ELLiptIC CURVES AND THEIR FROBENIUS FIELDS

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Abstract. Let $E_1$ and $E_2$ be two elliptic curves over a number field $K$. For a place $v$ of $K$ of good reduction for $E_1$ and for $E_2$, let $F(E_1, v)$ and $F(E_2, v)$ denote the splitting fields of the characteristic polynomials of the Frobenius automorphism at $v$ acting on the Tate modules of $E_1$ and $E_2$ respectively. $F(E_1, v)$ and $F(E_2, v)$ are called the Frobenius fields of $E_1$ and $E_2$ at $v$. Assume that at least one of the two elliptic curves is without complex multiplication. Then, we show that the set of places $v$ of $K$ of good reduction such that $F(E_1, v) = F(E_2, v)$ has positive upper density if and only if $E_1$ and $E_2$ are isogenous over some extension of $K$. We use this result to prove that, for an elliptic curve $E$ over a number field $K$, the set of finite places $v$ of $K$ such that $F(E, v)$ equals a fixed imaginary quadratic field $F$ has positive upper density iff $E$ has complex multiplication by $F$.

1. Introduction and statement of the main result

Let $K$ be a number field. Let $\Sigma_K$ denote the set of finite places of $K$. For a finite place $v \in \Sigma_K$, let $k_v := \mathcal{O}_K/p_v$, where $\mathcal{O}_K$ is the ring of integers of $K$, and $p_v$ is the prime ideal of $\mathcal{O}_K$ corresponding to $v$. Let $N_v$, the norm of $v$, denote the cardinality of the finite field $k_v$. Given an elliptic curve $E$ over $K$ and a finite place $v$ of $K$ of good reduction for $E$, let $E_v$, we denote the reduction of $E$ modulo $v$ defined over $k_v$, the residue field of $K$ at $v$.

Let $Frob_v$ be the Frobenius automorphism acting on the Tate module $T_\ell(E)$ for $\ell$ away from $v$. Let $\phi_v(t)$ be the characteristic polynomial of $Frob_v$. It has coefficients in $\mathbb{Z}$ and is independent of $\ell$. Thus, we have $\phi_v(t) := t^2 - a_v(E)t + N_v$. Then, we define the Frobenius field of $E$ at $v$ as the splitting field of $\phi_v(t)$ over $\mathbb{Q}$ and denote it by $F(E, v)$. Thus, $F(E, v) = \mathbb{Q}(\pi_v) = \mathbb{Q}(\sqrt{a_v^2 - 4N_v})$, where $\pi_v$ is a root of $\phi_v(t)$. If $E$ is an elliptic curve defined over a finite field $k$ of cardinality $q = p^n$, for some prime number $p$, then we define the Frobenius field of $E$ over $k$ as the splitting field of the characteristic polynomial of the Frobenius automorphism $x \mapsto x^q$ acting on the Tate module $T_\ell(E)$ for prime $\ell \neq p$ and by certain abuse of notation, we denote it by $F(E, k)$. Thus, for an elliptic curve $E$ defined over a number field $K$ and a place $v$ of $K$ of good reduction for $E$, $F(E, v) = F(E_v, k_v)$. Note that the Hasse bound $|a_v| \leq 2\sqrt{N_v}$ implies that $F(E, v)$ is either $\mathbb{Q}$ or an imaginary quadratic field.

Throughout the text, by CM elliptic curve or elliptic curve with CM, we mean an elliptic curve with complex multiplication.

Two elliptic curves $E_1$ and $E_2$ defined over a field $k$ are said to be potentially isogenous if $E_1$ and $E_2$ are isogenous over some extension of $k$.

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Given a set $S \subset \Sigma_K$ of finite places of $K$, recall that the upper density $ud(S)$ of $S$ is defined as the limit of the ratio given below:

$$ud(S) = \limsup_{x \to \infty} \frac{\#\{v \in \Sigma_K \mid Nv \leq x, v \in S\}}{\#\{v \in \Sigma_K \mid Nv \leq x\}}.$$ 

**Question:** Let $E$ be an elliptic curve over a number field $K$. Let $F$ be an imaginary quadratic field. Can $F$ be the Frobenius field for some finite place $v$ of $K$? If so, can $F$ occur infinitely often as a Frobenius field of $E$?

