The geometry of Schrödinger symmetry in non-relativistic CFT

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The non-relativistic conformal “Schrödinger” symmetry of some gravity backgrounds proposed recently in the AdS/CFT context, is explained in the “Bargmann framework”. The formalism incorporates the Equivalence Principle. Newton-Hooke conformal symmetries, which are analogs of those of Schrödinger in the presence of a negative cosmological constant, are discussed in a similar way. Further examples include topologically massive gravity with negative cosmological constant and the Madelung hydrodynamical description.

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I. INTRODUCTION

Non-relativistic conformal transformations have initially been discovered as those space-time transformations that permute the solutions of the free Schrödinger equation [1, 2]. In \( D + 1 \) dimensional non-relativistic space-time with position coordinates \( y \) and time \( t \) we have, in addition to the (one-parameter centrally extended) Galilean generators, also

\[
(y, t) \rightarrow (y^*, t^*) = (o y, o^2 t) \quad \text{dilatation}
\]

\[
(y, t) \rightarrow (y^*, t^*) = \left( \frac{y}{1 - \kappa t}, \frac{t}{1 - \kappa t} \right) \quad \text{expansions},
\]

referred to as “non-relativistic conformal transformations”. Added to the Galilean symmetries provides us with the Schrödinger group. Dilatations, expansions and time translations span an \( o(2, 1) \approx sl(2, \mathbb{R}) \) subalgebra.

These rather mysterious extra symmetries have been identified as the isomorphisms of the structure of non-relativistic space-time \[3, 4\]. For “empty” space, one gets, in particular, the (one-parameter centrally extended) Schrödinger group. Conformal transformations act as symmetries also when an inverse-square potential is introduced \[1\]. In \( D = 3 \) a Dirac monopole can be included \[5, 7\], and in \( D = 2 \) one can have instead a magnetic vortex \[8\]. Full Schrödinger symmetry is restored for a matter field interacting with a Chern-Simons gauge field \[9, 10\]. It can also be present in hydrodynamics \[11, 12\]. See also \[13\].

Recently, the AdS/CFT correspondence has been extended to non-relativistic field theory \[14, 15, 16, 17\]. The key point is to use the metric

\[
\bar{g}_{\mu\nu} dx^\mu dx^\nu = \frac{1}{r^2} \left[ dx^2 + dr^2 + 2dt ds - \frac{dt^2}{r^2} \right] = \frac{1}{r^2} g_{\mu\nu} dx^\mu dx^\nu,
\]

where \( x \) is an \( d \)-dimensional vector and \( r \) an additional coordinate. The metric \( \bar{g} \) is a \( d+3 \) dimensional relativistic spacetime, conformally related to the pp wave defined by \( g_{\mu\nu} \) on the same manifold. The interesting feature of the metric \( \bar{g} \) is that its isometries are the conformal transformations of \( d + 1 \) dimensional non-relativistic space-time, with coordinates \((x, t)\).

Below explain the construction and properties of this metric, and illustrate it on some physical examples.

II. SIKLOS SPACETIMES

The metric \( \bar{g} \) belongs to the class of Siklos spacetimes \[18, 19, 20\], interpreted as exact gravitational waves traveling along AdS,

\[
\bar{g}_{\mu\nu} dx^\mu dx^\nu = \frac{1}{r^2} \left[ dx^2 + dr^2 + 2dt ds - F(x, r, t) dt^2 \right].
\]

\( \xi = \partial_t \) is a null Killing vector. The metric \( \bar{g} \) can also be presented as \( \bar{g}_{\mu\nu} = g_{\mu\nu}^{\text{AdS}} - \nu^2 F \xi_{\mu} \xi_\nu \), generalizing the familiar Kerr-Schild transformation. For \( F = 0 \) \[3\] reduces to the anti-de Sitter metric.
The Einstein tensor of (3) satisfies
\[ \tilde{G}_{\mu \nu} + \Lambda \tilde{g}_{\mu \nu} = \rho \xi_\mu \xi_\nu, \]
\[ \rho = \frac{r^4}{2} \left( \frac{d+1}{r} \partial_t F - \partial_r F + \Delta_x F \right), \] (4)
where \( \Lambda = -(d+1)(d+2)/2 \). Hence, these spacetimes are solutions of gravity with a negative cosmological constant, coupled to lightlike fluid. The only non-vanishing component of the Einstein–de Sitter tensor is \( G_{tt} + \Lambda g_{tt} \).

