DIAMETER 2 PROPERTIES AND CONVEXITY

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Abstract. We present an equivalent midpoint locally uniformly rotund (MLUR) renorming $X$ of $C[0,1]$ on which every weakly compact projection $P$ satisfies the equation $\|I - P\| = 1 + \|P\|$ ($I$ is the identity operator on $X$). As a consequence we obtain an MLUR space $X$ with the properties D2P, that every non-empty relatively weakly open subset of its unit ball $B_X$ has diameter 2, and the LD2P+, that for every slice of $B_X$ and every norm 1 element $x$ inside the slice there is another element $y$ inside the slice of distance as close to 2 from $x$ as desired. An example of an MLUR space with the D2P, the LD2P+, and with convex combinations of slices of arbitrary small diameter is also given.

1. Introduction

Let $X$ be a Banach space. We say that $X$ (or its norm $\| \cdot \|$) is midpoint locally uniformly rotund [MLUR] (resp. weak midpoint locally uniformly rotund [weak MLUR]) if every $x$ in the unit sphere $S_X$ of $X$ is a strongly extreme point (resp. strongly extreme point in the weak topology), i.e. for every sequence $(x_n)$ in $X$, we have that $x_n \to 0$ in norm (resp. $x_n \to 0$ weakly) whenever $\|x + x_n\| \to 1$.

Let $x^* \in S_{X^*}$ and $\varepsilon > 0$. By a slice of the unit ball $B_X$ of $X$ we mean a set of the form

$S(x^*, \varepsilon) := \{ x \in B_X : x^*(x) > 1 - \varepsilon \}$.

Over the latest 15 years quite much has been discovered concerning Banach spaces with various kinds of diameter 2 properties (see e.g. [20], [2], [11], [12], [5], [6] to mention a few).

Definition 1.1. A Banach space $X$ has the

a) local diameter 2 property (LD2P) if every slice of $B_X$ has diameter 2.

b) diameter 2 property (D2P) if every non-empty relatively weakly open subset of $B_X$ has diameter 2.

c) strong diameter 2 property (SD2P) if every finite convex combination of slices of $B_X$ has diameter 2.

By [9, Lemma II.1 p. 26] c) implies b) and of course b) implies a). Non of the reverse implications hold (see [5] Theorem 2.4] and [11] Theorem 1] or
However, note Proposition 1.3 below which is an immediate consequence of Choquet’s lemma (see e.g. [8, Lemma 3.69 p. 111]).

**Lemma 1.2** (Choquet). Let $C$ be a compact convex set in a locally convex space $X$. Then for every $x \in \text{ext}(C)$, the extreme points in $C$, the slices of $C$ containing $x$ form a neighborhood base of $x$ in the relative topology of $C$.

**Proposition 1.3.** If $X$ is weak MLUR then the LD2P implies the D2P.

**Proof.** Simply recall that the points in $B_X$ which are strongly extreme in the weak topology are exactly the extreme points which continue to be extreme in $B_X^{**}$ (see e.g. [10]) and then use Lemma 1.2 on $B_X^{**}$ given the weak $*$ topology. □

It is not evident that weak MLUR spaces with the LD2P exist, but indeed they do. The quotient $C(T)/A$, where $C(T)$ is the space of continuous functions on the complex unit circle $T$ and where $A$ is the disc algebra, is such an example. Another example can be constructed as follows: Let $\Phi$ be a function on $c_0$, the space of real valued sequences which converges to 0, defined by $\Phi(x_n) = \sum_{n=1}^{\infty} x_n^2$. Define then a norm on $c_0$ by $|x| = \inf\{\lambda > 0 : \Phi(x/\lambda) \leq 1\}$ for every $x \in c_0$. Then the space $(c_0, |\cdot|)$ can be seen to be weak MLUR and to have the LD2P.

The two examples mentioned motivates the following question which we will address in this paper: How rotund can a Banach space be and still have diameter 2 properties? In [13, Remarks 4) p. 286] it is pointed out that $C(T)/A$ is M-embedded and that its dual norm is smooth (see also [19] and [14, p. 167]). Recall that $X$ is M-embedded provided we can write $X^{***} = X^* \oplus_1 X^\perp$ where $X^\perp \subset X^{***}$ is the annihilator of $X$ (a good source for the available theory of M-embedded spaces is the book [14]). It is well known that M-embedded spaces have the SD2P [2], and so $C(T)/A$ actually furnishes an example of a weak MLUR space with the SD2P. On can prove also that the space $(c_0, |\cdot|)$ mentioned above have the same properties. For still more examples see [23]. The unit ball of an M-embedded space cannot, however, contain strongly extreme points [14], so no MLUR M-embedded space exists. Still, one can ask if there exists an MLUR space with the LD2P (=D2P in this case). Until now, no such example has been known. But, in Section 5 of this paper we construct an equivalent MLUR renorming $X$ of $C[0,1]$ for which every slice $S$ of $B_X$ and every $x \in S \cap S_X$ there exists $y \in S$ of distance as close to 2 from $x$ as we want, i.e. $X$ has the local diameter 2 property + (LD2P+). In particular this renorming has the LD2P and thus the D2P as it is MLUR. Using this renorming we also construct in Section 2 an example of an MLUR space which has the D2P, the LD2P+, and has convex combinations of slices with arbitrary small diameter.

In Section 5 we characterize the property LD2P+. We show that a space $X$ has the LD2P+ if and only if the dual has the weak$^*$ LD2P+ if and only if the equation $\|I - P\| = 1 + \|P\|$ holds for every weakly compact projection on $P$ on $X$. It is also proved that the LD2P+ is inherited by ai-ideals (see p. 11 for the definition of this concept).

In Section 4 we show that if a Banach lattice contains a strongly extreme point $x$ which can be approximated by finite rank projections at $x$, then $x$ is already a denting point. From this we see that it might not be so easy to construct an equivalent MLUR norm on $c_0$ with the D2P.

