The moduli space of curves is rigid

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Abstract

We prove that the moduli stack $\mathcal{M}_{g,n}$ of stable curves of genus $g$ with $n$ marked points is rigid, that is, has no infinitesimal deformations. This confirms the first case of a principle proposed by Kapranov. It can also be viewed as a version of Mostow rigidity for the mapping class group.

1 Introduction

Kapranov has proposed the following informal statement [Kapranov97]. Given a smooth variety $X = X(0)$, consider the moduli space $X(1)$ of varieties obtained as deformations of $X(0)$, the moduli space $X(2)$ of deformations of $X(1)$, and so on. Then this process should stop after $n = \dim X$ steps, i.e., $X(n)$ should be rigid (no infinitesimal deformations). Roughly speaking, one thinks of $X(1)$ as $H^1$ of a sheaf of non-abelian groups on $X(0)$. Indeed, at least the tangent space to $X(1)$ at $[X]$ is identified with $H^1(T_X)$, where $T_X$ is the tangent sheaf, the sheaf of first order infinitesimal automorphisms of $X$. Then one regards $X(m)$ as a kind of non-Abelian $H^m$, and the analogy with the usual definition of Abelian $H^m$ suggests the statement above.

In particular, the moduli space of curves should be rigid. In this paper, we verify this in the following precise form: the moduli stack of stable curves of genus $g$ with $n$ marked points is rigid for each $g$ and $n$.

On the other hand, moduli spaces of surfaces should have non-trivial deformations in general. A simple example (for surfaces with boundary) is given in Sec. 6. It seems plausible that there should be a non-trivial deformation of a moduli space of surfaces whose fibres parametrise “generalised surfaces” in some sense, for example non-commutative surfaces. From this point of view the result of this paper says that the concept of a curve cannot be deformed.
Let us also note that our result can be thought of as a version of Mostow rigidity for the mapping class group. Recall that the moduli space \( M_g \) of smooth complex curves of genus \( g \) is the quotient of the Teichmüller space \( T_g \) by the mapping class group \( \Gamma_g \). The space \( T_g \) is a bounded domain in \( \mathbb{C}^{3g-3} \), which is homeomorphic to a ball, and \( \Gamma_g \) acts discontinuously on \( T_g \) with finite stabilisers. We thus obtain \( M_g \) as a complex orbifold with orbifold fundamental group \( \Gamma_g \). The space \( T_g \) admits a natural metric, the Weil–Petersson metric, which has negative holomorphic sectional curvatures. So, roughly speaking, \( M_g \) looks like a quotient of a complex ball by a discrete group \( \Gamma \) of isometries, with finite volume. Mostow rigidity predicts that such a quotient is uniquely determined by the group \( \Gamma \) up to complex conjugation. (This is certainly true if \( \Gamma \) acts freely with compact quotient, see [Siu80].) In particular, it should have no infinitesimal deformations. Unfortunately I do not know a proof along these lines.

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## 2 Statements

We work over an algebraically closed field \( k \) of characteristic zero. Let \( g \) and \( n \) be non-negative integers such that \( 2g - 2 + n > 0 \). Let \( \overline{M}_{g,n} \) denote the moduli stack of stable curves of genus \( g \) with \( n \) marked points. The stack \( \overline{M}_{g,n} \) is a smooth proper Deligne–Mumford stack of dimension \( 3g-3+n \).

**Theorem 2.1.** The stack \( \overline{M}_{g,n} \) is rigid, that is, has no infinitesimal deformations.

Let \( \partial \overline{M}_{g,n} \subset \overline{M}_{g,n} \) denote the boundary of the moduli stack, that is, the complement of the locus of smooth curves (with its reduced structure). The locus \( \partial \overline{M}_{g,n} \) is a normal crossing divisor in \( \overline{M}_{g,n} \).

**Theorem 2.2.** The pair \( (\overline{M}_{g,n}, \partial \overline{M}_{g,n}) \) has no locally trivial deformations.

Let \( \overline{M}_{g,n} \) denote the coarse moduli space of the stack \( \overline{M}_{g,n} \). The space \( \overline{M}_{g,n} \) is a projective variety with quotient singularities.

