A HAMILTON-JACOBI THEORY ON POISSON MANIFOLDS

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ABSTRACT. In this paper we develop a Hamilton-Jacobi theory in the setting of almost Poisson manifolds. The theory extends the classical Hamilton-Jacobi theory and can be also applied to very general situations including nonholonomic mechanical systems and time dependent systems with external forces.

1. Introduction. The standard formulation of the Hamilton-Jacobi problem is to find a function $S(t, q^i)$ (called the principal function) such that
\[ \frac{\partial S}{\partial t} + h(q^i, \frac{\partial S}{\partial q^i}) = 0, \]
where $h = h(q^i, p_i)$ is the hamiltonian function of the system. If we put $S(t, q^i) = W(q^i) - tE$, where $E$ is a constant, then $W$ satisfies
\[ h(q^i, \frac{\partial W}{\partial q^i}) = E; \]
$W$ is called the characteristic function.

Equations (1) and (2) are indistinctly referred as the Hamilton-Jacobi equation (see [1, 2, 22]). The Hamilton-Jacobi equation helps to solve the Hamilton equations for $h$
\[ \frac{dq^i}{dt} = \frac{\partial h}{\partial p_i}, \quad \frac{dp_i}{dt} = -\frac{\partial h}{\partial q^i}. \]

Indeed, if we find a solution $W$ of the Hamilton-Jacobi equation (2) then any solution $(q^i(t))$ of the first set of equations (3) gives a solution of the Hamilton equations by taking $p_i(t) = \frac{\partial W}{\partial q^i}$.

This result can be founded in [1]. Moreover, one can rephrase the above result by stating that if $W$ is a solution of the Hamilton-Jacobi equation, then $dW$ (a 1-form on $Q$) transforms the integral curves of the vector field $X_h^dW = T\pi_Q \circ X_h \circ dW$ into the integral curves of $X_h$; here, $X_h$ is the Hamiltonian vector field defined by the hamiltonian $h$ and $\pi_Q : T^*Q \to Q$ is the canonical projection. Of course we can

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121
think in a more general situation where we look for general 1-forms on $Q$ that play a similar role to $dW$.

This geometrical procedure has been successfully applied to many other different contexts, including nonholonomic mechanics (see [7, 8, 11, 13]), singular lagrangian systems [15, 16], and even classical field theories [12, 17, 14]. Notice that in these frameworks, we don’t have a symplectic framework; for instance, in nonholonomic mechanics the natural geometric framework is provided by a $(2,0)$-tensor field (an almost Poisson tensor) on the constraint submanifold that it is not integrable (that is, it is not satisfies the Jacobi identity). The almost-Poisson bracket in nonholonomic mechanics has been firstly introduced by A. van der Schaft and B.M. Mashke ([24]). All these scenarios are just the motivation for the investigation developed in this paper.

Our goal is to develop a Hamilton-Jacobi theory in a unifying and more general setting, say hamiltonian systems on an almost-Poisson manifold, that is, a manifold equipped with a skew-symmetric $(2,0)$-tensor field which does not necessarily satisfy the Jacobi identity. We also assume that the almost-Poisson manifold has a fibered structure over another manifold. The Hamilton-Jacobi problem now is to find a section of the fibered manifold such that its image is a lagrangian submanifold and the differential of the given hamiltonian vanishes on the tangent vectors to the section and belonging to the characteristic distribution.

The theory includes the case of classical hamiltonian systems on the cotangent bundle of the configuration manifold as well as the case of nonholonomic mechanical systems. We also apply the theory to time-dependent hamiltonian systems and systems with external forces.

We also discuss the existence of complete solutions and prove that if a complete solution exists then we obtain first integral in involution, which is a remarkable fact since our framework is just almost-Poisson.

Along the paper, all the manifolds are real, second countable and $C^\infty$. The maps are assumed to be also $C^\infty$. Sum over all repeated indices is understood.

2. Hamilton-Jacobi theory on almost-Poisson manifolds.

2.1. Hamiltonian systems on almost-Poisson manifolds. Assume that $(E,\Lambda)$ is an almost-Poisson manifold, that is, $E$ is a manifold equipped with an almost-Poisson structure $\Lambda$, which means that $\Lambda$ is a skew-symmetric $(2,0)$-tensor field on $E$. Notice that $\Lambda$ does not necessarily satisfy the Jacobi identity; in that case, we will have a Poisson tensor, and $E$ will be a Poisson manifold. For the moment, one only needs to ask $(E,\Lambda)$ be an almost-Poisson manifold.

Therefore, $\Lambda$ defines a vector bundle morphism

$$\sharp : T^*E \longrightarrow TE$$

by

$$\langle \sharp(\alpha), \beta \rangle = \Lambda(\alpha, \beta)$$

for all $\alpha, \beta \in T^*E$. Of course, we shall also denote by $\sharp$ the induced morphism of $C^\infty$-modules between the spaces of 1-forms and vector fields on $E$. Notice that we will use the notation $\sharp_\Lambda$ if there is danger of confusion.

We denote by $C$ the characteristic distribution defined by $\Lambda$, that is

$$C_p = \sharp(T^*_p E)$$
for all \( p \in E \) (in other terms, \( C_p = \text{Im} \sharp_p \), where \( \sharp_p = T_p^*E \to T_pE \)). The rank of the almost-Poisson structure at \( p \) is the dimension of the space \( C_p \). Notice that \( C \) is a generalized distribution and, moreover, is not (in general) integrable since \( \Lambda \) is not Poisson in principle.

The following lemma will be useful

**Lemma 2.1.** Let \((E, \Lambda)\) be an almost-Poisson manifold, then we have
\[
C^\circ = \ker(\sharp),
\]
where \( C^\circ \) denotes the annihilator of \( C \).

**Proof.** Observe that
\[
(\text{Im} \sharp_p)^\circ = \{ \mu \in T_p^*E \mid \langle \mu, \sharp_p(\alpha) \rangle = 0, \forall \alpha \in T_p^*E \}
= \{ \mu \in T_p^*E \mid \langle \sharp_p(\mu), \alpha \rangle = 0, \forall \alpha \in T_p^*E \}
= \ker \sharp_p
\]
and thus, the result holds. \( \square \)

We also have the following definition

**Definition 2.2.** ([19, 23]) A submanifold \( N \) of \( E \) is said to be a lagrangian submanifold if the following equality holds
\[
\sharp(\text{TN}^\circ) = \text{TN} \cap C.
\]

To have dynamics we need to introduce a hamiltonian function \( h : E \to \mathbb{R} \), and thus we obtain the corresponding hamiltonian vector field
\[
X_h = \sharp(\mu).
\]

**2.2. Hamilton-Jacobi theory on almost-Poisson manifolds.** Assume now that the almost-Poisson manifold \( E \) with almost-Poisson tensor \( \Lambda \) fibres over a manifold \( M \), say \( \pi : E \to M \) is a surjective submersion (in other words, a fibration).

Assume that \( \gamma \) is a section of \( \pi : E \to M \), i.e. \( \pi \circ \gamma = id_M \). Define the vector field \( X_h^\gamma \) on \( M \) by
\[
X_h^\gamma = T\pi \circ X_h \circ \gamma.
\]

The following diagram summarizes the above construction:

\[
\begin{array}{ccc}
E & \xrightarrow{X_h} & TE \\
\downarrow \gamma & \quad & \downarrow T\pi \\
M & \xrightarrow{X_h^\gamma} & TM
\end{array}
\]

The following result relates the integral curves of \( X_h \) and \( X_h^\gamma \).

**Theorem 2.3.** Assume that \( \text{Im}(\gamma) \) is a lagrangian submanifold of \((E, \Lambda)\). Then the following assertions are equivalent:
1. \( X_h \) and \( X_h^\gamma \) are \( \gamma \)-related, i.e. \( T\gamma \circ X_h^\gamma = X_h \circ \gamma \),
2. \( dh \in (T\text{Im}(\gamma) \cap C)^\circ \).
Proof. “(i) $\Rightarrow$ (ii)"

Assume that $X_h$ and $X_\gamma^h$ are $\gamma$-related. Then $X_h = T\gamma(X_\gamma^h)$ and since $X_h \in C$, we have $X_h \in \text{Im}(\gamma) \cap C$. But $\text{Im}(\gamma)$ is a lagrangian submanifold, so there exists $\beta \in (\text{Im}(\gamma))^\circ$ such that

$$X_h = \sharp(\beta),$$

along the image of $\gamma$.

Using that $X_h = \sharp(dh)$, we have $\sharp(dh - \beta) = 0$, so $dh - \beta \in \text{Ker}\sharp = C^\circ$.