Section 5 contains a list of open questions.
The notation we use is mostly standard and is, if considered necessary, explained as the text proceeds.

2. MLUR renormings of $C[0, 1]$ with the D2P

Let $D = (D_n)_{n=1}^\infty$ be a base of neighborhoods in $[0, 1]$. For each $x \in C[0, 1]$ put $\|x\|_n = \sup_{d \in D_n} |x(d)|$ and note that each $\| \cdot \|_n$ defines a semi-norm on $C[0, 1]$. Now define a norm on $C[0, 1]$ by

$$\|x\|_D := \left(\sum_{n=1}^\infty 2^{-n}\|x\|_n^2\right)^{1/2}.$$  

By compactness there exists $b > 0$ such that $b\|x\|_\infty \leq \|x\|_D \leq \|x\|_\infty$ so the norm $\| \cdot \|_D$ is equivalent to the max-norm $\| \cdot \|_\infty$ on $C[0, 1]$. The idea to introduce this norm goes back to [17].

In what follows let $X_D = (C[0, 1], \| \cdot \|_D)$.

**Proposition 2.1.** For any base $D = (D_n)_{n=1}^\infty$ of neighborhoods in $[0, 1]$ the space $X_D$ is MLUR.

**Proof.** Let $x$ and $(y_k)_{k=1}^\infty$ be such that $\lim_{k \to \infty} \|x \pm y_k\|_D = \|x\|_D$. We will show that $\|y_k\|_D \to 0$ to establish that $\| \cdot \|_D$ is MLUR. By a convexity argument (see e.g. [17] Fact II. 2. 3) we have

$$(1) \lim_{k \to \infty} \|x \pm y_k\|_n = \|x\|_n, \quad n = 1, 2, \ldots$$

Let $\varepsilon > 0$. We will first make three simple observations:

a) By uniform continuity of $x$, we can find $\delta = \delta(\varepsilon) > 0$ such that the oscillation over $A$, sup$_{t, s \in A} (x(t) - x(s))$, is less than $\varepsilon$ whenever $A \subset [0, 1]$ is of length less then $\delta$.

b) Proceeding with the $\delta$ above, since $(D_n)_{n=1}^\infty$ is a base for the topology on the compact space $[0, 1]$, there is a finite subset $M \subset \mathbb{N}$ such that $\bigcup_{m \in M} D_m$ covers $[0, 1]$ and the length of any $D_m, m \in M$ is less than $\delta$.

c) Proceeding with $M$ and $\delta$ as above, having in mind [11] which of course is true for each $m \in M$, we can find $K \in \mathbb{N}$ such that $\|x \pm y_k\|_m \leq \|x\|_m + \varepsilon$ whenever $k \geq K$ and $m \in M$.

We are now ready to finish the proof. To this end, let $t_0$ be an arbitrary point in $[0, 1]$. We will show that $|y_k(t_0)| \leq 2\varepsilon$ for $k \geq K$. Choose $\sigma_k$ from $\{-1, 1\}$ such that

$$|x(t_0) + \sigma_k y_k(t_0)| = |x(t_0)| + |y_k(t_0)|.$$

Since $\bigcup_{m \in M} D_m$ covers $[0, 1]$ there is $m' \in M$ such that $t_0 \in D_{m'}$. Now remember that the length of all the $D_m$’s are $< \delta$, such that the oscillation of $x$ over $D_{m'}$ is less than $\varepsilon$. We get

$$|y_k(t_0)| = |x(t_0) + \sigma_k y_k(t_0)| - |x(t_0)|$$

$$\leq \sup_{t \in D_{m'}} |x(t) + \sigma_k y_k(t)| - |x(t_0)|$$

$$\leq \|x + \sigma_k y_k\|_{m'} - (\|x\|_{m'} - \varepsilon)$$

$$\leq \|x\|_{m'} + \varepsilon - \|x\|_{m'} + \varepsilon = 2\varepsilon,$$

provided $k \geq K$.  

\[\Box\]
Definition 2.4. For a Banach space \( X \) we say that (the norm on) \( X \) is

\[
\text{MLUR and has the D2P and the LD2P+}.\]

Theorem 2.3. For any base \( D = (D_n)_{n=1}^\infty \) of neighborhoods in \([0,1]\) the space \( X_D \) has the LD2P+.

Proof. We know that the dual of \( X \) is isomorphic to \( rca[0,1] \), the space of regular and countably additive Borel measures on \([0,1]\). Let \( \lambda \in rca[0,1] \) be the Lebesgue measure. By Lebesgue’s decomposition theorem, any measure \( m \in rca[0,1] \) can be decomposed as \( m = \mu + \nu \), where \( \mu \) is absolutely continuous with respect to \( \lambda \) and \( \nu \) and \( \lambda \) are singular.

Now, let \( m \in S_{X^*} \), \( \varepsilon > 0 \), and denote by \( S \) the slice

\[
\{ x \in B_X : \int_{[0,1]} x \, dm > 1 - \varepsilon \}.\]

Let \( x \in S \) and find \( 1 - \|x\| \leq \delta < \varepsilon \) and \( N \in \mathbb{N} \) such that

\[
\left( \sum_{n=1}^{N} 2^{-n} \|x\|_n^2 \right)^{1/2} > 1 - \delta > 1 - \varepsilon.
\]

There exist open intervals \( E_n = (r_n, t_n) \) inside \( D_n \) with \( s_n = \frac{r_n + t_n}{2} \) such that

a) \( E_i \cap E_j = \emptyset \) for every \( i \neq j \),

b) \( \langle \sum_{n=1}^{N} 2^{-n} |x(e_n)|^2 \rangle^{1/2} > 1 - \delta \) whenever \( e_n \in E_n \),

c) \( \nu(\{s_n\}) = 0 \) for every \( 1 \leq n \leq N \),

d) \( b^{-1} \sum_{n=1}^{N} m(E_n) < \eta \) where \( E = \bigcup_{n=1}^{N} E_n \), and \( \int_{[0,1] \setminus E} x \, dm - \eta > 1 - \varepsilon \).

