OBSERVABILITY AND CONTROLLABILITY ANALYSIS OF BLOOD FLOW NETWORK

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Abstract. In this paper, we consider the initial-boundary value problem of a binary bifurcation model of the human arterial system. Firstly, we obtain a new pressure coupling condition at the junction based on the mass and energy conservation law. Then, we prove that the linearization system is interior well-posed and \(L^2\) well-posed by using the semigroup theory of bounded linear operators. Further, by a complete spectral analysis for the system operator, we prove the completeness and Riesz basis property of the (generalized) eigenvectors of the system operator. Finally, we present some results on the boundary exact controllability and the boundary exact observability for the system.

1. Introduction. In recent years, many doctors and researchers have devoted themselves to the prevention and treatment of cardiovascular diseases. For the blood flow in a single vessel shown as Figure 1(a), many models were used to obtain the physical property of the blood vessel and the interaction between blood flow and vessel in the previous literatures (e.g., see [1, 3]). And some mathematical issues and the well posedness analysis of these multiscale models were presented (see, [2, 4, 5]). In fact, the arterial circulation is often regarded as a network of compliant vessels. Wave propagation in the arterial tree is very important to the understanding of arterial circulation, especially at bifurcation (see,[6, 8, 10, 11]). Medical experiments and technology (see, e.g., [7, 12]) also were devoted to the study and found that geometry played a key role in determining the nature of haemodynamic patterns at the human carotid arterial bifurcation.

In this paper, we consider a new problem. The mathematical models previously used can be regarded as an initial and boundary value problem of partial differential equations. The solution including its numerical solution makes sense only when the initial data and boundary value are given. In the practical situation, we have no initial state. In most cases, we have some measure-datum. Therefore, how to find some property of the blood flow network from measurement, or the observability of the system briefly, is an interesting study. In the present paper, we will pay our attention to the observability and controllability of a simple vascular bifurcation system shown as Figure 1(b).

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As usual, we describe every vessel by the following one-dimensional quasilinear system:
\[
\begin{align*}
A_t + Q_x &= 0, \\
Q_t + \eta \frac{Q^2}{A} x + \frac{\rho}{\rho} P_x &= -K \frac{Q}{A},
\end{align*}
\]
(1)
that describes conservation of mass and balance of axial momentum in terms of the cross-section area \(A(x,t)\), internal pressure over the cross-section \(P(x,t)\), average velocity \(u(x,t)\) and the flow rate \(Q(x,t) = A(x,t)u(x,t)\), where \(\eta\) stands for the kinetic energy coefficient, \(x\) denotes the axial direction, \(\rho\) is the blood density and \(K\) is related to the viscosity assumed in the vessel (see, e.g., [4, 5, 6]).

To simplify the model, we introduce the following constitutive law for the pressure to system (1) (see, e.g., [4, 13, 14]):
\[
P = \beta A_0 \left(\sqrt{A} - \sqrt{A_0}\right), \quad \beta = \frac{\sqrt{\pi h E}}{1 - \sigma^2},
\]
(2)
where \(A_0, E, h\) and \(\sigma\) are the cross-section constant at rest \((u = 0)\), the Young modulus, the thickness and Poisson coefficient of the vessel wall, respectively. This expression entails that the pressure is a linear function of the vessel radius (see, e.g., [13, 15]). Using the pressure law (2), we get
\[
P'(A) = \frac{\beta}{2A_0 \sqrt{A}}.
\]
And we choose \(P\) and \(Q\) as variables. By linearizing the system (1) around a constant state \((A, u) = (A_0, 0)\), we obtain a linearization model of the single vessel as follows
\[
\begin{align*}
P_i(x, t) + \frac{\beta}{2A_0 \sqrt{A_0}} Q_i(x, t) &= 0, \\
Q_i(x, t) + \frac{\rho}{\rho} P_i(x, t) &= -\frac{K}{A_0} Q_i(x, t).
\end{align*}
\]
(3)
As pointed out in [6], the linearization model is reasonable as it can reproduce most of the essential feature of the blood flow. In particular, it is used to model the whole systemic tree.

For the binary vascular bifurcation (see, Figure 1(b)) under consideration, let \(P_i\) and \(Q_i\) be the pressure and flow rate, where \(i = 1\) denotes the first branch and \(i = 2, 3\) are the binary branches. Without loss of generality, we can assume that \(P_i\) and \(Q_i\) are the real measurable functions on interval \([0, 1]\) and the direction of parameter increasing coincides with that of blood flow for \(i \in \{1, 2, 3\}\). For the connective condition at the bifurcation, the mass balance condition implies that we can choose constant \(\alpha \in (0, 1)\) as the mass partition coefficient (see, e.g., [6, 9]). Then the flow rates satisfy conditions \(Q_2(0, t) = \alpha Q_1(1, t), Q_3(0, t) = (1 - \alpha)Q_1(1, t)\). Thus the
model under consideration can be described as follows
\[
\begin{align*}
P_{i,t}(x,t) + \frac{\beta_i}{2A_{i0}^3} Q_{i,x}(x,t) &= 0, \ x \in (0, 1), \ t > 0, \\
Q_{i,t}(x,t) + \frac{A_{i0}}{\rho P_{i,x}(x,t)} &= -\frac{K}{A_{i0}} Q_i(x,t), \ x \in (0, 1), \ t > 0, \\
Q_2(0, t) &= \alpha Q_1(1, t), \\
Q_3(0, t) &= (1 - \alpha)Q_1(1, t).
\end{align*}
\] (4)

In [3, 14], the energy functional is defined by
\[
E(t) = \frac{1}{2} \sum_{i=1,2,3} \int_0^1 \left[ \frac{\rho}{A_{i0}} Q_i^2(x,t) + \frac{2A_{i0}\sqrt{A_{i0}}}{\beta_i} P_i^2(x,t) \right] dx,
\] (5)

that provides energy estimates for the linearization system as the nonlinear entropy function guarantees the stability of the solution to the nonlinear system. Taking the derivative of (5) with respect to \( t \) and using the equations (4), we get
\[
\frac{dE(t)}{dt} = Q_1(t)(\alpha P_2(0, t) + (1 - \alpha)P_3(0, t) - P_1(t)) + P_1(0, t)Q_1(0, t)
- P_2(1, t)Q_2(1, t) - P_3(1, t)Q_3(1, t) - \sum_{i=1}^3 \int_0^1 \frac{\rho K A_{i0}^2}{\sqrt{A_{i0}}} Q_i^2 dx,
\] (6)

where the term \( P_1(0, t)Q_1(0, t) \) is the input energy, the terms \( P_2(1, t)Q_2(1, t) \) and \( P_3(1, t)Q_3(1, t) \) are the output energy, and the term \( \sum_{i=1}^3 \int_0^1 \frac{\rho K A_{i0}^2}{\sqrt{A_{i0}}} Q_i^2 dx \) is the energy dissipation rate due to the blood viscosity. Suppose that there is no energy loss at the bifurcation, we find out the pressure relation
\[
P_1(1, t) = \alpha P_2(0, t) + (1 - \alpha)P_3(0, t).
\] (7)

It’s worth pointing out that the result (7) is different from that used before (see, e.g., [6, 8, 10, 11]). Based on the flow rate distribution and pressure coupling contiguity condition, we can model the arteries at the bifurcation. Furthermore, supplementing the initial and boundary conditions, we get a complete description for a one-dimensional distributed parameter model of binary vascular network
\[
\begin{align*}
P_{i,t}(x,t) + \frac{\beta_i}{2A_{i0}^3} Q_{i,x}(x,t) &= 0, \ x \in (0, 1), \ t > 0, \\
Q_{i,t}(x,t) + \frac{A_{i0}}{\rho P_{i,x}(x,t)} &= -\frac{K}{A_{i0}} Q_i(x,t), \ x \in (0, 1), \ t > 0, \\
Q_2(0, t) &= \alpha Q_1(1, t), \\
P_1(1, t) &= \alpha P_2(0, t) + (1 - \alpha)P_3(0, t), \\
Q_1(x,0) &= u_1(t), \ P_1(x,0) = P_2(1, t) = u_2(t), \ P_3(1, t) = u_3(t),
\end{align*}
\] (8)

where \( u_1(t) \) is input flow rate and \( u_2(t), u_3(t) \) are the output pressures, that are regarded as the control variables. Moreover, we have observation data about the pressure input \( P_1(0, t) \) and the flow rate outputs \( Q_2(1, t) \) and \( Q_3(1, t) \), and we set
\[
Y(t) = (P_1(0, t), Q_2(1, t), Q_3(1, t)).
\] (9)

So far we have obtained a complete control-observation system described by (8) and (9).

The rest of this paper is organized as follows. In Section 2, we formulate the system (8) into a Hilbert space \( H \) and then prove the interior well-posedness and \( L^2 \) well-posedness of the network system by using the semigroup theory of bounded linear operators. In Section 3, we carry out a complete spectral analysis for the system operator. And we prove that the spectrum consists of all isolated eigenvalues, that is a union of finite many separated sets located in a strip parallel to the imaginary axis. In particular, we obtain some results on the multiplicity and
separability of eigenvalues. In Section 4, we present the completeness and Riesz basis property of (generalized) eigenvectors of the system operator \( \mathcal{A} \). Finally, in Section 5, we discuss the exact controllability and exact observability of the system and then give the main result in Theorem 5.3.

2. Well-posedness. In this section, we study the well-posedness of the linearization blood flow network (8). At first we reformulate system (8) into an appropriate Hilbert state space. For the sake of simplicity, we introduce some notations used later.

Set 3 × 3 diagonal matrices

\[
L = \text{diag}\{L_1, L_2, L_3\}, \quad M = \text{diag}\{M_1, M_2, M_3\}, \quad N = \text{diag}\{N_1, N_2, N_3\}
\]

where \( L_i = \frac{b_i}{2A_{10}} \), \( M_i = \frac{\Delta u_i}{\rho} \), \( N_i = \frac{\alpha_i}{A_{10}} \), \( i = 1, 2, 3 \), and define a matrix \( C \) by

\[
C = \begin{pmatrix} 0 & 0 & 0 \\ \alpha & 0 & 0 \\ 1 - \alpha & 0 & 0 \end{pmatrix}.
\]

Set \( X = L^2[0, 1] \times L^2[0, 1] \times L^2[0, 1] \), and set

\[
\mathcal{H} = X \times X
\]

equipped with an inner product

\[
\langle (f(x), g(x)), ((\varphi(x), \psi(x))) \rangle_{\mathcal{H}} = \int_0^1 [(L^{-1}f(x), \varphi(x))_{C^1} + (M^{-1}g(x), \psi(x))_{C^1}]dx
\]

\[
= \sum_{i=1,2,3} \int_0^1 L_i^{-1}f_i(x)\overline{\varphi_i(x)} + M_i^{-1}g_i(x)\overline{\psi_i(x)}dx
\]

for all \( f(x), g(x), \varphi(x), \psi(x) \in X \). Obviously, \( \mathcal{H} \) is a Hilbert space.

Define the system operator \( \mathcal{A} \) in \( \mathcal{H} \) by

\[
\mathcal{A}(f(x), g(x)) = -(Lg(x), Mf(x)) - (0, Ng(x))
\]

\[
D(\mathcal{A}) = \{ (f, g) \in \mathcal{H} \mid f', g' \in [L^2(0, 1)]^3; f(1) = C^Tf(0), g(0) = Cg(0) \}.
\]

It is easy to check that \( \mathcal{A} \) is a closed and densely defined linear operator in \( \mathcal{H} \).

A simple calculation gives that the dual operator \( \mathcal{A}^* \) of \( \mathcal{A} \) is

\[
\mathcal{A}^*(\varphi(x), \psi(x)) = (L\varphi'(x), M\varphi'(x)) - (0, N\psi(x))
\]

\[
D(\mathcal{A}^*) = \{ (\varphi, \psi) \in \mathcal{H} \mid \varphi', \psi' \in [L^2(0, 1)]^3; \varphi(1) = C^T\varphi(0), \psi(0) = C\psi(1) \}.
\]

(12)

We take the control functions space as \( L^2_{loc}(\mathbb{R}^+, \mathbb{U}) = L^2_{loc}(\mathbb{R}^+, \mathbb{C}^3) \). For any \( T > 0 \), and for \( U(t) = (u_1(t), u_2(t), u_3(t)) \in L^2_{loc}(\mathbb{R}^+, \mathbb{U}) \), its local norm is defined by

\[
\|U(\cdot)\|^2_{L^2([0, T], \mathbb{U})} = \int_0^T \|u_1(t)\|^2_{M_1} + \|u_2(t)\|^2_{M_2} + \|u_3(t)\|^2_{M_3} dt.
\]

Furthermore, we define control input operator \( \mathcal{B} : \mathbb{U} = \mathbb{C}^3 \rightarrow \mathcal{H}_{-1} \) by

\[
\mathcal{B}(U) = \left( \frac{\delta(x)u_1}{L_1}, 0, 0, -\frac{\delta(x-1)u_2}{M_2}, -\frac{\delta(x-1)u_3}{M_3} \right)^T \in \mathcal{H}_{-1}, U = (u_1, u_2, u_3) \in \mathbb{C}^3,
\]

where \( \mathcal{H}_{-1} = (D(\mathcal{A}), \|R(\lambda, \mathcal{A})\|) \) with \( \lambda \) being a resolvent point of the system operator \( \mathcal{A} \), and \( \delta(x), \delta(x-1) \) are the Dirac Delta function.
Lemma 2.2. Let \( H \) be the set of all \( \sigma \)-eigenvalues of finite multiplicity, i.e.,

\[
A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}
\]

Proof. For any \( (f(x), g(x))^T \) \( D(A) \), it follows from (6), (7) and (8) that

\[
\Re(A(f, g), (f, g))_{\mathcal{H}} = -\sum_{i=1}^{3} \frac{N_i}{M_i} \int_{0}^{1} |g_i(x)|^2 \, dx \leq 0.
\]

So, \( A \) is a dissipative operator in \( \mathcal{H} \).

Lemma 2.1. Let \( \mathcal{H} \) and \( \mathcal{A} \) be defined by (10) and (11) respectively. Then \( \mathcal{A} \) is dissipative in \( \mathcal{H} \).

Proof. For any \( (f(x), g(x))^T \) \( D(A) \), it follows from (6), (7) and (8) that

\[
\Re(A(f, g), (f, g))_{\mathcal{H}} = -\sum_{i=1}^{3} \frac{N_i}{M_i} \int_{0}^{1} |g_i(x)|^2 \, dx \leq 0.
\]

So, \( A \) is a dissipative operator in \( \mathcal{H} \).

Lemma 2.2. Let \( \mathcal{H} \) and \( \mathcal{A} \) be defined by (10) and (11) respectively. Then \( 0 \in \rho(A) \).

Furthermore, \( A^{-1} \) is compact on \( \mathcal{H} \), and hence the spectra of \( A \) consist of all isolated eigenvalues of finite multiplicity, i.e., \( \sigma(A) = \sigma_p(A) \).

Proof. For any fixed \( (u(x), v(x))^T \) \( \mathcal{H} \), we consider the solvability of the equations

\[
\mathcal{A}(f(x), g(x))^T = (u(x), v(x))^T, \quad (f(x), g(x))^T \in D(A),
\]

that is,

\[
\begin{cases}
-Lg'(x) = u(x), \\
-Mf'(x) - Ng(x) = v(x), \\
g(0) = Cg(1), \\
f(1) = C^T f(0).
\end{cases}
\]
Solving the differential equations above we find out the formal solutions
\[ g(x) = g(1) + \int_0^1 L^{-1} u(s) ds, \]
\[ f(x) = f(0) - \int_0^x M^{-1} [v(s) + Ng(s)] ds. \]

Inserting above into the formal solutions, we can get a unique solution pair to (16)
\[ (I - C)g(1) = - \int_0^1 L^{-1} u(s) ds, \]
\[ (I - C) f(0) = \int_0^1 M^{-1} (v(s) + N g(s)) ds. \]

A direct calculation shows that \( (I - C)^{-1} = I + C \) and \( (I - C^T)^{-1} = I + C^T \). Thus,
\[ g(1) = -(I + C) \int_0^1 L^{-1} u(s) ds, \]
\[ f(0) = (I + C^T) \int_0^1 M^{-1} (v(s) + N g(s)) ds. \]

Inserting above into the formal solutions, we can get a unique solution pair to (16)
\[ g(x) = -C \int_0^1 L^{-1} u(s) ds - \int_0^x L^{-1} u(s) ds, \]
\[ f(x) = C^T \int_0^1 M^{-1} (v(s) + N[-C \int_0^1 L^{-1} u(r) dr] ds - \int_0^1 L^{-1} u(r) dr] ds) \]
\[ + \int_x^1 M^{-1} (v(s) + N[-C \int_0^1 L^{-1} u(r) dr] ds - \int_0^1 L^{-1} u(r) dr] ds. \]

The closed operator Theorem asserts that \( 0 \in \rho(A) \). By the expression (17), \( (f, g) = A^{-1}(u, v) \) is an integral operator of continuous kernel function and so \( A^{-1} \) is compact on \( H \). Thus, the spectra of \( A \) consist of all isolated eigenvalues of finite multiplicity (see [16]) and \( \sigma(A) = \sigma_p(A) \). The proof is then complete.

