Stability of Heteroclinic Cycles: A New Approach Based on a Replicator Equation

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Abstract
This paper analyses the stability of cycles within a heteroclinic network formed by six cycles lying in a three-dimensional manifold, for a one-parameter model developed in the context of polymatrix replicator equations. We show the asymptotic stability of the network for a range of parameter values compatible with the existence of an interior equilibrium and we describe an asymptotic technique to decide which cycle (within the network) is visible in numerics. The technique consists of reducing the relevant dynamics to a suitable one-dimensional map, the so-called projective map. The stability of the fixed points of the projective map determines the stability of the associated cycles. The description of this new asymptotic approach is applicable to more general types of networks and is potentially useful in computational dynamics.

Keywords Asymptotic stability · Polymatrix replicator · Heteroclinic network · Heteroclinic cycle · Projective map

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1 Introduction

Recent studies in several areas have emphasized ways in which heteroclinic cycles and networks may be responsible for intermittent dynamics in nonlinear systems. They may be seen as the skeleton for the understanding of complicated dynamics (Hofbauer and Sigmund 1987; Gaunersdorfer and Hofbauer 1995).

A heteroclinic cycle is the union of hyperbolic equilibria and solutions that connect them in a cyclic fashion (Field and Swift 1991; Podvigina and Ashwin 2011; Krupa and Melbourne 1995; Rodrigues 2013). A heteroclinic network is a connected union of heteroclinic cycles (possibly infinite in number), such that for any pair of nodes in the network, there is a sequence of heteroclinic connections connecting them.

Heteroclinic cycles or networks do not exist in a generic dynamical system, because small perturbations break connections between saddles. However, they may exist in systems where some constraints are imposed and are robust with respect to perturbations that respect these restrictions. Typically, these constraints create flow-invariant subspaces where the connection is of saddle-sink type (Krupa and Melbourne 1995; Field 2020).

For Lotka–Volterra systems in \((\mathbb{R}^+_0)^n, n \in \mathbb{N}\), the Cartesian hyperplanes, also called extinction subspaces, are flow-invariant. Such hyperplanes are invariant subspaces for systems on a simplex, a usual state space in Evolutionary Game Theory (EGT) (Hofbauer and Sigmund 1987, 1998; Gaunersdorfer and Hofbauer 1995) and prompt the occurrence of heteroclinic networks associated to hyperbolic equilibria.

When a network is asymptotically stable, the transition times between saddles increase geometrically (Gaunersdorfer and Hofbauer 1995; Labouriau and Rodrigues 2017). Within a heteroclinic network, no individual heteroclinic cycle can be asymptotically stable. However, the cycles can exhibit intermediate levels of stability, namely essential and fragmentary asymptotic stability, important to decide the visibility of cycles in numerical simulations (Podvigina and Chossat 2015; Podvigina et al. 2020; Melbourne 1991). Useful conditions for asymptotic stability of some types of heteroclinic cycles have been established in Podvigina and Chossat (2015), Podvigina et al. (2020), Podvigina (2012).

A classification of the complex networks as simple, pseudo-simple and quasi-simple (among others) has been proposed by several authors, namely Krupa and Melbourne (1995), Podvigina and Chossat (2017), Garrido-da Silva and Castro (2019), Podvigina et al. (2020). A fruitful tool for quantifying stability of heteroclinic cycles is the local stability index of Podvigina and Ashwin (2011) and Lohse (2015).

Given a heteroclinic cycle, the derivation of conditions for its stability involves the construction of an appropriate first return map, which typically is a highly non-trivial problem. The existence of various itineraries along a network that can be followed by nearby trajectories makes the study of the stability of networks a hard problem. This is why there are few instances of networks whose asymptotic stability was proven.
1.1 Classical Method: An Overview

The classical method to analyse the stability of cycles and networks is based in the following procedure: assuming a non-resonance condition on the spectrum of the linearization of the vector field at the equilibria, we approximate the behaviour of nearby trajectories by composing local and global maps associated to a finite set of cross-sections. For compact networks, global maps may be linear whose coefficients are bounded from above. The estimates for local maps near the saddles involve exponents of eigenvalues’ ratio.

According to their role on the network, the eigenvalues can be classified as radial, contracting, expanding and transverse (Podvigina and Chossat 2015, 2017). A network is stable if certain products of the exponents appearing in the expression of the first return map to a cross section are larger than one. The estimates for local maps depend on the local structure of the network near the equilibria. In the presence of symmetry (or other constraints), the application of the method is slightly different since the fixed-point subspaces may be seen as borders that cannot be crossed.

1.2 Novelty

In this paper, we describe a new method to study the asymptotic dynamics of a continuous dynamical system arising in the context of a polymatrix game (Hofbauer and Sigmund 1987; Gaunersdorfer and Hofbauer 1995). We consider a one-parameter family of Ordinary Differential Equations (ODE) modelling the dynamics of a population divided in three groups, each one with two possible competitive strategies. Interactions between individuals of any two groups are allowed, including those in the same group. The differential equation associated to a polymatrix game, the so called polymatrix replicator, is defined in a product of three simplices. Examples of such dynamical systems arise naturally in the context of Evolutionary Game Theory (EGT) developed by Smith and Price (1973), Peixe and Rodrigues (2022)—see also references therein.

We focus on a one-parameter class of vector fields whose flow contains a heteroclinic network associated to the edges on the unit cube, whose eigenvalues of the Jacobian matrix are all real, a feature always valid in the replicator equation. By making use of the theory developed in Peixe and Rodrigues (2022), Alishah et al. (2019), we start by showing the asymptotic stability of the network (containing six cycles), where the parameter lies in an interval compatible with the existence of an interior equilibrium.

As in Podvigina (2012), Garrido-da Silva and Castro (2019), we consider a logarithmic quasi-change of coordinates (near the network) to compute the preferred attracting cycle of the network. The basin of attraction of each cycle defines a sector in the dual set, whose asymptotic dynamics may be analysed through a piecewise smooth one-dimensional map on an interval—the projective map. Using the classical Perron-Frobenius Theory applied to linear operators, we conclude about the existence of a bijection between stable periodic points of the projective map and stable heteroclinic cycles. We also discuss the geometry of the two-dimensional invariant manifold of the interior equilibrium.
As far as we know, the analysis of the stability of cycles within a network through the study of the projective map is new. Our techniques are computationally applicable not only to networks in the EGT context (Lotka–Volterra systems), but to “mean-field” equations borrowed from physics, where the connections are one-dimensional—see Section 2 of Barendregt and Thomas (2023) where the authors derive mean-field equations for heteroclinic cycling.

1.3 Structure

This article is organised as follows. In Sect. 2 we introduce the one-parameter family of polymatrix replicators that we will be focused on. Once defined the main concepts, in Sect. 4 we concentrate our analysis on a parameter interval where a single interior equilibrium exists, and we describe completely the dynamics on the boundary of the phase space. In particular, we describe an attracting heteroclinic network \( \mathcal{H} \) formed by the edges and vertices of the cube.

We present in Sect. 5 a piecewise linear model from where we analyse the asymptotic dynamics near the network \( \mathcal{H} \) introduced in Sect. 4. In Sect. 6, we apply the previously established machinery to study the stability of all cycles in \( \mathcal{H} \). Our method is algorithmic and in Sect. 7 we refer the reader to the Mathematica code for studying polymatrix replicators.

Finally, in Sect. 8 we relate our main results with others in the literature. We have endeavoured to make a self contained exposition bringing together all topics related to the method and the proofs.

In Appendices A and B, we add some tables that will help the reader to understand our article, as well as the notation for constants, sets and auxiliary functions.

2 Model

We analyse a particular case of a polymatrix game whose phase space may be identified with a cube in \( \mathbb{R}^3 \). Consider a population divided in three groups where individuals of each group have two strategies to interact with other members of the population. The model that we will consider to study the time evolution of the chosen strategies is the polymatrix game and may be formalised as:

\[
\dot{x}_i^\alpha(t) = x_i^\alpha(t) \left( (Px(t))^i - \sum_{j=1}^{2} (x_j^\alpha(t))(Px(t))^j \right), \alpha \in \{1, 2, 3\}, i \in \{1, 2\}, \quad (1)
\]

where \( \dot{x}_i^\alpha(t) \) represents the time derivative of \( x_i^\alpha(t) \), \( P \in M_{6 \times 6}(\mathbb{R}) \) is the payoff matrix,

\[
x(t) = (x_1^1(t), x_2^1(t), x_1^2(t), x_2^2(t), x_1^3(t), x_2^3(t))
\]
and

\[ x_1^1(t) + x_2^1(t) = x_1^2(t) + x_2^2(t) = x_1^3(t) + x_2^3(t) = 1. \]

The indices may be interpreted as:

\[ \alpha : \text{subgroup of the population}; \]
\[ i : \text{strategy of the associated subgroup}. \]

For simplicity of, we will write \( x \) instead of \( x(t) \). The payoff matrix \( P \) can be represented as a matrix,

\[
P = \begin{pmatrix}
  p^{1,1} & p^{1,2} & p^{1,3} \\
  p^{2,1} & p^{2,2} & p^{2,3} \\
  p^{3,1} & p^{3,2} & p^{3,3}
\end{pmatrix}
\]

where each block \( P^\alpha, \alpha, \beta \in \{1, 2, 3\} \), represents the payoff of the individuals of the group \( \alpha \) when interacting with individuals of the group \( \beta \), and where each entry \( p^{\alpha, \beta}_{i,j} \) represents the average payoff of an individual of the group \( \alpha \) using strategy \( i \) when interacting with an individual of the group \( \beta \) using strategy \( j \).

System (1) is designated as a polymatrix replicator (Peixe and Rodrigues 2022; Alishah and Duarte 2015; Alishah et al. 2015; Peixe 2019). Assuming random encounters between individuals of the population, for each group \( \alpha \in \{1, 2, 3\} \), the average payoff for a strategy \( i \in \{1, 2\} \), is given by

\[
(Px)^\alpha_i = \sum_{\beta=1}^{3} (P^{\alpha, \beta})^\alpha_i x^\beta = \sum_{\beta=1}^{3} \sum_{k=1}^{2} p^{\alpha, \beta}_{i,k} x^\beta_k,
\]

the average payoff of all strategies in \( \alpha \) is given by

\[
\sum_{i=1}^{2} x_i^\alpha (Px)^\alpha_i = \sum_{\beta=1}^{3} (x^\alpha)^T P^{\alpha, \beta} x^\beta.
\]
and the growth rate \( \frac{\dot{x}_i}{x_i} \) of the frequency of each strategy \( i \in \{1, 2\} \) is equal to the payoff difference

\[
(Px)^{x_i} - \sum_{\beta=1}^{3} (x^{x_i})^T P^{x_i, x_{i+1}} x_{i+1}.
\]

If \( x = (x_1, x_2, x_3, x_4, x_5, x_6) \) is such that

\[
x_1 + x_2 = x_3 + x_4 = x_5 + x_6 = 1,
\]

the system (1) may be written as

\[
\begin{cases}
\dot{x}_i = x_i \left( (Px)_i - x_i (Px)_i - x_{i+1} (Px)_{i+1} \right), & i \in \{1, 3, 5\}.
\end{cases}
\]

By Lemma 1 of Peixe and Rodrigues (2022), system (3) is equivalent to

\[
\begin{cases}
\dot{x}_1 = x_1 (1 - x_1) ((Px)_1 - (Px)_2), \\
\dot{x}_3 = x_3 (1 - x_3) ((Px)_3 - (Px)_4), \\
\dot{x}_5 = x_5 (1 - x_5) ((Px)_5 - (Px)_6)
\end{cases}
\]

where \( \dot{x}_2 = -\dot{x}_1, \dot{x}_4 = -\dot{x}_3, \) and \( \dot{x}_6 = -\dot{x}_5 \). Its phase space is

\[
\Gamma_{2.2,2} := \Delta^1 \times \Delta^1 \times \Delta^1 \subset \mathbb{R}^6,
\]

a three-dimensional submanifold of \( \mathbb{R}^6 \), where

\[
\Delta^1 = \{(x_i, x_{i+1}) \in \mathbb{R}^2 | x_i + x_{i+1} = 1, x_i, x_{i+1} \geq 0\}, \quad i \in \{1, 3, 5\}.
\]

Fixing a referential on \( \mathbb{R}^3 \), by (2) we define a bijection between \( \Gamma_{2.2,2} \subset \mathbb{R}^6 \) and \([0, 1]^3 \subset \mathbb{R}^3 \). In Table 1 (left) we associate each vertex of the cube \([0, 1]^3 \) with a vertex on \( \Gamma_{2.2,2} \), where \((1, 0, 1, 0, 1, 0) \in \Gamma_{2.2,2} \) and \((0, 0, 0) \in [0, 1]^3 \) are identified.

Given the polymatrix replicator (1), by Alishah et al. (2015, Proposition 1), we may obtain an equivalent game with another payoff matrix whose second row of each group has 0’s in all of its entries. From now on, we will consider system (4) with payoff matrix

\[
P_{\mu} = \begin{pmatrix}
102 & \mu & 0 & -158 & -18 & -9 \\
0 & 0 & 0 & 0 & 0 & 0 \\
-51 & 51 & 0 & 0 & -9 & 18 \\
0 & 0 & 0 & 0 & 0 & 0 \\
-102 & -153 & 237 & 0 & 27 & 9 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}.
\]
Remark The explicit expression for $P_{\mu}$ has been motivated by comments about the work (Peixe and Rodrigues 2022) and its finding has been possible due to the numerical expertise of the first author in previous works (Alishah et al. 2015; Peixe 2019, 2015; Alishah et al. 2023).

From now on, let $(2, 2, 2, P_{\mu})$ be the polymatrix game associated to (4). For $P = P_{\mu}$, system (4) becomes

$$
\begin{align*}
\dot{x}_1 &= x_1(1-x_1)(P_{\mu}x_1) \\
\dot{x}_3 &= x_3(1-x_3)(P_{\mu}x_3) \\
\dot{x}_5 &= x_5(1-x_5)(P_{\mu}x_5)
\end{align*}
$$

(5)

Considering $x = x_2$, $y = x_4$, $z = x_6$ and using (2), Eq. (5) is equivalent to the following equation defined on the cube $[0, 1]^3$:

$$
\begin{align*}
\dot{x} &= x(1-x) (-84 + (102 - \mu)x + 158y - 9z) \\
\dot{y} &= y(1-y) (60 - 102x - 27z) \\
\dot{z} &= z(1-z) (-162 + 51x + 237y + 18z)
\end{align*}
$$

(6)

Vertices, edges and faces of the cube are flow-invariant. In order to lighten the notation, when there is no risk of misunderstanding, the one-parameter vector fields associated to (5) and (6) will be denoted by $f_{\mu}$ and its flow by $\varphi(t, u_0), t \in \mathbb{R}_0^+, u_0 \in \Gamma_{(2,2,2)}$ (for (5)) and $u_0 \in [0, 1]^3$ (for (6)). When there is no risk of misunderstanding, we omit the dependence on $\mu$.

Remark As performed in Peixe and Rodrigues (2022), in the transition from (5) to (6), we have identified the point $(1, 0, 1, 0, 1, 0) \in \Gamma_{(2,2,2)}$, associated to a pure strategy in the original polymatrix replicator, with $(0, 0, 0) \in \mathbb{R}^3$ (cf. Table 1).

### Table 1

| Vertex | $\mathbb{R}^3$ | $\mathbb{R}^6$ | Face | Vertices |
|--------|----------------|----------------|------|----------|
| $v_1$  | $(0, 0, 0)$    | $(1, 0, 1, 0, 1, 0)$ | $\sigma_1$ | $\{v_5, v_6, v_7, v_8\}$ |
| $v_2$  | $(0, 0, 1)$    | $(1, 0, 1, 0, 0, 1)$ | $\sigma_2$ | $\{v_1, v_2, v_3, v_4\}$ |
| $v_3$  | $(0, 1, 0)$    | $(1, 0, 0, 1, 1, 0)$ | $\sigma_3$ | $\{v_3, v_4, v_5, v_6\}$ |
| $v_4$  | $(0, 1, 1)$    | $(1, 0, 0, 1, 0, 1)$ | $\sigma_4$ | $\{v_1, v_2, v_5, v_6\}$ |
| $v_5$  | $(1, 0, 0)$    | $(0, 1, 1, 0, 0, 1)$ | $\sigma_5$ | $\{v_2, v_4, v_6, v_8\}$ |
| $v_6$  | $(1, 0, 1)$    | $(0, 1, 0, 0, 1, 0)$ | $\sigma_6$ | $\{v_1, v_3, v_5, v_7\}$ |
| $v_7$  | $(1, 1, 0)$    | $(0, 1, 0, 1, 1, 0)$ |      |          |
| $v_8$  | $(1, 1, 1)$    | $(0, 1, 0, 1, 0, 1)$ |      |          |

It defines a polynomial vector field on the compact flow-invariant set $\Gamma_{(2,2,2)}$. By compactness of $\Gamma_{(2,2,2)}$, the flow associated to system (4) is complete, i.e. all solutions are defined for all $t \in \mathbb{R}$.
Notation

The following terminology will be used throughout the manuscript:

- \( \mathcal{V} : \{v_1, ..., v_8\} \)
- \( \mathcal{F} \): set of all faces of the cube \([0, 1]^3\)
- \( \mathcal{F}_v \): set of faces \(\sigma_j\), for which the component \(x_j\) of \(v\) are zero, \(v \in \mathcal{V}\).

