Double chiral logs in the $\pi\pi$ scattering amplitude

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Abstract

By using the Renormalization Group Equations in Chiral Perturbation Theory, one can calculate the double chiral logs that appear at two loops in any matrix element. We calculate them in the $\pi\pi$ scattering amplitude, where they represent the potentially largest two loop contribution. It is shown that their correction is reasonably small.

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1 Introduction

The $\pi\pi$ scattering reaction is of fundamental importance for understanding low energy hadronic physics. Since the sixties the most fruitful way to look at it has been to consider the constraints given by the symmetry properties of the strong interactions Hamiltonian. The first predictions for the $\pi\pi$ scattering amplitude according to this method were worked out by Weinberg in 1966 [1]. In recent years all the theoretical work that goes under the name of Current Algebra has been incorporated in a more systematic and rigorous framework called Chiral Perturbation Theory (CHPT) [2]. Within this approach one assumes that the pions are the (pseudo) Goldstone bosons of QCD, and that the singularities generated by their exchanges dominate the Green functions at low energy. After taking into account these singularities, CHPT works as a systematic expansion of their residues in powers of momenta and quark masses. Current Algebra corresponds to the leading term in this expansion.

Gasser and Leutwyler have fully worked out the theory up to the one loop level, and in particular have determined all the new constants occurring in the effective Lagrangian at this level by comparison with experimental data [3, 4]. After this determination one may calculate corrections to any Current Algebra prediction. For $\pi\pi$ scattering this was done by the same authors [5] who found rather large corrections to Weinberg’s predictions, even near threshold. It is interesting to try to understand why this happens. In fact, if we stick to threshold, we have only one expansion parameter, the average of $u$ and $d$ quark masses, which is known to be very small in comparison with the typical QCD scale. After expressing this parameter through the mass and decay constant of the pion it turns out to be $(M_\pi/4\pi F_\pi)^2 \sim 0.01$. With such a small expansion parameter one would expect the corrections to the leading term to be rather small. However, the $S$-wave, $I = 0$ scattering length, for example, gets at next order a 28% correction to the Weinberg prediction $a_0^0 = 0.16$. While at first sight this may look surprising, the reason for having such large corrections is well known [6]. Beyond leading order the unitarity property requires the appearance of nonanalytic functions such as $M^2_\pi \log M^2_\pi$, which, if $M^2_\pi$ is small, may be significantly larger. In fact, for $a_0^0$, 90% of the correction is given by the term proportional to $M^2_\pi \log M^2_\pi/\mu^2$, at a scale $\mu = 1\text{GeV}$.

A natural question then arises, whether higher orders which contribute terms like $(M^2_\pi \log M^2_\pi)^n$ may still give large corrections. This is especially interesting because the comparison with experimental data is still not completely satisfactory, as can be seen in Table [1]. For $a_0^0$ in particular, large corrections would still be needed in order to obtain agreement with the present experimental central value, while Gasser and Leutwyler assigned a very small uncertainty (of the order of 5%, if one considers only the uncertainty in the value of the low energy constants) to their one loop calculation.

On the other hand Stern and collaborators [7] have shown that by allowing the quark condensate to take an unexpectedly small value, and modifying accordingly the chiral expansion [8], one could find agreement with the measurement of Rosselet et al. [9] of $a_0^0$ already at one loop (see Table 1 and Figure 2 in Ref. [7]). This makes
this quantity even more interesting, since the possibility for the quark condensate to be numerically near to $F_3^2$ (and not much larger) has not yet been clearly excluded, and this would be one of the few places where one could find experimental evidence against or in favour of it.

Our knowledge about this quantity will most probably improve both on the experimental and theoretical side in the near future. A full two loop calculation is now in progress [10], and a new, high statistics measurement of $K_{e4}$ decays which will be made at DAΦNE, should sizeably reduce the present experimental error [11].

