The soft stadium’s classical dynamics

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Abstract. Billiards are physical models employed to probe experiments that measure the conductivity of quantum dots. In this context, the stadium billiard have been adopted as an standard model for realizations. We study the effect of softening in a classical mechanics context, pursuing a more realistic model. This classical approach is a first step towards the truly quantum or semiclassical case. We define the soft stadium as a monomial potential with an exponent $\alpha \in \mathbb{R}$ as a parameter, such that for $\alpha = 1$ the system is integrable and the $\alpha \to \infty$ limit converges to the hard billiard. Then, and for computational simplicity, we set up the construction of the classical Poincare map in such a way that it only depends on the partial separability of the system which holds for all $\alpha$’s. We present numerical results describing the classical transition from the integrable regime towards the chaotic regime.

1. Introduction

The study of classical mechanics has received special attention with regard to the qualitative behavior of physical systems at long times. Billiards are simple systems useful in the classic and quantum mechanics to model the behavior of chaotic systems. Planar billiards are constructed by closed curves inside which a point particle moves freely, subjected to mirror reflections at the boundaries. For these class of systems their classical behavior is strongly influenced by the boundaries. This is the case of the stadium with rigid walls introduced as an standard example of chaoticity \cite{1}. Note that, these systems have been used as physical models aiming to understand experiments measuring the conductivity of quantum dots \cite{2}. However, more realistic models should take into account the effect of softening the billiards’ boundaries \cite{3}.

In this work we examine a classical Hamiltonian system that can be considered a generalization of the hard wall stadium. Analogously to the billiard, this system has a $C_{2v}$ symmetry, so the quarter stadium becomes the fundamental domain. In this domain the system is described by the Hamiltonian with the potential,

$$\mathcal{H} = \frac{p_x^2}{2} + \frac{p_y^2}{2} + V(x, y); \quad V(x, y) = \begin{cases} \frac{1}{2}y^{2\alpha} & y \geq 0, x \in (-a, 0) \text{ (region 1)}; \\ \frac{1}{2}(x^2 + y^2)^\alpha & x, y \geq 0 \text{ (region 2)}; \\ \infty & (x = -a; \forall y \geq 0), (\forall x \geq -a; y = 0); \end{cases}$$

(1)

with real and positive parameters $a$ and $\alpha$. Notice that, the equipotential lines of the soft stadium match with those of the corresponding billiards’ boundaries. By varying $\alpha$ the system cover all the possibilities: from fully integrable motion when $\alpha = 1$ to the fully chaotic motion when $\alpha \to \infty$ (the quarter stadium billiard).

It is worthwhile to mention that, for any value of $\alpha$ the system persists to be partially separable by a plane located at $x = 0$. This feature has been explored in the quantum
Then, in region 1 with rectangular symmetry; the classical dynamics can be described by a longitudinal motion governed by the monomial potential while its transverse motion is free. In region 2 with radial symmetry; the dynamics is described in polar coordinates, the angular motion is free and the radial motion is governed by the monomial potential. The complete classical dynamics is the composition of both the zigzagging and the rotating motion, thus the classical Poincare map is readily obtained.

In this work, we present substantial numerical results describing the path from the integrable to the chaotic regime for the soft stadium. In section 2 we present the construction of the mapping, a brief discussion related to the system’s global dynamics is present in section 3. Some results are shown in section 4 and finally our conclusions are presented in section 5.

2. The mapping
We establish the Poincare section at $x = 0$, it takes the trajectory of a given point $Q(y_1, P_{y_1})$ onto some point $Q(y_2, P_{y_2})$ on the section, thus the orbit in the map is generated as a sequence of successive section crossings. For a given energy $E$ (constant) and a starting initial condition $(y_1, P_{y_1})$, in region 1 the respective flight’s times are: $t_x = 2a/P_x$ with $P_x = \sqrt{2E - P_{y_1}^2 - y_1^{2\alpha}}$ and $dt_y = dy/P_y$; with $P_y = \sqrt{2E - P_x^2 - y_2^{2\alpha}}$. The orbit coming onto this region evolves as $Q_i : (y_1, P_{y_1}) \rightarrow (Y, P_Y)$;

$$\pm \int_{y_1}^{Y} \frac{dy}{P_y} = 2 \left( \frac{P_x}{P_y} - \Lambda_1 \int_{y_1}^{y_2} \frac{dy}{P_y} \right) + sgn(P_{y_1}) \int_{y_1}^{y_2} \frac{dy}{P_y},$$

where $y_1$ is the turning point and the longitudinal bouncing number is $\Lambda_1 = \left\lfloor \frac{t_x}{2} \right\rfloor$, yielding $Q(Y, P_Y)$.

