Thermodynamic uncertainty relation for time-dependent driving

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Thermodynamic uncertainty relations yield a lower bound on entropy production in terms of the mean and fluctuations of a current. We derive its general form for systems under arbitrary time-dependent driving from arbitrary initial states and extend this relation beyond currents to state variables. The quality of the bound is discussed for various types of observables using simple model systems. Since the input for evaluating these bounds does not require specific knowledge of the system or its coupling to the time-dependent control, they should become widely applicable tools for thermodynamic inference in time-dependently driven systems.

Introduction. In a rough classification of non-equilibrium systems, one can distinguish non-equilibrium steady states (NESSs), periodically driven systems and systems relaxing into equilibrium or a NESS from the vast class of systems that are driven in some time-dependent way starting from an arbitrary initial state. A common characteristic for all these classes is the fact that they inevitably lead to entropy production which is arguably the most characteristic feature that separates non-equilibrium from thermal equilibrium. Without having detailed knowledge of the system, however, it is not easy to determine quantitively the entropy production associated with an experimentally explored non-equilibrium process beyond the linear response regime.

The Harada-Sasa relation as one prominent tool for such a quantitative inference requires to measure the response of a NESS to an external perturbation [1]. It has successfully been applied to, e.g., molecular motors [2] and living cells [3]. Alternatively, from the measurement of currents in phase space the entropy production can be inferred provided the relevant phase space is indeed accessible which, in complex systems, is a quite stringent requirement [4, 5]. Another strategy is to exploit operationally accessible lower bounds on entropy production that do not require access to all relevant degrees of freedom like the one based on the temporal asymmetry of fluctuating trajectories [6–10].

For a NESS, a lower bound on entropy production that can be obtained from the observation of any current and its fluctuations has recently been established [11–14]. This so-called thermodynamic uncertainty relation (TUR) holds for any system that, on possibly some deeper unobserved level, obeys a time-continuous Markovian dynamics on discrete states or an overdamped Markovian dynamics on a continuous configuration space. As one immediate striking consequence, the efficiency of molecular motors can be bounded from above without knowledge of the specific chemo-mechanical cycles that drive the motor by observing the speed and its fluctuations when the motor runs against a controlled external force [15–17].

For periodically driven systems, inferring the entropy production, or at least an upper bound for it is somewhat more complex. There exist variants that either require time-symmetric driving [18] or need input from the time-reversed protocol [19]. In addition, there are a number of more formal versions that cannot easily be applied under experimentally realistic conditions [20–22]. An operationally accessible version for arbitrary periodic driving has recently been found that requires the response of the current to a change of the driving frequency as an additional input [23]. Finally, for systems relaxing either to equilibrium or to a NESS, entropy production can be bounded by measuring the fluctuations of a current and its mean value at the end of the observation time [24, 25].

In this Letter, we present the thermodynamic uncertainty relation for the remaining huge class of time-dependently driven systems mentioned at the very beginning. We will show how by measuring an observable, its fluctuations and its change under speeding up the driving parameter(s) a lower bound on the entropy production can be obtained. The observable needs not to be a current; it could also be, e.g., a binary variable characterizing the state of the system at the final time or the integrated time spent in a subset of states.

The line-up of the genuine uncertainty relations just recalled should be distinguished from related inequalities, called generalized thermodynamic uncertainty relations that are a consequence of the fluctuation theorem [26, 27]. These GTURs typically yield weaker bounds on entropy production than the TURs described above and they become trivial in the long-time limit. A pertinent issue with all these relations is to determine the current or observable that leads to the best bound [28–33].

The discovery of the TUR has inspired the derivation of similar relations not necessarily involving overall entropy production for a variety of systems including the role of finite observation times [34, 35], underdamped dynamics [36–39], ballistic transport between different terminals [40], heat engines [41–43], stochastic field theo-
ries [44], for the response to perturbing fields [45], for observables that are even under time-reversal [46–48], for first-passage times [49, 50] and for arbitrary driving [51]. Last but certainly not least, several works have addressed how to generalize these concepts to the quantum realm, see, e.g., [40, 52–59].

Main result for a current. We consider a system prepared in an arbitrary initial state. This system is then driven through an arbitrary control \( \lambda(t) \) with speed parameter \( v \) from \( t = 0 \) to a final time \( t = T \). As a consequence, the system exhibits a mean current \( J(T, v) \) and corresponding current fluctuations characterized by a diffusion coefficient \( D_J(T, v) \), both defined more precisely below. Our first main result relates these quantities with the mean entropy production rate \( \sigma(T, v) \) in the interval \( T \) through

\[
\left[ J(T, v) + \Delta J(T, v) \right]^2 / D_J(T, v) \leq \sigma(T, v). \tag{1}
\]

In comparison with the ordinary TUR for NESSs [11, 12], there is first the dependence on the speed parameter \( v \), and, second, the crucial additional term \( \Delta J(T, v) \) with differential operator

\[
\Delta \equiv \mathcal{T} \partial_T - v \partial_v \tag{2}
\]

that describes the response of the current with respect to a slight change of the speed of driving \( v \) as well as with respect to the observation time \( T \). Consequently, all quantities entering the left-hand side of eq. (1) are physically transparent and thus provide an operationally accessible lower bound on entropy production. This result is valid for driven overdamped Langevin dynamics of an arbitrary number of coupled degrees of freedom and for driven Markovian systems on a discrete set of states [60].

A first illustration: Moving trap. The role of the additional response term can be illustrated with an overdamped particle dragged by a harmonic trap with stiffness \( k \). The system is initially prepared in equilibrium. The center of the trap is moved from \( \lambda_0 = 0 \) to \( \lambda_T = v T \) in time \( t = T \) with a constant velocity \( v \) leading to a potential

\[
V(x, \lambda(t)) = k[x - \lambda(vt)]^2 / 2 \tag{3}
\]

with protocol \( \lambda(vt) \equiv vt \).

One current of interest in this system is the time-averaged velocity

\[
\nu_T \equiv [x(T) - x(0)] / T \tag{4}
\]

which is still a stochastic quantity. Its mean, \( \nu(T, v) \equiv \langle \nu_T \rangle \), depends obviously on the observation time \( T \) and on the speed of the protocol \( v \) which yields the response \( \Delta \nu(T, v) \).

For a generic current \( J \), the quality of bounds like (1) will be quantified throughout the paper by plotting the quality factor

\[
Q_J \equiv \frac{\left[ J(T, v) + \Delta J(T, v) \right]^2}{D_J(T, v) \sigma(T, v)} \leq 1. \tag{5}
\]

For the moving trap, \( Q_J \) is shown in Fig. 1 for different stiffness parameters \( k \) and different observation times \( T \). The bound (1) becomes strongest for \( T \ll 1/(\mu k) \), i.e., for observation times smaller than the relaxation time. Remarkably, an estimate that yields up to \( \sim 80\% \) of the total entropy production is obtained by just observing the travelled distance of the particle without knowing the strength of the trap. In the long-time limit, the dispersion of the velocity become negligible, while heat is continuously dissipated into the surrounding medium. As a consequence, the quality factor \( Q_J \) decreases monotonically.

