On the approximation of the principal eigenvalue for a class of nonlinear elliptic operators

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joint work with

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Consider the eigenvalue problem

\[
\begin{cases}
L\varphi + \lambda \varphi = 0 & \text{in } \Omega, \\
\varphi = 0 & \text{on } \partial \Omega,
\end{cases}
\]

where \( L\varphi(x) = \text{div}(A(x)\nabla \varphi(x)) + b \cdot D\varphi + c\varphi \) for smooth functions \( A, b, c \) in \( \Omega \) with \( \xi^t A\xi \geq \alpha |\xi|^2 \) for some \( \alpha > 0 \).

Let \( \lambda_1 \) be the \textbf{principal eigenvalue} (i.e. the smallest eigenvalue in modulus), then

- \( \lambda_1 \) is real and simple
- There is an eigenfunction \( \varphi_1 \in H_0^1(\Omega) \) which is positive.
- \( \lambda_1 \leq \text{Re}(\lambda) \) for any other eigenvalue \( \lambda \).
- In the symmetric case (\( b = 0 \)), \( \lambda_1 \) is given by the Rayleigh-Ritz variational formula

\[
\lambda_1 = \inf_{\varphi \in H_0^1(\Omega), \varphi \neq 0} \frac{-\int_{\Omega} \nabla \varphi^t A \nabla \varphi + c\varphi^2 \, dx}{\| \varphi \|_{L^2(\Omega)}^2}
\]
Consider the eigenvalue problem

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Linear operator in divergence form

Consider the eigenvalue problem

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\lambda_1 = \inf_{\varphi \in H^1_0(\Omega), \varphi \neq 0} \left\{ -\int_{\Omega} \nabla \varphi^t A \nabla \varphi + c\varphi^2 \, dx \right\} / \|\varphi\|_{L^2(\Omega)}^2
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Consider the eigenvalue problem

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\]
Approximation:
An approximation of \( \lambda_1 \) can be computed via discretization of the weak formulation

\[
\int_\Omega \left( -A \nabla \varphi \nabla \psi + b \nabla \varphi \psi + c \varphi \psi \right) \, dx = \lambda_1 \int_\Omega \varphi \psi \, dx \quad \forall \psi \in H_0^1(\Omega)
\]

by means of standard \( P^1 \) finite elements resulting in the linear equation

\[
A_h \Phi = \lambda_1^h B_h \Phi
\]

where
- \( A_h \) is the stiffness matrix;
- \( B_h \) the mass matrix.

Convergence:
It can be proved that \( \lambda_1^h \to \lambda_1 \) as \( h \to 0 \) and the convergence is of order \( h^2 \).
Note that is $Lu(x) = \text{div} (A(x) \nabla u(x))$, then $\lambda^h_1$ is given by

$$\lambda^h_1 = \inf_{x \in \mathbb{R}^N \setminus \{0\}} \frac{-x^t A_h x}{\|x\|^2}$$

which is the finite dimensional analogous of the Rayleigh-Ritz variational formula

$$\lambda_1 = \inf_{\varphi \in H^1_0(\Omega), \varphi \neq 0 \setminus \{0\}} \frac{-\int_{\Omega} \nabla \varphi^t A \nabla \varphi \ dx}{\|\varphi\|_{L^2(\Omega)}^2}$$

Some references:
- Weinberger, *Variational methods for eigenvalue approximation*, CBMS, 15
- Babuska-Osborn, Math. Comp., 1989
- Boffi, Acta Numerica 2010
- Huang, J. Comput. Phys., 2014
Nonlinear operator

Let \( F[u] := F(x, u, Du, D^2u) \) be a uniformly elliptic operator, positive homogenous of degree 1 (i.e. \( F[tu] = tF[u] \) for \( t \geq 0 \)). Then the principal eigenvalue for \( F \) is defined by means of the formula

\[
\lambda_1 := \sup\{\lambda \in \mathbb{R} : \exists \varphi > 0 \text{ in } \Omega, \ F[\varphi] + \lambda \varphi \leq 0 \};
\]

(from now on all the (in)equalities have to be intended in viscosity sense).

