Universal Vassiliev invariants of links in coverings of 3-manifolds

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Abstract
We study Vassiliev invariants of links in a 3-manifold $M$ by using chord diagrams labeled by elements of the fundamental group of $M$. We construct universal Vassiliev invariants of links in $M$, where $M = P^2 \times [0, 1]$ is a cylinder over the real projective plane $P^2$, $M = \Sigma \times [0, 1]$ is a cylinder over a surface $\Sigma$ with boundary, and $M = S^1 \times S^2$. A finite covering $p : N \longrightarrow M$ induces a map $\pi_1(p)^*$ between labeled chord diagrams that corresponds to taking the preimage $p^{-1}(L) \subset N$ of a link $L \subset M$. The maps $p^{-1}$ and $\pi_1(p)^*$ intertwine the constructed universal Vassiliev invariants.

Introduction

Let $\Sigma$ be an oriented surface with non-empty boundary and let $I = [0, 1]$. The universal Vassiliev invariant of links in $I^2 \times I$ has been generalized to links in $\Sigma \times I$ by Andersen, Mattes, and Reshetikhin ([AMR]). Their universal invariant takes values in a space of chord diagrams, where by a chord diagram they mean a homotopy class of certain maps from a usual chord diagram into the surface $\Sigma$. By another approach a universal Vassiliev invariant of braids in $\Sigma \times I$ is constructed in [GIM], where $\Sigma$ is a closed surface of genus $\geq 1$. This invariant is universal for Vassiliev invariants with values in an abelian group and separates braids. Vassiliev invariants of links in a closed oriented 3-manifold $M$ are obtained by using Dehn surgery presentations of $M$ ([LMQ], [BGR]). In this general situation, a universal Vassiliev invariant of links is not known.

In this paper we start by studying Vassiliev invariants of links in an arbitrary connected 3-manifold $M$. For this purpose we use a vector space $\tilde{A}(M)$ spanned by chord diagrams that are labeled by elements of the fundamental group of $M$. We construct a universal Vassiliev invariant of links in $M$, where $M$ is one of the following manifolds:
- $M = \Sigma \times I$, where $\Sigma$ is a connected compact surface with non-empty boundary,
- $M = P^2 \times I$, where $P^2$ is the real projective plane,
- $M = S^1 \times S^2$.

In the first two cases the universal Vassiliev invariant takes values in a completion of $\bar{A}(M)$, whereas in the third case it takes values in a quotient of the completion. In the construction of the universal Vassiliev invariant of links in $\Sigma \times I$ we use a decomposition of the surface $\Sigma$ into an oriented disk and bands that are glued to the boundary of this disk. In difference to [AMR] we include the case of non-orientable surfaces $\Sigma$. The cases $M = P^2 \times I$ and $M = S^1 \times S^2$ are treated by representing links in $M$ by links in $\Sigma \times I$, where $\Sigma$ is the Möbius strip or $\Sigma = S^1 \times I$, respectively.

We extend the definition of the universal Vassiliev invariant of links in $\Sigma \times I$ to $I$-bundles over ribbon graphs $\Sigma$ which allows a more flexible choice of diagrams of links in $\Sigma \times I$. For a finite connected covering $p : N \to M$ of 3-manifolds we define a map $\pi_1(p)^* : \bar{A}(M) \to \bar{A}(N)$ that corresponds to taking the preimage $p^{-1}(L) \subset N$ of a link $L \subset M$. For coverings of $\Sigma \times I$, $P^2 \times I$, and $S^1 \times S^2$ the maps $p^{-1}$ and $\pi_1(p)^*$ intertwine the universal Vassiliev invariants that are constructed in this paper.

The paper is organized as follows. In Section 1 we define the Vassiliev filtration on the space $L(M)$ of links in a 3-manifold $M$ with associated graded space $\text{gr}L(M)$ and the algebra of Vassiliev invariants $V(M)$ of links in $M$. In Section 2 we introduce the graded coalgebra of $G$-labeled chord diagrams, where $G$ is a group together with a homomorphism $\sigma : G \to \{\pm 1\}$. In the case where $G = \pi_1(M, \ast)$ and $\sigma$ is the orientation character of $M$, we denote this coalgebra by $\bar{A}(M)$. In Section 3, we show that for every connected 3-manifold $M$ the coalgebra $\bar{A}(M)$ maps surjectively onto $\text{gr}L(M)$. In Section 4 we state our main results about the relation between the graded algebra $\bar{A}(M)^*$ dual to $\bar{A}(M)$ and $V(M)$: for $M = \Sigma \times I$ with $\partial \Sigma \neq \emptyset$ or $\Sigma = P^2$ we have $\bar{A}(M)^* \cong V(M)$. For $M = S^1 \times S^2$ the space $V(M)$ is isomorphic to a subalgebra of $\bar{A}(M)^*$, which we determine. Sections 5 to 11 are devoted to the proofs of these results. In Section 12 we discuss variations on the definition of universal Vassiliev invariants including the case of links in $I$-bundles over ribbon graphs $\Sigma$. In Section 13 we prove that the universal Vassiliev invariants constructed in this paper are compatible with finite coverings.

1 Links in a 3-manifold

Throughout this paper we work in the piecewise linear category. A link is an oriented closed 1-dimensional submanifold of a 3-manifold. A knot is a link that has only
one connected component. A singular link may have transversal double points but no other singularities. Let $\mathcal{L}(M)$ be the vector space freely generated by isotopy classes of links in $M$. We associate an element of $\mathcal{L}(M)$ to a singular link $L$ in the following way: we choose an arbitrary local orientation of $M$ at each double point of $L$ and then desingularize $L$ to a linear combination of links without double points by applying the local replacement rule shown in Figure 1 to each double point of $L$.

Figure 1: Desingularization of a singular link

This linear combination is determined by $L$ and the chosen local orientations, and it is determined by $L$ up to a sign. It follows that we have a well-defined filtration

$$\mathcal{L}(M) = \mathcal{L}_0(M) \supseteq \mathcal{L}_1(M) \supseteq \mathcal{L}_2(M) \ldots$$

where $\mathcal{L}_i(M)$ is spanned by the desingularizations of links in $M$ with $i$ double points. Let $\text{gr}\mathcal{L}(M) = \bigoplus_{n=0}^{\infty} \mathcal{L}_n(M)/\mathcal{L}_{n+1}(M)$ be the associated graded vector space. The subspace of $\text{gr}\mathcal{L}(M)$ generated by the desingularizations of singular links with $\ell$ components and $n$ double points is denoted by $\text{gr}\mathcal{L}(M)^\ell_n$.

The following definitions and results are adapted from $M = \mathbb{R}^3$ (see [Vog], [BN1] for details) to the case of an arbitrary connected 3-manifold $M$. The space $\mathcal{L}(M)$ has a coalgebra structure. The comultiplication is given by mapping a link $L$ (without double points) to $L \otimes L$. This coalgebra structure induces a coalgebra structure on $\text{gr}\mathcal{L}(M)$.

**Definition 1** A linear map $v : \mathcal{L}(M) \rightarrow \mathbb{Q}$ is called a Vassiliev invariant of degree $n$ if $v(\mathcal{L}_{n+1}(M)) = 0$. Let $V_n(M)$ be the vector space of all Vassiliev invariants of degree $n$. Define $V(M) = \bigcup_{i=0}^{\infty} V_i(M)$.

The spaces $V_i(M)$ form an increasing sequence

$$V_0(M) \subseteq V_1(M) \subseteq V_2(M) \subseteq \ldots \subseteq V(M).$$

The product $v_1v_2$ of Vassiliev invariants $v_1$ and $v_2$ is given by $(v_1v_2)(L) = v_1(L)v_2(L)$, where $L$ is a link in $M$. With this product $V(M)$ is a subalgebra of $\mathcal{L}(M)^*$. If $v_i$ has degree $n_i$, then $v_1v_2$ is a Vassiliev invariant of degree $n_1 + n_2$. 

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2 Labeled chord diagrams

Let $G$ be a group. Let $\Gamma$ be a compact one-dimensional oriented manifold together with a partition of $\partial \Gamma$ into two ordered sets $s(\Gamma)$ and $t(\Gamma)$ called lower and upper boundary of $\Gamma$ respectively. For a set $X$ we denote the set of finite words in the letters $X$ by $X^\ast$. One assigns a symbol $+$ or $-$ (resp. $-$ or $+$) to a lower (resp. upper) boundary point of $\Gamma$ according to whether $\Gamma$ directs toward this point or not. Doing this for all elements of $s(\Gamma)$ and $t(\Gamma)$ we obtain two elements of $\{+, -\}^\ast$ called source($\Gamma$) and target($\Gamma$) respectively. A $G$-labeled chord diagram $D$ with skeleton $\Gamma$ consists of the following data:

- a finite set $S$ of mutually distinct points on $\Gamma \setminus \partial \Gamma$,
- a subset $C \subset S$ called chord endpoints,
- a partition of $C$ into sets of cardinality two called chords,
- a map from $S \setminus C$ to $G$ that assigns labels to elements of $S \setminus C$.

Sometimes we call $G$-labeled chord diagrams simply chord diagrams. We consider two chord diagrams $D$ and $D'$ with skeleton $\Gamma$ and $\Gamma'$ respectively as being equal, if there exists a homeomorphism between $\Gamma$ and $\Gamma'$ that preserves all additional data. For a chord diagram with skeleton $\Gamma$ we define source($D$) = source($\Gamma$) and target($D$) = target($\Gamma$). Define the degree $\deg(D)$ of a chord diagram $D$ as the number of its chords.

We represent a chord diagram graphically as follows: we represent the 1-manifold $\Gamma$ by drawing the oriented circles and intervals of $\Gamma$ inside of the strip $\mathbb{R} \times [0, 1]$ such that the lower (resp. upper) boundary points of $\Gamma$ lie on the horizontal line $\mathbb{R} \times 0$ (resp. $\mathbb{R} \times 1$) and such that the elements of $s(\Gamma)$ and $t(\Gamma)$ are arranged in increasing order from left to right. The orientation of the intervals of $\Gamma$ is determined by source($\Gamma$) and target($\Gamma$) or indicated by arrows and the circle components of $\Gamma$ are oriented counterclockwise in the pictures. The chords of a chord diagram are represented graphically by connecting its two endpoints by a line. The 1-manifold $\Gamma$ is drawn with a thicker pencil than the chords of $D$. The labels of a chord diagram are represented by marking the points of $S \setminus C$ and by writing the labels close to these marked points. An example of a picture of a $\mathbb{Z}$-labeled chord diagram $D$ of degree 5 with source($D$) = $+$ and target($D$) = $- + +$ is shown in Figure 2.

In the following we define vector spaces by generators and graphical relations. We use the convention that all diagrams in a graphical relation coincide everywhere except for the parts we show, and that all configurations of the hidden parts are possible.
Figure 2: An example of a picture of a $\mathbb{Z}$-labeled chord diagram

Definition 2 Let $G$ be a group, $\sigma : G \rightarrow \{\pm 1\}$ a homomorphism of groups, and let $s, t \in \{+, -\}^*$. Define $\mathcal{A}(G, \sigma, s, t)$ as the graded $\mathbb{Q}$-vector space generated by $G$-labeled chord diagrams $D$ with source$(D) = s$ and target$(D) = t$ modulo the following relations:

- The four-term relation ($4T$):

- The Relation ($\sigma$-Nat):

- The Relations (Rep):

where $e$ is the neutral element of $G$ and $ab$ denotes the product of $a$ and $b$ in the group $G$.

Let $\tilde{\mathcal{A}}(G, \sigma, s, t)$ be the quotient of $\mathcal{A}(G, \sigma, s, t)$ by the framing independence relation ($FI$):

Define the composition $D_1 \circ D_2$ of two chord diagrams with source$(D_1) = \text{target}(D_2)$ graphically by placing $D_1$ onto the top of $D_2$ and by sticking them together. For a fixed pair $(G, \sigma)$ this operation is compatible with all relations of Definition 2. For a $G$-labeled chord diagram $D$ of degree $n > 0$ define

$$\Delta(D) = \sum_{D = D' \cup D''} D' \otimes D'', \quad (3)$$
where the sum is taken over the $2^n$ pairs of diagrams $(D', D'')$ such that $D'$ and $D''$ have the same labeled skeleton as $D$ and the chords of $D$ are the disjoint union of the chords of $D'$ and the chords of $D''$. For $D$ with $\text{deg}(D) = 0$ define $\Delta(D) = D \otimes D$.

As in the case $G = \{e\}$ the spaces $\mathcal{A}(G, \sigma, s, t)$ and $\bar{\mathcal{A}}(G, \sigma, s, t)$ have a coalgebra structure with comultiplication $\Delta$.

Define the category $\text{Gr}^\pm$ as follows. The objects of $\text{Gr}^\pm$ are pairs $(G, \sigma)$ of a group $G$ and a homomorphism $\sigma : G \to \{\pm 1\}$. A morphism $\varphi : (G, \sigma) \to (H, \sigma')$ in $\text{Gr}^\pm$ is a homomorphism of groups $\varphi : G \to H$ such that $\sigma' = \sigma \circ \varphi$.

Let $s, t \in \{+, -\}^*$ be given. We define a functor $\mathcal{F}$ from $\text{Gr}^\pm$ to the category of coalgebras for objects by $\mathcal{F}(G, \sigma) = \bar{\mathcal{A}}(G, \sigma, s, t)$. For morphisms $\varphi$ we define $\mathcal{F}(\varphi) = \varphi_*$ by applying $\varphi$ to the labels of a chord diagram.

### 3 Mapping chord diagrams to singular links

Denote the empty word by $\emptyset$. We denote $\bar{\mathcal{A}}(G, \sigma, \emptyset, \emptyset)$ simply by $\bar{\mathcal{A}}(G, \sigma)$. Let $M$ be a 3-manifold and $\ast \in M$. Define a map

$$\sigma : \pi_1(M, \ast) \to \{\pm 1\}$$

as follows. Choose a local orientation of $M$ at the point $\ast$. If the local orientation stays the same when we push it along a generic representative of an element $w \in \pi_1(M, \ast)$, then define $\sigma(w) = 1$, and define $\sigma(w) = -1$ otherwise. The map $\sigma$ is called the orientation character of $M$. We denote $\bar{\mathcal{A}}(\pi_1(M, \ast), \sigma)$ by $\bar{\mathcal{A}}(M)$. In this section we relate $\bar{\mathcal{A}}(M)$ with $\text{gr} \mathcal{L}(M)$.

