SU(1,2) invariance in two-dimensional oscillator

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Performing the Hamiltonian analysis we explicitly established the canonical equivalence of the deformed oscillator, constructed in arXiv:1607.03756, with the ordinary one. As an immediate consequence, we proved that the SU(1,2) symmetry is the dynamical symmetry of the ordinary two-dimensional oscillator. The characteristic feature of this SU(1,2) symmetry is a non-polynomial structure of its generators written it terms of the oscillator variables.

I. INTRODUCTION

It is a well-known fact that the invariance with respect to the \( \ell > 1/2 \)-conformal Galilei algebra demands the appearance of high-derivative terms in the Lagrangians of the corresponding mechanical systems. The important fact is that standard methods of nonlinear realizations work quite nicely for these algebras being equipped with the Inverse Higgs phenomenon constraints result in the corresponding Pais-Uhlenbeck oscillators. The exceptional case with \( \ell = 1/2 \) corresponds to the Shrödinger algebra, and the mechanical system possessing this symmetry is just a standard \( d \)-dimensional oscillator. It was demonstrated in the recent paper that the \( su(1,2) \) algebra admits a reduction to the two-dimensional Shrödinger algebra and, therefore, the system possessing the \( SU(1,2) \) symmetry reduces to the ordinary two-dimensional oscillator. Such deformed oscillator has been constructed within the Lagrangian formalism.

As for a possible relation of the deformed oscillator with the ordinary one, one should note that it seems to be impossible to relate these systems within the Lagrangian approach. Contrary, within the Hamiltonian approach the freedom to relate these systems is much wider, because the admitted change of variables includes arbitrary (but invertible) functions defined on the phase-space. That is why we provide a Hamiltonian description of the \( su(1,2) \) oscillator in the present paper. It turns out that the standard procedure to pass to the Hamiltonian formalism is not much suitable for the present case resulting in a rather complicated Hamiltonian. The basic explanation of this is that the canonically defined momenta have rather complicated transformation properties with respect to the \( SU(1,2) \) group. On the other hand, within the nonlinear realization approach applied to this system, we have proper variables \( v, \bar{v} \) in the coset space which can be used as proper momenta with transparent transformation properties. Interestingly enough, one of the Cartan forms, used as the Lagrangian in, is capable of providing the symplectic structure as well as the Hamiltonian in terms of the initial variables \( u, \bar{u}, v, \bar{v} \). The complicated structure of the Poisson brackets in this basis is compensated by the simple form of the Hamiltonian and the generators of \( su(1,2) \) algebra.

Having at hand all ingredients in the initial variables, we succeeded in finding the canonical variables in which the Hamiltonian of deformed \( su(1,2) \)-invariant oscillator coincides with the Hamiltonian of the ordinary two-dimensional oscillator. Thus, we proved that these two systems are canonically equivalent. However, in these canonical variables the generators of \( su(1,2) \) algebra have a non-polynomial structure, so it is problematic to state about their quantum equivalence.

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II. DEFORMED OSCILLATOR IN THE LAGRANGIAN APPROACH

In [12], the Lagrangian of the deformed oscillator

\[ \mathcal{L} = \frac{\dot{u} \ddot{u} - \omega^2 u \dddot{u}}{1 + \frac{\omega^2}{4} (u \dddot{u} - \dddot{u}) + \frac{\gamma}{2} \omega^2 u^2 \ddot{u}^2} \]  

(2.1)

was constructed within nonlinear realization of the \( SU(1,2) \) group. The structure relations of the corresponding algebra \( su(1,2) \) were chosen as

\[ i [L_n, L_m] = (n - m) L_{n+m}, \quad i [L_n, G_r] = \left( \frac{n}{2} - r \right) G_{n+r}, \quad i [L_n, \overline{G}_r] = \left( \frac{n}{2} - r \right) \overline{G}_{n+r}, \]

\[ [U, G_r] = G_r, \quad [U, \overline{G}_r] = -\overline{G}_r, \]

\[ i [G_r, \overline{G}_s] = \gamma \left( \frac{3}{2} (r - s) U - i L_{r+s} \right), \quad n, m = -1, 0, 1, \ r, s = -1/2, 1/2. \]  

(2.2)

In this form, in the limit \( \gamma = 0 \) these relations coincide with the relations of the \( \ell = 1/2 \) conformal Galilei algebra in three dimensions \([1][2]\), and so they can be viewed as the deformation of the conformal Galilei algebra with the parameter of deformation \( \gamma \). The exact value of \( \gamma \) is inessential: if nonzero, it can be put to unity by a re-scaling of the generators \( G_r \) and \( \overline{G}_r \).

