Concentration inequalities for the hydrodynamic limit of a two-species stochastic particle system

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Abstract

We study a stochastic particle system which is motivated from grain boundary coarsening in two-dimensional networks. Each particle lives on the positive real line and is labeled as belonging to either Species 1 or Species 2. Species 1 particles drift at unit speed toward the origin, while Species 2 particles do not move. When a particle in Species 1 hits the origin, it is removed, and a randomly selected particle mutates from Species 2 to Species 1. The process described is an example of a high-dimensional piecewise deterministic Markov process (PDMP), in which deterministic flow is punctuated with stochastic jumps. Our main result is a proof of exponential concentration inequalities of the Kolmogorov-Smirnoff distance between empirical measures of the particle system and solutions of limiting nonlinear kinetic equations. Our method of proof involves a time and space discretization of the kinetic equations, which we compare with the particle system to derive recurrence inequalities for comparing total numbers in small intervals. To show these recurrences occur with high probability, we appeal to a state dependent Hoeffding type inequality at each time increment.

Keywords: piecewise deterministic Markov process, concentration inequality, kinetic theory, grain boundary coarsening, functional law of large numbers

1 Introduction

An important topic in material science is the coarsening of network microstructures such as polycrystalline metals and soap froths. Through heating of metals or gas diffusion of foams, coarsening is induced from the migration of network boundaries to minimize interfacial surface energy. While tracking individual boundaries remains a multifaceted and active field of research in numerical [3] and geometric [8] analysis, in the 1950's von Neumann [15] and Mullins [13] proved a simple relation between the topology and geometry of a single cell in two-dimensional networks with isotropic surface tension evolving through curve-shortening flow. Specifically, a cell with area $A$ and $n$ sides has a constant growth rate

$$\frac{dA}{dt} = c(n - 6),$$

where $c$ is a material constant. When a cell with fewer than six sides shrinks to a point, neighboring cells may change their number of sides to maintain the topological requirement that exactly three edges meet at each junction. Therefore, a grain will typically change its number of sides, and therefore its rate of growth, several times during the coarsening process.

Several physicists [6, 5, 12, 1] used (1) as a starting point in writing kinetic equations for densities $u_n(a, t)$ of cells having $n$ sides ($n$-gons) and area $a$ at time $t$. These take the form of constant

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convection transport equations with intrinsic flux terms, given by

\[
\partial_t u_n + c(n-6)\partial_x u_n = \sum_{l=2}^{5} (l-6)u_l(0,t) \left( \sum_{m=2}^{M} A_{lm}(t)u_m(a,t) \right), \quad n \geq 2.
\]  

(2)

Models diverge in their choice of the matrix \(A_{lm}\), which prescribes mean field rules for how networks change topology when a grain vanishes. In [10], Menon, Pego, and the author presented a stochastic particle system, the \(M\)-species model, as an intermediate between kinetic equations and direct simulations of grain boundary coarsening. The focus of [10] was with well-posedness of the limiting kinetic equations and simulations. It remained, however, to provide estimates for convergence rates of the particle systems to their hydrodynamic limits. A study was conducted in [9] on a simplified model of one species, in which total loss of particles is shown to be equivalent to a diminishing urn process similar to Pittel’s model of cannibalistic behavior [14].

In this paper, we build the groundwork for establishing concentration inequalities for the \(M\)-species model by restricting our attention to a model of two species. Specifically, each particle lives in \(\mathbb{R}_+ = [0, \infty)\) and is tagged as belonging to either Species 1 or Species 2. Particles in Species 2 do not drift, while particles in Species 1 drift at unit speed toward the origin. When a particle reaches the origin, it is removed, and a particle from Species 2 immediately mutates into Species 1. For a visual representation of the process in which mutations are represented as vertical jumps, see Fig. 2. The process just described can be interpreted as a minimal model of network coarsening, with the behavior of particles in Species 1 analogous to the constant area decrease of cells with fewer than six sides. The removal of Species 1 and mutation of Species 2 are similar to the vanishing of faces and subsequent reassignment of neighboring cell topologies.

The hydrodynamic limits for densities \(f_j(x,t)\) of particles in Species \(j\) at position \(x > 0\) and time \(t \geq 0\) are transport equations with nonlinear intrinsic source terms, with

\[
\partial_t f_1(x,t) - \partial_x f_1(x,t) = \frac{f_1(0,t)f_2(x,t)}{N_2(t)}, \quad (3)
\]

\[
\partial_t f_2(x,t) = -\frac{f_1(0,t)f_2(x,t)}{N_2(t)}, \quad (4)
\]

\[
f_1(x,0) = f_1(x), \quad f_2(x,0) = f_2(x), \quad (5)
\]

where \(N_j(t) = \int_0^\infty f_j(x,t)dx\) is the total number of Species \(j\). To allow for nondifferentiable initial conditions, we will exclusively work with the integral form of (3–5), written with Duhamel’s formula as

\[
f_1(x,t) = \hat{f}_1(x+t) + \int_0^t f_2(x+t-s,s)\frac{f_1(0,s)}{N_2(s)}ds, \quad (6)
\]

\[
f_2(x,t) = \hat{f}_2(x) - \int_0^t f_2(x,s)\frac{f_1(0,s)}{N_2(s)}ds. \quad (7)
\]

The main result for this paper is the convergence of empirical measures of the particle system to limiting kinetic equations (6–7). In Section 3, we give a rigorous description of the particle system as a piecewise deterministic Markov process (PDMP), lay out a deterministic discretization of the kinetic equations, and present the main results. Section 4 gives a proof for Theorem 2 which shows that the discretization converges to the kinetic equations at rate \(O(\delta + \omega(\delta,0))\), where \(\delta\) is both the spatial and temporal step size of the scheme, and \(\omega(\delta,0)\) is the modulus of continuity in the initial conditions. This is achieved through writing recurrence relations which compare total numbers restricted to intervals of size \(\delta\). Section 5 is a proof of Theorem 3 on an exponential concentration inequality (with respect to the
initial total number of particles) between the discretization and particle system. This involves similar recurrence inequalities seen in Section 3, but now with an added task of showing that the inequalities occur under high probability. We will need to apply a generalization of Hoeffding’s inequality [7], a fundamental concentration inequality for sampling without replacement, for establishing estimates on the total number of mutations occurring in intervals.

Theorems 2 and 3 can be combined to produce our main result, Theorem 4, which gives a concentration inequality between empirical measures and solutions of the kinetic equation. For sufficiently small $\varepsilon > 0$ and $n > n(\varepsilon)$, the inequality takes the form

$$
\mathbb{P}_n \left( \sup_{t \leq T'} d((\mu_1(t), \mu_2(t)), (\mu_1^n(t), \mu_2^n(t))) \geq \varepsilon \right) \leq \frac{C}{\varepsilon^2} \exp(-\tilde{C}\varepsilon^5 n). \tag{8}
$$

Here, for $j = 1, 2$ and time $0 \leq t \leq T'$, $\mu_j^n(t)$ is the $n$-particle empirical measure for positions of Species $j$, and $\mu_j(t, dx) = f_j(t, x) \, dx$ where $f_j(x, t)$ is the solution of (6)-(7). The metric $d$ is a sum of Kolmogorov-Smirnov metrics between measures of each species. The constants $C$, $\tilde{C}$, and $T'$ all depend on the initial conditions $\bar{f}$.

We conclude with Section 5, in which we derive an explicit solution of the kinetic equations and prove the well-posedness stated in Theorem 1, relying on several well-known facts from renewal theory. We stress that explicit solutions are not used in either the proofs of Theorems 2 and 3 as we hope to extend the methods used here to the $M$-species model which have no known explicit solutions.

2 Particle model and statement of results

2.1 A two-species particle system and its kinetic limit

We now formally define the stochastic process $\{X^n(t)\}_{t \geq 0}$ for an initial system of $n$ particles. Each particle lives in one of two ordered copies of $\mathbb{R}_+ = (0, \infty)$, which we refer to as Species 1 and Species 2. Since particles may be removed during the process, the state space $E^n$ consists of states

$$
\mathbf{x} = \{(x_i, s_i) : i = 1, \ldots, |\mathbf{x}|, \ |\mathbf{x}| \leq n\}, \tag{9}
$$

with particle locations $x_i \in [0, \infty)$ and labels $s_i \in \{1, 2\}$ denoting each particle’s species. The state space can be expressed as a disjoint union of positive orthants

$$
E^n = \coprod_{l+m \leq n} E^n_{(l,m)}, \tag{10}
$$
with $E^n_{l,m} = \mathbb{R}^{l}_+ \times \mathbb{R}^{m}_+$ denoting positions for $l$ particles in Species 1 and $m$ particles in Species 2.

Fix an initial state $X^n(0) = \{(x^0_1, s^0_1), \ldots, (x^n_n, s^n_n)\} \in E^n$, and denote $\alpha$ as an index for a particle in Species 1 closest to the origin, meaning $x^0_\alpha \leq x^0_i$ for $i = 1, \ldots, n$ and $s^0_\alpha = 1$. Now let $\tau_1 = x^0_\alpha$ denote the time until a particle reaches the origin. Define $X^n(t) \in E^n$ for $t \in [0, \tau_1)$ deterministically by advecting particles in Species 1 toward the origin at unit speed while keeping particles in Species 2 fixed:

$$s_i(t) = s^0_i, \quad x_i(t) = \begin{cases} x^0_i - t, & s^0_i = 1, \\ x^0_i, & s^0_i = 2 \end{cases} \quad i = 1, \ldots, n. \quad (11)$$

Randomness is introduced with a mutation at time $t = \tau_1$. At this time, the smallest particle in Species 1 has reached the origin, and is removed from the system. Furthermore, a particle $(x_i(\tau_1^-), 2)$ selected with uniform probability from Species 2 mutates while keeping its position, meaning

$$(x_i(\tau_1), s_i(\tau_1)) = (x_i(\tau_1^-), 1). \quad (12)$$

Finally, particle indices $i > \alpha$ decrement by one so that the index set is $\{1, \ldots, n-1\}$. The (now stochastic) process then repeats deterministic drift until a particle from Species 1 reaches the origin at some time $\tau_2$, again triggering a random mutation, and the process continues until there are no particles left in Species 2. For instances in which multiple particles reach the origin simultaneously, particles in Species 2 are selected to mutate by sampling without replacement. The process $\{X(t)\}_{t \geq 0}$ is an example of the general $M$-species model, which was shown in \cite{10} to be a class of piecewise deterministic Markov processes (PDMPs). The stochastic process induces a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}_n, \{\mathcal{F}(t)\}_{t \geq 0})$, where $\mathcal{F}$ is the natural filtration. Davis \cite{2} established that PDMPs are in fact strong Markov, which we will use in Section 4 when considering events before and after certain mutation times.

At time $t \geq 0$, denote the number of particles in Species $j$ as $N^n_j(t)$, and the total number as $N^n(t) = N^n_1(t) + N^n_2(t)$. Empirical densities for species with $n$ initial particles are then defined as

$$\mu_j^n(t) = \frac{1}{n} \sum_{i=1}^{N^n(t)} \delta(x_i) \cdot 1(s_i = j), \quad j = 1, 2. \quad (13)$$

The differential form of the limiting equations of the infinite particle limit $n \to \infty$ are given by equations \cite{3}-\cite{5}, in which for each species $j$ the limiting measures $\lim_{n \to \infty} \mu^n_j \to \mu_j$ are deterministic with densities $f_j(x, t)$. We will require that that $N_1(0) + N_2(0) = 1$ and $N_2(0) > 0$. The left hand sides of \cite{3} and \cite{4} represent the constant drift of Species 1 toward the origin and zero drift in Species 2. The right hand sides give the intrinsic flux arising from mutations selected from a normalized density of $f_2$, occurring at a frequency of $f_1(0, t)$. To allow for nondifferentiable initial data and solutions, we will use the integral form \cite{6}-\cite{7} with initial data $(f_1, f_2) \in Z^2$, where $Z$ denotes the cone of positive, continuous, and locally bounded functions under the $L^1(\mathbb{R}_+)$ norm topology. Equations \cite{6}-\cite{7} reach a singularity when $N_2 = 0$, corresponding to when there are no more Species 2 particles to mutate. This occurs at time

$$T(\bar{f}) = \sup_{t > 0} \{N_2(t) > 0\}. \quad (14)$$

The derivation of an explicit solution of \cite{6}-\cite{7} first relies on explicitly solving for the removal rate, which we write as $a(t) = f_1(0, t)$, and subsequently the total loss

$$L(t) = \int_0^t a(s)ds, \quad (15)$$

which may be interpreted as an “internal clock” to the system, counting normalized total visits to the origin or, equivalently, total mutations.
Theorem 1. Let \( \vec{f} = (\vec{f}_1, \vec{f}_2) \in Z^2 \) with \( N_2(0) > 0 \),

(a) The removal rate \( a(t) = f(0, t) \) and \( N_2(t) \) may be written in terms of the initial conditions as

\[
a(t) = \sum_{j=0}^{\infty} \left( \frac{\vec{f}_2}{N_2(0)} \right)^j (t) \ast \vec{f}_1(t),
\]

(16)

\[
N_2(t) = N_2(0) - L(t).
\]

(17)

Here, an exponent of \( j \)-fold self convolution (with \( f^{(0)} = 1 \)). For \( 0 \leq t < T(\vec{f}) \), the solution \( (f_1(x, t), f_2(x, t)) \in Z^2 \) of \((\vec{d} + \vec{g})\) is unique and has the explicit form

\[
f_1(x, t) = \vec{f}_1(x + t) + \int_0^t \frac{\vec{f}_2(x + t - s)}{N_2(0)} a(s) ds,
\]

(18)

\[
f_2(x, t) = \frac{N_2(t)}{N_2(0)} \vec{f}_2(x).
\]

(19)

(b) For \( \vec{f}_1, \vec{f}_2 \in Z \) and \( 0 \leq t < T(\vec{f}) \), \((\vec{d} + \vec{g})\) defines a continuous dynamical system in \( Z^2 \), so that the map \((\vec{f}_1, \vec{f}_2, t) \mapsto (f_1(\cdot, t), f_2(\cdot, t))\) is in \( C(Z^2 \times [0, T(\vec{f})], Z^2) \).

