Boundary-induced Exotic Bulk Entropy in Strongly-correlated Systems at Finite Temperatures

Ding-Zu Wang,1 Guo-Feng Zhang,1,∗ Maciej Lewenstein,2,3,† and Shi-Ju Ran‡
1School of Physics, Beihang University,100191,Beijing, China
2ICFO - Institut de Ciencies Fotoniques, The Barcelona Institute of Science and Technology,
Av. Carl Friedrich Gauss 3, 08860 Castelldefels (Barcelona), Spain
3ICREA, Pg. Lluıs Companys 23, 08010 Barcelona, Spain
4Department of Physics, Capital Normal University, Beijing 100048, China
(Dated: April 15, 2022)

Exploiting the bulk-boundary correspondences and the boundary-induced phenomena in the strongly-correlated systems belongs to the most fundamental topics of condensed matter physics. In this work, we show that the entanglement-bath Hamiltonian (EBH) can induce exotic thermodynamic properties in the bulk of a quantum spin chain from the boundaries, analogous to the heat bath. The EBH is defined as the local Hamiltonian located on the boundary of a finite-size system, which approximately generates the bulk entanglement Hamiltonian of the translational-invariant system in the thermodynamic limit. A “boundary quench point” (BQP) is recognized by the discontinuity in the coefficients of the EBH and the bulk entropy. The BQP distinguishes the point, below which the thermal effects become insignificant and the bulk properties are dominated by the ground state. More interestingly, for the topologically non-trivial Haldane chain, an entropy trench emerges due to the competitions between the topological degeneracy of the ground states and the boundary effects from the EBH. We show that the trench is a typical boundary effect from the EBH, and can be a thermal signature of the ground-state topological degeneracy. Our work provides new opportunities on theoretically or experimentally investigating the boundary-induced physics in quantum many-body thermodynamics and on the possible applications in quantum technologies.

I. INTRODUCTION

The boundary effects in quantum many-body systems has witnessed a surge of interest, since it is more realistic and can provide novel opportunities of inducing exotic phenomena. It plays an central role in studying a wide range of issues, such as quantum entanglement [1–4], phase transition [5–8], and topological superconductor [9–15]. Impressive progresses have been achieved in utilizing the boundary effects on the artificial platform for probing the interesting but complex strongly-correlated systems [8, 16–18]. Such schemes also shed light on studying the nonequilibrium physics, e.g., with anomalous transport processes [19, 20].

An inspiring achievement in the non-trivial boundaries of the strongly-correlated systems concerns the entanglement Hamiltonian (EH) [21–25]. The EH is defined as the negative logarithm of the reduced density matrix, which allows to describe the bulk as a thermal state. It offers insights into the intriguing problems such as the subsystem thermalization [26–29] and quantum simulation [30]. How the EH could be applied to investigate many-body thermodynamics by affecting the boundaries for, e.g., the purposes of controlling, is still missing.

In this work, we investigate the bulk entropy of quantum spin chains with tunable interactions on the boundaries [see Fig. 1(a)]. Specifically, the Hamiltonians on the boundaries are given by the entanglement-bath Hamiltonian (EBH), which was originally proposed for constructing a finite-size simulator to access the properties of the infinite-size translational-invariant (TI) system [31, 32]. In other words, the EBH is variational determined so that the bulk entanglement Hamiltonian of the finite-size simulator mimics that of the infinite-size system. We show that the EBH can effective control the thermodynamic properties, analogous to the heat bath. The coefficients of the EBH and the bulk entropy show discontinuity at the “boundary quench point” (BQP), which distinguishes the region dominated by the ground-state physics. For the topologically non-trivial Haldane chain, we reveal that the entropy firstly drops then rises before passing the BQP, forming an entropy “trench”. This is due to the competitions between the topological degeneracy of the ground states and the boundary effects from the EBH. Our results suggest that the trench is a typical boundary effect from the EBH as it is invisible to the simulation of the infinite-size TI system. We speculate the trench to be a signature of the ground-state topological degeneracy.

II. ENTANGLEMENT-BATH HAMILTONIAN

Consider the finite-temperature bulk properties of a quantum spin chain whose interaction strengths are block-wise inhomogeneous. In specific, the coupling strengths inside a finite-size block (namely the bulk denoted by $B$) of the target inhomogeneous system are set to be one as the energy scale, and the strengths in the rest of the system (i.e., the left and right environments denoted by $L$ and $R$, respectively) as $J$. The total Hamiltonian reads

$$\hat{H}_{\text{inh}}(J) = \sum_{n\in B} \hat{h}_{n,n+1} + J \sum_{m\in L\cup R} \hat{h}_{m,m+1}, \tag{1}$$

with $\hat{h}_{n,n+1}$ the local two-body interaction, as illustrated in Fig. 1(a). Note we take the coupling strengths between the
we have
ationally construct the boundary interaction \( \hat{H}_{\text{TD}} = \sum_{n} \hat{h}_{n,n+1} \).

For \( J = 1 \), the spin chain is translationally invariant (TI), whose Hamiltonian reads

\[
\hat{H}_{\text{TI}} = \sum_{n} \hat{h}_{n,n+1}.
\]

The thermodynamics of the infinite-size TI chain can be efficiently simulated by the existing tensor network (TN) methods [33–37] such as transfer-matrix renormalization group [38] and linearized tensor renormalization group [39]. For \( J \neq 1 \), the translational invariance is broken. The existing approaches become unstable when the system size is infinite. We here propose to access the thermodynamics of the bulk \( B \) by constructing a finite-size model known as the quantum entanglement simulation (QES), as illustrated in Fig. 1 (b).