- There is an eigenfunction \( \varphi_1 \) which is positive, i.e. a solution of

\[
\begin{cases}
  F[\varphi] + \lambda_1 \varphi = 0 & \text{in } \Omega, \\
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- \( \varphi_1 \) is the only eigenfunction that does not change sign.
- \( \lambda_1 \) is simple.
- For any \( \lambda < \lambda_1 \) the maximum principle holds for \( F[\cdot] + \lambda \), i.e.

  \[ \text{If } u \text{ is such that } F[u] + \lambda u \geq 0 \text{ in } \Omega, \ u \leq 0 \text{ on } \partial \Omega \text{ then } u < 0 \text{ in } \Omega. \]
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\]
Observe that by definition for any $\lambda \leq \lambda_1$ there exists $\varphi > 0$ such that

$$F[\varphi](x) + \lambda \varphi(x) \leq 0 \quad \forall x \in \Omega$$

Hence

$$\lambda \leq \inf_{x \in \Omega} - \frac{F[\varphi](x)}{\varphi(x)} = -\sup_{x \in \Omega} \frac{F[\varphi](x)}{\varphi(x)}$$

and therefore

$$\lambda \leq \sup_{\varphi > 0} \left( -\sup_{x \in \Omega} \frac{F[\varphi](x)}{\varphi(x)} \right) = -\inf_{\varphi > 0} \sup_{x \in \Omega} \frac{F[\varphi](x)}{\varphi(x)}.$$  

For $\lambda = \lambda_1$ the equality holds in the previous formula and therefore

$$\lambda_1 = -\inf_{\varphi > 0} \sup_{x \in \Omega} \frac{F[\varphi](x)}{\varphi(x)}.$$
The previous formula is similar to an identity characterizing the effective Hamiltonian, i.e.

$$\overline{H}(P) = \inf_{\varphi \in C_{\text{per}}^1} \sup_{x \in \mathbb{T}^n} H(P + D_x \varphi, x)$$

This formula was used by Gomes-Oberman (Sicon 2004) to get the following approximation of $\overline{H}$

$$\overline{H}_h(P) = \inf_{\varphi \in C(T_h)} \sup_{x \in T_h} H(P + D_x \varphi, x)$$

where $T_h$ is a triangulation of $\mathbb{T}^n$ with cells of diameter smaller than $h$ and $C(T_h)$ is the collection of continuous piecewise linear grid functions which interpolate given nodal values. The computation of $\overline{H}_h(P)$ is given by a finite dimensional optimization problem.
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A general class of difference operators

Let $h\mathbb{Z}^n$ be the orthogonal lattice in $\mathbb{R}^n$ where $h > 0$ is a discretization parameter and $C_h$ the space of the mesh functions defined on $\Omega_h = \Omega \subset \mathbb{Z}_h^n$. Consider a discrete operator $F_h$ defined by

$$F_h[u](x) := F_h(x, u(x), [u]_x)$$

where

- $h > 0$ is the discretization parameter ($h$ is meant to tend to 0),
- $x \in \Omega_h$ is a grid point,
- $u \in C_h$
- $[\cdot]_x$ represents the stencil of the scheme, i.e. the points in $\Omega_h\setminus\{x\}$ where the value of $u$ are computed for writing the scheme at the point $x$ (we assume that $[w]_x$ is independent of $w(y)$ for $|x - y| > Mh$ for some fixed $M \in \mathbb{N}$).
Following Kuo-Trudinger (Siam J.Num.Analysis 1992) we introduce some basic assumptions for the difference operator $F_h$:

