Particle dynamics in non-rotating Konoplya and Zhidenko black hole immersed in an external uniform magnetic field

Aqeela Razzaq\textsuperscript{1,a}, Rehana Rahim\textsuperscript{2,b}, Bushra Majeed\textsuperscript{3,c}, Javlon Rayimbaev\textsuperscript{4,5,6,7,8,d}

\textsuperscript{1} Department of Mathematics and Statistics, Riphah International University, Islamabad, Pakistan
\textsuperscript{2} Department of Mathematics, The Rawalpindi Women University, Rawalpindi, Pakistan
\textsuperscript{3} College of Electrical and Mechanical Engineering (CEME), National University of Sciences and Technology, Islamabad 44000, Pakistan
\textsuperscript{4} Institute of Fundamental and Applied Research, National Research University TIIAME, Kori Niyoziy 39, 100000 Tashkent, Uzbekistan
\textsuperscript{5} Akfa University, Kichik Halqa Yuli Street 17, 100095 Tashkent, Uzbekistan
\textsuperscript{6} Tashkent State Technical University, 100095 Tashkent, Uzbekistan
\textsuperscript{7} National University of Uzbekistan, University Street 4, 100174 Tashkent, Uzbekistan
\textsuperscript{8} School of Mathematics and Natural Sciences, New Uzbekistan University, Mustaqillik Ave. 54, 100007 Tashkent, Uzbekistan

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Abstract This article investigates dynamics of particles in the background of non-rotating Konoplya and Zhidenko black hole, immersed in an external uniform magnetic field. The work involves the circular motion of electrically and magnetically charged particles. Analysis of the effective potential, specific energy, angular momentum of the particles are obtained and discussed graphically. We obtained the center of mass energy of the colliding particles in the vicinity of the KZ black hole. Energy efficiency of charged and magnetized particles is also obtained.

1 Introduction

Black holes (BHs) are an interesting and important prediction of Einstein’s theory of general relativity (GR). Recent developments in the fields of gravitation and astrophysics led to the successful tests of GR in strong gravitational fields [1]. This became possible because of the unprecedented high quality observational data. The Event Horizon Telescope (EHT) collaboration recently publicized a shadow and image of supermassive BH in Messier 87 (M87) galaxy [2]. In the LIGO-Virgo experiment, gravitational waves have been detected for merging of BHs and neutron stars in compact binaries [3]. Motion of photons and the matter in close vicinity of a BH can help in direct and indirect observation of event horizon. The geometric structure of a spacetime can be studied through the analysis of particle dynamics around a BH [4–7]. The motion of charged particles is affected by the presence of a test uniform magnetic field in the near vicinity of a BH. As a black hole does not have a magnetic field, an external magnetic field can be taken into account. Wald gave the solution of the electromagnetic field equations for a Kerr black hole surrounded by an asymptotically uniform magnetic filed [8]. Afterward, many studies have been devoted to investigation of electromagnetic fields around black holes surrounded by the external uniform and dipolar magnetic fields [9–12]. The strength of the magnetic field is assumed to be weak so that it does not affect geometry of spacetime.

In GR, Kerr black hole describes the geometry around an astrophysical black hole. Kerr metric contains two parameters, which are mass and spin (and charge in case of Kerr-Newman metric). In alternate theories of gravity, numerous metrics have been developed which contain deviations from Kerr [13–18]. In the present study, the modified Kerr metric developed in Ref. [18] by Konoplya and Zhidenko (referred in this work as KZ black hole) has been considered. The main aim behind this metric is to see if detection of gravitational waves lead to the possibility of modified theories of gravity [19]. Some studies also suggest that a KZ spacetime might describe a real astrophysical black hole [20, 21]. It can be expected that non-Kerr metrics may affect many strong gravity signatures and show changes in the observations. The static KZ metric has been studied for absorption cross-section of massless spin-0 waves in Ref. [22] where the absorption has been studied for the dependence on the deformation parameter. The particle dynamics around black holes with deformation parameters have been studied in [13, 14, 16, 23]. The studies show potentially observable effects induced by the deformation parameters. The aim of present study is also to look for effects of deformation parameter $\eta$ on particle dynamics.
In fact, the general relativity has been well tested in both weak and strong gravity regimes in various astrophysical observations. However, in large scale observation the GR does not work well and black hole solutions in it, such as Schwarzschild and Kerr black holes, have singularity problem. Thus, one may test alternative and/or modified gravity theories which may mimic the GR in astrophysical observations in which regular black hole solution can be obtained. On the other hands, it is quite complicated (in some case it is impossible) to get analytical solutions of gravitational field equation of some alternative gravity theories. In that cases, one may get approximate or parametrized solutions of the field equation. In astrophysical scenarios, determining the type of black hole using observational data is also impossible. Therefore, one way to get information about spacetime around black holes to test various gravity theories and get limits (constraints on) to the gravity and black hole parameters.

