Phase boundaries in algebraic conformal QFT

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Dedicated to Detlev Buchholz on the occasion of his 70th birthday.

Abstract

We study the structure of local algebras in relativistic conformal quantum field theory with phase boundaries. Phase boundaries (or “defects”) are instances of a more general notion of boundaries, that give rise to quite a variety of algebraic structures. These can be formulated in a common framework originating in Algebraic QFT, with the principle of Einstein Causality playing a prominent role. We classify the phase boundary conditions by the centre of a certain universal construction, which produces a reducible representation in which all possible boundary conditions are realized. While the classification itself reproduces results obtained in a different framework by other authors before (because the underlying mathematics turns out to be the same), the physical interpretation is quite different.

1 Introduction

Boundaries and defects are omnipresent phenomena in condensed matter systems and have been extensively studies in the setting of Euclidean quantum field theory. In contrast, boundaries and
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defects of relativistic quantum systems are less studied, especially the impact of the principle of Causality that must also be valid across the boundary. In view of the close (Osterwalder-Schrader) relation between Euclidean continuum field theory and relativistic quantum field theory, the two issues are expected to be closely connected, but the precise statement is not yet clear.

It is the aim of this work to develop a setup for the description of boundaries of relativistic quantum systems in the kinematically very simple case of two-dimensional conformal quantum field theory. Indeed, the expected relation with the Euclidean setting shows up in the relevant classification results.

To fix the ideas, let us introduce our terminology. A boundary is a timelike plane of co-dimension 1 in Minkowski spacetime, such as the plane $x = 0$. (In 2D, this is just the time axis.) While a given boundary itself necessarily breaks Poincaré invariance, in a relativistic description it can be moved to any other timelike plane.

“The physics” (i.e., the field content and the algebraic relations) may or may not be the same on both sides of the boundary. The extreme case is a “system in a (one-sided) box” with “no physics” on the other side of the boundary. We shall refer to such a boundary as a “hard boundary”. The opposite extreme has “the same physics” on both sides, with some discontinuity of the fields at the boundary. We shall refer to this as a “defect” [16, 17]. Between these extremes, there are many intermediate possibilities, among them the “phase boundaries” (also called “transmissive boundaries” [15]) which share the same stress-energy tensor (or other distinguished chiral fields) on both sides, while the bulk field content may change.

Because the two theories share the same stress-energy tensor, they share the same unitary representation of the conformal group on the Hilbert space. So the field operators of the theory on one side of the phase boundary can be translated to the other side by the translation unitaries – where they will in general fail to coincide with the fields of the theory on the other side. It is important to notice that the Hilbert space necessarily supports both sets of fields everywhere in Minkowski spacetime, but on either side of the boundary only one of them correspond to physical observables. The algebra generated by both sets of fields will in general fail to be local.

The first aim of this paper is to develop the algebraic framework to describe quantum field theories with phase boundaries that are compatible with the Einstein Principle of Causality. We shall address this issue in the case of 2D conformal QFT, where the available mathematical tools are most powerful, and where the most interesting classification results can be established.

The ordinary way to approach boundary problems is to impose boundary conditions, and study their implications. As it turns out in 2D CFT, one may indeed impose a continuum of boundary conditions on the chiral fields. E.g., energy and / or momentum may be required to be conserved. (A system in an energetically isolated box has a hard boundary where energy is conserved, but momentum is not.)

However, once a choice of the chiral boundary conditions is made, one can no longer freely choose the boundary conditions for the non-chiral bulk fields. Instead, these are “quantized” in a highly nontrivial way. It is the main purpose of this paper to quantitatively describe the quantization of boundary conditions for phase boundaries (including defects). The answer will be given in terms of a generalized Verlinde equation pertaining to the bimodule category of a modular tensor category. The case of hard boundaries has been treated earlier [6, 27, 28].

Our work is strongly influenced by, but at the same time largely complementary to the work by Fuchs, Fröhlich, Kong, Runkel, Schweigert, notably [16, 17, 13, 14, 15, 21]. The latter line
of research starts from a TFT-interpretation of partition functions and correlation functions of 2D Euclidean CFT in terms of certain topological amplitudes assigned to a three-dimensional volume bounded by two Riemann surfaces that carry the CFT degrees of freedom \cite{16}. The mathematical analysis is based on the modular tensor category of the chiral algebra. The properties of such categories discovered along the way and giving rise to many highly non-trivial and exciting mathematical structures, are explored, e.g., in \cite{14} \cite{21} \cite{7}. Independently, an approach to defects and boundaries in the setting of nets of von Neumann algebras recently appeared in \cite{1}.

Our work is influenced by these results, in that these structures become instrumental for many of our constructions and classifications, where the underlying modular tensor category is the DHR category of chiral superselection sectors \cite{8} \cite{11} \cite{20}. It is complementary, because the physical interpretation is completely different. Frobenius algebras in the category describe extensions (e.g., how to supplement chiral fields by bulk fields), and operations with Frobenius algebras (braided products and decompositions, the “centre” and the “full centre” \cite{14}) correspond to surprisingly elementary algebraic operations like quotients of free products, central decompositions and relative commutants of (nets of) local algebras.

The common feature is the association of modules and bimodules with boundary conditions. Because the relevant computation of central projections, that completes our classification of boundary conditions for phase boundaries, is quite hidden in \cite{16} \cite{14} \cite{21}, we present here a self-contained and comparably short new proof that benefits from some simplifications that occur in C* tensor categories, as opposed to the more general situation of tensor categories covered in the cited work.

Namely, in our application to relativistic QFT, the relevant category can be concretely realized as a category $\text{End}_0(N)$ of finite-index endomorphisms of a type $\text{III}$ von Neumann algebra $N$, and inherits the C* structure and notion of positivity from the latter. In compliance with earlier work on the classification of subfactors in terms of a Frobenius algebra in this C* tensor category \cite{25}, we shall use the term “Q-system” for a Frobenius algebra in $\text{End}_0(N)$ (or related C* tensor categories). In all that follows, the important thing to be kept in mind is the one-to-one correspondence between extensions of a local QFT and Q-systems in its DHR category.

The article is organized as follows. In Sect. 2, we present some heuristics in a field theoretic language. In particular, we recall the relativistic version of the critical Ising model. This model involves a pair of fields, called the “order parameter” $\sigma$ and the “disorder parameter” $\mu$, which each satisfy local commutation relations with itself, but which satisfy certain non-local commutation relations with each other \cite{35}. They can therefore not arise simultaneously in a QFT that satisfies the Principle of Causality. However, the commutation relations are such that $\sigma(x)$ commutes with $\mu(y)$ if $x$ is in the left spacelike complement of $y$ (“left locality”). This situation is prototypical for a defect: if $\sigma$ belongs to the field content to the left of the boundary, and $\mu$ to the right, then Causality is satisfied. Since $\sigma$ and $\mu$ are algebraically isomorphic, one has isomorphic field algebras (“the same physics”) on both sides of the boundary, but a discontinuity at the boundary.

The bottom-line of these considerations is that one has to study simultaneous extensions of a pair of local quantum field theories. Since every extension of a given field algebra yields a representation of the latter by restriction, the issues of interest are of a representation-theoretic nature. The best framework to address them is therefore Algebraic Quantum Field Theory, which captures the positive-energy representation theory in terms of the DHR category of superselection sectors. This is a braided C* tensor category which we shall briefly review in Sect.
3. At this point, all the powerful categorical machinery becomes available. Ultimately, we will be able to describe the two-dimensional boundary theory in terms of the DHR category of the underlying chiral algebra. The latter is known to be modular whenever the chiral algebra is completely rational [20].

In Sect. 4, we introduce the basic operations with possibly nonlocal extensions of local quantum field theories: The passage to maximal local intermediate extensions, and the “braided products” of extensions which amounts to the merging of the charged fields of two extensions into a single extension. We present these operations as operations on local nets of von Neumann algebras in the AQFT setting, and relate them to the corresponding operations with Frobenius algebras (Q-systems).

The braided product can be understood as a quotient of a free product of two extensions of the same underlying chiral algebra by the obvious relations due to the identification of the common chiral algebra, and by a single additional relation imposing left or right locality. It therefore necessarily violates “one half” of locality. New local models can be obtained by passing to one of the maximal local intermediate extensions.

In particular, a simple algebraic understanding of the previous “α-induction construction” of [33] emerges: Its local algebras are relative commutants of algebras of a nonlocal braided product, associated with wedge regions. This puts the α-induction construction on the same footing with the construction of hard boundary theories in terms of relative commutants [27].

Finally, in Sect. 5, we address the issue of boundary conditions at phase boundaries. Recall that the bulk algebras on both sides of the boundary are two local extensions of the common chiral subtheory, defined on the same Hilbert space, with the left extension being left-local w.r.t. the right extension. But these are precisely the relations defining the braided product of two local extensions (obtained by the appropriate product of Q-systems) as a quotient of the free product, and there is no need to descend to a maximal local intermediate extension. Instead, the boundary QFT is a representation of the product algebra, restricted to the relevant local subtheories on either side of the boundary.

It turns out that the braided product of two local extensions in general has a centre, and its irreducible representations are in one-to-one correspondence with its minimal central projections. Thus, in order to classify boundary conditions, the task is to identify the minimal central projections in a braided product of commutative Q-systems.

This can be completely accomplished in case the two Q-systems arise by the α-induction construction from two chiral Q-systems $A$, $B$ in a modular tensor category, as above. (Here is in fact the only point where modularity of the category is required.) The result is that the minimal projections are in one-to-one correspondence with $A$-$B$-bimodules between the pair of chiral Q-systems. The central projections implement certain relations between the fields on both sides of the boundary (e.g., they may coincide, flip sign, or stand at certain angles to each other). These are the “quantized boundary conditions” mentioned earlier. The precise values of the angles are given by matrix elements of generalized Verlinde matrices that diagonalize the bimodule fusion rules.

This classification in terms of chiral bimodules was in fact previously discovered in the formulation of two-dimensional Euclidean CFT in terms of three-dimensional topological CFT [16, 13, 14, 15, 21]. We shall present a streamlined proof that takes advantage of the simplifications occurring in $C^*$ tensor categories, by using a positivity argument in the decisive step.
2 Heuristic considerations

We rephrase in a Lorentzian setting the characterization of [15], who are working in a Euclidean setting. Spacetime is two-dimensional Minkowski space with coordinates \((t, x)\). Without loss of generality, we place the boundary at \(x = 0\), so the boundary is the time axis in \(\mathbb{M}_2\).

2.1 Boundary conditions for chiral fields

We denote the subspace \(x > 0\) by \(\mathbb{M}_R\), and the subspace \(x < 0\) by \(\mathbb{M}_L\). On both sides of the boundary, we have left- and right-moving chiral fields, for which we reserve the labels + and -.

Let us accordingly denote the chiral components of the stress-energy tensor on either side of the boundary by \(T^L_+(t + x)\), \(T^L_-(t - x)\), \(T^R_+(t + x)\), \(T^R_-(t - x)\) (where the former two are defined at \(x < 0\), and the latter two at \(x > 0\)). Then the energy conservation at the boundary \((T^L_0(0) = T^R_0(0))\) reads

\[
T^L_+(t) + T^R_-(t) = T^R_+(t) + T^L_-(t). \tag{2.1}
\]

There are several types of solutions to this condition: the reflective solution

\[
T^L_+(t) = T^L_-(t), \quad T^R_+(t) = T^R_-(t),
\]

the transmissive solution

\[
T^L_+(t) = T^R_+(t), \quad T^L_-(t) = T^R_-(t), \tag{2.2}
\]

or combinations of both, like

\[
T^L_+ = T_1 + T_2, \quad T^R_+ = T_3 + T_4, \quad T^R_+ = T_1 + T_3, \quad T^L_+ = T_2 + T_4,
\]

where \(T_k\) are independent chiral stress-energy tensors, and \(T_1\) and \(T_4\) are transmitted while \(T_2\) and \(T_3\) are reflected. There are more solutions which are of neither type, eg, if \(j_1\) and \(j_2\) are two independent chiral currents, then

\[
T^L_+ = j_2^2, \quad T^R_+ = j_2^2, \quad T^R_+ = (\cos \alpha j_1 + \sin \alpha j_2)^2, \quad T^L_+ = (- \sin \alpha j_1 + \cos \alpha j_2)^2
\]

for any angle \(\alpha\) also solve the condition Eq. (2.1). In this case, the boundary effect is a gauge transformation of the \(O(2)\)-symmetric doublet \((j_1, j_2)\).

Since causality should also hold in the presence of a boundary, one obtains conditions on the local commutativity among the four stress-tensors involved. (The argument is standard, exploiting the fact that a chiral field, say \(\phi_+(u)\), is localized at any point \((t, x)\) such that \(t + x = u\), with the restriction that, say for \(T^R_+\), \(x\) must be positive. Whenever one can find pairs of two such points which are spacelike separated, then the two field operators must commute.) First considering points at equal time in \(\mathbb{M}_R\), one finds that both \(T^R_+\) must be local fields (i.e., commute with themselves except at coinciding points), and that \(T^R_+\) must be forward local w.r.t. \(T^R_-\) (i.e., \([T^R_+(u), T^R_-(v)] = 0\) whenever \(u > v\)). Similarly, \(T^L_\pm\) must be local, and \(T^L_\pm\) must be

\[1\] corresponding to analytic and anti-analytic fields in the Euclidean framework. The renaming of “\(M_+\)” in [27] into \(\mathbb{M}_R\) in this paper is made in order to prevent confusion.

\[2\] This condition is called “conformal defect” in [15], whereas the more special case Eq. (2.2) is called “topological”. In [15], Eq. (2.2) is referred to as “conformal”.

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forward local w.r.t. $T^L_L$. Considering points at equal time on opposite sides of the boundary, one finds that $T^R_R$ must commute with $T^L_L$, and $T^L_L$ must commute with $T^R_R$ (that is: at all points, also coinciding ones).

But requiring also Möbius covariance of the chiral fields, forward locality implies backward locality because the fields on $\mathbb{R}$ extend to the compactification $S^1$, so the two mutually commuting fields $T^R_R$, $T^L_L$ must in fact be relatively local w.r.t. to the two mutually commuting fields $T^L_L$, $T^R_R$ (that is, the commutator is supported at equal points). Thus, all four fields are mutually local w.r.t. each other, and $T^L_L$ commutes with $T^R_R$, and $T^R_R$ commutes with $T^L_L$. These conditions are met in all the examples above.

In the reflective case, it follows that the two sides completely decouple, and the theory may be regarded as a tensor product of two hard boundary theories on either side, living on independent Hilbert spaces.

On the other hand, the transmissive solution Eq. (2.2) may also be characterized by the second condition supplementing Eq. (2.1)

$$T^L_L(t) - T^R_R(t) = T^R_R(t) - T^L_L(t). \quad (2.3)$$

This equation just expresses the conservation of momentum at the boundary. (Clearly, momentum is not conserved at a hard boundary.) While we always want to impose energy conservation by default, momentum conservation is equivalent to the transmissivity condition for the stress-energy tensors. Notice that the commutativity between $T^L_L$ and $T^R_R$ then implies that $T^L_L = T^R_R$.

In general, there may be other chiral fields. In fact, every two-dimensional conserved rank-$n$ symmetric traceless tensor current splits into a pair of two chiral fields of scaling dimension $n$. Thus one can imagine that by imposing or not imposing conservation of the associated charge at the boundary, one obtains a vast multitude of possibilities, among which hard boundaries (reflective for all chiral fields) and phase boundaries (transmissive for all chiral fields) are only extremal cases.

In general, the boundary will be transmissive for a subset of chiral fields if it preserves the corresponding symmetries, so that the two field theories on both sides of the boundary share the transmitted chiral fields.

### 2.2 Bulk fields

Apart from the transmitted (hence common) chiral fields, the quantum field theories on each side may have more chiral fields that are not transmitted, and each side may have further local fields that are not chiral. Thus both sides are extensions of quantum field theories of the form $\mathcal{A}_+ \otimes \mathcal{A}_-$, standing for the algebras of the transmitted left- and right-moving chiral fields on the respective lightrays $\mathbb{R}$.

Extensions of local field theories have been studied in the operator-algebraic setting, starting with [8], and further developed for chiral conformal quantum field theories in [26] generalizing and conceptualizing the notion of “conformal embeddings” [34]. The crucial assumption is that the relative commutant of the subtheory in the extension is trivial; this ensures that the subtheory contains the total stress-energy tensor of the full theory, since otherwise the relative commutant is a QFT of its own (the coset theory [18] in the chiral case) which commutes with the given subtheory. This assumption is certainly not true in the case of a hard boundary, where the
transmitted part of the stress-energy tensor is zero. On the other hand, if \( \mathcal{A}_+ \otimes \mathcal{A}_- \) is nontrivial, there may exist different extensions with trivial relative commutant which qualify as different “phases” describing the possible local quantum field theories on either side of a transmissive boundary. For this reason, we also refer to transmissive boundaries as phase boundaries.

In particular, the theory of local extensions [26] is applicable in the case of phase boundaries, including all energy and momentum conserving boundaries.

The transmitted left- and right-moving chiral fields \( \mathcal{A}_+ \) and \( \mathcal{A}_- \) may be specified independently. Most of the literature refers to the case when they are in fact isomorphic. This may be justified by the option to have several boundaries, including hard ones: As we have seen, a hard boundary on one side forces the right-moving and left-moving fields to coincide in the bulk between this boundary and a transmissive boundary on the other side.

Only for special issues: the canonical local extension (Sect. 4.6), the \( \alpha \)-induction construction (Sect. 4.10), and the complete classification of boundary conditions in terms of bimodules (Sect. 5.4), we shall limit our treatment to the case with isomorphic left- and right-moving chiral fields, i.e., we shall regard the bulk CFT on both sides of the boundary as (different) local extensions of the same chiral subtheory of the form \( \mathcal{A} \otimes \mathcal{A} \) with a fixed chiral theory \( \mathcal{A} \). In this case, it is well known that there is always a “canonical” local extension of \( \mathcal{A} \otimes \mathcal{A} \) (Prop. 4.5), while more general extensions can be obtained by the \( \alpha \)-induction construction [33]. In fact, every extension with trivial relative commutant is intermediate between the trivial extension \( \mathcal{B}_2 = \mathcal{A} \otimes \mathcal{A} \) and one of the \( \alpha \)-induction extensions [28].

### 2.3 Left locality

In the presence of a transmissive boundary, denote the observable fields to the left and to the right of the boundary by \( \Psi^L \) and \( \Psi^R \). Both field algebras contain the common chiral stress-energy tensor \( T_\pm \).

Since the stress-energy tensor provides the densities for the generators of space and time translations, the fields \( \Psi^L \) and \( \Psi^R \), defined on either side of the boundary, in fact extend to the full Minkowski spacetime by means of translations. However, on the “wrong side”, they are not necessarily observables.