The RHS of (4) is traceless, since it is the energy-momentum tensor of some relativistic fluid made of massless particles. When \( F \) satisfies the Siklos equation [19],
\[ \partial_2^2 F - \frac{d+1}{r} \partial_r F + \Delta_x F = 0, \] (5)
then \( \rho = 0 \), and (3) is the AdS\(_{d+3}\) metric.

The effect of a conformal redefinition of the metric, \( g_{\mu \nu} \rightarrow \tilde{g}_{\mu \nu} = \Omega^2 g_{\mu \nu} \), has been studied by Brinkmann [2].

Applied to our case we see that the Einstein equation of the pp wave \( g_{\mu \nu} \) in (1) goes over, for \( \Omega = r^{-1} \), to that with negative cosmological constant, appropriate for \( g_{\mu \nu} \).

III. Bargmann Space and NR Symmetries

The best way to understand non-relativistic symmetries is to use a Kaluza-Klein-type framework [3, 4].

The carrier space, \( M \) (called “Bargmann space”), is a \((D + 1,1)\) dimensional spacetime, endowed with a “relativistic” metric \( g_{\mu \nu} \) and a covariantly constant null vector, \( \xi \). Such a metric can always be written as
\[ g_{ij}dy^i dy^j + 2dt(ds + A \cdot dy) - 2U dt^2, \] (6)
where \( g_{ij} = g_{ij}(y,t) \) is a metric on \( D \) dimensional space, \( A = A(y,t) \) is a vector and \( U = U(y,t) \) is a scalar [2].

Then \( \xi = \partial_s \) is a covariantly constant vector. In the sequel, we only consider the special case \( g_{ij} = \delta_{ij} \). We also assume, for simplicity, that \( A \) and \( U \) do not depend explicitly on time.

Factoring out the curves generated by \( \xi \) and represented by the “vertical” coordinate \( s \) yields a \((D+1)\) dimensional manifold. The relativistic metric does not induce a metric on ordinary space-time with coordinates \( y, t \). The contravariant tensor \( g^{\mu \nu} \) does project, however, together with the closed one-form \( \xi_\mu \), endowing the quotient with a Newton-Cartan structure — the structure of non-relativistic spacetime [3], with position \( y \) and non-relativistic time \( t \).

Non-relativistic conformal symmetries are those conformal transformations \( f : M \rightarrow M \) which preserve the vertical vector \( \xi \).
\[ f^* g_{\mu \nu} = \Omega^2 g_{\mu \nu}, \quad f_* \xi = \xi \] (7)
where \( f^* \) and \( f_* \) denote “pull-back” and “push-forward”, respectively. The conformal factor, \( \Omega \), is seen to depend only on \( t \).

The origin of the terminology is that when \( A = 0 \) and \( U = 0 \) in addition to \( g_{ij} = \delta_{ij} \), Bargmann space is simply Minkowski space written in light-cone coordinates. Then the \( \xi \)-preserving isometries form precisely the one-parameter central extension of the Galilei group called the Bargmann group: those conformal transformations which preserve \( \xi \) span in turn the Schrödinger group, he latter acts on Bargmann space, \((y, t, s) \rightarrow (y^*, t^*, s^*)\), according to

\[ \begin{aligned}
    y^* &= \frac{R y + b t + c}{ft + g} \\
    t^* &= \frac{dt + e}{ft + g} \\
    s^* &= s + \frac{f (R y + b t + c)^2}{2 (ft + g)} - b \cdot \frac{R y - t}{2} b^2 + h,
\end{aligned} \] (8)
where \( R \in SO(D) \); \( b, c \in \mathbb{R}^D ; d, e, f, g, h \in \mathbb{R} \) and \( dg - ef = 1 \). The corresponding conformal factor in (8) is \( \Omega = (ft + g)^{-1} \). In particular, \( s^* = s \) for a dilatation, and
\[ s^* = s - \frac{\kappa |y|^2}{2(1 - \kappa t)} \] (9)
for an expansion.

