Product rules are optimal for numerical integration in classical smoothness spaces

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Dedicated to the memory of Joseph F. Traub

Abstract

We mainly study numerical integration of real valued functions defined on the $d$-dimensional unit cube with all partial derivatives up to some finite order $r \geq 1$ bounded by one. It is well known that optimal algorithms that use $n$ function values achieve the error rate $n^{-r/d}$, where the hidden constant depends on $r$ and $d$. Here we prove explicit error bounds without hidden constants and, in particular, show that the optimal order
of the error is $\min\{1, d n^{-r/d}\}$, where now the hidden constant only depends on $r$, not on $d$. For $n = m^d$, this optimal order can be achieved by (tensor) product rules.

We also provide lower bounds for integration defined over an arbitrary open domain of volume one. We briefly discuss how lower bounds for integration may be applied for other problems such as multivariate approximation and optimization.

1 Introduction

Multivariate integration is nowadays a popular research problem especially when the number of variables $d$ is huge. In this paper we mainly study numerical integration of $r \geq 1$ times continuously differentiable periodic and nonperiodic functions defined over the $d$-dimensional unit cube whose partial derivatives up to order $r \geq 1$ are bounded by one. Already in 1959, Bakhvalov [1] proved that the minimal number $n = n(\varepsilon, d, r)$ of function values which is needed to achieve an error at most $\varepsilon > 0$ satisfies

$$c_{d,r} \varepsilon^{-d/r} \leq n(\varepsilon, d, r) \leq C_{d,r} \varepsilon^{-d/r}$$

for some positive $c_{d,r}$ and $C_{d,r}$ and the upper bounds are achieved by product rules. Note that for fixed $d$ and $r$ we have a sharp behaviour with respect to $\varepsilon^{-d/r}$ and $n(\varepsilon, d, r) = \Theta(\varepsilon^{-d/r})$.

For large $d$, we would like to know how $c_{d,r}$ and $C_{d,r}$ depend on $d$. Unfortunately up to 2014, the knowledge on the dependence on $d$ was quite limited since the known lower bound on $c_{d,r}$ was exponentially small in $d$ whereas the known upper bound on $C_{d,r}$ was exponentially large in $d$. In [4], we proved for the nonperiodic case that there exists a positive $c_r$ such that for all $d$ and $\varepsilon \in (0, 1)$ we have

$$n(\varepsilon, d, r) \geq c_r (1 - \varepsilon) d^{d/(2r+3)}.$$  \hspace{1cm} (1)

Hence, we have a super-exponential dependence on $d$. This means that numerical integration suffers from the so-called curse of dimensionality for fixed $r$.

However, the exponent $d$ in (1) is $d/(2r + 3)$, whereas in Bakhvalov’s lower bound it is larger and equals $d/r$. Furthermore, there is really no dependence on $\varepsilon^{-1}$ in (1), although we expect from Bakhvalov’s bounds that it should be $\varepsilon^{-d/r}$.

This is the point of departure of the current paper. We improve the lower bound (1) and find a matching upper bound. Furthermore we will do it also for the periodic case which was not studied in [4]. The lower bound is found similarly as in [4] but instead of working with balls in the $\ell_2$-norm we switch to balls in the $\ell_1$-norm which yields a better result. The upper
bound is achieved by product rules of \( d \) copies of the rectangle (or trapezoidal) quadrature for the periodic case and of the Gaussian quadrature for the nonperiodic case.

We need a few definitions to formulate our results. We mainly study the problem of numerical integration, i.e., of approximating the integral

\[
S_d(f) = \int_{[0,1]^d} f(x) \, dx
\]

for integrable functions \( f: [0,1]^d \to \mathbb{R} \).

The function class under consideration is the unit ball in the space of all \( r \)-times continuously differentiable functions on \([0,1]^d\), i.e.,

\[
C^r_d = \{ f \in C^r([0,1]^d): \| D^\beta f \|_\infty \leq 1 \text{ for all } \beta \in \mathbb{N}_0^d \text{ with } |\beta|_1 \leq r \}
\]

equipped with the norm

\[
\| f \|_{C^r_d} := \max_{\beta: |\beta|_1 \leq r} \| D^\beta f \|_\infty.
\]

Here, \( D^\beta \) denotes the usual (weak) partial derivative of order \( \beta \in \mathbb{N}_0^d \). Moreover, the sup-norm of a bounded function \( f \) is given by \( \| f \|_\infty = \sup_{x \in [0,1]^d} |f(x)| \).

We consider algorithms for approximating \( S_d(f) \) that use finitely many function values. More precisely, the general form of an algorithm that uses \( n \) function values is

\[
A_n(f) = \varphi_n(f(x_1), f(x_2), \ldots, f(x_n)) \quad \text{for all } f \in C^r_d,
\]

where \( \varphi_n : \mathbb{R}^d \to \mathbb{R} \) may be a nonlinear mapping and the sample points \( x_i \in [0,1]^d \) may be chosen adaptively, that is, the choice of \( x_i \) may depend on the already computed \( f(x_1), f(x_2), \ldots, f(x_{i-1}) \). Nonadaptation means that the choice of \( x_i \) is independent of \( f \), i.e., it is the same for all functions \( f \) from \( C^r_d \). Obviously even for the nonadaptive case, \( x_i \) may depend on \( n \) and the class \( C^r_d \).