3 Preliminaries

In this section we define the main concepts used throughout the article. For \(n \in \mathbb{N}\), we are considering the Banach space \(\mathbb{R}^n\) endowed with the usual norm \(\|\cdot\|\) and the usual Euclidean metric \(\text{dist}\). The symbol \(\ell\) denotes the Lebesgue measure of \(\mathbb{R}^n\).

3.1 Admissible Path and Heteroclinic Cycle

For \(n \in \mathbb{N}\), we consider a smooth one-parameter family of vector fields \(f_\mu\) on \(\mathbb{R}^n\), with flow given by the unique solution \(u(t) = \varphi(t, u_0)\) of

\[
\dot{u} = f_\mu(u), \quad \varphi(0, u_0) = u_0,
\]

where \(\dot{u} = \frac{du}{dt}\), \(u_0 \in \mathbb{R}^n\), \(t \in \mathbb{R}\) and \(\mu\) is a real parameter. If \(A \subseteq \mathbb{R}^n\), we denote by \(\text{int}(A)\), \(A\) and \(\partial A\) the topological interior, closure and boundary of \(A\), respectively.

3.1.1 \(\alpha\) and \(\omega\)-Limit Set

For a solution of (7) passing through \(u_0 \in \mathbb{R}^n\), the set of its accumulation points as \(t\) goes to \(+\infty\) is the \(\omega\)-limit set of \(u_0\) and will be denoted by \(\omega(u_0)\). More formally,

\[
\omega(u_0) = \bigcap_{T=0}^{+\infty} \left( \bigcup_{t>T} \varphi(t, u_0) \right).
\]

The set \(\omega(u_0)\) is closed and flow-invariant, and if the \(\varphi\)-trajectory of \(u_0\) is contained in a compact set, then \(\omega(u_0)\) is non-empty (Guckenheimer and Holmes 2013). If \(Y \subset \mathbb{R}^n\), we define \(\omega(Y)\) as the union of all \(\omega\)-limits of \(y \in Y\). We define analogously, the \(\alpha\)-limit set by reversing the evolution of \(t \in \mathbb{R}\). We set the stable and unstable sets of \(Y\) as:

\[
W^s(Y) = \left\{ x \in \mathbb{R}^n : \lim_{t \to +\infty} \text{dist}(\varphi(t, x), Y) = 0 \right\}
\]
and

$$W^u(Y) = \left\{ x \in \mathbb{R}^n : \lim_{t \to -\infty} \text{dist}(\varphi(t, x), Y) = 0 \right\},$$

respectively, which have the structure of smooth manifold.

### 3.1.2 Heteroclinic Cycles

We introduce the concept of heteroclinic connection, heteroclinic path, heteroclinic cycle and network associated to a finite set of hyperbolic equilibria. We address the reader to Field (2020) for more information on the subject.

**Definition 3.1** For $m \in \mathbb{N}$, given two hyperbolic equilibria of saddle-type $A$ and $B$ associated to the flow of (7), an $m$-dimensional heteroclinic connection from $A$ to $B$, denoted $[A \to B]$, is an $m$-dimensional connected and flow-invariant manifold contained in $W^u(A) \cap W^s(B)$.

**Definition 3.2** For $k \in \mathbb{N}$, given a sequence of 1-dimensional heteroclinic connections $\{\gamma_0, ..., \gamma_k\}$ for (7), we say that it is an admissible path if for all $j \in \{0, 1, ..., k-1\}$, we have $\omega(\gamma_j) = \alpha(\gamma_{j+1})$. If $\omega(\gamma_k) = \alpha(\gamma_0)$, this sequence is called a heteroclinic cycle. A heteroclinic network is a connected union of (at least two) different heteroclinic cycles.

When there is no risk of misunderstanding, we represent the cycles and networks by the ordered set of their associated saddles as in Definitions 6.7 and 6.22 of Field (2020). In general, heteroclinic networks are represented by directed graphs where the vertices represent the equilibria and the oriented edges represent heteroclinic connections.

### 3.2 Stability

We recall the following definitions that can be found in Podvigina et al. (2020), Podvigina et al. (2019). In what follows $X, Y \subset \mathbb{R}^n$ are compact flow-invariant sets for the system (7).

**Definition 3.3** (1) The set $X$ is Lyapunov stable if for any neighbourhood $U$ of $X$, there exists a neighbourhood $V$ of $X$ such that

$$\forall x \in V, \quad \forall t \in \mathbb{R}^+, \quad \varphi(t, x) \in U.$$  

(2) The set $X$ is asymptotically stable if it is Lyapunov stable and in addition the neighbourhood $V$ can be chosen such that:

$$\forall x \in V, \quad \lim_{t \to +\infty} \text{dist}(\varphi(t, x), X) = 0.$$

(3) The set $X$ is globally asymptotically stable in $Y$ if it attracts all trajectories starting at $Y$.
(4) The set $X$ is unstable if it is not Lyapunov stable.

A heteroclinic cycle that belongs to a network cannot be asymptotically stable because it does not contain the entire unstable manifolds of all its equilibria (according to Podvigina et al. 2020, it is not clean). Various intermediate notions of stability have been introduced over the last decades—we refer the reader to Podvigina et al. (2019, 2020)\(^1\) for a nice description of these different levels of stability.

### 3.3 Likely Limit-Set

With respect to (7), we introduce two concepts that will be used throughout the article.

**Definition 3.4** If $X$ is a compact invariant subset of $\mathbb{R}^n$, the *basin of attraction* of $X$, denoted by $B(X)$, is the set

$$\{ x \in \mathbb{R}^n : \omega(x) \subset X \}.$$

**Definition 3.5** (Milnor 1985) If $Y \subset \mathbb{R}^n$ is a measurable forward invariant set with $\ell(Y) > 0$, the *likely limit set* of $Y$, denoted by $\mathcal{L}(Y)$, is the smallest closed invariant subset of $Y$ that contains all $\omega$-limit sets except for a subset of $Y$ of zero Lebesgue measure.

When we restrict the flow to a compact set, $\mathcal{L}(Y)$ is non-empty, compact and forward invariant (Milnor 1985).

### 3.4 Switching Node

The next definition is adapted from Rodrigues (2013), Castro et al. (2010). Let $A, B, X_1$ and $X_2$ be four saddle-equilibria of (7). Given a neighbourhood $V_A, V_B$ of $A$ and $B$, respectively, we say:

1. a point $p$ follows the 1-dimensional connection $[A \rightarrow B]$ at a distance $\varepsilon > 0$ if there is a $\tau > 0$ such that $\varphi(0, p) \equiv p \in V_A$, $\varphi(\tau, p) \in V_B$, and such that for all $t \in [0, \tau]$ the trajectory $\varphi(t, p)$ lies at a distance less than $\varepsilon$ from the connection $\gamma := [A \rightarrow B]$ (Fig. 1).

2. there is switching at the node $B$ (or $B$ is a switching node) if given a neighbourhood $V_B$ of $B$, for any $\varepsilon > 0$, and for any $(n-1)$-dimensional disk $D$ that meets the connection $\gamma := [A \rightarrow B]$ transversely, there are points in $D$ that follow each of the connections $\gamma_1 := [B \rightarrow X_1]$ and $\gamma_2 := [B \rightarrow X_2]$ at a distance $\varepsilon$ (Fig. 1).

3. a point $p \in V_A$ follows the admissible path $\{\gamma_0, ..., \gamma_k\}$, $k \in \mathbb{N}$ at a distance $\varepsilon > 0$ if there exist $q \in \mathbb{R}^n$ and two monotonically increasing sequences of times $(t_i)_{i \in \{0, 1, ..., k+1\}}$ and $(s_i)_{i \in \{0, 1, ..., k\}}$ such that for all $i \in \{0, ..., k\}$ we have $t_i < s_i < t_{i+1}$ and

   - $\varphi(t, p)$ lies in a $\varepsilon$-tubular neighbourhood of $\{\gamma_0, ..., \gamma_k\}$ for all $t \in [t_1, t_{k+1}]$;

---

\(^1\) There is an abundance of references in the literature. We choose to mention only two, based on our personal preferences. The reader interested in further detail may use the references within those we mention.
Fig. 1 Illustration of a switching node (B): for \(i \in \{1, 2\}\), there are initial conditions in \(D_i \subset D\) whose trajectories follow \(\gamma_i\) • \(\varphi(t_i, q) \in V_{\alpha(\gamma_i)}\) and \(\varphi(s_i, q)\) lies in a \(\varepsilon\)-tubular neighbourhood of \(\gamma_i\) disjoint from \(V_{\alpha(\gamma_i)}\) and \(V_{\omega(\gamma_i)}\); • for all \(t \in [s_i, s_{i+1}]\), the trajectory \(\varphi(t, p)\) does not visit the neighbourhood of any other saddle except that of \(\omega(\gamma_i)\).

Under the previous hypotheses, as depicted in Fig. 1, if \(B\) is a switching node we may define \(D_1, D_2 \subset D\) such that initial conditions within \(D_1, D_2\) follow the connections \(\gamma_1 = [B \rightarrow X_1]\) and \(\gamma_2 = [B \rightarrow X_2]\), respectively. These regions may be explicitly determined in the local coordinates of the cross section (Rodrigues 2017, Lemma 6).

4 Bifurcation Analysis

We proceed to the analysis of the one-parameter family of differential equations (6) in \([0, 1]^3\). Our analysis will be focused on \(\mu \in \mathcal{I} := \left[\frac{850}{11}, \frac{544}{3}\right]\) since for all \(\mu \in \text{int} (\mathcal{I})\) there exists a unique equilibrium in \(\text{int} [0, 1]^3\).

4.1 Boundary Dynamics

We describe a list of equilibria that appear on \(\partial [0, 1]^3\), as function of the parameter \(\mu\). We emphasise the bifurcations the equilibria undergo.

From now on, all figures with numerical plots of the flow of (6) on \([0, 1]^3\) are in the same position as Fig. 2 where \(v_1 = (0, 0, 0)\) is the vertex in light blue located in the lower left front corner. The cube has six faces defined, for \(i \in \{1, 2, 3\}\), by

\[
\sigma_{2i-1} = \{(x_1, x_2, x_3) \in \partial [0, 1]^3 : x_i = 1\},
\]

\[
\sigma_{2i} = \{(x_1, x_2, x_3) \in \partial [0, 1]^3 : x_i = 0\}.
\]

2 If \(\gamma = [A \rightarrow B]\) then \(\alpha(\gamma) = A\) and \(\omega(\gamma) = B\).
Fig. 2 The phase space and the corresponding equilibria of (6): the eight vertices \( v_1, \ldots, v_8 \) (blue), two equilibria on faces, \( B_1, B_2 \) (green), and the interior equilibrium \( O_\mu \) (in red), for \( \mu = 85 \) (left), \( \mu = 97 \) (center) and \( \mu = 106 \) (right). The interior equilibrium \( O_\mu \) lies on the line segment \( r \) that connects \( B_1 \) to \( B_2 \) (Lemma 5).

In Table 1 we identify the vertices that belong to each face. As suggested in Fig. 2, we denote by \( B_1 \) and \( B_2 \) the equilibria on the interior of the faces \( \sigma_5 \) and \( \sigma_6 \), respectively. These equilibria depend on \( \mu \) but we omit their dependence on the parameter.

**Lemma 1** For \( \mu \in \mathcal{I} \), the vertices \( v_1, \ldots, v_8 \) and

\[
B_1 = \left( \frac{11}{34}, \frac{2040 + 11\mu}{5372}, 1 \right) \quad \text{and} \quad B_2 = \left( \frac{10}{17}, \frac{204 + 5\mu}{1343}, 0 \right)
\]

are the unique equilibria of (6) on the cube’s boundary.

The proof of Lemma 1 is straightforward by computing zeros of \( f_\mu \) and taking into account that equilibria lie in \( \partial[0,1]^3 \). The eigenvalues and eigendirections of the vertices and the \( B \)’s are summarised in Tables 8 and 9 in Appendix A, respectively.

**Definition 4.1** If \( A \) is a hyperbolic saddle-focus for system (6), we say that it is of type \((1,2)\) if \( Df_\mu (A) \) has a pair of non-real eigenvalues and \( \dim W^s(A) = 1 \), \( \dim W^u(A) = 2 \).

The evolution of the eigenvalues’ sign as function of \( \mu \) (Tables 8 and 9) allows us to formulate the following result.

**Lemma 2** For \( \mu \in \mathcal{I} \), the following assertions hold for (6):

1. if \( \frac{850}{11} < \mu < 102 \), then \( B_1 \) and \( B_2 \) are saddle-foci of type \((1,2)\);
2. if \( \mu = 102 \), then \( B_1 \) and \( B_2 \) are non-hyperbolic when restricted to the corresponding faces;
3. if \( 102 < \mu < \frac{544}{5} \), then \( B_1 \) and \( B_2 \) are sinks.

Since there are no more invariant sets on the faces (for \( \mu \neq 102 \)), besides \( B_1, B_2 \) and the vertices, we may conclude that:

---

\(^3\) For \( \mu = 102 \), the equilibria \( B_1 \) and \( B_2 \) are centers when restricted to the corresponding faces. The dynamics corresponds to a non-generic bifurcation as described in Section 4 of Alishah et al. (2015).
Lemma 3 With respect to system (6), the following assertions hold:

1) For $\mu \in \mathcal{I}$, if $p \in \text{int}(\sigma_i)$, $i = 1, 2, 3, 4$, then $\omega(p)$ is a vertex.

2) For $\mu \in [\frac{850}{11}, 102[$:
   
   (a) if $p \in \text{int}(\sigma_5)$, then $\omega(p)$ is the cycle defined by $\{v_2, v_4, v_8, v_6\};$
   
   (b) if $p \in \text{int}(\sigma_6)$, then $\omega(p)$ is the cycle defined by $\{v_1, v_3, v_7, v_5\}.$

3) For $\mu \in ]102, \frac{544}{5}[$:
   
   (a) if $p \in \text{int}(\sigma_5)$, then $\omega(p) = \{B_1\};$
   
   (b) if $p \in \text{int}(\sigma_6)$, then $\omega(p) = \{B_2\}.$

4.2 Interior Equilibrium

We focus our attention on the interior equilibrium and its relation to others on the cube's boundary.

Lemma 4 For $\mu \in \text{int}(\mathcal{I})$, system (6) has a unique interior equilibrium, whose expression is

$$O_\mu := \left(\frac{68}{442 - 3\mu}, \frac{2(9180 - 61\mu)}{79(442 - 3\mu)}, \frac{4(544 - 5\mu)}{1326 - 9\mu}\right).$$

The proof is immediate by computing the zeros of $f_\mu$ lying outside $\partial[0, 1]^3.$ Taking into account that $O_\mu$, $B_1$ and $B_2$ depend on $\mu$, it is worth to notice that

$$\lim_{\mu \to \frac{850}{11}} O_\mu = \lim_{\mu \to \frac{850}{11}} B_1 = \left(\frac{11}{34}, \frac{85}{158}, 1\right) \in \sigma_5$$

and

$$\lim_{\mu \to \frac{544}{5}} O_\mu = \lim_{\mu \to \frac{544}{5}} B_2 = \left(\frac{10}{17}, \frac{44}{79}, 0\right) \in \sigma_6,$$

which means that along $\mathcal{I}$, the point $O_\mu$ travels from the face $\sigma_5$ to $\sigma_6$. The following result shows an elegant relative position of the equilibria $B_1$, $B_2$ and $O_\mu$ (see Fig. 2).

Lemma 5 For $\mu \in \mathcal{I}$, the interior equilibrium $O_\mu$ belongs to the segment $[B_1 B_2]$.

Proof Let $r$ be the segment $[B_1 B_2]$ defined by

$$r : (x, y, z) = B_1 + k B_1 B_2, \quad \text{for } k \in [0, 1].$$

By a simple computation we have that

$$O_\mu \in r \iff k = \frac{850 - 11\mu}{9\mu - 1326} \in [0, 1] \iff \mu \in \mathcal{I}. $$
Fig. 3 Graph of the real eigenvalue of $Df_{\mu}(O_{\mu})$ (left) and graph of the imaginary part of the complex eigenvalues of $Df_{\mu}(O_{\mu})$ (right) where $\mu \in I$, for system (6)

Lemma 6 There exists $\mu_2 \in I$ such that the equilibrium $O_{\mu}$ undergoes a supercritical Hopf bifurcation as $\mu$ increases.

Proof For $\mu \in I$, $Df_{\mu}(O_{\mu})$ depends on $\mu$ and is explicitly given by

\[
\begin{pmatrix}
\frac{68(\mu-102)(3\mu-374)}{(442-3\mu)^2} & -\frac{10744(3\mu-374)}{(442-3\mu)^2} & \frac{612(3\mu-374)}{(442-3\mu)^2} \\
-\frac{204(61\mu-9180)(115\mu-16558)}{6241(442-3\mu)^2} & 0 & -\frac{54(61\mu-9180)(115\mu-16558)}{6241(442-3\mu)^2} \\
-\frac{68(5\mu-544)(11\mu-850)}{3(442-3\mu)^2} & -\frac{316(5\mu-544)(11\mu-850)}{3(442-3\mu)^2} & -\frac{8(5\mu-544)(11\mu-850)}{442(442-3\mu)^2}
\end{pmatrix},
\]

whose characteristic polynomial has three roots. Although they have an intractable expression, it is possible to show that for $\mu_2 \approx 105.04 \in I$, $Df_{\mu}(O_{\mu})$ has a pair of complex (non-real) eigenvalues of the type $\alpha(\mu) \pm i\beta(\mu)$ such that $\alpha, \beta$ are $C^1$ maps, depend on $\mu$ and:

1. $\beta(\mu_2) > 0$ (Fig. 3);
2. $\alpha$ is positive for $\mu < \mu_2$;
3. $\alpha$ is negative for $\mu > \mu_2$ (Fig. 4).