The only time-independent potential which is consistent with the non-relativistic conformal symmetries [1]-[14] is homogeneous of order \((-2)\), \( U(\lambda y) = \lambda^{-2} U(y) \). Such a potential clearly reduces the Schrödinger symmetry to its \( SL(2, \mathbb{R}) \) subgroup (plus, perhaps, other residual symmetries, depending on the concrete form of \( U(y) \)).

Returning to the general case [14], we remind the Reader that non-relativistic mechanics can be discussed by considering null geodesics [2, 4]. Turning to quantum mechanics, “ordinary” wave functions lift to Bargmann space as functions, \( \psi \), that are equivariant with respect to vertical translations, viz.,
\[ \xi^\mu \partial_\mu \psi = im \psi, \] (10)
where \( m \) is some real constant, namely the Galilean mass. For such a function \( \Psi = e^{-ims} \psi(y, t, s) \) is \( s \)-independent, \( \Psi = \Psi(y, t) \), and can therefore be identified with an “ordinary” wave function. Including the “vertical” coordinate incorporates the gauge degree of freedom — just like for “monopoles without string”.

Let us now consider the massless wave equation
\[ \left[ -\frac{D}{4(D+1)} R \right] \psi = 0, \] (11)
where \( \square \) is the Laplace-Beltrami operator of the Bargmann metric. The inclusion of the scalar curvature, \( R \), insures the invariance of (11) under conformal rescalings \( g_{\mu \nu} \rightarrow \tilde{g}_{\mu \nu} = \Omega^2 g_{\mu \nu} \), with \( \psi \rightarrow \psi^* = \Omega^{D/2} \Psi \). The equations (10) and (11) yield the covariant Schrödinger equation on Newton-Cartan space-time [3]. In the special case \( g_{ij} = \delta_{ij} \) and \( A = 0 \), the ordinary Schrödinger
equation with potential $U$ is recovered,
\[ i\partial_t \Psi = \left[ -\frac{\Delta}{2m} + U \right] \Psi, \tag{12} \]
where $\Delta \equiv \Delta_x$ is the usual $D$-dimensional Laplacian.

The natural action of $\xi$-preserving conformal transformation of $M$ induces one acting on an ordinary wave function,
\[ \Psi \rightarrow \Psi^* = \Omega^{D/2} e^{im(\xi^*-s)} \Psi(y^*, t^*). \tag{13} \]
Owing to masslessness, eqn. (11), and hence also (14), are invariant w.r.t. such transformations. In the free case we get, in particular, the Schrödinger symmetry.

**IV. THE EQUIVALENCE PRINCIPLE**

Let us now return to the general case, (10) with non-vanishing $A$ and $U$. For an equivariant function, (10), our wave equation (11) reduces to
\[ i\partial_t \Psi = \left[ -\frac{1}{2m} D^2 + U \right] \Psi, \quad D = \nabla - imA. \tag{14} \]
This provides us with a natural interpretation of the components of the metric (6) : $A$ and $U$ represent the vector and scalar potential of an 'electromagnetic' field,
\[ \mathbf{E} = -\nabla U, \quad \mathbf{B} = \nabla \times \mathbf{A}. \tag{15} \]
(14) is plainly the minimally coupled Schrödinger equation in this 'electromagnetic' field. Note, however, that the coupling constant here is the mass, $m$, not the electric charge.

