Abstract

We provide generalizations of a class of stochastic Gronwall inequalities that has been studied by von Renesse and Scheutzow (2010), Scheutzow (2013), Xie and Zhang (2020) and Mehr and Scheutzow (2021). This class of stochastic Gronwall inequalities is a useful tool for SDEs.

Our focus are convex generalizations of the Bihari-LaSalle type. The constants we obtain are sharp. In particular, we provide new sharp constants for the stochastic Gronwall inequalities. The proofs are connected to a domination inequality by Lenglart (1977), an inequality by Pratelli (1976) and a characterization of Lenglart’s concept of domination via the Snell envelope.

The inequalities we study appear for example in connection with exponential moments of solutions to path-dependent SDEs: For non-path-dependent SDEs, criteria for the finiteness of exponential moments are known. To be able to extend these proofs to the path-dependent case, a convex generalization of a stochastic Gronwall inequality seems necessary. Using the results of this paper, we obtain a criterion for the finiteness of exponential moments which is similar to that known for non-path-dependent SDEs.

Stochastic Gronwall inequalities can also be applied to study other types of SDEs than path-dependent SDEs: An estimate of this paper is applied by Agresti and Veraar (2023) to prove global well-posedness for reaction-diffusion systems with transport noise.

Keywords: stochastic Gronwall inequality, stochastic Bihari-LaSalle inequality, Lenglart’s domination inequality, Snell envelope, sharp constants, exponential moments of path-dependent SDEs

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1 Introduction

In this article we provide sharp generalizations of two types of stochastic Gronwall inequalities. In particular, we establish new sharp constants for the stochastic Gronwall inequalities we generalize. The inequalities we study appear for example in connection with path-dependent SDEs: For non-path-dependent SDEs, criteria for the finiteness of exponential moments are known. To be able to extend these results to the path-dependent case, a convex generalization of a stochastic Gronwall inequality seems necessary. Using the results of this paper, we obtain a criterion for the finiteness of exponential moments that complements the criterion known for non-path-dependent SDEs. Stochastic Gronwall inequalities can also be applied to study other types of SDEs than path-dependent SDEs: An estimate of this paper is applied by Agresti and Verea [2] to prove global well-posedness for reaction-diffusion systems with transport noise.

Results in the literature on stochastic Gronwall inequalities: The following stochastic Gronwall inequality with supremum is due to von Renesse and Scheutzow [40, Lemma 5.4] and was generalized by Mehr and Scheutzow [30, Theorem 2.1]: Let $(X_t)_{t \geq 0}$ be a non-negative stochastic process that satisfies

$$X_t \leq \int_{(0,t]} X_s^* \, dA_s + M_t + H_t \quad \text{for all } t \geq 0,$$

where $X^*_t = \sup_{s \leq t} X_s$ denotes the running supremum. Here, $M$ is a càdlàg local martingale that starts in 0, and $H$ and $A$ are suitable non-decreasing stochastic processes. Then, for all $T > 0$ and $p \in (0,1)$ there exists an explicit upper bound for $\mathbb{E}[\sup_{t \in [0,T]} X^p_t]$ which does not depend on the local martingale $M$.

There is also a stochastic Gronwall inequality without supremum which is closely connected to the previous inequality with supremum. This result is due to Scheutzow [37, Theorem 4] and was generalized by Xie and Zhang [42, Lemma 3.7] from continuous local martingales to càdlàg local martingales: If we assume instead of (1) the slightly stronger assumption that

$$X_t \leq \int_{(0,t]} X_s - dA_s + M_t + H_t \quad \text{for all } t \geq 0,$$

sharper bounds can be obtained.

Both previously mentioned inequalities are useful tools for SDEs. The stochastic Gronwall inequality with supremum is applied to study SDEs with memory, see for example [4], [5], [6], [7], [20], [30], [38] and [40]. The stochastic Gronwall inequality without supremum is applied to study various SDEs without memory, see e.g. [13], [17], [18], [21], [25], [29], [35], [36], [39], [41] and [43].

Also other stochastic Gronwall inequalities have been studied, see e.g. Glatt-Holtz and Ziane [16, Lemma 5.3] and Agresti and Verea [1, Lemma A.1].

Results in the literature on generalizations of stochastic Gronwall inequalities: Also nonlinear extensions of the stochastic Gronwall inequalities have been studied. Makasu [28, Theorem 2.2] and Le and Ling [26, Lemma 3.8] studied a generalization where in the assumption the term $\int_{(0,t]} X_s \, dA_s$ is replaced by $(\int_{(0,t]} (X_s^\theta \, dA_s)^{1/\theta})^{\theta}$ for $\theta > 0$. Under similar assumptions as above, estimates for $\mathbb{E}[\sup_{t \in [0,T]} X^p_t]$ were obtained.

A further extension of the stochastic Gronwall inequalities has been studied by Mekki, Nieto and Ouahab [31, Theorem 2.4]: For continuous local martingales a so-called stochastic Henry Gronwall’s inequality with upper bounds that do not depend on the local martingale $M$ can be proven.

In addition, also extensions of the stochastic Gronwall inequalities have been studied, where the upper bounds depend on the quadratic variation of the martingale $M$, see e.g. Makasu [28], Makasu [27] and Mekki, Nieto and Ouahab [31]. In the present paper, we focus on bounds which do not depend on the local martingale $M$. Furthermore, Hudde, Hutzenhaller and Mazzonetto [19] have extended the stochastic
Gronwall inequality without supremum to the setting of Itô processes which satisfy a suitable one-sided affine-linear growth condition.

In this paper, we study the following generalization of the above mentioned stochastic Gronwall inequalities: We replace the assumptions (1) and (2) by

\[ X_t \leq \int_{(0,t]} \eta(X_{s-}) dA_s + M_t + H_t \quad \text{for all } t \geq 0, \quad (3) \]

and

\[ X_t \leq \int_{(0,t]} \eta(X_{s-}) dA_s + M_t + H_t \quad \text{for all } t \geq 0, \quad (4) \]

respectively, where \( \eta : [0, \infty) \to [0, \infty) \) is a convex non-decreasing function.

For continuous local martingales and \( \eta \) that satisfy \( \int_0^\infty \frac{du}{\eta(u)} = +\infty \), (3) is studied by von Renesse and Scheutzow [40, Lemma 5.1] in the context of global solutions of stochastic functional differential equations. For càdlàg martingales and concave \( \eta \), and \( \eta \) that satisfy either \( \int_0^1 \frac{du}{\eta(u)} = +\infty \) or \( \int_1^\infty \frac{du}{\eta(u)} = +\infty \), these inequalities are studied in [14].

The Bihari-LaSalle inequality provides an upper bound for \( X_t \) in the deterministic case (i.e. \( M \equiv 0 \)), see [8], [22]. More general versions than Lemma 1.1 are known in the literature, see also e.g. [14, Lemma 1.1] for a version which allows \( x \) and \( A \) to be càdlàg.

**Lemma 1.1** (Deterministic Bihari-LaSalle inequality). \( \) Let \( x, \varphi : [0, \infty) \to [0, \infty) \) be positive continuous functions and \( T, H \geq 0 \) constants. Further, let \( \eta : [0, \infty) \to [0, \infty) \) be a non-decreasing continuous function and set \( A(t) := \int_0^t \varphi(u) du \) for all \( t \in [0, T] \). Then, the inequality

\[ x(t) \leq \int_0^t \eta(x(s)) dA(s) + H \quad \forall t \in [0, T] \quad (5) \]

implies the inequality

\[ x(t) \leq G^{-1}(G(H) + A_t) \quad \forall t \in [0, T']. \]

where \( G(u) := \int_0^u \frac{du}{\eta(u)} \) for some constant \( c > 0 \). Here, \( T' \) is chosen small enough such that \( G(H) + A_t \in \text{domain}(G^{-1}) \) is ensured for all \( t \in [0, T'] \).

Note that the upper bound on \( x \) does not depend on the choice of the constant \( c > 0 \) used in the definition of \( G \). We obtain the well-known Gronwall inequality by choosing \( c = 1 \) and \( \eta(u) \equiv u \) in Lemma 1.1 so that \( G(u) \equiv \log(u) \), which implies the upper bound

\[ x(t) \leq H \exp(A(t)). \]

In [14, Theorem 3.1] it is shown for (3) and (4), concave \( \eta \) and deterministic \( A \) that inequalities of the form

\[ \mathbb{E}[(X_T^+)^{1/p}] \leq \tilde{c}_p G^{-1}(G(\bar{c}_p M_T^{1/p}) + \bar{c}_p A_T) \quad \forall T > 0 \quad (6) \]

hold true, where \( G \) is defined as in Lemma 1.1 and \( \tilde{c}_p, \bar{c}_p \) are constants which only depend on \( p \in (0, 1) \).

**Results of this article:** For the stochastic Bihari-LaSalle assumptions (3) and (4) with convex \( \eta \), estimates of the form (6) do not hold true in general: Let for example \( Y \) be a Cox-Ingersoll-Ross process and set \( X = \exp(A Y) \) for some \( \lambda > 0 \). For suitably chosen parameters, \( X \) satisfies (4) for some convex \( \eta \) and \( \|X_T^+\|_p \) explodes at a finite time (see e.g. [3, Proposition 3.1] or [11, Proposition 3.2]). Alternatively, see Counterexample 7.2 for a simple counterexample.

Instead, the following types of estimates can be shown for example for stochastic Bihari-LaSalle assumption without supremum (4), see Theorem 3.1 for the complete theorem:

**Theorem 1.2** (Convex stochastic Bihari-LaSalle inequality without supremum). \( \) Let

- \( (X_t)_{t \geq 0} \) be an adapted càdlàg process with \( X \geq 0 \),
- \( (A_t)_{t \geq 0} \) be a predictable non-decreasing càdlàg process with \( A_0 = 0 \),
\textbf{Corollary 6.7} Lemma 4.8 can also be applied to prove concave.

Theorem 3.1 by extending an inequality by Lemma 4.5.

complements.

Table 2

Then, for all $T \geq 0$, $p \in (0, 1)$ and $u \geq c_0$ and $w, R > 0$ the expressions $G^{-1}(G(X^*_T) - \varepsilon_p A_T)$ and $G^{-1}(G(u) - R)$ are well-defined and the following estimates hold:

$$
\|G^{-1}(G(X^*_T) - \varepsilon_p A_T)\|_p \leq \varepsilon_p \|H_T\|_p,
$$

$$
P \left[ \sup_{t \in [0,T]} X_t > u \right] \leq \frac{\mathbb{E}[H_T \wedge w]}{G^{-1}(G(u) - R)} + P[H_T \geq w] + P[A_T > R],
$$

where $\varepsilon_p := 1$ and $\varepsilon_p := (1 - p)^{-1/p} \varepsilon^{-1}$.

In particular, for the stochastic Gronwall inequality without supremum (i.e. $\eta(x) = x$), we obtain the new estimate

$$
P \left[ \sup_{t \in [0,T]} X_t > u \right] \leq \frac{e^{R}}{u} \mathbb{E}[H_T \wedge w] + P[H_T \geq w] + P[A_T > R].
$$

Estimate (9) complements [37, Theorem 4] and [42, Lemma 3.7], where under similar assumptions upper bounds for $\mathbb{E}[\sup_{t \in [0,T]} X^*_t]$ for $p \in (0, 1)$ are proven. The formulation (9) is useful when $A$ or $H$ do not have suitable integrability properties to satisfy the assumptions of [37, Theorem 4] and [42, Lemma 3.7].

A main result of this paper is that for the stochastic Bihari-LaSalle inequality with supremum (3) we also obtain a bound of the form (7) (see Theorem 3.1). Another main result of this paper is the sharpness of the estimates and constants. In particular, we show that for (3) the estimate (8) does not hold true in general. As an application of the sharpness results, we obtain that the tail behaviour of path-dependent and non-path-dependent SDEs differ, see Corollary 6.7.

An overview of the stochastic Bihari-LaSalle inequalities obtained can be found in Table 2, see Section 2.3.

We prove the stochastic Bihari-LaSalle inequalities (Theorem 3.1) by extending an inequality by Lenglart [24, Théorème I] and using a characterization of Lenglart’s concept of domination by the Snell envelope. The main lemmas of this paper (Lemma 4.5, Lemma 4.8) can also be applied to prove concave and other generalizations of the stochastic Gronwall inequalities, see [14] or Table 2.

Moreover, in addition, our proofs are inspired by the proof of an inequality by Pratelli [33, Proposition 1.2] and by the proofs of the stochastic Gronwall inequalities by Mehri and Scheutzow [30, Theorem 2.1] and Xie and Zhang [42, Lemma 3.7].

**Example where a stochastic Gronwall inequality instead of the deterministic Gronwall inequality is needed:** Deterministic Gronwall inequalities are widely used to study SDEs. We provide a simple toy example when stochastic generalizations are useful to shorten calculations, for a proper application example see e.g. [2, Lemma 5.3, Lemma 6.7], in which energy estimates for reaction-diffusion systems with transport noise are proven, which in turn are applied to obtain global well-posedness. For a toy example assume $(\tilde{X}_t)_{t \geq 0}$ to be a global solution of the following $d$-dimensional (not path-dependent) SDE driven by an $m$-dimensional Brownian motion $B$

$$
d\tilde{X}_t = f(t, \tilde{X}_t)dt + g(t, \tilde{X}_t)dB_t \quad \forall t \geq 0,
$$

and assume that the random coefficients $f$ and $g$ satisfy the one-sided coercivity assumption $2(\tilde{f}(t, x), x) + \|\tilde{g}(t, x)\|_F^2 \leq K_1(x)^2$ for all $x \in \mathbb{R}^d$, $t \geq 0$ where $(K_1)_{t \geq 0}$ is an adapted non-negative stochastic process. Then, by Itô’s formula we obtain

$$
|\tilde{X}_t|^2 \leq |\tilde{X}_0|^2 + \int_0^t |\tilde{X}_s|^2 K_s ds + 2 \int_0^t \langle \tilde{X}_s, \tilde{g}(t, \tilde{X}_t)dB_s \rangle \quad \forall t \geq 0.
$$
If \((K_t)_{t \geq 0}\) is random, it is not possible to take expectations and then directly apply the Gronwall inequality to \(t \mapsto \mathbb{E}[|\tilde{X}_t|^2]\). In this case stochastic Gronwall inequalities are useful, as they can be applied directly. Moreover, without further work they immediately give estimates for the running supremum of \((\tilde{X}_t)_{t \geq 0}\). For example by (9) (see Corollary 5.4) we have for all \(u > 0, w > 0\) and \(R > 0\)

\[
\mathbb{P}\left[ \sup_{s \in [0,t]} |\tilde{X}_s|^2 > u \right] \leq \frac{\alpha R}{u} \mathbb{E}[|X_0^2| \wedge w] + \mathbb{P}[|X_0^2| \geq w] + \mathbb{P}\left[ \int_0^t K_s > R \right].
\]

**Application example of the convex Bihari-LaSalle inequality:** The convex stochastic Bihari-LaSalle inequality with supremum is a useful tool to derive exponential moment estimates for path-dependent SDEs. For non-path-dependent SDEs, these types of estimates are known, see e.g. Cox, Hutzenthaler and Jentzen [10], Hudde, Hutzenthaler and Mazzonetto [19] and the references therein. We sketch a simple case of [10, Corollary 2.4] or [19, Corollary 3.3]. Afterwards, we shortly explain why our convex stochastic Bihari-LaSalle inequality succeeds in extending these types of results to path-dependent SDEs.

Consider the following \(d\)-dimensional (not path-dependent) SDE driven by an \(m\)-dimensional Brownian motion \(B\)

\[
d\tilde{X}_t = \tilde{f}(t, \tilde{X}_t)dt + \tilde{g}(t, \tilde{X}_t)dB_t, \quad \tilde{X}_0 = x_0 \in \mathbb{R}^d,
\]

and assume suitable measurability and integrability assumptions on the coefficients \(\tilde{f}\) and \(\tilde{g}\). Fix some \(U \in C^2(\mathbb{R}^d, \mathbb{R}), \) and assume that the coefficients \(\tilde{f}\) and \(\tilde{g}\) satisfy

\[
\tilde{g}_{\tilde{f}, \tilde{g}} U(x) + \frac{1}{2\alpha^2} (\nabla U(x))^T \tilde{g}(t, x) \tilde{g}(t, x) \leq \alpha U(x) \quad \forall t \geq 0, \quad \forall x \in \mathbb{R},
\]

(10) for some \(\alpha > 0\). Here \(\tilde{g}_{\tilde{f}, \tilde{g}}\) denotes the infinitesimal generator of the SDE. Then, [19, Corollary 3.3] (see also [10, Corollary 2.4]) implies for any (say global) solution \(\tilde{X}\) that for all \(p \in (0,1)\)

\[
\mathbb{E}[\sup_{t \in [0,T]} \exp(pU(\tilde{X}_t)e^{-\alpha t})] \leq c(p)\mathbb{E}[\exp(pU(x_0))],
\]

where \(c(p)\) is a constant that only depends on \(p\). This can be proven by applying Itô’s formula to compute \(\tilde{Z}_t := \exp(U(\tilde{X}_t)e^{-\alpha t})\) and then applying the assumption (10):

\[
d\tilde{Z}_t = \tilde{Z}_t e^{-\alpha t}\left[\tilde{g}_{\tilde{f}, \tilde{g}} U(t, x) - \alpha U(t, x) + \frac{1}{2\alpha^2} (\nabla U(x))^T \tilde{g}(t, x) \tilde{g}(t, x)\right] dt + \tilde{Z}_t e^{-\alpha t} \nabla U(t, x) \tilde{g}(t, x) dB_t
\]

(11)

By Fatou’s lemma, we have for all bounded stopping times \(\tau\) the inequality \(\mathbb{E}[\exp(U(\tilde{X}_\tau)e^{-\alpha \tau})] \leq \mathbb{E}[\exp(U(x_0))]\). This implies e.g. by Lenglart’s domination inequality (see e.g. Lemma 4.2) the claim.

This argument does not work for path-dependent SDEs. Let \(r > 0\) be some constant and \(x_0 \in C([-r,0]; \mathbb{R}^d)\) the initial value. Consider

\[
dX_t = f(t, X_{-r:t})dt + g(t, X_{-r:t})dB_t, \quad \forall t > 0
\]

\[
X_t = x_0(t) \quad \forall t \in (-r,0],
\]

where we use the notation \(X_{-r:t}\) for the path segment \(\{X(s), s \in [-r,t]\}\). (For details on the assumptions on the coefficients see Section 6.) Correspondingly, we define for all \(U \in C^2(\mathbb{R}^d, \mathbb{R}), x \in C([-r, \infty); \mathbb{R}^d)\) and all \(t \geq 0\)

\[
(G_{\tilde{f}, \tilde{g}} U)(t, x_{-r:t}) := (\nabla U(t))^T f(t, x_{-r:t}) + \frac{1}{2} \text{trace}(g(t, x_{-r:t})g(t, x_{-r:t})^T (\text{Hess} U)(x(t))).
\]

It seems reasonable to weaken the condition (10) to

\[
(G_{\tilde{f}, \tilde{g}} U)(t, x_{-r:t}) + \frac{1}{2\alpha^2} (\nabla U(x(t)))^T g(t, x_{-r:t}) g(t, x_{-r:t}) \leq \alpha \sup_{s \in [-r,t]} U(x(s)) \quad \forall t \in C([-r, \infty); \mathbb{R}^d), \forall t \geq 0,
\]

(12)

as the terms on the left-hand side depend on \(x_{-r:t}\), not only \(x(t)\). However, when computing \(Z_t := \exp(U(t, X_t)e^{-\alpha t})\), the terms in the integral w.r.t \(dt\) fail to cancel out after applying (12) due to the supremum in our condition on the coefficients:

\[
dZ_t = Z_t e^{-\alpha t}[(G_{\tilde{f}, \tilde{g}} U)(t, X_{-r:t}) - \alpha U(X_t) + \frac{1}{2\alpha^2} (\nabla U(X_t))^T g(t, X_{-r:t}) g(t, X_{-r:t})] dt + dM_t
\]

\[
\leq \alpha Z_t e^{-\alpha t} \sup_{s \in [-r,t]} (U(X_s) - U(X_t)) dt + dM_t
\]
where \( M_t := Z_t e^{-\alpha t} \nabla U(X_t) \nabla (g(t, X_{-t,t}) dB_t \). Assuming in addition that \( U \geq 0 \) and noting that \( U(X) = \log(Z_t) e^{\alpha t} \), we obtain a convex stochastic Bihari-LaSalle inequality for \( Z_t \) and \( \eta(x) := \alpha x \log(x) + \sup_{u \in [-r, 0]} U(x_u(s)) \)

\[
\text{d}Z_t \leq \eta(Z_t^*) \text{d}t + M_t, \quad \forall t \in [0, T].
\]

Applying our main theorem (a stochastic Bihari-LaSalle inequality) gives us a similar estimate as in the non-path-dependent case. We generalize and improve this approach in Section 6.

## 2 Notation, assumptions and overview

We assume that all processes are defined on an underlying filtered probability space \((\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t \geq 0})\) satisfying the usual conditions, i.e. which is complete and right-continuous.

### 2.1 Assumptions

We study the following two cases:

**Definition 2.1** (Assumption \( \mathcal{A}_{\text{sup}} \)). Let

- \((X_t)_{t \geq 0}\) be an adapted right-continuous process with \( X \geq c_0 \) for some \( c_0 \geq 0 \),
- \((A_t)_{t \geq 0}\) be a predictable non-decreasing càdlàg process with \( A_0 = 0 \),
- \((H_t)_{t \geq 0}\) be an adapted non-negative non-decreasing càdlàg process,
- \((M_t)_{t \geq 0}\) be a càdlàg local martingale with \( M_0 = 0 \),
- \( \eta : [c_0, \infty) \to [0, \infty) \) be a continuous non-decreasing function with \( \eta(x) > 0 \) for all \( x > c_0 \).

We say the processes \((X, A, H, M)\) satisfy \( \mathcal{A}_{\text{sup}} \) if they satisfy the inequality below for all \( t \geq 0 \):

\[
X_t \leq \int_{[0, t]} \eta(X_s^*) \text{d}A_s + M_t + H_t \quad \mathbb{P}\text{-a.s.,}
\]

where \( X_{s}^* = \sup_{u \leq s} X_u \).

