Stochastic mechanics and the Feynman integral

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The Feynman integral is given a stochastic interpretation in the framework of Nelson’s stochastic mechanics employing a time-symmetric variant of Nelson’s kinematics recently developed by the author.

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I. INTRODUCTION

In 1964, Nelson, exploiting results of Kato and Trotter, established the following important result [1].

Theorem I.1 Let $V$ be a real function on $\mathbb{R}^n$ belonging to the Kato class, let $\psi_0 \in L^2(\mathbb{R}^n)$, and let $H = -\frac{\hbar^2}{2m} \Delta + V(x)$ be the Hamiltonian operator. Then, with $x = x_0$,

$$
\psi(t, x) := \left( \exp\left[ -\frac{i}{\hbar} t H \right] \psi_0 \right) (x) = \mathcal{L}^2 \lim_{t \to \infty} \left[ \frac{2\pi \hbar t}{t m} \right]^{-\frac{n}{2}} \int \cdots \int \exp \left[ -\sum_{j=1}^l \frac{i}{\hbar} \left( -\frac{m}{2} |x_j - x_{j-1}|^2 \frac{t}{l} + V(x_j) \frac{t}{l} \right) \right] \psi_0(x_l) dx_l \cdots dx_1.
$$

This result gives a precise meaning to the Feynman integral [2]. There exists, by now, a large body of literature investigating various aspects of the Feynman integral and its generalization, see [3]-[15] and references therein. Two years later, Nelson, elaborating on previous work of Fényes and others, laid the foundations of a quantization procedure for classical dynamical systems based on diffusion processes [16].

The purpose of this paper is to show that there is a connection between [1] and [16]. More explicitly, we shall exhibit a natural interpretation of Theorem I.1 within Nelson’s stochastic mechanics [16]-[20].

As is well known, a close formal analogy between Feynman and Wiener integrals was observed very early. In order to emphasize the crucial difficulty in making this analogy complete, we recall a few well known facts. Let us consider the free case $V \equiv 0$. Then,

$$
\psi(t, x) := \left( \exp\left[ -\frac{i}{\hbar} t H \right] \psi_0 \right) (x) = \left[ \frac{2\pi \hbar t}{m} \right]^{-\frac{n}{2}} \int \exp \left[ -\frac{i}{\hbar} \left( -\frac{m}{2} \frac{|x-y|^2}{t} \right) \right] \psi_0(y) dy,
$$

since

$$
K(s, y, t, x) := \left[ \frac{2\pi \hbar (t-s)}{m} \right]^{-\frac{n}{2}} \exp \left[ \frac{im}{2\hbar} \frac{|x-y|^2}{t - s} \right]
$$

is the fundamental solution of

$$
\frac{\partial \psi}{\partial t} = \frac{i\hbar}{2m} \Delta \psi.
$$
Consider the heat equation

\[
\frac{\partial u}{\partial t} = \frac{1}{2} \Delta u, \tag{I.3}
\]

whose fundamental solution is

\[
p(s, y, t, x) := \left[2\pi(t - s)\right]^{-\frac{n}{2}} \exp\left(-\frac{|x - y|^2}{2(t - s)}\right), \quad s < t. \tag{I.4}
\]

The solution of (I.3), with initial condition \(u_0\) at time \(t = 0\), is then given by

\[
u(t, x) = \int p(0, y, t, x)u_0(y)\,dy.
\]

On the other hand, \(p(s, y, t, x)\) is also the transition density of a standard, \(n\)-dimensional Wiener process \(W\). Hence, we immediately get the probabilistic representation

\[
u(t, x) = \mathbb{E}\{u_0(W(0))|W(t) = x\}. \tag{I.5}
\]

Moreover, the kernel (I.4) may be employed to construct Wiener measure on path space via the Riesz-Markov representation theorem. Formula (I.3) may be then replaced by

\[
u(t, x) = \int_{\Omega} u_0(\omega(0))d\mathbb{W}_{tx}(\omega), \tag{I.6}
\]

where \(\Omega := C([0, t]; \mathbb{R}^n)\). With the help of the Trotter product formula, it is then possible to derive the Feynman-Kac formula for the semigroup \(\exp[-t(-\frac{1}{2}\Delta + V)]\) [1].

In 1956, Gelfand and Yaglom suggested that the same route could be followed in order to give sense to the Feynman integral as a path-integral [21]. However, as argued by Cameron [22], kernel (I.2) cannot be employed to construct a countably additive path-space measure. In particular, even in the free case \(V \equiv 0\), and differently from the diffusion case, there is no probabilistic interpretation of formula (I.1), as we don’t have a probabilistic interpretation of kernel (I.2).

In this paper, we show that a probabilistic interpretation of (I.2) is possible in the framework of Nelson’s stochastic mechanics. More explicitly, it is possible to connect the kernel (I.2) to the bi-directional generator \(L_b\) of the Nelson process (Proposition VIII.2) very much the same way that the kernel (I.4) is connected to the usual generator of the Markov process in the diffusion case (Proposition VII.2). The bi-directional generator of the Nelson process (see (III.32) for the definition) originates from a certain time-symmetric differential for finite-energy diffusions that has been used in [23]–[25] to develop elements of Lagrangian and Hamiltonian dynamics within Nelson’s stochastic mechanics. Moreover, as we showed in [26], this time-symmetric kinematics permits to derive the collapse of the wave function after a position measurement through a stochastic variational principle. The connection between the operators \(\left(\frac{\partial}{\partial t} + L_b\right)\) and \(\left(\frac{\partial}{\partial t} + \frac{i}{\hbar}H\right)\), where \(H\) is the Hamiltonian operator, is given in Theorem VI.4. The latter generalizes a well-known unitary correspondence between the usual generator and the Hamiltonian operator through the so-called ground state transformation.
II. NELSON-FÖLLMER KINEMATICS OF FINITE-ENERGY DIFFUSIONS

In this section, we review some basic results of the kinematics of diffusion processes. More information and the proofs may be found in [17]-[19], [27] and [28]. Let $(\Omega, \mathcal{E}, \mathbb{P})$ be a probability space. A stochastic process \( \{\xi(t); t_0 \leq t \leq t_1\} \) mapping \([t_0, t_1]\) into \(L^2_n(\Omega, \mathcal{E}, \mathbb{P})\) is called a finite-energy diffusion with constant diffusion coefficient \(I_n\sigma^2\) if the increments admit the representation

\[
\xi(t) - \xi(s) = \int_s^t \beta(\tau) d\tau + \sigma[w_+(t) - w_+(s)], \quad t_0 \leq s < t \leq t_1, \tag{II.7}
\]

where the forward drift \(\beta(t)\) is at each time \(t\) a measurable function of the past \(\{\xi(\tau); 0 \leq \tau \leq t\}\), and \(w_+()\) is a standard, n-dimensional Wiener process with the property that \(w_+(t) - w_+(s)\) is independent of \(\{\xi(\tau); 0 \leq \tau \leq s\}\). Moreover, \(\beta\) must satisfy the finite-energy condition

\[
E\left\{\int_{t_0}^{t_1} \beta(t) \cdot \beta(t) dt\right\} < \infty. \tag{II.8}
\]

In [27], Föllmer has shown that a finite-energy diffusion also admits a reverse-time differential. Namely, there exists a measurable function \(\gamma(t)\) of the future \(\{\xi(\tau); t \leq \tau \leq t_1\}\), called backward drift, and another Wiener process \(w_-\) such that

\[
\xi(t) - \xi(s) = \int_s^t \gamma(\tau) d\tau + \sigma[w_-(t) - w_-(s)], \quad t_0 \leq s < t \leq t_1. \tag{II.9}
\]