Now, define a continuous function \( y \) on \([0,1]\) by letting \( y(r_n) = x(r_n) \), \( y(s_n) = -x(s_n) \), \( y(t_n) = x(t_n) \), linear on \((r_n, s_n)\) and \((s_n, t_n)\), and otherwise equal to \( x \). Then \( y \in X \) with \( \sup_{d \in E_n} |y(d)| \leq \sup_{d \in E_n} |x(d)| \) and \( y(d) = x(d) \) for every \( d \in [0,1] \setminus E \). Therefore \( \|y\| \leq \|x\| \leq 1 \). Moreover, we have

\[
\int_{[0,1]} y \, dm = \int_{[0,1] \setminus E} y \, dm + \int_{E} y \, dm
\]

\[
\geq \int_{[0,1] \setminus E} x \, dm - \sum_{n=1}^{N} b^{-1} m(E_n) > \int_{[0,1] \setminus E} x \, dm - \eta > 1 - \varepsilon,
\]

and

\[
\|x - y\| \geq \left( \sum_{n=1}^{N} 2^{-n} \|x - y\|_n^2 \right)^{1/2}
\]

\[
\geq \left( \sum_{n=1}^{N} 2^{-n} |x(s_n) - y(s_n)|^2 \right)^{1/2} = 2 \left( \sum_{n=1}^{N} 2^{-n} |x(s_n)|^2 \right)^{1/2} > 2 - 2\delta.
\]

From the propositions 2.1, 2.2 and 1.3 we obtain the following result.

Theorem 2.3. For any base \( D = (D_n)_{n=1}^\infty \) of neighborhoods in \([0,1]\) the space \( X_D \) is MLUR and has the D2P and the LD2P+.

In [12] dual characterizations of the diameter 2 properties in Definition 1.1 were obtained. To formulate these we need to introduce some concepts.

Definition 2.4. For a Banach space \( X \) we say that (the norm on) \( X \) is
Definition 2.5. A dual Banach space $X$ is a Banach space $X^*$.

Theorem 2.6. The set of the form $D = \{ x^* \in X^* : x^*(x) > 1 - \varepsilon \}$.

Definition 2.5. A dual Banach space $X^*$ has the

a) **locally octahedral** if for every $\varepsilon > 0$ and every $x \in S_X$ there exists $y \in S_X$ such that $\|x \pm y\| > 2 - \varepsilon$.

b) **octahedral** if for every $\varepsilon > 0$ and every finite set of points $(x_i)_{i=1}^n \subset S_X$ there exists $y \in S_X$ such that $\|x_i + y\| > 2 - \varepsilon$ for every $1 \leq i \leq n$.

For a Banach space $X$, $x \in S_X$, and $\varepsilon > 0$ we mean by a weak*-slice of $B_{X^*}$ a set of the from $S(x, \varepsilon) := \{ x^* \in B_{X^*} : x^*(x) > 1 - \varepsilon \}$.

Theorem 2.6. [12, Theorems 3.1, 3.3, and 3.5] For a Banach space $X$ we have

a) $X$ is locally octahedral $\iff$ $X^*$ has the weak*-LD2P.

b) $X$ is octahedral $\iff$ $X^*$ has the weak*-SD2P.

It follows from Theorem 2.3, Theorem 3.6 below, and Theorem 2.6 that for any base $D = (D_n)_{n=1}^\infty$ of neighborhoods in $[0,1]$ the space $X_D$ is locally octahedral. However, every such space $X_D$ fails to be octahedral. To see this we will use the following lemma.

Lemma 2.7. Let $u$ and $v$ be continuous functions on the unit interval. Suppose $\|u\|_n = \|v\|_n$ for every $n \in \mathbb{N}$. Then $|u(t)| = |v(t)|$ for every $t \in [0,1]$.

Proof. Let $\varepsilon, \delta > 0$ such that

\[ |u(s') - u(s'')| < \varepsilon \text{ and } |v(s') - v(s'')| < \varepsilon \]

whenever $|s' - s''| < \delta$. Fix $t \in [0,1]$. There exists $n \in \mathbb{N}$ such that $t$ belongs to $D_n$ and diam$(D_n) < \delta$. Now find $t', t''$ in $D_n$ such that $\|u\|_n - |u(t')| < \varepsilon$ and $\|v\|_n - |v(t'')| < \varepsilon$. Then $|u(t')| - |v(t'')| < 2\varepsilon$, and thus by (2) we have $|u(t)| - |v(t)| < 4\varepsilon$.

Proposition 2.8. For any base $D = (D_n)_{n=1}^\infty$ of neighborhoods in $[0,1]$ the space $X_D$ fails to be octahedral.

Proof. Choose two different non negative norm 1 functions $u$ and $v$ in $X_D$. Assume there exists a sequence $(y_k)_{k=1}^\infty \subset S_X$ such that

\[ \lim_{k \to \infty} \|u + y_k\|_D = 2 \quad \text{and} \quad \lim_{k \to \infty} \|v + y_k\|_D = 2. \]

Using (3) and [7, Fact II. 2.3] we have for every $n \in \mathbb{N}$ that

\[ \|u\|_n = \lim_k \|y_k\|_n = \|v\|_n. \]

Now we get from Lemma 2.7 a contradiction as $u$ and $v$ are non negative and different.

The final part of this section will be devoted to showing that there exists a Banach which is MLUR, has the D2P, the LD2P+, and has convex combinations of slices with arbitrarily small diameter. First we will show that for any given $\delta > 0$ there exists a base $D = (D_n)_{n=1}^\infty$ of neighborhoods in $[0,1]$ for which $B_{X_D}$
contains convex combinations of slices with diameter < \delta. In that respect the following lemma will come in to use.

Let \( t \in [0, 1] \). Put \( J(t) = \{ n : t \in D_n \} \) and let \( w(t) = \sum_{n \in J(t)} 2^{-n} \).

