q-Exponential Distribution in Urban Agglomeration

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Usually, the study of city population distribution has been reduced to power laws. In such analysis, a common practice is to consider cities with more than one hundred thousand inhabitants. Here, we argue that the distribution of cities for all ranges of populations can be well described by using a q-exponential distribution. This function, which reproduces the Zipf-Mandelbrot law, is related to the generalized nonextensive statistical mechanics and satisfies an anomalous decay equation.

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In several areas in nature, besides the complexities, it is possible to identify macroscopic regularities that can be well described by simple laws. For example, frequency of words in a long text [1], forest fires [2], distribution of species lifetimes for North American breeding bird populations [3], scientific citations [4,5], www surfing [6], ecology [7], solar flares [8], football goal distribution [9], economic index [10], epidemics in isolated populations [11], among others.

In particular, recently, the interest in the study of city population distribution has been increased. Such interest is related to the analysis of data and to models that presents the asymptotic power law behavior [12–16]. However, in such analysis, only cities with more than one hundred thousand inhabitants have been considered. This power law behavior can be identified in terms of the distribution

\[ N(x)dx \propto x^{-\alpha}dx , \]  

that gives the number of cities with \( x \) and \( x + dx \) inhabitants, where \( \alpha \) is a positive constant. Another way to express the same relation is in terms of the relative number (rank or cumulative distribution) of cities with a population larger than a certain value \( x \),

\[ r(x) = \int_x^\infty N(y)dy \propto x^{1-\alpha} . \]

By expressing the population \( x(n) \) of the cities in descending order (\( x(1) \) being the city with the highest population, \( x(2) \) the city with the second highest population, and so on), it follows from (3) that

\[ x(n) \propto n^{1/(1-\alpha)} . \]

The plot of \( x(n) \) on a double logarithmic scale is called a “Zipf plot” [4] and leads to a straight line with slop \( 1/(1-\alpha) \). Note that the Zipf plot (from Eq. (1)) and cumulative plot (from Eq. (2)) are equivalent, except when regarding the weight related to the rare (largest) elements.

![Zipf-plot](image1)

FIG. 1. (a) Zipf-plot for cities with population bigger than one hundred thousand and, in inset plot, the cumulative Zipf plot to the same cities in Europe. (b) Zipf-plot for all cities in USA and Brazil. In the above graphics, \( x \) is the population of the cities, \( n \) is the descending rank and \( r \) is the cumulative rank.

The Zipf plot for cities with more than one hundred thousand inhabitants [17] for some countries and Europe is illustrated in Fig. (1-a). These graphics enable us to
visualize how good the power law is at describing the city population distribution for large cities. In inset plot of Fig. (4), we show the cumulative plot for the same cities in Europe. However, there is a little fraction of cities with more than a hundred thousand inhabitants. For instance, these cities represent about 15% of American cities and 4% of Brazilian cities. Furthermore, if we take into account all cities [13] in the country, and by using the Zipf plot, Fig. (4b), we can identify a notorious deviation from the asymptotic power law when cities with small populations are considered. Thus, an analysis that considers all cities is an important task. In this direction, this work is dedicated to an empirical analysis of this question.

An alternative approach to incorporate the deviation from power-law is employed in Ref. [20] by considering the stretched distribution (Weibull distribution), \( N(x) = N_0 x^{-1} \exp(-Ax^c) \), to fit data of some complex systems. In particular, for city formation, they also show an adjustment to cities with population bigger than a hundred thousand inhabitants, by using a kind of Zipf plot for \( x^c \) versus \( \ln(n) \), where \( c \) is an adjustable parameter. However, the Weibull distribution leads to a poor adjustment for the complete set of data, i.e., this distribution gives us a satisfactory adjustment only for a restricted range of data. Furthermore, it is clear that the stretched function does not lead to an asymptotic straight line in a log-log plot, i.e., a power law.

