A new approach is proposed for the quantum mechanical problem of the falling of a particle to a singularly attracting center, basing on a black-hole concept of the latter.

The singularity $\sim r^{-2}$ in the potential of the radial Schrödinger equation is considered as an emitting/absorbing center. The two solutions oscillating in the origin are treated as asymptotically free particles, which implies that the singular point $r = 0$ in the Schrödinger equation is treated on the same physical ground as the singular point $r = \infty$. To make this interpretation possible, it is needed that the norm squared of the wave function $\int |\psi(r)|^2 d\mu(r)$ should diverge when $r \to 0$, in other words, the measure used in definition of scalar products should be singular in the origin. Such measure comes into play if the Schrödinger equation is written in the form of the generalized (Kamke) eigenvalue problem for either of two - chosen differently depending on the sign of the energy $E$ - operators, other than Hamiltonian. The Hilbert spaces where these two operators act are used to classify physical states, which are: i) states of "confinement" - continuum of solutions localized near the origin, $E < 0$ - and ii) the states corresponding to the inelastic process of reflection/transmission, i.e. to transitions in-between states localized near the origin and in the infinitely remote region, $E > 0$. The corresponding unitary $2 \times 2$ $S$-matrix is written in terms of the Jost functions. The complete orthonormal sets of eigen-solutions of the two operators are found using "quantization in a box" $(r_L, r_U)$, followed by the transition to the limit $r_L \to 0, r_U \to \infty$. The corresponding expansions of the unity are written.
1 Introduction

We study the radial Schrödinger equation

\[ H \psi(r) = k^2 \psi(r) \]  
\[ H = -\frac{d^2}{dr^2} + \frac{\lambda^2 - \frac{1}{2}}{r^2} + V(r), \quad 0 \leq r < \infty, \]  

where \( k^2 = E \) is the energy, and \( \lambda \), when taken imaginary, is a coupling constant of singular attraction. This parameter may be also thought of as connected with “complex angular momentum” \( l = \lambda - \frac{1}{2} \). The potential \( V \) is assumed real and well-behaved, so as no extra troubles be introduced:

\[ \int_c^\infty |V(r)|dr < \infty, \quad \int_0^{c'} r|V(r)|dr < \infty, \quad c, c' > 0. \]  

The case of \( V(r) = 0 \) is explicitly solvable in terms of cylindrical functions.

The problem under consideration is usually addressed with the help of self-adjoint extension of the Hamiltonian \( \text{[1]} \). This was first done by K.Meetz \( \text{[1]} \), who grounded in this way the conjecture of K.M. Case \( \text{[3]} \) (see also \( \text{[4]} \)). The spectrum of the Hamiltonian, accurately defined via the extension procedure within the von Neuman theory (see e.g.\( \text{[5]} \) and the comprehensive physical survey in \( \text{[6]} \), is discrete and unlimited from below.

This circumstance is often recognized as physically unsatisfactory and became the motivation for issuing a different approach \( \text{[7]} \), which respects the principle of correspondence with classical mechanics, where a particle, placed in the field of the singular center, performs a spiral motion diverging from the center or converging towards it and making an infinite number of revolutions around it. A quantization, taking into account the correspondence principle, should describe a center, which emits and absorbs particles that are free near the origin, since in the classics the motion of a particle is sort of unbounded in this vicinity, similarly to their motion in the region remote to infinity. In short, the most characteristic feature of the approach of Ref.\( \text{[7]} \), which we are continuing in the present publication, is that, in it, mathematical conditions are provided to make it possible to treat the singular point \( r = 0 \) of the differential equation \( \text{[1]}, \text{[2]} \) on the same physical footing, as the singular point \( r = \infty \) is usually treated.
To be more precise, instead of $H$, eq. (2), we use two differential operations $H^{IV}$ and $H^{III}$

$$
H^{IV} = -\frac{d^2}{dr^2} - \frac{1}{4r^2} + V(r),
$$

and

$$
H^{III} = -\frac{d^2}{dr^2} - \frac{1}{4r^2} + V(r) - k^2,
$$

(one appropriate, if $E > 0$, and the other, if $E < 0$) for classifying physical states as vectors in the Hilbert spaces of the correspondingly defined self-adjoint operators. These operators are associated with the same equation (1), (2), but do not have the above unwanted properties of the Hamiltonian. The Schrödinger equation (1), (2) becomes a generalized eigenvalue problem of the type, studied by Kamke [8], for either of these operators:

$$
H^{IV} \psi_{\lambda,R}(r) = (\text{Im}\lambda)^2 \left( \frac{1}{R^2} + \frac{1}{r^2} \right) \psi_{\lambda,R}(r),
$$

$$
R = \frac{-\text{Im}\lambda}{k}, \quad k^2 > 0, \quad \lambda^2 < 0 \quad (\text{consider } R > 0 \text{ for definiteness})
$$

and

$$
H^{III} \psi_{\lambda,k}(r) = \frac{(\text{Im}\lambda)^2}{r^2} \psi_{\lambda,k}(r), \quad k^2 < 0, \quad \lambda^2 < 0.
$$

In eq. (7) the eigenvalue is $(\text{Im}\lambda)^2$, with the energy $k^2$ kept fixed as a negative parameter. In eq. (6) the eigenvalue is $(\text{Im}\lambda)^2$, with the ratio $R = -\text{Im}\lambda/k$ kept as a real parameter. Alternatively, writing the factor in front of $\psi$ in the r.-h. side of eq. (6) in the form $k^2(1 + R^2/r^2)$, one may consider the energy $k^2$ as an eigenvalue, with $R$ being a fixed parameter. These problems introduce new measures

$$
d\mu^{IV}(r) = \left( \frac{1}{R^2} + \frac{1}{r^2} \right) dr,
$$

$$
and

$$
d\mu^{III}(r) = \frac{1}{r^2} dr,
$$

which are to be used later in defining scalar products in the corresponding Hilbert spaces of eigenvectors of $H^{IV}$ and $H^{III}$. The measures (8) and (9) are both singular in the origin $r = 0$. This fact is of crucial importance for providing the possibility of treating the solutions, oscillating in the origin, as corresponding to free particles. We comment on this point a little later.
For defining \( H^{IV} \) and \( H^{III} \) as self-adjoint operators, and studying their spectra we use the physically straightforward procedure of quantization in a box. When doing so we introduce the lower boundary \( r_L \) of the box and let it later tend to the point of singularity of the differential equation \( r_L \to 0 \), exactly in the same manner, as we introduce the upper boundary \( r_U \) and let it tend to the other singularity point \( r_U \to \infty \), in accord with the customary procedure. We impose zero boundary conditions in the point \( r = r_L \) for problem (7), and (anti)periodic boundary conditions at the ends of the box \( r = r_L, r_U \) for problem (6). The spectra, discrete as long as \( r_L \) and \( r_U \) are finite, turn into continua in the limit \( r_L = 0, r_U = \infty \).

Certainly, for a self-adjoint definition of the operators \( H^{IV} \) and \( H^{III} \), the von Neuman technique might be applied, as well. As long as the finite interval \((r_L, r_U)\) is concerned, this makes no difference with the quantization in a box. The special power of von Neuman technique lies in its ability to handle directly the intervals with singular ends. We insist, however, that the corresponding results may make physical sense only to the extent, to which these reproduce the limit \( r_L \to 0, r_U \to \infty \), since, in the physical reality, there are not infinitely large boxes, as well as there are not point-like sources of force: there may only be very large boxes and very small-sized sources (as compared to other parameters with the dimensionality of length involved in the problem).

Transformations are known [8], [5] (see eqs. (10), (11) and eqs. (109), (110) below) that reduce the Kamke eigenvalue problems (7), (6) to the standard Liouville forms (see eq. (25) and eq. (111) below), to which the theory of self-adjoint differential operators is directly applicable.

Both of the transformations in the asymptotic region \( r \to 0 \) reduce to the change of the variable (\( r_0 \) is a free dimensional parameter)

\[
    r_* = r_0 \ln \frac{r}{r_0},
\]

accompanied with the transformation of the wave function

\[
    \left( \frac{r_0}{r} \right)^{\frac{1}{2}} \psi(r) = \tilde{\psi}(r_*).
\]

The origin \( r = 0 \) is mapped onto \( r_* = -\infty \). The singularity in the point \( r = 0 \) gives rise to solutions with the oscillating asymptotic behavior at \( r \to 0, r_* \to -\infty \)

\[
    \psi(r) \propto r^{\pm i \Im \lambda + \frac{1}{2}}
\]

\[
    \tilde{\psi}(r_*) \propto \exp \left( \pm i r_* \frac{\Im \lambda}{r_0} \right) r_0^{\Im \lambda}.
\]
The latter equation is a free wave. This observation alone is not yet enough to make (13) correspond to a free particle. To do this consider the behavior of the norm, associated with the transformed Schrödinger equation (see Sections 2 and 3 below), at the lower integration limit

$$
\int_{-L}^{-\infty} |\tilde{\psi}(r_*)|^2 dr_* = r_0^2 \int_{r_0 \exp(-L/r_0)}^{r_0} |\psi(r)|^2 \frac{dr}{r^2}.
$$

Eq. (14) diverges linearly with the box size in the $r_*$-space $L = r_0 \ln(r_0/r_L) \to \infty$. This is just what is needed to argue that the particle spends most part of its life in the form of a free particle in the asymptotic region $r \to 0$, the singularity in the integration measure $dr/r^2$ in (14) providing a sufficiently ample volume for doing this. This statement establishes the more precise meaning of the phrase [9] "falling of a particle down to the center". Unlike [9] we do not attribute this phenomenon to the fact that there is a point $E = -\infty$ in the energy spectrum. On the contrary, the states of $H^{III}$, asymptotically free near the origin, make a continuum (the situation is the same as for the usual continuum of states free at $r = \infty$ for real $k$).

After reduced to the standard Liouville form (see eq. 25 below), problem (7) describes particles, issued at negative infinity and totally reflected by impenetrable potential, resulting from the above transformation, back to negative infinity. This is a complete analogue of the elastic scattering of particles, belonging to the continuous spectrum of $H$, but now in the "inner" world near the singularity. The scattering is characterized by one scattering phase. The probability flux to or from the singularity is zero. We refer to this situation that occurs in the domain $k^2 < 0$, $\lambda^2 < 0$, called sector III in [7], as inner elastic scattering in continuum of asymptotically-free confined states, since the eigenfunctions of $H^{III}$ are concentrated near the origin $r = 0$, belong to a continuum and oscillate like a free exponential (with diverging norm) when approaching the point $r = 0$.

