Optimal rounding under integer constraints

Rama Cont* and Massoud Heidari†

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Abstract

Given $N$ real numbers whose sum is an integer, we study the problem of finding $N$ integers which match these real numbers as closely as possible, in the sense of $L^p$ norm, while preserving the sum. We describe the structure of solutions for this integer optimization problem and propose an algorithm with complexity $O(N \log N)$ for solving it. In contrast to fractional rounding and randomized rounding, which yield biased estimators of the solution when applied to this problem, our method yields an exact solution, which minimizes the relative rounding error across the set of all solutions for any value of $p \geq 1$, while avoiding the complexity of exhaustive search. The proposed algorithm also solves a class of integer optimization problems with integer constraints and may be used as the rounding step of relaxed integer programming problems, for rounding real-valued solutions.

Keywords: integer optimization, optimal rounding, randomized rounding, rounding heuristic, Apportionment problem, quota methods.

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*Dept of Mathematics, Imperial College London. E-mail: Rama.Cont@imperial.ac.uk
†tPlus Technology, 410 Park Avenue, New York, NY 10022. Email: massoud.heidari@tplustechnology.com
Introduction

The rounding, or integer approximation, of real numbers is a key step in many algorithms used in integer optimization [5, 8, 9, 20], whereby

(i) an optimization problem over integers is replaced by a corresponding problem over real numbers (‘relaxation’ step);

(ii) the real solution obtained in (i) is ’rounded’ to obtain an integer solution satisfying the constraints of the original problem (rounding step).

If the objective function of the relaxed problem is linear, convex or has some other special property, step (i) may be solved using efficient numerical methods with polynomial complexity in the dimension $N$ of the problem. The rounding step, on the other hand, may be described as a constrained integer optimization problem over $\{0, 1\}^N$ for which no generic polynomial algorithm is known.

Given the large dimensionality of integer programming problems arising in many applications, exhaustive search on $\{0, 1\}^N$ is certainly not an option. Randomized rounding has been extensively used and analyzed as an alternative [16]. Yet in many instances randomized rounding may result in a substantial loss in accuracy [20]. Thus, in practice, the final rounding step sets a lower bound on the overall accuracy of many integer optimization algorithms, which illustrates the importance of good rounding algorithms.

Many integer optimization problems, such as scheduling of tasks on parallel machines [15, 9, 17], or the proportional allocation of seats to political parties in elections with party list voting systems [1, 11], are naturally subject to integer constraints. The latter problem, known as the apportionment problem, has a long history in mathematics, dating back to Polya [14, 13], and continues to generate a lot of interest [4, 7]. Relaxation to real variables followed by rounding each component to the nearest integer may typically fail to satisfy such constraints and many different –and non-equivalent– methods exist for obtaining approximate or exact solutions satisfying the integer constraints [1, 16, 11, 20]. Most of these methods are based on heuristics which provide approximate solutions. Some algorithms, based on randomized rounding may yield a bias in finite samples, which may or may not vanish asymptotically [2, 7]. Even in the case where they yield exact solutions, the mathematical properties of such algorithms have not been systematically analyzed until recently [11].

The present work is an attempt to complement this picture by demonstrating the link between various families of such optimal rounding problems, proposing a polynomial algorithm for solving them and giving a systematic analysis of the optimality and convergence properties of this algorithm and comparing it to some popular methods.

Outline In this paper we formulate and study the problem of optimal rounding of a set of real numbers under integer constraints. The formulation of the problem is given in Section 1. In Section 2 we study the structure of the problem and show how it can be reduced to a sequence of unconstrained rounding
problems. Based on this insight, in Section 3 we propose an algorithm with polynomial complexity for solving this problem and analyze its convergence properties and computational complexity. Finally, in Section 4 we discuss how commonly used rounding methods perform when applied to our problem setting; in particular, we show that randomized rounding yields a systematically biased solution when applied to the problem at hand.

