Multi-black strings in five-dimensional Einstein-Maxwell theory

Ken Matsuno\textsuperscript{1}, Hideki Ishihara\textsuperscript{2}, Masashi Kimura\textsuperscript{3}, and Takamitsu Tatsuoka\textsuperscript{4}

Department of Mathematics and Physics, Graduate School of Science, Osaka City University, 3-3-138 Sugimoto, Sumiyoshi-ku, Osaka 558-8585, Japan

Abstract
We consider dyonically charged static black strings in the five-dimensional Einstein-Maxwell theory. We obtain three types of analytic solutions, i.e., electrically charged ones, magnetically charged ones and dyonically charged ones with a constant extra dimension. We also construct dyonically charged multi-black strings with a constant extra dimension.

1 Introduction
In the context of string theory and brane world models, investigations on black hole solutions in higher dimensions have attracted a lot of attention (see [1] for review). For example, in five-dimensional spacetimes, unlike in four-dimensional spacetimes, where the only allowed horizon topology is a two-sphere, we can have different more interesting horizon topologies such as black holes with a horizon topology of a three-sphere, black rings with a horizon topology of a direct product of a two-sphere and a circle, and a black lens in which the horizon geometry is a Lens space $L(p; q)$.

The five-dimensional vacuum black string, a direct product of the four-dimensional Schwarzschild black hole and a circle, is one of simple examples of higher-dimensional black holes. In this report, we generalize such vacuum black strings to the charged ones. Especially, we consider dyonically charged static black strings in the five-dimensional Einstein-Maxwell theory. We obtain three types of analytic solutions, i.e., electrically charged ones, magnetically charged ones and dyonically charged ones with a constant extra dimension. We also construct dyonically charged multi-black strings with a constant extra dimension.

2 Solutions
We start from the five-dimensional Einstein and the Maxwell equations,

\begin{equation}
R_{\mu\nu} = 2 \left( T_{\mu\nu} - \frac{T}{3} g_{\mu\nu} \right), \quad \nabla_{\mu} F^{\mu\nu} = 0.
\end{equation}

(1)

We consider five-dimensional dyonically charged static spacetimes with a compactified extra dimension. The ansatz of the metric and the gauge potential are

\begin{align*}
    ds^2 &= -\alpha^2(\rho)dt^2 + \beta^2(\rho)d\rho^2 + \gamma^2(\rho)d\Omega^2_{S^2} + \delta^2(\rho)dw^2, \\
    A &= A_t(\rho)dt + A_\phi(\theta)d\phi,
\end{align*}

(2)

(3)

where $d\Omega^2_{S^2} = d\theta^2 + \sin^2 \theta d\phi^2$ denotes a metric of the unit two-sphere, the functions $\alpha$, $\beta$, $\gamma$, $\delta$ and $A_t$ depend only on $\rho$, and the function $A_\phi$ depends only on $\theta$. When we take a coordinate condition such as $\beta = \alpha \delta \gamma^2$, then the gauge potential $A = Q \left( \int \alpha^2 d\rho \right) dt + P \cos \theta d\phi$ solves the Maxwell equation, where $Q$ and $P$ denote an electric and a magnetic charges, respectively. Then the Einstein equation gives the equations of motion,

\begin{equation}
\left( \frac{\alpha'}{\alpha} + \frac{\delta'}{\delta} + \frac{\gamma'}{\gamma} + \frac{\alpha'}{\alpha} + \frac{\delta'}{\delta} + \frac{\gamma'}{\gamma} \right) + \alpha^2 \left( P^2 \delta^2 + Q^2 \right) - (\alpha \delta \gamma)^2 = 0,
\end{equation}

(4)

\textsuperscript{1}Email address: matsuno@sci.osaka-cu.ac.jp
\textsuperscript{2}Email address: ishihara@sci.osaka-cu.ac.jp
\textsuperscript{3}Email address: mkimura@sci.osaka-cu.ac.jp
\textsuperscript{4}Email address: tatsuoka@sci.osaka-cu.ac.jp
\[
\left( \frac{\alpha'}{\alpha} \right)' = \frac{2\alpha^2}{3} \left( P^2 \delta^2 + 2Q^2 \right),
\]
\[
\left( \frac{\delta'}{\delta} \right)' = \frac{2\alpha^2}{3} \left( P^2 \delta^2 - Q^2 \right),
\]
\[
\left( \frac{\gamma'}{\gamma} \right)' = (\alpha \delta \gamma)^2 - \frac{2\alpha^2}{3} \left( 2P^2 \delta^2 + Q^2 \right),
\]
respectively, where the prime denotes the derivative with respect to \( \rho \). Combing Eqs. (5)-(7), we obtain an analytic solution as
\[
\alpha \delta \gamma = c_1 / \sinh(c_1 \rho + c_2),
\]
where \( c_i \) (i = 1, 2) are constants. For general \( Q \) and \( P \), we could not find a set of analytic solutions to Eqs. (5)-(7). In the following subsections, we study some special cases, i.e., electrically charged \((Q \neq 0, P = 0)\) solutions, magnetically charged \((Q = 0, P \neq 0)\) ones and dyonically charged \((Q, P \neq 0)\) ones with a constant extra dimension.