(i) The operator $F_h$ is of **positive type**, i.e. for all $x \in \Omega_h$, $z, \tau \in \mathbb{R}$, $u, \eta \in C_h$ satisfying $0 \leq \eta(y) \leq \tau$ for each $y \in \Omega_h$, then

$$F_h(x, z, [u + \eta]_x) \geq F_h(x, z, [u]_x) \geq F_h(x, z + \tau, [u + \eta]_x)$$

(ii) The operator $F_h$ is **positive homogeneous**, i.e. for all $x \in \Omega_h$, $z \in \mathbb{R}$, $u \in C_h$ and $t \geq 0$, then

$$F_h(x, tz, [tu]_x) = tF_h(x, z, [u]_x).$$

(iii) The family of operator $\{F_h, 0 < h \leq h_0\}$, where $h_0$ is a positive constant, is **consistent** with operator $F$ on the domain $\Omega \subset \mathbb{R}^n$, i.e. for each $u \in C^2(\Omega)$

$$\sup_{\Omega_h} \left| F(x, u(x), Du(x), D^2 u(x)) - F_h(x, u(x), [u]_x) \right| \to 0 \quad \text{as} \ h \to 0,$$

uniformly on compact subset of $\Omega$. 

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As in the continuous case, we define a principal eigenvalue for $F_h$ by means of the formula

$$\lambda_1^h := \sup\{ \lambda \in \mathbb{R} : \exists \varphi \in C_h, \varphi > 0 \text{ in } \Omega_h, \, F_h[\varphi] + \lambda \varphi \leq 0 \};$$

- There is an eigenfunction $\varphi_1^h$ which is positive, i.e. a solution of

$$\begin{cases}
F[\varphi] + \lambda_1^h \varphi = 0 & \text{in } \Omega_h, \\
\varphi = 0 & \text{on } \Omega_h^C,
\end{cases}$$

- For any $\lambda < \lambda_1$ the maximum principle holds for $F_h + \lambda$, i.e. if $u$ is such that $F_h[u] + \lambda u \geq 0$ in $\Omega_h$ and $u \leq 0$ on $\Omega_h^C$, then $u \leq 0$ in $\Omega_h$.

- $\lambda_1^h$ is given by the finite dimensional optimization problem

$$\lambda_1^h = - \inf_{\varphi \in C_h, \varphi > 0} \sup_{x \in \Omega_h} \frac{F_h[\varphi](x)}{\varphi(x)}$$
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$$\lambda_h^1 := \sup\{\lambda \in \mathbb{R} : \exists \varphi \in C_h, \varphi > 0 \text{ in } \Omega_h, F_h[\varphi] + \lambda \varphi \leq 0 \};$$

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$$\lambda_h^1 = -\inf_{\varphi \in C_h, \varphi > 0} \sup_{x \in \Omega_h} \frac{F_h[\varphi](x)}{\varphi(x)}$$
Convergence of $\lambda^h_1$ to $\lambda_1$ for $h \to 0$

The convergence result cannot rely on standard stability results in viscosity solution theory (Barles-Souganidis’ method) since the limit eigenvalue problem

$$\begin{cases} F[\varphi] + \lambda_1 \varphi = 0 & \text{in } \Omega, \\ \varphi = 0 & \text{on } \partial \Omega, \end{cases}$$

does not satisfy a comparison principle ($\varphi \equiv 0$ and the principal eigenfunction $\varphi_1$ are two distinct solutions of the problem).

