Dempsterian-Shaferian Belief Network From Data

Mieczysław A. Klopotek

Institute of Computer Science, Polish Academy of Sciences
e-mail: klopotek@ipipan.waw.pl

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Abstract

Shenoy and Shafer [11] demonstrated that both for Dempster-Shafer Theory and probability theory there exists a possibility to calculate efficiently marginals of joint belief distributions (by so-called local computations) provided that the joint distribution can be decomposed (factorized) into a belief network. A number of algorithms exists for decomposition of probabilistic joint belief distribution into a bayesian (belief) network from data. For example Spirtes, Glymour and Scheines [13] formulated a Conjecture that a direct dependence test and a head-to-head meeting test would suffice to construe bayesian network from data in such a way that Pearl’s concept of d-separation [4] applies.

This paper is intended to transfer Spirtes, Glymour and Scheines [13] approach onto the ground of the Dempster-Shafer Theory (DST). For this purpose, a frequentionistic interpretation of the DST developed in [7] is exploited. A special notion of conditionality for DST is introduced and demonstrated to behave with respect to Pearl’s d-separation [4] much the same way as conditional probability (though some differences like non-uniqueness are evident). Based on this, an algorithm analogous to that from [13] is developed. The notion of a partially oriented graph (pog) is introduced and within this graph the notion of p-d-separation is defined. If direct dependence test and head-to-head meeting test are used to orient the pog then its p-d-separation is shown to be equivalent to the Pearl’s d-separation for any compatible dag.

Keywords: Dempster-Shafer belief network; learning from data; conditional beliefs; d-separation; partially oriented belief networks.
1 Introduction

Many researchers consider the theory of evidence a proper tool for representation of uncertainty. It has been developed by Dempster [3] and Shafer [10] and possesses several interesting properties. One of them is that if we are able to factorize a joint belief distribution into a hyper-graph structure (that is to represent it as combination of simpler belief functions) then there exists the possibility of local computations of marginals of the joint belief distribution as well as conditionals of some variables on various events without actually calculating this joint belief distribution [11]. This is of tremendous importance if we imagine how much space in computer memory would be required to represent a joint belief distribution in, say, 20 variables. The Shenoy-Shafer theory of local computations makes marginalizations and calculation of conditionals for belief functions more feasible.

Actually to exploit this fine property we need a tool for factorizing the joint belief distribution into such factors. From the experience with probability distributions it is known that they may be represented in form of bayesian networks, that is directed acyclic graphs reflecting dependencies between variables. These directed acyclic graphs (dag) may be transformed directly to hypergraphs being precisely frameworks for factorizations required by the Shenoy-Shafer theory of local computations. A number of techniques of decomposition of a joint probability distribution into bayesian networks have been developed. In this paper we want to find an analogous decomposition of a joint belief distribution following the outlines of Spirtes et al approach [13], which has been developed for probability distributions.

In Section 2 we will briefly recall basic definitions of the DS theory of evidence. Section 3 will recall the Shenoy-Shafer requirements for local computations of a joint distribution. Section 4 is devoted to selection of proper conditional belief definition. Section 5 introduces my own sense of conditionality. Section 6 develops the algorithm. Section 7 contains some conclusions.

2 Formal Definition of the Dempster-Shafer Theory of Evidence

Let us make the remark that if an object is described by a set of discrete attributes (features, variables) $X_1, X_2, ..., X_n$ taking values from their respective domains $\Xi_1, \Xi_2, ..., \Xi_n$ then we can think of it as being described by a complex attribute $X$ having vector values, that is the domain $\Xi$ of $X$ is equal:

$$\Xi = \{(x_1, x_2, ..., x_n)|x_i \in \Xi_i \forall i = 1, ..., n\}$$

However sometimes we will treat $X$ as a set of attributes applying set-theoretic operations $\cap, \cup, -$ understanding, that there exists some natural ordering among the attributes constituting $X$ so that any non-empty "subset" $Y$ of $X$ $Y \subseteq X$ is a Cartesian product of its "components": $Y = X_{k_1} \times X_{k_2} \times \ldots \times X_{k_m}$, where indices $k_1, k_2, ..., k_m$ are an increasing subsequence from the sequence $1, 2, ..., n$. 
So in definitions below let us assume that we are talking of objects described by a single attribute $X$ taking its values from the domain $\Xi$. We say that $\Xi$, the domain of $X$ is our space of discernment spanned by the attribute $X$. We shall also briefly say that $X$ is our space of discernment instead.

The function $m$ (Mass Function, or basic probability assignment function bpa) is defined as

**Definition 1** The Pseudo-Mass Function in the sense of the DS-Theory is defined as $m:2^{\Xi} \rightarrow [-1, 1]$ with

$$\sum_{A \in 2^{\Xi}} |m(A)| = 1$$

$$m(\emptyset) = 0$$

$$\forall A \in 2^{\Xi} \quad 0 \leq \sum_{A \subseteq B} m(B)$$

where $|.|$ means the absolute value operator. If also

$$\forall A \in 2^{\Xi} \quad m(A) \geq 0$$

holds, then we talk of (intrinsic) Mass Function.

**Definition 2** Whenever $m(A) > 0$, we say that $A$ is the focal point of the Bel-Function.

For the purpose of this paper we define the Bel-function as follows.

**Definition 3** The Pseudo-Belief Function in the sense of the DS-Theory is defined as $Bel:2^{\Xi} \rightarrow [-1, 1]$ with

$$Bel(A) = \sum_{B \subseteq A} m(B)$$

for any nonempty $A \in 2^{\Xi}$ where $m(A)$ is a Pseudo-Mass Function in the sense of the DS-Theory (see Def. 1 above). If $m(A)$ is an (intrinsic) Mass Function, then Bel is called the (intrinsic) Belief Function (the word "intrinsic" is omitted subsequently).

Let us also introduce the Pl-Function (Plausibility) for non-empty sets $A$ as:

**Definition 4** The Pseudo-Plausibility Function in the sense of the DS-Theory is defined as $Pl:2^{\Xi} \rightarrow [-1, 1]$ with

$$\forall A \in 2^{\Xi} \quad Pl(A) = 1 - Bel(\Xi - A)$$

If $Bel$ is a Belief Function then $Pl$ is a Plausibility Function.

For completeness let us recall also the Q-Function of the DS-Theory.
Definition 5 The Q-Function in the sense of the DS-Theory is defined as $Q : 2^\Xi \to [0, 1]$ with

$$\forall A \in 2^\Xi \quad Q(A) = \sum_{A \subseteq B} m(B)$$

where $m(A)$ is a Pseudo-Mass Function in the sense of the DS-Theory (see Def.1 above and notice the difference to Def.3 in that the sum is taken over supersets, not subsets of $A$).

Please pay attention to the fact, that domains of values of Mass Function, Belief Function and Plausibility Function are always $[0, 1]$, whereas those of their Pseudo-Counterparts are $[-1, 1]$. However, the domain of the Q-Function is always $[0, 1]$ independently of whether $m$ is a Pseudo-Mass or (intrinsic) Mass Function.

Please pay attention to the fact that knowing one of the functions $m$, Bel, Pl or $Q$ suffices to derive any other of them.

Beside the above definition a characteristic feature of the DS-Theory is the so-called DS-rule of combination of (independent) evidence:

Definition 6 Let $Bel_{E_1}$ and $Bel_{E_2}$ represent independent information over the same space of discernment. Then:

$$Bel_{E_1,E_2} = Bel_{E_1} \oplus Bel_{E_2}$$

calculated as:

$$m_{E_1,E_2}(A) = c \cdot \sum_{B,C;A=B\cap C} m_{E_1}(B) \cdot m_{E_2}(C)$$

($c$ - normalizing constant, $c = \left(\sum_{B,C;B\cap C \neq \emptyset} m_{E_1}(B) \cdot m_{E_2}(C)\right)^{-1}$) represents the Combined Pseudo-Belief-Function of Two Independent Pseudo-Beliefs.

Let us also introduce the marginalization and extension operations: first for sets.

Definition 7 Let $X = X_1 \times X_2 \times \ldots \times X_n$ and $\Xi = \Xi_1 \times \Xi_2 \times \ldots \times \Xi_n$. Let $A$ be a set $A \in 2^\Xi$. Let $Y = X_{i_1} \times X_{i_2} \times \ldots \times X_{i_k}$, where indices $\{i_1, \ldots, i_k\}$ are all distinct and are subset of $\{1, \ldots, n\}$. The set $B$ is called the projection (marginalization) of the set $A$ onto the (sub)space $Y$ (denoted $B = A^Y$) iff for every element $(v_1, \ldots, v_n) \in A$ the element $(v_{i_1}, v_{i_2}, \ldots, v_{i_k})$ belongs to $B$.

We shall say also say that $A$ is an extension of $B$.

We shall distinguish one special extension: the empty extension.

Definition 8 Let $B \subseteq \Xi_1$. Let $A \subseteq \Xi_1 \times \Xi_2$ such that $A = B \times \Xi_2$. Then we say that $A$ is the empty extension of $B$, denoted $A = B^X$. ...
Now let us define the marginalization and extension for Bel-Functions:

**Definition 9** Let \( X = X_1 \times X_2 \) be our space of discernment for which the \( m \), and its \( \text{Bel}, \text{Pl} \) and \( Q \) functions are defined.

The \( m \) function marginalized (projected) onto the subspace \( X_1 \), denoted as \( m^{\downarrow X_1} \), is defined as:

\[
\forall B; B \subseteq \Xi_1 \quad m^{\downarrow X_1}(B) = \sum_{A; B = A^{\downarrow X_1}} m(A)
\]

The functions \( \text{Bel}^{\downarrow X_1}, \text{Pl}^{\downarrow X_1} \) and \( Q^{\downarrow X_1} \) are defined accordingly to \( \text{Bel}, \text{Pl} \) and \( Q \) definitions above with respect to \( m^{\downarrow X_1} \) as their (pseudo-)mass function.

**Definition 10** Let \( X_1 \) be our space of discernment for which the \( m \), and its \( \text{Bel}, \text{Pl} \) and \( Q \) functions are defined.