**Theorem 2.3.** The variety \( \overline{M}_{g,n} \) has no locally trivial deformations if \( (g,n) \neq (1,2), (2,0), (2,1), (3,0) \).
Remark 2.4. In the exceptional cases, the projection $\overline{\mathcal{M}}_{g,n} \to \overline{\mathcal{M}}_{g,n}$ is ramified in codimension one over the interior of $\overline{\mathcal{M}}_{g,n}$, and an additional calculation is needed to relate the deformations of the stack and the deformations of the coarse moduli space (cf. Prop. 5.2). Presumably the result still holds.

3 Proof of Theorem 2.2

Write $\mathcal{B}$ for the boundary of $\overline{\mathcal{M}}_{g,n}$. Let $\Omega_{\overline{\mathcal{M}}_{g,n}}(\log \mathcal{B})$ denote the sheaf of 1-forms on $\overline{\mathcal{M}}_{g,n}$ with logarithmic poles along the boundary, and $T_{\overline{\mathcal{M}}_{g,n}}(-\log \mathcal{B})$ the dual of $\Omega_{\overline{\mathcal{M}}_{g,n}}(\log \mathcal{B})$. The sheaf $T_{\overline{\mathcal{M}}_{g,n}}(-\log \mathcal{B})$ is the subsheaf of the tangent sheaf $T_{\overline{\mathcal{M}}_{g,n}}$ consisting of vector fields on $\overline{\mathcal{M}}_{g,n}$ which are tangent to the boundary. In other words, it is the sheaf of first order infinitesimal automorphisms of the pair $(\overline{\mathcal{M}}_{g,n}, \mathcal{B})$. Hence the first order locally trivial deformations of the pair $(\overline{\mathcal{M}}_{g,n}, \mathcal{B})$ are identified with the space $H^1(T_{\overline{\mathcal{M}}_{g,n}}(-\log \mathcal{B}))$. To prove Thm. 2.2 we show $H^1(T_{\overline{\mathcal{M}}_{g,n}}(-\log \mathcal{B})) = 0$.

Let $\pi : \mathcal{U}_{g,n} \to \overline{\mathcal{M}}_{g,n}$ denote the universal family over $\overline{\mathcal{M}}_{g,n}$. That is, $\mathcal{U}_{g,n}$ is the stack of $n$-pointed stable curves of genus $g$ together with an extra section (with no smoothness condition). Let $\Sigma$ denote the union of the $n$ tautological sections of $\pi$. We define the boundary $\mathcal{B}_U$ of $\mathcal{U}_{g,n}$ as the union of $\pi^*\mathcal{B}$ and $\Sigma$.

Let $\nu : \mathcal{B}^\nu \to \mathcal{B}$ be the normalisation of the boundary $\mathcal{B}$ of $\overline{\mathcal{M}}_{g,n}$, and $\mathcal{N}$ the normal bundle of the map $\mathcal{B}^\nu \to \overline{\mathcal{M}}_{g,n}$. Then we have an exact sequence

$$0 \to T_{\overline{\mathcal{M}}_{g,n}}(-\log \mathcal{B}) \to T_{\overline{\mathcal{M}}_{g,n}} \to \nu_*\mathcal{N} \to 0.$$ 

Let $\omega_\pi$ denote the relative dualising sheaf of the morphism $\pi$.

Lemma 3.1. There is a natural isomorphism

$$\delta : T_{\overline{\mathcal{M}}_{g,n}}(-\log \mathcal{B}) \xrightarrow{\sim} R^1\pi_*(\omega_\pi(\Sigma)^\vee).$$

Proof. For a pointed stable curve $(C, \Sigma_C = x_1 + \cdots + x_n)$, the space of first order deformations is equal to $\text{Ext}^1(\Omega_C(\Sigma_C), \mathcal{O}_C)$. See [DM69] p.79–82.