Moreover, \(\gamma\) also satisfies

\[
E\left\{\int_{t_0}^{t_1} \gamma(t) \cdot \gamma(t) dt\right\} < \infty, \tag{II.10}
\]

and \(w_-(t) - w_-(s)\) is independent of \(\{\xi(\tau); t \leq \tau \leq t_1\}\). Let us agree that \(dt\) always indicates a strictly positive variable. For any function \(f\) defined on \([t_0, t_1]\), let

\[
d_+f(t) := f(t + dt) - f(t)
\]

be the forward increment at time \(t\), and

\[
d_-f(t) = f(t) - f(t - dt)
\]

be the backward increment at time \(t\). For a finite-energy diffusion, Föllmer has also shown in [27] that the forward and backward drifts may be obtained as Nelson’s conditional derivatives, namely

\[
\beta(t) = \lim_{dt \searrow 0} E\left\{\frac{d_+\xi(t)}{dt} | \xi(\tau), t_0 \leq \tau \leq t\right\}, \tag{II.11}
\]

and

\[
\gamma(t) = \lim_{dt \searrow 0} E\left\{\frac{d_-\xi(t)}{dt} | \xi(\tau), t \leq \tau \leq t_1\right\}, \tag{II.12}
\]

the limits being taken in \(L^2_n(\Omega, \mathcal{B}, \mathbb{P})\). It was finally shown in [27] that the one-time probability density \(\rho(\cdot, t)\) of \(\xi(t)\) (which exists for every \(t > t_0\)) is absolutely continuous on \(\mathbb{R}^n\), and the following relation holds a.s. \(\forall t > 0\)
\[ E\{\beta(t) - \gamma(t)|\xi(t)\} = \sigma^2 \nabla \log \rho(\xi(t), t). \]  

(II.13)

Let \( \xi \) be a finite-energy diffusion satisfying (II.7) and (II.9). Let \( f : \mathbb{R}^n \times [t_0, t_1] \to \mathbb{R} \) be twice continuously differentiable with respect to the spatial variable and once with respect to time. Then, we have the following change of variables formulas:

\[
\begin{align*}
 f(\xi(t), t) - f(\xi(s), s) &= \int_s^t \left( \frac{\partial}{\partial \tau} + \beta(\tau) \cdot \nabla + \frac{\sigma^2}{2} \Delta \right) f(\xi(\tau), \tau) d\tau \\
 & \quad + \int_s^t \sigma \nabla f(\xi(\tau), \tau) \cdot d_+ w_+(\tau), \tag{II.14}
\end{align*}
\]

\[
\begin{align*}
 f(\xi(t), t) - f(\xi(s), s) &= \int_s^t \left( \frac{\partial}{\partial \tau} + \gamma(\tau) \cdot \nabla - \frac{\sigma^2}{2} \Delta \right) f(\xi(\tau), \tau) d\tau \\
 & \quad + \int_s^t \sigma \nabla f(\xi(\tau), \tau) \cdot d_- w_-(\tau). \tag{II.15}
\end{align*}
\]

The stochastic integrals appearing in (II.14) and (II.15) are a (forward) Ito integral and a backward Ito integral, respectively, see [29] for the details. Let us introduce the current drift \( v(t) := (\beta(t) + \gamma(t))/2 \) and the osmotic drift \( u(t) := (\beta(t) - \gamma(t))/2 \). Notice that, when \( \sigma \) tends to zero, \( v \) tends to \( \dot{\xi} \), and \( u \) tends to zero. The semi-difference of (II.7) and (II.9) gives the relation between the two driving "noises"

\[ 0 = \int_s^t u(\tau) d\tau + \frac{\sigma}{2} [w_+(t) - w_+(s) - w_-(t) + w_-(s)]. \tag{II.16} \]

The finite-energy diffusion \( \xi(\cdot) \) is called Markovian if there exist two measurable functions \( b_+(\cdot, \cdot) \) and \( b_-(\cdot, \cdot) \) such that \( \beta(t) = b_+(\xi(t), t) \) a.s. and \( \gamma(t) = b_-(\xi(t), t) \) a.s., for all \( t \) in \([t_0, t_1]\). The duality relation (II.13) now reduces to Nelson’s relation \([30,17]\)

\[ b_+(\xi(t), t) - b_-(\xi(t), t) = \sigma^2 \nabla \log \rho(\xi(t), t). \tag{II.17} \]

This immediately gives the osmotic equation

\[ u(x, t) = \frac{\sigma^2}{2} \nabla \log \rho(x, t), \tag{II.18} \]

where \( u(x, t) := (b_+(x, t) - b_-(x, t))/2 \). The probability density \( \rho(\cdot, \cdot) \) of \( \xi(t) \) satisfies (at least weakly) the Fokker-Planck equation

\[ \frac{\partial \rho}{\partial t} + \nabla \cdot (b_+ \rho) = \frac{\sigma^2}{2} \Delta \rho. \]

The latter can also be rewritten, in view of (II.17), as the equation of continuity of hydrodynamics

\[ \frac{\partial \rho}{\partial t} + \nabla \cdot (v \rho) = 0, \tag{II.19} \]

where \( v(x, t) := (b_+(x, t) + b_-(x, t))/2 \).

III. THE QUANTUM DRIFT, THE QUANTUM NOISE AND THE BI-DIRECTIONAL GENERATOR

We recall now the basic facts from the time-symmetric kinematics employed in [23]-[24]. In order to develop stochastic mechanics as a generalization of classical mechanics a salient difficulty is that the
finite-energy diffusion \( \{x(t); t_0 \leq t \leq t_1 \} \) representing position of the nonrelativistic particle has two natural velocities, namely the pair \((\beta(t), \gamma(t))\) or, equivalently, the pair \((v(t), u(t))\). It seems therefore natural to replace the pair of real velocities by a unique complex-valued velocity. Since in the semiclassical limit we want to recover the classical velocity, we only have the two choices \(v - iu\). As observed in [24], \(v - iu\) leads through a variational principle to the Schrödinger equation and \(v + iu\) to the conjugate of the Schrödinger equation, respectively. For a general, finite-energy diffusion \(\{\xi(t); t_0 \leq t \leq t_1\}\), how can we view the process \(v - iu\) as a drift? Let us multiply (II.7) by \(\frac{1-i}{2}\) and (II.9) by \(\frac{1+i}{2}\), respectively, and then add. We get

\[
\xi(t) - \xi(s) = \int_s^t \left[ \frac{1-i}{2} \beta(\tau) + \frac{1+i}{2} \gamma(\tau) \right] d\tau + \sigma \frac{1}{2} \left[ (1-i)(w_+(t) - w_+(s)) + (1+i)(w_-(t) - w_-(s)) \right].
\]  

We call

\[
v_q(t) := \frac{1-i}{2} \beta(t) + \frac{1+i}{2} \gamma(t) = v(t) - iu(t)
\]

the quantum drift, and

\[
w_q(t) := \frac{1-i}{2} w_+(t) + \frac{1+i}{2} w_-(t)
\]

the quantum noise. Hence, we can rewrite (III.20) as

\[
\xi(t) - \xi(s) = \int_s^t v_q(\tau) d\tau + \sigma [w_q(t) - w_q(s)].
\]  

