DENSEST PACKINGS OF TRANSLATES
OF STRINGS AND LAYERS OF BALLS

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Abstract. Let \( L \subset \mathbb{R}^3 \) be the union of unit balls, whose centres lie on the \( z \)-axis,
and are equidistant with distance \( 2d \in [2, 2\sqrt{2}] \). Then a packing of unit balls in \( \mathbb{R}^3 \)
consisting of translates of \( L \) has a density at most \( \pi/(3d\sqrt{3} - d^2) \), with equality for
a certain lattice packing of unit balls. Let \( L \subset \mathbb{R}^4 \) be the union of unit balls, whose
centres lie on the \( x_3x_4 \) coordinate plane, and form either a square lattice or a regular
triangular lattice, of edge length 2. Then a packing of unit balls in \( \mathbb{R}^4 \) consisting
of translates of \( L \) has a density at most \( \pi^2/16 \), with equality for the densest lattice
packing of unit balls in \( \mathbb{R}^4 \). This is the first class of non-lattice packings of unit
balls in \( \mathbb{R}^4 \), for which this conjectured upper bound for the packing density of balls
is proved. Our main tool for the proof is a theorem on \( (r, R) \)-systems in \( \mathbb{R}^2 \). If
\( R/r \leq 2\sqrt{2} \), then the Delone triangulation associated to this \( (r, R) \)-system has the
following property. The average area of a Delone triangle is at least \( \min\{V_0, 2r^2\} \),
where \( V_0 \) is the infimum of the areas of the non-obtuse Delone triangles. This general
theorem has applications also in other problems about packings: namely for \( 2r^2 \geq V_0 \)
it is sufficient to deal only with the non-obtuse Delone triangles, which is in general a
much easier task. Still we give a proof of an unpublished theorem of L. Fejes Tóth and
E (=J.) Székely: for the 2-dimensional analogue of our question about equidistant
strings of unit balls, we determine the densest packing of translates of an equidistant
string of unit circles with distance \( 2d \), for the first non-trivial interval \( 2d \in (2\sqrt{3}, 4) \).
The well known Kepler conjecture states that in $\mathbb{R}^3$ a packing of unit balls has density at most the density of the densest lattice packing of balls in $\mathbb{R}^3$, i.e., $\pi/\sqrt{18}$ (cf. [FT72], [Ro], [FT64], [GL], [BKJ]).

As a special case, the first named author posed in [Bo], in 1975, the following problem. Prove that the density of a packing of unit balls in $\mathbb{R}^3$, consisting of parallel strings of unit balls, whose centers are equally spaced on a straight line, at distances 2, is at most $\pi/\sqrt{18}$.

[BKM91], in 1991, solved this question in the affirmative. For some time their result was the most general result about a class of packings of unit balls in $\mathbb{R}^3$, for whose density the sharp upper bound $\pi/\sqrt{18}$ was proved.

Now of course this result is superceded by T. C. Hales, S. Ferguson [HF], who proved Kepler’s conjecture in full generality. A revision of their original proof is given in [HHMNOZ]. A complete formal proof of the Kepler conjecture that can be verified by automated proof checking software (cited from [V]) is given in [HABD...].

L. Fejes Tóth [FT] generalized the question of [Bo], requiring that the distance of the neighbouring centres of the balls in a string should be $2d$ rather than 2, where $d \in [1, \sqrt{2}]$, and asked for the maximal density under this more general condition. We will prove a general theorem, which contains as a special case the solution of L. Fejes Tóth’s question. We will obtain that among the extremal packings there are lattice packings, which we will concretely determine. We note that the authors of [BKM91] were aware that their paper gave possibility to prove L. Fejes Tóth’s conjecture for $d$ sufficiently close to 1, but this was not included in their paper.

Observe that the result of [HF] does not solve this problem, except for those values of $d$ for which $2d$ is the distance of some centres of unit balls in a packing of unit balls consisting of closely packed regular hexagonal layers which is periodic with some period $k$ (i.e., some fixed translation carries the $n$’th hexagonal layer to the $(n + k)$’th layer, for each integer $n$) and $2d$ is the distance of the centres of some balls from the first and $(k + 1)$’th hexagonal layers. Such values are, e.g., $d = 1$, $d = \sqrt{2}$ (for the densest lattice packing) and $d = \sqrt{8/3}$ (for the densest regular non-lattice packing).

The analogous question for the plane is that of the maximum density of a packing of unit circles in $\mathbb{R}^2$, consisting of parallel strings of unit circles, whose centers are equally spaced on a straight line, at distances $2d$ (where $d \geq 1$). This question is solved for $1 \leq d \leq \sqrt{3}$ in [FT62] (in the form of packing geodesic unit circles on the surface of a circular cylinder with base of perimeter $2d$): a densest packing is lattice-like, with the corresponding point lattice spanned by the vertices of a triangle of sides 2, 2, $2d$. Further, this question is solved for $\sqrt{3} \leq d \leq 2$ [FT62] (he did not give the densest packing) and [Sz], oral communication. In the second part
of the introduction we will give a proof for this case. This question is unsolved for all values $d > 2$, except those for which $2d$ is the distance of two centres of circles in the densest lattice packing of unit circles in $\mathbb{R}^2$.

[Sz] conjectured that for any $d$ one of the densest packings for the planar case is obtained in the following way. We choose a natural number $k$, and consider $k$ neighbouring strings (of touching unit circles) of a densest lattice packing of unit circles. We translate this block of $k$ strings periodically so that any two neighbouring blocks touch each other and for a circle of the first string of a block and some circle of the first string of the following block the distance of the centres is $2d$. This number $k$ equals 1 for $1 < d < \sqrt{3}$ (cf. [FT62]), and 2 for $\sqrt{3} < d < 2$ (cf. [Sz], oral communication). By this definition we get in fact a packing of unit circles, and also a packing of equidistant strings of unit circles with distance $2d$. The next interesting interval is $2 < d < \sqrt{7}$. Then still we should have $k = 2$ (for $k \geq 4$ the symmetry axes of the $k$'th neighbour strings have a distance at least $2\sqrt{12}$, and for $k = 3$ the centre of any circle touching the 3'rd neighbour string from the “other side” has a distance at least $2\sqrt{7}$ from the centre of any circle in the original string. The proof of the theorem of L. Fejes Tóth and E. (=J.) Székely in the second half of the introduction does not seem to be suitable for the case $2 < d < \sqrt{7}$.

A related problem is investigated in J. Molnár [M78]. Namely the maximal packing density of unit balls in a parallel slab in $\mathbb{R}^3$, of width $w \in [2, 2 + \sqrt{2}]$ was determined, as a function of $w$. The analogous problem of the maximal packing density of unit circles in a parallel strip in $\mathbb{R}^2$, of width $w \in [2, 2 + 2\sqrt{2}]$ was determined, as a function of $w$, by G. Kertész [Ke], unpublished, and for width $w \in [2, 2 + 2\sqrt{3}]$ by Z. Füredi [Fu].

The idea of our proofs is taken from [BKM]. Our problem is reduced essentially to a planar problem, cf. Theorem 2.9. There we have a nice system of points on the plane (an $(r, R)$-system), and consider the Delone triangulation associated to it. The acute or right Delone triangles present no problem, the problem is only with obtuse Delone triangles. However, an obtuse Delone triangle has a neighbour at its longest side, for which the common side is longer than guaranteed by the problem, and its vertex opposite to the common side is outside of the circumcircle of the obtuse triangle. These two properties will have the consequence that “the smaller area of the obtuse triangle is compensated by the surplus of the area of its neighbouring Delone triangle at its longest side”. Analogously: if some Delone triangle shares the longest sides of two or three obtuse triangles, then “its area surplus compensates the small areas of the mentioned two or three obtuse triangles”. Details cf. in §4.

Still we note that the analogue of the Kepler conjecture was most recently solved in $\mathbb{R}^8$ by M. Viazovska [V] (the solution is the densest lattice packing of unit balls in $\mathbb{R}^8$) and by H. Cohn, A. Kumar, S.D. Miller, D. Radchenko, M. Viazovska [CKMRV]
in \( \mathbb{R}^{24} \) (the solution is the Leech lattice, which was proved to be the densest lattice packing by H. Cohn, A. Kumar [CK] in 2009).

Since for the problem of the densest packing of translates of equidistant strings of unit circles with equal distances \( 2d \), for \( d \in (\sqrt{3}, 2) \), no proof seems to have been published, not even the densest packing or the packing density seems to have been published, we give the not complicated proof of the respective result.

**Theorem.** (L. Fejes Tóth and E. (=J.) Székely). For \( d \in (\sqrt{3}, 2) \), the maximum density of packings of translates of equidistant strings of unit circles, with equal distances \( 2d \), is \( 2\pi/[d(\sqrt{4 - d^2} + d\sqrt{3})] \). This density is attained for the packing of translates of our equidistant string of unit circles described in the next paragraph.

We consider the axes of symmetry of the strings (later called axes of the strings) as horizontal, and then the strings constituting our packing have a natural order: namely according to the \( y \)-coordinates of the centres of the circles in the strings. If the 0’th string has two neighbourly centres of circles \( A_0, A_1 \), with \( |A_0A_1| = 2d \), then let the 1’st string lie above the 0’th string and have a circle with centre \( B_0 \) where \( |A_0B_0| = |A_1B_0| = 2 \). Similarly, let the 2’nd string lie above the 1’st string, and have a circle with centre \( C_1 \), where \( |A_1C_1| = |B_0C_1| = 2 \). Then let the translation carrying the 0’th string to the 2’nd string carry the \( i \)’th string to the \( (i + 2) \)’nd string for each integer \( i \). (This system is identical with the one conjectured by E. (=J.) Székely, with \( k = 2 \). In particular, the above construction gives a packing of unit circles, and also a packing of equidistant strings of unit circles with distance \( 2d \).

**Proof of the theorem of L. Fejes Tóth and E. (=J.) Székely.** It will be sufficient to prove that the distance of the axes of the 0’th and 2’nd strings is at least the distance in the above construction, i.e., \( (\sqrt{4 - d^2} + \sqrt{3}d)/2 \). Indirectly suppose that this distance is less than \( (\sqrt{4 - d^2} + \sqrt{3}d)/2 \). Then this distance is also at most \( (\sqrt{4 - d^2} + \sqrt{3}d)/2 \), and we are going to show that

\[
(0.1) \quad \begin{cases} 
\text{under this new (weaker) assumption the distance of the axes} \\
\text{of the 0’th and 2’nd strings is equal to } (\sqrt{4 - d^2} + \sqrt{3}d)/2. 
\end{cases}
\]

This will prove our indirect statement.

We take \( A_0 \) as the origin, and then \( A_1 = (2d, 0) \), \( B_0 = (d, \sqrt{4 - d^2}) \), and \( C_1 = ((3d + \sqrt{3}\sqrt{4 - d^2})/2, (\sqrt{3}d + \sqrt{4 - d^2})/2) \). Let \( u \) and \( v \) be the vectors of translations carrying the 0’th string to the 1’st string and the 1’st string to the 2’nd string, where \( u \) and \( v \) lie in the strip \( 0 \leq x \leq 2d \). Then by the packing property and our hypothesis
\[
(0.2) \quad \begin{cases} 
  u, v \in K := \{(x, y) \in \mathbb{R}^2 \mid 0 \leq x \leq 2d, \ 0 \leq y \leq \\ 
  (\sqrt{4 - d^2} + \sqrt{3d})/2, \ \|(x, y)\| \geq 2, \ \|(x, y) - (2d, 0)\| \geq 2 \}. 
\end{cases}
\]

Then translation by \( w := u + v \) carries the 0'th string to the 2'nd string, hence analogously to (0.2), by the packing property and our hypothesis

\[
(0.3) \quad w \in K + (2id, 0) \text{ for some integer } i.
\]

Since \( w = u + v \) lies in the strip \( 0 \leq x \leq 4d \), so actually

\[
(0.4) \quad w \in K \text{ or } w \in K + (2d, 0).
\]

Also \( w = u + v \in K + K \). Hence by (0.4) we have

\[
(0.5) \quad w = u + v \in K \cap (K + K) \neq \emptyset \text{ or } w = u + v \in (K + (2d, 0)) \cap (K + K) \neq \emptyset.
\]

The set \( K \) is bounded by the segment \([C_0, C_1]\), where \( C_0 \) is the symmetric image of \( C_1 \) with respect to the line \( x = d \), and by two concave circular arcs \( \overline{B_0C_0} \) and \( \overline{B_0C_1} \) of radii \( 2 \), central angles \( \pi/3 \) and centres \( A_0 \) and \( A_1 \). Thus \( K \) is a “concave arc triangle”, consisting of the Jordan curve \([C_0, C_1] \cup \overline{B_0C_0} \cup \overline{B_0C_1} \) and its interior. Its convex hull \( \text{conv}K \) is the triangle \( \Delta B_0C_0C_1 \).

Since \( B_0, C_0, C_1 \in K \), we have \( 2B_0, 2C_0, 2C_1, B_0 + C_0, B_0 + C_1, C_0 + C_1 \in K + K \).

Since \( K \subset \Delta B_0C_0C_1 \), we have \( K + K \subset 2 \cdot \Delta B_0C_0C_1 \). We are going to bound \( K + K \) from outside better.

It will be more convenient to deal with \((K + K)/2\). From above we have

\[
(0.6) \quad \begin{cases} 
  \{B_0, C_0, C_1, (B_0 + C_0)/2, (B_0 + C_1)/2, (C_0 + C_1)/2 \} \\
  \subset (K + K)/2 \subset \Delta B_0C_0C_1.
\end{cases}
\]

Diminishing \( K \) from any of its vertices in ratio \( 1/2 \), we see that \((K + K)/2\) contains \([C_0, (C_0 + C_1)/2]\), \([(C_0 + C_1)/2, C_1]\) and and four (concave) circular arcs of radii \( 1 \) and central angles \( \pi/3 \), say, \( B_0 ((\overline{B_0 + C_0})/2), ((B_0 + C_0)/2) C_0, B_0 ((B_0 + C_1)/2), (\overline{(B_0 + C_1)})/2) C_1 \), which six arcs altogether form a Jordan curve \( J \), such that \( (\text{int} J) \cup J \supset K \), where \( \text{int} J \) denotes the interior of \( J \). Of course all three diminished copies of \( \text{int} K \), from any of its vertices in ratio \( 1/2 \), are contained in \( \text{int} ((K + K)/2) \).
Let \( x, y \in K \), and let one of \( x \) and \( y \) be an interior point of \( K \). Then \((x+y)/2 \in \text{int} ((K+K)/2)\). The case when one of \( x \) and \( y \) is a vertex of \( K \) was investigated in the last paragraph.

There remains the case when \( x, y \in K \) and both are relative inner points of some arc sides of \( K \). The slope of \( \partial K \) at relative inner points of \([C_0, C_1], \overline{B_0C_0} \) and \( \overline{B_0C_1} \) is 0, negative and positive, respectively. Therefore if \( x \) and \( y \) are relative inner points of different arc sides of \( K \), then these slopes are different, hence \((x+y)/2 \in \text{int} ((K+K)/2)\). If \( x, y \in [C_0, C_1] \), then \((x+y)/2 \in [C_0, C_1] \). If \( x, y \) are different and both belong to the relative interior of either \( \overline{B_0C_0} \) or \( \overline{B_0C_1} \), then the slopes of \( \partial K \) at \( x, y \) are different, hence, like above, \((x+y)/2 \in \text{int} ((K+K)/2)\). If \( x = y \) belongs to the relative interior of either \( \overline{B_0C_0} \) or \( \overline{B_0C_1} \), then \( x = y = (x'+y')/2 \) for some \( x', y' \in \text{int} K \), hence \( x = y \in \text{int} ((K+K)/2) \).

Summing up: boundary points of \((K+K)/2\) lie either on \( J \), or on the relative interiors of the diminished opposite arc sides of \( K \), from its three vertices in ratio \( 1/2 \) (which bound a convex arc-triangle \( T \) with vertices the side midpoints of \( \text{conv} K \)).

Next we show that \( J \subset \partial ((K+K)/2) \). We show this for the subarcs \( B_0((B_0+C_0)/2) \) and \( B_0((B_0+C_1)/2) \) of \( J \). (For the other subarcs the argument is the same.) These bound (partially) a diminished copy of \( K \) from \( B_0 \), in ratio \( 1/2 \), whose interior is a subset of \( \text{int} ((K+K)/2) \). If one of our considered two arcs were not contained in \( \partial ((K+K)/2) \), then some inner point of \((K+K)/2\) (namely in the diminished copy of \( \text{int} K \), from \( B_0 \), in ratio \( 1/2 \)) could be connected by an arc to infinity, avoiding \( \partial ((K+K)/2) \), contradicting \((K+K)/2 \subset \text{conv} K \).

Then by \( J \subset \partial ((K+K)/2) \) the relative interiors of the diminished copies of the opposite arc sides of \( K \) (i.e., the relative interiors of the sides of the above arc triangle \( T \)) cannot have points on the outer boundary of \((K+K)/2\), i.e., on the boundary of the unbounded connected component of \( \mathbb{R}^2 \setminus ((K+K)/2) \). Therefore the outer boundary of \((K+K)/2\) equals \( J \).

If \((K+K)/2\) had some point outside \( J \), then it would have also some outer boundary point outside \( J \), a contradiction. Therefore \((K+K)/2 \subset \text{int} J \cup J \). Therefore \( K+K \subset L := 2((\text{int} J) \cup J) \). (Actually here equality holds — but we do not need this. Namely, \( \emptyset \neq (\text{int} K) \cap (\text{int} T) \subset (\text{int} ((K+K)/2)) \cap (\text{int} T) \). Further, since \( (\text{int} T) \cap (\partial ((K+K)/2)) = \emptyset \), therefore \( \text{int} T \subset \text{int} ((K+K)/2) \subset (K+K)/2 \). Also the diminished copies of \( K \) from its three vertices, in ratio \( 1/2 \), lie in \((K+K)/2 \) as well. These together prove our claim.) Then (0.5) implies

\[
(0.7) \quad w \in K \cap L \neq \emptyset \text{ or } w \in (K+(2d,0)) \cap L \neq \emptyset .
\]

We are going to show that

\[
(0.8) \quad \begin{cases} \text{any point (e.g., } w \text{) of } K \cap L \text{ or of } (K+(2d,0)) \cap L \text{ has } y\text{-coordinate } \left( \sqrt{4-d^2} + \sqrt{3d} \right)/2, \end{cases}
\]
as promised in (0.1) (observe that the distance of the axes of the 0’th and 2’nd strings is the y-coordinate of \(w\)).

Observe that we have a symmetric trapezoid \(A_0A_1C_1C_0\), with \(A_0\) the origin, which contains \(K\). Both \(A_0A_1C_1C_0\) and \(K\) have the line \(x = d\) as symmetry axis. The vertices of \(K\) are \(B_0, C_0, C_1\), and the boundary of \(K\) is \(\overline{B_0C_0} \cup \overline{B_0C_1} \cup [C_0, C_1]\). Then \(L\) is bounded by \([2C_0, C_0 + C_1], [C_0 + C_1, 2C_1]\) and four concave circular arcs of radius 2 and central angles \(\pi/3\), say, \((2B_0)(B_0 + C_0), (B_0 + C_0)(2C_0)\) — which are translates of the boundary arc \(\hat{B_0C_0}\) of \(K\) — and \((2B_0)(B_0 + C_1), (B_0 + C_1)(2C_1)\) — which are translates of the boundary arc \(\hat{B_0C_1}\) of \(K\). Let the images of \(A_1, C_0, C_1\) by the translation through \((2d, 0)\) be \(A_2, C_2, C_3\). Then the line \(x = 2d\) is an axis of symmetry of \(L\), and the axially symmetric images of \(A_0A_1C_1C_0\) and \(K\) with respect to this line are \(A_1A_2C_3C_2\) and \(K + (2d, 0)\). Therefore also \(K \cap L\) and \((K + (2d, 0)) \cap L\) are axially symmetric images of each other with respect to the line \(x = 2d\). Hence, rather than (0.8), it suffices to show that

\[
(0.9) \quad \text{any point (e.g., } w\text{) of } K \cap L \text{ has } y\text{-coordinate } (\sqrt{4 - d^2 + \sqrt{3d}})/2.
\]

The line \((B_0 + C_0)(B_0 + C_1)\) cuts \(L\) into two closed parts, say \(L_1\) is the lower part, and \(L_2\) is the upper part. Since the \(y\)-coordinate of \(B_0\) is positive, the \(y\)-coordinate of \(B_0 + C_0\) is larger than the \(y\)-coordinate of \(C_0\), which equals \((\sqrt{4 - d^2} + \sqrt{3d})/2\). Also \(K\) lies (not strictly) below the line \(y = (\sqrt{4 - d^2} + \sqrt{3d})/2\). Thus \(K \cap L_2 = \emptyset\), and

\[
(0.10) \quad K \cap L = K \cap L_1.
\]

The set \(L_1\) is a translate of \(K\), through the vector \(B_0\). The leftmost point of \(L_1\) is \(B_0 + C_0\), whose \(x\)-coordinate is greater than the \(x\)-coordinate of \(B_0\) which equals \(\sqrt{4 - d^2}\). The part \(K_1\) of \(K\) (not strictly) to the left hand side of the line \(x = \sqrt{4 - d^2}\) is therefore disjoint to \(L_1\). Therefore only the part \(K_2\) of \(K\) (not strictly) to the right hand side of this line can intersect \(L_1\), i.e.,

\[
(0.11) \quad K \cap L_1 = K_2 \cap L_1.
\]

Hence

\[
(0.12) \quad \begin{cases} K_2 \subset K \subset \text{conv } K \subset \text{ the circle with centre } B_0 \text{ and radius } 2, \text{ and} \\ \text{the only point of } K_2 \text{ lying on the boundary of this circle is } C_1. \end{cases}
\]

On the other hand,

\[
(0.13) \quad \begin{cases} L_1 \text{ lies (not strictly) to the right hand side} \\ \text{of the arc } (2B_0)(B_0 + C_0) \text{ of the circle in (0.12).} \end{cases}
\]
Then (0.10), (0.11), (0.12) and (0.13) imply

\[(0.14) \quad K \cap L = K \cap L_1 = K_2 \cap L_1 = \{C_1\},\]

and the \(y\)-coordinate of \(C_1\) is \((\sqrt{4 - d^2} + \sqrt{3}d)/2\), as was promised in (0.9). Hence also (0.8) and (0.1) hold, thus the theorem is proved. ■

§2 Results

We begin with some notations. We write \(B^n \subset \mathbb{R}^n\) for the closed unit ball centred at 0. \(B((x_1, \ldots, x_n), R) \subset \mathbb{R}^n\) is the closed ball of centre \((x_1, \ldots, x_n)\) and radius \(R\). An analogous notation will be applied for closed balls in lower dimensional subspaces (the coordinates of the centre of the ball will indicate the subspace). For points \(A, B \in \mathbb{R}^n\) we write \([A, B]\) for the segment with endpoints \(A\) and \(B\), \(|AB|\) for the length of \([A, B]\), and for a vector \(v \in \mathbb{R}^n\) we write \(|v|\) for its norm.