The surjection

$$\text{Ext}^1(\Omega_C(\Sigma_C), \mathcal{O}_C) \to H^0(\mathcal{E}xt^1(\Omega_C(\Sigma_C), \mathcal{O}_C)) = \bigoplus_{q \in \text{Sing} C} \mathcal{E}xt^1(\Omega_C(\Sigma_C), \mathcal{O}_C)_q$$

sends a global deformation of $(C, \Sigma_C)$ to the induced deformations of the nodes. Étale locally at the point $[(C, \Sigma_C)] \in \overline{\mathcal{M}}_{g,n}$, the boundary $\mathcal{B}$ is a normal crossing divisor with components $B_q$ indexed by the nodes $q$ of $C$.
(the divisor $B_q$ is the locus where the node $q$ is not smoothed). The Kodaira–Spencer map identifies the fibre of the normal bundle of $B_q$ at $[(C, \Sigma_C)]$ with the stalk of $\mathcal{E}xt^1(\Omega_C(\Sigma_C), \mathcal{O}_C)$ at $q$.

We now work globally over $\mathcal{M}_{g,n}$. We omit the subscripts $g, n$ for clarity.

Consider the exact sequence

$$0 \to \pi^*\Omega_{\mathcal{M}} \to \Omega_{\mathcal{U}}(\log \Sigma) \to \Omega_{\mathcal{U}/\mathcal{M}}(\Sigma) \to 0. \quad (1)$$

For a sheaf $\mathcal{F}$ on $\mathcal{U}$, let $\mathcal{E}xt_i^\pi(\mathcal{F}, \cdot)$ denote the $i$th right derived functor of $\pi_* \circ \mathcal{H}om(\cdot, \mathcal{O}_U)$. Applying $\pi^* \circ \mathcal{H}om(\cdot, \mathcal{O}_U)$ to the exact sequence (1), we obtain a long exact sequence with connecting homomorphism

$$\rho: T_{\mathcal{M}} \to \mathcal{E}xt^1_\pi(\Omega_{\mathcal{U}/\mathcal{M}}(\Sigma), \mathcal{O}_U).$$

The map $\rho$ is the Kodaira–Spencer map for the universal family over $\mathcal{M}$ and thus is an isomorphism. (Note that, for a point $p = [(C, \Sigma_C)] \in \mathcal{M}$, the base change map

$$\mathcal{E}xt^1_\pi(\Omega_{\mathcal{U}/\mathcal{M}}(\Sigma), \mathcal{O}_U) \otimes k(p) \to \mathcal{E}xt^1(\Omega_C(\Sigma_C), \mathcal{O}_C)$$

is an isomorphism. Indeed, by relative duality [Kleiman80, Thm. 21], it suffices to show that $\pi_*(\Omega_{\mathcal{U}/\mathcal{M}}(\Sigma) \otimes \omega_\pi)$ commutes with base change. This follows from cohomology and base change.)

Consider the two exact sequences

$$0 \to T_{\mathcal{M}}(-\log \mathcal{B}) \to T_{\mathcal{M}} \to \nu_*\mathcal{N} \to 0$$

and

$$0 \to R^1\pi_*(\Omega_{\mathcal{U}/\mathcal{M}}(\Sigma)^\vee) \to \mathcal{E}xt^1_\pi(\Omega_{\mathcal{U}/\mathcal{M}}(\Sigma), \mathcal{O}_U) \to \pi_*\mathcal{E}xt^1(\Omega_{\mathcal{U}/\mathcal{M}}(\Sigma), \mathcal{O}_U) \to 0$$

The Kodaira–Spencer map $\rho$ identifies the middle terms, and induces an identification of the right end terms determined by the deformations of the singularities of the fibres of $\pi$. We thus obtain a natural isomorphism $\delta$ of the left end terms. Finally, note that $\Omega_{\mathcal{U}/\mathcal{M}}(\Sigma)^\vee = \omega_\pi(\Sigma)^\vee$ because $\omega_\pi(\Sigma)$ is invertible and agrees with $\Omega_{\mathcal{U}/\mathcal{M}}(\Sigma)$ in codimension 1. This completes the proof.

