Lyra black holes and Hawking radiation

Habiba Bouhallouf
Laboratoire de Physique Mathématique et Subatomique, Frères Mentouri
University of Constantine 25000, Algeria
E-mail: habiba.bouhalouf@umc.edu.dz

Abstract. Sen and Dunn constructed a field equation which is analogous to Einstein’s field equations based on the Lyra’s geometry and gave series type solution to the static vacuum field equations which correspond to black holes. In this work, the Hawking radiation near the trapping horizon of Lyra black holes is derived and the obtained results are discussed.

1. Introduction

Einstein developed [1] the general theory of relativity to unify gravity with other fundamental forces. However, in this theory, gravitation is described by a Riemannian geometry, which could not help for this unification program [2]. For that reason, Lyra [3] proposed in 1951 a modification on a Riemannian geometry (Lyra’s geometry) by introducing a gauge or scale function which removes the non-instability condition of a vector under a parallel transport. Soon after, Sen in 1951 [4] and Sen and Dun in 1971 [5] have constructed an analog of the Einstein field equation based on Lyra’s geometry as:

\[ R_{ij} - \frac{1}{2}g_{ij}R + \frac{3}{2}\phi_i\phi_j - \frac{3}{4}g_{ij}\phi_k\phi^k = -8\pi G T_{ij} \]  

(1)

where \( R_{ij} \), \( g_{ij} \), \( R \), \( T_{ij} \) and \( \phi_i \) are Ricci tensor, metric, scaler curvature, energy momentum stress tensor and the displacement field vector respectively. Here \( \phi_i = (\beta, 0, 0, 0) \) and \( G \) stands for the Newton gravitational constant. Note that \( \beta \) can be taken as a constant or a time-dependent function.

Furthermore, Sen and Dunn [5] gave a series type solutions to the static vacuum field equations. Retaining only a few terms in their solutions, they find that their solutions correspond to black holes (Lyra black holes). The goal of this paper is to study the Hawking radiation [6] of Lyra black hole. For that we proceed to analyze the Dirac equation in Lyra space-time and use the tunneling method since the Hawking’s effect is a phase phenomenon. Such a tunneling approach uses the fact that the WKB aproximation of the tunneling probability...
for the classical forbidden trajectory from inside to outside the horizon is:

\[ \Gamma \propto \exp(-\frac{2}{\hbar} \Im I) \]  

where \( \Im I \) stands for the imaginary part of the classical action of the trajectory.

In section 2, we give the mathematical formalism of the Lyra black holes [4]. In section 3, we present the Dirac equation in a curved Lyra space-time. In section 4, we study the Hawking temperature using the tunneling effect approach within the WKB approximation [9]. Finally, we derive our conclusion in the last section.

2. Lyra black holes

Let us consider a static spherically symmetric metric:

\[ ds^2 = e^\nu dt^2 - e^\lambda dr^2 - r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \]  

Sen and Dunn [4] have defined \( e^\nu \) and \( e^\lambda \) to obtain solutions to the field equations Eq. (1) as:

\[ e^\nu = \delta + \zeta \phi(r), \quad e^\lambda = \frac{\alpha r^4 (\phi')^2}{\delta + \zeta \phi(r)} \]  

where \( \phi = \sum_{r=0}^{\infty} a_n r^{-n} \) (\( \alpha, \delta, \zeta \) are arbitrary constants). The coefficients \( a_n \) are given by:

\[ 0 = a_{n-1} ((\delta + \zeta a_0)(n-1)(n-4)) - \alpha a_1 \sum_{k=3}^{n-1} (k-1)(n-k+1)a_{k-1}a_{n-k+1} \]
\[ -\alpha \sum_{l=3}^{n-1} ((l-1)a_{l-1}) \left( \sum_{k=2}^{n-l+2} (k-1)(n-l-k+3)a_{k-1}a_{n-l-k+3} \right) \]
\[ -\sum_{l=2}^{n-1} (n-l)(2l-n-1)a_{n-l}a_{l-1} \]  