The paper is arranged as: section 2 describes metric of Konoplya and Zhidenko black hole, event horizon and the scalar invariants. In Sect. 3, the magnetic field components are determined. Sections 4 and 5 are dedicated to describe motion of electrically and magnetically charged particles, respectively. Energy efficiency of the charged and magnetized particle is studied in section 6. Center of mass energy of two colliding particles is studied in section 7. The work has been concluded in the last section.

2 The Konoplya and Zhidenko black hole

The metric of Konoplya–Zhidenko rotating non-Kerr black hole in the Boyer-Lindquist coordinates is given as [18]

\[
ds^2 = -\left(1 - \frac{2Mr^2 + \eta}{r^3}\right)dt^2 + \frac{\Sigma}{\Delta}dr^2 + \Sigma d\theta^2
\]
\[
+ \left(a^2 + r^2 + \frac{(2Mr^2 + \eta)a^2 \sin^2 \theta}{r^3}\right) \sin^2 \theta d\phi^2
\]
\[
- \frac{2(2Mr^2 + \eta)a \sin^2 \theta}{r^3} dt d\phi,
\]

(1)

with

\[
\Delta = r^2 + a^2 - 2Mr - \eta/r^3,
\]
\[
\Sigma = r^2 + a^2 \cos^2 \theta
\]

(2)

where \(a\) is spin parameter and \(M\) is mass of black hole. This spacetime has been obtained by deforming Kerr metric, a static deformation is involved which results in the change of black hole mass and its event horizon, but asymptotic nature of Kerr metric remains the same i.e. \(M \to M + \eta/2r^2\). The parameter \(\eta\) denotes a deformation parameter which measures possible deviations from Kerr black hole. When \(\eta \to 0\) Eq. (1) becomes Kerr metric. To obtain the non-rotating form, the case of \(a = 0\) is considered. This gives

\[
ds^2 = -f dt^2 + \frac{1}{f} dr^2 + r^2 d\Omega^2,
\]

(3)

with

\[
d\Omega^2 = d\theta^2 + \sin^2 \theta d\phi^2,
\]

(4)

and

\[
f = 1 - \frac{2M}{r} - \frac{\eta}{r^3}.
\]

(5)

This paper deals with the non-rotating form of metric (1) shown in Eq. (3).

2.1 The event horizon and scalar invariants

The solution of \(f = 0\) leads to event horizon of KZ metric [22]. This gives one real and two complex roots. The real root is of the form (this will be taken as the event horizon)

\[
r_h = \frac{1}{3} \left(2M + \frac{4M^2}{y} + y\right),
\]

(6)

where

\[
y^3 = \frac{3}{2} \left(9\eta + \sqrt{3\eta(27\eta + 32M^3)}\right) + 8M^3.
\]

(7)

From Eq. (7), one can see that the event horizon exists for \(\eta \geq -32/27\) at \(r_h/M = 4/3\) and when (Schwarschild limit) \(\eta = 0\), \(r_h = 2M\). The graph of \(f\) and event horizon given in Eq. (6) has been plotted in Fig. 1.

The plot of \(f(r)\) shows that it decreases with increasing \(\eta\) with plots becoming close as the radial distance increases. The plot of event horizon shows that the it increases with increasing \(\eta\).
A better understanding of important characteristics of a spacetime can be obtained through the study of scalar invariants like Ricci tensor and its square and Kretschmann scalar. The current section deals with the scalar invariants of metric (3).