**Lemma 2.9.** Let \( D = (D_n)_{n=1}^{\infty} \) be a base of neighborhoods in \([0, 1], t \in [0, 1], \) and \( \delta_t \) the point measure in \( X_D^* \). If \( \overline{D}_n \cap \{ t \} = \emptyset \) for every \( n \notin J(t) \), then

\[
\| \delta_t \|_{D} = \frac{1}{\sqrt{w(t)}},
\]

where \( \| \cdot \|_{D} \) is the norm in \( X_D^* \).

**Proof.** Let \( x \in X_D \) with norm 1. Then

\[
1 = \sum_{n=1}^{\infty} 2^{-n} \| x \|_n^2 \geq \sum_{n \in J(t)} 2^{-n} |x(t)|^2 = w(t) \| \delta_t \| (x)^2.
\]

Thus \( \| \delta_t \|_{D} \leq \frac{1}{\sqrt{w(t)}} \). Moreover, by the assumptions it is always possible to find for \( i \notin J(t) \) an open set which contains \( t \) and which does not intersect \( \overline{D}_i \). Thus we can always find an \( x \in S_{X_D} \) which takes its maximum value at \( t \) and which is zero on \( D_i \). From this it follows that for any \( \varepsilon > 0 \) we can find \( x \in S_{X_D} \) which takes its maximum value at \( t \) such that \( \sum_{n \notin J(t)} 2^{-n} \| x \|_n^2 < \varepsilon \). From the inequality

\[
1 = \| x \|_D = \sum_{n \in J(t)} 2^{-n} x(t)^2 + \sum_{n \notin J(t)} 2^{-n} \| x \|_n^2 < \sum_{n \in J(t)} 2^{-n} x(t)^2 + \varepsilon
\]

we get that \( \delta_t^2(x) > \frac{1 - \varepsilon}{w(\delta)} \). Thus we can conclude that \( \| \delta_t \|_{D} = \frac{1}{\sqrt{w(t)}} \). \( \square \)

Let \((\varepsilon_n)_{n=1}^{\infty}\) (with \( \varepsilon_1 \) small!) be a strictly decreasing sequence of positive real numbers converging fast to 0. For each \( i \in \mathbb{N} \) let us define a base of neighborhoods \((D_i,n)_{n=1}^{\infty}\) in \([0, 1] \): Let \( i = 1 \) and

\[
D_{1,1} = [0, 2^{-1} + \varepsilon_1], D_{1,2} = (2^{-1} - \varepsilon_2, 1].
\]

We call this the first level. For the second level put

\[
D_{1,3} = [0, 2^{-2} + \varepsilon_3], D_{1,4} = (2^{-2} - \varepsilon_4, 2 \cdot 2^{-2} + \varepsilon_4),
\]

\[
D_{1,5} = (2 \cdot 2^{-2} - \varepsilon_5, 3 \cdot 2^{-2} + \varepsilon_5), D_{1,6} = (3 \cdot 2^{-2} - \varepsilon_6, 1].
\]

Continue in this fashion to obtain the base \((D_{1,n})_{n=1}^{\infty}\) consisting of open intervals in \([0, 1] \). Finally let \( D_i = (D_{i,n})_{n=1}^{\infty} \) be the base of \([0, 1] \) consisting of the intervals in \((D_{1,n})_{n=1}^{\infty}\) starting from level \( i \).

We will prove that for \( i \geq 2 \) the space \( X_{D_i} \) fails to have the SD2P. In fact, we will prove the following.

**Proposition 2.10.** For each \( i \geq 2 \) let \( X_{D_i} \) be the space \( C[0, 1] \) with the norm \( \| \cdot \|_{D_i} \). Then for every \( \varepsilon > 0 \) there exists finite convex combinations of slices of \( B_{X_{D_i}} \) with diameter at most \( \frac{\sqrt{1+\varepsilon}}{i} \).
Proof. First suppose $i = 2$ and choose $t_1 = 0$ and $t_2 = 1$ and note that $J(t_1) \cap J(t_2) = \emptyset$, $\{t_1\} \cap D_n = \emptyset$ for every $n \notin J(t_1)$, and $\{t_2\} \cap D_n = \emptyset$ for every $n \notin J(t_2)$. Put $M = \sup \{\|x\|_\infty : x \in B_{X_{D_2}}\} < \infty$. By a similar argument as in the last part of the proof of Lemma 2.9 it is possible to choose, for any $\varepsilon > 0$, a $\eta > 0$ such that
\[
\sum_{n \notin J(t_1)} 2^{-n}(2M\|x\|_2,n + \|x\|_2^2,n) < \varepsilon/3, \quad \sum_{n \notin J(t_2)} 2^{-n}(2M\|y\|_2,n + \|y\|_2^2,n) < \varepsilon/3,
\]
and
\[
\sum_{n \notin J(t_1) \cup J(t_2)} 2^{-n}(\|x\|_2^2,n + 2\|x\|_2,n\|y\|_2,n + \|y\|_2^2,n) < \varepsilon/3.
\]
whenever $x$ and $y$ are elements in the slices $S(\delta_1/\|\delta_1\|_D^1, \eta)$ and $S(\delta_2/\|\delta_2\|_D^2, \eta)$ of $B_{X_{D_2}}$, respectively. Now, if we put $h = \frac{1}{2}x + \frac{1}{2}y$ we get
\[
2^2\|h\|_D^2 = \sum_{n=1}^{\infty} 2^{-n}\|x + y\|_2^2,n
\]
\[
\leq \sum_{n \notin J(t_1)} 2^{-n}(\|x\|_2^2,n + 2\|x\|_2,n\|y\|_2,n + \|y\|_2^2,n)
\]
\[
+ \sum_{n \notin J(t_2)} 2^{-n}(\|x\|_2^2,n + 2\|x\|_2,n\|y\|_2,n + \|y\|_2^2,n)
\]
\[
+ \sum_{n \notin J(t_1) \cup J(t_2)} 2^{-n}(\|x\|_2^2,n + 2\|x\|_2,n\|y\|_2,n + \|y\|_2^2,n)
\]
\[
\leq 2 + \varepsilon.
\]
For an arbitrary $i \geq 2$ we can in $[0,1]$ choose $i$ points $(t_k)_{k=1}^i$ such that $J(t_j) \cap J(t_k) = \emptyset$ for any $j \neq k$ and such that $\{t_k\} \cap D_n = \emptyset$ for every $n \notin J(t_k)$. Using a similar argument as for $i = 2$ we get that for any $\varepsilon > 0$ there exists for every $k = 1, \ldots, i$ a slice $S(\delta_k, \eta)$ of $B_{X_{D_k}}$ such that the convex combination
\[
\sum_{k=1}^i \frac{1}{i} S(\delta_k, \eta)
\]
has diameter at most $\frac{2\varepsilon}{i}$. \qed