While waiting for the full two loop calculation, which, incidentally, is rather long and tedious, there is the possibility to evaluate at a rather low cost the potentially more dangerous part of the order $O(p^6)$, i.e. the double chiral logs. The calculation can be done in a relatively simple way by using equations that follow from general properties of the renormalization procedure. The content of these equations is exactly the same as that of the Renormalization Group Equations (RGE) of a renormalizable field theory, i.e. that (considering for example dimensional regularization) the residues of the double pole of two loop graphs are proportional to the residues of the single pole of one loop graphs.

The use of these equations with a nonrenormalizable Lagrangian is not new. Weinberg derived them for the chiral effective Lagrangian in 1979 [12], while several authors have used them (under the name of pole equations) in the framework of the two dimensional $\sigma$–model up to the four or five loop level [13].

2 Renormalization Group Equations in CHPT

As far as we know, the only application of RGE with an effective Lagrangian since Weinberg’s original derivation [12], has been Ref. [14], where these equations have been used as a check on a full two loop calculation.

Before deriving the equations we shortly review the basic ideas and notation of CHPT. Consider the QCD Lagrangian with two flavours in the isospin symmetry limit $m_u = m_d = \hat{m}$, with external fields coupled to quark bilinears:

$$\mathcal{L} = \mathcal{L}_{QCD}^0 + \bar{q}\gamma^\mu (v_\mu + a_\mu \gamma_5)q - \bar{q}(s - i\gamma_5 p)q ,$$

where $\mathcal{L}_{QCD}^0$ is the QCD Lagrangian with massless quarks – the masses can be absorbed in the external field $s$.

Under the assumption that the pions, because of their small mass, dominate the low energy phenomena described by this Lagrangian, one can construct a good representation of the generating functional of QCD at low energy by means of an effective Lagrangian:

$$\int [dq][dA]e^{i\mathcal{L}} = e^{i\mathcal{Z}[v,a,s,p]} = \int [dU]e^{i\mathcal{L}_{eff}} ,$$

where $[dA]$ stands for the integration over the gluon fields. The effective Lagrangian must be constructed with the only requirements of being the most general symmetric
Lagrangian containing the pions (through the matrix \( U \)) and the external fields. It can be expanded in a series of terms with increasing number of derivatives and powers of quark masses:

\[
\mathcal{L}_{\text{eff}} = \mathcal{L}_2 + \mathcal{L}_4 + \mathcal{L}_6 + \ldots .
\]  

(2.3)

Each term \( \mathcal{L}_{2n} \) contains a number of monomials with a coefficient which is not fixed by symmetry constraints. At present the best way to determine their value is by comparison with experimental data. \( \mathcal{L}_2 \) contains only two arbitrary parameters, \( F \) and \( B \), which are easily related to the decay constant and mass of the pion:

\[
F_\pi = F(1 + O(\hat{m})), \quad M_\pi^2 = M^2(1 + O(\hat{m}))
\]  

(2.4)

where \( M^2 = 2B\hat{m} \). At next order there are seven new free parameters. The relation between these constants called \( l_i \)'s, \( i = 1, \ldots, 7 \), and measurable quantities has been worked out by Gasser and Leutwyler in Ref. [3].

This expansion of the Lagrangian generates a corresponding series for the generating functional:

\[
Z[v, a, s, p] = Z_2 + Z_4 + Z_6 + \ldots .
\]  

(2.5)

As Weinberg has shown [12], the contribution of loops to the generating functional is suppressed with respect to tree diagrams: \( Z_2 \) is given only by tree diagrams from \( \mathcal{L}_2 \); \( Z_4 \) receives contributions from tree diagrams from \( \mathcal{L}_4 \) and one loop diagrams with \( \mathcal{L}_2 \); \( Z_6 \) contains tree diagrams from \( \mathcal{L}_6 \), one loop diagrams with one vertex from \( \mathcal{L}_4 \), and two loop diagrams containing only \( \mathcal{L}_2 \) vertices; etc. For more details about CHPT we refer the reader to the original article by Gasser and Leutwyler [3] and to recent reviews [2].

