On the facet pivot simplex method for linear programming I: algorithms and numerical test

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January 5, 2022

Abstract

The Hirsch Conjecture stated that any d-dimensional polytope with n facets has a diameter at most equal to $n - d$. This conjecture was disproven by Santos (A counterexample to the Hirsch Conjecture, Annals of Mathematics, 172(1) 383-412, 2012). The implication of Santos’ work is that all vertex pivot algorithms cannot solve the linear programming problem in the worst case in $n - d$ vertex pivot iterations.

In this paper, the first part in this series of papers, we propose a facet pivot method and perform some numerical tests to demonstrate its superiority to the existing vertex pivot method. In the second part of this series, we show that the proposed facet pivot method can solve the canonical linear programming problem in the worst case in at most $n - d$ facet pivot iterations. This series of the papers was inspired by Smale’s Problem 9 (Mathematical problems for the next century, In Arnold, V. I.; Atiyah, M.; Lax, P.; Mazur, B. Mathematics: frontiers and perspectives, American Mathematical Society, 271-294, 1999).

Keywords: Facet pivot algorithm, simplex method, linear programming, Klee-Minty cube, cycling problem.

MSC classification: 90C05 90C49.

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1 Introduction

Since Dantzig invented the simplex method in the 1940s [5], the linear programming (LP) problem has been investigated extensively. As the simplex method is very efficient, many researches focused on this strategy until Klee and Minty [15] discovered that the number of iterations used to find an optimal solution in the worst case for Dantzig’s vertex simplex method increases exponentially as a function of the problem size. Searching for efficient polynomial algorithms motivated the interior point method [14], which became very popular starting in the 1980s [28]. Even so, there is still sporadic research on novel simplex methods, for example, the recently proposed double pivot simplex method [25, 29] which updates two vertices at a time.

Almost all simplex algorithms are vertex simplex algorithms, including dual simplex algorithms. The vertex simplex algorithms utilize vertex geometry and the base is formed from a subset of columns (a vertex) of the constraint matrix $A$ or a subset of the rows of $A^T$; every iterate is a basic primal (or dual) feasible solution; and the iterate moves from a vertex to an adjacent vertex. Very recently, Liu et. al. [16] published their brilliant idea, which is based on the facet (of the convex polytope) geometry.

It is worthwhile to point out that even though the facet pivot simplex method deals with a similar (but more general) format of linear programming problem that the dual simplex method considers, the two methods are significantly different in basic ideas and in detailed steps. The dual simplex method checks the feasibility of the primal solution to determine if an optimal solution is found. If it is not, it determines leaving vector in the base and then finds the entering vector so that the new basic solution is feasible for the dual problem. The facet pivot method considers only one general problem with equality, inequality and boundary constraints (not primal and dual problems). Note that the optimal solution of LP can always be found in a vertex and a vertex is always formed by a set of $d$ independent facets ($d$ is the dimension of the polytope). The facet pivot algorithm starts from an initial base which is composed of $d$ facets (the constraints), it then finds an entering facet and a leaving facets to form a new base. Therefore, every iterate improves feasibility by replacing one facet (constraint) with a different facet (constraint), and all iterates are basic (to be defined later) but not feasible until an optimal solution is found.

1.1 facet pivot vs. vertex pivot

Conventional vertex (primal or dual) pivot algorithms maintain the feasibility which means the iterate path is moving from one vertex to the next vertex along the edges of the polytope. The famous Hirsch conjecture [6] states that the shortest path length (less than the diameter of the polytope) is bounded by $n - d$ (where $n$ is the number of facets and $d$ is the dimension of the polytope) which is disproven by Santos [22]. The best upper bounds for the polytope diameter [12, 24, 23] are sub-exponential, which means that vertex pivot algorithms may be very expensive in the worst case. Since facet pivot method does not move along the edges of the polytope, instead, the facet pivot iterate jumps among basic solutions, the sub-exponential bounds do not apply to
facet pivot method. In the second part of this series of papers, we will show that the facet pivot method will be much more efficient in the worst case than the vertex pivot method. As a matter of fact, the facet pivot method will find an optimizer in the worst case in at most \( n - d \) pivot iterations.

Another obvious advantage of using the general formulation considered by the facet pivot method, from a computational point of view, is that one does not need to convert inequality constraints into equality constraints and avoids introducing unnecessary and possibly many slack variables. Using this formulation, the problem size is significantly smaller than the problem size of the standard LP problem. It is also more natural to handle lower and upper bound constraints and free variables. In addition, numerical tests show that the facet pivot simplex method is more efficient than Dantzig’s most negative rule algorithm for general LP problems.

1.2 Summary of the work in this paper

Although the idea of [16] is brilliant, the paper is not well-organized; it has quite some inaccurate statements and sloppy proofs; and the algorithm is presented in a way that is only suitable for a tabular procedure. In this paper, we present a facet pivot simplex algorithm that is suitable for computer code implementation. We provide a series of technical results to show that the algorithm is well defined and it needs finite iterations for the algorithm to find the optimal solution. We implement the algorithm as a Matlab function and report test results for various benchmark problems, including a small set of known cycling LP problems [30], two variants of Klee-Minty problems [9, 13] with different sizes, and Netlib benchmark problems [4]. These tests demonstrate the efficiency and effectiveness of the facet pivot simplex algorithm.

In the remainder of this paper, we use small letters with bold font for vectors and capital letters with bold font for matrices. For a vector \( \mathbf{x} \), we denote by \( x_i \) its \( i \)-th component. Also, we use superscript \( k \) to denote the iteration count. Therefore, the scalar, vector, set, and matrix at the \( k \)-th iteration are denoted, for example, as \( c^k \), \( \mathbf{x}^k \), \( B^k \), and \( A_{B^k} \). To save space, we write the column vector \( \mathbf{x} = \begin{bmatrix} x_1^T, x_2^T \end{bmatrix}^T \) as \( \mathbf{x} = (x_1, x_2) \). The remainder of the paper is organized as follows. Section 2 describes the standard general form of the linear programming problem. Section 3 presents some mathematical results that will be used to justify the development of the facet pivot simplex algorithm. Section 4 provides the detailed steps of the facet pivot simplex algorithm and the reasoning behind every step, including a proof of the claim that the algorithm finds the optimal solution in finite steps. Section 5 discusses some important implementation details for readers who are interested in repeating the reported results given in this paper. Section 6 concludes the paper with some remarks.
2 The standard general form of the LP problem

We consider the general linear programming problem presented as follows:

\[ \begin{align*}
\min & \quad c^T x, \\
\text{subject to} & \quad A^I x = b^I, \\
& \quad A^J x \geq b^J, \\
& \quad u \geq x \geq \ell,
\end{align*} \]

where \( A^I \in \mathbb{R}^{m \times d}, A^J \in \mathbb{R}^{n \times d}, b^I \in \mathbb{R}^m, b^J \in \mathbb{R}^n, \ell \in \mathbb{R}^d, u \in \mathbb{R}^d, \) and \( c \in \mathbb{R}^d \) are given matrices and vectors respectively. Vector \( x \in \mathbb{R}^d \) is composed of variables to be optimized. We say that an \( x \) is a feasible solution of LP if \( x \) satisfies all the constraints of (1). We refer (1) as the general form of LP because the standard form of the LP problem

\[ \begin{align*}
\min & \quad c^T x, \\
\text{subject to} & \quad A^I x = b^I, \\
& \quad x \geq 0,
\end{align*} \]

and the canonical form of the LP problem

\[ \begin{align*}
\min & \quad c^T x, \\
\text{subject to} & \quad A^J x \geq b^J, \\
& \quad x \geq 0,
\end{align*} \]

are special cases of the general form of the LP. Clearly, all LP problems can be written as either a standard form which is useful for the unified theoretical analyses for the vertex simplex method, or a canonical form which is a better description from geometry point of view (convex polytope) for the LP problem (every row of the constraint forms a facet of the convex polytope). But using the general form of the LP, as we will show, normally leads to a more efficient and convenient (without some extra conversion work) algorithm for the general LP problems which are frequently met in real applications, for example, the facet enumeration problem [27]. We emphasize that neither the primal (vertex) simplex method nor the dual (vertex) simplex method can directly solve the general form of LP described in (1), but the proposed facet pivot simplex method can. Moreover, although the general form of LP can be converted to either standard LP or canonical LP, the facet pivot simplex method is more efficient than the traditional vertex simplex method.

