Numerical Study of the movement of a mobile object in different trajectories with a coupled pendulum.

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Abstract. Coupled systems are studied within classical mechanics, we study and analyse numerically the movement of the mobile object when it moves in a single dimension (y direction) with the pendulum oscillations: 1) in 2D (zy-plane) and 2) 3D (surface xyz). Subsequently, the movement of the object is studied when it moves in a circle of radius $R$ with the same oscillation conditions referred in the pendulum. We obtained the numerical solutions of this system as well as the analysis for the different phase diagrams and the numerical solution for the intersection of the trajectory and surface, to begin with the study of chaos of this system.

1. Introduction

In a previous article we presented the theoretical behavior of an object when it moves with an attached pendulum; the object moved in one and two dimensions, this time we show the equations of motion for different cases. Our final goal is to understand how a three-dimensional (3D) movement for a given object is disturbed by the oscillation of a second object subject to the first. This kind of movements occur in real life and this research may have applications for practical cases, for example: when a helicopter by means of a cable, carries a certain mass, like a sculpture, or moves granular material to put out a fire, etc. Based on these mentioned cases, we obtain the numerical solution of the equation of movement and analyse the movement of two coupled objects. The studied system is made up of a mobile object of mass $M$, to which is attached a pendulum of length $l$ and mass $m$.

We review the cases corresponding to: 1.- One-dimensional movement of the object of mass $M$, in the $y$ direction, with the restrictions of the pendulum movement: a) in two dimensions, (plane $yz$), b) in 3 dimensions (surface $xyz$).

2.- Two-dimensional movement of the object of mass $M$, in the $xy$ plane, describing a circumference of radius $R$, with the restrictions of the pendulum movement: a) in two dimensions, (plane "$xy$, $z$") and b) in 3 dimensions (xyz surface). We obtained the equations of motion in a previous work as a first delivery, in this work we use the typical methodologies of classical mechanics such as the Lagrangian formulation [1-2] and we will solve these equations numerically to analyze the behavior of the system.
2. Simulation.
In order to carry out the simulations of the described systems, a change of variable is usually suggested to reduce the system to a first order, the results obtained by the simulation correspond to the values $q_i$ and $\dot{q}_i$ since $q_i$ are the dynamical variables known, which is described by equations for the case of linear motion and, for circular motion. The numerical method used to solve the differential equations is the Runge-Kutta method of order 4 (RK4), which has been used for the numerical solution of other problems and dynamic systems that we have studied.

This method has been explained and applied for simple systems in books of numerical methods [3], as well as in books of differential equations [4]. The error due to the use of the method is of the order of the step size, which we denote by $\delta h$ raised to the 4, that is, $err = \Delta h^4$. The step size is defined as the partition of the integration interval $\delta h = (b - a)/n$, where $b$ is the upper limit of the interval, $a$ is the lower limit, $n$ is the number of parts into which the interval is divided, which in fact corresponds to the number of iterations that will have to be done to cover the entire proposed integration interval $(b-a)$. For case a) we have $\delta h = 0.0011$ and for case b) $\delta h = 0.0019$, throwing an error, $err = 1.4 \times 10^{-12}$, respectively. Table 1 contains the values of the initial condition (IC) used in the simulation: $g$ is the value of the acceleration of gravity, equal to the value reported for Mexico City, $g = 9.78 \text{ m/s}^2$. The upper limit of the simulation was taken as 30, 50, 60. The length of the pendulum is denoted by $l = 1.5 \text{ m}$, the pendular mass is $m = 3.5 \text{ kg}$, and $M = 6.0 \text{ kg}$ is the mass of the car, in the case of circular motion the radius of circle was $R = 2.0 \text{ m}$, $N = 20,000$ is the number of steps we used to integrate the numerical experiment; each repetition of the experiment was performed with his IC’s. To observe the changes produced by the new IC’s, it was decided to only make small changes systematically, for this purpose changes were proposed only in the variables related to speed.

| Table 1. Values of initial condition used in the simulations for all systems working |
|---------------------------------|-----|----------------|-----|
| value  | unit | value  | unit |
| $\omega_y$  | 0.00 | $s^{-1}$ | $\dot{\omega}_y$  | 1.50 |
| $\theta$  | 1.60 | $\text{rad}$ | $\dot{\theta}$  | 1.50 |
| $\phi$  | 0.40 | $\text{rad}$ | $\dot{\phi}$  | 0.50 |
| $\Omega$  | 0.90 | $\text{rad}$ | $\dot{\Omega}$  | 3.50 |
| $\theta$  | 1.30 | $\text{rad}$ | $\dot{\theta}$  | 1.50 |
| $\phi$  | 0.60 | $\text{rad}$ | $\dot{\phi}$  | 1.70 |
| $\Omega$  | 1.30 | $\text{rad}$ | $\dot{\Omega}$  | 1.50 |
| $\theta$  | 0.60 | $\text{rad}$ | $\dot{\theta}$  | 1.70 |
| $\phi$  | 0.90 | $\text{rad}$ | $\dot{\phi}$  | 3.50 |

3. Results.
There are several results obtained by simulation, because for each trial we get: time, position of the car, position of the pendulum, velocity and acceleration of the object of mass $M$, as well as angular position, velocity and acceleration for the pendular mass $m$.