Therefore

$$dh \in \beta + C^\circ \subset (\text{Im}(\gamma))^\circ + C^\circ = (\text{Im}(\gamma) \cap C)^\circ.$$

“(ii) $\Leftarrow$ (i)”

If $dh \in (\text{Im}(\gamma) \cap C)^\circ = T\text{Im}(\gamma)^\circ + C^\circ$, then $dh = \alpha_1 + \alpha_2$ where $\alpha_1 \in T\text{Im}(\gamma)^\circ$ and $\alpha_2 \in C^\circ$.

Then, along $\text{Im}(\gamma)$ we have

$$X_h = X_{\alpha_1} + X_{\alpha_2},$$

where $\sharp(\alpha_i) = X_{\alpha_i}, i = 1, 2$. Using Lemma 2.1 we get $X_h = X_{\alpha_1} + X_{\alpha_2} = X_{\alpha_1} + 0 = X_{\alpha_1}$, where $\alpha_1 \in T\text{Im}(\gamma)^\circ$.

Since $\text{Im}(\gamma)$ is a lagrangian submanifold, we have

$$\sharp(T\text{Im}(\gamma)^\circ) = T\text{Im}(\gamma) \cap C$$

and then

$$X_h = X_{\alpha_1} \in T\text{Im}(\gamma) \cap C.$$

Therefore we deduce that $X_h$ and $X_\gamma^h$ are $\gamma$-related since both are tangent to the submanifold $\text{Im} \gamma$.

Assume that $(E, \Lambda)$ is a transitive Poisson manifold (a symplectic manifold), that is, $C = TE$. Then, we have that a submanifold $N$ of a transitive Poisson manifold $(E, \Lambda)$ is a lagrangian submanifold if and only if

$$\sharp(TN^\circ) = TN.$$

Therefore, the above Theorem 2.3 takes the following classical form.

**Theorem 2.4.** Assume that $\text{Im}(\gamma)$ is a lagrangian submanifold of a transitive Poisson manifold $(E, \Lambda)$. Then the following assertions are equivalent:

1. $X_h$ and $X_\gamma^h$ are $\gamma$-related;
2. $d(h \circ \gamma) = 0$.

3. **Computations in local coordinates.** Assume that $(x^i, y^a)$ are local coordinates adapted to the fibration $\pi : E \to M$, that is, $\pi(x^i, y^a) = (x^i)$, where $(x^i)$ are local coordinates in $M$.

Therefore, the tensor $\Lambda$ can be locally expressed as follows

$$\Lambda = \frac{1}{2} \Lambda^{ij} \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j} + \Lambda^{ib} \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial y^b} + \frac{1}{2} \Lambda^{ab} \frac{\partial}{\partial y^a} \wedge \frac{\partial}{\partial y^b},$$

where $\Lambda_{ij} = -\Lambda_{ji}$, $\Lambda_{ab} = -\Lambda_{ba}$ due to the antisymmetry of $\Lambda$. Observe that

$$\Lambda^{ij} = \Lambda(dx^i, dx^j), \quad \Lambda^{ib} = \Lambda(dx^i, dy^b),$$

$$-\Lambda^{ja} = \Lambda(dy^a, dx^j), \quad \Lambda^{ab} = \Lambda(dy^a, dy^b).$$
The above local expressions implies that
\begin{align}
\sharp(dx^i) &= \Lambda^{ij} \frac{\partial}{\partial x^j} + \Lambda^{ib} \frac{\partial}{\partial y^b}, \quad (4) \\
\sharp(dy^a) &= -\Lambda^{ja} \frac{\partial}{\partial x^j} + \Lambda^{ab} \frac{\partial}{\partial y^b}. \quad (5)
\end{align}

Using (4) we deduce that a hamiltonian vector field $X_h$ for a hamiltonian function $h \in C^\infty(E)$ is locally expressed by
\begin{align}
X_h &= \left( \frac{\partial h}{\partial x^i} \Lambda^{ij} - \frac{\partial h}{\partial y^a} \Lambda^{ja} \right) \frac{\partial}{\partial x^j} \quad (6) \\
&\quad + \left( \frac{\partial h}{\partial x^i} \Lambda^{ib} + \frac{\partial h}{\partial y^a} \Lambda^{ab} \right) \frac{\partial}{\partial y^b}. \quad (7)
\end{align}

Now, let $\gamma : M \rightarrow E$ be a section of $\pi : E \rightarrow M$. If
\begin{equation}
\gamma(x^i) = (x^i, \gamma^a(x^i))
\end{equation}
we obtain
\begin{equation}
X_h^\gamma = \left( \frac{\partial h}{\partial x^i} \Lambda^{ij} - \frac{\partial h}{\partial y^a} \Lambda^{ja} \right) \circ \gamma \frac{\partial}{\partial x^j}. \quad (8)
\end{equation}

**Proposition 1.** Im($\gamma$) is a lagrangian submanifold of $(E, \Lambda)$ if and only if
\begin{equation}
\Lambda^{ab} - \Lambda^{ja} \frac{\partial \gamma^a}{\partial x^j} + \Lambda^{a} \frac{\partial \gamma^b}{\partial x^j} + \Lambda^{j} \frac{\partial \gamma^a}{\partial x^j} - \Lambda^{ab} = 0. \quad (9)
\end{equation}

**Proof.** First of all, let us observe that $T\text{Im}(\gamma)$ is locally generated by the local vector fields
\begin{equation}
\left\{ \frac{\partial}{\partial x^i} + \frac{\partial \gamma^a}{\partial x^i} \frac{\partial}{\partial y^a} \right\}
\end{equation}
since
\begin{equation}
T\gamma \left( \frac{\partial}{\partial x^i} \right) = \frac{\partial}{\partial x^i} + \frac{\partial \gamma^a}{\partial x^i} \frac{\partial}{\partial y^a}
\end{equation}
Therefore, if a 1-form on $E$
\begin{equation}
\alpha = \alpha_i dx^i + \alpha_a dy^a
\end{equation}
annihilates $T\text{Im}(\gamma)$ (recall that this is only valid along the points of $\text{Im}(\gamma))$ we deduce the following conditions on the coefficients:
\begin{equation}
\alpha_i = -\alpha_a \frac{\partial \gamma^a}{\partial x^i}. \quad (10)
\end{equation}

Now, a simple computation shows that
\begin{equation}
\sharp(\alpha) = \left( \alpha_i \Lambda^{ij} - \alpha_a \Lambda^{ja} \right) \frac{\partial}{\partial x^j} + \left( \alpha_i \Lambda^{ib} + \alpha_a \Lambda^{ab} \right) \frac{\partial}{\partial y^b}.
\end{equation}

Assume now that $\text{Im}(\gamma)$ is a lagrangian submanifold. Then, $\sharp(\alpha) \in T\text{Im}(\gamma)$, with $\alpha \in T\gamma(M)^{\circ}$, and using (10) we deduce that
\begin{align}
\sharp(\alpha) &= \left( \alpha_i \Lambda^{ij} - \alpha_a \Lambda^{ja} \right) \frac{\partial}{\partial x^j} + \left( \alpha_i \Lambda^{ib} + \alpha_a \Lambda^{ab} \right) \frac{\partial}{\partial y^b} \\
&= \alpha_i \left( -\frac{\partial \gamma^a}{\partial x^j} \Lambda^{ij} - \Lambda^{ja} \right) \frac{\partial}{\partial x^j} + \alpha_a \left( -\frac{\partial \gamma^a}{\partial x^j} \Lambda^{ib} + \Lambda^{ab} \right) \frac{\partial}{\partial y^b} \\
&= \lambda^j \left( \frac{\partial}{\partial x^j} + \frac{\partial \gamma^a}{\partial x^j} \frac{\partial}{\partial y^a} \right)
\end{align}
which implies
\[ \lambda^i = -\alpha_a \frac{\partial \gamma^a}{\partial x^i} \Lambda^{ij} - \alpha_a \Lambda^i, \] (11)
and
\[ \lambda^j \frac{\partial \gamma^b}{\partial x^j} = -\alpha_a \frac{\partial \gamma^a}{\partial x^i} \Lambda^{ib} + \alpha_a \Lambda^{ab}. \] (12)
Substituting the values of \( \lambda^j \) given by (11) in equation (12) we obtain
\[ \Lambda^{ab} - \Lambda^{ib} \frac{\partial \gamma^a}{\partial x^i} + \Lambda^{ja} \frac{\partial \gamma^b}{\partial x^j} = 0. \]
Assume now that equations (9) hold. Then, given an element \( \alpha \in (T\text{Im}(\gamma))^o \)
\[ \sharp(\alpha) = \alpha_a \left( -\frac{\partial \gamma^a}{\partial x^i} \Lambda^{ij} - \Lambda^i \right) \frac{\partial}{\partial x^i} + \alpha_a \left( -\frac{\partial \gamma^a}{\partial x^i} \Lambda^b + \Lambda^{ab} \right) \frac{\partial}{\partial y^b} \]
and therefore
\[ \sharp(T\text{Im}(\gamma))^o = T\text{Im}(\gamma) \cap C. \]

4. Complete solutions. The essential idea in the standard Hamilton-Jacobi theory consists in finding a complete family of solutions to the problem (not only one particular solution). Therefore, in this section we shall discuss the notion of complete solutions for the Hamilton-Jacobi equation in this general framework of almost-Poisson manifolds. As a consequence we recover in a really simple and geometric way the results about complete solutions proved in [7].

