Universal Results for Correlations of Characteristic Polynomials: Riemann-Hilbert Approach

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Abstract We prove that general correlation functions of both ratios and products of characteristic polynomials of Hermitian random matrices are governed by integrable kernels of three different types: a) those constructed from orthogonal polynomials; b) constructed from Cauchy transforms of the same orthogonal polynomials and finally c) those constructed from both orthogonal polynomials and their Cauchy transforms. These kernels are related with the Riemann-Hilbert problem for orthogonal polynomials. For the correlation functions we obtain exact expressions in the form of determinants of these kernels. Derived representations enable us to study asymptotics of correlation functions of characteristic polynomials via Deift-Zhou steepest-descent/stationary phase method for Riemann-Hilbert problems, and in particular to find negative moments of characteristic polynomials. This reveals the universal parts of the correlation functions and moments of characteristic polynomials for arbitrary invariant ensemble of $\beta = 2$ symmetry class.

1. Introduction

Correlation functions of characteristic polynomials for various ensembles of random matrices were investigated by a number of authors in a series of recent papers. Keating and Snaith \cite{KS}, Hughes, Keating and O’Connell \cite{HKO1, HKO2} demonstrated that averages of characteristic polynomials over ensembles of random matrices can be useful to make predictions about moments of Riemann zeta function, and other L-functions. These authors consider ensembles of matrices associated with compact groups (the simplest case in the family of ensembles of $\beta = 2$ symmetry class) and derive moments of characteristic polynomials \cite{KS}. In subsequent papers (Conrey, Farmer, Keating, Rubinstein and Snaith \cite{CFKRS1, CFKRS2}) compute more general (autocorrelations or "shifted moments") correlation functions of products of characteristic polynomials.

Brezin and Hikami \cite{BH} (and also Mehta and Normand \cite{MN}) considered correlation functions of products of characteristic polynomials for an arbitrary unitary invariant ensemble of Hermitian matrices. This family of ensembles is characterized by the weight $\exp[-N\text{Tr}V(H)]$ in the corresponding probability measure ( $V(H)$ is an essentially arbitrary potential function, $N$ is the dimension of the matrix $H$). Using the method of orthogonal polynomials, they found both exact and asymptotic (large $N$) expressions for the correlation functions, and for the positive moments. This enabled them to investigate universality of results and find dependence on density of states. Namely, the asymptotic expressions were proved to be factorized in product of universal and non-universal (ensemble-dependent) parts. It was found that the universal numerical pre-factors of positive moments of characteristic polynomials coincide asymptotically with those for the unitary random matrices obtained by Keating and Snaith \cite{KS}. Thus it was rather naturally to expect that these universal pre-factors should appear in the positive moments of Riemann zeta function.
While Brezin and Hikami compare the positive moments of characteristic polynomials with the positive moments of Riemann zeta function, it is similarly worth to compare negative moments of characteristic polynomials with the negative moments of zeta function. Note that such comparison makes sense only if the degree of universality of the negative moments is established. Indeed, it is clear that only universal parts of moments of characteristic polynomials (universal pre-factors, for example) may be related with the corresponding moments of zeta function.

For the negative moments of zeta function a conjecture is available due to the work by Gonek \[32\]. Fyodorov in \[25\] performed the calculations for negative moments of characteristic polynomials for the simplest case of Hermitian random matrices with the Gaussian potential function $V(H) = H^2/2$, known as the Gaussian Unitary ensemble (GUE). The result in \[25\] agreed with those by Gonek \[32\]. However, a full comparison was still not possible since the universal results were unavailable.

In this paper we both find the negative moments of characteristic polynomials for any unitary invariant ensemble of Hermitian matrices and compare them with moments of zeta function.

A study of the negative moments of characteristic polynomials is further motivated by the recent observation by Berry and Keating in \[4\]. These authors argue that divergences of the negative moments could be determined by degeneracies in the spectrum, or clusters of eigenvalues. It is an interesting assertion as clusters should be rare events for a random matrix due to the level repulsion. Berry and Keating show that the question of whether the influence of clusters is dominant is related to that how the negative moments diverge. More precisely, while the negative moments of characteristic polynomials are divergent this divergence can be removed once we agree that we consider these moments off the real axis (i.e. on the line shifted from the real axis by small parameter $\delta$). Then the moments are well defined and, in principle, can be computed. According to Berry and Keating scenario it follows that the $2K$ negative moments are proportional to $\delta^{-K^2}$ as $\delta$ goes to zero for unitary invariant ensembles of Hermitian matrices. In \[25\] it was shown that indeed the negative moments diverge as $\delta^{-K^2}$ for the case of GUE. In this paper we prove the universality of this result, i.e. negative moments of characteristic polynomials diverge as $\delta^{-K^2}$ for all unitary invariant ensembles of Hermitian matrices.

Another important class of correlation functions includes (product of) ratios of characteristic polynomials. As is well known those correlation functions can be used to extract more conventional n-point correlators of the spectral densities (see, for example, \[33\] and the references therein). As a simple illustration of this statement we compute correlation functions which include ratios of characteristic polynomials for an arbitrary unitary invariant ensemble, and then reproduce a well known asymptotic result for the two-point correlation function of the resolvents.

An even more general class of correlation functions are those that include both products and ratios of characteristic polynomials (i.e. when the numbers of characteristic polynomials in the numerator is different from that in the denominator). These functions provide a very detailed information about spectra of random matrices. For this reason such correlation functions (together with autocorrelation functions of characteristic polynomials) are pervasively used in the field of Quantum Chaos see \[1\] \[19\] \[33\] \[29\] and references therein. For example, they are used for extracting generating functions for such physically interesting characteristics as the distributions of the ”local”’ density of states and of the ”level curvatures” (see e.g. Andreev and Simons \[1\] for more detail).

For the particular case of the Gaussian Unitary Ensemble (GUE) the large $N$ asymptotics of such correlation functions is known. A straightforward way to study the asymptotics is to exploit the supersymmetry technique and its modifications \[1\] \[19\] \[25\] \[29\]. For example, in \[25\] one can find asymptotic values for negative moments for the GUE, and Andreev and Simons were the first who obtained the asymptotics for other correlation functions. Moreover, Brezin and Hikami in their recent paper \[8\] have made an effort to apply the supersymmetric technique to correlations of ratios of characteristic polynomials for more complicated Gaussian Orthogonal and Gaussian Symplectic ensembles.

However, the rigorous application of the supersymmetry technique (and its various modifications) is limited to Gaussian ensembles only. Another common disadvantage of these approaches is that they are rather robust. While they work well for the computation of asymptotics, investigation of the correlation functions for finite size matrices is hardly possible. Moreover, those methods hid a nice determinantal structure (revealed in our recent paper \[31\]) of the exact expressions and only yielded the asymptotic result in a form of a sum over permutations \[1\] \[29\].
Thus, for understanding the correlation functions of more general (non-Gaussian) ensembles a different procedure is required. In the present paper we solve the above mentioned problems and provide a unified approach to general correlation functions of characteristic polynomials for unitary invariant ensembles of Hermitian matrices.

2. Statement of the Problem and the Main Results

Let $H$ denote a $N \times N$ random Hermitian matrix which is an element of a unitary invariant ensemble (i.e. that of $\beta = 2$ symmetry class). Introducing $N$-dimensional vector $\hat{x}$ of eigenvalues of the matrix $H$, one defines the ensemble by the eigenvalue density function $P_N(\hat{x})$ (see Mehta [13]),

$$P_N(\hat{x}) = [Z_N]^{-1} \exp \left[ -N \sum_{i=1}^{N} V(x_i) \right] \triangle^2(\hat{x}) \tag{2.1}$$

The symbol $\triangle(\hat{x})$ stands for the Vandermonde determinant, $V(x)$ is a potential function and $Z_N$ is a normalization constant.

A characteristic polynomial which corresponds to the matrix $H$ is defined as $Z_N[\epsilon,H] = \det (\epsilon - H)$. This object is a building block for constructing various correlation functions of interest, such as correlation functions of products and ratios of characteristic polynomials. Our main goal is to provide a systematic method for computing all these correlation functions for non-Gaussian ensembles, i.e. for the potential $V(x)$ which is more general than $x^2$.

From this end we first consider the correlation functions of the following types:

$$F^K_I(\lambda, \mu) = \left\langle \prod_{j=1}^{K} Z_N[\lambda_j, H] Z_N[\mu_j, H] \right\rangle_H, \tag{2.2}$$

$$F^K_{II}(\mu, \epsilon) = \left\langle \prod_{j=1}^{K} Z_N[\mu_j, H] Z_N[\epsilon_j, H] \right\rangle_H, \tag{2.3}$$

$$F^K_{III}(\epsilon, \omega) = \left\langle \prod_{j=1}^{K} \frac{1}{Z_N[\epsilon_j, H]} Z_N[\omega_j, H] \right\rangle_H, \tag{2.4}$$

where the averages are understood as integrals with respect to the measure $d\mu(\hat{x}) = P_N(\hat{x}) d\hat{x}$, with $P_N(\hat{x})$ being defined by the equation (2.1). If the components of vectors $\epsilon$ and $\omega$ have nonzero imaginary parts the correlation functions above are well defined.

Our first result is that each of the above correlation functions are essentially governed by two-point kernels constructed from monic orthogonal polynomials $p_k(x)$ and their Cauchy transforms, $h_k(\epsilon)$. The monic polynomials, $\pi_j(x) = x^j + \ldots$, orthogonal with respect to the measure $d\mu(x) = e^{-NV(x)} dx$, are defined by

$$\int \pi_k(x) \pi_m(x) e^{-NV(x)} dx = c_{km} \delta_{km} \tag{2.5}$$

and their Cauchy transforms are determined in accordance with the following expression

$$h_k(\epsilon) = \frac{1}{2\pi i} \int \frac{e^{-NV(x)} \pi_k(x) dx}{x - \epsilon}, \quad \epsilon \in \mathbb{C}/\mathbb{R} \tag{2.6}$$

The correspondence between the types of correlation functions and different kernels obtained in this paper is summarized in Table 1.

One of those kernels, the kernel $W_{I,N+K}(\lambda, \mu)$, is well known in the theory of random matrices. It is related to the familiar kernel $\gamma_N(\lambda, \mu)$ which is known to determine completely the n-point correlation functions of eigenvalue densities as well as spacing distributions between eigenvalues. The kernel $\gamma_N(\lambda, \mu)$ is defined by

$$\gamma_N(x, y) = -\frac{1}{2\pi i} e^{-y} V(x) \pi_N(x) \pi_{N-1}(y) - \pi_N(x) \pi_{N-1}(y) e^{-y} V(y) \tag{2.7}$$
where
\[ \gamma_{n-1} = -\frac{2\pi i}{e_{n-1}} \]  
(2.8)

However two other kernels \( W_{I,N}(\epsilon, \mu) \), \( W_{III,N-K}(\epsilon, \omega) \) responsible for the correlation functions of characteristic polynomials have not been previously considered, to the best of our knowledge.

Apart from the correlation functions discussed above we investigate also the correlation functions containing non-equal number of characteristic polynomials in the numerator and the denominator, such as
\[
\mathcal{F}_{IV}^{K,M}(\hat{\epsilon}, \hat{\mu}) = \prod_{j=1}^{M} \mathcal{Z}_{j} \prod_{i=1}^{K} \mathcal{Z}_{i} \left\langle \prod_{i=1}^{K} \mathcal{Z}_{i} \right\rangle_H, \quad 0 < M < K
\]  
(2.9)

and
\[
\mathcal{F}_{V}^{K,M}(\hat{\epsilon}, \hat{\mu}) = \prod_{j=1}^{M} \mathcal{Z}_{j} \prod_{i=1}^{K} \mathcal{Z}_{i} \left\langle \prod_{i=1}^{K} \mathcal{Z}_{i} \right\rangle_H, \quad 0 < K < M
\]  
(2.10)

where \( K + M \) is an even number, i.e. \( K + M = 2L \). We reveal that these functions can be expressed in terms of determinants of size \( L \times L \). The entries of the determinant for the function \( \mathcal{F}_{IV}^{K,M}(\hat{\epsilon}, \hat{\mu}) \) are kernels \( W_{I,N-M+L} \) and \( W_{I,N-M+L} \), while the entries of the corresponding determinant for the function \( \mathcal{F}_{V}^{K,M}(\hat{\epsilon}, \hat{\mu}) \) are the kernels \( W_{I,N-M+L} \) and \( W_{III,N-M+L} \).

It is important to note that all the kernels introduced above belong to the family of the so-called "integrable" kernels, i.e. they correspond to integrable operators first distinguished as a class by Its, Izergin, Korepin and Slavnov [38, 39, 40]. According to that theory the kernels corresponding to the integrable operators are defined as follows: Let \( \Sigma \) be oriented contour in \( \mathbb{C} \). An operator \( L \) acting in \( L^2(\Sigma, \mathbb{C}) \) is called integrable if its kernel has the form
\[
L(z, z') = \sum_{j=1}^{M} f_j(z) g_j(z') \quad (2.11)
\]
for some functions \( f_i, g_j; \ i, j = 1, \ldots, M \). The formalism of integrable operators is described by Deift in [1].

Our kernels obviously satisfy the above definition. This is an important observation as integrable kernels have a variety of useful properties that can be exploited (see a review paper by Deift, Zhou and Its [3]). For example, it is known that the Fredholm determinants of integrable kernels satisfy non-linear differential equations. In particular, Tracy and Widom [5, 6] obtained differential equations (of the Painlevé type) for the Fredholm determinants associated with the kernel \( K_N(x, y) \). In the present paper we use another important property of integrable kernels, namely, their relation with Riemann-Hilbert problems. In a similar way as the

| Correlation function | Kernel |
|----------------------|--------|
| \( \mathcal{F}_I^K(\hat{\lambda}, \hat{\mu}) \) | \( W_{I,N+K}(\lambda, \mu) = \frac{\pi_{N+K}(\lambda)\pi_{N+K-1}(\mu) - \pi_{N+K-1}(\lambda)\pi_{N+K}(\mu)}{\lambda - \mu} \) |
| \( \mathcal{F}_{II}^K(\hat{\epsilon}, \hat{\mu}) \) | \( W_{II,N}(\epsilon, \mu) = \frac{h_{N}(\epsilon)\pi_{N-1}(\mu) - h_{N-1}(\epsilon)\pi_{N}(\mu)}{\epsilon - \mu} \) |
| \( \mathcal{F}_{III}^K(\hat{\epsilon}, \hat{\omega}) \) | \( W_{III,N-K}(\epsilon, \omega) = \frac{h_{N-K}(\epsilon)h_{N-K-1}(\omega) - h_{N-K-1}(\epsilon)h_{N-K}(\omega)}{\epsilon - \omega} \) |

Table 1. Correlation functions and kernels
Riemann-Hilbert technique was applied to the kernel $K_N(x, y)$ to demonstrate its universality in the Dyson’s scaling limit (see [12], [13] and [3]). We exploit the Riemann-Hilbert approach to find Dyson’s scaling limit of kernel functions $W_{I,N+K}(\lambda, \mu)$, $W_{II,N}(\epsilon, \mu)$ and $W_{III,N-K}(\epsilon, \omega)$. As a result, we obtain associated universal kernels summarized in Table 2.

| Finite $N$ kernel functions | Associated limiting kernels |
|-----------------------------|----------------------------|
| $W_{I,N+K}(\lambda, \mu)$  | $S_I(\zeta - \eta) = \frac{\sin[\pi(\zeta - \eta)]}{\pi(\zeta - \eta)}$ |
| $W_{II,N}(\epsilon, \mu)$  | $S_{II}(\zeta - \eta) = \begin{cases} \frac{e^{i\pi(\zeta - \eta)}}{\zeta - \eta} & \Im \zeta > 0 \\ \frac{e^{-i\pi(\zeta - \eta)}}{\zeta - \eta} & \Im \zeta < 0 \end{cases}$, $\Im \epsilon \neq 0$ |
| $W_{III,N-K}(\epsilon, \omega)$ | $S_{III}(\zeta - \eta) = \begin{cases} 1 & \Im \zeta > 0, \Im \eta < 0 \\ -1 & \Im \zeta < 0, \Im \eta > 0 \\ 0 & \text{otherwise} \end{cases}$, $\Im \epsilon \neq 0, \Im \omega \neq 0$ |

Table 2. Finite and associated limiting kernels

The representation of the correlation functions in terms of determinants of the kernels (see section 4) enables us to give explicit asymptotic formulae for all five correlation functions of characteristic polynomials discussed in the text above. We give a summary of these results below.