In contrast to sector III, the eigenvalue problem (6), appropriate in the domain $k^2 > 0$, $\lambda^2 < 0$, called sector IV in [7], becomes, after the corresponding transformation reduces it to the standard Liouville form (see eq. 111 below), a barrier penetration problem on the whole axis ($-\infty, \infty$). Particles, incoming from positive (negative) infinity are partially reflected back by the barrier potential, and partially penetrate through this potential to outgo to negative (positive) infinity. In accord with the (anti)periodic boundary conditions, imposed in problem (6), the total probability flux is, generally, nonvanishing. This confinement/deconfinement process - we call it this way, because particle outgoing to or incoming from the negative infinity represent transitions to
or from the continuum of asymptotically free, $\delta$-function normalized states, confined by the center - is described by a $2 \times 2$ unitary scattering matrix of an inelastic interchannel process, determined by two scattering phases and one inelasticity angle.

In Sec. 2 the spectral problem (7) is studied in sector III, the complete set of orthonormal eigenfunctions is found in a finite box and in the continuum limit, as well as the scattering phase for the elastic scattering of particles, emitted by the center - back to the center. In Sec. 3 we fulfil the same program for the spectral problem (6) in sector IV. The complete orthonormal set of eigenfunctions is found, which behave as standing wave both near the center and near infinity, the corresponding expansion of unity is written. The $2 \times 2$ scattering matrix elements are expressed in terms of the Jost functions.

2 Continuum of confined states

In the domain of parameters $\lambda^2 < 0$, $k^2 < 0$, called sector III in [7], only one fundamental solution is appropriate, the boundary conditions are to be imposed at one end-point and belong to Sturm-Liouville type, the probability flux is zero.

Define a new differential operation $H^{III}$ according to eq. (5) so that the Schrödinger equation (1) take the form (7). We consider this equation in the half-box

$$r_L \leq r < \infty.$$ (15)

The lower limit of the half-box $r_L$ is meant to tend to zero afterwards. The differential equation (7), defined on the interval (15) make the general eigenvalue problem of Kamke [8], for which it is peculiar that the eigenfunction in the r.-h. side is taken with a variable-depending factor, here $\lambda^2/r^2$, not just the constant eigenvalue $\lambda^2$. If boundary conditions are chosen in such a way that $H^{III}$ is symmetric (Hermitian), it is also self-adjoint and the set of eigenfunctions of the problem (7) is complete in a Hilbert space. To see this, it is sufficient to perform the transformation, which reduces the problem (7) to the normal Liouville form. However, the orthonormality of the eigenfunctions with nonflat measure can be derived already before we make this transformation.

Impose the ”zero boundary conditions” at the both ends of the half-box

$$\psi(\infty) = 0, \quad \psi(r_L) = 0.$$ (16)

The first one means in fact that the domain $D_{III}$ where $H^{III}$ acts consists of functions $\psi(r)$ that decrease fast enough when $r \to \infty$. This implies that
only one - out of the two - fundamental solutions is appropriate, the deficiency
index for the problem (7) with one regular, \( r = r_L \), and one singular \( r = \infty \).
Physically, the zero boundary conditions (16) guarantee that the overall probability flux carried by functions \( \psi \in \mathcal{D}_{III} \)

\[
P_{\psi} = i \left( \psi(r) \frac{d\psi^*(r)}{dr} - \psi^*(r) \frac{d\psi(r)}{dr} \right)
\]  

(17)
to/from the center \( r = r_L \) be zero, as well as the probability flux to/from the
infinity \( r = \infty \). This holds true also when \( r_L \to 0 \). We discuss later, whether
the second condition in (16) is the most general choice or not.

The operator (13) is symmetric (Hermitian)

\[
(H^{III})^* = (H^{III})^T,
\]

(18)
provided that its matrix element is defined as

\[
H_{ij}^{III} = \int_{r_L}^{\infty} \psi_i^*(r) H^{III} \psi_j(r) dr,
\]

(19)
where \( \psi_{i,j}(r) \) are any two square-integrable functions, sufficiently smooth in
the interval (15), subject to conditions (16). The asterisk in (19) designates
complex conjugation, and \( T \) indicates transposition. (Remind, that \( V(r), k^2 \)
are both real.)

As long as the lower box end \( r_L \) is finite, the Kamke eigenvalue problem
(7), (16) has a discrete spectrum. When \( r_L \ll r_0 \), where \( r_0 \) is a dimensional pa-
rameter, the discrete eigenvalues of \( H^{III} \) behave like \( \lambda_n = i\pi n (\ln(r_0/r_L))^{-1} \)
and do not depend on \( k \). (We come back to this point below in this section).
In the limit \( r_L = 0 \) they condense [7], as is usually the case with quantization
in a box, and we are left with a continuum of states, which we call confined,
since the functions from \( \mathcal{D}_{III} \) are concentrated in a finite domain.

Owing to the Hermiticity property (18), any two solutions \( \psi_{\lambda_1,k} \), \( \psi_{\lambda_2,k} \)
of the Kamke eigenvalue problem (7), (16) obey the relation

\[
(\lambda_1^2 - \lambda_2^2) \int_{r_L}^{\infty} \psi_{\lambda_1,k}^*(r) \psi_{\lambda_2,k}(r) \frac{dr}{r^2} = 0,
\]

(20)
which implies that these be orthogonal with the measure \( dr/r^2 \), provided the
(real) eigenvalues \( \lambda^2 \) are different, \( \lambda_1^2 \neq \lambda_2^2 \), while the energy \( k^2 \) is the same.
The equality of the \( k \)'s in the two functions \( \psi_{\lambda_1,k}^*(r) \) ans \( \psi_{\lambda_2,k}(r) \) in (20) is
dictated by the fact that $H^{III}$, eq. (5), contains $k$. The derivation of (20) is standard: one should left-multiply the equation

$$H^{III} \psi_{\lambda_1,k}(r) = -\frac{\lambda_1^2}{r^2} \psi_{\lambda_1,k}(r)$$

by $\psi_{\lambda_2,k}(r)$, and the equation

$$H^{III} \psi_{\lambda_2,k}(r) = -\frac{\lambda_2^2}{r^2} \psi_{\lambda_2,k}(r).$$

by $\psi_{\lambda_1,k}^*(r)$. The difference of these products, when integrated over $dr$, vanishes due to (18) to give (20). The ortho-normality relations, which follow from (20) in the continuum limit $r_L = 0$ are

$$\frac{2(\text{Im}\lambda)^2}{|f(\lambda, -k)|^2} \int_0^\infty f_{\lambda, -k}(r)f_{\lambda', -k}(r)\frac{dr}{r^2} = \pi\delta(\text{Im}\lambda - \text{Im}\lambda').$$

(23)

Here $f_{\lambda, -k}(r)$ designates the exact solution to the Schrödinger equation (11), valid in the whole domain $r \in (0, \infty)$, which decreases for $r \to \infty$ as $\exp(-r\text{Im}k)$ (consider $\text{Im}k > 0$ for definiteness), and $f(\lambda, k)$ is the Jost function, defined as the Wronsky determinant

$$f(\lambda, k) = f_{\lambda, k}(r)\frac{d\phi_{\lambda,k}(r)}{dr} - \frac{df_{\lambda,k}(r)}{dr}\phi_{\lambda,k}(r).$$

(24)

between the solution $f_{\lambda, k}(r)$ and another solution, called $\phi_{\lambda,k}(r)$, which oscillates like $r^{i\text{Im}\lambda + 1/2}$ near $r = 0$. For the case $V = 0$ eq.(23) becomes an orthogonality relation for McDonald functions $K_{i\text{Im}\lambda}(r)$ with (different or coinciding) imaginary indices $K_{i\text{Im}\lambda}(r)$ (see [7]).

Now it is time to perform the advertised transformation. This is the transformation of the coordinate (10) and of the wave function (11) made in the whole domain (15) (not only near $r = 0$, as it was discussed in Introduction). After this transformation, the Schrödinger equation (11) or (7) acquires the standard Liouville form

$$\left(-\frac{d^2}{dr_*^2} + U_{\text{con}}(r_*)\right)\tilde{\psi}(r_*) = \frac{(\text{Im}\lambda)^2}{r_0^2}\tilde{\psi}(r_*)$$

(25)

with

$$U_{\text{con}}(r_*) = -\exp\left(\frac{2r_*}{r_0}\right)\left(k^2 - V(r_0 \exp\frac{r_*}{r_0})\right).$$

(26)
The transformation (10) of the coordinate maps the half-box (15) to the half-box
\[-L \leq r_* < \infty, \quad r_L = e^{-\frac{L}{r_0}},\] (27)
where \(L = r_0 \ln(r_0/r_L)\). The left wall \(r_* = -L\) of the half-box (27) in the \(r_*\)-space tends to negative infinity as the core radius tends to zero, \((r_L/r_0) \to 0\).

The boundary conditions (16) now become
\[\tilde{\psi}(\infty) = 0, \quad \tilde{\psi}(-L) = 0.\] (28)

The transformation of the wave function (11) is intended to meet the requirement that there should be no linear-derivative term in (25). Equation (25) with the boundary conditions (28) has the form of a usual eigenvalue problem, with the potential \(U_{\text{con}}(r_*)\) containing \(k^2 < 0\) as a parameter. It proposes the customary measure \(dr_*\) to be used in defining the norm. The following relation between the norms in the \(r\)- and \(r_*\)-spaces
\[\int_{-L}^{\infty} |\tilde{\psi}(r_*)|^2 dr_* = r_0^2 \int_{r_L}^{\infty} |\psi(r)|^2 \frac{dr}{r^2}\] (29)
takes place.

The effective potential \(U_{\text{con}}(r_*)\) is plotted in Fig. 1 for the case of \(V = 0\).

