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Topological classification of complex vector bundles over 8-dimensional spin$^c$ manifolds

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Abstract. In this paper, complex vector bundles of rank $r$ over 8-dimensional spin$^c$ manifolds are classified in terms of the Chern classes of the complex vector bundles, where $r = 3$ or $4$. As an application, we see that two rank 3 complex vector bundles over 4-dimensional complex projective space $\mathbb{C}P^4$ are isomorphic if and only if they have the same Chern classes. Moreover, the Chern classes of rank 3 complex vector bundles over $\mathbb{C}P^4$ are determined. Together with results of Thomas and Switzer, this completes the classification of complex vector bundles of any rank over $\mathbb{C}P^4$.

1. Introduction

It is a classical topic in geometry and topology to classify vector bundles over manifolds (cf. [2–4,6,11,17,22]). For complex vector bundles, one may find from the beginning of Schneider [16] that in order to classify holomorphic vector bundles over a fixed complex manifold (for example, the $n$-dimensional complex projective space $\mathbb{C}P^n$), the first step is to classify smooth complex vector bundles over it. See Atiyah and Rees [2, 2], Okonek, Schneider and Spindler [14, Ch 1, 6] and Schneider [16, 1], for instance. This motivates us to investigate the topological classification of smooth complex vector bundles over manifolds.

For a closed oriented smooth $2n$-manifold $M$, denote by $\text{Vect}^r_{\mathbb{C}}(M)$ the isomorphism classes of rank $r$ smooth complex vector bundles over $M$, and by $H^*(M)$ the integral cohomology ring of $M$. It is well known that there is a natural bijection between $\text{Vect}^r_{\mathbb{C}}(M)$ and $\text{Vect}^n_{\mathbb{C}}(M)$ for any $r \geq n$. If $r = 1$, it is known that the rank 1 complex vector bundles over $M$ are classified by their first Chern class $c_1$, hence we have a bijection between $\text{Vect}^1_{\mathbb{C}}(M)$ and $H^2(M)$ for any $n \geq 0$. For $r = n = 2$, complex vector bundles are determined by their Chern classes $c_1$ and $c_2$, see Peterson [15, Theorem 3.2]. Moreover, a classical result of Wu [23, Proposition 7] asserts that there is a bijection between $\text{Vect}^2_{\mathbb{C}}(M)$ and $H^2(M) \times H^4(M)$. For $n = 3$, the classification of rank $r$ complex vector bundles over 3-dimensional complex manifolds is obtained by Bănică and Putinar [3,4, Theorem 1] for any $r \geq 1$. 

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For the $n$-dimensional complex projective space $\mathbb{C}P^n$, Thomas [20] proved that the Schwarzenberger condition [9, Appendix I] classifies rank $n$ complex bundles on $\mathbb{C}P^n$. Moreover, rank 2 complex vector bundles over $\mathbb{C}P^n$ with $3 \leq n \leq 6$ are classified by Atiyah and Rees [2] and Switzer [17].

In this paper, the classification of complex vector bundles over any 8-dimensional spin$^c$ manifold $M$ will be investigated. Since the classification of rank 2 complex vector bundles over $M$ has been studied by Switzer [18] and [17], we will focus mainly on the classification of complex vector bundles over $M$ with rank 3 and 4. Our main results are stated as Theorems 1.1 and 1.2 below. As applications, the complex vector bundles over $\mathbb{C}P^4$ with rank 3 and 4 are classified as in Corollaries 1.6 and 1.7 below.

We first recall some necessary notation and concepts. For a pathwise-connected CW-complex $X$, we will denote by

$$\cdots \rightarrow H^i(X) \xrightarrow{x^2} H^i(X) \xrightarrow{\rho_2} H^i(X; \mathbb{Z}/2) \xrightarrow{\beta} H^{i+1}(X) \cdots \tag{1.1}$$

the long exact Bockstein sequence associated to the coefficient sequence $0 \rightarrow \mathbb{Z} \xrightarrow{x^2} \mathbb{Z} \rightarrow \mathbb{Z}/2 \rightarrow 0$, where $\rho_2$ and $\beta$ are the mod 2 reduction and Bockstein homomorphisms respectively, and denote by $\mathrm{Sq}^2 : H^i(X; \mathbb{Z}/2) \rightarrow H^{i+2}(X; \mathbb{Z}/2)$ the Steenrod square.

Throughout this paper, $M$ will be an 8-dimensional closed oriented spin$^c$ manifold. We will fix an element $c \in H^2(M)$ satisfying $\rho_2(c) = w_2(M)$, where $w_2(M)$ is the second Stiefel-Whitney class of $M$. We call $c$ a spin$^c$ characteristic class of $M$. Let

$$I_* : \mathrm{Vect}_{\mathbb{C}}^3(M) \rightarrow \mathrm{Vect}_{\mathbb{C}}^4(M)$$

be the map given by $I_*(\alpha) = \alpha \oplus \epsilon$, the Whitney sum of $\alpha \in \mathrm{Vect}_{\mathbb{C}}^3(M)$ and $\epsilon$, where $\epsilon$ is the trivial complex vector bundle over $M$ with rank 1. Denote by

$$C_{14} : \mathrm{Vect}_{\mathbb{C}}^3(M) \rightarrow H^2(M) \times H^4(M) \times H^6(M) \times H^8(M)$$

$$C_{13} : \mathrm{Vect}_{\mathbb{C}}^2(M) \rightarrow H^2(M) \times H^4(M) \times H^6(M)$$

the maps given by

$$C_{14} (\eta) = (c_1(\eta), c_2(\eta), c_3(\eta), c_4(\eta)), \quad \text{for any } \eta \in \mathrm{Vect}_{\mathbb{C}}^3(M),$$

$$C_{13} (\eta) = (c_1(\eta), c_2(\eta), c_3(\eta)), \quad \text{for any } \eta \in \mathrm{Vect}_{\mathbb{C}}^2(M),$$

respectively, where $c_i(\eta)$ is the $i$-th Chern class of $\eta$. For a triple $u = (u_1, u_2, u_3) \in \text{Im } C_{13}$, let $\mathcal{B}_M$ and $\mathcal{T}_{M,u}$ be the quotient groups defined as

$$\mathcal{B}_M := \frac{\beta(H^5(M; \mathbb{Z}/2))}{\beta(\mathrm{Sq}^2(\rho_2(H^3(M))))},$$

$$\mathcal{T}_{M,u} := \frac{H^7(M)}{\{f^*(\gamma_7) + u_1 f^*(\gamma_5) + u_2 f^*(\gamma_3) + u_3 f^*(\gamma_1) \mid f \in [M, U]\}},$$

where $[M, U]$ is set of homotopy classes of maps of $M$ into the stable unitary group $U$, and where $\gamma_1$, $\gamma_3$, $\gamma_5$ and $\gamma_7$ are generators of the exterior algebra

$$H^*(U) \cong \Lambda(\gamma_1, \gamma_3, \gamma_5, \gamma_7, \cdots).$$
Theorem 1.1. Let $M$ be an 8-dimensional closed oriented spin$^c$ manifold.