Lemma 2.1 and Lemma 2.2 together with the Lumer-Phillips Theorem in [17] assert the following result.

**Theorem 2.3.** Let \( H \) and \( A \) be defined by (10) and (11) respectively. Then \( A \) generates a \( C_0 \) semigroup of contractions on \( H \), and hence the system (15) is interior well-posed.

2.2. \( L^2 \) well-posedness of system (15). \( L^2 \) well-posedness of a system means that if a system has local \( L^2 \) input, then its output (observation) also is \( L^2 \) locally. Sometimes, it also is called the exterior well-posedness of the system. In this section, we shall discuss the \( L^2 \) well-posedness of the system (15).

We begin with introducing the following lemma that gives a sufficient and necessary condition for \( L^2 \) well-posedness of a system(see [18, section 4, pp.281-284]).

**Lemma 2.4.** Let \( X, U \) and \( Y \) be Hilbert spaces, \( L^2_{loc}(\mathbb{R}_+, U) \) and \( L^2_{loc}(\mathbb{R}_+, Y) \) be abstract function spaces consisting of all local square integrable functions. Suppose that \( A \) generates a \( C_0 \) semigroup \( T(t), t \geq 0 \). Let \( B \) be a linear operator from \( U \) to \( X_{-1} \) and \( C \) also be a linear operator from \( D(A) \) to \( Y \). A linear system \( S \)
\[
\begin{cases}
\frac{dx(t)}{dt} = Ax(t) + Bu(t), & x(0) = x_0, \\
y(t) = Cx(t)
\end{cases}
\]
is \( L^2 \) well-posed if and only if, for any given \( t > 0 \), there exists a constant \( M_t > 0 \), which depends on the time \( t \), such that
\[
\|x(t)\|^2 + \int_0^t \|y(s)\|^2 ds \leq M_t \left( \|x_0\|^2 + \int_0^t \|u(s)\|^2 ds \right), \quad \forall (x_0, u) \in X \times L^2_{loc}(\mathbb{R}_+, U).
\]

\( L^2 \) well-posedness implies that the control operator \( B \) and the observation operator \( C \) are admissible for \( C_0 \) semigroup \( T = (T(t))_{t \geq 0} \). \( B \) is an admissible control operator for \( T \) means that for \( \forall t > 0 \) and \( \forall u \in L^2_{loc}(\mathbb{R}_+, U) \), \( \int_0^t T(t-s) Bu(s) ds \in X \),
the mild solution to (13) is given by \( x(t) = T(t)x_0 + \int_0^t T(t-s)Bu(s)ds \); \( C \) is an admissible observation operator for \( T \) implies that, for \( \forall x_0 \in D(A) \), the mapping \( x_0 \to CT(t)x_0 \) can be extended to a bounded mapping from \( X \) to \( L^2_{\text{loc}}(\mathbb{R}_+; Y) \) (see [19, chapter 4, pp.128–131]).

Using the multiplier method, we can prove the \( L^2 \) well-posedness of control-observation system (15).

**Theorem 2.5.** The linear system (15) is \( L^2 \) well-posed.

Since the proof of Theorem 2.5 has long and complicated calculation, we postpone the detail in Appendix A.

3. **Spectral analysis of \( A \).** In this section, we shall discuss the spectral property of the system operator and its eigenvectors. Firstly, we observe that \( D(A) = D(A^*) \) (see (11) and (12)) and for \((f,g) \in \mathcal{H}\),

\[
\frac{1}{2} [A(f, g) + A^*(f, g)] = -(0, Ng), \quad \frac{1}{2} [A(f, g) - A^*(f, g)] = -(Lg', Mf').
\]

So we can define two operators in \( \mathcal{H} \) by

\[
A_0(f, g)^T = -(Lg', Mf')^T, \quad D(A_0) = D(A) \tag{18}
\]

and

\[
A_1(f, g)^T = (0, Ng)^T, \quad D(A_1) = \mathcal{H}. \tag{19}
\]

Obviously, the following result is true.

**Theorem 3.1.** Let \( A_0 \) and \( A_1 \) be defined by (18) and (19) respectively. Then \( A_0 \) is a skew adjoint operator and \( A_1 \) is a bounded linear operator on \( \mathcal{H} \). In particular, \( A = A_0 + A_1 \).

Since \( A_0 \) is a skew adjoint operator, for any \( \lambda \notin i\mathbb{R} \), we have \( ||R(\lambda, A_0)|| \leq \frac{1}{|\Re \lambda|} \). So when \( |\Re \lambda| > ||A_1|| \), we have \( \lambda \in \rho(A) \) and

\[
||R(\lambda, A)|| \leq \frac{1}{|\Re \lambda| - ||A_1||}, \quad \forall |\Re \lambda| > ||A_1||.
\]

From Lemma 2.1 we know that \( \rho(A) \subset \{ \lambda \in \mathbb{C} \mid \Re \lambda \leq 0 \} \). Therefore, we have the following result.

**Theorem 3.2.** Let \( \mathcal{H} \) and \( A \) be defined by (10) and (11) respectively. Then there exists a positive constant \( h > 0 \) such that \( \sigma(A) \subset \{ \lambda \in \mathbb{C} \mid -h \leq \Re \lambda \leq 0 \} \).

3.1. **Eigenvalues of \( A \).** In this subsection, we shall determine the distribution of spectra \( \sigma(A) \). In Lemma 2.2, we have proved that \( \sigma(A) = \sigma_p(A) \). Therefore, we only need to discuss the eigenvalue problem of \( A \).

Let \( \lambda \in \mathbb{C} \) be an eigenvalue of \( A \) and \((P, Q)^T\) be the corresponding eigenvector. Then we have

\[
\begin{align*}
-\lambda Q'(x) &= \lambda P(x), \quad x \in (0, 1), \\
-MP'(x) &= (\lambda I + N)Q(x), \quad x \in (0, 1), \\
Q(0) &= CQ(1), \quad P(1) = CT^*P(0).
\end{align*}
\]

(20)

Since \( M \), \( L \) and \( N \) are the diagonal matrices, we denote \( B(\lambda) \) by the diagonal matrix

\[
B(\lambda) = \text{diag}\{b_1(\lambda), b_2(\lambda), b_3(\lambda)\} = (ML)^{-\frac{1}{2}}(\lambda^2 + N\lambda)^{-\frac{1}{2}},
\]

and denote

\[
e^{\pm B(\lambda)x} = \text{diag}\{e^{\pm b_1(\lambda)}, e^{\pm b_2(\lambda)}, e^{\pm b_3(\lambda)}\}
\]
where $b_i(\lambda), i = 1, 2, 3,$ have explicate expression as follows:

$$b_i(\lambda) = (M_i L_i)^{-\frac{1}{2}} \left( \lambda^2 + N_i \lambda \right)^{\frac{1}{2}} = \sqrt{\frac{2 \rho A_{i0}^{\frac{1}{2}}}{\beta_i}} \left( \lambda^2 + \frac{k}{A_{i0}} \lambda \right), \quad i = 1, 2, 3. \quad (21)$$

We set the solution of (20) be

$$P(x) = \sinh(x B(\lambda)) C_1 + \cosh(x B(\lambda)) P(0),$$

$$Q(x) = -M(\lambda + N)^{-1} P'(x)$$

$$= -MB(\lambda)(\lambda + N)^{-1} \left[ \cosh(x B(\lambda)) C_1 + \sinh(x B(\lambda)) P(0) \right].$$

Then we have

$$Q(0) = -MB(\lambda)(\lambda + N)^{-1} C_1$$

and hence

$$P(x) = \cosh(x B(\lambda)) P(0) - (\lambda + N)(MB(\lambda))^{-1} \sinh(x B(\lambda)) Q(0),$$

$$Q(x) = \cosh(x B(\lambda)) Q(0) - MB(\lambda)(\lambda + N)^{-1} \sinh(x B(\lambda)) P(0).$$

Using the boundary conditions $Q(0) = CQ(1)$ and $P(1) = C^T P(0)$, we get algebraic equations:

$$(I - C \cosh B(\lambda)) Q(0) + CMB(\lambda)(\lambda + N)^{-1} \sinh B(\lambda) P(0) = 0,$$

$$\left( \cosh B(\lambda) - C^T \right) P(0) - (\lambda + N)(MB(\lambda))^{-1} \sinh B(\lambda) Q(0) = 0. \quad (22)$$

We denote the coefficient matrix of (22) by $D(\lambda)$, i.e.,

$$D(\lambda) = \begin{pmatrix} I - C \cosh B(\lambda) & CMB(\lambda)(\lambda + N)^{-1} \sinh B(\lambda) \\ -(\lambda + N)(MB(\lambda))^{-1} \sinh B(\lambda) & \cosh B(\lambda) - C^T \end{pmatrix}. \quad (23)$$

Obviously, Eqs.(22) has a nonzero solution $(Q(0), P(0))^T$ if and only if $\det(D(\lambda)) = 0$. Therefore, $\lambda \in \sigma(A)$ if and only if $\det(D(\lambda)) = 0$.

In what follows, we calculate the $\det(D(\lambda))$. Since

$$C = \begin{pmatrix} 0 & 0 & 0 \\ \alpha & 0 & 0 \\ (1 - \alpha) & 0 & 0 \end{pmatrix}, \quad e^{B(\lambda)} = \begin{pmatrix} e^{b_1(\lambda)} & 0 & 0 \\ 0 & e^{b_2(\lambda)} & 0 \\ 0 & 0 & e^{b_3(\lambda)} \end{pmatrix},$$

and $e^{xB}$ is a diagonal matrix, so $I - C \cosh B(\lambda)$ is invertible and $(I - C \cosh B(\lambda))^{-1} = I + C \cosh B(\lambda)$. Thus we can write $D(\lambda)$ as a product of two matrices

$$D(\lambda) = \begin{pmatrix} I - C \cosh B(\lambda) & 0 \\ -(\lambda + N)(MB(\lambda))^{-1} \sinh B(\lambda) & d(\lambda) \end{pmatrix}$$

$$\bullet \begin{pmatrix} I + C \cosh B(\lambda) & CMB(\lambda)(\lambda + N)^{-1} \sinh B(\lambda) \\ 0 & I \end{pmatrix}$$

where

$$d(\lambda) = \cosh B(\lambda) - C^T$$

$$+ (\lambda + N)(MB(\lambda))^{-1} \sinh B(\lambda)[CMB(\lambda)(\lambda + N)^{-1}] \sinh B(\lambda).$$
Obviously, \( \det(D(\lambda)) = 0 \) if and only if \( \det(d(\lambda)) = 0 \). Since

\[
(\lambda + N)(MB(\lambda))^{-1} \sinh B(\lambda)CMB(\lambda)(\lambda + N)^{-1} \sinh B(\lambda)
\]

\[
= \begin{pmatrix}
\frac{(\lambda+N_1) \sinh b_1(\lambda)}{M_1 b_1(\lambda)} & 0 & 0 \\
0 & \frac{(\lambda+N_2) \sinh b_2(\lambda)}{M_2 b_2(\lambda)} & 0 \\
0 & 0 & \frac{(\lambda+N_3) \sinh b_3(\lambda)}{M_3 b_3(\lambda)}
\end{pmatrix}
\times \begin{pmatrix}
0 & 0 & 0 \\
\alpha & 0 & 0 \\
1 - \alpha & 0 & 0
\end{pmatrix}
\]

\[
= \begin{pmatrix}
\frac{\alpha (\lambda+N_2) \sinh b_2(\lambda)}{M_2 b_2(\lambda)} \frac{M_2 b_2(\lambda)}{\lambda+N_2} & 0 & 0 \\
0 & \frac{\alpha (\lambda+N_3) \sinh b_3(\lambda)}{M_3 b_3(\lambda)} & 0 \\
0 & 0 & \frac{\lambda+N_4}{M_3 b_3(\lambda)}
\end{pmatrix},
\]

and

\[
cosh B(\lambda) - C^T = \begin{pmatrix}
cosh b_1(\lambda) & -\alpha & -(1-\alpha) \\
0 & \cosh b_2(\lambda) & 0 \\
0 & 0 & \cosh b_3(\lambda)
\end{pmatrix}.
\]

Thus we find out

\[
d(\lambda) = \begin{pmatrix}
d_1(\lambda) & d_2(\lambda) & d_3(\lambda) \\
d_4(\lambda) & d_6(\lambda) & 0 \\
d_5(\lambda) & 0 & d_7(\lambda)
\end{pmatrix}
\]

where

\[
d_1(\lambda) = \cosh b_1(\lambda), \quad d_2(\lambda) = -\alpha, \quad d_3(\lambda) = -(1-\alpha), \quad d_4(\lambda) = \alpha \frac{(\lambda+N_2) \sinh b_2(\lambda)}{M_2 b_2(\lambda)} \frac{M_2 b_2(\lambda)}{\lambda+N_2}, \quad d_5(\lambda) = (1-\alpha) \frac{(\lambda+N_3) \sinh b_3(\lambda)}{M_3 b_3(\lambda)} \frac{M_2 b_2(\lambda)}{\lambda+N_3}, \quad d_6(\lambda) = \cosh b_2(\lambda), \quad d_7(\lambda) = \cosh b_3(\lambda).
\]

Note that \( B(\lambda) = (ML)^{-\frac{1}{2}}(\lambda^2 + N\lambda)^{\frac{1}{2}} \) and \((\lambda + N)(MB(\lambda))^{-1} = \frac{LB(\lambda)}{\lambda}\). Thus we get an explicate expression for \( \det(d(\lambda)) \):

\[
\det(d(\lambda)) = \cosh b_1(\lambda) \cosh b_2(\lambda) \cosh b_3(\lambda) + \alpha^2 \frac{(\lambda+N_2) \sinh b_1(\lambda) \sinh b_2(\lambda)}{M_2 b_2(\lambda)} \frac{M_2 b_2(\lambda)}{\lambda+N_2} \cosh b_3(\lambda)
\]

\[
+ (1 - \alpha)^2 \frac{(\lambda+N_3) \sinh b_1(\lambda)}{M_2 b_2(\lambda)} \frac{M_2 b_2(\lambda)}{\lambda+N_3} \cosh b_2(\lambda) \sinh b_3(\lambda)
\]

\[
= \cosh b_1(\lambda) \cosh b_2(\lambda) \cosh b_3(\lambda)
\]

\[
+ \alpha^2 \frac{L_2 b_2(\lambda)}{L_1 b_1(\lambda)} \sinh b_1(\lambda) \cosh b_2(\lambda) \cosh b_3(\lambda)
\]

\[
+ (1 - \alpha)^2 \frac{L_3 b_3(\lambda)}{L_1 b_1(\lambda)} \sinh b_1(\lambda) \cosh b_2(\lambda) \sinh b_3(\lambda).
\]

Clearly, the spectra of \( A \) are determined by \( \det(d(\lambda)) = 0 \).

When \( \Re \lambda \to +\infty \), it holds that \( e^{-B(\lambda)} \to 0 \), and hence

\[
\lim_{\Re \lambda \to \infty} \frac{\det(D(\lambda))}{\det(e^{B(\lambda)})} = \lim_{\Re \lambda \to \infty} \frac{\det(I - C \cosh B) \det(d(\lambda))}{\det(I + B(\lambda))} = \alpha^2 \frac{L_2}{8L_1} + (1-\alpha)^2 \frac{L_3}{8L_1} + \frac{1}{8}.
\]
When $\Re \lambda \to -\infty$, it has $e^{B(\lambda)} \to 0$ and
\[
\lim_{\Re \lambda \to -\infty} \frac{\det(D(\lambda))}{\det(e^{-B(\lambda)})} = \lim_{\Re \lambda \to -\infty} \frac{\det(I - C \cosh B) \det(d(\lambda))}{e^{-b_1(\lambda) - b_2(\lambda) - b_3(\lambda)}} = \frac{\alpha^2 L_2}{8L_1} + (1 - \alpha)^2 \frac{L_3}{8L_1} + \frac{1}{8}.
\]
The above two equalities show that there exist positive constants $h_1, h_2$ and $h$ such that
\[
h_1 e^{\Re \lambda |\text{tr}(ML)|^{-1/2}} \leq |\det(D(\lambda))| \leq h_2 e^{\Re \lambda |\text{tr}(ML)|^{-1/2}}, \quad |\Re \lambda| > h
\]
where $\text{tr}(ML)^{-1/2}$ is the trace of matrix $(ML)^{-1/2}$, that implies $\det(D(\lambda))$ is sine type function (see [21],[22]). Therefore we have the following result.

