Infrared factorization of tree-level QCD amplitudes at the next-to-next-to-leading order and beyond

Stefano Catani
Theory Division, CERN, CH 1211 Geneva 23, Switzerland

and

Massimiliano Grazzini
Institute for Theoretical Physics, ETH-Hönggerberg, CH 8093 Zurich, Switzerland

Abstract

We study the infrared behaviour of tree-level QCD amplitudes and we derive infrared-factorization formulae that are valid at any perturbative order. We explicitly compute all the universal infrared factors that control the singularities in the various soft and/or collinear limits at $\mathcal{O}(\alpha_S^2)$.

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1 Introduction

Infrared (soft and collinear) singularities appear in the calculation of multiparton QCD matrix elements. Although the singularities cancel in the evaluation of inclusive cross sections, their factorization properties are at the basis of many important tools in perturbative QCD applications to hard-scattering processes \[1\].

At the leading order in the QCD coupling, $\alpha_s$, the structure of the infrared singularities is well known to be universal. It is embodied in process-independent factorization formulae of tree-level \[2, 4\] and one-loop \[4, 14\] amplitudes. These factorization formulae have played an essential role in the setting up of completely general algorithms \[5, 7, 11, 12\] to handle and cancel infrared singularities, when combining tree-level and one-loop contributions in the evaluation of jet cross sections at the next-to-leading order (NLO) in perturbation theory.

The extension of these general algorithms at the next-to-next-to-leading order (NNLO) is at present one of the main goals to improve and precisely quantify the theoretical accuracy of perturbative QCD predictions. To this purpose we need to compute two-loop matrix elements \[13–15\] and to understand the structure of the infrared singularities of two-loop, one-loop and tree-level amplitudes at $\mathcal{O}(\alpha_s^2)$. The singular behaviour of two-loop QCD amplitudes has been discussed in Ref. \[16\]. The soft and collinear limits of one-loop amplitudes have been derived in Refs. \[17, 18\]. The soft, collinear and soft–collinear singularities of tree-level amplitudes have been studied in Refs. \[19, 20\], \[21, 22\] and \[21\], respectively.

The purpose of this paper is twofold. We consider tree-level matrix elements and present general techniques to compute their infrared singularities and to derive infrared-factorization formulae to any order in $\alpha_s$. We apply these techniques to the explicit calculation of all the relevant infrared factors at $\mathcal{O}(\alpha_s^2)$.

Our general method exploits the universality properties of soft and collinear emission and consists in directly computing process-independent Feynman subgraphs in a physical gauge. We use power-counting arguments \[23, 24\] and the eikonal approximation \[3\] to treat the collinear and soft limits, respectively. We show how the coherence properties of QCD radiation \[23\] can be used to deal with the mixed soft–collinear limit in terms of the collinear and soft factorization formulae.

Most of the explicit results at $\mathcal{O}(\alpha_s^2)$ presented in this paper were first obtained by Campbell and Glover \[21\]. The strategy followed in Ref. \[21\] was to take universal factorization for granted and thus to extract the $\mathcal{O}(\alpha_s^2)$-singular factors by performing the corresponding limits of a set of known matrix elements. We confirm their calculations by using a completely independent method. We also extend their results by considering the emission of a soft fermion pair and by fully taking into account spin (azimuthal) correlations in the collinear limit. The extension to azimuthal correlations is essential to apply some general methods to perform exact NNLO calculations of jet cross sections. For instance, the subtraction method \[3, 12\] works by regularizing the infrared singularities of the tree-level matrix element by identifying and subtracting a proper local counterterm. Thus, the study of the azimuthally averaged collinear limit \[21\] is not sufficient for this purpose.
The knowledge of the infrared structure of multiparton amplitudes is also important for other perturbative QCD applications. The leading-logarithmic (LL) parton showers, which are implemented in Monte Carlo event generators [1] to describe the exclusive structure of hadronic final states, are based on the $O(\alpha_S)$-factorization formulae supplemented with ‘jet calculus’ techniques [26] and colour-coherence properties [3, 25]. Analytical techniques to perform all-order resummation of logarithmically enhanced contributions at next-to-leading logarithmic (NLL) accuracy [27] rely on the factorization properties of soft and collinear emission. The results on infrared factorization presented in this paper can be useful to improve parton-shower algorithms and resummed calculations beyond their present logarithmic accuracy.

The outline of the paper is as follows. We start in Sect. 2 by studying the collinear behaviour. After reviewing the known factorization formulae at $O(\alpha_S)$, we discuss the kinematics of the triple collinear limit. Then, in Sect. 2.3, we present our derivation of factorization for the multiple collinear limit at any perturbative order. Finally, in Sect. 2.4, we perform the explicit calculation of the spin-dependent splitting functions at $O(\alpha_S^2)$. Our results for the splitting functions were anticipated in Ref. [22]. In Sect. 3 we study the soft behaviour. We first review the known results at $O(\alpha_S)$ and then, in Sect. 3.2, we compute the emission of a soft $q\bar{q}$ pair at $O(\alpha_S^2)$. Section 3.3 is devoted to double gluon emission: we present the corresponding soft current and obtain a compact expression for its square. Factorization for the mixed soft–collinear limit at $O(\alpha_S^2)$ and at higher perturbative orders is discussed in detail in Sects. 3.4 and 3.5. In Sect. 4 we summarize our results. In general, soft factorization formulae involve colour correlations. At $O(\alpha_S^2)$ these correlations cancel in four- and five-parton matrix elements. The explicit expressions for these particular cases are given in the Appendix.

2 The collinear behaviour

2.1 Notation and collinear factorization at $O(\alpha_S)$

We consider a generic scattering process involving final-state QCD partons (massless quarks and gluons) with momenta $p_1, p_2, \ldots$. Non-QCD partons ($\gamma^*, Z^0, W^\pm, \ldots$), carrying a total momentum $Q$, are always understood. The corresponding tree-level matrix element is denoted by

$$M^{c_1,c_2,\ldots;s_1,s_2,\ldots}(p_1, p_2, \ldots),$$

where $\{c_1, c_2, \ldots\}$, $\{s_1, s_2, \ldots\}$ and $\{a_1, a_2, \ldots\}$ are respectively colour, spin and flavour indices. The matrix element squared, summed over final-state colours and spins, will be denoted by $|M^{c_1,c_2,\ldots;s_1,s_2,\ldots}(p_1, p_2, \ldots)|^2$. If the sum over the spin polarizations of the parton $a_1$ is not carried out, we define the following ‘spin-polarization tensor’

$$\mathcal{T}^{s_1,s_1'}(p_1, \ldots) \equiv \sum_{\text{spins} \neq s_1,s_1'} \sum_{\text{colours}} M^{c_1,c_2,\ldots;s_1,s_2,\ldots}(p_1, p_2, \ldots) \left[ M^{c_1,c_2,\ldots;s_1',s_2,\ldots}(p_1, p_2, \ldots) \right]^\dagger.$$

\footnote{The case of incoming partons can be recovered by simply crossing the parton indices (flavours, spins and colours) and momenta.}
We work in \( d = 4 - 2\varepsilon \) space-time dimensions and consider two helicity states for massless quarks and \( d - 2 \) helicity states for gluons. This defines the conventional dimensional-regularization (CDR) scheme of both ultraviolet [28] and infrared [29] divergences. Thus, the fermion spin indices are \( s = \pm 1 \), while to label the gluon spin it is convenient to use the corresponding Lorentz index \( \mu = 1, \ldots, d \). The \( d \)-dimensional average of the matrix element over the polarizations of a parton \( a \) is obtained by means of the factors

\[
\frac{1}{2} \delta_{ss'}
\]

for a fermion, and (the gauge terms are proportional either to \( p^\mu \) or to \( p'^\mu \))

\[
\frac{1}{d - 2} d_{\mu\nu}(p) = \frac{1}{2(1 - \varepsilon)} (-g_{\mu\nu} + \text{gauge terms})
\]

with

\[
-g_{\mu\nu} d_{\mu\nu}(p) = d - 2, \quad p^\mu d_{\mu\nu}(p) = d_{\mu\nu}(p) p'^\nu = 0,
\]

for a gluon with on-shell momentum \( p \).

The singular collinear limit at \( \mathcal{O}(\alpha_s) \) is approached when the momenta of two partons, say \( p_1 \) and \( p_2 \), become parallel. This limit can be precisely defined as follows:

\[
p_1^\mu = z p^\mu + k_\perp^\mu - \frac{k_\perp^2}{2p \cdot n} n^\mu, \quad p_2^\mu = (1 - z)p^\mu - k_\perp^\mu - \frac{k_\perp^2}{1 - z} \frac{n^\mu}{2p \cdot n},
\]

\[
s_{12} \equiv 2p_1 \cdot p_2 = -\frac{k_\perp^2}{z(1 - z)}, \quad k_\perp \to 0.
\]

In Eq. (3) the light-like \( (p^2 = 0) \) vector \( p^\mu \) denotes the collinear direction, while \( n^\mu \) is an auxiliary light-like vector, which is necessary to specify the transverse component \( k_\perp \) \((k_\perp^2 < 0) \) \((k_\perp \cdot p = k_\perp \cdot n = 0) \) or, equivalently, how the collinear direction is approached. In the small-\( k_\perp \) limit (i.e. neglecting terms that are less singular than \( 1/k_\perp^2 \)), the square of the matrix element in Eq. (3) fulfils the following factorization formula (4):

\[
|M_{a_1,...}(p_1, p_2, \ldots)|^2 \simeq \frac{2}{s_{12}} 4\pi \mu^{2\varepsilon} \alpha_s \mathcal{T}^{ss'}_{a_1,...}(p_1, \ldots) \hat{P}_{a_1a_2}(z, k_\perp; \varepsilon),
\]

where \( \mu \) is the dimensional-regularization scale. The spin-polarization tensor \( \mathcal{T}^{ss'}_{a_1,...}(p_1, \ldots) \) is obtained by replacing the partons \( a_1 \) and \( a_2 \) on the right-hand side of Eq. (2) with a single parton denoted by \( a \). This parton carries the quantum numbers of the pair \( a_1 + a_2 \) in the collinear limit. In other words, its momentum is \( p^\mu \) and its other quantum numbers (flavour, colour) are obtained according to the following rule: anything + gluon gives anything, and quark + antiquark gives gluon.

The kernel \( \hat{P}_{a_1a_2} \) in Eq. (7) is the \( d \)-dimensional Altarelli–Parisi splitting function [2]. It depends not only on the momentum fraction \( z \) involved in the collinear splitting \( a \to a_1 + a_2 \), but also on the transverse momentum \( k_\perp \) and on the helicity of the parton \( a \) in the matrix element \( \mathcal{M}^{a_1a_2,...}(p_1, \ldots) \). More precisely, \( \hat{P}_{a_1a_2} \) is in general a matrix acting on the spin indices \( s, s' \) of the parton \( a \) in the spin-polarization tensor \( \mathcal{T}^{ss'}_{a_1,...}(p_1, \ldots) \). Because of these spin correlations, the spin-average square of the matrix element \( \mathcal{M}^{a_1a_2,...}(p_1, \ldots) \) cannot be simply factorized on the right-hand side of Eq. (7).
The explicit expressions of $\hat{P}_{a_1a_2}$, for the splitting processes
\[ a(p) \to a_1(zp + k_\perp + O(k_\perp^2)) + a_2((1 - z)p - k_\perp + O(k_\perp^2)) \] (8)
depend on the flavour of the partons $a_1, a_2$ and are given by
\[ \hat{P}^{ss'}_{qq}(z, k_\perp; \epsilon) = \hat{P}^{ss'}_{qq}(z, k_\perp; \epsilon) = \delta_{ss'} C_F \left( \frac{1 + z^2}{1 - z} - \epsilon(1 - z) \right) \] (9)
\[ \hat{P}^{ss'}_{qq}(z, k_\perp; \epsilon) = \delta_{ss'} C_F \left( \frac{1 + (1 - z)^2}{z} - \epsilon z \right) \] (10)
\[ \hat{P}^{\mu\nu}_{qq}(z, k_\perp; \epsilon) = \hat{P}^{\mu\nu}_{qq}(z, k_\perp; \epsilon) = T_R \left[ -g^{\mu\nu} + 4z(1 - z) \frac{k^{\mu}_z k^{\nu}_z}{k_\perp^2} \right] \] (11)
\[ \hat{P}^{\mu\nu}_{gg}(z, k_\perp; \epsilon) = 2C_A \left[ -g^{\mu\nu} \left( \frac{z}{1 - z} + \frac{1 - z}{z} \right) - 2(1 - \epsilon)z(1 - z) \frac{k^{\mu}_z k^{\nu}_z}{k_\perp^2} \right] \] (12)
where the $SU(N_c)$ QCD colour factors are
\[ C_F = \frac{N_c^2 - 1}{2N_c} \quad , \quad C_A = N_c \quad , \quad T_R = \frac{1}{2} \] (13)
and the spin indices of the parent parton $a$ have been denoted by $s, s'$ if $a$ is a fermion and $\mu, \nu$ if $a$ is a gluon.

Note that when the parent parton is a fermion (see Eqs. (9) and (10)) the splitting function is proportional to the unity matrix in the spin indices. Thus, in the factorization formula (9), spin correlations are effective only in the case of the collinear splitting of a gluon. Owing to the $k_\perp$-dependence of the collinear splitting functions in Eqs. (11) and (12), these spin correlations produce a non-trivial azimuthal dependence with respect to the directions of the other momenta in the factorized matrix element.

Equations (9)–(12) lead to the more familiar form of the $d$-dimensional splitting functions only after average over the polarizations of the parton $a$. The $d$-dimensional average is obtained by means of the factors in Eqs. (9) and (10). Denoting by $\langle \hat{P}_{a_1a_2} \rangle$ the average of $\hat{P}_{a_1a_2}$ over the polarizations of the parent parton $a$, we have:
\[ \langle \hat{P}_{qq}(z; \epsilon) \rangle = \langle \hat{P}_{qq}(z; \epsilon) \rangle = C_F \left( \frac{1 + z^2}{1 - z} - \epsilon(1 - z) \right) \] (14)
\[ \langle \hat{P}_{qq}(z; \epsilon) \rangle = \langle \hat{P}_{qq}(z; \epsilon) \rangle = C_F \left( \frac{1 + (1 - z)^2}{z} - \epsilon z \right) \] (15)
\[ \langle \hat{P}_{qq}(z; \epsilon) \rangle = \langle \hat{P}_{qq}(z; \epsilon) \rangle = T_R \left[ 1 - \frac{2z(1 - z)}{1 - \epsilon} \right] \] (16)

The $\epsilon$ dependence on the right-hand side of Eqs. (9)–(12) refers to the CDR scheme used throughout the paper. A detailed discussion of the regularization-scheme dependence of the collinear splitting functions at $O(\alpha_S)$, including the corresponding explicit expressions, can be found in Ref. [30].
\[
\langle \hat{P}_{gg}(z; \epsilon) \rangle = 2C_A \left[ \frac{z}{-z} + \frac{1}{-z} + z(1-z) \right].
\]

In the rest of this section we are mainly interested in the collinear behaviour of the tree-level matrix element \( M(p_1, \ldots) \) in Eq. (11) at \( O(\alpha_s^2) \). At this order there are two different collinear limits to be considered [21].

The first limit is approached when two pairs of parton momenta, say \( \{p_1, p_2\} \) and \( \{p_3, p_4\} \), become parallel independently. In this case collinear factorization follows from the straightforward iteration of Eq. (7): the ensuing factorization formula simply contains the product of the two splitting functions \( \hat{P}_{s_1 s_1}^{a_1 a_2} \) and \( \hat{P}_{s_3 s_3}^{a_3 a_4} \).