By a point lattice in \(\mathbb{R}^n\) we mean an inhomogeneous lattice, i.e., a translate of a homogeneous lattice (i.e., of a discrete subgroup of \(\mathbb{R}^n\), possibly not full dimensional). We denote point lattices by \(\Lambda\). By a lattice vector we mean a vector (point) in the corresponding homogeneous lattice. For \(L \subset \mathbb{R}^n\), a two-dimensional lattice packing of translates of \(L\) is a packing of the form \(\{L + \lambda | \lambda \in \Lambda\}\), where \(\Lambda \subset \mathbb{R}^n\) is a two-dimensional point lattice.

Our Theorem 2.1 about \(\mathbb{R}^3\) generalizes the theorem of [BKM91], which is the special case \(d = 1\) of our Theorem 2.1.

**Theorem 2.1.** Let \(L \subset \mathbb{R}^3\) be the union of a string of unit balls in \(\mathbb{R}^3\), whose centres lie on the \(z\)-axis, and are equidistant with distance \(2d\), where \(1 \leq d \leq \sqrt{2}\). Then any packing of translates of \(L\) in \(\mathbb{R}^3\) (i.e., any packing of unit balls in \(\mathbb{R}^3\), consisting of entire translates of \(L\)) has a density at most \(\pi/(3d\sqrt{3 - d^2})\), with equality e.g. for the lattice generated by the vertices of a tetrahedron with five edges of length 2 and one edge of length \(2d\). Another way of giving this lattice is the following: it is generated by the vertices of a rectangular right pyramid with base edges of lengths 2 and \(2d\), and lateral edges of length 2. Among lattice packings of translates of \(L\) the above lattice is the unique lattice with this maximal density.

**Remark 2.2.** Consider this packing as one consisting of layers generated by the balls corresponding to a face of the tetrahedron in the Theorem with sides 2, 2, 2d. Then we obtain packings of translates of \(L\) of the same density when these layers are packed parallely and closely, two neighbouring layers joining in any of the two congruent ways (as in case of the densest packing of unit balls in \(\mathbb{R}^3\) we may put on a regular hexagonal layer a next one in two different ways). As particular cases, we obtain the lattice packing from Theorem 2.1 and the non-lattice-like regular packing (in analogy with the densest lattice packing of unit balls and the densest non-lattice-like regular packing of unit balls in \(\mathbb{R}^3\)).
The next theorem is an analogue of Theorem 2.1 for $\mathbb{R}^4$. It is a well known conjecture that in $\mathbb{R}^4$ a densest packing of unit balls is lattice like, with packing lattice the space-centred cubic lattice, and this ball packing has a density $\pi^2/16$, cf. [Ro], [GL], [BKJ]. Our next Theorem 2.2 about $\mathbb{R}^4$ gives the first class of non-lattice packings of unit balls in $\mathbb{R}^4$, for which the conjectured density estimate is proved. We hope that this result will be soon superceded by the solution of the densest ball packing in $\mathbb{R}^4$, like it happened with [BKM91] and [HF] in $\mathbb{R}^3$.

**Theorem 2.3.** Let $L \subset \mathbb{R}^4$ be the union of the two-dimensional lattice packing of translates of the unit ball in $\mathbb{R}^4$, where the corresponding two-dimensional point lattice is either the square lattice with edge length 2, or the regular triangular lattice with edge length 2. Then any packing of translates of $L$ in $\mathbb{R}^4$ (i.e., any packing of unit balls in $\mathbb{R}^4$, consisting of entire translates of $L$) has a density at most $\pi^2/16$, i.e., the density of a densest lattice packing of unit balls in $\mathbb{R}^4$.

Returning to $\mathbb{R}^3$, we need some notations. Let $L$ be the union of a string of unit balls in $\mathbb{R}^3$, whose centres lie on the $z$-axis. Let the distance of the neighbouring centres of balls in $L$ be always at least 2, further let them satisfy that the average distance of the neighbouring balls is $2d$, and this holds uniformly. By this we mean the following.

\[
(2.1) \quad \begin{cases} 
\text{for any segment } [(0, 0, z - R), (0, 0, z + R)] \text{ in the } z\text{-axis} \\
\text{the quotient of } 2R \text{ and the number of ball centres from } L
\text{ in } [(0, 0, z - R), (0, 0, z + R)] \text{ tends for } R \to \infty \text{ to } 2d, \\
\text{and this convergence is uniform for all } z \in \mathbb{R}.
\end{cases}
\]

The next Proposition has a weaker hypothesis (more general sets $L$) and also a weaker conclusion (the extremal lattices are not described) than Theorem 2.1. We say that a two-dimensional point lattice $\Lambda \subset \mathbb{R}^n$ projects orthogonally on the $x_1x_2$-coordinate plane injectively, if for distinct $\lambda_1, \lambda_2 \in \Lambda$ their images by this projection are also distinct.

**Proposition 2.4.** Let $L \subset \mathbb{R}^3$ be the union of a string of unit balls in $\mathbb{R}^3$, whose centres lie on the $z$-axis. Let the distance of the neighbouring centres of balls in $L$ be always at least 2, and let the average distance of neighbouring centres of unit balls on the $z$-axis exist uniformly and equal $2d$ in the sense of (2.1), where $1 \leq d \leq \sqrt{2}$. Then any packing of translates of $L$ in $\mathbb{R}^3$ (i.e., any packing of unit balls in $\mathbb{R}^3$, consisting of entire translates of $L$) has a density at most the supremum of the densities of those two-dimensional lattice packings $\{L + \lambda \mid \lambda \in \Lambda\}$ of translates of $L$ in $\mathbb{R}^3$, for which the following holds. The point lattice $\Lambda$ projects orthogonally to the $xy$-plane injectively, onto a two-dimensional point lattice in the $xy$-plane.

Now we turn to $\mathbb{R}^n$. We again need some notations. Let $L$ be the union of a packing of unit balls in $\mathbb{R}^n$, whose centres lie on the
Further (as a generalization of (2.1)), let us suppose
\[
\begin{align*}
\text{(2.2)} \quad & \text{for any (0,0,x_3,\ldots,x_n) in the } x_3\ldots x_n\text{-coordinate hyperplane} \\
& \text{and for } R \to \infty \text{ the quotient of the number of ball centres from } L \\
\quad & \text{in } B((0,0,x_3,\ldots,x_n),R) \text{ and of } R^{n-2} \text{ tends to some positive number, and this convergence is uniform for all } (0,0,x_3,\ldots,x_n) \\
& \text{in the } x_3\ldots x_n\text{-coordinate hyperplane}
\end{align*}
\]

The inequalities in both Theorems 2.1 and 2.3 follow from the next Proposition.

**Proposition 2.5.** Let \( n \geq 2 \), and let \( L \) be the union of a packing of translates of \( B^n \), with centres in the \( x_3\ldots x_n\)-coordinate plane, such that the concentric balls of radius \( \sqrt{2} \) form a covering of the \( x_3\ldots x_n\)-coordinate plane. Let us suppose that (2.2) holds. Then any packing of translates of \( L \) in \( \mathbb{R}^n \) (i.e., any packing of unit balls in \( \mathbb{R}^n \), consisting of entire translates of \( L \)) has a density at most the supremum of the densities of those two-dimensional lattice packings \( \{L + \lambda | \lambda \in \Lambda \} \) of translates of \( L \) in \( \mathbb{R}^n \), for which the following holds. The point lattice \( \Lambda \) projects orthogonally to the \( x_1x_2\)-coordinate plane injectively, onto a two-dimensional point lattice in the \( x_1x_2\)-coordinate plane.

We say that \( X \subset \mathbb{R}^n \) has rotational symmetry about the \( x_3\ldots x_n\)-coordinate plane (of dimension \( n-2 \)), if \((x_1,x_2,x_3,\ldots,x_n) \in X \) implies \((x_1 \cos \varphi + x_2 \sin \varphi, -x_1 \sin \varphi + x_2 \cos \varphi, x_3,\ldots,x_n) \in X \) for every \( \varphi \in [0,2\pi] \).

**Notation.** If \( X \subset \mathbb{R}^n \) has rotational symmetry about the \( x_3\ldots x_n\)-coordinate plane, and also is open, then let
\[
\begin{align*}
\text{(2.3)} \quad & \{ m(X) := \inf \{ \sqrt{x_1^2 + x_2^2}/2 \mid (x_1,x_2,x_3,\ldots,x_n) \in \mathbb{R}^n, \\
& X \cap (X + (x_1,x_2,x_3,\ldots,x_n)) = \emptyset \} , \\
\text{(2.4)} \quad & M(X) := \sup \{ \sqrt{x_1^2 + x_2^2} \mid (x_1,x_2,x_3,\ldots,x_n) \in X \} .
\end{align*}
\]

Clearly, \( m(X) \leq M(X) \).

**Proposition 2.6.** Let \( L \) be the union of an \( (n-2)\)-dimensional lattice packing of translates of some convex body \( K \subset \mathbb{R}^n \), where \( K \) has rotational symmetry about the \( x_3\ldots x_n\)-coordinate plane and the corresponding \( (n-2)\)-dimensional point lattice lies in the \( x_3\ldots x_n\)-coordinate plane. With the notations (2.3) and (2.4), let \( (M(L) =) M(K) = 1 \) and let \( 1/\sqrt{2} \leq m(L) \). Then any packing of translates of \( L \) in \( \mathbb{R}^n \) (i.e., any packing of translates of \( K \) in \( \mathbb{R}^n \), consisting of entire
translates of $L$) has a density at most the density of the densest lattice packing of translates of $K$ in $\mathbb{R}^n$.

**Remark 2.7.** To get many examples where Proposition 2.6 can be applied, we consider the following examples. We suppose $M(K) = 1$, and ensure $m(L) \geq 1/\sqrt{2}$ by letting $\{(x_i) \in \mathbb{R}^n \mid \sqrt{x_1^2 + x_2^2} \leq 1/\sqrt{2}\} \subset L$. Let us consider a lattice-tiling on the $x_3 \ldots x_n$-coordinate plane, by translates of some convex $(n-2)$-polytope $P$, by the vectors of some $(n-2)$-dimensional lattice $\Lambda$. Let $K := \{(x_1, x_2, x_3, \ldots, x_n) \in \mathbb{R}^n \mid (x_3, \ldots, x_n) \in P, \sqrt{x_1^2 + x_2^2} \leq f(x_3, \ldots, x_n)\}$, where $f : P \to [1/\sqrt{2}, 1]$ is a concave function with maximum 1. Then $K$ is a convex body, rotationally symmetric about the $x_3 \ldots x_n$-coordinate plane, whose translates by the vectors in $\Lambda$ have a union satisfying the required properties for $L$.

Observe that in Theorems 2.1, 2.3, and Proposition 2.6 we had packings of lower dimensional lattices of some convex bodies, where the existence of density was automatic. On the other hand, in Propositions 2.4, 2.5 we had to take care for the densities.

All the above statements will turn out to be rather simple consequences of the following Theorem 2.8. Of course, also in this theorem we have to take care for the density.

For this theorem we have to introduce some notations. Let $L \subset \mathbb{R}^n$ be rotationally symmetric with respect to the $x_3 \ldots x_n$-coordinate plane and be open, with $1/\sqrt{2} \leq m(L)$ and $M(L) = 1$.

\[
\begin{align*}
\text{Two translates of } L, \text{ say, } L + (x_1, x_2, x_3, \ldots, x_n) \text{ and } L + (y_1, y_2, y_3, \ldots, y_n) \text{ are said to touch each other if they are disjoint,} \\
\text{but for every } (x_1', x_2') \text{ and } (y_1', y_2') \text{ with } (y_1' - x_1')^2 + (y_2' - x_2')^2 < (y_1 - x_1)^2 + (y_2 - x_2)^2 \text{ we have } (L + (x_1', x_2', x_3, \ldots, x_n)) \cap (L + (y_1', y_2', y_3, \ldots, y_n)) \neq \emptyset.
\end{align*}
\]

\[
\begin{align*}
\text{We define the function } g(x_3, \ldots, x_n) \text{ as the minimal value of } |x_1|/2 \\
\text{equivalently: of } \sqrt{x_1^2 + x_2^2/2} \text{ such that } L \cap (L + (x_1, 0, x_3, \ldots, x_n)) = \emptyset \text{ (equivalently, such that } L \cap (L + (x_1, x_2, x_3, \ldots, x_n)) = \emptyset).
\end{align*}
\]

(This minimum exists for each $(x_3, \ldots, x_n) \in \mathbb{R}^{n-2}$, by openness of $L$.) Clearly for “touching” translates of $L$ the distance of their axes of rotation (translates of the $x_3 \ldots x_n$-coordinate plane) is at most 2. This is sharp: we have
\( m(L) = \inf g \leq \sup g = g(0, \ldots, 0) = M(L) = 1 \)

For \( L \subset \mathbb{R}^n \) with the above properties we consider some three translates of \( L \) mutually “touching” each other in the sense of (2.5) (provided these exist). Observe that here the 3’rd, \( \ldots \), \( n' \)th coordinates of the translation vectors \( (x_1, x_2, x_3, \ldots, x_n) \) and \( (x_1, x_2, x_3, \ldots, x_n) \) are fixed. This condition determines the distances of the axes of rotation of these three translates of \( L \).

\begin{align*}
\text{(2.8)} \quad \begin{cases}
\text{Let } V_0(L) \text{ denote the infimum of the areas of the triangles (provided these exist), with vertices the points of intersection of the axes of rotation of these three mutually touching translates of } L \text{ with the } \\
x_1x_2\text{-coordinate plane. Here the infimum is taken for all three } \\
\text{translates of } L \text{ with 3’rd, } \ldots, d' \text{th coordinates of the respective } \\
\text{translation vectors chosen arbitrarily in the } x_3 \ldots x_d\text{-coordinate plane.}
\end{cases}
\end{align*}

(Later it will turn out that in our paper the triangles in question will always exist.)

We bound the densities of packings of translates of \( L \) in \( \mathbb{R}^n \) from above.

For \( m(L) \geq 1/\sqrt{2} > 0 \) and a two-dimensional lattice packing \( \{L + \lambda \mid \lambda \in \Lambda\} \) of translates of \( L \), naturally no non-zero lattice vector of \( \Lambda \) is in the \( x_3 \ldots x_n\)-coordinate plane, and even in its open \( m(L)\)-neighbourhood, i.e., the point lattice \( \Lambda \) projects orthogonally to the \( x_1x_2\)-plane injectively onto a two-dimensional point lattice in the \( x_1x_2\)-plane.

We need a generalization of hypotheses (2.1) and (2.2) about density.

\begin{align*}
\text{(2.9)} \quad \begin{cases}
\text{Let } L \subset \mathbb{R}^n \text{ be rotationally symmetric with respect to the } \\
x_3 \ldots x_n\text{-coordinate plane and be open, with } \\
1/\sqrt{2} \leq m(L) \text{ and } M(L) = 1.
\end{cases}
\end{align*}

Then \( L \) lies in the cylinder \( L_0 \) with base a circle of radius 1 and centre the origin in the \( x_1x_2\)-coordinate plane, and with axis the \( x_3 \ldots x_n\)-coordinate plane (and here the radius 1 cannot be replaced by any smaller number). Then it makes sense to speak about the density of \( L \) with respect to \( L_0 \) (if it exists). We write \( V \) for volume (Lebesgue measure). Let us suppose
for any \((0,0,x_3,\ldots,x_n)\) in the \(x_3\ldots x_n\)-coordinate hyperplane, and for \(R \to \infty\), the quotient 

\[
V \left( \left\{ (\xi_1, \xi_2, \xi_3, \ldots, \xi_n) \in L \mid \sqrt{(\xi_3 - x_3)^2 + \cdots + (\xi_n - x_n)^2} \leq R \right\} \right) \\
/V \left( \left\{ (\xi_1, \xi_2, \xi_3, \ldots, \xi_n) \in L_0 \mid \sqrt{(\xi_3 - x_3)^2 + \cdots + (\xi_n - x_n)^2} \leq R \right\} \right)
\]
tends to some positive number \(d(L)\), and this convergence is uniform for all \((0,0,x_3,\ldots,x_n)\) in the \(x_3\ldots x_n\)-coordinate hyperplane.

**Theorem 2.8.** With the above notations, suppose the above hypotheses (2.9) and (2.10) about \(L \subset \mathbb{R}^n\). Then the supremum of the densities of all packings of translates of \(L\) in \(\mathbb{R}^n\) is the supremum of the densities of the two-dimensional lattice packings of translates of \(L\) — the corresponding point lattice orthogonally projecting to the \(x_1x_2\)-coordinate plane injectively onto a two-dimensional point lattice in the \(x_1x_2\)-coordinate plane — spanned by three mutually touching translates of \(L\) (in the sense of (2.5)). The maximal density of two-dimensional lattice packings of translates of \(L\), with corresponding point lattice as described above, is at least \(\pi d(L)/(2\sqrt{4m(L)^2 - 1})\).

Theorem 2.8 is itself a rather straightforward consequence of the following general theorem. Before stating it, we recall the definition of an \((r,R)\)-system of points in \(\mathbb{R}^n\), where \(0 < r < R < \infty\). We say that \(P = \{p_1, p_2, \ldots \} \subset \mathbb{R}^n\) is an \((r,R)\)-system, if the balls with these centres and radii \(r\) form a packing in \(\mathbb{R}^n\), and the balls with these centres and radii \(R\) form a covering of \(\mathbb{R}^n\).

**Theorem 2.9.** Let \(P \subset \mathbb{R}^2\) be an \((r,R)\)-system in \(\mathbb{R}^2\). Let \(R/r \leq 2\sqrt{2}\). Let us consider a Delone (Delaunay) triangulation of \(\mathbb{R}^2\) associated to \(P\) (if there are Delone-polygons with more than three sides, we triangulate them arbitrarily). Then the Delone triangles can be grouped in groups, such that a group can be

1. a single non-obtuse triangle \(T\), or
2. an obtuse triangle \(T\) of area at least \(2r^2\), or
3. can consist of one non-obtuse triangle \(T\) and one, two or three obtuse triangles \(T_i\), whose longest sides coincide with some sides of \(T\), these two, three or four triangles having an average area at least \(2r^2\), or
4. can consist of an obtuse triangle \(T\) and one or two further obtuse triangles \(T_i\), whose longest sides coincide with some sides of \(T\) but not with the longest side of \(T\), these two or three triangles having an average area at least \(2r^2\).

In particular, if the infimum of the areas of the non-obtuse Delone triangles is \(V_0\), then the average area of the Delone triangles is at least \(\min\{V_0, 2r^2\}\). This bound is sharp.
Conjecture 2.10. The next distance that can occur after 2 and $2\sqrt{2}$ inside one (hexagonal, square, or 2 by $2\sqrt{2}$ rectangular) layer or between different such layers is $2\sqrt{8/3}$.