**Theorem 3.3.** Let $\mathcal{H}$ and $A$ be defined by (10) and (11) respectively and $D(\lambda)$ be defined by (22). Then $\sigma(A) = \{\lambda \in \mathbb{C} \mid \det(D(\lambda)) = 0\}$. Furthermore, $\det(D(\lambda))$ is an sine type function, and hence $\sigma(A)$ is a union of finite many separated sets.

### 3.2. Eigenvectors.

In this subsection, we shall discuss the eigenvectors of $A$, that is equivalent to determine the vectors $P(0)$ and $Q(0)$ from equation
\[
D(\lambda) \begin{pmatrix} Q(0) \\ P(0) \end{pmatrix} = 0.
\]
This leads to $d(\lambda)P(0) = 0$ and $Q(0) = H(\lambda)P(0)$ where
\[
H(\lambda) = -(I - C \cosh B(\lambda))^{-1} CM B(\lambda)(\lambda + N)^{-1} \sinh B(\lambda)
\]
\[
= \begin{pmatrix} 0 & 0 & 0 \\ -\alpha^2 M_1 b_5(\lambda) \sinh b_1(\lambda) & 0 & 0 \\ -(1 - \alpha)^2 M_1 b_5(\lambda) \sinh b_1(\lambda) / \lambda + N_1 & 0 & 0 \end{pmatrix}.
\]
(25)
The equations (22) have nonzero solutions if and only if $d(\lambda)P(0) = 0$ have nonzero solutions.

The equations $d(\lambda)P(0) = 0$ can be rewritten as
\[
\begin{cases}
d_1(\lambda)P_1(0) + d_2(\lambda)P_2(0) + d_3(\lambda)P_3(0) = 0, \\
d_4(\lambda)P_1(0) + d_6(\lambda)P_2(0) = 0, \\
d_5(\lambda)P_1(0) + d_7(\lambda)P_2(0) = 0.
\end{cases}
\]
(26)
If $\lambda \in \mathbb{C}$ satisfies $\det d(\lambda) = 0$, then (26) have nonzero solutions. In what follows, we classify four cases to determine the solution according to whether or not $\lambda$ makes $d_6(\lambda) = 0$ and $d_7(\lambda) = 0$ respectively.

#### 3.2.1. Case 1: $d_6(\lambda) \neq 0$ and $d_7(\lambda) \neq 0$.

In this case, according to (26) we have
\[
P_2(0) = -\frac{d_4(\lambda)}{d_6(\lambda)} P_1(0), \quad P_3(0) = -\frac{d_5(\lambda)}{d_7(\lambda)} P_1(0).
\]
Substituting above into the first equation in (26) leads to
\[
(d_1(\lambda) - \frac{d_2(\lambda)d_4(\lambda)}{d_6(\lambda)}) P_1(0) = 0.
\]
Due to $\det(d(\lambda)) = 0$, we have $d_1(\lambda) - \frac{d_2(\lambda)d_4(\lambda)}{d_6(\lambda)} = \frac{d_5(\lambda)d_6(\lambda)}{d_7(\lambda)} = 0$, which means that $P_1(0)$ is an arbitrary constant. Thus the nonzero solution of (26) is
\[
P(0) = \begin{pmatrix} 1 \\ -\frac{d_4(\lambda)}{d_6(\lambda)} \\ -\frac{d_5(\lambda)}{d_7(\lambda)} \end{pmatrix}_\xi = \tilde{p}_1(\lambda) \xi, \quad \xi \in \mathbb{C}.
\]
and the vector \((Q(0), P(0))^T\) is given by
\[
\begin{pmatrix}
Q(0) \\
P(0)
\end{pmatrix}
= \begin{pmatrix}
H(\lambda)\tilde{p}_1(\lambda) \\
\tilde{p}_1(\lambda)
\end{pmatrix} \xi.
\]

Since there is only one arbitrary parameter \(\xi\), so the eigen-space of \(\mathcal{A}\) corresponding to \(\lambda\) is one-dimensional (see, Eqs. (20)).

3.2.2. Case 2: \(d_6(\lambda) = 0\) and \(d_7(\lambda) \neq 0\). Since \(d_7(\lambda) \neq 0\), we have \(P_3(0) = -\frac{d_6(\lambda)}{d_7(\lambda)} P_1(0)\). Substituting it into the first equation in (26), we obtain the equations
\[
\begin{cases}
\left( d_1(\lambda) - \frac{d_3(\lambda)d_6(\lambda)}{d_7(\lambda)} \right) P_1(0) + d_2(\lambda) P_2(0) = 0, \\
\left( d_4(\lambda) P_1(0) + d_6(\lambda) P_2(0) = 0. \\
\end{cases}
\]

Since \(d_6(\lambda) = \cosh b_2(\lambda) = 0\) implies \(e^{b_2(\lambda)} = -e^{-b_2(\lambda)}\), so when \(d_6(\lambda) = 0\), we have
\[
d_6(\lambda) = \alpha \frac{(\lambda + N_2) \sinh b_2(\lambda) M_1 b_1(\lambda) \sinh b_1(\lambda)}{\lambda + N_1}
= \alpha e^{b_2(\lambda)} \frac{L_2 b_2(\lambda)}{L_1 b_1(\lambda)} \sinh b_1(\lambda)
\]
and \(d_2(\lambda) = -\alpha\). Obviously, if \(e^{-b_1(\lambda)} - e^{b_1(\lambda)} \neq 0\), then \(P_1(0) = 0\) and hence \(P_2(0) = 0\) and \(P_3(0) = 0\). This is impossible.

Therefore, in this case, only if \(e^{-b_1(\lambda)} - e^{b_1(\lambda)} = 0\), the equations (26) have nonzero solutions. And the nonzero solution to (26) is
\[
P(0) = \begin{pmatrix}
1 \\
-\frac{d_3(\lambda) d_6(\lambda)}{d_7(\lambda)}
\end{pmatrix} \xi = \tilde{p}_2(\lambda) \xi.
\]

Thus,
\[
\begin{pmatrix}
Q(0) \\
P(0)
\end{pmatrix}
= \begin{pmatrix}
H(\lambda)\tilde{p}_2(\lambda) \\
\tilde{p}_2(\lambda)
\end{pmatrix} \xi.
\]

So the eigen-space of \(\mathcal{A}\) corresponding to \(\lambda\) also is one-dimensional.

Note that if \(e^{b_2(\lambda)} + e^{-b_2(\lambda)} = 0\) and \(e^{b_1(\lambda)} - e^{-b_1(\lambda)} = 0\), then we have \(\det(d(\lambda)) = 0\) (see (27)), and hence \(\det(D(\lambda)) = 0\). Now we determine the zeros of function equations
\[
\begin{cases}
e^{b_2(\lambda)} + e^{-b_2(\lambda)} = 0, \\
e^{b_1(\lambda)} - e^{-b_1(\lambda)} = 0.
\end{cases}
\]

Clearly, the \(\lambda\) satisfies
\[
b_1(\lambda) = \sqrt{\frac{2 \rho A_{10}^{\frac{1}{2}}}{\beta_1}} \left( \lambda^2 + \frac{k}{A_{10} \lambda} \right)^{\frac{1}{2}} = \frac{2m + 1}{2} \pi i, \ m \in \mathbb{N}
\]
and
\[
b_2(\lambda) = \sqrt{\frac{2 \rho A_{20}^{\frac{1}{2}}}{\beta_2}} \left( \lambda^2 + \frac{k}{A_{20} \lambda} \right)^{\frac{1}{2}} = \frac{2n \pi}{2} i, \ n \in \mathbb{N}.
\]

If \(A_{10} = A_{20}\), then we have
\[
\frac{\sqrt{\beta_2}}{\sqrt{\beta_1}} = \frac{2m + 1}{2n}.
\]

This means that only when \(\beta_1\) and \(\beta_2\) satisfy the above equality the function equations have a common zero. Otherwise, there is no solution of the function equations.
If $A_{10} \neq A_{20}$, by a complex calculation we see that only if all parameters satisfy the following equality

$$\frac{n^2 \pi^2 \beta_2}{2 \rho A_{20}^2} = \frac{(2m + 1)^2 \pi^2 \beta_1}{8 \rho A_{10}^2} + \frac{K(A_{10} - A_{20})}{A_{20} A_{10}} \left[ -\frac{K}{A_{10}} \pm \sqrt{\frac{K^2}{4 A_{10}^2} - \frac{(2m + 1)^2 \pi^2 \beta_1}{8 \rho A_{10}^2}} \right]$$

the function equations have a solution. Otherwise, the function equations have no solution.

3.2.3. Case 3: $d_0(\lambda) \neq 0$ and $d_7(\lambda) = 0$. Since $d_0(\lambda) \neq 0$, we have $P_2(0) = -\frac{d_3(\lambda)}{d_6(\lambda)} P_1(0)$. Substituting it into the first equation in (26), we obtain the equations

$$\begin{cases}
(d_1(\lambda) - \frac{d_3(\lambda) d_4(\lambda)}{d_6(\lambda)}) P_1(0) + d_3(\lambda) P_2(0) = 0, \\
5 d_5(\lambda) P_1(0) + 7 d_7(\lambda) P_3(0) = 0.
\end{cases}$$

Note that $d_7(\lambda) = \cosh b_3(\lambda) = 0$ implies $e^{b_3(\lambda)} = -e^{-b_3(\lambda)}$. So when $d_7(\lambda) = 0$, we have

$$d_5(\lambda) = (1 - \alpha) \frac{(\lambda + N_3) \sinh b_3(\lambda)}{M_2 b_3(\lambda)} \frac{M_1 b_1(\lambda) \sinh b_1(\lambda)}{N \lambda + N_1} = (1 - \alpha) e^{b_3(\lambda)} \frac{L_3 b_3(\lambda)}{L_1 b_1(\lambda)} \sinh b_1(\lambda)$$

and $d_3(\lambda) = -(1 - \alpha)$. Obviously, if $e^{-b_1(\lambda)} - e^{b_1(\lambda)} \neq 0$, then $P_1(0) = 0$ and hence $P_3(0) = 0$ and $P_2(0) = 0$. This is impossible.

Therefore, in this case, only when $e^{-b_1(\lambda)} - e^{b_1(\lambda)} = 0$, the equations (26) have nonzero solutions. The nonzero solutions to (26) are

$$P(0) = \begin{pmatrix} \frac{1}{d_3(\lambda)} \\ -\frac{d_3(\lambda) d_4(\lambda) - d_2(\lambda) d_4(\lambda)}{d_3(\lambda) d_6(\lambda)} \end{pmatrix} \xi = \hat{\rho}_3(\lambda) \xi.$$

Thus,

$$\begin{pmatrix} Q(0) \\ P(0) \end{pmatrix} = \begin{pmatrix} H(\lambda) \hat{\rho}_3(\lambda) \\ \hat{\rho}_3(\lambda) \end{pmatrix} \xi.$$

So the eigen-space of $\mathcal{A}$ corresponding to $\lambda$ also is one-dimensional.

If $\lambda \in \mathbb{C}$ such that $e^{b_3(\lambda)} + e^{-b_3(\lambda)} = 0$ and $e^{b_1(\lambda)} - e^{-b_1(\lambda)} = 0$, then we have $\det(D(\lambda)) = 0$. Similar to previous discussion, the function equations

$$\begin{cases}
e^{b_3(\lambda)} + e^{-b_3(\lambda)} = 0, \\
e^{b_1(\lambda)} - e^{-b_1(\lambda)} = 0
\end{cases}$$

have a common zero if and only if

$$\sqrt{\frac{\beta_3}{\beta_1}} = \frac{2m + 1}{2n}, \quad \text{if} \quad A_{10} = A_{30}$$

and

$$\frac{n^2 \pi^2 \beta_3}{2 \rho A_{30}^2} = \frac{(2m + 1)^2 \pi^2 \beta_1}{8 \rho A_{10}^2} + \frac{K(A_{10} - A_{30})}{A_{30} A_{10}} \left[ -\frac{K}{A_{10}} \pm \sqrt{\frac{K^2}{4 A_{10}^2} - \frac{(2m + 1)^2 \pi^2 \beta_1}{8 \rho A_{10}^2}} \right]$$

if $A_{10} \neq A_{30}$. 

Remark 1. In the cases 3.2.2 and 3.2.3, the conditions
\[ \sqrt{\frac{\beta_j}{\beta_1}} = \frac{2m+1}{2n}, \quad \text{if} \quad A_{10} = A_{j0}, \ j=2,3 \]
and for \( j = 2,3, \) if \( A_{10} \neq A_{j0}, \) for some \( n, m \in \mathbb{N}, \)
\[
-\frac{n^2 \pi^2 \beta_j}{2 \rho A_{j0}^2} = \frac{(2m+1)^2 \pi^2 \beta_1}{8 \rho A_{10}^2} + \frac{K(A_{10} - A_{j0})}{A_{j0} A_{10}} \left[ -\frac{K}{A_{10}} \pm \sqrt{\frac{K^2}{4 A_{10}^2} - \frac{(2m+1)^2 \pi^2 \beta_1}{8 \rho A_{10}^2}} \right]
\]
are subtle. So in general we can assume that \( d_6(\lambda)d_7(\lambda) \neq 0 \) if \( \lambda \) satisfies the equation \( \det(D(\lambda)) = 0. \)

3.2.4. Case 4: \( d_6(\lambda) = 0, \ d_7(\lambda) = 0. \) When \( d_6(\lambda) = 0 \) and \( d_7(\lambda) = 0, \) it always holds \( \det(D(\lambda)) = 0. \) We have
\[ e^{2b_2(\lambda)} = -1, \quad e^{2b_3(\lambda)} = -1. \]
That means
\[
\lambda^2 + \frac{K}{A_{20}} \lambda = -\frac{(2m+1)^2 \pi^2}{4 \rho A_{20}^2} \beta_2, \quad \lambda^2 + \frac{K}{A_{30}} \lambda = -\frac{(2m+1)^2 \pi^2}{4 \rho A_{30}^2} \beta_3.
\]
From above we get
\[
\lambda + \frac{K}{2 A_{20}} = \pm \sqrt{\frac{K^2}{4 A_{20}^2} - \frac{(2m+1)^2 \pi^2}{4 \rho A_{20}^2} \beta_2},
\]
and
\[
\lambda + \frac{K}{2 A_{30}} = \pm \sqrt{\frac{K^2}{4 A_{30}^2} - \frac{(2m+1)^2 \pi^2}{4 \rho A_{30}^2} \beta_3}.
\]
Hence \( m \) and \( n \) satisfy condition
\[
-\frac{K}{2 A_{20}} \pm \sqrt{\frac{K^2}{4 A_{20}^2} - \frac{(2m+1)^2 \pi^2}{4 \rho A_{20}^2} \beta_2} = -\frac{K}{2 A_{30}} \pm \sqrt{\frac{K^2}{4 A_{30}^2} - \frac{(2m+1)^2 \pi^2}{4 \rho A_{30}^2} \beta_3}.
\]
If this condition is fulfilled, then it holds that
\[
d_4(\lambda) = \alpha e^{b_2(\lambda)} L_2 b_2(\lambda) \sinh b_1(\lambda), \quad d_5(\lambda) = (1 - \alpha) e^{b_3(\lambda)} L_3 b_3(\lambda) \sinh b_1(\lambda).
\]
According to the discussions of case 2 and case 3, we can assume \( d_4(\lambda) \neq 0, d_5(\lambda) \neq 0. \) From (26) we get
\[
P(0) = \begin{pmatrix} 0 \\ 1 \\ \frac{d_5(\lambda)}{d_4(\lambda)} \end{pmatrix} \xi = \tilde{p}_4(\lambda) \xi.
\]
Hence
\[
\begin{pmatrix} Q(0) \\ P(0) \end{pmatrix} = \begin{pmatrix} H(\lambda) \tilde{p}_4(\lambda) \\ \tilde{p}_4(\lambda) \end{pmatrix} \xi.
\]
The above discussions show that the following result is true.

**Theorem 3.4.** Let \( \mathcal{H} \) and \( \mathcal{A} \) be defined by (10) and (11) respectively. Then for each \( \lambda \in \sigma(\mathcal{A}), \) the corresponding eigen-space \( \mathcal{N}(\lambda I - \mathcal{A}) \) is one dimensional.
Let \( p(\lambda) = (p_1(\lambda), p_2(\lambda), p_3(\lambda)) \) be a solution to (26). Then \( P(0) = p(\lambda)\xi \), and \( Q(0) = H(\lambda)\eta(\lambda)\xi \). For simplicity, we take \( \xi = 1 \).

