ON PROFINITE POLYADIC GROUPS

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Abstract. We study the structure of profinite polyadic groups and we prove that a polyadic topological group \((G, f)\) is profinite, if and only if, it is compact, Hausdorff, totally disconnected, and for any open congruence \(R \subseteq G \times G\), the quotient polyadic group \((G/R, f_R)\) is finite.

1. Introduction

In this article, we study the structure of profinite polyadic groups: polyadic groups which are the inverse limit of a system of finite polyadic groups. A polyadic group is a natural generalization of the concept of group to the case where the binary operation of group replaced with an \(n\)-ary associative operation, one variable linear equations in which have unique solutions. So, in this article, polyadic group means an \(n\)-ary group for a fixed natural number \(n \geq 2\). These interesting algebraic objects are introduced by Kasner and Dörnte ([8] and [2]) and studied extensively by Emil Post during the first decades of the last century, [12]. During decades, many articles are published on the structure of polyadic groups. Already homomorphisms and automorphisms of polyadic groups are studied in [10]. A characterization of the simple polyadic groups is obtained by them in [11]. Also, the representation theory of polyadic groups is studied in [9] and the complex characters of finite polyadic groups are also investigated in [13]. The structure of free polyadic groups is determined in [1], [?], and [9].

It is easy to define topological polyadic groups, and so, one can ask which topological polyadic groups are profinite. In this paper, we study this problem and as the main result, we prove that a polyadic topological group \((G, f)\) is profinite, if and only if, it is compact, Hausdorff, totally disconnected, and for any open congruence \(R \subseteq G \times G\), the quotient polyadic group \((G/R, f_R)\) is finite.

2. Polyadic groups

A polyadic group is a pair \((G, f)\) where \(G\) is a non-empty set and \(f : G^n \rightarrow G\) is an \(n\)-ary operation, such that
i- the operation is associative, i.e.
\[ f(x_1^{i-1}, f(x_i^{n+i-1}, x_{n+i}^{2n-1})) = f(x_j^{i-1}, f(x_j^{n+j-1}, x_{n+j}^{2n-1})) \]
for any \( 1 \leq i < j \leq n \) and for all \( x_1, \ldots, x_{2n-1} \in G \), and

ii- for all \( a_1, \ldots, a_n, b \in G \) and \( 1 \leq i \leq n \), there exists a unique element \( x \in G \) such that
\[ f(a_1^{i-1}, x, a_{i+1}^n) = b. \]

Note that, here we use the compact notation \( x_j^i \) for every sequence \( x_i, x_{i+1}, \ldots, x_j \)
of elements in \( G \), and in the special case when all terms of this sequence are equal to a fixed \( x \), we denote it by \( x^t \), where \( t \) is the number of terms.

Clearly, the case \( n = 2 \) is exactly the definition of ordinary groups. During this article, we assume that \( n \) is fixed. Note that an \( n \)-ary system \((G, f)\) of the form \( f(x_1^n) = x_1 x_2 \ldots x_n b \), where \((G, \cdot)\) is a group and \( b \) a fixed element belonging to the center of \((G, \cdot)\), is a polyadic group, which is called \( b \)-derived from the group \((G, \cdot)\) and it is denoted by \( \text{der}_b^n(G, \cdot) \). In the case when \( b \) is the identity of \((G, \cdot)\), we say that such a polyadic group is reduced to the group \((G, \cdot)\) or derived from \((G, \cdot)\) and we use the notation \( \text{der}_b^n(G, \cdot) \) for it. For every \( n > 2 \), there are \( n \)-ary groups which are not derived from any group. A polyadic group \((G, f)\) is derived from some group if and only if, it contains an element \( a \) (called an \( n \)-ary identity) such that
\[ f^{(i-1)}(a, x, a^i) = x \]
holds for all \( x \in G \) and for all \( i = 1, \ldots, n \), see [3].

From the definition of an \( n \)-ary group \((G, f)\), we can directly see that for every \( x \in G \), there exists only one \( y \in G \), satisfying the equation
\[ f^{(n-1)}(x, y) = x. \]

This element is called skew to \( x \) and it is denoted by \( \overline{x} \). As Dörnte [2] proved, the following identities hold for all \( x, y \in G \), \( 2 \leq i \leq n \),
\[ f^{(i-2)}(x, \overline{x}, y) = f^{(n-i)}(x, \overline{x}, y) = y. \]

These identities together with the associativity identities, axiomatize the variety of polyadic groups in the algebraic language \((f, -)\).