The inclusion of \(V \neq 0\) cannot change the asymptotic forms \(U(-\infty) = 0, \quad U(\infty) = \infty\) due to the condition (3). Solutions of eq. (25) are free waves (13) near the negative infinity and are totally reflected by the effective potential (26) to the left side. This strictly forbids their penetration into the outer world \(r_* \to \infty\). We face the process of elastic scattering of particles, incoming from the negative infinity of the \(r_*\)-axis \((\text{i.e.}, \text{emitted by the center})\) back to the negative infinity \((\text{to be absorbed by the center})\). The free parameter of dimension of length \(r_0\) plays the role of the size of the system. As mentioned before, the total probability flux is zero. This means that the center absorbs all what it emits. Note, that the flux (17) is invariant under the transformation (10), (11), \(\text{i.e.},\) it does not change if one replaces \(\psi(r)\) by \(\tilde{\psi}(r)\) and \(r\) by \(r_*\).

This elastic scattering process in the inner world may be described exactly in the same terms as the usual one. The solution \(f_{\lambda,-k}(r)\), defined above, is a linear combination
\[f_{\lambda,-k}(r) = C\phi_{\lambda,k}(r) + D\phi_{-\lambda,k}(r)\] (30)
of two solutions, \(\phi_{\pm \lambda,k}(r)\), that oscillate like (12) near \(r = 0\). The coefficients \(C\) and \(D\) here are expressed (10) in terms of the Jost function (24). Define
Figure 1: The effective potential in the confining sector for the case of $V = 0$ (see eq. (26)). It prevents particles free near the singularity $r_* = -\infty$ ($r = 0$) from escaping to the outer world ($r_* \gg r_0$).

The dimensionless solution $\phi_{\pm \lambda, k}(r)$

$$\phi_{\pm \lambda, k}(r) = |k|^{\pm \text{Im} \lambda + 1/2} \phi_{\pm \lambda, k}(r) \sim \frac{r_k}{r_{\rightarrow \infty}} |r_k|^{\pm \text{Im} \lambda + 1/2}$$

and its Wronsky determinant with $f_{\pm \lambda, k}(r)$

$$f(\pm \lambda, k) = |k|^{\pm \text{Im} \lambda + 1/2} f(\pm \lambda, k).$$

The dimensionality of $f$ is $[\text{length}]^{-1}$. Then (33) is represented as

$$f_{\lambda, -k}(r) = \frac{f(\lambda, -k)}{2\lambda|k|} \left[ \exp(\text{i} \delta^{III}) \phi_{\lambda, k}(r) + \phi_{-\lambda, k}(r) \right],$$

where

$$\delta^{III}(\lambda, k) = \pi - 2 \arg f(\lambda, -k).$$

The boundary conditions (16) or (28) are satisfied provided that the spectral equation

$$\lambda_n = n \pi \frac{r_0}{L} + \frac{r_0}{L} \left( \lambda_n \ln |kr_0| - \text{i} \arg f(\lambda_n, -k) \right), \quad n = 0, \pm 1, \pm 2...$$
is solved with \( n \) being an arbitrary integer. When \( L \to \infty \) the second term in (35) should be neglected and the spectrum becomes \( \lambda_n = \frac{\pi nr_0}{L} \), as stated above.

The normalized solution of the eigenvalue problem (7), (16)

\[
\psi_{\lambda,k}(r) = \frac{|\text{Im}\lambda| \sqrt{2}}{\sqrt{\pi}|f(\lambda, -k)|} f_{\lambda,-k}(r)
\]

has the form

\[
\psi_{\lambda,k}(r) = \frac{i}{\sqrt{2\pi}} \left( e^{-i \arg f(\lambda,-k)} \phi_{\lambda,k}(r) - e^{i \arg f(\lambda,-k)} \phi_{-\lambda,k}(r) \right) =
\]

\[
= \frac{i}{\sqrt{2\pi|k|}} \left( e^{-i \arg f(\lambda,-k)} \phi_{\lambda,k}(r) - e^{i \arg f(\lambda,-k)} \phi_{-\lambda,k}(r) \right).
\] (37)

Near the singularity point \( r = 0 \) the eigenfunctions behave as

\[
\psi_{\lambda,k}(r) \bigg|_{r \to 0} \approx \frac{\sqrt{2r}}{\sqrt{\pi}} \sin \left( \arg f(\lambda, -k) - \text{Im} \lambda \ln(r|k|) \right)
\] (38)

Correspondingly, the eigenfunctions \( \tilde{\psi}_{\lambda,k}(r_*) \) of the problem (25) - (28), which are the functions (36), transformed according to (10), (11), behave near the singularity point \( r_* = -\infty \) as

\[
\tilde{\psi}_{\lambda,k}(r_*) \bigg|_{r_* \to -\infty} \approx \frac{\sqrt{2r_0}}{\sqrt{\pi}} \sin \left( \arg f(\lambda, -k) - \text{Im} \lambda \ln(r_0|k|) - \text{Im} \lambda r_* \right)
\] (39)

For \( V = 0 \), the dimensionless phase \( \arg f(\lambda, -k) \) cannot depend on \( k \), since there is no other dimensional parameter in the Schrödinger equation (1) or (7) in this special case. From the known exact solution of the Schrödinger equation one finds:

\[
\arg f_0(\lambda, -k) = - \arg \Gamma(1 - i\text{Im}\lambda)
\] (40)

The form (39) proposes the definition of the scattering matrix (just a unit-length complex number) describing the internal elastic scattering:

\[
S^{III} = e^{2i(\arg f(\lambda,-k))(r_0|k|)} - 2i\text{Im}\lambda
\]

(41)

The orthonormality relations (23), written for the eigenfunctions \( \tilde{\psi}_{\lambda,k}(r_*) \) of the problem (25) - (28) or for (36), take the form

\[
\frac{1}{r^2_0} \int_{-\infty}^{\infty} \tilde{\psi}_{\lambda,k}(r_*) \tilde{\psi}^*_{\lambda',k}(r_*) dr_* = \int_{0}^{\infty} \psi_{\lambda,k}(r) \psi^*_{\lambda',k}(r) \frac{dr}{r^2} = \delta (\text{Im}\lambda - \text{Im}\lambda')
\] (42)
Note, that as long as the parameter $\text{Im}\lambda$ can be viewed upon as a strength of the singular attraction, the ortho-normality relation (42) expresses spectral properties with respect to a "coupling constant".

The eigenfunctions (36) or (37) that belong to the continuum make a complete system, unless the eigenvalue problem (25) - (16) has extra discrete solutions (this depends upon the potential $V(r)$) for negative values of $(\text{Im}\lambda)^2$, i.e. beyond sector III, namely, in the domain $\lambda^2 > 0$, $k^2 < 0$, called sector I in [7]. This is the sector of bound states. Unlike sector III, in sector I the Hilbert spaces, where Hamiltonian $H$ (2) and operator $H_{III}$ (5) act, consist of the same functions, since one of the two solutions $\phi_{\pm\lambda,k}(r)|_{r \to 0} \propto r^{\pm\lambda+1/2}$ is ruled out as not belonging to $L^2_{\mu}(0, \infty)$, the space of functions, square integrable with the measure $d\mu_{III}(r)$ (9) on the interval $(0, \infty)$: out of the two integrals

$$\int_0^\infty r^{\pm\lambda+1/2} \frac{dr}{r^2} = \int_0^\infty \frac{dr}{r^{1+2\lambda}}$$

(43)

one is and the other is not equal to infinity. The $L^2$-solution satisfies the second boundary condition (16), extended to the limit $r_L = 0$, so we are within the same eigenvalue problem (7). On the other hand, the same solutions are also ruled out by imposing artificial condition, that the wave function should decrease fast enough in the origin, - the procedure, accepted when the spectrum of bound states is considered in physical text-books (see e.g. [9]). Thus, the (finite number of) discrete states of $H$ and $H_{III}$ are the same and can be presented in the form of trajectories $k = k_s(\lambda)$ or $\lambda = \lambda_s(k)$, $s = 1, 2, ..., s_0$. Thus the generalized Fourier expansion in the continuous limit may be written primarily for an arbitrary function $F(r) \in L^2(-\infty, \infty)$ in the $r_*$-representation as

$$\tilde{F}(r_*) = \int_{-\infty}^\infty C(\lambda, k) \tilde{\psi}_{\lambda,k}(r_*) d\text{Im}\lambda + \sum_{s=1}^{s_0} C(\lambda_s(k), k) \tilde{\psi}_{\lambda_s,k}(r_*)$$

$$C(\lambda, k) = \frac{1}{r_0^2} \int_0^\infty F(r) \psi_{\lambda,k}^*(r) dr_*,$$

(44)

and secondary in the initial variable representation for arbitrary function $F(r) = (r/r_0)^{1/2} \tilde{F}(r_*)$, $F(r) \in L^2_{\mu}(0, \infty)$ as

$$F(r) = \int_{-\infty}^\infty C(\lambda, k) \psi_{\lambda,k}(r) d\text{Im}\lambda + \sum_{s=1}^{s_0} C(\lambda_s(k), k) \psi_{\lambda_s,k}(r),$$

$$C(\lambda, k) = \int_0^\infty F(r) \psi_{\lambda,k}^*(r) \frac{dr}{r^2}.$$

(45)
It remains to comment on a generality of the boundary conditions (16). As the arbitrary dimensional parameter \( r_0 \) involved in the transformation (10), (11) varies, the point \( r_L = e^{-L/r_0} \), where the boundary condition (16) is imposed moves, provided that \( L \) is fixed. This effectively changes the boundary conditions, considered in an unmoving point. Thus, the arbitrariness in \( r_0 \) reflects the arbitrariness in choosing self-adjoint boundary conditions, or in other words in fixing the self-adjoint extension. According to a general theorem [5], the spectrum in the continuum limit does not depend on this arbitrariness. What does depend, is the \( S \)-matrix (41). The kinematic unitary factor, containing \( r_0 \) in it, is connected with the known \( U(1) \) arbitrariness in fixing the self-adjoint extension.

3 Two-channel sector

Now consider the domain of parameters \( \lambda^2 < 0, k^2 > 0 \), called sector IV in [7]. There, both fundamental solutions of the Schrödinger equation are appropriate. The probability flux to/from the center (from/to the infinity) may be nonzero. Correspondingly, the boundary conditions should be of non-Sturm-Liouville type: they should interconnect values of the wave function, taken at the opposite ends of the interval, like periodic or antiperiodic.