The following assumption is slightly stronger:

**Definition 2.2** (Assumption \( \mathcal{A}_{\text{nosup}} \)). Under the same assumptions on the processes as in the previous definition, we say that the processes satisfy \( \mathcal{A}_{\text{nosup}} \) if in addition \( X \) has left limits and the processes satisfy the following inequality for all \( t \geq 0 \):

\[
X_t \leq \int_{[0, t]} \eta(X_s^*) \text{d}A_s + M_t + H_t \quad \mathbb{P}\text{-a.s.}
\]

We also use the two definitions above for processes defined on a finite time interval \([0, T]\) and correspondingly adapt the definition in this case.

### 2.2 Notation and constants

**Constants**: For \( p \in (0, 1) \) define the following constants:

\[
\beta = (1 - p)^{-1}, \quad \alpha_1 = (1 - p)^{-1/p}, \quad \alpha_2 = p^{-1}.
\]

**Quasinorms**: We denote by \(| \cdot |\) the Euclidean norm and \(| \cdot |_F\) the Frobenius norm. Let \( Y \) be a random variable. We use for \( p \in (0, 1] \) the notation (if well-defined)

\[
\mathbb{E}_{\mathcal{F}_0}[Y] := \mathbb{E}[|Y| | \mathcal{F}_0], \quad ||Y||_p := \mathbb{E}[|Y|^p]^{1/p}, \quad ||Y||_{p, \mathcal{F}_0} := \mathbb{E}[|Y|^p | \mathcal{F}_0]^{1/p}.
\]

**Running supremum**: Let \( X \) be a non-negative stochastic process with right-continuous paths. We use the following notation for the running supremum and its left limits:

\[
X_{t}^* := \sup_{0 \leq s \leq t} X_s \quad \forall t \geq 0,
\]

\[
X_{t}^- := \lim_{s \uparrow t} X_{s}^* = \sup_{s < t} X_s \quad \forall t > 0.
\]
As usual, we set \( X^+_t := X_0 \). If \( X \) is càdlàg, then also \( X^+_t = \sup_{s \leq t} X_s^- \) holds true. If \( X \) is only right-continuous then \( X^+_t \) and \( X^-_t \) take values in \([0, +\infty)\).

**Functions:** For \( \eta : [c_0, \infty) \to [0, \infty) \) from Assumption \( A_{\sup} \) or Assumption \( A_{\no\sup} \) we choose some \( c > c_0 \) and define the following functions for \( p \in (0, 1) \).

\[
G(x) := \int_c^x \frac{du}{\eta(u)} \quad \forall x \in [c_0, \infty),
\]

(16)

\[
\eta_p(x) := \frac{p}{1 - p} \eta(x^{1/p}) x^{1-1/p} \quad \forall x \in [c_0^p, \infty) \cap (0, \infty),
\]

(17)

\[
\tilde{G}_p(x) := \int_c^x \frac{du}{\eta_p(u)} \quad \forall x \in [c_0^p, \infty) \cap (0, \infty).
\]

(18)

The functions have the following properties:

- The function \( G(c_0) \in [-\infty, 0) \). Moreover, \( G \) is increasing and concave. In particular, it has a well-defined increasing inverse \( G^{-1} : \text{range}(G) \cap (-\infty, \infty) \to [c_0, \infty) \). If \( G(c_0) = -\infty \), then we set \( G^{-1}(G(c_0) + a) := c_0 \) for \( a \in (-\infty, \infty) \). If \( \eta \) is continuous, then \( G \) is continuously differentiable on \((c_0, \infty)\).

- For any \( h \geq 0 \) and any \( a \in \mathbb{R} \) such that \( \int_{c_0}^h \frac{du}{\eta(u)} < a < \int_{c_0}^{\infty} \frac{du}{\eta(u)} \) the expression \( G^{-1}(G(h) + a) \) is well-defined and does not depend on the choice of \( c \in [c_0, \infty) \) used in the definition of \( G \). In particular, the upper bound given in the Bihari-LaSalle inequality Lemma 1.1 does not depend on the choice of \( c \).

- The functions \( G \) and \( \tilde{G}_p \) satisfy for all \( x \in (c_0^p, \infty) \)

\[
\tilde{G}_p(x) = \frac{1 - p}{p} \int_{c^p}^x \frac{du}{\eta_p(u)} = (1 - p) \int_{c^p}^{x^{1/p}} \frac{dv}{\eta(v)} = (1 - p)G(x^{1/p})
\]

(19)

and for all \( x \in \text{domain}(\tilde{G}_p^{-1}) \)

\[
\tilde{G}_p^{-1}(x) = (G^{-1}(\frac{x}{1-p}))^p.
\]

(20)

### 2.3 Overview of the results

The following two tables summarize some of the results in the literature and the results of this paper. The first table contains the stochastic Gronwall inequalities, the second table contains generalizations of Bihari-LaSalle type. The constant \( \alpha_1 \alpha_2 \) is the sharp constant from Lenglart’s inequality, \( \alpha_2 \) is the sharp constant from a monotone version of Lenglart’s inequality [15].
\begin{table}[h]
\centering
\begin{tabular}{|c|c|}
\hline
Assumption $\mathcal{A}_{\alpha \sup}$, $\eta(x) \equiv x$ & Assumption $\mathcal{A}_{\sup}$, $\eta(x) \equiv x$ \\
\hline
\text{(Special case of $\mathcal{A}_{\sup}$)} & von Renesse and Scheutzow \cite[Lemma 5.4]{RenesseScheutzow}, Mehri and Scheutzow \cite[Theorem 2.1]{MehriScheutzow}: \\
\hline
\text{A deterministic, $p \in (0, 1)$} & $H$ predictable or $\Delta M \geq 0$: \\
\hline
 & \begin{itemize}
\item $H$ predictable or $\Delta M \geq 0$:
\begin{align*}
\|X_T^p\|_p & \leq \alpha_1 \alpha_2 \|H_T\|_p e^{\Delta M T} \\
\mathbb{P}[X_T^p > u] & \leq e^{\Delta M T} \mathbb{E}[H_T \land w] + \mathbb{P}[H_T \geq w]
\end{align*}
See Corollary 5.4
\end{itemize}
\hline
\text{A random, $0 < q < p < 1$} & $H$ predictable or $\Delta M \geq 0$:
\begin{itemize}
\item $\|X_T^p\|_q \leq \alpha_1 \alpha_2 \|H_T\|_p e^{\Delta M T} \|_{qp/(p-q)}$
\item $\mathbb{P}[X_T^p > u] \leq e^{\Delta M T} \mathbb{E}[H_T \land w] + \mathbb{P}[H_T \geq w] + \mathbb{P}[\Delta T > R]
\end{itemize}
See Corollary 5.4
\hline
\end{tabular}
\caption{Summary of stochastic Gronwall inequalities (i.e. $\eta(x) \equiv x$)}
\end{table}
Table 2: Summary of stochastic Bihari-LaSalle inequalities obtained in [40], [14] and this paper

| Assumption $\mathcal{A}_{\sup}$ (Special case of $\mathcal{A}_{\sup}$) | Assumption $\mathcal{A}_{\sup}$ |
|---------------------------------------------------------------------|---------------------------------|
| $\eta$ random, $A$ deterministic, $p \in (0, 1)$                    | $\eta$ random, $A$ deterministic, $p \in (0, 1)$ |
| $H$ predictable or $\Delta M \geq 0$: $\frac{d}{dt} \mathbb{E}X_T^p \leq \alpha_1 G^{-1}(G(\alpha_2 \mathbb{E}H_T^p) + A_T)$ |
| $\mathbb{E}[H_T] < \infty$: $\frac{d}{dt} \mathbb{E}X_T^p \leq \alpha_1 G^{-1}(G(\mathbb{E}H_T^p) + A_T)$ |
| $\mathbb{P}[X_T^p > u] \leq \frac{G^{-1}(G(\mathbb{E}H_T^p) + A_T)}{G^{-1}(G(u) - R)} + \mathbb{P}[A_T > R]$ |
| $\mathbb{E}[H_T] < \infty$: $\frac{d}{dt} \mathbb{E}X_T^p \leq \alpha_1 \mathbb{E}H_T^p$ |
| $\mathbb{P}[X_T^p > u] \leq \frac{G^{-1}(G(\mathbb{E}H_T^p))}{G^{-1}(G(u) - R)}$ |
| $\mathbb{E}[H_T] < \infty$: $\frac{d}{dt} \mathbb{E}X_T^p \leq \alpha_1 \mathbb{E}H_T^p$ |
| $\mathbb{P}[X_T^p > u] \leq \frac{G^{-1}(G(\mathbb{E}H_T^p))}{G^{-1}(G(u) - R)}$ |

| Constants | $\beta = (1 - p)^{-1}$, $\alpha_1 = (1 - p)^{-1/p}$, $\alpha_2 = p^{-1}$ |
| Notation | $\|Y\|_p := \mathbb{E}[|Y|^p]^{1/p}$ for random variables $Y$, $G(x) := \int_c^x \frac{du}{\eta(u)}$, $\forall x \geq c_0$ |

3 Main results

In this section we provide generalizations of stochastic Gronwall inequalities and study the sharpness of the constants and estimates. The proofs are contained in Section 4. In Section 5 we compare the results of this paper with the literature, and in particular formulate the results for the Gronwall case $\eta(x) \equiv x$.

Recall the following definition from (15) for $p \in (0, 1)$:

$$\beta = (1 - p)^{-1}, \quad \alpha_1 = (1 - p)^{-1/p}, \quad \alpha_2 = p^{-1}.$$

### 3.1 Stochastic Bihari-LaSalle inequalities for convex $\eta$

For Assumption $\mathcal{A}_{\sup}$ or Assumption $\mathcal{A}_{\sup}$, concave $\eta$, deterministic $(A_t)_{t \geq 0}$ and suitable additional assumptions, estimates of the type

$$\|X_T^p\|_p \leq \tilde{c}_1 G^{-1}(G(\tilde{c}_2 \mathbb{E}H_T^p) + \tilde{c}_3 A_T)$$

for all $T \geq 0$

can be shown for $p \in (0, 1)$, see [14]. Here, $\tilde{c}_1$, $\tilde{c}_2$ and $\tilde{c}_3$ denote constants that only depend on $p$ and

$$G(x) := \int_c^x \frac{du}{\eta(u)}$$

is the function from the deterministic Bihari-LaSalle inequality Lemma 1.1. Hence, in this case estimates with a similar structure as in the deterministic case are possible. However, for convex $\eta$, even when $(H_t)_t \equiv H$ is a constant and $A_t \equiv t$, estimates of the type

$$\|X_T^p\|_p \leq c_1 G^{-1}(c_2 G(c_3 H) + c_4 A_T)$$  (21)
(where $c_1, c_2, c_3$ and $c_4$ are constants which depend only on $p \in (0, 1)$) are in general false: For processes $X$ which satisfy Assumption $A_{n0\sup}$ or Assumption $A_{\sup}$ for convex $\eta$, the quantity $\|X_t^\eta\|_p$ may explode at finite time. This type of behaviour cannot be captured by a bound of the type (21), for a counterexample see Section 1 or Counterexample 7.2.

However, the estimate of the deterministic Bihari-LaSalle inequality (see e.g. Lemma 1.1) can be rearranged to

$$G^{-1}(G(x(t)) - A(t)) \leq H.$$ 

This rearranged inequality can be generalized to the stochastic case for convex $\eta$. The following theorem can be used to study the finiteness of exponential moments of path-dependent SDEs, see Section 1 and Section 6.

**Theorem 3.1** (A sharp stochastic Bihari-LaSalle inequality for convex $\eta$).

a) Let $(X, A, H, M)$ and $\eta$ satisfy Assumption $A_{\sup}$ and assume that $\eta_p \equiv \frac{\mathbb{P}}{\mathbb{P}}\eta(x^{1/p})x^{1-1/p}$ (defined in (17)) is convex and $C^1$, and $\eta_p(c_0^\eta) := \lim_{x \to c_0^\eta} \eta_p(x) = 0$. Then, for all $p \in (0, 1)$ and $T \geq 0$, the expression $G^{-1}(G(X_t^\eta) - \beta A_t)$ is well-defined and the following estimates hold:

$$\left\|G^{-1}(G(X_t^\eta) - \beta A_t)\right\|_{p, \mathcal{F}_0} \leq \begin{cases} 
\alpha_1 \alpha_2 \|H_t\|_{p, \mathcal{F}_0} & \text{if } \mathbb{E}[H_p^p] < \infty \text{ and } H \text{ is predictable}, \\
\alpha_1 \alpha_2 \|H_t\|_{p, \mathcal{F}_0} & \text{if } \mathbb{E}[H_p^p] < \infty \text{ and } \Delta M \geq 0, \\
\alpha_1 \|H_t\|_{1, \mathcal{F}_0} & \text{if } \mathbb{E}[H] < \infty,
\end{cases}$$

where $\beta := (1 - p)^{-1}$, $\alpha_1 := (1 - p)^{-1/p}$ and $\alpha_2 := p^{-1}$. We use the notation $(\Delta M_t)_{t \geq 0} := (M_t - M_{t-})_{t \geq 0}$ and $\|Y\|_{p, \mathcal{F}_0} := \mathbb{E}[|Y|^p | \mathcal{F}_0]^{1/p}$ for random variables $Y$.

b) Let $(X, A, H, M)$ and $\eta$ satisfy Assumption $A_{n0\sup}$ and assume that $\eta$ is convex and $C^1$ and $\eta(c_0) = 0$. Then, for all $p \in (0, 1), T \geq 0, t \in [0, T]$ and $u, w > 0$, the expressions $G^{-1}(G(X_t^\eta) - A_t)$ and $G^{-1}(G(X_t^\eta) - A_t)$ are well-defined and the following estimates hold:

$$\left\|G^{-1}(G(X_t^\eta) - A_t)\right\|_{p, \mathcal{F}_0} \leq \left\| \sup_{t \in [0, T]} G^{-1}(G(X_t^\eta) - A_t) \right\|_{p, \mathcal{F}_0} \leq \begin{cases} 
\alpha_1 \alpha_2 \|H_t\|_{p, \mathcal{F}_0} & \text{if } \mathbb{E}[H_p^p] < \infty \text{ and } H \text{ is predictable}, \\
\alpha_1 \alpha_2 \|H_t\|_{p, \mathcal{F}_0} & \text{if } \mathbb{E}[H_p^p] < \infty \text{ and } \Delta M \geq 0, \\
\alpha_1 \|H_t\|_{1, \mathcal{F}_0} & \text{if } \mathbb{E}[H] < \infty.
\end{cases}$$

and

$$\mathbb{P}\left[ \sup_{t \in [0, T]} G^{-1}(G(X_t^\eta) - A_t) > u \left| \mathcal{F}_0 \right. \right] \leq \begin{cases} 
\frac{1}{u} \mathbb{E} \left[ H_t \wedge w \right] + \mathbb{P}[H_t \geq w | \mathcal{F}_0] & \text{if } H \text{ is predictable or } \Delta M \geq 0, \\
\frac{1}{u} \mathbb{E} \left[ H_t \wedge u \right] & \text{if } \mathbb{E}[H] < \infty.
\end{cases}$$

The latter can be reformulated as follows: For all $u > c_0, w, R > 0$ we have:

$$\mathbb{P}\left[ \sup_{t \in [0, T]} \mathbb{E} \left[ H_t \wedge u \right] \right] \leq \begin{cases} 
\frac{\mathbb{E} \left[ H_t \wedge u \right]}{\mathbb{E} \left[ H_t \wedge w \right]} + \mathbb{P}[H_t \geq w | \mathcal{F}_0] + \mathbb{P}[A_t > R | \mathcal{F}_0] & \text{if } H \text{ is predictable or } \Delta M \geq 0, \\
\left( \frac{\mathbb{E} \left[ H_t \wedge 0 \right]}{\mathbb{E} \left[ H_t \wedge 0 \right]} \right) \wedge 1 + \mathbb{P}[A_t > R | \mathcal{F}_0] & \text{if } \mathbb{E}[H] < \infty.
\end{cases}$$

We prove Theorem 3.1 b) by further developing the proof idea of Xie and Zhang [42, Lemma 3.7]. Theorem 3.1 a) is more difficult to prove and requires new techniques.

The constants $\alpha_1, \alpha_1 \alpha_2$ and $\beta$ are sharp, for details see Section 3.2. Under the stronger assumption $A_{n0\sup}$ (23) provides an upper bound for the weak $L^1$ norm, Theorem 3.8 shows that under $A_{\sup}$ the weak $L^1$ norm may be infinite.

**Remark 3.2** (On the relation between the convexity of $\eta_p$ and $\eta$). Assume that $c_0 = 0$ and $\eta_p(0) = 0$. Then, convexity of $\eta_p$ implies that $\eta$ is convex: If $\eta_p$ is convex, then it is almost everywhere differentiable. For any $x > 0$ in which $\eta_p$ is differentiable, we have for $y = x^{1/p}$

$$\eta(y) = \eta_p(y^p) y^{1-1/p} \text{ and } \eta'(y) = mp_p'(y^p) + (1-p)\eta_p(y^p) y^{-p}.$$
Due to $\eta_p$ being convex and $\eta_p(0) = 0$, we have that $z \mapsto \frac{2z}{\sqrt{\pi}}$ is non-decreasing. In particular, $\eta$ is almost everywhere differentiable and $\eta'$ is non-decreasing (on its domain). As convexity of $\eta_p$ implies, that $\eta_p$ (and hence also $\eta$) is locally Lipschitz continuous, we have that $\eta$ is absolutely continuous. Together, this implies that $\eta$ is convex.

However, convexity of $\eta$ does not imply that $\eta_p$ is convex: For example $\eta(x) = x \arctan(x)$ is convex (due to $\eta''(x) = 2(x^2 + 1)^{-2} > 0$) and $\eta_{1/2}(x) = x \arctan(x^2)$ is not convex (as $\eta''_{1/2}(x) = -2x(x^4 - 3)(x^4 + 1)^{-2}$).

**Remark 3.3.** The function $\eta_p$ appears (upto the factor $1 - p$) naturally in connection with the deterministic Bihari-LaSalle equalities: Let $x$ be as in Lemma 1.1 and assume that $x$ is non-decreasing. Then, it can be shown, that $x^p$ (for some $p \in (0,1)$) satisfies:

$$x(t)^p \leq (1 - p) \int_0^t \eta_p(x(s)^p)ds + H^p \quad \text{for all } t \in [0,T].$$  \hspace{1cm} (25)

Moreover, for any continuous $\eta$ such that $x(t) = G^{-1}(G(H) + t)$ is well-defined for $t \in [0,T]$, we have that $x$ satisfies

$$x(t) = \int_0^t \eta(x(s))ds + H$$

and $x^p$ satisfies

$$x^p(t) = (1 - p) \int_0^t \eta_p(x^p(s))ds + H^p.$$

**Corollary 3.4.** In the case Assumption $A_{\sup}$ and $\mathbb{E}[H_T] < \infty$ of the previous theorem we get the following estimates.

a) For $\eta(x) = x$ for all $x \geq 0$, we have

$$\|e^{-\beta A_T}X_T^++\|_{\mathcal{F}_t} \leq \alpha_1\|H_T\|_{1,\mathfrak{F}_\alpha}.$$  

b) If $X \geq 1$, $H \geq 1$ and $\eta(x) = x \log(x)$ for $x \geq 1$, we have

$$\|(X_T^+)^{e^{-\beta A_T}}\|_{\mathcal{F}_t} \leq \alpha_1\|H_T\|_{1,\mathfrak{F}_\alpha}.$$  

c) If $X \geq e$, $H \geq e$ and $\eta(x) = x \log(x) \log(\log(x))$ for all $x \geq e$, we have

$$\|e^{(\log(X_T^+))^{e^{-\beta A_T}}}\|_{\mathcal{F}_t} \leq \alpha_1\|H_T\|_{1,\mathfrak{F}_\alpha}.$$  

**Proof of Corollary 3.4.**  
a) For $\eta(x) = x$ and $c = 1$ we have $\eta_p(x) = \frac{1}{1-p}x$,

$$G(x) = \log(x) \text{ and } G^{-1}(x) = e^x.$$  \hspace{1cm} (26)
b) For $\eta(x) = x \log(x)$ and $c = e$ we have $\eta_p(x) = \frac{1}{1-p}x \log(x)$,

$$G(x) = \log(\log(x)) \text{ and } G^{-1}(x) = \exp(\exp(x)).$$
c) For $\eta(x) = x \log(x) \log(\log(x))$ and $c = e^e$ we have $\eta_p(x) = \frac{1}{1-p}x \log(x) \log(\log(x^{1/p}))$, 

$$G(x) = \log(\log(\log(x))) \text{ and } G^{-1}(x) = \exp(\exp(\exp(x))).$$

\square

### 3.2 Sharpness of the constants and estimates

In this section we study the sharpness of the constants $\beta$, $\alpha_1\alpha_2$ and $\alpha_1$. These constants also appear in other generalizations of stochastic Gronwall inequalities, see [14]. Moreover, we prove that the assumption $A_{\sup}$ does not imply a upper bound on the weak $L^1$ norm (in contrast to the stronger assumption $A_{\sup\sup}$).

To study the sharpness of the stochastic Gronwall inequality with supremum we will use the following lemma. The idea to study a process $(X_t)_{t \geq 0}$ of the following type to prove sharpness of the constant $\beta$ is due to Michael Scheutzow.
**Lemma 3.5.** Let \( \varepsilon, \delta \in (0, 1) \) and let \( l : [0, \delta \varepsilon) \to [0, \infty) \) be increasing, continuous, bijective and such that \( \int_0^{\delta \varepsilon} l^2(u)du = \infty \). We denote

\[
g_{\varepsilon, \delta} : [0, \infty) \to \{0, 1\}, \quad g_{\varepsilon, \delta}(s) := \sum_{k=0}^{\infty} \mathbb{I}_{[(k+\varepsilon)\delta, (k+1)\delta]}(s).
\]

Define for all \( x \in C([0, \infty), \mathbb{R}) \)

\[
b(s, x) := g_{\varepsilon, \delta}(s) \sup_{0 \leq u \leq s} x(u) \quad \forall s \geq 0,
\]

\[
\sigma(s, x) := \mathbb{I}_{\{x(s) > 0\}} \mathbb{I}_{(k\delta, (k+1)\delta]}(s)(s - k\delta) \quad \forall s \in (k\delta, (k+1)\delta], k \in \mathbb{N}.
\]

Let \( (W_t)_{t \geq 0} \) be a one-dimensional Wiener process on some underlying filtered probability space satisfying the usual conditions. Then there exists an adapted continuous non-negative process \((X_t)_{t \geq 0}\) enjoying the following properties:

a) For any (deterministic) \( t > 0 \) we have:

\[
\int_0^t |b(s, X)|ds + \int_0^t |\sigma(s, X)|^2ds < \infty \quad \mathbb{P}\text{-a.s.} \tag{27}
\]

b) The process \((X_t)_{t \geq 0}\) is a solution of the path-dependent SDE

\[
X_t = 1 + \int_0^t b(s, X)ds + \int_0^t \sigma(s, X)dW_s \quad \forall t \geq 0. \tag{28}
\]

In particular, \((X_t)_{t \geq 0}\) satisfies

\[
X_t \leq \int_0^t X_s^*ds + M_t + 1 \quad \forall t \geq 0
\]

for \(M_t := \int_0^t \sigma(s, X)dW_s\).

c) For all \( p \in (0, 1), k \in \mathbb{N}_0\)

\[
\mathbb{E}[(X_{k\delta+\varepsilon\delta})^p] = \frac{1}{1-p} \left(1 + \frac{p}{1-p}(1-\varepsilon)\delta\right)^k
\]

holds true.

