J_{k,k-\ell} \). Let \( J \) be a set of positive integers and define \( p_{dt}(n; J) \) as the number of partitions of the integer \( n \) using distinct parts only from the set \( J \). Denote the generating function for \( p_{dt}(n; J) \) as \( f_1(q) \) as found in [2]. The length of a partition is the number of parts and it is possible to divide partitions into classes with an even or odd length. Define \( r_{dt}(n; J) \) as the difference between the number of the above partitions with even length and odd length, which has integer entries and \( r_{dt}(0; J) = 1 \). For negative integers the number of partitions is always zero and denote the generating function for \( r_{dt}(n; J) \) as \( g_1(q) \). In general, generating functions for partition sequences that are the difference between even and odd length can be found substituting \( z = -1 \) for the length parameter in the two variable generating functions found in Chapter 2 of [2].

Consider now the modified result of Hermite, (1.4). One substitution gives an identity equal to the generating function of \( p_{dt}(n; J_{k,\ell,s}) \) and the other substitution gives an identity equal to the generating function of \( r_{dt}(n; J_{k,\ell,s}) \). Both involve the modular figurate numbers \( M_{k,\ell}(j) = \frac{k}{2}j(j-1) + \ell j \).

\[
\sum_{n \in \mathbb{N}} p_{dt}(n; J_{k,\ell,s})q^n = \sum_{j=-s}^{s} \left[ \frac{2s}{s+j} \right] q^{M_{k,\ell}(j)}
\]

\[
\sum_{n \in \mathbb{N}} r_{dt}(n; J_{k,\ell,s})q^n = \sum_{j=-s}^{s} (-1)^j \left[ \frac{2s}{s+j} \right] q^{M_{k,\ell}(j)}
\]

It is now possible to write the above as a single formula, if some other notation is introduced. Identify ±1 with ± and let \( p_{dt}(n; J) = p_{dt}^+(n; J) \) and \( r_{dt}(n; J) = p_{dt}^-(n; J) \). This notation should not be confused with notation relating partitions to even or odd permutations. It is possible to add or subtract the generating functions to get expressions equivalent to the generating functions for partitions on these sets of even or odd length.

For a positive integer \( c \), define \( cJ = \{ ci \mid i \in J \} \). Define a further set of parts, \( J_{k,\ell} = k\mathbb{I} \cup J_{k,\ell} \). Then \( J_{k,\ell} \) is the set of all positive integers congruent to either 0 or ±\( \ell \) modulo \( k \), and \( J_{k,\ell} = J_{k,k-\ell} \). From this set of distinct parts \( [1,3] \) gives the generating
function for partition function \( r_{ad}(n; I_{k,\ell}) \).

\[
g_1(q) = \sum_{n \in \mathbb{N}} r_{ad}(n; I_{k,\ell}) q^n = \sum_{j \in \mathbb{Z}} (-1)^j q^{\frac{x}{2}j(j-1)+\ell j}
\]

Restating the identity as a result on integer partitions is this proposition for \( k \geq 3 \), \( 0 < \ell < k \) and \( \ell \neq \frac{k}{2} \).

**Proposition 2.1.** Let \( n \) be an integer.

\[
r_{ad}(n; I_{k,\ell}) = \begin{cases} (-1)^j & \text{if } n = M_{k,\ell}(j), \\ 0 & \text{otherwise.} \end{cases}
\]

This result is similar to Theorem 2.11 of [2]. It appears that this proposition is due to Sylvester as it is referenced in [11] and included there among combinatorial proofs of the triple product identity. For the positive integers \( I = J_{3,1} \), Franklin's combinatorial proof of the partition result derives the pentagonal number identity, Theorem 1.6 of [2].