In this sector we take for definiteness \( \text{Im} \lambda < 0, k > 0 \) throughout. It is not adequate to try to extend equation (7), or (25) beyond sector III into sector IV by including positive \( k^2 \) into consideration. In that case equation (25) would correspond to negative, exponentially growing in absolute value with \( r \to \infty \), potential. Such a problem has a discrete spectrum, unlimited from below (cf the example considered in Section 5.8 of the textbook [11] and Appendix II of its Russian edition), similar to the energy spectrum of [3], [4], [1]. Our study of sector IV will be done using an operator \( H^{IV} \), coinciding with \( H^{III} \) for small \( r \) and with \( H \) for large \( r \).

3.1 Kamke eigenvalue problem

In sector IV define the differential operation \( H^{IV} \) so that the Schrödinger equation (1) take the form

\[
H^{IV} \psi(r) = \left( k^2 - \frac{\lambda^2}{r^2} \right) \psi(r).
\]

Let us introduce the new dimensional parameter \( R \), real in sector IV, according to the relation \( R = -\text{Im} \lambda / k \). In what follows the couple \( \lambda, R \) will be used to
parameterize the phase space instead of the couple \( \lambda , k \). Then eq. (46) turns into equation (6) We consider this equation in the box

\[ r_L \leq r \leq r_U. \] (47)

The lower limit of the box \( r_L \) is meant to tend to zero, whereas the upper limit \( r_U \) to infinity

\[ r_L = Re^{-\xi_l}, \quad r_U = R\xi_u. \] (48)

These limits contain the dependence on the ratio \( R \) - but not on the eigenvalue \( \text{Im} \lambda \), whereas \( \xi_{L,U} \) are independent numbers, which will be taken infinite later. The differential equation (6), defined on the interval (47) and supplemented with necessary boundary conditions make again the general eigenvalue problem of Kamke [8]. In the case of interest here the choice of the boundary conditions is restricted by the requirement that these should survive the limiting process \( r_L \to 0, r_U \to \infty \). This requirement is met, for example, by the following conditions imposed at the walls of the box (47)

\[ \frac{\psi(r_L)}{r_L^{1/2}} = \pm R^{1/2} \psi(r_U), \]

\[ (r_L)^{1/2} \frac{d\psi(r)}{dr} \bigg|_{r=r_L} - \frac{\psi(r_L)}{2(r_L)^{1/2}} = \pm R^{1/2} \frac{d\psi(r)}{dr} \bigg|_{r=r_U}. \] (49)

It is important, that the coefficients in (49), as well as the limits \( r_{L,R} \) (48) do not depend on the eigenvalue \( \text{Im} \lambda \) but only contain the ratio \( R \). The matrix elements, defined as

\[ H^{IV}_{ij} = \int_{r_L}^{r_U} \psi_{\lambda_i,R}(r) H^{IV} \psi_{\lambda_j,R}(r) dr, \] (50)

do satisfy the Hermiticity condition

\[ (H^{IV}_{ji})^* - H^{IV}_{ij} = \left( \psi_{\lambda_j,R}(r) \frac{d\psi_{\lambda_i,R}(r)}{dr} - \psi_{\lambda_i,R}(r) \frac{d\psi_{\lambda_j,R}(r)}{dr} \right) \bigg|_{r_L}^{r_U} = 0, \] (51)

once eqs. (49) are fulfilled for each of the functions \( \psi_{\lambda_j,R}(r) \). The origin of the boundary conditions (49) will become clear below in this section. The special choice (48) of dependence of \( r_{L,U} \) on \( R \) is not important for providing the Hermiticity.
Certainly, the boundary conditions (49) are not the most general conditions, meeting the above requirements. A more general choice might be provided, if one introduced an arbitrary parameter with the dimensionality of length $r_0$ in place of $R = -\text{Im} \lambda/k$ in (49). This would yield unreasonable complications in handling the spectra and eigenfunctions without, however, affecting the important conclusions about the condensation of eigenvalues into the continuum in the limit $r_L = 0$, $R_U = \infty$. Therefore, unlike the previous treatment in Section 2, we do not keep arbitrary $r_0$ in this Section. Its possible effect for the scattering matrix will be discussed in Subsection 3.3.

The Hermiticity condition (51), when taken with $i = j$, reads that the probability flux (17) is the same at the two opposite walls of the box (47). Thus, the boundary conditions (19) agree with the probability conservation, but, unlike the boundary conditions (16) imposed in sector III, admit that the probability flux (17) be nonzero. This means that the overall probability may flow either into or out of the system, depending upon the solution selected.

Following the spectral theory [8], [5], we conclude that the special eigenvalue problem (6), (49) should for every $R$ have two countable manifolds of infinitely growing eigenvalues $(\text{Im} \lambda_m(R))^2$ and $(\text{Im} \lambda_m(R))^2$, $m = 0, 1, 2, \ldots$, which alternate:

$$\text{(Im} \lambda_0)^2 \leq \text{(Im} \lambda_m)^2 < \text{(Im} \lambda_{m+1})^2. \quad (52)$$

(The lowest value $(\text{Im} \lambda_0)^2$ only exists for the upper sign in (49).) Thus, we face a discrete spectrum, as long as the box wall positions (47) are finite, $r_L \neq 0$, $r_U \neq \infty$. The spectrum is constituted by discrete trajectories $\text{Im} \lambda_n(k)$ labelled by the integer $n$. The trajectories are expected to condense to form a continuum of states, when $r_L \to 0$ and $r_U \to \infty$. The spectral theory predicts that, at least for large $(\text{Im} \lambda)^2$, the spacings between neighboring eigenvalues within one manifold are

$$\text{Im} \lambda_{n+1} - \text{Im} \lambda_n = \frac{2\pi}{N}, \quad N = \int_{r_L}^{r_U} \sqrt{\frac{1}{R^2} + \frac{1}{r^2}} \, dr, \quad (53)$$

and the same for $\text{Im} \lambda_n$. The integral $N$ here plays the role of the size of the box. It diverges both at the lower and the upper limits as $r_L \to 0$ and $r_U \to \infty$, $N = \xi_L + \xi_U$, thus providing the vanishing of the spacings. We shall study the spectrum specifically in this limiting case of interest in the next subsection to see that the spectral trajectories do condense everywhere throughout sector IV, not only for large $(\text{Im} \lambda)^2$.

By reducing the spectral problem (6), (49) to the standard Liouville form we shall explicitly see below that its eigenfunctions are a complete set for
every $R$. If all the eigenvalues in (6) are positive, i.e. belong to sector IV, the complete set in the limiting case is exhausted by the functions belonging to the continuum. If there are also several negative eigenvalues $\text{Im}\lambda^2$ (real $\lambda_n$), this means that there exist usual bound states, since, with $R$ fixed, the corresponding values of $k$ become imaginary, and we enter the sector of bound states (called sector I in [7]) along the ray $(-\text{Im}\lambda/k) = R = \text{const}$. The discussion of this point presented in Section 2, might be repeated here, with the only reservation that there we entered sector I along the ray $\text{Im}k = \text{const}$.

The corresponding (finite number of) eigenfunctions make the complete set when taken together with the eigenfunctions, which belong to the continuum.

The analog of Eq. (20) is the following orthogonality relation in sector IV

$$\int_{r_L}^{r_U} \psi^*_{\lambda,R}(r)\psi_{\lambda',R'}(r) \left( (k')^2 - k^2 + \frac{(\text{Im}\lambda)^2 - (\text{Im}\lambda')^2}{r^2} \right) \, dr = 0. \quad (54)$$

Any two solutions that belong to the same ray in the $(\text{Im}\lambda, k)$-plane (i.e. have common value of $R = -\text{Im}\lambda/k$) are mutually orthogonal with the universal measure (8) provided that the eigenvalues $(\text{Im}\lambda)^2$ do not coincide, $( (\text{Im}\lambda)^2 \neq (\text{Im}\lambda')^2)$:

$$((\text{Im}\lambda)^2 - (\text{Im}\lambda')^2) \int_{r_L}^{r_U} \psi^*_{\lambda,R}(r)\psi_{\lambda',R}(r) \left( \frac{1}{R^2} + \frac{1}{r^2} \right) \, dr = 0. \quad (55)$$

### 3.2 Spectrum

In sector IV every solution of the Schrödinger equation is meaningful, since it oscillates at the both ends of the interval. (We exclude the value $\lambda = 0$, which requires a special treatment to be done below in this Subsection.) Let $\phi_{\pm\lambda,k}(r)$ be the solutions, that behave in the origin like eq. (12):

$$\phi_{\pm\lambda,k}(r) \sim r^{\pm\text{Im}\lambda + \frac{1}{2}}. \quad (56)$$

These are expressed in sector IV as [10]

$$\phi_{\lambda,k}(r) = Ef_{\lambda,k}(r) + Gf_{\lambda,-k}(r),$$
$$\phi_{-\lambda,k}(r) = G^*f_{\lambda,k}(r) + E^*f_{\lambda,-k}(r) \quad (57)$$

in terms of the solutions $f_{\pm\lambda,k}(r)$ that behave like $\exp(\mp ikr)$ at infinity:

$$f_{\lambda,\pm k}(r) \sim \exp(\mp ikr). \quad (58)$$
The constants $E, G$ are connected with the Jost functions (24) as

$$f(\lambda, k) = 2ikG, \quad f(\lambda, -k) = -2ikE$$  \hspace{1cm} (59)

and are in sector IV subject to the relation \cite{10} (we corrected the obvious dimension-violating misprint in eq.(5.13) of \cite{10})

$$|E|^2 - |G|^2 = \frac{-\text{Im} \lambda}{k} \equiv R.$$  \hspace{1cm} (60)

Let us look for solution to the problem (6), (49) in the form of a linear combination of fundamental solutions of the differential equation (41)

$$a \phi_{\lambda,k}(r) + b \phi_{-\lambda,k}(r).$$  \hspace{1cm} (61)

We restrict ourselves to the case when $\text{Im} \lambda$ and $k$ are of opposite signs, $R > 0$. Using (57) and the asymptotic forms (56), (58) one writes the boundary conditions (49) in the form of the set of equations for the coefficients $a, b$ in (61), valid provided that $r_L$ is much less, while $r_U$ is much greater than all dimensional parameters in the problem, i.e. than $R$ and other dimensional parameters, on which the potential $V(r)$ may depend:

$$a \left( \mp R^{1/2} r_L^{\text{Im} \lambda} - E \exp(-ikr_U) - G \exp(ikr_R) \right) + b \left( \mp R^{1/2} r_L^{-\text{Im} \lambda} - E^* \exp(ikr_U) - G^* \exp(-ikr_U) \right) = 0,$$

$$a \left( \mp R^{1/2} r_L^{\text{Im} \lambda} + E \exp(-ikr_U) - G \exp(ikr_U) \right) - b \left( \mp R^{1/2} r_L^{-\text{Im} \lambda} + E^* \exp(ikr_U) - G^* \exp(-ikr_U) \right) = 0.$$  \hspace{1cm} (62)