(5)

retaining only a few terms, we find that:

\[ \zeta^2 = \frac{2a_3}{a_1}, \quad \zeta + \delta a_0 = 1, \quad a_1 = \pm \frac{1}{\sqrt{\alpha}}, \quad \frac{\zeta}{\sqrt{\alpha}} = 2M' = M \]  

(6)

where \( M' \) is the mass of black hole. Thus one can write \( e^\nu \) and \( e^\lambda \) as:

\[ e^\nu = 1 - \frac{M}{r} + \frac{Mp}{r^3} + \frac{M^2 p}{r^4}, \quad e^\lambda = \frac{\sigma^2}{e^\nu} \]  

(7)

where \( \sigma^2 = 1 - \frac{6p}{r^2} - \frac{8Mp}{r^4} + \frac{9p^2}{r^6} \) and \( p = a_3 \sqrt{\alpha} \).
If we take the time component of the Eq. (3), we can easily prove that the singularity is located at $r = 0$, and the horizons correspond to $e^{\nu} = 0$; at that moment we get an equation in $4^{th}$ order,

$$r^4 - Mr^3 + pMr + pM^2 = 0 \quad (8)$$

This means that we will obtain four roots:

$$r_{1,2} = \frac{1}{2} \left( -\sqrt{\Omega_2} \pm \sqrt{-\Omega_3 - \Omega_4 + M} \right) \quad (9)$$

$$r_{3,4} = \frac{1}{2} \left( \sqrt{\Omega_2} \pm \sqrt{-\Omega_3 + \Omega_4 + M} \right) \quad (10)$$

where

$$Y = \left( -\frac{Q}{2} + \sqrt{\Omega_1} \right)^{\frac{3}{2}} + \left( -\frac{Q}{2} - \sqrt{\Omega_1} \right)^{\frac{3}{2}} + \frac{A}{6} \quad (11)$$

is the real solution of the following $3^{rd}$ order equation:

$$-8Y^3 + 4AY^2 + 8CY - 4AC + B^2 = 0 \quad (12)$$

with

$$A = -\frac{3}{2}M^2, \quad B = \frac{1}{2}M^3 + pM, \quad C = \frac{5}{4}pM^2 - \frac{3}{16}M^4 \quad (13)$$

$$P = -\frac{5}{4}pM^2, \quad Q = \frac{3}{32}M^6 - \frac{3}{4}pM^4 - \frac{1}{8}p^2M^2 \quad (14)$$

and

$$\Omega_1 = \frac{Q^2}{4} + \frac{p^3}{27}, \quad \Omega_2 = 2Y - A, \quad \Omega_3 = 2Y + A, \quad \Omega_4 = \frac{2B}{\sqrt{\Omega_2}} \quad (15)$$

In order to get real and positive roots Eqs. (9) and (10), one has to put some conditions and the obtained results are summarized in the table 1. (" + " means that the quantity is positive, " − " means that it is negative and the symbol "/" denotes a meaningless case).