Remark 2.1 Let \( x_i \) with \( i = 1, \ldots, d \) be the \( i \)-th variable of \( x \). For any \( x_i \), a free or a constrained variable on one side, we can always represent it as the general form. For example, if \( x_i \) is a free variable, we can add two trivial constraints as \(-M \leq x_i \leq M\), where \( M \) is a big positive constant; if \( x_i \geq 0 \), we can add a trivial constraint so that it meets \( M \geq x_i \); if \( x_i \leq 0 \), we can add a trivial constraint so that it meets \(-M \leq x_i \leq 0\). This conversion seems increasing the complexity of the problem, but it provides us a convenient initial basic solution which avoids the expensive Phase 1 computation in traditional simplex method. We will discuss this shortly.
From Remark 2.1, any LP problem can always be written as the general form of (1). We use \( c_k \) with \( k = 1, \ldots, d \) for the component of \( \mathbf{c} \); \( \mathbf{a}_i \) with \( i = 1, \ldots, m \) for the \( i \)-th row (facet) of \( \mathbf{A}^I \); \( \mathbf{a}_j \) with \( j = m + 1, \ldots, m + n \) for the \( j \)-th row (facet) of \( \mathbf{A}' \). Similar notations are used for \( b_i \) with \( i = 1, \ldots, m; b_j \) with \( j = m + 1, \ldots, m + n; \ell_k \) with \( k = m + n + 1, \ldots, m + n + d; \) and \( u_k \) with \( k = m + n + d + 1, \ldots, m + n + 2d \). Noticing that \( \mathbf{u} \geq \mathbf{x} \iff -\mathbf{x} \geq -\mathbf{u} \), we denote

\[
\begin{align*}
\sigma_i(\mathbf{x}) &= \mathbf{a}_i \mathbf{x} - b_i, \quad i = 1, \ldots, m, \\
\sigma_j(\mathbf{x}) &= \mathbf{a}_j \mathbf{x} - b_j, \quad j = m + 1, \ldots, m + n, \\
\sigma_k(\mathbf{x}) &= x_k - \ell_k, \quad k = m + n + 1, \ldots, m + n + d, \\
\sigma_k(\mathbf{x}) &= -x_k + u_k, \quad k = m + n + d + 1, \ldots, m + n + 2d,
\end{align*}
\]

which will be used to measure the constraint violations. To keep the notation simple, we often omit \( \mathbf{x} \) and simply use \( \sigma_i, \sigma_j, \sigma_k, \) and \( \sigma_k \), but remember that they are functions of \( \mathbf{x} \).

Without the loss of the generality, we make the following assumptions throughout this paper.

**Assumption**

1. \( \text{rank}(\mathbf{A}^I) < d \).

This assumption means that the optimization problem is not trivial, otherwise the feasible solution is either unique or does not exist.

Let \( \mathbf{e}_i^T \) be the \( i \)-th row of the \( d \)-dimensional identity matrix \( \mathbf{I} \). The standard general form LP is given as follows:

\[
\begin{align*}
\min & \quad \sum_{i=1}^{n} \bar{c}_i \mathbf{e}_i^T \mathbf{x}, \quad \bar{c}_i \geq 0 \quad (5a) \\
\text{subject to} & \quad \mathbf{A}^I \mathbf{x} = \mathbf{b}^I, \quad (5b) \\
& \quad \mathbf{A}^I \mathbf{x} \geq \mathbf{b}^I, \quad (5c) \\
& \quad \mathbf{E} \mathbf{x} \geq \mathbf{b}_L, \quad (5d) \\
& \quad \mathbf{F} \mathbf{x} \geq \mathbf{b}_U, \quad (5e)
\end{align*}
\]

where (5b) and (5c) are the same as (1b) and (1c); some conversions from (1) to (5) will be needed for (1a) and (1d), which are described as follows. The \( i \)-th row (or loosely speaking, the facet) of \( \mathbf{E} \) is denoted as \( \mathbf{e}_i^T \) which is either \( \mathbf{e}_i^T \) or \( -\mathbf{e}_i^T \) depending on the sign of \( c_i \): (i) if \( c_i \geq 0 \), then, \( \bar{c}_i = c_i, \mathbf{e}_i^T = \mathbf{e}_i^T \); we set \( \mathbf{b}_{L_i} \leftarrow \ell_i \) in (5d) to represent \( x_i \geq \ell_i \); the \( i \)-th row (facet) of \( \mathbf{F} \) is given as \( \mathbf{f}_i^T = -\mathbf{e}_i^T \) and set \( \mathbf{b}_{U_i} \leftarrow (-u_i) \) to represent the constraint \(-x_i \geq -u_i \) in (5e); and (ii) if \( c_i > 0 \), then, this item in the objective function can be written as \( c_i x_i = (-c_i)(-\mathbf{e}_i^T) \mathbf{x} = \bar{c}_i \mathbf{e}_i^T \mathbf{x} \), we set \( \bar{c}_i \leftarrow (-c_i) \) in the objective function (5a), and rewrite the corresponding inequalities in (1d) as \(-u_i \leq -x_i \leq -\ell_i \) or \(-u_i \leq -\mathbf{e}_i^T \mathbf{x} \leq -\ell_i \), therefore, we set \( \mathbf{b}_{L_i} \leftarrow (-u_i) \) to represent \(-\mathbf{e}_i^T \mathbf{x} \geq -u_i \) in (5d), and \( \mathbf{f}_i^T = \mathbf{e}_i^T \) and \( \mathbf{b}_{U_i} \leftarrow \ell_i \) to represent \( x_i \geq \ell_i \) in (5e). This
completes the process of converting the general form LP to the standard general form LP of (5). To keep the notation simple, we will use $c_i$ for $\bar{c}_i$ in the remainder of the paper. Let

$$A = \begin{bmatrix} A^I \\ A^J \\ E \\ F \end{bmatrix}, \quad b = \begin{bmatrix} b^I \\ b^J \\ b_L \\ b_U \end{bmatrix}, \quad \text{(6)}$$

where $E \in \mathbb{R}^{d \times d}$ and $F \in \mathbb{R}^{d \times d}$ are diagonal and full rank matrices. We denote by $A_B$, a sub-matrix of $A$, which is composed of $d$ linear independent row vectors of $A$, as a base of $A$. The rows (facets) of $A_B$ are named as basic facets. We denote by $A_N$, which is composed of the remaining $m + n + d$ row (facet) vectors of $A$, as a non-base sub-matrix of $A$. The rows (facets) of $A_N$ are named as non-basic rows (facets). Similarly, we denote by $b_B$, a $d$-dimensional sub-vector of $b$, corresponding to the rows of $A_B; \quad \text{and by } b_N \text{ a } (m + n + d)-\text{dimensional sub-vector of } b, \text{corresponding the rows of } A_N.$

We say that a vector $x$ is a basic solution of (5) if it satisfies $A_B x = b_B$. Similarly, we say that a vector $x$ is a basic feasible solution of Problem (5) if $x$ is both a basic and a feasible solution. Another good feature of the standard general form is that we may choose the rows (facets) of $E$ as the initial base, i.e., $A_{B^0} = E$. It is worthwhile to note that $A_{B^0} = E$ corresponds to the inequality constraints, and $c_i, \quad i = 1, \ldots, d$, are the coefficients such that $c = \sum_{i=1}^{d} c_i \bar{e}_i, \text{ moreover, } c_i^0 = c_i \geq 0$. In addition, the initial basic solution $x^0$ is obtained by solving $Ex^0 = b_L$.

A very expensive step in traditional vertex simplex method is the so-called Phase-I step, which is aimed at finding a feasible initial point. By rewriting the general LP to the standard general form LP, the facet pivot simplex method does not need this expensive step. The key ideas is to represent objective vector $c$ as the linear combination of the basic facets $B^k$, i.e., $c^T = \sum_{i \in B^k} c_i^k a_i$ at the $k$-th iteration, such that $c_i^k$ corresponds to the $i$-th facet of the base at the $k$-th iteration, and keep $c_i^k \geq 0$ for all $i$ and $k$ corresponding to the inequality constraint facets in the base of the $k$-th iteration. We will see that keeping $c_i^k \geq 0$ for all inequality constraints in the base for all iterations $k$ is very important for us to use Farkas Lemma to justify the facet pivot simplex algorithm.

## 3 Mathematical results useful for the facet pivot simplex method

The facet pivot simplex method is based on Farkas Lemma which is presented as follows (see, for example, reference [28, Lemma 1.1]).

**Theorem 3.1 (Farkas Lemma)** Let $c \in \mathbb{R}^d$, $x \in \mathbb{R}^d$, $y \in \mathbb{R}^n$, and $A \in \mathbb{R}^{n \times d}$. Then, exact one of the following systems holds but not both.

(i) $Ax \geq 0$ and $c^T x < 0$. 

(ii) $A^T y = c$ and $y \geq 0$.
Let \( \mathcal{I} \) be the index set of all (including lower and upper boundary) inequality constraints in (5), \( \mathcal{E} \) be the index set of all equality constraints in (5), \( \mathcal{B} \) be the index set of the base \( B \), and \( \mathcal{N} \) be the index set of the non-base \( N \). We will use \( A_I, A_E, A_B, \) and \( A_N \) to denote sub-matrices corresponding to the index sets \( \mathcal{I}, \mathcal{E}, \mathcal{B}, \) and \( \mathcal{N} \) respectively. We will use the same partitions for \( I, E \) and \( B \). Finally, we denote by \( A_{I_0}, A_{E_0}, A_{I_1}, A_{E_1} \) the sub-matrices corresponding to \( I_0, E_0, I_1, \) and \( E_1 \); and by \( b_{I_0}, b_{E_0}, b_{I_1}, \) and \( b_{E_1} \) for the same partitions of \( b \).

