SPECIAL BENT AND NEAR-BENT FUNCTIONS

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Abstract. Starting from special near-bent functions in dimension $2t - 1$ we construct bent functions in dimension $2t$ having a specific derivative. We deduce new families of bent functions.

1. Introduction

Let $F_2$ be the finite field of order 2. An $m$-boolean function (or boolean function in $m$ dimensions) is a map $F$ from $F_2^m$ to $F_2$.

Bent functions are the boolean functions whose Fourier coefficients have constant magnitude. They were introduced by Rothaus in [8]. An $m$-boolean function $F$ is bent if all its Fourier coefficients are in $\{-2^{m/2}, 2^{m/2}\}$. Bent functions are of interest for Coding Theory, Cryptology and well-correlated binary sequences. For example, they have the maximum possible Hamming distance to the set of affine boolean functions. They were the topic of a lot of works (see [2, 3, 6, 7, 9, 10]) but the complete classification of bent functions and other questions are still open.

By definition, a $m$-boolean function $F$ is near-bent if all its Fourier coefficients are in $\{-2^{(m+1)/2}, 0, 2^{(m+1)/2}\}$. Since the Fourier coefficients are in $\mathbb{Z}$ the bent functions in $m$ dimensions exist only when $m$ is even and near-bent functions in $m$ dimensions exist only when $m$ is odd.

1.1. A two-variable representation. Assume $m = 2t$. We identify $F_2^t$ with $F_2^{2t-1}$ and $F_2^{2t}$ with

$$F_2^{2t-1} \times F_2 = \{ X = (u, \nu) \mid u \in F_2^{2t-1}, \nu \in F_2 \}.$$ 

This decomposition can be used for a two-variable representation of $(2t)$-boolean functions as follows

$$F_2^{(0)} = \{ (u, 0) \mid u \in F_2^{2t-1} \} \quad \text{and} \quad F_2^{(1)} = \{ (u, 1) \mid u \in F_2^{2t-1} \}.$$ 

The restrictions of a $(2t)$-boolean function $F$, respectively to $F_2^{(0)}$ and to $F_2^{(1)}$ induce two $(2t - 1)$-boolean functions $f_0$ and $f_1$ defined by $f_0(u) = F(u, 0)$ and $f_1(u) = F(u, 1)$.

The two-variable representation (TVR) of $F$ is then defined by

$$\phi_F(x, y) = (y + 1)f_0(x) + yf_1(x).$$

Note that $F(u, 0) = f_0(u) =$ $\phi_F(u, 0)$ and $F(u, 1) = f_1(u) =$ $\phi_F(u, 1)$. Hence if $X = (u, \nu)$ then $F(X) =$ $\phi_F(u, \nu)$.

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It can be proved that if $F$ is a $(2t)$-bent function then $f_0$ and $f_1$ are $(2t - 1)$-near-bent functions (see 3.1.4). This leads to considering the following inverse problem: construct $(2t)$-bent functions from $(2t - 1)$-near-bent functions. This was the purpose of [7] and in a certain sense of [10] where connections with cyclic codes were established.

Using the above representation of $\mathbb{F}_{2^t}$, it could be easily checked that $f_0 + f_1$ is the derivative of $F$ with respect to $(0, 1)$ (see next section for the definition of the derivative). A possible way to study bent functions is to classify the $m$ regarding the degree (as a boolean function) of this derivative. The most simple case to consider is when this degree is one. In the present paper we are concerned with bent functions such that $f_0 + f_1 = \text{tr} + \xi$ where $\text{tr}$ is the trace of $\mathbb{F}_{2^{t-1}}$ and $\xi \in \{0, 1\}$. We introduce the new notion of pseudo-duality and as application we present new families of bent functions. By the way we present a generalization of a result on the Gold function (Lemma 15 in Section 3.2).

2. Results

The proofs of the Theorems of this section will be given in Section 3 while the other results are proven in the present section.

First recall some usual definitions of a $m$-boolean function $F$.

- If $e \in \mathbb{F}_2^m$ then the Derivative of $F$ with respect to $e$ is the $m$-boolean function $D_e F$ defined by
  $$D_e F(X) = F(X) + F(X + e).$$

**Remark.** In this paper we consider the following special cases. If $F$ is a $(2t)$-boolean function whose two-variable representation is given in (1) then direct calculation shows that
  $$D_{(0,1)} F = f_0 + f_1.$$
If $f$ is a $(2t - 1)$-boolean function then
  $$D_1 f(x) = f(x + 1) + f(x).$$

- The Fourier transform (or the Walsh transform) $\hat{F}$ of $F$ is the map from $\mathbb{F}_2^m$ into $\mathbb{Z}$ defined by
  $$\hat{F}(v) = \sum_{X \in \mathbb{F}_2^m} (-1)^{F(X) + <v,X>},$$
  where $<,>$ denotes any inner product of $\mathbb{F}_2^m$ over $\mathbb{F}_2$. $\hat{F}(v)$ is called the Fourier coefficient of $v$.

**Remark.** The set of $\hat{F}(v)$ when $v$ runs through $\mathbb{F}_2^m$ is independent of the choice of the inner product $<,>$.

- $F$ is bent if all its Fourier coefficients are in $\{-2^{m/2}, 2^{m/2}\}$. $F$ is near-bent if all its Fourier coefficients are in $\{-2^{(m+1)/2}, 0, 2^{(m+1)/2}\}$.

- If $m = 2t$ and if $F$ is a bent function then the dual $\hat{F}$ of $F$ is the $(2t)$-boolean function defined by
  $$\hat{F}(v) = (-1)^{\hat{F}(v)} 2^t,$$
  where $\hat{F}$ is the Fourier transform of $F$. It is well-known, and easy to prove that the dual of a bent function is a bent function (see [3] or [8]).

- The algebraic degree, or more simply “the degree” of an $m$-boolean function is the degree of its $m$-variable polynomial representation. If it is expressed as $\text{Tr} (\pi(X))$ where $\text{Tr}$ is the trace function of $\mathbb{F}_{2^m}$ then its degree is the maximum
of the binary weight of the monomial exponents of \( \pi(X) \). Recall that the binary weight of an integer is the number of non-zero coefficients of the binary expansion of this integer.

See References section for details on the previous definitions and results.