The group \( SU(1,2) \) itself was realized by the left multiplication of the coset \( g = SU(1,2)/H \) with the stability subgroup \( H \propto (U, L_0, L_1) \) parameterized as

\[ g = e^{i \Omega (L_{-1} + \omega L_1)} e^{i (u \overline{G}_{-1/2} + \bar{u} \overline{G}_{-1/2})} e^{i (v G_{1/2} + \bar{v} \overline{G}_{1/2})}. \]  

(2.3)

Using the Cartan forms defined in a standard way as

\[ g^{-1} d g = i \sum_{n=-1}^{1} \Omega_n L_n + i \sum_{\alpha=1/2}^{1/2} (\omega_\alpha G_\alpha + \bar{\omega}_\alpha \overline{G}_\alpha) + i \Omega_U U, \]  

(2.4)

one may eliminate the unessential Goldstone fields \( v, \bar{v} \) via the fields \( u, \bar{u} \) by imposing the constraints \([15]\)

\[ \omega_{-1/2} = \bar{\omega}_{-1/2} = 0 \quad \Rightarrow \quad v = \frac{\dot{u} + i \frac{\omega^2}{4} u^2 \dddot{u}}{1 + i \frac{\omega^2}{2} (u \dddot{u} - \dddot{u}) + \frac{\gamma}{2} \omega^2 u^2 \ddot{u}^2}, \quad \bar{v} = \frac{\dot{\bar{u}} - i \frac{\omega^2}{4} u^2 \dddot{\bar{u}}}{1 + i \frac{\omega^2}{2} (\bar{u} \dddot{\bar{u}} - \dddot{\bar{u}}) + \frac{\gamma}{2} \omega^2 u^2 \ddot{\bar{u}}^2}. \]  

(2.5)

The action for the deformed oscillator is provided by the Cartan form \( \Omega_U \) \((2.4)\), which explicitly reads \([12]\)

\[ \Omega_U = \frac{3}{4} \gamma \left[ v \bar{v} \left( (1 + \frac{\omega^2}{4} \gamma \omega^2 u^2 \ddot{u}^2) dt + \frac{\gamma}{2} (u d\bar{u} - \bar{u} du) \right) - v (d\bar{u} - \frac{i}{2} \gamma \omega^2 u \ddot{u} \bar{u} dt) - \bar{v} (du + \frac{i}{2} \gamma \omega^2 u \ddot{u} \bar{u} dt) + \omega^2 u \bar{u} dt \right]. \]  

(2.6)

Finally, upon the substitution of \((2.6)\) into \((2.4)\) one may get

\[ S = -\frac{2}{3 \gamma} \int \Omega_U = \int \mathcal{L} dt \]  

(2.7)

where the Lagrangian is given by \((2.1)\). One has to stress again, that the action \((2.7)\) is invariant with respect to \( SU(1,2) \) symmetry.

Finally, note that the action

\[ S = -\frac{2}{3 \gamma} \int \Omega_U \]  

(2.8)

with the form \( \Omega_U \) given by expression \((2.6)\) is sufficient to describe the deformed oscillator without any references to the inverse Higgs constraints \((2.3)\). Indeed, varying the action \((2.8)\) over the variables \( v, \bar{v} \) we immediately reproduce the constraints \((2.5)\). Thus, the action \((2.8)\) contains all needed information to describe the deformed oscillator.
III. HAMILTONIAN FORMULATION

To provide the Hamiltonian description of the deformed oscillator with the Lagrangian (2.1), one may perform Legendre transformation and get the system with canonical Poisson brackets with the momenta $\pi, \bar{\pi}$ canonically conjugated with $u, \bar{u}$ variables. However, the Hamiltonian system written in these terms not very convenient for further analyzes.

