A universal time-dependent control scheme for realizing arbitrary bosonic unitaries

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We study the implementation of arbitrary unitary transformations between two sets of $N$ stationary bosonic modes, which are connected through a photonic quantum channel. By controlling the individual couplings between the modes and the channel, an initial $N$-partite quantum state in register A can be released as a multi-photon wavepacket and, successively, be reabsorbed in register B. Here we prove that there exists a set of control pulses that implement this transfer with arbitrarily high fidelity and, simultaneously, realize a pre-specified $N \times N$ unitary transformation between the two sets of modes. Moreover, we provide a numerical algorithm for constructing these control pulses and discuss the scaling and robustness of this protocol in terms of several illustrative examples. By being purely control-based and not relying on any adaptions of the underlying hardware, the presented scheme is extremely flexible and can find widespread applications, for example, for boson-sampling experiments, multi-qubit state transfer protocols or in continuous-variable quantum computing architectures.

Unitary transformations of bosonic modes play an integral part in many quantum information processing applications. For example, by sending a multi-mode photonic Fock state through a network of linear optical elements—thereby implementing such a unitary transformation—the output distribution of the photons is exponentially hard to predict on a classical computer [11], but can be simulated efficiently in a quantum experiment [2–9]. This problem, known as boson sampling, is one of the simplest examples for which a quantum computational advantage can be obtained. When combined with single photon sources and detectors, the same unitary transformations can even be used to realize a universal quantum computer according to the KLM scheme [10, 11]. Further, by encoding quantum information in continuous-variable degrees of freedom, one can benefit from efficient bosonic error correction schemes [12–15], a strategy that is currently explored in superconducting circuits [16–18] and trapped ion systems [19]. In larger networks, state transfer operations between such oscillator-encoded qubits require again the implementation of large unitary transformations between distant bosonic modes.

In most of these applications, unitary operations are realized by sending photons through an application-specific network of beam-splitters and phase shifters [20], with a limited amount of tunability. In this Letter, we describe a universal alternative strategy to achieve the same task through a controlled multi-photon emission and reabsorption process. The basic idea behind this approach is summarized in Fig. 1. Here two quantum registers A and B, each contain a set of $N$ bosonic modes, are connected by a unidirectional quantum channel. By controlling the coupling strength between the channel and each mode, an initial quantum state stored in register A is released as a multi-photon wavepacket into the channel and reabsorbed in register B. In the following analysis we demonstrate that, for any given $N \times N$ unitary matrix $U$, there exists a set of control pulses such that (i) the reabsorption of the emitted photons can be achieved with arbitrarily high fidelity and (ii) the whole process implements the transformation

$$b_j(t_f) = \sum_{k=1}^{N} U_{jk} a_k(t_0).$$

Here the $a_j(t_0)$ are the bosonic annihilation operators for the modes of register A at the initial time $t_0$ and the

![FIG. 1. (a) Sketch of the quantum network considered in this work. Two quantum registers A and B, each represented by $N$ bosonic modes, are connected via a unidirectional waveguide. By controlling the couplings $g_{A,j}(t)$ and $g_{B,j}(t)$ between the modes and the waveguide, a multi-photon wavepacket can be emitted from register A and successively be reabsorbed by the modes in register B. (b) A generic unitary transformation $U$ between the modes, which is conventionally implemented by (I) sending photons through a network of $N(N-1)/2$ beam-splitters, can be realized with our scheme in a time $t_p \sim N$ by (II) applying an appropriate choice of control pulses.](https://arxiv.org/)

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b_j(t_f) are the corresponding operators for the modes of register B at the final time \( t_f \) of the transfer. Moreover, we provide a numerical recipe for constructing the appropriate control pulses and show that even for completely random unitaries the overall protocol time, \( t_p = t_f - t_0 \sim N \), only scales linearly with the number of modes. Therefore, the current approach offers an efficient and very flexible way to realize unitary transformations, where the targeted operation is fully specified by the shape of the control pulses and not by the network layout.

Quantum network dynamics.—For the following analysis we focus on the quantum network shown in Fig. 1(a), where two sets of \( N \) bosonic modes, representing two separate registers A and B, are coupled to a unidirectional waveguide. We assume that all the modes have the same oscillation frequency \( \omega_0 \) and that they are coupled to the waveguide with tunable couplings \( g_{A,j}(t) \) and \( g_{B,j}(t) \), respectively, where \( j = 1, \ldots, N \) labels the modes within each register. Note that various schemes for realizing tunable couplings to waveguides have already been demonstrated, both in the optical [21–23] and in the microwave regime [24,31]. When combined with coherent circulators [22,33], chiral waveguides [34] or other types of directional couplers [35,37], a fully cascaded network, as assumed in this work, can be implemented.