(A) For any even dimensional cohomology classes $u_i \in H^{2i}(M)$, $1 \leq i \leq 4$,

$$(u_1, u_2, u_3, u_4) \in \text{Im } C_{14}$$

if and only if they satisfy the following three conditions

1. $\text{Sq}^2(\rho_2(u_2)) = \rho_2(u_3 + u_1 u_2),$
2. $\langle u_4, [M] \rangle \equiv \langle p_1(M) u_2 - u_1^2 u_2 + u_1 u_3 - u_2^2, [M] \rangle \mod 3,$
3. $\langle u_4, [M] \rangle \equiv \langle -u_1^2 u_2 + u_1 u_3 + [2u_2^2 + p_1(M)u_2 - 3c^2 u_2]/4 + c(u_1 u_2 - u_3)/2, [M] \rangle \mod 2,$

where $p_1(M)$ is the first Pontrjagin class of $M$, $[M]$ is the fundamental class of $M$ and $(\cdot, \cdot, \cdot)$ is the Kronecker product.

(B) For any $(u_1, u_2, u_3, u_4) \in \text{Im } C_{14}$, there is a bijection between

$$C^{-1}_{14}(u_1, u_2, u_3, u_4) \text{ and } \mathcal{B}_M,$$

where $C^{-1}_{14}(u_1, u_2, u_3, u_4)$ is the pre-image of $(u_1, u_2, u_3, u_4)$ under the map $C_{14}$. It follows that there is a one-to-one correspondence between

$$\text{Vect}^4_{c}(M) \text{ and } \mathcal{B}_M \times \text{Im } C_{14}.$$

Theorem 1.2. Let $M$ be an 8-dimensional closed oriented spin$^c$ manifold. (A) For any $\eta \in \text{Vect}^4_{c}(M)$, the necessary and sufficient condition for $\eta$ to lie in the image of $I_*$ is

$$c_4(\eta) = 0.$$ 

Therefore, for any cohomology classes $u_i \in H^{2i}(M; \mathbb{Z})$, $1 \leq i \leq 3$,

$$(u_1, u_2, u_3) \in \text{Im } C_{13} \text{ if and only if } (u_1, u_2, u_3, 0) \in \text{Im } C_{14}.$$

(B) If $u = (u_1, u_2, u_3) \in \text{Im } C_{13}$, then $C^{-1}_{13}(u_1, u_2, u_3)$ is equivalent, as a set, to $\mathcal{B}_M \times \mathfrak{T}_{M,u}$.

Remark 1.3. Under the condition (1) of Theorem 1.1 (A), one may find from the proof of Theorem 1.1 (A) that: (a) the rational number

$$\langle -u_1^2 u_2 + u_1 u_3 + [2u_2^2 + p_1(M)u_2 - 3c^2 u_2]/4 + c(u_1 u_2 - u_3)/2, [M] \rangle$$

is an integer, so it make sense to take congruence classes modulo 2; (b) the congruence (3) in Theorem 1.1 (A) does not depend on the choice of $c$.

Remark 1.4. Suppose that $M$ is a 4-dimensional complex manifold. Then we have $p_1(M) = c_1^2(M) - 2c_2(M)$ and one can take $c = c_1(M)$. One may find that Theorem 1.1 (A) was obtained by Bănică and Putinar [5, Proposition 3.1] under the assumptions that $H^6(M)$ and $H^7(M)$ have no 2-torsion.

Peterson [15, Theorem 3.2] tells us that if $H^6(M)$ has no 2-torsion, then the map $C_{14}$ is injective. Obviously, as a corollary of Theorem 1.1, this statement can be generalized and strengthened as
Corollary 1.5. Let $M$ be an 8-dimensional closed oriented spin$^c$ manifold. Then the map $\mathcal{C}_{14}$ is injective if and only if $\mathcal{B}_M = 0$. \hfill \Box

Finally, as applications, let us consider the classification of complex vector bundles over $\mathbb{C}P^4$. Set $t = -c_1(\gamma) \in H^2(\mathbb{C}P^4)$, where $\gamma$ is the canonical line bundle over $\mathbb{C}P^4$. It is well known that the cohomology ring $H^*(\mathbb{C}P^4) = \mathbb{Z}[t]/(t^5)$ and the total Chern class $c(\mathbb{C}P^4) = (1+t)^5$. Hence $p_1(\mathbb{C}P^4) = 5t^2$, and we can take $c = c_1(\mathbb{C}P^4) = 5t$. Therefore, since $\mathcal{B}_{\mathbb{C}P^4} = \mathcal{B}_{\mathbb{C}P^4,u} = 0$ for any $u \in \text{Im} \mathcal{C}_{13}$, it can be deduced easily from Theorem 1.1 and a straightforward calculation that

Corollary 1.6. The map

$$\mathcal{C}_{14} : \text{Vect}^4_\mathbb{C}(\mathbb{C}P^4) \to H^2(\mathbb{C}P^4) \times H^4(\mathbb{C}P^4) \times H^6(\mathbb{C}P^4) \times H^8(\mathbb{C}P^4),$$

given by $\mathcal{C}_{14}(\eta) = (c_1(\eta), c_2(\eta), c_3(\eta), c_4(\eta))$, is injective.

Moreover, $(a_1t, a_2t^2, a_3t^3, a_4t^4) \in \text{Im} \mathcal{C}_{14}$, if and only if the integers $a_i \in \mathbb{Z}$, $1 \leq i \leq 4$, satisfy the following two conditions

1. $2a_4 \equiv a_2^2 + a_2 + a_1(a_1a_2 - a_3) \mod 3$,
2. $2a_4 \equiv a_2^2 + a_2 + a_1a_2 - a_3 \mod 4$. \hfill \Box

Corollary 1.7. The map

$$\mathcal{C}_{13} : \text{Vect}^3_\mathbb{C}(\mathbb{C}P^4) \to H^2(\mathbb{C}P^4) \times H^4(\mathbb{C}P^4) \times H^6(\mathbb{C}P^4),$$

given by $\mathcal{C}_{13}(\eta) = (c_1(\eta), c_2(\eta), c_3(\eta))$, is an injection.

Furthermore, $(a_1t, a_2t^2, a_3t^3) \in \text{Im} \mathcal{C}_{13}$, if and only if the integers $a_i \in \mathbb{Z}$, $1 \leq i \leq 3$, satisfy the following two conditions

1. $a_2^2 + a_2 + a_1(a_1a_2 - a_3) \equiv 0 \mod 3$,
2. $a_2^2 + a_2 + a_1a_2 - a_3 \equiv 0 \mod 4$. \hfill \Box

Remark 1.8. One may find that the conditions (1) and (2) in Corollaries 1.6 and 1.7 are just the Schwarzkenberger condition ([9, Appendix I]).

Remark 1.9. In fact, Corollary 1.6 has been obtained by Thomas [20, Theorem A]. Recall from Switzer [17, Theorem 2] that the complex vector bundles with rank 2 over $\mathbb{C}P^4$ have also been classified. Combining their results with Corollary 1.7, yields that the classification of complex vector bundles over $\mathbb{C}P^4$ has been settled.

After some preliminaries in Sect. 2, Theorems 1.1 and 1.2 will be proved in Sects. 3 and 4 respectively.

2. Preliminaries

Some preliminaries are needed to prove the main results of this paper.