This set is simplified by linearly combining the equations as follows:

$$aG \exp(ikr_U) + b(\mp R^{1/2} r_L^{-\text{Im} \lambda} + E^* \exp(ikr_U)) = 0$$

$$a(\mp R^{1/2} r_L^{\text{Im} \lambda} + E \exp(-ikr_U)) + bG^* \exp(-ikr_U) = 0.$$  \hspace{1cm} (63)

The spectrum is obtained by equalizing the determinant of this set with zero. Using (60) one gets

$$(Ew_+ + E^*w_-) = \pm 2R^{1/2},$$  \hspace{1cm} (64)
where

\[ w_- = r_L^{-i\text{Im} \lambda} e^{-ikr_U} = R^{-i\text{Im} \lambda} e^{i\text{Im} \lambda (\xi_L + \xi_U)}. \] (65)

Define the three real angles \( \delta_{1,2}, \alpha \) (we shall need \( \delta_1 \) later)

\[
\cos \alpha = \left| \frac{f(\lambda, k)}{f(\lambda, -k)} \right|, \quad \sin \alpha = \frac{2\text{Im} \lambda}{R^{1/2}|f(\lambda, -k)|}, \quad -\frac{\pi}{2} \leq \alpha \leq \frac{\pi}{2}.
\] (66)

Eq. (60) guarantees the fulfillment of the necessary equality \( \sin^2 \alpha + \cos^2 \alpha = 1 \).

Bearing in mind (59), (60) we write equation (64) in the form

\[
\pm \sin \alpha - \sin (\delta_2 - \text{Im} \lambda \ln R + \text{Im} \lambda (\xi_L + \xi_U)) = 0.
\] (67)

This equation reduces to two infinite series of equations, wherein \( \alpha \) and \( \delta_2 \) are functions of \( R \) and of \( \text{Im} \lambda_n \), the latter being set equal to \( \text{Im} \lambda_n^{(1)} \) in the first, and to \( \text{Im} \lambda_n^{(2)} \) in the second equation:

\[
\text{Im} \lambda_n^{(1)} = \frac{2n\pi}{\xi_L + \xi_U} + \frac{1}{\xi_L + \xi_U} (\pm \alpha - \delta_2 + \text{Im} \lambda_n^{(1)} \ln R),
\]

\[
\text{Im} \lambda_n^{(2)} = \frac{(2n + 1)\pi}{\xi_L + \xi_U} + \frac{1}{\xi_L + \xi_U} (\mp \alpha - \delta_2 + \text{Im} \lambda_n^{(2)} \ln R),
\]

\[ n = 0, \pm 1, \pm 2, \ldots. \] (68)

Remind that the double sign here corresponds to that in (49). From these the eigenvalues \( \text{Im} \lambda_n^{(1,2)} \) are to be found for each \( n \). This result agrees with (53).

Finally, the asymptotic form of the discrete spectrum of the boundary problem (46), (49) in the limit \( r_L \to 0, r_U \to \infty \) in sector IV is given as

\[
\text{Im} \lambda_n^{(1)} = \frac{2n\pi}{\xi_L + \xi_U}, \quad \text{Im} \lambda_n^{(2)} = \frac{(2n + 1)\pi}{\xi_L + \xi_U}.
\] (69)

This does not depend on details of the interaction \( V(r) \) and on the choice of the sign in (49).

For the further analysis we shall need the relations [10], valid in sector IV:

\[
\phi_{\lambda,k}^*(r) = \phi_{-\lambda,k}(r), \quad f_{\lambda,k}^*(r) = f_{-\lambda,k}(r), \quad f^*(\lambda, k) = f(-\lambda, -k),
\] (70)

supplemented with the relations

\[
\phi_{\lambda,k}(r) = \phi_{-\lambda,-k}(r), \quad f_{\lambda,k}(r) = f_{-\lambda,k}(r),
\] (71)
which are a consequence of the asymptotic behavior (56), (58) and the evenness of (46) with respect to reflection of $k$ or $\lambda$.

Let us introduce the common enumeration of the eigenvalues (68), (69)

$$\text{Im} \lambda_m = \text{Im} \lambda^{(1)}_{m/2} \quad \text{for } m \text{ even,}$$

$$\text{Im} \lambda_m = \text{Im} \lambda^{(2)}_{(m-1)/2} \quad \text{for } m \text{ odd.}$$

(72)

From the second line of (70) it follows that $\alpha$ and $\delta_2$ in (66) are odd with respect to $\text{Im} \lambda$ (keeping $R$ invariant). This implies that the eigenvalues $\text{Im} \lambda^{(1,2)}_m$, defined as solutions of equations (68), obey the relations:

$$\text{Im} \lambda^{(1)}_n = -\text{Im} \lambda^{(1)}_{-n}, \quad \text{Im} \lambda^{(2)}_n = -\text{Im} \lambda^{(2)}_{-1-n}$$

or

$$\text{Im} \lambda_m = -\text{Im} \lambda_{-m}.$$  

(73)

This means that each eigenvalue $(\text{Im} \lambda_m)^2 \neq 0$ of the operator $H^{IV}$, eq. (4), is two-fold degenerate. If one identifies $\text{Im} \lambda_{-m}$ with $\text{Im} \lambda_m$, introduced in Subsection 3.1, one sees that this degeneracy corresponds to the equality signs in the chain of weak inequalities (52).

Eqs. (73) do not hold true for $m = 0$. Indeed, eqs. (73) would imply, that $\text{Im} \lambda_0 \equiv \text{Im} \lambda^{(1)}_0 = 0$. This is not the case, however: the oddness of $\alpha$ and $\delta_2$ does not yet provide that $\text{Im} \lambda^{(1)}_0 = 0$ be a solution of equation (68), since $\delta_2 = \arg f(\lambda, -k)$ in it is, generally, not a continuous function in the point $\lambda = 0$, $R \neq 0$. For instance, in the free case $V = 0$, when the Jost function is known, it can be written in sector IV as:

$$f^{(0)}(\lambda, -k) = \left(\lambda \frac{2}{2R}\right)^{-\lambda+\frac{1}{2}} \Gamma(1 + \lambda) = \left(\frac{|\lambda|}{2R}\right)^{\frac{1}{2}} |\Gamma(1 + \lambda)| \exp \left(|\lambda| \frac{\pi}{2} + i\delta^{(0)}_2\right),$$

$$\delta^{(0)}_2 = \arg f^{(0)}(\lambda, -k) = \frac{\pi}{4} \text{sgn}(\text{Im} \lambda) - \text{Im} \lambda \ln \left|\frac{\text{Im} \lambda}{2R}\right| + \arg \Gamma(1 + i\text{Im} \lambda),$$

(74)

where $\Gamma$ is the Euler gamma-function, and $\text{sgn}(\text{Im} \lambda)$ is 1 for positive and -1 for negative arguments. Here the phase $\delta^{(0)}_2$ contains the discontinuity $\text{sgn}(\text{Im} \lambda)$, and hence $\arg f^{(0)}(\lambda, -k)$ is not defined in the point $\text{Im} \lambda = 0$. Correspondingly, the boundary problem (4), (49) with $V = 0$ in (4) has no solutions for $\lambda = 0$, i.e. the boundary conditions (49) cannot be satisfied by combining the fundamental solutions of the differential equation (3), which in this case are $\sqrt{r}$ and $\sqrt{r} \ln r$. The said does not rule out the possibility that the point $\lambda = 0$ might belong to the spectrum. This may happen for dynamical reasons, for
some $V(r)$. The statement above only means that the general consideration alone are not enough for establishing the existence of the zero mode.

It can be demonstrated that any other self-adjoint boundary conditions used in place of (49) would lead to the same result. The most important conclusion about the spectrum (69) is: in the domain of interest $r \in (0, \infty)$ the spectral trajectories condense to make a continuum and to densely cover the space of quantum numbers $(\text{Im}\lambda, k)$ of sector IV.

3.3 Orthonormal solutions

To obtain the wave functions corresponding to the eigenvalues of the limiting spectral problem found in the previous Subsection, consider the first equation in (63). When taken on solutions of equations (68), it becomes

$$G a + (E^* \mp \varepsilon^{(j)} e^{i(\pm \varepsilon^{(j)} \alpha - \delta_2) R^{1/2}}) b = 0, \quad j = 1, 2, \quad (75)$$

where the factor $\varepsilon$ takes two different values: $\varepsilon^{(1)} = 1$, $\varepsilon^{(2)} = -1$ respective to whether the first or the second equation in (68) is used. According to (72), $\varepsilon = (-1)^m$. The moduli of the complex coefficients in front of $a$ and $b$ in this equation are the same

$$|E \mp \varepsilon^{(j)} e^{-i(\pm \varepsilon^{(j)} \alpha - \delta_2) R^{1/2}}| = |G|, \quad (76)$$

while their phases are expressed as follows

$$\arg G = \frac{\pi}{2} \text{sgn}(\text{Im}\lambda) + \delta_1, \quad \arg(E^* \mp \varepsilon^{(j)} e^{i(\pm \varepsilon^{(j)} \alpha - \delta_2) R^{1/2}}) = \pm \varepsilon^{(j)} \alpha - \delta_2 + \frac{\pi}{2} \text{sgn}(\text{Im}\lambda). \quad (77)$$

Relations (75), (77) are direct consequences of the definitions (59), (66). To derive them, eq.(60) and the relation

$$E^* e^{-i(\pm \varepsilon^{(j)} \alpha - \delta_2)} = \pm \varepsilon^{(j)} R^{1/2} + \frac{|f(\lambda, k)|}{2\text{Im}\lambda}, \quad (78)$$

are useful.