**Theorem 2.11.** The space $\ell_2 - \bigoplus_{i=1}^\infty X_{D_i}$ is MLUR, has the D2P, the LD2P+, and has convex combinations of slices of arbitrary small diameter.

**Proof.** The properties of being MLUR, having the D2P, and having the LD2P+ are all stable by taking $\ell_2$-sums (see [2, Theorem 3.2] and [15, Theorem 3.2] for the latter two). Thus the space $\ell_2 - \bigoplus_{i=1}^\infty X_{D_i}$ has to possess all these properties as well since each $X_{D_i}$ does. So, what is left to prove is that the unit ball of $\ell_2 - \bigoplus_{i=1}^\infty X_{D_i}$ has finite convex combinations of slices with arbitrary small diameter. To this end let $Z = X_{D_i} \oplus_2 Y_i$ where $Y_i = \ell_2 - \bigoplus_{k \neq i} X_{D_k}$. Let $x_i^* \in S_{X_{D_i}}, S_i(x_i^*, \delta)$ a slice of $B_{X_{D_i}}$, and let $0 < \delta < \eta$. Now, if $(x_i,y_i)$ is in the slice $S((x_i^*,0), \delta)$ of $B_Z$, then $x_i^*(x_i) > 1 - \delta$, and so $\|x_i\| > 1 - \delta$. Thus $\|y_i\|^2 \leq 2\delta - \delta^2$. But this means that
\[
S((x_i^*,0), \delta) \subset S_i(x_i^*, \delta) \times (2\delta - \delta^2)^{1/2} B_{Y_i}.
\]
From this we see that if \( z \in \sum_{j=1}^i \frac{1}{i} S_j(B_Z, (x_{i,j}^*, 0), \delta) \), then we can write \( z = x + y \) where \( x \in \sum_{j=1}^i \frac{1}{i} S_i,j(B_{X_{D_i}}, x_{i,j}^*, \delta) \) and \( y \in (2\delta - \delta^2)^{1/2} B_{Y_i} \). Now, if the convex combination \( \sum_{j=1}^i \frac{1}{i} S_i,j(B_{X_{D_i}}, x_{i,j}^*, \delta) \) is chosen so that its diameter is at most \( \frac{\sqrt{2}}{i} \), which is possible by Proposition 2.10, we get that \( \|x\| \leq \frac{\sqrt{2}}{i} \) and \( y \in (2\delta - \delta^2)^{1/2} B_{Y_i} \). As \( \|z\| \leq \|x\| + \|y\| \) and \( i \) can be chosen as big as desired and \( \delta > 0 \) as small as desired, we are done. \( \square \)

3. The local diameter 2 property +

Let \( X \) be a Banach space and \( I \) the identity operator on \( X \). Recall that \( X \) has the Daugavet property if the equation

\[
\|I + T\| = 1 + \|T\|
\]

holds for every rank 1 operator \( T \) on \( X \). The Daugavet property can be characterized as follows (see [24] or [21]):

**Theorem 3.1.** Let \( X \) be a Banach space. Then the following statements are equivalent.

a) \( X \) has the Daugavet property.

b) The equation \( \|I + T\| = 1 + \|T\| \) holds for every weakly compact operator \( T \) on \( X \).

c) For every \( \varepsilon > 0 \), every \( x \in S_X \), and every \( x^* \in S_{X^*} \), there exists \( y \in S(x^*, \varepsilon) \) such that \( \|x + y\| \geq 2 - \varepsilon \).

d) For every \( \varepsilon > 0 \), every \( x^* \in S_{X^*} \), and every \( x \in S_X \), there exists \( y^* \in S(x, \varepsilon) \) such that \( \|x^* + y^*\| \geq 2 - \varepsilon \).

e) For every \( \varepsilon > 0 \) and every \( x \in S_X \) we have \( B_X = \overline{\text{conv}}(\Delta_x(x)) \) where \( \Delta_x(x) = \{ y \in B_X : \|x - y\| \geq 2 - \varepsilon \} \).

Let us recall from the Introduction the definition of the LD2P+ and at the same time introduce its weak* version.

**Definition 3.2.** We say that a Banach space \( X \) has the local diameter 2 property + (LD2P+) if for every \( x^* \in S_{X^*} \), every \( \varepsilon > 0 \), every \( \delta > 0 \), and every \( x \in S(x^*, \varepsilon) \cap S_X \) there exists \( y \in S(x^*, \varepsilon) \) with \( \|x - y\| > 2 - \delta \). If \( X \) is a dual space and the above holds for weak* slices \( S(x^*, \varepsilon) \), then \( X \) is said to have the weak* local diameter 2 property + (weak*-LD2P+).

From [15, Theorem 1.4] and [24, Open problem (7) p. 95] the following is known.