First of all, we shall introduce the notion of complete solution.

Assume that we have a hamiltonian system given by a hamiltonian function \( h : E \rightarrow \mathbb{R} \) on an almost Poisson manifold \( (E, \Lambda) \) fibered over a base manifold \( M \). We assume that \( \dim M = n \) and \( \dim E = n + k \).

**Definition 4.1.** Consider \( U \subset \mathbb{R}^k \) an open set, where \( k \) is the dimension of the fiber of the bundle \( \pi : E \rightarrow M \). A map \( \Phi : M \times \mathbb{R}^k \rightarrow E \) is a complete solution if
1. \( \Phi \) is a local diffeomorphism.
2. For any \( \lambda = (\lambda_1, \ldots, \lambda_k) \in \mathbb{R}^k \), the map
   \[ \Phi_\lambda : M \rightarrow E \]
   \[ x \rightarrow \Phi_\lambda(x) = \Phi(x, \lambda_1 \ldots, \lambda_k) \]
is a solution of the Hamilton-Jacobi problem for \( h \).

For the sake of simplicity we will assume that \( \Phi \) is a global diffeomorphism. Consider \( f_a : E \rightarrow \mathbb{R} \) given the composition of \( \Phi^{-1} \) with the projection over the \( a \)-th component of \( \mathbb{R}^k \). We obtain

**Proposition 2.** The functions \( f_a, 1 \leq a \leq k \) are in involution, that is, \( \{ f_a, f_b \} = 0, \) for all \( a, b, \) where \( \{ \cdot, \cdot \} \) is the bracket determined by \( \Lambda \).

**Proof.** Given \( p \in E \) we will show that \( \{ f_a, f_b \}(p) = 0. \) Suppose that \( f_a(p) = \lambda_a, \) for each \( a = 1, \ldots, k, \) and observe that \( p \in \text{Im}(\Phi_\lambda) = \cap_a f_a^{-1}(\lambda_a). \) Therefore, we deduce that
\[ df_a(p)|_{T\text{Im}(\Phi_\lambda)} = 0 \]
since \( f_a \circ \Phi_\lambda \) is a constant function. By the hypothesis, \( \text{Im}(\Phi_\lambda) \) is a lagrangian submanifold in the sense previously explained, so

\[
\mathcal{z}(T\text{Im}(\Phi_\lambda)^\circ) = T\text{Im}(\Phi_\lambda) \cap C
\]

and then

\[
X_{f_a}(p) = \mathcal{z}_p(df_a(p)) \in T\text{Im}(\Phi_\lambda) \cap C.
\]

The result now follows since

\[
\{f_a, f_b\}(p) = df_a(p)(X_{f_b}(b))
\]

and the fact that \( df_a \in (T\text{Im}(\Phi_\lambda) \cap C)^\circ \) and \( X_{f_b} \in T\text{Im}(\Phi_\lambda) \cap C \). \( \Box \)

5. Applications.

5.1. Classical hamiltonian systems. (see [1, 2, 18])

A classical hamiltonian system is given by a hamiltonian function \( h \) defined on the cotangent bundle \( T^*Q \) of the configuration manifold \( Q \).

If it is the case, then \( E = T^*Q \) and \( \Lambda \) is the canonical Poisson structure \( \Lambda_Q \) on \( T^*Q \) provided by the canonical symplectic form \( \omega_Q \) on \( T^*Q \). Recall that now we can take canonical bundle coordinates \((q^i, p_i)\), where \( \pi_Q(q^i, p_i) = (q^i) \), and \( \pi_Q : T^*Q \to Q \) is the canonical projection.

Since in bundle coordinates

\[
\omega_Q = dq^i \wedge dp_i
\]

then

\[
\Lambda_Q = \frac{\partial}{\partial p_i} \wedge \frac{\partial}{\partial q^i}.
\]

Therefore,

\[
X_h = \frac{\partial h}{\partial p_i} \frac{\partial}{\partial q^i} - \frac{\partial h}{\partial q^i} \frac{\partial}{\partial p_i}
\]

and if a section \( \gamma : Q \to T^*Q \) (that is, a 1-form on \( Q \)) is locally expressed by

\[
\gamma(q^i) = (q^i, \gamma_i(q))
\]

we obtain

\[
X^\gamma_h = \left( \frac{\partial h}{\partial p_i} \circ \gamma \right) \frac{\partial}{\partial q^i}.
\]

The notion of lagrangian submanifold defined in Section 2 in the almost-Poisson setting reduces to the well-known in the symplectic setting, that is, it is isotropic and coisotropic with respect to the symplectic form \( \omega_Q \).

If we compute the condition (9) for the current case we obtain

\[
\frac{\partial \gamma_i}{\partial q^j} = \frac{\partial \gamma_j}{\partial q^i}
\]

which just means that \( \gamma \) is a closed form, i.e., \( d\gamma = 0 \). So we recover the classical result (see [1, 2]).

**Proposition 3.** Given a 1-form \( \gamma \) on \( Q \), we have that \( \text{Im}(\gamma) \) is a lagrangian submanifold of \( (T^*Q, \Lambda_Q) \) if and only if \( \gamma \) is closed.

As a consequence, we deduce the classical result directly from Theorem 2.4:

**Theorem 5.1.** Let \( \gamma \) be a closed 1-form on \( Q \). Then the following assertions are equivalent:

1. \( X_h \) and \( X^\gamma_h \) are \( \gamma \)-related;
2. \( d(h \circ \gamma) = 0 \).

5.2. **Nonholonomic mechanical systems.** In this section we will recover the results obtained in two previous papers [11, 13] (see also [8, 10, 21]).

A nonholonomic mechanical system is given by a lagrangian function \( L : TQ \rightarrow \mathbb{R} \) subject to constraints determined by a linear distribution \( D \) on the configuration manifold \( Q \). We will denote by \( D \) the total space of the corresponding vector sub-bundle \( (\tau Q)_D : D \rightarrow Q \) defined by \( D \), where \( (\tau Q)_D \) is the restriction of the canonical projection \( \tau Q : TQ \rightarrow Q \).

We will assume that the lagrangian \( L \) is defined by a Riemannian metric \( g \) on \( Q \) and a potential energy \( V \in C^\infty(Q) \), so that
\[
L(v_q) = \frac{1}{2} g(v_q, v_q) - V(q)
\]
or, in bundle coordinates \((q^i, \dot{q}^i)\)
\[
L(q^i, \dot{q}^i) = \frac{1}{2} g_{ij} \dot{q}^i \dot{q}^j - V(q^i).
\]

If \( \{\mu^a\}, 1 \leq a \leq k \) is a local basis of the annihilator \( D^\circ \) of \( D \), then the constraints are locally expressed as
\[
\mu^a_i(q) \dot{q}^i = 0,
\]
where \( \mu^a_i = \mu^a_i(q) dq^i \).

The nonholonomic equations can be written as
\[
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}^i} \right) - \frac{\partial L}{\partial q^i} = \lambda^i \mu^a_i(q)
\]
and
\[
\mu^a_i(q) \dot{q}^i = 0,
\]
for some Lagrange multipliers \( \lambda^i \) to be determined.

Let \( S \) (respectively, \( \Delta \)) be the canonical vertical endomorphism (respectively, the Liouville vector field) on \( TQ \). In local coordinates, we have
\[
S = dq^i \otimes \frac{\partial}{\partial q^i}, \quad \Delta = \dot{q}^i \frac{\partial}{\partial q^i}.
\]

Therefore, we can construct the Poincaré-Cartan 2-form \( \omega_L = -dS^*(dL) \) and the energy function function \( E_L = \Delta(L) - L \), such that the equation
\[
i_{\xi_L} \omega_L = dE_L \quad (13)
\]
has a unique solution, \( \xi_L \), which is a SODE on \( TQ \) (that is, \( S(\xi_L) = \Delta \)). Furthermore, its solutions coincide with the solutions of the Euler-Lagrange equations for \( L \):
\[
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}^i} \right) - \frac{\partial L}{\partial q^i} = 0.
\]

If we modify (13) as follows:
\[
i_X \omega_L - dE_L \in S^*((TD)^o) \quad (14)
\]
\[
X \in TD \quad (15)
\]
the unique solution \( X_{nh} \) is again a SODE whose solutions are just the ones of the nonholonomic equations.