The line bundle $\omega_\pi(\Sigma)$ is ample on fibres of $\pi$. Hence $\pi_*(\omega_\pi(\Sigma)^\vee) = 0$. Also $R^i\pi_*(\omega_\pi(\Sigma)^\vee) = 0$ for $i > 1$ by dimensions. So $H^{i+1}(\omega_\pi(\Sigma)^\vee) = H^i(R^1\pi_*(\omega_\pi(\Sigma)^\vee))$ for all $i$ by the Leray spectral sequence. Hence the isomorphism $\delta$ induces an isomorphism

$$H^i(T_{\mathcal{M}_{g,n}}(-\log \mathcal{B})) \xrightarrow{\sim} H^{i+1}(\omega_\pi(\Sigma)^\vee) \quad (2)$$
for each i.

Let $U_{g,n}$ denote the coarse moduli space of the stack $\mathcal{U}_{g,n}$ and $p : U_{g,n} \to U_{g,n}$ the projection. The line bundle $\omega_{\pi}(\Sigma)$ on the stack $\mathcal{U}_{g,n}$ defines a $\mathbb{Q}$-line bundle $\pi_\ast \omega_{\pi}(\Sigma)$ on the coarse moduli space $U_{g,n}$ (see Sec. 7). We use the following important result, which is essentially due to Arakelov [Arakelov71, Prop. 3.2, p. 1297]. We refer to [Keel99, Sec. 4] for the proof.

**Theorem 3.2.** The $\mathbb{Q}$-line bundle $\pi_\ast \omega_{\pi}(\Sigma)$ is big and nef on $U_{g,n}$.

It follows by Kodaira vanishing (see Thm 7.1) that $H^i(\omega_{\pi}(\Sigma)^\vee) = 0$ for $i < \dim U_{g,n}$. Combining with (2), we deduce

**Proposition 3.3.** $H^i(T_{\overline{M}_{g,n}}(- \log \mathcal{B})) = 0$ for $i < \dim \overline{M}_{g,n}$.

In particular, $H^1(T_{\overline{M}_{g,n}}(- \log \mathcal{B})) = 0$ if $\dim \overline{M}_{g,n} > 1$. The remaining cases are easy to check. This completes the proof of Theorem 2.2.

### 4 Proof of Theorem 2.1

We now prove that $\overline{M}_{g,n}$ is rigid. Since $\overline{M}_{g,n}$ is a smooth Deligne–Mumford stack, its first order infinitesimal deformations are identified with the space $H^1(T_{\overline{M}_{g,n}})$, and we must show that $H^1(T_{\overline{M}_{g,n}}) = 0$. Consider the exact sequence

$$0 \to T_{\overline{M}_{g,n}}(- \log \mathcal{B}) \to T_{\overline{M}_{g,n}} \to \nu_\ast \mathcal{N} \to 0$$

and the associated long exact sequence of cohomology

$$\cdots \to H^i(T_{\overline{M}_{g,n}}(- \log \mathcal{B})) \to H^i(T_{\overline{M}_{g,n}}) \to H^i(\mathcal{N}) \to \cdots$$

We prove below that $H^i(\mathcal{N}) = 0$ for $i < \dim \mathcal{B}$. Now $H^i(T_{\overline{M}_{g,n}}(- \log \mathcal{B})) = 0$ for $i < \dim \overline{M}_{g,n}$ by Prop. 3.3, so we deduce

**Proposition 4.1.** $H^i(T_{\overline{M}_{g,n}}) = 0$ for $i < \dim \overline{M}_{g,n} - 1$.

In particular, $H^1(T_{\overline{M}_{g,n}}) = 0$ if $\dim \overline{M}_{g,n} > 2$. In the remaining cases it is easy to check that $H^1(\mathcal{N}) = 0$, so again $H^1(T_{\overline{M}_{g,n}}) = 0$.