Interestingly enough, the non-linear realization approach allows to give Hamiltonian formulation of the system in suitable phase space coordinates, without referring to Legendre transformation. The key observation is that the form $\Omega_U$ (2.6) provide us with first-order Lagrangian which is variationally equivalent to (2.1):

$$\tilde{L} dt = -\frac{2}{\gamma} \Omega_U = \alpha - \mathcal{H} dt,$$

where

$$\alpha = v d\bar{u} + \bar{v} du + \frac{i\gamma}{2} v \bar{v} (\bar{u} d\bar{u} - u d\bar{u}),$$

is the symplectic one-form and

$$\mathcal{H} = v \bar{v} + \omega^2 u \bar{u} \left( 1 + \frac{i\gamma}{2} \bar{u} \bar{v} \right) \left( 1 - \frac{i\gamma}{2} u \bar{v} \right).$$

is the Hamiltonian. The external differential of the symplectic one-form yields the symplectic structure

$$\Omega = d\alpha = \left( 1 - \frac{i\gamma}{2} u \bar{v} \right) dv \wedge d\bar{u} + \left( 1 + \frac{i\gamma}{2} \bar{u} \bar{v} \right) d\bar{v} \wedge du + \frac{i\gamma}{2} (\bar{u} \bar{v} dv \wedge du - u v d\bar{v} \wedge d\bar{u}) + i\gamma v \bar{v} d\bar{u} \wedge du. \tag{3.3}$$

The respective Poisson brackets are defined by the following non-zero relations variables $u, \bar{u}, v, \bar{v}$

$$\{v, \bar{u}\} = \frac{1 + \frac{i\gamma}{2} v \bar{u}}{1 - \frac{i\gamma}{2} (u \bar{v} - \bar{u} v)}, \quad \{v, \bar{v}\} = -i\gamma \frac{v \bar{v}}{1 - \frac{i\gamma}{2} (u \bar{v} - \bar{u} v)}, \quad \{v, u\} = \frac{\frac{i\gamma}{2} v u}{1 - \frac{i\gamma}{2} (u \bar{v} - \bar{u} v)}, \tag{3.4}$$

and their complex conjugated ones. Let us notice that from the (3.1) one can immediately get the expression the canonical momenta $\pi, \bar{\pi}$ which arise within Legendre transformation of the Lagrangian (2.1)

$$\pi = v - \frac{i\gamma}{2} v \bar{v} u, \quad \bar{\pi} = \bar{v} + \frac{i\gamma}{2} v \bar{v} \bar{u} : \quad \{\pi, \bar{u}\} = \{\bar{\pi}, u\} = 1. \tag{3.5}$$

To complete this Section, let us write down, in terms of $u, \bar{u}, v, \bar{v}$, the Hamiltonian realization of the $su(1,2)$ generators:

$$L_{-1} = v \bar{v}, \quad L_0 = -\frac{1}{2} (u \bar{v} + \bar{u} v), \quad L_1 = u \bar{u} \left( 1 + \frac{i\gamma}{2} \bar{u} \bar{v} \right) \left( 1 - \frac{i\gamma}{2} u \bar{v} \right), \tag{3.6}$$

$$U = i (\bar{u} \bar{v} - u \bar{v}) + \gamma u \bar{u} v \bar{v}, \quad G_{-1/2} = -\bar{v} (1 + i\gamma \bar{u} v), \quad \overline{G}_{-1/2} = -v (1 - i\gamma u \bar{v}), \tag{3.7}$$

$$G_{1/2} = \bar{u} (1 + i\gamma \bar{u} v) \left( 1 - \frac{i\gamma}{2} u \bar{v} \right), \quad \overline{G}_{1/2} = u (1 - i\gamma u \bar{v}) \left( 1 + \frac{i\gamma}{2} \bar{u} \bar{v} \right). \tag{3.8}$$

These generators form the $su(1,2)$ algebra with respect to the Poisson brackets (3.4)

$$\{L_n, L_m\} = (n - m) L_{n+m}, \quad \{L_n, G_r\} = \left( \frac{n}{2} - r \right) G_{n+r}, \quad \{L_n, \overline{G}_r\} = \left( \frac{n}{2} - r \right) \overline{G}_{n+r}, \tag{3.9}$$

$$\{U, G_r\} = i G_r, \quad \{U, \overline{G}_r\} = -i \overline{G}_r, \quad \{G_r, \overline{G}_s\} = -i \gamma L_{r+s} + \frac{3}{2} \gamma (r - s) \left( U + \frac{2}{3\gamma} \right),$$

where $n, m = -1, 0, 1$, $r, s = -1/2, 1/2$.