## 1 Optimal rounding under integer constraints

### 1.1 Problem set-up

Denote by $\mathbb{Z}$ the set of integers, $\mathbb{N}$ the set of non-negative integers and $\mathbb{R}$ the set of real numbers. For a vector $x = (x_1, ..., x_N) \in \mathbb{R}^N$ and $q \geq 1$, denote

$$\|x\|_q = \left(\sum_{i=1}^{N} |x_i|^q \right)^{1/q}$$

For any real number $y \in \mathbb{R}$, we denote

$$\text{floor}(y) = \sup\{m \in \mathbb{Z}, \ m \leq y\} \quad \text{cei}(y) = \inf\{m \in \mathbb{Z}, \ m \geq y\}$$

The problem of optimal rounding under integer constraints can be formulated in the following way:

**Problem 1 (Optimal rounding under integer constraints)** Let $q \geq 1$. Given positive real numbers $x = (x_1, ..., x_N) \in \mathbb{R}^N_+$ with

$$\sum_{i=1}^{N} x_i = M \in \mathbb{N} \quad \text{(1)}$$

find a set of integers $m = (m_1, ..., m_N) \in \mathbb{N}^N$ which minimizes

$$\|x - m\|_q = \left(\sum_{i=1}^{N} |x_i - m_i|^q \right)^{1/q} \quad \text{under} \quad \sum_{i=1}^{N} m_i = M. \quad \text{(2)}$$

We denote the corresponding ($L^q$) rounding error

$$V_q(x) = \inf\{\|x - m\|_q, \ m \in \mathbb{N}^N, \sum_{i=1}^{N} m_i = M\}. \quad \text{(3)}$$

The non-trivial feature of the problem is the presence of the integer constraint. As the example $x = (2.25, 3.4, 4.35)$ shows, componentwise rounding to the nearest integer may fail to satisfy such a constraint.

Problem 1 is a special case of the following integer programming problem.
Problem 2 Given a continuous function \( f : \mathbb{R}^N \rightarrow \mathbb{R} \) find a set of integers \( m = (m_1, ..., m_N) \in \mathbb{N}^N \) which minimizes

\[
\inf_{m \in \mathbb{N}^N} f(m) \quad \text{under} \quad \sum_{i=1}^{N} m_i = M \in \mathbb{N}.
\]  

This is an optimization problem over the finite set

\[
\{ m \in \mathbb{N}^N, \sum_{i=1}^{N} m_i = M \}
\]

so the infimum is always attained. We denote by

\[
V(f) = \min \{ f(m), \quad m \in \mathbb{N}^N, \sum_{i=1}^{N} m_i = M \}
\]

the value of this minimum. However, the size of this set increases exponentially with \( N \), so at first sight, Problems [1] and [2] appear to be integer optimization problems with exponential complexity.

Our contribution is to study the structure of these problems and show that they can be solved using an algorithm with polynomial complexity in \( N \). We first show in Section [2] that, notwithstanding the constraint, the solution necessarily consists in rounding each component either up or down (Proposition [1]). We can thus reformulate the problem as an optimization problem on \( \{0, 1\}^N \). Next, we propose in Section [3] an algorithm which solves the problem with complexity \( N \log N \).

1.2 Related problems and ramifications

Problem [1] is a 'pure integer programming' problem in the sense that the relaxation to the case where \( m \in \mathbb{R}^N \) is trivial. As such, it enters as a building block in many integer and mixed-integer programming problems in which one first solves a relaxation of the problem to real variables then projects back the solution of the relaxed problem onto \( \mathbb{Z}^N \).

The following problem arises e.g. in rounding problems encountered in accounting, where one rounds \( N \) entries while leaving the total unchanged up to the nearest dollar:

Problem 3 (Decimal approximation with a given precision on the sum)

Given positive real numbers \( x = (x_1, ..., x_N) \in \mathbb{R}_+^N \), find decimal approximations with \( k \) decimal digits \( y = (y_1, ..., y_N) \in 10^{-k}\mathbb{N}^N \) which minimize

\[
\sum_{i=1}^{N} |x_i - y_i|^q \quad \text{under} \quad \sum_{i=1}^{N} x_i - \sum_{i=1}^{N} y_i < 10^{-k}.
\]
It is clear that Problem 1 may have multiple solutions in the case where at least two components \( i, j \) have equal fractional components. In this case one might consider minimizing the relative rounding error among all solutions of Problem 1.