We consider the worst case setting in which the error of \( A_n \) is defined as

\[
e(A_n) = \sup_{f \in C^r_d} |S_d(f) - A_n(f)|.
\]

The \( n \)th minimal (worst case) error is given by

\[
e_n(C^r_d) = \inf_{A_n} e(A_n),
\]

where the infimum is taken over all algorithms \( A_n \), i.e., over all mappings \( \varphi_n \) and adaptive choices of sample points \( x_1, x_2, \ldots, x_n \) from \([0,1]^d\).
It is known by the result of Bahvalov [2] on adaption and the result of Smolyak [12] on nonlinear algorithms, see also [9] or [14], that without loss of generality we may restrict ourselves to linear algorithms and nonadaptive sample points, i.e., to algorithms of the form

\[ A_n(f) = \sum_{i=1}^{n} a_i f(x_i) \]

for some real \( a_i \) and some \( x_i \in [0,1]^d \). Furthermore,

\[ e_n(C_d^r) = \inf_{a_i, x_i} \sup_{f \in C_d^r} \left| S_d(f) - \sum_{i=1}^{n} a_i f(x_i) \right| = \inf_{x_i} \sup_{f \in C_d^r, f(x_1)=-\ldots=f(x_n)=0} \left| S_d(f) \right|. \tag{3} \]

Note that for \( n = 0 \) we do not sample functions and

\[ e_0(C_d^r) = \|S_d\| = 1. \]

We can now formally define the minimal number of function values needed to compute an \( \varepsilon \)-approximation as

\[ n(\varepsilon, d, r) = \min \{ n : e_n(C_d^r) \leq \varepsilon \}. \]

Clearly, for \( \varepsilon \geq 1 \) we have \( n(\varepsilon, d, r) = 0 \) and, therefore, we always assume that \( \varepsilon \in (0,1) \).

We briefly discuss the results obtained in this paper and start with simplified results that might be easier to digest.

**Theorem 1.** For all \( r \in \mathbb{N} \) there exist constants \( c_{r,1}, c_{r,2} > 0 \) such that for all \( d, n \in \mathbb{N} \) with \( n = m^d \) for some \( m \in \mathbb{N} \), we have

\[ c_{r,1} \left( \frac{d}{\varepsilon} \right)^{d/r} \leq n(\varepsilon, d, r) \leq c_{r,2} \left( \frac{d}{\varepsilon} \right)^{d/r}. \tag{4} \]

Moreover, the lower bound holds when \( S_d \) is defined as integration over an arbitrary open set in \( \mathbb{R}^d \) of volume 1, and the upper bound holds for arbitrary \( \varepsilon \in (0,1) \).

For the errors, the corresponding result is the following.

**Theorem 2.** For all \( r \in \mathbb{N} \) there exist constants \( c_{r,1}, c_{r,2} > 0 \) such that for all \( d, n \in \mathbb{N} \) with \( n = m^d \) for some \( m \in \mathbb{N} \), we have

\[ \min \{ \frac{1}{2}, c_{r,1} \, d \, n^{-r/d} \} \leq e_n(C_d^r) \leq \min \{ 1, c_{r,2} \, d \, n^{-r/d} \}. \tag{5} \]

The lower bound holds for all \( n \in \mathbb{N} \), whereas the upper bound has to be replaced by \( \min \{ 1, c_{r,2} \, d \, (n^{1/d} - 1)^{-r} \} \) for general \( n \in \mathbb{N} \).
Now we go more into the details and present bounds without any hidden constants. We start with the periodic case. To stress periodicity, we denote $n(\varepsilon, d, r)$ by $n_{\text{per}}(\varepsilon, d, r)$. For $\varepsilon \in (0, d/(d + r)]$ we prove

$$\frac{r}{r + d} \left( \frac{d}{d + r} \frac{1}{4e^{r - 1}} \right)^{d/r} \leq n_{\text{per}}(\varepsilon, d, r) \leq \left( \frac{1}{2} \frac{d}{(2\pi)^{r}} \right)^{1/r} \frac{d}{\varepsilon}. \quad (6)$$

In fact, the assumption that $\varepsilon \in (0, d/(d + r)]$ is only needed for the lower bound, whereas the upper bound holds for all $\varepsilon \in (0, 1)$.