Two interesting features of the relation (1) can be illustrated using as current the time-averaged power

\[
P(T, v) = \frac{1}{T} \int_0^T dt \int dx p(x, t; v) \partial_x V(x, \lambda) \partial_t \lambda(vt). \tag{6}
\]

First, due to the Gaussian nature of the work fluctuations, it follows that \( D_P(T, v) = P(T, v) / \beta \). Moreover, the entropy production is bounded from above as \( \beta P(T, v) / \sigma(T, v) \geq 1 \) [60]. Consequently, the TUR for steady-state systems [11, 12] is always violated except in the long-time limit, where the mean power converges to the mean total entropy production rate. In contrast, our result (1) provides a lower bound on the mean total entropy production rate which in this case is obviously quite different from the ordinary TUR.

Second, as shown in Fig. 1, the quality of the bound derived from power becomes better the longer the observation time, reaching 1 for \( T \to \infty \). In conjunction with
the above result for the velocity that is best for $T \to 0$, this finding suggests the strategy to explore, where experimentally accessible, different currents for the same system and to choose the best resulting bound.

General set-up for overdamped Langevin dynamics. We consider a system described by an overdamped Langevin equation for the position $x(t)$ in a thermal environment with inverse temperature $\beta$. The system is driven out of equilibrium by a force $F(x, \lambda(\nu t))$ which depends on an external protocol $\lambda(\nu t)$ that contains a speed parameter $\nu$. The protocol is started at $t = 0$ and runs until $t = T$. The initial state of the system is prepared in an arbitrary distribution $p(x,0)$. The dynamics of the position of the particle is described by an overdamped Langevin equation

$$\partial_t x(t) = \mu F(x(t), \lambda(\nu t)) + \zeta(t),$$

(7)

where $\mu$ denotes the mobility and $\zeta(t)$ is Gaussian white noise with strength $2D \equiv 2\mu/\beta$. Equivalently, the time evolution of the probability density $p(x,t;\nu)$ is determined by the Fokker-Planck equation

$$\partial_t p(x,t;\nu) = -\partial_x j(x,t;\nu),$$

(8)

with the probability current

$$j(x,t;\nu) \equiv [\mu F(x, \lambda(\nu t)) - D\partial_x] p(x,t;\nu).$$

(9)

On the level of individual trajectories, we distinguish state variables from (still fluctuating) currents. Specifically, given a function $a(x, \lambda)$, we define an instantaneous state variable as

$$a_T \equiv a(x(T), \lambda(\nu T))$$

(10)

which depends on the final value of position and control. Another observable is its time-averaged variant that reads

$$A_T \equiv \frac{1}{T} \int_0^T dt \, a(x(t), \lambda(\nu t)).$$

(11)

The ensemble average of these stochastic quantities will be denoted by $\langle a(T, \nu) \rangle \equiv \langle a_T \rangle$ and $\langle A(T, \nu) \rangle \equiv \langle A_T \rangle$, where we make the dependence on the two crucial parameters explicit.

For time-dependently driven systems there exist two kinds of currents. Both are odd under time-reversal. The first type of current is called jump current and is of the form

$$J_T^I = \frac{1}{T} \int_0^T dt \, d^I(x(t), \lambda(\nu t)) \circ \dot{x}(t).$$

(12)

Here, $\circ$ denotes the Stratonovich product. The second type is a state current given by

$$J_T^II = \frac{1}{T} \int_0^T dt \, d^{II}(x(t), \lambda(\nu t)).$$

(13)

For jump currents, $d^I(x(t), \lambda(\nu t))$ is an arbitrary increment, whereas for state currents

$$d^{II}(x(t), \lambda(\nu t)) \equiv \dot{x}(t)\partial_\nu b(x(t), \lambda)|_{\lambda=\lambda(\nu t)}$$

(14)

involves the derivative of a state function $b(x, \lambda)$ with respect to the time-dependent driving in contrast to the quantity defined in eq. (11).

The mean value of the first type is given by

$$J^I(T, \nu) \equiv \langle J^I_T \rangle = \frac{1}{T} \int_0^T dt \int dx \, d^I(x, \lambda(\nu t)) j(x,t;\nu)$$

(15)

and that of the second by

$$J^{II}(T, \nu) \equiv \langle J^{II}_T \rangle = \frac{1}{T} \int_0^T dt \int dx \, d^{II}(x, \lambda(\nu t)) p(x,t;\nu).$$

(16)

A prominent example for the first type is the mean rate of entropy production in the medium [61]

$$\sigma_m(T, \nu) \equiv \frac{1}{T} \int_0^T dt \int dx \, \beta F(x(\nu t), \lambda(\nu t)) j(x,t;\nu)$$

(17)

with increment $d^I(x, \lambda) = \beta F(x(t), \lambda(\nu t))$. The mean total entropy production rate

$$\sigma_{\text{tot}}(T, \nu) \equiv \frac{1}{T} \int_0^T dt \int dx \, \frac{j^2(x,t;\nu)}{Dp(x,t;\nu)}$$

(18)

contains additionally the entropy production rate of the system [61]. The power applied to a system as in eq. (6) belongs to the second type of currents and is obtained by choosing $b(x, \lambda) = V(x, \lambda)$, where $V(x, \lambda)$ is an external potential.

Fluctuations of all these observables can be quantified by the effective diffusion coefficient

$$D_X(T, \nu) \equiv T \left( \langle X^2_T \rangle - \langle X_T \rangle^2 \right) / 2$$

(19)

and $X_T \in \{a_T, A_T, J^{II}_T \}$. For both types of current observables defined in eqs. (12) and (13), the TUR (1) holds true [60].

Uncertainty relation for state variables. Our second main result is a thermodynamic uncertainty relation for end-point and time-integrated state observables as defined in eqs. (10) and (11). For both types of observables, it reads [60]

$$\frac{[\Delta X(T, \nu)]^2}{D_X(T, \nu)} \leq \sigma(T, \nu),$$

(20)

where $X(T, \nu) \in \{a(T, \nu), A(T, \nu)\}$. Applied to the end-point observable, this relation shows that a lower bound for the mean total entropy production rate can be obtained by just observing the final state of the system. There is neither information required about the initial distribution nor information about the forces acting on
the particle. It is especially useful for finite-time or relaxation processes where the total entropy production is not necessarily time-extensive.

Generalization to multiple control parameters and discrete states. Our two main results (1) and (20) hold not only for overdamped Langevin systems but also for systems with discrete states. Moreover, for both types of dynamics, they hold if the system is driven by a set of control parameters \( \{\lambda_{\alpha}(v_{\alpha}t)\} \). In this case, the operator defined eq. (2) must be replaced by [60]

\[
\Delta = \mathcal{T} \partial_{\mathcal{T}} - \sum_{\alpha} v_{\alpha} \partial_{v_{\alpha}},
\]

For a simple paradigmatic illustration, we consider a two-state system initially prepared in equilibrium and driven through time-dependent energy levels

\[
E_i(\lambda_\alpha) \equiv E_0^i \left[ 1 - \exp(-v_{\alpha} t) \right],
\]

with \( v_{\alpha} \) the speed control parameter where \( \alpha = i \) and \( E_0^i \) the amplitude of driving for state \( i \in \{1,2\} \). We choose the rates between two state \( i \) and \( j \) as

\[
k_{ij}(\lambda_t) = k_0 \exp\left\{ -0.5\beta_i [E_j(\lambda_t) - E_i(\lambda_t)] \right\},
\]

with \( k_0 \) as basic time-scale. In this model, the protocol depends on two speed parameter, i.e., \( \lambda_t \equiv \{\lambda_1(v_1 t), \lambda_2(v_2 t)\} \). We keep the final value of the protocol fixed, i.e., \( v_1 T = \text{const} \) and \( v_2 T = \text{const} \).

