Fixed Duration Pursuit-Evasion Differential Game with Integral Constraints

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Abstract. We investigate a pursuit-evasion differential game of countably many pursuers and one evader. Integral constraints are imposed on control functions of the players. Duration of the game is fixed and the payoff of the game is infimum of the distances between the evader and pursuers when the game is completed. Purpose of the pursuers is to minimize the payoff and that of the evader is to maximize it. Optimal strategies of the players are constructed, and the value of the game is found. It should be noted that energy resource of any pursuer may be less than that of the evader.

1. Statement of the problem.

The study of two person zero-sum differential games was initiated by Isaacs [16]. Berkovitz [3], Fleming [5], Friedman [6], Hajek [8], Krasovskii, [19], Petrosyan [22], Pontryagin [23], Subbotin [26] and others developed mathematical foundations for the theory of differential games.

Many investigations were devoted to study the differential games with integral constraints; e.g., [1], [2], [4], [7], [9]-[15], [18], [20], [21], [25], [27].

Constructing the optimal strategies, and finding the value of the game are of interest in differential games, e.g., see [10], [11], [17], [24], [26], [27]. Such problems in the case of many pursuers were studied, for example, in [10] and [11]. In [10], a differential game of optimal approach of countably many pursuers to one evader was studied in Hilbert space with geometric constraints on controls of players. In [11], such differential game was studied for inertial players with integral constraints under the assumption that the control resource of the evader less than that of each pursuer.

In the present paper, we also discuss an optimal pursuit problem with countably many pursuers and one evader in Hilbert space $l_2$, and control resource of the evader $\sigma$ can be greater than that of any pursuer. In the space $l_2$ with elements

$$\alpha = (\alpha_1, \alpha_2, ..., \alpha_k, ...), \quad \sum_{k=1}^{\infty} \alpha_k^2 < \infty,$$

and inner product and norm

$$(\alpha, \beta) = \sum_{k=1}^{\infty} \alpha_k \beta_k, \quad ||\alpha|| = \left(\sum_{k=1}^{\infty} \alpha_k^2\right)^{1/2},$$
motions of the pursuers \( P_i \) and the evader \( E \) are described by the equations

\[
P_i : \quad \dot{x}_i = u_i, \quad x_i(0) = x_{i0}, \quad E : \quad \dot{y} = v, \quad y(0) = y_0, \tag{1}
\]

where \( x_i, \ x_0, \ u_i, \ y, \ y_0, \ v \in l_2, \ u_i = (u_{i1}, u_{i2}, ..., u_{ik}, ...) \) is control parameter of the pursuer \( P_i \), and \( v = (v_1, v_2, ..., v_k, ...) \) is that of the evader \( E \); throughout, \( i = 1, 2, ..., m, \ldots \) Let \( \vartheta \) be a fixed time, \( I = \{1, 2, ..., m, ..., \} \), and

\[
H(x_0, r) = \{ x \in l_2 : ||x - x_0|| \leq r \}; \quad S(x_0, r) = \{ x \in l_2 : ||x - x_0|| = r \}
\]

**Definition 1.** A function \( u_i = u_i(t), \ 0 \leq t \leq \vartheta \), with the Borel measurable coordinates \( u_{ik} : [0, \vartheta] \rightarrow R^1, \ k = 1, 2, ..., \), subjected to the condition

\[
\left( \int_0^\vartheta ||u_i(t)||^2 dt \right)^{1/2} \leq \rho_i,
\]

is called an admissible control of the pursuer \( P_i \), where \( \rho_i \) are given positive numbers.

**Definition 2.** A function \( v = v(t), \ 0 \leq t \leq \vartheta \), with the Borel measurable coordinates \( v_k : [0, \vartheta] \rightarrow R^1, \ k = 1, 2, ..., \), subjected to the condition

\[
\left( \int_0^\vartheta ||v(t)||^2 dt \right)^{1/2} \leq \sigma,
\]

is called an admissible control of the evader \( E \), where \( \sigma \) is a given positive number.