We now turn to the nonperiodic case. Since the class of nonperiodic functions is larger than the class of periodic functions we clearly have $n(\varepsilon, d, r) \geq n_{\text{per}}(\varepsilon, d, r)$. As already mentioned, to obtain upper bounds on $n(\varepsilon, d, r)$ we use product rules based on univariate Gaussian quadratures. We use the error estimates of Gaussian quadratures proved by Köhler [5] who studied the class $W_{s}^{\infty}$ of functions $f$ for which $f^{(s-1)}$ is absolutely continuous and $\|f^{(s)}\|_{\infty} \leq 1$. Obviously, our class is a subset of $W_{s}^{\infty}$ for all $s \leq r$ and we can apply Köhler’s estimates which hold if the number of sample points is at least equal to $s + 1$. For the $d$-variate case, we use the result of Haber [3] for product rules and obtain

$$n(\varepsilon, d, r) \leq \min_{s=1,2,\ldots,r} \max \left\{ (s + 1)^{d}, \left[ \frac{\pi}{2} \left( \frac{e}{6\sqrt{3}} \right)^{s} \frac{d}{\varepsilon} \right]^{1/s} \right\}. \quad (7)$$

Obviously, for large $d/\varepsilon$ relative to $r$, more precisely, for

$$\frac{d}{\varepsilon} \geq \frac{2}{\pi} \left( \frac{6\sqrt{3}(r + 1)}{e} \right)^{r}$$

we have

$$n(\varepsilon, d, r) \leq \left[ \frac{\pi}{2} \left( \frac{e}{6\sqrt{3}} \right)^{r} \frac{d}{\varepsilon} \right]^{1/r} \frac{d}{\varepsilon}. \quad (8)$$

In this case, the lower and upper bounds similarly depend on $d, \varepsilon$ and are exponentially large in $d/\varepsilon$ and exponentially small in $r$. This proves optimality of product rules also for the nonperiodic case.

The estimates (6)–(8) hold if the domain of integration is $[0, 1]^{d}$. Interestingly enough, our proof technique for lower bounds works for more general integration domains. We can integrate over an open set $D_{d} \subset \mathbb{R}^{d}$ of volume one. Then, for the same smoothness class of
real nonperiodic functions defined over $D_d$, we obtain a lower bound on the minimal number $n(\varepsilon, D_d, r)$ of function values needed to compute an $\varepsilon$ approximation of the form

$$n(\varepsilon, D_d, r) \geq \frac{r}{r + d} \left( \frac{d}{d + r} \frac{1}{6^r r^{r-1}} \frac{d}{\varepsilon} \right)^{d/r}.$$  

(9)

This is a similar lower bound as (6) with $4^r$ replaced by $6^r$.

Assume first that $r$ is fixed. Then $n(\varepsilon, d, r) \asymp n_{\text{per}}(\varepsilon, d, r) = \Theta((d/\varepsilon)^{d/r})$ which agrees with the bounds of Bakhvalov. We also have an exponential dependence on $d$ which results in the curse of dimensionality, although it is delayed for large $r$. Observe that the dependence of the lower and upper bounds in (6)–(9) is exponentially small with respect to $r$.

Now let $r$ be a function of $d$, i.e., $r = r(d)$. Although it is not a subject of this paper it is easy to show that the curse of dimensionality still holds both in the nonperiodic and the periodic case if $r(d) \ln r(d) = o(\ln d)$. It seems to be an interesting open problem to characterize how fast $r(d)$ must go to infinity with $d$ to break the curse of dimensionality or to obtain various notions of tractability. We are not sure if the bounds (6) and (9) are sufficiently sharp to solve such questions.

It is well known that many multivariate problems are at least as hard as multivariate integration. In particular, this holds for multivariate approximation and optimization. That is why for the same smoothness class of $r$-times continuously differentiable functions we have the curse of dimensionality also for multivariate approximation and optimization. Details are provided in Section 4.

2 Nonperiodic smooth functions

In this section we study lower and upper bounds for numerical integration of nonperiodic functions from the class $C^r_d$. Lower bounds will be presented for more general integration domains than $[0,1]^d$, whereas we show upper bounds only for the integration domain $[0,1]^d$.

2.1 Lower bounds

In this section we present a lower bound on $n(\varepsilon, d, r)$. Interestingly enough, our proof technique can be applied not only for the domain $[0,1]^d$ but also for an arbitrary open subset $D_d \subset \mathbb{R}^d$ of volume $\text{vol}_d(D_d) = 1$. All definitions presented in the previous section for $[0,1]^d$ (or, which is the same, for $(0,1)^d$) readily generalize for $D_d$. In this case, we denote the class of functions as $C^r(D_d)$ with the norm

$$\|f\|_{C^r} = \max_{\beta: \|\beta\|_1 \leq 1} \max_{x \in D_d} |(D^\beta f)(x)|.$$
To stress the role of $D_d$ we denote the minimal number of function values needed to compute an $\varepsilon$-approximation as $n(\varepsilon, D_d, r)$.

Let $\mathcal{P} = \{x_1, x_2, \ldots, x_n\} \subset D^d$ be a collection of $n$ points. We construct a so-called fooling function $f$ from $C^r(D_d)$ with $f(x_i) = 0$, $i = 1, 2, \ldots, n$, and as large as possible integral. Due to [3], this allows us to get lower bounds on $e_n(C^r(D_d))$ and on $n(\varepsilon, D_d, r)$.