Choose an oriented neighborhood $U$ of $\ast$ with $U \cong \mathbb{R}^3$. We identify homotopy classes of paths $w$ from $p_1 \in U$ to $p_2 \in U$ with elements in $\pi_1(M, \ast)$ by pulling $p_1$ and $p_2$ to $\ast$ along paths inside of $U$. We will use this identification throughout the rest of this paper. If $D$ is a $\pi_1(M, \ast)$-labeled chord diagram with source$(D) = \text{target}(D) = \emptyset$, then it is easy to see that one can construct a singular link $L_D \subset M$ with the following properties:

1) All double points of $L_D$ lie inside of $U$.
2) There is an immersion of the skeleton of $D$ onto $L_D$ that fails to be injective only at the preimages of the double points of $L_D$. The preimages of double points of $L_D$ correspond to the pairs of points of $D$ connected by a chord.
3) Let $w \in \pi_1(M, \ast)$ be the product of the markings of $D$ along a part of $D$ between two consecutive (not necessarily different) chord endpoints. The empty product is considered as neutral element. By 1) and 2) the segment of $D$ between these two
chord endpoints is mapped to a part \( p \) of \( L_D \) connecting two double points in \( U \). We require that the path \( p \) represents \( w \in \pi_1(M, \ast) \).

4) Let \( K \) be a component of \( D \) without chord endpoints. Let \( A \subseteq \pi_1(M, \ast) \) be the conjugacy class represented by the product of the labels on \( K \) (\( A \) is independent of the choice of the starting point for this product). Then \( K \) is mapped to an unbased loop in \( M \) representing \( A \).

Let us consider an example. Let \( M = S^1 \times I^2 \) and identify \( \pi_1(M, \ast) \) with \( \mathbb{Z} \). In Figure 3 we see a \( \mathbb{Z} \)-labeled chord diagram \( D \) and a possible choice of the singular link \( L_D \subset M \). The neighborhood \( U \) of \( \ast \) consists of the right half of \( S^1 \times I^2 \) in Figure 3.

![Figure 3: Mapping a chord diagram \( D \) to a singular link \( L_D \)](image)

Notice that the singular link \( L_D \) is not uniquely determined by \( D \), but we call \( L_D \) every singular link with properties 1)–4). By property 1) we may desingularize all double points of \( L_D \) with the local orientations at double points determined by the orientation of \( U \). We call this desingularization \([L_D]\). If \( D \) has \( n \) chords, then we have \([L_D] \in \mathcal{L}_n(M)\).

**Proposition 3**  
Mapping a \( \pi_1(M, \ast) \)-labeled chord diagram \( D \) of degree \( n \) to  

\[
\psi_M(D) = [L_D] \mod \mathcal{L}_{n+1}(M)
\]

induces a graded surjective morphism of coalgebras \( \psi_M : \bar{\mathcal{A}}(M) \rightarrow \text{gr}\mathcal{L}(M) \).

**Proof:**  
a) \( \psi_M \) is well-defined: Consider two singular links \( L \) and \( L' \) with the properties 1)–4) of \( L_D \). Then one can pass from \( L \) to \( L' \) by crossing changes and isotopies. In other words \([L] - [L'] \in \mathcal{L}_{n+1}(M)\). So \( \psi_M \) is well-defined for labeled chord diagrams. We have to check that \( \psi_M \) respects the defining relations of \( \bar{\mathcal{A}}(M) \).
The map $\psi_M$ respects the Relations (4T) and (FI) because these relations are a consequence of the Reidemeister moves in $\mathbb{R}^3 \cong U$ and properties 1) and 2) of $L_D$ (compare [BNI]). It follows from property 3) of $L_D$ that $\psi_M$ respects the Relations (Rep).

Call the part of a chord diagram from the left side of the Relation $(\sigma\text{-Nat}) P$ and the part from the right side $P'$. Let $D$ and $D'$ be chord diagrams with $P \subset D$, $P' \subset D'$ and $D \setminus P = D' \setminus P'$. Let $\hat{s}$ be an annulus $S^1 \times [-1,1]$ in $M$ whose core $S^1 \times 0$ represents $s \in \pi_1(M,*)$. We may choose $L_D$ such that the oriented arcs of $P \subset D$ are mapped entirely into $\hat{s}$ and no other part of $L_D$ intersects $\hat{s}$. Now we push the double point of $L_D$ corresponding to the chord in $P$ along $\hat{s}$ until we obtain a singular link $L_D'$. Since the desingularization of singular links is defined with respect to the orientation of $U$, we have $\psi_M(D) = \sigma(s) \psi_M(D')$.

(b) $\psi_M$ is surjective: Let $L$ be a singular link with $n$ double points. Push all double points of $L$ into the neighborhood $U$ of $*$ by an isotopy and call the resulting singular link $\tilde{L}$. Let $D$ be a chord diagram such that $\tilde{L}$ has all properties of $L_D$ (If we replace labels of $D$ between consecutive chord endpoints by unique labels using the Relation (Rep) and if we fix a choice of representatives of conjugacy classes as labels of components of $D$ without chord endpoints, then the diagram $D$ is uniquely determined by $\tilde{L}$). Then $\psi_M(D)$ is equal to $\pm 1$ times the desingularization of $L$ with freely chosen local orientations at double points modulo $L_{n+1}(M)$.

(c) It follows as in the case of $M = \mathbb{R}^3$ that $\psi_M$ is a morphism of coalgebras (see [Vog], Proposition 4). ☐

As an immediate application of Proposition 3 we obtain the following corollary.

**Corollary 4** (1) If $\pi_1(M,*)$ is finite, then $\text{gr} \mathcal{L}(M)_{\ell}^n$ is finite dimensional.

(2) If there exists an element $s$ with $\sigma(s) = -1$ in the center of $\pi_1(M,*)$, then we have $\mathcal{L}_{2n-1}(M) = \mathcal{L}_{2n}(M)$ for all $n > 0$.

**Proof:** (1) If $\pi_1(M,*)$ is finite, then there exists only a finite number of $\pi_1(M,*)$-labeled chord diagrams of degree $n$ with $\ell$ circles as skeleton and without neighbored labels that would allow an application of Relation (Rep). It follows from Proposition 3 that $\text{gr} \mathcal{L}(M)_{\ell}^n$ is finite dimensional in this case.

(2) Let $D$ be a $\pi_1(M,*)$-labeled chord diagram. Let $s$ be in the center of $\pi_1(M,*)$ with $\sigma(s) = -1$. Using the Relations (Rep) of $\bar{A}(M)$ we insert the labels $s$ and $s^{-1}$ between each pair of consecutive chord endpoints. Then we commute these labels with the other labels of $D$ such that near the endpoints of each chord we have labels as shown in Figure 4.

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We apply the Relation \((\sigma\text{-Nat})\) for every chord. Using (Rep) we forget the pairs \((s^{-1}, s)\) again. This procedure implies that in \(\bar{A}(M)\) we have \(D = (-1)^{\deg D} D\). Hence all elements of odd degree in \(\bar{A}(M)\) are 0. The surjectivity of \(\psi_M\) stated in Proposition 3 implies part (2) of Corollary 4.

The reduction of the coefficient of \(z^{2n+1}\) modulo 2 of the Conway polynomial of links in \(P^2 \times I\) (see [Lie]) is an example of a Vassiliev invariant of degree \(2n+1\) that is not a Vassiliev invariant of degree \(2n\). Corollary 4 implies that this invariant cannot be lifted to a \(\mathbb{Z}\)-valued Vassiliev invariant.

Now let us clarify how \(\psi_M\) depends on the choices we made. Let \(p_1, p_2 \in M\). Choose oriented neighborhoods \(U_i \cong \mathbb{R}^3\) of \(p_i\). Then two maps \(\psi_{M,i} : \bar{A}(M, p_i) \rightarrow \text{gr}\mathcal{L}(M)\) are defined. If \(M\) is orientable and if \(U_1\) and \(U_2\) induce different orientations of \(M\), then let \(\epsilon = -1\). Let \(\epsilon\) be 1 otherwise.

**Proposition 5** There exists a path \(w\) from \(p_2\) to \(p_1\), such that the following diagram is commutative for \(\varphi : \pi_1(M, p_1) \rightarrow \pi_1(M, p_2)\) defined by \(\varphi(u) = wuw^{-1}\).

\[
\begin{array}{ccc}
\bar{A}(M, p_1) & \xrightarrow{D \mapsto \epsilon^{\deg D} \varphi_*(D)} & \bar{A}(M, p_2) \\
\downarrow \psi_{M,1} & & \downarrow \psi_{M,2} \\
\text{gr}\mathcal{L}(M) & & \\
\end{array}
\]

The proof of the proposition is easy and is left to the reader.

**4 Main results: universal Vassiliev invariants**

Consider the completion \(\hat{A}(M) = \prod_{i=0}^{\infty} \bar{A}(M)_i\) of \(\bar{A}(M)\). We write elements of \(\hat{A}(M)\) as formal series. The comultiplication \(\Delta\) of \(\bar{A}(M)\) extends to a map

\[
\hat{\Delta} : \hat{A}(M) ightarrow \hat{A}(M) \hat{\otimes} \hat{A}(M) = \bigoplus_{i=0}^{\infty} \bigotimes_{j=0}^{i} \bar{A}(M)_i \otimes \bar{A}(M)_{j-i}.
\]
In Section 3 we saw that there exists a surjective map \( \psi_M : \mathring{A}(M) \to \text{gr}\mathcal{L}(M) \) for every connected manifold \( M \). In this section we state stronger results for particular manifolds \( M \). The following lemma generalizes the invariant of [AMR] to non-orientable surfaces \( \Sigma \).

**Lemma 6** Let \( \Sigma \) be a compact connected surface with non-empty boundary or let \( \Sigma \) be the real projective plane. Then there exists a linear map

\[
Z_{\Sigma \times I} : \mathcal{L}(\Sigma \times I) \to \mathring{A}(\Sigma \times I)
\]

such that for every \( \pi_1(\Sigma \times I, \ast) \)-labeled chord diagram \( D \) of degree \( n \) with \( \text{source}(D) = \text{target}(D) = \emptyset \) we have

\[
Z_{\Sigma \times I}([L_D]) = D + \text{terms of degree} > n,
\]

and for every link \( L \) in \( \Sigma \times I \) we have

\[
\hat{\Delta}(Z_{\Sigma \times I}(L)) = Z_{\Sigma \times I}(L) \hat{\otimes} Z_{\Sigma \times I}(L).
\]

The lemma shall be proven in Sections 9 and 10.

Let \( \mathring{A}(\Sigma \times I)^* \) be the graded dual algebra of the coalgebra \( \mathring{A}(\Sigma \times I) \). The map \( Z_{\Sigma \times I} \) is called universal Vassiliev invariant for a reason given by the second statement of the following theorem.

**Theorem 1** Let \( \Sigma \) be a compact surface with non-empty boundary or let \( \Sigma \) be the real projective plane. Then the map \( \psi_{\Sigma \times I} : \mathring{A}(\Sigma \times I) \to \text{gr}\mathcal{L}(\Sigma \times I) \) is an isomorphism of coalgebras. The formula \((Z_{\Sigma \times I}^*(w))(L) = w(Z_{\Sigma \times I}(L))\) defines an isomorphism of algebras

\[
Z_{\Sigma \times I}^* : \mathring{A}(\Sigma \times I)^* \to \mathcal{V}(\Sigma \times I).
\]

**Proof:** In view of Proposition 9, it suffices to show that \( \psi_{\Sigma \times I} \) is injective for the first statement. Equation (3) implies that \( Z_{\Sigma \times I} \) induces a map from \( \text{gr}\mathcal{L}(\Sigma \times I) \) to \( \mathring{A}(\Sigma \times I) \) which is the left inverse of \( \psi_{\Sigma \times I} \). This implies that \( \psi_{\Sigma \times I} \) is injective.

If \( w \in \mathring{A}(\Sigma \times I)^* \) has degree \( n \), then equation (3) implies that \( Z_{\Sigma \times I}^*(w) \) is a Vassiliev invariant of degree \( n \). If \( v \) is a Vassiliev invariant of degree \( n \), then the map \( W(v) \) defined for chord diagrams \( D \) of degree \( n \) by \( W(v)(D) = v([L_D]) \) is in \( \mathring{A}(\Sigma \times I)^*_n \). Equation (3) implies that \( w = W(Z_{\Sigma \times I}^*(w)) \) and that \( v - Z_{\Sigma \times I}^*(W(v)) \) is a Vassiliev invariant of degree \( n - 1 \). This implies the injectivity of \( Z_{\Sigma \times I}^* \) and its surjectivity by induction. Equation (8) implies that \( Z_{\Sigma \times I}^* \) is an isomorphism of algebras. \( \square \)
Let $s$ be an element of a group $G$. In Figure 5 we introduce a notation for a part of a $G$-labeled chord diagram that depends on the orientations of strands with an unspecified orientation in the figure.

![Figure 5: A notation for oriented labeled strands](image)

Similar to Figure 5, we introduce the notation shown in Figure 6. The sum in the figure has one term for each strand of the shown part of a chord diagram. The free chord end can lead to any point on the chord diagram but this point must be the same for all terms in the sum.

![Figure 6: A notation for certain sums of chord diagrams](image)

We also use the notation obtained by reflection of the diagrams in Figure 6 in a vertical axis and by combinations of the notations as shown in the example of Figure 7.

![Figure 7: An example for the notation introduced in Figures 5 and 6](image)

Now let $s$ be a generator of $\pi_1(S^1 \times S^2, \ast) \cong (\mathbb{Z}, +)$. Let $m$ be the number of strands in Figure 8. As usual, we assume that the $m$ terms of the sum of elements of $\bar{A}(S^1 \times S^2)$ in this figure coincide except for the shown parts. We further assume that the hidden part of the diagram in Figure 8 has no labels.

Let $\mathcal{E}(S^1 \times S^2)$ be the quotient of $\bar{A}(S^1 \times S^2)$ by the Relation $(S^2$-slide$)$ shown in Figure 8. The canonical projection $p : \mathcal{A}(S^1 \times S^2) \longrightarrow \mathcal{E}(S^1 \times S^2)$ has a non-trivial

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1Lemma 7 implies that Relation $(S^2$-slide$)$ from Figure 8 does not depend on the choice of $s$. 

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Figure 8: The Relation \((S^2\text{-slide})\)

kernel. For example, the diagram \(D\) in Figure 9 is equal to 0 in \(\mathcal{E}(S^1 \times S^2)\) and it is easy to see that \(D \neq 0\) in \(\bar{\mathcal{A}}(S^1 \times S^2)\).

![Figure 9: A non-trivial element in \(\text{Ker}(p)\)](image)

In Section 11 we will construct a map \(Z_{S^1 \times S^2}\) from \(\mathcal{L}(S^1 \times S^2)\) to the completion \(\hat{\mathcal{E}}(S^1 \times S^2)\) of \(\mathcal{E}(S^1 \times S^2)\) with properties as in Lemma 6. This will imply the following theorem.

