ON THE CHIRAL PERTURBATION THEORY FOR TWO-FLAVOR TWO-COLOR QCD AT FINITE CHEMICAL POTENTIAL

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We construct the chiral perturbation theory for two-color QCD with two quark flavors as an effective theory on the $SO(6)/SO(5)$ coset space. This formulation turns out to be particularly useful for extracting the physical content of the theory when finite baryon and isospin chemical potentials are introduced, and Bose–Einstein condensation sets on.

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1. Introduction

The phase diagram of quantum chromodynamics (QCD) has attracted much attention in recent years. The region of high baryon density and low temperature is relevant for the description of deconfined quark matter, which can be found in the centers of neutron stars. It is expected to exhibit a variety of color-superconducting phases.

Unfortunately, there is only very little firm knowledge concerning the behavior of the cold and dense quark matter. At very high densities, asymptotic freedom of QCD allows one to use weak-coupling methods to determine the structure of the ground state. On the other hand, the phenomenologically interesting region of densities corresponds to the strong-coupling regime where ab initio calculations within QCD are not available.

At the same time, current techniques of lattice numerical computations are not able to reach sufficiently high densities, due to the complexity of the fermionic Dirac operator that occurs in the Euclidean path-integral measure. This gave rise to the interest in QCD-like theories that are amenable to lattice simulations, in particular the two-color QCD with fundamental quarks and three-color QCD with adjoint quarks.

It turns out that these theories may also be studied by the means of a low-energy effective field theory similar to the chiral perturbation theory of QCD, and non-
trivial information about their phase diagram thus obtained. Within this approach, the structure of the phase diagram has been investigated in detail, including the effects of finite temperature. The model-independent predictions of the low-energy effective field theory have been complemented by lattice computations and calculations within several models.

The aim of the present letter is to provide an alternative low-energy effective formulation of the simplest of this class of theories — the two-color QCD with two quark flavors. While the general description of the whole class is based on the extended $SU(2N_f)$ chiral symmetry of the underlying Lagrangian with $N_f$ quark flavors, we construct the effective Lagrangian by exploiting the Lie algebra isomorphism $SU(4) \cong SO(6)$. We show that such a picture displays more transparently the physical content of the theory and at the same time allows for an easy determination of the true ground state, which has been sought by a convenient ansatz previously.

The paper is organized as follows. In the next section we summarize the basic features of two-color QCD to set the stage for the following considerations. Next we work out the mapping between the coset space $SO(6)/SO(5)$ that we use, and the $SU(4)/Sp(4)$ used in the literature. The rest of the paper is devoted to the construction of the effective Lagrangian and its detailed analysis.

2. Two-color QCD

In this section we recall the basic properties of two-color QCD, following closely the treatment of Kogut et al. The distinguishing feature of two-color QCD is the pseudoreality of the gauge group generators, the Pauli matrices, $T_k^* = -T_2 T_k T_2$. Consider now a set of $N_f$ quark flavors in the fundamental representation of the gauge group. As an immediate consequence, we may trade the right-hand ed component of the quark field, $\psi_R$ (flavor and color indices are suppressed), for the left-handed conjugate spinor $\tilde{\psi}_R = \sigma_2 T_2 \psi_R^*$. (The Pauli matrices $\sigma_k$ act in the Dirac space.) Instead of the usual Dirac spinor, $\psi = (\psi_L \psi_R)^T$, we now work with the left-handed spinor, $\Psi = (\psi_L \tilde{\psi}_R)^T$, in terms of which the quark Euclidean Lagrangian of the massive two-color QCD at finite chemical potential becomes

$$L = i\Psi^\dagger \sigma_\nu (D_\nu - \Omega_\nu) \Psi - m \left[ \frac{1}{2} \Psi^T \sigma_2 T_2 M \Psi + H.c. \right].$$

Here $D_\nu$ is the gauge-covariant derivative that includes the $SU(2)$ gluon field. $\Omega_\nu$ is the static uniform external $U(1)$ gauge field that incorporates the chemical potential. In the two-flavor case we shall deal with both the baryon number and the isospin chemical potential, $\mu_B$ and $\mu_I$, respectively, so that $\Omega_\nu$ will eventually be set to $\Omega_\nu = \delta_\nu^0 (\mu_B B + \mu_I I)$. Here,

$$B = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad I = \frac{1}{2} \begin{pmatrix} \tau_3 & 0 \\ 0 & -\tau_3 \end{pmatrix}.$$

The determinant of the Dirac operator is always real in two-color QCD, it is, however, positive only for an even number of flavors.
are the baryon number and isospin generators, respectively. (The Pauli matrices $\tau_k$ act in the flavor space.) Finally,

$$M = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

denotes the mass matrix in the basis of the spinor $\Psi$ and $\sigma_\nu$ stands for the four-vector of spin matrices, $\sigma_\nu = (-i, \sigma)$.