Note that
\[
P(x) = \cosh(xB(\lambda))P(0) - \sinh(xB(\lambda))(\lambda + N)(MB(\lambda))^{-1}Q(0),
\]
\[
Q(x) = \cosh(xB(\lambda))Q(0) - MB(\lambda)(\lambda + N)^{-1}\sinh(xB(\lambda))P(0).
\]

Therefore, the eigenvector \((P(x, \lambda), Q(x, \lambda))^T\) can be rewritten by
\[
P(x, \lambda) = \begin{pmatrix}
cosh(xb_1(\lambda))p_1(\lambda) \\
cosh(xb_2(\lambda))p_2(\lambda) + \alpha \sinh(xb_2(\lambda)) \sinh(b_1(\lambda)) \frac{L_2 b_2(\lambda)}{L_1 b_1(\lambda)} p_1(\lambda) \\
cosh(xb_3(\lambda))p_3(\lambda) + (1 - \alpha) \sinh(xb_3(\lambda)) \sinh(b_1(\lambda)) \frac{L_3 b_3(\lambda)}{L_1 b_1(\lambda)} p_1(\lambda)
\end{pmatrix},
\]
\[
Q(x, \lambda) = \begin{pmatrix}
0 - \frac{M_1 b_1(\lambda)}{L_1 b_1(\lambda)} \sinh(xb_1(\lambda)) p_1(\lambda) \\
- \frac{\alpha \cosh(xb_2(\lambda)) M_1 b_1(\lambda) \sinh(b_1(\lambda)) p_1(\lambda)}{L_1 b_1(\lambda)} - \frac{M_2 b_2(\lambda)}{L_1 b_1(\lambda)} \sinh(xb_2(\lambda)) p_2(\lambda) \\
- \frac{(\alpha - 1) \cosh(xb_3(\lambda)) M_1 b_1(\lambda) \sinh(b_1(\lambda)) p_1(\lambda)}{L_1 b_1(\lambda)} - \frac{M_3 b_3(\lambda)}{L_1 b_1(\lambda)} \sinh(xb_3(\lambda)) p_3(\lambda)
\end{pmatrix}.
\]

3.3. Multiplicity and separability of eigenvalues. In this subsection, we shall discuss the algebraic multiplicity and separability of eigenvalues of \( A \). In the sequel, we simply denote \( G(\lambda) = \det(d(\lambda)) \). At first, we have the following result.

**Theorem 3.5.** Let \( H \) and \( A \) be defined by (10) and (11) respectively. The for any \( \lambda \in \rho(A) \), and \( (f, g) \in H \), we have the resolvent expression
\[
R(\lambda, A)(f, g)^T = \frac{E(\lambda)(f, g)}{G(\lambda)}
\]
where \( E(\lambda)(f, g) \) is an exponential-type \( H \)-valued entire function with respect to \( \lambda \).

This result is a direct verification (we omit the details). As a direct consequence of Theorem 3.5, we have the following corollary.

**Corollary 1.** Let \( H \) and \( A \) be defined by (10) and (11) respectively. Then for \( \lambda \in \sigma(A) = \sigma_p(A) \), the multiplicity of the \( \lambda \) equal to the multiplicity of zeros of \( G(\lambda) \).

In what follows, we shall discuss the multiplicity and separability of eigenvalues of \( A \) based on Corollary 1. We classify three cases according to whether the three vessels are the same or not.

3.3.1. Three vessels are the same. At first, we consider the case that the three vessels for the bifurcation are the same, i.e., \( \beta_1 = \beta_2 = \beta_3 \) and \( A_{10} = A_{20} = A_{30} \). In this case, it holds that \( b_1(\lambda) = b_2(\lambda) = b_3(\lambda) \). According to formula (24), we have
\[
G(\lambda) = \alpha^2 \frac{L_2 b_2(\lambda)}{L_1 b_1(\lambda)} \sinh b_1(\lambda) \sinh b_2(\lambda) \cosh b_3(\lambda)
\]
\[
+ (1 - \alpha)^2 \frac{L_2 b_3(\lambda)}{L_1 b_1(\lambda)} \sinh b_1(\lambda) \cosh b_2(\lambda) \sinh b_3(\lambda)
\]
\[
+ \cosh b_1(\lambda) \cosh b_2(\lambda) \cosh b_3(\lambda)
\]
\[
= \cosh b_1(\lambda) \left[ \alpha^2 + (1 - \alpha)^2 (\sinh b_1(\lambda))^2 + (\cosh b_1(\lambda))^2 \right].
\]

Obviously, its zeros are determined by \( \cosh b_1(\lambda) = 0 \), or
\[
[\alpha^2 + (1 - \alpha)^2 (\sinh b_1(\lambda))^2 + (\cosh b_1(\lambda))^2] = 0.
\]
If $\cosh b_1(\lambda) = 0$, then the zeros are determined by

$$b_1(\lambda) = \frac{(2n + 1)\pi i}{2}, \quad \forall n \in \mathbb{Z}.$$ 

Clearly, they are simple zeros and separable.

If $[\alpha^2 + (1 - \alpha)^2](\sinh b_1(\lambda))^2 + (\cosh b_1(\lambda))^2 = 0$, then the zeros are determined by

$$e^{2b_1(\lambda)} = \frac{\pm i\sqrt{\alpha^2 + (1 - \alpha)^2} - 1}{\pm i\sqrt{\alpha^2 + (1 - \alpha)^2} + 1}.$$ 

Also, they are simple zeros and separable.

**Theorem 3.6.** Let $\mathcal{H}$ and $\mathcal{A}$ be defined by (10) and (11) respectively, if three vessels are the same for system (15), all eigenvalues of $\mathcal{A}$ are simple and separable.

3.3.2. Two sub-branches are the same. We consider the case that $A_{20} = A_{30}$ and $\beta_2 = \beta_3$. Thus $b_2(\lambda) = b_3(\lambda)$ and

$$G(\lambda) = \cosh b_2(\lambda)$$

$$= \frac{L_2 b_2(\lambda)}{L_1 b_1(\lambda)} (\alpha^2 + (1 - \alpha)^2) \sinh b_1(\lambda) \sinh b_2(\lambda) + \cosh b_1(\lambda) \cosh b_2(\lambda)$$

whose zeros are given by

$$e^{2b_2(\lambda)} = -1$$

or

$$b_2(\lambda) \sinh b_1(\lambda) \sinh b_2(\lambda) = - \frac{L_1}{[\alpha^2 + (1 - \alpha)^2]L_2} = -\eta.$$

Clearly, if $e^{2b_2(\lambda)} = -1$, the zeros of $G(\lambda)$ are separable and simple (at most finite number non-simple). We consider the second case.

Set

$$G(\lambda) = b_2(\lambda) \sinh b_1(\lambda) \sinh b_2(\lambda) + \eta b_1(\lambda) \cosh b_1(\lambda) \cosh b_2(\lambda),$$

then

$$G'(\lambda) = b_2'(\lambda) \sinh b_1(\lambda) \sinh b_2(\lambda) + \eta b_1'(\lambda) \cosh b_1(\lambda) \cosh b_2(\lambda)$$

$$+ b_1'(\lambda) [b_2(\lambda) \cosh b_1(\lambda) \sinh b_2(\lambda) + \eta b_1(\lambda) \sinh b_1(\lambda) \cosh b_2(\lambda)]$$

$$+ b_2'(\lambda) [b_1(\lambda) \cosh b_2(\lambda) \sinh b_1(\lambda) \cosh b_2(\lambda) + \eta b_1(\lambda) \cosh b_1(\lambda) \sinh b_2(\lambda)]$$

$$= [b_2'(\lambda) - b_1'(\lambda) \frac{b_1(\lambda)}{b_2(\lambda)}] \eta \cosh b_1(\lambda) \cosh b_2(\lambda)$$

$$+ [b_2(\lambda) b_2'(\lambda) + \eta b_1(\lambda) b_1'(\lambda)] \sinh b_1(\lambda) \cosh b_2(\lambda)$$

$$+ [b_2(\lambda) b_1'(\lambda) + \eta b_1(\lambda) b_2'(\lambda)] \cosh b_1(\lambda) \sinh b_2(\lambda).$$

Since

$$b_i(\lambda) = \sqrt{\frac{2\rho A_{i0}^\frac{2}{\beta_i}}{\beta_i}} \left( \lambda^2 + \frac{K}{A_{i0}} \right)^\frac{i}{2}, \quad i = 1, 2, 3,$$

we have asymptotic expression for sufficiently large $|\lambda|$

$$b_i(\lambda) = \sqrt{\frac{2\rho A_{i0}^\frac{2}{\beta_i}}{\beta_i}} \left( \lambda + \frac{K}{2A_{i0}} - \frac{1}{4} \frac{K}{2A_{i0}}^2 \lambda^{-1} + o(\lambda^{-1}) \right).$$

Set

$$\ell_i = \sqrt{\frac{2\rho A_{i0}^\frac{2}{\beta_i}}{\beta_i}}, \quad \delta_i = \frac{k}{2A_{i0}} \sqrt{\frac{2\rho A_{i0}^\frac{2}{\beta_i}}{\beta_i}}.$$
Then
\[ b_i(\lambda) = \ell_i \lambda + \delta_i + O(\lambda^{-1}). \]
Thus we have
\[ \frac{b_i(\lambda)}{b_i(\lambda)} = \frac{\ell_i}{\ell_1} + O(\lambda^{-1}), \quad i = 2, 3, \]
\[ \cosh b_i(\lambda) = \cosh(\ell_i \lambda + \delta_i + O(\lambda^{-1})) = \cosh(\ell_i \lambda + \delta_i) + \sinh(\ell_i \lambda + \delta_i)O(\lambda^{-1}), \]
\[ \sinh b_i(\lambda) = \sinh(\ell_i \lambda + \delta_i + O(\lambda^{-1})) = \sinh(\ell_i \lambda + \delta_i) + \cosh(\ell_i \lambda + \delta_i)O(\lambda^{-1}). \]
When \( \lambda \in \mathbb{C} \) satisfies \(|\Re \lambda| \leq r, \cosh(\ell_i \lambda + \delta_i) \) and \( \sinh(\ell_i \lambda + \delta_i) \) are bounded functions, so we have
\[ \begin{cases} \mathcal{G}(\lambda) = (\ell_2 \lambda + \delta_2) \sinh(\ell_1 \lambda + \delta_1) \sinh(\ell_2 \lambda + \delta_2) \\
+ \eta(\ell_1 \lambda + \delta_1) \cosh(\ell_1 \lambda + \delta_1) \cosh(\ell_2 \lambda + \delta_2) + O(\lambda^{-1}), \end{cases} \]
\[ \mathcal{G}'(\lambda) = [(\ell_2 \lambda + \delta_2)\ell_1 \eta(\ell_1 \lambda + \delta_1) \cosh(\ell_1 \lambda + \delta_1) \cosh(\ell_2 \lambda + \delta_2) \\
+ (\ell_2 \lambda + \delta_2)\ell_1 \eta(\ell_1 \lambda + \delta_1) \ell_2 \cosh(\ell_1 \lambda + \delta_1) \sinh(\ell_2 \lambda + \delta_2) + O(\lambda^{-1}). \]
So the asymptotic zeros of \( \mathcal{G}(\lambda) \) are given by
\[ \sinh(\ell_1 \lambda + \delta_1) \sinh(\ell_2 \lambda + \delta_2) + \eta \frac{\ell_1}{\ell_2} \cosh(\ell_1 \lambda + \delta_1) \cosh(\ell_2 \lambda + \delta_2) = 0. \]
To show the separability of eigenvalues, we consider function equations:
\[ \begin{cases} \sinh(\ell_1 \lambda + \delta_1) \sinh(\ell_2 \lambda + \delta_2) + \eta \frac{\ell_1}{\ell_2} \cosh(\ell_1 \lambda + \delta_1) \cosh(\ell_2 \lambda + \delta_2) = 0, \\
\sinh(\ell_1 \lambda + \delta_1) \cosh(\ell_2 \lambda + \delta_2) + \eta \frac{\ell_1}{\ell_2} \sinh(\ell_1 \lambda + \delta_1) \sinh(\ell_2 \lambda + \delta_2) = 0. \end{cases} \]
Solving the equations yields
\[ \frac{\sinh(\ell_2 \lambda + \delta_2)}{\cosh(\ell_2 \lambda + \delta_2)} = \sqrt{\eta} \sqrt{\frac{\ell_1^2 + \eta \ell_2^2}{\ell_2^2 + \eta \ell_1^2}} = \sqrt{\eta} \xi_1 \]
and
\[ \frac{\sinh(\ell_1 \lambda + \delta_1)}{\cosh(\ell_1 \lambda + \delta_1)} = -\sqrt{\eta} \sqrt{\frac{\ell_1^2 + \eta \ell_2^2}{\ell_2^2 + \eta \ell_1^2}} = -\sqrt{\eta} \xi_2. \]
Thus \( \lambda \in \mathbb{C} \) is of the form
\[ \lambda + \frac{K}{2A_{20}} = \frac{1}{2} \sqrt{\frac{\beta_2}{2\rho A_{20}^2}} \left[ \ln \frac{1 + \xi_1 \sqrt{\eta}}{1 - \xi_1 \sqrt{\eta}} + 2n\pi i \right], \quad n \in \mathbb{R} \]
and
\[ \lambda + \frac{K}{2A_{10}} = \frac{1}{2} \sqrt{\frac{\beta_1}{2\rho A_{10}^2}} \left[ \ln \frac{1 - \xi_2 \sqrt{\eta}}{1 + \xi_2 \sqrt{\eta}} + 2m\pi i \right], \quad m \in \mathbb{R}. \]
Comparing the imaginary parts of \( \lambda \) leads to
\[ \frac{\sqrt{\frac{\beta_2}{2\rho A_{20}^2}}}{\sqrt{\frac{\beta_1}{2\rho A_{10}^2}}} = \frac{m}{n} \]
and comparing the real parts yields
\[ - \frac{K}{2A_{20}} + \frac{1}{2} \sqrt{\frac{\beta_2}{2\rho A_{20}^2}} \ln \frac{1 + \xi_1 \sqrt{\eta}}{1 - \xi_1 \sqrt{\eta}} = - \frac{K}{2A_{10}} + \frac{1}{2} \sqrt{\frac{\beta_1}{2\rho A_{10}^2}} \ln \frac{1 - \xi_2 \sqrt{\eta}}{1 + \xi_2 \sqrt{\eta}}. \] (29)
Therefore, only when the above two equations are satisfied simultaneously, the function equations have a solution $\lambda$. Otherwise, the function equations have no solution. Thus, the zeros are simple and separable.

**Theorem 3.7.** Let $\mathcal{H}$ and $A$ be defined by (10) and (11) respectively, when the two sub-branch vessels are the same for system (15) but (29) does not hold, all eigenvalues of $A$ are separable and only at most finite number of eigenvalues are not simple.