The following theorem (see [16, Corollary 1]), which can easily be derived from Farkas Lemma, is useful in the development of the facet pivot simplex algorithm. Denote
\[
A_B = \begin{bmatrix} A_{E_0} \\ A_{I_0} \end{bmatrix} \in \mathbb{R}^{d \times d}.
\]
Let vectors \( y = \begin{bmatrix} y_{E_0} \\ y_{I_0} \end{bmatrix} \in \mathbb{R}^d \) and \( b_B = \begin{bmatrix} b_{E_0} \\ b_{I_0} \end{bmatrix} \in \mathbb{R}^d \), where the row (facet) indices of \( A_{E_0}, y_{E_0}, \) and \( b_{E_0} \) are identical, and the row (facet) indices of \( A_{I_0}, y_{I_0}, \) and \( b_{I_0} \) are identical.

**Theorem 3.2** Let \( x \in \mathbb{R}^d \) be a basic solution of (5), i.e., \( A_B x = b_B \). Let \( \bar{x} \in \mathbb{R}^d \) be a feasible solution of (5), i.e., \( A_E \bar{x} = b_E \) and \( A_I \bar{x} \geq b_I \). Then, exact one of the following systems holds but not both.

(i) \( A_{I_0} x \geq b_{I_0}, A_{E_0} x = b_{E_0}, \) and \( a_r x < a_r \bar{x} \).

(ii) \( A_{E_0}^T y_{E_0} - A_{I_0}^T y_{I_0} = a_r^T \) and \( y_{I_0} \geq 0 \).

**Proof:** Since \( A_{E_0} x = b_{E_0} \) is equivalent to \( b_{E_0} \geq A_{E_0} x \geq b_{E_0} \), which can be written as \( A_{E_0} x \geq b_{E_0} \) and \( -A_{E_0} x \geq -b_{E_0} \). Using the fact that \( A_{E_0} \bar{x} = b_{E_0} \), we have \( A_{E_0} (x - \bar{x}) \geq 0 \) and \( -A_{E_0} (x - \bar{x}) \geq 0 \). Combining \( A_{I_0} x = b_{I_0} \) and \( A_{I_0} x \geq b_{I_0} \), we have \( -A_{I_0} (x - \bar{x}) \geq 0 \). Therefore, system (i) is equivalent to

\[
\begin{align*}
A_{E_0} (x - \bar{x}) &\geq 0, \\
-A_{E_0} (x - \bar{x}) &\geq 0, \\
-A_{I_0} (x - \bar{x}) &\geq 0, \\
a_r (x - \bar{x}) &< 0.
\end{align*}
\]

Let \( A^T = [A_{E_0}^T, -A_{E_0}^T, -A_{I_0}^T] \) and \( c^T = a_r \), then, system (i) of this theorem is equivalent to the relations of (7), which is the same as system (i) of the Farkas Lemma.
System (ii) of the Farkas Lemma \((A^T y = c \text{ and } y \geq 0)\) is equivalent to

\[
A^T y = \begin{bmatrix} A_{E_0}^T, & -A_{E_0}^T, & -A_{I_0}^T \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_{I_0} \end{bmatrix} = A_{E_0}^T (y_1 - y_2) - A_{I_0}^T y_{I_0} := A_{E_0}^T y_{E_0} - A_{I_0}^T y_{I_0} = a_r^T
\]

with \(y_1 - y_2 = y_{E_0}\) and \(y_{I_0} \geq 0\), which is system (ii) of this theorem.

**Corollary 3.1** Let \(x \in \mathbb{R}^d\) be a basic solution of (5), i.e., \(A_B x = b_B\). Let \(\bar{x} \in \mathbb{R}^d\) be a feasible solution of (5), i.e., \(A_E \bar{x} = b_E\) and \(A_I \bar{x} \geq b_I\). Then, exact one of the following systems holds but not both.

(i) \(A_{I_0} x \geq b_{I_0}, A_{E_0} x = b_{E_0}\), and \(a_r x > a_r \bar{x}\).

(ii) \(A_{E_0}^T y_{E_0} + A_{I_0}^T y_{I_0} = a_r^T\) and \(y_{I_0} \geq 0\).

**Proof:** Since \(A_{E_0} x = b_{E_0}\) is equivalent to \(b_{E_0} \geq A_{E_0} x \geq b_{E_0}\), which can be written as \(A_{E_0} x \geq b_{E_0}\) and \(-A_{E_0} x \geq -b_{E_0}\). Using the fact that \(A_{E_0} \bar{x} = b_{E_0}\), we have \(A_{E_0} (x - \bar{x}) \geq 0\) and \(-A_{E_0} (x - \bar{x}) \geq 0\), which are equivalent to \(-A_{E_0} (\bar{x} - x) \geq 0\) and \(A_{E_0} (\bar{x} - x) \geq 0\). Combining \(A_{I_0} x = b_{I_0}\) and \(A_{I_0} \bar{x} \geq b_{I_0}\), we have \(-A_{I_0} (x - \bar{x}) \geq 0\), which is equivalent to \(A_{I_0} (\bar{x} - x) \geq 0\). Therefore, system (i) of this Corollary is equivalent to

\[
\begin{align*}
A_{E_0} (\bar{x} - x) & \geq 0, \\
-A_{E_0} (\bar{x} - x) & \geq 0, \\
A_{I_0} (\bar{x} - x) & \geq 0, \\
a_r (\bar{x} - x) & < 0.
\end{align*}
\]

Let \(A^T = [A_{E_0}^T, -A_{E_0}^T, A_{I_0}^T]\) and \(c^T = a_r\), then, following the same argument as we have done in Theorem 3.2 proves the corollary.

The optimality condition of (5) can be derived either from Theorem 3.2 or from the KKT conditions given in [28, Theorem 1.6]. This condition is presented as the following theorem.

**Theorem 3.3** ([16]) Let \(x\) be a feasible solution of (5). Assume that \(A_{I_0}\) is a matrix satisfying \(A_{I_0} x = b_{I_0}\) whose index is a subset of \(\mathcal{I}\), where the components of \(b_{I_0}\) correspond to \(A_{I_0}\), i.e., the rows (facets) of \(A_{I_0}\) reach the boundaries at \(x\). If

\[
c = A_{E_0}^T y_{E_0} + A_{I_0}^T y_{I_0}, \quad y_{I_0} \geq 0, \quad (10)
\]

then, \(x\) is an optimal solution of (5).
Clearly, for the standard general form of LP (5), there is an initial point \( x^0 \) satisfying \( Ex^0 = b_L \) (i.e., \( Ex^0 = A_{B^0}x = b_{B^0} = b_L \)) and equation (10) holds (\( c = E^Tc \) and \( \bar{c} \geq 0 \)). Assuming that \( x^0 \) is not feasible, otherwise, an optimal solution is found according to Theorem 3.3. In general, at the \( k \)-th iteration, the facet pivot simplex method have an \( x^k \) and a \( y^k_c = (y_{E^k_c}, y_{I^k_c}) \) that satisfy

\[
A_{B^k}x^k = b_{B^k}, \quad c = A_{E^k_0}^T y_{E^k_c} + A_{I^k_0}^T y_{I^k_c}, \quad y_{I^k_c} \geq 0.
\]

The facet pivot simplex method will then select an entering row (facet) from \( A_{X^k} \) to replace one of the rows (facets) in \( A_{B^k} \) to form a new base \( A_{B^{k+1}} \) such that a new \( x^{k+1} \) and a new \( y^{k+1}_c = (y_{E^{k+1}_c}, y_{I^{k+1}_c}) \) satisfy

\[
A_{B^{k+1}}x^{k+1} = b_{B^{k+1}}, \quad c = A_{E^{k+1}_0}^T y_{E^{k+1}_c} + A_{I^{k+1}_0}^T y_{I^{k+1}_c}, \quad y_{I^{k+1}_c} \geq 0.
\]

This process is repeated until a feasible condition is found. We will show that this process will be ended in finite steps, i.e., the algorithm will find the optimal solution in finite iterations.

The next theorem states that if an optimal solution exists, then an optimal basic solution exists, which means that the process described above is indeed well-defined.

**Theorem 3.4** For the standard general form of LP (5), the following claims hold.

(i) If there is a feasible solution of (5), then there is a basic feasible solution of (5).

(ii) If there is an optimal solution of (5), then there is a basic optimal solution of (5).

The proof of this theorem follows exactly the same argument of [17, Page 19]. Now we are ready to describe the details of the facet pivot simplex algorithm.

4 The facet pivot simplex algorithm

This section describes major steps of the facet pivot simplex algorithm. It is largely based on the brilliant ideas of Liu et. al. [16] but has corrections, improvements, and additional materials.