So for the approximating operators, we consider a specific class of finite difference schemes $F_h$ introduced by Kuo-Trudinger which satisfy some crucial pointwise estimates which are the discrete analogues of those valid for a general class of fully nonlinear, uniformly elliptic equations.
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We assume that the stencil \([\cdot]_x\) of the scheme is given by \(x + hY\) where \(Y = \{y_1, \ldots, y_k\} \subset \mathbb{Z}^n\) is a finite set containing all the vectors of the canonical basis of \(\mathbb{R}^n\). We consider a finite difference operator of the form

\[
F_h[u] = \mathcal{F}(x, u, \delta_h u, \delta^2_h u)
\]

where \(\mathcal{F} : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^Y \times \mathbb{R}^Y \rightarrow \mathbb{R}\) and

\[
\delta_{h,y} u(x) = \frac{u(x+hy) - u(x-hy)}{2h|y|}
\]

\[
\delta^2_{h,y} u(x) = \frac{u(x+hy) + u(x-hy) - 2u(x)}{h^2|y|^2}
\]

\[
\delta_h u = \{\delta_{h,y} u : y \in Y\}
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\[
\delta^2_h u = \{\delta^2_{h,y} u : y \in Y\}.
\]

Moreover \(\mathcal{F}\) satisfies the following assumptions

\[
\frac{\partial \mathcal{F}}{\partial s_y} - \frac{|hy|}{2} \left| \frac{\partial \mathcal{F}}{\partial q_y} \right| \geq \alpha_0, \quad \frac{\partial \mathcal{F}}{\partial s_y} \leq a_0, \quad \left| \frac{\partial \mathcal{F}}{\partial q_y} \right| \leq b_0
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where \(\alpha_0, a_0, b_0\) are given constants.
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where \(\alpha_0, a_0, b_0\) are given constants.
If $F$ is uniformly elliptic, then it is always possible to find a scheme of the previous type which is of positive type and consistent with $F$. For this class of schemes we have

**Theorem**

Let $(\lambda_1^h, \varphi_1^h)$ be the sequence of the discrete eigenvalues and of the corresponding eigenfunctions associated to $F_h$. Then

$$\lambda_1^h \to \lambda_1, \quad \varphi_1^h \to \varphi_1$$

uniformly in $\overline{\Omega}$ as $h \to 0$, where $\lambda_1$ and $\varphi_1$ are respectively the principal eigenvalue and the corresponding eigenfunction associated to $F$. 

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**Theorem**

Let $(\lambda^h_1, \varphi^h_1)$ be the sequence of the discrete eigenvalues and of the corresponding eigenfunctions associated to $F_h$. Then

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Main ingredients of the proof are

- The semi-relaxed limits in viscosity solution sense;
- A maximum principle for the limit problem (rather than the comparison principle);
- The following local Hölder estimate proved by Kuo-Trudinger:

If \( u_h \) is a solution of \( F_h[u] = f \), then for any \( x, y \in \Omega_h \)

\[
|u_h(x) - u_h(y)| \leq C \frac{|x - y|^\delta}{R} \left( \max_{B^h_R} u_h + \frac{R}{\alpha_0} \left\{ \sum_{x \in \Omega_h} h^n |f(x)|^n \right\}^{\frac{1}{n}} \right),
\]

where \( R = \min\{\text{dist}(x, \partial \Omega_h), \text{dist}(x, \partial \Omega_h)\} \), \( B^h_R = B(0, R) \cap \Omega_h \), \( \delta \), \( \alpha_0 \) and \( C \) are positive constants independent of \( h \).
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where $R = \min\{\text{dist}(x, \partial \Omega_h), \text{dist}(x, \partial \Omega_h)\}$, $B_R^h = B(0, R) \cap \Omega_h$, $\delta$, $\alpha_0$ and $C$ are positive constants independent of $h$. 
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- The following local Hölder estimate proved by Kuo-Trudinger:

If $u_h$ is a solution of $F_h[u] = f$, then for any $x, y \in \Omega_h$

$$|u_h(x) - u_h(y)| \leq C \frac{|x - y|^{\delta}}{R} \left( \max_{B_R^h} u_h + \frac{R}{\alpha_0} \left\{ \sum_{x \in \Omega_h} h^n |f(x)|^n \right\}^{\frac{1}{n}} \right),$$

where $R = \min\{\text{dist}(x, \partial \Omega_h), \text{dist}(x, \partial \Omega_h)\}$, $B_R^h = B(0, R) \cap \Omega_h$, $\delta$, $\alpha_0$ and $C$ are positive constants independent of $h$. 
Recall the formula

\[
\lambda_1^h = - \inf_{u \in C_h, u > 0} \sup_{x \in \Omega_h} \frac{F(x, u, \delta_h u, \delta^2_h u)}{u(x)}
\]

