The dependence of the nuclear charge form factor on short range correlations and surface fluctuation effects

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Abstract

We investigate the effects of fluctuations of the nuclear surface on the harmonic oscillator elastic charge form factor of light nuclei, while simultaneously approximating the short-range correlations through a Jastrow correlation factor. Inclusion of surface-fluctuation effects within this description, by truncating the cluster expansion at the two-body part, is found to improve somewhat the fit to the elastic charge form-factor of $^{16}O$ and $^{40}Ca$. However, the convergence of the cluster expansion is expected to deteriorate. An additional finding is that the surface-fluctuation correlations produce a drastic change in the asymptotic behavior of the point-proton form factor, which now falls off quite slowly (i.e. as $\text{const.} \cdot q^{-4}$) at large values of the momentum transfer $q$. 

1
1 Introduction

The calculation of the charge form factors $F_{ch}(q)$ of nuclei is a challenging and appealing problem [1]. A possibility to face this problem is by means of an independent-particle model. In this approach, which is particularly attractive because of its simplicity, the choice of the single particle potential has to be suitably made. In fact a short range repulsion in this potential seems advisable for light nuclei (see ref. [2] and references therein). For example, with an harmonic oscillator (HO) potential having in addition an infinite soft core of the form $\frac{B}{r^2}$ ($B > 0$) the $F_{ch}(q)$ of $^4$He can be well reproduced, but for the heavier nuclei, such as $^{12}$C and $^{16}$O, state dependent potentials seem necessary and even then the fit is not so good for higher q-values [2]. Furthermore, the correction of the centre of mass motion cannot be made exactly and unambiguously. An approach which is rather similar is the approach of Ripka and Gillespie [3] and Gaudin et al [4]. It was shown by these authors that if a Jastrow wave function consisting of HO orbitals and of simple state-independent correlation functions of the form $f(r) = (1 - e^{-\beta r^2})^{1/2}$ is used, one can construct a Slater determinant which yields a density "very similar" to that of such a Jastrow wave function in the two-body approximation. This is done by diagonalizing the density matrix:

$$\rho_{ij} = <\Psi | a_j^+ a_i | \Psi >$$

that is by using the so-called "natural orbitals". It is clear, however, that the actual density matrix is unknown and one obtains it approximately, usually by means of a Jastrow wave function, as is the case in the above references (see expression (7.7) of the paper by Gaudin et al [4]). Therefore, even with such an approach one still has in practice to use a complicated wave function such as a Jastrow one. From the above discussion it is clear that a Jastrow wave function is useful even if one is only interested in calculating the form factor and density distribution of nuclei and not necessarily other quantities, such as the momentum distribution [5].

In a series of papers [6, 7, 8] an expression of the elastic charge form factor, $F_{ch}(q)$, truncated at the two body term, was derived using the factor cluster expansion of Ristig et al [9, 10]. This expression, which is a sum of one-body and two-body terms, depends on the harmonic oscillator (HO) parameter $b_1$ and the correlation parameter $\lambda$ through a Jastrow type correlation function which introduces the short range correlations (SRC). The
use of the $HO$ orbitals (as well as of the particular and popular form of the Jastrow correlation function used) stems mainly from the considerable simplification they imply. There are, however, additional advantages, such as: i. The correction of the centre of mass motion can be done exactly by means of the Tassie and Barker factor \[11\]. ii. One can obtain analytically the asymptotic behaviour of the form factor $F_{ch}(q)$. In principle, of course, one should start with a Hartree-Fock independent-particle model assuming an effective interaction and then introduce correlations. Such an approach is, however, computation-wise quite demanding. Although it is more sophisticated and more satisfactory, it lacks the above mentioned two advantages.

The fit of $F_{ch}(q)$ to the experimental data with the above mentioned procedure was very good both for low and high values of momentum transfer except for the values around the last maximum for $^{16}O$ and $^{40}Ca$. Better fit can be obtained if the parameter $\lambda$ is taken to be state dependent but in this case there is a big number of parameters, six for $^{16}O$ \[3\] and ten for $^{40}Ca$ \[12\].

Another possible way to make the agreement between theory and experiment better might be to introduce, in addition, other types of ground state ("long range") correlations which have been the subject of previous investigations by a number of authors (see, for example, \[13, 14, 15, 16, 17, 18, 19\]). We focus our attention, as in ref.\[17\], on fluctuations of nuclear surface due to the zero point motion of collective surface vibrations \[20, 21\] which can affect the ground state charge density. The presence of surface fluctuation correlations (SFC) introduce another fitting parameter in addition to the HO and the SRC parameters. Thus, it appears to be of interest to develop the relevant formalism and to investigate what would be the effect, if any, of this additional parameter to the best fit values of the other parameters and to the quality of the fit. The aim of this paper is to report on results of certain investigations towards this direction.