For three different observables, we consider the quality of the resulting bound. One estimate for the total entropy production using eq. (1) is obtained by observing the current between state 1 and 2

\[
\nu_{12}^T \equiv \frac{\Delta}{\mathcal{T}} \left[ m_{12}(\mathcal{T}) - m_{21}(\mathcal{T}) \right],
\]

where the variable \( m_{ij}(\mathcal{T}) \) counts the total number of transitions from state \( i \) to state \( j \). Two more bounds are obtained using \( \alpha(i,j,\lambda) = \delta_{i,j} \) in eqs. (10) and (11), which corresponds to the characteristic function of state 2 either at the end of the observation time or time-averaged [62]. The first choice corresponds to the probability to be in state 2 and the latter one to the fraction of time the system spends in this state. We denote the corresponding quality factors by \( \mathcal{Q}_a \) and \( \mathcal{Q}_\lambda \), respectively. The quality factors obtained from monitoring the mean, the fluctuation and the response of these three observables are shown in Fig. 2. For fast driving \( \mathcal{T} \ll 1 \), the current observable \( \nu_{12}^T \) yields the best estimate for the total entropy production, whereas for intermediate speeds of driving \( \mathcal{T} \sim 1/k_0 \), the observable based on the final state yields the best bound. In the limit of quasi-static driving, the fraction of time spent in state 2 yields up to 90\% of the total entropy production rate. Throughout the whole range of driving speeds, the bounds based on these three observables yield at least 60\% of the total entropy production rate.

Concluding perspective. We have derived a universal thermodynamic uncertainty relation that holds for current and for state variables in systems that are time-dependently driven from an arbitrary initial state over any finite time-interval. In all cases, the relevant mean and fluctuations yield a lower bound on the overall entropy production. As demonstrated with simple paradigmatic examples, depending on the conditions the observables leading to the relative best bound may change. For observables based on currents, our relation becomes the established ones for the very special cases of time-independent driving, of periodic driving and of relaxation at constant control parameters as summarized in table I.

With these relations we have provided universally applicable tools that will allow thermodynamic inference in time-dependently driven systems. We emphasize that it
is neither necessary to know the precise coupling between the system and the control nor to know the interactions within the system. It suffices that the experimentalist can change the overall speed of the control slightly and measure the resulting response of an observable. These rather weak demands should facilitate the application to systems beyond the obvious colloidal particles and single molecules manipulated with time-dependent optical traps. Finally, as a challenge to theory, it will be intriguing to explore whether and how these relations can be extended to time-dependently driven open quantum systems.

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[62] Note that $a(i, \lambda)$ is a straightforward generalization of the state variable $a(x, \lambda)$ defined in eq. (10) to systems with discrete degrees of freedom, where the continuous state $x$ is replaced by the discrete state $i$ and the integral $\int dx$ becomes a sum $\sum_i$. 


Supplemental Material for "Thermodynamic uncertainty relation for time-dependent driving"

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This supplemental material contains three sections. In Sect. I, we derive the main results for current-like observables (eq. (1) in the main text) and for state variables (eq. (20) in the main text) for systems with continuous degrees of freedom. Sect. II contains the derivation of these main results for systems with discrete degrees of freedom. Sect. III provides more details for the example of the moving trap.

I. DERIVATION OF THE MAIN RESULTS I: CONTINUOUS DEGREES OF FREEDOM

Setup

We derive the main results for time-dependently driven systems with \( N_x \) interacting continuous degrees of freedom. The coordinate vector \( \mathbf{x} \equiv \{x_1, ..., x_{N_x}\} \) describes the state of the system which is driven by multiple protocols \( \lambda_i \equiv \lambda_i(\{v_\alpha\}) \equiv \{\lambda_1(v_1t), ..., \lambda_{N_x}(v_\alpha t, t)\} \) with \( N_\lambda \) speed parameter \( v_\alpha \) and \( \alpha \in [1, N_\lambda] \). The dynamics obeys the Langevin equation with multiplicative noise

\[
\dot{x}(t) = B(x(t), \lambda_i) + \nabla D(x(t), \lambda_i) - [G^T(x(t), \lambda_i) \nabla] G(x(t), \lambda_i) + \sqrt{2} G(x(t), \lambda_i) \circ \zeta(t). \tag{1}
\]

Here, we used the Stratonovich convention, where \( \circ \) denotes the Stratonovich product, \( (\cdot)^T \) denotes transposition, \( \nabla \equiv \{\partial_{x_1}, ..., \partial_{x_{N_x}}\} \) is the Nabla operator and \( \zeta(t) \equiv \{\zeta_1(t), ..., \zeta_{N_x}(t)\} \) is a Gaussian white noise vector describing the random forces with mean and correlations

\[
\langle \zeta_i(t) \rangle = 0, \tag{2}
\]

\[
\langle \zeta_i(t) \zeta_j(t') \rangle = \delta_{ij} \delta(t - t'). \tag{3}
\]

Furthermore, using the \( N_x \times N_x \) matrix \( G(x, \lambda_i) \) we define the state- and time-dependent symmetric diffusion matrix by \( D(x, \lambda_i) \equiv G(x, \lambda_i) G^T(x, \lambda_i) \). The diffusion matrix obeys the Einstein relation, i.e., \( D(x, \lambda_i) = \mu(x, \lambda_i) / \beta \), where \( \mu(x, \lambda_i) \) denotes the \( N_x \times N_x \) mobility matrix and \( \beta \) is the inverse temperature of the heat bath. The drift vector

\[
B(x, \lambda_i) = \mu(x, \lambda_i) F(x, \lambda_i) \equiv \mu(x, \lambda_i) (-\nabla V(x, \lambda_i) + f(x, \lambda_i)) \tag{4}
\]

contains the forces \( F(x, \lambda_i) \) driving the system out of equilibrium. They comprise a conservative force generated by a potential \( V(x, \lambda_i) \) and a non-conservative force \( f(x, \lambda_i) \). Both, the drift and diffusion term are controlled by the protocol \( \lambda_i \). The additional drift term \( -\nabla^T D(x, \lambda_i) - [G^T(x, \lambda_i) \nabla] G(x, \lambda_i) \) arises due to the Stratonovich convention and makes sure that a non-driven system evolves to the Boltzmann distribution for \( t \to \infty \) [1].

