C*-tensor categories and subfactors for totally disconnected groups

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Abstract

We associate a rigid C*-tensor category C to a totally disconnected locally compact group G and a compact open subgroup K < G. We characterize when C has the Haagerup property or property (T), and when C is weakly amenable. When G is compactly generated, we prove that C is essentially equivalent to the planar algebra associated by Jones and Burstein to a group acting on a locally finite bipartite graph. We then concretely realize C as the category of bimodules generated by a hyperfinite subfactor.

1 Introduction

Rigid C*-tensor categories arise as representation categories of compact groups and compact quantum groups and also as (part of) the standard invariant of a finite index subfactor. They can be viewed as a discrete group like structure and this analogy has lead to a lot of recent results with a flavor of geometric group theory, see [PV14, NY15a, GJ15, NY15b, PSV15].

In this paper, we introduce a rigid C*-tensor category C canonically associated with a totally disconnected locally compact group G and a compact open subgroup K < G. Up to Morita equivalence, C does not depend on the choice of K. The tensor category C can be described in several equivalent ways, see Section 2. Here, we mention that the representation category of K is a full subcategory of C and that the “quotient” of the fusion algebra of C by Rep K is the Hecke algebra of finitely supported functions on K\G/K equipped with the convolution product.

When G is compactly generated, we explain how the C*-tensor category C is related to the planar algebra P (i.e. standard invariant of a subfactor) associated in [J98, B10] with a locally finite bipartite graph G and a closed subgroup G< Aut(G). At the same time, we prove that these planar algebras P can be realized by a hyperfinite subfactor.

Given a finite index subfactor N ⊂ M, the notions of amenability, Haagerup property and property (T) for its standard invariant G_{N,M} were introduced by Popa in [P94a, P99, P01] in terms of the associated symmetric enveloping algebra T ⊂ S (see [P94a, P99]) and shown to only depend on G_{N,M}. Denoting by C the tensor category of M-M-bimodules generated by the subfactor, these properties were then formulated in [PV14] intrinsically in terms of C, and in particular directly in terms of G_{N,M}. We recall these definitions and equivalent formulations in Section 4. Similarly, weak amenability and the corresponding Cowling-Haagerup constant for the standard invariant G_{N,M} of a subfactor N ⊂ M were first defined in terms of the symmetric enveloping inclusion in [Br14] and then intrinsically for rigid C*-tensor categories in [PV14], see Section 5. Reinterpreting [DFY13, A14], it was proved in [PV14] that the representation

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category of SU_q(2) (and thus, the Temperley-Lieb-Jones standard invariant) is weakly amenable and has the Haagerup property, while the representation category of SU_q(3) has property (T).

For the C*-tensor categories C that we associate to a totally disconnected group G, we characterize when C has the Haagerup property or property (T) and when C is weakly amenable. We give several examples and counterexamples, in particular illustrating that the Haagerup property/weak amenability of G is not sufficient for C to have the Haagerup property or to be weakly amenable. Even more so, when C is the category associated with $G = \text{SL}(2, \mathbb{Q}_p)$, then the subcategory $\text{Rep} \ K$ with $K = \text{SL}(2, \mathbb{Z}_p)$ has the relative property (T). When $G = \text{SL}(n, \mathbb{Q}_p)$ with $n \geq 3$, the tensor category C has property (T), but we also give examples of property (T) groups G such that C does not have property (T).

Our main technical tool is Ocneanu’s tube algebra [O93] associated with any rigid C*-tensor category, see Section 3. When C is the C*-tensor category of a totally disconnected group G, we prove that the tube algebra is isomorphic with a canonical dense *-subalgebra of $C_0(G) \ast_{\text{Ad}} G$, where G acts on G by conjugation. We can therefore express the above mentioned approximation and rigidity properties of the tensor category C in terms of G and the dynamics of the conjugation action $G \rtimes_{\text{Ad}} G$.

In this paper, all locally compact groups are assumed to be second countable. We call totally disconnected group every second countable, locally compact, totally disconnected group.

2 C*-tensor categories of totally disconnected groups

Throughout this section, fix a totally disconnected group G. For all compact open subgroups $K_1, K_2 < G$, we define

$C_1$ : the category of $K_1$-$K_2$-$L^\infty(G)$-modules, i.e. Hilbert spaces $H$ equipped with commuting unitary representations $(\lambda(k_1))_{k_1 \in K_1}$ and $(\rho(k_2))_{k_2 \in K_2}$ and with a normal *-representation $\Pi : L^\infty(G) \to B(H)$ that are equivariant with respect to the left translation action $K_1 \curvearrowright G$ and the right translation action $K_2 \curvearrowleft G$;

$C_2$ : the category of $K_1$-$L^\infty(G/K_2)$-modules, i.e. Hilbert spaces $H$ equipped with a unitary representation $(\pi(k_1))_{k_1 \in K_1}$ and a normal *-representation $\Pi : L^\infty(G/K_2) \to B(H)$ that are covariant with respect to the left translation action $K_1 \curvearrowright G/K_2$;

$C_3$ : the category of $G$-$L^\infty(G/K_1)$-$L^\infty(G/K_2)$-modules, i.e. Hilbert spaces $H$ equipped with a unitary representation $(\pi(g))_{g \in G}$ and with an $L^\infty(G/K_1)$-$L^\infty(G/K_2)$-bimodule structure that are equivariant with respect to the left translation action of G on $G/K_1$ and $G/K_2$;

and with morphisms given by bounded operators that intertwine the given structure.

Let $K_3 < G$ also be a compact open subgroup. We define the tensor product $H \otimes_{K_2} K$ of a $K_1$-$K_2$-$L^\infty(G)$-module $H$ and a $K_2$-$K_3$-$L^\infty(G)$-module $K$ as the Hilbert space $H \otimes_{K_2} K = \{ \xi \in H \otimes K \mid (\rho(k_2) \otimes \lambda(k_2))\xi = \xi \text{ for all } k_2 \in K_2 \}$ equipped with the unitary representations $(\lambda(k_1) \otimes 1)_{k_1 \in K_1}$ and $(1 \otimes \rho(k_3))_{k_3 \in K_3}$ and with the representation $(\Pi_H \otimes \Pi_K) \circ \Delta$ of $L^\infty(G)$, where $\Delta : L^\infty(G) \to L^\infty(G) \otimes L^\infty(G)$ is the comultiplication given by $(\Delta(F))(g, h) = F(gh)$ for all $g, h \in G$. 

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The tensor product of a $G$-$L^\infty(G/K_1)$-$L^\infty(G/K_2)$-module $\mathcal{H}$ and a $G$-$L^\infty(G/K_2)$-$L^\infty(G/K_3)$-module $\mathcal{K}$ is denoted as $\mathcal{H} \otimes_{L^\infty(G/K_2)} \mathcal{K}$ and defined as the Hilbert space

$$\mathcal{H} \otimes_{L^\infty(G/K_2)} \mathcal{K} = \{ \xi \in \mathcal{H} \otimes \mathcal{K} \mid \xi(1_{gK_2} \otimes 1) = (1 \otimes 1_{gK_2})\xi \text{ for all } gK_2 \in G/K_2 \}$$

with the unitary representation $(\pi_\mathcal{H}(g) \otimes \pi_\mathcal{K}(g))_{g \in G}$ and with the $L^\infty(G/K_1)$-$L^\infty(G/K_3)$-bimodule structure given by the left action of $1_{gK_1} \otimes 1$ for $gK_1 \in G/K_1$ and the right action of $1 \otimes 1_{hK_3}$ for $hK_3 \in G/K_3$.

We say that objects $\mathcal{H}$ are of finite rank $\mathcal{C}_1$ : if $\mathcal{H}_{K_2} := \{ \xi \in \mathcal{H} \mid \rho(k_2)\xi = \xi \text{ for all } k_2 \in K_2 \}$ is finite dimensional ; as we will see in the proof of Proposition 2.2, this is equivalent with requiring that $K,\mathcal{H}$ is finite dimensional ;

$\mathcal{C}_2$ : if $\mathcal{H}$ is finite dimensional ;

$\mathcal{C}_3$ : if $1_{cK_1} \cdot \mathcal{H}$ is finite dimensional ; as we will see in the proof of Proposition 2.2, this is equivalent with requiring that $\mathcal{H} \cdot 1_{cK_2}$ is finite dimensional.

Altogether, we get that $\mathcal{C}_1$ and $\mathcal{C}_3$ are $C^*$-2-categories. In both cases, the 0-cells are the compact open subgroups of $G$. For all compact open subgroups $K_1, K_2 < G$, the 1-cells are the categories $\mathcal{C}_i(K_1, K_2)$ defined above and $\mathcal{C}_i(K_1, K_2) \times \mathcal{C}_i(K_2, K_3) \to \mathcal{C}_i(K_1, K_3)$ is given by the tensor product operation that we just introduced. Restricting to finite rank objects, we get rigid $C^*$-2-categories.

Another typical example of a $C^*$-2-category is given by Hilbert bimodules over $\mathbb{II}_1$ factors: the 0-cells are $\mathbb{II}_1$ factors, the 1-cells are the categories $\text{Bimod}_{M_1-M_2}$ of Hilbert $M_1-M_2$-bimodules and $\text{Bimod}_{M_1-M_2} \times \text{Bimod}_{M_2-M_3} \to \text{Bimod}_{M_1-M_3}$ is given by the Connes tensor product. Again, restricting to finite index bimodules, we get a rigid $C^*$-2-category.

**Remark 2.1.** The standard invariant of an extremal finite index subfactor $N \subset M$ can be viewed as follows as a rigid $C^*$-2-category. There are only two 0-cells, namely $N$ and $M$; the 1-cells are the $N$-$N$, $N$-$M$, $M$-$N$ and $M$-$M$-bimodules generated by the subfactor; and we are given a favorite and generating 1-cell from $N$ to $M$, namely the $N$-$M$-bimodule $L^2(M)$.

Abstractly, a rigid $C^*$-2-category $\mathcal{C}$ with only two 0-cells (say + and −), irreducible tensor units in $\mathcal{C}_{++}$ and $\mathcal{C}_{--}$, and a given generating object $\mathcal{H} \in \mathcal{C}_{+-}$ is exactly the same as a standard $\lambda$-lattice in the sense of Popa [P94b, Definitions 1.1 and 2.1]. Indeed, for every $n \geq 0$, define $\mathcal{H}_{+n}$ as the $n$-fold alternating tensor product of $\mathcal{H}$ and $\mathcal{H}$ starting with $\mathcal{H}$. Similarly, define $\mathcal{H}_{-n}$ by starting with $\mathcal{H}$. For $0 \leq j$, define $A_{ij} = \text{End}(\mathcal{H}_{+j})$. When $0 \leq i \leq j < \infty$, define $A_{ij} \subset A_{0j}$ as $A_{ij} := 1^i \otimes \text{End}(\mathcal{H}_{(-1)^{i}j^{1-i}})$ viewed as a subalgebra of $A_{0j} = \text{End}(\mathcal{H}_{+j})$ by writing $\mathcal{H}_{+j} = \mathcal{H}_{+j} \mathcal{H}_{(-1)^{j}} \mathcal{H}_{(-1)^{i}j^{1-i}}$. The standard solutions for the conjugate equations (see Section 3) give rise to canonical projections $e_{+} \in \text{End}(\mathcal{H}\mathcal{H})$ and $e_{-} \in \text{End}(\mathcal{H}\mathcal{H})$ given by

$$e_{+} = d(\mathcal{H})^{-1}s_{\mathcal{H}}s_{\mathcal{H}}^{\ast}$$
$$e_{-} = d(\mathcal{H})^{-1}t_{\mathcal{H}}t_{\mathcal{H}}^{\ast},$$

and thus to a representation of the Jones projections $e_{j} \in A_{kl}$ (for $k < j < l$). Finally, if we equip all $A_{ij}$ with the normalized categorical trace, we have defined a standard $\lambda$-lattice in the sense of [P94b, Definitions 1.1 and 2.1]. Given two rigid $C^*$-2-categories with fixed generating objects as above, it is straightforward to check that the associated standard $\lambda$-lattices are isomorphic if and only if there exists an equivalence of $C^*$-2-categories preserving the generators. Conversely given a standard $\lambda$-lattice $\mathcal{G}$, by [P94b, Theorem 3.1], there exists
an extremal subfactor \( N \subset M \) whose standard invariant is \( G \) and we can define \( \mathcal{C} \) as the C*-2-category of the subfactor \( N \subset M \), generated by the \( N\cdot M \)-bimodule \( L^2(M) \) as in the beginning of this remark. One can also define \( \mathcal{C} \) directly in terms of \( G \) (see e.g. [MPS08, Section 4.1] for a planar algebra version of this construction).

Thus, also subfactor planar algebras in the sense of [J99] are “the same” as rigid C*-2-categories with two 0-cells and such a given generating object \( \mathcal{H} \in \mathcal{C}_{+\ldots} \).

For more background on rigid C*-tensor categories, we refer to [NT13].

**Proposition 2.2.** The C*-2-categories \( \mathcal{C}_1 \) and \( \mathcal{C}_3 \) are naturally equivalent. In particular, fixing \( K_1 = K_2 = K \), we get the naturally equivalent rigid C*-tensor categories \( \mathcal{C}_{1,i}(K < G) \) and \( \mathcal{C}_{3,i}(K < G) \). Up to Morita equivalence\(^3\), these do not depend on the choice of compact open subgroup \( K < G \).

**Proof.** Using the left and right translation operators \( \lambda_g \) and \( \rho_g \) on \( L^2(G) \), one checks that the following formulæ define natural equivalences and their inverses between the categories \( \mathcal{C}_1 \), \( \mathcal{C}_2 \) and \( \mathcal{C}_3 \).

- \( \mathcal{C}_1 \to \mathcal{C}_2 : \mathcal{H} \mapsto \mathcal{H}_{K_2} \), where \( \mathcal{H}_{K_2} \) is the space of right \( K_2 \)-invariant vectors and where the \( K_1 \)-L\( ^\infty \)-\( (G/K_2) \)-module structure on \( \mathcal{H}_{K_2} \) is given by restricting the corresponding structure on \( \mathcal{H} \).
- \( \mathcal{C}_2 \to \mathcal{C}_1 : \mathcal{H} \mapsto \mathcal{H} \otimes_{L^\infty(G/K_2)} L^2(G) \) given by
  \[
  \{ \xi \in \mathcal{H} \otimes L^2(G) \mid (1_{gK_2} \otimes 1)\xi = (1 \otimes 1_{gK_2})\xi \text{ for all } g \in G \} = \bigoplus_{g \in G/K_2} 1_{gK_2} \cdot \mathcal{H} \otimes L^2(gK_2)
  \]
  and where the \( K_1 \)-\( K_2 \)-L\( ^\infty \)-\( G \)-module structure is given by \( (\lambda_G(k_1) \otimes \lambda_{k_1})_{k_1 \in K_1}, (1 \otimes \rho_{k_2})_{k_2 \in K_2} \) and multiplication with \( 1 \otimes F \) when \( F \in L^\infty(G) \).
- \( \mathcal{C}_3 \to \mathcal{C}_2 : \mathcal{H} \mapsto 1_{eK_1} \cdot \mathcal{H} \) and where the \( K_1 \)-L\( ^\infty \)-\( (G/K_2) \)-module structure on \( 1_{eK_1} \cdot \mathcal{H} \) is given by restricting the corresponding structure on \( \mathcal{H} \).
- \( \mathcal{C}_2 \to \mathcal{C}_3 : \mathcal{H} \mapsto L^2(G) \otimes_{K_1} \mathcal{H} = \{ \xi \in L^2(G) \otimes \mathcal{H} \mid (\rho_{k_1} \otimes \pi(k_1))\xi = \xi \text{ for all } k_1 \in K_1 \} \) and where the \( G \)-L\( ^\infty \)-\( (G/K_1) \)-L\( ^\infty \)-\( (G/K_2) \)-module structure is given by \( (\lambda_g \otimes 1)_{g \in G} \), multiplication with \( 1 \otimes F \) for \( F \in L^\infty(G/K_1) \) and multiplication with \( (\text{id} \otimes \Pi)\Delta(F) \) for \( F \in L^\infty(G/K_2) \).

By definition, if \( \mathcal{H} \in \mathcal{C}_1 \) has finite rank, the Hilbert space \( \mathcal{H}_{K_2} \) is finite dimensional. Conversely, if \( K \in \mathcal{C}_2 \) and \( K \) is a finite dimensional Hilbert space, then the corresponding object \( \mathcal{H} \in \mathcal{C}_1 \) has the property that both \( K \cdot \mathcal{H} \) and \( \mathcal{H}_{K_2} \) are finite dimensional. Therefore, \( \mathcal{H} \in \mathcal{C}_1 \) has finite rank if and only if \( K_1 \mathcal{H} \) is a finite dimensional Hilbert space. A similar reasoning holds for objects in \( \mathcal{C}_3 \).

It is straightforward to check that the resulting equivalence \( \mathcal{C}_1 \leftrightarrow \mathcal{C}_3 \) preserves tensor products, so that we have indeed an equivalence between the C*-2-categories \( \mathcal{C}_1 \) and \( \mathcal{C}_3 \).

To prove the final statement in the proposition, it suffices to observe that for all compact open subgroups \( K_1, K_2 < G \), we have that \( L^2(K_1, K_2) \) is a nonzero finite rank \( G \)-\( K_1 \cdot K_2 \)-L\( ^\infty \)-\( G \)-module and that \( L^2(G/(K_1 \cap K_2)) \) is a nonzero finite rank \( G \)-L\( ^\infty \)-\( (G/K_1) \)-L\( ^\infty \)-\( (G/K_2) \)-module, so that \( \mathcal{C}_{1,i}(K_1 < G) \) and \( \mathcal{C}_{3,i}(K_2 < G) \) are Morita equivalent for \( i = 1, 3 \). \( \square \)

\(^3\)In the sense of [M01, Section 4], where the terminology weak Morita equivalence is used; see also [PSV15, Definition 7.3] and [NY15b, Section 3].
The rigid C*-2-categories \( C_1 \) and \( C_2 \) can as follows be fully faithfully embedded in the category of bimodules over the hyperfinite \( II_1 \) factor. We construct this embedding in an extremal way in the sense of subfactors (cf. Corollary 2.4).

To do so, given a totally disconnected group \( G \), we fix a continuous action \( G \curvearrowleft \alpha P \) of \( G \) on the hyperfinite \( II_\infty \) factor \( P \) that is strictly outer in the sense of [V03, Definition 2.1]: the relative commutant \( P' \cap P \rtimes G \) equals \( \mathbb{C}1 \). Moreover, we should choose this action in such a way that \( \text{Tr} \circ \alpha_g = \Delta(g)^{-1/2} \text{Tr} \) for all \( g \in G \) (where \( \Delta \) is the modular function on \( G \)) and such that there exists a projection \( p \in P \) of finite trace with the property that \( \alpha_k(p) = p \) whenever \( k \) belongs to a compact subgroup of \( G \). Such an action indeed exists: write \( P = R_0 \boxtimes R_1 \) where \( R_0 \) is a copy of the hyperfinite \( II_1 \) factor and \( R_1 \) is a copy of the hyperfinite \( II_\infty \) factor. Choose a continuous trace scaling action \( \mathbb{R}^+_0 \curvearrowleft \alpha \) \( R_1 \). By [V03, Corollary 5.2], we can choose a strictly outer action \( G \curvearrowleft \alpha R_0 \). We then define \( \alpha_g = (\alpha_0)_g \circ (\alpha_1)_{\Delta(g)}^{-1/2} \) and we take \( p = 1 \boxtimes p_1 \), where \( p_1 \in R_1 \) is any projection of finite trace. Whenever \( k \) belongs to a compact subgroup of \( G \), we have \( \Delta(k) = 1 \) and thus \( \alpha_k(p) = p \).

Whenever \( K_1, K_2 < G \) are compact open subgroups of \( G \), we write

\[
[K_1 : K_2] = [K_1 : K_1 \cap K_2][K_2 : K_1 \cap K_2]^{-1}.
\]

Fixing a left Haar measure \( \lambda \) on \( G \), we have \( [K_1 : K_2] = \lambda(K_1)^{-1}\lambda(K_2) \). Therefore, we have that \( [K : gKg^{-1}] = \Delta(g) \) for all compact open subgroups \( K < G \) and all \( g \in G \).

**Theorem 2.3.** Let \( G \) be a totally disconnected group and choose a strictly outer action \( G \curvearrowleft \alpha P \) on the hyperfinite \( II_\infty \) factor \( P \) and a projection \( p \in P \) as above. For every compact open subgroup \( K < G \), write \( R(K) = (pPp)^K \). Then each \( R(K) \) is a copy of the hyperfinite \( II_1 \) factor.