Euler used the values of \( r_{ad}(n; I) \) to derive a recursive formula for the partition function \( p(n) \), Corollary 1.8 of [2]. A similar idea is now possible for unrestricted partitions on a specific set of positive integers parts. Define the number of partitions of an integer \( n \) using only parts from a set \( J \) as \( p(n; J) \) with generating function \( f(q) \) as in [2] and similarly define \( r(n; J) \) with generating function \( g(q) \). The generating functions for \( r_{ad}(n; J) \) and \( p(n; J) \) as infinite products are reciprocals, consequently the generating functions are product inverses, \( g_1(q) f(q) = 1 \). The values for \( r_{ad}(n; J_{k,\ell}) \) lead to a recursive formula for \( p(n; J_{k,\ell}) \) as the number of partitions for negative integers is zero and \( p(0; J_{k,\ell}) = 1 \).

**Theorem 2.2.** Let \( n \) be a positive integer.

\[
p(n; J_{k,\ell}) = \sum_{j \in \mathbb{Z} - \{0\}} (-1)^{j-1} p(n - M_{k,\ell}(j); J_{k,\ell})
\]

\[
= \sum_{j \in \mathbb{I}} (-1)^{j-1} p(n - M_{k,\ell}(j); J_{k,\ell}) + \sum_{j \in \mathbb{I}} (-1)^{j-1} p(n - M_{k,k-\ell}(j); J_{k,\ell})
\]

For a positive integer \( c \), \( p(cn; J_{ck,\ell}) = p(cn; c J_{k,\ell}) = p(n; J_{k,\ell}) \) and the theorem gives a distinct recursive formula when \( k \) and \( \ell \) are relatively prime. Considering other sets of positive integer parts, it is also possible to study the set of parts \( k \mathbb{I} \), \( k \geq 1 \), using the results of Euler. If \( I_{k,\ell} \) is the set of positive integers congruent to \( \ell \) modulo \( k \), substitutions into Corollary 2.2 of [2] give results for the partition functions: \( p(n) \), \( r(n) \), \( p_{ad}(n) \), \( r_{ad}(n) \) and the parts \( I_{k,\ell} \).
3. Recursive formulas

The previous recursive formula is derived from the triple product identity and it is possible to identify other recursive formulas. Consider the product of two generating functions equaling a third, \( a(q)b(q) = c(q) \), then there is a relationship between their coefficients:

\[
\sum_{k=0}^{n} a(k)b(n-k) = c(n)
\]

If both \( a(q) \) and \( c(q) \) are generating functions with known formulas from either (1.2) or (1.3), then a recursive formula for \( b(n) \) can be derived by isolating the term \( b(n-0) \). There are two cases for the coefficients \( c(n) \), nonzero or zero, and only the nonzero \( a(k) \) summands are used. If only \( a(q) \) has a known formula, then there is an identity relating the sequence terms \( b(n) \) and \( c(n) \). In [13], the triple product identity is used to derive specific recursive formulas and a similar procedure will be used focusing on examples that exclude the boundary cases. By excluding examples that use triangular and square numbers, the remaining modular figurate numbers are all indexed uniquely.

Let \( \gamma_1 = \pm 1 \), \( \gamma_2 = \pm 1 \) and consider a function that is the quotient of two formulas derived from (1.1). Assume neither product is identically zero. The function \( H(q) \) converges for \( |q| < 1 \) and is a generating function indexed by \( \mathbb{N} \).

\[
H(q) = \frac{(q^{k_2}, -\gamma_2 q^{k_2-\ell_2}, -\gamma_2 q^{\ell_2} : q^{k_2})_{\infty}}{(q^{k_1}, \gamma_1 q^{k_1-\ell_1}, \gamma_1 q^{\ell_1} : q^{k_1})_{\infty}}
\]

For the pairs \((k_1, \ell_1)\) and \((k_2, \ell_2)\), let the associated modular figurate numbers be \( M_1(Z) \) and \( M_2(Z) \), respectively. Denote the sequence that has \( H(q) \) as its generating function, \( s_{\gamma_1, \gamma_2}(n) \). Consider \( H(q) \) multiplied by the product indexed by one and the related generating functions. Using the values derived from either (1.2) or (1.3) gives a recursive formula for the sequence \( s_{\gamma_1, \gamma_2}(n) \) as \( s_{\gamma_1, \gamma_2}(0) = 1 \) and consider \( s_{\gamma_1, \gamma_2}(n) = 0 \) for negative integers. For the following theorem, assume that neither pair \((k_1, \ell_1)\), \((k_2, \ell_2)\) is from a boundary case.