In section 2, the above SFC are introduced to the HO densities and analytic expressions of the nuclear density, elastic form factor and of the n-th moment of the density distribution are given for the nuclei $^4He$, $^{16}O$ and $^{40}Ca$. In section 3 the introduction of the SFC in addition to the SRC is studied. Numerical results are reported and discussed in section 4.
2 The effect of the surface fluctuations on the harmonic oscillator density and form factor

Our starting point is the expression for the proton (or charge) density of a nucleus which has been deformed through the zero-point motions of the collective surface vibrations. This expression, according to ref. [17] (see also ref. [13, 14, 15] for a rather similar expression) is the following:

$$\rho_{1\sigma}(r) = \frac{1}{\sqrt{2\pi\sigma}} \int_{-\infty}^{\infty} \rho_1(r - \xi) \exp \left[ - \frac{(\xi - s_0)^2}{2\sigma^2} \right] d\xi$$

(1)

where $\rho_1(r)$ is the uncorrelated density, $s_0$ is a correction needed to conserve the number of particles in the correlated ground state and $\sigma$ is a measure of the effect of the zero point fluctuations. The value of $\sigma$ is related to $\beta_\lambda$, the deformation parameters for the states of multipolarity $\lambda$, with the relation

$$\sigma^2 \approx \frac{R_0^2}{4\pi} \sum \beta_\lambda^2(\tau = 0)$$

while the $\beta_\lambda$ parameters can be determined from the values of $B(E_\lambda)$ [17, 19].

In (1) we consider for $\rho_1(r)$ the HO proton density, in which the centre of mass correction has been taken into account, for nuclei $^4He$ to $^{40}Ca$. This is given by the expression:

$$\rho_1(r) = \frac{1}{Z\pi^{3/2}} \frac{1}{b_1^2} \exp \left[ - \frac{r^2}{b_1^2} \right] \sum_{k=0}^{2} N_{2k} \left( \frac{r}{b_1} \right)^{2k}$$

(2)

where

$$\begin{align*}
N_0 &= 2\eta_{1s} + 6 \left( 1 - \frac{b_1^4}{b_1^2} \right) \eta_{1p} + \left( 10 - 20 \frac{b_1^2}{b_1^2} + 10 \frac{b_1^4}{b_1^2} \right) \eta_{1d} + \\
N_2 &= 4\eta_{1p} \frac{b_1^2}{b_1^2} + \left( \frac{4 b_1^2}{3 b_1^2} - \frac{20 b_1^4}{3 b_1^4} \right) \eta_{2s} + \left( \frac{40 b_1^2}{3 b_1^2} - \frac{40 b_1^4}{3 b_1^4} \right) \eta_{1d} \\
N_4 &= \left( \frac{8}{3} \eta_{1d} + \frac{4}{3} \eta_{2s} \right) \frac{b_1^4}{b_1^2}
\end{align*}$$

(3)
and \( \bar{b}_1^2 = b_1^2 \left( 1 - \frac{1}{A} \right) \), \( A \) is the mass number and \( b_1 = \sqrt{\frac{n}{m \omega}} \) the harmonic oscillator parameter. \( \eta_{nl} \) is the occupation probability (0 or 1 in the present treatment) of the \( nl \) state. It is easily checked that when \( \bar{b}_1 = b_1 \), that is when the centre of mass correction is not taken into account, the coefficients of the polynomial in 2 are reduced to the well known expressions:

\[
N_0 = 2\eta_{1s} + 3\eta_{2s}, \quad N_2 = 4\eta_{1p} - 4\eta_{2s}, \quad N_4 = \frac{8}{3}\eta_{1d} + \frac{4}{3}\eta_{2s}
\]

General expressions of similar structure for the density and the form factor in the HO model have been given in ref \([22]\).

From (1) and (2) an analytic expression of \( \rho_1(\sigma)(r) \) can be derived. This is:

\[
\rho_1(\sigma)(r) = \frac{1}{Z} \frac{\pi^{3/2}}{\bar{b}_1^2} \frac{1}{\sqrt{\bar{b}_1^2 + 2\sigma^2}} \exp \left[ \frac{(r - s_0)^2}{(\bar{b}_1^2 + 2\sigma^2)} \right] \sum_{k=0}^4 C_k r^k
\]

where the coefficients \( C_k \) depend on \( N_0, N_2, N_4, \sigma, s_0 \) and \( \bar{b}_1 \) and are given by the following formulae:

\[
\begin{align*}
C_0 &= N_0 + B^2 \left( \bar{b}_1^2 \sigma^2 + 2\sigma^4 + \bar{b}_1^2 s_0^2 \right) N_2 + \\
&\quad \left( 3B^2 \sigma^4 + 6B^3 \bar{b}_1^2 \sigma^2 s_0^2 + B^4 \bar{b}_1^4 s_0^4 \right) N_4 \\
C_1 &= -2B^2 \bar{b}_1^2 s_0 N_2 - 4B^4 \bar{b}_1^2 s_0 \left( 3\bar{b}_1^2 \sigma^2 + 6\sigma^4 + \bar{b}_1^2 s_0^2 \right) N_4 \\
C_2 &= B^2 \bar{b}_1^2 N_2 + 6B^4 \bar{b}_1^2 \left( \bar{b}_1^2 \sigma^2 + 2\sigma^4 + \bar{b}_1^2 s_0^2 \right) N_4 \\
C_3 &= -4B^4 \bar{b}_1^4 s_0 N_4 \\
C_4 &= B^4 \bar{b}_1^4 N_4
\end{align*}
\]

and \( B = 1/(\bar{b}_1^2 + 2\sigma^2) \)