To every \( K_1-K_2-L^\infty(G) \)-module \( \mathcal{H} \), we associate the Hilbert \( R(K_1)-R(K_2) \)-bimodule \( \mathcal{K} \) given by (2.1) below. Then \( \mathcal{H} \mapsto K \) is a fully faithful 2-functor. Also, \( \mathcal{H} \) has finite rank if and only if \( K \) is a finite index bimodule. In that case,

\[
\dim_{R(K_1)}(\mathcal{K}) = [K_1 : K_2]^{1/2} \dim_{C_1}(\mathcal{H}) \quad \text{and} \quad \dim_{R(K_2)}(\mathcal{K}) = [K_2 : K_1]^{1/2} \dim_{C_1}(\mathcal{H}),
\]

where \( \dim_{C_1}(\mathcal{H}) \) is the categorical dimension of \( \mathcal{H} \in C_1 \).

**Proof.** Given a \( K_1-K_2-L^\infty(G) \)-module \( \mathcal{H} \), turn \( \mathcal{H} \otimes L^2(P) \) into a Hilbert \( (P \rtimes K_1)-(P \rtimes K_2) \)-bimodule via

\[
\begin{align*}
  u_k \cdot (\xi \otimes b) \cdot u_r &= \lambda(k)\rho(r)^*\xi \otimes \alpha^{-1}(b) & \text{for all} \ k, r \in K_1, \xi \in \mathcal{H}, b \in L^2(P), \\
  a \cdot \xi \cdot d &= (\Pi \otimes \text{id})\alpha(a)\xi (1 \otimes d) & \text{for all} \ a, d \in P, \xi \in \mathcal{H} \otimes L^2(P),
\end{align*}
\]

where \( \alpha : P \to L^\infty(G) \otimes P \) is given by \( (\alpha(a))(g) = \alpha^{-1}_g(a) \).

Whenever \( K < G \) is a compact open subgroup, we define the projection \( p_K \in L(G) \) given by

\[
p_K = \lambda(K)^{-1}\int_K \lambda_k dk.
\]

We also write \( e_K = pp_K \) viewed as a projection in \( P \rtimes K \). Since \( P \subset P \rtimes K \subset P \rtimes G \), we have that \( P' \cap (P \rtimes K) = \mathbb{C}1 \), so that \( P \rtimes K \) is a factor. So, \( P \rtimes K \) is a copy of the hyperfinite \( II_\infty \) factor and \( e_K \in P \rtimes K \) is a projection of finite trace. We identify \( R(K) = e_K(P \rtimes K)e_K \) through the bijective \(*\)-isomorphism \( (pP)p)^K \to e_K(P \rtimes K)e_K : a \mapsto ap_K \). In particular, \( R(K) \) is a copy of the hyperfinite \( II_1 \) factor.
So, for every $K_1$-$K_2$-$L^\infty(G)$-module $\mathcal{H}$, we can define the $R(K_1)$-$R(K_2)$-bimodule

$$\mathcal{K} = e_{K_1} \cdot (\mathcal{H} \otimes L^2(P)) \cdot e_{K_2}. \tag{2.1}$$

We claim that $\text{End}_{R(K_1)-R(K_2)}(\mathcal{K}) = \text{End}_{c_1}(\mathcal{H})$ naturally. More concretely, we have to prove that

$$\text{End}_{(P \rtimes K_1)-(P \rtimes K_2)}(\mathcal{H} \otimes L^2(P)) = \text{End}_{c_1}(\mathcal{H}) \otimes 1, \tag{2.2}$$

where $\text{End}_{c_1}(\mathcal{H})$ consists of all bounded operators on $\mathcal{H}$ that commute with $\lambda(K_1)$, $\rho(K_2)$ and $\Pi(L^\infty(G))$. To prove (2.2), it is sufficient to show that

$$\text{End}_{P-P}(\mathcal{H} \otimes L^2(P)) = \Pi(L^\infty(G))' \otimes 1. \tag{2.3}$$

Note that the left hand side of (2.3) equals $(\Pi \otimes \text{id})\alpha(P)' \cap B(\mathcal{H}) \overline{\otimes} P$. Assume that $T \in (\Pi \otimes \text{id})\alpha(P)' \cap B(\mathcal{H}) \overline{\otimes} P$. In the same was as in [V03, Proposition 2.7], it follows that $T \in \Pi(L^\infty(G))' \cap 1$. For completeness, we provide a detailed argument. Define the unitary $W \in L^\infty(G) \overline{\otimes} L(G)$ given by $W(g) = \lambda_g$. We view both $T$ and $(\Pi \otimes \text{id})(W)$ as elements in $B(\mathcal{H}) \overline{\otimes} (P \times G)$. For all $a \in P$, we have

$$(\Pi \otimes \text{id})(W) T (\Pi \otimes \text{id})(W)^* (1 \otimes a) = (\Pi \otimes \text{id})(W) T (\Pi \otimes \text{id})\alpha(a) (\Pi \otimes \text{id})(W)^* = (1 \otimes a) (\Pi \otimes \text{id})(W) T (\Pi \otimes \text{id})(W)^* .$$

Since the action $\alpha$ is strictly outer, we conclude that $(\Pi \otimes \text{id})(W) T (\Pi \otimes \text{id})(W)^* = S \otimes 1$ for some $S \in B(\mathcal{H})$. So,

$$T = (\Pi \otimes \text{id})(W)^* (S \otimes 1) (\Pi \otimes \text{id})(W) .$$

The left hand side belongs to $B(\mathcal{H}) \overline{\otimes} P$, while the right hand side belongs to $B(\mathcal{H}) \otimes L(G)$, and both are viewed inside $B(\mathcal{H}) \overline{\otimes} (P \times G)$. Since $P \cap L(G) = \mathbb{C}1$, we conclude that $T = T_0 \otimes 1$ for some $T_0 \in B(\mathcal{H})$ and that

$$T_0 \otimes 1 = (\Pi \otimes \text{id})(W)^* (S \otimes 1) (\Pi \otimes \text{id})(W) .$$

Defining the normal $\ast$-homomorphism $\Psi : L(G) \to L(G) \overline{\otimes} L(G)$ given by $\Psi(\lambda_g) = \lambda_g \ast \lambda_g$ for all $g \in G$, we apply id $\otimes \Psi$ and conclude that

$$T_0 \otimes 1 \otimes 1 = (\Pi \otimes \text{id})(W)^*_{13} (\Pi \otimes \text{id})(W)^*_{12} (S \otimes 1) (\Pi \otimes \text{id})(W)_{12} (\Pi \otimes \text{id})(W)_{13} = (\Pi \otimes \text{id})(W)^*_{13} (T_0 \otimes 1 \otimes 1) (\Pi \otimes \text{id})(W)_{13} .$$

It follows that $T_0$ commutes with $\Pi(L^\infty(G))$ and (2.2) is proven.

It is easy to check that $\mathcal{H} \mapsto \mathcal{K}$ naturally preserves tensor products. So, we have found a fully faithful 2-functor from $\mathcal{C}_1$ to the $C^*$-2-category of Hilbert bimodules over hyperfinite $\Pi_1$ factors.

To compute $\dim_{-R(K_2)}(\mathcal{K})$, observe that for all $k \in K_1$, $r \in K_2$ and $g \in G$, we have $\alpha_{kr}(p) = \alpha_k(p) = \alpha_g(\alpha_{g^{-1}k}(p)) = \alpha_g(p)$. Therefore, as a right $(P \rtimes K_2)$-module, we have

$$e_{K_1} \cdot (\mathcal{H} \otimes L^2(P)) \cong \bigoplus_{g \in K_1 \setminus G/K_2} (\mathcal{L}_g \otimes L^2(p_g P)) ,$$

where $p_g = \alpha_g^{-1}(p)$, where the Hilbert space $\mathcal{L}_g := \Pi(1_{K_1} g K_2)(K_1, \mathcal{H})$ comes with the unitary representation $(\rho(r))_{r \in K_2}$ and where the right $(P \rtimes K_2)$-module structure on $\mathcal{L}_g \otimes L^2(p_g P)$ is given by

$$(\xi \otimes b) \cdot (du_r) = \rho(r)^* \xi \otimes \alpha_r^{-1}(bd) \quad \text{for all } \xi \in \mathcal{L}_g, b \in L^2(p_g P), d \in P, r \in K_2 .$$
Since $p_gP_g \ltimes K_2 = p_g(P \ltimes K_2)p_g$ is a factor (actually, $K_2 \simeq p_gP_g$ is a so-called minimal action), it follows from [W88, Theorem 12] that there exists a unitary $V_g \in B(L_g) \otimes p_gP_g$ satisfying

$$(\text{id} \otimes \alpha_r)(V_g) = V_g(\rho(r) \otimes 1) \quad \text{for all} \quad r \in K_2.$$ 

Then left multiplication with $V_g$ intertwines the right $(P \ltimes K_2)$-module structure on the Hilbert space $L_g \otimes L^2(p_gP)$ with the right $(P \ltimes K_2)$-module structure given by

$$(\xi \otimes b) \cdot (du_r) = \xi \otimes \alpha_{r}^{-1}(bd) \quad \text{for all} \quad \xi \in L_g, b \in L^2(p_gP), d \in P, r \in K_2.$$

Therefore,

$$\dim_{-R(K_2)}L_g \otimes L^2(p_gP) \cdot e_{K_2} = \dim(L_g) \dim_{-(p_gP)K_2}(L^2(p_gP^{K_2}p)) = \dim(L_g) \frac{\text{Tr}(p_g)}{\text{Tr}(p)} = \dim(L_g) \Delta(g)^{1/2}.$$ 

So, we have proved that

$$\dim_{-R(K_2)}(K) = \sum_{g \in K_1 \setminus G/K_2} \dim(\Pi(1_{K_1gK_2})(K_1\mathcal{H})) \Delta(g)^{1/2}.$$ 

We similarly get that

$$\dim_{R(K_1)-}(K) = \sum_{g \in K_1 \setminus G/K_2} \dim(\Pi(1_{K_1gK_2})(\mathcal{H}K_2)) \Delta(g)^{-1/2}.$$ 

To make the connection with the categorical dimension of $\mathcal{H}$, it is useful to view $\mathcal{H}$ as the image of a $G$-$L^\infty(G/K_1)$-$L^\infty(G/K_2)$-module $\mathcal{H}'$ under the equivalence of Proposition 2.2. This means that we can view $\mathcal{H}$ as the space of $L^2$-functions $\xi : G \to \mathcal{H}'$ with the property that $\xi(g) \in 1_{eK_1} \cdot \mathcal{H}' \cdot 1_{gK_2}$ for a.e. $g \in G$. The $L^\infty(G)$-module structure of $\mathcal{H}$ is given by pointwise multiplication, while the $K_1$-$K_2$-module structure on $\mathcal{H}$ is given by

$$(k \cdot \xi \cdot r)(g) = \pi(k)\xi(k^{-1}gr^{-1}) \quad \text{for all} \quad k \in K_1, r \in K_2, g \in G.$$ 

With this picture, it is easy to see that

$$\Pi(1_{K_1gK_2})(\mathcal{H}K_2) \cong 1_{eK_1} \cdot \mathcal{H}' \cdot 1_{gK_2}.$$ 

The map $\xi \mapsto \tilde{\xi}$ with $\tilde{\xi}(g) = \pi(g)^*\xi(g)$ is an isomorphism between $\mathcal{H}$ and the space of $L^2$-functions $\eta : G \to \mathcal{H}'$ with the property that $\eta(g) \in 1_{g^{-1}K_1} \cdot \mathcal{H}' \cdot 1_{K_2}$ for a.e. $g \in G$. The $L^\infty(G)$-module structure is still given by pointwise multiplication, while the $K_1$-$K_2$-module structure is now given by

$$(k \cdot \eta \cdot r)(g) = \pi(r)^*\eta(k^{-1}gr^{-1}).$$ 

In this way, we get that

$$\Pi(1_{K_1gK_2})(K_1\mathcal{H}) \cong 1_{K_2g^{-1}K_1} \cdot \mathcal{H}' \cdot 1_{eK_2}.$$ 

It thus follows that

$$\dim_{-R(K_2)}(K) = \sum_{g \in K_1 \setminus G/K_2} \dim(1_{K_2g^{-1}K_1} \cdot \mathcal{H}' \cdot 1_{eK_2}) \Delta(g)^{1/2} \quad \text{and} \quad (2.4)$$

$$\dim_{R(K_1)-}(K) = \sum_{g \in K_1 \setminus G/K_2} \dim(1_{eK_1} \cdot \mathcal{H}' \cdot 1_{K_1gK_2}) \Delta(g)^{-1/2}. \quad (2.5)$$
Also note that for every $g \in G$, we have
\[
\dim(1_{K_{2g^{-1}K_1}} \cdot \mathcal{H}^' \cdot 1_{eK_2}) = [K_2 : g^{-1}K_1 \cap gK_2] \dim(1_{g^{-1}K_1} \cdot \mathcal{H}^' \cdot 1_{eK_2})
\]
\[
= [K_2 : g^{-1}K_1 \cap gK_2] \dim(1_{eK_1} \cdot \mathcal{H}^' \cdot 1_{gK_2})
\]
\[
= [K_2 : g^{-1}K_1 \cap gK_2] \dim(1_{eK_1} \cdot \mathcal{H}^' \cdot 1_{1}gK_2)
\]
\[
= [K_2 : K_1] \Delta(g)^{-1} \dim(1_{eK_1} \cdot \mathcal{H}^' \cdot 1_{1}gK_2).
\]

It follows that
\[
\dim_{R(K_2)}(\mathcal{K}) = [K_2 : K_1] \sum_{g \in K_1 \setminus G/K_2} \dim(1_{eK_1} \cdot \mathcal{H}^' \cdot 1_{1}gK_2) \Delta(g)^{-1/2}
\]
\[
= [K_2 : K_1] \dim_{R(K_1)^{-1}}(\mathcal{K}).
\]

If $\mathcal{H}$ has finite rank, also $\mathcal{H}^'$ has finite rank so that $1_{eK_1} \cdot \mathcal{H}^' \cdot 1_{1}gK_2$ are finite dimensional Hilbert spaces. It then follows that $\mathcal{K}$ is a finite index bimodule.

Conversely, assume that $\mathcal{K}$ has finite index. For every $g \in G$, write
\[
\kappa(g) := \dim(1_{K_{2g^{-1}K_1}} \cdot \mathcal{H}^' \cdot 1_{eK_2}) \Delta(g)^{1/2} = [K_2 : K_1] \dim(1_{eK_1} \cdot \mathcal{H}^' \cdot 1_{1}gK_2) \Delta(g)^{-1/2}.
\]

So,
\[
\kappa(g)^2 = [K_2 : K_1] \dim(1_{K_{2g^{-1}K_1}} \cdot \mathcal{H}^' \cdot 1_{eK_2}) \dim(1_{eK_1} \cdot \mathcal{H}^' \cdot 1_{1}gK_2).
\]

Thus, whenever $\kappa(g) \neq 0$, we have that $\kappa(g) \geq [K_2 : K_1]^{1/2}$. Since
\[
\dim_{R(K_2)}(\mathcal{K}) = \sum_{g \in K_1 \setminus G/K_2} \kappa(g),
\]
we conclude that there are only finitely many $g \in K_1 \setminus G/K_2$ for which $1_{K_{2g^{-1}K_1}} \cdot \mathcal{H}^' \cdot 1_{eK_2}$ is nonzero and for each of them, it is a finite dimensional Hilbert space. This implies that $\mathcal{H}^' \cdot 1_{eK_2}$ is finite dimensional, so that $\mathcal{H}^'$ has finite rank.

We have proved that $\mathcal{H} \mapsto \mathcal{K}$ is a fully faithful 2-functor from $\mathcal{C}_{1,f}$ to the finite index bimodules over hyperfinite $\Pi_1$ factors. Moreover, for given compact open subgroups $K_1, K_2 < G$, the ratio between $\dim_{R(K_1)^{-1}}(\mathcal{K})$ and $\dim_{R(K_2)}(\mathcal{K})$ equals $[K_1 : K_2]$ for all finite rank $K_1 \cdot K_2$-modules $\mathcal{H}$. Since the functor is fully faithful, this then also holds for all $R(K_1)$-$R(K_2)$-subbimodules of $\mathcal{K}$. It follows that the categorical dimension of $\mathcal{K}$ equals
\[
[K_2 : K_1]^{1/2} \dim_{R(K_1)}(\mathcal{K}) = [K_1 : K_2]^{1/2} \dim_{R(K_2)}(\mathcal{K}).
\]

Since the functor is fully faithful, the categorical dimensions of $\mathcal{H} \in \mathcal{C}_{1,f}$ and $\mathcal{K} \in \text{Bimod}_{f}$ coincide, so that
\[
[K_2 : K_1]^{1/2} \dim_{R(K_1)}(\mathcal{K}) = \dim_{\mathcal{C}_{1}}(\mathcal{H}) = [K_1 : K_2]^{1/2} \dim_{R(K_2)}(\mathcal{K}). \tag{2.6}
\]

\[\square\]

**Corollary 2.4.** Let $G$ be a totally disconnected group with compact open subgroups $K_\pm < G$ and assume that $\mathcal{H}$ is a finite rank $G\cdot L^\infty(\mathcal{G}/K_\pm)\cdot L^\infty(\mathcal{G}/K_\pm)$-module. Denote by $\mathcal{C} = (\mathcal{C}_+, \mathcal{C}_-, \mathcal{C}_+, \mathcal{C}_-)$ the $C^*$-2-category of $G\cdot L^\infty(\mathcal{G}/K_\pm)\cdot L^\infty(\mathcal{G}/K_\pm)$-modules (with 0-cells $K_+$ and $K_-$) generated by the alternating tensor products of $\mathcal{H}$ and its adjoint.

Combining Proposition 2.2 and Theorem 2.3, we find an extremal hyperfinite subfactor $N \subset M$ whose standard invariant, viewed as the $C^*$-2-category of $N\cdot N$, $N\cdot M$, $M\cdot N$ and $M\cdot M$-bimodules generated by the $N\cdot M$-bimodule $L^2(M)$, is equivalent with $(\mathcal{C}, \mathcal{H})$ (cf. Remark 2.1).
Proof. A combination of Proposition 2.2 and Theorem 2.3 provides the finite index $R(K_+)$-$R(K_-)$-bimodule $\mathcal{H}$ associated with $\mathcal{H}$. Take nonzero projections $p_\pm \in R(K_\pm)$ such that writing $N = p_+ R(K_+ p_+)$ and $M = p_- R(K_- p_-)$, we have that $\dim_{-\mathcal{H}}(p_+ \cdot \mathcal{H} \cdot p_-) = 1$. We then view $N \subset M$ in such a way that $L^2(M) \cong p_+ \cdot \mathcal{H} \cdot p_-$. The $C^*$-category of $N-N$, $N-M$, $M-N$ and $M-M$-bimodules generated by the $N-M$-bimodule $L^2(M)$ is by construction equivalent with the rigid $C^*$-category of $R(K_+)$-$R(K_-)$-bimodules generated by $\mathcal{H}$. Since the 2-functor in Theorem 2.3 is fully faithful, this $C^*$-2-category is equivalent with $\mathcal{C}$ and this equivalence maps the $N-M$-bimodule $L^2(M)$ to $\mathcal{H} \in \mathcal{C}_{++}$. □

From Corollary 2.4, we get the following result.

**Proposition 2.5.** Let $\mathcal{P}$ be the subfactor planar algebra of $[J98,B10]$ associated with a connected locally finite bipartite graph $G$, with edge set $\mathcal{E}$ and source and target maps $s : \mathcal{E} \to V_+$, $t : \mathcal{E} \to V_-$, together with a closed subgroup $G < \text{Aut}(\mathcal{G})$ acting transitively on $V_+$ as well as on $V_-$. Fix vertices $v_\pm \in V_\pm$ and write $K_\pm = \text{Stab}(v_\pm)$.

There exists an extremal hyperfinite subfactor $N \subset M$ whose standard invariant is isomorphic with $\mathcal{P}$. We have $[M : N] = \delta^2$ where

$$\delta = \sum_{w \in V_-} \#\{ e \in \mathcal{E} \mid s(e) = v_+, t(e) = w \} \frac{\text{Stab} w : \text{Stab} v_+}{\text{dim} \mathcal{H}}^{1/2} = \sum_{w \in V_+} \#\{ e \in \mathcal{E} \mid s(e) = w, t(e) = v_- \} \frac{\text{Stab} w : \text{Stab} v_-}{\text{dim} \mathcal{H}}^{1/2}.$$

Moreover, $\mathcal{P}$ can be described as the rigid $C^*$-2-category $\mathcal{C}_{3,f}(G,K_\pm,K_\mp)$ of all finite rank $G$-$L^\infty(G/K_-)$-$L^\infty(G/K_\pm)$-modules together with the generating object $\ell^2(\mathcal{E}) \in \mathcal{C}_{3,f}(G,K_+,K_-)$ (cf. Remark 2.1).