If it has been chosen admissible controls \( u_i(\cdot), \ v(\cdot) \) of players, then corresponding to them motions \( x_i(\cdot), \ y(\cdot) \) are defined by formulas

\[
x_i(t) = (x_{i1}(t), x_{i2}(t), ..., x_{ik}(t), ...), \quad y = (y_1(t), y_2(t), ..., y_k(t), ...),
\]

\[
x_{ik}(t) = x_{i0} + \int_0^t u_{ik}(s)ds, \quad y_k(t) = y_0 + \int_0^t v_k(s)ds.
\]

It is not difficult to verify that \( x_i(\cdot), y(\cdot) \in C(0, \vartheta; l_2) \), where \( C(0, \vartheta; l_2) \) is the space of continuous functions \( f(t) = (f_1(t), f_2(t), ..., f_k(t), ...) \in l_2, \ 0 \leq t \leq \vartheta \), with absolutely continuous components \( f_k(t), \ 0 \leq t \leq \vartheta \).

**Definition 3.** A function

\[
U_i(\xi_i, v), \quad U_i : [0, \rho_i^2] \times l_2 \rightarrow l_2,
\]

is called a strategy of the pursuer \( P_i \) if for any admissible control \( v = v(t), \ 0 \leq t \leq \vartheta \), of evader \( E \), the system of equations

\[
\dot{x}_i = U_i(\xi_i, v), \quad x_i(0) = x_{i0}, \quad \dot{\xi}_i = -||U_i(\xi_i, v)||^2, \quad \xi_i(0) = \rho_i^2,
\]

\[
\dot{y} = v, \quad y(0) = y_0,
\]

has a unique solution \( \{x_i(\cdot), \xi_i(\cdot), y(\cdot)\} \), where \( x_i(\cdot), y(\cdot) \in C(0, \vartheta; l_2) \), and \( \xi_i(\cdot) \) is absolutely continuous scalar function on \([0, \vartheta]\). A strategy \( U_i \) is said to be admissible, if every control generated by \( U_i \) is admissible.
Definition 4. Strategies \( U_{i0} \) of the pursuers \( P_i \) are referred to as the optimal strategies if

\[
\inf_{U_1, \ldots, U_m} \Gamma_1(U_1, \ldots, U_m, \ldots) = \Gamma_1(U_{i0}, \ldots, U_{m0}, \ldots),
\]

where

\[
\Gamma_1(U_1, \ldots, U_m, \ldots) = \sup_{v(\cdot)} \inf_{s(\cdot)} ||x_i(\vartheta) - y(\vartheta)||,
\]

\( U_i \) are admissible strategies of the pursuers \( P_i \), and \( v(\cdot) \) is an admissible control of the evader \( E \).

Definition 5. A function

\[
V(x_1, \ldots, x_m, \ldots, y), \ V : l_2 \times \ldots \times l_2 \times \ldots \times l_2 \rightarrow l_2,
\]

is called a strategy of the evader \( E \), if for any admissible controls \( u_i = u_i(t) \), \( 0 \leq t \leq \vartheta \), of the pursuers \( P_i \), the system of equations

\[
\begin{align*}
\dot{x}_k &= u_k, \\
\dot{y} &= V(x_1, \ldots, x_m, \ldots, y), \quad y(0) = y_0,
\end{align*}
\]

has a unique solution \((x_1(\cdot), \ldots, x_m(\cdot), \ldots, y(\cdot))\), \( x_i(\cdot), y(\cdot) \in C(0, \vartheta, l_2) \). A strategy \( V \) is said to be admissible, if every control generated by \( V \) is admissible.

Definition 6. Strategy \( V_0 \) of the evader \( E \) is said to be optimal, if

\[
\sup_V \Gamma_2(V) = \Gamma_2(V_0),
\]

where

\[
\Gamma_2(V) = \inf_{u_1(\cdot), \ldots, u_m(\cdot)} \inf_{s(\cdot)} ||x_i(\vartheta) - y(\vartheta)||,
\]

\( u_i(\cdot) \) are admissible controls of pursuers \( P_i \), \( V \) is admissible control of evader \( E \).

If \( \Gamma_1(U_{i0}, \ldots, U_{m0}, \ldots) = \Gamma_2(V_0) = \gamma \), then we say that the game has the value \( \gamma \) [26].

The problem is to find the optimal strategies \( U_{i0}, V_0 \) of the players \( P_i \) and \( E \), respectively, and the value of the game.