Representation (III.22) enjoys the time reversal invariance property [24]. It has been employed in [23]-[25] in order to develop elements of Lagrangian and Hamiltonian dynamics in the frame of Nelson’s stochastic mechanics. In particular, to derive the second form of Hamilton’s principle, the key tool has been a change of variables formula related to representation (II.22). In order to recall such a formula, we need first to define stochastic integrals with respect to the quantum noise \(w_q\). Let us denote by

\[
d_b f(t) := \frac{1-i}{2} d_+ f(t) + \frac{1+i}{2} d_- f(t)
\]

the bilateral increment of \(f\) at time \(t\). From (III.21) and (II.16), we get

\[
d_+ w_q(t) = \frac{1+i}{\sigma} u(t) dt + d_+ w_+ + o(dt),
\]  

\[
d_- w_q(t) = \frac{-1+i}{\sigma} u(t) dt + d_- w_- + o(dt).
\]

These in turn give immediately the important relation:

\[
d_b w_q(t) := \frac{1-i}{2} d_+ w_+(t) + \frac{1+i}{2} d_- w_-(t) + o(dt).
\]  

**Proposition III.1**

Let \(f(x, t)\) be a measurable, \(\mathbb{C}^n\)-valued function such that

\[
P \left\{ \omega : \int_0^T f(\xi(t), t) \cdot \overline{f(\xi(t), t)} dt < \infty \right\} = 1.
\]  

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In view of (III.23), we define
\[
\int_t^s f(\xi(\tau), \tau) \cdot d_\theta w_q(\tau) := \frac{1 - i}{2} \int_t^s f(\xi(\tau), \tau) \cdot d_+ w_+(\tau) + \frac{1 + i}{2} \int_t^s f(\xi(\tau), \tau) \cdot d_- w_-(\tau).
\]
Thus, integration with respect to the bilateral increments of \(w_q\) is defined through a linear combination with complex coefficients of a forward and a backward Ito integral. Let \(f(x, t)\) be a complex-valued function with real and imaginary parts of class \(C^{2,1}\). Then, multiplying (II.14) by \(\frac{1 - i}{2}\) and (II.15) by \(\frac{1 + i}{2}\), respectively, and then adding, we get the change of variables formula
\[
f(\xi(t), t) - f(\xi(s), s) = \int_s^t \left( \frac{\partial}{\partial \tau} + v_q(\tau) \cdot \nabla - \frac{i \sigma^2}{2} \Delta \right) f(\xi(\tau), \tau) d\tau \\
+ \int_s^t \sigma \nabla f(\xi(\tau), \tau) \cdot d_\theta w_q(\tau).
\]
(III.26)

Rewriting (II.14)-(II.15) in differential form, and exploiting (III.25), we get the differential form of (III.26)
\[
d_f(\xi(t), t) = \left( \frac{\partial}{\partial t} + v_q(t) \cdot \nabla - \frac{i \sigma^2}{2} \Delta \right) f(\xi(t), t) dt + \sigma \nabla f(\xi(t), t) \cdot d_\theta w_q(t) + o(dt).
\]
(III.27)

Finally, specializing (III.27) to \(f(x, t) = x\), we get the differential form of (III.22)
\[
d_f(\xi(t)) = v_q(t) dt + \sigma d_\theta w_q(t) + o(dt).
\]

A few remarks are now in order. As it is apparent from (III.23)-(II.24), there are profound differences between the representations (II.7)-(II.9) and representation (II.22) for the increments of \(\xi\).

- The distribution of the quantum noise \(w_q\) depends on the stochastic process \(\xi\);
- Let \(\mathcal{F}_t^-\) and \(\mathcal{F}_t^+\) denote the \(\sigma\)-fields induced by the past \(\{\xi(\tau); t_0 \leq \tau \leq t\}\) and the future \(\{\xi(\tau); t \leq \tau \leq t_1\}\) of \(\xi\), respectively. The quantum noise \(w_q\) is not a forward \(\{\mathcal{F}_t^-\}\)-martingale neither a reverse-time \(\{\mathcal{F}_t^+\}\)-martingale;
- The quantum noise \(w_q\) is not Markovian even when \(\xi\) is Markovian.

The increments of the quantum noise \(w_q\) are, nevertheless, adapted both to the increasing filtration \(\mathcal{F}^+ := \{\mathcal{F}_t^+\}\), and to the decreasing filtration \(\mathcal{F}^- := \{\mathcal{F}_t^-\}\). Moreover, \(w_q\) is mean-forward differentiable with respect to the filtration \(\mathcal{F}^-\) and the corresponding mean-forward derivative is
\[
(D_+^- w_q)(t) = \lim_{dt \searrow 0} E \left\{ \frac{d_+ w_q(t)}{dt} | \mathcal{F}_t^- \right\} = \frac{1 + i}{\sigma} u(t).
\]
Similarly, \(w_q\) is mean-backward differentiable with respect to the filtration \(\mathcal{F}^+\) and the corresponding mean-backward derivative is
\[
(D_+^- w_q)(t) = \lim_{dt \searrow 0} E \left\{ \frac{d_- w_q(t)}{dt} | \mathcal{F}_t^+ \right\} = \frac{-1 + i}{\sigma} u(t).
\]

We then have the following remarkable result.

**Proposition III.2** The quantum drift of \(w_q\) with respect to \((\mathcal{F}^-, \mathcal{F}^+)\) is zero, i.e.
\[
v_q(\mathcal{F}^-, \mathcal{F}^+)(w_q)(t) := \frac{1 - i}{2} (D_+^- w_q)(t) + \frac{1 + i}{2} (D_+^- w_q)(t) = 0, \quad \forall t \in [t_0, t_1].
\]
Observing that, for all \( t \in [t_0, t_1] \), we have \((DF^- w_+)(t) = 0\) and \((DF^+ w_-)(t) = 0\), we see that there is in fact a deep analogy between the three driving processes in the representations \([I.7]\), \([I.3]\) and \([III.22]\). It follows from this result and \([III.25]\), that the quantum noise for \( w_q \) corresponding to the pair of filtrations \((\mathcal{F}^-, \mathcal{F}^+)\) is \( w_q \) itself. From now on, we consider the case where \( \{\xi(t); t_0 \leq t \leq t_1\} \) is Markovian. The analogy between the three driving noise can then also be seen in the following result \([31]\).

\[
E\{dbw_q(t)|\xi(t)\} = 0 \quad \text{(III.28)}
\]

\[
E\{dbw_q(t)dbw_q(t)^T|\xi(t)\} = -i\alpha dt. \quad \text{(III.29)}
\]

**Proposition III.3**

Now let \( L_+ \) and \( L_- \), defined by

\[
L_+ := b_+ \cdot \nabla + \frac{\sigma^2}{2} \Delta, \quad L_- := b_- \cdot \nabla - \frac{\sigma^2}{2} \Delta,
\]

be the forward and the backward generator of \( \xi \), respectively. Then \([29]\), for a scalar \( f \) of class \( C^2 \) with compact support in \( \mathbb{R}^n \), we have

\[
\lim_{dt \searrow 0} E \left\{ \frac{d_t f(\xi(t))}{dt}|\xi(t) = x\right\} = [L_+ f](x), \quad \text{(III.30)}
\]

\[
\lim_{dt \searrow 0} E \left\{ \frac{d_- f(\xi(t))}{dt}|\xi(t) = x\right\} = [L_- f](x) \quad \text{(III.31)}
\]

Let \( C^2_b(\mathbb{R}^n; \mathbb{C}) \) denote the complex, twice continuously differentiable, functions with compact support in \( \mathbb{R}^n \). For \( f \in C^2_b(\mathbb{R}^n; \mathbb{C}) \), in view of \([III.28]\), we define the bi-directional generator \( L_b \) of \( \{\xi(t)\} \) by

\[
L_b f = v_q \cdot \nabla f - \frac{i\sigma^2}{2} \Delta f = \left[ \frac{1-i}{2} L_+ + \frac{1+i}{2} L_- \right] f, \quad \text{(III.32)}
\]

where the quantum drift field is

\[
v_q(x, t) := \frac{1-i}{2} b_+(x, t) + \frac{1+i}{2} b_-(x, t).
\]

Motivation for this definition is provided also by the following result:

**Proposition III.4**

\[
\lim_{dt \searrow 0} E \left\{ \frac{db f(\xi(t))}{dt}|\xi(t) = x\right\} = [L_b f](x).
\]

Notice that the operator \( L_b \) is completely different from the generator of the bi-directional Markov semigroup \( \hat{L} \) in \([32, \text{Section 2}]\).