Proof. We are given $G \subset \mathcal{E}$ and $G \subset V_+ \subset V_- \subset V_-$ such that the source and target maps $s,t$ are $G$-equivariant and such that $G$ acts transitively on $V_+$ and on $V_-$. Put $K_\pm = \text{Stab} v_\pm$ and note that $K_\pm < G$ are compact open subgroups. We identify $G/K_\pm = V_\pm$ via the map $gK_\pm \mapsto g \cdot v_\pm$. In this way, $\mathcal{H} := \ell^2(\mathcal{E})$ naturally becomes a finite rank $G$-$L^\infty(G/K_\pm)$-$L^\infty(G/K_-)$-module. Denote by $\mathcal{C}$ the $C^*$-2-category of $G$-$L^\infty(G/K_-)$-$L^\infty(G/K_\pm)$-modules generated by the alternating tensor products of $\mathcal{H}$ and its adjoint.

In the 2-category $\mathcal{C}_3$, the $n$-fold tensor product $\mathcal{H} \otimes \overline{\mathcal{H}} \otimes \cdots$ equals $\ell^2(\mathcal{E}_{+,n})$, where $\mathcal{E}_{+,n}$ is the set of paths in the graph $\mathcal{G}$ starting at an even vertex and having length $n$. Similarly, the $n$-fold tensor product $\overline{\mathcal{H}} \otimes \mathcal{H} \otimes \cdots$ equals $\ell^2(\mathcal{E}_{-,n})$, where $\mathcal{E}_{-,n}$ is the set of paths of length $n$ starting at an odd vertex. So by construction, under the equivalence of Remark 2.1, $\mathcal{C}$ together with its generator $\mathcal{H} \in \mathcal{C}_{++}$ corresponds exactly to the planar algebra $\mathcal{P}$ constructed in $[B10,J98]$.

By Corollary 2.4, we get that $(\mathcal{C},\mathcal{H})$ is the standard invariant of an extremal hyperfinite subfactor $N \subset M$. In particular, $[M : N] = \delta^2$ with $\delta = \dim_{\mathcal{C}_3}(\mathcal{H})$. Combining (2.6) with (2.4), and using that

$$\Delta(g)^{-1/2} = [gK_+ g^{-1} : K_+]^{1/2} = [\text{Stab}(g \cdot v_+) : K_+]^{1/2},$$

Note that in $[B10]$, also a weight function $\mu : V_+ \cup V_- \to \mathbb{R}_+$ scaled by the action of $G$ is part of the construction. But only when we take $\mu$ to be a multiple of the function $v \mapsto [\text{Stab} v : \text{Stab} v_+]^{1/2}$, we actually obtain a subfactor planar algebra, contrary to what is claimed in $[B10,\text{Proposition 4.1}]$. 

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we get that

\[ \delta = [K_+: K_-]^{1/2} \sum_{g \in G / K_+} \dim(1_{gK_+} \cdot \mathcal{H} \cdot 1_cK_-) \Delta(g)^{-1/2} \]

\[ = \sum_{g \in G / K_+} \# \{ e \in \mathcal{E} \mid s(e) = g \cdot v_+, t(e) = v_- \} [\text{Stab}(g \cdot v_+) : K_+]^{1/2} [K_+ : K_-]^{1/2} \]

\[ = \sum_{w \in V_+} \# \{ e \in \mathcal{E} \mid s(e) = w, t(e) = v_- \} [\text{Stab} w : \text{Stab} v_-]^{1/2} . \]

Combining (2.6) with (2.5), we similarly get that

\[ \delta = \sum_{w \in V_-} \# \{ e \in \mathcal{E} \mid s(e) = v_+, t(e) = w \} [\text{Stab} w : \text{Stab} v_+]^{1/2} . \]

To conclude the proof of the proposition, it remains to show that \( \mathcal{C} \) is equal to the \( C^* \)-2-category of all finite rank \( G \cdot L^\infty(G / K_+) \cdot L^\infty(G / K_-) \)-modules. For the \( G \cdot L^\infty(G / K_+) \cdot L^\infty(G / K_-) \)-modules, this amounts to proving that all irreducible representations of \( K_+ \cap K_- \) appear in

\[ L^2(\text{paths starting at } v_+ \text{ and ending at } v_-) . \]

Since the graph is connected, the action of \( K_+ \cap K_- \) on this set of paths is faithful and the result follows. The other cases are proved in the same way. \( \square \)

**Remark 2.6.** Note that the subfactors \( N \subset M \) in Proposition 2.5 are irreducible precisely when \( G \) acts transitively on the set of edges and there are no multiple edges. This means that the totally disconnected group \( G \) is generated by the compact open subgroups \( K_+ < G \) and that we can identify \( \mathcal{E} = G / (K_+ \cap K_-) \), \( V_+ = G / K_+ \) with the natural source and target maps \( G / (K_+ \cap K_-) \to G / K_+ \). The irreducible subfactor \( N \subset M \) then has integer index given by \([M : N] = [K_+ : K_+ \cap K_-] [K_- : K_+ \cap K_-]\).

We finally note that the rigid \( C^* \)-tensor categories \( \mathcal{C}_{1,f}(K < G) \) and \( \mathcal{C}_{3,f}(K < G) \) also arise in a different way as categories of bimodules over a \( II_1 \) factor in the case where \( K < G \) is the Schlichting completion of a Hecke pair \( \Lambda < \Gamma \), cf. [DV10, Section 4].

Recall that a Hecke pair consists of a countable group \( \Gamma \) together with a subgroup \( \Lambda < \Gamma \) that is almost normal, meaning that \( g\Lambda g^{-1} \cap \Lambda \) has finite index in \( \Lambda \) for all \( g \in \Gamma \). The left translation action of \( \Gamma \) on \( \Gamma / \Lambda \) gives a homomorphism \( \pi \) of \( \Gamma \) to the group of permutations of \( \Gamma / \Lambda \). The closure \( G \) of \( \pi(\Gamma) \) for the topology of pointwise convergence is a totally disconnected group and the stabilizer \( K \) of the point \( e\Lambda \in \Gamma / \Lambda \) is a compact open subgroup of \( G \) with the property that \( \Lambda = \pi^{-1}(K) \). One calls \((G, K)\) the Schlichting completion of the Hecke pair \((\Gamma, \Lambda)\). Note that there is a natural identification of \( G / K \) and \( \Gamma / \Lambda \).

**Proposition 2.7.** Let \( \Lambda < \Gamma \) be a Hecke pair with Schlichting completion \( K < G \). Choose an action \( \Gamma \curvearrowright \nabla P \) of \( \Gamma \) by outer automorphisms of a \( II_1 \) factor \( \mathcal{P} \). Define \( N = \mathcal{P} \rtimes \Lambda \) and \( M = \mathcal{P} \rtimes \Gamma \). Note that \( N \subset M \) is an irreducible, quasi-regular inclusion of \( II_1 \) factors. Denote by \( \mathcal{C} \) the tensor category of finite index \( N \cdot N \)-bimodules generated by the finite index \( N \cdot \text{sub} \mathcal{C} \) modules of \( L^2(M) \).

Then, \( \mathcal{C} \) and the earlier defined \( \mathcal{C}_{1,f}(K < G) \) and \( \mathcal{C}_{3,f}(K < G) \) are naturally equivalent rigid \( C^* \)-tensor categories.

**Proof.** Define
\(C_4\) : the category of \(\Lambda\)-\(\mathcal{L}^\infty(\Gamma)\)-modules, i.e. Hilbert spaces \(\mathcal{H}\) equipped with two commuting unitary representations of \(\Lambda\) and a representation of \(\mathcal{L}^\infty(\Gamma)\) that are covariant with respect to the left and right translation actions \(\Lambda \curvearrowright \Gamma\);

\(C_5\) : the category of \(\Lambda\)-\(\mathcal{L}^\infty(\Gamma/\Lambda)\)-modules, i.e. Hilbert spaces equipped with a unitary representation of \(\Lambda\) and a representation of \(\mathcal{L}^\infty(\Gamma/\Lambda)\) that are covariant with respect to the left translation action \(\Lambda \curvearrowright \Gamma/\Lambda\):

with morphisms again given by bounded operators that intertwine the given structure.

To define the tensor product of two objects in \(C_4\), it is useful to view \(\mathcal{H} \in C_4\) as a family of Hilbert spaces \((\mathcal{H}_g)_{g \in \Gamma}\) together with unitary operators \(\lambda(k) : \mathcal{H}_g \to \mathcal{H}_{kg}\) and \(\rho(k) : \mathcal{H}_g \to \mathcal{H}_{gk^{-1}}\) for all \(k \in \Lambda\), satisfying the obvious relations. The tensor product of two \(\Lambda\)-\(\mathcal{L}^\infty(\Gamma)\)-modules \(\mathcal{H}\) and \(\mathcal{K}\) is then defined as

\[
(\mathcal{H} \otimes_\Lambda \mathcal{K})_g = \left\{(\xi_h)_{h \in \Gamma} \left| \xi_h \in \mathcal{H}_h \otimes \mathcal{K}_{h^{-1}g}, \xi_hk^{-1} = (\rho_h(k) \otimes \lambda_k(k))(\xi_h) \text{ for all } h \in \Gamma, k \in \Lambda, \sum_{h \in \Gamma/\Lambda} \|\xi_h\|^2 < \infty \right\}
\]

with \(\lambda(k) : (\mathcal{H} \otimes_\Lambda \mathcal{K})_g \to (\mathcal{H} \otimes_\Lambda \mathcal{K})_{kg}\) given by \((\lambda(k)\xi)_h = (\lambda_h(k) \otimes 1)\xi_{k^{-1}h}\) and \(\rho(k) : (\mathcal{H} \otimes_\Lambda \mathcal{K})_g \to (\mathcal{H} \otimes_\Lambda \mathcal{K})_{gk^{-1}}\) given by \((\rho(k)\xi)_h = (1 \otimes \rho_k(k))\xi(h)\) for all \(k \in \Lambda, h \in \Gamma\). Of course, choosing a section \(i : \Gamma/\Lambda \to \Gamma\), we have

\[
(\mathcal{H} \otimes_\Lambda \mathcal{K})_g \cong \bigoplus_{k \in \Gamma/\Lambda} (\mathcal{H}_{i(h)} \otimes \mathcal{K}_{i(h)^{-1}g}),
\]

but this isomorphism depends on the choice of the section.

As in Proposition 2.2, \(C_4\) and \(C_5\) are equivalent C\(^*\)-categories, where the equivalence and its inverse are defined as follows.

- \(C_4 \to C_5 : \mathcal{H} \mapsto \mathcal{K}\), with

\[
\mathcal{K}_{g\Lambda} = \left\{(\xi_h)_{h \in g\Lambda} \left| \xi_h \in \mathcal{H}_h, \xi_hk^{-1} = \rho(k)\xi_h \text{ for all } h \in g\Lambda, k \in \Lambda \right\}
\]

and with the natural \(\Lambda\)-\(\mathcal{L}^\infty(\Gamma/\Lambda)\)-module structure. Note that \(\mathcal{K}_{g\Lambda} \cong \mathcal{H}_g\), but again, this isomorphism depends on a choice of section \(\Gamma/\Lambda \to \Gamma\).

- \(C_5 \to C_4 : \mathcal{K} \mapsto \mathcal{H}\), with \(\mathcal{H}_g = \mathcal{K}_{g\Lambda}\) and the obvious \(\Lambda\)-\(\mathcal{L}^\infty(\Gamma)\)-module structure.

We say that an object \(\mathcal{H} \in C_5\) has finite rank if \(\mathcal{H}\) is a finite dimensional Hilbert space. This is equivalent to requiring that all Hilbert spaces \(\mathcal{H}_{g\Lambda}\) are finite dimensional and that there are only finitely many double cosets \(\Lambda g\Lambda\) for which \(\mathcal{H}_{g\Lambda}\) is nonzero. Similarly, we say that an object \(\mathcal{H} \in C_4\) has finite rank if all Hilbert spaces \(\mathcal{H}_g\) are finite dimensional and if there are only finitely many double cosets \(\Lambda g\Lambda\) for which \(\mathcal{H}_g\) is nonzero. Note here that an algebraic variant of the category of finite rank objects in \(C_4\) was already introduced in [Z98].

In this way, we have defined the rigid C\(^*\)-tensor category \(C_{4,f}(\Lambda < \Gamma)\) consisting of the finite rank objects in \(C_4\). Note that, in a different context, this rigid C\(^*\)-tensor category \(C_{4,f}(\Lambda < \Gamma)\) already appeared in [DV10, Section 4].

Denote by \(\pi : \Gamma \to G\) the canonical homomorphism. Identifying \(G/K\) and \(\Gamma/\Lambda\) and using the homomorphism \(\pi : \Lambda \to K\), every \(K\)-\(\mathcal{L}^\infty(G/K)\)-module \(\mathcal{H}\) also is a \(\Lambda\)-\(\mathcal{L}^\infty(\Gamma/\Lambda)\)-module.
This defines a functor $C_2(K < G) \to C_3(\Lambda < \Gamma)$ that is fully faithful because $\pi(\Lambda)$ is dense in $K$. Note however that this fully faithful functor need not be an equivalence of categories: an object $\mathcal{H} \in C_3(\Lambda < \Gamma)$ is isomorphic with an object in the range of this functor if and only if the representation of $\Lambda$ on $\mathcal{H}$ is of the form $k \mapsto \lambda(\pi(k))$ for a (necessarily unique) continuous representation $\lambda$ of $K$ on $\mathcal{H}$.

Composing with the equivalence of categories in Proposition 2.2, we have found the fully faithful $C^*$-tensor functor $\Theta : C_3(K < G) \to C_4(\Lambda < \Gamma)$, sending finite rank objects to finite rank objects. By construction, $\Theta$ maps the $G$-$L^\infty(G/K)$-$L^\infty(G/K)$-module $L^2(G/K) \otimes L^2(G/K)$ (with $G$-action given by $(\lambda_g \otimes \lambda_g)_{g \in G}$ and obvious left and right $L^\infty(G/K)$-action) to the $\Lambda$-$\Lambda$-$\ell^\infty(\Gamma)$-module $\ell^2(\Gamma)$.

Next, given the outer action $\Gamma \curvearrowright \Lambda$, we write $N = P \rtimes \Lambda$ and $M = P \rtimes \Gamma$. Consider the category $\text{Bimod}(N)$ of Hilbert $N$-$N$-bimodules. We define the natural fully faithful $C^*$-tensor functor $C_4(\Lambda < \Gamma) \to \text{Bimod}(N) : \mathcal{H} \mapsto \mathcal{K}$ where $\mathcal{K} = L^2(P) \otimes \mathcal{H}$ and where the $N$-$N$-bimodule structure on $\mathcal{K}$ is given by

$$(au_k) \cdot (b \otimes \xi) \cdot (du_r) = a\alpha_k(b)\alpha_{kh}(d) \otimes \lambda(k)\rho(r^{-1})\xi$$

for all $a, b, d \in P$, $k, r \in \Lambda$, $h \in \Gamma$ and $\xi \in \mathcal{H}_h$. By construction, this functor maps the $\Lambda$-$\Lambda$-$\ell^\infty(\Gamma)$-module $\ell^2(\Gamma)$ to the $N$-$N$-bimodule structure on $\mathcal{K}$.

Denoting by $\mathcal{C}$ the tensor category of finite index $N$-$N$-bimodules generated by the finite index $N$-subbimodules of $L^2(M)$, it follows that $\mathcal{C}$ is naturally monoidally equivalent to the tensor subcategory $\mathcal{C}_0$ of $C_3, f(K < G)$ generated by the finite rank subobjects of $L^2(G/K) \otimes L^2(G/K)$. So, it remains to prove that $\mathcal{C}_0 = C_3, f(K < G)$. Taking the $n$-th tensor power of $L^2(G/K) \otimes L^2(G/K)$ and applying the equivalence between the categories $\mathcal{C}_3, f(K < G)$ and $C_2, f(K < G)$, it suffices to show that every irreducible $K$-$L^\infty(G/K)$-module appears in one of the $K$-$L^\infty(G/K)$-modules $L^2(G/K) \otimes \cdots \otimes L^2(G/K)$ with diagonal $G$-action and action of $L^\infty(G/K)$ on the last tensor factor. Reducing with the projections $1_{gK}$, this amounts to proving that for every $g \in G$, every irreducible representation of the compact group $K \cap gKg^{-1}$ appears in a tensor power of $L^2(G/K)$. Because $K < G$ is a Schlichting completion, we have that $\bigcap_{h \in G} hKh^{-1} = \{e\}$ so that the desired conclusion follows.

3 The tube algebra of $\mathcal{C}(K < G)$

Recall from [O93] the following construction of the tube $*$-algebra of a rigid $C^*$-tensor category $\mathcal{C}$ (see also [GJ15, Section 3] where the terminology annular algebra is used, and see as well [PSV15, Section 3.3]). Whenever $I$ is a full\(^5\) family of objects in $\mathcal{C}$, one defines as follows the $*$-algebra $\mathcal{A}$ with underlying vector space

$$\mathcal{A} = \bigoplus_{i,j \in I, \alpha \in \text{Irr}(\mathcal{C})} (i\alpha, \alpha j).$$

Here and in what follows, we denote the tensor product in $\mathcal{C}$ by concatenation and we denote by $(\beta, \gamma)$ the space of morphisms from $\gamma$ to $\beta$. By definition, all $(\beta, \gamma)$ are finite dimensional Banach spaces. Using the categorical traces $\text{Tr}_\beta$ and $\text{Tr}_\gamma$ on $(\beta, \beta)$, resp. $(\gamma, \gamma)$, we turn $(\beta, \gamma)$ into a Hilbert space with scalar product

$$\langle V, W \rangle = \text{Tr}_\beta(V W^*) = \text{Tr}_\gamma(W^*V).$$

\(^5\)Fullness means that every irreducible $i \in \text{Irr}(\mathcal{C})$ appears as a subobject of one of the $j \in I$. 

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For every $\beta \in \mathcal{C}$, the categorical trace $\text{Tr}_\beta$ is defined by using a standard solution for the conjugate equations for $\beta$, i.e. morphisms $s_\beta \in (\beta^*, \varepsilon)$ and $t_\beta \in (\beta, \varepsilon)$ satisfying
\[
(s_\beta^* \otimes 1)(1 \otimes t_\beta) = 1, \quad (1 \otimes s_\beta^*)(t_\beta \otimes 1) = 1, \quad t_\beta^*(1 \otimes V)t_\beta = s_\beta^*(V \otimes 1)s_\beta
\]
for all $V \in (\beta, \beta)$. Then, $\text{Tr}_\beta(V) = t_\beta^*(1 \otimes V)t_\beta = s_\beta^*(V \otimes 1)s_\beta$ and $d(\beta) = \text{Tr}_\beta(1)$ is the categorical dimension of $\beta$.

We will also make use of the partial traces
\[
\text{Tr}_\beta \otimes \text{id} : (\beta \alpha, \beta \gamma) \to (\alpha, \gamma) : (\text{Tr}_\beta \otimes \text{id})(V) = (t_\beta^* \otimes 1)(1 \otimes V)(t_\beta \otimes 1).
\]
Whenever $\mathcal{K}$ is a Hilbert space, we denote by $\text{onb}(\mathcal{K})$ any choice of orthonormal basis in $\mathcal{K}$.

The product in $\mathcal{A}$ is then defined as follows: for $V \in (i \alpha, \alpha j)$ and $W \in (j' \beta, \beta, k)$, the product $V \cdot W$ equals 0 when $j \neq j'$ and when $j = j'$, it is equal to
\[
V \cdot W = \sum_{\gamma \in \text{Irr}(\mathcal{C})} \sum_{U \in \text{onb}(\alpha, \gamma)} d(\gamma)(1 \otimes U^*)(V \otimes 1)(1 \otimes W)(U \otimes 1).
\]

The $^*$-operation on $\mathcal{A}$ is denoted by $V \mapsto V^*$ and defined by
\[
V^* = (t_\alpha^* \otimes 1)(1 \otimes V^* \otimes 1)(1 \otimes s_\alpha)
\]
for all $V \in (i \alpha, \alpha j)$.

The $^*$-algebra $\mathcal{A}$ has a natural positive faithful trace $\text{Tr}$ and for $V \in (i \alpha, \alpha j)$, we have that $\text{Tr}(V) = 0$ when $i \neq j$ or $\alpha \neq \varepsilon$, while $\text{Tr}(V) = \text{Tr}_i(V)$ when $i = j$ and $\alpha = \varepsilon$, so that $V \in (i, i)$.

