How to construct wavelets on local fields of positive characteristic.

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Abstract: We present an algorithm for construction step wavelets on local fields of positive characteristic.

Key words: Local field, scaling function, wavelets, multiresolution analysis.

Introduction

In 2004 H.Jiang, D.Li, and N.Jin [10] introduced the notion of multiresolution analysis (MRA) on local fields $F^{(s)}$ of positive characteristic $p$, proved some properties and constructed "Haar MRA" and corresponding "Haar wavelets". The wavelet theory developed in [11][2][3][4][11]. Construction of non-Haar wavelets is the a basic problem in this theory. The problem of constructing orthogonal MRA on the field $F^{(1)}$ is studied in detail in the works [6][7][8][12][16][17]. S.F.Lukomskii, A.M.Vodolazov [15][18] considered local field $F^{(s)}$ as a vector space over the finite field $GF(p^s)$ and constructed non-Haar wavelets. In [15] the authors construct the mask $m^{(0)}$ and correspondent refinable function $\varphi$ using some tree with zero as a root. In this case wavelets $\Psi = (\psi^{(l)})_{l \in GF(p^s)}$ may be found from the equality

$$\hat{\psi}^{(l)}(\chi) = m^{(l)}(\chi)\hat{\varphi}(\chi A^{-1})$$

where $A$ is a dilation operator, $m^{(l)}(\chi) = m^{(0)}(\chi r_0^{-1})$, and $r_k$ are Rademacher functions. In the article [13], the concept of $N$-valid tree was introduced and an algorithm for constructing the mask $m^{(0)}$ and correspondent refinable function $\varphi$ was indicated in the field $F^{(1)}$. In the articles [14], [5] the mask $m^{(0)}$ and correspondent refinable function $\varphi$ were constructed using graph which is obtained from $N$-valid tree by adding new arcs. But in this case we cannot define "masks" $m^{(l)}(\chi)$ by the equation $m^{(l)}(\chi) = m^{(0)}(\chi r_0^{-1})$.

In this article we give an algorithm for construction of "masks" $m^{(l)}(\chi)$ in general case.

1 Basic concepts

Let $p$ be a prime number, $s \in \mathbb{N}$, $GF(p^s)$ – finite field. Local field $F^{(s)}$ of positive characteristic $p$ is isomorphic (Kovalski-Pontryagin theorem [9]) to the...
set of formal power series

\[ a = \sum_{i=k}^{\infty} a_i t^i, \quad k \in \mathbb{Z}, \ a_i \in GF(p^s). \]

Addition and multiplication in the field \( F^{(s)} \) are defined as sum and product of such series. Therefore we will consider local field \( F^{(s)} \) of positive characteristic \( p \) as the field of sequences infinite in both directions

\[ a = (\ldots, 0_{n-1}, a_n, a_{n+1}, \ldots), \ a_j \in GF(p^s) \]

which have only finite number of elements \( a_j \) with negative \( j \) nonequal to zero, and the operations of addition and multiplication are defined by equalities

\[ a \dot{+} b = ((a_i \dot{+} b_i))_{i \in \mathbb{Z}}, \]
\[ ab = (\sum_{i,j: i+j=l} (a_i b_j))_{l \in \mathbb{Z}}, \]

(1)

where "\( \dot{+} \)" and "\( \cdot \)" are respectively addition and multiplication in \( GF(p^s) \).

The norm of the element \( a \in F^{(s)} \) is defined by the equality

\[ ||a|| = ||(\ldots, 0_{n-1}, a_n, a_{n+1}, \ldots)|| = \left( \frac{1}{p^s} \right)^n, \text{ если } a_n \neq 0. \]

Therefore

\[ F^{(s)}_n = \{ a = (a_j)_{j \in \mathbb{Z}} : a_j \in GF(p^s); \ a_j = 0, \ \forall j < n \} \]

is a ball of radius \( p^{-ns} \).