### 2.1 Electrically charged solutions

When the magnetic charge vanishes, \( P = 0 \), we obtain a set of analytic solutions to Eqs. (5)-(7) as
\[
\alpha^2(\rho) = \frac{c_1^2}{u^2 \sinh^2(c_1 \rho)}, \quad \delta^2(\rho) = \frac{c_5 u}{c_1} e^{2\rho} \sinh(c_1 \rho), \quad \gamma^2(\rho) = \frac{c_5^2 u c_3^2 \sinh(c_1 \rho)}{c_1 c_5 \sinh^2(c_3 \rho + c_4)},
\]
where \( u^2 = 4Q^2/3 \) and \( c_i \) (i = 1, ... , 5) are constants. Substituting the solution (9) into Eq. (4), we have
\[
c_1^2 = (3c_1^2 + c_2^2)/4.
\]
It would be easy to see the geometry of the solution (9) in a new coordinate system. We introduce the coordinate and the constants as \( e^\theta = \nu (1 - 2M/r) \), \( a = -2c_1 \), \( b = c_1 + c_2 \), \( \alpha = u/(8c_1 c_5 M^2 \nu^b) \), \( b = u/(8c_1 c_5 M^2 \nu^b) \), \( c_4 = \nu^{-1/2} \) and choose the constant as \( c_3 = 1/2 \) without loss of generality. After rescaling the coordinates as \( t \rightarrow 4c_3 M^2 \nu^{a/2+b} t, \quad w \rightarrow w/ \left(2c_5 M \nu^{a/2+b}\right)\), we have the metric and the gauge potential in the forms
\[
ds^2 = -f^2 dt^2 + h \left( f^{-a-b} dr^2 + f^{1-a-b} r^2 d\Omega_5^2 + f^b dw^2 \right),
\]
\[
A = \pm \sqrt{3\alpha \over 4\beta \bar{h}},
\]
where the functions \( f, \ h \) and the relation between parameters \( a \) and \( b \) are, respectively, given by
\[
f(r) = 1 - 2M/r, \quad h(r) = \alpha - \beta f^\theta(r), \quad a^2 + b^2 + ab = 1.
\]
When the Maxwell field (11) vanishes, \( \alpha \rightarrow 1 \) and \( \beta \rightarrow 0 \), the function \( h \rightarrow 1 \) and the metric (10) reduces to that of vacuum Kaluza-Klein solitons [2]. As same as vacuum Kaluza-Klein solitons, for general \( a \) and \( b \), the charged solution (10) has a naked curvature singularity at \( r = 2M \).

To obtain black string solutions, we choose the parameters such that \( a = 1, \ b = 0 \). In this case, introducing the coordinate and the constants as \( R = r \sqrt{\alpha - \beta + 2M \hat{\beta} \left( \alpha - \hat{\beta} \right)^{-1/2}}, \quad R_1 = 2M \hat{\alpha} \left( \alpha - \hat{\beta} \right)^{-1/2}, \quad R_2 = 2M \hat{\beta} \left( \alpha - \hat{\beta} \right)^{-1/2}, \) and rescaling the coordinates as \( t \rightarrow t \left( \alpha - \hat{\beta} \right), \quad w \rightarrow w \left( \alpha - \hat{\beta} \right)^{-1/2}, \) the metric (10) and the gauge potential (11) are rewritten as
\[
ds^2 = -\frac{(R - R_1)(R - R_2)}{R^2} dt^2 + \frac{R}{R - R_1} dR^2 + R (R - R_2) d\Omega_5^2 + \frac{R}{R - R_2} dw^2,
\]
\[
A = \pm \frac{\sqrt{3R_1 R_2}}{2R} dt.
\]
We see that the metric (13) and the gauge potential (14) coincide with those of five-dimensional electrically charged static black strings in [3]. When \( R_1 > R_2 > 0 \), the horizon is located at \( R = R_1 \), and the curvature singularity at \( R = R_2 \) is concealed behind it. At infinity, \( R = \infty \), the metric (13) asymptotically approaches a direct product of the four-dimensional Minkowski spacetime and a compactified extra dimension, i.e., the black string (13) is asymptotically locally flat.
2.2 Magnetically charged solutions