The asymptotic behaviour of the set of places $v$ for which the associated Frobenius field is a given imaginary quadratic field $F$ has been studied by various authors. Lang and Trotter in 1976 [1] conjectured that if $E$ is an elliptic curve defined over the field of rational numbers without complex multiplication, then, as $x \to \infty$,

$$S(x, E, F) := \#\{p \leq x \mid F(E, p) = F\} \sim C(E, F) \frac{x^{1/2}}{\log x}$$

for some positive constant $C(E, F)$ depending on $E$ and $F$.

It is natural to ask how often the Frobenius fields of two elliptic curves coincide. Our main result is the following multiplicity-one type theorem under the assumption that the set of primes $v$ for which the Frobenius fields coincide has positive upper density.

**Theorem 1.1.** Let $E_1$ and $E_2$ be two elliptic curves over a number field $K$. Assume that one of them is without complex multiplication. Let

$$S(E_1, E_2) := \{v \in \Sigma_K \mid F(E_1, v) = F(E_2, v)\}.$$

Then, $E_1$ and $E_2$ are isogenous over a finite extension of $K$ if and only if $S(E_1, E_2)$ has positive upper density.

Using Theorem 1.1, we prove:

**Theorem 1.2.** Let $E$ be an elliptic curve over a number field $K$. Let $F$ be an imaginary quadratic field. Let $S(E, F) := \{v \in \Sigma_K \mid F(E, v) = F\}$. Then, $S(E, F)$ has positive upper density if and only if $E$ has complex multiplication by $F$.

As a consequence, we prove:

**Corollary 1.1.** Let $E$ be an elliptic curve over a number field $K$. Let $F(E)$ be the compositum of the Frobenius fields $F(E, v)$ as $v$ varies over places of good ordinary reduction for $E$. Then, $F(E)$ is a number field if and only if $E$ is an elliptic curve with complex multiplication.

The above corollary follows from a set of exercises in Serre’s book [4] Chapter IV, pages 13-14. Thus, Theorem 1.2 can be considered as a strengthening of this corollary.

**Remark 1.1.** The proof of Theorem 1.2 is indeed via Theorem 1.1 using a CM elliptic curve, however, since elliptic curves with CM by an imaginary quadratic field $F$ are defined only over $\mathbb{Q}(j(O_F))$ and thus over the Hilbert class fields $H(F)$, and since the assertion of Theorem 1.2 is not invariant under field extensions, in the proof of Theorem 1.2 we need to invoke a crucial property of Hilbert class fields, viz. that the principal ideals of $F$ split completely in $H(F)$.
2. Results we use

We now state a few well-known results and facts.

Proposition 2.1. Let $E$ be an elliptic curve over a number field $K$ without complex multiplication. Then, the set of places $v \in \Sigma_K$ such that $E$ has supersingular reduction at $v$ has upper density 0.

This is well-known and follows from the set of exercises in Serre’s book [4], Chapter IV, pages 13-14.

We also need the following result on the classification of the isogeny classes of elliptic curves defined over a finite field.

Proposition 2.2. (Waterhouse, Theorem (4.1) [5]) Let $k$ be a finite field with $q = p^n$ elements. The isogeny classes of elliptic curves over $k$ are in one-to-one correspondence with the rational integers $a$ such that $|a| \leq 2\sqrt{q}$ and satisfying one of the following conditions:

1. $\gcd(a, p) = 1$: $E$ ordinary and $F(E, k)$ is imaginary quadratic field in which $p$ splits completely.
2. $n$ even: $a = \pm 2\sqrt{q}$, $E$ supersingular, $F(E, k) = \mathbb{Q}$.
3. $n$ even and $p \neq 1 \mod 3$: $a = \pm \sqrt{q}$, $E$ supersingular, $F(E, k) = \mathbb{Q}(\sqrt{-3})$.
4. $n$ odd and $p = 2$: $a = \pm p^{1/2}$, $E$ supersingular, $F(E, k) = \mathbb{Q}(\sqrt{-1})$.
5. $n$ odd and $p = 3$: $a = \pm p^{1/2}$, $E$ supersingular, $F(E, k) = \mathbb{Q}(\sqrt{-3})$.
6. $n$ odd: $a = 0$, $E$ supersingular, $F(E, k) = \mathbb{Q}(\sqrt{-p})$.
7. $n$ even and $p \neq 1 \mod 4$: $a = 0$, $E$ supersingular, $F(E, k) = \mathbb{Q}(\sqrt{-1})$.