Under the assumption that the spectrum of the waveguide is sufficiently broad and approximately linear, we can adiabatically eliminate the dynamics of the propagating photons and derive a set of cascaded quantum Langevin equations for the register modes [35,39]. In a frame rotating with \( \omega_0 \), we obtain

\[
\dot{c}_\mu(t) = -\frac{|g_\mu(t)|^2}{2} c_\mu(t) - g_\mu(t) f_{in,\mu}(t),
\]

(2)

together with the input-output relations

\[
f_{out,\mu}(t) = f_{in,\mu}(t) + g_\mu(t)c_\mu(t).
\]

(3)

Here, in order to simplify the notation, the index \( \mu \) runs over all \( 2N \) modes and we have made the identifications \( c_\mu \equiv a_\mu \) and \( g_\mu \equiv g_{A,\mu} \) for \( \mu = 1, \ldots, N \) and \( c_\mu \equiv b_{\mu-N} \) and \( g_\mu \equiv g_{B,\mu-N} \) for \( \mu = N+1, \ldots, 2N \). For the first node, the input field \( f_{in,1}(t) \equiv f_{in}(t) \) is a \( \delta \)-correlated noise operator, which satisfies \([f_{in}(t), f_{in}'(t')] = \delta(t-t')\). All other in-fields are determined by the relation \( f_{in,\mu}(t) = f_{out,\mu-1}(t) \), which captures the directional nature of the quantum channel. By iterating this relation and adopting a vector notation, \( \vec{c} = (c_1, \ldots, c_{2N})^T \) and \( \vec{g} = (g_1, \ldots, g_{2N})^T \), the set of quantum Langevin equations can be written in a compact form as

\[
\dot{\vec{c}}(t) = -\mathcal{M}(t)\vec{c}(t) - \vec{g}(t) f_{in}(t),
\]

(4)

couplings \( g_\mu(t) \) are complex numbers and constrained to \(|g_\mu(t)| \leq 1\). A detailed derivation of Eq. (4) can be found in the supplementary material [40].

The general solution of Eq. (4) can be written as

\[
\vec{c}(t) = \mathcal{G}(t, t_0)\vec{c}(t_0) - \int_{t_0}^t ds \mathcal{G}(t, s)\vec{g}(s) f_{in}(s),
\]

(5)

where the Green’s function \( \mathcal{G}(t, t_0) \) obeys \( \partial_t \mathcal{G}(t, t_0) = -\mathcal{M}(t)\mathcal{G}(t, t_0) \) and \( \mathcal{G}(t, t_0) = \mathbb{I}_{2N} \). The cascaded structure imposed by the unidirectional waveguide implies that both \( \mathcal{M} \) and \( \mathcal{G} \) have a lower-triangular form, i.e., \( \mathcal{M}_{\mu\nu}, \mathcal{G}_{\mu\nu} = 0 \) for \( \mu < \nu \). Moreover, each row \( \mu \) of these matrices only depends on the couplings \( g_\mu(t) \) associated with that and previous modes \( \nu \leq \mu \). This allows us to write the Green’s function as

\[
\mathcal{G} = \begin{pmatrix} G_{AA} & 0 \\ G_{BA} & G_{BB} \end{pmatrix} \rightarrow \begin{pmatrix} 0 & 0 \\ U & 0 \end{pmatrix},
\]

(6)

where the expression to the right indicates the targeted evolution at \( t = t_f \), as specified in Eq. (1).

Control pulses.—To realize the desired dynamics, we first choose a set of time-dependent control pulses for the couplings \( g_{A,j}(t) \) in register A. These pulses do not have to be of any specific shape, but they must be mutually overlapping and satisfy [40]

\[
\int_{t_0}^{t_f} ds \, |g_{A,j}(s)|^2 \geq 1.
\]

(7)

This last condition ensures that all the initial excitations in register A decay into the waveguide and \( G_{AA}(t_f, t_0) \approx 0 \) up to exponentially small corrections.

In a second step we must identify a set of control pulses \( g_{B,j}(t) \), which achieve the non-trivial part of the dynamics, \( G_{BA}(t_f, t_0) \rightarrow U \). To do so we assume for now that the whole network is initially prepared in the single excitation state \(|\psi_k\rangle = \Psi_k |\text{vac}\rangle\), where \(|\text{vac}\rangle\) is the vacuum state and \( \Psi_k = \sum_{k=1}^N U_{k\ell} a_k(t_0) \). Given this initially state, we then define the set of amplitudes \( F_{j,\ell}(t, t_0) = \langle \text{vac} | f_{out,N+j}(t) |\psi_\ell\rangle \), which correspond to the amplitudes of the channel field right after the \( j \)-th mode of register B. In terms of the Green’s function, we obtain

\[
F_{j,\ell}(t, t_0) = \sum_{k=1}^N g_{k}(t) \left[ \mathcal{G}(t, t_0) U \right]_{k,\ell},
\]

(8)

where \( U = \text{diag}(U, 0_N) \) is a block-diagonal matrix.

