Lie Symmetries and Exact Solutions of the Generalized Thin Film Equation

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Abstract

A symmetry group classification for fourth-order reaction-diffusion equations, allowing for both second-order and fourth-order diffusion terms, is carried out. The fourth order equations are treated, firstly, as systems of second-order equations that bear some resemblance to systems of coupled reaction-diffusion equations with cross diffusion, secondly, as systems of a second-order equation and two first-order equations. The paper generalizes the results of Lie symmetry analysis derived earlier for particular cases of these equations. Various exact solutions are constructed using Lie symmetry reductions of the reaction-diffusion systems to ordinary differential equations. The solutions include some unusual structures as well as the familiar types that regularly occur in symmetry reductions, namely self-similar solutions, decelerating and decaying traveling waves, and steady states.

1. Introduction

We consider the fourth order nonlinear partial differential equation (PDE) of the form

\[ u_t = -[K(u)u_{xxx}]_x + [D(u)u_x]_x + F(u), \tag{1} \]

where \( K, D \) and \( F \) are arbitrary smooth functions (hereafter the subscripts \( t \) and \( x \) denote differentiation with respect to these variables). Eq. (1) generalizes a wide range of the known scalar reaction-diffusion equations arising in applications. The case with \( K \) identically zero and \( D(u) > 0 \) for almost all \( u \), is the case of second-order reaction-diffusion which has already been widely studied in many practical contexts including combustion, population dynamics, population genetics, neurobiology, biological cellular growth and adsorptive porous media, e.g. \cite{1}-\cite{5}. Hereinafter we assume that \( K(u) \) is not identically zero, so that the governing equation is of the fourth order, including a fourth-order diffusion term when \( K \) is non-negative. The simplest equation of the form (1), with \( F = 0 \) and \( D = 0 \), follows from the approximations of lubrication theory to describe thin films of a Newtonian liquid dominated by surface tension.
The thin film equations are an active area of research (see [6]–[8] and papers cited therein). The equation

\[ u_t = -[u^\gamma u_{xxx}]_x \]  \hspace{1cm} (2)

with the non-negative parameter \( \gamma \) was introduced in [9]. \( \gamma = 3 \) describes a classical thin film of Newtonian fluid, as reviewed in [10]. \( \gamma = 1 \) occurs in the dynamics of a Hele-Shaw cell [11] and \( \gamma = 2 \) arises in a study of wetting films with a free contact line between film and substrate [6].

One important generalization of Eq. (2), which is also a particular case of Eq. (1), reads as

\[ u_t = -[u^\gamma u_{xxx}]_x + [u^\mu u_x]_x, \]  \hspace{1cm} (3)

where \( \mu \) is a positive parameter (arbitrary positive coefficients of each term can be set to one by rescaling variables). Eq. (3) with \( \gamma = 0 \) can be considered as a semilinear limit of the classical Cahn-Hilliard model of phase separation [12], which is also widely studied (see [13] and the papers cited therein). The linear case \( \gamma = \mu = 0 \) also follows from a small-slope approximation to metal surface evolution, with surface-diffusion and evaporation-condensation represented by fourth-order and second-order diffusion terms [14, 15]. The case \( \gamma = \mu = 3 \) arises in a study of capillary instability of axisymmetric thin films [16].

Several papers are devoted to the construction of exact solutions of the thin film equations (2) by Lie symmetry reductions and their generalizations [17]–[24], or by searching for special invariant finite vector spaces of solutions [25]. The symmetry classification is extended here to include a reaction term. In some circumstances, the reaction term \( F \) should naturally arise in fourth-order transport equations with a role similar to that in second-order reaction-diffusion. For example, a particular case of Eq. (1) with \( F(u) = u \) occurs as the limiting case of the unstable Cahn-Hilliard equation [26, 27]. In fabricated metal surface evolution, a positive source term may represent ion beam sputtering [28] and a negative source term may represent chemical decay or evaporation [29]. Other examples of equations of the form (1) arising in applications and having a reaction term, are presented in [27].

The first aim of this paper is to describe all possible Lie symmetries, which Eq. (1) can admit depending on the function triplets \((K, D, F)\), i.e. to solve the so-called group classification problem, which was formulated and solved for a class of non-linear heat equations in the pioneering work in [30]. This problem for the second-order reaction-diffusion equation was solved in [31] (see also [32, 33] where the problem is solved for the general reaction-diffusion-convection equation). Note that the most general results concerning non-classical (\( Q \)-conditional) symmetries of reaction-diffusion equations were obtained in [34]–[36].

It should be noted that we shall not directly search for Lie symmetries of Eq. (1) but replace one scalar equation by an equivalent cross-diffusion system of equations. Using the symmetries found, we construct exact solutions of Eq. (1) with such triplets \((K, D, F)\), which arise in applications and compare the results obtained with those derived earlier.

Ovsiiannikov’s method of Lie symmetry classification of differential equations [37] is based on the classical Lie scheme and a set of equivalence transformations of a given equation. The formal application of this method to equations containing several arbitrary functions (Eq. (1)
contains three arbitrary functions) usually leads to a large number of equations admitting non-trivial Lie algebras of invariance. Our approach of Lie symmetry classification of differential equations is based on the classical Lie scheme and on finding and then making systematic use of the sets of local transformations that reduce any differential equation with a Lie algebra of invariance, to one given in the relevant list, that is representative of each equivalence class. This approach has earlier been applied also for reaction-diffusion systems [38–40].

The paper is organized as follows. Section 2 is devoted to a complete description of Lie symmetries of Eq. (1), i.e. all possible Lie symmetries, which this equation can admit depending on the form the functions $K$, $D$ and $F$, are found. In most applications, $K$ and $D$ are non-negative so that the diffusive transport processes are dissipative [41]. However, in solving a group classification problem, it is more common to allow the coefficient functions to be arbitrary smooth functions. In Section 3, the symmetry reductions and some exact solutions are constructed for particular cases of Eq. (1) that are likely to be useful in applications. The main results of the paper are summarized in the last section.

2. Lie symmetry of Eq. (1)

Firstly, we note that Eq. (1) can be reduced to the system

$$
\begin{align*}
  u_t &= -[K(u)v_x]_x + [D(u)u_x]_x + F(u), \\
  0 &= u_{xx} - v
\end{align*}
$$

by the substitution $v = u_{xx}$, $v = v(t, x)$. Since $u_x$ is not relabeled as a separate variable, point symmetries of this system do not include contact symmetries of the original single equation (1). Since second-order contact symmetries do not exist [42], point symmetries of this system could include nothing more than extensions (prolongations) of point symmetries of the single fourth order equation for $u(x,t)$. However, it is convenient to analyse system (4) as a cross-diffusion system, in which the second equation contains the time variable as a parameter. The motivation follows from the well-known fact that application of Lie’s algorithm in the case of high-order equations leads to very cumbersome formulae. Moreover, each Lie symmetry of Eq. (1) can be easily established from one of system (1) and there is one-to-one correspondence between solutions of (4) and (1). Thus, we shall investigate system (1) instead of Eq. (1).