The key idea of the QES is that the entanglement induced by the boundaries couplings optimally mimics that in the infinite-size system, as in Fig. 1 (c). This can be understood from the perspective of the entanglement Hamiltonian (EH). Generally speaking, the EH of the bulk in the QES optimally mimics the EH of the infinite-size TI chain. The QES consists of \((N+2)\) spins (with \( N \) the bulk size), and its Hamiltonian reads

\[
\hat{H}_{\text{QES}}(T') = \hat{H}_L(T') + \hat{H}_B + \hat{H}_R(T').
\]

\( \hat{H}_L(R)(T') \) gives the interaction between the two spins on the left (right) boundary, and is called the entanglement-bath Hamiltonian (EBH), as illustrated in Fig. 1(b). The spins on the left and right boundaries are dubbed as the entanglement bath sites (bath in short).

The EBHs are determined variationally. Let us consider an infinite-size TI chain with the Hamiltonian \( \hat{H}_{\text{TD}} = \sum_{n} \hat{h}_{n,n+1} \). Taking the temperature of the canonical ensemble to be \( T' \), the EH of the bulk satisfies

\[
\hat{H}^E_{\text{TD}}(T') = -\ln \text{Tr}_{\text{EB}} \exp \left( -\frac{\hat{H}_{\text{TD}}}{T'} \right),
\]

with \( \text{Tr}_{\text{EB}} \) the trace of all degrees of freedom except for those of the bulk. At \( T' \), the EH of the bulk for the QES satisfies

\[
\hat{H}^E_{\text{QES}}(T') = -\ln \text{Tr}_{\text{EB}} \exp \left( -\frac{\hat{H}_{\text{QES}}(T')}{T'} \right).
\]

The EBHs are the solution to the following minimization problem

\[
\min_{\hat{H}_L(R)(T') \geq 0} \left| \hat{H}_{\text{TD}}(T') - \hat{H}^E_{\text{QES}}(T') \right|.
\]

In other words, the EBHs are taken so that the EH of the QES optimally mimics that of the infinite-size TI model [Fig. 1(c)].

One shall note that the EBHs \( \hat{H}_L(R)(T') \) should depend on \( T' \), due to the \( T' \)-dependence of the EHs.

The central point of this work is that by embedding in the EBHs, the bulk of the QES described by \( \hat{H}_{\text{QES}}(T') \) [Eq. (6)] would qualitatively exhibit similar physics as the bulk of the system described by \( \hat{H}_{\text{TD}}(J) \) [Eq. (1)]. Note when obtaining the EBHs, we take the coupling strength in the infinite-size

\[\text{(a)}\]

\[\text{(b)}\]

\[\text{(c)}\]

FIG. 1. (Color online) Our goal is to construct a finite-size model whose bulk physics mimics that of the inhomogeneous system [Eq. (1)] shown in (a). Specifically, the interaction strengths in the bulk are taken as one and those outside the bulk as \( J \). The idea is to vari-
ationally construct the boundary interaction \( \hat{H}_L(T') \) and \( \hat{H}_R(T') \) as shown in (b), where \( T' = T/J \) is the effective environment temperature. The boundary interactions are known as the entanglement-bath

\[\text{EHs} \] whose bulk physics mimics that of the inhomogeneous system [Eq. (4)].

The key idea of the QES is that the entanglement induced by the boundaries couplings optimally mimics that in the infinite-size system, as in Fig. 1 (c). This can be understood from the perspective of the entanglement Hamiltonian (EH). Generally speaking, the EH of the bulk in the QES optimally mimics the EH of the infinite-size TI chain. The QES consists of \((N+2)\) spins (with \( N \) the bulk size), and its Hamiltonian reads

\[
\hat{H}_{\text{QES}}(T') = \hat{H}_L(T') + \hat{H}_B + \hat{H}_R(T').
\]

\( \hat{H}_L(R)(T') \) gives the interaction between the two spins on the left (right) boundary, and is called the entanglement-bath Hamiltonian (EBH), as illustrated in Fig. 1(b). The spins on the left and right boundaries are dubbed as the entanglement bath sites (bath in short).

The EBHs are determined variationally. Let us consider an infinite-size TI chain with the Hamiltonian \( \hat{H}_{\text{TD}} = \sum_{n} \hat{h}_{n,n+1} \). Taking the temperature of the canonical ensemble to be \( T' \), the EH of the bulk satisfies

\[
\hat{H}^E_{\text{TD}}(T') = -\ln \text{Tr}_{\text{EB}} \exp \left( -\frac{\hat{H}_{\text{TD}}}{T'} \right),
\]

with \( \text{Tr}_{\text{EB}} \) the trace of all degrees of freedom except for those of the bulk. At \( T' \), the EH of the bulk for the QES satisfies

\[
\hat{H}^E_{\text{QES}}(T') = -\ln \text{Tr}_{\text{EB}} \exp \left( -\frac{\hat{H}_{\text{QES}}(T')}{T'} \right).
\]

The EBHs are the solution to the following minimization problem

\[
\min_{\hat{H}_L(R)(T') \geq 0} \left| \hat{H}_{\text{TD}}(T') - \hat{H}^E_{\text{QES}}(T') \right|.
\]

In other words, the EBHs are taken so that the EH of the QES optimally mimics that of the infinite-size TI model [Fig. 1(c)].

One shall note that the EBHs \( \hat{H}_L(R)(T') \) should depend on \( T' \), due to the \( T' \)-dependence of the EHs.

The central point of this work is that by embedding in the EBHs, the bulk of the QES described by \( \hat{H}_{\text{QES}}(T') \) [Eq. (6)] would qualitatively exhibit similar physics as the bulk of the system described by \( \hat{H}_{\text{TD}}(J) \) [Eq. (1)]. Note when obtaining the EBHs, we take the coupling strength in the infinite-size

\[\text{(a)}\]

\[\text{(b)}\]

\[\text{(c)}\]
TI chain as one [see Eq. (5)]. Therefore, the $T'$ in Eqs. (6)-(9) can be considered as the $T'$ of the inhomogeneous system appearing in Eqs. (3) and (4).