The \( m \) function empty-extended onto the superspace \( X = X_1 \times X_2 \), denoted as \( m^{\uparrow X} \), is defined as:

\[
\forall A; A \subseteq \Xi, A = (A^{\downarrow X_1})^{\uparrow X} \quad m^{\uparrow X}(A) = m(A^{\downarrow X_1})
\]

and

\[
\forall A; A \subseteq \Xi, A \neq (A^{\downarrow X_1})^{\uparrow X} \quad m^{\uparrow X}(A) = 0
\]

otherwise. The functions \( \text{Bel}^{\uparrow X}, \text{Pl}^{\uparrow X} \) and \( Q^{\uparrow X} \) are defined accordingly to \( \text{Bel}, \text{Pl} \) and \( Q \) definitions above with respect to \( m^{\uparrow X} \) as their (pseudo-)mass function.

Please notice that the operator \( \oplus \) is defined for combination of Bel’s only for the same space of discernment. Should it happen, however that \( \text{Bel}_1 \) is defined over the space \( X_1 \times X_3 \), and \( \text{Bel}_2 \) over \( X_2 \times X_3 \), then instead of writing:

\[
\text{Bel}_{1,2} = \text{Bel}_1^{\uparrow X_1 \times X_2 \times X_3} \oplus B_2^{\uparrow X_1 \times X_2 \times X_3}
\]

we will simply write

\[
\text{Bel}_{1,2} = \text{Bel}_1 \oplus \text{Bel}_1
\]

whenever no misunderstandings may occur.

### 3 Prerequisites for Shenoy-Shafer Propagation

We cite below extensively the paper [11] of Shenoy and Shafer to recall some basic notions and to show usefulness of decomposition of the DS joint belief distribution in terms of a belief network.
Variables and valuations: Let $V$ be a finite set. The elements of $V$ are called variables. For each $h \subseteq V$ there is a set $VV_h$. The elements of $VV_h$ are called valuations. Let $VV = \bigcup \{VV_h | h \subseteq V\}$ be called the set of all valuations.

In case of probabilities a valuation on $h$ will be a non-negative, real-valued function on the set of all configurations of $h$ (a configuration on $h$ is a vector of possible values of variables in $h$). In the belief function case a valuation is a non-negative, real-valued function on the set of all subsets of configurations of $h$.

Proper valuation: for each $h \subseteq V$ there is a subset $P_h$ of $VV_h$ elements of which are called proper valuations on $h$. Let $P$ be the set of all proper valuations.

Combination: We assume that there is a mapping $\odot : VV \times VV \rightarrow VV$ called combination such that:

(i) if $G$ and $H$ are valuations on $g$ and $h$ respectively, then $G \odot H$ is a valuation on $g \cup h$
(ii) if either $G$ or $H$ is not a proper valuation then $G \odot H$ is not a proper valuation
(iii) if both $G$ and $H$ are proper valuations then $G \odot H$ may be or not be a proper valuation

In case of probabilities, combination is value-by-value multiplication. In case of DS-theory - it is the Dempster rule operator $\oplus$ (previous section).

Marginalization: We assume that there is a mapping $\downarrow h : \bigcup \{VV_g | g \subseteq h\} \rightarrow VV_h$ called marginalization to $h$ such that:

(i) if $G$ is a valuation on $g$ and $h \subseteq g$ then $G^{lh}$ is a valuation on $h$.
(ii) if $G$ is a proper valuation then $G^{lh}$ is a proper valuation
(iii) if $G$ is a not proper valuation then $G^{lh}$ is not a proper valuation

In case of probabilities, marginalization is the summation over the dropped dimension(s). In case of DS-theory - it is the Dempster-Shafer marginalization.

Axiom A1: (Commutativity and associativity of combination). Suppose $G,H,K$ are valuations on $g$, $h$, $k$ respectively. Then $G \odot H = H \odot G$ and $(G \odot H) \odot K = G \odot (H \odot K)$.

Axiom A2: (Consonance of marginalization) Suppose $G$ is a valuation on $g$, and suppose $k \subseteq h \subseteq g$. Then $(G^{lh})^{lk} = G^{lk}$

Axiom A3: (Distributivity of marginalization over combination) Suppose $G$ and $H$ are valuations on $g$ and $h$, respectively. Then $(G \odot H)^{\downarrow g} = G \odot (H^{\downarrow g \cap h})$
Hypergraph: We call a non-empty set $HV$ of non-empty subsets of a finite set of $V$ a hypergraph on $V$. We call the elements of $HV$ hyperedges. We call the elements of $V$ nodes.

Factorization: Suppose $A$ is a valuation on a finite set of variables $V$, and suppose $HV$ is a hypergraph on $V$. If $A$ is equal to the combination of valuations of all hyperedges $h$ of $HV$ then we say that $A$ factorizes on $HV$.

The axiom A3 states that to compute $(G \odot H)^{1g}$ it is not necessary to compute $G \odot H$ first.

Shenoy and Shafer consider it unimportant whether or not the factorization should refer to conditional probabilities in case of probabilistic belief networks. We shall make at this point the remark that for expert system inference engine it is of primary importance how the contents of the knowledge base should be understood by the user as beside computation an expert system is expected at least to justify its conclusions and it can do so only referring to elements of the knowledge base. So if a belief network (or a hypergraph) is to be used as the knowledge base, as much elements as possible have to refer to experience of the user.

In our opinion, the major reason for this remark of Shenoy and Shafer is that in fact the Dempster-Shafer belief function cannot be decomposed in terms of conditional belief functions as they are defined in the literature. This claim is demonstrated in the next section. Thereafter we introduce our own definition of conditionality for the DS theory.

4 Definitions of Conditionality in Literature

The probability update function $\text{cond}_{B}$ with respect to the event (evidence) $B$ is defined (e.g. in [6]) as a partial function from the set of probability functions into the set of probability functions. As usual, let $\text{cond}_{B}(Pr) = Pr(\cdot \mid B) \ (Pr \text{ stands for "probability"}). It is known that then if $\circ$ denotes the operator of update combination then the following holds:

$$\text{cond}_{C} \circ \text{cond}_{B} = \text{cond}_{B \cap C} = \text{cond}_{B} \circ \text{cond}_{C}$$

The belief update function be defined (after e.g. [6]) with respect to evidence $B$ ($\text{cond}_{B}$) as a partial function from the set of belief functions into the set of belief functions.

4.1 Dempsterian Interpretation of Conditional Belief

Dempster [3] defined conditional belief function for a Bel function conditioned on the event $B$ as:

$$\text{Bel}(\cdot \mid |B|) = \text{Bel} \oplus \text{Bel}_{B}$$
(notation after [6]) with $Bel_B$ being the deterministic function of belief into validity of the event $B$, that is $m_B(B) = 1$ and $m_B(A) = 0$ for every $A \neq B$. It has been shown that

$$Bel(A||B) = \frac{Bel(A \cup B^c) - Bel(B^c)}{1 - Bel(B^c)}$$

which implies directly:

$$Pl(A||B) = \frac{Pl(A \cap B)}{Pl(B)}$$

It is easy to show that for $cond_B = Bel(.||B)$ the following holds:

$$cond_C \circ cond_B = cond_{B \cap C} = cond_B \circ cond_C$$

### 4.2 Halpernian Interpretation of Conditional Belief

Halpern and Fagin [6] insisted on treating the belief function as generalized probability.

Let $P$ be a set of probability functions defined over a sample space $\Xi$. A lower envelop of $P$ is defined as a function $f$ such that for every $A \subseteq \Xi$ $f(A) = \inf\{Pr(A); Pr \subseteq P\}$ (Pr means probability), $A$ is measurable with respect to $Pr$). The upper envelop is defined respectively (as supremum). Let $Bel$ be a belief function defined over $\Xi$ and let $(\Xi, A, Pr)$ be a probability space. We say that $Pr$ is consistent with $Bel$, if $Bel(A) \leq Pr(A) \leq Pl(A)$ for every $A \in A$. This reflects the intuition that $Pr$ is consistent with $Bel$, if the probabilities assigned by $Pr$ are consistent with intervals $[Bel(A), Pl(A)]$ set by $Bel$. For consistency it suffices that $Bel(A) \leq Pr(A)$, as then immediately $Pr(A) \leq Pl(A)$. Hence $Bel$ is the lower envelop for $P$, and $Pl$ the upper envelop for $P$.

Though every belief function is a lower envelop, not every lower envelop is a belief function.

Let for the belief function $Bel P_{Bel}$ denote the set of all probability functions consistent with $Bel$. It has been shown [6] that if $Bel$ is a belief function over $\Xi$, then for every $A, A \subseteq \Xi$ we have:

$$Bel(A) = \inf_{Pr \in P_{Bel}} Pr(A)$$

$$Pl(A) = \sup_{Pr \in P_{Bel}} Pr(A)$$

Halpern and Fagin define conditional belief function with respect to the event $B$ $Bel(A|B)$ with respect to the belief function $Bel()$ as:

$$Bel(A|B) = \inf_{Pr \in P_{Bel}} Pr(A|B)$$

$$Pl(A|B) = \sup_{Pr \in P_{Bel}} Pr(A|B)$$

It has been shown [6] that if $Bel()$ is a belief function such that $Bel(B) > 0$, then

$$Bel(A|B) = \frac{Bel(A \cap B)}{Bel(A \cap B) + Pl(A^c \cap B)}$$
It has been shown further [6] that then \( Bel(\cdot|B) \) is a belief function and \( Pl(\cdot|B) \) the respective plausibility function.

One may be tempted by analogy to probability update function to define \( cond_B = Bel(\cdot|B) \). However, it has been shown that in general case for the belief functions:

\[
cond_C \circ cond_B \neq cond_{B \cap C} \neq cond_B \circ cond_C
\]

### 4.3 Kyburgian Definitions of Conditionality

Kyburg [8] was probably the first to demonstrate that a belief function may be represented as an envelop of a family of traditional probability functions. He proved also that the following holds:

\[
Bel(A|B) \leq Bel(A||B) \leq Pl(A||B) \leq Pl(A|B)
\]

He considered also non-deterministic evidence. He defined non-deterministic conditional probability with respect to event \( B \) with probability \( p \) as (Jeffry-rule):

\[
P(X|pB) = P(X|B) \cdot p + P(X|B^c) \cdot (1-p)
\]

(Notice that: \( P(B|B) = 1 \), and \( P(B|pB) = p \)).