Lie’s classical algorithm has been implemented within several computer-algebra routines. The classification problem may be solved by iterating a symmetry-finding program such as the REDUCE-based program DIMSYM [43] or MAPLE-based program DESOLV [44] that flag special cases of the coefficient functions that may give rise to additional symmetries. The REDUCE-based program CRACK [45] can fully solve some classification problems automatically. However, these programs may produce many special cases of equations with additional symmetries, which are equivalent up to the correctly specified local substitutions. Finding and systematically using the sets of such substitutions often leads to a significant practical reduction in the number of special cases that admit additional invariance. Thus, the problem under investigation here will be tractable without the assistance of a computer.

Now we formulate the main theorem, which presents the classification of special forms of system (1) (with K not identically 0) having additional symmetries.
Theorem 1  All possible maximal algebras of invariance (up to equivalent representations generated by transformations of the form (5)) of system (4) for any fixed triplet \((K, D, F)\) are presented in Table 1. Any other system of the form (4) with non-trivial Lie symmetry is reduced by a local substitution of the form

\[
\begin{align*}
t &\to C_0 t + C_1 e^{C_2 t}, \\
x &\to C_3 x, \\
u &\to C_4 + C_5 t + C_6 e^{C_7 t} u, \\
v &\to C_8 + C_9 e^{C_10 t} v
\end{align*}
\]

(5)
to one of those given in Table 1 (the constants \(C\) with subscripts are determined by the form of the system in question, some of them necessarily being zero in all particular cases).

Remark 1. All the systems listed in Table 1 are inequivalent up to arbitrary local substitution (not only of the form (5)!). This can be shown using the same approach as that used in [32] for the general reaction-diffusion-convection equation.

Proof. According to the classical Lie scheme [46–48], we consider system (4) as the manifold \((S_1, S_2)\) determined by the restrictions:

\[
\begin{align*}
S_1 &\equiv -u_t - [K(u) v_x]_x + [D(u) u_x]_x + F(u) = 0, \\
S_2 &\equiv u_{xx} - v = 0
\end{align*}
\]

(6)
in the space of the variables \(t, x, u, v, u_t, u_x, v_x, u_{xx}, v_{xx}\). The maximal algebra of invariance (MAI) of this system is generated by infinitesimal operators of the form

\[
X = \xi^0(t, x, u, v) \partial_t + \xi^1(t, x, u, v) \partial_x + \eta^1(t, x, u, v) \partial_u + \eta^2(t, x, u, v) \partial_v,
\]

(7)

where the functions \(\xi^0, \xi^1, \eta^1, \eta^2\) are to be determined. In order to determine these unknown functions one needs to use the invariance conditions

\[
\begin{align*}
X^2 S_1 &\equiv X^2(-u_t - [K(u) v_x]_x + [D(u) u_x]_x + F(u)) \big|_{S_1=0, S_2=0} = 0, \\
X^2 S_2 &\equiv X^2(u_{xx} - v) \big|_{S_1=0, S_2=0} = 0,
\end{align*}
\]

(8)

where \(X^2\) is the second prolongation of the operator \(X\):

\[
X^2 = \xi^0(t, x, u, v) \partial_t + \xi^1(t, x, u, v) \partial_x + \eta^1(t, x, u, v) \partial_u + \eta^2(t, x, u, v) \partial_v + \\
\tau_{10}^1 \partial_{u_t} + \tau_{10}^2 \partial_{u_x} + \tau_{10}^3 \partial_{u_{xx}} + \\
\tau_{20}^1 \partial_{v_t} + \tau_{20}^2 \partial_{v_x} + \tau_{20}^3 \partial_{v_{xx}} + \\
\tau_{11}^2 \partial_{u_{tt}} + \tau_{11}^3 \partial_{u_{tx}} + \tau_{11}^4 \partial_{u_{xx}} + \tau_{20}^2 \partial_{v_{tt}} + \tau_{11}^5 \partial_{v_{tx}} + \tau_{11}^6 \partial_{v_{xx}}.
\]

(9)
Table 1. Lie symmetries of \((\text{4})\)

| Case | Systems of the form \((\text{4})\) | Restrictions | Basic operators of MAI |
|------|----------------------------------|--------------|----------------------|
| 1.   | \(u_t = -[K(u)v_x]_x\) \(u_{xx} - v = 0\) | \(P_t, P_x, D_1 = 4t \frac{\partial}{\partial t} + x \frac{\partial}{\partial x} - 2v \frac{\partial}{\partial v}\) |
| 2.   | \(u_t = -[e^{\gamma u} v_x]_x + d[e^{\mu u} u_x]_x + \lambda e^{(2 \mu - \gamma) u} u_{xx} - v = 0\) | \(\gamma^2 + \mu^2 \neq 0\) \(d^2 + \lambda^2 \neq 0\) \(D_2 = 2(\gamma - 2\mu)t \frac{\partial}{\partial t} + (\gamma - \mu)x \frac{\partial}{\partial x} + 2(\partial_u - (\gamma - \mu)v \frac{\partial}{\partial v})\) |
| 3.   | \(u_t = -[v^2 u_x]_x + d[u^2 u_x]_x + \lambda e^{2 \mu - \gamma + 1} u_{xx} - v = 0\) | \(\gamma^2 + \mu^2 \neq 0\) \(d^2 + \lambda^2 \neq 0\) \(D_3 = 2(\gamma - 2\mu)t \frac{\partial}{\partial t} + (\gamma - \mu)x \frac{\partial}{\partial x} + 2(u \frac{\partial}{\partial u} + (\mu - \gamma + 1)v \frac{\partial}{\partial v})\) |
| 4.   | \(u_t = -v_{xx} + du_{xx} + \lambda u \ln u\) \(u_{xx} - v = 0\) | \(\lambda \neq 0\) \(Q_1 = e^\mu (u \frac{\partial}{\partial u} + v \frac{\partial}{\partial v})\) |
| 5.   | \(u_t = -[e^{\gamma u} v_x]_x\) \(u_{xx} - v = 0\) | \(\gamma \neq 0\) \(P_t, P_x, D_1, D_2\) with \(\mu = \gamma\) |
| 6.   | \(u_t = -[u^2 v_x]_x\) \(u_{xx} - v = 0\) | \(\gamma \neq 0\) \(P_t, P_x, D_1, D_2\) with \(\mu = \gamma\) |
| 7.   | \(u_t = -v_{xx} + du_{xx}\) \(u_{xx} - v = 0\) | \(d \neq 0\) \(P_t, P_x, I = u \frac{\partial}{\partial u} + v \frac{\partial}{\partial v}, X^\infty = P(t, x) \frac{\partial}{\partial u} + P_{xx}(t, x) \frac{\partial}{\partial v},\) where \(P_t + P_{xxx} - dP_{xx} = 0\) |
| 8.   | \(u_t = -v_{xx}\) \(u_{xx} - v = 0\) | \(P_t, P_x, I, D_1, X^\infty = P(t, x) \frac{\partial}{\partial u} + P_{xx}(t, x) \frac{\partial}{\partial v},\) where \(P_t + P_{xxx} = 0\) |

The coefficients with relevant subscripts \(\tau_1^1, \tau_2^1, \tau_1^{10}, \tau_2^{10}, \tau_1^{20}, \tau_1^{21}, \tau_2^{01}, \tau_2^{20}, \tau_1^{22}, \tau_2^{22}\) are calculated by the well-known prolongation formulae (see, e.g. [46]–[48]).