In the second limit, three parton momenta can simultaneously become parallel. This triple collinear limit is discussed in the following subsections.

### 2.2 Kinematics in the triple collinear limit

We denote by \( p_1, p_2 \) and \( p_3 \) the momenta of the three collinear partons. The most general parametrization of these collinear momenta is

\[
p_i^\mu = x_i p^\mu + k_i^\perp - \frac{k_i^2}{x_i} n^\mu, \quad i = 1, 2, 3,
\]

where, as in Eq. (11), the light-like vector \( p^\mu \) denotes the collinear direction and the auxiliary light-like vector \( n^\mu \) specifies how the collinear direction is approached (\( k_i^\perp \cdot p = k_i^\perp \cdot n = 0 \)). Note that no other constraint (in particular \( \sum_i x_i \neq 1 \) and \( \sum_i k_i^\perp \neq 0 \)) is imposed on the longitudinal and transverse variables \( x_i \) and \( k_i^\perp \). Thus, we can easily consider any (asymmetric) collinear limit at once.

Note, however, that the triple collinear limit is invariant under longitudinal boosts along the direction of the total momentum \( p_{123}^\mu = p_1^\mu + p_2^\mu + p_3^\mu \). Thus, the relevant kinematical variables are the following boost-invariant quantities

\[
z_i = \frac{x_i}{\sum_{j=1}^3 x_j},
\]

\[
\tilde{k}_i^\mu = k_i^\perp - \frac{x_i}{\sum_{k=1}^3 x_k} \sum_{j=1}^3 k_j^\perp.
\]

Note that these variables automatically satisfy the constraints \( \sum_{i=1}^3 z_i = 1 \) and \( \sum_{i=1}^3 \tilde{k}_i = 0 \), so that only four of them are actually independent.

In terms of the longitudinal and transverse variables introduced so far, the two-particle sub-energies \( s_{ij} \) are written as

\[
s_{ij} = (p_i + p_j)^2 = -x_i x_j \left( \frac{k_{ij}}{x_i} - \frac{k_{ii}}{x_i} \right)^2 = -z_i z_j \left( \frac{\tilde{k}_i}{z_i} - \frac{\tilde{k}_i}{z_i} \right)^2.
\]

It is also convenient to define the following variable \( t_{ij,k} \)

\[
t_{ij,k} \equiv 2 \frac{z_i s_{jk} - z_j s_{ik}}{z_i + z_j} + \frac{z_i - z_j}{z_i + z_j} s_{ij}.
\]
2.3 Power counting and tree-level factorization at any order

In the triple-collinear limit, the matrix element squared \( |M_{a_1,a_2,a_3,...}(p_1,p_2,p_3,...)|^2 \) has the singular behaviour \( |M_{a_1,a_2,a_3,...}(p_1,p_2,p_3,...)|^2 \sim 1/(ss') \), where \( s \) and \( s' \) can be either two-particle \( (s_{ij} = (p_i + p_j)^2) \) or three-particle \( (s_{123} = (p_1 + p_2 + p_3)^2) \) sub-energies. To define the collinear limit more precisely, we can rescale the transverse momenta \( k_{\perp i} \) by an overall factor \( \lambda \):

\[
k_{\perp i} \rightarrow \lambda \ k_{\perp i}
\]

and then perform the limit \( \lambda \rightarrow 0 \). In this limit the matrix element squared behaves as

\[
|M_{a_1,a_2,a_3,...}|^2 \rightarrow \mathcal{O}(1/\lambda^4) + \ldots,
\]

where the dots stand for less singular contributions when \( \lambda \rightarrow 0 \). We are interested in explicitly evaluating the dominant singular contributions at the tree level (i.e. at \( \mathcal{O}(\alpha_S^{m-1}) \)). Thus, we shall discuss the most general case with \( m \) collinear partons.

We shall show that in the multiple collinear limit \( a \rightarrow a_1...a_m \), the matrix element squared \( |M_{a_1,...,a_m,...}(p_1,...,p_m,...)|^2 \) still fulfils a factorization formula analogous to Eq. (23), namely

\[
|M_{a_1,...,a_m,...}(p_1,...,p_m,...)|^2 \sim \left( \frac{8\pi\mu^2\alpha_S}{s_{1...m}} \right)^{m-1} T_{a,...,}(xp,...) \hat{P}_{a_1...a_m}^{ss'},
\]

where \( s_{1...m} = (p_1 + \ldots + p_m)^2 \) is the \( m \)-particle sub-energy and \( x = \sum_{i=1}^{m} x_i \).

As in Eq. (1), the spin-polarization tensor \( T_{a,...,}(xp,...) \) is obtained by replacing the partons \( a_1,...,a_m \) with a single parent parton, whose flavour \( a \) is determined by flavour conservation in the splitting process. More precisely, \( a \) is a quark (antiquark) if the set \( \{a_1,...,a_m\} \) contains an odd number of quarks (antiquarks), and \( a \) is a gluon otherwise.

The factorization formula (25) takes into account all the dominant singular contributions in the multiple collinear limit, that is, all the contributions that have the scaling behaviour \( (1/\lambda^2)^{m-1} \) under the scale transformation in Eq. (23). Relative corrections of \( \mathcal{O}(\lambda) \) are systematically neglected on the right-hand side of Eq. (25).

The \( m \)-parton splitting functions \( \hat{P}_{a_1...a_3} \) are dimensionless functions of the parton momenta \( p_1,...,p_m \) and generalize the Altarelli–Parisi splitting functions in Eq. (7). Owing to their invariance under longitudinal boosts along the collinear direction, the splitting functions can depend in a non-trivial way only on the sub-energy ratios \( s_{ij}/s_{1...m} \) and on the longitudinal- and transverse-momentum variables \( z_i \) and \( \vec{k}_i \) defined by the generalization of Eqs. (19) and (20) to the \( m \)-parton case.

The spin correlations produced by the collinear splitting are taken into account by the splitting functions in a universal way, i.e. independently of the specific matrix element on the right-hand side of Eq. (25).
In the case of the splitting processes that involve a fermion as parent parton, we find that spin correlations are absent. We can thus write the corresponding spin-dependent splitting function in terms of its average \( \langle \hat{P}_{a_1...a_m} \rangle \) over the polarizations of the parent fermion \( a \):

\[
\hat{P}^{ss'}_{a_1...a_m} = \delta^{ss'} \langle \hat{P}_{a_1...a_m} \rangle .
\] (26)

This feature is completely analogous to the \( O(\alpha_s) \) case and follows from helicity conservation in the quark–gluon vector coupling.

In the case of collinear decays of a parent gluon, however, spin correlations are highly non-trivial.

Note also that the splitting functions for the collinear decay of an antiquark can be simply obtained by charge-conjugation invariance from those of the corresponding quark decay process, i.e. \( \langle \hat{P}_{a_1...a_m} \rangle = \langle \hat{P}_{\bar{a}_1...\bar{a}_m} \rangle \).

The method used to derive these results exploits the basic observation [23] that interfering Feynman diagrams obtained by squaring the amplitude \( M(p_1, \ldots, p_m, \ldots) \) are collinearly suppressed when computed in a physical gauge. Thus, in the evaluation of the multiple collinear limit we can write

\[
|M_{a_1,...,a_m,...}(p_1, \ldots, p_m, \ldots)|^2 \simeq \left[ M^{(n)}_{a_1,...,a_m}(p_1 + \ldots + p_m, \ldots) \right]^\dagger V^{(n)}_{a_1,...,a_m}(p_1, \ldots, p_m) \\
\cdot M^{(n)}_{a_1,...,a_m}(p_1 + \ldots + p_m, \ldots) + \ldots .
\] (27)

The first term on the right-hand side of Eq. (27) corresponds to the non-interfering Feynman diagrams in Fig. 1, while the dots stand for subdominant contributions coming from interferences (see e.g. the diagram in Fig. 2). The superscripts \( (n) \) denote that the various terms are evaluated in a physical gauge. To simplify the calculation it is convenient to choose the axial gauge \( n \cdot A = 0 \), where the gauge vector \( n^\mu \) coincides with the auxiliary light-like vector used in Eq. (18) to parametrize the collinear kinematics. The corresponding gluon polarization tensor \( d^{\mu\nu}_{(n)} \) is

\[
d^{\mu\nu}_{(n)}(q) = -g^{\mu\nu} + \frac{n^\mu q^\nu + q^\mu n^\nu}{n \cdot q} ,
\] (28)

where \( q \) is the gluon momentum.

The summation over spin and colour indices is understood on the right-hand side of Eq. (27). The function \( V_{a_1,...,a_m} \) in Eq. (27) is the \( m \)-parton dispersive contribution to the two-point function of the parent parton \( a \). Being a two-point function, it is proportional to the unity matrix in the colour indices of the parton \( a \). Thus, we can sum over the colours of the partons in the tree-level amplitudes, and we can rewrite Eq. (27) in terms of the spin-polarization tensor \( T^{(n)}_{a_1,...,a_m} \) introduced in Eq. (2):

\[
|M_{a_1,...,a_m,...}(p_1, \ldots, p_m, \ldots)|^2 \simeq \left( \frac{8\pi\mu^2 G_F}{s_{1...m}} \right)^{m-1} T^{(n)}_{a_1,...,a_m}(p_1 + \ldots + p_m, \ldots) V^{(n)}_{a_1,...,a_m}(p_1, \ldots, p_m) .
\] (29)

The function \( V^{(n)}_{a_1,...,a_m} \) is simply obtained from \( V^{(n)}_{a_1,...,a_m} \) in Eq. (27) by performing the average over the colours of the parent parton \( a \) and extracting the factor in the round bracket.
on the right-hand side of Eq. (29). Thus the tree-level function $V^{(n)}_{a_1...a_m}(p_1,\ldots,p_m)$ does not contain any additional power of the QCD coupling $\alpha_S$. Note also that the spin tensor $T^{(n)}_{a_1...}(p_1 + \ldots + p_m,\ldots)$ is not yet exactly the physical polarization tensor of Eq. (25). In fact, the momentum of the parton $a$ is off-shell ($(p_1 + \ldots + p_m)^2 = s_1...m \neq 0$) and, thus, $T^{(n)}_{a_1...}(p_1 + \ldots + p_m,\ldots)$ is gauge-dependent.

To proceed, we should consider separately the two cases in which the parton $a$ is either a quark (or antiquark) or a gluon.

**Quark splitting processes**

It is convenient to include the spin-polarization matrices $\not{p}_1 + \ldots + \not{p}_m$ of the decaying quark $a = q$ in the definition of the Dirac matrix $V^{(n)}_{a_1...a_m}(p_1,\ldots,p_m)$. The most general decomposition of $V^{(n)}_{a_1...a_m}(p_1,\ldots,p_m)$ is

$$V^{(n)}_{a_1...a_m}(p_1,\ldots,p_m) \sim \sum \left( \text{scalar amplitude} \right) \cdot \left( \text{string of gamma matrices} \right). \quad (30)$$

Any string of gamma matrices is obtained by multiplying an arbitrary number of terms $\not{p}_l$ with $l = 1,\ldots,m+1$, where $\not{p}$ can be either $\not{p}_i = \not{p}_1$ or $\not{p}_{m+1} = \not{s}_1...m/n \cdot (p_1 + \ldots + p_m)$. The matrices $\not{p}_l$, like $V^{(n)}_{a_1...a_m}(p_1,\ldots,p_m)$, are homogeneous functions of $n^\mu$ with vanishing homogeneity degree. Thus, by Lorentz covariance, the amplitudes on the right-hand side
of Eq. \[30\] are scalar functions of the sub-energies \(s_{ij}\) and the longitudinal-momentum fractions \(z_i = n \cdot p_i / n \cdot (p_1 + \ldots + p_m)\). Moreover, they are rational functions of the variables \(s_{ij}, z_i\) and thus, by dimensional analysis, the corresponding strings can contain only an odd number of gamma matrices.

We can now exploit the hermiticity properties of \(V_{a_1\ldots a_m}^{(n)}(p_1, \ldots, p_m)\). Since the scalar amplitudes are real, the strings of gamma matrices appear in the form

\[
\left(\frac{1}{s_{1\ldots m}}\right)^{(k-1)/2} \left[ \slashed{p}_{i_1}\slashed{p}_{i_2} \cdots \slashed{p}_{i_k} + \frac{\lambda}{v} \shat{p}_{i_k} \cdots \shat{p}_{i_2} \shat{p}_{i_1} \right],
\]

(31)

where the normalization by the overall power of \(1/s_{1\ldots m}\) has been introduced to make the scalar amplitudes on the right-hand side of Eq. \[30\] dimensionless. Owing to the fact that \(k\) is odd, the terms with \(k = 3, 7, 11, \ldots\) in Eq. \[31\] can in turn be reduced to strings that contain \(k = 1, 5, 9, \ldots\) gamma matrices by using the anticommuting properties of the Dirac algebra. This is the simplest form in which we can write the general decomposition of Eq. \[30\].

We can now discuss separately the cases that involve the collinear decay of less or more than four partons.

From the previous discussion we conclude that, when \(m \leq 3\), the functions \(V_{a_1\ldots a_m}^{(n)}\) can be written as follows

\[
V_{a_1\ldots a_m}^{(n)}(p_1, \ldots, p_m) = \sum_{i=1}^{m} A_i^{(q)}(\{s_{jl}, z_j\}) \slashed{p}_i + B^{(q)}(\{s_{jl}, z_j\}) \frac{\lambda}{n \cdot (p_1 + \ldots + p_m)}, \quad (m \leq 3),
\]

(32)

while, when \(m = 4\), we have

\[
V_{a_1\ldots a_4}^{(n)}(p_1, \ldots, p_4) = \sum_{i=1}^{4} A_i^{(q)}(\{s_{jl}, z_j\}) \slashed{p}_i + C^{(q)}(\{s_{jl}, z_j\}) \frac{\lambda}{s_{1\ldots 4} n \cdot (p_1 + \ldots + p_4)} \nonumber
\]

(33)

\[
+ \frac{\lambda}{s_{1\ldots 4} n \cdot (p_1 + \ldots + p_4)}.
\]

(34)

Then we can proceed to single out the dominant singular behaviour of Eqs. \[32\] and \[33\] in the multiple collinear limit. Since the scalar functions \(A_i^{(q)}(\{s_{jl}, z_j\})\), \(B^{(q)}(\{s_{jl}, z_j\})\) and \(C^{(q)}(\{s_{jl}, z_j\})\) are dimensionless, they are invariant under the scale transformation \[23\]. Moreover, since we can write

\[
p_i^\mu = z_i(p_1 + \ldots + p_m)^\mu + \tilde{k}_i^\mu + \mathcal{O}(k_i^2),
\]

(35)

by rescaling the transverse momenta as in Eq. \[23\] we obtain the following scaling behaviour

\[
V_{a_1\ldots a_m}^{(n)}(p_1, \ldots, p_m) = (\slashed{p}_1 + \ldots + \slashed{p}_m) \sum_{i=1}^{m} z_i A_i^{(q)}(\{s_{jl}, z_j\}) [1 + \mathcal{O}(\lambda)], \quad (m \leq 4).
\]

(36)

Thus, inserting Eq. \[36\] into Eq. \[29\], we can use the spin polarization factor \(\slashed{p}_1 + \ldots + \slashed{p}_m\) to reconstruct the matrix element squared \(|\mathcal{M}_{q_n}(p_1 + \ldots + p_m, \ldots)|^2\). Having already factorized the dominant singular term, we can now replace \(p_1 + \ldots + p_m \to xp\) in \(|\mathcal{M}_{q_n}(p_1 + \ldots + p_m, \ldots)|^2\), so that its gauge dependence disappears, and we obtain the factorization formula \[23\].
Moreover, we also obtain an explicit expression for the quark splitting function in terms of the scalar amplitudes $A^{(q)}_i$ in Eqs. (32) and (33):

\[ \hat{P}^{ss'}_{a_1 \ldots a_m} = \delta^{ss'} \sum_{i=1}^{m} z_i A^{(q)}_i (\{ s_{jl}, z_j \}) , \quad (m \leq 4). \] (37)

This argument to prove collinear factorization is based on the fact that a single spin structure (see Eq. (30)) dominates the collinear limit of the quark decay function $V^{(n)}_{a_1 \ldots a_m}$. In particular, this implies that spin correlations are absent from the collinear decay of a fermion, independently of the number of its spin polarizations. However, the argument works only for the cases with $m \leq 4$. When $m > 4$ collinear factorization still applies but, as shown below, spin correlations cancel only if we use a dimensional-regularization prescription in which the massless fermion has two spin polarizations.