3.3.3. **Three vessels are different.** Finally, we discuss such $\lambda \in \mathbb{C}$ that $\det(d(\lambda)) = 0$ but $d_6(\lambda)d_7(\lambda) \neq 0$ which implies $\cosh b_2(\lambda) \cosh b_3(\lambda) \neq 0$. In this case, by (24), we also have $e^{b_1(\lambda)} - e^{-b_1(\lambda)} \neq 0$. Since

$$b_1(\lambda) = \sqrt{\frac{2\rho A_{56}}{\beta_1}} \left( \lambda^2 + \frac{K}{A_{10}} \right)^{\frac{1}{2}}, \quad i = 1, 2, 3,$$

we have asymptotic expression for sufficiently large $|\lambda|

$$G(\lambda) = \alpha^2 \frac{L_2 b_2(\lambda)}{L_1 b_1(\lambda)} \sinh b_1(\lambda) \sinh b_2(\lambda) \cosh b_3(\lambda) + (1 - \alpha)^2 \frac{L_3 b_3(\lambda)}{L_1 b_1(\lambda)} \sinh b_1(\lambda) \cosh b_2(\lambda) \sinh b_3(\lambda) + \cosh b_1(\lambda) \cosh b_2(\lambda) \cosh b_3(\lambda)$$

$$= \alpha^2 \frac{L_2 \ell_2}{L_1 \ell_1} \sinh(\ell_1 \lambda + \delta_1) \sinh(\ell_2 \lambda + \delta_2) \cosh(\ell_3 \lambda + \delta_3) + (1 - \alpha)^2 \frac{L_3 \ell_3}{L_1 \ell_1} \sinh(\ell_1 \lambda + \delta_1) \cosh(\ell_2 \lambda + \delta_2) \sinh(\ell_3 \lambda + \delta_3) + \cosh(\ell_1 \lambda + \delta_1) \cosh(\ell_2 \lambda + \delta_2) \cosh(\ell_3 \lambda + \delta_3) + O(\lambda^{-1}).$$

Set

$$\tilde{G}(\lambda) = \alpha^2 \frac{L_2 \ell_2}{L_1 \ell_1} \sinh(\ell_1 \lambda + \delta_1) \sinh(\ell_2 \lambda + \delta_2) \cosh(\ell_3 \lambda + \delta_3)$$

$$+ (1 - \alpha)^2 \frac{L_3 \ell_3}{L_1 \ell_1} \sinh(\ell_1 \lambda + \delta_1) \cosh(\ell_2 \lambda + \delta_2) \sinh(\ell_3 \lambda + \delta_3) + \cosh(\ell_1 \lambda + \delta_1) \cosh(\ell_2 \lambda + \delta_2) \cosh(\ell_3 \lambda + \delta_3).$$

(30)

Obviously, the asymptotic zeros of $G(\lambda)$ are determined by $\tilde{G}(\lambda) = 0$. For simplicity, we set

$$\Gamma(\lambda) = \sinh(\ell_1 \lambda + \delta_1) \cosh(\ell_2 \lambda + \delta_2) \cosh(\ell_3 \lambda + \delta_3).$$

Clearly, $\Gamma(\lambda) \neq 0$ as $\tilde{G}(\lambda) = 0$. Then we have

$$F(\lambda) = \frac{\tilde{G}(\lambda)}{\Gamma(\lambda)} = \frac{\cosh(\ell_1 \lambda + \delta_1)}{\sinh(\ell_1 \lambda + \delta_1)} + \alpha^2 \frac{L_2 \ell_2}{L_1 \ell_1} \sinh(\ell_2 \lambda + \delta_2) \cosh(\ell_3 \lambda + \delta_3)$$

$$+(1 - \alpha)^2 \frac{L_3 \ell_3}{L_1 \ell_1} \sinh(\ell_3 \lambda + \delta_3) \cosh(\ell_2 \lambda + \delta_2)$$

and

$$F'(\lambda) = \frac{-\ell_1}{\sinh^2(\ell_1 \lambda + \delta_1)} + \alpha^2 \frac{L_2 \ell_2}{L_1 \ell_1} \frac{\ell_2}{\cosh^2(\ell_2 \lambda + \delta_2)} + (1 - \alpha)^2 \frac{L_3 \ell_3}{L_1 \ell_1} \frac{\ell_3}{\cosh^2(\ell_3 \lambda + \delta_3)}.$$
Obviously, the zeros set of $\Gamma(\lambda)$ is

$$\sigma_\Gamma = \{ \lambda \in \mathbb{C} \mid \lambda = -\delta_1 + \frac{k\pi}{\ell_1}, -\delta_2 + \frac{m\pi}{\ell_2}, -\delta_3 + \frac{n\pi}{\ell_3}, \forall k, m \in \mathbb{Z} \},$$

and $\lambda \in \mathbb{C}\setminus\sigma_\Gamma$ makes $F(\lambda) = 0$ if and only if $\hat{G}(\lambda) = 0$.

We rewrite $F(\lambda)$ as

$$F(\lambda) = \frac{1}{2} \sinh 2(\ell_1 \lambda + \delta_1) + \frac{\alpha^2}{2} \frac{L_2 \ell_2 \sinh 2(\ell_2 \lambda + \delta_2)}{L_1 \ell_1 \cosh^2(\ell_2 \lambda + \delta_2)} + \frac{(1 - \alpha)^2}{2} \frac{L_3 \ell_3 \sinh 2(\ell_3 \lambda + \delta_3)}{L_1 \ell_1 \cosh^2(\ell_3 \lambda + \delta_3)}.$$

For any $\eta \in \mathbb{C}$, we consider the function equations

$$\begin{cases}
\alpha^2 \frac{L_2 \ell_2}{L_1 \ell_1 \cosh^2(\ell_2 \lambda + \delta_2)} \frac{\ell_2}{\sinh^2(\ell_1 \lambda + \delta_1)} = \eta, \\
(1 - \alpha)^2 \frac{L_3 \ell_3}{L_1 \ell_1 \cosh^2(\ell_3 \lambda + \delta_3)} \frac{\ell_3}{\sinh^2(\ell_1 \lambda + \delta_1)} = -(1 + \eta).
\end{cases} \quad \lambda \in \mathbb{C}\setminus\sigma_\Gamma.$$

Obviously, for some $\eta \in \mathbb{C}$, $\lambda \in \mathbb{C}\setminus\sigma_\Gamma$ is a solution of the above equations if and only if $F(\lambda) = 0$ and hence $\hat{G}(\lambda) = 0$. We assume that $\eta \in \mathbb{C}$ makes the function equations have at least one solution. For such an $\eta$, $\lambda$ is a corresponding solution, then

$$\begin{align*}
\alpha^2 \frac{L_2 \ell_2}{L_1 \ell_1 \cosh^2(\ell_2 \lambda + \delta_2)} \frac{\ell_2}{\sinh^2(\ell_1 \lambda + \delta_1)} &= \frac{\eta \ell_2}{\sinh^2(\ell_1 \lambda + \delta_1) \sinh 2(\ell_2 \lambda + \delta_2)}, \\
(1 - \alpha)^2 \frac{L_3 \ell_3}{L_1 \ell_1 \cosh^2(\ell_3 \lambda + \delta_3)} \frac{\ell_3}{\sinh^2(\ell_1 \lambda + \delta_1)} &= \frac{(1 + \eta) \ell_3}{\sinh^2(\ell_1 \lambda + \delta_1) \sinh 2(\ell_3 \lambda + \delta_3)}.
\end{align*}$$

So we have

$$\begin{align*}
\alpha^2 \frac{L_2 \ell_2}{L_1 \ell_1 \cosh^2(\ell_2 \lambda + \delta_2)} + (1 - \alpha)^2 \frac{L_3 \ell_3}{L_1 \ell_1 \cosh^2(\ell_3 \lambda + \delta_3)} &= \frac{\sinh 2(\ell_1 \lambda + \delta_1)}{\sinh^2(\ell_1 \lambda + \delta_1)} \left[ \frac{\eta \ell_2}{\sinh^2(\ell_1 \lambda + \delta_1) \sinh 2(\ell_2 \lambda + \delta_2)} - \frac{(1 + \eta) \ell_3}{\sinh^2(\ell_1 \lambda + \delta_1) \sinh 2(\ell_3 \lambda + \delta_3)} \right] \\
&= \frac{\sinh 2(\ell_1 \lambda + \delta_1)}{\sinh^2(\ell_1 \lambda + \delta_1)} \frac{\eta \ell_2 \sinh 2(\ell_3 \lambda + \delta_3) - (1 + \eta) \ell_3 \sinh 2(\ell_2 \lambda + \delta_2)}{\sinh 2(\ell_2 \lambda + \delta_2) \sinh 2(\ell_3 \lambda + \delta_3)}.
\end{align*}$$

We can assume that $\ell_3 \sinh 2(\ell_2 \lambda + \delta_2) - \ell_2 \sinh 2(\ell_3 \lambda + \delta_3) \neq 0$. Clearly, only if the pair $(\eta, \lambda)$ satisfies the following equality

$$\frac{\eta \ell_2 \sinh 2(\ell_3 \lambda + \delta_3) - (1 + \eta) \ell_3 \sinh 2(\ell_2 \lambda + \delta_2)}{\sinh 2(\ell_2 \lambda + \delta_2) \sinh 2(\ell_3 \lambda + \delta_3)} = \frac{\ell_1}{\sinh 2(\ell_1 \lambda + \delta_1)}$$

or

$$\eta = \frac{\sinh 2(\ell_2 \lambda + \delta_2)[\ell_3 \sinh 2(\ell_1 \lambda + \delta_1) + \ell_1 \sinh 2(\ell_3 \lambda + \delta_3)]}{\sinh 2(\ell_1 \lambda + \delta_1)[\ell_2 \sinh 2(\ell_3 \lambda + \delta_3) - \ell_3 \sinh 2(\ell_2 \lambda + \delta_2)]} = 0,$$

we have $F'(\lambda) = 0$. In this case, it holds that

$$\alpha^2 \frac{L_2 \ell_2}{L_1 \ell_1 \cosh^2(\ell_2 \lambda + \delta_2)} \frac{\ell_2}{\sinh^2(\ell_1 \lambda + \delta_1)} - \frac{\ell_3 \sinh(\ell_1 \lambda + \delta_1) + \ell_1 \sinh 2(\ell_3 \lambda + \delta_3)}{\ell_2 \sinh 2(\ell_1 \lambda + \delta_1) - \ell_3 \sinh 2(\ell_2 \lambda + \delta_2)} = 0. \quad (31)$$

Therefore, we get the result of simplicity and separability in this case.

**Theorem 3.8.** Let $\mathcal{H}$ and $\mathcal{A}$ be defined by (10) and (11) respectively and the three vessels be different for system (15). If $\forall \lambda \in \mathbb{C}$ with $\hat{G}(\lambda) = 0$ makes (31) do not hold, then all eigenvalues of $\mathcal{A}$ are simple and separable.
4. Completeness and Riesz basis property of (generalized) eigenvectors of $\mathcal{A}$. In this section, we shall discuss the completeness and Riesz basis property of (generalized) eigenvectors of $\mathcal{A}$. Firstly, we establish the completeness of the (generalized) eigenvectors of $\mathcal{A}$.

**Lemma 4.1.** Let $\mathcal{H}$ be a Hilbert space. Suppose that $\mathcal{A}_0$ is a skew adjoint operator in $\mathcal{H}$ with compact resolvent, and there exist two positive constants $M$ and $\beta$ such that its resolvent satisfies $\|R(\lambda, \mathcal{A}_0)\| \leq Me^{\beta |\lambda|}$ for all $\lambda$ satisfying $\text{dist}(\lambda, \sigma(\mathcal{A}_0)) \geq \delta$, and $\mathcal{A}_1$ is a bounded linear operator on $\mathcal{H}$. Then the root vectors of operator $\mathcal{A} = \mathcal{A}_0 + \mathcal{A}_1$ is complete in $\mathcal{H}$.

**Proof.** Denote the spectral subspace of $\mathcal{A}$ by

$$Sp(\mathcal{A}) = \text{span}\{ \sum y_k, y_k \in E(\lambda_k, \mathcal{A})\mathcal{H}, \forall \lambda_k \in \sigma(\mathcal{A}) \}$$

(32) where $E(\lambda_k, \mathcal{A})$ is the Riesz projector corresponding to $\lambda_k$. We shall prove $Sp(\mathcal{A}) = \mathcal{H}$.

Let $u_0 \in \mathcal{H}$ such that $u_0 \perp Sp(\mathcal{A})$. Then $R^*(\lambda; \mathcal{A})u_0$ is an $\mathcal{H}$-valued entire function on $\mathbb{C}$. For any $u \in \mathcal{H}$, we denote a scalar function on $\mathbb{C}$ by

$$F(\lambda) = (u, R^*(\lambda; \mathcal{A})u_0).$$

Note that for any $u \in \mathcal{H}$, the equation $(\lambda I - \mathcal{A})f = u$ is solvable if and only if the equation

$$f - R(\lambda, \mathcal{A}_0)\mathcal{A}_1 f = R(\lambda, \mathcal{A}_0)v$$

is solvable. Clearly, when $|\Re \lambda| > ||\mathcal{A}_1||$, it holds that $||R(\lambda, \mathcal{A}_0)\mathcal{A}_1|| < 1$. Hence $\lambda \in \rho(\mathcal{A})$ satisfies

$$||R(\lambda, \mathcal{A})|| \leq \frac{1}{|\Re \lambda| - ||\mathcal{A}_1||}.$$ Therefore, the entire function $F(\lambda)$ satisfies the following properties:

1) Set $\lambda = w + i\tau$, then for $|\Re \lambda| > ||\mathcal{A}_1||$, $|F(\lambda)| \leq \frac{|u||u_0|}{|\Re \lambda| - ||\mathcal{A}_1||}$, which implies $\lim_{|\Re \lambda| \to \infty} F(\lambda) = 0$ and $F(w + i\tau)$ is uniformly bounded for fixed $w \in \mathbb{R}$ with $|w| > ||\mathcal{A}_1||$.

2) $F(\lambda)$ is an entire function of finite exponential type due to $||R(\lambda, \mathcal{A}_0)|| \leq Me^{\beta |\lambda|}$.

Based on the above properties, the Phragmén-Lindelöf Theorem (see [23]) asserts that $F(\lambda)$ is a bounded function on complex plane $\mathbb{C}$, and then the Liouville’s Theorem says that $F(\lambda) \equiv 0$. So $R^*(\lambda; \mathcal{A})u_0 \equiv 0$, which implies $u_0 = 0$. Hence $Sp(\mathcal{A}) = \mathcal{H}$. This completes the proof.

As a direct consequence of Lemma 4.1, we have the following result.

**Theorem 4.2.** Let $\mathcal{H}$, $\mathcal{A}$, $\mathcal{A}_0$ and $\mathcal{A}_1$ be defined by (10), (11), (18) and (19), respectively. Then the root vectors of $\mathcal{A}$ is complete in $\mathcal{H}$.

In what follows, we shall discuss the Riesz basis property of eigenvector and generalized eigenvectors of $\mathcal{A}$. At first we introduce the conception of general divide difference of exponentials.

**Definition 4.3.** Let $\sigma = \{\mu_1, \mu_2, \cdots, \mu_k\}$ be a set of complex numbers. The zero-order divide difference of exponential function $e^{\mu t}$ is defined by $[\mu_1](t) := e^{\mu_1 t}$, and the first-order divide difference is defined by

$$[\mu_1, \mu_2](t) = \frac{e^{\mu_2 t} - e^{\mu_1 t}}{\mu_2 - \mu_1}.$$
If $\mu_1 = \mu_2$, the first-order generalized divide difference

$$[\mu_1, \mu_1](t) = \lim_{z \to \mu_1} \frac{e^{zt} - e^{\mu_1 t}}{z - \mu_1} = te^{\mu_1 t}.$$  

In general, the $k$-order Generalized divide difference is defined by

$$[\mu_1, \mu_2, \cdots, \mu_k](t) = \lim_{z \to \mu_k} \frac{[\mu_1, \mu_2, \cdots, \mu_k](t)[z - \mu_1]}{\mu_k - \mu_k}$$

The following result gives the subspace Riesz basis of the eigenvectors and the family of exponentials (see e.g. [26, 27, 29]).

**Lemma 4.4.** Let $\mathcal{H}$ be a separable Hilbert space, and $A$ be the generator of a $C_0$ semigroup $T(t)$ on $\mathcal{H}$. Suppose that

1. $\sigma(A) = \sigma_1(A) \cup \sigma_2(A)$, where $\sigma_2(A) = \{\lambda_k\}_{k=1}^{\infty}$ consists of isolated eigenvalues of $A$ with finite multiplicity;
2. $\sup_{k \geq 1} m_\alpha(\lambda_k) < \infty$, where $m_\alpha(\lambda_k) = \dim E(\lambda_k, A) \mathcal{H}$ and $E(\lambda_k, A)$ is the Riesz projector associated with $\lambda_k$;
3. There is a constant $\alpha$ such that
   $$\sup \{\Re \lambda \mid \lambda \in \sigma_1(A)\} \leq \alpha \leq \inf \{\Re \lambda \mid \lambda \in \sigma_2(A)\}.$$  
4. The set $\sigma_2(A)$ is a union of finite separate sets.

Then the following assertions are true.

(a) There exists a decomposition for $\sigma(A)$

$$\sigma(A) = \bigcup_{p \in \mathbb{Z}} \Lambda^{(p)},$$

where each $\Lambda^{(p)}$ consists of at most finite eigenvalues of $A$, saying $M^{(p)}$, and $\sup_{p} M^{(p)} < \infty$, and the family of exponentials

$$\mathcal{E}^{(p)} = \{\lambda_1^{(p)}(t), \lambda_2^{(p)}(t), \cdots, \lambda_1^{(p)}, \lambda_2^{(p)}, \cdots, \lambda_{M^{(p)}-1}^{(p)}, \lambda_{M^{(p)}}^{(p)}(t)\}$$

forms a Riesz basis sequence in $L^2(0, T)$ for sufficiently large $T$.

(b) There exist two $T(t)$-invariant closed subspaces $\mathcal{H}_1$ and $\mathcal{H}_2$ with the property that $\sigma(A|_{\mathcal{H}_1}) = \sigma_1(A)$, $\sigma(A|_{\mathcal{H}_2}) = \sigma_2(A)$, $\{E(\lambda_k, A)\mathcal{H}_2\}_{k=1}^{\infty}$ forms a subspace Riesz basis for $\mathcal{H}_2$ (the definition of subspace Riesz basis can be found in [16]), and $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$.