According to Eq. (1), the initial excitation created by \( \Psi_k \) is mapped onto the corresponding excitation of mode \( b_j \) in register B. To achieve this mapping, during the whole protocol, the photon emitted from state \(|\psi_\ell\rangle\) must not propagate beyond the \( \ell \)-th node of register B, as otherwise it would be impossible to recapture it at a later time. Therefore, a necessary requirement for a perfect
transfer is that the dark state condition \( F_{r,\ell}(t, t_0) = 0 \) is satisfied for all times \( t \in [t_0, t_f] \), or equivalently,
\[
g_{B,\ell}(t) \left[ G(t, t_0) U^{\dagger} \right]_{N+\ell,\ell} = -F_{r,\ell}(t, t_0).
\] (9)

For \( N = 1 \) and \( U = 1 \), Eq. (9) reduces to the dark-state condition employed for single-qubit quantum state transfer schemes \([14,15]\) (see also Ref. [17] for a preliminary extension to multi-mode setups). In the supplementary material [18] we show in more detail that satisfying this generalized set of dark state conditions for all \( \ell = 1, \ldots, N \) is not only necessary, but also sufficient to obtain \( G_{BA}(t_f, t_0) \simeq U \) and \( G_{BB}(t_f, t_0) \simeq 0 \) for sufficiently long \( t_f \). Moreover, we show that the implicit equation for \( g_{B,\ell}(t) \) in Eq. (9) can be converted into the following explicit expression for the control pulses,
\[
g_{B,\ell}(t) = \frac{F_{r,\ell}(t, t_0)}{\sqrt{\int_{t_0}^{t} ds \left| F_{r,\ell}(s, t_0) \right|^2}}.
\] (10)

Due to the cascaded structure of \( G \), the amplitudes \( F_{r,\ell}(t, t_0) \) depend on the known control pulses \( g_{A,j}(t) \) and on the previously obtained pulses \( g_{B,j}(t) \) for \( j < \ell \) only. Therefore, Eq. (10) can be iteratively applied to compute all control pulses \( g_{B,j}(t) \) for register B.

Equation (10) proves the existence of a solution to our control problem by an explicit construction of the coupling pulses, which is the main result of this paper. We still need to show, however, that this formal result does not lead to solutions that violate the constraints \( |g_j(t)|^2 \leq 1 \), are unbounded in time or otherwise unphysical. In the following we achieve this conclusion by simply applying the protocol for engineering generic \( N \times N \) unitary transformations. This approach will also allow us to deduce the scaling and the robustness of the scheme under realistic conditions.

Two-by-two unitaries.—In a first step we illustrate the application of the protocol for the simplest non-trivial scenario, \( N = 2 \), shown in Fig. 2 (a). For this setup we discuss the implementation of the following four unitary operations
\[
U_T = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad U_S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},
\]
\[
U_H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad U_C = \frac{1}{\sqrt{2}} \begin{pmatrix} i & 1 \\ 1 & i \end{pmatrix}.
\] (11)

Here, \( U_T \) corresponds to a simple state transfer between the two registers, \( U_S \) additionally swaps the two modes and the Hadamard operation \( U_H \) and the unitary \( U_C \) create superpositions between the modes with real and complex coefficients.

To calculate the appropriate control pulses for realizing each of these unitaries, we set \( t_0 = 0 \) and within the transfer interval \( t \in [0, t_f] \) we fix the control pulses for the modes in register A to be of the form
\[
g_{A,j}(t) = \frac{\eta_j}{\sqrt{e^{(t_c-t)/\tau} + 1}}, \quad \eta_j = \sqrt{\frac{1 + (N - j)\delta}{1 + (N - 1)\delta}}.
\] (12)

The parameters \( \delta, t_c \) and \( \tau \) can be used to optimize the protocol for a given application, but none of the following findings depends crucially on this specific pulse shape nor on a specific set of parameters. The actual pulses \( g_{A,j}(t) \) used in the following examples are depicted in Fig. 2 (b).

Given \( g_{A,j}(t) \) and the transformation \( U \), we evaluate numerically Eq. (10) to obtain the control pulses \( g_{B,j}(t) \), and evaluate the fidelity of the operation \([18,19]\).

\[
\mathcal{F}(t) = \frac{\text{Tr} \{ U^\dagger G_{BA}(t, t_0) U G_{BA}^\dagger(t, t_0) \}}{N(N + 1)}.
\] (13)

It reaches a value of \( \mathcal{F}(t_f) \simeq 1 \), if the protocol was successful. For the four unitaries given in Eq. (11), the shape of the control pulses as well as the resulting fidelities are plotted in Fig. 2 (c)-(f). We see that for all examples the numerical algorithm provides the correct control pulses and that the unitary transformation is implemented with close to unit fidelity, as long as the protocol time \( t_f = t_f - t_0 \) is long enough. We emphasize that while the shape of the waveform released from register A depends on the initial quantum state [see Fig. 2 (b)], the protocol implements the unitary transformation \( U \) independently of which state the involved modes are prepared in.

**Scalability.**—Using the construct from Ref. [20], a sequential combination of \( O(N^2) \) of the \( 2 \times 2 \) unitary operations demonstrated above is sufficient to recreate any
possible $N \times N$ unitary transformation $U$ in a time $t_p \sim \mathcal{O}(N^2)$. This strategy is usually employed for implementing bosonic unitaries with photons or atoms \cite{40,50,51}. However, in the current approach, already in a single run, each of the emitted photons interacts with multiple modes in register B. This intrinsic parallelization allows us to improve over the scheme by Beck et al. \cite{20} and obtain protocol times that only scale linearly with the number of modes, $t_p \sim \mathcal{O}(N)$.