2.1 Dyson’s Limit for $F^K_I(\hat{x}, \hat{\mu}) = \left\langle \prod_{j=1}^{K} Z_N[\lambda_j, H] Z_N[\mu_j, H] \right\rangle_H$

Define $K$-dimensional vectors, $\hat{x} = (x_1, \ldots, x_K)$, where $x_j$ belongs to the support of the equilibrium measure for the potential function $V(x)$ (see section 5 for the definitions), $\hat{\zeta} = (\zeta_1, \ldots, \zeta_K)$ and $\hat{\eta} = (\eta_1, \ldots, \eta_K)$. Then for the correlation function of products of characteristic polynomials we obtain

$$F^K_I(\hat{x} + \hat{\zeta}/N \rho(x), \hat{x} + \hat{\eta}/N \rho(x)) = \left[ c_N \right]^{2K} e^{KNV(x)} [N \rho(x)]^K e^{\alpha(x) \sum_{j=1}^{K} (\zeta_j + \eta_j)} \frac{\Delta(\hat{\zeta}) \Delta(\hat{\eta})}{\Delta(\zeta) \Delta(\eta)} \det \left[ S_I(\zeta_i - \eta_j) \right]_{1 \leq i, j \leq K}$$

(2.12)

where $\rho(x)$ stands for the density of states, and we have introduced the notation

$$\alpha(x) = \frac{V’(x)}{2 \rho(x)}$$

(2.13)

2.2 Dyson’s Limit for $F^K_{II}(\hat{x}, \hat{\mu}) = \left\langle \prod_{j=1}^{K} Z_N[\mu_j, H] \right\rangle_H$

Let $\hat{x}, \hat{\zeta}, \hat{\eta}$ be $K$-dimensional vectors. Assume that the components of $\hat{\zeta}$ have non-zero imaginary parts. Then we find

$$F^K_{II}(\hat{x} + \hat{\zeta}/N \rho(x), \hat{x} + \hat{\eta}/N \rho(x))$$

(2.14)
2.3 Dyson’s limit of $F_{H}^{K}(\hat{\omega}, \hat{\omega}) = \left\langle \prod_{i=1}^{K} \frac{z_N(\omega_i, H) z_N(\omega_i, H)}{1} \right\rangle_H$

It is convenient to introduce $2K$ dimensional vector, $\hat{\epsilon} = (\hat{\sigma}, \hat{\omega})$. The new coordinates appropriate for investigation of Dyson’s asymptotic limit of the correlation function are defined so that $\hat{\epsilon} = \hat{x} + \hat{\zeta}/N\rho(x)$, $\dim \hat{x} = \dim \hat{\zeta} = 2K$. Here the vector $\hat{x}$ has $2K$ equal components, and $x$ belongs to the support of the equilibrium measure for the potential function $V(x)$. As for the components of the vector $\hat{\zeta}$, we assume that they have non-zero imaginary parts. We find

$$F_{H}^{K}(\hat{x} + \hat{\zeta}/N\rho(x)) = (-)^K [\gamma_N]^K [N\rho(x)]^{K^2} e^{-K NV(x)} e^{-\alpha(x) \sum (\hat{\zeta} + \hat{\zeta} + K)}$$

$$\times \frac{1}{(2K)!} \sum_{\pi \in \mathbb{S}_{2K}} \frac{\det [S_{II}(\zeta_{\pi(i)} - \zeta_{\pi(j) + K})]}{\Delta(\zeta_{\pi(1)}, \ldots, \zeta_{\pi(K)}) \Delta(\zeta_{\pi(K+1)}, \ldots, \zeta_{\pi(2K)})} \tag{2.15}$$

2.4 Dyson’s limit for $F_{V}^{K}(\hat{\epsilon}, \hat{\mu}) = \left\langle \prod_{i=1}^{K} \frac{z_N(\mu_i, H)}{1} \right\rangle_H$, $K > M$

In this case new coordinates are introduced so that $\hat{\epsilon} = \hat{x} + \hat{\zeta}/N\rho(x)$, $\hat{\mu} = \hat{x} + \hat{\eta}/N\rho(x)$. It is clear that $\dim \hat{\zeta} = M$, $\dim \hat{\eta} = K$, $\exists \eta \neq 0$. We define $2L = K + M$ (i.e. we consider the correlation function of an even number of characteristic polynomials). With these definitions we find

$$F_{V}^{K}(\hat{x} + \hat{\zeta}/N\rho(x), \hat{x} + \hat{\eta}/N\rho(x))$$

$$= (-)^{\frac{M(M-1)}{2}} [\gamma_N]^{K-M} [N\rho(x)]^{(L-M)^2} e^{N(L-M)V(x)} \tag{2.16}$$

$$\times e^{-\alpha(x) \sum (\zeta - \zeta)} \frac{\Delta(\zeta_1, \ldots, \zeta_M; \eta_{L-M+1}, \ldots, \eta_K)}{\Delta^2(\zeta) \Delta^2(\eta_{L-M+1}, \ldots, \eta_K)}$$

$$\times \frac{\det}{\det} \begin{vmatrix} S_{II}(\zeta_1 - \eta_{L-M+1}) & \ldots & S_{II}(\zeta_1 - \eta_K) \\ \vdots & \ddots & \vdots \\ S_{II}(\zeta_M - \eta_{L-M+1}) & \ldots & S_{II}(\zeta_M - \eta_K) \\ S_{I}(\eta_1 - \eta_{L-M+1}) & \ldots & S_{I}(\eta_1 - \eta_K) \\ \vdots & \ddots & \vdots \\ S_{I}(\eta_{L-M} - \eta_{L-M+1}) & \ldots & S_{I}(\eta_{L-M} - \eta_K) \end{vmatrix}$$

2.5 Dyson’s limit for $F_{V}^{K}(\hat{\epsilon}, \hat{\mu}) = \left\langle \prod_{i=1}^{K} \frac{z_N(\nu_i, H)}{1} \right\rangle_H$, $K < M$

We introduce new coordinates $\hat{x}$, $\hat{\zeta}$, $\hat{\eta}$ as in the previous case. Let $2L = K + M$. Then the Dyson’s limit of the correlation function $F_{V}^{K}(\hat{x} + \hat{\zeta}/N\rho(x), \hat{x} + \hat{\eta}/N\rho(x))$ is

$$F_{V}^{K}(\hat{x} + \hat{\zeta}/N\rho(x), \hat{x} + \hat{\eta}/N\rho(x))$$

$$= (-)^{\frac{M(M-1)}{2}} [\gamma_N]^{M-L} [N\rho(x)]^{(L-K)^2} e^{N(L-K)V(x)} \frac{1}{\Delta^2(\eta)} \frac{1}{M!} \tag{2.17}$$
Theorem 2.1. Equation (2.14) implies that the following theorem is valid:

\[
\sum_{\pi \in S_M} \Delta(\eta_1, \ldots, \eta_K; \zeta_{\pi(1+\frac{M-K}{2})}, \ldots, \zeta_{\pi(M)})
\]

2.6 The average of the resolvent

An interesting observation is that the non-universal functions emerging in above expressions for the correlations of characteristic polynomials can be expressed in terms of the large \( N \) limit of the averaged resolvent defined as

\[
R_N^+(x) = \left\langle \text{Tr} \frac{1}{x-H} \right\rangle_H
\]

Here we assume that the parameter \( x \) has an infinitesimal positive imaginary part. It is then straightforward to observe that \( R_N^+(x) \) is expressed in terms of the correlation function \( F_H^{1}(%(\epsilon, \mu) \)). We have therefore the following expression for the large \( N \) limit of the averaged resolvent:

\[
R_N^+(x) = i\pi \rho(x) - \frac{N V(x)}{2}
\]

attributing a particular meaning to the non-universal factors:

\[
\rho(x) = \frac{1}{\pi} \Im \left[ R_N^+(x) \right], \quad \alpha(x) = -\pi \frac{\Re \left[ R_N^+(x) \right]}{\Im \left[ R_N^+(x) \right]}
\]

2.7 Universality of \( F_H^{1}(%(\epsilon, \mu) \) at the center of the spectrum

Equation (2.14) implies that the following theorem is valid

**Theorem 2.1.** Assume that \( \alpha(x) \equiv V'(x)/2\rho(x) = 0 \) (the center of the spectrum, for example), where \( V(x) \) is the potential function, \( \rho(x) \) is the density of states and \( x \) belongs to the bulk of the spectrum. Then correlation functions of ratios of characteristic polynomials of random Hermitian matrices are universal in the Dyson scaling limit.

**Remark.** It can be observed that for the Gaussian case \( V(x) = x^2/2 \) our results for the correlation functions (at the centre of the spectrum \( x = 0 \)) are reduced to the formula by Andreev and Simons [1]. A detailed derivation of the large \( N \) asymptotics for the Gaussian case can be found in [29].

2.8 Negative moments of characteristic polynomials

Positive moments of characteristic polynomials are determined by the following asymptotic expression (Brezin and Hikami [B]):

\[
\langle Z_N^{2K} | x, H \rangle = [cN]^{-2K} e^{KNV(x)} [\rho(x)]^{K^2} \Upsilon_K^+
\]

where

\[
\Upsilon_K^+ = \prod_{l=0}^{K-1} l!/(l+K)!
\]
is a universal coefficient which also appears in positive moments of zeta-function. In the present paper we, in particular, obtain an asymptotic result for the negative moments of characteristic polynomials $M_{x,N}^{K}(\delta)$. These moments are defined as

$$M_{x,N}^{K}(\delta) = \left\langle Z_{N}^{-K}(x^{+},H)Z_{N}^{-K}(x^{-},H) \right\rangle_{H}$$

where $x^\pm = x \pm \frac{i\delta}{2N\rho(x)}$, $x$ and $\delta$ are real parameters and $\rho(x)$ is the density of states. For the large $N$ limit we find that the negative moments behave asymptotically as

$$M_{x,N}^{K}(\delta) = \left[\frac{2\pi}{K} \right]^{cN} e^{-KN\nu(x)} \left[ \frac{N\rho(x)}{\delta} \right]^{K^{2}}$$

Formula (2.24) should be compared with Gonek’s conjecture for the negative moments of Riemann zeta function, which states that

$$\lim_{T \to \infty} \left\{ \frac{1}{T} \int_{1}^{T} \left| \zeta \left( \frac{1}{2} + \frac{\delta}{\log T} + it \right) \right|^{-2K} dt \right\} \sim \left( \frac{\log T}{\delta} \right)^{K^{2}}$$

A similarity between (2.24) and (2.25) becomes apparent if we put

$$N\rho(x) = \log T$$

in accord with the known expression for the mean density of Riemann zeroes. The pre-factor $[2\pi]^{K} [cN]^{-2K}$ in front of the exponent in equation (2.24) is not universal, and as such it is irrelevant for the comparison with the moments of zeta function (e.g. for the Gaussian case it is equal to $e^{NK}$ in the large $N$ limit). On the other hand, the analogue of the universal coefficient $\Upsilon_{K}^{+}$ for the negative moments is given by

$$\Upsilon_{K}^{-} = 1$$

as is immediately evident from the formula (2.24). In other words, the universal coefficient for negative moments of characteristic polynomials which should also appear in the negative moments of zeta function is 1.

Furthermore, we can see from the expression (2.24) that the negative moments diverge at $\delta \to 0$ as $\delta^{-\nu(K)}$, with the exponent $\nu(K)$ being equal to $K^{2}$. This fact fully agrees with the behaviour conjectured by Berry and Keating for all unitary invariant ensembles of $\beta = 2$ symmetry class (see the discussion in the Introduction).

3. Lagrange Interpolation Formula and Identities for Characteristic Polynomials

In this section we discuss some consequences of the Lagrange interpolation formula (see Szegö [50]). The obtained relations enable us to derive exact expressions for the correlation functions (2.2)-(2.4), (2.9), (2.10).

Let $x_{1}, x_{2}, \ldots, x_{N}$ be eigenvalues of the matrix $H$. Let us associate with the characteristic polynomial $Z_{N}[\epsilon, H]$ of the matrix $H$

$$Z_{N}[\epsilon, H] = (\epsilon - x_{1})(\epsilon - x_{2}) \ldots (\epsilon - x_{N})$$

the fundamental polynomials of the Lagrange interpolation:

$$l_{\nu}(\epsilon) = \frac{Z_{N}[\epsilon, H]}{Z_{N}'[x_{\nu}, H](\epsilon - x_{\nu})}, \quad \nu = 1, 2, \ldots, N$$

From equation (3.1) it is easy to observe that

$$Z_{N}'[x_{\nu}, H] = \prod_{j \neq \nu} (x_{\nu} - x_{j}), \quad \nu = 1, 2, \ldots, N$$