The proof for Theorem 1 is postponed until Section 5, as neither the explicit solution nor its derivation will be used in future results. The well-posedness of \((\vec{d} + \vec{g})\) is invoked for defining constants in the convergence analysis of Sections 3 and 4. Note, however, that the particle system in this paper is a special case of the \( M \)-species model developed in [10], in which well-posedness is derived through a Banach fixed point argument, rather than appealing to an explicit solution.

2.2 Discretization scheme of kinetic equations

To enable us to write down recurrence inequalities involving total numbers restricted to an interval, we will construct a deterministic scheme for \((\vec{d} + \vec{g})\). We do so with measures \( \tilde{\mu}_1(t, \cdot; \delta), \tilde{\mu}_2(t, \cdot; \delta) \in M(\mathbb{R}_+) \) at time \( t > 0 \) which are piecewise constant in \( \delta > 0 \) sized time intervals \( \Delta t_k = [\delta(k - 1), \delta k) \) for \( k \geq 1 \). Note that while these measures depend on \( \delta \), we will often suppress this argument in the notation for simplicity in presentation.

Initial measures are given by

\[
\tilde{\mu}_j(t, \cdot) = \mu_j, \quad t \in [0, \delta), \quad j = 1, 2,
\]

(20)

with the requirement that \( (\tilde{\mu}_1 + \tilde{\mu}_2)(0, \infty) = 1 \). For each time step \( t_k = k\delta \), we define the incremental loss over a time interval as

\[
\Delta \tilde{L}(t_k) = \begin{cases} 0, & k = 0, \\ \tilde{\mu}_1(t_{k-1}, [0, \delta)), & k \geq 1. \end{cases}
\]

(21)

Total number for Species 2 then decreases by the incremental loss, with

\[
\tilde{N}_2(t_k) = \begin{cases} \tilde{\mu}_2(0, [0, \infty)), & k = 0, \\ \tilde{N}_2(t_{k-1}) - \Delta \tilde{L}(t_k), & k \geq 1. \end{cases}
\]

(22)

Measures update by a shift of distance \( \delta \) toward the origin in Species 1 followed by mutation in
Species 2 of total number $\Delta \tilde{L}(t_k)$. Therefore, for $t \in [t_k, t_{k+1})$, $k \geq 1$, we update with

$$\begin{align*}
\tilde{\mu}_1(t) &= S_h(\tilde{\mu}_2(t_{k-1})) + \frac{\Delta \tilde{L}(t_k)}{\tilde{N}_2(t_{k-1})} \tilde{\mu}_2(t_{k-1}), \\
\tilde{\mu}_2(t) &= \tilde{\mu}_2(t_{k-1}) \left( 1 - \frac{\Delta \tilde{L}(t_k)}{\tilde{N}_2(t_{k-1})} \right).
\end{align*}$$

Here $S_h$ is the left translation operator acting on measures, defined through the cumulative function $F_\mu$ of a measure $\mu \in \mathcal{M}(\mathbb{R}_+)$ by

$$S_h(F_\mu(x)) = F_\mu(x + h) - F_\mu(h), \quad h \geq 0.$$  

(25)

Since Species 1 shifts before adding mutated particles from Species 2, we have a conserved quantity $\tilde{N}_1(t) = \tilde{N}_1(0)$, and thus the total number

$$\tilde{N}(t_k) := \tilde{N}_1(t_k) + \tilde{N}_2(t_k) = 1 - \sum_{i=1}^{k-1} \Delta \tilde{L}(t_i).$$

(26)

This scheme remains well-defined as long as $\tilde{N}_2(t) > 0$. It is clear that

$$\tilde{N}_2(t) \geq \tilde{N}_2(0) - (\tilde{\mu}_1([0,t]) + \tilde{\mu}_2([0,t])),$$

(27)

so for initial measures in $Z$ we are always able to find a nonzero length interval of existence $[0, T_1(\tilde{f})]$, with

$$T_1(\tilde{f}) = \sup \{ t : \tilde{N}_2(t) > \tilde{N}_2(0)/2 \}.$$  

(28)

### 2.3 Main results: convergence rates and concentration inequalities

Our first main result gives a comparison between the deterministic discretization and solutions of (6)-(7) using the Kolmogorov-Smirnov (KS) metric. For two measures $\nu, \eta \in \mathcal{M}(\mathbb{R}_+)$, with associated cumulative functions $F_\nu(x) = \nu([0,x])$ and $F_\eta(x) = \eta([0,x])$ the KS metric is defined as

$$d_{KS}(\nu, \eta) = \sup_{x \in \mathbb{R}} |F_\nu(x) - F_\eta(x)|.$$  

(29)

For handling convergence of both species, we define a metric on $\mathcal{M}(\mathbb{R}_+) \times \mathcal{M}(\mathbb{R}_+)$ between $\nu = (\nu_1, \nu_2)$ and $\eta = (\eta_1, \eta_2)$ by

$$d(\nu, \eta) = d_{KS}(\nu_1, \eta_1) + d_{KS}(\nu_2, \eta_2).$$  

(30)

As we are often working with measures, we define measures associated to solutions of (6)-(7) by

$$\mu_j(t, dx) = f_j(x, t) dx \quad j = 1, 2.$$  

(31)

We will also need to track the modulus of continuity for solutions. For densities $(f_1(x, t), f_2(x, t))$, we let

$$\omega(\delta, t) = \sup_{x \in \mathbb{R}} \sum_{j=1}^{2} |f_j(x + \delta, t) - f_j(x, t)|.$$  

(32)

We will impose that initial conditions have compact support for the sake of clarity, as calculations relating convergence and the decay of initial conditions can become rather technical. Since initial conditions are continuous, compact support implies that $\omega(\delta, 0) \to 0$ as $\delta \to 0$.  

6
Theorem 2. Let \( \tilde{f} = (f_1, f_2) \in Z^2 \) have compact support with \( N_2(0) > 0 \). For \( t < T(\tilde{f}) \), let \( \mu(t) = (\mu_1(t), \mu_2(t)) \) be measures for the unique solution to (6)-(7). Let measures \( \tilde{\mu}(t) = (\tilde{\mu}_1(t), \tilde{\mu}_2(t)) \) for the discretization scheme with step size of \( \delta > 0 \) also have initial conditions \( \tilde{f} \). Then there exist positive constants \( \delta^d \) and \( C^d \) such that for all \( \delta \in (0, \delta^d) \) and \( T' \in [0, T(\tilde{f})) \),

\[
\sup_{t \in [0, T']} d(\tilde{\mu}(t), \mu(t)) \leq C_d(\delta + \omega(\delta, 0)).
\tag{33}
\]

The constants \( C^d \) and \( \delta^d \) are dependent on \( L^\infty \) and \( L^1 \) bounds of solutions in Theorem 1 and the compact support bound \( M = \sup\{x : \sum_{j=1,2} f_j(x) > 0\} \).

To set up for the next main result, we generate initial conditions \( \tilde{\mu}_n = (\tilde{\mu}_1^n, \tilde{\mu}_2^n) \) for the particle system with uniform spacing through the cumulative distribution functions \( F_j(x) = \mu_j([0, x]) \) for \( j = 1, 2 \). The explicit particle positions are

\[
(x_i(0), s_i(0)) = \begin{cases}
(F_1^{-1}(i/n), 1) & i = 1, \ldots, \lfloor nN_1(0) \rfloor, \\
(F_2^{-1}((nN_1(0) - i)/n), 2) & i = \lfloor nN_1(0) \rfloor + 1, \ldots, n,
\end{cases}
\tag{34}
\]

where \( F^{-1} \) is the quantile function of a cumulative function \( F \). For initial conditions \( (f_1, f_2) \in Z^2 \) with \( \tilde{\mu}_j(dx) = f_j dx \), it is easy to show that \( d(\tilde{\mu}_n, \tilde{\mu}) \to 0 \) in law as \( n \to \infty \), and that there exists a positive integer \( n_0(\tilde{f}) \) such that if \( n > n_0(\tilde{f}) \),

\[
\tilde{\mu}_1^n([0, T_1(\tilde{f}))] + \tilde{\mu}_2^n([0, T_1(\tilde{f})]) \leq 2N_2^n(0)/3 \quad \text{for } n > n_0(\tilde{f}),
\tag{35}
\]

meaning that there is always a particle available to mutate at each jumping time in \([0, T_1(\tilde{f})] \), and the process therefore does not reach its cemetery state.

The next theorem, which we show in Section 4, gives an exponential concentration inequality between the deterministic discretization and the particle system.

Theorem 3. Let \( \tilde{f} = (f_1, f_2) \in Z^2 \) have compact support with \( N_2(0) > 0 \). For \( t < T(\tilde{f}) \), let \( \mu^n(t) = (\mu_1^n(t), \mu_2^n(t)) \) be empirical measures for \( X^n(t) \) generated from \( \tilde{f} \) with \( f \in Z^2 \). Let measures \( \tilde{\mu}(t) = (\tilde{\mu}_1(t), \tilde{\mu}_2(t)) \) for the discretization scheme with step size of \( \delta > 0 \) have initial conditions \( \tilde{f} \). Then there exist positive constants \( \delta^p, C^p, C_1^p, C_2^p \), such that for all \( \delta \in (0, \delta^p) \) there exists \( n^p(\delta) > 0 \) such that for all integers \( n > n^p(\delta) \) and \( T' < T(\tilde{f}) \),

\[
\mathbb{P}_n \left( \sup_{t \leq T'} d(\tilde{\mu}(t), \mu^n(t)) \geq C^p(\delta + \omega(\delta, 0)) \right) \leq \frac{C^p}{\delta^p} \exp(-C_2^p \delta^5 n).
\tag{36}
\]

The constants \( \delta^p, C^p, C_1^p, C_2^p \) are all dependent on \( L^\infty \) and \( L^1 \) bounds of solutions in Theorem 1 with initial conditions of \( \tilde{f} = (f_1, f_2) \), and the compact support bound \( M = \sup\{x : \sum_{j=1,2} f_j(x) > 0\} \).

From Theorems 2 and 3 it is straightforward to obtain our main concentration inequality for initial conditions which are also locally Lipschitz.

Theorem 4. Let \( \tilde{f} = (f_1, f_2) \in Z^2 \) with \( N_2(0) > 0 \) be both compactly supported and locally Lipschitz, so that \( \omega(0, \delta) \leq C^\omega \delta \). For \( C^p \) and \( \delta^p \) determined from Theorem 3 let \( \varepsilon \in (0, 2C^p C^\omega \delta^p) \). Then there exist positive constants \( C, \tilde{C} \) and \( N(\varepsilon) > 0 \) such that for all integers \( n > N(\varepsilon) \) and \( T' < T(\tilde{f}) \),

\[
\mathbb{P}_n \left( \sup_{t \leq T'} d(\mu(t), \mu^n(t)) \geq \varepsilon \right) \leq \frac{C}{\varepsilon^2} \exp(-\tilde{C} \varepsilon^5 n).
\tag{37}
\]
Proof. Since initial conditions are locally Lipschitz and compactly supported, we may replace the $C^d(\delta + \omega(\delta,0))$ and $C^p(\delta + \omega(\delta,0))$ terms in Theorem 2 and with $\hat{C}^d\delta$ and $\hat{C}^p\delta$, respectively, where $\hat{C}^d = C^dC^\omega$ and $\hat{C}^p = C^pC^\omega$.

Let $\epsilon \in (0,2\hat{C}^p\delta_p)$ and choose $\delta = \epsilon/(2\hat{C}^d)$. We then have

$$\mathbb{P}\left(\sup_{t \leq T'} d(\mu(t), \mu^n(t)) \geq \epsilon \right) \leq \mathbb{P}\left(\sup_{t \leq T'} d(\tilde{\mu}(t), \mu^n(t)) + d(\mu(t), \tilde{\mu}(t)) \geq \epsilon \right) \leq \mathbb{P}\left(\sup_{t \leq T'} d(\tilde{\mu}(t), \mu^n(t)) \geq \epsilon - \hat{C}^d\delta \right) = \mathbb{P}\left(\sup_{t \leq T'} d(\tilde{\mu}(t), \mu^n(t)) \geq \epsilon/2 \right)$$

$$\leq \frac{4(\hat{C}^p)^2\hat{C}^p}{\epsilon^2} \exp\left(-\frac{C_3^p}{32(\hat{C}^p)^5}\epsilon^5n\right).$$

The second inequality uses Theorem 2 and the third uses Theorem 3. We obtain (37) with $C = 4(\hat{C}^p)^2\hat{C}^p$ and $\tilde{C} = C_3^p/(32(\hat{C}^p)^5)$. \hfill \Box

From Theorem 4, an application of the Borel-Cantelli lemma then gives us a strong law of large numbers.

