Dynamical matrix for arbitrary quadratic fermionic bath Hamiltonians and non-Markovian dynamics of one and two qubits in an Ising model environment

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(Dated: 13th April 2017)

We obtain the analytical expression for the Kraus decomposition of the quantum map of an environment modeled by an arbitrary quadratic fermionic Hamiltonian acting on one or two qubits, and derive simple functions to check the non-positivity of the intermediate map. These functions correspond to two different sufficient criteria for non-Markovianity. In the particular case of an environment represented by the Ising Hamiltonian, we discuss the two sources of non-Markovianity in the model, one due to the finite size of the lattice, and another due to the kind of interactions.

PACS numbers: 05.50.+q, 03.65.Ta, 03.65.Yz, 05.70.Jk

I. INTRODUCTION

The need to fight decoherence, to guarantee the proper working of the quantum enhanced technologies of information and computation [11], has renewed the motivation for the in-depth study of system-environment interaction dynamics. In particular, the Markovian or non-Markovian nature of the dynamics is of great interest [23]. Several witnesses and quantifiers have been proposed in order to characterize the non-Markovianity of a quantum process [13]. For instance, the information flow between system and environment, quantified by the distinguishability of any two quantum states [6–8], or by the Fisher information [5], or mutual information [16]. Another interesting quantifier is the entanglement based measure of non-Markovianity [11]. It is related to the classical information flow between system and environment [12]. The physical interpretation of these quantifiers, however, remains an open question. The behavior of the quantifiers depend on the kind of interactions and size of the system, as is discussed in [13].

In this work we wish to obtain explicitly the Choi representation of the quantum map of an arbitrary quadratic fermionic Hamiltonian acting on qubits, and perform a comparative exploration of its dynamics from the point of view of (non-)divisibility [14] [15]. After obtaining the analytical expression of the dynamical matrix, we specialize to the case of an environment represented by the quantum one-dimensional Ising model acting on one central qubit, which in the case of finite size lattices can be solved analytically by means of the well known Jordan-Wigner and Bogoliubov transformations [16] [17]. The availability of an analytical solution for this representative critical model is the reason why this system is recurrently investigated in many instances. The study we perform here is complementary to previous investigations and, besides its pedagogical purpose, reveals functional dependencies among different indicators of non-Markovianity, and also stresses that there are two sources of non-divisibility in the dynamics, one intrinsic to the kind of interactions, and another due to the finite size of the lattice. The divisibility criterion consists in checking if an intermediate quantum map is not Complete Positive (CP) for some time instant, which amounts to checking the non-positivity of the corresponding dynamical matrix [18]. We will show that the non-positivity of the dynamical matrix, measured by its eigenvalues, in this case is a simple function of the Loschmidt echo [20], a quantity that indicates decoherence induced by perturbations. We will also investigate the action of a trivial extension of the map on the decay of entanglement of the system coupled to an ancilla. We shall see that the intermediate map is not contractive, and entanglement is again a function of the Loschmidt echo which is not monotonically decreasing, signaling non-Markovianity and information flux from the environment to the system [12]. Finally we wish to know if the number of particles in the system has some influence on the dynamics of the environment. Thus we derive the map acting on a system composed of two qubits, concluding that the results do not have any change.

The paper is organized as follows. We briefly revise the formalism of dynamical maps and the divisibility criterion in Sec. II. Our first result appears in Sec. III, where we present the exact Kraus decomposition for general quadratic fermionic Hamiltonians, and introduce a measure of non-Markovianity. In Sec. IV, we introduce the model we shall investigate numerically, and relate it to the formalism of Sec. III. In Sec. V, we obtain the map for a system of two-qubits, showing that the results related to non-Markovianity do not change in

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relation to the one-qubit case. Our results for the dynamics of a qubit interacting with an environment governed by the Ising model are presented in Secs. VI and VII, where we investigate the non-Markovianity both at and outside of the critical point. In Sec.VI we investigate the non-Markovianity using the most negative eigenvalue of the intermediate map as a quantifier, while in Sec. VII we use the increase of entanglement under local CP maps as a quantifier. Our final remarks are in Sec. VIII.

II. QUANTUM DYNAMICAL MAPS AND THE DIVISIBILITY CRITERION

The evolution of an open quantum system ($\rho' = \Phi(\rho)$) can be written in the well known operator sum representation as \[9,18]\:

$$\rho' = \sum_{\mu} K_{\mu} \rho K_{\mu}^\dagger, \quad \sum_{\mu} K_{\mu}^\dagger K_{\mu} = \mathbb{I},$$

(1)

where the $K_{\mu}$ are the Kraus operators related to the quantum map $\Phi$, and $\mathbb{I}$ is the identity in the Hilbert space of the system. Using the $\text{vec}$ operation, defined by \[18]\:

$$\text{vec}(|x\rangle\langle y|) = |x\rangle \otimes |y\rangle,$$

(2)

and the corresponding inverse operation,

$$\text{vec}^{-1}(|x\rangle \otimes |y\rangle) = |x\rangle\langle y|,$$

(3)

the following matrix product $(ABC)$ can be cast as:

$$ABC = \text{vec}^{-1}[(A \otimes C^T)\text{vec}(B)],$$

(4)

Therefore, a product of three matrices can be thought of as a super-operator (or map) $A \otimes C^T$ acting on the linear operator $B$. Now Eq. (1) can be conveniently rewritten as \[18]\:

$$|\rho'\rangle = \Phi |\rho\rangle, \quad \Phi = \sum_{\mu} K_{\mu} \otimes K_{\mu}^*,$$

(5)

where $|\rho\rangle \equiv \text{vec}(\rho)$.