**IV. DISCUSSION**

We come now to a crucial point. Consider the forward driving noise \( w_+ \) in \([I.7]\). Strictly speaking, \( w_+ \) is originally only defined as an-dimensional Wiener difference process \( w_+(s, t) \), see \([17, \text{Chapter 11}]\).
and [24, Section 1]. It is namely a process such that \( w_+(t, s) = -w_+(s, t) \), \( w_+(s, u) + w_+(u, t) = w_+(s, t) \), and \( w_+(s, t) \) is Gaussian distributed with mean zero and variance \( I_n|s - t| \). Moreover, (the components of) \( w_+(s, t) \) and \( w_+(u, v) \) are independent whenever \([s, t]\) and \([u, v]\) don’t overlap. Of course, \( w_+(t) := w_+(t_0, t) \) is a standard Wiener process such that \( w_+(s, t) = w_+(t) - w_+(s) \) and \( w_+(t_0) = 0 \). The fact that \( w_+(t_0) = 0 \) is important. It makes so that the past \( \sigma \)-fields generated by \( w_+ \) and by the increments of \( w_+ \) coincide. Similarly, we can define \( w_- \) of (II.9) so that \( w_-(t_1) = 0 \). Hence, the future \( \sigma \)-fields generated by \( w_- \) and by the increments of \( w_- \) are made to coincide. Now let \( f : \mathbb{R}^n \times [t_0, t_1] \rightarrow \mathbb{C} \) be of class \( C^{2,1} \).

Then, we have:

\[
\begin{align*}
    f(w_+(t), t) - f(w_+(s), s) &= \int_s^t \left( \frac{\partial}{\partial \tau} - \frac{1}{2} \Delta \right) f(w_+(\tau), \tau) d\tau + \int_s^t \nabla f(w_+(\tau), \tau) \cdot d_+ w_+ (\tau) \\
    f(w_-(t), t) - f(w_-(s), s) &= \int_s^t \left( \frac{\partial}{\partial \tau} - \frac{1}{2} \Delta \right) f(w_-(\tau), \tau) d\tau + \int_s^t \nabla f(w_-(\tau), \tau) \cdot d_- w_- (\tau).
\end{align*}
\]

Thus, the forward generator of \( w_+ \) is \( \frac{1}{2} \Delta \), and the backward generator of \( w_- \) is \(- \frac{1}{2} \Delta \). It would be nice if we could argue along the same lines that, for \( f \in C_b^{2} (\mathbb{C}^n; \mathbb{C}) \), the bi-directional generator of the quantum noise is the operator

\[
\frac{1}{2} - i \left( \frac{1}{2} \Delta \right) + \frac{1 + i}{2} \left( - \frac{1}{2} \Delta \right) = - \frac{i}{2} \Delta.
\]

But this is not possible because of measurability problems. Let us see why. Instead of definition (III.21), we could start by defining \( w_q \) only as a difference process by

\[
w_q(s, t) := \frac{1 - i}{2} w_+(s, t) + \frac{1 + i}{2} w_-(s, t).
\]

For a difference process \( \theta(s, t) \), we define \( d_+ \theta(t) := \theta(t, t + dt) \) and \( d_- \theta(t) := \theta(t - dt, t) \). We can then derive as before formulas (III.23)-(III.25). Then, we would need to define the quantum noise \( w_q \) at some time \( t \) so that the process \( w_q(t) := w_q(t) + w_q(t, t) \) is simultaneously adapted to the two filtrations induced by its past and future increments. But this is clearly impossible. Hence, an object such as

\[
\int_s^t \nabla f(w_q(\tau), \tau) \cdot d_+ w_q (\tau) = \frac{1 - i}{2} \int_s^t \nabla f(w_q(\tau), \tau) \cdot d_+ w_q (\tau) + \frac{1 + i}{2} \int_s^t \nabla f(w_q(\tau), \tau) \cdot d_- w_q (\tau)
\]

cannot be given a meaning, since at least one of the two \( \text{Ito} \) integrals in the right-hand side cannot be defined.

V. STOCHASTIC MECHANICS

Nelson’s stochastic mechanics \([18 - 29]\) may be based, since the important paper by Guerra and Morato \([9, Section 1]\). It is namely a process such that \( x(t); t_0 \leq t \leq t_1 \) with diffusion coefficient \( \frac{\Delta}{m} \) to which it is naturally associated a quantum evolution \( \{\psi(x, t); t_0 \leq t \leq t_1\} \), namely a solution of the Schrödinger equation

\[
\frac{\partial \psi}{\partial t} = i \frac{\hbar}{2m} \Delta \psi - i \frac{\hbar}{\hbar} V(x) \psi,
\]

(V.33)
The probability density $\rho(\cdot, t)$ of $x(t)$ satisfies $\rho(x, t) := |\psi(x, t)|^2$, and the quantum drift field is given by

$$v_q(x, t) = \frac{\hbar}{mi} \nabla \log \psi(x, t).$$

Conversely, given a solution of the Schrödinger equation $\{\psi(x, t); t_0 \leq t \leq t_1\}$ satisfying the finite action condition (V.34), a probability measure $P$ may be constructed on path space under which the coordinate process is a finite-energy Markov diffusion with quantum drift as in (V.35), cf. [34], [20, Chapter IV].

**VI. RELATION BETWEEN THE BI-DIRECTIONAL GENERATOR AND THE HAMILTONIAN OPERATOR**

In order to establish the relation in the section title, we need first the following elementary result.

**Lemma VI.1** Let $a$ and $b$ be two complex numbers, and let $V : \mathbb{R}^n \to \mathbb{R}$ be a measurable function. Let $u : \mathbb{R}^n \times [t_0, t_1] \to \mathbb{C}$ be a never vanishing solution of the p.d.e.

$$\frac{\partial u}{\partial t} = a \Delta u + b V u,$$

on $[t_0, t_1]$. Then $\theta := u \phi$ is another solution of (VI.36) on $[t_0, t_1]$ if and only if $\phi$ satisfies on the same time interval

$$\frac{\partial \phi}{\partial t} = 2a \nabla \log u \cdot \nabla \phi + a \Delta \phi.$$

**Proof.** We have the following chain of equalities

$$\frac{\partial(u \phi)}{\partial t} = \frac{\partial u}{\partial t} \phi + u \frac{\partial \phi}{\partial t} = a \Delta u \phi + b V u \phi + u \frac{\partial \phi}{\partial t} = a (\Delta u \phi + 2 \nabla u \cdot \nabla \phi + u \Delta \phi) + b V u \phi + u \frac{\partial \phi}{\partial t} - 2a \nabla u \cdot \nabla \phi - au \Delta \phi$$

$$= a \Delta (u \phi) + b V (u \phi) + u \left( \frac{\partial \phi}{\partial t} - 2a \frac{\nabla u}{u} \cdot \nabla \phi - a \Delta \phi \right).$$