The Principle of Causality requires that observables at spacelike separation commute. Hence \( \Psi^L(x) \) must commute with \( \Psi^R(y) \) whenever \( x \) is to left of the boundary, \( y \) is to the right of the boundary, and \( (x - y)^2 < 0 \). Using translations, the same must be true whenever \( x \) is to the spacelike left of \( y \). This motivates the

**Definition 2.1.** A quantum field theory with fields \( \Psi^L \) is called left-local w.r.t. a QFT with fields \( \Psi^R \) defined on the same Hilbert space, if

\[
[\Psi^L(x), \Psi^R(y)] = 0
\]

whenever \( x \) is to the spacelike left of \( y \).

Then, the construction of a transmissive boundary between two given quantum field theories amounts to finding a representation of both algebras of fields \( \Psi^L \) and \( \Psi^R \) on the same Hilbert space such that \( \Psi^L \) is left-local w.r.t. \( \Psi^R \), and both stress-energy tensors are represented by the same operators.
2.4 Example 1: The Ising model

The prototype of a phase boundary may occur in the Ising model. We briefly review the basic results for the relativistic Ising model, as studied by [35]. Recall that the local fields of the Ising model are the two chiral components of stress-energy tensor with \( c = \frac{1}{2} \), a local field \( \varepsilon(t, x) \) with scaling dimensions \( (\frac{1}{2}, \frac{1}{2}) \), and another local field \( \sigma(t, x) \) with scaling dimensions \( (\frac{1}{16}, \frac{1}{16}) \). The theory may be extended by the chiral Fermi fields \( \psi_+(t + x) \) and \( \psi_-(t - x) \) on a larger Hilbert space, such that

\[
\varepsilon(t, x) = -i\psi_+(t + x)\psi_-(t - x). \tag{2.4}
\]

(The imaginary factor is necessary to make \( \varepsilon \) hermitean when \( \psi_\pm \) are hermitean.) In particular, both \( \psi_\pm \) are relatively local w.r.t. to \( \varepsilon \). In contrast, they are not relatively local w.r.t. to \( \sigma(t, x) \): one has instead the commutation relations at spacelike distance (displayed at equal times \( t = 0 \))

\[
\sigma(0, x)\psi_+(y) = \pm \psi_+(y)\sigma(0, x), \tag{2.5}
\]

\[
\sigma(0, x)\psi_-(y) = \pm \psi_-(y)\sigma(0, x),
\]

with the sign given by \( \pm = \text{sign}(x - y) \). (Notice that by our coordinate conventions, \( \psi_\pm(\pm y) \) are both localized at the point \( (0, y) \).) One may finally introduce the (short-distance regularized) operator product

\[
\mu(t, x) = \sqrt{i} :\psi_+(t + x)\sigma(t, x) : = \sqrt{-i} :\psi_-(t - x)\sigma(t, x) :. \tag{2.6}
\]

This field is known as the dual field. Indeed, the fields \( T_\pm, \varepsilon, \mu \) have exactly the same correlation functions as the fields \( T_\pm, \varepsilon, \sigma \). (In the Euclidean version of the Ising model, \( \mu \) is known as the order parameter, indicating the phase transition at the critical temperature.) Conversely,

\[
\sigma(t, x) = \sqrt{i} :\psi_-(t - x)\mu(t, x) : = \sqrt{-i} :\psi_+(t + x)\mu(t, x) :. \tag{2.7}
\]

Now, a quantum field theory with a phase boundary at \( x = 0 \) is given as follows. Its generating local fields on the right halfplane are \( T_\pm, \varepsilon, \sigma \), and its local fields on the left halfplane are \( T_\pm, \varepsilon, \mu \). By the commutation relations Eq. (2.5), the field \( \mu \) is left-local w.r.t. the field \( \sigma \). In particular,

\[
\mu(x)\sigma(y) = \pm \sigma(y)\mu(x) \quad ((x - y)^2 < 0, x \in \mathbb{M}_L, y \in \mathbb{M}_R). \tag{2.8}
\]

Thus, the theory satisfies causality also across the phase boundary.

Because both \( \sigma \) and \( \mu \) must be represented on the same Hilbert space, the common Hilbert space is the enlarged Hilbert space that accommodates also the chiral Fermi fields. (A detailed analysis of the structure of this Hilbert space has been presented in [12]. As a representation of the two chiral Virasoro algebras, it has the form

\[
\mathcal{H} = (\mathcal{H}_0 \oplus \mathcal{H}_\frac{1}{2}) \otimes (\mathcal{H}_0 \oplus \mathcal{H}_\frac{1}{2}) \oplus 2(\mathcal{H}_\frac{1}{16} \otimes \mathcal{H}_\frac{1}{16}), \tag{2.9}
\]

where the representation \( \frac{1}{16} \otimes \frac{1}{16} \) occurs twice.)

Finally, by the equality of the correlation functions as stated above, the local field algebras on both sides of the boundary are perfectly isomorphic, though different subalgebras of the nonlocal algebra generated by the order and disorder fields Eq. (2.6)–Eq. (2.7).
3 Theory of superselection sectors

Our aim is the description and construction of local quantum observables in the presence of a phase boundary. The classification of the possible boundary conditions is an issue of representation theory. For this kind of questions, the setting of algebraic quantum field theory is the most appropriate \[8\]. Apart from the most powerful description of representations in terms of endomorphisms, which allows to exploit the full power of the theory of subfactors, it also has the benefit of sparing the necessity of short-distance regularizations of operator products, and other issues of technical nature.

We shall give a brief review of the underlying DHR theory and the use of Q-systems in order to characterize extensions of a given QFT. A Q-system is essentially the same thing as a Frobenius algebra in a tensor category, with the additional qualification that the tensor category carries a natural C* structure, inherited from the local von Neumann algebras. This feature simplifies some of the general results about Frobenius algebras. For us, the important point is that a Q-system allows to grasp all algebraic features of an extension (commutation relations, operator product expansion) in terms of finitely many data, as we will explain below.

In the sequel of the paper (Sect. \[4\]), we shall review various operations with Q-systems, known from purely categorical approaches to CFT, and establish their meaning in terms of simple algebraic operations like relative commutants. It then turns out (Sect. \[5\]) that boundaries are described by braided product Q-systems which precisely ensure the Principle of Causality for observables on the two sides of the boundary. The hard part is the identification of the irreducible representations of the resulting “product algebra”, which amounts to the computation of its centre.

3.1 Algebraic QFT

Algebraic quantum field theory starts from the net of local algebras. This means the assignment \(O \mapsto \mathcal{A}(O)\) where \(O\) is a spacetime region, and \(\mathcal{A}(O)\) the algebra of observables localized in \(O\). The net is subject to the Haag-Kastler axioms, which just “transcribe” the Wightman axioms without referring to individual fields as generators of the local algebras. Instead, the latter may be taken as von Neumann algebras (provided the causal complement of \(O\) is nontrivial). In the case at hand, we assume two nets of (left- and right-moving) chiral algebras, indexed by the open intervals \(I \subset \mathbb{R}\) of the real line (= future-directed light rays). A two-dimensional quantum field theory is then an extension

\[
\mathcal{A}_+(I) \otimes \mathcal{A}_-(J) \equiv \mathcal{A}_2(O) \subset \mathcal{B}_2(O),
\]

where

\[
O = I \times J = \{(t, x) : t + x \in I, t - x \in J\}
\]

are the double cones in Minkowski spacetime \(\mathbb{M}_2\). Extensions are assumed to be covariant, relatively local, i.e., \(\mathcal{B}_2(O_1)\) commutes with \(\mathcal{A}_2(O_2)\) if \(O_1\) and \(O_2\) are spacelike separated, and irreducible, i.e., the relative commutant \(\mathcal{A}_2(O)' \cap \mathcal{B}_2(O)\) is trivial. If \(\mathcal{B}_2\) is itself local, we call it a local extension.

At some point, we shall be forced to admit finite-dimensional relative commutants. This obviously still excludes the trivial possibility of extensions via tensor products with independent theories. In particular, the “coset stress-energy tensor” \[15\] is trivial, because it is affiliated with
the relative commutant, hence $B_2$ and $A_2$ share the same stress-energy tensor (i.e., a common Virasoro subnet), and the generators of the local diffeomorphisms of $A_2$ also generate the local diffeomorphisms of $B_2$.

It is clear that every irreducible (positive-energy) representation of $B_2$ restricts to a (positive-energy) representation of $A_2$. The latter may be reducible. This explains why extensions are an issue of representation theory.

### 3.2 Review of DHR theory

By default, we shall assume Haag duality for the chiral observables. This means $\mathcal{A}(I) = \mathcal{A}(I')'$, where $I'$ is the complement in $\mathbb{R}$ (= the union of two halfrays). (On the conformal completion $S^1$ of $\mathbb{R}$, Haag duality is automatic, whereas on $\mathbb{R}$ it is a slightly stronger property, equivalent to “strong additivity”.) This implies [5] that all positive-energy representations are given by “DHR endomorphisms” $\rho$ of the net $\mathcal{A}$, namely

$$\pi \equiv \pi_0 \circ \rho,$$

where $\pi_0$ is the vacuum representation. The DHR endomorphisms are localized in some interval $I$ in the sense that $\rho(a) = a$ for all $a \in \mathcal{A}(I)$. The localization interval may be varied arbitrarily, namely for every other interval $\tilde{I}$, one can chose unitary “charge transporters” such that $\tilde{\rho}(\cdot) = Ad_{\rho}u(\cdot) = u\rho(\cdot)u^*$ is localized in $\tilde{I}$.

Intertwiners between localized endomorphisms $\rho, \sigma$ are operators $t$ satisfying $t\rho(a) = \sigma(a)t$ for all $a \in \mathcal{A}$. We write $t : \rho \rightarrow \sigma$. By Haag duality, every intertwiner $t : \rho \rightarrow \sigma$ between two localized endomorphisms is an element of some local algebra, hence $t_1\rho_1(t_2) = \sigma_1(t_2)t_1$ is an intertwiner : $\rho_1\rho_2 \rightarrow \sigma_1\sigma_2$.

The composition of DHR endomorphisms defines a “fusion” product of representations. The composition of localized endomorphisms is commutative when the localization intervals do not overlap. In general, it is commutative only up to unitary equivalence, implemented by unitary statistics operators (“braiding”)

$$\varepsilon_{\rho,\sigma} : \rho\sigma \rightarrow \sigma\rho.$$

In chiral theories, the braiding is defined [11] by

$$\varepsilon_{\rho,\sigma} := \sigma(u_\rho^*u^\sigma_\rho\rho(u_\sigma)) : \rho\sigma \rightarrow \rho\tilde{\sigma} \rightarrow \tilde{\rho}\tilde{\sigma} = \tilde{\sigma}\tilde{\rho} \rightarrow \sigma\rho,$$

where $u_\rho, u_\sigma$ are unitaries such that the auxiliary DHR endomorphism $\tilde{\rho} = Ad_{u_\rho}\rho$ and $\tilde{\sigma} = Ad_{u_\sigma}\sigma$ are localized in auxiliary intervals $I_\rho, I_\sigma$ such that $I_\sigma < I_\rho$ (“$\sigma$ is localized in the past of $\rho$”). The braiding is independent of the auxiliaries with the given specifications. In particular, one has

$$\varepsilon_{\rho,\sigma} = 1 \quad \text{whenever $\sigma$ is localized in the past of $\rho$} \quad (3.1)$$

(because one may just take $u_\rho = u_\sigma = 1$). Actually, Eq. (3.1) may be regarded as an “initial condition” which completely fixes the braiding. The opposite braiding $\varepsilon^{-}_{\rho,\sigma} \equiv \varepsilon^{*}_{\sigma,\rho}$ would have resulted with the opposite ordering of the auxiliary intervals.

These structures and data turn the DHR representation theory of $\mathcal{A}$ into a braided C* tensor category $\mathcal{C} = \mathcal{C}^\text{DHR}(\mathcal{A})$. This category is simple ($\text{Hom}(\text{id}, \text{id}) = \mathbb{C}$), strict (the composition is associative), and direct sums and subobjects corresponding to projections $p \in \text{Hom}(\rho, \rho)$ are defined.
3.3 Example 2: The Virasoro net with $c = \frac{1}{2}$ (= the “chiral Ising model”)

The theory has three sectors: $\pi_0$ (the vacuum representation), $\pi_{\tau}$ and $\pi_{\sigma}$ satisfying the fusion rules $\tau \times \tau \simeq \text{id}$, $\tau \times \sigma \simeq \sigma \times \tau \simeq \sigma$, $\sigma \times \sigma \simeq \text{id} \oplus \tau$. These are realized by localized endomorphisms $\text{id}$, $\tau$, $\sigma$ which can be chosen in their unitary equivalence class such that $\tau^2 = \text{id}$, $\tau \sigma = \sigma$, $\sigma \tau = \text{Ad}_u \sigma$, where $u \in \text{Hom}(\sigma, \sigma \tau) = \text{Hom}(\sigma \tau, \sigma)$ is a unitary with $u^2 = 1$. One may choose a pair of intertwiners $r \in \text{Hom}(\text{id}, \sigma^2)$, $t \in \text{Hom}(\tau, \sigma^2)$ such that

$$
\tau(r) = t, \quad \tau(t) = r, \quad \tau(u) = -u,
\sigma(r) = (r + t)/\sqrt{2}, \quad \sigma(t) = (r - t)u/\sqrt{2}, \quad \sigma(u) = rt^* + tr^*.
$$

The braiding is given by

$$
\varepsilon_{\tau,\tau} = -1, \quad \varepsilon_{\tau,\sigma} = \varepsilon_{\sigma,\tau} = -iu, \quad \varepsilon_{\sigma,\sigma} = \kappa_\sigma^{-1}(rr^* + itt^*) \quad (\kappa_\sigma = e^{2\pi i/16}).
$$

3.4 Two dimensions

In two dimensions, the analysis is essentially the same, replacing intervals by double cones, and taking $O'$ to be the causal complement in $M_2$ (= the union of two wedge regions). The braiding is defined such that

$$
\varepsilon_{\rho,\sigma} = 1 \quad \text{whenever } \sigma \text{ is localized in the spacelike left of } \rho.
$$

In particular, the irreducible localized endomorphisms of $A_2 = A_+ \otimes A_-$ are tensor products $\rho = \rho^+ \otimes \rho^-$, and, in view of Eq. (3.1), the convention Eq. (3.2) means that the two-dimensional braiding is

$$
\varepsilon_{\rho,\sigma} = \varepsilon_{\rho^+,\sigma^+} \otimes \varepsilon_{\sigma^-,\rho^-}.
$$

The DHR category of $A_2 = A_+ \otimes A_-$ is the completion under direct sums $C^\text{DHR}(A_2) = C^\text{DHR}(A_+) \boxtimes C^\text{DHR}(A_-)^{\text{opp}}$ of the tensor product of categories $C^\text{DHR}(A_+) \otimes C^\text{DHR}(A_-)^{\text{opp}}$.

3.5 Extensions

In Sect. 4 we shall describe extensions $A(O) \subset B(O)$ of local quantum field theories $A$ in terms of the DHR category $C^\text{DHR}(A)$. Notably, we shall construct “charged fields” $\Psi$ (along with their algebraic relations) that together with the subalgebra $A$ generate $B$.

For this approach to be applicable, we may assume complete rationality, which implies that $A$ possesses only finitely many inequivalent positive-energy representations, and all of them have conjugates and finite statistical dimension (to be defined below) [31]. However, complete rationality becomes truly essential only for the final classification of the boundary conditions, because it ensures modularity of the representation category.

Thus, if we do not want to assume complete rationality, we may restrict to the full subcategory of endomorphisms with finite statistical dimension. In either case, all extensions have finite index.
3.6 Phase boundaries

A phase boundary is given by specifying two local extensions $B^L_2$ and $B^R_2$, such that

$$B_2(O) = \begin{cases} B^L_2(O) & \text{if } O \text{ lies to the left of the boundary.} \\ B^R_2(O) & \text{if } O \text{ lies to the right of the boundary.} \end{cases}$$

Since both extensions contain the common subnet $A_2$ of chiral observables by definition Eq. (2.2) of a phase boundary, and $A_2$ contains the generators of the local diffeomorphisms, the local algebras of both nets $B^L_2$ and $B^R_2$ are in fact defined for every region $O \subset \mathbb{M}_2$. They just do not qualify as observables on the “wrong” side of the boundary. Causality requires that the algebras $B^L_2(O_1)$ and $B^R_2(O_2)$ commute whenever $O_1 \subset \mathbb{M}^L$ and $O_2 \subset \mathbb{M}^R$ are spacelike separated. By diffeomorphism covariance, the same must be true whenever $O_1$ is to the spacelike left of $O_2$.

In fact, the actual location of the boundary is perfectly arbitrary, and it may be any timelike curve. This feature is referred to as “topological” in [16].

As in Sect. 2.3 we define “left locality”:

**Definition 3.1.** For two nets of local algebras $B_1(O)$ defined on the same Hilbert space, we say that $B_1$ is left-local w.r.t. $B_2$, if $B_1(O_1)$ commutes with $B_2(O_2)$ whenever $O_1$ is in the left causal complement of $O_2$. $B_1$ is right-local w.r.t. $B_2$, iff $B_2$ is left-local w.r.t. $B_1$. (In the chiral situation, the more appropriate terms “past-local” and “future-local” may be used instead [29].)

If $B^L$ and $B^R$ are given on the same Hilbert space, one may consider the local algebras

$$C_2(O) = B^L_2(O) \vee B^R_2(O).$$

$C_2$ is another extension of $A_2$, but $C_2$ will in general be nonlocal, because $B^L_2$ may (and will) not be also right-local w.r.t. $B^R_2$, cf. Example Eq. (2.4). A boundary condition is the precise “relative algebraic position” of the local subtheories $B^L_2$ and $B^R_2$ in $C_2$.

Our strategy (cf. Sect. 5) will not assume $B^L$ and $B^R$ to be given on the same Hilbert space from the outset. Instead, we shall present a “universal construction” for $C_2$ along with a reducible vacuum representation, in such a way that every irreducible sub-representation (each containing a vacuum vector) contains both $B^L$ and $B^R$, and implements one boundary condition.