As suggested by Fig. 4 (right), the complex (non-real) eigenvalues cross the imaginary axis with positive speed as $\mu$ passes through $\mu_2$, confirming that:

\[
\frac{d \alpha}{d \mu} \bigg|_{\mu=\mu_2} \neq 0.
\]

This means that at $\mu = \mu_2$, the equilibrium $O_{\mu}$ undergoes a supercritical Hopf bifurcation as $\mu$ increases, destroying an attracting periodic solution, say $C_{\mu}$. ☐

4.3 Terminology

For $\mu < \mu_2$, the interior equilibrium $O_{\mu}$ is a source and the tangent space $\mathbb{R}^3$ may be decomposed as two $Df_{\mu}(O_{\mu})$–invariant subspaces $E_1^u$ and $E_2^u$ (in direct sum) such that $\dim E_j^u = j$, for $j = 1, 2$. ☝ Springer
Fig. 4   Illustration of $\frac{d\alpha}{d\mu}\bigg|_{\mu=\mu_2} \neq 0$. Graph of the real part of the non-real eigenvalues of $Df_\mu (O_\mu)$ for $\mu \in I$ (left) and its zoom around $\mu_2$, with $\mu \in [104.99, 105.09]$ (right), for system (6)

The set $E^{u}_1$ is the eigendirection associated to the real positive eigenvalue and $E^{u}_2$ is associated to the complex (non-real) eigenvalues. We denote by $W^{u}_2(O_\mu)$ the part of the invariant manifold whose tangent space at $O_\mu$ is $E^{u}_2$. For $\mu > \mu_2$, $W^{u}_2(O_\mu)$ stands for the stable manifold of $O_\mu$. Let

$$I_1 = \left[ \frac{850}{11}, \mu_1 \right], \quad I_2 = [\mu_1, \mu_2[, \quad \text{and} \quad I_3 = \left] \mu_2, \frac{544}{5} \right]$$

where $\mu_1 = 102$ and $\mu_2 \approx 105.04$. Therefore, we may identify the following local bifurcations:

$$I = I_1 \cup \{\mu_1\} \cup I_2 \cup \{\mu_2\} \cup I_3$$

- $\mu_1 = 102$ : Change of stability of $B_1$ and $B_2$ (Lemma 2);
- $\mu_2 \approx 105.04$ : Supercritical Hopf bifurcation of $O_\mu$ destroying a periodic solution $C_\mu$ (Lemma 6).

### 4.4 Heteroclinic Network

In this subsection, we show that the flow of (6) exhibits a heteroclinic network formed by six cycles.

**Lemma 7** For $\mu \in I$, the flow associated to (6) has six heteroclinic cycles whose connections are associated to the following set of hyperbolic equilibria (Fig. 5):

1. $\mathcal{H}_1 := \{v_2, v_4, v_8, v_6, v_2\}$
2. $\mathcal{H}_2 := \{v_1, v_3, v_4, v_8, v_6, v_2, v_1\}$
3. $\mathcal{H}_3 := \{v_1, v_3, v_4, v_8, v_6, v_5, v_1\}$
4. $\mathcal{H}_4 := \{v_1, v_3, v_7, v_8, v_6, v_2, v_1\}$
5. $\mathcal{H}_5 := \{v_1, v_3, v_7, v_8, v_6, v_5, v_1\}$
6. $\mathcal{H}_6 := \{v_1, v_3, v_7, v_6, v_1\}$
Proof Since there are no equilibria on the edges besides the vertices, analysing the eigenvalues of system (6) at the vertices (see Table 8), the result follows.

From now on, denote by $\mathcal{H}$ the heteroclinic network $\mathcal{H}_1 \cup \cdots \cup \mathcal{H}_6$.

Lemma 8 For $\mu \in \mathcal{I}$, the equilibria $v_2, v_3, v_6, v_7$ are switching nodes for system (6).

Proof The proof follows from observing Table 8. At these equilibria there are two positive real eigenvalues ($\iff$ two arrows leave the equilibrium in the corresponding graph).

4.5 Numerics

We list some numerical evidence, hereafter called by Facts, about system (6).

Fact 1 For $\mu \in \mathcal{I}\{102\}$, there exists an open 2-dimensional invariant manifold $\mathcal{M}_\mu$ containing:

(i) $W^u_2(O_\mu)$ for $\mu < \mu_2$

(ii) $W^s_2(O_\mu)$ for $\mu > \mu_2$.

such that $\partial \mathcal{M}_\mu \subset \mathcal{H}$ and there are no more compact sets in $\text{int } ([0,1]^3) \setminus \mathcal{M}_\mu$.

Fact 2 For $\mu \in \mathcal{I}$, there are two 1-dimensional heteroclinic connections $[O_\mu \to B_1]$ and $[O_\mu \to B_2]$.

It is possible to observe numerically that $\mathcal{M}_\mu$ referred in Fact 1:
• coincides with $W^u_2(\mathcal{O}_\mu)$ for $\mu \in \mathcal{I}_1$;
• contains $\{\mathcal{O}_\mu\} \cup \tilde{W}^s(\mathcal{C}_\mu)$ for $\mu \in \mathcal{I}_2$, where $\mathcal{C}_\mu$ is the periodic solution associated to the Hopf Bifurcation described in Lemma 6;
• coincides with $W^s(\mathcal{O}_\mu)$ for $\mu \in \mathcal{I}_3$.

4.6 Stability of $\mathcal{H}$

The next result asserts that the network $\mathcal{H}$ is globally asymptotically stable in $[0, 1]^3 \setminus \{\mathcal{M}_\mu, [\mathcal{O}_\mu \rightarrow B_1], [\mathcal{O}_\mu \rightarrow B_2]\}$, for $\mu \in \mathcal{I}_1$.

**Lemma 9** For $B(\mathcal{H}) = [0, 1]^3 \setminus \{\mathcal{M}_\mu, [\mathcal{O}_\mu \rightarrow B_1], [\mathcal{O}_\mu \rightarrow B_2]\}$.

**Proof** If $u_0 \in \partial [0, 1]^3$, the result follows from Lemma 3.

If $u_0 \in \text{int}([0, 1]^3) \setminus \{\mathcal{M}_\mu, [\mathcal{O}_\mu \rightarrow B_1], [\mathcal{O}_\mu \rightarrow B_2]\}$, then $\varphi(t, u_0)$ accumulates on a compact invariant set. Since there are no more invariant sets in $\text{int}([0, 1]^3) \setminus \mathcal{M}_\mu$ (Fact 1), then $\varphi(t, u_0)$ accumulates on the boundary’s cube. Since the equilibria $B_1$ and $B_2$ are sources in the corresponding faces (Lemma 2), the result follows. □

4.7 Questions

At the moment, motivated by numerical simulations, there are questions that are worth to be answered concerning the dynamics of (6).

1st: For $\mu \in \mathcal{I}_1$, the network $\mathcal{H}$ is globally asymptotically stable in $[0, 1]^3 \setminus \{\mathcal{M}_\mu, [\mathcal{O}_\mu \rightarrow B_1], [\mathcal{O}_\mu \rightarrow B_2]\}$. What is the likely limit set of $\mathcal{H}$? In other words, is there some preferred cycle to where Lebesgue-almost all solutions are attracted?

2nd: For $\mu \in \mathcal{I}_2 \cup \mathcal{I}_3$, the network $\mathcal{H}$ is not asymptotically stable, Lebesgue-almost all points in $[0, 1]^3 \setminus \mathcal{M}_\mu$ are attracted either to $B_1$ or $B_2$, and trajectories starting in $\mathcal{M}_\mu$ seems to accumulate on a cycle of $\mathcal{H}$. Could we describe which one?

In the following sections, we develop a general method to answer the previous questions. Although we describe a technique implemented to study the model of Sect. 2, the (affirmative) answers to the questions are given as a series of results that are applicable to more general types of networks.

5 Asymptotic Dynamics: The Theory

We describe a piecewise linear model from where we may analyse the dynamics associated to the asymptotic dynamics near the heteroclinic network $\mathcal{H}$ of Lemma 7.

This piecewise linear map is easily computed. We study system (5) bearing in mind that it is $C^1$–equivalent to (6), as observed at the end of Sect. 2.

5.1 Non-resonance Hypothesis

Let $\mathcal{H} \subset \Gamma_{(2,2,2)} \subset \mathbb{R}^6$ be a heteroclinic network associated to the set of hyperbolic saddles $\mathcal{V} = \{v_1, ..., v_8\}$ and one-dimensional heteroclinic connections $\mathcal{E}$.
Given \( v = (x_1, x_2, x_3, x_4, x_5, x_6) \in \mathcal{V} \), we denote by \( \mathcal{F}_v \) the set of three faces \( \sigma_j \) with \( j \in \{1, \ldots, 6\} \), for which the component \( x_j \) of \( v \) are zero. Geometrically, this means that for each \( v \in \mathcal{V} \), \( \mathcal{F}_v \) is the set of the three faces whose intersection is \( v \). From now on, we assume the following technical hypothesis:

**TH** For \( v \in \mathcal{V} \), the eigenvalues of \( Df_\mu(v) \) are non-resonant in the terminology of Ruelle (2014):

\[
\lambda_i = \lambda_j + \lambda_k,
\]

where \( \lambda_i, \lambda_j \) and \( \lambda_k \) are the nonzero eigenvalues of the linear part of the vector field (5) evaluated at the equilibrium \( v \in \mathcal{V} \) (cf. Table 8).

The necessary and sufficient conditions for \( C^1 \)-linearization of Ruelle show that linearization is not possible for subsets of points on the lines described by the restrictions above. These restrictions correspond to a set of zero Lebesgue measure in parameter space and place no serious constraint on the analysis that follows.

### 5.2 Local and Global Maps in the Phase Space

Since \( v \in \mathcal{V} \) is hyperbolic, assuming the non-resonance condition (TH) of \( Df_\mu(v) \), there exists \( \varepsilon > 0 \) (small) such that it is possible to define an open \( \varepsilon \)-cubic neighbourhood of \( v \), say \( N_v \), such that the flow associated to (5) is \( C^1 \)-conjugated to that of

\[
\dot{x} = Df_\mu(v)(x - v), \quad x \in \mathbb{R}^6.
\]

In particular, it is possible to define two cross sections, \( \text{In}(v) \subset \overline{N_v} \) and \( \text{Out}(v) \subset \overline{N_v} \), such that solutions starting in \( \text{In}(v) \setminus W^s(v) \) enter in \( N_v \) in positive time, spend

---

\[^4\] This hypothesis is equivalent to the Condition (c) of Definition 3.1 of Alishah et al. (2019).
some time there and leave the cube through $Out(v)$—see Fig. 6. It induces the local diffeomorphism:

$$P_v : In(v) \setminus W^s(v) \to Out(v).$$

Using local coordinates associated to system (5), the cubic neighbourhood $N_v$ may be defined by:

$$N_v := \{ p \in \Gamma_{(2,2,2)} : 0 < x_j(p) < \varepsilon \quad \text{for} \quad 1 \leq j \leq 6 \}$$  \hspace{1cm} (8)

where $(x_1, x_2, x_3, x_4, x_5, x_6)$ is a system of linear coordinates around $v$ which assigns coordinates $(0, 0, 0, 0, 0, 0)$ to $v$.

For $v^*, v \in \mathcal{V}$, given a one-dimensional heteroclinic connection of the type $\gamma := [v^* \to v]$, we may also define an invertible map from a small neighbourhood of $Out(v^*) \cap \gamma$ to $In(v) \cap \gamma$, that is called the global map and will be denoted by $P_{\gamma}$. This map is a diffeomorphism (Palis and de Melo 1982, Ch. 2) and is depicted in Fig. 7.

Let $T_\varepsilon$ a tubular neighbourhood of $\mathcal{H}$. It can be written as the “system of connected pipes”:

$$T_\varepsilon = \left( \bigcup_{v \in \mathcal{V}} N_v \right) \bigcup \left( \bigcup_{\gamma \in \mathcal{E}} N_\gamma \right)$$  \hspace{1cm} (9)

where:

- $N_v$ is the neighbourhood of $v$ (see (8));
- $N_\gamma$ is the tubular neighbourhood of $\gamma \in \mathcal{E}$ (of radius $\varepsilon$) defined by:

$$N_\gamma = \{ q \in \Gamma_{(2,2,2)} \setminus (N_v^* \cup N_v) : x_j \leq \varepsilon \quad \text{for all} \quad j \quad \text{such that} \quad \gamma \subset \sigma_j \quad \text{and} \quad x_j = 0 \quad \text{for all} \quad j \quad \text{such that} \quad \sigma_j \not\subset \gamma \}.$$

**Remark** In order to define correctly the set $T_\varepsilon$ we might need to shrink either the cubic neighbourhoods of the saddles or the tubular neighbourhoods of the connections. This is possible by decreasing $\varepsilon$ finitely many times (if necessary).

### 5.3 Quasi-Change of Coordinates

We describe a rescaling change of coordinates $\Psi_\varepsilon$, depending on the parameter $\varepsilon > 0$. Since the tubular neighbourhood $T_\varepsilon$ may be written as in (9), the map $\Psi_\varepsilon$ acts in different ways according to the point $q$ lies on $N_v$ or in $N_\gamma$, where $v \in \mathcal{V}$, $\gamma \in \mathcal{E}$. The variable $\varepsilon$ plays the role of blow-up parameter as we proceed to explain. The examples are related with system (5) and the index $j$ runs over $\{1, \ldots, 6\}$. 
5.3.1 Action of $\Psi_\varepsilon$ on $N_v$

In the first case, if $q \in N_v$, the rescaling change of coordinates $\Psi_\varepsilon$ takes points $q = (x_1, x_2, x_3, x_4, x_5, x_6)$ to points in the sector $\{(y_j)_{\sigma_j \in F} \}$ according to the law:

- $y_j = -\varepsilon^2 \log x_j(q) \geq 0$ if $j$ is such that the face $\sigma_j$ contains $v$;
- $y_j = 0$, otherwise.

The logarithmic change of coordinates is similar to that performed in Podvigina (2012), Garrido-da Silva and Castro (2019).

**Example:** The faces through $v = (1, 0, 1, 0, 1, 0)$ are $\sigma_2, \sigma_4, \sigma_6$; in other words, $F_v = \{\sigma_2, \sigma_4, \sigma_6\}$. Hence, the map $\Psi_\varepsilon$ is defined on the neighbourhood $N_v \setminus \Gamma(2,2,2)$ by

$$\Psi_\varepsilon(q) = (0, -\varepsilon^2 \log x_2(q), 0, -\varepsilon^2 \log x_4(q), 0, -\varepsilon^2 \log x_6(q))$$

where $(x_1, x_2, x_3, x_4, x_5, x_6)$ stands for the system of affine coordinates introduced above.

**Notation:** $\Pi_v := \Psi_\varepsilon(N_v) \subset \{(u_j)_{j \in \mathbb{R}_+^6} : u_j = 0, j \text{ such that } \sigma_j \notin F_v \}$ is a 3-dimensional subset of $(\mathbb{R}_+^6)^6$.

5.3.2 Action of $\Psi_\varepsilon$ on $N_\gamma$

Similarly, given an edge $\gamma = [v^* \to v]$, the map $\Psi_\varepsilon$ takes points $q = (x_1, x_2, x_3, x_4, x_5, x_6)$ in the neighbourhood $N_\gamma$ of $\gamma$ to points in the sector $\{(y_j)_{\sigma_j \in F} \}$ such that:

- $y_j = -\varepsilon^2 \log x_j(q) \geq 0$ if $j$ is such that the face $\sigma_j$ contains $\gamma$;
- $y_j = 0$, otherwise.

**Example:** For $\gamma_5 = [v_1 \to v_3]$ we know that $\gamma_5 = \sigma_2 \cap \sigma_6$. If $q \in N_\gamma \setminus \Gamma(2,2,2)$, then the expression of the map $\Psi_\varepsilon$ is given by:

$$\Psi_\varepsilon(q) = (0, -\varepsilon^2 \log x_2(q), 0, 0, 0, -\varepsilon^2 \log x_6(q)),$$
where \((x_1, x_2, x_3, x_4, x_5, x_6)\) stands for the system of affine coordinates introduced above.

**Notation:** \(\Pi_\gamma := \Psi_\varepsilon (N_\gamma)\) is a 2-dimensional submanifold of \((\mathbb{R}_0^+)^6\). Observe that

\[
\text{rank}(\Psi_\varepsilon (N_\gamma)) = 3 \quad \text{and} \quad \text{rank}(\Psi_\varepsilon (N_\gamma)) = 2. \tag{10}
\]

In particular, the map \(\Psi_\varepsilon\) is not injective when restricted to \(N_\gamma\). We know precisely how the loss of injectivity is performed; the map \(\Psi_\varepsilon |_{N_\gamma}\) identifies all points in the same trajectory on \(\Gamma_{(2,2,2)}\). This loss of injectivity will not affect the validity of our results. This is why we say that the map is \(\Psi_\varepsilon\) is a *quasi-change of coordinates*. We now introduce the stage where the asymptotic piecewise linear dynamics plays its role.