The two corresponding series of eigenfunctions (61) have the form

$$\psi_{\lambda,R}^{(j)}(r) \equiv \psi_{\lambda(j),R}(r) = \frac{i}{2\sqrt{\pi}} \left( e^{-i(\mp \varepsilon^{(j)} \alpha + \delta_1 + \delta_2)/2} \phi_{\lambda,k}(r) - e^{i(\mp \varepsilon^{(j)} \alpha + \delta_1 + \delta_2)/2} \phi_{-\lambda,k}(r) \right), \quad k = -\text{Im}\lambda/R. \quad (79)$$
It is understood, that Im$\lambda$ in $\psi^{(j)}_{\lambda,R}(r)$ is Im$\lambda^{(j)}_n$, the solution of the first ($j = 1$) and the second ($j=2$) equation in (68). In the continuum limit the two series $\psi^{(1,2)}_{\lambda,R}(r)$ only differ due to the factor $\varepsilon^{(1,2)}$ in them. The common phase factor has been chosen in such a way that the reality of the eigenfunction (79) be provided. This follows from the oddness of the angles $\alpha$, $\delta_{1,2}$ (66) under reflection of sign of $\lambda$, with $R$ kept constant. The latter property is proved using (70) and (71). The oddness of $\alpha$, $\delta_{1,2}$ also leads to that of the eigenfunctions (79):

$$(\psi^{(j)}_{\lambda,R}(r))^* = \psi^{(j)}_{-\lambda,R}(r) = -\psi^{(j)}_{-\lambda,R}(r). \quad (80)$$

Note the important difference with the standard Fourier analysis, based on the eigenvalue problem $-d^2y/d\xi^2 = p^2y$, $y(-\xi_L) = y(\xi_U)$, $y'(-\xi_L) = y'(\xi_U)$, where there are two independent eigenfunctions $\exp(\pm i\phi_{\lambda,k}(r)\xi)$ (connected by the complex conjugation operation), related to the same eigenvalue $p^2$. In that case the corresponding coefficients $a$ and $b$ are not subject to an equation like (75), and remain arbitrary. On the contrary, in our case the complex conjugation operation, when applied to an arbitrarily normalized eigenfunction, does not create any new, independent solution of the boundary problem, but only multiplies it by a unit complex factor. This explains why there is only one, sine-like, eigenfunction (79), whereas the other, cosine-like, eigenfunction is absent.

The probability flux (17) calculated with the wave function (79) is zero. Certainly, a linear combination of eigenfunctions with complex coefficients carries nonzero flux to or from the center. In this respect the situation is different from the bound states, the confined states of sector III, or from the elastic scattering states in what is called sector II in [7], where not only the eigenfunctions do not carry probability flux, but any their linear combinations do not either, since all the wave functions disappear at the both ends of the box, in accord with the Sturm-Liouville boundary conditions, used in these sectors. Remind, that the self-adjoint boundary conditions (49) are not of the Sturm-Liouville type.

With the use of (57) and of definitions (66) the same eigenfunction (79) may be also presented as

$$
\psi^{(j)}_{\lambda,R}(r) = \pm i\varepsilon^{(j)}_k \left( \frac{R}{\pi} \right)^{1/2} \left( e^{i(\mp \varepsilon^{(j)}_k \alpha_2 \mp \varepsilon^{(j)}_2 \delta_2)/2} f_{\lambda,k}(r) - e^{-i(\mp \varepsilon^{(j)}_k \alpha_2 \mp \varepsilon^{(j)}_2 \delta_2)/2} f_{\lambda,-k}(r) \right),
$$

$$k = -\text{Im} \lambda/R. \quad (81)$$
The eigenfunction \((79), (81)\) behaves near \(r = 0\) as
\[
\psi^{(j)}_{\lambda, R}(r) \propto \frac{r^{1/2}}{\pi} \sin \left( \mp \frac{\varepsilon^{(j)}}{\pi} \alpha + \delta_1 + \delta_2 - \frac{2}{\pi} \ln r \text{Im}\lambda \right) \tag{82}
\]
and near \(r = \infty\) as
\[
\psi^{(j)}_{\lambda, R}(r) \propto \mp \varepsilon^{(j)} \left( \frac{R}{\pi} \right)^{1/2} \sin \left( \mp \frac{\varepsilon^{(j)}}{\pi} \alpha - \delta_1 + \delta_2 - kr \right), \quad k = -\text{Im}\lambda/R. \tag{83}
\]
These are standing waves, that do not vanish at any of the end points, unlike the standing wave \((38)\) in sector III, which vanishes at \(r = r_L\), or the standing wave, corresponding to the usual elastic scattering in sector II, which vanishes at the remote end of the box \(r = r_U\).

The eigensolutions of the self-adjoint boundary problem under consideration in sector IV in the quadrant \(\text{sgn}(\text{Im}\lambda) = -1\), taken in any of the forms \((79)\) or \((81)\), satisfy the following orthonormality relation (the scalar product with the measure \((8)\)) in the asymptotic regime \(\xi_L, \xi_U \to \infty\) \((r_L \to 0, r_U \to \infty)\)
\[
\left( \psi^{(j)}_{\lambda, R}(r), \psi^{(i)}_{\lambda', R}(r) \right) \equiv \int_{r_L}^{r_U} (\psi^{(j)}_{\lambda, R}(r))^* \psi^{(i)}_{\lambda', R}(r) \left( \frac{1}{R^2} + \frac{1}{r^2} \right) \, dr = \\
= \delta_{ij} \delta_{nn'} \frac{\xi_L + \xi_U}{2\pi}, \quad i, j = 1, 2, \quad n = 1, 2, 3, \ldots. \tag{84}
\]
In the limit \(\xi_L = \xi_U = \infty\) this is
\[
\left( \psi^{(j)}_{\lambda, R}(r), \psi^{(i)}_{\lambda', R}(r) \right) \equiv \int_0^{\infty} (\psi^{(j)}_{\lambda, R}(r))^* \psi^{(i)}_{\lambda', R}(r) \left( \frac{1}{R^2} + \frac{1}{r^2} \right) \, dr = \\
= \delta_{ij} \delta(\text{Im}\lambda - \text{Im}\lambda'). \tag{85}
\]
The complex conjugation sign may be omitted here, since the eigenfunctions \(\psi^{(j)}_{\lambda, R}(r)\) have been chosen real. To see, that eqs.\((84), (85)\) hold true, we follow the standard procedure \([9]\). First note, that the scalar product of eigenfunctions with \(n \neq n', i \neq j\) is zero according to \((55)\). Then, we only need to calculate the contributions into \((84), (85)\), originating from the coinciding values \(n = n', i = j\). These are divergent due to integration near the end points. To find these contributions, the asymptotic expressions \((82), (83)\) are sufficient. When integrating near \(r = r_L\), we take \((82)\), neglect \(1/R^2\) as compared to \(1/r^2\) in the measure, and use the new integration variable \(\xi = \ln r/R\). When integrating near \(r = r_U\), we take \((83)\), neglect \(1/r^2\) as compared with \(1/R^2\) in the measure, and use the new integration variable \(\xi = r R\). In this way integral \((84)\) is reduced to \((1/2\pi) \int_{-\xi_L}^{\xi_U} \exp\{i(\text{Im}\lambda' - \text{Im}\lambda)\} d\xi\), where the limits are given as \((18)\).
3.4 Generalized Fourier expansion

According to the spectral theory \cite{8,5}, the sets of eigenfunctions associated with the self-adjoint boundary problem (6), (49) should be complete. We shall see this directly later, after we reduce Eq. (46) to the normal Liouville form. Now, using the family of orthonormalized eigenfunctions (79) or, which is the same, (81) found in the previous subsection for the case \( \text{sgn}(\text{Im } \lambda_k) = -1 \), we may write the Fourier expansion of a sufficiently smooth arbitrary function \( F(r) \) in the Hilbert space of functions with finite norm \( \int_0^\infty |f(r)|^2 d\mu_{IV}(r) < \infty \), under the assumption that there are no bound states in the problem, \textit{i.e.} that the eigenvalues \((\text{Im } \lambda)^2 \) of \( H_{IV} \) eq.(6) are all positive. Negative eigenvalues, if they exist, are not covered by the analysis above, since they fall out of sector IV and the corresponding fundamental solutions do not behave like (56), (58), contrary to what was assumed in Subsection 3.2.

In the asymptotic regime when the upper position \( r_U \) of the box wall tends to infinity, while its lower position \( r_L \) tends to zero, this expansion is:

\[
F(r) = \frac{2\pi}{\xi_L + \xi_U} \sum_{j=1,2} \sum_{n=1}^\infty C(\lambda^{(j)}_n, R) \psi^{(j)}_{\lambda^{(j)}_n, R}(r). \tag{86}
\]

In the continuum limit \( r_L = 0, r_U = \infty \) this becomes

\[
F(r) = \sum_{j=1,2} \int_0^\infty C^{(j)}(\lambda, R) \psi^{(j)}_{\lambda, R} d\text{Im } \lambda. \tag{87}
\]

We introduced \( d\text{Im } \lambda \equiv \text{Im } \lambda^{(j)}_{n+1} - \text{Im } \lambda^{(j)}_n = 2\pi/(\xi_L + \xi_U) \) in accord with (89). It is meant throughout, that \( R = -\text{Im } \lambda/k \) is positive and kept constant while summing or integrating over \( \text{Im } \lambda \). The expansion coefficients follow from (81), (85) to be in (86)

\[
C(\lambda^{(j)}_n, R) = \int_{r_L}^{r_U} (\psi^{(j)}_{\lambda^{(j)}_n, R}(r))^* F(r) d\mu(r) \tag{88}
\]

and in (87)

\[
C^{(j)}(\lambda, R) = \int_0^\infty (\psi^{(j)}_{\lambda, R}(r))^* F(r) d\mu(r) \tag{89}
\]

The measure \( d\mu(r) \) is defined as (8). We should have marked \( C \), as well as \( \psi \), here with the label \( \pm \) to indicate the dependence of these quantities upon the
choice of sign in the boundary conditions (49), which we did not, however, to
avoid an excessive complexity of notations.