Consider the evolution of the system from an initial time $t_0$ to a final time $t_f$,

$$|\rho(t_f)\rangle = \Phi(\rho(t_0)) |\rho(t_0)\rangle.$$

(6)

Suppose this evolution is broken in two steps with an intermediate time, $t_f > t_m > t_0$, namely:

$$|\rho(t_f)\rangle = \Phi(\rho(t_m)) \Phi(\rho(t_0)) |\rho(t_0)\rangle.$$

(7)

Whereas $\Phi(\rho(t_0))$ is a completely positive (CP) map for arbitrary $t_f$ \[18\], the map corresponding to the intermediate step, $\Phi(\rho(t_m))$, may be non-CP for some $t_m$. As realizable maps are always CP, $\Phi(\rho(t_m))$ being non-CP for the particular time $t_m$ witnesses the fact that such a division is not possible. A trivial case in which any intermediate division is possible corresponds to unitary evolution. Markovian evolutions also admit arbitrary intermediate steps. The intermediate map may fail to be CP only in the case of non-Markovian evolution. This divisibility criterion \[13\] is therefore a sufficient condition to detect non-Markovianity.

In order to check the complete positivity of a map, we use the well known duality between CP maps and positive operators, expressed by the Choi’s theorem \[18,19\]. First we define the unique dynamical matrix associated to the map:

$$D_{\mu\nu}^{mn} = \Phi_{mn}^{\mu\nu} = \langle m|\mu|\Phi|n\nu\rangle,$$

(8)

where Latin and Greek indices correspond to system and environment Hilbert spaces, respectively. The Choi’s theorem states that the map $(\Phi)$ is CP if and only if its dynamical matrix $(D)$ is a positive semi-definite operator. Finally, to check the complete positivity of the intermediate map, we form the matrix of its super-operator by means of the product:

$$\Phi(\rho(t_f, t_m)) = \Phi(\rho(t_f, t_0)) \Phi^{-1}(\rho(t_m, t_0)).$$

(9)

Note that $\Phi(t_f, t_0)$ is the matrix representation of the map that evolves the system from the initial time $t_0$ to any time $t$. $\Phi^{-1}(t_m, t_0)$ is the pseudo-inverse of $\Phi(t_m, t_0)$, and thus evolves the system from $t_m$ to $t_0$. Therefore the matrix product in Eq. (9) defines a matrix representation for the intermediate map. While the dynamical matrix $(D(t_f, t_0))$ corresponding to $\Phi(t_f, t_0)$ is always positive semi-definite, the one $(D(t_f, t_m))$ related to $\Phi(t_f, t_m)$ may happen to be non-positive, and in this case it witnesses a non-Markovian evolution.

III. DYNAMICAL MATRIX FOR A GENERAL FERMIONIC QUADRATIC HAMILTONIAN

In the previous section, we reviewed the formalism of quantum maps and the divisibility criterion. We now apply such formalism to environments described by general fermionic quadratic Hamiltonians, interacting with a qubit. We will show how to obtain the exact expression for the Kraus decomposition of the dynamical matrix.

Let us then consider a general fermionic quadratic Hamiltonian, namely,

$$H_q = \sum_{m,n=1}^{L} (x_m a_m^\dagger a_n + y_m a_m^\dagger a_n + h.c.).$$

(10)

where $L$ is the lattice size, and $x_m, y_m$ are arbitrary complex numbers. $a_m^\dagger$ is the creation (annihilation) operator, satisfying the usual anti-commutation relations:

$$\{a_i, a_j^\dagger\} = \delta_{ij}, \{a_i, a_j\} = 0.$$
For the interaction of the qubit with this environment, we consider the following Hamiltonian:

\[ H_{int} = -\delta |e\rangle \langle e| \otimes V_e, \]

where \(|g\rangle\) and \(|e\rangle\) are the qubit ground and excited states, respectively, and \(V_e\) is a fermionic quadratic Hamiltonian. We consider that the qubit and environment are initially uncorrelated, and they are in an arbitrary pure initial state,

\[ |\psi(0)\rangle = |\chi(0)\rangle \otimes |\varphi(0)\rangle = (c_g |g\rangle + c_e |e\rangle) \otimes |\varphi(0)\rangle, \]

where \(|\chi(0)\rangle = c_g |g\rangle + c_e |e\rangle\), with \(c_g^2 + c_e^2 = 1\), is the initial qubit state. The evolution under the total Hamiltonian,

\[ H = H_g + H_{int}, \]

is given by:

\[ |\psi(t)\rangle = e^{-iH t/\hbar} |\chi(0)\rangle \otimes |\varphi(0)\rangle, \]

\[ |\varphi(t)\rangle = c_g |g\rangle \otimes e^{-iE_g t/\hbar} |\varphi(0)\rangle + c_e |e\rangle \otimes e^{-iE_e t/\hbar} |\varphi(0)\rangle, \]

where

\[ H_e = H_g - \delta V_e. \]

Such Hamiltonians, \(H_e\) and \(H_g\), can be easily diagonalized by a Bogoliubov transformation \((\ref{bogoliubov})\), namely:

\[ B_{\pm k} \equiv \cos \left( \frac{\theta_k}{2} \right) a_{\pm k} \mp i \sin \left( \frac{\theta_k}{2} \right) a^\dagger_{\mp k}, \]

\[ A_{\pm k} \equiv \cos \left( \frac{\phi_k}{2} \right) a_{\pm k} \mp i \sin \left( \frac{\phi_k}{2} \right) a^\dagger_{\mp k}. \]