Up to strong Morita equivalence, the tube $^*$-algebra $\mathcal{A}$ does not depend on the choice of the full family $I$ of objects in $\mathcal{C}$, see [NY15b, Theorem 3.2] and [PSV15, Section 7.2]. Also note that for an arbitrary object $\alpha \in \mathcal{C}$ and $i, j \in I$, we can associate with $V \in (i \alpha, \alpha j)$ the element in $\mathcal{A}$ given by
\[
\sum_{\gamma \in \text{Irr}(\mathcal{C})} \sum_{U \in \text{onb}(\alpha, \gamma)} d(\gamma)(1 \otimes U^*)V(U \otimes 1).
\]

Although this map $(i \alpha, \alpha j) \to \mathcal{A}$ is not injective, we will view an element in $V \in (i \alpha, \alpha j)$ as an element of $\mathcal{A}$ in this way.

Formally allowing for infinite direct sums in $\mathcal{C}$, one defines the $C^*$-tensor category of ind-objects in $\mathcal{C}$. Later in this section, we will only consider the rigid $C^*$-tensor category $\mathcal{C}$ of finite rank $G\text{-}L^\infty(G/K)\text{-}L^\infty(G/K)$-modules for a given totally disconnected group $G$ with compact open subgroup $K < G$. In that case, the ind-category precisely\(^6\) is the $C^*$-tensor category of all $G\text{-}L^\infty(G/K)\text{-}L^\infty(G/K)$-modules. Whenever $\mathcal{K}_1, \mathcal{K}_2$ are ind-objects, we denote by $(\mathcal{K}_1, \mathcal{K}_2)$ the vector space of finitely supported morphisms, where a morphism $V : \mathcal{K}_2 \to \mathcal{K}_1$ is said to be finitely supported if there exist projections $p_i$ of $\mathcal{K}_i$ onto a finite dimensional subobject (i.e. an object in $\mathcal{C}$) such that $V = p_iV = Vp_2$.

We say that an ind-object $\mathcal{H}_0$ in $\mathcal{C}$ is full if every irreducible object $i \in \text{Irr}(\mathcal{C})$ is isomorphic with a subobject of $\mathcal{H}_0$. We define the tube $^*$-algebra of $\mathcal{C}$ with respect to a full ind-object $\mathcal{H}_0$ as the vector space
\[
\mathcal{A} = \bigoplus_{\alpha \in \text{Irr}(\mathcal{C})} (\mathcal{H}_0 \alpha, \alpha \mathcal{H}_0)
\]
\(^6\)Using Proposition 2.2, every $G\text{-}L^\infty(G/K)\text{-}L^\infty(G/K)$-module is a direct sum of finite rank modules because every $K\text{-}L^\infty(G/K)$-module is a direct sum of finite dimensional modules, which follows because every unitary representation of a compact group is a direct sum of finite dimensional representations.
on which the $*$-algebra structure is defined in the same way as above. Note that $(\mathcal{H}_0, \mathcal{H}_0)$ naturally is a $*$-subalgebra of $\mathcal{A}$, given by taking $\alpha = \varepsilon$ in the above description of $\mathcal{A}$. In particular, every projection of $p$ of $\mathcal{H}_0$ on a finite dimensional subobject of $\mathcal{H}_0$ can be viewed as a projection $p \in \mathcal{A}$. These projections serve as local units: for every finite subset $\mathcal{F} \subset \mathcal{A}$, there exists such a projection $p$ satisfying $p \cdot V = V \cdot p$ for all $V \in \mathcal{F}$.

Whenever $p_\varepsilon$ is the projection of $\mathcal{H}_0$ onto a copy of the trivial object $\varepsilon$, we identify $p_\varepsilon \cdot A \cdot p_\varepsilon$ with the fusion $*$-algebra $\mathbb{C}[\mathcal{C}]$ of $\mathcal{C}$, i.e. the $*$-algebra with vector space basis $\text{Irr}(\mathcal{C})$, product given by the fusion rules and $*$-operation given by the adjoint object.

To every full family $I$ of objects in $\mathcal{C}$, we can associate the full ind-object $\mathcal{H}_0$ by taking the direct sum of all $i \in I$. The tube $*$-algebra of $\mathcal{C}$ associated with $I$ is then naturally a $*$-subalgebra of the tube $*$-algebra of $\mathcal{C}$ associated with $\mathcal{H}_0$. If every irreducible object of $\mathcal{C}$ appears with finite multiplicity in $\mathcal{H}_0$, then this inclusion is an equality and both tube $*$-algebras are naturally isomorphic.

For the rest of this section, we fix a totally disconnected group $G$ and a compact open subgroup $K < G$. We denote by $\mathcal{C}$ the rigid $C^*$-tensor category of all finite rank $G$-$L^\infty(G/K)$-$L^\infty(G/K)$-modules, which we denoted as $\mathcal{C}_{\lambda,f}(K < G)$ in Section 2. We determine the tube $*$-algebra $\mathcal{A}$ of $\mathcal{C}$ with respect to the following full ind-object.

$$\mathcal{H}_0 = L^2(G \times G/K) \quad \text{with}$$

$$\langle F \cdot \xi \rangle(g, hK) = F(gK)\xi(g, hK), \quad \langle \xi \cdot F \rangle(g, hK) = \xi(g, hK)F(ghK) \quad \text{and}$$

$$(\pi(x)\xi)(g, hK) = \xi(x^{-1}g, hK)$$

for all $\xi \in L^2(G \times G/K)$, $F \in L^\infty(G/K)$, $x, g \in G$, $hK \in G/K$. Note that every irreducible object of $\mathcal{C}$ appears with finite multiplicity in $\mathcal{H}_0$.

We denote by $(\text{Ad} g)_{g \in G}$ the action of $G$ on $G$ by conjugation: $(\text{Ad} g)(h) = ghg^{-1}$. In the rest of this paper, we will make use of the associated full and reduced $C^*$-algebras

$$\mathcal{C}_b(G) \rtimes^f_{\text{Ad}} G \quad \text{and} \quad \mathcal{C}_b(G) \rtimes^\lambda_{\text{Ad}} G,$$

as well as the von Neumann algebra $L^\infty(G) \rtimes_{\text{Ad}} G$. We fix the left Haar measure $\lambda$ on $G$ such that $\lambda(K) = 1$. We equip $L^\infty(G) \rtimes_{\text{Ad}} G$ with the canonical normal semifinite faithful trace $\text{Tr}$ given by

$$\text{Tr}(F \lambda_f) = f(\varepsilon) \int_G F(g)\Delta(g)^{-1/2} \, dg . \quad (3.2)$$

Note that the modular function $\Delta$ is affiliated with the center of $L^\infty(G) \rtimes_{\text{Ad}} G$, so that $L^\infty(G) \rtimes_{\text{Ad}} G$ need not be a factor. Also note that the measure used in (3.2) is half way between the left and the right Haar measure of $G$.

We consider the dense $*$-algebra $\text{Pol}(L^\infty(G) \rtimes_{\text{Ad}} G)$ defined as

$$\text{Pol}(L^\infty(G) \rtimes_{\text{Ad}} G) = \text{span}\{1_U u_x p_L \mid U \subset G \text{ compact open subset} , \ x \in G , \ L < G \text{ compact open subgroup}\}$$

and where $p_L \in L(G)$ denotes the projection onto the $L$-invariant vectors, i.e.

$$p_L = \lambda(L)^{-1} \int_L u_k dk .$$

Note that $\text{Pol}(L^\infty(G) \rtimes_{\text{Ad}} G)$ equals the linear span of all $F \lambda_f$ where $F$ and $f$ are continuous, compactly supported, locally constant functions on $G$. 

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We now identify the tube ∗-algebra of C with Pol(L∞(G) × Ad G). For every \( x \in G \) and every irreducible representation \( π : K \cap xKx^{-1} \to U(K) \), we denote by \( ℜ(π, x) \in \text{Irr}(C) \) the irreducible \( G\text{-}L_{\infty}(G/K)\text{-}L_{\infty}(G/K) \)-module such that \( π \) is isomorphic with the representation of \( K \cap xKx^{-1} \) on \( 1_{xK} \cdot ℜ(π, x) \cdot 1_{eK} \). Note that this gives us the identification

\[
\text{Irr}(C) = \left\{ (π, x) \mid x \in K \setminus G/K , \; π \in \text{Irr}(K \cap xKx^{-1}) \right\} \tag{3.4}
\]

We denote by \( χ_π \) the character of \( π \), i.e. the locally constant function with support \( K \cap xKx^{-1} \) and \( χ_π(k) = ℜ(π(k)) \) for all \( k \in K \cap xKx^{-1} \).

**Theorem 3.1.** The \( G\text{-}L_{\infty}(G/K)\text{-}L_{\infty}(G/K) \)-module \( ℜ_0 \) introduced in (3.1) is full. There is a natural ∗-anti-isomorphism \( T \) of the associated tube ∗-algebra \( A \) onto \( Pol(L_{\infty}(G) \times Ad G) \). The ∗-anti-isomorphism \( T \) is trace preserving.

Denoting by \( p_e \) the projection in \( A \) that corresponds to the unique copy of the trivial object \( e \) in \( ℜ_0 \) and identifying \( p_e \cdot A \cdot p_e \) with the fusion ∗-algebra of \( C \), we have that \( T(p_e) = 1_{K} p_{K} \) and that the restriction of \( T \) to \( C[C] \) is given by

\[
d(π, x)^{-1} T(π, x) = p_{K} \text{ dim}(π)^{-1} χ_π u_x p_{K} \tag{3.5}
\]

where \( d(π, x) \) denotes the categorical dimension of \( (π, x) \in \text{Irr}(C) \) and \( \text{dim}(π) \) denotes the ordinary dimension of the representation \( π \).

**Proof.** To see that \( ℜ_0 \) is full, it suffices to observe that for every \( h \in G \), the unitary representation of \( K \cap hKh^{-1} \) on \( 1_{eK} \cdot ℜ_0 \cdot 1_{hK} \) contains the regular representation of \( K \cap hKh^{-1} \).

Assume that \( \Psi : C_0(G) \times_{Ad} G \to B(K) \) is any nondegenerate ∗-representation. As follows, we associate with \( \Psi \) a unitary half braiding\(^7\) on \( \text{ind-}C \). Whenever \( ℜ \) is a \( G\text{-}L_{\infty}(G/K)\text{-}L_{\infty}(G/K) \)-module, we consider a new \( G\text{-}L_{\infty}(G/K)\text{-}L_{\infty}(G/K) \)-module with underlying Hilbert space \( K \otimes ℜ \) and structure maps

\[
π_K \otimes ℜ(g) = \Psi(g) \otimes π_K(h) \quad , \quad λ_K \otimes ℜ = (\Psi \otimes λ_K)∆(F) \; , \; \rho_K \otimes ℜ(F) = 1 \otimes ρ_K(F) \tag{3.6}
\]

for all \( g \in G \), \( F \in L_{\infty}(G/K) \), with \( ∆(F)(g, hK) = F(ghK) \).

We similarly turn \( ℜ \otimes ℜ \) into a \( G\text{-}L_{\infty}(G/K)\text{-}L_{\infty}(G/K) \)-module with structure maps

\[
π_{H \otimes K} = π_H \otimes \Psi \quad , \quad λ_{H \otimes K} = λ_H(1) \otimes 1 \; , \; ρ_{H \otimes K} = (ρ_H \otimes \Psi)ʻ(F) \tag{3.7}
\]

where \( ʻ(F)(g, h) = F(h^{-1}gK) \).

Defining the unitary \( U \in M(C_0(G) \otimes K(\text{H})) \) given by \( U(x) = π_H(x) \) for all \( x \in G \) and denoting by \( Σ : ℜ \otimes ℜ \to ℜ \otimes ℜ \) the flip map, one checks that \( Σ(Ψ \otimes \text{id})(U) \) is an isomorphism between the \( G\text{-}L_{\infty}(G/K)\text{-}L_{\infty}(G/K) \)-modules \( ℜ \otimes ℜ \) and \( ℜ \otimes ℜ \). So, defining

\[
K_1 := ℜ \otimes L^2(G/K) \cong L^2(G/K) \otimes ℜ \tag{3.8}
\]

we have found the \( G\text{-}L_{\infty}(G/K)\text{-}L_{\infty}(G/K) \)-module \( K_1 \) with the property that for every \( G\text{-}L_{\infty}(G/K)\text{-}L_{\infty}(G/K) \)-module \( ℜ \), there is a natural unitary isomorphism

\[
σ_ℜ : ℜK_1 \to K_1 ℜ \tag{3.9}
\]

\(^7\)Formally, a unitary half braiding is an object in the Drinfel’d center of \( \text{ind-}C \). More concretely, a unitary half braiding consists of an underlying \( \text{ind-} \)object \( K_1 \) together with natural unitary isomorphisms \( ℜK_1 \to K_1 ℜ \) for all objects \( ℜ \). We refer to [NY15a, Section 2.1] for further details.

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Here and in what follows, we denote by concatenation the tensor product in the category of $G$-$L^\infty(G/K)$-$L^\infty(G/K)$-modules. So, $\sigma$ is a unitary half braiding for ind-$\mathcal{C}$.

Using the ind-object $\mathcal{H}_0$ defined in (3.1) and recalling that $\mathcal{K}_1 \mathcal{H}_0 = \mathcal{K} \otimes \mathcal{H}_0$ as Hilbert spaces, we define the Hilbert space

$$\mathcal{K}_2 = (\mathcal{K} \otimes \mathcal{H}_0, \varepsilon)$$

and we consider the tube $\ast$-algebra $\mathcal{A}$ associated with $\mathcal{H}_0$. Using standard solutions for the conjugate equations, there is a natural linear bijection

$$V \in (\mathcal{H}_0 \mathcal{H}, \mathcal{H} \mathcal{H}_0) \mapsto \tilde{V} \in (\mathcal{H} \mathcal{H}_0, \mathcal{H}_0 \mathcal{H})$$

between finitely supported morphisms.

By [PSV15, Proposition 3.14] and using the partial categorical trace $\text{Tr}_\mathcal{H} \otimes \text{id} \otimes \text{id}$, the unitary half braiding $\sigma$ gives rise to a nondegenerate $\ast$-anti-homomorphism $\Theta : \mathcal{A} \rightarrow B(\mathcal{K}_2)$ given by

$$\Theta(V) \xi = (\text{Tr}_\mathcal{H} \otimes \text{id} \otimes \text{id})((\sigma^*_\mathcal{H} \otimes 1)(1 \otimes \tilde{V})(\xi \otimes 1))$$

(3.6)

for all $\mathcal{H} \in \mathcal{C}$, $\xi \in \mathcal{K}_2$ and all finitely supported $V \in (\mathcal{H}_0 \mathcal{H}, \mathcal{H} \mathcal{H}_0)$.

We now compute the expression in (3.6) more concretely. Whenever $h \in G$ and $K_0 < K$ is an open subgroup such that $hK_0h^{-1} \subset K$, we define the finite rank $G$-$L^\infty(G/K)$-$L^\infty(G/K)$-module $L^2(G/K_0)_h$ with underlying Hilbert space $L^2(G/K_0)$ and structure maps

$$(x \cdot \xi)(gK_0) = \xi(x^{-1}gK_0) \ , \ (F_1 \cdot \xi \cdot F_2)(gK_0) = F_1(gK_0)\xi(gK_0)F_2(gh^{-1}K) \ .$$

Note that there is a natural isomorphism $\overline{L^2(G/K_0)_h} \cong L^2(G/K_0)_{h^{-1}}$. Letting $K_0$ tend to $\{e\}$, the direct limit of $L^2(G/K_0)_{h^{-1}}$ becomes $L^2(G)_{h^{-1}}$. Since $\mathcal{H}_0 = \bigoplus_{h \in G/K} L^2(G)_{h^{-1}}$, we identify

$$\overline{\mathcal{H}_0} = \bigoplus_{h \in G/K} L^2(G)_h$$

and we view $L^2(G/K_0)_h \subset \overline{\mathcal{H}_0}$ whenever $h \in G$ and $K_0 < K \cap h^{-1}Kh$ is an open subgroup.

The Hilbert space $\mathcal{K}_2$ equals the space of $K$-invariant vectors in $1_{eK} \cdot (\mathcal{K} \otimes \overline{\mathcal{H}_0}) \cdot 1_{eK}$. In this way, the space of $K$-invariant vectors in $1_{eK} \cdot (\mathcal{K} \otimes L^2(G/K_0)_h) \cdot 1_{eK}$ naturally is a subspace of $\mathcal{K}_2$. But this last space of $K$-invariant vectors can be unitarily identified with $\Psi(1_{K^{-1}p_hK_0})\mathcal{K}$ by sending the vector $\xi_0 \in \Psi(1_{K^{-1}p_hK_0})\mathcal{K}$ to the vector

$$\Delta(h)^{-1/2} \sum_{k \in K/hK_0h^{-1}} \Psi(k)\xi_0 \otimes 1_{khK_0} \in \mathcal{K} \otimes L^2(G/K_0) \ .$$

We now use that for every $\mathcal{H} \in \mathcal{C}$, the categorical trace $\text{Tr}_\mathcal{H}$ on $(\mathcal{H}, \mathcal{H})$ is given by

$$\text{Tr}_\mathcal{H}(V) = \sum_{x \in G/K, \eta \in \text{emb}(1_{xK} \cdot \mathcal{H} \cdot 1_{xK})} \Delta(x)^{-1/2} \langle V\eta, \eta \rangle$$

$$= \sum_{y \in G/K, \eta \in \text{emb}(1_{yK} \cdot \mathcal{H} \cdot 1_{yK})} \Delta(y)^{-1/2} \langle V\eta, \eta \rangle \ .$$

A straightforward computation then gives that for all $\mathcal{H} \in \mathcal{C}$ and all

$$V \in (L^2(G/K_0)_h, \mathcal{H}, L^2(G/K_1)_h)$$

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with \( g, h \in G \) and \( K_0 < K \cap g^{-1}Kg \), \( K_1 < K \cap h^{-1}Kh \) open subgroups, we have

\[
\Theta(V) = \Delta(g)^{-1/2} \Delta(h)^{1/2} [K : K_1] \sum_{x \in G / gK_0g^{-1}} \Delta(x)^{-1/2} \Psi(1_{K_2yg^{-1}h^{-1}} u_x p_{gK_0g^{-1}}) \left\langle \tilde{V}(1_{xgK_0} \otimes \eta), \pi_{\mathcal{H}}(hy)\eta \otimes 1_{hK_1} \right\rangle, \tag{3.7}
\]

whenever \( K_2 < K \) is a small enough open subgroup such that \( \pi_{\mathcal{H}}(k) \) is the identity on \( \mathcal{H} \cdot 1_eK \) for all \( k \in K_2 \). Note that because \( \mathcal{H} \) has finite rank, such an open subgroup \( K_2 \) exists. Also, there are only finitely many \( x \in G / K \) such that \( 1_{xK} \cdot \mathcal{H} \cdot 1_eK \) is nonzero. Therefore, the sum appearing in (3.7) is finite.

Applying this to the regular representation \( C_0(G) \rtimes_{\text{Ad}} G \to B(L^2(G \times G)) \), we see that (3.7) provides a \(*\)-anti-homomorphisms \( \Theta \) from \( \mathcal{A} \) to \( \text{Pol}(L^\infty(G) \rtimes_{\text{Ad}} G) \). A direct computation gives that \( \Theta \) is trace preserving, using the trace \( \text{Tr} \) on \( L^\infty(G) \rtimes_{\text{Ad}} G \) defined in (3.2). In particular, \( \Theta \) is injective.

We now prove that \( \Theta \) is surjective. Fix elements \( g, h, \alpha \in G \) satisfying \( \alpha g = h \alpha \). Choose any open subgroup \( K_0 < K \) such that \( gK_0g^{-1}, \alpha K_0\alpha^{-1} \) and \( K_1 := h^{-1}\alpha K_0\alpha^{-1}h \) are all subgroups of \( K \). Put \( \mathcal{H} = L^2(G/K_0)_g \) and note that \( \mathcal{H}, L^2(G/K_0)_g \) and \( L^2(G/K_1)_h \) are well defined objects in \( \mathcal{C} \). For every \( k \in K \), we consider the vectors

\[
1_{kagK_0} \otimes 1_{kaK_0} \in 1_{kagK} \cdot (L^2(G/K_0)_g \mathcal{H}) \cdot 1_eK \quad \text{and} \quad 1_{kh\alpha K_0} \otimes 1_{khK_1} \in 1_{kagK} \cdot (\mathcal{H} \cdot L^2(G/K_1)_h) \cdot 1_eK.
\]

In both cases, we get an orthogonal family of vectors indexed by

\[
k \in K / (K \cap \alpha K_0\alpha^{-1} \cap \alpha gK_0(\alpha g)^{-1})\).
\]

So, we can uniquely define \( \hat{V} \in (L^2(G/K_0)_g \mathcal{H}, \mathcal{H} \cdot L^2(G/K_1)_h) \) such that the restriction of \( \hat{V} \) to \( (L^2(G/K_0)_g \mathcal{H}) \cdot 1_eK \) is the partial isometry given by

\[
1_{kagK_0} \otimes 1_{kaK_0} \mapsto \Delta(\alpha)^{-1/2} \Delta(h)^{-1/2} 1_{kh\alpha K_0} \otimes 1_{khK_1} \quad \text{for all } k \in K.
\]

A direct computation gives that \( \Theta(V) \) is equal to a nonzero multiple of

\[
1_{\alpha K_0\alpha^{-1}h^{-1}} u_{\alpha} p_{gK_0g^{-1}} \cdot \tag{3.8}
\]

From (3.7), we also get that \( \Theta \) maps \((\mathcal{H}_0, \mathcal{H}_0) \subset \mathcal{A} \) onto \( \text{Pol}(L^\infty(K \setminus G) \rtimes K) \), defined as the linear span of all

\[
1_{Kx} u_k p_L
\]

with \( x \in G, k \in K \) and \( L < K \) an open subgroup. In combination with (3.8), it follows that \( \Theta \) is surjective.