According to our definition, the scalar amplitudes on the right-hand side of Eq. (30) are dimensionless and, hence, they are invariant under the scale transformation (23). The prescription in which the massless fermion has two spin polarizations.

\[
\left( \frac{1}{s_{1 \ldots m}} \right)^k \left[ (p_1 + \ldots + p_m) \bar{\kappa}_{i_1} \bar{\kappa}_{i_2} \cdots \bar{\kappa}_{i_{2k}} + \bar{\kappa}_{i_2} \cdots \bar{\kappa}_{i_{2k}} \bar{\kappa}_{i_1} (p_1 + \ldots + p_m) \right]
\]

\[ = \left( \frac{1}{s_{1 \ldots m}} \right)^k x \left[ \bar{\kappa}_{i_1} \bar{\kappa}_{i_2} \cdots \bar{\kappa}_{i_{2k}} + \bar{\kappa}_{i_2} \cdots \bar{\kappa}_{i_{2k}} \bar{\kappa}_{i_1} \right] [1 + O(\lambda)] , \] (38)

where the dots stand for the product of $\bar{\kappa}_i$ factors. We can now multiply Eq. (38) by unity in the form $1 = (\bar{\rho} \bar{\phi} + \bar{\phi} \bar{\rho})/(2p \cdot n)$, and, using $\{ \bar{\rho}, \bar{\kappa}_i \} = 0$ and $\bar{\rho}^2 = 0$, we obtain

\[ \left( \frac{1}{s_{1 \ldots m}} \right)^k x \left[ \frac{\bar{\rho} \bar{\kappa}_{i_1} \bar{\kappa}_{i_2} \cdots \bar{\kappa}_{i_{2k}} \bar{\rho} \bar{\phi} + \bar{\phi} \bar{\rho}}{2p \cdot n} + \frac{\bar{\rho} \bar{\phi} + \bar{\phi} \bar{\rho}}{2p \cdot n} \bar{\kappa}_{i_2} \cdots \bar{\kappa}_{i_{2k}} \bar{\kappa}_{i_1} \right] \]

\[ = \left( \frac{1}{s_{1 \ldots m}} \right)^k \frac{x \bar{\rho} \bar{\kappa}_{i_1} \bar{\kappa}_{i_2} \cdots \bar{\kappa}_{i_{2k}} \bar{\phi} + \bar{\phi} \bar{\kappa}_{i_2} \cdots \bar{\kappa}_{i_{2k}} \bar{\kappa}_{i_1}}{2xp \cdot n} \bar{\rho}. \] (39)

Denoting by $\chi_s(p)$ the spinor of an on-shell fermion with momentum $p$ and spin polarization $s$, we then replace the polarization matrices $x \bar{\rho}$ in Eq. (39) by using the identity $x \bar{\rho} = \sum_s \chi_s(xp) \bar{\chi}_s(xp)$ and we can rewrite the string in Eq. (39) as follows

\[ \sum_{s, s'} \left[ \chi_s(xp) \bar{\chi}_{s'}(xp) \right] \left( \frac{1}{s_{1 \ldots m}} \right)^k \bar{\kappa}_{i_1} \bar{\kappa}_{i_2} \cdots \bar{\kappa}_{i_{2k}} \bar{\phi} + \bar{\phi} \bar{\kappa}_{i_2} \cdots \bar{\kappa}_{i_{2k}} \bar{\kappa}_{i_1} \frac{2xp \cdot n}{2} \chi_{s'}(xp) . \] (40)

When inserted in Eqs. (30) and (29), the factor in the square bracket reconstructs the polarization matrix of the decaying quark and, thus, the spin-polarization tensor $T_{q \ldots s}(xp, \ldots)$ in the factorization formula (25). The remaining factor in Eq. (10) gives the contribution of the string in Eq. (38) to the quark splitting function $\hat{P}^{ss'}_{a_1 \ldots a_m}$.

By explicit construction we see that in general the splitting function $\hat{P}^{ss'}_{a_1 \ldots a_m}$ is not diagonal with respect to the spin indices. Nonetheless, the spin correlations are absent within the dimensional-regularization prescription used throughout the paper. Since we are
considering only two helicity states for massless quarks, we have \( \chi_{s=\pm 1}(p) = \frac{1}{2}(1 \pm \gamma_5)\chi(p) \), where \( \chi(p) \) is a generic Dirac spinor. Thus, using the general properties of the Dirac algebra, the contribution of Eq. (30) to the splitting function can straightforwardly be recast in a form that explicitly shows the cancellation of the spin correlations:

\[
\left( \frac{1}{s_{1...m}} \right)^k \chi_s(xp) \frac{\bar{k}_{i_1} \bar{k}_{i_2} \cdots \bar{k}_{i_{2k}} \gamma^\mu + \bar{k}_{i_1} \bar{k}_{i_2} \cdots \bar{k}_{i_{2k}} \gamma^\nu}{2xp \cdot n} \chi_{s'}(xp) \\
= \delta_{ss'} \left( \frac{1}{s_{1...m}} \right)^k \frac{1}{4p \cdot n} \text{Tr} \left[ \gamma^\mu (\gamma^\mu \bar{p} \bar{k}_{i_1} \bar{k}_{i_2} \cdots \bar{k}_{i_{2k}} + \bar{k}_{i_1} \bar{k}_{i_2} \cdots \bar{k}_{i_{2k}} \gamma^\mu \bar{p} \gamma^\nu) \right], \quad s, s' = \pm 1 , \quad (41)
\]

where \( \text{Tr} \) denotes the trace of the Dirac matrices. The identity in Eq. (41) relies on the definition and the properties of the chiral projectors \( \frac{1}{2}(1 \pm \gamma_5) \) and the absence of spin correlations ultimately follows from helicity conservation in the quark–gluon vector coupling.

This method to derive collinear factorization also provides us with an expression of the (spin-averaged) quark splitting function in terms of the dispersive part \( V_{a_1...a_m}^{(n)}(p_1, \ldots, p_m) \) of the two-point quark amplitude. From Eqs. (30) and (40), we find

\[
\langle \tilde{P}_{a_1...a_m} \rangle = \text{Tr} \left( \frac{\gamma^\mu V_{a_1...a_m}^{(n)}(p_1, \ldots, p_m)}{4n \cdot (p_1 + \ldots + p_m)} \right) . \quad (42)
\]

This equation is useful for a straightforward evaluation of the splitting function for the multiple collinear decay of a quark.

**Gluon splitting processes**

Unlike the quark case, it is convenient to define the gluon two-point function \( V_{a_1...a_m}^{(n)}(p_1, \ldots, p_m) \) without including in it the spin-polarization tensors \( d_{a_1...a_m}^{(n)}(p_1 + \ldots + p_m) \) of the two external gluons. Because of Lorentz covariance and the vanishing degree of homogeneity with respect to \( n^\mu \), the spin tensor \( V_{a_1...a_m}^{(n)} \) can be decomposed in terms of dimensionless scalar amplitudes as

\[
V_{a_1...a_m}^{(n)}(p_1, \ldots, p_m) = A^{(g)}(\{s_{ji}, z_j\}) g^{\mu\nu} + \sum_{i,j=1}^m B_i^{(g)}(\{s_{kl}, z_k\}) \frac{p_i^\mu p_j^\nu}{s_{1...m}}
+ \sum_{i=1}^m C_i^{(g)}(\{s_{ji}, z_j\}) \frac{p_i^\mu n^\nu + n^\mu p_i^\nu}{n \cdot (p_1 + \ldots + p_m)} + D^{(g)}(\{s_{ji}, z_j\}) \frac{n^\mu n^\nu s_{1...m}}{(n \cdot (p_1 + \ldots + p_m))^2} . \quad (43)
\]

Then, we have to multiply \( V_{a_1...a_m}^{(n)} \) by the gluon polarization tensors as follows

\[
d_{\nu}^{(n)}(p_1 + \ldots + p_m) V_{a_1...a_m}^{(n)}(p_1, \ldots, p_m) d_{\rho}^{(n)}(p_1 + \ldots + p_m) . \quad (44)
\]

Inserting Eq. (43) into Eq. (44), we immediately see that the scalar amplitudes \( C_i^{(g)}(\{s_{ji}, z_j\}) \) and \( D^{(g)}(\{s_{ji}, z_j\}) \) give a vanishing contribution because the gauge vector \( n^\mu \) is orthogonal to the polarization tensors. As for the second term on the right-hand side of Eq. (43), we can extract its dominant collinear contribution by simply performing the replacement \( p_i^\mu \rightarrow \tilde{k}_i^\mu \). Indeed, using Eq. (33) and

\[
(p_1 + \ldots + p_m)_\mu d_{\nu}^{(n)}(p_1 + \ldots + p_m) = O(s_{1...m}) , \quad (45)
\]
we have
\[ d_{\nu(n)}^\mu(p_1 + \ldots + p_m) p_i^\nu p_j^\rho d_{\rho(n)}^\sigma(p_1 + \ldots + p_m) = k_i^\mu k_i^\nu + \mathcal{O}(\lambda^3) \] 
so that the longitudinal component of \( p_i^\mu \) is suppressed in the multiple collinear limit \( \lambda \to 0 \).

We can now safely perform the collinear limit and we obtain the factorization formula (24) and an explicit expression of the gluon splitting function in terms of the scalar amplitudes in Eq. (43):
\[ \hat{P}_{a_1 \ldots a_m}^{\mu\nu} = A^{(g)}(\{s_{ij}, z_j\}) g^{\mu\nu} + \sum_{i,j=1}^m B_{ij}^{(g)}(\{s_{kl}, z_k\}) \frac{k_i^\mu k_j^\nu}{s_{1 \ldots m}} . \] 

The splitting function can be averaged over the spin polarizations of the parent gluon according to Eq. (3), and we obtain
\[ \frac{1}{2(1 - \epsilon)} d_{\rho(n)}^\mu(p_1 + \ldots + p_m) \hat{P}_{a_1 \ldots a_m}^{\mu\nu} = -A^{(g)}(\{s_{ij}, z_j\}) - \frac{1}{2(1 - \epsilon)} \sum_{i,j=1}^m B_{ij}^{(g)}(\{s_{kl}, z_k\}) \frac{k_i^\mu k_j^\nu}{s_{1 \ldots m}} , \] 
where
\[ 2\vec{k}_i \cdot \vec{k}_j = s_{ij} - \sum_{k=1}^m (z_is_{jk} + z_js_{ik}) + 2z_iz_js_{1 \ldots m} . \] 

Note that, since \( 2\vec{k}_i \cdot \vec{k}_j = 2(p_i \cdot d_{(n)}(p_1 + \ldots + p_m) \cdot d_{(n)}(p_1 + \ldots + p_m) \cdot p_j) \), the spin-averaged splitting function can also be expressed in terms of the Lorentz trace of Eq. (14):
\[ \langle \hat{P}_{a_1 \ldots a_m} \rangle = \frac{1}{2(1 - \epsilon)} d_{\rho(n)}^\mu(p_1 + \ldots + p_m) V_{a_1 \ldots a_m}^{\rho\nu(n)}(p_1, \ldots, p_m) d_{\nu(n)}(p_1 + \ldots + p_m) . \]

In the following subsection we present the explicit calculation of the quark and gluon splitting functions in the triple collinear limit.

### 2.4 Collinear splitting functions at \( \mathcal{O}(\alpha_S^2) \)

The list of (non-vanishing) splitting processes that we have to consider is as follows:
\[ q \to q_1' + q_2' + q_3 \quad , \quad (\bar{q} \to \bar{q}_1' + q_2' + \bar{q}_3) , \] 
\[ q \to q_1 + q_2 + q_3 \quad , \quad (\bar{q} \to \bar{q}_1 + q_2 + \bar{q}_3) , \] 
\[ q \to g_1 + g_2 + q_3 \quad , \quad (\bar{q} \to g_1 + g_2 + \bar{q}_3) , \] 
\[ g \to g_1 + g_2 + \bar{q}_3 \quad , \] 
\[ g \to g_1 + g_2 + g_3 . \] 

The superscripts in \( q', \bar{q}' \) denote fermions with different flavour with respect to \( q, \bar{q} \). As already mentioned in Sect. 2.3, the splitting functions for the processes in parenthesis in Eqs. (51) and (52) can be simply obtained by charge-conjugation invariance, i.e. \( \hat{P}_{q'q\bar{q}_3} = \hat{P}_{\bar{q}'q_3\bar{q}} \) and \( \hat{P}_{q_1q_2\bar{q}_3} = \hat{P}_{\bar{q}_1q_2\bar{q}_3} \). In summary, we have to compute five independent splitting functions.
Figure 3: The diagram for the collinear decay \( q \to \bar{q}' q_2 q_3 \).

To illustrate our calculation, we first consider the process in Eq. (51), that is, the case in which a quark–antiquark pair \( \bar{q}' q_2 \) and a quark \( q_3 \) with different flavour become collinear. This is the simplest case, because the two-point function \( V_{\bar{q}' q_2 q_3}^{(n)}(p_1, p_2, p_3) \) for the corresponding splitting process is obtained by squaring the sole Feynman diagram in Fig. 3. According to the definition in Eqs. (29) and (30), we extract the overall factor \( \left( \frac{8 \pi \mu^2 \alpha_s}{s_{123}} \right)^2 \) and, performing the average over the colours of the decaying quark, we find

\[
V_{\bar{q}' q_2 q_3}^{(n)}(p_1, p_2, p_3) = \frac{1}{2} C_F T_R \frac{s_{123}}{s_{12}} \left[ -\frac{2 z_3}{z_1 + z_2} - \left( \frac{t_{123}^2}{s_{12}} + 1 - 2 \epsilon \right) \frac{s_{12}}{s_{123}} \right] (\hat{p}_1 + \hat{p}_2 + \hat{p}_3)
\]

\[
+ \frac{2}{z_1 + z_2} \hat{p}_3 + (1 - 2 \epsilon)(\hat{p}_1 + \hat{p}_2) + \frac{z_1 - z_2}{z_1 + z_2} (\hat{p}_1 - \hat{p}_2)
\]

\[
+ \frac{2 t_{123}}{(z_1 + z_2)s_{12}} (z_1 \hat{p}_2 - z_2 \hat{p}_1) + \frac{1}{z_1 + z_2} \left( \frac{s_{12}}{s_{123}} - 1 \right) \frac{\hat{p}_3}{n \cdot (p_1 + p_2 + p_3)} \right] ,
\]

where \( t_{123} \) is the kinematical variable defined in Eq. (22). Note that Eq. (56) has the general structure obtained in Eq. (32). Using Eq. (42) to compute the splitting function \( \langle \hat{P}_{\bar{q}' q_2 q_3} \rangle \), the last two terms on the right-hand side of Eq. (56) give a vanishing contribution and we obtain the final result:

\[
\langle \hat{P}_{\bar{q}' q_2 q_3} \rangle = \frac{1}{2} C_F T_R \frac{s_{123}}{s_{12}} \left[ -\frac{t_{123}^2}{s_{12}s_{123}} + \frac{4 z_3 + (z_1 - z_2)^2}{z_1 + z_2} + (1 - 2 \epsilon) \left( z_1 + z_2 - \frac{s_{12}}{s_{123}} \right) \right] .
\]

The calculation of the splitting functions for the other processes in Eqs. (52)–(55) can be performed exactly in the same manner, by using the general procedure discussed in Sect. 2.3. We first compute the corresponding two-point functions \( V_{\bar{q}' q_2 q_3}^{(n)}(p_1, p_2, p_3) \) and then, using Eqs. (42) and (47), we evaluate the splitting functions. Since the intermediate expressions for the two-point functions are quite cumbersome, in the following we limit ourselves to showing the relevant Feynman diagrams and to presenting the final results for the splitting functions.