Suppose \((G, f)\) is a polyadic group and \( a \in G \) is a fixed element. Define a binary operation
\[ x \ast y = f^{(n-2)}(x, a, y). \]

Then \((G, \ast)\) is an ordinary group, called the retract of \((G, f)\) over \( a \). Such a retract will be denoted by \( \text{ret}_a(G, f) \). All retracts of a polyadic group are
isomorphic \[5\]. The identity of the group \((G, \ast)\) is \(e\). One can verify that the inverse element to \(x\) has the form

\[ y = f(e, (n-3)x, x, x). \]

One of the most fundamental theorems of polyadic group is the following, now known as Hosszú-Gloskin’s theorem. We will use it frequently in this article and the reader can use \[4\], \[7\], \[14\] for detailed discussions.

**Theorem 2.1.** Let \((G, f)\) be a polyadic group. Then there exists an ordinary group \((G, \cdot)\), an automorphism \(\theta\) of \((G, \cdot)\) and an element \(b \in G\) such that

1. \(\theta(b) = b\),
2. \(\theta^{n-1}(x) = bxb^{-1}\), for every \(x \in G\),
3. \(f(x_1^n) = x_1\theta(x_2)\theta^2(x_3) \cdots \theta^{n-1}(x_n)b\), for all \(x_1, \ldots, x_n \in G\).

According to this theorem, we use the notation \(\text{der}_{\theta,b}(G, \cdot)\) for \((G,f)\) and we say that \((G, f)\) is \((\theta, b)\)-derived from the group \((G, \cdot)\).

There is one more important ordinary group associated to a polyadic group which we call it the Post’s cover. This is the first fundamental theorem concerning polyadic groups. The proof can be find in \[12\].

**Theorem 2.2.** Let \((G, f)\) be a polyadic group. Then, there exists a unique group \((G^*, \circ)\) such that

1- \(G\) is contained in \(G^*\) as a coset of some normal subgroup \(K\).
2- \(K\) is isomorphic to a retract of \((G, f)\).
3- We have \(G^*/K \cong \mathbb{Z}_{n-1}\).
4- Inside \(G^*\), for all \(x_1, \ldots, x_n \in G\), we have \(f(x_1^n) = x_1 \circ x_2 \circ \cdots \circ x_n\).
5- \(G^*\) is generated by \(G\).

The group \(G^*\) is also universal in the class of all groups having properties 1, 4. More precisely, if \(\beta : (G, f) \to \text{der}^n(H, \ast)\) is a polyadic homomorphism, then there exists a unique ordinary homomorphism \(h : G^* \to H\), such that \(h_{|G} = \beta\). This universal property characterizes \(G^*\) uniquely. The explicit construction of the Post’s cover can be find in \[13\].

Finally, we have to mention that the structure of polyadic homomorphisms will be needed in what follows. The reader can see \[10\] for details.

**Theorem 2.3.** Suppose \((G, f) = \text{der}_{\theta,b}(G, \cdot)\) and \((H, h) = \text{der}_{\eta,c}(H, \ast)\) are two polyadic groups. Let \(\psi : (G, f) \to (H, h)\) be a homomorphism. Then there exists a \(a \in H\) and an ordinary homomorphism \(\phi : (G, \cdot) \to (H, \ast)\), such
that \( \psi = R(a)\phi \), where \( R(a) \) denotes the map \( x \mapsto x \ast a \). Further \( a \) and \( \phi \) satisfy the following conditions:

\[
h^{(n)}(a) = \phi(b) \ast a \quad \text{and} \quad \phi \theta = I_a \eta \phi,
\]

where, \( I_a \) denotes the inner automorphism \( x \mapsto a \ast x \ast a^{-1} \). Conversely, if \( a \) and \( \phi \) satisfy the above two conditions, then \( \psi = R_a \phi \) is a homomorphism \( (G, f) \to (H, h) \).