To demonstrate this scaling, we numerically evaluate the minimal protocol time $t_{\text{min}}$, required to implement a given $N \times N$ unitary with a fidelity $\mathcal{F} \geq 0.99$. Specifically, for this study we compare the implementation of the $N$-mode state transfer operation, $U_T(N) = 1_N$, the $N$-dimensional Hadamard transformation, $U_H(N)$, and $U_R(N)$, as well as generic complex unitaries $U_R(N)$ with randomly drawn matrix elements. The results are summarized in Fig. 3(a) and demonstrate, first of all, that the protocol works perfectly even for a large number of modes and for arbitrary classes of unitaries. As an illustrative example, Fig. 3(b) shows the numerically generated control pulses $g_{B,j}(t)$ for the case $U_R(N=8)$ and qualitatively similar pulse shapes are obtained for other unitaries as well. See \cite{10} for further details about the numerical procedure that has been used to obtain these results.

The key observation from Fig. 3 (a) is that the minimal protocol time scales only linearly with the number of modes, $t_{\text{min}} \sim N$, and that this scaling does not rely on any specific properties of $U$. The remaining limit on the protocol time can be roughly understood as follows. Because the maximal coupling strength is bounded, $|g_j(t)| \leq \sqrt{\gamma_{\text{max}}}$, the local modes can emit or absorb photons only on timescales longer than $\gamma_{\text{max}}^{-1}$. Therefore, in order to emit (absorb) photons into (from) $N$ spatio-temporally distinct modes, the total pulse duration must increase proportionally to $N$. The prefactor for this scaling is the same for all the tested unitaries, but it is still factor of $\sim 3$ higher than what one would obtain from implementing $N$ times a single-mode transfer. We attribute this overhead to the non-optimal choice of control pulses $g_{A,j}(t)$ in Eq. (12). Indeed, for the transfer unitary $U_T(N)$ a substantial reduction of the protocol times can already be obtained by simply optimizing the pulse parameter $\delta$ \cite{40}. This suggests that also for other unitaries, a similar improvement of the scaling prefactor can be achieved by optimizing the shape of the control pulses in register A.


t_{\text{min}} \sim \mathcal{O}(N^2).

Imperfections.—In our discussion so far we have assumed ideal conditions, while in a real system waveguide losses, the decay of the local modes and other imperfections will degrade the maximally achievable fidelities. The effect of photon propagation losses with a probability $p_{\text{loss}}$ or an additional unwanted decay of each mode with a rate $\gamma$ can be approximately taken into account by $\mathcal{F} \approx \mathcal{F}_{\text{id}}(1-p_{\text{loss}}) - \gamma t_p$, where $\mathcal{F}_{\text{id}}$ is the fidelity of the ideal implementation. Further, the linearity of the transformation makes the protocol insensitive to input noise, such as residual thermal excitations in the channel \cite{45,46}.

Most relevant for the current discussion is the sensitivity of the protocol with respect to imperfections in the control pulses. To investigate this aspect, we show in Fig. 4 the protocol fidelities for the case where a noisy signal is added to the ideal control pulses, i.e., $g_j(t) = g_j(t)|_{\text{id}} + \delta g_j(t)$, where

$$\delta g_j(t) = \sqrt{\varepsilon \Omega} \int_{-\infty}^{t} ds e^{-\Omega (t-s)/2} \xi_j(s).$$

Here the $\xi_j(t)$ is independent white noise processes with $\langle \xi_j(t) \xi_k(t') \rangle = \delta_{jk} \delta(t-t')$, and $\varepsilon$ and $\Omega$ characterize the strength and the bandwidth of the noisy signal. From Fig. 4(a) we find that the protocol is extremely robust with respect to pulse imperfections. The infidelity, $1 - \mathcal{F}$, scales sub-linearly with the strength of the noise for rather high values of $\varepsilon$ and fluctuations that are faster than $\gamma_{\text{max}}^{-1}$, are further suppressed. Importantly, as shown in Fig. 4(b), the fidelity of the operation also does not degrade significantly when the number of modes is increased and again a rather weak dependence on $N$ is observed.
for the state transfer unitary, $U$. The solid and the dashed lines show the results as obtained from splitters and phase shifters, the transformation is implemented through a multi-photon emission and reabsorption process. Therefore, arbitrary unitaries can be realized by simply changing the control pulses and without changing the network configuration. The protocol is robust with respect to the main sources of imperfections and even for very complex unitaries the protocol time only increases linearly with the number of modes.

While the protocol can be implemented with various physical platforms, we envision important near-term applications in the context of circuit QED, where tunable and directional couplers are already experimentally available. Here, the protocol can be used for parallel state transfer and entanglement distribution schemes, or, when combined with local nonlinearities, for large-scale quantum computing architectures based on continuous variable encoded qubits.

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Discussion and conclusions.—In summary, we have presented a universal protocol for implementing unitary transformations between $N$ bosonic modes, where, instead of sending photons through a fixed network of beam splitters and phase shifters, the transformation is implemented through a multi-photon emission and reabsorption process. Therefore, arbitrary unitaries can be realized by simply changing the control pulses and without changing the network configuration. The protocol is robust with respect to the main sources of imperfections and even for very complex unitaries the protocol time only increases linearly with the number of modes.

