Transport studies in three-terminal microwave graphs with orthogonal, unitary, and symplectic symmetry

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The Landauer-Büttiker formalism establishes an equivalence between the electrical conduction through a device, e. g. a quantum dot, and the transmission. Guided by this analogy we perform transmission measurements through three-port microwave graphs with orthogonal, unitary, and symplectic symmetry thus mimicking three-terminal voltage drop devices. One of the ports is placed as input and a second one as output, while a third port is used as a probe. Analytical predictions show good agreement with the measurements in the presence of orthogonal and unitary symmetries, provided that the absorption and the influence of the coupling port are taken into account. The symplectic symmetry is realized in specifically designed graphs mimicking spin 1/2 systems. Again a good agreement between experiment and theory is found. For the symplectic case the results are marginally sensitive to absorption and coupling strength of the port, in contrast to the orthogonal and unitary case.

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Wave transport and wave scattering phenomena have been of great interest in the last decades, both from experimental and theoretical points of view (see for instance Ref. [1]). Apart from the intrinsic importance in the complex scattering in a particular medium, the interest also comes from the equivalence between physical systems belonging to completely different areas, in which the dimensions of the systems may differ by several orders of magnitude [2]. One of these equivalences occurs in mesoscopic quantum systems, where the electrical conduction reduces to a scattering problem through the Landauer-Büttiker formalism [3–5]. Following this line, classical analogies of quantum systems have been used as auxiliary tools to understand the properties of the conductance of electronic devices in two-terminal configurations [6–10]. A plethora of chaotic scattering experiments in presence of time reversal invariance (TRI) and no spin 1/2 have been performed [7, 8, 10–16], while very few experimental studies regarding absence of TRI are reported [7, 8, 17, 18]. Furthermore, due to its intrinsic complexity, there are no scattering experiments up to now for systems with TRI and spin 1/2, where the signatures of the symplectic ensemble are expected, though there is one study of the spectral statistics in Au nanoparticles obeying this symmetry [19]. Moreover, very recently the appearance of a microwave experiment showing the signatures of the symplectic symmetry [20, 21] for eigenvalue statistics has opened the possibility to study transport in the presence of this symmetry.

Multiterminal devices are good candidates to provide experimental realizations for the three symmetry classes: orthogonal, unitary and symplectic. Alternatively to the most used two-terminal configuration, three terminal systems provide information of nonlocal effects of transport observables. In the present paper, we make theoretical predictions for coherent transport in a three-terminal quantum device. In the spirit of the mentioned classical analogy, we propose experimental realizations with microwave graphs, which represent the first experiments of transport in three-terminal systems for the three symmetry classes and the first experiment with the symplectic symmetry.

The electrical current \( I_i \) on the terminal \( i \) of an electronic device, as given by Büttiker’s formula, can be written as [22]

\[
I_i = \sum_j G_{ij} (V_i - V_j), \quad \text{with} \quad G_{ij} = \frac{e^2}{h} T_{ij}, \quad (1)
\]

where \( V_i \) is the voltage at terminal \( i \), and \( G_{ij} \) and \( T_{ij} \) are the conductance and transmission coefficient, respectively, from terminal \( j \) to terminal \( i \). In a three-terminal

FIG. 1. Sketch of a three-terminal setting that allows the measurement of the voltage along a device. The device carries a current while the vertical wire measures the voltage drop. Thin lines represent perfect conductors connected to sources of voltages \( V_1 \), \( V_2 \), and \( V_3 \).
configuration, one of the ports, let’s say terminal 3, can be used as a probe by tuning its voltage to zero current. This voltage $V_3$, is a weighted average of the voltages in the other terminals, the weight being determined by the conductance coefficients from the other terminals to the probe [22]. It can be written as

$$V_3 = \frac{1}{2} (V_1 + V_2) + \frac{1}{2} (V_1 - V_2) f, \quad (2)$$

where

$$f = \frac{T_{31} - T_{32}}{T_{31} + T_{32}}, \quad (3)$$

see Fig. 1. This equation shows that $V_3$ varies around the average of the voltages producing the bias, $V_1$ and $V_2$. Hence, the quantity $f$ takes values in the interval $[-1, 1]$ and contains all the information about the system. For instance, a three-terminal setting was considered in Refs. [23, 24] to study the voltage drop along a disordered quantum wire.