The calculation of the splitting function for the case of final-state fermions with identical flavour involves a diagram analogous to that in Fig. 3 plus its crossed diagram (see Fig. 4). Thus, the result can be written in terms of that in Eq. (57), as follows:

\[
\langle \hat{P}_{\bar{q}' q_2 q_3} \rangle = \left[ \langle \hat{P}_{\bar{q}' q_2 q_3} \rangle + \langle \hat{P}_{\bar{q}' q_2 q_3}^{\text{id}} \rangle \right] + \langle \hat{P}_{\bar{q}' q_2 q_3}^{\text{id}} \rangle ,
\]

(58)
where the interference contribution is given by

$$
\langle \hat{P}_{\text{id}}^{(id)} \rangle = C_F \left( C_F - \frac{1}{2} C_A \right) \left\{ (1 - \epsilon) \left( \frac{2 s_{23}}{s_{12}} - \epsilon \right) + \frac{s_{123}}{s_{12}} \left( \frac{1 + z_2^2}{1 - z_2} - \frac{2 z_2}{1 - z_3} - \epsilon \left( \frac{1 - z_3}{1 - z_2} + 1 + z_1 - \frac{2 z_2}{1 - z_3} \right) - \epsilon^2 (1 - z_3) \right) \right\} + (2 \leftrightarrow 3). \quad (59)
$$

The splitting function of the remaining quark-decay subprocess is obtained by squaring the diagrams in Fig. 5. It can be decomposed according to the different colour coefficients:

$$
\langle \hat{P}_{\text{g1g2q3}} \rangle = C_F^2 \langle \hat{P}_{\text{g1g2q3}}^{(ab)} \rangle + C_F C_A \langle \hat{P}_{\text{g1g2q3}}^{(nah)} \rangle, \quad (60)
$$

and the abelian and non-abelian contributions are

$$
\langle \hat{P}_{\text{g1g2q3}}^{(ab)} \rangle = \left\{ \frac{s_{123}}{2 s_{13} s_{23}} \left[ \frac{1 + z_3^2}{1 - z_3} - \epsilon \left( \frac{z_1^2 + z_2^2}{z_1 z_2} - \epsilon (1 + \epsilon) \right) \right] + \frac{s_{123}}{s_{13}} \left[ z_3 (1 - z_1) + (1 - z_2)^3 \right] + \epsilon^2 (1 + z_3) - \epsilon \left( \frac{z_1^2 + z_1 z_2 + z_2^2}{z_1 z_2} \right) \left( 1 - \frac{z_2}{z_1 z_2} \right) \right\} + (1 \leftrightarrow 2), \quad (61)
$$
\[(\hat{P}_{gq_{2}q_{3}}^{(nab)}) = \left\{ (1 - \epsilon) \left( t_{12,3}^{2} + \frac{1}{4} - \frac{\epsilon}{2} \right) + \frac{s_{123}^{2}}{2s_{12}s_{13}} \left[ (1 - z_{3})^{2}(1 - \epsilon) + 2z_{3} \right] \right. \\
+ \frac{z_{2}^{2}(1 - \epsilon) + 2(1 - z_{2})}{1 - z_{3}} \right\} - \frac{s_{123}^{2}}{4s_{13}s_{23}} \left[ (1 - z_{3})^{2}(1 - \epsilon) + 2z_{3} \right] + \epsilon(1 - \epsilon) \right\} + s_{123} \left[ (1 - \epsilon) \left( z_{1}(2 - 2z_{1} + z_{2}^{2}) - z_{2}(6 - 6z_{2} + z_{2}^{2}) \right) + 2\epsilon z_{3}(z_{1} - 2z_{2} - z_{2}) \right] \right.$$ \\
+ \left. \frac{s_{123}}{2s_{12}} \left[ (1 - \epsilon)z_{1}(2 - 2z_{1} + z_{2}^{2}) - z_{2}(6 - 6z_{2} + z_{2}^{2}) \right] + 2\epsilon z_{3}(z_{1} - 2z_{2} - z_{2}) \right) \right\} + (1 \leftrightarrow 2). \right\}

As discussed in Sect. 2.3, in the case of collinear decays of a gluon (see Eqs. (54, 53)), spin correlations are highly non-trivial.

\[ \text{Figure 6: The diagrams for the collinear decay } g \rightarrow g_{1}q_{2}q_{3}. \]

To compute the splitting function for the decay into a \( q\bar{q} \) pair plus a gluon, we have to evaluate the square of the diagrams in Fig. 4. The colour-factor decomposition of the splitting function is

\[ \hat{P}_{gq_{2}q_{3}}^{\mu\nu} = C_{F}T_{R} \hat{P}_{gq_{2}q_{3}}^{\mu\nu (ab)} + C_{A}T_{R} \hat{P}_{gq_{2}q_{3}}^{\mu\nu (nab)}, \]

where the abelian and non-abelian terms are given by

\[ \hat{P}_{gq_{2}q_{3}}^{\mu\nu (ab)} = -g^{\mu\nu} \left[ -2 + \frac{2s_{123}s_{23}}{s_{12}s_{13}} \right] \right\} + \frac{4s_{123}}{s_{12}s_{13}} \left( \bar{k}_{2}^{\mu}k_{2}^{\nu} + \bar{k}_{3}^{\mu}k_{3}^{\nu} - (1 - \epsilon)\bar{k}_{1}^{\mu}k_{1}^{\nu} \right), \]

\[ \hat{P}_{gq_{2}q_{3}}^{\mu\nu (nab)} = \frac{1}{4} \left[ \frac{s_{123}}{s_{23}} \left[ g^{\mu\nu}t_{23,1}^{2} - 16 \frac{z_{2}z_{3}}{z_{1}(1 - z_{1})} \left( \frac{\bar{k}_{2}^{\mu} - \bar{k}_{3}^{\mu}}{z_{2} - z_{3}} \right) \frac{\bar{k}_{2}^{\nu} - \bar{k}_{3}^{\nu}}{z_{2} - z_{3}} \right) \right] \right.$$
\\+ \left. \frac{s_{123}}{s_{12}s_{13}} \left[ 2s_{123}g^{\mu\nu} - 4(\bar{k}_{2}^{\mu}\bar{k}_{3}^{\nu} + \bar{k}_{3}^{\mu}k_{2}^{\nu} - (1 - \epsilon)\bar{k}_{1}^{\mu}k_{1}^{\nu}) \right] \right.$$
\\- \left. g^{\mu\nu} \left[ -(1 - 2\epsilon) + 2s_{123} \frac{1 - z_{3}}{z_{1}(1 - z_{1})} + 2s_{123} \frac{1 - z_{1} + 2z_{2}^{2}}{z_{1}(1 - z_{1})} \right] \right. \]
\[
+ \frac{s_{123}}{s_{12}s_{23}} \left[ -2s_{123}g^{\mu\nu}z_2(1 - 2z_1) \frac{z_2^2}{z_1(1 - z_1)} - 16k_3^{\mu}k_3^{\nu} \frac{z_2^2}{z_1(1 - z_1)} + 8(1 - \epsilon)k_2^{\mu}k_2^{\nu} \right] + (2 \leftrightarrow 3) .
\]

Figure 7: The diagrams for the collinear decay \( g \rightarrow g_1g_2g_3 \).

In the case of gluon decay into three collinear gluons we have to consider the diagrams in Fig. 7. Note that the contribution of the four-gluon vertex cannot be neglected. The result for the splitting function is quite involved. Its expression can be simplified by exploiting the complete symmetry under the six permutations of the gluon momenta. We obtain

\[
\hat{P}_{g_1g_2g_3}^{\mu\nu} = C_A^2 \left\{ \frac{(1 - \epsilon)}{4s_{12}^2} \left[ -g^{\mu\nu}t_{12,3}^2 + 16s_{123} \frac{z_2^2 z_3^2}{z_3(1 - z_3)} \left( \frac{k_2}{z_2} - \frac{k_1}{z_1} \right)^\mu \left( \frac{k_2}{z_2} - \frac{k_3}{z_3} \right)^\nu \right] + \frac{3}{4}(1 - \epsilon)g^{\mu\nu} + \frac{s_{123}}{s_{12}s_{13}} \left[ 2z_1 - \frac{2z_3}{1 - z_3} + \frac{2z_2}{1 - z_2} \right] \right\} + (5 \text{ permutations}) .
\]

The splitting functions in Eqs. (65)-(66) can be averaged over the spin polarizations of the parent gluon according to Eq. (67): 

\[
\langle \hat{P}_{a_1a_2g_3} \rangle = \frac{1}{2(1 - \epsilon)} d_{\mu\nu}(p) \hat{P}_{a_1a_2g_3}^{\mu\nu} .
\]

Performing the average, we obtain

\[
\langle \hat{P}_{g_1g_2g_3}^{(ab)} \rangle = -2 - (1 - \epsilon)s_{23} \left( \frac{1}{s_{12}} + \frac{1}{s_{13}} \right) + 2 \frac{s_{123}^2}{s_{12}s_{13}} \left( 1 + \frac{z_2^2}{1 - z_1 - 2z_2 z_3} \frac{z_2^2}{1 - z_1} \right)
- \frac{s_{123}}{s_{12}} \left( 1 + 2z_1 + \epsilon - \frac{z_1(1 + 2z_2 z_3)}{1 - \epsilon} \right) - \frac{s_{123}}{s_{13}} \left( 1 + 2z_1 + \epsilon - \frac{z_1(1 + 2z_2 z_3)}{1 - \epsilon} \right) ,
\]

(68)
The $O(\alpha_S^2)$-collinear limit of tree-level QCD amplitudes has been independently considered by Campbell and Glover [21]. They have computed only the spin-averaged splitting functions. The comparison with their results has been discussed in detail in Ref. [22] and we do not repeat it here. Our results agree with those of Ref. [21]. Since the methods used by the two groups are completely different (cf. the discussion in Sect. 1), this agreement can be regarded as an important cross-check of the calculations.

The expressions of the spin-dependent splitting functions $\hat{P}_{a_1 a_2 a_3}^{s s' t}$ derived in this section refer to the CDR scheme. Other dimensional-regularization schemes can be used. We mention two of them, which differ from CDR only by the number of spin-polarizations of quarks and gluons.

The dimensional-reduction (DR) scheme [31] works by considering two spin-polarization states for quarks and two for gluons. The corresponding spin-dependent splitting functions $\hat{P}_{a_1 a_2 a_3}^{s s' t}$ are simply obtained from those in the CDR scheme by setting $\epsilon = 0$.

The ‘toy’ dimensional-regularization (TDR) scheme introduced in Ref. [30] considers $d - 2 = 2(1 - \epsilon)$ spin-polarization states for quarks as for gluons. Its practical implementation is very simple. When computing traces of gamma matrices, we should use the relation $\text{Tr} \, 1 = 4(1 - \epsilon)$, where $1$ is the unity matrix in the spinor space. The corresponding spin-dependent splitting functions $\hat{P}_{a_1 a_2 a_3}^{s s' t}$ are obtained from those in the CDR scheme by the simple replacement $T_R \rightarrow T_R(1 - \epsilon)$.

The QCD results presented in this section can also be extended in a straightforward way to the abelian and supersymmetric cases.

In the case of QED, we have to perform the replacement $\alpha_S \rightarrow \alpha$ in the factorization formula (25), and the relevant splitting functions, $\hat{P}_{q_1 q_2 q_3}^{(QED)}$, for the triple collinear limit are obtained from the QCD splitting functions as

$$\hat{P}_{q_1 q_2 q_3}^{(QED)} = e^2 q_1^\nu q_2^\nu \left( \hat{P}_{q_1 q_2 q_3} \right)_{ab} ,$$
where \( e_q \) is the quark electric charge and the notation \((\ldots)_{ab}\) means that we have to set \( C_F = T_R = 1 \) and \( C_A = 0 \) in the QCD expression inside the round bracket.

The supersymmetric version of QCD, namely \( N = 1 \) supersymmetric Yang–Mills theory, is obtained by replacing the quark with the gluino \( g \), a Majorana fermion in the adjoint representation of the gauge group. To obtain the corresponding splitting functions, we have to change the colour factors accordingly, and we have to identify \( q = \bar{q} = g \) after having summed over the different permutations of the final-state fermions. We have

\[
\hat{P}_{g_1 g_2 g_3}^{(QED)} = e_q \left( \hat{P}_{g_1 g_2 g_3} + \hat{P}_{g_1 g_2 g_3} + \hat{P}_{g_1 g_2 g_3} \right)_{SQCD},
\hat{P}_{g_1 g_2 g_3}^{(QED)} = e_q \left( \hat{P}_{g_1 g_2 g_3} \right)_{SQCD},
\hat{P}_{g_1 g_2 g_3}^{(QED)} = e_q \left( \hat{P}_{g_1 g_2 g_3} \right)_{SQCD},
\]

where the notation \((\ldots)_{SQCD}\) means that we have to set \( C_F = 2T_R = C_A \) in the QCD expression inside the round bracket.

Gluino and gluon amplitudes are related by supersymmetry transformations. In the collinear limit, these transformations relate the total splitting functions \( \hat{P}_{g \to 2}^{ss'} \) and \( \hat{P}_{g \to 2}^{\mu \nu} \) for gluino and gluon decays, which are defined as

\[
\hat{P}_{g \to 3}^{ss'} \equiv \hat{P}_{g_1 g_2 g_3}^{ss'} + \left[ \hat{P}_{g_1 g_2 g_3}^{ss'} + (3 \leftrightarrow 1) + (3 \leftrightarrow 2) \right] = \delta^{ss'} \langle \hat{P}_{g \to 3} \rangle,
\hat{P}_{g \to 3}^{\mu \nu} \equiv \hat{P}_{g_1 g_2 g_3}^{\mu \nu} + \left[ \hat{P}_{g_1 g_2 g_3}^{\mu \nu} + (1 \leftrightarrow 2) + (1 \leftrightarrow 3) \right].
\]

In the four-dimensional supersymmetric theory, gluon and gluino have the same decay probability. Provided supersymmetry is not broken by the dimensional-regularization procedure, we thus have the following supersymmetric Ward identity:

\[
\hat{P}_{g \to 3}^{\mu \nu} = -g^{\mu \nu} \langle \hat{P}_{g \to 3} \rangle.
\]

Note that the Ward identity holds for the spin-dependent splitting functions. Since spin correlations are absent in the gluino splitting function, they cancel in the right-hand side of Eq. (74), and \( \hat{P}_{g \to 3}^{ss'} \) and \( \hat{P}_{g \to 3}^{\mu \nu} \) differ only by the overall spin-factors \( \delta^{ss'} \) and \( -g^{\mu \nu} \). As is well known, the splitting functions \( \hat{P}_{g \to 2}^{ss'} \) and \( \hat{P}_{g \to 2}^{\mu \nu} \) are related by a similar Ward identity at \( \mathcal{O}(\alpha_S) \).