In fact, we will construct a fooling function $f$ from the Sobolev space $W^{r+1}_\infty(D_d) = \{f : D_d \to \mathbb{R} : \|D^\beta f\|_\infty < \infty \text{ for all } |\beta|_1 \leq r + 1\}$ with $\|f\|_{C^r} \leq 1$. Here $\|D^\beta f\|_\infty = \text{ess sup}_{x \in D_d} |(D^\beta f)(x)|$. Clearly, such a function $f$ belongs to $C^r(D_d)$.

To define the fooling function for the given point set $\mathcal{P}$, we choose some $\varrho > 0$, to be specified later, and define the function

$$h_\varrho(x) = \min \left\{ 1, \frac{\text{dist}(x, \mathcal{P}_\varrho)}{\varrho} \right\} \quad \text{for all } x \in \mathbb{R}^d,$$

where for $A \subseteq \mathbb{R}^d$ we have

$$\text{dist}(x, A) := \min_{y \in A} \|x - y\|_1 = \min_{y \in A} \sum_{j=1}^d |x_j - y_j|$$

and

$$\mathcal{P}_\varrho = \bigcup_{i=1}^n (\varrho B^d_1(x_i)).$$

Here, $B^d_1(x_i)$ is the $\ell_1$-unit ball around $x_i$. We write $B^d_1$ for $B^d_1(0)$.

To treat functions of higher smoothness we consider the $r$-fold convolution of the function $h_\varrho$ with the normalized indicator functions

$$g_\varrho(x) = \frac{1_{\varrho B^d_1}(x)}{\text{vol}_d(\varrho B^d_1)} = \frac{1}{\text{vol}_d(\varrho B^d_1)} \begin{cases} 1 & \text{if } x \in \varrho B^d_1, \\ 0 & \text{otherwise}, \end{cases} \quad (10)$$

where $\varrho_r = \varrho / r$. The convolution of a function $h$ with $g_\varrho$ is given by

$$(h * g_\varrho)(x) = \frac{1}{\text{vol}_d(\varrho B^d_1)} \int_{\varrho B^d_1} h(x + t) \, dt \quad \text{for all } x \in \mathbb{R}^d.$$

The fooling function we consider in the sequel is therefore given by

$$f_\varrho := h_\varrho * g_\varrho * \cdots * g_\varrho := h_\varrho *_{r \text{-fold}} g_\varrho.$$
It is clear that the support of the $r$-fold convolution of the function $g_\varrho$ is the $r$-fold Minkowski sum of the sets $\varrho B^d_1$, i.e., it is $\varrho B^d_1$. This shows that the function $f_\varrho$ satisfies $f_\varrho(x) = 0$ for all $x \in \mathcal{P}$.

For the integral of $f_\varrho$, we only need to observe that $h_\varrho(x) = 1$ if $\text{dist}(x, \mathcal{P}) > 2\varrho$ and that $\int_{\mathbb{R}^d} g_\varrho(x) \, dx = 1$. This implies that $f_\varrho(x) = 1$ if $\text{dist}(x, \mathcal{P}) > 3\varrho$, and hence

$$\int_{D_d} f_\varrho(x) \, dx \geq \text{vol}_d(D_d \setminus \mathcal{P}_{3\varrho}) \geq 1 - \text{vol}_d(\mathcal{P}_{3\varrho}) \geq 1 - n \cdot \text{vol}_d(3\varrho B^d_1(0)) = 1 - n \frac{(6\varrho)^d}{d!} \quad (11)$$

We stress that this bound holds for arbitrary collections of points $\mathcal{P} = \{x_1, x_2, \ldots, x_n\}$ and sets $D_d$. As we will see, $f_\varrho \in W^{r+1}_{\infty}(D_d)$ and the norm of the function $f_\varrho$ only depends on $\varrho$, $r$ and $d$. Since we are interested in a fooling function from the unit ball $C^r(D_d)$ it remains to normalize $f_\varrho$. Hence we define

$$f_\varrho^*(x) := f_\varrho(x)/\|f_\varrho\|_{C^r}. \quad (12)$$

Using (11) we obtain that $\int_{D_d} f_\varrho^*(x) \, dx \leq \varepsilon$ implies that

$$n \geq (1 - \varepsilon \cdot \|f_\varrho\|_{C^r}) \left(\frac{6\varrho}{d}\right)^{-d}. \quad (13)$$

To finish our lower bound, we will choose $\varrho$ such that $\|f_\varrho\|_{C^s} \leq \delta/\varepsilon$ for some $\delta \in (0, 1)$.