The density operator of the QES at the physical temperature $T$ satisfies the canonical distribution as

$$
\hat{\rho}_{\text{QES}}(T; T') = \exp\left(-\frac{\mathcal{H}_{\text{QES}}(T')}{T}\right).
$$

By tracing over all the degrees of freedom except for the bulk, the bulk properties are given by the reduced density matrix (RDM)

$$
\hat{\rho}^B_{\text{QES}}(T; T') = \text{Tr}_f\hat{\rho}_{\text{QES}}(T; T').
$$

The thermodynamic quantities, such as the von Neumann entropy per site we are interested in here, can be obtained as

$$
S_n = -\text{Tr}_n\hat{\rho}_n(T; T') \ln \hat{\rho}_n(T; T'),
$$

where $\hat{\rho}_n(T; T') = \text{Tr}_f\hat{\rho}_{n,\text{QES}}^B(T; T')$ is the $n$-site RDM, with $\text{Tr}_f$ the trace of all degrees of freedom except for the $n$-th site.

To obtain the EBHs for the ground states, i.e., $\mathcal{H}_{L(R)}(T' = 0)$, one can use the existing methods such as infinite density matrix renormalization group and tensor ring encoding. Non-trivial phenomena of the ground state are exhibited in the bulk by tuning the EBHs [8]. For the EBHs at finite temperatures, we employ the TN Tailoring approach [40], which gives the stay-of-the-art accuracy in simulating the thermodynamics of one-dimensional many-body systems. Specifically, we construct the TN that represents the density operator $\exp\left(-\frac{\mathcal{H}_{\text{QES}}}{T}\right)$. Then we calculate the boundary matrix product states (MPS) $|L\rangle$ and $|R\rangle$, which are the left and right dominant eigenstates of the transfer matrix $\hat{\mathcal{T}}$ of the TN, respectively [Fig. 2(a)]. We have $\lambda^+|L\rangle = \hat{\mathcal{T}}^+|L\rangle$ and $\lambda|R\rangle = \hat{\mathcal{T}}|R\rangle$ with $\lambda$ the eigenvalue. In other words, the free energy per site $f$ is maximized so that the boundary MPS’s are optimized, with

$$
f = \max_{|L\rangle, |R\rangle} (-T' \ln \lambda).
$$

![FIG. 2. (Color online) (a) The thermal tensor network (TN) represents the density matrix of the TI chain at the temperature $T'$. The boundary MPS $|L\rangle$ and $|R\rangle$ can be calculated by the TN Tailoring approach. (b) The process of obtaining the EBHs.](image)

The boundary MPS’s are uniform (namely formed by the copies of one inequivalent tensor, denoted by $A^L$ and $A^R$, respectively), and possess periodic boundary condition along the temporal direction. The length of the boundary MPSs satisfies $K = \frac{1}{\tau T'}$ with $\tau$ a small positive number known as the Trotter slice. The EBHs are obtained by $A^L(R)$ and $V^L(R)$ as illustrated in Fig. 2(b), where $V^L(R)$ are obtained by decomposing the local imaginary-time evolution operator $\exp\left(-\tau h_{n+1}\right)$.

![FIG. 3. (Color online) (a) The couplings strength and (b) the magnetic fields of the EBHs versus the inverse environment temperatures $1/T'$. The dash lines give the couplings at the zero environment temperature $T' = 0$ [8]. The discontinuous point indicates the thermal quench point $T'_{Q}$.

III. EXOTIC BULK ENTROPY INDUCED BY ENTANGLEMENT-BATH HAMILTONIANS

A. Interactions in the entanglement-bath Hamiltonians

We take the quantum Ising model as an example, where the local Hamiltonian satisfies

$$
\hat{h}_{n,n+1} = \hat{S}_n^z \hat{S}_{n+1}^z - \frac{h}{2} (\hat{S}_n^x + \hat{S}_{n+1}^x),
$$

with $h$ the transverse magnetic field. Our results show that the EBHs possess the following forms

$$
\hat{H}_{L}(T') = \sum_{\alpha = x, z} [J^L_{\alpha}\hat{S}_n^\alpha \hat{S}_{n+1}^\alpha - \frac{1}{2} h^L_{\alpha}\hat{S}_n^\alpha],
$$

$$
\hat{H}_{R}(T') = \sum_{\alpha = x, z} [J^R_{\alpha}\hat{S}_n^\alpha \hat{S}_{n+1}^\alpha - \frac{1}{2} h^R_{\alpha}\hat{S}_n^\alpha].
$$
The coupling terms are consistent with those obtained for the ground states. Except for the coupling constants $S^x S^x$ and the transverse field $\hat{S}^z$ that originally exist in the quantum Ising model, the $S^z S^z$ coupling and a vertical field $\hat{S}^z$ emerge in the EBHs. Due to the symmetries of the system, we also obtain

$$J_{L}^{zz} = -J_{R}^{zz} \equiv J^{zz}, \quad h_{L}^{z} = h_{R}^{z} \equiv h^{z}, \quad (17)$$

$$J_{L}^{zz} = -J_{R}^{zz} \equiv J^{zz}, \quad h_{L}^{z} = h_{R}^{z} \equiv h^{z}. \quad (18)$$

The coupling strengths have odd parity while magnetic fields exhibit even parity when changing from the left to the right environment. Same as the ground-state cases, this is possibly because that couplings are antiferromagnetic and the magnetic fields are uniform.

The difference from the ground-state cases is that the strengths of the couplings here are temperature adaptive and vary with $T'$. See Fig. 3 for the coupling strengths and magnetic fields in the EBHs at $h = 0.5$ [Eq. (14)] with different $T'$. Note in principle, the spins-1/2’s on the boundaries of the QES can be replaced by the high-dimensional spins [31, 32].