Finally, by restricting (3.7) to the cases where \( g = h = e \) and \( K_0 = K_1 = K \), we find that (3.5) holds. \( \square \)

We recall from [PV14] the notion of a completely positive \((cp)\) multiplier on a rigid \(C^*\)-tensor category \( \mathcal{C} \). By [PV14, Proposition 3.6], to every function \( \varphi : \text{Irr}(\mathcal{C}) \to \mathbb{C} \) is associated a system of linear maps

\[
\Psi_{\alpha_1,\beta_1,\alpha_2,\beta_2} : (\alpha_1,\beta_1,\alpha_2,\beta_2) \to (\alpha_1,\beta_1,\alpha_2,\beta_2) \quad \text{for all } \alpha_i, \beta_i \in \mathcal{C} \quad \tag{3.9}
\]
satisfying
\[
\Psi_{\alpha_1|\beta_3,\alpha_4|\beta_4}^\varphi((X \otimes Y)V(Z \otimes T)) = (X \otimes Y) \Psi_{\alpha_1|\beta_1,\alpha_2|\beta_2}^\varphi(V)(Z \otimes T)
\]
for all \(X \in (\alpha_3, \alpha_1), Y \in (\beta_3, \beta_1), Z \in (\alpha_2, \alpha_4), T \in (\beta_2, \beta_4)\), as well as
\[
\Psi_{\alpha_1|\beta_3,\alpha_4|\beta_4}^\varphi(s_\alpha) = \varphi(\alpha) s_\alpha \quad \text{and} \quad \Psi_{\alpha_1|\beta_3,\alpha_4|\beta_4}^\varphi(1 \otimes V \otimes 1) = 1 \otimes \Psi_{\alpha_2|\beta_2,\alpha_4|\beta_4}^\varphi(V) \otimes 1
\]
for all \(V \in (\alpha_2, \alpha_4, \beta_2, \beta_4)\).

**Definition 3.2** ([PV14, Definition 3.4]). Let \(\mathcal{C}\) be a rigid \(\mathbb{C}^*\)-tensor category.

- A **cp-multiplier** on \(\mathcal{C}\) is a function \(\varphi : \text{Irr}(\mathcal{C}) \to \mathbb{C}\) such that the maps \(\Psi_{\alpha|\beta,\alpha|\beta}^\varphi\) on \((\alpha, \beta)\) are completely positive for all \(\alpha, \beta \in \mathcal{C}\).

- A cp-multiplier \(\varphi : \text{Irr}(\mathcal{C}) \to \mathbb{C}\) is said to be \(c_0\) if the function \(\varphi : \text{Irr}(\mathcal{C}) \to \mathbb{C}\) tends to zero at infinity.

- A **cb-multiplier** on \(\mathcal{C}\) is a function \(\varphi : \text{Irr}(\mathcal{C}) \to \mathbb{C}\) such that
\[
\|\varphi\|_{cb} := \sup_{\alpha, \beta \in \mathcal{C}} \|\Psi_{\alpha|\beta,\alpha|\beta}^\varphi\|_{cb} < \infty.
\]

A function \(\varphi : \text{Irr}(\mathcal{C}) \to \mathbb{C}\) gives rise to the following linear functional \(\omega_\varphi : \mathcal{A} \to \mathbb{C}\) on the tube algebra \(\mathcal{A}\) of \(\mathcal{C}\) with respect to any full family of objects containing once the trivial object \(\varepsilon\):
\[
\omega_\varphi : \mathcal{A} \to \mathbb{C} : \omega_\varphi(V) = \begin{cases} \langle \alpha \rangle \varphi(\alpha) & \text{if } V = 1_{\alpha} \in (\varepsilon, \alpha), \\ 0 & \text{if } V \in (i\alpha, \alpha) \text{ with } i \neq \varepsilon \text{ or } j \neq \varepsilon. \end{cases}
\]

By [GJ15, Theorem 6.6], the function \(\varphi : \text{Irr}(\mathcal{C}) \to \mathbb{C}\) is a cp-multiplier in the sense of Definition 3.2 if and only if \(\omega_\varphi\) is positive on \(\mathcal{A}\) in the sense that \(\omega_\varphi(V \cdot V^\#) \geq 0\) for all \(V \in \mathcal{A}\). In Proposition 5.1, we prove a characterization of cb-multipliers in terms of completely bounded multipliers of the tube algebra.

From Theorem 3.1, we then get the following result. We again denote by \(\mathcal{C}\) be the rigid \(\mathbb{C}^*\)-tensor category of finite rank \(G\)-\(L^\infty(G/K)\)-\(L^\infty(G/K)\)-modules and we identify \(\text{Irr}(\mathcal{C})\) as in (3.4) with the set of pairs \((\pi, x)\) where \(x \in K \backslash G / K\) and \(\pi\) is an irreducible representation of the compact group \(K \cap xKx^{-1}\). In order to identify the \(c_0\) cp-multipliers on \(\mathcal{C}\), we introduce the following definition.

**Definition 3.3.** We say that a complex measure \(\mu\) on \(G\) (i.e. an element of \(C_0(G)^*\)) is \(c_0\) if
\[
\lambda(\mu) := \int_G \lambda_g \, d\mu(g) \in L(G)
\]
belongs to \(C^*_r(G)\).

We say that a positive functional \(\omega\) on \(C_0(G) \rtimes_{\text{Ad}} G\) is \(c_0\) if for every \(x \in G\), the complex measure \(\mu_x\), defined by \(\mu_x(F) = \omega(Fu_x)\) for all \(F \in C_0(G)\) is \(c_0\) and if the function \(G \to C^*_r(G) : x \mapsto \lambda(\mu_x)\) tends to zero at infinity, i.e. \(\lim_{x \to \infty} \|\lambda(\mu_x)\| = 0\).
Proposition 3.4. The formula
\[ \varphi(\pi, x) = \omega(p_K \dim(\pi)^{-1} \chi_{\pi} u_x p_K) \]  
(3.10)
gives a bijection between the cp-multipliers \( \varphi \) on \( \text{Irr}(\mathcal{C}) \) and the positive functionals \( \omega \) on the \( C^* \)-algebra \( q(C_0(G) \rtimes_{Ad}^f G)q \), where \( q = 1_{qK} \).

The cp-multiplier \( \varphi \) is \( c_0 \) if and only if the positive functional \( \omega \) is \( c_0 \) in the sense of Definition 3.3.

Using the notations \( C_u(\mathcal{C}) \) and \( C_r(\mathcal{C}) \) of [PV14, Definition 4.1] for the universal and reduced \( C^* \)-algebra of \( \mathcal{C} \), we have the natural anti-isomorphisms \( C_u(\mathcal{C}) \to q(C_0(G) \rtimes_{Ad}^f G)q \) and \( C_r(\mathcal{C}) \to q(C_0(G) \rtimes_{Ad}^f G)q \).

Proof. Note that the \( G-L^\infty(G/K)-L^\infty(G/K) \)-module \( \mathcal{H}_0 \) in (3.1) contains exactly once the trivial module. The first part of the proposition is then a direct consequence of Theorem 3.1 and the above mentioned characterization [GJ15] of cp-multipliers as positive functionals on the tube \( * \)-algebra. The isomorphisms for \( C_u(\mathcal{C}) \) and \( C_r(\mathcal{C}) \) follow in the same way.

Fix a positive functional \( \omega \) on \( q(C_0(G) \rtimes_{Ad}^f G)q \) with corresponding cp-multiplier \( \varphi : \text{Irr}(\mathcal{C}) \to \mathbb{C} \) given by (3.10). We extend \( \omega \) to \( C_0(G) \rtimes_{Ad}^f G \) by \( \omega(T) = \omega(qTq) \). For every \( x \in G \), define \( \mu_x \in C_0(G)^* \) given by \( \mu_x(F) = \omega(F u_x) \) for all \( F \in C_0(G) \). Note that \( \mu_x \) is supported on \( K \cap xKx^{-1} \) and that \( \mu_x \) is \( \text{Ad}(K \cap xKx^{-1}) \)-invariant. Therefore, \( \lambda(\mu_x) \in Z(L(K \cap xKx^{-1})) \).

For every \( \pi \in \text{Irr}(K \cap xKx^{-1}) \), denote by \( z_\pi \in Z(L(K \cap xKx^{-1})) \) the corresponding minimal central projection. From (3.10), we get that
\[ \lambda(\mu_x)z_\pi = \varphi(\pi, x)z_\pi \]  
for all \( x \in G, \pi \in \text{Irr}(K \cap xKx^{-1}) \).

(3.11)

For a fixed \( x \in G \), an element \( T \in Z(L(K \cap xKx^{-1})) \) belongs to \( C_0^r(K \cap xKx^{-1}) \) if and only if \( T \in C_0^r(K \cap xKx^{-1}) \) if and only if \( \lim_{\pi \to \infty} \|Tz_\pi\| = 0 \). Also, \( \|T\| = \sup_{\pi \in \text{Irr}(K \cap xKx^{-1})} \|Tz_\pi\| \). So by (3.11), we get that \( \mu_x \) is \( c_0 \) if and only if
\[ \lim_{\pi \to \infty} \|\varphi(\pi, x)\| = 0 \]  
(3.12)
and that \( \omega \) is a \( c_0 \) functional if and only if (3.12) holds for all \( x \in G \) and we moreover have that
\[ \lim_{x \to \infty} \left( \sup_{\pi \in \text{Irr}(K \cap xKx^{-1})} |\varphi(\pi, x)| \right) = 0 . \]

Altogether, it follows that \( \omega \) is a \( c_0 \) functional in the sense of Definition 3.3 if and only if \( \varphi \) is a \( c_0 \)-function.

For later use, we record the following lemma.

Lemma 3.5. Let \( \mu \) be a probability measure on \( G \) that is \( c_0 \) in the sense of Definition 3.3. Then every complex measure \( \omega \in C_0(G)^* \) that is absolutely continuous with respect to \( \mu \) is still \( c_0 \).

Proof. Denote by \( C_c(G) \) the space of continuous compactly supported functions on \( G \). Since \( C_c(G) \subseteq L^1(G, \mu) \) is dense, it is sufficient to prove that \( F \cdot \mu \) is \( c_0 \) for every \( F \in C_c(G) \). Denote by \( \omega_F \in C_0^r(G)^* \) the functional determined by \( \omega_F(\lambda_x) = F(x) \) for all \( x \in G \). Denote by \( \hat{\Delta} : C_0^r(G) \to M(C_0^r(G) \otimes C_0^r(G)) \) the comultiplication determined by \( \hat{\Delta}(\lambda_x) = \lambda_x \otimes \lambda_x \). Recall that for every \( \omega \in C_0^r(G)^* \) and every \( T \in C_0^r(G) \), we have that \( (\omega \otimes \text{id})\hat{\Delta}(T) \in C_0^r(G) \). Since
\[ \lambda(F \cdot \mu) = (\omega_F \otimes \text{id})\hat{\Delta}(\lambda(\mu)) , \]
the lemma is proven.
4 Haagerup property and property (T) for $\mathcal{C}(K < G)$

In Definition 3.2, we already recalled the notion of a cp-multiplier $\varphi : \text{Irr}(\mathcal{C}) \to \mathbb{C}$ on a rigid $\mathcal{C}^*$-tensor category $\mathcal{C}$. In terms of cp-multipliers, amenability of a rigid $\mathcal{C}^*$-tensor category, as defined in [P94a, LR96], amounts to the existence of finitely supported cp-multipliers $\varphi_n : \text{Irr}(\mathcal{C}) \to \mathbb{C}$ that converge to 1 pointwise, see [PV14, Proposition 5.3]. Following [PV14, Definition 5.1], a rigid $\mathcal{C}^*$-tensor category $\mathcal{C}$ has the Haagerup property if there exist $c_0$ cp-multipliers $\varphi_n : \text{Irr}(\mathcal{C}) \to \mathbb{C}$ that converge to 1 pointwise, while $\mathcal{C}$ has property (T) if all cp-multipliers converging to 1 pointwise, must converge to 1 uniformly.

Similarly, when $\mathcal{C}_1$ is a full $\mathcal{C}^*$-tensor subcategory of $\mathcal{C}$, we say that $\mathcal{C}_1 \subset \mathcal{C}$ has the relative property (T) if all cp-multipliers on $\mathcal{C}$ converging to 1 pointwise, must converge to 1 uniformly on $\text{Irr}(\mathcal{C}_1) \subset \text{Irr}(\mathcal{C})$.

We now turn back to the rigid $\mathcal{C}^*$-tensor category of finite rank $G$-$L^\infty(G/K)$-$L^\infty(G/K)$-modules, where $G$ is a totally disconnected group and $K < G$ is a compact open subgroup. Note that $\text{Rep} K$ is a full $\mathcal{C}^*$-tensor subcategory of $\mathcal{C}$, consisting of the $G$-$L^\infty(G/K)$-$L^\infty(G/K)$-modules $\mathcal{H}$ with the property that $1_{xK} \cdot \mathcal{H} \cdot 1_{eK}$ is zero for all $x \notin K$.

Recall from Definition 3.3 the notion of a $c_0$ complex measure on $G$. We identify the space of complex measures with $C_0(G)^*$ and we denote by $S(C_0(G)) \subset C_0(G)^*$ the state space of $C_0(G)$, i.e. the set of probability measures on $G$.

**Theorem 4.1.** Let $G$ be a totally disconnected group and $K < G$ a compact open subgroup. Denote by $\mathcal{C}$ the rigid $\mathcal{C}^*$-tensor category of finite rank $G$-$L^\infty(G/K)$-$L^\infty(G/K)$-modules.

1. $\mathcal{C}$ has the Haagerup property if and only if $G$ is amenable.

2. $\mathcal{C}$ has the Haagerup property if and only if $G$ has the Haagerup property and there exists a sequence of $c_0$ probability measures $\mu_n \in S(C_0(G))$ such that $\mu_n \to \delta_e$ weakly* and such that $\|\mu_n \circ \text{Ad} x - \mu_n\| \to 0$ uniformly on compact sets of $x \in G$.

3. $\mathcal{C}$ has property (T) if and only if $G$ has property (T) and every sequence sequence of $\text{Ad} G$-invariant probability measures $\mu_n \in S(C_0(G))$ that converges to $\delta_e$ weakly* must converge in norm.

4. $\text{Rep} K \subset \mathcal{C}$ has the relative property (T) if and only if every sequence of probability measures $\mu_n \in S(C_0(G))$ such that $\mu_n \to \delta_e$ weakly* and $\|\mu_n \circ \text{Ad} x - \mu_n\| \to 0$ uniformly on compact sets of $x \in G$ satisfies $\|\mu_n - \delta_e\| \to 0$.

**Proof.** Denote by $\epsilon : C_0(G) \rtimes_{\text{Ad}}^f G \to \mathbb{C}$ the character given by $\epsilon(F \lambda_f) = F(\epsilon) \int_G f(x) dx$. Write $q = 1_{KPK}$.

1. Combining Proposition 3.4 and [PV14, Proposition 5.3], we get that $\mathcal{C}$ is amenable if and only if the canonical *-homorphism $q(C_0(G) \rtimes_{\text{Ad}}^f G)q \to q(C_0(G) \rtimes_{\text{Ad}}^0 G)q$ is an isomorphism. This holds if and only if $G$ is amenable.

2. First assume that $\mathcal{C}$ has the Haagerup property. By Proposition 3.4, we find a sequence of states $\omega_n$ on $q(C_0(G) \rtimes_{\text{Ad}}^f G)q$ such that $\omega_n \to \epsilon$ weakly* and such that every $\omega_n$ is a $c_0$ state in the sense of Definition 3.3. For every $x \in G$, define $\mu_n(x) \in C_0(G)^*$ given by $\mu_n(x)(F) = \omega_n(F_u x)$.

Using the strictly continuous extension of $\omega_n$ to the multiplier algebra $M(C_0(G) \rtimes_{\text{Ad}}^f G)$, we get that $x \mapsto \omega_n(1_{u_x})$ is a sequence of continuous positive definite functions converging to 1 uniformly on compact subsets of $G$. We claim that for every fixed $n$, the function $x \mapsto \omega_n(x)$
tends to 0 at infinity. Denote by \( \epsilon_K : C_r^*(G) \to \mathbb{C} \) the state given by composing the conditional expectation \( C_r^*(G) \to C_r^*(K) \) with the trivial representation \( \epsilon : C_r^*(K) \to \mathbb{C} \). Then,

\[
\omega_n(x) = \epsilon_K(\lambda(\mu_n(x)))
\]

and the claim is proven. So, \( G \) has the Haagerup property.

The restriction of \( \omega_n \) to \( C_0(G) \) provides a sequence of \( c_0 \) probability measures \( \mu_n \in \mathcal{S}(C_0(G)) \) such that \( \mu_n \to \delta_c \) weakly* and \( \|\mu_n \circ \text{Ad } x - \mu_n\| \to 0 \) uniformly on compact sets of \( x \in G \).

Conversely assume that \( G \) has the Haagerup property and that \( \mu_n \) is such a sequence of probability measures. By restricting \( \mu_n \) to \( K \), normalizing and integrating \( \int_K (\mu_n \circ \text{Ad } k) \, dk \), we may assume that the probability measures \( \mu_n \) are supported on \( K \) and are \( \text{Ad } K \)-invariant. Fix a strictly positive right \( K \)-invariant function \( w : G \to \mathbb{R}_0^+ \) with \( \int_G w(g) \, dg = 1 \). Define the probability measures \( \tilde{\mu}_n \) on \( G \) given by

\[
\tilde{\mu}_n = \int_G w(g) \, \mu_n \circ \text{Ad } g \, dg.
\]

Note that \( \tilde{\mu}_n \) is still \( \text{Ad } K \)-invariant. Also,

\[
\lambda(\tilde{\mu}_n) = \int_G w(g) \, \lambda_\mu(\mu_n) \, \lambda_g \, dg
\]

so that each \( \tilde{\mu}_n \) is a \( c_0 \) probability measure.

By construction, for every \( x \in G \), the measure \( \tilde{\mu}_n \circ \text{Ad } x \) is absolutely continuous with respect to \( \tilde{\mu}_n \). We denote by \( \Delta_n(x) \) the Radon-Nikodym derivative and define the unitary representations

\[
\theta_n : G \to \mathcal{U}(L^2(G, \tilde{\mu}_n)) : \theta_n(x)\xi = \Delta_n(x)^{1/2} \xi \circ \text{Ad } x^{-1}.
\]

We also define \( \theta_n : C_0(G) \to B(L^2(G, \tilde{\mu}_n)) \) given by multiplication operators and we have thus defined a nondegenerate \( * \)-representation of \( C_0(G) \rtimes_{\text{Ad}} G \) on \( L^2(G, \tilde{\mu}_n) \).

Note that \( \mu_n \) is absolutely continuous with respect to \( \tilde{\mu}_n \). We denote by \( \zeta_n \in L^2(G, \tilde{\mu}_n) \) the square root of the Radon-Nikodym derivative of \( \mu_n \) with respect to \( \tilde{\mu}_n \). Since both \( \mu_n \) and \( \tilde{\mu}_n \) are \( \text{Ad } K \)-invariant, we get that \( \theta_n(p_K)\zeta_n = \zeta_n \). Since \( \mu_n \) is supported on \( K \), also \( \zeta_n \) is supported on \( K \) meaning that \( \theta(1_K)\zeta_n = \zeta_n \).