(c) If $\sup_{k \geq 1} \|E(\lambda_k, A)\| < \infty$, then $D(A) \subset \mathcal{H}_1 \oplus \mathcal{H}_2 \subset \mathcal{H}$.

(d) $\mathcal{H}$ has the decomposition

$$\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2 \text{ (topological direct sum)}$$

if and only if $\sup_{n \geq 1} \sum_{k=1}^{n} E(\lambda_k, A)\| < \infty$.

Applying Lemma 4.4 to our model, we have the following result.

**Theorem 4.5.** Let $A$ and $\mathcal{H}$ be defined as before. Then the following statements are true.

1. There is a decomposition $\sigma(A) = \bigcup_{p \in \mathbb{Z}} \Lambda^{(p)}$, where $\Lambda^{(p)}$ has $M^{(p)}$ eigenvalues and the family of exponentials

$$\mathcal{E}^{(p)} = \{\lambda_1^{(p)}(t), \lambda_2^{(p)}(t), \cdots, \lambda_1^{(p)}, \lambda_2^{(p)}, \cdots, \lambda_{M^{(p)}-1}^{(p)}, \lambda_{M^{(p)}}^{(p)}(t)\}$$

2. Suppose that $A|_{\mathcal{H}_1}$ is the first-order generalized divide difference
forms a Riesz basis sequence in $L^2(0, T)$ for $T$ large enough.

(2) There is a sequence of eigenvectors and generalized eigenvectors of $A$ that forms a Riesz basis with parentheses for $H$.

Proof. Set $\sigma_1(A) = \{-\infty\}$ and $\sigma_2(A) = \sigma_p(A)$. Theorem 2.3 says that $A$ generates a $C_0$ semigroup $T(t)$. Lemma 2.2 shows the condition (1) in Lemma 4.4 is satisfied. Theorem 3.6, 3.7 and 3.8 show the condition (2) in Lemma 4.4 is satisfied. In particular, the completeness of the root vectors of $A$ in Theorem 4.2 shows that $H_2$ in Lemma 4.4 equals to $H$. So the results of Theorem 4.5 are true.

Remark 2. In Theorem 4.5, we do not use the exact multiplicity of eigenvalues of $A$. Set $\sigma(A) = \{\lambda_n, n \in \mathbb{Z}^+\}$. Under certain conditions, the eigenvalues of $A$ are simple and separable. In this case, the exponentials $\{e^{\lambda_n t}, n \in \mathbb{Z}^+\}$ forms a Riesz basis sequence in $L^2(0, T)$ for some sufficiently large $T$, and there is a sequence of eigenvectors of $A$ that is a Riesz basis for $H$. These facts will be used in the next section.

5. The exact observability and the exact controllability. In this section, we shall discuss the exact observability and the exact controllability of the system (15) in finite time. We first recall the concepts of the exact observability and exact controllability.

Definition 5.1. (see [19, chapter 6, pp.184]) Let $H$ and $Y$ be Hilbert spaces. Let $A: D(A) \subset H \to H$ be the generator of a $C_0$ semigroup $T(t)$ and $C: D(A) \to Y$ be admissible for $T(t)$. A linear dynamic system

$$
\begin{align*}
\frac{dX(t)}{dt} &= AX(t), \\
X(0) &= x_0, \\
Y(t) &= CX(t)
\end{align*}
$$

is said to be exactly observable in finite time $\tau$ if there exists a positive constant $m_\tau > 0$ such that

$$
\int_0^\tau \|CT(t)x_0\|^2 dt \geq m_\tau \|x_0\|^2, \quad \forall x_0 \in D(A).
$$

Obviously, the mild solution of (33) is $X(t) = T(t)x_0$. If $x_0 \in D(A)$, then $T(t)x_0 \in D(A)$. So $Y(t) = CT(t)x_0$ makes sense. The admissibility of operator $C$ for $T(t)$ means that the operator $\Psi: H \to L^2_{loc}(\mathbb{R}^+, Y)$ defined by

$$
\Psi_t(x_0) = Y(s) = CT(s)x_0, \quad s \in (0, t), \quad x_0 \in D(A)
$$

can be extended to a bounded linear operator from $H$ to $L^2_{loc}(\mathbb{R}^+, Y)$. The exact observability of the system means that the operator $\Psi_\tau$ is injective and has closed range in $L^2((0, \tau), Y)$. Hence, we have

$$
x_0 = \Psi_\tau^{-1}(Y(\cdot)),
$$

that means that we can find out the initial state $x_0$ of the system from the observation $Y(t), t \in (0, \tau)$ and hence for all state $X(t)$. Therefore, the exact observability in finite time is an important property of the system.
Definition 5.2. (see [19, chapter 11, pp.364]) Let $\mathcal{H}$ and $\mathbb{U}$ be Hilbert spaces. Let $\mathcal{A} : \mathcal{D}(\mathcal{A}) \subset \mathcal{H} \to \mathcal{H}$ be the generator of a $C_0$ semigroup $T(t)$ and $\mathcal{B} : \mathbb{U} \to \mathcal{H}_{-1}$ be admissible for $T(t)$. A linear dynamic system

$$
\begin{cases}
\frac{dX(t)}{dt} = AX(t) + Bu(t) \\
X(0) = x_0
\end{cases}
$$

(35)

is said to be exactly controllable in finite time $\tau$ if for any $x_0, x_1 \in \mathcal{H}$, there exists a control function $u(t) \in L^2_{loc}(\mathbb{R}_+, \mathbb{U})$ such that

$$
x_1 = X(\tau) = T(\tau)x_0 + \int_0^\tau T(\tau - s)Bu(s)ds.
$$

(36)

Note that, the admissibility of operator $\mathcal{B}$ for $T(t)$ means that, for each $u(\cdot) \in L^2_{loc}(\mathbb{R}_+, \mathbb{U})$, the operator $\Phi : L^2_{loc}(\mathbb{R}_+, \mathbb{U}) \to \mathcal{H}$ defined by

$$
\Phi_\tau(u) = \int_0^\tau T(t - s)Bu(s)ds
$$

is a bounded linear operator from $L^2_{loc}(\mathbb{R}_+, \mathbb{U})$ to $\mathcal{H}$. Thus the mild solution of (35) is

$$
X(t, u) = T(t)x_0 + \int_0^t T(t - s)Bu(s)ds.
$$

The exact controllability of the system means that for any given final state $x_1$, we can find out a control $u(t)$ such that the mild solution $X(t, u)$ of the system arrives at $x_1$ at finite time $\tau$. So the exact controllability also is an important property of the system.

Remark 3. (see [19, chapter 11, pp.365]) Usually it is difficult to verify (36). Note that (36) implies that the range of operator $\Phi_\tau$ is the whole space $\mathcal{H}$. By the duality theory, we can transfer the controllability into the observability of the dual system. Then the system (35) is exactly controllable in finite time $\tau$ if and only if its dual system

$$
\begin{cases}
\frac{dZ(t)}{dt} = A^*Z(t), \\
Z(0) = x_0, \\
Y(t) = B^*Z(t)
\end{cases}
$$

(37)

is exactly observable in finite time $\tau$, or equivalently, there exists a positive constant $m_\tau$ such that

$$
\int_0^\tau ||B^*T^*(t)z_0||^2 dt \geq m_\tau ||z_0||^2, \forall z_0 \in \mathcal{D}(A^*).
$$

5.1. The exact observability of system (15). In this subsection, we shall discuss the exact observability of system (15) in finite time. Without loss of generality, we assume that all eigenvalues of $\mathcal{A}$ are separable and simple(also available for at most finite number non-simple eigenvalues). As pointed in Remark 2, for sufficiently large $\tau$, the family of exponentials $\{e^{\lambda_n t}, n \in \mathbb{Z}\}$ forms a Riesz basis sequence in $L^2(0, \tau)$, and there exists a sequence of eigenvectors of $\mathcal{A}$, saying $\{\Phi_n(x, \lambda_n), n \in \mathbb{Z}\}$, which forms a Riesz basis for $\mathcal{H}$.

The property of the Riesz basis sequence $\{e^{\lambda_n t}, n \in \mathbb{Z}\}$ in $L^2(0, \tau)$ for sufficiently large $\tau$ implies that there exist two positive constants $m_\tau$ and $M_\tau$ such that

$$
m_\tau \sum_{n \in \mathbb{Z}} |a_n|^2 \leq \int_0^\tau \left| \sum_{n \in \mathbb{Z}} a_n e^{\lambda_n t} \right|^2 dt \leq M_\tau \sum_{n \in \mathbb{Z}} |a_n|^2.
$$
And the Riesz basis property of eigenvectors \( \{\Phi_n(x, \lambda_n), n \in \mathbb{Z}\} \) of \( A \) means that for each \( F = (f, g) \in \mathcal{H}, \)

\[
F = \sum_{n \in \mathbb{Z}} a_n \Phi_n(x), \quad a_n = (F, \Psi_n(\lambda_n)),
\]

where the sequence \( \{\Psi_n(x, \lambda_n), n \in \mathbb{Z}\} \) is bi-orthogonal to \( \{\Phi_n(x, \lambda_n), n \in \mathbb{Z}\} \) and there exist \( m_1 \) and \( m_2 \) such that

\[
m_1 \sum_{n \in \mathbb{Z}} |a_n|^2 ||\Phi_n||^2 \leq ||F||^2 \leq m_2 \sum_{n \in \mathbb{Z}} |a_n|^2 ||\Phi_n||^2. \tag{38}
\]

Now let \( A \) be defined by in (11) and \( T(t) \) be the semigroup generated by \( A \). The \( C \) is the observation operator defined in Section 2. Then \( T(t) \) has an analytic expression as

\[
T(t)F = \sum_{n \in \mathbb{Z}} a_n e^{\lambda_n t} \Phi_n(x), \quad \forall F \in \mathcal{H}.
\]

In particular, for \( F \in D(A) \), the observation \( Y(t) \) satisfy

\[
Y(t) = CT(t)F = \sum_{n \in \mathbb{Z}} a_n e^{\lambda_n t} C \Phi_n(x).
\]

Using the result of [28, Lemma 1, pp.971], we have

\[
m_\tau \sum_{n \in \mathbb{Z}} |a_n|^2 ||C \Phi_n||^2 \leq \int_0^\tau ||Y(t)||^2 dt \leq M_\tau \sum_{n \in \mathbb{Z}} |a_n|^2 ||C \Phi_n||^2, \quad 0 < m_\tau < M_\tau < +\infty. \tag{39}
\]

Comparing (36), (38) and (39), the key point of proving the exact observability is to find out a constant \( \zeta \in \mathbb{R}^+ \) such that \( ||C \Phi_n||_X^2 \geq \zeta ||\Phi_n||^2 \) for all eigenvectors \( \Phi_n \).

Note that we have obtained the expression of eigenvectors of \( A \) in subsection 3.2, whose components are given by (27) and (28), respectively. Set the eigenvector corresponding to \( \lambda_n \) be

\[
\Phi_n = \left( \begin{array}{c} P(x, \lambda_n) \\ Q(x, \lambda_n) \end{array} \right).
\]

Then we have

\[
C \Phi_n = \left( \begin{array}{c} P_1(0, \lambda_n) \\ Q_2(1, \lambda_n) \\ Q_3(1, \lambda_n) \end{array} \right).
\]

Since for the case \( d_6(\lambda) = 0, d_7(\lambda) \neq 0 \) or the case \( d_6(\lambda) \neq 0, d_7(\lambda) = 0 \), there is at most finite number eigenvalues of \( A \) satisfying the conditions, so we can choose \( \zeta_1 \in \mathbb{R}^+ \) such that

\[
||C \left( \begin{array}{c} P(\lambda) \\ Q(\lambda) \end{array} \right)||^2 \geq \zeta_1 ||(P(\lambda), Q(\lambda))||^2, \quad d_6(\lambda) = 0 \quad \text{or} \quad d_7(\lambda) = 0.
\]

Therefore, we only need to consider the cases that \( d_6(\lambda) = d_7(\lambda) = 0 \) and \( d_6(\lambda) \neq 0, d_7(\lambda) \neq 0. \)
5.1.1. $d_6(\lambda) \neq 0, d_7(\lambda) \neq 0$. We shall consider the case $d_6(\lambda) \neq 0, d_7(\lambda) \neq 0$ first. Note that

$$d_1(\lambda) = \cosh b_1(\lambda), \quad d_2(\lambda) = -\alpha, \quad d_3(\lambda) = -(1 - \alpha),$$

$$d_4(\lambda) = \alpha \frac{(\lambda + N_2) \sinh b_2(\lambda) M_1 b_1(\lambda) \sinh b_1(\lambda)}{M_2 b_2(\lambda),}$$

$$d_5(\lambda) = (1 - \alpha) \frac{(\lambda + N_2) \sinh b_3(\lambda) M_1 b_3(\lambda) \sinh b_1(\lambda)}{M_2 b_3(\lambda),}$$

$$d_6(\lambda) = \cosh b_2(\lambda), d_7(\lambda) = \cosh b_3(\lambda).$$

A solution of (26) is

$$p(\lambda) = \left(\begin{array}{c} 1 \\ \frac{-d_4(\lambda)}{d_6(\lambda)} \\ \frac{-d_5(\lambda)}{d_7(\lambda)} \end{array}\right) = \left(\begin{array}{c} 1 \\ -\alpha L_2 b_2(\lambda) \sinh b_1(\lambda) \sinh b_2(\lambda) \\ -(1 - \alpha) L_2 b_1(\lambda) \cosh b_1(\lambda) \sinh b_2(\lambda) \sinh b_3(\lambda) \cosh b_1(\lambda) \cosh b_3(\lambda) \end{array}\right).$$

Substituting this expression into (27) and (28) yields

$$P(x, \lambda) = \left(\begin{array}{c} \alpha \sinh(x b_2(\lambda) \sinh(b_1(\lambda)) L_2 b_2(\lambda) \cosh(x b_1(\lambda)) \\ \alpha L_2 b_2(\lambda) \sinh(b_1(\lambda)) L_2 b_2(\lambda) \sinh(x b_2(\lambda)) \\ (1 - \alpha) \sinh(x b_3(\lambda) \sinh(b_1(\lambda)) L_3 b_3(\lambda) \cosh(x b_3(\lambda)) \end{array}\right) = \left(\begin{array}{c} \alpha \sinh(b_3(\lambda)) L_2 b_2(\lambda) \\ \alpha \sinh(b_1(\lambda)) L_2 b_2(\lambda) \cosh(x b_1(\lambda)) \sinh((x - 1) b_2(\lambda)) \\ \alpha \sinh(b_1(\lambda)) L_2 b_2(\lambda) \cosh(x b_1(\lambda)) \sinh((x - 1) b_3(\lambda)) \end{array}\right)$$

and

$$Q(x, \lambda) = \left(\begin{array}{c} \frac{-\lambda}{L_1 b_1(\lambda)} \sinh(x b_3(\lambda)) \\ -\alpha \cosh(x b_2(\lambda)) \sinh(b_1(\lambda)) L_1 b_1(\lambda) \cosh(x b_1(\lambda)) \\ (\alpha - 1) \cosh(x b_2(\lambda)) \sinh(b_1(\lambda)) L_1 b_1(\lambda) \cosh(x b_1(\lambda)) \end{array}\right) = \left(\begin{array}{c} -\frac{\lambda}{L_1 b_1(\lambda)} \sinh(x b_1(\lambda)) \\ -\frac{\lambda}{L_1 b_1(\lambda)} \sinh(x b_1(\lambda)) \cosh((x - 1) b_2(\lambda)) \\ -\frac{\lambda}{L_1 b_1(\lambda)} \sinh(x b_1(\lambda)) \cosh((x - 1) b_3(\lambda)) \end{array}\right).$$

Hence

$$P(0, \lambda) = \left(\begin{array}{c} P_1(0) \\ P_2(0) \\ P_3(0) \end{array}\right) = \left(\begin{array}{c} \frac{1}{\alpha \sinh(b_2(\lambda)) L_2 b_2(\lambda)} \sinh(b_2(\lambda)) \\ \frac{1}{\alpha \sinh(b_3(\lambda)) L_2 b_2(\lambda)} \sinh(b_3(\lambda)) \end{array}\right).$$
and

\[
Q(1, \lambda) = \begin{pmatrix}
Q_1(1) \\
Q_2(1) \\
Q_3(1)
\end{pmatrix} = \begin{pmatrix}
-\frac{\lambda b_1(\lambda)}{L_1 b_1(\lambda)} \sinh(b_1(\lambda)) \\
\alpha \lambda \sinh b_1(\lambda) \\
\frac{\alpha \lambda \sinh b_1(\lambda)}{L_1 b_1(\lambda)} \cosh b_1(\lambda) \\
-\frac{\lambda b_1(\lambda)}{L_1 b_1(\lambda)} \sinh(b_1(\lambda)) \\
\alpha \lambda \sinh b_1(\lambda) \\
\frac{\alpha \lambda \sinh b_1(\lambda)}{L_1 b_1(\lambda)} \cosh b_1(\lambda)
\end{pmatrix}.
\]