2.3 The Ricci scalar and square of Ricci tensor

Ricci scalar $R$ and square of Ricci tensor ($R_{\alpha\beta}$) are defined as $R = g^{\alpha\beta}R_{\alpha\beta}$ and $R^{\alpha\beta}R_{\alpha\beta}$, respectively. For the metric (3), these scalars are

$$R = \frac{2\eta}{r^5}, \quad R^{\alpha\beta}R_{\alpha\beta} = \frac{26\eta^2}{r^{10}}. \quad (8)$$

In the limit when $\eta = 0$ (Schwarzschild case), both the invariants vanish showing that the metric (3) is Ricci flat. Also, the presence of singularity is evident at $r = 0$, for a non-zero $\eta$.

2.4 The Kretschmann scalar

The Kretschmann scalar is defined as

$$K = R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta}. \quad (9)$$

For the metric (3),

$$K = \frac{8(23\eta^2 + 6M^2r^4 + 20\eta Mr^2)}{r^{10}}. \quad (10)$$

When $\eta = 0$, we get the Schwarzschild’s Kretschman scalar. At the center of the black hole, Eq. (10) shows divergent behaviour.
This section deals with the KZ black hole surrounded by an external uniform magnetic field of strength $B$. The magnetic field is taken to be axially symmetric, static and homogeneous at spatial infinity. The strength of magnetic field is not so strong that it can affect the spacetime geometry around the black hole. Electromagnetic 4-potential defined by Wald is [8]

$$A_\mu = \left(0, 0, 0, \frac{1}{2}B r^2 \sin^2 \theta \right).$$

(11)

The Maxwell tensor in terms of $A_\mu$ is

$$F_{\alpha\beta} = A_{\beta,\alpha} - A_{\alpha,\beta},$$

(12)

with non-zero components

$$F_{r\phi} = Br \sin^2 \theta,$$

(13)

$$F_{\theta\phi} = B r^2 \sin \theta \cos \theta.$$  

(14)

The orthonormal components of the magnetic field around the black hole are

$$B^\hat{r} = B \cos \theta,$$

(15)

$$B^\hat{\theta} = \sqrt{f} B \sin \theta.$$  

(16)

The plot of $B^\hat{\theta}$ against various values of $\eta$ and $\theta$ has been shown in Fig. 2. It is observed that $B^\hat{\theta}$ increase with decreasing value of $\eta$. As the distance becomes large, the dependence of $\eta$ one $B^\hat{\theta}$ becomes very small.

4 Dynamics of electrically charged particles

Let us consider an electrically charged particle of mass $m$ and charge $e$ moving in circular orbits around a weakly magnetized KZ black hole. We use Hamilton-Jacobi equation to study its dynamics. It is given as

$$g^{\mu\nu} \left( \frac{\partial S}{\partial x^\mu} + eA_\mu \right) \left( \frac{\partial S}{\partial x^\nu} + eA_\nu \right) = m^2,$$

(17)

here $S$ is the Hamilton-Jacobi action taken as

$$S = -Et + L\phi + S_r(r) + S_\theta(\theta),$$

(18)

with $E$ being energy and $L$ being angular momentum of the particle. For motion on the equatorial plane ($\theta = \pi/2$), Eq. (17) after putting values becomes

$$-f^{-1}E^2 + f g^{rr} \dot{r}^2 + \frac{1}{r^2} \left(L + eA_\phi \right)^2 = m^2.$$  

(19)

Further simplification, leads to

$$\dot{r}^2 = E^2 - f \left( 1 + \left( \frac{L}{r} + \omega_B r \right)^2 \right).$$

(20)
Table 1 The $\omega_B$ for various cases

| Particle       | Mass m (kg) | Charge e | $\omega_B$ (1/cm) |
|----------------|-------------|----------|-------------------|
| Proton         | $1.6 \times 10^{-27}$ | $e$     | $5 \times 10^7$   |
| Electron       | $9.11 \times 10^{-31}$ | $e$     | $8.78 \times 10^{10}$ |
| Fe+            | $9.27 \times 10^{-26}$ | $e$     | 862999           |
| Dust charge    | $1 \times 10^{-18}$ | $e$     | 0.08              |

Here, the strength of the magnetic field $B$ has been set to $10^4$ Gausses

where $E = E/m$, $L = L/m$ be energy and angular momentum, per unit mass, respectively, and

$$\omega_B = \frac{eB}{2m},$$

(21)

is the cyclotron frequency. It accounts for the magnetic interaction between an electric charge and external magnetic field. The values of $\omega_B$ for protons, electrons, Fe+ and positively charged dust particle of mass around $10^{-15}$ g are shown in the Table 1.