By using expression (4) one can find an analytic expression for the \( n \)th moment of the density. This is the following:

\[
<r^n>_{1\sigma} = \frac{2\bar{b}_1^2}{Z} \sqrt{\frac{\pi}{n}} \exp[-s_0^2 B] \sum_{k=0}^4 C_k \bar{b}_1^k \left( 1 + \frac{2\sigma^2}{\bar{b}_1^2} \right)^{(k+n+2)/2} \times
\]

\[
\left[ \Gamma \left( \frac{k+n+3}{2} \right) \right] \left( \frac{k+n+3}{2} ; \frac{1}{2} ; s_0^2 B \right) + \\
2 s_0 \sqrt{B} \Gamma \left( \frac{k+n+4}{2} \right) \left( \frac{k+n+4}{2} ; \frac{3}{2} ; s_0^2 B \right)
\]

(6)
An approximate expression for \( < r^n >_{1\sigma} \) may be derived by truncation of the series at the second power for \( \sigma \) and the first power for \( s_0 \):

\[
< r^n >_{1\sigma} \simeq \frac{2\bar{b}_1^2}{Z\sqrt{\pi}} \sum_{k=0}^{4} C_k \bar{b}_1^k \times \\
\left[ \left( 1 + (k + n + 2)\frac{\sigma^2}{b_1^2} \right) \Gamma\left( \frac{k + n + 3}{2} \right) + \right. \\
\left. \frac{2s_0}{b_1} \left( 1 + (k + n + 1)\frac{\sigma^2}{b_1^2} \right) \Gamma\left( \frac{k + n + 4}{2} \right) \right]
\]

(7)

By taking into account that

\[
< r^0 >_{1\sigma} = < r^0 >_{1} = 1
\]

the approximate expression for the parameter \( s_0 \) is:

\[
s_0 \simeq -\frac{\sqrt{\pi}}{4} \frac{\sigma^2}{\bar{b}_1} \frac{2N_0 + N_2 + \frac{3}{2}N_4}{N_0 + N_2 + 2N_4}
\]

(8)

That expression was used as a first approximation in our calculations. More accurate values were obtained by varying \( s_0 \) until normalization of \( \rho_{1\sigma}(r) \) was achieved to a good approximation. From expressions (7) and (8) and from the known expression of the moments of the HO density one can find the approximate expression of the contribution of the SFC, \( \Delta < r^2 >_{1\sigma} \), to the mean square radius for nuclei \(^4\)He to \(^{40}\)Ca. This is given by the following expression:

\[
\Delta < r^2 >_{1\sigma} \simeq \frac{2}{Z} \sigma^2 \left[ 3(N_0 + \frac{3}{2}N_2 + \frac{15}{4}N_4) - \\
\left( 2N_0 + N_2 + \frac{3}{2}N_4 \right) \left( N_0 + 2N_2 + 6N_4 \right) \right] / N_0 + N_2 + 2N_4
\]

(9)

Finally, for the elastic point proton form factor the well known expression in Born approximation

\[
F_{1\sigma}(q) = 4\pi \int_{0}^{\infty} \rho_{1\sigma}(r) \frac{\sin(qr)}{qr} r^2 dr
\]

(10)
is used. Substitution of $\rho_1(r)$ from (3) leads to the following analytic expression of $F_1(\sigma)$ in terms of the confluent hypergeometric function

$$F_1(\sigma) = \frac{1}{Z} \sqrt{\frac{B}{b_1^2}} \frac{1}{q} \sum_{k=0}^{4} C_k I_k$$

where

$$I_k = \frac{1}{B^{k+2/2}} \exp[-s_0^2 B] \times$$

$$\text{Im} \left[ 2\Gamma\left(\frac{k+2}{2}\right) \text{i} F_1\left(\frac{k+2}{2}; \frac{1}{2}; z^2\right) + 4\Gamma\left(\frac{k+3}{2}\right) z F_1\left(\frac{k+3}{2}; \frac{3}{2}; z^2\right) \right]$$

The complex quantity $z$ is given by: $z = \sqrt{B}s_0 + iq/(2\sqrt{B})$.

Expression (12) may be reduced to a somewhat more convenient form:

$$F_1(\sigma) = \frac{1}{Z} \exp[-\frac{q^2}{4B}] \sum_{n=0}^{2} \left[ \tilde{C}_{2n} \cos(qs_0) + \tilde{\tilde{C}}_{2n} \frac{\sin(qs_0)}{q} \right] \left( \frac{q}{2\sqrt{B}} \right)^{2n} +$$

$$\frac{2}{\sqrt{\pi}Zb_1^2 \sqrt{B}} \exp[-s_0^2 B] \frac{1}{q} \text{Im}[I]$$

where

$$I = \sum_{n=0}^{2} \frac{C_{2n}}{B^n} \Gamma(n+1) F_1(n+1; \frac{1}{2}; z^2) + 2z \sum_{n=0}^{1} \frac{C_{2n+1}}{B^{n+\frac{3}{2}}} \Gamma(n+2) F_1(n+2; \frac{3}{2}; z^2)$$