Equivalently to the Langevin equation (1), we can describe the dynamics for the probability density \( \rho(x, t; \{v_\alpha\}) \) by the Fokker-Planck equation [2]

\[
\partial_t \rho(x, t; \{v_\alpha\}) = -\nabla j(x, t; \{v_\alpha\}) \tag{5}
\]

with probability current vector

\[
j(x, t; \{v_\alpha\}) \equiv (B(x, \lambda_i) - D(x, \lambda_i) \nabla) \rho(x, t; \{v_\alpha\}). \tag{6}
\]

Likewise, the probability density for an individual trajectory \( x(t) \) of length \( T \) is given by the path weight

\[
\mathcal{P}[x(t)] \equiv N \exp \left(-S[x(t), \lambda_i]\right) \rho(x(0), 0) \tag{7}
\]
for an arbitrary initial condition \( p(x(0),0) \) and the action

\[
S[x(t),\lambda_t] \equiv \frac{1}{4} \int_0^T dt \left( \dot{x}(t) - B(x(t),\lambda_t) \right)^T D(x(t),\lambda_t)^{-1} (\dot{x}(t) - B(x(t),\lambda_t)) + \frac{1}{2} \int_0^T dt \nabla B(x(t),\lambda_t).
\]

Here, \( D(x(t),\lambda_t)^{-1} \) denotes the inverse of the diffusion matrix and

\[
\mathcal{N} \equiv \left( \prod_{l=1}^{N_t} \frac{1}{\sqrt{4\pi \text{det} \left[ G(x(t_l),\lambda_{t_l}) \right]}} \right)
\]

is a normalization factor with respect to the measure of integration

\[
\int d[x(t)] \equiv \prod_{l=0}^{N_t} \int dx(t_l),
\]

i.e., \( \int d[x(t)] \mathcal{P}[x(t)] = 1 \), with \( \text{det} [\cdot] \) denotes the determinant of a matrix. The path weight in eq. (7) is well defined for a suitable discretization in time, i.e., the trajectory is sliced into \( N_t \) discrete values \( \{x(t_0), x(t_1), ..., x(t_{N_t})\} \) with \( t_l \equiv l\Delta t \), where \( \Delta t \) is a small enough time step and chosen such that \( N_t \Delta t = T \) is fulfilled. The last term in eq. (8) arises from a Stratonovich discretization scheme [1].

**Bound on the diffusion coefficient**

For deriving our main result, we use a method introduced by Dechant and Sasa [3] which bounds the cumulant generating function or short generating function

\[
\lambda(z) \equiv \frac{1}{T} \langle \exp(z T X_T) \rangle \equiv \int d[x(t)] \mathcal{P}[x(t)] \exp(z T X_T [x(t)])
\]

for a fluctuating observable \( X_T = X_T [x(t)] \in \{a_T, A_T, J_T^{(1)}\} \) by introducing an auxiliary path weight \( \mathcal{P}^\dagger [x(t)] \) describing an auxiliary dynamics obeying a Langevin equation of type (1). The first two derivatives of \( \lambda(z) \) at \( z = 0 \) yield the mean and diffusion coefficient of \( X_T \), i.e.,

\[
\lambda'(z)|_{z=0} = \langle X_T \rangle,
\]

\[
\lambda''(z)|_{z=0} = 2D_X(T, \{v_\alpha\}).
\]

Writing the expectation value in (11) in terms of the auxiliary path weight \( \mathcal{P}^\dagger [x(t)] \) and using Jensen’s inequality yields the lower bound

\[
\lambda(z) = \int d[x(t)] \frac{\mathcal{P}^\dagger [x(t)]}{\mathcal{P}[x(t)]} \exp(z T X_T [x(t)]) \mathcal{P}^\dagger [x(t)] \geq z \langle X_T \rangle^\dagger - \frac{1}{T} \left\langle \ln \left( \frac{\mathcal{P}^\dagger [x(t)]}{\mathcal{P}[x(t)]} \right) \right\rangle^\dagger
\]

(14)

on the generating function, where \( \langle \cdot \rangle^\dagger \) denotes the expectation value in the auxiliary dynamics. For a suitable choice of the path weight \( \mathcal{P}^\dagger [x(t)] \) the bound (14) implies a bound on the diffusion coefficient as we will show below.

We require the auxiliary path weight \( \mathcal{P}^\dagger [x(t)] \) to follow the Fokker-Planck equation

\[
\partial_t p^\dagger (x,t;\{v_\alpha^\dagger\}) = - \nabla j^\dagger (x,t;\{v_\alpha^\dagger\})
\]

(15)

with auxiliary density \( p^\dagger (x,t;\{v_\alpha^\dagger\}) \) and auxiliary current

\[
 j^\dagger (x,t;\{v_\alpha^\dagger\}) = \left( B^\dagger (x,\lambda_t^\dagger) - D^\dagger (x,\lambda_t^\dagger) \nabla \right) p^\dagger (x,t;\{v_\alpha^\dagger\}).
\]

Here, the auxiliary diffusion process is generated by the drift vector \( B^\dagger (x,\lambda_t^\dagger) \) and diffusion matrix \( D^\dagger (x,\lambda_t^\dagger) \), where we introduced the auxiliary protocol \( \lambda_t^\dagger \equiv \lambda_t^\dagger (\{v_\alpha^\dagger\}) \equiv \{\lambda_t^\dagger (v_\alpha^\dagger t), ..., \lambda_t^\dagger (v_{N_\lambda}^\dagger t)\} \) with auxiliary speed parameter \( v_\alpha^\dagger \) and \( \alpha \in [1, N_\lambda] \).

We choose the auxiliary dynamics such that it describes the original dynamics that evolves slower or faster in time, i.e., \( v_\alpha \to v_\alpha (1 + \epsilon) \) and \( t \to t/(1 + \epsilon) \) and is driven with the same protocol functions \( \{\lambda_\alpha\} \). Here, \( \epsilon = O(z) \) is assumed
to be a small parameter, i.e., the auxiliary dynamics is in a linear response regime around the original dynamics [4]. Hence, the protocol and the speed parameter of the auxiliary dynamics read
\[ \lambda^\dagger_i = \lambda_i/(1 + \epsilon), \quad \{v^\dagger_\alpha\} = \{\lambda_i(v^1 t), \ldots, \lambda_N(v^1 t)\} = \lambda_i, \]  
\[ v^\dagger_\alpha = v_\alpha(1 + \epsilon), \]
i.e., the protocol is the same as for the original dynamics. The auxiliary density and current are consequently given by
\[ p^\dagger(x, t; \{v^\dagger_\alpha\}) \equiv p(x, t/(1 + \epsilon); \{v^\dagger_\alpha\}), \]  
\[ j^\dagger(x, t; \{v^\dagger_\alpha\}) \equiv (1 + \epsilon)j(x, t/(1 + \epsilon); \{v^\dagger_\alpha\}), \]
where we assume that both processes start in the same initial condition \( p(x(0), 0) \). Furthermore, we assume that the initial condition does not depend on the speed parameter \( v_\alpha \). If this was not the case, like, e.g., for a system in a periodic steady-state, an additional marginal term would occur for finite observation times (see [5]). The above introduced auxiliary dynamics describing a “time-scaled” diffusion process can be considered as a process generated by an additional drift vector and the original diffusion matrix
\[ B^\dagger(x, \lambda^\dagger_i) = B(x, \lambda_i) + \epsilon Y(x, t/(1 + \epsilon); \{v^\dagger_\alpha\}), \]  
\[ D^\dagger(x, \lambda^\dagger_i) = D(x, \lambda_i). \]
The drift term \( B^\dagger(x, \lambda^\dagger_i) \) contains the small additional force
\[ \epsilon Y(x, t/(1 + \epsilon); \{v^\dagger_\alpha\}) = \epsilon j(x, t/(1 + \epsilon); \{v^\dagger_\alpha\})/p(x, t/(1 + \epsilon); \{v^\dagger_\alpha\}) \]
and is called a virtual perturbation [4]. The diffusion matrix \( D(x, \lambda_i) \) is the same as for the original process such that the normalization constant (9) is identical.