For convergence of (88), (89) it is needed that
\[ \lim_{r \to 0} \frac{F(r)}{\sqrt{r}} = 0, \quad F(\infty) = 0. \]

(90)
The expansion coefficients \( C \), initially defined for \( n \geq 1, \Im \lambda > 0 \), can be
extended to negative values of \( n \) or \( \Im \lambda \), using (80), and to \( \lambda = 0 \) as
\[ C(\lambda_n^{(j)}, R) = C(-\lambda_n^{(j)}, R) = -C(\lambda_n^{(j)}, R) \]
\[ C^{(j)}(-\lambda, R) = -C^{(j)}(\lambda, R), \quad C(0) = 0. \]
(91)
Then, with the use of (79), or (81), we can reduce the direct (86) and inverse
(87) Fourier transformations to a transformation, referring to the fundamental
solutions \( \phi_{\lambda,k}(r) \) and \( f_{\lambda,k}(r) \). One has
\[ F(r) = \frac{i}{\sqrt{\pi}} \sum_{j=1,2} \sum_{n=-\infty}^{\infty} C(\lambda_n^{(j)}, R) e^{-i(\mp \varepsilon(j) \alpha + \delta_1 + \delta_2)/2} \phi_{\lambda,k}(r) = \]
\[ = \frac{(\pi R)^{1/2}}{\xi_L + \xi_U} \sum_{j=1,2} \sum_{n=-\infty}^{\infty} \mp i \varepsilon(j) C(\lambda_n^{(j)}, R) e^{i(\mp \varepsilon(j) \alpha - \delta_1 + \delta_2)/2} f_{\lambda,k}(r) \]
(92)
in the asymptotic regime, and
\[ F(r) = \frac{1}{2\sqrt{\pi}} \sum_{j=1,2} \int_{-\infty}^{\infty} C^{(j)}(\lambda, R) e^{-i(\mp \varepsilon(j) \alpha + \delta_1 + \delta_2)/2} \phi_{\lambda,k}(r) d\lambda = \]
\[ = \mp \frac{i}{2} \left( \frac{R}{\pi} \right)^{1/2} \sum_{j=1,2} C^{(j)}(\lambda, R) e^{i(\mp \varepsilon(j) \alpha - \delta_1 + \delta_2)/2} f_{\lambda,k}(r) d\lambda \]
(93)
in the continuum limit.

Writing the latter expansion in the form of a transformation with respect
to the fundamental solution \( \phi_{\lambda,k}(r) \):
\[ F(r) = \int_{-\infty}^{\infty} D(\lambda, R) \phi_{\lambda,k}(r) d\Im \lambda, \quad k = -\frac{\Im \lambda}{R}, \]
(94)
where
\[ D(\lambda, R) = \frac{i\sqrt{\pi}}{2} e^{-i\delta_1 + \delta_2} \sum_{j=1,2} C^{(j)}(\lambda, R) e^{\pm \varepsilon(j) \alpha / 2} = \]
\[ = -\frac{1}{2} \int_{0}^{\infty} (\phi_{\lambda,k}(r) \cos \alpha \ e^{-i(\delta_1 + \delta_2)} - \phi_{-\lambda,k}(r)) F(r) d\mu(r), \]
(95)
with the help of (77) we finally obtain the transformation, inverse to (94),
\[ D(\lambda, R) = \frac{i\text{Im}\lambda}{f(\lambda, -k)} \int_{0}^{\infty} F(r) f_{\lambda, -k}(r) d\mu(r), \quad k = -\frac{\text{Im}\lambda}{R}, \quad (96) \]
wherein \( f_{\lambda, -k}(r) \) is the other, independent fundamental solution.

We can also write a transformation, dual to (94), (96). Writing the second line of (93) in the form
\[ F(r) = \int_{-\infty}^{\infty} B(\lambda, R) f_{\lambda, k}(r) d\text{Im}\lambda, \quad k = -\frac{\text{Im}\lambda}{R}, \quad (97) \]
where
\[ B(\lambda, R) = \frac{\pm i}{2} \left( \frac{R}{\pi} \right)^{\frac{1}{2}} \left( \sum_{j=1,2} \varepsilon^{(j)} e^{\mp \varepsilon^{(j)} \alpha/2} C^{(j)}(\lambda, R) \right) e^{i(\delta_2 - \delta_1)/2} = \]
\[ = -i \frac{\sqrt{R}}{2\pi} e^{\delta_2} \sin \alpha \int_{0}^{\infty} F(r) \phi_{-\lambda, k}(r) d\mu(r). \quad (98) \]
We used (79) in the expression for \( C \) (89). With the help of (77) we finally obtain the transformation, inverse to (97) in the form
\[ B(\lambda, R) = \frac{-i\text{Im}\lambda}{\pi f(-\lambda, k)} \int_{0}^{\infty} F(r) \phi_{-\lambda, k}(r) d\mu(r), \quad k = -\frac{\text{Im}\lambda}{R}. \quad (99) \]

It is remarkable, that the transforms \( D(\lambda, R) \) and \( B(\lambda, R) \) do not depend upon choice of sign in (49), in other words the validity of transformations (94), (96) and (97), (99) could be established with the help of any of the two self-adjoint limiting boundary problems.

If there are some bound states, after equation (6) is continued beyond sector IV, the expansion (86) should be supplemented by the sum, with \( R \) being the same positive parameter as in the rest of the expansion (86),
\[ \sum_{s=1}^{s_0} C_s \phi_{\lambda_s, k_s}(r), \quad \text{Im} k_s = \frac{\text{Re}\lambda_s}{R} > 0, \quad \text{Re} k = \text{Im}\lambda = 0 \quad (100) \]
where
\[ \phi_{\lambda_s, k_s}(r) = \frac{-R}{2\lambda_s} f(\lambda_s, k_s) f_{\lambda_s, -k_s}(r) \quad (101) \]
is the solution of equation (8), taken on the zeros of the Jost function \( f(\lambda_s, -k_s) = 0 \) (this means that \( E = 0 \) in (57)). It decreases, when \( r \to 0 \), as \( r^{-\lambda+1/2} \), and
as \( \exp\{-\text{Im}kr\} \), when \( r \to \infty \). Hence, the both sides in eqs. (49), expressing the boundary conditions, vanish in the limit \( r_L = 0, r_U = \infty \), and these conditions are satisfied. Correspondingly, two eigenfunctions \( \phi_{\lambda_s,k_s}(r) \) are orthogonal with the measure \( (8) \), provided that the values of \( \lambda_s \) for them are different, but the values of \( R = \text{Re}\lambda_s/\text{Im}k_s \) are coinciding. Therefore, (101) make solutions to the eigenvalue problem (6), (49).

### 3.5 S-matrix

Restrict ourselves to the case \( \text{Im}\lambda < 0, \text{Re}k > 0 \) for definiteness. By equalizing the probability fluxes near the origin and the infinitely remote point we get from (30) and (57) in a standard way the probability conservation relations

\[
\left| \frac{|G|^2}{|E|^2} \right| - \frac{\text{Im}\lambda}{k|E|^2} = 1, \quad \left| \frac{|C|^2}{|D|^2} \right| - \frac{k}{\text{Im}\lambda |D|^2} = 1.
\]  

(102)

The first eq. (102) means that the coefficient of reflection of the wave, incoming from infinity, plus the coefficient of transmission of this wave into the inner world makes unity. The second eq. (102) reads: the coefficient of reflection of the wave, emitted from the origin, back into the origin plus the coefficient of transmission of the emitted wave to infinity is unity.

Define the two two-component columns

\[
\Lambda = \begin{pmatrix} f_{\lambda,-k}(r) \\ \sigma \phi_{\lambda,k}(r) \end{pmatrix}, \quad \Lambda' = \begin{pmatrix} f_{\lambda,k}(r) \\ \sigma \phi_{-\lambda,k}(r) \end{pmatrix}, \quad \text{with} \quad \sigma = \left( \frac{k}{\text{Im}\lambda} \right)^\frac{1}{2}.
\]  

(103)

Then relations (30) and the first one of (57) can be expressed using a square \( 2 \times 2 \) matrix \( S \)

\[
\Lambda' = S \Lambda.
\]  

(104)

The unitarity of the \( S \)-matrix

\[
SS^\dagger = 1,
\]  

(105)

where \( S^\dagger \) is Hermitian conjugate to \( S \), follows from the relations (102) and from the complex conjugation rules (10) for the Jost functions:

\[
f^*(\lambda, k) = f(\lambda^*, k^* \exp(-i\pi)).
\]  

(106)
The $S$-matrix in (104) is (we used the relation $D = -E\sigma^2$)

$$S = \begin{pmatrix} \frac{-G}{E} & \frac{1}{\sigma E} \\ -\frac{1}{\sigma E} & \frac{C}{E} \end{pmatrix} = e^{-i\delta_2} \begin{pmatrix} \exp(i\delta_1) \cos \alpha & \sin \alpha \\ -\sin \alpha & \exp(-i\delta_1) \cos \alpha \end{pmatrix},$$

(107)

where

$$\delta_1 = \arg f(\lambda, k), \quad \delta_2 = \arg f(\lambda, -k),$$

$$\tan \alpha = \frac{2\sqrt{-k\Im \lambda}}{|f(\lambda, k)|}.$$ (108)

The two scattering phases $\delta_1, \delta_2$ and the channel-mixing nonelasticity angle $\alpha$ are real in sector IV, $\Im \lambda < 0, k > 0$. The above definition of the $S$-matrix is subject to an arbitrariness. Without affecting the unitarity and the meaning of the $S$-matrix elements we can change the normalization by multiplying $\sigma$ by a unit length complex number $\exp(i\delta_3)$. Then, the off-diagonal elements $S_{12}$ and $S_{21}$ in (107) are multiplied by $\exp(\mp i\delta_3)$, resp. with the phase $\delta_3$ arbitrary. The values the angles (66) take when $V = 0$ may be found in [7].

In sector IV the $S$-matrix elements $S_{11}$ and $S_{22}$ are no longer unit length complex numbers, as they are in the cases of elastic scattering in the outer world when $\lambda^2 > 0, k^2 > 0$ ($S_{22} = S_{12} = 0, |S_{11}| = 1$) or elastic scattering in the inner world of Sec. 2 ($S_{11} = S_{12} = 0, |S_{22}| = 1$). Here the unitarity can only be formulated with the inclusion of the elements $S_{12}, S_{21}$ responsible for transitions between the two channels as it was done above. It would be inappropriate to be taking into account the nonelasticity of the scattering process in sector IV by analytic continuation with respect to $k$ or $\lambda$ from any of the elastic sectors, as it is prescribed within the approach of Ref. [12]. The analytic continuation makes the corresponding scattering phase complex but is unable to create the lacking phase and the nonelasticity angle: a description of the system with a greater number of degrees of freedom cannot be achieved by mere analytic continuation. We have seen in this Section that the singular Schrödinger equation is no longer a single-particle equation.