**Definition 5.1** The dual cone associated to the network \(\mathcal{H}\) is the set \(\bigcup_{v \in V} \Pi_v\).

Embedding the dual cone in \(\mathbb{R}^6\) is formally and computationally convenient; it may be seen as the union of sectors, one for each vertex of \(\mathcal{H}\).

The map \(\Psi_\varepsilon\) is not well defined in \(\partial \Gamma_{(2,2,2)}\). When a trajectory is approaching the network \(\mathcal{H}\), the non-zero coordinates of its image under \(\Psi_\varepsilon\) go to \(\infty\) in the dual cone. This is why we say that \(\varepsilon > 0\) plays the role of *blow-up parameter*.

**5.4 Skeleton Character**

For \(v \in V\), the main result of this subsection relates the asymptotic dynamics of \((\Psi_\varepsilon)_* f\), the push-forward of \(f\) by \(\Psi_\varepsilon\) (restricted to \(N_v\)), with a constant vector field on the dual cone associated to the network \(\mathcal{H}\). We omit the dependence of \(f\) on \(\mu \in \mathcal{I}\) to lighten the notation.

**Definition 5.2** For \(v \in V\), we define the map \(\chi^v\) as:

\[
\chi^v_j = \begin{cases} 
-\text{eigenvalue of } Df (v) \text{ in the orthogonal direction to } \sigma_j, & \text{if } \sigma_j \in \mathcal{F}_v \\
0, & \text{otherwise}
\end{cases} \tag{11}
\]

where \(j \in \{1, \ldots, 6\}\) is the component of the vector. For an equilibrium \(v \in V\), the vector field \(\chi^v = (\chi^v_j)_{j \in \{1, \ldots, 6\}}\) is called the *skeleton character* at \(v\). Note that for each \(v \in V\), three components of this map are zero.

The next result asserts that the vector field \((\Psi_\varepsilon)_* f\), rescaled by the factor \(\varepsilon^{-2}\), converges to the constant vector field \(\chi^v\). In particular, the trajectories associated to the push-forward vector field \(\varepsilon^{-2} (\Psi_\varepsilon)_* f\) are asymptotically linearised to lines i.e. there exists \(T > 0\) such that the solution with initial condition \(y \in \Pi_v\) is the segment defined by

\[
y + t \chi^v, \quad t \in [0, T], \quad y \in \Pi_v.
\]

In order to state precise results, we introduce the following definition.
Definition 5.3. For $\tau > 0$, let $(F_\lambda)_{\lambda \in [0, \tau]}$ be a one-parameter family of maps defined on $D \subset (\mathbb{R}_0^+)^6$, and $F$ be another function with the same domain. We say that $F_\lambda$ converges in the $C^1$-topology to $F$, as $\lambda$ tends to $0$, and we write

$$\lim_{\lambda \to 0} F_\lambda = F,$$

to mean that for every compact set $K \subset D$, the following equality holds:

$$\lim_{\lambda \to 0} \max_{\lambda \to 0^+} \left\{ \sup_{u \in K} [F_\lambda(u) - F(u)], \sup_{u \in K} D[F_\lambda(u) - F(u)] \right\} = 0,$$

where $D$ denotes the usual first order Fréchet derivative.

If a map is the composition of finitely many maps, the domain should be understood as the domain where the composition is well defined. From now on, let us define the sets (in the dual cone):

$$\Pi_{\nu}(\varepsilon) = \{ y \in \Pi_{\nu} : y_j \geq \varepsilon, \forall j : \sigma_j \in \mathcal{F}_\nu \},$$

$$\Pi_{\gamma}(\varepsilon) = \{ y \in \Pi_{\gamma} : y_j \geq \varepsilon, \forall j : \sigma_j \in \mathcal{F}_{\nu^*} \cap \mathcal{F}_\nu \}.$$

We omit the dependence on $\varepsilon$ of $\Pi_{\nu}(\varepsilon)$ and $\Pi_{\gamma}(\varepsilon)$ to lighten the reading. The main result of this subsection is:

Lemma 10 (Alishah et al. 2019, Lemma 5.6) The following equality holds for $v \in \mathcal{V}$:

$$\lim_{\varepsilon \to 0} \varepsilon^{-2} (\Psi_{\varepsilon})_* f |_{\Pi_{\nu}(\varepsilon)} = \chi^v.$$

5.5 Global Map in the Dual Cone

For $v_*, v \in \mathcal{V}$ and $\gamma := [v^* \rightarrow v]$, let

$$P_\gamma : \text{Out}(v_*) \rightarrow \text{In}(v)$$

be the diffeomorphism defined in Sect. 5.2. Define the map:

$$H^\varepsilon := \Psi_{\varepsilon} \circ P_\gamma \circ (\Psi_{\varepsilon})^{-1} : \Psi_{\varepsilon}(\text{Out}(v_*)) \rightarrow \Psi_{\varepsilon}(\text{In}(v)).$$

The next result ensures that, although the original global map $P_\gamma$ is given by an invertible linear map (cf. Sect. 5.2), the map $H^\varepsilon$ converges, in the $C^1$-topology, to the Identity map (denoted by $I$) as $\varepsilon \to 0$.

Lemma 11 (Alishah et al. 2019, Lemma 7.2) The following equality holds for $v, v^* \in \mathcal{V}$:

$$\lim_{\varepsilon \to 0} H^\varepsilon |_{\Psi_{\varepsilon}(\text{Out}(v^*)) \cap \Pi_{\gamma}(\varepsilon)} = I.$$

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Lemma 11 says that, for any heteroclinic connection of the type $\gamma = [v^* \to v]$, we can asymptotically identify the two sections $\Psi_\varepsilon(Out(v^*))$ and $\Psi_\varepsilon(In(v))$. We will refer to the identified sections as the two-dimensional manifold $\Pi_\varepsilon(\gamma)$; it may be seen as $\Psi_\varepsilon(\Sigma_\gamma)$, where $\Sigma_\gamma$ is (any) cross section to $\gamma$, as depicted in Fig. 8.

**Remark** In order to get an approximation of Lemmas 10 and 11 in topology $C^r, r > 1$, we might need to rescale the radius of $T_\varepsilon$ defined in (9). This is not necessary to the scope of the present work since conclusions about the stability of cycles hold in the $C^1$–topology.

### 5.6 Shadowing a Path of Order 2

As depicted in Fig. 8, define $D_{\gamma,\gamma'}$ as the set of points within $Out(v^*)$ whose trajectories follow the connection $\gamma' = [v \to v']$ at a distance $\varepsilon > 0$ and set

$$D_{\gamma,\gamma'}^* = \Psi_\varepsilon(D_{\gamma,\gamma'}) \subset \Pi_\varepsilon(\gamma).$$

Let $P_{\gamma,\gamma'}$ be the map that carries points from $D_{\gamma,\gamma'} \subset \Sigma_\gamma$ to $Out(v) \cap \gamma'$. For the admissible path $\{\gamma, \gamma'\}$ defined as above, define:

$$F_{\gamma,\gamma'} = \Psi_\varepsilon \circ P_{\gamma,\gamma'} \circ (\Psi_\varepsilon)^{-1}|_{D_{\gamma,\gamma'}^*}.$$

Denote by $j^*$ the index of the face $\sigma^*$ within $F_v$ orthogonal to $\gamma'$.—see Fig. 6. Consider the sector $\Pi_{\gamma,\gamma'} \subset int(\Pi_\gamma)$ defined as

$$\Pi_{\gamma,\gamma'} := \left\{ y \in int(\Pi_\gamma) : y_j > \frac{\chi_j^v}{\chi_j^v} y_{j^*}, \forall j : \sigma_j \in F_v, \sigma_j \neq \sigma_{j^*} \right\}, \quad (12)$$
containing all points in $\text{int}(\Pi_\gamma)$ whose images by $(\Psi_\varepsilon)^{-1}$ follow the admissible path $\{\gamma, \gamma'\}$ at a given positive (small) distance.

**Lemma 12** The following equality holds for the admissible path $\{\gamma, \gamma'\}$:

$$\lim_{\varepsilon \to 0} F_{\gamma, \gamma'} = L_{\gamma, \gamma'}$$

where $L_{\gamma, \gamma'} : \Pi_{\gamma, \gamma'} \to \Pi_{\gamma'}$ is the linear map defined by:

$$L_{\gamma, \gamma'}(y) = \left( y_j - \frac{x_j^\gamma}{x_j} y_{j_\sigma} \right)_{\sigma_j \in \mathcal{F}}.$$

**Proof** We consider in $\Sigma_\gamma$ (section transverse to $\gamma$), the points that follow the chain of heteroclinic connections

$$\gamma = [v^* \to v], \quad \gamma' = [v \to v'].$$

Observe that the equilibrium $v$ is a switching node of $\mathcal{H}$. This means that $Df(v)$ has two positive real eigenvalues, say $E_2, E_1$ where $E_2 > E_1$, and one negative, say $-C$.

Let us consider a neighbourhood $N_v$ and the nonzero coordinates $(x, y, z)$ introduced in 8, in such a way that $v \mapsto (0, 0, 0)$, the axis $Ox$ is associated to the eigenvalue $E_1$, the axis $Oy$ is associated to the eigenvalue $E_2$, and the $Oz$ is associated to the eigenvalue $-C < 0^5$. Therefore, using the non-resonance eigenvalue condition (TH)

---

5 The values of $E_1, E_2$ and $C$ depend on $\mu \in \mathcal{I}$; we omit this dependence in order to lighten the notation.
of Sect. 5.1, by Lemma 10, the system of ODEs that locally describes the vector field in \( N_v \), is given by

\[
\begin{align*}
\dot{x} &= E_1 x, \\
\dot{y} &= E_2 y, \\
\dot{z} &= -Cz
\end{align*}
\]  

(13)

and the solution of (13) is:

\[
\begin{align*}
x(t) &= x_0 e^{E_1 t} \\
y(t) &= y_0 e^{E_2 t} \\
z(t) &= z_0 e^{-Ct}
\end{align*}
\]  

(14)

where \((x_0, y_0, z_0) \in \mathbb{R}^3_+\). The local map from the cross section \( \text{In}(v) = \{ z = \varepsilon \} \) to the connected component of \( \text{Out}(v) \) defined by \( \{ y = \varepsilon \} \) is given (in local coordinates \((x, y, \varepsilon) \equiv (x, y)\)) by

\[
P_v(x, y) = \left( \frac{x_0 y_0 - E_1 E_2}{E_2}, \frac{C}{E_2} y_0 \right)
\]

and the associated time of flight is

\[
\frac{1}{E_2} \ln \left( \frac{\varepsilon}{y_0} \right).
\]

The line defined by \( x = 1 \wedge y = 1 \) is the intersection of the two connected components of \( \text{Out}(v) \). Noticing that \( x_0 y_0^{-E_1 E_2} > 1 \) is equivalent to \( x_0 > y_0^{\frac{E_1}{E_2}} \), one may define the region of points in \( \{ z = \varepsilon \} \) that follow the admissible path \( \{ y, y' \} \) as

\[
y_j > \frac{E_1}{E_2} y_j^* \quad \text{(5.3)} \quad \frac{\chi_j^v}{\chi_j^v} y_j^*
\]

and the result is proved.

The 6 \times 6 matrix associated to \( L_{\gamma, \gamma'} \) (of Lemma 12) may be represented by

\[
M_{\gamma, \gamma'} = \left( \delta_{ik} - \frac{\chi_i^v}{\chi_j^v} \delta_{j,k} \right)_{\sigma_i, \sigma_k \in \mathcal{F}},
\]

(15)

where \( \delta \) represents the usual Kronecker delta operator.

5.7 Digestive Remarks

We point out some important remarks about the dynamics associated to \( F_{\gamma, \gamma'} \).
(1) Lemma 12 is not saying that $F_{\gamma, \gamma'} = L_{\gamma, \gamma'}$; it says that, in the $C^1$–topology, the nonlinear terms of $F_{\gamma, \gamma'}$ converge to zero as $\epsilon$ goes to zero.

(2) For $v^*, v, v' \in \mathcal{V}$, we concentrate our attention in the following chain of heteroclinic connections:

$$\gamma = [v^* \to v], \quad \gamma' = [v \to v'] \quad \text{and} \quad \gamma'' = [v \to v'']$$

where $v$ is a switching node. Since $v$ is a switching node and $Df_\mu(v)$ has real eigenvalues, up to a set of zero Lebesgue measure, the cross section $\Sigma_{\gamma}$ is divided in two regions containing initial conditions that follow $\gamma'$ and $\gamma''$ at a small distance. These regions are disjoint cusps whose topological closure contains the origin (Rodrigues 2017). The map $\Psi_\epsilon$ sends these cusps into triangles where the origin is one vertex (Fig. 9).

(3) If $v$ is not a switching node, then there are two incoming directions to $v$ and one outcoming from $v$, which means that the inequality of (12) does not impose any additional condition.

### 5.8 Shadowing a Path of Any Order

For $m \in \mathbb{N}$, given an admissible path $\xi = \{\gamma_0, \gamma_1, ..., \gamma_m\}$, with $v_0 = \alpha(\gamma_0)$ and $v_m = \alpha(\gamma_m)$, the map

$$P_\xi := P_{\gamma_{m-1}, \gamma_m} \circ P_{\gamma_{m-2}, \gamma_{m-1}} \circ ... \circ P_{\gamma_1, \gamma_2} \circ P_{\gamma_0, \gamma_1} : D_0 \to \text{Out}(v_m)$$

is the first return map to $\text{Out}(v_m)$ of solutions of (5) starting at $D_0 \subset D_{\gamma_0, \gamma_1}$ which follow $\xi$ at a distance $\epsilon > 0$. It is the composition of local and global maps, when they are well defined. The following result generalizes Lemma 12 and asserts that the quasi-change of coordinates $\Psi_\epsilon$ transforms the map $P_\xi$ into a linear map.

**Corollary 13** For $m \in \mathbb{N}$, given an admissible path $\xi = \{\gamma_0, \gamma_1, ..., \gamma_m\}$, let

$$F_\xi = \Psi_\epsilon \circ P_\xi \circ (\Psi_\epsilon)^{-1} : \Pi_\xi \to \Pi_{\gamma_m},$$

where

$$\Pi_\xi := \text{int}(\Pi_{\gamma_0}) \cap \bigcap_{j=1}^{m} (L_{\gamma_{j-1}, \gamma_j} \circ ... \circ L_{\gamma_0, \gamma_1})^{-1}(\text{int}(\Pi_{\gamma_j})).$$

Then

$$\pi_\xi := \lim_{\epsilon \to 0} F_\xi = L_{\gamma_{m-1}, \gamma_m} \circ ... \circ L_{\gamma_0, \gamma_1}.$$

For every $y \in \Pi_\xi$, we have $\pi_\xi(y) \in \text{int}(\Pi_{\gamma_m})$ and then there exists a solution of (5) from $(\Psi_\epsilon)^{-1}(y)$ to $(\Psi_\epsilon)^{-1}(\pi_\xi(y))$ following the heteroclinic path $\xi$. The map

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\( \pi_\xi = L_{\gamma_{m-1}, \gamma_m} \circ \ldots \circ L_{\gamma_0, \gamma_1} \) is designated by the skeleton map along \( \xi \). The \( 6 \times 6 \) matrix associated to \( \pi_\xi \) casts as:

\[
M_\xi := M_{\gamma_m-1, \gamma_m} \cdots M_{\gamma_0, \gamma_1},
\]

(17)

where \( M_{\gamma_j-1, \gamma_j} \) is defined as in (15). This matrix \( M_\xi \) induces a linear endomorphism on \( \mathbb{R}^6 \) whose restriction to the sector \( \Pi_\xi \) corresponds to the skeleton flow map \( \pi_\xi \). In particular, it gives a representation of \( \pi_\xi \) useful for computational purposes.

Before going further, recall that:

- \( \Pi_\xi \) : subset of \( \Pi_{\gamma_0} \) of initial conditions whose images under \( \Psi_e^{-1} \) follow the heteroclinic path \( \xi \) at a distance \( \varepsilon > 0 \) (sector);
- \( \pi_\xi \) : linear map from \( \Pi_{\gamma_0} \) to \( \Pi_{\gamma_m} \) (skeleton map along \( \xi \)).

The subset of \( \Sigma_\gamma \) that realise a nested “increased” chain of heteroclinic connections in the sense of Aguiar et al. (2010) give rise to a sequence of nested cusps containing the origin (in the phase space) and to a sequence of nested triangles (in the dual cone), as suggested by Figs. 8 and 9.

5.9 Dynamics of a Linear Operator

For the sake of completeness, we review the dynamics associated to a linear two-dimensional operator, which follows from the **Perron-Frobenius Theory**—we address the reader to Chapter 1.9 of Katok and Hasselblatt (1997) for more information on the subject. Suppose that \( A \) is a linear map defined in \( \mathbb{R}^2 \) whose eigenvalues are real, different and positive, say \( \lambda_1 < \lambda_2 \in \mathbb{R}^+ \), with eigenspaces \( E_1 \) and \( E_2 \), respectively. Then:

**Lemma 14** If \( v \in \mathbb{R}^2 \setminus E_1 \), then \( \lim_{n \in \mathbb{N}} \frac{A^n(v)}{\|A^n(v)\|} \in E_2 \).