The coefficients $\tilde{C}_{2n}$ and $\tilde{\tilde{C}}_{2n}$ depend also on $N_0$, $N_2$, $N_4$, $\sigma$, $s_0$ and $b_1$ and are given by the following expressions:

$$\tilde{C}_0 = N_0 + \frac{3}{2} N_2 + \frac{15}{4} N_4 + \frac{\sigma^2}{b_1^2} (2N_0 + N_2 + \frac{3}{2} N_4)$$

$$\tilde{C}_2 = -N_2 - (5 - 4\sigma^2 B)N_4$$

$$\tilde{C}_4 = b_1^2 B N_4$$

$$\tilde{\tilde{C}}_0 = \frac{s_0}{b_1^2} (2N_0 + N_2 + \frac{3}{2} N_4)$$

$$\tilde{\tilde{C}}_2 = -s_0 B(2N_2 + 6N_4)$$

$$\tilde{\tilde{C}}_4 = 2s_0 b_1^2 B^2 N_4$$

(15)
It may be easily checked from expression (13) that when the SFC are switched off, that is when the limiting case \( \sigma \to 0 \) is considered, expression (13) for \( F_{1\sigma}(q) \) goes over to the well known harmonic oscillator one, as should be the case, on the basis of expressions (10) and (1). Furthermore, by using the asymptotic expansion of the confluent hypergeometric function, we find that the behaviour for \( F_{1\sigma}(q) \) at large values of the momentum transfer is the following:

\[
F_{1\sigma}(q) \simeq \frac{1}{\sqrt{\pi Z}} \frac{\exp[-s_0^2 B]}{b_1^2 \sqrt{B}} \left( A_4 \left( \frac{q}{2\sqrt{B}} \right)^{-4} + A_6 \left( \frac{q}{2\sqrt{B}} \right)^{-6} + A_8 \left( \frac{q}{2\sqrt{B}} \right)^{-8} \right)
\]

(16)

where

\[
A_4 = -s_0 C_0 - \frac{1}{2B} C_1
\]

\[
A_6 = (-3 s_0 + 2 B s_0^3) C_0 + (3 s_0^2 - \frac{3}{2B}) C_1 + \frac{3s_0}{B} C_2 + \frac{3}{2B^2} C_3
\]

\[
A_8 = \left( \frac{45 s_0}{4} + 15 B s_0^3 - 597 B^2 s_0^5 \right) C_0 + \left( \frac{45}{8B} + \frac{45s_0^2}{2} + \frac{315}{2} B s_0^4 \right) C_1 + \left( \frac{45 s_0}{2B} - 15 s_0^3 \right) C_2 + \left( \frac{45}{4B^2} - \frac{45s_0^2}{2B} \right) C_3 - \frac{45s_0}{2B^2} C_4
\]

(17)

Thus, it is seen that for sufficiently large values of \( q \), the form factor tends to zero rather slowly, namely as the inverse fourth power of the momentum transfer. On the contrary, the HO form factor goes rapidly to zero for large \( q \), namely as a Gaussian or as a Gaussian times an even power of \( q \) (depending on the nucleus).

The value of \( q \) at which the \( F_{1\sigma}(q) \) approaches the value given by the asymptotic expression (16) does not seem to depend very strongly on the nucleus, at least when the values of the parameters \( b_1 \) and \( \sigma \) are determined in the way described in the following two sections. In Fig. 1 the \( F_{1\sigma}(q) \) has been plotted for the \(^{16}O\) nucleus, using the values \( b_1 = 1.563 \text{fm} \) and \( \sigma = 0.414 \text{fm} \) (see section 4), together with its asymptotic behaviour \( \text{const.} \cdot q^{-4} \) and the improved asymptotic expression (16), respectively. It is seen that \( F_{1\sigma}(q) \) becomes close to the asymptotic behaviour \( \text{const.} \cdot q^{-4} \) at quite large values of the momentum transfer (larger than \( 10 \text{fm}^{-1} \)) while essential convergence to the behavior (16) is achieved at much smaller \( q \) values. It might also be of interest to note that people have assumed in the past [23] a decrease
of the form factor of the type $Cq^{-4}$ in the region where no measurements are performed: $q > q_{max}$ (measured), in order to obtain error envelopes on the densities of nuclei. The present analysis indicates that the inclusion of additional terms of the type mentioned above seems appropriate in this sort of analysis.

There are two parameters in expression (13), the HO parameter $b_1$ and the SFC parameter $\sigma$ which can be determined from the deformation parameters $\beta_\lambda$ associated with the low lying collective states of the nucleus or can be treated for example as a free parameter. In the latter case, the fit of the form factor (13) (after correcting it for the finite proton size [6]) to the experimental data (refs. [24, 25] for $^4$He, [26] for $^{16}$O and [27] for $^{40}$Ca) leads to zero value for the parameter $\sigma$ except for $^4$He. For $^4$He, $\sigma$ is different from zero ($\sigma = 0.706 \text{fm}$ and $b_1 = 1.252 \text{fm}$) and the value of $\chi^2$ is smaller compared to the one obtained with the HO model. However, the diffraction minimum is not reproduced. Because of these reasons the introduction of short range correlations is advisable. This is done in the next section.