We work in the category of $CW$-complexes with basepoints. For any $CW$-complex $X$, we denote the space of loops on $X$ by $\Omega X$. We will endow $\Omega X$ with a standard $H$-structure, given by composition of loops. We denote by $[X, Y]$ the set
of homotopy classes of maps of $X$ into a $CW$-complex $Y$. For a map $f: Y \to Z$ between $CW$-complexes $Y$ and $Z$, denote by

$$f_*: [X, Y] \to [X, Z] \quad \text{and} \quad f^*: [Z, X] \to [Y, X]$$

the maps induced by $f$. It is known that the standard $H$-structures on $\Omega Y$ and $\Omega Z$ make $[X, \Omega Y]$ and $[X, \Omega Z]$ into groups, and the map $\Omega f: \Omega Y \to \Omega Z$ induces a homomorphism $(\Omega f)_*: [X, \Omega Y] \to [X, \Omega Z]$.

Let $K(G, n)$ be the Eilenberg-MacLane space of type $(G, n)$ with $G = \mathbb{Z}$ or $\mathbb{Z}/2$. It is known that $K(G, n)$ is a topological group and $\Omega K(G, n) = K(G, n - 1)$. For any $CW$-complex $X$, we will identify $[X, K(G, n)] = H^n(X; G)$ in the usual way. By this identification, we will regard the homotopy class of a map $k: X \to K(G, n)$ as a cohomology class, and denote it by the same symbol $k \in H^n(X; G)$.

In order to classify and enumerate complex vector bundles over manifolds, the Postnikov resolution of a map and some facts of James and Thomas [10] are needed. Let $p: E \to B$ be a fibration where $E$ and $B$ are pathwise-connected $CW$-complexes. Denote by $F$ the homotopy fiber of $p$. Suppose that both $B$ and $F$ are simply connected. If the first two non-trivial homotopy groups of $F$ are $\pi_{n_1}(F)$ and $\pi_{n_2}(F)$, where $n_1$ and $n_2$ are two integers with $1 < n_1 < n_2$, Then the Postnikov resolution of the map $p$ through dimension $(n_2 + 1)$ is given below (cf. Thomas [19])

$$
\begin{array}{c}
E_1 \quad h_1 \quad K(\pi_{n_2}(F), n_2 + 1) \\
E \quad \downarrow q_1 \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \qu
Then, for a pathwise-connected $CW$-complex $X$, we have an exact sequence of sets (cf. [21, p. 141])

$$\cdots \to [X, \Omega B] \xrightarrow{(\Omega k)_*} [X, \Omega C] \xrightarrow{j_*} [X, E] \xrightarrow{q_*} [X, B] \xrightarrow{k_*} [X, C],$$

where $k_*, q_*$ and $j_*$ are the induced maps, which are generally not homomorphisms, and where $(\Omega k)_*$ is the induced homomorphism. Recall from [10, p. 495] or [21, p. 140] that there is a group action of $[X, \Omega C]$ on $[X, E]$ in the usual way. We will denote this action by

$$*: [X, \Omega C] \times [X, E] \to [X, E].$$

**Lemma 2.1.** (Lemma 3.1 in James and Thomas [10]) Let $\alpha, \alpha' \in [X, E]$. Then $q_*(\alpha) = q_*(\alpha')$ if and only if there exists an element $g \in [X, \Omega C]$ such that $g*\alpha = \alpha'$. \hfill $\Box$

For any $\eta \in [X, B]$, a homomorphism

$$\Delta(k, \eta): [X, \Omega B] \to [X, \Omega C],$$

was defined by James and Thomas [10]. We will denote by $\text{Cok} \Delta(k, \eta)$ the cokernel of $\Delta(k, \eta)$. One may find from the definition of $\Delta(k, \eta)$ (see [10, page 486] for more details) and [10, Corollary 1.4] that:

**Lemma 2.2.** For the trivial element $1 \in [X, B]$, we have $\Delta(k, 1) = (\Omega k)_*$. \hfill $\Box$

**Lemma 2.3.** (Corollary 1.4 in [10]) Take $C$ to be the Eilenberg-MacLane space $K(G, n)$ with $n \geq 2$. If the classifying map $k: B \to K(G, n)$ satisfies

$$\mu^*(k) = k \times 1 + 1 \times k + \sum x_i \times y_i \in H^n(B \times B; G),$$

where the degrees of $x_i$ and $y_i$ are positive, then

$$\Delta(k, \eta)(f) = f^*(\sigma(k)) + \sum f^*(\sigma(x_i))\eta^*(y_i) \in H^{n-1}(X; G),$$

for any $f \in [X, \Omega B]$, where $\sigma$ is the cohomology suspension defined below. \hfill $\Box$

**Remark 2.4.** One may find that the degrees of $x_i$ and $y_i$ are all greater than or equal to 1.

Now suppose that $k_*(\eta) = 0 \in [X, C]$ is the trivial element. It follows from the exact sequence above that $q_*^{-1}(\eta) \neq \emptyset$, where $q_*^{-1}(\eta) \subset [X, E]$ is the preimage of $\eta$ under $q_*$. Let $\alpha_\eta \in q_*^{-1}(\eta)$ be a fixed lifting of $\eta$ in $[X, E]$. Then for any $\alpha \in q_*^{-1}(\eta)$, there exists $g_\alpha \in [X, \Omega C]$ such that $g_\alpha \star \alpha_\eta = \alpha$ by Lemma 2.1. Let

$$\Phi_\eta: q_*^{-1}(\eta) \to \text{Cok} \Delta(k, \eta)$$

be the map given by $\Phi_\eta(\alpha) = [g_\alpha]$ for any $\alpha \in q_*^{-1}(\eta)$, where $[g_\alpha]$ is the coset containing $g_\alpha$. Lemma 2.1 and [10, Theorems 2.9, 3.2 and 3.3] imply that
Lemma 2.5. (Theorem 1.2 in James and Thomas [10]) The map $\Phi_\eta$ is a bijection. □

In particular, it follows from Lemmas 2.2 and 2.5 or the exact sequence above that

Lemma 2.6. There is a bijection between $\text{Ker } q_* = q_*^{-1}(1)$ and $\text{Cok } (\Omega k)_*$. □

In order to prove our main results, the following two lemmas are also needed.

Lemma 2.7. (Lemma 4.1 in [10]) Let $f : Y \to Z$ be a map between two pathwise-connected CW-complexes. Suppose that the induced homomorphism between homotopy groups $f_* : \pi_r(Y) \to \pi_r(Z)$ is an isomorphism for $1 \leq r \leq m$. Then the induced map

$$f_* : [X, Y] \to [X, Z]$$

is bijective, where $X$ is a CW-complex with $\dim X \leq m$. □

Let $\sigma : H^n(X; G) \to H^{n-1}(\Omega X; G)$ be the cohomology suspension defined as: for any $h \in H^n(X; G)$ represented by the map $h : X \to K(G, n)$, $\sigma(h) \in H^{n-1}(\Omega X; G)$ is the class represented by the map $\Omega h : \Omega X \to K(G, n - 1)$. Recall from [21, p. 382, Theorem (3.1)] that we have

Lemma 2.8. Suppose that $X$ is $m$-connected. Then the cohomology suspension $\sigma : H^n(X; G) \to H^{n-1}(\Omega X; G)$ is an isomorphism for $n \leq 2m$. □

We conclude this section with the following facts about the complex vector bundles.