In the chiral limit and the absence of the chemical potential, the Lagrangian Eq. (1) is invariant under the extended global symmetry $SU(2N_f)$, which includes the naive chiral group $SU(N_f)_L \times SU(N_f)_R$ and additional symmetry transformations due to the pseudoreality of the gauge group generators. The global symmetry is spontaneously broken by the standard chiral condensate down to its $Sp(2N_f)$ subgroup.

The low-energy effective field theory for the Goldstone bosons of the broken symmetry is thus naturally constructed on the coset space $SU(2N_f)/Sp(2N_f)$. This is parametrized by an antisymmetric unimodular unitary matrix $\Sigma$, in terms of which the leading-order effective Lagrangian reads

$$\mathcal{L}_{\text{eff}} = \frac{F^2}{2} \text{Tr}(\nabla_\nu \Sigma \nabla_\nu \Sigma^\dagger) - G \text{Re Tr}(J\Sigma).$$

(3)

The $\nabla$'s denote the covariant derivatives,

$$\nabla_\nu \Sigma = \partial_\nu \Sigma - (\Omega_\nu \Sigma + \Sigma \Omega_\nu^T),$$
$$\nabla_\nu \Sigma^\dagger = \partial_\nu \Sigma^\dagger + (\Sigma^\dagger \Omega_\nu + \Omega_\nu^T \Sigma^\dagger),$$

while $J$ serves as a source field for $\Sigma$, and is eventually set to $mM$. The quark mass $m$ is connected to the Goldstone boson mass squared $m_\pi^2$ by the Gell-Mann–Oakes–Renner relation

$$mG = F^2 m_\pi^2.$$

It is worth emphasizing that the incorporation of the chemical potential into the effective theory involves no extra free parameters — the way the chemical potential enters the Lagrangian is fixed by the form of the covariant derivatives.

3. The $SO(6)/SO(5)$ coset space

From now on we shall restrict our attention to the case $N_f = 2$. In that case, note the Lie algebra isomorphisms $SU(4) \simeq SO(6)$ and $Sp(4) \simeq SO(5)$. This allows us to recast the low-energy effective field theory on the $SO(6)/SO(5)$ coset. There are altogether five degrees of freedom, or Goldstone bosons, corresponding to the five independent entries of the antisymmetric unimodular unitary matrix $\Sigma$. 

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3.1. Matrix basis

The mapping to the $SO(6)/SO(5)$ coset space is now provided by the formula

$$\Sigma = n_i \Sigma_i,$$  (4)

where $\vec{n}$ is a six-dimensional real unit vector and $\Sigma_i$ is a convenient set of independent antisymmetric $4 \times 4$ matrices. For $\Sigma$ to be unitary, the basis matrices must satisfy the constraint

$$\Sigma_i^\dagger \Sigma_j + \Sigma_j^\dagger \Sigma_i = 2\delta_{ij}. \quad (5)$$

Such a relation is fulfilled for instance by the matrices

$$\Sigma_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \Sigma_2 = \begin{pmatrix} \tau_2 & 0 \\ 0 & \tau_2 \end{pmatrix}, \quad \Sigma_3 = \begin{pmatrix} 0 & i\tau_1 \\ -i\tau_1 & 0 \end{pmatrix},$$  $$\Sigma_4 = \begin{pmatrix} i\tau_2 & 0 \\ 0 & -i\tau_2 \end{pmatrix}, \quad \Sigma_5 = \begin{pmatrix} 0 & i\tau_2 \\ i\tau_2 & 0 \end{pmatrix}, \quad \Sigma_6 = \begin{pmatrix} 0 & i\tau_3 \\ -i\tau_3 & 0 \end{pmatrix}. $$

This particular set has been chosen to comply with existing literature. In fact, Kogut et al. use the notation $\Sigma_c$ and $\Sigma_d$ for our $\Sigma_1$ and $\Sigma_2$, respectively, while Splittorff et al. denote our $\Sigma_1$, $\Sigma_2$ and $\Sigma_3$ by $\Sigma_M$, $\Sigma_B$ and $\Sigma_I$, respectively.