Consistently with [6, 8], the $SL(2, \mathbb{R})$ survives the inclusion of a Dirac monopole [23], or of an Aharonov-Bohm vector potential into the components of the metric [6, 8, 23].

The components in (6) admit, however, another interpretation : $A$ and $U$ can also be view as associated with non-inertial coordinates. Let us explain this using an example. Consider the metric,
\[ dy^2 + 2dtds + 2(g \cdot y)dt^2, \tag{16} \]
of a uniform gravitational field, $g$. Let us now switch to an uniformly accelerating coordinate system, which amounts to using accelerated coordinates,
\[ Y = y - \frac{1}{2}gt^2, \quad T = t, \quad S = s + (g \cdot y)t - \frac{1}{6}g^2 t^3. \tag{17} \]
When expressed in these coordinates, the metric (16) is simply that of a free particle,
\[ dY^2 + 2dTdS. \tag{18} \]
Thus, the inertial forces compensate the uniform gravitational field : a particle in Einstein’s freely falling lift is free [22].

Conversely, one can start with a free system, (18), and switch to coordinates in a system which rotates with angular velocity $\omega = \omega \mathbf{Z}$ around the axis perpendicular to the $X - Y$ plane,
\[ \left( \begin{array}{c} X \\ Y \end{array} \right) = \left( \begin{array}{c} \cos \omega t x + \sin \omega t y \\ -\sin \omega t x + \cos \omega t y \end{array} \right). \tag{19} \]
Completed with $T = t$ and $S = s$ and dropping the coordinate $Z$, the free metric (18) reads, in these new terms,
\[ dy^2 + 2dt[ds + (\omega \times y) \cdot dy] + \omega^2 y^2 dt^2 \tag{20} \]
($y = (x, y)$). Switching to a rotating coordinate system amounts, therefore, to generating a ‘magnetic’ field twice the angular momentum, $B = 2\omega$, and an ‘electric’ field representing the centrifugal force,
\[ \begin{array}{c}
\text{electric charge } e & \longleftrightarrow & \text{mass } m \\
\text{vector potential } A & \longleftrightarrow & \text{Coriolis potential } 2\omega \times y \\
\text{scalar potential } U & \longleftrightarrow & \text{centrifugal potential } \frac{1}{2} \omega^2 y^2 
\end{array} \tag{21} \]
We conclude that the “Bargmann” framework incorporates Einstein’s *Equivalence Principle* into non-relativistic physics, in that it allows to use *any* coordinate system.

The correspondence (21) has been used in explaining the Sagnac effect, [an Aharonov-Bohm type interference experiment on a turntable [23]]. It has yet another, rather striking, consequence. Let us rotate a bulk of Type II superconductor. According to (21), this should be equivalent to putting it into a combined electric and magnetic field. But Type II superconductors react to a (sufficiently weak) magnetic field by *expelling it* (Meissner effect) – and this should also happen when rotating it. This has actually been experimentally observed by Zimmermann and Mercereau [24] : rotating a superconductor generates in fact an electromagnetic field called the *London moment* [25], which precisely compensates for the inertial forces [26] – this is the “inertial Meissner effect”.

One could also start with slowly rotating our Type II superconductor. Initially, the London moment excludes the flux. Passing a critical value of the angular velocity, however, it should create “Abrikosov” vortices.

The experiment can also be performed using by rotating a superfluid. Then one should again create vortices – which have been indeed observed [27].

In what follows, we restrict our attention to $A = 0$.

**V. PROPERTIES OF THE METRIC (2)**

Let us now turn to the metric (2). The bracketed part, $g_{\mu\nu}$, is a Bargmann space with $D = d + 1$. Let us call $M$ the same manifold, endowed with the rescaled metric $\bar{g}_{\mu\nu}$. One of the coordinates, denoted by $r$, has been
distinguished, \( y = (x, r) \). The potential term, \( U = r^{-2} \), breaks the full, \( D \)-dimensional Schrödinger symmetry to \( o(2,1) \). It plainly has also the \( d = (D - 1) \) dimensional Schrödinger symmetry of \((x,t)\) space.