Neighborhoods \( F^{(s)}_n \) are compact subgroups of the group \( F^{(s)+} \). We will denote them as \( F^{(s)+}_n \). They have the following properties:

1) \( \ldots \subset F^{(s)+}_1 \subset F^{(s)+}_0 \subset F^{(s)+}_{-1} \subset \ldots \)

2) \( F^{(s)+}_n / F^{(s)+}_{n+1} \cong GF(p^s)\) и \( \#(F^{(s)+}_n / F^{(s)+}_{n+1}) = p^s. \)

It is noted in [15] that the field \( F^{(s)} \) can be described as a linear space over \( GF(p^s) \). Using this description one may define the multiplication of element \( a \in F^{(s)} \) on element \( \lambda \in GF(p^s) \) coordinatewise, i.e. \( \lambda a = (\ldots 0_{n-1}, \lambda a_n, \lambda a_{n+1}, \ldots) \), and the modulus \( \lambda \in GF(p^s) \) can be defined as

\[ |\lambda| = \begin{cases} 1, & \lambda \neq 0, \\ 0, & \lambda = 0. \end{cases} \]

It is also proved there, that the system \( g_k \in F^{(s)}_k \setminus F^{(s)}_{k+1} \) is a basis in \( F^{(s)} \), i.e. any element \( a \in F^{(s)} \) can be represented as:
\[ a = \sum_{k \in \mathbb{Z}} \lambda_k g_k, \ \lambda_k \in GF(p^s). \]

From now on we will consider \( g_k = (..., 0_{k-1}, (1^{(0)}), 0^{(1)}, ..., 0^{(s-1)}), 0_{k+1}, ...). \) In this case \( \lambda_k = a_k. \) Let us define the sets
\[
H_0^{(s)} = \{ h \in G : h = a_{-1}g_{-1}^1 + a_{-2}g_{-2}^1 + \ldots + a_{-s}g_{-s} \}, \ s \in \mathbb{N}.
\]
\[
H_0 = \{ h \in G : h = a_{-1}g_{-1}^1 + a_{-2}g_{-2}^1 + \ldots + a_{-s}g_{-s}, \ s \in \mathbb{N} \}.
\]

The set \( H_0 \) is the set of shifts in \( F^{(s)} \). It is an analogue of the set of nonnegative integers.

We will denote the collection of all characters of \( F^{(s)+} \) as \( X \). The set \( X \) generates a commutative group with respect to the multiplication of characters:
\[
(\chi \ast \phi)(a) = \chi(a) \cdot \phi(a).
\]
Inverse element is defined as \( \chi^{-1}(a) = \overline{\chi(a)} \), and the neutral element is \( e(a) \equiv 1 \).

Following [15] we define characters \( r_n \) of the group \( F^{(s)+} \) in the following way. Let \( x = (..., 0_{k-1}, x_k, x_{k+1}, ...), \ x_j = (x_j^{(0)}, x_j^{(1)}, \ldots, x_j^{(s-1)}) \in GF(p^s) \). The element \( x_j \) can be written in the form \( x_j = (x_j^{s+0}, x_j^{s+1}, \ldots, x_j^{s+(s-1)}) \).