Here we want to apply the RGE to the \( \pi\pi \) scattering amplitude obtained from \( Z_6 \). We derive the equations in a way which is much similar to the one followed in Ref. [14]. We use dimensional regularization and the minimal subtraction scheme. Before renormalization \( \text{ (i.e. before including the contribution of tree diagrams from } \mathcal{L}_6) \), the \( O(p^6) \) \( \pi\pi \) scattering amplitude looks like this:

\[
A^{(6)}(s, t, u) = F^{-6} \left[ M^{2\epsilon} G(s, t, u) + M^\epsilon \sum_i l_i H_i(s, t, u) \right],
\]  

(2.6)

where \( \epsilon = d - 4 \). In this expression we have simply made evident the distinction between real two loop graphs, all incorporated into the function \( G(s, t, u) \), and the one loop graphs with one vertex from the \( \mathcal{L}_4 \) Lagrangian, that are included into the functions \( H_i(s, t, u) \). The dimension of the product \( F^6 A^{(6)} \) is \( \text{(mass)}^{6+2\epsilon} \), then, to carry out the renormalization program we write

\[
A^{(6)} = \mu^{2\epsilon} \left\{ \text{Laurent series of } \mu^{-2\epsilon} F^6 A^{(6)} \text{ in } \epsilon \right\}.
\]  

(2.7)

The curly brackets contain a function with dimension \( \text{(mass)}^{6} \), that will be provided by powers of external momenta and masses. The poles in \( \epsilon \) appearing in the Laurent
series will be cancelled by the contribution of tree graphs from $\mathcal{L}_6$. The ingredients we need to analyze the structure of the two loop amplitude are the following:

$$G(s,t,u) = \frac{g_2(s,t,u)}{\epsilon^2} + \frac{g_1(s,t,u)}{\epsilon} + \overline{G}(s,t,u) + O(\epsilon),$$

$$H_i(s,t,u) = \frac{h_i(s,t,u)}{\epsilon} + \overline{H}_i(s,t,u) + +O(\epsilon),$$

$$l_i = \mu \left( \frac{\delta_i}{\epsilon} + l_{i,\text{MS}} + O(\epsilon) \right) \quad (2.8)$$

Note that $G$ and the $H_i$’s do not depend at all on $\mu$, while the constants $l_i$’s are by definition $\mu$–independent; i.e. we have the equations:

$$\mu \frac{d}{d\mu} l_{i,\text{MS}} = -\delta_i, \quad \mu \frac{d}{d\mu} l_{i,\text{MS}}^1 = -l_{i,\text{MS}}^1 \quad (2.9)$$

From general arguments on the structure of the divergences of loop graphs (see e.g. Ref. [15]), we know that the residues $g_2$ and $h_i$’s are polynomials in external momenta and masses. On the other hand, $g_1$ is in general a nonlocal function of the same variables, due to the presence of divergent subgraphs in the two loop graphs. Despite the fact that CHPT is a nonrenormalizable theory, the renormalization program can be carried out in complete analogy with the case of a renormalizable theory. In particular, all the poles in $\epsilon$ must have residues which are polynomials in the external momenta and masses, in such a way that one can remove them by adding counterterms to the original Lagrangian. If one writes down explicitly the Laurent series, one immediately realizes that for the double pole this is already obvious. In order to satisfy this property in the case of the single pole, the nonlocal functions appearing inside its residue must cancel. In other words, the following two equations must be satisfied:

$$g_2(s,t,u) = -\frac{1}{2} \sum_i \delta_i h_i(s,t,u),$$

$$g_1(s,t,u) = -\sum_i \delta_i \overline{H}_i(s,t,u) + \text{polynomial in } s,t,u \text{ and } M^2 \quad (2.10)$$

As a consequence of the first equation the residue of the double pole, and the coefficient of the double log in the finite part, turn out to be proportional to the polynomial $\sum_i \delta_i h_i(s,t,u)$.