Note, that in the present subsection we did not refer to any definite choice of boundary conditions. The only necessary condition for the $S$-matrix to be defined is that the fundamental solutions in (103) may be interpreted as asymptotically-free particles at the both end of the interval $(0, \infty)$, which is guaranteed in our procedure with any boundary conditions from the self-adjoint class. When defining (104) we acted differently from what we did when handling sector III. Namely, unlike (11), we took in (103) the whole
\( \phi_{\pm \lambda k}(r) \), and not its \( \delta \)-function-normalizable part. The two differ by the factor \( R^{\pm \text{Im} \lambda} \). For a more general choice of boundary conditions, resulting from the substitution of arbitrary parameter \( r_0 \) for \( R \) in (49), we would encounter the arbitrariness due to the factor \( r_0^{\pm \text{Im} \lambda} \) in the "noninvariant" \( S \)-matrix, defined like (41). This is only one-parameter dependence on the way of self-adjoint extension, whereas the general case is \( U(2) \).

3.6 Standard Liouville form

Let us perform the following transformation of the wave function and of the variable in the Schrödinger equation (cf. [8]) in sector IV

\[
\tilde{\psi}(\xi) = \psi(r) \left( \frac{1}{r^2} + \frac{1}{R^2} \right)^{\frac{1}{4}},
\]
(109)

\[
\xi(r) = \int_{\frac{i \text{Im} \lambda}{k}}^{r} \left( \frac{1}{r^2} + \frac{1}{R^2} \right)^{\frac{1}{2}} dr = \int_{1}^{y} \frac{dy}{y} (1 + y^2)^{\frac{1}{4}},
\]
\[
y = r \left( \frac{-k}{\text{Im} \lambda} \right) = \frac{r}{R} > 0.
\]
(110)

Note, that unlike [8], [5] the lower limit of integration in the transformation (110) is not the lower end of the interval, on which the differential equation is defined. This transformation cannot be used in sector III, because the expression under the root signs in (109), (111) may vanish for \( r \in (0, \infty) \) there.

The probability flux (17) is form-invariant under the transformations (109), (110).

The Schrödinger equation (11), (2) takes the standard Liouville form

\[- \frac{d^2 \tilde{\psi}(\xi)}{d\xi^2} + U(R; \xi) \tilde{\psi}(\xi) = (\text{Im} \lambda)^2 \tilde{\psi}(\xi). \]
(111)

The potential in (111) is

\[
U(R; \xi) = U_0(\xi) + U_V(R; \xi),
\]

\[
U_0(\xi) = \frac{1 + 6y^2}{4(1 + y^2)^3} - \frac{1}{4(1 + y^2)};
\]

\[
U_V(R; \xi) = \frac{y^2}{1 + y^2} R^2 V(yR).
\]
(112)
Here $y$ is a function of $\xi$ to be obtained by inverting (110). This function $y(\xi)$ does not depend on the parameters $k$, $\lambda$. Consequently, when $V = 0$, the potential $U(R, \xi) = U_0(\xi)$ does not depend on them either. If, however, $V \neq 0$, the potential $U(R, \xi)$ depends upon the combination of the parameters $|k/\text{Im}\lambda|$, denoted as $|R|$ in the previous subsections.

For the case $V(r) = 0$ the effective barrier potential $U(\xi)$ is plotted in Fig. 2. The inclusion of the potential $V \neq 0$, subject to conditions (3), does not affect the asymptotic values $U(\pm\infty)$. Near $r = \infty$ the variable $\xi$ becomes

Figure 2: The effective barrier potential for confinement/deconfinement processes in the case of $V = 0$. $U(\infty) = U(-\infty) = 0$. The maximum is achieved at the value of $r$, determined by the dimensional parameter $|R|$: $r_{\text{max}} = (5 - 21^{1/2})|R|$ ($\xi_{\text{max}} = -0.56$)

$\xi = r/|R|$, $U(\infty) = 0$, and equation (111) takes the usual asymptotic form of the Schrödinger equation

$$-\frac{d^2\tilde{\psi}}{dr^2} = k^2\tilde{\psi}, \quad \tilde{\psi} = \sqrt{\frac{|k|}{\text{Im}\lambda}} \psi,$$

(113)

describing particles, free in the infinitely remote region. Near $r = 0$ the barrier potential is $U(-\infty) = 0$, the variable $\xi$ is $\xi = \ln(r/|R|)$, and equation (111) is an equation for particles, free in the region remote to the negative infinity,
with the solution given as (13).

\[-d^2\tilde{\psi}(\xi) = (\text{Im} \lambda)^2\tilde{\psi}(\xi), \quad \tilde{\psi}(\xi) = \frac{\psi}{\sqrt{r}}.\]  \hspace{1cm} (114)

Thus, we face one-dimensional Schrödinger equation within infinite limits with the barrier potential that decreases at the both sides. It introduces the pattern of reflection and transmission, which we have studied above in this section directly in the primary variable \(r\), without appealing to the transformation \((109), (110)\).

For large \(r_U\) and small \(r_L\) the transformation \((110)\) maps the box \((47)\) into the box in the \(\xi\)-space:

\[-\xi_L \leq \xi \leq \xi_U, \quad \xi_L \gg 1, \quad \xi_U \gg 1,\]  \hspace{1cm} (115)

where the limits \(\xi_L = -\xi(r_L) = -\ln(r_L/|R|)\), \(\xi_U = \xi(r_U) = r_U/|R|\) are meant to be independent of the parameters \(\lambda\) and \(k\), and are connected with the limits \(r_L, r_U\) in the initial space according to \((48)\). In the same asymptotic regime the boundary conditions \((49)\) are transformed into the following periodic (antiperiodic) boundary conditions in the box \((115)\):

\[\tilde{\psi}(\xi_L) = \pm \tilde{\psi}(\xi_U),\]

\[
\frac{d\tilde{\psi}(\xi)}{d\xi} \bigg|_{\xi=-\xi_L} = \pm \frac{d\tilde{\psi}(\xi)}{d\xi} \bigg|_{\xi=\xi_U}.
\]  \hspace{1cm} (116)

The self-adjointness of the eigenvalue problem \((111)\) in the box \((115)\) with the periodic (antiperiodic) boundary conditions \((116)\), with the potential \((112)\), which may at the most depend on the ratio \(\lambda/k\), and with the ends of the interval independent of \(\lambda\) and \(k\), is evident.

The eigenfunctions of the boundary problem \((111), (116)\) are expressed in terms of the eigenfunctions \((79), (81)\) as

\[\tilde{\psi}^{(j)\lambda, R}(\xi) = \left(\frac{1}{R^2} + \frac{1}{r^2}\right)^{\frac{1}{2}} \psi^{(j)\lambda, R}(r),\]  \hspace{1cm} (117)

the spectrum being given by solutions of equations \((68)\). The functions are real. The set \((117)\) is complete. The scalar product in the corresponding Hilbert space is defined with the plane measure \(d\xi\):

\[
(\tilde{\psi}_1(\xi), \tilde{\psi}_2(\xi)) \equiv \int_{\xi_L}^{\xi_U} (\tilde{\psi}_1(\xi))^* \tilde{\psi}_2(\xi) d\xi =
\]

\[
= \int_{r_L}^{r_U} (\psi_1(r))^* \psi_2(r) d\mu(r) \equiv (\psi_1(r), \psi_2(r)).
\]  \hspace{1cm} (118)
The intermediate equality in (118) is proved using (109), (110), (8). An arbitrary sufficiently smooth function \( \tilde{F}(\xi) \) of \( \xi \) from \( L^2(-\infty, \infty) \), i.e. such that \( \int_{-\infty}^{\infty} |\tilde{F}(\xi)|^2 d\xi < \infty \) is expanded into the generalized Fourier series over eigenfunctions (117) as

\[
\tilde{F}(\xi) = \frac{2\pi \sqrt{R}}{\xi_L + \xi_U} \sum_{j=1,2} \sum_{n=1}^{\infty} C(\lambda^{(j)}_n, R) \tilde{\psi}^{(j)}_{\lambda_n, R}(\xi). 
\] (119)

Owing to the orthogonality relations (84), which are the same for the eigenfunctions \( \tilde{\psi}^{(j)}_{\lambda_n, R}(\xi) \) due to (118), the expansion coefficients above are

\[
C(\lambda^{(j)}_n, R) = \frac{1}{\sqrt{R}} \int_{-\xi_L}^{\xi_U} \tilde{F}(\xi)(\tilde{\psi}^{(j)}_{\lambda_n, R})^* d\xi. 
\] (120)

This is equal to (88), provided that one identifies

\[
\tilde{F}(\xi) = F(r) \left(1 + \frac{R^2}{r^2}\right)^{\frac{1}{4}}. 
\] (121)

Both functions \( \tilde{F}(\xi) \) and \( F(r) \) are arbitrary and cannot depend upon the parameter \( R \), which only characterizes the set of eigenfunctions used in the expansion. This statement does not contradict to (121), since the factor \( (1+R^2/r^2) \) does not contain \( R \) after it is transformed to the variable \( \xi \) according to (110).

The eigenfunction (117) behaves near \( \xi = -\infty \) as

\[
\tilde{\psi}^{(j)}_{\lambda_n, R}(\xi) \simeq \frac{1}{\sqrt{\pi}} \sin \left( \frac{\pm \varepsilon^{(j)} \alpha + \delta_1 + \delta_2}{2} - \ln R \text{ Im} \lambda - \xi \text{ Im} \lambda \right), 
\] (122)

and near \( \xi = \infty \) as

\[
\tilde{\psi}^{(j)}_{\lambda_n, R}(\xi) \simeq \frac{\pm \varepsilon^{(j)} \alpha + \delta_1 - \delta_2}{\sqrt{\pi}} \sin \left( \frac{\pm \varepsilon^{(j)} \alpha + \delta_1 - \delta_2}{2} - \xi \text{ Im} \lambda \right). 
\] (123)

The logarithm of the dimensional parameter \( R \) is cancelled by an analogous logarithm that is contained in the sum \( \delta_1 + \delta_2 \), involved in (122), while the difference \( \delta_1 - \delta_2 \), involved in (123), does not contain such logarithm.

For convergence of (119) it is needed that \( \tilde{F}(\pm \infty) = 0 \). In view of (121) this requirement is in agreement with (10).

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