In comparison with the realization of the bulk thermodynamics of the inhomogeneous system $\hat{H}_{\text{inh}}(J)$ [Eq. (1)], an advantage of employing the QES scheme is that it concerns the bounded strengths of the couplings. For $\hat{H}_{\text{inh}}(J)$, $J$ increases linearly as $T'$ lowers [Eq. (3)]. Therefore, low $T'$’s requires large coupling strength $J$ which is challenging for the experiment realization. With the QES, the coupling strengths in the EBHs are bounded. Down to the low $T'$, the strengths of all terms in the EBHs converge to some finite values with the order of magnitude $O(10^{-1})$. Moreover, a discontinuous point is observed at $1/T' \approx 31.46(0)$, which we dub as the boundary quench point (BQP) (denoted by $T'_Q$). A straightforward property is that below BQP, the EBHs becomes approximately identical to those of the ground state (see the dash lines in Fig. 3). We will come back to the BQP below with more properties from the perspective of bulk entropy.

**B. Bulk entropy and boundary quench point**

Fig. 4 compares the average bulk entropy $S = \sum_{n=1}^{N} S_n / N$ of the inhomogeneous system $\hat{H}_{\text{inh}}(J)$ and that of the QES $\hat{H}_{\text{QES}}(T')$. The $T'$ in the EBHs of the QES and the $J$ in the environments of the inhomogeneous system satisfy Eq. (3). We use the exact diagonalization (ED) to obtain the entropy of the inhomogeneous system by taking the total size from $N_{\text{tot}}$ from 6 to 14, as shown in Fig. 4(a). The bulk size is fixed to be $N = 4$. The average bulk entropy $S$ decreases as the physical temperature $T$ lowers, as expected. The results with fixed $N = 4$ and $N_{\text{tot}} = 12$ are given in Fig. 4(b), $J$ is varied from 1 to 10. For all sizes, we observe that $S$ is suppressed by increasing $J$ (meaning lowering the environment temperature). This is analogous to the phenomenon when lowering the environment temperature in an open system. No essential change of such a phenomenon is expected if we further increase the size of the environments.

For the $\hat{H}_{\text{QES}}(T')$ [Fig. 4(c)], similar suppression of $S$ by $T'$ is observed. This supports our conjecture that adjusting the parameters in the EBHs by following the results given in Fig. 3 will qualitatively realize the bulk physics induced by tuning $J$ in the inhomogeneous system.

The difference between inhomogeneous system $\hat{H}_{\text{inh}}(J)$ and QES $\hat{H}_{\text{QES}}(T')$ is the smoothness of $S$ while lowering $T'$ in the EBHs. In specific, for $\hat{H}_{\text{QES}}(T')$, the drop of $S$ is much more drastic for $T' < T'_Q$ compared with those for $T' > T'_Q$, with $T'_Q$ the BQP. The reason is that the EBHs for $T' < T'_Q$ jumps to those of the ground state (analog to the zero-temperature bath), while the EBHs for $T' > T'_Q$ possess obvious $T'$-dependence (see Fig. 3). In contrast, for $\hat{H}_{\text{inh}}(J)$, entropy $S$ behaves more smoothly as $J$ varies.

Fig. 5 demonstrates the $S$ in the QES versus the physical and environment temperatures ($T$ and $T'$, respectively). We take the bulk size $N = 8$. We shall stress that even for $T \neq T'$, the temperature defining the canonical distribution is the physical temperature $T$ [Eq. (10)]. The environment temperature $T'$ is a hyper-parameter of the couplings in the EBHs (see Eq. (15) and Fig. 3).

When $T'$ and $T$ are both low, the bulk entropy $S$ is sup-
FIG. 5. (Color online) The average bulk entropy $S$ of the QES for the quantum Ising model at different physical temperatures $T$ and environment temperatures $T'$. We take the transverse field at the critical point $h_c = 0.5$. $S$ is suppressed to $S = 0.188$ for about $T < 0.022$ and $T' < 0.026$.

FIG. 6. (Color online) The on-site entropy $S_n$ of the QES for the quantum Ising model on different site $n$ of the bulk at different physical temperatures $T$. Taking the left and right bath temperatures as $T'_L \approx 8$ and $T'_R \approx 10^{-3}$, respectively, a slope of the spatial distribution of the fluctuations is observed.

pressed to a low value with $S \simeq S_0 \simeq 0.188$, as demonstrated by the blue area. Define the thermal cross-over point (TCP) $T_C$ as the temperature where we have $S \simeq S_0$ for $T < T_C$. Our result show $T'_Q \simeq T_C + O(10^{-3})$, which indicates the underlying equivalence of the scalings between these two temperatures. This can be explained by considering $T = T'$. In this case, the QES optimally mimics the infinite-size TI model $\tilde{H}_T$, where the cross-over temperature is accurately predicted by the drop of $S$. Therefore, the BQP and TCP must be unified to the same value.

The EBHs with different left and right environment temperatures can be taken. Fig. 6 shows the on-site entropy $S_n$ [Eq. (12)] at different sites ($n$) with the left and right environment temperatures as $T'_L \approx 8$ and $T'_R \approx 10^{-3}$, respectively. The physical temperature is still uniformly $T$ and the system is still described by the canonical distribution as Eq. (10). The spatial distribution of $S_n$ shows non-zero gradient in the bulk of the QES. It indicates that the EBH with a low (high) $T'$ tends to drive the system into an ordered (disordered) state, similar to a heat bath with a low (high) temperature. It is interesting to note such a system should be generally described by a grand canonical ensemble [41], while the above results are obtained from the canonical ensemble.

