Gauge Equivalence, Supersymmetry and Classical Solutions of the ospu(1, 1/1) Heisenberg Model and the Nonlinear Schrödinger Equation

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Abstract. An integrable generalization of the continuous classical O(2, 1) pseudospin Heisenberg model to the case of the ospu(1, 1/1) superalgebra is constructed. The gauge equivalence of the constructed model and the related NLSE is established. We indicate a method of generating classical solutions using the global ospu(1, 1/1) supersymmetry. The relationship between solutions of O(2, 1) HM and 'superpartners' of NLSE is obtained.

1. Introduction

During the last few years, nonlinear σ-models with noncompact symmetry groups and their supersymmetric extensions have attracted considerable interest [1]. They arise in gravity theory [2], extended supergravity [3], in the theory of Anderson localisation [4], the Kaluza–Klein theory [5], and in the theory of strings [6] and superstrings [7]. The simplest version of the nonlinear σ-model is a continuous classical spin Heisenberg model (HM) and its extensions to higher spins. Then, in a stationary limit, Landau–Lifshitz equations of the corresponding models coincide with the nonlinear σ-model equations. As demonstrated in [8, 9], the one-dimensional isotropic HM on a noncompact manifold of the constant negative curvature $S^{1,-1}$ is gauge equivalent to a nonlinear Schrödinger equation (NLSE) of the repulsive type (as is well known, the attractive-type NLSE corresponds to the O(3) HM defined on the sphere $S^2$ [10]). At the same time, the formulation of the Zakharov–Shabat–AKNS scheme on the superalgebra osp(2/1) [11, 12] shows the superextension to be allowed only for the repulsive NLSE. The problem naturally arises of how to construct a σ-model associated with super-NLSE** on the superalgebra osp(2/1), which will be a supergeneralisation of the O(2, 1) HM. In this Letter, we construct an integrable generalisation of the continuous classical O(2, 1) pseudospin Heisenberg model [8, 9] to the case of the ospu(1, 1/1) superalgebra. The gauge equivalence of the constructed model and the related NLSE is established. We indicate a method of generating classical solutions using the global supersymmetry ospu(1, 1/1). The relationship between solutions to O(2, 1) HM and the superpartners of NLSE is obtained.

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** At this point and below, the terms super-NLSE and super-HM mean that the corresponding Lax pairs are defined on superalgebra.
2. NLSE on the Superalgebra osp(1, 1/1)

The linear problem of the corresponding super-NLSE [12]

\[ \Phi_\chi = U \Phi, \quad \Phi_\psi = V \Phi \]

is given by the 3 x 3 operators

\[ U = -i \begin{pmatrix} \lambda & -\bar{\phi} & -\kappa \bar{\psi} \\ \bar{\phi} & -\lambda & \psi \\ \kappa \bar{\psi} & \bar{\psi} & 0 \end{pmatrix} \]

\[ V = -2\lambda U + i \begin{pmatrix} |\phi|^2 - \rho + 2\kappa \bar{\psi}\psi & -i\bar{\phi}_s & -2i\kappa \bar{\psi}_s \\ -i\phi_s & -|\phi|^2 + \rho - 2\kappa \bar{\psi}\psi & -2\psi_s \\ -2i\psi_s & 2\kappa i \bar{\psi}_s & 0 \end{pmatrix} \]

where \( \phi(x, t) \) and \( \psi(x, t) \) are the complex boson and fermion fields, respectively, taking values in the Grassmann algebra \( \kappa = \pm 1 \). The matrices \( U_0 = iU \) and \( V_0 = -iV \) satisfy the pseudo-Hermiticity condition

\[ \Gamma_\chi U_0^\dagger \Gamma_\chi = U_0, \quad \Gamma_\psi V_0^\dagger \Gamma_\psi = V_0, \]

where \( \Gamma_\chi = \text{diag}(1, -1, -\kappa) \) and, therefore, are elements of the superalgebra \( \text{su}_\kappa(1, 1/1) \). The conjugation conditions (3) show that in contrast to the superalgebra \( \text{su}(2/1) \) [14], in our case there exist two real superalgebras \( \text{su}_\kappa(1, 1/1) \) corresponding to the values \( \kappa = \pm 1 \) and \( \Gamma_\chi = \Gamma_\psi \). In fact, our \( U, V \) pair belongs to some subalgebra of \( \text{su}_\kappa(1, 1/1) \). In order to describe it, we introduce a vector space \( \mathcal{V}(2/1) \) with two bosonic and one fermionic dimensions [15]. The orthosymplectic metric tensor

\[ G = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \]

defines the scalar product

\[ (x, y) = G_{\hat{a}\hat{b}} x^{\hat{a}} y^{\hat{b}} = -x^1 y^2 + x^2 y^2 + x^3 y^3 \]

of two superspinors \( x, y \in \mathcal{V}(2/1) \). The transformations \( R \) in the superspace \( \mathcal{V}(2/1) \), \( x' = R x, y' = R y \), conserving the scalar product (4), \( (x', y') = (Rx, Ry) = (x, y) \), form the supergroup \( \text{OSP}(2/1) \). The corresponding generators \( (R = e^{iA}) \) form the superalgebra \( \text{osp}(2/1) \) and satisfy the condition

\[ A^{\dagger A} = -GAG^{-1}, \]