**Corollary 1.** Under the product measure $\mathbb{Q} = \prod_{n \geq 2} \mathbb{P}_n$, for $T' < T(\bar{f})$,

$$\lim_{n \to 0} \sup_{t \leq T'} d(\mu^n(t), \mu(t)) = 0 \quad \text{almost surely.} \quad (39)$$

## 3 Comparison of kinetic equations and deterministic scheme

In this section, we present a proof of Theorem 2. The discretization scheme outlined in Section 2.2 allows us to write recursive formulas at time steps $t_k$ related to measures restricted to size $\delta$ intervals, which we denote as

$$I_l = [(l-1)\delta, l\delta), \quad l \geq 1. \quad (40)$$

In Section 3.1, we collect estimates related to growth of quantities for solutions of the kinetic equations and the discretization scheme, including the modulus of continuity $\omega(\delta, t)$ and total number contained in an interval. Estimates related to the comparison between solutions of the kinetic equations and the discretization are presented in Section 3.2. The main quantity of interest is the difference of total number in intervals $I_l$ at times $t_k$. Through constructing a closed recurrence inequality, we show differences are of order $\delta^2 + \delta\omega(\delta, 0)$. In Section 3.3, these differences are then summed over $[0, M]$ to establish Theorem 2.

### 3.1 Growth estimates

Our estimates for solutions of (0)-(7) and iterations (23)-(24) will make frequent use of the constant bounds

$$C_\infty(\bar{f}) = \max_{j=1,2} \sup_{x \in [0,T_1(\bar{f})]} \|f_j(x,s)\|_\infty, \quad (41)$$

$$C_b(\bar{f}) = \max\{1/N_2(T_1(\bar{f})), 1/\tilde{N}_2(T_1(\bar{f}))\}, \quad (42)$$

which are dependent upon the initial conditions $\bar{f} = (\bar{f}_1, \bar{f}_2)$ and time of existence $T(\bar{f})$. That $C_\infty(\bar{f}), C_b(\bar{f})$ are finite follows from well-posedness of $(f_1(x,t), f_2(x,t))$ in Theorem 1 and the existence of $T_1(\bar{f})$ established from (27). For simplicity, in future estimates we will refer to these constants.
simply as $C_\infty$ and $C_b$. As we will see in Lemma 2, it will be necessary to further restrict our interval of existence to $t \in [0, T_2(\bar{f})]$, with

$$T_2(\bar{f}) = T_1(\bar{f}) \wedge 1/(8C_\infty C_b).$$

We will work with solutions $(f_1(x,t), f_2(x,t))$ of (6)-(7) having initial conditions $\bar{f}_1, \bar{f}_2 \in \mathbb{Z}$ with compact support in some interval $[0, M] \subset \mathbb{R}_+$. For the deterministic scheme, measures have identical initial conditions as the kinetic equations, with $\bar{\mu}_j(dx) = \bar{f}_j(x)dx$.

We begin with a simple estimate on the propagation of the modulus of continuity.

**Lemma 1.** For all $\delta > 0$ and $t \leq T_2(\bar{f})$,

$$\omega(\delta, t) \leq C_1 \omega(\delta, 0),$$

where $C_1 = 2 \exp(2C_b)$.

**Proof.** Since $N_2(t)$ is decreasing, we use (6)-(7) to show

$$\sum_{j=1}^{2} |f_j(x+\delta,t) - f_j(x,t)|$$

$$\leq |f_1(x+\delta+t,0) - f_1(x+t,0)| + |f_2(x+\delta,t) - f_2(x,0)|$$

$$+ C_b \int_0^t |f_2(x+\delta+t-s,s) - f_2(x+t-s,s)| + |f_2(x+\delta,s) - f_2(x,s)|dL(s),$$

where $L(s) = \int_0^s f_1(0,s)ds$ is the total loss. By taking the supremum of the above inequality over $x \in \mathbb{R}_+$, from the definition of $\omega(\delta, t)$ given in (32),

$$\omega(\delta, t) \leq 2\omega(\delta, 0) + 2C_b \int_0^t \omega(\delta, s)dL(s).$$

From Gronwall’s inequality,

$$\omega(\delta, t) \leq 2 \exp(2C_b L(t))\omega(\delta, 0).$$

Since $L(t) \leq 1$, we obtain (44). \qed

Next, we turn to studying the maximum total number of a measure on length $\delta$ intervals, denoted as

$$m_\delta^j(t) = \sup_{I:|I|=\delta} \mu_j(t, I), \quad j = 1, 2, \quad m_\delta(t) = \sum_{j=1}^{2} m_\delta^j(t).$$

We define $\tilde{m}_\delta(t)$ similarly.

**Lemma 2.** For $\delta > 0$ and $t < T_2(\bar{f})$,

$$m_\delta(t) \leq C_2 \delta,$$

$$\tilde{m}_\delta(t_k) \leq \tilde{C}_2 \delta,$$

$$\tilde{N}(t_k) \geq 1 - \tilde{C}_2 \delta k.$$
Proof. To show (49), for an interval $I$ with $|I| \leq \delta$, integrate (6)-(7) over $I$ to obtain

$$
\mu_1(t, I) = \mu_1(0, I) + \int_0^t \frac{\mu_2(s, I + t - s)}{N_2(s)} dL(s), \tag{53}
$$

$$
\mu_2(t, I) = \mu_2(0, I) - \int_0^t \frac{\mu_2(s, I)}{N_2(s)} dL(s). \tag{54}
$$

By taking the supremum over all length $\delta$ intervals, we find

$$
m_1^1(t) \leq m_1^1(0) + C_b \int_0^t m_2^2(s) dL(s), \tag{55}
$$

$$
m_2^2(t) \leq m_2^2(0). \tag{56}
$$

We then obtain (49) by summing (55)-(56), applying Gronwall’s lemma, and observing from initial conditions that $m_3(0) \leq 2C_\infty \delta$.

To show (50), we will work with $\hat{m}_j^\delta(t) = \sup_{I \geq 1} \tilde{\mu}_j(t, I)$, $j = 1, 2$,

$$
\hat{m}_\delta(t) = \sum_{j=1}^2 \hat{m}_j^\delta(t). \tag{57}
$$

Note that

$$
\hat{m}_\delta^2(t) \leq 2\hat{m}_\delta^2(t). \tag{58}
$$

From evaluating measures on $I_t$, the recursion (23)-(24) implies that for $l \geq 1$ and $s \in [t_k, t_{k+1})$,

$$
\tilde{\mu}_1(s, I_l) = \tilde{\mu}_1(t_{k-1}, I_{l+1}) + \frac{\Delta L(t_k)}{N_2(t_{k-1})} \tilde{\mu}_2(t_{k-1}, I_l), \tag{59}
$$

$$
\tilde{\mu}_2(s, I_l) = \tilde{\mu}_2(t_{k-1}, I_l) \left(1 - \frac{\Delta L(t_k)}{N_2(t_{k-1})}\right). \tag{60}
$$

We now take the supremum over $j \geq 1$ in (59)-(60) and sum to obtain

$$
\hat{m}_\delta(t_k) \leq \hat{m}_\delta(t_{k-1}) + C_b (\hat{m}_\delta(t_{k-1}))^2. \tag{61}
$$

For simplicity, we rescale by writing $\hat{m}_\delta(t_k) = A(t_k) \delta$. From (61), and noting

$$
\hat{m}_\delta(0) \leq 2\hat{m}_\delta(0) = 2m_3(0) \leq 4C_\infty \delta, \tag{62}
$$

we obtain the recurrence inequalities

$$
A(t_k) \leq A(t_{k-1}) + \delta C_b (A(t_{k-1}))^2, \tag{63}
$$

$$
A(0) \leq 4C_\infty. \tag{64}
$$

In the case of equality, (63)-(64) is an Euler scheme for the differential equation $g'(t) = C_b g^2(t)$ with $g(0) = C_\infty$. We may check directly that $A(t_k) \leq g(t_k)$ before blowup, or

$$
A(t_k) < \frac{A(0)}{1 - kA(0)C_b \delta} \tag{65}
$$

for $t_k \leq T_2(\bar{f})$. We then use (65), (64), and (58) to obtain (50).
Finally, \([51]\) follows immediately, since
\[
\tilde{N}_2(t_k) = 1 - \sum_{i=1}^k \Delta \tilde{L}(t_i) \geq 1 - \sum_{i=1}^k \tilde{m}_\delta(t_i). 
\] (66)

We finish this subsection with one more estimate related to the incremental loss and total number in the kinetic limit.

**Lemma 3.** For \(t_k < T_2(\hat{f})\),
\[
\Delta L(t_k) := L(t_k) - L(t_{k-1}) \leq C_2 \delta + C_6 C^2_\infty \delta^2, 
\] (67)
\[
N(t_k) \geq 1 - C_2 k \delta - C_6 C^2_\infty k \delta^2. 
\] (68)

**Proof.** We use \([6]\) on the removal rate \(f_1(0, t)\) to obtain
\[
\Delta L(t_k) = \int_0^\delta f_1(0, t_{k-1} + s) ds 
\] (69)
\[
= \int_0^\delta \left( f_1(s, t_{k-1}) + \int_0^s \frac{f_1(0, t_{k-1} + r)}{N_2(t_{k-1} + r)} f_2(s - r, t_{k-1} + r) dr \right) ds 
\]
\[
\leq \mu_1(t_{k-1}, [0, \delta]) + C_6 C^2_\infty \delta^2.
\]
From \([49]\), we obtain \([67]\). Since \(N(t_k) = 1 - \sum_{i=1}^k \Delta L(t_i)\), \([68]\) also follows. \(\square\)

### 3.2 Convergence estimates

We now use estimates from the previous subsection to establish asymptotics for the differences of total number between the solution of \([3]-[5]\) and its discretization. We begin with a simple result which follows immediately from Lemma 3 comparing incremental losses and total numbers of species.

**Corollary 2.** For \(t_k < T_2(\hat{f})\),
\[
|\Delta L(t_k) - \Delta \tilde{L}(t_k)| \leq |\mu_1(t_{k-1}, [0, \delta]) - \tilde{\mu}_1(t_{k-1}, [0, \delta])| + C_6 C^2_\infty \delta^2. 
\] (70)
\[
|N(t_k) - \tilde{N}(t_k)| \leq \sum_{i=1}^k |\mu_1(t_i, [0, \delta]) - \tilde{\mu}_1(t_i, [0, \delta])| + T(\hat{f}) C_6 C^2_\infty \delta. 
\] (71)

To compare behavior on an interval \(I_j\), we will use a formula similar to \([59]-[60]\) for evolving densities over a time step \(\Delta t_k = [t_{k-1}, t_k)\). It follows directly from \([6]-[7]\) that
\[
f_1(x, t_k) = f_1(x + \delta, t_{k-1}) + \int_{\Delta t_k} \frac{f_2(x + t_k - s, s)}{N_2(s)} f_1(0, s) ds, 
\] (72)
\[
f_2(x, t_k) = f_2(x, t_{k-1}) - \int_{\Delta t_k} \frac{f_2(x, s)}{N_2(s)} f_1(0, s) ds. 
\] (73)

To arrive at an estimate for the difference of total number in an interval, we use \([23]-[24]\) and \([72]-[73]\) to express the difference of total number of an interval in Species 1 as
\[
|\mu_1(t_k, I_I) - \tilde{\mu}_1(t_k, I_I)| \leq |\mu_1(t_{k-1}, I_{I+1}) - \tilde{\mu}_1(t_{k-1}, I_{I+1})| 
\]
\[
+ \int_{\Delta t_k} \int_{\Delta t_k} \frac{f_2(x + t_k - s, s)}{N_2(s)} dL(s) dx - \frac{\Delta \tilde{L}(t_k)}{N_2(t_{k-1})} \tilde{\mu}_2(t_{k-1}, I_I). 
\] (74)
For Species 2,\n
\[ |\mu_2(t_k, I_l) - \tilde{\mu}_2(t_k, I_l)| \leq |\mu_2(t_{k-1}, I_l) - \tilde{\mu}_2(t_{k-1}, I_l)| \leq (75) \]

\[ + \left| \int_{I_l} \int_{\Delta t_k} \frac{f_2(x, s)}{N_2(s)} dL(s) dx - \frac{\Delta \tilde{L}(t_k)}{N_2(t_{k-1})} \tilde{\mu}_2(t_{k-1}, I_l) \right|. \]

The following lemma allows us to estimate the integrals in (74) and (75) by quantities at discretized times \( t_k \).

**Lemma 4.** For \( l \geq 1 \) and \( t_k < T_2(\bar{f}) \),

\[ \left| \int_{I_l} \int_{\Delta t_k} \frac{f_2(x + t_k - s, s)}{N_2(s)} dL(s) dx - \frac{\Delta \tilde{L}(t_k)}{N_2(t_{k-1})} \mu_2(t_{k-1}, I_l) \right| = O(\delta^3 + \delta^2 \omega(\delta, 0)). \]  

(76)

\[ \left| \int_{I_l} \int_{\Delta t_k} \frac{f_2(x, s)}{N_2(s)} dL(s) dx - \frac{\Delta \tilde{L}(t_k)}{N_2(t_{k-1})} \mu_2(t_{k-1}, I_l) \right| = O(\delta^3). \]  

(77)

**Proof.** We show (76). The proof for (77) is similar. Denote the left hand side of (76) as \( \mathcal{I} \). Through the triangle inequality, we have

\[ \mathcal{I} = \left| \int_{I_l} \int_{\Delta t_k} \frac{f_2(x + t_k - s, s)}{N_2(s)} dL(s) dx - \frac{f_2(x, t_{k-1})}{N_2(t_{k-1})} dL(s) dx \right| \]

(78)

\[ \leq \int_{I_l} \int_{\Delta t_k} \left| \frac{f_2(x + t_k - s, s) - f_2(x, t_{k-1})}{N_2(s)} dL(s) dx \right| 
+ \int_{I_l} \int_{\Delta t_k} \left| f_2(x, t_{k-1}) \left( \frac{1}{N_2(s)} - \frac{1}{N_2(t_{k-1})} \right) dL(s) dx \right| \]

=: \mathcal{I}_1 + \mathcal{I}_2.