### 3. Profinite polyadic groups

A profinite polyadic group is the inverse limit of an inverse system of finite polyadic groups. More precisely, let \((I, \leq)\) be a directed set and suppose \(\{(G_i, f_i), \varphi_{ij}, I\}\) is an inverse system of finite polyadic groups. This means that for every pair \((i, j)\) of elements of \(I\) with \(j \leq i\), we are given a polyadic homomorphism

\[
\varphi_{ij} : (G_i, f_i) \to (G_j, f_j)
\]

such that the equality \( \varphi_{jk} \varphi_{ij} = \varphi_{ik} \) holds for all \( k \leq j \leq i \). Now, assume that

\[
(G, f) = \varprojlim_i (G_i, f_i).
\]

Then \((G, f)\) is called a profinite polyadic group. Note that, for \((G, f)\) to be a profinite group, it requires that \( G \) is non-empty, and we will see that it is indeed so. From now on, we consider the pair \((G, f)\) which is the above mentioned inverse limit. A realization of this pair can be given as follows: Let \( \prod_i (G_i, f_i) \) be the direct product of the family \(\{(G_i, f_i)\}_{i \in I}\). This is a polyadic group with the \( n \)-ary operation

\[
(\prod_i f_i)((x_{i1}), (x_{i2}), \ldots, (x_{in})) = (f_i(x_{i1}, x_{i2}, \ldots, x_{in}))_{i \in I}.
\]

Here of course, we denoted an arbitrary element of the direct product as sequence \((a_i)_{i \in I}\) or simply \((a_i)\). Now, we have

\[
G = \{(x_i)_{i \in I} : \forall j \leq i \quad \varphi_{ij}(x_i) = x_j \},
\]

and hence

\[
f((x_{i1}), (x_{i2}), \ldots, (x_{in})) = (f_i(x_{i1}, x_{i2}, \ldots, x_{in}))_{i \in I}.
\]

This realization allows us to consider the \textit{natural projection} maps \( \varphi_j : G \to G_i \) defined by

\[
\varphi_j((x_i)_{i \in I}) = x_j,
\]

which are obviously polyadic homomorphisms.

Note that, as each \( G_i \) is finite, being a closed subspace of the direct product of a family of finite sets, \((G, f)\) is compact, Hausdorff, and totally disconnected topological polyadic group, of course, if it has been shown that \( G \) is non-empty. Indeed, using standard topological arguments, we can prove that \( G \neq \emptyset \) as every \( G_i \) is compact.
Recall that, according to Hosszú-Gloskin’s theorem, we have 
\((G_i, f_i) = \text{der}_{\theta_i, b_i}(G_{i}, \bullet_i)\), for some ordinary group \((G_i, \bullet_i)\), an element \(b_i \in G_i\), and an automorphism \(\theta_i\), satisfying the conclusions of Theorem 2.1. We will prove 
that in some sense, there exists a binary operation \(\bullet\) on \(G\) such that 

\[(G, \bullet) = \varprojlim_i (G_i, \bullet_i),\]

and hence \((G, \bullet)\) will be proved to be profinite. Consider the polyadic homomorphism \(\varphi_{ij}\). According to Theorem 2.3, there exist an element \(a_{ij} \in G_j\), and a group homomorphism \(\psi_{ij} : (G_i, \bullet_i) \to (G_j, \bullet_j)\), such that 

\[\varphi_{ij} = R(a_{ij})\psi_{ij}.
\]

Further, we have the following equalities:

1. \(f_j(a_{ij}, a_{ij}, \ldots, a_{ij}) = \psi_{ij}(b_i) \bullet_j a_{ij},\)
2. \(\psi_{ij} \theta_i = I(a_{ij}^{-1})\theta_j \psi_{ij}.
\]

For any triple of indices \(k \leq j \leq i\), we have

\[\varphi_{ij} = R(a_{ij})\psi_{ij}, \quad \varphi_{ik} = R(a_{ik})\psi_{ik}, \quad \varphi_{jk} = R(a_{jk})\psi_{jk},\]

therefore

\[a_{ij} = \varphi_{ij}(1), \quad a_{ik} = \varphi_{ik}(1), \quad a_{jk} = \varphi_{jk}(1).
\]

Note that in each equality, 1 is the identity element of the corresponding group. Since \(\varphi_{ik} = \varphi_{jk}\varphi_{ij}\), so we have

\[a_{ik} = \varphi_{jk}(a_{ij}).\]

Now, let \(Y_i\) be the set of all sequences \((x_j)\) (in the direct product) such that for any \(j \leq i\), we have \(\varphi_{jk}(x_j) = x_k\). This set is non-empty, because we can consider a sequence where \(x_j = a_{ij}\), for \(j \leq i\), and for all other \(j\), \(x_j\) is arbitrary. This sequence will be an element of \(Y_i\). The set \(Y_i\) is closed and if \(i \leq s\), then \(Y_s \subseteq Y_i\). As the direct product is compact, and the family \(\{Y_i\}\) has finite intersection property, we have

\[\bigcap_i Y_i \neq \emptyset,
\]

showing that \(G\) is not empty.