**Theorem 2** The map \(\psi_{S^1 \times S^2}\) factors through an isomorphism of coalgebras

\[
\tilde{\psi}_{S^1 \times S^2} : \mathcal{E}(S^1 \times S^2) \longrightarrow \text{gr} \mathcal{L}(S^1 \times S^2).
\]

The map \(Z_{S^1 \times S^2}\) induces an isomorphism of algebras

\[
Z^*_{S^1 \times S^2} : \mathcal{E}(S^1 \times S^2)^* \longrightarrow \mathcal{V}(S^1 \times S^2).
\]

It was shown in [Eis] that there exist knots in \(S^2 \times S^1\) that can be distinguished by \(\mathbb{Z}/2\)-valued Vassiliev invariants but that cannot be distinguished by \(\mathbb{Q}\)-valued Vassiliev invariants.

In Section 13 we state and prove Theorems 3 and 4. By Theorem 3 a suitably defined universal Vassiliev invariant of links in \(I\)-bundles over surfaces with boundary is compatible with finite coverings. In Theorem 4 the invariant \(Z_{P^2 \times I}\) is related to the usual universal Vassiliev invariant of links in \(\mathbb{R}^3\). Theorem 4 is used in [Lie] to prove a refinement of a theorem of Hartley and Kawauchi about the Alexander polynomial of strongly positive amphicheiral knots in \(\mathbb{R}^3\).

Let us overview the rest of this paper: in Section 5 we derive some general consequences of the defining relations of \(\mathcal{A}(G, \sigma)\) and some individual properties of \(\bar{\mathcal{A}}(P^2 \times\)
We introduce a category of tangles in thickened decomposed surfaces in Section 6. In Section 7 we recall the definition and some properties of a functor $Z$ from a category of non-associative tangles in $I^2 \times I$ to chord diagrams. In Section 8 we represent a tangle in $\Sigma \times I$ as the composition of a standard tangle in $\Sigma \times I$ with a tangle in $I^2 \times I$. By defining non-associative standard tangles in $\Sigma \times I$ and by choosing labeled chord diagrams as their values, we extend the functor $Z$ to a functor $Z_S$ from non-associative tangles in $\Sigma \times I$ to labeled chord diagrams in Section 9. This will imply Lemma 6 for surfaces with non-empty boundary. In Section 10 we prove Lemma 6 for $\Sigma = P^2$. Section 11 is devoted to the proof of Theorem 2. The main purpose of Section 12 is to generalize the invariant $Z_{\Sigma \times I}$ to links in $I$-bundles over ribbon graphs. In Section 13 we use this generalization to prove Theorems 3 and 4 about universal Vassiliev invariants and coverings.

5 Computations with chord diagrams

For later use we provide some combinatorial identities for labeled chord diagrams in this section. Throughout this section $G$ is a group and $\sigma : G \rightarrow \{\pm 1\}$ is a homomorphism. The following lemma is a consequence of the four-term relation.

**Lemma 7** The relation shown in Figure 10 holds in $A(G, \sigma)$, where $g_1, \ldots, g_m \in G$ are all labels of the diagrams.

Figure 10: A relation for chord diagrams without boundary

**Proof:** Let $D$ be a $G$-labeled chord diagram with $\text{source}(D) = \text{target}(D) = \emptyset$ that coincides with the diagrams in Figure 10 except for the chord shown in this figure. Keep in mind the common endpoint $c \in D$ of the chord in Figure 10. Consider a picture of $D$ that is decomposed by horizontal lines into parts as in Figure 11 (if the label $g$ is equal to $e$, we may omit it). Let the non-trivial labels be the maxima of the picture. We obviously have the relations shown in Figure 12. The second relation is implied by the $(4T)$-relation in $A(G, \sigma)$. 

13
Figure 11: Parts of a labeled chord diagram

Figure 12: Relations for parts of a labeled chord diagram

Figure 13: A consequence of the relations from Figure 12

Starting with the empty sum at the bottom of the picture of \( D \) and applying the relations from Figure 12 for chords leading to \( c \in D \), we arrive near the top of the diagram as shown in Figure 13.

This relation is equivalent to the relation from the lemma. \( \square \)

The proof of the next lemma is easy. It follows from the Relations (Rep) and (\( \sigma \)-Nat) along the same lines as the proof of Lemma 8.

**Lemma 8** Let \( D \) be a chord diagram without labels. Let \( s \in G \). Define \( \bar{\sigma}(s) \in \{0, 1\} \) by \( \sigma(s) = (-1)^{\bar{\sigma}(s)} \). Then the relation in Figure 14 holds in \( \mathcal{A}(G, \sigma, a, b) \), where \( a = \text{source}(D) \) and \( b = \text{target}(D) \).

A special case of Lemma 8 is shown in Figure 15.

By the Relations (Rep) we can leave out a label with the neutral element of a group. In the case of the projective plane, we represent the non-trivial element in \( \pi_1(P^2 \times I, \ast) \cong \mathbb{Z}/(2) \) by a marking without label. We derive the following particular relations in \( \mathcal{A}(P^2 \times I) \).

**Lemma 9** In \( \mathcal{A}(P^2 \times I) \) the relations shown in Figure 17 hold, where the markings in the second relation represent all non-trivial labels of the diagram.
Figure 14: A generalization of the Relation (σ-Nat)

\[ D \cdot \ldots \cdot (s-1) \sigma(s) \cdot \ldots \cdot s \cdot D = (-1)^{\deg(D)\sigma(s)} \]

Figure 15: A version of Relation (σ-Nat)

\[ s^{-1} \cdot \sigma(s) \cdot s^{-1} = s \cdot \sigma(s) \cdot s \]

Figure 16: Special relations for the thickened projective plane

\[ 0 = 2 \quad \text{and} \quad -2 \]

Figure 17: Proof of a "labeled framing independence" relation

\[
\begin{align*}
(1) & \quad \ldots \cdot (s-1) \sigma(s) \cdot \ldots \cdot s \cdot D \cdot \ldots = \ldots + \ldots \\
(2) & \quad \ldots \cdot (s-1) \sigma(s) \cdot \ldots \cdot s \cdot D \cdot \ldots = \ldots + \ldots
\end{align*}
\]

Figure 18: Special relations for the thickened projective plane

We can apply Lemma 8 no matter how the strands on the right side of part (2) of Figure 17 are oriented because $x^{-1} = x$ for $x \in \pi_1(P^2 \times I, \ast)$. So when we add the two equations in Figure 17 the right sides of the equations cancel and we obtain the first relation of Figure 16. The second relation follows from Figure 18.

In this figure the first equality follows from Lemma 8 and the second one follows from the first part of this proof, the third one follows by $m - 1$ applications of Lemma 7, and the last one follows by Relation (σ-Nat), the relation in Figure 15.

Proof: By using Relation (FI) and by applying Lemma 7 in two different ways, we obtain the equations from Figure 17.
Figure 18: Freeing the strand on the right side from its chords

and the first part of this proof. □

6 Tangles in a thickened decomposed surface

Recall that the definition of the map $\psi_M$ only depends on the choice of a basepoint $*$ and an oriented neighborhood $* \in U \cong \mathbb{R}^3 \subset M$. The definition of the universal Vassiliev invariant $Z_{\Sigma \times I}$ will depend on more choices. We collect the necessary data in the following definition.

**Definition 10** A decomposed surface is a surface $\Sigma$ together with a tuple of distinguished subsets $(B_0, B_1, \ldots, B_k, J_1, J_2)$, where $I^2 \cong B_0 \subset \Sigma$ is oriented, $I \cong J_i \subset B_0 \cap \partial \Sigma$ ($i = 1, 2$), and $I^2 \cong B_i \subset \Sigma$ ($i > 0$) such that $\bigcup_{i=0}^k B_i = \Sigma$ and equations (7) to (9) are satisfied.

1. $B_0 \cap B_i \cong I \times \{0, 1\}$ for $i \geq 1$,  
   \hspace{1cm} (7)
2. $B_i \cap B_j = \emptyset$ for $i \neq j, i, j \geq 1$ and $J_1 \cap J_2 = \emptyset$,  
   \hspace{1cm} (8)
3. $\exists J_3 \cong I, J_3 \subset B_0 \cap \partial \Sigma$ such that $J_3 \cap J_i = \{P_i\}$ ($i = 1, 2$) and $J_3$ with the orientation induced by $\partial B_0$ directs from $P_1$ to $P_2$.  
   \hspace{1cm} (9)

For every connected compact surface $\Sigma$ there exist subsets $(B_0, \ldots, B_k, J_1, J_2)$ that equip $\Sigma$ with the structure of a decomposed surface. The index $k = \text{rank}(H_1(\Sigma))$ is uniquely determined by $\Sigma$. We call $B_0$ the disk of $\Sigma$ and $B_i$ the bands of the decomposition. We equip $J_1$ with the orientation induced by $\partial B_0$ and $J_2$ with the opposite of the orientation induced by $\partial B_0$. An isomorphism of decomposed surfaces $\Sigma$ and $\Sigma'$ is a homeomorphism of the surfaces that respects the additional data.
We represent decomposed surfaces graphically as shown in Figure 19 by an example. By convention the orientation of $B_0$ is counterclockwise in this figure. The intervals $J_i$ direct from left to right. If $B_0 \cup B_i$ is non-orientable, then the 2-dimensional representation of the band $B_i$ is drawn with a singular point. We will also represent $\Sigma \times I$ as in Figure 19 where we assume that on $B_0$ the interval $I$ directs towards the reader.

![Figure 19: A picture of a decomposed surface](image)

Given two decomposed surfaces $\Sigma$ and $\Sigma'$ with distinguished subspaces 

$$(B_0, \ldots, B_k, J_1, J_2) \text{ and } (B'_0, \ldots, B'_{k'}, J'_1, J'_2)$$

respectively, we choose an orientation preserving homeomorphism $\varphi : J_1 \to J'_2$. We call the surface $\Sigma \cup \varphi \Sigma'$ with its natural decomposition 

$$(B_0 \cup B'_0, B_1, \ldots, B_k, B'_1, \ldots, B'_{k'}, J'_1, J_2)$$

the sum of $\Sigma$ and $\Sigma'$. Let $\mathcal{S}$ be the set of isomorphism classes of decomposed surfaces. With the sum of decomposed surfaces as binary operation $\mathcal{S}$ becomes a monoid with neutral element represented by an arbitrary decomposition of $I^2$. See Figure 20 for an example.

![Figure 20: The sum of decomposed surfaces](image)

**Definition 11** Let $\Sigma$ be a decomposed surface. A one-dimensional compact oriented submanifold $T \subset \Sigma \times I$ is called a tangle if
\[ \partial T = T \cap \partial \Sigma \times I \subset ((J_1 \setminus \partial J_1) \cup (J_2 \setminus \partial J_2)) \times 1/2. \]

We call the ordered sets \( T \cap (J_1 \times 1/2) \) the lower boundary and \( (T \cap J_2 \times 1/2) \) the upper boundary of \( T \). We assign a symbol + or − (resp. − or +) to a lower (resp. upper) boundary point of the tangle \( T \) according to whether \( T \) directs towards the boundary point or not. This way we assign words in the letters \{ +, − \} to the lower and to the upper boundary of \( T \) called source(\( T \)) and target(\( T \)) respectively.

Let \( \Sigma \) and \( \Sigma' \) be decomposed surfaces. We say that two tangles \( T \subset \Sigma \times I \) and \( T' \subset \Sigma' \times I \) are isotopic, if there exists an isotopy \( f_t : \Sigma \times I \rightarrow \Sigma' \times I \) such that

\[
\begin{align*}
f_0 &= \alpha \times \text{id}_I, \text{ where } \alpha : \Sigma \rightarrow \Sigma' \text{ is an isomorphism of decomposed surfaces}, \\
f_{t}(\partial(\Sigma \times I)) &= (\alpha \times \text{id}_I)_{|\partial(\Sigma \times I)} \text{ for all } t \in [0, 1], \\
f_1(T) &= T'.
\end{align*}
\]

A link in \( \Sigma \times I \) is the same as a tangle \( T \subset \Sigma \times I \) with \( \partial T = \emptyset \).

Let \( \Sigma \) be a decomposed surface. A tangle \( T \subset \Sigma \times I \) is called a standard tangle if it is contained in \( \Sigma \times 1/2 \), \( T \) has no circle components, every strand in \( T \cap (B_i \times I) \) \((i > 0)\) connects the two components of \((B_i \cap B_0) \times I\), and every strand in \( T \cap (B_0 \times I) \) connects \( J_1 \times I \) with \((\partial B_0 \setminus J_1) \times I\). See Figure 21 for an example.

\[
\text{Figure 21: A standard tangle}
\]

Define a category \( \mathcal{T}(\mathcal{S}) \) as follows. The class of objects of \( \mathcal{T}(\mathcal{S}) \) is the set of words \{ +, − \} \* and the morphisms are isotopy classes of tangles in a decomposed surface. The composition \( T \circ T' \) of tangles \( T, T' \) with source(\( T \)) = target(\( T' \)) is given by \( T_r \cup T'_r \subset (\Sigma \cup_{\varphi} \Sigma') \times I \), where \( T_r \) and \( T'_r \) are representatives of \( T \) and \( T' \), such that \( T_r \cup T'_r \) is a tangle in \((\Sigma \cup_{\varphi} \Sigma') \times I\). The identity morphisms of \( \mathcal{T}(\mathcal{S}) \) are represented by the standard tangles in \( I^2 \times I \).

Let \( \mathcal{T} \) be the subcategory of \( \mathcal{T}(\mathcal{S}) \) that has the same objects as \( \mathcal{T}(\mathcal{S}) \), but whose morphisms are only the isotopy classes of tangles in \( I^2 \times I \). We represent morphisms of \( \mathcal{T} \) always as tangles in \([a, b] \times I \times I\) for suitable numbers \( a < b \), where the surface
For a non-associative word \( r \) by the in \( C \) catenation of words, then one defines a functor \( C \) if \( \hom \) Let \( \mathbf{A} \) category \( T \) in \( C \) is given by
\[ (\bar{y}, \sigma, r) = \text{id} \left( r_1 \bar{r}_2 \bar{r}_3 \right) \] for some \( k \in \mathbb{N} \), where \( y \in \mathbf{A}(F_k, \sigma, r_1, r_2) \), \( F_k \) is the free group in \( k \) generators \( x_1, \ldots, x_k \), and \( \sigma : F_k \to \{\pm 1\} \) is given by \( \sigma(x_i) = \sigma_i \). The composition of two morphisms is given by
\[ (y, \sigma_1, \ldots, \sigma_k) \circ (z, \sigma_1', \ldots, \sigma_k') = (i_*(y) \circ j_*(z), \sigma_1, \ldots, \sigma_k, \sigma_1', \ldots, \sigma_k') \]
where \( i : F_k \to F_{k+k'} \) and \( j : F_{k'} \to F_{k+k'} \) are the inclusions given by \( i(x_i) = x_i \) and \( j(x_i) = x_{k+i} \).

7 The functor \( Z \)

In this section we recall the definition and some properties of a functor \( Z \) from tangles in \( I^2 \times I \) to chord diagrams. Some properties of this functor can be formulated most naturally by using so-called non-associative tangles that we introduce first.