Special conformal transformations [expansions] act in particular, as \((x, r, t, s) \rightarrow (x^*, r^*, t^*, s^*)\),

\[
(x^*, r^*, t^*, s^*) = \left( \Omega x, \Omega r, \Omega t, s - \Omega^2 \frac{r^2}{2} \right),
\]

\[
\Omega = \frac{1}{1 - \kappa t}.
\]

Similar formulæ hold for an arbitrary conformal transformation \[40\]. For any of them, the bracketed part in \[22\], \( g_{\mu\nu} \), gets multiplied by the conformal factor \( \Omega^2 \); but this is exactly compensated for by the action on the pre-factor \( r^{-2} \), leaving us with an isometry of the full metric \( \bar{g}_{\mu\nu} \). Hence, passing from \( g_{\mu\nu} \) to \( \bar{g}_{\mu\nu} \) converts conformal Bargmann transformations into isometries.

The massless wave equation \[11\] on \( D = d + 1 \) dimensional Bargmann space, reduced to \[13\], reads now

\[
\left[ i\partial_t + \frac{\Delta}{2m} - \frac{m}{2r^2} \right] \Psi = 0,
\]

where \( \Delta \equiv \Delta_{\chi} \) is the Laplacian of \( D = d + 1 \) dimensional space with coordinate \( y = (x, r) \) \[41\]. Its symmetry w.r.t. non-relativistic conformal transformations is obtained when the latter are implemented according to \[13\].

What happens when we trade the pp-wave \( \bar{g}_{\mu\nu} \) for \( g_{\mu\nu} \)? By analogy with \[11\] we postulate,

\[
\left[ \Box - \frac{d + 1}{4(d + 2)} R \right] \phi = 0,
\]

where \( \phi \) is equivariant. \[10\], and \( R \) is the curvature of \( M \). The curvature of the rescaled metric does not vanish, but is rather \( R = -(d + 3)(d + 2) \). Reduced equation reads, therefore,

\[
\left[ i\partial_t + \frac{\Delta}{2m} - \frac{d + 1}{2mr^2} \partial_r - \frac{M^2}{2mr^2} \right] \Phi = 0,
\]

where

\[
M^2 = m^2 - \frac{(d + 1)(d + 3)}{4(d + 2)^2}.
\]

In the \( M \) framework the conformal transformations are implemented without the conformal factor, \( \phi \rightarrow \phi^* = \phi(x^*, r^*, t^*, s^*) \), i.e.,

\[
\Phi^*(x, r, t) = e^{im(s^* - s)} \Phi(x^*, r^*, t^*).
\]

The conformal factor has been absorbed into the new terms in the modified equation \[24\]. Alternatively, direct calculation shows that equation \[21\] becomes \[26\] under the redefinition, \( \Psi = r^{\frac{d+1}{2}} \Phi \). The weight \(-(d + 1)/2\) in the exponent is precisely the one necessary to assure the conformal invariance of \[25\].

As seen before, a non-relativistic conformal transformation acts on \((\bar{\Omega}, \bar{g}_{\mu\nu})\) by isometries. Implementing them according to \[28\] yields, once again, symmetries. Does we have more conformal transformations than just the mere isometries? The answer is negative: a tedious calculation whose details will not be reproduced here shows, in fact, that any \( \xi \)-preserving conformal transformation of the \( \bar{g}_{\mu\nu} \) metric is necessarily an isometry.

VI. CONFORMAL EXTENSION OF THE NEWTON-HOOKE GROUP

In the same spirit, we can consider the Newton-Hooke spacetime, which can be obtained as the limit of a relativistic spacetime with cosmological constant \( \Lambda \) when \( c^2 \Lambda / 3 \rightarrow r^{-2} = \text{const.} \) \[29\]. The extended space,

\[
ds^2 = dy^2 + 2dtds - \frac{|y|^2}{\tau^2} dt^2.
\]

is a homogenous plane wave. It is is again a Bargmann space, and plays a role analogous to that of Minkowski space for the Galilei group.