**Theorem 3.1.** Let \( n \) be a positive integer. Let \( \gamma_1 = \pm 1 \) and \( \gamma_2 = \pm 1 \).

\[
s_{\gamma_1, \gamma_2}(n) = \begin{cases} 
\sum_{j \in \mathbb{Z}-\{0\}} (-\gamma_1)^j s_{\gamma_1, \gamma_2}(n - M_1(j)) + (\gamma_2)^i & \text{if } n = M_2(i), \\
\sum_{j \in \mathbb{Z}-\{0\}} (-\gamma_1)^j s_{\gamma_1, \gamma_2}(n - M_1(j)) & \text{otherwise}.
\end{cases}
\]

If the function \( H(q) \) involves a boundary case, that is the triangular or square numbers, the argument is subtly different as evidenced by Theorem 1 or Theorem 3 of [13]. Two consequences of the theorem for partition sequences, \( k \geq 3 \) and
0 < \ell < k, \ell \neq \frac{k}{2}, now follow. First, let \((k_2, \ell_2) = (k, \ell)\) and \((k_1, \ell_1) = (3k, k)\) with \(\gamma_1 = 1\). Then \(M_1(\mathbb{Z}) = M_{3k,k}(\mathbb{Z}) = kM_{3,1}(\mathbb{Z}) = k\mathbb{I}, \gamma = \pm 1\) and for function \(H(q)\) the following are equal.

\[
\frac{(q^k, -q^{k-\ell}, -q^\ell; q^k)_\infty}{(q^{3k}, q^{2k}, q^k; q^{3k})_\infty} = \frac{(q^k; q^k)_\infty}{(q^k; q^k)_\infty} \frac{(-q^{k-\ell}, -q^\ell; q^k)_\infty}{(q^k; q^k)_\infty}
\]

Then \(H(q)\) equals generating functions for partition functions involve the sets \(J_{k,\ell}\).

**Proposition 3.2.** Let \(n\) be a positive integer and \(\gamma = \pm 1\).

\[
p^\gamma(n; J_{k,\ell}) = \left\{ \begin{array}{ll}
\sum_{j \in \mathbb{Z}-\{0\}} (-1)^{j-1} p^\gamma(\gamma n \gamma k \omega(j); J_{k,\ell}) + (\gamma)^j \quad & \text{if } n = M_{k,\ell}(i), \\
\sum_{j \in \mathbb{Z}-\{0\}} (-1)^{j-1} p^\gamma(n - k \omega(j); J_{k,\ell}) \quad & \text{otherwise.}
\end{array} \right.
\]

By a similar procedure, consider \(\gamma_2 = -1\) and \(M_2(\mathbb{Z}) = M_{3k_1,k_1}(\mathbb{Z}) = k_1M_{3,1}(\mathbb{Z}) = k_1\mathbb{I}\). This leads to a recursive formula for unrestricted partitions and the set \(J_{k,\ell}\).

**Proposition 3.3.** Let \(n\) be a positive integer and \(\gamma = \pm 1\).

\[
p^\gamma(n; J_{k,\ell}) = \left\{ \begin{array}{ll}
\sum_{j \in \mathbb{Z}-\{0\}} (-\gamma)^2 p^\gamma(n - M_{k,\ell}(j); J_{k,\ell}) + (-1)^i \quad & \text{if } n = k \omega(i), \\
\sum_{j \in \mathbb{Z}-\{0\}} (-\gamma)^2 p^\gamma(n - M_{k,\ell}(j); J_{k,\ell}) \quad & \text{otherwise.}
\end{array} \right.
\]

Again, by adding or subtracting a pair of generating functions gives a generating function for partitions of even and odd length. If instead, for \(\gamma_1 = 1\), the relationship between the generating functions is \(H(q) = g_1(q)f(q)\), then the following identities are possible.