**Theorem 3.6** (Sharpness of the constant \( \beta \)). Let \( p \in (0, 1) \) and assume that \( \tilde{\alpha}, \tilde{\beta} \) are positive constants (depending on \( p \)) such that for any non-negative adapted continuous process \((X_t)_{t \geq 0}\) which satisfies

\[
X_t \leq \int_0^t X_s^*ds + M_t + H, \quad \forall t \geq 0 \tag{29}
\]

for some continuous local martingale \((M_t)_{t \geq 0}\) starting in 0 and some constant \( H > 0 \), we have for all \( t \geq 0 \)

\[
\|X_t^*\|_p \leq \tilde{\alpha} H \exp(\tilde{\beta}t). \tag{30}
\]

Then, \( \tilde{\beta} \geq \beta := (1-p)^{-1} \) holds true.

**Corollary 3.7.** The constant \( \beta \) in Theorem 3.1 a) is sharp. It is already sharp when \( \eta(x) \equiv x, A_t \equiv t \), \( H \) is a constant and \( M \) and \( X \) are continuous processes.

**Theorem 3.8** (No \( A_{\text{no sup}} \), no tail estimate of order \( O(1/u) \)). For any \( T > 0 \) let \( \varepsilon, \delta \in (0, 1) \) and \( k \in \mathbb{N} \) be chosen such that \( T = k\delta + \varepsilon\delta \). Let \((X_t)_{t \geq 0}\) and \((M_t)_{t \geq 0}\) denote the process from Lemma 3.5. Then \((X_t)_{t \geq 0}\) is a non-negative adapted continuous process and \((M_t)_{t \geq 0}\) a continuous local martingale starting in 0 which satisfy

\[
X_t \leq \int_0^t X_s^*ds + M_t + 1, \quad \forall t \geq 0 \tag{31}
\]

and

\[
\sup_{u \geq 0} \left( u \mathbb{P}\left[ \sup_{t \in [0, T]} X_t \right] \right) = \infty. \tag{32}
\]

In particular, estimates of the form (23) do not hold in the case of Assumption \( A_{\sup} \).
The next theorem studies the sharpness of the constants $\alpha_1$ and $\alpha_1\alpha_2$ which appear in Theorem 3.1. The constant $\alpha_1\alpha_2$ is the sharp constant of Lenglart’s domination inequality (see Lemma 4.2). In particular, Theorem 3.9 a) and b) are closely connected to [15, Theorem 2.1]. The upper bound given in Theorem 3.9 a) is by Fatou’s Lemma a special case of Lenglart’s domination inequality. Assertion c) of Theorem 3.9 is known in the literature, see for example [32, Theorem 7.6, p. 300].

**Theorem 3.9** (Sharpness of the constants $\alpha_1$ and $\alpha_1\alpha_2$). Assume Assumption $\mathcal{A}_{\text{sup}}$ and $A \equiv 0$, i.e. let $(X_t)_{t \geq 0}$ be an adapted non-negative right-continuous process, $(H_t)_{t \geq 0}$ be an adapted non-negative non-decreasing càdlàg process, $(M_t)_{t \geq 0}$ be a càdlàg local martingale with $M_0 = 0$. Assume that for all $t \geq 0$

$$X_t \leq M_t + H_t \quad \mathbb{P}\text{-a.s.}$$

Then the following assertions hold for $p \in (0,1)$.

a) If $H$ is predictable and $\mathbb{E}[H_T^p] < \infty$, then $\|X_t^*\|_{p,F_0} \leq \alpha_1\alpha_2\|H_t\|_{p,F_0}$ for all $t \in [0,T]$ and the constant $\alpha_1\alpha_2 = (1 - p)^{-1/p} \|H_T\|_{p,F_0}$ is sharp. The constant is already sharp if $X$ and $H$ are continuous and $M$ has no negative jumps.

b) If $M$ has no negative jumps and $\mathbb{E}[H_T^p] < \infty$, then $\|X_t^*\|_{p,F_0} \leq \alpha_1\alpha_2\|H_t\|_{p,F_0}$ for all $t \in [0,T]$ and the constant $\alpha_1\alpha_2$ is sharp. The constant is already sharp if $X$ and $H$ are continuous.

c) If $\mathbb{E}[H_T] < \infty$, then $\|X_t\|_{p,F_0} \leq \alpha_1\|H_t\|_{1,F_0}$ for all $t \in [0,T]$ and the constant $\alpha_1 = (1 - p)^{-1/p}$ is sharp. The constant is already sharp if $X$ and $M$ are continuous and $H$ is a constant.

**Corollary 3.10.** The constants $\alpha_1\alpha_2$ and $\alpha_1$ in Theorem 3.1 are sharp and they are already sharp when $A \equiv 0$.

## 4 Proofs of the results of Section 3

### 4.1 Lenglart’s concept of domination and the Snell envelope

We prove the stochastic Bihari-LaSalle inequality Theorem 3.1 a) by extending the proof technique of Lenglart’s domination inequality, which we recall here for the convenience of the reader.

The following concept of domination was introduced by Lenglart in [24, Définition II] and slightly generalized by Lenglart, Lépine, and Pratelli [23, Lemma 1.4], see also Ren and Shen [34] and Mehri and Scheutzow [30].

**Definition 4.1** (Lenglart’s concept of domination). Let

- $(X_t)_{t \geq 0}$ be an adapted right-continuous non-negative process,
- and $(H_t)_{t \geq 0}$ be a predictable càdlàg non-negative non-decreasing process,

such that for all bounded stopping times $\tau$

$$\mathbb{E}[X_\tau \mid F_0] \leq \mathbb{E}[H_\tau \mid F_0] \leq \infty$$

holds. Then we call $X$ dominated by $H$.

By Fatou’s lemma (33) also holds for all finite stopping times $\tau$.

The following lemma is [30, Lemma 2.2], which is a sharpened generalisation of [24, Théorème I, Corollaire II]. See also the references listed in [30].

**Lemma 4.2** (Lenglart’s domination inequality). Let $X$ be dominated by $H$. Then, we have

a) for all $u > 0$, $\lambda > 0$ and $T > 0$:

$$\mathbb{P}[X_T^* > u \mid F_0]u \leq \lambda \mathbb{E}[(H_T - \lambda) \wedge u \mid F_0] + \mathbb{P}[H_T \geq u \mid F_0]u. \quad (34)$$

b) for all $p \in (0,1)$ and $T > 0$:

$$\mathbb{E} \left[ \left( \sup_{t \in [0,T]} X_t \right)^p \mid F_0 \right]^{1/p} \leq \alpha_1\alpha_2 \mathbb{E} \left[ \left( \sup_{t \in [0,T]} H_t \right)^p \mid F_0 \right]^{1/p},$$

where $\alpha_1\alpha_2 = (1 - p)^{-1/p}$. 

Note that Lemma 4.2 b) follows from a) by integrating equation (34) w.r.t. $pu^{p-1}du$, e.g. using the formulas of Remark 4.3 and choosing $\lambda = p$.

**Remark 4.3** (Calculation of $Z^p$, $p \in (0,1)$). Let $Z$ be a non-negative random variable and $p \in (0,1)$. Then $Z^p$ can be calculated using the three formulas below.

\[
Z^p = p \int_0^\infty 1_{\{Z \geq u\}}u^{p-1}du \\
Z^p = (1 - p) \int_0^\infty Z 1_{\{Z \leq u\}}u^{p-2}du \\
Z^p = p(1 - p) \int_0^\infty (Z \wedge u)u^{p-2}du
\] (35)

The third equality follows e.g. by using the first and second equality. In particular, we also have $Z^{p-1} = (1 - p) \int_0^\infty 1_{\{Z \leq u\}}u^{p-2}du$ for $Z > 0$. The third equality exists more generally also for concave functions, see Burkholder [9, Theorem 20.1, p.38-39] and Pratelli [33, Proposition 1.2].

We use the Snell envelope contained in [12, Appendix 1: (22), p.416-417]. This version uses optional strong supermartingales. An optional strong supermartingale $(Z_t)_{t \geq 0}$ is an optional process such that for any bounded stopping time $\tau$ the random variable $Z_\tau$ is integrable and such that for any pair of bounded stopping times $\sigma \leq \tau$ the inequality $\mathbb{E}[X_\tau \mid \mathcal{F}_\sigma] \leq X_\sigma$ holds almost surely (see [12, Appendix 1: Definition I, p.393-394]). Note that càdlàg supermartingales are optional strong supermartingales.

The following corollary of the Snell envelope is useful to prove the convex Bihari-LaSalle inequality Theorem 3.1 a). Alternatively, with some more work the Snell envelope could also be directly applied in the proof Theorem 3.1 a). However, we prefer to state the following corollary as it yields in addition also a characterization of Lenglart’s concept of domination. As the author did not find this corollary in the literature, a proof is provided in the appendix.

**Corollary 4.4** (Characterization of Lenglart’s concept of domination). Let $X$ be dominated by $H$ and assume that $\mathbb{E}[H_0] < \infty$. Then there exists a càdlàg local supermartingale $(N_t)_{t \geq 0}$ with $N_0 \leq 0$ such that $X_t \leq H_t + N_t$ for all $t \geq 0$.

By the general Doob-Meyer decomposition theorem we could also replace 'local supermartingale' by 'local martingale' in the corollary.

### 4.2 Main Lemma: A Lenglart type estimate

An extension of Lemma 4.2 a) is provided in Lemma 4.5 for the cases Assumption $\mathcal{A}_{\sup}$ and Assumption $\mathcal{A}_{\no\sup}$. This lemma is one of the key steps of the proof of Theorem 3.1 a). Moreover, Lemma 4.5 can be used to prove other generalizations of stochastic Gronwall inequalities, which is done in the closely connected paper [14].

**Lemma 4.5** (Lenglart type estimates). Fix some $T > 0$ and $p \in (0,1)$ and let $(X,A,H,M)$ satisfy Assumption $\mathcal{A}_{\sup}$ or Assumption $\mathcal{A}_{\no\sup}$. We consider the following 6 cases, which arise from combining $\mathcal{A}_{\sup}$ or $\mathcal{A}_{\no\sup}$ with one of the following three assumptions:

a) $H$ is predictable,

b) $M$ has no negative jumps,

c) $\mathbb{E}[HT] < \infty$.

Fix arbitrary $u, \lambda > 0$ and set:

\[
\tau_u := \tau := \inf\{s \geq 0 \mid H_s \geq \lambda u\}, \quad \sigma_u := \sigma := \inf\{s \geq 0 \mid X_s > u\},
\]

where $\inf \emptyset := +\infty$. Then, the following estimate holds true for all $t \in [0,T]$:

\[
1_{\{X_t > u\}}u \leq X_{t \wedge \sigma_u} \wedge u \leq L^{\lambda u}_t + M^{\lambda u}_t + H^{\lambda u}_t.
\] (36)
Here \((I^L_t)_{t \geq 0}\) is a non-decreasing process containing the integral term from (13) and (14) respectively with an additional indicator function

\[
I^L_t := I^L_t(u) := \begin{cases} 
\int_{[0,t]} \eta(X_{s-}) \mathbb{1}_{\{X_s^* \leq u\}} \, dA_s & \text{for Assumption } A_{\sup}, \\
\int_{[0,t]} \eta(X_{s-}) \mathbb{1}_{\{X_s^* \leq u\}} \, dA_s & \text{for Assumption } A_{\no\sup},
\end{cases}
\]

the process \((M^L_t(u))_{t \geq 0}\) is a local martingale with càdlàg paths starting in 0 defined by

\[
M^L_t := M^L_t(u) := \begin{cases} 
\lim_{n \to \infty} M_{t \wedge \tau(n) \wedge \sigma} & \text{if } H \text{ is predictable,} \\
M_{t \wedge \tau(n) \wedge \sigma} & \text{if } M \text{ has no negative jumps,} \\
\mathbb{E}_{F_0}[H_t] \mathbb{1}_{\{H_t \leq u\}} & \text{if } \mathbb{E}[H_t] < \infty,
\end{cases}
\]

(where \(\tau(n)\) denotes an announcing sequence of \(\tau\) and \(\bar{M}_t := M_t + \mathbb{E}[H_T \mid F_t] - \mathbb{E}_{F_0}[H_T]\) for \(t \in [0,T]\)), and \((H^L_t)_{t \geq 0}\) is a non-decreasing process depending on \(H\):

\[
H^L_t := H^L_t(u) := \begin{cases} 
H_t \wedge (\lambda u) + u \mathbb{1}_{\{H_t \geq \lambda u\}} & \text{if } H \text{ is predictable,} \\
H_t \wedge (\lambda u) + u \mathbb{1}_{\{H_t \geq \lambda u\}} & \text{if } M \text{ has no negative jumps,} \\
\mathbb{E}_{F_0}[H_t] \wedge u & \text{if } \mathbb{E}[H_t] < \infty.
\end{cases}
\]

**Remark 4.6** (Connection of Lemma 4.5 and Lenglart’s inequality). Lemma 4.5 is connected to Lenglart’s inequality as follows: Let \((X, A, H, M)\) satisfy Assumption \(A_{\sup}\) or Assumption \(A_{\no\sup}\). Moreover, assume that \(A \equiv 0\) and that \(H\) is predictable. Then, we can apply Lemma 4.5 yielding

\[
\mathbb{1}_{\{X_t > u\}} u \leq M^L_t(u) + H_t \wedge (\lambda u) + u \mathbb{1}_{\{H_t \geq \lambda u\}}.
\]

Under the assumptions listed above, \(X\) is also dominated by \(H\). Hence, we can also apply Lenglart’s inequality and obtain:

\[
P[X^*_t > u \mid F_0] u \leq \lambda \mathbb{E}[(H_T \lambda^{-1} \wedge u) \wedge u \mid F_0] + \mathbb{P}[H_T \lambda^{-1} \geq u \mid F_0] u,
\]

Taking the conditional expectation given \(F_0\) of inequality (37) implies by monotone convergence inequality (38).

**Proof of Lemma 4.5.** The inequality \(\mathbb{1}_{\{X_t > u\}} u \leq X_t \wedge u \wedge u\) follows easily from the right-continuity and non-negativity of \(X\). It remains to prove the second inequality of (36).

We denote by \(Y\) the upper bound for \(X\) which is given by Assumption \(A_{\sup}\) or Assumption \(A_{\no\sup}\) respectively, i.e. for all \(t \geq 0\)

\[
X_t \leq Y_t := M_t + H_t + \begin{cases} 
\int_{[0,t]} \eta(X_{s-}) \, dA_s & \text{for } A_{\sup}, \\
\int_{[0,t]} \eta(X_{s-}) \, dA_s & \text{for } A_{\no\sup}.
\end{cases}
\]

**Step (a):** We first prove the inequality for the case that \(H\) is predictable. Fix some \(t \in [0,T]\). Because \(H\) is predictable there exists a sequence of stopping times \((\tau(n))_{n \in \mathbb{N}}\) that announces \(\tau\). In particular, due to \(H\) being non-decreasing, we have on \(\{H_0 < \lambda u\}\) the inequality \(H_{t \wedge (\tau(n) \wedge \sigma)} \leq H_t \wedge (\lambda u)\). Moreover, by definition of \(\sigma\) the equality

\[
\{s \leq \sigma\} = \{s > \sigma\}^c = \{\exists r < s \mid X_r > u\}^c = \{X_s^* > u\}^c = \{X_s^* \leq u\}
\]

holds true for all \(s > 0\). Therefore, using (13) or (14) respectively, we have on \(\{H_0 < \lambda u\}\):

\[
X_{t \wedge \tau(n) \wedge \sigma} \leq M_{t \wedge \tau(n) \wedge \sigma} + H_{t \wedge \tau(n) \wedge \sigma} + \begin{cases} 
\int_{[0,t]} \eta(X_{s-}) \mathbb{1}_{\{s \leq \sigma\} \wedge (s \leq \tau(n))} \, dA_s & \text{for } A_{\sup}, \\
\int_{[0,t]} \eta(X_{s-}) \mathbb{1}_{\{s \leq \sigma\} \wedge (s \leq \tau(n))} \, dA_s & \text{for } A_{\no\sup}.
\end{cases}
\]

\[
\leq M_{t \wedge \tau(n) \wedge \sigma} + H_t \wedge (\lambda u) + I^L_t.
\]
Moreover, note that due to non-negativity of $X$, we have
\[ X_{t \land \sigma} \land u - X_{t \land \tau(n) \land \sigma} \land u \leq u \mathbf{1}_{\{t < \tau\}}. \]

The previous two inequalities and the definitions of $\tau$ and $(\tau^{(n)})_n$ imply on $\{H_0 < \lambda u\}$:
\[
X_{t \land \sigma} \land u \leq \limsup_{n \to \infty} X_{t \land \tau^{(n)} \land \sigma} \land u + \limsup_{n \to \infty} (X_{t \land \sigma} \land u - X_{t \land \tau^{(n)} \land \sigma} \land u)
\leq I^L_t + \limsup_{n \to \infty} M_{t \land \tau^{(n)} \land \sigma} + H_t \land (\lambda u) + \lim_{n \to \infty} u \mathbf{1}_{\{\tau^{(n)} < t\}}
\leq I^L_t + \limsup_{n \to \infty} M_{t \land \tau^{(n)} \land \sigma} + H_t \land (\lambda u) + u \mathbf{1}_{\{H_t \geq \lambda u\}}.
\]

Hence, we have proven that (36) holds true on $\{H_0 < \lambda u\}$. On $\{H_0 \geq \lambda u\}$ we bound $X_{t \land \sigma} \land u$ by $u$, i.e. we have
\[
X_{t \land \sigma} \land u \mathbf{1}_{\{H_0 \geq \lambda u\}} \leq u \mathbf{1}_{\{H_0 \geq \lambda u\}} \leq u \mathbf{1}_{\{H_t \geq \lambda u\}}.
\]

Noting that on $\{H_0 \geq \lambda u\}$ we have $\tau^{(n)} = \tau = 0$ and $M_0 = 0$, this implies by non-negativity of $I^L_t$ the inequality (36) on $\{H_0 \geq \lambda u\}$.

**Step** (b): Next we prove the inequality for the case that $M$ has no negative jumps. Fix again some $t \in [0, T]$. In the proof of the previous assertion we used $\lim_{n \to \infty} H_{t \land \tau^{(n)}} \leq H_t \land (\lambda u)$ on $\{H_0 < \lambda u\}$. As our filtration satisfies by assumption the usual conditions, we may assume w.l.o.g. that $\tilde{M}$ is càdlàg. Using again, that $\{s \leq \sigma\} = \{X^*_s \leq u\}$, we obtain for all $u > 0$ on $\{\mathbb{E}_0[H_T] \leq u\}$:
\[
X_{t \land \sigma} \land u \leq I^L_t + \tilde{M}_{t \land \sigma} + \mathbb{E}_0[H_T] \land u.
\]

Noting that $X_{t \land \sigma} \land u \leq \mathbb{E}[H_T | F_0] \land u$ on $\{\mathbb{E}_0[H_T] \geq u\}$ implies the claim.
For the proof of Theorem 3.1b) we need the following lemma, which is an immediate consequence of combining Remark 4.3 with Lemma 4.5. For predictable $H$ Lemma 4.7 is an immediate corollary of Lenglart’s domination inequality. Lemma 4.7 provides the upper bounds of Theorem 3.9.

**Lemma 4.7.** Let Assumption $A_{sup}$ and $A \equiv 0$ hold and let $p \in (0, 1)$.

- a) If $H$ is predictable or $M$ has no negative jumps and $E[H_T^p] < \infty$, we have for all $T > 0$:
  \[ \|X_T^p\|_{p,F_0} \leq \alpha_1 \alpha_2 \|H_T\|_{p,F_0}. \]

- b) If $E[H_T] < \infty$ we have for all $T > 0$:
  \[ \|X_T^p\|_{p,F_0} \leq \alpha_1 \|H_T\|_{1,F_0}. \]

**Proof of Lemma 4.7.** Assume w.l.o.g. that $M$ is a martingale, and hence also $M^{L,u}$ for any $u > 0$. The inequalities can be proven by taking the conditional expectation of (36) given $F_0$ and integrating w.r.t $pu^{p-2}du$. This gives (due to $A \equiv 0$):

\[
\mathbb{E}_{F_0}\left[\int_0^\infty \mathbb{1}_{(X_t^p > u)}pu^{p-1}du\right] \leq \int_0^\infty \mathbb{E}_{F_0}[M_t^{L,u}]pu^{p-2}du + \mathbb{E}_{F_0}\left[\int_0^\infty H_t^{L,u}pu^{p-2}du\right] \leq \mathbb{E}_{F_0}\left[\int_0^\infty H_t^{L,u}pu^{p-2}du\right].
\]

It remains to compute the single terms. By Remark 4.3 we have:

\[ p \int_0^\infty \mathbb{1}_{(X_t^p > u)}u^{p-1}du = (X_t^p)^p \]

If $E[H_T^p] < \infty$ and either $H$ is predictable or $\Delta M \geq 0$, we have (choosing $\lambda = p$ and applying Remark 4.3 and recalling $\alpha_1 = (1 - p)^{-1/p}$, $\alpha_2 = p^{-1}$):

\[
\int_0^\infty H_t^{L,u}pu^{p-2}du = \int_0^\infty (H_t \wedge (\lambda u) + u \mathbb{1}_{(H_t \leq \lambda u)})pu^{p-2}du = \lambda(1 - p)^{-1}(H_t\lambda^{-1})^p + (H_t\lambda^{-1})^p = \alpha_1^p \alpha_2^p H_t^p.
\]

If $E[H_T] < \infty$, we have:

\[
\int_0^\infty H_t^{L,u}pu^{p-2}du = \int_0^\infty (\mathbb{E}_{F_0}[H_T] \wedge u)pu^{p-2}du = (1 - p)^{-1}\mathbb{E}_{F_0}[H_T]^p = \alpha_1^p \mathbb{E}_{F_0}[H_T]^p.
\]

Combining the calculations above gives the claim. □

### 4.3 Proof of the convex stochastic Bihari-LaSalle inequality

The following lemma allows us to assume w.l.o.g. that the integrator $A$ is continuous and adapted instead of predictable. It can be proven by a time change argument, by smoothing out the large jumps of $A$. The proof is hidden away in the appendix. Recall that in Assumption $A_{no sup}$ and Assumption $A_{sup}$, the process $A$ is assumed to be predictable. In general, the assertion of the lemma is false if $A$ is not predictable, for a counterexample see Counterexample 7.1.