Substituting operator \((9)\) into system \((8)\) and carrying out the relevant calculations, we obtain so called system of determining equations for finding the functions \(\xi^0, \xi^1, \eta^1, \eta^2\). This is an overdetermined system of PDEs that can be written in the explicit form:

\[
\xi^0_x = \xi^0_u = \xi^0_v = \xi^1_t = \xi^1_u = \xi^1_v = \xi^1_{xx} = 0
\]  \(10\)
\[ \eta_1^1 = \eta_{1u}^1 = \eta_{uu}^1 = 0 \] (11)

\[ \eta^2 = \eta_{xx}^1 + (\eta_u^1 - 2\xi_x^1)v \] (12)

\[ K(u)(\xi_t^0 - 4\xi_x^1) + K'(u)\eta^1 = 0 \] (13)

\[ D(u)(\xi_t^0 - 2\xi_x^1) + D'(u)\eta^1 = 0 \] (14)

\[ \eta_1^1 + K(u)\eta_{xxx}^1 + F(u)(\eta_u^1 - \xi_t^0) - D(u)\eta_{xx}^1 - F'(u)\eta^1 = 0 \] (15)

Of course, Eqs. (10)–(12) are linear and don’t depend on the functions \( K, D \) and \( F \), hence their general solution can be easily constructed:

\[ \xi^0 = \xi_t^0(t), \]
\[ \xi^1 = \alpha x + x_0, \]
\[ \eta^1 = R(t)u + P(t,x), \]
\[ \eta^2 = P_{xx}(t,x) + (R(t) - 2\alpha)v, \]

where \( P(t,x) \), \( R(t) \) are arbitrary smooth functions, \( \alpha \) and \( x_0 \) are arbitrary constants. Eqs. (13)–(15) form the system of classification equations. Its general solution under assumption of arbitrarily given functions \( K, D \) and \( F \) generates the invariance algebra that consists only of generators of translations in \( x \) and \( t \) with the basis

\[ P_t = \frac{\partial}{\partial t} \equiv \partial_t, \quad P_x = \frac{\partial}{\partial x} \equiv \partial_x. \] (17)

The algebra with this basis is called the trivial Lie algebra of the system (11) (note that other authors, instead use ‘kernel of the basic Lie groups’[37] or ‘the principal Lie algebra’[24] in this context). Thus, we aim to find all triplets \( (K, D, F) \) that lead to extensions of the trivial Lie algebra generated by (17). This means that one needs to solve Eqs. (13)–(15) with coefficients (16). The crucial step in solving this task is to analyze differential consequences of Eqs. (13)–(14) with respect to the variable \( x \). Since these consequences take the form \( K'(u)P_x(t,x) = 0 \) and \( D'(u)P_x(t,x) = 0 \), respectively, one arrives at two basic cases

\[ i \quad [K'(u)]^2 + [D'(u)]^2 \neq 0 \quad \text{ii} \quad K'(u) = D'(u) = 0. \]

Consider case i. In this case \( P_x(t,x) = 0 \), so that Eqs. (13)–(15) take the form

\[ K(u)[\xi_t^0(t) - 4\alpha] + K'(u)[R(t)u + P(t)] = 0, \] (18)

\[ D(u)[\xi_t^0(t) - 2\alpha] + D'(u)[R(t)u + P(t)] = 0, \] (19)
\[ R_t(t)u + P_t(t) + F(u)[R(t) - \xi^0_t(t)] - F'(u)[R(t)u + P(t)] = 0. \]

(20)

Setting \( R(t) = P(t) = 0 \), we immediately arrive at case 1 of Table 1. In fact, Eqs. (18)-(20) with \( R(t) = P(t) = 0 \) and non-zero \( K(u) \) are equivalent to

\[ \xi^0_t(t) = 4\alpha \neq 0, \quad D(u) = 0, \quad F(u) = 0 \]

and \( K(u) \) is an arbitrary smooth function. This means that the triplet \((K(u), 0, 0)\) forms the system from case 1 of Table 1 and the coordinates of the infinitesimal operator (7) take the form

\[
\begin{align*}
\xi^0 & = 4\alpha t + t_0, \\
\xi^1 & = \alpha x + x_0, \\
\eta^1 & = \eta^2 = 0,
\end{align*}
\]

(21)

where \( \alpha, t_0 \) and \( x_0 \) are arbitrary parameters. Operator (7) with coordinates (21) generates exactly the operators \( P_t \) (for \( t_0 = 1, \alpha = x_0 = 0 \)), \( P_x \) (for \( t_0 = \alpha = 0, x_0 = 1 \)) and \( D_1 \) (for \( \alpha = 1, x_0 = t_0 = 0 \)) listed in case 1 of Table 1.

If \( R^2(t) + P^2(t) \neq 0 \) then two possible subcases arise: ia. \( R(t) \neq 0 \) and ib. \( R(t) = 0, \) \( P(t) \neq 0. \)

Consider subcase ia. Integrating Eq. (18) as an ODE on the function \( K(u) \), one obtains

\[
K(u) = k\left[u + \frac{P(t)}{R(t)}\right]^\frac{4\alpha - \xi^0_t(t)}{R(t)},
\]

(22)

where \( k \) is a nonzero constant, which can be reduced to \( k = 1 \) (by scaling time \( t \rightarrow kt \)) without losing generality. Since the function \( K \) must depend only on \( u \), the restrictions

\[
\begin{align*}
\xi^0(t) & = 4\alpha - \gamma R(t), \\
P(t) & = \gamma_0 R(t)
\end{align*}
\]

(23)

are obtained. Here \( \gamma \) and \( \gamma_0 \) are arbitrary constants. Thus, solving Eq. (18), we arrive at the power function

\[
K(u) = (u + \gamma_0)^\gamma
\]

(24)

and restrictions (23).

To solve Eqs. (19)-(20), we need to analyze two subcases ia1. \( D(u) \neq 0 \) and ia2. \( D(u) = 0. \)

In subcase ia1, Eq. (19) can be solved as an ODE on the function \( D(u) \), so that one obtains

\[
D(u) = d(u + \gamma_0)^\mu
\]

(25)

and the condition

\[
\xi^0_t(t) = 2\alpha - \mu R(t),
\]

(26)
where \( d \neq 0 \) and \( \mu \) are arbitrary constants.