Now, we can prove our first main result:

**Theorem 3.1.** Let \((G, f) = \text{der}_{\theta, b}(G, \bullet)\) be a profinite polyadic group. Then the ordinary group \((G, \bullet)\) is profinite.

**Proof.** Let \((G, f)\) be the inverse limit of the inverse system \:\{\((G_i, f_i), \varphi_{ij}, I\)\} where every \((G_i, f_i)\) is a finite polyadic group. As \(G \neq \emptyset\), we choose an arbitrary element \((v_i) \in G\). We know that all retracts of a polyadic group are isomorphic to each other. So, we consider the retract

\[(G_i, \bullet_i) = \text{ret}_{v_i}(G_i, f_i).\]
By the construction of Sokolov (see [14]), we have
\[ \theta_i(x) = f_i(\tau_i, x, \nu_i), \]
for any \( x \). Also we have
\[ b_i = f_i(\tau_i, \ldots, \tau_i). \]
Using this special form of the retract, we see that the maps \( \varphi_{ij} \) are group homomorphisms as well, because
\[
\varphi_{ij}(x \cdot_i y) = \varphi_{ij}(f_i(x, \nu_i, y)) = f_j(\varphi_{ij}(x), \varphi_{ij}(v_i), \varphi_{ij}(y)) = f_j(\varphi_{ij}(x), \nu_j, \varphi_{ij}(y)) = \varphi_{ij}(x) \cdot_j \varphi_{ij}(y).
\]
Note that, here we use the fact \( \varphi_{ij}(v_i) = v_j \) as we assumed that \( (v_i) \in G \).
This shows that the maps \( \varphi_{ij} \) are in the same time, group homomorphisms and \( \{(G_i, \bullet_i), \varphi_{ij}, I\} \) is an inverse system of finite groups. Obviously, \((G, \bullet)\) is the inverse limit of this system and so it is a profinite group.

Note that in some sense, the inverse of the above theorem is also true: if we consider a profinite group \((G, \bullet)\) together with a continuous automorphism \( \theta \) and an element \( b \) satisfying the requirement of 2.1, then the polyadic group \( \text{der}_{\theta,b}(G, \bullet) \) will be profinite. One may ask also about the automorphism \( \theta \) in the above proof. The above construction shows that, for any \((x_i) \in G\), we have
\[
\theta((x_i)_{i \in I}) = (\theta_i(x_i))_{i \in I}.
\]
As a result of the above theorem, we can see that the Post’s cover of a profinite polyadic group is also profinite.

**Corollary 3.2.** Let \((G, f)\) be profinite and \(G^*\) be the Post’s cover. Then \(G^*\) is also profinite.

**Proof.** We know that there exists a normal subgroup \( K \) of the Post’s cover which has the index \( n - 1 \) and \( K \) is isomorphic to the retract \((G, \bullet)\). Hence, \( K \) is profinite. Now, being a finite extension of a profinite group, \(G^*\) is also profinite. \(\square\)

We are ready now, to give a characterization of the profinite polyadic groups.

**Theorem 3.3.** A polyadic topological group \((G, f)\) is profinite, if and only if, it is compact, Hausdorff, totally disconnected, and for any open congruence \( R \subseteq G \times G \), the quotient polyadic group \((G/R, f_R)\) is finite.
Proof. We already saw that a profinite polyadic group is compact, Hausdorff and totally disconnected. Suppose $R \subseteq G \times G$ is an open congruence of the polyadic group $(G, f)$. In [11], it is proved that in this case, the equivalence relation $R$ is a subgroup of the ordinary group $G \times G$ (the letter $G$ here stands for the group $(G, \bullet)$). This means that $R$ is an open subgroup of the ordinary profinite group $G \times G$, and hence, the quotient $(G \times G)/R$ is finite. Let $[x]_R$ denotes the equivalence class of $x \in G$. So, we define a map