**Lemma 4.8 (Continuity of integrator by time change).** Assume that $(X_t)_{t \geq 0}$, $(A_t)_{t \geq 0}$, $(H_t)_{t \geq 0}$ and $(M_t)_{t \geq 0}$ satisfy Assumption $A_{no sup}$ (or Assumption $A_{sup}$) on some filtered probability space $(\Omega, F, \mathbb{P}, (F_t)_{t \geq 0})$. Denote by $\tilde{A}^r$ the continuous part of $A$ i.e. $\tilde{A}^r_t := A_t - \sum_{s \leq t} \Delta A_s$ for all $t \geq 0$. Assume that $A^r$ is strictly increasing and $A^r_{\infty} = +\infty$. Then, there exists a time-changed version of $(X, A, H, M)$ such that the integrator is continuous, i.e. a family of stochastic processes $(\tilde{X}, \tilde{A}, \tilde{H}, \tilde{M})$ satisfying Assumption $A_{no sup}$ (or Assumption $A_{sup}$) on a filtered probability space $(\tilde{\Omega}, \tilde{F}, \tilde{\mathbb{P}}, (\tilde{F}_t)_{t \geq 0})$ such that:

- a) $\tilde{A}_t = t$ for all $t \geq 0$,
b) for all $t \geq 0$:
\[ \tilde{X}_{A_t} = X_t, \quad \tilde{M}_{A_t} = M_t, \quad \tilde{H}_{A_t} = H_t, \]
and $A_t$ is a $(\tilde{F}_t)_{t \geq 0}$ stopping time for every $s \geq 0$,

c) if $H$ is predictable then $\tilde{H}$ is predictable,

d) if $M$ has no negative jumps then $\tilde{M}$ has no negative jumps,

e) and $(\Omega, F, P, (\tilde{F}_t)_{t \geq 0})$ satisfies the usual conditions.

**Remark 4.9.** The assertion of Lemma 4.8 is not trivial if $A$ has jumps: Simply extending $A^{-1}(\omega) : \text{Image}(A)(\omega) \mapsto [0, \infty)$ to $A^{-1}(\omega) : [0, \infty) \mapsto [0, \infty)$ and then setting $\tilde{Y} := Y_{A_t}$ for $Y \in \{X, M, H\}$ will not work in general, because these processes will not satisfy inequality (14) of Assumption $A_{\text{nosup}}$ for $t \notin \text{Image}(A)(\omega)$. For more details see the first step of the proof.

**Remark 4.10.** Lemma 4.8 can be also applied if $A^c$ is not strictly increasing and if $A^c_{\infty} < \infty$ by setting $A_t := A_t + \delta t$ for some $\delta > 0$ and applying Lemma 4.8 to $(X, \tilde{A}, \tilde{H}, M)$.

**Remark 4.11.** In the proofs of the stochastic Bihari-LaSalle inequalities, we will always assume $X \geq \varepsilon + c_0$ and $H \geq \varepsilon + c_0$ for some $\varepsilon > 0$ instead of $X \geq c_0$ and $H \geq c_0$ (where $c_0$ is the constant from $\eta : [c_0, \infty) \to [0, \infty)$ in Assumption $A_{\text{sup}}$ and Assumption $A_{\text{nosup}}$). We do this to ensure that terms like $G(X_t)$ or $G(H_t)$ are finite. We may do this without loss of generality because we can add an arbitrary $\varepsilon > 0$ to (13) (or similarly (14)) and slightly weaken (13) (using that $\eta$ is non-decreasing) to obtain for all $t \in [0, T]$:
\[
(X_t + \varepsilon) \leq \int_{(0,t]} \eta(X_{t -} + \varepsilon) dA_t + M_t + (H_t + \varepsilon) \quad \mathbb{P}\text{-a.s.}
\]
Proving the assertions of the theorems for the processes $(X_t + \varepsilon)_{t \geq 0}$, $(A_t)_{t \geq 0}$, $(M_t)_{t \geq 0}$ and $(H_t + \varepsilon)_{t \geq 0}$ and then taking the limit $\varepsilon \to 0$ will imply the assertions for the general case $X \geq c_0$, $H \geq c_0$.

**Proof of Theorem 3.1. Proof of b):** We first prove the claim for continuous $A$. We assume w.l.o.g. that $H \geq c_0 + \varepsilon$ and $X \geq c_0 + \varepsilon$ on $\Omega$ for some constant $\varepsilon > 0$.

We define
\[ f : (c_0, \infty) \times [0, \infty) \mapsto (c_0, \infty), \quad (x, a) \mapsto G^{-1}(G(x) - a), \]
noting that $\|f(X^*_T, A_T)\|_p$ is the quantity we need to find an upper bound for. The function $f$ is indeed well-defined: Due to the convexity of $\eta$ and $\eta(c_0) = 0$ there exists some $K > 0$ s.t. $\eta(x) \leq K(x - c_0)$ for all $x \in [c_0, \infty)$ where $c$ denotes the constant from the definition of $G$, see (16). This implies $\lim_{x \to c_0} G(x) = -\infty$ and therefore $\text{domain}(G^{-1}) = \text{range}(G) = (\infty, \lim_{x \to \infty} G(x))$. Moreover, we have $\text{range}(G^{-1}) = \text{domain}(G) = (c_0, \infty)$. Therefore, $f$ is well-defined.

Moreover, we have for all $x \in (c_0, \infty)$, $a \in [0, \infty)$:
\[
\frac{\partial}{\partial x} f(x, a) = \frac{G'(x)}{G'(G^{-1}(G(x) - a))} = \frac{\eta(f(x, a))}{\eta(x)},
\]
\[
\frac{\partial}{\partial a} f(x, a) = -\eta(f(x, a)),
\]
\[
\frac{\partial^2}{\partial x^2} f(x, a) = \frac{\eta(f(x, a))}{(\eta(x))^2} (\eta'(f(x, a)) - \eta'(x)).
\]

Denote by $(Y_t)_{t \geq 0}$ to be the right-hand side of (14). Instead of finding an upper bound for $\|f(X^*_T, A_T)\|_p$, it suffices by $X_t \leq Y_t$ to find an upper bound for $\|f(Y^*_T, A_T)\|_p$. To this end we will estimate $(f(Y_t, A_t))_{t \geq 0}$ using Itô’s formula.

We first show that the jump term
\[
\sum_{s \leq t} f(Y_s, A_s) - f(Y_{s^-}, A_s) - \frac{\partial}{\partial x} f(Y_{s^-}, A_s) \Delta Y_s
\]
that occurs in the Itô formula for $(f(Y_t, A_t))_{t \geq 0}$ is non-positive. (Recall that we assumed that $A$ is continuous.) By assumption $\eta'$ is non-decreasing and $f(x, a) \leq x$, therefore $\frac{\partial^2}{\partial x^2} f(x, a) \leq 0$ holds. This implies by a Taylor’s expansion that for all fixed $a > 0$ and for all $x, x + \Delta x \in (c_0, \infty)$
\[
f(x + \Delta x, a) - f(x, a) - \frac{\partial}{\partial x} f(x, a) \Delta x \leq 0,
\]
and therefore the jump term in Itô formula for \( (f(Y_t, A_t))_{t\geq 0} \) is non-positive. Hence, Itô’s formula implies:

\[
\begin{align*}
    df(Y_t, A_t) & = \frac{\partial}{\partial x} f(Y_t, A_t) dY_t + \frac{\partial^2}{\partial u^2} f(Y_t, A_t) dA_t + \frac{1}{2} \frac{\partial^2}{\partial u^2} f(Y_t, A_t) d(Y^u)^2_t \\
    & \leq \eta(f(Y_t, A_t)) \eta(Y_t) dB_t + \frac{\eta(f(Y_t, A_t))}{\eta(Y_t)} \Delta f(Y_t, A_t) dM_t - \eta(f(Y_t, A_t)) dB_t \\
    & \leq dM_t + dH_t,
\end{align*}
\]

where \( \Delta M_t := \int_{(0,t]} \frac{\eta(f(Y_t, A_t))}{\eta(Y_t)} dM_t \) is a local martingale starting in \( 0 \). Note that \( \Delta M_t \geq 0 \) implies \( \Delta M_t \geq 0 \) for all \( t \geq 0 \). Due to \( \eta \geq 0 \), the family of processes \( (f(Y, A)) \), the process which is constant 0, \( H, M \) satisfy Assumption \( \mathbb{A}_{\text{sup}} \) e.g. for \( \eta(x) \equiv x \).

For predictable \( H \) the estimate \( (23) \) follows immediately by applying Lenglart’s inequality to \( f(Y_t, A_t) \leq \tilde{M}_t + H_t \). In the other cases, we apply Lemma \( 4.5 \) to \( (f(Y, A)) \), the process which is constant 0, \( H, M \) and obtain (using the notation of the lemma)

\[
\mathbb{P}[ \sup_{t \in [0,T]} f(Y_t, A_t) > u \mid \mathcal{F}_0 ] \leq \frac{1}{u} \mathbb{E}[\tilde{M}^{L,u}_T + H^{L,u}_T \mid \mathcal{F}_0].
\]

Noting that \( \tilde{M}^{L,u} \) is a local martingale, \( \tilde{M}^{L,u}_T + H^{L,u}_T \geq 0 \), we obtain by Fatou’s lemma

\[
\mathbb{P}[ \sup_{t \in [0,T]} f(X_t, A_t) > u \mid \mathcal{F}_0 ] \leq \mathbb{P}[ \sup_{t \in [0,T]} f(Y_t, A_t) > u \mid \mathcal{F}_0 ] \leq \frac{1}{u} \mathbb{E}[\tilde{M}^{L,u}_T + H^{L,u}_T]
\]

which is \( (23) \). Inequality \( (23) \) implies \( (24) \) by the following calculation. For all \( u > 0 \), \( w, R > 0 \) we have:

\[
\mathbb{P}\left[ \sup_{t \in [0,T]} X_t > u \mid \mathcal{F}_0 \right] \leq \mathbb{P}\left[ \sup_{t \in [0,T]} G^{-1}(G(X_t) - R) > G^{-1}(G(u) - R), \mathcal{A}_T \leq R \right] \mid \mathcal{F}_0 \right] + \mathbb{P}[\mathcal{A}_T > R \mid \mathcal{F}_0]
\]

\[
= \left\{ \begin{array}{ll}
    \mathbb{E}[H_T \wedge w] & \text{if } H \text{ is predictable or } \mathbb{D} \geq 0 \\
    \mathbb{E}[H_T] \wedge 1 + \mathbb{P}[\mathcal{A}_T > R \mid \mathcal{F}_0] & \text{if } \mathbb{E}[H_T] < \infty.
\end{array} \right.
\]

To obtain \( (22) \), apply Lemma \( 4.7 \) to \( (f(Y, A)) \), the process which is constant 0, \( H, \tilde{M} \):

\[
\| G^{-1}(G(X_t^T) - \mathcal{A}_T) \|_{p, \mathcal{F}_0} \leq \| G^{-1}(G(X_t) - A_t) \|_{p, \mathcal{F}_0} \leq \| G^{-1}(G(Y_t) - A_t) \|_{p, \mathcal{F}_0}
\]

\[
= \| \sup_{t \in [0,T]} f(Y_t, A_t) \|_{p, \mathcal{F}_0} \leq \left\{ \begin{array}{ll}
    \alpha_1 \| H_T \|_{1, \mathcal{F}_0} & \text{if } \mathbb{E}[H_T] < \infty,
    \\
    \alpha_1 \alpha_2 \| H_T \|_{p, \mathcal{F}_0} & \text{if } \mathbb{D} \geq 0, \mathbb{E}[H_T^p] < \infty,
    \\
    \alpha_1 \alpha_2 \| H_T \|_{p, \mathcal{F}_0} & \text{if } H \text{ predictable, } \mathbb{E}[H_T^p] < \infty.
\end{array} \right.
\]

This proves the assertion for continuous \( A \).

Now we prove the assertion for non-continuous (but predictable) \( A \). We may assume w.l.o.g. that the continuous part of \( A \) is strictly increasing and \( A^\infty = \infty \), for details see Remark 4.10. Let \( (\bar{X}, \bar{A}, \bar{H}, \bar{M}) \) the family of processes and \( (\bar{F}_t)_{t \geq 0} \) the filtration we obtain by applying Lemma 4.8 to \( (X, A, H, M) \). Fix some arbitrary \( T > 0 \). As \( \mathcal{A}_T \) is a \( (\bar{F}_t)_{t \geq 0} \) stopping time, \( (\mathcal{A}_T \wedge T)_{t \geq 0} \) is a continuous adapted process, so we may apply the first part of this proof to \( (\bar{X} \wedge \mathcal{A}_T)_{t \geq 0}, (\mathcal{A}_T \wedge \mathcal{A}_T)_{t \geq 0}, (\bar{M} \wedge \mathcal{A}_T)_{t \geq 0} \), to obtain for all \( T > 0 \):

\[
\mathbb{P}\left[ \sup_{t \in [0,T \wedge \mathcal{A}_T]} G^{-1}(G(\bar{X}_t) - t) > u \mid \mathcal{F}_0 \right] \leq \left\{ \begin{array}{ll}
    \frac{1}{u} \mathbb{E}[\bar{H}_T \wedge (\lambda u)] + \mathbb{P}[\bar{H}_T \geq \lambda u \mid \mathcal{F}_0] & \text{if } H \text{ is predictable or } \mathbb{D} \geq 0
    \\
    \frac{1}{u} \mathbb{E}[\bar{H}_T] \wedge u & \text{if } \mathbb{E}[H_T] < \infty.
\end{array} \right.
\]
and
\[ \| \sup_{t \in [0, A_T \wedge T]} G^{-1}(G(\tilde{X}_t) - t) \|_{p, \mathcal{F}_0} \leq \begin{cases} 
\alpha_1 \| \tilde{H}_{A_T \wedge T} \|_{1, \mathcal{F}_0} & \text{if } \mathbb{E}[H_T] < \infty, \\
\alpha_1 \alpha_2 \| \tilde{H}_{A_T \wedge T} \|_{p, \mathcal{F}_0} & \text{if } \Delta M \geq 0, \ \mathbb{E}[H_T^p] < \infty, \\
\alpha_1 \alpha_2 \| \tilde{H}_{A_T \wedge T} \|_{p, \mathcal{F}_0} & \text{if } H \text{ predictable, } \mathbb{E}[H_T^p] < \infty. 
\end{cases} \]

Here we used that Lemma 4.8 ensures that $\tilde{H}$ is predictable if $H$ is predictable, and $\Delta \tilde{M} \geq 0$ if $\Delta M \geq 0$.

By letting $T \to \infty$, monotone convergence, $\tilde{X}_{A_t} = X_t$ and $\tilde{H}_{A_t} = H_t$ for all $t \geq 0$, this implies the assertion for non-continuous $A$.

**Proof of a):** It suffices to prove Theorem 3.1 a) for continuous $A$, as the assertion can be extended by Lemma 4.8 to predictable $A$ as in the proof of Theorem 3.1 b). We assume w.l.o.g. that $H \geq c_0 + \varepsilon$ and $X \geq c_0 + \varepsilon$ on $\Omega$ for some constant $\varepsilon > 0$.

We first sketch the idea of the proof, then provide the details. Denote by $(Y_t)_{t \geq 0}$ the right-hand side of inequality (13) of Assumption $A_{\sup}$ and let $f$ be as in the proof of b). Under Assumption $A_{\sup}$, the inequality $\|f(Y_t, A_t)\| \leq dM_t + dH_t$ does not hold because terms of Itô’s formula fail to cancel out. However, for any $p \in (0, 1)$, it can be shown that $((X_t^*)^p)_{t \in [0, T]}$ satisfies Assumption $A_{\no sup}$ for $\eta_p(x) = \frac{1}{1-p} \eta(x^{1/p})^{x^{1/p}}$. Thus, similarly as in the proof of b), an application of Itô’s formula implies an estimate for $\tilde{G}_p((X_t^*)^p - A_T)$ where $\tilde{G}_p(x) := \int_0^x \eta_p(s)^{-1} ds$ for some $c > c_0$.

We first show that $((X_t^*)^p)_{t \in [0, T]}$ satisfies Assumption $A_{\no sup}$ for $\eta_p(x) = \frac{1}{1-p} \eta(x^{1/p})^{x^{1/p}}$. We apply Lemma 4.5 and obtain e.g. for the case $\mathbb{E}[H_T] < \infty$ for all $u > 0$, $t \in [0, T]$

\[ \mathbb{I}_{(X_t^*)^p > u} \leq \int_{(0, t]} \eta((X_{s-})^p)^p u \mathbb{I}_{(X^*_s)^p \leq u} dA_s + M_{t\wedge u}^{L,u} + \mathbb{E}_\mathcal{F}_0 [H_T] \wedge u. \]  \hspace{1cm} (42)

Let $\tau$ be a bounded stopping time. The right-hand side of (42) is non-negative and $M_{\tau \wedge u}^{L,u}$ is a local martingale. Hence, taking the conditional expectation, Fatou’s lemma and monotone convergence imply

\[ \mathbb{P}(X_{\tau \wedge T}^* > u | u) \leq \mathbb{E}_\mathcal{F}_0 \left[ \int_{(0, \tau \wedge T]} \eta((X_{s-})^p)^p u \mathbb{I}_{(X^*_s)^p \leq u} dA_s \right] + \mathbb{E}_\mathcal{F}_0 [H_T] \wedge u. \]  \hspace{1cm} (43)

We integrate this inequality w.r.t. $p u^{p-2} du$ and apply the formulas (35). To this end we first compute the single terms:

\[ \int_0^\infty \mathbb{P}(X_{\tau \wedge T}^* > u | u) p u^{p-1} du = \mathbb{E}_\mathcal{F}_0 \left[ \int_0^\infty \mathbb{I}_{(X_{\tau \wedge T}^*)^p > u} p u^{p-1} du \right] \overset{(35)}{=} \mathbb{E}_\mathcal{F}_0 [(X_{\tau \wedge T}^*)^p] \]

\[ \int_0^\infty \mathbb{E}_\mathcal{F}_0 \left[ \int_{(0, \tau \wedge T]} \eta((X_{s-})^p)^p u \mathbb{I}_{(X^*_s)^p \leq u} dA_s \right] p u^{p-2} du = \mathbb{E}_\mathcal{F}_0 \left[ \int_{(0, \tau \wedge T]} \eta((X_{s-})^p)^p u \mathbb{I}_{(X^*_s)^p \leq u} p u^{p-2} du dA_s \right] \]

\[ \overset{(35)}{=} \mathbb{E}_\mathcal{F}_0 \left[ \int_{(0, \tau \wedge T]} \eta((X_{s-})^p)^p u \mathbb{I}_{(X^*_s)^p \leq u} dA_s \right] \]

Moreover, the calculations of the proof of Lemma 4.7 imply:

\[ \mathbb{E}_\mathcal{F}_0 \left[ \int_0^\infty H_{\tau \wedge T}^{L,u} p u^{p-2} du \right] \leq \mathbb{E}_\mathcal{F}_0 \left[ \int_0^\infty H_{\tau \wedge T}^{L,u} p u^{p-2} du \right] = \begin{cases} 
\alpha_1^p \alpha_2^p \mathbb{E}_\mathcal{F}_0 [H_T^p] & \text{if } H \text{ predictable or } \Delta M \geq 0, \\
\alpha_1^p \mathbb{E}_\mathcal{F}_0 [H_T^p] & \text{if } \mathbb{E}[H_T] < \infty \end{cases} \]

Combining the calculations implies for any bounded stopping time $\tau$

\[ \mathbb{E}_\mathcal{F}_0 [(X_{\tau \wedge T}^*)^p] \leq \mathbb{E}_\mathcal{F}_0 \left[ \int_{(0, \tau \wedge T]} \eta((X_{s-})^p)^p dA_s + \tilde{H}_{\tau \wedge T} \right]. \]  \hspace{1cm} (44)
By applying Corollary 4.4 and the general Meyer-Doob decomposition to \((X_{t,T})^p\)\(\geq 0\) and 
\[(\int_{(0,t,T]} \eta_p((X_{s,T-})^p) dA_s + \tilde{H}_t)_{t \geq 0}\]
we obtain that there exists a càdlàg local martingale \((M^S_t)_{t \geq 0}\) with \(M^S_0 = 0\) such that
\[
(X_{t,T})^p \leq \int_{(0,t,T]} \eta_p((X_{s,T-})^p) dA_s + \tilde{H}_t + M^S_t \quad \forall t \geq 0.
\]
(45)

Hence, \(((X_{t,T})^p)_{t \geq 0}, A, \tilde{H}, M^S)\) and \(\eta_p\) satisfy Assumption \(A_{\text{no sup}}\).