Along the way, we shall completely work out the example of the Ising model. The local field content of the two-dimensional QFT $B_2$ without a boundary is given by the chiral observables and the primary fields $\Psi_{\tau \sigma}$ and $\Psi_{\sigma \tau}$ (the bounded operator AQFT counterparts of the fields $\varepsilon$ and $\sigma$ fields above). The universal algebra $C_2$ is generated by two copies of them, $B^L_2$ and $B^R_2$ sharing the same chiral observables $A_2$. There are three inequivalent subrepresentations: in the first $\Psi^L = \Psi^R$ for both primary fields. This is just the theory without a boundary. In the second, we have $\Psi^L_{\tau \sigma} = \Psi^R_{\tau \sigma}$, but $\Psi^L_{\sigma \tau} = -\Psi^R_{\sigma \tau}$, ie, the field $\sigma$ changes sign at the boundary. In the third, we have $\Psi^L_{\tau \sigma} = -\Psi^R_{\tau \sigma}$, whereas $\Psi^L_{\sigma \tau}$ and $\Psi^R_{\sigma \tau}$ remain independent fields. The product $\Psi^L_{\sigma \tau} \cdot \Psi^R_{\sigma \tau}$ turns out to split into two new primary fields: a leftmoving and a rightmoving chiral Fermi field. Of course, the Fermi fields cannot be constructed out of observables on either side. (The detailed computations are sufficiently simple to be carried out by hand, see Example 6 below.)
4 Extensions and Q-systems

We review the characterization of extensions of quantum field theories in terms of Q-systems [26], in particular, the construction of the extended local algebras $B(O)$ from $A(O)$ and the Q-system. We then show that the centre of a Q-system corresponds to a maximal local intermediate extension, which can be obtained in a simple algebraic way.

4.1 Q-systems and subfactors

Let us start with a review of the necessary facts from subfactor theory.

The local algebras in (conformal) quantum field theory are known to be type III factors $M$. This implies that every projection $e \in M$ is equivalent to the identity in the sense that there is an operator $s \in M$ such that $ss^* = e$, $s^*s = 1_M$.

This feature allows the notions of “sub-homomorphisms” and “conjugate homomorphisms”. Let $N$ and $M$ by two type III factors, and $\varphi : N \to M$ a homomorphism. (Homomorphisms are always understood to be unital, i.e., $\varphi(1_N) = 1_M$.) Let $e \in \varphi(N)' \cap M$ be a projection in the relative commutant. Let $e = ss^*$ with an isometry $s \in M$. Then $\varphi_s(\cdot) := s^*(\varphi(\cdot))s$ is another homomorphism $\varphi_s : N \to M$. We write $\varphi_s \prec \varphi$, because if $M$ is represented on a Hilbert space $H$ and $N$ is represented on $H$ via $\varphi$, then $s$ is a unitary operator from $H$ carrying the representation $\varphi_s$ to the subspace $eH$ carrying the representation $\varphi$.

$s$ satisfies the intertwining relation $s\varphi_s(n) = \varphi(n)s$. More generally, we call $M \ni t : \varphi_1 \to \varphi_2$ an intertwiner between two homomorphisms $\varphi_1 : N \to M$, if

$$t\varphi_1(n) = \varphi_2(n)t.$$  

The set of intertwiners $t : \varphi_1 \to \varphi_2$ is a linear space called $\text{Hom}(\varphi_1, \varphi_2) \subset M$. If $\varphi_1$ is irreducible $(\text{Hom}(\varphi_1, \varphi_1) \equiv \varphi_2(N)' \cap M = \mathbb{C} \cdot 1_M)$, then $t^*t \in \text{Hom}(\varphi_1, \varphi_1)$ is a multiple of $1_M$, thus turning $\text{Hom}(\varphi_1, \varphi_2)$ into a Hilbert space (every element is a multiple of an isometry).

The direct sum of two homomorphisms $\varphi_i : N \to M$ is defined up to unitary equivalence as

$$\varphi(\cdot) := s_1\varphi_1(\cdot)s_1^* + s_2\varphi_2(\cdot)s_2^*,$$

with any pair of isometries such that $s_1s_1^* + s_2s_2^* = 1_M$. Conversely, the decomposition of $1_M$ into minimal projections in $\varphi(N)' \cap M$ gives rise to a decomposition of $\varphi$ as a direct sum of irreducibles.

A pair of homomorphisms $\varphi : N \to M$ and $\varphi : M \to N$ is called conjugate, if $\text{id}_N \prec \varphi \varphi$ and $\varphi \varphi \prec \varphi$, and if there are intertwiners $N \ni w : \text{id}_N \to \varphi \varphi$, and $M \ni v : \text{id}_M \to \varphi \varphi$ satisfying the conjugacy relations

$$v^* \varphi(w) = 1_M, \quad w^* \varphi(v) = 1_N.$$  

One may normalize the pair such that $w^*w = d \cdot 1_N$ and $v^*v = d \cdot 1_M$. Then the infimum of $d$ among all solutions to the normalized conjugacy relations is called the dimension $d(\varphi) = \text{dim}(\varphi) \geq 1$. If a solution of the conjugacy relations exists at all, then there exist solutions taking the infimum value. Such solutions are called standard pairs. (Any two standard solutions differ by a pair of unitaries $\in \text{Hom}(\varphi, \varphi)$ and $\in \text{Hom}(\varphi, \varphi)$.) If an endomorphism does not have a conjugate, one puts $\text{dim}(\varphi) := \infty$. 

The notion of conjugacy is stable under unitary equivalence. More remarkably \cite{30}, the dimension is multiplicative under the composition
\[
\dim(\varphi_2 \circ \varphi_1) = \dim(\varphi_2) \cdot \dim(\varphi_1)
\]
and additive under direct sums
\[
\dim(\varphi_1 \oplus \varphi_2) = \dim(\varphi_1) + \dim(\varphi_2).
\]

If \( N \subset M \) is a (type \( III \)) subfactor, denote by \( \iota : N \to M \) the embedding homomorphism. Then the square of the dimension \( \dim(\varphi) = [M : N] \) (the type \( III \) version \cite{22, 23} of the Jones index for type \( II \)). Thus, for finite-index subfactors, there exists a conjugate homomorphism \( \tau : M \to N \) with standard pair \((w, v)\). It also follows that \([N : \tau(M)]\) has the same index.

Standard pairs give rise to bijective Frobenius maps \( \text{Hom}(\varphi_1, \varphi_2, \varphi_3) \to \text{Hom}(\varphi_2, \varphi_1, \varphi_3) \) and \( \text{Hom}(\varphi_1, \varphi_2, \varphi_3) \to \text{Hom}(\varphi_1, \varphi_3, \varphi_2) \), as well as to positive maps \( \text{Hom}(\varphi_1, \varphi_2) \to \text{Hom}(\varphi_1, \varphi_2) \) \( \text{Hom}(\varphi_1, \varphi_2) \to \text{Hom}(\varphi_1, \varphi_2) \) with tracial properties \cite{30, 3}, which we shall freely use in the rest of the paper.

These structures turn the homomorphisms between type \( III \) factors into a C* tensor 2-category (with 1-objects the factors, 2-objects the homomorphisms, and 2-morphisms the intertwiners). It is again simple, strict, and has direct sums and subobjects. The endomorphisms of a given factor \( N \) form just a C* tensor category \( \text{End}(N) \) (with objects the endomorphisms, and morphisms the intertwiners). We denote the full subcategory of endomorphisms of finite dimension as \( \text{End}_0(N) \).

For many purposes below,
\[
\mathcal{C} \subset \text{End}_0(N)
\]
will stand for any full subcategory of \( \text{End}_0(N) \) that without loss of generality can be assumed to have direct sums and subobjects. For more specific results, \( \mathcal{C} \) may be assumed to be braided (which is automatically the case for the representation category of local quantum field theory, \cite{3, 11}, cf. Sect. \[4.3\]), and even modular (which is automatically the case for completely rational conformal chiral quantum field theory, \cite{20}).

Conjugates and the dimension of \( \varphi : N \to M \) can also be defined when the condition that \( M \) is a factor, is relaxed and one admits \( M = \bigoplus_i M_i \) to be a finite direct sum of factors. In this case, \( \varphi(n) = \bigoplus_i \varphi_i(n) \), and \( \varphi(\bigoplus_i m_i) = \sum_i s_i \varphi_i(m_i) s_i^* \), where \( s_i \in N \) are a complete system of orthonormal isometries as before. It turns out \cite{3} Sect. 2.3] that the dimension \( \dim(\varphi) \) (defined as an infimum over solutions of the conjugate relations as before), is no longer additive, but
\[
\dim(\varphi) = \left( \sum_i \dim(\varphi_i) \right)^{1/2}.
\]

For the embedding homomorphisms \( \iota : N \to M \), one calls \( \gamma := \iota \circ \tau : M \to M \) the canonical endomorphism of \( M \), and \( \theta := \tau \circ \iota : N \to N \) the dual canonical endomorphism of \( N \). By definition, \( w \in \text{Hom}(\text{id}_N, \theta) \) and \( x := \tau(v) \in \text{Hom}(\theta, \theta^2) \), and these intertwiners satisfy
\[
w^* x = \theta(w^*) x = 1_N, \quad x x^* = \theta(x^*) x, \quad w^* w = x^* x = 1_N,
\]
where \( d = \dim(\iota) = \sqrt{\dim(\theta)} \). The first relation is referred to as \textit{unit property}, the second as \textit{associativity}, the third as \textit{Frobenius property}, and the last as \textit{standardness}. If \( \iota \) is irreducible \((N' \cap M = \mathbb{C} \cdot 1)\), then \( \text{id}_N \) is contained in \( \theta \) with multiplicity 1.
This system of equations only refers to the algebra $N$ and its endomorphism $\theta$. The triple
\[ A = (\theta, w, x) \quad (\theta \in \text{End}(N), w : \text{id}_N \to \theta, x : \theta \to \theta^2) \]  \tag{4.2}
is called the Q-system associated with the subfactor $N \subset M$, and the number $d = \dim(\iota)$ is called the dimension $d_A$ of the Q-system.

The basic result for our purposes is that the Q-system, i.e., the triple Eq. (4.2) satisfying Eq. (4.1) with $d \equiv d_A = \sqrt{\dim(\theta)}$, allows to reconstruct the subfactor $N \subset M$ up to isomorphism:

**Theorem 4.1.** [23] Let $N$ be a type III factor, and $A = (\theta, w, x)$ a Q-system in $\text{End}_0(N)$ of dimension $d_A$, such that $\text{id}_N \prec \theta$ with multiplicity 1. Then there is a (unique up to isomorphism) type III factor $M$ and an irreducible embedding $\iota : N \to M$ such that $A$ is the Q-system associated with this subfactor. The dimension $\dim(\iota)$ equals the dimension $d_A$.

The construction of $M$ from the given data is rather simple. Namely, $M$ is the algebra generated by $N$ (regarded as a subalgebra written as $\iota(N) \subset M$) and a single generator $v \in M$, with the defining relations
\[ v^* = \iota(w^*x^*)v, \quad vv = \iota(x)v, \quad \iota(w^*)v = 1_M, \quad \iota(n)v = \iota(\theta(n))v \quad (n \in N). \]  \tag{4.3}
Obviously, every element of $M$ is of the form $\iota(n)v$ with $n \in N$, and $\iota(n)v = 0$ implies $n = 0$.

The conjugate homomorphism $\tau : M \to N$ is given by
\[ \tau : \iota(n)v \mapsto \theta(n)x, \]  \tag{4.4}
and $(w, v)$ is a standard solution of the conjugacy relations. The relative commutant $\iota(N)' \cap M$ is given by elements of the form $\iota(q)v$ with $q \in \text{Hom}(\theta, \text{id}) = \mathbb{C} \cdot w^*$ by assumption, hence $\iota(N)' \cap M = \mathbb{C} \cdot 1_M$. Thus, the inclusion is irreducible, and in particular $M$ is a factor, $M' \cap M = \mathbb{C} \cdot 1_M$. Because $M$ is finitely generated from $\iota(N)$, it is closed in the weak topology inherited from $N$. One can also show the type III property.

Notice that a braiding is not required in this result.

### 4.2 Decompositions of Q-systems

One can easily generalize Thm. [4.1] by relaxing the condition that $\text{Hom}(\text{id}_N, \theta)$ is one-dimensional, see [3, Sect. 2.3]. In this case, the irreducibility $N' \cap M = \mathbb{C}$ will fail, and $M$ may even fail to be a factor.

**Theorem 4.2.** Let $N$ be a type III factor, and $A = (\theta, w, x)$ a Q-system in $\text{End}_0(N)$ of dimension $d_A$. Then there is a (unique up to isomorphism) type III von Neumann algebra $M$ and an embedding $\iota : N \to M$ with conjugate $\tau : M \to N$ such that $A$ is the Q-system associated with this conjugate pair. The dimension $\dim(\iota) = \sqrt{\dim(\theta)}$ equals the dimension $d_A$.

If $\dim(\text{Hom}(\text{id}_N, \theta)) > 1$, the inclusion $\iota(N) \subset M$ reconstructed from the Q-system fails to be irreducible, and the von Neumann algebra $M$ may fail to be a factor. Namely $\iota(N)' \cap M$ is given by elements of the form $\iota(q)v$ with $q \in \text{Hom}(\theta, \text{id}_N)$. These are central in $M$ if in addition $qx = \theta(q)x$ holds. Correspondingly, we may call a Q-system **factorial** if $qx = \theta(q)x$ for $q \in \text{Hom}(\theta, \text{id})$ implies that $q$ is a multiple of $w^*$. This is the same as “simple” in the literature on Frobenius algebras, cf. [3, Sect. 3.7].
In [3 Sect. 4], we develop different decompositions of Q-systems pertinent to these cases.

More precisely, certain projections in Hom(θ,θ) give rise to reduced Q-systems. We characterize the precise properties of these projections which correspond (i) to the central decomposition of M when M is not a factor, (ii) to the irreducible decomposition when M is a factor but N' ∩ M ≠ ℂ, and (iii) to intermediate inclusions N ⊂ L ⊂ M.

4.3 Q-systems in QFT

We now “transfer” the basic result Thm. 4.1 to quantum field theory [26]. Let 𝐴 be a Haag dual net of local von Neumann algebras 𝐴(𝒪), and 𝐶^{DHR}(𝐴) the category of its DHR endomorphisms. The net may be two-dimensional or chiral; in the latter case “O” is understood to stand for an interval ⊂ ℍ.

The transfer from subfactor theory to quantum field theory is possible, essentially because if ρ is a DHR endomorphism localized in �喳, then ρ maps 𝐴(𝒪) into itself, and ρ(𝐴(𝒪)) ⊂ 𝐴(𝒪) is a type $\text{III}$ subfactor. Likewise, for an extension 𝐴 ⊂ 𝐵, every local inclusion 𝐴(𝒪) ⊂ 𝐵(𝒪) is a subfactor.

DHR endomorphisms localized in O, when restricted to 𝐴(𝒪), are in fact endomorphisms of 𝐴(𝒪), and they have the same intertwiners as endomorphisms of the net and as elements of End(𝐴(𝒪)) [19]. Therefore, they are the objects of a C* tensor category $\mathcal{C}^{DHR}(𝐴)|𝒪$, which is a full subcategory of $\mathcal{C}^{DHR}(𝐴)$ and of End(𝒪), 𝑁 = 𝐴(𝒪).

In other words, (θ, 𝑤, 𝑥) is a Q-system in $\mathcal{C}^{DHR}(𝐴)|𝒪$ as subcategory of End(𝒪) if and only if θ is the restriction of a DHR endomorphism (also denoted by θ) localized in 𝒖, and 𝑤, 𝑥 ∈ 𝐴 satisfy the relations Eq. (4.1). It is therefore safe to drop the distinction altogether.

Since dim(ρ) was defined in terms of intertwiners, one may assign the same dimension to ρ as a DHR endomorphism, and the same properties (additivity and multiplicativity) remain valid. This definition coincides [24] with the “statistical dimension” originally defined in terms of the statistics operators [3-11].

DHR endomorphism with infinite dimension exist, so we shall from now on restrict ourselves to the full subcategory $\mathcal{C}^{DHR}_{₀}(𝐴)$ of DHR endomorphisms with finite dimension.

Let now $𝐴 = (θ, 𝑤, 𝑥)$ be a Q-system in $\mathcal{C}^{DHR}_{₀}(𝐴)$, i.e., θ is a DHR endomorphism of 𝐴, and the intertwiners 𝑤 and 𝑥 are elements of 𝐴(𝒪) if θ is localized in 𝐴(𝒪).

Define an algebra 𝐵 by adding one generator 𝑣 with the relations

\[ v^* = ι(w^*x^*)v, \quad vv = ι(x)v, \quad ι(w^*)v = 1_𝐵, \quad vv(a) = ι(θ(a))v \quad (a ∈ 𝐴). \quad (4.5) \]

Again, every element is of the form ι(𝑎)𝑣 with 𝑎 ∈ 𝐴, and ι(𝑎)𝑣 = 0 implies 𝑎 = 0.

The local subalgebras of 𝐵 are defined as

\[ 𝐵(𝒪) := ι(𝐴(𝒪)) \cdot \hat{𝑣} \quad (4.6) \]

where $\hat{𝑣} = ι(u)v$ with u unitary such that $\hat{θ} = Ad_u θ$ is localized in 𝒖. Because $\hat{𝑤} = uw ∈ \text{Hom}(id, \hat{θ})$ is in $𝐴(𝒪)$ and $ι(\hat{𝑤})^* \hat{𝑣} = ι(w^*)v = 1$, it follows that $𝐴(𝒪) ⊂ 𝐵(𝒪)$. Then, one has

\[ ι(u)vι(a) = ι(uθ(a))v = ι(\hat{θ}(a))v = ι(a)v \]

whenever $a ∈ 𝐴(𝒪')$, hence the extended net 𝐵 is relatively local w.r.t. 𝐴. Moreover, 𝐵 is itself local if $ι(u_i)v$ localized in spacelike separated $𝒪_i$ commute, which turns out to be equivalent to
the condition
\[ \varepsilon_{\theta,\theta} x = x. \]

The conjugate homomorphism \( \tau \) is again defined by Eq. (4.4). There is a conditional expectation \( \mu : \mathcal{B} \to \mathcal{A} \) (i.e., a faithful positive unit-preserving linear map satisfying \( \mu(a_1 \cdot b \cdot \mu(a_2)) = a_1 \cdot \mu(b) \cdot a_2 \), generalizing the properties of an average over an automorphic group action), defined by
\[ \mu(b) = d_A^{-1} \cdot w^* \tau(b)w. \] (4.7)
The conditional expectation respects the local structure, namely \( \mu(\mathcal{B}(O)) = \mathcal{A}(O) \).

With the help of the conditional expectation Eq. (4.7), the vacuum state \( \omega_0 \) of \( \mathcal{A} \) extends to a vacuum state \( \omega^0 := \omega_0 \circ \mu \) of \( \mathcal{B} \). The GNS representation of the restriction \( \omega^0|_{\mathcal{A}} \) as a state on \( \mathcal{A} \) is unitarily equivalent to the representation \( \pi_\theta = \pi_0 \circ \theta \) given by the DHR endomorphism \( \theta \).

