Banach–Mazur Distance from the Parallelogram to the Affine-Regular Hexagon and Other Affine-Regular Even-Gons

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Abstract. We show that the Banach–Mazur distance between the parallelogram and the affine-regular hexagon is $\frac{3}{2}$ and we conclude that the diameter of the family of centrally-symmetric planar convex bodies is just $\frac{3}{2}$. A proof of this fact does not seem to be published earlier. Asplund announced this without a proof in his paper proving that the Banach–Mazur distance of any planar centrally-symmetric bodies is at most $\frac{3}{2}$. Analogously, we deal with the Banach–Mazur distances between the parallelogram and the remaining affine-regular even-gons.

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1. Introduction
Denote by $\mathcal{M}^d$ the family of all centrally symmetric convex bodies of the Euclidean space $E^d$ centered at the origin $o$ of $E^d$. Below always when we say on a homothety, we mean that its center is at $o$. For any $K \in \mathcal{M}^d$ and any positive $\lambda$, by $\lambda K$ we denote the homothety image of $K$ with ratio $\lambda$. 
The Banach–Mazur distance (or shortly, the BM-distance) of \( C, D \in \mathcal{M}^d \) is defined as

\[
\delta_{BM}(C, D) = \inf_{a, \lambda} \{ \lambda; a(C) \subset D \subset \lambda a(C) \},
\]

where \( a \) stands for an affine transformation.

This definition is presented by Banach [3] in behalf of him and Mazur. From the role of the affine transformation in this definition we see that a different formulation of this definition is formally more precise, namely when we consider the Banach–Mazur distance of the equivalence classes of centrally symmetric bodies with respect to affine transformations. But in this note it is more convenient to deal with the BM-distance of convex bodies from \( \mathcal{M}^d \). It is well known that the Banach–Mazur distance is a multiplicative metric, i.e., that \( \log \delta_{BM} \) is a metric. In particular, we have \( \delta_{BM}(C, D) = \delta_{BM}(D, C) \) and the multiplicative triangle inequality \( \delta_{BM}(C, D) \cdot \delta_{BM}(D, E) \geq \delta_{BM}(C, E) \) for every \( C, D, E \in \mathcal{M}^d \). Later the notion of the BM-distance has been extended to BM-distance of pairs of arbitrary convex bodies \( C, D \). For a survey of results on the BM-distance we propose the books by Tomczak-Jaegerman [8], Toth [9], and Aubrun and Szarek [2].

By \( P_n \) we denote the regular \( n \)-gon with vertices \((\cos \frac{2j}{n} \pi, \sin \frac{2j}{n} \pi)\) for \( j = 0, \ldots, n - 1 \).

Our main aim is to show that \( \delta_{BM}(P_4, P_6) = \frac{3}{2} \) and to establish all the optimal positions of the parallelograms \( a(P_4) \) with respect to the hexagon \( P_6 \). Recall that the value \( \frac{3}{2} \) is stated in Theorem 1 of [1] by Asplund. It says that \( \delta_{BM}(P_4, C) \leq \frac{3}{2} \) for any \( C \in \mathcal{M}^2 \) with equality only for \( C = P_6 \). But the proof of Lemma 3 of [1] that \( \delta_{BM}(P_4, P_6) = \frac{3}{2} \) is omitted. This statement is quoted by Stromquist on p. 206 of [7] and illustrated in Fig. 1 there. Since the hexagon from this Fig. 1 is not inscribed in the larger parallelogram, then looking to the second thesis of our Proposition we get further questions. The basic task of [7] is to construct a center \( R \) of \( \mathcal{M}^2 \) (see p. 207 and compare Fig. 2). With the help of Fig. 3 it is shown that both \( P_4 \) and \( P_6 \) are in the distance \( \sqrt{3/2} \) from \( R \). This implies \( \delta_{BM}(P_4, P_6) \leq \frac{3}{2} \), but does not imply the equality. These doubts mobilized the writer to present a detailed proof of the equality \( \delta_{BM}(P_4, P_6) = \frac{3}{2} \) in the present note in order to be sure that the diameter of \( \mathcal{M}^2 \) is \( \frac{3}{2} \).