**Problem 4 (Optimal rounding with smallest relative error)**

*Given positive real numbers \( x = (x_1, \ldots, x_N) \in \mathbb{R}^N_+ \), find \( m^* = (m^*_1, \ldots, m^*_N) \in \mathbb{N}^N \) such that*

\[
\sum_{i=1}^{N} \left| \frac{x_i - m_i}{|x_i|^q} \right|^q = \min \left\{ \sum_{i=1}^{N} \left| \frac{x_i - m_i}{|x_i|^q} \right|^q, \quad V_q(m) = \min_{z \in \mathbb{N}^N, \sum z_i = M} V_q(z) \right\}. \tag{7}
\]

Another problem which is equivalent to Problem 1 is that of rounding of a vector of decimal numbers under the constraint that their sum is conserved to within a given precision, often expressed in terms of number of significant digits after the decimal. This problem arises, for example, in accounting, where each figure contributing to a total is rounded, say, to the nearest cent. Financial statements of many companies routinely carry the warning that “numbers may not add up due to rounding.” This problem may be formulated as follows:

**Problem 5 (Decimal approximation with a given precision on the sum)**

*Given positive real numbers \( x = (x_1, \ldots, x_N) \in \mathbb{R}^N_+ \), find a decimal number with \( k \) decimal digits \( y = (y_1, \ldots, y_N) \in 10^{-k}\mathbb{N}^N \) which minimizes*

\[
\sum_{i=1}^{N} |x_i - m_i|^q \quad \text{under} \quad \sum_{i=1}^{N} |x_i - \sum_{i=1}^{N} y_i| < 10^{-k}. \tag{8}
\]

Although the precision constraint is an inequality, it is readily observed that this problem is in fact equivalent to Problem 1 applied to \( 10^k x \).

Finally, let us mention a formulation of the optimal rounding problem based on relative errors, in which the sum of the absolute rounding errors is replaced by a product of relative rounding errors:

**Problem 6**

*Given positive real numbers \( x = (x_1, \ldots, x_N) \in \mathbb{R}^N_+ \), find \( (m_1, \ldots, m_N) \in \mathbb{N}^N \) which minimizes*

\[
\prod_{i=1}^{N} \left| \frac{x_i - m_i}{|x_i|^q} \right|^q \quad \text{under} \quad \sum_{i=1}^{N} m_i = M. \tag{9}
\]

### 1.3 Applications

Problem 1 arises in many applications involving allocation of indivisible resources. Many of these applications involve solving an integer optimization problem, which is generally done by first relaxing it to a continuous optimization problem, then rounding the solution to find an approximate integer solution which needs to conserve the sum. Here, we point out a few applications which appear prominently in various fields.
Example 1 (Apportionment problem) A first example is the apportionment problem which arises in proportional election systems, for allocating a total of $M$ seats to $N$ political parties such that each party gets a number of seats that is proportional to the number of votes received. Given that seats are indivisible, this is a rounding problem and the optimality criterion in [1] reflects the fact that one tries, in the seat allocation, to stay as close as possible to the true proportions observed in the votes. The constraint on the sum arises through the fixed number of seats to be allocated.

The same mathematical problem arises if seats are to be apportioned before the election between constituencies according to their populations. See Balinski and Young [1] for a detailed exposition and Grimmett [4] for a discussion in the context of the European Parliament. Here $N$ is typically small while $M$ may be very large, which means that the rounding step affects the allocation significantly.

This problem has a long history in mathematics, dating back to Polya [13, 14], and continues to generate a lot of interest [11, 4, 7].

Another example is the asset allocation problem: the problem of determining the optimal mix of $N$ assets in a portfolio. This problem is classically treated as a continuous optimization problem, but in fact assets are purchased in indivisible shares; thus, the implementation of such allocations involves a rounding step which is usually ignored in the theoretical analysis:

Example 2 (Asset allocation problem) Given $N$ investment opportunities available in indivisible units (shares), the asset allocation problem is to find allocation to each investment to optimize a (convex) risk-return tradeoff. Whether this is done through a simple mean-variance criterion [10], more complex variants involving transaction costs [12], or expected utility maximization [18], it leads to a problem of the form in which the dimension $N$ may range from hundreds to thousands. While classical treatment of this problem [10, 12, 18] has ignored the rounding error, in practice this rounding error—and the related 'integrality gap'—need not be small and may result in a portfolio which lies at a finite distance from the efficient frontier, even when $N$ is large.

Example 3 (Scheduling of unrelated tasks on parallel machines) Another instance of rounding problem with integer constraint arises in the problem of scheduling $M$ unrelated tasks across $N$ parallel machines, which is itself related to the classical 'assignment problem' [9, 19]. Each task can be executed by at most one machine, at a certain cost. Minimizing the total execution cost leads to a Linear Programming (LP) relaxation of the problem, which is followed by a rounding operation to obtain the final allocation. Randomized rounding has been proposed as a method for this last step [17].