In this case
\[
x = (..., 0, x_{ks+0}, x_{ks+1}, \ldots, x_{ks+s-1}, x_{(k+1)s+0}, x_{(k+1)s+1}, \ldots, x_{(k+1)s+s-1}, \ldots)
\]
and the collection of all such sequences \( x \) is Vilenkin group. Thus the equality
\[
r_n(x) = r_{ks+l}(x) = e^ {\frac{2\pi i}{p}(x_{ks+l})}
\]
defines Rademacher function of \( F^{(s)+} \) and every character \( \chi \in X \) can be described in the following way:
\[
\chi = \prod_{n \in \mathbb{Z}} r_n^{a_n}, \ a_n = 0,p - 1.
\]
(2)
The equality (2) can be rewritten as
\[
\chi = \prod_{k \in \mathbb{Z}} r_k^{a_k^{(0)}} r_k^{a_k^{(1)}} \ldots r_k^{a_k^{(s-1)}}
\]
(3)
and let us define
\[
r_k^{a_k^{(0)}} r_k^{a_k^{(1)}} \ldots r_k^{a_k^{(s-1)}} = r_k^{a_k}
\]
where \( a_k = (a_k^{(0)}, a_k^{(1)}, \ldots, a_k^{(s-1)}) \in GF(p^s) \). Then (3) takes the form
\[
\chi = \prod_{k \in \mathbb{Z}} r_k^{a_k}.
\]
(4)
We will refer to \( r_k^{(1,0,...,0)} = r_k \) as the Rademacher functions. By definition we set
\[(r_k^{a_k})^{b_k} = r_k^{a_k b_k}, \quad \chi^b = (\prod r_k^{a_k})^b = \prod r_k^{a_k b_k}, \quad a_k, b_k, b \in GF(p^s).\]

It follows that if \(x = ((x_k^{(0)}, x_k^{(1)}, \ldots, x_k^{(s-1)}))_{k \in \mathbb{Z}}\) and \(u = (u^{(0)}, u^{(1)}, \ldots, u^{(s-1)}) \in GF(p^s)\) then

\[(r_k^u, x) = \prod_{l=0}^{s-1} e^{\frac{2\pi i}{p} u^{(l)} x_k^{(l)}}.\]

In [15] the following properties of characters are proved

1) \(r_k^{u+v} = r_k^u r_k^v, \quad u, v \in GF(p^s).\)

2) \((r_k^u, u g_j) = 1, \quad \forall k \neq j, \quad u, v \in GF(p^s).\)

3) The set of characters of the field \(GF(p^s)\) is a linear space \((X, *, \cdot, GF(p^s))\) over the finite field \(GF(p^s)\) with multiplication being an inner operation and the power \(u \in GF(p^s)\) being an outer operation.

4) The set of Rademacher functions \((r_k)\) is a basis in the space \((X, *, \cdot, GF(p^s))\). The dilation operator \(A\) in local field \(GF(p^s)\) is defined as \(A x := \sum_{n=0}^{+\infty} a_n g_{n-1},\) where \(x = \sum_{n=0}^{+\infty} a_n g_n \in F^s\). In the group of characters it is defined as \((\chi A, x) = (\chi, A x)\).

## 2 Step Wavelets

We will consider a case of scaling function \(\varphi\), which generates an orthogonal MRA, being a step function. The set of step functions constant on cosets of a subgroup \(F^{(s)}_M\) with the support \(\text{supp}(\varphi) \subset F^{(s)}_N\) will be denoted as \(D_M(F^{(s)}_N)\), \(M, N \in \mathbb{N}\). Similarly, \(D_{-N}(F^{(s)}_M)\) is a set of step functions, constant on the cosets of a subgroup \(F^{(s)}_M\) with the support \(\text{supp}(\varphi) \subset F^{(s)}_M\).

Let \(\varphi \in D_M(F^{(s)}_{-N})\) generate an orthogonal MRA \(\{V_n\}\), satisfies the refinement equation \(\varphi(x) = \sum_{h \in H_0^{(N+1)}} \beta_h \varphi(A x - h)\) [15], which we rewrite in a frequency from

\[\hat{\varphi}(\chi) = m^{(0)}(\chi) \hat{\varphi}(\chi A^{-1}),\]  \hspace{1cm} (5)

where

\[m^{(0)}(\chi) = \frac{1}{p^s} \sum_{h \in H_0^{(N+1)}} \beta_h \overline{\chi (\chi A^{-1}, h)}\]

is the mask of equation [15]. There exist methods for constructing \(m^{(0)}(\chi)\) and \(\hat{\varphi}(\chi)\) (see e.g.[5]). We want to construct wavelets \(\psi(l), \quad l \in GF(p^s), l \neq 0\) from
refinable function \( \varphi \). We will find these wavelets \( \psi^{(l)} \) from the equations

\[
\hat{\psi}^{(l)}(\chi) = m^{(l)}(\chi)\hat{\varphi}(\chi A^{-1}),
\]

and will call the functions \( m^{(l)}(\chi) \) masks, too. It is evident that \( \hat{\psi}^{(0)}(\chi) = \hat{\varphi}(\chi) \).