On the other hand, Zipf-Mandelbrot law [21], \( N(x) = b/(c+x)^\alpha \) (\( b, c, \) and \( \alpha \) all being positive constants), gives a curvature in a log-log plot, presents an asymptotic power law behavior and is normalizable for \( \alpha > 1 \). In this way, the Zipf-Mandelbrot distribution is a natural generalization of an inverse power law. This distribution has been applied in many contexts; in particular, it was recently employed in the discussion of scientific citations [3] and football goal distribution [4]. Another important aspect of the Zipf-Mandelbrot’s distribution is that it arises naturally in the context of a generalized statistical mechanics proposed some years ago [22,23]. In this framework, the above distribution is usually rewritten as a q-exponential function,

\[
N(x) = N_0 \exp_q(-ax) \equiv N_0[1 - (1-q)ax]^{1/(1-q)},
\]

where \( N_0 = bc^{-\alpha} \), \( a = \alpha/c \), and \( q' = 1 + 1/\alpha \) are positive parameters. Moreover, the above distribution has been largely used with \( q' < 1 \) in other contexts [20]. In this case, Eq. (4) is defined equal to zero when \( 1 - (1-q')ax < 0 \) in order to overcome imaginary values for \( N(x) \). Thus, the distribution (4) is equivalent to Zipf-Mandelbrot law only for \( q' > 1 \) and gives an extension for such law when \( q' < 1 \) is employed. Note also that \( \exp_{q'}(-x) \) reduces to the usual exponential function, \( \exp(-x) \), in the limit \( q' \to 1 \). In addition, Eq. (4) satisfies an anomalous decay equation,

\[
\frac{d}{dx} \left( \frac{N(x)}{N_0} \right) = -a \left( \frac{N(x)}{N_0} \right)^{q'},
\]

independently of the \( q' \) value. Since this equation reduces to the usual decay one in the limit \( q' \to 1 \), the parameter \( q' \) can be interpreted as a measure of how anomalous the decay is. These aspects put the Zipf-Mandelbrot law in a broad context, motivating us to employ the generalized Tsallis exponential, Eq. (4), instead of the Zipf-Mandelbrot form to study the city population distribution.

The cumulative distribution, for \( 1 < q' < 1.5 \), is

\[
r(x) = r_0 \left[ 1 - \left( \frac{1-q}{q} ax \right)^{1/(1-q)} \right],
\]

where \( r_0 = N_0 q/\alpha \), and \( q = (2-q')^{-1} \). Usually, to compare this cumulative distribution with that obtained from data, it is employed a log-log plot. Here, we introduce another possible way to analyze data by using a generalized mono-log plot based on the generalized logarithm function, \( \ln_q(x) \equiv (x^{1-q} - 1)/(1-q) \). This generalized function arises naturally in the framework of Tsallis statistics [24,25] and reduces to the usual logarithm, \( \ln(x) \), for \( q \to 1 \). It is easy to verify that the plot of \( \ln_q(r(x)) \) versus \( x \) leads to a straight line. So, if the data are well described by the distribution (4), we can obtain the \( q \)-value that gives the best linear fit in the generalized mono-log plot, independently of other parameters.

![FIG. 2. Fit of cumulative distribution for all cities in USA. The parameters are \( q = 1.7 \), \( r_0 = 2919.4 \) and \( a = 0.00008 \). The coefficient of determination in non-linear fit is \( R^2 = 0.99 \).
Inset plot: generalized mono-log plot for American cities.
](image)

Here we used this generalized mono-log plot analysis and we found that \( q \approx 1.7 \) gives a good adjustment to all American and Brazilian cities. Inset plots of Fig. (4) and (5) show this adjust for American and Brazilian cities respectively. Note that in Fig. (3) the two biggest cities are above the straight line formed by all other cities. This fact is known as “king” effect [20,27], and occurs because a few cities in some of the countries, by a specific cause
Comparing the deviation \( \Delta p \) just adjusted by a more, when one deals with a distribution that can be adapted, results, Tsallis statistics and anomalous decay. Furthermore, when one deals with a distribution that can be adjusted by a \( q \)-exponential, the generalized mono-log plot introduced here gives a practical way to determine the \( q \) value, independently of other parameters of the distribution.

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