Let
\[
FL : TQ \rightarrow T^*Q
\]
be the Legendre transformation given by
\[ FL(q^i, \dot{q}^i) = (q^i, p_i = \frac{\partial L}{\partial \dot{q}^i} = g_{ij} \dot{q}^j) \]

\( FL \) is a global diffeomorphism which permits to reinterpret the nonholonomic mechanical system in the Hamiltonian side. Indeed, we denote by \( h = E_L \circ F_L^{-1} \) the Hamiltonian function and by \( M = F_L(D) \) the constraint submanifold of \( T^*Q \).

The nonholonomic equations are then given by
\[
\begin{align*}
\frac{dq^i}{dt} &= \frac{\partial h}{\partial p_i}, \\
\frac{dp_i}{dt} &= -\frac{\partial h}{\partial q^i} + \bar{\lambda}^l \mu^a_i,
\end{align*}
\]

where \( \bar{\lambda}^l \) are new Lagrange multipliers to be determined.

As above, the symplectic equation
\[ i_{X_h} \omega_Q = dh \]

which gives the Hamiltonian vector field \( X_h \) should be modified as follows to take into account the nonholonomic constraints:
\[
\begin{align*}
i_X \omega_Q - dh &\in F^o \\
X &\in TM
\end{align*}
\]

where \( F \) is a distribution along \( M \) whose annihilator \( F^o \) is obtained from \( S^*(TD)^o \) through \( FL \). Equations (16) and (17) have a unique solution, the nonholonomic vector field \( X_{nh} \).

An alternative way to obtain \( X_{nh} \) is to consider the Whitney sum decomposition
\[ T(T^*Q)|_M = TM \oplus F \]

where the complement is taken with respect to \( \omega_Q \). If
\[ P : T(T^*Q)|_M \rightarrow TM \]

is the canonical projection onto the first factor, one easily proves that
\[ X_{nh} = P(X_h) \]

Moreover, one can introduce an almost-Poisson tensor \( \Lambda_{nh} \) on \( M \) by
\[ \Lambda_{nh}(\alpha, \beta) = \Lambda_Q(P^*\alpha, P^*\beta); \]

its associated bracket is called the nonholonomic bracket. Let us recall that this bracket has been firstly introduced and studied by A. van der Shaft and B.M. Mashke ([24]).

Obviously, we have
\[ X_{nh} = \sharp_{nh}(dh), \]

where \( \sharp_{nh} \) stands for \( \sharp_{\Lambda_{nh}} \).

An alternative way to define the nonholonomic bracket is as follows. Consider the distribution
\[ TM \cap F \]

along \( M \). A direct computation shows that the subspace
\[ T_pM \cap F_p \]

is symplectic within the symplectic vector space \( (T_p(T^*Q), \omega_Q(p)) \), for every \( p \in M \) (see [4, 6]).
Thus we have a second Whitney sum decomposition
\[ T(T^*Q)|_M = (TM \cap F) \oplus (TM \cap F)^\perp \]
where the complement is taken with respect to \( \omega_Q \).

If
\[ \tilde{P} : T(T^*Q)|_M \longrightarrow TM \cap F \]
is the canonical projection onto the first factor, one easily proves that
\[ X_{nh} = \tilde{P}(X_h). \]

Moreover, it is possible to write \( \Lambda_{nh} \) in terms of the projection \( \tilde{P} \) as follows
the nonholonomic almost-Poisson tensor \( \Lambda_{nh} \) on \( M \) is now rewritten as
\[ \Lambda_{nh}(\alpha, \beta) = \Lambda_Q(\tilde{P}^*\alpha, \tilde{P}^*\beta) = \omega_Q(\tilde{P}(X_\alpha), \tilde{P}(X_\beta)) \]
(see [6] for a proof).

Consider now the fibration
\[ (M, \Lambda_{nh}) \]
\[ \pi_{Q|_M} \]
\[ Q \]
and the hamiltonian \( h_{|M} \) (also denoted by \( h \) for sake of simplicity).

We can easily prove that
\[ C_p = T_pM \cap F_p. \]
Indeed, we have
\[ \langle \sharp_{nh}(\alpha), \beta \rangle = -\omega_Q(\tilde{P}X_\alpha, X_\beta) = \omega_Q(X_\beta, \tilde{P}X_\alpha) \]
\[ = (i_{X_\alpha} \omega_Q)(\tilde{P}X_\alpha) = \langle \beta, \tilde{P}X_\alpha \rangle \]
which implies
\[ \sharp_{nh}(\alpha) = \tilde{P}(X_\alpha). \]

Furthermore, the symplectic structure \( \Omega_p \) on \( C_p \) at any point \( p \in M \) is given by the restriction of the canonical symplectic structure \( \omega_Q \) on \( T^*Q \) to \( C_p \).

**Proposition 4.** Let \( \gamma : Q \rightarrow M \) be a section of \( \pi_{Q|_M} : M \rightarrow Q \), then \( \text{Im}(\gamma) \) is a lagrangian submanifold of \( (M, \Lambda_{nh}) \) if and only if \( d \gamma(X,Y) = 0 \) for all \( X,Y \in D \).

**Proof.** We notice that \( F = \{ v \in T(T^*Q) \text{ such that } T_{\pi_Q}(v) \in D \} \), and an easy computation in local coordinates shows that \( \dim(F \cap TM) = 2 \dim(D) \). Thus, we have
\[ T\text{Im}(\gamma) \cap C = T\gamma(D). \]
On the other hand, it is clear that our definition of lagrangian submanifold is equivalent to \( T\text{Im}(\gamma) \cap C \) be lagrangian with respect to the simplectic structure \( \Omega \) on the vector space \( C \). Since \( \Omega \) is the restriction of \( \omega_Q \), given \( X, Y \in D \) we have
\[ \Omega(T\gamma(X), T\gamma(Y)) = \Omega_Q(T\gamma(X), T\gamma(Y)) = d \gamma(X,Y). \]
So, after a careful counting of dimensions, we deduce that \( \text{Im}(\gamma) \) is lagrangian with respect to \( \Lambda_{nh} \) if and only if \( d \gamma(X,Y) = 0 \) for all \( X, Y \in D \).

Using this proposition we can recover the Nonholonomic Hamilton-Jacobi Theorem as a consequence of Theorem 2.3 (see [11, 13, 21]).
Theorem 5.2. [Nonholonomic Hamilton-Jacobi] Given a hamiltonian $h : M \to \mathbb{R}$, and $\gamma$ a $1$-form on $Q$ taking values in $M$, such that $d\gamma(X, Y) = 0$ for all $X, Y \in D$, then the following conditions are equivalent

1. $X_{nh}$ and $X_{nh}'$ are $\gamma$-related.
2. $dh \in (T\gamma(D))^o$ (which is in turns equivalent to $d(h \circ \gamma) \in D^o$).

We will get a suitable expression for the nonholonomic almost-Poisson tensor $\Lambda_{nh}$ defined on the constraint submanifold $M$ of $T^*Q$ (see [24, 6]). This local representation can be also used to prove Proposition 4.

Let us recall that the constraints were defined through a distribution $D$ on $Q$. Let $D'$ a complementary distribution of $D$ in $TQ$ and assume that $\{X_\alpha\}, 1 \leq \alpha \leq n-k$ is a local basis of $D$ and that $\{Y_a\}, 1 \leq a \leq k$ is a local basis of $D'$. Notice that $\mu^a(X_\alpha) = 0$.

Next we introduce new coordinates in $T^*Q$ as follows:

$$\tilde{p}_\alpha = X^i_\alpha p_i, \tilde{p}_{n-k+a} = Y^i_a p_i$$

where $\{\mu^a\}, 1 \leq a \leq k$ is a local basis of the annihilator $D^o$ of $D$.

In these new coordinates we deduce that the constraints become

$$\tilde{p}_{n-k+a} = 0.$$

Therefore, we can take local coordinates $(q^i, \tilde{p}_\alpha)$ on $M$.