The upshot of this discussion is

**Theorem 4.3.** [26] Let \( \mathcal{A} \) be a (conformal) QFT, and \( \mathcal{A} = (\theta, w, x) \) a Q-system in \( \mathcal{C}_0^{\text{DHR}}(\mathcal{A}) \) of dimension \( d_A \). Then there is a (unique up to isomorphism) QFT \( \mathcal{B} \) with local algebras Eq. (4.6) and an irreducible embedding \( \iota : \mathcal{A} \to \mathcal{B} \) such that \( \mathcal{A} \) is the Q-system associated with this extension. The dimension \( \dim(\iota) \) equals the dimension \( d_A \). \( \mathcal{B} \) is relatively local w.r.t. \( \mathcal{A} \), and \( \mathcal{B} \) is local iff
\[ \varepsilon_{\theta,\theta} x = x \] (4.8)
holds. The conditional expectation \( \mu : \mathcal{B} \to \mathcal{A} \) extends the vacuum state on \( \mathcal{A} \) to a vacuum state on \( \mathcal{B} \), such that the GNS representation of its restriction to \( \mathcal{A} \) is equivalent to \( \theta \).

Here, \( \mathcal{C}_0^{\text{DHR}}(\mathcal{A}) \) is the full braided subcategory of DHR endomorphisms of finite dimension. Notice that the objects of \( \mathcal{C}_0^{\text{DHR}}(\mathcal{A}) \) are just the finitely reducible objects of \( \mathcal{C}^{\text{DHR}}(\mathcal{A}) \) if \( \mathcal{A} \) is completely rational. For a more “physical” interpretation of the extension, see the Sect. 4.4 below.

Thm. 4.3 was proven in [26] under the additional assumption that \( \dim\Hom(\id_{\mathcal{A}}, \theta) = 1 \), in which case the extension is irreducible, and the local algebras are factors. As in Thm. 4.2 this condition can be relaxed without difficulty, thus admitting also reducible extensions and extensions whose local algebras are not factors. This generalization is necessary because in the analysis of boundary conditions, it will turn out that one has to consider Q-systems that give rise to extensions that have a nontrivial centre, \( \mathcal{B}' \cap \mathcal{B} \neq \mathbb{C} \cdot 1_{\mathcal{B}} \).

The central decomposition of \( \mathcal{B} \) will give rise to inequivalent representations (and inequivalent boundary conditions). It is therefore necessary to characterize the centre of the extension, and its central projections.

**Lemma 4.4.** (i) (With assumptions as in Theorem 4.2 admitting \( \id < \theta \) with multiplicity \( > 1 \)) The relative commutant is given by \( \iota(N)\prime \cap M = \iota(\Hom(\theta, \id_N))v \). The centre of \( M \) is given by \( \iota(q)v, q \in \Hom(\theta, \id_N) \) satisfying
\[ qx = \theta(q)x. \] (4.9)
\( \iota(q)v \) is selfadjoint iff \( q^* = \theta(q)xw, \) and idempotent iff \( q\theta(q)x = qwx = q \).

(ii) (With assumptions as in Thm. 4.3 admitting \( \id < \theta \) with multiplicity \( > 1 \)) The relative commutant is given by \( \iota(A)\prime \cap \mathcal{B} = \iota(\Hom(\theta, \id_G))v \). The centre of \( \mathcal{B} \) is given by \( \iota(q)v, q \in \Hom(\theta, \id_G) \) satisfying Eq. (4.9). \( \iota(q)v \) is selfadjoint iff \( q^* = \theta(q)xw, \) and idempotent iff \( q\theta(q)x = qwx = q \).

The proof proceeds by direct computation, using the relations Eq. (4.3), Eq. (4.5).
4.4 Charged fields

The reconstruction presented above by adjoining an element \( v \) to the algebra \( \mathcal{A} \) of local observables and imposing its relations, can be given a physical interpretation. Namely, the DHR endomorphism \( \theta \) gives the restriction of the vacuum representation of the larger net \( \mathcal{B} \) as a representation of the original net \( \mathcal{A} \). If \( \rho \prec \theta \) is an irreducible subrepresentation localized in \( O \), then \( H_\rho := \iota(\text{Hom}(\rho, \theta))^* v \) is a Hilbert space of isometries in \( \mathcal{B}(O) \), and its elements \( \psi_\rho = \iota(w_\rho) v \in \mathcal{B}(O) \) with \( w_\rho \in \text{Hom}(\rho, \theta) \) satisfy the commutation relation

\[
\psi_\rho a = \rho(a) \psi_\rho
\]

(4.10)

with the observables in \( \mathcal{A} \). Because \( \rho \) is localized in \( O \), it follows that \( \psi_\rho \) commutes with \( \mathcal{A}(O') \), i.e., the extension is relatively local w.r.t. \( \mathcal{A} \).

By choosing a complete irreducible decomposition of \( \theta \) by isometries \( w_\rho \) (possibly with multiplicities) such that \( \sum_\rho w_\rho w_\rho^* = 1 \), every element of \( \mathcal{B} \) has a representation

\[
b = \iota(a) v = \sum_\rho \iota(aw_\rho) \cdot \psi_\rho,
\]

so that the operators \( \psi_\rho \) form a basis of \( \mathcal{B} \) (as a vector space) with coefficients in \( \mathcal{A} \). They may therefore be regarded as the analogue of the primary fields in AQFT, such that a general ("descendant") field is obtained by left multiplication ("OPE") of \( \psi_\rho \) with some element of \( \mathcal{A} \).

Because \( \mu(\psi_\rho \psi_\rho^*) \in \text{Hom}(\rho, \rho) = \mathbb{C} \cdot 1 \), one may normalize \( \psi_\rho \) such that \( \mu(\psi_\rho \psi_\rho^*) = 1 \). Then \( \omega_\rho(a) = \omega_0(\psi_\rho \psi_\rho^*) \equiv \omega_0 \circ \mu(\psi_\rho \psi_\rho^*) \) is a state on \( \mathcal{A} \). Its GNS representation is equivalent to the representation \( \pi_\rho = \pi_0 \circ \rho \), justifying the interpretation of \( \psi_\rho \) as "charged field operators".

The formulae given for the product \( v^2 \) and the adjoint \( v^* \) determine the multiplication and adjoint of the charged field operators \( \psi_\rho \). In other words, the Q-system is a generating functional for the coefficients of the operator product expansion of the charged field operators.

In four dimensions, the Doplicher-Roberts reconstruction [9] theorem states (among other things) that there is an orthonormal basis of charged field operators satisfying the Cuntz relation \( \sum_i \psi_{\rho,i} \psi_{\rho,i}^* = 1 \), such that the endomorphism \( \rho \) is implemented by the formula \( \rho(a) = \sum_i \psi_{\rho,i} a \psi_{\rho,i}^* \).

In low dimensions, the non-triviality of the braiding poses an obstruction, so that the range projections of \( H_\rho \) in general do not exhaust the unit operator; but one has still the implementation via the conditional expectation: \( \rho(a) = \mu(\psi_\rho a \psi_\rho^*) \). More details can be found, e.g., in [27].

Let us for later use compute the proper normalization of the charged fields. We have

\[
\mu(\psi_\rho \psi_\rho^*) = d_A^{-1} \cdot w^* \iota(\psi_\rho \psi_\rho^*) w = d_A^{-1} \cdot w^* \theta(w_\rho^*) x x^* \theta(w_\rho) w = d_A^{-1} \cdot w_\rho^* w_\rho = d_A^{-1} \cdot w^* w_\rho.
\]

Thus, \( \mu(\psi_\rho \psi_\rho^*) = 1 \) if \( w_\rho^* w_\rho = d_A \). Later, we shall prefer a normalization such that \( \psi_\rho \) are isometries. This is a reasonable option if \( \iota \) is irreducible because \( \psi_\rho^* \psi_\rho \in \mathcal{A}(O') \cap \mathcal{B}(O) \) by Eq. (4.10), hence is automatically a multiple of 1.

In this case, it equals its image under \( \mu \). We compute \( \mu(\psi_\rho^* \psi_\rho) = d_A^{-1} \cdot w^* x \theta(w_\rho^*) x w = d_A^{-1} \cdot \overline{\theta}(w_\rho^*) w \overline{\theta}(w_\rho) \) by the properties of standard pairs [30]. Because \( \overline{\theta}(w_\rho) = \dim(\rho) \), we conclude that, in the irreducible case, \( \psi_\rho = \iota(w_\rho^*) v \) are isometries if

\[
w_\rho^* w_\rho = \frac{d_A}{\dim(\rho)}.
\]

(4.11)
Charged fields $\psi_\rho$ are localized in $B(O)$ if $\rho$ is localized in $O$. They can be transported to any other region $O_1$ with the help of unitary intertwiners $u \in \text{Hom}(\rho, \rho_1)$ where $\rho_1$ is localized in $O_1$, hence $w_\rho = w_\rho u^* \in \text{Hom}(\rho_1, \theta)$:

$$
\psi_{\rho_1} = \iota(w_\rho) v = \iota(u) \psi_\rho.
$$

(4.12)

4.5 Example 3: The fermionic extension of the chiral Ising model

Apart from the trivial Q-system $(\text{id}, 1, 1)$ with $B = A$, the chiral Ising model $(= \text{Vir}(c = \frac{1}{2}))$ has one other irreducible Q-system with $\theta \sim \text{id} \oplus \tau$ of dimension $d_A = \sqrt{2}$. We may choose $(\theta = \sigma^2, w = 2^{1/4} \cdot r, x = 2^{1/4} \cdot \sigma(r) = 2^{-1/4}(r + t))$. The nontrivial charged field is $\psi = 2^{1/4} \cdot \iota(t^*) v$, satisfying $\psi u(a) = \iota(\tau(a)) \psi$, $\psi^* = \psi$, $\psi^2 = 1$.

Shifting the localization of $\psi$ by a unitary intertwiner $\psi(x) = u_x \psi$ where $u_x \in \text{Hom}(\tau, \tau_x)$, one finds $\psi(x) \psi(y) = u_x \tau(u_y)$, and in particular, the local anti-commutativity $\psi(x) \psi(y) = -\psi(y) \psi(x)$ whenever the localization regions $I+x$ and $I+y$ are disjoint, because $\tau(u_x)^* u_y^* u_{x \tau(u_y)}$ equals the statistics operator $\varepsilon_{\tau, \tau} = -1$. Indeed, $\psi$ is the (smeared) real chiral Fermi field.

The extension $B$ is graded local, with grading automorphism $\alpha : \psi \mapsto -\psi$, such that $A = B^\alpha$, i.e., $\alpha$ is the gauge transformation with fixed points $A$.

4.6 The canonical extension

In two dimensions, the same results can be applied to $A_2 = A_+ \otimes A_-$ with $C^{\text{DHR}}(A_+ \otimes A_-) = C^{\text{DHR}}(A_+) \boxtimes C^{\text{DHR}}(A_-)^{\text{opp}}$. In order to distinguish it from the chiral case, we adopt capital letters for the two-dimensional case.

A Q-system is a triple $(\Theta, W, X)$, where $\Theta$ has the general form

$$
\Theta \simeq \bigoplus_{\rho, \sigma} Z_{\sigma, \tau} \cdot \sigma \otimes \tau.
$$

(4.13)

The multiplicity matrix $Z$, coupling the left- and the right-moving representations, is also called the coupling matrix.

Thus, from a Q-system $(\Theta, W, X)$, one can construct an extension $A_2 \subset B_2 = \iota(A_2) V$. The charged fields are of the form $\Psi_{\sigma \otimes \tau} = \iota(W_{\sigma \otimes \tau}^*) V$ satisfying

$$
\Psi_{\sigma \otimes \tau} a = (\sigma \otimes \tau)(a) \Psi_{\sigma \otimes \tau}
$$

where $\sigma \otimes \tau \prec \Theta$ are the irreducible subrepresentations of $\Theta$. The charged field $\Psi_{\sigma \otimes \tau}$ are isometries if $W_{\sigma, \tau} \in \text{Hom}(\sigma \otimes \tau, \Theta)$ are normalized as $W_{\sigma, \tau}^* W_{\sigma, \tau} = \sqrt{\text{dim}(\Theta)/\text{dim}(\sigma) \text{dim}(\tau)}$ in accordance with Eq. (4.11).

In the case $A_+ = A_-$, there is a distinguished extension of $A_2 = A \otimes A$.

Proposition 4.5. Let $C$ be a tensor category with finitely many inequivalent irreducible objects $\rho$, then there is a canonical Q-system $R$ in $C \boxtimes C^{\text{opp}}$ with

$$
\Theta \simeq \bigoplus \rho \otimes \rho^*.
$$

where the sum extends over the inequivalent irreducible objects of $C$. More precisely, choosing isometries $T_\rho \in \text{Hom}(\rho \otimes \rho^*, \Theta)$, one has $W = d_R^{1/2} \cdot T_\text{id}$ and

$$
X = d_R^{-1/2} \sum_{\rho, \sigma, \tau} \left( \frac{d_\rho d_\sigma}{d_\tau} \right)^{1/2} \cdot \Theta(T_\sigma) T_\rho \circ \left( \sum_{\alpha} t_\alpha \otimes t_\alpha^* \right) \circ T_\tau^*.
$$
where the second sum extends over an orthonormal basis of isometries \( t_a \in \text{Hom}(\tau, \rho \sigma) \) and \( t_\bar{\tau} \in \text{Hom}(\bar{\tau}, \bar{\rho} \bar{\sigma}) = j(t) \) with a suitable antilinear conjugation such that \( \bar{\tau} = j \circ \rho \circ j \) for all representatives. The dimension is \( d_R = \sqrt{\dim(\Theta_{\text{can}})} = \sqrt{\sum \dim(\rho)^2} \).

The canonical Q-system is also referred to as regular because it shares certain properties with the regular representation of a group. In other contexts, the corresponding extensions are also known as “Cardy extension” or “Longo-Rehren subfactor”.

For the present form of the result, a braiding is not needed. If \( C \) is braided, and \( C \otimes C^{\text{opp}} \) is equipped with the braiding \( \varepsilon \otimes \varepsilon^{\text{opp}} \), turning it into the braided category \( C \otimes C^{\text{opp}} \), then one has in addition:

**Proposition 4.6.** [26] The canonical Q-system satisfies

\[
\varepsilon_{\Theta, \Theta} \cdot X = X
\]

as a Q-system in \( C \otimes C^{\text{opp}} \).

Thus, with \( C = C_0^{\text{DHR}}(A) \) for a chiral QFT \( A \), the canonical Q-systems describes a local two-dimensional extension \( A \otimes A \subset \mathcal{B}_2 \) with charged fields \( \Psi_{\rho \otimes \bar{\tau}} \).

### 4.7 Example 4: The local two-dimensional Ising model

The Ising model (without phase boundary) is a local extension of the theory \( A_2 = A \otimes A \), where \( A \) is the Virasoro theory with \( c = \frac{1}{2} \). The Q-system is the canonical Q-system (Prop. 4.5) given as follows. Let \( \tau \) and \( \sigma \) be localized in \( I \), and \( O = I \times I \). Choose any three isometries \( T_0, T_1, T_2 \in (A \otimes A)(O) \) satisfying \( T_i^j T_j = \delta_{ij} \cdot 1 \), \( \sum_i T_i T_i^* = 1 \), and define

\[
\Theta(a) = T_0 a T_0^* + T_1 (\tau \otimes \tau)(a) T_1^* + T_2 (\sigma \otimes \sigma)(a) T_2^*.
\]

Set \( W := \sqrt{2} \cdot T_0 \) and

\[
X = \left( \Theta(T_0) T_0 T_0^* + \Theta(T_0) T_1 T_1^* + \Theta(T_0) T_2 T_2^* + \Theta(T_1) T_0 T_2^* + \Theta(T_2) T_0 T_1^* + \Theta(T_1) T_1^* T_2^* \right) / \sqrt{2}.
\]

Properly normalizing \( W_{\alpha \circ \alpha} = \sqrt{2} / \dim(a) \cdot T_a \), one obtains isometric operators \( \Psi_{\tau \otimes \tau} = \iota(W_{\tau \otimes \tau}^*) V \) and \( \Psi_{\sigma \circ \sigma} = \iota(W_{\sigma \circ \sigma}) V \) satisfying the relations

\[
\Psi_{\tau \otimes \tau}^* a = (\tau \otimes \tau)(a) \Psi_{\tau \otimes \tau}, \quad \Psi_{\sigma \circ \sigma}^* a = (\sigma \circ \sigma)(a) \Psi_{\sigma \circ \sigma}
\]

\[
\Psi_{\tau \otimes \tau}^* = \Psi_{\tau \otimes \tau}, \quad \Psi_{\sigma \circ \sigma}^* = \sqrt{2} (r^* \otimes r^*) \Psi_{\sigma \circ \sigma},
\]

\[
\Psi_{\tau \otimes \tau}^2 = 1, \quad \Psi_{\tau \otimes \tau} \Psi_{\sigma \circ \sigma} = \Psi_{\sigma \circ \sigma} \Psi_{\tau \otimes \tau} = (u \otimes u) \Psi_{\sigma \circ \sigma},
\]

\[
(\Psi_{\sigma \circ \sigma})^2 = ((r \otimes r) + (t \otimes t)) \Psi_{\tau \otimes \tau} / \sqrt{2}.
\]

Then \( \mathcal{B}_2(I \times I) \) is generated by \( A_2(I \times I) \) and \( \Psi_{\tau \otimes \tau} \) and \( \Psi_{\sigma \circ \sigma} \). Shifting the localization by unitary intertwiners, one can explicitly verify local commutativity among all fields at spacelike distance.

One may add also a chiral Fermi field to this algebra, giving rise to a nonlocal extension \( A_2 \subset C_2 \) with \( \Theta \simeq (\text{id} \otimes \tau) \otimes (\text{id} \otimes \tau) \otimes 2\sigma \otimes \sigma \) containing the canonical local extension \( \mathcal{B}_2 \) as an intermediate extension. The relations of this extension will be presented in Example 5.

---

\[\text{E.g., \{t \otimes 1, rr \otimes 1, rt \otimes 1\} or \{r \otimes 1, t \otimes r, t \otimes t\}, or any other choices which need not be functions of \( r \) and \( t \).}\]
4.8 Maximal local subtheories of nonlocal extensions

We introduce two maximal local extensions contained in a nonlocal extension.

In the following, \( O \) may be a double cone in two dimensions, or an interval \( \subset \mathbb{R} \), and \( O' \) the causal complement, rep. the complement in the chiral case. In the two-dimensional case, we denote the connected components of \( O' \) as \( W_L \) (the “left wedge”) and \( W_R \) (the “right wedge”). In the chiral case, the two connected components are in fact halfrays, but we shall adopt the same notation \( W_L \) (for the negative=past halfray) and \( W_R \) (for the positive=future halfray).