It should be noted the appearance of the constant central charge in the Poisson brackets $\{G_r, \overline{G}_s\}$. If $\gamma \neq 0$, it can be absorbed by redefinition of the generator $U \rightarrow \tilde{U} = U + \frac{2}{3\gamma}$. But if $\gamma = 0$, this central charge survives and we have at hand the central charge extension of the $\ell = 1/2$ conformal Galilei algebra.
Time-dependent extensions of the generators, defining the isometries of the Lagrangian of the deformed oscillator, are given by the expressions

\[
\begin{align*}
L_{-1}^t &= \cos^2(\omega t) (L_{-1} + \omega^2 L_1) - \omega \sin(2\omega t) L_0 - \omega^2 \cos(2\omega t) L_1, \\
L_0^t &= \cos(2\omega t) L_0 + \frac{\sin(2\omega t)}{2\omega} (L_{-1} - \omega^2 L_1), \\
L_1^t &= \frac{\sin^2(\omega t)}{\omega^2} (L_{-1} + \omega^2 L_1) + \cos(2\omega t) L_1 + \frac{\sin(2\omega t)}{\omega} L_0, \\
G_{-1/2}^t &= \cos(\omega t) G_{-1/2} - \omega \sin(\omega t) G_{1/2}, \\
G_{1/2}^t &= \cos(\omega t) G_{1/2} + \frac{\sin(\omega t)}{\omega} G_{-1/2}.
\end{align*}
\]

IV. CANONICAL VARIABLES

Within the Hamiltonian description of the given system, we have a much more possibilities to redefine the variables than in the Lagrangian approach. In this Section we will demonstrate that the deformed oscillator with the Hamiltonian (3.2) and the symplectic structure (3.3) is canonically equivalent to the ordinary oscillator. To simplify the presentation, we start with the deformed free particle (i.e. with \( \omega = 0 \)) and then will consider the deformed oscillator in a full generality.

A. Free particle

In the free particle case, i.e. when \( \omega = 0 \), the Hamiltonian is given by the generator \( L_{-1} \)

\[
H_0 = v \ddot{v}.
\]

It has three constants of motion given by the generators \( G_{-1/2}, \overline{G}_{-1/2} \) and \( U \)

\[
\{G_{-1/2}, H_0\} = \{\overline{G}_{-1/2}, H_0\} = \{U, H_0\} = 0
\]

It immediately follows from relations (2.2) that

\[
\{G_{-1/2}, \overline{G}_{-1/2}\} = -i \gamma L_{-1} = -i \gamma H_0, \quad \{U, G_{-1/2}\} = i G_{-1/2}, \quad \{U, \overline{G}_{-1/2}\} = -i \overline{G}_{-1/2}.
\]

The Hamiltonian can be written down in terms of these constants of motion

\[
H_0 = \frac{G_{-1/2} \overline{G}_{-1/2}}{1 + \gamma U}.
\]

It is slightly unexpected that the evident definitions of the new variables \( p, \bar{p} \)

\[
p = - \frac{G_{-1/2}}{\sqrt{1 + \gamma U}}, \quad \bar{p} = - \frac{\overline{G}_{-1/2}}{\sqrt{1 + \gamma U}}, \quad H_0 = p \bar{p}
\]

provide us with proper momenta because \( \{p, \bar{p}\} = 0 \). To get complete correspondence with free particle, we have to find the coordinates \( x, \bar{x} \) canonically conjugated with the momenta \( p, \bar{p} \). Explicitly, they read

\[
x = \bar{u} \frac{2 + \gamma U + i \gamma \bar{u} v}{2 \sqrt{1 + \gamma U}}, \quad \bar{x} = u \frac{2 + \gamma U - i \gamma u \bar{v}}{2 \sqrt{1 + \gamma U}},
\]

\[
\{p, \bar{x}\} = \{\bar{p}, x\} = 1, \quad \{p, \bar{p}\} = \{p, x\} = \{\bar{p}, \bar{x}\} = \{x, \bar{x}\} = 0.
\]

