Physical-mathematical model for determination of variation of internal forces in the simplified analysis of a beam

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Abstract. Springs are often taught in subjects of physics such as statics and solid mechanics belonging to civil engineering programs and mechanical engineering. This knowledge can be applied successfully in the modeling of structures and the consequent development of structural analysis. This paper presents the results of an investigation on physical-mathematical models which uses springs to replace complex connective conditions attempting to simplify the structural analysis process. The work focuses on the analysis of beams supported upon masonry walls, applying variations to the span lengths, sections and loads on them and considering realistic variations of the stiffness conditions required in the supports to meet the demands that these variations impose. For this, continuous beams with two spans with three types of section that are supported by walls that support different levels of restriction from different heights of the building of which they are part are modeled. It is concluded that there is an important influence of the slenderness of the beams and the degree of confinement of the supports upon the precision of the simplified model.

1. Introduction
In basic college and high school physics courses, one of the first relationships between force and strain taught is that shown by springs in their elastic range. This teaching is perhaps one of the most important to introduce the student to realistic logical thinking. The relationship described shows that it is necessary to consider most materials as deformable solids. In mechanical and civil engineering programs, the teaching of springs is usually presented in statics and solid mechanics subjects. Later, in some courses of structural analysis, springs are used to describe conditions of connection or support of linear elements in which it is required to introduce a deformability condition [1-3].

In civil engineering, when a beam is supported on soft soils or upon other flexible beams, models which consider articulated or roller-type supports can induce errors that may show a different balance of forces to that measured in the reality. In this case, it is convenient to propose and solve models of beams resting on springs attempting to describe the conditions of internal forces that could appear in reality [4,5]. In fact, the balance of forces can change substantially when comparing a classical model with a deformable support model.

A special case of beams modeled with classic supports (roller, articulated, embedded) can be given when they are supported or fixed on masonry walls. These walls tend to have high stiffness for some force-generating physical effects that could substantially change the force balance configuration predicted by a classical model [6]. There is lack of information in the literature on these special support conditions, so it is convenient to carry out research on the subject.

This work studies the effects that supports formed by masonry walls can have on the beams. For this purpose, classic models and simplified models using springs of beams supported on masonry walls are
proposed and solved to study the variations in internal forces. The results of the research could be used to propose recommendations on the convenience of using springs in some cases and on the need for research in other cases.

2. Modeling of beams with classical supports and with deformable supports (springs)

In teaching of physics subjects related to statics and solid mechanics, it is shown that a simple and effective way to model supports in beams is to use special nodes that behave as rollers, articulations, or fixed supports [7,8]. It is convenient to clarify that this is applicable to cases in which the supports can guarantee small deformations in the degrees of freedom that are not allowed to move. For example, in the case of an articulated support, the horizontal and vertical displacements are expected to be so small that they can be neglected. However, in some cases real measurements that consider the materials, and the real dimensions can show that these displacements are far from that hypothesis. For this reason, this work presents a model that considers the possible variations in the stiffness of the supports and their effects on the internal forces. This research considers structures that are contained in a single vertical plane, so the force vectors (V) are located on that same plane, while the moment vectors (M) have their rotation axis perpendicular to it [4,5].

A two-section beam is idealized that rests on masonry walls that have different degrees of confinement. This degree of confinement depends on the number of stories and total load applied upon the supports. It produces variable restrictions on the displacements generated in the supports and is modeled using springs with a stiffness that is calculated using the results of a conventional analysis of a frame-structure. In any case, only axial forces are developed in the fictitious wire of the springs. Therefore, there are two types of springs which are shown in Figure 1.

- Rectilinear springs. As can be seen in Figure 1(a), in this case the response forces of the spring only develop in a straight line. They are used to model \( \frac{\text{force (V)}}{\text{displacement (U)}} \) relationships that occur in supports which replace roller or articulated supports. The forces and displacements have the same unit vector and are usually vertical or horizontal or in some cases with a constant angle of inclination [7-10].

- Torsion springs. As can be seen in Figure 1(b), in this case the response forces of the spring only develop in a curved line that varies according to the tension in the fictitious wire. The vectorial product of the arm of the tangent with respect to the spring axis by the tangent force in the wire gives as a result the moment (M) that develops in the spring. This spring is used to model moment (M) / rotation (R) relationships that occur in supports that replace fixed supports. Moments and rotations have the same unit vector and always rotate around an axis perpendicular to the plane containing the structure [7-10].