C. Bulk entropy with topological degeneracy

We consider the spin-1 Heisenberg chain [42] with the Hamiltonian $\hat{h}_{n,n+1} = \hat{S}^x_n \hat{S}^x_{n+1} + \hat{S}^y_n \hat{S}^y_{n+1} + \hat{S}^z_n \hat{S}^z_{n+1}$, where $\hat{S}_n^\alpha (\alpha = x, y, z)$ denotes the spin-1 operator on the $n$-th site. The ground states are in the Haldane phase with two-fold degenerate and possess non-trivial topological properties [43].

Fig. 7 demonstrates the bulk entropy per site $S$ at different $T$ and $T'$ with the bulk size $N = 4$. When $T$ and $T'$ are both low, the bulk entropy is suppressed to $S \rightarrow S_0 \simeq 0.937$. Due to the topological degeneracy, the low-temperature entropy $S_0$ is much larger than that of the topologically-trivial quantum Ising model. The equivalence between the TCP and BQP still stands, with $T_C \simeq T'_Q \simeq 0.101$.

With $T$ fixed, we find that $S$ does not monotonously decrease with $T'$. A “trench” of $S$ (see the blue region in Fig. 7) emerges before the system enters the region of $S = S_0$. It means the quantum fluctuations from the degenerate ground states and the thermal fluctuations are simultaneously suppressed by tuning $T'$ to this range. The trench appears only in the left-down region in the $T' - T$ map, i.e., $T' > T$. Such an entropy trench is a typical boundary effects caused by tuning the EBHs. One cannot observe the trench in the normal infinite TI model, because such a model is optimally predicted by the QES with $T' = T$, indicated by the dash-dot line that does
not cross the trench. We speculate that the entropy trench can be a thermal signature of the ground-state topological degeneracy, which we will verify in more topological systems in the future.

IV. EXPERIMENTAL IMPLEMENTATIONS

Reducing \( \hat{H}_{\text{qho}}(J) \) to the QES is advantageous to its experimental realization. The EBH contains just one-body and two-body terms, and their strengths are bounded even for the extremely low environment temperatures. For the case of spin-1/2 models, one needs the controlled coupling between the boundary spins and auxiliary (non-necessarily 1/2) spins. We foresee four possible platforms to realize these tasks. Each of these platforms contributes to the pillars of contemporary Quantum Technologies: quantum computing, quantum simulation and quantum metrology [44]. We list and comment on them below:

- **Ultracold trapped ions.** This is probably the most promising platform, so we focus on it. These systems reduce to spin models, in which ions’ internal states serve a spin states, and couplings are mediated by phonons/lasers (cf. [45 and 46] and references therein; for overview of underlying theory and experiments see [47–57]). One could design additional traps at the edges of the system with the same or even different ions and couple them in the desired controlled way to the bulk. Such approach would work even in two dimensions.

- **Trapped Rydberg atoms.** Similarly, one could use arrays of trapped Rydberg atoms that may serve as simulators of spin models with long-range couplings [58–61]. Again, the idea is to design additional traps with auxiliary (generally different) Rydberg atoms, and design couplings of these atoms to the bulk by, e.g., following the data in Fig. 2.

- **Ultracold atoms in optical lattices.** Spin models can be realized with ultracold atoms in optical lattices employing for instance super-exchange interactions (cf. [62]). Using contemporary super-lattice/holographic mask techniques, one can design a lattice, in which atoms in other internal states are trapped, and are brought to interact with the bulk in a desired way.

- **Ultracold atoms in nano-structures.** Such systems realize spin models with controlled long range interactions [63–65]. Again, an appropriate design of nano-structures allows one to add additional traps and atoms at the edges.

V. SUMMARY AND DISCUSSION

In summary, we expose the exotic boundary-induced thermodynamic properties with the entanglement bath Hamiltonian (EBH). The EBH is variationally determined as the Hamiltonian that reproduces the bulk entanglement Hamiltonian of the infinite-size translational-invariant system at finite temperatures. We show the bulk entropy can be controlled by tuning the EBH. A discontinuous point is found on the coefficients of the EBH and the bulk entropy. It indicate a low-entropy region where the thermal fluctuations are suppressed to be insignificant and the bulk properties are dominated by the ground-state physics. For the Haldane chain with non-trivial topological properties, an entropy trench is observed as a typical boundary effect that cannot be observed on the normal translational-invariant system, which could be a signature of ground-state topological degeneracy. The possible experimental realizations of the uncovered boundary physical phenomena are discussed.

ACKNOWLEDGMENTS

The authors are grateful to Gang Su, Wei Li, Han Li, Kai Xu, Han-Jie Zhu, Bin-Bin Chen and Leticia Tarruell for stimulating discussions. This work is supported by NSFC (No. 12004266, No. 11834014 and Grant No. 12074027), Beijing Natural Science Foundation (Grant No. 1192005 and No. Z180013), Foundation of Beijing Education Committees (No. KM202010028013), and the key research project of Academy for Multidisciplinary Studies, Capital Normal University. M.L. acknowledges support from ERC AdG NOQIA, Agenzia Estatal de Investigación (“Severo Ochoa” Center of Excellence CEX2019-000910-S, Plan National FIDUEIA PID2019-106901GB-I00/10.13039/501100011033, FPI), Fundació Privada Cellex, Fundació Mir-Puig, and from Generalitat de Catalunya (AGAUR Grant No. 2017 SGR 1341, CERCA program, QuantumCAT U16-011424, co-funded by ERDF Operational Program of Catalonia 2014-2020), MINECO-EU QUANTERA MAQS (funded by State Research Agency (AEI) PCI2019-111828-2/10.13039/501100011033, EU Horizon 2020 FET-OPEN OPTOLogic (Grant No 899794), and the National Science Centre, Poland-Symphonia Grant No. 2016/20/W/ST4/00314, Marie Sklodowska-Curie Grant STRETCH No. 101029393.