In analogy to the above definitions, he defined also two new types of non-deterministic conditional belief functions: First of them in analogy to Dempsterian conditional belief: Let \( Bel^{p,B} \), be the so-called simple belief function such that \( m^{p,B}(B) = p \) and \( m^{p,B}(\Xi) = 1 - p \), and for other subsets of \( \Xi \) \( m \) is equal 0, with \( \Xi \) being the universe (the space of discernment). Then conditional belief \( Bel(\cdot||pB) \) is defined as

\[
Bel(\cdot||pB) = Bel \oplus Bel^{p,B}
\]

that is as Dempsterian combination of belief function and the simple belief function.

On the other hand, he introduced also a generalized "envelop" definition of the conditional belief function:

\[
Bel(A|pB) = \inf_{P \in P_{a.i}} Pr(A|pB)
\]

and subsequently he has shown that

\[
Bel(A|pB) \leq Bel(A||pB) \leq Pl(A||pB) \leq Pl(A|pB)
\]
4.4 Criticism of the Notions of Conditionality

The "envelop" definitions of conditionality ($\text{Bel}(.|B)$, $\text{Bel}(.|pB)$ above) share one property making then not suitable for Shenoy-Shafer propagation: the belief update function is in general not cumulative, hence the sequence of usage of evidence proves to be of importance, and hence the Axiom A1 is violated.

It should also be mentioned that Smets [12] sharply criticizes envelop interpretations proposed by Kyburg, Halpern and Fagin as well as earlier by Dempster (see above) as misleading and not compatible with the spirit of the DST.

On the other hand, we would expect of a conditional belief function that :
$$\text{Bel} = \text{Bel}(.|B) \oplus \text{Bel}_B,$$
but obviously this is not the case (in general it is not true that: $\text{Bel} = \text{Bel} \oplus \text{Bel}_B \oplus \text{Bel}_B$).

Hence it is impossible that the factorization using $\text{Bel}(.|B)$ may reflect the function $\text{Bel}$.

Under these circumstances one should plainly ask: why not to try the expression

\textbf{Definition 11} If $\text{Bel}$ is a DS belief function defined over some space of discernment, and $h$ is a set of variables spanning this space, then the conditional belief function $\text{Bel}^h$ should be any function meeting the condition $\text{Bel} = \text{Bel}^h \oplus \text{Bel}^h$
as the defining expression for conditioning on the set of variables $h$.

There are several severe reasons why this amazingly simple idea may have not been exploited in the past:
(i) in general, $\text{Bel}^h$ is not unique.
(ii) in general, $\text{Bel}^h$ is not a belief function (it is only a pseudo-belief function, with Q-values being non-negative).

In order to exploit still the results of Shenoy-Shafer, it must be stated that neither the dempsterian combination nor dempsterian marginalization leads outside the set of pseudo-belief function set. Hence it may be easily shown that Shenoy-Shafer axioms apply also to pseudo-belief functions. Given this prerequisite, we develop in the subsequent sections a theory of identification of DS belief networks following results obtained by Spirtes, Glymour and Scheines [13] for probabilistic belief networks.

5 An Alternative Definition of Conditionality and Implications

In [7] we have introduced a new frequentist interpretation of DS Belief function. Under this interpretation we introduced the notion of composite measurement of two variables $X_1, X_2$, a notion of statistical independence
of Bel functions closely related to the empty extension and we introduced a new notion of conditionality. Below we briefly summarize this interpretation:

F. Bacchus in his paper [1] on axiomatization of probability theory and first order logic shows that probability should be considered as a quantifier binding free variables in first order logic expressions just like universal and existential quantifiers do. So if e.g. $\alpha(x)$ is an open expression with a free variable $x$ then $[\alpha(x)]_x$ means the probability of truth of the expression $\alpha(x)$. (The quantifier $[\ ]_x$ binds the free variable $x$ and yields a numerical value ranging from 0 to 1 and meeting all the Kolmogorov axioms). Within the expression $[\alpha(x)]_x$ the variable $x$ is bound. While sharing Bacchus’ view, we find his notation a bit cumbersome so we change it to be similar to the universal and existential quantifiers throughout this paper. Furthermore, Morgan [9] insisted that the probabilities be always considered in close connection with the population they refer to. Bacchus’ expression $[\alpha(x)]_x$ we rewrite as:

\[ \text{Prob}^P_x (\alpha(x)) \text{ - the probability of } \alpha(x) \text{ being true within the population } P. \]

The $P$ (population) is a unary predicate with $P(x) = \text{TRUE}$ indicating that the object $x \in \Omega$, that is element of a universe of objects) belongs to the population under considerations. If $P$ and $P'$ are populations such that $\forall x P'(x) \rightarrow P(x)$ (that is membership in $P'$ implies membership in $P$, or in other words: $P'$ is a subpopulation of $P$), then we distinguish two cases:

**case 1:** $(\text{Prob}^P_x (\alpha(x)) = 0$ (that is probability of membership in $P'$ with respect to $P$ is equal 0) - then (according to [9] for any expression $\alpha(x)$ in free variable $x$ the following holds for the population $P'$:

\[ (\text{Prob}^P_x (\alpha(x))) = 1 \]

**case 2:** $(\text{Prob}^P_x (\alpha(x))) > 0$ then (according to [9] for any expression $\alpha(x)$ in free variable $x$ the following holds for the population $P'$:

\[
(\text{Prob}^{P'}_x (\alpha(x))) = \frac{\text{Prob}^P_x (\alpha(x) \land P'(x))}{(\text{Prob}^P_x (P'(x)))}
\]

We also use the following (now traditional) mathematical symbols:

$\forall x \alpha(x)$ - always $\alpha(x)$ (universal quantifier)

$\exists x \alpha(x)$ - there exists an $x$ such that $\alpha(x)$ (existential quantifier)
Definition 12 Let $X$ be a set valued attribute taking its values as subsets of a finite domain $\Xi$. By a measurement method we understand a function:

$$M : \Omega \times 2^\Xi \rightarrow \{TRUE, FALSE\}$$

where $\Omega$ is the set of objects, (or population of objects) such that

- $\forall \omega : \omega \in \Omega \implies M(\omega, \Xi) = TRUE$ (X takes at least one of values from $\Xi$)
- $\forall \omega : \omega \in \Omega \implies M(\omega, \emptyset) = FALSE$
- whenever $M(\omega, A) = TRUE$ for $\omega \in \Omega$, $A \subseteq \Xi$ then for any $B$ such that $A \subset B$ $M(\omega, B) = TRUE$ holds,
- whenever $M(\omega, A) = TRUE$ for $\omega \in \Omega$, $A \subseteq \Xi$ and if $\text{card}(A) > 1$ then there exists $B$, $B \subset A$ such that $M(\omega, B) = TRUE$ holds.
- for every $\omega$ and every $A$ either $M(\omega, A) = TRUE$ or $M(\omega, A) = FALSE$ (but never both).

$M(\omega, A)$ indicates that for the object $\omega$ the value of the attribute $X$ has a non-empty intersection with the set $A$.

Definition 13 A label of an object $\omega \in \Omega$ is a subset of the domain $\Xi$ of the attribute $X$.

A labeling under the measurement method $M$ is a function $l : \Omega \rightarrow 2^\Xi$ such that for any object $\omega \in \Omega$ either $l(\omega) = \emptyset$ or $M(\omega, l(\omega)) = TRUE$.

Each labelled object (under the labeling $l$) consists of a pair $(O_j, L_j)$, $O_j$ - the $j^{th}$ object, $L_j = l(O_j)$ - its label.

By a population under the labeling $l$ we understand the predicate $P : \Omega \rightarrow \{TRUE, FALSE\}$ of the form $P(\omega) = TRUE$ if $l(\omega) \neq \emptyset$ (or alternatively, the set of objects for which this predicate is true).

If for every object of the population the label is equal to $\Xi$ then we talk of an unlabeled population (under the labeling $l$), otherwise of a pre-labelled one.
Definition 14 Let \( l \) be a labeling under the measurement method \( M \). Let us consider the population under this labeling. The modified measurement method

\[
M_l : \Omega \times 2^\Xi \rightarrow \{ \text{TRUE, FALSE} \}
\]

where \( \Omega \) is the set of objects, is is defined as

\[
M_l(\omega, A) = M(\omega, A \cap l(\omega))
\]

(Notice that \( M_l(\omega, A) = \text{FALSE} \) whenever \( A \cap l(\omega) = \emptyset \).)

Definition 15 Let \( P \) be a population and \( l \) its labeling. Then

\[
\begin{align*}
\text{Bel}^M_l (A) &= \Pr(\omega) \neg M_l(\omega, \Xi - A) \\
\text{Pl}^M_l (A) &= \Pr(\omega) M_l(\omega, A) \\
m^M_l (A) &= \Pr(\omega) \left( \bigwedge_{B \subseteq A} M_l(\omega, B) \land \bigwedge_{B : B \neq \emptyset, B \subseteq A} \neg M_l(\omega, B) \right) \\
Q^M_l (A) &= \Pr(\omega) \left( \bigwedge_{B : B \neq \emptyset, B \subseteq A} M_l(\omega, B) \right)
\end{align*}
\]

Theorem 1 \( m^M_l \), \( \text{Bel}^M_l \), \( \text{Pl}^M_l \), \( Q^M_l \) are Mass, Belief, Plausibility and Q Functions in the sense of the Dempster-Shafer Theory resp.

Definition 16 Let \( M \) be a measurement method, \( l \) be a labeling under this measurement method, and \( P \) be a population under this labeling (Note that the population may also be unlabeled). Let us take a set of (not necessarily disjoint) nonempty sets of attribute values \( \{ L^1, L^2, ..., L^k \} \) and let us define the probability of selection as a function \( m^{LP,L^1,L^2, ..., L^k} : 2^\Xi \rightarrow [0, 1] \) such that

\[
\sum_{A : A \subseteq \Xi} m^{LP,L^1,L^2, ..., L^k}(A) = 1
\]

\[
\forall A : A \subseteq \Xi \land k \land \exists \epsilon \in (0, 1) m^{LP,L^1,L^2, ..., L^k}(A) > 0
\]

\[
\forall A : A \subseteq \Xi \land k \land \exists \epsilon \in (0, 1) m^{LP,L^1,L^2, ..., L^k}(A) = 0
\]

The (general) labelling process on the population \( P \) is defined as a (randomized) functional \( LP : 2^\Xi \times \Delta \times \Gamma \rightarrow \Gamma \), where \( \Gamma \) is the set of all possible labelings under \( M \), and \( \Delta \) is a set of all possible probability of selection functions, such that for the given labeling \( l \) and a given set of (not necessarily disjoint) nonempty sets of
attribute values \( \{L^1, L^2, \ldots, L^k\} \) and a given probability of selection \( m^{LP,L^1,L^2,\ldots,L^k} \) it delivers a new labeling \( l'' \) such that for every object \( \omega \in \Omega \):

1. a label \( L \), element of the set \( \{L^1, L^2, \ldots, L^k\} \) is sampled randomly according to the probability distribution \( m^{LP,L^1,L^2,\ldots,L^k} \); This sampling is done independently for each individual object,

2. if \( M_l(\omega, L) = \text{FALSE} \) then \( l''(\omega) = \emptyset \) (that is \( l'' \) discards an object \( (\omega, l(\omega)) \) if \( M_l(\omega, L) = \text{FALSE} \)

3. otherwise \( l''(\omega) = l(\omega) \cap L \) (that is \( l'' \) labels the object with \( l(\omega) \cap L \) otherwise.)