Let us in addition show a simple argument that suggests how to choose in general a set of matrices satisfying Eq. (5). Recall that six independent antisymmetric Hermitian $4 \times 4$ matrices generate the real Lie algebra $SO(4) \cong SO(3) \times SO(3)$. This means that we deal with two sets of three matrices, which can be shown to fulfill the usual anticommutator of Pauli matrices, $\{\tau_i, \tau_j\} = 2\delta_{ij}$. By multiplying the matrices from one of the sets by $i$, we arrive at three Hermitian matrices, $H_i = \{\Sigma_2, \Sigma_3, \Sigma_6\}$, and three anti-Hermitian ones, $A_i = \{\Sigma_1, \Sigma_4, \Sigma_5\}$. These satisfy the relations

$$\{H_i, H_j\} = 2\delta_{ij}, \quad \{A_i, A_j\} = -2\delta_{ij}, \quad [H_i, A_j] = 0,$$

that are equivalent to Eq. (5).

3.2. Structure of the coset

It remains to prove that Eq. (4) provides a one-to-one parametrization of the coset $SU(4)/Sp(4)$. To that end, note that any antisymmetric $4 \times 4$ matrix $U$ may be expanded in the basis $\Sigma_i$, $U = z_i \Sigma_i$, where $z_i$ are in general complex coefficients. The unitarity of $U$ constrains these coefficients as

$$1 = U^\dagger U = \sum_i |z_i|^2 + i \sum_{i \neq j} (x_i y_j - x_j y_i) \Sigma_i^\dagger \Sigma_j,$$

the $x_i$ and $y_i$ being the real and imaginary parts of $z_i$, respectively.

It is now crucial to observe that the products $i \Sigma_i^\dagger \Sigma_j$ for $i \neq j$ span the set of 15 linearly independent generators of $SU(4)$ so that the unitarity of $U$ requires separately $\sum_i |z_i|^2 = 1$ and $x_i y_j = x_j y_i$ for all pairs of $i, j$. 

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The latter condition means that the complex phases of all the $z_i$’s must be equal so that $z_i = n_i e^{i\varphi}$ with real $n_i$, while the former one requires $\sum_i n_i^2 = 1$. It is a matter of simple algebra to calculate the determinant of $U$,

$$\det U = e^{4i\varphi} \left( \sum_i n_i^2 \right)^2 = e^{4i\varphi}.$$ 

Since the elements of the coset $SU(4)/Sp(4)$ are unimodular matrices, we are left with two distinct possibilities, $\varphi = 0$ or $\varphi = \pi/2$. (The next solution, $\varphi = \pi$, already corresponds to $\varphi = 0$ with just the sign of all the $n_i$’s inverted.)

In conclusion, every antisymmetric unimodular unitary matrix $\Sigma$ may be cast in the form $\bar{n}$, where $\bar{n}$ is either real, or pure imaginary vector. However, in the standard coset construction of the effective Lagrangian, the global symmetry group is required to act transitively on the parameter space of the Goldstone fields that is, the actual coset space must be connected. As the chiral condensate, above which we build our effective theory, is described by the matrix $\Sigma_1$ (note that $M = -\Sigma_1$), we have to choose the connected component with real $\bar{n}$, as in Eq. (4).

**3.3. Physical content of the basis matrices**

It is instructive to look at the transformation properties of the matrix $\Sigma$. This will allow us to classify the Goldstone modes by their baryon and isospin quantum numbers.

Recall that $\Sigma$ is an antisymmetric tensor under $SU(4)$ that is, it transforms as $\Sigma \rightarrow U \Sigma U^T$ for $U \in SU(4)$. For an infinitesimal transformation generated by the baryon number or the third component of the isospin we get

$$\delta_\varepsilon \Sigma = i\varepsilon (Q \Sigma + \Sigma Q^T) = i\varepsilon \{Q, \Sigma\}, \quad Q = B, I.$$ 

Let $\Sigma$ be a general block matrix of the form $\begin{pmatrix} K & L \\ M & N \end{pmatrix}$. Then

$$\{B, \Sigma\} = \begin{pmatrix} K & 0 \\ 0 & -N \end{pmatrix}, \quad \{I, \Sigma\} = \begin{pmatrix} \frac{1}{2}\{\tau_3, K\} & \frac{1}{2}\{\tau_3, L\} \\ -\frac{1}{2}\{\tau_3, M\} & \frac{1}{2}\{\tau_3, N\} \end{pmatrix}.$$ 

The quantum numbers of the particular components of $\Sigma$ are summarized in Fig. 1.