Next, we insert the auxiliary path weight
\[ \mathcal{P}^\dagger[x(t)] \equiv \mathcal{N} \exp \left( -\mathcal{S}^\dagger[x(t), \lambda_i] \right) \]  
with action
\[ \mathcal{S}^\dagger[x(t), \lambda_i] = \frac{1}{4} \int_0^T dt \left( \dot{x}(t) - B^\dagger(x(t), \lambda_i) \right)^T D(x(t), \lambda_i)^{-1} \left( \dot{x}(t) - B^\dagger(x(t), \lambda_i) \right) + \frac{1}{2} \int_0^T dt \nabla B^\dagger(x(t), \lambda_i), \]
and the auxiliary drift term defined in eq. (21) into eq. (14) and obtain the bound
\[ \lambda(z) \geq z \langle X_T \rangle^\dagger - \frac{c^2}{4} \sigma(T^\dagger, \{v^\dagger_\alpha\}) \]
with the total entropy production rate
\[ \sigma(T^\dagger, \{v^\dagger_\alpha\}) = \frac{1}{T^\dagger} \int_0^{T^\dagger} dt' j(x, t'; \{v^\dagger_\alpha\})^T D(x, \lambda_i) \left( \{v^\dagger_\alpha\} \right)^{-1} j(x, t'; \{v^\dagger_\alpha\})/p(x, t'; \{v^\dagger_\alpha\}) \]
in a system with observation time \( T^\dagger \equiv T/(1 + \epsilon) \) and speed parameter \( v^1 = v_\alpha(1 + \epsilon) \), where we used the substitution \( t' = t/(1 + \epsilon) \).

We take the limit \( \epsilon \to 0 \) and calculate the leading orders of the two terms in eq. (26). The first term in eq. (26) depends on the observable \( X_T \) and is given by one of the four expectation values depending on the choice of \( X_T \in \{a_T, A_T, J^\dagger_T, J^\ddagger_T\} \)
\[ \langle a_T \rangle^\dagger = a(T^\dagger, \{v^\dagger_\alpha\}) = \int dx a(x, \lambda_i) p(x, T^\dagger; \{v^\dagger_\alpha\}), \]  
\[ \langle A_T \rangle^\dagger = A(T^\dagger, \{v^\dagger_\alpha\}) = \frac{1}{T^\dagger} \int_0^{T^\dagger} dt' \int dx a(x, \lambda_i) p(x, t'; \{v^\dagger_\alpha\}), \]  
\[ \langle J^\dagger_T \rangle^\dagger = J^\dagger(T^\dagger, \{v^\dagger_\alpha\}) = \frac{1}{T^\dagger} \int_0^{T^\dagger} dt' \int dx (1 + \epsilon) d^\dagger(x, \lambda_i) \cdot j(x, t'; \{v^\dagger_\alpha\}), \]  
\[ \langle J^\ddagger_T \rangle^\dagger = J^\ddagger(T^\dagger, \{v^\dagger_\alpha\}) = \frac{1}{T^\dagger} \int_0^{T^\dagger} dt' \int dx (1 + \epsilon) d^\ddagger(x, \lambda_i) p(x, t'; \{v^\dagger_\alpha\}), \]
where the increments
\[ d^{II}(x, \lambda_t) \equiv \partial_t \lambda_t \cdot \nabla \lambda b(x, \lambda_t), \quad (32) \]
which involve the time-derivative of a state function \( b(x, \lambda_t) \), and vector \( d^I(x, \lambda_t) \) are arbitrary. Calculating the leading order in \( \epsilon = \mathcal{O}(\varepsilon) \) in eq. (26) via eqs. (27)–(31) and optimizing with respect to \( \epsilon \) leads to a local quadratic bound on the generating function that implies with eqs. (12) and (13) our main results, i.e., the bounds on the diffusion coefficients
\[
D_{ij}(\mathcal{T}, \{v_\alpha\}) \geq \frac{[J(\mathcal{T}, \{v_\alpha\}) + \Delta J(\mathcal{T}, \{v_\alpha\})]^2}{\sigma(\mathcal{T}, \{v_\alpha\})}, \quad (33)
\]
\[
D_{\lambda,\alpha}(\mathcal{T}, \{v_\alpha\}) \geq \frac{[\Delta A_{\alpha}(\mathcal{T}, \{v_\alpha\})]^2}{\sigma(\mathcal{T}, \{v_\alpha\})}, \quad (34)
\]
with \( J(\mathcal{T}, \{v_\alpha\}) \in \{J^I(\mathcal{T}, \{v_\alpha\}), J^{II}(\mathcal{T}, \{v_\alpha\})\} \), \( A_{\alpha}(\mathcal{T}, \{v_\alpha\}) \in \{a(\mathcal{T}, \{v_\alpha\}), A(\mathcal{T}, \{v_\alpha\})\} \) and the differential operator
\[
\Delta \langle X_T \rangle \equiv \left( T \partial_T - \sum_\alpha v_\alpha \partial_{v_\alpha} \right) \langle X_T \rangle. \quad (36)
\]
These relations prove the inequalities eqs. (1) and (20) in the main text for continuous degrees of freedom.

**II. DERIVATION OF THE MAIN RESULT: DISCRETE DEGREES OF FREEDOM**

**Setup**

Here, we consider systems with discrete degrees of freedom with \( N_\alpha \) states. These systems obey a Markovian dynamics described by the master equation
\[
\partial_t p_i(t; \{v_\alpha\}) = - \sum_j j_{ij}(t; \{v_\alpha\}) \quad (37)
\]
with probability current
\[
j_{ij}(t; \{v_\alpha\}) \equiv p_i(t; \{v_\alpha\}) k_{ij}(\lambda_t) - p_j(t; \{v_\alpha\}) k_{ji}(\lambda_t). \quad (38)
\]
Here, \( k_{ij}(\lambda_t) \) denotes the transition rate from state \( i \) to state \( j \) at time \( t \), when the system is driven by the protocols \( \lambda_t \equiv \lambda_t(\{v_\alpha\}) \equiv \{\lambda_1(\alpha_t), \ldots, \lambda_{N_\lambda}(v_{N_\lambda}, t)\} \) with \( N_\lambda \) speed parameter \( v_{N_\lambda} \) and \( \alpha \in [1, N_\lambda] \). In order to model the system thermodynamically consistent the rates \( k_{ij}(\lambda_t) \) must fulfill the so-called local detailed balance condition [6]
\[
\frac{k_{ij}(\lambda_t)}{k_{ji}(\lambda_t)} = \exp(-\beta \Delta E_{ij}(\lambda_t) + A_{ij}(\lambda_t)), \quad (39)
\]
where \( \beta \) is the inverse temperature of the heat bath, \( \Delta E_{ij}(\lambda_t) \equiv E_j(\lambda_t) - E_i(\lambda_t) \) is the energy difference between state \( i \) and \( j \) and \( A_{ij}(\lambda_t) \) is a driving affinity, e.g., a non-conservative force which drives the system additionally to the time-dependent energies. Both, the energies and the driving affinities depend on the protocol \( \lambda_t \).