**THEOREM 2** \( m^{LP,L^1,\ldots,L^k} \) is a Mass Function in sense of DS-Theory.

Let \( Bel^{LP}; L^1,\ldots,L^k \) be the belief and \( Pl^{LP,L^1,\ldots,L^k} \) be the Plausibility corresponding to \( m^{LP,L^1,\ldots,L^k} \). Now let us pose the question: what is the relationship between \( Bel^{M_l}; p \), \( Bel^{M_l}; p \), and \( Bel^{LP,L^1,\ldots,L^k} \). It is easy to show that

**THEOREM 3** Let \( M \) be a measurement function, \( l \) a labeling, \( P \) a population under this labeling. Let \( LP \) be a generalized labeling process and let \( l'' \) be the result of application of the \( LP \) for the set of labels from the set \( \{L^1, L^2, \ldots, L^k\} \) sampled randomly according to the probability distribution \( m^{LP,L^1,L^2,\ldots,L^k} \); Let \( P'' \) be a population under the labeling \( l'' \). Then the expected value over the set of all possible resultant labelings \( l'' \) (and hence populations \( P'' \)) (or, more precisely, value vector) of \( Bel^{M_l}; p \) is a combination via DS Combination rule of \( Bel^{M_l}; p \), and \( Bel^{LP,L^1,\ldots,L^k} \), that is:

\[
E(Bel^{M_l}; p) = Bel^{M_l}; p \oplus Bel^{LP,L^1,\ldots,L^k}
\]

Let us assume that our space of discernment consists of the attributes \( X_1, X_2 \) \( (X = X_1 \times X_2) \) ranging over \( \Xi_1 = \{v_{11}, v_{12}, \ldots, v_{1n_1}\} \), \( \Xi_2 = \{v_{21}, v_{22}, \ldots, v_{2n_2}\} \).

Let us understand the marginal distribution of \( X_i \) as follows: the measurement method for the subspace \( X_i \) be equal to the logical sum of all the measurements on sets from \( X \) compatible with the given set in \( X_i \).

\[
M_i^{X_i}(object, A) = \bigvee_{B: A = B \upharpoonright X_i} M_i(object, B)
\]

This implies immediately that

\[
m_i^{M_i^{X_i}}(A) = (m_i^{M_i})^{X_i}(A)
\]

for \( A \subseteq \Xi_i \)

As the above relationship holds and we are subsequently concerned with only one population \( P \), we will drop indices referring to the measurement method and the population relying only on projections.

Let us now introduce the notion of quantitative independence for DS-Theory.
Definition 17 Two variables $X_1, X_2$ are (mutually, marginally) quantitatively independent when for objects of the population knowledge of the truth value of $M_i^{X_1}(\text{object}, A^{iX_1})$ for all $A \subseteq \Xi_1 \times \Xi_2$ does not change our prediction capability of the values of $M_i^{X_2}(\text{object}, B^{iX_2})$ for any $B \subseteq \Xi_1 \times \Xi_2$, that is

$$
\mathbb{P}_{\omega}^{\omega} M_i^{X_2}(\omega, B^{iX_2}) = \mathbb{P}_{\omega}^{M_i^{X_1}(\omega, A^{iX_1})\land \omega} M_i^{X_2}(\omega, B^{iX_2})
$$

Theorem 4 If variables $X_1, X_2$ are quantitatively independent, then for any $B \subseteq \Xi_2$, $A \subseteq \Xi_1$

$$
m^{iX_2}(B) \cdot m^{iX_1}(A) = \sum_{F:F^{iX_1}A \land F^{iX_2}B} m(F)
$$

Definition 18 Two variables $X_1, X_2$ are measured compositely iff for $A \subseteq \Xi_1, B \subseteq \Xi_2$ for every object $\omega$:

$$
M(\omega, A \times C) = M(\omega, A \times \Xi_2) \land M(\omega, \Xi_1 \times C)
$$

and whenever $M(\omega, B)$ is sought,

$$
M(\omega, B) = \bigvee_{A,C:A \subseteq \Xi_1, C \subseteq \Xi_2, A \times C \subseteq B} M(\omega, A \times B)
$$

Under these circumstances, it is easily shown that whenever $m(B) > 0$, then there exist $A$ and $C$ such that: $B = A \times C$.

So we obtain:

Theorem 5 If variables $X_1, X_2$ are quantitatively independent and measured compositely, then

$$
m(A \times C) = m^{iX_1}(A) \cdot m^{iX_2}(C)
$$

Hence the Belief function can be calculated from Belief functions of independent variables under composite measurement:

Theorem 6 If variables $X_1, X_2$ are quantitatively independent and measured compositely, then

$$
Bel = Bel^{iX_1} \oplus Bel^{iX_2}
$$

Let us justify now the notion of empty extension:

Definition 19 The joint distribution over $X = X_1 \times X_2$ in variables $X_1, X_2$ is quantitatively independent of the variable $X_1$ when for objects of the population for every $A, A \subseteq \Xi_1 \times \Xi_2$ knowledge of the truth value of $M_i^{X_1}(\text{object}, A^{iX_1})$ does not change our prediction capability of the values of $M_i(\text{object}, A)$, that is

$$
\mathbb{P}_{\omega}^{\omega} M_i(\omega, A) = \mathbb{P}_{\omega}^{M_i^{X_1}(\omega, A^{iX_1})\land \omega} M_i(\omega, A)
$$
THEOREM 7 The joint distribution over \( X = X_1 \times X_2 \) in variables \( X_1, X_2 \), measured compositely, is independent of the variable \( X_1 \) only if \( m^{X_1}(\Xi_2) = 1 \) that is the whole mass of the marginalized distribution onto \( X_2 \) is concentrated at the only focal point \( \Xi_2 \).

THEOREM 8 If for \( X = X_1 \times X_2 \) \( Bel = (Bel^{X_2})^X \) that is \( Bel \) is the empty extension of some \( Bel \) defined only over \( X_2 \), then the \( Bel \) is independent of the variable \( X_2 \).

If for a \( Bel \) over \( X = X_1 \times X_2 \) with \( X_1, X_2 \) measured compositely \( Bel \) is independent of \( X_2 \), then \( Bel = (Bel^{X_2})^X \).

In the light of the above theorem, and taking into account that a belief function which is an empty extension of another function may be stored in a compressed manner, we shall say

Definition 20 Let \( Bel \) be defined over \( X_1 \times X_2 \). We shall speak that \( Bel \) is compressibly independent of \( X_2 \) iff \( Bel = (Bel^{X_1})^{X_2} \).

REMARK: \( m^{X_1}(\Xi_1) = 1 \) does not imply empty extension as such, especially for non-sigleton values of the variable \( X_2 \). As previously with marginal independence, it is the composite measurement that makes the empty extension a practical notion.

Let us consider now the conditional independence:

Let us introduce a concept of conditionality related to the above definition of independence. Traditionally, conditionality is introduced to obtain a kind of independence between variables de facto on one another. So let us define that:

Definition 21 For discourse spaces of the form \( X = X_1 \times ... \times X_n \) we define conditional belief function \( Bel^{X|X_i}(A) \) as

\[
Bel = Bel^{X_i} \oplus Bel^{X|X_i}
\]

Let us notice at this point that the conditional belief as defined above does not need to be unique, hence we have here a kind of pseudoinversion of the \( \oplus \) operator. Furthermore, the conditional belief does not need to be a belief function at all, because some focal points \( m \) may be negative. But it is then the pseudo-belief function in the sense of the DS-theory as the Q-measure remains positive. Please recall the fact that if \( Bel_{12} = Bel_1 \oplus Bel_2 \) then \( Q_{12}(A) = c \cdot Q_1(A) \cdot Q_2(A) \), \( c \) being a proportionality factor (as all supersets of a set are contained in all intersections of its supersets and vice versa). Hence also for our conditional belief definition:

\[
Q(A) = c \cdot (Q^{X_i})^X(A) \cdot Q^{X|X_i}(A)
\]
We shall talk later of unnormalized conditional belief $Q_{*|X}^{X}$ iff

$$Q_{*|X}^{X}(A) = Q(A)/(Q^{X}X^{*})^{X}(A)$$

Let us now reconsider the problem of independence, this time of a conditional distribution of $(X_1 \times X_2 \times X_3|X_1 \times X_3)$ from the third variable $X_3$.