We bound the derivatives of $f_\varrho$ by induction. First of all note that $\|h_\varrho\|_{\infty} \leq 1$ and $\|D^\beta h_\varrho\|_{\infty} \leq 1/\varrho$ for $|\beta|_1 = 1$. Here, $D^\beta h_\varrho$ is the weak partial derivative of the Lipschitz-continuous function $h_\varrho$. Let $e_j = (0, \ldots, 0, 1, 0, \ldots, 0)$ be the $j$th unit vector. Then, for every $f \in L_{\infty}(\mathbb{R}^d)$, we have

$$D^{e_j}[f \ast g_\varrho](x) = \frac{1}{\text{vol}_d(\varrho B^d_1)} \int_{\mathbb{R}^d} f(x + t) \mathbb{1}_{\varrho B^d_1}(t) \, dt = \frac{1}{\text{vol}_d(\varrho B^d_1)} D^{e_j} \left( \int_{\mathbb{R}^d} f(x + s + he_j) \mathbb{1}_{\varrho B^d_1}(s + he_j) \, dh \, ds \right) = \frac{1}{\text{vol}_d(\varrho B^d_1)} \int_{e_j^1} D^{e_j} \left( \int_{\mathbb{R}^d} f(x + s + he_j) \mathbb{1}_{\varrho B^d_1}(s + he_j) \, dh \right) \, ds,$$
where \( e^\perp_j \) is the hyperplane orthogonal to \( e_j \). For any function \( \tilde{f} \) on \( \mathbb{R} \) of the form

\[
\tilde{f}(x) = \int_{x-a}^{x+a} g(y) \, dy
\]

with some continuous function \( g \) we have

\[
\tilde{f}'(x) = g(x + a) - g(x - a).
\]

Therefore

\[
\left| D^{e_j}[f \ast g_\rho](x) \right| = \left| \frac{1}{\text{vol}_d(\rho \cdot B^d_1)} \int_{(\rho/\rho)B^d_1 \cap e^\perp_j} \left[ f \left( x + s + h_{\max}(s) e_j \right) 
- f \left( x + s - h_{\max}(s) e_j \right) \right] \, ds \right|
\]

\[
\leq \frac{2 \text{vol}_{d-1}(\rho \cdot B^d_1 \cap e^\perp_j)}{\text{vol}_d(\rho \cdot B^d_1)} \| f \|_\infty = \frac{2 \text{vol}_{d-1}(\rho \cdot B^{d-1}_1)}{\text{vol}_d(\rho \cdot B^d_1)} \| f \|_\infty
\]

\[
= \frac{d}{\rho} \| f \|_\infty = \frac{d \rho}{\rho} \| f \|_\infty
\]

with

\[
h_{\max}(s) = \max\{ h \geq 0 \mid s + h e_j \in \rho \cdot B^d_1 \}.
\]

Moreover, using Young’s inequality, we obtain

\[
\| D^{e_j}[f \ast g_\rho] \|_\infty = \| (D^{e_j}f) \ast g_\rho \|_\infty \leq \| (D^{e_j}f) \|_\infty.
\]

Using these inequalities recursively, we see that \( f_\rho \in W^{r+1}_\infty(D^d) \) with

\[
\forall 1 \leq \ell \leq r + 1 : \max_{\beta \in \mathbb{N}^d : |\beta|_1 = \ell} \| D^\beta f_\rho \|_\infty \leq \rho^{-\ell} (d \rho)^{\ell-1}.
\]

This shows that

\[
\| f_\rho \|_{C^r} \leq \max \{ 1, \rho^{-r}(d \rho)^{r-1} \},
\]

if \( \rho \leq d \rho \). Note that we simply ignore the bounds on \( \| D^\beta f_\rho \|_\infty \) for \( |\beta|_1 = r + 1 \). We now choose \( \rho = (\varepsilon/\delta)^{1/r} (d \rho)^{1-1/r} \) to obtain \( \| f_\rho \|_{C^r} \leq \delta/\varepsilon \) if \( \varepsilon \leq \delta \). This already implies the lower bound in our main result.
**Theorem 3.** For any $r, d \in \mathbb{N}, \delta \in (0, 1)$ and $\varepsilon \in (0, \delta]$ we have

\[ n(\varepsilon, D_d, r) \geq (1 - \delta) c_r^d \left( \frac{\delta d}{\varepsilon} \right)^{d/r} \quad \text{with} \quad c_r = \frac{1}{6 \varepsilon r^{1-1/r}}. \]

Taking $\delta = d/(d + r)$, which maximizes the last bound, we obtain

\[ n(\varepsilon, D_d, r) \geq \frac{r}{r + d} \left( \frac{d}{d + r} \frac{1}{6 \varepsilon r^{1-1/r}} \right)^{d/r}. \]

**Proof.** Using the construction of the fooling function with $\rho = (\varepsilon / \delta)^{1/r} (dr)^{1-1/r}$ as above, we obtain from (13) that

\[ n \geq (1 - \delta) \left( \frac{6 \varepsilon \rho}{d} \right)^{-d} = (1 - \delta) \left( 6 \varepsilon r^{1-1/r} \right)^{-d} \left( \frac{\delta d}{\varepsilon} \right)^{d/r}. \]

Clearly, the function $f(\delta) = (1 - \delta)\delta^{d/r}$ is maximized for $\delta = d/(d + r)$ and substituting this $\delta$ to the previous bound we complete the proof.