**Remark VI.2** We shall apply Lemma [VI.1] to both the diffusion and the quantum case. Particularly for the latter application, it would be desirable to have a more general result where $u$ may vanish. In order to avoid obscuring ideas with technicalities, we shall be content here with discussing the non singular case. It appears quite feasible, however, that applying ideas and results of Carlen and others, see [20, Chapter IV] and references therein, some of these applications may be suitably extended to the singular case.
Lemma VI.3 Let $u$ and $V$ be as in Lemma [VI.1]. Let $C^{2,1}_b(\mathbb{R}^n \times [t_0,t_1]; \mathbb{C})$ denote the complex-valued functions of class $C^{2,1}$ with compact support in $\mathbb{R}^n \times [t_0,t_1]$. On this domain, we consider the operators

$$A := \frac{\partial}{\partial t} - a\Delta - bM_V,$$

where $M_V$ denotes the operator of multiplication by the function $V$, and

$$B := \frac{\partial}{\partial t} - 2a\nabla \log u \cdot \nabla - a\Delta.$$

Then, for $f \in C^{2,1}_b(\mathbb{R}^n \times [t_0,t_1]; \mathbb{C})$, we have

$$Bf = M_{u^{-1}}AM_u f.$$  \hfill (VI.38)

Let $\mathcal{L}^{2,1}_c$ denote the Hilbert space of complex-valued functions $f$ satisfying

$$\int_{t_0}^{t_1} \|f\|^2_{\mathcal{L}^2_c(\mathbb{R}^n)} dt < \infty.$$

Theorem VI.4 Let $a = \frac{i}{2m}$ and $b = -\frac{i}{\hbar}$ in Lemma [VI.1]. Let $\{\psi(x,t); t_0 \leq t \leq t_1\}$ be a never vanishing solution of the Schrödinger equation (V.33) satisfying (V.34). Let $L_b$ denote the bi-directional generator of the associated Nelson process as defined in (III.32), and let $H = -\frac{\hbar^2}{2m}\Delta + V(x)$ denote the quantum Hamiltonian operator. We consider the operator $(\frac{\partial}{\partial t} + \frac{i}{\hbar}H)$ defined in $\mathcal{L}^{2,1}_c$, and let $\mathcal{L}^{2,1}_c(|\psi|^2)$ denote the Hilbert space of functions $g$ such that $(g\psi) \in L^2(\mathbb{R}^n; |\psi|^2 dx)$. Then, $(\frac{\partial}{\partial t} + L_b)$ defined in $\mathcal{L}^{2,1}_c(|\psi|^2)$ and $(\frac{\partial}{\partial t} + \frac{i}{\hbar}H)$ are unitarily equivalent. Indeed, it follows from (VI.39) that

$$\frac{\partial}{\partial t} + L_b = M_{\psi}^{-1} \left( \frac{\partial}{\partial t} + \frac{i}{\hbar}H \right) M_{\psi}. \hfill (VI.39)$$

Remark VI.5 Relation (VI.39) supports the choice of the kinematics of Section 3 to study quantum-mechanical problems. It may be viewed as a generalization of a well-known result relating the usual generator to the Hamiltonian operator through the ground state transformation, see e.g. [35,36,8]. Indeed, for $\psi(x,t) = \psi_0(x)$ the ground state of the Hamiltonian ($H\psi_0 = 0$), and $f \in L^2(\mathbb{R}^n; |\psi_0|^2 dx)$, (VI.33) reads

$$\frac{\hbar}{im} \nabla \log \psi_0 \cdot \nabla f - \frac{i\hbar}{2m}\Delta f = \frac{i}{\hbar}M_{\psi_0}^{-1}HM_{\psi_0} f.$$

This immediately gives

$$- \frac{\hbar^2}{m} \left( \nabla \log \psi_0 \cdot \nabla f + \frac{1}{2}\Delta f \right) = M_{\psi_0}^{-1}HM_{\psi_0} f. \hfill (VI.40)$$

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VII. THE FEYNMAN-KAC FORMULA

Let \( h : \mathbb{R}^n \times [t_0, t_1] \to \mathbb{R} \) be a classical, never vanishing solution of the terminal value problem
\[
\frac{\partial h}{\partial t} + \frac{1}{2} \Delta h = V(x)h, \quad h(x, t_1) = h_1(x), \tag{VII.41}
\]
where \( V \) is a nonnegative, measurable function on \( \mathbb{R}^n \). A simple calculation shows that \( \log h \) satisfies
\[
\frac{\partial \log h}{\partial t} + \nabla \log h \cdot \nabla \log h + \frac{1}{2} \Delta \log h = \frac{1}{2} \nabla \log h \cdot \nabla \log h + V(x). \tag{VII.42}
\]
Assume that there exists a weak solution \( P \) on \([t_0, t_1]\) of the stochastic differential equation
\[
dx = \nabla \log h \, dt + dw.
\]
Namely, the coordinate process \( \{x(t); t_0 \leq t \leq t_1\} \) under \( P \) admits the above forward differential. Applying Lemma VI.1 to the diffusion case, we get a different generalization of (VI.40). Let \( L^2, 1 \) denote the Hilbert space of real-valued functions \( f \) satisfying
\[
\int_{t_0}^{t_1} \|f\|^2_{L^2(\mathbb{R}^n)} dt < \infty
\]

**Proposition VII.1** Let \( a = -\frac{1}{2} \) and \( b = 1 \) in Lemma VI.1. Let \( h(x, t); t_0 \leq t \leq t_1 \) be a never vanishing solution of equation (VII.41). Let
\[
L_+ = \nabla \log h \cdot \nabla + \frac{1}{2} \Delta
\]
denote the generator of the measure \( P \) and let \( H = -\frac{1}{2} \Delta + V(x) \) denote the Hamiltonian operator. We consider the operator \( \left( \frac{\partial}{\partial t} - H \right) \) defined in \( L^{2, 1} \). Let \( L^{2, 1}(h^2) \) denote the Hilbert space of functions \( g \) such that \( (gh) \in L^{2, 1} \). Then, \( \left( \frac{\partial}{\partial t} + L_+ \right) \) defined in \( L^{2, 1}(h^2) \) and \( \left( \frac{\partial}{\partial t} - H \right) \) are unitarily equivalent. Indeed, it follows from (VI.38), that
\[
\frac{\partial}{\partial t} + L_+ = Mh^{-1} \left( \frac{\partial}{\partial t} - H \right) Mh. \tag{VII.43}
\]

We recall below three derivations of the Feynman-Kac formula, see e.g. [3]. These will serve for the purpose of comparison in the following section. Hence, no effort will be made for maximal generality.

**Derivation 1.**
Suppose now that, under \( P \), \( \{x(t)\} \) is a finite energy diffusion. Under \( P \), we have
\[
h(x(t), t) = h_1(x(t_1)) \exp \left[ - \int_t^{t_1} d\log h(x(\tau), \tau) d\tau \right].
\]
By Ito’s rule, and (VII.42), we get
\[
h(x(t), t) = h_1(x(t_1)) \exp \left\{ - \int_t^{t_1} \left( \frac{1}{2} \nabla \log h \cdot \nabla \log h + V \right) d\tau - \int_t^{t_1} \nabla \log h \cdot dw(\tau) \right\}. \tag{VII.44}
\]
Let us introduce the random variable
Let Derivation 2. which is the Feynman-Kac formula. The above derivation of (VII.48), based on the Girsanov transformation, indeed, gives

\[ P \] with respect to \( \Omega = E \).

Then, observing that \( \exp \) is bounded, we conclude that the stochastic integral on the right-hand side is a martingale. Taking the conditional expectation \( E \cdot |w(t) = x \) on both sides, we get (VII.48).