The rest of the proof is very similar to the proof of assertion b). Denote by \(Z_t\) the right-hand side of (45). We redefine \(f\) now using \(\tilde{G}_p := \int_x^y \eta_p(d\eta)\) instead of \(G:\n\[
\begin{align*}
\eta_p((X_{s,T-})^p) dA_s + \tilde{H}_t + M^S_t.
\end{align*}
\]

By the same calculation as in the proof of b) we have by Itô’s formula:
\[
f(Z_0, A_0) = H_0, \quad df(Z_t, A_t) \leq \frac{\eta_p(f(Z_{t-}, A_t))}{\eta_p(Z_{t-})} dA_t + \frac{\eta_p(f(Z_{t-}, A_t))}{\eta_p(Z_{t-})} dM^S_t.
\]

So, using that \(f \geq 0\), an application of Fatou’s lemma implies
\[
\mathbb{E}_{\mathcal{F}_T}[f(Z_T, A_T)] = \mathbb{E}_{\mathcal{F}_T}[\tilde{G}_p(Z_T) - A_T]) \leq \mathbb{E}_{\mathcal{F}_T}[\tilde{H}_T].
\]
(46)

We have for all \(x^{1/p} > c_0\) and \(y \in \text{domain}(\tilde{G}^{-1}_p)\)
\[
\tilde{G}_p(x) = (1 - p)G(x^{1/p}), \quad \tilde{G}_p^{-1}(y) = (G^{-1}(\frac{y}{1-p}))^p
\]
(see (19) for details). Together, using \((X_T^p) \leq Z_T\), we have
\[
\mathbb{E}_{\mathcal{F}_T}[G^{-1}(\tilde{G}_p(Z_T)) - \beta A_T]^p = \mathbb{E}_{\mathcal{F}_T}[\tilde{G}_p((X_T^p) - \beta A_T)]
\leq \mathbb{E}_{\mathcal{F}_T}[\tilde{G}_p(Z_T)]
\leq \mathbb{E}_{\mathcal{F}_T}[\tilde{H}_T]
\]
which is assertion b).

\[\square\]

4.4 Proof of sharpness (Lemma 3.5, Theorem 3.6, Theorem 3.8)

**Proof of Lemma 3.5.** We first prove that (28) has a non-negative solution exploiting that either \(b\) or \(\sigma\) are always 0. More precisely, on the time intervals \((0, \varepsilon \delta), (\delta, \delta + \varepsilon \delta), (2\delta, 2\delta + \varepsilon \delta), \ldots\), the coefficient \(b\) is identically 0. On \((\delta, \delta + \varepsilon \delta), (\delta + \varepsilon \delta, 2\delta), (2\delta + \varepsilon \delta, 3\delta), \ldots\) the coefficient \(\sigma\) is 0.

To simplify the notation we define the processes \((B^i_t)_{t \in [0, \varepsilon \delta]}\) by
\[
B^i_t := \begin{cases} 0 & \forall t \in [0, \delta i] \\
\int_{\delta i}^{t} l(u - i \delta) dW_u & \forall t \in [i \delta, i \delta + \delta \varepsilon) \\
\tilde{B}^i_{h(t - i \delta)} & \forall t \in [i \delta, i \delta + \delta \varepsilon].
\end{cases}
\]
where \(\{\tilde{B}^i_t\}_{t \geq 0, i \in \mathbb{N}}\) is a family of independent Brownian motions and \(h(t) := \int_0^t l^2(u) du\) for all \(t \in [0, \varepsilon \delta]\). Note that \(h(0) = 0, h\) is increasing, continuous and \(h(\varepsilon \delta) = +\infty\). Therefore, \((\tilde{B}^i_{h(t)})_{t \in [0, \varepsilon \delta]}\) can be seen as a sped-up Brownian motion.

**Step 1: Construction of \(X_t\) for \(t \in [0, \delta]\)**

We define
\[
\tau_0 := \inf\{t \in [0, \varepsilon \delta] \mid 1 + B^0_t = 0\} \quad \text{setting here } \inf \emptyset := 0.
\]
As we assumed that the underlying filtered probability space satisfies the usual conditions, i.e. is in particular complete, \(\tau_0\) is indeed a stopping time. As on \([0, \delta \varepsilon]\) we have \(g_{\delta, \varepsilon} = 0\), i.e. \(b = 0\), the path-dependent SDE (28) corresponds for all \(t \in [0, \delta \varepsilon]\) to:
\[
X_t = 1 + \int_0^t b(s, X) ds + \int_0^t \sigma(s, X) dW_s = 1 + \int_0^t \mathbb{1}_{\{X_s > 0\}} l(s) dW_s.
\]
Hence, $X_t = 1 + B_{t \wedge \tau_0}^0$ satisfies (28) for $t \in [0, \varepsilon \delta]$. Note, that $\tau_0 < \varepsilon \delta$ on $\Omega$. By construction, we have $X_{\varepsilon \delta} = X_{\tau_0} = 0$ $P$-almost surely. For all $t \in [\delta \varepsilon, \delta]$ we have (due to $\sigma$ being 0 here):

$$X_t = X_{\varepsilon \delta} + \int_{\varepsilon \delta}^t b(s, X)ds + \int_{\varepsilon \delta}^t \sigma(s, X)dw_s = \int_{\varepsilon \delta}^t X_s^* ds = (t - \delta \varepsilon)X_{\tau_0}^*.$$

In particular, we have $X_{\delta \varepsilon} = \gamma X_{\tau_0}^*$ $P$-almost surely where $\gamma := (1 - \varepsilon)\delta < 1$. So, we have constructed the following non-negative solution (upto a null set) on the time interval $[0, \delta]$:

$$X_t = \begin{cases} 1 + B_{t \wedge \tau_0}^0 & \forall t \in [0, \varepsilon \delta] \\ (t - \delta \varepsilon)X_{\tau_0}^* & \forall t \in (\varepsilon \delta, \delta]. \end{cases}$$

Due to $\tau_0 < \varepsilon \delta$ and $\int_0^\delta l^2(u)du < \infty$ for all $t < \varepsilon \delta$, we have

$$\int_0^\delta |b(s, X)|ds + \int_0^\delta |\sigma(s, X)|^2ds = \int_0^{\varepsilon \delta} b(s, X)ds + \int_0^{\delta - \varepsilon} |\sigma(s, X)|^2ds < \infty \quad P\text{-a.s.}$$

**Step 2: Construction of $X_t$ for $t \in (k\delta, (k + 1)\delta]$**

Assume we have constructed $(X_t)_{t \in [0, k\delta]}$ for some $k \in \mathbb{N}$. Now we construct a solution on $(k\delta, (k + 1)\delta]$. Similarly as before, we set $\tau_k := \inf\{t \in [k\delta, k\delta + \varepsilon \delta] \mid X_{k\delta} + B_t^k = 0\}$ setting here $\inf \emptyset := 0$.

Due to completeness of the underlying filtered probability space $\tau_k$ is a stopping time. By definition, we have $g_{\varepsilon \delta} = 0$ i.e. $b = 0$ on $(k\delta, k\delta + \varepsilon \delta)$, and hence (28) corresponds for $t \in [k\delta, k\delta + \varepsilon \delta]$ to

$$X_t = X_{k\delta} + \int_{k\delta}^t b(s, X)ds + \int_{k\delta}^t \sigma(s, X)dw_s = X_{k\delta} + \int_{k\delta}^t \mathbb{1}_{\{X_s > 0\}}(s - k\delta)dw_s$$

and therefore $X_t = X_{k\delta} + B_{t \wedge \tau_k}^k$ is a solution of (28) for $t \in (k\delta, k\delta + \varepsilon \delta]$. As before we have $\tau_k < k\delta + \varepsilon \delta$ on $\Omega$ and $X_{(k + 1)\delta} = X_{\tau_k} = 0$ $P$-almost surely. For all $t \in (k\delta + \varepsilon \delta, (k + 1)\delta]$ we have (due to $\sigma$ being 0 here):

$$X_t = X_{k\delta + \varepsilon \delta} + \int_{k\delta + \varepsilon \delta}^t b(s, X)ds + \int_{k\delta + \varepsilon \delta}^t \sigma(s, X)dw_s = \int_{k\delta + \varepsilon \delta}^t X_s^* ds = (t - k\delta - \varepsilon \delta)X_{\tau_k}^* \leq \gamma X_{\tau_k}^*$$

and $X_{(k + 1)\delta} = \gamma X_{\tau_k}^*$. Hence,

$$X_t = \begin{cases} X_{k\delta} + B_{t \wedge \tau_k}^k & \forall t \in [k\delta, k\delta + \varepsilon \delta] \\ (t - (k\delta + \varepsilon \delta))X_{\tau_k}^* & \forall t \in (k\delta + \varepsilon \delta, (k + 1)\delta] \end{cases}$$

is a non-negative solution of (28). By the same calculation as in step 1 it satisfies (27).

**Step 3: Proof of $E[(X_{k\delta + \varepsilon \delta}^*)^p] = E[(X_{\tau_k}^*)^p] = (1 - p)^{-1}(1 + p(1 - p)^{-1}\gamma)^k$ for all $p \in (0, 1), k \in \mathbb{N}$**

We prove the equality by induction over $k$. To this end, note that if $(X_{\tau_k}^*)^p > (X_{\tau_{k-1}}^*)^p$, then the supremum of $X^p$ on $[0, \tau_k]$ must occur on $[k\delta, \tau_k]$, since by construction

$$X_t = \begin{cases} 0 & \text{if } t \in [\tau_{k-1}, (k - 1)\delta + \varepsilon \delta] \\ \int_{(k - 1)\delta + \varepsilon \delta}^t X_s^* ds \leq \gamma X_{\tau_{k-1}}^* & \text{if } t \in [(k - 1)\delta + \varepsilon \delta, k\delta] \\ X_{k\delta} + B_{t \wedge \tau_k}^k = \gamma X_{\tau_{k-1}}^* + B_{t \wedge \tau_k}^k & \text{if } t \in [k\delta, \tau_k] \end{cases}$$

and $\gamma = (1 - \varepsilon)\delta < 1$. Define for some fixed $c > 0$:

$$\sigma := \inf\{t \in [k\delta, k\delta + \varepsilon \delta] \mid (B_t^k + X_{k\delta})^p = (X_{\tau_{k-1}}^*)^p + c\}$$

$$= \inf\{t \in [k\delta, k\delta + \varepsilon \delta] \mid B_t^k = ((X_{\tau_{k-1}}^*)^p + c)^{1/p} - X_{k\delta}\},$$

setting $\inf \emptyset = k\delta + \varepsilon \delta$. Due to $X_{k\delta} = \gamma X_{\tau_{k-1}}$ we have $((X_{\tau_{k-1}}^*)^p + c)^{1/p} - X_{k\delta} \geq 0$. By the definition of $\sigma$ we have

$$\{(X_{\tau_k}^*)^p > (X_{\tau_{k-1}}^*)^p + c\} = \{\sigma < \tau_k\}.$$
By the independence of the Brownian motions $\tilde{B}_k$, $k \in \mathbb{N}_0$ we have:

$$\mathbb{P}[(X^*_{t\delta})^p \geq (X^*_{t\delta-1})^p + c \mid \mathcal{F}_{t\delta}] = \mathbb{P}[\sigma < \tau_k \mid \mathcal{F}_{t\delta}] = \frac{X^*_{k\delta}}{((X^*_{t\delta-1})^p + c)^{1/p}},$$

as $\mathbb{P}[\sigma < \tau_k \mid \tilde{F}_{k-1}]$ is the conditional probability that the Brownian motion $B^k$ hits $((X_{t\delta-1})^{\ast}p + c)^{1/p} - X_{k\delta}$ before $-X_{k\delta}$. Applying the previous equation gives:

$$\mathbb{E}[(X^*_{t\delta})^p] = \mathbb{E}[(X^*_{t\delta-1})^p + \int_0^\infty \mathbb{P}[(X^*_{t\delta})^p \geq (X^*_{t\delta-1})^p + u \mid \mathcal{F}_{k-1}]du]
= \mathbb{E}[(X^*_{t\delta-1})^p] + \int_0^\infty \frac{X_{k\delta}}{(u)^{1/p}}d\mu
= \mathbb{E}[(X^*_{t\delta-1})^p] + \frac{p}{1 - p} \mathbb{E}[X_{k\delta}]^{(X^*_{t\delta-1})^{1+p}}.$$

Applying that $X_{k\delta} = \gamma X^*_{t\delta-1}$ implies

$$\mathbb{E}[(X^*_{t\delta})^p] = (1 + p(1 - p)^{-1})\mathbb{E}[(X^*_{t\delta-1})^p].$$

Noting that $\mathbb{E}[(X^*_{t\delta})^p] = \mathbb{E}[\sup_{s \in [0, t]} (1 + B_s^p)] = \frac{1}{t}$ and iterating (47) yields:

$$\mathbb{E}[(X^*_{t\delta})^p] = (1 - p)^{-1} \left(1 + p(1 - p)^{-1}\right)^k.$$

\begin{proof}[Proof of Theorem 3.6] For $\epsilon, \delta \in (0, 1)$ let $(X^\epsilon_{\delta})_{t \geq 0}$ denote the process from Lemma 3.5. By Lemma 3.5 the process $(X^\epsilon_{\delta})_{t \geq 0}$ is non-negative, adapted and continuous. Moreover, it satisfies

$$X^\epsilon_{t\delta} \leq \int_0^t (X^\epsilon_{s\delta})^\ast ds + M^\epsilon_{t\delta} + 1 \quad \forall t \geq 0$$

for $M^\epsilon_{t\delta} := \int_0^t \sigma(s, X^\epsilon_{\delta})dW_s$, which is a continuous local martingale starting in 0. Moreover, by Lemma 3.5 we have

$$\mathbb{E}[(X^\epsilon_{t\delta})^\ast_{k\delta + \epsilon^\delta}]^p = \beta (1 + p\beta \gamma)^k.$$

Due to $X^\epsilon_{\delta}$, $M^\epsilon_{\delta}$ and $H = 1$ satisfying the assumptions of Theorem 3.6, the inequality

$$\|X^\epsilon_{t\delta}\|^p \leq \tilde{\alpha} H \exp(\tilde{\beta} t)$$

holds true by assumption (30). Rearranging the inequality and choosing $t = (k + 1)\delta$ implies for all $k \in \mathbb{N}$:

$$\tilde{\beta} \geq ((k + 1)\delta)^{-1} \log(\|X^\epsilon_{(k+1)\delta}\|^p) - ((k + 1)\delta)^{-1} \log(\tilde{\alpha}).$$

By inserting (48), we have for all $k \in \mathbb{N}$:

$$\tilde{\beta} \geq \frac{k}{k + 1} \log \left( 1 + p\beta \gamma \right)^{1/\delta} + \frac{1}{\delta} \log(1 + (k + 1)\delta)^{-1} \log(\tilde{\alpha}),$$

i.e. taking the limits $k \to \infty$, $\epsilon \to 0$, $\delta \to 0$ gives

$$\tilde{\beta} \geq \lim_{\delta \to 0, \epsilon \to 0} \frac{1}{\delta} \log \left( 1 + \beta \delta (1 - \epsilon) \right) = \lim_{\delta \to 0} \frac{1}{\delta} \log(1 + \beta (p \delta)) = \frac{\partial}{\partial x} \log(1 + \beta x) \bigg|_{x = 0} = \beta$$

which implies the assertion.
\end{proof}

\begin{proof}[Proof of Theorem 3.8] Fix some $\epsilon, \delta \in (0, 1)$ and $k \in \mathbb{N}$ such that $T = k\delta + \epsilon \delta$. Let $(X_t)_{t \geq 0}$ be the process from Lemma 3.5, which satisfies (31) for $M_t := \int_0^t \sigma(s, X)dW_s$. We have

$$\sup_{p \in (0, 1)} \left( (1 - p)\mathbb{E}[(X^\ast_t)^p] \right) = \sup_{p \in (0, 1)} \left( 1 + \frac{p}{1 - p} (1 - \epsilon) \right)^k = \infty.$$
Assume that there exists a $0 < C < \infty$ such that $\mathbb{P}[X_T^* > u] \leq C \frac{1}{u}$ for all $u > 0$. Then e.g. by (35) we have
\[
\mathbb{E}[(X_T^*)^p] = 1^p + p \int_1^\infty \mathbb{P}[X_t > u] u^{p-1} du \leq 1 + pC \int_1^\infty u^{p-2} du = 1 + C \frac{p}{1-p}
\]
which implies
\[
\sup_{p \in (0,1)} (1-p)\mathbb{E}[(X_T^*)^p] \leq \sup_{p \in (0,1)} (1-p) + Cp \leq 1 + C < \infty,
\]
which is a contradiction. This proves $\sup_{u>0} (u\mathbb{P}[X_T^* > u]) = \infty$. □

5 Special case: Sharp stochastic Gronwall inequalities

In this section we summarize the results in the literature for the linear case $\eta(x) = x$ and compare them to the inequalities of this paper.

Von Renesse and Scheutzow [40, Lemma 5.4] developed a stochastic Gronwall inequality for continuous martingales to study stochastic functional differential equations. This result was further generalized by Mehri and Scheutzow [30, Theorem 2.1], who applied Lenglart’s domination inequality in the proof.

**Theorem 5.1** (Mehri and Scheutzow: A stochastic Gronwall inequality for $A_{\text{sup}}$). Let Assumption $A_{\text{sup}}$ hold and assume that $A$ is deterministic and $\eta(x) \equiv x$. Then, the following estimates hold for $p \in (0,1)$ and $T > 0$

\[
\|X_T^*\|_{p,\mathcal{F}_T} \leq \begin{cases} 
\frac{p-1}{p}c_{p}\|H_T\|_{p,\mathcal{F}_T}E_{\mathbb{F}}^{-1}e^{\beta A_T} & \text{if } \mathbb{E}[H_T^p] < \infty \text{ and } H \text{ is predictable}, \\
\frac{p-1}{p}(c_{p}^2 + 1)^{1/p}\|H_T\|_{p,\mathcal{F}_T}e^{\beta A_T} & \text{if } \mathbb{E}[H_T^p] < \infty \text{ and } \Delta M \geq 0, \\
\frac{p-1}{p}c_{p}\|H_T\|_{1,\mathcal{F}_T} & \text{if } \mathbb{E}[H_T] < \infty,
\end{cases}
\]

where $c_{p} = \alpha_1 \alpha_2 = (1-p)^{-1/p}p^{-1}$.

The following is a corollary of Theorem 3.1 a), Theorem 3.6 and Theorem 3.9. It slightly sharpens the result above and extends it to predictable integrators $A$:

**Corollary 5.2** (A sharp stochastic Gronwall inequality for $A_{\text{sup}}$). Let Assumption $A_{\text{sup}}$ hold and assume $\eta(x) \equiv x$ and $p \in (0,1)$. Then, the following estimates hold for all $T > 0$

\[
\|e^{-\beta A_T}X_T^*\|_{p,\mathcal{F}_T} \leq \begin{cases} 
\alpha_1 \|H_T\|_{p,\mathcal{F}_T} & \text{if } \mathbb{E}[H_T^p] < \infty \text{ and } H \text{ predictable}, \\
\alpha_1 \|H_T\|_{1,\mathcal{F}_T} & \text{if } \mathbb{E}[H_T^p] < \infty \text{ and } \Delta M \geq 0, \\
\alpha_1 \|H_T\|_{1,\mathcal{F}_T} & \text{if } \mathbb{E}[H_T] < \infty,
\end{cases}
\]

The constants $\alpha_1 = (1-p)^{-1/p}, \alpha_2 = (1-p)^{-1/p}p^{-1}$ and $\beta = (1-p)^{-1}$ are sharp. If $\|e^{\beta A_T}\|_{q/(p-q),\mathcal{F}_T}$ is integrable, we have for $0 < q < p < 1$ and all $T > 0$

\[
\|X_T^*\|_{q,\mathcal{F}_T} \leq \begin{cases} 
\alpha_1 \|H_T\|_{p,\mathcal{F}_T}e^{\beta A_T} \|e^{\beta A_T}\|_{q/(p-q),\mathcal{F}_T} & \text{if } \mathbb{E}[H_T^p] < \infty \text{ and } H \text{ predictable}, \\
\alpha_1 \|H_T\|_{1,\mathcal{F}_T}e^{\beta A_T} & \text{if } \mathbb{E}[H_T^p] < \infty \text{ and } \Delta M \geq 0, \\
\alpha_1 \|H_T\|_{1,\mathcal{F}_T}e^{\beta A_T} & \text{if } \mathbb{E}[H_T] < \infty.
\end{cases}
\]

In Assumption $A_{\text{sup}}$, it is assumed that $A$ is predictable. For an example that this assumption cannot be dropped see Counterexample 7.1.

A stochastic Gronwall lemma (in a setting nearly identical to $A_{\text{nosup}}$ with $\eta(x) \equiv x$) was proven for continuous martingales $M$ by Scheutzow [37, Theorem 4]. This result was extended by Xie and Zhang [42, Lemma 3.7] to càdlàg martingales:

**Theorem 5.3** (Xie and Zhang: A stochastic Gronwall inequality for $A_{\text{nosup}}$). Let Assumption $A_{\text{nosup}}$ hold. Furthermore, assume that $\eta(x) \equiv x$ and that $A$ is continuous. Then, for any $0 < q < \tilde{p} < 1$ and $t \geq 0$, we have:

\[
\|X_t^*\|_{q} \leq \left(\frac{\tilde{p}}{\tilde{p} - q}\right)^{1/q} \|H_t\| \|e^{A_t}\|_{\tilde{p}/(1-\tilde{p})}.
\]

The following corollary of Theorem 3.1 and Theorem 3.9 slightly extends and marginally sharpens [42, Lemma 3.7]. Recall that Theorem 3.1 b) was proven by further developing the proof idea of [42, Lemma 3.7].
Corollary 5.4 (A sharp stochastic Gronwall inequality for $A_{\text{no sup}}$). Let Assumption $A_{\text{no sup}}$ (see Definition 2.2) hold and assume $\eta(x) \equiv x$ and $p \in (0, 1)$. Then, the following estimates hold.

a) $(L^p$ estimates, $p \in (0, 1))$ For all $T > 0$ we have:

$$
\|e^{-A_t} X_T^\tau\|_{K, F_0} \leq \begin{cases}
\alpha_1 \|H_T\|_{K, F_0} + \|e^{A_T}\|_{q/p, q, F_0} & \text{if } E[H_T^p] < \infty \text{ and } H \text{ predictable,} \\
\alpha_1 \|H_T\|_{K, F_0} & \text{if } E[H_T] < \infty.
\end{cases}
$$

The constants $\alpha_1 = (1 - p)^{-1/p}$ and $\alpha_1 \alpha_2 = (1 - p)^{-1/p} - 1$ are sharp. If $\|e^{A_T}\|_{q/p, q, F_0}$ is integrable, we have for $0 < q < p < 1$

$$
\|X_T^\tau\|_{q, F_0} \leq \begin{cases}
\alpha_1 \|H_T\|_{K, F_0} + \|e^{A_T}\|_{q/p, q, F_0} & \text{if } E[H_T^p] < \infty \text{ and } H \text{ predictable,} \\
\alpha_1 \|H_T\|_{K, F_0} & \text{if } E[H_T] < \infty.
\end{cases}
$$

b) $(L^{1, w}$ estimates) We have for all $T > 0, u > 0, w > 0$ and $R > 0$

$$
\mathbb{P}[e^{-A_T} X_T^\tau > u \mid F_0] \leq \begin{cases}
\frac{1}{w} E_{F_0}[H_T \wedge (\lambda u)] + \mathbb{P}[H_T \geq \lambda u \mid F_0] & \text{if } H \text{ is predictable,} \\
\frac{1}{w} E_{F_0}[H_T \wedge (\lambda u)] + \mathbb{P}[H_T \geq \lambda u \mid F_0] & \text{if } \Delta M \geq 0, \quad \text{(49)}
\end{cases}
$$

and

$$
\mathbb{P}[X_T^\tau > u \mid F_0] \leq \begin{cases}
\frac{1}{w} E_{F_0}[H_T \wedge w] + \mathbb{P}[H_T \geq w \mid F_0] + \mathbb{P}[A_T > R \mid F_0] & \text{if } H \text{ is predictable,} \\
\frac{1}{w} E_{F_0}[H_T \wedge w] + \mathbb{P}[H_T \geq w \mid F_0] + \mathbb{P}[A_T > R \mid F_0] & \text{if } \Delta M \geq 0, \quad \text{(50)}
\end{cases}
$$

Remark 5.5. Theorem 3.8 shows that under $A_{\text{sup}}$ the $L^{1, w}$ norm may be infinite, hence the estimates (49) and (50) do not hold under the weaker assumption $A_{\text{no sup}}$.