To gain more insight into the nature of the effective field $\Sigma$, let us assign to it a composite field,

$$\Sigma \rightarrow \frac{1}{2} \Psi^T \sigma_2 T_2 \Sigma \Psi + \text{H.c.},$$
which corresponds to the form of the mass term in Eq. (1). Such a composite operator may be regarded as an interpolating field for the Goldstone boson.

With the explicit knowledge of the matrices $\Sigma_i$ it is now straightforward to find the particle content of the corresponding interpolating fields, cf. also Fig. 1.

\[
\begin{align*}
\Sigma_2 &\to -\frac{i}{2} \psi^T C \gamma_5 T_2 \tau_2 \psi + \text{H.c.}, \\
\Sigma_4 &\to -\frac{i}{2} \psi^T C \gamma_5 T_2 \tau_3 \psi + \text{H.c.}, \\
\Sigma_3 &\to -i \bar{\psi} \tau_1 \gamma_5 \psi, \\
\Sigma_5 &\to i \bar{\psi} \tau_2 \gamma_5 \psi, \\
\Sigma_6 &\to -i \bar{\psi} \tau_3 \gamma_5 \psi, \\
\Sigma_1 &\to \bar{\psi} \psi.
\end{align*}
\]

4. Chiral perturbation theory

We are now ready to write down the leading-order effective Lagrangian and use it to analyze the phase diagram of the theory. First, we have to minimize the static part of the Lagrangian in order to determine the ground state at nonzero chemical potential.

4.1. Global minimum of the static Lagrangian

From Eq. (3) we can immediately infer the static part,

\[
\mathcal{L}_{\text{stat}} = -\frac{F^2}{2} \text{Tr} \left( (\Omega_\nu \Sigma + \Sigma \Omega_\nu^T)(\Sigma^\dagger \Omega_\nu + \Omega_\nu^T \Sigma^\dagger) \right) - G \text{Re} \text{Tr}(J \Sigma). \tag{6}
\]

We include the external source $J$ in the general form

\[ J = j_i \Sigma_i^\dagger, \]

with real $j_i$. Note that setting $j_1 = m$, we reproduce the quark mass contribution to the effective Lagrangian.

The other sources can be taken as infinitesimally small, since they essentially serve to generate the ground-state condensates,

\[ \langle \Sigma_i \rangle = -\frac{\partial \mathcal{L}_{\text{stat}}}{\partial j_i}. \]

From the orthogonality property, $\text{Tr}(\Sigma_i^\dagger \Sigma_j) = 4 \delta_{ij}$, we find $\text{Re} \text{Tr}(J \Sigma) = 4 \vec{j} \cdot \vec{n}$ so that we have

\[ \langle \vec{\Sigma} \rangle = 4G \vec{n}. \]

It is obvious that the vacuum condensate rotates on a sphere in the six-dimensional space, with coordinates corresponding to the six basis matrices $\Sigma_i$. It remains to calculate the vector $\vec{n}$ minimizing the static Lagrangian (6).

Note first that in the absence of chemical potential, $\Omega_\nu = 0$, the static Lagrangian is minimal when the condensate is aligned with the external source $\vec{j}$.

When baryon and isospin chemical potentials are switched on, we shall for simplicity assume that only the sources $j_1, j_2, j_3$ are present. This is sufficient to include
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the quark mass effects and calculate both the diquark and the isospin (pion) condensate. Taking into account the explicit form of the charge matrices, Eq. (2), the static Lagrangian becomes

$$L_{\text{stat}} = -2F^2 \left[ \mu_B^2 (n_2^2 + n_4^2) + \mu_I^2 (n_3^2 + n_5^2) \right] - 4G(j_1 n_1 + j_2 n_2 + j_3 n_3).$$

The first term is invariant under $SO(2) \times SO(2)$ rotations in the planes $(2, 4)$ and $(3, 5)$. In the absence of the external sources, this symmetry may be exploited to set $n_4 = n_5 = 0$. The source $J$ breaks the symmetry and, in fact, prefers the solutions with $n_4 = n_5 = 0$. The problem of finding the ground state thus reduces to minimizing the expression,

$$L_{\text{stat}} = -2F^2 (\mu_B^2 n_2^2 + \mu_I^2 n_3^2) - 4G(j_1 n_1 + j_2 n_2 + j_3 n_3),$$

(7)
on the sphere $S^5$:

$$\vec{n}^2 = 1.$$

The Lagrangian now does not depend on $n_4, n_5, n_6$ so that we are actually looking for a minimum on the ball, $n_2^2 + n_3^2 + n_4^2 \leq 1$. It is clear that at the global minimum, both terms on the right hand side of Eq. (7) are negative (otherwise we could lower the energy by the inversion, $\vec{n} \to -\vec{n}$). By the same token, the global minimum must lie on the surface of the ball, since if this were not the case, we could lower the energy by scaling up the vector: $\vec{n} \to t\vec{n}, t > 1$.