Here, we perform measurements of the quantity $f$ through microwave graphs connected to three single channel ports: an input port, an output port, and a probe port. We focus here on the particular situation where the probe port is on one side of the microwave graph, see Fig. 1. We study graphs with chaotic dynamics characterized by the orthogonal, unitary, and symplectic symmetries; labeled by $\beta = 1$, 2, and 4 in Dyson’s scheme [25], respectively. The $\beta = 4$ case is realized in a network with specific properties that mimics a spin $1/2$ system [20, 21]. The fluctuations of $f$, that arise when the frequency is varied, are analyzed by means of random matrix theory (RMT) calculations. Analytical expressions for the distribution of $f$, that describe the experiments for the three symmetry classes, are verified by the measurements.

The experimental setup for $\beta = 2$ (with a small modification also for $\beta = 1$) is shown in Fig. 2. A chaotic microwave graph is formed by coaxial semirigid cables (Huber & Suhner EZ-141) with SMA connectors, coupled by T-junctions at the nodes. An additional T-junction at the exit port forms the three-terminal setting. For $\beta = 1$ all bonds were connected by T-junctions, for $\beta = 2$ one of the T-junctions was replaced by a circulator to break TRI. In both cases the found spectral level spacing distributions were in perfect agreement with the Wigner distributions for the Gaussian orthogonal ensemble (GOE), $\beta = 1$, and the Gaussian unitary ensemble (GUE), $\beta = 2$, respectively, see e.g. Chapter 4.4 of Ref. [26]. The measurements were restricted to the operating range of the circulators (Aerotek I70-1FFF) from 6 to 12 GHz. To realize graphs showing the signatures of the Gaussian symplectic ensemble (GSE), $\beta = 4$, two copies of the graph shown in Fig. 2 are needed, where the implemented circulators lead to an opposite sense of rotation. They are coupled by two bonds in an inversion symmetric geometry, with a phase shift of $\pi$ in one of the bonds but not the other one, see Fig. 3. The whole graph obeys an antunitary symmetry $T$, squaring to -1, thus mimicking a spin 1/2, see Ref. [20]. Transmission measurements were performed with an Agilent 8720ES vector network analyzer (VNA).

With respect to the quantity $f$, its fluctuations can be described by the scattering approach of RMT. Appealing to an ergodic hypothesis, fluctuations on the frequency are replaced by fluctuations on an ensemble of chaotic graphs, represented by an ensemble of scattering matrices. In the two-channel situation, the scattering matrix
of the graph has the structure

\[ S_g = \begin{pmatrix} r_g & t_g' \\ t_g & r_g' \end{pmatrix}, \quad (4) \]

where \( r_g \) (\( r_g' \)) and \( t_g \) (\( t_g' \)) are the reflection and transmission amplitudes, for incidence from the left (right). Depending on the symmetry class, \( S_g \) belongs to one of the Circular Ensembles: Orthogonal (COE) for \( \beta = 1 \), Unitary (CUE) for \( \beta = 2 \), and Symplectic (CSE) for \( \beta = 4 \). The \( S_g \) matrix can be written in the polar representation as \[ S_g = \begin{bmatrix} -\sqrt{1 - \tau e^{2i\phi'}} & a^{-1} \sqrt{\tau} e^{i(\phi + \phi')} \\
\tau a \sqrt{\tau} e^{i(\phi + \phi')} & \sqrt{1 - \tau e^{2i\phi}} \end{bmatrix}, \quad (5) \]

where \( 0 \leq \tau \leq 1, 0 \leq \phi, \phi' \leq \pi \), and \( a \) is a real, complex, or real quaternion number of modulus 1 for \( \beta = 1, 2 \) or 4, respectively. The probability density distribution of \( S_g \) is given by [27]

\[ dP_\beta(S_g) = \frac{\beta}{2} \tau^{-1+\beta/2} \frac{d\tau}{\pi} \frac{d\phi}{\pi} \frac{d\phi'}{\pi} da. \quad (6) \]

The scattering matrix associated to the three-terminal setup of Fig. 1, where the probe is at the right of the graph, is given by [28]

\[ S = S_{PP} + S_{PQ} S_0 \frac{1}{1 - S_{QQ} S_0} S_{QP}, \quad (7) \]

where \( S_0 \) is the scattering matrix for the junction (see Fig. 1), \( 1_3 \) stands for the unit matrix of dimension 3, and

\[ S_{PP} = \begin{pmatrix} r_g & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \end{pmatrix}, \quad S_{PQ} = \begin{pmatrix} r_g' & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \end{pmatrix}, \quad (8) \]

\[ S_{QP} = \begin{pmatrix} t_g & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \end{pmatrix}, \quad S_{QQ} = \begin{pmatrix} t_g' & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \end{pmatrix}. \quad (9) \]

Equation (7) is a general combination rule for scattering matrices which appears in several scattering problems. The first term, \( S_{PP} \), represents reflections on the terminals (only the Terminal 1 presents reflection for the present case). The second term comes from multiple scattering in the system. Reading from right to left, \( S_{QP} \) represents the transmissions to the inside region, passing through the graph and the junction, \( (1_3 - S_{QQ} S_0)^{-1} \) contains the multiple reflections between the junction and the graph, and \( S_{QP} \) gives the transmissions from the internal region to the terminals.