Remark 5.6 (Comparison of constants). Choose any $0 < q < p < 1$ and set $\tilde{p} := qp/(q + p - q)$. Then, $q < \tilde{p} < 1$ holds and Theorem 5.3 implies:

$$
\|X_T^\tau\|_q \leq \left(\frac{p}{q} \frac{1}{1 - \frac{1}{p}}\right)^{1/q} \|H_T\|_{1} \|e^{A_T}\|_{q/p, q}.
$$

Noting that, due to $0 < q < p < 1$ we have

$$
\left(\frac{p}{q} \frac{1}{1 - \frac{1}{p}}\right)^{1/q} \geq \left(\frac{1}{1 - \frac{1}{p}}\right)^{1/q} \geq \left(\frac{1}{1 - \frac{1}{p}}\right)^{1/p},
$$

the constant in Corollary 5.4 is slightly sharper than that in Theorem 5.3. However, for deterministic $A$, Theorem 5.3 yields the sharp constant. The choice of the norm $\|e^{A_T}\|_{\tilde{p}/(1 - \tilde{p})} = e^{A_t}$ has no effect, so we may take the limit $\tilde{p} \to 1$ implying the (optimal) constant $(1 - q)^{-1/q}$.

Remark 5.7. Under assumption $A_{\text{no sup}}$, random $A$, deterministic $H$ and $p \in (0, 1)$ there exists no finite constant $\tilde{c}$ such that

$$
\|X_T^\tau\|_p \leq \tilde{c} \|H e^{A_T}\|_p.
$$

holds. For a counterexample, see [14, Example 6.1].

6 Application: Path-dependent SDEs

The convex stochastic Bihari-LaSalle inequalities are, like the stochastic Gronwall inequalities, useful tools to study SDEs. We provide in the following two applications of Section 3 to path-dependent SDEs driven by Brownian motion. Alternatively also more general (e.g. Levy-driven) path-dependent SDEs could be studied by the same approach.
For solutions of non-path-dependent SDEs, exponential integrability bounds are known, see e.g. Cox, Hu, and Jentzen [10], Hude, Hu and Mazzonetto [19] and the references therein. As explained in the introduction, these proofs do not extend to the case of path-dependent SDEs, because terms fail to cancel out in the path-dependent case. Using the stochastic Bihari-LaSalle inequality for \( \eta(x) = x(\log(x) + c) \) we provide a similar result for path-dependent SDEs.

The second application is connected to tail estimates of path-dependent SDEs: We obtain as a corollary of Lemma 3.5 (which is the key lemma to show the sharpness of the constant \( \beta \)) that path-dependent SDEs may have tails of a different order than than non-path-dependent SDEs.

Assume an underlying filtered probability space \((\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t \geq 0})\) satisfying the usual conditions. Let \( B \) be an \( m \)-dimensional Brownian motion and let \(| \cdot |\) denote the Frobenius norm on \( \mathbb{R}^{d \times m} \). We study the path-dependent SDE with random coefficients driven by the Brownian motion \( B \):

\[
\begin{aligned}
dX_t &= f(t, X_t)dt + g(t, X_t)dB_t \\
X_t &= z_t, \quad t \in [-r, 0],
\end{aligned}
\]

where \( r > 0 \) is some constant and the initial condition \((z_t)_{t \in [-r, 0]}\) has continuous paths and is \( \mathcal{F}_0 \) measurable. Denote by \( \mathcal{P} \) the predictable \( \sigma \)-field on \([0, \infty) \times \Omega\). Let \( \mathcal{B}(C([-r, \infty) ; \mathbb{R}^d)) \) denote the Borel \( \sigma \)-field on the continuous functions \( C([-r, \infty) ; \mathbb{R}^d) \) induced by convergence in the uniform norm on compacts sets. Assume that the coefficients

\[
\begin{align*}
f : ([0, \infty) \times \Omega \times C([-r, \infty); \mathbb{R}^d), \mathcal{P} \otimes \mathcal{B}(C([-r, \infty); \mathbb{R}^d))) &\to (\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d)), \\
g : ([0, \infty) \times \Omega \times C([-r, \infty); \mathbb{R}^d), \mathcal{P} \otimes \mathcal{B}(C([-r, \infty); \mathbb{R}^d))) &\to (\mathbb{R}^{d \times m}, \mathcal{B}(\mathbb{R}^{d \times m}))
\end{align*}
\]

are measurable mappings. For every \( t \in [0, \infty) \), \( \omega \in \Omega \), \( \xi \in U \) assume that \( f(t, \omega, x) \) and \( g(t, \omega, x) \) only depend on the path segment \( x(s), s \in [-r, t] \). We denote by \( f(t, x) \) and \( g(t, x) \) the corresponding random variables.

### 6.1 Exponential moment estimates

For \( U = (U(s, y))_{s \in [0,T], y \in \mathbb{R}^d} \in C^{1,2}([0, T] \times \mathbb{R}^d, \mathbb{R}) \) we define for all \( x \in C([-r, \infty); \mathbb{R}^d) \) and all \( t \geq 0 \)

\[
(\mathcal{G}_{f,g}U)(t, x) := (\frac{\partial}{\partial y_{1}}U)(t, x(t)) + (D_y U)(t, x(t))f(t, x) + \frac{1}{2}\text{trace}(g(t, x)g(t, x)^T)\text{Hess}_y U(t, x(t)),
\]

where \( D_y := (\frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial y_d}) \).

One difference of the following corollary to corresponding results for non-path-dependent SDEs, see [10, Corollary 2.4] and [19, Corollary 3.3], is, that we assume that \( U \) is non-negative.

See also [20, Theorem 2.1], in which moment estimates for path-dependent SDEs are proven using the stochastic Gronwall inequality [30, Theorem 2.1].

**Corollary 6.1** (Exponential moment estimates for path-dependent SDEs). Let \( X \) be a (strong) solution of the SDE (51) satisfying \( \int_0^t |f(s, X)|ds + \int_0^t |g(s, X)|^2ds < \infty \mathbb{P}\text{-a.s. for all } t \in [0, T] \). Let \( U = (U(s, y))_{s \in [0,T], y \in \mathbb{R}^d} \in C^{1,2}([0, T] \times \mathbb{R}^d, [0, \infty)) \), and let \( \gamma, \kappa \geq 0 \). Assume that for all \( x \in C([-r, \infty); \mathbb{R}^d), t \in [0, T] \)

\[
(\mathcal{G}_{f,g}U)(t, x) + \frac{1}{2}((D_y U)(t, x(t))g(t, x))^2 \leq \gamma \sup_{s \in [-r, t]} U(s, x(s)) + \gamma \kappa.
\]

Then, for all \( p \in (0, 1) \) and all \( t \in [0, T] \)

\[
\mathbb{E}_{\mathcal{F}_0} \left[ \sup_{t \in [0,T]} \exp(pU(t, X_t)e^{-\gamma \beta T}) \right]^{1/p} \leq \alpha_1 (\alpha_2 \exp(U(0, X_0) + (\kappa + U_0)(1 - e^{-\gamma \beta T})).
\]

where \( U_0 := \sup_{s \in [-r, 0]} U(s, z_s) \) and \( \alpha_1, \alpha_2 \) and \( \beta \) only depend on \( p \) (see (15)).

**Proof of Corollary 6.1.** We apply Itô’s formula to compute \( (Y_t)_{t \in [0,T]} := (U(t, X_t))_{t \in [0,T]} : \)

\[
dY_t = dU(t, X_t) = (G_{f,g}U)(t, X_t)dt + (D_y U)(t, X_t)g(t, X)dB_t.
\]
We apply Itô’s formula to compute \((Z_t)_{t \in [0,T]} := (\exp(Y_t))_{t \in [0,T]}:\)
\[
dZ_t = d\exp(Y_t) = \exp(Y_t) dY_t + \frac{1}{2} \exp(Y_t) d(Y,Y)_t
\]
\[
= \exp(Y_t) \left( (\mathcal{G}_{f,g} U)(t,X) + \frac{1}{2} |(D_u U)(t,X_t)|^2 \right) dt + d\tilde{M}_t
\]
\[
\leq \gamma \exp(Y_t)(Y_t^* + \kappa + \sup_{s \in [-r,0]} U(s,z_s)) dt + d\tilde{M}_t
\]
where \(\tilde{M} := \int_0^t \exp(Y_s) |(D_u U)(s,X_s)|^2 ds\) is a local martingale which starts in 0. By assumption \(U \geq 0\), and therefore \(Z \geq 1\). Hence, we have for \(\eta(x,u_0) = x(\log(x) + \kappa + u_0)\) for \(x \geq 1\) and \(u_0 \geq 0:\)
\[
Z_t \leq Z_0 + \int_0^t \eta(Z_t,U_0) \gamma ds + \tilde{M}_t.
\]
Using that \(U_0\) is \(\mathcal{F}_0\) measurable, we obtain by approximating \(U_0\) from above by a sequence of random variables, where each random variable only takes countably many values, that Theorem 3.1 implies for \(A_t := \gamma t\)
\[
\|G^{-1}(G(Z_t^*) - \beta A_t)\|_{\mathcal{F}_0} \leq \alpha_1 \alpha_2 \|Z_0\|_{\mathcal{F}_0}
\]
where
\[
G(x,\omega) = \log(\kappa + U_0(\omega) + \log(x)), \quad G^{-1}(x,\omega) = \exp(\kappa - \kappa - U_0(\omega)),
\]
while the inverse \(G^{-1}\) is w.r.t. to the variable \(x\) for fixed \(x \in \Omega\). We compute the left-hand side:
\[
\|G^{-1}(G(Z_t^*) - \beta A_t)\|_{\mathcal{F}_0} = \|\exp(\exp(\log(\kappa + U_0 + \log(Z_t^*)) - \beta \gamma t) - \kappa + U_0)\|_{\mathcal{F}_0}
\]
\[
= \|\exp(\exp(\log(\kappa + U_0 + \log(Z_t^*)))e^{-\beta \gamma T} - \kappa + U_0)\|_{\mathcal{F}_0}
\]
\[
= \|\exp((\kappa + U_0 + \log(Z_t^*))e^{-\beta \gamma T})\|_{\mathcal{F}_0} - \|\kappa - U_0\|_{\mathcal{F}_0}
\]
\[
= \|G(Z_t^*)e^{-\beta \gamma T}\|_{\mathcal{F}_0} - \|\kappa - U_0\|_{\mathcal{F}_0}.
\]
Rearranging the terms and recalling the definition \(Z_t = \exp(U(t,X_t))\) yields:
\[
\sup_{t \in [0,T]} \exp(U(t,X_t)e^{-\beta \gamma T}) \leq \alpha_1 \alpha_2 \exp(U(0,X_0)e^{(\kappa + U_0)(1 - e^{-\beta \gamma T})}.
\]

\[\square\]

Remark 6.2. Note that (52) is a weaker assumption than assuming
\[
(\mathcal{G}_{f,g} U)(t,x) \leq \gamma \sup_{s \in [-r,t]} U(s,x(s)) + \gamma \kappa,
\]
and
\[
\frac{1}{2} |(D_u U)(t,x(t))|^2 \leq \gamma \sup_{s \in [-r,t]} U(s,x(s)) + \gamma \kappa,
\]
(53)
since \((\mathcal{G}_{f,g} U)(t,x)\) can also be negative, and hence (52) allows for suitable \(f\) a considerably weaker assumption on \(g\). Alternative proofs, using e.g. the pathwise BDG inequality, Young’s inequality and then e.g. Itô’s formula, only yield estimates under the stronger assumption (53) in path-dependent case. See Example 6.4 for an example, where (52) is satisfied, but not (53).

We use in the following examples (as before) the notation \(\beta = (1 - p)^{-1}\), \(\alpha_1 = (1 - p)^{-1/p}\) and \(\alpha_2 = p^{-1}\).

Example 6.3. Let \(X\) be a solution of the SDE (51) satisfying \(\int_0^t |f(s,X)| ds + \int_0^t |g(s,X)|^2 ds < \infty\) \(\mathbb{P}\)-a.s. for all \(t \geq 0\). For \(U(t,x) := R|x|^2\) for some constant \(R > 0\) the assumption (52) can be slightly strengthened to
\[
2R|x(t)|^2 + R|g(t,x)|^2 + 2R^2 |x(t)|^2 |g(t,x)|^2 \leq \gamma R \sup_{s \in [-r,t]} |x(s)|^2 + \gamma \kappa.
\]
and obtain the estimate
\[
\mathbb{E}_{X_0} \left[ \sup_{t \in [0,T]} \exp(pR|X_t|^2 e^{-\gamma \beta T}) \right]^{1/p} \leq \alpha_1 \alpha_2 \exp(R|z_0|^2 e^{(\kappa + \sup_{s \in [-r,t]} R|z_t|^2)(1 - e^{-\gamma \beta T})}.
\]
In particular, if we assume there exist constants $\gamma_1 \geq 0$, $\gamma_2 \geq 0$ such that for all $\omega \in \Omega$, $t \in [0, T]$ and $x \in C([-r, \infty); \mathbb{R}^d)$
\[ \langle x(t), f(t, x) \rangle \leq \gamma_1 \sup_{s \in [-r, t]} |x(s)|^2, \quad |g(t, x)|_F^2 \leq \gamma_2. \]
Then, we have
\[ \mathbb{E} \left[ \sup_{t \in [0, T]} \exp(pR|X_t|^2e^{-\gamma \beta T}) \right]^{1/p} \leq \alpha_1 \alpha_2 \exp(R|X_0|^2)e^{(\kappa + U_0)(1 - e^{-\gamma \beta T})} \]
where $\gamma := 2\gamma_1 + 2R\gamma_2$, $\kappa := \frac{R\gamma_2}{2\gamma_1 + 2R\gamma_2}$ and $U_0 := \sup_{u \in [-r, 0]} R|z_u|^2$.

**Example 6.4.** Choose $U$ as in the previous example, $d = 1$, $R = 1$. Then, the following coefficients satisfy (52) but not (53):
\[ f(t, x) = -\frac{1}{2}x(t) - x(t)^3 + \sup_{s \leq t} |x(s)|, \quad g(t, x) = x(t)^2 + (\sup_{s \leq t} |x(s)| \wedge 1) \quad \forall x \in C([-r, \infty); \mathbb{R}^d), \forall t \geq 0. \]

**Example 6.5.** Let $X$ be a solution of the SDE (51) satisfying $f^i(t, x, Y) = f_0^i|f(s, X)|ds + f_0^i |g(s, X)|^2 ds < \infty \mathbb{P}$-a.s. for all $t \geq 0$. Choose $U(t, x) := R(|x|^2 + 1)^{1/2}$ for some constant $R > 0$. Let the coefficients $f$ and $g$ satisfy
\[ \langle x(t), f(t, x) \rangle + \frac{1}{2}g(t, x)^2_0 + \frac{1}{2}R(1 + |x(t)|^2)^{-1/2} |g(t, x)|^2 |x(t)|^2 \leq \gamma \sup_{s \in [-r, t]} (|x(s)|^2 + 1)^{1/2}(1 + |x(t)|^2)^{1/2}, \quad (54) \]
then Corollary 6.1 implies for all $p \in (0, 1)$ and all $T \geq 0$
\[ \mathbb{E} \left[ \sup_{t \in [0, T]} \exp(pR(|X_t|^2)^{1/2}e^{-\gamma \beta T}) \right]^{1/p} \leq \alpha_1 \alpha_2 \exp \left( R(|z_0|^2 + 1)^{1/2} + \kappa + R \sup_{t \in [-r, 0]} |z|^2 (1 + |x|^2)^{1/2} \right)(1 - e^{-\gamma \beta T}). \]

This can be seen as follows: The function $y \mapsto R(|y|^2 + 1)^{1/2}$ is convex, hence Hess$_y U(t, y)$ is positive semidefinite for any $t \geq 0$, $y \in \mathbb{R}^d$. Moreover, we have trace(Hess$_y U(t, x(t))) \leq Rd(1 + |x(t)|^2)^{-1/2}$. Using that trace$(A_1A_2)$ $\leq$ trace$(A_1)$trace$(A_2)$ for symmetric positive semidefinite matrices $A_1, A_2$, the assumption (52) can be strengthened to
\[ (1 + |x(t)|^2)^{-1/2} R \langle x(t), f(t, x) \rangle + \frac{1}{2} \text{trace}(g(t, x)g(t, x)^T) Rd(1 + |x(t)|^2)^{-1/2} \]
\[ + \frac{1}{2}(1 + |x(t)|^2)^{-1} R^2 |g(t, x)|^2 |x(t)|^2 \leq \gamma R \sup_{s \in [-r, t]} (|x(s)|^2 + 1)^{1/2}. \]

Multiplying with $(1 + |x(t)|^2)^{1/2} R^{-1}$ implies (54).

In particular, if there exist constants $\gamma_1 \geq 0$, $\gamma_2 \geq 0$ such that for all $\omega \in \Omega$, $t \in [0, T]$ and $x \in C([-r, \infty); \mathbb{R}^d)$
\[ \langle x(t), f(t, x) \rangle \leq \gamma_1 \sup_{s \in [-r, t]} (|x(s)|^2 + 1)^{1/2}(1 + |x(t)|^2)^{1/2}, \quad |g(t, x)|_F^2 \leq \gamma_2 \sup_{s \in [-r, t]} (|x(s)|^2 + 1)^{1/2}, \]
then, we have
\[ \mathbb{E} \left[ \sup_{t \in [0, T]} \exp(pR(|X_t|^2 + 1)^{1/2}e^{-\gamma \beta T}) \right]^{1/p} \leq \alpha_1 \alpha_2 \exp \left( R(|z_0|^2 + 1)^{1/2} + U_0(1 - e^{-\gamma \beta T}) \right) \]
where $\gamma := \gamma_1 + \frac{d}{2} \gamma_2 + \frac{R}{2} \gamma_2$ and $U_0 := \sup_{u \in [-r, 0]} R(|z|^2 + 1)^{1/2}$.

**6.2 Tail estimates**

**Remark 6.6** (Estimate for non-path-dependent SDEs). Let $Y$ be a global solution of the following (not path-dependent) Brownian-driven SDE
\[ dY_t = \tilde{f}(t, Y_t)dt + \tilde{g}(t, Y_t)dB_t, \quad Y_0 = y_0, \]

28
where $y_0 \in \mathbb{R}^d$ is a deterministic initial value and $\tilde{f}$ and $\tilde{g}$ satisfy suitable measurability conditions to make the SDE well-defined. It can be easily seen that if the coefficients satisfy the following one-sided coercivity condition (for some $K > 0$)

$$2\langle y, \tilde{f}(t, y) \rangle + |\tilde{g}(t, y)|_F^2 \leq K|y|^2 \quad \forall y \in \mathbb{R}^d, \forall t \geq 0$$

then we have (e.g. by applying Theorem 3.1 b) or Corollary 5.4 to $(\langle Y^2 \rangle)_{t \geq 0}$ for all $u > 0$

$$\mathbb{P}\left[ \sup_{t \in [0,T]} |Y^2| > u \right] \leq \frac{e^{KT}}{u|y_0|^2}, \quad (55)$$

i.e. in particular $\|\langle Y^2 \rangle\|_{L^1} \leq e^{KT}|y_0|^2 < \infty$ for all $T \geq 0$.

The following corollary of Lemma 3.5 and Theorem 3.8 shows that SDEs with a path-dependent drift coefficient enjoy in general a faster growth in $u$.

**Corollary 6.7.** Let $d = 1$, $r > 0$ and $T > 0$. For the path-dependent SDE (51) with initial value $z_t = \sqrt{2} \forall t \in [-r, 0]$ there exist coefficients $f$ and $g$ and a global strong solution $(Y_t)_{t \in [-r, \infty)}$ satisfying

b) $\forall t \geq 0, y \in C([-r, \infty); \mathbb{R}^d)$

$$2\langle y(t), f(t, y) \rangle + |g(t, y)|_F^2 \leq \sup_{s \in [0,t]} |y(s)|^2,$$

c) and $\sup_{u > 0} \{u\mathbb{P}[\sup_{t \in [0,T]} Y_t^2 > u]\} = \infty$.

In particular an estimate of the type (55) does not hold.