The isometries of \[29\] which preserve \( \xi = \partial_s \) form the Newton-Hooke group. The latter act on \( D + 1 \) non-relativistic space-time, \((y, t)\), obtained by factoring out the vertical direction \( s \), according to \((t, y) \rightarrow (t^*, y^*)\),

\[
\left\{ \begin{array}{l}
t^* = t + \epsilon, \\
y^* = R y + a \cos(\frac{t}{\tau}) + b \tau \sin(\frac{t}{\tau}),
\end{array} \right.
\]

where \( R \) is an \( SO(D) \) matrix, \( \epsilon \in \mathbb{R} \), \( a, b \in \mathbb{R}^D \) are parameters associated with spatial rotations, time translation, pseudo translations and pseudo Galileo boosts, respectively.

Assuming equivariance with unit mass \( m = 1 \), the wave equation \[11\] becomes the Schrödinger equation of a harmonic oscillator,

\[
\left[ i\partial_t + \frac{1}{2} \Delta - \frac{|y|^2}{2\tau^2} \right] \Phi = 0,
\]

which reduces to the free Schrödinger equation when \( \tau \rightarrow \infty \).

Newton-Hooke transformations act as symmetries. This can also be seen directly, using that their natural implementation on Bargmann space reads, when expressed on non-relativistic objects,

\[
\Phi^*(t, y) = \Phi(t^*, y^*) e^{i\Theta(t, y)},
\]

where \( \Theta(t, y) \) is

\[
\frac{1}{\tau} (a \cdot y) \sin\left(\frac{t}{\tau}\right) + \frac{a^2}{4\tau} \sin\left(\frac{2t}{\tau}\right) - (b \cdot y) \cos\left(\frac{t}{\tau}\right) - \frac{b^2}{4\tau} \sin\left(\frac{2t}{\tau}\right).
\]
Beyond Newton-Hooke transformations, equation (31) admits two further symmetries, analogous to dilatations and expansions. The first of these is

\[
\begin{align*}
t^* &= \tau \arctan \left( \frac{\alpha}{1 - \kappa \tan \left( \frac{t}{\tau} \right)} \right), \\
y^* &= \frac{\alpha \cos \left( \arctan \left( \frac{\alpha}{1 - \kappa \tan \left( \frac{t}{\tau} \right)} \right) \right)}{\cos \left( \frac{t}{\tau} \right) (1 - \kappa \tan \left( \frac{t}{\tau} \right))} y,
\end{align*}
\tag{33}
\]

where \( \alpha \) is the parameter of the transformation. We also have an analog of expansions,

\[
\begin{align*}
t^* &= \tau \arctan \left( \frac{\tan \left( \frac{t}{\tau} \right)}{1 - \kappa \tan \left( \frac{t}{\tau} \right)} \right), \\
y^* &= \frac{\cos \left( \arctan \left( \frac{\tan \left( \frac{t}{\tau} \right)}{1 - \kappa \tan \left( \frac{t}{\tau} \right)} \right) \right)}{\cos \left( \frac{t}{\tau} \right) (1 - \kappa \tan \left( \frac{t}{\tau} \right))} y,
\end{align*}
\tag{34}
\]

These transformations are implemented by involving both phase and “conformal” factors,

\[
\Phi^*(t,y) = \Omega^{D/2} \Phi(t^*,y^*) e^{i\Theta(t,y)},
\tag{35}
\]

where

\[
\Omega^2(t) = \frac{1 + \tan \left( \frac{t}{\tau} \right)^2}{1 - (1 + \kappa) \tan \left( \frac{t}{\tau} \right)^2 - 2 \kappa \tan \left( \frac{t}{\tau} \right)}
\tag{36}
\]