2. An auxiliary game.

The attainability set of the pursuer \( P_i \) from the initial position \( x_{i0} \) up to the time \( \vartheta \), i.e., the set of all points \( x(\vartheta) = x_{i0} + \int_0^\vartheta u_i(s)ds \), where \( u_i(\cdot) \) is an admissible control of the \( i \)th pursuer, is the ball \( H(x_{i0}, \rho_i\sqrt{\vartheta}) \).

Indeed, by the Cauchy-Schwartz inequality we have

\[
||x_i(\vartheta) - x_{i0}|| = || \int_0^\vartheta u_i(s)ds || \leq \int_0^\vartheta ||u_i(s)||ds
\]

\[
\leq \left( \int_0^\vartheta ds \cdot \int_0^\vartheta ||u_i(s)||^2ds \right)^{1/2} \leq \rho_i\sqrt{\vartheta}.
\]

On the other hand, if \( \bar{x} \in H(x_{i0}, \rho_i\sqrt{\vartheta}) \), then for the control

\[
u_i(s) = (\bar{x} - x_{i0})/\vartheta, \quad 0 \leq s \leq \vartheta,
\]
of the pursuer we get \( x_i(\vartheta) = \bar{x} \). Admissibility of this control follows from the relations

\[
\int_{0}^{\vartheta} ||u_i(s)||^2 ds = \int_{0}^{\vartheta} \frac{1}{\vartheta^2} ||\bar{x} - x_{i0}||^2 ds \leq \frac{1}{\vartheta} \cdot \rho_i^2 \vartheta = \rho_i^2.
\]

In a similar fashion we can show that the attainability set of the evader \( E \) from the initial position \( y_0 \) up to the time \( \vartheta \) is the ball \( H(y_0, \sigma \sqrt{\vartheta}) \).

For the simplicity in this section we will drop index \( i \), i.e., we use designations \( \rho_i = \rho, x_{i0} = x_0, x_i = x \). Let \( x_0 \neq y_0 \) and

\[
X = \{ z \in l_2 : 2(y_0 - x_0, z) \leq (\rho^2 - \sigma^2)\vartheta + ||y_0||^2 - || x_0 ||^2 \}, \quad e = \frac{y_0 - x_0}{||y_0 - x_0||}.
\]

Consider the following game with one pursuer

\[
P : \dot{x} = u, \quad x(0) = x_0, \quad E : \dot{y} = v, \quad y(0) = y_0.
\]

The goal of the pursuer \( P \) is to realize the equality \( x(\tau) = y(\tau) \) at some \( \tau, \quad 0 \leq \tau \leq \vartheta \), and that of the evader \( E \) is opposite. Construct the strategy of the pursuer as follows:

\[
u(t) = \frac{1}{\vartheta} (y_0 - x_0) + v(t), \quad 0 \leq t \leq \vartheta.
\]

**Lemma 1.** If \( y(\vartheta) \in X \), then the strategy (3) of the pursuer \( P \) is admissible and ensures the equality \( x(\vartheta) = y(\vartheta) \) in the game (2).

We show that if \( y(\vartheta) \in X \), then the strategy (3) is admissible. As

\[
y(\vartheta) = y_0 + \int_{0}^{\vartheta} v(s) ds,
\]

then from the inequality

\[
2(y_0 - x_0, y(\vartheta)) \leq (\rho^2 - \sigma^2)\vartheta + ||y_0||^2 - || x_0 ||^2
\]

we have

\[
2 \int_{0}^{\vartheta} (y_0 - x_0, v(s)) ds \leq (\rho^2 - \sigma^2)\vartheta - || x_0 - y_0 ||^2.
\]

Hence from (3) we have

\[
\int_{0}^{\vartheta} ||u(s)||^2 ds = \frac{1}{\vartheta} ||y_0 - x_0||^2 + \frac{2}{\vartheta} \int_{0}^{\vartheta} (y_0 - x_0, v(s)) ds + \int_{0}^{\vartheta} ||v(s)||^2 ds
\]

\[
\leq \frac{1}{\vartheta} ||y_0 - x_0||^2 + \frac{1}{\vartheta} \left( (\rho^2 - \sigma^2)\vartheta - || x_0 - y_0 ||^2 \right) + \sigma^2 = \rho^2,
\]
and so strategy (3) is admissible. Then
\[ x(\vartheta) = x_0 + \int_0^\vartheta u(s)ds = x_0 + y_0 - x_0 + \int_0^\vartheta v(s)ds = y(\vartheta). \]

This proves lemma.