A direct computation shows now that the nonholonomic almost-Poisson tensor $\Lambda_{nh}$ on $M$ is given by [6]

$$\Lambda_{nh} (dq^i, dq^j) = X^i_\alpha,$$
$$\Lambda_{nh} (dp_\alpha, dp_\beta) = X^i_\alpha \frac{\partial X^j_\beta}{\partial q^i} - X^i_\beta \frac{\partial X^j_\alpha}{\partial q^i}.$$

Summarizing the above discussion we can apply the results obtained in Section 2 to the hamiltonian system given by $h$ on the almost-Poisson manifold $(M, \Lambda_{nh})$.

Assume that $\gamma : Q \to M$ is a section of $\pi_{Q|M} : M \to Q$. Then, we have

$$\gamma(q^i) = (q^i, \tilde{\gamma}_\alpha(q^i))$$

Since $\gamma$ is a $1$-form on $Q$ taking values on $M$ we deduce

$$\gamma(q^i) = (q^i, \gamma_i(q^i))$$

and since it takes values in $M$ we get

$$\tilde{\gamma}_\alpha = X^i_\alpha \gamma_i.$$

A direct computation from equation (9) gives

$$\Lambda_{nh}^{\alpha\beta} + \Lambda_{nh}^{\beta\gamma} \frac{\partial \tilde{\gamma}_\alpha}{\partial q^j} - \Lambda_{nh}^{\alpha\gamma} \frac{\partial \tilde{\gamma}_\beta}{\partial q^j} = X^i_\beta \frac{\partial X^j_\alpha}{\partial q^i} - X^i_\alpha \frac{\partial X^j_\beta}{\partial q^i} - X^j_\beta \frac{\partial X^i_\alpha}{\partial q^j} - X^j_\alpha \frac{\partial X^i_\beta}{\partial q^j} (X^i_\alpha \gamma_i) - X^j_\alpha \frac{\partial \tilde{\gamma}_\beta}{\partial q^j} (X^i_\beta \gamma_i) = X^i_\alpha X^j_\beta \left( \frac{\partial \tilde{\gamma}_j}{\partial q^i} - \frac{\partial \tilde{\gamma}_i}{\partial q^j} \right) = 0.$$
which can be equivalently written as
\[ d\gamma(X_{\alpha}, X_{\beta}) = 0 \] (18)

Therefore, \( \gamma(Q) \) is a lagrangian submanifold of \((M, \Lambda_{nh})\) if and only if \( d\gamma \in I(Do) \), where \( I(Do) \) denotes the ideal of forms generated by \( Do \). Indeed, notice that (18) holds if and only if \( d\gamma = \sum a \xi_a \wedge \mu^a \), for some 1-forms \( \xi_a \).

5.3. Time dependent systems. In this section we will follow [18] for the description of time dependent mechanical systems. Now we are going to develop a time-dependent version of the previous construction. If we have the fibration \( E \to M \) such that \( E \) is equipped with an almost-Poisson structure \( \Lambda \), we can construct the following fibration in the obvious way

\[ \mathbb{R} \times E \xrightarrow{\pi_R} \mathbb{R} \times M \] (19)

where now \( \mathbb{R} \times E \) is equipped with the almost-Poisson structure \( \Lambda \) obtained in a natural way extending \( \Lambda \) in the trivial manner, that is, if \( f \) is a function on \( \mathbb{R} \times E \), then \( \sharp(df) = \sharp(df_0) \), where \( f_0 : E \to \mathbb{R} \) is defined by \( f_0(p) = f(0, p) \), and where we are using the natural identifications for the tangent and cotangent vector spaces of product manifolds. Here, \( \sharp = \sharp_\Lambda \) and \( \tilde{\sharp} = \tilde{\sharp}_\Lambda \).

We can consider the “extended” version of this diagram, that is, consider \( T^*\mathbb{R} \times E \), equipped with the almost-Poisson structure \( \Lambda_{ext} \) given by the addition of the canonical Poisson structure on \( T^*\mathbb{R} \) and \( \Lambda \). Notice that if we consider global coordinates \((t, e)\) on \( T^*\mathbb{R} \cong \mathbb{R} \times \mathbb{R} \), then

\[ \Lambda_{ext} = \frac{\partial}{\partial t} \wedge \frac{\partial}{\partial e} + \Lambda, \]

and the canonical projection is

\[ \mu : T^*\mathbb{R} \times E \to \mathbb{R} \times E \]
\[ (t, e, p) \to \mu(t, e, p) = (t, p). \] (20)

According to the above notation, diagram (19) becomes

\[ \begin{array}{ccc}
\mathbb{R} \times E & \xrightarrow{\mu} & T^*\mathbb{R} \times E \\
\downarrow & & \downarrow \\
\mathbb{R} \times M & \xrightarrow{\pi_R} & \mathbb{R} \times E
\end{array} \]

where \( \tilde{\pi} = \pi_R \circ \mu \).

Given a time dependent hamiltonian \( h : \mathbb{R} \times E \to \mathbb{R} \), the dynamics are given by the evolution vector field \( \frac{\partial}{\partial t} + X_h \in \mathfrak{X}(\mathbb{R} \times E) \). We can introduce the extended hamiltonian \( h_{ext} : T^*\mathbb{R} \times E \to \mathbb{R} \) given by \( h_{ext} = \mu^*h + e \) and the respective hamiltonian vector field \( X_{h_{ext}} = \sharp_{\Lambda_{ext}}(dh_{ext}) \). Notice that \( \mu_*(X_{h_{ext}}) = \frac{\partial}{\partial t} + X_h \).

We will denote by \( \mathcal{C}_{ext} \) the characteristic distribution of \( \Lambda_{ext} \). Notice that \( \mathcal{C}_{ext}(t, e, p) = (\frac{\partial}{\partial t}, \frac{\partial}{\partial e}) + \mathcal{C}_p \), under the obvious identifications.
If $\gamma$ is a section of $\tilde{\pi}$, we can consider the section of $\pi_R$ given by $\mu \circ \gamma$ and define the vector field $(\frac{\partial}{\partial t} + X_h)^\gamma$ on $\mathbb{R} \times M$ as follows:

$$(\frac{\partial}{\partial t} + X_h)^\gamma = T\pi_R \circ (\frac{\partial}{\partial t} + X_h) \circ (\mu \circ \gamma)$$

Now, we can state the time-dependent version of Theorem 2.3.

**Theorem 5.3.** If $\text{Im}(\gamma)$ is a lagrangian manifold in $(T^*\mathbb{R} \times E, \Lambda_{\text{ext}})$, then the following assertions are equivalent.

1. $(\frac{\partial}{\partial t} + X_h)$ and $(\frac{\partial}{\partial t} + X_h)^\gamma$ are $\mu \circ \gamma$-related
2. $dh_{\text{ext}} \in (\text{Im}(\gamma) \cap \mathcal{C}_{\text{ext}})^o + \langle dt \rangle$

**Proof.** “(i) $\Rightarrow$ (ii)”

Assume that $(\frac{\partial}{\partial t} + X_h)$ and $(\frac{\partial}{\partial t} + X_h)^\gamma$ are $\mu \circ \gamma$-related, which means that given $m \in M$, then

$$T\mu \circ T\gamma((\frac{\partial}{\partial t} + X_h)^\gamma(m)) = (\frac{\partial}{\partial t} + X_h)(\mu \circ \gamma(m))$$

or equivalently, there exists $B \in \mathbb{R}$ such that

$$T\gamma((\frac{\partial}{\partial t} + X_h)^\gamma(m)) = (X_{h_{\text{ext}}} + B \frac{\partial}{\partial e})(\gamma(m))$$

since any tangent vector in $T\gamma(m)T^*\mathbb{R} \times E$ which projects by $\mu$ onto $\frac{\partial}{\partial t} + X_h$ is of the form

$$X_{h_{\text{ext}}} + B \frac{\partial}{\partial e}, \quad B \in \mathbb{R}.$$ 

Using the same argument that we used in Theorem 2.3 we can conclude that

$$dh_{\text{ext}}(\gamma(m)) + Bdt \in (T\gamma(m)\text{Im}(\gamma) \cap \mathcal{C}_{\text{ext}}(\gamma(m)))^o$$

and so

$$dh_{\text{ext}} \in (\text{Im}(\gamma) \cap \mathcal{C}_{\text{ext}})^o + \langle dt \rangle.$$ 

“(ii) $\Leftarrow$ (i)”

Assume that $dh_{\text{ext}} \in (\text{Im}(\gamma) \cap \mathcal{C}_{\text{ext}})^o + \langle dt \rangle$; therefore, given any point $u \in \text{Im}(\gamma)$, there exists a real number $B$ such that

$$dh_{\text{ext}}(u) + Bdt(u) \in (T_u\text{Im}(\gamma) \cap (\mathcal{C}_{\text{ext}})_u)^o.$$ 

Now we can deduce

$$\sharp_{\Lambda_{\text{ext}}}(dh_{\text{ext}}(u) + Bdt(u)) \in T_u\text{Im}(\gamma),$$

where $\sharp_{\Lambda_{\text{ext}}}(dh_{\text{ext}}(u) + Bdt(u)) = X_{h_{\text{ext}}}(u) + B \frac{\partial}{\partial e}(u)$

Obviously, the last statement implies that $T\mu_*(X_{h_{\text{ext}}}(u) + B \frac{\partial}{\partial e}(u)) \in T_{\mu(x)}\text{Im}(\mu \circ \gamma)$, but

$$T\mu_*(X_{h_{\text{ext}}}(u) + B \frac{\partial}{\partial e}(u)) = T\mu_*(X_{h_{\text{ext}}}(u)) + T\mu_*(B \frac{\partial}{\partial e}(u))$$

which implies that $(\frac{\partial}{\partial t} + X_h)$ and $(\frac{\partial}{\partial t} + X_h)^\gamma$ are $\mu \circ \gamma$-related. $\square$
5.4. **External forces.** In this section we will apply the above general scheme to time-dependent systems and systems with external forces (see [9, 5]).