Equation (20) can also be written as

$$\dot{r}^2 = E^2 - V_{eff},$$

(22)

where

$$V_{eff} = f \left(1 + \left(\frac{L}{r} + \omega_B r\right)^2\right).$$

(23)

The radial plot of $V_{eff}$ has been shown in Fig. 3. The plots show that due to magnetic field, as $\omega_B$ increases, $V_{eff}$ also gets higher values. Also for non-zero value of black hole parameter $\eta$, $V_{eff}$ of the particle moving in geometry of weakly magnetized KZ black hole is higher than the $V_{eff}$ in geometry of a Schwarzschild black hole ($\eta = 0, \omega_B = 0$).

For circular motion of the particles, one needs the conditions

$$\dot{r} = 0, \quad dV_{eff}/dr = 0.$$

(24)

Equation (24) leads to

$$V_{eff} = E^2,$$

(25)

Equation (24) leads to angular momentum $L$ as

$$L = \omega_B f^3 + r^{3/2} \sqrt{4r\omega_B^2 f^2 - r f'^2 + 2f f'^2}/2 f - rf'$$

(26)

The energy $E$ is obtained as

$$E^2 = f \left[1 + \left(\frac{\omega_B^2 f^3 + r^{3/2} \sqrt{4r\omega_B^2 f^2 - r f'^2 - 2f f'^2 + \omega_B r}}{2 rf - r^2 f'}\right)^2\right].$$

(27)

Graphical behaviour of $L$ of the particle is shown in Fig. 4. Note that the angular momentum is high in presence of magnetic field with $\omega_B > 0$ (top left panel), while in absence of magnetic field, and for $\eta = 0$ angular momentum of the particle become very less and both the curves coincide. In the bottom left panel, $L$ increases with increasing $\eta$ and the values are higher than the Schwarzschild case. The behaviour of specific energy is shown in Fig. 5. Energy has high values for $\omega_B < 0$ (top right panel) as compared to non-negative values while in bottom right panel, $E$ increases as $\eta$ increases.

4.1 Innermost stable circular orbits (ISCO)

The condition for the particle to move in the innermost stable circular orbit or the ISCO is $d^2V_{eff}/dr^2 = 0$, which leads to

$$k' + \frac{1}{(rf'' - 2f)^2} \left[k^3/r^{3/2} \omega_B f' + p\right] \times \left[rf''(\sqrt{r}\omega_B(4f - rf') + p) - f'(\sqrt{r}\omega_B(7rf' - 12f) + p)\right] = 0,$$

(28)
The graph on the upper panel shows the behavior of $V_{\text{eff}}$ for varying cyclotron frequency with other parameters being fixed. The plot in the lower panel shows $V_{\text{eff}}$ for varying $\eta$.

$$k = f' r^3 (1 + r^2 \omega_B^2) + 2 f \omega_B r^4 \quad (29)$$

$$p^2 = 4r \omega_B^2 f^2 - r (f')^2 - 2 ff'. \quad (30)$$

The graphical behaviour of the ISCO is shown in Fig. 6, note that as the values of $\eta$ increase it results in larger values of ISCO ($r$), same is the case for increasing values of $\omega_B$. This shows that in presence of magnetic field, $\omega_B$, the particle gets closer to the black hole. Figure 6 (right panel) shows that for both $\omega_B < 0$ and $\omega_B > 0$ the radius of ISCO increases as $\eta$ increases, while the numerical value of ISCO approaches to the value for Schwarzschild black hole $r = 6$, when $\eta = 0$, $\omega_B = 0$ (left panel of Fig. 6).

Since it is difficult to obtain exact solution for $r_{\text{ISCO}}$, therefore, the solution is obtained numerically in the tabular form. The numerical solution is shown for various values of $\omega_B$ and $\eta$ in Table 2. The table shows for a fixed value of $\omega_B$, the $r = r_{\text{ISCO}}$ increases by increasing deformation parameter $\eta$.