Derivation 3.
We shall now look at the derivation of the Feynman-Kac formula based on the Trotter product formula. We consider first the case $V \equiv 0$. Let $q(t, x, t_1, y)$ be the transition density of the measure $P$. Taking $a = -\frac{1}{2}$ in Lemma VI.1, we get that
\[
\frac{h(x, t)}{h_1(y)}q(t, x, t_1, y)
\]
is the fundamental solution of
\[
\frac{\partial u}{\partial t} + \frac{1}{2}\Delta u = 0.
\]

Proposition VII.2 The kernel
\[
\frac{h(x, t)}{h_1(y)}q(t, x, t_1, y)
\]
does not depend on $\{h(x, t); t_0 \leq t \leq t_1\}$. Indeed,
\[
\frac{h(x, t)}{h_1(y)}q(t, x, t_1, y) = p(t, x, t_1, y) = [2\pi(t_1 - t)]^{-\frac{n}{2}} \exp \left[-\frac{|x - y|^2}{2(t_1 - t)}\right]. \tag{VII.52}
\]
Notice that relation (VII.52) between transition densities mirrors the corresponding relation between probability measures that, in view of (VII.46), here reads
\[
\frac{h(x(t), t)}{h_1(x(t_1))} dP_{tx} = dW_{tx}.
\]
From (VII.52), we immediately get
\[
h(x, t) = E\{h_1(w(t_1)|w(t) = x\} = \int_{\Omega} h_1(\omega(t_1))dW_{tx}(\omega).
\]
Consider now the case where $V$ is any continuous function. An interesting consequence of Lemma VI.1 is the following. Let $\{h_2(x, \tau); t_0 \leq t \leq t_1\}$ be another solution of (VII.41). Let
\[
\varphi(x, t) := \frac{h_2(x, t)}{h(x, t)}.
\]

Corollary VII.3 Under $P$, the stochastic process $\varphi(x(t), t)$ satisfies
\[
\varphi(x(t), t) - \varphi(x(s), s) = \int_s^t \nabla \varphi(x(\tau), \tau) \cdot dw(\tau), \quad s < t. \tag{VII.53}
\]
Proof. By Lemma VI.1,
\[
\left[ \frac{\partial}{\partial t} + \nabla \log h(x, t) \cdot \nabla + \frac{1}{2}\Delta \right] \varphi = 0.
\]
By Itô's rule, we now get (VII.53). \qed

Now let $q(t, x, t_1, y)$ be the transition density of the measure $P$. Taking $a = -\frac{1}{2}$ and $b = 1$ in the Lemma VI.1 we get that $w(t, x, t_1, y)$ defined by
\[
w(t, x, t_1, y) := \frac{h(t, x)}{h_1(y)}q(t, x, t_1, y)
\]
is another solution of equation (VII.41). Let us find some heuristic connection between \( w(t, x, t_1, y) \) and the kernel \( p(t, x, t_1, y) \) in (I.4). Let

\[
\begin{align*}
    r(t, x, t_1, y, x_1) := w(t, x, t_1, y) \exp [V(x_1)(t_1 - t)].
\end{align*}
\]

\( r \) satisfies

\[
\frac{\partial r}{\partial t} + \frac{1}{2} \Delta r = [V(x) - V(x_1)] r.
\]

Then, for \( |x_1 - x| \) small, the function \( r(t, x, t_1, y, x_1) \) is close to \( p(t, x, t_1, y) \). Now let \( \omega(\cdot) \) be a continuous curve on \([t, t_1]\), and let \( x_j = \omega(t + (t_1 - t)j/l), j = 0, 1, \ldots, l\). Iterating, we then get

\[
\begin{align*}
    h(t, x) &= \lim_{l \to \infty} \frac{[2\pi(t_1 - t)/l]^{-\frac{1}{4}} - n l^2}{\int \cdots \int \exp \left[ -\sum_{j=1}^l \frac{t_1 - t}{l} \left( \frac{|x_j - x_{j-1}|^2}{2(t_1 - t)/l} + V(x_j) \right) \right] h(t_1, x_1) dx_1 \cdots dx_1.}
\end{align*}
\]

(VII.54)

Observing that

\[
\int_t^{t_1} \frac{1}{2} \dot{\omega}(\tau)^2 d\tau
\]

may be viewed as the density of Wiener measure with respect to a (fictitious) uniform measure on \( \mathbb{R}^\infty \), we recognize that (VII.54) coincides with the Feynman-Kac formula (VII.48). For \( V \) in the Kato class, this heuristic argument can be turned into the rigorous one of Theorem I.1 by means of the Trotter formula [1].

**VIII. FEYNMAN INTEGRALS**

Let \( \{\psi(x, t); t_0 \leq t \leq t_1\} \) be the solution of the Schrödinger equation (V.33) with initial condition \( \psi(x, t_0) = \psi_0(x) \). We suppose that \( \psi \) never vanishes and satisfies

\[
\int_{t_0}^{t_1} \int_{\mathbb{R}^n} \left[ \nabla \psi(x, t) \cdot \nabla \psi(x, t) \right] dx dt < \infty.
\]

(VIII.55)

Hence, the finite energy condition of [34] is satisfied, and there exists a probability measure \( P \) on path space under which the coordinate process has forward drift field

\[
\psi(x, t) + u(x, t) = \frac{\hbar}{m} \nabla \left[ 3 \log \psi(x, t) + \Re \log \psi(x, t) \right],
\]

and quantum drift field \( v_q(x, t) = \frac{\hbar}{m} \nabla \log \psi(x, t) \). Let \( \{x(t); t_0 \leq t \leq t_1\} \) denote the coordinate process with the Nelson measure \( P \). Observe that \( \log \psi(x, t) \) satisfies

\[
\frac{\partial \log \psi}{\partial t} + \frac{\hbar}{2im} \nabla \log \psi \cdot \nabla \log \psi + \frac{i}{\hbar} V(x) - \frac{i\hbar}{2m} \Delta \log \psi = 0.
\]

(VIII.56)

We now seek to derive a path-integral representation for \( \psi(x, t) \) adapting to the present setting the first derivation of the Feynman-Kac formula in the previous section. Under the Nelson measure \( P \), we have
\[ \psi(x(t), t) = \psi_0(x(0)) \exp \left[ \log \psi(x(t), t) - \log \psi_0(x(0)) \right]. \quad (VIII.57) \]

By the change of variables formula (III.26), we get
\[
\psi(x(t), t) = \psi_0(x(0)) \times 
\exp \left\{ \int_0^t \left[ \frac{\partial}{\partial \tau} + [iV(x(\tau), \tau) - iu(x(\tau), \tau)] \cdot \nabla - \frac{i\hbar}{2m} \Delta \right] \log \psi(x(\tau), \tau) d\tau + \int_0^t \nabla \log \psi(x(\tau), \tau) \cdot d_b w_q(\tau) \right\}. \quad (VIII.58)
\]

By equation (VIII.56), and recalling that
\[ v(x(\tau), \tau) - iu(x(\tau), \tau) = \frac{\hbar}{im} \nabla \log \psi(x(\tau), \tau), \]
we get
\[
\psi(x(t), t) = \psi_0(x(0)) \times 
\exp \left\{ \int_0^t \left[ \frac{im}{2\hbar} [v(x(\tau), \tau) - iu(x(\tau), \tau)] \cdot [v(x(\tau), \tau) - iu(x(\tau), \tau)] - \frac{i}{\hbar} V(x(\tau)) \right] \right. \\
\left. + \int_0^t \nabla \log \psi(x(\tau), \tau) \cdot d_b w_q(\tau) \right\}. \quad (VIII.59)
\]