The identity (75) is violated in the CDR scheme, because gluinos and gluons have a different number of spin-polarization states. The Ward identity is recovered in the \( \epsilon \to 0 \) limit or, equivalently, in the DR scheme, which is known to explicitly preserve supersymmetry. Our results for the spin-dependent splitting functions fulfill Eq. (75), and this is an important check of our calculation. As pointed out in Ref. [30], the Ward identity at \( \mathcal{O}(\alpha_S^2) \) is fulfilled also in the TDR scheme. We have verified that this remains true at \( \mathcal{O}(\alpha_S^2) \), as expected from the fact that in the TDR scheme the number of gluino states is the same as the number of gluon states.
3 The soft behaviour

The tree-level matrix elements $\mathcal{M}(p_1, p_2, \ldots)$ are singular not only when parton momenta become collinear but also when one or more of them become soft. In QCD calculations of physical cross sections at NLO, the soft limit is approached when the momentum of a single gluon vanishes. At NNLO we have to consider three different types of soft configurations:

- the emission of a soft quark–antiquark pair;
- the emission of two soft gluons;
- the emission of a soft gluon and a pair of collinear partons.

The behaviour of the tree-level matrix elements in these singular limits is considered in this section. We also discuss the generalization of the corresponding factorization formulae to higher perturbative orders.

3.1 Colour correlations and eikonal current at $\mathcal{O}(\alpha_s)$

The emission of a soft gluon does not affect the kinematics (momenta and spins) of the radiating partons. However, it does affect their colour because the gluon always carries away some colour charge, no matter how soft it is. Unlike the case of soft-photon emission in QED, soft-gluon emission thus does not factorize exactly and leads to colour correlations.

To take into account the colour structure (as well as the spin and flavour structures), it is useful to introduce a basis $\{|c_1, \ldots, c_n\rangle \otimes |s_1, \ldots, s_n\rangle\}$ in colour + helicity space in such a way that the tree-level matrix element in Eq. (1) with $n$ final-state partons can be written as

$$\mathcal{M}_{a_1, \ldots, a_n}(p_1, \ldots, p_n) \equiv \langle c_1, \ldots, c_n | s_1, \ldots, s_n \rangle |M_{a_1, \ldots, a_n}(p_1, \ldots, p_n)\rangle. \quad (76)$$

Thus $|\mathcal{M}_{a_1, \ldots, a_n}(p_1, \ldots, p_n)\rangle$ is a vector in colour + helicity space.

According to this notation, the matrix element squared (summed over final-state colours and spins) $|\mathcal{M}|^2$ can be written as

$$|\mathcal{M}_{a_1, \ldots, a_n}(p_1, \ldots, p_n)|^2 = \langle \mathcal{M}_{a_1, \ldots, a_n}(p_1, \ldots, p_n) | \mathcal{M}_{a_1, \ldots, a_n}(p_1, \ldots, p_n) \rangle. \quad (77)$$

To describe the colour correlations produced by soft-gluon emission, it is convenient to associate a colour charge $T_i$ with the emission of a gluon from each parton $i$. If the emitted gluon has colour index $c$ ($c = 1, \ldots, N_c^2 - 1$), the colour-charge operator is:

$$T_i \equiv \langle c | T_i^c \rangle \quad (78)$$

and its action onto the colour space is defined by

$$\langle c_1, \ldots, c_i, \ldots, c_m, c | T_i | b_1, \ldots, b_i, \ldots, b_m \rangle = \delta_{c_1 b_1} \cdots \delta_{c_i b_i} \cdots \delta_{c_m b_m}, \quad (79)$$
where \( T^a_{cb} \equiv if_{cab} \) (colour-charge matrix in the adjoint representation) if the emitting particle \( i \) is a gluon and \( T^a_{\alpha\beta} \equiv t^a_{\alpha\beta} \) (colour-charge matrix in the fundamental representation with \( \alpha, \beta = 1, \ldots, N_c \)) if the emitting particle \( i \) is a quark (in the case of an emitting antiquark \( T^a_{\alpha\beta} \equiv \bar{t}^a_{\alpha\beta} = -t^a_{\beta\alpha} \)).

The colour-charge algebra is

\[
T^c_i T^c_j \equiv T_i \cdot T_j = T_j \cdot T_i \quad \text{if} \quad i \neq j; \quad T^2_i = C_i,
\]

where \( C_i \) is the Casimir operator, that is, \( C_i = C_A = N_c \) if \( i \) is a gluon and \( C_i = C_F = (N_c^2 - 1)/2N_c \) if \( i \) is a quark or antiquark.

Note that, by definition, each vector \( |M_{a_1,\ldots,a_n}(p_1,\ldots,p_n)\rangle \) is a colour-singlet state. Therefore colour conservation is simply

\[
\sum_{i=1}^{n} T_i |M_{a_1,\ldots,a_n}(p_1,\ldots,p_n)\rangle = 0.
\]

Let us now consider the tree-level matrix element \( \mathcal{M}_{g,a_1,\ldots,a_n}(q,p_1,\ldots,p_n) \) in the limit where the momentum \( q \) of the gluon becomes soft. Denoting by \( c \) and \( \mu \) the colour and spin indices of the soft gluon, the matrix element fulfils the following factorization formula

\[
\langle c; \mu | \mathcal{M}_{g,a_1,\ldots,a_n}(q,p_1,\ldots,p_n) \rangle \simeq g \mu^{\epsilon} J^c\mu(q) |M_{a_1,\ldots,a_n}(p_1,\ldots,p_n)\rangle,
\]

where \( |M_{a_1,\ldots,a_n}(p_1,\ldots,p_n)\rangle \) is obtained from the original matrix by simply removing the soft gluon \( q \). The factor \( J^c\mu(q) \) is the eikonal current

\[
J^c\mu(q) = \sum_{i=1}^{n} T_i \frac{p_i^\mu}{p_i \cdot q},
\]

which depends on the momenta and colour charges of the hard partons in the matrix element on the right-hand side of Eq. (82). The symbol ‘\( \simeq \)’ means that on the right-hand side we have neglected contributions that are less singular than \( 1/q \) in the soft limit \( q \to 0 \). Note that Eq. (82) is valid in any number \( d = 4 - 2\epsilon \) of space-time dimensions, and the sole dependence on \( d \) is in the overall factor \( \mu^{\epsilon} \).

The factorization formula (82) can be derived in a simple way by working in a physical gauge and using the following soft-gluon insertion rules. The coupling of the gluon to any internal (i.e. highly off-shell) parton in the amplitude \( \mathcal{M}_{g,a_1,\ldots,a_n}(q,p_1,\ldots,p_n) \) is not singular in the soft limit; it can thus be neglected. The soft-gluon coupling to any external or, in general, nearly on-shell parton with colour charge \( T \) and momentum \( p \) can be factorized by extracting the contribution \( g_s \mu^{\epsilon} 2p^\mu T \) for the vertex and the contribution \( 1/(p+q)^2 \simeq 1/(p^2 + 2p \cdot q) \) for the propagator.

An important property of the eikonal current is current conservation. Multiplying Eq. (82) by \( q^\mu \), we obtain

\[
q^\mu J^c\mu(q) = \sum_{i=1}^{n} T_i,
\]

More details on the colour algebra and useful colour-matrix relations can be found in Appendix A of Ref. [3].
and thus
\[ q_\mu J^\mu(q) | M_{a_1, \ldots, a_n}(p_1, \ldots, p_n) \rangle = \sum_{i=1}^{n} T_i | M_{a_1, \ldots, a_n}(p_1, \ldots, p_n) \rangle = 0 , \] (85)

where the last equality follows from colour conservation as in Eq. (81).

Although Eq. (82) is most easily derived in a physical gauge, the conservation of the eikonal current implies the gauge invariance of the squared amplitude summed over the soft-gluon polarizations. Squaring the eikonal current and introducing the gluon polarization tensor \( d_{\mu \nu}(q) = (-g_{\mu \nu} + \text{gauge terms}) \) in Eq. (4), we have
\[ [J^\mu(q)]^\dagger d_{\mu \nu}(q) J^\nu(q) = - \sum_{i,j=1}^{n} T_i \cdot T_j \frac{p_i \cdot p_j}{(p_i \cdot q)(p_j \cdot q)} + \ldots , \] (86)

where we have used the fact that the gauge terms in \( d_{\mu \nu}(q) \) are due to longitudinal polarizations proportional either to \( q^\mu \) or to \( q^\nu \). Thus, the dots on the right-hand side stand for gauge-dependent contributions that are proportional to the total colour charge \( \sum_{i=1}^{n} T_i \) and, hence, that cancel when they are inserted in \( | M_{a_1, \ldots, a_n}(p_1, \ldots, p_n) \rangle \).

Using Eq. (86), the soft-gluon factorization formula at \( O(\alpha_s) \) for the squared amplitude is
\[ |M_{g,a_1, \ldots, a_n}(q,p_1, \ldots, p_n)|^2 \simeq -4\pi\alpha_s \mu^{2\epsilon} \sum_{i,j=1}^{n} S_{ij}(q) |M_{a_1, \ldots, a_n}^{(i,j)}(p_1, \ldots, p_n)|^2 , \] (87)

where the scalar eikonal function \( S_{ij}(q) \) for the emission of a single gluon can be written in terms of two-particle sub-energies \( s_{ij} = (p_i + p_j)^2 \) as follows
\[ S_{ij}(q) = \frac{p_i \cdot p_j}{(p_i \cdot q)(p_j \cdot q)} = \frac{2s_{ij}}{s_{iq} s_{jq}} . \] (88)
The colour correlations produced by soft-gluon emission are taken into account by the square of the colour-correlated tree-amplitude \( |M^{(i,j)}|^2 \) on the right-hand side. This is defined by
\[ |M_{a_1, \ldots, a_n}^{(i,j)}(p_1, \ldots, p_n)|^2 = \langle M_{a_1, \ldots, a_n}(p_1, \ldots, p_n) | T_i \cdot T_j | M_{a_1, \ldots, a_n}(p_1, \ldots, p_n) \rangle \] (89)
\[ = [M_{a_1, \ldots, a_n}^{c_1, \ldots, c_n}(p_1, \ldots, p_n)]^* T_{b_i d_i}^c T_{b_j d_j}^c M_{a_1, \ldots, a_n}^{c_1, \ldots, c_n}(p_1, \ldots, p_n) , \]
where the sum over the spin indices is understood.

The soft-gluon factorization formula is often presented [21] in an equivalent way by decomposing the matrix element in terms of colour subamplitudes [1]. In this formalism, the eikonal function \( S_{ij}(q) \) in Eq. (88) controls the factorization properties of the square of the colour-connected subamplitudes.

### 3.2 Emission of a soft q\bar{q}-pair

We now consider the tree-level matrix element \( M_{q,\bar{q},a_1, \ldots, a_n}(q_1, q_2, p_1, \ldots, p_n) \) when the momenta \( q_1 \) and \( q_2 \) of the quark \( q \) and the antiquark \( \bar{q} \) become soft \( (q_1, q_2 \rightarrow 0 \text{ at fixed } q_1/q_2) \).
In this limit the matrix element squared has the dominant behaviour:

\[ |M_{q,q,a_1,...,a_n}(q_1, q_2, p_1, \ldots, p_n)|^2 \sim \frac{1}{(q_1 \cdot q_2) [p_i \cdot (q_1 + q_2)] [p_j \cdot (q_1 + q_2)]}. \tag{90} \]

When integrated over the phase space of the quark-antiquark pair, this behaviour gives rise to a single-logarithmic soft singularity, in addition to possible single- and double-logarithmic collinear singularities.

The soft singularity arises when the $q\bar{q}$-pair is produced by the decay of a gluon that carries the soft momentum $q_1 + q_2$ (Fig. 3). Thus, using the soft-gluon insertion rules described in Sect. 3, we can straightforwardly derive the following factorization formula:

\[ |M_{q,q,a_1,...,a_n}(q_1, q_2, p_1, \ldots, p_n)|^2 \simeq (4\pi \mu^2 \alpha_S)^2 \cdot \left< M_{a_1,...,a_n}(p_1, \ldots, p_n) | I_{(q\bar{q})}(q_1, q_2) | M_{a_1,...,a_n}(p_1, \ldots, p_n) \right>, \tag{91} \]

where

\[
I_{(q\bar{q})}(q_1, q_2) = \left[J^\lambda(q_1 + q_2)\right] d_{\lambda\mu}(q_1 + q_2) \Pi^{\mu\nu}(q_1, q_2) \cdot J^\nu(q_1 + q_2), \tag{92}
\]

\[
= \left[J_\mu(q_1 + q_2)\right] \Pi^{\mu\nu}(q_1, q_2) \cdot J^\nu(q_1 + q_2) + \ldots. \tag{93}
\]

The insertion operator $I_{(q\bar{q})}(q_1, q_2)$ depends on the colour charges of the fast partons $a_1, \ldots, a_n$ and it is given in terms of the soft-gluon current $J^\mu(q_1 + q_2)$ in Eq. (83) and of $\Pi^{\mu\nu}(q_1, q_2)$, which is the $q\bar{q}$-contribution to the discontinuity of the gluon propagator:

\[ \Pi^{\mu\nu}(q_1, q_2) = \frac{T_R}{(q_1 \cdot q_2)^2} \left\{ -g^{\mu\nu}q_1 \cdot q_2 + q_1^\mu q_2^\nu + q_2^\mu q_1^\nu \right\}. \tag{94} \]

The dots on the right-hand side of Eq. (94) denote the gauge-dependent contribution to the insertion operator $I_{(q\bar{q})}(q_1, q_2)$. This term is due to the longitudinal polarizations (proportional to $(q_1 + q_2)^\alpha$ or $(q_1 + q_2)^\beta$) of the polarization tensors $d_{\alpha\beta}(q_1 + q_2)$ in Eq. (92).

Since $\Pi^{\mu\nu}(q_1, q_2)$ is transverse ($q_1 \cdot \Pi^{\mu\nu}(q_1, q_2) = 0$) and the soft current $J^\mu(q_1 + q_2)$ is conserved (see Eq. (84)), the contribution of the longitudinal polarizations is either vanishing or proportional to the total colour charge of the fast partons. Because of colour conservation (see Eq. (71)), we thus conclude that the gauge-dependent part of $I_{(q\bar{q})}(q_1, q_2)$ does not contribute to the factorization formula (91).

Inserting Eq. (93) into Eq. (71) and performing the Lorentz algebra, we obtain the final factorization formula

\[ |M_{q,q,a_1,...,a_n}(q_1, q_2, p_1, \ldots, p_n)|^2 \simeq (4\pi \mu^2 \alpha_S)^2 T_R \sum_{i,j=1}^n \mathcal{I}_{ij}(q_1, q_2) |M_{a_1,...,a_n}^{(ij)}(p_1, \ldots, p_n)|^2, \tag{95} \]

where $|M_{a_1,...,a_n}^{(ij)}|^2$ is the colour-correlated tree-amplitude of Eq. (89) and the soft function $\mathcal{I}_{ij}(q_1, q_2)$ is given by

\[
\mathcal{I}_{ij}(q_1, q_2) = \frac{(p_i \cdot q_1) (p_j \cdot q_2) + (p_j \cdot q_1) (p_i \cdot q_2) - (p_i \cdot p_j) (q_1 \cdot q_2)}{(q_1 \cdot q_2)^2 [p_i \cdot (q_1 + q_2)] [p_j \cdot (q_1 + q_2)]} \tag{96}
\]

\[
= - \frac{2(p_i \cdot p_j) (q_1 \cdot q_2) + [p_i \cdot (q_1 - q_2)] [p_j \cdot (q_1 - q_2)]}{2(q_1 \cdot q_2)^2 [p_i \cdot (q_1 + q_2)] [p_j \cdot (q_1 + q_2)]} + \ldots. \tag{97}
\]
Note that both the expressions (96) and (97) can equivalently be used to compute Eq. (95). The difference between the two expressions, denoted by the dots on the right-hand side of Eq. (97), gives a vanishing contribution to the factorization formula (95) because of colour conservation (see Eq. (81)).