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I. DERIVATION OF THE QUANTUM LANGEVIN EQUATIONS

We consider the network shown in Fig. 1 (a) in the main text, which consists of a set of $2N$ bosonic modes that are weakly coupled to a unidirectional photonic waveguide. In a frame rotating with the bare oscillator frequency $\omega_0$, the coupling between the local oscillators and the waveguide is described by the Hamiltonian

$$H_{\text{int}}(t) = i\hbar \sum_{j=1}^{N} \left[ \tilde{g}_j(t) c_j \bar{f}(x_j, t) - \tilde{g}_j(t) c_j^\dagger f(x_j, t) \right],$$  

(S1)

where the $x_j$ denote the positions of the resonators along the waveguide with $x_j > x_k$ for $j > k$. In Eq. (S1), the bosonic field operator $\bar{f}(x, t)$ represents the photons in the waveguide. For a unidirectional channel with a group velocity $v$, this field operator is given by

$$\bar{f}(x, t) \simeq \frac{1}{\sqrt{2\pi}} \int_{\omega_0 - \Delta}^{\omega_0 + \Delta} d\omega e^{i\omega_0 t} e^{-i\omega(t-x/v)} b_{\omega}(t),$$  

(S2)

where $\Delta$ is the bandwidth of the channel and the $b_{\omega}(t)$ are slowly varying bosonic operators obeying $[b_{\omega}, b_{\omega'}] = \delta(\omega - \omega')$. Note that with these conventions the couplings $\tilde{g}_j$ have dimensions of $\sqrt{\text{Hz}}$.

The equations of motion for the Heisenberg operators derived from $H_{\text{int}}(t)$ are

$$\dot{c}_j(t) = -\tilde{g}_j(t) f(x_j, t),$$  

(S3)

$$\dot{\bar{b}}_{\omega}(t) = \frac{1}{\sqrt{2\pi}} \sum_j \tilde{g}_j^*(t) c_j(t) e^{-i\omega_0 t} e^{i\omega(t-x_j/v)}. $$  

(S4)

The formal solution of the field operator is

$$f(x, t) = \tilde{f}_0(x, t) + \sum_j \int_0^t ds \tilde{g}_j^*(s) c_j(s) \delta_{\Delta} \left( t - s - \frac{x-x_j}{v} \right) e^{i\omega_0(x-x_j)/v}.$$  

(S5)

Here $\tilde{f}_0(x, t)$ is the free field operator and we have used

$$\int_{\omega_0 - \Delta}^{\omega_0 + \Delta} d\omega e^{-i(\omega-\omega_0)(t-s-\tau)} = 2\pi \delta_{\Delta}(t-s-\tau),$$  

(S6)

where $\delta_{\Delta}(t)$ is the $\delta$-function on timescales that are long compared to the inverse of the channel bandwidth. By reinserting the result for $f(x, t)$ into the equations of motion for the $c_j$, we finally obtain

$$\dot{c}_j = -\tilde{g}_j(t) \tilde{f}_0(x_j, t) - \frac{1}{2} \tilde{g}_j(t)^2 c_j(t) - \sum_{k<j} \tilde{g}_j(t) \tilde{g}_k(t-\tau_{jk}) c_k(t-\tau_{jk}) e^{i\omega_0 \tau_{jk}},$$  

(S7)

where $\tau_{jk} = (x_j - x_k)/v$.

In a final step we switch to rescaled units of time, $t \gamma_{\text{max}} \rightarrow t$, where $\gamma_{\text{max}}$ is the maximal decay rate, and define the dimensionless couplings $g_j(t) = \tilde{g}_j(t)/\sqrt{\gamma_{\text{max}}}$.

We also define $\tilde{f}_{\text{in}}(t) = \tilde{f}_0(x_1, t)/\sqrt{\gamma_{\text{max}}}$. As a result, we obtain the quantum Langevin equations from the main text

$$\dot{c}_j(t) = -\frac{g_j(t)^2}{2} c_j(t) - g_j(t) \tilde{f}_{\text{in},j}(t), $$  

(S8)

together with $\tilde{f}_{\text{in},1}(t) \equiv \tilde{f}_{\text{in}}(t)$ and $\tilde{f}_{\text{out},j}(t) = \tilde{f}_{\text{in},j}(t) + g_j(t) c_j(t)$. Note, however, that the relation between incoming and outgoing fields,

$$\tilde{f}_{\text{in},j}(t) = \tilde{f}_{\text{out},j-1}(t - \tau_{j,j-1}) e^{i\omega_0 \tau_{j,j-1}},$$  

(S9)

still contains propagation delays, which make the set of differential equations nonlocal. Because of the unidirectional nature of the channel, these delays can be eliminated by setting $x_1 = 0$ and defining the time-advanced operators and couplings,

$$\tilde{c}_j(t) = e^{-i\omega x_j/v} c_j(t + x_j/v),$$  

(S10)

$$\tilde{g}_j(t) = g_j(t + x_j/v).$$  

(S11)

As a result we obtain a set of equations given in Eq. (4) in the main text, where we omitted the bar on top of the operators and coupling constants for notational simplicity.