| $\Omega_1$ | $\Omega_2$ | $\Omega_4 - \Omega_3$ | $\Omega_5$ | $\Omega_5^+$ | $r_1$ | $r_2$ | $\Omega_4 + \Omega_3$ | $\Omega_6^+$ | $\Omega_6^-$ | $r_3$ | $r_4$ |
|------------|------------|----------------------|------------|------------|------|------|----------------------|------------|------------|------|------|
| +          | +          | +                    | +          | +          | yes  | yes  | -                    | /          | /          | no   | no   |
| +          | +          | +                    | -          | -          | yes  | no   | -                    | /          | /          | no   | no   |
| +          | +          | +                    | -          | +          | no   | yes  | -                    | /          | /          | no   | no   |
| +          | +          | -                    | -          | -          | no   | no   | -                    | /          | /          | no   | no   |
| +          | -          | /                    | /          | /          | no   | no   | +                    | +          | +          | yes  | yes  |
| +          | -          | /                    | /          | /          | no   | no   | +                    | -          | +          | no   | yes  |
| +          | -          | /                    | /          | /          | no   | no   | +                    | -          | yes       | no   | no   |
| +          | -          | /                    | /          | /          | no   | no   | +                    | -          | no        | no   | no   |
where
\[ \Omega_5 = \sqrt{\Omega_4 - \Omega_3}, \quad \Omega_5^+ = \Omega_5 + \sqrt{\Omega_4 - \Omega_3}, \quad \Omega_5^- = \Omega_5 - \sqrt{\Omega_4 + \Omega_3} \]
\[ \Omega_6 = \sqrt{-(\Omega_4 + \Omega_3)}, \quad \Omega_6^+ = \Omega_6 + \sqrt{\Omega_3 - \Omega_4}, \quad \Omega_6^- = \Omega_6 - \sqrt{\Omega_3 + \Omega_4} \]

It is easy to note that we are dealing with three cases; two real positive, one real positive or no real positive radii:

(i) If \( \Omega_4 - \Omega_3 > 0 \) and \( \Omega_4 + \Omega_3 < 0 \):
- If \( \Omega_5^- > 0 \) and \( \Omega_5^+ > 0 \), we have two event horizons at \( r_1 \) and \( r_2 \)
- If \( \Omega_5^- > 0 \) and \( \Omega_5^+ < 0 \), we have one event horizon situated at \( r_1 \)
- If \( \Omega_5^- < 0 \) and \( \Omega_5^+ > 0 \), we have one event horizon situated at \( r_2 \)
- If \( \Omega_5^- < 0 \) and \( \Omega_5^+ < 0 \), we have no horizons.

(ii) If \( \Omega_4 - \Omega_3 < 0 \) and \( \Omega_4 + \Omega_3 > 0 \):
- If \( \Omega_6^- > 0 \) and \( \Omega_6^+ > 0 \), we have two event horizons at \( r_3 \) and \( r_4 \)
- If \( \Omega_6^- > 0 \) and \( \Omega_6^+ < 0 \), we have one event horizon situated at \( r_4 \)
- If \( \Omega_6^- < 0 \) and \( \Omega_6^+ > 0 \), we have one event horizon situated at \( r_3 \)
- If \( \Omega_6^- < 0 \) and \( \Omega_6^+ < 0 \), we have no horizons.

3. Dirac equation

Now we calculate the fermion’s Hawking radiation from the apparent horizons of the Lyra black holes via the tunneling formalism [6]-[7]. Let us start with the massless spinor field \( \Psi(t, r, \theta, \phi) \) obeyed the general covariant Dirac equation:
\[ i\gamma^\mu D_\mu \Psi(t, r, \theta, \phi) = 0 \]  \hfill (16)
where \( D_\mu \) is the spinor covariant derivative is defined by
\[ D_\mu = \partial_\mu + \frac{i}{2} \omega_\mu^{\ ab} \Sigma_{ab} \]  \hfill (17)
and \( \omega_\mu^{\ ab} \) is the spin connection, which can be given in terms of the tetrad \( e_\alpha^\mu \).

The matrices \( \gamma^\mu = \gamma^a e_a^\mu \) satisfy the Clifford algebra,
\[ [\gamma_a, \gamma_b]_+ = 2\eta_{ab}I_{4\times4} \]  \hfill (18)
and they are selected as
\[ \gamma^0 = i \begin{pmatrix} 1_{2\times2} & 0 \\ 0 & -1_{2\times2} \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} 0 & \sigma^3 \\ \sigma^3 & 0 \end{pmatrix}, \quad \gamma^2 = \begin{pmatrix} 0 & \sigma^1 \\ \sigma^1 & 0 \end{pmatrix}, \quad \gamma^3 = \begin{pmatrix} 0 & \sigma^2 \\ \sigma^2 & 0 \end{pmatrix} \]  \hfill (19)
with
\[ \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \]  \hfill (20)
\( \sigma^i (i, j, k = 1, 2, 3) \) are the Pauli matrices satisfying the usual relation:

\[
\sigma_i \sigma_j = I_{2 \times 2} \delta_{ij} + i \varepsilon_{ijk} \sigma_k
\]