3 The effect of the surface fluctuations and short range correlations on the charge form factor and density

A general expression for the charge form factor $F_{ch}(q)$ of light closed shell nuclei was derived [6, 8] using the factor cluster expansion of Ristig, Ter Low and Clark [9, 10]. This formula was further simplified by using normalized correlated wave functions of the relative motion which were parameterized through a Jastrow type relative two-body wave function of the form:

$$\psi_{nlS}(r) = N_{nlS}[1 - \exp(-\lambda r^2/b^2)]\phi_{nl}(r)$$

where $N_{nlS}$ are the normalization factors, $\phi_{nl}(r)$ the harmonic oscillator wave functions and $b = \sqrt{2}b_1$ is the HO parameter for the relative motion. The expression for $F(q)$ is of the form

$$F(q) = F_1(q) + F_2(q)$$
$F_1(q)$ is the contribution of the one-body term to $F(q)$:

$$F_1(q) = \frac{1}{Z} \exp \left[-\frac{b_1^2 q^2}{4}\right] \sum_{k=0}^{2} \tilde{N}_{2k} \left(-\frac{b_1 q}{2}\right)^{2k}$$

where

$$\tilde{N}_0 = 2(\eta_{1s} + \eta_{2s} + 3\eta_{1p} + 5\eta_{1d}) , \quad \tilde{N}_2 = -\frac{4}{5}(2\eta_{2s} + 3\eta_{1p} + 10\eta_{1d})$$

$$\tilde{N}_4 = \frac{1}{3}(4\eta_{2s} + 8\eta_{1d})$$

while $F_2(q)$ is the contribution of the two-body term to $F(q)$ and is a function of $q^2$ through the matrix elements

$$A_{nl's'}^{j_k}(j_k) = \langle \psi_{nl's}|j_k(qr/2)|\psi_{n'l's'}\rangle$$

It consists of simple polynomials and exponential functions of $q^2$.

The point proton density can be obtained from (19) by Fourier transforming $F(q)$. The density is also separated into two parts:

$$\rho(r) = \rho_1(r) + \rho_2(r)$$

$\rho_1(r)$ and $\rho_2(r)$ are the Fourier transforms of $F_1(q)$ and $F_2(q)$, respectively. $\rho_1(r)$ is given by expression (20) (with $\tilde{b}_1 = b_1$) while $\rho_2(r)$ is calculated numerically because of the complexity of $F_2(q)$ mainly for $^{40}Ca$.

The correlation parameter $\lambda$ and the HO parameter $b_1$ were determined by fitting $F_{ch}(q) = f_p(q)f_{CM}(q)F(q)$ to the experimental charge form factor. $f_p(q)$, $f_{CM}(q) = \exp\left[\frac{b_1^2 q^2}{4}\right]$ are the corrections due to the finite proton size [6] and the centre of mass motion [11], respectively.

As it was pointed out in the introduction, a possible way of improving the quality of the fit in the approach outlined previously, should be to take into account the correlations originating from the fluctuation of the nuclear surface. Thus in (22), $\rho_1(r)$ is substituted by $\rho_{1\sigma}(r)$, while $\rho_2(r)$ by $\rho_{2\sigma}(r)$, which results by using in expression (20) instead of $\rho_1(r)$ the Fourier transform of $\tilde{F}_2(q) = f_{CM}(q)F_2(q)$. Therefore, instead of $\rho(r)$ given by (22) we have now $\rho_\sigma$:

$$\rho_\sigma(r) = \rho_{1\sigma}(r) + \rho_{2\sigma}(r)$$
The (total) point-proton form factor $F_\sigma(q)$ is then obtained by summing the Fourier transforms of $\rho_{1\sigma}(r)$, $F_{1\sigma}$ and of $\rho_{2\sigma}$, $F_{2\sigma}$, that is:

$$F_\sigma = F_{1\sigma} + F_{2\sigma} \quad (24)$$

It should be noted that because of the assumed form of the correlation function the asymptotic behaviour of $F_\sigma(q)$ for large $q$ is expected to be of the same functional form as that of $F_{1\sigma}(q)$. The coefficients, however, in the various even negative power terms of $q$ will be different. Thus, the introduction of the SFC effects in the correlated (through Jastrow correlations), H.O. wave function leads to less steep decrease of $F_\sigma(q)$ at large $q$. It might be of interest to point out that a slow decrease of the form factor at large $q$ is also suggested from our numerical results.

Finally, the charge form factor is obtained by multiplying $F_\sigma(q)$ by the charge form factor of the proton, $f_p(q)$:

$$F_{ch}(q) = f_p(q) F_\sigma(q) \quad (25)$$

Expression (25) was used in fitting to the experimental values of the charge form factor, by treating $b_1$, $\lambda$, and $\sigma$ as fitting parameters.

4 Numerical results and discussion

The best fit values of the three parameters in the form factor, as well as the values of $\chi^2$, for the nuclei $^{16}O$ and $^{40}Ca$ are displayed in table 1 where three cases are considered. In case 1 there are no correlations of any kind while in case 2 the SRC are included. These two cases have been studied in previous works [6, 7, 8]. Finally, in case 3, both the SFC and SRC are included. In the case of $^4He$ the best fit value of the parameter $\sigma$ goes to zero in the least-squares fitting procedure. Therefore, the present approach does not seem to be applicable to this nucleus.

The value $\sigma = 0.293 \text{ fm}$ for $^{40}Ca$ may be compared with the value $\sigma = 0.638 \text{ fm}$ which is given in [18]. It is seen that the value of $\sigma$ is considerably smaller in the present approach in comparison to that of ref. [18]. An analogous remark holds for the value of this parameter for $^{16}O$, if comparison is made with the value obtained in ref. [30].