In particular, the equations (3.2) and (3.3) imply that the fundamental polynomials of the Lagrange interpolation have the following property:

$$l_{\nu}(x_{\mu}) = \delta_{\nu\mu}.$$
As each polynomial $P(x)$ of degree $N-1$ is determined uniquely by its value in $N$ points, we have
\[
P(x) = P(x_1)l_1(x) + P(x_2)l_2(x) + \ldots + P(x_N)l_N(x) \tag{3.5}
\]
From the expression (3.5) it follows that
\[
1 = \sum_{\nu=1}^{N} l_{\nu}(\epsilon) \\
\epsilon = \sum_{\nu=1}^{N} x_{\nu} l_{\nu}(\epsilon) \\
\epsilon^{N-1} = \sum_{\nu=1}^{N} x_{\nu}^{N-1} l_{\nu}(\epsilon)
\]
We immediately conclude from the above expressions and the equation (3.2) that the following algebraic identities must hold:
\[
\sum_{\nu=1}^{N} x_{\nu}^{K} / Z_{N}[x_{\nu}, H] = 0, \quad 0 \leq K \leq N - 2 \tag{3.7}
\]
and
\[
\frac{\epsilon^{K}}{Z_{N}[\epsilon, H]} = \sum_{\nu=1}^{N} \frac{x_{\nu}^{K}}{\epsilon - x_{\nu}} \frac{1}{Z'_{N}[x_{\nu}, H]}, \quad \forall K = 0, \ldots, N - 1 \tag{3.8}
\]
With these equations in mind it is not difficult to obtain a representation for the Cauchy transform $h_{N-1}(\epsilon)$ of monic orthogonal polynomial $\pi_{N-1}(x)$ defined by formula (2.6) in terms of a multi-variable integral (for a derivation see [31]):
\[
\gamma_{N-1} h_{N-1}(\epsilon) = \frac{1}{Z_{N}} \int \prod_{j=1}^{N} (\epsilon - x_{j})^{-1} e^{-N \sum_{j=1}^{N} V(x_{j})} \Delta^{2}(x_{1}, \ldots, x_{N}) dx_{1} \ldots x_{N} \tag{3.9}
\]
The right hand side of this formula can be looked at as the average of $Z_{N}^{-1}(\epsilon, H)$ taken over the ensemble of unitary invariant Hermitian matrices. In other words, equation (3.9) implies that
\[
\langle Z_{N}^{-1}[\epsilon, H] \rangle_{H} = \gamma_{N-1} h_{N-1}(\epsilon) \tag{3.10}
\]
In what follows we compute more complicated correlation functions of characteristic polynomials. Those correlation functions include products of characteristic polynomials both in the numerator and the denominator. The algebraic identity which enables us to average the product of the characteristic polynomials in the denominator is
\[
\prod_{l=1}^{M} \frac{\epsilon_{N-M}^{i}}{Z_{N}[\epsilon_{l}, H]}
\]
\[
= \sum_{\sigma} \left( \prod_{i,j=1}^{M} \frac{x_{\sigma(i)}^{N-M}}{\epsilon_{\sigma(i)} - x_{\sigma(j)}} \right) \frac{\Delta(x_{\sigma(1)}, \ldots, x_{\sigma(M+1)}, \ldots, x_{\sigma(N)})}{\Delta(x_{\sigma(1)}, \ldots, x_{\sigma(N)})} \tag{3.11}
\]
where $\sigma \in S_{N}/S_{N-M} \times S_{M}$, $S_{N}$ is the permutation group of the full index set $1, \ldots, N$, whereas $S_{M}$ is the permutation group of the first $M$ indices and $S_{N-M}$ is the permutation group of the remaining $N - M$ indices. Identity (3.11) was proved in [31] and follows as a consequence of the Cauchy-Littlewood formula [17]
\[
\prod_{j=1}^{N} (1 - x_{i}y_{j})^{-1} = \sum_{\lambda} s_{\lambda}(x_{1}, \ldots, x_{N}) s_{\lambda}(y_{1}, \ldots, y_{M}) \tag{3.12}
\]
and the Jacobi-Trudi identity 

\[ s_\lambda(x_1, \ldots, x_N) = \frac{\det \left( x_i^{\lambda_j - j + N} \right)}{\Delta(x_1, \ldots, x_N)} \]  

(3.13)

where the Schur polynomial \( s_\lambda(x_1, \ldots, x_N) \) corresponds to a partition \( \lambda \), and the indices \( i, j \) take their values from 1 to \( N \).

4. Finite Correlation Functions

4.1 Correlation function \( \mathcal{F}_I^K(\hat{\lambda}, \hat{\mu}) = \left\langle \prod \mathcal{Z}_N[\lambda_j, H] \mathcal{Z}_N[\mu_j, H] \right\rangle_H \)

The correlation function of products of characteristic polynomials \( \mathcal{F}_I^K(\hat{\lambda}, \hat{\mu}) \) was investigated in detail by Brezin and Hikami \([6, 7]\). For finite size \( N \) these authors have demonstrated that the correlation function could be rewritten in a determinant form. Namely,

\[ \mathcal{F}_I^K(\hat{\lambda}, \hat{\mu}) = \frac{1}{\Delta(\hat{\lambda}, \hat{\mu})} \det \left| \begin{array}{cccc} \pi_N(\lambda_1) & \pi_{N+1}(\lambda_1) & \ldots & \pi_{N+2K-1}(\lambda_1) \\
\pi_N(\lambda_2) & \pi_{N+1}(\lambda_2) & \ldots & \pi_{N+2K-1}(\lambda_2) \\
\vdots & \vdots & \ddots & \vdots \\
\pi_N(\lambda_K) & \pi_{N+1}(\lambda_K) & \ldots & \pi_{N+2K-1}(\lambda_K) \\
\pi_N(\mu_1) & \pi_{N+1}(\mu_1) & \ldots & \pi_{N+2K-1}(\mu_1) \\
\pi_N(\mu_2) & \pi_{N+1}(\mu_2) & \ldots & \pi_{N+2K-1}(\mu_2) \\
\vdots & \vdots & \ddots & \vdots \\
\pi_N(\mu_K) & \pi_{N+1}(\mu_K) & \ldots & \pi_{N+2K-1}(\mu_K) \\
\end{array} \right| \]  

(4.1)

The same correlation function \( \mathcal{F}_I^K(\hat{\lambda}, \hat{\mu}) \) has also an alternative representation in terms of a determinant of a kernel constructed from monic orthogonal polynomials.

Proposition 4.1. \([6, 7]\): The correlation function of products of characteristic polynomials is governed by a two-point kernel function constructed from monic orthogonal polynomials,

\[ \mathcal{F}_I^K(\hat{\lambda}, \hat{\mu}) = \frac{C_{N,K}}{\Delta(\hat{\lambda})\Delta(\hat{\mu})} \det [W_{I,N+K}(\lambda_i, \mu_j)]_{1 \leq i, j \leq K} \]  

(4.2)

where the kernel \( W_{I,N+K}(x, y) \) is given by the formula

\[ W_{I,N+K}^{\lambda, \mu}(\lambda, \mu) = \frac{\pi_{N+K}(\lambda)\pi_{N+K-1}(\mu) - \pi_{N+K}(\mu)\pi_{N+K-1}(\lambda)}{\lambda - \mu} \]  

(4.3)

The constant \( C_{N,K} \) can be expressed in terms of the coefficients \( \gamma_l \) defined by equation \([2, 3]\)

\[ C_{N,K} = [c_{N+K-1}]^{-2K} \prod_{N}^{N+K-1} (c_l)^2 = \sum_{\gamma_l=1}^{K+1} \gamma_l \]  

(4.4)

Proof. To prove formula \((4.2)\) we observe that the correlation function \( \mathcal{F}_I^K(\hat{\lambda}, \hat{\mu}) \) can be represented as the integral

\[ \mathcal{F}_I^K(\hat{\lambda}, \hat{\mu}) = \frac{Z_{N}^{-1}}{\Delta(\hat{\lambda})\Delta(\hat{\mu})} \int d^{N+K} \hat{x} e^{-N \sum_{i=1}^{N} V(x_i)} \Delta(\hat{\lambda}, \hat{\mu}) \]  

(4.5)

This integral can be evaluated using the method of orthogonal polynomials. Namely, we rewrite the Vandermonde determinants as determinants of monic orthogonal polynomials. Then the product of the Vandermonde determinants in the integrand above can be rewritten as a sum over permutations, i.e.

\[ \Delta(\hat{\lambda}, \hat{\mu}) \]  

\[ = \sum_{\sigma, \rho \in \mathcal{S}_{N+K}} (-1)^{\nu + \nu'} [\pi_{\sigma(1)-1}(\lambda_1) \ldots \pi_{\sigma(K)-1}(\lambda_K)] \]  

(4.5)
\[
\prod_{i=1}^{K} \left( x_1 - x_i \right) \prod_{j=1}^{K} \left( x_j - x_N \right) \]

We insert the above formula into the integrand of \( F_{II}^K (\hat{\lambda}, \hat{\mu}) \) and integrate over the variables \( x_1, \ldots, x_N \). The orthogonality of monic polynomials leads to the expression

\[
F_{II}^K (\hat{\lambda}, \hat{\mu}) = \frac{Z^{-1}_N \prod_{i=0}^{N+K-1} c_i^2}{\Delta(\hat{\lambda}) \Delta(\hat{\mu})} \sum_{\sigma, \rho} (-)^{\sigma + \rho} q_{\sigma(1)-1}(\lambda_1) \cdots q_{\sigma(K)-1}(\lambda_K) \times q_{\rho(1)-1}(\mu_1) \cdots q_{\rho(K)-1}(\mu_K) \times \delta_{\sigma(K+1)\rho(K+1)} \cdots \delta_{\sigma(K+N)\rho(K+N)}.
\]

Here we introduced the polynomials \( q(x) = c_x^{-1} \pi(x) \) normalized with respect to the measure \( dp(x) = \exp(-NV(x)) \). The sum in the equation above can be further transformed to a determinant of the kernel \( W_{II,N}(\epsilon, \mu) \) and we end up with the following expression

\[
F_{II}^K (\hat{\lambda}, \hat{\mu}) = \frac{N! \prod_{i=0}^{N+K-1} c_i^2}{Z_N \Delta(\hat{\lambda}) \Delta(\hat{\mu})} \det \left[ \sum_{0}^{N+K-1} q_i(\lambda_i) q_i(\mu_j) \right]_{1 \leq i, j \leq K}.
\]

Applying to the Christoffel-Darboux formula (see, for example, Szegö [50]) we recover the expression (4.2). \( \square \)

### 4.2 Correlation function \( F_{II}^K (\hat{\epsilon}, \hat{\mu}) \)

Here we derive an exact formula representing \( F_{II}^K (\hat{\epsilon}, \hat{\mu}) \) as a determinant of the kernel \( W_{II,N}(\epsilon, \mu) \).

**Proposition 4.2.** Let \( \Im \epsilon_j \neq 0, j = 1, \ldots, K \). Then the correlation function of ratios of characteristic polynomials is determined by a two-point kernel constructed from monic orthogonal polynomials and their Cauchy transforms. More precisely, the following formula holds

\[
F_{II}^K (\hat{\epsilon}, \hat{\mu}) = \left( - \right)^{\frac{K(K-1)}{2}} \left( \gamma_{N-1} \right)^K \frac{\Delta(\hat{\epsilon}) \Delta(\hat{\mu})}{\Delta^2(\hat{\epsilon}) \Delta^2(\hat{\mu})} \det \left[ W_{II,N}(\epsilon_i, \mu_j) \right]_{1 \leq i, j \leq K}.
\]

where the kernel \( W_{II,N}(\epsilon, \mu) \) is given by

\[
W_{II,N}(\epsilon, \mu) = \frac{h_N(\epsilon) \pi_{N-1}(\mu) - h_{N-1}(\epsilon) \pi_N(\mu)}{\epsilon - \mu}
\]

and the constant \( \gamma_{N-1} \) is defined by the equation (2.8).

**Proof.** We propose a "reduction procedure". The idea is to reduce computation of the correlation functions containing ratios of characteristic polynomials to that of the correlation function which contains only products of characteristic polynomials. Namely, we exploit the identity (3.11) to express the denominator, \( \prod_{i=1}^{K} Z^{-1}_N [\epsilon, H] \), as a sum over permutations. \( F_{II}^K (\hat{\epsilon}, \hat{\mu}) \) is a multi-variable integral with the measure defined by the eigenvalue density function (2.4), so the integrand is symmetric under permutations of the variables of the integration. (Recall that \( x_1, \ldots, x_N \) denote eigenvalues of the Hermitian matrix \( H, \dim H = N \)). It means that each permutation gives
the same contribution to the correlation function. The total number of those permutations is \(\frac{N!}{(N-K)!K!}\). We then find that

\[
\mathcal{F}_I^K(\hat{\epsilon}, \hat{\mu}) = \frac{N!}{(N-K)!K!} \left[ \prod_{i=1}^{K} \epsilon_i^{K-N} \right] ^{\frac{K}{K!}} \left[ \prod_{i,j=1}^{K} \left( \frac{x_i^{N-K} (\mu_j - x_i)}{\epsilon_j - x_i} \right) \right] ^{\frac{K}{K!}} \left[ \prod_{j=1}^{N} \frac{\mu_j - x_j}{x_j - x_j} \right] ^{\frac{K}{K!}} H
\]

(4.11)

The next step is to decompose the integration measure in accordance with the following expression for the eigenvalue density function

\[
P^{(N)}(x_1, \ldots, x_N) = \frac{Z_{N-K} Z_K}{Z_N} \left[ \prod_{i=1}^{K} \prod_{s=K+1}^{N} (x_i - x_s)^2 \right] \times p^{(K)}(x_1, \ldots, x_K) p^{(N-K)}(x_{K+1}, \ldots, x_N)
\]

(4.12)

which allows one to rewrite the correlation function as

\[
\mathcal{F}_I^K(\hat{\epsilon}, \hat{\mu}) = \frac{N!}{(N-K)!K!} \frac{Z_{N-K}}{Z_N} \left[ \prod_{i=1}^{K} \epsilon_i^{K-N} \right] \times \int dx_1 \ldots dx_K \Delta^2(x_1, \ldots, x_K)
\]

(4.13)

\[
\times \left[ \prod_{i,j=1}^{K} \frac{e^{-NV(x_i)}}{\epsilon_j - x_i} \right] \left[ \prod_{j=1}^{N} \frac{\mu_j - x_j}{x_j - x_j} \right] \left[ \prod_{r=1}^{K} Z_{N-K} [\mu_r, \hat{H}] Z_{N-K} [x_i, \hat{H}] \right] \frac{\epsilon_i^{K-N}}{\Delta(\hat{\mu})}
\]

(4.14)

where \(\hat{H} = N - K, \hat{H}^\dagger = \hat{H}\). We then observe that the original integration over \(N\) variables is replaced by an integration over \(K\) variables. Moreover, we notice that the correlation function of products of characteristic polynomials emerges in the integrand of the formula (4.13). Then the equation (4.13) yields

\[
\frac{CN_{-K,K}}{\Delta(\hat{\mu})} \det [W_{I,N}(\mu_i, x_j)]_{1 \leq i,j \leq K}
\]

which leads to essential simplifications in the expression for \(\mathcal{F}_I^K(\hat{\epsilon}, \hat{\mu})\):

\[
\mathcal{F}_I^K(\hat{\epsilon}, \hat{\mu}) = C_{N-K,K} \frac{N!}{(N-K)!K!} \frac{Z_{N-K}}{Z_N} \left[ \prod_{i=1}^{K} \epsilon_i^{K-N} / \Delta(\hat{\mu}) \right] \times \int dx_1 \ldots dx_K \Delta(x_1, \ldots, x_K)
\]

(4.15)

\[
\times \left[ \prod_{i,j=1}^{K} \frac{e^{-NV(x_i)}}{\epsilon_j - x_i} \right] \det [W_{I,N}(\mu_i, x_j)]_{1 \leq i,j \leq K}
\]

The first two terms in the integrand can be further rewritten as \(K \times K\) determinant,

\[
\Delta(x_1, \ldots, x_K) \left[ \prod_{i,j=1}^{K} \frac{e^{-NV(x_i)}}{\epsilon_j - x_i} \right]
\]

(4.16)
Now it is not difficult to calculate the last integral. Let us rewrite determinants as sums over monic polynomials (for details see equation (4.27)),

\[ f_i(x) = x^{N-K+i-1}e^{-NV(x)} \left[ \prod_{\ell=1}^{K} \frac{\mu_{\ell} - x}{\epsilon_{\ell} - x} \right] \]

\[ = \frac{x^{i-1}g(x)}{\prod_{\ell=1}^{K} (\epsilon_{\ell} - x)} \]  

(4.17)

i.e.

\[ g(x) = x^{N-K}e^{-NV(x)} \prod_{\ell=1}^{K} (\mu_{\ell} - x) \]  

(4.18)