Since \( G \) has the Haagerup property, we can also fix a unitary representation \( \pi : G \to \mathcal{U}(\mathcal{H}) \) and a sequence of \( \pi(K) \)-invariant unit vectors \( \xi_n \in \mathcal{H} \) such that \( \|\pi(x)\xi_n - \xi_n\| \to 0 \) uniformly on compact sets of \( x \in G \) and, for every fixed \( n \), the function \( x \mapsto \langle \pi(x)\xi_n, \xi_n \rangle \) tends to zero at infinity.

The formulas \( \psi(x) = \theta_n(x) \otimes \pi(x) \) and \( \psi(F) = \theta(F) \otimes 1 \) define a nondegenerate \( * \)-representation of \( C_0(G) \rtimes_{\text{Ad}} G \) on \( L^2(G, \tilde{\mu}_n) \otimes \mathcal{H} \). We define the states \( \omega_n \) on \( C_0(G) \rtimes_{\text{Ad}} G \) given by \( \omega_n(T) = \langle \psi(T)(\zeta_n \otimes \xi_n), \xi_n \otimes \zeta_n \rangle \). By construction, \( \omega_n(q) = 1 \) for all \( n \) and \( \omega_n \to \epsilon \) weakly*. It remains to prove that each \( \omega_n \) is a \( c_0 \) state. Proposition 3.4 then gives that \( C \) has the Haagerup property.

Fix \( n \). Defining \( \mu_n(x) \in C_0(G)^* \) given by \( \mu_n(x)(F) = \omega_n(Fu_x) \), we get that

\[
\mu_n(x)(F) = \langle \theta_n(F) \theta(x) \zeta_n, \zeta_n \rangle \langle \pi(x)\xi_n, \xi_n \rangle.
\]

Since the function \( x \mapsto \langle \pi(x)\xi_n, \xi_n \rangle \) tends to zero at infinity, we get that even \( x \mapsto \|\mu_n(x)\| \) tends to zero at infinity. So, we only have to show that for every fixed \( x \), the complex measure given by \( F \mapsto \langle \theta_n(F) \theta(x) \zeta_n, \zeta_n \rangle \) is \( c_0 \). By construction, this complex measure is absolutely continuous with respect to \( \tilde{\mu}_n \). The conclusion then follows from Lemma 3.5.
3. Note that it follows from [PV14, Proposition 5.5] that $C$ has property $(T)$ if and only if every sequence of states on $q(C_0(G) \rtimes_{\text{Ad}} G)q$ converging weakly$^*$ to $\epsilon$ must converge to $\epsilon$ in norm.

First assume that $C$ has property $(T)$. Both states on $C^*(G)$ and $\text{Ad}$-$G$-invariant states on $C_0(G)$ give rise to states on $C_0(G) \rtimes_{\text{Ad}} G$. One implication of 3 thus follows immediately. Conversely assume that $G$ has property $(T)$ and that every sequence of $\text{Ad}$-$G$-invariant probability measures $\mu_n \in \mathcal{S}(C_0(G))$ converging weakly$^*$ to $\delta_\epsilon$ must converge in norm to $\delta_\epsilon$. Let $\omega_n$ be a sequence of states on $q(C_0(G) \rtimes_{\text{Ad}} G)q$ converging to $\epsilon$ weakly$^*$. Let $p \in C^*(G)$ be the Kazhdan projection. Replacing $\omega_n$ by $\omega_n(p)^{-1} p \cdot \omega_n \cdot p$, we may assume that $\omega_n$ is left and right $G$-invariant. This means that $\omega_n(Fu_x) = \mu_n(F)$ for all $F \in C_0(G)$, $x \in G$, where $\mu_n$ is a sequence of states on $G$-invariant probability measures on $G$ converging weakly$^*$ to $\delta_\epsilon$. Thus $\|\mu_n - \delta_\epsilon\| \to 0$ so that $\|\omega_n - \epsilon\| \to 0$.

4. First assume that $\text{Rep} \ K \subset C$ has the relative property $(T)$ and take a sequence of probability measures $\mu_n \in \mathcal{S}(C_0(G))$ such that $\mu_n \to \delta_\epsilon$ weakly$^*$ and $\|\mu_n \circ \text{Ad} x - \mu_n\| \to 0$ uniformly on compact sets of $x \in G$. We must prove that $\|\mu_n - \delta_\epsilon\| \to 0$. As in the proof of 2, we may assume that $\mu_n$ is supported on $K$ and that $\mu_n$ is $\text{Ad} \ K$-invariant, so that we can construct a sequence of states $\omega_n$ on $C_0(G) \rtimes_{\text{Ad}} G$ such that $\omega_n \to \delta_\epsilon$ weakly$^*$, $\omega_n = q \cdot \omega_n \cdot q$ and $\omega_n|_{C_0(G)} = \mu_n$ for all $n$.

The formula (3.10) associates to $\omega_n$ a sequence of cp-multipliers $\varphi_n$ on $C$ converging to 1 pointwise. Since $\text{Rep} \ K \subset C$ has the relative property $(T)$, we conclude that $\varphi_n(\pi, e) \to 1$ uniformly on $\pi \in \text{Irr}(K)$. Using [PV14, Lemma 5.6], it follows that $\|\omega_n|_{C_0(G)} - \delta_\epsilon\| \to 0$. So, $\|\mu_n - \delta_\epsilon\| \to 0$.

To prove the converse, let $\varphi_n : \text{Irr}(C) \to C$ be a sequence of cp-multipliers on $C$ converging to 1 pointwise. Denote by $\omega_n$ the states on $q(C_0(G) \rtimes_{\text{Ad}} G)q$ associated with $\varphi_n$ in Proposition 3.4. Since $\omega_n \to \epsilon$ weakly$^*$, the restriction $\mu_n := \omega_n|_{C_0(G)}$ is a sequence of probability measures on $G$ such that $\mu_n \to \delta_\epsilon$ weakly$^*$ and $\|\mu_n \circ \text{Ad} x - \mu_n\| \to 0$ uniformly on compact sets of $x \in G$. By our assumption, $\|\mu_n - \delta_\epsilon\| \to 0$. For every $\pi \in \text{Irr}(K)$, the function $\dim(\pi)^{-1} \chi_\pi$ has norm 1. Therefore, $\omega_n(\dim(\pi)^{-1} \chi_\pi) \to 1$ uniformly on $\text{Irr}(K)$. By (3.10), this means that $\varphi_n \to 1$ uniformly on $\text{Irr}(K)$.

The following proposition gives a concrete example where $G$ has the Haagerup property, while $C(K \lt G)$ does not and even has $\text{Rep} \ K$ as a full C$^*$-tensor subcategory with the relative property $(T)$.

**Proposition 4.2.** Let $F$ be a non-archimedean local field with characteristic $\neq 2$. Let $k \geq 2$ and define $G = \text{SL}(k, F)$. Let $K \subset G$ be a compact open subgroup, e.g. $K = \text{SL}(k, \mathcal{O})$, where $\mathcal{O}$ is the ring of integers of $F$. Denote by $C$ the rigid C$^*$-tensor category of finite rank $G$-$L^\infty(G/K)$-$L^\infty(G/K)$-modules.

1. $\text{Rep} \ K \subset C$ has the relative property $(T)$. In particular, $C$ does not have the Haagerup property, although for $k = 2$, the group $G$ has the Haagerup property.

2. $C$ has property $(T)$ for all $k \geq 3$.

**Proof.** We denote by $\mathbb{I}$ the identity element of $G = \text{SL}(k, F)$. Let $\mu_n \in \mathcal{S}(C_0(G))$ be a sequence of probability measures on $G$ such that $\mu_n \to \delta_\mathbb{I}$ weakly$^*$ and $\|\mu_n \circ \text{Ad} x - \mu_n\| \to 0$ uniformly on compact sets of $x \in G$. Assume that $\|\mu_n - \delta_\mathbb{I}\| \not\to 0$. Passing to a subsequence and replacing $\mu_n$ by the normalization of $\mu_n - \mu_n(\{\mathbb{I}\})\delta_\mathbb{I}$, we may assume that $\mu_n(\{\mathbb{I}\}) = 0$ for all $n$. Since $\mu_n \to \delta_\mathbb{I}$ weakly$^*$ and since there are at most $k$ of $k$'th roots of unity in $F$, we may also assume that $\mu_n(\{\lambda\}) = 0$ for all $n$ and all $k$'th roots of unity $\lambda \in F$. 

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Every $\mu_n$ defines a state $\Omega_n$ on the $C^*$-algebra $\mathcal{L}(G)$ of all bounded Borel functions on $G$. Choose a weak$^*$-limit point $\Omega \in \mathcal{L}(G)^*$ of the sequence $(\Omega_n)$. Then, $\Omega$ induces an Ad$G$-invariant mean on the Borel sets of $G$. In particular, $\Omega$ defines an Ad$G$-invariant mean $\Omega$ on the Borel sets of the space $M_n(F)$ of $n \times n$ matrices over $F$. By Lemma 4.5 below, $\Omega$ is supported on the diagonal $F^1 \subset M_n(F)$. Since $\Omega$ is also supported on $G$, it follows that $\Omega$ is supported on the finite set of $\lambda \bar{l}$ where $\lambda$ is a $k'$th root of unity in $F$. But by construction, $\Omega\{\lambda \bar{l}\} = 0$ for all $k'$th roots of unity $\lambda \in F$. We have reached a contradiction. So, $\|\mu_n - \delta_l\| \to 0$.

By Theorem 4.1, Rep $K \subset \mathcal{C}$ has the relative property (T). For $k \geq 3$, the group $\text{SL}(k, F)$ has property (T) and it follows from Theorem 4.1 that $\mathcal{C}$ has property (T).

The following example of [C05] illustrates that $G$ may have property (T), while the category $\mathcal{C}$ of finite rank $G$-$L^\infty(G/K)$-$L^\infty(G/K)$-modules does not.

**Example 4.3.** Let $F$ be a non-archimedean local field and $k \geq 3$. Define the closed subgroup $G < \text{SL}(k+2, F)$ given by

$$G = \left\{ \begin{pmatrix} 1 & b_1 & \cdots & b_k & c \\ 0 & a_{11} & \cdots & a_{1k} & d_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & a_{k1} & \cdots & a_{kk} & d_k \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix} \right\},$$

where

$$A = (a_{ij}) \in \text{SL}(k, F), b_i, c, d_j \in F.$$ As in [C05], we get that $G$ has property (T). Also, the center of $G$ is isomorphic with $F$ (sitting in the upper right corner) and since $F$ is non discrete, we can take a sequence $g_n \in \mathcal{Z}(G)$ with $g_n \neq e$ for all $n$ and $g_n \to e$. Using the Ad$G$-invariant probability measures $\delta_{g_n}$, it follows from Theorem 4.1 that $\mathcal{C}$ does not have property (T).

Finally, we also include a nonamenable example having the Haagerup property.

**Example 4.4.** Let $2 \leq |m| < n$ be integers. Define the totally disconnected compact abelian group $K = \mathbb{Z}_m$, as the profinite completion of $\mathbb{Z}$ with respect to the decreasing sequence of finite index subgroups $(n^k \mathbb{Z})_{k \geq 0}$. We have open subgroups $mK < K$ and $nK < K$, as well as the isomorphism $\varphi : mK \to nK : \varphi(mk) = nk$ for all $k \in K$. We define $G$ as the HNN extension of $K$ and $\varphi$. Alternatively, we may view $K < G$ as the Schlichting completion of the Baumslag-Solitar group

$$B(m, n) = \langle a, t \mid ta^mt^{-1} = a^n \rangle$$

and the almost normal subgroup $\langle a \rangle$.

Since $G$ is acting properly on a tree, $G$ has the Haagerup property. Also, $G$ is nonamenable. For all positive integers $k, l \geq 0$, we denote by $\mu_{k,l}$ the normalized Haar measure on the open subgroup $n^km^lK$. Note that $\varphi_*(\mu_{k,l}) = \mu_{k+1, l-1}$ whenever $k, l \geq 1$. Then the probability measures

$$\mu_n := \frac{1}{n+1} \sum_{k=0}^n \mu_{n+k, 2n-k}$$

are absolutely continuous with respect to the Haar measure of $G$, and thus $c_0$ in the sense of Definition 3.3, and they satisfy $\mu_n \to \delta_e$ weakly$^*$ and $\|\mu_n \circ \text{Ad} x - \mu_n\| \to 0$ uniformly on compact sets of $x \in G$. By Theorem 4.1, $\mathcal{C}$ has the Haagerup property.

**Lemma 4.5.** Let $F$ be a local field with characteristic $\neq 2$. Let $k \geq 2$ and define $G = \text{SL}(k, F)$. Every Ad$G$-invariant mean on the Borel sets of the space $M_k(F)$ of $k \times k$ matrices over $F$ is supported on the diagonal $F^1 \subset M_k(F)$.

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Proof. We start by proving the lemma for \( k = 2 \). So assume that \( m \) is an \( \text{Ad SL}(2, F) \)-invariant mean on the Borel sets of \( M_2(F) \).

In the proof of [BHvO8, Proposition 1.4.12], it is shown that if \( m \) is a mean on the Borel sets of \( F^2 \) that is invariant under the transformations \( \lambda \cdot (x, y) := (x + \lambda y, y) \) for all \( \lambda \in F \), then

\[
m(\{(x, y) \mid (x, y) \neq (0, 0), |x| \leq |y|\}) = 0.
\]

Define \( g_\lambda := \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} \) and notice that

\[
g_\lambda \begin{pmatrix} a & b \\ c & d \end{pmatrix} g_\lambda^{-1} = \begin{pmatrix} a + \lambda c & -\lambda a + b - \lambda^2 c + \lambda d \\ c & -\lambda c + d \end{pmatrix}.
\]

Hence, the map \( \theta : M_2(F) \to F^2 : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto (a - d, c) \) satisfies \( \theta(g_\lambda A g_\lambda^{-1}) = (2\lambda) \cdot \theta(A) \).

Therefore, \( m(\Omega_0) = 0 \) for \( \Omega_0 := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(F) \mid |a - d| \leq |c| \text{ and } (a - d, c) \neq (0, 0) \right\} \).

Taking the adjoint by \( \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \) for \( |\lambda| \geq 2 \), we get that \( m(\Omega_1) = 0 \) for \( \Omega_1 := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(F) \mid |a - d| \leq 4|c| \text{ and } (a - d, c) \neq (0, 0) \right\} \).

For the same reason, we get that \( m(\Omega_1') = 0 \) for \( \Omega_1' := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(F) \mid |a - d| \leq 4|b| \text{ and } (a - d, b) \neq (0, 0) \right\} \).

Write \( X = M_2(F) \setminus F\mathbb{I} \). The matrices with \( (a - d, c) = (0, 0) \) belong to \( \Omega_1' \) unless they are diagonal. Similarly, the matrices with \( (a - d, b) = (0, 0) \) belong to \( \Omega_1 \) unless they are diagonal. So, we find that \( m(\Omega) = 0 = m(\Omega') \) for \( \Omega = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in X \mid |a - d| \leq 4|c| \right\} \) and \( \Omega' = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in X \mid |a - d| \leq 4|b| \right\} \).

Put \( \Omega'' := g_1 \Omega g_1^{-1} \), so that \( m(\Omega'') = 0 \). To conclude the proof in the case \( k = 2 \), it suffices to show that \( \Omega \cup \Omega' \cup \Omega'' = X \).

Take \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in X \setminus (\Omega \cup \Omega') \). So, \( \frac{1}{4}|a - d| > |b|, |c| \). We claim that

\[
\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} := g_1^{-1} \begin{pmatrix} a & b \\ c & d \end{pmatrix} g_1 = \begin{pmatrix} a - c & a + b - c - d \\ c & c + d \end{pmatrix}
\]

belongs to \( \Omega \). Since

\[
|a' - d'| = |a - d - 2c| \leq |a - d| + 2|c| < \frac{3}{2}|a - d| \quad \text{and} \quad |b'| \geq |a - d| - |c| - |b| > \frac{1}{2}|a - d|,
\]

we indeed get that \( |a' - d'| < 3|b'| \). The claim follows and the lemma is proved in the case \( k = 2 \).
For an arbitrary \( k \geq 2 \) and fixed \( 1 \leq p < q \leq k \), the map

\[
M_k(F) \rightarrow M_2(F) : (x_{ij}) \mapsto \begin{pmatrix} x_{pp} & x_{pq} \\ x_{qp} & x_{qq} \end{pmatrix}
\]

is \( \text{Ad SL}(2, F) \)-equivariant. So, an \( \text{Ad SL}(k, F) \)-invariant mean \( m \) on \( M_k(F) \) is supported on \( \{(x_{ij}) \in M_k(F) \mid x_{pp} = x_{qq}, x_{pq} = x_{qp} = 0\} \). Since \( F \mathbb{I} \) is the intersection of these sets, \( m \) is supported on \( F \mathbb{I} \).

\[ \square \]

## 5 Weak amenability of rigid C*-tensor categories

Following [PV14, Definition 5.1], a rigid C*-tensor category is called weakly amenable if there exists a sequence of completely bounded (cb) multipliers \( \varphi_n : \text{Irr}(\mathcal{C}) \rightarrow \mathbb{C} \) (see Definition 3.2) converging to 1 pointwise, with \( \limsup_n \| \varphi_n \|_{\text{cb}} < \infty \) and with \( \varphi_n \) finitely supported for every \( n \).

Recall from the first paragraphs of Section 3 the definition of the tube \(*\)-algebra \( \mathcal{A} \) of \( \mathcal{C} \) with respect to a full family of objects in \( \mathcal{C} \). To every function \( \varphi : \text{Irr}(\mathcal{C}) \rightarrow \mathbb{C} \), we associate the linear map

\[
\theta_\varphi : \mathcal{A} \rightarrow \mathcal{A} : \theta_\varphi(V) = \varphi(\alpha) V \quad \text{for all } V \in (i\alpha, \alpha j) .
\]

We define \( \| \theta_\varphi \|_{\text{cb}} \) by viewing \( \mathcal{A} \) inside its reduced C*-algebra, i.e. by viewing \( \mathcal{A} \subset B(L^2(\mathcal{A}, \text{Tr})) \), where \( \text{Tr} \) is the canonical trace on \( \mathcal{A} \). We also consider the von Neumann algebra \( \mathcal{A}'' \) generated by \( \mathcal{A} \) acting on \( L^2(\mathcal{A}, \text{Tr}) \).

In the following result, we clarify the link between the complete boundedness of \( \varphi \) in the sense of Definition 3.2 and the complete boundedness of the map \( \theta_\varphi \).

**Proposition 5.1.** Let \( \mathcal{C} \) be a rigid C*-tensor category. Denote by \( \mathcal{A} \) the tube \(*\)-algebra of \( \mathcal{C} \) with respect to a full family of objects in \( \mathcal{C} \). Let \( \varphi : \text{Irr}(\mathcal{C}) \rightarrow \mathbb{C} \) be any function.

Then, \( \| \varphi \|_{\text{cb}} = \| \theta_\varphi \|_{\text{cb}} \). If this cb-norm is finite, we can uniquely extend \( \theta_\varphi \) to a normal completely bounded map on \( \mathcal{A}'' \) having the same cb-norm.

**Proof.** For any family \( J \) of objects, we can define the tube \(*\)-algebra \( \mathcal{A}_J \) and the linear map \( \theta^J_\varphi : \mathcal{A}_J \rightarrow \mathcal{A}_J \). By strong Morita equivalence, we have \( \| \theta^J_\varphi \|_{\text{cb}} = \| \theta_\varphi \|_{\text{cb}} \) whenever \( J \) is full and we have \( \| \theta^J_\varphi \|_{\text{cb}} \leq \| \theta_\varphi \|_{\text{cb}} \) for arbitrary \( J \). Also, using standard solutions for the conjugate equations, we get natural linear maps \( (i\alpha, \alpha j) \rightarrow (\overline{\varphi}, \varphi) \) and they define a trace preserving \(*\)-anti-isomorphism of \( \mathcal{A}_J \) onto \( \mathcal{A}_J \). Defining \( \overline{\varphi} : \text{Irr}(\mathcal{C}) \rightarrow \mathbb{C} \) by \( \overline{\varphi}(\alpha) = \varphi(\overline{\alpha}) \) for all \( \alpha \in \text{Irr}(\mathcal{C}) \), it follows that \( \| \theta_\varphi \|_{\text{cb}} = \| \theta_{\overline{\varphi}} \|_{\text{cb}} \) and it follows that \( \theta_\varphi \) extends to a normal completely bounded map on \( \mathcal{A}'' \) if and only if \( \theta_{\overline{\varphi}} \) extends to \( \mathcal{A}'' \).