### 5 Magnetized particle motion

This section deals with motion of magnetized particles around a KZ black hole that is immersed in an external asymptotically uniform magnetic field. Hamilton-Jacobi equation in this case is

$$g^{\alpha\beta} \frac{\partial S}{\partial x^\alpha} \frac{\partial S}{\partial x^\beta} = -\left( m - \frac{1}{2} F_{\alpha\beta} F_{\alpha\beta} \right)^2. \quad (31)$$
with $m$ being particle’s mass, $S$ denotes action for magnetized particle in the curved spacetime. $D^{\mu\nu}$ represents the polarization tensor with the form

$$D^{\alpha\beta} = \eta^{\alpha\beta\mu\nu} u_{\mu} u_{\nu},$$  \hfill (32)

and has the following constraint

$$D^{\alpha\beta} u_{\beta} = 0.$$  \hfill (33)

Here $u_{\nu}$ and $u_{\nu}$ denote the 4-velocity of magnetic dipole moment and particles in an arbitrary observer’s rest frame of reference, respectively. The product of $D^{\mu\nu} F_{\mu\nu}$ accounts for interaction between the external magnetic field and magnetized particles. The Maxwell tensor can be written as

$$F_{\alpha\beta} = 2u_{[\mu} E_{\nu]} - \eta_{\alpha\beta\mu\nu} B^{\mu} u^{\nu},$$  \hfill (34)

where $E_{\nu}$ and $B_{\nu}$ are the electric and magnetic field, respectively, and $\eta_{\alpha\beta\mu\nu}$ is obtained from the Levi-civita symbol $\epsilon_{\alpha\beta\mu\nu}$ as

$$\eta_{\alpha\beta\mu\nu} = \epsilon_{\alpha\beta\mu\nu} \sqrt{-g},$$  \hfill (35)

with $g$ being the determinant of the metric. Taking into account Eqs. (32–35) one gets

$$D^{\alpha\beta} F_{\alpha\beta} = 2\mu^{\alpha} B_{\alpha} = 2\mu^{2} \hat{B}_{\alpha} = 2\mu B \sqrt{f(r)},$$  \hfill (36)

where $f(r)$ is given in Eq. (5). The radial equation of motion obtained from Eq. (31) is

$$-\frac{\mathcal{E}^2}{f} + \frac{\dot{r}^2}{r^2} + \frac{1}{r^2} \mathcal{L}^2 = -\left(1 - \frac{\mu B \sqrt{f(r)}}{m}\right)^2,$$  \hfill (37)
The graph on the upper panel shows the behaviour of energy for varying cyclotron frequency with other parameters being fixed. The plot in the lower panel shows energy for some values of $\eta$.

\[ \dot{r}^2 = \mathcal{E}^2 - f \left[ (1 - \beta \sqrt{f})^2 + \frac{L^2}{r^2} \right], \]  

(38)

where $\beta = \frac{\mu B}{m}$ is the magnetic coupling parameter. It characterizes the electromagnetic interaction between magnetic dipole and the external magnetic field. The dynamics of a neutron star having dipole moment $\mu = (1/2)B_{NS}R_{NS}^3$, can be studied by this approach, treating it as a magnetized particle orbiting around a supermassive black hole. The parameter $\beta$ can be calculated by employing the
neutron star’s observational parameters and the external magnetic field where the neutron star move, around the supermassive black hole

\[ \beta = \frac{B_{NS} R_{NS}^3 B_{ext}}{m_{NS}} \approx \frac{\pi}{10^3} \left( \frac{B_{NS}}{10^{12} G} \right) \left( \frac{B_{ext}}{10^9 G} \right) \left( \frac{R_{NS}}{10^6 \text{cm}} \right)^3 \left( \frac{m_{NS}}{1.4M_{\odot}} \right)^{-1}. \]  

Equation (38) can be written as

\[ \frac{\dot{r}^2}{\varepsilon^2} = \varepsilon^2 - V_{\text{eff}}, \]  

with

\[ V_{\text{eff}} = f \left[ (1 - \beta \sqrt{f})^2 + \frac{\ell^2}{r^2} \right]. \]  

The effective potential of the magnetized particle decreases with increasing values of \( \beta \) and \( \eta \).