An additional aim is explained here. The idea of the proof of our Theorem 1 that \( \delta_{BM}(P_4, P_6) = \frac{3}{2} \) encouraged the author to consider the analogous task for all the regular even-gons with more vertices in place of \( P_6 \). Theorem 2 establishes the Banach–Mazur distances from \( P_4 \) to all \( P_{8j} \) and \( P_{8j+4} \). This task for the remaining even-gons, so to \( P_{8j+2} \) and \( P_{8j+6} \) appears to be more complicated. We estimate and conjecture the values of these distances.
2. Pairs of Homothetic Parallelograms Contained and Containing a Centrally Symmetric Convex Body

By an inscribed parallelogram in a convex body $C$ we mean a parallelogram with all vertices in the boundary of $C$ and by a circumscribed parallelogram about $C$ we mean a parallelogram containing $C$ whose all sides have non-empty intersections with $C$.

By the width $\text{width}(S)$ of a strip $S$ between two parallel hyperplanes of $E^d$ we mean the distance of these hyperplanes. We omit an easy proof of the following lemma, whose two-dimensional version is applied in the proofs of Theorems 1 and 2.

**Lemma.** Let $H_1^+, H_1^-$ and $H_2^+, H_2^-$ be two pairs of hyperplanes of $E^d$ and let all of them be parallel. Assume that each pair is symmetric with respect to $o$ and that all these hyperplanes do not contain $o$. Let $L$ be a straight line through $o$ not parallel to these hyperplanes. Assume that $H_2^+, H_1^+, H_1^-, H_2^-$ intersect $L$ in this order. Denote by $(a_1^i, \ldots, a_n^i)$ the points of intersection of $L$ with $H_1^i$, where $i \in \{1, 2\}$. Consider the strips $S^i = \text{conv}(H_1^+ \cup H_1^-)$, where $i \in \{1, 2\}$. We claim that $\text{width}(S^2)/\text{width}(S^1) = a_j^2/a_j^1$ for any $j \in \{1, \ldots, n\}$ if $a_j^1 \neq 0$.

The following proposition is applied later a few times when evaluating particular BM-distances.

**Proposition.** Let $C \in \mathcal{M}^2$. Assume that $P \subset C \subset \mu P$ for a parallelogram $P \in \mathcal{M}^2$ and a positive $\mu$. Then there exists a parallelogram $P'$ inscribed in $C$ such that $\mu P'$ is circumscribed about $C$ for a $\mu' \leq \mu$.

**Proof.** Since the thesis is obvious for $\mu = 1$, below we assume that $\mu > 1$.

If $P$ in the part of $P'$ fulfills the thesis, there is nothing to prove. In the opposite case we intend to construct the required $P'$.

Of course, there is a homothetic image $P_\alpha \subset P$ of $P$ such that for least one pair of opposite sides $Z^+, Z^-$ of $P_\alpha$ the sides $\mu Z^+, \mu Z^-$ of $\mu P_\alpha$ touch $C$ from outside. Denote the other pair of sides of $P_\alpha$ by $W^+, W^-$ and assume that their order is $W^+, Z^+, W^-, Z^-$ when we move counterclockwise.

If $\mu W^-, \mu W^+$ do not touch $C$, then we lessen $P_\alpha$ up to $P_\beta$ by moving $W^+, W^-$ such that their vertices remain in $Z^+, Z^-$ symmetrically closer to $o$ up to the position $F^+, F^-$ when both $\mu F^+$ and $\mu F^-$ touch $C$. The other two sides of the parallelogram $P_\beta$ are denoted by $G^+, G^-$. Clearly, $G^+ \subset Z^+$ and $G^- \subset Z^-$. The parallelogram $\mu P_\beta$ is circumscribed about $C$. Of course, when we go counterclockwise, then the order of the sides of $P_\beta$ is $F^+, G^+, F^-, G^-$. Clearly $P_\beta$ is a subset of $C$, but it may be not inscribed in $C$.

Take the homothetic copy $P_\gamma \subset C$ of $P_\beta$ such that at least one pair of opposite vertices $t, v$ of $P_\gamma$ is in $\text{bd}(C)$. Denote the other two vertices of $P_\gamma$ by $u, w$ such that $t, u, v, w$ be in this order on $\text{bd}(C)$.

Let $a, b \in \text{bd}(C)$ be the points such that $w \in ta$ and $w \in vb$
If \( u, w \) are not in \( \text{bd}(C) \), let us cleverly enlarge \( P_\gamma \) up to a parallelogram \( P_\delta \) (not obligatory homothetic) inscribed in \( C \) such that a homothetic copy of \( P_\delta \) with a ratio at most \( \mu \) is circumscribed about \( C \).