Similar to systems with continuous degrees of freedom, we can define a probability density for a discrete trajectory \( n(t) \) of length \( T \) with initial condition \( p_{n(0)}(0) \)
\[
P[n(t)] = \exp \left( - \int_0^T dt \sum_i r_i(\lambda_t) \delta_{n(t),i} + \int_0^T dt \sum_{ij} \ln[k_{ij}(\lambda_t)] m_{ij}(t) \right) p_{n(0)}(0), \quad (40)
\]
where \( r_i(\lambda_t) \equiv \sum_j k_{ij}(\lambda_t) \) is the escape or exit rate of state \( i \), \( \delta_{n(t),i} \) is a variable that is 1 if state \( i \) is occupied and 0, otherwise, and \( m_{ij}(T) \) counts the total number of transitions from state \( i \) to state \( j \). The time derivative of the latter one
\[
m_{ij}(t) \equiv \sum_i \delta(t - t_{ij}), \quad (41)
\]
depends on the times \( t_{ij} \) at which transitions from \( i \) and \( j \) occur. The path probability (40) is normalized, i.e., \( \sum_{n(t)} P[n(t)] = 1 \), where \( \sum_{n(t)} \) denotes the summation over all paths. The mean values of the variables \( \delta_{n(t),i} \) and \( \bar{m}_{ij}(t) \) describing a trajectory are given by

\[
\langle \delta_{n(t),i} \rangle \equiv \sum_{n(t)} P[n(t)] \delta_{n(t),i} = p_i(t; \{ v_\alpha \}),
\]

(42)

\[
\langle \bar{m}_{ij}(t) \rangle \equiv \sum_{n(t)} P[n(t)] \bar{m}_{ij}(t) = p_i(t; \{ v_\alpha \}) k_{ij}(\lambda_t).
\]

(43)

The analogues of observables \( a_T, A_T \) and \( J^{11}_T \) defined in eqs. (10)–(13) in the main text for systems with discrete degrees of freedom are given by

\[
a_T \equiv a(n(T), \lambda_T),
\]

(44)

\[
A_T \equiv \frac{1}{T} \int_0^T dt a(n(t), \lambda_t),
\]

(45)

\[
J^1_T \equiv \frac{1}{T} \int_0^T dt \sum_{ij} d^1_{ij}(\lambda_t) \bar{m}_{ij}(t),
\]

(46)

\[
J^{11}_T \equiv \frac{1}{T} \int_0^T dt d^{11}(n(t), \lambda_t),
\]

(47)

where \( a(n(t), \lambda_t) \) is an arbitrary state variable, \( d^1_{ij}(\lambda_t) = -d^1_{ji}(\lambda_t) \) are anti-symmetric increments and

\[
d^{11}(n(t), \lambda_t) \equiv \partial_t \lambda_t \cdot \nabla b(n(t), \lambda_t)
\]

(48)

can be written as a time-derivative of a state variable \( b(n(t), \lambda_t) \). The mean values of eqs. (44)-(47) are given by

\[
\langle a_T \rangle \equiv a(T, \{ v_\alpha \}) = \sum_i a(i, \lambda_T) p_i(T, \{ v_\alpha \}),
\]

(49)

\[
\langle A_T \rangle \equiv A(T, \{ v_\alpha \}) = \frac{1}{T} \int_0^T dt \sum_i a(i, \lambda_t) p_i(t; \{ v_\alpha \}),
\]

(50)

\[
\langle J^1_T \rangle \equiv J^1(T, \{ v_\alpha \}) = \frac{1}{T} \int_0^T dt \sum_{i>j} d^1_{ij}(\lambda_t) j_{ij}(t; \{ v_\alpha \}),
\]

(51)

\[
\langle J^{11}_T \rangle \equiv J^{11}(T, \{ v_\alpha \}) = \frac{1}{T} \int_0^T dt \sum_i d^{11}(n(t), \lambda_t) p_i(t; \{ v_\alpha \}).
\]

(52)

**Bound on the diffusion coefficient**

We use the same formalism as above for systems with continuous degrees of freedom to bound the generating function (11) for systems with discrete degrees of freedom. The generating function for an observable \( X_T = X_T[n(t)] \in \{ a_T, A_T, J^{11}_T \} \) is bounded analogously to eq. (14) by introducing an auxiliary process with path weight \( P^{\dagger}[n(t)] \), where the integral \( \int d[\xi(t)] \) is replaced by the summation over all paths \( \sum_{n(t)} \) in eq. (14).

We require the auxiliary path weight \( P^{\dagger}[n(t)] \) to describe a master equation

\[
\partial_t p_i^\dagger(t; \{ v_\alpha^i \}) = -\sum_j j_{ij}^\dagger(t; \{ v_\alpha^i \})
\]

(53)

with probability current

\[
j_{ij}^\dagger(t; \{ v_\alpha^i \}) \equiv p_i^\dagger(t; \{ v_\alpha^i \}) k_{ij}^\dagger(\lambda_t^i) - p_j^\dagger(t; \{ v_\alpha^i \}) k_{ji}^\dagger(\lambda_t^j)
\]

(54)

and auxiliary transition rates \( k_{ij}^\dagger(\lambda_t^i) \). Here, we introduced the auxiliary protocol \( \lambda_t^i \equiv \lambda_t^i \{ v_\alpha^i \} \equiv \{ \lambda_1^i(v_1^i t), ..., \lambda_N^i(v_N^i t) \} \) with auxiliary speed parameter \( v_\alpha^i \) and \( \alpha \in [1, N_\lambda] \). Inserting the auxiliary path weight \( P^{\dagger}[n(t)] \) via definition (40)
with transition rates $k_{ij}^\dagger(\lambda_i^\dagger)$ into the discrete version of eq. (14) yields

$$\lambda(z) \geq z \langle X_T \rangle \frac{dt}{\tau} \sum_{ij} \left( p_i^\dagger (t; \{v_\alpha\}) k_{ij}^\dagger(\lambda_i^\dagger) \ln \left[ \frac{k_{ij}^\dagger(\lambda_i^\dagger)}{k_{ij}(\lambda_i)} \right] - p_i^\dagger (t; \{v_\alpha\}) \left[ k_{ij}^\dagger(\lambda_i^\dagger) - k_{ji}(\lambda_i) \right] \right), \quad (55)$$

where we assume that both processes, i.e., the original and the auxiliary process start with the same initial condition $p_{\alpha(0)}(0)$, which we require to be independent of the speed parameters $\{v_\alpha\}$ (see discussion for systems with continuous degrees of freedom given above).