If \( \gamma \neq \mu \) then solving Eqs. (23) and (26) and substituting the found functions \( R(t), P(t) \) and \( \xi^0 \) into Eq. (20), we arrive at the ODE

\[
(u + \gamma_0)F'(u) + (\gamma - 2\mu - 1)F(u) = 0.
\]

(27)

The general solution of Eq. (27) and the functions \( K(u) \) and \( D(u) \) found above form the system

\[
\begin{align*}
  u_t &= -[(u + \gamma_0)^\gamma v_x]_x + d[(u + \gamma_0)^\mu u_x]_x + \lambda(u + \gamma_0)^{2\mu-\gamma+1} \\
  u_{xx} - v &= 0,
\end{align*}
\]

(28)

which is reduced to the system listed in case 3 of Table 1 by renaming \( u + \gamma_0 \to u \). Substituting the functions \( R(t), P(t) \) and \( \xi^0 \) into (16), we obtain the infinitesimal operator (7), which generates three basic operators listed in case 3 of Table 1.

If \( \gamma = \mu \) then Eqs. (23) and (26) are compatible only under restriction \( \alpha = 0 \). It turns out that this restriction doesn’t lead to any new cases but to case 3 of Table 1 with \( \gamma = \mu \). In fact, Eqs. (18) and (19) have identical structure, hence \( D(u) = d(u + \gamma_0)^\gamma, \ d = \text{const} \). The general solution of (20) has the form

\[
F(u) = \lambda(u + \gamma_0)^{1+\gamma} + \lambda_1(u + \gamma_0),
\]

where \( \lambda \) and \( \lambda_1 \neq 0 \) are arbitrary constants. The relevant coordinates of infinitesimal operator (7) take the form

\[
\begin{align*}
  \xi^0 &= \frac{r_1}{\lambda_1} e^{-\lambda_1 \gamma t} + t_0, \ \xi^1 = x_0, \\
  \eta^1 &= r_1 e^{-\lambda_1 \gamma t}(u + \gamma_0), \ \eta^2 = r_1 e^{-\lambda_1 \gamma t}v,
\end{align*}
\]

(29)

where \( t_0, r_1 \) and \( x_0 \) are arbitrary parameters. Thus, the system

\[
\begin{align*}
  u_t &= -[(u + \gamma_0)^\gamma v_x]_x + d[(u + \gamma_0)^\gamma u_x]_x + \lambda(u + \gamma_0)^{1+\gamma} + \lambda_1(u + \gamma_0) \\
  u_{xx} - v &= 0
\end{align*}
\]

(30)

is invariant under three-dimensional MAI with the basic operators

\[
P_t, P_x, \\
Q^* = e^{-\gamma \lambda_1 t}((u + \gamma_0)\partial_u + v\partial_v).
\]

(31)

However, we found the local substitution

\[
\begin{align*}
  t^* &= \frac{1}{\lambda_1 \gamma} e^{\lambda_1 \gamma t}, \\
  x^* &= x, \\
  u^* &= (u + \gamma_0)e^{-\lambda_1 t}, \\
  v^* &= v e^{-\lambda_1 t},
\end{align*}
\]

(32)

which reduces system (30) and Lie algebra (31) to the system and Lie algebra listed in case 3 of Table 1 with \( \gamma = \mu \).

The analysis of subcase ia2 is straightforward because Eq. (19) vanishes for \( D(u) = 0 \) while Eq. (20) can be treated in a similar way. In conclusion, we found that the system

\[
\begin{align*}
  u_t &= -[(u + \gamma_0)^\gamma v_x]_x + \lambda_1(u + \gamma_0), \\
  u_{xx} - v &= 0,
\end{align*}
\]

(33)
is invariant under four-dimensional MAI with the basic operators
\[ P_t, P_x, Q^*, D^* = x \partial_x + \frac{4}{7} (u + \gamma_0) \partial_u + (\frac{4}{7} - 2) v \partial_v. \]

(34)

Direct checking shows that system (33) and Lie algebra (34) are reduced to the system and Lie algebra listed in case 6 of Table 1 if one applies substitution (32).

Examination of subcase \( \text{ib} \) is much simpler because Eqs. (18)–(20) with \( R(t) = 0 \) can be easily integrated. Finally, one obtains cases 2 and 5 of Table 1.

To complete the proof we need to examine case \( \text{ii} \) \( K'(u) = D'(u) = 0 \), i.e. \( K(u) = k = \text{const}, \ D(u) = d = \text{const} \). Since \( K(u) \neq 0 \), we can again set \( k = 1 \) without losing generality.

Thus, the classification equations (13)–(14) with coefficients (16) can be essentially simplified and one obtains
\[ \xi^0(t) = 4\alpha t + t_0, \quad \alpha d = 0. \]

(35)

The third classification equation (15) takes the form
\[ R_t(t)u + P_t(t, x) + P_{xxx}(t, x) - dP_{xx}(t, x) + \\
+ F(u)[R(t) - 4\alpha] - F'(u)[R(t)u + P(t, x)] = 0. \]

(36)

Differentiating Eq. (36) with respect to \( x \) and \( u \), we find the condition \( F''(u)P_x(t, x) = 0 \).

Hence two different subcases should be examined:

\( \text{iiia} \) \( F'' = 0 \) \quad \( \text{iiib} \) \( F'' \neq 0, \ P_x(t, x) = 0 \).

Consider subcase \( \text{iiia} \). Since \( F''(u) = 0 \) we immediately obtain \( F(u) = \lambda_1 u + \lambda_0 \) so that the triplet of functions \( (K, D, F) \) is known.

Substituting the function \( F \) into Eq. (36) and splitting the obtained expression into two equations (with the variable \( u \) and without it) we arrive at \( R(t) = 4\alpha \lambda_1 t + r_1 \) and the linear PDE
\[ P_t(t, x) + P_{xxx}(t, x) - dP_{xx}(t, x) - \lambda_1 P(t, x) + \lambda_0 (4\alpha \lambda_1 t + r_1 - 4\alpha) = 0 \]

(37)

to find the function \( P(t, x) \). Thus, the coordinates of infinitesimal operator (7) take the form
\[ \xi^0 = 4\alpha t + t_0, \xi^1 = \alpha x + x_0, \]
\[ \eta^1 = (4\alpha \lambda_1 t + r_1)u + P(t, x), \eta^2 = (4\alpha \lambda_1 t + r_1 - 2\alpha) v + P_{xx}(t, x), \]

(38)

where \( P(t, x) \) is the general solution of Eq. (37).

The last step is to take into account the second condition from (35). If \( d \neq 0 \) then \( \alpha = 0 \) and, applying the relevant simplifications, we arrive at the case 7 of Table 1.

If \( d = 0 \) then Eq. (37) takes the form
\[ P_t(t, x) + P_{xxx}(t, x) - \lambda_1 P(t, x) + \lambda_0 (4\alpha \lambda_1 t + r_1 - 4\alpha). \]

(39)
The corresponding system is
\begin{align*}
  u_t &= -v_{xx} + \lambda_1 u + \lambda_0, \\
  u_{xx} - v &= 0.
\end{align*}
(40)

It turns out that system (39) is reduced to the form listed in case 8 of Table 1 if one applies the local substitutions
\begin{align*}
  u^* &= u - \lambda_0 t, \quad \lambda_1 = 0 \\
  v^* &= v
\end{align*}
and
\begin{align*}
  u^* &= e^{-\lambda_1 t}(u + \frac{\lambda_0}{\lambda_1}), \quad \lambda_1 \neq 0 \\
  v^* &= e^{-\lambda_1 t}v.
\end{align*}
(42)

Simultaneously, operator (7) with coordinates (38) is transformed in such a way that the basic operators $P_t, P_x, I, D_1$ and $X^\infty = P(t,x)\partial_u + P_{xx}(t,x)\partial_v$, listed in case 8 of Table 1, can be easily derived.