A force is represented by a semi-basic 1-form \( F(t, v_q) = \alpha_i(t, q, \dot{q}) \, dq^i \), which is equivalent to give a bundle mapping

\[
\begin{array}{c}
\mathbb{R} \times TQ \xrightarrow{F} T^*Q \\
\downarrow \text{Id}_x \times \pi_Q \\
\mathbb{R} \times Q \xrightarrow{\text{pr}_Q} Q
\end{array}
\]

(see [9] for details). Assuming that our dynamical system is described by a hyper-regular lagrangian \( L : TQ \rightarrow \mathbb{R} \) and the force \( F \), then using the Legendre transformation \( FL : TQ \rightarrow \mathbb{R} \) we can transport \( F \) to the hamiltonian side and define \( \tilde{F} = F \circ (\mathbb{R}L)^{-1} \).

We have

\[
\begin{array}{c}
\mathbb{R} \times T^*Q \xrightarrow{\tilde{F}} T^*Q \\
\downarrow \text{Id}_x \times \pi_Q \\
\mathbb{R} \times Q \xrightarrow{\text{pr}_Q} Q
\end{array}
\]

where \( \text{pr}_Q(t, q) = q \).

Given a hamiltonian \( h : \mathbb{R} \times T^*Q \rightarrow \mathbb{R} \), then the evolution of the system with external force \( \tilde{F} \) is now given by

\[
\frac{\partial}{\partial t} + X_h + V_{\tilde{F}}
\]

where \( V_{\tilde{F}} \) is the vector field determined by

\[
V_{\tilde{F}}(t, \alpha_q) = \sharp \Lambda_Q (\pi_Q^* (\tilde{F}(t, \alpha_Q))),
\]

\( \Lambda_Q \) being the canonical Poisson structure on \( T^*Q \).

In bundle coordinates \( \frac{\partial}{\partial t} + X_h + V_{\tilde{F}} \) provides the differential equation

\[
\dot{q}_i = \frac{\partial h}{\partial p_i}, \\
\dot{p}_i = -\frac{\partial h}{\partial q_i} - \tilde{F}_i.
\]

We can equip \( T^*(\mathbb{R} \times Q) \) with the almost-Poisson structure \( \tilde{\Lambda} \) given by \( \tilde{\Lambda} = \Lambda_{\mathbb{R} \times Q} + V_{\tilde{F}} \wedge \frac{\partial}{\partial e} \) (recall the definition of \( e \) in the previous section). Here \( \Lambda_{\mathbb{R} \times Q} \) denotes the canonical Poisson tensor on \( T^*(\mathbb{R} \times Q) \).

In local coordinates

\[
\tilde{\Lambda} = \tilde{F}_i \frac{\partial}{\partial e} \wedge \frac{\partial}{\partial p_i} + \frac{\partial}{\partial t} \wedge \frac{\partial}{\partial e} + \frac{\partial}{\partial q_i} \wedge \frac{\partial}{\partial p_i}.
\]

It is easy to see that the characteristic distribution of \( \tilde{\Lambda} \) is the whole space. Indeed, using the local expression of \( \tilde{\Lambda} \) we have

\[
\begin{align*}
\sharp \tilde{\Lambda}(dt) &= -\frac{\partial}{\partial e} \\
\sharp \tilde{\Lambda}(de) &= -\tilde{F}_i \frac{\partial}{\partial p_i} + \frac{\partial}{\partial t} \\
\sharp \tilde{\Lambda}(dq_i) &= -\frac{\partial}{\partial p_i} \\
\sharp \tilde{\Lambda}(dp_i) &= \frac{\partial}{\partial q_i} + \tilde{F}_i \frac{\partial}{\partial e}.
\end{align*}
\]
We can define \( h_{ext} = \mu^* h + \epsilon \), where \( \mu \) is defined in the same way that in 20, and construct the hamiltonian vector field \( X_{h_{ext}} = \tilde{\Lambda} (dh_{ext}) \). Due to the definition of \( \tilde{\Lambda} \) it is easy to see that \( \mu_*(X_{h_{ext}}) = \frac{\partial}{\partial t} + X_h + V_F \).

The following diagram summarizes our construction

\[
\begin{array}{ccc}
T^*(\mathbb{R} \times Q) & \xrightarrow{\mu} & \mathbb{R} \times T^*Q \\
\downarrow \pi & & \downarrow \pi_{\times Q} \\
\mathbb{R} \times Q & \xrightarrow{\mu \circ \gamma} & \mathbb{R} \times Q.
\end{array}
\]

If \( \gamma \) is a section of \( \pi_{\mathbb{R} \times Q} \) (a 1-form on \( \mathbb{R} \times Q \)) we can consider the section of \( \pi \) given by \( \mu \circ \gamma \) and define the vector field \( \partial_t + X_h + V_F \) on \( \mathbb{R} \times M \)

\[
\frac{\partial}{\partial t} + X_h + V_F = T\pi \circ (\frac{\partial}{\partial t} + X_h + V_F) \circ (\mu \circ \gamma)
\]

and we can state the following.

**Theorem 5.4.** If \( \text{Im}(\gamma) \) is a lagrangian manifold in \( (T^*(\mathbb{R} \times Q), \tilde{\Lambda}) \), then the following assertions are equivalent.

1. \( (\frac{\partial}{\partial t} + X_h + V_F) \) and \( (\frac{\partial}{\partial t} + X_h + V_F) \gamma \) are \( \mu \circ \gamma \)-related

2. \( dh_{ext} \in T\text{Im}(\gamma) + \langle dt \rangle \)

**Proof.** The proof is analogous to that in Theorem 5.3.

Next, we shall characterize when a section \( \gamma \) is lagrangian.

**Proposition 5.** Let \( \gamma \) be a 1-form on \( \mathbb{R} \times Q \); then the image of \( \gamma \) is a lagrangian submanifold with respect to \( \tilde{\Omega} \) if and only if

\[
d\gamma = (\tilde{F} \circ \mu \circ \gamma) \wedge dt.
\]

**Proof.** Using (21), it is easy to see that \( \tilde{\Omega} \) is an isomorphism, and so we can define the corresponding almost-symplectic structure \( \Omega \), that is \( (\tilde{\Omega})^{-1} = \tilde{\Omega} \), and thus

\[
\begin{align*}
\tilde{\Omega} (\frac{\partial}{\partial t}) &= -\tilde{F}_i dq^i + de \\
\tilde{\Omega} (\frac{\partial}{\partial e}) &= dt \\
\tilde{\Omega} (\frac{\partial}{\partial q^i}) &= dp_i + \tilde{F}_i dt \\
\tilde{\Omega} (\frac{\partial}{\partial p_i}) &= -dq^i.
\end{align*}
\]

Here \( \tilde{\Omega} \) denotes the induced mapping from tangent vector to 1-forms defined by the 2-form \( \tilde{\Omega} \).

So we can conclude that

\[
\Omega = dq^i \wedge dp_i + dt \wedge de + \tilde{F}_i dq_i \wedge dt.
\]

The image of the 1-form \( \gamma \) will be lagrangian for \( \tilde{\Omega} \) if and only if

\[
0 = \gamma^* (\tilde{\Omega}) = \gamma^* (dq^i \wedge dp_i + dt \wedge de + \tilde{F}_i dq_i \wedge dt) = \gamma^* (dq^i \wedge dp_i + dt \wedge de) + \gamma^* (\tilde{F}_i dq_i \wedge dt) = -d\gamma + (\tilde{F}_i \circ \mu \circ \gamma) dq^i \wedge dt
\]

and the result follows.
Remark 1. Our result generalizes the Hamilton-Jacobi theorem derived in [3] for the case of linear forces and time-dependent systems [20].