To determine the energy and angular momentum, Eqs. (24, 25) are again employed, giving angular momentum and energy as

\[ \mathcal{L}^2 = \frac{r^3 f' (\beta \sqrt{f} - 1)(1 - 2\beta \sqrt{f})}{r f' - 2 f}, \]  

The expression for energy is

\[ \varepsilon^2 = f(r) \left[ (1 - \beta \sqrt{f})^2 + \frac{f' (\beta \sqrt{f} - 1)(1 - 2\beta \sqrt{f})}{r f' - 2 f} \right]. \]  

The graphs of angular momentum and energy are displayed in Figs. 8 and 9, respectively. Note that angular momentum of a magnetized particle moving in the vicinity of KZ black hole, due to magnetic field with \( \beta > 0 \) (Fig. 8 top right panel), is low as compared to the angular momentum in the absence of magnetic field, and for \( \eta = 0 \).

For fixed values of \( \beta = 0.2 \), \( \mathcal{L} \) is high in Schwarzschild geometry, while it reduces for the particles in the KZ spacetime. The behaviour of specific energy is displayed in Fig. 9. In the top left panel, the same trend is observed as for angular momentum for fixed \( \eta \) while in the lower left panel, it is clear that the energy of the magnetized particle, moving around the KZ black hole increases as \( \eta \) increases, same as in the case of \( \mathcal{L} \). Note that the energy of the particle moving around the Schwarzschild black hole is higher than the energy of the particle moving in KZ geometry (lower left panel). The ISCO is given by the equation

\[ \chi f'' + \frac{A}{r^4} \left( f'' r^2 - 4 r f' + 6 f \right) \geq 0, \]  

with

\[ \chi = 1 + \frac{2 \beta^2 f''}{f'} + 2 \beta^2 f - 3 \beta f^{1/2} - \frac{3 \beta f'^2}{2 f'' f^{1/2}}, \]

\[ A = \left( \frac{r^3}{r f' - 2 f} \right) \left( \beta \sqrt{f} - 1 \right) \left( 1 - 2 \beta \sqrt{f} \right). \]  

The tendency of ISCO of magnetized particle around the weakly magnetized KZ black hole is given in the Table 3. It can be observed that as \( \beta \) increases ISCO of the magnetized particle also increases, the same is the situation, for \( \eta \). This shows that due to presence of magnetic field and \( \eta \), orbits in which particles can move with stability, get away from the centre of the black hole.
6 The energy efficiency

Following the Novikov-Thorne model, the Keplerian accretion surrounding the astrophysical black hole is considered as the geometrically thin disks which are specified by characteristics of the spacetime circular geodesics [24]. The energy efficiency of the accretion disk surrounding the black hole is taken as the maximum energy that can be taken out as radiation of the falling matter into the central black hole from the disk. The expression for energy efficiency is

\[
\alpha = 1 - E \bigg|_{r=\text{ISCO}}.
\]  

(45)

For the charged particle, the energy at ISCO is given in Eq. (27). The plot of \(\alpha\) in this case has been shown in Fig. 10. The graph shows that energy efficiency increases with increase in \(\omega_B\).

For the magnetized particle, the energy at the ISCO has been calculated in Eq. (43). Behaviour of energy efficiency of magnetized particle in the KZ spacetime is shown in Fig. 11. The graph shows increasing trend in \(\alpha\) with increasing \(\beta\).

7 Center of mass energy in the equatorial plane

This section deals with center of mass energy for the collision of particles. The particles are assumed to be having equal masses and are assumed to be coming from infinity with the same initial energy as \(E_1/m_1 = E_2/m_2 = 1\) but with different angular momenta. The center of mass energy for the collision of two particles given by Bañados, Silk and West (BSW) [25] is

\[
E_{\text{cm}} = \frac{E_{\text{cm}}^2}{2m_0} = 1 - g_{\mu\nu}u_{1\mu}u_{2\nu},
\]

(46)
where $v_i^\mu = (\dot{t}_i, \dot{r}_i, \dot{\theta}_i, \dot{\phi}_i)$ for $i = 1, 2$ represent the velocity of the particles.