It is further noted that the value of $\sigma$ for $^{16}O$ is larger than that of $^{40}Ca$. This is in accordance with what one expects on the basis of the variation
of \( \sigma^2 \) with the mass number, which was studied in ref. [30] and is further elaborated in the Appendix of the present paper. We also observe that if the parameter \( k \) (see expression (A.1)) is determined from our value of \( \sigma \) for \( ^{16}O \), the value of \( \sigma \) we find for \( ^{40}Ca \) is not far from the value predicted on the basis of expression (A.7). It is smaller than that value by less than 30%.

We observe also from table 1 that for \( ^{16}O \) and \( ^{40}Ca \), the introduction of the SFC and the SRC decreases the values of parameters \( b_1 \) and \( \lambda \) while the value of \( (b_1^2/\lambda)^{1/2} \) (which is the ”actual correlation parameter”, since small values of \( (b_1^2/\lambda)^{1/2} \) imply values of the correlation factors closer to unity) is increased. That would deteriorate the convergence of the cluster expansion. This is indicated by the values of the so-called ”healing” or ”wound” integrals (see ref. [31]) for the various states:

\[
\eta_{nl}^2 = \int_0^\infty |\psi_{nl}(r) - \phi_{nl}(r)|^2 dr
\]  

(26)

It is expected that, the larger the value of \( \eta^2 \) the worse the convergence of the cluster expansion of the density. With the correlated relative, two-body wave function (18), the healing integral is given simply by

\[
\eta_{nl}^2 = 2[1 + N_{nl}(I_{nl} - 1)]
\]  

(27)

where the normalization factors \( N_{nl} \) and the integrals \( I_{nl} = \int_0^\infty e^{-\lambda r^2/\eta_{nl}^2} \phi_{nl}^2 dr \) can be obtained analytically for the various states. For the lowest state, its expression is fairly simple:

\[
\eta_{00}^2 = 2 \left\{ \left[ 1 - (1 + \lambda)^{-3/2} \right] \left[ 1 - 2(1 + \lambda)^{-3/2} + (1 + 2\lambda)^{-3/2} \right]^{-1/2} \right\}
\]  

(28)

and can be easily used for calculations. We have dropped the index \( S \), since we consider state-independent correlation functions. For \( ^{16}O \), in case 2 we find \( \eta_{00}^2 = 0.00717 \) while in case 3, \( \eta_{00}^2 = 0.01379 \), indicating deterioration of the convergence of the cluster expansion. Similar are the results for the higher states (e.g. for the 1p state we have in case 2, \( \eta_{01}^2 = 0.00027 \) and \( \eta_{01}^2 = 0.00083 \) in case 3). Analogous are the results for \( ^{40}Ca \). In this case, however, the increase of the values of \( \eta_{nl}^2 \) is smaller in comparison with the corresponding one for \( ^{16}O \), indicating milder deterioration of the convergence of the cluster expansion.

It is clear regarding the results of case 3 in table 1 that the SFC effects on both, the one-body and the two-body part of the cluster expansion of the
form factor have been taken into account. Quite a simpler approach would be to take into account these effects only on the one-body part. Such an approach, however, would not be appropriate since then both terms would not be treated on the same footing. It might have been acceptable if the effect of the SFC on the two-body part had been sufficiently small, which unfortunately does not appear to be the case. This is indicated by the change in the best-fit values of the parameters. If the SFC effects are taken into account only for the one-body part of the form factor, then the best fit values for $^{16}\text{O}$ are: $b_1 = 1.647 \text{ fm}$, $\lambda = 11.440$, $\sigma = 0.224 \text{ fm}$ and for $^{40}\text{Ca}$: $b_1 = 1.814 \text{ fm}$, $\lambda = 11.786$, $\sigma = 0.364 \text{ fm}$. It is seen that there is a noticeable change in the values of these parameters. Furthermore, the value of $\sigma$ for $^{16}\text{O}$ is smaller than the value of $\sigma$ for $^{40}\text{Ca}$, which contradicts our expectations (see Appendix).

For the two nuclei we have considered, the introduction of the SFC has the effect of improving the fit of $F_{ch}(q)$ to the experimental data. Although the introduction of SFC does not decrease the value of $\chi^2$ too much, its relative decrease for $^{16}\text{O}$ being larger than that of $^{40}\text{Ca}$, there is an improvement of the fit in the region of the large $q$-values. This can be seen in figures 2a and 3a where the $F_{ch}(q)$ for $^{16}\text{O}$ and $^{40}\text{Ca}$ have been plotted with the best fit values of the parameters and compared with the experimental $F_{ch}(q)$. Another interesting observation one can make regarding these figures is the slow decrease of $F_{ch}(q)$ at large $q$ in case 3 (in which the SRC and SFC are included in the form factor), in comparison with cases 1 and 2.

It seems also appropriate to point out that the putative roles of the mean-field, short range correlations and surface-fluctuation effects get mixed up to some degree. Thus, (see table 1) the introduction of SRC changes the value of the HO parameter and the additional introduction of SFC not only does it change further this value but also the value of the parameter determing the short range correlations. Furthermore, the SFC effects which should predominantly influence the low and medium $q$ behaviour of the form factor produce a substantial change in its values and its functional behavior at very large values of the momentum transfer. One could perhaps say that the introduction of the SFC “relieves the burden” assumed by the SRC (in the two-body approximation) in correcting the independent particle model, so that they can better perform the function for which they were designed.