These new fermionic operators are related according to

\[ B_{\pm k} = \cos(\alpha_k) A_{\pm k} \mp i \sin(\alpha_k) A^\dagger_{\pm k}, \]

where \(\alpha_k = (\theta_k^2 - \phi_k^2)/2\). The Hamiltonians in diagonal form read:

\[ H_g = \sum_k c_{g}^2 (B_k^\dagger B_k + C_g), \quad H_e = \sum_k c_{e}^2 (A_k^\dagger A_k + C_e), \]

where \(C_g\) and \(C_e\) are both real constants, and \(c_{g(e)}^2\) are the single-particle eigenvalues. The ground states of \(H_g\) (\(G_g\)) and \(H_e\) (\(G_e\)) are related by:

\[ |G_g\rangle = \prod_{k > 0} \left[ \cos(\alpha_k) + i \sin(\alpha_k) A_k^\dagger A_{-k}^\dagger \right] |G_e\rangle. \]

Now we derive the Kraus decomposition of the map super-operator \((\Phi)\). The Kraus operators of the evolution are:

\[ K_i = (\mathbb{I}_S \otimes \langle i|) e^{-iH t/\hbar} (\mathbb{I}_S \otimes |\varphi(0)\rangle), \]

with \(\mathbb{I}_S = |g\rangle \langle g| + |e\rangle \langle e|\). Assuming, without loss of generality (the map does not depend on the initial states of the qubit-environment), that the environment is initially in its ground state, \(|\varphi(0)\rangle = |G_e\rangle\), and using Eq.\((\ref{bogoliubov})\), we obtain:

\[ K_i = \mathbb{I}_S \otimes \langle i| \left[ |g\rangle \langle g| \otimes |\varphi_g(t)\rangle + |e\rangle \langle e| \otimes |\varphi_e(t)\rangle \right]. \]

The environment states \(|\varphi_g(t)\rangle\) and \(|\varphi_e(t)\rangle\) are given by:

\[ |\varphi_g(t)\rangle = e^{-iE_g t/\hbar} |G_g\rangle = e^{-iE_g t/\hbar} \prod_{k > 0} \left[ \cos(\alpha_k) + i \sin(\alpha_k) A_k^\dagger A_{-k}^\dagger \right] |G_e\rangle, \]

where \(E_g\) is the ground state energy of \(H_g\). Likewise, using Eq.\((\ref{bogoliubov})\), we obtain:

\[ |\varphi_e(t)\rangle = e^{-iE_e t/\hbar} \times \prod_{k > 0} \left[ \cos(\alpha_k) + i \sin(\alpha_k) A_k^\dagger A_{-k}^\dagger \right] |G_e\rangle. \]

In order to obtain the Kraus operators, it is enough to calculate the overlaps \((\langle i|\varphi_g(t)\rangle)\) and \((\langle i|\varphi_e(t)\rangle)\) for a given environment basis \(\{|i\}\), as shown in Eq.\((\ref{bogoliubov})\). A convenient basis is formed by the eigenstates of \(H_e\), namely:

\[ \{|i\}\} = \{|G_e\rangle, A_{k_1}^\dagger |G_e\rangle, \ldots \}

where \(k_N = (k_1, k_2, \ldots, k_N)\) is the vector representing the momentum of the \(N(= 1, \ldots, L)\) excitations, and \(A_{k_N}^\dagger = A_{k_1}^\dagger A_{k_2}^\dagger \ldots A_{k_N}^\dagger\). It is easy to see that the only non null elements for “\((\langle i|\varphi_g(t)\rangle)\)”, using Eq.\((\ref{bogoliubov})\), are given by:

\[ \langle G_e| \varphi_g(t)\rangle = e^{-iE_g t/\hbar} (\prod_{k > 0} \cos(\alpha_k)), \]

and

\[ a_{k_N}(t) \equiv \langle G_e| A_{-k_N} A_{k_N} \varphi_g(t)\rangle = e^{-iE_g t/\hbar} \prod_{k \in k_N} (i \sin(\alpha_k)) \prod_{k > 0, k \neq k_N} \cos(\alpha_k), \]

where \(N\) varies from 1 to \(L/2\). Analogously, the non null terms for “\((\langle i|\varphi_e(t)\rangle)\)”, using Eq.\((\ref{bogoliubov})\), are given by:

\[ b_{k_N}(t) \equiv \langle G_e| A_{-k_N} A_{k_N} \varphi_e(t)\rangle = e^{-iE_e t/\hbar} \prod_{k \in k_N} (i \sin(\alpha_k) \exp(-i(\epsilon_k^e + \epsilon_{-k}^e)/\hbar)) \times \prod_{k > 0, k \neq k_N} \cos(\alpha_k). \]
It is easy to check the following relation:

\[ b_{K_N}(t) = a_{K_N}(t) f_{K_N}(t), \quad (31) \]

where

\[ f_{K_N}(t) = e^{-i\epsilon_{K_N}^2 t} \exp(-i \sum_{k \in K_N}^N (\epsilon_k^2 + \epsilon_k^- t) / \hbar). \quad (32) \]

Finally, we reach the first result of this work, obtaining a simple expression for the Kraus operators of the quantum map,

\[ K_{K_N} = a_{K_N}(t) (\ket{g} \bra{g} + f_{K_N}(t) \ket{e} \bra{e}). \quad (33) \]

Note that \( |a_{K_N}(t)|^2 \) is not a time dependent variable, and

\[ \sum_{\{K_N\}} |a_{K_N}(t)|^2 = Tr(|\varphi_{ls}(t) \rangle \langle \varphi_{ls}(t)|) = 1. \quad (34) \]

By using this fact, we can then write the quantum map in terms of the Kraus operators as follows,

\[ \Phi(t, 0) = \sum_{\{K_N\}} K_{K_N} \otimes K_{K_N}^* = |\varphi_{ls}(t) \rangle \langle \varphi_{ls}(t)| f_{K_N}(t) + (35) \]

If we define the following variable,

\[ x(t) \equiv \sum_{\{K_N\}} |a_{K_N}(t)|^2 f_{K_N}(t), \quad (36) \]

the quantum map can be rewritten as,

\[ \Phi(t, 0) = |\varphi_{ls}(t) \rangle \langle \varphi_{ls}(t)| f_{K_N}(t) + (37) \]

As expected, the quantum map consists in a decoherence channel, and thus we can identify the variable \( x(t) \) with the known Loschmidt echo \( L(t) \).