2.1. MAIN THEOREMS. From now on, we use the definitions and notations of the introduction. Thus \( \mathbb{F}_2^{2t} \) is identified with \( \mathbb{F}_{22t} \) and \( \mathbb{F}_{22t} \) with \( \mathbb{F}_{22t-1} \times \mathbb{F}_2 \). The two-variable representation (TVR) of a \((2t)\)-boolean function \( F \) is defined by \( \phi_F(x, y) = (y + 1)f_0(x) + yf_1(x) \) where \( f_0 \) and \( f_1 \) are defined in Section 1. Obviously, \( \phi_{F_1 + F_2} = \phi_{F_1} + \phi_{F_2} \).

Definition. \( f_0 \) and \( f_1 \) are called the components of \( F \).

Notation. The trace of \( \mathbb{F}_{22t-1} \) is denoted by \( tr \) and defined by

\[
tr(x) = \sum_{i=0}^{2t-2} x^{2^i}.
\]

The next theorem sets a condition on a near-bent function to be the first component of a bent function.

Theorem 1. Let \( f_0 \) be a \((2t-1)\)-near-bent function. If the derivative \( D_1f_0 \) is a constant function then the \((2t)\)-boolean function \( F \) such that

\[
\phi_F(x, y) = (y + 1)f_0(x) + yf_1(x)
\]

with \( f_0 + f_1 = tr \) is a bent function.

We now present a pseudo-reciprocal theorem.

Theorem 2. Let \( F \) be a bent function and \( \tilde{F} \) be its dual bent function with

\[
\phi_{\tilde{F}}(x, y) = (y + 1)f_0(x) + y\tilde{f}_1(x)
\]

and \( \phi_F(x, y) = (y + 1)f_0(x) + y\tilde{f}_1(x) \).

If \( f_0 + f_1 = tr \), then \( D_1\tilde{f}_0 = 0 \) and \( D_1\tilde{f}_1 = 1 \).

The converse of Theorem 1 is not true. In other words, it is not true that if \( f_0 + f_1 = tr \) then \( D_1f_0 \) is a constant function, as it will be seen in Example 2. However, there is a special case given by the next corollary which follows immediately from Theorem 2.

Corollary 3. With the above notation, let \( F \) be a bent such that \( f_0 + f_1 = tr \). If \( F \) is self dual, say \( F = \tilde{F} \), then \( D_1f_0 = 0 \).

Relations between the components of the dual \( \tilde{F} \) of a bent function \( F \) such that \( f_0 + f_1 = tr \) are now presented in the next result.

Theorem 4. Let \( F \) be a bent function and let \( \tilde{F} \) be its dual bent function with

\[
\phi_{\tilde{F}}(x, y) = (y + 1)f_0(x) + y\tilde{f}_1(x)
\]

and \( \phi_F(x, y) = (y + 1)f_0(x) + y\tilde{f}_1(x) \).

Assume \( f_0 + f_1 = tr \). Let \( \tilde{f}_0 \) be the Fourier transform of \( f_0 \). Define \( S = \{ v \in \mathbb{F}_{22t-1} \mid \tilde{f}_0(v) = -2^t \} \) and \( S_1 = \{ u + 1 \mid u \in S \} \). Let \( G = \{ v \in \mathbb{F}_{22t-1} \mid \tilde{f}_0(v) = 0 \} \). Let \( g \) be the characteristic function of \( G \).

1) The support of \( \tilde{f}_0 \) is \( S \cup S_1 \);
2) \( \tilde{f}_1(x) = \tilde{f}_0(x) + g(x) \).

The next theorem states properties of a bent function in the case when the hypothesis of Theorems 1 and 2 are both satisfied.
Theorem 5. Let $H$ be a bent function and let $\tilde{H}$ be its dual bent function with
\[ \phi_H(x, y) = (y + 1)h_0(x) + yh_1(x) \quad \text{and} \quad \phi_{\tilde{H}}(x, y) = (y + 1)\tilde{h}_0(x) + y\tilde{h}_1(x). \]
Assume $h_0 + h_1 = tr$.
\begin{itemize}
  \item[a)] If $D_1h_0 = 0$ then $\tilde{h}_0 + \tilde{h}_1 = tr$;
  \item[b)] If $D_1h_0 = 1$ then $\tilde{h}_0 + \tilde{h}_1 = tr + 1$.
\end{itemize}

2.2. Pseudo-duality. The previous results lead to the introduction of a new def-
inition (notations are above).

Definition 6. Let $G$ be a $(2t)$-bent function and let $\tilde{G}$ be its dual bent function with
\[ \phi_G(x, y) = (y + 1)g_0(x) + yg_1(x). \]
The Pseudo-duals of $G$ are the two $(2t)$-boolean function $G_0$ and $G_1$ defined by
\[ \phi_{G_0}(x, y) = (y + 1)\tilde{g}_0(x) + y(\tilde{g}_0(x) + tr(x)); \]
\[ \phi_{G_1}(x, y) = (y + 1)\tilde{g}_1(x) + y(\tilde{g}_1(x) + tr(x)). \]

The meaning of this definition is given by the next theorem

Theorem 7. Define the following two conditions on a $(2t)$-bent functions $G$ with
\[ \phi_G(x, y) = (y + 1)g_0(x) + yg_1(x). \]
\begin{itemize}
  \item [(T)] $g_0 + g_1 = tr + \xi$ with $\xi \in \{0, 1\}$;
  \item [(C)] $D_1g_0 = 0$.
\end{itemize}
If $F$ is a $(2t)$-bent function meeting condition (T) then
\begin{itemize}
  \item [A)] The pseudo-duals $\tilde{F}_0$ and $\tilde{F}_1$ are bent functions;
  \item [B)] The dual $\tilde{F}_0$ of $\tilde{F}_0$ meets (C) and (T) with $\xi = 0$;
  \item [C)] The dual $\tilde{F}_1$ of $\tilde{F}_1$ meets (C) and (T) with $\xi = 1$.
\end{itemize}

2.3. New families of bent functions. Let $F$ be a family of $(2t)$-boolean functions, define $\mathcal{F}_0 = \{F_0 \mid F \in \mathcal{F}\}$ and $\mathcal{F}_1 = \{F_1 \mid F \in \mathcal{F}\}$.

By applying the previous theorem, if $F$ is a family of bent functions meeting
(T), then $\mathcal{F}_0$ and $\mathcal{F}_1$ are new families of bent functions.

This is the case in the next proposition by using a family introduced in [7, Theorem 9].

