An exponential kernel associated with operators that have one-dimensional self-commutators

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Abstract

The exponential kernel

\[ \exp \left( -\frac{1}{\pi} \int_{\mathbb{C}} \frac{g(u)}{u-w(u-\lambda)} da(u) \right), \]

where the compactly supported bounded measurable function \( g \) satisfies \( 0 \leq g \leq 1 \), and suitably defined for all complex \( \lambda, w \), plays a role in the theory of Hilbert space operators with one-dimensional self-commutators and in the theory of quadrature domains. This article studies continuity and integral representation properties of \( E_g \) with further applications of this exponential kernel to operators with one-dimensional self-commutator.

1 Introduction

For \( g \) a compactly supported bounded measurable function defined on the complex plane \( \mathbb{C} \) that satisfies \( 0 \leq g \leq 1 \), let \( E_g(\lambda, w) = E(\lambda, w) \) be defined by

\[ E(\lambda, w) = \exp \left( -\frac{1}{\pi} \int_{\mathbb{C}} \frac{g(u)}{u-w(u-\lambda)} da(u) \right) = \]

\[ \exp \left( -\frac{1}{\pi} \int_{\mathbb{C}} \frac{g(u)}{u-w(u-\lambda)} da(u) \right) \]  

(1.1)

for \( \lambda \neq w \), with \( E(w, w) \) defined to be 0 if \( \frac{1}{\pi} \int_{\mathbb{C}} g(u)|u-w|^{-2} = \infty \) and equal to

\[ \exp \left( -\frac{1}{\pi} \int_{\mathbb{C}} \frac{g(u)}{|u-w|^2} da(u) \right) \]  

(1.2)

when

\[ \frac{1}{\pi} \int_{\mathbb{C}} \frac{g(u)}{|u-w|^2} da(u) < \infty. \]

Here \( a \) denotes area measure.

The function \( E_g \) first appeared in the study of bounded linear operators on a Hilbert space with one-dimensional self-commutator. To show this connection,
let $T$ be a bounded linear operator on a Hilbert space $H$ satisfying $T^*T - TT^* = \varphi \otimes \varphi$. Throughout the following it will be assumed that $T$ is irreducible, which in this case is equivalent to the statement that there are no non-zero subspaces of $H$ reducing $T$ where $T$ restricts to a normal operators. Pincus [17] established that there is a one-to-one correspondence between the unitary equivalence classes of the collection of such operators with the collection of equivalence classes of compactly supported Lebesgue measurable functions $g$ satisfying $0 \leq g \leq 1$. The (equivalence class of the) function $g_T$ associated with $T$ is called the principal function of $T$.

The principal function $g_T$ (the subscript $T$ will usually not be included on the principal function) first appeared in [18] in the study of spectral theory of self-adjoint singular integral operators on the real line. To continue the story of the connection of $E_{g_T}$ with $T$, we introduce the local resolvent. It develops that for $\lambda$ in $C$ there is a unique solution of the equation $T^* \lambda \varphi = \varphi$ orthogonal to the kernel of $T^* \lambda = (T - \lambda)^*$. This solution is denoted $T^*_{\lambda}^{-1} \varphi$. The $H$-valued function $T^*_{\lambda}^{-1} \varphi$ defined for $\lambda \in C$ was first investigated by Putnam [21] and Radjabalipour [22] and will be called the global-local resolvent associated with the operator $T$. The following result from [6] shows the connection between the function $E_{g_T}$ and the $H$-valued function $T^*_{\lambda}^{-1} \varphi$.

**Theorem 1.1.** Let $T$ be an irreducible operator with one-dimensional self-commutator $T^*T - TT^* = \varphi \otimes \varphi$. Let $g = g_T$ be the associated principal function and $T^*_{\lambda}^{-1} \varphi$, $\lambda \in C$ the associated global-local-resolvent. Then for $\lambda$ and $w$ in $C$

$$1 - (T^*_{w}^{-1} \varphi, T^*_{\lambda}^{-1} \varphi) = E_{g}(\lambda, w) = \exp \left(-\frac{1}{\pi} \int_{C} \frac{u-w}{u-\lambda} \frac{g(u)}{|u-w|^2} du(u) \right).$$

(1.3)

For $|\lambda|$ and $|w|$ larger than $\|T\|$, the result in the above theorem follows from early work on the principal function as presented in [4]. The identity (1.3) has consequences for both the operator $T$ and the function $E$. For example, it follows easily from the weak continuity of the global-local resolvent and (1.3) that the function $E$ is separately continuous. However, the fact that $E$ is separately continuous without assuming the identity is far from obvious (see, [15, p. 260]).

In descending chronological order, expository accounts of the relevant operator theory can be found in [15], [26], [27], [5], and [20]. We also refer to these sources for historical accounts and many of the references to the area.

In an unexpected direction, when the function $g$ is the characteristic function of a planar domain $\Omega$, Putinar [19] made connections between the exponential kernel $E_g$, quadrature domains, and operators with one dimensional self-commutator. A recent account of these connections, including a discussion of properties of $E$, and citations of earlier work can be found in the book [12].

The main focus here is on the continuity and integral representation properties of $E_g$. This is the content of the first part of this paper. We will study the continuity properties function $E_g$ without any reference to the associated operator $T$. In the second part, we will offer some comments about the operator $T$ that can be gleaned from the identity (1.3) and results in the first part.
2 Continuity properties of E

In this section we will establish the sectional continuity of the function $E_g$ and show this function is locally Lipschitz at points of positive density of the measure $gda$. A study of Cauchy transform representations of $E_g$ will also be presented. It should be remarked that the sectional continuity of $|E_g|$ was established in [6] using methods similar to but less exact than those employed here. It will be assumed that $w$ is fixed and $E_g(\lambda, w)$ will be considered as a function of $\lambda$. Unless stated otherwise, the function $g$ will be assumed to satisfy $0 \leq g \leq 1$.

2.1 Sectional continuity

Fix the point $w$. For $\lambda \neq w$, the continuity of the function

$$f_w(\lambda) = -\frac{1}{\pi} \int_{\mathbb{C}} \frac{g(u)}{(u-w)(u-\lambda)} da(u)$$

(2.1)

can be established by elementary means. In particular, this follows from basic properties of the Cauchy transform that will be introduced below. Thus for $w$ fixed, a study of the sectional continuity of $E_g(\lambda, w)$ reduces to investigating the continuity at $w$. This will accomplished by first studying the continuity of (2.1) in the case $\frac{1}{\pi} \int_{\mathbb{C}} \frac{g(u)}{|u-w|} da(u) < \infty$.

We first derive some integral formulas for the case where the function $g$ in (2.1) is the characteristic function of well-chosen discs.

The linear fractional mapping

$$T_{w,\lambda}(u) = \frac{u-w}{u-\lambda}$$

has invariant properties relative to the measure $\frac{1}{|u-w|^2} da(u)$. This property can be used to compute the real and imaginary parts of the integral

$$-\frac{1}{\pi} \int_{D} \frac{u-w}{u-\lambda} da(u)$$

over specific discs.