$$\psi : \frac{G}{R} \rightarrow \frac{G \times G}{R}$$

by $\psi([x]_R) = (x, 1)_R$. This map is well-defined as if we suppose $[x]_R = [y]_R$, then $(x, y) \in R$ and so $(x, y) \bullet R = R$. This means that

$$(x, 1)_R = (1, y^{-1})_R,$$

and as in the quotient we have $(1, y^{-1})_R = (y, 1)_R$, so the map is well-defined. Also, it is injective, since if $(x, 1)_R = (y, 1)_R$, then $(x^{-1}y, 1) \in R$, so, we have also

$$(y, x) = (x, x) \bullet (x^{-1}y, 1) \in R.$$  

This shows that $\psi$ is injective and hence the polyadic group $G/R$ is finite.

Conversely, suppose that the polyadic group $(G, f) = \text{der}_{\theta, b}(G, \bullet)$ is compact, Hausdorff, and totally disconnected. Further, suppose for any open congruence $R \subseteq G \times G$, the quotient polyadic group $G/R$ is finite. We prove that the polyadic group is profinite. To do this, we consider the retract $(G, \bullet)$. This group is compact, Hausdorff, and totally disconnected as well.

So, we show that for any open normal subgroup $K \trianglelefteq (G, \bullet)$, the group $G/K$ is finite. This will show that the retract is profinite and hence, according to what we said already, the polyadic group will be profinite. Note that, in general, there is no a simple correspondence between congruences of $(G, f)$ and normal subgroups of $(G, \bullet)$ (see [11]). For this reason, we argue directly:

consider an open normal subgroup $K \trianglelefteq (G, \bullet)$. Let

$$R_0 = \bigcup_{i=0}^{n-1} \theta^i(K \times K).$$

Obviously, $R_0$ is a $\theta$-invariant normal subgroup of $G \times G$. Note that again here we use $G$ for the retract $(G, \bullet)$. This subgroup is open as $K \times K \subseteq R_0$.

Let $R$ be the congruence generated by $R_0$. A simple argument shows that $R$ is also an open subgroup of $G \times G$ (we will need this fact in what follows).

By our assumption, the polyadic group $G/R$ is finite. Define a map

$$\lambda : \bigcup_{i=0}^{n-1} \frac{G}{\theta^i(K)} \rightarrow \frac{G}{R}$$

by

$$\lambda(x\theta^i(K)) = [x]_R.$$
This map is well defined because if $x\theta^i(K) = y\theta^i(K)$, then $x^{-1}y \in \theta^i(K)$ and hence $(x^{-1}y, 1) \in \theta^i(K \times K) \subseteq R$. Since $R$ is a subgroup of $G \times G$, and $(x, x) \in R$, so

$$(y, x) = (x, x)(x^{-1}y, 1) \in R,$$

which means $[x]_R = [y]_R$, proving that the map $\lambda$ is well-defined. We show that this map is $n$ to 1. Suppose $[x]_R = [y]_R$. So $(x, y) \in R$ and again we see that $(x^{-1}y, 1) \in R$. This means that there are elements

$$c_0, c_1, \ldots, c_s$$

such that $c_0 = x^{-1}y$, $c_s = 1$, and for every $0 \leq i \leq s - 1$, we have $(c_i, c_{i+1}) \in R_0$. Starting from $i = s - 1$, we see that $(c_{s-1}, 1) \in \theta^{i_1}(K \times K)$, for some $i_1$. In other words, $(\theta^{-i_1}(c_{s-1}), 1) \in K \times K$. Similarly $(c_{s-2}, c_{s-1}) \in \theta^{i_2}(K \times K)$, for some $i_2$, and hence $(\theta^{-i_2}(c_{s-2}), \theta^{-i_2}(c_{s-1})) \in K \times K$. This shows that $(\theta^{-i_2}(c_{s-2}), 1) \in K \times K$ and the similar statement is true for all smaller indices, especially, $(\theta^{-i_0}(x^{-1}y), 1) \in K \times K$, for some $i_0$. This means that for some $r$, we have $x^{-1}y \in \theta^r(K)$, and hence $x\theta^r(K) = y\theta^r(K)$. Therefore, the map $\lambda$ is $n$ to 1. This proves that $G/K$ is finite and hence $(G, \bullet)$ is profinite. \hfill \Box

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