**Proposition 3.4.** Let \(n\) be a natural number and \(\gamma = \pm 1\).

\[
p^\gamma(n; J_{k,\ell}) = \sum_{j \in \mathbb{Z}} (\gamma)^j p(n - M_{k,\ell}(j); k\mathbb{I})
\]

\[
p(n; J_{k,\ell}) = \sum_{j \in \mathbb{Z}} (-1)^j p(n - k \omega(j); J_{k,\ell})
\]
Theorem 5 of [13] contains the generating function for partitions excluding the parts congruent to 0 modulo \(d+1\). This is also equal to the partitions where no part is repeated more than \(d\) times, Corollary 1.3 of [2]. It is the later form that will be generalized. For a set of positive integers \(J\), the number of partitions of an integer \(n\) where no part appears more than \(d\) times is denoted by \(p_d^J(n; J)\), \(d \geq 1\). In this way, \(p_1(n; J) = p_a(n; J)\). The generating function for \(p_d^J(n; J)\) is found in [2].

Consider \(\gamma_1 = 1, \gamma_2 = -1\) and \((k_2, \ell_2) = ((d+1)k, (d+1)\ell)\), then \(H(q)\) equals the following:

\[
H(q) = \frac{(q^{k_2}, q^{k_2-\ell_2}, q^{\ell_2}; q^{k_2})_\infty}{(q^k, q^{k-\ell}, q^{\ell}; q^k)_\infty} = \prod_{m \in J_{k,\ell}} \frac{1 - q^{(d+1)m}}{1 - q^m}
\]

\(H(q)\) is the generating function for \(p_d^J(n; J_{k,\ell})\) which gives the following recursive formula.

**Proposition 3.5.** Let \(n\) be a positive integer.

\[
p_d^J(n; J_{k,\ell}) = \begin{cases} 
\sum_{j \in \mathbb{Z}} (-1)^{j-1} p_d(n - M_{k,\ell}(j); J_{k,\ell}) + (-1)^i & \text{if } n = (d+1)M_{k,\ell}(i), \\
\sum_{j \in \mathbb{Z}} (-1)^{j-1} p_d(n - M_{k,\ell}(j); J_{k,\ell}) & \text{otherwise.}
\end{cases}
\]

The case of \(J_{3,1} = \mathbb{I}\) is Theorem 5 of [13] and there is also an identity between partition values.

**Proposition 3.6.** Let \(n\) be a natural number.

\[
p_d^J(n; J_{k,\ell}) = \sum_{j \in \mathbb{Z}} (-1)^j p(n - (d+1)M_{k,\ell}(j); J_{k,\ell})
\]

Chapter 12 of [15] contains three relationships between partition functions and the sum of divisors function, \(\sigma(n)\). These are a recursion for \(p(n)\) using \(\sigma(n)\), a recursion for \(\sigma(n)\) using \(r_a(n; \mathbb{I})\) and the general pentagonal numbers, and an identity for \(\sigma(n)\) using \(p(n)\) and the general pentagonal numbers. It is possible to generalize these three concepts for partitions in this article. Theorem 14.8 of [3] provides formulas to derive recursions involving \(p(n; J)\) or \(r_a(n; J)\), and the following divisors function.