**Proof of Corollary 6.7.** Fix some $\varepsilon, \delta \in (0,1)$ and $k \in \mathbb{N}$ such that $T = k\delta + \varepsilon \delta$. Let $(X_t)_{t \geq 0}$ be the process from Lemma 3.5. Define $Y_t := \sqrt{X_t + 1}$ for all $t > 0$ and $Y_t := \sqrt{2} = z_t \forall t \in [-r, 0]$. Since $X_t \geq 0$ for all $t$ we have $Y_t \geq 1$ for all $t$ and $X_t = Y_t^2 - 1$ for all $t \geq 0$. Moreover, by Ito’s formula we have

$$dY_t = \frac{1}{2}(X_t + 1)^{-1/2}dX_t - \frac{1}{8}(X_t + 1)^{-3/2}d(X, X)_t = f(t, Y)dt + g(t, Y)dW_t$$

where $\forall t \geq 0, \forall y \in C([-r, \infty), \mathbb{R})$

$$f(t, y) := \left( \frac{1}{2} \frac{b(t, y^2 - 1)}{y_t} - \frac{1}{8} \frac{\sigma^2(s, y^2 - 1)}{y_t^2} \right) \mathbb{1}_{\{y_t > 1/2\}}$$

$$g(t, y) := \frac{1}{2} \sigma(t, y^2 - 1) \mathbb{1}_{\{y_t > 1/2\}}$$

where $b$ and $\sigma$ are defined as in Lemma 3.5. $b(t, y^2 - 1)$ denotes $b(t, (y(t)^2 - 1))_{t \in (0, \infty)}$.

a) We have

$$\int_0^T |f(s, Y)|ds + \int_0^T |g(s, Y)|_F^2 ds \leq \int_0^T \frac{1}{2} |b(s, X)|ds + \left( \frac{1}{8} + \frac{1}{4} \right) \int_0^T |\sigma(s, X)|_F^2 ds$$

which is finite $\mathbb{P}$-a.s. by Lemma 3.5.

b) We have for $y \in C([-r, \infty), \mathbb{R})$ and $t \geq 0$ such that $y(t) > 1/2$:

$$2\langle y(t), f(t, y) \rangle + |g(t, y)|_F^2 = b(t, y^2 - 1) - \frac{1}{8} \sigma^2(t, y^2 - 1) + \frac{1}{4} \sigma^2(t, y^2 - 1)$$

$$= b(t, y^2 - 1) \leq \sup_{u \in [0,t]} y^2(u)$$

C) By Theorem 3.8 we have $\sup_{u > 0} \{u\mathbb{P}[X_T > u]\} = \infty$, which implies

$$\sup_{v > 1} \{v\mathbb{P}[Y_T^2 > v]\} = \sup_{v > 1} \{v\mathbb{P}[X_T > v - 1]\} = \sup_{u > 0} \{(u + 1)\mathbb{P}[X_T > u]\} = \infty.$$
7 Appendix

7.1 Proof of the sharpness of $\alpha_1$ and $\alpha_1\alpha_2$ (Theorem 3.9)

Sketch of proof of Theorem 3.9. The inequalities are proven in Lemma 4.7. Here we only shortly discuss the sharpness of the constants:

Proof of c): Let $B$ be a Brownian motion on a suitable underlying filtered probability space. Choose $H_t \equiv 1$. Let $\tau$ be the time $B$ first hits $-1$ and set $M_t := B_{t,\tau}$ and $X_t := M_t + H_t$ for all $t \geq 0$. The stopping times $\tau$ ensures $X \geq 0$. An easy calculation (which was also used in step 4 of the proof of Theorem 3.6 to compute $\mathbb{E}[(X_{t,\delta})^{+}] = \frac{1}{e^{\varepsilon \delta}}$) gives $\sup_{t \geq 0} X_t \leq (1 - p)^{-1/p}$ for $p \in (0,1)$, which implies the assertion of c).

Proof of a) and b): Note that the assertions of a) and b) concerning sharpness are identical. Fix some $p \in (0,1)$. In the proof of [15, Theorem 2.1]) families of continuous processes $X^{(n)}, n \in \mathbb{N}$ and $H^{(n)}, n \in \mathbb{N}$ are defined, satisfying:

(i) $X^{(n)}$ is non-negative, adapted, continuous,
(ii) $H^{(n)}$ is non-negative, adapted, continuous, non-decreasing,
(iii) $\mathbb{E}[X^{(n)}_{\tau}] \leq \mathbb{E}[H^{(n)}_{\tau}]$ for all bounded stopping times $\tau$,
(iv) and

$$\alpha_1\alpha_2 = \lim_{n \to \infty} \frac{\sup_{t \geq 0} X^{(n)}_t}{\sup_{t \geq 0} H^{(n)}_t}.$$ 

To prove the sharpness assertion of a) and b), it remains to show the existence of a family of local martingales $M^{(n)}, n \in \mathbb{N}$ with no negative jumps and $M^{(n)}_0 = 0$ such that $X^{(n)}_t \leq H^{(n)}_t + M^{(n)}_t$ a.e. for all $t \geq 0, n \in \mathbb{N}$.

To this end, we first shortly recall the definition of $X^{(n)}$ and $H^{(n)}$ from [15, Theorem 2.1]: Let $Z$ be an exponentially distributed random variable on a complete probability space $(\Omega, F, P)$ with $\mathbb{E}[Z] = 1$. Set

$$a : [0, \infty) \to [0, \infty), \ t \mapsto \exp(t/p).$$

Define for all $t \geq 0$

$$\tilde{X}_t := a(Z)1_{[Z, \infty)}(t), \quad \tilde{H}_t := \int_0^{t \wedge Z} a(s) ds.$$ 

Choose $\tilde{\mathcal{F}} := \sigma \{ [Z \leq r] \mid 0 \leq r \leq t \}$ for all $t \geq 0$. The compensator of $\tilde{X}$ is $\tilde{H}$ due to $Z$ being exponentially distributed. Now we use $\tilde{X}$ and $\tilde{H}$ to construct the families of processes $X^{(n)}, n \in \mathbb{N}$ and $H^{(n)}, n \in \mathbb{N}$. Assume w.l.o.g. that there exists a Brownian motion $B$ on $(\Omega, F, P)$ such that $B$ is independent of $Z$. Denote by $(\mathcal{F}_t)_{t \geq 0}$ the smallest filtration satisfying the usual conditions which contains $(\tilde{\mathcal{F}}_t)$, and with respect to which $B$ is a Brownian motion. Denote by $g_{n,n+1} : [0, \infty) \to [0,1]$ a continuous non-decreasing function such that

$$g_{n,n+1}(t) = 0 \quad \forall t \in [0, n], \text{ and } g_{n,n+1}(t) = 1 \quad \forall t \in [n+1, \infty).$$

Define:

$$\tau^{(n)} := \inf \{ t \geq n + 1 \mid \tilde{X}_n + (B_{t} - B_{n+1})1_{[t \geq n+1]} = 0 \},$$

$$X^{(n)}_t := g_{n,n+1}(t) \tilde{X}_n + (B_{t \wedge \tau^{(n)}} - B_{t \wedge (n+1)}),$$

$$H^{(n)}_t := \tilde{H}_{t \wedge n}.$$ 

The stopping time $\tau^{(n)}$ ensures that $X^{(n)}$ is non-negative. Define

$$M^{(n)}_t := \tilde{X}_{t \wedge n} - \tilde{H}_{t \wedge n} + B_{t \wedge \tau^{(n)}} - B_{t \wedge (n+1)},$$

which is a local martingale (recall that $\tilde{H}$ is the compensator of $\tilde{X}$). Moreover,

$$X^{(n)}_t \leq \tilde{X}_{t \wedge n} + (B_{t \wedge \tau^{(n)}} - B_{t \wedge (n+1)}) \leq H^{(n)}_t + M^{(n)}_t.$$ 

It is easily seen that $M$ only has non-negative jumps, as $\tilde{X}$ only has non-negative jumps. 

\[\square\]
7.2 Proof of time change lemma (Lemma 4.8)

Proof of Lemma 4.8. We prove the lemma for Assumption $\mathcal{A}_{\text{nosup}}$, the proof for Assumption $\mathcal{A}_{\text{sup}}$ is up to one minor difference (mentioned in the proof below) identical.

To keep the notation simple, we first prove the claim for the special case that $A$ has at most one jump. Afterwards we extend the same technique to prove the assertion for $A$ having finitely many jumps on each path. By approximation we then can prove the assertion for general $A$.

Step 1: We prove the lemma under the additional assumption that $A$ has at most one jump.

We will first smoothen out the jump of $A$ yielding $(\hat{X}, \hat{A}, \hat{H}, \hat{M})$ and $(\tilde{F}_t)_{t \geq 0}$ enjoying properties $\mathcal{A}_{\text{nosup}}$ and (c) - (e) and such that $\hat{A}$ is continuous and strictly increasing. By a time shift the family $(\hat{X}, \hat{A}, \hat{H}, \hat{M})$ will yield $(\check{X}, \check{A}, \check{H}, \check{M})$ i.e. the claim of the lemma.

More precisely, define $\tau_1 := \inf\{t > 0 \mid \Delta A_t > 0\}$ where $\inf\emptyset := \infty$. We smoothen the jump of $A$ by inserting at time $\tau_1$ a time interval of length $1$:

$$\hat{A}_t := \begin{cases} A_t & \forall t \in [0, \tau_1) \\ \text{linear interpolation between } A_{\tau_1 -} \text{ and } A_{\tau_1} & \forall t \in [\tau_1, \tau_1 + 1) \\ A_{\tau_1 -} & \forall t \in [\tau_1 + 1, \infty). \end{cases}$$

We also define correspondingly time-changed processes $\check{X}, \check{A}, \check{H}, \check{M}$ such that $(\check{X}, \check{A}, \check{H}, \check{M})$ satisfy $\mathcal{A}_{\text{nosup}}$. Note that the following definition would not work unless $M$ has only non-negative jumps, which is also the reason why the choice $\check{X}_t := X_{A_t^{-1}}$ (for a generalized inverse $A^{-1}$) does not work in general:

$$\check{Y}_t := \begin{cases} Y_t & \forall t \in [0, \tau_1) \\ Y_{\tau_1} & \forall t \in [\tau_1, \tau_1 + 1) \\ Y_{\tau_1 -} & \forall t \in [\tau_1 + 1, \infty) \end{cases} \quad \text{for } Y \in \{X, M, H\}.$$

The processes $\check{X}, \check{A}, \check{H}, \check{M}$ will not satisfy (14) on the interval $[\tau_1, \tau_1 + 1)$ in general: At time $t = \tau_1$ the local martingale $M$ might have a large negative jump $\Delta M_{\tau_1} \ll 0$ such that the right-hand side of (14) becomes negative, hence in particular less than $X_{\tau_1}$. Instead, we define in the case of Assumption $\mathcal{A}_{\text{nosup}}$:

$$\tilde{Y}_t := \begin{cases} Y_t & \forall t \in [0, \tau_1) \\ Y_{\tau_1 -} & \forall t \in [\tau_1, \tau_1 + 1) \\ Y_{\tau_1 -} & \forall t \in [\tau_1 + 1, \infty) \end{cases} \quad \text{for } Y \in \{X, M, H\}.$$

By this definition, $(\check{X}, \check{A}, \check{H}, \check{M})$ satisfy (14): For $t \in (0, \tau_1)$ this is trivial; for $t = \tau_1$ this follows by taking the left limits, for $t \in (\tau_1, \tau_1 + 1)$ only the integral term $\int_{[0,t]} \eta(X_{s-})dA_s$ is increasing, hence it follows from (14) holding for $t = \tau_1$. For $t \geq \tau_1 + 1$ (14) corresponds to (14) for $(X, A, H, M)$ at time $t - 1$.

In the case of Assumption $\mathcal{A}_{\text{sup}}$ we slightly modify the definition of $\check{X}$ as we do not assume that $X$ has left limits in this case:

$$\tilde{X}_t := \begin{cases} X_t & \forall t \in [0, \tau_1) \\ \liminf_{s \uparrow \tau_1} X_s & \forall t \in [\tau_1, \tau_1 + 1) \\ X_{\tau_1 -} & \forall t \in [\tau_1 + 1, \infty). \end{cases}$$

Due to the supremum in the integral $\int_{[0,t]} \eta(\tilde{X}_{s-})dA_s$ the processes $(\check{X}, \check{A}, \check{H}, \check{M})$ satisfy (13).

Now we define a suitable filtration which ensures that $\check{M}$ is a local martingale. Due to $A$ being predictable, there exists an announcing sequence $\tau_1^{(n)}$ of $\tau_1$. Define the following stopping times for each $n \in \mathbb{N}$ and $t \geq 0$:

$$\sigma_n(t) := \begin{cases} t \land \tau_1^{(n)} & t \in [0, \tau_1^{(n)} + 1) \\ t - 1 & t \in [\tau_1^{(n)} + 1, \infty). \end{cases}$$

Note that $\sigma_n(t)$ in non-decreasing in $n$ (for each $t \geq 0$ and each $\omega \in \Omega$). Moreover, we have $\lim_{n \to \infty} Y_{\sigma_n(t)} = \tilde{Y}_t$ for $Y = X, H, M$ in the case of Assumption $\mathcal{A}_{\text{nosup}}$. (In the case of Assumption $\mathcal{A}_{\text{sup}}$ this only holds for $Y = H, M$ however the following argumentation is still possible.) We choose $\tilde{F}_t := \ell_{\in \mathbb{N}} \sigma_n(t)$ and
denote by \((\hat{F}_t)_{t \geq 0}\) the smallest filtration that contains \((\hat{F}_t)_{t \geq 0}\) and satisfies the usual conditions. Clearly, \(\hat{X}, \hat{H}\) and \(\hat{M}\) are adapted w.r.t. to \((\hat{F}_t)_{t \geq 0}\). The property d) is immediate and by definition e) is satisfied.

We show that \(\hat{M}\) is a local martingale w.r.t. \((\hat{F}_t)_{t \geq 0}\) (and hence also \((\hat{F}_t)_{t \geq 0}\)):

By change of variables we obtain

This implies by the definition of \(\hat{F}_t\) for all \(t < s < \tau\). Hence the \(\sigma\)-system \(\cup_{n \in \mathbb{N}} \mathcal{F}_{\sigma_n(t)}\) is a subset of the Dynkin system \(\mathcal{D}\), so the \(\lambda\)-\(\pi\) theorem implies that \(\mathcal{F}_s \subseteq \mathcal{D}\). This proves that \(\hat{M}\) is indeed a local martingale w.r.t. \((\hat{F}_t)_{t \geq 0}\) (and hence also \((\hat{F}_t)_{t \geq 0}\)).

We show that predictability of \(H\) implies predictability of \(\hat{H}\): Continuous adapted processes remain continuous and adapted by the transformation \(Y \rightarrow \hat{Y}\). So we apply the monotone class theorem to \(\mathcal{H} = \{Y : [0, \infty) \times \Omega \rightarrow \mathbb{R} | \hat{Y}\text{ is predictable and bounded}\}\), noting that \(\mathcal{M} = \{Y : [0, \infty) \times \Omega \rightarrow \mathbb{R} | \hat{Y}\text{ is continuous, adapted, bounded}\}\) is closed under multiplication and contained in \(\mathcal{H}\).

Now we use \((\hat{X}, \hat{A}, \hat{H}, \hat{M})\) and \((\hat{F}_t)_{t \geq 0}\) to construct \((X, \hat{A}, \hat{H}, \hat{M})\) and \((\Omega, \mathcal{F}, \mathcal{F}, (\hat{F}_t)_{t \geq 0})\), i.e. prove Step 1. Noting that \(\hat{A}\) is continuous, strictly increasing and \(\hat{A}_\infty = +\infty\), \(\hat{A}_\infty^{-1}(\omega) : [0, \infty) \rightarrow [0, \infty)\) is a well-defined mapping for every \(\omega \in \Omega\). Note that \(\hat{A}_\infty^{-1}\) is a \((\hat{F}_t)_{t \geq 0}\) stopping time for all \(t \geq 0\). We define for all \(t \geq 0\)

We verify that the family \((\hat{X}, \hat{A}, \hat{H}, \hat{M})\) satisfies inequality (14) of \(\mathcal{A}_{\text{no sup}}^\text{st}\):

By change of variables we obtain

By change of variables we obtain

We verify that \(A_{s}^\tau)_{t \geq 0}\) is a \((\hat{F}_t)_{t \geq 0}\) stopping time for any \(s \geq 0\): For this we first show that \(\tau_{1}^{(n)}\) is a \((\hat{F}_t)_{t \geq 0}\) stopping time. We have due to \(t \land \tau_{1}^{(n)} \leq \sigma_{n}(t)\) that \(\mathcal{F}_{t \land \tau_{1}^{(n)}} \subseteq \mathcal{F}_{\sigma_{n}(t)}\). This implies:

This implies by the definition of \(\hat{A}\) and by using that \(\hat{A}_\tau^{-1}\) is a \((\hat{F}_t)_{t \geq 0}\) stopping time:

Hence b) is satisfied.

As \(\hat{A}_\tau^{-1}\) is continuous, it is clear that c) and d) hold true. The right-continuity of \((\hat{F}_t)_{t \geq 0}\) and \((\hat{A}_\tau^{-1})_{t \geq 0}\) implies that \((\hat{F}_t)_{t \geq 0}\) is a right-continuous filtration, therefore e) holds.

**Step 2:** We define \(\hat{X}, \hat{H}, \hat{M}\) for general \(A\).

The paths of \(A\) are strictly increasing. We define the generalized inverse \(A_\tau^{-1}(\omega) : [0, \infty) \rightarrow [0, \infty)\) pathwise by

\[
A_\tau^{-1}(\omega) = \inf\{r \geq 0 \mid A_r(\omega) \geq t\}
\]
which satisfies $A^{-1}(\omega) \circ A(\omega) = \text{id}_{[0,\infty)}$ and $A(\omega) \circ A^{-1}(\omega)|_{\text{range}(A(\omega))} = \text{id}_{\text{range}(A(\omega))}$ on each path. Note that due to $A$ being càdlàg, we have $(A(\omega) \circ A^{-1}(\omega))_t \geq t$. Note that this implies
\[
\forall t_1 \in \text{range}(A(\omega)) \text{ and } t_2 > t_1 \text{ we have } A^{-1}_{t_2}(\omega) > A^{-1}_{t_1}(\omega), \tag{56}
\]
hence $A^{-1}$ is not strictly increasing, only non-decreasing. We define
\[
\tilde{Y}_t = \begin{cases} 
Y_{A^{-1}_t} & \text{if } t \in \text{range}(A) \\
\liminf_{r \nearrow A^{-1}_t} Y_r & \text{if } t \notin \text{range}(A) 
\end{cases} \quad \text{for all } Y \in \{X, H, M\}. \tag{57}
\]
This definition generalizes the definition of Step 1. It is easily checked that $\tilde{Y}$ is right-continuous: Let $t \notin \text{range}(A)$. Then, $A_s = A_t \leq t$ for $s = A^{-1}_s$. This implies $(t, A_s) \cap \text{range}(A) = \emptyset$ and that we have $A^{-1}_s = A_r$ for all $r \in (t, A_s)$. Together this implies that $\tilde{Y}$ is right-continuous in $t$. Now consider $t \in \text{range}(A)$. $A^{-1}$ is continuous and non-decreasing, hence for $t_n \searrow t$ we have $A^{-1}_{t_n} \searrow A^{-1}_t$, and hence the right-continuity of $Y$ and (56) imply $\tilde{Y}$ is right-continuous in $t$. By similar arguments, using that $A^{-1}$ is continuous and non-decreasing, it can be verified that $\tilde{Y}$ has left limits if $Y$ has left limits.

Moreover, for $t \in \text{range}(A)$ we have
\[
\tilde{X}_t \leq \int_{[0,t]} \eta(\tilde{X}_s) \, ds + \tilde{M}_t + \tilde{H}_t \quad \mathbb{P}\text{-a.s}
\]
due to change of variables (using $\tilde{Y}_s = Y_s \circ A^{-1}_s$ and $\{ x \in [0,t] \mid A^{-1}_x \in [a,b] \} = [A_a, A_b]$ due to (56). For $t \notin \text{range}(A)$ it follows from the same argument as in Step 1.

However, defining a suitable filtration $(\tilde{F}_t)_{t \geq 0}$ seems to be non-trivial: The choice $\tilde{F}_t = \mathcal{F}_{A_t}$ is not possible since $\tilde{M}$ is not a martingale with respect to this filtration. Instead, we first find a filtration for the special case that $A$ has at most finitely many jumps on each path (Step 3) and then obtain the desired result by an approximation argument (Step 4). For Step 3 it is crucial that $A$ is predictable.

**Step 3:** We prove the lemma under the following additional assumption: Assume that the jumps of $A$ be bounded from below, i.e. that $\exists c > 0$ such that for all $t \geq 0$ and all $\omega \in \Omega$ it holds that $\Delta A_t(\omega) \notin (0, c)$.

We construct the filtration $(\tilde{F}_t)_{t \geq 0}$ needed to complete the construction of Step 2 by repeating the construction of Step 1. Using that $A$ has on each path on every bounded time interval at most finitely many jumps, we may smoothen the jumps of $A$ by inserting at the $k$-th jump of $A$ a time interval of length $2^{-k}$. To this end, we set $\tau_0 := 0$ and denote by $\tau_k$ the time of the $k$-th jump of $A$, i.e. $\tau_k := \inf\{ t > \tau_{k-1} \mid \Delta A_t \neq 0 \}$ using $\inf \emptyset := \infty$. Moreover, set $s_0 := 0$ and $s_k := \sum_{i\leq k} 2^{-i}$. Noting that for any $T > 0$ we have finitely many $k$ with $\tau_k < T$ and $s_\infty = 1$, we define:
\[
\hat{A}_t := \begin{cases} 
A_{t-s_i} & \text{for } t \in [\tau_i + s_i, \tau_{i+1} + s_i), i \in \mathbb{N}_0, \\
\text{linear interpolation between } A_{\tau_{i+1}} \text{ and } A_{\tau_i} & \text{for } t \in [\tau_{i+1} + s_i, \tau_{i+1} + s_{i+1}), i \in \mathbb{N}_0.
\end{cases}
\]
\[
\hat{Y}_t := \begin{cases} 
Y_{t-s_i} & \text{for } t \in [\tau_i + s_i, \tau_{i+1} + s_i), i \in \mathbb{N}_0, \\
Y_{\tau_{i+1}} & \text{for } t \in [\tau_{i+1} + s_i, \tau_{i+1} + s_{i+1}), i \in \mathbb{N}_0,
\end{cases} \quad \text{for } Y \in \{X, M, H\}.
\]

We can construct a sequence of announcing times $\tau_i^{(n)}$, $n \in \mathbb{N}$, $i \in \mathbb{N}$ such that each $(\tau_i^{(n)})_n$ announces $\tau_i$ and
\[
0 = \tau_0 < \tau_1 < \tau_2 < \cdots < \tau_i < \tau_{i+1} < \cdots
\]
on $\{\tau_{i+1} < \infty\}$. Moreover, we define $\tau_0^{(n)} := 0$ for all $n \in \mathbb{N}$. We define analogously as before:
\[
\sigma_n(t) := \begin{cases} 
t - s_i & \text{for } t \in [\tau_i^{(n)} + s_i, \tau_{i+1}^{(n)} + s_i), i \in \mathbb{N}_0, \\
\tau_{i+1}^{(n)} & \text{for } t \in [\tau_{i+1}^{(n)} + s_i, \tau_{i+1}^{(n)} + s_{i+1}), i \in \mathbb{N}_0.
\end{cases}
\]
Set $\tilde{F}_t := \bigvee_{n \in \mathbb{N}} \mathcal{F}_{\sigma_n(t)}$ and denote by $(\hat{F}_t)_{t \geq 0}$ the smallest filtration that contains $(\tilde{F}_t)_{t \geq 0}$ and satisfies the usual conditions. By the same arguments as in the first part of the proof, $(\Omega, \mathcal{F}, \mathbb{P}, (\hat{F}_t)_{t \geq 0})$ and $(\hat{X}_t)_{t \geq 0}$, $(\hat{A}_t)_{t \geq 0}$, $(\hat{H}_t)_{t \geq 0}$, $(\hat{M}_t)_{t \geq 0}$ satisfy $\mathcal{A}_{\text{nosup}}$ and c-o). As in Step 1 we define for all $t \geq 0$
\[
\tilde{Y}_t := \tilde{Y}_{\hat{A}_t} \quad \text{for } Y \in \{X, A, H, M\} \quad \text{and} \quad \hat{F}_t := \hat{F}_{\hat{A}_t}^{-1},
\]
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which satisfy by the same arguments the assertion of this lemma. Moreover, this definition yields the same processes as definition (57).