The phase diagrams show the behaviour of dynamic variables, these are of interest because the temporal parameter is eliminated and allow the observation of the existing changes between position and speed for each dynamic variable.
3.1. First system, case $a$)

The Figure 1 shows the phase diagram for the angle to the first system and the case $a$), the equation of movement for $\theta$. In this graph, it is observed that the trajectory in the phase plane is represented by a closed curve, the change of ICs is also appreciated, by the different colours plotted, which shows the different trajectories which are quite similar to each other.

![Figure 1](image)

**Figure 1.** Diagram phase for $\theta$, was obtained for the pendulum, in the graph we showed two different CI, the second simulation is very close to the first trajectory.

3.2. First system, case $b$)

Figures 3 to figure 6 show the phase diagram for the different dynamic variables, the graphs were plotted for two different ICs, the initial positions are: $\omega_y = 0.0$, $\theta = 0.4$, $\phi = 0.2$, and the initial velocities are $\dot{\omega}_y = 0.9$, $\dot{\theta} = 2.1$, $\dot{\phi} = 0.3$ for the orange curve. The initial positions $\omega_y = 0.0$, $\theta = 1.4$, $\phi = 1.2$, and the initial velocities $\dot{\omega}_y = 1.9$, $\dot{\theta} = 0.3$, $\dot{\phi} = 0.4$, are for the blue curve. Chaotic behaviour is observed in all graphs when the car's speed increases. The shape on the charts is essentially the same.

![Figure 3](image)

**Figure 3.**

- a) Phase diagram for distance $z$.
- b) Phase diagram for the pendulum is not the same as for a simple pendulum.

![Figure 4](image)

**Figure 4.**

- a) Shows the behaviour of the car’s acceleration in relation to the angular displacement of the pendulum.
- b) The relationship between the acceleration of the car and the angular velocity of the pendulum.

In all figures shown for this case, the relationships between the dynamic properties of both objects are shown. As the speed increases for the pendulum, the behaviour of both objects tends to be chaotic. However, if the speed of the car increases, the behaviour of the pendulum is practically the same. This situation is important for the applications mentioned in the introduction. There are many results for these changes in initial conditions, so they need to be explored in more detail.
Figure 5. a) Phase diagram for the x coordinate, b). Relationship between position and acceleration for the x coordinate of the pendulum. From left to right the changes to the initial condition are shown. There are systematic changes in the speed of the car and the angular speeds for the pendulum, these systematic changes are from 0.3 to 1.8, with an increase of $i \times 10$ for $i = 1 \ldots 14$. All different CI’s are shown. The left one shows the changes for the car's speed $\omega_y$. The centre one shows the changes for the angular velocity $\theta$. The right one shows the changes for the angular velocity $\phi$. When the angular velocities are higher, the behaviour shown tends to be chaotic.

A test was performed to verify that the answer was correct and it consisted in drawing the trajectory of both objects in motion. figure 6 shows this test for two different considerations, a) for changes in the speed of the car, and b) for changes in the angular velocity of the pendulum. The line parallel to the horizontal line, is the path followed by the car, the path of the pendulum is shown in the curves. We made an animation that can be obtained in reference [5], this animation was made for this work.

Figure 6. The trajectories for two simulations are shown, figure a) shows the first and last initial conditions, with the changes given for $\omega_y$, b) shows the changes given by $\dot{\theta}$. The shape of the path is similar, the same structure is shown.

3.3. Second system, case a)

The restriction imposed on the system allows to observe well-determined figures in their shape, the figure 7 shows the diagrams for the pendulum's behaviour, up figures correspond to xy-plane, the first case is observed from left to right, where there were systematic changes in the angular velocity of the pendulum, again these changes were from 0.3 to 1.8 in increase given by the $i \times 10$ ratio, for $i = 1 \ldots 14$. In the centre, the changes for the angular speed of the car are shown, with the same systematic change, finally in the figure on the right the same systematic changes are shown but random changes are added for the angular speed of the pendulum. The shape of the pendulum trajectory has a good behaviour, the pendulum trajectory is very similar to the curves that are traced with spirographs.
Figure 7. a) The trajectories for the pendulum movement, for different initial conditions. From left to right, there is an increase in the angular velocity of the car, angular velocity of the pendulum and both angular velocities increase, only three initial conditions of the fourteen are graphed. b) The phase diagram for the x coordinate is shown, from left to right the same considerations are used in a), these graphs show all the curves for all the initial conditions obtained. Like in the previous case, as a test we make the graph corresponding to the movement of the car and the pendulum in the xyz space. Figure 10 shows the graph obtained for two initial conditions. The structure that describes the pendulum are segments of the sphere and has to do with the coordinate system used.