With a relatively simple calculation we are then able to get interesting information about the two loops. To have the complete $O(p^6)$ result one still has to calculate part of the coefficient of the single log, and the finite contribution.

### 3 Double logs in the $\pi\pi$ scattering amplitude

We have calculated the polynomials $h_i$ partly using Feynman diagrams and partly with the Heat Kernel technique. In some cases we have performed the calculation
with both methods in order to have a cross check. More details about the calculation can be found in Ref. [16]. Here we simply give the results:

\[
\begin{align*}
    h_1(s,t,u) &= \frac{1}{16\pi^2} \left[ \frac{-21}{2} s^3 - \frac{5}{6} s(t-u)^2 + \frac{200}{3} M^2 s^2 - \frac{4}{3} M^2(t-u)^2 -164 M^4 s + \frac{376}{3} M^6 \right], \\
    h_2(s,t,u) &= \frac{1}{16\pi^2} \left[ \frac{-107}{12} s^3 - \frac{25}{12} s(t-u)^2 + 44 M^2 s^2 - \frac{4}{3} M^2(t-u)^2 -296 \frac{3}{3} M^4 s + \frac{208}{3} M^6 \right], \\
    h_3(s,t,u) &= \frac{1}{16\pi^2} \left[ -4 M^4 s - 2 M^6 \right].
\end{align*}
\]

(3.11)

All the other \(h_i\)'s are zero. Before using these expressions to get numerical results, we need to shuffle part of the two loop amplitude \(A^{(6)}\) into the lower order amplitudes in order to express everything in terms of the physical pion mass and the physical decay constant. Since we are calculating only the divergent part of one loop graphs containing one \(l_i\) in \(A^{(6)}\), we need to calculate the same contribution to \(M^2_\pi\) and \(F^2_\pi\):

\[
\begin{align*}
    M^2_\pi &= M^2 \left\{ 1 + \frac{M^2}{F^2} \left[ 2 l_3 + \frac{1}{2i} \frac{\Delta(0)}{M^2} \right] \right. \\
    &+ \left. \frac{M^4}{F^4} \left[ (-14 l_1 - 8 l_2 - 3 l_3) \frac{1}{i} \frac{\Delta(0)}{M^2} + \ldots \right] + O(M^6) \right\}, \\
    F^2_\pi &= F^2 \left\{ 1 + \frac{M^2}{F^2} \left[ 2 l_4 - \frac{2}{i} \frac{\Delta(0)}{M^2} \right] \right. \\
    &+ \left. \frac{M^4}{F^4} \left[ (14 l_1 + 8 l_2 - 4 l_3 - 3 l_4) \frac{1}{i} \frac{\Delta(0)}{M^2} + \ldots \right] + O(M^6) \right\},
\end{align*}
\]

(3.12)

where \(\Delta(z)\) is the Feynman propagator of a scalar field with mass \(M\): \(\Delta(0) = 2iM^2/(16\pi^2\epsilon) + O(1)\).

In order to evaluate the size of these corrections to the \(\pi\pi\) scattering amplitude, we calculate their contribution to threshold parameters, following the conventions of Ref. [3]{\footnote{The ellipses between parentheses stand for the one loop contributions that we do not display here in order not to put too many formulae in this note. We refer the reader again to Ref. [3] (note that the corrections there are usually expressed in terms of \(M\) and \(F\) and not with the physical mass and decay constant, as we do here).}}:

\[
\begin{align*}
    a_0^0 &= \frac{7 M^2_\pi}{32\pi F^2_\pi} \left\{ 1 + (\ldots) + \frac{M^4_\pi}{F^4_\pi} \left[ -\frac{58}{7} k_1 - \frac{96}{7} k_2 - 5 k_3 - \frac{11}{2} k_4 + \ldots \right] + O(M^6_\pi) \right\}, \\
    b_0^0 &= \frac{1}{4\pi F_\pi^2} \left\{ 1 + (\ldots) + \frac{M^4_\pi}{F^4_\pi} \left[ -\frac{86}{3} k_1 - \frac{73}{2} k_2 - \frac{5}{2} k_3 - \frac{29}{3} k_4 + \ldots \right] + O(M^6_\pi) \right\},
\end{align*}
\]
\[a_0^2 = -\frac{M_\pi^2}{16\pi F_\pi^2} \left\{ 1 + \ldots + \frac{M_\pi^4}{F_\pi^4} \left[ 2k_1 + 6k_2 + k_3 + \frac{1}{2}k_4 + \ldots \right] + O(M_\pi^6) \right\},\]

\[b_0^2 = -\frac{1}{8\pi F_\pi^2} \left\{ 1 + \ldots + \frac{M_\pi^4}{F_\pi^4} \left[ -\frac{20}{3}k_1 + \frac{3}{2}k_2 - \frac{5}{2}k_3 + \frac{7}{3}k_4 + \ldots \right] + O(M_\pi^6) \right\},\]

\[a_1^2 = \frac{1}{24\pi F_\pi^2} \left\{ 1 + \ldots + \frac{M_\pi^4}{F_\pi^4} \left[ -12k_1 - \frac{119}{6}k_2 - \frac{5}{2}k_3 - 3k_4 + \ldots \right] + O(M_\pi^6) \right\},\]

\[b_1^2 = \frac{1}{F_\pi^4}(\ldots) + \frac{M_\pi^2}{18\pi F_\pi^6} \left[ -23k_1 - \frac{109}{4}k_2 + \ldots \right] + O(M_\pi^4),\]

\[a_2^0 = \frac{1}{F_\pi^4}(\ldots) + \frac{M_\pi^2}{10\pi F_\pi^6} \left[ \frac{11}{18}k_1 - \frac{8}{3}k_2 - \frac{5}{9}k_4 + \ldots \right] + O(M_\pi^4),\]

\[a_2^2 = \frac{1}{F_\pi^4}(\ldots) + \frac{M_\pi^2}{10\pi F_\pi^6} \left[ \frac{16}{9}k_1 + \frac{3}{4}k_2 - \frac{2}{9}k_4 + \ldots \right] + O(M_\pi^4),\]  \tag{3.13}

where
\[k_i = \left( 4l_i - \delta_i \log \frac{M_\pi^2}{\mu^2} \right) \frac{1}{16\pi^2} \log \frac{M_\pi^2}{\mu^2}, \quad i = 1, \ldots, 4. \tag{3.14}\]

Note that with this definition of the \(k_i\)’s we are also taking into account in Eqs. (3.13) the part of the single log which is proportional to the renormalized constants. From our previous considerations one can see that this contribution is unambiguously determined by the polynomials \(h_i\). Note also that the renormalized constants we are using here are defined in the renormalization scheme adopted by Gasser and Leutwyler in Ref. [3], and not in the MS scheme that we used in the previous section to simplify the notation.

Before analyzing the part of the two loop contribution evaluated here it is interesting to investigate in some more details the structure of the \(O(p^4)\) correction. In the third column of Table 1 we show how much of that correction is due to the chiral log evaluated at a scale \(\mu = 1\text{GeV}\). It is clear that for the \(S\)-wave parameters the chiral log is responsible for almost all the correction. On the other hand, for the \(P\)-wave, the chiral log arises only through \(F_\pi\) renormalization – had one used the pion decay constant in the chiral limit \(F\), there would not have been any log (and in fact for \(b_1^1\), which does not contain any tree level contribution, so that the use of \(F\) or \(F_\pi\) is equivalent, there is no chiral log at this order). Moreover, in this case we expect the \(\rho\) to give an important contribution, and it is well known that the resonances manifest themselves through the \(l_i^\prime\)’s. Hence the fact that the chiral logs are not the main part of the one loop correction does not come out as a surprise.