**THEOREM 9** Let $X = X_1 \times X_2 \times X_3$ and let Bel be defined over $X$. Furthermore let $Bel^{X|X_1 \times X_3}$ be a conditional Belief conditioned on variables $X_1, X_3$. Let this conditional distribution be compressibly independent of $X_3$. Let $Bel^{X|X_1 \times X_2}$ be the projection of Bel onto the subspace spanned by $X_1, X_2$. Then there exists $Bel^{X|X_1 \times X_2|X_1}$ being a conditional belief of that projected belief conditioned on the variable $X_1$ such that this $Bel^{X|X_1 \times X_3}$ is the empty extension of $Bel^{X|X_1 \times X_2|X_1}$

$$Bel^{X|X_1 \times X_3} = (Bel^{X|X_2|X_1})^{X}$$

Let us notice that under the conditions of the above theorem

$$Bel = Bel^{X|X_1 \times X_3} \oplus Bel^{X_1 \times X_3} = Bel^{X_1 \times X_2|X_1} \oplus Bel^{X_1 \times X_3}$$

and hence for any $Bel^{X_1 \times X_3|X_1}$

$$Bel = Bel^{X_1 \times X_2|X_1} \oplus Bel^{X_1 \times X_3} \oplus Bel^{X_1 \times X_3|X_1}$$

and therefore

$$Bel = Bel^{X_1 \times X_2} \oplus Bel^{X_1 \times X_3|X_1}$$

This means that whenever the conditional $Bel^{X_1 \times X_2 \times X_3|X_1 \times X_3}$ is compressibly independent of $X_3$, then there exists a conditional $Bel^{X_1 \times X_2 \times X_3|X_1 \times X_2}$ compressibly independent of $X_2$. But this fact combined with the previous theorem results in:

**THEOREM 10** Let $X = X_1 \times X_2 \times X_3$ and let Bel be defined over $X$. Furthermore let $Bel^{X|X_1 \times X_3}$ be a conditional Belief conditioned on variables $X_1, X_3$. Let this conditional distribution be compressibly independent of $X_3$. Then the empty extension onto $X$ of any $Bel^{X_1 \times X_2|X_1}$ being a conditional belief of projected belief conditioned on the variable $X_1$ is a conditional belief function of $X$ conditioned on variables $X_1, X_3$. Hence for every $A \subseteq \Xi$

$$\frac{Q(A)}{Q^{X_1 \times X_3}(A^{X_1 \times X_3})} = \frac{Q^{X_1 \times X_2}(A^{X_1 \times X_2})}{Q^{X_1}(A^{X_1})}$$

In this way we obtained some sense of conditionality suitable for decomposition of a joint belief distribution.
Above we defined precisely what is meant by marginal independence of two variables in terms of the relationship between marginals and the joint distribution, as well as concerning the independence of a joint distribution from a single variable.

For the former case we can establish frequency tables with rows and columns corresponding to cardinalities of focal points of the first and the second marginal, and inner elements being cardinalities from the respective sum on DS-masses of the joint distribution. Clearly, cases falling into different inner categories of the table are different and hence $\chi^2$ test is applicable. The match can be $\chi^2$-tested. The following formula should be followed for calculation

$$\sum_{A; A \subseteq \Xi, m^{X_1}(A) > 0} \sum_{B; B \subseteq \Xi, m^{X_2}(B) > 0} \frac{((\sum_{C; C \subseteq \Xi, A = C; x_1, B = C; x_2} m(C)) - m^{X_1}(A) \cdot m^{X_2}(B))^2}{m^{X_1}(A) \cdot m^{X_2}(B)}$$

The number of df is calculated as

$$\left(\text{card} \{ A; A \subseteq \Xi, m^{X_1}(A) > 0 \} \right) - 1 \cdot \left(\text{card} \{ B; B \subseteq \Xi, m^{X_2}(B) > 0 \} \right) - 1$$

In case of independence of a distribution from one variable one needs to calculate the marginal of the distribution of that variable, say $X_i$. Then the measure of discrepancy from the assumption of independence is given as:

$$1 - m^{X_i}(\Xi_i)$$

Statistically we can test, based on Bernoullie distribution, what is the lowest possible and the highest possible value of $1 - m^{X_i}(\Xi_i)$ for a given significance level of the true underlying distribution.

In case of independence between the conditional distribution and one of conditioning variables, however, it is useless to calculate the pseudoinversion of $\oplus$, as we are working then with a population and a sample the size of which is not properly defined (by the "anti-labeling"). But we can build the contingency table of the unconditional joint distribution for the independent variable on the one hand and the remaining variables on the other hand, and compare the respective cells on how do they match the distribution we would obtain assuming the independence. So let $m$ be a Mass Function for the variable $X = X_1 \times X_2 \times X_3$. Composite measurement of $X_1, X_2, X_3$ is to be assumed. We want to show that $X_1$ conditioned on $X_2$ is independent of $X_3$. We calculate $Q$ of $m$, and $Q^{X_2}, Q^{X_1 \times X_2}, Q^{X_2 \times X_3}$. We define $Q_t$ to be a Q-function calculated as follows:

$$Q_t = c \cdot Q^{X_2 \times X_3}, \frac{Q^{X_1 \times X_2}}{Q^{X_2}}$$

$c$ - a normalizing constant. Let $m_t$ be the (Pseudo-)Mass Function corresponding to $Q_t$. Should $m_t(A)$ have a negative mass for any set $A \subseteq \Xi$, so the hypothesis of independence should be viewed as statistically
rejected. Also if \( m_t(A) = 0 \) and \( m(A) > 0 \) holds for any \( A \subseteq \Xi \), then hypothesis of independence should be viewed as statistically rejected. Otherwise we calculate the following \( \chi \)-statistics:

\[
\sum_{A: A \subseteq \Xi, m_t(A) \neq 0} \frac{(m(A) - m_t(A))^2}{|m_t(A)|}
\]

The number of degrees of freedom for the \( \chi^2 \) test would then be the product: (the number of focal points of the projection of the joint distribution onto \( X_1 \times X_2 \) minus one) \( * \) (the number of focal points of the projection of the joint distribution onto \( X_2 \times X_3 \) minus one).

6 DS Belief Network - Definition and Properties

**Definition 22** A DS Belief network is a pair \( (D, \text{Bel}) \) where \( D \) is a dag (directed acyclic graph) and \( \text{Bel} \) is a DS belief distribution called the underlying distribution. Each node \( i \) in \( D \) corresponds to a variable \( X_i \) in \( \text{Bel} \), a set of nodes \( I \) corresponds to a set of variables \( X_I \) and \( x_i, x_I \) denote values drawn from the domain of \( X_i \) and from the (cross product) domain of \( X_I \) respectively. Each node in the network is regarded as a storage cell for any distribution \( \text{Bel}^{\{X_i\} \cup X_{\pi(i)} | X_{\pi(i)}} \) where \( X_{\pi(i)} \) is a set of nodes corresponding to the parent nodes \( \pi(i) \) of \( i \). The underlying distribution represented by a DS belief network is computed via:

\[
\text{Bel} = \bigoplus_{i=1}^{n} \text{Bel}^{\{X_i\} \cup X_{\pi(i)} | X_{\pi(i)}}
\]

Please pay attention to the expression *any distribution* in front of the conditional distribution as more than one conditional distribution is possible. We may well imagine a situation where the decomposition of a joint belief distribution may be valid for some and not for the other set of conditional beliefs. Some important properties follow from this definition:

**THEOREM 11** Let \( \text{DSN}=(D, \text{Bel}) \) be a belief network with \( \text{Bel} \) equal to

\[
\text{Bel} = \bigoplus_{i=1}^{n} \text{Bel}^{\{X_i\} \cup X_{\pi(i)} | X_{\pi(i)}}
\]

Let \( j \) be a node in \( D \) without any outcoming edges (a terminal node). Then the following holds:

\[
\text{Bel}^{\{X_1, \ldots, X_n\} \setminus \{X_j\}} = \bigoplus_{i=1}^{n} \text{Bel}^{\{X_i\} \cup X_{\pi(i)} | X_{\pi(i)}}
\]

**PROOF:** Let \( \text{Bel}_2 \) be a pseudo-belief function defined over the set of variables \( g \cup h \), and \( \text{Bel}_1 \) be a pseudo-belief function defined over \( g \cup \{X\} \), \( X \not\in g \), \( X \not\in h \), \( g \cap h = \emptyset \). Let \( \text{Bel}_{12} = \text{Bel}_1 \oplus \text{Bel}_2 \) be a belief function (defined over \( g \cup \{X\} \cup h \). Let us make the projection \( \text{Bel}_{12}^{g \cup \{X\} \cup h} = (\text{Bel}_1 \oplus \text{Bel}_2)^{g \cup \{X\} \cup h} \)
If we investigate the m-values we will find out that: \( Bel_{12}^{i_{g}j_{h}} = Bel_{1}^{i_{g}} \oplus Bel_{2}^{i_{g}j_{h}} \)

On the other hand we know that a complete set of Q-values in the DS theory determines completely the Belief Function.

Let us assume \( Bel_{1} \) be a conditional belief \( Bel_{12}^{i_{g}j_{(X)}} \), and let us assume that the equation holds for any such conditional belief. Then \( Bel_{12}^{i_{g}} = Bel_{1}^{i_{g}} \oplus Bel_{2}^{i_{g}j_{h}} \)

It is easily checked that for non-zero Q-points of \( Bel_{1}^{i_{g}} \) (in this case focal points and their subsets) \( Q_{1}^{i_{g}} \) must be equal 1. This implies, however, that \( (Q_{1}^{i_{g}})^{i_{g}j_{h}} \) must be equal 1 also for non-zero Q-points of As any conditional belief is allowed then also ones with non-zero Q-points at zero-Q-points of \( Bel_{12}^{i_{g}j_{h}} \).

But then \( Bel_{2}^{i_{g}j_{h}} = Bel_{2} \) is completely determined to be identical with \( Bel_{12}^{i_{g}j_{h}} \) both at zero and non-zero Q-points.

What is more, \( Bel_{2} \) is a belief function (and not only a pseudo-belief function).

This is an important theorem as it states that a properly chosen subnetwork reflects a sub Belief function.

**Definition 23** [4] A trail in a dag is a sequence of links that form a path in the underlying undirected graph. A node \( \beta \) is called a head-to-head node with respect to a trail \( t \) if there are two consecutive links \( \alpha \rightarrow \beta \) and \( \beta \leftarrow \gamma \) on that \( t \).

**Definition 24** [4] A trail \( t \) connecting nodes \( \alpha \) and \( \beta \) is said to be active given a set of nodes \( L \), if (1) every head-to-head-node wrt \( t \) either is or has a descendent in \( L \) and (2) every other node on \( t \) is outside \( L \). Otherwise \( t \) is said to be blocked (given \( L \)).

**Definition 25** [4] If \( J, K \) and \( L \) are three disjoint sets of nodes in a dag \( D \), then \( L \) is said to d-separate \( J \) from \( K \), denoted \( I(J,K|L)_{D} \) iff no trail between a node in \( J \) and a node in \( K \) is active given \( L \).

It has been shown in [5] that

**THEOREM 12** Let \( L \) be a set of nodes in a dag \( D \), and let \( \alpha, \beta \notin L \) be two additional nodes in \( D \). Then \( \alpha \) and \( \beta \) are connected via an active trail (given \( L \)) iff \( \alpha \) and \( \beta \) are connected via a simple (i.e. not possessing cycles in the underlying undirected graph) active trail (given \( L \)).