Here we explain how to provide this task. For every boundary point \( c \) of \( \text{bd}(C) \) on the arc \( \widetilde{ab} \) provide the straight lines \( K_1^+(c), K_2^+(c) \) containing \( cv \) and \( ct \), respectively. Let \( K_1^-(c), K_2^-(c) \) be the line symmetric to \( K_1^+(c), K_2^+(c) \), respectively. Moreover, for \( i = 1, 2 \) provide the two pairs of the straight lines \( L_i^+(c) \) and \( L_i^-(c) \) supporting \( C \) and parallel to the pairs \( K_i^+(c) \) and \( K_i^-(c) \) such that the order of them is \( L_i^+(c), K_i^+(c), K_i^-(c), L_i^-(c) \).

Let \( L_i(c) \) be the strip between \( L_i^+(c) \) and \( L_i^-(c) \) for \( i = 1, 2 \) and \( K_i(c) \) be the strip between \( K_i^+(c) \) and \( K_i^-(c) \) for \( i = 1, 2 \).

Provide the straight line through \( o \) and points \( d^+, d^- \) of support of \( C \) by \( L_1^+(c) \) and \( L_1^-(c) \), respectively. Denote by \( f^+, f^- \) its intersections with \( F^+, F^- \), respectively. Denote by \( k^+, k^- \) its intersections with \( K_1^+(c), K_1^-(c) \), respectively. Denote by \( h^+, h^- \) its intersections with \( \mu F^+, \mu F^- \). Since \( h^+, d^+, k^+, f^+, f^-, k^-, d^-, h^-, h^- \) are in this order on this straight line, we obtain \( |h^+h^-|/|f^+f^-| \leq |d^+d^-|/|k^+k^-| \). Hence by Lemma we obtain that

\[
\text{width}(L_i(c))/\text{width}(K_i(c)) \leq \mu
\]

for \( i = 1 \). Analogously, we conclude this for \( i = 2 \).

If \( c \) is sufficiently close to \( a \), then

\[
\text{width}(L_1(c))/\text{width}(K_1(c)) > \text{width}(L_2(c))/\text{width}(K_2(c)).
\]

If \( c \) is sufficiently close to \( b \), then we have the opposite inequality. Moreover, observe that when \( c \) moves on \( \widetilde{ab} \) from \( a \) to \( b \), then the strips \( L_i(c) \) and \( K_i(c) \), where \( i = 1, 2 \), change continuously. Consequently, there is at least one position \( c_0 \) of \( c \) for which

\[
\text{width}(L_1(c_0))/\text{width}(K_1(c_0)) = \text{width}(L_2(c_0))/\text{width}(K_2(c_0)).
\]

Therefore the thesis of our proposition is true for the parallelograms \( P' = K_1(c_0) \cap K_2(c_0) \). Still the homothetic parallelogram \( \mu'P = L_1(c_0) \cap L_2(c_0) \), where \( \mu' \leq \mu \), is circumscribed about \( C \).

\[\square\]

**Corollary.** Denote the Banach–Mazur distance between \( C \in \mathcal{M}^2 \) and \( P \) by \( \mu \). Assume that \( P \subset C \subset \mu P \) for a particular affine image \( P \in \mathcal{M}^2 \) of \( P_4 \). Then the parallelogram \( P \) is inscribed in \( C \) and \( \mu P \) is circumscribed about \( C \).

The author does not know if Proposition holds true in higher dimensions for the parallelepoid or the cross-polytope in place of the parallelogram.

### 3. Banach–Mazur Distance Between the Parallelogram and the Affine-Regular Hexagon

**Theorem 1.** We have \( \delta_{BM}(P_4, P_6) = \frac{3}{2} \).
**Proof.** The parallelogram with the four vertices at \( (\pm 1, 0) \) and \( (0, \pm \frac{1}{2} \sqrt{3}) \) is contained in \( P_6 \) and its homothetic image with ratio \( \frac{3}{2} \) contains \( P_6 \). Consequently, \( \delta_{BM}(P_4, P_6) \leq \frac{3}{2} \).

Having in mind Proposition, in order to show that \( \delta_{BM}(P_4, P_6) \geq \frac{3}{2} \), it is sufficient to consider only any parallelogram \( P = a(P_4) \) inscribed in \( P_6 \) such that a positive homothetic copy \( \lambda P \) of \( P \) is circumscribed about \( P_6 \), and to show that this homothety ratio \( \lambda \) is at least \( \frac{3}{2} \).

Denote by \( \mathcal{P} \) the class of all such parallelograms \( P \).