So, it suffices to prove that \( \| \varphi \|_{\text{cb}} = \| \theta_{\overline{\varphi}} \|_{\text{cb}} \) and that in the case where \( \| \varphi \|_{\text{cb}} < \infty \), we can extend \( \theta_{\overline{\varphi}} \) to a normal completely bounded map on \( \mathcal{A}'' \). First assume that \( \| \theta_{\overline{\varphi}} \|_{\text{cb}} \leq \kappa \). Fix arbitrary objects \( \alpha, \beta \in \mathcal{C} \) and write \( \Psi^{\varphi}_{\alpha|\beta} := \Psi^{\overline{\phi}}_{\alpha|\beta} \). We prove that \( \| \Psi^{\varphi}_{\alpha|\beta} \|_{\text{cb}} \leq \kappa \). Since \( \alpha, \beta \) were arbitrary, it then follows that \( \| \varphi \|_{\text{cb}} \leq \kappa \).

Note that \((\alpha|\beta, \alpha|\beta)\) is a finite dimensional C*-algebra. Consider the following three bijective
Consider the projection \(\eta_1: \bigoplus_{\pi \in \text{Irr}(C)} ((\alpha, \alpha \pi) \otimes (\pi \beta, \beta)) \to (\alpha \beta, \alpha \beta) : \eta_1(V \otimes W) = (V \otimes 1)(1 \otimes W)\),
\[\eta_2: \bigoplus_{\pi \in \text{Irr}(C)} ((\alpha, \alpha \pi) \otimes (\pi \beta, \beta)) \to \bigoplus_{\pi \in \text{Irr}(C)} ((\alpha \pi, \alpha) \otimes (\beta, \pi \beta)) : \eta_2(V \otimes W) = (V \otimes 1)(1 \otimes s_\pi \otimes (t_\pi^* \otimes 1)(1 \otimes W)\),
\[\eta_3: \bigoplus_{\pi \in \text{Irr}(C)} ((\alpha \pi, \alpha) \otimes (\beta, \pi \beta)) \to \mathcal{A}_{\beta \alpha} : \eta_3(V \otimes W) = (1 \otimes V)(W \otimes 1)\).

A direct computation shows that \(\eta := \eta_3 \circ \eta_2 \circ \eta_1^{-1}\) is a unital faithful \(*\)-homomorphism of \((\alpha \beta, \alpha \beta)\) to the tube \(*\)-algebra \(\mathcal{A}_{\beta \alpha}\). One also checks that \(\theta_{\bar{\varphi}}^{\beta \alpha} \circ \eta = \eta \circ \Psi_{\alpha \beta}^\varphi\). So, we get that
\[\|\Psi_{\alpha \beta}^\varphi\|_{cb} \leq \|\theta_{\bar{\varphi}}^{\beta \alpha}\|_{cb} \leq \|\theta_{\bar{\varphi}}\|_{cb} \leq \kappa.\]

Conversely, assume that \(\|\varphi\|_{cb} \leq \kappa\). Define the ind-objects \(\rho_1\) and \(\rho_2\) for \(\mathcal{C}\) given by
\[
\rho_1 = \bigoplus_{\alpha, i \in \text{Irr}(C)} \alpha i \quad \text{and} \quad \rho_2 = \bigoplus_{\alpha \in \text{Irr}(C)} \alpha.\]

Define the type I von Neumann algebra \(\mathcal{M}\) of all bounded endomorphisms of \(\rho_1 \rho_2\). Note that for all \(\alpha, i, \beta \in \text{Irr}(\mathcal{C})\), we have the natural projection \(p_\alpha \otimes p_i \otimes p_\beta \in \mathcal{M}\) and we have the identification
\[(p_\alpha \otimes p_i \otimes p_\beta) \mathcal{M}(p_\gamma \otimes p_j \otimes p_\delta) = (\alpha i \beta, \gamma j \delta)\]
for all \(\alpha, i, \beta, \gamma, j, \delta \in \text{Irr}(\mathcal{C})\). By our assumption, there is a normal completely bounded map \(\Psi: \mathcal{M} \to \mathcal{M}\) satisfying
\[\Psi(V) = \Psi_{\alpha \beta, \gamma j \delta}^\varphi(V) \quad \text{for all} \ V \in (\alpha i \beta, \gamma j \delta).\]

We have \(\|\Psi\|_{cb} \leq \kappa\).

Consider the projection \(q \in \mathcal{M}\) given by
\[q = \sum_{\alpha, i \in \text{Irr}(\mathcal{C})} p_\alpha \otimes p_i \otimes p_\alpha.\]

Since \(\Psi(qTq) = q\Psi(T)q\) for all \(T \in \mathcal{M}\), the map \(\Psi\) restricts to a normal completely bounded map on \(q \mathcal{M} q\) with \(\|\Psi|_{q \mathcal{M} q}\|_{cb} \leq \kappa\).

Denote by \(\mathcal{A}\) the tube \(*\)-algebra associated with \(\text{Irr}(\mathcal{C})\) itself as a full family of objects. We construct a faithful normal \(*\)-homomorphism \(\Theta: \mathcal{A}'' \to q \mathcal{M} q\) satisfying \(\Psi \circ \Theta = \Theta \circ \theta_{\bar{\varphi}}\). Once we have obtained \(\Theta\), it follows that \(\|\theta_{\bar{\varphi}}\|_{cb} \leq \kappa\) and that \(\theta_{\bar{\varphi}}\) extends to a normal completely bounded map on \(\mathcal{A}''\).

To construct \(\Theta\), define the Hilbert space
\[\mathcal{H} = \bigoplus_{\alpha, i, j \in \text{Irr}(\mathcal{C})} (\pi i \alpha, j)\]
and observe that we have the natural faithful normal \(*\)-homomorphism \(\pi: q \mathcal{M} q \to \mathcal{B}(\mathcal{H})\) given by left multiplication. Also consider the unitary operator
\[U: L^2(\mathcal{A}, \text{Tr}) \to \mathcal{H}: U(V) = d(\alpha)^{-1/2} (1 \otimes V)(t_\alpha \otimes 1) \quad \text{for all} \ V \in (i \alpha, \alpha j).\]
We claim that $\Theta$ can be constructed such that $\pi(\Theta(V)) = UVU^*$ for all $V \in \mathcal{A}$. To prove this claim, fix $i, \alpha, j \in \operatorname{Irr}(\mathcal{C})$ and $V \in (\phi, \alpha, j)$. For all $\gamma, \beta \in \operatorname{Irr}(\mathcal{C})$, define the element $W_{\gamma,\beta} \in \mathcal{C}(\mathcal{C}(\gamma, \beta))$ given by the finite sum

$$W_{\gamma,\beta} = \sum_{Z \in \operatorname{comp}(\gamma, \beta)} d(\beta)^{1/2} d(\gamma)^{1/2} (1 \otimes 1 \otimes \tilde{Z}) (1 \otimes V \otimes 1) (Z \otimes 1 \otimes 1), \quad (5.1)$$

where $\tilde{Z} = (1 \otimes t^*_\beta)(1 \otimes Z^* \otimes 1)(s_\gamma \otimes 1)$ belongs to $(\gamma, \alpha \beta)$. A direct computation shows that

$$\langle \pi(W_{\gamma,\beta}) U(X), U(Y) \rangle = \langle V \cdot X, Y \rangle$$

for all $X \in (j, \beta, k)$ and $Y \in (i, \gamma, l)$. So, there is a unique element $\Theta(V) \in (1 \otimes p_i \otimes 1)q \mathcal{M}q(1 \otimes p_j \otimes 1)$ satisfying

$$(p_\gamma \otimes p_i \otimes p_\gamma) \Theta(V) (p_\gamma \otimes p_j \otimes p_\beta) = W_{\gamma,\beta}$$

for all $\gamma, \beta \in \operatorname{Irr}(\mathcal{C})$ and $\pi(\Theta(V)) = UVU^*$.

We have defined a faithful normal $*$-homomorphism $\Theta : \mathcal{A}^\omega \to q \mathcal{M}q$. It remains to prove that $\Psi \circ \Theta = \Theta \circ \theta_\pi$. Using (5.1), it suffices to prove that

$$\Psi_{\gamma,\beta, \alpha, j}^\phi (1 \otimes V \otimes 1) = \varphi(\pi) 1 \otimes V \otimes 1. \quad (5.2)$$

The left hand side of (5.2) equals $1 \otimes \Psi_{\gamma,\beta, \alpha, j}^\phi (V) \otimes 1$. Writing $V = (T \otimes 1)(1 \otimes 1 \otimes s_\pi)$ with $T \in (i, \alpha \beta \pi)$, we have

$$\Psi_{\gamma,\beta, \alpha, j}^\phi (V) = (T \otimes 1)\Psi_{\gamma,\beta, \alpha, j}^\phi (1 \otimes 1 \otimes s_\pi) = (T \otimes 1)(1 \otimes 1 \otimes \Psi_{\gamma,\beta, \alpha, j}^\phi (s_\pi))$$

$$= \varphi(\pi) (T \otimes 1)(1 \otimes 1 \otimes s_\pi) = \varphi(\pi) V.$$ 

So (5.2) holds and the proposition is proven. \hfill $\square$

6 Weak amenability of $\mathcal{C}(K < G)$

**Theorem 6.1.** Let $G$ be a totally disconnected group and $K < G$ a compact open subgroup. Denote by $\mathcal{C}$ the rigid $C^*$-tensor category of finite rank $G$-$L^\infty(G/K)$-$L^\infty(G/K)$-modules.

Then $\mathcal{C}$ is weakly amenable if and only if $G$ is weakly amenable and there exists a sequence of probability measures $\omega_n \in C_0(G)$ such that are absolutely continuous with respect to the Haar measure and such that $\omega_n \to \delta_e$ weakly* and $\|\omega_n \circ \text{Ad} x - \omega_n\| \to 0$ uniformly on compact sets of $x \in G$.

In that case, the Cowling-Haagerup constant $\Lambda(\mathcal{C})$ of $\mathcal{C}$ equals $\Lambda(G)$.

In order to prove Theorem 6.1, we must describe the cb-multipliers on $\mathcal{C}$ in terms of completely bounded multipliers on the $C^*$-algebra $C_0(G) \rtimes \text{Ad} G$.

We denote by $\operatorname{Pol}(G)$ the $*$-algebra of locally constant, compactly supported functions on $G$. Note that $\operatorname{Pol}(G)$ is the linear span of the functions of the form $1_L y$ where $y \in G$ and $L < G$ is a compact open subgroup. Also note that for any compact open subgroup $K_0 < G$, $\operatorname{Pol}(K_0)$ coincides with the $*$-algebra of coefficients of finite dimensional unitary representations of $K_0$.

We define $\mathcal{E}(G) = \operatorname{Pol}(G)^*$ as the space of all linear maps from $\operatorname{Pol}(G)$ to $\mathbb{C}$. Note that $\mathcal{E}(G)$ can be identified with the space of finitely additive, complex measures on the space $\mathcal{F}(G)$ of compact open subsets of $G$. 

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When \( K_0 < G \) is a compact open subgroup, we say that a map \( \mu : G \to \mathcal{E}(G) \) is \( K_0 \)-equivariant if
\[
\mu(kxk') = \mu(x) \circ \text{Ad} k^{-1} \quad \text{for all } k, k' \in K_0.
\]
Note that this implies that \( \mu(x) \) is \( \text{Ad}(K_0 \cap xK_0x^{-1}) \)-invariant for all \( x \in G \).

As in (3.4), we associate to every \( x \in G \) and \( \pi \in \text{Irr}(K \cap xKx^{-1}) \) the irreducible object \((\pi, x) \in \text{Irr}(\mathcal{C})\) defined as the irreducible \( G-L^\infty(G/K)-L^\infty(G/K) \)-module \( \mathcal{H} \) such that \( \pi \) is isomorphic with the representation of \( K \cap xKx^{-1} \) on \( 1_{xK} \cdot \mathcal{H} \cdot 1_{eK} \). The formula
\[
\varphi(\pi, x) = \dim(\pi)^{-1} \mu(x)(\chi_x) \quad (6.1)
\]
then gives a bijection between the set of all functions \( \varphi : \text{Irr}(\mathcal{C}) \to \mathbb{C} \) and the set of all \( K \)-equivariant maps \( \mu : G \to \mathcal{E}(G) \) with the property that \( \mu(x) \) is supported on \( K \cap xKx^{-1} \) for every \( x \in G \).

Denote by \( \mathcal{P} = \text{Pol}(L^\infty(G) \rtimes \text{Ad} G) \) the dense \(*\)-subalgebra defined in (3.3). We always equip \( \mathcal{P} \) with the operator space structure inherited from \( \mathcal{P} \subset L^\infty(G) \rtimes \text{Ad} G \). As in Section 5, to every function \( \varphi : \text{Irr}(\mathcal{C}) \to \mathbb{C} \) is associated a linear map \( \theta_\varphi : \mathcal{A} \to \mathcal{A} \) on the tube \(*\)-algebra \( \mathcal{A} \) of \( \mathcal{C} \). We now explain how to associate to any \( K_0 \)-equivariant map \( \mu : G \to \mathcal{E}(G) \) a linear map \( \Psi_\mu : \mathcal{P} \to \mathcal{P} \). When \( \varphi \) and \( \mu \) are related by (6.1) and \( \Theta : \mathcal{A} \to \mathcal{P} \) is the \(*\)-anti-isomorphism of Theorem 3.1, it will turn out that \( \Psi_\mu \circ \Theta = \Theta \circ \theta_\varphi \), so that in particular, \( \|\theta_\varphi\|_{cb} = \|\Psi_\mu\|_{cb} \). We will further prove a criterion for \( \Psi_\mu \) to be completely bounded and that will be the main tool to prove Theorem 6.1.

Denote \( \Delta : L^\infty(G) \to L^\infty(G \times G) : \Delta(F)(g, h) = F(gh) \). For every \( \mu \in \mathcal{E}(G) \), the linear map
\[
\psi_\mu : \text{Pol}(G) \to \text{Pol}(G) : \psi_\mu(F) = (\mu \otimes \text{id})\Delta(F)
\]
is well defined. When \( \mu : G \to \mathcal{E}(G) \) is \( K_0 \)-equivariant with respect to the compact open subgroup \( K_0 < G \), we define
\[
\Psi_\mu : \mathcal{P} \to \mathcal{P} : \Psi_\mu(Fu_x p_L) = \psi_\mu(x)(F) u_x p_L
\]
for every \( F \in \text{Pol}(G), \ x \in G \) and open subgroup \( L < K_0 \).

**Lemma 6.2.** Denote by \( \Theta : \mathcal{A} \to \mathcal{P} \) the \(*\)-anti-isomorphism constructed in Theorem 3.1 between the tube \(*\)-algebra \( \mathcal{A} \) and \( \mathcal{P} = \text{Pol}(L^\infty(G) \rtimes \text{Ad} G) \). Let \( \varphi : \text{Irr}(\mathcal{C}) \to \mathbb{C} \) be any function and denote by \( \mu : G \to \mathcal{E}(G) \) the associated \( K \)-equivariant map given by (6.1) with \( \mu(x) \) supported in \( K \cap xKx^{-1} \) for all \( x \in G \). Then, \( \Psi_\mu \circ \Theta = \Theta \circ \theta_\varphi \).

**Proof.** The result follows from a direct computation using (3.7). \( \square \)

We prove the following technical result in exactly the same way as [J91].

**Lemma 6.3.** Let \( K_0, K < G \) be compact open subgroups and \( \mu : G \to \mathcal{E}(G) \) a \( K_0 \)-equivariant map. Let \( \kappa \geq 0 \). Then the following conditions are equivalent.

1. \( \Psi_\mu \) extends to a completely bounded map on \( C_0(G) \rtimes_{\text{Ad}}^* G \) with \( \|\Psi_\mu\|_{cb} \leq \kappa \).
2. \( \Psi_\mu \) extends to a normal completely bounded map on \( L^\infty(G) \rtimes_{\text{Ad}} G \) with \( \|\Psi_\mu\|_{cb} \leq \kappa \).
3. There exists a nondegenerate \(*\)-representation \( \pi : C_0(G) \rtimes_{\text{Ad}} I G \to B(\mathcal{K}) \) and bounded maps \( V, W : G \to \mathcal{K} \) such that
• $V(kxk') = \pi(k) V(x)$ and $W(kxk') = \pi(k) W(x)$ for all $x \in G$, $k \in K_0$ and $k' \in K$,
• $\mu(zy^{-1})(F) = \langle \pi(F) \pi(zy^{-1}) V(y), W(z) \rangle$ for all $F \in \text{Pol}(G)$ and $y, z \in G$,
• $\|V\|_\infty \|W\|_\infty \leq \kappa$.

In particular, every $\mu(x)$ is an actual complex measure on $G$, i.e. $\mu(x) \in C_0(G)^*$. 

**Proof.** 1 $\Rightarrow$ 3. Denote $P = C_0(G) \rtimes_{\text{Ad}}^f G$ and consider the (unique) completely bounded extension of $\Psi_\mu$ to $P$, which we still denote as $\Psi_\mu$. Define the nondegenerate $*$-representation

$$\zeta : P \rightarrow B(L^2(G)) : \zeta(F) = F(e)1 \quad \text{and} \quad \zeta(u_x) = \lambda_x$$

for all $F \in C_0(G)$, $x \in G$. Then $\zeta \circ \Psi_\mu : P \rightarrow B(L^2(G))$ has cb norm bounded by $\kappa$ and satisfies

$$(\zeta \circ \Psi_\mu)(u_k S u_{k'}) = \lambda_k (\zeta \circ \Psi_\mu)(S) \lambda_{k'}$$

for all $S \in P$, $k, k' \in K_0$. By the Stinespring dilation theorem proved in [BO08, Theorem B.7], we can choose a nondegenerate $*$-representation $\pi : P \rightarrow B(K)$ and bounded operators $\mathcal{V}, \mathcal{W} : L^2(G) \rightarrow K$ such that

• $(\zeta \circ \Psi_\mu)(S) = \mathcal{W}^* \pi(S) \mathcal{V}$ for all $S \in P$,
• $\mathcal{V}\lambda_k = \pi(k) \mathcal{V}$ and $\mathcal{W}\lambda_k = \pi(k) \mathcal{W}$ for all $k \in K_0$,
• $\|\mathcal{V}\| \|\mathcal{W}\| = \|\Psi_\mu\|_{cb} \leq \kappa$.

We normalize the left Haar measure on $G$ such that $\lambda(K) = 1$ and define the maps $V, W : G \rightarrow K$ given by $V(y) = V(1_{yK})$ and $W(z) = W(1_{zK})$. By construction, 3 holds.

3 $\Rightarrow$ 2. Write $P'' = L^\infty(G) \rtimes_{\text{Ad}} G$. Denote by $\pi_r : P'' \rightarrow B(L^2(G \times G))$ the standard representation given by

$$(\pi_r(F) \xi)(g, h) = F(hg \tau^{-1}) \xi(g, h) \quad \text{and} \quad (\pi_r(u_x) \xi)(g, h) = \xi(g, x^{-1}h)$$

for all $g, h, x \in G$, $F \in L^\infty(G)$. For every nondegenerate $*$-representation $\pi : C_0(G) \rtimes_{\text{Ad}}^f G \rightarrow B(K)$, there is a unique normal $*$-homomorphism $\tilde{\pi} : P'' \rightarrow B(K \otimes L^2(G \times G))$ satisfying

$$\tilde{\pi}(F) = (\pi \otimes \pi_r)(\Delta(F)) \quad \text{and} \quad \tilde{\pi}(u_x) = \pi(x) \otimes \pi_r(x)$$

for all $F \in C_0(G)$, $x \in G$. Given $V$ and $W$ as in 3, we then define the bounded operators $\mathcal{V}, \mathcal{W} : L^2(G \times G) \rightarrow K \otimes L^2(G \times G)$ by

$$(\mathcal{V}\xi)(g, h) = \xi(g, h)V(h) \quad \text{and} \quad (\mathcal{W}\xi)(g, h) = \xi(g, h)W(h)$$

for all $g, h \in G$. Note that $\|\mathcal{V}\| = \|V\|_\infty$ and $\|\mathcal{W}\| = \|W\|_\infty$. Since $\Psi_\mu(T) = \mathcal{W}^* \tilde{\pi}(T) \mathcal{V}$ for all $T \in \mathcal{P}$, it follows that 2 holds.

2 $\Rightarrow$ 1 is trivial. □

We are now ready to prove Theorem 6.1. We follow closely the proof of [O10, Theorem A].

**Proof of Theorem 6.1.** We define $\mathcal{Q}(G)$ as the set of all maps $\mu : G \rightarrow \mathcal{E}(G)$ satisfying the following properties:

• there exists a compact open subgroup $K_0 < G$ such that $\mu$ is $K_0$-equivariant,
• for every $x \in G$, we have that $\mu(x) \in C_0(G)^*$, $\mu(x)$ is compactly supported and $\mu(x)$ is absolutely continuous with respect to the Haar measure,

• $\|\Psi\mu\|_{cb} < \infty$.