[1] Nicolas Laflorencie, Erik S. Sørensen, Ming-Shyang Chang, and Ian Affleck, “Boundary effects in the critical scaling of entanglement entropy in 1d systems,” Phys. Rev. Lett. 96, 100603 (2006).

[2] Ian Affleck, Nicolas Laflorencie, and Erik S. Sørensen, “Entanglement entropy in quantum impurity systems and systems with boundaries,” Journal of Physics A Mathematical General 42, 504009 (2009).
[3] Dmitry V. Pursuev and Sergey N. Solodukhin, “Anomalies, entropy, and boundaries,” Phys. Rev. D 93, 084021 (2016).

[4] Clément Berthiere and William Witzczak-Krempa, “Relating bulk to boundary entanglement,” Phys. Rev. B 100, 235112 (2019).

[5] Massimo Campostrini, Andrea Pelissetto, and Ettore Vicari, “Quantum transitions driven by one-bond defects in quantum ising rings,” Phys. Rev. E 91, 042123 (2015).

[6] Massimo Campostrini, Andrea Pelissetto, and Ettore Vicari, “Quantum Ising chains with boundary fields,” Journal of Statistical Mechanics: Theory and Experiment 2015, 11015 (2015).

[7] Jiao Wang and Giulio Casati, “One-dimensional self-organization and nonequilibrium phase transition in a hamiltonian system,” Phys. Rev. Lett. 118, 040601 (2017).

[8] Shi-Ju Ran, Cheng Peng, Gang Su, and Maciej Lewenstein, “Controlling the phase diagram of finite spin-$\frac{1}{2}$ chains by tuning the boundary interactions,” Phys. Rev. B 98, 085111 (2018).

[9] B. Andrei Bernevig, Taylor L. Hughes, and Shou-Cheng Zhang, “Quantum Spin Hall Effect and Topological Phase Transition in HgTe Quantum Wells,” Science 314, 1757 (2006).

[10] M. Z. Hasan and C. L. Kane, “Colloquium: Topological insulators,” Rev. Mod. Phys. 82, 3045–3067 (2010).

[11] Tony E. Lee, “Anomalous edge state in a non-hermitian lattice,” Phys. Rev. Lett. 116, 133903 (2016).

[12] Yang Cao, Yang Li, and Xiaosen Yang, “Non-hermitian bulk-boundary correspondence in a periodically driven system,” Phys. Rev. B 103, 075126 (2021).

[13] Xiao-Liang Qi and Shou-Cheng Zhang, “Topological insulators and superconductors,” Rev. Mod. Phys. 83, 1057–1110 (2011).

[14] Adrien Bouhon and Manfred Sigrist, “Current inversion at the edges of a chiral $p$-wave superconductor,” Phys. Rev. B 90, 220511 (2014).

[15] Andrea Benfenati, Albert Samoilienka, and Egor Babaev, “Boundary effects in two-band superconductors,” Phys. Rev. B 103, 144512 (2021).

[16] I. M. Georgescu, S. Ashhab, and Franco Nori, “Quantum simulation,” Rev. Mod. Phys. 86, 153–185 (2014).

[17] Shi-Ju Ran, Bin Xi, Cheng Peng, Gang Su, and Maciej Lewenstein, “Efficient quantum simulation for thermodynamics of infinite-size many-body systems in arbitrary dimensions,” Phys. Rev. B 99, 205132 (2019).

[18] Viacheslav Kuzmin, Torsten V. Zache, Christian Kokail, Lorenzo Pastori, Alessio Celi, Mikhail Baranov, and Peter Zoller, “Probing infinite many-body quantum systems with finite-size quantum simulators,” PRX Quantum 3, 020304 (2022).

[19] Jin-Wu Jiang, Jie Chen, Jian-Sheng Wang, and Baowen Li, “Edge states induce boundary temperature jump in molecular dynamics simulation of heat conduction,” Phys. Rev. B 80, 052301 (2009).

[20] Henk van Beijeren, “Exact results for anomalous transport in one-dimensional hamiltonian systems,” Phys. Rev. Lett. 108, 180601 (2012).

[21] Joseph J. Bisognano and Eyvind H. Wichmann, “On the duality condition for a Hermitian scalar field,” Journal of Mathematical Physics 16, 985–1007 (1975).

[22] Joseph J. Bisognano and Eyvind H. Wichmann, “On the duality condition for quantum fields,” Journal of Mathematical Physics 17, 303–321 (1976).

[23] Francesco Parisen Toldin and Fakher F. Assaad, “Entanglement hamiltonian of interacting fermionic models,” Phys. Rev. Lett. 121, 200602 (2018).

[24] W. Zhu, Zhoushen Huang, and Yin-Chen He, “Reconstructing entanglement hamiltonian via entanglement eigenstates,” Phys. Rev. B 99, 235109 (2019).

[25] Mahdieh Pourjafarabadi, Hanieh Najafzadeh, Mohammad-Sadegh Vaezi, and Abolhassan Vaezi, “Entanglement hamiltonian of interacting systems: Local temperature approximation and beyond,” Phys. Rev. Research 3, 013217 (2021).

[26] Pasquale Calabrese and John Cardy, “Evolution of entanglement entropy in one-dimensional systems,” Journal of Statistical Mechanics: Theory and Experiment 2005, 2005 (2005).

[27] Anatoli Polkovnikov, Krishnendu Sengupta, Alessandro Silva, and Mukund Vengalattore, “Colloquio: Nonequilibrium dynamics of closed interacting quantum systems,” Rev. Mod. Phys. 83, 863–883 (2011).

[28] Joseph J. Bisognano and Eyvind H. Wichmann, “On the duality condition for quantum fields,” Journal of Statistical Mechanics: Theory and Experiment 11, 113103 (2018).