![Figure 1. Springs for modelling.](image)

The use of the springs showed in Figure 1 is restricted to the elastic zone behavior, *i.e.*, the springs follow a proportionality law that is expressed according to Equation (1) and Equation (2) [7].

\[
V = K_d \cdot U, \tag{1}
\]

\[
M = K_r \cdot R, \tag{2}
\]
where force $V$ developed when a rectilinear spring is subjected to an elongation $U$, computed using a proportionality constant called $k_d$. Equation (2) states that moment $M$ developed when a torsion spring is subjected to a rotation $R$, can be computed using a proportionality constant called $k_r$; if $V$, $U$, $M$, $R$ are known, $K_d$ and $K_r$ values can be calculated. Using the obtained values from the conventional analysis of a reference structure which considers several degrees of freedom per node, it is possible to make a simplified model supported by springs. The study of this process is shown in detail in the following section.

3. Methodology
In a simplified model based on the use of springs it is necessary to determinate the values described in the previous section; to do this, first a structural model containing several degrees of freedom per node is used to compute the stiffnesses of springs that could generate the same forces and moments in a simplified model. This process is repeated with different conditions of load and geometric configurations to generate a comparison in different working ranges of the structure; conclusions about the variability of the simplified model can be drawn from this comparison. The related methodology is summarized below.

A frame-type structure (reference structure) is proposed and analyzed considering load and stiffness conditions of two-section beams supported by masonry walls; to perform the analysis, a code based on the stiffness method was written and run in MATLAB® [11,12]. The code was validated using the SAP2000® program that uses the finite element method [13]; as it is a flat structure, three degrees of freedom are used per node: a horizontal displacement, a vertical displacement, and a rotation around an axis perpendicular to the plane. The load characteristics and geometry of the reference structures are presented in Figure 2(a), Figure 2(b), and Table 1.

The supports of the simplified model are named 1, named 2 and named 3 for the left border, the center, and the right respectively Figure 2(b). When there is total symmetry, i.e., there is symmetric load and symmetric stiffness (mirror condition), the rotations and horizontal displacements of the central support are equal to zero, which resembles a partial fixing condition that allows only vertical displacements of the central support are null [4,5]. For this reason, in practice, a single span simplified structure that is fixed in the continuous edge of the beam is usually used as shown in Figure 3; to verify if this simplification is adequate, its results are compared with those described in the previous paragraphs.

![Figure 2. General model: (a) frame structure, (b) equivalent structure supported on springs.](image)

| Span L (m) | Stories* | b (m) | h (m) | t (m) | $L_w$ (m) | $W_l$ (kN/m) | P (kN) |
|------------|----------|-------|-------|-------|-----------|-------------|--------|
| 2          | 2        | 0.28  | 0.28  | 0.12  | 0.76      | 38          | 113    |
| 3          | 4        | 0.33  | 0.33  | 0.24  | 1.29      | 66          | 592    |
| 6          | 6        | 0.36  | 0.36  | 0.36  | 1.80      | 92          | 1377   |
| 2          | 2        | 0.44  | 0.44  | 0.12  | 0.92      | 46          | 137    |
| 6          | 6        | 0.51  | 0.51  | 0.24  | 1.47      | 75          | 675    |
| 6          | 6        | 0.56  | 0.56  | 0.36  | 2.00      | 102         | 1530   |

*Number of stories supported by the wall above the level of beam.
4. Results

The results obtained for the stiffness values of the equivalent springs $K_d$ and $K_r$ are shown in Table 2. To define a comparison pattern, the displacements and rotations obtained from the simplified model were divided on the values given by the reference model was computed. This quotient is called SA ratio regardless of the nature of the referred parameter. Table 3 shows the Sa ratio for cases of symmetric structures like the that shown in Figure 3. The values of columns 3 to 8 correspond to results obtained for the free (see Figure 3) while that the $V_{fixed}$ and $M_{fixed}$ values refer to those computed in the fixed support.

Table 2. Stiffness of equivalent springs.

| Load | Support | 2 | 4 | 6 |
|------|---------|---|---|---|
|      | Stories | Span | Kd | Kr | Kd | Kr | Kd | Kr | Kd | Kr |
|      |         | 3m  | MN/m | MN * m/rad | 6m | 3m  | MN/m | MN * m/rad | 6m | 3m  | MN/m | MN * m/rad |
| 100% | 1       | 66  | 102 | 2 | 2 | 104 | 181 | 29 | 29 | 144 | 279 | 119 | 163 |
| 1    | 2       | 86  | 125 | N/A | N/A | 117 | 205 | N/A | N/A | 151 | 261 | N/A | N/A |
| 3    | 6       | 66  | 102 | 2 | 2 | 104 | 181 | 29 | 29 | 144 | 279 | 119 | 163 |
| 50%  | 1       | 69  | 106 | 2 | 2 | 106 | 186 | 29 | 29 | 149 | 266 | 135 | 135 |
| 2    | 86      | 125 | 35  | 23 | 117 | 205 | 90  | 86  | 145 | 277 | 204 | 260 |
| 3    | 60      | 94  | 2   | 2 | 100 | 172 | 29  | 29  | 153 | 258 | 135 | 135 |

Table 3. Results for SA ratio in cases of symmetrical structures.