Set \( U_i = u(x_i) \). If \( F(x, z, q, s) \) is linear or more generally convex in \((q, s)\), then the functions \( G_i : \mathbb{R}^{Nh} \rightarrow \mathbb{R}^{Nh} \)

\[
G_i(x, U_1, \ldots, U_{Nh}) = F \left( x_i, 1, \frac{U_{i+1} - U_{i-1}}{2hU_i}, \frac{U_{i+1} + U_{i-1}}{h^2 U_i} - \frac{2}{h^2} \right).
\]

are either linear or respectively convex in \( U_i, U_{i+1}, U_{i-1} \) and therefore \( G : \mathbb{R}^{Nh} \rightarrow \mathbb{R} \) defined by

\[
G(U_1, \ldots, U_{Nh}) = \max_{i=1,\ldots,Nh} G_i(x_i, U_1, \ldots, U_{Nh})
\]

is either linear or convex.
Hence the computation of $\lambda^h_1$ is equivalent to the convex minimization problem

$$\min_{U \in \mathbb{R}^{N_h}_+} \left[ \max_{i=1,\ldots,N_h} G_i(x_i, U_1, \ldots, U_{N_h}) \right]$$

and this problem can be solved by means of standard algorithms in convex optimization. Note that

- The vector $U$ attaining the minimum is *unique* (up to a multiplicative constant)
- The map is sparse, in the sense that the value of $G_i$ at $U_i$ depends only on the values at $U_{i-1}$ and $U_{i+1}$.
- The algorithm also computes an approximation of the eigenfunction.
- Similar considerations hold for eigenvalue problem in $\mathbb{R}^n$. 
Hence the computation of $\lambda_1^h$ is equivalent to the convex minimization problem

$$\min_{U \in \mathbb{R}^{N_h}_+} \left[ \max_{i=1, \ldots, N_h} G_i(x_i, U_1, \ldots, U_{N_h}) \right]$$

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Hence the computation of $\lambda^h_1$ is equivalent to the **convex minimization problem**

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- Similar considerations hold for eigenvalue problem in $\mathbb{R}^n$. 
Hence the computation of $\lambda_1^h$ is equivalent to the convex minimization problem

$$
\min_{U \in \mathbb{R}^{N_h}_+} \left[ \max_{i=1, \ldots, N_h} G_i(x_i, U_1, \ldots, U_{N_h}) \right]
$$

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- The algorithm also computes an approximation of the eigenfunction.

- Similar considerations hold for eigenvalue problem in $\mathbb{R}^n$. 
Some (very simple) examples

Example: eigenvalue of the second order derivative

\[
\begin{align*}
\varphi'' + \lambda \varphi &= 0 \quad \text{in } (0, 1), \\
\varphi(0) &= \varphi(1) = 0
\end{align*}
\]

In this case

\[
\lambda_1 = \pi^2, \quad \varphi_1(x) = \sin \pi x
\]

Given a discretization step \( h \) and the corresponding grid points \( x_i = ih \), \( i = 0, \ldots, Nh + 1 \), the max-min problem is

\[
\lambda_1^h = - \min_{U \in \mathbb{R}^{Nh}} \left[ \max_{i=1,\ldots,Nh} \frac{U_{i+1} + U_{i-1} - 2U_i}{h^2 U_i} \right]
\]