Finally we have the two \(D\)-wave scattering lengths, for which the order \(O(p^6)\) is the next to leading correction. In this case we feel that it is more hazardous to estimate it only through the double chiral logs; also because, a posteriori, the part of the single chiral logs that we get turns out to be as important as the double logs. Moreover
the $I = 2$ scattering length at leading order is unnaturally small because of a strong cancellation between the chiral log and the finite part.

From all these observations we conclude that for $P$ and $D$ waves the assumption that the double chiral logs could be a good estimate of the full two loop correction is poorly justified.

The numerical results are displayed in Table 1, together with the one loop predictions and experimental data available. Since the numbers we produce are $\mu$-dependent we have calculated them for two reasonable values of $\mu$. Moreover, in Table 1 we show both the double logs and the complete $k_i$'s contributions.

We observe that:

1. For the $S$-wave parameters this $O(p^6)$ correction is, in the extreme case, of the order of 10%. To judge whether the size of this correction is reasonable or not, we may compare the coefficients of the single and double logs in the $O(p^4)$ and $O(p^6)$ corrections, respectively:

\[
\begin{align*}
    a_0^0 &= \frac{7M_\pi^2}{32\pi F_\pi^2} \left\{ 1 - \frac{9}{2} L + \frac{857}{42} L^2 + \ldots \right\}, \\
    b_0^0 &= \frac{1}{4\pi F_\pi^2} \left\{ 1 - \frac{26}{3} L + \frac{1871}{36} L^2 + \ldots \right\}, \\
    a_0^2 &= -\frac{M_\pi^2}{16\pi F_\pi^2} \left\{ 1 + \frac{3}{2} L - \frac{31}{6} L^2 + \ldots \right\}, \\
    b_0^2 &= -\frac{1}{8\pi F_\pi^2} \left\{ 1 + \frac{10}{3} L - \frac{83}{18} L^2 + \ldots \right\},
\end{align*}
\]  

where $L \equiv (M_\pi^2/16\pi^2 F_\pi^2) \log M_\pi^2/\mu^2$. In first place one can see that the absolute value of the $L^2$ coefficient is numerically quite similar to the square of the $L$ coefficient, in all cases but $b_0^2$. [Incidentally, had $b_0^2$ respected this rule of thumb for the absolute value, with the sign it has, the distance from the present experimental central value would have increased sizeably.] The second interesting point is that they show a very regular sign pattern. In order to have some more insight into it, it is useful to disentangle two effects of comparable importance: the correction to the amplitude $A(s, t, u)$ expressed in terms of $F$, and the effect of the renormalization of $F$ into $F_\pi$. This is done in Table 3, where one can see that whilst the two effects add up for the $I = 0$ parameters, they partially cancel in the case of the $I = 2$ parameters. In particular, in the latter case one can trace back the minus sign of the coefficient of $L^2$ to the interference of the two one loop corrections.

2. As far as the $P$- and $D$-wave parameters are concerned, we have displayed here the double chiral logs contribution, essentially for the sake of completeness. As we stressed already, we see no special reasons why the numbers we find here should be representative of the full two loop correction.

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3. As we mentioned in the introduction, it is hoped that there will be an improvement in the experimental knowledge of $a_0^0$ in the near future, arising from a new measurement of $K_{e4}$ decays. On the theoretical side the full two loop calculation, which is now in progress\cite{11}, will improve the one loop result of Gasser and Leutwyler\cite{3,5}. Our results show that the contribution from the two loops is expected to be reasonably small, and confirm that it looks very unlikely that CHPT could at any order reproduce the present experimental central value.