The irreducible components of the normalisation $\overline{\mathcal{B}}$ of the boundary $\mathcal{B}$ of $\overline{M}_{g,n}$ are finite images of the following stacks [Knudsen83a, Def. 3.8, Cor. 3.9]:

(1) $\overline{M}_{g_1,S_1\cup\{n+1\}} \times \overline{M}_{g_2,S_2\cup\{n+2\}}$ where $g_1 + g_2 = g$ and $S_1, S_2$ is a partition of $\{1, \ldots, n\}$. 

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Here $\overline{M}_{h,S}$ denotes the moduli stack of stable curves of genus $h$ with marked points labelled by a finite set $S$. In each case the map to $B^\nu$ is given by identifying the points labelled by $n+1$ and $n+2$. The map is an isomorphism onto the component of $B^\nu$ except in case (1) for $g_1 = g_2$ and $n = 0$ and case (2), when it is étale of degree 2.

For $\overline{M}_{h,S}$ a moduli stack of pointed stable curves as above, let $\pi : U_{h,S} \to \overline{M}_{h,S}$ denote the universal family, and $x_i : \overline{M}_{h,S} \to U_{h,S}$, $i \in S$, the tautological sections of $\pi$. Define $\psi_i = x_i^* \omega_\pi$, the pullback of the relative dualising sheaf of $\pi$ along the section $x_i$. The following result is well-known, see for example [HMo98, Prop. 3.32].

**Lemma 4.2.** The pullback of $N^\vee$ to $\overline{M}_{g_1,S_1 \cup \{n+1\}} \times \overline{M}_{g_2,S_2 \cup \{n+2\}}$ is identified with $\text{pr}_1^* \psi_{n+1} \otimes \text{pr}_2^* \psi_{n+2}$. Similarly, the pullback of $N^\vee$ to $\overline{M}_{g-1,n+2}$ is identified with $\psi_{n+1} \otimes \psi_{n+2}$.

There is an isomorphism of stacks $c : \overline{M}_{g,n+1} \to U_{g,n}$ which identifies the morphism $p_{n+1} : \overline{M}_{g,n+1} \to \overline{M}_{g,n}$ given by forgetting the last point with the projection $\pi : U_{g,n} \to \overline{M}_{g,n}$ [Knudsen83a, Sec. 1–2].

**Lemma 4.3.** [Knudsen83b, Thm. 4.1(d), p. 202] The line bundle $\psi_{n+1}$ on $\overline{M}_{g,n+1}$ is identified with the pullback of the line bundle $\omega_\pi(\Sigma)$ under the isomorphism $c : \overline{M}_{g,n+1} \to U_{g,n}$.

**Corollary 4.4.** The $\mathbb{Q}$-line bundle on the coarse moduli space of $B^\nu$ defined by $N^\vee$ is big and nef on each component.

**Proof.** This follows immediately from Lem. 4.2, Lem. 4.3 and Thm. 3.2. $\square$

We deduce that $H^i(N) = 0$ for $i < \dim B$ by Thm. 7.1. This completes the proof of Theorem 2.1.

### 5 Proof of Theorem 2.3

We first prove a basic result which relates the deformations of a smooth Deligne–Mumford stack and its coarse moduli space.

Let $\mathcal{X}$ be a smooth proper Deligne–Mumford stack, $X$ the coarse moduli space of $\mathcal{X}$, and $p : \mathcal{X} \to X$ the projection. Let $T_\mathcal{X}$ denote the tangent sheaf of $\mathcal{X}$. Let $D \subset X$ be the union of the codimension one components of the branch locus of $p : \mathcal{X} \to X$ (with its reduced structure). Let $T_\mathcal{X}(-\log D)$ denote the subsheaf of the tangent sheaf $T_\mathcal{X}$ consisting of derivations which
preserve the ideal sheaf of $D$. It is the sheaf of first order infinitesimal automorphisms of the pair $(X, D)$.