The proposition below is a slightly general version of Corollary (2.2) of Theorem (2.1) from Patankar-Rajan [2] or Corollary 2 of Rajan [3].

Proposition 2.3. Let $E_1$ and $E_2$ be two elliptic curves over a number field $K$. Assume that $E_1$ does not have complex multiplication. Suppose that $T$ is a set of places of $K$ of positive upper density consisting of places of good reduction for $E_1$ and $E_2$ and such that for every $v \in T$, the reductions $E_{1v}$ and $E_{2v}$, modulo $v$ of the elliptic curves $E_1$ and $E_2$ respectively, are isogenous over a finite extension of the residue field $k_v$ of $K$ at $v$. Then $E_1$ and $E_2$ are isogenous over a finite extension $L$ of $K$.

Proof (Sketch) The statement and its proof is a slight modification of corollary (2.2) [2]. This is proved by applying Theorem (2.1) of [2] to the $\ell$-adic galois representations attached to $E_1$ and $E_2$. To ensure that Theorem (2.1) [2] is applicable, one needs to use the well-known result of Serre [5] that for large enough prime $\ell$, the image of the galois group $\text{Gal}(\overline{K}/K)$ is open in $GL_2(\mathbb{Z}_\ell) \simeq \text{Aut}(T_\ell(E_1))$. □

3. Proofs

3.1. Proof of Theorem (1.1). ($\Rightarrow$) Let $L$ be a finite extension of $K$ such that $E_1$ and $E_2$ are isogenous over $L$. Since one of them is without complex multiplication, both the elliptic curves are without complex multiplication. This implies that the set of finite places $w$ of $L$ such that $E_{1w}$ or $E_{2w}$ are supersingular at $w$ has upper density 0. Let $S_L$ be the set of places $w$ of $L$ such that both $E_1$ and $E_2$ have good ordinary reduction at $w$. $S_L$ has upper density 1. By Tate’s well-known isogeny
theorem, it follows that for \( w \in S \), the characteristic polynomials of the Frobenius automorphism at \( w \) are equal. Thus, for \( w \in S \), \( F(E_1, w) = F(E_2, w) \). Note that the density of \( S_{E_2} \) is equal to the upper density of the set of degree 1 places \( w \in S_1 \).

Let \( v \) be a place of \( K \) that lies below \( w \). Since \( w \) is of degree 1, it follows that the Frobenius endomorphism at \( w \) equals the Frobenius endomorphism at \( v \). Thus, for a place \( v \) of \( F \) that lies below \( w \) of degree 1, \( F(E_1, v) = F(E_2, v) \). This implies that \( S(E_1, E_2) \) has upper density \( \frac{1}{[L:K]} \).

\((\Leftarrow )\) Suppose that the upper density of \( S := S(E_1, E_2) \) is positive. Since \( E_1 \) is without complex multiplication, the set of places \( v \in \Sigma_K \) such that \( E_{1v} \) is ordinary at \( v \) has upper density 1. Let \( S_1 := \{ v \in S \mid E_{1v} \) is ordinary \( \} \). Thus, \( ud(S_1) = ud(S) \). Recall that if \( v \) is finite place of good reduction for an elliptic curve \( E \) defined over a number field \( K \), then \( E \) has ordinary reduction at \( v \) iff \( F(E, v) \) is an imaginary quadratic field and \( p_v \) splits in \( F(E, v) \), where \( p_v \) is the prime of \( \mathbb{Q} \) that lies below \( v \). This implies that \( p_v \) splits in \( F(E_1, v) = F(E_2, v) \). As a consequence, every \( v \in S_1 \) is a place of good ordinary reduction for both \( E_1 \) and \( E_2 \). Let us consider the subset \( T \) of \( S_1 \) consisting of places \( v \in S_1 \) with \( \deg v = 1 \). Thus, \( ud(T) = ud(S_1) > 0 \). For \( v \in T \), let \( \pi_1, \pi_1 \) and \( \pi_2, \pi_2 \) be respectively the roots of the characteristic polynomials \( \phi_v(E_1, t) \) and \( \phi_v(E_2, t) \). Thus,

\[
\pi_1 \pi_1 = \pi_2 \pi_2 = p_v
\]