4.1 Initial point

As we have explained that the initial base can be taken directly from the standard general form with \( A_{B^0}x^0 = Ex^0 = b_L \), and \( c = A_{E^0}^T y_{B^0} = E^T y_{B^0} \), and \( y_{B^0} = \bar{c} \geq 0 \). Clearly, \( rank(A_{B^0}) = rank(E) = d \) and \( x^0 \) can easily be obtained by solving \( Ex = b_L \) because \( E \) is a diagonal full rank matrix.
4.2 Criterion to check for optimal solution

The facet pivot simplex algorithm maintains two properties: (a) every iterate $x^k$ for $k \geq 0$ is a basic solution, and (b) the condition given in (10) holds. From (4), if $\sigma_i(x^k) = 0, \sigma_j(x^k) \geq 0, \tau_k(x^k) \geq 0,$ and $\tau_k(x^k) \geq 0$ hold, then, $x^k$ is feasible, according to Theorem 3.3, we find the optimal solution. If at least one of the conditions in (4) is not met, $x^k$ is infeasible, we will continue the iteration.

4.3 Remove redundant constraints

Before we find the entering and leaving rows (facets) to update the base of problem (5), we may want to remove redundant constraints to simplify the problem. Let $r \in I_1 \cup E_1,$ we can represent any non-base row (facet) $a_r$ in $A_N$ as

$$a_r^T = \sum_{j \in B^k} y_{rj} a_j^T = A_{B^k}^T y_r := A_{I_0}^T y_{I_r} + A_{E_0}^T y_{E_r},$$

(11)

where the base matrix $A_{B^k}$ at the $k$-th iteration is partitioned as inequality constraints $A_{I_0}$ and equality constraints $A_{E_0},$ i.e., $A_{B^k}^T = \left[ A_{I_0}^T, A_{E_0}^T \right].$

**Theorem 4.1** Assume that $x^k$ is a basic solution of (5) at the $k$-th iteration and $a_r \in A_N$ can be expressed as (11), then, the following claims hold.

(1). If the row vector $a_r$ corresponds to an equality constraint which satisfies $a_r x^k = b_r$ and $y_{rj} = 0$ for all $j \in I_0,$ i.e., $y_{I_r} = 0,$ then, $a_r x^k = b_r$ is a redundant constraint.

(2). If the row vector $a_r$ corresponds to an inequality constraint which satisfies $a_r x^k > b_r,$ and $y_{rj} \geq 0$ for $\forall j \in I_0,$ then, $a_r x^k \geq b_r$ is a redundant constraint.

**Proof:** We first show that condition (1) implies the equality constraint $a_r x = b_r$ is redundant. Since $y_{I_r} = 0,$ from (11), we have $a_r^T = A_{E_0}^T y_{E_r}.$ Multiplying both sides of (11) by $x^k$ yields $b_r = a_r x^k = y_{E_r}^T A_{E_0} x^k = y_{E_r}^T b_{E_0}.$ Therefore, $y_{E_r}^T \left[ A_{E_0}^T, b_{E_0}^T \right] = [a_r, b_r].$ This proves that if condition (1) holds, the equality constraints $a_r x = b_r$ is redundant. Note that Condition (2) is equivalent to say that system (ii) of Corollary 3.1 is true, this means that system (i) of Corollary 3.1 is not true, therefore, for any feasible solution $\bar{x}$ of (5), we have $a_r \bar{x} \geq a_r x^k > b_r.$ This shows that the inequality constraint $a_r x > b_r$ holds for all feasible solutions $\bar{x},$ Therefore, it is redundant.

Once we identify the redundant constraints, then we can remove these constraints from the original problem.
4.4 General rules on entering/leaving row (facet) selection

The proposed facet pivot simplex method is based on Theorem 3.3, which keeps all the iterates to meet conditions (10) and

\[ A_{B^k} x^k = b_{B^k}, \]

and strikes to find a feasible solution of \( x \) by iteration. This implies that equality constraints should be selected to the base before the inequality constraints are selected. Therefore, the first rule in considering the entering row (facet) is to select rows (facets) of \( A_E \) with \( \sigma_i \neq 0 \) before the rows (facets) of \( A_I \) with \( \sigma_j > 0 \). Once the rows (facets) of \( A_E \) are selected to the base, they will never leave the base.

**Remark 4.1** *In traditional vertex simplex method, we cannot determine what columns will be part of the optimal base until the optimal base is found. Therefore, some optimal columns enter and leave the base multiple times during the iteration. This wastes a lot of computational time due to lack of intuition and using short sighted strategies [29].*

4.5 Specific rules for entering row (facet) selection

Several rules are proposed in [16]. Since we would like to increase feasibility or identify infeasibility as soon as possible, the first specific entering rule is “the maximal deviation rule” which is described below.

**Entering base rule 1:** Among all the rows (facets) in the non-basic equality/inequality constraints, select the row (facet) \( a_p \) in \( A_N \) which has the maximal deviation from the constraint, i.e.,

\[ |\sigma_p| = \max\{|\sigma_i|, |\sigma_j|, |\sigma_k|, |\sigma_k| : \sigma_i \neq 0, \sigma_j < 0, \sigma_k < 0, i, j, k \in I_1 \cup E_1}\].

(13)

The second specific entering rule is “the maximal normalized deviation rule”.

**Entering base rule 2:** Among all the rows (facets) in the non-basic equality/inequality constraints, select the row (facet) \( a_p \) in \( A_N \) which has the maximal normalized deviation from the constraint, i.e.,

\[ |\sigma_p|/\|a_p\| = \max\{|\sigma_i|/\|a_i\|, |\sigma_j|/\|a_j\|, |\sigma_k|/\|a_k\|, |\sigma_k|/\|a_k\| : \sigma_i \neq 0, \sigma_j < 0, \sigma_k < 0, i, j, k \in I_1 \cup E_1\}. \]

(14)

It is worthwhile to mention that this rule finds the facet that has the maximum distance from the current iterate among all infeasible constraints.

The third specific entering rule is “the least/lowest index rule”.

**Entering base rule 3:** Assume that all equality constraints have been selected. Among all the vectors in the non-basic inequality constraints, select the row (facet) \( a_p \) in \( A_N \) which has the least/lowest index in \( I_1 \), i.e., the least/lowest index in the following set

\[ \{\sigma_j < 0, \sigma_k < 0, \sigma_k < 0, j, k \in I_1\}. \]

(15)
Remark 4.2 Assume that the optimal solution is not found, as we have discussed at the end of the section 4.2, at least one of the relations $\sigma_i \neq 0$, $\sigma_j < 0$, $\sigma_k < 0$, $\sigma_k < 0$ holds. Therefore, the rules based on (13), (14) and (15) are well-defined.

Remark 4.3 The ideas of the first two entering rules are to examine the most restrictive constraints so that we can remove as many redundant constraints or identify infeasibility as early as possible.

Assume that $A_{B^k}$ is full rank, which is true for $k = 0$ ($A_{B^0} = E$) and will be shown to be true for $k > 0$ in Theorem 4.5, then the candidate entering row (facet) can be expressed as

$$a_p^T = \sum_{j \in B^k} y_{pj}^k a_j^T = A_{B^k}^T y_p^k := A_{I_0}^T y_{I_p} + A_{E_0}^T y_{E_p}. \quad (16)$$

Again, the base matrix $A_{B^k}$ at iteration $k$ is partitioned as inequality constraints $A_{I_0}$ part and equality constraints $A_{E_0}$ part, i.e., $A_{B^k}^T = [A_{I_0}^T, A_{E_0}^T]$.

Remark 4.4 In view of (13) and (14), for the selected entering row (facet) $a_p$, it must meet one of the following conditions but not both.

1. $a_p x^k < b_p$ with $p \in I_1 \cup E_1$.
2. $a_p x^k > b_p$ with $p \in E_1$.

4.6 Identify infeasible solution

After we select an entering row (facet) and get $y_p$ by solving linear systems of equations (16), we can check if Problem (5) is infeasible by the following theorem.

Theorem 4.2 Let $B^k$ be the base of (5) at the $k$-th iteration, denote $x^k$ the basic (but infeasible) solution of (5), i.e., $A_{B^k} x^k = b_{B^k}$. If either condition

(1). $(a)$ $a_p x^k < b_p$ for the entering row (facet) $p \in I_1 \cup E_1$, and
(b) $y_{pj}^k \leq 0$ for all $j \in I_0$ in (16)

or condition

(2). $(a)$ $a_p x^k > b_p$ for the entering row (facet) $p \in E_1$, and
(b) $y_{pj}^k \geq 0$ for all $j \in I_0$ in (16)

holds, then, there is no feasible solution for Problem (5).

