Abstract: We use the time-dependent mean-field Gross-Pitaevskii equation to study the formation of a dynamically-stabilized dissipation-managed bright soliton in a quasi-one-dimensional Bose-Einstein condensate (BEC). Because of three-body recombination of bosonic atoms to molecules, atoms are lost (dissipated) from a BEC. Such dissipation leads to the decay of a BEC soliton. We demonstrate by a perturbation procedure that an alimentation of atoms from an external source to the BEC may compensate for the dissipation loss and lead to a dynamically-stabilized soliton. The result of the analytical perturbation method is in excellent agreement with mean-field numerics. It seems possible to obtain such a dynamically-stabilized BEC soliton without dissipation in laboratory.

Dissipation-managed soliton in a quasi-one-dimensional Bose-Einstein condensate

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Solitons are solutions of wave equation where localization is obtained due to a nonlinear interaction. Solitons have been noted in optics [1], high-energy physics and water waves [1], and also in Bose-Einstein condensates (BEC) [2–6]. The bright solitons of BEC represent local maxima [5–9], whereas dark solitons represent local minima [2–4]. In addition to the observation of an isolated bright soliton in an expulsive potential [6], a number of bright solitons forming a soliton train was observed by Strecker et al. [5], where they suddenly turned a repulsive BEC of 7Li atoms attractive by manipulating an external magnetic field near a Feshbach resonance [10]. Consequently, the BEC collapsed, exploded and generated a soliton train which was studied in detail. Also, a bright vector soliton in a repulsive BEC supported by interspecies attraction has been studied [11,12].

A soliton or solitary wave by definition propagates over large time intervals without visible modification of shape which makes it of special interest. However, a soliton of BEC, or a BEC in general, suffer loss of atoms due to three-body recombination leading to formation of molecules [13–15]. This means that a BEC soliton will decay and eventually disappear as it propagates. It would be of interest if an artificial situation could be created in laboratory with a supply of atoms so as to compensate for the three-body recombination loss of a BEC soliton to generate a dynamically-stabilized soliton. To the best of our knowledge we demonstrate for the first time, using the mean-field Gross-Pitaevskii (GP) equation [16], that such a dynamically stabilized soliton could indeed be prepared in a radially trapped and axially free BEC. As a strict soliton appears only in one dimension, we shall be concerned in this paper with a quasi-one-dimensional BEC soliton in a cigar-shaped trap in an axially symmetric configuration.

To demonstrate the presence of a dynamically-stabilized dissipation-managed BEC soliton, we employ both time-dependent and time-independent analytic perturbation techniques and a complete numerical solution of the GP equation. The numerical result is found to be in ex-
cellent agreement with that obtained from the perturbation techniques.

Bright solitons are really eigenfunctions of the one-dimensional nonlinear Schrödinger (NLS) equation. However, the experimental realization of bright solitons in trapped attractive cigar-shaped BECs has been possible under strong transverse binding which, in the case of weak or no axial binding, simulates the ideal one-dimensional situation for the formation of bright solitons. The dimensionless NLS equation in the attractive or self-focusing case [1]

\[ \frac{\partial \phi}{\partial t} + \frac{1}{2} \frac{\partial^2 \phi}{\partial y^2} + |\phi|^2 \phi = 0 \]  

sustains the following bright soliton [1]:

\[ \phi(y, t) = a \sech(\sigma(y - vt)) \times \exp[i(\nu y - \frac{1}{2}(v^2 - a^2)t) + i\sigma], \]

with three parameters.

The parameter \( a \) represents the amplitude as well as pulse width, \( v \) represents velocity, the parameter \( \sigma \) is a phase constant. The bright soliton profile is easily recognized for \( v = 0 \) as \( \phi(y, t) = a \sech(\sigma y) \). There have been experimental [5,6] and theoretical [7–9] studies of the formation of bright solitons in a BEC. In view of this, here we study for the first time the possibility of a dynamically stabilized dissipation-managed bright BEC soliton in a quasi-one-dimensional configuration.