As ideals of \( F(v) := F(E_1, v) = F(E_2, v) \), we have:

\[
(\pi_1)(\pi_1) = (\pi_2)(\pi_2) = (p_v).
\]

By unique factorization theorem for ideals, it follows that \( \pi_1 = u \pi_2 \) or \( \pi_1 = u \pi_2 \), where \( u \) is unit of \( F(v) \). Since, \( F(v) \) is an imaginary quadratic field, \( u = \zeta_n \) is, in fact, a root of unity of order \( n = 2, 3 \) or 4. It thus follows that, for \( v \in T \)

\[
\#E_{1v}(\mathbb{F}_{p_v}) = p_v^n + 1 - (\pi_1^n + \pi_1^n) = p_v^n + 1 - (\pi_2^n + \pi_2^n) = \#E_{2v}(\mathbb{F}_{p_v}).
\]

This implies that \( E_{1v} \) and \( E_{2v} \) are potentially isogenous for every \( v \in T \). Applying Proposition (2.3) as above, it follows that \( E_1 \) and \( E_2 \) are isogenous over some finite extension of \( K \), proving the theorem. Note that the Proposition (2.3) is applicable as one of the elliptic curves is without complex multiplication.

\( \square \)

**Remark 3.1.** In the above theorem, it is necessary to assume that at least one of the two elliptic curves is without complex multiplication. Let \( E_1 \) and \( E_2 \) be the elliptic curves over \( K = \mathbb{Q} \) defined by \( y^2 = x^3 - x \) and \( y^2 = x^3 - 1 \) respectively. \( E_1 \) and \( E_2 \) have complex multiplication by \( \mathbb{Q}(\sqrt{-1}) \) and \( \mathbb{Q}(\sqrt{-3}) \) respectively. Note that for every prime \( p \equiv 11 \) mod 12, \( F(E_1, p) = F(E_2, p) = \mathbb{Q}(\sqrt{-p}) \). This set of primes \( p \equiv 11 \) mod 12 has upper density \( \frac{1}{4} \). However, \( E_1 \) and \( E_2 \) are non-isogenous.

Using Proposition (2.3), it is not difficult to show that any counterexample to the weakened version of Theorem (1.1) is of the above type. We provide another example to illustrate this.

Let \( F_1 = \mathbb{Q}(\sqrt{-5}) \) and \( F_2 = \mathbb{Q}(\sqrt{-1}) \). Note that the class numbers of \( F_1 \) and \( F_2 \) are 2 and 1, respectively. Furthermore, it can be shown that \( j(\sqrt{-5}) \in \mathbb{Q}(\sqrt{5}) \setminus \mathbb{Q} \), where \( j \) stands for the \( j \)-function. Let \( K := \mathbb{Q}(j(\sqrt{-5})) = \mathbb{Q}(\sqrt{5}) \). Thus, there exists an elliptic \( E_1 \) defined over \( K \) with complex multiplication by \( F_1 \). For example, we can take \( E_1 \) defined by \( y^2 = x^3 + (-3\sqrt{5}(-5-1728))x + (-2\sqrt{5}(-5-1728)^2) \). Let \( E_2 \) be the elliptic curve given by \( y^2 = x^3 - x \) over \( K \). \( E_2 \) has complex multiplication by \( F_2 \). Let \( S \) be the set of primes \( p \) of \( \mathbb{Q} \) that are inert in both \( F_1 \) and \( F_2 \) and
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which split completely in \( K \). Thus, \( S := \{ p \mid p \equiv 1 \text{ mod } 20 \text{ or } 9 \text{ mod } 20 \} \). Let \( S_K \) be the set of primes \( v \) of \( K \) that lie above \( p \in S \). By the very construction, \( F(E_1, v) = \mathbb{Q}(\sqrt{-p}) = F(E_2, v) \) for all primes \( v \in S_K \). Note that \( S_K \) has positive density. However, \( E_1 \) is not isogenous to \( E_2 \) over \( \mathbb{Q} \).