6. Examples.

6.1. Hamilton-Jacobi equation for the nonholonomic particle. Let a particle of unit mass be moving in space $Q = \mathbb{R}^3$, with lagrangian

$$L = K - V = \frac{1}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - V(x, y, z),$$

and subject to the constraint

$$\Phi = \dot{z} - y\dot{x} = 0.$$

Passing to the hamiltonian point of view we define the hamiltonian

$$h(x, y, z, p_x, p_y, p_z) = \frac{1}{2}(p_x^2 + p_y^2 + p_z^2) + V(x, y, z),$$

and the submanifold $M$ of $T^*Q$ given by $p_z - yp_x = 0$. We have the following decomposition along $M$

$$T(T^*Q)|_M = TM \oplus F^\perp$$

where

$$F^\perp = \text{span}\left\{ \frac{\partial}{\partial p_z} - y \frac{\partial}{\partial p_x} \right\}.$$ 

Therefore, the projector $P: T(T^*Q)|_M \rightarrow TM$ is given by

$$P = \text{id}_{T(T^*Q)|_M} - \frac{1}{1 + y^2} \left( \frac{\partial}{\partial p_z} - y \frac{\partial}{\partial p_x} \right) \otimes (dp_z - y dp_x - p_x dy)$$

and the solution of the nonholonomic dynamics is the following vector field

$$\bar{X}_h = P(X_h)$$

$$= p_x \frac{\partial}{\partial x} + p_y \frac{\partial}{\partial y} + p_z \frac{\partial}{\partial z} - \frac{1}{1 + y^2} \left( \frac{\partial V}{\partial x} + y \frac{\partial V}{\partial z} + yp_x p_y \right) \frac{\partial}{\partial p_x} - \frac{\partial V}{\partial y} \frac{\partial}{\partial p_y} - \frac{1}{1 + y^2} \left( y \frac{\partial V}{\partial x} + y^2 \frac{\partial V}{\partial z} - p_x p_y \right) \frac{\partial}{\partial p_z}$$

restricted to $M$.

Taking noncanonical coordinates $(x, y, z, \tilde{p}_1, \tilde{p}_2, \tilde{p}_3)$ on $T^*Q$ where $\tilde{p}_1 = p_x + yp_z$, $\tilde{p}_2 = p_y$ and $\tilde{p}_3 = p_z - yp_x$ then $(x, y, z, \tilde{p}_1, \tilde{p}_2)$ are coordinates for $M$. Therefore, the nonholonomic bracket defined on $M$ is given by

$$\{x, \tilde{p}_1\}_{nh} = 1, \quad \{y, \tilde{p}_2\}_{nh} = 1, \quad \{z, \tilde{p}_1\}_{nh} = y,$$

$$\{\tilde{p}_1, \tilde{p}_2\}_{nh} = \frac{1}{1 + y^2},$$

and the remaining brackets are zero.

Let $\gamma : \mathbb{R}^3 \rightarrow M$ be a section of $\pi_Q|_M$. Im($\gamma$) is a lagrangian submanifold of $(M, \Lambda_{nh})$ if and only if

$$d\gamma \left( \frac{\partial}{\partial x} + y \frac{\partial}{\partial z} \right) = 0$$

with $\gamma_3 - y\gamma_1 = 0$. If $\gamma = \gamma_1 dx + \gamma_2 dy + y\gamma_1 dz$ then this condition is equivalent to

$$(1 + y^2) \frac{\partial \gamma_1}{\partial y} + y\gamma_1 - \frac{\partial \gamma_2}{\partial x} - y \frac{\partial \gamma_2}{\partial z} = 0.$$
The nonholonomic Hamilton-Jacobi equations are in this case
\[ \gamma_1 \frac{\partial \gamma_1}{\partial x} + \gamma_2 \frac{\partial \gamma_2}{\partial x} + \gamma_3 \frac{\partial \gamma_3}{\partial x} + y \left( \gamma_1 \frac{\partial \gamma_1}{\partial z} + \gamma_2 \frac{\partial \gamma_2}{\partial z} + \gamma_3 \frac{\partial \gamma_3}{\partial z} \right) + \frac{\partial V}{\partial x} + y \frac{\partial V}{\partial z} = 0 \]
\[ \gamma_1 \frac{\partial \gamma_1}{\partial y} + \gamma_2 \frac{\partial \gamma_2}{\partial y} + \gamma_3 \frac{\partial \gamma_3}{\partial y} + \frac{\partial V}{\partial y} = 0. \]

Since the distribution generated by \( \{ X_1 = \frac{\partial}{\partial x} + y \frac{\partial}{\partial z}, X_2 = \frac{\partial}{\partial y} \} \) is completely nonholonomic, i.e. \( \text{span} \{ X_1, X_2, [X_1, X_2] \} = TQ \), then the nonholonomic Hamilton Jacobi equations can be written in the form:
\[ \frac{1}{2}((1 + y^2) \gamma_1^2 + \gamma_2^2) + V(x, y, z) = \text{constant} \]
with
\[ (1 + y^2) \frac{\partial \gamma_1}{\partial y} + y \gamma_1 - \frac{\partial \gamma_2}{\partial x} - y \frac{\partial \gamma_2}{\partial z} = 0. \]

In the case \( V \equiv 0 \), then the solutions are \( \gamma_1 = \frac{k_1}{\sqrt{1 + y^2}} \) and \( \gamma_2 = k_2 \), where \( k_1, k_2 \) are constants.

Thus, the map
\[ \Phi : Q \times \mathbb{R}^2 \longrightarrow M \]
\[ (x, y, x, k_1, k_2) \longmapsto (x, y, z, \frac{k_1}{\sqrt{1 + y^2}}, k_2, \frac{yk_1}{\sqrt{1 + y^2}}) \]
is a complete solution of the nonholonomic problem. Therefore it is only necessary to integrate the vector field
\[ \dot{x} = k_1 \]
\[ \dot{y} = k_2 \]
\[ \dot{z} = \frac{yk_1}{\sqrt{1 + y^2}} \]
to obtain all the solutions of the nonholonomic problem.

6.2. Complete Hamilton-Jacobi equation for the snakeboard. The configuration manifold that modelizes the snakeboard (see figure) is \( Q = SE(2) \times T^2 \) with coordinates \((x, y, \theta, \psi, \phi)\).

![Diagram of the snakeboard configuration manifold]

The system is described by a Lagrangian
\[ L(q, \dot{q}) = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2) + \frac{1}{2} (J + 2 J_1) \dot{\theta}^2 + \frac{1}{2} J_0 (\dot{\theta} + \psi)^2 + J_1 \dot{\phi}^2 \]
where \( m \) is the total mass of the board, \( J > 0 \) is the moment of inertia of the board, \( J_0 > 0 \) is the moment of inertia of the rotor of the snakeboard mounted on the body’s center of mass and \( J_1 > 0 \) is the moment of inertia of each wheel axles. The distance between the center of the board and the wheels is denoted by \( r \). For simplicity, we assume that \( J + J_0 + 2J_1 = mr^2 \).

Since the wheels are not allowed to slide in the sideways direction, we impose the constraints

\[
\begin{align*}
-\dot{x} \sin(\theta + \phi) + \dot{y} \cos(\theta + \phi) - r \dot{\theta} \cos \phi &= 0 \\
-\dot{x} \sin(\theta - \phi) + \dot{y} \cos(\theta - \phi) + r \dot{\theta} \cos \phi &= 0
\end{align*}
\]

To avoid singularities of the distribution defined by the previous constraints we will assume, in the sequel, that \( \phi \neq \pm \pi/2 \) and the remaining brackets are zero.