For $\alpha \in \mathbb{R}$ let $D_{\lambda,\alpha}$ be the disc with center $c_\alpha = \frac{w+\lambda+(\lambda-w)}{2}$ of radius $r_\alpha = \frac{|(\lambda-w)(1-\alpha)|}{2}$. For $\alpha < 1$,

$$D_{\lambda,\alpha} = \{ u : \text{Re} \left[ \frac{u-w}{u-\lambda} \right] < \frac{\alpha}{\alpha-1} \}$$

and for $\alpha > 1$

$$D_{\lambda,\alpha} = \{ u : \text{Re} \left[ \frac{u-w}{u-\lambda} \right] > \frac{\alpha}{\alpha-1} \}.$$

We note that for $N > 1$

$$D_{\lambda,\infty} = \{ u : \text{Re} \left[ \frac{u-w}{u-\lambda} \right] < -N \}$$
and
\[ D_{\lambda, N} = \{ u : \text{Re} \left( \frac{u-w}{u-\lambda} \right) > N \}. \]

For \( 0 \leq \alpha \), a direct computation using a change of variables and polar coordinates shows
\[ -\frac{1}{\pi} \int_{D_{\lambda, \alpha}} \text{Re} \frac{u-w}{u-\lambda} \frac{1}{|u-w|^2} da(u) = -\frac{1}{\pi} \int_0^{\pi} \log(\alpha^2 \cos^2 \theta + \sin^2 \theta) d\theta = \ln \frac{2}{1+\alpha}, \]
and for \( \alpha < 0 \)
\[ -\frac{1}{\pi} \int_{D_{\lambda, \alpha}} \text{Re} \frac{u-w}{u-\lambda} \frac{1}{|u-w|^2} da(u) = \ln \frac{2}{1+|\alpha|}. \tag{2.2} \]

In a similar manner, for \( \beta \neq 0 \) in \( \mathbb{R} \), let \( \Delta_{\lambda, \beta} \) be the disc of radius \( r_\beta = \frac{|\beta(\lambda-w)|}{2} \) centered at \( c_\beta = \lambda + \frac{i(\lambda-w)\beta}{2} \).
For \( \beta > 0 \),
\[ \Delta_{\lambda, \beta} = \{ u : \text{Im} \frac{u-w}{u-\lambda} < -\frac{1}{\beta} \}, \]
and for \( \beta < 0 \)
\[ \Delta_{\lambda, \beta} = \{ u : \text{Im} \frac{u-w}{u-\lambda} > -\frac{1}{\beta} \}. \]

Another polar-coordinates computation shows that for \( \beta \neq 0 \)
\[ -\frac{1}{\pi} \int_{\Delta_{\lambda, \beta}} \text{Im} \frac{u-w}{u-\lambda} \frac{1}{|u-w|^2} da(u) = \frac{1}{\pi} \int_0^{\pi} \arctan(\beta + \cot \theta) d\theta = \arctan\left( \frac{\beta}{2} \right). \tag{2.3} \]

At the end of this paper it will be shown that the identities (2.2) and (2.3) are closely aligned and derivable from (1.3) in the case where the operator \( T \) is the unilateral shift.

One consequence of the identities (2.2) and (2.3) is the following. Given \( \varepsilon > 0 \), there is an \( M = M(\varepsilon) \) independent of \( \lambda \) and \( w \) such that for \( N > M \)
\[ -\frac{1}{\pi} \int_{|\text{Re} \frac{u-w}{u-\lambda}| > N} \text{Re} \left( \frac{u-w}{u-\lambda} \right) \frac{1}{|u-w|^2} da(u) < \varepsilon \]
and
\[ -\frac{1}{\pi} \int_{|\text{Im} \frac{u-w}{u-\lambda}| > N} \text{Im} \left( \frac{u-w}{u-\lambda} \right) \frac{1}{|u-w|^2} da(u) < \varepsilon. \tag{2.4} \]

For technical reasons, it is required that \( M > 1 \).

Using the above, one can directly establish the sectional continuity of the function \( E \).

**Theorem 2.1.** For \( w \) fixed in \( \mathbb{C} \) the function \( E_g(\cdot, w) \) is continuous on \( \mathbb{C} \).
Proof. Case 1. \( \frac{1}{\pi} \int_{C} \frac{g(u)}{|u-w|^2} da(u) < \infty. \)

Let \( \varepsilon > 0. \) Let \( N > M \) be fixed so that the inequalities (2.4) and (2.5) hold for all \( \lambda \neq w. \) Choose \( \delta_0 > 0 \) such that for \( 0 < \delta \leq \delta_0 \) one has

\[
\frac{1}{\pi} \int_{|u-w| < \delta} \frac{g(u)}{|u-w|^2} da(u) < \frac{\varepsilon}{N}. \tag{2.6}
\]

There exists a \( \delta_1 < \delta_0 \) such that for \( |\lambda - w| < \delta_1, \) one has

\[
\frac{|u - w|}{u - \lambda} - 1 < \varepsilon \text{ for } |u - w| \geq \delta_0.
\]

Note for \( \lambda \) sufficiently close to \( w, \) say for \( |\lambda - w| < \delta_2, \) the set

\[
U_N^{\lambda} = \{ u : |Re \frac{u - w}{u - \lambda}| \geq N \} = D_{\lambda, \frac{N}{\varepsilon}} \cup \Delta_{\lambda, \frac{\pi}{\varepsilon}, N}^{\lambda}
\]

and the set

\[
V_N^{\lambda} = \{ u : |Im \frac{u - w}{u - \lambda}| \geq N \} = \Delta_{\lambda, \frac{1}{\varepsilon}, N} \cup \Delta_{\lambda, \frac{\pi}{\varepsilon}, N}^{\lambda}
\]

will be in the disc \( \{ u : |u-w| < \delta_1 \}. \) For \( |\lambda - w| < \delta_2, \) we estimate

\[
D(\lambda) := \left| \frac{1}{\pi} \int_{C} \frac{u - w}{u - \lambda} \frac{g(u)}{|u-w|^2} da(u) - \frac{1}{\pi} \int_{C} \frac{g(u)}{|u-w|^2} da(u) \right|
\]

separately over the sets

\[ A = \{ u : |u-w| \geq \delta_1 \}, \quad B = \{ u : |u-w| < \delta_1 \}\setminus(U_N^{\lambda} \cup V_N^{\lambda}), \quad C = U_N^{\lambda} \cup V_N^{\lambda}. \]

On \( A \) we have the estimate

\[
\left| \frac{1}{\pi} \int_{A} \frac{u - w}{u - \lambda} \frac{g(u)}{|u-w|^2} da(u) - \frac{1}{\pi} \int_{A} \frac{g(u)}{|u-w|^2} da(u) \right| \leq \frac{\varepsilon}{\pi} \int_{C} \frac{g(u)}{|u-w|^2} da(u).
\]

On \( B \)

\[
\left| \frac{1}{\pi} \int_{B} \frac{u - w}{u - \lambda} \frac{g(u)}{|u-w|^2} da(u) - \frac{1}{\pi} \int_{B} \frac{g(u)}{|u-w|^2} da(u) \right| \leq \frac{\varepsilon}{\pi} \int_{C} \frac{g(u)}{|u-w|^2} da(u) \leq 2N \frac{1}{\pi} \int_{B} \frac{g(u)}{|u-w|^2} da(u) + \frac{\varepsilon}{N} \leq 3\varepsilon.
\]

On \( C \)
The last integral is less than the corresponding integral over \( \{ u : |u - w| < \delta_1 \} \) and consequently less than \( \varepsilon \). The first of these last two integrals is less than \( \lambda \) established in [6]. The result follows once it is shown that \( \lim_{\lambda \to 1} [\pi w] \) is less than \( \delta_2 \).

This completes the proof in Case 1.