We first show that

\[ \mathcal{I}_1 \leq C_0^2 C_2^\infty C_2^2 \delta^3 + C_1 C_2 C_0 \omega(\delta, 0) \delta^2 + O(\delta^4 + \omega(\delta, 0) \delta^3). \]  

(79)

This is done by another use of the triangle inequality to align arguments in time and space,

\[ \mathcal{I}_1 \leq C_0 \left( \int_{\Delta t_k} \int_{I_l} |f_2(x + t_k - s, t_{k-1}) - f_2(x, t_{k-1})| dx dL(s) \right) \]

(80)

\[ + \int_{I_l} \int_{\Delta t_k} |f_2(x + t_k - s, s) - f_2(x + t_k - s, t_{k-1})| dL(s) dx. \]

From (44), (49) and (67), the first integral may be bounded by

\[ \int_{\Delta t_k} \int_{I_l} |f_2(x + t_k - s, t_{k-1}) - f_2(x, t_{k-1})| dx dL(s) \leq C_1 C_2 \omega(\delta, 0) \delta^2 + O(\omega(\delta, 0) \delta^3). \]

(81)

For the last integrand for (80), we may use (7) to obtain

\[ |f_2(x + t_k - s, s) - f_2(x + t_k - s, t_{k-1})| \]

\[ = \int_{t_{k-1}}^s f_2(x + t_k - s, r) \frac{f_1(0, r) dr}{N_2(r)} \leq C_1 C_2^\infty. \]

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Substituting (81) and (82) into (80) then gives (79).

By a similar calculation we may use (77) to obtain

\[ I_2 \leq C_{\infty}C_{2}\delta (\Delta L(t_k))^2 \leq C_{\infty}C_{2}\delta^3 + O(\delta^4). \]  

(83)

Finally, (76) then comes from collecting estimates for \( I = I_1 + I_2 \).

We are now ready to compare total numbers over length \( \delta \) intervals by defining

\[ h_{\delta}^j(t_k) = \sup_{t \geq 1} |\mu_j(t_k, I_t) - \bar{\mu}_j(t_k, I_t)|, \quad h_{\delta}(t_k) = \sum_{j=1}^{2} h_{\delta}^j(t_k). \]  

(84)

Our next lemma gives closed recurrence inequalities for \( h_{\delta}(t_k) \).

**Lemma 5.** There exists a \( C_3(\bar{f}) \) dependent on initial conditions such that for \( t_k < T_{\delta}(\bar{f}) \), \( h_{\delta}(t_k) \) satisfies the recurrence inequality

\[ h_{\delta}(t_k) \leq h_{\delta}(t_{k-1}) + C_3 \left( \delta^2 \sum_{i=1}^{k-1} h_{\delta}(t_i) + \delta h(t_{k-1}) + \omega(\delta, 0)\delta^2 + \delta^3 \right). \]  

(85)

**Proof.** From Lemma 4 and (74), for \( l \geq 1 \),

\[ \begin{align*}
|\mu_1(t_k, I_t) - \bar{\mu}_1(t_k, I_t)| & \leq |\mu_1(t_{k-1}, I_{t+1}) - \bar{\mu}_1(t_{k-1}, I_{t+1})| \quad \text{(86)} \\
& + \left| \frac{\Delta L(t_k)}{N_2(t_{k-1})} \mu_2(t_{k-1}, I_t) - \bar{\mu}_2(t_{k-1}, I_t) \right| + O(\delta^3 + \delta^2 \omega(\delta, 0)).
\end{align*} \]

Two applications of the triangle inequality yield

\[ \begin{align*}
& \left| \frac{\Delta L(t_k)}{N_2(t_{k-1})} \mu_2(t_{k-1}, I_t) - \bar{\mu}_2(t_{k-1}, I_t) \right| \\
& \leq \left| \left( \frac{1}{N_2(t_{k-1})} - \frac{1}{N_2(t_{k-1})} \right) \Delta L(t_k) \mu_2(t_{k-1}, I_t) \right| \\
& + \left| \frac{1}{N_2(t_{k-1})} \mu_2(t_{k-1}, I_t)(\Delta L(t_k) - \bar{\Delta} L(t_k)) \right| \\
& + \left| \frac{\bar{\Delta} L(t_k)}{N_2(t_{k-1})} (\mu_2(t_{k-1}, I_t) - \bar{\mu}_2(t_{k-1}, I_t)) \right|.
\end{align*} \]  

(87)

We use Lemmas 2 and 3, (86), and (87) to obtain

\[ \begin{align*}
|\mu_1(t_k, I_t) - \bar{\mu}_1(t_k, I_t)| & \leq |\mu_1(t_{k-1}, I_{t+1}) - \bar{\mu}_1(t_{k-1}, I_{t+1})| \\
& + C_3^2 C_2^2 \delta^2 |N_2(t_{k-1}) - \bar{N}_2(t_{k-1})| + C_6 C_2 \delta |\Delta L(t_k) - \bar{\Delta} L(t_k)| \\
& + C_6 C_2 \delta |\mu_2(t_{k-1}, I_t) - \bar{\mu}_2(t_{k-1}, I_t)| + O(\delta^3 + \delta^2 \omega(\delta, 0)).
\end{align*} \]  

(88)
Similar bounds hold for Species 2, with
\[
\begin{align*}
|\mu_2(t_k, I_1) - \bar{\mu}_2(t_k, I_1)| & \leq |\mu_2(t_{k-1}, I_1) - \bar{\mu}_2(t_{k-1}, I_1)| \\
& + C_b^2 C_b^3 \delta^2 |\sum_{i=1}^{k-1} h_\delta(t_i) + \delta h_\delta(t_0) + \omega(\delta, 0)\delta^2 + \delta^3|,
\end{align*}
\]
where \(h_\delta(t)\) is defined in Lemma 6.

We complete the proof by taking the supremum over \(l\) for (88) and (89), using (71), and then adding to show that for some \(C_3\) which depends on \((f_1, f_2)\),
\[
h_\delta(t_k) \leq h_\delta(t_{k-1}) + C_3 \left( \delta^2 \sum_{i=1}^{k-1} h_\delta(t_i) + \delta h_\delta(t_{k-1}) + \omega(\delta, 0)\delta^2 + \delta^3 \right).
\]

We now show that the recurrence inequality (88) implies asymptotics for \(h_\delta(t_k)\).

**Lemma 6.** For \(t_k < T_2(f)\),
\[h_\delta(t_k) = O(\delta^2 + \delta \omega(\delta, 0)).\]

**Proof.** Let \(b_\delta(t)\) satisfy the recurrence equation
\[
b_\delta(t_k) = b_\delta(t_{k-1}) + C_3 \left( \delta^2 \sum_{i=1}^{k-1} b_\delta(t_i) + \delta b_\delta(t_{k-1}) + \delta \right),
\]
with initial condition \(b_\delta(0) = h_\delta(0) = 0\). It follows immediately from induction that \(h(t_k) \leq (\delta^2 + \omega(\delta, 0)\delta)b_\delta(t_k)\) for all \(k \geq 0\). Thus, it is sufficient to show that \(b_\delta(t_k) = O(1)\) for all \(t_k < T_2(f)\). To see that this holds, note that as \(\delta \to 0\), (92) converges to the linear integro-differential equation
\[
\ddot{b}(t) = C_3 \left( \ddot{b}(t) + \int_0^t \dot{b}(s)ds + 1 \right), \quad t \in [0, T_2(f)],
\]
with initial condition \(\ddot{b}(0) = h_\delta(0)\). This can be solved through elementary second order methods to obtain the locally bounded solution
\[
\ddot{b}(t) = A_1 e^{\gamma_1 t} + A_2 e^{\gamma_2 t},
\]
where \(A_1 = -A_2 = C_3/(r_2 - r_1)\), and \(r_1, r_2\) are the two real solutions of \(r^2 - C_3 r - C_3 = 0\).

We remark that this theorem also holds when \(h_\delta(0) = O(\delta^2 + \delta \omega(\delta, 0)).\) This is important for extending the time interval of existence.

### 3.3 Proof of Theorem 2

With bounds on differences of total numbers restricted to an interval, we are finally ready to show Theorem 2. We will require two more lemmas, which are straightforward to show. First, we give a formula computing KS distances when only given information about cumulative functions on grid points and bounds on growth between grid points.
Lemma 7. For $j = 1, 2$, suppose we have measures $\nu_j \in \mathcal{M}(\mathbb{R}_+)$ with corresponding cumulative functions $F_j(x) = \nu_j([0, x])$, each with compact support $[0, M]$. For $\delta > 0$, we can bound the KS metric by

$$d_{KS}(\nu_1, \nu_2) \leq \frac{[M/\delta]}{\sum_{i=1}^{[M/\delta]|\nu_i(I_i) - \nu_2(I_i)| + \sup_{[x-y]=\delta}(|F_1(x) - F_1(y)| + |F_2(x) - F_2(y)|)}.$$

We will also require a lemma comparing differences of solutions of kinetic equations under small changes in time.

Lemma 8. If $|t_1 - t_2| \leq \delta$, then

$$d((\mu_1(t_1), \mu_2(t_1)), (\mu_1(t_2), \mu_2(t_2))) = O(\delta + \omega(\delta, 0)).$$

Proof. Similar to the proof of Lemma 3.

Proof of Theorem 2 Let $t \in [0, T_2(f)]$, $\delta > 0$, and $K = \lfloor \frac{1}{\delta} \rfloor$, which means $t_K$ is the largest discretized time which is at most $t$. From Lemmas 2, 6, 7, 8 and recalling that $\tilde{\mu}_j(t)$ is constant in intervals $t \in [t_{k-1}, t_k)$, we compute

$$d((\mu_1(t), \mu_2(t)), (\tilde{\mu}_1(t), \tilde{\mu}_2(t))) \leq \sum_{j=1}^{[M/\delta]} \sum_{i=1}^{[M/\delta]} |\mu_j(t, I_i) - \tilde{\mu}_j(t, I_i)| + O(\delta + \omega(\delta, 0))$$

Finally, let us argue for extending the time interval of existence to any $T' < T(f)$. Consider initial conditions $\mu^{(2)}(0) = \mu(T_2(f))$ and $\tilde{\mu}^{(2)}(0) = \tilde{\mu}(T_2(f))$, and new time bounds

$$T_1^{(2)}(f) = \sup\{t : N_2^{(2)}(t) > 2N_2^{(2)}(0)/3\}, \quad T_2^{(2)} = T_1^{(2)} + 1/(8\check{C}_\infty \check{C}_b),$$

with

$$\check{C}_\infty = \max_{j=1,2} \sup_{s \in [0, T_1^{(2)}(f)]} \|f_j(x, s)\|_\infty, \quad \check{C}_b = 2/\inf_{t < T_1^{(2)}} N_2(t).$$

Then for sufficiently small $\delta$, it follows that Lemmas 1, 6 hold, with possibly larger constants, and therefore Theorem 2 holds for the time interval $[0, T_2^{(2)} + T_1^{(2)}]$ from stitching solutions. This argument may be repeated to produce $T^{(K)}$. Each new time interval either adds the constant $1/(8\check{C}_\infty \check{C}_b)$ (which does not depend on $k$) or reduces $N_2$ by at least $4/5$. After finitely many extensions we will reach a time $T^{(K)}$ at which $N_2(T^{(K)})$ is arbitrarily small, so that $t$ is arbitrarily close to $T(f)$. This completes the proof of Theorem 2.

4 Comparison of particle system and deterministic scheme

4.1 Stochastic analogues of Section 3

We now compare the discretized measures described in the Section 3 with the $n$-particle PDMP $\{X^a(t)\}_{t \geq 0}$. To do so, it will help to express evolution of $\mu_0^a$ from $t_k$ to $t_{k+1}$ in a form that is similar
to the iterative formulas \([23]-[24]\). For defining analogous notation to the the discretization scheme, we denote the number of mutations occurring in the time interval \( t \in \Delta t_k = [t_{k-1}, t_k) \) as \( \Delta L^n(t_k) \).

At each \( i \in 1, \ldots, \Delta L^n(t_k) \), mutation time \( t^i_k \) denote when a particle at position \( x_i \geq 0 \) in Species 2 mutates into Species 1. For particles in Species 2 which mutate in interval \( I \) during \( \Delta t_k \), the empirical measure of their positions at mutation times is defined by

\[
\pi^n_1(t_k, I) = \frac{1}{n} \sum_{i=1}^{\Delta L^n(t_k)} 1(x_i \in I).
\]

We define \( \pi^n_1 \) as the empirical measure for the positions of mutated particles at time \( t_k \), with

\[
\pi^n_1(t_k, I) = \frac{1}{n} \sum_{i=1}^{\Delta L^n(t_k)} 1(x_i \in I + \tau_i - t_{k-1})
\]

\[
:= \frac{1}{n} \sum_{i=1}^{\Delta L^n(t_k)} Q^n_i(I).
\]

Updates for measures on intervals during a time step may then be succintly written as

\[
\mu_1^n(t_k, I) = \mu_1^n(t_{k-1}, I + \delta) + \pi^n_1(t_k, I)
\]

\[
\mu_2^n(t_k, I) = \mu_2^n(t_{k-1}, I) - \pi^n_2(t_k, I).
\]

From \([35]\), the time interval of existence, under sufficiently many particles \( n > n_0(\bar{f}) \), before reaching the cemetery state is at least \( T_1(\bar{f}) \), but since we are comparing the particle system with the deterministic discretization, we will work with the smaller time interval \( [0, T_2(\bar{f})] \).