Consider a parallelogram \( P = pqrst \) from \( \mathcal{P} \). Some two consecutive vertices of \( P \) must be in two consecutive sides of \( P_6 \). The reason is that in the opposite case no positive homothetic image of our \( P \) is circumscribed about \( P_6 \) in contradiction to \( P \in \mathcal{P} \). In order to fix attention, thanks to the symmetries of \( P_6 \), we do not make our considerations narrower assuming that \( p \in v_0v_1 \) and \( q \in v_1v_2 \). For the same reason, we may additionally assume that \( p \in v_0m \), where \( m \) denotes the middle of the side \( v_0v_1 \).
Take the line $y = bx$ passing through $p$. It is easy to show that $p$ has the form $(\frac{\sqrt{3}}{b+\sqrt{3}}, \frac{\sqrt{3}b}{b+\sqrt{3}})$, where $b$ belongs to the interval $[0, \frac{\sqrt{3}}{3}]$. Here $b = 0$ generates $v_0$, while $b = \frac{\sqrt{3}}{3}$ generates $m$.

Our $q$ is the intersection of the segment $v_1v_2$ with a straight line $x = cy$, where $c \in [-\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}]$. So $q$ has the form $(\frac{\sqrt{3}c}{2}, \frac{\sqrt{3}c}{2})$. An easy calculation shows that the directional coefficient of the straight line containing $pq$ is $\sigma = \frac{b-\sqrt{3}}{2-bc-\sqrt{3c}}$, and that the directional coefficient of the straight line containing $sp$ is $\varsigma = \frac{3b+\sqrt{3}}{2+bc+\sqrt{3c}}$.

Denote by $S$ the strip between the straight lines containing $pq$ and $rs$, and by $S^+$ the narrowest strip parallel to $S$ which contains $P_0$. Denote by $T$ the strip between the straight lines containing $qr$ and $sp$, and by $T^+$ the narrowest strip parallel to $T$ which contains $P_0$.

We see that $P = S \cap T$. Of course, $S^+ \cap T^+$ is the parallelogram with sides parallel to the sides of $P$ which is circumscribed about $P_0$. Since we are looking only for parallelograms $P \in \mathcal{P}$, the parallelogram $S^+ \cap T^+$ should be a positive homothetic copy of $P$.

Now, our task is to describe such parallelograms $P$.

We find the first coordinate of the intersection of the straight line through $pq$ with the straight line $y = \sqrt{3}x$. It is $x_0 = \frac{\sqrt{3}}{2} \cdot \frac{1-c\varsigma}{\sqrt{3}-\sigma}$. Next we evaluate the ratio of the first coordinate of $v_1$ to $x_0$ which is $\frac{\sqrt{3}}{3} \cdot \frac{\sqrt{3}-\sigma}{1-c\varsigma}$.

By Lemma this ratio equals to $\frac{\text{width}(S^+)/\text{width}(S)}{\text{width}(T^+)/\text{width}(T)}$. By the substitution of $\sigma$ we get $\frac{\text{width}(S^+)/\text{width}(S)}{\text{width}(T^+)/\text{width}(T)} = \frac{\sqrt{3}}{3} \cdot \frac{3\sqrt{3}-\sqrt{3bc}-3c-b}{2-2bc}$. In a similar way we obtain that $\frac{\text{width}(T^+)/\text{width}(T)}{\text{width}(S^+)/\text{width}(S)}$ we conclude that only for $c = \frac{2b}{\sqrt{3b+3}}$, its both sides are equal. We see that $q \in nv_2$, where $n$ is the midpoint of $v_1v_2$. Since our $P$ is a function of $b$, we denote it by $P(b)$. Substituting $c = \frac{2b}{\sqrt{3b+3}}$ into $\frac{\text{width}(S^+)/\text{width}(S)}{\text{width}(T^+)/\text{width}(T)}$ we obtain that the common value of $w(S^+)/w(S)$ and $\frac{\text{width}(T^+)/\text{width}(T)}{\text{width}(S^+)/\text{width}(S)}$ is $h(b) = \frac{b^2+4\sqrt{3b+9}}{4b^2+2\sqrt{3b+6}}$, where $h(b)$ stands in place of $h(P(b))$. This ends our task.

Every $P(b)$ is inscribed in $P_0$ and every $h(b)P(b)$ is circumscribed about $P_0$ (see Fig. 1). The vertices of $h(b)P(b)$ being the images of vertices $p, q, r, s$ of $P(b)$ are denoted by $p', q', t', u'$, respectively. Of course, the straight line containing the side $p'q'$ of $h(b)P(b)$ supports $P_0$ at $v_1$.