It is easy to see that one can produce many such examples.

**Remark 3.2.** We have stated and proved our theorem under the assumption that one of the two elliptic curves is without complex multiplication. From the above remark, it is easy to see that we can modify and prove the theorem under the assumption that the upper density of the set of finite places \( v \) of \( K \) for which both the elliptic curves have good ordinary reduction at \( v \) is positive.

3.2. **Proof of Theorem (1.2).** Given a place \( v \) of \( K \), we will denote by \( p_v \) the rational prime of \( \mathbb{Q} \) that lies below \( v \).

\((\Leftarrow)\) We want to prove that if \( E \) has complex multiplication by an imaginary quadratic field \( F \), then the set \( S(E, F) := \{ v \in \Sigma_K \mid F(E, v) = F \} \) has positive upper density.

It is well-known that if \( v \) is a place of \( K \) of good reduction for \( E \) with CM by \( F \), then \( E \) has ordinary reduction modulo \( v \) if and only if \( F(E, v) = F \) if and only if \( p_v \) splits in \( F \). Let \( K \) be the Galois closure of \( K \) over \( \mathbb{Q} \). Let \( L \) be the compositum of \( K \) and \( F \). \( L \) is Galois over \( \mathbb{Q} \). Let \( \text{Spl}(L/\mathbb{Q}) \) be the set of all primes \( p \) that split completely in \( L \). Let \( S := \{ v \in \Sigma_K \mid v \text{ lies over } p \in \text{Spl}(L/\mathbb{Q}) \} \).

Thus, for a finite place \( v \in S \), \( \deg v = 1 \). Since every prime \( p \in \text{Spl}(L/\mathbb{Q}) \) also splits in \( F \), it follows that \( F(E, v) = F \) for \( v \in S \). By the very construction, \( S \subseteq S(E, F) \). Since every place \( v \in S \) is of degree 1 and lies over the primes of \( \text{Spl}(L/\mathbb{Q}) \),

\[
ud(S(E, F)) \geq ud(S) \geq ud(\text{Spl}(L/\mathbb{Q})) = \frac{1}{[L: \mathbb{Q}]} > 0.
\]

\((\Rightarrow)\) We want to prove that if for some imaginary quadratic field \( F \), \( ud(S(E, F)) > 0 \) then \( E \) has complex multiplication by \( F \).

**Case 1:** Suppose \( E \) has complex multiplication by an imaginary quadratic field \( F' = \mathbb{Q}(\sqrt{-d}) \). We want to prove that \( F' = F \). Let \( S = S(E, F) \).

Claim 1: There exists \( v \in S \) such that \( E \) has good reduction modulo \( v \) and \( p_v \) is not ramified in \( F' \). This is true because \( S \) is an infinite set.

Claim 2: There exists \( v \in S \) such that \( p_v \) is not inert in \( F' \).

Proof of Claim 2: Assume otherwise. Then, for every \( v \in S \), \( E \) has supersingular reduction modulo \( v \). We are using the well-known criterion that \( E \) defined over a number field \( K \) with complex multiplication by \( F' \) has supersingular reduction modulo \( v \) if and only if \( p_v \) is inert in \( F' \). Using the classification of the isogeny classes of elliptic curves as stated in Proposition (2.2), it follows that for \( v \in S \), \( F(E, v) \) is either \( \mathbb{Q}, \mathbb{Q}(\sqrt{-1}), \mathbb{Q}(\sqrt{-3}) \) or \( \mathbb{Q}(\sqrt{-p}) \).