3. Main Result.

Now we consider the game (1). Define

\[ \gamma = \inf\{l \geq 0 : H(y_0, \sigma \sqrt{\vartheta}) \subset \bigcup_{i=1}^{\infty} H(x_{i0}, \rho_i \sqrt{\vartheta} + l)\}. \]  

**Theorem.** If there exists a non-zero vector \( p_0 \in l^2 \) such that \( (y_0 - x_{i0}, p_0) \geq 0 \) for all \( i \in I \), then the number \( \gamma \) defined by the formula (4) is the value of the game (1).

**Proof.** To prove this theorem we need the following statement (see, [10], Assertions 4 and 5 in Appendix)

**Lemma 2.** Suppose there exists a non-zero vector \( p_0 \in l^2 \) such that \( (y_0 - x_{i0}, p_0) \geq 0 \) for all \( i \in I \).

(a) If \( H(y_0, r) \subset \bigcup_{i=1}^{\infty} H(x_{i0}, R_i) \) then

\[ H(y_0, r) \subset \bigcup_{i=1}^{\infty} \{y : 2(y_0 - x_{i0}, y) \leq R_i^2 - r^2 + ||y_0||^2 - ||x_{i0}||^2\}. \]

(b) If for any \( \varepsilon > 0 \)

\[ H(y_0, r) \nsubseteq \bigcup_{i \in I} H(x_{i0}, (R_i - \varepsilon)_+) \ (a_+ = \max\{0, a\}), \]

then there exists a point \( \bar{y} \in S(y_0, r) \) such that \( ||\bar{y} - x_{i0}|| \geq R_i \) for all \( i \in I \).

1°. **Construction the strategies of the pursuers.** We introduce fictitious pursuers (FPs) \( z_i \), whose motions are described by the equations

\[ \dot{z}_i = w_i^\varepsilon, \quad z_i(0) = x_{i0}, \quad \left( \int_0^{\vartheta} ||w_i^\varepsilon(s)||^2 ds \right)^{1/2} \leq \bar{p}_i(\varepsilon) = \rho_i + \frac{\gamma}{\sqrt{\vartheta}} + \frac{\varepsilon}{k_i \sqrt{\vartheta}}, \]  

where \( \varepsilon \) is an arbitrary positive number, \( k_i = \max\{1, \rho_i\} \). It can be shown easily that the attainability set of the FP \( z_i \) from the initial position \( x_{i0} \) up to the time \( \vartheta \) is the ball

\[ H(x_{i0}, \bar{p}_i(\varepsilon) \sqrt{\vartheta}) = H(x_{i0}, \rho_i \sqrt{\vartheta} + \gamma + \varepsilon/k_i). \]

We define strategies for FPs \( z_i \) on the time interval \([0, \vartheta]\) as follows.

\[ w_i^\varepsilon(t) = \begin{cases} \frac{1}{\vartheta}(y_0 - x_{i0}) + v(t), & 0 \leq t \leq \tau_i^\varepsilon, \\ 0, & \tau_i^\varepsilon < t \leq \vartheta, \end{cases} \]  

where \( \tau_i^\varepsilon \), \( 0 \leq \tau_i^\varepsilon \leq \vartheta \), is the time for which
\[
\int_0^{\tau_i^\varepsilon} ||w_i^\varepsilon(s)||^2 ds = \bar{\rho}_i^2(\varepsilon),
\]

if such a time exists. We define strategies of the pursuers \(x_i\) by the strategies of the FPs as follows.

\[
u_i(t) = \frac{\bar{\rho}_i}{\bar{\rho}_i} w_i(t), \quad 0 \leq t \leq \vartheta, \quad (7)
\]

where \(\bar{\rho}_i = \bar{\rho}_i(0) = \rho_i + \frac{\gamma}{\sqrt{\vartheta}}\),

\[
w_i(t) = \begin{cases} \frac{1}{\vartheta} (y_0 - x_i^0) + v(t), & 0 \leq t \leq \tau_i, \\ 0, & \tau_i < t \leq \vartheta, \end{cases} \quad (8)
\]

where \(\tau_i, \quad 0 \leq \tau_i \leq \vartheta\), is the time for which

\[
\int_0^{\tau_i^\varepsilon} ||w_i^\varepsilon(s)||^2 ds = \bar{\rho}_i^2(\varepsilon),
\]

that is, \(w_i(t)\) is obtained from (6) at \(\varepsilon = 0\).