2.3.1. The Kasami-Welch example. This definition comes from the description of the near-bent function $f_0$ introduced in this example which is a classical object in
the theory of boolean functions.

Proposition 8. Let $t, s, d$ be integers such that
\begin{itemize}
  \item $2t - 1$ is not divisible by 3, $3s \equiv \pm 1 \mod (2t - 1)$;
  \item $s < t$, $d = 4^s - 2^s + 1$.
\end{itemize}
Let $F$ be the $(2t - 1)$-boolean function with
\[ \phi_F(x, y) = (y + 1)tr(x^d) + ytr(x^d + x). \]
The assertions A), B), C) of Theorem 7 hold for $F$.

Proof. In [7, Theorem 9] it is proved that $F$ is a bent function. Since $F$ satisfies
(T) then Theorem 7 applies and gives the expected result. \hfill \Box

Remark. It is proved in [4] that in the Kasami-Welch case, if $g$ is the function introduced in Theorem 4 then $g(x) = 1 + tr(x^{2^s+1})$. 

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Example. \( t = 4, s = 2 \).

\( F: \) \( f_0(x) = \text{tr}(x^{13}), \) \( f_1(x) = f_0(x) + \text{tr}(x), \)

\( \tilde{F}: \) \( \tilde{f}_0(x) = \text{tr}(x^7 + x^{11} + x^{19} + x^{21}), \) \( \tilde{f}_1(x) = \tilde{f}_0(x) + \text{tr}(x^5 + 1). \)

New bent functions

\( \tilde{F}_0: \) \( \tilde{\tilde{f}}_0(0) = \tilde{\tilde{f}}_0(0), \) \( \tilde{\tilde{f}}_1(0) = \tilde{\tilde{f}}_0(x) + \text{tr}(x), \)

\( \tilde{F}_1: \) \( \tilde{\tilde{f}}_0(1) = \tilde{\tilde{f}}_1(x), \) \( \tilde{\tilde{f}}_1(1) = \tilde{\tilde{f}}_1(x) + \text{tr}(x), \)

\( \tilde{\tilde{F}}_0: \) \( \tilde{\tilde{f}}_0(0) = \text{tr}(x + x^3 + x^{11} + x^{19} + x^{21}), \)

\( \tilde{\tilde{F}}_1: \) \( \tilde{\tilde{f}}_0(1) = \text{tr}(1 + x^5 + x^7 + x^9 + x^{11} + x^{19} + x^{21}), \)

\( \tilde{\tilde{f}}_1(1) = \tilde{\tilde{f}}_0(0) + \text{tr}(x + 1). \)

Remark. In the above example \( D_1 f_0 \) is not a constant function.

2.3.2. The quadratic case. In the next proposition we study the case where \( f_0 \) is quadratic and such that \( f_0(x) = \text{tr}(\pi(x)) \) where all the coefficients of \( \pi(x) \) are in \( \mathbb{F}_2 \).

Proposition 9. Let \( f_0 \) be a \((2t-1)\)-near-bent function such that

1) Then the \((2t)\)-boolean function \( F \) such that

\[ \phi_F(x, y) = (y + 1)f_0(x) + yf_1(x) \quad \text{with} \quad f_1(x) = f_0(x) + \text{tr}(x) \]

is a bent function.

2) The dual \( F \) of \( F \) meets \((T)\).

3) The assertions \( A), B), C) \) of Theorem 7 hold for \( F \).

Proof. 1) First notice that \( r \) and \( s \) are in \( \mathbb{N} \), then for a convenient exponent \( l \),

\[ \text{tr}(x^r + x^s) = \text{tr}[(x^r + x^s)^2] = \text{tr}(x^{2r+1}) \]

for some \( j \). If follows that we can express \( f \) as \( f_0(x) = \xi + \sum_J \text{tr}(x^{2j+1}) \) where \( J \) is a subset of \( \mathbb{N} \) with \( J \neq \{0\} \) and \( \xi \in \mathbb{F}_2 \). Now, it is easy to check that \( x^{2j+1} + (x + 1)^{2j+1} = x^{2j+1} + x + 1 \) and thus \( \text{tr}(x^{2j+1}) + \text{tr}((x + 1)^{2j+1}) = 1 \). Therefore \( f_0(x) + f_0(x+1) = \sum_J 1 \) and \( f_0(x) + f_0(x+1) = 0 \) if \( |J| \) is even and \( f_0(x) + f_0(x+1) = 1 \) if \( |J| \) is odd. In both cases \( D_1 f_0 \) is a constant function then Part 1) is a consequence of Theorem 1.

2) Since \( f_0 + f_1 = \text{tr} \) and \( D_1 f_0 \) is a constant function then Theorem 5 applies.

3) This is a direct consequence of Theorem 7.

Remark. Observe that for any \((2t)\)-boolean function \( G \) whose components are \( g_0 \) and \( g_1 \), the TVR of \( G \phi_G(x, y) \) can be rewritten as

\[ \phi_G(x, y) = y(g_0(x) + g_1(x)) + g_0(x). \]

If \( G = F \) then \( \phi_F(x, y) = y \text{tr}(x) + f_0(x) \). Since the degree of \( f_0 \) is \( 2 \) we deduce that the degree of \( F \) also is \( 2 \). Similarly for \( \tilde{F}: \phi_{\tilde{F}}(x, y) = y(\tilde{f}_0(x) + \tilde{f}_1(x)) + \tilde{f}_0(x). \)

We know from \([6, \text{Lemma 2.5}]\) that if the degree of \( F \) is \( 2 \) then \( \tilde{F} \) also has degree \( 2 \). From \( \phi_{\tilde{F}}(x, y) = y(\tilde{f}_0(x) + \tilde{f}_1(x)) + \tilde{f}_0(x) \) we deduce that degree of \( \tilde{f}_0 \) is \( 2 \). It follows that for every bent function among \( F, \tilde{F}, \tilde{F}_0, \tilde{F}_1, \tilde{\tilde{F}}_0, \tilde{\tilde{F}}_1 \), the degree is \( 2 \) and
then the components have degree 2. Obviously, \( \tilde{F}_0 = \tilde{F} \), \( \tilde{F}_0 = F \). It is shown in [7] that if \( \deg f_0 = 2 \) there exists \( e \) in \( \mathbb{F}_{2t-1} \) such that \( f_0(x) + f_1(x) = tr(ex) \). Another way to prove 1) is to show that \( e = 1 \) when \( \pi(x) \) is binary.