We are considering here only the situation when the straight line containing the side $s'p'$ of $h(b)P(b)$ supports $P_0$ at $v_5$. This holds true if and only if the directional coefficient $\varsigma$ of the straight line containing $s'p'$ is at most the directional coefficient of the straight line containing $v_5v_0$, so when it is at most $\sqrt{3}$. Recall that $\varsigma = \frac{3b+\sqrt{3}}{2(bc+\sqrt{3c})}$, which after substituting $c = \frac{-2b}{\sqrt{3b+3}}$ gives $\varsigma = \frac{3\sqrt{3b+3}}{2(\sqrt{3}-b)}$. Solving $\frac{3\sqrt{3b+3}}{2(\sqrt{3}-b)} = \sqrt{3}$ we obtain $b = \frac{\sqrt{3}}{5}$.
So the straight line containing the side $h(b)ur$ of $h(b)P(b)$ supports $P_6$ at $v_5$ if and only if $b$ is in the interval $[0, \frac{\sqrt{3}}{5}]$. In other words, if and only if $r \in v_0k$, where $k$ is the point of intersection of $y = \frac{\sqrt{3}}{5}x$ with $v_0v_1$.

The derivative of the function $h(b)$ is $-\frac{7\sqrt{3}b^2 - 30b + 3\sqrt{3}}{4(b^2 + \sqrt{3}b + 6)^2}$. An evaluation shows that this derivative is 0 if and only if $b = \frac{1}{14}(-10\sqrt{3} \pm \sqrt{384})$.

Only $b = \frac{1}{14}(-10\sqrt{3} + \sqrt{384}) \approx 0.1625$ in the interval $[0, \frac{\sqrt{3}}{5}]$. The value of $h(b)$ for this $b$ is approximately 1.5224, so over $\frac{3}{2}$. Since $h(0) = \frac{3}{2}$ and $h(\frac{\sqrt{3}}{5}) = \frac{3}{2}$, we conclude that the global minimum of $h(b)$ in the interval $[0, \frac{\sqrt{3}}{5}]$ is $\frac{3}{2}$ and that it is attained only for $b = 0$ and $b = \frac{\sqrt{3}}{5}$. In Fig. 2 we see the two pairs $P(b), \frac{3}{2}P(b)$ for these two values of $b$.

It remains to explain what happens for $b \in [\frac{\sqrt{3}}{5}, \frac{\sqrt{3}}{3}]$, so when $p \in km$. Observe that then the boundary of the smallest positive homothetic copy of $P$ containing $P_6$ touches it at points $v_1, v_3, v_4, v_0$. Hence this situation is symmetric to the preceding one with respect to the straight line through $o$ perpendicular to $v_3v_0$. Considering $b$ only in the interval $[0, \frac{\sqrt{3}}{5}]$ is sufficient since we have in mind rotations of $P(b)$ by $60^\circ$ and $120^\circ$ and the axial symmetries with respect to the lines containing $v_0v_3, v_1v_4, v_2v_5$. □
Let us add that when $b$ changes from $0$ to $\frac{\sqrt{2}}{2}$ in the paragraph before the last of the proof, the point $p$ changes from $v_0 = (0,1)$ to $k = (\frac{5}{6}, \frac{\sqrt{3}}{6})$ beating $\frac{1}{3}$ of the unit. Simultaneously $q$ changes from $(0, \frac{\sqrt{3}}{2})$ to $(-\frac{1}{6}, \frac{\sqrt{3}}{2})$ beating $\frac{1}{6}$ of the unit.

Remark. The only two positions of $P \subset P_6$ such that $P_6 \subset \frac{3}{2} P$ (besides their rotations by $60^\circ$ and $120^\circ$ and axial symmetries with respect to the lines containing $v_0 v_3, v_1 v_4, v_2 v_5$) are the parallelograms with vertices $(1,0), (0, \frac{\sqrt{3}}{2}), (-1,0), (0, -\frac{\sqrt{3}}{2})$ and with vertices $(\frac{5}{6}, \frac{\sqrt{3}}{6}), (-\frac{5}{6}, \frac{\sqrt{3}}{6}), (\frac{\sqrt{3}}{6}, -\frac{\sqrt{3}}{6})$. Again see Fig. 2.

4. Banach–Mazur Distance Between the Parallelogram and the Affine-Regular Even-Gons

In this section we consider the BM-distances from $P_4$ to the regular even-gons with more than six vertices. In Theorem 2 we find these distances to $P_{8j}$ and $P_{8j+4}$, and we present the estimates from above from $P_4$ to $P_{8j+2}$ and $P_{8j+6}$.

Next we conjecture that the values of these two upper estimates are just the BM-distances from $P_4$ to $P_{8j+2}$ and $P_{8j+6}$.