In recent times there have been routine experimental studies on the formation of BEC in the presence of a periodic axial optical-lattice potential [17–19] formed by a polarized standing-wave laser beam. This leads to a different condition of trapping from the harmonic trap and generates a BEC of distinct modulation. Hence we also consider in this paper the modulations of a dissipation-managed bright BEC soliton in the presence of an optical-lattice potential.

The time-dependent Bose-Einstein condensate wave function \( \Psi(\mathbf{r}, \tau) \) at position \( \mathbf{r} \) and time \( \tau \) may be described by the following mean-field nonlinear GP equation [16,20]

\[ \left[ -i\hbar \frac{\partial}{\partial \tau} - \frac{\hbar^2 \nabla^2}{2m} + V(\mathbf{r}) - gn + \frac{\Gamma}{2} - \right. \]

\[ \left. - \frac{\hbar^2}{2} \kappa n^2 \right] \Psi(\mathbf{r}, \tau) = 0, \]

with normalization \( \int d\mathbf{r} |\Psi(\mathbf{r}, \tau)|^2 = N \). Here \( m \) is the mass and \( N \) the number of bosonic atoms in the condensate, \( n = |\Psi(\mathbf{r}, \tau)|^2 \) is the boson probability density, \( g = 4\pi \hbar^2 a/m \) the strength of inter-atomic attraction, with \(-a \) the atomic scattering length. The trap potential with axial symmetry may be written as \( V(\mathbf{r}) = \frac{1}{2} \mu \omega_\perp^2 (\rho^2 + \nu^2 z^2) \), where \( \omega_\perp \) and \( \nu \) are the angular frequencies in the radial (\( \rho \)) and axial (\( z \)) directions with \( \nu \) the anisotropy parameter. The term \( K_3 \) denotes the three-body recombination loss-rate coefficient and \( I \) denotes the constant almentation of atoms from an external source.

For the study of bright solitons we shall reduce Eq. (3) to a minimal one-dimensional form under the action of stronger radial trapping. The one-dimensional form is appropriate for studying bright solitons in the so-called cigar-shaped quasi-one-dimensional geometry where \( \nu \ll 1 \). For radially-bound and axially-free solitons we eventually set \( \nu = 0 \).

For \( \nu = 0 \), Eq. (3) can be reduced to an effective one-dimensional form by considering solutions of the type \( \Psi(\mathbf{r}, \tau) = \Phi(z, \tau)\psi(0)(\rho) \), where

\[ |\psi(0)(\rho)|^2 = \frac{m \omega}{\pi \hbar} \exp \left( -\frac{m \omega \rho^2}{\hbar} \right). \]

The expression (4) corresponds to the ground state wave function in the radial variable in the absence of nonlinear interactions and satisfies

\[ -\frac{\hbar^2}{2m} \nabla^2 \psi(0) + \frac{1}{2} m \omega^2 \rho^2 \psi(0) = \hbar \omega \psi(0), \]

with normalization

\[ 2\pi \int_0^\infty |\psi(0)(\rho)|^2 \rho d\rho = 1. \]

Now the dynamics is carried by \( \Phi(z, \tau) \) and the radial dependence is frozen in the ground state \( \psi(0)(\rho) \).

Averaging over the radial mode \( \psi(0)(\rho) \), i.e., multiplying Eq. (3) by \( \psi(0)^*(\rho) \) and integrating over \( \rho \), we obtain the following one-dimensional dynamical equation [21]:

\[ \left[ -i\hbar \frac{\partial}{\partial \tau} - \frac{\hbar^2 \partial^2}{2m} - \frac{m \omega}{2\pi \hbar} \phi^2 + \right. \]

\[ \left. + \frac{\Gamma}{2} - \right] \Phi(z, \tau) = 0, \]

In Eq. (6) the normalization is given by

\[ \int_{-\infty}^{\infty} |\Phi(z, \tau)|^2 dz = N \]

and we have set the anisotropy parameter \( \nu = 0 \) to remove the axial trap and thus to generate an axially-free quasi-one-dimensional soliton.