**Example.** \( t = 4 \).

\[
F : \quad f_0(x) = tr(x^3 + x^9), \quad f_1(x) = f_0(x) + tr(x).
\]

\[
\tilde{F} : \quad \tilde{f}_0(x) = tr(x^9 + x), \quad \tilde{f}_1(x) = \tilde{f}_0(x) + tr(x) = tr(x^9).
\]

\[
\tilde{F}_0 = \tilde{F}, \quad \tilde{F}_0 = F.
\]

\[
\tilde{F}_1 : \quad \tilde{f}_0^{(1)}(x) = \tilde{f}_1(x), \quad \tilde{f}_1^{(1)}(x) = \tilde{f}_1(x) + tr(x) = \tilde{f}_0(x).
\]

\[
\tilde{F}_1 : \quad \tilde{\tilde{f}}_0^{(1)}(x) = tr(x + x^3 + x^9), \quad \tilde{\tilde{f}}_1^{(1)}(x) = \tilde{\tilde{f}}_0^{(1)}(x) + tr(x + 1).
\]

2.4. **Comments.**

- Note that starting from a near-bent function \( f \) whose derivative \( D_1 f \) is a constant function and applying Theorem 1 and Theorem 7, we are in position to construct six bent functions.
- It is easy to check that, for every boolean function \( f \),

\[
D_1(f + 1) = D_1(f), \quad D_1(f + tr) = D_1(f) + 1,
\]

if \( f(0) = 1 \) then \( (f + 1)(0) = 0 \).

Now assume that \( f \) is a near-bent function. As it will be shown by (R2) in Section 3.1.2, \( f + 1 \), \( f + tr \) and \( f + tr + 1 \) are also near-bent functions. From the above-mentioned results \( D_1(f + 1) = 0 \) or \( D_1(f + tr) = 0 \).

If \( f(0) = 0 \) then \( (f + 1)(0) = 1 \) and \( (f + 1 + tr)(0) = 0 \). We deduce that among \( f \), \( f + 1 \), \( f + tr \), \( f + tr + 1 \) there always exists a near-bent function \( h \) such that \( D_1(h) = 0 \) and \( h(0) = 0 \). Therefore, in order to apply Theorem 1 it is sufficient to find a near-bent function \( f \) such that \( D_1(f) = 0 \) and \( f(0) = 0 \).

In this case, consider the following polynomial \( p(X) = \sum_{i=0}^{2t-2} f(\alpha^i)X^i \) where \( \alpha \) is a primitive root of \( \mathbb{F}_{2t-1} \). As pointed out in [10], this is the representation of a word of a special cyclic code of length \( 2t - 1 \) over \( \mathbb{F}_2 \) which depends on the degree of \( f \).

- The map sending a bent function to its pseudo-dual is not injective. For example the bent function defined by \( f_0(x) = tr(x^7 + x^{13} + x^{19} + x^{21}) \) and \( f_1(x) = f_0(x) + tr(x) \) and the bent function such that \( f_0(x) = tr(x^3 + x^{11}) \) and \( f_1(x) = f_0(x) + tr(x) \) have different duals, but the same pseudo-dual.
- Starting from a bent function \( F \) which fulfill (C) or (T) it is possible to find other bent functions with the same properties. The bent functions \( F \), the dual \( \tilde{F} \), the pseudo-duals \( \tilde{F}_0 \) and \( \tilde{F}_1 \) and the duals of these pseudo-duals could be either distinct or not. The examples in Section 4 show different situations.

- In a very interesting paper [7] by Leander and McGuire the authors consider the two-variable representations of boolean functions with other notations. In particular, they introduce a characterization of near-bent functions \( f \) such that \( f \) and \( f + tr \) are the components of a bent function (Theorem 3, \( e = 1 \)). This could be used to obtain an alternative proof of Theorem 1.

3. **Proofs**

3.1. **Preliminaries.**
3.1.1. Notation. Let $F$ be an $m$-boolean function.

- The weight of $F$ is defined by $w(F) = \sharp \{v \in \mathbb{F}_2^m \mid F(v) = 1\}$.
- The TVR of $F$ is defined as in Section 1 by
  $$\phi_F(x, y) = (y + 1)f_0(x) + yf_1(x).$$
- The Fourier transform of $F$ is defined as in Section 2.
- $T_v$ denotes the linear form of $\mathbb{F}_2^m$ defined by $T_v(X) = \langle v, X \rangle$.
- If $m = 2t - 1$ then $tr$ denotes the trace function of $\mathbb{F}_{2^{2t-1}}$ and the map $x \rightarrow tr(ax)$ is denoted by $t_a$.

3.1.2. Elementary and known results. We begin by summarizing some of the elementary or classical results on bent and near-bent functions (see \cite{1, 2, 9}).

(R1) $\tilde{F}(v) = 2^m - 2w(F + T_v)$.

(R2) Let $F$ be a $m$-boolean function and let $L$ be an affine linear form of $\mathbb{F}_2^m$. $F$ is a bent function if and only if $F + L$ is a bent function. $F$ is a near-bent function if and only if $F + L$ is a near-bent function.

(R3) $w(F) = w(f_0) + w(f_1)$.

(R4) $F$ is bent if and only if $\forall V \in \mathbb{F}_{2^n}$ $D_V F$ is balanced. That is
$$\sharp \{U \in \mathbb{F}_{2^n} \mid D_V F(U) = 1\} = \sharp \{U \in \mathbb{F}_{2^n} \mid D_V F(U) = 0\}.$$

(R5) Let $\tilde{F}$ be the dual of a $(2t)$-bent function $F$. Then $\tilde{F}(v) = 1$ if and only if $\tilde{F}(v) = -2^t$.

(R6) If $m = 2t - 1$ and $f$ is a near-bent function then the distribution of Fourier coefficients is as follows

- $\hat{f}(v) = 2^t$, number of $v : 2^{2t-3} + (-1)^{f(0)}2^{t-2}$;
- $\hat{f}(v) = 0$, number of $v : 2^{2t-2}$;
- $\hat{f}(v) = -2^t$, number of $v : 2^{2t-3} - (-1)^{f(0)}2^{t-2}$.

Comment. (R1), (R2) and (R5) follow immediately from the definitions. (R3) is obtained with straightforward calculations and (R4) is classical. The distribution given in (R6) is a special cases of Proposition 4 in \cite{2}.