In [13], the authors, using the \( SE(2) \)-symmetry of the problem, show that this nonholonomic problem is described by a vector subbundle \( E \) of \( T^*T^2 \times \mathbb{R}^3 \rightarrow T^2 \). This vector subbundle is equipped with coordinates \((\phi, \psi, \tilde{p}_1, \tilde{p}_2, \tilde{p}_3)\) and almost-Poisson determined by the bracket relations:

\[
\begin{align*}
\{\phi, \tilde{p}_1\}_{nh} &= \frac{1}{\sqrt{2J_1}}, & \{\psi, \tilde{p}_2\}_{nh} &= \frac{1}{\sqrt{J(\phi)}} \\
\{\tilde{p}_1, \tilde{p}_2\}_{nh} &= \frac{J_0 \cos \phi}{r \sqrt{2J_1 mf(\phi)}} \tilde{p}_3 \\
\{\tilde{p}_1, \tilde{p}_3\}_{nh} &= -\frac{J_0 \cos \phi}{r \sqrt{2J_1 mf(\phi)}} \tilde{p}_2
\end{align*}
\]

where \( f(\phi) = J_0 - \frac{J_0^2 \sin^2 \phi}{m r^2} \). The hamiltonian of the nonholonomic system is now written as

\[
h(\phi, \psi, \tilde{p}_1, \tilde{p}_2, \tilde{p}_3) = \frac{1}{2}(\tilde{p}_1^2 + \tilde{p}_2^2 + \tilde{p}_3^2) .
\]

A section \( \gamma \) of \( E \rightarrow T^2 \), that is, \((\phi, \psi) \rightarrow (\phi, \psi, \gamma_1(\phi, \psi), \gamma_2(\phi, \psi), \gamma_3(\phi, \psi))\) verifies that \( \text{Im}(\gamma) \) is Lagrangian if and only if verifies Equations (9); that is,

\[
\begin{align*}
0 &= \frac{J_0 \cos \phi}{r \sqrt{2J_1 mf(\phi)}} \gamma_3 + \frac{1}{\sqrt{2J_1}} \frac{\partial \gamma_2}{\partial \phi} - \frac{1}{\sqrt{J(\phi)}} \frac{\partial \gamma_1}{\partial \psi} \\
0 &= \frac{1}{\sqrt{2J_1}} \frac{\partial \gamma_3}{\partial \phi} - \frac{J_0 \cos \phi}{r \sqrt{2J_1 mf(\phi)}} \gamma_2 \\
0 &= \frac{1}{\sqrt{J(\phi)}} \frac{\partial \gamma_3}{\partial \psi} .
\end{align*}
\]

In order to find solutions of the Hamilton-Jacobi equation, we can try to solve the equation \( h \circ \gamma = \text{constant} \), that is

\[
(\gamma_1(\phi, \psi))^2 + (\gamma_2(\phi, \psi))^2 + (\gamma_3(\phi, \psi))^2 = \text{constant} \tag{23}
\]

A complete solution of the Hamilton-Jacobi equation is given by the map

\[
\Phi : \quad T^2 \times \mathbb{R}^3 \rightarrow E \quad (\phi, \psi, \lambda_1, \lambda_2, \lambda_3) \mapsto (\phi, \psi, \Phi_1, \Phi_2, \Phi_3)
\]
where
\[ \Phi_1 = \lambda_1 \sqrt{2J_1}, \]
\[ \Phi_2 = \lambda_2 \sqrt{f(\phi)} + \frac{J_0 \lambda_3}{r \sqrt{m}} \sin \phi, \]
\[ \Phi_3 = \frac{J_0 \lambda_2}{r \sqrt{m}} \sin \phi - \lambda_3 \sqrt{f(\phi)} \]
and the functions
\[ f_1(\phi, \psi, \tilde{p}_1, \tilde{p}_2, \tilde{p}_3) = \frac{1}{\sqrt{2J_1} \tilde{p}_1} \]
\[ f_2(\phi, \psi, \tilde{p}_1, \tilde{p}_2, \tilde{p}_3) = \frac{1}{J_0} \left( \tilde{p}_2 \sqrt{f(\phi)} + \frac{J_0 \tilde{p}_3}{r \sqrt{m}} \sin \phi \right) \]
\[ f_3(\phi, \psi, \tilde{p}_1, \tilde{p}_2, \tilde{p}_3) = \frac{1}{J_0} \left( \frac{J_0 \tilde{p}_2}{r \sqrt{m}} \sin \phi - \tilde{p}_3 \sqrt{f(\phi)} \right) \]
are in involution.

REFERENCES
[1] R. Abraham and J. E. Marsden, Foundations of Mechanics, 2nd ed., Benjamin-Cummings, Reading (Ma), 1978.
[2] V. I. Arnold, Mathematical Methods of Classical Mechanics, Second edition. Graduate Texts in Mathematics, 60. Springer-Verlag, New York, 1989.
[3] P. Balseiro, J. C. Marrero, D. Martín de Diego and E. Padrón, A unified framework for mechanics: Hamilton-Jacobi equation and applications, Nonlinearity, 23 (2010), 1887–1918.
[4] L. Bates and J. Sniatycki, Nonholonomic reduction, Rep. Math. Phys., 32 (1993), 99–115.
[5] F. Cantrijn, Vector fields generating invariants for classical dissipative systems, J. Math. Phys., 23 (1982), 1589–1595.
[6] F. Cantrijn, M. de León and D. Martín de Diego, On almost-Poisson structures in nonholonomic mechanics, Nonlinearity, 12 (1999), 721–737.
[7] J. F. Cariñena, X. Gracia, G. Marmo, E. Martínez, M. Muñoz-Lecanda and N. Román-Roy, Geometric Hamilton-Jacobi theory, Int. J. Geom. Meth. Mod. Phys., 3 (2006), 1417–1458.
[8] J. F. Cariñena, X. Gracia, G. Marmo, E. Martínez, M. Muñoz-Lecanda and N. Román-Roy, Geometric Hamilton-Jacobi theory for nonholonomic dynamical systems, Int. J. Geom. Meth. Mod. Phys., 7 (2010), 431–454.
[9] C. Godbillon, Géométrie Différentielle et Mécanique Analytique, Hermann, Paris, 1969.
[10] M. Leok, T. Ohsawa and D. Sosa, Hamilton-Jacobi Theory for Degenerate Lagrangian Systems with Holonomic and Nonholonomic Constraints, Journal of Mathematical Physics, 53 (2012), 072905 (29 pages).
[11] M. de León, D. Iglesias-Ponte and D. Martín de Diego, Towards a Hamilton-Jacobi theory for nonholonomic mechanical systems, Journal of Physics A: Math. Gen., 41 (2008), 015205, 14 pp.
[12] M. de León, J. C. Marrero and D. Martín de Diego, A geometric Hamilton-Jacobi theory for classical field theories, In: Variations, geometry and physics, 129–140, Nova Sci. Publ., New York, (2009).
[13] M. de León, J. C. Marrero and D. Martín de Diego, Linear almost Poisson structures and Hamilton-Jacobi equation. Applications to nonholonomic mechanics, J. Geom. Mech., 2 (2010), 159–198.
[14] M. de León, D. Martín de Diego, J. C. Marrero, M. Salgado and S. Vilarino, Hamilton-Jacobi theory in k-symplectic field theories, Int. J. Geom. Meth. Mod. Phys., 7 (2010), 1491–1507.
[15] M. de León, J. C. Marrero, D. Martín de Diego and M. Vaquero, A Hamilton-Jacobi theory for singular lagrangian systems, J. Math. Phys., 54 (2013), 032902, 32 pp.
[16] M. de León, D. Martín de Diego and M. Vaquero, A Hamilton-Jacobi theory for singular lagrangian systems in the Skinner and Rusk setting, Int. J. Geom. Meth. Mod. Phys., 9 (2012), 1250074, 24 pp.
[17] M. de León, D. Martín de Diego, C. Martínez-Campos and M. Vaquero, A Hamilton-Jacobi theory in infinite dimensional phase spaces, In preparation.
[18] M. de León and P. R. Rodrigues, Methods of differential geometry in analytical mechanics, North-Holland Mathematics Studies, 158. North-Holland Publishing Co., Amsterdam, 1989.
[19] P. Libermann and Ch. M. Marle, Symplectic Geometry and Analytical Mechanics, D. Reidel Publishing Co., Dordrecht, 1987.
[20] J. C. Marrero and D. Sosa, The Hamilton-Jacobi equation on Lie affgebroids, Int. J. Geom. Methods Mod. Phys., 3 (2006), 605–622.
[21] T. Oshawa and A. M. Bloch, Nonholonomic Hamilton-Jacobi equations and integrability, J. Geom. Mech., 1 (2009), 461–481.
[22] H. Rund, The Hamilton-Jacobi Theory in the Calculus of Variations, Hazell, Watson and Viney Ltd., Aylesbury, Buckinghamshire, U.K. 1966.
[23] I. Vaisman, Lectures on the Geometry of Poisson Manifolds, Progress in Mathematics, 118. Birkhäuser Verlag, Basel, 1994.
[24] A. J. van der Schaft and B. M. Maschke, On the Hamiltonian formulation of nonholonomic mechanical systems, Rep. Math. Phys., 34 (1994), 225–233.

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