\textbf{Step 4:} We prove the assertion of the lemma (without additional assumptions on $A$).

In (57) of Step 2 we already defined $\tilde{X}, \tilde{H}, \tilde{M}$. It remains to find a suitable filtration and prove that $\tilde{M}$ is indeed a local martingale. To this end we use Step 3 and approximations. We define

\[ A_t^{n, small} := \sum_{s \leq t} \Delta A_s \mathbb{1}_{[\Delta A_s \leq \frac{1}{n}]}, \quad A_t^{(n)} := A_t - A_t^{n, small}, \quad \forall t \geq 0, \]

i.e. for all $t \geq 0$ we have $\Delta A_t^{(n)} \notin (0, \frac{1}{n})$. We have

\[ X_t \leq \int_0^t \eta(X_s^-)dA_s^{(n)} + M_t + H_t^{(n)}, \quad \text{where} \quad H_t^{(n)} := H_t + \int_0^t \eta(X_s^-)dA_s^{n, small} \quad \forall t \geq 0. \quad (58) \]

We apply Step 3 to $(X, A^{(n)}, H^{(n)}, M)$ and $(\Omega, F, \mathbb{P}, (\mathcal{F}_t)_{t \geq 0})$ so that we obtain a sequence $(\tilde{X}^{(n)}, \tilde{A}^{(n)}, \tilde{H}^{(n)}, \tilde{M}^{(n)})$, $n \in \mathbb{N}_0$ and $(\Omega, F, \mathbb{P}, (\tilde{\mathcal{F}}_t^{(n)})_{t \geq 0})$. We define

\[ \tilde{\mathcal{F}}_t := \bigcap_{n \in \mathbb{N}_0} \tilde{\mathcal{F}}_t^{(n)}. \]

Proving claims 4a-4d finishes Step 4:

\textbf{Claim 4a:} We have $\tilde{Y}_t = \lim_{n \to \infty} \tilde{Y}_t^{(n)}$ for $Y \in \{X, H, M\}$. In particular we have that $\tilde{X}, \tilde{H}, \tilde{M}$ are $(\tilde{\mathcal{F}}_t)_{t \geq 0}$-adapted.

We first show $\tilde{Y}_t = \lim_{n \to \infty} \tilde{Y}_t^{(n)}$ for $t \in \text{range}(A)$: For $t \in \text{range}(A)$ we have $\tilde{Y}_t = Y_{A_t^{-1}}$. By construction we have $A_t^{(n)} \leq A_t^{(n+1)} \leq A_s$ for all $n \in \mathbb{N}$, $r \geq 0$, which implies $(A_t^{(n)})^{-1} \geq (A_t^{(n+1)})^{-1} \geq A_s^{-1}$ for all $r \geq 0$. This implies $(A_t^{(n)})^{-1}$ is non-increasing in $n$. It can be verified that $(A_t^{(n)})^{-1} \uparrow A_t^{-1}$ for $n \not\to \infty$. For $t$ such that $(A_t^{(n)})^{-1} = A_t^{-1}$ and $t \in \text{range}(A)$ it holds that $t \in \text{range}(A^{(n)})$. So we obtain $\lim_{n \to \infty} \tilde{Y}_t^{(n)} = \lim_{n \to \infty} A_{t^{-1}} Y_t = \tilde{Y}_t$.

Now we show $\tilde{Y}_t = \lim_{n \to \infty} \tilde{Y}_t^{(n)}$ for $t \notin \text{range}(A)$: In this case $\exists s \geq 0, \epsilon > 0 \text{ s.t. } A_s \leq t < t + \epsilon < A_s$. This implies that $A_s^{(n)} \leq A_s \leq t + \epsilon \leq A_t^{(n)} \leq A_s$ for sufficiently large $n$, and hence $(A_t^{(n)})^{-1} = s = A_s^{-1}$ and $r \notin \text{range}(A^{(n)})$ for sufficiently large $n$. This implies $\tilde{Y}_t = \lim_{n \to \infty} \tilde{Y}_t^{(n)}$.

Noting that $\tilde{\mathcal{F}}_t^{(n+1)} \subseteq \tilde{\mathcal{F}}_t^{(n)}$ for all $n \in \mathbb{N}$ and using $\tilde{Y}_t = \lim_{n \to \infty} \tilde{Y}_t^{(n)}$ implies that $\tilde{X}, \tilde{H}, \tilde{M}$ are $(\tilde{\mathcal{F}}_t)_{t \geq 0}$ adapted.

\textbf{Claim 4b:} $\tilde{M}$ is a local $(\tilde{\mathcal{F}}_t)_{t \geq 0}$ martingale.

Let $(\tau_n^M)_n$ be a localizing sequence of $M$, then define a new localizing sequence with respect to $(\tilde{\mathcal{F}}_t)_{t \geq 0}$ by

\[ \tau_n^{\tilde{M}} := \inf\{t \geq 0 \mid (M_t^{\tau_n^M})^{-1} - \tilde{M}_t \neq 0\} \geq A_{\tau_n^M}, \]

where $M_t^{\tau_n^M}$ denotes the stopped process. The inequality $\tau_n^{\tilde{M}} \geq A_{\tau_n^M}$ follows from $M_t^{\tau_n^M}$ and $M$ coinciding up to time $\tau_n^M$. By the Debut theorem $\tau_n^{\tilde{M}}$ are indeed $(\tilde{\mathcal{F}}_t)_{t \geq 0}$ stopping times. Due to $A_{\infty} = \infty$ it is indeed a localizing sequence.

Hence, we may assume w.l.o.g. that $M$ is a martingale with $\mathbb{E}[|M_{\infty}|] < \infty$. Recall that by Step 3 $M^{(n)}$ is a $(\tilde{\mathcal{F}}_t^{(n)})_{t \geq 0}$ (local) martingale. Fix some $0 \leq s < t$. We want to prove

\[ \tilde{\mathcal{F}}_s \subseteq \{A \in \mathcal{F} \mid \mathbb{E}[(\tilde{M}_t - \tilde{M}_s)1_A] = 0\}. \]

Let $A \in \tilde{\mathcal{F}}_s = \bigcap_{n \in \mathbb{N}} \tilde{\mathcal{F}}_s^{(n)}$. Then we have

\[ \mathbb{E}[(\tilde{M}_t - \tilde{M}_s)1_A] \overset{\text{Claim 4a}}{=} \mathbb{E} \left[ \lim_{n \to \infty} (\tilde{M}_t^{(n)} - \tilde{M}_s^{(n)})1_A \right] = \lim_{n \to \infty} \mathbb{E}[(\tilde{M}_t^{(n)} - \tilde{M}_s^{(n)})1_A] = 0. \]

For the second equality we used $\tilde{M}_t^{(n)} = \mathbb{E}[M_{\infty} \mid \tilde{\mathcal{F}}_t^{(n)}]$ (since $\tilde{M}_t^{(n)} = M_{\infty}$), i.e. uniform integrability with respect to $n$. For the third equality we used that $A \in \tilde{\mathcal{F}}_s^{(n)}$ and that $M^{(n)}$ is a $(\tilde{\mathcal{F}}_t^{(n)})_{t \geq 0}$ martingale.

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This implies that $\tilde{M}$ is a martingale.

**Claim 4c:** If $H$ is predictable then $\tilde{H}$ is predictable.

This follows from applying the monotone class theorem to $\mathcal{H} = \{Y : [0, \infty) \times \Omega \to \mathbb{R} \mid Y \text{ is predictable and bounded}\}$, noting that $\mathcal{M} = \{Y : [0, \infty) \times \Omega \to \mathbb{R} \mid Y \text{ continuous, adapted, bounded}\}$ is closed under multiplication and contained in $\mathcal{H}$.

**Claim 4d:** For any $s \geq 0$ we have that $A_s$ is $(\mathcal{F}_t)_{t \geq 0}$ stopping time.

This follows from

$$\{A_s \leq t\} = \bigcap_{n \in \mathbb{N}} \{A_s^{(n)} \leq t\} = \bigcap_{n=m}^{\infty} \{A_s^{(n)} \leq t\} \in \mathcal{F}_t^{(m)} \quad \forall t \geq 0, m \in \mathbb{N}.$$  

### 7.3 Proof of the Snell corollary (Corollary 4.4)

**Proof.** We first prove the assertion for the special case that $H_0 \leq n_0$ for some $n_0 \in \mathbb{N}$. We start by defining a suitable localization sequence of bounded stopping times $(\sigma_n)_{n \geq n_0}$ which ensures that $\{X_{t \wedge \sigma_n} - H_{t \wedge \sigma_n} \mid \tau \text{ finite stopping time}\}$ is bounded from below by $-n$ and a uniformly integrable family of random variables. To this end let $(\tau_n)_{n \geq n_0}$ be a localizing sequence such that $H_{\tau_n} \leq n$ for all $n \in \mathbb{N}, n \geq n_0$, which exists due to the predictability of $H$ and the assumption that $H_0 \leq n_0$. For all $n \geq n_0$ set

$$\sigma_n := \inf\{t \geq 0 \mid X_t \geq n\} \wedge \tau_n \wedge n.$$  

The family $\{X_{t \wedge \sigma_n} - H_{t \wedge \sigma_n} \mid \tau \text{ finite stopping time}\}$ is uniformly integrable due to

$$\mathbb{E}[\sup_{t \geq 0} |X_{t \wedge \sigma_n} - H_{t \wedge \sigma_n}|] \leq \mathbb{E}[\sup_{t \geq 0} X_{t \wedge \sigma_n} + H_{t \wedge \sigma_n}] \leq \mathbb{E}[X_{t \wedge \sigma_n} + n + H_{\tau_n}] \leq n + 2\mathbb{E}[H_{\tau_n}] < \infty,$$

where we used that $H \geq 0$, $X \geq 0$ for the first inequality and (33) for the third inequality.

Due to the choice of $\sigma_n$ we may apply the Snell envelope theorem [12, Appendix 1: (22), p.416-417] to $Y_t^{(m,n)} = X_{t \wedge \sigma_n} - H_{t \wedge \sigma_n} + m$, where $m \in \mathbb{N}$ is a new parameter. We define $N_t^{(m,n)} := Z_t^{(m,n)} - m$ where $Z^{(m,n)}$ denotes the optional strong supermartingale given by Snell envelope theorem. Note that $N^{(m,n)}$ is by definition the minimal optional strong supermartingale such that

$$\max\{X_{t \wedge \sigma_n} - H_{t \wedge \sigma_n} - m\} \leq N_t^{(m,n)} \quad \forall t \geq 0.$$  

Hence, due to $\max\{X_{t \wedge \sigma_n} - H_{t \wedge \sigma_n} - m\} \leq N_t^{(m+1,n)}$ and $\max\{X_{t \wedge \sigma_n} - H_{t \wedge \sigma_n} - (m + 1)\} \leq N_t^{(m,n)}$, we have

$$N_t^{(m,n)} \leq N_t^{(m+1,n)} \quad \text{and} \quad N_t^{(m+1,n)} \leq N_t^{(m,n)} \quad \forall t \geq 0, \; n, \; m \in \mathbb{N}, \; n \geq n_0. \quad (59)$$  

Let $I$ denote the set of all $(\mathcal{F}_t)_{t \geq 0}$ stopping times. In the Snell envelope theorem the convention $Y^{(m,n)}_t := 0$ is used and the essential supremum runs over all (not necessarily finite) stopping times $S$. However, due to $Y^{(m,n)} \geq 0$ for all $m \geq n$ we have $Z_t^{(m,n)} \leq \mathbb{E}[Y_S^{(m,n)} \mid \mathcal{F}_t]$, and therefore $Z_0^{(m,n)} \leq 0 + m$ i.e. $N_0^{(m,n)} \leq 0$ for all $m \geq n \geq n_0$ by using (33). Due to $N^{(m,n)}$ being an optional strong supermartingale and $0 \leq N_t^{(m,n)} + H_{t \wedge \tau_k}$ for all $k, m, n$, we have

$$\mathbb{E}[|N_t^{(m,n)}|] \leq 2\mathbb{E}[(N_t^{(m,n)})^{-}] \leq 2\mathbb{E}[H_{\tau_k}]. \quad (60)$$  

We define $N^{(n)} := \lim_{m \to \infty} N_t^{(m,n)}$ for all $t \geq 0$, which converges P-a.s. by (59) and in $L^1$ to a supermartingale due to (60). Clearly $N_0^{(n)} \leq 0$ and $X_{t \wedge \sigma_n} \leq H_{t \wedge \sigma_n} + N_t^{(m,n)}$ for all $t$.

We define $\tilde{N}_t := \lim_{n \to \infty} N_t^{(n)}$ for all $t \geq 0$ which converges pointwise due to (59) and $\tilde{N}_{t \wedge \tau_k} = L^1\lim_{n \to \infty} N_t^{(n)}$, due to (60). Hence, $\tilde{N}$ is a local supermartingale with localizing sequence $(\tau_k)_{k \in \mathbb{N}}$. Clearly $\tilde{N}_0 \leq 0$ and $X_t \leq H_t + \tilde{N}_t$. As $X$ and $H$ are right-continuous, also the right-limits process $(\tilde{N}_{t+})_{t \geq 0}$ satisfies $X_t \leq H_t + \tilde{N}_{t+}$. Since the filtration satisfies the usual conditions, $(\tilde{N}_{t+})_{t \geq 0}$ is a local
supermartingale with a càdlàg modification. Setting $N_t := \tilde{N}_{t+}$ for all $t \geq 0$ and noting $N_0 \mathbb{1}_{\{\tau_n > 0\}} = \mathbb{E}[N_0 \mathbb{1}_{\{\tau_n > 0\}} \mid \mathcal{F}_0] \leq \liminf_{k \to \infty} \mathbb{E}[\tilde{N}_{\tau_n \wedge n} \mathbb{1}_{\{\tau_n > 0\}} \mid \mathcal{F}_0] \leq 0$ proves the claim for the special case.

For the general case that $\mathbb{E}[H_0] < \infty$ we define $A_k := \{\omega \in \Omega \mid H_0(\omega) \in [k, k+1)\}$. As $(X_t \mathbb{1}_{A_k})_{t \geq 0}$ is dominated by $(H_t \mathbb{1}_{A_k})_{t \geq 0}$ and $H_0 \mathbb{1}_{A_k} \leq k + 1$, we can apply the first part of the proof to obtain local càdlàg supermartingales $(N_t^{(k)} \mathbb{1}_{A_k})_{t \geq 0}$. Due to $H$ being predictable and $H_0$ integrable, there exists a localizing sequence for $(\tau_n)_{n \geq 0}$ such that $\mathbb{E}[H_{\tau_n}] < \infty$ for all $n$. Define $N_t := \sum_{k \leq n} N_t^{(k)} \mathbb{1}_{A_k}$. Using $0 \leq N_t + H_t$ for all $t \geq 0$, $\mathbb{E}[H_{\tau_n}] < \infty$ implies that $\mathbb{E}[\sum_{k \leq n} N_t^{(k)} \mathbb{1}_{A_k}] < \infty$ for all $n$, i.e. $(N_t)_{t \geq 0}$ is indeed a local supermartingale.

### 7.4 Counterexample: Predictability of integrator $A$ necessary

The following example is similar to the proof of Theorem 3.9 i.e. [15, Theorem 2.1].

**Counterexample 7.1.** We provide an example that the assumption, that $A$ is predictable, cannot be dropped in Corollary 5.2 and Corollary 5.4. As we used in the proofs of these corollaries the predictability of $A$ solely when applying Lemma 4.8, it implies in particular, that also Lemma 4.8 is false if the predictability assumption is dropped.

More precisely, we provide an example of an adapted continuous process $(X_t)_{t \geq 0}$, a càdlàg martingale $(M_t)_{t \geq 0}$ and an adapted càdlàg non-decreasing process $(A_t)_{t \geq 0}$ which satisfy

\[
0 \leq X_t \leq \int_0^t X_s - dA_s + M_t + 1.
\]

with the property that $\mathbb{E}[\sup_{t \geq 0} X_t^p] = +\infty$ and $\sup_{t \geq 0} \mathbb{E}[e^{qA_t}] < \infty$ for $p \in (0, 1), q > 0$.

Let $Z$ be an exponentially distributed random variable on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with $\mathbb{E}[Z] = 1$. Define for all $t \geq 0$, $p \in (0, 1)$

\[
X_t := \int_0^{t \wedge Z} p^{-1} \exp(s/p)ds + 1 = \exp((Z \wedge t)/p)
\]

\[
M_t := \int_0^{t \wedge Z} p^{-1} \exp(s/p)ds - p^{-1} \exp(Z \wedge t)/p \mathbb{1}_{[Z, \infty)}(t)
\]

\[
H_t := 1
\]

\[
A_t := p^{-1} \mathbb{1}_{[Z, \infty)}(t)
\]

Choose $\tilde{\mathcal{F}}_t := \sigma(\{Z \leq r \mid 0 \leq r \leq t\}$ for all $t \geq 0$ and denote by $(\mathcal{F}_t)_{t \geq 0}$ the smallest filtration satisfying the usual conditions, that contains $(\tilde{\mathcal{F}}_t)_{t \geq 0}$. $M$ is a martingale because $(\mathbb{1}_{[Z, \infty)}(t) - Z \wedge t)_{t \geq 0}$ is a martingale. We have for all $t \geq 0$

\[
X_t = p^{-1} \exp(p^{-1} (Z \wedge t)) \mathbb{1}_{[Z, \infty)}(t) + M_t + 1
\]

\[
= \int_0^t X_s - dA_s + M_t + 1.
\]

For all $q \geq 0$ and $t \geq 0$ we have:

\[
\mathbb{E}[X_t^p] = \mathbb{E}[\exp(Z \wedge t)] = \int_0^t \exp(z) \exp(-z)dz + \exp(t) \exp(-t) = t + 1
\]

\[
\mathbb{E}[\exp(qA_t)] = \mathbb{E}[\exp(qp^{-1} \mathbb{1}_{[Z, \infty)}(t))] \leq \exp(qp^{-1}).
\]

Noting that $\lim_{t \to \infty} \mathbb{E}[X_t^p] = +\infty$ and $\sup_{t \geq 0} \mathbb{E}[\exp(qA_t)] \leq \exp(qp^{-1})$ for any $q > 0$ implies the assertion.

### 7.5 Counterexample: Structure of upper bounds for convex and concave $\eta$ differ

We provide a counterexample which shows that bounds of the type (21) are in general not true for convex $\eta$ under Assumption $\mathcal{A}_{\text{nosup}}$. 

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Counterexample 7.2. Let \((W_t)_{t \geq 0}\) be a Brownian motion on a suitable underlying filtered probability space and let \(x_0 > 0\) and \(\gamma > 0\) be constants. Define:

\[
X_t := e^{\gamma t(x_0 + W_t)^2} \quad \forall t \geq 0.
\]

An application of Itô’s formula implies that \(X\) satisfies the following equation:

\[
dX_t = 2\gamma(x_0 + W_t)X_t dW_t + (\gamma + 2\gamma^2(x_0 + W_t)^2)X_t dt = \eta(X_t)dt + dM_t
\]

where \(dM_t := 2\gamma(x_0 + W_t)X_t dW_t, t \geq 0\) is a local martingale starting in 0, and \(\eta(x) := \gamma(1 + 2 \log(x))x\) for \(x \geq 1\). Note, that \(X_t \geq 1\). The function \(\eta\) can be extended to a convex non-decreasing function on \([0, \infty)\) with \(\eta(0) = 0\). Hence, \(X\) satisfies

\[
X_t = \int_0^t \eta(X_s)ds + M_t + e^{\gamma x_0^2}, \quad \forall t \geq 0.
\]

It can be shown that \(\|X_t\|_p\) explodes at time \(T = \frac{1}{2p\gamma}\), but we only prove here that it explodes at some finite time point. For \(p\gamma\gamma \geq 1/2\) we have

\[
\|X_t\|_p^p \geq \mathbb{E}[X_T^p] = \int_{-\infty}^{\infty} \exp(p\gamma(x_0 + \sqrt{t}w)^2) \exp(-w^2/2)dw \\
= \int_{-\infty}^{\infty} \exp((p\gamma - 1/2)w^2 + 2x_0w\sqrt{t}w + p\gamma x_0^2)dw = +\infty.
\]

We apply Theorem 3.1 to verify that the quantity \(\|X_t\|_p\) is finite for small \(T\). To this end we first compute \(G\) and \(G^{-1}\). We have for all \(x \geq 1, y \geq 0:\n
\[
G(x) := \int_0^x \frac{d\eta(u)}{\eta(u)} = \frac{1}{2\gamma} \log(2 \log(x) + 1), \quad G^{-1}(y) = e^{2\gamma y/2^{1/2}}.
\]

Theorem 3.1 implies for all \(q \in (0, 1)\):

\[
\|G^{-1}(G(X_T^p) - T))\|_q = \|(X_T^p)^{\exp(-2\gamma y)}\|_q \exp(1/2e^{-2\gamma y} - 1) \leq (1 - q)^{-1}e^{\gamma x_0^2}
\]

yielding that \(\|X_t\|_p\) is finite for \(p < \exp(-2\gamma T)\).

As \(G\) and \(G^{-1}\) are bounded on bounded subintervals of \([1, \infty)\), the explosion of the quantity \(\|X_t\|_p\) at a finite time \(t\) is a contradiction to \(X\) having an upper bound of the type (21).

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