Note that the second bound on $n(\varepsilon, D_d, r)$ proves (9).

### 2.2 Upper bounds

We describe a known upper error bound for numerical integration for functions defined over the cube $[0, 1]^d$ from the class $C_r^d = C^r([0, 1]^d)$, see [3]. We start with quadrature formulas $Q_m$, $m \in \mathbb{N}$, for the univariate case $d = 1$,

\[ Q_m(f) = \sum_{i=1}^{m} a_i f(x_i) \]

with $a_i \in \mathbb{R}, x_i \in [0, 1]$ and $f : [0, 1] \to \mathbb{R}$.

Then the (tensor) product rule $Q_m^d$ uses $m^d$ function values and is defined by

\[ Q_m^d(f) = \sum_{i_1=1}^{m} \cdots \sum_{i_d=1}^{m} a_{i_1} \cdots a_{i_d} f(x_{i_1}, x_{i_2}, \ldots, x_{i_d}), \]

where $f : [0, 1]^d \to \mathbb{R}$. 

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We now compare the worst case error $e(Q^d_m, C^r([0, 1]^d))$ with the error $e(Q_m, C^r([0, 1]))$. It is known that

$$e(Q^d_m, C^r([0, 1]^d)) \leq \left( \sum_{j=0}^{d-1} A^2 \right)^{1/2} e(Q_m, C^r([0, 1])) \quad \text{with} \quad A = \sum_{i=1}^m |a_i|. \quad (15)$$

Of course, we may use positive quadrature formulas with $a_i \geq 0$ for which $A = 1$.

For example, we may use the standard Gaussian formulas for the class $C^s([0, 1])$ with $s = 1, 2, \ldots, r$. Obviously, $C^r([0, 1]) \subseteq C^s([0, 1])$. Then we obtain

$$e(Q_m, C^r([0, 1])) \leq e(Q_m, C^s([0, 1])) \leq c_s m^{-s},$$

where for $m > s$ we have

$$c_s = \frac{\pi}{2} \left( \frac{e}{6\sqrt{3}} \right)^s.$$

This was proved in [5], where the interval $[-1, 1]$ was used instead of $[0, 1]$. Hence, we need to rescale the problem and multiply the estimate of [5] by $2^{-(s+1)}$. Observe that $c_s$ is exponentially small in $s$, but this works only if $m > s$. It is also known, see [14] p.127, that the $m$th minimal error for algorithms that use not necessarily positive coefficients satisfies

$$e_m(C^s([0, 1])) = K_s \left( 1 + o(1) \right) \quad \text{as} \quad m \to \infty,$$

where $K_s = 4/\pi \sum_{k=0}^{\infty} (-1)^k (2k+1)^{-s-1} \in [1, \pi/2]$ is the Favard constant. Hence $(e/(6\sqrt{3}))^s = (0.26\ldots)^s$ cannot be improved asymptotically more than $1/(2\pi)^s = (0.159\ldots)^s$.

For $s \in \{1, 2\}$ we do not need to assume that $m \geq s$. For $s = 1$ we can take the optimal midpoint rule, see the discussion below. For $s = 2$ a better bound was proved in [11].

From (15) and the discussion of the Gaussian error bounds we obtain for $s = 1, 2, \ldots, r$ and $m \geq s + 1$ that

$$e(Q^d_m, C^r([0, 1]^d)) \leq c_s d m^{-s}.$$

Then $e(Q^d_m, C^r([0, 1]^d)) \leq \varepsilon$ if we take

$$m = m_s = \max \left\{ s + 1, \left\lceil \left( \frac{c_s d}{\varepsilon} \right)^{1/s} \right\rceil \right\}.$$

Since $Q^d_m$ uses $m_s^d$ function values we have that $n(\varepsilon, C^r([0, 1]^d)) \geq \min_{s=1,2,\ldots,r} m_s^d$. This proves the following theorem.
Theorem 4. For any \( r, d \in \mathbb{N} \) and \( \varepsilon \in (0, 1) \) we have

\[
n(\varepsilon, d, r) \leq \min_{s=1,2,\ldots,r} \max \left\{ (s+1)^d \left( \left( \frac{\pi}{2} \left( \frac{e}{6\sqrt{3}} \right)^s \frac{d}{\varepsilon} \right)^{1/s} \right)^d \right\}.
\]

Note that this proves (7).

It is interesting to notice that if we want to consider all \( m \leq r \) then to have an estimate \( e(Q_m, C^r([0,1])) \leq c_r m^{-r} \) we need to chose \( c_r \) (super-)exponentially large in \( r \). Indeed, take \( m = 2 \) and consider the minimal worst case error \( e_2 \) of any two-point quadrature formula on the subset of the unit ball of \( C^r([0,1]) \) consisting of polynomials of degree at most 4. Since for any two points \( x_1, x_2 \), a suitable polynomial \( c(x - x_1)^2(x - x_2)^2 \) is in this unit ball for all \( r \) (with a positive \( c \) independent of \( r \)), we find by a compactness argument that \( e_2 > 0 \). Hence,

\[
e_2 \leq e(Q_2, C^r([0,1])) \leq c_r 2^{-r}
\]

implies \( c_r \geq e_2 2^r \) for all \( r \). Extending this argument to any fixed \( m_0 \) shows that there exist constants \( e_{m_0} \) such that \( c_r \geq e_{m_0} m_0^r \) for all \( r \).