For calculational purpose it is convenient to reduce the set (6) to dimensionless form by introducing convenient dimensionless variables. For this purpose we consider the dimensionless variables \( t = \tau \omega, y = z/l, \phi = \sqrt{(2a)} \Psi, \gamma = \Gamma/\langle \hbar \omega \rangle, \) and \( \xi = K_3/(24\pi^2 a^2 l^4 \omega) \) with \( l = \sqrt{\hbar/\langle \omega m \rangle} \), so that

\[ \left[ -i\frac{\partial}{\partial \tau} + \frac{1}{2} \frac{\partial^2}{\partial y^2} + |\phi|^2 \right] \phi(y, t) = i\epsilon(\phi)\phi(y, t), \]

where

\[ \epsilon(\phi) = \frac{\gamma}{2} - \xi |\phi|^4. \]
and the normalization is given by
\[ \int_{-\infty}^{\infty} |\phi(y, t)|^2 \, dy = \frac{2aN}{t}. \]

In addition to studying an axially-free dissipation-managed soliton, we also consider a dissipation-managed soliton in a periodic optical-lattice potential \( V(y) = V_0 \sin^2(2\pi y/\lambda) \) in the axial direction [17–19], so that the NLS equation of interest becomes
\[ i \frac{\partial \phi}{\partial t} + \frac{1}{2} \frac{\partial^2 \phi}{\partial y^2} + |\phi|^2 \phi - V(y) \phi = i\epsilon \phi \phi(t), \quad (9) \]
where \( V_0 \) is the strength and \( \lambda \) is the wavelength of the laser used to generate the optical-lattice potential. Eq. (9) will generate a dissipation-managed soliton with an axial periodic modulation.

In Eq. (7) \( \epsilon(\phi) \) is a small perturbation. In the absence of a perturbation \( \epsilon = 0 \) the soliton profile is known. The fate of the soliton when \( \epsilon \neq 0 \) can be obtained by a time-dependent perturbation technique [1] which assumes that the functional form of the soliton remains unchanged in the presence of a small perturbation. However, the soliton parameters change with propagation. The most general form of the perturbed soliton is taken as [1]
\[ \phi(y, t) = a(t) \text{sech}\{a(t)(y - q(t))\} \exp(i\sigma(t) - iy\delta(t)). \quad (10) \]

In the absence of a perturbation, \( a \) and \( \delta \) are constants but \( q(t) \) and \( \sigma(t) \) are determined by the following ordinary differential equations [1]:
\[ \frac{da}{dt} = -\delta, \quad \frac{d\sigma}{dt} = \frac{1}{2}(a^2 - \delta^2). \quad (11) \]

The use of the perturbation technique leads to the following ordinary differential equations for the soliton parameters \( a \) and \( \delta \) [1]:
\[ \frac{da}{dt} = \text{Re} \int_{-\infty}^{\infty} \epsilon(\phi) \phi^*(y) \, dy, \quad (12) \]
\[ \frac{d\delta}{dt} = -\text{Im} \int_{-\infty}^{\infty} \epsilon(\phi) \text{tanh}[a(y - q)] \phi^*(y) \, dy. \quad (13) \]

In this work we shall be concerned only with the profile of the soliton with zero velocity: \( q(t) = 0 \). This can be obtained by solving Eq. (12). With the \( \epsilon \) of Eq. (8) and the \( \phi \) of Eq. (10), Eq. (12) becomes
\[ \frac{da}{dt} = \int_{-\infty}^{\infty} \frac{\gamma}{2} |\phi|^2 \, dy - \int_{-\infty}^{\infty} \xi|\phi|^6 \, dy = \gamma a - \frac{16}{15} \xi a^5. \quad (14) \]