(22)

In order to get the Dirac \( \gamma^\mu \) matrices which are expressed in terms of the tetrad, we first define a tetrad of orthogonal vector \( e^a_\mu \) where:

\[
\eta_{ab} e^a_\mu e^b_\nu = g_{\mu \nu}
\]

(23)

here \((a, b) \equiv (0, 1, 2, 3)\) and \((\mu, \nu) \equiv (t, r, \theta, \phi)\). The simplest choice of tetrads leads to the following matrix:

\[
e^\mu_a = \begin{pmatrix}
e^{-\nu/2} & 0 & 0 & 0 \\
0 & e^{-\lambda/2} & 0 & 0 \\
0 & 0 & \frac{1}{r} & 0 \\
0 & 0 & 0 & \frac{1}{r \sin \theta}
\end{pmatrix}
\]

(24)

With these tetrads, it turns out that:

\[
\gamma^t = e^{-\nu/2} \gamma^0, \quad \gamma^r = e^{-\lambda/2} \gamma^1, \quad \gamma^\theta = \frac{1}{r} \gamma^2, \quad \gamma^\phi = \frac{1}{r \sin \theta} \gamma^3
\]

(25)

we can also write the matrix \( \gamma^5 \) in this way:

\[
\gamma^5 \overset{\text{def}}{=} i \gamma^t \gamma^r \gamma^\theta \gamma^\phi = \frac{i e^{-(\nu+\lambda)/2}}{r^3 \sin \theta} \gamma^0 \gamma^1 \gamma^2 \gamma^3
\]

(26)

4. Hawking temperature

To calculate the Hawking temperature, let us use the following ansatz for the spin-up Dirac field (same results remain to valid for the spin-down case):

\[
\Psi^\uparrow (t, r, \theta, \phi) = \begin{pmatrix}
\Lambda (t, r, \theta, \phi) \\
\Xi (t, r, \theta, \phi) \\
0
\end{pmatrix} \exp \left[ \frac{i}{\hbar} I^\uparrow (t, r, \theta, \phi) \right]
\]

(27)

In order to apply the WKB approximation, we have to plug the ansatz into the general covariant Dirac Eq. (16) [6], it turns out that the term in square brackets is of order \( O(\hbar) \). Thus we do not need to work out its precise form, since in the \( \hbar \to 0 \) limit it vanishes. So the equation becomes:

\[
\hbar \frac{\partial \Psi^\uparrow (t, r, \theta, \phi)}{\partial (t, r, \theta, \phi)} + O(\hbar) = 0
\]

(28)

Direct simplifications lead to the expression:

\[
e^{-\nu/2} \left( \begin{array}{c}
\Lambda \partial_t I^\uparrow \\
0 \\
-\Xi \partial_t I^\uparrow
\end{array} \right) + e^{-\lambda/2} \left( \begin{array}{c}
0 \\
0 \\
i \Lambda \partial_r I^\uparrow
\end{array} \right) + \frac{1}{r} \left( \begin{array}{c}
i \Xi \partial_\theta I^\uparrow \\
0 \\
i \Lambda \partial_\phi I^\uparrow
\end{array} \right) + \frac{1}{r \sin \theta} \left( \begin{array}{c}
0 \\
-\Xi \partial_\phi I^\uparrow \\
-\Lambda \partial_\phi I^\uparrow
\end{array} \right) = 0
\]