Now we simplify \( \det [f_i(x)] \). In order to do this we notice that from (3.8) follows that,

\[ x^{i-1} \prod_{\ell=1}^{K} \frac{g(x)}{\epsilon_{\ell} - x} = (-)^K \sum_{\nu=1}^{K} \frac{\epsilon_{\nu}^{i-1}}{x - \epsilon_{\nu}} \prod_{\ell\neq \nu} \left( \frac{g(x)}{\epsilon_{\ell} - \epsilon_{\nu}} \right) \]

and thus we can write

\[ \det [f_i(x)]_{1 \leq i,j \leq K} = (-)^{K^2} \det \left[ \sum_{\nu=1}^{K} \frac{\epsilon_{\nu}^{i-1}}{x - \epsilon_{\nu}} \prod_{\ell\neq \nu} \left( \frac{g(x)}{\epsilon_{\ell} - \epsilon_{\nu}} \right) \right]_{1 \leq i,j \leq K} \]

\[ = (-)^{K^2} \det \left( \epsilon_{\nu}^{i-1} \right) \det \left[ \frac{1}{x - \epsilon_{\nu}} \prod_{\ell\neq \nu} \left( \frac{g(x)}{\epsilon_{\ell} - \epsilon_{\nu}} \right) \right]_{1 \leq j,\nu \leq K} \]

\[ = (-)^{K^2} (-)^{K(K-1)} \frac{1}{\Delta(\epsilon)} \det \left[ \frac{g(x)}{x - \epsilon_{\nu}} \right]_{1 \leq j,\nu \leq K} \]  

(4.19)

Therefore,

\[ F_{II}^{K}(\hat{\epsilon}, \hat{\mu}) = (-)^{K+1} C_{N-K,K} \frac{N!}{(N-K)!K!} \frac{Z_{N-K}}{Z_N} \prod_{\ell=1}^{K} \frac{1}{\epsilon_{\ell}^{N-K}} \]

\[ \times \int dx_1 \ldots dx_K \det \left[ \frac{g(x)}{x - \epsilon_{\nu}} \right]_{1 \leq i,j \leq K} \det [W_{I,N}(\mu_i, x_j)]_{1 \leq i,j \leq K} \]  

(4.20)

Now it is not difficult to calculate the last integral. Let us rewrite determinants as sums over permutations

\[ \int dx_1 \ldots dx_K \det \left[ \frac{g(x)}{x - \epsilon_{\nu}} \right]_{1 \leq i,j \leq K} \det [W_{I,N}(\mu_i, x_j)]_{1 \leq i,j \leq K} \]

\[ = \sum_{\sigma, \rho \in S_K} (-)^{\nu_\sigma + \nu_\rho} \int dx_1 \left[ \frac{g(x)}{x - \epsilon_{\sigma(1)}} \right] W_{I,N}(\mu_{\rho(1)}, x_1) \]

\[ \times \ldots \times \int dx_K \left[ \frac{g(x)}{x - \epsilon_{\sigma(K)}} \right] W_{I,N}(\mu_{\rho(K)}, x_K) \]  

(4.21)

We compute the integrals above using orthogonality of monic polynomials (for details see equation (4.27), where a similar integral is calculated). This yields

\[ \int dx_1 \ldots dx_K \det \left[ \frac{g(x)}{x - \epsilon_{\nu}} \right]_{1 \leq i,j \leq K} \det [W_{I,N}(\mu_i, x_j)]_{1 \leq i,j \leq K} \]

\[ = \sum_{\sigma, \rho \in S_K} (-)^{\nu_\sigma + \nu_\rho} \epsilon_{\sigma(1)}^{N-K} \left[ \prod_{\ell=1}^{K} (\mu_{\ell} - \epsilon_{\sigma(1)}) \right] 2\pi i W_{II,N}(\epsilon_{\sigma(1)}, \mu_{\rho(1)}) \]

\[ \times \ldots \times \epsilon_{\sigma(K)}^{N-K} \left[ \prod_{\ell=1}^{K} (\mu_{\ell} - \epsilon_{\sigma(K)}) \right] 2\pi i W_{II,N}(\epsilon_{\sigma(K)}, \mu_{\rho(K)}) \]  

(4.22)

\[ = (2\pi i)^{K} K! \prod_{\ell=1}^{K} \epsilon_{\ell}^{N-K} \left[ \prod_{i,j=1}^{K} (\mu_i - \epsilon_j) \right] \det [W_{II,N}(\epsilon_i, \mu_j)]_{1 \leq i,j \leq K} \]
We insert the obtained expression to (4.21) and with some simple algebra prove the proposition.

\[ \square \]

### 4.3 Correlation function \( F_{III}^K(\hat{\omega}, \hat{\omega}) \)

Similar to the previous cases, the correlation functions that contain an even number of characteristic polynomials in the denominator are governed by a (different) two-point kernel. However, such correlation functions are not readily expressible as a determinant of a kernel divided by two Vandermonde determinants. Now an exact formula will be slightly more complicated. In fact, we prove that the correlation function \( F_{III}^K(\hat{\omega}, \hat{\omega}) \) is a sum over permutations.

**Proposition 4.3.** Define \( 2K \) dimensional vector \( \hat{\epsilon} = (\hat{\omega}, \hat{\epsilon}) \), \( \Im \epsilon_j \neq 0 \)

Then the correlation function which contains an even number of characteristic polynomials in the denominator, \( F_{III}^K(\hat{\epsilon}) \equiv F_{III}^K(\hat{\omega}, \hat{\epsilon}) \), can be expressed as the following sum over permutations

\[ F_{III}^K(\hat{\epsilon}) = (-)^K \frac{[N-1]}{(2K)!} \sum_{\pi \in S_{2K}} \det \left[ [W_{III,N-K}(\epsilon_{\pi(i)}, \epsilon_{\pi(K+j)})]_{1 \leq i,j \leq K} \right] \]

The two-point kernel \( W_{III,N-K}(\epsilon, \omega) \) is constructed from the Cauchy transforms of monic orthogonal polynomials,

\[ W_{III,N-K}(\epsilon, \omega) = h_{N-K}(\epsilon)h_{N-K-1}(\omega) - h_{N-K-1}(\epsilon)h_{N-K}(\omega) \]

The constant \( \gamma_{N-1} \) is determined by equation (3).

**Proof.** We follow the procedure applied previously to the correlation function \( F_{II}^K(\hat{\epsilon}, \hat{\mu}) \). Instead of equation (4.20) we obtain

\[ F_{III}^K(\hat{\epsilon}) = (-)^K C_{N-2K,K} \frac{N!}{(N-2K)!(2K)!} \frac{Z_{N-2K}}{Z_N} \frac{1}{\Delta(\epsilon)} \]

\[ \times \int dx_1 \ldots dx_{2K} \left[ \prod_{i=1}^{2K} (x_i - x_l) \right] \]

\[ \times \det \left[ \frac{x_i^{N-2K} e^{-NV(x_i)}}{x_i - \epsilon_j} \right]_{1 \leq i,j \leq 2K} \]

To compute the integral above we proceed as follows

\[ \int dx_1 \ldots dx_{2K} \left[ \prod_{i=1}^{2K} (x_i - x_l) \right] \]

\[ \times \det \left[ \frac{x_i^{N-2K} e^{-NV(x_i)}}{x_i - \epsilon_j} \right]_{1 \leq i,j \leq 2K} \]

\[ = \sum_{\pi \in S_{2K}} (-)^{\nu_{\pi}} \sum_{\sigma \in S_{2K}} (-)^{\nu_{\sigma}} \]

\[ \times \int dx_1 \ldots dx_K \left[ \frac{x_i^{N-2K} e^{-NV(x_i)}}{x_i - \epsilon_{\pi(1)} - x_1} \right] \times \ldots \times \left[ \frac{x_i^{N-2K} e^{-NV(x_i)}}{x_i - \epsilon_{\pi(K)} - x_K} \right] \]
This yields
\[ \times \int dy_1 \frac{y_1^{N-2K} e^{-NV(y_1)} \prod_1^K (x_1 - y_1)}{\epsilon_i(y_1)} W_{I,N-K}(x_{\sigma(1)}, y_1) \]
\[ \times \ldots \times \]
\[ \times \int dy_K \left[ \frac{y_K^{N-2K} e^{-NV(y_K)} \prod_1^K (x_1 - y_K)}{\epsilon_i(y_K)} \right] W_{I,N-K}(x_{\sigma(K)}, y_K) \]
We have
\[ \int dy \left[ \frac{y^{N-2K} e^{-NV(y)} \prod_1^K (x_1 - y)}{\epsilon_i - y} \right] W_{I,N-K}(x, y) \]
\[ = \int dy \frac{y^{N-2K} e^{-NV(y)} \prod_1^{j-1} (x_1 - y) \prod_{j+1}^K (x_1 - y)}{\epsilon_i - y} \]
\[ \times \left[ \pi_{N-K}(x_j) \pi_{N-K-1}(y) - \pi_{N-K}(y) \pi_{N-K-1}(x_j) \right] \]
\[ = \left[ \prod_1^{j-1} (x_1 - \epsilon_i) \right] \left[ \prod_{j+1}^K (x_1 - \epsilon_i) \right] \]
\[ \times \int dy \left[ \frac{y^{N-2K} e^{-NV(y)} \prod_1^K (x_1 - y)}{\epsilon_i - y} \right] W_{I,N-K}(x, y) \]
where the orthogonality of the monic polynomials with respect to the weight function \( \exp \left[ -NV(x) \right] \) is used. Therefore,
\[ \int dy \left[ \frac{y^{N-2K} e^{-NV(y)} \prod_1^K (x_1 - y)}{\epsilon_i - y} \right] W_{I,N-K}(x, y) \]
\[ = (-2\pi i)^{N-2K} \left[ \prod_1^K (x_1 - \epsilon_i) \right] W_{II,N-K}(\epsilon_i, x_j) \]
This yields
\[ \int dx_1 \ldots dx_{2K} \left[ \prod_{s=1}^{2K} (x_s - x_l) \right] \]
\[ \times \det \left[ \frac{x^{N-2K} e^{-NV(x)}}{x_l - \epsilon_j} \right]_{1 \leq i \leq 2K} \]
\[ \times \det [W_{I,N-K}(x_l, x_j)]_{1 \leq i, j \leq K} \]
\[ = (-2\pi i)^K \sum_{\pi \in S_{2K}} (-)^{\nu_{\pi}} \left[ \prod_{s=1}^K e^{\epsilon_i N-2K} \right] \sum_{\sigma \in S_K} (-)^{\nu_{\pi}} \]
\[ \int dx_{\sigma(1)} \frac{y^{N-2K} e^{-NV(x_{\sigma(1)})} \prod_{1}^K (x_{\sigma(1)} - \epsilon_{\pi(1)})}{\epsilon_{\pi(1)} - x_{\sigma(1)}} W_{II,N-K}(\epsilon_{\pi(1)+\sigma(1)}, x_{\sigma(1)}) \]
\[ \times \ldots \times \]
\[ \int dx_{\sigma(K)} \frac{x^{N-2K} e^{-NV(x_{\sigma(K)})} \prod_{1}^K (x_{\sigma(K)} - \epsilon_{\pi(K)})}{\epsilon_{\pi(K)} - x_{\sigma(K)}} W_{II,N-K}(\epsilon_{\pi(K)+\sigma(K)}, x_{\sigma(K)}) \]
where M < K following formula holds

\[ \text{Proposition 4.4.} \]



\[ \text{We insert the above expression to the formula (4.26) and prove the proposition.} \]

\[ \text{□} \]

\[ \text{(4.28)} \]

\[ \text{4.4 Correlation function } F^K_{IV}(\hat{\epsilon}, \hat{\mu}) = \left( \frac{\prod_{i=1}^{K} H_n[\mu_i, H]}{\prod_{i=1}^{K} H_n[\epsilon_i, H]} \right), \text{ } K > M \]

As a result of the fact that the numbers of characteristic polynomials in the denominator and
the numerator are not equal to each other, the correlation function of characteristic polynomials
turns out to be determined by two kernel functions.

\[ \text{Proposition 4.4.} \text{ Let } \Im \epsilon_j \neq 0. \text{ For the correlation function which contains } K \text{ characteristic polynomials in the numerator and } M < K \text{ characteristic polynomials in the denominator the following formula holds} \]

\[ F^K_{IV}(\hat{\epsilon}, \hat{\mu}) = (-)^{\frac{M(K-1)}{2}} \left( \frac{\prod_{i=1}^{K} \gamma_{N-M-L-1}^{L-i} \gamma_i}{\prod_{i=1}^{K} \gamma_{N-M-L-1}^{L-i}} \right) \]

\[ \times \frac{\Delta(\epsilon_1, \ldots, \epsilon_M; \mu_{L-M+1}, \ldots, \mu_K)}{\Delta^2(\epsilon^*(1), \ldots, \epsilon^*(K)) \Delta(\epsilon^*_1, \ldots, \epsilon^*_K) \Delta(\mu_1, \ldots, \mu_{L-M})} \]

\[ \times \det \begin{vmatrix} W_{I,I,N-M+L}(\epsilon_1, \mu_{L-M+1}) & \cdots & W_{I,I,N-M+L}(\epsilon_1, \mu_K) \\ \vdots & \ddots & \vdots \\ W_{I,N-M+L}(\epsilon_1, \mu_{L-M+1}) & \cdots & W_{I,N-M+L}(\epsilon_1, \mu_K) \\ V_{I,I,N-M+L}(\mu_1, \mu_{L-M+1}) & \cdots & V_{I,N-M+L}(\mu_1, \mu_K) \\ \vdots & \ddots & \vdots \\ V_{I,N-M+L}(\mu_1, \mu_{L-M+1}) & \cdots & V_{I,N-M+L}(\mu_1, \mu_K) \end{vmatrix} \]

where 2L = K + M (i.e. the total number of characteristic polynomials is even) and the kernel functions are defined by equations \[ \text{(4.13)} \] and \[ \text{(4.14)} \].
4.5 Correlation function $F^K_V(\hat{\epsilon}, \hat{\mu}) = \left( \prod_{j=1}^{K} Z_N[\mu_j, H] \right) \left( \prod_{j=1}^{M} Z_N[\epsilon_j, H] \right)_H$, $K < M$

We also have found the representation for the correlation function $F^K_V(\hat{\epsilon}, \hat{\mu})$ in terms of the kernels. Similar to the correlation function $F^K_{\mathcal{I}}(\hat{\epsilon}, \hat{\mu})$ the correlation function $F^K_V(\hat{\epsilon}, \hat{\mu})$ is determined by two kernels. Both these kernels now include Cauchy transforms of the orthogonal polynomials. Here we present the formula without a proof.

**Proposition 4.5.** Let $3m \epsilon_j \neq 0$. Then the following formula holds

\[
F^K_V(\hat{\epsilon}, \hat{\mu}) = (-\frac{M(M-1)}{2}) \left[ \frac{\gamma^L_{N-M+L-1}}{\prod_{N-M+L} \gamma_{\lambda}} \right] \frac{1}{\Delta^2(\mu)} \frac{1}{M!} 
\times \sum_{\pi \in S_M} \triangle \left( \epsilon_1, \epsilon_2, \ldots, \epsilon_M, \epsilon_{M+1}, \ldots, \epsilon_{N-M+L} \right) \triangle \left( \epsilon_{\pi(1)}, \epsilon_{\pi(2)}, \ldots, \epsilon_{\pi(M)} \right) 
\times \det 
\begin{vmatrix}
W_{II_1,N-M+L}(\epsilon_{\pi(1)}, \mu_1) & \ldots & W_{II_1,N-M+L}(\epsilon_{\pi(M)}, \mu_1) \\
\vdots & & \vdots \\
W_{III_1,N-M+L}(\epsilon_{\pi(1)}, \mu_K) & \ldots & W_{III_1,N-M+L}(\epsilon_{\pi(M)}, \mu_K) \\
\end{vmatrix}
\]

where $2L = K + M$ (i.e. the total number of the characteristic polynomials is even) and the kernel functions are defined by equations (4.10) and (4.23).