Consider subcase iib. Since $P_x(t,x) = 0$, Eq. (36) takes the form
\begin{equation}
  R_t(t)u + P_t(t) + F(u)[R(t) - 4\alpha] - F'(u)[R(t)u + P(t)] = 0.
\end{equation}
(43)

Differentiating Eq. (43) with respect to the variables $t$ and $u$, we find the equation
\begin{equation}
  F''(u)[R_t(t)u + P_t(t)] = R_{tt}(t).
\end{equation}
(44)

Because Eq. (44) has a simple structure we prefer to solve this equation and check when the solution obtained will satisfy Eq. (43). We note that the special case $R_t(t) = P_t(t) = 0$ doesn’t lead to new results, so that $R_{tt}(t) = P_{tt}(t) \neq 0$. Moreover, since the function $F$ depends only on $u$, the relations
\begin{align*}
  P_t(t) &= \gamma R_t(t), \\
  R_{tt}(t) &= \lambda R_t(t),
\end{align*}
(45)

where $\gamma$ and $\lambda \neq 0$ are arbitrary constants, should take place. The general solution of Eq. (44) with the coefficients (45) is
\begin{equation}
  F(u) = \lambda(u + \gamma) \ln(u + \gamma) + \lambda_1 u + \lambda_0,
\end{equation}
(46)

where $\lambda_0$ and $\lambda_1$ are arbitrary constants. Now we substitute (46) and the general solution of the linear ODEs system (45) into Eq. (43) and find conditions when the obtained expression can be fulfilled. The simple calculations give
\begin{align*}
  R(t) &= r_1 e^{\lambda t}, \\
  P(t) &= \gamma r_1 e^{\lambda t} \\
  \alpha &= 0, \quad \lambda_0 = \lambda_1 \gamma.
\end{align*}
(47)
So the system
\[
\begin{align*}
  u_t &= -v_{xx} + d u_{xx} + \lambda (u + \gamma) \ln(u + \gamma) + \lambda_1 (u + \gamma), \\
  u_{xx} - v &= 0
\end{align*}
\] (48)
admits MAI generated by operator (7) with coordinates
\[
\begin{align*}
  \xi^0 &= t_0, \quad \xi^1 = x_0, \\
  \eta^1 &= r_1 e^{\lambda t} (u + \gamma), \quad \eta^2 = r_1 e^{\lambda t} v.
\end{align*}
\] (49)
Finally, the system (48) and operator (7) with (49) are simplified by the substitution
\[
\begin{align*}
  u^* &= e^{\frac{\lambda}{\mu} \lambda t} (u + \gamma), \\
  v^* &= e^{\frac{\lambda}{\mu} \lambda t} v,
\end{align*}
\] (50)
so that the system and MAI listed in case 4 of Table 1 are obtained.

Thus, the system of determining equations (10) — (15) is completely solved and eight different systems of the form (4) have been found, which admit three- and higher-dimensional Lie algebras. Simultaneously, we have shown that all other systems admitting non-trivial Lie algebra are reduced to those listed in Table 1 by the substitutions of the form (32), (41), (42) and (50). One notes that all of these substitutions can be united to the form (5).

The proof is now completed.

One easily notes that cases 2 and 3 of Table 1 generalize the results of Lie symmetry analysis for the Cahn-Hilliard equation derived in [19, 21]. For example, the systems
\[
\begin{align*}
  u_t &= -v_{xx} + d \left[ e^{\mu u} u_x \right]_x, \\
  u_{xx} - v &= 0
\end{align*}
\] (51)
and
\[
\begin{align*}
  u_t &= -v_{xx} + d [u^\mu u_x]_x, \\
  u_{xx} - v &= 0
\end{align*}
\] (52)
which are the particular cases of the corresponding systems from Table 1, admit the Lie symmetry operators
\[
4 \mu t \partial_t + \mu x \partial_x - 2 (\mu u + \mu v \partial_v) \] (53)
and
\[
4 \mu t \partial_t + \mu x \partial_x - 2 (\mu u + (\mu + 1) v \partial_v) \] (54)
respectively. The table also includes the results obtained in [21] for the equation Eq. (11) with \( K(u) = const \) and \( F(u) = 0 \).
To finish the Lie symmetry description we note that Eq. (1) can be reduced to an equivalent system of PDEs in different ways. One sees that the system

\[ \begin{align*}
  u_t & = -[K(u)v_x]_x + [D(u)u_x]_x + F(u), \\
  0 & = u_x - w, \\
  0 & = w_x - v
\end{align*} \]  

by introducing new unknown functions \( v = v(t, x) \), \( w = w(t, x) \) can be obtained from Eq. (1). Since the system includes the first-order variable \( w = u_x \), point symmetries of this system include contact symmetries of the original single fourth-order equation. System (55) is nothing else but a cross-diffusion system, in which the second and third equations contain the time variable \( t \) as a parameter. Thus, we may investigate also system (55) instead of Eq. (1). There is an essential difference between systems (4) and (55) because the second system contains the equations of different orders. In fact, according to the classical Lie scheme, MAI of this system is generated by the infinitesimal operator

\[ X = \xi^0(t, x, u, v, w) \partial_t + \xi^1(t, x, u, v, w) \partial_x + \eta^1(t, x, u, v, w) \partial_u + \eta^2(t, x, u, v, w) \partial_v + \eta^3(t, x, u, v, w) \partial_w, \]  

where the functions \( \xi^0, \xi^1, \eta^1, \eta^2, \eta^3 \) are to be determined. Applying the second prolongation of the operator (56) to system (55) and using the invariance conditions one can derive the determining equations for finding the functions \( \xi^0, \xi^1, \eta^1, \eta^2, \eta^3 \). It should be stressed that the relevant invariance conditions must take into account the differential consequences (with respect to the variable \( x \)) of the second and third equations of system (55). We omit cumbersome calculations and present the final result in the explicit form:

\[ \begin{align*}
  \xi^0_x & = \xi^0_u = \xi^0_v = \xi^1_t = \xi^1_u = \xi^1_v = \xi^1_{xx} = 0 \\
  \eta^1_v & = \eta^1_u = \eta^1_{uu} = 0 \\
  \eta^2 & = n^1_{xx} + (\eta^1_u - 2\xi^1_x)v \\
  \eta^3 & = n^1_x + (\eta^1_u - \xi^1_x)w \\
  K(u)(\xi^0_t - 4\xi^1_x) + K'(u)\eta^1 & = 0 \\
  D(u)(\xi^0_t - 2\xi^1_x) + D'(u)\eta^1 & = 0 \\
  \eta^1_t + K(u)\eta^1_{xxx} + F(u)(\eta^1_u - \xi^0_t) - D(u)\eta^1_{xx} - F'(u)\eta^1 & = 0
\end{align*} \]  

Now we note that the determining equations obtained (of course, without Eq. (60)) are equivalent to those for the system (1) so that no new Lie point symmetries or contact symmetries can be found.
3. Symmetry reduction and exact solutions

Some of the systems presented in Table 1 are equivalent to known fourth order PDEs arising in applications. For example, the system listed in case 6 is nothing else but the thin film equation \(\frac{2}{4}\). Since the motivation to this study is to consider this type of equation with non-zero reaction terms, which naturally arise in some processes \([26]-[29]\), henceforth we restrict our attention mainly to case 3 of Table 1.