[29] Christian Kokail, Rick van Bijnen, Andreas Elben, Benoît Vermersch, and Peter Zoller, “Entanglement Hamiltonian tomography in quantum simulation,” Nature Physics 17, 936–942 (2021).

[30] Shi-Ju Ran, “Ab initio optimization principle for the ground states of translationally invariant strongly correlated quantum lattice models,” Phys. Rev. E 93, 053301 (2016).

[31] Shi-Ju Ran, Angelo Piga, Cheng Peng, Gang Su, and Maciej Lewenstein, “Few-body systems capture many-body physics: Tensor network approach,” Phys. Rev. B 96, 155120 (2017).

[32] Jacob C. Bridgeman and Christopher T. Chubb, “Hand-waving and interpretive dance: an introductory course on tensor networks,” Journal of Physics A Mathematical General 50, 223001 (2017).

[33] Xiaoqun Wang and Tao Xiang, “Transfer-matrix density-matrix renormalization group methods for quantum spin systems,” Advances in Physics 57, 143–224 (2008).

[34] J. Ignacio Cirac and Frank Verstraete, “Renormalization and tensor product states in spin chains and lattices,” J. Phys. A: Math. Theor. 42, 504004 (2009).

[35] Y. R. Vaezi and Mahdieh Pourjafarabadi, “Efficient tensor renormalization group algorithm for the calculation of thermodynamic properties of quantum lattice models,” Phys. Rev. B 99, 151102 (2019).

[36] J. Ignacio Cirac, David Pérez-García, Norbert Schuch, and Frank Verstraete, “Matrix product states and projected entangled pair states: Concepts, symmetries, theorems,” Rev. Mod. Phys. 93, 045003 (2021).

[37] Xueda Wen, Shinsei Ryu, and Andreas W. W. Ludwig, “Entanglement Hamiltonian evolution during thermalization in conformal field theory,” Journal of Statistical Mechanics: Theory and Experiment 11, 113103 (2018).

[38] Mahdieh Pourjafarabadi and Christian Kokail, “Colloquio: Nonequilibrium dynamics of closed interacting quantum systems,” Rev. Mod. Phys. 83, 863–883 (2011).

[39] John Cardy and Erik Tonni, “Entanglement Hamiltonians in two-dimensional conformal field theory,” Journal of Statistical Mechanics: Theory and Experiment 12, 123103 (2016).

[40] Joseph J. Bisognano and Eyvind H. Wichmann, “On the duality condition for quantum fields,” Journal of Statistical Mechanics: Theory and Experiment 11, 113103 (2018).

[41] Christian Kokail, Rick van Bijnen, Andreas Elben, Benoît Vermersch, and Peter Zoller, “Entanglement Hamiltonian tomography in quantum simulation,” Nature Physics 17, 936–942 (2021).

[42] Jacob C. Bridgeman and Christopher T. Chubb, “Hand-waving and interpretive dance: an introductory course on tensor networks,” Journal of Physics A Mathematical General 50, 223001 (2017).

[43] Xiaoqun Wang and Tao Xiang, “Transfer-matrix density-matrix renormalization-group theory for thermodynamics of one-dimensional quantum systems,” Phys. Rev. B 56, 5061–5064 (1997).

[44] Wei Li, Shi-Ju Ran, Shou-Shu Geng, Yang Zhao, Bin Xi, Fei Ye, and Gang Su, “Linearized tensor renormalization group algorithm for the calculation of thermodynamic properties of quantum lattice models,” Phys. Rev. Lett. 106, 127202 (2011).

[45] Mahdieh Pourjafarabadi, Hanieh Najafzadeh, Mohammad-Sadegh Vaezi, and Abolhassan Vaezi, “Entanglement hamiltonian of interacting systems: Local temperature approximation and beyond,” Phys. Rev. Research 3, 013217 (2021).

[46] Francesco Parisen Toldin and Fakher F. Assaad, “Entanglement hamiltonian of interacting fermionic models,” Phys. Rev. Lett. 121, 200602 (2018).

[47] Michael C. Mackey, “The dynamic origin of increasing entropy,” Rev. Mod. Phys. 61, 981–1015 (1989).
[42] Steven R. White and David A. Huse, “Numerical renormalization-group study of low-lying eigenstates of the antiferromagnetic S=1 Heisenberg chain,” Phys. Rev. B 48, 3844–3852 (1993).

[43] Chetan Nayak, Steven H. Simon, Ady Stern, Michael Freedman, and Sankar Das Sarma, “Non-abelian anyons and topological quantum computation,” Rev. Mod. Phys. 80, 1083–1159 (2008).

[44] Antonio Acín, Immanuel Bloch, Harry Buhrman, Tommaso Calarco, Christopher Eichler, Jens Eisert, Daniel Esteve, Nicolas Gisin, Steffen J Glaser, Fedor Jelezko, Stefan Kuhr, Maciej Lewenstein, Max F Riedel, Piet O Schmidt, Rob Thew, Andreas Wallraff, Ian Walmsley, and Frank K Wilhelm, “The quantum technologies roadmap: a European community view,” New Journal of Physics 20, 080201 (2018).

[45] Tobias Graß, David Raventós, Bruno Juliá-Díaz, Christian Gogolin, and Maciej Lewenstein, “Quantum annealing for the number-partitioning problem using a tunable spin glass of ions,” Nature Communications 7, 11524 (2016).

[46] David Raventós, Tobias Graß, Bruno Juliá-Díaz, and Maciej Lewenstein, “Semiclassical approach to finite-temperature quantum annealing with trapped ions,” Phys. Rev. A 97, 052310 (2018).

[47] Florian Mintert and Christof Wunderlich, “Ion-trap quantum logic using long-wavelength radiation,” Phys. Rev. Lett. 87, 257904 (2001).

[48] D. Porras and J. I. Cirac, “Effective quantum spin systems with trapped ions,” Phys. Rev. Lett. 92, 207901 (2004).