**Lemma 5.1.** $p_*T_X = T_X(-\log D)$

*Proof.* The sheaves $p_*T_X$ and $T_X(-\log D)$ satisfy Serre’s $S_2$ condition, and are identified over the locus where $p$ is étale. So it suffices to work in codimension 1. We reduce to the case $X = \mathbb{A}^1_x/\mu^e_x$, where $\mu^e_x \ni \zeta : x \mapsto \zeta x$. Then $X = \mathbb{A}^1_y$, where $y = x^e$, and $D = (y = 0) \subset X$. Let $\pi : \mathbb{A}^1_x \to \mathbb{A}^1_x/\mu^e_x$ be the quotient map. We compute

$$p_*T_X = \left( \pi_* \mathcal{O}_{\mathbb{A}^1_x} \cdot \frac{\partial}{\partial x} \right)^{\mu_e} = \mathcal{O}_{\mathbb{A}^1_y} \cdot x \frac{\partial}{\partial x} = \mathcal{O}_{\mathbb{A}^1_y} \cdot y \frac{\partial}{\partial y} = T_X(-\log D),$$

as required. \qed

**Proposition 5.2.** The first order deformations of the stack $X$ are identified with the first order locally trivial deformations of the pair $(X, D)$.

*Proof.* By the Lemma, $H^1(T_X) = H^1(p_*T_X) = H^1(T_X(-\log D))$. \qed

We now apply this result to relate deformations of the stack $\mathcal{M}_{g,n}$ and its coarse moduli space $\overline{M}_{g,n}$.

A stable $n$-pointed curve of genus 0 has no non-trivial automorphisms. Hence the stack $\overline{M}_{0,n}$ is equal to its coarse moduli space $\overline{M}_{0,n}$, and $\overline{M}_{0,n}$ is rigid by Thm. 2.1. Also, recall that $\overline{M}_{1,1}$ is isomorphic to $\mathbb{P}^1$ and therefore rigid. So, in the following, we assume that $g \neq 0$ and $(g, n) \neq (1, 1)$.

Let $D \subset \overline{M}_{g,n}$ be the component of the boundary whose general point is a curve with two components of genus 1 and $g - 1$ meeting in a node, with each of the $n$ marked points on the component of genus $g - 1$. Note that each point of $D$ has a non-trivial automorphism given by the involution of the component of genus 1 fixing the node. Let $p : \overline{M}_{g,n} \to \overline{M}_{g,n}$ be the projection, and $D \subset \overline{M}_{g,n}$ the coarse moduli space of $D$.

**Lemma 5.3.** [HMu82, §2] If $g + n \geq 4$ then the automorphism group of a general point of $\overline{M}_{g,n}$ is trivial, and the divisor $D \subset \overline{M}_{g,n}$ is the unique codimension 1 component of the branch locus of $p$.

Assume $g + n \geq 4$. Let $\nu : \mathcal{D}^{\nu} \to \mathcal{D}$ denote the normalisation of $\mathcal{D}$, so $\mathcal{D}^{\nu} = \overline{M}_{1,1} \times \overline{M}_{g-1,n+1}$. Let $\mathcal{N}_D$ denote the normal bundle of the map $\mathcal{D}^{\nu} \to \overline{M}_{g,n}$.

**Lemma 5.4.** There is an exact sequence

$$0 \to T_{\overline{M}_{g,n}}(-\log D) \to T_{\overline{M}_{g,n}} \to p_*\nu_*\mathcal{N}_D^\otimes 2 \to 0.$$
Proof. This is a straightforward calculation similar to [HMn82, Lemma, p. 52].

We have \( H^1(T_{\mathcal{M}_{g,n}}(- \log D)) = H^1(T_{\mathcal{M}_{g,n}}) = 0 \) by Prop. 5.2 and Thm. 2.1. Also, \( H^1(N_{D}^{\otimes 2}) = 0 \) by Thm. 7.1 because the \( \mathbb{Q} \)-line bundle defined by \( N_{D} \) on the coarse moduli space of \( \mathcal{D}^\nu \) is big and nef by Cor. 4.4. So \( H^1(T_{\mathcal{M}_{g,n}}) = 0 \) by Lem. 5.4, that is, \( \mathcal{M}_{g,n} \) has no locally trivial deformations. This concludes the proof of Thm. 2.3.