Let \(J\) be a set of positive integers and \(n \in \mathbb{I}\).

\[
f_J(n) = \sum_{d \mid n, d \in J} d
\]

For \(n \in \mathbb{Z}\), \(f_J(n)\) is zero for negative integers and zero. Denote the generating function for \(f_J(n)\) as \(F(q)\). The exposition for Theorem 14.8 of [3] gives the relationships between the generating functions, \(qf^J(q) = f(q)F(q)\) and \(qg_1^J(q) = -F(q)g_1(q)\).
through logarithmic differentiation. These relationships can be used to give a recursive formulas using \( f_J(n) \) for \( p(n; J) \) and \( r_{a(J_k, \ell)} \), respectively. In order to use the values for \( r_{a(J_k, \ell)} \), Proposition 2.1 define \( f_{k, \ell}(n) = f_J(n) \) for the sets \( J_{k, \ell} \).

Such a divisors function is found in [10] using a different notation. Theorem 14.8 of [3] gives the following equivalence.

\[
n \cdot r_{a(J_k, \ell)}(n) = -f_{k, \ell}(n) - \sum_{j=1}^{n-1} r_{a(J_k, \ell)}(n-j)
\]

Isolating \( f_{k, \ell}(n) \), using the values for \( r_{a(J_k, \ell)}(n) \) and indexing over the nonzero integers gives a finite recursive formula for this sum of divisors function for \( k \geq 3 \) and \( 0 < \ell < k, \ell \neq \frac{k}{2} \).

**Theorem 3.7.** Let \( n \) be a positive integer.

\[
f_{k, \ell}(n) = \begin{cases} 
\sum_{j \in \mathbb{Z} - \{0\}} (-1)^{j-1} f_{k, \ell}(n - M_{k, \ell}(j)) + (-1)^{j-1} M_{k, \ell}(i) & \text{if } n = M_{k, \ell}(i), \\
\sum_{j \in \mathbb{Z} - \{0\}} (-1)^{j-1} f_{k, \ell}(n - M_{k, \ell}(j)) & \text{otherwise}.
\end{cases}
\]

Chapter 12 of [15] provides the process to find an identity that relates \( f_{k, \ell}(n) \) and \( p(n; J_{k, \ell}) \). For this remaining identity, \( F(q) = -qg'_1(q)f(q) \) is the relationship between the generating functions. This results in an identity of S. Kim found in [10]. There the identity was proven without the use of Proposition 2.1 and the notation differs from the notation in this article. As with the above recursions, it is possible to state this identity of Kim using the modular figurate numbers \( M_{k, \ell}(j) \).

4. **Further Concepts**

Returning to the triple product identity (1.1), it is possible to consider a theta function and related functions from the identity. Chapter 10 of [15] provides a guide to the motivation to define a theta function. Consider for complex \( \nu, \tau \) with \( \text{Im}(\tau) > 0 \).

\[
\Theta(\nu; \tau) = \sum_{n \in \mathbb{Z}} \exp\left(2\pi i\frac{n(n-1)}{2} + 2\pi i\nu \right)
\]

The function \( \Theta \) has the properties that \( \Theta(\nu + \tau; \tau) = \exp(-2\pi i\nu)\Theta(\nu; \tau) \) and \( \Theta(\nu + 1; \tau) = \Theta(\nu; \tau) \). With the substitutions \( q = \exp(2\pi i\tau) \) and \( z = \exp(2\pi i\nu) \) this gives the form in the triple product identity above.

\[
\Theta(z | q) = \sum_{n \in \mathbb{Z}} q^{\frac{n(n-1)}{2}} z^n
\]
It is then true that $\Theta(qz \mid q) = z^{-1}\Theta(z \mid q)$ and it is possible to define auxiliary theta functions similar to the Jacobi theta functions.

\[
\begin{align*}
\theta_a(z \mid q) &= \Theta(z \mid q) \\
\theta_b(z \mid q) &= \Theta(-z \mid q) \\
\theta_c(z \mid q) &= q^{\frac{1}{8}}z^{\frac{1}{2}}\Theta(q^{\frac{1}{2}}z \mid q) \\
\theta_d(z \mid q) &= q^{\frac{1}{8}}z^{\frac{1}{2}}\Theta(-q^{\frac{1}{2}}z \mid q)
\end{align*}
\]