Given a set \( X \), the set of non-associative words \( X^{na} \) in \( X \) is defined as the smallest set such that the empty word \( \emptyset \) is a non-associative word, each element \( a \in X \) is a non-associative word, and if \( r_1 \) and \( r_2 \) are non-empty non-associative words, then so is \( (r_1r_2) \). We also use the notations \( (\emptyset r) \) and \( (r\emptyset) \) for the non-associative word \( r \).

For a non-associative word \( r \) we define the word \( \bar{r} \in X^* \) by forgetting the brackets of \( r \). For a category \( C \) whose objects are words in \( X \), we denote by \( C^{na} \) the category whose objects are non-associative words in \( X \) and whose sets of morphisms are given by

\[
\hom_{C^{na}}(r_1, r_2) = \hom_C(\bar{r}_1, \bar{r}_2). \tag{12}
\]

If \( C \) is a strict tensor category whose tensor product is given for objects by the concatenation of words, then one defines a functor \( \boxtimes : C^{na} \times C^{na} \to C^{na} \) by defining \( r_1 \boxtimes r_2 = (r_1r_2) \) for objects of \( C^{na} \) and by defining the tensor product of morphisms as in \( C \). We can define associativity constraints \( a_{r_1,r_2,r_3} \in \hom_{C^{na}}((r_1r_2)r_3), (r_1(r_2r_3)) \) by \( a_{r_1,r_2,r_3} = \text{id} \left( r_1 \bar{r}_2 \bar{r}_3 \right) \). Then \( (C^{na}, \boxtimes, \emptyset, a, \text{id}, \text{id}) \) is a tensor category (see Definition XI.2.1 of [Kas]). Starting from the category \( T \) of tangles we consider the category \( T^{na} \) of non-associative tangles as a tensor category in this way.

Let \( \mathbf{A}(S) \) be the category whose objects are elements of \( \{+, -\}^* \) and whose morphisms in \( \hom_{\mathbf{A}(S)}(r_1, r_2) \) are \((k+1)\)-tuples \((y, \sigma_1, \ldots, \sigma_k)\) for some \( k \in \mathbb{N} \), where \( y \in \mathbf{A}(F_k, \sigma, r_1, r_2) \), \( F_k \) is the free group in \( k \) generators \( x_1, \ldots, x_k \), and \( \sigma : F_k \to \{\pm 1\} \) is given by \( \sigma(x_i) = \sigma_i \). The composition of two morphisms is given by

\[
(y, \sigma_1, \ldots, \sigma_k) \circ (z, \sigma_1', \ldots, \sigma_k') = (i_*(y) \circ j_*(z), \sigma_1, \ldots, \sigma_k, \sigma_1', \ldots, \sigma_k'),
\]

where \( i : F_k \to F_{k+k'} \) and \( j : F_{k'} \to F_{k+k'} \) are the inclusions given by \( i(x_i) = x_i \) and \( j(x_i) = x_{k+i} \).
Let \( \hat{A} \) be the subcategory of \( \hat{\mathcal{A}}(\mathcal{S}) \) that has the same objects as \( \hat{\mathcal{A}}(\mathcal{S}) \), but whose morphisms are only the 1-tuples \( (y) \in \text{Hom}_{\hat{\mathcal{A}}}(r_1, r_2) \). The category \( \hat{A} \) is a strict tensor category. The tensor product \( \boxtimes \) of objects is given by the concatenation of words and the tensor product of morphisms is induced by the concatenation of their sources and targets.

The category \( \hat{A}^{na} \) has many associativity constraints turning \( \hat{A}^{na} \) into a tensor category. Recall from \([Dri]\) that a Drinfeld associator is a power series \( \Phi(A, B) \) in two non-commuting indeterminates \( A \) and \( B \) and constant term 1 satisfying certain properties. One of these properties ensures that the morphism \( a'_{r_1, r_2, r_3} (r_i \in \{+, -\}^{na}) \) defined in Figure 22 is an associativity constraint of \( \hat{A}^{na} \) and that \((\hat{A}^{na}, \boxtimes, \emptyset, a', id, id)\) is a tensor category. In Figure 22 the product of indeterminates is replaced by the composition of chord diagrams as defined in Section 4.

![Figure 22: Substitution of the non-commuting indeterminates A and B of \( \Phi \)](image)

Let \( c_{r_1, r_2} \in \text{Hom}_{\mathcal{T}^{na}}((r_1 r_2), (r_2 r_1)) \), \( b_\epsilon \in \text{Hom}_{\mathcal{T}^{na}}(\emptyset, (\epsilon - \epsilon)) \), \( d_\epsilon \in \text{Hom}_{\mathcal{T}^{na}}((-\epsilon \epsilon), \emptyset) \) be the morphisms of \( \mathcal{T}^{na} \) shown in Figure 23.

![Figure 23: Morphisms in \( \mathcal{T}^{na} \)](image)

The element \( \nu \in \text{End}_{\hat{A}}(+) \) is defined as shown in Figure 24. In degree 0 the element \( \nu \) is equal to \( id_+ \). Hence there exists a unique square root \( \nu^{-1/2} \) of \( \nu^{-1} \) that is equal to \( id_+ \) in degree 0. Define \( c'_{r_1, r_2} \in \text{Hom}_{\hat{\mathcal{A}}}(r_1 r_2, r_2 r_1) \), \( b'_{\epsilon} \in \text{Hom}_{\hat{\mathcal{A}}}(\emptyset, (\epsilon - \epsilon)) \), and \( d'_\epsilon \in \text{Hom}_{\hat{\mathcal{A}}}((-\epsilon \epsilon), \emptyset) \) as shown in Figure 25, where \( \exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} \) is regarded as a formal power series.
Recall the following fact (see Theorem 2 of [LM1], Theorem 2 of [BN2]) that plays a fundamental role in the theory of Vassiliev invariants.

**Fact 12** There exists a unique functor $Z : T^a \rightarrow \hat{A}^a$ preserving the tensor product $\otimes$ such that $Z(r) = r$ for objects, and $Z(c_{+,+}) = c'_{+,+}$, $Z(d_\epsilon) = d'_\epsilon$, and $Z(a_{r_1,r_2,r_3}) = a'_{r_1,r_2,r_3}$ for $\epsilon \in \{+,-\}$, $r,r_i \in \{+,-\}^a$. We have $Z(b_\epsilon) = b'_\epsilon$ and $Z(c_{+1}^{\pm} r_1,r_2) = c'^{\pm} r_1,r_2$.

Recall that for an $\{\epsilon\}$-labeled chord diagram $D$ one can define a formal linear combination $[T_D]$ of tangles in $I^2 \times I$ similar to our definition of $[L_D]$ in Section 3. Extend $Z$ by linearity to formal linear combinations, and extend the comultiplication to a map $\hat{\Delta} : \text{Hom}_{\hat{A}^a}(r_1,r_2) \rightarrow \text{Hom}_{\hat{A}^a}(r_1,r_2) \otimes \text{Hom}_{\hat{A}^a}(r_1,r_2)$. Using this notation we recall the following fact (see [LM2]).

**Fact 13** (1) For a chord diagram $D \in \text{Hom}_{\hat{A}^a}(r_1,r_2)$ with $\text{deg}(D) = n$ we have

$$Z([T_D]) = D + \text{terms of degree} > n.$$  

(2) For a non-associative tangle $T \subset I^2 \times I$ we have

$$\hat{\Delta}(Z(T)) = Z(T) \otimes Z(T).$$

For $r \in \{+,-\}^a$ define $-r$ by interchanging the symbols $+$ and $-$, and define $r^\dagger$ by reading the word $r$ from the right side to the left side. For a tangle
$T \in \text{Hom}_{\mathcal{T}^{na}}(r_1, r_2)$ we define $T^- \in \text{Hom}_{\mathcal{T}^{na}}(-r_2, -r_1)$ by turning $T$ around a horizontal axis by an angle $\pi$, we define $T^\dagger \in \text{Hom}_{\mathcal{T}^{na}}(r_1, r_2)$ by turning $T$ around a vertical axis by an angle $\pi$, and we define $T^*$ by reflection of $T$ in $I^2 \times 1/2$. 

For $t = \sum_{i=0}^{\infty} t_i \in \text{Hom}_{\hat{\mathcal{A}}^{na}}(r_1, r_2)$ with $\deg(t_i) = i$ we define $t^* \in \text{Hom}_{\hat{\mathcal{A}}^{na}}(r_1, r_2)$ by $t^* = \sum_{i=0}^{\infty} (-1)^i t_i$. We define $t^- \in \text{Hom}_{\hat{\mathcal{A}}^{na}}(-r_2, -r_1)$ and $t^\dagger \in \text{Hom}_{\hat{\mathcal{A}}^{na}}(r_1, r_2)$ like $T^-$ and $T^\dagger$ by applying the respective rotations to the graphical representation of chord diagrams.

A Drinfeld associator $\Phi$ is called even if it is 0 in odd degrees. By Proposition 5.4 of [Dri] there exists an even associator with rational coefficients that is an exponential of a Lie series. We will fix a choice of an associator with these properties until the end of Section 11. With this choice for the definition of the associativity constraint of the tensor category $\hat{\mathcal{A}}^{na}$ the invariant $Z$ has the following symmetry properties (see Proposition 3.1 of [LM2]).

**Fact 14** For an associator $\Phi$ as chosen above one has

$$Z(T^-) = Z(T)^- \quad , \quad Z(T^\dagger) = Z(T)^\dagger \quad \text{and} \quad Z(T^*) = Z(T)^*.$$ 

**8 Pairs and diagrams of tangles in $\Sigma \times I$**

The existence of the functor $Z$ of the previous section can be proved by using a presentation of the category $\mathcal{T}$ by generators and relations. In this section we extend this presentation to $\mathcal{T}(S)$. Let $\Sigma$ be a decomposed surface. Let $T$ be a tangle in $\Sigma \times I$. Then $T$ is isotopic to the composition $T_1 \circ T_2$ of a standard tangle $T_1 \subset \Sigma \times I$ and a tangle $T_2 \subset I^2 \times I$.

**Definition 15** (1) A pair $(T_1, T_2)$ of a standard tangle $T_1 \subset \Sigma \times I$ and of a tangle $T_2 \subset I^2 \times I$ with source($T_1$) = target($T_2$) is said to be a representing pair of the tangle $T_1 \circ T_2 \subset \Sigma \times I$.

(2) Two representing pairs $(T_1, T_2)$ and $(T'_1, T'_2)$ are isotopic if $T_1$ is isotopic to $T'_1$ and $T_2$ is isotopic to $T'_2$.

The graphical representation of the composition $T_1 \circ T_2$ of a representing pair as shown in Figure 20 by an example is called a diagram of a tangle in $\Sigma \times I$. Notice that $T_2 \subset I^2 \times I$ lies in an oriented manifold. In a diagram, the tangles $T_1 \subset \Sigma \times I$ and $T_2 \subset I^2 \times I$ are given by the projection of representatives in generic position to $\Sigma$ (resp. to $I^2$) along $I$ together with the information saying which one is the overcrossing strand at double points of the projection of $T_2$. 

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In Figure 26, we see four moves between different representing pairs \((T_1, T_2)\) of a tangle \(T\). The two parallel strands at the left and at the right side of each diagram represent a (possibly empty) bundle of strands. We assume an arbitrary compatible choice of orientations of the strands.

Isotopies of tangles in \(\Sigma \times I\) can be described by sequences of Reidemeister moves and by isotopies of tangle diagrams. This can be translated into the following proposition.

**Proposition 16**  Let \((T_1, T_2)\) and \((T'_1, T'_2)\) be two representing pairs of tangles in \(\Sigma \times I\). Then \(T_1 \circ T_2\) is isotopic to \(T'_1 \circ T'_2\), if and only if one can pass from \((T_1, T_2)\) to \((T'_1, T'_2)\) by a finite sequence of the following moves:

1. An isotopy of representing pairs.
2. One of the moves shown in Figure 27, and their images under reflection in a vertical axis.


9 The functor \( Z_S \)

In addition to the subcategory \( \mathcal{T}^{\text{na}} \), the category \( \mathcal{T}(S)^{\text{na}} \) has a further important subcategory \( \mathcal{U} \). The objects of \( \mathcal{U} \) are elements of \( \{+,−\}^{\text{na}} \), and its morphisms are standard tangles \( T \) in a thickened decomposed surface \( \Sigma \) satisfying the following conditions (NA1) and (NA2). To state these conditions we first introduce some notation:

Let \( (B_0, \ldots, B_k, J_1, J_2) \) be the distinguished subsets of the decomposed surface \( \Sigma \). Let \( T \) be a standard tangle in \( \Sigma \times I \) with \( s = \text{source}(T) \). On the upper boundary of \( B_0 \) lie the disjoint intervals \( B_i \cap B_0 \) and \( J_2 \). We denote these intervals by \( I_1, \ldots, I_{2k+1} \) numbered from the left side to the right side. Subsets of the lower boundary of \( T \) correspond to subwords of \( \bar{s} \) and vice versa. The word \( \bar{s} \) can be subdivided into (possibly empty) subwords \( r_1, \ldots, r_{2k+1} \) such that when we travel along \( T \) starting at a boundary point of \( T \) belonging to \( r_\nu \) we arrive at the interval \( I_\mu \times 1/2 \) before traveling across any other interval \( I_\mu \times 1/2 (\mu \neq \nu) \).

(NA1) There exist non-associative words \( s_\nu \) with \( \bar{s} = r_\nu \), and

\[
s = (s_2(s_2 \ldots (s_{2k} (s_{2k} s_{2k+1})) \ldots )))
\]

with all the \( 2k \) closing brackets to the extreme right.

(NA2) We have \( s_{2k+1} = \text{target}(T) \). For some \( i \leq k \), let \( I_\nu \) and \( I_\mu \) be the intervals \( B_i \cap B_0 \). If \( B_i \cup B_0 \) is orientable, then we have \( s_\nu = -s_\mu \), and \( s_\nu = -s_\mu \) otherwise.

For the standard tangle \( T_1 \) of the representing pair in Figure 26 an example of parenthesis on \( \text{source}(T_1) \) satisfying (NA1) and (NA2) is the following:

\[
(r_1(r_2(-r_1(-r_2+)))) = ((+-)(-)(--)(+-)(-+-)(+++)(--))),
\]

where \( r_1 = (+++) \) and \( r_2 = (+-+) \).

We define a functor \( Z_{st} : \mathcal{U} \rightarrow \hat{\mathcal{A}}(S)^{\text{na}} \) as follows. For objects we have \( Z_{st}(r) = r \).