II. CONTROL PULSES

In this section we present a step-by-step derivation of Eq. (10) in the main text. This equation provides an explicit solution for the control pulses $g_{A,j}(t)$, which implement a given unitary transformation $U$ for a given set of control pulses $g_{A,j}(t)$.

A. Control pulses for emission

As a first step in the protocol, we must choose a set of control pulses $g_{A,j}(t)$ for the modes in register $A$. These pulses do not have to be of any specific shape, but for the implementation of a generic unitary transformation the following three conditions must be satisfied:

1. The pulses $g_{A,j}(t)$ have a sufficiently large pulse area,

$$\int_{t_0}^{t_f} ds |g_{A,j}(s)|^2 > 1.$$  

(S12)

2. The pulses $g_{A,j}(t)$ are mutually overlapping,

$$\int_{t_0}^{t_f} ds g_{A,j}(s) g_{A,k}(s) \neq 0.$$  

(S13)
3. The pulses $g_{A,j}(t)$ are nonidentical. As already mentioned in the main text, the first condition ensures that all the initial excitations in register $A$ decay up to exponentially small corrections. For example, for the first mode we find

$$
(a_{1}^\dagger a_{1})(t_f) = \langle a_{1}^\dagger a_{1}(t_0) e^{-\int_{t_0}^{t_f} ds |g_{A,1}(s)|^2}.
$$

(S14)

For the successive modes the dynamics is more complicated, but the unidirectional propagation of the emitted photons ensures that all the initial excitation eventually leave register $A$, as long as all the couplings $g_{A,j}(t)$ are switched on for a sufficiently long time.

The second condition is necessary to implement unitary transformations where, for example, a superposition of two or more modes in register $A$ is mapped onto a single mode in register $B$. This is not possible when the emitted photons have no overlap.

The third condition of non-identical initial pulses is actually not strictly necessary, but it is included for practical reasons. When applying the numerical algorithm for calculating the optimal pulse shapes as described below, we typically find that non-identical pulse lead to shorter overall protocol times. Also when using identical pulses it is more likely to obtain unphysical oscillations due to numerical errors.

### B. Generalized dark state conditions

To construct the control pulses $g_{B,j}(t)$, we proceed as outlined in the main text and assume that the whole network is initially prepared in the single excitation state $|\psi_i\rangle = \Psi_i |\text{vac}\rangle$. Here $|\text{vac}\rangle$ is the $2N$-mode vacuum state and

$$
\Psi_i = \sum_{k=1}^{N} U_{i,k} a_k(t_0).
$$

(S15)

This approach has the conceptual advantage that the whole transfer process can be divided into individual processes, where in each step an initial excitation $|\psi_i\rangle$ is mapped onto the single-photon state $b_i^\dagger |\text{vac}\rangle$ at the end of the protocol. From a physical perspective it is then clear that this specific process must only involve excitations of the first $N$ modes in register $A$ and the first $\ell$ modes of register $B$. Mathematically, this can be expressed as

$$
P_i(t) = \sum_{\mu=1}^{N+\ell} \left| [G(t,f_{0})U^\dagger]_{\mu,\ell} \right|^2 = 1,
$$

where

$$
U = \begin{pmatrix} U & 0_N \\ 0_N & 0_N \end{pmatrix}.
$$

(S17)

Eq. (S16) follows from the fact that, given the initial state $|\psi_i\rangle$, the population of the $\mu$-th mode of the whole network can be expressed in terms of the Green’s function as $\langle c_i^\dagger c_i(t) \rangle = \left| [G(t,f_{0})U^\dagger]_{\mu,\ell} \right|^2$.

The constraint on the excitation probability given in Eq. (S16) can also be rewritten in a differential form as a conservation law,

$$
\dot{P}_i(t) = -|\tilde{F}_{N+\ell,i}(t_f,t_0)|^2 = 0.
$$

(S18)

Note that compared to the out-field amplitudes defined in the main text, we use here the slightly different convention

$$
\tilde{F}_{\mu,\ell}(t_f,t_0) = \langle \text{vac}|f_{\text{out},\mu}(t)|\psi_i\rangle = \sum_{\mu=1}^{\mu} g_{\mu}^*(t) [G(t,t_0)U^\dagger]_{\mu,\ell},
$$

(s19)

such that the first index can assume any value $\mu = 1, \ldots, 2N$. The result in Eq. (S18) can be verified by evaluating both sides of the equation. This is most conveniently done by making use of the relation (omitting the time variables)

$$
\left[ G U^\dagger \right]_{\mu,\ell} = -g_{\mu} \left( \frac{g_{\mu}^2}{2} \left[ G U^\dagger \right]_{\mu,\ell} + \tilde{F}_{\mu-1,i} \right).
$$

(S20)

Therefore, keeping in mind that $\tilde{F}_{N+\ell,i}(t_f,t_0) = \tilde{F}_{\ell,i}(t_f,t_0)$, the dark state condition stated in Eq. (9) in the main text can be directly derived from the conservation of the excitation probability within the first $N + \ell$ modes of the network.