3.3 Soft current for double gluon emission

The limit of QCD tree-amplitudes when the momenta of two gluons simultaneously become soft was independently studied by Berends and Giele [19] and by one of the authors [20]. The singular behaviour of the matrix elements can be described in terms of factorization formulae given in terms of process-independent two-gluon currents acting either on colour-ordered subamplitudes [19] or on the colour space of the hard partons [20].

The formalism of the colour subamplitudes was used in Ref. [21] to derive explicit soft-gluon factors for the square of colour-connected and colour-unconnected subamplitudes. In this section we recall the formalism and the results of Ref. [20] and we present the corresponding factorization formula for the square of the matrix elements.

We consider the tree-level matrix element $M_{g,g,a_1,...,a_n}(q_1, q_2, , p_1, \ldots, p_n)$ when the momenta $q_1$ and $q_2$ of the two gluons become soft. The limit is precisely defined by rescaling the gluon momenta by an overall factor $\lambda$:

$$q_1 \rightarrow \lambda q_1, \quad q_2 \rightarrow \lambda q_2,$$

and then performing the limit $\lambda \rightarrow 0$. The matrix element thus behaves as

$$M_{g,g,...} \rightarrow \mathcal{O}(1/\lambda^2) + \ldots$$

(99)

where the dots stand for less singular contributions as $\lambda \rightarrow 0$. We are interested in explicitly evaluating the dominant singular term $\mathcal{O}(1/\lambda^2)$. 
Note that the double soft limit is more general (accurate) than the soft limit in the strong-ordering approximation, that is, when $p_i \gg q_1 \gg q_2$. The strongly-ordered limit describes only the double-logarithmic soft singularity of the matrix elements. The double soft limit reproduces consistently the double-logarithmic behaviour and correctly evaluates also the single-logarithmic soft singularity.

We denote by $a_1, a_2$ and $\mu_1, \mu_2$ the colour and Lorentz indices of the two gluons, respectively. In the double soft limit the matrix element fulfils the following factorization formula \cite{20}

$$
\langle a_1, a_2; \mu_1, \mu_2 | \mathcal{M}_{g,g,a_1,\ldots,a_n}(q_1, q_2, p_1, \ldots, p_n) \rangle \simeq g_S^2 \mu^2 e J_{\mu_1 \mu_2}^{a_1 a_2}(q_1, q_2) | \mathcal{M}_{a_1,\ldots,a_n}(p_1, \ldots, p_n) \rangle,
$$

(100)

where the two-gluon soft current $J_{\mu_1 \mu_2}^{a_1 a_2}(q_1, q_2)$ is the generalization of the eikonal current in Eq. (83).

The explicit expression of the soft current is \cite{20}

$$
J_{\mu_1 \mu_2}^{a_1 a_2}(q_1, q_2) = \sum_{i \neq j} T_i^{a_1} \frac{p_i^{\mu_1}}{p_i \cdot q_1} T_j^{a_2} \frac{p_j^{\mu_2}}{p_j \cdot q_2} + \\
+ \sum_i \left[ \left( \delta^{a_1 a} T_i^{a_2} \frac{p_i^{\mu_2}}{p_i \cdot q_2} - i f_{a_2 a_1 a} q_i^{\mu_2} \frac{q_1^{\mu_1}}{q_1 \cdot q_2} \right) T_i^{a_1} \frac{p_i^{\mu_1}}{p_i \cdot (q_1 + q_2)} + \\
+ \left( \delta^{a_2 a} T_i^{a_1} \frac{p_i^{\mu_1}}{p_i \cdot q_1} - i f_{a_1 a_2 a} q_i^{\mu_1} \frac{q_2^{\mu_2}}{q_1 \cdot q_2} \right) T_i^{a_2} \frac{p_i^{\mu_2}}{p_i \cdot (q_1 + q_2)} + \\
+ \frac{1}{2} i f_{a_1 a_2} T_i^{a} g^{\mu_1 \mu_2} \frac{p_i \cdot (q_2 - q_1)}{q_1 \cdot q_2} \frac{p_i \cdot (q_2 + q_1)}{p_i \cdot (q_2 + q_1)} \right].
$$

(101)

It can be derived by working in a physical gauge and using the soft-gluon insertion technique described in Sect. 3.1. We have to consider the diagrams in Fig. [4]. The contribution on the first line of Eq. (101) comes from the eikonal emission of the two soft gluons from two different external partons (diagrams (a) in Fig. [4]). The first term on the second and third lines come from the eikonal emission of the two gluons from the same external parton (diagrams (b) in Fig. [4]). The remaining contributions in Eq. (101) are proportional to $f_{aa_1 a_2}$ and originate from the non-abelian diagrams of Fig. [4](c). Note that the three-gluon vertex has to be treated exactly, without introducing any soft approximation.

![Figure 9: Soft-gluon insertion diagrams used to evaluate the two-gluon current $J_{a_1 a_2}^{\mu_1 \mu_2}(q_1, q_2)$.](image)
The two-gluon current in Eq. (101) can be recast in the following equivalent form

\[ J_{a_1a_2}^{\mu_1\mu_2}(q_1, q_2) = \frac{1}{2} \left\{ J_{a_1}^{\mu_1}(q_1), J_{a_2}^{\mu_2}(q_2) \right\} + \sum_{i=1}^{n} T_i \left( \frac{p_i \cdot q_1^{\mu_1} q_2^{\mu_2} - p_i^{\mu_1} q_2^{\mu_2}}{(q_1 \cdot q_2) [q_i \cdot (q_1 + q_2)]} \right) \left( \frac{p_i \cdot (q_1 - q_2)}{2[q_i \cdot (q_1 + q_2)]} + \frac{g^{\mu_1\mu_2}}{q_1 \cdot q_2} \right) \right\}, \]

where the first term on the right-hand side is the colour anticommutator of the single-gluon eikonal currents of Eq. (83). This is the only contribution that survives in the abelian case, where it reduces itself to the product of two independent single-gluon currents. The second term on the right-hand side is typical of the non-abelian theory.

Note that, as in the single-gluon case, the expressions (101) and (102) for the two-gluon current do not explicitly depend on the number \( d = 4 - 2\epsilon \) of space-time dimensions. However, because of the contribution proportional to \( g^{\mu_1\mu_2} \) in \( J_{a_1a_2}^{\mu_1\mu_2} \), an explicit dependence on the number \( d - 2 = 2(1 - \epsilon) \) of gluon polarizations appears (see Eqs. (108) and (109)) by squaring the factorization formula (100).

The current \( J_{\mu_1\mu_2}^{a_1a_2}(q_1, q_2) \) fulfils the following properties.

- It is symmetric under the exchange of the two soft gluons,

\[ J_{\mu_1\mu_2}^{a_1a_2}(q_1, q_2) = J_{\mu_2\mu_1}^{a_2a_1}(q_2, q_1). \]

- Its divergence is proportional to the total colour charge of the hard partons:

\[ q_1^{\mu_1} J_{\mu_1\mu_2}^{a_1a_2}(q_1, q_2) = \left( J_{\mu_2}^{a_2}(q_2) \delta_{a_1a} + \frac{i}{2} f_{a_1a_2a} q_1^{\mu_2} \right) \sum_{i=1}^{n} T_i, \]

\[ q_2^{\mu_2} J_{\mu_1\mu_2}^{a_1a_2}(q_1, q_2) = \left( J_{\mu_2}^{a_1}(q_1) \delta_{a_2a} + \frac{i}{2} f_{a_2a_1a} q_2^{\mu_2} \right) \sum_{i=1}^{n} T_i. \]

This property is analogous to Eq. (84) for the single-gluon emission and implies that the two-gluon current is conserved when it acts on a colour singlet state:

\[ q_1^{\mu_1} J_{\mu_1\mu_2}^{a_1a_2}(q_1, q_2) | M_{a_1, \ldots, (p_1, \ldots)} = q_2^{\mu_2} J_{\mu_1\mu_2}^{a_1a_2}(q_1, q_2) | M_{a_1, \ldots, (p_1, \ldots)} = 0. \]

Thus, the factorization formula (100) is gauge-invariant.

- In the strong-ordered limit \( q_2 \ll q_1 \), the third and fourth lines in Eq. (101) give subleading contributions and the current becomes

\[ J_{a_1a_2}^{\mu_1\mu_2}(q_1, q_2) \rightarrow \left( J_{a_2}^{\mu_2}(q_2) \delta_{a_1a} + \frac{i f_{a_1a_2a}}{q_1 \cdot q_2} q_1^{\mu_2} \right) J_{a_1}^{\mu_1}(q_1). \]

Thus, the current correctly factorizes into the product of the two eikonal currents corresponding to the iterative application of the leading-order factorization formula (82).

The double soft limit of \( |M_{a_1, a_2, \ldots, a_n}(q_1, q_2, p_1, \ldots, p_n)|^2 \) is obtained by squaring Eq. (100) and by summing over the soft-gluon polarizations. The square of the two-gluon current

\[ 25 \]
involves a quite cumbersome colour algebra. Nonetheless, we find that the final result can be recast in a relatively simple form:

\[
\left[ J^{a_1 a_2}_{\mu \rho}(q_1, q_2) \right]^\dagger \, d^{\mu \nu}(q_1) \, d^{\rho \sigma}(q_2) \, J^{a_1 a_2}_{\nu \sigma}(q_1, q_2) = \frac{1}{2} \left\{ J^2(q_1), J^2(q_2) \right\} - C_A \sum_{i,j=1}^n T_i \cdot T_j \, S_{ij}(q_1, q_2) + \ldots , (108)
\]

where, as in Eq. (86), the dots stand for gauge-dependent terms. These are proportional to the total colour charge of the hard partons and, thus, give a vanishing contribution when inserted on \( |\mathcal{M}_{a_1, \ldots, a_n}(p_1, \ldots, p_n)\rangle \).

The first term on the right-hand side of Eq. (108) is the only one that survives in the abelian case. It is given in terms of the colour anticommutator of the squares of the single-gluon currents in Eq. (86). The second term is proportional to \( C_A \) and, hence, is purely non-abelian. It is given in terms of the two-gluon soft function \( S_{ij}(q_1, q_2) \):

\[
S_{ij}(q_1, q_2) = \frac{(1 - \epsilon)}{(q_1 \cdot q_2)^2} \, \frac{p_i \cdot q_1 \, p_j \cdot q_2 + p_i \cdot q_2 \, p_j \cdot q_1}{p_i \cdot (q_1 + q_2) \, p_j \cdot (q_1 + q_2)} - \frac{(p_i \cdot p_j)^2}{2p_i \cdot q_1 \, p_j \cdot q_2 \, p_i \cdot q_2 \, p_j \cdot q_1} \left[ 2 - \frac{p_i \cdot q_1 \, p_j \cdot q_2 + p_i \cdot q_2 \, p_j \cdot q_1}{p_i \cdot (q_1 + q_2) \, p_j \cdot (q_1 + q_2)} \right] + \frac{2}{2p_i \cdot q_1} \left[ \frac{2}{p_i \cdot q_1} \right] \left[ \frac{2}{p_i \cdot q_1} \right] - \frac{1}{p_i \cdot (q_1 + q_2) \, p_j \cdot (q_1 + q_2)} \left[ 4 + \frac{(p_i \cdot q_1 \, p_j \cdot q_2 + p_i \cdot q_2 \, p_j \cdot q_1)^2}{p_i \cdot q_1 \, p_j \cdot q_2 \, p_i \cdot q_2 \, p_j \cdot q_1} \right]. (109)
\]

Expression (103) can also be written as

\[
S_{ij}(q_1, q_2) = S_{ij}^{(s,o)}(q_1, q_2) + \frac{p_i \cdot q_1 \, p_j \cdot q_2 + p_i \cdot q_2 \, p_j \cdot q_1}{p_i \cdot (q_1 + q_2) \, p_j \cdot (q_1 + q_2)} \left[ \frac{(1 - \epsilon)}{(q_1 \cdot q_2)^2} - \frac{1}{2} S_{ij}^{(s,o)}(q_1, q_2) \right] - \frac{2p_i \cdot p_j}{q_1 \cdot q_2 \, p_i \cdot (q_1 + q_2) \, p_j \cdot (q_1 + q_2)}, (110)
\]

where \( S_{ij}^{(s,o)} \) is the approximation of the soft function \( S_{ij}(q_1, q_2) \) in the strong-ordering limit (either \( q_1 \ll q_2 \) or \( q_2 \ll q_1 \)):

\[
S_{ij}^{(s,o)}(q_1, q_2) = \frac{p_i \cdot p_j}{q_1 \cdot q_2} \left( \frac{1}{p_i \cdot q_1 \, p_j \cdot q_2} + \frac{1}{p_j \cdot q_1 \, p_i \cdot q_2} \right) - \frac{(p_i \cdot p_j)^2}{p_i \cdot q_1 \, p_j \cdot q_2 \, p_i \cdot q_2 \, p_j \cdot q_1}. (111)
\]

Using Eq. (108), we can write the soft-gluon factorization formula for the square of the matrix element as follows:

\[
|\mathcal{M}_{g,g,a_1,\ldots,a_n}(q_1, q_2, p_1, \ldots, p_n)|^2 \simeq \left( 4\pi \alpha_S \mu^2 \right)^2 \cdot \left[ \frac{1}{2} \sum_{i,j,k,l=1}^n S_{ij}(q_1) \, S_{kl}(q_2) \, |\mathcal{M}_{a_1,\ldots,a_n}^{(i,j,k,l)}(p_1, \ldots, p_n)|^2 - C_A \sum_{i,j=1}^n S_{ij}(q_1, q_2) \, |\mathcal{M}_{a_1,\ldots,a_n}^{(i,j)}|^2 \right], (112)
\]
where $S_{ij}(q)$ is the soft function in Eq. (88) and $|\mathcal{M}_{a_1,\ldots,a_n}^{(i,j)}|^2$ is the colour-correlated amplitude in Eq. (89). We can see that the double soft limit involves colour correlations that are more cumbersome than those appearing in the case of single-gluon emission. Indeed, the amplitude $|\mathcal{M}_{a_1,\ldots,a_n}^{(i,j)(k,l)}|^2$ on the right-hand side of Eq. (112) is defined by

$$|\mathcal{M}_{a_1,\ldots,a_n}^{(i,j)(k,l)}(p_1,\ldots,p_n)|^2 \equiv \langle \mathcal{M}_{a_1,\ldots,a_n}(p_1,\ldots,p_n) \{T_i \cdot T_j , T_k \cdot T_l \} \mathcal{M}_{a_1,\ldots,a_n}(p_1,\ldots,p_n) \rangle,$$

and leads to irreducible correlations among four different hard partons.