In figures 2b and 3b the (corresponding to $F_{ch}(q)$) charge densities $\rho_{ch}(r)$ of $^{16}\text{O}$ and $^{40}\text{Ca}$ have been plotted. In the same figures certain other relevant
quantities have also been plotted, namely:

$\alpha$. The two-body part of the density $\rho_2(r)$ corrected for the finite proton size: $\rho'_2(r)$ without SFC. The dashed line $\rho'_2(r)$ was obtained with the parameters of case 2 and the solid line $\rho'_2(r)$ was obtained with the parameters of case 3. It is seen that if the two-body part of the cluster expansion of the charge density with the Jastrow correlation function and HO orbitals is calculated with the parameters of case 3, its absolute values for various $r$ increase rather considerably (see solid line) in comparison with its absolute values calculated with the parameters of case 2.

$\beta$. The difference of the charge densities $\Delta \rho_{\text{ch}}(r) = \rho_{\text{ch}}(\text{HO} + \text{SRC} + \text{SFC}) - \rho_{\text{ch}}(\text{HO} + \text{SRC})$, that is the difference between the (HO+SRC) charge densities with and without SFC, calculated with the parameters of case 3. It is seen that $\Delta \rho_{\text{ch}}(r)$ is quite small and characterized by oscillations.

In summary, the present analysis suggests that the inclusion of the correlations originating from the fluctuations of the nuclear surface in the usual cluster expansion (truncated at the two-body part) of the charge form factor of $^{16}O$ and $^{40}Ca$ leads to some improvement in the quality of the fit to the experimental data. It is also found that the convergence of the cluster expansion is expected to deteriorate. Furthermore, the inclusion of these correlations has a drastic effect on the asymptotic behaviour of the point-proton form factor, which now falls off for large $q$ quite slowly, that is as $\text{const.} \, q^{-4}$.

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A Appendix

In this appendix we discuss the variation of the parameter $\sigma^2$ with the mass number $A$ on the basis of a phenomenological analysis. We assume that $\sigma^2$ is proportional to the ratio: $(\text{surface of the nucleus})/(\text{volume of the nucleus})$. This is suggested by the results and discussion of ref. [30]. Thus, we may approximately write:

$$\sigma^2 = \frac{k}{R_{eq}} \quad \text{(A.1)}$$

where $k$ is a proportionality constant and $R_{eq}$ the equivalent uniform radius of the nucleus, defined by

$$<r^2>_{eq} = \frac{3}{5} R_{eq}^2 = <r^2> \quad \text{(A.2)}$$

We consider that the nuclear density distribution is approximated by a symmetrized Fermi function [32]:

$$\rho_{SF}(r) = \rho_0 \left[ \left( 1 + \exp \left[ \frac{r - c}{a} \right] \right)^{-1} + \left( 1 + \exp \left[ -\frac{r - c}{a} \right] \right)^{-1} \right] \quad \text{(A.3)}$$

where $c$ is determined by the normalization condition and given by [33, 34]:

$$c = \left( \frac{1}{2} \right)^{\frac{2}{5}} r_0 A^{\frac{1}{5}} \left\{ 1 + \left[ 1 + \frac{4}{27} \left( \frac{\pi a}{r_0 A^{\frac{1}{5}}} \right)^6 \right]^{\frac{1}{2}} \right\} \left\{ 1 - \left[ 1 + \frac{4}{27} \left( \frac{\pi a}{r_0 A^{\frac{1}{5}}} \right)^6 \right]^{\frac{1}{2}} \right\}$$

$$= \left( \frac{1}{2} \right)^{\frac{2}{5}} r_0 A^{\frac{1}{5}} \left\{ 1 - \frac{1}{3} \left( \frac{\pi a}{r_0 A^{\frac{1}{5}}} \right)^2 + \frac{1}{81} \left( \frac{\pi a}{r_0 A^{\frac{1}{5}}} \right)^6 + \frac{1}{243} \left( \frac{\pi a}{r_0 A^{\frac{1}{5}}} \right)^8 + \cdots \right\} \quad \text{(A.4)}$$

The parameter $r_0$ is expressed in terms of $\rho_0$: $r_0 = \left( \frac{3}{4 \pi \rho_0} \right)^{1/3}$.

Thus, we may write:

$$<r^2> \simeq <r^2>_{SF} = \frac{3}{5} c^2 \left[ 1 + \frac{7}{3} \left( \frac{\pi a}{c} \right)^2 \right] \quad \text{(A.5)}$$

On the basis of expressions (A.1), (A.2) and (A.5), we find:

$$\sigma^2 = \frac{k}{c} \left[ 1 + \left( \frac{\pi a}{c} \right)^2 \right]^{-1/2} \quad \text{(A.6)}$$
Except for the very light nuclei, we may use only the first terms of the expansion of the above expression in powers of $A$ and we may therefore write:

$$\sigma^2 \simeq kr_0^{-1}A^{-1/3} \left[ 1 - \frac{5}{6} \left( \frac{\pi a}{r_0A^{1/3}} \right)^2 + \frac{71}{72} \left( \frac{\pi a}{r_0A^{1/3}} \right)^4 \right]$$

(A.7)

The parameters $r_0$ and $a$ appearing in $c$ may be determined by a least squares fit of the RMS radius of the symmetrized Fermi distribution to the experimental values of the RMS radius of nuclei: $<r^2>_{exp}$. Considering the nuclei of table 2 of ref. [30] and the corresponding values of $<r^2>_{exp}$ cited there, we find $r_0 = 1.147$ fm and $a = 0.507$ fm. The quality of the fit is very satisfactory.