Thus the norm of the eigenvector \( (P(x, \lambda), Q(x, \lambda))^T \) is

\[
norm{\begin{pmatrix}
P \\
Q
\end{pmatrix}}_\mathcal{H}^2 = \int_0^1 (L^{-1} P(x), P(x))_{\mathcal{C}^3} + (M^{-1} Q(x), Q(x))_{\mathcal{C}^3} dx
\]

\[
= \int_0^1 L_1^{-1} |P_1(0)|^2 \cosh(x b_1(\lambda)) |^2 + M_1^{-1} |\frac{\lambda}{L_1 b_1(\lambda)} |^2 \sinh(x b_1(\lambda)) |^2 dx
\]

\[
+ \int_0^1 L_2^{-1} \frac{\alpha \lambda b_2(\lambda) \sinh b_1(\lambda)}{L_1 b_1(\lambda) \cosh b_2(\lambda)} \sinh((x-1) b_2(\lambda)) |^2 dx
\]

\[
+ \int_0^1 L_3^{-1} \frac{\alpha \lambda b_3(\lambda) \sinh b_1(\lambda)}{L_1 b_1(\lambda) \cosh b_3(\lambda)} \sinh((x-1) b_3(\lambda)) |^2 dx,
\]

and the norm of observation \( C(P, Q) \) is given by

\[
norm{C \begin{pmatrix}
P \\
Q
\end{pmatrix}}_\mathcal{V}^2 = M_2^{-1} |Q_2(1)|^2 + M_3^{-1} |Q_3(1)|^2 + L_1^{-1} |P_1(0)|^2.
\]

Note that

\[
\lim_{|\lambda| \to \infty} M_1^{-1} |\frac{\lambda}{L_1 b_1(\lambda)} |^2 = L_1^{-1} |P_1(0)|^2,
\]

\[
\lim_{|\lambda| \to \infty} L_2^{-1} \frac{\alpha \lambda b_2(\lambda) \sinh b_1(\lambda)}{L_1 b_1(\lambda) \cosh b_2(\lambda)} |^2 = M_2^{-1} \frac{\alpha \lambda b_1(\lambda)}{L_1 b_1(\lambda) \cosh b_2(\lambda)} |^2 = M_2^{-1} |Q_2(1)|^2,
\]

\[
\lim_{|\lambda| \to \infty} L_3^{-1} \frac{\alpha \lambda b_3(\lambda) \sinh b_1(\lambda)}{L_1 b_1(\lambda) \cosh b_3(\lambda)} |^2 = M_3^{-1} \frac{\alpha \lambda b_1(\lambda)}{L_1 b_1(\lambda) \cosh b_3(\lambda)} |^2 = M_3^{-1} |Q_3(1)|^2.
\]

Then in order to verify observability inequality, we only need to successively compare formulas \((40), (41), (42)\) with \(L_1^{-1} |P_1(0)|^2, M_2^{-1} |Q_2(1)|^2, M_3^{-1} |Q_3(1)|^2\).

Clearly, we can choose a constant \(\omega_1 \in \mathbb{R}^+\) such that \((40)\) has estimate

\[
\int_0^1 L_1^{-1} |P_1(0)|^2 \cosh(x b_1(\lambda)) |^2 + M_1^{-1} |\frac{\lambda}{L_1 b_1(\lambda)} |^2 \sinh(x b_1(\lambda)) |^2 dx
\]

\[
\leq \omega_1 \int_0^1 L_1^{-1} |P_1(0)|^2 \cosh(x b_1(\lambda) + \overline{b_1(\lambda)}) dx
\]

\[
= \omega_1 \left\{ L_1^{-1} |P_1(0)|^2, b_1(\lambda) + \overline{b_1(\lambda)} = 0 \\
L_1^{-1} |P_1(0)|^2 |\frac{\sinh(x b_1(\lambda) + \overline{b_1(\lambda)})}{b_1(\lambda) + \overline{b_1(\lambda)}}|, b_1(\lambda) + \overline{b_1(\lambda)} \neq 0\right\};
\]

\((41)\) has estimate

\[
\int_0^1 L_2^{-1} \frac{\alpha \lambda b_2(\lambda)}{L_1 b_1(\lambda) \cosh(b_2(\lambda))} \sinh((x-1)b_2(\lambda)) |^2
\]
and \((42)\) has estimate

\[
\int_0^1 L_3^{-1} \left[ \frac{\alpha \sinh(b_1(\lambda)) L_3 b_3(\lambda)}{L_1 b_1(\lambda) \cosh(b_1(\lambda))} \right] \sinh((x - 1)b_3(\lambda)) dx.
\]

It is easy to check that

\[
\frac{\sinh(b_j + \beta_j)}{b_j(\lambda) + \beta_j(\lambda)} = 1, \quad i = 1, 2, 3.
\]

And recall that Theorem 3.2 asserts that there exist two constants \(h_1, h_2 \in \mathbb{R}\) such that

\[
h_1 \leq b_1(\lambda) + \beta_1(\lambda), \quad b_2(\lambda) + \beta_2(\lambda), \quad b_3(\lambda) + \beta_3(\lambda) \leq h_2.
\]

Thus there exists a constant \(\varsigma \in \mathbb{R}^+\) such that

\[
0 < \frac{\sinh(b_j + \beta_j)}{b_j(\lambda) + \beta_j(\lambda)} \leq \varsigma, \quad j = 1, 2, 3, \quad \forall \lambda \in \sigma(A).
\]

Therefore, there exists a constant \(\varsigma_2 \in (0, \omega_1 \varsigma)\), such that

\[
\|C \left( \begin{array}{c} P(\lambda) \\ Q(\lambda) \end{array} \right) \|_Y^2 \geq \varsigma_2 \|P(\lambda), Q(\lambda)\|^2, \quad \forall \lambda \text{ satisfying } d_6(\lambda) \neq 0, d_7(\lambda) \neq 0.
\]  \hfill (43)

### 5.1.2. \(d_6(\lambda) = 0, d_7(\lambda) = 0\).

In this case, a solution to (26) is

\[
p(\lambda) = \begin{pmatrix} p_1(\lambda) \\ p_2(\lambda) \\ p_3(\lambda) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ -d_2(\lambda)/d_3(\lambda) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ \frac{\alpha}{1 - \alpha} \end{pmatrix}.
\]

Substituting it into (27) and (28) yields

\[
P(x, \lambda) = \begin{pmatrix} \cosh(xb_1(\lambda)) p_1(\lambda) \\ \cosh(xb_2(\lambda)) p_2(\lambda) + \alpha \sinh(xb_2(\lambda)) \sinh(b_1(\lambda)) \frac{L_2 b_1(\lambda)}{L_1 b_1(\lambda)} p_1(\lambda) \\ \cosh(xb_3(\lambda)) p_3(\lambda) + (1 - \alpha) \sinh(xb_3(\lambda)) \sinh(b_1(\lambda)) \frac{L_1 b_1(\lambda)}{L_3 b_1(\lambda)} p_1(\lambda) \end{pmatrix}
\]

\[
= \begin{pmatrix} 0 \\ \cosh(xb_2(\lambda)) \frac{\alpha}{1 - \alpha} \\ -\cosh(xb_3(\lambda)) \frac{\alpha}{1 - \alpha} \end{pmatrix}.
\]
and

\[ Q(x, \lambda) = \begin{pmatrix}
0 \\
-\frac{M_2b_2(\lambda)}{\alpha M_2b_2(\lambda)} \sinh(xb_2(\lambda)) \\
-\frac{M_2b_3(\lambda)}{(1-\alpha)(\lambda+N_3)} \sinh(xb_3(\lambda))
\end{pmatrix}. \]

Hence

\[ P(0, \lambda) = \begin{pmatrix}
P_1(0) \\
P_2(0) \\
P_3(0)
\end{pmatrix} = \begin{pmatrix}
0 \\
1 \\
-\frac{\alpha}{1-\alpha}
\end{pmatrix}, \]

and

\[ Q(1, \lambda) = \begin{pmatrix}
Q_1(1) \\
Q_2(1) \\
Q_3(1)
\end{pmatrix} = \begin{pmatrix}
0 \\
-\frac{M_2b_2(\lambda)}{\alpha M_2b_2(\lambda)} \sinh(b_2(\lambda)) \\
-\frac{M_2b_3(\lambda)}{(1-\alpha)(\lambda+N_3)} \sinh(b_3(\lambda))
\end{pmatrix}. \]

The norm of \((P(x), Q(x))^T\) is

\[
\left\| \begin{pmatrix} P \\ Q \end{pmatrix} \right\|_H^2 = \int_0^1 \left( L^{-1} P(x), P(x) \right)_{C^3} + \left( M^{-1} Q(x), Q(x) \right)_{C^3} \, dx
\]

\[
= \int_0^1 L_2^{-1} |\cosh(xb_2(\lambda))|^2 + M_2^{-1} \frac{M_2b_2(\lambda)}{\lambda + N_2} \sinh(xb_2(\lambda))|^2 \, dx \tag{44}
\]

\[
+ \int_0^1 L_3^{-1} \frac{M_3b_3(\lambda)}{(1-\alpha)(\lambda+N_3)} \sinh(xb_3(\lambda))|^2 \, dx. \tag{45}
\]

Note that \(d_6(\lambda) = 0, d_7(\lambda) = 0\) imply \(e^{b_2(\lambda)} = \pm i, e^{b_3(\lambda)} = \pm i\), and

\[
\lim_{|\lambda| \to \infty} M_2^{-1} \frac{|Q_2(1)|^2}{2i} = \lim_{|\lambda| \to \infty} M_2^{-1} \frac{M_2b_2(\lambda)}{\lambda + N_2} \sinh(xb_2(\lambda))|^2 = L_2^{-1},
\]

\[
\lim_{|\lambda| \to \infty} M_3^{-1} \frac{M_3b_3(\lambda)}{(1-\alpha)(\lambda+N_3)} \sinh(xb_3(\lambda))|^2 = L_3^{-1}. \]

So we can choose a constant \(\omega_2 \in \mathbb{R}^+\) such that \((44)\) has estimate

\[
\int_0^1 L_2^{-1} |\cosh(xb_2(\lambda))|^2 + M_2^{-1} \frac{M_2b_2(\lambda)}{\lambda + N_2} \sinh(xb_2(\lambda))|^2 \, dx
\]

\[
\leq \omega_2 \int_0^1 M_2^{-1} \frac{|Q_2(1)|^2}{2i} \cosh(x(b_2(\lambda) + b_2(\lambda))) \, dx
\]

\[
= \omega_2 \begin{cases}
\frac{|Q_2(1)|^2}{M_2}, & b_2(\lambda) + b_2(\lambda) = 0, \\
\frac{|Q_2(1)|^2}{M_2} \left[ \frac{\sinh(b_2(\lambda) + b_2(\lambda))}{(b_2(\lambda) + b_2(\lambda))} \right], & b_2(\lambda) + b_2(\lambda) \neq 0
\end{cases}
\]

and \((45)\) has estimate

\[
\int_0^1 L_3^{-1} \frac{M_3b_3(\lambda)}{(1-\alpha)(\lambda+N_3)} \sinh(xb_3(\lambda))|^2 \, dx
\]

\[
\leq \omega_2 \int_0^1 M_3^{-1} \frac{|Q_3(1)|^2}{2i} \cosh(x(b_3(\lambda) + b_3(\lambda))) \, dx
\]

\[
= \omega_2 \begin{cases}
\frac{|Q_3(1)|^2}{M_3}, & b_3(\lambda) + b_3(\lambda) = 0, \\
\frac{|Q_3(1)|^2}{M_3} \left[ \frac{\sinh(b_3(\lambda) + b_3(\lambda))}{(b_3(\lambda) + b_3(\lambda))} \right], & b_3(\lambda) + b_3(\lambda) \neq 0.
\end{cases}
\]
While the norm of $C(P, Q)^T$ is
\[
\|C\left( \begin{array}{c} P \\ Q \end{array} \right) \|_{Y}^2 = M_2^{-1}|Q_2(1)|^2 + M_3^{-1}|Q_3(1)|^2.
\]
By a similar discussion, we can find out a constant $\zeta_3 \in \mathbb{R}^+$ such that
\[
\|C\left( \begin{array}{c} P \\ Q \end{array} \right) \|_{Y}^2 \geq \zeta_3\| \left( \begin{array}{c} P \\ Q \end{array} \right) \|_2^2, \forall \lambda \text{ satisfying } d_0(\lambda) = d_T(\lambda) = 0. \tag{46}
\]
Summarizing discussion above, we can choose a constant $\zeta \in (0, \min \{\zeta_1, \zeta_2, \zeta_3\})$ such that
\[
\|C\Phi_n\|_{Y}^2 \geq \zeta\|\Phi_n\|^2
\]
for all eigenvalue $\lambda_n$ of $A$. Substituting the above inequality into (39) and using (38), we get
\[
\int_0^T \|Y(t)\|_{Y}^2 dt \geq \zeta m r \sum_n \|a_{nn}\|^2 \|\Phi_n\|^2 \geq \zeta \frac{m r}{m_2} ||F||^2.
\]
So the system (15) is exactly observable in finite time $\tau$.

5.2. The exact controllability in finite time. In this subsection, we shall discuss the exact controllability of system (15) in finite time. Thanks to the duality, we only need to discuss the exact observability of the dual system (37). We complete the proof by the following three steps.

**Step 1.** Calculating the conjugate operator $B^*$.

Firstly we notice that $L^2_{loc}(\mathbb{R}^+, \mathbb{C}^3) = L^2_{loc}(\mathbb{R}^+, \mathbb{C}^3)$ with the norm
\[
\|U\|_{L^2_{loc}}^2 = \int_0^T M_1|u_1(t)|^2 + L_2|u_2(t)|^2 + L_3|u_3(t)|^2 dt, \quad U(t) = (u_1(t), u_2(t), u_3(t)). \tag{47}
\]
According to the definition of $B$ (see, section 2), we have for all $(\varphi(x), \psi(x)) \in D(A^*)$ and $U = (u_1, u_2, u_3) \in \mathbb{C}^3$,
\[
\langle B^*(\varphi, \psi), U \rangle_{Y^*, Y} = ((\varphi, \psi), BU)_{H^*_{-1}, H_{-1}} = \int_0^1 u_1 \varphi_1(x)\delta(x) - u_2\delta(x-1)\psi_2(x) - u_3\delta(x-1)\psi_3(x) dx,
\]
where $H_{-1} = \frac{D(A), ||R(\lambda_1, A)^* ||}{|| \lambda \lambda_1 \frac{A^*}{2, (A^*)^*} ||}$, $H^*_{-1} = \frac{D(A^*), ||(\lambda_2 I - A^*)^* ||}{|| \lambda \lambda_2 \frac{A}{2, (A^*)^*} ||}$, and $\lambda_1, \lambda_2$ are the resolvent point of operator $A$ and $A^*$, respectively.

So, $B^* : H^*_{-1} \to Y^* = \mathbb{C}^3$ is given by
\[
B^*(\varphi(x), \psi(x)) = (\varphi_1(0), -\psi_2(1), -\psi_3(1))^T \in \mathbb{C}^3, \quad (\varphi(x), \psi(x)) \in H^*_{-1}.
\]

**Step 2.** Calculating the eigenvectors of $A^*$.

Since $H$ is a Hilbert space, we have $\sigma(A^*) = \overline{\sigma(A)} = \sigma(A)$. Notice that
\[
A^*(\varphi(x), \psi(x))^T = (L\psi'(x), M\varphi'(x))^T - (0, N\psi(x))^T
\]
\[
D(A^*) = \{ (\varphi, \psi) \in H \mid \varphi', \psi' \in [L^2(0, 1)]^3; \varphi(1) = C^T \varphi(0), \psi(0) = C \psi(1) \}.
\]

Then $D(A) = D(A^*)$ and $\sigma(A^*) = \sigma(A^*)$. Let $\mu \in \mathbb{C}$ be an eigenvalue of $A^*$ and $\Phi^*(x, \mu) = (\varphi(x), \psi(x))^T$ be a corresponding eigenvector. Then we have
\[
\begin{cases}
L \psi'(x) = \mu \varphi(x), x \in (0, 1), \\
M \varphi'(x) = (\mu I + N) \psi(x), x \in (0, 1), \\
\psi(0) = C \psi(1), \varphi(1) = C^T \varphi(0).
\end{cases}
\]
Compare this equation with (20), we see that if $\psi(x, \mu)$ satisfies the equation

$$ML\psi''(x) = \mu(\mu + N)\psi(x), \quad x \in (0, 1)$$

and boundary condition

$$\psi(0) = C\psi(1), \quad Lu'(1) = CTu'(0),$$

then the pair $(\varphi, \psi)$ is an eigenvector of $A^*$ corresponding to $\mu$ where $\varphi(x) = \mu^{-1}Lu'(x)$, and the pair $(\varphi, -\psi)$ is an eigenvector of $A$ corresponding to $\mu$.