For wedge algebras (halfray algebras), Haag duality holds, namely \( \mathcal{A}(W') = \mathcal{A}(W)' \).

Let \( \mathcal{A} \subset \mathcal{B} \) be an extension. We define

\[
\mathcal{B}_{\text{loc}}^+(O) := \mathcal{B}(W_L)' \cap \mathcal{B}(W_R)', \quad \text{resp.} \quad \mathcal{B}_{\text{loc}}^-(O) := \mathcal{B}(W_R)' \cap \mathcal{B}(W_L)'.
\]

These algebras are the relative commutants of the pair of algebras associated with the nested left resp. right wedges (past resp. future halfrays) defined by the double cone (interval) \( O \). If \( \mathcal{B} \) is local, then these constructions just give back \( \mathcal{B} \).

**Lemma 4.7.** (i) \( \mathcal{B}_{\text{loc}}^\pm \) are Poincaré covariant isotonous nets of von Neumann algebras.
(ii) \( \mathcal{B} \) is left-local w.r.t. \( \mathcal{B}_{\text{loc}}^+ \), and right-local w.r.t. \( \mathcal{B}_{\text{loc}}^- \). Equivalently, \( \mathcal{B}_{\text{loc}}^+ \) is right-local, and \( \mathcal{B}_{\text{loc}}^- \) is left-local w.r.t. \( \mathcal{B} \).
(iii) The nets \( \mathcal{B}_{\text{loc}}^\pm \) are both local, and they are intermediate between \( \mathcal{A} \) and \( \mathcal{B} \):

\[
\mathcal{A}(O) \subset \mathcal{B}_{\text{loc}}^+(O) \subset \mathcal{B}(O).
\]

In particular, one has

\[
\mathcal{B}_{\text{loc}}^+(O) = \mathcal{B}(W_L)' \cap \mathcal{B}(O), \quad \text{resp.} \quad \mathcal{B}_{\text{loc}}^-(O) = \mathcal{B}(W_R)' \cap \mathcal{B}(O), \tag{4.14}
\]

where the wedges \( W_L \) and \( W_R \) are the left and right connected components of \( O' \).
(iv) The nets \( \mathcal{B}_{\text{loc}}^\pm \) are the maximal intermediate nets with the properties stated in (ii).

**Proof:** (i) Enlarging the double cone \( O \), certainly decreases \( W_L \) and/or \( W_R \) and increases \( W_L' \) and/or \( W_R' \). Therefore \( \mathcal{B}_{\text{loc}}^\pm(O) \) increase with \( O \) (“isotony”). The Poincaré covariance of the constructions is manifest.
(ii) Clear by definition.
(iii) To prove locality, it suffices to note that for every pair of spacelike separated double cones, one is contained in the left component of the complement of the other, and the other is contained in the right component of the complement of the first. Locality then follows because the algebras are defined as relative commutants.

The inclusion \( \mathcal{A}(O) \subset \mathcal{B}_{\text{loc}}^+(O) \) is obvious from relative locality of \( \mathcal{B} \) w.r.t. \( \mathcal{A} \). The inclusion \( \mathcal{B}_{\text{loc}}^+(O) \subset \mathcal{B}(O) \) is not immediate because \( \mathcal{B}(W_R') \) is larger than \( \mathcal{B}(O) \).

Consider an element \( b \) of \( \mathcal{B}_{\text{loc}}^+(O) = \mathcal{B}(W_L)' \cap \mathcal{B}(W_R') \). By Eq. (4.14), \( \mathcal{B}(W_R') = \mathcal{A}(W_R')\widehat{\nu} \) where \( \widehat{\nu} \) can be chosen in \( \mathcal{B}(O) \subset \mathcal{B}(W_R') \). Thus \( b = a\widehat{\nu} \) with \( a \in \mathcal{A}(W_R') \). Because \( b \) commutes with \( \mathcal{B}(W_L) \), it commutes with \( \mathcal{A}(W_L) \). But \( \widehat{\nu} \) commutes with \( \mathcal{A}(W_L) \) by relative locality, hence \( a \) must commute with \( \mathcal{A}(W_L) \). Then, \( a \in \mathcal{A}(W_L') \) by Haag duality, and hence \( a \in \mathcal{A}(W_R') \cap \mathcal{A}(W_L') = \mathcal{A}(O) \). Thus \( b \in \mathcal{A}(O)\widehat{\nu} = \mathcal{B}(O) \). The argument is the same for \( \mathcal{B}_{\text{loc}}^- \).

(iv) Is now clear from Eq. (4.14). \( \square \)
If $B$ is itself local, then obviously $B_{\text{loc}} = B$. Also the other extreme, $B_{\text{loc}} = A$, may occur; e.g., the fermionic extension of the chiral Ising model (Example 3) does not admit any local intermediate extension.

Let $A \subset B_1 \subset B$ be any intermediate extension with Q-system $(\theta_1, w_1, x_1)$. Then $\iota = \iota_2 \circ \iota_1$ and $\theta_1 = \tilde{\iota}_1 \iota_1 \ll \tilde{\iota}_l = \theta$. Let $p \in \text{Hom}(\theta, \theta)$ be the projection corresponding to $\theta_1 \ll \theta$. It is easy to see that

$$pw = w, \quad p\theta(p)x = pxp = \theta(p)xp.$$ \hspace{1cm} (4.15)

In fact, every projection $p \in \text{Hom}(\theta, \theta)$ satisfying Eq. (4.15) comes from an intermediate extension

$$A \subset B_p \subset B$$

where $B_p = A \cdot pv$. This is proven in [3 Sect. 4.4] for von Neumann algebras, and translates to local nets by $B_p(O) := A \cdot s^+v$ where $p = ss^+$, $s^+s = 1$, and $s^+\theta(\cdot)s$ is localized in $O$.

**Lemma 4.8.** $B$ is left-local w.r.t. $B_1$ if and only if $p$ satisfies

$$\theta(p)x = \varepsilon_{\theta, \theta} \cdot px.$$ \hspace{1cm} (4.16)

$B$ is right-local w.r.t. $B_1$ if and only if $p$ satisfies the same relation with $\varepsilon_{\theta, \theta}$ replaced by $\varepsilon_{\theta, \theta}^\perp$.

**Proof:** For left locality, we have to consider the commutativity between $B(W_L)$ and $B_1(O)$ where $O$ is some double cone (interval) and $W_L$ is the left connected component of $O'$. Let $\theta$ be localized in $O$, and choose a unitary $u \in \text{Hom}(\theta, \tilde{\theta})$ with $\tilde{\theta}$ localized in $W_L$. Then $B(W_L) = A(W_L)uv$ as in Eq. (4.10). Let $p = ss^*$ with an isometry $s \in A(O)$, hence $\theta_1$ is localized in $O$. Then we also have $B_1(O) = A(O)v_1 = A(O)s^+v = A(O)s^+pv = A(O)pv$.

Now, $A(O)$ commutes with $B(W_L)$ by relative locality, and $A(W_L)$ commutes with $B_1(O)$ by relative locality. Thus $B_1(O)$ commutes with $B(W_L)$ if and only if $uv$ commutes with $pv$. We have

$$uvpv = u\theta(p)xv, \quad puvv = p\theta(u)xv = \theta(u)pxv.$$  

Equality holds iff $\theta(p)x = u^*\theta(u)px$. But $u^*\theta(u) = \varepsilon_{\theta, \theta}$, so the claim follows. \hfill $\square$

We now claim that in terms of Q-systems, the above construction $B_{\text{loc}}^\pm$ corresponds to the **centre of the Q-system** [14]. The centre of a Q-system $(\theta, w, x)$ (which in general does not satisfy the commutativity condition $\varepsilon_{\theta, \theta}x = x$) is given by a projection $p$ in $\text{Hom}(\theta, \theta)$, such that the associated intermediate Q-system $(\theta_p, w_p, x_p)$ satisfies $\varepsilon_{\theta_p, \theta_p}x_p = x_p$ and is maximal with this property. There are in fact two such projections, $p^\pm$, defined by

$$p^\pm := r^\theta(\varepsilon_{\theta, \theta}^\pm)x^{(2)} = \varepsilon_{\theta, \theta}^\mp \theta(r^\pm)x^{(2)},$$ \hspace{1cm} (4.17)

where $x^{(2)} = xx = \theta(x)x \in \text{Hom}(\theta, \theta^3)$ and $r = xv \in \text{Hom}(\text{id}, \theta^2)$. They satisfy the relations Eq. (4.15), and in addition

$$\theta(p^+\!x) = \varepsilon_{\theta, \theta} \cdot p^+\!x, \quad \theta(p^-\!x) = \varepsilon_{\theta, \theta}^* \cdot p^-\!x,$$ \hspace{1cm} (4.18)

and they are maximal to satisfy Eq. (4.18), i.e., for every other projection $p$ satisfying the first resp. the second of Eq. (4.18), one has $pp^+ = p$ resp. $pp^- = p$. The intermediate Q-system associated with $p^+$ is called the **left centre** $C^+[A]$ of $A = (\theta, w, x)$, the Q-system associated with $p^-$ is called the **right centre** $C^-[A]$.

Eq. (4.18) entails $\varepsilon_{\theta_p, \theta_p}x_p = x_p$ [14]. The proof can also be found in [3]. It follows
Proposition 4.9. The intermediate extensions associated with the left resp. right centre of the
Q-system for \( A \subset B \) are \( A \subset B^+_{\text{loc}} \subset B \) resp. \( A \subset B^-_{\text{loc}} \subset B \).

Proof: By Lemma 4.7, \( B^+_{\text{loc}} \) is intermediate between \( A \subset B \) and right-local w.r.t. \( B \Leftrightarrow B \) is
left-local w.r.t. \( B^+_{\text{loc}} \), and by definition, they are maximal with these properties. By Lemma 4.8, the associated intermediate projection \( p \) is the maximal projection satisfying Eq. (4.18). But these properties characterize the centre projection, hence \( p = p^+ \).

This result gives a simple interpretation of the “centre of a Q-system” in terms of local
algebras, namely as the relative commutant of local algebras associated with nested wedges (in
two dimensions) or lightrays (in chiral theories).

It also provides us a simple formulation of the locality condition required at a phase boundary,
namely, the extension \( C_2 \) of \( A_2 = A^+_\text{loc} \otimes A^-_\text{loc} \) generated by \( B^L_2 \) and \( B^R_2 \) must fulfill
\[
B^L_2 \subset (C_2)^\text{loc} \subset C_2 \supset (C_2)^\text{loc} \supset B^R_2.
\]

4.9 The braided product of two extensions

Suppose we have two extensions \( A \subset B_1 \) and \( A \subset B_2 \) with generating charged fields \( \psi_{1,\rho} \) (\( \rho < \theta_1 \))
and \( \psi_{2,\sigma} \) (\( \sigma < \theta_2 \)). The two extensions in general “live” on different Hilbert spaces, that are
both representations of the underlying algebra \( A \).

We want to construct an extension \( A \subset C \) containing both \( B_1 \) and \( B_2 \) as intermediate
extensions. Because of the commutation relations \( \psi_{1,\rho}a = \rho(a)\psi_{1,\rho} \) and \( \psi_{2,\sigma}a = \sigma(a)\psi_{2,\sigma} \) that
must also hold in \( C \), one may expect algebraic consistency problems. The following proposition
shows that such a construction is canonically possible.

Proposition 4.10. Let \( A_1 = (\theta_1, w_1, x_1) \) and \( A_2 = (\theta_2, w_2, x_2) \) be two Q-systems. Then the
triples
\[
(\theta_1, w_1, x_1) \pm (\theta_2, w_2, x_2) = (\theta = \theta_1\theta_2, w = \theta_1(w_2)w_1, x^\pm = \theta_1(\varepsilon_{\theta_1,\theta_2}^\pm) x_1 \theta_1(x_2)) \quad (4.19)
\]
define a pair of Q-systems denoted \( A_1 \pm A_2 \). We call them (braided) product Q-systems,
and denote the corresponding (braided) product of extensions by \( A \subset B_1 \pm B_2 \equiv C^\pm \).

Both product Q-systems \( A_1 \pm A_2 \) contain both \( A_1 \) and \( A_2 \) as intermediate Q-systems, hence \( C^\pm \)
contain \( B_1 \) and \( B_2 \) as intermediate extensions:
\[
A \subset B_1 \subset B_2 \subset C^\pm.
\]

\( C^\pm \) can be characterized as the quotient of the free product of the algebras \( B_1 \) and \( B_2 \) by the
relations \( \iota_1(A) = \iota_2(A) \) and the commutation relations
\[
v_2v_1 = \varepsilon_{\theta_1,\theta_2}^\pm \cdot v_1v_2. \quad (4.20)
\]

Proof: The projections \( p_1 = d^{-1}_2\theta_1(w_2w_2^*) \in \text{Hom}(\theta,\theta) \) and \( p_2 = d^{-1}_1w_1w_1^* \in \text{Hom}(\theta,\theta) \)
define intermediate Q-systems according to \[7\ Sect. 4.4\]. The reduced Q-systems coincide with
\( (\theta_1, w_1, x_1) \) and \( (\theta_2, w_2, x_2) \), hence both \( B_1 \) and \( B_2 \) are intermediate extensions with \( v_1 = \theta_1(w_2)w_1 \)
and \( v_2 = w_1w_2 \). Conversely,
\[
v_1v_2 = \theta_1(w_2^*) \cdot \theta_1\theta_2(w_1^*) \cdot \varepsilon_{\theta_1,\theta_2}^\pm x_1 \theta_1(x_2) = v,
\]
by the unit property of Q-systems. Hence \( C^\pm \) is generated by \( \mathcal{A} \) and \( v_1 \) and \( v_2 \). The product \( v_2 v_1 \) is given by
\[
v_2 v_1 = w_1^* \cdot \theta_1 \theta_2 (\theta_1 (w_2^*)) \cdot \varepsilon_{\theta_1, \theta_2} v_1 x_1 \theta_1 (x_2) v = \varepsilon_{\theta_1, \theta_2} v.
\]
Comparing these identities, Eq. (4.20) follows. \( \square \)

**Remark 4.11.** \( A_1 \times^+ A_2 \) is unitarily equivalent to \( A_2 \times^- A_1 \) by the unitary \( \varepsilon_{\theta_1, \theta_2} \).

Rewriting Eq. (4.20) in terms of the charged field decomposition of \( v_1 \) and \( v_2 \), one has equivalently
\[
\psi_{2, \sigma} \psi_{1, \rho} = \varepsilon_{\rho, \sigma}^{\pm} \cdot \psi_{1, \rho} \psi_{2, \sigma}.
\]

**Remark 4.12.** In view of Eq. (4.12), these commutation relations are compatible with the transport, i.e., if they hold for any \( \rho, \sigma \), then they hold for every unitarily equivalent pair. In particular, as the statistics operators \( \varepsilon_{\rho, \sigma} = 1 \) whenever \( \sigma \) is localized to the left of \( \rho \), left (right) locality of \( \psi^1 \) w.r.t. \( \psi^2 \) implies Eq. (4.21) with the \( - \) sign (\( + \) sign) for arbitrary localization of \( \rho \) and \( \sigma \).

Therefore, we have

**Proposition 4.13.** Within \( C^+ = B_1 \times^+ B_2 \), \( B_1 \) is right-local w.r.t. \( B_2 \), whereas within \( C^- = B_1 \times^- B_2 \), \( B_1 \) is left-local w.r.t. \( B_2 \). Every irreducible extension containing both \( B_1 \) and \( B_2 \) as intermediate extensions such that \( B_1 \) is right-local (left-local) w.r.t. \( B_2 \), is a quotient of \( C^+ \) (\( C^- \)).

**Proof:** The first statement follows from Eq. (4.21) and the stated triviality of the statistics operator for charged fields with the respective relative localization, along with the fact that \( \mathcal{A} \) is local and both \( B_i \) are relatively local w.r.t. \( \mathcal{A} \). The second statement is due to the fact that Eq. (4.21), or equivalently Eq. (4.20) is the only independent relation among the generators \( v_1 \) and \( v_2 \) of \( C \), using the multiplication law of the product Q-system. \( \square \)

Notice that even if both \( B_i \) are local, neither \( C^+ \) nor \( C^- \) will be local in general. However, we know how to construct maximal local subtheories in \( C^\pm \) by the construction in the Sect. 4.8. Namely, one will have to determine the center(s) of the braided product Q-systems. The following partial result is true without model specific knowledge.

**Lemma 4.14.** If \( B_2 \) is local, then the right centre \((C^+)_{\text{loc}}^- \) of \( C^+ = B_1 \times^+ B_2 \) contains at least \( B_2 \). The same is true for the left centre \((C^-)_{\text{loc}}^+ \) of \( C^- = B_1 \times^- B_2 \):
\[
B_2 \subset (C^+)_{\text{loc}}^- , \quad B_2 \subset (C^-)_{\text{loc}}^+ .
\]

Similarly, if \( B_1 \) is local, then
\[
B_1 \subset (C^+)_{\text{loc}}^+ , \quad B_1 \subset (C^-)_{\text{loc}}^- .
\]

Similar statements are in general not true for the other combinations, e.g., \( B_2 \) local will not belong to the left centre \((C^+)_{\text{loc}}^+ \) (= relative commutant of left wedges) of the \( \times^+ \) braided product, or to \((C^-)_{\text{loc}}^- \). See, however, the Sect. 4.10 for the local subtheories \((C^+)_{\text{loc}}^+ \) when \( B_1 \) is chiral and \( B_2 \) is the canonical local extension.

**Proof:** This can be seen by inspection of Fig. 1: an element of \( B_2 \) in the double cone commutes with every element of \( B_2 \) in the right wedge because \( B_2 \) is local, and with every element of \( B_1 \) in the right wedge by Prop. 4.13. \( \square \)
Lemma 4.15. Let $A_2 = A \otimes A$, where $C^{DHR}(A)$ is modular. If $A_1$ is a chiral Q-system of the form $A \otimes 1$, i.e., $B_1$ is chiral extension $A_2 \subset B \otimes A$, and $A_2$ is the canonical Q-system, i.e., $B_2$ is the canonical extension, then one has equality in Lemma 4.14:

$$(C^+)_{\text{loc}}^{-} = (C^-)_{\text{loc}}^{+} = B_2.$$

Proof: One can compute the trace (cf. Sect. 4.1) $\text{Tr}(p^\pm) \equiv R^* p^\pm R = R^* \Theta(p^\pm) R$ of the centre projection. By using the fact that the braiding is non-degenerate, i.e., $\varepsilon_{\sigma, \rho} \varepsilon_{\rho, \sigma} = 1$ for all $\rho$ implies $\sigma = \text{id}$, one finds $\text{Tr}(p^\pm) = d^2_R = \dim(R)$ for the left center of the $\times^-$ product, and for the right center of the $\times^+$ product. There for the canonical extension exhausts the centre projection. □

![Fig. 1](image-url)

**Fig. 1**: This figure visualizes the statement of Lemma 4.14: If $B_2$ is local, then the right center (computed by relative commutants of right wedge algebras) of the braided product of $B_1$ and $B_2$ contains $B_2$ if and only if $B_2$ is left-local w.r.t. $B_1$ (i.e., the braided product is the $\times^+$ product by Prop. 4.13).