(with \( U_0 = U_{Nh+1} = 0 \)).
|   |   |   |   |
|---|---|---|---|
| $h$  | $Err(\lambda_1)$ | $Order(\lambda_1)$ | $Err_\infty(w_1)$ | $Err_2(w_1)$ |
| $1.00 \cdot 10^{-1}$ | $8.0908 \cdot 10^{-2}$ | | $3.3662 \cdot 10^{-11}$ | $5.7732 \cdot 10^{-11}$ |
| $5.00 \cdot 10^{-2}$ | $2.0277 \cdot 10^{-2}$ | $1.9964$ | $1.4786 \cdot 10^{-10}$ | $3.8119 \cdot 10^{-10}$ |
| $2.50 \cdot 10^{-2}$ | $5.0723 \cdot 10^{-3}$ | $1.9991$ | $6.6613 \cdot 10^{-16}$ | $1.8731 \cdot 10^{-15}$ |
| $1.25 \cdot 10^{-2}$ | $1.2683 \cdot 10^{-3}$ | $1.9998$ | $1.5543 \cdot 10^{-15}$ | $6.2524 \cdot 10^{-15}$ |
| $6.25 \cdot 10^{-3}$ | $3.1708 \cdot 10^{-4}$ | $1.9999$ | $1.2212 \cdot 10^{-15}$ | $7.1576 \cdot 10^{-15}$ |

**Table:** Space step (first column), eigenvalue error (second column), eigenfunction error in $L^\infty$ (fourth column), eigenfunction error in $L^2$ (last column)
Example: A linear equation with a discontinuous coefficient

\[
\begin{cases}
a(x)\varphi'' + \lambda \varphi = 0 & x \in (0, \pi) \\
\varphi(x) = 0 & x = 0, \pi
\end{cases}
\]  \tag{2}

where

\[
a(x) = \begin{cases} 
1 & \text{for } x \in [0, \frac{\pi}{2k}) \\
2 & \text{for } x \in \left[\frac{\pi}{2k}, \pi\right]
\end{cases}
\]

and \( k := \frac{2+\sqrt{2}}{2\sqrt{2}} > 1 \). The principal eigenvalue \( \lambda_1 \) is given by \( k^2 \). Note that the principal eigenfunction

\[
\varphi_1(x) = \begin{cases} 
\sin(kx) & \text{for } x \in [0, \frac{\pi}{2k}), \\
b \sin\left(\frac{kx}{\sqrt{2}} + c\right) & \text{for } x \in \left[\frac{\pi}{2k}, \pi\right].
\end{cases}
\]

is not \( C^2 \).
The scheme is

\[
\lambda_{1,h} = - \min_{U \in \mathbb{R}^{N_h}} \left[ \max_{i=1,\ldots,N_h} a(ih) \frac{U_{i+1} + U_{i-1} - 2U_i}{h^2 U_i} \right]
\]

(with \( U_0 = U_{N_h+1} = 0 \)).

| \( h \)   | \( \text{Err}(\lambda_1) \) | \( \text{Err}_\infty(w_1) \) | \( \text{Err}_2(w_1) \) |
|----------|------------------|----------------|----------------|
| 0.1571   | 0.1197           | 0.0213         | 0.0563         |
| 0.0785   | 0.0476           | 0.0090         | 0.0383         |
| 0.0393   | 0.0347           | 0.0065         | 0.0391         |
| 0.0196   | 0.0157           | 0.0030         | 0.0264         |
| 0.0098   | 0.0061           | 0.0012         | 0.0149         |

**Table:** Space steps (first column), Error eigenvalue (second column), Error eigenfunction in \( L^\infty \) (fourth column), Error eigenfunction in \( L^2 \) (last column)
Figure: Exact and approximate eigenfunctions for $h = 10^{-1}$
Example: The $p$-Laplacian
Consider the $p$-Laplace operator (Birindelli-Demengel, CPAA, 2006)

$$D(|D\varphi|^{p-2}D\varphi) + \lambda_p |\varphi|^{p-2} \varphi = 0.$$ 

Even if the operator is not uniformly elliptic, the formula

$$\lambda_p^h := - \inf_{\varphi > 0} \sup_{x \in \Omega_h} \left\{ \frac{F_{h,p}[\varphi](x)}{\varphi(x)^{p-1}} \right\}$$

where $F_{h,p}$ is a finite-difference approximations of $F_p$ gives an approximation of the principal eigenvalue $\lambda_p = \left( \frac{2\pi \sqrt{p-1}}{(b-a)p \sin(\frac{\pi}{p})} \right)^{1/p}$.