3.1.3. Representation of $(2t)$-linear forms. The purpose of this part is to express linear forms and the inner product $\langle \cdot, \cdot \rangle$ used in the calculation of the Fourier coefficients in such a way which is consistent with the decomposition of $\mathbb{F}_{2^n}$ as $\mathbb{F}_{2^{2t-1}} \times \mathbb{F}_2$.

Definition 10. For every $(a, \eta)$ in $\mathbb{F}_{2^n}$ the map $L_{(a, \eta)}$ from $\mathbb{F}_{2^n}$ into $\mathbb{F}_2$ is defined by
$$L_{(a, \eta)}(x, \nu) = tr(ax) + \eta \nu.$$

Remark. $L_{(a, \eta)}(x, \nu)$ is nothing but $tr(ax) + t_r^{(1)}(\eta \nu)$ where $t_r^{(1)}$ is the trace of $\mathbb{F}_2$ since $t_r^{(1)}(\mu) = \mu$ for every $\mu$ in $\mathbb{F}_2$.

It can be easily checked that

(*) $L_{(a, \eta)}$ is a linear form of $\mathbb{F}_{2^n}$.

(**) The map $(a, \eta) \rightarrow L_{(a, \eta)}$ from $\mathbb{F}_{2^n}$ to $\{L_{(a, \eta)} \mid (a, \eta) \in \mathbb{F}_{2^n}\}$ is injective.

(***) The map $(a, \eta), (x, \nu) \rightarrow L_{(a, \eta)}(x, \nu)$ is a non-degenerate symmetric bilinear form of $\mathbb{F}_{2^n}$.
We immediately deduce from (⋆) and (⋆, ⋆) that \( \{ L_{(a, \eta)} \mid (a, \eta) \in \mathbb{F}_{2^n} \} \) is the set of linear forms of \( \mathbb{F}_{2^n} \). On the other hand, (⋆, ⋆, ⋆) leads to a choice of the inner product \( T_v \) as defined in section 2.

**Definition 11.** The inner product \(<, >\) such that \( T_v(X) = < v, X > \) is now defined by \( T_{(a, \eta)} = L_{(a, \eta)} \). In other words,
\[
<(a, \eta), (x, \nu) > = tr(ax) + \eta \nu.
\]

We immediately deduce

**Proposition 12.** \((⋆)\) \( \phi_{T_{(a, \eta)}}(x, y) = (y + 1)tr(ax) + y(tr(ax) + \eta) \).

Let \( F \) be a \((2t)\)-boolean function such that \( \phi_F(x, y) = (y + 1)f_0(x) + yf_1(x) \). Then
\((⋆⋆)\) \( \phi_{F + T_{(a, \eta)}}(x, y) = (y + 1)(f_0(x) + tr(ax)) + y(f_1(x) + tr(ax) + \eta) \).

3.1.4. **Representation of bent functions.** As before, in this subsection we consider a \((2t)\)-boolean \( F \) function such that \( \phi_F(x, y) = (y + 1)f_0(x) + yf_1(x) \).

**Lemma 13.**

\( a) \) \( \hat{F}(u, 0) = \hat{f}_0(u) + \hat{f}_1(u) \).

\( b) \) \( \hat{F}(u, 1) = \hat{f}_0(u) - \hat{f}_1(u) \).

\( c) \) If \( f_0 + f_1 = \text{tr} \) then \( \hat{f}_1(u) = \hat{f}_0(u + 1) \)

**Proof.** From \((⋆⋆)\) and \((R_3)\),

- If \( \eta = 0 \) : \( w(F + T_{(u, 0)}) = w(f_0 + t_u) + w(f_1 + t_u) \). According to \((R_1)\) this means
  \[
  2^{2t-1} - \frac{1}{2} \hat{F}(u, 0) = 2^{2t-2} - \frac{1}{2} \hat{f}_0(u) + 2^{2t-2} - \frac{1}{2} \hat{f}_1(u),
  \]
  and this leads to a).

- If \( \eta = 1 \) : first notice that \( w(f_1 + t_u + 1) = 2^{2t-1} - w(f_1 + t_u) \). Hence
  \[
  w(F + T_{(u, 0)}) = w(f_0 + t_u) + 2^{2t-1} - w(f_1 + t_u).
  \]
  By using \((R_1)\) as above, we obtain the result of b).

- \( \hat{f}_0(u) = 2^{2t-1} - 2w(f_0 + t_u) \) and \( \hat{f}_1(u) = 2^{2t-1} - 2w(f_0 + t_{u+1}) \) whence
  \( \hat{f}_1(u) = \hat{f}_0(u + 1) \).

The next proposition is a version of a classical result which can be found in several papers on bent functions \([2, 9]\). We give now a proof for sake of convenience.

**Proposition 14.** \( F \) is a bent function if and only if

- \( a) \) \( f_0 \) and \( f_1 \) are near-bent.
- \( b) \) \( \forall a \in \mathbb{F}_{2t-1} \mid f_0(a) \mid + \mid f_1(a) \mid = 2^t \).

**Remark.** \( b) \) means that one of \( \mid f_0(a) \mid \) and \( \mid f_1(a) \mid \) is equal to \( 2^t \) and the other one is equal to \( 0 \).

**Proof.** Let \( (a, \eta) \) be in \( \mathbb{F}_{2^n} \).

Assume \( F \) is bent. From Lemma 13, \( \hat{f}_0(a) = \frac{1}{2}[\hat{F}(a, 0) + \hat{F}(a, 1)] \) and \( \hat{f}_1(a) = \frac{1}{2}[\hat{F}(a, 0) - \hat{F}(a, 1)] \). Since \( F \) is bent, \( \hat{f}_0(a) \) and \( \hat{f}_0(a) \) are in \( \{-2^t, 2^t\} \). By inspection of all possible case we see that \( \hat{f}_0(a) \) and \( \hat{f}_1(a) \) are in \( \{-2^t, 0, -2^t\} \) for every \( a \), which means that \( \hat{f}_0(a) \) and \( \hat{f}_1(a) \) are near-bent. Furthermore, we can check that in every case only one of \( \hat{f}_0(a) \) and \( \hat{f}_1(a) \) is 0.
Conversely, now assume (a) and (b). By Lemma 13, this immediately implies that for every \((a, \eta)\) in \(\mathbb{F}_{2^t}\) the weight of \(\hat{F}(a, \eta) = c2^t\) with \(c \in \{-1, +1\}\) and this proves that \(F\) is bent.