Let us introduce the random variable
\[
\tilde{Z}_{t_0}^t := \exp \left\{ \int_0^t \left[ \frac{im}{2\hbar} [v(x(\tau), \tau) - iu(x(\tau), \tau)] \cdot [v(x(\tau), \tau) - iu(x(\tau), \tau)] d\tau \\
+ \int_0^t \frac{im}{\hbar} [v(x(\tau), \tau) - iu(x(\tau), \tau)] \cdot d_b w_q(\tau) \right\}
\]
\[= \exp \left\{ \int_0^t \left[ -\frac{im}{2\hbar} [v(x(\tau), \tau) - iu(x(\tau), \tau)] \cdot [v(x(\tau), \tau) - iu(x(\tau), \tau)] d\tau \\
+ \int_0^t \frac{im}{\hbar} [v(x(\tau), \tau) - iu(x(\tau), \tau)] \cdot d_b x(\tau) \right\}, \quad (VIII.60)\]
and rewrite (VIII.59) as
\[
\psi(x(t), t) = \psi_0(x(0)) \exp \left\{ \int_0^t \left[ -\frac{i}{\hbar} V(x(\tau)) \right] d\tau \right\} \tilde{Z}_{t_0}^t. \quad (VIII.61)
\]

Let \( P_{tx} \) denote the conditional Nelson measure \( P[\omega(t) = x] \) on \( \Omega = C([0, t]; \mathbb{R}^n) \). Taking expectations of both sides of (VIII.61) with respect to \( P_{tx} \), we get
\[
\psi(x, t) = \int_{\Omega} \psi_0(\omega(0)) \exp \left\{ \int_0^t \left[ -\frac{i}{\hbar} V(\omega(\tau)) \right] d\tau \right\} \tilde{Z}_{t_0}^t dP_{tx}(\omega). \quad (VIII.62)
\]

This representation appears similar to representation (VII.47) for the solution \( h(x, t) \) of the antiparabolic equation of the previous section. What made (VII.47) useful was the relation \( Z_{t_0}^t dP_{tx} = dW_{tx} \) showing that the product \( Z_{t_0}^t dP_{tx} \) is a universal measure on path space independent of the particular solution \( h(x, t) \). It is apparent that \( \tilde{Z}_{t_0}^t \) cannot be a Radon-Nikodym derivative between two probability measures on path space since it is complex-valued. We are then led to the following two crucial questions:

1. Is \( \tilde{Z}_{t_0}^t dP_{tx} \) a bona fide complex measure of bounded variation (see Appendix A) on \( C([0, T]; \mathbb{R}^n) \)?
2. Is \( \hat{Z}_{t_0}^t dP_{tx} \) in some appropriate sense independent from the particular solution \( \{ \psi(x,t) \} \), i.e. is it independent of \( \psi_0(x) \) and of \( V(x) \)?

Obviously, we expect a negative answer to the second question as the quantum noise, to which the “measure” \( \hat{Z}_{t_0}^t dP_{tx} \) should correspond, does depend on the particular solution \( \{ \psi(x,t) \} \).

**Proposition VIII.1** Let \( \{ \psi(x,t); t_0 \leq t \leq t_1 \} \) be a never vanishing solution of the Schrödinger equation \( \{V.3\} \) with initial condition \( \psi(x,t_0) = \psi_0(x) \), and satisfying \( \{VIII.53\} \). Assume that \( \psi_0 \in L^1(\mathbb{R}^n) \).

Let \( P_{tx} \) be the conditional Nelson measure associated to \( \{ \psi(x,s); 0 \leq s \leq t \} \) and let \( \hat{Z}_{t_0}^t \) be defined by \( \{VIII.60\} \). Then, \( \hat{Z}_{t_0}^t \in L^1(P_{tx}) \). It follows that \( d\mu := \hat{Z}_{t_0}^t dP_{tx} \) is a complex measure of bounded variation on \( C([t_0,t];\mathbb{R}^n) \).

**Proof.** Taking absolute values on both sides of \( \{VIII.61\} \), and recalling Born’s relation \( |\psi(x,\tau)|^2 = \rho(x,t) \) relating the wave function to the probability density of the Nelson process at time \( t \), we get

\[
|\hat{Z}_{t_0}^t| = \frac{\rho^{1/2}(x(t),t)}{\rho_0^{1/2}(x(0))} \tag{VIII.63}
\]

where \( \rho_0(x) = |\psi_0(x)|^2 \). Hence

\[
\int |\hat{Z}_{t_0}^t| dP_{tx} = \int \left( \frac{\rho^{1/2}(x(t),t)}{\rho_0^{1/2}(x(0))} \right) dP_{tx} = \rho^{1/2}(x,t) \int_{\mathbb{R}^n} \rho_0^{1/2}(x) dx = \rho^{1/2}(x,t) \int_{\mathbb{R}^n} |\psi_0(x)| dx < \infty
\]

Thus, under the hypothesis and in the notation of the above proposition, we can rewrite \( \{VIII.62\} \) in the form

\[
\psi(x,t) = \int_{\Omega} \psi_0(\omega(0)) \exp \left\{ \int_0^t \left[ -\frac{i}{\hbar} V(\omega(\tau)) \right] d\tau \right\} d\mu(\omega). \tag{VIII.64}
\]

It follows, however, from \( \{VIII.63\} \) that the total variation \( |\mu| \) of \( \mu \) satisfies

\[
d|\mu|(\omega) = |\hat{Z}_{t_0}^t| dP_{tx}(\omega) = \frac{\rho^{1/2}(x,t)}{\rho_0^{1/2}(\omega(0))} dP_{tx}(\omega).
\]

Thus, the measure \( \mu \) does depend on the particular solution \( \{ \psi(x,t) \} \). An attempt to derive a path-integral representation for \( \psi(x,t) \) along the lines of the second derivation of the Feynman-Kac formula appears hopeless because \( \psi(w_q(t),t) \) makes no sense since \( w_q \) has complex values and, more importantly, because of the considerations made in Section 4. We turn, therefore, to the third derivation. Consider first the case \( V = 0 \). In view of the change of variable formula \( \{III.27\} \), we take \( p_q(t_0,y,t,x) \) to be the fundamental solution of the equation

\[
\left( \frac{\partial}{\partial t} + v_q(x,t) \cdot \nabla - \frac{i\hbar}{2m} \Delta \right) u = 0, \tag{VIII.65}
\]

where, as usual, \( v_q(x,t) = \frac{\hbar}{im} \nabla \log \psi(x,t) \). Taking \( a = \frac{\hbar}{2m} \) in Lemma \( \{V1.1\} \), we get that

\[
\frac{\psi(t,x)}{\psi_0(y)} p_q(t_0,y,t,x)
\]

is the fundamental solution of
\[ \frac{\partial u}{\partial t} - \frac{\hbar}{2m} \Delta u = 0. \]  

(VIII.66)

Hence, we get the counterpart of Proposition VII.2.