For a morphism \( T \) we have \( Z_{st}(T) = (z(T), \sigma_1, \ldots, \sigma_k) \), where \( \sigma_i = 1 \) if \( B_0 \cup B_i \) is orientable and \( \sigma_i = -1 \) otherwise, and where \( z(T) \) is the \( F_k \)-labeled chord diagram of degree 0 defined as follows: the skeleton \( \Gamma \) of the chord diagram \( z(T) \) consists of oriented intervals connecting the same points as the strands of \( T \). An interval of \( \Gamma \) is labeled by \( x_i \) (resp. \( x_i^{-1} \)) if and only if the corresponding strand of \( T \) passes through a band \( B_i \) in the counterclockwise (resp. clockwise) sense in a diagram of \( T \). Using the notation of Figure 28 we show an example for \( z(T) \) in Figure 28.

**Theorem 17** There exists a unique functor \( Z_S : \mathcal{T}(S)^{\text{na}} \rightarrow \hat{\mathcal{A}}(S)^{\text{na}} \) that makes the following diagram commutative, where the functors \( i_\nu \) are given by inclusion of subcategories.
Proof of Theorem 17: For objects we necessarily have $Z_S(r) = r$. For morphisms $T$ the definition of $Z_S(T)$ will involve some choices. Let $(T_1, T_2)$ be a representing pair of the tangle $T$. Choose $s \in \{+, -\}^{*_{na}}$ such that $T_1$ regarded as a non-associative tangle with target($T_1$) = target($T$) and source($T_1$) = $s$ is a morphism of $\mathcal{U}$. Regard $T_2$ as a non-associative tangle with source($T_2$) = source($T$) and target($T_2$) = $s$. Define

$$Z_S(T) = Z_{st}(T_1) \circ Z(T_2).$$

We have to show that $Z_S$ is well-defined. Therefore we verify in part (a) of this proof that $Z_S(T)$ does not depend on the choice of the non-associative structure of $s$, and in part (b) that $Z_S$ is compatible with the moves between representing pairs of a tangle in Proposition 16.

(a) As a special case of Fact 14 we have the equalities

$$a_{r_1,r_2,r_3}^d = a_{-r_1,-r_2,-r_3}^d \quad \text{and} \quad a_{r_1,r_2,r_3}^l = a_{-r_1,-r_2,-r_3}^l$$

where $r_i \in \{+, -\}^{*_{na}}$. This implies the two equations of Figure 29, where the source of the first morphism is
\[((\,(r_1 r_2) r_3) (\,(r_1^1 - r_2^1) - r_1^1))\]
and the source of the second morphism is \(((\,(r_1 r_2) r_3) (\,-r_1 (\,-r_2 - r_3)))\).

Figure 29: Symmetries of an even associator

By Lemma 8 the equation in Figure 30 holds because the associator $\Phi$ is even and therefore the sign in Figure 14 is always equal to 1.

Figure 30: Commuting an even associator with the labels $x_i$

Combining the equations from Figure 29 and 30 we see that for a word $s$ satisfying $\text{(NA1)}$ and $\text{(NA2)}$ we can rearrange the parenthesis of $s_\nu$ and $s_\mu$ (with the notation of Condition $\text{(NA2)}$) without changing the value of $Z_S(T)$. But by $\text{(NA1)}$ the parenthesis of the subwords $s_\nu$ ($\nu \in \{1, \ldots, 2k\}$) of $s$ are the only indeterminacy in the choice of $s$.

(b) The invariance under Move (1) of Proposition 16 is clear because $Z$ and $Z_{st}$ are well-defined. Let us check the invariance under the third move of Figure 27. We can assume that the parenthesis belonging to the lower boundary of the shown diagrams have the following properties: Condition $\text{(NA1)}$ is satisfied, Condition $\text{(NA2)}$ is satisfied for the bands $B_i$ different from the one shown in the picture, and the source of the diagrams is of the form $((r_1 (\gamma - \gamma)) r_2) \ldots (\,-r_1 - r_2)$, where $\gamma \in \{+, -\}$ and $r_1, r_2 \in \{+,-\}^{*na}$. Then the values of $Z_S$ on the parts of the diagrams are shown in Figure 31.

By definition $\nu$ and therefore also $\nu^{-1/2}$ vanish in odd degrees. Now Lemma 8 implies that the two diagrams in Figure 31 represent equal elements.

With the assumption on the chosen bracketing similar to the one made before, the values of $Z_S$ on the parts of the diagrams in the fourth move of Figure 27 are shown in Figure 32.
where $X = \exp (\phi / 2)$. Again an application of Lemma 8 shows that the diagrams of Figure 32 represent equal elements. The proof of the compatibility with the first two moves of Figure 27 is similar. The proofs are the same when the diagrams are reflected in a vertical axis. This implies that $Z_S$ is well-defined. □

Now we come to the first part of the proof of Lemma 6.

Proof of Lemma 6 for $\partial \Sigma \neq \emptyset$: Let $\Sigma$ be a compact connected surface with non-empty boundary. We choose a decomposition $(B_i, J_i)$ of $\Sigma$ as in Definition 10 and $* \in B_0 \times I$. Let $g_\nu \in \pi_1(\Sigma \times I, *)$ be the element given by a path that passes one time through the band $B_\nu$ in the counterclockwise sense in our pictures. Let $\varphi : F_k \longrightarrow \pi_1(\Sigma \times I, * )$ be the isomorphism mapping $x_\nu$ to $g_\nu$. Let $L$ be a link in $\Sigma \times I$. We can regard $L$ as an element of $\text{End}_{F_k(\Sigma \times I)}(\emptyset )$. Then we define

$$Z_{\Sigma \times I}(L) = \varphi_*(Z_S(L)),$$

where the map $\bar{F}(\varphi) = \varphi_*$ defined in Section 2 is extended to completions. For the definition of $[L_D]$ we choose the oriented neighborhood $U$ of $*$ inside of $B_0 \times I$. Then equation (5) follows from part (1) of Fact 13 and equation (6) follows from part (2) of Fact 13 and from the definition of $\Delta(D)$ for diagrams $D$ of degree 0. □
10 The invariant $Z_{P^2 \times I}$

Let $P^2$ be the real projective plane. We choose $B, X \subset P^2$, where $X$ is a Möbius strip, $B \cong I^2$, $B \cup X = P^2$, and $B \cap X = \partial B = \partial X$. By an isotopy we can push every link in $P^2 \times I$ into $X \times I$. We choose a decomposition $(B_0, B_1, J_i)$ of $X$. Then the diagrams of links in $X \times I$ shown in Figure 33 represent isotopic links in $P^2 \times I$.

\[ \text{Figure 33: Pushing a strand of a link across the disk } B \times 1/2 \]

The two strands leaving the box labeled $T$ in Figure 33 represent a possibly empty bunch of strands. The two diagrams on the right hand side of Figure 33 represent isotopic links in $X \times I$ (see Proposition 16). A general position argument implies the following lemma.

**Lemma 18** Two links in $X \times I$ represent isotopic links in $P^2 \times I$ if and only if one can pass from one to the other by isotopies in $X \times I$ and by the move shown in Figure 33.

The key result for the completion of the proof of Lemma 6 is the following lemma.

**Lemma 19** Let $\varphi : \pi_1(X \times I, \ast) \longrightarrow \pi_1(P^2 \times I, \ast)$ and $p : \mathcal{L}(X \times I) \longrightarrow \mathcal{L}(P^2 \times I)$ be the maps induced by the inclusion $X \times I \subset P^2 \times I$. Then there exists a unique map $Z_{P^2 \times I} : \mathcal{L}(P^2 \times I) \longrightarrow \hat{\mathcal{A}}(P^2 \times I)$ that makes the following diagram commutative.

\[ \begin{array}{ccc} \mathcal{L}(X \times I) & \longrightarrow & \mathcal{A}(X \times I) \\
\downarrow p & & \downarrow \varphi_* \\
\mathcal{L}(P^2 \times I) & \longrightarrow & \hat{\mathcal{A}}(P^2 \times I) \\
\end{array} \]

**Proof:** We divide each diagram in Figure 33 into two parts by cutting along a horizontal line directly above the box labeled $T$. Let $T_1$ (resp. $T_2$) be the upper part.
of the diagram on the left (resp. right) side in Figure 33. We turn $T_1$ and $T_2$ into non-associative tangles by choosing $\text{source}(T_1) = \text{source}(T_2) = \left(((-\epsilon r) - r) \right)$ with appropriate $\epsilon \in \{+,-\}$ and $r \in \{+,-\}^{\text{na}}$. The values $t_i = \varphi_* \circ Z_{X \times I}(T_i)$ are shown in Figure 34.

$$t_1 = \Phi^{-1} \exp\left( \begin{array}{c} \Phi \end{array} \right) / 2$$

$$t_2 = \Phi^{-1} \exp\left( \begin{array}{c} \Phi \end{array} \right) / 2$$

Figure 34: The values $t_i = \varphi_* \circ Z_{X \times I}(T_i)$

We first consider the associator on the top of these diagrams. The noncommuting indeterminates $A$ and $B$ of $\Phi$ are replaced by the chord diagrams $\Phi$, respectively. The equalities in Figure 35 follow by Lemma 7 and Figure 12.

Figure 35: Replacing the indeterminates in the first associator

We see by this figure that a monomial in $A$ and $B$ is replaced by a diagram that only depends on the number of occurrences of $A$ and $B$ in the monomial. But when $A$ and $B$ are replaced by commuting elements then the result vanishes in degree $> 0$ because the associator $\Phi$ is an exponential of a Lie series in $A$ and $B$. We can show by a similar argument using Lemma 8 that the associator in the second box counting
from the top in Figure 34 vanishes in degree > 0 (see Figure 36 for the replacement of the indeterminate $B$ of $\Phi^{-1}$).

\[ \cdots = \cdots = \frac{1}{2} \]

Figure 36: Freeing the strand on the left side from its chords

We can calculate the contribution of the third box of Figure 34 using Figure 36. It is easy to see that the associator in the fourth box of Figure 34 vanishes in degree > 0. Then Lemma 9 implies that the contribution from the last box of $t_2$ in Figure 34 also vanishes in degree > 0. Hence we have the desired equality shown in Figure 37.

\[ t_1 = \exp\left(\frac{\text{\textendash\textendash}4}{4}\right) \cdots = t_2 \]

Figure 37: Invariance of $\varphi_* \circ Z_{X \times I}$ under the move of Figure 33

By Lemma 18 this completes the proof. □

Now it is easy to complete the proof of Lemma 3.

**Proof of Lemma 3 for $\Sigma = P^2$:** For a $\pi_1(P^2 \times I, \ast)$-labeled chord diagram $D$ of degree $n$ we can choose the singular link $L_D$ lying inside of $X \times I$. The property

\[ Z_{X \times I}(L_D) = D + \text{terms of degree > } n \]

implies the same property for $Z_{P^2 \times I}$ in view of Lemma 13. Since $\varphi_*$ is a morphism of coalgebras the equation

\[ \hat{\Delta}(Z_{X \times I}(L)) = \hat{\Delta}(Z_{X \times I}(L)) \otimes \hat{\Delta}(Z_{X \times I}(L)) \]

implies the same equation for $Z_{P^2 \times I}$. □
11 The invariant $Z_{S^1 \times S^2}$

Let $B \subset S^2$, $B \cong I \times I$. Choose a decomposition $(B_0, B_1, J_i)$ of the surface $S^1 \times I$. We can represent links in $S^1 \times B \cong (S^1 \times I) \times I$ by diagrams. The following lemma follows from general position arguments.

**Lemma 20** A link in $S^1 \times S^2$ can be represented by a link in $S^1 \times B$. Two links in $S^1 \times B$ represent isotopic links in $S^1 \times S^2$ if and only if they are the same by isotopies in $S^1 \times B$ and by the move shown in Figure 38.

![Figure 38: The second Kirby move for links in $S^1 \times S^2$](image)

The number of the strands passing through $B_1 \times I$ and the orientations of the strands in Figure 38 are arbitrary. Recall the definition of $\mathcal{E}(S^1 \times S^2)$ from Section 4.

**Lemma 21** The map $\psi_{S^1 \times S^2}$ factors through a map

$$\tilde{\psi}_{S^1 \times S^2} : \mathcal{E}(S^1 \times S^2) \rightarrow \text{gr}\mathcal{L}(S^1 \times S^2).$$

**Proof:** Assume that the diagrams in Figure 38 are parts of a singular link having $n - 1$ double points and assume that the strands passing through $B_1 \times I$ in Figure 38 direct from the right to the left. Then the difference of the desingularizations of the two isotopic singular links of Figure 38 can be expressed as a linear combination of desingularizations of singular links with $n$ singularities as shown in Figure 39.

Since every part of the link may be pulled to the position of the maximum on the right hand side of Figure 38 it follows that the Relation ($S^2$-slide) with all strands pointing upwards is mapped to 0 by $\psi_{S^1 \times S^2}$. Considering what happens when the orientations of some strands in Figure 39 are changed we see that all Relations ($S^2$-slide) are mapped to 0 by $\psi_{S^1 \times S^2}$. □

The next lemma will imply that the Relations ($S^2$-slide) generate all additional relations in the case $M = S^1 \times S^2$. 

31
Lemma 22 There exists a unique map $Z_{S^1 \times S^2}$ that makes the following diagram commutative, where $q : \mathcal{L}(S^1 \times I \times I) \rightarrow \mathcal{L}(S^1 \times S^2)$ and $\varphi : \pi_1(S^1 \times I \times I, *) \rightarrow \pi_1(S^1 \times S^2, *)$ are induced by the inclusion $S^1 \times I \times I \cong S^1 \times B \subset S^1 \times S^2$ and $p$ denotes the canonical projection.

$$
\begin{array}{ccc}
\mathcal{L}(S^1 \times I \times I) & \xrightarrow{Z_{(S^1 \times I) \times I}} & \hat{A}(S^1 \times I \times I) \\
q & & \downarrow \varphi_* \\
\mathcal{L}(S^1 \times S^2) & \xrightarrow{Z_{S^1 \times S^2}} & \hat{E}(S^1 \times S^2)
\end{array}
$$

**Proof:** Turn the two tangles $T_1$ and $T_2$ shown in Figure 38 into non-associative tangles by choosing $\text{source}(T_1) = \text{source}(T_2) = (r((-r^* \epsilon)) - \epsilon)$ with appropriate $\epsilon \in \{+, -\}$ and $r \in \{+, -\}^{\text{na}}$. The values $t_i = \varphi_* \circ Z_{(S^1 \times I) \times I}(T_i)$ are shown in Figure 40, where $s$ is the generator of $\pi_1(S^1 \times S^2, *)$ that can be represented by a clockwise oriented circle in our pictures.

$$
\begin{array}{c}
t_1 = s \bullet \nu^- \top \\
\Phi
\end{array} \quad \begin{array}{c}
t_2 = s \bullet \nu^- \top \\
\exp(\frac{\pi^* \cdot \Phi}{2})
\end{array}
$$

**Figure 40:** The values $t_i = \varphi_* \circ Z_{(S^1 \times I) \times I}(T_i)$

The associator in the diagrams of $t_1$ and $t_2$ vanishes in degree $> 0$ by the Relation $(S^2\text{-slide})$. Then the expression in the box at the bottom of $t_2$ also vanishes in degree $> 0$.\[\text{Figure 39: A relation in gr}\mathcal{L}(S^1 \times S^2)\]
degree > 0 by the Relation (S^2-slide). This implies the compatibility with the move shown in Figure 38 and completes the proof. □

Proof of Theorem 2: The theorem follows by the same arguments as in the proof of Lemma 6 and Theorem 1. Notice that Proposition 3 implies that Ker(\psi_{S_1 \times S^2}) is a coideal. Therefore \( \mathcal{E}(S_1 \times S^2) \) is a coalgebra. □

Alternatively to the proof above it may be verified directly that Relation (S^2-slide) generates a coideal of \( \bar{A}(S_1 \times S^2) \) by using Lemma 7.