We claim that:

**THEOREM 13** Let \( n \) be a node in a dag \( D \). Let \( D' \) be a subgraph of \( D \) such that all (and only) outcoming edges of \( n \) are removed. Let \( L \) be a set of nodes in the dag \( D \) (and hence \( D' \)) containing \( n \) (\( n \in L \)), and let
Figure 1: Both dags a) and b) represent the same independence information if $X_6$ is in the d-separating set. In b) directed edges $(X_6, X_3)$ and $(X_6, X_8)$ have been removed.

$\alpha, \beta \not\in L$ be two additional nodes in $D$. Then $\alpha$ and $\beta$ are connected in $D$ via an active trail given $L$ iff $\alpha$ and $\beta$ are connected via an active trail given $L$ in $D'$.

**Proof:** As $n$ is in $L$, the removal of outgoing edges does not influence any active trail as $n$ would block any trail containing them.

**Definition 26** If $X_J, X_K, X_L$ are three disjoint sets of variables of a distribution $Bel$, then $X_J, X_K$ are said to be conditionally independent given $X_L$ (denoted $I(X_J, X_K | X_L)_Bel$) iff

$$Bel \downarrow X_J \cup X_K \cup X_L | X_L \oplus Bel \downarrow X_J \cup X_K \cup X_L | X_L \oplus Bel \downarrow X_K \cup X_L | X_L \oplus Bel \downarrow X_L$$

$I(X_J, X_K | X_L)_Bel$ is called a (conditional independence) statement.

**Theorem 14** Let $Bel_D = \{Bel | (D, Bel)$ be a DS belief network$\}$. Then:

$I(J, K|L)_D$ iff $I(X_J, X_K | X_L)_Bel$ for all $Bel \in Bel_D$.

**Proof:** The "only if" part (soundness) states that whenever $I(J, K|L)_D$ holds in $D$, it must represent an independence that holds in every underlying distribution. We prove it as follows: Let us take a node l
in L having no predecessor in L. Let us try to calculate the conditional distribution on l. If n were a root node (without incoming edges) then simply

$$\text{Bel}^{[X]} = \bigoplus_{i=1,...,n,i \neq l} \text{Bel}^{[X_1 \cup X_{\pi(i)} | X_{\pi(i)}]}$$

Otherwise we have to transform the node l into such one. First let us exploit the previous theorem and remove all nodes not being predecessors of l and not l itself from the graph. The remaining dag represents the joint distribution projected onto the remaining nodes. Let us take the youngest predecessor of l (that is a node l’ not having a successor which were predecessor of l). Let us consider the two factors: $\text{Bel}^{[X,l} \cup X_{\pi(l)} | X_{\pi(l)}]} \oplus \text{Bel}^{[X,l'} \cup X_{\pi(l')} | X_{\pi(l')}].$ It is easy to check that the above DS combination is equal to:

$$\text{Bel}^{[X_l,X_l'] \cup X_{\pi(l)} | X_{\pi(l)}]} \cup X_{\pi(l')}. \sum \text{Bel}^{[X_l]} \cup X_{\pi(l)} | X_{\pi(l)}] - \{l'\} \cup X_{\pi(l')}.$$ This means that the node l’ can be made now a terminal node and the process of node removal may be continued until l becomes a root node (that is a node without predecessors).

Let us consider now the respective graph transformations. The outgoing edges of the node l can be removed as shown previously without deactivation of active trails and without introducing new ones (see e.g. Fig.1). The change of direction of the edge (l’ l) with introduction of new edges does not affect active trails either (some of them are only shortened, a head = to-head-meeting is by-passed - see e.g. Fig.2). Hence we can move all the nodes of L to become either root nodes or to have only nodes from L as predecessors. If we remove now these nodes from the transformed graph, then the remaining graph will represent the conditional distribution on these nodes. And all the active trails will not contain any head-to-head meeting. Hence two nodes α and β not connected by an active trail will neither possess a common predecessor nor be a successor of one another. Let us remove stepwise terminal nodes not being α,β. A graph consisting of two disjoint graphs with α,β as solely terminal nodes of each. Then obviously their calculations of marginals may be separated in the remaining dag. Hence missing active trail implies independence statement.

The ”if” part (completeness) asserts that any independence that is not detected by d-separation cannot be shared by all distributions in $P_D$ and hence cannot be revealed by non-numeric methods. we prove it by construction of an example as follows: If there exists an active p-trail connecting nodes i,j then by graph transformations (edge removal and edge reversal, thereafter terminal node removal) as described above we obtain a final graph for which either i and j have a common predecessor k or there exists an oriented path connecting both i,j. Let $X_i, X_j, X_k$ be variables associated with nodes i,j,k.

In the first case (common predecessor) let $Y_1,...,Y_m$ be variables associated with nodes on the directed path from k to i, let $Z_1,...,Z_n$ be variables associated with nodes on the directed path from k to j.
Figure 2: Both dags a) and b) represent the same independence information if $X_6$ is in the d-separating set. In b) the direction of edge $(X_2, X_6)$ has been reversed and edges $(X_1, X_6)$, $(X_5, X_2)$ and $(X_7, X_2)$ have been added.

Furthermore, let the conditional beliefs associated with the nodes be constructed as follows: the only focal points are (c - normalizing constants)

Node k:

$$m_{\downarrow \{X_k\} \cup X_{\pi(k)} | X_{\pi(k)}}(X_k = \{v_1\}, ...) = p/c_X$$

$$m_{\downarrow \{X_k\} \cup X_{\pi(k)} | X_{\pi(k)}}(X_k = \{v_2\}, ...) = (1 - p)/c_X$$

Nodes on the path from k to i (r=1,...,m+1, $Y_0$ means $X_{k_i}$, $Y_{m+1}$ means $X_i$):

$$m_{\downarrow \{Y_r\} \cup Y_{\pi(r)} | Y_{\pi(r)}}(Y_r = \{v_1\}, Y_{r-1} = \{v_1\}, ...) = 1/c_{Y_r}$$

$$m_{\downarrow \{Y_r\} \cup Y_{\pi(r)} | Y_{\pi(r)}}(Y_r = \{v_2\}, Y_{r-1} = \{v_2\}, ...) = 1/c_{Y_r}$$

Nodes on the path from k to j (r=1,...,n+1, $Z_0$ means $X_{k_j}$, $Z_{n+1}$ means $X_j$):

$$m_{\downarrow \{Z_r\} \cup Z_{\pi(r)} | Z_{\pi(r)}}(Z_r = \{v_1\}, Z_{r-1} = \{v_1\}, ...) = 1/c_{Z_r}$$

$$m_{\downarrow \{Z_r\} \cup Z_{\pi(r)} | Z_{\pi(r)}}(Z_r = \{v_2\}, Z_{r-1} = \{v_2\}, ...) = 1/c_{Z_r}$$
It is immediately visible that the joint belief distribution of $X_i$ and its predecessors in the remaining graph can be expressed as the only focal points:

$$m_{\downarrow X_i \text{anditspredecessors}}(X_i = \{v_1\}, X_k = \{v_1\}, ... ) = p \cdot m_{\downarrow \text{predecessorsof} X_i}(X_k = \{v_1\}, ... ) / c$$

$$m_{\downarrow X_i \text{anditspredecessors}}(X_i = \{v_2\}, X_k = \{v_2\}, ... ) = (1 - p) \cdot m_{\downarrow \text{predecessorsof} X_i}(X_k = \{v_2\}, ... )$$

Hence obviously:

$$m_{\downarrow X_i}(X_i = \{v_1\}) = p$$

$$m_{\downarrow X_i}(X_i = \{v_2\}) = (1 - p)$$

In the same way we show that:

$$m_{\downarrow X_j}(X_j = \{v_1\}) = p$$

$$m_{\downarrow X_j}(X_j = \{v_2\}) = (1 - p)$$

and that

$$m_{\downarrow \{X_i, X_j\}}(X_i = \{v_1\}, X_j = \{v_1\}) = p$$

$$m_{\downarrow \{X_i, X_j\}}(X_i = \{v_2\}, X_j = \{v_2\}) = (1 - p)$$

But we see immediately, that if

$$Bel_{\text{prod}} = Bel_{\downarrow X_i} \oplus Bel_{\downarrow X_j}$$

then focal points of $Bel_{\text{prod}}$ are

$$m_{\text{prod}}(X_i = \{v_1\}, X_j = \{v_1\}) = p^2$$

$$m_{\text{prod}}(X_i = \{v_1\}, X_j = \{v_2\}) = p \cdot (1 - p)$$

$$m_{\text{prod}}(X_i = \{v_2\}, X_j = \{v_1\}) = p \cdot (1 - p)$$

$$m_{\text{prod}}(X_i = \{v_2\}, X_j = \{v_2\}) = (1 - p)^2$$

which is obviously different from $Bel_{\downarrow \{X_i, X_j\}}$. This means, however, that for any dependence in the sense of d-separation we are actually capable to construct a joint belief distribution from the family of compatible distributions such that there is a dependence in the distribution corresponding to the
d-separation dependence.

In the same manner we can proceed in case of a direct oriented path from i to j or from j to i. Again we will manage to construct a belief function where missing d-separation at a given point indicates dependence in the distribution. Q.e.d.

7 Principles for Construction of dag from Data

Many writers have connected causality with statistical dependence. We parallel here [13] in formulating the following principles, while understanding independence as given by definitions Def.17 and 19

**Definition 27** Let $\mathbf{V}$ be a set of random variables with a joint DS-belief distribution. We say that variables $X, Y \in \mathbf{V}$ are directly causally dependent if and only if there is a causal dependency between $X, Y$ (either the value of $X$ influences the value of $Y$ or the value of $Y$ influences the value of $X$ or the value of a third variable not in $\mathbf{V}$ influences the values of both $X$ and $Y$) that does not involve any other variable in $\mathbf{V}$.

**Principle I:** For all $X, Y$ in $\mathbf{V}$, $X$ and $Y$ are directly causally dependent if and only if for every subset $S$ of $\mathbf{V}$ not containing $X$ or $Y$, $X$ and $Y$ are not statistically independent conditional on $S$.

**Definition 28** We say that $B$ is directly causally dependent on $A$ provided that $A$ and $B$ are causally dependent and the direction of causal influence is from $A$ to $B$.