Case 2. \( \frac{1}{\pi} \int_{C} \left| \frac{u - w}{u - \lambda} \right| \frac{g(u)}{|u - w|^2} da(u) = \infty \). This case is easier then the first case and was established in [6]. The result follows once it is shown that \( \lim_{\lambda \to w} |E(\lambda, w)| = 0 \).

For completeness, we include the details. The notation \( D(w, r) \) will be used for the disc in \( \mathbb{C} \) centered at \( w \) of radius \( r \). By the monotone convergence theorem

\[
\lim_{\lambda \to w} \frac{1}{\pi} \int_{D(w, |\lambda - w|)} \frac{g(u)}{|u - w|^2} da(u) = \infty.
\]

In the notation introduced above, the disc \( D(w, |\lambda - w|) \) coincides with \( D_{\lambda,-1} \), which can be written as the disjoint union \( (D_{\lambda,-1} \setminus D_{\lambda,0}) \cup D_{\lambda,0} \). Then

\[
|E(\lambda, w)| = \exp \left( -\frac{1}{\pi} \int_{C} \left| \frac{u - w}{u - \lambda} \right| \frac{g(u)}{|u - w|^2} da(u) \right) \left( \exp \left( -\frac{1}{\pi} \int_{D_{\lambda,0}} \left| \frac{u - w}{u - \lambda} \right| \frac{g(u)}{|u - w|^2} da(u) \right) \right) \leq 2 \exp \left( -\frac{1}{2\pi} \int_{C \setminus D_{\lambda,-1}} \frac{g(u)}{|u - w|^2} da(u) \right).
\]

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Here we used the facts that $\text{Re} \left\{ \frac{u - w}{u - \lambda} \right\}$ is greater than $\frac{1}{2}$ on $\mathbb{C} \setminus D(w, |\lambda - w|)$ and non-negative on $D_{\lambda, -1} \setminus D_{\lambda, 0}$ as well as the result that
\[
-\frac{1}{\pi} \int_{D_{\lambda, 0}} \text{Re} \left\{ \frac{u - w}{u - \lambda} \right\} \frac{1}{|u - w|^2} da(u) = \ln 2.
\]
As noted above, the desired result now follows from the monotone convergence theorem.

\[\square\]

**Remark 2.2.** Assuming, as is the case here, that $0 \leq g \leq 1$, one consequence of the last integral inequality is the inequality
\[|E_g(\lambda, w)| \leq 2, \quad \text{for all } \lambda, w\]
with equality holding if and only if $g$ is the characteristic function of a disc and where $\lambda, w$ are antipodal boundary points.

**Remark 2.3.** With $w$ fixed, with minor modifications, the proof of Case 1 in Theorem 2.1 establishes the continuity of the integral in (2.1) as a function of $\lambda$ for any compactly supported bounded measurable function $g$ under the assumption
\[
\frac{1}{\pi} \int_{C} \frac{|g(u)|}{|u - w|^2} da(u) < \infty.
\]

### 2.2 Local Lipschitz continuity

The function $E_g$ is Lipschitz at almost every point in the support of $g$. To establish this we will use the following elementary lemmas.

**Lemma 2.4.** Let $h = h(x)$ be continuous on the interval $[0, R]$ with $h(0) = 0$ and $R > 0$ is fixed. For $t \in (0, R]$ define
\[
H(t) = \int_{t}^{R} \frac{h(x)}{x} dx.
\]
Given $0 < \varepsilon$ there exists a $\delta > 0$ and constant $K = K(\varepsilon)$ such that for $0 < t < \delta$ one has the estimate
\[|H(t)| \leq K - \varepsilon \ln t\]

**Proof.** Let $\varepsilon > 0$ and find $\delta > 0$ such that $0 < t < \delta$ implies $|h(t)| < \varepsilon$. Then for $0 < t < \delta$
\[
|H(t)| \leq \int_{t}^{\delta} |h(x)| \frac{dx}{x} + \int_{\delta}^{R} |h(x)| \frac{dx}{x} \leq \varepsilon \int_{t}^{\delta} \frac{dx}{x} + M \ln R - M \ln \delta = M \ln R + (\varepsilon - M) \ln \delta - \varepsilon \ln t,
\]
where $M$ is the maximum of $|h|$ on $[0, R]$. The result follows with $K(\varepsilon) = M \ln R + (\varepsilon - M) \ln \delta$. \[\square\]
We continue to assume that \( g \) is a measurable function with compact support satisfying \( 0 \leq g \leq 1 \) and introduce the notation \( \mathbf{L}_g \) for the set of points of positive Lebesgue density of \( g \). Thus \( \mathbf{L}_g \) is the set of points \( w \) that satisfy

\[
\lim_{R \to 0} \frac{1}{\pi R^2} \int_{D(w, r)} g(u) da(u) = \gamma > 0.
\]

**Lemma 2.5.** Let \( g = g(u) \) be a bounded non-negative measurable function defined in a neighborhood of \( D(0, R) \) of where \( 0 \in \mathbf{L}_g \), that is,

\[
\lim_{r \to 0} \frac{1}{\pi r^2} \int_{D(0, r)} g(u) da(u) = \gamma > 0.
\]

Given \( \varepsilon > 0 \) there is a \( \delta = \delta(\varepsilon) \) and a constant \( K = K(\varepsilon) \) such that with \( 0 < t < \delta \) one has the estimate

\[
\frac{1}{2\pi} \int_{D(0, R) \setminus D(0, t)} \frac{g(u)}{|u|^2} da(u) < K - (\gamma - \varepsilon) \ln t.
\]

**Proof.** For \( 0 \leq s \leq R \), let \( G(s) = \frac{1}{2\pi} \int_{0}^{2\pi} g(se^{i\theta}) d\theta \) and for \( 0 < t \leq R \) set

\[
f(t) = \int_{t}^{R} \frac{G(s)}{s^2} ds.
\]

Note that

\[
f(t) = \frac{1}{2\pi} \int_{D(0, R) \setminus D(0, t)} \frac{g(u)}{|u|^2} da(u).
\]

Applying integration by parts for Lebesgue-Stieltjes integration, see, for example, [13], to this first integral above for \( f \) one obtains the identity

\[
f(t) = \left[ \frac{\int_{t}^{x} G(s) ds}{x^2} \right]_{t}^{R} + \int_{t}^{R} \left\{ \frac{2}{x^2} \int_{0}^{x} G(s) ds \right\} dx = \frac{1}{R^2} \int_{0}^{R} G(s) ds - \frac{1}{2t^2} \int_{0}^{t} \int_{0}^{x} G(s) ds - \gamma \right\} dx + \gamma \ln R - \gamma \ln t.
\]

We note that as \( t \to 0 \) the second term in this last expression approaches \( \frac{1}{2} \gamma \) and as a consequence is bounded on \([0, R]\). If one applies the preceding lemma to the third integral in the right side of this last identity with

\[
h(x) = \frac{2}{x^2} \int_{0}^{x} G(s) ds - \gamma
\]

the result follows. \( \square \)

**Theorem 2.6.** Let \( w \) be in \( \mathbf{L}_g \), that is, assume

\[
\lim_{\lambda \to w} \frac{1}{\pi |\lambda - w|^2} \int_{D(w, |\lambda - w|)} g(u) da(u) = \gamma > 0,
\]

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so that, \( E(w, w) = 0. \) Then the function \( E_g(\lambda, w) \) is Lipschitz at \( w. \) More specifically, given \( \varepsilon < \gamma \) there is a disc \( D(w, \delta) \) with

\[
|E(\lambda, w)| \leq K|\lambda - w|^\gamma - \varepsilon, \quad \lambda \in D(w, \delta). \tag{2.7}
\]