6 Nonrigidity of moduli of surfaces

We exhibit a moduli space of surfaces with boundary that is not rigid.

Let \( P_1, \ldots, P_4 \) be 4 points in linear general position in \( \mathbb{P}^2 \). Let \( l_{ij} \) be the line through \( P_i \) and \( P_j \). Let \( l \) be a line through the point \( Q = l_{12} \cap l_{34} \) such that \( l \) does not pass through \( l_{13} \cap l_{24} \) or \( l_{14} \cap l_{23} \) and is not equal to \( l_{12} \) or \( l_{34} \). Let \( S \to \mathbb{P}^2 \) be the blowup of the points \( P_1, \ldots, P_4, Q \), and \( B \) the sum of the strict transforms of \( l \) and the \( l_{ij} \) and the exceptional curves. Then \((S, B)\) is a smooth surface with normal crossing boundary such that \( K_S + B \) is very ample. We fix an ordering \( B_1, \ldots, B_{12} \) of the components of \( B \). The moduli stack \( \mathcal{M} \) of deformations of \((S, B)\) is isomorphic to \( \mathbb{P}^1 \setminus \{q_1, \ldots, q_4\} \) where the \( q_i \) are distinct points. Indeed, it suffices to observe that all deformations of \((S, B)\) are obtained by the construction above. The moduli space \( \mathcal{M} \) has a modular compactification \((\overline{\mathcal{M}}, \partial \overline{\mathcal{M}})\), the Kollár–Shepherd-Barron–Alexeev moduli stack of stable surfaces with boundary, which is isomorphic to \((\mathbb{P}^1, \sum q_i)\). In particular, the pair \((\overline{\mathcal{M}}, \partial \overline{\mathcal{M}})\) has non-trivial deformations.

Remark 6.1. The compact moduli space \( \overline{\mathcal{M}} \) is an instance of the compactifications of moduli spaces of hyperplane arrangements described in [Lafforgue03] (cf. [HKT06]).

7 Appendix: Kodaira vanishing for stacks

Let \( \mathcal{X} \) be a smooth proper Deligne–Mumford stack, \( X \) the coarse moduli space of \( \mathcal{X} \), and \( p: \mathcal{X} \to X \) the projection. Étale locally on \( X \), \( p: \mathcal{X} \to X \) is of the form \( p: [U/G] \to U/G \), where \( U \) is a smooth affine variety and \( G \) is a finite group acting on \( U \) [AV02, Lemma 2.2.3, p. 32]. A sheaf \( \mathcal{F} \) on \([U/G]\) corresponds to a \( G \)-equivariant sheaf \( \mathcal{F}_U \) on \( U \), and \( p_* \mathcal{F} = (\pi_* \mathcal{F}_U)^G \), where \( \pi: U \to U/G \) is the quotient map.
Let $\mathcal{L}$ be a line bundle on $\mathcal{X}$. Let $n \in \mathbb{N}$ be sufficiently divisible so that for each open patch $[U/G]$ of $\mathcal{X}$ as above and point $q \in U$ the stabilizer $G_q$ of $q$ acts trivially on the fibre of $\mathcal{L}_U^\otimes n$ over $q$. Then the pushforward $p_* (\mathcal{L}^\otimes n)$ is a line bundle on $X$. We define $p_*^Q \mathcal{L} = \frac{1}{n} p_* (\mathcal{L}^\otimes n) \in \text{Pic}(X) \otimes \mathbb{Q}$, and call $p_*^Q \mathcal{L}$ the $\mathbb{Q}$-line bundle on $X$ defined by $\mathcal{L}$.

**Theorem 7.1.** Assume that the coarse moduli space $X$ is an algebraic variety. If the $\mathbb{Q}$-line bundle $p_*^Q \mathcal{L}$ on $X$ is big and nef then $H^i (\mathcal{L}^\vee) = 0$ for $i < \dim \mathcal{X}$.