4.6 Formula for the general correlation function

It is possible to derive an exact expression which is valid for all the five cases considered above. Namely, in [31] we have proved that

\[
\left( \prod_{j=1}^{K} Z_N[\mu_j, H] \right) \left( \prod_{j=1}^{M} Z_N[\epsilon_j, H] \right)_H = \frac{\prod_{j=N-M}^{N-1} \gamma_j}{\Delta(\bar{\mu}) \Delta(\bar{\epsilon})} 
\times \det 
\begin{vmatrix}
h_{N-M}(\epsilon_1) & h_{N-M+1}(\epsilon_1) & \ldots & h_{N+K-1}(\epsilon_1) \\
\vdots & & & \vdots \\
h_{N-M}(\epsilon_M) & h_{N-M+1}(\epsilon_M) & \ldots & h_{N+K-1}(\epsilon_M) \\
\pi_{N-M}(\mu_1) & \pi_{N-M+1}(\mu_1) & \ldots & \pi_{N+K-1}(\mu_1) \\
\vdots & & & \vdots \\
\pi_{N-M}(\mu_K) & \pi_{N-M+1}(\mu_K) & \ldots & \pi_{N+K-1}(\mu_K) \\
\end{vmatrix}
\]

(4.30)

Here $K$ and $M$ are arbitrary positive integers. Thus formula (4.30) is also valid when the total number of characteristic polynomial is odd. However, the new formulae obtained in this section reveal a kernel structure which makes them more convenient for investigations of the asymptotic behaviour.
5. Correlation Functions of Characteristic Polynomials and the Riemann-Hilbert Problem

In this section we establish a relation between the correlation functions of characteristic polynomials and the Riemann-Hilbert problem. This relation is crucial as it enables us to study the asymptotic behaviour for the non-Gaussian case. Below we review some aspects of the steepest-descent/stationary phase method for Riemann-Hilbert problems introduced by Deift and Zhou (a detailed presentation can be found in the book by Deift [12]).

The method is then applied in section 6 for extracting the asymptotics of the kernels $W_{I,N}(\lambda, \mu)$, $W_{II,N}(\epsilon, \mu)$, and $W_{III,N+K}(\epsilon, \omega)$. These results combined with the propositions proved in the previous section give us access to the asymptotic values for all five correlation functions of characteristic polynomials (2.13)- (2.17).

5.1 Relation to the Riemann-Hilbert problem.

A technique of integrable systems (called the Riemann-Hilbert problem technique) is exploited in a large number of problems in mathematics and mathematical physics (see, for example, [20]). The works of Fokas, Its and Kitaev [21, 22] relate orthogonal polynomials and their Cauchy transforms with an appropriate $2 \times 2$ matrix Riemann-Hilbert problem thus opening a possibility to apply the Riemann-Hilbert techniques to orthogonal polynomials and to the theory of random matrices. The observation of Fokas, Its and Kitaev (combined with semi-classical methods for the analysis of Riemann-Hilbert problems [5, 12]-[18]) enabled one to understand the semi-classical asymptotics of the orthogonal polynomials and to provide an elegant proof of the Dyson universality conjecture for the Random Matrix Theory [16, 18, 5].

The exact formulas obtained in the previous sections show that only monic orthogonal polynomials and their Cauchy transforms enter various correlation functions of characteristic polynomials. For this reason the Riemann-Hilbert problem technique arise quite naturally in the study of correlation functions of that type.

All asymptotic questions we are going to address turn out to be tractable in the framework of the Riemann-Hilbert problem for orthogonal polynomials (Fokas, Its and Kitaev [21, 22]). Assume that the contour $\sum = \mathbb{R}$ is oriented from left to right. The upper side of the complex plane with respect to the contour will be called the positive side and the lower part - the negative side. Once the integer $n \geq 0$ is fixed, the Riemann-Hilbert problem is to find a $2 \times 2$ matrix valued function $Y = Y^{(n)}(z)$ such that the following conditions are satisfied

- $Y^{(n)}(z)$ is analytic in $\mathbb{C}/\mathbb{R}$
- $Y^{(n)}_+(z) = Y^{(n)}_-(z) \left( \begin{array}{cc} 1 & e^{-nV(z)} \\ 0 & 1 \end{array} \right)$, $z \in \mathbb{R}$
- $Y^{(n)}(z) \to (I + \mathcal{O}(z^{-1})) \left( \begin{array}{cc} z^n & 0 \\ 0 & z^{-n} \end{array} \right)$ as $z \to \infty$

Here $Y^{(n)}_+(z)$ denotes the limit of $Y^{(n)}(z')$ as $z' \to z \in \mathbb{R}$ from the positive/negative side. It was proved by Fokas, Its and Kitaev [21, 22] that the solution of this Riemann-Hilbert problem is unique and is expressed as

$$Y^{(n)}(z) = \left( \begin{array}{cc} \pi_n(z) & h_n(z) \\ \gamma_{n-1} \pi_{n-1}(z) & \gamma_{n-1} h_{n-1}(z) \end{array} \right), \quad z \in \mathbb{C}/\mathbb{R} \quad (5.1)$$

where $\pi_n(z)$ and $h_n(z)$ are monic orthogonal polynomials and their Cauchy transforms, respectively, and the constant $\gamma_{n-1}$ is defined by equation (2.3).

From the propositions of the previous section we immediately observe that the correlation functions of characteristic polynomials are expressible in terms of the elements of $Y^{(n)}(z)$. Moreover, the constant $\gamma_{N-1}$ which determines pre-factors in the exact expressions for the correlation functions (see the propositions (4.1)-(4.5) and the formula (4.30)) emerges in the solution of the Riemann-Hilbert problem as well.
To demonstrate the relation between the correlation functions of characteristic polynomials and the Riemann-Hilbert problem proposed by Fokas, Its and Kitaev it is instructive to consider the following example. Let us define the function
\[
\mathcal{F}(z) = \frac{\det(\mu - H)}{\det(\epsilon - H)} , \quad z \in \mathbb{C}
\] (5.2)
The function \( \mathcal{F}(z) \) is analytic in the whole complex plane, and \( \mathcal{F}(z) \equiv 1, \forall z \in \mathbb{C} \). On one hand, proposition [4.2] implies that
\[
\operatorname{det} Y = \pi_N(z) \frac{h_N(z)}{\gamma_N-1} \frac{h_{N-1}(z)}{\gamma_N-1} (5.3)
\]
where \( Y^{(N)}(z) \) solves the Riemann-Hilbert problem for the orthogonal polynomials defined above. On the other hand, the fact that \( \det [Y^{(N)}(z)] = 1, \forall z \in \mathbb{C} \) can be directly obtained from the definition of the Riemann-Hilbert problem (Bleher and Its [4]). Indeed, if \( Y^{(N)}(z) \) solves the Riemann-Hilbert problem for the orthogonal polynomials it should be that
\[
\left[ \operatorname{det} Y^{(N)} \right]_+(z) = \left[ \operatorname{det} Y^{(N)} \right]_-(z) \det \begin{pmatrix} 1 & e^{-NV(z)} \\ 0 & 1 \end{pmatrix} \] (5.4)
Therefore, \( \det [Y^{(N)}(z)] \) is analytic in \( \mathbb{C} \) and \( \det [Y^{(N)}(z)] = 1 + \mathcal{O}(z^{-1}) \) as \( z \to \infty \). Then we have \( \det [Y^{(N)}(z)] \equiv 1 \).

5.2 The Deift-Zhou deformations of Riemann-Hilbert problems.

A key ingredient of the Deift-Zhou approach to extracting the asymptotics of the Riemann-Hilbert problem is the notion of the equilibrium measure (see [12, 17, 18]). The equilibrium measure is the solution of the following energy minimization problem. Assume that the value \( E^V \) is defined by
\[
E^V = \int V(s) d\mu(s) + \int \int \log |s-t|^{-1} d\mu(s) d\mu(t) \] (5.5)
Then the energy minimization problem is to find a measure \( d\mu(s) \) which minimizes \( E^V \). On the real line the equilibrium measure \( d\mu(x) = \psi(x) dx \) can be uniquely determined and satisfies the following Euler-Lagrange variational conditions:

There exists a real constant \( l \) such that
\[
2 \int \log |x-y|^{-1} d\mu(y) + V(x) \geq l, \quad \forall x \in \mathbb{R} \] (5.6)
\[
2 \int \log |x-y|^{-1} d\mu(y) + V(x) = l, \quad \psi(x) > 0 \] (5.7)
For the potential \( V(x) = x^{2m} \) the solution of the energy minimization problem described above is given by the equations:
\[
d\mu(x) = \psi(x) dx, \quad \psi(x) = \frac{m}{4\pi} (x^2 - a^2)^{1/2} h_1(x) \chi_{(-a,a)} \] (5.8)
\[
h_1(x) = x^{2m-2} + \sum_{j=1}^{m-1} x^{m-2j} a^{2j} \prod_{l=1}^{j} \frac{2l - 1}{2l} , \] (5.9)
\[
a = \left( m \prod_{l=1}^{m} \frac{2l - 1}{2l} \right)^{-1/2} \]
where \( \chi_{(-a,a)} \) is the characteristic function of the interval \( (-a,a) \).
When the equilibrium measure is found we can define the following function:

$$g(z) = \int \log(z - s) \psi(s) ds, \ s \in \mathbb{R}, \exists m \ z \neq 0$$

(5.10)

Here we take the principal branch of the logarithm, i.e.

$$\log(z - s) = \log |z - s| + i \arg(z - s)$$

(5.11)

where

$$0 < \arg(z - s) < \pi, \ s \in \mathbb{R}, \ \exists m \ z > 0$$

(5.12)

$$-\pi < \arg(z - s) < 0, \ s \in \mathbb{R}, \ \exists m \ z < 0$$

(5.13)

The function $g(z)$ has the following analytical properties:

- $g(z)$ is analytic in $\mathbb{C} \setminus (-\infty, a]$.
- $g_{\pm}(z) = \int \log |z - s| \psi(s) ds \pm i\pi, \ z < -a$.
- $g_{\pm}(z) = \int \log |z - s| \psi(s) ds \pm i\pi \int \psi(s) ds, \ -a < z < a$.
- $g(z) = \log z + O(z^{-1}), \ z \to \infty, \ z \in \mathbb{C} \setminus (-\infty, a]$.

Once the function $g(z)$ and the constant $l$ are specified, the following transformation is introduced:

$$Y^{(N)}(z) = e^{-\frac{N}{2} \sigma_3} m^{(1)}(z) e^{\frac{N}{2} \sigma_3} e^{Ng(z) \sigma_3}$$

(5.14)

We note that $e^{Ng(z)}$ is analytic in $\mathbb{C} \setminus [-a, a]$ and $e^{Ng(z)} = z^N(1 + O(z^{-1}))$ as $z \to \infty$. It then follows that the matrix function $m^{(1)}(z)$ satisfies the conditions:

- $m^{(1)}(z)$ is analytic in $\mathbb{C} \setminus \mathbb{R}$.
- $m^{(1)}(z) = m^{(1)}(z) \begin{pmatrix} e^{Ng_z(z)g_+(z)} & e^{Ng_z(z)g_+(z)+l-V(z)} \\ e^{Ng_z(z)g_-(z)} & e^{Ng_z(z)g_-(z)} \end{pmatrix}$.
- $m^{(1)}(z) = I + O(z^{-1})$ as $z \to \infty$.

Now the analytical properties of the function $g(z)$, the Euler-Lagrange variational conditions, and the explicit form of the function $\psi(s)$ can be exploited altogether to derive the following representation for the jump matrix $\psi^{(1)}(z)$ of the Riemann-Hilbert problem above:

$$\psi^{(1)}(z) = \begin{pmatrix} 1 & -2mN \int_{-a}^{z} (t^2 - a^2)^{1/2} h_1(t) dt \\ 0 & 1 \\ e^{-2mN \int_{-a}^{z} (t^2 - a^2)^{1/2} h_1(t) dt} & 1 \\ 0 & e^{-2mN \int_{-a}^{z} (t^2 - a^2)^{1/2} h_1(t) dt} \\ 1 & -2mN \int_{-a}^{z} (t^2 - a^2)^{1/2} h_1(t) dt \\ 0 & 1 \end{pmatrix}$$

(5.15)

>In the above formula $(t^2 - a^2)^{1/2}$ has positive (negative) value when $z \geq a$ ($z \leq a$). Let us set

$$\varphi(z) = m \int_{a}^{z} (t^2 - a^2)^{1/2} h_1(t) dt, \ z \in \mathbb{C} \setminus [-a, a]$$

(5.15)

The function $\varphi(z)$ is not well-defined as it depends on the path of integration, but $e^{\pm N\varphi(z)}$ is well-defined and analytic in $\mathbb{C} \setminus [-a, a]$. With help of the function $\varphi(z)$ we obtain the following
factorization of the jump matrix $v^{(1)}(z)$:

$$v^{(1)}(z) = \begin{cases} 
(1 \ e^{-2N\varphi(z)}) & , z \leq -a \\
(0 \ 1) & \\
(1 \ 0) & (0 \ 1) \\
(e^{2N\varphi(z)} \ 1) & , z \in [-a, a] \\
1 & e^{-2mN \int_{-a}^{a} (t^2-a^2)^{1/2} h(t) dt} & , z \geq a \\
0 & 1 \\
1 & e^{-2N\varphi(z)} & 0 & 1 \\
0 & 1 & e^{2N\varphi(z)}
\end{cases}$$

Once the jump matrix $v^{(1)}(z)$ is factorized we can define the new matrix-valued function $m^{(2)}(z)$ as shown in Figure 1.

$$m^{(2)}(z) = m^{(1)}(z) \begin{pmatrix} 1 & 0 \\ -e^{2N\varphi(z)} & 1 \end{pmatrix}$$

Fig. 1 Definition of $m^{(2)}(z)$

Fig. 2 The R-H problem for $m^{(2)}(z)$
The matrix \( m^{(2)}(z) \) is the solution of the Riemann-Hilbert problem on the extended contour \( \Sigma_2 \) (see Fig. 2):

- \( m^{(2)}(z) \) is analytic in \( \mathbb{C}/\Sigma_2 \)
- \( m^{(2)}_m(z) = m^{(2)}_m(z) \) \( m^{(2)}(z) \)
- \( m^{(2)}(z) \to I \) as \( z \to \infty \)

The solution \( m^{(2)}(z) \) of the Riemann-Hilbert problem defined above has the following property:

**Proposition 5.1.** Let \( x \in (-a,a) \). Then for \( z \) in the vicinity of \( x \) the solution of the Riemann-Hilbert problem \( m^{(2)}(z) \) and its derivative \( dm^{(2)}(z)/dz \) are bounded as \( N \to \infty \).