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Table 1: Numerical values of the threshold parameters according to Eqs. (3.13), for $\mu = 0.5, 1\text{GeV}$. In the column containing the one loop results, the numbers between parentheses show how much of the one loop contribution is due to the chiral log evaluated at the scale $\mu = 1\text{GeV}$. In the calculation we have used $M_\pi = 139.6\text{MeV}$ and $F_\pi = 93.2\text{MeV}$. To calculate the $k_i$’s we have used the $l_i$’s given in Table 2. Scattering lengths $a_i^l$, and effective ranges $b_i^l$ are given in the appropriate powers of $M_\pi$. The value of $\mu$ is expressed in GeV.

|       | 1 loop (log) $\mu = 1$ | 1 loop +log$^2 M_\pi^2$ $\mu = 0.5$ $\mu = 1$ | 1 loop+k$_i$’s $\mu = 0.5$ $\mu = 1$ | experiment [17] |
|-------|------------------------|-----------------------------------------------|----------------------------------|----------------|
| $a_0^0$ | 0.201 (88%)             | 0.205 0.211                                   | 0.211 0.213                      | 0.26 ± 0.05    |
| $b_0^0$ | 0.248 (125%)            | 0.260 0.277                                   | 0.275 0.279                      | 0.25 ± 0.03    |
| $-10a_0^2$ | 0.418 (133%)        | 0.415 0.411                                   | 0.409 0.407                      | 0.28 ± 0.12    |
| $-10b_0^2$ | 0.726 (100%)            | 0.721 0.713                                   | 0.701 0.691                      | 0.82 ± 0.08    |
| $10a_1^2$ | 0.370 (46%)             | 0.379 0.390                                   | 0.391 0.395                      | 0.38 ± 0.02    |
| $10^2b_1$ | 0.485 (0%)              | 0.619 0.805                                   | 0.749 0.780                      |                |
| $10^2a_0^0$ | 0.181 (122%)            | 0.206 0.241                                   | 0.273 0.302                      | 0.17 ± 0.03    |
| $10^3a_2^0$ | 0.205 (432%)            | 0.145 0.061                                   | 0.267 0.335                      | 0.13 ± 0.3     |
Table 2: Numerical values of the coupling constants $l_i$’s and the $k_i$’s used in Table 1. The values of $l_3$ and $l_4$ are taken from Ref. [3], while for the other two we have used the more recent ones obtained in Ref. [18]. The value of $\mu$ is expressed in GeV.

| $i$ | $10^3 l_i$ | $10^3 k_i$ |
|-----|------------|------------|
|     | $\mu = 0.5$ | $\mu = 1$ |
|     | $\mu = 0.5$ | $\mu = 1$ |
| 1   | -4.5       | 0.20       |
| 2   | 7.5        | -0.65      |
| 3   | -0.5       | 0.17       |
| 4   | 11.1       | -1.24      |

Table 3: Breakdown of the coefficient of $L$ and $L^2$ into contributions arising from two different effects: the corrections to the amplitude $A(s, t, u)$, and the renormalization of $F$ into $F_\pi$. For the $L^2$ coefficient we have also separated out the contribution due to the interference between the two one loop effects. [The small effect due to mass renormalization is included into the columns tagged by $A(s, t, u)$.]

|       | $L$-coefficient | $L^2$-coefficient |
|-------|-----------------|-------------------|
|       | $A(s, t, u)$    | $F_\pi$ total     | $A(s, t, u)$ interference | $F_\pi$ total |
| $a_0^0$ | $-\frac{5}{2}$  | $-2$              | $-\frac{9}{2}$            | 145            | 10             | $\frac{7}{2}$ | 857            | $\frac{2}{42}$ |
| $b_0^0$ | $-\frac{20}{3}$ | $-2$              | $-\frac{26}{3}$           | 785            | 80             | $\frac{7}{2}$ | 1871           |
| $a_0^2$ | $\frac{7}{2}$   | $-2$              | $\frac{3}{2}$             | 16             | $-14$          | $\frac{7}{2}$ | $-\frac{31}{6}$ |
| $b_0^2$ | $\frac{16}{3}$  | $-2$              | $\frac{10}{3}$            | 476            | $-\frac{64}{3}$| $\frac{7}{2}$ | $-\frac{83}{18}$ |