Denote by $BU(n)$ and $BU$ the classifying spaces of the unitary group $U(n)$ and the stable unitary group $U$ respectively. Let $I_n : BU(n) \to BU$ be the canonical map. For any pathwise-connected CW-complex $X$, we will identify

$$\text{Vect}^0(X) = [X, BU(n)], \quad (2.1)$$

in the usual way. Suppose that $\dim X \leq 2n$. Since $I_n^* : \pi_r(BU(n)) \to \pi_r(BU)$ is an isomorphism for $r \leq 2n$, it follows from Lemma 2.7 that the induced map $I_n^* : [X, BU(n)] \to [X, BU]$ is bijective. Therefore, we will identify

$$\text{Vect}^0(X) = [X, BU], \quad (2.2)$$

when $\dim X \leq 2n$.

Let $KU(X)$ denote the reduced $KU$-group of $X$, which is the set of stable isomorphism classes of complex vector bundles over $X$, and let $\widetilde{KSU}(X)$ denote the subgroup of $\widetilde{KU}(X)$ which consists of the elements with trivial first Chern class. Denote by $BSU$ the classifying space of the stable special unitary group $SU$. We know that $BSU$ fits into the fibration $BSU \to BU \overset{c_1}{\to} K(\mathbb{Z}, 2)$, where $c_1$ is the first universal Chern class. Then, we can identify

$$\widetilde{KU}(X) = [X, BU], \quad (2.3)$$
$$\widetilde{KSU}(X) = [X, BSU], \quad (2.4)$$

in the usual way, and will regard a homotopy class of a map $\eta : X \to BU$ (resp. $\eta : X \to BSU$) as a stable complex vector bundle $\eta$ (resp. $\eta$ with $c_1(\eta) = 0$) over $X$, and vice versa. In particular, the identifications above make $[X, BU]$ and $[X, BSU]$ into groups under the Whitney sum operation.
3. Proof of theorem 1.1

In this section, Theorem 1.1, restated as Theorem 3.1 below, will be proved by using the differential Riemann-Roch theorem, and analysing the Postnikov resolution of a certain fibration.

**Theorem 3.1.** (Theorem 1.1) Let $M$ be an 8-dimensional closed oriented spin$^c$ manifold.

(A) For any even dimensional cohomology classes $u_i \in H^{2i}(M)$, $1 \leq i \leq 4$,

$$(u_1, u_2, u_3, u_4) \in \text{Im} \, C_{14}$$

if and only if they satisfy the following three conditions

1. $\text{Sq}^2(\rho_2(u_2)) = \rho_2(u_3 + u_1u_2)$,
2. $\langle u_4, [M] \rangle \equiv \langle p_1(M)u_2 - u_1^2u_2 + u_1u_3 - u_3^2, [M] \rangle \mod 3$,
3. $\langle u_4, [M] \rangle \equiv \langle -u_1^2u_2 + u_1u_3 + [2u_2^2 + p_1(M)u_2 - 3c^2u_2]/4 + cu_1u_2 - u_3^2, [M] \rangle \mod 2$.

(B) For any $(u_1, u_2, u_3, u_4) \in \text{Im} \, C_{14}$, there is a bijection between $C_{14}^{-1}(u_1, u_2, u_3, u_4)$ and $\mathcal{B}_M$.

It follows that there is a one-to-one correspondence between $\text{Vect}_C^4(M)$ and $\mathcal{B}_M \times \text{Im} \, C_{14}$.

Recall that $c \in H^2(M)$ is a fixed spin$^c$ characteristic class of $M$, i.e., it satisfies $\rho_2(c) = w_2(M)$.

Let us first prove Theorem 3.1 (A). Note that we have $c_1(\eta) = 0$ for any $\eta \in [M, BSU]$. We will consider the map

$$C_{24} = (c_2, c_3, c_4) : [M, BSU] \to H^4(M) \times H^6(M) \times H^8(M)$$

given by $C_{24}(\eta) = (c_2(\eta), c_3(\eta), c_4(\eta))$ for any $\eta \in [M, BSU]$.

**Theorem 3.2.** Let $M$ be an 8-dimensional closed oriented spin$^c$ manifold. For any even dimensional cohomology classes $u_i \in H^{2i}(M)$, $2 \leq i \leq 4$,

$$(u_2, u_3, u_4) \in \text{Im} \, C_{24}$$

if and only if they satisfy the following three conditions

1. $\text{Sq}^2(\rho_2(u_2)) = \rho_2(u_3)$,
2. $\langle u_4, [M] \rangle \equiv \langle p_1(M)u_2 - u_1^2, [M] \rangle \mod 3$,
3. $\langle u_4, [M] \rangle \equiv \langle [2u_2^2 + p_1(M)u_2 - 3c^2u_2]/4 - cu_3, [M] \rangle \mod 2$.

**Remark 3.3.** Under the condition (1) of Theorem 3.2, one may find from the proof of Lemma 3.4 below that: (a) the rational number $\langle [2u_2^2 + p_1(M)u_2 - 3c^2u_2]/4 - cu_3, [M] \rangle$ is an integer, so it makes sense to take congruence classes modulo 2; (b) the congruence (3) in Theorem 3.2 does not depend on the choice of $c$. 

**Proof of Theorem 3.1 (A).** For any 4-dimensional complex vector bundle $\eta$ over $M$, by the identifications (2.2) and (2.3), we will regard it as a stable complex vector bundle. Denote by $l_\eta$ the complex line bundle over $M$ with $c_1(l_\eta) = c_1(\eta)$. Set $\eta' = \eta - l_\eta$. Then the total Chern classes of $\eta$ and $\eta'$ satisfy the identity

$$1 + c_1(\eta) + c_2(\eta) + c_3(\eta) + c_4(\eta) = [1 + c_1(\eta') + c_2(\eta') + c_3(\eta') + c_4(\eta')][1 + c_1(\eta)].$$

This identity tells us that $c_1(\eta') = 0$, $c_2(\eta') = c_2(\eta)$, $c_3(\eta') = c_3(\eta) - c_1(\eta)c_2(\eta)$ and $c_4(\eta') = c_4(\eta) - c_1(\eta)c_3(\eta) + c_1^2(\eta)c_2(\eta)$. In particular, we have $\eta' \in [M, BSU]$.

The discussion above yields that for any even dimensional cohomology classes $u_i \in H^{2i}(M)$, $1 \leq i \leq 4$, $(u_1, u_2, u_3, u_4) \in \text{Im } C_{14}$ if and only if $(u_2, u_3 - u_1u_2, u_4 - u_1u_3 + u_1^2u_2) \in \text{Im } C_{24}$. Then Theorem 3.1 (A) can be deduced easily from Theorem 3.2 by a straightforward calculation.

In order to prove Theorem 3.2, we need the following lemmas.