**Principle II:** if $A$ and $B$ are directly causally dependent and $B$ and $C$ are directly causally dependent, but $A$ and $C$ are not, then: $B$ is causally dependent on $A$, and $B$ is causally dependent on $C$ if and only if $A$ and $C$ are statistically dependent conditional on any set of variables containing $B$ and not containing $A$ or $C$.

**Principle III:** A directed acyclic graph represents a DS-belief distribution on the variables that are vertices of the graph if and only if for all vertices $X, Y$ and all sets $S$ of vertices in the graph ($X, Y \notin S$), $S$ d-separates $X$ and $Y$ if and only if $X$ and $Y$ are independent conditional on $S$.

Please notice that Principle III bears close resemblance with theorem 14. It actually transfers a fine property of family of belief networks into a criterion for building a belief network. A weak point of such a criterion is that a decision whether or not an edge is to be included into the underlying dag is based on the
whole (at the moment of edge inclusion unknown) structure of the target dag. Fortunately, as we will show below, Principle III implies both Principles I and II. This means that Principles I and II, being local with respect to the target dag, may provide useful initial hints for construction of the target dag. What is more, Principles I and II combined with dag definition imply Principle III which means that we can indeed construct the whole dag structure exploiting only local properties of the target dag.

Let us introduce some notions. First let us define a partially oriented graph (pog) as a structure \((V, E, O)\), with \(V\) being the set of nodes, \(E\) being the set of edges with an edge being a subset of \(V\) with cardinality 2, \(O: E \rightarrow 2^{V \times V}\) being the orientation function of edges assigning each edge \(\{X_i, X_j\}\) in \(E\) either the orientation \(\{\}\) (no orientation) or \(\{(X_i, X_j)\}\) (from \(X_i\) to \(X_j\)), or \(\{(X_j, X_i)\}\) (from \(X_j\) to \(X_i\)) or \(\{(X_i, X_j), (X_j, X_i)\}\) (both from \(X_i\) to \(X_j\) and from \(X_j\) to \(X_i\)). The last orientation is an unpleasant one, but may occur in processes described below. If the first (empty) orientation is assigned, the edge is called unoriented, otherwise it is called oriented.

Furthermore let us call two edges neighbouring edges iff they share a vertex. Let \(\{X_i, X_j\}\) and \(\{X_k, X_j\}\) be neighbouring edges (they share \(X_j\) so they are neighbouring at \(X_j\)). We call them bridged edges iff there exists an edge \(\{X_i, X_k\}\) in \(E\). Otherwise they are called unbridged. The edge \(\{X_i, X_j\}\) (with respect to the neighbouring pair of edges) is said to be head-to-neighbour oriented iff \((X_i, X_j) \in O(\{X_i, X_j\})\). The edge \(\{X_i, X_j\}\) (with respect to the neighbouring pair of edges) is said to be tail-to-neighbour oriented iff \((X_j, X_i) \in O(\{X_i, X_j\})\).

We claim the following:

**THEOREM 15** Let \(\text{Bel}\) be a DS-belief distribution represented by an acyclic directed graph \(G\) according to Principle III. Then \(G\) is an orientation (\(G\) has the undirected structure) of the undirected graph \(U\) that represents \(\text{Bel}\) according to Principle I.

**PROOF:** If two nodes/variables \(X_i\) and \(X_j\) are connected via an undirected edge within the U-graph generated by Principle I, then there exists no set of variables \(Y_1, ..., Y_k\) such that for every combination of values \(\text{Bel}^\downarrow\{X_i, X_j, Y_1, ..., Y_k\}\) \(\oplus\) \(\text{Bel}^\downarrow\{Y_1, ..., Y_k\} = \text{Bel}^\downarrow\{X_i, Y_1, ..., Y_k\}\) \(\oplus\) \(\text{Bel}^\downarrow\{X_j, Y_1, ..., Y_k\}\) \(\oplus\) \(\text{Bel}^\downarrow\{Y_1, ..., Y_k\}\) as otherwise the edge would not be inserted. Assume for a moment Principle III would not generate a directed edge connecting both variables in a directed graph \(D\). Then in this \(D\)-graph a d-separation of both variables can be found: take simply the set of nodes which directly precede any of the variables. But this would enforce conditional independence in contradiction with the result established previously. So any edge generated by Principle I is also present in every graph generated by Principle III.
Figure 3: An Example of a Directed Acyclic Graph (dag)

Figure 4: An undirected graph obtained by application of Principle I
On the other hand if Principle I establishes that there is no undirected edge connecting both variables then there exists a set of variables on which these two are conditionally independent. But then Principle III cannot establish an edge between them as there would exist no d-separation between them. So whenever Principle I establishes no edge between variables, no edge will be established by Principle III. Q.e.d.

**THEOREM 16** Principle III implies Principle II.

**PROOF:** Let us consider the graph U generated by Principle I. Let us consider partial orientations of the graph U generated from it by Principle II. It is easily seen that there may be only one such orientation.

Let us turn our attention to Theorem 16. Let us consider a head-to-head meeting of directed edges \((X_i, X_l), (X_j, X_l)\) generated by Principle II, that is \(X_i, X_j\) not being directly connected in U, \(X_i, X_l\) being directly connected in U, \(X_j, X_l\) being directly connected in U, no set containing \(X_l\) rendering \(X_i, X_j\) independent. Then Principle III has also to generate this head-to-head meeting as the existence of the trail of directed edges \((X_i, X_l), (X_j, X_l)\) guarantees in this case that no d-separation containing \(X_l\) exists. So every head-to-head-meeting generated by Principle II occurs also in every graph generated by Principle III. On the other hand, if during testing independence by means of Principle II for the edges \((X_i, X_l), (X_j, X_l)\) a set containing \(X_l\) was detected such that it renders \(X_i, X_j\) independent, then head-to-head meeting of these edges must not occur if Principle III is applied. Q.e.d.

In this way we have established that: if there exists a dag of the distribution generated by Principle III, then application of Principles I and II will deliver its undirected structure and orientation of all those unbridged pairs of arcs which meet head-to-head at a node. (So if the intrinsic graph is given by Fig.3 then Principle II yields a graph given by Fig.5).

Let us now discuss which orientations of other arcs are established rigidly by Principle III. Pearl’s definition of d-separation refers to arc orientation at following nodes: (1) head-to-head nodes (2) direct and indirect descendants of head-to-head nodes

So let us establish the following principle:

**Principle IIc** Let \(G\) be a partially oriented graph generated by Principles I and II. Whenever \(\{X_i, X_j\}\) and \(\{X_k, X_j\}\) are neighbouring unbridged edges, with \(\{X_i, X_j\}\) being head-to-neighbour oriented and \(\{X_k, X_j\}\) being unoriented, orient \(\{X_k, X_j\}\) tail-to-neighbour.

Please notice that Principle IIc is a kind of operationalization of Principle II, as it is a direct consequence of the ”if and only if” expression in Principle II. It has been introduced because the formulation of Principle
II directs our attention to orienting edges head-to-head, but it is less obvious that it also implies some head-to-tail orientations.

Obviously, the following theorem holds:

**THEOREM 17** Principle III implies Principle II

The Theorem is obvious if we consider the previous ones. (So if the intrinsic graph is given by Fig.3 then Principle II yields a graph given by Fig.6).

Furthermore let us introduce the following principle:

**Principle IV:** Let $H$ be a partially oriented graph generated by Principles I and II and $II^c$. Let the subgraph $H'$ of $H$ contain only oriented edges in $H$. Let $\{X_i, X_j\}$ be an unoriented edge in $H$. If $X_j$ is a descendent of $X_i$ in $H'$, then orient this edge from $X_i$ to $X_j$. Apply thereafter Principle II exhaustively.

Obviously

**THEOREM 18** Dag-structure and Principle III imply Principle IV.

(So if the intrinsic graph is given by Fig.3 then Principle IV yields a graph given by Fig.7).
Figure 6: A partially oriented graph due to Principle IIc (arrow \((X_6, X_8)\))

Figure 7: A partially oriented graph due to Principle IV (arrows \((X_7, X_8), (X_5, X_8)\))
Figure 8: A partially oriented graph due to Principle V (arrow $(X_1, X_9)$). Nodes $X_1$, $X_3$, $X_8$, $X_9$ are legitimately removable.

both edges $\{X_k, X_l\}$, $\{X_j, X_l\}$, or all the edges $\{X_i, X_l\}$, $\{X_k, X_l\}$, $\{X_j, X_l\}$ be left unoriented in the process. Then orient $\{X_j, X_l\}$ as from $X_l$ to $X_j$. Apply thereafter Principles II\textsuperscript{c}, IV exhaustively.

(If the intrinsic graph is given by Fig.3 then Principle V yields a graph given by Fig.8).

**THEOREM 19** *Dag-structure and Principle III imply Principle V.*

**PROOF:** The edges $\{X_i, X_l\}$, $\{X_k, X_l\}$ (see Fig.9) are unbridged (because $\{X_i, X_j\}$, $\{X_k, X_j\}$ are unbridged), hence their orientation head-to-head is excluded (as Principle II didn’t orient them). Hence

Figure 9: Visualisation to the Proof of the Theorem on Principle V
either we have orientation \((X_i, X_i)\) or \((X_i, X_k)\)

Let us assume the orientation \((X_i, X_i)\) of \(\{X_i, X_j\}\). Then if \(\{X_i, X_j\}\) would be oriented \((X_j, X_i)\) then \(X_j, X_i, X_i\) would form an oriented cycle, hence \(H\) would not be a dag. So this is impossible.

Let us assume the orientation \((X_i, X_k)\) of \(\{X_k, X_i\}\). Then if \(\{X_i, X_j\}\) would be oriented \((X_j, X_i)\) then \(X_j, X_i, X_k\) would form an oriented cycle, hence \(H\) would not be a dag. So this is impossible.

Hence \(\{X_i, X_j\}\) must be oriented \((X_i, X_j)\)

Q.e.d.\(\Box\)

We conjecture furthermore that

Let \(\Gamma\) be the set of directed graphs that represent DS-belief distribution \(\text{Bel}\) according to Principle III. Then \(\Gamma\) is also the set of directed graphs obtained from \(P\) by Principles I and II.

This conjecture will be proven after showing some intermediate results.

We shall introduce first the notion of p-d-separation.