[49] A. Friedenauer, H. Schmitz, J. T. Glueckert, D. Porras, and T. Schaeetz, “Simulating a quantum magnet with trapped ions,” Nature Physics 4, 757–761 (2008).

[50] Manuel Mielzen, Henning Kalis, Matthias Wittener, Frederick Hakelberg, Ulrich Warring, Roman Schmied, Matthew Blain, Peter Maunz, David L. Moehring, Dietrich Leibfried, and Tobias Schaeetz, “Arrays of individually controlled ions suitable for two-dimensional quantum simulations,” Nature Communications 7, 11839 (2016).

[51] Henning Kalis, Frederick Hakelberg, Matthias Wittener, Manuel Mielzen, Ulrich Warring, and Tobias Schaeetz, “Motional-mode analysis of trapped ions,” Phys. Rev. A 94, 023401 (2016).

[52] P. Jurcevic, H. Shen, P. Hauke, C. Maier, T. Brydges, C. Hempel, B. P. Lanyon, M. Heyl, R. Blatt, and C. F. Roos, “Direct observation of dynamical quantum phase transitions in an interacting many-body system,” Phys. Rev. Lett. 119, 080501 (2017).

[53] Hao-Kun Li, Erik Urban, Crystal Noel, Alexander Chuang, Yang Xia, Anthony Ransford, Boerge H¨ubnerling, Yuan Wang, Tongcang Li, Hartmut Haefner, and Xiang Zhang, “Realization of translational symmetry in trapped cold ion rings,” Phys. Rev. Lett. 118, 030501 (2017).

[54] J. Zhang, P. W. Hess, A. Kyprianidis, P. Becker, A. Lee, J. Smith, G. Pagano, I. D. Potirniche, A. C. Potter, A. Vishwanath, N. Y. Yao, and C. Monroe, “Observation of a discrete time crystal,” Nature (London) 543, 217–220 (2017), arXiv:1609.08684 [quant-ph].

[55] J. Zhang, G. Pagano, P. W. Hess, A. Kyprianidis, P. Becker, H. Kaplan, A. V. Gorshkov, Z. X. Gong, and C. Monroe, “Observation of a many-body dynamical phase transition with a 53-qubit quantum simulator,” Nature (London) 551, 601–604 (2017), arXiv:1708.01044 [quant-ph].

[56] A. Safavi-Naini, R. J. Lewis-Swan, J. G. Bohnet, M. Gartner, K. A. Gilmore, E. Jordan, J. Cohn, J. K. Freericks, A. M. Rey, and J. J. Bollinger, “Exploring adiabatic quantum dynamics of the Dicke model in a trapped ion quantum simulator,” arXiv e-prints, arXiv:1711.07392 (2017), arXiv:1711.07392.

[57] W. L. Tan, P. Becker, F. Liu, G. Pagano, K. S. Collins, A. De, L. Feng, H. B. Kaplan, A. Kyprianidis, R. Lundgren, W. Mrog, S. Whitsitt, A. V. Gorshkov, and C. Monroe, “Domain-wall confinement and dynamics in a quantum simulator,” Nature Physics (2021), 10.1038/s41567-021-01194-3.

[58] Hannes Bernien, Sylvain Schwartz, Alexander Keesling, Harry Levine, Ahmed Omran, Hannes Piechler, Soonwoo Choi, Alexander S. Zibrov, Manuel Endres, Markus Greiner, Vladan Vuleti´c, and Mikhail D. Lukin, “Probing many-body dynamics on a 51-atom quantum simulator,” Nature (London) 551, 579–584 (2017).

[59] Henning Labuhn, Daniel Barredo, Sylvain Ravets, Sylvain de Léscelleuc, Tommaso Macri, Thierry Lahaye, and Antoine Browaeys, “Tunable two-dimensional arrays of single Rydberg atoms for realizing quantum Ising models,” Nature (London) 534, 667–670 (2016).

[60] Vincent Lienhard, Pascal Scholl, Sebastian Weber, Daniel Barredo, Sylvain de Léscelleuc, Rukmani Bai, Nicolai Lang, Michael Fleischhauer, Hans Peter Büchler, Thierry Lahaye, and Antoine Browaeys, “Realization of a density-dependent peierls phase in a synthetic, spin-orbit coupled rydberg system,” Phys. Rev. X 10, 021031 (2020).

[61] D. Bluvstein, A. Omran, H. Levine, A. Keesling, G. Semeghini, S. Ebadi, T. T. Wang, A. A. Michailidis, N. Maskara, W. W. Ho, S. Choi, M. Serbyn, M. Greiner, V. Vuleti´c, and M. D. Lukin, “Controlling quantum many-body dynamics in driven Rydberg atom arrays,” Science 371, 1355–1359 (2021).

[62] M. Lewenstein, A. Sanpera, and V. Ahufinger, Ibracold atoms in Optical Lattices (Oxford University Press, 2017).

[63] D. E. Chang, J. I. Cirac, and H. J. Kimble, “Self-Organization of Atoms along a Nanophotonic Waveguide,” Phys. Rev. Lett. 110, 113606 (2013).

[64] A. Goban, C. L. Hung, S. P. Yu, J. D. Hood, J. A. Muniz, J. H. Lee, M. J. Martin, A. C. McClung, K. S. Choi, D. E. Chang, O. Painter, and H. J. Kimble, “Atom-light interactions in photonic crystals,” Nature Communications 5, 3808 (2014).

[65] J. S. Douglas, H. Habibian, C. L. Hung, A. V. Gorshkov, H. J. Kimble, and D. E. Chang, “Quantum many-body models with cold atoms coupled to photonic crystals,” Nature Photonics 9, 326–331 (2015).

[66] The official website of PyTorch is at https://pytorch.org/