**Proof.** First note that the disc \( D(w, |\lambda - w|) \) is precisely \( D_{\lambda, 0} \) and the complement of this disc is precisely the set where \( \text{Re} \left[ \frac{w - \lambda}{u - \lambda} \right] \geq 1. \) As a consequence

\[
|E(\lambda, w)| = \exp \left( -\frac{1}{\pi} \int_{C \setminus D(\lambda, w)} \frac{g(u)}{u - \lambda} \frac{\text{Re} u - w}{|u - w|^2} \, da(u) \right) \leq 2 \exp \left( -\frac{1}{2\pi} \int_{C \setminus D(\lambda, w)} \frac{g(u)}{|u - w|^2} \, da(u) \right). \tag{2.8}
\]

This last identity is obtained by writing the integral

\[-\frac{1}{\pi} \int_{D(\lambda, w)} \frac{\text{Re} u - w}{u - \lambda} \frac{g(u)}{|u - w|^2} \, da(u)\]

over \( D(w, |\lambda - w|) = D_{\lambda, -1} \) as the sum of the integrals over \( D_{\lambda, -1} \setminus D_{\lambda, 0} \) and \( D_{\lambda, 0}. \) The first of these integral being negative contributes nothing to the last inequality. In the second integral one can replace \( g \) by 1 and use the identity

\[\frac{1}{\pi} \int_{D_{\lambda, 0}} \frac{\text{Re} u - w}{u - \lambda} \frac{1}{|u - w|^2} \, da(u) = \ln 2\]

to obtain the above bound. The result then follows from (2.8) and Proposition 2.5. \( \square \)

**Example 2.7.** A simple example to keep in mind is the case where \( g \) is the characteristic function \( \mathbb{1}_D \) of the unit disc \( D. \) In this case, we denote the function \( E_g \) by \( E_D. \) One computes

\[
E_D(\lambda, w) = \begin{cases} \frac{|\lambda - w|^2}{1 - \lambda w} & \text{if } \lambda, w \in D \\ \frac{|w - \lambda|^2}{1 - \lambda w} & \text{if } \lambda \in D \text{ and } w \notin D \\ 1 - \frac{1}{\lambda w} & \text{if } \lambda, w \notin D. \end{cases} \tag{2.9}
\]

We remark that this example suggests that, in general, the local Lipschitz order of \( E_g(w, \cdot) \) at \( w \) in (2.7) should be \( 2(\gamma - \varepsilon). \) This is the case if \( g \) is smooth in a neighborhood of \( w. \)

### 2.3 Cauchy Transform Representations of \( E_g \)

Given a compactly supported measure \( \mu \) on \( \mathbb{C}, \) the Cauchy Transform \( \hat{\mu} \) is the locally integrable function defined for a.e. \( \lambda \) in \( \mathbb{C} \) by

\[
\hat{\mu}(\lambda) = \frac{1}{\pi} \int_{\mathbb{C}} \frac{d\mu(u)}{u - \lambda}. \tag{2.10}
\]
For a measure of the form \(d\mu = fda\), where \(f\) is a compactly supported integrable function, the Cauchy transform will be denote \(\hat{f}\). A good place to read about the Cauchy Transform is the monograph of Garnett [11]. In the sense of distributions

\[-\bar{\partial}\hat{\mu} = \mu.\]

Formally, for \(w\) fixed and \(\lambda \neq w\), one expects in the sense of distributions

\[\bar{\partial}_\lambda E_g(\lambda, w) = \frac{E_g(\lambda, w)}{\lambda - w}g(\lambda)\]  

(2.11)

and, consequently,

\[E_g(\lambda, w) = 1 - \left(\frac{E_g(\lambda, w)}{\lambda - w}g(\lambda)\right)^\wedge.\]  

(2.12)

Here, \(\hat{\cdot}\) denotes the distributional Cauchy transform, given for a distribution \(S\) with compact support on the test function \(\phi\) by

\[< \phi, \hat{S}> = -< \hat{\phi}, S >.\]

Taking into account behavior at infinity and Weyl’s Lemma, for \(w\) fixed and \(\lambda \neq w\), formally, one further expects

\[E(\lambda, w) = 1 - \frac{1}{\pi} \int_\mathbb{C} \frac{E(u, w)}{u - w}g(u)\frac{da(u)}{u - \lambda}.\]  

(2.13)

All of the above distributional identities are subtle. Although, for \(w\) fixed, the function \(\frac{E_g(\lambda, w)}{\lambda - w}g(\lambda)\) is integrable, it is unclear whether the identity (2.13) holds for all \(\lambda\). The goal of this subsection is to establish circumstances where, for \(w\) fixed, (2.13) holds for all \(\lambda\). More specifically, it will be shown that this is the case when \(w \in L_g\) or when \(\frac{1}{\pi} \int_\mathbb{C} \frac{g(u)}{|u-w|^2}da(u) < \infty\). We will take a somewhat ad hoc path to the results.

We will first consider an easy case. Let \(G\) denote the essential support of the function \(g\). Thus \(z\) is in \(G\) if and only if every neighborhood of \(z\) intersects the set \(\{u : g(u) \neq 0\}\) in a set of positive measure. For the case \(w \notin G\) one can give a direct proof that (2.13) holds for all \(\lambda\). To this end, recall that for \(h\) and \(k\) bounded measurable functions with compact support in \(\mathbb{C}\) one has

\[\hat{hk} = \hat{h}\hat{k} + \hat{\hat{k}}\]

see, for example, [11] p.107]. We remark that fact that \(h\) and \(k\) are bounded with compact support implies that \(\hat{h}\) and \(\hat{k}\) are continuous. As a consequence, for \(h\) bounded with compact support and \(N \geq 1\) one has

\[\hat{h}^N = N\hat{h}^{N-1}h.\]  

(2.14)
Proposition 2.8. Let \( w \notin G \). Then (2.13) holds for all \( \lambda \) in \( \mathbb{C} \).

Proof. Applying the identity (2.14) with \( h(u) = \frac{g(u)}{u-w} \) one sees

\[
1 - E(\lambda, w) = \sum_{N=1}^{\infty} \frac{(-1)^{N-1}}{N!} \left( \frac{1}{\pi} \int_{\mathbb{C}} \frac{g(u)}{u-w} \frac{da(u)}{u-\lambda} \right)^N =
\]

\[
\sum_{N=1}^{\infty} \frac{(-1)^{N-1}}{(N-1)!} \frac{1}{\pi} \int_{\mathbb{C}} \left( \frac{1}{\pi} \int_{\mathbb{C}} \frac{g(v)}{v-w} \frac{da(v)}{v-u} \right)^{N-1} \frac{g(u)}{u-w} \frac{da(u)}{u-\lambda} =
\]

\[
\frac{1}{\pi} \int_{\mathbb{C}} E(u, w) \frac{g(u)}{u-w} \frac{da(u)}{u-\lambda}.
\]

where, for \( \lambda \) fixed, the interchange of summation and integration to produce the last expression follows from the uniform convergence of the series for \( E_g(u, w) \) in the variable \( u \) on \( G \), which is the support of the finite measure \( \frac{g(u)}{u-w} da \).

Remark 2.9. The extent of the validity of (2.14) for arbitrary planar integrable \( h \) is unclear; however, when \( N = 2 \) it does hold a.e. when \( \hat{h} \) is integrable with respect to \( |h| da \) (see, [10]).