### 4.10 The $\alpha$-induction construction and the full centre

A special case of the centre is the **full centre** [14]. If the braided category has only finitely many irreducible objects, then the canonical Q-system $R$ (Prop. 4.5) is a commutative Q-system in $C \boxtimes \overline{C}$. Every Q-system $A$ in $C$ lifts trivially to a Q-system in $C \boxtimes \overline{C}$ by tensoring with the identity. Then the (left) full centre of $A$ is defined [14] as

$$Z^+ [A] = C^+[(A \otimes 1) \times^+ R],$$

i.e., the left centre of the $\times^+$-braided product of the lifted Q-system $A$ with the canonical Q-system. The right full centre can be defined analogously as $Z^- [A] = C^-[(A \otimes 1) \times^- R]$; in the sequel, “full centre” will always mean the left full centre.

The full centre has many interesting properties as a mathematical object of its own, in particular, if $C$ is modular, then two Q-systems have the same full centre if and only if they are Morita equivalent [21].

Here, we shall be interested only in its relevance for quantum field theory. (Recall that $C^{DHR}(A)$ is modular [20] for every completely rational theory, hence the full centre is indeed a Morita invariant.) The full centre of a Q-system $A$ in the DHR category of a chiral net $A$ is a Q-system in $C^{DHR}(A \otimes A)$, hence it describes a local extension of $A_2 = A \otimes A$. By considering the algebraic interpretation Props. 4.10 and 4.9 of the two steps: braided product and left centre, involved in the definition Eq. (4.22), the full centre extension is obtained by the corresponding steps:

1. (i) Construction of a 2D nonlocal CFT as a braided product of a chiral extension with the canonical 2D local QFT, and
2. (ii) restriction to a maximal local subtheory by taking relative commutants of the algebras associated with nested wedge regions.

It is therefore an instance of the combinations “(+,+)” and “(−,−)” not covered by Lemma 4.14.

Fig. 2a: Same as Fig. 1, with $B_1$ some chiral extension, and $B_2$ the canonical local extension. By nondegeneracy of the braiding, the right centre of the braided product (= relative commutant of right wedges) is exactly the canonical extension (Lemma 4.15).

Fig. 2b: The full centre of the chiral theory is the left centre (= relative commutant of left wedges) of the braided product with the canonical extension chiral extension. It equals the $\alpha$-induction construction from the chiral extension. It is in general different from, but may happen to be isomorphic to the canonical local extension. Being contained in the braided product, it is generated by certain products of canonical and chiral fields.

Fig. 3: Another view of Fig. 2.

It was recently recognized [2] that the full centre of the Q-system for the chiral extension $\mathcal{A} \subset \mathcal{B}$ coincides with the “$\alpha$-induction construction”.

The latter is a generalization of the canonical Q-system (Prop. 4.5) for the construction of two-dimensional (2D) extensions [33]. It was found by solving the defining identities of a Q-system in $\mathcal{C}^{\text{DHR}}(\mathcal{A} \otimes \mathcal{A})$ with the help of $\alpha$-induction. $\alpha$-induction is a prescription to extend DHR endomorphisms $\rho$ of the underlying chiral theory $\mathcal{A}$ to endomorphisms $\alpha_{\rho}^{\pm}$ of a chiral extension $\mathcal{B}$ of $\mathcal{A}$. The latter depend on the braiding $\varepsilon^{\pm}$, and are in general not DHR endomorphisms. From these data, one obtains Q-systems with coupling matrix (Eq. 4.13) $Z_{\sigma,\tau} = \dim(\text{Hom}(\alpha^{+}_{\sigma}, \alpha^{-}_{\tau}))$. If $\mathcal{A}$ is completely rational, then $Z$ is a modular invariant matrix [4].

This “arithmetic” construction may be regarded as not very satisfactory from the AQFT point of view. However, combining the above algebraic interpretation of the full centre, as well as the results in [2] and [28], we conclude:

**Corollary 4.16.** The following constructions of two-dimensional local extensions of $\mathcal{A}_2 = \mathcal{A} \otimes \mathcal{A}$ are all equivalent:
(1) The extension corresponding to the “full centre” of a chiral Q-system.

(2) The “α-induction construction” of a 2D local CFT associated with a (possibly nonlocal) chiral extension [27].

(3) The construction of a 2D local CFT by “removing the boundary” via a limit state on a (hard) boundary CFT [28].

(4) The construction of a 2D local CFT by the two steps (i) and (ii) above.

Proof: The equivalence (1) = (2) is established in [2], by identification of the Q-systems. The equivalence (2) = (3) was proven in [28]. The equivalence (1) = (4) is an immediate consequence of the definition of the full centre and the algebraic interpretation of the two operations involved, as discussed before.

Now, (4) provides a more satisfactory direct algebraic understanding (and in fact, a surprisingly simple one) of the previous indirect constructions (2) and (3) of two-dimensional extensions with modular-invariant coupling matrices Eq. (4.13).

Moreover, the construction (3) together with the maximality result for hard boundaries in [27] implies that every maximal irreducible local extension is of this form.

4.11 The relative commutant of product extensions

The braided product $A_1 \times \pm A_2$ of two irreducible Q-systems (dimHom(id, $\theta_1$) = 1) will in general be reducible, because dimHom(id, $\theta_1 \theta_2$) = dimHom($\theta_1, \theta_2$) > 1, cf. Sect. 4.2. For the corresponding extensions, this means $\mathcal{A'} \cap C^\pm \neq C \cdot 1$ because, for $\rho < \theta_1$ and $\rho < \theta_2$, operators of the form $\psi_{1,\rho}^*, \psi_{2,\rho}$ commute with every $a \in \mathcal{A}$ by Eq. (4.10).

Proposition 4.17. The relative commutant $\mathcal{A'} \cap C^\pm$ of the braided product extension $C^\pm$ is spanned by all operators of the form $\psi_{1,\rho}^* \psi_{2,\rho}$, where $\rho < \theta_1$ and $\rho < \theta_2$. If both Q-systems $A_1$ and $A_2$ are commutative (hence the corresponding extensions $\mathcal{B}_i$ are local), then the relative commutant $\mathcal{A'} \cap C^\pm$ coincides with the centre $\mathcal{C}^\pm \cap \mathcal{C}$.

Proof: In Lemma 4.4 we have characterized the relative commutant by the elements $i(q)v$ with $q \in \text{Hom}(\theta, \text{id})$, where $\theta = \theta_1 \theta_2$ for a product Q-system. By Frobenius reciprocity, we have $\text{Hom}(\theta_1, \theta_2, \text{id}) = r_{\theta_1}^*(\text{Hom}(\theta_2, \theta_1))$. A basis of $\text{Hom}(\theta_2, \theta_1)$ is given by $w_{1,\rho} w_{2,\rho}^*$ with $w_{i,\rho} \in \text{Hom}(\rho, \theta_i)$. With $\psi_{1,\rho} = w_{1,\rho}^* \theta_1(w_2)^* v^\pm$ and $\psi_{2,\rho} = w_{2,\rho}^* \theta_1(w_1)^* v^\pm$ (suppressing the symbols $i$ for simplicity), one computes

$$\psi_{1,\rho}^* \psi_{2,\rho} = v^{\pm*} \theta_1(w_2) w_{1,\rho} w_{2,\rho}^* \theta_1(w_2)^* v^\pm = r_{\theta_1}^* \theta_2(\theta_1(w_2) w_{1,\rho} w_{2,\rho}^* \theta_1(w_2)^*) x^\pm v^\pm = q v^\pm$$

with $q = r_{\theta_1}^* \theta_1(w_{1,\rho} w_{2,\rho}^*)$. Since $q \in \text{Hom}(\theta_1 \theta_2, \text{id})$ arises by Frobenius reciprocity from a basis $w_{1,\rho} w_{2,\rho}^*$ of $\text{Hom}(\theta_1, \theta_2)$, the operators $\psi_{1,\rho}^* \psi_{2,\rho}$ form a basis of $\text{Hom}(\theta_1 \theta_2, \text{id}) v = \mathcal{A'} \cap \mathcal{C}$.

In Lemma 4.4 we have characterized the centre by $q \in \text{Hom}(\theta, \text{id})$ satisfying the supplementary relation Eq. (4.9). For $q \in \text{Hom}(\theta_1 \theta_2, \text{id})$ one computes

$$qx^\pm = \theta_1(q) x_{\theta_1,\theta_2} x_1 \theta_1(x_2) = \theta_1(q) x_1 \theta_1(x_2) = \theta_1(q) x_1 \theta_1(x_2) = \theta_1(q) x_1 \theta_1(x_2) = \theta_1(q) x_1 \theta_1(x_2)$$

if both Q-systems are commutative, using Eq. (4.9). Thus the supplementary condition Eq. (4.9) for the product Q-system is satisfied by every $q \in \text{Hom}(\theta, \text{id})$. \hfill \square
This result will become important in the treatment of phase boundaries (cf. Sect. 5).

Every minimal projection $e$ in the center of $\mathcal{C}$ defines an irreducible representation $\pi_e(\mathcal{C}) = e\mathcal{C}$ of $\mathcal{C}$, and hence also of the two intermediate local extensions $\mathcal{A} \subset \mathcal{B}_1$ and $\mathcal{A} \subset \mathcal{B}_2$.

5 Phase boundaries

As discussed in Sect. 3.6, causality at a phase boundary requires only that the local QFT to the left of the boundary is left-local w.r.t. the local QFT to the right of the boundary. Such situations are precisely constructed by the braided products of extensions.

This is in contrast to previous constructions, like Sect. 4.10, where braided product extensions were used as intermediate nonlocal constructions from which one has to descend to the left or right centre in order to get (globally) local subtheories.

5.1 The universal construction

A phase boundary is a transmissive boundary with chiral observables $\mathcal{A}_2 = \mathcal{A}_+ \otimes \mathcal{A}_-$. The phases on both sides of the boundary are given by a pair of Q-systems $A^L = (\Theta^L, W^L, X^L)$ and $A^R = (\Theta^R, W^R, X^R)$ in $\mathcal{C}^{DHR}(\mathcal{A}_2)$, describing local 2D extensions $\mathcal{A}_2 \subset \mathcal{B}_2^L$ and $\mathcal{A}_2 \subset \mathcal{B}_2^R$.

Now consider the braided product Q-systems (cf. Sect. 4.9)
\[
(\Theta = \Theta^L \circ \Theta^R, W = W^L \times W^R, X = (1 \times \varepsilon^\pm_{\Theta^L, \Theta^R} \times 1) \circ (X^L \times X^R))
\]
and the corresponding extensions $\mathcal{A}_2 \subset \mathcal{C}_2^\pm$. The original extensions $\mathcal{B}_2^L$, $\mathcal{B}_2^R$ are intermediate, cf. Sect. 4.9
\[
\mathcal{A}_2 \subset \mathcal{B}_2^L \subset \mathcal{C}_2^\pm \quad \mathcal{A}_2 \subset \mathcal{B}_2^R \subset \mathcal{C}_2^\pm,
\]
and the nets $\mathcal{C}_2^\pm$ are generated by $\mathcal{A}_2$ and two sets of charged fields $\Psi_{\sigma \otimes \tau}^L (\sigma \otimes \tau \prec \Theta^L)$ and $\Psi_{\sigma \otimes \tau}^R (\sigma \otimes \tau \prec \Theta^R)$. The braided product Q-system determines their commutation relations among each other:
\[
\Psi_{\sigma' \otimes \tau'}^R \Psi_{\sigma \otimes \tau}^L = \varepsilon^\pm_{\sigma' \otimes \tau', \sigma \otimes \tau} \Psi_{\sigma' \otimes \tau'}^L \Psi_{\sigma \otimes \tau}^R.
\]

By Eq. (5.2), $\varepsilon^-_{\sigma' \otimes \tau', \sigma \otimes \tau} = 1$ whenever $\sigma' \otimes \tau'$ is localized to the spacelike left of $\sigma \otimes \tau$. Thus, the choice of $\varepsilon^-$ in Eq. (5.3) ensures that $\mathcal{B}^L$ is left-local w.r.t. $\mathcal{B}^R$, as required by causality. This dictates our choice of the braided product Q-system to be
\[
(\Theta^L, W^L, X^L) \times^- (\Theta^R, W^R, X^R),
\]
or – equivalently – $(\Theta^R, W^R, X^R) \times^+ (\Theta^L, W^L, X^L)$.

In view of 4.13, we consider Eq. (5.4) as a universal construction for the algebras Eq. (5.2) for a phase boundary connecting phases with local observables $\mathcal{B}_2^L$ and $\mathcal{B}_2^R$. Namely, the left and right charged fields are algebraically independent generators, only subject to the commutation relations Eq. (5.3) with $\varepsilon^-$ that ensure left locality.

5.2 Boundary conditions

The treatment in this subsection is quite general. It applies to boundaries between two-dimensional conformal QFTs described by local extensions $\mathcal{B}_2^L$ and $\mathcal{B}_2^R$ of $\mathcal{A} = \mathcal{A}_+ \otimes \mathcal{A}_-$, as
Phase boundaries in algebraic conformal QFT

As to “boundaries” between two chiral QFTs, i.e., two chiral QFTs $B^L$ and $B^R$ which are local chiral extensions of an underlying local chiral net $A$, separated by a point.

In Sect. 5.4, we shall present a more powerful result, Thm. 5.2, which only pertains to two dimensions. Namely, it covers the case when the underlying net is $A \otimes A$ and $B^L_2$ and $B^R_2$ are maximal two-dimensional local extensions, namely $A^L$ and $A^R$ both are full centres of chiral Q-systems.

For the remainder of this section, we drop the subscript distinguishing two-dimensional nets.

Let $A \subset B^L$ and $A \subset B^R$ be the local extensions of $A$ defining the QFT to the left and to the right of the boundary. Let $C \equiv C^-$ be the universal construction, namely the (nonlocal) extension obtained by the braided product $A^L \times^- A^R$ of the corresponding commutative Q-systems $A^L$ and $A^R$.

By Prop. 4.17, the extension $A \subset C$ is in general reducible, and the relative commutant coincides with the centre $C' \cap C$. The central decomposition gives irreducible representations of $C$, restricting to representations of both $B^L$ and $B^R$ on a common Hilbert space.

Let $\Theta^L = \bigoplus \rho Z^L_\rho \cdot \rho$ and $\Theta^R = \bigoplus \rho Z^R_\rho \cdot \rho$ be the irreducible decomposition. (In the two-dimensional case, each $\rho$ is of the form $\rho^+ \otimes \rho^-$. ) Then the dimension of the center is $\dim \text{Hom}(\Theta^L, \Theta^R) = \sum \rho Z^L_\rho \cdot Z^R_\rho$, and there are as many irreducible representations = different boundary conditions.

From Prop. 4.17, we know the centre as a linear space. A basis of this space (see below) are the products of charged field operators of the form

$$B_\rho = \Psi^L_\rho \Psi^R_\rho,$$

where $\rho = \sigma \otimes \tau \prec \Theta^L$ and $\prec \Theta^R$. These operators commute with $\iota(A)$ and hence belong to the centre by Prop. 4.17. In an irreducible representation, these operators are multiples of 1. Since $\Psi^L_\rho$ and $\Psi^R_\rho$ are isometries with the normalization convention as in Eq. (4.11), the numerical values of $B_\rho$ mean a linear relation between the charged fields.

Thus, every representation comes with a characteristic set of relations between the charged fields $\Psi^L_\rho$ and $\Psi^R_\rho$. These relations are the boundary conditions.

In order to determine the values of the central operators $B_\rho$ in each representation, we need to compute $B_\rho$ as linear combinations of the minimal central projections $E_m$ in $C_2$:

$$B_\rho = \sum_m \pi_m(B_\rho) \cdot E_m,$$

with $\pi_m(B_\rho) \in C$.

In order to determine the minimal central projections $E_m$, we need to know the centre as an algebra. At this point, a reformulation of the task is convenient.

We introduce the linear bijection

$$\chi : \text{Hom}(\Theta^R, \Theta^L) \rightarrow C' \cap C, \quad T \mapsto \iota(R^{L*} \Theta^L(T)) \cdot V.$$  (5.7)

Then one has

$$\chi(T_1) \cdot \chi(T_2) = \chi(T_1 * T_2)$$  (5.8)

where the associative $*$-product in $\text{Hom}(\Theta^R, \Theta^L)$ is defined as

$$T_1 * T_2 := X^{L*} \cdot \Theta^L(T_1)T_2 \cdot X^R \equiv X^{L*} \cdot T_2 \Theta^R(T_1) \cdot X^R.$$  (5.9)
This associative product is also commutative, because the Q-systems $A^L$ and $A^R$ are commutative. The unit of the $*$-product is $W^L W^{R*}$. Furthermore,

$$\chi(T^*) = \chi(F(T^*)) \quad (5.10)$$

where $F: \text{Hom}(\Theta^L, \Theta^R) \to \text{Hom}(\Theta^R, \Theta^L)$ is the Frobenius conjugation $T^* \mapsto R^{R*} \Theta^R (T^* R^L) = \Theta^L (R^{R*} T^*) R^L$.

Thus, the task is to find $I_m = \chi^{-1}(E_m)$ which are the minimal projections in $\text{Hom}(\Theta^R, \Theta^L)$ with respect to the $*$-product, and expand $\chi^{-1}(B_\rho)$ in terms of $I_m$.

Let $\Psi^L_\rho = W^L \Theta^L (W^R)^* V, \Psi^R_\rho = W^R \chi \chi W \chi^L$ with $W^Y_\rho \in \text{Hom}(\rho, \Theta^Y)$ ($Y = L, R$), as in Sect. 4.4. We suppress possible multiplicity indices if these spaces are more than one-dimensional, and assume the charged field operators to be isometries, i.e., $W^Y_\rho$ are normalized as $W^Y_\rho W^Y_\rho = d_Y/\dim(\rho)$ in accordance with Eq. (4.11). $d_Y = \sqrt{\dim(\Theta^Y)}$ are the dimensions of the factor Q-systems.

By Frobenius reciprocity $W^Y_\rho := R^T_\rho (W^Y_\rho R^*) = \Theta^Y (R^T_\rho W^Y_\rho R^*) R^Y \in \text{Hom}(\rho, \Theta^Y)$ with the same normalization, and $\Psi^Y_\rho$ and $B_\rho$ are the corresponding charged fields and central operators.