Next, in region 2, for the angular motion $\dot{\theta} = \nu/r^2$, where the angular momentum is $\nu = Y P_Y$ and the independent radial momentum is $P_r = \sqrt{2E - \nu^2/r^2 - r^{2\alpha}}$. Relating both

![Figure 1. Typical trajectories in the configuration space for different values of $\alpha = 0.25, 0.75, 1.0$ at the top, $\alpha = 1.25, 1.75, 2.0$ in the middle and $\alpha = 4.0, 15.0, 30.0$ at the bottom of the figure.](image-url)
angular and radial momentum, the orbit evolves as $Q_{i+1} : (Y, P_Y) \rightarrow (y'', P_{yy})$; and we obtain

$$
\pm \int_{r_1}^{r_2} \frac{dr}{r^2 P_r} = 2 \left( \frac{\pi}{2E} - \Lambda_2 \int_{r_1}^{r_2} \frac{dr}{r^2 P_r} \right) + \text{sgn}(P_Y) \int_{r_1}^{Y} \frac{dr}{r^2 P_r},
$$

where the radial bouncing number is $\Lambda_2 = \frac{\pi}{2E} - \frac{1}{\sqrt{2E - y^2 / r^2 - r^2 \alpha}}$, and $r_{1,2}$ are the turning points (real solutions of $P_r = 0$). Then, by considering physical classical trajectories and by preserving the direction of the flux leaving the section into region 1, the mapping is the result of the composed motion,

$$
Q_{i+1}(y'', P_{yy}) = R \mathcal{T} Q_i(y', P_{y'}),
$$

3. The global dynamics

3.1. The phase space volume

The dynamical orbits are bounded for any appropriate initial conditions satisfying any given energy value $E$. In region 1, for $Q(y', P_y)$: $P_y^2 + y^{2\alpha} \leq 2E' = P_{y'}^2 + y'^{2\alpha}$, $0 \leq y \leq y_1 = (\sqrt{2E'})^{\frac{1}{\alpha}}$; $\forall x \in (-\alpha, 0)$. The partial time of flight $t_x = \frac{2\alpha}{\sqrt{2(E-E')}}$ and $\Lambda_1 = \frac{(2\alpha)(2\alpha)}{\sqrt{2(E-E')}} \frac{1}{\frac{1}{2\alpha} - \frac{1}{2\alpha} - \frac{1}{2\alpha}}$. In region 2, defined the point where the effective potential is a minimum as $\tilde{r} = \left( \frac{1}{2} \right)^{1+\frac{1}{\alpha}}$;

$$
\Lambda_2 \approx \left\{ \frac{\arctan\left( \frac{\sqrt{E}-2}{2} \right) - \arctan\left( \frac{\sqrt{E}+2}{2} \right) + \sqrt{2E} (1+2\alpha)}{2E} \left\{ B_z(r_2) \left( \frac{1}{2\alpha}, \frac{1}{2} \right) - B_z(r) \left( \frac{1}{2\alpha}, \frac{1}{2} \right) \right\} \right\},
$$

$$
t_\theta \approx \frac{\Lambda_2}{2E} \left\{ \left| \frac{\sqrt{E}-2}{2} \right| + \sqrt{2E} + 1 + \frac{1}{2} \left\{ B_z(r_2) \left( \frac{1}{2\alpha}, \frac{1}{2} \right) - B_z(r) \left( \frac{1}{2\alpha}, \frac{1}{2} \right) \right\} \right\},
$$

where $\chi(r) = \sqrt{2E} r^2 - y^2$, $z(r) = r^{2\alpha} / (2E)$, the functions $B(\pm \frac{1}{2\alpha}, \frac{1}{2})$ and $B_z(\pm \frac{1}{2\alpha}, \frac{1}{2})$ are the complete and incomplete beta function [5]; respectively.
Figure 3. The trajectories’ spreading on phase space are shown for different values of the parameter $\alpha = 0.25, 0.8, 1.75, 2, 5$ and $30$; respectively.