Figure 8. Trajectories for the car and the pendulum; the red ring is the trajectory of the car, the blue curve is the first initial condition when both angular speeds are switched, the orange curve is the last initial condition with the same considerations. The structure is very systematic.

The red ring observed corresponds to the trajectory of the car. Only the initial conditions of the first and last iteration of the fourteen CIs worked are shown.

3.4. Second system, case b)
Figures 7 show that the trajectories of the pendulum behave well to make well-defined geometric structures. Figure 9 shows the phase diagrams for the x coordinate (figure 9a) and the z coordinate (figure 13b), the path trace is similar to the second pendulum when the “Double Pendulum” dynamic system is solved [6-7], its similarity is because our system somehow turns out to be a “double pendulum”. However, in this case, the first pendulum rotates in one plane, while the other can swing freely. Figures 11a to 11c show the symmetry of the trajectories in the phase spaces with respect to the speed variables, not in relation to the positions, their effect on the phase space of the z coordinate is observed. Figure 10a shows the final trajectory in 3D space. The figures 10a to 10c, show the view frontal, lateral, and aerial, of the trajectory of the system, the orange curve, in the figure 10b, is the orange line in figures 10a and 10c, it represents the trajectory of the car.

Figure 9a. Phase diagram for x coordinate, three different IC’s are plotted. Figure 9b. Phase diagram for y coordinate, the same IC’s in figure 11a are plotted. Figure 9c. Phase diagram for z coordinate, the same IC’s in figure 11a are plotted.

Figure 10. In this case it is simply to justify the caption so that it is as the same width as the graphic.
4. Poincare Sections

4.1. Second system, case b)

Poincare maps are used to analyze the stability of the orbit. This section only shows some preliminary results obtained when the studied surface for this map is proposed. We define different surfaces, the condition used to measure the intersection between surface and trajectory was \((\theta_0, t = 2\pi)\) and any pair of situations are taken, for example \((\theta \text{ vs } \dot{\theta})\). Figure 11a, \((\Omega \text{ vs } \dot{\Omega})\), figure11b, \((\dot{\Omega} \text{ vs } \dot{\theta})\), figure11c, figure 11d, and figures 11e and f, \((\Omega \text{ vs } \dot{\theta})\) figure 11f, all figures were generated for the movement of the car in circular motion and the oscillation of the pendulum in 3D. Different structures in the iterations to change the initial conditions in the numerical solution are shown.

![Figure 11](image11)

**Figure 11.** a) Intersection of trajectories in a surface \((\theta \text{ vs } \dot{\theta})\). b) Intersection in a surface \((\Omega \text{ vs } \dot{\Omega})\). c) Intersection in a surface \((\dot{\Omega} \text{ vs } \dot{\theta})\). d) Intersection in a surface \((\dot{\Omega} \text{ vs } \dot{\theta})\), with different CI that c), e) Intersection in a surface \((\dot{\Omega} \text{ vs } \dot{\theta})\) with different IC that c) and e) f) Intersection in a surface \((\Omega \text{ vs } \dot{\theta})\).

![Figure 12](image12)

**Figure 12.** a) Intersection of trajectories in a surface \((\theta \text{ vs } \dot{\theta})\). b) Intersection in a surface \((\dot{\phi} \text{ vs } \dot{\theta})\). c) Intersection in a surface \((\dot{\phi} \text{ vs } \dot{\theta})\). d) Intersection in a surface \((\dot{\phi} \text{ vs } \dot{\theta})\), with different CI that c), e) Intersection in a surface \((\theta \text{ vs } \dot{\phi})\) with different IC that c) and e) f) Intersection in a surface \((\theta \text{ vs } \dot{\phi})\). All pictures are for the object in motion in one dimension and the pendulum free in 3D.

5. Conclusions.

A set of numerical simulations was developed to obtain the solutions of the differential equations. From the analysis of the numerical results, it is concluded that the system, both in linear and circular motion, is chaotic; thus, considering the possibility of studying them through the methodology used to analyse chaos, such as: Poincare sections and Lyapunov coefficients, for example. In this way, it is possible to offer a more precise quantitative evaluation of the chaotic behaviour of the system.

References.

[1] H Goldstein, Charles Poole, John Safko 2000 Classical Mechanics. *New York Addison Wesley.*
[2] Jorge V José and Eugene J Saletan 1998 Classical Dynamics. *England Cambridge University Press*
[3] Timothy Sauer 2013 Numerical Methods *Chicago PEARSON*
[4] W E Boyce and R C Di Prima 2004 Elementary Differential Equations and Boundary Value Problems *New York Wiley*

[5] https://roeshe.wixsite.com/misitio

[6] R Kwiatkowski 2014 Movement of Double Mathematical Pendulum with Variable Mass, *Machine Dynamics Research* **38** 47–58.

[7] R Espíndola, G Del Valle, G Hernández, I Pineda, D Muciño, P Díaz and S Guijosa 2019 *Journal of Physics: Conf. Series* **1221** 012049

[8] Wolfram Research, Inc., Mathematica 2020 Version 12.1.1.0 Student Edition *Champaign, IL*