Once the proposition (5.1) is proved one can observe that \( K_N(x,x) \) defined by the equation (2.7) is equal to \( N\psi(x) \) to the leading order, i.e.

\[
K_N(x,x) = N\psi(x) + O(1) \quad \text{as} \quad N \to \infty
\]

(5.16)

### 6. Asymptotics of the Kernels

In this section we use the results outlined above to determine the asymptotic behaviour of three kernels \( W_{I,N+K}(\lambda,\mu), \) \( W_{II,N}(\lambda,\mu), \) and \( W_{III,N-K}(\lambda,\mu) \) in the Dyson’s limit. This is achieved by three subsequent transformations. The first step is to express the kernels \( W_{I,N+K}(\lambda,\mu) \) and \( W_{III,N-K}(\lambda,\mu) \) in terms of matrix elements of \( Y^{(3)}(z) \). We then rewrite them in terms of \( m^{(2)}(z) \) and, finally, in terms of \( m^{(2)}(z) \) defined by Fig. 1 and Fig. 2. The reason for these transformations is that \( m^{(2)}(z) \) and its derivative are bounded matrix valued functions as \( N \to \infty \). It is this fact and the equation \( \det m^{(2)}(z) = 1 \) that enable us to find the large \( N \) asymptotic of kernels in the Dyson’s scaling limit. The obtained asymptotic formulae are summarized in Table 3.

| Kernel | Large \( N \) asymptotic |
|-------|--------------------------|
| \( W_{I,N+K}(x,\zeta,\eta) \) | \( c_N^2 [N\rho(x)] e^{NV(x)} e^{\alpha(x)(\zeta+\eta)} S_I(\zeta - \eta) \) |
| \( W_{II,N}(x,\zeta,\eta) \) | \( -\frac{c_N^2}{2\pi i} [N\rho(x)] e^{-\alpha(x)(\zeta-\eta)} S_{II}(\zeta - \eta) \) |
| \( W_{III,N-K}(x,\zeta,\eta) \) | \( -\frac{c_N^2}{2\pi i} [N\rho(x)] e^{-NV(x)} e^{-\alpha(x)(\zeta+\eta)} S_{III}(\zeta - \eta) \) |

**Table 3.** Asymptotic of kernels

#### 6.1 Large \( N \) limit of \( W_{I,N+K}(\lambda,\mu) \)

We fix a point \( x \in (-a,a) \). The interval \((-a,a)\) is the support of the equilibrium measure for the given potential function \( V(x) \). For simplicity we assume that the support of the equilibrium measure includes only one interval. This is the case for the potential function \( V(x) = x^{2m} \), \( m \geq 1 \). Introduce new coordinates \( \zeta, \eta \)

\[
\lambda = x + \frac{\zeta}{N\rho(x)}, \quad \mu = x + \frac{\eta}{N\rho(x)}
\]

(6.1)

In what follows we consider the Dyson’s scaling limit. In such a limit the difference between points \( \lambda \) and \( \mu \) goes to zero, the size \( N \) goes to infinity, the product \( N(\lambda - \mu) \) remains finite. In equation (6.1) \( \rho(x) = K_N(x,x)/N \) is the density of states, with the kernel \( K_N(x,x) \) being given by the
equation \((2.7)\). In terms of the new coordinates the kernel \(W_{I,N+K}(\lambda, \mu) \equiv W_{I,N+K}(\omega, \gamma)\) is expressed as a determinant of monic orthogonal polynomials,

\[
W_{I,N+K}(\omega, \gamma) = \left[ N\rho(x) \right] [\gamma - \eta]^{-1}
\]

\[
\times \det \begin{bmatrix}
\pi_N(x + \frac{\omega}{N\rho(x)}) & \pi_N(x + \frac{\gamma}{N\rho(x)}) \\
\pi_{N-1}(x + \frac{\omega}{N\rho(x)}) & \pi_{N-1}(x + \frac{\gamma}{N\rho(x)})
\end{bmatrix}
\]

(6.2)

Thus the problem about asymptotics of the two-point kernel function \(W_{I,N+K}(\lambda, \mu)\) is reduced to the investigation of the large \(N\) asymptotics of the determinant in the equation (6.2).

**Proposition 6.1.** (Large \(N\) asymptotics of the kernel \(W_{I,N+K}(\omega, \gamma)\)).

Let \(x \in (-a, a), -\theta \leq \omega, \gamma \leq \theta, \omega \neq \gamma\) and \(N \gg K\). Then in the large \(N\) limit the kernel function \(W_{I,N+K}(\omega, \gamma)\) is related with the universal kernel \(S_1(\zeta - \eta)\) as

\[
W_{I,N+K}(\omega, \gamma) = [c_N]^{2K} \left[ N\rho(x) \right] e^{N\rho(x)} e^{\rho(x)(\zeta + \gamma)} [S_1(\zeta - \eta) + \mathcal{O}(1/N)]
\]

(6.3)

where the universal kernel \(S_1(\zeta - \eta)\) is

\[
S_1(\zeta - \eta) = \frac{\sin \left( \pi(\zeta - \eta) \right)}{\pi(\zeta - \eta)}
\]

(6.4)

**Proof.** Equation (6.1) enables us to write the kernel \(W_{I,N+K}(\omega, \gamma)\) in the form:

\[
W_{I,N+K}(\omega, \gamma) = \left[ \gamma_{N+K-1} \right]^{-1} \left[ N\rho(x) \right] [\gamma - \eta]^{-1}
\]

\[
\times \det \begin{bmatrix}
Y_{11}^{(N+K)}(x + \frac{\omega}{N\rho(x)}) & Y_{11}^{(N+K)}(x + \frac{\gamma}{N\rho(x)}) \\
Y_{21}^{(N+K)}(x + \frac{\omega}{N\rho(x)}) & Y_{21}^{(N+K)}(x + \frac{\gamma}{N\rho(x)})
\end{bmatrix}
\]

(6.5)

The large \(N\) asymptotics is completely determined by the determinant\(^1\) in the equation above. Large \(N\) limit of this determinant is considered in Deift’s book [2], Chapter 8. Here we reproduce his derivation. Equation (6.14) gives

\[
\det \begin{bmatrix}
Y_{11}^{(N+K)}(x + \frac{\omega}{N\rho(x)}) & Y_{11}^{(N+K)}(x + \frac{\gamma}{N\rho(x)}) \\
Y_{21}^{(N+K)}(x + \frac{\omega}{N\rho(x)}) & Y_{21}^{(N+K)}(x + \frac{\gamma}{N\rho(x)})
\end{bmatrix}
\]

\[
= e^{N[g_+(x+\omega/N\rho(x))+g_+(x+\gamma/N\rho(x))+\ell]}
\]

(6.6)

\[
\times \det \begin{bmatrix}
m_{11}^{(1)}(x + \frac{\omega}{N\rho(x)}) & m_{11}^{(1)}(x + \frac{\gamma}{N\rho(x)}) \\
m_{21}^{(1)}(x + \frac{\omega}{N\rho(x)}) & m_{21}^{(1)}(x + \frac{\gamma}{N\rho(x)})
\end{bmatrix}
\]

where \(m^{(1)}(z)\) is the solution of the transformed Riemann-Hilbert problem (see the previous section). Rewrite the determinant in terms of \(m^{(2)}(z)\) (see Fig. 1 and Fig. 2 where \(m^{(2)}(z)\) is related with \(m^{(1)}(z)\)). This leads to the following expression for the kernel

\[
W_{I,N+K}(\omega, \gamma) = \left[ \gamma_{N+K-1} \right]^{-1} \left[ N\rho(x) \right] [\gamma - \eta]^{-1}
\]

\[
\times e^{N[g_+(x+\omega/N\rho(x))+g_+(x+\gamma/N\rho(x))+\ell]}
\]

(6.7)

\[
\times \det \begin{bmatrix}
m_{11}^{(2)}(x) & \m_{11}^{(2)}(x) + \m_{22}^{(2)}(x) + e^{2N\phi(x)} \m_{12}^{(2)}(x) + \e^{2N\phi(x)} \m_{22}^{(2)}(x) \\
m_{21}^{(2)}(x) & \m_{21}^{(2)}(x) + \m_{22}^{(2)}(x) + e^{2N\phi(x)} \m_{21}^{(2)}(x) + \e^{2N\phi(x)} \m_{22}^{(2)}(x)
\end{bmatrix}
\]

where \(x = x + \frac{\omega}{N\rho(x)}\), \(x = x + \frac{\gamma}{N\rho(x)}\). Remember that \(m^{(2)}(z)\) and its derivative are bounded matrix valued functions in the vicinity of the point \(x\) and \(\det m^{(2)}(z) = 1\). This observation

---

\(^1\)Since \(N \gg K\), we do not distinguish between \(N\) and \(N + K\) when studying the asymptotics.
enables one to expand all functions in the determinant in the vicinity of the point \( x \), and to find the leading term:

\[
W_{I,N+K}(x,\zeta,\eta) = [\gamma_{N+K-1}]^{-1} [N\rho(x)] [\zeta - \eta]^{-1}
\]

\[
\times e^{N[g_+(x+\zeta/N\rho(x))+g_+(x+\eta/N\rho(x))+I]}
\]

\[
\times \left[ e^{2N\varphi_+(x+\eta/N\rho(x))} - e^{2N\varphi_+(x+\zeta/N\rho(x))} \right] [1 + \mathcal{O}(N^{-1})]
\]

The next step is to express the functions \( g_+(x+\zeta/N\rho(x)) \), \( g_+(x+\eta/N\rho(x)) \) in terms of the functions \( \varphi_+(x_\zeta) \), \( \varphi_+(x_\eta) \). It can be done as follows. Let \( z \in (-a,a) \). When \( z \in (-a,a) \) the following equations hold (see properties of the function \( g_\pm(z) \) summarized in the previous section):

\[
g_+(z) + g_-(z) = 2\int \log |z - s| \psi(s) ds
\]

\[
g_+(z) - g_-(z) = 2i\pi \int_a^a \psi(s) ds
\]

We use the Euler-Lagrange variational condition \((5.7)\) together with the equations above and find that the functions \( g_\pm(z) \) are completely determined by the function \( \varphi_+(z) \), by the potential \( V(z) \) and the constant \( I \). Namely,

\[
\begin{cases}
g_+(z) = -\varphi_+(z) + V(z) - \frac{I}{2} \\
g_-(z) = \varphi_+(z) + V(z) - \frac{I}{2}
\end{cases}
\]

This gives

\[
W_{I,N+K}(x,\zeta,\eta) = [\gamma_{N+K-1}]^{-1} [N\rho(x)] [\zeta - \eta]^{-1} e^{NV(x)} e^{\alpha(x)(\zeta + \eta)}
\]

\[
\times \left[ e^{N[\varphi_+(x+\zeta/N\rho(x)) - \varphi_+(x+\eta/N\rho(x))] - e^{-N[\varphi_+(x+\eta/N\rho(x)) - \varphi_+(x+\zeta/N\rho(x))]} \right]
\]

In the vicinity of the point \( x \) we have

\[
\varphi_+(x + \frac{\zeta}{N\rho(x)}) = \varphi_+(x) + i\pi\psi(x) \frac{\zeta}{N\rho(x)} + \mathcal{O}\left(\frac{1}{N^2}\right)
\]

A similar expression is obtained for \( \varphi_+(x + \frac{\eta}{N\rho(x)}) \). We complete the proof using the equation \((5.16)\) and the relation \( K_N(x,x) = N\rho(x) \).

\[\square\]

### 6.2 Large \( N \) limit of \( W_{II,N}(\epsilon,\mu) \)

In this subsection we investigate Dyson’s scaling limit of the kernel \( W_{II,N}(\epsilon,\mu) \) constructed from monic orthogonal polynomials and their Cauchy transforms. Similar to the procedure applied to the kernel function \( W_{I,N+K}(\epsilon,\mu) \) we introduce new coordinates,

\[
\epsilon = x + \frac{\zeta}{N\rho(x)}, \quad \mu = x + \frac{\eta}{N\rho(x)}
\]

In the coordinates \( \zeta, \eta \) the kernel \( W_{II,N}(\epsilon,\mu) \equiv W_{II,N}(x,\zeta,\eta) \) has the form:

\[
W_{II,N}(x,\zeta,\eta) = [N\rho(x)] [\eta - \zeta]^{-1}
\]

\[
\times \det \begin{vmatrix}
\pi_N(x + \frac{\eta}{N\rho(x)}) & h_N(x + \frac{\zeta}{N\rho(x)}) \\
\pi_{N-1}(x + \frac{\eta}{N\rho(x)}) & h_{N-1}(x + \frac{\zeta}{N\rho(x)})
\end{vmatrix}
\]

We can see that in order to derive an asymptotic expression for the kernel \( W_{II,N}(x,\zeta,\eta) \) we need the asymptotics of the determinant which contains monic orthogonal polynomials and their Cauchy transform. The Riemann-Hilbert technique proves to be a convenient tool in this case as well.
Proposition 6.2. (Large $N$ asymptotics for the kernel $W_{11,N}(x, \zeta, \eta)$)

Let $x \in (-a, a)$, $-\theta \leq \zeta, \eta \leq \theta$, $\zeta \neq \eta$, $\Im \zeta \neq 0$. Then the following asymptotic expression for the kernel $W_{11,N}(x, \zeta, \eta)$ holds:

$$W_{11,N}(x, \zeta, \eta) = [\gamma_N]^{-1} [N \rho(x)] e^{-\alpha(x)(\zeta - \eta)} [S_{11}(\zeta - \eta) + O(1/N)]$$  \hspace{1cm} (6.16)

The universal two-point kernel $S_{11}(\zeta - \eta)$ is expressed as

$$S_{11}(\zeta - \eta) = \left\{ \begin{array}{ll}
\frac{e^{i\pi(\zeta - \eta)}}{\zeta - \eta} & \Im \zeta > 0, \\
\frac{\zeta - \eta}{\zeta - \eta} & \Im \zeta < 0
\end{array} \right.$$  \hspace{1cm} (6.17)

Proof. Express the determinant in the equation (6.15) in terms of $Y^{(N)}(z)$ which is the matrix valued solution of the Riemann Hilbert problem for the orthogonal polynomials. We have

$$W_{11,N}(x, \zeta, \eta) = [N \rho(x)] [\eta - \zeta]^{-1} [\gamma_{N-1}]^{-1}$$

$$\times \det \begin{bmatrix}
Y^{(N)}_{11}(x + \frac{\eta}{N \rho(x)}) & Y^{(N)}_{12}(x + \frac{\zeta}{N \rho(x)}) \\
Y^{(N)}_{21}(x + \frac{\eta}{N \rho(x)}) & Y^{(N)}_{22}(x + \frac{\zeta}{N \rho(x)})
\end{bmatrix}$$  \hspace{1cm} (6.18)

Equation (6.14) enables one to replace the elements of $Y^{(N)}$ by the elements of $m^{(1)}(z)$ (we remind that $m^{(1)}(z)$ is the solution of the transformed Riemann-Hilbert problem). Then we obtain

$$W_{11,N}(x, \zeta, \eta) = [N \rho(x)] [\eta - \zeta]^{-1} [\gamma_{N-1}]^{-1} e^{N[g(x+\eta/N \rho(x)) - g(x+\zeta/N \rho(x))]}$$

$$\times \det \begin{bmatrix}
m^{(1)}_{11}(x + \frac{\eta}{N \rho(x)}) & m^{(1)}_{12}(x + \frac{\zeta}{N \rho(x)}) \\
m^{(1)}_{21}(x + \frac{\eta}{N \rho(x)}) & m^{(1)}_{22}(x + \frac{\zeta}{N \rho(x)})
\end{bmatrix}$$  \hspace{1cm} (6.19)

The elements of $m^{(1)}(z)$ can be further replaced by the elements of $m^{(2)}(z)$ (see Fig. 1 and Fig. 2 where the Riemann-Hilbert problem for $m^{(2)}(z)$ is specified and the relation between $m^{(1)}(z)$ and $m^{(2)}(z)$ is shown). Again, $m^{(2)}(z)$ and its derivative are bounded near the point $x$, and det $m^{(2)}(z) = 1$. Thus we can expand around the point $x$ and show that the determinant in equation (6.19) is equal to one to the leading order. Therefore,

$$W_{11,N}(x, \zeta, \eta) = [N \rho(x)] [\eta - \zeta]^{-1} [\gamma_{N-1}]^{-1} e^{N[g(x+\eta/N \rho(x)) - g(x+\zeta/N \rho(x))]} [1 + O(1/N)]$$