\(\square\)

3.2. A fundamental lemma. We need the following lemma which is important for the next proofs and is a generalization of a classical result on the Gold function (see [5]), since if \(f\) is the Gold function then \(D_1f\) is a constant function.

**Lemma 15.** Let \(f\) be a \((2t-1)\)-near-bent function.

- If \(D_1f = 0\), then \(\hat{f}(u) = 0\) if and only if \(tr(u) = 1\).
- If \(D_1f = 1\), then \(\hat{f}(u) = 0\) if and only if \(tr(u) = 0\).

**Proof.** Assume that \(D_1f = \omega\) with \(\omega \in \mathbb{F}_2\) which means that \(f(x + 1) = f(x) + \omega\). The transform \(\tau : x \rightarrow x + 1\) is a permutation of \(\mathbb{F}_{2t-1}\) and then preserves the weight of every \((2t-1)\)-boolean function. Thus

\[\sharp\{x \mid f(x) + tr(ux) = 1\} = \sharp\{x \mid f(x + 1) + tr(u(x + 1)) = 1\},\]

\((E)\) \[\sharp\{x \mid f(x) + tr(ux) = 1\} = \sharp\{x \mid f(x) + \omega + tr(ux) + tr(u) = 1\}.\]

If \(tr(u) + \omega = 1\) the right hand member of \((E)\) is

\[\sharp\{x \mid f(x) + tr(ux) = 0\} = 2^{2t-1} - \sharp\{x \mid f(x) + tr(ux) = 1\}\]

Hence \((E)\) becomes

\[\sharp\{x \mid f(x) + tr(ux) = 1\} = 2^{2t-1} - \sharp\{x \mid f(x) + tr(ux) = 1\} - \sharp\{x \mid f(x) + tr(ux) = 1\}.\]

In other words \(w(f + tu) = 2^{2t-1} - w(f + tu)\) and thus

If \(tr(u) + \omega = 1\) \(w(f + tu) = 2^{2t-2}\) which is equivalent to \(\hat{f}(u) = 0\).

For \(\omega = 0\) or \(\omega = 1\) the number of \(u\) such that \(tr(u) + \omega = 1\) is \(2^{2t-2}\) and \((R_6)\) claims that this is also the number of \(u\) such that \(\hat{f}(u) = 0\). Then, immediately \(\hat{f}(u) = 0\) if and only if \(tr(u) + \omega = 1\). Finally if \(\omega = 0\) then \(\hat{f}(u) = 0\) if and only if \(tr(u) = 1\) and if \(\omega = 1\) then \(\hat{f}(u) = 0\) if and only if \(tr(u) = 0\).

\(\square\)

3.3. Proof of Theorem 1. Let \(f_0\) be a \((2t-1)\)-near bent function. Let \(F\) be the \((2t)\)-boolean function whose components are \(f_0\) and \(f_1\) such that \(f_1 = f_0 + tr\). Our task is to prove that if \(D_1f_0\) is a constant function, then \(F\) is a bent function.

First notice that \(f_1\) also is a near-bent function. This means that \(f_0\) and \(f_1\) take their values in \(\{-2^t, 0, 2^t\}\). Since \(f_1 = f_0 + tr\) and \(a \in \mathbb{F}_{2t-1}\), according to Lemma 13, (c): \(\hat{f}_1(a) = \hat{f}_0(a + 1)\).

Because \(2t - 1\) is odd, observe that \(tr(1) = 1\). Therefore, one element of \(\{tr(a), tr(a + 1)\}\) is 0 and the other one is 1.

Lemma 15 shows that if \(D_1f_0\) is a constant function then \(\hat{f}_0(a)\) and \(\hat{f}_0(a + 1)\) are not 0 in the same time. Hence, one element of \(\{f_0(a), f_1(a)\}\) is 0 and the other one is \(2^t\) or \(-2^t\).

Finally, according to Proposition 14, this is the proof that \(F\) is bent.

3.4. Proof of Theorem 2. First, we need the following proposition

**Proposition 16.** Let \(f_0\) and \(f_1\) be the components of a bent function \(F\). Let \(\omega \in \mathbb{F}_2\).

Then we have

\[D_1f_0 = \omega \text{ if and only if } D_1f_1 = \omega + 1.\]
Proof. $D_{(0,1)}F(X) = F(X + (0,1)) + F(X)$. The TVR of $D_{(0,1)}F$ is

$$(y + 1)(f_0(x + 1) + f_0(x)) + y(f_1(x + 1) + f_1(x)) = (y + 1)D_1f_0(x) + yD_1f_1(x).$$

From $(R_3): w(D_{(0,1)}F) = w(D_1f_0) + w(D_1f_1)$. Furthermore, $(R_4)$ shows that $w(D_{(0,1)}F) = 2^{2t-1}$. Thus

$$(\dagger) \quad 2^{2t-1} = w(D_1f_0) + w(D_1f_1).$$

On the other hand, if $f$ is any $(2t-1)$-boolean function, then $w(D_1f) = 2^{2t-1}$ is equivalent to $D_1f = 1$, while $w(D_1f) = 0$ is equivalent to $D_1f = 0$. Therefore, $(\dagger)$ proves that if one of the two derivatives $D_1f_0$ and $D_1f_1$ is 0, then the other one is 1.

Now we go back to the proof of Theorem 2. $D_1\tilde{f}_0 = 0$ means $\tilde{f}_0(u) = \tilde{f}_0(u + 1)$ for all $u$ in $\mathbb{F}_{2^{2t-1}}$. Since $\tilde{f}_0$ is the restriction of $\tilde{F}$ to $\mathbb{F}^{(0)}_{2t} = \{ (u,0) \mid u \in \mathbb{F}_{2^{2t-1}} \}$, then in order to prove that $D_1\tilde{f}_0 = 0$ it suffices to show that

$$\forall u \in \mathbb{F}_{2^{2t-1}} : \tilde{F}(u,0) = \tilde{F}(u + 1,0).$$

Using Lemma 13, since $f_0 + f_1 = tr$, we have successively

$$\hat{f}_0(u) = f_0(u + 1).$$

$$\hat{F}(u,0) = \hat{f}_0(u) + \hat{f}_1(u) = \hat{f}_0(u) + \hat{f}_0(u + 1).$$

$$\hat{F}(u + 1,0) = \hat{f}_0(u + 1) + \hat{f}_1(u + 1) = \hat{f}_0(u + 1) + \hat{f}_0(u + 1).$$

$$\hat{F}(u,0) = \hat{F}(u + 1,0).$$

Following $(R_5)$ we deduce that $\hat{F}(u,0) = 1$ if and only if $\hat{F}(u + 1,0) = 1$ for all $u \in \mathbb{F}_{2^{2t-1}}$ and then $\hat{F}(u,0) = \hat{F}(u + 1,0)$. $D_1\hat{f}_1 = 1$ is a direct consequence of Proposition 16.