As \(\bar{\rho}_i(\varepsilon) > \bar{\rho}_i\), then

\[
\int_0^{\tau_i^\varepsilon} ||w_i^\varepsilon(s)||^2 ds = \bar{\rho}_i^2(\varepsilon) > \bar{\rho}_i^2 = \int_0^{\tau_i} ||w_i(s)||^2 ds,
\]

that is,

\[
\int_0^{\tau_i^\varepsilon} \left\| \frac{y_0 - x_i^0}{\vartheta} + v(t) \right\|^2 dt > \int_0^{\tau_i} \left\| \frac{y_0 - x_i^0}{\vartheta} + v(t) \right\|^2 ds.
\]

Hence \(\tau_i^\varepsilon > \tau_i\).

20. **Guaranteed result for the pursuers.** We shall show that strategies (7) of pursuers guarantee that

\[
\sup_{v(\cdot)} \inf_{\varepsilon \in I} ||y(\vartheta) - x_i(\vartheta)|| \leq \gamma. \quad (9)
\]

In accordance with the definition of the number \(\gamma\) we have

\[
H(y_0, \sigma \sqrt{\vartheta}) \subset \bigcup_{i=1}^{\infty} H(x_i^0, \rho_i \sqrt{\vartheta} + \gamma + \varepsilon/k_i). \quad (10)
\]

Denote

\[
X_i = \{ z : 2(y_0 - x_i^0, z) \leq (\rho_i \sqrt{\vartheta} + \gamma + \varepsilon/k_i)^2 - (\sigma \sqrt{\vartheta})^2 + ||y_0||^2 - ||x_i^0||^2 \}.
\]
According to the condition of the theorem \((y_0 - x_{i0}, p_0) \geq 0\) for all \(i \in I\). Then by Lemma 2 it follows from (10) that
\[
H(y_0, \sigma \sqrt{\vartheta}) \subset \bigcup_{i=1}^{\infty} X_i.
\]
Consequently for the point \(y(\vartheta) \in H(y_0, \sigma \sqrt{\vartheta})\) at some \(s \in I\) we have
\[
2(y_0 - x_{s0}, y(\vartheta)) \leq (\rho_s \sqrt{\vartheta} + \gamma + \varepsilon/k_s)^2 - (\sigma \sqrt{\vartheta})^2 + \|y_0\|^2 - \|x_{s0}\|^2. \tag{11}
\]
If \(x_{s0} = y_0\), then according to (6) the strategy of \(s\)th FP takes the form
\[
w^s_{\tau}(t) = v(t), \; 0 \leq t \leq \vartheta
\]
and from (11) we get \(\rho_s \sqrt{\vartheta} + \gamma + \varepsilon/k_s \geq \sigma \sqrt{\vartheta}\), and so
\[
\int_0^\vartheta \|w^s_{\vartheta}(t)\|^2 dt = \int_0^\vartheta \|v(t)\|^2 dt \leq \sigma^2 \leq \left(\rho_s + \frac{\gamma}{\sqrt{\vartheta}} + \frac{\varepsilon}{k_s \sqrt{\vartheta}}\right)^2,
\]
that is, \(w^s_{\tau}(\cdot)\) is admissible and, moreover, \(z_s(\vartheta) = y(\vartheta)\).