| $h$       | $\text{Err}(\lambda_4)$ | $\text{Order}(\lambda_4)$ |
|-----------|-------------------------|-----------------------------|
| $1.00 \cdot 10^{-1}$ | 2.6770                  |                             |
| $5.00 \cdot 10^{-2}$  | 0.6210                  | 2.1079                      |
| $2.50 \cdot 10^{-2}$  | 0.1457                  | 2.0912                      |
| $1.25 \cdot 10^{-2}$  | 0.0347                  | 2.0724                      |
| $6.25 \cdot 10^{-3}$  | 0.0083                  | 2.0581                      |

Table: Space step(1$^{st}$ column), eigenvalue error (2$^{nd}$ column), convergence order (3$^{rd}$ column) for the 4-Laplace operator
If $\Omega$ is a ball, $\varphi_p$ converges for $p \to \infty$ to $d(x, \partial \Omega)$. We draw approximations of $\varphi_p$ for various values of $p$ and we observe the convergence to $d(x, \{0, 1\})$ for $p$ increasing.

**Figure:** Approximate eigenfunction $\varphi_p^h$ for $p = 2, 4, 6, 8, 10$ and $h = 10^{-3}$
Example: A bi-dimensional example
Consider the eigenvalue problem for the Ornstein-Uhlenbeck operator

$$\Delta \varphi - x \cdot D\varphi + \lambda \varphi = 0, \quad x \in (0, 1)^2$$

with homogeneous boundary conditions. The eigenvalue and the corresponding eigenfunction are given by

$$\lambda_1 = 4, \quad \varphi_1(x_1, x_2) = (1 - x_1^2)(1 - x_2^2)$$

The Laplacian is discretized by a five-point formula.

| $h$         | $\text{Err}(\lambda_1)$ | $\text{Order}(\lambda_1)$ |
|-------------|--------------------------|-----------------------------|
| $4.00 \cdot 10^{-1}$ | 0.1524                   |                             |
| $2.00 \cdot 10^{-1}$ | 0.0392                   | 1.9592                      |
| $1.00 \cdot 10^{-1}$ | 0.0103                   | 1.9250                      |
| $5.00 \cdot 10^{-2}$ | 0.0027                   | 1.9580                      |
Ongoing work (with Simone Cacace)

It is known that

- Among all rectangles of same area, the one that minimizes the first Laplace-Dirichlet eigenvalue is the square.
- The Faber-Krahn’s inequality affirms that in any dimension, among all domains of same volume, the euclidean ball has the smallest first Laplace-Dirichlet eigenvalue.

The idea seems to be that the $\lambda_1(\Omega)$ is decreasing with respect to the symmetry of the domain $\Omega$. Is Faber-Krahn inequality true for the Pucci operator?

The idea is to investigate numerically this conjecture using the previous approximation scheme and, for the computation of the eigenvalue, the nonlinear least square method developed in

- S. CACACE, F. CAMILLI, Ergodic problems for Hamilton-Jacobi equations: yet another but efficient numerical method, 2016
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Fabio Camilli (“Sapienza”)
In the picture, approximation of the Laplace-Dirichlet eigenvalue $\lambda_1$ of the rectangle $[0, a] \times [0, 1/a]$ as a function of $a$. The minimum is attained for $a = 1$, i.e. by the square.
The Laplace-Dirichlet eigenfunction in a flower domain
(courtesy S. Cacace)

Thank You!