**Theorem 2.1** Let \( m^{(k)}(\chi) (k \in GF(p^s)) \) be a masks that are constant on the cosets of a subgroup \( F_{-N}^{(s)} \perp \) and periodic with any period \( r_1^{a_1}r_2^{a_2} \ldots r_p^{a_p}, a_j \in GF(p^s), \nu \in \mathbb{N} \). Define wavelets \( \psi^{(l)} \) by the equations

\[
\hat{\psi}^{(l)}(\chi) = m^{(l)}(\chi)\hat{\varphi}(\chi A^{-1}),
\]

where \( \varphi \in \mathcal{D}_M(F_{-N}^{(s)}) \) is a refinable function. The shifts system \( (\psi^{(l)}(x - h^{(l)})) \), \( l \in GF(p^s), h^{(l)} \in H_0 \) will be orthonormal iff for any \( a_{-N} \ldots a_{-1} \in GF(p^s) \)

\[
\sum_{a_0 \in GF(p^s)} m^{(k)}(F_{-N}^{(s)}^{-1}r_{-N}^{a_{-N}} \ldots r_{0}^{a_0})m^{(l)}(F_{-N}^{(s)}^{-1}r_{-N}^{a_{-N}} \ldots r_{0}^{a_0}) = \delta_{k,l}. \tag{6}
\]

**Proof.** The sufficiency. Let \( \hat{\varphi}(\chi) \in \mathcal{D}_{-N}(F_{-N}^{(s)}) \) Consider scalar product \( (\varphi(x - g), \psi^{(l)}(x - h)) \), where \( g, h \in H_0 \).

\[
(\varphi(x - g), \psi^{(l)}(x - h)) = \int_{F^{(s)}} \varphi(x - g)\overline{\psi^{(l)}(x - h)}d \mu(x) = \int_{X} \hat{\varphi}(\chi)\overline{\hat{\psi}^{(l)}(\chi)}(\chi, g)m^{(l)}(\chi)d \nu(\chi) = \\
= \int_{F_{M}^{(s)}\perp} |\hat{\varphi}(\chi A^{-1})|^2(\chi, h - g)m^{(0)}(\chi)m^{(l)}(\chi)d \nu(\chi) = \\
= |h - g = \bar{h} = h_{-1}g_{-1} + h_{-2}g_{-2} + \ldots| = \\
= \sum_{a_{-N}, \ldots, a_{-0}, \ldots, a_{M-1}} \int_{F_{-N}^{(s)}\perp r_{-N}^{a_{-N}} \ldots r_{0}^{a_0} \ldots r_{M-1}^{a_{M-1}}} |\hat{\varphi}(F_{-N}^{(s)}^{-1}r_{-N}^{a_{-N}} \ldots r_{M-1}^{a_{M-1}} A^{-1})|^2(\chi, \bar{h}) d \nu(\chi) = \\
m^{(0)}(F_{-N}^{(s)}^{-1}r_{-N}^{a_{-N}} \ldots r_{0}^{a_0})m^{(l)}(F_{-N}^{(s)}^{-1}r_{-N}^{a_{-N}} \ldots r_{0}^{a_0})d \nu(\chi) = \\
= \sum_{a_{-N}, \ldots, a_{0}} m^{(0)}(F_{-N}^{(s)}^{-1}r_{-N}^{a_{-N}} \ldots r_{0}^{a_0})m^{(l)}(F_{-N}^{(s)}^{-1}r_{-N}^{a_{-N}} \ldots r_{0}^{a_0})
\[
\sum_{a_1, a_2, \ldots, a_{M-1}} |\hat{\varphi}(F_N^{-s}) \cdot r_{a-N}^{a-N+1} \cdots r_{0}^{a_1} \cdots r_{M-2}^{a_{M-1}}|^2 \int_{F_N^{-s} \cdot r_{a-N}^{a-N} \cdots r_0^{a_0} \cdots r_{M-1}^{a_{M-1}}} (\chi, \tilde{h}) d\nu(\chi).
\]