If \(x_{i0} \neq y_0\) then by Lemma 1 for the strategies (6) of FP we get \(z_s(\vartheta) = y(\vartheta)\). Then taking into account (5) and (7), we get
\[
\|y(\vartheta) - x_s(\vartheta)\| = \|z_s(\vartheta) - x_s(\vartheta)\| = \left\| \int_0^\vartheta \left( w^s_{\vartheta}(t) - \frac{\rho_s}{\rho_s} w_s(t) \right) dt \right\|
\leq \int_0^\vartheta \|w^s_{\vartheta}(t) - w_s(t)\| dt + \int_0^\vartheta \|w_s(t) - \frac{\rho_s}{\rho_s} w_s(t)\| dt. \tag{12}
\]
We now estimate right-hand side of the inequality (12). Firstly, we show that
\[
\int_0^\vartheta \|w^s_{\tau}(t) - w_{\tau_i}(t)\| dt \leq K \sqrt{\varepsilon} \tag{13}
\]
for all \(i \in I\) and some constant \(K\). Indeed, as we noted above that \(\tau^s_{\tau_i} > \tau_i\) and according to (6) and (8) \(w^s_{\tau_i}(t) = w_{\tau_i}(t)\) for \(0 \leq t \leq \tau_i; w_{\tau_i}(t) = 0\) for \(t > \tau_i\), \(w^s_{\tau_i}(t) = 0\) for \(t > \tau^s_{\tau_i}\), then we have
\[
\int_0^\vartheta \|w^s_{\tau_i}(t) - w_{\tau_i}(t)\| dt = \int_0^{\tau_i} \|w^s_{\tau_i}(t) - w_{\tau_i}(t)\| dt + \int_{\tau_i}^{\tau^s_{\tau_i}} \|w^s_{\tau_i}(t) - w_{\tau_i}(t)\| dt
\]
\[
+ \int_{\tau^s_{\tau_i}}^{\vartheta} \|w^s_{\tau_i}(t) - w_{\tau_i}(t)\| dt = \int_0^{\tau_i} \|w^s_{\tau_i}(t) - w_{\tau_i}(t)\| dt \leq \sqrt{\tau^s_{\tau_i} - \tau_i} \left( \int_{\tau_i}^{\tau^s_{\tau_i}} \|w^s_{\tau_i}(t)\|^2 dt \right)^{1/2}
\]
≤ \sqrt{\vartheta} \left( \int_0^{\tau_i} ||w_i^e(t)||^2 dt - \int_0^{\tau_i} ||w_i^e(t)||^2 dt \right)^{1/2} = \sqrt{\vartheta} \left( \hat{\rho}_i^2(\varepsilon) - \hat{\rho}_i^2 \right)^{1/2}

\tau_i \int_0^{\tau_i} ||w_i^e(t)||^2 dt \leq \tau_i \int_0^{\tau_i} ||w_i^e(t)||^2 dt \leq K \sqrt{\varepsilon},

where \( K \) does not depend on \( i \). We note that the sequence \( \rho_1, ..., \rho_m, ... \) may be unbounded or their infimum may equal to 0. Thus, we have shown the inequality (13). For the second integral of (12) we have

\[ \left| \int_0^{\vartheta} \left( 1 - \frac{\rho_s}{\hat{\rho}_s} \right) w_s(t) dt \right| \leq \left( 1 - \frac{\rho_s}{\hat{\rho}_s} \right) \int_0^{\vartheta} ||w_s(t)|| dt \leq \left( 1 - \frac{\rho_s}{\hat{\rho}_s} \right) \sqrt{\vartheta} \hat{\rho}_s = \gamma. \]

It follows then from (12) that

\[ ||y(\vartheta) - x_s(\vartheta)|| \leq \gamma + K \sqrt{\varepsilon}. \]

Thus, if pursuers use strategies (7), then inequality (9) holds. So the result \( \gamma \) is guaranteed for the pursuers.

3. Guaranteed result for the evader. We construct a strategy for the evader, which provides the inequality

\[ \inf_{u_1(\cdot),...,u_m(\cdot),...} \inf_{\vartheta \in I} ||y(\vartheta) - x_i(\vartheta)|| \geq \gamma, \tag{14} \]

where \( u_1(\cdot),...,u_m(\cdot),... \) are arbitrary admissible controls of pursuers.