When the electric charge vanishes, $Q = 0$, we obtain a set of analytic solutions to Eqs. (5)-(7) as

$$
\alpha^2(\rho) = \frac{c_2 e^{-c_1 \rho}}{v \sinh(c_2 \rho + c_3)}, \quad \delta^2(\rho) = \frac{c_2 e^{c_1 \rho}}{v \sinh(c_2 \rho + c_3)}, \quad \gamma^2(\rho) = \frac{c_2^3 v^2 \sinh^2(c_2 \rho + c_3)}{c_2^2 \sinh^2(c_4 \rho + c_5)},
$$

(15)

where $v^2 = 4P^2/3$ and $c_i$ ($i = 1, \ldots, 5$) are constants. Substituting the solution (15) into Eq. (4), we have $c_4^2 = (c_1^2 + 3c_2^2)/4$. In order to see the geometry of the solution (15) easily, we introduce the coordinate and the constants as $e^\nu = 1 - 2M/r$, $a = -c_1 - c_2$, $b = c_1 - c_2$, $\tilde{\alpha} = ve^{c_3}/(4c_2M)$, $\tilde{\beta} = \nu/(4c_2Me^{c_3})$ and choose the constants as $c_4 = 1/2$, $c_5 = 0$ without loss of generality. After rescaling the coordinates as $t \rightarrow \sqrt{2}Mt$, $w \rightarrow \sqrt{2}Mw$, we have the metric and the gauge potential in the forms

$$ds^2 = h^{-1} \left( -f^a dt^2 + f^b dw^2 \right) + h^2 \left( f^{a-b} dr^2 + f^{1-a-b} r^2 d\Omega^2_{S^2} \right),
$$

(16)

$$A = \pm \sqrt{3\tilde{\alpha}\tilde{\beta}(a+b)M \cos \theta d\phi},
$$

(17)

where the functions $f$, $h$ and the relation between parameters $a$ and $b$ are, respectively, given by

$$f(r) = 1 - 2M/r, \quad h(r) = \tilde{\alpha} - \tilde{\beta} f^{a+b}(r), \quad a^2 + b^2 + ab = 1.
$$

(18)

When the Maxwell field (17) vanishes, $\tilde{\alpha} \rightarrow 1$ and $\tilde{\beta} \rightarrow 0$, the function $h \rightarrow 1$ and the metric (16) reduces to that of vacuum Kaluza-Klein solitons [2]. As same as vacuum Kaluza-Klein solitons, for general $a$ and $b$, the charged solution (16) has a naked curvature singularity at $r = 2M$.

To obtain black string solutions, we choose the parameters such that $a = 1$, $b = 0$. In this case, introducing the coordinate and the constants as $R = \left( \tilde{\alpha} - \tilde{\beta} \right) r + 2M\tilde{\beta}$, $R_1 = 2M\tilde{\alpha}$, $R_2 = 2M\tilde{\beta}$ and rescaling the coordinates as $t \rightarrow t\sqrt{\tilde{\alpha} - \tilde{\beta}}$, $w \rightarrow w\sqrt{\tilde{\alpha} - \tilde{\beta}}$, the metric (16) and the gauge potential (17) are rewritten as

$$ds^2 = -\frac{R - R_1}{R} dt^2 + \frac{R^2}{(R - R_1)(R - R_2)} dr^2 + R^2 d\Omega^2_{S^2} + \frac{R - R_2}{R} dw^2,
$$

(19)

$$A = \pm \sqrt{3R_1R_2/2} \cos \theta d\phi.
$$

(20)

We see that the metric (19) and the gauge potential (20) coincide with those of five-dimensional magnetically charged static black strings, which asymptote to locally flat spacetimes [4]. When $R_1 > R_2 > 0$, the horizon is located at $R = R_1$, and the curvature singularity at $R = 0$ is concealed behind it.

2.3 Dyonically charged solutions with a constant extra dimension

When the size of extra dimension becomes constant everywhere, $\delta = \text{const.}$, we obtain a set of analytic solutions to Eqs. (5)-(7) as

$$
\alpha^2(\rho) = \frac{c_1^2}{2Q^2 \sinh^2(c_1 \rho)}, \quad \delta = \left( \frac{Q}{P} \right)^2 = \text{const.}, \quad \gamma^2(\rho) = \frac{2P^2 \sinh^2(c_1 \rho)}{\sinh^2(c_2 \rho + c_3)},
$$