### C. Sufficiency of the dark state conditions

While we have argued that it is necessary to obey the set of dark state conditions $\tilde{F}_{N+\ell,i}(t_f,t_0) = 0$ at any time during the protocol, we now demonstrate that this is even a sufficient requirement for implementing the correct unitary operation. To do so we rearrange Eq. (S16) and show that

$$
\left| [G(t_f,t_0)U^\dagger]_{N+\ell,\ell} \right|^2 = 1 - \sum_{\mu=1}^{N+\ell-1} \left| [G(t_f,t_0)U^\dagger]_{\mu,\ell} \right|^2 = 1.
$$

(S21)

For $\ell = 1$ this result simply follows from the fact that $G_{AA}(t_f,t_0) = 0$ for sufficiently long pulses. Given that $\left| [G(t_f,t_0)U^\dagger]_{N+1,1} \right|^2 = 1$ and $U$ is unitary, it follows that $\left| [G(t_f,t_0)U^\dagger]_{N+1,2} \right|^2 = 0$, since the norm of each row of $G$ is bounded, i.e., $\sum_{\nu=1}^{2N} |g_{\mu,\nu}|^2 \leq 1$. This result then implies that Eq. (S21) holds also for $j = 2$, and so on. Therefore, as long as the control pulses $g_{A,j}(t)$ satisfy Eqs. (S12)-(S13) and the set of dark state conditions in Eq. (S18) is fulfilled during the whole duration of the protocol, $t \in [t_0, t_f]$, and for all $j = 1, \ldots, N$, we obtain

$$
[G_{BA}(t_f,t_0)U^\dagger]_{\ell,\ell} = e^{i\theta_{\ell}}.
$$

(S22)
The remaining phases can be eliminated by simple rotations of the pulses, $g_{B,ℓ}(t) \rightarrow e^{-iθ} g_{B,ℓ}(t)$, which leave the dark state conditions invariant. After this adjustment we obtain the desired result, $G_{BA}(t_f,t_0) = U$.

D. Implicit and explicit constructions of the control pulses

The arguments above show that the targeted unitary transfer operation can be implemented by imposing the set of generalized dark state conditions \(S18\) at all times, but they do not provide a way to construct the required pulses \(g_{B,j}(t)\) yet. To show how this can be done, we first use the input-output relation \(f_{out,N+ℓ}(t) = f_{out,N+ℓ-1}(t) + g_{B,ℓ}(t)b_ℓ(t)\) to rewrite the dark state condition \(\tilde{F}_{N+ℓ,ℓ}(t,t_0) = 0\) as

\[
  g_{B,ℓ}(t) = -\frac{\tilde{F}_{N+ℓ-1,ℓ}(t,t_0)}{[G(t,t_0)U]_{N+ℓ,ℓ}}. \tag{S23}
\]

This is still an implicit equation for the control pulses, since \([G(t,t_0)U]_{N+ℓ,ℓ}\) depends on \(g_{B,ℓ}(t)\) as well. To turn Eq. \(S23\) into an explicit equation for the \(g_{B,ℓ}(t)\), we take its time-derivative and make again use of Eq. \(S20\) and Eq. \(S23\) to simplify the result. After some manipulations we obtain the differential equation

\[
\frac{d}{dt}g_{B,ℓ}(t) = g_{B,ℓ}(t) \left[ \frac{d}{dt} \log \tilde{F}_{N+ℓ-1,ℓ}(t,t_0) - \frac{1}{2} |g_{B,ℓ}(t)|^2 \right], \tag{S24}
\]

with a nontrivial solution

\[
g_{B,ℓ}(t) = \frac{\tilde{F}_{N+ℓ-1,ℓ}(t,t_0)}{\sqrt{\int_{t_0}^{t}\int_{s}ds |\tilde{F}_{N+ℓ-1,ℓ}(s,t)|^2}}. \tag{S25}
\]

This is the result given in Eq. \(10\) in the main text.

III. NUMERICS

In this section we provide a more detailed description of the numerical methods that have been used to calculate the control pulses \(g_{B,j}(t)\) for all the examples discussed in the main text.

A. Control pulses for register A

As pointed out above, as long as the initial control pulses \(g_{A,j}(t)\) satisfy a few basic requirements, their precise shape is not important for the protocol to work. For all our examples in this work we use the pulses

\[
g_{A,j}(t) = \frac{η_j}{\sqrt{e^{-(t-t_0)^2}/τ + 1}}, \tag{S26}
\]

where

\[
η_j = \sqrt{\frac{1 + (N-j)δ}{1 + (N-1)δ}}. \tag{S27}
\]

B. Explicit method

The explicit expression for the control pulses given in Eq. \(S25\) is in principle enough to calculate all the \(g_{B,j}(t)\) through numerical integration. To do so, one first solves the Green’s function \(G_{AA}(t,t_0)\) (using, for example, a Runge-Kutta method), which only depends on the known control pulses \(g_{A,j}(t)\). With \(G_{AA}(t,t_0)\) known, one then obtains the out-field amplitude \(\tilde{F}_{N,ℓ}(t,t_0)\) from Eq. \(S19\) and the control pulse for the first mode in register B as

\[
g_{B,1}(t) = \frac{\tilde{F}_{N,1}(t,t_0)}{\sqrt{\int_{t_0}^{t}\int_{s}ds |\tilde{F}_{N,1}(s,t)|^2}}. \tag{S28}
\]

The knowledge of \(g_{B,1}(t)\) can then be used to obtain \(\tilde{F}_{N+1,2}(t,t_0)\) and \(g_{B,2}(t)\), etc. However, in practice this method is rather slow for large \(N\) as it involves many numerical integration steps and many integrations of the Green’s function.