Let \( S_1 \tilde{\times} S^2 \) be the quotient of \( S_1 \times S^2 \) by the fixed point free involution of \( S_1 \times S^2 \) given by \((x, y) \mapsto (-x, -y)\). Denote by \( p : S_1 \times S^2 \rightarrow S_1 \tilde{\times} S^2 \) the canonical projection. Let \( \alpha : I \rightarrow S_1 \) and \( \beta : I \rightarrow S^2 \) be paths that connect antipodal points. A closed tubular neighborhood of \( p((\alpha \times \beta)(I)) \) and the closure of the complement of this neighborhood are both homeomorphic to \( X \times I \), where \( X \) is the Möbius strip. Represent links in \( S_1 \tilde{\times} S^2 \) by links in \( X \times I \). Then the obvious versions of Lemmas 21 and 22 and of Theorem 2 hold. This establishes a universal Vassiliev invariant \( Z_{S_1 \tilde{\times} S^2} \) of links in \( S_1 \tilde{\times} S^2 \) with values in a space \( \hat{E}(S_1 \tilde{\times} S^2) \).

12 Variations on the definition of \( Z_{\Sigma \times I} \)

For some applications the definition of a decomposed surface \( \Sigma \) is too restrictive. Our first goal in this section will be to extend the definition of \( Z_{\Sigma \times I} \) to a more general structure on \( \Sigma \). We briefly recall the notion of a ribbon graph (see [Tur], Section 2.1, with the difference that we allow non-orientable ribbon graphs as well): a band \( B \) is homeomorphic to \( I \times I \), has a distinguished lower base \( I \times 0 \) and a distinguished upper base \( I \times 1 \), and its core \((1/2) \times I\) directs from \( I \times 1 \) to \( I \times 0 \). A coupon \( C \) is homeomorphic to \( I \times I \) and has a distinguished oriented base \( I \times 0 \). An (abstract) ribbon graph is a decomposition of a surface \( \Sigma \) into bands and coupons as shown in Figure 41 by an example.

We consider a locally trivial projection \( p : \Sigma' \times I \rightarrow \Sigma \) with typical fiber \( I \), where \( \Sigma' \) is a surface and \( \Sigma \) is a ribbon graph with bands \( B_i \ (i \in J_1) \), coupons \( C_j \ (j \in J_2) \), a basepoint \( * \in C_o \subset \Sigma \) for a distinguished coupon \( C_o \ (o \in J_2) \), and where \( p^{-1}(C_o) \) is oriented. We choose a maximal number of bands \( B_k \ (k \in J_3 \subset J_1) \) such that

\[
T = \bigcup_{i \in J_3} B_i \cup \bigcup_{j \in J_2} C_j
\]

is contractible. We extend the orientation of \( p^{-1}(C_o) \) to \( p^{-1}(T) \). The core of each band \( B \) leads from \( T \) to \( \bar{T} \) and therefore defines an element \( g_B \in \pi_1(\Sigma, *) \). For \( i \in J_3 \) the element \( g_{B_i} \) is the neutral element.
Orient $C_j$ ($j \in J_2$) such that the induced orientation of $\partial C_j$ coincides with the orientation of the distinguished base of $C_j$. We choose orientation preserving homeomorphisms $\varphi_j : C_j \times I \rightarrow p^{-1}(C_j)$ such that $p(\varphi(c, t)) = c$ for all $c \in C_j$, $t \in I$. Let $L \subset \Sigma' \times I$ be a link. We may assume that $L \subset \Sigma' \times I$ is in standard position, meaning that $L \cap p^{-1}(C_j)$ is a tangle in $p^{-1}(C_j) \cong C_j \times I$ for every $j \in J_2$, $p$ maps $L \cap p^{-1}(B_i)$ injectively onto $p(L) \cap B_i$ and every component of $p(L) \cap B_i$ connect the two different bases of the band $B_i$ ($i \in J_1$).

For a basepoint $*' \in p^{-1}(*)$ the map $p_* : \pi_1(\Sigma' \times I, *) \rightarrow \pi_1(\Sigma, *)$ is an isomorphism. To an oriented strand of $p(L) \cap B_i$ we associate the label $p_*^{-1}(g_{B_i})$ if its orientation coincides with the orientation of the core of $B_i$, and the label $p_*^{-1}(g_{B_i}^{-1})$ if the orientations do not coincide.

Let $J$ be the distinguished oriented base of a coupon $C$ of $\Sigma$. As in Section 6 we associate source$(L \cap p^{-1}(C)) \in \{+, -\}^*$ to $p(L) \cap J$ and target$(L \cap p^{-1}(C)) \in \{+, -\}^*$ to $p(L) \cap \partial C \setminus J$, where $\partial C$ carries the opposite of the orientation induced by $J$. The conditions (NA1) and (NA2) can be adapted to ribbon graphs in an obvious way, giving us non-associative tangles $L \cap p^{-1}(C_j)$. We define $Z_{\Sigma' \times I}^p(L)$ as in Section 3 by gluing bunches of labeled strands corresponding to $L \cap p^{-1}(B_i)$ to the invariants $Z(L \cap p^{-1}(C_j))$.

**Proposition 23** The definition of $Z_{\Sigma' \times I}^p$ does not depend on the choice of $T \subset \Sigma$. The map $Z_{\Sigma' \times I}^p$ is an isotopy invariant of links $L \subset \Sigma' \times I$.

**Proof:** In order to verify that $Z_{\Sigma \times I}$ does not depend on the choice of $T$, it is enough to consider $T'$ that coincides with $T$ except for two bands that are incident to the same coupon. This case follows by a single application of Lemma 8.
The rest of the proof proceeds along the same lines as the proof of Theorem 17. We discuss two differences:

1. We have to consider bands $B$ that connect the bottom of a coupon with the top of a coupon.

The independence of $Z_{\Sigma' \times I}(L)$ from the choice of parentheses coming from the intersection of $p(L)$ with the upper and lower boundary of $B$ is simpler because no symmetry with respect to rotation around a horizontal axis is needed. The compatibility with moving crossings and local minima or maxima along $B$ follows like in the proof of Theorem 17 where we again use less symmetry properties of the elements $c_{e_1, e_2}$, $b_{e_1}$, and $d_{e_1}$ in Figure 23 than before.

(2) Let $\sigma : \pi_1(\Sigma, *) \to \{\pm 1\}$ and $\sigma' : \pi_1(\Sigma' \times I, *) \to \{\pm 1\}$ be the orientation characters of $\Sigma$ and $\Sigma' \times I$, respectively. We have to consider bands $B \subset \Sigma$ with $-1 = \sigma(g_B) \neq \sigma'(p^{-1}(g_B)) = 1$.

Since orientations are only involved in the value of $Z$ on crossings the same arguments as before imply that the map $Z_{\Sigma' \times I}(L)$ does not depend on the choice of parentheses and is compatible with moving local minima and maxima of a diagram of $L$ across the band $B$. The map $Z_{\Sigma' \times I}$ is also compatible with moving crossings across the band $B$ because $p^{-1}(T \cup B)$ is orientable, the preimage $p^{-1}(C_i)$ of each coupon is equipped with the orientation induced by $p^{-1}(T)$, and $\hat{A}(\Sigma' \times I)$ is defined using $\sigma'$.

As before $Z_{\Sigma' \times I}$ is a universal Vassiliev invariant of links in $\Sigma' \times I$. The value $Z_{\Sigma' \times I}(L)$ depends non-trivially on the ribbon graph structure of $\Sigma$. There exists an extension of Proposition 23 that allows to consider certain non-continuous projections $p : \Sigma' \times I \to \Sigma$ as well. As an example this allows to represent links in $X \times I$ by diagrams on $S^1 \times I$. I do not know how to obtain a universal Vassiliev invariant of links in the non-trivial $I$-bundle over $P^2$. This invariant would give rise to a universal Vassiliev invariant of links in $P^3$.

Besides varying the projection of a link $L \subset \Sigma' \times I$ there are also different possible normalizations of $Z_{\Sigma' \times I}(L)$ for a fixed diagram of $L$: let $s = s_1 \ldots s_n \in \{+,-\}^n$ be a word with $n$ letters. Then there exists a map $d_s : \hat{A}(G, \sigma, +, +) \to \hat{A}(G, \sigma, s, s)$ defined as follows: the skeleton of $d_s(D)$ consists of a bunch of $n$ strands. When $s_i = +$ the labels and the orientation of the $i$-th strand coincide with the labels and orientation of $D$. When $s_i = -$ the labels are the inverse elements of the labels of $D$ and the orientation is opposite. The diagram $d_s(D)$ is defined by replacing each chord endpoint by the signed sum of $n$ ways of lifting that endpoint to the new skeleton, where the signs are determined by Figure 23.

After the choice of $T \subset \Sigma$ as in equation (14) we associate an orientation to a
band $B$ of $\Sigma$ as follows: Let $V$ be a small neighborhood of the lower base of $B$ such that the orientation of $p^{-1}(T)$ extends uniquely to $p^{-1}(T \cup V)$. Let $C$ be the coupon of $\Sigma$ with $C \cap V \neq \emptyset$. Orient $C$ such that the orientation of $\partial C$ coincides with the orientation of the base of $C$. Let $\varphi : (V \cup C) \times I \to p^{-1}(V \cup C)$ be a homeomorphism such that $\varphi|_{C \times I}$ is orientation preserving and $p(\varphi(x,t)) = x$ for all $x \in V \cup C$, $t \in I$. Then there exists a unique orientation of $B \supset V$ such that $\varphi|_{V \times I}$ is orientation preserving. We denote the band $B$ with the orientation from above by $\text{or}_T(B)$.

Let $\mathcal{B}^\text{or}(\Sigma)$ be the set of oriented bands of $\Sigma$. For $B \in \mathcal{B}^\text{or}(\Sigma)$ we denote by $B^*$ the band $B$ with the opposite orientation. Let $a$ be a map from $\mathcal{B}^\text{or}(\Sigma)$ to $\tilde{\mathcal{A}}(\{e\}, 1, +, +)$ satisfying $a(B^*) = a(B)^*$. We denote $a(\text{or}_T(B))$ simply by $a_T(B)$. For a link $L \subset \Sigma' \times I$ in standard position, let $s(B, L) \in \{+,-\}^*$ be the sequence of letters determined by the intersection of $p(L)$ with the lower base of $B$. Then we define $Z^p_{\Sigma' \times I}(L)$ in the same way as $Z^p_{\Sigma' \times I}(L)$ with the only difference that instead of gluing bunches of labeled intervals corresponding to $L \cap p^{-1}(B)$ we now glue the product of bunches of labeled intervals with $d_{s(B, L)}(a_T(B))$ to the invariants of the non-associative tangles $L \cap p^{-1}(C_j)$.

**Proposition 24** The definition of $Z^p_{\Sigma' \times I}$ does not depend on the choice of $T \subset \Sigma$. The map $Z^p_{\Sigma' \times I}$ is an isotopy invariant of links $L \subset \Sigma' \times I$. If for all $B \in \mathcal{B}^\text{or}(\Sigma)$ the elements $a(B)$ satisfy $\tilde{\Delta}(a(B)) = a(B) \hat{\otimes} a(B)$, then we also have

$$
\tilde{\Delta} \left( Z^p_{\Sigma' \times I}(L) \right) = Z^p_{\Sigma' \times I}(L) \hat{\otimes} Z^p_{\Sigma' \times I}(L).
$$

**Proof:** The first two statements of the proposition follow along the same lines as Proposition 23 and Theorem 17 by using the relations in Figure 12.

By part 2) of Fact 13 we have $\tilde{\Delta}(t_j) = t_j \hat{\otimes} t_j$ for $t_j = Z^p_{\Sigma' \times I}(L \cap p^{-1}(C_j))$ with $j \in J_2$. Equation (13) follows by applying properties 1) to 3) below to the building blocks $a_T(B_i)$ ($i \in J_1$) and $t_j$ ($j \in J_2$) of $Z^p_{\Sigma' \times I}(L)$.

1) Let $\kappa : \tilde{\mathcal{A}}(G, \sigma, s, t) \to \tilde{\mathcal{A}}(G, \sigma, s', t')$ be a map induced by gluing some $G$-labeled intervals to boundary points of a $G$-labeled diagram $D$ with source($D$) = $s$, target($D$) = $t$. Then $\tilde{\Delta}(\kappa(b)) = (\kappa \hat{\otimes} \kappa) \left( \tilde{\Delta}(b) \right)$.

2) Let $b_i \in \tilde{\mathcal{A}}(G, \sigma, s_i, t_i)$ ($i = 1, 2$) with $\tilde{\Delta}(b_i) = b_i \hat{\otimes} b_i$. Then

$$
\tilde{\Delta}(b_1 \hat{\otimes} b_2) = (b_1 \hat{\otimes} b_2) \hat{\otimes} (b_1 \hat{\otimes} b_2).
$$

3) For $b \in \tilde{\mathcal{A}}(\{e\}, 1, +, +)$ with $\tilde{\Delta}(b) = b \hat{\otimes} b$ and $s \in \{+,-\}^*$ we have

$$
\tilde{\Delta}(d_s(b)) = d_s(b) \hat{\otimes} d_s(b).
$$

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The properties 1) and 2) follow directly from the definitions. We give a sketch of a proof of property 3): the equation $\Delta^*(b) = b \otimes b$ implies that $b = \exp(P)$ for some $P \in \overline{A}({e}, 1, +, +)$ with $\Delta(P) = \id_s \otimes P + P \otimes \id_s$. By Theorem 8 of [BN1], the element $P$ can be written in terms of connected trivalent diagrams. The descriptions of the extensions of the maps $\Delta$ and $d_s$ to trivalent diagrams imply that $\Delta(d_s(P)) = \id_s \otimes d_s(P) + d_s(P) \otimes \id_s$. This implies property 3) and completes the proof.

Invariants of links in $\Sigma' \times I$ might be helpful in a definition of invariants of links in $I$-bundles over closed surfaces or of invariants of links in closed 3-manifolds defined in terms of Heegard splittings. For these purposes it might be useful to have different normalizations of $Z_{\Sigma' \times I}^p$ like provided by Proposition 24.