2 Structure of solutions

We first show that Problem 1 and Problem 2 are indeed equivalent to classical 'rounding' problems.
Proposition 1 (Restriction to term-by-term rounding)
Let \( m^* = (m^*_1, ..., m^*_N) \in \mathbb{N}^N \) be a solution to Problem\(^7\). Then
\[
\forall i = 1..N, \quad m^*_i \in \{\text{floor}(x_i), \text{ceil}(x_i)\}. \tag{10}
\]

Proof: First, note that the feasible set is non-empty and finite, so the infimum is attained at some \( m^* \in \mathbb{N}^N \) with \( \sum m^*_i = M \).

If \( M = \sum \text{floor}(x_i) \) or \( M = \sum \text{ceil}(x_i) \) then this means \( x \in \mathbb{N}^N \) and the feasible set is reduced to a single element. We exclude this case below and assume \( \sum \text{ceil}(x_i) > M > \sum \text{floor}(x_i) \).

Assume \( m^*_i < \text{floor}(x_i) \) for some \( i \in \{1, ..., N\} \). Then, given that \( \sum m^*_i = M \), there exists at \( j \neq i \) such that \( m^*_j > \text{ceil}(x_j) \). Then consider \( y \in \mathbb{N}^N \) defined by
\[
y_k = m^*_k \quad \text{for} \quad k \notin \{i,j\}, \quad y_i = m^*_i + 1, \quad y_j = m^*_j - 1 \geq \text{floor}(x_j) \quad (11)
\]
Then we have \( \sum y_i = \sum m^*_i = M \), \( |x_i - y_i| = |x_i - m^*_i| - 1 \), and \( |x_j - y_j| \leq |x_j - m^*_j| \) so
\[
\|x - y\|_q < \|x - m^*\|_q \tag{12}
\]
which contradicts the definition of \( m^* \).

Proposition 1 shows that Problem 1 can be reduced to a component-by-component rounding problem, i.e. an optimization problem on \( \{0, 1\}^N \).

We now turn to Problem\(^2\). In addition to continuity, we shall assume the following property for \( f \):

Assumption 1 (Directional convexity) For each \( i = 1..N \), and \( (x_1, x_{i-1}, x_{i+1}, ..., x_N) \in \mathbb{R}^{N-1} \) the partial function
\[
\mathbb{R} \rightarrow \mathbb{R}
\]
\[
u \rightarrow f(x_1, ..., x_{i-1}, u, x_{i+1}, ..., x_N)
\]
is strictly convex.

For example, if \( f(x) = \sum_{i=1}^N \phi_i(x_i) \) where \( \phi_i : \mathbb{R} \rightarrow \mathbb{R} \) is strictly convex then \( f \) satisfies the assumption. Note that \( f \) need not be strictly convex in a global sense.

Under this assumption, the relaxation to \( \mathbb{R}_+^N \) of Problem\(^2\) i.e.
\[
\inf_{y \in \mathbb{R}_+^N} f(y) \quad \text{under} \quad \sum_{i=1}^N y_i = M.
\]
has a unique solution \( x^* \in \mathbb{R}_+^N \).

Proposition 2 Under Assumption\(^7\), any solution \( m^* = (m^*_1, ..., m^*_N) \in \mathbb{N}^N \) to Problem\(^2\) verifies
\[
m^*_i \in \{\text{floor}(x^*_i), \text{ceil}(x^*_i)\} \quad (13)
\]
where \( x^* \in \mathbb{R}^N_+ \) is the unique solution of the constrained optimization problem

\[
\inf_{y \in \mathbb{R}^N_+} f(y) \quad \text{under} \quad \sum_{i=1}^N y_i = M. \tag{14}
\]

**Proof:** Let \( x^* \in \mathbb{R}^N_+ \) be the unique solution to the constrained optimization problem (14). Without loss of generality we may assume that \( f(x^*) = 0 \) by subtracting \( f(x^*) \) from \( f \) and consider the non-trivial case \( \sum \text{cei}(x_i) > M > \sum \text{floor}(x_i) \).