Let \( T := \{ v \in S \mid F(E, v) = \mathbb{Q}(\sqrt{-p}) \} \). We claim that \( ud(T) = ud(S) \). This follows from the Proposition (2.2) of section 2. The Proposition (2.2) says that \( F(E, v) = \mathbb{Q}(\sqrt{-1}) \) if \( p_v = 2 \) or \( p_v \not\equiv 1 \text{ mod } 4 \) and \( \deg(v) \) is even. The set of such places \( v \) has upper density 0, since \( \deg(v) > 1 \). Similarly, the Proposition (2.2) says
that $F(E, v) = \mathbb{Q}(\sqrt{-3})$ iff $p_v = 3$ or $p_v \not\equiv 1 \pmod{3}$ and $\deg(v)$ is even. By the same reasoning, the set of such places $v$ also has upper density 0. This proves that $ud(S) = ud(T) > 0$. Thus, it follows that $T \subseteq S$ is an infinite set and that for all $v \in T$, $F(E, v) = \mathbb{Q}(\sqrt{-p_v}) = F$. This is a contradiction. That proves Claim 2.

It thus follows that the set of places $v \in S$ such that $p_v$ splits in $F'$ has positive upper density. Let $v \in S$ such that $p_v$ splits in $F'$. This implies that the Frobenius field at $v$ equals the CM field, i.e. $F(E, v) = F'$. On the other hand, since $v \in S = S(E, F)$, we have $F(E, v) = F$. This proves $F = F'$.

**Case 2:** Let us now consider the case when $E$ is an elliptic curve over $K$ without complex multiplication. The idea is to construct an elliptic curve, say $E'$, over a suitable number field with complex multiplication by $F$ and to apply Theorem (1.1) to prove that $E$ and $E'$ are isogenous over some extension of $K$.

Let $O_F$ be the ring of integers of $F$. Let $E'$ be the elliptic curve over $\mathbb{C}$ such that $E'(\mathbb{C}) \cong \mathbb{C}/O_F$. The theory of complex multiplication implies that $E'$ is defined over $H := H(F)$, the Hilbert class field of $F$. Let $L := HK$ be the compositum of $H$ and $K$. We wish to apply Theorem (1.1) to the two elliptic curves $E$ and $E'$ considered as elliptic curves defined over $L$. Thus, we need to prove that the set of places $w$ of $L$ such that $F(E, w) = F(E', w)$ has positive upper density.

Let us denote by $S_K$ the set of degree 1 places $v \in S(E, F) \subseteq \Sigma_K$. Then, $ud(S_K) = ud(S(E, F))$. Let $S_Q$ be the set of primes $p_v$ of $\Sigma_Q$ that lie below the places of $v \in S_K$. $ud(S_Q)$ is also positive. Let $p \in S_Q$ and let $v$ be a place of $K$ that lies above $p = p_v$. By construction, the Frobenius field at $v$ equals $\mathbb{Q}(\pi_v) = F$. Since, $\pi_v\overline{\pi_v} = Nv = p = p_v$, the primes $p \in S_Q$ split in $F$. Let $S_F$ be the set of places of $F$ that lie over the set of places of $S_Q$. $S_F := \bigcup_{v \in S_K} \{ (\pi_v), (\overline{\pi_v}) \}$. Thus, $ud(S_F)$ is positive. The prime ideals of $S_F$ are principal. By class field theory, they split completely in the Hilbert class field $H$ of $F$. This implies that the primes $p \in S_Q$ split completely in $H$. Let $S_L$ be the set of primes of $L$ that lie above $S_K$. Let $w \in S_L$ be a place above $v \in S_K$. Since $p_v$ splits completely in $H$, it is easy to see that the prime $v$ of $K$ splits completely in $L$. This implies that $\deg(w) = \deg(v) = 1$, implying $ud(S_L) > 0$. By considering $E$ as an elliptic curve over $L$, it follows that $F(E, w) = F(E, v) = F$ where $w \in S_L$ and $v \in S_K$ that lies below $w$. Similarly, we have $F(E', w) = F$. By applying Theorem (1.1) to $E$ and $E'$ considered as elliptic curves over $L$, it follows that $E$ and $E'$ are isogenous over some finite extension of $L$, proving the theorem.

4. Generalization to Higher Dimensions, Galois Representations, Drinfeld Modules

We are currently exploring the possibility of extending our results in the context of $\ell$-adic Galois representations and thus to modular forms and Abelian varieties. We are also looking into proving analogous results for Drinfeld modules.

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