We have thus shown that in the global minimum, $n_1^2 + n_2^2 + n_3^2 = 1$ and $n_4 = n_5 = n_6 = 0$, and the ground-state condensate is given by the linear combination $\Sigma = n_1 \Sigma_1 + n_2 \Sigma_2 + n_3 \Sigma_3$. We stress the simplicity of the proof of this fact within the $SO(6)/SO(5)$ coset formulation of the chiral perturbation theory. Indeed, using the standard $SU(4)/Sp(4)$ formalism, Splittorff et al.\cite{4} only assumed such a form of $\Sigma$, and also did not prove that the minimum thus found was global.

To demonstrate the power of the formalism we have built so far, we shall next rederive the results of Kogut et al.\cite{3} for the case of nonzero baryon chemical potential $\mu_B$. We shall thus set $\mu_I = 0$ and $j_i = \delta_{i1} m$. Isospin chemical potential can be introduced along the same lines and the results of Splittorff et al.\cite{4} would be easily recovered.

With the assumptions made, the static Lagrangian becomes

$$L_{\text{stat}} = -2F^2 m^2_n (x^2 \sin^2 \alpha + 2 \cos \alpha),$$

where $x = \mu_B/m_n$ and $\alpha$ parametrizes the minimum, $\Sigma = \Sigma_1 \cos \alpha + \Sigma_2 \sin \alpha$. (The same argument as above tells us that when $\mu_I = 0$ and $j_3 = 0$, then $n_3 = 0$ in the global minimum.)

Now when $x < 1$, the minimum occurs at $\alpha = 0$ — only the chiral condensate is nonzero, this is the normal phase. When, on the other hand, $x > 1$, the Lagrangian is minimized by $\cos \alpha = 1/x^2$. In this case, the chiral condensate rotates into the diquark condensate as the angle $\alpha$ increases. This is the Bose–Einstein condensation phase.
4.2. Excitation spectrum

The spectrum of excitations above the ground state is determined by the bilinear part of the Lagrangian. Expanding Eq. (3) in terms of the components $n_i$, it acquires the form

$$L_{\text{eff}} = 2F^2(\partial_\nu \vec{n})^2 + 4iF^2\mu_B(n_2\partial_0n_4 - n_4\partial_0n_2) - 2F^2\mu^2_B(n_2^2 + n_4^2) - 4F^2m^2_\pi n_1.$$  (8)

To proceed, we have to deal separately with the two phases of the theory.

4.2.1. The normal phase

When $x < 1$, the ground state expectation values of $\vec{n}$ are $n_1 = 1$ and all other components zero. The independent excitations above the ground state may be identified with $n_i, i = 2, \ldots, 6$, while $n_1$ is expressed in terms of them via the constraint $\vec{n}^2 = 1$,

$$n_1 = \sqrt{1 - \sum_{i=2}^{6} n_i^2} = 1 - \frac{1}{2} \sum_{i=2}^{6} n_i^2 + \text{higher order terms.}$$

The bilinear part of the Lagrangian (8) becomes

$$\frac{\mathcal{L}_{\text{bilinear}}}{2F^2} = \sum_{i=3,5,6} (\partial_\nu n_i)^2 + (\partial_0 N - \mu_B N)(\partial_0 N^\dagger + \mu_B N^\dagger) + \nabla N \cdot \nabla N^\dagger + m^2_\pi \sum_{i=2}^{6} n_i^2,$$

where we have introduced $N = n_2 + i n_4$, a complex field that carries baryon number one. This field thus corresponds to the diquark, while $N^\dagger$ describes the antidiquark.

We find the following dispersion relations,

- $E = \sqrt{\rho^2 + m^2_\pi}$ pion triplet $n_3, n_5, n_6$,
- $E = \sqrt{\rho^2 + m^2_\pi} - \mu_B$ diquark $N$,
- $E = \sqrt{\rho^2 + m^2_\pi} + \mu_B$ antidiquark $N^\dagger$.