By the orthonormality criteria for the system of shifts \( (\varphi(x, h)) \) of the refinable function \( \varphi \in GF(p^s) \) the following equality holds:

\[
\sum_{a_1, a_2, \ldots, a_{M-1}} |\hat{\varphi}(F_N^{-s}) \cdot r_{a-N}^{a-N+1} \cdots r_{0}^{a_1} \cdots r_{M-1}^{a_{M-1}}|^2 = 1.
\]

Consider integral from (7)

\[
\int_{F_N^{-s} \cdot r_{a-N}^{a-N} \cdots r_0^{a_0} \cdots r_{M-1}^{a_{M-1}}} (\chi, \tilde{h}) d\nu(\chi) = \frac{1}{p^{sN}} l_{F_N^{-s} \cdot r_{a-N}^{a-N} \cdots r_0^{a_0} \cdots r_{M-1}^{a_{M-1}}}^{-1} e^{\frac{2\pi i}{p} ((h_j, a_j))},
\]

where \( (h_j, a_j) = h_j^{(0)} a_j^{(0)} + \ldots + h_j^{(s-1)} a_j^{(s-1)} \) is a scalar product.

Let us introduce the following notation:

\[
m_{(a-N), a_0}^{(0)} = m_{(a-N), a_0}^{(1)} (F_N^{-s} \cdot r_{a-N}^{a-N} \cdots r_0^{a_0} \cdots r_{M-1}^{a_{M-1}}), \quad m_{(a-N), a_0}^{(l)} = m_{(a-N), a_0}^{(l)} (F_N^{-s} \cdot r_{a-N}^{a-N} \cdots r_0^{a_0} \cdots r_{M-1}^{a_{M-1}}).
\]

Then we obtain

\[
(\varphi(\cdot, g), \psi^{(l)}(\cdot, h)) = \frac{1}{p^{sN}} l_{F_N^{-s} \cdot r_{a-N}^{a-N} \cdots r_0^{a_0} \cdots r_{M-1}^{a_{M-1}}}^{-1} e^{\frac{2\pi i}{p} ((h_j, a_j))} =
\]

\[
= \left\{
\begin{array}{ll}
0 & \text{if } \tilde{h} \notin F_N^{-s} \\
\frac{1}{p^{sN}} \sum_{a-N, a_0} m_{a-N, a_0}^{(0)} m_{a-N, a_0}^{(l)} \prod_{j=-N}^{-1} e^{\frac{2\pi i}{p} ((h_j, a_j))} & \text{if } \tilde{h} = 0;
\end{array}
\right.
\]

\[
= \left\{
\begin{array}{ll}
\frac{1}{p^{sN}} \sum_{a-N, a_0} \prod_{j=-N}^{-1} e^{\frac{2\pi i}{p} ((h_j, a_j))} m_{a-N, a_0}^{(0)} m_{a-N, a_0}^{(l)} & \text{if } \tilde{h} \neq 0, \tilde{h} \in F_N^{-s} \\
0 & \text{if } \tilde{h} \notin F_N^{-s}.
\end{array}
\right.
\]