(A) One can conjecture that for some $\varepsilon > 0$, for $d \in (\sqrt{2}, \sqrt{2} + \varepsilon]$ a densest packing of strings with distance $2d$ is obtained in the following way. Recall from Theorem 2.1 the densest packing of strings equidistant with distance $2\sqrt{2}$. This can be obtained as follows. We take a homogeneous point lattice in $\mathbb{R}^3$ with basis
\begin{align*}
(2, 0, 0), (0, 2, 0), (0, 0, 2\sqrt{2})
\end{align*}
whose basic cell is a $2 \times 2 \times 2\sqrt{2}$ rectangular box, say $B$. We add to the points of this lattice all centres of lattice translates of $B$, obtaining this way a new point lattice $\Lambda$. This is the point lattice corresponding to the densest lattice packing of unit balls in $\mathbb{R}^3$. In particular, the centres of the neighbours of the ball with centre $(0, 0, 0)$ are the following twelve points:
\begin{align*}
(\pm 2, 0, 0), (0, \pm 2, 0), (\pm 1, \pm 1, \pm \sqrt{2})
\end{align*}
(in the last case the plus-minus signs are independent of each other). Now let us apply the following linear transformation $T$ to our lattice $\Lambda$:
\begin{align*}
T(1, 0, 0) := (1, 0, 0),
T(0, 1, 0) := (0, 1, 0),
T(0, 0, \sqrt{2}) := (0, 0, d),
\end{align*}
with $d \in (\sqrt{2}, \sqrt{2} + \varepsilon]$ and $\varepsilon > 0$ being sufficiently small. Then the norms of all lattice vectors in the $xy$-plane are preserved, and the norms of all lattice vectors not in the $xy$-plane become longer. This means that the minimum vectors of the lattice $T\Lambda$ are
\begin{align*}
(\pm 2, 0, 0), (0, \pm 2, 0)
\end{align*}
and the norms of $T(\pm 1, \pm 1, \pm \sqrt{2})$ are just a bit larger than 2, while the norms of all other non-zero lattice vectors are greater than some number strictly greater than 2. Thus
\begin{align}
\begin{cases}
T\Lambda & \text{cannot be the point lattice corresponding to a densest lattice packing of the equidistant string of balls with distances of neighbouring ball centres } \sqrt{2}d.
\end{cases}
\end{align}

Namely, let us apply to the point lattice $T\Lambda$ another linear transformation $S$, defined as follows. We let $S(0, 0, 1) := (0, 0, 1)$, and $S(1, 0, 0) := (\cos \delta, \sin \delta, 0)$ and and $S(0, 1, 0) := (\sin \delta, \cos \delta, 0)$, where $\delta$ is small (depending on $\varepsilon$). Then $S$ carries the rectangular box $TB$ to a right prism $STB$ with base a rhomb with edge lengths 2 (which is close to a $2 \times 2$ square) and with height $2d$. Then the linear map $S$ carries the minimum vectors $(\pm 2, 0, 0)$ and $(0, \pm 2, 0)$ of the point lattice $T\Lambda$ to vectors of length 2, which we want to be minimum vectors of the lattice $STA$. Thus, for $\delta > 0$ sufficiently small (depending on $\varepsilon$) there arise no new minimum vectors, and $V(STB) < V(TB)$, showing (2.11). For $\varepsilon > 0$ sufficiently small, we want to investigate the images of $(\pm 1, \pm 1, \pm \sqrt{2})$ (said otherwise, the vectors pointing from the centre of $B$ to its vertices) by $ST$, to see how large $\delta > 0$ can be so that still these would have lengths at least 2. By symmetry in the $z$ coordinate, it suffices to investigate the images of $(\pm 1, \pm 1, \sqrt{2})$. By central symmetry of the base rhomb
it suffices to investigate only two neighbouring vertices of this rhomb. Actually, the vertex of the rhomb with an acute angle (e.g., \((0, 0, 0)\)) got farther from the string containing as one unit ball the one with centre the centre of \(STB\). So there no new touching can occur. However, the vertex of the rhomb with an obtuse angle (e.g., \(2(\cos \delta, \sin \delta, 0)\)) got closer to the above mentioned string. Namely, its distance to the axis of rotation of our string, i.e., to the centre of the rhomb, decreased to \((\cos \delta - \sin \delta)\sqrt{2}\). We want to have touching of the strings containing the unit ball with centre at \((0, 0, 0)\) and at the centre of \(STB\), i.e.,

\[
4 = 2(\cos \delta - \sin \delta)^2 + d^2, \quad \text{i.e.,} \quad 2\sin(2\delta) = d^2 - 2.
\]

In this case, the set of the intersection points of the axes of rotation of the strings with the \(xy\)-plane forms the vertices and centres of a lattice of the above rhombs. This is a \((\cos \delta + \sin \delta)\sqrt{2} \times (\cos \delta - \sin \delta)\sqrt{2}\) rectangular lattice. We conjecture that for \(\varepsilon > 0\) sufficiently small, the above constructed packing is the densest packing of translates of our string of unit balls, with centres equidistant with distances \(2d \in (2\sqrt{2}, 2(\sqrt{2} + \varepsilon))\). Its density is

\[
(2.13) \quad \frac{(4\pi/3)}{(4d\cos(2\delta))}.
\]

(B) For \(d = \sqrt{8/3}\) we have another picture. We take the non-lattice-like regular close packing of regular triangular lattices of unit balls with edge lengths 2, consisting of layers which are translates of one such layer with ball centres in the \(xy\)-plane. Then on each vertical line containing some ball centre the ball centres on them are equidistant, with distance \(\sqrt{8/3}\). If we take only every second layer, then the intersections of the rotation axes of the strings, which have a ball centre in these every second layers, with the \(xy\)-plane, form a lattice of regular triangles of edge length 2 in the \(xy\)-plane, joining along entire edges. This mosaic \(\{3, 6\}\) is also the Delone triangulation corresponding to these points in the \(xy\)-plane. If we consider the remaining every second layers, then the intersection points of the axes of rotation of strings with one ball centre in these other every second layers with the \(xy\)-plane will be the centres of all original regular triangles which are translates of each other (thus only of a “half” of all considered regular triangles in the \(xy\)-plane). Adding these new points, the triangles whose centres are not added have three edge-neighbour triangles whose centres are added. The triangle with centre not added remains an empty triangle, but actually its circumcircle passes also through the centres of its edge-neighbour triangles. Therefore the Delone triangulation of all these points consists of the vertices of a mosaic \(\{6, 3\}\), of regular hexagons with circumradius \(2/\sqrt{3}\).

(C) All this shows that for \(d\) in the interval \([\sqrt{2}, \sqrt{8/3}]\) the solution of the
densest packing of translates of our strings conjecturally will behave differently in several subintervals: it begins with a lattice packing, and ends with a non-lattice-like regular packing.

**Remark 2.11.** Analogously to the conjecture of [Sz] in §1, for L. Fejes Tóth’s question for arbitrary $d$ one has also the following construction. Let $L$ be a string of unit balls in $\mathbb{R}^3$, with centres on a straight line, which are equidistant with distance $2d$, where $d > \sqrt{2}$. (The case $d \in [1, \sqrt{2}]$ is settled in Proposition 2.5.) We choose a natural number $k$ and consider $k$ closely packed parallel hexagonal, or square, or $2 \times \sqrt{2}$ rectangular layers of unit balls, in the first case for any layer the following layer packed in any of the possible two ways (or possibly some other packings of unit balls with centres in some lattice plane of the densest lattice packing of balls). We translate this block of $k$ layers periodically so that the centre of a ball of the first layer of a block has a distance $2d$ from the centre of some ball of the first layer of the preceding block and has distances 2 from the centres of two balls of the last layer of the preceding block. Of course, this construction does not include the example from Conjecture 2.10 about $d \in (\sqrt{2}, \sqrt{2} + \varepsilon]$.

If $2d$ is the distance of the centres of two balls in some close packing of parallel hexagonal or square or $2 \times \sqrt{2}$ rectangular layers of unit balls (or possibly of some other packings of unit balls from the densest lattice packing of unit balls, with centres in some lattice plane), then evidently some periodic system of closely packed parallel such layers will be a densest packing. Of course, the problem for any $d$ is hopeful to be proved only for small values of $d$, since for $d \to \infty$ the solution of this problem would imply the theorem of T. C. Hales-S. Ferguson [HF] on the densest packing of unit balls in $\mathbb{R}^3$.

**Remark 2.12.** Also for some other problems, by estimating in some way the area of the non-obtuse Delone triangles, that depends on the particular problem, this theorem can be applied. As an example, we mention the following problem of L. Fejes Tóth [FT]. A *molecule in $\mathbb{R}^2$* is the union of a unit circle and of one or two circles of radius $r < 1$, such that they form a packing, and the unit circle touches the circle(s) of radius $r$. The position of two small circles is not determined, only the packing and touching properties are prescribed. For $r \leq 2/\sqrt{3} - 1 = 0.1547\ldots$ this problem is not interesting, since in the holes of a densest lattice packing of unit circles there is enough space for the small circles. By using some further ideas we can determine the densest packing of two-atom molecules, for $r \leq 0.1899\ldots$. (The so called $L^*$-decomposition introduced by J. Molnár [M77], [M78] also plays a role there.) This is attained, e.g., in the following case. We consider packings of unit circles, which consist of horizontal strings of touching copies of the unit circle. We enumerate these horizontal strings by the integers, so that the $(i + 1)$’st string is the upper neighbour of the $i$’th string. The neighbouring strings of unit circles touch. The $2i$’th and $(2i + 1)$’st strings are closely packed. The $(2i + 1)$’st
and \((2i + 2)\)’nd strings are not closely packed, but so that two neighbouring centres of the \((2i + 1)\)’st string of circles and some centre of the \((2i + 2)\)’nd string of circles form isosceles triangles with two sides of length 2, and with circumradius \(1 + r\). The small circles have centres exactly in the circumcentres of these last mentioned isosceles triangles. For three-atom molecules, for \(r - (2/\sqrt{3} - 1)\) sufficiently small, a densest packing consists of horizontal strings of touching unit circles, where any two neighbouring horizontal strings join to each other as the \((2i + 1)\)’st and \((2i + 2)\)’nd strings above. We hope to return to this problem later.

\section*{§ 3 Preparatory lemmas}

In this paragraph we will prove statements in \(\mathbb{R}^2\). We write \(\triangle ABC\) for the triangle \(ABC\) in \(\mathbb{R}^2\), and denote the area of a set in \(\mathbb{R}^2\) by \(V(\cdot)\). For a triangle \(\triangle ABC\) in \(\mathbb{R}^2\) the sides opposite to \(A, B, C\) will be denoted by \(a, b, c\), and the angles at \(A, B, C\) by \(\alpha, \beta, \gamma\).

We will frequently use the following elementary fact.

\[(3.0)\]

\[
\{ \begin{array}{l}
\text{For non-obtuse triangles the area is a strictly monotonous function of the side-lengths.}
\end{array}\]

In fact, if the side lengths are \(a \leq a', b \leq b', c \leq c'\) then by inflation we may attain, with the evident notations, that, e.g., \(c = c'\) (and \(A = A'\) and \(B = B'\)). Then \(C'\) has a distance from \(A\) and \(B\) at least \(b\) and \(a\), and projects orthogonally to a point of the side \(AB\), so its distance to the line \(AB\) is minimal if and only if \(C' = C\).

\textbf{Lemma 3.1.} Let \(p, c \in (0, \infty)\) with \(c \geq p\) be fixed. Let in a triangle \(\triangle ABC\) the side \(c\) be fixed, the vertex \(C\) be variable, and let \(\alpha \leq \pi/2, \beta \leq \pi/2, a \geq p, b \geq p\) and \(\gamma \leq \gamma_0\). If under these conditions the triangle \(\triangle ABC\) has a minimal area, then in two of the above five last inequalities we have equality.

Let us assume, in addition to the above hypotheses, that also \(\gamma \geq \pi/2\). If under these new conditions the triangle \(\triangle ABC\) has a minimal area, then in two of the inequalities \(a \geq p, b \geq p\) and \(\gamma \leq \gamma_0\) we have equalities.

\textbf{Proof.} By \(\alpha, \beta \leq \pi/2\) we have that our triangle lies in a parallel strip \(S\) with boundary lines passing through \(A, B\) and orthogonal to the side \(AB\). We may assume \(AB\) horizontal, and that \(C\) is above the side \(AB\) (the hypotheses imply that \(C\) lying on the line \(AB\) is impossible). The inequalities \(a \geq p, b \geq p\) and \(\gamma \leq \gamma_0\) express that the vertex \(C \in S\) does not lie strictly below the graphs of certain concave functions (for \(\gamma \leq \gamma_0\) we have to distinguish the cases of obtuse and non-obtuse \(\gamma_0\)). If \(C\) does not lie on any of these graphs, and also does not lie on the boundary lines of \(S\), then we can move it vertically downwards, decreasing \(V(\triangle ABC)\). If \(C\) lies only on one of these graphs, or boundary lines, then on this graph, or boundary line we can move \(C\) either increasing or decreasing its \(x\)-coordinate, or decreasing its \(y\)-coordinate, decreasing \(V(\triangle ABC)\).
Lemma 3.3. (Cf. also [Ka], proof of Theorem 14, formula 10.) Let the triangles $\triangle ABC$ and $\triangle A_1BC$ lie on the opposite sides of side $BC$, and let $\alpha + \alpha_1 \leq \pi$. Fixing $b, c, b_1, c_1$ decrease $\alpha$ and $\alpha_1$. Then the sum of the areas of $\triangle ABC$ and $\triangle A_1BC$ decreases.

Proof. We have $b^2 + c^2 - 2bc \cos \alpha = b_1^2 + c_1^2 - 2b_1c_1 \cos \alpha_1$, hence $d\alpha_1/d\alpha = (bc \sin \alpha)/(b_1c_1 \sin \alpha_1)$. Thus

$$d\alpha_1/d\alpha = (bc \sin \alpha + b_1c_1 \sin \alpha_1) = bc \sin \alpha (\cot \alpha + \cot \alpha_1) \geq 0.$$ (3.3.1)

Lemma 3.4. Let the triangles $\triangle ABC$ and $\triangle A_1BC$ lie on the opposite sides of their common side $BC$, and let $\alpha = \alpha_1 = \pi/2$. Fixing $b, c, b_1, c_1$ decrease $\alpha$ and $\alpha_1$ a bit, preserving $c_1 < a$. After this rotate side $A_1B$ about $B$, decreasing $\beta_1$, until $\alpha_1$ increases to $\pi/2$. Then $V(\triangle ABC) + V(\triangle A_1BC)$ decreases.
Proof. The first motion decreases the total area by Lemma 3.3. The second motion is possible if the first motion was sufficiently small, and during this second motion $V(\Delta BA_1C) = (1/2)a_1c_1\sin\beta_1$ decreases (and $V(\Delta ABC)$ remains constant).

Lemma 3.5. Let $p \in (0, \infty)$. Let the triangles $\Delta ABC$ and $\Delta A_1BC$ lie on the opposite sides of their common side $BC$, and let $c_1 = b_1 = p$, $p \leq b \leq p\sqrt{7/2}$, $\gamma \geq \pi/2$, $\alpha_1 \geq \pi/2$, $\alpha + \alpha_1 \leq \pi$, and the circumradius of $\Delta ABC$ is $p\sqrt{2}$. Then under these hypotheses the sum of the areas of our triangles is minimal only either for $\alpha_1 = \pi/2$ or for $\alpha + \alpha_1 = \pi$.

Proof. Let

\[(3.5.1) \quad \delta := \alpha_1/2 \in (\pi/4, \pi/2) \text{ and } \beta_0 = \arcsin\left(1/\sqrt{8}\right) = 20.7048 \ldots^{\circ}.
\]

We have $a/2 = p\sqrt{2}\sin\alpha$ and $a_1/2 = p\sin\delta$. But $a = a_1$, hence

\[(3.5.2) \quad \sin\delta = \sqrt{2} \cdot \sin\alpha.
\]

Also, by $\delta = \alpha_1/2 \geq \pi/4$ we have $a = a_1 = 2p\sin\delta \geq p\sqrt{2}$ and the circumradius of $\Delta ABC$ is $p\sqrt{2}$, hence

\[(3.5.3) \quad \alpha \geq \pi/6.
\]

Next we determine the upper bound for $\alpha$. By increasing $\alpha_1$, and thus $\delta$, but retaining the circumcircle of $\Delta ABC$ and $b$ (hence also $\beta$), also $a = a_1 = 2p\sin\delta < 2p$ increases. Then the line $BC$ gets closer to the circumcentre of $\Delta ABC$, but does not pass over the circumcentre, since $a < 2p$, while the diameter of the circumcircle is $2p\sqrt{2}$, thus is larger than $a$. Also the smaller half central angle $\alpha$ subtended by the side $AB$ increases. At the same time the distance of $A_1$ to the line $BC$, i.e., $p\cos\delta$ decreases, so $A_1$ gets ever closer to the circumcircle. For a certain value of $\alpha_1$, the point $A_1$ will lie on the boundary of the circumcircle, and then we cannot increase $\alpha_1$ this way any more (by $\alpha + \alpha_1 \leq \pi$). Then the quadrangle $BA_1CA$ has a circumcircle, $K$, say. During this motion the order of the points $A, B, C$ on $K$ does not change: namely, the smaller central angles subtended both by the sides $BC$ and $CA$ remain less than $\pi$. Moreover, the smaller central angles subtended by the sides $BA_1$ and $A_1C$ in $K$ (which are half the smaller central angle subtended by $AB$ in $K$) will be $2\arcsin\left((p/2)/(p\sqrt{2})\right) = 2\arcsin(1/\sqrt{8})$. Then $\alpha$ is the smaller central angle subtended by the chord $BA_1$ (or $A_1C$) of length $p$, therefore after this increasing $\alpha$ becomes $2\arcsin(1/\sqrt{8}) = 2\beta_0$. That is,

\[(3.5.4) \quad \pi/6 \leq \alpha \leq 2\beta_0.
\]

Similarly, by $b \geq p$ we have that $\beta$ is half the smaller central angle
subtended in $K$ by a chord of length in $[p, p\sqrt{7}/2]$. That is, $\beta$ is at least half the smaller central angle subtended in $K$ by a chord of length $p$, i.e., $\beta \geq \arcsin \beta_0$. Similarly, $\beta$ is at most half the smaller central angle subtended in $K$ by a chord of length $p\sqrt{7}/2$, i.e., $\beta \leq \arcsin(\sqrt{7}/4) = 2\beta_0$. That is,

$$(3.5.5) \quad \beta_0 \leq \beta \leq 2\beta_0.$$  

From the upper bounds for $\alpha, \beta$ in (3.5.4) and (3.5.5) we get

$$(3.5.6) \quad \gamma \geq \pi - 4\beta_0 > \pi/2.$$  

By (3.5.2) we have

$$(3.5.7) \quad d\delta/d\alpha = \sqrt{2} \cdot \cos \alpha/\cos \delta.$$  

The sum $S$ of the areas of the two triangles $\Delta ABC$ and $BA_1C$ is

$$(3.5.8) \quad \begin{cases} S = ab \sin(\alpha + \beta)/2 + p^2 (\sin(2\delta))/2 = \\ 2(p\sqrt{2})^2 \sin \alpha \sin \beta \sin(\alpha + \beta)/2 + p^2 (\sin(2\delta))/2 = \\ 2p^2 \sin \beta (\cos(\beta) - \cos(2\alpha + \beta)) + p^2 (\sin(2\delta))/2 \end{cases}$$

that depends on $\alpha$ and $\beta$ (recall that $\delta$ is a function of $\alpha$ by (3.5.10)). However, we will consider $\beta$ as fixed, and then $S$ is a function of $\alpha$ only. We have, also using (3.5.7),

$$(3.5.9) \quad \frac{dS}{d\alpha} = 4p^2 \sin \beta \sin(\beta + 2\alpha) + p^2 (\sin(2\delta)) \sqrt{2} \frac{\cos \alpha}{\cos \delta}.$$  

Once more using (3.5.2), we see that this has the same sign as

$$(3.5.10) \quad \begin{cases} T := 2\sqrt{2} \sin \beta \sin(\beta + 2\alpha) \cos \delta + \cos(2\delta) \cos \alpha = \\ 2\sqrt{2} \sin \beta \sin(\beta + 2\alpha) \sqrt{\cos(2\alpha)} - (4 \sin^2 \alpha - 1) \cos \alpha. \end{cases}$$  

Observe that under the square root we have by $2\alpha \leq 4\beta_0 < \pi/2$ a positive number, and by (3.5.4) and (3.5.5) $\beta_0 + \pi/3 \leq \beta + 2\alpha \leq 6\beta_0$, so $\sin(\beta + 2\alpha) > 0$.

We are going to show that $T$ is strictly monotonically decreasing for $\pi/6 \leq \alpha \leq 2\beta_0$ (cf. (3.5.4)). Thus either it has always positive or always negative sign, or below some value of $\alpha$ it is positive, and above this value of $\alpha$ it is negative. In any of these three cases the same holds for $dS/d\alpha$, so

$$(3.5.11) \quad S \text{ attains its minimum only for } \alpha = \pi/6 \text{ or for } \alpha = 2\beta_0.$$  

Taking in account (3.5.2) we have that the equalities $\pi/6 = \alpha$
and $\alpha = 2\beta_0$ are equivalent to $\alpha_1 = \pi/2$, and to $\alpha_1/2 = \arcsin \sqrt{7/8}$ when simultaneously $\sin(\alpha/2) = \sin \beta_0 = 1/\sqrt{8}$ and thus $\alpha_1 + \alpha = \pi$.

We write

\begin{equation}
T_1 := \sin^2(\beta + 2\alpha) \cos(2\alpha) = (1 - \cos(2(\beta + 2\alpha))) \cos(2\alpha)/2
\end{equation}

and

\begin{equation}
T_2 := (4\sin^2 \alpha - 1) \cos \alpha.
\end{equation}

Then

\begin{equation}
T = 2\sqrt{2}\sin \beta \sqrt{T_1 - T_2}.
\end{equation}

Since $\beta$ is considered as fixed, and only $\alpha$ varies,

\begin{equation}
\begin{cases}
\text{the strictly monotonically decreasing property of } T \\
\text{for } \pi/6 \leq \alpha \leq 2\beta_0 \text{ follows from the strictly monotonically decreasing property of } T_1 \text{ and from the} \\
\text{strictly monotonically increasing property of } T_2, \\
\text{also for } \pi/6 \leq \alpha \leq 2\beta_0.
\end{cases}
\end{equation}

We are going to show these strict monotonicity properties.

We have (using for the first term the second form of $T_1$ and for the second term the first form of $T_1$)

\begin{equation}
dT_1/d\alpha = 2 \sin(2(\beta + 2\alpha)) \cos(2\alpha) - 2\sin^2(\beta + 2\alpha) \sin(2\alpha).
\end{equation}

For $\beta + 2\alpha > \pi/2$ (also noting $\beta + 2\alpha \leq 6\beta_0 = 124.2288\ldots$) this is negative by $\pi/3 \leq 2\alpha \leq 4\beta_0 = 82.8192\ldots < \pi/2$. For $\beta + 2\alpha \leq \pi/2$ we have by $\pi/3 \leq 2\alpha \leq 4\beta_0 < \pi/2$ and $80.7048\ldots = \beta_0 + \pi/3 \leq \beta + \pi/3 \leq \beta + 2\alpha \leq \pi/2$ that

\begin{equation}
\begin{cases}
dT_1/d\alpha \leq 2 \cdot 1 \cdot \cos(\pi/3) - 2\sin^2(\beta + 2\alpha) \cdot \sin(\pi/3) \leq \\
1 - \sin^2(\beta_0 + \pi/3) \cdot \sqrt{3} = 1 - \sqrt{3} \cdot 0.9739\ldots < 0.
\end{cases}
\end{equation}

Hence $T_1$ strictly monotonically decreases for $\pi/6 \leq \alpha \leq 2\beta_0$.

For $T_2$ we have by $\pi/6 \leq \alpha \leq 2\beta_0 = 41.4096\ldots$ that

\begin{equation}
\begin{cases}
dT_2/d\alpha = 4 \sin(2\alpha) \cdot \cos \alpha - (4\sin^2 \alpha - 1) \sin \alpha \geq \\
4 \sin(\pi/3) \cdot \cos(2\beta_0) - (4\sin^2(2\beta_0) - 1) \cdot \sin(2\beta_0) = \\
2\sqrt{3} \cdot (3/4) - (3/4) \cdot (\sqrt{7}/4) > 0.
\end{cases}
\end{equation}
Hence $T_2$ strictly monotonically increases for $\pi/6 \leq \alpha \leq 2\beta_0$.

By (3.5.15) these strict monotonicity properties of $T_1$ and $T_2$ imply the strictly monotonically decreasing property of $T$ for $\pi/6 \leq \alpha \leq 2\beta_0$. This shows (3.5.11), that is just the statement of this lemma. ■

The following Lemmas 3.6 and 3.8 are related to [BKM], Lemma 3.1 — however, in our lemmas also the circumradius of the polygon is variable.