Besides different possible choices of diagrams and normalizations, one can also consider different objects inside of $\Sigma' \times I$. Extensions of the definition of $Z_{\Sigma' \times I}^{p,a}$ to framed (and half-framed) links, tangles and trivalent graphs with and without orientation can be adapted without problems from $\mathbb{R}^3$ (see [MuO], [BeS]) to $\Sigma' \times I$.

In view of an application in [Lie] we give some details concerning an adaptation of $Z_{\Sigma' \times I}^{p,a}$ to links with basepoints. By definition of a link $L$ with basepoints there has to be one distinguished point $P$ on each component of $L$. These basepoints have to lie inside of $p^{-1}(T)$ and must not leave $p^{-1}(T)$ during isotopies. On labeled chord diagrams with basepoints the basepoints have to be disjoint from labels and chord endpoints. We represent basepoints graphically by the symbol $\times$. We define a $\mathbb{Q}$-vector space $\overline{A}^b(G, \sigma)$ generated by $G$-labeled chord diagrams with basepoints modulo the relations (4T), (FI), (Rep), ($\sigma$-Nat) and modulo the Relation (Bas1) shown in Figure 42.

![Figure 42: The Relation (Bas1)](image)

For a 3-manifold $M$ we denote the completion of $\overline{A}^b(\pi_1(M, \ast), \sigma)$ by $\hat{A}^b(M)$. For a generic diagram of a link with basepoints $L \subset \Sigma' \times I$ we define $Z_{\Sigma' \times I}^{p,a}(L) \in \hat{A}^b(\Sigma' \times I)$ in the same way as for links without basepoints except that we map basepoints of links to basepoints of chord diagrams. We have the following lemma.

**Lemma 25** For a link $L$ with basepoints the element $Z_{\Sigma' \times I}^{p,a}(L) \in \hat{A}^b(\Sigma' \times I)$ is invariant under isotopies of links with basepoints.

**Proof:** Relation (Bas1) assures that $Z_{\Sigma' \times I}^{p,a}(L)$ is compatible with moving basepoints across local minima, local maxima, crossings and bands $B \subset T$ of $\Sigma$ in a generic...
diagram of $L$. The rest of the proof follows by the same arguments as in the proofs of Propositions \[23\] and \[24\]. □

Links and labeled chord diagrams with basepoints are useful to avoid problems with signs in the definition of invariants of links in non-orientable 3-manifolds. The adaptations of $Z_{P^2 \times I}$ and $Z_{S^1 \times S^2}$ to links with basepoints are straightforward.

13 Universal Vassiliev invariants and coverings

Let $H, G$ be groups and let $\sigma_H : H \to \{\pm 1\}$ and $\sigma_G : G \to \{\pm 1\}$ be homomorphisms. Let $i : H \to G$ be a monomorphism satisfying $\sigma_G(i(h)) = \sigma_H(h)$ for all $h \in H$. We identify $H$ with the subgroup $i(H)$ of $G$. The group $G$ acts from the right on the set of left cosets $H \setminus G$. Assume that the index $d = [G : H]$ is finite.

Then $i$ induces a morphism of coalgebras $i^* : \bar{A}(G, \sigma_G) \to \bar{A}(H, \sigma_H)$ (16) that we describe below. Let $D$ be a $G$-labeled chord diagram. Replace the skeleton $\Gamma$ of $D$ by a $d$-fold covering $i^*(\Gamma)$ of $\Gamma$ that is constructed as follows. Replace intervals on $\Gamma$ without labels by a bunch of $d$ intervals that are in bijection with $H \setminus G$. These bunches of intervals are glued according to the permutations on $H \setminus G$ induced by the labels $g \in G$ of $\Gamma$. The labeled points of $\Gamma$ are covered by points on $i^*(\Gamma)$ in front of and close to these permutations. In order to describe how to replace the labels and chords of $D$, we choose a set $R = \{g_a \mid a \in H \setminus G\}$ of representatives of left cosets. Then we lift each label $g \in G$ of $\Gamma$ to $d$ labels on $i^*(\Gamma)$, where the label on the interval belonging to $a \in H \setminus G$ is $g_ag_a^{-1}g \in H$ (or, more precisely, $j(g_ag_a^{-1}g) \in H$, where $j : i(H) \to H$ is the inverse of $i$). We complete the description of $i^*(D)$ by lifting the chords to $i^*(\Gamma)$ as follows. We replace each chord of $D$ by a linear combination of $d$ terms by summing over the lifts of the chord endpoints to intervals belonging to the same coset $a$ of $H \setminus G$ with coefficients $\sigma(g_a)$.

Lemma 26 The map $i^*$ is well-defined and does not depend on the choice of $R$.

Proof: First we have to verify that $i^*$ is compatible with the relations $(\sigma\text{-Nat})$, $(\text{Rep})$, $(4\text{T})$, and $(\text{FI})$: consider a relation $(\sigma\text{-Nat})$ of the form $D = \sigma(s)D'$ like in Definition \[2\]. Expand $i^*(D)$ (resp. $i^*(D')$) as a sum of $d$ terms $t_a$ (resp. $t'_a$) with $a \in H \setminus G$ according to which pair of strands the chord in the relation is lifted. Then by relation $(\sigma\text{-Nat})$ in $\bar{A}(H, \sigma_H)$ and by the signs belonging to the lifted chords we
have $\sigma(g_{as})i^*(t_{a,s}) = \sigma(g_{as}g_{a,s}^{-1})\sigma(g_a)i^*(t'_a)$ which implies $i^*(t_{a,s}) = \sigma(s)i^*(t'_a)$. We obtain $i^*(D) = \sigma(s)i^*(D')$ by summing over $a \in H \setminus G$.

For $g,h \in G$ and $a \in H \setminus G$ we have $g_ahg_{a,gh}^{-1} = (g_ag_{a,g}^{-1})(g_{a,g}h)$. Since $G$ acts on $H \setminus G$ we obtain compatibility of $i^*$ with the relations (Rep). The argument that $i^*$ is compatible with the relations (4T) and (FI) is the same as in the proof that the comultiplication $\Delta$ (see equation (3)) is compatible with these relations and is well-known from the case of diagrams without labels.

Define a map $i''$ in the same way as $i^*$, but by replacing one coset representative $g_a \in R$ by $g_a' = hg_a$ with $h \in H$. It remains to show that $i^* = i''$. Let $D$ be a $G$-labeled chord diagram. We compare $i^*(D)$ with $i''(D)$ term by term: we can eliminate all new labels $h,h^{-1}$ in $i''(D)$ by using relations (Rep), then $(\sigma$-Nat) for each chord $c$ in $i''(D)$ between strands corresponding to $a$, and again (Rep). This way we obtain the product of $\sigma(h)$ from relation $(\sigma$-Nat) with $\sigma(g_{a,s}')$ from the definition of $i''$ for all chords $c$ as above. This product equals $\sigma(g_{a})$ from the definition of $i^*(D)$. □

Let $p : E \to B$ be a finite covering of connected 3-manifolds. Taking the preimage of links in $B$ induces a map $p^* : \mathcal{L}(B) \to \mathcal{L}(E)$.

**Lemma 27** The map $p^*$ satisfies $p^*(\mathcal{L}_n(B)) \subseteq \mathcal{L}_n(E)$ for all $n \geq 0$.

**Proof:** When $L,L' \subset B$ differ by a crossing change, then $p^{-1}(L)$ can be obtained from $p^{-1}(L')$ by changing $d$ crossings. Therefore the preimage of the desingularization (see Figure 1) of a singular link in $B$ with $n$ double points can be written as a sum of desingularizations of $n^d$ singular links in $E$ with suitable local orientations at the double points. □

By Lemma 27 the map $p^*$ induces a map $\text{gr}(p^*) : \text{gr}\mathcal{L}(B) \to \text{gr}\mathcal{L}(E)$. Choose a basepoint $\ast \in p^{-1}(\ast)$. From now on we will consider the case $G = \pi_1(B,\ast)$ and $H = \pi_1(E,\hat{\ast})$, where $\sigma_G$ (resp. $\sigma_H$) is the orientation character of $B$ (resp. $E$), and the inclusion $H \to G$ is given by $p_* = \pi_1(p)$. We identify $H$ with $p_*(H) \subset G$. Choose oriented neighborhoods $U \subset B$ of $\ast$ and $\hat{U} \subset p^{-1}(U)$ of $\hat{\ast}$ with $U,\hat{U} \cong \mathbb{R}^3$ such that $p|_{\hat{U}} : \hat{U} \to U$ is orientation preserving. Then by the following proposition the maps $\pi_1(p)^*$ and $\text{gr}(p^*)$ are compatible with the realization of chord diagrams as singular links (see Section 3).

**Proposition 28** For a finite covering of connected 3-manifolds $p : E \to B$ the
following diagram commutes.

\[
\begin{array}{c}
\tilde{A}(E) & \xrightarrow{\psi_E} & \text{gr}\mathcal{L}(E) \\
\pi_1(p)^* & & \downarrow \text{gr}(p^*) \\
\tilde{A}(B) & \xrightarrow{\psi_B} & \text{gr}\mathcal{L}(B)
\end{array}
\]

**Proof:** Let \( D \) be a \( G \)-labeled chord diagram where \( G = \pi_1(B, \ast) \). By relations (Rep) we may assume that between any two consecutive chord endpoints of \( D \) there is exactly one label. Furthermore, we may assume that connected components of the skeleton of \( D \) without chord endpoints have exactly one label. Let \( L_D \subset B \) be a realization of \( D \) such that the complement of small intervals around the labels of \( D \) is mapped into \( U \). As in the proof of Lemma 27 we write \( p^*(L_D) \) as a sum of \( n^d \) singular links \( L_i \) by local modifications of \( p^{-1}(L_D) \) inside of \( p^{-1}(U) \). The connected components of \( p^{-1}(U) \) are in natural one-to-one correspondence with \( H \setminus G \). By a homotopy of singular links we pull \( L_i \cap p^{-1}(U) \) into \( \tilde{U} \) by following paths belonging to the lifts of elements of \( R \) with starting point \( \tilde{\ast} \). This way \( L_i \) becomes the realization of an \( H \)-labeled chord diagram that appears as one of the \( n^d \) terms in the definition of \( \pi_1(p)^*(D) \). Orient \( p^{-1}(U) \) such that \( p_{p^{-1}(U)} \) is orientation preserving. Then the path in \( E \) belonging to the lift of \( g_a \in R \) is orientation preserving iff \( \sigma_G(g_a) = 1 \). This shows that the diagram belonging to \( L_i \) appears in \( \pi_1(p)^*(D) \) with the correct sign and completes the proof. \( \square \)

Let \( \Sigma_1 \) be a ribbon graph with bands \( B_i \ (i \in J_1) \), coupons \( C_i \ (i \in J_2) \) and with a basepoint \( \ast \) in a distinguished coupon \( C_o \). Let \( p_1 : B \to \Sigma_1 \) be an \( I \)-bundle with an orientation of \( p_1^{-1}(C_0) \). Consider a finite connected covering \( p : E \to B \). Let \( \Sigma_2 = p^{-1}(s(\Sigma_1)) \) where \( s \) is a section of \( p_1 \). Then there exists a unique projection \( p_2 : E \to \Sigma_2 \) such that \( p_1 \circ p = p_1 \circ p_2 \) and \( E \) is an \( I \)-bundle over \( \Sigma_2 \). The surface \( \Sigma_2 \) has a structure of a ribbon graph with bands \( B^c_i \ (i \in J_1, c \in H \setminus G) \) and coupons \( C^c_i \ (i \in J_2, c \in H \setminus G) \) such that \( p_1 \circ p \) restricts to homeomorphisms from \( C^c_i \) to \( C_i \) and from \( B^c_i \) to \( B_i \). Given elements \( a(B_1) \in \tilde{A}(\{e\}, 1, +, +) \) satisfying \( a(B^c_1) = a(B_1)^* \) for all \( B_1 \in B^a(\Sigma_1) \), we define \( b(B_2) = a(p_1(p(B_2))) \) for all \( B_2 \in B^a(\Sigma_2) \). Choose \( \tilde{\ast} \in p^{-1}(s(C_0)) \). Let \( C \) be the unique coupon of \( \Sigma_2 \) with \( \tilde{\ast} \in C \). We orient \( p_2^{-1}(C) \) such that \( p_{|C} \) is orientation preserving. Then we have the following theorem.

**Theorem 3** Let \( p : E \to B \) be a covering of \( I \)-bundles \( p_1 : B \to \Sigma_1, \ p_2 : E \to \Sigma_2 \) over ribbon graphs \( \Sigma_1, \Sigma_2 \) as above. We assume that for all \( B_1 \in B^a(\Sigma_1) \) we have \( \tilde{\Delta}(a(B_1)) = a(B_1) \otimes a(B_1) \). Then the following diagram commutes.
Proof: Choose $T_1 \subset \Sigma_1$ as in equation (14). Let $L$ be a generic link in $B$. By definition of $Z_B^{p_1,a}$ we can express $Z_B^{p_1,a}(L)$ in the following form:

$$Z_B^{p_1,a}(L) = \kappa_1 \left( \bigotimes_{i \in J_2} t_i \otimes \bigotimes_{i \in J_1} d_{s_i} (I(\hat{g}_i) \circ a_{T_1}(B_i)) \right), \quad (17)$$

where the symbols $\kappa_1$, $t_i$, $s_i$, and $I(\hat{g}_i)$ are explained now: $\kappa_1$ is a gluing map defined in terms of the gluing pattern of the bands and coupons of $\Sigma_1$. The tangles $L \cap p_1^{-1}(C_i)$ are equipped with suitable parenthesis on their sources and targets in the definition of $t_i = Z(L \cap p_1^{-1}(C_i))$. Let $I_0^i$ be the lower base of the band $B_i$. Let $C^i$ be the unique coupon of $\Sigma_1$ with $C^i \cap I_0^i \neq \emptyset$. Then we define $s_i$ as the subword of source($L \cap p_1^{-1}(C^i)$) or target($L \cap p_1^{-1}(C^i)$) determined by $L \cap p_1^{-1}(I_0^i)$. Finally, for $\hat{g}_i = p_1^{-1}(g_{B_i})$ we define $I(\hat{g}_i) \in \hat{A}(G, \sigma_G, +, +)_0$ as a single strand without chords labeled by $\hat{g}_i$.