It is well-known that a Lie symmetry allows one to reduce the given PDE (system of PDEs) to an equation (system of equations) of lower dimensionality. Here we reduce systems arising in case 3 of Table 1 to systems of ordinary differential equations (ODEs), furthermore these ODE systems are solved in particular cases and exact solutions of the initial PDE systems are constructed. Finally, these solutions are compared with those obtained by other authors.

Consider the system arising in the case 3 of Table 1:

\[
\begin{align*}
    u_t &= -\left[u \gamma v_x\right]_x + d[\mu u_x]_x + \lambda u^{2\mu-\gamma+1}, \\
    u_{xx} - v &= 0,
\end{align*}
\]

where \(\gamma^2 + \mu^2 \neq 0\). The most general Lie symmetry operator of system \((64)\) has the form

\[
X = \alpha_1 P_t + \alpha_2 P_x + \alpha_3 D_3 =
\]

\[
= [\alpha_1 + 2\alpha_3(\gamma - 2\mu)t]\partial_t + [\alpha_2 + \alpha_3(\gamma - \mu)x]\partial_x + [2\alpha_3 u]\partial_u + [2\alpha_3(1 - \gamma + \mu)v]\partial_v,
\]

where \(\alpha_i, i = 1, 2, 3\) are arbitrary constants. To construct the relevant ansatz one needs to solve the Pfaffian system of characteristic equations

\[
\frac{dt}{\alpha_1 + 2\alpha_3(\gamma - 2\mu)t} = \frac{dx}{\alpha_2 + \alpha_3(\gamma - \mu)x} = \frac{du}{2\alpha_3 u} = \frac{dv}{2\alpha_3(1 - \gamma + \mu)v}.\]

The general solution of \((66)\) essentially depends on the parameters \(\alpha_1, \alpha_2, \alpha_3, \gamma\) and \(\mu\) and five different cases occur.

**Case 1** \(\alpha_3 = 0\) leads to the plane wave solutions of the form

\[
\omega = \alpha_1 x - \alpha_2 t, \quad u = \phi(\omega), \quad v = \psi(\omega),
\]

where \(\phi\) and \(\psi\) are new unknown functions. These functions should satisfy the ODE system

\[
\begin{align*}
    -\alpha_2 \phi' &= -\alpha_1^2 \gamma^{-1}[\gamma \phi' \psi' + \phi \psi''] + d\alpha_1^2 \phi^{2\mu - 1}[\mu(\phi')^2 + \phi \phi''] + \lambda \phi^{1-\gamma+2\mu}, \\
    \psi &= \alpha_1^2 \phi''.
\end{align*}
\]

It should be noted that this system is equivalent to the fourth-order ODE

\[
\alpha_1^4 \phi^\gamma \phi'' + \gamma \alpha_1^4 \phi^{\gamma-1} \phi' \phi''' - d\alpha_1^2 \phi^{2\mu} \phi'' - d\alpha_1^2 \mu \phi^{\mu-1}(\omega)(\phi')^2 - \alpha_2 \phi' - \lambda \phi^{1-\gamma+2\mu} = 0.
\]

This equation is not integrable because there are no general solutions for this equation in terms of elementary functions and known special functions \([49]\). However, it can be noted that the special case with \(\gamma = 3\mu\) possesses the particular solution

\[
\phi(\omega) = \alpha \omega^{\frac{1}{\mu}},
\]
where $\alpha$ is a solution of algebraic equation

$$\alpha^4(1-\mu)(1-2\mu)\alpha^{4\mu} - d\alpha^2\mu^2\alpha^{2\mu} - \alpha_2\mu^3\alpha^\mu - \lambda\mu^4 = 0.$$  \(\text{(71)}\)

Hence the system

$$u_t = - [u^3u_x]_x + du^\mu u_x + \lambda u^{1-\mu},$$
$$u_{xx} - v = 0$$  \(\text{(72)}\)

possesses the exact solution

$$u = \alpha(\alpha_1 x - \alpha_2 t)^{\frac{1}{\mu}},$$
$$v = \alpha_2^2 \alpha^{\frac{1}{\mu}} (1 - \mu) (\alpha_1 x - \alpha_2 t)^{\frac{1}{\mu} - 2},$$  \(\text{(73)}\)

where $\alpha$ satisfies (71).

**Case 2** $\alpha_3 \neq 0$, $\gamma = 2\mu \neq 0$, $\alpha_1 = 0$ leads to the ansatz

$$\omega = t, \quad u = \phi(t) x^{\frac{2}{\mu}}, \quad v = \psi(t) x^{\frac{2}{\mu} - 2}.$$  \(\text{(74)}\)

The corresponding ODE system takes the form

$$\phi' = -2(\frac{1}{\mu} - 1)(1 + \frac{2}{\mu})\phi^{2\mu}\phi + d\frac{2}{\mu}(\frac{2}{\mu} + 1)\phi^{\mu+1} + \lambda\phi,$$
$$\psi = \frac{2}{\mu}(\frac{2}{\mu} - 1)\phi$$  \(\text{(75)}\)

and can be rewritten as the single ODE:

$$\phi' = -\frac{4}{\mu}(\frac{1}{\mu} - 1)(\frac{4}{\mu^2} - 1)\phi^{2\mu+1} + d\frac{2}{\mu}(\frac{2}{\mu} + 1)\phi^{\mu+1} + \lambda\phi.$$  \(\text{(76)}\)

It should be noted that this ODE with $\lambda = 0$ coincides with one derived in the recently published paper [24] for the fourth-order PDE, which is equivalent to (64) with $\lambda = 0$. However, system (64) with $\lambda \neq 0$ cannot be reduced to one with $\lambda = 0$ so that the solutions presented below cannot be obtained from [24].