We choose the rates of the auxiliary process as

$$k_{ij}^\dagger(\lambda_i) \equiv k_{ij}(\lambda_i) (1 + \epsilon [1 - \eta_{ij}(\epsilon, t) \delta]), \quad (56)$$

where $\epsilon = \mathcal{O}(z)$ is a small parameter, $\delta$ is a free parameter which can be chosen as 1 or 0 and

$$\eta_{ij}(\epsilon, t) \equiv \frac{2p_j (t/(1 + \epsilon); \{v_\alpha(1 + \epsilon)\}) k_{ji}(\lambda_i)}{t_{ij}(t/(1 + \epsilon); \{v_\alpha(1 + \epsilon)\})}, \quad (57)$$

where

$$t_{ij}(t; \{v_\alpha\}) \equiv p_i (t; \{v_\alpha\}) k_{ij}(\lambda_i) + p_j (t; \{v_\alpha\}) k_{ji}(\lambda_i) \quad (58)$$

is the average dynamical activity at link $ij$ of the original process. Here, we have chosen the auxiliary protocol and speed parameter according to eqs. (17) and (18). This choice of rates corresponds to a “time-scaled” process with a probability and current given by

$$p_i^\dagger (t; \{v_\alpha\}) \equiv p_i (t/[1 + \epsilon]; \{v_\alpha[1 + \epsilon]\}), \quad (59)$$

$$j_{ij}^\dagger (t; \{v_\alpha\}) \equiv (1 + \epsilon) j_{ij} (t/[1 + \epsilon]; \{v_\alpha[1 + \epsilon]\}). \quad (60)$$

The first term in eq. (55) is given by

$$\langle a_T \rangle^\dagger \equiv a(T, \{v_\alpha\}) = \sum_i a(i, \lambda_T) p_i (T/[1 + \epsilon]; \{v_\alpha[1 + \epsilon]\}), \quad (62)$$

$$\langle A_T \rangle^\dagger \equiv A(T, \{v_\alpha\}) = \frac{1}{\tau} \int_0^T dt \sum_i a(i, \lambda_T) p_i (T/[1 + \epsilon]; \{v_\alpha[1 + \epsilon]\}), \quad (63)$$

$$\langle J_T^I \rangle^\dagger \equiv J^I(T, \{v_\alpha\}) = \frac{1}{\tau} \int_0^T dt \sum_{i>j} (1 + \epsilon) d_{ij}(\lambda_i) j_{ij} (T/[1 + \epsilon]; \{v_\alpha[1 + \epsilon]\}), \quad (64)$$

$$\langle J_T^II \rangle^\dagger \equiv J^{II}(T, \{v_\alpha\}) = \frac{1}{\tau} \int_0^T dt \sum_i (1 + \epsilon) d^{II}(i, \lambda_i) p_i (T/[1 + \epsilon]; \{v_\alpha[1 + \epsilon]\}) \quad (65)$$

depending on the choice of the observable $X_T$. The second term in eq. (55) is given by

$$- \frac{\epsilon^2}{2\tau} \int_0^T dt \sum_{ij} \left[ p_i (t; \{v_\alpha\}) k_{ij}(\lambda_i) + p_j (t; \{v_\alpha\}) k_{ji}(\lambda_i) (1 - 2\delta) \right]^2 \frac{t_{ij}(t; \{v_\alpha\})}{t_{ij}(t; \{v_\alpha\})} + \mathcal{O}(\epsilon^3). \quad (66)$$

An expansion for small $\epsilon$ of the r.h.s of eq. (55) and an optimization with respect to $\epsilon$ leads to the bounds on the diffusion coefficient

$$D_{J}(T, \{v_\alpha\}) \geq \frac{[J(T, \{v_\alpha\}) + \Delta J(T, \{v_\alpha\})]^2}{C_\delta(T, \{v_\alpha\})}, \quad (67)$$

$$D_{A^\alpha}(T, \{v_\alpha\}) \geq \frac{[\Delta A_\alpha(T, \{v_\alpha\})]^2}{C_\delta(T, \{v_\alpha\})}, \quad (68)$$

with the observables $J(T, \{v_\alpha\}) \in \{J^I(T, \{v_\alpha\}), J^{II}(T, \{v_\alpha\})\}$, $A_\alpha(T, \{v_\alpha\}) \in \{a(T, \{v_\alpha\}), A(T, \{v_\alpha\})\}$ and the cost term

$$C_\delta(T, \{v_\alpha\}) \equiv \frac{2}{\tau} \int_0^T dt \sum_{ij} \left[ p_i (t; \{v_\alpha\}) k_{ij}(\lambda_i) + p_j (t; \{v_\alpha\}) k_{ji}(\lambda_i) (1 - 2\delta) \right]^2 \frac{t_{ij}(t; \{v_\alpha\})}{t_{ij}(t; \{v_\alpha\})}. \quad (69)$$
For the choice $\delta = 1$ the cost term in eq. (69) is equal or smaller than the total entropy production rate
\[\sigma(T, \{v_\alpha\}) \equiv \frac{1}{T} \int_0^T dt \sum_{i>j} j_{ij}(t; \{v_\alpha\}) \ln \left[ \frac{p_i(t; \{v_\alpha\}) k_{ij}(\lambda_t)}{p_j(t; \{v_\alpha\}) k_{ji}(\lambda_t)} \right],\] (70)
i.e.,
\[C_{\delta=1}(T, \{v_\alpha\}) \leq \sigma(T, \{v_\alpha\}),\] (71)
which can be shown by using the log-mean inequality $(\gamma_1 - \gamma_2) \ln(\gamma_1 / \gamma_2) \geq 2(\gamma_1 - \gamma_2)^2/(\gamma_1 + \gamma_2)$ for arbitrary $\gamma_1, \gamma_2 > 0$ (see e.g. [4]). Together with eqs. (67) and (68) the inequality (71) proves our main results eqs. (1) and (20) in the main part for systems with discrete degrees of freedom.

### Bound on the total average dynamic activity

Additionally, we obtain bounds on the total average dynamic activity
\[A \equiv \frac{1}{T} \int_0^T dt \sum_{i>j} t_{ij}(t; \{v_\alpha\})\] (72)
from eqs. (67) and (68) which follows by choosing $\delta = 0$ in eq. (69), i.e., $C_{\delta=0}(T, \{v_\alpha\}) = A(T, \{v_\alpha\})$.

Finally, we show that the above derived bounds on the average dynamic activity $A(T, \{v_\alpha\})$ can also be applied to jump observables of type
\[\chi_T \equiv \frac{1}{T} \int_0^T dt \sum_{ij} d_{ij}^{III}(\lambda_t) \dot{m}_{ij}(t),\] (73)
where $d_{ij}^{III}$ are arbitrary increments that do not have necessarily to be symmetric or anti-symmetric for which we use $\delta = 0$ in our ansatz (56). The first term in eq. (55) is given by
\[\langle \chi_T \rangle^\dagger = \frac{1}{T} \int_0^T dt(1 + \epsilon)d_{ij}^{III}(\lambda_t)p_i(t/1 + \epsilon; \{v_\alpha[1 + \epsilon]\}) k_{ij}(\lambda_t)\] (74)
An expansion for small $\epsilon$ of the r.h.s of eq. (55) and an optimization with respect to $\epsilon$ leads to
\[D_\chi(T, \{v_\alpha\}) \equiv \frac{[\chi(T, v) + \Delta \chi(T, v)]^2}{2A(T, \{v_\alpha\})},\] (75)
where
\[D_\chi(T, \{v_\alpha\}) \equiv T \left( \langle \chi_T^2 \rangle - \langle \chi_T \rangle^2 \right)/2\] (76)
is the diffusion coefficient of jump observable $\chi_T$ and $A(T, \{v_\alpha\})$ is the total average dynamical activity defined in eq. (72). The bound in eq. (76) is a generalization of bounds obtained in [7] for steady-state systems, in [8] for relaxation processes and in [5] for periodically driven systems to arbitrary time-dependently driven systems.