**Theorem 3.3.** Let \( X \) be a Banach space. Then the following statements are equivalent.

a) The equation \( \|I - P\| \leq 2 \) holds for every norm-1 rank-1 projection \( P \) on \( X \).

b) For every \( \varepsilon > 0 \), every \( x^* \in S_{X^*} \) and every \( x \in S(x^*, \varepsilon) \) there exists \( y \in S(x^*, \varepsilon) \cap S_X \) with \( \|x - y\| > 2 - \varepsilon \).

c) For every \( x \in S_X \) and every \( \varepsilon > 0 \) we have \( x \in \overline{\text{conv}}(\Delta_x(x)) \) where \( \Delta_x(x) = \{ y \in B_X : \|x - y\| > 2 - \varepsilon \} \).

From Lemma 3.1 of Kadets and Ivakhno (see [15, Lemma 2.1]) stated below it is clear that the LD2P+ is equivalent to the statements in Theorem 3.3. Therefore
every Daugavet space has the LD2P+. Note, however, that the converse is not true as the LD2P+ is stable by taking unconditional sums of Banach spaces which fails for spaces with the Daugavet property (see e.g. [15, Corollary 3.3]).

**Lemma 3.4** (Kadets and Ivakhno). Let \( \varepsilon > 0 \) and \( x^* \in S_{X^*} \). Then for every \( x \in S(x^*, \varepsilon) \cap S_X \) and every positive \( \delta < \varepsilon \) there exist \( y^* \in S_{X^*} \) such that \( x \in S(y^*, \delta) \) and \( S(y^*, \delta) \subset S(x^*, \varepsilon) \).

In the proof of Proposition 3.6 below we will need the following weak*-version of Lemma 3.4. Its proof is more or less verbatim to that of Lemma 3.4 and will therefore be omitted.

**Lemma 3.5.** Let \( \varepsilon > 0 \) and \( x \in S_X \). Then for every \( x^* \in S(x, \varepsilon) \cap S_{X^*} \) which attains its norm and every positive \( \delta < \varepsilon \) there exist \( y \in S_X \) such that \( x^* \in S(y, \delta) \) and \( S(y, \delta) \subset S(x, \varepsilon) \).

We will now add to the list of statements in Theorem 3.3 statements similar to b) and d) in Theorem 3.1.

**Theorem 3.6.** Let \( X \) be a Banach space. Then the following statements are equivalent:

a) \( X \) has the LD2P+.

b) For every \( x \in S_X \), every \( \varepsilon > 0 \), every \( \delta > 0 \), and every \( x^* \in S(x, \varepsilon) \cap S_{X^*} \), there exists \( y^* \in S(x, \varepsilon) \) with \( \|x^* - y^*\| > 2 - \delta \).

c) The equation \( \|I - P\| = 1 + \|P\| \) holds for every weakly compact projection \( P \) on \( X \).

**Proof.** a) \( \Rightarrow \) b). By the Bishop-Phelps theorem we can assume without loss of generality that \( x^* \in S(x, \varepsilon) \cap S_{X^*} \) attains its norm. Let \( 0 < \eta < \min\{\varepsilon, \delta/2\} \) and find by Lemma 3.4 \( y \in S_X \) such that \( x^* \in S(y, \eta) \) and \( S(y, \eta) \subset S(x, \varepsilon) \). Note that \( y \in S(x^*, \eta) \) and thus, since \( X \) has the LD2P+, we can find \( z \in S(x^*, \eta) \) such that \( \|y - z\| > 2 - \eta \). Hence there is \( y^* \in S_{X^*} \) such that \( y(y^*) - z(y^*) = (y - z)(y^*) > 2 - \eta \).

From this we have \( y(y^*) > 1 - \eta \) and \( z(y^*) > 1 - \eta \). It follows that \( y^* \in S(x, \varepsilon) \) as \( S(y, \eta) \subset S(x, \varepsilon) \). Moreover, using that \( z \in S(x^*, \eta) \) and b), we have

\[
\|x^* - y^*\| \geq (x^* - y^*)(z) \\
= x^*(z) - y^*(z) \\
> 1 - \eta + 1 - \eta > 2 - \delta.
\]

b) \( \Rightarrow \) a). The proof is identical to the proof of the converse except that one does not have to use the Bishop-Phelp’s theorem and that one uses [13, Lemma 2.1] in place of Lemma 3.5.

a) \( \Rightarrow \) c). The proof is similar to that of [16, Theorem 2.3].

c) \( \Rightarrow \) a). This is clear as c) trivially implies a) in Theorem 3.3. \( \square \)

Note that \( c_0 \) does not have the LD2P+ as \( e_1 \in S(e_1, \varepsilon) \cap S_{c_0} \) for every \( 1 \geq \varepsilon > 0 \), but every point in \( S(e_1, \varepsilon) \) is of distance 1 or less from \( e_1 \). \( c_0 \) is the prototype of an M-embedded space. Since the dual is an M-embedded space has the RNP (see e.g. [14, III.3 Corollary 3.2]) we actually get from Proposition 3.6 that every M-embedded space fails the LD2P+.

**Corollary 3.7.** M-embedded spaces fail the LD2P+.
It is known that all the diameter 2 properties in Definition 1.1 as well as the Daugavet property are inherited by certain subspaces called ai-ideals (see [3] and [1]). We will end this section by showing that this is true for the LD2P+ as well.

A subspace $X$ of a Banach space $Y$ is called an ideal in $Y$ if there exists a norm 1 projection $P$ on $Y^*$ with $\ker P = X^\perp$. $X$ being an ideal in $Y$ is in turn equivalent to the $X$ being locally 1-complemented in $Y$, i.e. for every $\varepsilon > 0$ and every finite-dimensional subspace $E \subset Y$ there exists $T : E \to X$ such that

a) $Te = e$ for all $e \in X \cap E$.

b) $\|Te\| \leq (1 + \varepsilon)\|e\|$ for all $e \in E$.

Following [3] a subspace $X$ of a Banach space $Y$ is called an almost isometric ideal (ai-ideal) in $Y$ if $X$ is locally 1-complemented with almost isometric local projections, i.e., for every $\varepsilon > 0$ and every finite-dimensional subspace $E \subset Y$ there exists $T : E \to X$ which satisfies a) and

b’) $(1 - \varepsilon)\|e\| \leq \|Te\| \leq (1 + \varepsilon)\|e\|$ for all $e \in E$.