We are interested in finding the constant amplitude \( a \) of a dynamically-stabilized dissipation-managed soliton for large \( t \) from the solution of Eq. (14). It is assumed that the perturbation is switched on at \( t = 0 \). The initial soliton at \( t = 0 \) is taken to satisfy \( a(t = 0) = a_0 \). With this condition Eq. (14) can be integrated to yield
\[ \ln \left[ a \left( \frac{16}{15} \xi a^4 - \gamma \right) \right]^{-1/4} a = \gamma t, \quad (16) \]
the solution of which is
\[ a^4 = \frac{15\gamma}{16\xi} e^{4\gamma t} - (16\xi a_0^2 - 15\gamma), \quad (17) \]
which for finite \( \xi \) and \( \gamma \) leads for large time \( t \) to the amplitude
\[ a = \left( \frac{15\gamma}{16\xi} \right)^{1/4}. \quad (18) \]

The most interesting feature of this result is that the lowest-order perturbation-theory prediction for the amplitude of the dissipation-managed soliton is independent of the initial choice for the amplitude \( a_0 \) at \( t = 0 \). The dissipation-managed soliton is robust and depends only on the ratio \( \gamma/\xi \) of the dissipation parameters \( \gamma \) and \( \xi \) and is independent of the initial choice \( a_0 \) in the lowest-order perturbation theory.

The above result that a robust time-independent soliton of amplitude (18) can be formed at large times in the presence of a weak dissipation suggests that the same could be derivable from a time-independent perturbation analysis. For a “stationary state” \( \phi(y, t) = \phi(y) \exp(i\mu t) \), from Eq. (7) one could write the following time-independent equation
\[ -\mu + \frac{1}{2} \frac{\partial^2 \phi}{\partial y^2} + |\phi|^2 \phi(y) = i\epsilon(\phi)\phi(y), \quad (19) \]
where \( \mu \) is a chemical potential. Employing the usual time-independent perturbation theory, for Eq. (19) to have the soliton (10) with \( q(t) = 0 \) as a solution, one should have
\[ \langle \phi|\epsilon(\phi)|\phi \rangle \equiv \int_{-\infty}^{\infty} \epsilon(\phi)|\phi(y)|^2 \, dy = 0. \quad (20) \]

With \( \epsilon(\phi) \) given by Eq. (8) and \( \phi(y) = a \text{sech}(ay) \), the solution of Eq. (20) is given by Eq. (18). This demonstrates the equivalence between the results of time-dependent and time-independent perturbation theories. However, it should be recalled that perturbative result (18) is valid only for small dissipation \( \epsilon(\phi) \) or for small values of \( \gamma \) and \( \xi \).

We solve the NLS equation (7) for dynamically stabilized dissipation-managed bright solitons numerically using a time-iteration method based on the Crank-Nicholson discretization scheme elaborated in Ref. [22,23]. We discretize the mean-field-hydrodynamic equation using time step 0.01 and space step 0.1.
We performed the time evolution of Eq. (7) starting with the solution \( \phi(y) = 0.5 \text{sech}(y/2) \exp(it/8) \) with \( a_0 = 1/2 \) for \( \epsilon(\phi) = 0 \) at \( t = 0 \). The dissipation \( \epsilon(\phi) \) is introduced for \( t > 0 \) and we look for the dissipation-managed soliton for large time \( t \). If a converged solution is obtained, it corresponds to the dissipation-managed soliton. We also repeated the calculation with a different initial choice \( \phi(y) = \text{sech}(y) \exp(it/2) \) with \( a_0 = 1 \) to verify if the dissipation-managed soliton is independent of the initial choice as predicted by the perturbation treatment above.