In particular, Eq. (5.15) turns into

$$B_\rho \equiv \Psi^L_\rho \Psi^R_\rho = \chi(W^L W^{R*}_\rho). \quad (5.11)$$

Since $W^L W^{R*}_\rho \text{ span } \text{Hom}(\Theta^R, \Theta^L), B_\rho \text{ span } C' \cap C$.

The following proposition gives the algebraic relations among $B_\rho$ in terms of the pair of Q-systems $A^L, A^R$. Knowing their algebra, one can compute their decomposition into central projections.

**Proposition 5.1.** The algebraic relations of the centre $C' \cap C$ of the braided product of two local extensions (corresponding to two commutative Q-systems) are

$$B^*_\rho = B^T_\rho, \quad f_\rho \cdot B_\sigma = \sum_{\tau<\rho\sigma} f^T_{\rho,\sigma} \cdot B_\tau \quad (5.12)$$

where

$$f^T_{\rho,\sigma} = f^*_{\sigma,\rho} = \frac{\dim(\tau)^2}{d_L d_R} \cdot W^L_{\tau} \cdot \left( (W^L_\rho W^{R*}_\rho) * (W^L_\sigma W^{R*}_\sigma) \right) \cdot W^R_{\tau}. \quad (5.13)$$

The expressions given for $f^T_{\rho,\sigma}$ are self-intertwiners $\in \text{Hom}(\tau, \tau)$, hence multiples of 1.

**Proof:** We give only a sketch of the proof. The first relation follows, in view of Eq. (5.10), from $F(W^L W^{R*}_\rho) = W^L \chi \chi W^R$. The second relation is just the expansion of $(W^L W^{R*}_\rho) * (W^L W^{R*}_\sigma)$ in terms of $W^L W^{R*}_\tau$, by using the stated normalizations of $W^Y_\rho$, translated in view of Eq. (5.11). $\Box$

Up to normalization factors, the coefficients $f^T_{\rho,\sigma}$ are just products of the expansion coefficients $\zeta^R$ and $\zeta^L$ of the Q-systems, as in [26] or [33].

In the two-dimensional case, if $A_+ = A_-$ and both Q-systems in the proposition equal the canonical Q-system in the category $C_{\text{DHR}}(\mathcal{A}) \boxtimes C_{\text{DHR}}(\mathcal{A})^{\text{opp}}$, each sector is of the form $\rho \otimes \tau$, there are no multiplicities, and the coefficients simplify drastically:

$$f^T_{\rho \otimes \tau} = \frac{\dim(\tau)}{\dim(\rho) \dim(\sigma)} \quad (5.13)$$
Thus the central operators \( \dim(\rho)B_{\rho \otimes \overline{\rho}} \) represent the fusion rules. This implies that the various irreducible representations are labelled by the irreducible chiral sectors \( \sigma \in \mathcal{C}^{\text{DHR}}(\mathcal{A}) \), and Eq. (5.14) becomes

\[
\pi_{\sigma}(\Psi_{\rho \otimes \overline{\rho}}^L \Psi_{\rho \otimes \overline{\rho}}^R) = \frac{S_{\rho,\sigma}S_{\rho,0}}{S_{\rho,0}S_{\rho,\sigma}}
\]

(where \( S_{0,\sigma} = S_{\sigma,0} = \dim(\sigma)/d_R \)), thus giving certain angles between the isometric intertwiners \( \Psi^L \) and \( \Psi^R \). In particular, \( \sigma = \text{id} \) gives \( \Psi_{\rho \otimes \overline{\rho}}^L \Psi_{\rho \otimes \overline{\rho}}^R = 1 \), hence \( \Psi_{\rho \otimes \overline{\rho}}^L = \Psi_{\rho \otimes \overline{\rho}}^R \), i.e., the trivial boundary where the left and right fields coincide.

### 5.3 Example 5: Phase boundaries of the Ising model.

In the Ising model \( (\Theta^L = \Theta^R = \Theta_{\text{can}}) \) everything can be computed explicitly: \( \mathcal{C}_2 \) is generated by \( A_2 \) and \( \Psi_{\sigma \otimes \sigma}^Y \) and \( \Psi_{\sigma \otimes \sigma}^{Y^*} \) \( (Y = L, R) \) satisfying the relations as in Example 4 for both \( Y = L, R \) and mutual commutation relations

\[
\Psi_{r \otimes r}^R \Psi_{r \otimes r}^L = \Psi_{r \otimes r}^L \Psi_{r \otimes r}^R, \quad \Psi_{r \otimes r}^L \Psi_{r \otimes r}^L = (u \otimes u) \Psi_{\sigma \otimes \sigma}^L \Psi_{r \otimes r}^L, \quad \Psi_{r \otimes r}^L \Psi_{r \otimes r}^R = (u \otimes u) \Psi_{r \otimes r}^L \Psi_{r \otimes r}^R,
\]

\[
\Psi_{r \otimes r}^L \Psi_{r \otimes r}^L = \left((rr^* - itt^*) \otimes (rr^* + itt^*)\right) \Psi_{\sigma \otimes \sigma}^L \Psi_{r \otimes r}^R.
\]

The center of \( \mathcal{C}_2 \) is spanned by 1 and \( B_{\sigma \otimes \sigma} = \Psi_{\sigma \otimes \sigma}^L \Psi_{\sigma \otimes \sigma}^R \), and \( B_{\sigma \otimes \sigma} = \Psi_{\sigma \otimes \sigma}^L \Psi_{\sigma \otimes \sigma}^R \), satisfying \( B_{\omega \otimes \omega}^2 = 1 \), \( B_{\tau \otimes \tau} B_{\sigma \otimes \sigma} = B_{\sigma \otimes \sigma} \), and \( B_{\sigma \otimes \sigma} = 1/2 \). Therefore, the minimal central projections are \( \frac{1}{2}(1 + B_{\tau \otimes \tau}) \) and \( \frac{1}{2}(1 - B_{\tau \otimes \tau}) \), and the ensuing relations in the three different quotients are, respectively:

(i) \( \Psi_{\tau \otimes \tau}^L = \Psi_{\tau \otimes \tau}^R \), \( \Psi_{\sigma \otimes \sigma}^L = \Psi_{\sigma \otimes \sigma}^R \);

(ii) \( \Psi_{\tau \otimes \tau}^L = \Psi_{\tau \otimes \tau}^R \), \( \Psi_{\sigma \otimes \sigma}^L = -\Psi_{\sigma \otimes \sigma}^R \);

(iii) \( \Psi_{\tau \otimes \tau}^L = -\Psi_{\tau \otimes \tau}^R \).

The first case is the trivial boundary; the second the “fermionic” boundary where the field \( \Psi_{\sigma \otimes \sigma} \) changes sign, and the third the “dual” boundary, in which there are two independent fields \( \Psi_{\sigma \otimes \sigma}^L \) and \( \Psi_{\sigma \otimes \sigma}^R \) (corresponding to the order and disorder parameter \( \sigma \) and \( \mu \) in Example 1). Notice that as a representation of either \( B_2^L \) or \( B_2^R \) (which are both isomorphic to the unique maximal local 2D Ising model), the Hilbert space Eq. (2.9) splits into two inequivalent representations.

### 5.4 Classification of boundary conditions

In general, not only signs will appear as boundary conditions between two canonical extensions. E.g., in models with a chiral sector of dimension \( \dim(\sigma) = \gamma = \frac{1}{2}(1 + \sqrt{5}) \) (the golden ratio) satisfying the fusion rules \( \sigma^2 \sim \text{id} \oplus \sigma \), it follows from Eq. (5.14) that \( B_{\sigma \otimes \sigma}^2 = \gamma^{-2} \cdot 1 + \gamma^{-1} \cdot B_{\sigma \otimes \sigma} \). This implies that the spectrum of \( B_{\sigma \otimes \sigma} \) is \( \{1, -\gamma^{-2} = \gamma - 2\} \). In the eigenspace \( B_{\sigma} = 1 \) one has \( \Psi_{\sigma \otimes \sigma}^L = \Psi_{\sigma \otimes \sigma}^R \), while in the eigenspace \( B_{\sigma \otimes \sigma} = -\gamma^{-2} \) the two charged fields \( \Psi_{\sigma \otimes \sigma}^L \) and \( \Psi_{\sigma \otimes \sigma}^R \) stand at an angle of 112.5° (cos \( \varphi = -\gamma^{-2} \)) in a two-dimensional space of charged fields satisfying \( \Psi_{\sigma \otimes \sigma} a = (\sigma \otimes \sigma)(a) \Psi_{\sigma \otimes \sigma} \).

The following theorem provides a formula for the minimal central projections \( E \) for boundaries between two extensions of \( \mathcal{A}_2 = \mathcal{A} \otimes \mathcal{A} \) that arise as full centres (= \( \alpha \)-induction constructions) of (possibly different) chiral extensions. This formula allows to compute, in the corresponding irreducible representations, the numerical values \( \pi_m(\Psi_{\sigma \otimes \tau}^L \Psi_{\sigma \otimes \tau}^R) \), i.e., the angles
between a pair of charged fields at the boundary, without explicitly diagonalizing the algebra Eq. (5.12). In particular, explicit knowledge of the coefficients \( \zeta \) of the Q-systems is not needed.

For Thm. 5.2 we have to assume that \( \mathcal{A}_+ = \mathcal{A}_- \) and that its DHR category \( c^{\text{DHR}}(A) \) is modular. Recall that modularity is automatic if the chiral theory \( \mathcal{A} \) is completely rational [20], and is satisfied for all minimal \((c < 1 \text{ Virasoro})\) models and for many (presumably all) current algebra models associated with semisimple Lie algebras [31]. This theorem, including the special case Eq. (5.14), is the only place where we assume modularity. We also assume that the local extensions \( \mathcal{B}_2^Y \) \((Y = L, R)\) on both sides of the boundary are irreducible maximal extensions. These are precisely those extensions whose Q-systems are full centres of irreducible chiral Q-systems, cf. Sect. 4.10, thus we assume that the Q-systems for \( \mathcal{A}_2 = \mathcal{A} \otimes \mathcal{A} \subset \mathcal{B}_2^Y \) are given as

\[
(\theta^L, W^L, X^L) = Z[A], \quad (\theta^R, W^R, X^R) = Z[B].
\]

The chiral Q-systems \( A = (\theta^A, w^A, x^A) \) and \( B = (\theta^B, w^B, x^B) \) can be different, and they may belong to different Morita equivalence classes.

In order to find the minimal projections \( I_m \) in \( \text{Hom}(\Theta^R, \Theta^L) \) w.r.t. the \( \ast \)-product, such that \( E_m = \chi(I_m) \) are the minimal projections in the centre \( C'_2 \cap C_2 \), we have to solve the three equations

\[
\begin{align*}
\text{self-adjointness:} & \quad I_m^* = F(I_m) \\
\text{idempotency:} & \quad I_m \ast I_{m'} = \delta_{mm'} \cdot I_m, \\
\text{completeness:} & \quad \sum_m I_m = W^L W^R \ast.
\end{align*}
\]

Minimality is ensured if the number of \( I_m \) exhausts the dimension of \( \text{Hom}(\Theta^R, \Theta^L) = \text{Tr}(Z^L \ast Z^R) \).

Having found the intertwiners \( I_m \) corresponding to the central projections, we obtain the values (= boundary conditions) \( \pi_m(B_\rho) \) from the expansion

\[
W_\rho^L W_\rho^R = \sum_m \pi_m(B_\rho) \cdot I_m
\]

which is the same as Eq. (5.6) under the linear bijection \( \chi \).

**Theorem 5.2.** Let \( \mathcal{A} \) be a completely rational chiral CFT net, and \( \mathcal{C} \) its DHR category, which is modular [20]. Let \( \mathcal{B}^Y \) \((Y = L, R)\) be two local chiral extensions of \( \mathcal{A} \otimes \mathcal{A} \) whose commutative Q-systems \( (\Theta^Y, W^Y, X^Y) \) arise as full centers of chiral Q-systems \( A = (\theta^A, w^A, x^A) \) and \( B = (\theta^B, w^B, x^B) \). Then the minimal projections \( I_m \) in \( \text{Hom}(\Theta^R, \Theta^L) \) (i.e., the solutions to the system Eq. (5.15)) are given by the operators

\[
I_m = \frac{\dim(\beta)}{d_A d_B d_R} \cdot D_{R[m]} \ast
\]

where \( m = (\beta, m) \) run over the equivalence classes of irreducible \( A \)-\( B \)-bimodules.

We shall define the operators \( D_{R[m]} \ast \) below, but we shall not give a complete proof, which can be found in the compagnon paper [3, Sect. 4.11]. The result is actually implicit in [14] [21], but our proof in [3] is more streamlined and takes benefits from substantial simplifications, that apply in the present case of categories of homomorphisms of C*-algebras, but that are not assumed in [14] [21].
Here, we mainly want to explain the notions of the Theorem, and how it can be used to compute the boundary conditions, i.e., the values $\pi_m(B_\rho) \in \mathbb{C}$.

A-$B$-bimodules between Q-systems $A$ and $B$ in a $C^*$ tensor category $\mathcal{C}$ are, in terms of the category, pairs $m = (\beta, m)$ where $\beta$ is an object of $\mathcal{C}$ and $m \in \text{Hom}(\theta^A \beta B)$ subject to relations. In the case of $\mathcal{C}$ a full subcategory of $\text{End}_0(N)$, and $A$ and $B$ corresponding to extensions $N \subset M^A$ and $N \subset M^B$ according to Thm. 4.2 respectively, a bimodule is equivalent to a homomorphism $\varphi : M^B \to M^A$ such that $\beta = \tau^A \circ \varphi \circ \nu^A$ and $m = \tau(v^A \beta(v^B))$. Conversely, every homomorphism $\varphi : M^B \to M^A$ corresponds to a bimodule, provided $\varphi < \tau^A \rho \nu^B$ for some $\rho \in \mathcal{C}$. In particular, a Q-system $A$ is an $A$-$A$-bimodule in a natural way with $\varphi = \text{id}_M$, hence $m = x \cdot x$.

The $A$-$B$-bimodules for a given pair $A$, $B$ form again a category, i.e., one can define intertwiners $\in \text{Hom}(\mathbf{m}_1, \mathbf{m}_2)$, with which equivalence, inclusion, and direct sums of bimodules can be defined. $A$-$B$-bimodules can be tensored with $B$-$C$-bimodules giving rise to $A$-$C$-bimodules $\mathbf{m}_{1B} \otimes_B \mathbf{m}_{2C}$. One arrives at a (non-strict) bicategory whose 1-objects are the Q-systems, the 1-morphisms = 2-objects are the bimodules, and the 2-morphisms are the bimodule intertwiners. In fact, up to unitary equivalence, the notions of inclusion, direct sum, and tensor product of bimodules coincide with the corresponding notions of inclusion, direct sum, and composition in the (strict) 2-category of homomorphisms as in Sect. 4.1.

We now assume that $\mathcal{C}$ is braided, as is the case for $\mathcal{C} = \mathcal{C}^{\text{DHR}}(A)$. With an $A$-$B$-bimodule $m = (\beta, m)$, one can associate an intertwiner $D_m \in \text{Hom}(\theta^B, \theta^A)$ as follows. With $m \in \text{Hom}(\beta, \theta^A \beta B)$, one has $\varepsilon(\theta, \theta^A \beta \beta B)m \in \text{Hom}(\beta \beta B, \beta \theta^A)$. Then the intertwiners

$$D_m := r^\beta_m (\varepsilon(\theta, \theta^A \beta \beta B)m) r_\beta \in \text{Hom}(\theta^B, \theta^A)$$

with $r_\beta \in \text{Hom}(\theta, \beta \theta)$ (part of) a standard solution to the conjugacy relations for $\beta, \beta'$, do not depend on the standard solution, and depend on $m$ only through its equivalence class as a bimodule. Moreover, these operators form a “representation” of the tensor category of bimodules in the sense that one has $D_m D_m = D_m$ and

$$D_m D_m D_m = d_B \cdot D_m D_m D_m = d_B \cdot \sum_{\mathbf{m}_{AC}} \text{dimHom}(\mathbf{m}_{AC}, \mathbf{m}_{AB} \otimes_B \mathbf{m}_{BC}) \cdot D_{m_{AC}}$$

where the sum extends over the equivalence classes of irreducible $A$-$C$-bimodules.

Passing from $\mathcal{C}$ to $\mathcal{C} \otimes \mathcal{C}^{\text{opp}}$, a Q-system $A$ in $\mathcal{C}$ trivially defines a Q-system $A \otimes 1$ in $\mathcal{C} \otimes \mathcal{C}^{\text{opp}}$. Similarly, $A$-$B$-bimodules lift to $A \otimes 1$-$B$-$1$-bimodules. The braided product of Q-systems gives rise to a braided product of bimodules, so that one can define the $R[A]$-$R[B]$-bimodule

$$R[\mathbf{m}] := (\mathbf{m} \otimes 1) \times^+ \mathbf{R},$$

where $\mathbf{R}$ is the canonical Q-system $R$ in $\mathcal{C} \otimes \mathcal{C}^{\text{opp}}$ regarded as an $R$-$R$-bimodule, and $R[A] = (A \otimes 1) \times^+ \mathbf{R}$. The full centre $Z[A]$ is an intermediate Q-system of $R[A]$. One can then restrict $R[\mathbf{m}]$ to the full centres $Z[A]$ and $Z[B]$ giving rise to a $Z[A]$-$Z[B]$-bimodule denoted by $R[\mathbf{m}]|_{Z}$. Now, $D_{R[\mathbf{m}]|_{Z}} \in \text{Hom}(\Theta^R, \Theta^L)$ in Eq. (5.17) is just Eq. (5.18) for the $Z[A]$-$Z[B]$-bimodule $R[\mathbf{m}]|_{Z}$, where $\mathbf{m}$ is an irreducible $A$-$B$-bimodule.

The number of inequivalent irreducible $A$-$B$-bimodules is known [10]. Namely, if $\Theta^L = \bigoplus_{\sigma, \tau} Z^L_{\sigma, \tau} \cdot \sigma \otimes \tau$ and $\Theta^R = \bigoplus_{\sigma, \tau} Z^R_{\sigma, \tau} \cdot \sigma \otimes \tau$, then this number is $\text{Tr}(Z^{\text{L}} \cdot Z^{\text{R}})$ which equals the dimension of the centre $\text{dimHom}(\Theta^R, \Theta^L)$. Thus, provided $D_{R[\mathbf{m}]|_{Z}}$ are linearly independent, they form a basis of $\text{Hom}(\Theta^R, \Theta^L)$. 