(29)
and one ends up with the following equations system:

\[
\begin{align*}
    t & : \quad -e^{-\nu/2}\Lambda \partial_t I^\uparrow + ie^{-\lambda/2}\Xi \partial_r I^\uparrow = 0 \quad (30) \\
    r & : \quad \Xi \left( i \partial_\theta I^\uparrow - \frac{1}{\sin \theta} \partial_\phi I^\uparrow \right) = 0 \quad (31) \\
    \theta & : \quad e^{-\nu/2}\Xi \partial_t I^\uparrow + ie^{-\lambda/2}\Lambda \partial_r I^\uparrow = 0 \quad (32) \\
    \phi & : \quad \Lambda \left( i \partial_\theta I^\uparrow - \frac{1}{\sin \theta} \partial_\phi I^\uparrow \right) = 0 \quad (33)
\end{align*}
\]

Here the Killing vector $\chi = \partial_t$ is sufficient for this static black holes [7]. It plays the role of the Kodama vector for the dynamical ones.

To solve the above system, we use another ansatz for the action $I^\uparrow$ (using the spherical symmetry):

\[
I^\uparrow = \int Edt + R(r) + J(\theta, \phi) + C \quad (34)
\]

where $C$ is a constant. This choice of $I^\uparrow$ leads to the following equations:

\[
\begin{align*}
    t & : \quad -e^{-\nu/2}\Lambda \partial_t I^\uparrow + ie^{-\lambda/2}\Xi \partial_r I^\uparrow = 0 \quad (35) \\
    r & : \quad \Xi \left( i \partial_\theta I^\uparrow - \frac{1}{\sin \theta} \partial_\phi I^\uparrow \right) = 0 \quad (36) \\
    \theta & : \quad e^{-\nu/2}\Xi \partial_t I^\uparrow + ie^{-\lambda/2}\Lambda \partial_r I^\uparrow = 0 \quad (37) \\
    \phi & : \quad \Lambda \left( i \partial_\theta I^\uparrow - \frac{1}{\sin \theta} \partial_\phi I^\uparrow \right) = 0 \quad (38)
\end{align*}
\]

where:

\[
R'(r) = \frac{dR}{dr}, \quad J'_\theta(\theta, \phi) = \frac{\partial J(\theta, \phi)}{\partial \theta}, \quad J'_\phi(\theta, \phi) = \frac{\partial J(\theta, \phi)}{\partial \phi} \quad (39)
\]

Regarding Eqs. (36) and (38), one obtains the same results in the spin-down case [9]. They imply that $J(\theta, \phi)$ is a complex function. However, from the structure of Eqs. (35) and (37), we distinguish two cases:

1. $\Lambda = \pm i\Xi$, then we have:

\[
\left( \mp e^{-\nu/2}E + e^{-\lambda/2}R'(r) \right) \Xi = 0
\]

which implies that:

\[
R'(r) = \pm \frac{e^{-\nu/2}}{e^{-\lambda/2}E}
\]

2. $\Lambda = \pm \Xi$, then:

\[
R'(r) = 0
\]
The last equation corresponds to an absorbed incoming particle in the classical limit with probability $P_{\text{incident}} = 1$, while the first case describes the emission process with the probability:

$$\Lambda \propto e^{-2/\hbar \Im R(r)}$$ (40)

For that we need the imaginary part of the function $R(r)$. Using the fact that

$$\Im R(r) = \pm E \int \frac{r^4 \sigma(r)}{(r - r_+)(r - r_-) H(r)} dr$$ (41)

we distinguish two cases: one with two horizons and the other with one. Thus, if $\Omega_4 - \Omega_3 > 0$ and $\Omega_4 + \Omega_3 < 0$ and replacing $e^\nu$ by its expression, we can write:

$$\Im R(r) = \pm E \int \frac{r^4 \sigma(r)}{(r - r_+)(r - r_-) H(r)} dr$$ (42)

where

$$H(r) = r^2 + (r_+ + r_- - M)r + \frac{pM^2}{r_+ r_-} + pM^2$$ (43)

($r_+$ and $r_-$) are respectively ($r_1$ and $r_2$) or ($r_3$ and $r_4$).