Assume \( m^*_i < \text{floor}(x_i) \) for some \( i \in \{1, \ldots, N\} \). Then, given that \( \sum m^*_i = M \), there exists at \( j \neq i \) such that \( m^*_j > \text{cei}(x_j) \). Let \( z \in \mathbb{N}^N \) be given by

\[
z_k = m^*_k \quad \text{for} \quad k \neq i, \quad z_i = m^*_i + 1 \tag{15}
\]

Since \( m^*_i < z_i < x^*_i \), we can write \( z_i = (1 - \alpha)m^*_i + \alpha x^*_i \) with \( 0 < \alpha < 1 \). Noting that \( z \) is obtained by modifying a single component of \( m^* \), the directional convexity property of \( f \) implies that

\[
f(z) < (1 - \alpha)f(m^*) + \alpha f(x^*) = (1 - \alpha)f(m^*) < f(m^*)
\]

Now define

\[
y_k = m^*_k \quad \text{for} \quad k \neq \{i, j\}, \quad y_i = m^*_i + 1, \quad y_j = m^*_j - 1 \geq \text{cei}(x_j^*) \geq x_j^* \tag{16}
\]

Again, noting that \( y \) is obtained by modifying a single component of \( z \) and writing \( y_j = (1 - \beta)z_j + \beta x_j^* \) with \( 0 < \beta < 1 \) we can use the directional convexity property of \( f \) to obtain

\[
f(y) < (1 - \beta)f(z) + \beta f(x^*) = (1 - \beta)f(z) < (1 - \beta)f(m^*) < f(m^*)
\]

Since \( \sum y_i = \sum x_i^* - 1 + 1 = M, y \in \mathbb{N}^N \) is thus a solution to Problem 2 and \( f(y) < f(m^*) \) which contradicts the definition of \( m^* \). QED.

### 3 A polynomial algorithm for optimal rounding under integer constraints

Proposition 1 reduces the complexity of Problem 1 by confining the search to the finite set

\[
\prod_{i=1}^N \{\text{floor}(x_i), \text{cei}(x_i)\}
\]

which can be reparameterized as \( \{0, 1\}^N \) i.e. a componentwise rounding problem. But the size of this set grows exponentially with \( N \), so optimization through exhaustive search is not feasible, even if \( N \) is only moderately large. We now exhibit an algorithm which exploits the structure of the problem, in particular the constraint, to compute a solution with a polynomial number of operations.
Given the structure of the objective function in Problem \textsuperscript{1}, the idea is to optimize term by term, controlling for the constraint at each step.

We start by rounding all components downwards and compute the constraint shortfall $I = \sum_{i=1}^{N}(x_i - m_i) \leq N$. If $I = 0$ this means $x$ is already integer valued so one does not need to proceed further. If $I \geq 1$ we need to round upwards exactly $I$ components to meet the constraint. To choose these components optimally, we

1. sort the indices according to decreasing values of fractional part $x_i - \text{floor}(x_i) = x_i - m_i$.
2. sort indices with equal fractional part in decreasing order of their integer part $\text{floor}(x_i)$.
3. In the last step, we proceed to round upwards the first $I$ components sorted in this order.

Steps 1) and 2) may be done using a QuickSort algorithm \textsuperscript{6}.

This yields a polynomial algorithm for solving the optimal rounding problem:

**Optimal rounding under integer constraints (ORIC):**

1. Set $\forall i = 1..N, m_i = \text{floor}(x_i)$.
2. Compute constraint shortfall $I(x) = \sum_{i=1}^{N}(x_i - m_i) \geq 0$.
3. If $I = 0$ then end.
4. Sort indices in decreasing order of fractional part $(x_i - m_i)$:
   
   $$(x_1 - m_1) \geq ... \geq (x_N - m_N) \geq 0.$$ 

5. Sort each subsequence with equal fractional parts $(x_k - m_k) = ... = (x_k + j - m_k + j)$ in decreasing order of integer parts: $m_k \geq m_{k+1} \geq ... \geq m_{k+j}$.
6. for $k = 1, \ldots, I$ do
   7. $m_k = \text{cei}(x_k)$.
8. end for
9. end

There exists a symmetric version of the algorithm where one initializes with $m_i = \text{cei}(x_i)$ and then needs to round downwards $M - I$ of the components. It is readily verified that the two methods yield the same solutions.