7.1 The center of mass energy of neutral particles

Here the collision of two neutral particles having the same rest mass energies, moving in the equatorial plane is considered. The velocity components in this case are

\[
\dot{t} = \frac{E}{f}, \quad \dot{\phi} = \frac{l}{r^2}, \quad \dot{r}^2 = E^2 - f \left(1 + \frac{l^2}{r^2}\right).
\]

The $E_{\text{cm}}$ given in Eq. (46) becomes

\[
E_{\text{cm}} = 1 + \frac{1}{f} - \frac{l_1 l_2}{r^2} - \frac{1}{f} \sqrt{1 - f(r) \left(1 + \frac{l_1^2}{r^2}\right) \sqrt{1 - f \left(1 + \frac{l_2^2}{r^2}\right)}}.
\]
Fig. 9 Behaviour of energy versus $\beta$ (top left figure) and $\eta$ (bottom left figure)

Fig. 10 Graph of $\alpha$ against $\eta$ for some values of $\omega_B$

where $l_1$ and $l_2$ represent angular momentum of the colliding particles. A radial plot of $E_{\text{cm}}$ is shown in Fig. 12. Note that the $E_{\text{cm}}$ of the particles becomes high as the particles get closer to the black hole. For different values of $\eta$, the values of centre of mass energy remain the same for increasing $r$. 

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Fig. 11 Graph of $\alpha$ against $\eta$ for some values of $\beta$

![Graph of $\alpha$ against $\eta$ for some values of $\beta$]

Fig. 12 Center of mass energy for case of two neutral particles’ collision

![Center of mass energy for case of two neutral particles’ collision]

Table 3 ISCO radius for the magnetized particle for various values of $\eta$ and $\beta$

| $\beta$ ↓ / $\eta$ → | 0     | 0.05  | 0.1   | 0.15  | 0.2   |
|----------------------|-------|-------|-------|-------|-------|
| 0                    | 6.0000| 6.0262| 6.0521| 6.0777| 6.1031|
| 0.05                 | 6.0998| 6.1260| 6.1519| 6.1775| 6.2028|
| 0.1                  | 6.2190| 6.2451| 6.2710| 6.2966| 6.3219|

7.2 The center of mass energy of magnetized particles

The collision of two magnetized particles is considered in this subsection. The first two equations of motion in this case are exactly the same as that for neutral particle and the radial equation of motion is

$$\dot{r}^2 = \mathcal{E}^2 - f \left( (1 - \beta \sqrt{f})^2 + \frac{l_1^2}{r^2} \right).$$

The center of mass energy is

$$\mathcal{E}_{\text{cm}} = 1 + \frac{1}{f} \left( \frac{l_1 l_2}{r^2} \right)$$

$$- \frac{1}{f} \left( 1 - f \left( (1 - \beta_1 \sqrt{f})^2 + \frac{l_1^2}{r^2} \right) \right).$$
The graphical behaviour is shown in Fig. 13. The center of mass energy is high for small \( r \), and it decreases with increasing \( r \). A small increase in observed as values of deformation parameter decrease.

7.3 The center of mass energy for the collision of a neutral and a magnetized particle

In this subsection, collision of a magnetized and neutral particle has been considered. The equations of motion are given in Sect. 7.1 and 7.2. The particle 1 is taken to be magnetized and particle 2 is assumed to be neutral. Using these in center of mass energy expression

\[
E_{\text{cm}} = 1 + \frac{1}{f} - \frac{l_1 l_2}{r^2} - \frac{1}{f} \left( 1 - f \left( (1 - \beta \sqrt{f})^2 + \frac{l_2^2}{r^2} \right) \right) \times \sqrt{1 - f \left( 1 + \frac{l_2^2}{r^2} \right)}.
\]  

The radial profile of \( E_{\text{cm}} \) is shown in Fig. 14. If we increase values of \( \eta \) center of mass energy increases (top left panel) while opposite effect is seen for increasing \( \beta \).

In the limit \( r \rightarrow r_h \), center of mass energy reduces to

\[
E_{\text{cm}} = 2 + \frac{(-l_1 + l_2)^2}{2r_h^2}
\]

which is the same as for \( E_{\text{cm}} \) (at the event horizon) in the case of collision of a neutral particle discussed in section (7.1).