In the first numerical simulation we take \( \xi = 0.01 \), \( \gamma = 0.0002 \), and \( a_0 = 1/2 \). The \( t = 0 \) solution is taken as \( \phi(y) = 0.5 \text{sech}(y/2) \). Upon a time evolution of the NLS equation (7) the dissipation-managed soliton emerges at large times. In Fig. 1 we plot the numerically-calculated dissipation-managed soliton at times \( t = 2000, 4000, 6000 \), and 8000 as well as the result of the lowest-order perturbation theory. At large times the numerically-calculated dissipation-managed soliton exhibits excellent convergence properties and agrees very well with the result of the lowest-order perturbation theory. We also repeated this calculation with a different \( a_0 (= 1) \). In agreement with the result of perturbation theory, the dissipation-managed soliton for \( a_0 = 1 \) is indistinguishable from that for \( a_0 = 1/2 \). Although the dissipation-managed soliton appears like a stable solution of the NLS equation (7), it cannot be called a stationary solution of a time-independent Schrödinger-type equation with real eigenvalue. It is rather a dynamically-stabilized solution of the dissipative NLS equation (7) where there is a delicate cancellation between the two imaginary dissipative terms \( \xi \) and \( \gamma \) to maintain a stable profile of the soliton.

To study how the dissipation-managed soliton changes with a change of the dissipation parameters \( \xi \) and \( \gamma \) we repeat the calculation of Fig. 1 first with smaller values of these parameters: \( \xi = 0.002 \) and \( \gamma = 0.00025 \). A very similar scenario emerges for the dynamically-stabilized soliton which we exhibit in Fig. 2. However, in this case the dynamically-stabilized soliton has a larger amplitude \( a(= 0.58) \) governed by Eq. (18) than in Fig. 1, where \( a(= 0.37) \). In these numerical studies we verified that for small dissipation the dissipation-managed soliton is independent of \( a_0 \) and is solely determined by the ratio \( \gamma/\xi \). However, for large dissipation parameters this is not quite so and quite expectedly the numerical solution of the soliton does not agree with the result of lowest-order perturbation theory. Also, with the increase of dissipation parameters it becomes increasingly difficult numerically to find a stabilized soliton. With further increase in dissipation parameters we could not obtain a stabilized soliton. Numerically, no stabilized soliton could be found for \( \xi = 0.1 \).

Finally, we consider a dissipation-managed bright soliton formed on a periodic optical-lattice potential controlled by Eq. (9). In our simulation we take \( V_0 = 5 \) and \( \lambda = 2\pi \). To solve Eq. (9) with this optical-lattice potential we take the initial soliton at \( t = 0 \) to be the one calculated in Fig. 1 above for \( V(y) = 0 \) and consider the time evolution of Eq. (9). During this time evolution the optical-lattice potential is slowly introduced in an interval of time \( t \) of about 2000 units, so that a stable dissipation-managed soliton of Eq. (9) is obtained at large time. We study the stability of this dissipation-managed soliton by continuing the time evolution. Again a dynamically-stabilized dissipation-managed soliton is obtained which is independent of the input guess for \( a_0 \).

Now let us see to what values of the three-body recombination rate \( K_3 \) the parameters \( \xi = 0.01 \) and 0.002 considered in this paper correspond to for typical experimental values of \( a = 10 \) nm, \( l = 1 \) \( \mu \)m and \( \omega = 2\pi \times 50 \) Hz. Re-calculating that \( K_3 = 24\pi^2\xi a^2 l^2 \omega \) we find that for \( \xi = 0.01 \), \( K_3 = 7.5 \times 10^{-26} \) cm/s and for \( \xi = 0.002 \), \( K_3 = 1.5 \times 10^{-26} \) cm/s. Typical estimates of \( K_3 \) for different atoms [13–15] are comparable to or less than these values. Hence the present demonstration that dissipation-managed soliton can be obtained for typical values of \( \xi \) less than...
0.01 is compatible with experimental values of $K_3$. This leads to the conclusion that dissipation-managed solitons can be generated in laboratory for realistic values of the parameters.

To summarize, we have demonstrated the possibility of the generation of a dynamically-stabilized dissipation-managed robust soliton in a quasi-one-dimensional BEC under a typical experimental situation. This can be used in laboratory to generate robust BEC solitons where the loss of atoms due to three-body recombination could be compensated by alimentation of atoms from an external source.

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