We continue our study of the validity of (3.5) and (2.12) by first deriving and analogue of (2.14) for functions of the form

\[
h_w(u) = \frac{g(u)}{u-w}. \tag{2.15}
\]

Without loss of generality, it is sufficient to consider this last identity when \( w = 0 \). We begin by noting that for \( \lambda \neq 0 \)

\[
\int_{\mathbb{C}} \frac{g(u)}{u(u-\lambda)} da(u) = \frac{1}{\lambda} \left[ \int_{\mathbb{C}} \frac{ug(u)}{u-\lambda} da(u) - \int_{\mathbb{C}} \frac{g(u)}{u} da(u) \right]. \tag{2.16}
\]

We introduce the notations

\[
h_0(u) = \frac{g(u)}{u} \quad k_0(u) = \frac{ug(u)}{u} \quad C = -\frac{1}{\pi} \int_{\mathbb{C}} \frac{g(u)}{u} da(u).
\]

Thus equation (2.16) can be written in the compact form

\[
\hat{h}_0(\lambda) = \frac{1}{\lambda} \left[ \hat{k}_0(\lambda) + C \right] \quad \lambda \neq 0.
\]

Note that

\[
C = -\hat{k}_0(0).
\]

Since the function \( k_0 \) is bounded with compact support, we can apply (2.14) to this function. Using a binomialial expansion we see for \( \lambda \neq 0 \),
\( \hat{h}_0^N (\lambda) = \frac{1}{\lambda^N} \sum_{j=0}^{N} \frac{N!}{j!(N-j)!} \hat{k}_0^j (\lambda) C^{N-j} = \)
\[ \frac{C^N}{\lambda^N} + \frac{N}{\lambda^N} \sum_{j=1}^{N} \frac{(N-1)!}{(j-1)!(N-j)!} (\hat{k}_0)^{j-1} k_0(\lambda) C^{N-j} = \] 
(2.17)

Therefore we have the following:

**Proposition 2.10.** Let \( h_0 \) be the function defined by (2.15) with \( w = 0, C = -\frac{1}{\pi} \int_{\mathbb{C}} \frac{g(v)}{v-u} da(v) \), and \( \lambda \neq 0 \). For \( N \geq 1 \)
\[ \hat{h}_0^N (\lambda) = \frac{C^N}{\lambda^N} + \frac{N}{\lambda^N} \int_{\mathbb{C}} u^N \left( \frac{1}{\pi} \int_{\mathbb{C}} \frac{h_0(v)}{v-u} da(v) \right)^{N-1} \frac{h_0(u)}{u-\lambda} da(u). \] 
(2.18)

As a consequence, in the sense of distributions, on \( \mathbb{C} \setminus \{0\} \),
\[ -\mathcal{F} (\hat{h}_0^N) = Nh_0^{N-1} h_0. \] 
(2.19)

**Remark 2.11.** We remark that the function \( f_N(u) = u^N \left( \frac{1}{\pi} \int_{\mathbb{C}} \frac{h_0(v)}{v-u} da(v) \right)^{N-1} \)
appearing in this last integral extends to be continuous on \( \mathbb{C} \).

On \( \mathbb{C} \setminus \{0\} \), the series
\[ E(\lambda, 0) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \hat{h}_0^{-n} (\lambda) \]
converges in the sense of distributions. Consequently, using (2.19), on \( \mathbb{C} \setminus \{0\} \),
we have the distributional identity
\[ -\mathcal{F} E(\lambda, 0) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} (-\mathcal{F}) (\hat{h}_0^{-n}) (\lambda) = \]
\[ -\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(n-1)!} \hat{h}_0^{-n-1} h_0(\lambda) = \]
(2.20)

Thus, for fixed \( w \), we have established the distributional identity (2.11) which
we emphasize is in the sense of distributions on \( \mathbb{C} \setminus \{w\} \). The distribution
\[ \frac{E(u, w)}{u-w} g(u) \]
now considered as a locally integrable distribution on \( \mathbb{C} \) differs from the distribution

\[-\overline{\partial}[1 - E(u, w)],\]

on \( \mathbb{C} \), by a first-order distribution supported on \( \{w\} \). Consequently in the sense of distributions on \( \mathbb{C} \)

\[- \overline{\partial}[1 - E(u, w)] - \frac{E(u, w)}{u - w} g(u) = \alpha \delta_w + \beta \partial \delta_w + \gamma \overline{\partial}\delta_w, \tag{2.21}\]

for some constants \( \alpha, \beta, \gamma \). Let \( S_w \) be the locally integrable distribution

\[ S_w(u) = 1 - E(u, w) - \frac{1}{\pi} \int_{\mathbb{C}} \frac{E(u, w) g(u)}{u - w - \lambda} da(u) \tag{2.22} \]

on \( \mathbb{C} \). Then equation (2.21) can be written in the form

\[ -\overline{\partial}S_w = \alpha \delta_w + \beta \partial \delta_w + \gamma \overline{\partial}\delta_w. \]

We remark that when \( w \) is a Lebesgue point for the function \( g \), then the estimate (2.7) of Theorem 2.6 implies that \( \overline{E}(\lambda, w) \) is in \( L^{2+\sigma}(\mathbb{C}) \) for some \( \sigma > 0 \). As indicated in Remark 2.1 with \( w \) fixed and \( \frac{1}{\pi} \int_{\mathbb{C}} \frac{g(u)}{|u-w|^2} da(u) \) converges, the integral on the right side in (2.22) is continuous. Consequently, in these cases, the distribution \( S_w \) is a continuous function. It is an exercise in distribution theory to show that if \( S \) is continuous and

\[ -\overline{\partial}S = \alpha \delta_w + \beta \partial \delta_w + \gamma \overline{\partial}\delta_w, \]

then \( \alpha = \beta = \gamma = 0 \) and, therefore, \( S \) is an entire function. Since \( S_w \) vanishes at infinity, \( S \) must be zero. We have therefore established the following:

**Theorem 2.12.** Let \( w \) be in \( L_\sigma \) or satisfy \( \frac{1}{\pi} \int_{\mathbb{C}} \frac{g(u)}{|u-w|^2} da(u) < \infty \). Then (2.13) holds for all \( \lambda \) in \( \mathbb{C} \).

**Remark 2.13.** It would be interesting to know the extent to which (2.13) holds and, in particular, whether the integral in this equation always converges at \( w \).

We also remark that when \( f \) is a compactly supported function that belongs to \( L^q \) for \( q > 2 \), then \( f \) satisfies the Lipschitz condition \( |f(\lambda) - f(w)| \leq K|\lambda - w|^{1-\frac{2}{q}} \). For a proof of this last statement, see [1]. The results on the function \( E \) in Theorem 2.12 and Theorem 2.6 fall inline with this result.