**Lemma 3.4.** For any $\eta \in [M, BSU]$, we must have

$$\langle c_4(\eta), [M] \rangle \equiv \langle p_1(M)c_2(\eta) - c_2^2(\eta), [M] \rangle \mod 3,$$

$$\langle c_4(\eta), [M] \rangle \equiv \langle [2c_2^2(\eta) + p_1(M)c_2(\eta) - 3c_2^2(\eta)]/4 - c_3(\eta)/2, [M] \rangle \mod 2.$$

**Proof.** Denote by $\hat{A}(M)$ the $A$-class of $M$ (see for instance [1, p. 278]). Let $ch(\eta)$ be the Chern character of $\eta$. The differential Riemann-Roch Theorem (cf. Atiyah and Hirzebruch [1, Corollary 1]) tells us that the rational number $\langle \hat{A}(M) \cdot e^{\frac{\xi}{2}} \cdot ch(\eta), [M] \rangle$ is an integer. Therefore, we have

$$24 \langle \hat{A}(M) \cdot e^{\frac{\xi}{2}} \cdot ch(\eta), [M] \rangle \equiv 0 \mod 3,$$

$$6 \langle \hat{A}(M) \cdot e^{\frac{\xi}{2}} \cdot ch(\eta), [M] \rangle \equiv 0 \mod 2.$$

Note that $\eta \in [M, BSU]$, and $\hat{A}(M)$ and $ch(\eta)$ can be expressed as

$$\hat{A}(M) = 1 - \frac{p_1(M)}{24} + \frac{-4p_2(M) + 7p_1^2(M)}{5760},$$

$$ch(\eta) = -c_2(\eta) + \frac{1}{2}c_3(\eta) + \frac{1}{12}[c_2^2(\eta) - 2c_4(\eta)],$$

respectively by the definitions. Then a straightforward calculation shows that

$$\langle \hat{A}(M) \cdot e^{\frac{\xi}{2}} \cdot ch(\eta), [M] \rangle = -\langle c_4(\eta)/6, [M] \rangle$$

$$+ \langle [2c_2^2(\eta) + p_1(M)c_2(\eta) - 3c_2^2(\eta)]/24 + c_3(\eta)/4, [M] \rangle.$$

Thus, the lemma follows by substituting this identity into the congruences above. 

Denote by $M^\circ := M - \text{int}(D^8)$ the space obtained from $M$ by removing the interior of a small 8-disc in $M$. Let $p : M \to S^8$ be the map by collapsing $M^\circ$ to the basepoint, and $i : M^\circ \to M$ be the inclusion map.
Lemma 3.5. For any cohomology class $w \in H^8(M)$ with $\langle w, [M] \rangle \equiv 0 \mod 6$, there exists a stable complex vector bundle $\xi'$ over $S^8$, such that $\xi = p^*(\xi')$ is a stable complex vector bundle over $M$ satisfying $c_1(\xi) = c_2(\xi) = c_3(\xi) = 0$, and $c_4(\xi) = w$.

Proof. Since the degree of $p$ is one, for any $w \in H^8(M)$ with $\langle w, [M] \rangle \equiv 0 \mod 6$, there exists a class $w' \in H^8(S^8)$ such that $p^*(w') = w$, and

$$\langle w', [S^8] \rangle = \langle w, [M] \rangle \equiv 0 \mod 6.$$ 

Therefore, there must exist a stable complex vector bundle $\xi'$ over $S^8$ such that $c_4(\xi') = w'$ by Peterson [15, Theorem 5.1]. It follows that $\xi = p^*(\xi')$ is a stable complex vector bundle over $M$ satisfying $c_1(\xi) = c_2(\xi) = c_3(\xi) = 0$, and $c_4(\xi) = w$. \hfill $\square$

Proof of Theorem 3.2. Suppose that $(u_2, u_3, u_4) \in \text{Im} C_{24}$, i.e., there exists a stable complex vector bundle $\eta \in [M, BU]$ such that $c_1(\eta) = 0$, $u_2 = c_2(\eta)$, $u_3 = c_3(\eta)$ and $u_4 = c_4(\eta)$. As the second Stiefel-Whitney class $w_2(\eta) = 0$, condition (1) in Theorem 3.2 follows from Wu’s explicit formula

$$\text{Sq}^2(w_4) = w_2w_4 + w_6$$

referring to the universal Stiefel-Whitney classes (see Milnor and Stasheff [12, p. 94, Problem 8-A]), and the conditions (2) and (3) are deduced directly from Lemma 3.4.

Conversely, suppose that the cohomology classes $u_i \in H^{2i}(M)$, $2 \leq i \leq 4$ satisfy the conditions (1)-(3) in Theorem 3.2. We will consider the map

$$C_{23} = (c_2, c_3) : BU \rightarrow K(\mathbb{Z}, 4) \times K(\mathbb{Z}, 6)$$

given by the universal Chern classes $c_2$ and $c_3$. Let $F$ denote the homotopy fiber of $C_{23}$. It can be deduced easily from the homotopy sequence of this fiber space that

$$\pi_i(F) \cong \begin{cases} 0, & i \leq 7, i \neq 5; \\ \mathbb{Z}/2, & i = 5; \\ \pi_i(BSU), & i \geq 8. \end{cases}$$

Therefore, the Postnikov resolution of the map $C_{23}$ through dimension 9 is:

$$\begin{array}{ccc}
BSU & \xrightarrow{C_{23}} & K \\
\downarrow h & & \downarrow q \\
E & \xrightarrow{E} & K(\mathbb{Z}, 9) \\
\downarrow k & & \\
K & & K(\mathbb{Z}/2, 6),
\end{array}$$

where $K = K(\mathbb{Z}, 4) \times K(\mathbb{Z}, 6)$ and $q$ is a principal fibration with fiber $K(\mathbb{Z}/2, 5)$ and classifying map $k$. We will denote also by $k \in H^6(K; \mathbb{Z}/2)$ the class represented by the map $k$.

Denote by $l_n \in H^n(K(\mathbb{Z}, n))$ the fundamental class. Note that $K(\mathbb{Z}, 6)$ is 5-connected, and $H^6(K(\mathbb{Z}, 4); \mathbb{Z}/2) \cong \mathbb{Z}/2$ and $H^6(K(\mathbb{Z}, 6); \mathbb{Z}/2) \cong \mathbb{Z}/2$ are
generated by $\text{Sq}^2(\rho_2(l_4))$ and $\rho_2(l_6)$ respectively (see for instance [8, p. 568, Theorem 5.36]). It follows that $H^6(K; \mathbb{Z}/2) \cong (\mathbb{Z}/2)^2$ is generated by $\text{Sq}^2(\rho_2(l_4))$ and $\rho_2(l_6)$ by the Künneth formula. Wu’s explicit formula above implies that

$$k = \text{Sq}^2(\rho_2(l_4)) + \rho_2(l_6).$$

As $\dim M = 8$ and $k = 0$ by condition (1) in Theorem 3.2, it follows from the resolution above that there exists a stable complex vector bundle $\eta'$ over $M$ such that $c_1(\eta') = 0$, $c_2(\eta') = u_2$ and $c_3(\eta') = u_3$. And it follows from Lemma 3.4 that the fourth Chern class of $\eta'$ satisfies the following two congruences:

$$\langle c_4(\eta'), [M] \rangle \equiv \langle p_1(M)u_2 - u_2^2, [M] \rangle \mod 3,$$

$$\langle c_4(\eta'), [M] \rangle \equiv \langle [2u_2^2 + p_1(M)u_2 - 3c^2u_2]/4 - cu_3/2, [M] \rangle \mod 2.$$