for a dilatation and

\[
\Omega^2(t) = \frac{1 + \tan \left( \frac{t}{\tau} \right)^2}{1 - (1 + \kappa^2 \tau^2) \tan \left( \frac{t}{\tau} \right)^2 - 2 \kappa \tan \left( \frac{t}{\tau} \right)}
\tag{37}
\]

for an expansion. The phase change for the dilatation reads

\[
\Theta(t,y) = \frac{1 + \tan \left( \frac{t}{\tau} \right)^2}{2 \tau [1 + \cos(2t/\tau) + 1 - \cos(2t/\tau)]},
\tag{38}
\]

while for the expansion the expression is rather complicated, and is not reproduced here. A tedious computation shows that the transformations (35) and (36) act as symmetries for (31). Note for further reference that, in both cases, \( y^* = \Omega y \).

In the limit \( \tau \to \infty \), these symmetries become precisely the dilatation and the expansion of the Schrödinger group.

Not surprisingly, these “conformal” transformations lift to genuinely conformal transformation of the Bargmann metric (29). When completed with \( s^* = s + \Theta \), they become conformal transformation of the Bargmann space with conformal factor \( \Omega \).

Bargmann conformal transformations can, again, be converted into isometries following the same strategy as before. We detach one coordinate, i.e. we write \( y = (x, r) \) and rescale the metric (29) as

\[
ds^2 = \frac{1}{r^2} \left[ dx^2 + dr^2 + 2 dt ds - \frac{(x^2 + r^2)}{r^2} dt^2 \right].
\tag{39}
\]

Then the change of the bracketed quantity under a conformal transformation is, once again, compensated by the change of \( r^2 \).

The metric (39) is that of \( AdS \) as it should, since the function in front of \( dt^2 \) trivially satisfies (5).

The simplest way to understand the origin of Newton-Hooke symmetries is to realize that they are, in fact, “imported” Galilean symmetries [9, 28, 30, 31, 32]. Putting \( \omega = \tau^{-2} \), the time-dependent dilatation,

\[
y \to y = \frac{1}{\cos \omega t} y, \quad t \to T = \frac{\tan \omega t}{\omega},
\tag{40}
\]

implemented as

\[
\Psi(y,t) = \frac{1}{(\cos \omega t)^{D/2}} e^{-i \Phi(t) \tan \omega t} \Phi(Y, T),
\tag{41}
\]

transforms the Newton-Hooke (in fact, oscillator) background problem into a “free” one. When completed with

\[
S = s - \frac{\omega y^2}{2} \tan \omega t,
\tag{42}
\]

the Newton-Hooke Bargmann metric (29) is carried by (40-42) Bargmann-conformally into the free form, (18).

The conformal factor is \( \Omega(t) = |\cos \omega t|^{-1} \). Accordingly, this transformation maps (39) isometrically to AdS space.

VII. TOPOLOGICAL GRAVITY

At last, the metric (39) is also encountered in topologically massive gravity in (2 + 1) dimensions [33] with negative cosmological constant \( \Lambda = -1 \). This theory exhibits, among other properties, gravitons; it also admits black hole solutions [34]. The (third order) equations of motion are

\[
G_{\mu \nu} - g_{\mu \nu} + \frac{1}{\mu} C_{\mu \nu} = 0,
\tag{43}
\]

where \( C_{\mu \nu} \) is the Cotton tensor (the analog of the Weyl tensor in (3 + 1) dimensions) and the parameter \( \mu \) has the dimension of mass. There has been great recent interest in what is now known as chiral gravity [35].