Hence, we have shown that the deformed free particle introduced in [12] is canonically equivalent to the ordinary free particle. Respectively, the actions of both systems admit \( SU(1, 2) \) invariance, which is reduced in the \( \gamma = 0 \) limit to the \( \ell = 1/2 \)-extended conformal Galilei group.
It is instructive to write explicit realization of the generators of the $su(1,2)$ algebra in terms of the canonical variables $x, \bar{x}, p, \bar{p}$:

$$L_{-1} = \mathcal{H}_0 = p \bar{p}, \quad L_0 = \frac{1}{2} (p \ddot{x} + \bar{p} \ddot{\bar{x}}), \quad L_1 = x \ddot{x}, \quad U = i (x \ddot{\bar{p}} - \bar{x} \ddot{p}), \quad G_{-1/2} = -p \sqrt{1 + \gamma \bar{U}}, \quad G_{1/2} = x \sqrt{1 + \gamma \bar{U}}. \tag{4.8}$$

Time-dependent extensions of these generators, defining the isometries of the Lagrangian, are given by the expressions

$$L_{-1}^t = L_{-1}, \quad L_0^t = L_0 + t L_{-1}, \quad L_1^t = L_1 + 2 t L_0 + t^2 L_{-1}, \quad U^t = U, \quad G_{-1/2}^t = G_{-1/2}, \quad G_{1/2}^t = G_{1/2} + t G_{-1/2}, \quad G_{1/2} = G_{1/2} + t G_{-1/2}. \tag{4.9}$$

The respective Hamiltonian vector fields restricted to the second-order Lagrangian (parameterized by $x, \bar{x}$) define the following symmetry transformations

$$L_{-1} = -\dot{x} \frac{\partial}{\partial x} + \text{ c. c.}, \quad L_0 = \left( -\frac{1}{2} x + 2 t \dot{x} \right) \frac{\partial}{\partial x} + \text{ c. c.}, \quad L_1 = \left( -tx + t^2 \dot{x} \right) \frac{\partial}{\partial x} + \text{ c. c.} \quad \tag{4.10}$$

$$G_{-1/2} = -\frac{1 + \beta \dot{x}}{\sqrt{1 + \gamma \bar{x}}} \frac{\partial}{\partial x} - \frac{\beta \dot{\bar{x}}}{\sqrt{1 + \gamma \bar{x}}} \frac{\partial}{\partial \bar{x}}, \quad G_{1/2} = -\frac{2 t + \beta \dot{x}}{\sqrt{1 + \gamma \bar{x}}} + \frac{\beta \dot{\bar{x}}}{\sqrt{1 + \gamma \bar{x}}} \frac{\partial}{\partial x} + \frac{\beta \dot{\bar{x}} \dot{x} - t \dot{x}}{\sqrt{1 + \gamma \bar{x}}} \frac{\partial}{\partial \bar{x}} \tag{4.11}$$

It is worth noting that the explicit realization of the $su(1,2)$ algebra (4.8) makes evident the statement that the $su(1,2)$ algebra as well as its $\gamma = 0$ reduction (i.e. the $\ell = 1/2$ conformal Galilei algebra) can be constructed in terms of two oscillators $(x, \bar{p})$ and $(\bar{x}, p)$. Thus, both these algebras lie in the enveloping algebra of two oscillators.

### B. Oscillator

Now, let us consider the general case of deformed oscillator given by the Hamiltonian (4.2) which is simply $L_{-1} + \omega^2 L_1$. In addition to constant of motion $U$ given in (4.7) defining $U(1)$ symmetry, it possesses hidden symmetries given by the generalization of Fradkin tensor

$$A = G_{-1/2}^2 + \frac{1}{\omega^2} \left( \{ \mathcal{H}, G_{-1/2} \} \right)^2 = G_{-1/2}^2 + \omega^2 \left( u + \gamma u U - \frac{i}{2} \gamma \bar{u} G_{-1/2} \right)^2 \tag{4.12}$$

$$\bar{A} = \bar{G}_{-1/2}^2 + \frac{1}{\omega^2} \left( \{ \bar{\mathcal{H}}, \bar{G}_{-1/2} \} \right)^2 = \bar{G}_{-1/2}^2 + \omega^2 \left( \bar{u} + \gamma \bar{u} U + \frac{i}{2} \gamma \bar{u} \bar{G}_{-1/2} \right)^2.$$