### 3 Operators with one-dimensional self-commutator

As described in the introduction, there is a close connection between the function \( E \) and operators with one dimensional self-commutator. In this section, we will describe some of these connections and derive a few consequences of the discussion of \( E \) from the preceding section. Let \( T \) be a bounded operator on the Hilbert space \( \mathcal{H} \) satisfying \( T^*T - TT^* = \varphi \otimes \varphi \), where \( \varphi \) is an element of \( \mathcal{H} \). It will always be assumed that the operator \( T \) is irreducible, equivalently, there no non-trivial subspaces of \( \mathcal{H} \) reducing \( T \) where the restriction is a normal
operator. Note that we have elected to assume $T$ is hyponormal, that is, the self-commutator $[T^*, T] = T^*T - TT^*$ is non-negative. The spatial behaviors of the hyponormal operator $T$ and its cohyponormal adjoint $T^*$ are quite distinct. As noted in the introduction, for $\lambda \in \mathbb{C}$, there is a unique solution of the equation $T^*_\lambda x = \varphi$ orthogonal to the kernel of $T^*_\lambda = (T - \lambda)^*$, which will be denoted $T^*_\lambda^{-1}\varphi$. This follows easily from the range inclusion theorem of Douglas [10] when one notes that for all $\lambda \in \mathbb{C}$ one has $T^*_\lambda T^*_\lambda - T^*_\lambda T = \varphi \otimes \varphi$ and, consequently, $T^*_\lambda T^*_\lambda \geq \varphi \otimes \varphi$. The $\mathcal{H}$-valued function $T^*_\lambda^{-1}\varphi$ defined for all $\lambda \in \mathbb{C}$ is called the global-local resolvent associated with the operator $T^*$. Using the result of Douglas mentioned above one can also see that for all $\lambda \in \mathbb{C}$ there is a contraction operator $K(\lambda)$ satisfying $T^*_\lambda = K(\lambda)T^*_\lambda$, where $T^*_\lambda = T - \lambda$. The contraction operator $K(\lambda)$ is unique if one requires it to be zero on the orthogonal complement of the range of $T^*_\lambda$. The following identity, specialized here to the case of rank-one self-commutators, was first established for general hyponormal operators in [10]

$$I = T^*_\lambda^{-1}\varphi \otimes T^*_\lambda^{-1}\varphi + K(\lambda)K^*(\lambda) + P_\lambda, \ \lambda \in \mathbb{C},$$

where $P_\lambda$ denotes the orthogonal projection onto the (at most one-dimensional) kernel of $T^*_\lambda$. There is no equivalent of the global-local resolvent for the operator $T$. To see this, we recall the following:

**Proposition 3.1.** Let $T$ be an irreducible operator on the Hilbert space $\mathcal{H}$ with one-dimensional self-commutator $T^*T - TT^* = \varphi \otimes \varphi$. Fix $\lambda \in \mathbb{C}$. Then the following are equivalent:

(I) There is a solution of the equation $(T - \lambda)x = \varphi$

(ii) The operator $K(\lambda)$ is invertible

(III) $\|T^*_\lambda^{-1}\varphi\| < 1$

(IV) $\int_\mathbb{C} \frac{g(u)}{|u - \gamma|^2} da(u) < \infty$, where $g$ is the principal function of $T$.

Let $\rho_T(\varphi)$ be the set of $\lambda \in \mathbb{C}$ such that there is a solution of the equation $(T - \lambda)x = \varphi$. The last condition in the above proposition shows that condition (I) cannot hold at Lebesgue points of $g$. This implies the result of Putnam [21] which establishes that the interior of $\rho_T(\varphi) \cap \sigma(T)$, where $\sigma(T)$ denotes the spectrum of $T$, is empty and points out the significant difference between the local resolvents $T^*_\lambda^{-1}\varphi$ and $T^*_\lambda^{-1}\varphi$.

We will continue to use the notation $L_g$ for the set of points of positive Lebesgue density of a bounded measurable function $g$. It develops that the local resolvent function is locally Lipschitz on $L_g$. This is the content of the following:

**Proposition 3.2.** Let $g$ be the the principal function associated with the operator $T$ having one-dimensional self-commutator $T^*T - TT^* = \varphi \otimes \varphi$. Let $w$ be in the set $L_g$ with $0 < \gamma = \lim_{R \to 0} \frac{1}{\pi R^2} \int_{D(w, \delta)} g(u) da(u)$. Then there is a disc $D(w, \delta)$ such that

$$\|T^*_\lambda^{-1}\varphi - T^*_w^{-1}\varphi\| \leq K|\lambda - w|^{-\gamma},$$

(3.1)

for $\lambda \in D(w, \delta) \cap L_g$, where $K$ is a constant, and $\epsilon < \gamma$.  

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Proof. It follows from Theorem 2.6 that there is a disc \(D(w, \delta)\) so that for \(\lambda \in D(w, \delta)\) one has
\[|E_g(\lambda, w)| \leq K|\lambda - w|^{\gamma - \varepsilon}.\] (3.2)
Since,
\[\|T^* - 1\|\lambda \phi\| = 1\]
for \(\lambda \in L_g\)
\[\|T^* - 1\|\lambda \phi\| = \|T^* - 1\|\lambda \phi\| - \|T^* - 1\|w \phi\| + \|T^* - 1\|w \phi\| = E_g(\lambda, w) + E_g(\lambda, w) \leq 2|E_g(\lambda, w)|,
\](3.3)
where we have made use of Theorem 1.3. The result follows from (3.2).

Remark 3.3. The result in this last proposition will not be true at points \(w\) in the spectrum \(\sigma(T)\) where any of the conditions in Proposition 3.1 hold. It is easy to construct an example of an operator \(T\) with this property, so that, \(w \in \sigma(T)\) and \(\|T^* - 1\|\phi\| < 1\). By Putnam’s result mentioned above, in every neighborhood of \(w\), there will be points with \(\|T^* - 1\|\phi\| = 1\). Thus the conclusion of this last proposition cannot hold at the point \(w\).

3.1 Integral representations using the global-local resolvent

The connection between the global local resolvent and the principal function can be seen in the following result from [7]

Theorem 3.4. Let \(T\) be an operator with one-dimensional self-commutator \(T^* T - T T^* = \varphi \otimes \varphi\) and \(g\) the associated principal function. For \(r = r(u)\) a rational function with poles off the spectrum of \(T\) and \(\lambda \in \mathbb{C}\)
\[(r(T) \varphi, T^* - 1\) \varphi) = \frac{1}{\pi} \int_{\mathbb{C}} r(u) \frac{g(u)}{u - \lambda} da(u).\] (3.4)
As a consequence, in the sense of distributions,
\[-\bar{\partial}(\varphi, T^* - 1\) \varphi) = g.\] (3.5)

We record the following analogue of this last result for the operator \(T^*\).

Theorem 3.5. Let \(T\) be an operator with one-dimensional self-commutator \(T^* T - T T^* = \varphi \otimes \varphi\) and \(g\) the associated principal function. Assume \(\lambda \in L_g\).
Let \(p = p(\bar{u})\) be a polynomial in the variable \(\bar{u}\), then
\[(p(T^*) \varphi, T^* - 1\) \varphi) = \frac{1}{\pi} \int_{\mathbb{C}} p(\bar{u}) \frac{E_g(\lambda, u)}{u - \lambda} g(u) da(u).\] (3.6)
Proof. Since $\lambda \in L_g$, then by Theorems 1.3 and 2.12 for all $z$
\[< T^*_{\lambda}^{-1} \varphi, T^*_{\mu}^{-1} \varphi > = 1 - E(z, \lambda) = \frac{1}{\pi} \int_{\mathbb{C}} \frac{E(u, \lambda)}{u - \lambda} g(u) \frac{da(u)}{u - z} \] (3.7)
Equating powers of $z$ at infinity one obtains
\[< T^*_{\lambda}^{-1} \varphi, T^*_{\mu}^{-1} \varphi > = \frac{1}{\pi} \int_{\mathbb{C}} u^k \frac{E(u, \lambda)}{u - \lambda} g(u) da(u), \] (3.8)
for $k = 0, 1, \cdots$. The result follows by taking complex conjugates in this last identity.