For \( (\psi^{(k)}(x, g), \psi^{(l)}(x, h)) \) we can derive similar equality:

\[
(\psi^{(k)}(\cdot, g), \psi^{(l)}(\cdot, h)) = \frac{1}{p^{sN}} l_{F_N^{-s} \cdot r_{a-N}^{a-N} \cdots r_0^{a_0} \cdots r_{M-1}^{a_{M-1}}}^{-1} e^{\frac{2\pi i}{p} ((h_j, a_j))}
\]

\[
= \left\{
\begin{array}{ll}
0 & \text{if } \tilde{h} \notin F_N^{-s} \\
\frac{1}{p^{sN}} \sum_{a-N, a_0} m_{a-N, a_0}^{(k)} m_{a-N, a_0}^{(l)} \prod_{j=-N}^{-1} e^{\frac{2\pi i}{p} ((h_j, a_j))} & \text{if } \tilde{h} = 0;
\end{array}
\right.
\]

\[
= \left\{
\begin{array}{ll}
\frac{1}{p^{sN}} \sum_{a-N, a_0} \prod_{j=-N}^{-1} e^{\frac{2\pi i}{p} ((h_j, a_j))} m_{a-N, a_0}^{(k)} m_{a-N, a_0}^{(l)} & \text{if } \tilde{h} \neq 0, \tilde{h} \in F_N^{-s} \\
0 & \text{if } \tilde{h} \notin F_N^{-s}.
\end{array}
\right.
\]
Let us fix $\hat{h}$, consider the matrix $A$ where \( \det A \neq 0 \). Then the system of shifts \((\psi^{(1)}(x-h^{(1)})), l \in GF(p^s)\) is an orthonormal system. The necessity. Let us fix $k, l \in FG(p^s)$ and consider equalities \((8), (9)\) as a system of linear equation with unknowns $x_{a_{-N} \ldots a_{-1}}^{k l} = m_{a_{-N} \ldots a_{-1}}^{k (1)}$, consider the matrix $A$ of this system.

It is obvious that $A$ is a square matrix $p^{sN} \times p^{sN}$. Let us prove that its determinant is nonequal to zero.

Let us start with $N = 1, s = 1$. In this case

\[
A = \frac{1}{p} \begin{pmatrix}
1 & 1 & 1 & \ldots & 1 \\
1 & e^{2\pi i/p} & e^{2\pi i/2} & \ldots & e^{2\pi i(p-1)} \\
1 & e^{2\pi i/2} & e^{2\pi i/2} & \ldots & e^{2\pi i(p-1)/2} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & e^{2\pi i/(p-1)} & e^{2\pi i/(p-1)/2} & \ldots & e^{2\pi i/(p-1)(p-1)} \\
\end{pmatrix} = V,
\]

where $V$ is Vandermonde matrix, which is known to have nonzero determinant.

For the sake of clarity let us consider a case $N = 2, s = 1$. In this case the matrix $A$ may be represented as block matrix

\[
A = \frac{1}{p} \begin{pmatrix}
V & V & V & \ldots & V \\
V & e^{2\pi i/p}V & e^{2\pi i/2}V & \ldots & e^{2\pi i/(p-1)}V \\
V & e^{2\pi i/2}V & e^{2\pi i/2}V & \ldots & e^{2\pi i/(p-1)/2}V \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
V & e^{2\pi i/(p-1)}V & e^{2\pi i/(p-1)/2}V & \ldots & e^{2\pi i/(p-1)(p-1)}V \\
\end{pmatrix} = V \otimes V,
\]

where $\otimes$ symbol corresponds to Kronecker product. By the properties of Kronecker product $\det V \otimes V = (\det V)^p (\det V)^p = (\det V)^{2p} \neq 0$. Thus, again matrix $A$ is nonsingular.
For the case of arbitrary $N$, $s = 1$ matrix $A$ can be represented as $A = V \otimes V \otimes \ldots \otimes V$ $N$ times and will again have nonzero determinant by the properties of Kronecker product.