Using the non-degeneracy property of the braiding in a modular category, one proves that they are indeed linearly independent, and in fact orthogonal w.r.t. the scalar product
\[(D_1, D_2) := R^{R*} \cdot D_1^* D_2 \cdot R^R = R^{L*} \cdot D_2 D_1^* \cdot R^L,\] namely (e.g., [3 Sect. 4.11])
\[(D_{R[m_1]}|z), D_{R[m_2]}|z) = d_A^2 d_B^2 d_R^2 \cdot \delta_{m_1, m_2}.\]
Using this fact together with the observation that the scalar product Eq. (5.20) is related to the ∗-product by \((D_1, D_2) = W^{L*} \cdot (F(D_1^*) \ast D_2) \cdot W^R,\) we can also prove \([2]\) that, up to normalizations as given in Eq. (5.17), \(D_{R[m]|z}\) solve Eq. (5.15). (This last step is considerably simplified as compared to the general theory [14 21] by the use of a positivity argument.)

As an interesting side-result one obtains [21 3]

**Proposition 5.3.** All full centre Q-systems \(Z[A]\) (\(A\) an irreducible Q-system in \(C\) in a modular tensor category \(C\)) have the same dimension
\[d^2_{Z[A]} \equiv \dim(\Theta^Z[A]) \equiv \sum_{\sigma, \tau} Z_{\sigma, \tau} \dim(\sigma) \dim(\tau) = d_R^2 = \sum_{\rho} \dim(\rho)^2.\]
where the sum extends over the equivalence classes of irreducible objects of \(C\).

### 5.5 Computation of the boundary conditions

Now, we turn to computing the values \(\pi_m(B_\rho).\) W.r.t. the scalar product Eq. (5.20), also the basis \(T := W^L_\rho W^{R*}_\rho\) of \(\text{Hom}(\Theta^R, \Theta^L)\) (with \(W^L_\rho\) normalized as in Sect. 5.2) is orthogonal:
\[(T, T') = \frac{d_R^2}{\dim(\rho)} \cdot \delta_{T, T'}\]
(Where the label \(T\) is understood to include the data \(\rho\) and possible multiplicity indices). Therefore, the passage from the normalized basis \(\frac{1}{d_A d_B d_R^2} \cdot D_{R[m]|z}\) to the normalized basis \(\frac{\sqrt{\dim(\beta)}}{d_R} \cdot T\) defines a unitary matrix
\[S_{m,T}^{AB} := \frac{\sqrt{\dim(\rho)}}{d_A d_B d_R^2} \cdot (D_{R[m]|z}, T),\] and hence
\[W^L_\rho W^{R*}_\rho \equiv T = \frac{d_R}{\sqrt{\dim(\rho)}} \sum_m S_{m,T}^{AB} \cdot \frac{1}{d_A d_B d_R^2} \cdot D_{R[m]|z} = \frac{d_A d_B d_R}{\sqrt{\dim(\beta)}} \sum_m S_{m,T}^{AB} \cdot \frac{1}{\dim(\beta)} \cdot I_m.\]
This is the desired decomposition Eq. (5.16), i.e., the central operators \(B_\rho = \Psi^{L*}_\rho \Psi^R_\rho\) take the values
\[\pi_m(B_\rho) = \frac{d_A d_B d_R}{\dim(\beta) \sqrt{\dim(\rho)}} \cdot S_{m,T}^{AB} \quad (m = (\beta, m), \ T = W^L_\rho W^{R*}_\rho).\]

**Corollary 5.4.** The boundary conditions, i.e., the irreducible representations of \(C = B^L_2 \times B^R_2\) are labelled by the chiral bimodules between the chiral Q-systems whose full centres give rise to the extensions \(B^L_2\) and \(B^R_2\). The “angles” \(\pi_m(\Psi^{L*}_\rho \Psi^R_\rho)\) between left and right fields are given by the matrix elements of the generalized S-matrix.
The remaining task is to find an efficient way to compute the matrix elements $S_{m,\rho}^{AB}$. First, we list some rather trivial special cases.

We have

$$S_{m,\rho}^{AB} = \frac{\dim(\beta)}{d_Ad_B\rho}$$

for $T = W^L W^R$ the unit w.r.t. $\ast$-product. This just reflects the fact that $\Psi_{\text{id}}^L = \Psi_{\text{id}}^R = 1$, hence the corresponding operator $B_{\text{id}}$ must be $1$ in every representation.

For $A = B$ (i.e., the phase boundary is a defect), and $m = A$ as an $A$-$A$-module, one finds $D_{R(A)\mid Z} = d_A^2 d_R \cdot 1$, hence

$$S_{A,\rho}^{AA} = \sqrt{\dim(\rho)}.$$ 

Consequently, $\pi_A(B_{\rho}) = 1$ and hence $\Psi_{\rho}^L = \Psi_{\rho}^R$ for all $\rho$. Thus, the trivial bimodule describes the trivial boundary.

For $A = B = (\text{id}, 1, 1)$ the Q-system of the trivial chiral extension $A \subset A$, the bimodules are just the irreducible chiral sectors $\sigma \in \mathcal{C}$. Since $Z[A] = R$ is the canonical Q-system, $\rho^+ \otimes \rho^-$ are of the form $\rho \otimes \bar{\rho}$, and one recovers the example Eq. (5.14)

$$S_{\sigma, \rho \otimes \bar{\rho}}^{11} = S_{\sigma, \rho}.$$ 

One may use further special properties of the matrix Eq. (5.21) in order to compute its matrix elements more efficiently than by evaluation of the defining scalar products. Namely, the property Eq. (5.19) lifts to $R(m)|_Z$:

$$D_{R(m_{AB})|Z} D_{R(m_{BC})|Z} = d_B d_R^2 \cdot D_{R(m_{AB} \otimes B m_{BC})|Z} = d_B d_R^2 \cdot \sum_{m_{AC}} N_{m_{AB}, m_{BC}}^{m_{AC}} \cdot D_{R(m_{AC})|Z},$$ 

where we write the fusion rules of irreducible $A$-$B$-bimodules and $B$-$C$-bimodules as

$$m_1 \otimes_B m_2 \simeq \bigoplus_{m_3} N_{m_1, m_2}^{m_3} \cdot m_3.$$ 

If we now express $D_{R(m)|_Z}$ in terms of the bases $T = W^L W^R$, we arrive at

$$\sum_{T_1, T_2} S_{m_1, T_1}^{AB} S_{m_2, T_2}^{BC} \cdot \sqrt{\dim(\rho)} \cdot T_1 T_2 = \sum_{m_3} N_{m_1, m_2}^{m_3} \sum_{T_3} S_{m_3, T_3}^{AC} \cdot T_3,$$

which can be solved for the fusion rules:

$$N_{m_1, m_2}^{m_3} = \sum_{T_1, T_2, T_3} \frac{(\dim(\rho))^{3/2}}{d_R^2} (T_3, T_1 T_2) \cdot S_{m_1, T_1}^{AB} S_{m_2, T_2}^{BC} S_{m_3, T_3}^{AC}. \quad (5.23)$$

If there are no multiplicities (i.e., $\dim_{\text{Hom}}(\rho, \Theta^Y) = 0$ or $1$ for all $Y = A, B, C$), then $(T_3, T_1 T_2) = d_R^3 (\dim(\rho))^{-2} \cdot \delta_{T_1, T_3} \delta_{T_2, T_3}$, and Eq. (5.23) simplifies to the “generalized Verlinde formula”

$$N_{m_1, m_2}^{m_3} = \sum_T \frac{d_R}{\sqrt{\dim(\rho)}} \cdot S_{m_1, T}^{AB} S_{m_2, T}^{BC} S_{m_3, T}^{AC}. \quad (5.24)$$
In particular, for $B = C$, this becomes

$$N_{m_3}^{m_1,m_2} = \sum_T \frac{S_{m_1,T}^{AB} S_{m_2,T'}^{BB} S_{m_3,T}^{AB}}{S_{B,T}^{BB}},$$

(5.25)

and the unitary matrices $S_{m,T}^{AB}$ diagonalize the fusion rules:

$$\sum_{m_1,m_3} S_{m_1,T}^{AB} N_{m_1,m_2}^{m_3} S_{m_3,T'}^{AB} = \frac{S_{m_1,T}^{BB}}{S_{B,T}^{BB}} \delta_{T,T'}.$$

(5.26)

Because fusion rules are comparatively easy to compute with the help of Frobenius reciprocity, one may use Eq. (5.26) to determine the matrices $S_{m,T}^{AB}$ (with some complication if there are multiplicities).

Together with Eq. (5.22), this completes the computation of the boundary conditions.

We want to stress that in this way, the boundary conditions are ultimately characterized by chiral data only, namely the fusion category of chiral bimodules, which are in turn the irreducible subobjects of $\prec \iota_L \circ \rho \circ \tau_R$, $\rho \in \mathcal{C}$.

### 5.6 Example 6: Chiral boundary fields in the Ising model

Now let us consider, for the Ising model, the operators

$$\psi_+ := \sqrt{-i} \cdot \Psi_{\sigma \otimes \sigma}^L (u \otimes 1) \Psi_{\sigma \otimes \sigma}^R,$$

$$\psi_- := \sqrt{i} \cdot \Psi_{\sigma \otimes \sigma}^L (1 \otimes u) \Psi_{\sigma \otimes \sigma}^R,$$

(5.27)

which are chiral intertwiners:

$$\psi_+ a = (\tau \otimes \text{id})(a) \psi_+,$$

$$\psi_- a = (\text{id} \otimes \tau)(a) \psi_- \quad (a \in A_2).$$

Thus, they belong to the relative commutants $(1 \otimes A)' \cap C_2$ and $(A \otimes 1)' \cap C_2$, respectively, and in fact the latter are generated by $(A \otimes 1) \vee \psi_+ \text{ and } (1 \otimes A) \vee \psi_-$, respectively. They satisfy the relations

$$\psi_+^2 = \psi_+,$$

$$\psi_-^2 = \frac{1}{2} (1 - \Psi_{\tau \otimes \tau}^L \Psi_{\tau \otimes \tau}^R),$$

$$\psi_+ \psi_- = -\psi_- \psi_+ = \frac{i}{2} (\Psi_{\tau \otimes \tau}^R - \Psi_{\tau \otimes \tau}^L).$$

Thus, in the representations (i) and (ii) in Sect. 5.3, $\psi_\pm = 0$, while in the “dual” representation (iii), they are a pair of left- and right-moving selfadjoint unitary Fermi fields implementing the chiral automorphisms $\tau$, such that

$$\psi_+ \psi_- = -\psi_- \psi_+ = i \Psi_{\tau \otimes \tau}^R = -i \Psi_{\tau \otimes \tau}^L.$$

We also compute, eg, $\psi_+ \Psi_{\sigma \otimes \sigma}^R = \sqrt{-i} \cdot \frac{1}{2} \left( \Psi_{\sigma \otimes \sigma}^L - \Psi_{\tau \otimes \tau}^R \Psi_{\sigma \otimes \sigma}^L \right)$ which again vanishes in (i) and (ii) (because $\Psi_{\tau \otimes \tau}^R = \Psi_{\tau \otimes \tau}^L \text{ and } \Psi_{\tau \otimes \tau}^L \Psi_{\sigma \otimes \sigma}^L = \Psi_{\sigma \otimes \sigma}^L$), and equals $\sqrt{-i} \cdot \Psi_{\sigma \otimes \sigma}^L$ in (iii). We obtain

$$\Psi_{\sigma \otimes \sigma}^L = \sqrt{-i} \cdot \psi_\pm \Psi_{\sigma \otimes \sigma}^R,$$

$$\Psi_{\sigma \otimes \sigma}^R = \sqrt{+i} \cdot \psi_\pm \Psi_{\sigma \otimes \sigma}^L,$$

(5.28)

(valid in (iii)), thus expressing the left fields in terms of the right fields and the Fermi fields, and vice versa $R \leftrightarrow L$. This is the AQFT analogue of Eq. (2.6) and Eq. (2.7), after identifying $\Psi_{\sigma \otimes \sigma}^R$ with the order parameter $\sigma$ and $\Psi_{\sigma \otimes \sigma}^L$ with the disorder parameter $\mu$.

It would be interesting to understand in the general case, whether and how chiral (nonlocal) fields emerge as certain products of fields from $B_2^L$ and from $B_2^R$ (i.e., as a subalgebra of a representation of the nonlocal braided product), as exemplified in Eq. (5.27), and how, conversely, the local fields of $B_2^L$ can be represented as certain products of chiral fields and local fields from $B_2^R$ (and vice versa), as exemplified by Eq. (5.28).
6 Outlook: boundaries and local gauge transformations

6.1 Spacelike boundaries?

One could attempt to repeat the same analysis for spacelike boundaries, i.e., $t = 0$ lines in some Lorentz system, separating local QFTs $B^F_2$ (defined in the future of the boundary) and $B^P_2$ (defined in the past) with a common subtheory including the stress tensor. Again, both nets extend to the entire Minkowski spacetime.

But in this case, causality requires that $B^P_2$ and $B^F_2$ must be mutually local, whereas the “universal construction” $B^P_2 \times \pm B^F_2$ will in general not be local. An easy exercise shows that it is local if and only if $\varepsilon^+_\Theta^F, \Theta^P = \varepsilon^-_{\Theta^F, \Theta^P}$.

Otherwise, the braided product may become local in some representations, e.g., if $B^P_2 = B^F_2$ this happens for the central projection induced by the trivial bimodule, giving rise to a quotient in which $\Psi^F_\rho = \Psi^P_\rho$, as in Sect. 5.5. It would be an interesting question to characterize all central projections which give rise to a local net.

6.2 Four spacetime dimensions

Let us try and draw some lessons for four-dimensional QFT, although the situation departs from the present setting in several respects. This leads us to some speculative considerations concerning local gauge transformations in algebraic QFT.

In four dimensions, there is no distinction between “left” and “right”, and the braiding is in fact a symmetry, i.e., $\varepsilon^+_{\sigma, \rho} = \varepsilon^-_{\sigma, \rho}$. Consequently, there is only one braided product, and the braided product of two local extensions of a net of observables is indeed local, namely the commutation relations

$$\Psi^1_\rho \Psi^2_\sigma = \varepsilon_{\sigma, \rho} \Psi^2_\sigma \Psi^1_\rho$$

derived from the braided product imply that the generating charged fields $\Psi^1_\rho$ and $\Psi^2_\sigma$ commute whenever $\rho$ and $\sigma$ are localized at spacelike distance.

Causality at a timelike boundary requires “left locality” as before. But left locality implies mutual locality of the QFTs on either side of the boundary, and the braided product is again a universal construction. Also, one may as well admit spacelike boundaries, since (unlike in two dimensions, cf. Sect. 6.1) they do not require stronger locality properties than timelike boundaries.

Indeed, the original development of the DHR theory of superselection sectors was aimed at physical spacetime, and conformal symmetry was not assumed. Also in our previous analysis, conformal symmetry was not directly used, it only served to guarantee the existence of chiral observables so that the structure $C^{DHR}(A_+ \otimes A_-) = C^{DHR}(A_+) \boxtimes C^{DHR}(A_-)^{opp}$ arises which became essential for the classification of boundary conditions in Sect. 5.4.

Also, in general the tensor category $C^{DHR}(A)$ will not be rational. Instead, its irreducible objects are given by the unitary representations of an intrinsically determined compact gauge group $G$, and there is a distinguished graded local extension $A \subset B$ (the Doplicher-Roberts field algebra [9]) with a faithful action of $G$ by automorphisms (global gauge transformations) such that $A = B^G$. Since the index of $A \subset B$ equals the order of the group $G$, this extensions will not be described by a Q-system proper, unless $G$ finite. Otherwise, one would have to adapt the theory to Q-systems for extensions of infinite index with suitable regularity properties. Let
us for the sake of this outlook that $G$ is finite. The distinguished DR extension is maximal in the sense that every irreducible extensions is intermediate to $A \subset B$, and is given by $B^H$ with $H \subset G$ some subgroup of $G$.

For the braided product of the distinguished DR extension $B$ with itself, $\text{Hom}(\Theta, \Theta)$ is isomorphic to the group algebra $\mathbb{C}G$. Let us assume for the moment that $B$ is local, i.e., $A$ has no fermionic sectors. Then the centre of the braided product $B \times B$ is given by $\text{Hom}(\Theta, \Theta)$ w.r.t. the $\ast$-product, which is isomorphic to the algebra $C(G)$ of functions on the group, and the minimal projections are the $\delta$-functions $\delta_g \in C(G)$, cf. [22]. Hence, the boundary conditions are labelled by the elements $g \in G$, and the relations between the fields on both sides of the boundary are given by

$$\pi_g(\Psi^2_\rho) = \pi_g(\alpha_g(\Psi^1_\rho)) = u_\rho(g^{-1}) \cdot \pi_g(\Psi^1_\rho)$$

with $u_\rho$ being a unitary representation of $G$ acting in the $\dim(\rho)$-dimensional space of charged intertwiners $H_\rho = \text{Hom}(\iota, \iota\rho) \subset B$.

Thus, a boundary can be regarded as a vector bundle (with a two-point basis and the charged fields as fibre) in which a boundary condition defines a parallel transport by a gauge transformation. One may now imagine a lattice in Minkowski spacetime with spacelike and timelike edges, and its dual subdivision of spacetime into cells with timelike and spacelike faces as boundaries. The corresponding multiple braided product of field algebras has a huge center (one copy of $C(G)$ for each edge), and its minimal central projections are gauge configurations on the lattice. We suggest that this is the kinematical arena in which one should place algebraic QFT theory with local gauge symmetry.

Of course, several technical aspects have to be addressed. First, the braided product of extensions must be defined when $G$ is of infinite order, such that the center becomes $C(G)$. Second, the faces intersect in surfaces of co-dimension 2, thus there will arise defects of higher order. Third, the case of fermionic sectors has to be included, such as to cover the case of Quantum Electrodynamics where $A$ is to be thought of as the electrically neutral subalgebra of the Dirac algebra. The centre of the braided product of two graded-local extensions will be smaller than $C(G)$; we expect it to be $C(G/\mathbb{Z}_2)$ where $\mathbb{Z}_2 \subset G$ is the grading automorphism. Furthermore, a continuum limit has to be devised in which local algebras arise by refining the cells in a given region.

The most important element, however, still missing in our suggestion, is a formulation of the “gauge dynamics”, i.e., the dynamics of the gauge field itself and its “minimal coupling” to the gauge covariant fields. In the picture above, the infinite braided product algebra is just a direct sum of algebras, one for each gauge configuration.

Gauge curvature is described in terms of closed paths (plaquettes) in the lattice, which necessarily come along with intersecting faces. It may be expected that the associated defects of higher order carry additional degrees of freedom that play a dynamical role. Adding the defect degrees of freedom should embed the infinite braided product with its huge centre into a larger field algebra that may even be irreducible.

We hope to come to a better understanding of these (presently highly speculative) issues in future work.

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