The **Jordan decomposition Theorem** (Palis and de Melo 1982; Katok and Hasselblatt 1997) provides an unitary orthogonal basis of \( \mathbb{R}^2 \) such that the matrix of \( A \) with respect to that basis is diagonal. In this case, the basis consists of two non-zero unit vectors of \( E_1 \) and \( E_2 \), respectively. Then for \( (v_1, v_2) \in \mathbb{R}^2 \setminus \{(0, 0)\} \), we have:

\[
A^n \left( \begin{array}{c} v_1 \\ v_2 \end{array} \right) = \left( \begin{array}{cc} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{array} \right) \cdot \left( \begin{array}{c} v_1 \\ v_2 \end{array} \right) = \left( \begin{array}{c} \lambda_1^n v_1 \\ \lambda_2^n v_2 \end{array} \right).
\]

Since \( \lambda_2 > \lambda_1 \), we get:

\[
\lim_{n \in \mathbb{N}} \frac{(\lambda_1^n v_1, \lambda_2^n v_2)}{\sqrt{\lambda_1^{2n} v_1^2 + \lambda_2^{2n} v_2^2}} = (0, 1) \in E_2,
\]

and Lemma 14 follows.
5.10 Structural Set

We now define the concept of structural set, a definition emerging from the Isospectral Theory (Bunimovich and Webb 2012).

**Definition 5.4** A non-empty set of heteroclinic connections $S$ is said to be a structural set for the heteroclinic network $H$ if every heteroclinic cycle of $H$ contains an edge of $S$.

In general, the structural set associated to a heteroclinic network is not unique, but the results do not depend on this set (Alishah et al. 2019). From now on, we ask that the structural set is minimal. We are interested in admissible paths that start and end at an element of the structural set.

**Definition 5.5** For $m \in \mathbb{N}$, we say that the admissible heteroclinic path $\xi = \{\gamma_0, ..., \gamma_m\}$ is a $S$-branch for the network $H$ if:

1. $\gamma_0$ and $\gamma_m$ belong to $S$;
2. $\gamma_j \notin S$ for all $j \in \{1, ..., m-1\}$.

We denote by $B_S$ the set of all $S$-branches.

The next definition distinguishes between cycles that intersect $S$ just once from others.

**Definition 5.6** Let $H'$ be a cycle of the heteroclinic network $H$. We say that $H'$ is elementary if $H' \cap S$ contains just one element. Otherwise $H'$ is non-elementary.

If a cycle $H'$ is non-elementary, then it is the concatenation of a finite number of branches of $S$, say $\xi_0, \xi_1, ..., \xi_m$, $m \in \mathbb{N}$; in this case we write

$$H' = \xi_0 \oplus \xi_1 \oplus ... \oplus \xi_m.$$ 

Our next goal is the formal definition of skeleton map associated to a given structural set $S$. First, let us set:

$$\Pi_S := \bigcup_{\gamma \in S} \Pi_{\gamma} \quad \text{and} \quad D^*_S := \bigcup_{\xi \in B_S} \Pi_\xi.$$ 

If $\xi$ is a $S$-branch, as observed in (10), the set $\Pi_\xi \subset \mathbb{R}^6_+$ is a two-dimensional submanifold of $\mathbb{R}^6_+$ since four components of $\Pi_{\gamma}$ are zero. We introduce the skeleton map associated to a structural set $S$ using the concept of skeleton map along a path discussed in Corollary 13.

**Definition 5.7** Given a structural set $S$ associated to $H$, the skeleton map associated to $S$ is $\pi_S : D^*_S \rightarrow \Pi_S$ defined by

$$\pi_S(y) = \pi_\xi(y),$$

for $y \in \Pi_\xi$ and $\xi \in B_S$. 

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The following result says that Lebesgue almost all points in $\Pi_\mathcal{S}$ follow \textit{ad infinitum} either a prescribed $\mathcal{S}$-branch or an admissible concatenation of $\mathcal{S}$-branches.

\textbf{Proposition 15} If $\mathcal{H}$ is asymptotically stable, then the set $D^*_\mathcal{S}$ has full Lebesgue measure in $\Pi_\mathcal{S}$.

\textbf{Proof} Suppose that $\mathcal{H}$ is asymptotically stable. In particular, there are no more invariant and compact sets in $\Gamma_{(2,2,2)}$ in any small neighbourhood of $\mathcal{H}$. Define $D^*_\gamma = \bigcup_{\gamma_0,\gamma_1 \in \mathcal{E}} \Pi_{\gamma_0,\gamma_1}$ over any admissible heteroclinic path of the type $\{\gamma_0, \gamma_1\}$. The set $D^*_\gamma$ has full Lebesgue measure in $\Pi_{\gamma_0}$ because (Alishah et al. 2019):

$$\Pi_{\gamma_0} \setminus D^*_\gamma \subset \partial \Pi_{\gamma_0} \cup \left( \bigcup_{\gamma_0,\gamma_1 \in \mathcal{E}} L^{-1}_{\gamma_0,\gamma_1}(\partial \Pi_{\gamma_1}) \right).$$

Note that $L_{\gamma_0,\gamma_1}$ is a linear isomorphism carrying sets with zero Lebesgue measure into sets with the same property. Consider now any heteroclinic path of the type $\{\gamma_0, \gamma_1, \gamma_2\}$. Using the same line of argument, we get:

$$\Pi_{\gamma_0} \setminus D^*_\gamma \subset \partial \Pi_{\gamma_0} \cup \left( \bigcup_{\gamma_0,\gamma_1 \in \mathcal{E}} L^{-1}_{\gamma_0,\gamma_1}(\partial \Pi_{\gamma_1}) \right) \cup \left( \bigcup_{\gamma_0,\gamma_1,\gamma_2 \in \mathcal{E}} L^{-1}_{\gamma_0,\gamma_1} \circ L^{-1}_{\gamma_1,\gamma_2}(\partial \Pi_{\gamma_2}) \right),$$

and then $\Pi_{\gamma_0} \setminus D^*_\gamma$ has zero Lebesgue measure in $\Pi_{\gamma_0}$. Continuing the procedure a countable number of times, we may conclude that $D^*_\gamma$ has also full measure since it is a countable union of sets with full Lebesgue measure in $\Pi_{\gamma_0}$. \hfill $\Box$

For $m \in \mathbb{N}$, assume that $\xi = \{\gamma_0, \gamma_1, ..., \gamma_m\}$ is an elementary cycle with respect to a structural set $\mathcal{S}$, and the map $\pi_{\xi}$ of Corollary 13 has two different positive real eigenvalues. Given $\Pi_{\xi}$, we have three disjoint possibilities:

- the eigenvector associated to the greatest eigenvalue of $\pi_{\xi}$ lies on the corresponding sector $\Pi_{\xi}$ (Case A of Fig. 10);
- the eigenvector associated to the greatest eigenvalue of $\pi_{\xi}$ lies on another sector of $\Pi_{\gamma_0}$ and then the asymptotic dynamics is computed using the matrix associated to the sector to where the eigenvector has moved (Cases B and D of Fig. 10);
- the eigenvector associated to the greatest eigenvalue of $\pi_{\xi}$ lies outside the first quadrant. In this case, this analysis is valid just to the iterate where points hit on the boundary (Case C of Fig. 10). Dynamics accumulates on the boundary.

If $\xi$ is a non-elementary cycle, then the same conclusions hold by concatenating a finite number of $\mathcal{S}$–branches, provided the corresponding eigenvalues (for the composition of linear maps) are positive and different.

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Fig. 10 The dictionary between the dynamics of $\pi_\xi$ on the dual and the projective map for an elementary cycle $\xi$. 
5.11 Projective Map

Based on Peixe (2015, Section 3.6) we define a projective map defined on an interval of \(\mathbb{R}\) and study their periodic orbits, from where we are able to deduce the asymptotic dynamics for (5). The following notation simplifies the writing:

- \(v = \sum_{i=1}^{6} v_j\) for \(v = (v_1, ..., v_6) \in (\mathbb{R}^+)^6\);
- \(\Delta_{\gamma} := \{ u \in \text{int}(\Pi_{\gamma}) : \overline{u} = 1 \}\), where \(\gamma \in \mathcal{S}\);
- \(\Delta_{\xi} := \{ u \in \text{int}(\Pi_{\xi}) : \overline{u} = 1 \}\), where \(\xi\) is a \(\mathcal{S}\)-branch of \(\mathcal{H}\) with \(\text{int}(\Delta_{\xi}) \neq \emptyset\);
- \(\Delta \mathcal{S} := \bigcup_{\gamma \in \mathcal{S}} \Delta_{\gamma}\).

**Definition 5.8** For a structural set \(\mathcal{S}\) associated to the network \(\mathcal{H}\) and \(\xi = \{\gamma_0, ..., \gamma_m\}\) a \(\mathcal{S}\)-branch \((m \in \mathbb{N})\), we define:

1. *the projective map along \(\xi\)* as \(\hat{\pi}_\xi : \Delta_{\xi} \subseteq \Delta_{\gamma_0} \rightarrow \Delta_{\gamma_m}\) given by:

   \[
   \hat{\pi}_\xi(u) = \pi_\xi(u)/\pi_\xi(u);
   \]

2. *the projective \(\mathcal{S}\)-map* \(\hat{\pi}_\mathcal{S} : \Delta \mathcal{S} \rightarrow \Delta \mathcal{S}\) given by:

   \[
   \hat{\pi}_\mathcal{S}(u) = \hat{\pi}_\xi(u).
   \]

**Definition 5.9** Let \(n \in \mathbb{N}\). We say that \(u \in \Delta \mathcal{S}\) is a \(n\)-periodic point of \(\hat{\pi}_\mathcal{S}\) if \(n\) is the lowest \(m \in \mathbb{N}\) for which \(u = (\hat{\pi}_\mathcal{S})^m(u)\).

For \(n \in \mathbb{N}\), if \(u \in \Delta \mathcal{S}\) is a \(n\)-periodic point of \(\hat{\pi}_\mathcal{S}\), let us denote by \(\xi_k\) the unique \(\mathcal{S}\)-branch such that \((\hat{\pi}_\mathcal{S})^j(u) \in \Delta_{\xi_k}\) for all \(j, k = 0, 1, ..., n - 1\) and \(u \in \Delta_{\xi_j}\). Concatenating these branches, we obtain the cycle

\[
\Theta = \xi_{k_0} \oplus \xi_{k_1} \oplus ... \oplus \xi_{k_{n-1}}.
\]

We refer to this cycle \(\Theta\) as the *itinerary* of the periodic point \(u\).

**Definition 5.10** Let \(u \in \Delta \mathcal{S}\) be a periodic point of \(\hat{\pi}_\mathcal{S}\) whose itinerary is the cycle \(\Theta\). We say that:

1. \(u\) is an *eigenvector* of \(\hat{\pi}_\mathcal{S}\) if there is \(\lambda \in \mathbb{R}\setminus\{0\}\) such that \(\hat{\pi}_\mathcal{S}(u) = \lambda u\). The number \(\lambda = \lambda(u) > 0\) is the *Perron eigenvalue* of \(u\).
2. the *saddle-value* of \(u\), denoted by \(\sigma(u)\), is the maximum ratio \(\frac{\lambda'}{\lambda}\) where \(\lambda'\) ranges over all non-zero eigenvalues of \(D\hat{\pi}_\mathcal{S}\) different from \(\lambda\).

The next proposition follows straightforwardly and makes the bridge between the dynamics of a sector in the dual cone and that of the projective map.

**Proposition 16** Let \(u \in \Delta \mathcal{S}\) be a periodic point of \(\hat{\pi}_\mathcal{S}\) with itinerary \(\Theta\).

(a) If \(\sigma(u) < 1\) then \(u\) is an attracting periodic point of \(\hat{\pi}_\mathcal{S}\);

(b) If \(\sigma(u) > 1\) then \(u\) is a repelling periodic point of \(\hat{\pi}_\mathcal{S}\).
Proof We prove item (a). Let \( u \) be a \( n \)-periodic point of \( \hat{\pi}_S \) with itinerary \( \Theta \) and such that \( \sigma(u) < 1 \). Let \( \overline{u} \) be the corresponding vector in \( \Pi_\xi \subset \Pi_S \), where \( \xi \) is either a \( S \)-branch or a concatenation of \( S \)-branches associated to \( \mathcal{H} \), depending on whether the itinerary \( \Theta \) is elementary or not. Since \( \sigma(u) < 1 \), it means that the other eigenvalue of \( \pi_\xi \) is less than \( \lambda \). The result follows by Lemma 14 which says that initial conditions are attracted to the eigendirection associated to the greatest eigenvalue. The proof of (b) is analogous. \( \square \)

In order to study the projective map \( \hat{\pi}_S : \Delta_S \to \Delta_S \), we identify \( \Delta_\xi \) with \( J_k \), where \( k \) is over the number the \( S \)-branches. With these identifications, we define a map \( \varphi : [0, m] \to [0, m] \), where \( m := \#S \). This map describes the dynamics of \( \hat{\pi}_S \). We call this map as the 1-dimensional \textit{projective map}.

Remark The existence of an unstable invariant line within a sector of \( \Pi_S \) has two implications in terms of dynamics: first, the associated cycle is unstable; secondly, there is an invariant compact manifold of dimension two in the phase space “touching” the corresponding cycle. This will be used in Corollary 19 to show where the manifold \( \overline{\mathcal{M}}_\mu \) “touching” \( \mathcal{H} \).

6 Analysis of the Projective Map

In this section, we put together the established theory to study the stability of the heteroclinic cycles of \( \mathcal{H} \) listed in Lemma 7.

6.1 Procedure

We give a description of our method, locating its theoretical background in the previous section. Our starting point is the heteroclinic network \( \mathcal{H} \) given in Lemma 7 formed by 6 cycles, and the vector field \( f_\mu \) (5) defined in an interval \( \mathcal{I} \) where the interior equilibrium \( O_\mu \) exists.

(1) Compute the character map \( \chi^v \) of \( f_\mu \) and draw its flowing-edge graph (Definition 5.2);
(2) Find a structural set \( S \) associated to \( \mathcal{H} \) and determine all associated \( S \)-branches (Definition 5.4);
(3) Write explicitly the skeleton map \( \pi_S \) associated to all possible \( S \)-branches \( \xi \) with matrix \( M_\xi \) (see Definition 5.7 and Expression (17)). Note that \( M_\xi \) just depends on the eigenvalues of \( f_\mu \) at the equilibria and the architecture of \( \mathcal{H} \);
(4) For the periodic points of the skeleton map associated to \( S \), define all heteroclinic cycles \( \mathcal{H}' \) (given by possible concatenation of branches) and compute the eigenvalues and eigenvectors of \( M_{\mathcal{H}'} \). Every matrix \( M_{\mathcal{H}'} \) is a two-dimensional projection of \( \mathbb{R}^6 \) and has exactly 4 zero eigenvalues;
(5) For all \( S \)-branches, identify the eigenvectors associated to the greatest eigenvalues and, according to their location in the dual cone, use Lemma 14 to determine its stability;
Table 2: The dictionary between the dynamics of the projective map, the dual cone and the phase space, for \( \xi \in B_\mathcal{S} \) (elementary cycle) and \( \xi_1, \xi_2 \in B_\mathcal{S} \) (non-elementary cycles)

| Projective map \( \varphi \) | Sector in \( \Pi_\mathcal{S} \) | Phase space |
|-------------------------------|-----------------------------|-------------|
| Stable fixed point in \( \text{int}(J_\xi) \) | Stable eigendirection in \( \Pi_\xi \) | Stable elementary cycle |
| Unstable fixed point in \( \text{int}(J_\xi) \) | Unstable eigendirection in \( \Pi_\xi \) | Unstable elementary cycle |
| Stable fixed point in \( \text{int}(J_{\xi_1}) \cup \text{int}(J_{\xi_2}) \) | Stable eigendirection in \( \Pi_{\xi_1 \oplus \xi_2} \) | Stable cycle (concatenation of two branches) |
| Unstable fixed point in \( \text{int}(J_{\xi_1}) \cup \text{int}(J_{\xi_2}) \) | Unstable eigendirection in \( \Pi_{\xi_1 \oplus \xi_2} \) | Unstable cycle (concatenation of two branches) |
| No fixed point in \([0, m]\) | Strongest eigendirection in \( \Pi_\xi \) lies outside \( \Pi_\xi \) | Initial conditions are repelled |
Fig. 11 Terminology for the twelve different paths on $\mathcal{H}$ displayed in Table 3

Table 3 Edge labels

| $\gamma_1$ = $[v_2 \rightarrow v_1]$ | $\gamma_2$ = $[v_3 \rightarrow v_4]$ | $\gamma_3$ = $[v_5 \rightarrow v_6]$ | $\gamma_4$ = $[v_6 \rightarrow v_7]$ |
| $\gamma_5$ = $[v_1 \rightarrow v_7]$ | $\gamma_6$ = $[v_2 \rightarrow v_8]$ | $\gamma_7$ = $[v_7 \rightarrow v_8]$ | $\gamma_8$ = $[v_8 \rightarrow v_6]$ |

(6) Intersect the eigenvectors associated to the greatest eigenvalues with the hyperplane $\bar{u} = 1$ (Sect. 5.11);

(7) Define the projective map $\varphi$ for the corresponding $S$-branches and analyse their periodic points. Fixed points of $\hat{\pi}_S$ correspond to eigendirections of the corresponding matrices $M_{\xi}$. By computing the Perron eigenvalue associated to each periodic point (Definition 5.10), we determine its stability by applying Proposition 16.