Let us present variables to be used for the particle system which are similar to those found in Section 3. We begin with an analogue to \( m_n \) of length \( \delta \) intervals as

\[
m^{j,n}_\delta(t) = \sup_{|I| \leq \delta} \mu_j^n(t, I), \quad m^n_\delta(t) = \sum_{j=1,2} m^{j,n}_\delta(t).
\]

From \([101]-[102]\), it follows for all realizations of \( X^n(t) \) that

\[
m^n_\delta(t_k) \leq m^n_\delta(t_{k-1}) + \sum_{j=1}^{2} \sup_{I, |I| \leq \delta} \pi^n_j(t_k, I).
\]

There is also a particle system analog of \( h_\delta \), where we now compare total number in intervals between \( X^n(t) \) and the discretization scheme as

\[
h^{j,n}_\delta(t_k) = \sup_{I \subseteq [t_{k-1}, t_k]} \left| \mu^n_j(t_k, I) - \bar{\mu}_j(t_k, I) \right|, \quad h^n_\delta = \sum_{j=1}^{2} h^{j,n}_\delta(t_k).
\]

The use of measures \( \bar{\mu}(t_k) \) rather than \( \mu(t_k) \) comes from ability to use the recurrence \([59]-[60]\), along with \([101]-[102]\) and \([23]-[24]\), to write the recurrence inequality

\[
h^n_\delta(t_k) \leq h^n_\delta(t_k-1) + \sum_{j=1}^{2} \max_{I \subseteq [t_{k-1}, t_k]} \left| \pi^n_j(t_k, I) - \frac{\Delta \bar{L}_j(t_k)}{\bar{N}_2(t_k-1)} \bar{\mu}_2(t_{k-1}, I) \right|
\]

\[
:= h^n_\delta(t_k-1) + \Pi^n(t_k).
\]

From \([104]\) and \([106]\), the major task for controlling \( h^n \) and \( m^n \) will clearly hinge on appropriate estimates for \( \pi_j \) (and subsequently \( \Pi^n \)). These key bounds are provided in the next two lemmas.
4.2 Concentration bounds for $\pi_j$

We begin with an easy to establish generalization of the Hoeffding inequality, which states that for $n \geq 1$ and with $B_i(n, p) \sim \text{Binom}(n, p)$,

$$
P\left(\left|\frac{B_i(n, p)}{n} - p\right| > \varepsilon\right) \leq 2e^{-2n\varepsilon^2}.
$$

(107)

**Corollary 3.** For $n \geq 1$, let $X^n = \sum_{i=1}^n Z_i$, where $Z_i \sim \text{Ber}(p_i)$ are independent Bernoulli random variables with parameters $0 \leq p_i \leq p \leq 1$. Then for $\varepsilon > 0$,

$$
P\left(\frac{X^n}{n} - \overline{p} > \varepsilon\right) \leq 2e^{-2n\varepsilon^2} \quad \text{and} \quad P\left(\frac{X^n}{n} - p < -\varepsilon\right) \leq 2e^{-2n\varepsilon^2}.
$$

(108)

For the next two lemmas, we will be interested in cases where parameters $p_i$ are themselves $[0, 1]$-valued random variables with known lower and upper bounds. In particular, we will use the mutation probability $P^2_k(t, I)$ defined as the state-dependent probability that if a mutation occurs at time $t \in [t_{k-1}, t_k]$, then the mutated particle would be located in $I$ at time $t_k$. We also define $P^1_k(t, I)$ as the probability that a particle would mutate in $I$ from Species 2 at time $t$. Thus,

$$
P^1_k(t, I) = \frac{\mu^2(t^-\cdot, I + t_k - t)}{N^2_n(t^-)}, \quad P^2_k(t, I) = \frac{\mu^2(t^-\cdot, I)}{N^2_n(t^-)}.
$$

(109)

For an initial distribution $(\mu^1_0(0), \mu^2_0(0))$ with support $[0, M]$, tracking which intervals mutations occur in during a time interval $[t_{k-1}, t_k]$ can be represented through a random sum of multinomials of one draw with random selection probabilities. In particular, we perform a total of $\Delta L^n(t_k)$ draws with $\lceil M/\delta \rceil$ bins, in which for each draw the $i$th bin has the mutation probability $P^2_i(\tau^-_i, I)$ of being selected.

The following two lemmas comparing $\pi_j$ with bounds for $P^1_k(t, I)$ and $\Delta L^n(t_k)$ involve using Corollary 3 along with the strong Markov property of PDMPs introduced by Davis [2]. For the first lemma, we consider an initial state $X^n = X^n(0)$ which has pathwise bounds during $\Delta t_1$ for $\Delta L^n(t_1)$, and $P^1_1(t, I)$. The second lemma assumes these bounds occur with some probability which may be less than 1. Since $X^n(t)$ is homogeneous, both these lemmas are readily applicable when considering transitions during $\Delta t_k$ for $k > 1$.

We will use common notation for stochastic ordering: for real-valued random variables $X$ and $Y$, we write $X \leq_{st} Y$ if $\mathbb{P}(X > c) \leq \mathbb{P}(Y > c)$ for all real $c$. Also note in the next two lemmas, we will suppress time arguments when no confusion will arise. Finally, for simplicity with presentation, we will assume that $M/\delta$ is integral.

**Lemma 9.** Let $X^n(0) = \mathbf{x} \in E$ be an initial state such that for $l \in 1, \ldots, M/\delta$, $j \in \{1, 2\}$, and $t \in \Delta t_1$,

$$
\underline{L} \leq \Delta L^n \leq \bar{L} \quad \text{and} \quad \underline{p}_j \leq P^j_k(t, I) \leq \bar{p}_j,
$$

(110)

where $\underline{L}, \bar{L}, \underline{p}_j, \bar{p}_j$ are $[0, 1]$-valued constants. Then for $\varepsilon > 0$,

$$
P\left(\max_{l \leq M/\delta} \left(\pi^1_j(t_1, I_l) - \bar{L}\bar{p}_j\right) > \varepsilon\right) \leq \frac{2M}{\delta} \exp(-2n\varepsilon^2/\bar{L})
$$

(111)

and

$$
P\left(\min_{l \leq M/\delta} \left(\pi^2_j(t_1, I_l) - L\underline{p}_j\right) < -\varepsilon\right) \leq \frac{2M}{\delta} \exp(-2n\varepsilon^2/\underline{L}).
$$

(112)
Proof. We will show (111) for \( j = 1 \). The proofs for the other cases are similar. For the parameter \( q \in [0,1] \), denote \( \{ B_i(q) \}_{i \geq 1} \) as an iid stream of Bernoulli random variables with \( B_1 \sim \text{Ber}(q) \). We write

\[ Q_i := B_i(p_i^n), \quad \bar{Q}_i := \begin{cases} Q_i^1(I_l) & i \leq \Delta L^n n, \\ Q_i^1 & i > \Delta L^n n. \end{cases} \]

We use iterated conditioning to show that

\[ \mathbb{P}(Q_i^1 = 1) = \mathbb{E}[Q_i^1] = \mathbb{E}[\mathbb{E}[Q_i^1 | X(\tau_i^-)]] = \mathbb{E}[p_i^1(\tau_i^1, I_l)] \leq \bar{p}_l = \mathbb{P}(\bar{Q}_i = 1). \]  

This calculation implies the stochastic ordering \( Q_i^1 \leq_{ST} \bar{Q}_i \).

Next, we show

\[ \pi_i^1(I_l) \leq_{ST} \frac{1}{n} \sum_{i=1}^{L_n} Q_i^1 \leq_{ST} \frac{1}{n} \sum_{i=1}^{L_n} \bar{Q}_i. \]

The left inequality is immediate, and in fact holds for all paths in \( X^a(t) \). To show the right inequality, we use induction, assuming that for \( 1 \leq j < Ln \),

\[ \mathbb{P}\left( \sum_{i=1}^{j} Q_i^1 > c \right) \leq \mathbb{P}\left( \sum_{i=1}^{j} \bar{Q}_i > c \right). \]  

The base case holds trivially. For the inductive step, we use the law of total probability to show

\[ \mathbb{P}\left( \sum_{i=1}^{j+1} Q_i > c \right) = \mathbb{E}\left[ \mathbb{P}\left( \sum_{i=1}^{j} Q_i^1 + Q_i^{j+1} > c \mid X((\tau_i^{j+1})^-) \right) \right] \leq \mathbb{E}\left[ \mathbb{P}\left( \sum_{i=1}^{j} Q_i^1 + \bar{Q}_i^{j+1} > c \mid X((\tau_i^{j+1})^-) \right) \right] = \mathbb{P}\left( \sum_{i=1}^{j} Q_i^1 + \bar{Q}_i^{j+1} > c \right) \leq \mathbb{P}\left( \sum_{i=1}^{j+1} \bar{Q}_i > c \right). \]

The first inequality in (117) uses a well-known property for stochastic dominance when summing random variables: if \( X_1 \) and \( X_2 \) are independent, \( Y_1 \) and \( Y_2 \) are independent, and \( X_i \leq_{ST} Y_i \) for \( i = 1, 2 \), then \( X_1 + X_2 \leq_{ST} Y_1 + Y_2 \). From the strong Markov property of PDMPs, the \( \mathcal{F}(\tau_k^1) \)-measurable quantity \( \sum_{i=1}^{j} Q_i^1 \) and the \( \mathcal{F}(\tau_k^{j+1}) \)-measurable quantity \( \bar{Q}_i^{j+1} \) are conditionally independent under \( \mathbb{P}(\cdot \mid X((\tau_i^{j+1})^-)) \). The last inequality uses the same property of stochastic dominance along with the induction hypothesis.

With (117) we then obtain our result from Lemma 3 with

\[ \mathbb{P}\left( \max_{l \leq M/\delta} \left( \pi_i^1(I_l) - \bar{L}_{\bar{p}_l} \right) > \varepsilon \right) \leq \sum_{l \leq M/\delta} \mathbb{P}(\pi_i^1(I_l) - \bar{L}_{\bar{p}_l} > \varepsilon) \leq \frac{2M}{\delta} \exp(-2n\varepsilon^2/L). \]  

\[ \square \]
While Lemma 6 is useful for computing total numbers in Species 1 and 2, we find in Section 4.4 that for comparing the PDMP to our deterministic discretization, it is necessary to consider another estimate of $\pi_i$ in which the bounds on $\Delta L(t_k)$ and $P^\pi_{ik}$ may not hold with small probability. This differs from Lemma 9, in which we assume such bounds occur over all paths given an appropriate initial condition.

For the next lemma and in many other places, we will frequently rely on an elementary inequality derived from the law of total probability: for events $C$ and $D$,

$$P(C) = P(C|D)P(D) + P(C|\bar{D})P(\bar{D}) \leq P(C|D) + P(\bar{D}).$$

(119)

**Lemma 10.** Let $T$ be an event such that

$$T \subset \{L \leq \Delta L^n \leq \bar{L}\} \cap \{P^I(t, I_i) \in [\bar{p}_i, \tilde{p}_i] : l \leq M/\delta, j = 1, 2, t \in \Delta t_1\},$$

where $L, \bar{L}, \bar{p}_i$, and $\tilde{p}_i$ are $[0,1]$-valued constants. Suppose for some $r(\delta, n) \in [0,1)$ that

$$P(T^c) \leq r(\delta, n).$$

(120)

Then for $j = 1, 2$ and $\varepsilon > 0$,

$$P \left( \max_{l \leq M/\delta} \left( \pi_j^I(t_1, I_1) - \bar{L}\bar{p}_i \right) > \varepsilon \middle| T \right) \leq \frac{2M}{\delta} \exp(-2n\varepsilon^2/\bar{L}) + \frac{M}{\delta} \left( \frac{r}{1-r} + \frac{2rL}{(1-r)^2} \right)$$

(121)

and

$$P \left( \min_{l \leq M/\delta} \left( \pi_j^I(t_1, I_1) - L_\bar{p}_i \right) < -\varepsilon \middle| T \right) \leq \frac{2M}{\delta} \exp(-2n\varepsilon^2/L) + \frac{M}{\delta} (2r\bar{L} + r).$$

(122)

**Proof.** We show (121) and (122) for $j = 1$. We first show by induction that

$$P \left( \sum_{i=1}^j Q_i > c \right) \leq P \left( \sum_{i=1}^j \tilde{Q}_i > c \right) + \frac{2rj}{1-r}, \quad j = 1, \ldots, \bar{L} n. \quad \text{(123)}$$

We will condition on the $\mathcal{F}(X(\tau^{I^-}_i))$-measurable event

$$\mathcal{T}_i = \{p_j \leq P^I(\tau^I_i, I_i) \leq \tilde{p}_i\} \supseteq T.$$ \hspace{1cm} \text{(124)}

The base case for (123) follows from (119) and (120), since

$$P \left( Q_1 > c \right) \leq P \left( Q_1 > c \middle| T_1 \right) + P(T_1^c) \leq \frac{P \left( Q_1 > c \right)}{1-r} + r$$

(125)

$$\leq P \left( Q_1 > c \middle| T_1 \right) + \frac{r}{1-r} + r \leq P \left( Q_1 > c \right) + \frac{2r}{1-r}.$$ \hspace{1cm} \text{(126)}