Introduce $z_1 = x + \eta/N \rho(x)$, $z_2 = x + \zeta/N \rho(x)$. Then we have

$$N \left[ g(z_1) - g(z_2) \right] = N \left[ \int \log(z_1 - s) \psi(s) ds - \int \log(z_2 - s) \psi(s) ds \right]$$

$$= N \int \log \left[ \frac{z_1 - s}{z_2 - s} \right] \psi(s) ds$$

$$= N \int \log \left[ 1 + \frac{z_1 - z_2}{z_2 - s} \right] \psi(s) ds$$  \hspace{1cm} (6.20)

$$= \frac{\eta - \zeta}{\rho(x)} \lim_{\delta \to 0} \left[ \int_{-\infty}^{+\infty} \psi(s) ds \int_{x - s \pm i\delta} \right] + O(1/N)$$

$$= \frac{\eta - \zeta}{\rho(x)} \left[ \pi H \psi(x) \mp i\pi \psi(x) \right]$$

where $+(-)$ corresponds to the positive (negative) imaginary part of $\zeta$, and $H \psi(x)$ stands for the Hilbert transform of $\psi(x)$,

$$H \psi(x) = \frac{1}{\pi} P.V. \int \frac{\psi(s) ds}{x - s}$$  \hspace{1cm} (6.21)
Furthermore, from equation (5.7) (the second Euler-Lagrange condition) we observe that
\[ H \psi(x) = \frac{1}{2\pi} V'(x) \]  
(6.22)
Now use that \( \psi(x) \) is equal to the density of states in the large \( N \) limit to obtain
\[ N \left[ g(z_1) - g(z_2) \right] = \pm i\pi (\zeta - \eta) - \frac{V'(x)}{2\rho(x)} (\zeta - \eta) + \mathcal{O}\left(\frac{1}{N}\right) \]  
(6.23)
which completes the proof.  \( \square \)
6.3 Large \( N \) limit of \( W_{III,N-K}(\epsilon, \omega) \)

Finally we investigate the Dyson’s scaling limit for the kernel \( W_{III,N-K}(\epsilon, \omega) \). This kernel is constructed from the Cauchy transforms of monic orthogonal polynomials, and the transforms are not analytic. For this reason we need to consider different situations corresponding to different signs of the imaginary parts. Just as before we introduce new coordinates

\[
\epsilon = x + \zeta/N\rho(x), \quad \omega = x + \eta/N\rho(x)
\]

where \( \Im \zeta \neq 0, \Im \eta \neq 0 \). In terms of these coordinates we define \( W_{III,N-K}(x,\zeta,\eta) \equiv W_{III,N-K}(x + \frac{\zeta}{N\rho(x)}, x + \frac{\eta}{N\rho(x)}) \).

**Proposition 6.3.** (Large \( N \) asymptotics of the kernel \( W_{III,N-K}(x,\zeta,\eta) \)). Let \( x \in (-a,a), -\theta \leq \zeta, \eta \leq \theta, \Im \zeta \neq 0, \Im \eta \neq 0 \). Then the large \( N \) limit of the kernel function \( W_{III,N-K}(x,\zeta,\eta) \) is related to a universal kernel \( S_{III}(\zeta,\eta) \) as

\[
W_{III,N-K}(x,\zeta,\eta) = [\gamma_N]^{-1} [N\rho(x)] e^{-NV(x)} e^{-\alpha(x)(\zeta+\eta)} \left[ S_{III}(\zeta,\eta) + \mathcal{O}(N^{-1}) \right]
\]

where the universal kernel \( S_{III}(\zeta,\eta) \) is given by the formula

\[
S_{III}(\zeta,\eta) = \begin{cases} 
\frac{\zeta}{\eta} & \text{if } \Im \zeta > 0, \Im \eta < 0, \\
\frac{\eta}{\zeta} & \text{if } \Im \zeta < 0, \Im \eta > 0, \\
0 & \text{otherwise}
\end{cases}
\]

**Proof.** We give the proof only for the case \( \Im \zeta > 0, \Im \eta < 0 \). The other three cases with different signs of imaginary parts can be considered in a similar manner. The kernel \( W_{III,N-K}(x,\zeta,\eta) \) can be expressed as

\[
W_{III,N-K}(x,\zeta,\eta) = [N\rho(x)] [\zeta - \eta]^{-1} \times \det \begin{bmatrix} h_{N-K}(x + \zeta/N\rho(x)) & h_{N-K}(x + \eta/N\rho(x)) \\ h_{N-K-1}(x + \zeta/N\rho(x)) & h_{N-K-1}(x + \eta/N\rho(x)) \end{bmatrix}
\]

Now we exploit the relation to the Riemann-Hilbert problem for the orthogonal polynomials. We replace the Cauchy transforms in the determinant above by the corresponding elements of the matrix \( Y^{(N-K)} \) which is the solution of the Riemann-Hilbert problem. We have

\[
W_{III,N-K}(x,\zeta,\eta) = [N\rho(x)] [\zeta - \eta]^{-1} [\gamma_{N-K}]^{-1} \times \det \begin{bmatrix} Y_{12}^{(N-K)}(x + \zeta/N\rho(x)) & Y_{12}^{(N-K)}(x + \eta/N\rho(x)) \\ Y_{22}^{(N-K)}(x + \zeta/N\rho(x)) & Y_{22}^{(N-K)}(x + \eta/N\rho(x)) \end{bmatrix}
\]

Then we employ the transformation (equation (5.14)) from the solution \( Y^{(N-K)}(z) \) of the original Riemann-Hilbert problem to that of the new Riemann-Hilbert problem \( m^{(1)}(z) \) defined by the jump matrix \( v^{(1)}(z) \) (\( N \) should be replaced by \( N - K \)). This yields

\[
W_{III,N-K}(x,\zeta,\eta) = [N\rho(x)] [\zeta - \eta]^{-1} [\gamma_{N-K}]^{-1} \times \exp \left[-(N-K) \left[l + g_+(x + \zeta/N\rho(x)) + g_+(x + \eta/N\rho(x))\right]\right] \times \det \begin{bmatrix} m_{12}^{(1)}(x + \zeta/N\rho(x)) & m_{12}^{(1)}(x + \eta/N\rho(x)) \\ m_{22}^{(1)}(x + \zeta/N\rho(x)) & m_{22}^{(1)}(x + \eta/N\rho(x)) \end{bmatrix}
\]
In turn, the function \( m^{(1)}(z) \) is related to the matrix valued function \( m^{(2)}(z) \) (see Fig. 2) which is the solution of the deformed Riemann-Hilbert problem defined by Fig. 2. Correspondingly, we rewrite the kernel in terms of the elements of \( m^{(2)}(z) \),

\[
W_{III,N-K}(x, \zeta, \eta) = [N\rho(x)] [\zeta - \eta]^{-1} [\gamma_{N-K}]^{-1} \times \exp \left[ -(N - K) \left[ l + g_+(x + \zeta/N\rho(x)) + g_+(x + \eta/N\rho(x)) \right] \right]
\]

\[
\times \det \left| \begin{array}{cc}
m_{11}(x) & m_{12}(x) \\
m_{21}(x) & m_{22}(x)
\end{array} \right| + \det \left| \begin{array}{cc}
m_{11}(x) & m_{12}(x) \\
m_{21}(x) & m_{22}(x)
\end{array} \right| - + \mathcal{O}(1/N)
\]

As \( m^{(2)}(z) \) and \( dm^{(2)}(z)/dz \) are bounded matrix valued functions in the vicinity of the point \( x \), and \( g_+(z) + g_+(z) + l = V(z) \) for \( z \in (-a, a) \) (see equation (7.11)) we obtain the following large \( N \) limit for the kernel \( W_{III,N-K}(x, \zeta, \eta) \)

\[
W_{III,N-K}(x, \zeta, \eta) = [N\rho(x)] [\zeta - \eta]^{-1} [\gamma_{N-K}]^{-1} e^{-NV(x)e^{-a(x)(\zeta+\eta)} [1 + \mathcal{O}(1/N)]}
\]

However, in the vicinity of the point \( x \) we have (see Fig. 2)

\[
\left( \begin{array}{cc}
m_{121}(x) & m_{12}(x) \\
m_{21}(x) & m_{22}(x)
\end{array} \right) + \left( \begin{array}{cc}
m_{12}(x) & m_{12}(x) \\
m_{21}(x) & m_{22}(x)
\end{array} \right) = \left( \begin{array}{cc}
m_{11}(x) & m_{12}(x) \\
m_{21}(x) & m_{22}(x)
\end{array} \right) - \left( \begin{array}{cc}
m_{12}(x) & m_{12}(x) \\
m_{21}(x) & m_{22}(x)
\end{array} \right) \left( \begin{array}{cc}
0 & 1 \\
-1 & 0
\end{array} \right)
\]

i.e.

\[
\left[ m^{(2)}(x) \right]_- = - \left[ m^{(2)}(x) \right]_+, \quad \left[ m^{(2)}(x) \right]_- = - \left[ m^{(2)}(x) \right]_+
\]

Taking into account that

\[
\det \left| \begin{array}{cc}
m_{11}(x) & m_{12}(x) \\
m_{21}(x) & m_{22}(x)
\end{array} \right| = 1
\]

we finally obtain

\[
W_{III,N-K}(x, \zeta, \eta) = [N\rho(x)] [\zeta - \eta]^{-1} [\gamma_{N-K}]^{-1} e^{-NV(x)e^{-a(x)(\zeta+\eta)} [1 + \mathcal{O}(1/N)]}
\]

\[ \square \]

7. Negative Moments

In this section we derive the asymptotic expression (7.1) for the negative moments of characteristic polynomials \( \mathcal{M}_{e,N}^{K}(\delta) \) defined by the equation (2.23). The negative moments can be obtain as limiting values of the correlation function \( \mathcal{F}_{II1}^{K}(x + \zeta/N\rho(x)) \). The large \( N \) asymptotic for that function is given by the equation (2.13). We define three \( K \) dimensional vectors, \( \hat{\zeta}^+, \hat{\zeta}^− \), and \( \delta \). The components of the vectors \( \hat{\zeta}^+ (\hat{\zeta}^-) \) are pure imaginary and have positive (negative) imaginary parts. The vector \( \delta \) has all components equal to each other and equal to the real parameter \( \delta \) in the definition of the negative moments (equation (2.23)). We exploit our asymptotic formula (2.15) for the correlation function \( \mathcal{F}_{II1}^{K}(x + \zeta/N\rho(x)) \) and write the following expression for the negative moments:

\[
\mathcal{M}^{K}_{e,N}(\delta) = (-)^K [\gamma_N]^K [N\rho(x)]^{K^2} e^{-KNV(x)} \mathcal{F}_{K}(\delta) \quad (2K)!.
\]
where

\[ J_K(\delta) = \lim_{\xi \to \pm i\delta/2} \left[ \sum_{\pi \in S_{2K}} \det \left[ S_{III} \left( \zeta_{\pi(i)}, \zeta_{\pi(j+K)} \right) \right]_{1 \leq i, j \leq K} \right] \]

Following the method by Brezin and Hikami’s ([1] [2]) we represent the determinant of the kernel divided by two Vandermonde determinants as a contour integral, i.e.

\[ \sum_{\pi \in S_{2K}} \frac{\det \left[ S_{III} \left( \zeta_{\pi(i)}, \zeta_{\pi(j+K)} \right) \right]_{1 \leq i, j \leq K}}{\Delta(\zeta_{\pi(1)}, \ldots, \zeta_{\pi(K)}) \Delta(\zeta_{\pi(K+1)}, \ldots, \zeta_{\pi(2K)})} = \frac{1}{K!} \sum_{\pi \in S_{2K}} \oint \oint \prod_{i=1}^{K} \frac{du_i dv_i}{(2\pi i)^2} \frac{\Delta(\bar{u}) \Delta(\bar{v})}{\prod_{i,j=1}^{K} \left[ u_i - \zeta_{\sigma(j)} \right] \left[ v_i - \zeta_{\pi_K(j+1)} \right]} \]

where the contours of integration are chosen in such a way that all components of the vector \( \hat{\zeta} \) give rise to contributions to the integral as simple poles. We have to compute the integral above when

\[ \hat{\zeta} = (i\delta/2, \ldots, i\delta/2, -i\delta/2, \ldots, -i\delta/2) \equiv (\hat{\zeta}^+, \hat{\zeta}^-) \]

All those permutations \( \pi \in S_{2K} \) that lead to the same vector \( \pi \hat{\zeta} \) produce the same contribution to the integral. This observation permits us to rewrite the right-hand side of the equation (7.2) as

\[ K! \sum_{\pi \in S_{2K}} \oint \oint \prod_{i=1}^{K} \frac{du_i dv_i}{(2\pi i)^2} \frac{\Delta(\bar{u}) \Delta(\bar{v})}{\prod_{i,j=1}^{K} \left[ u_i - \zeta_{\sigma(j)} \right] \left[ v_i - \zeta_{\pi_K(j+1)} \right]} \]

where \( \sigma \in S_{2K}/S_K \times S_K \) should be understood as permutations exchanging elements between the two sets, \((i\delta/2, \ldots, i\delta/2)\) and \((-i\delta/2, \ldots, -i\delta/2)\) (the dimension of each set is \( K \)). Consider all such permutations that replace \( K_1 \leq K \) elements of the first set, \((i\delta/2, \ldots, i\delta/2)\), by \( K_1 \) elements of the second set, \((-i\delta/2, \ldots, -i\delta/2)\). The number of such permutations is equal to \( \frac{K!}{(K-K_1)!K_1!} \). A precise location of new elements in the first and the second sets does not affect the integral, so all \( \left[ \frac{K!}{(K-K_1)!K_1!} \right]^2 \) give the same contribution. Therefore,

\[ J_K(\delta) = K! \sum_{K_1=0}^{K} \left[ \frac{K!}{(K-K_1)!K_1!} \right]^2 I^K_{\delta, K_1} \]

where

\[ I^K_{\delta, K_1} = \oint \oint \prod_{i=1}^{K} \frac{du_i dv_i}{(2\pi i)^2} \frac{\Delta(\bar{u}) \Delta(\bar{v})}{\prod_{i,j=1}^{K_1} \left[ u_i + i\frac{\delta}{2} (v_i - i\frac{\delta}{2}) \right] \left[ (u_i - i\frac{\delta}{2})(v_i + i\frac{\delta}{2}) \right]^{K-K_1}} \]