3.5. **Proof of Theorem 4.** According to Lemma 13, for every $a$ in $\mathbb{F}_{2^{2t-1}}$, $\hat{f}_1(a) = \hat{f}_0(a + 1)$ whence $\hat{F}(a,0) = \hat{f}_0(a) + \hat{f}_0(a + 1)$. On the other hand, a remark after Proposition 14 says that one of $|\hat{f}_0(a)|$ and $|\hat{f}_1(a)|$ is equal to $2^t$ and the other one is equal to 0. It follows that every $a$ in $\mathbb{F}_{2^{2t-1}}$ belongs to one of the following sets

$$A_1 = \{ a \in \mathbb{F}_{2^{2t-1}} \mid \hat{f}_0(a) = -2^t \text{ and } \hat{f}_0(a + 1) = 0 \},$$

$$A_2 = \{ a \in \mathbb{F}_{2^{2t-1}} \mid \hat{f}_0(a) = 0 \text{ and } \hat{f}_0(a + 1) = -2^t \},$$

$$A_3 = \{ a \in \mathbb{F}_{2^{2t-1}} \mid \hat{f}_0(a) = 2^t \text{ and } \hat{f}_0(a + 1) = 0 \},$$

$$A_4 = \{ a \in \mathbb{F}_{2^{2t-1}} \mid \hat{f}_0(a) = 0 \text{ and } \hat{f}_0(a + 1) = 2^t \}.$$

The definition of the dual of $F$ induces that $(a,\eta)$ is in the support of $\hat{F}$ if and only if $\hat{F}(a,\eta) = -2^t$. If we notice

- $\hat{F}(a,0) = -2^t$ if $a \in A_1$ or $a \in A_2$;
- $\hat{F}(a,0) = 2^t$ if $a \in A_3$ or $a \in A_4$,

we deduce that, $(a,0)$ is in the support of $\hat{F}$ if and only if $a \in A_1 \cup A_2$. In other words the support of $\hat{f}_0$ is $S_0 = A_1 \cup A_2$.

From the descriptions of $A_1$ and $A_2$, if $a \in A_1$ then $a + 1 \in A_2$ and if $b \in A_2$ then $b = a + 1$ with $a = b + 1 \in A_1$. Hence $A_2 = \{ u + 1 \mid u \in A_1 \}$. Finally, by inspection we see that $A_1$ is nothing but the set $S = \{ v \in \mathbb{F}_{2^{2t-1}} \mid \hat{f}_0(v) = -2^t \}$ and this leads to result 1.
Since \( f_0 + f_1 = \text{tr} \) then by Lemma 13, \( \hat{f}_1(u) = \hat{f}_0(u + 1) \). Hence, \((a, 1)\) is in the support of \( \hat{F} \) if and only if \( a \) is in \( \mathcal{A}_3 \) or in \( \mathcal{A}_4 \). Consequently, the support of \( \hat{f}_1 \) is \( \mathcal{T}_1 = \mathcal{A}_3 \cup \mathcal{A}_4 \). It can be easily seen that the symmetric difference of the support of \( \hat{f}_0 \) and of \( \mathcal{G} = \{ v \in \mathbb{F}_{2^t-1} \mid f_0(v) = 0 \} \). This immediatly gives result 2).

### 3.6. Proof of Theorem 5
Since \( h_0 + h_1 = \text{tr} \) then Theorem 4 claims that \( \hat{h}_0 + \hat{h}_1 \) is the characteristic function of \( \mathcal{H} = \{ v \in \mathbb{F}_{2^t-1} \mid \hat{h}_0(v) = 0 \} \). We know from Lemma 15 that if \( D_1h_0 = 0 \) then \( \hat{h}_0(u) = 0 \) is equivalent to \( \text{tr}(u) = 1 \) and if \( D_1h_0 = 1 \) then \( \hat{h}_0(u) = 0 \) is equivalent to \( \text{tr}(u) = 0 \). This the same as saying that the characteristic function of \( \mathcal{H} \) is \( \text{tr} \) if \( D_1h_0 = 0 \) and is \( \text{tr} + 1 \) if \( D_1h_0 = 1 \) and this is the expected result.

### 3.7. Proof of Theorem 7
The components of the considered boolean functions are
- \( f_0, f_1 \) for \( F \) and \( \hat{f}_0, \hat{f}_1 \) for the dual \( \hat{F} \) of \( F \).
- \( \hat{f}_0^{(0)}, \hat{f}_1^{(0)} \) for the pseudo-dual \( \hat{F}_0 \) and \( \hat{f}_0^{(1)}, \hat{f}_1^{(1)} \) for the pseudo-dual \( \hat{F}_1 \).
- \( \hat{f}_0^{(0)}, \hat{f}_1^{(0)} \) for the dual \( \hat{F}_0 \) and \( \hat{f}_0^{(1)}, \hat{f}_1^{(1)} \) for the dual \( \hat{F}_1 \) of \( \hat{F}_1 \).

#### Proof of A)
Since \( f_0 + f_1 = \text{tr} \) then \( D_1\hat{f}_0 = 0 \) and \( D_1\hat{f}_1 = 1 \) (Theorem 2). We deduce from Theorem 1 that \( \hat{F}_0 \) and \( \hat{F}_1 \) are bent functions.

#### Proof of B) and C)
From the definitions of the duals,
- \( \hat{f}_0^{(0)} = \hat{f}_0, \hat{f}_1^{(0)} = \hat{f}_1, \hat{f}_0^{(0)} + \hat{f}_1^{(0)} = \text{tr}, \hat{f}_0^{(1)} + \hat{f}_1^{(1)} = \text{tr} \).

Hence, according to Theorem 2, \( D_1\hat{f}_0^{(0)} = 0 \) and \( D_1\hat{f}_1^{(1)} = 0 \) and thus \( \hat{F}_0 \) and \( \hat{F}_1 \) meet (C).