For the inductive step, assuming (123) holds for $1 \leq j < \bar{L} n$, we use the strong Markov property of PDMPs to show

$$P \left( \sum_{i=1}^{j+1} Q_i > c \right) \leq P \left( \sum_{i=1}^j Q_i + Q_{j+1} > c \middle| T_{j+1} \right) + P(T_{j+1}^c)$$

$$\leq P \left( \sum_{i=1}^j Q_i > c \middle| T_{j+1} \right) + \frac{r}{1-r} \leq P \left( \sum_{i=1}^j \tilde{Q}_i > c \right) + \frac{2r}{1-r}$$

$$\leq P \left( \sum_{i=1}^{j+1} \tilde{Q}_i > c \right) + \frac{2r(j+1)}{1-r} = P \left( \sum_{i=1}^{j+1} Q_i > c \right) + \frac{2r(j+1)}{1-r}.$$
From calculations similar to (114)-(117), we use (115) to show
\[
P(\pi_n(I_i) > c | T) \leq \mathbb{P}\left( \frac{1}{n} \sum_{i=1}^{L_n} Q_i > c \right)
\]
\[
\leq \frac{1}{1 - r} \mathbb{P}\left( \frac{1}{n} \frac{L_n}{n} \sum_{i=1}^{L_n} Q_i > c \right) \leq \frac{1}{1 - r} \mathbb{P}\left( \frac{1}{n} \sum_{i=1}^{L_n} Q_i > c \right) + \frac{2r \bar{L}_n}{(1 - r)^2}
\]
\[
\leq \mathbb{P}\left( \frac{1}{n} \frac{L_n}{n} \sum_{i=1}^{L_n} Q_i > c \right) + \frac{r}{1 - r} + \frac{2r \bar{L}_n}{(1 - r)^2}.
\]

For the lower bound, we note
\[
P(Q_i = 1) \geq P(Q_i = 1 | T_i) P(T_i) \geq P(Q_i = 1) - r(\delta, n).
\]

We also note, for an event \( A \) and \( j \leq \bar{L}_n \),
\[
P(A) - r \leq P(A) - P(T_j) P(A | T_j) = P(T_j) P(A | T_j) \leq P(A | T_j).
\]

We again use induction to show
\[
P\left( \sum_{i=1}^{j+1} Q_i > c \right) \geq P\left( \sum_{i=1}^{j} Q_i > c \right) - 2rj, \quad j = 1, \ldots, \bar{L}_n.
\]

Showing the base case is similar to (128). For the induction step,
\[
P\left( \sum_{i=1}^{j+1} Q_i > c \right) \geq P\left( \sum_{i=1}^{j} Q_i > c \right) - r \leq P\left( \sum_{i=1}^{j} Q_i > c | T_{j+1} \right) - r
\]
\[
\geq P\left( \sum_{i=1}^{j} Q_i + Q_{j+1} > c | T_{j+1} \right) - r \geq P\left( \sum_{i=1}^{j} Q_i + Q_{j+1} > c \right) - 2r
\]
\[
\geq P\left( \sum_{i=1}^{j+1} Q_i > c \right) - 2r(j + 1).
\]

With (130), we then can show
\[
P(\pi_n(I_i) > c | T) \geq \mathbb{P}\left( \frac{1}{n} \sum_{i=1}^{L_n} Q_i > c \right)
\]
\[
\geq \mathbb{P}\left( \frac{1}{n} \sum_{i=1}^{L_n} Q_i > c \right) - r \geq \mathbb{P}\left( \frac{1}{n} \sum_{i=1}^{L_n} Q_i > c \right) - 2r \bar{L}_n - r.
\]

Upon taking complements, we arrive at
\[
P(\pi_n(I_i) < c | T) \leq \mathbb{P}\left( \frac{1}{n} \sum_{i=1}^{L_n} Q_i < c \right) + 2r \bar{L}_n + r.
\]
Lemma 11. For all realizations of pathwise bound between time intervals.

In this section, we derive bounds for total number in an interval. Our first estimate is a simple PDMP lemmas on growth

$m$

For our next lemma, we compare recurrence

Taking the supremum over length $\delta$ intervals then yields (135).

Proof. For Species 2, note that $m^2_n(t)$ is decreasing in $t$. As for Species 1, a particle in an interval $I$ of size $\delta$ at time $s \in [t_{k-1}, t_k)$ must have been located, at time $t_{k-1}$, in $I$ in Species 2 or $I + \delta$ in either Species 1 or 2, so that

$$
\mu^2_n(s, I) \leq \mu^2_1(t_{k-1}, I) + \mu^2_1(t_{k-1}, I + \delta) + \mu^2_2(t_{k-1}, I + \delta).
$$

(136)

$$
\mu^2_n(s, I) \leq \mu^2_2(t_{k-1}, I).
$$

(137)

Taking the supremum over length $\delta$ intervals then yields (135).

Proof. For Species 2, note that $m^2_n(t)$ is decreasing in $t$. As for Species 1, a particle in an interval $I$ of size $\delta$ at time $s \in [t_{k-1}, t_k)$ must have been located, at time $t_{k-1}$, in $I$ in Species 2 or $I + \delta$ in either Species 1 or 2, so that

$$
\mu^2_n(s, I) \leq \mu^2_1(t_{k-1}, I) + \mu^2_1(t_{k-1}, I + \delta) + \mu^2_2(t_{k-1}, I + \delta).
$$

(136)

$$
\mu^2_n(s, I) \leq \mu^2_2(t_{k-1}, I).
$$

(137)

Taking the supremum over length $\delta$ intervals then yields (135).

For our next lemma, we compare $m^3_n(t_k)$ with its deterministic analog $m^3_d(t)$, defined through the recurrence

$$
\hat{m}^3_d(t_k) = \hat{m}^3_d(t_{k-1}) + 24C_4 \hat{m}^3_d(t_{k-1})^2, \quad \hat{m}^3_d(0) = m^3_3(0).
$$

(139)

We may use the same reasoning as in Lemma 2 to show that there exists $\delta^p > 0$ such that for $0 < \delta < \delta^p$, we can find $n^p(\delta, \hat{f}) > n^p_1(\hat{f})$ such that for $n > n^p(\delta, \hat{f})$ there exist positive constants $C_5, \hat{C}_5 > 0$ such that

$$
\hat{C}_5 \delta \leq \hat{m}_3(t_k) \leq C_5 \delta.
$$

(140)

For the remaining lemmas in this section, we will assume that $0 < \delta < \delta^p$ and $n > n^p(\delta)$. Our interest is in whether interval growth in the particle system exceeds that of $\hat{m}_3$. Whether this occurs is expressed in the sequence of events

$$
\mathcal{A}_k = \{m^3_n(t_k) > \hat{m}_3(t_k)\}, \quad 0 \leq t_k \leq T_2.
$$

(141)

Our next lemma shows that conditioned under $\mathcal{A}_k$, we can use Lemma 9 to obtain a concentration inequality for $\pi_j$ and consequently $m^3_n(t_k)$.

A similar calculation using (138) yields (122). □

4.3 PDMP lemmas on growth

In this section, we derive bounds for total number in an interval. Our first estimate is a simple pathwise bound between time intervals.

Lemma 11. For all realizations of $X^n(t)$, if $s \in [t_{k-1}, t_k)$, then

$$
m^3_n(s) \leq 3m^3_n(t_{k-1}).
$$

(135)

Proof. For Species 2, note that $m^2_n(t)$ is decreasing in $t$. As for Species 1, a particle in an interval $I$ of size $\delta$ at time $s \in [t_{k-1}, t_k)$ must have been located, at time $t_{k-1}$, in $I$ in Species 2 or $I + \delta$ in either Species 1 or 2, so that

$$
\mu^2_1(s, I) \leq \mu^2_1(t_{k-1}, I) + \mu^2_1(t_{k-1}, I + \delta) + \mu^2_2(t_{k-1}, I + \delta).
$$

(136)

$$
\mu^2_2(s, I) \leq \mu^2_2(t_{k-1}, I).
$$

(137)

Taking the supremum over length $\delta$ intervals then yields (135).

From (35), there is $\tau(f) \geq \tau_0(f)$ such that for $n > \tau(f)$, we can use the constant

$$
C_4 = \inf_{n \geq \tau} N^n_2(T_2(\hat{f})) > N_2(0)/2.
$$

(138)

For our next lemma, we compare $m^3_n(t_k)$ with its deterministic analog $m^3_d(t_k)$, defined through the recurrence

$$
\hat{m}_3(t_k) = \hat{m}_3(t_{k-1}) + 24C_4 \hat{m}_3(t_{k-1})^2, \quad \hat{m}_3(0) = m^3_3(0).
$$

(139)

We may use the same reasoning as in Lemma 2 to show that there exists $\delta^p > 0$ such that for $0 < \delta < \delta^p$, we can find $n^p(\delta, \hat{f}) > n^p_1(\hat{f})$ such that for $n > n^p(\delta, \hat{f})$ there exist positive constants $C_5, \hat{C}_5 > 0$ such that

$$
\hat{C}_5 \delta \leq \hat{m}_3(t_k) \leq C_5 \delta.
$$

(140)

For the remaining lemmas in this section, we will assume that $0 < \delta < \delta^p$ and $n > n^p(\delta)$. Our interest is in whether interval growth in the particle system exceeds that of $\hat{m}_3$. Whether this occurs is expressed in the sequence of events

$$
\mathcal{A}_k = \{m^3_n(t_k) > \hat{m}_3(t_k)\}, \quad 0 \leq t_k \leq T_2.
$$

(141)

Our next lemma shows that conditioned under $\mathcal{A}_k$, we can use Lemma 9 to obtain a concentration inequality for $\pi_j$ and consequently $m^3_n(t_k)$.

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**Lemma 12.** Let $j = 1, 2$. For $0 < t_k \leq T_2$, 

$$
P\left( \sup_{|t| \leq \delta} \pi^n_j(t_k, I) > 12C_4\bar{m}_\delta(t_k-1)^2|A^c_{k-1} \right) \leq \frac{2M}{\delta}\exp(-\tilde{C}_0\delta^3n), \quad (142)$$

where $\tilde{C}_0 = 72C_4^2\tilde{C}_0^2$.

**Proof.** We give a proof for $j = 1$, with the proof for $j = 2$ being nearly identical. First, observe that under $A^c_{k-1}$ there will be at most $\Delta L^n(t_k)n \leq \bar{m}_\delta(t_k-1)n$ mutations during $\Delta t_k = [t_{k-1}, t_k)$, since only particles contained in $[0, \delta)$ at $t_{k-1}$ for Species 1 and 2 may possibly reach the origin in Species 1. Under the event $A^c_{k-1}$, we use Lemma 11 to show mutation probabilities are uniformly bounded from above by the deterministic quantity

$$
P^1_k(\tau, I) = \frac{\mu^2_k(t^-, I + t_k - t)}{N_2(t^-)} \leq 3C_4\bar{m}_\delta(t_k-1). \quad (143)$$

Thus, using From (140), we can then use Lemma 9 with $\bar{C}_0 = 3C_4\bar{m}_\delta(t_k-1)$, and $\varepsilon = 6C_4(\bar{m}_\delta(t_k-1))^2$, which gives

$$
P \left( \max_{t \leq M/\delta} \pi^n_1(t_k, I_t) > 6C_4(\bar{m}_\delta(t_k-1))^2|A^c_{k-1} \right) \leq \frac{2M}{\delta}\exp(-\tilde{C}_0\delta^3n). \quad (144)$$

Note that to apply Lemma 9 we used the fact that $X^n(t)$ is homogeneous. Indeed, we can write the left hand side of (144) as

$$
P \left( \max_{t \leq M/\delta} \pi^n_1(t_k, I_t) > 6C_4(\bar{m}_\delta(t_k-1))^2|\bar{A}^c \right) = P \left( \max_{t \leq M/\delta} \pi^n_1(t_1, I_1) > 6C_4(\bar{m}_\delta(t_k-1))^2|\bar{A}^c \right), \quad (145)$$

where $\bar{A} = \{m^n_\delta(0) > \bar{m}_\delta(t_k-1)\}$ gives requirements on the initial condition so that $\Delta L(t_1) \leq \bar{L}$, and $P_k^1(\tau, I_t) \leq \tilde{\mu}$ pathwise in $\Delta t_1$.

We can extend (144) to hold over all intervals of size less than $\delta$, not just those on a grid. This is done by noting that for any measure $\mu$, if $\mu(I_k) \leq a$ on a uniform grid $I_k$ of size $\delta$, then for any $I$ with $|I| \leq \delta$, $\mu(I) \leq 2a$. Thus, (142) follows from (144) and (140), since

$$
P \left( \sup_{|t| \leq \delta} \pi^n_1(t_k, I_t) > 12C_4\bar{m}_\delta(t_k-1)^2|A^c_{k-1} \right) \leq P \left( \max_{t \leq M/\delta} \pi^n_1(t_k, I_t) > 6C_4(\bar{m}_\delta(t_k-1))^2|\bar{A}^c \right).$$

\[\square\]

We can now derive a concentration inequality which shows that $m^n_\delta(t_k) = O(\delta)$ with high probability.

**Lemma 13.** For $0 \leq t_k \leq T_2$,

$$
P(m^n_\delta(t_k) > C_5\delta) \leq P(A_k) \leq \frac{4Mk}{\delta}\exp(-\tilde{C}_0\delta^3n). \quad (146)$$

**Proof.** The proof follows from induction, in which we assume

$$
P(A_t) \leq \frac{4Mt}{\delta}\exp(-\tilde{C}_0\delta^3n) \quad (147)$$

22
holds for $0 \leq t < k$. The base case for when $k = 0$ follows since $\mathbb{P}(A_0) = 0$. To show the inductive step, we can use Lemma [12] and the recurrence inequalities [104] and [139] to show

\[
\mathbb{P}(A_k | A_{k-1}^c) 
\leq \mathbb{P}\left(\mu_\gamma^n(t_{k-1}) + \sum_{j=1}^{\bar{l}} \sup_{|t| \leq \delta} \pi_\gamma^n(t_k, I) > \bar{m}_\delta(t_{k-1}) + 24C_4\bar{m}_\delta(t_{k-1})^2|A_{k-1}^c) \right) \leq \sum_{j=1}^{\bar{l}} \mathbb{P}\left(\sup_{|t| \leq \delta} \pi_\gamma^n(t_k, I) > 12C_4\bar{m}_\delta(t_{k-1})^2|A_{k-1}^c) \right) \leq \frac{4M}{\delta} \exp(-\tilde{C}_6\delta^3n).
\]

We may then apply (119) with $C = A_k$ and $D = A_{k-1}^c$, and then apply (147) for $l = k - 1$, along with (148) and (140) to obtain the right inequality of (146) (the left inequality follows immediately from (140)).