**Lemma 3.6.** Let a convex pentagon $P$ have a circumcircle and let it contain its circumcentre. Let each side of $P$ have a length at least $p > 0$. Then the area of $P$ is at least $p^2 \cdot \left(\frac{5}{4}\right) \cot(\pi/10) = p^2 \cdot 1.7204\ldots$. Equality holds if and only if $P$ is a regular pentagon of side length $p$.

**Proof.** We may suppose that $P$ is an extremal pentagon (which exists, since $V(P)/p^2$ is similarity invariant).

Let us denote the circumradius of our pentagon $P$ by $R$, and the angles subtended by the sides of $P$ at its circumcentre $O$ by $\varphi_i$, where $1 \leq i \leq 5$. Since $P$ contains $O$, for $1 \leq i \leq 5$ we have $\varphi_i \in [0, \pi]$. Let us write

\[(3.6.1) \quad \varphi_0 := 2 \arcsin \left(\frac{p}{2R}\right) > 0, \text{ i.e., } R = \frac{p}{2 \sin(\varphi_0/2)}.\]

Since each side of $P$ has a length at least $p$, for $1 \leq i \leq 5$ we have $\varphi \geq \varphi_0$. Thus

\[(3.6.2) \quad \text{for } 1 \leq i \leq 5 \text{ we have } \varphi_i \in [\varphi_0, \pi] \text{ and } \sum_{i=1}^5 \varphi_i = 2\pi.\]

We have

\[(3.6.3) \quad \begin{cases} V(P)/p^2 = (R^2/2) \sum_{i=1}^5 \sin \varphi_i / (2R \sin(\varphi_0/2))^2 \\ = \left(\sum_{i=1}^5 \sin \varphi_i / (8 \sin^2(\varphi_0/2))\right). \end{cases}\]

Observe that the function $\sin \varphi$ is a strictly concave function on $[0, \pi]$. This implies that if $V(P)/p^2$ is maximal then

\[(3.6.4) \quad \text{there is at most one } \varphi_i \text{ in the open interval } (\varphi_0, \pi).\]

In fact if there were two such ones then preserving their sum we could increase their difference a bit, and then $V(P)/p^2$ would decrease.

There cannot be two $\varphi_i$'s equal to $\pi$, since then we would have $\sum_{i=1}^5 \varphi_i > 2\pi$. So either

1. there is one $i$ with $\varphi = \pi$, or
(2) there is no \( i \) with \( \varphi = \pi \).

1. We begin with the proof of case (1). Let, e.g., \( \varphi_5 = \pi \). Then

\[
\text{(3.6.5) for } 1 \leq i \leq 4 \text{ we have } \varphi_i \in [\varphi_0, \pi) \text{ and } \sum_{i=1}^{4} \varphi_i = \pi,
\]

and we have to minimize \( \sum_{i=1}^{5} \sin \varphi_i / (8 \sin^2(\varphi_0/2)) = \sum_{i=1}^{4} \sin \varphi_i / (8 \sin^2(\varphi_0/2)) \).

By (3.6.4) we have, e.g., that \( \varphi_1 = \varphi_2 = \varphi_3 = \varphi_0 \), and then \( \varphi_0 \leq \varphi_4 = \pi - \sum_{i=1}^{3} \varphi_i = \pi - 3\varphi_0 \), from which there follows

\[
\text{(3.6.6) } \varphi_0 \in (0, \pi/4].
\]

We have by (3.6.3)

\[
\text{(3.6.7) } \begin{cases}
V(P)/p^2 = (3 \sin \varphi_0 + \sin(\pi - 3\varphi_0)) / (8 \sin^2(\varphi_0/2)) \\
= [(3 \sin \varphi_0) / (8 \sin^2(\varphi_0/2))] + \\
[\sin \varphi_0 \cdot (3 - 4 \sin^2 \varphi_0) / (8 \sin^2(\varphi_0/2))] .
\end{cases}
\]

We investigate both summands in the last expression in (3.6.7). Its first summand is

\[
\text{(3.6.8) } (3 \sin \varphi_0) / (8 \sin^2(\varphi_0/2)) = (3/4) \cot(\varphi_0/2),
\]

which is a positive strictly decreasing function of \( \varphi_0 \in (0, \pi/4] \). Its second summand is

\[
\text{(3.6.9) } \begin{cases}
\sin \varphi_0 \cdot (3 - 4 \sin^2 \varphi_0) / (8 \sin^2(\varphi_0/2)) \\
= (1/8) (\sin \varphi_0 / \sin^2(\varphi_0/2)) (3 - 4 \sin^2 \varphi_0) \\
= (1/4) \cot(\varphi_0/2)(3 - 4 \sin^2 \varphi_0)
\end{cases}
\]

Here both \( \cot(\varphi_0/2) \) and \( (3 - 4 \sin^2 \varphi_0) \) are positive strictly decreasing functions of \( \varphi_0 \in (0, \pi/4] \), hence the last expression in (3.6.9) is a positive strictly decreasing function of \( \varphi_0 \in (0, \pi/4] \). Since we know the same for the last expression in (3.6.8), by (3.6.6) \( V(P)/p^2 \) in (3.6.7) attains its minimum exactly for \( \varphi_0 = \pi/4 \), and this minimum is \( 1 + \sqrt{2} = 2.4142 \ldots \). This is greater than \( (5/4) \cot(\pi/10) = 1.7204 \ldots \), and this ends the proof of case (1) of this Lemma.

2. We turn to the proof of case (2). In case (2)
for $1 \leq i \leq 5$ we have $\varphi_i \in [\varphi_0, \pi)$ and $\sum_{i=1}^{5} \varphi_i = 2\pi$,

and we have to minimize $\sum_{i=1}^{5} \sin \varphi_i / (8 \sin^2(\varphi_0/2))$.

By (3.6.4) we have, e.g., that $\varphi_1 = \varphi_2 = \varphi_3 = \varphi_4 = \varphi_0$, and then by (2) $\pi > \varphi_5 = 2\pi - \sum_{i=1}^{4} \varphi_i = 2\pi - 4\varphi_0$, from which there follows $\varphi_0 > \pi/4$. Also we have $2\pi = \sum_{i=1}^{5} \varphi_i \geq 5\varphi_0$, from which there follows $\varphi_0 \leq 2\pi/5$. That is, we have

$$\varphi_0 \in (\pi/4, 2\pi/5] .$$

We have (like in (3.6.8))

$$V(P)/p^2 = \frac{4 \sin \varphi_0 + \sin(2\pi - 4\varphi_0))}{8 \sin^2(\varphi_0/2)} =$$

$$\frac{4 \sin \varphi_0 + 4 \sin \varphi_0 \cdot (\cos \varphi_0 - 2 \cos^3 \varphi_0)}{8 \sin^2(\varphi_0/2)} =$$

$$\cot(\varphi_0/2) \cdot (1 + \cos \varphi_0 - 2 \cos^3 \varphi_0) =$$

$$\cot(\varphi_0/2) \cdot (1 - \cos \varphi_0)(1 + 2 \cos \varphi_0 + 2 \cos^2 \varphi_0) =$$

$$\sin \varphi_0 \cdot (1 + 2 \cos \varphi_0 + 2 \cos^2 \varphi_0) .$$

We are going to show that for $\varphi_0 \in (\pi/4, 2\pi/5]$ the last expression in (3.6.12) is decreasing. Its derivative with respect to $\varphi_0$ is the cosine polynomial

$$-2 - 3 \cos \varphi_0 + 4 \cos^2 \varphi_0 + 6 \cos^3 \varphi_0 = (2 \cos^2 \varphi_0 - 1)(2 + 3 \cos \varphi_0) .$$

By (3.6.11) the first factor of the last expression in (3.6.13) is negative, and the second factor is positive, hence the last expression in (3.6.13) is negative. That is, for $\varphi_0 \in (\pi/4, 2\pi/5]$ the last expression in (3.6.2) is strictly decreasing. Hence its minimum is attained exactly for $\varphi_0 = 2\pi/5$, and its value is $(5/4) \cot(\pi/10)$, as asserted in the Lemma. This ends the proof of case (2) of this Lemma.

The case of equality can be checked from the respective steps of the proof (actually one has to consider only case (2), and there we had a unique minimum). This ends the proof of this lemma. ■

**Lemma 3.7.** Let $ABA_1C$ be a convex quadrangle having a circumcircle, of radius at most $p\sqrt{2}$. Let the side $AB$ be a diameter of the circumcircle, and let $|BA_1|, |A_1C| \geq p$ and $|CA| \geq p\sqrt{2}$. Then the area of $ABA_1C$ is at least $p^2 \cdot 1.4048 \ldots.
Proof. Let $R$ denote the circumradius of $ABA_1C$. We have $\angle BA_1C > \angle BA_1A = \pi/2$, hence by $|BA_1|, |A_1C| \geq p$ we have $|BC| > p\sqrt{2}$. Then, also using the hypothesis of the lemma, in the right triangle $\Delta ABC$ both legs have lengths at least $p\sqrt{2}$, hence

$$(3.7.1) \quad 2R = |AB| \geq p\sqrt{2} \cdot \sqrt{2} = 2p.$$  

This implies, by the hypothesis of the Lemma that

$$(3.7.2) \quad p \leq R \leq p\sqrt{2}.$$  

If fixing $A, B$, we move $C$ on the circumcircle, so that $|BC|, |CA| \geq p\sqrt{2}$, then the minimum of $V(\Delta ABC)$ occurs if $\min\{|BC|, |CA|\} = p\sqrt{2}$, and the value of this minimum is

$$(3.7.3) \quad (1/2) \cdot p\sqrt{2} \cdot \sqrt{4R^2 - 2p^2}.$$  

On the other hand, the area of $\Delta BA_1C$ can be estimated from below by $R$ and $p$ from Lemma 3.2 ($\Delta ABC$ in Lemma 3.2 corresponding to $\Delta BA_1C$ here, $a_0$ and $b_0$ in Lemma 3.2 corresponding to $p$ here, and $R_0$ in Lemma 3.2 corresponding to $R$ here). Then the minimal area $V(\Delta BA_1C)$ occurs for $|BA_1| = |A_1C| = p$. Then $\Delta BA_1C$ is an isosceles triangle, with $\angle BA_1C = \pi - \varphi$, where $\varphi$ is the (smaller) central angle in the circle of radius $R$ corresponding to a chord of length $p$. That is, this minimum is

$$(3.7.4) \quad (1/2)p^2 \sin(\pi - \varphi),$$  

where, also using (3.7.2), we have

$$(3.7.5) \quad \sin(\varphi/2) = p/(2R) \in [1/(2\sqrt{2}), 1/2].$$  

This implies that for $R \in [p, p\sqrt{2}]$ we have that $\varphi \in (0, \pi/3]$ is a decreasing function of $R$, and thus also $\sin \varphi$ is a decreasing function of $R$.

Then

$$(3.7.6) \quad V(ABA_1C) = V(\Delta BA_1A) + V(\Delta A_1CA) \geq \sqrt{2R^2 - p^2} + (1/2)p^2 \sin \varphi.$$  

By homogeneity, we may assume for simplicity $p = 1$. Then by (3.7.4) it remains to prove

$$(3.7.7) \quad \sqrt{2R^2 - 1} + (1/2) \sin \varphi \geq 1.4048 \ldots.$$  

Here the first summand is an increasing, and the second summand is a decreasing function of $R \in [p, p\sqrt{2}] = [1, \sqrt{2}]$.

We cover the interval $[1, \sqrt{2}]$ by the intervals $[1 + (i - 1) \cdot 0.1, 1 + i \cdot 0.1]$, where $1 \leq i \leq 5$. We show the inequality (3.7.7) for $R$ in the larger interval $[1, 1.5]$.

In the $i$'th interval we estimate from below $\sqrt{2R^2 - 1}$ by its value in the left hand endpoint of this interval, and $(1/2)\sin \varphi$ by its value in the right hand endpoint of this interval. Thus for their sum we obtain five lower estimates in the five subintervals (sums of the lower estimates of the two summands), whose values are $1.4048 \ldots, 1.5704 \ldots, 1.7261 \ldots, 1.8763 \ldots, 2.0230 \ldots$. Then the smallest of these values, i.e., $1.4048 \ldots$ is a lower bound of the left hand side of (3.7.7) in the whole interval $[1, 1.5]$, thus also in the smaller interval $[1, \sqrt{2}]$. This proves the lemma. ■

**Lemma 3.8.** Let a convex pentagon $P$ have a circumscribed circle and let the circumcentre of $P$ be not an interior point of $P$. Let each side of $P$ have a length at least $p$ ($> 0$), and let the circumradius $R$ of $P$ be at most $p\sqrt{2}$. Then the area of $P$ is at least $p^2 \cdot 2.3977 \ldots$. Equality holds if and only if $R = p\sqrt{2}$, and four sides of $P$ have length $p$ (with the line spanned by the fifth side separating, not strictly, $P$ and its circumcentre).

**Proof.** We may suppose that $P = A_1A_2A_3A_4A_5$ is an extremal pentagon (which exists, since $V(P)/p^2$ is similarity invariant).

Let us denote the circumradius of $P$ by $R$, and the angles subtended by the sides $A_iA_{i+1}$ of $P$ at its circumcentre $O$ by $\varphi_i$, where $1 \leq i \leq 5$. We may suppose that the line spanned by the side $A_5A_1$ separates, not strictly, $P$ and $O$. This implies

$$\sum_{i=1}^{4} \varphi_i \leq \pi. \tag{3.8.1}$$

Since by hypothesis $R \leq p\sqrt{2}$, we write

$$\varphi_0 := 2 \arcsin \left( \frac{p}{2R} \right) \geq 2 \arcsin \left( \frac{1}{2\sqrt{2}} \right) = 41.4096 \ldots \circ, \quad \text{thus} \quad R = \frac{p}{2 \sin(\varphi_0/2)}. \tag{3.8.2}$$

Since each side has a length at least $p$, for $1 \leq i \leq 4$ we have

$$\varphi_1, \varphi_2, \varphi_3, \varphi_4 \geq \varphi_0 \quad \text{for} \quad 1 \leq i \leq 4, \tag{3.8.3}$$

which implies by (3.8.1)

$$\varphi_0 \leq \pi/4. \tag{3.8.4}$$
Thus, also using (3.8.1),

\[(3.8.5) \quad \text{for } 1 \leq i \leq 4 \text{ we have } \varphi_i \in [\varphi_0, \pi) \text{ and } \varphi_5 = 2\pi - \sum_{1 \leq i \leq 4} \varphi_i \in [\pi, 2\pi)\]

We have

\[(3.8.6) \quad V(P)/p^2 = (R^2/2) \left( \sum_{i=1}^{4} \sin \varphi_i - \sin \left( \sum_{i=1}^{4} \varphi_i \right) \right) / (2R \sin(\varphi_0/2))^2.\]

Our pentagon \( P \) is contained in a half-circle of its circumcircle, and on the corresponding half boundary of the circumcircle the Euclidean and angle distances depend strictly monotonically on each other. Therefore, supposing \( \varphi_1 > \varphi_0 \), we may move \( A_1 \) toward \( A_2 \) in the boundary of the circumcircle, and then \( V(P) = V(A_2A_3A_4A_5) + V(\Delta A_1A_2A_5) \) strictly decreases (since \( |A_1A_5| \) and \( \angle A_1A_5A_2 \) decrease). Observe that by permuting the angles \( \varphi_i \) (i.e., the sides \( A_iA_{i+1} \) of \( P \)) for \( 1 \leq i \leq 4 \) the area \( V(P) \) does not change. This implies that for our extremal \( P \) we have

\[(3.8.7) \quad \varphi_i = \varphi_0 \text{ for } 1 \leq i \leq 4,\]

and

\[(3.8.8) \quad V(P)/p^2 = (4 \sin \varphi_0 - \sin(4\varphi_0)) / (8 \sin^2(\varphi_0/2)).\]

From (3.6.12) we have

\[(3.8.9) \quad V(P)/p^2 = \sin \varphi_0 \cdot (1 + 2 \cos \varphi_0 + 2 \cos^2 \varphi_0).\]

The derivative of (3.8.9) with respect to \( \varphi_0 \) is from (3.6.13)

\[(3.8.10) \quad (2 \cos^2 \varphi_0 - 1)(2 + 3 \cos \varphi_0).\]

By (3.8.2) and (3.8.4) we have

\[(3.8.11) \quad \varphi_0 \in [2 \arcsin \left( 1/(2\sqrt{2}) \right), \pi/4],\]

hence (3.8.10) is positive for \( \varphi_0 \neq \pi/4 \). Thus (3.8.9) is a strictly increasing function of \( \varphi_0 \) on the interval in (3.8.11), i.e., is a strictly decreasing function of \( R \) (cf. (3.8.2)), hence its minimum is attained for the maximal value \( p\sqrt{2} \) of \( R \).
The uniqueness of the extremal pentagon $P$ follows from the proof. ■

**Lemma 3.9.** Let a packing consisting of the triangles $\Delta ABC$, $\Delta BA_1C$, $\Delta CB_2A$ satisfy the side hypothesis and the angular hypothesis, and let $c_1 = b_1 = a_2 = c_2 = p$. Let the circumradius of $\Delta ABC$ be some fixed $R \in [p, p\sqrt{2}]$. Let the (smaller) central angles corresponding to chords of the circumcircle of $\Delta ABC$ of lengths $a$, $b$ and $p$ be $\varphi_A$, $\varphi_B$ and $\varphi_0 = 2\arcsin(p/(2R))$. Let $\varphi_A, \varphi_B \in [\varphi_{\min}, \varphi_{\max}] := [2\arcsin((p\sqrt{2})/(2R)), 4\arcsin(p/(2R))]$. Then $V(\Delta ABC) + V(\Delta BA_1C) + V(\Delta CB_2A)$, as a function of the variables $\varphi_A, \varphi_B$, is a strictly concave function both of $\varphi_A$ and of $\varphi_B$. In particular, its minimum is attained at some vertex of the square $[\varphi_{\min}, \varphi_{\max}] \times [\varphi_{\min}, \varphi_{\max}]$, and only there.

**Proof.** We have

$$V(\Delta ABC) = (R^2/2) (\sin \varphi_A + \sin \varphi_B - \sin(\varphi_A + \varphi_B)),$$

(3.9.1)

$$V(\Delta BA_1C) = R \sin(\varphi_A/2) \sqrt{p^2 - R^2 \sin^2(\varphi_A/2)}$$

(3.9.2)

and

$$V(\Delta CB_2A) = R \sin(\varphi_B/2) \sqrt{p^2 - R^2 \sin^2(\varphi_B/2)},$$

(3.9.3)

where the expressions under the square root signs are positive. Adding these inequalities, and using the identity $\sin^2(t/2) = (1 - \cos t)/2$, and the notation

$$c := (2p^2 - R^2)/(2R^2) \ (\in [0, 1/2])$$

(3.9.4)

we get

$$\begin{cases}
(V(\Delta ABC) + V(\Delta BA_1C) + V(\Delta CB_2A)) / (R^2/2) = \sin \varphi_A + \sin \varphi_B \\
- \sin(\varphi_A + \varphi_B) + 2 \sin(\varphi_A/2) \sqrt{c + (\cos \varphi_A)/2} + 2 \sin(\varphi_B/2) \sqrt{c + (\cos \varphi_B)/2}.
\end{cases}$$

(3.9.5)

The partial derivative of the right hand side of (3.9.5) with respect to $\varphi_A$ is

$$\begin{cases}
\cos \varphi_A - \cos(\varphi_A + \varphi_B) + \cos(\varphi_A/2) \sqrt{c + (\cos \varphi_A)/2} - \\
\sin(\varphi_A/2) / \sqrt{c + (\cos \varphi_A)/2} = \cos \varphi_A - \cos(\varphi_A + \varphi_B) + \\
(\cos(\varphi_A/2) \cdot c + \cos(\varphi_A/2) \cdot (\cos \varphi_A)/2 - \sin(\varphi_A/2) \cdot (\sin \varphi_A)/2) / \sqrt{c + (\cos \varphi_A)/2} = (\cos \varphi_A - \cos(\varphi_A + \varphi_B)) + \\
(\cos(\varphi_A/2) \cdot c + (\cos(3\varphi_A)/2)) / \sqrt{c + (\cos \varphi_A)/2},
\end{cases}$$

(3.9.6)
where the expressions under the square root signs are positive. (The case of $\varphi_B$ is analogous.) We have to show that (3.9.6) is a strictly decreasing function of $\varphi_A$, i.e., that its partial derivative with respect to $\varphi_A$ is negative. The last expression of (3.9.6) is a sum of two summands, the second one being a quotient. Here $\varphi_A$ varies in the interval

\[(3.9.7) \ [\varphi_{\text{min}}, \varphi_{\text{max}}] = [2 \arcsin \left(\frac{p\sqrt{2}}{2R}\right), 4 \arcsin \left(\frac{p}{2R}\right)] \subset \left[\frac{\pi}{3}, \frac{2\pi}{3}\right].\]

The derivative of the first summand of the last expression of (3.9.6) with respect to $\varphi_A$ is

\[(3.9.8) \ \sin(\varphi_A + \varphi_B) - \sin \varphi_A .\]

Observe that (3.9.7) implies

\[(3.9.9) \ \varphi_A, \varphi_B \in \left[\frac{\pi}{3}, \frac{2\pi}{3}\right] \text{ and so } \varphi_A + \varphi_B \in \left[\frac{2\pi}{3}, \frac{4\pi}{3}\right],\]

which in turn implies

\[(3.9.10) \ \sin(\varphi_A + \varphi_B) \leq \sqrt{3}/2 \leq \sin \varphi_A ,\]

which in turn implies that (3.9.8) is nonpositive, and so

\[(3.9.11) \ \text{the first summand of the last expression of (3.9.6) is non-increasing.}\]

We turn to the derivative of the second summand of the last expression of (3.9.6) with respect to $\varphi_A$. This summand is a quotient of the form

\[(3.9.12) \frac{f(\varphi_A)}{\sqrt{g(\varphi_A)}},\]

with $g(\varphi_A) > 0$. Its derivative with respect to $\varphi_A$ is $(f'(\varphi_A)g(\varphi_A) - (1/2)f(\varphi_A)g'(\varphi_A)) / g(\varphi_A)^{3/2}$. So we have to show

\[(3.9.13) \ f'(\varphi_A)g(\varphi_A) - (1/2)f(\varphi_A)g'(\varphi_A) \leq 0 .\]

Writing out (3.9.13) in detail, we have to show

\[(3.9.14) \ \left\{ (-1/2) \sin(\varphi_A/2) \cdot c - (3/4) \sin(3\varphi_A/2) \right) \cdot (c + (\cos \varphi_A)/2) + (1/2) \cdot (\cos(\varphi_A/2) \cdot c + (\cos(3\varphi_A/2))/2) \cdot ((\sin \varphi_A)/2) \leq 0 .\]

Now we investigate the signs of the factors of the two summands of (3.9.14). By (3.9.9)
\[\varphi_A \in [\pi/3, 2\pi/3]\] and so \[3\varphi_A/2 \in [\pi/2, \pi],\]

which implies with (3.9.4) (and (3.9.7)) that the first factor of the first summand is nonpositive (is actually negative for \(\varphi_A \in (\varphi_{\text{min}}, \varphi_{\text{max}})\)) and the third factor of the second summand of (3.9.14) is positive.