Using now Eq. \( \Phi \), we have the following expression for the intermediate map:

\[ \Phi(t_f, t_m) = |\varphi_{ls}(t_f) \rangle \langle \varphi_{ls}(t_f)| f_{K_N}(t) + \quad (39) \]

The dynamical matrix of this quantum map is

\[ D_{\Phi(t_f, t_m)} = \begin{pmatrix} 1 & 0 & 0 & y(t_f, t_m)^* \\ 0 & 0 & 0 \end{pmatrix}. \quad (40) \]

Computing the minimum eigenvalue, we arrive at the following simple sufficient condition for the positive-semi-definiteness of the dynamical matrix:

\[ 1 - |y(t_f, t_m)| \geq 0. \quad (42) \]

Therefore we have obtained a simple function capable to witness the non-Markovianity of the dynamics, i.e., \( \Phi \) is non-Markovian if \(|y(t_f, t_m)| > 1 \).

IV. ISING MODEL AS AN ENVIRONMENT FOR A SYSTEM OF ONE QUBIT

In the previous section, we derived the dynamical matrix for an arbitrary quadratic fermionic Hamiltonian. In this section we focus on an environment described by the Ising Hamiltonian in a transverse field (\( H_{Ising} \)), with periodic boundary conditions (\( L + 1 = 1 \)). The interaction with the environment (\( H_{int} \)) is by means of the transverse magnetic field in the Z direction (see Fig.1), more precisely,

\[ H_{Ising} = -J \sum_{j=1}^L (\sigma_j^z \sigma_{j+1}^z + \lambda \sigma_j^z), \quad (43) \]

\[ H_{int} = -\delta |\varphi| \sigma \sum_{j=1}^L \sigma_j^z. \quad (44) \]

In order to employ the previous section’s results, we first do the identification:

\[ H_e = H_{Ising} - \delta \sum_{j=1}^L \sigma_j^z \quad (45) \]

\[ H_g = H_{Ising}. \quad (46) \]
We now diagonalize the Ising Hamiltonian [17]. First we use the usual Jordan-Wigner transformation,

\[
\sigma_j^+ = \exp\left(i\sum_{l<j} a_l^+ a_l\right) = \prod_{l<j} (1 - 2a_l^+ a_l), \quad a_j = \prod_{l<j} (\sigma_l^+)^1, \quad (47)
\]

The Ising Hamiltonian can then be rewritten in terms of quadratic fermionic operators:

\[
H_{\text{ising}} = \int \left[-\sum_{j=1}^{L-1} (a_j^+ a_{j+1} + a_j a_{j+1}^+) + 2\lambda \hat{N} - \lambda L\right], \quad (49)
\]

where \(\hat{N} = \sum_j a_j^+ a_j\). The Hamiltonian conserves the parity, \(H_{\text{e}} e^{i\pi \hat{N}} = 0\). Thus we can analyze its odd/even subspaces separately. The gap between the ground state energy of these two subspaces obviously closes in the thermodynamic limit. For simplicity, we shall proceed in the even sector, which leads to a simple quadratic Hamiltonian with anti-periodic boundary conditions. Using the momentum eigenstates,

\[
a_k = \frac{1}{\sqrt{L}} \sum_j e^{-ikj} a_j, \quad (50)
\]

with \(k = 2\pi q, q = \pm 1/2, \pm 3/2, \ldots, \pm(L-1)/2\), for \(L\) even, and the Bogoliubov transformation (Eq. 19), with phases

\[
\theta^\pm_k(\delta) = \arctan\left(\frac{-\sin(k)}{\cos(k) - (\lambda + \delta)}\right), \quad (51)
\]

the Hamiltonian assumes the desired diagonal form:

\[
H_{\text{c}} = \sum_k e^k(\delta) (A_k^k A_k - 1/2), \quad (52)
\]

with eigenvalues given by:

\[
e^k(\delta) = \sqrt{1 + (\lambda + \delta)^2 - 2(\lambda + \delta) \cos(k)}, \quad (53)
\]

V. ISING MODEL AS AN ENVIRONMENT FOR A SYSTEM OF TWO QUBITS

Now we determine the exact expression for the quantum map \(\Phi\), in the case of two qubits interacting with an environment described by an arbitrary quadratic fermionic Hamiltonian \(H_S\) (Eq. 10). The motivation is to investigate how the number of particles in the system affects the environment.

We assume two qubits described by the Hamiltonian

\[
H_S = -J_S (\sigma_1^x \sigma_2^x + \lambda_S (\sigma_1^y + \sigma_2^y)) , \quad (54)
\]

where \(\sigma^2 = |g\rangle\langle g| - |e\rangle\langle e|\), with \(|g\rangle\) and \(|e\rangle\) being the qubit ground and excited states. For the interaction with the environment, we consider the following Hamiltonian:

\[
H_{\text{int}} = - \left[\delta_1 |gg\rangle\langle gg| + \delta_2 (|ge\rangle\langle ge| + |eg\rangle\langle eg|) \right] \otimes V, \quad (55)
\]

where \(V\) is a fermionic quadratic Hamiltonian. We assume that the two qubits and the environment are initially uncorrelated, and they are in an arbitrary pure initial state,

\[
|\Psi(0)\rangle = |\chi(0)\rangle \otimes |\varphi(0)\rangle,
\]