The results discussed in this subsection can be presented in a different manner by using the colour subamplitude formalism. Considering the projection of Eq. (109) onto colour-ordered subamplitudes, it is straightforward to check that the colour current $J_{a_1 a_2}(q_1, q_2)$ leads to the colourless current derived by Berends and Giele (see Eqs. (3.11) and (3.18) in Ref. [21]). The square of this colourless current is denoted by $S_{iq,qqj}$ in Sect. 5.3 of Ref. [21] and is related to the soft function $S_{ij}(q_1, q_2)$ in Eq. (108). More precisely, using the following relation

$$\sum_{i,j=1}^n T_i \cdot T_j [S_{ij}(q_1, q_2) + S_{ij}(q_1) S_{ij}(q_2)] = \frac{1}{2} \sum_{i,j=1}^n T_i \cdot T_j S_{iq,qqj} + \ldots,$$

the second term on the right-hand side of Eqs. (108) and (112) can equivalently be written in terms of $S_{iq,qqj}$. The contribution denoted by the dots on the right-hand side of Eq. (114) is proportional to the total colour charge of the hard partons and, thus, it vanishes when inserted in the factorization formula (112).

### 3.4 Soft–collinear limit at $\mathcal{O}(\alpha_S^2)$ and at higher orders

We now consider the tree-level matrix element $\mathcal{M}_{g,a_1,\ldots,a_n}(q, p_1, \ldots, p_n)$ in the limit where the momentum $q$ of the gluon becomes soft ($q \to 0$) and, at the same time, two partons, say $p_1$ and $p_2$, become collinear. The collinear region is parametrized as in Eq. (3) and we are interested in the limit $k_\perp \to 0$.

Studying this soft–collinear limit we can neglect $i)$ contributions that are uniformly of $\mathcal{O}(q)$ when $q \to 0$, and $ii)$ contributions that are uniformly of $\mathcal{O}(k_\perp)$ when $k_\perp \to 0$. The terms in class $i)$ are not singular in the soft limit and their contribution in the collinear limit can thus be taken into account by supplementing the results of this section with the $\mathcal{O}(\alpha_S)$-collinear factorization discussed in Sect. 2.1. Analogously, the terms in class $ii)$ are not singular in the collinear limit and their contribution in the soft limit can be taken into account by supplementing the results of this section with the soft-gluon factorization formula at $\mathcal{O}(\alpha_S)$ presented in Sect. 3.1.

This comment can be summarized in a formal manner by writing the square of the matrix element as

$$|\mathcal{M}_{g,a_1,a_2,\ldots,a_n}(q, p_1, p_2, \ldots, p_n)|^2 = \frac{1}{s_{12}s_{1q}s_{2q}} F(q, p_1, p_2, \ldots, p_n).$$

The first factor on the right-hand side contains the correct scaling behaviour in the soft and collinear regions. Thus, the soft–collinear limit is defined by the soft ($q \to 0$) and collinear ($k_\perp \to 0$) approximations of the function $F(q, p_1, p_2, \ldots, p_n)$ at fixed ratio $q/k_\perp^2$. 

27
To compute the soft–collinear limit we perform first soft approximations and then collinear approximations.

The singular behaviour of the matrix element in the soft limit (and at fixed $q/k_\perp^2$) is given by a factorization formula analogous to Eq. (82), namely

$$
\langle \mu \rvert M_{g,a_1,a_2,\ldots,a_n}(q,p_1,p_2,\ldots,p_n) \rangle \simeq g \mu \epsilon J_{c(12)}(q) \rvert M_{a_1,a_2,\ldots,a_n}(p_1,p_2,\ldots,p_n) \rangle ,
$$

but now the soft current $J_{c(12)}(q)$ is no longer equal to the eikonal current in Eq. (83). In fact, since $p_1$ and $p_2$ can become collinear, the internal partonic line with momentum $p_1 + p_2$ in $M_{a_1,a_2,\ldots,a_n}(p_1,p_2,\ldots,p_n)$ is close to the mass shell ($\left(p_1 + p_2\right)^2 = s_{12} \to 0$). Near the mass shell, soft-gluon radiation from this internal line leads to soft singularities and it cannot be neglected.

![Soft-gluon insertion diagrams for the soft–collinear limit.](image)

Figure 10: Soft-gluon insertion diagrams for the soft–collinear limit.

The explicit expression of the gluon current $J_{c(12)}^\mu(q)$ can be derived by working in a physical gauge and using the soft-gluon insertion rules described in Sect. 3.1 (Fig. 10). We find

$$
J_{(12)}^\mu(q) = \sum_{i=3}^{n} T_i \frac{p_i^\mu}{p_i \cdot q} + \frac{(p_1 + p_2)^2}{(p_1 + p_2 + q)^2} \left[ T_1 \frac{p_1^\mu}{p_1 \cdot q} + T_2 \frac{p_2^\mu}{p_2 \cdot q} \right] + (T_1 + T_2) \frac{2(p_1^\mu + p_2^\mu)}{(p_1 + p_2 + q)^2}.
$$

(117)

We discuss the three contributions on the right-hand side in turn. The first contribution comes from the usual eikonal insertions on the external parton lines $i = 3,\ldots,n$ (the diagrams (a) in Fig. 10).
The second contribution comes from the soft-gluon emission from the external partons \( p_1 \) and \( p_2 \) (diagrams (b) and (c) in Fig. [10]). The factor in the square bracket is the usual contribution from the eikonal vertices and propagators of the lines \( p_1 + q \) and \( p_2 + q \). The factor in front of the square bracket has the following origin. In Eq. (116) we have already factorized the tree amplitude \( \mathcal{M}_{a_1,a_2,\ldots,a_n}(p_1,p_2,\ldots,p_n) \), which contains the propagator factor \( 1/(p_1 + p_2)^2 \). In diagrams (b) and (c) of Fig. [10] this propagator is instead absent, and it is replaced by the propagator \( 1/(p_1 + p_2 + q)^2 \) of the internal line with momentum \( p_1 + p_2 + q \). Thus, the rescaling propagator factor \( (p_1 + p_2)^2/(p_1 + p_2 + q)^2 \) has to be applied to the contribution to the current.

The third contribution is the eikonal factor due to the soft emission from the nearly on-shell internal line \( p_1 + p_2 + q \) (diagram (d) in Fig. [10]).

Note that we have neglected diagrams in which \( p_1 \) and \( p_2 \) are not produced by a single line with momentum \( p_1 + p_2 \). These diagrams are not collinearly singular (see the discussion in Sect. 2.3) in the physical gauge we are working on.

Note also that, as the eikonal current in Eq. (84), the soft current in Eq. (117) satisfies the property \( q_\mu J_{(12)}^\mu(q) = \sum_{i=1}^n T_i \). The ensuing current conservation, which follows from Eq. (81), guarantees the gauge invariance of the factorization formula (116).

Using Eq. (116) we could now perform the collinear limit of the tree-level matrix element \( \mathcal{M}_{a_1,a_2,\ldots,a_n}(p_1,p_2,\ldots,p_n) \) on the right-hand side. However, since we are eventually interested in the soft–collinear limit of the square of the matrix element \( \mathcal{M}_{g,a_1,\ldots,a_n}(q,p_1,\ldots,p_n) \), this procedure is not convenient for two reasons. First, we have to introduce collinear splitting functions for the various colour subamplitudes that contribute to the colour vector \( |\mathcal{M}_{a_1,a_2,\ldots,a_n}(p_1,p_2,\ldots,p_n)|^2 \). These splitting functions differ from the Altarelli–Parisi splitting functions of Sect. 2.1 (roughly speaking, the former are the square root of the latter) and, although they are well known [4], we shall show that they are not really necessary for the final result. Secondly, the colour-charge transformation produced by the soft current in Eq. (116) implies a non-trivial relation between the colour-subamplitude decomposition of the matrix element on the left-hand side and the corresponding decomposition for the matrix element on the right-hand side. This non-trivial relation complicates the colour structure and leads to mixed soft–collinear splitting functions [21], whose introduction can instead be avoided or, at least, simplified.

In other words, if we square the right-hand side of Eq. (116), the soft current \( J_{(12)} \) produces non-trivial colour correlations of the type \( T_1 \cdot T_i \) or \( T_2 \cdot T_i \) (with \( i = 3, \ldots, n \)) between \( \mathcal{M}(p_1,p_2,\ldots,p_n) \) and \( \mathcal{M}^l(p_1,p_2,\ldots,p_n) \). Thus, we cannot perform the collinear limit \( k_\perp \to 0 \) by simply using the known \( \mathcal{O}(\alpha_S) \) results of Sect. 2.1 for the colour-summed squared amplitude \( |\mathcal{M}(p_1,p_2,\ldots,p_n)|^2 \).

The whole procedure can be simplified by exploiting the QCD coherence properties of soft-gluon emission. We rewrite Eq. (117) by splitting the soft current in two terms as follows:

\[
J_{(12)}^\mu(q) = \sum_{i=3}^n T_i \frac{p_i^\mu}{p_i \cdot q} + (T_1 + T_2) \frac{p_1^\mu + p_2^\mu}{(p_1 + p_2) \cdot q} + \delta J_{(12)}^\mu(q),
\]

(118)
where

\[ \delta J^{\mu}_{(12)}(q) = \frac{(p_1 + p_2)^2}{(p_1 + p_2 + q)^2} \left[ T_1 \frac{p_1^\mu}{p_1 \cdot q} + T_2 \frac{p_2^\mu}{p_2 \cdot q} - (T_1 + T_2) \frac{p_1^\mu + p_2^\mu}{(p_1 + p_2) \cdot q} \right]. \tag{119} \]

The two terms, \( \delta J_{(12)} \) and the other contribution on the right-hand side of Eq. (118), are separately conserved and, thus, the decomposition in Eq. (118) does not spoil the gauge invariance.

Then we note that each of the two terms in Eq. (118) has the correct scaling behaviour of \( \mathcal{O}(1/q) \) when \( q \to 0 \). Their collinear behaviour is nonetheless quite different. Performing the limit \( k_\perp \to 0 \) at fixed \( k_\perp^2/q \) in Eq. (119), the propagator factor \( (p_1 + p_2)^2/(p_1 + p_2 + q)^2 \) is of \( \mathcal{O}(1) \) but the term in the square bracket is of \( \mathcal{O}(k_\perp/q) \). Thus, the contribution of \( \delta J_{(12)} \) to the soft current \( J_{(12)} \) is suppressed by a relative factor of \( \mathcal{O}(k_\perp) \) in the collinear region, and it can be neglected in the soft–collinear limit.

We conclude that in the factorization formula (116) we can consistently use the following approximation for the soft current in Eq. (117):

\[ J^{\mu}_{(12)}(q) \simeq \sum_{i=3}^n T_i \frac{p_i^\mu}{p_i \cdot q} + T_{(12)} \frac{p_1^\mu + p_2^\mu}{(p_1 + p_2) \cdot q}, \tag{120} \]

where \( T_{(12)} = T_1 + T_2 \). The subdominant effect of \( \delta J_{(12)} \) is due to the cancellation between the different contributions in the square bracket on the right-hand side of Eq. (119). The cancellation is a typical consequence of colour coherence. When the parton momenta \( p_1 \) and \( p_2 \) become collinear, they radiate soft gluon in a coherent way, i.e. as a single parton with momentum \( p_1 + p_2 \) and colour charge \( T_{(12)} = T_1 + T_2 \) (see the last term in Eq. (120)).

The expression in Eq. (120) is certainly simpler than that in Eq. (117). More importantly, it depends on the colour charge \( T_{(12)} \) rather than separately on the colour charges \( T_1 \) and \( T_2 \). This implies that, when we square the amplitude in Eq. (116), the partons \( p_1 \) and \( p_2 \) are no longer colour-correlated, and the collinear limit \( k_\perp \to 0 \) can be performed by using the collinear factorization formula (1). We obtain the final soft–collinear factorization formula:

\[ |M_{g,a_1,a_2,\ldots,a_n}(q,p_1,p_2,\ldots,p_n)|^2 \simeq -\frac{2}{s_{12}} \left( 4\pi \mu^2 \alpha_s \right)^2 \left\langle \frac{2}{s_{12}} \hat{P}_{a_1a_2} J^{\mu}_{(12)}(q) J^{\mu}_{(12)}(q) \right| M_{a,\ldots,a_n}(p,\ldots,p_n) \right\rangle, \tag{121} \]

where the matrix elements on the right-hand side are obtained by removing the soft gluon \( q \) and by replacing the partons \( a_1 \) and \( a_2 \) by the single parton \( a \) that leads to the collinear splitting process \( a \to a_1 + a_2 \). Since these matrix elements are vectors in the colour+helicity space, both spin and colour correlations are present in Eq. (121).

The spin correlations are exactly the same as in Eq. (11). The spin indices \( s, s' \) of the parent parton \( a \) are correlated by the Altarelli–Parisi splitting functions \( \hat{P}_{a_1a_2} = \hat{P}_{a_1a_2}^{ss'}(z, k_\perp; \epsilon) \) in Eqs. (1)–(12).

The colour correlations affect all the partons and are analogous to those in Eq. (87). They are produced by the square of the soft current:
\[ J^\mu_{(12)}(q) J^\nu_{(12)}(q) = \sum_{i,j=3}^n T_i \cdot T_j \frac{p_i \cdot p_j}{(p_1 \cdot q)(p_j \cdot q)} + 2 \sum_{i=3}^n T_i \cdot T_{(12)} \frac{p_i \cdot (p_1 + p_2)}{(p_i \cdot q)(p_1 + p_2) \cdot q} + T_{(12)}^2 \frac{(p_1 + p_2)^2}{((p_1 + p_2) \cdot q)^2} \]

\[ \simeq \sum_{i,j=3}^n T_i \cdot T_j S_{ij}(q) + 2 \sum_{i=3}^n T_i \cdot T_{(12)} S_{i(12)}(q) \]  

In Eq. (123) we have neglected the last term on the right-hand side of Eq. (122), because it is not collinearily singular. We have also introduced the eikonal functions \( S_{ij}(q) \) of Eq. (88) and the analogous eikonal function \( S_{i(12)}(q) \),

\[ S_{i(12)}(q) = \frac{2(s_{i1} + s_{i2})}{s_{iq}(s_{1q} + s_{2q})}. \]  

From Eqs. (121) and (123) we can see that the soft–collinear limit at \( \mathcal{O}(\alpha_s^2) \) is simply and fully described in terms of the same factors, namely, soft eikonal functions and Altarelli–Parisi splitting functions, which control the soft and collinear limits at \( \mathcal{O}(\alpha_s) \), respectively.

The soft–collinear limit at \( \mathcal{O}(\alpha_s^2) \) was first studied by Campbell and Glover [21]. They neglected spin correlations and considered the singular behaviour of the colour subamplitudes. This behaviour, which was extracted by directly performing the singular limit of known squared matrix elements, was given in terms of two different factors. The first factor (see Sect. 4.4 in Ref. [21]) refers to subamplitudes in which the collinear partons are not colour-connected and it corresponds exactly to the \( S_{ij}(q) \)-term in Eq. (123). The second factor regards the subamplitudes in which the collinear partons \( p_1 \) and \( p_2 \) are colour-connected. This factor is given in Sect. 5.2 of Ref. [21] and it can be written as

\[ S_{i1q12} = \frac{2(s_{i1} + s_{i2})}{s_{iq} s_{1q}} \left[ z + \frac{s_{1q} + z s_{12}}{s_{12q}} \right] \]

\[ = \frac{2(s_{i1} + s_{i2})}{s_{iq}} \left( \frac{2}{s_{1q} + s_{2q}} + \left( 1 + \frac{s_{12}}{s_{12q}} \right) \left[ \frac{z}{s_{1q}} - \frac{1}{s_{1q} + s_{2q}} \right] \right) \]

\[ \simeq \frac{4(s_{i1} + s_{i2})}{s_{iq}(s_{1q} + s_{2q})}. \]  

Since \( z \) is the longitudinal momentum fraction carried by \( p_1 \) in the collinear region, the term in the square bracket of Eq. (126) vanishes in the collinear limit and Eq. (127) follows. This simplification, which is due to colour coherence, was not performed in Ref. [21]. Taking it into account, we have \( S_{i1q12} \simeq 2S_{i(12)}(q) \), which, when inserted in Eq. (123), shows the equivalence of our results with those of Ref. [21].