(21)

where $c_i$ ($i = 1, 2, 3$) are constants. Substituting the solution (21) into Eq. (4), we have $c_1 = c_2$. In order to see the geometry of the solution (21) easily, we introduce the coordinates and the constants as $T = \sqrt{2}c_1 t/ |Q(e^{c_3} - e^{-c_3})|$, $R = \gamma = \sqrt{2}P \sinh(c_1 \rho)/ \sinh(c_1 \rho + c_3)$, $W = wQ/P$, $P_\pm = \sqrt{2}Pe^{\pm c_3}$. Then the metric and the gauge potential are rewritten as

$$ds^2 = -\frac{(R - R_+)(R - R_-)}{R^2} dt^2 + \frac{R^2}{(R - R_+)(R - R_-)} dr^2 + R^2 d\Omega^2_{S^2} + dw^2,
$$

(22)

$$A = \pm \sqrt{\frac{R_+ R_-}{2}} \left( \frac{dT}{R} + \cos \theta d\phi \right),
$$

(23)
For \( R_+ > R_+ > 0 \), the solution (22) describes a non-extremal dyonically charged static black string with a constant extra dimension, where the horizons and the curvature singularity are located at \( R = R_\pm \) and \( R = 0 \), respectively. When two horizons degenerate, \( R_- \rightarrow R_+ \), the solution (22) reduces to an extremal black string solution. In this case, we can generalize such single extremal black string to multi-black strings.

The forms of the metric and the gauge potential of charged static multi-black strings are

\[
\begin{align*}
 ds^2 &= -H^{-2}dT^2 + H^2ds_{E_3}^2 + dW^2, \\
 \sqrt{2}A &= H^{-1}dT + \omega,
\end{align*}
\]

where \( ds_{E_3}^2 = dx^2 + dy^2 + dz^2 \) is a metric on the three-dimensional Euclid space \( E^3 \), the function \( H \) and the 1-form \( \omega \) are, respectively, given by

\[
H = 1 + \sum_i \frac{M_i}{|R - R_i|},
\]

\[
\omega = \sum_i M_i \frac{z - z_i}{|R - R_i|} \frac{(x - x_i)dy - (y - y_i)dx}{(x - x_i)^2 + (y - y_i)^2},
\]

with the constants \( M_i \) and \( R = (x, y, z) \) denotes a position vector on \( E^3 \). The function \( H \) is a harmonic function on \( E^3 \) with point sources located at \( R = R_i = (x_i, y_i, z_i) \) and the 1-form \( \omega \) is determined by \( \nabla \times \omega = \nabla H \). To avoid the existence of singularities on and outside the black string horizons, we choose the parameters such that \( M_i > 0 \). Then the solution (24) describes dyonically charged multi-black strings, which are located at \( R = R_i \), in the spacetime with a constant extra dimension.

3 Summary and discussion

We have considered dyonically charged static black strings in the five-dimensional Einstein-Maxwell theory. In dyonically charged case, where both an electric charge \( Q \) and a magnetic charge \( P \) exist, we have not found an analytic general solution. In some special cases, we have obtained analytic solutions, i.e., electrically charged \( (Q \neq 0, P = 0) \) ones (10), magnetically charged \( (Q = 0, P \neq 0) \) ones (16) and dyonically charged \( (Q, P \neq 0) \) ones with a constant extra dimension (22). We have also constructed dyonically charged multi-black strings with a constant extra dimension (24).

We note that the multi-black string solutions in the five-dimensional Einstein-Maxwell theory (24) coincide with those uplifted from the four-dimensional equal charged dyonic Majumdar-Papapetrou solutions with constant dilaton fields [5]. By uplifting the same four-dimensional solutions to the five-dimensional Einstein system, we obtain different multi-black object solutions [6]. The metric is

\[
ds^2 = -H^{-2}dt^2 + H^2(dx^2 + dy^2 + dz^2) + \left( \sqrt{2}H^{-1}dt + dw + \sqrt{2}\omega \right)^2,
\]

where the function \( H \) and the 1-form \( \omega \) are given by Eqs. (26) and (27), respectively. We expect that the solution (28) describes rotating Kaluza-Klein vacuum multi-black holes with a twisted constant extra dimension. We leave the analysis for the future.

References

[1] R. Emparan and H.S. Reall, Living Rev. Rel. 11, 6 (2008).
[2] D.J. Gross and M.J. Perry, Nucl. Phys. B 226, 29 (1983).
[3] G.T. Horowitz and K. Maeda, Phys. Rev. D 65, 104028 (2002).
[4] U. Miyamoto, Phys. Lett. B 659, 380 (2008).
[5] H.J. Sheinblatt, Phys. Rev. D 57, 2421 (1998).
[6] R.R. Khuri and T. Ortin, Phys. Lett. B 373, 56 (1996).