where \(|\chi(0)\rangle = c_{gg} |gg\rangle + c_{ge} |ge\rangle + c_{eg} |eg\rangle + c_{ee} |ee\rangle\) (\(c_{gg}^2 + c_{ge}^2 + c_{eg}^2 + c_{ee}^2 = 1\)) is the initial two-qubit state. Therefore, the state of the composite system, at an arbitrary time \(t\), can be written as:

\[
|\Psi(t)\rangle = e^{-i(H_{\text{total}} + H_S)\hbar/t} |\chi(0)\rangle \otimes |\varphi(0)\rangle
= e^{-iJ_{\text{total}}\hbar/t} (c_{ge} |ge\rangle + c_{eg} |eg\rangle) |\varphi_2(t)\rangle +
+ c_{gg} e^{iJ_{\text{total}}(1+2\lambda)\hbar/t} |gg\rangle |\varphi_1(t)\rangle +
+ c_{ee} e^{iJ_{\text{total}}(1-2\lambda)\hbar/t} |ee\rangle |\varphi_0(t)\rangle,
\]

where \(|\varphi_\alpha(t)\rangle = e^{-iH_{\alpha}\hbar/t} |\varphi(0)\rangle\), with Hamiltonian \(H_{\alpha} = H_0 - \delta_\alpha V_c, \alpha = (0, 1, 2)\), and \(\delta_0 = 0\). With this notation, we have \(H_0 \equiv H_g\).

The Hamiltonian \(H_{\alpha}(a = (0, 1, 2))\) can be diagonalized by a Bogoliubov transformation,

\[
\eta^\pm_k = \cos\left(\frac{\theta^k_\alpha}{2}\right) a_{\pm k} \pm i \sin\left(\frac{\theta^k_\alpha}{2}\right) a_{\mp k}^+, \quad (57)
\]

These fermionic operators are related by:

\[
\eta^\pm_k = \cos(\theta^k_{a,b}) \eta_{\mp k}^\pm + i \sin(\theta^k_{a,b}) \bar{\eta}_{\mp k}^\pm, \quad (58)
\]

where \(\theta^k_{a,b} = (\theta^k_a - \theta^k_b)/2\). The Hamiltonian in diagonal form reads:

\[
H_\alpha = \sum_k e^k(\delta) (\eta^k_{a,b} \eta_{a,b}^k + C_\alpha), \quad (59)
\]
where \( C_0, C_1 \) and \( C_2 \) are real constants, \( e^{\Phi}_{0,a} e^{i\Phi}_{a} \) and \( e^{\Phi}_{a} \) are the single-particle eigenvalues. The ground states of \( H_0 \) \((G_0)\), \( H_1 \) \((G_1)\) and \( H_2 \) \((G_2)\) are related according to:

\[
|G_a\rangle = \prod_{k>0} \left[ \cos \left( a^k_{a,b} \right) + i \sin \left( a^k_{a,b} \right) \right] |G_b\rangle.
\]

Using the definition of Kraus operators in Eq. (23), and the Eq. [46], we can write:

\[
K_i = \langle i| \phi_2 \rangle e^{i|f/t} \left[ \langle ge| \langle ge| + |eg\rangle \langle eg| \right]
\]

\[
\langle i| \phi_1 \rangle e^{i(1+2h)/t} |gg\rangle \langle gg| + \langle i| \phi_0 \rangle e^{i(1-2h)/t} |ee\rangle \langle ee|,
\]

where \( \{ |i\rangle \} \) is an environment basis. Finally we obtain the quantum map:

\[
\Phi (t, 0) = \sum_i K_i \otimes K_i^t
\]

\[
= |gg\rangle \langle gg| \otimes |gg\rangle \langle gg| + |ee\rangle \langle ee| \otimes |ee\rangle \langle ee| + |ge\rangle \langle ge| \otimes |ge\rangle \langle ge| + |eg\rangle \langle eg| \otimes |eg\rangle \langle eg| + |ge\rangle \langle ge| \otimes |eg\rangle \langle eg| + |eg\rangle \langle eg| \otimes |ge\rangle \langle ge| + (|ee\rangle \langle ee| \otimes |ge\rangle \langle ge| + |ge\rangle \langle ge| \otimes |ee\rangle \langle ee|) \times
\]

\[
\times x_{0,2}(t)^* e^{i\Phi_{0,t} + t} + (|eg\rangle \langle eg| \otimes |ee\rangle \langle ee| + |ge\rangle \langle ge| \otimes |ee\rangle \langle ee|) \times
\]

\[
x_{0,2}(t) e^{-i\Phi_{0,t} + t} + (|gg\rangle \langle gg| \otimes |eg\rangle \langle eg| + |gg\rangle \langle gg| \otimes |ge\rangle \langle ge|)
\]

\[
x_{1,2}(t)^* e^{i\Phi_{1,t} + t} + (|eg\rangle \langle eg| \otimes |gg\rangle \langle gg| + |ge\rangle \langle ge| \otimes |gg\rangle \langle gg|)
\]

\[
x_{1,2}(t) e^{-i\Phi_{1,t} + t} + |ee\rangle \langle ee| \otimes |gg\rangle \langle gg| x_{0,1}(t)^* e^{-i\Phi_{0,t} + t} + |gg\rangle \langle gg| \otimes |ee\rangle \langle ee| x_{0,1}(t) e^{i\Phi_{0,t} + t},
\]