Next we investigate the second factor of the first summand of (3.9.14). By hypothesis of this lemma \(\varphi_A \leq 4 \arcsin(p/(2R))\) (\(\leq 4 \cdot \pi/6\)). Then, since the arcsin function is monotonically increasing on \([0, \sqrt{3}/2]\) (\(\subset [0, 1]\)), we have

\[
\begin{align*}
\cos^2 \theta &\geq \arcsin^2(p/(2R)) = \\
&= 1 - 2 \cdot \left(\frac{p}{2R}\right) \sqrt{1 - \left(\frac{p}{2R}\right)^2} = 1 - 2(p/R)^2 + (1/2)(p/R)^4.
\end{align*}
\]

Then, recalling the notation (3.9.4), the second factor of the first summand of (3.9.14) is positive, since we have, writing

\[d := (p/R)^2 \in [1/2, 1]\] (cf. the hypothesis of the lemma),

and using (3.9.16), that

\[
d + \cos \varphi_A/2 \geq d - 1/2 + (1 - 2d + d^2)/2 = d^2/4 \geq 1/16 > 0.
\]

Next we investigate the second factor of the second summand of (3.9.14). By hypothesis of this lemma and by (3.9.7) we have

\[
\pi/6 \leq \arcsin\left(p\sqrt{2}/(2R)\right) = \varphi_{\text{min}}/2 \leq \varphi_A/2 \leq \varphi_{\text{max}}/2 \leq \pi/3.
\]

Then, since the cosine function is monotonically decreasing on \([\pi/6, \pi/3]\) (\(\subset [0, \pi/2]\)), we have

\[
0 < 1/2 \leq \cos(\varphi_A/2) \leq \sqrt{1 - p^2/(2R^2)}.
\]

Now using the identity \(\cos(3t) = 4 \cos^3 t - 3 \cos t\) we have, from (3.9.17) and (3.9.20),

\[
\begin{align*}
\cos(\varphi_A/2) \cdot c + (\cos(3\varphi_A/2))/2 = \\
\cos(\varphi_A/2) \cdot (c + 4 \cos^2(\varphi_A/2) - 3)/2 \\
\leq \cos(\varphi_A/2) \cdot (c + (4(1 - p^2/(2R^2)) - 3)/2) = \\
\cos(\varphi_A/2) \cdot (d - 1/2 + (4(1 - d/2) - 3)/2) = 0.
\end{align*}
\]
Then, recalling the notation (3.9.4), the second factor of the second summand of (3.9.14) is nonpositive.

Summing up: for \( \varphi \in (\varphi_{\text{min}}, \varphi_{\text{max}}) \) in (3.9.14) the first summand is the product of a negative and a positive factor, and the second summand is 1/2 times the product of a nonpositive and a positive factor, hence (3.9.14) is negative. This shows that

\[
(3.9.22) \quad \left\{ \begin{array}{l}
\text{the second summand of the last expression} \\
\text{of (3.9.6) is strictly decreasing.}
\end{array} \right.
\]

Together with (3.9.11) this implies that (3.9.6) is strictly decreasing, i.e., (3.9.5) is a strictly concave function of \( \varphi_A \). Analogously, (3.9.5) is a strictly concave function of \( \varphi_B \) as well. This shows the statement of the lemma. (The statement about attaining of the minimum is immediate from these strict concavity properties.) \( \blacksquare \)

**Lemma 3.10.** Let the hypotheses of Lemma 3.9 hold. Additionally, let \( \varphi_A = \varphi_{\text{min}} \) and \( \varphi_B = \varphi_{\text{max}} \). If now we allow \( R \) to vary in the interval \([p, p\sqrt{2}]\), then \( V(\Delta ABC) + V(\Delta BA_1C) + V(\Delta CB_2A) \), as a function of \( R \), is always at least \( p^2 \cdot 1.8720 \ldots \).

**Proof.** By (3.9.2) and \( \varphi_A = \varphi_{\text{min}} := 2 \arcsin((p\sqrt{2})/(2R)) \) we have

\[
(3.10.1) \quad \left\{ \begin{array}{l}
V(\Delta BA_1C) = R \sin(\varphi_A/2)\sqrt{p^2 - R^2 \sin^2(\varphi_A/2)} \\
= R \cdot ((p\sqrt{2})/(2R)) \sqrt{p^2 - R^2 ((p^2/(2R)^2))} \\
= (p/\sqrt{2})\sqrt{p^2 - p^2/2} = p^2/2.
\end{array} \right.
\]

On the other hand, by \( \varphi_B = \varphi_{\text{max}} := 4 \arcsin(p/(2R)) \) we have that \( B_2 \) lies on the circumcircle of \( \Delta ABC \). In fact, let us consider the convex deltoid \( OCBA_2 \), where \( O \) is the circumcentre of \( \Delta ABC \). Its diagonal \( OB_2 \), which is also its axis of symmetry, is cut by its other diagonal into two parts. The part having as one endpoint \( O \) has length

\[
(3.10.2) \quad R \cos(\varphi_{\text{max}}/2) = R \cos(2 \arcsin(p/(2R))) = R - p^2/(2R),
\]

and the part having as one endpoint \( B_2 \) has length

\[
(3.10.3) \quad \sqrt{p^2 - R^2 \sin^2(\varphi_{\text{max}}/2)} = \sqrt{p^2 - R^2 \sin^2(2 \arcsin(p/(2R)))} = p^2/(2R),
\]

hence

\[
(3.10.4) \quad |OB_2| = R.
\]
as asserted.

Writing $O$ for the circumcentre of $\Delta ABC$, we have, also using that $B_2$ lies on the circumcircle of $\Delta ABC$,

$$
\begin{align*}
(3.10.5) & \left\{ 
\begin{aligned}
(V(\Delta ABC) + V(\Delta BA_1C) + V(\Delta CB_2A)) / p^2 &= (V(\Delta OBC) + V(\Delta OCB_2) + V(\Delta OB_2A) + V(\Delta OAB) + V(\Delta BA_1C)) / p^2 = \\
((R/p)^2/2) \left( \sin \varphi_{\min} + 2 \sin(\varphi_{\max}/2) - \sin(\varphi_{\min} + 2(\varphi_{\max}/2)) \right) \\
&+ 1/2 = ((R/p)^2/2) \left[ \sin \left(2 \arcsin \left((p/\sqrt{2})/(2R)\right)\right) + 2 \sin \left(2 \arcsin \left((p/2R)\right)\right) - \sin \left(2 \arcsin \left((p\sqrt{2})/(2R)\right) + 2 \cdot 2 \arcsin \left(p/(2R)\right)\right) \right] \\
&+ 1/2 .
\end{aligned}
\right.
\end{align*}
$$

Now recall $R \in [p, p\sqrt{2}]$. In (3.10.5), last expression, the first factor $((R/p)^2/2)$ of the first summand is a positive increasing function of $R$ on $[p, p\sqrt{2}]$, and the second factor of the first summand is positively proportional to $V(ABC B_2)$, hence is positive for $R \in [p, p\sqrt{2}]$ as well. We are going to show that

$$
\begin{align*}
(3.10.6) & \left\{ 
\begin{aligned}
\text{the second factor of the first summand of} \\
\text{the last expression in (3.10.5) is a positive} \\
\text{decreasing function of} R \text{ on} [p, p\sqrt{2}] .
\end{aligned}
\right.
\end{align*}
$$

We have $(p\sqrt{2})/(2R) \in [1/2, 1/\sqrt{2}]$, and on $[1/2, 1/\sqrt{2}]$ the arcsine function is increasing, has values there in $[\pi/6, \pi/4]$, whose double lie in $[\pi/3, \pi/2]$. Moreover, the sine function is increasing on this last interval. Hence $\sin \left(2 \arcsin \left((p\sqrt{2})/(2R)\right)\right)$ is a decreasing function of $R \in [p, p\sqrt{2}]$.

Similarly, we have $p/(2R) \in [1/\sqrt{8}, 1/2]$, and on $[1/\sqrt{8}, 1/2]$ the arcsine function is increasing, has values there in $[\arcsin(1/\sqrt{8}), \pi/6] = [20.7048\ldots^\circ, \pi/6]$, whose double lie in $[2 \arcsin(1/\sqrt{8}), \pi/3]$. Moreover, the sine function is increasing on this last interval. Hence $\sin \left(2 \arcsin \left(p/(2R)\right)\right)$ is a decreasing function of $R \in [p, p\sqrt{2}]$.

These imply that also $2 \arcsin \left((p\sqrt{2})/(2R)\right) + 4 \arcsin \left(p/(2R)\right)$ is a decreasing function of $R \in [p, p\sqrt{2}]$. Its values for $R \in [p, p\sqrt{2}]$ form the interval $[\pi/3 + 4 \arcsin(1/\sqrt{8}), \pi/2 + 2 \pi/3] = [142.8192\ldots^\circ, 210^\circ]$, on which the minus sine function is increasing. Hence $- \sin \left(2 \arcsin \left((p\sqrt{2})/(2R) + 4 \arcsin \left(p/(2R)\right)\right)\right)$ is a decreasing function of $R \in [p, p\sqrt{2}]$ as well.

Recapitulating: in the last expression of (3.10.5) the second factor of the first summand, being positive, and being equal to the sum of three decreasing functions of $R$ on $[p, p\sqrt{2}]$, is itself a positive decreasing function of $R$ on $[p, p\sqrt{2}]$. That is, (3.10.6) is proved.
Now we turn to the lower estimate of the last expression of (3.10.5). Its second summand \(1/2\) is constant. Its first summand is a product of two positive functions, the first factor being an increasing, the second factor being a decreasing function of \(R \in [p, p\sqrt{2}]\), by (3.10.6).

Now we subdivide the interval \([p, p\sqrt{2}]\) into five subintervals: \([p, p \cdot 1.1]\), \([p \cdot 1.1, p \cdot 1.2]\), \([p \cdot 1.2, p \cdot 1.3]\), \([p \cdot 1.3, p \cdot 1.4]\), \([p \cdot 1.4, p\sqrt{2}]\). On each of these subintervals we estimate from below the above mentioned first factor by its value at the left hand endpoint of the subinterval, and we estimate from below the above mentioned second factor by its value at the right hand endpoint of the subinterval. This way we obtain the following numerical values as lower estimates for the first summand of (3.10.5) in the above five subintervals: \(1.3730\ldots, 1.3949\ldots, 1.3912\ldots, 1.3720\ldots, 1.5528\ldots\) (observe that the last interval is much shorter than the previous ones, therefore do we have there a much higher value). The minimum of these five numbers, i.e., \(1.3720\ldots\) is a lower estimate of the first summand of the last expression in (3.10.5) on the whole interval \([p, p\sqrt{2}]\). The second summand of the last expression in (3.10.5) being the constant \(1/2\), we obtain the statement of the lemma. ■

§4 The main lemmas

In this paragraph we will prove statements in \(\mathbb{R}^2\).

We say that a packing of triangles, any two of which join by entire edges, or by common vertices, or being disjoint, satisfies the angular hypothesis, if for any two triangles joining with a common edge the sum of the angles opposite to the common side is at most \(\pi\). An example is the Delone triangulation corresponding to an \((r, R)\)-system in \(\mathbb{R}^2\), where \(0 < r < R < \infty\) (with Delone polygons with more than three sides triangulated in an arbitrary way). We say that this packing satisfies the side hypothesis if all the sides of these triangles are at least \(p\), where \(p \in (0, \infty)\) is fixed. In our lemmas there will occur also right and obtuse triangles. Then of course the side hypothesis is sufficient to be supposed for these triangles only for their sides adjacent to the right or obtuse angle.

In the proofs of our Lemmas we will use the notations \(a_i, b_i, c_i\) and \(\alpha_i, \beta_i, \gamma_i\) for \(1 \leq i \leq 3\) as introduced before Lemma 3.3.

**Lemma 4.1.** Let the packing consisting of the triangles \(\Delta ABC, \Delta BAC\) satisfy the angular and side hypotheses and let \(\alpha, \beta, \gamma \leq \pi/2, \alpha_1 \geq \pi/2\). Then the sum of the areas of the two triangles is at least \(p^2\). Equality holds if and only if \(b = c = c_1 = b_1 = p\) and \(\alpha_1 = \pi/2\).

**Proof.** Consider an extremal configuration (this exists).

We make a case distinction.
(1) $\alpha + \alpha_1 < \pi$ and $\beta, \gamma < \pi/2$.  
(2) $\alpha + \alpha_1 < \pi$, and, e.g., $\beta = \pi/2$ (the case when here $\gamma = \pi/2$ is analogous).  
(3) $\alpha + \alpha_1 = \pi$, i.e., the quadrangle $ABA_1C$ has a circumcircle.

1. In case (1) we apply Lemma 3.1 to $\triangle BCA_1$ and $\triangle BCA$ (rather than $\triangle ABC$ in Lemma 3.1), respectively. Then by $\beta_1, \gamma_1 < \pi/2$ and $\alpha_1 < \pi - \alpha$ we obtain $b_1 = c_1 = p$, and by $\alpha < \pi - \alpha_1$ and $\beta, \gamma < \pi/2$ we obtain $b = c = p$. Hence $ABC$ and $A_1BC$ are congruent triangles, thus $\pi/2 \geq \alpha = \alpha_1 \geq \pi/2$, thus $\alpha = \alpha_1 = \pi/2$, a contradiction to our hypothesis $\alpha + \alpha_1 < \pi$.

2. In case (2) fixing $\triangle CBA_1$ and the side length $c = |AB|$, decrease $\beta$ a bit. Then the hypotheses of the lemma remain valid, and the area of $\triangle ABC$ decreases, a contradiction.

3. In case (3), unless $\alpha_1 = \pi/2$, we can apply Lemma 3.3 (with the same notations there as here), obtaining $\beta = \pi/2$ or $\gamma = \pi/2$. Thus we have to investigate the cases
(a) $\alpha_1 = \pi/2$, and
(b) $\beta = \pi/2$ (the case $\gamma = \pi/2$ is analogous).

In case (a), observe that for $\alpha_1 = \pi/2$ we have by (3) also $\alpha = \pi/2$, and the total area of the triangles $\triangle ABC$ and $\triangle A_1BC$ is

\[(4.1.1) \quad (bc + b_1c_1)/2 \geq p^2,\]

proving the Lemma in case (a).

In case (b), let, e.g., $\beta = \pi/2$ (the case $\gamma = \pi/2$ is analogous). By (3) all four vertices of our two triangles lie on a circle, and by $\beta = \pi/2$ the side $AC$ is a diameter of this circle. Let $O$ be the centre of this circle. Then the quadrangle $ABA_1C$ is inscribed to this circle, and its area (i.e., the sum of the areas of our two triangles in the lemma) is

\[(4.1.2) \quad (b/2)^2 \cdot (\sin \angle AOB + \sin \angle BOA_1 + \sin \angle A_1OC)/2.\]

The longest side of this quadrangle is $b$, the other three sides are $c, c_1, b_1 \geq p$. The central angles corresponding to the sides $c, c_1, b_1$ will be denoted by $\varphi_1, \varphi_2, \varphi_3$, respectively. Then $\varphi_i \geq \varphi_0 := 2 \arcsin (p/(2R))$ for $1 \leq i \leq 3$ and $\varphi_1 + \varphi_2 + \varphi_3 = \pi$, which implies $\varphi_i \in [\varphi_0, \pi]$ for $1 \leq i \leq 3$. We will use concavity of the function $\sin x$ in $[0, \pi]$. Thus at most one $\varphi_i$ can be in the open interval $(\varphi_0, \pi)$, since else we decrease one and increase another one by a bit, preserving their sum and increasing their difference, and then $\sum_{1 \leq i \leq 3} \sin \varphi_i$ decreases (while $b$ is fixed), and then (4.1.2) decreases also, a contradiction. So two of the $\varphi_i$’s equal $\varphi_0$, and then the third one equals $\pi - 2\varphi_0$ ($\geq \varphi_0$). The last inequality implies $6 \arcsin (p/(2R)) = 3\varphi_0 \leq \pi$ thus $2 \arcsin (p/(2R)) = \varphi_0 \leq \pi/3$, which in turn implies $b = 2R \geq 2p$. 

Recapitulating, the area of our quadrangle is minimal, e.g., for a symmetric trapezoid of longer base of length \( b \), which is also the diameter of the circumscribed circle of our quadrangle, and equal sides of lengths \( p \), and other base \( c_1 \geq p \). Observe that here we have a free parameter \( b \), whose value determines our quadrangle \( ABA_1C \) uniquely, and we have to find the minimal area of our quadrangle when \( b \) varies.

We calculate the area of our symmetric trapezoid as the arithmetic mean of the two bases times the height corresponding to the bases. The equal sides of our symmetric trapezoid have lengths \( p \), and enclose with the longer base an angle \( \angle CAB = \angle ACA_1 = \arccos(p/b) \geq \pi/3 \). Then the longer base is \( b \geq 2p \), the shorter base is \( b-2p \cos \angle CAB \geq b-2p \cos(\pi/3) = b-p \geq p \), and the height corresponding to the bases is \( p \sin \angle CAB \geq p \sin(\pi/3) = p \sqrt{3}/2 \). Putting all these estimates together, we gain that the area of our quadrangle is at least \((2p + p)/2 \cdot p \sqrt{3}/2 = p^2 \cdot (3\sqrt{3}/4) > p^2 \), proving the lemma in this case as well. (This lower estimate is sharp for a “half of a regular hexagon or side length \( p \).”)

The case of equality follows from the proof. (All cases except the one investigated in (4.1.1) were contradictory, or gave better estimates.)

**Lemma 4.2.** Let the packing consisting of the triangles \( \Delta ABC, \Delta A_1B_1, \Delta C_1B_2 \) satisfy the angular and side hypotheses, let the circumradii of \( \Delta A_1B_1C \) and \( \Delta C_1B_2 \) be at most \( p \sqrt{2} \), and let \( \alpha, \beta, \gamma \leq \pi/2 \), \( \alpha_1, \beta_2 \geq \pi/2 \). Then the sum of the areas of the three triangles is at least \( \alpha + \beta + \gamma \leq \pi/2 \). If additionally \( c \geq p \sqrt{2} \), then the sum of the areas of the three triangles is at least \( p^2 \cdot (5/4) \cot(\pi/10) = p^2 \cdot 1.7204 \ldots \). The second lower estimate cannot be strengthened by replacing its right hand side by any number greater than \( p^2 (1 + \sqrt{3}/2) = p^2 \cdot 1.8660 \ldots \).

**Proof.** We begin with showing an example for the second inequality having total area \( p^2 (1 + \sqrt{3}/2) \). We let \( c_1 = b_1 = a_2 = c_2 = c = p \) and \( \alpha_1 = \beta_2 = \pi/2 \).

Now we turn to prove the two lower estimates. Consider an extremal configuration (this exists).

We have two angular hypotheses, namely \( \alpha + \alpha_1 \leq \pi \) and \( \beta + \beta_2 \leq \pi \). We distinguish three cases. Either

1. \( \alpha + \alpha_1 = \beta + \beta_2 = \pi \), or, e.g.,
2. \( \alpha + \alpha_1 = \pi > \beta + \beta_2 \), or
3. \( \alpha + \alpha_1, \beta + \beta_2 < \pi \).

**1.** We begin with case (1). That is, the pentagon \( ABA_1CB_2 \) has a circum-circle. Then by Lemma 3.6 we have \( V(\Delta ABC) + V(\Delta A_1BC) + V(\Delta AB_2C) = V(ABA_1CB_2) \geq p^2 \cdot (5/4) \cot(\pi/10) = p^2 \cdot 1.7204 \ldots > p^2 (\sqrt{7}/4 + 1) = p^2 \cdot 1.6614 \ldots \). This proves the first inequality of our Lemma, and also the second inequality (for \( c \geq p \sqrt{2} \)) in case 1.

**2.** We continue with case (2). We apply Lemma 3.3 to the triangles
$\Delta ABC$ and $\Delta BA_1 C$. Then $\alpha$ and $\alpha_1$ decrease while $\beta + \beta_2 < \pi$, so the angular hypotheses remain preserved (observe $\beta + \beta_2 < \pi$). The side hypothesis is preserved since by $\alpha_1 \geq \pi/2$

\begin{equation}
(4.2.1) \quad a = a_1 \geq \sqrt{b_1^2 + c_1^2} \geq p\sqrt{2} > p.
\end{equation}

For $\alpha_1 > \pi/2$ the circumradius hypothesis remains preserved, since $\Delta CB_2 A$ remains preserved, and the circumradius of $\Delta BA_1 C$ is $a/(2 \sin \alpha_1)$, where $a$ decreases and $\alpha_1 (> \pi/2)$ decreases. Since $\beta_2$ remains preserved, this motion can be prevented by

(a) $\alpha_1 = \pi/2$ and then by (2) also $\alpha = \pi/2$, or
(b) $\beta = \pi/2$, or
(c) $\gamma = \pi/2$.