Proof: Assume that condition (1.b) holds, then, we can rewrite (16) as

$$a_p^T = A_{E_0}^T y_{E_p} - A_{I_0}^T y_{I_p}, \quad y_{I_p} \geq 0.$$
Further, in view of Theorem 3.2, system (ii) of Theorem 3.2 is true, which means that system (i) of Theorem 3.2 does not hold. Because \( A_{B^k}x^k = b_{B^k} \) implies that \( A_{E^k}x^k = b_{E^k} \) and \( A_{I^k}x^k \geq b_{I^k} \) hold; it must have \( a_p x^k \geq a_p \bar{x} \) for the entering row (facet) \( p \in \mathcal{I}_1 \cup \mathcal{E}_1 \). Using assumption (1.a) \( b_p > a_p x^k \) for the entering row (facet) \( p \in \mathcal{I}_1 \cup \mathcal{E}_1 \), \( b_p > a_p x^k \geq a_p \bar{x} \) must hold, i.e., there is no feasible solution for Problem (5). This proves part (1).

To prove part (2), assume that condition (2.b) holds, then, we can rewrite (16) as

\[
a_p^T = A_{E^k}^T y_{E^p} + A_{I^k}^T y_{I^p}, \quad y_{I^p} \geq 0.
\]

Further, since \( y_{I_0} \geq 0 \), in view of Corollary 3.1, system (ii) of Corollary 3.1 is true, which means that system (i) of Corollary 3.1 does not hold. Because \( A_{B^k}x^k = b_{B^k} \) implies that \( A_{E^k}x^k = b_{E^k} \) and \( A_{I^k}x^k \geq b_{I^k} \) hold; it must have \( a_p x^k \leq a_p \bar{x} \) for the entering row (facet) \( p \in \mathcal{E}_1 \). Using assumption (2.a) \( b_p < a_p x^k \) for the entering row (facet) \( p \in \mathcal{E}_1 \), \( b_p < a_p x^k \leq a_p \bar{x} \) must hold, i.e., there is no feasible solution for Problem (5).

If Problem (5) does not have a feasible solution, the algorithm will stop here. Assume that the problem has a feasible solution, then we move forward to select the leaving row (facet). According to Remark 4.4 and Theorem 4.2, we need to consider the following two scenarios:

1. \( a_p x^k < b_p \) with \( p \in \mathcal{I}_1 \cup \mathcal{E}_1 \) and there is at least one \( y_{I_0}^{kj} > 0 \) for \( j \in I_0 \).

2. \( a_p x^k > b_p \) with \( p \in \mathcal{E}_1 \) and there is at least one \( y_{I_0}^{kj} < 0 \) for \( j \in I_0 \).

### 4.7 Rules for leaving row (facet) selection

As we mentioned earlier, the equality constraints will never leave the base once they are in the row (facet) base. Therefore, the leave row (facet) are always selected from inequality constraints. Denote

\[
c = \sum_{j \in B^k} y_{I_0}^{kj} a_j^T = A_{I_0}^T y_{I_0} + A_{E_0}^T y_{E_0},
\]

where \( y_{I_0} \geq 0 \) corresponding to the inequality constraints in \( B^k \). From the selection of the initial base, we have \( y_{I_0} = \bar{c} \geq 0 \). According to Theorem 3.3, we want to select the leaving row (facet) to maintain \( y_{I_0} \geq 0 \) in all iterations \( k \) so that if we find a feasible solution, then, we actually find an optimal solution according to Theorem 3.3. Let \( q \in \mathcal{I}_0 \) be the index of the leaving row (facet), the index of the new row (facet) base can be expressed as

\[
B^{k+1} = B^k \cup \{p\} \setminus \{q\}.
\]
Therefore, from (16), we can express the leaving row (facet) \( \mathbf{a}_q \) using the entering row (facet) \( \mathbf{a}_p \) and the rest rows (facets) in the base \( B^k \) as follows:

\[
\mathbf{a}_q^T = \frac{1}{y_{pq}} \mathbf{a}_p^T + \sum_{j \in B^k \setminus \{q\}} \left( -\frac{y_{pq}^j}{y_{pq}} \right) \mathbf{a}_j^T = \mathbf{A}_{I_0+B^k}^T \mathbf{y}_{I_0+B^k} + \mathbf{A}_{I_0+B^k}^T \mathbf{y}_{E_{k+1}} \\
\quad := \sum_{j \in B^k} \mathbf{a}_j^T y_{qj}^{k+1} = \mathbf{A}_{B^{k+1}}^T \mathbf{y}_{q}^{k+1}
\]

where the base matrix \( \mathbf{A}_{B^{k+1}} \) at the \((k+1)\)-th iteration is partitioned as inequality constraints \( \mathbf{A}_{I_0+B^k} \) and equality constraints \( \mathbf{A}_{I_0} \) and \( \mathbf{y}_{q}^{k+1} \) corresponding to the inequality constraints in \( B^{k+1} \). Substituting (19) into (17) yields

\[
c = y_{cq} \mathbf{a}_q^T + \sum_{j \in B^k \setminus \{q\}} y_{cj} \mathbf{a}_j^T \\
\quad = \frac{y_{cq}}{y_{pq}} \mathbf{a}_p^T - \sum_{j \in B^k \setminus \{q\}} \frac{y_{pq}^j}{y_{pq}} y_{cq} \mathbf{a}_j^T + \sum_{j \in B^k \setminus \{q\}} y_{cj} \mathbf{a}_j^T \\
\quad = \frac{y_{cq}}{y_{pq}} \mathbf{a}_p^T + \sum_{j \in B^k \setminus \{q\}} \left( \frac{y_{cq}}{y_{pq}} y_{pq}^j - y_{cq} \right) \mathbf{a}_j^T \\
\quad = \sum_{j \in B^k} y_{cj}^{k+1} \mathbf{a}_j^T := \mathbf{A}_{B^{k+1}}^T \mathbf{y}_{c}^{k+1} \\
\quad := \mathbf{A}_{I_0+B^k}^T \mathbf{y}_{I_0+B^k} + \mathbf{A}_{I_0+B^k}^T \mathbf{y}_{E_{k+1}}.
\]

Again, in (21), the base matrix \( \mathbf{A}_{B^{k+1}} \) at the \((k+1)\)-th iteration is partitioned as inequality constraints \( \mathbf{A}_{I_0+k+1} \) and equality constraints \( \mathbf{A}_{E_{k+1}} \). As discussed before, we want to maintain \( \mathbf{y}_{I_0+B^k} \geq \mathbf{0} \). We divide our discussion into two cases described at the end of the previous section.

**Case 1:** Assume that \( \mathbf{a}_p \mathbf{x}^k < \mathbf{b} \) with \( p \in I_1 \cup E_1 \) and there is at least one \( y_{pq}^k > 0 \) for \( j \in I_0 \).

For this case, we select the leaving row (facet) \( \mathbf{a}_q \) that satisfies the condition \( y_{pq}^k > 0 \) and the following rule:

\[
\frac{y_{cq}}{y_{pq}} = \min \left\{ \frac{y_{cj}^k}{y_{pq}^j} \mid \ y_{pq}^j > 0, \ j \in I_0 \right\}.
\]

Since \( q \in I_0 \), it must have \( y_{cq}^k \geq 0 \), which means \( \frac{y_{cq}}{y_{pq}} \geq 0 \) because \( y_{pq}^k > 0 \). Also, it must have \( \left( y_{cj}^k - \frac{y_{cj}^k y_{pq}^j}{y_{pq}^k} \right) \geq 0 \) for all \( j \in I_0 \) because of (22). This indicates that, according to (20), \( \mathbf{y}_{I_0+B^k} \geq \mathbf{0} \) and condition (10) holds.

The following theorem reveals several important facts.

**Theorem 4.3 ([16])** Let \( B^k \) be the base of (5) in the \( k \)-th iteration. Denote by \( \mathbf{x}^k \) the basic solution of (5) corresponding to \( B^k \), i.e., \( \mathbf{A}_{B^k} \mathbf{x}^k = \mathbf{b} \), and by \( \mathbf{x}^{k+1} \) the basic solution of (5) corresponding to \( B^{k+1} \), i.e., \( \mathbf{A}_{B^{k+1}} \mathbf{x}^{k+1} = \mathbf{b} \). Assume that
(a) $\mathbf{a}_p \mathbf{x}^k < b_p$ for the entering row (facet) $p \in \mathcal{I}_1 \cup \mathcal{E}_1$,

(b) there is an index $q \in \mathcal{I}_0$ such that $y_{pq}^k > 0$, and

(c) the leaving row (facet) $q$ is determined by (22),

then,

(i) $\mathbf{c}$ is given as (20) with

$$
\left( y_{cj}^k - y_{pj}^k \frac{y_{pq}^k}{y_{pq}} \right) \geq 0
$$

for all $j \in \mathcal{I}_0$. Moreover, $y_{I_1}^k \geq 0$ holds.

(ii) The following relation holds

$$
\mathbf{c}^T \mathbf{x}^{k+1} - \mathbf{c}^T \mathbf{x}^k = \frac{y_{cq}^k}{y_{pq}^k} \left( b_p - \mathbf{a}_p \mathbf{x}^k \right) \geq 0.
$$

(iii) If $y_{pq}^k > 0$ and $y_{pq}^k \leq 0$ for all $j \in \mathcal{I}_0 \setminus \{ q \}$, then, $\mathbf{a}_q \mathbf{x} \geq b_q$ is a redundant constraint.