7.4 The center of mass energy for the collision of a charged and a magnetized particle

Here, collision of a magnetized and a charged particle has been considered. The particle 1 is taken to be charged and particle 2 is assumed to be magnetized. Using these in center of mass energy expression, we obtain

\[
E_{\text{cm}} = 1 + \frac{1}{f} - \frac{l_1 l_2}{r^2} - \frac{1}{f} \left( 1 - f \left( 1 + \frac{l_2^2}{r^2} - \omega_B r \right)^2 \right) \times \sqrt{1 - f \left( 1 + \frac{l_2^2}{r^2} - \omega_B r \right)}.
\]
The center of mass energy for the collision of between a magnetized and a neutral particle. The top left panel shows the radial profile of $E_{cm}$ for changing $\eta$ while the lower left panel shows $E_{cm}$ for changing $\beta$.

\[ E_{cm} = 2 + \frac{(r^2 \omega_B - l_1 + l_2)^2}{2r_h^2} \]

The center of mass energy is shown in Fig. 15, for increasing values of $\eta$ the center-of-mass energy also increases while the opposite effect is seen for increasing $\beta$.

In the limit $r \to r_h$, the center of mass energy reduces to

\[ E_{cm} = 2 + \frac{(r^2 \omega_B - l_1 + l_2)^2}{2r_h^2} \]

notice that Eqs. (54) and (56) with $\omega_B = 0$, reduce to the Schwarzschild case in the limit $r \to 2M$.

8 Summary and conclusion

The Kerr black hole solution is an axisymmetric, stationary and vacuum solution of the Einstein theory of general relativity. All the astrophysical black holes are expected to be described by the Kerr metric. There has been a lot of interest in modified Kerr black hole solutions. Such solutions contain parameters which account for possible deviations from Kerr, which is obtained when deviations are set to zero. One such rotating metric has been developed by Konoplya and Zhidenko [18], whose non-rotating case has been discussed in the present article. In this work, dynamics of charged and magnetized particles (in the equatorial plane) have been discussed in the background of non-rotating Konoplya-Zhidenko metric immersed in an external magnetic field. First, the effective potential, angular momentum and specific energy for the circular motion of charged test particles have been studied for the dependence on deformation parameter $\eta$ and cyclotron frequency $\omega_B$. In Fig. 1 a graph of lapse function is plotted. It is noted
Fig. 15 $E_{\text{cm}}$ for the collision of between a magnetized and a charged particle. The top right panel shows the radial profile of $E_{\text{cm}}$ for changing $\eta$ while the lower right panel shows $E_{\text{cm}}$ for changing $\beta$.

that for higher values of $\eta$ the horizon of the black hole becomes larger. In Fig. 2 orthonormal component of $B^\hat{r}$ has been given, for varying $\eta$. The plots of magnetic field components show that in the surrounding of black hole $B^\hat{r}$ increases as value of $\eta$ decreases. The scalar invariants of the metric have been calculated. Clearly at $r = 0$ there is a curvature singularity. In Fig. 3, effective potential has been plotted for varying $\omega_B$ (top right panel) and $\eta$ (lower right panel). The upper panels show increasing trend with increasing the respective parameter and lower panel show decreasing trend with increasing $\eta$ parameter. This can be explained as the decrease in the least distance between the charged particles and the black hole with increase of $\omega_B$ and $\eta$. In Figs. 4 and 5, radial plots of angular momentum and energy are shown. Note that $E$ increases as $\eta$ increases. In the presence of magnetic field, angular momentum of the particle moving around the black hole becomes high. The inner stable circular orbits of the charged particle are discussed graphically and numerically, as analytical solution of $V''_{\text{eff}} = 0$ is not attainable. The Table 2 shows the values of $r_{\text{ISCO}}$ for different values of $\omega_B$ and $\eta$. Radius of ISCO increases for increasing deformation parameter $\eta$.

In section V dynamics of a magnetized particle in the background of non-rotating KZ black hole immersed in a magnetic field is studied. It is observed that energy of the magnetized particle decreases as the strength of magnetic field increases with the same happening for angular momentum. In section (VI) energy efficiency of the charged and magnetized particles moving in the ISCO have been calculated. It is noted that presence of $\omega_B$ and $\beta$ increases the strength of energy efficiency.

In the last section, center of mass energy of the colliding particles is calculated in the equatorial plane, we discussed all the cases (neutral, electrically charged, magnetically charged particles) separately. The exact expressions are obtained and observed graphically. It is noted that KZ black hole parameter $\eta$ results in higher values of centre of mass energy of the particles. The CME remains finite for all the cases, unless one of the particle gets infinite angular momentum, which is not possible physically.

Data availability  No Data associated in the manuscript.
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