Theorem 2. We have

(I) $\delta_{BM}(P_4, P_{8j}) = \sqrt{2}$,

(II) $\delta_{BM}(P_4, P_{8j+2}) \leq \frac{1}{2} \sec \frac{2j}{8j+2} \pi + \cos \frac{2j}{8j+2} \pi$,

(III) $\delta_{BM}(P_4, P_{8j+4}) = \sqrt{2} \cos \frac{1}{8j+4} \pi$,

(IV) $\delta_{BM}(P_4, P_{8j+6}) \leq \sin \frac{2j+2}{8j+6} \pi \cdot \csc \frac{4j+2}{8j+6} \pi + \cos \frac{2j+2}{8j+6} \pi$.

Proof. The inequalities showing that the left sides are at most the right sides result from positions of the inscribed parallelograms whose vertices are the intersections of the coordinate axes with the boundaries of $P_{8j}, P_{8j+2}, P_{8j+4}, P_{8j+6}$, respectively. Evaluating the ratio of the smallest homothetic copy of such an inscribed parallelogram which contains our polygon from amongst $P_{8j}, P_{8j+2}, P_{8j+4}, P_{8j+6}$, we find each value at the right sides of (I)–(IV).

Let us show the opposite inequalities for (I) and (III).

Ad (I). To prove that $\delta_{BM}(P_4, P_{8j}) \geq \sqrt{2}$, we have to show that for any parallelogram $P \in M^2$ contained in $P_{8j}$ such that $P_{8j} \subset \lambda P$, where $\lambda$ is positive, the inequality $\lambda \geq \sqrt{2}$ holds true. By Proposition, in order to show that $\delta_{BM}(P_4, P_{8j}) \geq \sqrt{2}$, it is sufficient to take into account only any $P = a(P_4)$ inscribed in $P_{8j}$ such that a positive homothetic copy $\lambda P$ is circumscribed about $P_{8j}$, and to show that $\lambda \geq \sqrt{2}$. Just we may disregard the other $P \subset P_{8j}$. This is realized in the following Parts (a) and (b).

(a) If a parallelogram $P \in M^2$ is inscribed in $P_{8j}$ and its homothetic image is circumscribed, then $P$ is a square.
Take into account a parallelogram $P \in \mathcal{M}^2$ inscribed in $P_{8j}$, which is not a square. Denote its successive vertices by $a, b, c, d$, when we go counterclockwise.

First let us explain that they must be in some $2j$-th sides of $P_{8j}$. In the opposite case, some of them, say $a$ and $b$ do not fulfill this. We may assume that $a \in v_0v_1$ and $b \notin v_2jv_{2j+1}$ (besides when $a = v_0$ or $b = v_{2j+1}$). Provide the straight line parallel to $ab$ which supports $P_{8j}$ at a vertex $v_k$ with $1 < k < 2j$. Provide the straight line parallel to $bc$ which supports $P_{8j}$ at a vertex $v_l$ with $2j < l < 4j$. Then the distances from $v_k$ to $ab$ and from $v_l$ to $bc$ are different. This means that no homothetic image of $abcd$ has a chance to be circumscribed about $P_{8j}$.

Hence consider the situation when successive vertices of $P$ are at every $2j$-th side of $P_{8j}$. Clearly, $a, c$ are symmetric with respect to $o$, and $o$ is in different distances from $a$ and $b$. Say, let $a \in v_0v_1$ and $b \in v_2jv_{2j+1}$. From the symmetry we see that $|v_0a| = |v_4j|$. Let $b^* \in v_3jv_{2j+1}$ fulfill $|v_oa| = |v_2j b^*|$ and let $d^*$ be opposite to $b^*$. Then $ab^*d^*$ is a square inscribed in $P_{8j}$. Clearly an enlarged homothetic image of it is circumscribed about $P_{8j}$. Thus the half-lines with origin at $o$ through $v_1$ and $v_{2j+1}$ intersect the sides $ab^*$ and $bc^*$, respectively, at points $w_1^+, w_{2j+1}^-$, respectively, in equal distances from $o$. These half-lines intersect the sides $ab$ and $cb$ at points $z_1^+, z_{2j+1}^-$, respectively. Since $b \neq b^*$, then $P$ is not a rhombus and thus $|oz_1^+| \neq |oz_{2j+1}^-|$. Since $|ov_1| = |ov_{2j+1}|$, we get $|z_1^+z_1^-| = |w_1^+w_1^-| = |z_{2j+1}^-z_{2j+1}^-| = |w_{2j+1}^+w_{2j+1}^-|$ where $w_i^+, z_i^-$ denote the points symmetric to $w_i^+, z_i^-$, respectively. From Lemma we conclude that the strip between the straight lines through $v_1$ and $v_{4j+1}$, parallel to $ab$ has a different width than the strip between the straight lines through $v_{2j+1}$ and $v_{6j+1}$, parallel to $bc$. Hence no homothetic image of $abcd$ is circumscribed about $P_{8j}$.