(62)

with, \( \Phi_0 = 2J_S(1 - \lambda_S)/\hbar \) and \( \Phi_0 = 4J_S\lambda_S/\hbar \), and \( x_{a,b}(t) = \langle \varphi_b \varphi_a \rangle \). Choosing the environment in its initial ground state, \( \langle \varphi(0) \rangle = |G_0\rangle \), and using equations (58-62), we have:

\[
x_{a,b}(t) = \langle \varphi_b \varphi_a \rangle = \prod_{k>0} \left[ \cos \left( a^k_{a,b} \right) \cos \left( a^k_{a,b} \right) \right] + \left[ \cos \left( a^k_{a,b} \right) \sin \left( a^k_{a,b} \right) e^{i(e^a_{a,k} + e^b_{a,k})/t}/h - \cos \left( a^k_{a,b} \right) \sin \left( a^k_{a,b} \right) e^{-i(e^a_{a,k} + e^b_{a,k})/t}/h \right] \times
\]

\[
\sin \left( a^k_{a,b} \right) + \sin \left( a^k_{a,b} \right) \sin \left( a^k_{a,b} \right) \cos \left( a^k_{a,b} \right) \times
\]

\[
e^{-i[(e^a_{a,k} + e^b_{a,k}) - (e^a_{a,k} + e^b_{a,k})]t}/h \right] e^{-i(e^a_{a,k} - e^b_{a,k})/t}/h,
\]

(63)

where \( E_a \) is the ground state energy of \( H_a \). Finally, we obtain the dynamical matrix of the intermediate map, namely:

\[
D_{\Phi(t,t_m)} = \left[ |gg\rangle \langle gg| \otimes |gg\rangle \langle gg| + |ee\rangle \langle ee| \otimes |ee\rangle \langle ee| + |ge\rangle \langle ge| \otimes |ge\rangle \langle ge| + |eg\rangle \langle eg| \otimes |eg\rangle \langle eg| + |ge\rangle \langle eg| \otimes |eg\rangle \langle ge| + |eg\rangle \langle ge| \otimes |eg\rangle \langle ge| + \langle ee\rangle \langle ee| \otimes |gg\rangle \langle gg| \langle gg\rangle \langle gg| + \langle gg\rangle \langle gg| \otimes |ee\rangle \langle ee| \langle ee\rangle \langle ee| +
\]

\[
y_{a,b}(t,t_m) = x_{a,b}(t_f)/x_{a,b}(t_m),
\]

(64)

where

\[
\]

Unlike the case of one qubit, where we presented a very simple expression for the minimum eigenvalue of the dynamical matrix (Eq. 43), directly related to the well known Loschmidt echo, in the case of two qubits the minimum eigenvalue is a non-trivial function of the parameters \( y_{a,b}(t_f, t_m) \). However, working numerically we learn that the two-qubit case does not present any new characteristic that would result in a different behavior of the non-Markovianity in relation to the one-qubit case.

VI. WITNESSING THE NON-MARKOVIANITY IN THE ISING MODEL: FINITE SIZE EFFECTS

Now we are equipped to characterize the dynamics of a qubit interacting with an environment governed by the Ising model (Fig. 1). We consider lattices up to \( L = 5 \times 10^5 \) sites, and investigate the non-Markovianity in the vicinity of the critical point of the quantum Ising model, which is well known to be equal to \( \lambda^* \equiv \lambda + \delta = 1 \).
Let us define a measure ($\eta$) of non-Markovianity as the minimum of the eigenvalues for the intermediate quantum dynamical matrix $D_{\Phi(\{t_f, t_m\})}$ over all final times $t_f$ and over all time partitions $t_m$, precisely:

$$\eta = \min_{\{t_f\}} \min_{\{t_m < t_f\}} \text{eig}\{D_{\Phi(\{t_f, t_m\})}\},$$  \hspace{1cm} (66)

where $\text{eig}$ is the set of eigenvalues of the intermediate dynamical matrix $D_{\Phi(\{t_f, t_m\})}$. In order to exemplify such a non-Markovianity measure, we plot, in Fig. 2, the smallest eigenvalue of the intermediate map as a function of the final ($t_f$) and intermediate ($t_m$) times, at the critical point of the Ising model, for a lattice with $L = 10$ sites. As the values of $t_m$ and $t_f$ are swept, the non-Markovian regions of the dynamics are revealed.

Notice that the previously defined non-Markovianity measure is only based on the minimum eigenvalue of the dynamical matrix. One might expect, however, that the number of negative eigenvalues could influence the strength of the non-Markovianity. For our models under analysis, however, it seems not to play any relevant effect: i) in the case of a single qubit it becomes trivial, since one can only have a single negative eigenvalue for the dynamical matrix; ii) and in the case of two-qubits we found that indeed there are cases where the dynamical matrix presents more than one negative eigenvalue, but its absolute value is always at least two orders of magnitude smaller than the absolute value of the minimum eigenvalue, and thus could be neglected.

In Fig. 3, the non-Markovianity, quantified by $\eta$ (Eq. (66)), is plotted against the transverse field ($\lambda$), in the vicinity of the Ising model critical point. We see that the larger the lattice, the larger the non-Markovianity. The most interesting feature shown in this figure is the maximum of non-Markovianity occurring precisely at the Ising model critical point. The behavior of this measure for larger lattice sizes, and in the thermodynamic limit, for the particular model studied in this section could also be inferred by the Loschmidt echo [8, 21], from Eqs. (38) and (42). Note, however, that this equivalence between $\eta$ and the Loschmidt echo is not necessarily true in general.

In Fig. 4, we see the behavior of the Loschmidt echo, for different lattice sizes, at the critical point ($\lambda^* = 1$). We highlight some of its features: (i) the Loschmidt echo has an abrupt decay followed by a revival, with a time
period “τ”, which is proportional to the lattice size, τ ∝ L; (ii) the difference between the minimum value of the decay (which we shall denote by $L_{\text{dec}}$) and the maximal of the revival ($L_{\text{rev}}$) becomes higher as we increase the lattice size. In this way, the non-Markovianity measure is simply given by $\eta = L_{\text{rev}} / L_{\text{dec}}$.