If $\mu = 1$ or $\mu = 2$ then the known solutions of the reaction-diffusion equations

$$u_t = d[u u_x]_x + \lambda u \quad \mu = 1,$$

and

$$u_t = d[u^2 u_x]_x + \lambda u \quad \mu = 2$$

are obtained because $u_{xxx} = 0$ (see (74)). If $\mu = -2$ then

$$\phi(t) = Ce^{\lambda t}.$$  \(\text{(77)}\)
and there follows an exact solution
\[ u = \frac{Ce^{\lambda t}}{x}, \]
\[ v = \frac{2Ce^{\lambda t}}{x^3} \]
(78)
of the system
\[ u_t = -\left[\frac{1}{u^2}v_x\right]_x + d\left[\frac{1}{u^2}u_x\right]_x + \lambda u, \]
\[ u_{xx} - v = 0. \]
(79)

If \( \mu \neq 1; \pm 2 \) then two subcases, \( \lambda = 0 \) and \( \lambda \neq 0 \), should be separately considered. Both of them lead to the function \( \phi(t) \) in the implicit form and one can be found from the transcendental equation
\[ \ln \left| 1 + \frac{d}{2(1 - \frac{1}{\mu})(\frac{2}{\mu} - 1)} \frac{1}{\phi(t)} \right| - \frac{d}{2(1 - \frac{1}{\mu})(\frac{2}{\mu} - 1)} \frac{1}{\phi(t)} = \frac{d^2(\frac{2}{\mu} + 1)}{(1 - \frac{1}{\mu})(\frac{2}{\mu} - 1)}(t - t_0) \]
if \( \lambda = 0 \) and from the equation
\[ \ln \left| \frac{\phi^{2\mu}(t)}{\phi^{2\mu}(t) + \alpha}\phi^{\mu}(t) + \beta \right| - \alpha \int_0^{\phi^{\mu}(t)} \frac{dz}{z^2 + \alpha z + \beta} = 2\lambda\mu(t - t_0), \]
(81)

\[ \alpha = \frac{d\mu^2}{2(1 - \mu)(\mu - 2)}, \quad \beta = \frac{\lambda\mu^4}{4(1 - \mu)(\mu^2 - 4)} \]
if \( \lambda \neq 0 \).

Thus, the system
\[ u_t = -\left[u^{2\mu}v_x\right]_x + d[u^{\mu}u_x]_x, \]
\[ u_{xx} - v = 0 \]
(82)
possesses the exact solution
\[ u = \phi(t)x^{\frac{2}{\mu}}, \]
\[ v = \frac{2}{\mu}(\frac{2}{\mu} - 1)\phi(t)x^{\frac{2}{\mu} - 2}, \]
(83)
where \( \phi(t) \) satisfies the equation \( (80) \). Note that the function \( \phi(t) \) tends to 0 if \( t \to \infty \) and this function blows up if \( t \to t_0 \). These properties follow from the simple analysis of \( (80) \).
Analogously, the system
\[ u_t = -\left[u^{2\mu}v_x\right]_x + d[u^{\mu}u_x]_x + \lambda u, \]
\[ u_{xx} - v = 0 \]
(84)
possesses the exact solution \( (83) \) with \( \phi(t) \) satisfying the equation \( (81) \).
Case 3 $\alpha_3 \neq 0, \gamma = 2\mu \neq 0, \alpha_1 \neq 0$ leads to the ansatz

$$\omega = xe^{-\frac{\alpha_3}{\alpha_1}t}, \quad u = \phi(\omega)e^{\frac{2\alpha_3}{\alpha_1}(1-\mu)t}, \quad v = \psi(\omega)e^{\frac{2\alpha_3}{\alpha_1}(1-\mu)t},$$

which reduces the initial system to the ODE system

$$-\alpha_3\mu\omega\phi' + 2\alpha_3\phi = -\alpha_1\phi^{2\mu-1}[2\mu\phi'\psi' + \phi\psi''] + d\alpha_1\phi^{\mu-1}\mu(\phi')^2 + \phi\phi'' + \alpha_1\lambda\phi,$$  
$$\psi = \phi''.$$  

System (86) is equivalent to the 4-th order equation

$$\alpha_1\phi^{2\mu}\phi^{iv} + 2\alpha_1\phi^{2\mu-1}\phi'\phi''' - d\alpha_1\phi^{\mu}\phi'' - d\alpha_1\mu\phi^{\mu-1}(\omega)(\phi')^2 -$$
$$-\alpha_3\mu\omega\phi' - (\alpha_1\lambda - 2\alpha_3)\phi = 0,$$  

which possesses the particular solution

$$\phi(\omega) = \alpha\omega^{\frac{2}{\mu}}.$$  

Here, $\alpha$ must be a solution of the algebraic equation

$$4(4-\mu^2)(1-\mu)\alpha^{2\mu} - 2d\mu^2(\mu + 2)\alpha^\mu - \lambda\mu^4 = 0,$$  

which is simply a quadratic equation for $\alpha^\mu$.

Thus, the cross-diffusion system

$$u_t = -[u^{2\mu}v_x]_x + d[u^\mu u_x]_x + \lambda u,$$  
$$u_{xx} - v = 0$$  

has the stationary solution

$$u = \alpha x^{\frac{2}{\mu}},$$  
$$v = 2\alpha^{\frac{2-\mu}{\mu^2}}x^{\frac{2}{\mu}-2},$$  

where $\alpha$ satisfies (89).

Case 4 $\alpha_3 \neq 0, \gamma = \mu \neq 0$ leads to the ansatz

$$\omega = x + \frac{\alpha_2}{2\mu\alpha_3} \ln t, \quad u = \phi(\omega)t^{-\frac{1}{\mu}}, \quad v = \psi(\omega)t^{-\frac{1}{\mu}}.$$  

The corresponding ODE system takes the form

$$\alpha_2\phi' - 2\alpha_3\phi = -2\alpha_3\mu\phi^{\mu-1}[\mu\phi'\psi' + \phi\psi''] + 2\alpha_3d\phi^{\mu-1}[\mu(\phi')^2 + \phi\phi''] + 2\alpha_3\lambda\phi^{\mu+1},$$  
$$\psi = \phi''$$  

and is equivalent to the 4-th order equation

\[
2\alpha_3\mu \phi^iv + 2\alpha_3\mu^2 \phi^{i-1} \phi'' - 2\alpha_3\mu d\phi^i \phi'' - 2d\alpha_3\mu^2 \phi^{i-1} (\phi')^2 + \\
+ \alpha_2 \phi' - 2\alpha_3\mu \lambda \phi^{i-1} - 2\alpha_3 \phi = 0.
\]

(94)

For \(\alpha_2 \neq 0\), the solutions \(u(x, t)\) are travelling waves whose speed decreases in proportion to \(t^{-1}\) and whose amplitude decreases in proportion to \(t^{-1/\mu}\).

Equation (94) is not integrable but we were able to find the particular solutions if \(\mu = 1, \alpha_2 = 0\):

\[
\phi(\omega) = -\frac{2}{3\lambda} + 2\frac{3\chi}{3\lambda^2} \sin(\theta \omega + \theta_0), \quad \theta^4 + d\theta^2 - \frac{1}{2} = 0
\]

(95)

and

\[
\phi(\omega) = -\frac{2}{3\lambda} + C_1 e^{\theta \omega} + 2\frac{3\chi}{9C_1\lambda^2} e^{-\theta \omega}, \quad \theta^4 + d\theta^2 - \frac{1}{2} = 0.
\]

(96)

Thus, using ansatz (92), we arrive at the solutions

\[
u = -\frac{2}{3\lambda} + \frac{3\chi}{3\lambda^2} \sin(\theta \omega + \theta_0),
\]

(97)

\[
v = -\frac{2}{3\lambda} \theta^2 \sin(\theta \omega + \theta_0),
\]

which is a spatial sinusoid for which amplitude varies in proportion to \(1/t\), and

\[
u = -\frac{2}{3\lambda} + \frac{3\chi}{3\lambda^2} \sin(\theta \omega + \theta_0),
\]

(98)

which is a spatial sinusoid for which amplitude varies in proportion to \(1/t\), and

\[
u = -\frac{2}{3\lambda} \theta^2 \sin(\theta \omega + \theta_0),
\]

of the system

\[
u_t = -[uv_x]_x + d[uv_x]_x + \lambda u^2,
\]

(99)

In formulae (97) and (98), we can also shift the time \(t\) to \(t - t_0\) or \(t + t_0\) with the positive parameter \(t_0\). The first shift leads to a solution having blow-up at \(t = t_0\), the second leads to a solution that avoids singularity at \(t = 0\) and tends to 0 as \(t\) approaches \(\infty\). Blow-up and extinction are interesting phenomena in some applications.