A similar algorithm, based on the sorting of fractional components, but without the sorting step (5), has been used for a long time in the context of proportional seat allocation for representative assemblies with party list voting systems \textsuperscript{1}, where it is known as the Hare-Niemeyer or 'largest remainder method'. The focus of the Hare-Niemeyer and related methods is to achieve a 'fair' allocation rather than to minimize a given objective function, so the notion of optimality considered here may or may not be relevant for such applications and more complex considerations apply; we refer to Balinski & Young \textsuperscript{1}. 
for a detailed discussion of seat allocation methods in elections. Nevertheless, the largest remainder method is a special case of the method considered here, when all fractional parts are distinct and the analysis of optimality bears many similarities \[11\].

To quantify the complexity of the algorithm, we first note that

\[ I = \sum_{i=1}^{N} (x_i - m_i) \leq N, \]

thus the sorting steps (Steps 4-5) may be achieved through a sorting algorithm such as QuickSort \[6\] with complexity \( \sim N \log N \) \[6, 3\]. Once the sorting has been done, the rounding of the sorted sequence (Steps 6-8) requires \( I \leq N - 1 \) operations. So, overall, the complexity is dominated by the sorting step:

**Proposition 3** The ORIC algorithm solves Problem \([7]\) for all \( q \geq 1 \), with a computational complexity of \( N \log N \) for typical input vectors. It converges to a vector \( m^* \in \mathbb{N}^N \) satisfying \( \sum_{i=1}^{N} m_i = M \) which is a solution to Problem \([4]\) and Problem \([1]\) for all values of \( q \geq 1 \).

**Proof** Due to the ordering of the indices in the sorting step 4 algorithm yields a vector \((x_i - m^*_i, i = 1..N)\) which is componentwise smaller than \((x_i - m_i, i = 1..N)\) for any other element of the feasible set. So for any componentwise increasing function, in particular \( V_q \), it yields the minimum. The ordering of indices in Step 5 guarantees that when fractional parts are equal, the algorithm picks the solution whose componentwise relative error is the smallest, hence the characterization above.

To quantify the complexity, denote by \( p \) the number of subsequences with lengths \( M_1, ..., M_p \) inside which the fractional parts are equal. This implies that, in addition to the sorting of Step 4, we need to do sort these \( p \) subsequences according to integer parts. We sort each of these batches using a QuickSort algorithm with complexity \( M_k \log M_k \), leading to an overall complexity of

\[ N \log N + \sum_{k=1}^{p} M_k \log M_k \]  \hspace{1cm} (17)

where the second term corresponds to the sorting of the \( p \) batches. Let \( \beta \leq 1 \) define the order or magnitude of the longest subsequence: \( \max(M_k, k = 1..p) \sim N^\beta \) and \( N^\alpha, \alpha \leq 1 \) denote the number of subsequences with length \( \sim N^\beta \). Then \( \alpha + \beta \leq 1 \) and the order of magnitude of the complexity of the second term in \((17)\) is given by

\[ N^\alpha N^\beta \log(N^\beta) = \beta N^{\alpha+\beta} \log N \leq \beta N \log N. \]

Thus, aside from a prefactor, the sorting of subsequences according to integer parts (Step 5) does not increase the order of the complexity, which is determined by the global sorting (Step 4).
Remark 1 The ORIC algorithm also provides a solution to Problem 6 for all \( q \geq 1 \).

4 Comparison with other rounding methods

The computational complexity of the ORIC algorithm is dominated by the sorting step, whose complexity for a typical vector \( x \in \mathbb{R}^N_+ \) is \( O(N \log N) \) \[6\]. However, this step is essential and any attempt to bypass it with simpler rounding methods, whether deterministic \[20\] or randomized \[16\], may fail to yield the optimal solution. The asymptotic bias of rounding methods has been studied by Diaconis & Freedman \[2\] and more recently by Janson \[7\]. The argument of vanishing asymptotic bias is often used to argue that these methods yield 'unbiased' solutions. We will argue here that, unlike what is suggested by the asymptotic properties, the finite sample solution is in fact systematically biased, in a way that has significant implications for the applications considered in Section 1.3.