Consequently, $P$ must be a square.

(β) If a square $P$ is inscribed in $P_{8j}$ and $\lambda P$ is circumscribed about $P_{8j}$, then $\lambda \geq \sqrt{2}$, and $\lambda = \sqrt{2}$ if and only if the vertices of $P$ are at every $2j$-th vertex of $P_{8j}$ or at the middle of every $2j$-th side of $P_{8j}$.

Having in mind the axial symmetries of $P_{8j}$, we may limit our considerations to the squares $P$ inscribed in $P_{8j}$ whose one vertex denoted by $p$ is in $v_0m$, where $m$ is the midpoint of $v_0v_1$.

Denote by $k$ the directional factor $\frac{\sin(\pi/4j)}{\cos(\pi/4j) - 1}$ of the straight line containing the side $v_0v_1$. So the line has equation $y = k(x - 1) + 1$. Clearly, $p$ is in the intersection of the segment $v_0m$ with a ray $y = bx$, where $x \geq 0$ and $0 \leq b \leq \tan \pi/8j$. We omit an easy calculation showing that $p = (\frac{k}{k-b}, \frac{kb}{k-b})$. Since the diagonals of $P$ are orthogonal, the successive vertex $q$ of $P$ is the intersection of the rotated by 90° ray $y = -\frac{1}{b}x$, where $x \geq 0$, with the side $v_{2j}v_{2j+1}$. We easily establish that $q = (\frac{-kb}{k-b}, \frac{k}{k-b})$.

We provide the straight line containing $pq$. Its equation is $y = b\frac{b-1}{b+1}x + \frac{k(b^2+1)}{(k-b)(b+1)}$. Consider its intersection point $u$ with the line $y = x$. An easy calculation shows that both coordinates of $u$ are equal to $x = \frac{k}{2}b^2 + 1$. Consequently,
the ratio of the first coordinate of \( v_j \) to the first coordinate of \( u \) equals to 
\[
h(b) = \frac{\sqrt{2}}{2} : \left( \frac{k}{2} \cdot \frac{b^2 + 1}{k-b} \right) = \sqrt{2} \cdot \frac{k-b}{k(b^2+1)}.\]
By Proposition, the homothetic square \( h(b) P \) is circumscribed about \( P_{8j} \).

Clearly, we consider the function \( h(b) \) in the interval \([0, \tan \frac{\pi}{8j}]\). Observe that \( h(0) = \sqrt{2} = h(\tan \frac{\pi}{8j}) \). Our aim is to show that \( h(b) \geq \sqrt{2} \) for every \( b \) from this interval. We find the derivative \( h'(b) = \frac{\sqrt{2} \cdot b^2 - 2bk - 1}{k(b^2 + 1)^2} \). An easy calculation shows that \( h'(b) = 0 \) in the interval \([0, \tan \frac{\pi}{8j}]\) if and only if \( b = k + \sqrt{k^2 + 1} \). We have \( h(k + \sqrt{k^2 + 1}) = -\frac{\sqrt{2}}{k} \cdot \frac{\sqrt{k^2 + 1}}{(k + \sqrt{k^2 + 1})^2 + 1} \). Applying the fact that \( k < 0 \) we show that this value is over \( \sqrt{2} \). Hence \( h(b) \geq \sqrt{2} \) for every \( b \in [0, \tan \frac{\pi}{8j}] \) with equality only for \( b = 0 \) and \( b = \tan \frac{\pi}{8j} \).

Consequently, the thesis of Part (\( \beta \)) holds true.

We conclude that \( \delta_{BM}(P_4, P_{8j}) \geq \sqrt{2} \), which confirms the required opposite inequality of (I).