C. Implicit method

As an alternative approach to calculate the pulses \(g_{B,j}(t)\), we can simply make use of the fact that the set of dark state conditions in Eq. \(S18\) must be satisfied at each point in time. This determines the value of the control pulses through the implicit relation in Eq. \(S23\).

In this method, in a first step we evaluate again the elements \(G_{AA}(t,n)\) of the Green’s function and the out-field amplitude \(\tilde{F}_{N,ℓ}(t_n)\) on a grid of time points \(t_n\) with spacing \(Δt\). The unknown couplings \(g_{B,1}(t_n)\) and the unknown elements \(G_{N+1,ℓ}(t_n,t_0)\) of the Green’s function can then be obtained via Euler integration, using Eq. \(S23\) to determine the value of \(g_{B,1}(t_{n+1})\) for the next time step. Once the values for \(g_{B,1}(t_n)\) are known, the same procedure can be iterated to obtain \(g_{B,2}(t_n)\), etc.

D. Initial conditions

A remaining issue for both the explicit and the implicit method is that at the beginning of the protocol the control pulses \(g_{B,j}(t)\) are undetermined or can become very large. This arises from the fact that at \(t = t_0\) there is a finite out-field from register A, but the population of, for example, the \(b_1\) mode is still vanishingly small. Therefore, the dark state condition can only be satisfied by a correspondingly large (diverging) coupling.

To deal with this complication in both protocols, whenever \(|g_{B,ℓ}(t)| > 1\) we set the control pulses to a fixed value of

\[
g_{B,ℓ}(t) \rightarrow \frac{\tilde{F}_{N,ℓ}(t_0,t_0)}{|\tilde{F}_{N,ℓ}(t_0,t_0)|}. \tag{S29}
\]

Although in this case the dark state condition is no longer fulfilled exactly, this occurs only in the very beginning
FIG. S1. Details of the numerical procedure to obtain control pulses $g_{B,j}(t)$ illustrated for the example $U_T(N = 2)$ shown in Fig. 2 (c) in the main text. Upper panel: Starting with the control pulses $g_{A,j}(t)$ (dotted lines) the control pulses $g_{B,j}(t)$ are calculated using either the explicit or implicit method for $t'_0 = 0$ and $t'_f = 48$ and enforcing the bound in Eq. (S29) at the initial stage. Lower panel: In a second step, keeping the same control pulses, the actual initial time $t_0$ is increased and the final time $t_f$ is decreased (yellow dashed lines) to avoid the initial non-analytic part of the control pulses and set the total protocol time to $t_p = t_f - t_0 = 40$. To find the minimal time $t_{min}$ plotted in Fig. 3 of the main text, $t_0$ and $t_f$ are further adjusted (purple dashed lines) up to the point where the fidelity drops below a value of $\mathcal{F} = 0.99$.

of the protocol and only causes an exponentially small error for the whole transfer. In the actual numerical simulations, smooth and well-behaved control pulse are obtained as follows: First, by choosing an initial time $t'_0$ and a sufficiently large final time $t'_f$, such that $t'_f - t'_0 > t_p$, the control pulses are evaluated according to the prescription in Eq. (S29). Then, keeping this set of control pulses fixed, the actual initial time $t_0 > t'_0$ and the actual final time $t_f < t'_f$ are chosen such that within the new time window all the pulses are well-behaved, while still reaching the targeted value of the fidelity. This whole procedure is illustrated in Fig. S1 for the example shown in Fig. 2 (c) in the main text.

E. Optimized protocol time

In Fig. 3 in the main text we evaluate the minimal protocol time $t_{min}$ that is required to achieve a fidelity of $\mathcal{F} \geq 0.99$. To do so, we use the implicit method described above and the parameters $\delta = 2$, $\tau = 1$ and $t_c = t_f/2$ for $t'_0 = 0$ and an initial protocol time of $t'_f \approx 20 \times [1 + (N - 1)/\delta]$. This results in fidelities of $\mathcal{F} > 0.99$ for all examples. Successively, we run the algorithm for a gradually adjusted $t_0$ and $t_f$, as described above, until the fidelity drops below the threshold (see Fig. S1 for an illustrative example).

In the case of the transfer operation $U_T(N)$ we repeat this search for the minimal time $t_{min}$ for different parameters $\delta$. The minimal value of $t_{min}$ obtained in this way, which, for example is reached at $\delta \approx 0.41$ for $N = 32$, is marked by the stars in Fig. (3).