Our route in this section is to pass from (1), (2) to the projective map defined in (7) to classify the stability of a given subcycle of $\mathcal{H}$. This is the main novelty of our article.

6.2 Structural Set

We see how the analysis on the dual allows us to draw conclusions about the stability of heteroclinic cycles in $[0, 1]^3$. For $\mu \in \mathcal{I}$, all twelve edges of $[0, 1]^3$ correspond to heteroclinic connections and will be called by $\gamma_1, \ldots, \gamma_{12}$, according to Table 3 and Fig. 11.

Looking at Fig. 11, we can see that

$$\mathcal{S} = \{ \gamma_5 = [v_1 \rightarrow v_3]; \gamma_8 = [v_8 \rightarrow v_6] \}$$

is a structural set for the heteroclinic network $\mathcal{H}$ in $[0, 1]^3$ (Definition 5.4), whose $S$-branches (Definition 5.5) are displayed in Table 4. We can see also that:
Fig. 12 Illustration of the cross sections $\Sigma_{\gamma_5}$ and $\Sigma_{\gamma_8}$ (in the phase space), and $\Pi_{\gamma_5}$ and $\Pi_{\gamma_8}$ (in the dual set). The letters $\xi$ and $\gamma$ are associated to $S$-branches and heteroclinic connections, respectively.

Table 4 $S$-branches associated to $\mathcal{H}$

| From $\gamma_5$ | $\gamma_5$ | $\gamma_8$ |
|----------------|-------------|-------------|
| $\gamma_5$     | $\xi_1$     | $\xi_2$, $\xi_3$ |
| $\gamma_8$     | $\xi_4$, $\xi_5$ | $\xi_6$ |

$\xi_1 = \{\gamma_5, \gamma_{11}, \gamma_7, \gamma_9, \gamma_5\}$  $\xi_2 = \{\gamma_5, \gamma_{12}, \gamma_8\}$  $\xi_3 = \{\gamma_5, \gamma_{11}, \gamma_4, \gamma_8\}$  $\xi_4 = \{\gamma_8, \gamma_{10}, \gamma_1, \gamma_5\}$  $\xi_5 = \{\gamma_8, \gamma_{10}, \gamma_6, \gamma_12, \gamma_8\}$  $\xi_6 = \{\gamma_8, \gamma_{10}, \gamma_6, \gamma_{12}, \gamma_8\}$

- there is only one path, $\xi_1$, that starts and ends at $\gamma_5$;
- there are two paths, $\xi_2$ and $\xi_3$, starting at $\gamma_5$ and ending at $\gamma_8$;
- there are two paths, $\xi_4$ and $\xi_5$, starting at $\gamma_8$ and ending at $\gamma_5$;
- there is only one path, $\xi_6$, that starts and ends at $\gamma_8$.

Define $\Pi_S = \Pi_{\gamma_5} \cup \Pi_{\gamma_8}$ where (Fig. 12)

$$\Pi_{\gamma_5} = \Pi_{\xi_1} \cup \Pi_{\xi_2} \cup \Pi_{\xi_3} \quad \text{and} \quad \Pi_{\gamma_8} = \Pi_{\xi_4} \cup \Pi_{\xi_5} \cup \Pi_{\xi_6}.$$  

6.3 Dynamics of the Skeleton Map Associated to $\mathcal{S}$

We consider now the skeleton map $\pi_S : D^*_\mathcal{S} \to \Pi_S$ whose domain is depicted in Fig. 13.

Because the remaining coordinates vanish, we consider the coordinates $(u_2, u_6)$ on $\Pi_{\gamma_5}$ and $(u_1, u_5)$ on $\Pi_{\gamma_8}$. Table 10 provides the matrix representation and the corresponding conditions for all the branches of the skeleton map $\pi_S$, with respect to the previous coordinates. In all domains $\Pi_{\xi_j}$, the inequalities $u_1 \geq 0$, $u_2 \geq 0$, $u_5 \geq 0$ and $u_6 \geq 0$ are implicit.

To represent the projective map $\hat{\pi}_S : \Delta_\mathcal{S} \to \Delta_\mathcal{S}$ (Definition 5.8), we identify $\Delta_{\xi_k}$ with $J_k$, where $k = 1, 2, 3$, and $1 + \Delta_{\xi_\ell}$ with $J_\ell$ and $\ell = 4, 5, 6$. Hence, we are iden-
Fig. 13 The domains \( \Pi_{\xi_1}, \Pi_{\xi_2}, \Pi_{\xi_3} \subset \Pi_{\gamma_5} \) (left), and \( \Pi_{\xi_1}, \Pi_{\xi_2}, \Pi_{\xi_4} \subset \Pi_{\gamma_8} \) (right), of the skeleton flow map \( \pi_S : \Pi_S \rightarrow \Pi_S \), with \( \mu = \frac{850}{11} \). Moreover, the domains \( \Delta_{\xi_1}, \Delta_{\xi_2}, \Delta_{\xi_3} \subset \Delta_{\gamma_5} \) (left), and \( \Delta_{\xi_4}, \Delta_{\xi_5}, \Delta_{\xi_6} \subset \Delta_{\gamma_8} \) (right), of the skeleton map \( \hat{\pi}_S : \Delta_S \rightarrow \Delta_S \).

The points 0, 1 and 2 correspond to the invariant boundary lines of the domains \( \Pi_{\gamma_5} \) and \( \Pi_{\gamma_8} \). They are fixed points for the projective map \( \varphi_{\mu} \), associated to initial conditions lying on the cube’s boundary.

Taking into account that systems (5) and (6) are equivalent, we are now in conditions to state the following result.

**Proposition 17** For system (5), there exist \( \mu_3, \mu_4, \mu_5 \in I_1 \) such that:

(a) for \( \mu \in \left] \frac{850}{11}, \mu_3 \right[ \), the projective map \( \varphi_{\mu} \) has a unique globally attracting fixed point in \( \text{int} \ (J_1) \);

\[
\varphi_{\mu}(x) = \begin{cases} 
\frac{588(\mu-176)x}{79(65\mu-6834)x-518(\mu-18)}, & x \in \left[0, \frac{74}{329}\right] = J_1 \\
\frac{2(74(\mu-78)+(131\mu-5529)x)}{74(\mu+34)+(131\mu-92146)x}, & x \in \left[\frac{74}{329}, \frac{74}{149}\right] = J_3 \\
\frac{324\mu+(226\mu-160251)x-24674}{162\mu+(113\mu-141380)x+2380}, & x \in \left[\frac{74}{149}, 1\right] = J_2 \\
\frac{-3476\mu+(2761\mu-412581)x+419016}{-2453\mu+158(11\mu-1530)x+248175}, & x \in \left[1, \frac{4137+22\mu}{4236+11\mu}\right] = J_6 \\
\frac{-4(-26\mu+(13\mu-2772)x+2889)}{-94\mu+(47\mu+49212)x-48789}, & x \in \left[\frac{4137+22\mu}{4236+11\mu}, \frac{75+2\mu}{84+\mu}\right] = J_4 \\
\frac{-42(-6\mu+(3\mu-530)x+626)}{(\mu+96270)x-2(\mu+52374)}, & x \in \left[\frac{75+2\mu}{84+\mu}, 2\right] = J_5 
\end{cases}
\]
The projective map \( \varphi_{\mu} : [0, 2] \rightarrow [0, 2] \) with \( \mu = \frac{850}{11} \) (left) and \( \mu = \frac{544}{5} \) (right), and the corresponding domains \( J_k \), for \( k = 1, \ldots, 6 \)

(b) for \( \mu \in ]\mu_3, \mu_4[ \), the projective map \( \varphi_{\mu} \) has:
- two attracting fixed points, one in \( \text{int}(J_1) \) and other in \( \text{int}(J_6) \);
- a repelling periodic point of period two in \( \text{int}(J_2) \) such that its image by \( \varphi_{\mu} \) is in \( \text{int}(J_4) \) (cf. case \( \mu = 96 \) in Table 5).

(c) for \( \mu \in ]\mu_4, \mu_5[ \), the projective map \( \varphi_{\mu} \) has:
- two attracting fixed points, one in \( \text{int}(J_1) \) and other in \( \text{int}(J_6) \);
- a repelling periodic point of period two in \( \text{int}(J_3) \) such that its image by \( \varphi_{\mu} \) is in \( \text{int}(J_4) \) (cf. case \( \mu = 99 \) in Table 5).

(d) for \( \mu \in ]\mu_5, 102[ \), the projective map \( \varphi_{\mu} \) has:
- two attracting fixed points, one in \( \text{int}(J_1) \) and other in \( \text{int}(J_6) \);
- a repelling periodic point of period two in \( \text{int}(J_3) \) such that its image by \( \varphi_{\mu} \) is in \( \text{int}(J_5) \) (cf. case \( \mu = 101 \) in Table 5).

(e) for \( \mu \in I_2 \cup I_3 \), the projective map \( \varphi_{\mu} \) has a repelling periodic point of period two in \( \text{int}(J_3) \) such that its image by \( \varphi_{\mu} \) is in \( \text{int}(J_5) \) (cf. case \( \mu = 103 \) in Table 5).

Proof: The eigenvector of \( M_{\xi_1} \) that depends on \( \mu \),

\[
\psi_1 \equiv \left\{ \frac{14(\mu - 102)}{51(\mu - 106)}, 1 \right\}
\]

lies in the interior of \( \Pi_{\xi_1} \) if and only if \( \mu \in I_1 \). For these values of the parameter, the eigenvector is the one associated to the greatest eigenvalue of \( M_{\xi_1} \). Hence, by Proposition 16 the point \( \frac{\psi_1}{\psi_1} \in \text{int}(\Delta_{\xi_1}) \) corresponds to the attracting fixed point

\[
x_1 := \frac{14\mu - 1428}{65\mu - 6834} \in \text{int}(J_1),
\]
of $\varphi_\mu$ for $\mu \in \mathcal{I}_1$. Moreover, for $\mu \in ]\frac{850}{11}, \mu_3 [$, where $\mu_3 = 94$, this is the unique periodic point of $\varphi_\mu$ and hence $\frac{\mu_3}{11} \in \Delta_{\xi_1}$ corresponds to the unique globally attracting fixed point $x_1$ of $\varphi_\mu$. This concludes the proof of $(a)$.

We can analogously see that the eigenvector of $M_{\xi_6}$ that depends on $\mu$, $v_6 \equiv \left\{ \frac{11(102 - \mu)}{408}, 1 \right\}$, lies in the interior of $\Pi_{\xi_6}$ if and only if $\mu \in ]\mu_3, 102 [$ for these values of the parameter, this eigenvector is the one associated to the greatest eigenvalue of $M_{\xi_6}$. By Proposition 16, the point $\frac{v_6}{\mu} \in \text{int} \left( \Delta_{\xi_6} \right)$ corresponds to the attracting fixed point $x_2 := \frac{22 \mu - 2652}{11 \mu - 1530} \in \text{int} (J_6)$, of $\varphi_\mu$ for $\mu \in ]\mu_3, 102 [$.

Furthermore, we can see that:

1. for $\mu \in ]\mu_3, \mu_4 [$, where $\mu_4 = \frac{85251}{869}$, the point $x_3 := 3 \sqrt{\frac{320140324 \mu^2 - 60787796412 \mu + 2893236225489}{(71941 \mu - 6256224)^2} - \frac{5(7486 \mu - 751455)}{71941 \mu - 6256224} \in \text{int} (J_2)$, is a repelling periodic point of period two, such that $\varphi_\mu (x_3) \in J_4$;

2. for $\mu \in ]\mu_4, \mu_5 [$, where $\mu_5 = \frac{85234}{849}$, the point $x_4 := 3 \sqrt{\frac{2615265201481 \mu^2 - 504216904560828 \mu + 24312026567983716}{(9965897 \mu - 946939158)^2} - \frac{5(533489 \mu - 52940718)}{9965897 \mu - 946939158} \in \text{int} (J_3)$, is a repelling periodic point of period two, such that $\varphi_\mu (x_4) \in J_4$;

3. for $\mu \in ]\mu_5, \frac{544}{5} [$, the point $x_5 := \frac{1}{8} \sqrt{\frac{3386009761 \mu^2 - 644714033868 \mu + 30745583285796}{(9157 \mu - 787158)^2} - \frac{5(8467 \mu - 845394)}{8(9157 \mu - 787158)} \in \text{int} (J_3)$, is a repelling periodic point of period two, such that $\varphi_\mu (x_5) \in J_5$;

which concludes the proof of $(b), (c), (d)$, and $(e)$.
Table 5  First 100 iterates of the $\phi_\mu$-orbit of the blue dots lying on int $(J_3)$ and int $(J_2)$ for $\mu = 96$ and $\mu = 99, 101, 103$, respectively

These schemes suggest the existence of an unstable 2–periodic orbit for $\phi_\mu$. 
6.4 Stability of the Heteroclinic Cycles

First of all, observe that

\[ \mathcal{H}_1 = \xi_6 \]
\[ \mathcal{H}_2 = \xi_2 \oplus \xi_4 \]
\[ \mathcal{H}_3 = \xi_2 \oplus \xi_5 \]
\[ \mathcal{H}_4 = \xi_3 \oplus \xi_4 \]
\[ \mathcal{H}_5 = \xi_3 \oplus \xi_5 \]
\[ \mathcal{H}_6 = \xi_1 \]

where the symbol \( \oplus \) means the concatenation between admissible paths. The entries of the matrix associated to an elementary cycle are all positive while those associated to a non-elementary cycle may be negative. Nevertheless those that correspond to the matrix of concatenated paths are positive. In particular, the Perron-Frobenius Theory may be applied. 6

Corollary 18 For system (6) there exist \( \mu_3, \mu_4, \mu_5 \in I_1 \) such that:

(a) for \( \mu \in ]850/1, \mu_3[ \) (cf. case \( \mu = 90 \) in Table 6):

- \( \mathcal{M}_\mu \cap \partial[0,1]^3 = \mathcal{H}_6 \);
- the cycle \( \mathcal{H}_6 \) is globally asymptotically stable in \( \text{int}(\partial[0,1]^3) \).

(b) for \( \mu \in ]\mu_3, \mu_4[ \), \( \mathcal{M}_\mu \cap \partial[0,1]^3 = \mathcal{H}_2 \) (cf. case \( \mu = 96 \) in Table 6);

(c) for \( \mu \in ]\mu_4, \mu_5[ \), \( \mathcal{M}_\mu \cap \partial[0,1]^3 = \mathcal{H}_4 \) (cf. case \( \mu = 99 \) in Table 6);

6 Note that the theory revisited in Sect. 5.9 (in particular, Lemma 14) is valid for different positive real eigenvalues.
Moreover, in Cases (b), (c), and (d), $\overline{M_\mu}$ divides the interior of the phase space in two regions, $\mathcal{U}_1$ and $\mathcal{U}_2$, such that for any initial condition in $\mathcal{U}_1$, its $\omega$-limit is the cycle $\mathcal{H}_1$, and for any initial condition in $\mathcal{U}_2$, its $\omega$-limit is the cycle $\mathcal{H}_6$.

**Proof** The proof follows from the analysis of the projective map performed in Proposition 17 and the theory developed in Sect. 5.

To conclude about the cycles stability we look at the eigenvectors of the matrices for each cycle. In the case $\mu \in [\mu_5, 102]$, we can see that the eigenvector of $M_{\mathcal{H}_1}$ that belongs to the interior of the sector/ is the greatest eigenvalue, and the same happens for $M_{\mathcal{H}_6}$. The eigenvector of $M_{\mathcal{H}_5}$ that belongs to the interior of the sector $\Pi_{\mathcal{H}_5}$ is the smallest eigenvalue (see case $\mu = 101$ in Table 6). For the other cases, the analysis is analogous.

The existence of an unstable periodic point for the projective map $\phi_\mu$ implies that there exists an invariant line for the corresponding dual cone $\Pi_{\mathcal{S}}$. Since the flow of system (5) may be seen as the the lift of the first return map to $\Pi_{\mathcal{S}}$ it implies that there exists a two-dimensional invariant manifold repelling all trajectories nearby. By Fact 1, there are no more invariant sets besides $M_\mu$. Therefore, this invariant line should correspond to the cycle within $\mathcal{H}$ containing the $\omega$-limit of all points of $M_\mu$. 

**Corollary 19** For $\mu \in \mathcal{I}_2 \cup \mathcal{I}_3$, the following assertions hold:

1. the set $\overline{M_\mu}$ contains on $\mathcal{H}_5$;
2. the set $\overline{M_\mu}$ divides $[0, 1]^3$ in two connected components, each one containing either $B_1$ or $B_2$;
3. for $z \in \text{int}([0, 1]^3) \setminus M_\mu$, $\omega(z)$ is either $\{B_1\}$ or $\{B_2\}$, according to the connected component where $z$ lies.