In case (a) we use the analogue of (4.2.1) for $b$, i.e., $b \geq p\sqrt{2}$, which yields

\begin{equation}
(4.2.2) \quad a = \sqrt{b^2 + c^2} \geq \sqrt{(p\sqrt{2})^2 + p^2} = p\sqrt{3}.
\end{equation}

Then

\begin{equation}
(4.2.3) \quad V(\Delta ABC) = (1/2) \cdot bc \geq (1/2) \cdot p\sqrt{2} \cdot p = p^2 \sqrt{2}/2
\end{equation}

Further, moving $A_1$ on the circumcircle of $\Delta ABC$, the minimal area $V(\Delta BA_1 C)$ occurs when $\min\{b_1, c_1\} = p$, and the minimal area is by (4.2.2)

\begin{equation}
(4.2.4) \quad V(\Delta BA_1 C) = (1/2) \cdot p\sqrt{a^2 - p^2} \geq (1/2) \cdot p\sqrt{3p^2 - p^2} = p^2 \cdot \sqrt{2}/2
\end{equation}

For $\Delta AB_2 C$ we use the circumradius hypothesis, which gives by Lemma 3.2, (3)

\begin{equation}
(4.2.5) \quad V(\Delta AB_2 C) \geq p^2 \sqrt{7}/8.
\end{equation}

Adding (4.2.3), (4.2.4) and (4.2.5) we have

\begin{equation}
(4.2.6) \quad \left\{ \begin{array}{l}
V(\Delta ABC) + V(\Delta BA_1 C) + V(\Delta AB_2 C) \geq p^2(\sqrt{2} + \sqrt{7}/8) \\
> p^2 \cdot 1.7449 \ldots > p^2 \cdot (5/4) \cot(\pi/10) = p^2 \cdot 1.7204 \ldots \\
> p^2(\sqrt{7}/4 + 1) = p^2 \cdot 1.6614 \ldots ,
\end{array} \right.
\end{equation}

which proves both lower estimates of the lemma in case (a).
In case (b), i.e., when $\beta = \pi/2$, we have by (2) $\beta_2 < \pi/2$, while by hypothesis of the lemma $\beta_2 \geq \pi/2$, a contradiction.

In case (c), i.e., when $\gamma = \pi/2$, the quadrangle $ABA_1C$ has a circumcircle, and its circumradius is at most $p\sqrt{2}$ (by the hypothesis about the circumradii in the lemma). By $\gamma = \pi/2$ the side $AB$ is a diameter of the circumcircle, and by the side hypothesis we have $|BA_1|, |A_1C| \geq p$ and also $|AC| \geq \sqrt{a_2^2 + c_2^2} \geq p\sqrt{2}$. Thus we may apply Lemma 3.7, yielding

$$V(ABA_1C) \geq p^2 \cdot 1.4048 \ldots$$

Then by the circumradius hypothesis and Lemma 3.2 we have

$$V(\Delta C_{B_2}A) \geq 0.3307 \ldots.$$ 

Adding (4.2.7) and (4.2.8) we get

$$V(\Delta ABC) + V(\Delta BA_1C) + V(\Delta C_{B_2}A) \geq p^2 \cdot 1.7355 \ldots$$

$$> p^2 \cdot (5/4) \cot(\pi/10) = p^2 \cdot 1.7204 \ldots$$

$$> p^2(\sqrt{7}/4 + 1) = p^2 \cdot 1.6614 \ldots,$$

proving both lower estimates of the lemma in case (c).

3. We turn to case (3). We distinguish the cases
(a) $\alpha, \beta, \gamma < \pi/2$, and
(b) $\beta = \pi/2$ (the case $\alpha = \pi/2$ is analogous)
(c) $\gamma = \pi/2$.

In case (a) we apply Lemma 3.3 to the triangles $\Delta ABC$ and $\Delta BA_1C$ (with the same notations here as in Lemma 3.3). The case of the triangles $\Delta ABC$ and $\Delta C_{B_2}A$ can be settled in an identical way.

By $\alpha_1 \geq \pi/2$ we have $a = a_1 \geq p\sqrt{2} > p$, so the side hypothesis is satisfied if we apply Lemma 3.3 (the other side lengths are unchanged). The angular hypothesis $\alpha + \alpha_1 \leq \pi$ plays no role here. For $\alpha_1 > \pi/2$ the circumradius hypotheses are preserved by the same reasoning as in 2. So by Lemma 3.3 either $\alpha_1 = \pi/2$ or $\beta + \beta_2 = \pi/2$, but the second case is impossible by case (3), so

$$\alpha_1 = \pi/2.$$ 

Analogously, by Lemma 3.3 applied to $\Delta ABC$ and $\Delta C_{B_2}A$, one obtains
Thus $\alpha_1 = \beta_2 = \pi/2$. Then $b_1, c_1 \geq p$ gives $a = a_1 \geq p\sqrt{2}$, and analogously, $b = b_1 \geq p\sqrt{2}$. Moreover, we have $c \geq p$. Then (3.0) yields

\[(4.2.12)\quad V(\Delta ABC) \geq p^2 \cdot \sqrt{7}/4.\]

Further,

\[(4.2.13)\quad V(\Delta BA_1C) = b_1 c_1 \sin \alpha_1/2 \geq p^2 / 2 \quad \text{and similarly} \quad V(\Delta CB_2A) \geq p^2 / 2,\]

Adding (4.2.12) and the two inequalities in (4.2.13) we obtain the first inequality of the lemma in case (a).

If moreover we have $c \geq p\sqrt{2}$, then (3.0) yields even

\[(4.2.14)\quad V(\Delta ABC) \geq p^2 \cdot \sqrt{3}/2,\]

and then adding (4.2.14) and the two inequalities in (4.2.13) the total area of the three triangles is at least $p^2 (1 + \sqrt{3}/2) \geq p^2 \cdot 1.8660 \ldots > p^2 \cdot (5/4) \cot(\pi/10) = p^2 \cdot 1.7204 \ldots$. This proves the second inequality of the lemma in case (a).

In case (b), i.e., when $\beta = \pi/2$, we have by (3) $\beta_2 < \pi/2$, while by hypothesis of the lemma $\beta_2 \geq \pi/2$, a contradiction.

In case (c) we have $\gamma = \pi/2$. We decrease $\gamma$ a bit, rotating $\Delta BA_1C$ and $\Delta CB_2A$ towards each other. Then $V(\Delta ABC)$ decreases, $V(\Delta BA_1C)$ and $V(\Delta CB_2A)$ remain constant, so their sum decreases. The circumradius hypotheses remain preserved. The side hypothesis is to be checked only for $c$, but that was originally by $a, b \geq p$ and $\gamma = \pi/2$ at least $p\sqrt{2}$, so the side hypothesis remains valid. Similarly, originally we had $\alpha, \beta < \pi/2$, so they remain acute angles. So all the hypotheses of this lemma (and also those of case (3)) remain preserved, and the total area of our three triangles decreased. This is a contradiction.

The case of equality in the first inequality of the lemma follows from the proof. (In 1 and 2 we had strict inequalities or contradiction, 3, cases (b) and (c) were contradictory, and 3, case (a) is easily discussed.) ■

**Lemma 4.3.** Let the packing consisting of the triangles $\Delta ABC$, $\Delta BA_1C$, $\Delta CB_2A$ and $\Delta AC_3B$ satisfy the angular and side hypotheses, let the circumradii of $\Delta BA_1C$, $\Delta CB_2A$ and $\Delta AC_3B$ be at most $p \cdot \sqrt{2}$, and let $\alpha, \beta, \gamma \leq \pi/2$ and $\alpha_1, \beta_2, \gamma_3 \geq \pi/2$. Then the sum of the areas of the four triangles is at least $p^2 \cdot [(5/4) \cot(\pi/10) + \sqrt{7}/8] = p^2 \cdot 2.0511 \ldots$. This estimate cannot be strengthened by replacing the right hand side of this inequality by any number greater than $p^2 (\sqrt{3}/2 + 3/2) = p^2 \cdot 2.366 \ldots$. 

Proof. We begin with showing an example having a total area \( p^2(\sqrt{3}/2 + 3/2) \). We let \( c_1 = b_1 = a_2 = c_2 = b_3 = a_3 = p \) and \( \alpha_1 = \beta_2 = \gamma_3 = \pi/2 \).

Now we turn to the lower estimate. By \( a_3, b_3 \geq p \) and \( \gamma_3 \geq \pi/2 \) we have \( c_3 \geq p \sqrt{2} \). Therefore we can apply Lemma 4.2 to the triangles \( \Delta ABC, \Delta BA_1C, \Delta CB_2A \), with the same notations here as in Lemma 4.2. Therefore the second inequality of Lemma 4.2 applies, and gives

\[
\begin{align*}
\{ & V(\Delta ABC) + V(\Delta BA_1C) + V(\Delta CB_2A) \\
& \geq p^2 \cdot (5/4) \cot(\pi/10) = p^2 \cdot 1.7204 \ldots .
\end{align*}
\]

Lemma 3.2, (3) implies

\[
V(\Delta AC_3B) \geq p^2 \cdot \sqrt{7}/8 = p^2 \cdot 0.3307 \ldots .
\]

Adding these we obtain

\[
\begin{align*}
\{ & V(\Delta ABC) + V(\Delta BA_1C) + V(\Delta CB_2A) + V(\Delta AC_3B) \\
& \geq p^2 \cdot [(5/4) \cot(\pi/10) + \sqrt{7}/8] = p^2 \cdot 2.0511 \ldots ,
\end{align*}
\]

proving the lemma.

Lemma 4.4. Let the packing consisting of the triangles \( \Delta ABC, \Delta BA_1C \) satisfy the angular and side hypotheses and let \( \gamma, \alpha_1 \geq \pi/2 \). Further let the circumradius of \( \Delta ABC \) be at most \( p \sqrt{2} \) (or \( p \sqrt{5}/2 \)). Then \( V(\Delta ABC) \) is at least \( p^2 \cdot (\sqrt{7} + \sqrt{3})/8 = p^2 \cdot 0.5472 \ldots \) (or \( p^2/2 \)). Equality holds if and only if \( c_1 = b_1 = b = p \), \( a = p \sqrt{2} \) and the circumradius of \( \Delta ABC \) is \( p \sqrt{2} \) (or \( p \sqrt{5}/2 \)).

Proof. We have \( b_1, c_1 \geq p \) and \( \alpha_1 \geq \pi/2 \), which imply \( a \geq p \sqrt{2} \). Moreover, we have \( b \geq p \). Now we can apply Lemma 3.2, (1) (or (2)) to \( \Delta ABC \) (with the same notation of vertices as in Lemma 3.2), which yields both statements of the Lemma. The case of equality follows from the proof of this lemma and from Lemma 3.2.

Lemma 4.5. Let the packing consisting of the triangles \( \Delta ABC, \Delta A_1BC \) satisfy the angular and side hypotheses and let \( \gamma, \alpha_1 \geq \pi/2 \). Further let the circumradius of \( \Delta ABC \) be at most \( p \sqrt{2} \). Then the sum of the areas of these two triangles is at least \( p^2 \cdot (\sqrt{7} + \sqrt{3} + 4)/8 = p^2 \cdot 1.0472 \ldots . \) Equality holds if and only if \( c_1 = b_1 = b = p \), \( \alpha_1 = \pi/2 \) and the circumradius of \( \Delta ABC \) is \( p \sqrt{2} \).

Proof. Consider an extremal configuration (this exists).

We apply Lemma 3.2 to \( \Delta ABC \) (with the same notation of vertices there as here), with \( a_0 := a \), \( b_0 := p \) and \( R_0 := p \sqrt{2} \). Observe that the angular hypothesis remains preserved by the application of Lemma 3.2, since in the proof of Lemma 3.2 both acute angles of \( \Delta ABC \) there do not increase. Thus we get that
(4.5.1) \( b = p \) and the circumradius of \( \Delta ABC \) is \( p\sqrt{2} \).

1. Now we make a case distinction.
(1) \( \alpha + \alpha_1 = \pi \), i.e., that the quadrangle \( ABA_1C \) has a circumcircle, and
(2) \( \alpha + \alpha_1 < \pi \).

First we deal with case (1). Then we apply Lemma 3.1 to \( \Delta BA_1C \) (rather than \( \Delta ABC \) in Lemma 3.1), with \( \gamma_0 \) there replaced by \( \alpha_1 \) here. Thus we have either \( b_1 = p \) or \( c_1 = p \). Although these are two different cases, the areas of the quadrangle \( ABA_1C \) coincide in these two cases. Thus for estimating the area of our quadrangle below, we may suppose that

(4.5.2) \( b_1 = p \).

The area of our quadrangle also equals \( V(\Delta ABA_1) + V(\Delta A_1CA) \). Observe that by \( \gamma \geq \pi/2 \) our quadrangle lies in a closed half-circle \( H \) of its circumcircle, partly bounded by a diameter of the circumcircle, parallel to the line \( AB \). The orthogonal bisector line of side \( AA_1 \) meets the shorter arc \( \overline{AA_1} \) of the circumcircle in the midpoint \( M \) of this arc, which lies in the relative interior of \( H \) (with respect to the entire circumcircle). The same orthogonal bisector meets the longer arc \( AA_1 \) of the circumcircle in the opposite point of \( M \) of the circumcircle, thus in the relative interior of the complementary half-circle to \( H \). Therefore moving \( B \) on the circumcircle of our quadrangle, towards \( A_1 \), \( V(\Delta ABA_1) \) strictly decreases, while \( V(\Delta A_1CA) \) remains constant, therefore the area of our quadrangle \( ABA_1C \) also strictly decreases. Therefore in the extremal configuration we have also

(4.5.3) \( c_1 = p \).

If we had, rather than (4.5.2),

(4.5.4) \( c_1 = p \),

then by equality of the areas in these two cases (which are in fact axially symmetric images of each other), in the extremal configuration we have also

(4.5.5) \( b_1 = p \).

We turn to case (2). Then we again apply Lemma 3.1 to \( \Delta BA_1C \) (rather than \( \Delta ABC \) in Lemma 3.1), with \( \gamma_0 \) there replaced by \( \pi - \alpha \) here. By \( \alpha + \alpha_1 < \pi \) we get that
Therefore (4.5.6) is valid in both cases (1) and (2).

2. Now we apply Lemma 3.5 (with the same notations there as here). All its hypotheses are satisfied by (4.5.1) (observe $p = b \leq p\sqrt{7/2}$) and (4.5.6). By the conclusion of Lemma 3.5 we may suppose at the lower estimate of $V(\Delta ABC) + V(BA_1C)$ that either

(1) $\alpha_1 = \pi/2$, or
(2) $\alpha + \alpha_1 = \pi$.

In case (1) we have $V(\Delta BA_1C) = b_1c_1/2 = p^2/2$. Moreover, by Lemma 4.4 we have $V(\Delta ABC) \geq p^2 \cdot (\sqrt{7} + \sqrt{3})/8$, hence their sum is at least $p^2 \cdot (\sqrt{7} + \sqrt{3} + 4)/8 = p \cdot 1.0472 \ldots$, as asserted in the Lemma, finishing the proof of case (1).

In case (2) the quadrangle $ABA_1C$ has a circumcircle, which has by (4.5.1) radius $p\sqrt{2}$. By (4.5.1) and (4.5.6) we have $c_1 = b_1 = b = p$. Thus the quadrangle $ABA_1C$ is completely determined, is a symmetric trapezoid inscribed to a circle of radius $p\sqrt{2}$, and a straightforward calculation gives that its area is $p^2 \cdot 7\sqrt{7}/16 = p^2 \cdot 1.1575 \ldots$ which is a larger value than asserted in the lemma, finishing the proof of case (2).

The case of equality follows from the proof. □

Lemma 4.6. Let the packing consisting of the triangles $\Delta ABC, \Delta BA_1C, \Delta CB_2A$ satisfy the angular and side hypotheses and let $\gamma, \alpha_1, \beta_2 \geq \pi/2$. Further let the circumradius of $\Delta ABC$ be at most $p\sqrt{2}$. Then the sum of the areas of these three triangles is at least $p^2 \cdot (1 + \sqrt{3}/2) = p^2 \cdot 1.8660 \ldots$. Equality holds if and only if $c_1 = b_1 = a_2 = c_2 = p$ and $\alpha_1 = \beta_2 = \pi/2$ and the circumradius of $\Delta ABC$ is $p\sqrt{2}$.

Proof. Consider an extremal configuration (this exists).

We have two angular hypotheses, namely $\alpha + \alpha_1 \leq \pi$ and $\beta + \beta_2 \leq \pi$. Analogously as in the proof of Lemma 4.2, we make the following case distinctions. Either

(1) $\alpha + \alpha_1 = \beta + \beta_2 = \pi$, or
(2) $\alpha + \alpha_1, \beta + \beta_2 < \pi$ or, e.g.,
(3) $\alpha + \alpha_1 = \pi > \beta + \beta_2$.

1. We begin with case (1). That is, the pentagon $ABA_1CB_2$ has a circumcircle, and by $\gamma \geq \pi/2$ its circumcentre is not an interior point of it. Applying Lemma 3.8 we get $V(ABA_1CB_2) \geq p^2 \cdot 2.3977 \ldots > p^2 \cdot (1 + \sqrt{3}/2) = p^2 \cdot 1.8660 \ldots$

2. We turn to case (2). We apply Lemma 3.1 to $\Delta BA_1C$ and $\Delta CB_2A$
(in place of $\triangle ABC$ in Lemma 3.1, with $\pi - \alpha$ and $\pi - \beta$ in place of $\gamma$ in Lemma 3.1), obtaining

\[(4.6.1) \quad c_1 = b_1 = a_2 = c_2 = p\]

We apply Lemma 3.2 to $\triangle ABC$ (with the same notations here and there, and $a_0 := a$ and $b_0 := b$) obtaining for the circumradius $R$ of $\triangle ABC$

\[(4.6.2) \quad R = p\sqrt{2}\]

(observe that now by (2) we need not care the angular hypotheses, and the side hypothesis remains preserved by $c \geq \sqrt{a^2 + b^2} \geq p\sqrt{2} > p$).

3. Leaving the further investigation of case (2) later, for a while we turn to case (3). Analogously as in the proof of (3.8.7), we have also here

\[(4.6.3) \quad b_1 = c_1 = p .\]

On the other hand, by Lemma 3.1 applied to $\triangle CB_2A$ (in place of $\triangle ABC$ in Lemma 3.1, with $\pi - \beta$ in place of $\gamma$ in Lemma 3.1), we have

\[(4.6.4) \quad a_2 = c_2 = p .\]

Hence (4.6.1) holds also now.

By

\[(4.6.5) \quad a \geq \sqrt{c_1^2 + b_1^2} \geq p\sqrt{2} \text{ and similarly } b \geq p\sqrt{2}\]

we have

\[(4.6.6) \quad 2R \geq c \geq \sqrt{a^2 + b^2} \geq 2p .\]

thus, also using the hypothesis of the lemma,

\[(4.6.7) \quad R \in [p, p\sqrt{2}]/.\]

In 2 we had (4.6.2), in 3 we have obtained (4.6.7). This means that (4.6.7) holds in both of these cases.

4. Both in 2 and 3 we denote the central angles belonging to the sides $a$ and $b$ in the circumcircle of $\triangle ABC$ by $\varphi_A$ and $\varphi_B$. We denote the (smaller) central angle corresponding to a chord of length $p$ of this circumcircle by $\varphi_0$; then

\[(4.6.8) \quad \sin(\varphi_0/2) = p/(2R) .\]
Then the minimum of $\alpha_1$ or of $\beta_2$ occurs when $a = p\sqrt{2}$ or $b = p\sqrt{2}$ (cf. (4.6.5)). By $2p \leq 2R$ we have $\varphi_A \leq \pi$. Then $\alpha_1, a_1 = a$ and $\varphi_A = 2\alpha$ increase together, so their maxima occur when $\alpha + \alpha_1$ increases to $\pi$, i.e., when $A_1$ gets to the boundary of the circumcircle of $\Delta ABC$. Analogously, the maximum of $\beta_2$ occurs when $B_2$ gets to the boundary of the circumcircle of $\Delta ABC$. That is,

$$(4.6.9) \quad \varphi_A, \varphi_B \in [2 \arcsin\left(\frac{p\sqrt{2}}{2R}\right), 4 \arcsin\left(\frac{p}{2R}\right)]$$

Thus we have obtained that both in 2 and in 3 all hypotheses of Lemma 3.9 are satisfied. Then the conclusion of Lemma 3.9 holds as well, i.e., the minimum of the total area of our three triangles is attained (only) for

$$(4.6.10) \quad \varphi_A, \varphi_B \in \{\varphi_{\text{min}}, \varphi_{\text{max}}\} = \{2 \arcsin\left(\frac{p\sqrt{2}}{2R}\right), 4 \arcsin\left(\frac{p}{2R}\right)\}$$

5. The case

$$(4.6.11) \quad \varphi_A = \varphi_B = \varphi_{\text{max}}$$

is covered by 1.

In the case

$$(4.6.12) \quad \varphi_A = \varphi_B = \varphi_{\text{min}}$$

we have that both $\Delta BA_1C$ and $\Delta CB_2A$ are isosceles right triangles of legs of length $p$, hence have total area $p^2$. On the other hand, by Lemma 3.2 (with the same notations there and here) $V(\Delta ABC)$ is minimal when $a$ and $b$ are minimal, i.e., $a = b = p\sqrt{2}$, and the circumradius $R$ is maximal, i.e., $R = p\sqrt{2}$, when $V(\Delta ABC) = \sqrt{3}/2$, and thus the total area of the three triangles is

$$(4.6.13) \quad p^2(1 + \sqrt{3}/2),$$

proving the inequality of the lemma, with case of equality only as given in the lemma, as follows from this proof.

There remains the case when one of $\varphi_A$ and $\varphi_B$ is $\varphi_{\text{min}}$, and the other one is $\varphi_{\text{max}}$. Then we apply Lemma 3.10, obtaining

$$(4.6.14) \quad \left\{ \begin{array}{l} V(\Delta ABC) + V(\Delta BA_1C) + V(\Delta CB_2A) \geq \\
p^2 \cdot 1.8720 \ldots > p^2(1 + \sqrt{3}/2) = p^2 \cdot 1.8660 \ldots \end{array} \right.$$
§5 Proofs of the results from §2

Now we prove the statements from §2, practically in the inverse order, as they were stated in §2.

Proof of Theorem 2.9. We consider an \((r, R)\)-system \(P \subset \mathbb{R}^2\) satisfying the hypotheses of the theorem. There will be no problems with the non-obtuse triangles in the Delone triangulation, then case (1) of Theorem 2.9 could work. However, as we will see later, they will not be always settled via case (1) of Theorem 2.9.

To handle the obtuse triangles in the Delone triangulation, we define an oriented graph, whose vertices are the triangles of the Delone triangulation of \(\mathbb{R}^2\). The oriented edges pass from an obtuse triangle to the triangle joining to it along the longest side of the obtuse triangle. Thus between any two triangles at most one oriented edge passes (since in a Delone triangulation the sum of the angles opposite to the common edge of two triangles is at most \(\pi\)). The number of edges starting from a triangle is at most 1 and the number of edges ending in a triangle is at most 3.