We finish this subsection with an estimate for total mutations during $\Delta t_k$ analogous to Lemma 3.

**Lemma 14.** For $0 < t_k \leq T_2$, we have

\[
\mathbb{P}(\left| \Delta L^n(t_k) - \mu^n_\gamma(t_{k-1}, [0, \delta]) \right| > C_0\delta^2) \leq \frac{4Mk}{\delta} \exp(-\tilde{C}_7\delta^3n),
\]

with $C_0 = 4C_2C^2_5$ and $\tilde{C}_7 = \tilde{C}_5 \wedge (18C_4^2C_3^3)$.

**Proof.** Denote $\mathcal{M}_\delta^n(t_k)$ as the normalized total number of mutations that affect Species 2 particles in the interval $[0, \delta]$ during time $[t_{k-1}, t_k)$. This may be written as

\[
\mathcal{M}_\delta^n(t_k) = \frac{1}{n} \sum_{i=1}^{\Delta t_k} M^k_i,
\]

where $M^k_i$ is an indicator random variable for the event that the $i$th mutation during $\Delta t_k$ occurs within $[0, \delta)$. All particles of Species 1 located in $[0, \delta)$ at $t_{k-1}$ with reach the origin and trigger a mutation by time $t_k$. The only other particles in the system which potentially hit the origin are those initially located in $[0, \delta)$ in Species 2 which have mutated during $\Delta t_k$. It follows that under all paths in $X^n(t)$,

\[
|\Delta L^n(t_k) - \mu^n_\gamma(t_{k-1}, [0, \delta])| \leq \mathcal{M}_\delta^n(t_k).
\]

Thus proving (149) follows from showing an equivalent estimate on $\mathcal{M}_\delta^n(t_k)$. Toward that end, note that under $A_{k-1}^c$ the number of mutations during $\Delta t_k$ is less than $Ln = C_5\delta n$. Also, the probability of selecting a Species 2 particle to mutate in $[0, \delta]$ at each mutation time during $\Delta t_k$ is bounded by $\tilde{p} = 3C_4C_5\delta$. Let $B_i(q)$ denote an iid stream of Bernoulli random variables with parameter $q \in [0, 1]$. It follows that under $\mathbb{P}(A_{k-1}^c)$, from arguments similar to Lemma 9 that

\[
\mathcal{M}_\delta^n(t_k) \leq ST \frac{1}{n} \sum_{i=1}^{Ln} B_i(\tilde{p}).
\]

For $C_0 = 6C_2C^2_5$, from the Hoeffding inequality,

\[
\mathbb{P}(\mathcal{M}_\delta^n(t_k) > C_0\delta^2 | A_{k-1}^c) \leq \mathbb{P}\left(\sum_{i=1}^{Ln} B_i(\tilde{p})/Ln > \tilde{p}\right) \leq 2\exp(-2\tilde{L}n\tilde{p}^2n) \leq 2\exp(-18C_4^2C_3^3\delta^3n).
\]

The lemma then follows from applying Lemma [13] [119], [147],and [153].
4.4 Difference of total number on an interval

We now begin our estimates comparing the deterministic discretization $\tilde{\mu}_j$ and empirical measures $\mu^n_j$. As in Section 3.2, our focus is on differences of measures restricted to length $\delta$ intervals.

For inequalities related to bounding $P_k^n$ with $h_j^{I,n}$, we will need to consider a modulus of continuity for the deterministic discretization, defined by

$$\tilde{\omega}(s, t_k) = \sup_{I: |I| = \delta} \sum_{j=1}^2 |\tilde{\mu}_j(t_k, I + s) - \tilde{\mu}_j(t_k, I)|, \quad \text{(154)}$$

Fortunately, we can compare $\tilde{\omega}(\delta, t_k)$ with $\omega(\delta, t_k)$, the modulus of continuity for the solutions to the kinetic equations, through the following lemma.

**Lemma 15.** There exists $C_{\delta} > 0$ such that for $0 < s \leq \delta$,

$$\tilde{\omega}(s, t_k) \leq C_{\delta}(\omega(\delta, 0)\delta + \delta^2). \quad \text{(155)}$$

**Proof.** For $j = 1, 2$,

$$|\tilde{\mu}_j(t_k, I + s) - \tilde{\mu}_j(t_k, I)| \leq |\tilde{\mu}_j(t_k, I + s) - \mu_j(t_k, I + s)| + |\mu_j(t_k, I + s) - \mu_j(t_k, I)| + |\mu_j(t_k, I) - \tilde{\mu}_j(t_k, I)|. \quad \text{(156)}$$

Summing over $j$ and taking suprema gives

$$\tilde{\omega}(s, t_k) \leq \omega(s, t_k)\delta + 2h_\delta(t_k). \quad \text{(157)}$$

The result then follows from Lemmas 13 and 16.

We now give a pathwise inequality over $\Delta t_k$ for comparing mutation probabilities.

**Lemma 16.** For $\tau \in \Delta t_k$, there exists $C_{\delta} > 0$ such that for all paths in $X^n(t)$,

$$|P^1_k(\tau, I) - P^1_k(t_k - 1, I)| \leq C_{\delta}(h^n(t_k - 1) + \pi^0_2(t_k) + (m^n(t_k - 1))^2 + \omega(\delta, 0)\delta + \delta^2). \quad \text{(158)}$$

**Proof.** For $j = 1$ (the proof for $j = 2$ is similar), we write

$$|P^1_k(\tau, I) - P^1_k(t_k - 1, I)| = \left| \frac{\mu^n_2(\tau, I + t_k - \tau)}{N^n(\tau)} + \frac{\mu^n_2(t_k - 1, I + \delta)}{N^n(t_k - 1)} \right| \quad \text{(159)}$$

$$\leq C_4(|\mu^n_2(\tau, I + t_k - \tau) - \mu^n_2(t_k - 1, I + \delta)| + |\mu^n_2(t_k - 1, I + \delta)(N^n(\tau) - N^n(t_k - 1))) \quad \text{(158)}$$

$$\leq C_4(|\mu^n_2(\tau, I + t_k - \tau) - \mu^n_2(t_k - 1, I + \delta)| + (m^n(t_k - 1))^2).$$

We may then break up terms further, with

$$|\mu^n_2(\tau, I + t_k - \tau) - \mu^n_2(t_k - 1, I + \delta)| \quad \text{(160)}$$

$$\leq |\mu^n_2(\tau, I + t_k - \tau) - \mu^n_2(t_k - 1, I + t_k - \tau)|$$

$$+ |\mu^n_2(t_k - 1, I + t_k - \tau) - \mu^n_2(t_k - 1, I + t_k - \tau)|$$

$$+ |\tilde{\mu}_2(t_k - 1, I + t_k - \tau) - \tilde{\mu}_2(t_k - 1, I + \delta)| + |\tilde{\mu}_2(t_k - 1, I + \delta) - \mu^n_2(t_k - 1, I + \delta)|$$

$$\leq \pi^n_2(t_k) + 2h^n(t_k - 1) + \tilde{\omega}(\delta - t_k - \tau, t_k - 1).$$

The result follows from applying Lemma 15.
Here we collect previous results to form a high-probability event under which we can bound $h^o_n(t_k)$.

**Lemma 17.** There exist positive constants $C_{10}, C_{11}, C_{12}$ such that the events

\[
\mathcal{C}(t_k) = \left\{ \sup_l \pi^o_l(t_k, I_l) > C_{10} \delta^2 \right\},
\]

\[
\mathcal{L}(t_k) = \cup_{t_k \leq T_2} \left\{ |\Delta L^n(t_k) - \mu^n_1(t_k-1, [0, \delta])| > C_{10} \delta^2 \right\},
\]

\[
\mathcal{D}(t_{k+1}) = \mathcal{C}(t_{k+1})^c \cap \mathcal{A}(t_k)^c \cap \mathcal{L}(t_{k+1})^c.
\]

satisfy

\[
\mathbb{P}(\mathcal{D}(t_k)^c | \mathcal{A}(t_{k-1})^c) \leq \frac{C_{12}}{\delta} \exp(-C_{11} \delta^3 n).
\]

**Proof.** This follows immediately from applying (142) with (151) and (153) to the event

\[
\mathbb{P}(\mathcal{D}(t_k)^c | \mathcal{A}(t_{k-1})^c) \leq \mathbb{P}(\mathcal{C}(t_k) | \mathcal{A}(t_{k-1})^c) + \sum_{k \leq [T_2/\delta]} \mathbb{P}(\mathcal{L}(t_k) | \mathcal{A}(t_{k-1})^c).
\]

The upshot of using $\mathcal{D}(t_k)$ is that we may bound selection probability and total losses by quantities which are $\mathcal{F}(t_{k-1})$ measurable. Furthermore, we may use $\mathcal{D}(t_k)$ for the event $\mathcal{T}$ in Lemma 10 to produce concentration bounds for $h(t_k)$.

**Lemma 18.** Under $\mathcal{D}(t_k)$, there exists $C_{13}$ such that for $t \in \Delta t_k$,

\[
P^n_k(t, I_l) \in [\tilde{p}_n(t_k-1, \tilde{P}_n(t_k-1)), \Delta L^n(t_k) \in [L^n(t_k), \bar{L}^n(t_k)]],
\]

where

\[
\tilde{p}_n(t_k-1) = P^n_k(t_k, I_l) + C_{13}(h^n(t_k-1) + \omega(\delta, 0)\delta + \delta^2),
\]

\[
\bar{p}_n(t_k-1) = P^n_k(t_k, I_l) - C_{13}(h^n(t_k-1) + \omega(\delta, 0)\delta + \delta^2),
\]

\[
\tilde{L}^n(t_k-1) = L^n(t_k) + C_{13} \delta^2,
\]

\[
\bar{L}^n(t_k-1) = L^n(t_k) - C_{13} \delta^2.
\]

**Proof.** We obtain (165)–(166) from Lemma 16 and (167)–(168) follows from Lemma 14.

**Lemma 19.** Let

\[
H^n(t_k) := \delta^3 + \omega(\delta, 0)\delta^2 + \delta h^n(t_k) + \delta^2 \sum_{i < k} h^n(i),
\]

and let $\Pi$ be defined as in (106). Then there are positive constants $C_{16}, \tilde{C}_{16}, C_{17}$ such that

\[
\mathbb{P}(\Pi^n(t_k) > C_{17} H^n(t_k)|\mathcal{D}(t_k)) \leq \frac{C_{16}}{\delta} \exp(-\tilde{C}_{16} \delta^3 n).
\]

**Proof.** We begin with breaking up $\Pi^n$ as

\[
\Pi^n(t_k) \leq \max_{l \leq M/\delta} \left| \frac{\tilde{p}_n(t_k-1, I_l)}{\tilde{N}_n(t_k-1)} \Delta \tilde{L}(t_k) - \mu^n_1(t_k-1, [0, \delta]) P^n_k(t_k-1, I_l) \right| + \sum_{j=1}^2 \max_{l \leq M/\delta} \left| \pi^n_j(t_k, I_l) - \mu^n_1(t_k-1, [0, \delta]) P^n_k(t_k-1, I_l) \right| := G^n(t_k) + \tilde{\Pi}^n(t_k).
\]
From Lemma 18 we can rewrite $\mu^n_i(t_{k-1}, [0, \delta])$ in terms of $L^n(t_{k-1})$ and $\bar{L}^n(t_{k-1})$, and also $P^n_k(t_{k-1}, I_1)$ in terms of $\bar{p}_i(t_{k-1})$ and $\bar{p}_i(t_{k-1})$, from which we can then bound $\bar{\Pi}^n(t_k)$, for some $C_{14} > 0$, by

$$
P(\bar{\Pi}^n(t_k) > \varepsilon | \mathcal{D}(t_k)) 
$$

(172)

$$
\leq \sum_{j=1}^{2} \mathbb{P}(\max_j |\pi^n_j(t_k, I_1) - \mu^n_i(t_{k-1}, [0, \delta]) P^n_k(t_{k-1}, I_1)| > \varepsilon/2 | \mathcal{D}(t_k)) 
$$

$$
\leq \sum_{j=1}^{2} \mathbb{P}(\max_j (\pi^n_j(t_k, I_1) - \mu^n_i(t_{k-1}, [0, \delta]) P^n_k(t_{k-1}, I_1)) > \varepsilon/2 | \mathcal{D}(t_k)) 
$$

$$
+ \mathbb{P}(\min_j (\pi^n_j(t_k, I_1) - \mu^n_i(t_{k-1}, [0, \delta]) P^n_k(t_{k-1}, I_1)) < -\varepsilon/2 | \mathcal{D}(t_k)) 
$$

$$
\leq \sum_{j=1}^{2} \left[ \mathbb{P}(\max_j (\pi^n_j(t_k, I_1) - \bar{L}^n(t_k) \bar{p}^n_i(t_{k-1}))) + C_{14} H^n(t_{k-1}) > \varepsilon/2 | \mathcal{D}(t_k)) 
$$

$$
+ \mathbb{P}(\min_j (\pi^n_j(t_k, I_1) - \bar{L}^n(t_k) \bar{p}^n_i(t_{k-1})) - C_{14} H^n(t_{k-1}) < -\varepsilon/2 | \mathcal{D}(t_k)) \right]. 
$$