Writing $\|\mu\|_{cb} := \|\Psi\mu\|_{cb}$, we call a sequence $\mu_n \in Q(G)$ a cbai (completely bounded approximate identity) if

• $\limsup_n \|\mu_n\|_{cb} < \infty$,

• for every $F \in C_0(G)$, we have that $\mu_n(x)(F) \to F(e)$ uniformly on compact sets of $x \in G$,

• for every $n$, we have that $\mu_n$ has compact support (i.e. $\mu_n(x) = 0$ for all $x$ outside a compact subset of $G$).

If a cbai exists, we define $\Gamma(G)$ as the smallest possible value of $\limsup_n \|\mu_n\|_{cb}$, where $(\mu_n)$ runs over all cbai. Note that this smallest possible value is always attained by a cbai.

First assume that $C$ is weakly amenable. By Proposition 5.1, we can take a sequence of finitely supported functions $\varphi_n : \text{Irr}(C) \to \mathbb{C}$ converging to 1 pointwise and satisfying $\limsup_n \|\varphi_n\|_{cb} = \Lambda(C)$ where $\theta_{\varphi_n} : \mathcal{A} \to \mathcal{A}$ as before. Define the $K$-equivariant maps $\mu_n : G \to \mathcal{E}(G)$ associated with $\varphi_n$ by (6.1).

For a fixed $n$ and a fixed $x \in G$, there are only finitely many $\pi \in \text{Irr}(K \cap xKx^{-1})$ such that $\varphi_n(\pi, x) \neq 0$. So, $\mu_n(x)$ is an actual complex measure on $K \cap xKx^{-1}$ that is absolutely continuous with respect to the Haar measure (and with the Radon-Nikodym derivative being in $\text{Pol}(K \cap xKx^{-1})$). By Lemma 6.2, $\|\Psi\mu_n\|_{cb} = \|\theta_{\varphi_n}\|_{cb} < \infty$. So, $\mu_n \in Q(G)$ and the sequence $(\mu_n)$ is a cbai with $\limsup_n \|\mu_n\|_{cb} \leq \Lambda(C)$. Thus, $\Gamma(G) \leq \Lambda(C)$. Write $\kappa = \Gamma(G)^{1/2}$.

For every map $\mu : G \to \mathcal{E}(G)$, we define

$$\overline{\mu} : G \to \mathcal{E}(G) : \overline{\mu}(x)(F) = (\mu(x^{-1}) \circ \text{Ad}(x^{-1}))(F).$$

If $\mu$ is $K_0$-equivariant, also $\overline{\mu}$ is $K_0$-equivariant and $\Psi\overline{\mu}(T) = (\Psi\mu(T^*))^*$ for all $T \in \mathcal{P}$. So, $\|\overline{\mu}\|_{cb} = \|\mu\|_{cb}$. Also, if $(\mu_n)$ is a cbai, then $(\overline{\mu}_n)$ is a cbai.

Since $\Gamma(G) = \kappa^2 < \infty$ and using Lemma 6.3, we can take a cbai $(\mu_n)$, a nondegenerate $*$-representation $\pi : C_0(G) \rtimes \text{Ad} G \to B(K)$ and bounded functions $V_n, W_n : G \to K$ as in Lemma 6.3.3 with

$$\lim_n \|V_n\|_\infty = \kappa = \lim_n \|W_n\|_\infty.$$

Replacing $\mu_n$ by $(\mu_n + \overline{\mu}_n)/2$, we may assume that $\mu_n = \overline{\mu}_n$ for all $n$. It then follows that both formulas

$$\mu_n(zy^{-1})(F) = \langle \pi(F)\pi(zy^{-1})V_n(y), W_n(z) \rangle \quad \text{and} \quad \mu_n(zy^{-1})(F) = \langle \pi(F)\pi(zy^{-1})W_n(y), V_n(z) \rangle$$

hold for all $F \in C_0(G)$ and $y, z \in G$.

Put $\eta_n := \mu_n(e)$. We prove that $\|\eta_n \circ \text{Ad} x - \eta_n\| \to 0$ uniformly on compact sets of $x \in G$. To prove this statement, fix an arbitrary compact subset $C \subset G$ and an arbitrary sequence $x_n \in C$. Define

$$\zeta_n : G \to \mathcal{E}(G) : \zeta_n(x) = \mu_n(x_n x) \circ \text{Ad} x_n .$$

Since $\Psi\zeta_n(T) = u_{x_n}^* \Psi\mu_n(u_{x_n} T)$, it follows that $(\zeta_n)$ is a cbai. Also note that for all $y, z \in G$ and $F \in C_0(G)$, we have

$$\zeta_n(zy^{-1})(F) = \langle \pi(\text{Ad} x_n)(F))\pi(x_n y^{-1})V_n(y), W_n(x_n z) \rangle = \langle \pi(F)\pi(y^{-1})V_n(y), W_n^*(z) \rangle ,$$

which completes the proof.
with $W'_n(z) = \pi(x_n)^*W_n(x_nz)$. Then also $(\mu_n + \zeta_n)/2$ is a cbai satisfying
\[
\frac{1}{2}(\mu_n + \zeta_n)(zy^{-1})(F) = \langle \pi(F)\pi(zy^{-1})V_n(y), (W_n(z) + W'_n(z))/2 \rangle
\]
for all $y, z \in G$ and $F \in C_0(G)$. We conclude that
\[
\kappa^2 \leq \liminf_n \|V_n\|_\infty \|\frac{(W_n + W'_n)}{2}\|_\infty = \kappa \liminf_n \|\frac{(W_n + W'_n)}{2}\|_\infty
\]
\[
\leq \kappa \limsup_n \|\frac{(W_n + W'_n)}{2}\|_\infty \leq \kappa \frac{1}{2} \limsup_n \|\frac{W_n}{2}\|_\infty = \kappa^2 .
\]
Therefore, $\lim_n \|\frac{(W_n + W'_n)}{2}\|_\infty = \kappa$. So, we can choose $z_n \in G$ such that $\lim_n \|\frac{(W_n(z_n) + W'_n(z_n))}{2}\| = \kappa$. Since also $\limsup_n \|W_n(z_n)\| \leq \kappa$ and $\limsup_n \|W'_n(z_n)\| \leq \kappa$, the parallelogram law implies that $\lim_n \|W_n(z_n) - W'_n(z_n)\| = 0$.

Since for all $F \in C_0(G)$,
\[
\zeta_n(\epsilon)(F) = \zeta_n(z_nz_n^{-1})(F) = \langle \pi(F)V_n(z_n), W'_n(z_n) \rangle
\]
\[
\mu_n(\epsilon)(F) = \mu_n(z_nz_n^{-1})(F) = \langle \pi(F)V_n(z_n), W_n(z_n) \rangle ,
\]
it follows that $\lim_n \|\zeta_n(\epsilon) - \mu_n(\epsilon)\| = 0$. This means that $\lim_n \|\mu_n(x_n) \circ \text{Ad} x_n - \mu_n(\epsilon)\| = 0$.

Since the sequence $x_n \in C$ was arbitrary, we have proved that $\lim_n \|\mu_n(x) - \mu_n(\epsilon) \circ \text{Ad} x^{-1}\| = 0$ uniformly on compact sets of $x \in G$.

Reasoning in a similar way with $\zeta_n : G \to \mathcal{E}(G) : \zeta_n(x) = \mu_n(x{x_n}^{-1})$, which satisfies
\[
\zeta_n(zy^{-1})(F) = \langle \pi(F)\pi(zy^{-1})V_n(y), W_n(z) \rangle
\]
with $V_n'(y) = \pi(x_n)^*V_n(x_ny)$, we also find that $\lim_n \|\mu_n(x) - \mu_n(\epsilon)\| = 0$ uniformly on compact sets of $x \in G$. Both statements together imply that $\|\eta_n \circ \text{Ad} x - \eta_n\| \to 0$ uniformly on compact sets of $x \in G$.

We next claim that for every $H \in \text{Pol}(G)$ with $H(e) = 1$ and $\|H\|_\infty = 1$, we have that $\lim_n \|\eta_n \cdot H - \eta_n\| = 0$. To prove this claim, define
\[
\zeta_n : G \to \mathcal{E}(G) : \zeta_n(x)(F) = \mu_n(x)(HF) .
\]
Since $\zeta_n(zy^{-1})(F) = \langle \pi(F)\pi(zy^{-1})V_n(y), W'_n(z) \rangle$ with $W'_n(z) = \pi(H)^*W_n(z)$ and because the function $H \in \text{Pol}(G)$ is both left and right $K_0$-invariant for a small enough compact open subgroup $K_0 < G$, it follows from Lemma 6.3 that
\[
\|\zeta_n\|_{cb} \leq \|V_n\|_\infty \|W'_n\|_\infty \leq \|V_n\|_\infty \|W_n\|_\infty = \|\mu_n\|_{cb} .
\]
So again, $(\zeta_n)$ and $(\mu_n + \zeta_n)/2$ are cbai. The same reasoning as above gives us a sequence $z_n \in G$ with $\lim_n \|W_n(z_n) - W'_n(z_n)\| = 0$, which allows us to conclude that $\lim_n \|\mu_n(\epsilon) - \zeta_n(\epsilon)\| = 0$, thus proving the claim.

Altogether, we have proved that $\eta_n \in C_0(G)^*$ is a sequence of complex measures that are absolutely continuous with respect to the Haar measure and that satisfy
\begin{itemize}
  \item $\|\eta_n - \eta_n \circ \text{Ad} x\| \to 0$ uniformly on compact sets of $x \in G$,
  \item $\|\eta_n \cdot 1_L - \eta_n\| \to 0$ for every compact open subset $L \subset G$ with $e \in L$,
  \item $\eta_n(F) \to F(e)$ for every $F \in C_0(G)$.
\end{itemize}
In particular, \( \liminf_n \| \eta_n \| \geq 1 \). But then \( \omega_n := \| \eta_n \|^{-1} |\eta_n| \) is a sequence of probability measures on \( G \) that are absolutely continuous with respect to the Haar measure and satisfy \( \omega_n \to \delta_e \) weakly* and \( \| \omega_n \circ \text{Ad} x - \omega_n \| \to 0 \) uniformly on compact sets of \( x \in G \).

By Lemma 6.3, the maps \( \Psi_{\mu_n} \) extend to normal cb maps on \( L^\infty(G) \rtimes_{\text{Ad}} G \). Restricting to \( L(G) \), we obtain the compactly supported Herz-Schur multipliers

\[
L(G) \to L(G) : u_x \mapsto \gamma_n(x) u_x \quad \text{for all } x \in G,
\]

where \( \gamma_n : G \to \mathbb{C} \) is the compactly supported, locally constant function given by \( \gamma_n(x) = \mu_n(x)(1) \). So, \( G \) is weakly amenable and

\[
\Lambda(G) \leq \limsup_n \| \Psi_{\mu_n} \|_{cb} \leq \limsup_n \| \Psi_{\mu_n} \| \leq \Lambda(G).
\]

Conversely, assume that \( G \) is weakly amenable and that there exists a sequence of probability measures \( \omega_n \in C_0(G)^* \) that are absolutely continuous with respect to the Haar measure and such that \( \omega_n \to \delta_e \) weakly* and \( \| \omega_n \circ \text{Ad} x - \omega_n \| \to 0 \) uniformly on compact sets of \( x \in G \).

Since \( G \) is weakly amenable, we can take a sequence of \( K \)-biinvariant Herz-Schur multipliers \( \zeta_n : G \to \mathbb{C} \) having compact support, converging to 1 uniformly on compacta and satisfying \( \limsup_n \| \zeta_n \|_{cb} = \Lambda(G) \).

Denote by \( \text{Pol}(G)^+ \) the set of positive, locally constant, compactly supported functions on \( G \). Denote by \( h \in C_0(G)^* \) the Haar measure on the compact open subgroup \( K < G \). Approximating \( \omega_n \), we may assume that \( \omega_n = h \cdot \xi_n^2 \), where \( \xi_n \) is a sequence of \( \text{Ad} K \)-invariant functions in \( \text{Pol}(K)^+ \). Define the representation \( \pi : C_0(G) \rtimes_{\text{Ad}}^F G \to B(L^2(G)) \) given by

\[
(\pi(F)\xi)(g) = F(g)\xi(g) \quad \text{and} \quad (\pi(x)\xi)(g) = \Delta(x)^{1/2} \xi(x^{-1}gx)
\]

for all \( F \in C_0(G) \), \( \xi \in L^2(G) \) and \( x, g \in G \). We then define the \( K \)-equivariant map

\[
\mu_n : G \to C_0(G)^* : \mu_n(x)(F) = \zeta_n(x) \langle \pi(F)\pi(x)\xi_n, \xi_n \rangle.
\]

Since \( \xi_n \) is an \( \text{Ad} K \)-invariant element of \( \text{Pol}(K) \) and \( \pi(x)\xi_n \) is an \( \text{Ad}(xKx^{-1}) \)-invariant element of \( \text{Pol}(xKx^{-1}) \), we get that \( \mu_n(x) \) is an \( \text{Ad}(K \cap xKx^{-1}) \)-invariant complex measure supported on \( K \cap xKx^{-1} \) and having a density in \( \text{Pol}(K \cap xKx^{-1}) \) with respect to the Haar measure. Since moreover \( \zeta_n \) is compactly supported, it follows that the functions \( \varphi_n : \text{Irr}(G) \to \mathbb{C} \) associated with \( \mu_n \) through (6.1) are finitely supported.

Since \( \| \omega_n \circ \text{Ad} x - \omega_n \| \to 0 \) for every \( x \in G \), we have that \( \| \pi(x)\xi_n - \xi_n \| \to 0 \) for every \( x \in G \). Since \( \omega_n \to \delta_e \) weakly*, we have that \( (\pi(F)\xi_n, \xi_n) \to F(e) \) for every \( F \in C_0(G) \). Both together imply that \( \varphi_n \to 1 \) pointwise.

To conclude the proof of the theorem, by Lemma 6.2, it suffices to prove that \( \limsup_n \| \mu_n \|_{cb} \leq \Lambda(G) \).

Since \( \zeta_n \) is a \( K \)-biinvariant Herz-Schur multiplier on \( G \), we can choose a Hilbert space \( \mathcal{K} \) and \( K \)-biinvariant functions \( V_n, W_n : G \to \mathcal{K} \) such that

\[
\| V_n \|_\infty \| W_n \|_\infty = \| \zeta_n \|_{cb} \quad \text{and} \quad \zeta_n(zy^{-1}) = \langle V_n(y), W_n(z) \rangle \quad (6.2)
\]

for all \( y, z \in G \). We equip \( L^2(G) \otimes \mathcal{K} \) with the \(*\)-representation of \( C_0(G) \rtimes_{\text{Ad}} G \) given by \( \pi(\cdot) \otimes 1 \). We define the bounded maps

\[
V_n : G \to L^2(G) \otimes \mathcal{K} : V_n(y) = \xi_n \otimes V_n(y) \quad \text{and} \quad W_n : G \to L^2(G) \otimes \mathcal{K} : W_n(y) = \xi_n \otimes W_n(y).
\]
One checks that
\[ \mu_n(zy^{-1})(F) = \langle \pi(F)\pi(zy^{-1}) \otimes 1 \rangle \mathcal{V}_n(y), W_n(z) \]
for all \( y,z \) and that all other conditions in Lemma 6.3.3 are satisfied, with \( \| V_n \|_\infty = \| V_n \|_\infty \) and \( \| W_n \|_\infty = \| W_n \|_\infty \). So, we conclude that
\[ \limsup_n \| \mu_n \|_{cb} \leq \limsup_n \| \zeta_n \|_{cb} = \Lambda(G) \]
and this ends the proof of the theorem.

**Example 6.4.** Taking \( G \) as in Example 4.4, the category \( C \) is weakly amenable with \( \Lambda(C) = 1 \). Indeed, \( G \) is weakly amenable with \( \Lambda(G) = 1 \) and the probability measures \( \mu_n \) constructed in Example 4.4 are absolutely continuous with respect to the Haar measure, so that the result follows from Theorem 6.1.

Taking \( G = \text{SL}(2,F) \) as in Proposition 4.2, we get that \( C \) is not weakly amenable, although \( G \) is weakly amenable with \( \Lambda(G) = 1 \).

**References**

[A14] Y. Arano, Unitary spherical representations of Drinfeld doubles. *J. Reine Angew. Math.*, to appear. arXiv:1410.6238

[BHV08] B. Bekka, P. de la Harpe and A. Valette, Kazhdan’s property (T). Cambridge University Press, Cambridge, 2008.

[Br14] A. Brothier, Weak amenability for subfactors. *Internat. J. Math.* 26 (2015), art. id. 1550048.

[BO08] N.P. Brown and N. Ozawa, \( \mathcal{C}^* \)-algebras and finite-dimensional approximations. *Graduate Studies in Mathematics* 88. American Mathematical Society, Providence, 2008.

[B10] R.D. Burstein, Automorphisms of the bipartite graph planar algebra. *J. Funct. Anal.* 259 (2010), 2384-2403.

[DFY13] K. De Commer, A. Freslon and M. Yamashita, CCAP for universal discrete quantum groups. *Comm. Math. Phys.* 331 (2014), 677-701.

[DV10] S. Deprez and S. Vaes, A classification of all finite index subfactors for a class of group-measure space II\(_1\) factors. *J. Noncommut. Geom.* 5 (2011), 523-545.

[C05] Y. de Cornulier, Finitely presentable, non-Hopfian groups with Kazhdan’s property (T) and infinite outer automorphism group. *Proc. Amer. Math. Soc.* 135 (2007), 951-959.

[GJ15] S. Ghosh and C. Jones, Annular representation theory for rigid \( \mathcal{C}^* \)-tensor categories. *J. Funct. Anal.* 270 (2016), 1537-1584.

[J91] P. Jolissaint, A characterization of completely bounded multipliers of Fourier algebras. *Colloq. Math.* 63 (1992), 311-313.

[J98] V.F.R Jones, The planar algebra of a bipartite graph. In *Knots in Hellas ’98*, World Scientific, 1999, pp. 94-117.

[J99] V.F.R Jones, Planar algebras, I. arXiv:math.QA/9909027

[LR96] R. Longo and J.E. Roberts, A theory of dimension. *K-Theory* 11 (1997), 103-159.

[MP08] S. Morrison, E. Peters and N. Snyder, Skein theory for the \( D_{2n} \) planar algebras. *J. Pure Appl. Algebra* 214 (2010), 117-139.

[M01] M. Müger, From subfactors to categories and topology, I. Frobenius algebras in and Morita equivalence of tensor categories. *J. Pure Appl. Algebra* 180 (2003), 81-157.

[NT13] S. Neshveyev and L. Tuset, Compact quantum groups and their representation categories. *Cours Spécialisés* 20. Société Mathématique de France, Paris, 2013.

[NY15a] S. Neshveyev and M. Yamashita, Drinfeld center and representation theory for monoidal categories. Preprint. arXiv:1501.07390
[NY15b] S. Neshveyev and M. Yamashita, A few remarks on the tube algebra of a monoidal category. Preprint. arXiv:1511.06332

[O93] A. Ocneanu, Chirality for operator algebras. In Subfactors (Kyuzeso, 1993), World Sci. Publ., River Edge, 1994, pp. 39-63.

[O10] N. Ozawa, Examples of groups which are not weakly amenable. Kyoto J. Math. 52 (2012), 333-344.

[P94a] S. Popa, Symmetric enveloping algebras, amenability and AFD properties for subfactors. Math. Res. Lett. 1 (1994), 409-425.

[P94b] S. Popa, An axiomatization of the lattice of higher relative commutants of a subfactor. Invent. Math. 120 (1995), 427-445.

[P99] S. Popa, Some properties of the symmetric enveloping algebra of a subfactor, with applications to amenability and property T. Doc. Math. 4 (1999), 665-744.

[P01] S. Popa, On a class of type II1 factors with Betti numbers invariants. Ann. of Math. 163 (2006), 809-899.

[PSV15] S. Popa, D. Shlyakhtenko and S. Vaes, Cohomology and L2-Betti numbers for subfactors and quasi-regular inclusions. Preprint. arXiv:1511.07329

[PV14] S. Popa and S. Vaes, Representation theory for subfactors, λ-lattices and C*-tensor categories, Commun. Math. Phys. 340 (2015), 1239-1280.

[V03] S. Vaes, Strictly outer actions of groups and quantum groups. J. Reine Angew. Math. 578 (2005), 147-184.

[W88] A. Wassermann, Ergodic actions of compact groups on operator algebras, I. Ann. of Math. 130 (1989), 273-319.

[Z98] Y. Zhu, Hecke algebras and representation ring of Hopf algebras. In First International Congress of Chinese Mathematicians (Beijing, 1998), AMS/IP Stud. Adv. Math. 20, Amer. Math. Soc., Providence, 2001, pp. 219-227.