Our derivation of the soft–collinear factorization formula (121) can straightforwardly be extended to higher orders. We can consider the limit where a single gluon with momentum \( q \) becomes soft and, at the same time, \( m \) partons, say \( p_1, \ldots, p_m \), become simultaneously collinear (see Sect. 2.3). In this limit the factorization formula is

\[ |\mathcal{M}_{g,a_1,a_2,\ldots,a_n}(q,p_1,\ldots,p_m,\ldots,p_n)|^2 \simeq -4\pi \mu^2 \alpha_s \left( \frac{8\pi \mu^2 \alpha_s}{s_{1\ldots m}} \right)^{m-1} \]

\[ \cdot \langle \mathcal{M}_{a_1,a_2,\ldots,a_n}(xp,\ldots,p_n) | \hat{P}_{a_1,a_2} \left[ J^\mu_{(1\ldots m)}(q) J^\nu_{(1\ldots m)}(q) \right] | \mathcal{M}_{a_1,a_2,\ldots,a_n}(xp,\ldots,p_n) \rangle \]  

(128)
where \( \hat{P}_{a_1\ldots a_m} \equiv \hat{P}_{a_1\ldots a_m}^{ss'} \) is the spin-dependent splitting function in Eq. (23), and the soft current \( J_{(1\ldots m)}(q) \) is:

\[
J^\mu_{(1\ldots m)}(q) \simeq \sum_{i=m+1}^n T_i \frac{p_i^\mu}{q} + T_{(1\ldots m)} \frac{p_1^\mu + \ldots + p_m^\mu}{(p_1 + \ldots + p_m) \cdot q},
\]

with \( T_{(1\ldots m)} = T_1 + \ldots + T_m \). Squaring the soft current as in Eqs. (122) and (123), we obtain

\[
J_1^{(1\ldots m)}(q) J_1^{(1\ldots m)}(q) \simeq \sum_{i,j=m+1}^n T_i \cdot T_j S_{ij}(q) + 2 \sum_{i=m+1}^n T_i \cdot T_{(1\ldots m)} S_{i(1\ldots m)}(q),
\]

where \( S_{i(1\ldots m)}(q) = 2(s_{i1} + \ldots + s_{im})/[s_{iq}(s_{1q} + \ldots + s_{mq})] \).

The proof of these results is very simple. The soft current \( J_{(1\ldots m)}(q) \) is derived by using the soft-gluon insertion rules as in Eq. (117). Then, the coherence argument used in Eqs. (118) and (119) can iteratively be applied to any vertex in the \( m \)-parton dispersive amplitude \( V_{a_1\ldots a_m} \) of Eq. (27). This leads to the expression in Eq. (129).

### 3.5 Multiple soft and soft–collinear limits

In Sects. (3.1) and (3.3) we have discussed in detail single and double soft-gluon emission. The generalization to multiple soft-gluon radiation is straightforward. If we consider the matrix element \( \mathcal{M}_{g,a_1\ldots a_n}(q_1, \ldots, q_k, p_1, \ldots, p_n) \) when the \( k \) gluons with momenta \( q_1, \ldots, q_k \) become soft simultaneously, we can still write a factorization formula similar to Eq. (101) by performing the simple replacement

\[
g_2^2 \mu^2 e J_{\mu_1\mu_2}^{a_1a_2}(q_1, q_2) \rightarrow (gs\mu^e)^k J_{\mu_1\ldots \mu_k}^{a_1\ldots a_k}(q_1, \ldots, q_k),
\]

where \( a_1, \ldots, a_k \) and \( \mu_1, \ldots, \mu_k \) denote the colour and Lorentz indices of the soft gluons.

As the two-gluon current \( J_{\mu_1\mu_2}^{a_1a_2}(q_1, q_2) \) in Eq. (101), the multigluon current on the right-hand side of Eq. (131) is obtained by working in a physical gauge and using the soft-gluon insertion rules described in Sect. 3.1. Of course, the explicit expression of \( J(q_1, \ldots, q_k) \) turns out to be quite involved, because all the possible interactions between the soft gluons have to be included without using any soft approximation.

It is also clear that the soft-gluon insertion rules can be used to derive a factorization formula analogous to Eq. (116) for the multiple soft–collinear limit in which \( k \) gluons are soft and \( m \) partons become collinear simultaneously. More importantly, it is worth while pointing out that the coherence argument leading to Eqs. (121) and (128) still applies. Thus, the factorization formula can be written as

\[
|\mathcal{M}_{g,a_1\ldots a_n}(q_1, \ldots, q_k, p_1, \ldots, p_m, \ldots, p_n)|^2 \simeq \left(-4\pi \mu^2 e\alpha_S\right)^k \left(\frac{8\pi \mu^2 e\alpha_S}{s_{1\ldots m}}\right)^{m-1} \quad (132)
\]

\[
\langle \mathcal{M}_{a_1\ldots a_n}(xp, \ldots, p_n) | \hat{P}_{a_1\ldots a_m} \left[ J_{(1\ldots m)}^+(q_1, \ldots, q_k) J_{(1\ldots m)}(q_1, \ldots, q_k) \right] | \mathcal{M}_{a_1\ldots a_n}(xp, \ldots, p_n) \rangle.
\]

Equation (132) does not involve any additional factor with respect to those that are necessary to deal with the multiple collinear and multiple soft limits separately. The spin-dependent splitting function \( \hat{P}_{a_1\ldots a_m} = \hat{P}_{a_1\ldots a_m}^{ss'} \) is exactly the same as that in Eq. (23).
The current $J_{(1...m)}(q_1, \ldots, q_k)$ is completely analogous to the soft current $J(q_1, \ldots, q_k)$ in Eq. (131). As the latter, the former is constructed by inserting the soft gluons only on the $(1 + n - m)$ external parton lines with momenta $p_1 + \ldots + p_m, p_{m+1}, \ldots, p_n$, and each insertion on the collinear parton $p_1 + \ldots + p_m$ is taken into account by the simple eikonal factor $(T_1 + \ldots + T_m)(p_1 + \ldots + p_m)^\mu/(p_1 + \ldots + p_m) \cdot q$, despite the fact that $(p_1 + \ldots + p_m)^2 \neq 0$ (see e.g. Eqs. (129)).

In the most general case, the infrared singularities of the tree-level amplitudes are produced by the multiple collinear decay of hard partons and by the associated radiation of soft gluons and $q\bar{q}$-pairs. The corresponding factorization formula can be constructed in a straightforward manner by using the rules derived and illustrated throughout the paper. Using a shorthand symbolic notation, we have (Fig. 11)

$$|\mathcal{M}|^2 \simeq \langle \mathcal{M}_{\text{hard}} | \left( \prod_i \hat{P}_i \right) S | \mathcal{M}_{\text{hard}} \rangle . \quad (133)$$

Here $\mathcal{M}_{\text{hard}}$ denotes the factorized amplitude that depends only on the momenta of the hard partons. The factor $\prod_i \hat{P}_i = \prod_i \hat{P}_{s_{i}i}$ is the product of the spin-dependent splitting functions for the collinear decay of $i = 1, \ldots, l$ hard partons. The factor $S$ is a colour matrix that takes into account the radiation of soft partons. It has to be computed exactly at the tree level, but its external gluon lines are coupled to the hard partons by using the eikonal approximation as in the calculation of the current $J_{(1...m)}(q_1, \ldots, q_k)$ in Eq. (132). Note that spin and colour correlations are factorized independently. This decoupling follows from colour coherence.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig11.png}
\caption{General structure of infrared factorization at any perturbative order.}
\end{figure}
4 Summary

In this paper we have studied the infrared structure of tree-level QCD amplitudes in all the possible soft and collinear limits.

We have first considered the collinear behaviour. We have shown that, in the limit in which \( m \) partons become parallel simultaneously, the singularities are given by the universal factorization formula \( \mathcal{O}(\alpha_S^2) \) and are controlled by process-independent splitting functions that generalize the customary Altarelli–Parisi splitting functions. These splitting functions fully take into account the azimuthal correlations produced in the collinear decay. We have presented a recipe to compute the splitting functions at any perturbative order and we have performed their explicit calculation at \( \mathcal{O}(\alpha_S^2) \).

Then we have studied the soft behaviour and shown how to construct soft factorization formulae at any order in \( \alpha_S \). We have considered the limit in which a \( q\bar{q} \) pair becomes soft and we have computed the corresponding singularity at \( \mathcal{O}(\alpha_S^2) \) in terms of a simple universal insertion factor. We have then recalled the known results about the limit in which two gluons become soft. This limit is controlled by an \( \mathcal{O}(\alpha_S^2) \) soft current that is tensor in colour space and generalizes the eikonal current at \( \mathcal{O}(\alpha_S) \). We have obtained a compact expression for the square of the two-gluon current that, in particular, shows the absence of colour correlations in the case of four- and five-parton amplitudes.

Finally, we have studied the mixed soft–collinear limit and pointed out that its description does not require the introduction of new infrared factors. Exploiting the coherence property of soft gluon radiation, we have been able to show that the singularities are given by a factorization formula written only in terms of the soft currents and of the splitting functions that control the soft and collinear limits, respectively.

These results are one of the necessary ingredients to extend QCD predictions at higher perturbative orders. In particular, our calculation of the \( \mathcal{O}(\alpha_S^2) \) singular factors is relevant to setting up general methods to compute QCD jet cross sections at NNLO.

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Appendix: Soft limits of four- and five-parton amplitudes

In general, soft factorization formulae involve colour correlations. As shown in Eqs. (95), (112) and (123), at \( \mathcal{O}(\alpha_S^2) \) the correlations are completely given in terms of products of colour-charge factors \( T_i \cdot T_j \).

In this appendix we collect the factorization formulae for the \( \mathcal{O}(\alpha_S^2) \) soft (and soft–collinear) limits of the square of the matrix elements with four and five partons plus an
it is possible (see e.g. the Appendix A in Ref. [5] ) to express the products of an arbitrary number of colourless particles. In these particular cases, using colour conservation, it is possible to express the products $T_i \cdot T_j$ in terms of the Casimir invariants $C_i$ ($C_i = C_R$ if $i = q, \bar{q}$ and $C_i = C_A$ if $i = g$) of the hard partons. Thus the colour algebra completely factorizes and colour correlations cancel.

Note that two of the hard partons in the four- and five-parton amplitudes necessarily form a particle–antiparticle pair $a, \bar{a}$. This further simplifies the combinations of Casimir invariants that appear in the factorization formulae.

**Emission of a soft $q\bar{q}$-pair**

We consider the four-parton amplitude $\mathcal{M}_{q,\bar{q},a,\bar{a}}(q_1, q_2, p_1, p_2)$ in the limit in which $q_1, q_2 \to 0$. From Eq. (95), we get

$$|\mathcal{M}_{q,\bar{q},a,\bar{a}}(q_1, q_2, p_1, p_2)|^2 \simeq (4\pi\mu^2a_S)^2 T_R C_a \left( I_{11} + I_{22} - 2I_{12} \right) |\mathcal{M}_{a,\bar{a}}(p_1, p_2)|^2. \quad (A.1)$$

In the case of five partons we get

$$|\mathcal{M}_{q,\bar{q},a,\bar{a},a_3}(q_1, q_2, p_1, p_2, p_3)|^2 \simeq (4\pi\mu^2a_S)^2 T_R \left[ C_a \left( I_{11} + I_{22} - 2I_{12} \right) + C_{a_3} \left( I_{33} + I_{12} - I_{13} - I_{23} \right) \right] |\mathcal{M}_{a,\bar{a},a_3}(p_1, p_2, p_3)|^2. \quad (A.2)$$

The soft function $I_{ij} = I_{ij}(q_1, q_2)$ is given in Eq. (96).

**Emission of two soft gluons**

We consider the amplitude $\mathcal{M}_{g,g,a,\bar{a}}(q_1, q_2, p_1, p_2)$ in the limit in which the two gluons become soft. Using Eq. (112), we get

$$|\mathcal{M}_{g,g,a,\bar{a}}(q_1, q_2, p_1, p_2)|^2 \simeq (4\pi\mu^2a_S)^2 C_a \left[ 4 C_a S_{12}(q_1) S_{12}(q_2) + C_A \left( 2 S_{12} - S_{11} - S_{22} \right) \right] \cdot \left| \mathcal{M}_{a,\bar{a}}(p_1, p_2) \right|^2. \quad (A.3)$$

In the case of five partons we get

$$|\mathcal{M}_{g,g,a,\bar{a},a_3}(q_1, q_2, p_1, p_2, p_3)|^2 \simeq (4\pi\mu^2a_S)^2 |\mathcal{M}_{a,\bar{a},a_3}(p_1, p_2, p_3)|^2 \cdot \left\{ 
\begin{array}{l}
(2C_a - C_{a_3}) S_{12}(q_1) + C_{a_3} (S_{13}(q_1) + S_{23}(q_1)) \\
(2C_a - C_{a_3}) S_{12}(q_2) + C_{a_3} (S_{13}(q_2) + S_{23}(q_2)) \\
C_a \left[ C_a (2S_{12} - S_{11} - S_{22}) + C_{a_3} (S_{13} + S_{23} - S_{33} - S_{12}) \right] .
\end{array}
\right\} \quad (A.4)$$

The soft functions $S_{ij}(q)$ and $S_{ij} = S_{ij}(q_1, q_2)$ are given in Eqs. (88) and (109), respectively.
Soft–collinear limit

We consider the amplitude $M_{g,a_1,a_2,a_3}(q,p_1,p_2,p_3)$ in the limit in which $q \to 0$ and $s_{12} \to 0$. Using Eq. (121), we get

$$|M_{g,a_1,a_2,a_3}(q,p_1,p_2,p_3)|^2 \approx \frac{4}{s_{12}} \left(4 \pi \mu^2 a_S \right)^2 C_{a_3} S_{3(12)}(q) \hat{P}_{a_1a_2}^{s's'} T_{aa4}(xp,p_3).$$  \hspace{1cm} (A.5)

In the case of five partons we get

$$|M_{g,a_1,a_2,a_3,a_4}(q_1,p_1,p_2,p_3,p_4)|^2 \approx \frac{2}{s_{12}} \left(4 \pi \mu^2 a_S \right)^2 T_{aa3a4}(xp,p_3,p_4) \hat{P}_{a_1a_2}^{s's'}$$

$$\cdot \left[(C_{a_3} + C_{a_4} - C_a) S_{34}(q) + (C_a + C_{a_3} - C_{a_4}) S_{3(12)}(q) + (C_a + C_{a_4} - C_{a_3}) S_{4(12)}(q) \right].$$ \hspace{1cm} (A.6)

Here $a$ denotes the parton that decays collinearly, $a \to a_1a_2$, $T_{aa...(xp,...)}$ is the spin-polarization tensor in Eq. (2) and $\hat{P}_{a_1a_2}^{s's'}$ is the spin-dependent splitting function in Eq. (7). The soft functions $S_{ij}(q)$ and $S_{i(12)}(q)$ are given in Eqs. (88) and (124), respectively.

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