4.1 Fractional rounding

A rounding algorithm, often used by default, is fractional rounding, which rounds to floor(\( x \)) (resp. ceil(\( x \))) if \( x_i - \text{floor}(x_i) \leq q \) (resp. \( x_i - \text{floor}(x_i) > q \)) where \( 0 \leq q < 1 \). \( q = 1/2 \) corresponds to mid-point rounding. Denote by

\[ N(q, x) = \{ i = 1..N, x_i - \text{floor}(x_i) \leq q \} . \]

Fractional rounding with parameter \( q \) yields the optimal solution only if the distance to the constraint shortfall \( I(x) = \sum_{i=1}^{N} (x_i - m_i) \) verifies

\[ I(x) = N(q, x) . \]

Obviously this condition depends on \( x \in \mathbb{R}^N_+ \) and it is easily observed that there cannot exist any \( 0 \leq q \leq 1 \) which verifies \( I(x) = N(q, x) \) for all \( x \in \mathbb{R}^N_+ \).

For a given \( x \), there is always a \( 0 \leq q(x) < 1 \) which verifies \( I(x) = N(q(x), x) \), but the value of \( q(x) \) depends on \( x \) and a fractional rounding algorithm with a fixed \( q \) (such as mid-point rounding) will yield the optimal solution to Problem \[1\] only for \( x \) satisfying \( I(x) = N(q, x) \) and not otherwise.

So, a fractional rounding rule fails to yield the optimal solution to Problem \[1\] for all \( x \in \mathbb{R}^N_+ \).

4.2 Randomized rounding

We now compare the algorithm described above with randomized rounding \[16\], in which each \( x_i \) is rounded up with probability \( p_i = x_i - \text{floor}(x_i) \). Randomized rounding thus yields a solution given by

\[ R = U + \text{floor}(x) \] (18)
where $U$ is a random variable with values in $\{0, 1\}^N$ and

$$
\mathbb{P}(U = x) = \prod_{i=1}^{N} p_i^{x_i}(1 - p_i)^{1-x_i}.
$$

Assume, without loss of generality, that $1 > p_1 \geq ... \geq p_N \geq 0$. Let $I = \sum_{i=1}^{N} (x_i - \text{floor}(x_i))$ and $r = \sum_{i=I+1}^{N} (x_i - \text{floor}(x_i))$. Then the optimal solution of Problem 1 corresponds to

$$
m^* = (1, 1, ..., 1, 0, ..., 0) + \text{floor}(x)
$$

The probability that randomized rounding gives an incorrect solution is $r/I$. This probability can be higher than 50%, as the following example shows.

**Example 4** Let $N = 3$ and $x - m = (0.4, 0.35, 0.25)$. The optimal solution is to round up 0.4 and round down the other components to zero. However, in randomized rounding there is a 60% probability that one of the other 2 get rounded up i.e. a 60% probability of a non-optimal solution.

Since $\sum p_i = I$, when $N$ is large the proportion of components rounded up is asymptotically equal to the optimal one, i.e. $I/N$ and, by the strong law of large numbers, the constraint is satisfied with probability 1 for $N \to \infty$. However, the randomized estimator (18) is *biased* and this bias remains systematic, even for $N$ large. In fact, in the case where none of the components are integers $x_i \notin \mathbb{Z}$, randomized rounding yields a systematic bias, whose sign is determined by the rank of $x_i - \text{floor}(x_i)$ in the ordering $x_1 - \text{floor}(x_1) \geq ... \geq x_N - \text{floor}(x_N)$:

**Proposition 4** Randomized rounding applied to Problem 1 yields a biased solution: if $1 > p_1 = x_1 - \text{floor}(x_1) \geq ... \geq p_N = x_N - \text{floor}(x_N) > 0$ then

$$
E[R] < m_i^* \quad \text{for} \quad i = 1..I, \quad \text{and} \quad E[R] > m_i^* \quad \text{for} \quad i = I + 1, ..., N.
$$

This follows from the form (19) of the optimal solution and the fact that $0 < p_i < 1$. Though the main justification often advanced for randomized rounding is that ‘it yields an unbiased estimator of the solution’, as Proposition 4 shows, randomized rounding leads to a systematic upward bias in components whose fractional part is small.

This leads to a solution that is non-optimal in ways that are not intended. For instance, in the asset allocation problem (Example 2), the resulting bias depends not on the allocation to each asset but only on the fractional part of this allocation. The result is a portfolio which lies at a random distance from the efficient frontier. By contrast, the solution obtained using the ORIC algorithm is guaranteed to minimize the distance to the efficient frontier: this distance is explicitly given by the value of the optimum in (3).
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