Because it is expected that the T-junction couples the terminals symmetrically, \( S_0 \) can be assumed to be symmetric. According to some measurements [29], it can be proposed as

\[ S_0 = \frac{1}{3} \begin{pmatrix} -1 & 2 & 2 \\
2 & -1 & 2 \\
2 & 2 & -1 \end{pmatrix}. \quad (10) \]

The general structure of \( S \) is of the form

\[ S = \begin{pmatrix} q_{11} & q_{12} & q_{13} \\
q_{21} & q_{22} & q_{23} \\
q_{31} & q_{32} & q_{33} \end{pmatrix}, \quad (11) \]

where \( q_{ij} = S_{ij} \) for \( \beta = 1 \) and 2, while for \( \beta = 4 \)

\[ q_{ij} = \begin{pmatrix} S_{ij} & S_{ij}' \\
S_{ij} & S_{ij}' \end{pmatrix}, \quad (12) \]

where the “bar” in the subscripts denotes the corresponding terminal in the second GUE subgraph needed for the construction of the GSE graph (see Fig. 3). Therefore, the transmission coefficient from terminal \( j \) to terminal \( i \) is given by \( T_{ij} = |S_{ij}|^2 \) for \( \beta = 1 \) and 2, and \( T_{ij} = \frac{1}{2} \text{tr}(q_{ij} q_{ij}') \) for \( \beta = 4 \).

Since \( q_{ij} \) is a quaternion real number, \( q_{ij} q_{ij}' \) is proportional to the \( 2 \times 2 \) unit matrix. However, in the experiment this cannot be achieved with arbitrary accuracy due to power losses. For the transmission measurements relevant to our study, they were realized within a 10% and 1% of error for \( q_{31} \) and \( q_{32} \), respectively.

By substituting the parametrization given in Eq. (5) into Eqs. (7) to (9), and extracting the transmission coefficients \( T_{31} \) and \( T_{32} \) from Eq. (11), Eq. (3) yields

\[ f = \frac{\tau - |1 + \sqrt{1 - \tau e^{2i\phi}}|^2}{\tau + |1 + \sqrt{1 - \tau e^{2i\phi}}|^2}, \quad (13) \]

where \( a \) and \( \phi' \) drop out.

Using the probability density distribution of Eq. (6) the distribution of \( f \) is obtained once the integration over all parameters is done; the result is

\[ p_\beta(f) = \frac{(\beta - 1)!!}{\beta |\Gamma(\beta/2)|^2} \frac{(1 - f)^{\beta/2}}{(1 + f)^{1-\beta/2}}. \quad (14) \]

This distribution dominates for negative \( f \) values in agreement with the physical intuition since the probe at the right of the graph is closer to port 2 (see Fig. 1), making the transmission \( T_{32} \) predominantly larger than the transmission \( T_{31} \). The width of the distribution is a signature of the nonlocal effects in the measurement of the probe port.

Equation (14) represents our main result which is valid in an ideal situation: It applies to quantum systems in the absence of any inelastic process and to classical wave systems in the absence of dissipation and imperfect coupling to the device. In Fig. 4 we show the transmissions \( T_{31} = |S_{31}|^2 \) and \( T_{32} = |S_{31}|^2 \) as a function of the frequency, obtained from the measurements of the elements of the scattering matrix \( S_{31} \) and \( S_{32} \) for \( \beta = 1, 2 \) and 4. We observe that they do not reach the value of 1 due to the losses of power. Their corresponding distribution are also shown.