Combining these congruences with the conditions (2) and (3) in Theorem 3.2, we get that

$$\langle u_4 - c_4(\eta'), [M] \rangle \equiv 0 \mod 3,$$

$$\langle u_4 - c_4(\eta'), [M] \rangle \equiv 0 \mod 2.$$

That is, $\langle u_4 - c_4(\eta'), [M] \rangle \equiv 0 \mod 6$. Thus, there exists a stable complex vector bundle $\xi$ over $M$ such that $c_1(\xi) = c_2(\xi) = c_3(\xi) = 0$ and $c_4(\xi) = u_4 - c_4(\eta')$ by Lemma 3.5. Let $\eta = \eta' \oplus \xi$ be the Whitney sum of $\eta'$ and $\xi$. Then we have $c_1(\eta) = 0$, $c_2(\eta) = u_2$, $c_3(\eta) = u_3$, and $c_4(\eta) = c_4(\eta') + c_4(\xi) = u_4$. It follows that $(u_2, u_3, u_4) \in \text{Im} C_{24}$ and the proof is complete. \hfill $\Box$

The remainder of this section will be devoted to the proof of Theorem 3.1 (B). Recall that we have identified $\text{Vect}_{\mathbb{C}}^9(M) = [M, BU]$ by (2.2). We will divide the proof into the following three lemmas.

For any CW-complex $X$, denote by $V_0(X)$ the set of stable isomorphism classes of complex vector bundles over $X$ with trivial Chern classes. Obviously, $V_0(X)$ is a subgroup of $[X, BU] \subset [X, BU]$.

**Lemma 3.6.** For any $(u_1, u_2, u_3, u_4) \in \text{Im} C_{14}$, there is a bijection between $C_{14}^{-1}(u_1, u_2, u_3, u_4)$ and $V_0(M)$. Therefore, there is a one-to-one correspondence between $[M, BU]$ and $V_0(M) \times \text{Im} C_{14}$.

**Proof.** For any $u = (u_1, u_2, u_3, u_4) \in \text{Im} C_{14}$, let $\eta_u$ be a fixed stable complex vector bundle over $M$ such that $C_{14}(\eta_u) = u = (u_1, u_2, u_3, u_4)$. Let $\Psi : C_{14}^{-1}(u_1, u_2, u_3, u_4) \to V_0(M)$ be the map given by $\Psi(\eta) = \eta - \eta_u$. Clearly, $\Psi$ is bijective. Note that $V_0(M)$ does not depend on $u = (u_1, u_2, u_3, u_4)$. It follows that there is a one-to-one correspondence between $[M, BU]$ and $V_0(M) \times \text{Im} C_{14}$. \hfill $\Box$

**Lemma 3.7.** The induced homomorphism $i^* : V_0(M) \to V_0(M^\circ)$ is bijective.
Proof. Consider the following commutative diagram:

\[
\begin{array}{ccc}
V_0(M) & \xrightarrow{i^*} & V_0(M^\circ) \\
\downarrow & & \downarrow \\
[S^8, BSU] & \xrightarrow{p^*} & [M, BSU] & \xrightarrow{i^*} & [M^\circ, BSU] \xrightarrow{} [S^7, BSU],
\end{array}
\]

where \( p^* \) and \( i^* \) are the induced homomorphisms, and the bottom sequence is exact.

For any \( \eta \in V_0(M) \) with \( i^*(\eta) = 0 \), there exists \( \xi \in [S^8, BSU] \) such that \( p^*(\xi) = \eta \) by the exactness of the bottom sequence. Since the degree of \( p \) is one, it deduced from \( c_4(\eta) = 0 \) that we must have \( c_4(\xi) = 0 \). Hence \( \xi \) is trivial by Peterson [15, Theorem 3.2], and so is \( \eta \). Thus \( i^*: V_0(M) \to V_0(M^\circ) \) is injective.

Secondly, we prove that \( i^*: V_0(M) \to V_0(M^\circ) \) is surjective. It follows from \([S^7, BSU] = 0 \) and the exactness of the bottom sequence that the induced homomorphism \( i^*: [M, BSU] \to [M^\circ, BSU] \) is surjective. Therefore, for any \( \eta_1 \in V_0(M^\circ) \subset [M^\circ, BSU] \), there exists \( \eta_2 \in [M, BSU] \) such that \( i^*(\eta_2) = \eta_1 \).

Note that the induced homomorphism \( i^*: H^q(M) \to H^q(M^\circ) \) is an isomorphism for any \( q < 7 \). Then \( c_i(\eta_1) = 0 \) for \( 1 \leq i \leq 3 \) implies that \( c_1(\eta_2) = c_2(\eta_2) = c_3(\eta_2) = 0 \). Now Theorem 3.1 (A) yields that \( (c_4(\eta_2), [M]) \equiv 0 \) mod 6, and it follows from Lemma 3.5 that there exists \( \xi \in [M, BU] \) such that \( \xi = p^*(\xi') \) for some \( \xi' \in [S^8, BU] \), and \( c_1(\xi) = c_2(\xi) = c_3(\xi) = 0, c_4(\xi) = -c_4(\eta_2) \). Set \( \eta = \eta_2 \oplus \xi \). It follows that \( \eta \in V_0(M) \), and \( i^*(\eta) = i^*(\eta_2) \oplus i^*p^*(\xi') = \eta_1 \), which implies that \( i^*: V_0(M) \to V_0(M^\circ) \) is surjective, and the proof is complete.  

\[\square\]

Lemma 3.8. There is a one-to-one correspondence between \( V_0(M^\circ) \) and \( \mathcal{B}_M = \frac{\beta(H^3(M;\mathbb{Z}/2))}{\beta(Sq^2(p_2(H^3(M))))} \).

Proof. Let us first recall from the proof of Theorem 3.2 that the Postnikov resolution of the fiber space \( C_{23} = (c_2, c_3): BSU \to K(\mathbb{Z}, 4) \times K(\mathbb{Z}, 6) \) through dimension 9 is as follows:

\[
\begin{array}{ccc}
E & \xrightarrow{h} & K(\mathbb{Z}, 9) \\
\downarrow q & & \downarrow \\
BSU & \xrightarrow{C_{23}} & K & \xrightarrow{k} & K(\mathbb{Z}/2, 6),
\end{array}
\]

where \( q \) is the principal fibration with fiber \( K(\mathbb{Z}/2, 5) \) and classifying map \( k \).