If \( \gamma = 0 \), then for any admissible control of the evader validity of (14) is clear. Let \( \gamma > 0 \). Then by definition of the number \( \gamma \), for any \( \varepsilon > 0 \), the set \( \bigcup_{i=1}^{\infty} H(x_{i0}, \rho_i \sqrt{\vartheta} + \gamma - \varepsilon) \) does not contain the ball \( H(y_0, \sigma \sqrt{\vartheta}) \). Consequently, by Lemma 2 there exists a point \( \tilde{y} \in S(y_0, \sigma \sqrt{\vartheta}) \) such that \( ||\tilde{y} - x_{i0}|| \geq \rho_i \sqrt{\vartheta} + \gamma \). Hence

\[ ||\tilde{y} - x_{i}(\vartheta)|| \geq ||\tilde{y} - x_{i0}|| - ||x_{i}(\vartheta) - x_{i0}|| \geq \rho_i \sqrt{\vartheta} + \gamma - \rho_i \sqrt{\vartheta} = \gamma. \]

The control

\[ v(t) = \frac{1}{\vartheta} (\tilde{y} - y_0), \quad 0 \leq t \leq \vartheta, \]

supplies validity of the inequality (14), since for this control we have

\[ y(\vartheta) = y_0 + \int_0^{\vartheta} v(s) ds = \tilde{y}. \]

Now, we give an illustrative example.
Example. Let $\rho_i = 1$, $v = 9$, $\sigma = 4$ in the game (1). We consider the following initial positions

$$x_{i0} = (0, ..., 0, 5, 0, ...), \quad y_0 = (0, 0, ...)$$

of the players, where number 5 is $i$th coordinate of the point $x_{i0}$. Note that $\rho_i \sqrt{v} = 3$, $\sigma \sqrt{v} = 12$. We show that the value of the game is $\gamma = 10$. It is sufficient to show that

1) for any $\varepsilon > 0$ the inclusion

$$H(O, 12) \subset \bigcup_{i=1}^{\infty} H(x_{i0}, 13 + \varepsilon),$$

holds, where $O$ is the origin;

2) the ball $H(O, 12)$ is not contained in the set $\bigcup_{i=1}^{\infty} H(x_{i0}, 13)$.

Indeed, let $z = (z_1, z_2, ...)$ be an arbitrary point of the ball $H(O, 12)$. So $\sum_{i=1}^{\infty} z_i^2 \leq 144$. Then either the vector $z$ has a nonnegative coordinate or all the coordinates of the vector $z$ are negative. In the former case, $z$ has a non negative coordinate $z_k$. Then

$$||z - x_{k0}|| = \left( z_1^2 + ... + z_{k-1}^2 + (5 - z_k)^2 + z_{k+1}^2 + ... \right)^{1/2}$$

$$= \left( \sum_{i=1}^{\infty} z_i^2 + 25 - 10z_k \right)^{1/2} \leq (169 - 10z_k)^{1/2} \leq 13 < 13 + \varepsilon.$$

Hence, $z \in H(x_{k0}, 13 + \varepsilon)$.

In the latter case, since $\sum_{i=1}^{\infty} z_i^2$ is convergent then $z_k \to 0$ as $k \to \infty$ and therefore

$$||z - x_{k0}|| = \left( \sum_{i=1}^{\infty} z_i^2 + 25 - 10z_k \right)^{1/2} \leq (169 - 10z_k)^{1/2} < 13 + \varepsilon$$

for an index $k$.

On the other hand, any point $z \in S(O, 12)$ with negative coordinates does not belong to the set

$$\bigcup_{i=1}^{\infty} H(x_{i0}, 13),$$

since for any number $i$

$$||z - x_{i0}|| = (169 - 10z_i)^{1/2} > 13.$$

Therefore by the theorem the number

$$\gamma = \inf\{l \geq 0 : H(y_0, \sigma \sqrt{v}) \subset \bigcup_{i=1}^{\infty} H(x_{i0}, \rho_i \sqrt{v} + l)\}$$

$$= \inf\{l \geq 0 : H(O, 12) \subset \bigcup_{i=1}^{\infty} H(x_{i0}, 3 + l)\} = 10$$

is the value of the game.
4. Conclusion
We have studied a simple motion pursuit-evasion differential game of fixed duration with countably many pursuers. Control functions of all players are subjected to integral constraints. The value of the game has been found, and the optimal strategies of players have been constructed. The main assumption in the game is existence of a non-zero vector $p_0 \in l_2$ such that $(y_0 - x_{i0}, p_0) \geq 0$ for all $i \in I$. If the initial positions of the players do not satisfy this condition then the stated problem is open. The advantage of the pursuers' strategies in this work with respect to those of the paper [11] is that $\sigma$ can be greater than $\rho_i$ for any $i \in I$.

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