**Remark 7.2.** If the coarse moduli space $X$ is smooth then Thm. 7.1 follows from [MO05, Thm. 2.1].

Theorem 7.1 is proved by reducing to the following generalisation of the Kodaira vanishing theorem.

**Theorem 7.3.** [KM98, Thm. 2.70, p. 73] Let $X$ be a proper normal variety and $\Delta$ a $\mathbb{Q}$-divisor on $X$ such that the pair $(X, \Delta)$ is Kawamata log terminal (klt). Let $N$ be a $\mathbb{Q}$-Cartier Weil divisor on $X$ such that $N \equiv M + \Delta$, where $M$ is a big and nef $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor. Then $H^i (X, \mathcal{O}_X (-N)) = 0$ for $i < \dim X$.

**Proof of Thm. 7.1.** Observe first that $X$ is a normal variety with quotient singularities. Consider the sheaf $p_* (\mathcal{L}^\vee)$ on $X$. If the automorphism group of a general point of $\mathcal{X}$ acts nontrivially on $\mathcal{L}$, then $p_* \mathcal{L}^\vee = 0$, and so $H^i (\mathcal{L}^\vee) = H^i (p_* \mathcal{L}^\vee) = 0$ for each $i$. Suppose now that the automorphism group of a general point acts trivially on $\mathcal{L}$. Then $p_* \mathcal{L}^\vee$ is a rank 1 reflexive sheaf on $X$. Write $p_* \mathcal{L}^\vee = \mathcal{O}_X (-N)$, where $N$ is a Weil divisor on $X$. Let $n \in \mathbb{N}$ be sufficiently divisible so that $p_*^Q (\mathcal{L}) = \frac{1}{n} p_* (\mathcal{L}^\otimes n)$ as above. Let $M$ be a $\mathbb{Q}$-divisor corresponding to the $\mathbb{Q}$-line bundle $p_*^Q \mathcal{L}$. There is a natural map $(p_* \mathcal{L}^\vee)^\otimes n \to p_* (\mathcal{L}^\otimes \otimes n)$, i.e., a map $\mathcal{O}_X (-nN) \to \mathcal{O}_X (-nM)$, which is an isomorphism over the locus where $p$ is étale. So $N \equiv M + \Delta$, where $\Delta$ is an effective $\mathbb{Q}$-divisor supported on the branch locus of $p$. Let $D_1, \ldots, D_r$ be the codimension 1 components of the branch locus. Let $e_i$ be the ramification index at $D_i$, and $a_i$ the age of the line bundle $\mathcal{L}^\vee$ along $D_i$. That is, after removing the automorphism group of a general point of $\mathcal{X}$, a transverse slice of $\mathcal{X}$ at a general point of $D_i$ is of the form $[\mathbb{A}^1_\mathbb{Q} / \mu_{e_i}]$, where $\mu_{e_i} \ni \zeta : x \mapsto \zeta \cdot x$, and $\mu_{e_i}$ acts on the fibre of $\mathcal{L}^\vee$ by the character $\zeta \mapsto \zeta^{-a_i}$, where $0 \leq a_i \leq e_i - 1$. We compute that $\Delta = \sum_{i=1}^r a_i D_i$.

We claim that $(X, \Delta)$ is klt. Let $\Delta' = \sum_{i=1}^r \frac{e_i - 1}{e_i} D_i$, then $K_X = p^* (K_X + \Delta')$, and $\mathcal{X}$ is smooth, so $(X, \Delta')$ is klt by [KM08, Prop. 5.20(4), p. 160].


Now $\Delta \leq \Delta'$ and $X$ is $\mathbb{Q}$-factorial, so $(X, \Delta)$ is also klt. We deduce that $H^i(L^\vee) = H^i(p_*L^\vee) = H^i(O_X(-N)) = 0$ for $i < \dim \mathcal{X}$ by Thm. 7.3.

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