Now we rewrite the contour integral \( I^K_{\delta, K_1} \) as a determinant of a kernel divided by two Vandermonde determinants. It follows from explicit expressions for limiting kernels summarized in table 2 that \( S_{III}(\alpha, \beta) = 0 \) to the leading order, if the variables \( \alpha \) and \( \beta \) have imaginary parts of different signs. Then we obtain

\[ I^K_{\delta, K_1} = \lim_{\delta \to \pm i\delta/2} \det \begin{array}{cccccc} S_{III}(i\frac{\delta}{2}, i\frac{\delta}{2}) & \cdots & S_{III}(-i\frac{\delta}{2}, -i\frac{\delta}{2}) & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \ddots & \vdots \\ S_{III}(i\frac{\delta}{2}, -i\frac{\delta}{2}) & \cdots & S_{III}(-i\frac{\delta}{2}, i\frac{\delta}{2}) & 0 & \cdots & 0 \\ 0 & \cdots & 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & S_{III}(i\frac{\delta}{2}, -i\frac{\delta}{2}) & \cdots & S_{III}(i\frac{\delta}{2}, -i\frac{\delta}{2}) \end{array} \]

\[ \Delta \left( -i\frac{\delta}{2}, \ldots, -i\frac{\delta}{2}, i\frac{\delta}{2}, \ldots, i\frac{\delta}{2} \right) \Delta \left( i\frac{\delta}{2}, \ldots, i\frac{\delta}{2}, -i\frac{\delta}{2}, \ldots, -i\frac{\delta}{2} \right) \]
We insert the explicit expression for the kernel $S_{III}(\zeta - \eta)$ (see Table 2) to the above formula and find that

$$I_{\delta}^{K,K_1} = \frac{(-i)^K K!}{(\delta)^{2K_1(K-K_1)}} \times \lim_{\delta \to 0} \left[ \frac{\det\left(\frac{2}{\delta_i + \delta_j + \delta_j} \right)}{\Delta^2} \right]_{1 \leq i,j \leq K_1} \times \lim_{\delta \to 0} \left[ \frac{\det\left(\frac{2}{\delta_{K_1+1} \cdots \delta_K} \right)}{\Delta^2} \right]_{1 \leq i,j \leq K} \quad (7.5)$$

Exploiting once again Brezin and Hikami’s representation in terms of a contour integral we find

$$\lim_{\delta \to 0} \left[ \frac{\det\left(\frac{2}{\delta_i + \delta_j} \right)}{\Delta^2} \right]_{1 \leq i,j \leq M} = \det (a_{mn})_{0 \leq m,n \leq M-1} \quad (7.6)$$

where

$$a_{mn} = \frac{1}{n! m!} \frac{\partial^n}{\partial u^n} \frac{\partial^m}{\partial v^m} \left[ \frac{1}{u + v + \delta} \right]_{u=v=0} = \frac{(-)^{m+n}}{\delta^{m+n}} \left( \frac{(m+n)!}{m! n!} \right)$$

Now it is straightforward to compute $\det (a_{mn})$. We obtain

$$\lim_{\delta \to 0} \left[ \frac{\det\left(\frac{2}{\delta_i + \delta_j} \right)}{\Delta^2} \right]_{1 \leq i,j \leq M} = \frac{1}{\delta^{M^2}}$$

This equation together with the equation (7.5) yields

$$I_{\delta}^{K,K_1} = (-i)^K K! \delta^{-K^2} \quad (7.7)$$

(i.e. $I_{\delta}^{K,K_1}$ does not depend on $K_1$). Taking into account that

$$\sum_{K_1=0}^{K} \left[ \frac{K!}{(K-K_1)!K_1!} \right] = \frac{(2K)!}{(K!)^2} \quad (7.8)$$

and equations (7.1), (7.4) we finally obtain our result (2.24) for the negative moments of the characteristic polynomials.

### 8. Two-point Correlation Function

The present section serves for illustrating the utility of the correlation functions constructed with help of the characteristic polynomials. We are going to demonstrate how one can derive the resolvent two-point correlation function from our asymptotic result for the average values of ratios of the characteristic polynomials. We consider the case when $\alpha(x) \equiv \psi''(x) = 0$. Then the correlation function $F_{II}^{K}(\hat{x} + \hat{\eta}/N\rho(x), \hat{x} + \hat{\eta}/N\rho(x))$ is universal in the Dyson scaling limit. As the result, the answer for the resolvent correlation function will be universal as well. As a by-product we provide a new proof of the universality of the two-point correlation function of eigenvalue densities.

The resolvent two-point correlation function is defined as

$$S_{2}(E, E') = N^{-2} \left\langle \frac{1}{E-H} \left[ \frac{1}{E' - H} \right] \right\rangle_{H}$$

where the (complex) energy $E$ has a positive imaginary part and the energy $E'$ has a negative imaginary part. To consider the scaling limit we introduce new coordinates

$$E = x + \eta_i / N \rho(x), \quad \Im \eta_i > 0$$

and equations (7.1), (7.4) we finally obtain our result (2.24) for the negative moments of the characteristic polynomials.
Equation (2.14) yields

\[
E' = x + \eta_2/N \rho(x), \quad \exists \eta_2 < 0
\]  

To connect the two-point correlation function \(S_2(x + \eta_1/N \rho(x), x + \eta_2/N \rho(x))\) with the correlation function \(F_{II}^{K=2}(\hat{x} + \hat{\zeta}/N \rho(x), \hat{x} + \hat{\eta}/N \rho(x))\) investigated above we exploit the following identity

\[
\text{Tr} \left[ \frac{1}{x + \eta/N \rho(x) - H} \right] = [N \rho(x)] \frac{\partial \mathcal{Z}_N[x_\eta, H]}{\mathcal{Z}_N[x_\eta, H]} \quad \text{(8.4)}
\]

where \(x_\eta \equiv x + \eta/N \rho(x)\). We have

\[
S_2(x + \eta_1/N \rho(x), x + \eta_2/N \rho(x))
\]

\[
= [\rho(x)]^2 \left[ \frac{\partial^2_{\eta_1, \eta_2}}{\partial \eta_1 \partial \eta_2} \left( \frac{\det(x_{\eta_1} - H) \det(x_{\eta_2} - H)}{\det(x_{\zeta_1} - H) \det(x_{\zeta_2} - H)} \right) \right]_{\zeta_1 = \eta_1, \zeta_2 = \eta_2}
\]

\[
= [\rho(x)]^2 \left[ \frac{\partial^2_{\eta_1, \eta_2} F_{II}^{K=2}(\hat{x} + \hat{\zeta}/N \rho(x), \hat{x} + \hat{\eta}/N \rho(x))}{\partial \eta_1 \partial \eta_2} \right]_{\zeta_1 = \eta_1, \zeta_2 = \eta_2}
\]

Equation (2.14) yields

\[
F_{II}^{K=2}(\hat{x} + \hat{\zeta}/N \rho(x), \hat{x} + \hat{\eta}/N \rho(x)) = \frac{e^{i\pi(\zeta_2 - \zeta_1)}}{\zeta_1 - \zeta_2}
\]

\[
\times \left[ e^{i\pi(\eta_1 - \eta_2)} \frac{(\eta_1 - \zeta_1)(\eta_2 - \zeta_2)}{\eta_1 - \eta_2} - e^{-i\pi(\eta_1 - \eta_2)} \frac{(\eta_1 - \zeta_2)(\eta_2 - \zeta_1)}{\eta_1 - \eta_2} \right]
\]

The next step is to compute derivatives. In particular, we find

\[
\frac{\partial^2_{\eta_1, \eta_2}}{\partial \eta_1 \partial \eta_2} \left( e^{i\pi(\eta_1 - \eta_2)} \frac{(\eta_1 - \zeta_1)(\eta_2 - \zeta_2)}{\eta_1 - \eta_2} \right)_{\zeta_1 = \eta_1, \zeta_2 = \eta_2} = \frac{e^{i\pi(\eta_1 - \eta_2)}}{\eta_1 - \eta_2}
\]  

\[
(8.5)
\]

and

\[
\frac{\partial^2_{\eta_1, \eta_2}}{\partial \eta_1 \partial \eta_2} \left( e^{-i\pi(\eta_1 - \eta_2)} \frac{(\eta_1 - \zeta_2)(\eta_2 - \zeta_1)}{\eta_1 - \eta_2} \right)_{\zeta_1 = \eta_1, \zeta_2 = \eta_2} = -\pi^2 (\eta_1 - \eta_2) e^{-i\pi(\eta_1 - \eta_2)} + \frac{e^{-i\pi(\eta_1 - \eta_2)}}{\eta_1 - \eta_2}
\]  

\[
(8.6)
\]

which gives the well-known result (see [43][8]) for the resolvent two-point correlation function:

\[
S_2(x + \eta_1/N \rho(x), x + \eta_2/N \rho(x))
\]

\[
= [\rho(x)]^2 \left[ 1 - 2i \frac{\sin \pi(\eta_2 - \eta_1) e^{-i\pi(\eta_2 - \eta_1)}}{[\pi(\eta_2 - \eta_1)]^2} \right]
\]  

\[
(8.7)
\]

9. Summary and Discussions

In this paper we prove three basic statements: 1) correlation functions of characteristic polynomials are governed by two-point kernels, 2) the kernels are "integrable" in the sense of the definition by Its, Izergin, Korepin and Slavnov [38][8][7], 3) the kernels are constructed from monic orthogonal polynomials and their Cauchy transforms. As a consequence, it becomes quite natural to exploit a relation to the Riemann-Hilbert problem for orthogonal polynomials proposed by Forrester, Its and Kitaev [24][22].

It is known that the simplest correlation functions, i.e. the moments of the characteristic polynomials, can be described in terms of non-linear differential equations (see works of Forrester and White [23][24], and also the paper by Kanzieper [11], and by Bittorf and Verbaarschot [18]). As for more complicated correlation functions a description in terms of differential equations is
unknown. While in the present paper we focus on asymptotic questions, the "integrability" of kernels suggests that such description should be possible.

In this paper the discussion is restricted to the ensemble of unitary invariant Hermitian matrices. However, the case of compact group ensembles (circular ensembles) can be approached in the same way. The circular ensembles are even simpler as the representation theory of compact groups (inapplicable for Hermitian random matrices) can be exploited there. Indeed, the method of "dual pairs" (Nonnenmacher and Zirnbauer [16, 55, 56]) enables one to find exact formulas for correlation functions of characteristic polynomials for the circular ensembles. However, this method is based on an interpretation of characteristic polynomials as characters of "spinor" group representations. As a result, its applicability is restricted to group ensembles. The method proposed in this paper is based neither on the representation theory, nor on specific features of Hermitian matrices. The only fact which is important is that the ensemble under considerations is of $\beta = 2$ symmetry class. For this reason it is more general and can be applied both to Hermitian random matrices and to the group ensembles.

Correlation functions of characteristic polynomials of random matrices for the unitary circular ensemble and the ensemble of Hermitian matrices are sometimes represented as Toeplitz / Hankel determinants, correspondingly. Results for Toeplitz determinants with rational generating functions can be found in Basor and Forrester [3]. However, Hankel determinants are less well studied. The correlation functions investigated in the present paper can be understood as Hankel determinants with rational generating functions. Thus, asymptotic and exact statements for Hankel determinants equivalent to our results should be possible to make.

We hope that the method proposed in this paper can be modified as to provide an access to other symmetry classes of invariant ensembles $\beta = 1, 4$ (for the gaussian case some attempts were undertaken recently in [8]). Another important goal is to apply them to ensembles of non-Hermitian random matrices [26, 28], which are certain deformations of the invariant class.

The most challenging problem is to investigate the conditions of universality of the discussed correlation functions for non-invariant non-gaussian ensembles with independent, identically distributed (i.i.d.) entries. So far the progress was rather limited and restricted to the specific choice of the probability measure (see Johansson [40]). At the same time, non-rigorous heuristic methods hint to a kind of universality covering also the so-called ensembles of sparse random matrices, see Mirlin, Fyodorov [15] and Fyodorov, Sommers [27].

We leave a detailed investigation of these issues for future research.

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APPENDIX A.

Let us consider the sum

$$ S = \sum_{\sigma, \pi \in S_{n+m}} (-)^{\nu_{\sigma+\pi}} [f_{\sigma_{(1)}}(\varphi_1) \cdots f_{\sigma_{(m)}}(\varphi_m)] $$

$$ \times g_{\pi_{(1)}}(\psi_1) \cdots g_{\pi_{(m)}}(\psi_m) $$

$$ \times \delta_{\sigma_{(m+1)} \pi_{(m+1)}} \cdots \delta_{\sigma_{(m+n)} \pi_{(m+n)}} \quad (A.1) $$

In order to reduce this expression to a determinant form (to a determinant of size \(m \times m\)) we proceed as follows. We fix the set of numbers \([k_1, \ldots, k_m]\) satisfying the following condition
\[
n + m \geq k_1 > k_2 > \ldots > k_m \geq 1
\] (A.2)

Let us denote by \(\tilde{\sigma}(1), \ldots, \tilde{\sigma}(m)\) (and by \(\tilde{\pi}(1), \ldots, \tilde{\pi}(m)\)) the permutations under which the numbers \(1, 2, \ldots, m\) end up inside the set \([k_1, k_2, \ldots, k_m]\). Once such notations are introduced, the sum \(S\) (equation (A.1)) can be rewritten as
\[
S = \sum_{k_1 > \cdots > k_m \geq 1} \sum_{\tilde{\sigma}, \tilde{\pi} \in S_{n+m}} (-)^{\nu_1 + \cdots + \nu_m} [f_{\tilde{\pi}(1)}(\varphi_1) \cdots f_{\tilde{\pi}(m)}(\varphi_m)]
\times g_{\tilde{\pi}(1)}(\psi_1) \cdots g_{\tilde{\pi}(m)}(\psi_m)
\times \delta_{\tilde{\sigma}(m+1)\tilde{\pi}(m+1)} \cdots \delta_{\tilde{\sigma}(m+n)\tilde{\pi}(m+n)}
\] (A.3)

Under permutations \(\tilde{\sigma}, \tilde{\pi}\) the numbers \(m+1, \ldots, m+n\) remain outside of the set \([k_1, \ldots, k_m]\). The sets \([1, \ldots, m], [m+1, \ldots, m+n]\) do not mix, and this leads to the following expression for \(S\):
\[
S = \sum_{k_1 > \cdots > k_m \geq 1} \left[ \sum_{\tilde{\sigma} \in S_m} (-)^{\nu_1} f_{\tilde{\sigma}(1)}(\varphi_1) \cdots f_{\tilde{\sigma}(m)}(\varphi_m) \right]
\times \left[ \sum_{\tilde{\pi} \in S_m} (-)^{\nu_2} g_{\tilde{\pi}(1)}(\psi_1) \cdots g_{\tilde{\pi}(m)}(\psi_m) \right]
\times \delta_{\tilde{\sigma}(m+1)\tilde{\pi}(m+1)} \cdots \delta_{\tilde{\sigma}(m+n)\tilde{\pi}(m+n)}
\] (A.4)

From the above representation we immediately conclude that
\[
S = n! \sum_{k_1 > \cdots > k_m \geq 1} \det [f_{k_i}(\varphi_j)] \det [g_{k_i}(\psi_j)]
\] (A.5)

where the indices \(i, j\) take values from 1 to \(m\). Now the sum \(S\) can be rewritten as a determinant \(m \times m\), as a consequence of the formula
\[
\sum_{k_1 > \cdots > k_m \geq 1} \det [f_{k_i}(\varphi_j)] \det [g_{k_i}(\psi_j)] = \det \left[ \sum_{\lambda=0}^{n+m-1} f_\lambda \varphi_1 g_\lambda \psi_1 \right]_{1 \leq i, j \leq m}
\] (A.6)

(This fact is a generalization of the theorem on determinants. This theorem (which can be found in the book of Hua [34]) states that
\[
\det \left[ \sum_{\lambda=0}^{n+m-1} A_{\lambda_i} t_1^\lambda \right]_{1 \leq i, j \leq m} = \sum_{\lambda_1 > \cdots > \lambda_m \geq 0} \det \left[ t_i^\lambda \right]_{1 \leq i, j \leq m}
\] (A.7)

A nice proof of the formula (A.6) can be found, for example, in the paper by Balantekin and Cassak [4].

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