Note that an ideal $X$ in $Y$ is an ai-ideal if $P(Y^*)$ is a 1-norming subspace of $X$. Ideals $X$ in $Y$ for which $P(Y^*)$ is a 1-norming subspace for $X$ are called strict ideals. An ai-ideal is, however, not necessarily strict (see [3]).

**Proposition 3.8.** Let $Y$ have the LD2P+ and assume $X$ is an ai-ideal in $Y$. Then $X$ has the LD2P+.

**Proof.** For $\delta > 0$, $Z$ a subspace of $Y$, and $x \in S_Z$ put

$$
\Delta^Z_\delta(x) = \{y \in B_Z : \|x - y\| > 2 - \delta\}.
$$

Let $x \in S_X$, $\varepsilon > 0$, and $\alpha > 0$. We will show that there exists $z \in \text{conv}\Delta^X_\varepsilon(x)$ with $\|x - z\| < \alpha$. First, since $Y$ enjoys the LD2P+, we know that for any positive $\beta < \varepsilon$ and any positive $\gamma < \alpha$ we can find $y = \sum_{n=1}^N \lambda_n y_n \in \text{conv}\Delta^X_\beta(x)$ with $(y_n)_{n=1}^N \subset \Delta^Y_\gamma(x)$ such that $\|x - y\| < \gamma$. Now let $E = \text{span}\{y_1, \ldots, y_N, x\}$ and pick a local projection $T : E \to X$ such that $T$ is a $(1 + \eta)$-isometry with $\eta > 0$ so small that $(1 + \eta)\gamma + \eta < \alpha$, and $(1 - \eta)(2 - \beta) - \eta > 2 - \varepsilon$. Put $z_n = \frac{Ty_n}{\|Ty_n\|}$ and $z = \sum_{n=1}^N \lambda_n z_n$. As $Tx = x$ we get

$$
\|x - z\| \leq \|x - Ty\| + \|Ty - z\|
\leq \|T(x - y)\| + \sum_{n=1}^N \lambda_n \left|1 - \|Ty_n\|\right|
\leq (1 + \eta)\gamma + \max_{1 \leq n \leq N} \left|1 - \|Ty_n\|\right|
\leq (1 + \eta)\gamma + \eta < \alpha.
$$
Moreover for every $1 \leq n \leq N$ we have
\[
\|x - z_n\| = \|T(x - \frac{y_n}{\|Ty_n\|})\|
\geq (1 - \eta)\|x - \frac{y_n}{\|Ty_n\|}\|
\geq (1 - \eta)(\|x - y_n\| - \|y_n - \frac{y_n}{\|Ty_n\|}\|)
\geq (1 - \eta)(2 - \beta - \frac{\eta}{1 - \eta}) > 2 - \varepsilon,
\]
Thus $(z_n)_{n=1}^N \subset \Delta_\varepsilon(x)$ and as $\alpha > 0$ is arbitrarily chosen, we are done. □

4. The difficulty of finding an MLUR norm on $c_0$ with the D2P

This section is motivated by the question whether it is possible to construct an MLUR norm on $c_0$ with the D2P. Actually this turns out to be much harder than in $C[0,1]$. From Proposition 4.2 below we see that if such a norm exists, it cannot be a lattice norm.

**Definition 4.1.** Let $X$ be a Banach lattice. A projection $P : X \to X$ is said to be a **lattice projection** if $u$ and $v$ are disjoint whenever $u \in P(X)$ and $v \in \ker P$, the kernel of $P$.

We say that the identity can be approximated by finite rank lattice projections at a point $x \in X$ if for all $\varepsilon > 0$ there exists a lattice projection $P$ with finite rank such that $\|x - Px\| < \varepsilon$.

**Proposition 4.2.** Let $X$ be a Banach lattice and $x \in B_X$ a strongly extreme point. If the identity can be approximated by finite rank lattice projections at $x$, then $x$ is a denting point.

To prove this we will use the following lemma.

**Lemma 4.3.** Let $\varepsilon > 0$. Then there exists $\delta > 0$ such that for any lattice projection $P : X \to X$, the condition $\|x - Px\| < \delta$, $u \in \ker P$, and $\|Px + u\| \leq 1 + \delta$ imply $\|u\| < \varepsilon$.

**Proof.** Since $x \in B_X$ is strongly extreme, there exists $\eta > 0$ such that $\|y\| < \varepsilon$ whenever $\|x \pm y\| \leq 1 + \eta$. Put $\delta = \eta/3$ and suppose $P$, $x$, and $u$ satisfy the assumptions. As $P$ is a lattice projection we have
\[
\|Px + u\| = \|Px - u\|.
\]
Thus
\[
\|x \pm u\| \leq \|Px \pm u\| + \|x - Px\| \leq 1 + 3\delta = 1 + \eta
\]
and hence $\|u\| \leq \varepsilon$. □

**Corollary 4.4.** Let $\varepsilon, \delta > 0$, let $P$ satisfy the assumptions in Lemma 4.3, and let
\[
W = \{w \in X : \|P(x - w)\| < \delta\}.
\]
Then
\[
diam(W \cap B_X) < 2\varepsilon + \delta.
\]
Proof. Pick \( w \in W \cap B_X \) and put \( u = w - Pw \). Then
\[
\|P(x + u)\| = \|P(x - w)\| + \|w\| \leq 1 + \delta.
\]
From Lemma 4.3 we get \( \|u\| < \varepsilon \). Thus
\[
\|x - w\| = \|x + u - Pw\| \leq \|P(x - w)\| + \|x - Px\| + \|u\| \leq 2\delta + \varepsilon,
\]
and so we are done. \( \square \)

Proof of Proposition 4.2. If \( \dim(PX) < \infty \), we get that \( W \) is weak open. This implies that \( x \) is a point of continuity for \( B_X \). Since every point of weak to \( \| \cdot \| \) continuity which is extreme is a denting point \cite{18} Theorem\], we get that \( x \) is denting. \( \square \)

5. Questions

Let us end the paper with some questions that is suggested by the current work:

Question 1. Does there exist an equivalent MLUR norm on \( c_0 \) with LD2P?