Therefore, we have the commutative diagram

\[
\begin{array}{ccc}
[M^\circ, BSU] & \xrightarrow{h_*} & [M^\circ, K(\mathbb{Z}/2, 5)] & \xrightarrow{q_*} & [M^\circ, K] & \xrightarrow{k_*} & [M^\circ, K(\mathbb{Z}/2, 6)],
\end{array}
\]

where \( h_*, q_*, C_{23*} \) and \( k_* \) are the induced maps, \( (\Omega k)_* \) is the induced homomorphism and where the bottom sequence is an exact sequence of sets.
Since the homomorphisms $h_* : \pi_r(BSU) \to \pi_r(E)$, $r \leq 7$, are isomorphic by the construction of the Postnikov resolution (see Section 2), note that $M^\circ$ can be regard as the $7$-skeleton of $M$, it follows from Lemma 2.7 that $h_* : [M^\circ, BSU] \to [M^\circ, E]$ is bijective. Then there is a one-to-one correspondence between $V_0(M^\circ) = \text{Ker } C_{23*}$ and $\text{Ker } q_*$, and hence a bijection between $V_0(M^\circ)$ and $\text{Cok } (\Omega k)_*$ by Lemma 2.6. Therefore, it remains to prove that $\text{Cok } (\Omega k)_*$ is isomorphic to $\mathcal{B}_M = \frac{\beta(H^5(M; \mathbb{Z}/2))}{\beta(\text{Sq}^2(\rho_2(H^3(M))))}$.

Recall that we have $k = \text{Sq}^2(\rho_2(l_4)) + \rho_2(l_6) \in H^6(K; \mathbb{Z}/2)$. Then since $K = K(\mathbb{Z}, 4) \times K(\mathbb{Z}, 6)$ is $3$-connected and $\Omega K = K(\mathbb{Z}, 3) \times K(\mathbb{Z}, 5)$, it follows from Lemma 2.8 that the cohomology suspension $\sigma : H^6(K; \mathbb{Z}/2) \to H^5(\Omega K; \mathbb{Z}/2)$ is an isomorphism. Therefore, it follows easily from the definition of cohomology suspension that

$$\Omega k = \sigma(k) = \text{Sq}^2(\rho_2(l_3)) + \rho_2(l_5).$$

This implies that the homomorphism $(\Omega k)_* : [M^\circ, \Omega K] \to [M^\circ, K(\mathbb{Z}/2, 5)]$ is just the homomorphism $(\Omega k)_* : H^3(M^\circ) \times H^5(M^\circ) \to H^5(M ; \mathbb{Z}/2)$ given by

$$(\Omega k)_*(y, z) = \text{Sq}^2(\rho_2(y)) + \rho_2(z) \quad (2.1)$$

for any $(y, z)$ in $H^3(M^\circ) \times H^5(M^\circ)$. Note that the homomorphism $i^* : H^r(M; G) \to H^r(M^\circ; G)$ is an isomorphism for any $r < 7$. Then it follows from (2.1) and the Bockstein sequence (1.1) that

$$\text{Cok } (\Omega k)_* = \frac{H^5(M^\circ; \mathbb{Z}/2)}{\rho_2(H^5(M^\circ)) + \text{Sq}^2(\rho_2(H^3(M^\circ)))}$$

which completes the proof. \qed

4. Proof of theorem 1.2

By the identifications (2.1) and (2.2), we will regard the map $I_* : \text{Vect}_C^3(M) \to \text{Vect}_C^4(M)$ defined in section 1 as the map $I_3_* : [M, BU(3)] \to [M, BU]$ induced by $I_3 : BU(3) \to BU$ (see Section 2). In this section, Theorem 1.2, restated as Theorem 4.1 below, will be proved by analysing the Postnikov resolution of $I_3$. 
Theorem 4.1. (Theorem 1.2) Let $M$ be an 8-dimensional closed oriented spin$^c$ manifold.

(A) For any $\eta \in [M, BU]$, the necessary and sufficient condition for $\eta$ to lie in the image of $I_3$ is

$$c_4(\eta) = 0.$$ 

(B) If $u = (u_1, u_2, u_3) \in \text{Im } C_{13}$, then $C_{13}^{-1}(u_1, u_2, u_3)$ is equivalent, as a set, to $\mathcal{B}_M \times \mathcal{Z}_{M,u}$.

Proof of Theorem 4.1 (A). Consider the Postnikov resolution of the canonical map $I_3 : BU(3) \to BU$.

Its homotopy fiber is $U/U(3)$. Note that the canonical homomorphism $\pi_r(U(5)) \to \pi_r(U)$ is an isomorphism for any $r \leq 9$. Then we have $\pi_r(U/U(3)) \cong \pi_r(U(5)/U(3))$ for $r \leq 9$. Now since the canonical homomorphism $\pi_r(U(3)) \cong \pi_r(U(5))$ is an isomorphism for any $r \leq 5$ and $\pi_6(U(5)) = \pi_6(U) = 0$, it follows from the homotopy sequence of the fibration $U(3) \to U(5) \to U(5)/U(3)$ that $\pi_r(U(5)/U(3)) = 0$ for $r \leq 6$. In addition, Gilmore [7, pp. 630 - 631] tells us that

$$\pi_r(U(5)/U(3)) \cong \begin{cases} \mathbb{Z}, & r = 7, 9; \\ 0, & r = 8. \end{cases}$$

Summarizing the facts above, we have

$$\pi_r(U/U(3)) \cong \begin{cases} 0, & r \leq 8, r \neq 7; \\ \mathbb{Z}, & r = 7, 9. \end{cases}$$

Therefore, the Postnikov resolution of the map $I_3$ through dimension 10 can be shown as

$$BU(3) \xrightarrow{I_3} BU \xrightarrow{k} K(\mathbb{Z}, 8).$$

Here $q$ is a principal fibration with fiber $K(\mathbb{Z}, 7)$ and $k$ as the classifying map. Since the kernel of the induced homomorphism $I_3^*: H^8(BU) \to H^8(BU(3))$ is generated by the 4-th universal Chern class $c_4$, it follows that $k = c_4 \in H^8(BU)$. Thus, recall that we have $\dim M = 8$, the proof is complete. \hfill \Box

Remark 4.2. I am very grateful to the referee for pointing to me that Theorem 1.2 (A) can also be proved as follows. For any $\eta \in \text{Vect}_{\mathbb{C}}^4(M)$, denote by $\eta_\mathbb{R}$ the underlying real vector bundle of $\eta$ and $e(\eta_\mathbb{R})$ the Euler class of $\eta_\mathbb{R}$. Since $e(\eta_\mathbb{R}) = 0$ if and only if $\eta_\mathbb{R}$ admits a nowhere-zero section, it follows from $c_4(\eta) = e(\eta_\mathbb{R})$ that $c_4(\eta) = 0$ if and only if $\eta$ has a nowhere-zero section, which is equivalent to $\eta \in \text{Im } I_*$.
Proof of Theorem 4.1 (B). Let us consider the induced map

\[ I_{3*} : [M, BU(3)] \to [M, BU]. \]

We claim that for any \( \eta \in \text{Im } I_{3*} \), there is a bijection between \( I_{3*}^{-1}(\eta) \) and

\[ \Xi_{M,u} := \frac{H^7(M)}{\{ f^*(\gamma_7) + u_1 f^*(\gamma_5) + u_2 f^*(\gamma_3) + u_3 f^*(\gamma_1) \mid f \in [M, U] \}}, \]

where \( u = (u_1, u_2, u_3) = (c_1(\eta), c_2(\eta), c_3(\eta)) \), and \( \gamma_{2r-1} \in H^{2r-1}(U), 1 \leq r \leq 4 \), are the generators of the exterior algebra

\[ H^*(U) \cong \Lambda(\gamma_1, \gamma_3, \gamma_5, \gamma_7, \cdots). \]