Using the Residus theorem,

$$R(r) = 2i\pi [\text{Res } (R(r), r_+) + \text{Res } (R(r), r_-)]$$ (44)

one can easily find $\Im R(r)$ where one has two poles located at the horizons located at $r_+$ and $r_-$:

$$\Im R(r) = \pm \frac{2\pi E}{r_+ - r_-} \left[ \frac{r_+^4 \sigma(r_+)}{H(r_+)} - \frac{r_-^4 \sigma(r_-)}{H(r_-)} \right]$$

with

$$\sigma(r_+) = \sqrt{1 - \frac{p}{r_+^2} - \frac{8pM}{r_+^3} + \frac{9p^2}{r_+^4}}$$ (45)

$$\sigma(r_-) = \sqrt{1 - \frac{p}{r_-^2} - \frac{8pM}{r_-^3} + \frac{9p^2}{r_-^4}}$$ (46)

Finally, if we take into account Eq. (2), one can write:

$$\frac{E}{T} = \pm \frac{2}{h} \Im R(r)$$ (47)

Notice that one can distinguish two cases to get the expression of the Hawking temperature $T_{\text{int}}$ (resp. $T_{\text{ext}}$) generated by $r_-$ (resp. $r_+$) (see Figure 1 and Figure 2) respectively:

$$T_{\text{ext}} = \pm \frac{\hbar}{4\pi} \frac{(r_+ - r_-)(r_+^2 - \frac{M}{2})(r_-^2 - \frac{M}{2})}{r_+^4 \sigma(r_+)(r_-^2 - \frac{M}{2}) - r_-^4 \sigma(r_-)(r_+^2 - \frac{M}{2})}$$ (48)

and

$$T_{\text{int}} = \frac{1}{\sigma(r_-)} \left[ \frac{1}{r_-} - \frac{M}{2r_-^2} \right]$$ (49)
5. Conclusion
We conclude that Hamilton-jacobi method is useful to study black holes radiation via the tunnelling effect. The main result was that the static black hole starts evaporating when the space-time is already static and the corresponding Hawking temperature depends only on the black hole properties (mass) and the structure of the Lyra space-time.

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References
[1] Einstein A 1905 Ann. Phys (Germany) \textbf{17} 891
[2] Wald R M 1992 "General Relativity" (Chicago. Chicago university Press)
[3] Lyra G 1951 Math Z \textbf{54} 52
[4] Rahaman F, Ghosh A and Kalam M 2006 Nuovo Cim. B \textbf{121} 649-659
[5] Sen D K and Dunn K A 1971 J. Math. Physi \textbf{12} 578
[6] Di Criscenzo R, Vanzo L 2008 arXiv. hep-th \textbf{0803.0435}
[7] Di Criscenzo R, Hayward S A, Nadalini M, Vanzo L and Zerbini S 2010 Class. Quant. Grav, \textbf{27} 015006
[8] Li R, Ren J R and Shi D F 2008 arXiv. gr-qc \textbf{0812.4217}
[9] Hayward S A 1998 Class. Quant. Grav \textbf{15} 3147
[10] Bouhalouf H, Mebarki N, Aissaoui A, 2010 Proc. Int, Conf on Algerian Astronomy and Astrophysics (Algeria) vol 1295 (American Institute of Physics) pp 201-209
[11] Mebarki M and Bouhalouf H 2010 Proc. Int, Conf on Auresian Astronomy and Astrophysics (Algeria) (African Skies / Cieux Africain Issue) p 61
[12] Aissaoui H, Mebarki M, Bouhalouf H, 2010 Proc. Int, Conf on Algerian Astronomy and Astrophysics (Algeria) vol 1295 (American Institute of Physics) pp164-175