### III. MOVING TRAP

In this section we derive expressions for the mean particle current $\nu(T, v)$, the mean power $P(T, v)$, their diffusion coefficients $D_\nu, P$ and the total entropy production rate $\sigma(T, v)$ for the moving trap model discussed in the main text.
Fokker-Planck equation and solution

The Fokker-Planck equation for the moving trap reads
\[ \partial_t p(x,t;v) = -\partial_x (-\mu_k [x - \lambda(t)] - D \partial_x) p(x,t;v), \] (77)
with protocol \( \lambda(t) = vt \) and diffusion constant \( D \equiv \mu/\beta \). The solution of eq. (77) is a Gaussian distribution
\[ p(x,t;v) \equiv \frac{1}{2\pi y_t^2} \exp\left(-\frac{[x - c_t]^2}{2y_t^2}\right) \] (78)
with mean and variance
\[ c_t \equiv c(t;v) \equiv \langle x(t) \rangle, \] (79)
\[ y_t^2 \equiv y^2(t;v) \equiv \langle x^2(t) \rangle - \langle x(t) \rangle^2. \] (80)
In general, both, mean and variance depend on the speed \( v \). The system is initially prepared in equilibrium, i.e., \( c_0 = 0 \) and \( y_t^2(0) = 1/(\beta k) \). Consequently, mean and variance are given by
\[ c_t = vt - \frac{v}{\mu k} \left[ 1 - \exp(-\mu_k t) \right], \] (81)
\[ y_t^2 = \frac{1}{\beta k}. \] (82)
With these expressions the probability current can be written as
\[ j(x,t;v) = v(1 - \exp[-\mu_k t])p(x,t;v). \] (83)

Mean values and response terms

Using eq. (81), the mean value of the velocity of the particle is given by
\[ \nu(T,v) = \langle x(T) \rangle / T = c_T / T = v \left( 1 - \frac{1}{\mu k T} \left[ 1 - \exp(-\mu_k T) \right] \right). \] (84)
The response term becomes
\[ \Delta \nu(T,v) \equiv (T \partial_T - v \partial_v) \nu(T,v) = v \left( \frac{2}{\mu k T} \left[ 1 - \exp(-\mu_k T) \right] - \exp(-\mu_k T) + 1 \right). \] (85)
The mean value of the power reads
\[ P(T,v) = \frac{1}{T} \int_0^T dt \left\{ -kv(x(t) - vt) \right\} = \frac{v^2}{\mu} \left( 1 - \frac{1}{\mu k T} \left[ 1 - \exp(-\mu_k T) \right] \right) \] (86)
with its response term
\[ \Delta P(T,v) \equiv (T \partial_T - v \partial_v) P(T,v) = -2P(T,v) - \frac{v^2}{\mu} \exp(-\mu_k T) + \frac{v^2}{\mu^2 k T} \left( 1 - \exp[-\mu_k T] \right). \] (87)
The mean total entropy production rate can be calculated by using eq. (83) and is given by
\[ \sigma(T,v) = \frac{1}{T} \int_0^T dt \int_{-\infty}^{\infty} dx \frac{j(x,t;v)^2}{Dp(x,t;v)} = \frac{v^2}{TD} \left( T - \frac{2}{\mu k} \left[ 1 - \exp(-\mu_k T) \right] + \frac{1}{2\mu k} \left[ 1 - \exp(-2\mu_k T) \right] \right). \] (88)
Diffusion coefficients

For calculating the diffusion coefficients of the form

\[ D_J(T, v) \equiv T \left( \langle J^2_v \rangle - \langle J_T \rangle^2 \right) / 2 \]  \tag{89} \]

the term \( \langle J^2_v \rangle \) must be calculated. The correlation function \( \langle \dot{x}(t) \dot{x}(t') \rangle \) which enters in the correlation \( \langle \nu^2_T \rangle \) for the velocity can be written in terms of correlations between state functions, i.e.,

\[
\langle \dot{x}(t) \dot{x}(t') \rangle = 2D\delta(t - t') + \langle [2\nu(x(t'), t') - \mu F(x(t'), \lambda(vt'))] \mu F(x(t), \lambda(vt)) \rangle \Theta(t - t') \\
+ \langle [2\nu(x(t), t) - \mu F(x(t), \lambda(vt))] \mu F(x(t'), \lambda(vt')) \rangle \Theta(t' - t).
\]  \tag{90} \]

Hence, in both correlation functions, i.e., in \( \langle \nu^2_T \rangle \) for the velocity and in \( \langle P^2_T \rangle \) for the power, correlation functions of the form \( \langle x(t)x(t') \rangle \) occur which can be directly evaluated by solving the Langevin equation and taking the average over all noise realizations. Inserting these expressions into eq. (89) yields the diffusion coefficient

\[ D_v(T, v) = \frac{D}{\mu kT} (1 - \exp [-\mu kT]) \]  \tag{91} \]

for the velocity and

\[ D_P(T, v) = \frac{v^2}{\beta \mu} \left( 1 - \frac{1}{\mu kT} [1 - \exp (-\mu kT)] \right) = P(T, v)/\beta \]  \tag{92} \]

for the power. The latter relation between the diffusion coefficient of the power and its mean value arises due to the Gaussian nature of the work statistics \[9\]. With eqs. (84)–(88), (91) and (92) the quality factors \( Q_{P,v} \) defined in eq. (5) in the main part can be obtained.

Violation of the TUR for steady-state systems

In this section, we will show that the TUR for steady-state systems

\[ \frac{D_J \sigma}{J^2} \geq 1 \]  \tag{93} \]

is strictly violated for the power, i.e., \( J = P(T, v) \). With eq. (92) the l.h.s of (93) can be written as \( \sigma(T, v) / [\beta P(T, v)] \) for the moving trap. Thus, if the TUR for steady-state systems was valid it would state \( \sigma(T, v) \geq \beta P(T, v) \). However, as we now show in the case for the moving trap the power fulfills the inequality

\[ \sigma(T, v) \leq \beta P(T, v). \]  \tag{94} \]

First, the mean entropy of the system \( \langle S_{sys} \rangle \) is constant in time due to the fact that the variance \( y^2_t \) is constant. Hence, the mean entropy change of the system \( \langle \Delta S_{sys} \rangle \) is zero. Second, using

\[ \langle \Delta V(x(t), t) \rangle = \langle V(x(t), t) - V(x(0), 0) \rangle = \frac{k}{2} \langle x^2(t) \rangle \geq 0 \]  \tag{95} \]

the first law of thermodynamics leads to

\[ \sigma(T, v) = \frac{1}{T} \langle \Delta S_{int} \rangle = \frac{1}{T} (\beta \langle Q \rangle + \langle \Delta S_{sys} \rangle) = \beta P(T, v) - \langle \Delta V(x(t), t) \rangle / T \leq \beta P(T, v), \]  \tag{96} \]

where \( \langle Q \rangle \) is the mean heat dissipated in the medium.

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