**Proof:** Most part of (i) has been proved before this theorem. Since the leaving row (facet) is an inequality constraint, from (17), we have $y_{cq}^k \geq 0$. This shows that $y_{I_1}^k \geq 0$ and therefore proves part (i). Since $\mathbf{x}^{k+1}$ is a basic solution, from (20), we have

$$
\mathbf{c}^T \mathbf{x}^{k+1} = \frac{y_{cq}^k}{y_{pq}^k} \mathbf{a}_p \mathbf{x}^{k+1} + \sum_{j \in B^k \setminus \{ q \}} \left( y_{cj}^k - y_{pj}^k \frac{y_{cq}^k}{y_{pq}^k} \right) \mathbf{a}_j \mathbf{x}^{k+1}
$$

$$
= \frac{y_{cq}^k}{y_{pq}^k} b_p + \sum_{j \in B^k \setminus \{ q \}} \left( y_{cj}^k - y_{pj}^k \frac{y_{cq}^k}{y_{pq}^k} \right) b_j
$$

$$
= \frac{y_{cq}^k}{y_{pq}^k} b_p + \sum_{j \in B^k} \left( y_{cj}^k - y_{pj}^k \frac{y_{cq}^k}{y_{pq}^k} \right) b_j.
$$

(24)

The last equation holds because $y_{cq}^k - y_{pq}^k \frac{y_{cq}^k}{y_{pq}^k} = 0$. From (17), we have

$$
\mathbf{c}^T \mathbf{x}^k = \sum_{j \in B^k} y_{cj}^k \mathbf{a}_j \mathbf{x}^k = \sum_{j \in B^k} y_{cj}^k b_j.
$$

(25)

Substituting (25) from (24) and invoking (16) yield

$$
\mathbf{c}^T \mathbf{x}^{k+1} - \mathbf{c}^T \mathbf{x}^k = \frac{y_{cq}^k}{y_{pq}^k} \left( b_p - \sum_{j \in B^k} y_{pj}^k b_j \right)
$$

$$
= \frac{y_{cq}^k}{y_{pq}^k} \left( b_p - \sum_{j \in B^k} y_{pj}^k \mathbf{a}_j \mathbf{x}^k \right)
$$

$$
= \frac{y_{cq}^k}{y_{pq}^k} \left( b_p - \mathbf{a}_p \mathbf{x}^k \right) \geq 0.
$$

(26)
the last inequality follows from assumption (a) and $y_{pq}^k \geq 0$. This proves part (ii).

Multiplying both sides of (19) by $x^{k+1}$, and using (16), assumption (c) ($y^k_{pq} > 0$), and assumption (a) ($b_p - a_p x^k > 0$) yield

$$a_q x^{k+1} = \frac{1}{y_{pq}^k} a_p x^{k+1} + \sum_{j \in B^k \setminus \{q\}} \left( -\frac{y_{pj}^k}{y_{pq}^k} \right) a_j x^{k+1}$$

$$= \frac{1}{y_{pq}^k} b_p + \sum_{j \in B^k \setminus \{q\}} \left( -\frac{y_{pj}^k}{y_{pq}^k} \right) b_j$$

$$= \frac{1}{y_{pq}^k} \left( b_p - \sum_{j \in B^k} y_{pj}^k b_j \right) + \frac{y_{pq}^k b_q}{y_{pq}^k}$$

$$= \frac{1}{y_{pq}^k} \left( b_p - a_p x^k \right) + b_q > b_q. \quad (27)$$

In view of the condition in (iii) and (19), it follows that

$$A_{l_0}^T x^{k+1} \geq b_{l_0}, \quad A_{E_0}^k x^{k+1} = b_{E_0}^k, \quad a_q x < a_q x^{k+1}. \quad (28)$$

holds. This indicates that system (ii) of Corollary 3.1 holds, therefore, system (i) of Corollary 3.1 is not true, i.e., at least one of the following relations does not hold for any feasible solution $x$

$$A_{l_0} x^{k+1} \geq b_{l_0}, \quad y_{l_0}^{k+1} = a_q, \quad y_{l_0}^{k+1} \geq 0$$

The first two relations hold because $x^{k+1}$ is a basic solution, it must have $a_q x \geq a_q x^{k+1}$. In view of (27), $a_q x^{k+1} > b_q$, this shows that $a_q x > b_q$ holds for all feasible $x$, therefore, the constraint is redundant. This proves part (iii).

**Case 2:** Assume that $a_p x^k > b_p$ with $p \in E_1$ and there is at least one $y^k_{pq} < 0$ for $j \in I_0$.

For this case, we select the leaving row (facet) $a_q$ that satisfies the condition $y^k_{pq} < 0$ and the following rule:

$$\frac{y_{cq}^k}{y_{pq}^k} = \max \left\{ \frac{y_{cj}^k}{y_{pj}^k} \left| y_{pq}^k \right| y_{pq}^k \right\} \left\{ y^k_{pj} < 0, \quad j \in I_0 \right\}. \quad (28)$$

Since $q \in I_0$, it must have $y_{cq}^k \geq 0$, which means $\frac{y_{cq}^k}{y_{pq}^k} \leq 0$ because $y^k_{pq} < 0$. Also, it must have $\left( y_{cj}^k - y_{pq}^k \frac{y_{cq}^k}{y_{pq}^k} \right) \geq 0$ for all $j \in I_0$ because of (28). This indicates that, according to (20), $y_{l_0}^{k+1} \geq 0$ and condition (10) holds.

The following theorem reveals several important facts.

**Theorem 4.4 ([16])** Let $B^k$ be the base of (5) at the $k$-th iteration. Denote by $x^k$ the basic solution of (5) corresponding to $B^k$, i.e., $A_B^k x^k = b_B^k$, and by $x^{k+1}$ the basic solution of (5) corresponding to $B^{k+1}$, i.e., $A_{B^{k+1}} x^{k+1} = B_{B^{k+1}}$. Assume that
(a) $a_p x^k > b_p$ for the entering row (facet) $p \in \mathcal{E}_1$,
(b) there is a $q \in \mathcal{I}_0$ such that $y^k_{pq} < 0$, and
(c) the leaving row (facet) $a_q$ is determined by (28),
then,

(i) $c$ is given as (20) with $\left( y^k_{cj} - y^k_{pj} y^k_{pq} y^k_{cq} y^k_{kpq} y^k_{bpq} \right) \geq 0$ for all $j \in \mathcal{I}_0$. Moreover, $y^k_{I_k+1} \geq 0$ holds.

(ii) The following relation holds

$$c^T x^{k+1} - c^T x^k = \frac{y^k_{cq}}{y^k_{pq}} (b_p - a_p x^k) \geq 0. \quad (29)$$

(iii) If $y^k_{pq} < 0$ and $y^k_{pq} \leq 0$ for all $j \in \mathcal{I}_0 \setminus \{q\}$, then, $a_q x \geq b_q$ is a redundant constraint.

**Proof:** Part (i) can easily be derived from (20) and (28). Therefore, we prove only Parts (ii) and (iii). Since $x^{k+1}$ is a basic solution, from (20), following the exact the same steps of the derivation of (24), we have

$$c^T x^{k+1} = \frac{y^k_{cq}}{y^k_{pq}} b_p + \sum_{j \in B^k} \left( y^k_{cj} - y^k_{pj} y^k_{cq} y^k_{kpq} y^k_{bpq} \right) b_j. \quad (30)$$

From (17), again, we have

$$c^T x^k = \sum_{j \in B^k} y^k_{cj} a_j x^k = \sum_{j \in B^k} y^k_{cj} b_j. \quad (31)$$

Substituting (31) from (30), invoking (16), and following the exactly same steps of the derivation of (26), we have

$$c^T x^{k+1} - c^T x^k = \frac{y^k_{cq}}{y^k_{pq}} (b_p - a_p x^k) \geq 0, \quad (32)$$

the last inequality follows from assumption (a) and $y^k_{pq} < 0$. This proves part (ii). Multiplying both sides of (19) by $x^{k+1}$, using (16), and assumptions (a) and (c), and following the exactly same steps in the derivation of (27), we have

$$a_q x^{k+1} = \frac{1}{y^k_{pq}} (b_p - a_p x^k) + b_q > b_q, \quad (33)$$

again, the last inequality follows from assumption (a) and $y^k_{pq} < 0$. In view of the condition in (iii) and (19), it follows that

$$A_{E_q \hat{E}_q}^T y_{E_q \hat{E}_q} \geq 0, \quad y_{I_q^k+1} \geq 0$$
holds. This indicates that system (ii) of Theorem 3.2 holds. Therefore, system (i) of Theorem 3.2 is not true, i.e., at least one of the following relations does not hold for any feasible solution \( x \)

\[
A_{j_0}^{k+1}x^{k+1} \geq b_{j_0}^{k+1}, \quad A_{E_{0}}^{k+1}x^{k+1} = b_{E_{0}}^{k+1}, \quad a_qx < a_qx^{k+1}.
\]

The first two relations hold because \( x^{k+1} \) is a basic solution, it must have \( a_qx \geq a_qx^{k+1} \). In view of (33), \( a_qx^{k+1} > b_q \), this shows that \( a_qx > b_q \) holds for all feasible \( x \), therefore, the constraint is redundant. This proves part (iii).