Remark 3.6. The integral kernel
\[T(\lambda, u) = \frac{E_g(\lambda, u)}{u - \lambda} g(u),\]
appearing in Theorem 3.5 a.e. $gda$ satisfies $|T(\lambda, u)| \leq |u - \lambda|^{-\sigma}$ near $\lambda$, where $0 < \sigma < 1$. In particular, has the advantage that $T(\lambda, \cdot)$ is in $L^2(\mathbb{C})$.

It is known that the closed span of $T^*_{\lambda}^{-1} \varphi$, $\lambda \in \sigma(T)$, is $\mathcal{H}$. Since the closure of $L_g$ equals $\sigma(T)$, it follows that the closed span of $T^*_{\lambda}^{-1} \varphi$, $\lambda \in L_g$, also is $\mathcal{H}$.

We conclude this subsection with a few examples of applications of Theorem 3.5. A test-function model for the operator $T^*$ was constructed in [14] (see also [15, p. 151, p.261]) using the map
\[\eta \in D(\mathbb{C}) \rightarrow U(\eta) = \frac{1}{\pi} \int_{\mathbb{C}} \partial \eta(\lambda) T^*_{\lambda}^{-1} \varphi da(\lambda),\]
so that $U(\bar{\eta}) = T^* U(\eta)$. This test function model is dual to the distributional model described in [8], where the map $V : \mathcal{H} \rightarrow \mathcal{E}'(\mathbb{C})$ defined by $V(f) = -\bar{\partial} < f, T^*_{\lambda}^{-1} \varphi >$ was studied. It is easily verified that for $\eta$ a test function and $f \in \mathcal{H}$
\[< U(\eta), f > = \frac{1}{\pi} < \eta, V(f) > .\]

Note that using (3.7) for $w \in L_g$
\[< T^{*-1}_{w} \varphi, U(\eta) > = \frac{1}{\pi} \int_{\mathbb{C}} \overline{\partial \eta}(\lambda) \left( \frac{1}{\pi} \int_{\mathbb{C}} \frac{E(u, w)}{u - w} \frac{g(u)}{u - \lambda} da(u) \right) da(\lambda) = \]
\[= \frac{1}{\pi} \int_{\mathbb{C}} \overline{\eta}(\lambda) \frac{E(\lambda, w)}{\lambda - w} g(\lambda) da(\lambda). \] (3.9)
Equivalently,
\[< U(\eta), T^{*-1}_{w} \varphi > = \frac{1}{\pi} \int_{\mathbb{C}} \overline{\eta}(\lambda) \frac{E(\lambda, w)}{\lambda - w} g(\lambda) da(\lambda). \] (3.10)
Example 3.7. In some cases, when combined with Theorem 3.5, this last equation allows one to identify the vector \( U(\eta) \). For example, if \( \eta(u) = \bar{u}^k \), \( k = 0, 1, 2, \ldots \) on the set where \( g \) is non-zero, then \( U(\eta) = T^*T \varphi \). It is noted that this result can also be obtained directly from the definition of \( U(\eta) \).

Example 3.8. For simplicity, suppose the essential support \( G \) of the principal function \( g \) is contained in the open unit disc \( \mathbb{D} \). Let

\[
\Phi_1 = \frac{1}{\pi} \int_{\mathbb{D}} T^{*\lambda}_1 \varphi d\lambda \quad \text{and} \quad \Phi_g = \frac{1}{\pi} \int_{\mathbb{C}} T^{*\lambda}_1 \varphi g d\lambda.
\]

For \( w \in L_g \), using (3.7) one computes

\[
< T^{*\lambda}_1 \varphi, \Phi_1 + \Phi_g > = < T^{*\lambda}_1 \varphi, T \varphi > .
\]

This yields

\[
T \varphi = \Phi_1 + \Phi_g = \frac{1}{\pi} \int_{\mathbb{D}} (1 + g) T^{*\lambda}_1 \varphi d\lambda,
\]

which gives a concrete representation of \( T \varphi \) in terms of the dense family \( \{ T^{*\lambda}_1 \varphi : \lambda \in \sigma(T) \} \). As is often the case, the unilateral shift provides an illuminating version of this last identity. This last identity can be viewed as a realization of the formula for \( T \varphi \) given in the test function model in [16, p. 261]. That is

\[
T \varphi = \frac{1}{\pi} \int_{\partial(\eta - \bar{\eta})} T^{*\lambda}_1 \varphi,
\]

where the test function \( \eta \) satisfies \( n(\lambda) = \lambda \) on the support of \( g \).

3.2 Non-cyclic vectors

Let \( T \) be an operator with one-dimensional self-commutator \( T^*T - TT^* = \varphi \otimes \varphi \) and \( T^{*\lambda}_1 \varphi \lambda \in \mathbb{C} \) the corresponding global-local resolvent. Given a compactly supported planar measure \( \mu \), one can define the vector

\[
\phi_\mu = \int_{\mathbb{C}} T^{*\lambda}_1 \varphi d\mu \quad (3.11)
\]

as a weak integral, that is, for \( f \in \mathcal{H} \)

\[
< \phi_\mu, f > = \int_{\mathbb{C}} < T^{*\lambda}_1 \varphi, f > d\mu(\lambda).
\]

In an extremely formal sense \( \phi_\mu = -(\hat{\mu}(T))^* \varphi \). For example, if \( \mu = \delta_w \), with \( \mu \notin \sigma(T) \), one has \( -\hat{\mu}(\lambda) = \frac{1}{\pi(\lambda - w)} \) and \( \phi_\mu = T^{*\lambda}_w \varphi \).

It follows from Theorem 3.4 that for \( r \) a rational function with poles off the spectrum of \( T \) we have

\[
< r(T) \varphi, \phi_\mu > = - \int_{\mathbb{C}} r(u) \hat{\mu}(u) g(u) da(u).
\]
We remark on the connection between this last identity and results concerning rational approximation in [23] and more recently [28]. For $X$ a compact set in the plane, let $P(X)$ respectively, $R(X)$ be the closure in the space $C(X)$ of continuous function on $X$ of the polynomials, respectively, the rational functions with poles off $X$. It was shown in [23] when $X$ is nowhere dense, then the closure of the module $\overline{zP(X)} + R(X)$ is $C(X)$ if and only if $R(X) = C(X)$. Here $z$ is the function $z(u) = u$. This is in contrast to the result in [24] that establishes when $X$ is a compact nowhere dense set, then the closure of $\overline{zR(X)} + R(X)$ is $C(X)$.

We are interested here in the case where the characteristic function $1_X$ “is” the principal function of an operator with one-dimensional self-commutator. Since the principal function of an operator with one-dimensional self commutator is only determined up to sets of measure zero, this has to be properly interpreted. If $g$ is the principal function associated with $T$, then $\sigma(T)$ is the essential closure of the set $\{u : g(u) \neq 0\}$. Consequently, we only consider the class of closed nowhere dense sets of positive measure that are essentially closed, i.e., equal their essential closure. For such a set $X$ there is a unique associated irreducible operator $T_X$ with one-dimensional self-commutator having principal function $1_X$. Moreover, different such sets $X$ correspond to different operators $T_X$ and $\sigma(T_X) = X$. It should also be noted that for a compact set $X$ the essential closure of $X$ is a closed subset of $X$ that differs from $X$ by a set of planar measure zero. Moreover, if $R(X) \neq C(X)$, the same is true for its essential closure [2].