Performing a finite-size scaling analysis, we see, in Fig. 5 that such a measure grows exponentially with lattice size, $\eta(\Lambda^*) \propto \exp(\alpha_\ast L)$, with $\alpha_\ast \sim 2.36 \times 10^{-3}$. Notice however that, despite such exponentially increasing behavior, at the thermodynamic limit the period $\tau$ diverges, and there is no revival of the function, consequently, the non-Markovianity pointed by this measure must be null: $\eta(\Lambda^*) = 0$ for $L \to \infty$. It should be clear by now, that the non-Markovianity we have observed so far is due to the finite size of the lattice and the periodical dynamical revivals thereof. The behavior of the Loschmidt echo outside of the critical point is plotted in Fig. 6. We highlight some of its features: (i) due to finite size effects, we see that after a certain time ($\Gamma$), which increases with the lattice size ($\Gamma \propto L$), the function has a chaotic behavior; (ii) the “shape” of the function before the chaotic behavior is invariant with the lattice size, only its amplitude is changed.

![Figure 5: Finite size scaling analysis: ln(−η) as a function of L, for L = 100 to L = 10^5 sites, at the critical point $\Lambda^* = 1$, with $\delta = 10^{-2}$. The linear fit reveals an exponential divergence of the non-Markovianity with the lattice size.](image)

![Figure 6: The Loschmidt echo L (Eq. (37)) as a function of the time, outside of the critical point; more precisely, for $\lambda = \lambda^* - 0.1$, and $\delta = 10^{-2}$. The behavior for $\lambda = \lambda^* + 0.1$ is completely similar to this one.](image)

Performing then a finite-size scaling analysis, we see, in Fig. 7 that the non-Markovianity measure grows exponentially with lattice size, $\eta(\Lambda^* - 0.1) \propto \exp(\beta_1 L)$, with $\beta_1 \sim 1.43 \times 10^{-5}$, and $\eta(\Lambda^* + 0.1) \propto \exp(\beta_2 L)$, with $\beta_2 \sim 1.29 \times 10^{-5}$. Notice that although the measure also has an exponential scaling, as in the critical point, its exponential factors are much smaller, namely, $\beta_1(\tau) / \alpha_\ast \sim 10^{-2}$.

In summary, we see that the non-Markovianity measure, for finite size systems, reaches its maximal at the critical point, whereas in the thermodynamic limit it is zero exactly at the critical point, and it diverges outside of the critical point.

![Figure 7: Finite size scaling analysis: ln(−η) as a function of L, for L = 100 to L = 5 \times 10^5 sites, outside of the critical point, more precisely, for $\lambda = \lambda^* \pm 0.1$, and $\delta = 10^{-2}$. The linear fit reveals an exponential divergence of the non-Markovianity with the lattice size, $(−\eta) \propto e^{\beta L}$.](image)
Hamiltonian, the measure \((\eta)\) (Eq.66) and the witness \((\mathcal{N})\) (Eq.69) for the non-Markovian dynamics for the two qubits have exactly the same behavior of the non-Markovian dynamics for one qubit. Here we will just highlight that the results do not depend on the parameters \(J_S\) and \(\lambda_S\), and the choice of a Hamiltonian \(H_S\) for the open system (two spins) just adds a relative phase in its initial state, \(|\chi(0)\rangle\), do not affecting \((\eta)\) nor \((\mathcal{N})\).

VII. ENTANGLEMENT AS A WITNESS OF NON-MARKOVIANITY IN THE ISING MODEL: BEYOND FINITE SIZE EFFECTS

In the previous section, we characterized the non-Markovianity by means of the non-positivity of the dynamical matrix expressed as a simple function of the Loschmidt echo. Now we will further explore the dynamics using a witness of non-Markovianity. Different non-Markovianity witnesses based on entanglement, or on bipartite correlations, have recently appeared in the literature [22–24]. We based our witness on the entanglement between the central qubit coupled to an ancilla. Our main concern shall be to detect the non-Markovianity that is not due to the finite lattice size. To see how this works, we assume a system \(S\), with dynamics described by a map \(\Phi\), and a static ancillary system \(A\). The system-ancilla evolution is given by,

\[
\rho_{SA}(t_f) = \Phi(t_f, t_0) \otimes I_A \left[ \rho_{SA}(t_0) \right].
\]

(67)

Note that we have trivially extended the map to a separable one, with no local action over the ancilla.

Entanglement cannot be generated by a local CP map. Assuming that the map \((\Phi(t_f, t_0))\) is divisible, in the sense discussed in section II, i.e., the intermediate map \((\Phi(t_f, t_m))\) is CP, \(t_f > t_m > t_0\), we have:

\[
E \left[ \rho_{SA}(t_f) \right] = E \left[ (\Phi(t_f, t_m) \Phi(t_m, t_0) \otimes I_A \left[ \rho_{SA}(t_0) \right] \right]
\]

\[
= E \left[ (\Phi(t_f, t_m) \otimes I_A \left[ \rho_{SA}(t_m) \right] \right]
\]

\[
\leq E \left[ \rho_{SA}(t_m) \right],
\]

(68)

where \(E[\rho_{SA}(t)]\) is some quantifier of bipartite entanglement. The above equation expresses the fact that entanglement is monotonically decreasing under local CP maps.

In order to simplify notation, from now on we shall write \(E[\rho_{SA}(t)] = E_{SA}(t)\). From Eq.(68) we have that a local CP divisible map leads to a monotonic decrease \(\left( \frac{d}{dt} E_{SA}(t) \leq 0 \right)\) of an entanglement measure of the system and ancilla. Therefore any violation of this monotonicity \(\left( \frac{d}{dt} E_{SA}(t) > 0 \right)\) is a sufficient criterion to witness non-Markovianity. Based on this idea, we can consider a witness \((\mathcal{N})\) of non-Markovianity in the form [22]:

\[
\mathcal{N} = \int_{(d/dt)E_{SA} > 0} dtdE_{SA}(t),
\]

(69)

such that \(\mathcal{N} > 0\) for non-Markovian dynamics.