For \( r = 1 \) we know more, see [13]. For \( d = 1 \) we take the optimal midpoint rule \( Q_m \) with error \( 1/(4m) \) and obtain

\[
e(Q_m, W^1_{\infty}([0,1])) \leq \frac{1}{4} \frac{1}{n^{1/d}}
\]

for \( n = m^d \). This is not quite optimal, but almost. It is known that, asymptotically for large \( d \), the optimal constant is \( d/(2e) \) instead of \( d/4 \).

3 Periodic smooth functions

In this section we study numerical integration over the domain \( D_d = [0,1]^d \) for the classes of periodic functions.

\[
C^r_\pi = \left\{ f \in C^r(\mathbb{R}^d): \text{ f is 1-periodic and } \|f\|_{C^r} \leq 1 \right\}.
\]

Hence, for all \( f \in C^r_\pi \) we have \( f(x+e) = f(x) \) for all \( x \in \mathbb{R}^d \) and all \( e = (e_1, e_2, \ldots, e_d) \) with \( e_j \in \{0,1\} \). Since \( C^r_\pi \subset C^r([0,1]^d) \), all lower bounds for the class of periodic functions also hold for the class \( C^r([0,1]^d) \). We present slightly larger lower bounds and smaller upper bounds for \( C^r_\pi \) compared to the results for the class \( C^r([0,1]^d) \) that are provided in the last section.
3.1 Lower bounds

We will follow the same arguments as for nonperiodic functions. For this let

\[ \mathcal{P} = \{x_1, x_2, \ldots, x_n\} \subset [0, 1]^d \]

be a collection of \( n \) points. Since we want to construct a periodic fooling function, we consider the extended infinite point set \( \tilde{\mathcal{P}} := \bigcup_{m \in \mathbb{Z}^d} (\mathcal{P} + m) \), and define for some \( \rho > 0 \) the function

\[ \tilde{h}_\rho(x) = \min \left\{ 1, \frac{\text{dist}(x, \tilde{\mathcal{P}}_\rho)}{\rho} \right\} \quad \text{for all } x \in \mathbb{R}^d, \]

where

\[ \tilde{\mathcal{P}}_\rho = \bigcup_{m \in \mathbb{Z}^d} \bigcup_{i=1}^n (\rho B_1^d(x_i + m)). \]

Again, \( B_1^d(x) \) is the \( \ell_1 \)-unit ball in \( \mathbb{R}^d \) around \( x \) and we write \( B_1^d \) for \( B_1^d(0) \). Clearly, \( \tilde{h}_\rho \) is a \( 1 \)-periodic function from \( W_1^\infty(\mathbb{R}^d) \).

Again, we consider the \( r \)-fold convolution of the function \( \tilde{h}_\rho \) with the normalized indicator functions \( g_\rho \) from (10), which we denote by \( \tilde{f}_\rho \). This function is obviously also \( 1 \)-periodic and satisfies the same bounds on the derivatives as given in (14). However, the lower bound on the integral of \( \tilde{f}_\rho \) over \( [0, 1]^d \) can be improved by the following argument.

Let \( g = g_\rho * \cdots * g_\rho \) be the \( r \)-fold convolution of the functions \( g_\rho \) such that \( \tilde{f}_\rho = \tilde{h}_\rho * g \). Then, using periodicity of \( \tilde{h}_\rho \), we obtain

\[
\int_{[0,1]^d} \tilde{f}_\rho(x) \, dx = \int_{[0,1]^d} \int_{\mathbb{R}^d} \tilde{h}_\rho(x-y) g(y) \, dy \, dx = \int_{[0,1]^d} \tilde{h}_\rho(x) \, dx \int_{\mathbb{R}^d} g(y) \, dy
\]

\[
= \int_{[0,1]^d} \tilde{h}_\rho(x) \, dx.
\]

Now we only use that \( \tilde{h}_\rho \) satisfies \( \tilde{h}_\rho(x) = 1 \) if \( \text{dist}(x, \tilde{\mathcal{P}}) > 2 \rho \) and obtain, similarly to (11), that

\[
\int_{[0,1]^d} \tilde{f}_\rho(x) \, dx \geq 1 - n \left( \frac{4 \epsilon \rho}{d} \right)^d.
\]

Note that there is an improvement of \( 2^{-d} \) in the “volume term”. Finishing the proof as for the nonperiodic case we obtain for any \( \delta \in (0, 1) \) and \( \varepsilon \in (0, \delta] \)

\[
n^{\text{per}}(\varepsilon, d, r) \geq (1 - \delta) c_r \left( \frac{\delta d}{\varepsilon} \right)^{d/r}
\]
with $c_r = 1/(4e^{r-1/r})$. We now choose $\delta$ to maximize the lower bounds and obtain for $\delta = d/(d+r)$ the following proposition.