Choose $T_2 \subset \Sigma_2$ as in equation (14) such that $(p_1 \circ p)^{-1}(T_1) \subset p_2^{-1}(T_2)$. The connected components of $(p_1 \circ p)^{-1}(T_1)$ are in a natural one-to-one correspondence with $H \setminus G$, so we may assume that coupons are indexed in a way such that $C_i^c$ lies in the part of $(p_1 \circ p)^{-1}(T_1)$ belonging to $c \in H \setminus G$. Furthermore, we may assume that the bands of $\Sigma_2$ are indexed in a way such that the band $B'_k$ leads from $C_i^c \cdot \hat{g}_k$ to $C_j$ whenever the band $B_k$ of $\Sigma_1$ leads from $C_i$ to $C_j$. There exist unique representatives $g_c$ of cosets $c \in H \setminus G$ such that the lifts of the elements $g_c$ with starting point $\tilde{s}$ can be represented by paths inside of $T_2 \subset E$. Using Fact 14 we can relate $Z_E^{p_2,b}(p^{-1}(L))$ to equation (17) as follows

$$Z_E^{p_2,b}(p^{-1}(L)) = \kappa_2 \left( \bigotimes_{c \in H \setminus G} \left( \bigotimes_{i \in J_2} t^c_i \otimes \bigotimes_{i \in J_1} d_{s_i} (I(\hat{g}^c_i) \circ b_{T_2}(B'_i)) \right) \right), \quad (18)$$

where $\kappa_2$ is defined in terms of the gluing pattern of the bands and coupons of $\Sigma_2$, $t^c_i = t_i$ if $\sigma_G(g_c) = 1$ and $t^c_i = t^*_i$ otherwise, and $\hat{g}^c_i = p_2^{-1}(g_{B'_i})$. We have $p_*(\hat{g}^c_k) = g_c \cdot \hat{g}_k \cdot g_c^{-1}$ because the lift of $\hat{g}_k$ with starting point $\tilde{s}$, $g_c \cdot \hat{g}_k$ can be represented by a
path inside of \((p_1 \circ p)^{-1}(T_1) \cup B_k^c \subset p_2^{-1}(T_2) \cup B_k^c\) that travels through \(B_k^c\) exactly once in the positive direction.

Let \(d = |G : H|\). Define the \(d\)-th iterated comultiplication \(\hat{\Delta}_d\) by \(\hat{\Delta}_2 = \hat{\Delta}\) and by \(\hat{\Delta}_d = (\hat{\Delta}_{d-1} \otimes \text{id}) \circ \hat{\Delta}\). Let \(H \setminus G = \{c_1, \ldots, c_d\}\). For a \(G\)-labeled chord diagram \(D\) we define \(\psi_\nu(D) = D\) if \(\sigma_G(g_{c_\nu}) = 1\), and \(\psi_\nu(D) = D^*\) otherwise. Using the definition of \(\pi_1(p)^*\) with respect to the set \(\mathcal{R} = \{g_c \mid c \in H \setminus G\}\) we construct \(\pi_1(p)^* (Z_{B_k}^{p_1, a}(L))\) from the elements \(t_i (i \in J_2)\) and \(d_i = d_i (I(\hat{g}_i) \circ a_{T_i}(B_i)) (i \in J_1)\) in the following three steps:

(1) for \(j \in J_2\) let \(T_j = (\psi_1 \otimes \ldots \psi_d) \left( \hat{\Delta}_d(t_{j}) \right)\),

(2) define \(D_i (i \in J_1)\) by replacing the labels \(\hat{g}_i\) in the \(\nu\)-th tensor factor of \((\psi_1 \otimes \ldots \psi_d) \left( \hat{\Delta}_d(d_{i}) \right)\)

with \(q(g_{c_\nu}\hat{g}_i g_{c_\nu}^{-1}) \in H \ (\nu = 1, \ldots, d)\) where \(q : p_*(H) \to H\) is the inverse of \(p_*\),

(3) assume that the band \(B_k\) of \(\Sigma_1\) is leading from \(C_i\) to \(C_j\). Let \(\pi_k\) be the permutation determined by \(c_\nu \cdot \hat{g}_k = c_{\pi_k(\nu)}\). Then the \(\nu\)-th tensor factor of \(D_k\) is glued between the \(\pi_k(\nu)\)-th tensor factor of \(T_i\) and the \(\nu\)-th tensor factor of \(T_j\) in the same way as \(d_k\) is glued to \(t_i\) and to \(t_j\) by the map \(\kappa_1\).

Part (2) of Fact \([3]\) implies \(\hat{\Delta}_d(t_{j}) = t_{j}^{\otimes d}\), so we obtain

\(T_j = t_{j_1}^{a_1} \otimes \ldots \otimes t_{j_d}^{a_d}\).

By the assumptions on the elements \(a(B_1)\) made in the theorem we have \(\hat{\Delta}_d(d_{i}) = d_i^{\otimes d}\) and therefore

\[ D_i = \bigotimes_{\nu = 1}^d d_i (I(q(g_{c_\nu}\hat{g}_i g_{c_\nu}^{-1})) \circ \psi_\nu(a_{T_i}(B_i))) = \bigotimes_{\nu = 1}^d d_i (I(\hat{g}_i^{c_\nu}) \circ b_{T_i}(B_i^{c_\nu})). \]

From part (3) of the description of \(\pi_1(p)^* (Z_{B_k}^{p_1, a}(L))\) from above we see that the elements \(T_j\) and \(D_i\) are stuck together according to the gluing map \(\kappa_2\). This way we obtain the expression in equation \([13]\) for \(\pi_1(p)^* (Z_{B_k}^{p_1, a}(L))\) which completes the proof. \(\square\)

Let \(p : S^2 \times I \to P^2 \times I\) be the universal covering of \(P^2 \times I\). Recall the decomposition of \(P^2\) into a coupon \(B_0\), a band \(B_1\) and a disk \(B\) from Section \([10]\). The preimages \(p^{-1}(B_0 \times 1/2), p^{-1}(B_1 \times 1/2),\) and \(p^{-1}(B \times 1/2)\) constitute a decomposition of \(S^2\) into two coupons \(B_0^i\), two bands \(B_1^i\) and two disks \(B_i^i\) \((i = 0, 1)\). For the ribbon graph \(S = \bigcup_{i,j = 0}^1 B_1^j\) a universal Vassiliev invariant \(Z_{S \times I} = Z_{S \times I}^d\) with \(q(x, t) = x\) is defined in Section \([12]\). Denote the inclusion map \(S \times I \subset S^2 \times I\)
The map $\pi_1(j)_*$ was associated to $\pi_1(j)$ at the end of Section 2. It replaces all labels of chord diagrams by the neutral element. For links $L \subset S \times I \subset S^2 \times I$ we define $Z_{S^2 \times I}(L) = \pi_1(j)_*(Z_{S \times I}(L))$. Any link in $S^2 \times I$ is isotopic to a link in $S \times I$. As a corollary to part (1) of the following theorem we see that the definition of $Z_{S^2 \times I}$ leads to an isotopy invariant of links in $S^2 \times I$.

Theorem 4
(1) Let $i : S^2 \times I \rightarrow \mathbb{R}^3$ be the inclusion defined by $i(x, t) = (t + 1)x$. Then for links $L \subset S \times I \subset S^2 \times I$ we have

$$Z_{S^2 \times I}(L) = Z(i(L)).$$

(2) For the universal covering $p : S^2 \times I \rightarrow P^2 \times I$ we have

$$\pi_1(p)^* \circ Z_{P^2 \times I} = Z_{S^2 \times I} \circ p^*.$$

Proof: (1) Let $L \subset S \times I$ be a link in standard position. The inclusion $i|_{S \times I} : S \times I \rightarrow \mathbb{R}^3$ is depicted on the left side of Figure 43.

![Figure 43: The ribbon graph $S \subset S^2$ and the abstract ribbon graph $S$](image)

We can regard $i(L \cap B_{\nu}^\mu \times I)$ ($\nu, \mu = 0, 1$) as non-associative tangles such that

$$Z(i(L)) = Z(i(L \cap B_{\nu}^\mu \times I)) \circ (Z(i(L \cap B_{\nu}^0 \times I)) \otimes Z(i(L \cap B_{1}^1 \times I))) \circ Z(i(L \cap B_{0}^1 \times I)),$$

the non-associative tangle $i(L \cap B_{\nu}^{\mu} \times I)$ (resp. $i(L \cap B_{\nu}^{1} \times I)$) consists of a left-handed (resp. right-handed) half twist of a bunch of strands, and we have

$$s_{\nu} = \text{source}(i(L \cap B_{\nu}^{\mu} \times I)) = \text{target}(i(L \cap B_{\nu}^{\mu} \times I)) \upharpoonright (\nu = 0, 1).$$

We compute the invariants $Z(i(L \cap B_{\nu}^1 \times I))$ using the analytic definition of $Z$: from Example 1.4, Property 1.12 with $A = -1$, and Section 3.3 of [Les] we obtain the left side of equation (19) for some $u \in \mathcal{A} \langle \{e\}, 1, s_{\nu}, s_{\nu} \rangle$. 

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where \( t_\nu = -(-1)^\nu \frac{1}{4} \). The right side of equation (19) follows because we can use the relations in Figure 12 to cancel \( u \) with \( u^{-1} \). Let \( b_0^\nu \) be the part of degree 0 of \( Z(i(L \cap B_1^\nu \times I)) \). Again by applying the relations in Figure 12 we obtain

\[
Z(i(L)) = Z(i(L \cap B_0^1 \times I)) \circ (b_0^0 \boxtimes b_0^1) \circ Z(i(L \cap B_0^0 \times I)).
\]  

(20)

Recall that in the definition of \( Z_{S^2 \times I}(L) \) the invariants of non-associative tangles \( Z(L \cap B_0^0 \times I) \) are sticked together using a gluing map determined by a diagram of the ribbon graph \( S \) shown on the right side of Figure 13. This implies

\[
Z_{S^2 \times I}(L) = Z(L \cap B_0^1 \times I)^{-1} \circ (b_0^0 \boxtimes b_0^1) \circ Z(L \cap B_0^0 \times I),
\]  

(21)

where the symmetry operations \( - \) and \( | \) have been defined at the end of Section 4.

Since \( i(L \cap B_0^1 \times I) = (L \cap B_0^1 \times I)^{-1} \) and \( i(L \cap B_0^0 \times I) = L \cap B_0^0 \times I \) we obtain the first part of the theorem from Fact 14 and equations (20) and (21).

(2) Let \( X = B_0 \cup B_1 \subset P^2 \). Let \( q = p_{S \times I} : S \times I \rightarrow X \times I \). For a link \( L \) in \( X \times I \) Theorem 3 implies

\[
(\pi_1(q)^* \circ Z_{X \times I}) (L) = Z_{S \times I} \left( p^{-1}(L) \right).
\]  

(22)

Let \( \pi_1(j)_* \) be the map from above given by forgetting labels of a chord diagram. Denote the inclusion map \( X \times I \subset P^2 \times I \) by \( k \). Then \( \pi_1(k)_* \) is given by reduction modulo 2 of the labels of a \( \mathbb{Z} \)-labeled chord diagram, where \( \mathbb{Z} \cong \pi_1(X \times I, *) \). Since it is easy to verify

\[
\pi_1(j)_* \circ \pi_1(q)^* = \pi_1(p)^* \circ \pi_1(k)_*
\]

we obtain part (2) of the theorem by applying \( \pi_1(j)_* \) to both sides of equation (22). \( \Box \)

It is easy to show that the definition of \( Z_{S^1 \times S^2} \) (resp. \( Z_{S^1 \times \tilde{S}^2} \)) in Section 14 extends to the case where \( Z_{S^1 \times I \times I} \) (resp. \( Z_{X \times I} \)) is defined using an arbitrary ribbon graph structure on \( S^1 \times I \) (resp. \( X \times I \)). Let \( B = S^1 \times S^2 \) or \( B = S^1 \times \tilde{S}^2 \). For a
finite covering \( p : E \to B \) with a connected space \( E \) we have \( E \cong S^1 \times S^2 \) if \( B = S^1 \times S^2 \) and \( E \cong S^1 \times S^2 \) or \( E \cong S^1 \tilde{\times} S^2 \) otherwise. The map \( \pi_1(p)^* \) descends to a map \( \mathcal{E}(B) \to \mathcal{E}(E) \) and a statement similar to Theorem 3 holds for finite coverings of \( B \).

References

[AMR] J. E. Andersen, J. Mattes and N. Reshetikhin, *Quantization of the algebra of chord diagrams*, Math. Proc. Camb. Phil. Soc. 124, No. 3 (1998), 451–467.

[BN1] D. Bar-Natan, *On the Vassiliev knot invariants*, Topology 34 (1995), 423–472.

[BN2] D. Bar-Natan, *Non–associative tangles*, Geometric topology proceedings of the Georgia International Topology Conference, (W. H. Kazez, ed.), Amer. Math. Soc. and international Press, Providence (1997), 139–183.

[BGRT] D. Bar-Natan, S. Garoufalidis, L. Rozansky and D. P. Thurston, *The Århus invariant of rational homology 3-spheres II: Invariance and universality*, Selecta Math., to appear.

[BeS] A.-B. Berger and I. Stassen, *The skein relation for the \((g_2, V)\)-link invariant*, Comment. Math. Helv. 75, No. 1 (2000), 134–155.

[Dri] V. G. Drinfeld, *On quasitriangular quasi-Hopf algebras and a group closely connected with \( \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \)*, Algebra i Analiz 2:4 (1990), 149–181. English transl.: Leningrad Math. J. 2 (1991), 829–860.

[Eis] M. Eisermann, *Les invariants rationnels de type fini ne distinguent pas les nœuds dans \( S^2 \times S^1 \)*, Comptes Rendus de l’Académie des Sciences de Paris, Serie I, 332 (2001), 51–55.

[GMP] J. González-Meneses and L. Paris, *Vassiliev invariants of braids on surfaces*, Universidad de Sevilla and Université de Bourgogne preprint (2000), math.GT/0003187

[Kas] C. Kassel, *Quantum groups*, GTM 155, Springer-Verlag, New York 1995.

[Les] C. Lescop, *Introduction to the Kontsevich integral of framed tangles*, CNRS Institut Fourier preprint, 1999.
[Lie] J. Lieberum, *Skein modules of links in cylinders over surfaces*, preprint, math.QA/9911174.

[LM1] T. Q. T. Le and J. Murakami, *The universal Vassiliev-Kontsevich invariant for framed oriented links*, Comp. Math. 102 (1996), 41–64.

[LM2] T. Q. T. Le and J. Murakami, *Parallel version of the universal Vassiliev-Kontsevich invariant*, J. Pure and Appl. Algebra 121 (1997), 271–291.

[LMO] T. Q. T. Le, J. Murakami and T. Ohtsuki, *On a universal perturbative invariant of 3-manifolds*, Topology 37-3 (1998), 539–574.

[MuO] J. Murakami and T. Ohtsuki, *Topological quantum field theory for the universal quantum invariant*, Commun. Math. Phys. 188, No. 3 (1997), 501–520.

[Tur] V. G. Turaev, *Quantum invariants of Knots and 3-Manifolds*, De Gruyter studies in math. 18 (1994).

[Vog] P. Vogel, *Invariants de Vassiliev des nœuds*, Séminaire Bourbaki 769 (1993), 1–17, Astérisque 216 (1993), 213–232.

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