Therefore, if the corresponding non-oriented graph contained a cycle, our oriented graph would contain an oriented cycle. In fact, if we have a non-oriented cycle \(T_1 \ldots T_m\) (with cyclic notation), then to an oriented edge \(\overrightarrow{T_iT_{i+1}}\) there cannot join an oriented edge \(\overrightarrow{T_iT_{i-1}}\), but only an oriented edge \(\overrightarrow{T_{i-1}T_i}\). Repeat these considerations for the oriented edge \(\overrightarrow{T_{i-1}T_i}\), etc., and we get our claim.

However, an oriented cycle in our graph is impossible. Namely, if an obtuse triangle \(T_1\) joins by its longest side to a side of another obtuse triangle \(T_2\), then the common side is opposite to an acute angle in \(T_2\). Then the square of the longest side of \(T_2\) is at least the square of the longest side of \(T_1\) (which is at least \((2r)^2\)) plus \((2r)^2\). Thus a cycle in our graph is impossible, since passing on the oriented edges the length of the longest side of the triangle strictly increases, and returning to the starting triangle we obtain a contradiction.

Since each triangle in the Delone triangulation has a circumradius at most \(R\), hence longest side at most \(2R\), similarly we get that an oriented path has a length at most 7. In fact, if there were an oriented path \(\overrightarrow{T_1T_2\ldots T_9}\), then by induction the longest side of \(T_i\) would have square at least \(i(2r)^2\). In particular for \(i = 9\) the square of the longest side were at least \(9(2r)^2\), but on the other hand, as we have just seen, it is at most \((2R)^2\). Thus \(9(2r)^2 \leq (2R)^2\), i.e., \(3 \leq R/r \leq 2\sqrt{2}\), a contradiction to the hypotheses of our theorem.

By the above facts we see that the connected components of the corresponding non-oriented graph contain a bounded number of triangles. In fact, each oriented edge path has an endpoint, and that is a non-obtuse triangle \(T_0\). Then \(T_0\) is the endpoint of at most three oriented edges, from obtuse triangles \(T_{1i}\). Repeating, each \(T_{1i}\) is the endpoint of at most two oriented edges, from
obtuse triangles $T_{2ij}$. Continuing similarly, since the lengths of oriented paths are at most 7, a connected component of the corresponding unoriented graph can have a size at most $1 + 3 \cdot (1 + 2 + \ldots + 2^6)$.

A non-trivial connected component (i.e., which contains at least two triangles) of the corresponding non-oriented graph (later we will say just “non-trivial component”) is thus a tree, and contains just one non-obtuse triangle, namely the one from which no edge of the graph starts. On the other hand, there are triangles in the non-trivial component, at which no edge ends, but there starts just one edge. These will be divided in classes, two such triangles being in the same class if the oriented edge starting from them end at the same triangle.

Next we divide all triangles of a non-trivial component in classes. An above class of triangles, in which no edge ends, together with the triangle which is the common endpoint of the edges starting from them will form a new class. All other triangles in the same non-trivial component, not contained in any of the new classes, i.e., which are the end-points of oriented paths of length (at least) 2, with the exception of the triangle from which no edge starts (unless if it is in a new class), will form one-element new classes. (In fact, we will use only that they are endpoints of an oriented edge, cf. Lemma 4.4, which will be applied to these triangles.)

Each Delone triangle, either the triangle in a non-trivial component from which no edge starts (provided that it is not in a new class of triangles), or a triangle not contained in any non-trivial component of our graph, which triangle is therefore in both cases non-obtuse, will form a one element class. Thus now all the Delone triangles are divided to classes.

Observe that the diameter of the union of any class is at most twice the maximal possible diameter of a Delone triangle, i.e., is at most $4R$. By boundedness of the diameters, one can estimate from below the average area of a Delone triangle in each class separately, and the same lower bound will be valid on the whole plane.

The hypotheses of our theorem imply the hypotheses of each of Lemmas 4.1 to 4.6, with $p := 2r$. By these lemmas, the average area of Delone triangles in one class is at least $p^2/2 = 2r^2$. In fact, one element classes in a non-trivial component, except the triangle from which no edge starts (unless if it is in a new class), are settled by Lemma 4.4, giving cases (1) and (2) of the theorem. Two element classes are settled in Lemmas 4.1 and 4.5, giving cases (3) and (4) of the theorem. Three element classes are settled in Lemmas 4.2 and 4.6, giving cases (3) and (4) of the theorem. Four element classes are settled in Lemma 4.3, giving case (3) of the theorem. (Except Lemma 4.1 we have even that the average area is strictly greater than $2r^2$.) Last, if a class is a single non-obtuse triangle, either the triangle in a non-trivial component from which no edge starts (if it is not in a new class), or a triangle not contained in any non-trivial component of our graph, then the area of this triangle is at least $V_0$. Therefore the average area of all
Delone triangles is at least \( \min\{V_0, 2r^2\} \).

This bound is sharp. In fact, for any \( \varepsilon > 0 \) there is a non-obtuse Delone triangle \( T \) with area less than \( V_0 + \varepsilon \). Now consider the point lattice generated by the vertices of \( T \). Then in the Delone triangulation of this point lattice all Delone triangles are congruent to \( T \), hence all of them have area less than \( V_0 + \varepsilon \), so the average area is also less than \( V_0 + \varepsilon \). The average area \( 2r^2 \), even all Delone triangles having area \( 2r^2 \) can be attained for a square lattice of side length \( 2r \), where the respective covering radius \( R \) is \( r\sqrt{2} \) (< \( r \cdot 2\sqrt{2} \)).

\[ \square \]

**Proof of Theorem 2.8.** 1. By hypothesis we have

\[ (2.8.1) \quad 1/\sqrt{2} \leq m(L) \leq M(L) = 1. \]

Clearly we may suppose that our packing of translates of \( L \) in \( \mathbb{R}^n \) is saturated, i.e., we cannot add any new translate of \( L \) to the packing without violating the packing property. This implies that any empty circle associated to the point system on the \( x_1x_2 \)-coordinate plane, consisting of the intersection points of the axes of rotation of the translates of \( L \) in our packing with the \( x_1x_2 \)-coordinate plane, has a radius strictly smaller than 2. In fact, if one such radius would be at least 2, then we could add to our packing of translates of \( L \) even a translate of the cylinder \( L_0 \) with base a circle of radius 1 and centre the origin in the \( x_1x_2 \)-coordinate plane, and with axis the \( x_3 \ldots x_n \)-coordinate plane, without violating the packing property. Then this translate of \( L_0 \) would contain some translate of \( L \), and addition of this translate of \( L \) to our packing of translates of \( L \) would not violate the packing property, contradicting saturatedness of our packing of translates of \( L \).

Later we will use only that each above empty circle has a radius at most 2.

So we may restrict our attention to saturated packings in this sense. Now consider the intersection points of the axes of rotation of the translates of \( L \) consisting our packing with the \( x_1x_2 \)-coordinate plane. We consider the Delone triangulation of the \( x_1x_2 \)-coordinate plane belonging to this system of points (if there are Delone-polygons with larger numbers of sides, they are triangulated in an arbitrary way). By saturatedness any triangle in this triangulation has a circumradius at most (actually, smaller than) 2. The number density of this system of points is \( 1/(2\overline{V}) \), where \( \overline{V} \) is the average area of the Delone-triangles. (For this observe that the total angle sum of all Delone triangles in some large circle about the origin is about \( 2\pi \) times the total number of vertices.) So we have to estimate \( \overline{V} \) from below.

2. Let \( \Delta ABC \) be a non-obtuse triangle of our Delone triangulation. We are going to show that

\[ (2.8.2) \quad V(\Delta ABC) \geq V_0(L) \ (\text{cf. } (2.8)). \]
The points $A, B, C$ are the intersection points of the axes of rotation of suitably translated copies of $L$ with the $x_1x_2$-coordinate plane. Let these translated copies be $L+x_1, L+x_2, L+x_3$. Then by the definition of the function $g$ we have

\begin{align}
|AB| &= \sqrt{(x_{21} - x_{11})^2 + (x_{22} - x_{12})^2} \\
&\geq 2g(x_{23} - x_{13}, \ldots, x_{2n} - x_{1n}).
\end{align}

The analogous inequalities hold also for $|BC|$ and $|CA|$.

That is, the side lengths of the triangle $\triangle ABC$ are at least certain values of the function $g$. By (2.8.1) and (2.7) the values of the function $g$ lie in the interval $[1/\sqrt{2}, 1]$. We define the triangle $T$ as follows.

\begin{align}
\text{(2.8.4) } \\
\text{Denote } T \text{ a triangle with side lengths } 2g(x_{23} - x_{13}, \ldots, x_{2n} - x_{1n}), \\
&\quad 2g(x_{33} - x_{23}, \ldots, x_{3n} - x_{2n}) \text{ and } 2g(x_{13} - x_{33}, \ldots, x_{1n} - x_{3n}).
\end{align}

Observe that a triangle with any given side lengths, with maximal side/minimal side $\leq \sqrt{2}$ exists (by the triangle inequality) and is non-obtuse (by the cosine law), thus $T$ is a non-obtuse triangle. Then using (3.0) we have

\begin{align}
\text{(2.8.5) } V(\triangle ABC) &\geq V(T).
\end{align}

By rotational symmetry of $L$ with respect to the $x_3 \ldots x_n$-coordinate plane, we can change the first two coordinates of the translation vectors $v_1 := (x_{11}, x_{12}, \ldots, x_{1n}), v_2 := (x_{21}, x_{22}, \ldots, x_{2n})$ and $v_3 := (x_{31}, x_{32}, \ldots, x_{3n})$, thus obtaining new translation vectors $v'_1 := (x'_{11}, x'_{12}, x_{13}, \ldots, x_{1d}), v'_2 := (x'_{21}, x'_{22}, x_{23}, \ldots, x_{2n})$ and $v'_3 := (x'_{31}, x'_{32}, x_{33}, \ldots, x_{3n})$ such that the respective new translated copies of $L$ are pairwise touching (in the sense prevised in (2.5)). The projections of $v'_1, v'_2, v'_3$ to the $x_1x_2$-coordinate plane are denoted by $A', B', C'$. Let $T' := \Delta v'_1v'_2v'_3$; its projection to the $x_1x_2$-coordinate plane is $\Delta A'B'C'$, which is congruent to $T$. We may and will suppose that actually

\begin{align}
\text{(2.8.6) } \Delta A'B'C' = T.
\end{align}

By the definition of $V_0(L)$ (otherwise said, varying the coordinates $x_{13}, \ldots, x_{1n}, x_{23}, \ldots, x_{2n}$ and $x_{33}, \ldots, x_{3n}$ of the translation vectors in an arbitrary way, and then taking infimum over these coordinates), we obtain that

\begin{align}
\text{(2.8.7) } V(T) &\geq V_0(L).
\end{align}

Then (2.8.5) and (2.8.7) together give (2.8.2).
3. Clearly

\[
\text{the isosceles right triangle with legs of length } 2m(L) \text{ has at least such an area, as the isosceles triangle with sides } 2m(L), 2m(L), 2M(L).
\]

In formula,

\[
(2.8.9) \quad 2m(L)^2 \geq M(L)\sqrt{(2m(L))^2 - M(L)^2}
\]

(this also follows from the arithmetic-geometric mean inequality).

Now we show that,

\[
(2.8.10) \quad \begin{cases}
\text{for any } \varepsilon > 0, \text{ for a suitable two-dimensional lattice packing of} \\
\text{translates of } L \text{ generated by three mutually touching translates of} \\
L \text{ (in the sense of (2.5)), a non-obtuse Delone triangle on the} \\
x_1x_2\text{-coordinate plane can have an area at most}
\end{cases}
\]

\[
M(L)\sqrt{(2m(L))^2 - M(L)^2} + \varepsilon.
\]

Let us consider the following translates of \( L: L + (M(L), 0, \ldots, 0), L - (M(L), 0, \ldots, 0) \) (these are touching translates in the sense of (2.5)) and \( L + (0, x_2, x_3, \ldots, x_n) \), where \( x_2 \) is chosen so that \( L \) and any of \( L \pm (0, x_2, x_3, \ldots, x_n) \) should be touching (in the sense of (2.5)), for any given \( (x_3, \ldots, x_n) \in \mathbb{R}^{n-2} \). (This is possible by the rotational symmetry of \( L \) about the \( x_3 \ldots x_n\)-coordinate plane.) Then the intersection points of the axes of rotation (translates of the \( x_3 \ldots x_n\)-coordinate plane) of these three translates of \( L \) with the \( x_1x_2\)-coordinate plane are the vertices of an isosceles triangle \( T(\delta) \), with base of length \( 2M(L) \) and equal sides of lengths \( 2g(x_3, \ldots, x_n) \). By a suitable choice of \( x_3, \ldots, x_n \) we can attain that \( g(x_3, \ldots, x_n) - m(L) \in [0, \delta) \), which for suitably small \( \delta > 0 \) implies

\[
(2.8.11) \quad V(T(\delta)) - M(L)\sqrt{(2m(L))^2 - M(L)^2} \in [0, \varepsilon),
\]

proving the inequality in (2.8.10), provided \( T(\delta) \) is a Delone triangle, for a suitable packing of translates of \( L \).

Still we have to show that \( T(\delta) \) is a non-obtuse Delone triangle, for a suitable packing of translates of \( L \). The base of \( T(\delta) \) has length \( 2M(L) \), and its equal sides have length in \([2m(L), 2(m(L) + \delta)]\). By \( M(L)/m(L) \leq \sqrt{2} \) this triangle is non-obtuse. There remains to show that \( T(\delta) \) is a Delone triangle for a suitable packing of translates of \( L \).
4. We consider the (inhomogeneous) two-dimensional lattice of translates of \( L \) spanned by the translates of \( L \) by the three new translation vectors \( v'_1, v'_2, v'_3 \) from 2. Any two of these three translates of \( L \) are touching. Then, in the spanned body lattice, any two translates of \( L \), which are simultaneous translates of any two of these three mutually touching translates, are also touching. If \( L \) and \( L + v' \) are two translates of \( L \) from this body lattice which are not simultaneous translates of any two of our three mutually touching translates \( L + v'_1, L + v'_2, L + v'_3 \), then the following holds. If we decompose our two-dimensional lattice of translates of \( L \) to one-dimensional lattices of translates of \( L \) spanned by suitable two of our three mutually touching translates of \( L \), then \( L \) and \( L + v' \) are not in the same above one-dimensional lattice of translates of \( L \), but also not in neighbourly above one-dimensional lattices of translates of \( L \).

The distance of the orthogonal projections to the \( x_1x_2 \)-coordinate plane of the straight lines spanned by homologous points in translates of \( L \) for two neighbourly above one-dimensional lattices of translates of \( L \), is the height of a triangle with vertices the points of intersection of the axes of rotation of three mutually touching copies of \( L \), with sides in \([2m(L), 2M(L)] = [2m(L), 2] \subset [\sqrt{2}, 2]\) with the \( x_1x_2 \)-coordinate plane. The minimal height belongs to the maximal side (for which the adjacent angles are acute), and for given length of the maximal side, it is attained if the other two sides are minimal possible, i.e., have length \( 2m(L) \). Hence the height is at least \( \sqrt{(2m(L))^2 - M(L)^2} = \sqrt{(2m(L))^2 - 1} \geq \sqrt{2-1} = 1 \).

Therefore the distance of the orthogonal projections to the \( x_1x_2 \)-coordinate plane of the straight lines spanned by homologous points in translates of \( L \) for two neighbourly above one-dimensional lattices of translates of \( L \) is at least 1. This implies that the distance of the orthogonal projections to the \( x_1x_2 \)-coordinate plane of the straight lines spanned by homologous points in translates of \( L \) for two at least second neighbour above one-dimensional lattices of translates of \( L \) is at least 2 \( (= \sup 2g \), cf. (2.7)). Therefore the translates \( L \) and \( L + v' \) are disjoint. This shows that our two-dimensional lattice arrangement of translates of \( L \) is in fact a two-dimensional lattice packing, spanned by some three mutually touching translates of \( L \) (in the sense of (2.5)).

Then the points of intersection of the axes of rotation of all translates of \( L \) in the two-dimensional lattice packing of translates of \( L \) is the point lattice in the \( x_1x_2 \)-coordinate plane generated by the projections of the three new translation vectors \( v'_1, v'_2, v'_3 \) from 2 to the \( x_1x_2 \)-coordinate plane, i.e., the points \( A', B', C' \), which are the vertices of \( T(\delta) \) by (2.8.6) (observe that the role of
T is now taken over by $T(\delta)$.

The corresponding two-dimensional point lattice projects orthogonally to the $x_1x_2$-plane injectively onto a two-dimensional point lattice, generated by $A', B', C'$ in the $x_1x_2$-coordinate plane, since $m(L) > 0$. Moreover, the Delone triangulation of the $x_1x_2$-coordinate plane corresponding to this point lattice has as Delone triangles $T(\delta) = A'B'C'$ and all its lattice translates (by this point lattice) and $A' + B' - T(\delta)$ and all its lattice translates (by this point lattice). Therefore the number density of this point lattice in the $x_1x_2$-coordinate plane is

$$\frac{1}{2V(T(\delta)))} \in \left(\frac{2m(L)\sqrt{(2m(L))^2 - M(L)^2} + 2\varepsilon}{2M(L)\sqrt{(2m(L))^2 - M(L)^2}}\right)^{-1},$$

(2.8.12)

$$\left(\frac{2m(L)\sqrt{(2m(L))^2 - M(L)^2}}{2M(L)\sqrt{(2m(L))^2 - M(L)^2}}\right)^{-1} = \left(\frac{2\sqrt{4(m(L))^2 - 1 + 2\varepsilon}}{2\sqrt{4(m(L))^2 - 1}}\right)^{-1}, \left(\frac{2\sqrt{4(m(L))^2 - 1}}{2\sqrt{4(m(L))^2 - 1}}\right)^{-1}$$

All this shows the second statement of the theorem, taking in consideration (2.9).

5. (2.8.8), (2.8.9) and (2.8.10) imply that

$$2m(L)^2 \geq M(L)\sqrt{(2m(L))^2 - M(L)^2} = \inf_{\delta > 0} V(T(\delta)) \geq V_0(L),$$

which in turn implies

$$\min\{V_0(L), 2m(L)^2\} = V_0(L).$$

Hence, for any packing of translates of $L$, the number density of the intersections of the axes of rotation of the translates of $L$ in the packing is by Theorem 2.9 at most

$$1/ \min\{2V_0(L), 4m(L)^2\} = 1/ (2V_0(L)),$$

(2.8.14)

which implies by hypothesis (2.9) the theorem. ■

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Proof of Proposition 2.6. This follows from Proposition 2.2. We claim that the packing density of translates of $L$ is at most the density of a body lattice of translates of $K$. Here the corresponding point lattice is the sum
of the \((n-2)\)-dimensional point lattice in the \(x_3 \ldots x_n\)-coordinate plane from this Proposition, and a two dimensional point lattice that intersects the \(x_3 \ldots x_n\)-coordinate plane only in \(\{0\}\) (by inf \(g > 0\)). Still recall that there exists a densest lattice packing of translates of \(K\). 

**Proof of Proposition 2.5.** Let us consider the Dirichlet-Voronoj (DV) decomposition of the \(x_3 \ldots x_n\)-coordinate plane associated to the centres of our balls occurring in the packing with union \(L\). Then the DV-cell of any of these unit balls is contained in the concentric ball of radius \(\sqrt{2}\).

Clearly we have sup \(g = 1\). Next we show that inf \(g \geq 1/\sqrt{2}\). Let \(L\) and \(L + (x_1, \ldots, x_n)\) be two touching copies of translates of \(L\). Let \(B_0\) be one of the balls constituting the packing \(L + (x_1, \ldots, x_n)\), with centre \((x_{10}, \ldots, x_{n0})\). We project this centre orthogonally to the \(x_3 \ldots x_n\)-coordinate plane, obtaining the point \((0, 0, x_{30}, \ldots, x_{n0})\). This last point is in the DV-cell of the centre \((0, 0, x_{31}, \ldots, x_{n1})\) of one of the balls constituting the packing \(L\). Hence by hypothesis of the Proposition the distance of \((0, 0, x_{30}, \ldots, x_{n0})\) and \((0, 0, x_{31}, \ldots, x_{n1})\) is at most \(\sqrt{2}\). By Pythagoras theorem, the distance of the \(x_3 \ldots x_n\)-coordinate plane and \((x_{10}, \ldots, x_{n0})\) is at least \(\sqrt{2}\). Hence inf \(g \geq \sqrt{2}\), and therefore \(1 = M(L) \geq m(L) \geq 1/\sqrt{2}\) (cf. (2.7)). Now applying Theorem 2.8 this proposition is proved. 

**Proof of Proposition 2.4.** 1. By hypothesis (2.2) the density of the packing consisting of some translated copies of \(L\) is proportional to the number density of the intersection of their rotation axes with the \(xy\)-plane. We have sup \(g = 1\). We are going to show that inf \(g \geq 1/\sqrt{2}\). Let us consider two different translates of \(L\) from our packing of translates of \(L\), say, \(L + (x_1, y_1, z_1)\) and \(L + (x_2, y_2, z_2)\). Let us consider a segment \([z - R, z + R] \subset \mathbb{R}\). Let us project the centres of the balls consisting \(L + (x_1, y_1, z_1)\) and \(L + (x_2, y_2, z_2)\) to the \(z\)-axis. Then the number of the projection points in \([z - R, z + R]\) is \(2R/(2d) + o(R)\), where \(o(R)\) is uniform for all \(z\)'s and all translates \(L + (x_1, y_1, z_1)\) and \(L + (x_2, y_2, z_2)\) consisting our packing of translates of \(L\). Then the number of projection points (possibly with multiplicities) of the centres of balls consisting \(L + (x_1, y_1, z_1)\) and \(L + (x_2, y_2, z_2)\) is \(2R/d + o(R)\). Hence some neighbouring projections have a distance is at most \(d + o(1) \leq \sqrt{2} + o(1)\), with \(o(1)\) uniform (as above with \(o(R)\)). Then both projections cannot come from \(L + (x_1, y_1, z_1)\), or from \(L + (x_2, y_2, z_2)\), since they are unions of packings of unit balls, and then this distance should be at least 2. Hence these neighbouring projections come one from \(L + (x_1, y_1, z_1)\), and the other one from \(L + (x_2, y_2, z_2)\).