**Remark 4.5** Once a constraint is identified as a redundant one, there is no need to consider it in the remaining iterations.

**Remark 4.6** In case that there is a tie in the selection of the leaving row (facet) by using (22), the row (facet) with the least/lowest index should be selected.

**Theorem 4.5** Assume that the rows of \( A_{B^k} \) are independent, then the rows of \( A_{B^{k+1}} \) are also independent.

**Proof:** Denote \( u = (u_1, \ldots, u_d) \) and \( v = (v_1, \ldots, v_d) \). Using (16), we have the following equivalent expressions:

\[
A_{B^{k+1}}^Tv = 0 \quad \iff \quad \sum_{j \in B^{k+1}} a_j^Tv_j = 0
\]

\[
\iff \quad \sum_{j \in B^{k+1}\{p\}} a_j^Tv_j + a_p^Tv_p = 0
\]

\[
\iff \quad \sum_{j \in B^{k+1}\{p\}} a_j^Tv_j + \sum_{j \in B^k} a_j^T y_j v_p = 0
\]

\[
\iff \quad \sum_{j \in B^{k+1}\{p\}} a_j^T(v_j + y_j^k v_p) + a_q^T y_q^k v_p = 0
\]

\[
\iff \quad \sum_{j \in B^k} a_j^Tu_j = 0 \quad (34)
\]

where \( u_q = y_{pq}^k v_p \), and for \( j \neq q, u_j = v_j + y_{pq}^k v_p \). Since the rows of \( B^k \) are independent, it follows that \( \sum_{j \in B^k} a_j^Tu_j = 0 \) holds if and only if \( u_j = 0 \) for \( j \in B^k \). Since \( y_{pq}^k > 0 \) \( u_q = 0 \) implies that \( v_p = 0 \), which in turn implies the \( v_j = 0 \) for \( j = 1, \ldots, d \). Therefore, \( A_{B^{k+1}}^Tv = 0 \) if and only if \( v = 0 \), i.e., the rows of \( B^{k+1} \) are also independent.

**4.8 Unbounded solution**

There are cases that linear programming problems have unbounded solutions. The following theorem provides the criteria to identify these cases.
Theorem 4.6 ([16]) Let $M$ be the artificial bound introduced in Remark 2.1. If at least one basic row (facet) in $A_B$ reaches its (if it has an) artificial bound $M$ or $-M$ at the end of the iteration, then the linear programming problem is unbounded.

Proof: The claim is obvious and the proof is omitted.

4.9 The facet pivot simplex algorithm

Summarizing the results discussed in this section, the facet pivot simplex algorithm is given as follows:

Algorithm 4.1

1: Data: Matrices $A^I$, $A^J$, $E$, $F$, vectors $b^I$, $b^J$, $u$, $\ell$, and $c$.
2: Form the standard general LP problem and $y_0^I = \bar{c}_j$.
3: Compute the initial basic solution $x^0$ from $Ex^0 = b_L$ (i.e., $A_{B^0}x^0 = b_L$).
4: Compute the constraints violation determinants $\sigma_i$, $\sigma_j$, $\sigma_k$, and $\bar{\sigma}_k$ using (4).
5: while $\sigma_i \neq 0$ or $\sigma_j < 0$ or $\sigma_k < 0$ or $\bar{\sigma}_k < 0$ do
6: if some constraints are redundant (Theorem 4.1) then
7: Remove the redundant equality constraints.
8: end if
9: Select the entering row (facet) $a_p$ using (13) or (14) or least/lowest index rule.
10: Given $a_p$, compute $y_{I_k}^p$ (i.e., $y_{pj}^k$) by solving linear systems of equations (16).
11: if there is no feasible solution (Theorem 4.2) then
12: Exit while loop and report “there is no feasible solution”.
13: end if
14: if $a_p x_k < b_p$ with $p \in I_1 \cup E_1$ and there is at least one $y_{pj}^k > 0$ for $j \in I_0$ then
15: Select leaving row (facet) $a_q$ by using (22).
16: Update base using (18).
17: else if $a_p x_k > b_p$ with $p \in E_1$ and there is at least one $y_{pj}^k < 0$ for $j \in I_0$ then
18: Select leaving row (facet) $a_q$ by using (28).
19: Update base using (18).
20: end if
21: Update $c$ (i.e., $y_{cj}^{k+1}$) using (20), i.e., $y_{cp}^{k+1} = \frac{y_{cp}}{y_{pq}}$ and $y_{cj}^{k+1} = \left( y_{cj}^k - y_{pj}^k \frac{y_{cq}}{y_{pq}} \right)$.
22: if leaving row (facet) $a_q$ is redundant (Theorems 4.3 and 4.4) then
23: Remove the $q$-th constraint from $I$.
24: end if
25: Compute the updated solution $x^{k+1}$ from $A_{B^{k+1}}x = b_{B^{k+1}}$.
26: Compute the constraints violation determinants $\sigma_i$, $\sigma_j$, and $\sigma_k$ using (4).
27: $k \leftarrow k + 1$.
28: end while
4.10 Finite iterations of the facet pivot simplex algorithm

Convergence of conventional (vertex) simplex method depends on if cycling occurs or not, which has been realized [10] shortly after Dantzig published his brilliant work. It is well-known that the conventional vertex simplex method will find the optimal solution in finite iterations if cycling will not occur, which is true if Bland’s rule is used [3]. The cycling problem for conventional vertex simplex method has been studied by a number of authors, for example, [2, 32]. Cycling is also observed for facet pivot simplex method while we test Netlib benchmark problems. Cycling is a phenomenon that the iterates move in a cycle. For the conventional vertex simplex method, when a basic feasible solution is degenerate, after a few iterations using a vertex simplex algorithm, it may returns to a previously constructed basic feasible solution. For the facet pivot simplex method, we observed that a set of base constraints may be repeated after some iterations. For both conventional and facet pivot simplex methods, when cycling happens, there is no improvement in objective function and they may stay away from the optimal solution.

Similar to Bland’s rule [3] for conventional simplex method, if (a) the least/lowest index rule is applied in the selection of the entering constraint, and (b) the least/lowest index rule is applied when a tie occurs in the selection of leaving constraints, then the facet pivot simplex algorithm will find the optimal solution in finite steps. This can be shown by the following arguments.

First, the number of bases of the linear programming Problem (5) is finite. Let \(N = m + n + 2d\), and denote the number \(n\)-combinations in a set of \(N\) elements as \(C(N, d)\), it is straightforward to see that the number of bases of Problem (5) is at most \(C(N, d)\). Second, in every iteration, we have seen from (23) and (32) that the objective function is monotonically non-decreasing. Third, Liu et. al. showed [16] the following result which is similar to Bland’s theorem.

**Theorem 4.7 ([16])** If the least/lowest index rule is used in the selection of row (facet) base in Algorithm 4.1, then cycling will not happen.

Since there are finite many bases for Problem (5), and the objective function is monotonically non-decreasing, in addition, cycling will never happen if the least/lowest index rule is used in Algorithm 4.1, in conclusion, the base of constraints will never repeat. Therefore, we have the following Theorem:

**Theorem 4.8** If the least/lowest index rule is used in the selection of row (facet) base in Algorithm 4.1, then the algorithm will find the optimal solution in finite iterations.

5 Some implementation details and numerical test

Algorithm 4.1 has been implemented in Matlab. Numerical tests for the proposed algorithm have been done for two purposes. First, we would like to verify that the facet pivot simplex method indeed solves some specially designed hard LP problems
effectively, including benchmark cycling problems [30] and Klee-Minty cube problems [15]. Second, we would like to know if this method is competitive to the Dantzig’s most negative pivot rule for general and benchmark testing LP problems, for example, Netlib benchmark LP problems [4], as we know that Dantzig’s most negative rule has been one of the most efficient deterministic pivot rules for LP problems [21]. All tests are done on a personal computer (Intel(R) Core(TM) i5-4440 CPU @3.10GHz, Installed RAM 12.0GB, 64-bit operating system, x64-based processor).

5.1 Some implementation details

We provide some important implementation details in this section in case some readers are interested in repeating the numerical tests reported in this paper. Since least/lowest index rule is normally not very efficient, we implement maximal deviation rule in Step 9 of Algorithm 4.1. All the testing results reported in this section are based on this implementation.

Although removing the redundant constraints using Theorem 4.1 in Steps 6-8 makes the problems smaller, our experience shows that the cost associated with this constraint reduction is too high, which significantly increases the computational time. Although an option of constraint reduction is available, we believe it is better to not use this option.

In Algorithm 4.1, we need to calculate $y_{kj}^k$ in Step 9 by solving linear system equations (16), and $x^{k+1}$ in Step 24 by solving linear system equations (12). Computation in these two steps can be done by using the same LU decomposition for $A_{B^k}$, which will save significant amounts of CPU time in every iteration.