| Stories | Span | Kd | Kr | U | R | V | M | V_{fixed} | M_{fixed} |
|---------|------|----|----|---|---|---|---|-----------|-----------|
| 2       | 3m   | 95%| 100%| 98%| 64%| 94%| 64%| 104%      | 120%      |
| 4       | 6m   | 89%| 100%| 98%| 56%| 87%| 56%| 111%      | 141%      |
| 6       | 99%  | 100%| 94%| 64%| 96%| 94%| 64%| 100%      | 102%      |

The vertical axis of Figure 4, Figure 5 and Figure 6 shows the results obtained for the SA ratio for the different parameters. The horizontal axis shows the type of support 1, 2, 3 (see Figure 2(b)). Each Figure presents two conditions associated with span length and the load level of the right section ($W_d$). In addition, the degree of confinement of the support has been discriminated according to the number $X$ of floors (stories) that confine it 2, 4 and 6. Vertical displacement $U_X$ and rotation $R_X$ can be related to...
the main movement measurements which allow to determine if simplified model offers similar stiffness support efficiency. On the other hand, force VX and moment MX can be associated to the global distribution of internal forces achieved by using the simplified model. In general, the higher the value the higher the distribution of internal forces.

According to Table 2, the stiffness of equivalent springs depends strongly on the length of the connected beam spans. For example, for beams configured using 3 m-spans, the stiffness of equivalent rectilinear springs beams supports confined by 6-story is approximately twice that of those that are confined by 2 stories (Kd from 69 MN/m to 146 MN/m). For 6 m-spans, this proportion is about 2.5 times (Kd from 106 MN/m to 274 MN/m). In contrast, the stiffness of the equivalent torsion springs increases more than 60 times (Kr from 2 MN*m/rad to 127 MN*m/rad) when confinement increases from 2 to 6 stories. These findings indicate that the characteristics of the equivalent springs should be assigned considering the relative stiffness of the beams that depends on the slenderness relationship given by the quotient moment of inertia of the section/span length.

It is important to note the variation in the stiffness of springs depending on their position; from Table 2, it is observed that the ratio of stiffness between the middle and extreme supports oscillates around 118% coefficient of variation (CV) CV = 9%, for rectilinear springs and is higher than 405% (CV = 71%) in the case of torsion springs. This means that the continuity of the beam in the central support provides an important rotational stiffness which gives it the ability to develop bending moment at that site.

![Figure 4](image1.png)

**Figure 4.** SA ratio for simplified model when Wd = Wl: (a) span L = 3m, (b) span L = 6m.

![Figure 5](image2.png)

**Figure 5.** SA ratio for simplified model when Wd = 0.50*Wl: (a) span L = 3m, (b) span L = 6m.
From Figure 4, Figure 5, and Figure 6, it is evident that the change in degree of confinement and the load asymmetry impose important variations on the stiffness of equivalent springs. When the load is symmetric, the variation is slight since the value of the internal forces and displacements of the simplified model tends to be of the order of 95% to 108% of the values obtained from the reference model (SA ratio in Figure 4). When right span load is equal to 50% of the left span load, this range increases until reaching values between 43% and 132% (SA ratio in Figure 5). The critical case occurs when the load on the right span is equal to zero, that is, when the largest load asymmetry is simulated, and variations oscillate between 72% and 267% (SA ratio in Figure 6). Curves marked as not representative (N.R) in Figure 6(a) are related to negligible values of displacements and forces that do not provide new information. In such case, U, R, M, V values are set to zero at the support 3. These findings confirm that the allocation of springs must consider not only the slenderness of the beams but also the asymmetry of the load.

In practice, for symmetrical structures, the structural analysis is usually simplified using only one of its halves as shown in Figure 3. In the case of beams supported by masonry walls, it is necessary to verify if this simplification is valid using results reported in Table 3. As can be seen, the values obtained from the simplified model for Kd and Kr, U and V are around 97% (CV = 8%) of those obtained for the free support (1) of the reference model, which indicates a good coincidence. In contrast, for the same support, the R and M values show a proportion of about 77% (CV = 27%) indicating an important variation that must be considered. On the other hand, in the fixed support, the values of the simplified model oscillate around 106% (CV = 6%), and 122% (CV = 20%) for V and M respectively. Then it is convenient to fit the values predicted by the model for M.

5. Conclusions
This work presents a study of the variations in internal forces in a beam supported on masonry walls when a simplified analysis model is used. Variations are expressed as a relationship of the results between the simplified and the reference model.

The use of a simplified spring-based model is suitable for symmetrical structure conditions where the load and geometry are symmetric with respect to the mid-axis of the structure. On the contrary, the simplified model should be used with caution in the cases of load asymmetry since the variations of the calculated moments can be about a fourth part of those reference results.

In symmetrical structures formed by beams supported by masonry walls, the typical simplified model that considers only half of the structure should be used with caution because the important variations in bending moment and shear force.
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