The actual measurements for \( f \) (see Eq. (3)) are shown in Fig. 5 for experiments for the three symmetry classes:
\[ H \] being the Hamiltonian that describes the closed microwave graph with mean level spacing \( \Delta \) and it is taken from the Gaussian ensembles corresponding to the symmetry present in the graph. The imaginary part of \( \hat{H} \) mimics the absorption quantified by the parameter \( \gamma \). It can be extracted from the experimental data through the autocorrelation function, \( C^{(\beta)}_{11}(t) \), between the elements of the scattering matrix \( S_{ab} \). The corresponding expression for the GOE is given in Ref. 30, while for all \( \beta \) in Ref. 31. After some mathematics, they can be written for the element \( S_{11} \) as

\[
C^{(\beta)}_{11}(t) = \begin{cases} 
\frac{3}{(1+2T_1t)^2} - \frac{b_{1,1}(t)}{(1+T_1t)^2} \ e^{-\gamma t} & \text{for } \beta = 1, \\
\frac{2}{(1+T_1t)^2} - \frac{b_{2,2}(t)}{(2+T_1t)^2} \ e^{-\gamma t} & \text{for } \beta = 2, \\
\frac{6}{(1+T_1t)^6} - \frac{2^{12} b_{2,2}(t)}{(2+T_1t)^{10}} \ e^{-2\gamma t} & \text{for } \beta = 4,
\end{cases}
\]

(16)

where \( b_{\beta,2}(t) \) is the two-level form factor \([32]\) and \( T_1 \) is the coupling strength, which is also extracted from the experiment via \( T_1 = 1 - |\langle S_{11} \rangle|^2 \) with the average \( \langle S_{11} \rangle \) taken over the frequency.

In Fig. 6 we show the autocorrelation function \( C^{(\beta)}_{11}(t) \) of the experimental data. The best fit yields \( T_1 = 0.98 \) and \( \gamma = 1.9 \), for \( \beta = 1 \), \( T_1 = 0.96 \) and \( \gamma = 0.5 \), for \( \beta = 2 \), and \( T_1 = 0.97 \) and \( \gamma = 0.2 \), for \( \beta = 4 \), and they are plotted as dashed lines. As expected the coupling parameters are almost the same for the three symmetries but the absorption parameter is significantly different from one symmetry to another. In particular, we notice that the value of \( \gamma \) for \( \beta = 2 \) is almost twice the value for \( \beta = 4 \). This may be due to the interplay between reflection and absorption \([33]\), i.e., the higher the reflection the smaller the absorption, and also due to the fact that \( \gamma \) is given in units of \( \Delta \) which is not the same for all graphs. This is the situation of the \( \beta = 4 \) case which presents twice the reflection than that of the \( \beta = 2 \) case (two subgraphs). Also, the circulators introduce more reflections for \( \beta = 2 \) and 4 in comparison with the \( \beta = 1 \) case with no circulators. The parameters \( T_1 \) and \( \gamma \) are used in Eq. (15), from which we obtain \( T_{31} \) and \( T_{32} \), and finally compute \( f \). The results are shown in Fig. 5 (lower panels) as dashed lines. A good agreement with the experimental distribution is observed. For the symplectic case the agreement between experiment and theory is good even without the correction due to absorption and imperfect coupling; since \( \gamma \) is relatively small, \( p_4(f) \) depends only weakly on the port couplings which are almost perfect.

To conclude, we used three-terminal chaotic microwave graphs to measure the different transmissions to extract the quantity \( f \), that accounts for the voltage drop in an equivalent quantum device and exhibits its nonlocal effects. We successfully described the experimentally obtained distribution \( p(f) \) providing analytical expressions for the ideal case, for the three symmetry classes. Devi-
FIG. 5. $f$ as a function of the frequency is shown in the upper panels, and its corresponding distribution in the lower ones, for $\beta = 1$ (left), 2 (middle), and 4 (right). In the lower panels the continuous lines represent the analytical result for the ideal case, Eq. (14), while the dashed lines correspond to RMT simulations with power losses and imperfect coupling of the T-junctions to the graph, where all parameters were fixed beforehand using the autocorrelation functions (see Fig. 6). In the insets the difference between the numerical and the experimental distribution, $\delta p_\beta(f) = p_\beta(f)_{\text{num}} - p_\beta(f)_{\text{exp}}$, are presented for comparison purposes. For the statistical analysis we used an ensemble of $5 \times 10^4$ realizations.

Our experimental realizations are the first experiments of transport in three-terminal systems for the three symmetry classes and the first experiment with symplectic symmetry. We expect that our results motivate further studies for a successful explanation of robustness of symplectic symmetry for imperfect couplings and higher dissipation.

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FIG. 6. Fitting of the autocorrelation function, Eq. (16), to the experimental data. The parameters are $T_1 = 0.98$ and $\gamma = 1.9$ for $\beta = 1$, $T_1 = 0.96$ and $\gamma = 0.5$ for $\beta = 2$, and $T_1 = 0.97$ and $\gamma = 0.2$ for $\beta = 4$.

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