Question 2. Does there exist a Banach space with the LD2P and which is weakly locally uniformly rotund?

Regarding this question we note that there does exist a Banach space \( X \) which is weakly uniformly rotund (wUR) and which has the property that for every \( \varepsilon > 0 \) and every \( \text{weak}^{*} \) null sequence \( (f_n) \subset S_X \), the diameter of the slices \( S(f_n, \varepsilon) \) tends to 2. Such a Banach space can be constructed as follows: Let \( 1 < p_1 < p_2 < \ldots \) a sequence such that
\[
\prod_{i \in \mathbb{N}} \|I: \ell_\infty(2) \to \ell_{p_i}(2)\| < 2
\]
(operator norms of the formal identity mappings between 2-dimensional \( \ell_p \) spaces). Then one can form a Banach sequence space as follows:
\[
X = \mathbb{R} \oplus_{p_1} (\mathbb{R} \oplus_{p_2} (\mathbb{R} \oplus_{p_3} (\ldots \ldots)))
\]
where \( \mathbb{R} \) is considered a 1-dimensional Banach space and the space is normed by first defining semi-norms in finite-dimensional initial parts according to the above schema and then taking a limit of the semi-norms in a similar way as in the construction of the variable exponent spaces introduced in [22]. We will now show that this space \( X \) has the above mentioned properties.

Proof. Put \( Y = \text{span}(e_n: n \in \mathbb{N}) \subset X \) and \( Y_k := \text{span}(e_n: n \in \mathbb{N}, n \geq k) \subset X \). It can be seen from arguments in [22] that \( X \) and \( Y \) are isomorphic to \( \ell_\infty \) and \( c_0 \) respectively. Also the tail spaces \( Y_k \) become asymptotically isometric to \( c_0 \), i.e. for each \( \varepsilon > 0 \) there is \( k \in \mathbb{N} \) such that the tail spaces \( Y_j, j \geq k \), are \( 1 + \varepsilon \)-isomorphic to \( c_0 \) via a linear mapping which identifies the canonical unit vector bases of \( Y_j \) and \( c_0 \).

The wUR part. Let \( (x_n), (y_n) \in B_Y \) be such that \( \|x_n + y_n\|_Y \to 2 \). Denote by \( P_n \) the basis projection to the first \( n \)-coordinates and let \( Q_n = I - P_n \) be the
coprojection to the rest of the coordinates. Then according to the definition of the space

\[(|P_1(x_n + y_n)|^{p_1} + \|Q_1(x_n + y_n)\|^{p_1})^{\frac{1}{p_1}} \to 2,\]

so by the triangle inequality

\[((|P_1(x_n)| + |P_1(y_n)|)^{p_1} + (\|Q_1(x_n)\| + \|Q_1(y_n)\|)^{p_1})^{\frac{1}{p_1}} \to 2,\]

and by the uniform convexity of $\ell_{p_1}(2)$ we get that

\[(|P_1(x_n)| - |P_1(y_n)| \to 0, \quad \|Q_1(x_n)\| - \|Q_1(y_n)\| \to 0).\]

By inspecting (6) we obtain $|P_1(x_n - y_n)| \to 0$. By continuing inductively, using the right-hand side of (7), we get that $P_k(x_n - y_n) \to 0$ for each $k$. Recall that $Y$ is isomorphic to $c_0$, thus $Y^*$ is isomorphically $\ell_1$. Therefore $x_n - y_n \to 0$ weakly.

The large slices part. First note that if $(f_n) \subset Y^*$ is a normalized sequence then $\|f_n\|_{\ell_1} \geq 1$ because $\|\cdot\|_{c_0} \leq \|\cdot\|_{Y}$. Fix $\varepsilon > 0$. Let $k \in \mathbb{N}$ be such that

\[\sum_{i=1}^{\infty} a_i e_{k+i} \mapsto \sum_{i=1}^{\infty} a_i e_i\]

defines a $(1 + \varepsilon/4)$-isomorphism $Y_k \to c_0$. Note that then

\[\frac{1}{(1 + \varepsilon/4)} \|f \circ Q_k\|_{\ell_1} \leq \|f \circ Q_k\|_{Y} \leq (1 + \varepsilon/4)\|f \circ Q_k\|_{\ell_1}, \quad f \in \ell_1.\]

Because $(f_n)$ is weak-star null we may choose $m_0 \in \mathbb{N}$ such that sufficiently large part of the mass is supported on the domain of $Q_k$, more precisely,

\[\frac{1}{(1 + \varepsilon/3)} \|f_m \circ Q_k\|_{\ell_1} < \frac{1}{1 + \varepsilon/3}\]

for $m \in \mathbb{N}$, $m \geq m_0$.

Put $g = \frac{f_m \circ Q_k}{\|f_m \circ Q_k\|_{\ell_1}}$. Then

\[\left\{x \in c_0 : g(x) > \frac{1}{1 + \varepsilon/3}\right\} \subset \left\{x \in c_0 : (f_m \circ Q_k)(x) > 1 - \varepsilon\right\}.\]

Note that $\frac{1}{(1 + \varepsilon/4)}B_{c_0} \cap Y_k \subset B_{Y_k}$. Therefore the above inclusion yields that we may pick

\[x, y \in \{z \in B_{Y_k} : f_m(z) > 1 - \varepsilon\}\]

with

\[\|x - y\|_{Y} \geq \|x - y\|_{\infty} > \frac{2}{(1 + \varepsilon/3)}.\]

\[\square\]

**Question 3.** Does there exist a Banach space with the LD2P+ which fails the D2P?

**Question 4.** Does every Banach space with the LD2P+ contain a copy of $\ell_1$?
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