5.2 Test on small size cycling problems

A set of small size cycling problems is collected in [30]. Algorithm 4.1 has successfully solved all 30 problems in this benchmark test set. Cycling does not happen for this set of testing problems. However, this does not mean that the facet pivot simplex algorithm using the maximal deviation rule will prevent the cycling problem from happening.

5.3 Test on Klee-Minty cube problems

Klee-Minty cube and its variants have been used to prove that several popular (vertex) simplex algorithms need an exponential number of iterations in the worst case to find the optimal solution [1, 7, 8, 11]. In this section, two variants of Klee-Minty cube [9, 13] are used to test the facet pivot simplex algorithm.
The first variant of Klee-Minty cube is given in [9]:

\[
\begin{align*}
\min \quad & -\sum_{i=1}^{d} 2^{d-i} x_i \\
\text{subject to} \quad & \begin{bmatrix} 1 & 0 & 0 & \ldots & 0 & 0 \\ 2^2 & 1 & 0 & \ldots & 0 & 0 \\ 2^3 & 2^2 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 2^{d-1} & 2^{d-2} & 2^{d-3} & \ldots & 1 & 0 \\ 2^d & 2^{d-1} & 2^{d-2} & \ldots & 2^2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{d-1} \\ x_d \end{bmatrix} & \leq \begin{bmatrix} 5 \\ 25 \\ \vdots \\ 5^{d-1} \\ 5^d \end{bmatrix} \\
x_i & \geq 0 \quad i = 1, \ldots, d.
\end{align*}
\] 

(35)

The optimizer is \([0, \ldots, 0, 5^d]\) with optimal objective function \(-5^d\).

The second variant of Klee-Minty cube is given in [13]:

\[
\begin{align*}
\min \quad & -\sum_{i=1}^{d} x_i \\
\text{subject to} \quad & x_1 \leq 1, \\
& 2 \sum_{i=1}^{k-1} x_i + x_k \leq 2^k - 1 \quad k = 2, \ldots, d, \\
x_i & \geq 0 \quad i = 1, \ldots, d.
\end{align*}
\] 

(36)

The optimizer is \([0, \ldots, 0, 2^d - 1]\) with optimal objective function \(-(2^d - 1)\).

A Matlab code for Dantzig’s most negative method is implemented in [29], which is used to compare the performance of the Matlab code that implements Algorithm 4.1. The proposed facet pivot simplex algorithm is much more efficient than Dantzig’s most negative simplex method for these Klee-Minty cube problems. The iteration counts for Dantzig’s most negative simplex method and facet pivot simplex algorithm are listed in Table 1. When the problem size is bigger than 16, the Dantzig’s most negative rule needs so many iterations that the computer cannot handle it, but the facet pivot simplex algorithm can easily solve these problems.

| Problem size | Klee-Minty Variant 1 [9] | Klee-Minty Variant 2 [13] |
|--------------|--------------------------|--------------------------|
|              | Dantzig’s rule | Alg. 4.1 | Dantzig’s rule | Alg. 4.1 |
| 3            | 7             | 3         | 2^3 – 1       | 3         |
| 4            | 15            | 4         | 2^4 – 1       | 4         |
| 5            | 31            | 5         | 2^5 – 1       | 5         |
| 6            | 63            | 6         | 2^6 – 1       | 6         |
| 7            | 127           | 7         | 2^7 – 1       | 7         |
| 8            | 255           | 8         | 2^8 – 1       | 8         |
| 9            | 511           | 9         | 2^9 – 1       | 9         |
| 10           | 1023          | 10        | 2^10 – 1      | 10        |
| 11           | 2^11 – 1      | 11        | 2^11 – 1      | 11        |
| 12           | 2^12 – 1      | 12        | 2^12 – 1      | 12        |
| 13           | 2^13 – 1      | 13        | 2^13 – 1      | 13        |
| 14           | 2^14 – 1      | 14        | 2^14 – 1      | 14        |
| 15           | 2^15 – 1      | 15        | 2^15 – 1      | 15        |

22
16 \quad 2^{16} - 1 \quad 16 \quad 2^{16} - 1 \quad 16
17 \quad - \quad 17 \quad 2^{17} - 1 \quad 17
18 \quad - \quad 18 \quad 2^{18} - 1 \quad 18
19 \quad - \quad 19 \quad 2^{19} - 1 \quad 19

Table 1: Iteration count comparison for Dantzig pivot and facet pivot algorithms for the two Klee-Minty variants

5.4 Test on Netlib benchmark problems

Netlib problems have been widely used for testing linear programming algorithms/codes, see for example, [18, 19, 20, 26]. In this section, we test the Netlib problems that have lower and upper bounds, which is more general than the standard problems but less general than the problem discussed in (1). This problem can be expressed as

$$\min \quad c^T x$$
subject to \quad $A x = b$, \quad $\ell \leq x \leq u$. \quad (37)

It can be converted to the following standard problem

$$\min \quad [c^T \ 0^T \ 0^T](x, \ y, \ z)$$
subject to \quad $\begin{bmatrix} A & 0 & 0 \\ I & I & 0 \\ I & 0 & -I \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} b \\ u \\ \ell \end{bmatrix}$ \quad (38b)

$(x, y, z) \geq 0$. \quad (38c)

Therefore, Problem (38) can be solved by Dantzig’s most negative pivot rule algorithm. Both Dantzig’s most negative pivot rule algorithm and the facet pivot simplex method are used to solve the Netlib benchmark problems. This test does not include big problems that cannot be handled by both methods on the old PC because of its limited memory. We also stop the program if it reaches 100000 iterations or it runs more than 20 hours. As one can see from Table 2 that the facet pivot simplex algorithm performs very well in solving these problems because it uses less CPU time than Dantzig’s most negative pivot rule algorithm for all problems except grow15, grow7 for which the facet pivot simplex method finds better solutions.
| Problem | Type | Status | Best Value | Feasible Value | Iterations | CPU Time |
|---------|------|--------|------------|---------------|------------|----------|
| finnis  |      | 0      | 5.7269     | 2.0172e-05    | 1.7279e+05| 78.8601  |
| fit1d   |      | 0      | 9.1042     | -9.1803e+03   | 123.2869  | 9454     |
| fit1p   |      | 0      | 9.1042     | -9.1803e+03   | 123.2869  | 9454     |
| fit2d   |      | 0      | 9.1042     | -9.1803e+03   | 123.2869  | 9454     |
| fit2p   |      | 0      | 9.1042     | -9.1803e+03   | 123.2869  | 9454     |
| ganges  |      | 0      | 20.0904    | -1.0959e+05   | 94.8308   | 6561     |
| gfrd_pnc|      | 0      | 4.4859     | 6.9022e+06    | 55.6054   | 5429     |
| grow15* |      | 0      | 14.0927    | -4.8627e+07   | 3.9425    | 1105     |
| grow7*  |      | 0      | 14.0927    | -4.8627e+07   | 3.9425    | 1105     |
| kb2     |      | 0      | 0.1572     | 0.3447         | -1.7499e+03| 320      |
| ken_07  |      | 0      | 191.1106   | -6.7952e+08   | -          | 100000   |
| pds_02  |      | 0      | 412.1377   | 2.8791e+10    | -         | 100000   |
| recipe  |      | 0      | 0.1446     | -266.6160     | 1.3981    | 682      |
| scorpion|      | 0      | 0.1265     | 1.8781e+03    | 10.5318   | 2251     |
| shell   |      | 0      | 8.1693     | 1.2088e+09    | -         | 100000   |
| sierra  |      | 0      | 8.1693     | 1.2088e+09    | -         | 100000   |
| standata|      | 0      | 1.3567     | 1.2577e-03    | 124.0870  | 8410     |
| standgub|      | 0      | 1.3280     | 123.3777      | 8410      | 1.2577e+03|
| standmps|      | 0      | 2.4103     | 1.4060e-03    | 117.6300  | 8341     |

Table 2: Test of Algorithm 2.1 on Netlib problems

### 6 Conclusion

In this paper, we proposed a facet pivot simplex algorithm. It is proven that the facet pivot simplex algorithm finds the optimal solution in finite iterations if the least index rule is used in the selection of entering/leaving rows (facets). A Matlab function is developed to implement the facet pivot simplex algorithm. Numerical test is performed. The test result shows that the facet pivot simplex algorithm is more efficient than Dantzig’s most negative pivot rule algorithm for general LP problems that have equality, inequality, and boundary constraints. Besides, the facet pivot simplex algorithm is more efficient than Dantzig’s most negative pivot rule algorithm for some specially designed hard problems, such as cycling LP problems and Klee-Minty problems. In the second part of this series [31], we will show that the facet pivot simplex method solves the standard general form of the linear programming problem in a number of iterations bounded by a linear function of the problem size in the worse case.

### 7 Acknowledgment

The author would like to thank Dr. Y. Liu for sharing his excellent paper which is very helpful in the preparation of this work.
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