The value of $k$ may also be determined by fitting to the values of $\sigma^2$ obtained in [30], shown in Table 2. The corresponding best fit value is $k = 2.028 \text{ fm}^3$.

Our results for the RMS radii are displayed in table 2, where also the results for $\sigma^2$ obtained with (A.6) and (A.7) are given, as well as those with the expression $\sigma^2 = k_1 A^{-\beta}$. The best fit values of the parameters in the latter expression are $k_1 = 1.415 \text{ fm}^3$ and $\beta = 0.312$, which are rather close to the corresponding values of ref. [30] ($\sigma^2 = 1.64 A^{-0.35}$), shown also in table 2.

We finally observe that expression (A.7) gives results very close to those obtained with (A.6). There is only a small difference for the lighter elements.
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Table 1: The values of the parameters $b_1$, $\lambda$, $(b_1^2/\lambda)^{1/2}$ and $\sigma$, the $\chi^2$ and the RMS radius and the contributions to it from the HO density and the various correlations for nuclei $^{16}O$ and $^{40}Ca$ found by fitting to the experimental form factors. Case 1 is referred to the HO form factor, Case 2 when the SRC are included in the form factor and Case 3 when the SRC and SFC are included in the form factor. All the quantities except $\lambda$ and $\chi^2$ are in fm.

| Case | Nucleus | $b_1$ | $\lambda$ | $\sqrt{\frac{b_1^2}{\lambda}}$ | $\sigma$ | $\chi^2$ | $<r_{ch}^2>_{\text{total}}$ | HO | SRC | SFC | Exper. |
|------|---------|-------|-----------|-------------------------------|---------|--------|-----------------|-----|-----|-----|--------|
| 1    | $^{16}O$ | 1.786 |           |                               |         |        |  9013 | 2.728 | 2.728 |        |
| 2    | $^{16}O$ | 1.679 | 12.768    | 0.470                         | 6226    | 2.659  | 2.577 | 2.577 | 0.654 |        |
| 3    | $^{16}O$ | 1.563 | 7.989     | 0.553                         | 6002    | 2.655  | 2.433 | 2.433 | 0.655 | 2.728$^a$ |
| 1    | $^{40}Ca$ | 1.950 |           |                               |         |        | 26847 | 3.439 | 3.439 |        |
| 2    | $^{40}Ca$ | 1.860 | 13.915    | 0.499                         | 19930   | 3.420  | 3.289 | 3.289 | 0.936 |        |
| 3    | $^{40}Ca$ | 1.849 | 10.356    | 0.575                         | 19588   | 3.505  | 3.272 | 3.272 | 1.165 | 0.484  | 3.482$^b$ |

Table 2: The values of RMS radii (in fm) and of the parameter $\sigma^2$ (in $fm^2$) for a number of nuclei using the results of ref.[30] and the phenomenological expressions of the Appendix.

| $^A$ | $<r^2>_{\exp}^{1/2}$ | $<r^2>_{\exp}^{1/2}$ | $\sigma^2$ Ref. [30] (Table 1) | $\sigma^2$ Ref. [30] (1.64$A^{-1/3}$) | $\sigma^2$ Ref. [30] (1.415$A^{-0.312}$) | $\sigma^2_{\text{expr. A.6}}$ | $\sigma^2_{\text{expr. A.7}}$ |
|------|----------------|----------------|---------------------------------|---------------------------------|---------------------------------|----------------|----------------|
| 16   | 2.73           | 2.76           | 0.576                           | 0.621                           | 0.597                           | 0.570          | 0.588          |
| 40   | 3.49           | 3.43           | 0.487                           | 0.451                           | 0.449                           | 0.458          | 0.460          |
| 90   | 4.26           | 4.29           | 0.360                           | 0.340                           | 0.348                           | 0.366          | 0.367          |
| 208  | 5.50           | 5.50           | 0.235                           | 0.253                           | 0.268                           | 0.286          | 0.286          |
Figure Caption

Fig. 1. The elastic point form factor in the HO model with SFC: $F_{1\sigma}(q)$ for $^{16}\text{O}$ with $b_1 = 1.563\text{fm}$ and $\sigma = 0.414\text{fm}$ (solid line) and its asymptotic behaviour $\text{const}.q^{-4}$ (long dashed line) together with the values of the asymptotic expression (16) (short dashed line).

Fig. 2. The charge form factor (3a) and density distribution (3b) of $^{16}\text{O}$. The experimental points of the form factor are from ref. [26], while for the density from refs. [28, 29]. (For various cases and notation see text).

Fig. 3. The charge form factor (4a) and density distribution (4b) of $^{40}\text{Ca}$. The experimental points of the form factor are from ref. [27], while for the density from refs. [28, 29]. (For various cases and notation see text).