By Pythagoros theorem, the distance of the axes of rotation of \(L + (x_1, y_1, z_1)\) and \(L + (x_2, y_2, z_2)\) is a least \(\sqrt{2} + o(1)\). Letting \(R \to \infty\), we get that the distance of these axes of rotation is at least \(\sqrt{2}\).

Thus the hypotheses of Theorem 2.8 are satisfied, hence also its conclusion is satisfied, i.e., the statement of Proposition 2.4 is proved.
Proof of Theorem 2.1. By \( d \leq \sqrt{2} \) we have that the concentric 1-dimensional balls of radius \( \sqrt{2} \) cover \( \mathbb{R} \), hence we may apply Proposition 2.5. Thus the density of our packing of translates of \( L \) is at most the maximal density of the two-dimensional lattice packings of \( L \), where the corresponding point lattice projects orthogonally to the \( xy \)-plane injectively, onto a two-dimensional point lattice in the \( xy \)-plane. (This maximum exists.) We have to prove that these two-dimensional lattice packings of translates of \( L \) have a density at most \( \pi/(3\sqrt{3} - d^2) \), and the unique densest such lattice packing is the one given in this Theorem.

Evidently in a densest lattice packing of translates of \( L \) there are two translates of \( L \) touching each other. These together generate a two-dimensional lattice \( B^3 + \Lambda := \{B^3 + \lambda \mid \lambda \in \Lambda \} \) of balls (recall that \( B^3 \) is the closed unit ball centred at the origin). We may assume that the 2-plane \( \Pi \) spanned by the centres of these balls is the \( xy \)-plane.

Join the consecutive centres of the balls in translates of \( L \) belonging to \( B^3 + \Lambda \) by segments, and also join the centres of each pair of balls belonging to two neighbouring translates of \( L \) in \( B^3 + \Lambda \) and touching each other. Thus the \( xy \)-plane is subdivided to a lattice of parallelograms of sides \( 2d \) and 2.

Still draw the diagonals of these parallelograms joining two obtuse angles, thus cutting these parallelograms into two triangles. If the parallelogram is a rectangle, we draw one of its diagonals (chosen parallel to each other). Thus we obtain a tiling \( \mathcal{T} \) of the \( xy \)-plane to triangles which are translates and centrally symmetric images of some fixed triangle from \( \mathcal{T} \).

Let \( \Delta ABC \) be one of the triangles from \( \mathcal{T} \), with \( A \) and \( B \) being the centres of balls belonging to the same translate of \( L \), and with \( |BC| = 2 \). Then, denoting the angles of \( \Delta ABC \) at \( A, B, C \) by \( \alpha, \beta, \gamma \), we have

\[
(2.1.1) \quad \alpha, \beta \leq \pi/2.
\]

We have also

\[
(2.1.2) \quad \gamma \leq \pi/2
\]

since else \( \sqrt{2} < |AB|/\min\{|BC|, |CA|\} = |AB|/|BC| = d \), contradicting the hypothesis of the theorem.

Our lattice of unit balls decomposes to horizontal layers which are translates of \( \{B^3 + \lambda \mid \lambda \in \Lambda \} \). We determine how close a neighbouring horizontal layer can be to \( B^3 + \Lambda \) (i.e., how close their mid-planes can be). Let \( E \) be the centre of a unit ball in a neighbouring horizontal layer. Then its distance to the centre of each ball in \( \Lambda \) is at least 2. Let \( E' \) be the orthogonal projection of \( E \) to the \( xy \)-plane. We want to determine the minimum of \( |EE'| \).
We may suppose that $E'$ lies in the above triangle $\Delta ABC \in \mathcal{T}$. Then

\[(2.1.3) \quad |EA|, |EB|, |EC| \geq 2.\]

Observe that we have right triangles $\Delta AEE'$, $\Delta BEE'$, $\Delta CEE'$, whose hypotenuses $EA$, $EB$, $EC$ have lengths at least 2 (by (2.1.3)). Then, by Pythagoras' theorem,

\[(2.1.4) \quad \left\{ \begin{array}{l} |EE'| \geq \max\{\sqrt{2^2 - |E'A|^2}, \sqrt{2^2 - |E'B|^2}, \sqrt{2^2 - |E'C|^2}\} \\
= \sqrt{4 - \left(\min\{|E'A|, |E'B|, |E'C|\}\right)^2}. \end{array} \right.\]

Since by (2.1.1) and (2.1.2) $\Delta ABC$ is a non-obtuse triangle, with the circumradius $R$ of $\Delta ABC$ we have

\[(2.1.5) \quad \left\{ \begin{array}{l} \min\{|E'A|, |E'B|, |E'C|\} \leq R, \text{ with equality if and only if } \\
E' \text{ is the circumcentre } c \text{ of } \Delta ABC. \end{array} \right.\]

Thus (2.1.4) implies

\[(2.1.6) \quad \left\{ \begin{array}{l} |EE'| \geq \sqrt{4 - R^2}, \text{ with equality if and only if } \\
E' = c \text{ and } |EA| = |EB| = |EC| = 2. \end{array} \right.\]

In what follows we assume that

\[(2.1.7) \quad \left\{ \begin{array}{l} E \text{ and } E' \text{ are the points for which (2.1.5) and (2.1.6)} \\
\text{become equalities, i.e., } E' = c \text{ and } |EA| = |EB| = |EC| = 2. \end{array} \right.\]

Still we have to show that

\[(2.1.8) \quad A, B, C \text{ and the above chosen } E \text{ generate a lattice packing.}\]

In a horizontal layer the corresponding open unit balls are disjoint. Two open unit balls in neighbourly horizontal layers are disjoint by the following reason. The minimal distance of $E$ and any lattice point in $\Lambda$ is attained if and only if the minimal distance of $E'$ and any lattice point in $\Lambda$ is attained. Observe that the DV-cells of $\Lambda$ in the $xy$-plane satisfy the following: $\Delta ABC$ is covered by the DV-cells of $A, B, C$, whose common vertex is $c$. Thus $c$ is closest among any points of $\Lambda$ to $A, B, C$, and this minimal distance is $R$. In particular, all other lattice points of $\Lambda$ have a distance at least $R$ from $E' = c$. Then also $E$ has a distance at least $\sqrt{R^2 + |EE'|^2} = 2$ from any lattice points of $\Lambda$.

It remains to show that the open unit balls in at least second neighbour horizontal layers are disjoint. For this it is sufficient to show that the height $|EE'|$ of the tetrahedron $ABCE$ corresponding to the vertex $E$ is at least 1.
Since $|EE'| = \sqrt{2^2 - R^2}$, therefore

\begin{equation}
(2.1.9) \quad \text{we have to prove that } R \leq \sqrt{3}.
\end{equation}

Observe that $c$ lies on the perpendicular bisectors of the sides $AB$ and $BC$ of $\Delta ABC$. If $|AC|$ increases from its minimum 2 till its maximum $2\sqrt{d^2 + 1}$ then the angle $\beta = \angle ABC$ strictly increases, and by elementary geometric considerations $c$ moves on the perpendicular bisector of side $AB$ farther from side $AB$. Then $c$ will be the farthest from side $AB$ when $|AC|$ attains its maximum $2\sqrt{d^2 + 1}$, which happens if $\Lambda$ is a rectangular lattice on the $xy$-plane, when by $(2.1.10)$

\begin{equation}
R = \sqrt{d^2 + 1} \leq \sqrt{3},
\end{equation}

proving our claim $(2.1.9)$ and thus also $(2.1.8)$.

Now we investigate the volume of the basic parallelepiped of our lattice. We consider the tetrahedron $ABCE$. Its face $\Delta BCE$ is a regular triangle of side 2, i.e., $|BC| = |CE| = |EB| = 2$, and $|AE| = 2$, $|AB| = 2d$ and lastly $2 \leq |AC| \leq 2\sqrt{d^2 + 1}$. Thus the faces $\Delta BCE$ and $\Delta ABE$ are given up to congruence. They join at their common edge $BE$. If their angle is $\varphi$, then the volume $V(ABCE)$ of our tetrahedron is proportional to $\sin \varphi$. Now we calculate $|AC|$. Let us denote the projections of $A$ and $C$ to the line $BE$ by $A'$ and $C'$. Then $|AC|^2 = |A'C'|^2 + |A'A|^2 + |C'C|^2 - 2 |A'A| \cdot |C'C| \cdot \cos \varphi$, thus $|AC|$ is a strictly increasing function of $\varphi$. We have $2 \leq |AC| \leq 2\sqrt{d^2 + 1}$. We denote the values of $\varphi$ belonging to $|AC| = 2$ and $|AC| = 2\sqrt{d^2 + 1}$ by $\varphi_{\text{min}}$ and $\varphi_{\text{max}}$. Then we have $\varphi_{\text{min}} \leq \varphi_{\text{max}}$. Since the sine function is strictly concave on $[0, \pi]$, on the interval $[\varphi_{\text{min}}, \varphi_{\text{max}}]$ the function $\sin \varphi$, and thus also $V(ABCE)$ attains its minimum either at $\varphi = \varphi_{\text{min}}$, or at $\varphi = \varphi_{\text{max}}$, but not at any $\varphi \in (\varphi_{\text{min}}, \varphi_{\text{max}})$.

For $\varphi = \varphi_{\text{max}}$ we have $|AC| = 2\sqrt{d^2 + 1}$, hence our parallelogram lattice on the $xy$-plane is a lattice of $2 \times 2d$ rectangles. Then our lattice, $\Lambda_1$, say, is generated by the vertices of a rectangular pyramid $ABCDE$ with basis edges of lengths 2 and $2d$, and lateral edges 2, which is one of the forms of giving the lattice in the Theorem. The density of the corresponding unit ball packing is

\begin{equation}
(2.1.11) \quad (4\pi/3)/[2 \cdot 2d \cdot \sqrt{2^2 - (1 + d^2)}],
\end{equation}

as asserted in the Theorem.

For $\varphi = \varphi_{\text{min}}$ the triangulation of the $xy$-plane consists of isosceles triangles of base $2d$ and lateral sides 2. Then as above, the next horizontal layer has a centre $E$ of a unit ball such that $|AE| = |BE| = |CE| = 2$, where $\Delta ABC$ is the triangle of the triangulation $T$ of the $xy$-plane, containing the
projection $E'$ of $E$ to the $xy$-plane. The remaining three edges of the tetrahedron $ABCE$ have lengths $|AC| = |BC| = 2$ and $|AB| = 2d$. It remains to show that this lattice, $\Lambda_2$, say, is, up to congruence, the same lattice as $\Lambda_1$.

For convenience, we identify the vertices with the respective vectors. Then $\triangle ABC$ and $\triangle ACE$ and $\triangle BCE$ can be completed to parallelograms with vertices $A, C, B, A' := A + E - C$ and $A, C, E, B' := B + E - C$ and $B, C, E, C' := A + B - C$. The new vertices $A', B', C'$ belong to the respective point lattice. Then $A' - A = B' - B = E - C$, hence the quadrangle $ABB'A'$ is a parallelogram, with edge lengths $|AA'| = |BB'| = |CE| = 2$ and $|AB| = |B'A'| = 2d$. However, this parallelogram is in fact a rectangle, since $\langle B - A, A' - A \rangle = \langle B - A, E - C \rangle = 0$, since $|AC| = |BC| = |AE| = |BE| = 2$, and thus $[E, C]$ lies on the orthogonal bisector plane of $[A, B]$. We assert that the distance of $C'$ from each vertex of the parallelogram $ABB'A'$ is 2. In fact, we have $C' - A = B - C$ while $|BC| = 2$, and similarly $C' - B$ has length 2. Further, $C' - A' = B - E$ while $|BE| = 2$, and similarly $C' - B'$ has length 2. Hence the lattice $\Lambda_1$ is a sublattice of this new lattice $\Lambda_2$. Now we assert that

$$\begin{cases}
\text{the volume of the basic tetrahedron } ABCE \text{ of } \Lambda_1 \\
\text{and of the lattice tetrahedron } AC'A' \text{ of } \Lambda_2 \\
AC'A' = AC(A + E - C)(A + B - C) \text{ of } \Lambda_2
\end{cases}$$

are equal, from which the equality of the lattices $\Lambda_1$ and $\Lambda_2$ (up to congruence) follows.

By a translation we achieve $E = 0$ Then we have tetrahedra $0ABC$ and $AC(A - C)(A + B - C)$, or by a translation, $0(C - A)(-C)(B - C)$. The second tetrahedron is the linear image of the first one by the matrix

$$\begin{pmatrix}
-1 & 0 & 0 \\
0 & 0 & 1 \\
1 & -1 & -1
\end{pmatrix}$$

which has determinant $-1$. Hence we have showed (2.1.12), which ends the proof of the theorem. ■

Proof of Theorem 2.3. For the two-dimensional square lattice $\Lambda'$ ($\subset x_3x_4$-coordinate plane) with edge length 2 the concentric balls of radius $\sqrt{2}$ cover the $x_3x_4$-coordinate plane. For the two-dimensional regular triangular lattice $\Lambda''$ ($\subset x_3x_4$-coordinate plane) with edge length 2 the concentric balls of radius $2/\sqrt{3}$ cover the $x_3x_4$-coordinate plane (and $2/\sqrt{3} < \sqrt{2}$). Then we may apply Proposition 2.5, since a lattice packing of unit balls satisfies its hypothesis (2.2). Thus any packing of translates of these sets $L$ has a density at most the
supremum of the densities of the lattice packings of $B^4$, with corresponding point lattices $\Lambda + \Lambda'$, or $\Lambda + \Lambda''$, respectively, where $\Lambda$ is like in Proposition 2.5. Now recall that for lattice packings of $B^4$ in $\mathbb{R}^4$ the maximal density is $\pi^2/16$. $lacksquare$

§6 Some remarks about the Main Lemmas in §4

Observe that in Lemma 4.1 we did not use the hypothesis about the upper bound of the circumradii of the triangles. Probably also in the first inequality of Lemma 4.2 and in Lemma 4.3 we may omit the circumradius hypotheses and we have even then $p^2(\sqrt{7}/4 + 1)$ and $p^2(\sqrt{3}/2 + 3/2)$ as lower bounds. In Lemmas 4.4-4.6, i.e., when $\Delta ABC$ was right or obtuse angled, we used the upper bound of the circumradius only for $\Delta ABC$. However, the hypotheses in Lemmas 4.4-4.6 imply that the circumradii of the triangles $\Delta A_1 BC$, $\Delta AB_2 C$, $\Delta ABC_3$ are bounded above by the circumradius of $\Delta ABC$, so actually the upper bound of the circumradii is valid for all triangles in Lemmas 4.4-4.6.

Moreover, in Lemmas 4.2-4.6 the average value of the areas of the considered triangles is strictly greater than $p^2/2$ (in Lemma 4.4 in the first case). This means that we could relax the hypothesis that the circumradii are at most $p\sqrt{2}$ a bit, and still statements corresponding to Lemmas 4.2-4.6 would give that the average values of the areas of the triangles in these lemmas are at most $p^2/2$, which would still make it possible to prove our Theorem 2.9 (as we saw in §5, its proof in §5 only used Lemmas 4.1-4.6).

Of course, this would not imply a sharpening of our packing theorems in §2. We show this on the example of Theorem 2.1. If in Theorem 2.1 we would allow $d = \sqrt{2} + \varepsilon > \sqrt{2}$, then we consider the second way of describing the densest lattice packing of translates of $L$. It is generated by the vertices of a rectangular right pyramid with base edges of lengths 2 and $2d$, and lateral edges of length 2. The vertices of the base form a $2 \times 2d$ rectangular lattice in a plane, and closely packed translates of this rectangular lattice form the densest packing of translates of $L$. Observe that the height of our pyramid is $\sqrt{3 - d^2}$. This is at least 1 for $1 \leq d \leq \sqrt{2}$, but for $d > \sqrt{2}$ it is less than 1, which means that the closely packed translates of the rectangular lattice being just above and just below the plane of the original rectangular lattice overlap. Locally we may attain the density corresponding to two closely packed neighbourly translates of our original rectangular lattice, but this cannot be continued over the whole space. Thus we cannot obtain a sharp upper bound for the packing densities in Theorem 2.1 for $d > \sqrt{2}$. This implies that also Propositions 2.4-2.6 and Theorem 2.8 cannot be extended to any $d > \sqrt{2}$, and any covering radius greater than $\sqrt{2}$, and any $m(L) < 1/\sqrt{2}$, and any $m(L) < 1/\sqrt{2}$, respectively.

Now we return to Lemmas 4.4-4.6. Clearly Lemma 4.4 is
false for any $R > p\sqrt{5/2} = p \cdot 1.5811\ldots$. Probably in Lemma 4.5 we could admit such circumradii $R$, that the convex quadrangle $ABA_1C$ satisfying that the circumradius of $\Delta ABC$ is $R$ and $c_1 = b_1 = b = p$ and $\alpha_1 = \pi/2$ should have an area at least $p^2$, which gives that $R \leq p\sqrt{5/2}$, like for Lemma 4.4. Similarly, probably also in Lemma 4.6 we could admit circumradii $R \leq p\sqrt{5/2}$. As numerical evidence, we give for $R = p\sqrt{5/2}$ and $c_1 = b_1 = a_2 = c_2 = p$ and either for (1) $\alpha_1 = \beta_2 = \pi/2$, or for (2) $\alpha_1 = \pi/2$ and $\beta + \beta_2 = \pi$, or for (3) $\alpha + \alpha_1 = \beta + \beta_2 = \pi$ the numerical values of the total area of the three triangles. These are (1) $p^2 \cdot 1.8$, or (2) $p^2 \cdot 1.7$ (exact values!), or (3) $p^2 \cdot 1.8624\ldots??\ldots??$, all of which are greater than $p^2 \cdot 1.5$. (Observe that for $R = p\sqrt{2}$ the corresponding values decreased in this order, so the eventual proofs for $R = p\sqrt{5/2}$ should be more complicated than those in our paper.) However, because of the rather technical proofs in §3, where we essentially used for our numerical calculations the inequality that the circumradius of $\Delta ABC$ is at most $p\sqrt{2}$, it is not clear whether the proofs of Lemmas 4.3, 4.5 and 4.6, with average area of the triangles at least $p^2/2$, can be done for all circumradii at most $p\sqrt{5/2}$.

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References

[BKM91] A. Bezdek, W. Kuperberg, E. Makai, Jr., Maximum density space packing with parallel strings of spheres, Discrete Comput. Geom. 6 (1991), 277-283. MR 92a:52024.

[BFT] M. N. Bleicher, L. Fejes Tóth, Circle-packings and circle coverings on a cylinder, Michigan Math. J. 11 (1964), 337-341. MR 29#6393.

[BF] T. Bonnesen, W. Fenchel, Theorie der konvexen Körper, Berichtigter reprint, Springer, Berlin-New York, 1974. MR 49#9736.

[Bo] K. Böröczky, Problem 12, Period. Math. Hungar. 6 (1975), 109. MR 1553591.

[BKM99] Böröczky, K., Kertész, G., Makai, E. Jr., The minimum area of a simple polygon with given side lengths. Discrete Geom. and Rigidity (Budapest, 1999). Period. Math. Hungar. 39 (1999), 33-49. MR 2001e:51016.

[BKJ] K. Böröczky, Jr., Finite packing and covering, Cambridge Tracts in Math. 154, Cambridge Univ. Press, Cambridge, 2004. MR 2005g:52045.

[CK] H. Cohn, A. Kumar, Optimality and uniqueness of the Leech lattice among lattices, Ann. Math. (2) 170 (2009), 1003-1050.

[CKMRV] H. Cohn, A. Kumar, S.D. Miller, D. Radchenko, M. Viazovska, The sphere packing problem in dimension 24, arXiv:1603.16518 (2016).

[FT62] L. Fejes Tóth, Dichteste Kreispackungen auf einem Zylinder, Elem. Math. 17 (1962), 30-33. MR 24#A3563.

[FT64] L. Fejes Tóth, Reguläre Figuren; in English: Regular figures, Akad. Kiadó; A Pergamon Press book, Macmillan, Budapest; New York, 1965; 1964. MR 30#3408, 29#2705.

[FT72] L. Fejes Tóth, Lagerungen in der Ebene, auf der Kugel und im Raum, Zweite verbesserte und erweiterte Auflage, Grundlehren Math. Wiss. 65. Springer, Berlin-New York, 1972. MR 50#5603.
L. Fejes Tóth, oral communication.

Z. Füredi, The densest packing of equal circles into a parallel strip, Discrete Comput. Geom. 6 (1991), 95-106. MR 92e:52025.

P.M. Gruber, C.G. Lekkerkerker, Geometry of numbers, 2-nd ed., North-Holland Math. Library, 37, North-Holland, Amsterdam etc., 1987. MR 88j:11034.

T. C. Hales, with S. Ferguson, The Kepler conjecture, Special issue, Discr. Comput. Geom. 36 (2006 (1)), 5-269. MR 2007d:52021, 52022, 52023, 52024, 52026.

T. C. Hales, J. Harrison, S. McLaughlin, T. Nipkow, S. Obua, R. Zunkelel, A revision of the proof of the Kepler conjecture, Discr. Comput. Geom. 44 (2010), 1-34. MR 2012h:52044.

A. Heppes, Some densest two-size disc packings in the plane, U.S.-Hungarian Workshops on Discr. Geom. and Convexity (Budapest, 1999/Auburn, AL, 2000), Discr. Comput. Geom. 30 (2003), 241-262. MR 2004h:52020.

N. D. Kazarinoff, Geometric inequalities, New Math. Library 4, Random House and Yale Univ., New York-Toronto, 1961. MR 24#A1.

G. Kertész, unpublished paper (1980).

J. Molnár, On the $p$-system of unit circles, Ann. Univ. Sci. Budapest. Eötvös Sect. Math. 20 (1977), 195-203. MR 58#12733.

J. Molnár, Packings of congruent spheres in a strip, Acta Math. Acad. Sci. Hungar. 31 (1978), 173-183. MR 58#7406.

C. A. Rogers, Packing and covering, Cambridge Tracts in Math. and Math. Phys. 54, Cambridge Univ. Press, Cambridge, 1964. MR 30#2405.

R. Schneider, Convex bodies: the Brunn-Minkowski theory., Second expanded ed., Enc. of Math. and its Appl., 151, Cambridge Univ. Press, Cambridge, 2014. MR 3155183.

E. Székely (=J. Székely), oral communication.

M. Viazovska, The sphere packing problem in dimension 8, arXiv:1603.04246 (2016).