The proof of Corollary 19 runs along the same arguments of Corollary 18. We have $\mathcal{L}(\text{int}([0, 1]^3) \setminus M_\mu) = \{B_1, B_2\}$ because $M_\mu \cup \mathcal{H}$ is repelling and there are no more compact invariant sets candidates for $\omega$-limit sets.

**7 Implemented Software Code**

We provide in https://www.iseg.ulisboa.pt/aquila/homepage/telmop/investigacao/flows-on-polytopes-mathematica-code the Mathematica code we developed to explore the dynamics of polymatrix replicators for low dimensional polytopes (First author’s personal webpage).

**8 Discussion**

We have considered a one-parameter polymatrix game with three groups and two strategies for each one ($\Gamma_{(2,2,2)}$). The associated one-parameter polymatrix replicator,
Table 6  Eigenvectors of $M_{\mathcal{H}}$ and corresponding sectors $\Pi_{\mathcal{H}}$ in $\Pi_{\gamma^5}$ and $\Pi_{\gamma^8}$ (for each cycle $\mathcal{H}$, the color of the eigenvectors of $M_{\mathcal{H}}$ is the same of the corresponding sector $\Pi_{\mathcal{H}}$)

| $\mu$ | Eigenvectors of $M_{\mathcal{H}}$ in $\Pi_{\gamma^5}$ | Eigenvectors of $M_{\mathcal{H}}$ in $\Pi_{\gamma^8}$ | Phase space $[0, 1]^3$ |
|-------|--------------------------------------------------|--------------------------------------------------|------------------|
| $\mu = 90$ | ![Eigenvectors of $M_{\mathcal{H}}$ in $\Pi_{\gamma^5}$](image1) | ![Eigenvectors of $M_{\mathcal{H}}$ in $\Pi_{\gamma^8}$](image2) | ![Phase space $[0, 1]^3$](image3) |
| $\mu = 96$ | ![Eigenvectors of $M_{\mathcal{H}}$ in $\Pi_{\gamma^5}$](image4) | ![Eigenvectors of $M_{\mathcal{H}}$ in $\Pi_{\gamma^8}$](image5) | ![Phase space $[0, 1]^3$](image6) |
| $\mu = 99$ | ![Eigenvectors of $M_{\mathcal{H}}$ in $\Pi_{\gamma^5}$](image7) | ![Eigenvectors of $M_{\mathcal{H}}$ in $\Pi_{\gamma^8}$](image8) | ![Phase space $[0, 1]^3$](image9) |
| $\mu = 101$ | ![Eigenvectors of $M_{\mathcal{H}}$ in $\Pi_{\gamma^5}$](image10) | ![Eigenvectors of $M_{\mathcal{H}}$ in $\Pi_{\gamma^8}$](image11) | ![Phase space $[0, 1]^3$](image12) |

The third column is the plot of an orbit of (6) with initial condition in the interior of the phase space near $M_{\mu}$, for different values of $\mu = 90, 96, 99, 101$. In order to find the preferred attracting cycle, it is enough to check the “strongest” directions of $M_{\mathcal{H}}$ and the sector where they lie.

defined by systems (5) and (6), exhibits an attracting heteroclinic network $\mathcal{H}$ where the theory developed in Alishah et al. (2019) may be applied. The network $\mathcal{H}$ lies on the boundary of the three-dimensional cube and is formed by six one-dimensional cycles involving hyperbolic switching nodes whose eigenvalues of the Jacobian matrix are all real. The parameter $\mu$ is related to the average payoffs in the context of EGT.

In the present paper, we have developed a method to study the asymptotic dynamics near $\mathcal{H}$ by using the theory introduced in Alishah et al. (2019), as the parameter $\mu$. Springer
Table 7 Summary of concluding results for system (6), where $\mu_3 = 94$, $\mu_4 = \frac{85251}{869}$ and $\mu_5 = \frac{85234}{849}$

| Parameter interval | $L(B(H))$ | Basin of attraction of $L(B(H))$ | $\mathcal{M}_\mu$ “touches” $H$ at |
|--------------------|------------|----------------------------------|----------------------------------|
| $[850/11, \mu_3]$ | $H_6$      | $[0, 1]^3 \setminus \mathcal{M}_\mu$ | $H_6$                           |
| $[\mu_3, \mu_4]$ | $H_1 \cup H_6$ | Each CC of $[0, 1]^3 \setminus \mathcal{M}_\mu$ accumulates either on $H_1$ or $H_6$ | $H_2$                           |
| $[\mu_4, \mu_5]$ | $H_1 \cup H_6$ | Each CC of $[0, 1]^3 \setminus \mathcal{M}_\mu$ accumulates either on $H_1$ or $H_6$ | $H_4$                           |
| $[\mu_5, 102]$   | $H_1 \cup H_6$ | Each CC of $[0, 1]^3 \setminus \mathcal{M}_\mu$ accumulates either on $H_1$ or $H_6$ | $H_5$                           |
| $I_2 \cup I_3$   | $\{B_1, B_2\}$ | Each CC of $[0, 1]^3 \setminus \mathcal{M}_\mu$ accumulates either on $B_1$ or $B_2$ | $H_5$                           |

CC, connected component
varies. More specifically, we have described a general way to compute the likely limit set associated to the basin of attraction of \( \mathcal{H} \).

For all \( \mu \in \left[ \frac{850}{11}, \frac{544}{5} \right] \), the associated dynamics is non-chaotic and a given set of strategies dominates. Our results extend to other models that preserve the invariance of coordinate lines and hyperplanes. This study contributes to a deeper understanding of the results obtained in Alishah and Duarte (2015), Alishah et al. (2015), Afraimovich et al. (2016), where numerical simulations evidenced the visibility of two cycles.

Our method has similarities with the *transitions matrices technique* used by Krupa and Melbourne (1995) and more recently by Garrido-da Silva and Castro (2019), Castro et al. (2022). The main advantage of our method is twofold. First, the dynamics in a given cross section may be seen as a piecewise linear map where the classical Perron-Frobenius theory of linear operators may be easily used. The analysis is computationally much more amenable than the classical method.

Secondly, the reduction to a one-dimensional projective map allows us to construct a direct bridge between its periodic points and the existence of heteroclinic cycles (in the flow), as well as their stability. In contrast to the findings of Podvigina et al. (2020), we do not need neither the assumption on the cleanness of the network nor conditions about symmetry.

Our class of examples is related to the dynamical systems represented by ODEs that support the dynamics of the Rock-Scissors-Paper-Lizard-Spock game (Postlethwaite and Rucklidge 2021) and Lotka–Volterra systems constructed using the methods of Field (2020), Ashwin and Postlethwaite (2013). See also Podvigina et al. (2020), Aguiar (2011).

### 8.1 Summary of the Technique

Let \( \mathcal{H} \) be an attracting heteroclinic network defined on the boundary of a three-dimensional cube, consisting of a finite number one-dimensional cycles involving hyperbolic equilibria whose eigenvalues of the Jacobian matrix are all real. Let \( \mathcal{S} \) be the structural set associated to \( \mathcal{H} \); it has the minimal number of connections for which every cycle of \( \mathcal{H} \) contains at least one connection in \( \mathcal{S} \).

Given a structural set \( \mathcal{S} \), we denote by \( \Sigma \) the union of cross sections to \( \mathcal{S} \), one at each heteroclinic connection of \( \mathcal{S} \). The flow induces a return map to \( \Sigma \), say \( P_{\mathcal{S}} \), designated as the \( \mathcal{S} \)-Poincaré map associated to each possible itinerary that starts and ends at \( \mathcal{S} \). This map captures well the global dynamics near \( \mathcal{H} \).

Using the quasi-change of coordinates of Sect. 5.3, we obtain a return map \( \pi_{\mathcal{S}} \) well defined on \( \Pi_{\mathcal{S}} \) (union of sectors in the dual cone) up to a set with zero Lebesgue zero.

After making explicit the piecewise linear skeleton map \( \pi_{\mathcal{S}} \) in Proposition 15, we use an algorithm to compute the associated matrix \( M_\xi \) for each \( \xi \in B_{\mathcal{S}} \) (itineraries starting and ending at \( \mathcal{S} \)). All solutions approach the eigendirection associated to the greatest eigenvalue, as a consequence of the Perron-Frobenius Theory.

The map \( \pi_{\mathcal{S}} \) carries the asymptotic behaviour of \( P_{\mathcal{S}} \) along the different paths in the sense that, after a rescaling change of coordinates \( \Psi_\varepsilon \), \( \pi_{\mathcal{S}} \) is the limit of \( \Psi_\varepsilon \circ P_{\mathcal{S}} \circ (\Psi_\varepsilon)^{-1} \) as \( \varepsilon \) tends to 0+ (in the \( C^1 \)-topology).
Because the map $\pi_S$ is easily computable, we can run an algorithm to find the $\pi_S$-invariant linear algebraic structures, provided their eigenvalues are two different positive real numbers. If these structures are invariant under small non-linear perturbations, they will persist as invariant geometric structures for $P_S$, and hence for the flow.

The intersection of each iterate of $\pi_S$ with the line $\bar{u} = 1$ generates the projective map $\hat{\pi}_S$. The saddle-value given by the ratio of the eigenvalues of $D\pi_S$ at the corresponding fixed point determines its stability.

The connection between the stability of periodic points for the projective map and the stability of the original heteroclinic cycles constitutes a novelty of this article and is summarized in Table 2.

Results of Sect. 5.4 also provides an important breakthrough in the study of stability for networks in replicator systems; the local and global maps are stated according to the architecture of the network. They depend on the coordinates of the system allowing a systematic study of all subcycles of $H$. This technique may be generalized for other vector fields defined on a manifold isomorphic to $[0, 1]^n$, $n \in \mathbb{N}$, containing quasi-simple heteroclinic networks on the boundary in the sense of Garrido-da Silva and Castro (2019).

### 8.2 Limitations and Future Work

The theory developed in Sects. 5 and 6, does not hold if the network either contains a saddle-focus or one heteroclinic connection with dimension greater than 1. Furthermore, if $H$ is not asymptotically stable then we cannot use Proposition 15 and then our results just rely on a partial portion of the phase space.

The natural continuation of the work presented in this article is the application of our method to higher dimensions. The most intriguing question concerns how the switching properties of the network may be realised in “switching properties” of the skeleton map in the sense of Rodrigues (2013), Aguiar et al. (2010). According to Castro and Garrido-da Silva (2022), switching is ruled out if the initial network is asymptotically stable. This is why our next research focuses on Hamiltonian systems.

We plan to apply this theory to an explicit polymatrix game studied by the first author in Peixe (2015) and in Alishah et al. (2019, Section 10), whose flows exhibit a non-attracting network and a suspended horseshoe near it. We would like to explore the shift dynamics associated to this hyperbolic set and investigate the existence of “switching properties”. Another question is the relation between the $\mu_3, \mu_4, \mu_5$ values and the eigenvalues of $Df_{\mu}$ at the equilibria. These questions are deferred for a future work.

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**Data availability** Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

## Appendix A

See Tables 8, 9, 10, 11 and 12.

### Table 8

The eigenvalues of system (6) at the vertices, where the entry at line $i$ and row $j$ is the eigenvalue of the vertex $v_j$ in the orthogonal direction to the face $\sigma_i$, and the symbol $*$ means that the vertex $v_i$ does not belong to the face $\sigma_j$ of $[0, 1]^3$.

| Eq./Eignv | $\sigma_1$ | $\sigma_2$ | $\sigma_3$ | $\sigma_4$ | $\sigma_5$ | $\sigma_6$ |
|-----------|------------|------------|------------|------------|------------|------------|
| $v_1$     | *          | −84        | *          | 60         | *          | −162       |
| $v_2$     | *          | −93        | *          | 33         | 144        | *          |
| $v_3$     | *          | 74         | −60        | *          | *          | 75         |
| $v_4$     | *          | 65         | −33        | *          | −93        | *          |
| $v_5$     | $\mu$ − 18 | *          | *          | −42        | *          | −111       |
| $v_6$     | $\mu$ − 9  | *          | *          | −69        | 93         | *          |
| $v_7$     | $\mu$ − 176| *          | 42         | *          | *          | 126        |
| $v_8$     | $\mu$ − 167| *          | 69         | *          | −144       | *          |

### Table 9

Eigenvalues of equilibria $B_1$ and $B_2$, depending on $\mu$, at the corresponding faces and pointing to the interior of the cube, where the signs (−), (0), and (+) mean that the eigenvalues are real negative, zero or positive, respectively.

| Eq   | Eigenvalues | $\mu$ | On face    | On the interior |
|------|-------------|-------|------------|-----------------|
| $B_1$ | $\left\{ \frac{2550−33\mu}{68}, z_1, \bar{z}_1 \right\}$ | $\frac{850}{17} < \mu < 102$ | (+, +)$_C$ | (−) |
|      | $\mu = 102$ | $0 < \mu < \frac{544}{5}$ | (0, 0)$_C$ | (−) |
| $B_2$ | $\left\{ \frac{15\mu−1632}{17}, z_2, \bar{z}_2 \right\}$ | $\frac{850}{17} < \mu < 102$ | (+, +)$_C$ | (−) |
|      | $\mu = 102$ | $0 < \mu < \frac{544}{5}$ | (0, 0)$_C$ | (−) |
|      | $102 < \mu < \frac{544}{5}$ | (−, −)$_C$ | (−) |

The notation (+, +)$_C$, (−, −)$_C$ and (0, 0)$_C$ means that the eigenvalues are conjugate (non-real) with positive, negative and zero real part.

\[
\begin{align*}
  z_1 &= \frac{2038674 - 19987\mu + \sqrt{19987^2 + 44671\mu^2 - 6976596\mu - 1178700372}}{182648}, \\
  z_2 &= \frac{282030 - 2765\mu + \sqrt{2765^2 + 12965\mu^2 - 2471460\mu - 66034188}}{22831}
\end{align*}
\]
Table 10  Branches of $\pi_\xi$: defining equations of $\Pi_\xi$, the matrix of $\pi_\xi$, and their eigenvalues and eigenvectors, for $\xi \in \{\xi_1, \xi_2, \ldots, \xi_6\}$

| $\xi$ | Coordinates of $\Pi_\xi$ | Matrix of $\pi_\xi$ | Eigenvalues | Eigenvectors |
|-------|-------------------------|---------------------|-------------|--------------|
| $\xi_1$ | $255u_2 - 74u_6 < 0$ | $\begin{pmatrix} 42(-176 - \mu) & 0 \\ 37(18 - \mu) & -106(18 - \mu) \end{pmatrix}$ | $-14(\mu - 102)$ | $\{1; 0, 1\}$ |
| $\xi_2$ | $75u_2 - 74u_6 > 0$ | $\begin{pmatrix} 111(167 - \mu) & 162(167 - \mu) \\ 1495 & -32434 \end{pmatrix}$ | $a_2 - b_2, a_2 + b_2$ | $\{c_2 - b_2, 1; c_2 + b_2, 1\}$ |
| $\xi_3$ | $75u_2 - 74u_6 < 0$ | $\begin{pmatrix} 5(6502 - 41\mu) & 190 - \mu \\ 2519 & 859 \end{pmatrix}$ | $a_3 - b_3, a_3 + b_3$ | $\{c_3 - b_3, 1; c_3 + b_3, 1\}$ |
| $\xi_4$ | $93u_1 - (\mu - 9)u_5 < 0$ | $\begin{pmatrix} 531 & 13 \\ 7805 & -99 \end{pmatrix}$ | $3(\mu - 3\sqrt{3}b_4), 3(\mu + 3\sqrt{3}b_4)$ | $\{c_4 - \sqrt{3}b_4, 1; c_4 + \sqrt{3}b_4, 1\}$ |
| $\xi_5$ | $93u_1 - (\mu - 9)u_5 > 0$ | $\begin{pmatrix} 254 & 63(\mu - 32) \\ 5(\mu - 18) & 155(\mu - 18) \end{pmatrix}$ | $a_5 - b_5, a_5 + b_5$ | $\{c_5 - b_5, 1; c_5 + b_5, 1\}$ |
| $\xi_6$ | $4335u_1 - 11(\mu - 9)u_5 > 0$ | $\begin{pmatrix} 93(167 - \mu) & 0 \\ 65(\mu - 9) & 71(\mu - 9) \end{pmatrix}$ | $93(167 - \mu), 1$ | $\{11(102 - \mu), 1; 102 - \mu, 1\}$ |

$a_2 = 16491 - 275\mu, \quad b_2 = \sqrt{75625\mu^2 - 101760050\mu + 15751183081}, \quad c_2 = 75359 - 275\mu,$
$a_3 = 15934 - 205\mu, \quad b_3 = \sqrt{42025\mu^2 - 37032780\mu + 5621864196}, \quad c_3 = 49086 - 205\mu,$
$a_4 = 3837 - 33\mu, \quad b_4 = \sqrt{363\mu^2 + 371906\mu + 80063}, \quad c_4 = 32433 + 33\mu,$
$a_5 = 13782 - 127\mu, \quad b_5 = \sqrt{16129\mu^2 + 40819452\mu - 607817916}, \quad c_5 = 22674 - 127\mu,$
### Table 11