Similarly, when $N$ and $s$ are both arbitrary $A = V \otimes V \otimes \ldots \otimes V$ $sN$ times. Thus, the system is nonsingular and has a unique solution, which proves the necessity. \[ \square \]

Theorem \[2.1\] can be reformulated in the following way: $m^{(k)}(\chi)$ are the masks of corresponding step compactly supported orthonormal wavelets $\psi^{(l)}(\chi)$ if and only if for each $a_{-N} \ldots a_{-1} \in GF(p^s)$ matrix $M(a_{-N} \ldots a_{-1})$ with elements

$$M_{1, a_0}(a_{-N}, \ldots, a_{-1}) = m^{(l)}(F_{-N}^{(s)} r_{-N}^{a_{-N}} \ldots r_0^{a_0})$$

is unitary. The sufficiency of this theorem was proved in \[10\] (theorem 3). For step refinable functions the condition \[1\] is necessary and sufficient. If the condition \[3\] is fulfilled then the functions $\hat{\psi}^{(l)}(\chi) = m^{(l)}(\chi) \hat{\varphi}(\chi A^{-1})$ form a wavelet system \[10\]. For a step refinable function we can describe an algorithm for constructing masks $m^{(l)}$ and wavelets $\psi^{(l)}$, $l \in GF(p^s)$.

Let us assume we have all the values of $m^{(0)}(\chi)$. We may obtain them using an algorithm presented in \[5\]. Recall the notation:

$$m^{(0)}_{a_{-N} \ldots a_0} = m^{(0)}(F_{-N}^{(s)} r_{-N}^{a_{-N}} \ldots r_0^{a_0}), \quad m^{(1)}_{a_{-N} \ldots a_0} = m^{(1)}(F_{-N}^{(s)} r_{-N}^{a_{-N}} \ldots r_0^{a_0}).$$

1) For each $a_{-N} \ldots a_{-1}$ we construct a matrix $M(a_{-N} \ldots a_{-1}) \in Mat_{p^x \times p^x}(\mathbb{C})$ with elements $M_{1, a_0}(a_{-N} \ldots a_{-1})$ the following way. The first row consists of all the values

$$m^{(0)}_{a_{-N} \ldots a_{-1}, 0}, m^{(0)}_{a_{-N} \ldots a_{-1}, 1}, \ldots, m^{(0)}_{a_{-N} \ldots a_{-1}, p^s - 1}$$

where $a_{-N} \ldots a_{-1}$ are fixed and $j = a_0^{(0)} + a_0^{(1)} p + \ldots + a_0^{(s-1)} p^{s-1}$ calculated from $a_0 = (a_0^{(0)}, a_0^{(1)}, \ldots, a_0^{(s-1)})$. Supplement this matrix to unitary in the following way.

If $m^{(0)}_{a_{-N} \ldots a_{-1}, 0} \neq 0$ then we make $M_{1, 1} = 1$ for $l \neq 0$ and $M_{1, a_0} = 0$ for $l \neq 0, l \neq a_0$.

If $m^{(0)}_{a_{-N} \ldots a_{-1}, 0} = 0$ then there exists number

$$j = j(a_0) = a_0^{(0)} + a_0^{(1)} p + \ldots + a_0^{(s-1)} p^{s-1}$$

for which $m^{(0)}_{a_{-N} \ldots a_{-1}, j} \neq 0$. This nonzero value exists by the property of $m^{(0)}$ (see e.g. \[10\]) In this case we make $M_{j, 0} = 1$, $M_{i, 1} = 1$ for $l \neq 0, l \neq j$, and $M_{1, a_0} = 0$ in another case.

2) Run the Gram-Schmidt process on each matrix in order to make them unitary.
3) Now for each \( l \in GF(p^s), \ l \neq 0 \) we find the values of the mask \( m^{(l)} \) from the equalities

\[
m^{(l)}(F^{-s}_{-N} r_{-N}^{a_{-N}} \ldots r_0^{a_0}) = M_{l, a_0}(a_{-N} \ldots a_{-1}).
\]

4) The wavelets \( \psi^{(l)} \) can be obtained using the formula

\[
\hat{\psi}^{(l)}(\chi) = m^{(l)}(\chi)\hat{\phi}(\chi A^{-1})
\]

and performing inverse Fourier transform.

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