Based on the result of Thomson [23], as noted in [7], one can establish the following:

**Proposition 3.9.** Suppose $T$ is an operator with one-dimensional self-commutator $T^*T - TT^* = \varphi \otimes \varphi$ associated with the principal function $1_X$, where $X$ is a compact essentially closed nowhere dense set of positive measure. If the closure of $\overline{zP(X)} + R(X)$ is not equal to $C(X)$, equivalently, the closure of $\overline{zP(\sigma(T))} + R(\sigma(T))$ is not equal to $C(\sigma(T))$, then the vector $\varphi$ is not cyclic for the operator $T$.

**Proof.** Let $\mu$ be a non-zero measure on $X$ that annihilates $\overline{zP(X)} + R(X)$. Let $\phi_\mu$ be given by (3.11) with $\mu$ replacing $\mu$. Then for $r$ a rational function with poles off $X$ the last equation results in the identity

$$< r(T)\varphi, \phi_\mu > = -\int_X r(u)\mu(u)da(u) = \int_X \bar{r}(u)d\mu(u).$$

(3.12)

By the result of [24] the closure of $\overline{zR(X)} + R(X)$ is $C(X)$ and therefore for some $r \neq 0$ the right side of this last equation is non-zero. This implies $\phi_\mu$ is non-zero. Since $\int_X \zbar p d\mu = 0$ for all polynomials $p$ it follows that $\varphi$ is not (polynomially) cyclic for the operator $T$.

**Remark 3.10.** In the context of the last result, a natural example to consider is that of a Swiss cheese $X$, that is, where $X$ is a closed nowhere dense...
set of positive planar measure obtained by removing a collection of open discs
\( D(w_n, r_n), \ n = 1, 2 \cdots \) with disjoint closures from the closed unit disc. If one
assumes \( \sum r_n < \infty \), then \( R(X) \neq C(X) \). Let \( T \) be the irreducible operator
with one-dimensional self-commutator associated with the principal function \( 1_X \),
where \( X \) is a Swiss cheese with \( \sum r_n < \infty \). It is known that the \( \varphi \) is rationally
cyclic for both \( T \) and \( T^* \), see [8]. The result above shows \( \varphi \) is not (polynomially) cyclic for \( T \).

In a fundamental paper, Brown [3] established the existence of invariant
subspaces for a hyponormal operators \( H \) whenever there is a closed disc \( D \) such
that \( R(D \cap \sigma(H)) \neq C(D \cap \sigma(H)) \). Thus the result in the last proposition does
not advance the theory of invariant subspaces. However, the fact that \( \varphi \) is not cyclic for \( T \) is new, albeit depending on the deep result of Thomson [23].

The result in the last proposition can be extended to the case, again assuming
\( X \) is nowhere dense, where for some closed disc \( D \) the closure of \( \bar{z}P(X \cap D) + R(X \cap D) \) is not equal to \( C(X \cap D) \). To see this, note if \( \mu \) is a non-zero measure
on \( X \cap D \) annihilating \( \bar{z}P(X \cap D) + R(X \cap D) \), then one has

\[
< r(T) \varphi, \phi_\mu > = \int_{X \cap D} \overline{u} r(u) d\mu(u)
\]

for \( r \) a rational function with poles off \( X \) and the last integral will be zero for
\( r = p \) a polynomial. In order to see that \( \phi_\mu \) is not zero, note that the last integral
will be non-zero for some rational function \( r \) with poles off \( X \cap D \). If one does a
partial fractions decomposition of \( r \) the part of this decomposition with poles in
\( X \setminus (X \cap D) \) can be approximated by a polynomial on \( X \cap D \). In this way, \( r \) can be replaced by a rational function \( r_0 \) with poles off \( X \) where \( \int_{X \cap D} \overline{u} r_0(u) d\mu(u) \neq 0 \).

It would be interesting to see if the results in [23], [23] and [28] appropriately
extend so that the above techniques can be used to establish that the vector \( \varphi \) is
not cyclic under the conditions that \( \sigma(T) \) is nowhere dense and there is a closed
disc \( D \) such that \( R(D \cap \sigma(H)) \neq C(D \cap \sigma(H)) \).

3.3 Some definite integral values computed using \( E \) and
the unilateral shift

In the case, where the operator \( T \) is the unilateral shift \( Uf(z) = zf(z) \) acting
on the Hardy space \( H^2 \) consisting of analytic functions \( f \) on the open unit disc
\( D \) with norm

\[
\|f\| = \left( \lim_{r \to 1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^2 d\theta \right)^{\frac{1}{2}},
\]

the principal function \( g_T \) is the characteristic function of the disc \( D \). It is easy
to verify that for \( |\lambda| < 1 \),

\[
U_{\lambda}^{-1} 1(z) = \frac{z - \lambda}{1 - \lambda z}
\]

and for \( |\lambda| \geq 1 \),

\[
U_{\lambda}^{-1} 1(z) = -\frac{1}{\lambda}
\]
where we are using the notation \( \mathbf{1} \) for the constant function \( \mathbf{1}(z) = 1 \) that appears in the self-commutator \( U^*U - UU^* = \mathbf{1} \otimes \mathbf{1} \). A straightforward computation can be used to verify

\[
1 - (U_w^{* -1} \mathbf{1}, U_\lambda^{* -1} \mathbf{1}) = E_D(\lambda, w) = \exp \frac{1}{\pi} \int_D \frac{u - w}{u - \lambda} \frac{1}{|u - w|^2} \, da(u), \quad (3.15)
\]

where \( E_D \) is given by (2.9).

Using this last formula one can directly verify the integral formulas (2.2) and (2.3). For example, consider the formula (2.2) where \( \alpha < 1 \). The map \( u = (\lambda - c_\alpha)z + c_\alpha \) maps \( D \) onto \( D_{\lambda, \alpha} \) sending \(-1\) to \( \lambda_\alpha = w + \alpha(\lambda - w) \) and \( 1 \) to \( \lambda \). This change of variables results in the equality

\[
-\frac{1}{\pi} \int_{D_{\lambda, \alpha}} \frac{u - w}{u - \lambda} \frac{1}{|u - w|^2} \, da(u) = -\frac{1}{\pi} \int_D \frac{z - s_\alpha}{z - 1} \frac{1}{|z - s_\alpha|^2} \, da(z), \quad (3.16)
\]

where \( s_\alpha = \frac{1 + \alpha}{\alpha - 1} \) is the image of \( w \) under the inverse map \( z = z(u) \). The right side of this last equation is recognized as the exponent in the following special case of equation (3.15):

\[
1 - (U_{s_\alpha}^{* -1} \mathbf{1}, U_1^{* -1} \mathbf{1}) = E_D(s_\alpha, 1) = \exp \frac{1}{\pi} \int_D \frac{u - s_\alpha}{u - 1} \frac{1}{|u - s_\alpha|^2} \, da(u). \quad (3.17)
\]

Depending on whether \( s_\alpha \) is inside the open unit disc \( (\alpha < 0) \) or outside the open unit disc \( (0 \leq \alpha < 1) \) one uses (3.13) or (3.14) to compute the left side of this last equality. For example, in the case where \( s_\alpha \) is inside the open unit disc

\[
1 - (U_{s_\alpha}^{* -1} \mathbf{1}, U_1^{* -1} \mathbf{1}) = \frac{2}{1 + |\alpha|}.
\]

Combining this last identity with (3.16) equation (2.2) follows. Similar arguments using the unilateral shift can be used to obtain the other instances of (2.2) and (2.3).

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