Similar to the discussion in the previous section, we can get that if $\varphi(0) = (\varphi_1(0), \varphi_2(0), \varphi_3(0))$ satisfies the algebraic equation $d(\mu)\varphi(0) = 0$, i.e.,

$$\begin{align*}
d_1(\mu)&\varphi_1(0) + d_2(\mu)\varphi_2(0) + d_3(\mu)\varphi_3(0) = 0, \\
d_4(\mu)&\varphi_1(0) + d_6(\mu)\varphi_2(0) = 0, \\
d_5(\mu)&\varphi_1(0) + d_7(\mu)\varphi_3(0) = 0,
\end{align*}$$

then $\psi(0) = H^*(\mu)\varphi(0)$ where

$$H^*(\mu) = \begin{pmatrix}
0 & 0 & 0 \\
\alpha & 0 & 0 \\
0 & 1 - \alpha & 0
\end{pmatrix}.$$

The eigenvector $(\varphi(x, \mu), \psi(x, \mu))^T$ is

$$\varphi(x, \mu) = \begin{pmatrix}
\cosh(xb_1(\mu))\varphi_1(\mu) \\
\cos(xb_2(\mu))\varphi_2(\mu) + \alpha \sinh(xb_2(\mu)) \sinh(b_1(\mu)) \frac{L \sinh(\mu)}{L \sinh(b_1(\mu))} \varphi_1(\mu) \\
\cosh(xb_3(\mu))\varphi_3(\mu) + (1 - \alpha) \sinh(xb_3(\mu)) \sinh(b_1(\mu)) \frac{L \sinh(\mu)}{L \sinh(b_1(\mu))} \varphi_1(\mu)
\end{pmatrix},$$

$$\psi(x, \mu) = \begin{pmatrix}
\frac{\alpha \cosh(xb_2(\mu))M_1}{\mu + N_1} \sinh(xb_1(\mu))\varphi_1(\mu) + \frac{M_2}{\mu + N_2} \sinh(xb_2(\mu))\varphi_2(\mu) + \frac{M_3}{\mu + N_3} \sinh(xb_3(\mu))\varphi_3(\mu)
\end{pmatrix},$$

where $(\varphi_1(\mu), \varphi_2(\mu), \varphi_3(\mu))$ is a solution to (48).

**Step 3.** Verifying the controllability, i.e., the observation inequality of the dual system (37).

Similarly, we only need to estimate a constant $\zeta^* \in \mathbb{R}^+$ such that $\|B^*\Phi^*(x, \mu)\|_2^2 \geq \zeta^* \|\Phi^*(x, \mu)\|_2^2$ for any eigenvalue $\mu$ and its eigenvector $\Phi^*(x, \mu)$.

We note that for any $\lambda \in \sigma_p(A)$, $p(\lambda) = (p_1(\lambda), p_2(\lambda), p_3(\lambda))$ is a solution to (48), the vector $(P(x, \lambda), Q(x, \lambda))$ is an eigenvector of $A$, and $(\varphi(x, \lambda), \psi(x, \lambda)) = (P(x, \lambda), -Q(x, \lambda))$ is an eigenvector of $A^*$. Therefore,

$$\|P(x, \lambda), Q(x, \lambda)\| = \|(\varphi(x, \lambda), \psi(x, \lambda))^T\|,$$

and $C = B^*$. Using the estimates about $C$ we can get the estimates of $B^*(\varphi, \psi)$. So there exists a constant $\zeta^* \in \mathbb{R}^+$ such that

$$\|B^*(\varphi(x, \lambda), \psi(x, \lambda))^T\|_2^2 \geq \zeta^* \|\varphi(x, \lambda), \psi(x, \lambda)^T\|_2^2, \quad \forall \lambda \in \sigma(A^*).$$

So the dual system (37) is exactly observable and then the system (15) is exactly controllable in finite time.

**Remark 4.** (14) and (47) show that the norms of $L_{loc}^2(\mathbb{R}^+, U^*)$ and $L_{loc}^2(\mathbb{R}^+, \mathbb{Y})$ are equivalent. So the controllability follows from the observability in finite time.

Recall the results of the simplicity and separability of the eigenvalues of the system operator $A$ in Theorem 3.6, Theorem 3.7 and Theorem 3.8. Then we have the following result.
Theorem 5.3. The system (15) are exactly controllable and exactly observable in finite time in the following three cases:

(1) The three vessels are the same;
(2) The two sub-branch vessels are the same but (29) does not hold;
(3) The three vessels are different and if $\forall \lambda \in \mathbb{C}$ with $\hat{G}(\lambda) = 0$ makes (31) do not hold.

6. Conclusion. In the previous several sections, we have carried out a complete system analysis for a linearization blood flow network described by (8) and (9). We have proved that the system is $L^2$ well-posed, exactly observable and exactly controllable in finite time. Note that this system is the simplest model of blood flow network, and we can extend this method to more complicated ones. The key point in the proof of observability is to show the Riesz basis sequence property of exponential family of eigenvalues for the corresponding system. And the main difficulty we encounter is the proof of separability of eigenvalues of the system operator. In the future, we shall study the observability and the controllability of more complicated blood flow networks.

Appendix A: The proof of Theorem 2.5.

Proof. Let $P_i(x, t)$ and $Q_i(x, t)$, $i = 1, 2, 3$, satisfy equations

\[
\begin{aligned}
    P_i(x, t) + L_i Q_i(x, t) &= 0, \quad x \in (0, 1), t > 0, \\
    Q_i(x, t) + M_i P_i(x, t) + N_i Q_i(x, t) &= 0, \quad x \in (0, 1), t > 0, \\
    Q_2(0, t) &= \alpha Q_1(1, t), \quad Q_3(0, t) = (1 - \alpha)Q_1(1, t), \\
    P_1(t) &= \alpha P_2(0, t) + (1 - \alpha)P_3(0, t), \\
    Q_1(0, t) &= u_1(t), \quad P_2(1, t) = u_2(t), \quad P_3(1, t) = u_3(t) \\
    y_1(t) &= P_1(0, t), \quad y_2(t) = Q_2(1, t), \quad y_3(t) = Q_3(1, t)
\end{aligned}
\]  

(49)

For $i = 1$, multiplying $\frac{(x-1)Q_i(x,t)}{L_i M_i}$ to the first equation and $\frac{(x-1)P_i}{L_i M_i}$ to the second equation in (49) respectively, and then integrating over $[0, 1] \times [0, T]$, we find

\[
\begin{aligned}
    \int_0^T \int_0^1 (x - 1)Q_1(x, t) \left( \frac{P_{1,x}(x, t)}{L_1 M_1} + \frac{Q_{1,x}(x, t)}{M_1} \right) dx dt &= 0 \tag{50} \\
    \int_0^T \int_0^1 (x - 1)P_1(x, t) \left( \frac{Q_{1,x}(x, t)}{L_1 M_1} + \frac{P_{1,x}(x, t)}{L_1} + \frac{N_1 Q_1(x, t)}{L_1 M_1} \right) dx dt &= 0. \tag{51}
\end{aligned}
\]

For $i \in \{2, 3\}$, multiplying $\frac{x Q_i(x, t)}{L_i M_i}$ to the first equation and $\frac{x P_i(x, t)}{L_i M_i}$ to the second equation in (49), respectively, and then integrating over $[0, 1] \times [0, T]$, we get

\[
\begin{aligned}
    \int_0^T \int_0^1 x Q_i(x, t) \left( \frac{P_{i,x}(x, t)}{L_i M_i} + \frac{Q_{i,x}(x, t)}{M_i} \right) dx dt &= 0, \tag{52} \\
    \int_0^T \int_0^1 x P_i(x, t) \left( \frac{Q_{i,x}(x, t)}{L_i M_i} + \frac{P_{i,x}(x, t)}{L_i} + \frac{N_i Q_i(x, t)}{L_i M_i} \right) dx dt &= 0. \tag{53}
\end{aligned}
\]

Adding (50)–(53) up leads to

\[
\begin{aligned}
    \int_0^T \int_0^1 (x - 1) \left[ \frac{(Q_1 P_{1,t} + P_1 Q_{1,t})}{L_1 M_1} + \frac{Q_1 Q_{1,x}}{M_1} + \frac{P_{1,x}}{L_1} + \frac{N_1 P_1 Q_1}{L_1 M_1} \right] dx dt \\
    + \sum_{i=2}^{3} \int_0^T \int_0^1 x \left[ \frac{(Q_i P_{i,t} + P_i Q_{i,t})}{L_i M_i} + \frac{Q_i Q_{i,x}}{M_i} + \frac{P_{i,x}}{L_i} + \frac{N_i P_i Q_i}{L_i M_i} \right] dx dt &= 0. \tag{54}
\end{aligned}
\]
Since

\[ \int_0^T \int_0^1 (x-1) \frac{(Q_1 P_{1,t} + P_{1} Q_{1,t})}{L_1 M_1} dx dt + \sum_{i=2,3} \int_0^T \int_0^1 x \frac{(Q_1 P_{1,t} + P_{i} Q_{1,t})}{L_i M_1} dx dt \]

\[ = \int_0^1 (x-1) \frac{Q_1 P_1}{L_1 M_1} \bigg|_0^T dx + \sum_{i=2,3} \int_0^1 x \frac{Q_1 P_i}{L_i M_i} \bigg|_0^T dx. \quad (55) \]

and

\[ \int_0^T \int_0^1 (x-1) \frac{Q_{1} Q_{1,x}}{M_1} + \frac{P_{1} P_{1,x}}{L_1} + \frac{N_{1} P_{1} Q_{1}}{L_1 M_1} dx dt 
+ \sum_{i=2,3} \int_0^T \int_0^1 x \left( \frac{Q_{i} Q_{i,x}}{M_i} + \frac{P_{i} P_{i,x}}{L_i} + \frac{N_{i} P_{i} Q_{i}}{L_i M_i} \right) dx dt 
= \int_0^T (x-1) \left( \frac{Q_1^2}{2 M_1} + \frac{P_1^2}{2 L_1} \right) \bigg|_0^T dx dt + \int_0^T \int_0^1 \frac{(x-1) N_1 P_1 Q_1}{L_1 M_1} dx dt 
+ \int_0^T \sum_{i=2,3} \left( x \frac{Q_i^2}{2 M_i} + x \frac{P_i^2}{2 L_i} \right) \bigg|_0^T dx dt - \int_0^T E(t) dt 
\]

\[ = \frac{1}{2} \left( \|U(T)\|^2 + \|Y(T)\|^2 - 2 \int_0^T E(t) dt \right) + \sum_{i=2,3} \int_0^T \int_0^1 \frac{x N_i P_i Q_i}{L_i M_i} dx dt 
+ \int_0^T \int_0^1 \frac{(x-1) N_1 P_1 Q_1}{L_1 M_1} dx dt \quad (56) \]

where \( E(t) \) is defined as (5).

Thus, from (54), (55) and (56) we derive

\[ \|U(T)\|^2 + \|Y(T)\|^2 
\]

\[ = 2 \int_0^T E(t) dt - 2 \left( \int_0^1 (x-1) \frac{Q_1 P_1}{L_1 M_1} \bigg|_0^T dx + \sum_{i=2,3} \int_0^1 x \frac{Q_i P_i}{L_i M_i} \bigg|_0^T dx \right) 
-2 \left( \sum_{i=2,3} \int_0^T \int_0^1 x \frac{N_i P_i Q_i}{L_i M_i} dx dt + \int_0^T \int_0^1 \frac{(x-1) N_1 P_1 Q_1}{L_1 M_1} dx dt \right). \quad (57) \]

Set \( \mathcal{N} = \max_{1 \leq i \leq 3} \frac{1}{L_i M_i} \). We can get an estimate

\[ \left| -2 \left( \int_0^1 (x-1) \frac{Q_1 P_1}{L_1 M_1} \bigg|_0^T dx \sum_{i=2,3} \int_0^1 x \frac{Q_i P_i}{L_i M_i} \bigg|_0^T dx \right) \right| 
\leq 2 \sum_{i=1,2,3} \int_0^1 \left| \frac{Q_i P_i}{L_i M_i} \right|_{t=0} \left| \frac{Q_i P_i}{L_i M_i} \right|_{t=T} dt \]

\[ \leq 2 \mathcal{N} (E(T) - E(0)). \quad (58) \]
Thus, we have
\[
\|U(t)\|_{L^2((0,T], U)}^2 + \|Y(t)\|_{L^2((0,T], Y)}^2 \leq 2(L+1) \int_0^T E(t) dt + 2N(E(T) + E(0)).
\]

Set \( M = \max_{1 \leq i \leq 3} \sqrt{L_i M_i} \). Using (6) we obtain the following estimate
\[
E(T) = \int_0^T \left( u_1 y_1 - u_2 y_2 - u_3 y_3 - \sum_{i=1}^3 \int_0^1 \frac{N_i Q_i^2}{M_i} dx \right) dt + E(0)
\]

\[
\leq \sum_{i=1}^3 \int_0^T |u_1 y_1| + |u_2 y_2| + |u_3 y_3| dt + E(0)
\]

\[
\leq \frac{M}{2} (\gamma \|Y(t)\|_{L^2((0,T], Y)}^2 + \frac{1}{\gamma} \|U(t)\|_{L^2((0,T], U)}^2) + E(0)
\]

\[
\leq \frac{M}{2} (\gamma \|Y(t)\|_{L^2((0,T], Y)}^2 + \gamma \|U(t)\|_{L^2((0,T], U)}^2 + \frac{1}{\gamma} \|U(T)\|^2) + E(0)
\]

\[
\leq M(L+1) \gamma \int_0^T E(t) dt + MN \gamma E(T)
\]

\[
+ (MN \gamma + 1) E(0) + \frac{M}{2 \gamma} \|U(T)\|_{L^2((0,T], U)}^2.
\]

Choose \( \gamma > 0 \) such that \( 1 - MN \gamma > 0 \), then (61) can be rewritten as
\[
E(T) \leq \frac{M(L+1) \gamma}{1 - MN \gamma} \int_0^T E(t) dt + \frac{MN \gamma + 1}{1 - MN \gamma} E(0) + \frac{M}{2(1 - MN \gamma) \gamma} \|U(T)\|_{L^2((0,T], U)}^2.
\]

The Gronwall inequality asserts that
\[
\int_0^T E(t) dt \leq \frac{1 - MN \gamma}{M(L+1) \gamma} e^{\frac{M(L+1) \gamma}{1 - MN \gamma} T} \left[ (MN \gamma + 1) E(0) + \frac{M}{2(1 - MN \gamma) \gamma} \|U(T)\|_{L^2((0,T], U)}^2 \right].
\]

Thus, we have
\[
E(T) \leq (1 + e^{\frac{M(L+1) \gamma}{1 - MN \gamma} T}) \left[ (MN \gamma + 1) E(0) + \frac{M}{2(1 - MN \gamma) \gamma} \|U(T)\|_{L^2((0,T], U)}^2 \right].
\]

Also,
\[
\|Y(t)\|_{L^2((0,T], Y)}^2 \leq \|U(t)\|_{L^2((0,T], U)}^2 + \|Y(t)\|_{L^2((0,T], Y)}^2
\]

\[
\leq 2 \left[ 1 - MN \gamma \right] e^{\frac{M(L+1) \gamma}{1 - MN \gamma} T} \left[ (MN \gamma + 1) E(0) + \frac{M}{2(1 - MN \gamma) \gamma} \|U(T)\|_{L^2((0,T], U)}^2 \right]
\]

\[
+ 2N(1 + e^{\frac{M(L+1) \gamma}{1 - MN \gamma} T}) \left[ (MN \gamma + 1) E(0) + \frac{M}{2(1 - MN \gamma) \gamma} \|U(T)\|_{L^2((0,T], U)}^2 \right]
\]

\[
+ 2N E(0)
\]

\[
= 2N \left[ 1 + e^{\frac{M(L+1) \gamma}{1 - MN \gamma} T} \right] \left[ (MN \gamma + 1) E(0) + \frac{M}{2(1 - MN \gamma) \gamma} \|U(T)\|_{L^2((0,T], U)}^2 \right]
\]
Therefore, there exists a positive $M_T$ depending on $T$ such that
\[ E(T) + \|Y(t)\|_{L^2([0,T],Y)}^2 \leq M_T \left( E(0) + \|U(t)\|_{L^2([0,T],U)}^2 \right). \]
Therefore the linear system (15) is $L^2$ well-posed. The proof is then complete. □

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