Cycles in $\Pi_{\gamma}$; defining equations of $\Pi_{\gamma} \subset \Pi_{\gamma}$, the matrix of $\pi_{\gamma}$, and their eigenvalues and eigenvectors, for each $\gamma \in \{\gamma_2, \gamma_4, \gamma_5, \gamma_6\}$ in $\Pi_{\gamma}$

| $\gamma$ | Coordinates of $\Pi_{\gamma} \subset \Pi_{\gamma}$ | Matrix of $\pi_{\gamma}$ | Eigenvalues | Eigenvectors |
|----------|-----------------------------------------------|--------------------------|-------------|--------------|
| $\gamma_2$ | $75u_2 - 74u_6 > 0$ \[1107975(\mu - 94)u_2 + 2(94624\mu - 28591281)u_6 < 0$ | $\begin{pmatrix} 3(3199\mu + 740274) & 3(3199\mu - 238203) \\ 996620(\mu - 94) & 398871(\mu - 94) \end{pmatrix}$ | $\begin{pmatrix} 3(\mu - 79b) & 3(\mu + 79b) \\ 790000(\mu - 9) & 790000(\mu - 9) \end{pmatrix}$ | $\begin{pmatrix} c_2 - b \gamma_4 & c_2 + b \\ 42075(\mu - 94) & 1 \end{pmatrix}$ |
| $\gamma_4$ | $75u_2 - 74u_6 < 0$ \[15(235834 - 5079\mu)u_2 + 74(15654 + 131\mu)u_6 < 0$ | $\begin{pmatrix} 3(1321\mu + 5308734) & 3(1321\mu - 5567\mu) \\ 702496(\mu - 9) & 459509(\mu - 9) \end{pmatrix}$ | $\begin{pmatrix} 3(c_3 - b_4) & 3(c_3 + b_4) \\ 3812400(\mu - 9) & 3812400(\mu - 9) \end{pmatrix}$ | $\begin{pmatrix} c_3 - b_4 & c_3 + b_4 \\ 7657994(\mu - 94864) & 1 \end{pmatrix}$ |
| $\gamma_5$ | $255u_2 - 74u_6 > 0$ \[15(235834 - 5079\mu)u_2 + 74(15654 + 131\mu)u_6 > 0$ | $\begin{pmatrix} 21(1177\mu + 123226) & 21(1177\mu - 9650\mu) \\ 369334(\mu - 18) & 40292(\mu - 18) \end{pmatrix}$ | $\begin{pmatrix} a_5 & a_5 + 79b_4 \\ 369334(\mu - 18) & 369334(\mu - 18) \end{pmatrix}$ | $\begin{pmatrix} c_5 - b_4 & c_5 + b_4 \\ 510000(\mu - 299145) & 1 \end{pmatrix}$ |
| $\gamma_6$ | $255u_2 - 74u_6 < 0$ | $\begin{pmatrix} 42(\mu - 176) & 42(\mu - 106) \\ 518(\mu - 18) & 0 \end{pmatrix}$ | $\begin{pmatrix} a_7 - b & a_7 + b \\ 510(\mu - 106) & 1 \end{pmatrix}$ | $\begin{pmatrix} c_6 & c_6 + b \\ 510(\mu - 299145) & 1 \end{pmatrix}$ |

$a_2 = 122787543 - 123922\mu, \quad b_2 = \sqrt{320140324\mu^2 - 60787964412\mu + 2893236225489}, \quad c_2 = 6172117 - 5618\mu$

$a_4 = 142050954 - 75491\mu, \quad b_4 = \sqrt{2615265204181\mu^2 - 50421690456082\mu + 24312026567983716}, \quad c_4 = 35786274 - 339871\mu$

$a_5 = 353512794 + 104449\mu, \quad b_5 = \sqrt{3386009761\mu^2 - 644714033868\mu + 30745583285796}, \quad c_5 = 2181906 - 20579\mu$
Table 12  Cycles in $\Pi_{y_8}$: defining equations of $\Pi_{\mathcal{H}} \subset \Pi_{y_8}$, the matrix of $\pi_\mathcal{H}$, and their eigenvalues and eigenvectors, for $\mathcal{H} \in \{\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_4, \mathcal{H}_5\}$ in $\Pi_{y_8}$

| $\mathcal{H}$ | Coordinates of $\Pi_{\mathcal{H}} \subset \Pi_{y_8}$ | Matrix of $\pi_\mathcal{H}$ | Eigenvalues | Eigenvectors |
|----------------|-----------------------------------------------------|-----------------------------|-------------|--------------|
| $\mathcal{H}_1$ 4335$u_1 - 11(\mu - 9)u_5 < 0$ | $\begin{pmatrix} 93(\mu - 167) & 0 \\ \frac{65(\mu - 9)}{715(\mu - 19)} & 1 \end{pmatrix}$ | $\{ 93(167 - \mu), 1 \} \quad \{ \frac{111(102 - \mu)}{408}, 1 \}$ | $\{ 0, 1 \}$ |
| $\mathcal{H}_2$ 4335$u_1 - 11(\mu - 9)u_5 > 0$ | $\begin{pmatrix} 924093(\mu - 167) & 869(\mu - 167) \\ -389000(\mu - 9) & 2990000 \end{pmatrix}$ | $\{3(\mu - 79) b_2, \frac{3(\mu + 79) b_2}{2990000} \}$ | $\{ c_2 - b_2, c_2 + b_2 \}$ | $\{ \frac{337365}{c_2}, 1 \}$ |
| $\mathcal{H}_3$ 93$u_1 - (\mu - 9)u_5 < 0$ | $\begin{pmatrix} 93(\mu - 167) & 0 \\ \frac{65(\mu - 9)}{715(\mu - 19)} & 1 \end{pmatrix}$ | $\{ 93(167 - \mu), 1 \} \quad \{ \frac{111(102 - \mu)}{408}, 1 \}$ | $\{ 0, 1 \}$ |
| $\mathcal{H}_4$ 348435$u_1 - 1871(\mu - 9)u_5 > 0$ | $\begin{pmatrix} 3(\mu - 167) & 0 \\ \frac{65(\mu - 9)}{715(\mu - 19)} & 1 \end{pmatrix}$ | $\{ 3(\mu - 167), 1 \} \quad \{ \frac{111(102 - \mu)}{408}, 1 \}$ | $\{ 0, 1 \}$ |
| $\mathcal{H}_5$ 93$u_1 - (\mu - 9)u_5 > 0$ | $\begin{pmatrix} 93(\mu - 167) & 0 \\ \frac{65(\mu - 9)}{715(\mu - 19)} & 1 \end{pmatrix}$ | $\{ 93(167 - \mu), 1 \} \quad \{ \frac{111(102 - \mu)}{408}, 1 \}$ | $\{ 0, 1 \}$ |

$\mu = 122787543 - 123922\mu, \quad b_2 = \sqrt{320140324 \mu^2 - 60787796412 \mu + 2893236225489}, \quad c_2 = 17927 \mu - 1701498,$
$\mu = 142050954 - 75491\mu, \quad b_4 = \sqrt{2615265201481 \mu^2 - 504216904560828 \mu + 24312026567983716}, \quad c_4 = 1612579 \mu - 155884746,$
$\mu = 353512794 + 104449\mu, \quad b_5 = \sqrt{3386009761 \mu^2 - 644714033868 \mu + 30745583258796}, \quad c_5 = 57553 \mu - 5466054.$
Appendix B. Notation

We list the main notation for constants and auxiliary functions used in this paper in order of appearance with the reference of the section containing their definition Table 13.

Table 13 Notation

| Notation | Definition/meaning | Section |
|----------|--------------------|---------|
| $V, E$   | Set of equilibria (vertices), set of edges of the cube | §2      |
| $F_v$    | Set of three faces $x_j$ defined by the component $x_j = 0$ at $v \in V$ | §2      |
| $F$      | Set of all faces of the cube | §2      |
| $I, I_1, I_2, I_3$ | $I_1 = \left[ \frac{850}{11}, \mu_1 \right], I_2 = \left[ \mu_1, \mu_2 \right], I_3 = \left[ \mu_2, \frac{544}{5} \right]$, $\mu_1 = 102$ and $\mu_2 \approx 105.04$ | §4.2    |
| $F$      | Heteroclinic network, heteroclinic cycle | §4.4    |
| $N_v$    | Cubic neighbourhood of $v \in V$ where the flow may be $C^1$–linearized | §5.2    |
| $N_\gamma$ | Tubular neighbourhood of $\gamma \in \mathcal{E}$ | §5.2    |
| $\Psi_\varepsilon$ | Quasi-change of coordinates ($\varepsilon$: blow-up parameter) | §5.3    |
| $\Pi_\varepsilon, \Pi_\gamma$ | $\Psi_\varepsilon(N_v), \Psi_\varepsilon(N_v) = \Pi_\varepsilon \cap \Pi_\gamma$ | §5.3    |
| $\chi^v, \chi^v_j$ | Character vector field at $v$; $j$-component $\sigma$ of $\chi^v$ where $j \in \{1, \ldots, 6\}$ | §5.4    |
| $\Pi_\gamma, \gamma'$ | $\Pi_\gamma(\gamma'), \Pi_\gamma(\gamma') = \Pi_\varepsilon \cap \Pi_\gamma$ | §5.5    |
| $P_{\gamma, \gamma'}$ | Map carrying points from $D_{\gamma, \gamma'}$ to $Out(v) \cap \gamma'$ | §5.5    |
| $F_{\gamma, \gamma'}$ | $\Psi_\varepsilon \circ P_{\gamma, \gamma'} \circ (\Psi_\varepsilon)^{-1} | D_{\gamma, \gamma'}^*$ | §5.5    |
| $\Pi_{\gamma, \gamma'}$ | $\left\{ y \in \text{int}(\Pi_\gamma): y_\sigma > \frac{\chi^v_j}{\chi^v_{\ast}} y_\sigma, \forall \sigma \in F_\sigma, \sigma \neq \sigma_\ast \right\}$ | §5.5    |
| $L_{\gamma, \gamma'}$, $M_{\gamma, \gamma'}$ | Induced linear map from $\Pi_{\gamma, \gamma'}$ to $\Pi_\gamma$, associated matrix | §5.5    |
| $\pi_\xi$ | Sector int($\Pi_{\gamma \varepsilon}$) $\cap \bigcap_{j=1}^{m} \left( L_{\gamma_{j-1}, \gamma_j} \circ \cdots \circ L_{\gamma_{0}, \gamma_1} \right)^{-1} \left( \text{int}(\Pi_\gamma) \right) \subset (\mathbb{R}_+^0)^6$ | §5.8    |
| $\pi_\xi, M_\xi$ | Linear map $L_{\gamma \varepsilon_{m-1}, \gamma_m} \circ \cdots \circ L_{\gamma_{0}, \gamma_1}$, associated 2-dim matrix | §5.8    |
| $\pi_\xi \xi$ | Skeleton map defined in any sector of $\Pi_{S}$ ($S$: structural set) | §§5.10  |
| $\Delta_{\xi}$ | $\{ u = (u_1, \ldots, u_6) \in \text{int}(\Pi_{\xi}): \overline{u} = 1 \}$ where $\overline{u} = \sum_{j=1}^{6} u_j$ | §5.10   |
| $\hat{\pi}_{\xi}$ | Projective map along the $S$-branch $\xi$ | §5.11   |
References

Afraimovich, V.S., Moses, G., Young, T.: Two-dimensional heteroclinic attractor in the generalized Lotka–Volterra system. Nonlinearity 29(5), 1645 (2016)

Aguiar, M.A.D.: Is there switching for replicator dynamics and bimatrix games? Physica D 240(18), 1475–1488 (2011)

Aguiar, M.A.D., Labouriau, I.S., Rodrigues, A.A.P.: Switching near a network of rotating nodes. Dyn. Syst. 25(1), 75–95 (2010)

Alishah, H.N., Duarte, P.: Hamiltonian evolutionary games. J. Dyn. Games 2(1), 33 (2015)

Alishah, H.N., Duarte, P., Peixe, T.: Conservative and dissipative polymatrix replicators. J. Dyn. Games 2(2), 157 (2015)

Alishah, H.N., Duarte, P., Peixe, T.: Asymptotic Poincaré maps along the edges of polytopes. Nonlinearity 33(1), 469 (2019)

Alishah, H.N., Duarte, P., Peixe, T.: Asymptotic dynamics of hamiltonian polymatrix replicators. Nonlinearity 36(6), 3182 (2023)

Ashwin, P., Postlethwaite, C.: On designing heteroclinic networks from graphs. Physica D 265, 26–39 (2013)

Barendregt, N.W., Thomas, P.J.: Heteroclinic cycling and extinction in May–Leonard models with demographic stochasticity. J. Math. Biol. 86(2), 30 (2023)

Bunimovich, L.A., Webb, B.Z.: Isospectral compression and other useful isospectral transformations of dynamical networks. Chaos Interdiscip. J. Nonlinear Sci. 22, 3 (2012)

Castro, S.B.S.D., Garrido-da Silva, L: Finite switching near heteroclinic networks. arXiv preprint arXiv:2211.04202 (2022)

Castro, S.B.S.D., Labouriau, I.S., Podvigina, O.: A heteroclinic network in mode interaction with symmetry. Dyn. Syst. 25(3), 359–396 (2010)

Castro, S.B., Ferreira, A., Garrido-da-Silva, L., Labouriau, I.S.: Stability of cycles in a game of Rock-Scissors-Paper-Lizard-Spock. SIAM J. Appl. Dyn. Syst. 21(4), 2393–2431 (2022)

Field, M.J.: Lectures on Bifurcations, Dynamics and Symmetry. CRC Press (2020)

Field, M., Swift, J.W.: Stationary bifurcation to limit cycles and heteroclinic cycles. Nonlinearity 4(4), 1001 (1991)

Garrido-da Silva, L., Castro, S.B.S.D.: Stability of quasi-simple heteroclinic cycles. Dyn. Syst. 34(1), 14–39 (2019)

Gaunersdorfer, A., Hofbauer, J.: Fictitious play, Shapley polygons, and the replicator equation. Games Econ. Behav. 11(2), 279–303 (1995)

Guckenheimer, J., Holmes, P.: Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields, vol. 42. Springer (2013)

Hofbauer, J., Sigmund, K.: Permanence for replicator equations. In: Dynamical Systems, pp. 70–91. Springer (1987)

Hofbauer, J., Sigmund, K., et al.: Evolutionary Games and Population Dynamics. Cambridge University Press (1998)

Katok, A., Hasselblatt, B.: Introduction to the Modern Theory of Dynamical Systems, vol. 54. Cambridge University Press (1997)

Krupa, M., Melbourne, I.: Asymptotic stability of heteroclinic cycles in systems with symmetry. Ergodic Theory Dyn. Syst. 15(1), 121–147 (1995)

Labouriau, I.S., Rodrigues, A.A.P.: On Takens’ last problem: tangencies and time averages near heteroclinic networks. Nonlinearity 30(5), 1876 (2017)

Lohse, A.: Unstable attractors: existence and stability indices. Dyn. Syst. 30(3), 324–332 (2015)

Melbourne, I.: An example of a nonasymptotically stable attractor. Nonlinearity 4(3), 835 (1991)

Milnor, J.: On the concept of attractor. In: The Theory of Chaotic Attractors, pp. 243–264. Springer (1985)

Palis, J., de Melo, W.: Local stability. In: Geometric Theory of Dynamical Systems, pp. 39–90. Springer (1982)

Peixe, T.: Lotka–Volterra Systems and Polymatrix Replicators. ProQuest LLC, Ann Arbor. Thesis (Ph.D.)–Universidade de Lisboa (Portugal) (2015)

Peixe, T.: Permanence in polymatrix replicators. J. Dyn. Games (2019)

Peixe, T., Rodrigues, A.: Persistent strange attractors in 3d polymatrix replicators. Physica D 438, 133346 (2022)

Podvigina, O.: Stability and bifurcations of heteroclinic cycles of type z. Nonlinearity 25(6), 1887 (2012)
Podvigina, O., Ashwin, P.: On local attraction properties and a stability index for heteroclinic connections. Nonlinearity 24(3), 887 (2011)
Podvigina, O., Chossat, P.: Simple heteroclinic cycles. Nonlinearity 28(4), 901 (2015)
Podvigina, O., Chossat, P.: Asymptotic stability of pseudo-simple heteroclinic cycles in $\mathbb{R}^4$. J. Nonlinear Sci. 27(1), 343–375 (2017)
Podvigina, O., Castro, S.B.S.D., Labouriau, I.S.: Stability of a heteroclinic network and its cycles: a case study from Boussinesq convection. Dyn. Syst. 34(1), 157–193 (2019)
Podvigina, O., Castro, S.B.S.D., Labouriau, I.S.: Asymptotic stability of robust heteroclinic networks. Nonlinearity 33(4), 1757 (2020)
Postlethwaite, C.M., Rucklidge, A.M.: Stability of cycling behaviour near a heteroclinic network model of Rock-Paper-Scissors-Lizard-Spock. Nonlinearity 35, 1702 (2021)
Rodrigues, A.A.P.: Persistent switching near a heteroclinic model for the geodynamo problem. Chaos Solitons Fractals 47, 73–86 (2013)
Rodrigues, A.A.P.: Attractors in complex networks. Chaos Interdiscip. J. Nonlinear Sci. 27(10), 103105 (2017)
Ruelle, D.: Elements of Differentiable Dynamics and Bifurcation Theory. Elsevier (2014)
Smith, J.M., Price, G.R.: The logic of animal conflict. Nature 246(5427), 15–18 (1973)

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