4.2.2. The Bose–Einstein condensation phase

For $x > 1$, the chiral condensate alone is no longer the proper ground state and the Bose–Einstein condensation sets. We therefore parametrize the field $\vec{n}$ as

$$\vec{n} = (\rho \cos \varphi, \rho \sin \varphi, n_3, n_4, n_5, n_6).$$

The ground state corresponds to $\rho = 1$, $\varphi = \alpha$ and $n_3 = n_4 = n_5 = n_6 = 0$. We set $\varphi = \alpha + \theta$ so that the five independent degrees of freedom are now $\theta$ and $n_i, i = 3, \ldots, 6$. The radial parameter $\rho$ is given by

$$\rho = \sqrt{1 - \sum_{i=3}^{6} n_i^2} = 1 - \frac{1}{2} \sum_{i=3}^{6} n_i^2 + \text{higher order terms.}$$
The orientation of $\theta$ is perpendicular to the direction of the ground-state condensate, which is represented by the dotted line. The coordinates $n_1, n_2$ are labeled schematically by the chiral and the diquark condensate, respectively.

The bilinear Lagrangian reads in this case,

$$\frac{\mathcal{L}_{\text{bilinear}}}{2F^2} = \sum_{i=3}^{6} (\partial_\nu n_i)^2 + (\partial_\nu \theta)^2 + 2i \mu_B (\theta \partial_0 n_4 - n_4 \partial_0 \theta) \cos \alpha + \mu_B^2 \sum_{i=3,5,6} n_i^2 + \theta^2 \sin^2 \alpha.$$  

Three of the degrees of freedom, $n_3, n_5, n_6$, again represent the pion triplet, now with the dispersion relation $E = \sqrt{\vec{p}^2 + \mu_B^2}$. The dispersions of the remaining two excitations are obtained by a diagonalization of the inverse propagator in the $(\theta, n_4)$ sector. The result is

$$E^2_{\pm} = \vec{p}^2 + \frac{\mu_B^2}{2} (1 + 3 \cos^2 \alpha) \pm \frac{\mu_B}{\mu_B^2} \sqrt{\mu_B^2 (1 + 3 \cos^2 \alpha)^2 + 16 \vec{p}^2 \cos^2 \alpha},$$

in accord with previous work. The masses of these modes are given by

$$m^2_{\pm} = \mu_B^2 (1 + 3 \cos^2 \alpha) = \frac{3m_\pi^2}{\mu_B^2}, \quad m^2_0 = 0.$$  

In contrast to the normal phase, there is always one truly massless Goldstone boson stemming from the exact baryon number $U(1)$ symmetry, which is spontaneously broken by the diquark condensate $\Sigma_2$.

It is, however, worth emphasizing that the nature of this Goldstone boson, as well as of the massive mode, changes as the chemical potential increases. There are two reasons — the rotation of the ground state in the $(n_1, n_2)$ plane, and the balance between the mass term in the bilinear Lagrangian and the term with a single time derivative.

In the limit $\alpha \to 0$, the parameter $\theta$ is, to the lowest order, equal to $n_2$ and the parametrization of the case $x < 1$ is recovered. Here the Goldstone boson is the diquark $N = \theta + i n_4$.

As the angle $\alpha$ grows, the orientation of $\theta$ also changes in the $(n_1, n_2)$ plane so that it is always perpendicular to the direction of the condensate, see Fig. 2. In the limit $\alpha \to \pi/2$ that is, $\mu_B \gg m_\pi$, the condensate is purely diquark and $\theta$ has the quantum numbers of $n_1$, i.e. the $\sigma$ field. The Goldstone boson is now $n_4$. Note also that it is a linear combination of the diquark and the antidiquark so that it has no
definite baryon number. This is, of course, not surprising since the baryon number is spontaneously broken and thus it cannot be used to label the physical states.

5. Conclusions

We have constructed the chiral perturbation theory for two-color QCD with two quark flavors on the $SO(6)/SO(5)$ coset. We have provided an explicit mapping between this formulation and that used previously in literature, based on the $SU(4)/Sp(4)$ coset space.

The virtue of the present approach is that the orthogonal rotations, in contrast to the unitary symplectic transformations, can be easily visualized and the physical content of the theory thus made manifest. We were also able to give a simple proof of the fact that the condensate taken previously as an ansatz is indeed the true ground state, and we thus justified the assumptions made in the older work.

Since the $SO(N)$-symmetric nonlinear sigma model is known to great detail, the connection provided in this letter can hopefully lead to the improvement in the understanding of the phase diagram of the two-color QCD at low energies, at least in the two-flavor case.

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