**Proposition VIII.2** The kernel

\[ \frac{\psi(t, x)}{\psi_0(y)} p_q(t, y, t, x) \]

does not depend on \{\( \psi(x,t); t_0 \leq t \leq t_1 \}\}. Indeed,\[ \frac{\psi(t, x)}{\psi_0(y)} p_q(t, y, t, x) = K(t_0, y, t, x) = \left[ \frac{2\pi \hbar (t - t_0)}{m} \right]^{-\frac{1}{2}} \exp \left[ \frac{im|x - y|^2}{2\hbar(t - t_0)} \right]. \]  

(VIII.67)

Consider now the case where \( V \) is any continuous function. Let \{\( \psi(x,t); t_0 \leq t \leq t_1 \}\} be a never vanishing solution of the Schrödinger equation (V.33) with initial condition \( \psi(x,t_0) = \psi_0(x) \), and satisfying (VIII.5), and let \{\( \psi_2(x,t); t_0 \leq t \leq t_1 \}\} be another solution of (V.33). Let\[ \tilde{\psi}(x,t) := \frac{\psi_2(t, x)}{\psi(t, x)} \]

**Corollary VIII.3** Under the Nelson measure \( P \) associated to \{\( \psi(x,t); t_0 \leq t \leq t_1 \}\}, the stochastic process \( \tilde{\psi}(x,t) \) satisfies

\[ \tilde{\psi}(x,t) - \tilde{\psi}(x,s) = \int_s^t \sqrt{\frac{\hbar}{m}} \nabla \tilde{\psi}(x,\tau) \cdot dw_q(\tau), \quad s < t. \]  

(VIII.68)

**Proof.** By Lemma VI.1,

\[ \left[ \frac{\partial}{\partial t} + v_q(x,t) \cdot \nabla - \frac{i\hbar}{2m} \Delta \right] \tilde{\psi} = 0, \]  

(VIII.69)

where \( v_q(x,t) = \frac{\hbar}{im} \nabla \log \psi(x,t) \). By (III.26), we now get (VIII.68). \( \square \)

This result is the counterpart of Corollary VII.3. Notice that, since the ratio of two solutions of the Schrödinger equation satisfies (VIII.69), the function

\[ \theta(x,t) := \log \frac{\psi_2(x,t)}{\psi(x,t)} \]

satisfies the nonlinear equation

\[ \frac{\partial \theta}{\partial t} + v_q(x,t) \cdot \nabla \theta(x,t) - \frac{i\hbar}{2m} \Delta \theta(x,t) = \frac{i\hbar}{2m} \nabla \theta(x,t) \cdot \nabla \theta(x,t). \]

This is precisely the Hamilton-Jacobi-type equation associated to the variational problem that produces the new Nelson process after a position measurement, causing the “collapse of the wave function”, see [26, Section VI].

Now, let \( p_q(t_0, y, t, x) \) be the fundamental solution of (VIII.63). Taking \( a = \frac{\hbar}{2m} \) and \( b = -\frac{1}{m} \) in Lemma VI.1 we get that \( \tilde{w}(t_0, y, t, x) \) defined by\[ \tilde{w}(t_0, y, t, x) := \frac{\psi(t, x)}{\psi_0(y)} p_q(t_0, y, t, x) \]
is another solution of the Schrödinger equation (V.33). Let us find some heuristic connection between \( \tilde{w}(t_0, y, t, x) \) and the kernel \( K(t_0, y, t, x) \) in (I.2). Let \( \tilde{r}(t_0, y, t, x, x_1) := \tilde{w}(t_0, y, t, x) \exp[i\hbar V(x_1)(t - t_0)] \). Then \( \tilde{r} \) satisfies
\[
\frac{\partial \tilde{r}}{\partial t} - \frac{i\hbar}{2m} \Delta \tilde{r} = \frac{i\hbar}{\hbar} [V(x_1) - V(x)] \tilde{r}.
\]
For \( |x_1 - x| \) small, the function \( \tilde{r}(t_0, y, t, x, x_1) \) is close to \( K(t_0, y, t, x) \). Now let \( \omega(\cdot) \) be a continuous curve on \([t_0, t]\), and let \( x_j = \omega(t_0 + (t - t_0)j/l), j = 0, 1, \ldots, l \). Iterating, we then get
\[
\psi(t, x) = \lim_{l \to \infty} \left[ \frac{2\pi \hbar (t - t_0)}{lm} \right]^{-\frac{|x|}{2}} \int \cdots \int \exp \left[ -\sum_{j=1}^{l} \frac{i \hbar}{2m} \frac{|x_j - x_{j-1}|^2}{(t - t_0)/l} + V(x_j)(t - t_0) \right] \psi_0(x_l) dx_l \cdots dx_1.
\]
This heuristics can be turned into the rigorous argument of Theorem I.1 by means of the Kato-Trotter formula [1].

IX. CLOSING COMMENTS

We have shown that, employing the time-symmetric kinematics of Section 3, it is possible to establish a link between Nelson’s stochastic mechanics and the Feynman integral. Not surprisingly, we do have the following negative result. It is not possible to view the operator \( -\frac{i}{\hbar} \Delta \) as the bi-directional generator of the quantum noise, as argued in Section 4. Moreover, the complex measure \( \mu \) in Proposition VIII.1 does depend on the particular solution of the Schrödinger equation. Nevertheless, the results in the second part of the previous section show that the analogy with the diffusion case goes far beyond what was believed, provided the time-symmetric kinematics of stochastic mechanics is employed in the quantum case.

In [18], concerning the Feynman integral and stochastic mechanics, Guerra writes: “The full clarification of the deep connection between the two approaches will be a major step toward a better understanding of the physical foundations of quantum mechanics”. We hope that this paper will stimulate new research in this direction.

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**APPENDIX A: COMPLEX MEASURES**

We collect in this appendix a few basic facts about complex measures. We refer the reader to Chapter 6 for the proofs and more information.

Let \( \Omega \) be a set and \( \mathcal{B} \) a \( \sigma \)-algebra of subsets of \( \Omega \). A complex function \( \mu \) on \( \mathcal{B} \), i.e. \( \mu : \mathcal{B} \rightarrow \mathbb{C} \), is called a complex measure on \( \mathcal{B} \) if, for every \( B \in \mathcal{B} \),

\[
\mu(B) = \sum_{i=1}^{\infty} \mu(B_i)
\]

holds whenever \( \{B_i\}_{i=1}^{\infty} \) is a countable partition of the set \( B \). It is implicit in this definition that every such series must converge.

Let \( \mu \) be a complex measure. Then, among all positive, i.e. usual, measures \( \lambda \) satisfying \( |\mu(B)| \leq \lambda(B) \), \( \forall B \in \mathcal{B} \), there exists a least one called total variation of \( \mu \) and denoted by \( |\mu| \). The measure \( |\mu| \) is
minimal among all positive measures \( \lambda \) described above in the sense that \(|\mu|(B) \leq \lambda(B)\) for all \( B \in \mathcal{B} \).

The measure \(|\mu|\) has the remarkable property that \(|\mu(\Omega)| < \infty\). Thus the range of every complex measure \( \mu \) lies in a disc of finite radius. It is then usual to say that \( \mu \) is of bounded variation.

**Theorem A.1** Let \( \lambda \) be a positive, \( \sigma \)-finite measure on \( \mathcal{B} \). Let \( \mu \) be a complex measure on \( \mathcal{B} \). Suppose that \( \mu \) is absolutely continuous with respect to \( \lambda \), namely \( \mu(B) = 0 \) for every \( B \in \mathcal{B} \) for which \( \lambda(B) = 0 \). Then there exists a unique function \( h \in L^1(\lambda) \) such that

\[
\mu(B) = \int_B h \, d\lambda
\]

for every \( B \in \mathcal{B} \).

A consequence of this theorem taking \( \lambda = |\mu| \), is the following result.

**Theorem A.2** Let \( \mu \) be a complex measure on \( \mathcal{B} \). Then there exists a unimodular function \( h \), i.e. \( |h(\omega)| = 1 \) for all \( \omega \in \Omega \), such that the following polar decomposition of \( \mu \) holds

\[
d\mu = h \, d|\mu|.
\]

We also have the following result.

**Theorem A.3** Suppose \( \lambda \) is a positive measure on \( \mathcal{B} \), \( h \in L^1(\lambda) \), and \( \mu \) is the complex measure on \( \mathcal{B} \) defined by

\[
\mu(B) = \int_B h \, d\lambda.
\]

Then, for all \( B \in \mathcal{B} \), we have

\[
|\mu|(B) = \int_B |h| \, d\lambda.
\]