Finally, case 5 \(\alpha_3 \neq 0, \gamma \neq 2\mu, \gamma \neq \mu\) leads to the similarity reduction

\[
\omega = xt^{\frac{\mu-\gamma}{\gamma-2\mu}}, \quad u = \phi(\omega)t^{\frac{1}{\gamma-2\mu}}, \quad v = \psi(\omega)t^{\frac{1-\gamma+\mu}{\gamma-2\mu}}
\]

(100)

and to the ODE system

\[
(\mu - \gamma)\omega \phi' + 2\phi = -2(\gamma - 2\mu)\phi^{\gamma-1}[\gamma \phi' \psi + \phi \psi''] + 2(\gamma - 2\mu)d\phi^{\mu-1}[\mu(\phi')^2 + \phi \phi''] + \\
+2(\gamma - 2\mu)\lambda \phi^{1-\gamma+2\mu},
\]

(101)

\[
\psi = \phi''.
\]
The equivalent 4-th order equation has the form

\[
2(\gamma - 2\mu)\phi^\gamma \phi^{iv} + 2(\gamma - 2\mu)\gamma \phi^{\gamma-1} \phi' \phi'' - 2(\gamma - 2\mu)d\phi^\mu \phi'' - 2d(\gamma - 2\mu)\mu \phi^{\mu-1}(\phi')^2 + \\
+ (\mu - \gamma)\omega \phi' - 2(\gamma - 2\mu)\lambda \phi^{1-\gamma + 2\mu} + 2\phi = 0. 
\]  
(102)

whose solutions are self-similar by a scaling invariance. Although this equation is again not integrable, one may try to construct particular solutions in the form of a high-order polynomial. For example, setting \(d = 0, \ \gamma = 1, \ \mu = 0\), the exact solution

\[
\phi(\omega) = \frac{1}{120}\omega^4 + \frac{5}{6} \lambda 
\]

is obtained. Thus, we arrive at the solution

\[
u = \frac{1}{10} x^2 + \frac{5}{6} \lambda t, \\
u_{xx} - v = 0. 
\]

(104)

Remark 2. Using the ad hoc ansatz

\[
u = \phi_0(t) + \phi_1(t)x + \phi_2(t)x^2 + \phi_3(t)x^3 + \phi_4(t)x^4, \\
u = 2\phi_2(t) + 6\phi_3(t)x + 12\phi_4(t)x^2, 
\]

solution (104) can be generalized to the form

\[
u = \frac{C_0}{t^{\frac{5}{6}}} + 30\frac{C_2 C_4}{t^{\frac{7}{6}}} + 900\frac{C_2 C_4}{t^{\frac{7}{6}}} + 6750\frac{C_1}{t^{\frac{5}{6}}} + \frac{5}{6} \lambda t + \\
+ \left(\frac{C_4}{t^{\frac{5}{6}}} + 60\frac{C_2 C_4}{t^{\frac{7}{6}}} + 900\frac{C_2 C_4}{t^{\frac{7}{6}}}\right)x + \left(\frac{C_4}{t^{\frac{5}{6}}} + 45\frac{C_2 C_4}{t^{\frac{7}{6}}}\right)x^2 + \frac{C_4}{t}x^3 + \frac{1}{120} \frac{1}{t^2}x^4, \\
u = 2\left(\frac{C_4}{t^{\frac{5}{6}}} + 45\frac{C_2 C_4}{t^{\frac{7}{6}}}\right)x^2 + \frac{6C_4}{t}x^2 + \frac{1}{10} \frac{1}{t}x^3, 
\]

(107)

where \(C_i (i = 0, ..., 3)\) are arbitrary constants.

Similarly, setting \(d = 0, \ \gamma = 1, \ \mu = \frac{1}{4}\), the solution

\[
\phi(\omega) = \left(\frac{1}{\sqrt{120}}\omega^2 + \frac{5}{11} \lambda\right)^2 
\]

(108)

can be derived. Thus, the system

\[
u_t = -[uv_x]_x + \lambda \sqrt{u}, \\
u_{xx} - v = 0 
\]

(109)
possesses the exact solution
\[
 u = \left( \frac{1}{\sqrt{120}} x^2 + \frac{5}{11} \lambda t \right)^2,
 \]
\[
 v = \frac{1}{10} x^2 + \frac{10}{11 \sqrt{30}} \lambda \sqrt{t}.
\] (110)

We note that solutions (104) and (110) have been earlier obtained in [25] via the method of invariant subspaces. Indeed, the formulas (3.29) and (3.76) [25] contain (104) and (110), respectively.

4. Conclusions

We have carried out a symmetry group classification for fourth-order reaction-diffusion equations, allowing for both second-order and fourth-order diffusion terms. The fourth order equation has been treated, firstly, as a system of second-order equations that bears some resemblance to a system of coupled reaction-diffusion equations with cross diffusion, secondly, as a system of a second-order equation and two first-order equations. It turns out that both systems lead to the same result of symmetry group classification. Our paper generalizes the results of Lie symmetry analysis derived earlier for particular cases of Eq. 1. Moreover, we were able to construct all possible Lie symmetries, which Eq. 1 can admit depending on the function triplets \((K, D, F)\). This distinguishes our investigation from those that have focussed on Lie symmetry of particular cases of the given fourth-order evolution equation. On the other hand, our result is analogous to that derived in [31] and [32] for second-order evolution equations.

To the best of our knowledge, there is only the recently published book [25], where exact solutions have been found for some equations of the form 1 with \(F \neq 0\). Thus, a fourth-order nonlinear equation with the non-zero reaction term in the form of system (64) was examined by applying the Lie symmetry reduction. Where possible, we have constructed exact solutions to the ordinary differential equations that were obtained from this reaction-diffusion system. The solutions include some unusual structures as well as the familiar types that regularly occur in symmetry reductions, namely self-similar solutions, decelerating and decaying traveling waves, and steady states. Many of the functional relationships between the two symmetry invariants are quite simple, involving polynomials, algebraic functions, logarithms, exponentials and sinusoids. However, there are some that have been reduced only to the solutions of transcendental equations (see formulas (80) and (81)).

Finally, it should be also noted that the nonlinear fourth order ODEs obtained in section 3 can be solved by numerical methods and therefore solutions of the relevant generalized thin film equations will be constructed.

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