**Definition 29** A p-trail in a pog is a sequence of links that form a path in the underlying undirected graph. A node \(\beta\) is called a head-to-head node with respect to a p-trail \(t\) if there are two consecutive links \(\alpha \rightarrow \beta\) and \(\beta \leftarrow \gamma\) on that \(t\). A p-trail is minimal iff no two of its succeeding links on the p-trail are bridged in the graph.

**Definition 30** A p-descendent of a node \(n\) in a pog is any node \(m\) such that there exists a minimal p-trail from \(n\) to \(m\) such that every oriented link on the p-trail is oriented from \(n\) to \(m\) and an oriented edge \((m, n)\) does not exist in the graph.

**Definition 31** A p-trail \(t\) connecting nodes \(\alpha\) and \(\beta\) is said to be active given a set of nodes \(L\), if (1) every head-to-head-node wrt \(t\) either is or has a p-descendent in \(L\) and (2) every other node on \(t\) is outside \(L\). Otherwise \(t\) is said to be blocked (given \(L\)).

**Definition 32** If \(J, K\) and \(L\) are three disjoint sets of nodes in a pog \(H\), then \(L\) is said to p-d-separate \(J\) from \(K\), denoted \(I(J, K | L)_H\) iff no minimal p-trail between a node in \(J\) and a node in \(K\) is active given \(L\).

We claim that

**THEOREM 20** Let \(L\) be a set of nodes in a pog \(H\), and let \(\alpha, \beta \notin L\) be two additional nodes in \(H\). Then \(\alpha\) and \(\beta\) are connected via an active p-trail (given \(L\)) iff \(\alpha\) and \(\beta\) are connected via a simple (i.e. not possessing cycles in the underlying undirected graph) active p-trail (given \(L\)).

Now let us formulate the central theorem of this paper.
**THEOREM 21** Let $D$ be a dag generated by Principle III. Let $H$ be a pog generated by Principles I, II, $IF$, IV and V. Then $I(J,K|L)_H$ iff $I(J,K|L)_D$

**PROOF:** To show this, let us consider an active minimal p-trail. We claim that there exists then an active trail.

If after final orientation no head-to-head meeting occurs on the p-trail then this is also the interesting active trail. Otherwise if there exists a head-to-head-meeting on the underlying trail then two cases are possible: (1) it existed on the original p-trail, (2) it did not exist on the original p-trail. The second case is impossible since then it must have been generated by Principle II (the meeting edges are unbridged). So we have had also a head-to-head-meeting on the original p-trail. So let us consider the p-descenders of the head-to-head-meeting. No head-to-head-meeting could have been generated on the path as the p-trail to the descendent was minimal. p-descendants of head-to-head meetings connected by unoriented links form a kind of equivalence class in that if the edges $(A,B)$, $(C,B)$ are there and $D$ is a p-descendent of $B$ on a totally unoriented path then oriented edges $(A,D)$ and $(C,D)$ are also present. So p-descendants are either descendants (OK) or are such predecessors, that they form together with the nodes of the primary p-trail but the discussed head-to-head node a minimal p-trail containing that predecessor as a head-to-head node and which proves to be an active trail in the dag (see Fig.10).

Let us consider an active minimal trail. We claim that then there exists a minimal p-trail. First of
all the successors are also p-successors. Second, a minimal trail is also a minimal p-trail. Now the question is whether or not it is also active. As the trail is minimal, no head-to-head meeting will vanish on the p-trail. Hence also the successor requirement is met. So the proof is complete. Q.e.d. ✷

This theorem actually corresponds straight forwardly to our conjecture. The only difference to it is the extensive use of Principle II<sup>c</sup> which is actually a kind of exploitation of Principle II. Furthermore, it is to some extent constructive: it states how it is possible to uncover the d-separations applying only Principles I, II, II<sup>c</sup>, IV and V for construction of a pog, and without actually instantiating a single dag. It is immediately visible, that any dag compatible with the pog expresses exactly all the independences Principle III dag does and hence is a Principle III dag.

We can however be still more constructive and formulate the construction algorithm for generation of all the dags according to Principle III based only on the results of Principles I, II, II<sup>c</sup>, IV and V and the definition of a dag.

Let us define the legitimate removal of a node from the pog graph: a node can be removed legitimately from a pog iff all the oriented edges it meets are oriented towards it, and all pairs edges meeting at it for which at least one is unoriented, are bridged.

**Pog-to-dag algorithm:**

1. find a legitimately removable node in the pog, remove it with edges meeting it while marking the edges as oriented towards this node.
2. Proceed with Step 1 until all the nodes are removed.
3. Orient the edges of the original pog so as they were marked in step 1.

(Compare Fig.8, Fig.11). We claim that:

**THEOREM 22** Let there exist a dag obtainable from Principle III. Let G be a pog generated from Principles I, II, II<sup>c</sup>, IV and V. Then every dag obtained from the pog B by the above algorithm is a Principle III dag. Every Principle III dag for this population is a dag obtainable from G by means of the above algorithm.

**PROOF:** This is easily seen as on the one hand every dag has a legitimately removable node, and on the other hand the orientations generated by the above algorithm do not lead to any conflict with Principles I,II,II<sup>c</sup>, IV and V, if a dag exists.

Q.e.d. ✷

In this way we hope to have also shown the usefulness of Theorem 21 definitely, giving a constructive algorithm to generate the dag out of a pog which is necessary for belief network applications.
Figure 11: After legitimate removal of nodes $X_3$, $X_8$ and $X_9$. (The arrow $(X_3, X_4)$ was inforced). Nodes $X_1$, $X_4$, $X_6$ are legitimately removable.

This is actually the main result of this paper with respect to structuring joint DS-belief distributions. It may be stated as follows:

**THEOREM 23** Let $\Gamma$ be the set of directed graphs that represent DS-belief distribution $Bel$ according to Principle III. Then $\Gamma$ is also the set of directed graphs obtained from $P$ by Principles I and II.

**PROOF:** Let us look closely at Theorem 21. From Theorems 15 and 16 we know that any dag $D$ in $\Gamma$ must have been generated also by Principles I and II. As Principles II', IV and V follow from Principles I and II and from the property of being a dag (look at Theorems 17, 18, 19), then any dag in $\Gamma$ as generated by Principle III would also be generated by Principles I, II, II', IV and V. Let us take now any of these dags in $\Gamma$, say $D$. Let us assume that from the respective pog $H$ generated by Principles I, II, II', IV and V (that is in fact from the only such pog $H$) a different dag $D'$ may be derived beside $D$. From Theorem 21 we have: $I(J, K|L)_H$ iff $I(J, K|L)_D$, but also: $I(J, K|L)_H$ iff $I(J, K|L)'_D$. Hence also $I(J, K|L)_D$ iff $I(J, K|L)'_D$. But then $D'$ must also have been generated by Principle III as both $D$ and $D'$ carry the same independence information.

So we see immediately that any dag in $\Gamma$ must have been generated by Principles I and II and all the dags derived via Principles I and II must be in $\Gamma$. Q.e.d.
Figure 12: An effect of variable hiding (a) an original dag (b) after "hiding" $X_{23}$ - double orientation of an edge (connecting $X_{22}$ with $X_{24}$)

The only open question remains whether a DS-belief distribution can be represented using Principle III.

8 When Principle III Fails

In our theorem on equivalence of d-separation and conditional independence it is explicitly stated that d-separation is maximal only with respect to the whole population of distributions. For a single distribution, there may exist independences not covered by the d-separation. Hence for such a distribution, Principle III may fail.

The non-existence of a dag can in general be visible when:

1. the application of Principle II leads to double oriented edges.
2. the oriented subgraph generated by Principle II contains a cycle.
3. the application of Principles II*, IV and/or V leads to head-to-head-meetings which were forbidden under Principle II.

Then no dag exists reflecting all the independences for a given population. So if we need a dag, we have to resign from some of the independences.
Figure 13: An effect of variable hiding (a) an original dag (b) hiding \(X_{33}\) leads to an oriented cycle

The non-existence of a dag may be attributed in the first two cases to existence of hidden variables, as we can see from examples in Fig.12 (double orientation of an edge) and in Fig.13 (a directed cycle). We shall, however, not discuss this issue at length here. It can be, however, easily checked that introduction of these additional hidden variables as indicated in both figures will not give rise to emerging of new independences, not present in the population.

But the third case is hard to resolve. Unless there exist information outside the data permitting to assume that the unexpected head-to-head-meeting will not introduce unjustified independences in the dag, there may exist the necessity to make a complete subgraph out of that part of the graph which leads to the unwished head-to-head meeting.

9 Summary and Outlook

In this paper, a general framework for recovery of a dag structure of a joint DS-belief distribution from data has been established, paralleling the work of Spirtes et al [13] on probabilistic networks. The proven theorems imply that it is possible to infer causal structure from data if this structure has the form of a directed acyclic graph. Strictly speaking: The statistical inference allows for deducing a set of such candidate causal structures with indication which fragment of the causal structure is shared by all the candidates.

Specifically: In this paper the notion of DS-belief network was introduced along with a new notion of con-
ditional independence of variables. The applicability of Pearl’s notion of d-separation for such a belief network was demonstrated, especially its relationship to conditional independence in DS-belief networks. Principles I and II were introduced to uncover partial dependency structure of the joint belief distribution. An algorithm was given allowing for derivation of all the dags having identical dependence/independence information as the partially oriented graph derived from Principles I and II, provided at least one dag exists. If it does not exist, a partial procedure transposing the partial dependence structure into one with dag-representation via introduction of hidden variables is also suggested. The new notion of p-d-separation paralleling d-separation of Geiger, Verma and Pearl [4], being applicable to partially oriented graphs was introduced and has been shown to carry the same dependence/independence information as all the d-separations of all compatible dags.

Over the last years a number of alternative methods to the algorithm of Spirtes et al [13] (both general and specialized) for construction of probabilistic belief networks has been proposed (compare the method described in [2] and other discussed in last sections therein). However, they were hardly transferable into the domain of DS-belief functions. For some special case, [7] offers solutions. The method investigated here deserves special attention because it relates the oriented structure of a directed acyclic graph representation to the causal relationship in the described part of reality. Two essential complementary conclusions can be drawn from proving theorem 21: (i) if one recovers a dag structure for the DS-belief distribution one derives more than just a formal description and (ii) for proper construction of a dag causality is essential.

Further research on the subject is needed, especially concerning approximations binding combinatorial explosion with the number of variables considered.

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