Ad (III). The vertices \( v_0, v_{2h+1}, v_{4h+2}, v_{6h+3} \) of \( P_{8j+4} \) are the vertices of \( P_4 \). Observe that every side of \( P_4 \) is parallel to every \( h \)-th side of \( P_{8j+4} \). So the side \( v_0v_{2h+1} \) of \( P_4 \) is parallel to the side \( v_jv_{j+1} \) of \( P_{8j+4} \), and so on. An evaluation shows that the ratio of the first coordinates of the points of the intersection of the line \( y = (\tan \frac{\pi}{8j})x \) with these two sides (the points are centers of these two sides) is \( \cos \frac{1}{8j+4} \pi \cdot \sec \frac{1}{4} \pi \). Analogous is true for every side of \( P_4 \) and the corresponding parallel side of \( P_{8j+4} \). Consequently, by Lemma the homothetic copy of \( P_4 \) with ratio \( \cos \frac{1}{8j+4} \pi \cdot \sec \frac{1}{4} \pi \) contains \( P_{8j+4} \). Consequently, \( \delta_{BM}(P_4, P_{8j+4}) \leq \cos \frac{1}{8j+4} \pi \cdot \sec \frac{1}{4} \pi \).

From Part (\( \beta \)) we see the only positions of \( P \) for which \( \delta_{BM}(P_4, P_{8j}) = \sqrt{2} \) is realized. Observe that the only positions of \( P \) for which \( \delta_{BM}(P_4, P_{8j+4}) \) is realized are these with the vertices at every \((2j + 1)\)-th vertex of \( P_{8j+4} \).

Generalizing (III) we observe that \( \delta_{BM}(P_n, P_{hn}) = \cos \frac{1}{hn} \pi \cdot \sec \frac{1}{n} \pi \) for every even \( n \geq 4 \) and every odd \( h \geq 3 \).

In connection with (II) and (IV) we conjecture that \( \delta_{BM}(P_4, P_{8j+2}) = \frac{1}{2} \sec \frac{2j}{8j+2} \pi + \cos \frac{2j}{8j+2} \pi \) and \( \delta_{BM}(P_4, P_{8j+6}) = \sin \frac{2j}{8j+6} \pi \cdot \csc \frac{2j}{8j+6} \pi + \cos \frac{2j}{8j+6} \pi \).
We also do not know the BM-distances between \( P_4 \) and the regular odd-gons. Besides some special cases, the task of finding the extended BM-distances (see “Appendix”) \( \delta_{BM}(P_m, P_n) \) seems to be very complicated.

5. Appendix

According to the recommendation of the referee we present the notion of the extended BM-distance and explain why it extends the original one. The extended Banach–Mazur distance of convex bodies \( C, D \subset E^d \) is the number
\[
\delta_{BM}(C, D) = \inf_{a, h_\lambda} \{ \lambda; a(C) \subset D \subset h_\lambda(a(C)) \},
\]
where $a$ is an affine transformation and $h_\lambda$ is a homothety with a positive ratio $\lambda$. The superscript $e$ is added here only for the purpose of the proof of the below Claim, which permits to omit this superscript. This notion is considered in many papers and books, e.g., in [2,4–6,9]. We present Claim since we cannot find this somehow obvious fact in the literature.

**Claim.** For every $C, D \in \mathcal{M}^d$ the original and the extended Banach–Mazur distance of $C$ and $D$ are equal.

**Proof.** We can assume that $D$ is centered at $o$ in the definition of $\delta_{BM}^e(C, D)$.

Since every $\lambda(C)$ from the definition of $\delta_{BM}(C, D)$ is a homothetical image of $C$ in the definition of $\delta_{BM}^e(C, D)$, we get $\delta_{BM}^e(C, D) \leq \delta_{BM}(C, D)$.

Show the reverse inequality. In the definition of $\delta_{BM}^e(C, D)$ the infimum is realized by an affine image $a'(C)$ of $C$ and by a homothetical image $I' = h_\lambda a'(C)$ of it. Denote by $a^0(C)$ the translate of $a'(C)$ and by $I^o$ the translate of $I'$, both with center $o$. By $D \in \mathcal{M}^d$ we get $-a'(C) \subset D$ and $D \subset -I'$. So $a^0(C) \subset \text{conv}(a'(C) \cup (-a'(C))) \subset D$ and $D \subset I' \cap (-I') \subset I^o$. Thus the infimum of $\delta_{BM}^e(C, D)$ is realized by $a^0(C)$ and $I^o$ in parts of $a(C)$ and $h_\lambda(a(C))$. Sets $a^0(C)$ and $I^o$, as centered at $o$, play the part of $a(C)$ and $\lambda a(C)$ in the definition of $\delta_{BM}(C, D)$. Hence the infimum of $\lambda$ in this definition is at most the infimum of $\lambda$ in the one of $\delta_{BM}^e(C, D)$. So $\delta_{BM}(C, D) \leq \delta_{BM}^e(C, D)$. \qed

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