These constants of motion form the deformation of $su(2)$ algebra

$$\{ A, \bar{A} \} = -4 i \omega^2 (U + 3 \gamma U^2 + 2 \gamma^2 U^3) + 4 i \gamma (1 + \gamma U) \mathcal{H}^2, \quad \{ U, A_\pm \} = \mp 2 i A_\pm. \tag{4.13}$$

The Hamiltonian is the Casimir of this algebra. It expresses via above constants of motion as follows

$$\mathcal{H}^2 = \frac{A \bar{A}}{(1 + \gamma U)^2} + \omega^2 U^2. \tag{4.14}$$

One may may directly check that the Hamiltonian (4.2), being rewritten in terms of canonical variables $x, \bar{x}, p, \bar{p}$ (4.6), (4.10), acquires the form

$$\mathcal{H} = p \bar{p} + \omega^2 x \bar{x}. \tag{4.14}$$
as it should be. To get the canonical formulation of symmetry algebra, we redefine the Fradkin tensors as

\[
A = \frac{A}{1 + \gamma U} = p^2 + \omega^2 x^2, \quad \overline{A} = \frac{\bar{A}}{1 + \gamma U} = \bar{p}^2 + \omega^2 \bar{x}^2 : \quad \{A, \overline{A}\} = -4i\omega^2 U, \quad \{U, A\} = -2i A.
\] (4.15)

where \(p, \bar{p}, x, \bar{x}\) are given by (4.5), (4.6). In terms of these tensors the Hamiltonian of the oscillator reads

\[
\mathcal{H}^2 = A\overline{A} + \omega^2 U^2.
\] (4.16)

Hence, the deformed oscillator is canonically (classically) equivalent to non-deformed one. Since its action admits the invariance under \(su(1, 2)\) algebra, we conclude that non-deformed oscillator action possesses the same invariance as well.

\section{V. CONCLUSION}

In this paper, we provided the Hamiltonian description of the deformed two dimensional oscillator \cite{12}, i.e. oscillator possessing dynamical \(SU(1, 2)\) symmetry. One of the interesting features of this system is the fact that its first-order Lagrangian is nothing but one of the Cartan forms defined on the coset \(SU(1, 2)/H\) with a quite unusual choice of the stability subgroup \(H\), which includes the dilatation and conformal boosts together with \(U(1)\) rotation. On the other hand, this one-form is a source of the symplectic form and the Hamiltonian, both written in terms of the initial variables. In this basis, the Hamiltonian of the deformed oscillator is simple, while the Poisson brackets are more involved. Analysing the structure of the Hamiltonian we succeeded in finding the canonical variables in which the Hamiltonian of the deformed oscillator coincides with the Hamiltonian of the ordinary two dimensional oscillator. Thus, we proved that these two systems, deformed and ordinary two dimensional oscillators, are canonically equivalent.

Proving the canonical equivalence of these systems, we have explicitly constructed the currents spanning \(su(1, 2)\) algebra in terms of the ordinary oscillator. The main feature of this realization is a non-polynomial structure of \(su(1, 2)\) currents. Probably just this property was the obstacle preventing from immediate visualization of the \(su(1, 2)\) algebra within the enveloping algebra of the two-dimensional oscillator.

The established equivalence of the deformed and ordinary oscillators within the Hamiltonian approach does not mean their equivalence as the Lagrangian systems. Indeed, the transformations between these two systems are forbidden at the Lagrangian level. Thus, the Hamiltonian formulation is more suitable for analysis of this type of systems, as compared to the Lagrangian one.

Concerning the further developments, one has to note that the \(su(1, 2)\) algebra is not a unique one which admits reduction to the conformal Galilei algebra and thus can be viewed as its deformation. The immediate example of possible algebras having the proper structure is provided by the wedge subalgebras in the \(U(n)\) quasi-superconformal algebras \cite{12}. A preliminary analysis shows that the extension of the approaches of \cite{12} and the Hamiltonian formalism of the present paper to these algebras will result in the \(SU(n + 2)\) invariant \(d = n + 1\) dimensional oscillators. Another interesting algebra is \(so(2, 3)\) which may be viewed as a deformation of the three-dimensional \(\ell = 1\) conformal Galilei algebra \cite{14}.

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