We will give the proof of this claim later. Note that we have the following commutative diagram

\[
\begin{array}{ccc}
[M, BU(3)] & \xrightarrow{C_{13}} & H^2(M) \times H^4(M) \times H^6(M) \\
\downarrow I_{3*} & & \downarrow J \\
[M, BU] & \xrightarrow{C_{14}} & H^2(M) \times H^4(M) \times H^6(M) \times H^8(M)
\end{array}
\]

by the definitions of \( C_{13}, C_{14} \) and \( I_{3*} \), where \( J \) is the homomorphism given by

\[ J(x_1, x_2, x_3) = (x_1, x_2, x_3, 0) \]

for any \( x_i \in H^{2i}(M), 1 \leq i \leq 3 \). For any \((u_1, u_2, u_3) \in \text{Im } C_{13}, \) we have \( C_{14}^{-1}(u_1, u_2, u_3, 0) \subset \text{Im } I_{3*} \) by Theorem 4.1 (A), and

\[ C_{13}^{-1}(u_1, u_2, u_3) = I_{3*}^{-1}(C_{14}^{-1}(u_1, u_2, u_3, 0)) \]

by the commutative diagram above. Then since there is a bijection between \( C_{14}^{-1}(u_1, u_2, u_3, 0) \) and \( \mathcal{B}_M \) by Theorem 1.1, and the quotient group \( \Xi_{M,u} \) depends only on \( u = (u_1, u_2, u_3) \), it follows from the claim above that there is a bijection between

\[ C_{13}^{-1}(u_1, u_2, u_3) \quad \text{and} \quad \mathcal{B}_M \times \Xi_{M,u}, \]

which completes the proof.

Now, let us prove the claim above.

According to the Postnikov resolution of \( I_3 \) as in the proof of Theorem 4.1 (A), we have the following commutative diagram

\[
\begin{array}{ccc}
[M, BU(3)] & \xrightarrow{(\Omega c_4)_*} & [M, K(\mathbb{Z}, 7)] \\
\downarrow h_* & & \downarrow q_* \\
[M, U] & \xrightarrow{(\Omega c_4)_*} & [M, K(\mathbb{Z}, 8)]
\end{array}
\]
where the bottom sequence is exact. By the construction of the Postnikov resolution (see Section 2), the induced homomorphism \( h_r : \pi_r(\text{BU}(3)) \to \pi_r(E) \) is an isomorphism for any \( r \leq 8 \). Hence, it follows from Lemma 2.7 that \( h_r : [M, \text{BU}(3)] \to [M, E] \) is a bijection. Therefore, for any \( \eta \in [M, \text{BU}] \) with \( \eta \in \text{Im} \ I_3 \), i.e., \( c_4(\eta) = 0 \), there is a one-to-one correspondence between \( I_3^{-1}(\eta) \) and \( q_r^{-1}(\eta) \). Moreover, since \( q \) is a principal fibration with fiber \( K(\mathbb{Z}, 7) \) and \( c_4 \) as the classifying map, it follows from Lemma 2.5 that there is a bijection between

\[
q_r^{-1}(\eta) \quad \text{and} \quad \text{Cok} \ \Delta(c_4, \eta).
\]

Hence it remains to prove that \( \text{Cok} \ \Delta(c_4, \eta) = \mathcal{I}_{M, u} \).

Let \( \mu : \text{BU} \times \text{BU} \to \text{BU} \) be the canonical \( H \)-structure of \( \text{BU} \), determined by the Whitney sum construction. The Whitney sum formula tell us that

\[
\mu^*(c_4) = c_4 \times 1 + 1 \times c_4 + c_1 \times c_3 + c_2 \times c_2 + c_3 \times c_1.
\]

Now recall that we always have (cf. [13, p. 148, Theorem 6.5])

\[
\sigma(c_r) = \gamma_{2r-1},
\]

for any positive integer \( r \). Then, by Lemma 2.3, \( \Delta(c_4, \eta) : [M, U] \to [M, K(\mathbb{Z}, 7)] \) is the homomorphism given by

\[
\Delta(c_4, \eta)(f) = f^*(\sigma(c_4)) + f^*(\sigma(c_1))\eta^*(c_3) + f^*(\sigma(c_2))\eta^*(c_2) + f^*(\sigma(c_3))\eta^*(c_1)
\]

\[
= f^*(\gamma_7) + f^*(\gamma_1) c_3(\eta) + f^*(\gamma_3) c_2(\eta) + f^*(\gamma_5) c_1(\eta),
\]

for any \( f \in [M, U] \). It follows that

\[
\text{Cok} \ \Delta(c_4, \eta) = \mathcal{I}_{M, u} := \frac{H^7(M)}{\{f^*(\gamma_7) + u_1 f^*(\gamma_5) + u_2 f^*(\gamma_3) + u_3 f^*(\gamma_1) | f \in [M, U]\}}.
\]

The claim is proved. \( \square \)

**Remark 4.3.** In the proof of Theorem 4.1, the Postnikov resolution of the map \( I_3 : \text{BU}(3) \to \text{BU} \) is investigated. One may ask why we do not consider the Postnikov resolution of the canonical map \( I : \text{BU}(3) \to \text{BU}(4) \). In fact, since the homotopy fiber of \( I \) is \( U(4)/U(3) = S^7 \) and we have \( \pi_7(S^7) \cong \mathbb{Z} \) and \( \pi_8(S^7) \cong \pi_9(S^7) \cong \mathbb{Z}/2 \), the Postnikov resolution of \( I \) through dimension 10 is:

\[
\begin{array}{cccc}
E_2 & \longrightarrow & K(\mathbb{Z}/2, 10) \\
\downarrow q_2 & & \\
E_1 & \longrightarrow & K(\mathbb{Z}/2, 9) \\
\downarrow q_1 & & \\
\text{BU}(3) & \longrightarrow & \text{BU}(4) & \longrightarrow & K(\mathbb{Z}, 8),
\end{array}
\]

with \( h_1 \) and \( h_2 \) are 8- and 9-equivalences respectively. Note that we have \( \text{dim} \ M = 8 \). Although Theorem 4.1 (A) can be deduced from this resolution, it may not be easy to classify rank 3 complex vector bundles over 8-dimensional spin\(^c\) manifolds by using this Postnikov resolution.
Remark 4.4. Consider the canonical map \( I_2 : BU(2) \to BU \). Let \( M \) be an 8-dimensional manifold, and \( \eta : M \to BU \) be a map. One may conjecture that there must exist a lift \( \xi : M \to BU(2) \) under the assumption that \( c_3(\eta) = c_4(\eta) = 0 \). Unfortunately, this is not true. In fact, the Postnikov resolution of \( I_2 \) has been studied by Switzer [18], and [18, Theorem 1] tells us that besides \( c_3 \) and \( c_4 \), there are two more obstructions to finding a lift \( \xi \) of \( \eta \). Moreover, there are five obstructions to finding a homotopy between two lifts of \( \eta \).

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Data availability Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

Declarations

Conflict of interest I states that there is no conflict of interest.

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