A supersymmetric extension of the equations (43) have been proposed [36]. The most general supersymmetric solutions [37] read

\[
ds^2 = \frac{1}{\tau^2} \left[ dr^2 + 2 dt ds - f(t) r^{1-\nu} dt^2 \right],
\tag{44}
\]

where \( \mu^2 \neq 1 \), and \( f \) is an arbitrary function of \( t \). These solutions have also been obtained from another point of view, namely by means of a correspondence between conformal gravity with conformal source and the equations [43, 38].
When $f$ is constant and $\mu = 3$, we clearly recover the metric \([2]\). For $\mu \neq 3$ the conformal symmetry is broken (unless the arbitrary function $f$ is chosen as $f(t) = t^{-3+\mu}$). This particular “point” $\mu = 3$ should exhibit interesting features worth studying, just like at the chiral point $\mu = 1$. For $f(t) = a(bt + c)^{-1}$ and $\mu = 2$ we get Dirac’s theory of a variable gravitational constant, conformal with the Kepler problem \([4]\).

\section*{VIII. MADELUNG TRANSCRIPTION}

The equations of motion of a perfect fluid in $(d+1+1)$ dimensions are given by

\[
\begin{align*}
\partial_t \rho + \nabla \cdot (\rho \mathbf{v}) &= 0, \\
\rho \left[ \partial_t + \mathbf{v} \cdot \nabla \right] \mathbf{v} &= -\nabla p + \rho \mathbf{f},
\end{align*}
\]

where $\rho$ is the fluid density, $\mathbf{v}$ the velocity, $p$ the pressure and $\mathbf{f}$ represents the force density. For a perfect fluid with polytropic equations of motion, i.e., $p \propto \rho^\gamma$ with polytropic exponent $\gamma = 1 + 2/(d+1)$, equations \([15]\) are Schrödinger symmetric \([11, 12]\).

Decomposing the wave function $\Phi$ of the modified Schrödinger equation \([20]\) as $\Phi = r^{d+1} \sqrt{\rho} e^{i\Theta}$ yields the hydrodynamical system known as the Madelung fluid,

\[
\begin{align*}
\partial_t \rho + \frac{1}{m} \nabla \cdot (\rho \nabla \Theta) &= 0, \\
\partial_t \Theta + \frac{1}{2m} \left| \nabla \Theta \right|^2 &= -\frac{1}{4m^2} \left[ \frac{1}{2\rho} |\nabla \rho|^2 - \Delta \rho \right] - \frac{\alpha}{8m^2}
\end{align*}
\]

where $\alpha = 4M^2 + (d + 3)(d + 1)$ and the gradient is w.r.t. $\mathbf{y} = (x, r)$. These equations can be interpreted as the equations of a perfect fluid \([15]\), whose motion is irrotational, $\mathbf{v} = \nabla \Theta$. It is submitted to an external potential force, $\mathbf{f} = -\nabla \left( \frac{\alpha}{8m^2} \right)$, and has enthalpy $\omega = \frac{1}{4m^2} \left[ \frac{1}{2\rho} |\nabla \rho|^2 - \Delta \rho \right]$. For the Madelung system \([17]\) the non-relativistic conformal symmetry is implemented as,

\[
\begin{align*}
\Theta(t, \mathbf{y}) &\to \Theta\left( \frac{t}{1-\kappa t}, \frac{\mathbf{y}}{1-\kappa t} \right) - \kappa \frac{\mathbf{y}^2}{2(1-\kappa t)}, \\
\rho(t, \mathbf{y}) &\to \frac{1}{(1-\kappa t)^{d+1}} \rho\left( \frac{t}{1-\kappa t}, \frac{\mathbf{y}}{1-\kappa t} \right).
\end{align*}
\]

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[41] For a monopole, ds + A · dy is a U(1) connection form on the Hopf bundle.

[42] The action (23) is natural when x and r are viewed as parts of y. It also corresponds to viewing the d = D − 1 dimensional Schrödinger group as a subgroup of the D-dimensional one. It is not the only implementation, though: we could have also extended it by having it act trivially, i.e., leaving r invariant. Then y2 = x2 + r2 would be replaced by x2. Such an action would again be an isometry of the rescaled metric. It seems that this is not the action people are interested in, however.

[43] If we argue that only x is “physical” and the coordinate r has been added as an auxiliary variable, we should require that the wave function be independent of r. Then the Laplacian in (23) reduces to △x.