**Proposition 1.** For any $r, d \in \mathbb{N}$, $\delta \in (0, 1)$ and $\varepsilon \in (0, \delta]$ we have

$$n_{\text{per}}(\varepsilon, d, r) \geq (1 - \delta) c_r^d \left( \frac{\delta d}{\varepsilon} \right)^{d/r}$$

with $c_r = \frac{1}{4e^{r-1/r}}$.

Taking $\delta = d/(d+r)$, which maximizes the last bound, we obtain

$$n_{\text{per}}(\varepsilon, d, r) \geq \frac{r}{r + d} \left( \frac{d}{d + r} \frac{1}{4e^{r-1}} \frac{d}{\varepsilon} \right)^{d/r}.$$ 

Note that this proves the lower bound of (6).

### 3.2 Upper bounds

For the univariate case, it was proved by Motornyı [6, 7], see also [14] pp. 119–122, that the rectangle (or trapezoidal) quadrature

$$Q_m(f) = \frac{1}{m} \sum_{i=0}^{m-1} f \left( \frac{i}{m} \right) \quad \text{for all} \quad f \in C_r^\pi$$

is optimal for numerical integration over the class $\tilde{W}_\infty^r$ of periodic functions whose $(r-1)$ derivatives are absolutely continuous and $\|f^{(r)}\|_\infty \leq 1$. Furthermore, the $n$th minimal error is $1/(2(2\pi)^r n^r)$. The domain of integration considered by Motornyı was $[0, 2\pi]$. Therefore we have to rescale the problem to switch to $[0, 1]$ which corresponds to multiplying the bound of Motornyı by $(2\pi)^{-(r+1)}$.

Since the class $C_r^\pi$ is a subset of $\tilde{W}_\infty^r$ this implies that

$$e_m(Q_m, C_r^\pi) \leq \frac{1}{2(2\pi)^r m^r}.$$ 

For the $d$-variate case, we take the tensor product rule $Q_m^d$ of $d$ copies of $Q_m$ that uses $m^d$ function values and is defined by

$$Q_m^d(f) = \frac{1}{m^d} \sum_{i_1=0}^{m-1} \cdots \sum_{i_d=0}^{m-1} f \left( \frac{i_1}{m}, \ldots, \frac{i_d}{m} \right) \quad \text{for all} \quad f \in C_r^\pi.$$ 

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The upper bound on the error of $Q^d_m$ follows again from [3] and we obtain

$$e(Q^d_m, C^r_\pi) \leq d e(Q_m, C^r_\pi) \leq \frac{d}{2 (2\pi)^r m^r}.$$  

Equating the upper bound to $\varepsilon$ we get the following proposition.

**Proposition 2.** For any $r, d \in \mathbb{N}$ and $\varepsilon \in (0, 1)$ we have

$$n_{\text{per}}(\varepsilon, d, r) \leq \left[ \frac{1}{2 (2\pi)^r \varepsilon} \right]^{1/r} d.$$  

Note that this proves the upper bound of (6).

4 Multivariate approximation and optimization

We worked with the formula (3) which says that, for the integration problem, the $n$th error bound $e_n(C^r(D_d))$ is given by

$$e_n(C^r(D_d), \text{INT}) = \inf_{x_1, \ldots, x_n} \sup_{f \in C^r(D_d), f(x_1) = \cdots = f(x_n) = 0} \left| \int_{D_d} f(x) \, dx \right|.  \tag{16}$$

Observe that we added the symbol INT since we now discuss two more problems, APP and OPT. For the approximation problem we have

$$\text{APP}(f) = S(f) = f$$

with $S : C^r(D_d) \to L_\infty$ and we measure the error in the norm of $L_\infty$. Then we obtain the analogous formula

$$e_n(C^r(D_d), \text{APP}) = \inf_{x_1, \ldots, x_n} \sup_{f \in C^r(D_d), f(x_1) = \cdots = f(x_n) = 0} \| f \|_\infty.  \tag{17}$$

Clearly we obtain

$$e_n(C^r(D_d), \text{INT}) \leq e_n(C^r(D_d), \text{APP})$$

and hence all lower bounds for integration also hold for the approximation problem.

It is less known that also for the problem of (global) optimization,

$$S_d(f) = \text{OPT}(f) = \sup_{x \in D_d} f(x),$$

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we get a very similar bound, namely

$$\frac{1}{2} e_n(C^r(D_d), \text{APP}) \leq e_n(C^r(D_d), \text{OPT}) \leq e_n(C^r(D_d), \text{APP}),$$

see Wasilkowski [15] and [8, 10] for this and similar results. Hence all the lower bounds of this paper are also true (after a trivial modification because of the factor $1/2$) for the problem of global optimization. Actually it would not be difficult to improve the constants slightly for the problems APP and OPT, but we do not go into the details.

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