For all paths in $\mathcal{D}(t_k)$, by calculations similar to (87)-(90), there exists a positive constant $C_{15} > C_{14}$ such that

$$
G^n(t_k) \leq C_{15} H^n(t_{k-1}). 
$$

(173)

and thus

$$
\mathbb{P}(\Pi^n(t_k) > 8C_{15} H^n(t_{k-1}) | \mathcal{D}(t_k)) \leq \mathbb{P}(\bar{\Pi}^n(t_k) > 4C_{15} H^n(t_{k-1}) | \mathcal{D}(t_k)). 
$$

(174)

Consider a path $\omega : [0, t_{k-1}] \to E$ such that $\omega \in \mathcal{A}(t_{k-1})^c$. We now invoke (174), (172), and Lemma 10 with $\varepsilon(t_{k-1}) = C_{15} H^n(t_{k-1})$, $C_{17} = 8C_{15}$, and $r(\delta, n) = \frac{C_{12} \delta^2}{C_{16}} \exp(-C_{11} \delta^3 n)$ from (163) to obtain

$$
\mathbb{P}(\Pi^n(t_k) > C_{17} H^n(t_{k-1}; \omega) | \mathcal{D}(t_k)) 
$$

(175)

$$
\mathbb{P}(\max_j (\pi^n_j(t_k, I_1) - \bar{L}^n(t_{k-1}; \omega) \bar{p}^n_i(t_{k-1}; \omega)) > \varepsilon^n(t_{k-1}; \delta) | \mathcal{D}(t_k)) 
$$

$$
+ \mathbb{P}(\min_j (\pi^n_j(t_k, I_1) - \bar{L}^n(t_{k-1}; \omega) \bar{p}^n_i(t_{k-1}; \omega)) < -\varepsilon^n(t_{k-1}; \delta) | \mathcal{D}(t_k)) 
$$

$$
\leq \sum_{j=1}^{2} \mathbb{P} \left[ \exp(-2C_{15} H^n(t_{k-1}; \omega)^2 n/\bar{L}^n(t_{k-1}; \omega)) + \frac{2M}{\delta} \left( \frac{r}{1-r} + \frac{2rn\bar{L}^n(t_{k-1}; \omega)}{1-r} + 2rn\bar{L}^n(t_{k-1}; \omega) + r \right) \right] 
$$

(177)

:= J_1(t_{k-1}, n; \omega) + J_2(t_{k-1}, n; \omega). 

(178)

By increasing $n^p$ used in obtain (140), if necessary, elementary calculations show that for $n > n^p$,

$$
J_2(t_{k-1}, n; \omega) \leq \frac{18M}{\delta^2} \exp(-C_{11} \delta^3 n/2) \cdot \bar{L}^n(t_{k-1}; \omega). 
$$

(179)

On the other hand, since $\mathcal{D}(t_k) \subset \mathcal{A}(t_{k-1})^c$, for $X(t_{k-1}) = \omega' \in \mathcal{A}(t_{k-1})$ we have
\[ \mathbb{P}(\Pi^n(t_k) > C_{17} H^n(t_{k-1}; \omega')|\mathcal{D}(t_k)) = 0. \]  

(180)

From the law of total probability,

\[ \mathbb{P}(\Pi^n(t_k) > C_{17} H^n(t_{k-1})|\mathcal{D}(t_k)) \leq \mathbb{E}[J_1(t_{k-1}, n)|\mathcal{A}(t_{k-1})] + \mathbb{E}[J_2(t_{k-1}, n)|\mathcal{A}(t_{k-1})^c] \leq \frac{C_{16}}{\delta} \exp(-\tilde{C}_{16}\delta^5 n) \]

(181)

for a sufficiently small \( \tilde{C}_{16} > 0 \) and sufficiently large \( C_{16} > 0 \). In the last inequality, we used the simple pathwise bound of \( H^n(t_k) \geq \delta^5 \) and that \( L = O(\delta) \) under \( \mathcal{A}(t_{k-1})^c \).

\[ \square \]

4.4.1 Proof of Theorem 3

We consider the events

\[ \mathcal{H}(t_k) = \cup_{t \leq k} \{ h^n(t_k) > h^n(t_{k-1}) + C_{17} H^n(t_{k-1}) \}, \]

\[ \mathcal{B}(t_k; C) = \{ d(\bar{\mu}(t_k), \mu^n(t_k)) > C(\delta + \omega(\delta, 0)) \}. \]

(182)

(183)

Using the same argument given in Lemma 6 under \( \cap_{t_k \leq T_2} \mathcal{H}(t_k) \) and a sufficiently large \( C_{18} \),

\[ h^n(t_k) \leq C_{18}(\delta^2 + \delta \omega(\delta, 0)) \quad t_k \leq T_2. \]

(184)

We may then use (184) with Lemma 7 to show that for a sufficiently large \( C_{19} \),

\[ \cap_{t_k \leq T_2} \mathcal{H}(t_k)^c \subseteq \cap_{t_k \leq T_2} \mathcal{B}(t_k; C_{19})^c \]

(185)

and that for sufficiently large \( C_{20} \) and small \( \tilde{C}_{20} \),

\[ \mathbb{P} \left( \max_{t_k \leq T_2} d(\bar{\mu}(t_k), \mu^n(t_k)) > C_{19}(\delta + \omega(\delta)) \right) = \mathbb{P}(\cup_{t_k \leq T_2} \mathcal{B}(t_k; C_{19})) \]

(186)

\[ \leq \sum_{t_k \leq T_2} \mathbb{P}(\mathcal{H}(t_k)|\mathcal{D}(t_k)) + \mathbb{P}(\mathcal{D}(t_k)^c) \]

\[ \leq \sum_{t_k \leq T_2} \mathbb{P}(\Pi^n(t_k) > C_{17} H^n(t_{k-1})|\mathcal{D}(t_k)) + \mathbb{P}(\mathcal{D}(t_k)^c) \]

\[ \leq \frac{C_{20}}{\delta^2} \exp(-\tilde{C}_{20}\delta^5 n). \]

To conclude, we may replace the maximum in (186) with a supremum. Indeed, since \( \bar{\mu}(t) \) is constant during time intervals \( \Delta t_k \), for \( t \leq T_2 \) and \( K = \left[ \frac{t}{\delta} \right] \),

\[ d(\bar{\mu}(t), \mu^n(t)) \leq d(\bar{\mu}(t_K), \mu^n(t_K)) + d(\mu^n(t), \mu^n(t_K)). \]

(187)

During \( \Delta t_k \), an interval can change its total number by at most \( \sum_{j=1,2} \pi_j(t_k, I) \), and thus

\[ d(\mu^n(t), \mu^n(t_K)) \leq \sum_{I \leq M/\delta} \pi_j(t_k, I) \leq \frac{M}{\delta} \sup_{I \leq M/\delta} \pi_j(t_k, I). \]

(188)
Then for sufficiently large $C_p > 2C_{19}$ and $C_p^2$, and sufficiently small $C_p^3$.

\[
P \left( \sup_{t \leq T_2} d(\tilde{\mu}(t), \mu^n(t)) > C_p^2(\delta + \omega(\delta)) \right) \leq \mathbb{P} \left( \max_{t \leq T_2} d(\tilde{\mu}(t_k), \mu^n(t_k)) > C_{19}(\delta + \omega(\delta)) \right) + \sum_{j=1}^{2} \mathbb{P} \left( \sup_{t \leq M/\delta} \pi_j(t_k, I_t) > \frac{C_p^2 \delta^2}{2M} \right) \leq \frac{C_p^2}{\delta^2} \exp(-C_p^3 \delta^5 n).
\]

Finally, through a similar stitching argument argument appealed to in the proof of Theorem 2 at the end of Section 3, we may extend (189) to any $T' < T(\bar{f})$, which yields Theorem 3.

5 Proof of Theorem 1

For deriving an explicit solution for (6)-(7), we assume $(f_1(x, t), f_2(x, t)) \in Z^2$ for $t \in T(\bar{f})$, and show that such a solution must be given by the explicit expressions (16)-(19), and later verify that this solution is in fact in $Z^2$.

We begin by integrating (7) over space, giving

\[
N_2(t) = N_2(0) - L(t).
\]

This implies that $N_2(t)$ is differentiable, with

\[
\dot{N}_2(t) = -a(t).
\]

Substituting (191) into (7) yields the simple form

\[
f_2(x, t) = \frac{N_2(t)}{N_2(0)} \bar{f}_2(x).
\]

Another substitution of (192) into (6) then gives

\[
f_1(x, t) = \bar{f}_1(x + t) + \int_0^t \frac{\bar{f}_2(x + t - s)}{N_2(0)} a(s) ds.
\]

It remains to express $a(t)$ and $N_2(t)$ in terms of initial conditions. Setting $x = 0$, we arrive at the closed equation

\[
a(t) = \bar{f}_1(t) + \int_0^t \frac{\bar{f}_2(x + t - s) a(s) ds}{N_2(0)}.
\]

Denoting the probability density $\hat{f}_2(s) = \frac{f_2(s)}{N_2(0)}$, we may rewrite (194) as the integral equation

\[
a(t) = \bar{f}_1(t) + \int_0^t a(t - s) \hat{f}_2(s) ds.
\]

Equation (195) is a renewal equation, which has been studied extensively in probability theory (see [4, Chapter XI] for an introduction). It is well-known that there exists a unique solution for (195) given by

\[
a(t) = \sum_{j=0}^{\infty} \hat{f}_2^{(j)}(t) \ast \bar{f}_1(t) := Q \hat{f}_2(t) \ast \bar{f}_1(t),
\]

28
where the exponent \( * (k) \) denotes \( k \)-fold self-convolution. Then \((17)\) follows directly from \((190)\). For a locally bounded density \( p \), the operator \( Q_p(t) \) is also locally bounded, (see Thm. 3.18 of \((11)\)). Thus, it is clear that \( a(t) \) and \( N(t) \) are both positive, locally bounded, and continuous for \( 0 \leq t < T(\tilde{f}) \), and subsequently that \((f_1(x, t), f_2(x, t)) \in \mathbb{Z}^2 \). This completes the derivation for part (a) for Theorem \((11)\).

For showing part (b), the only ambiguity is in establishing for a fixed \( t \in [0, T(\tilde{f})) \), the map \( p \mapsto Q_p(t) \) is in \( C(L^1([0, T(\tilde{f})), \mathbb{R}^+)) \). We will use a probabilistic argument. Let \( X_i^{(p)} \) \( i = 1, 2, \ldots \) denote a sequence of \([0, \infty)\) valued, iid random variables, each with a probability density \( p \in L^1(\mathbb{R}^+) \). The number of renewals up to time \( t < \infty \) is given by

\[
N_p(t) = \sup \left\{ k : \sum_{i=1}^k X_i^{(p)} \leq t \right\}.
\]

(197)

In renewal theory, \( Q_p(t) \) is the well-known renewal density, satisfying

\[
\int_0^t Q_p(t) = \mathbb{E}[N_p(t)] - 1.
\]

(198)

Each term in the sum of \( Q_p(t) \) also has a probabilistic interpretation, with

\[
c_k^{(p)}(t) := \int_0^t f^{* (k)}(t) = \mathbb{P} \left( \sum_{i=1}^k X_i^{(p)} \leq t \right), \quad k \geq 1.
\]

(199)

Estimates for the decay of \( c_k^{(p)}(t) \) as \( k \to \infty \) can be obtained from Markov’s inequality, with

\[
c_k^{(p)}(t) = \mathbb{P} \left( \exp \left( - \sum_{i=1}^k X_i^{(p)} \right) \geq e^{-t} \right) \leq e^{t \mathbb{E}[\exp(-X_1^{(p)})]^k}.
\]

(200) \hspace{1cm} (201)

As \( X^{(p)} \) is a non-deficient random variable, \( \mathbb{P}(X = 0) \neq 1 \). Then \( \mathbb{E}[\exp(-X_1^{(p)})] < 1 \), and thus \( c_k^{(p)}(t) \) decays exponentially as \( k \to \infty \).

To show continuity, fix \( p \in L^1(\mathbb{R}^+) \), \( t \in [0, T(\tilde{f})) \), and let \( \varepsilon > 0 \). From \((200)-(201)\), \( c_k^{p}(t) \) is summable in \( k \), so we may choose a \( K > 0 \) such that

\[
\sum_{i=K}^{\infty} c_i^{(p)}(t) < \varepsilon / 6.
\]

(202)

Since \( \mathbb{E}[\exp(-X_1^{(p)})] \) varies continuously with respect with \( p \) in \( L^1(\mathbb{R}^+) \), a similar calculation to \((200)-(201)\) implies that the map \( p \mapsto c_k^{(p)}(t) \) is also in \( C(L^1(\mathbb{R}^+), \mathbb{R}^+) \) for all \( k \geq 1 \). Furthermore, tail sums of \( c_k^{(p)}(t) \) also vary continuously in the \( p \) variable. Thus, there exists \( \delta > 0 \) such that if \( \bar{p} \in L^1(\mathbb{R}^+) \) satisfies \( \| p - \bar{p} \|_{L^1(\mathbb{R}^+)} < \delta \), then both

\[
\sum_{i=K}^{\infty} c_i^{(\bar{p})}(t) < \varepsilon / 3 \quad \text{and} \quad \sum_{i=1}^{K-1} |c_i^{(p)}(t) - c_i^{(\bar{p})}(t)| < \varepsilon / 2.
\]

(203)

hold. It then follows that

\[
|Q_p(t) - Q_{\bar{p}}(t)| \leq \sum_{i=1}^{K-1} |c_i^{(p)}(t) - c_i^{(\bar{p})}(t)| + \sum_{i=K}^{\infty} c_i^{(p)}(t) + \sum_{i=K}^{\infty} c_i^{(\bar{p})}(t) < \varepsilon.
\]

(204)
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