3.2. Computation of the maximum Lyapunov characteristic exponent
Let us to consider the state of the system at a certain time $t$, which is given by the vector $Q$ evolving according to Eqn 4. We consider two neighboring trajectories, i.e. $Q_i$ and $Q_i + \Delta Q_i$ at time $t_i$, then after each iteration of the map are computed the magnitude of the successive displacements as $d_i = \sqrt{\Delta Q_i}$. In this way, similarly to [6], the exponent is readily computed as,

$$\lambda = \sum_{i=1}^{n} \frac{1}{t_i} \ln \left( \frac{d_i}{d_0} \right).$$

4. Results
The system’s dynamics is numerically computed as a function of $\alpha$, we fix the size of the rectangular region at $a = 1$ and the energy at $E = 1$. We execute map-iterations of the order

Figure 4. The Lyapunov exponents ($\lambda$) computed numerically for different values of the parameter $\alpha$. It is found $\lambda < 0.15$ for $\alpha < 1$ and $\lambda < 0.5$ for $\alpha > 1$. 
of 10,000. Given arbitrary initial conditions \( Q_0 \), we integrate numerically Eqn. 2 to find the crossing point \((Y, P_Y)\) and subsequently numerically integrate Eqn. 3 to find the first iterated point on the map \( Q_1 \). In Fig.(1) are shown typical trajectories in the configuration space for values of \( \alpha = 0.25, 0.75 \) and 1 at the top of the panel, following for \( \alpha = 1.25, 1.75, 2 \) in the middle and for \( \alpha = 4, 15 \) and 30 at the bottom of the same panel; respectively. Next, the Poincare map is shown in Fig 2 for \( \alpha = 0.25, 0.75, 1 \) at the top of the panel and for \( \alpha = 0.9, 0.95, 1.0 \) at the bottom of the left panel. Next, on the right one: for \( \alpha = 1.25, 1.5, 1.75 \) at the top and for \( \alpha = 2, 15 \) and 30 at the bottom of the same panel. Here we can see the breaking of the regular trajectories on phase space.

It is instructive to show the spreading of the trajectories in phase space, which give us some idea about the dynamical instability of the manifolds. This is shown in Fig.3 for \( \alpha = 0.25, 0.8, 1.75 \) at the top and for \( \alpha = 2, 5, 30 \) at the bottom of the figure. Finally, the quantitative measure of the instability using Eqn 6 is shown in Fig. 4 for different values of the parameter \( \alpha \).

### 5. Conclusions

We notice that the study of the soft stadium has been restricted to values of \( \alpha \geq 1 \). Nevertheless we find interesting physics for values of \( \alpha < 1 \), namely the mixed chaotic region displayed in Figure 2. The hardness of the potential increases with \( \alpha \) and destroys the classical constant of motion. The Poincare’s maps in Fig.2 show different routes from the integrable behavior (\( \alpha = 1 \)) to the mixed regime by decreasing the value of the parameter and a route to the hard chaos regime by increasing the value of the parameter \( \alpha \). The spreading of different manifolds shown in Fig.3 qualitatively support these expectations. Finally a quantitative measure of the maximum Lyapunov exponent confirms these observations, since we find that they are smaller than 0.5 when the parameter \( \alpha > 1 \) and they are smaller than 0.15 when the parameter \( \alpha < 1 \). Thus we present important predictions for the classical transition from regular, mixed and chaotic regimes for the soft stadium’s family. These results are important because they must have strong influence on the quantum counterpart as well as in the semiclassical limit.

### Acknowledgments

The authors would like to thank the support of the Goiás Research Foundation - FAPEG.

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