Now applying Theorem 5 to \( \hat{F}_0 \) with \( h_0 = \hat{f}_0^{(0)} \) and \( h_1 = \hat{f}_1^{(0)} \), we get \( h_0 + h_1 = \text{tr} \). Since \( D_1\hat{f}_0^{(0)} = 0 \) then \( \hat{h}_0 + \hat{h}_1 = \hat{f}_0^{(0)} + \hat{f}_1^{(0)} = \text{tr} \). That is \( \hat{F}_0 \) meets \( (T) \) with \( \xi = 0 \).

Similarly, for \( \hat{F}_1 \) with \( h_0 = \hat{f}_0^{(1)} \) and \( h_1 = \hat{f}_1^{(1)} \) we have \( h_0 + h_1 = \text{tr} \). Remark \( D_1\hat{f}_0^{(1)} = D_1\hat{f}_1 \) whence, again from Theorem 2, \( D_1\hat{f}_0^{(1)} = 1 \) and Theorem 5 gives \( \hat{f}_0^{(0)} + \hat{f}_1^{(0)} = \text{tr} + 1 \) and thus \( \hat{F}_1 \) meets \( (T) \) with \( \xi = 1 \).

### 4. Examples
If \( (C) \) or \( (T) \) hold for a bent function \( F \), the bent functions \( F \), the dual \( \hat{F} \), the pseudo-duals \( \hat{F}_0, \hat{F}_1 \) and the duals of these pseudo-duals can be distinct or not. The examples below show different situations.

For every following example, condition \( (T) \) is satisfied for the initial bent function \( F, t = 4 \) and \( \text{tr} \) is the trace function of \( \mathbb{F}_{2^t} \).

#### Example 1
\( F : f_0(x) = \text{tr}(x^7 + x^{13}) \) not \( (C), (T) \)
- \( f_1(x) = f_0(x) + \text{tr}(x) \)
- \( \hat{F} : \hat{f}_0(x) = \text{tr}(x^5 + x^7 + x^9 + x^{13} + x^{19} + x^{21}) \)
- \( \hat{f}_1(x) = f_0(x) + \text{tr}(x + x^5 + x^9) \)
- \( \hat{F}_0 : \hat{f}_0^{(0)}(x) = \hat{f}_0(x), \hat{f}_1^{(0)}(x) = \hat{f}_0(x) + \text{tr}(x) \)
- \( \hat{F}_1 : \hat{f}_0^{(1)}(x) = \hat{f}_1(x), \hat{f}_1^{(1)}(x) = \hat{f}_1(x) + \text{tr}(x) \)
- \( \hat{F}_0 : \hat{f}_0^{(0)}(x) = \text{tr}(x + x^7 + x^9 + x^{13} + x^{19} + x^{21}), \hat{f}_1^{(0)}(x) = \hat{f}_0^{(0)}(x) + \text{tr}(x) \)
we obtain six bent functions. An open question now is to describe

Example 2. \( F : f_0(x) = tr(x^{15} + x^{27} + x^{29} + x^{43}) \), not \((C), (T)\)
\( f_1(x) = f_0(x) + tr(x) \)
\( \tilde{F} : \tilde{f}_0(x) = tr(x + x^3 + x^7 + x^{11} + x^{19} + x^{21}) \)
\( \tilde{f}_1(x) = \tilde{f}_0(x) + tr(x) \)
\( \tilde{F}_0 : \tilde{f}_0^{(0)}(x) = \tilde{f}_0(x), \tilde{f}_1^{(0)}(x) = \tilde{f}_0(x) + tr(x) \)
\( \tilde{F}_1 : \tilde{f}_0^{(1)}(x) = \tilde{f}_1(x), \tilde{f}_1^{(1)}(x) = \tilde{f}_1(x) + tr(x) \)
\( \tilde{F}_0 = \tilde{F}_0 \)
\( \tilde{F}_1 : \tilde{f}_0^{(1)}(x) = tr(x + x^3 + x^5 + x^7 + x^9 + x^{11} + x^{19} + x^{21}) \)
\( \tilde{f}_1^{(1)}(x) = \tilde{f}_{0,1}(x) + tr(x + 1) \)

Example 3. \( F : f_0(x) = tr(x + x^3 + x^7 + x^{11} + x^{19} + x^{21}) \), \((C), (T)\)
\( f_1(x) = f_0(x) + tr(x) \)
\( \tilde{F} : \tilde{f}_0(x) = tr(x^7 + x^{11} + x^{19} + x^{21}) \)
\( \tilde{f}_1(x) = \tilde{f}_0(x) + tr(x) \)
\( \tilde{F}_0 = \tilde{F} \) and \( \tilde{F}_0 = F \) \( \tilde{F}_1 = \tilde{F} \) and \( \tilde{F}_1 = F \)

Example 4. \( F : f_0(x) = tr(x^3 + x^5 + x^7 + x^{11} + x^{19} + x^{21}) \), \((C), (T)\)
\( f_1(x) = f_0(x) + tr(x) \)
\( \tilde{F} = F = \tilde{F}_0 = \tilde{F}_1 = \tilde{F}_0 = \tilde{F}_1 \)

4.1. A special example and an open question. In [7] the authors recall the definition of non-weakly-normal bent function and they introduce (Fact 13) an example of such a function \( F \) in dimension 12 defined by \( \phi_F(x, y) = (y + 1)tr(x^{241} + x) + ytr(x^{241}) \) where \( tr \) is the trace of \( \mathbb{F}_{2^{11}} \) over \( \mathbb{F}_2 \). We see that \( f_1 + f_0 = tr(x) \) and thus Theorem 7 holds.

An interesting open question is: Are \( \tilde{F}, \tilde{F}_0, \tilde{F}_1, \tilde{F}_0, \tilde{F}_1 \) also non-weakly normal?

5. Conclusion

We have introduced a way to construct bent functions starting from a near-bent functions which fulfill the special condition of Theorem 1. Applying this Theorem and Theorem 7 we obtain six bent functions. An open question now is to describe explicitly the near-bent functions \( f \) such that \( D_1 f \) is a constant function, for example by means of the trace function.

Another question is to express the characteristic function of the set \( S \) which appears in Theorem 4, by using the trace function.

The results of this work could probably be generalized by replacing 1 with \( e \) such that \( tr(e) = 1 \) and \( tr(x) \) with \( tr(ex) \) in \((C)\) and \((T)\).
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