Now consider system and ancilla as two qubits in an initial maximally entangled state, \(|\phi^+\rangle = (|gg\rangle + |ee\rangle) / \sqrt{2}\). The system is under the action of the map given by Eq.(57), and the ancilla is let alone. We resume the study of our problem (Fig. 1) under this new perspective.

![Figure 8: The entanglement measure \(E_{SA}\), quantified by the negativity as a function of time, at the critical point \(\lambda^* = \lambda + \delta = 1\), and \(\delta = 10^{-2}\), for different lattice sizes.](image-url)

In Fig.4, we saw that at the critical point \(\Lambda^* = 1\), the revival of the Loschmidt echo, i.e. the revival of the coherence (recoherence), occurs in a time \(\tau\) proportional to the lattice size. This non-Markovianity, due to the finite size of the lattice, allows for the open system to regain coherence and information from the environment. It is shown in Fig. 8, where the entanglement measure \((E_{SA})\) is the negativity, for different lattice sizes at the critical point. The period of time in which the negativity increases is proportional to the lattice size, as expected. However, looking at outside of the critical point, in a time before the detection of non-Markovianity due to the size effect, we can witness non-Markovianity related to the characteristic features of the environment. This fact was observed before by means of the distinguishability of two quantum states [8]. In Fig. 9, we plot the negativity, for different lattice sizes, outside of the critical point, with fixed interaction coupling constant \(\delta = 0.01\), in a time interval excluding the finite size effect. We see that even for different lattice sizes the negativity presents the same behavior, i.e. the period of time in which \(E_{SA}\) monotonically increases is the same. The degree of non-Markovianity, quantified by \(N\) (Eq.(69)), becomes higher as we increase the lattice size, \(N = \sum \left( E_{SA}(\tau_{\text{max}}) - E_{SA}(\tau_{\text{min}}) \right)\), where \(E_{SA}(\tau_{\text{max}})\) and \(E_{SA}(\tau_{\text{min}})\) are the set of local maximum and minimum values of \(E_{SA}(t)\). At this point one can note that the behavior of the negativity is similar to the
Loschmidt echo, more precisely, in this specific case we have the interesting result:

\[ E_{SA} = \sqrt{L} \]  \hspace{1cm} (70)

The above equation follows from the definition of negativity, \( E_{SA} = \sum_i (|p_i| - p_i) \), where the \( p_i \) are the four eigenvalues \( \frac{1}{2}(-\langle x(t) \rangle_1, \langle x(t) \rangle_1, 1, 1) \) of \( \rho_{SA}^x(t) \), which is the partial trace of \( \rho_{SA}(t) = \Phi(t, 0) \otimes 1_A (|\phi^+ \rangle \langle \phi^+|) = (\langle g | g \otimes | g \rangle \langle g | + \langle e | e \otimes | e \rangle \langle e | + \langle g | e \otimes | g \rangle \langle e | x(t)^* + \langle e | g \otimes | e \rangle \langle g | x(t) \rangle )/2 \). In Fig. 10, we see the witness of non-Markovianity, against the effective transverse field \( \lambda_{ef} = \lambda + \delta \), for two different lattice sizes, in an interval that avoids finite size effects. Increasing the field from small values, the witness decreases, until it gets close to the critical point, where it starts to increase, and suddenly drops to zero, exactly at the critical point (\( N(\lambda^*) = 0 \)). This is a very nice result to conclude this section, for the dynamics is known to be Markovian at the critical point.

![Figure 9: The negativity \( E_{SA} \) as a function of time, outside of the critical point, for \( \lambda = \lambda^* - 0.1 \), and \( \delta = 10^{-2} \), for different lattice sizes.](image)

![Figure 10: The witness of non-Markovianity \( N \) as a function of the effective field \( \lambda_{ef} = \lambda + \delta \), for \( \delta = 0.01 \), and in a time window excluding finite size effects.](image)

**VIII. CONCLUSION**

We derived the analytical expression for the Kraus representation of the map corresponding to the evolution of one and two qubits interacting with an environment represented by a general quadratic fermionic Hamiltonian. We concluded that the non-Markovian dynamics of two qubits interacting with the Ising environment does not present any new feature in relation to the dynamics of one qubit. We introduced simple functions to check the non-Markovianity of the dynamics. For the particular case of the Ising environment, we investigated the dynamics of one qubit interacting with lattices up to \( 10^5 \) sites. We quantified the non-Markovianity by the most negative eigenvalue \( \eta \) (Eq. 66) of the dynamical matrix, and obtained that, for finite size systems, it reaches its maximum at the critical point, whereas in the thermodynamic limit it is zero exactly at the critical point, diverging outside of the critical point. We also quantified the non-Markovianity using an entanglement based approach \( N \) (Eq. 69). We showed, in the case of one qubit interacting with Ising model, that the non-Markovianity measures we introduced are simple functions of the Loschmidt echo. Finally, we clearly identified two kinds of non-Markovianity, one due to the finite size of the environment, and another intrinsic of the Ising Hamiltonian, and we were able to quantify both.

**Acknowledgments**

We acknowledge financial support by the Brazilian agencies INCT-IQ (National Institute of Science and Technology for Quantum Information), FAPEMIG, and...
Author contribution statement

Fernando Iemini and Leonardo da Silva Souza are the main authors and equally contributed with development of algorithms, numerical calculations, analytical calculations, data analysis and the art of figures. T. Debarba, A.T. Cesário, T. Maciel and R.O. Vianna helped with the calculations, data analysis, interpretation of the results.

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