Each of these four auxiliary functions, when thought of as functions of $\nu$ and $\tau$, is a solution to a specific second order partial differential equation. If the four functions are considered one class of theta functions, then the substitutions $q$ replaced by $q^k$ and $z$ replaced by $q^\ell z$ give another class of theta functions. The pair $(k, l)$ equal to $(2, 1)$ gives a form of the Jacobi theta functions. In general, the functions $\theta_a$ and $\theta_b$ have infinite product forms derived from (1.1) and product forms can also be determined for $\theta_c$ and $\theta_d$.

Each class of theta functions potentially could be useful in the study of the number of representations as sums of general polygonal numbers or general modular figurate numbers. In Chapter 3 of [12], the explanation of the Jacobi theta functions includes their use in study of the number of representations as sums of squares. This involves identities of the null values of the theta functions. The functions most similar to those null values would be functions such as $\theta_a(\tau ; 2\tau)$.

The above theta functions are more closely related to the triangular numbers. The use of theta functions in the study of the number of representations as sums of triangular numbers can be found in [14]. In order to use such functions of $\tau$ to study sums of general polygonal numbers or general modular figurate numbers, identities of these functions would first need to be derived. That is to consider identities of the functions either for a specific class of theta functions or in general subject to the parameters $k$ and $\ell$.

References

[1] G. E. Andrews and K. Eriksson. Integer Partitions. Cambridge University Press, Cambridge, U.K., 2004.
[2] G. E. Andrews. The Theory of Partitions. Addison-Wesley, Reading, Mass, 1976.
[3] T. M. Apostol. Introduction to Analytic Number Theory. Springer-Verlag, New York, 1976.
[4] A. Berger. Recherches sur les nombres polygonaux. Nova Acta Reg. Soc. Sc. Upsaliensis, Ser. III, 17 (1898).
[5] L. E. Dickson. History of the Theory of Numbers: Volume II. Chelsea Publishing Company, New York, 1971.
[6] L. Euler. Introduction to Analysis of the Infinite: Book I. Springer-Verlag, New York, 1988.
[7] E. Grosswald. Representations of Integers as Sums of Squares. Springer-Verlag, New York, 1985.
[8] G. H. Hardy and E. M. Wright. *An Introduction to the Theory of Numbers, fifth edition*. Oxford University Press, Oxford, U.K., 1979.

[9] S. Jitman and C. Phongthai. On the characterization and enumeration of some generalized trapezoidal numbers. *Int. J. Math. Math. Sci.* (2017).

[10] S. Kim. Bijective proofs of partition identities and covering systems. Ph.D. Dissertation, University of Illinois at Urbana-Champaign, 2010.

[11] L. W. Kolitsch and S. Kolitsch. A combinatorial proof of Jacobi’s triple product identity. *Ramanujan J.*, 45 (2018), 483-489.

[12] H. McKeen and V. Moll. *Elliptic Curves*. Cambridge University Press, Cambridge, U.K., 1997.

[13] K. Ono, N. Robbins and B. Wilson. Some recurrences for arithmetical functions. *J. Indian Math. Soc. (N.S.)*, 62 (1996), 29-50.

[14] K. Ono, S. Robins and P. T. Wahl. On the representation of integers as sums of triangular numbers. *Aequationes Math.*, 50 (1995), 73-94.

[15] H. Rademacher. *Topics in Analytic Number Theory*. Springer-Verlag, New York, 1973.

[16] J. J. Sylvester. A constructive theory of partitions, arranged in three acts, an interact and an exodiant. *Amer. J. Math.*, 5 (1882), 231-330. Corrections. *Amer. J. Math.*, 6 (1884), 334-336.

Department of Computer Science and Mathematical Sciences, Penn State University Harrisburg, Middletown, PA 17057

E-mail address: cwc15@psu.edu