Neutrino mass hierarchy and three-flavor spectral splits of supernova neutrinos

Basudeb Dasgupta,1 Alessandro Mirizzi,2 Irene Tamborra,1,3,4 and Ricard Tomàs2
1 Max-Planck-Institut für Physik (Werner-Heisenberg-Institut), Föhringer Ring 6, 80805 München, Germany
2II Institut für Theoretische Physik, Universität Hamburg, Luruper Chaussee 149, 22761 Hamburg, Germany
3 Dipartimento Interateneo di Fisica “Michelangelo Merlin”, Via Amendola 173, 70126 Bari, Italy
4 INFN, Sezione di Bari, Via Orahona 4, 70126 Bari, Italy
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It was recently realized that three-flavor effects could peculiarly modify the development of spectral splits induced by collective oscillations, for supernova neutrinos emitted during the cooling phase of a protoneutron star. We systematically explore this case, explaining how the impact of these three-flavor effects depends on the ordering of the neutrino masses. In inverted mass hierarchy, the solar mass splitting gives rise to instabilities in regions of the (anti)nuetrrino energy spectra that were otherwise stable under the leading two-flavor evolution governed by the atmospheric mass splitting and by the 1-3 mixing angle. As a consequence, the high-energy spectral splits found in the electron (anti)neutrino spectra disappear, and are transferred to other flavors. Imperfect adiabaticity leads to smearing of spectral swap features. In normal mass hierarchy, the three-flavor and the two-flavor instabilities act in the same region of the neutrino energy spectrum, leading to only minor departures from the two-flavor treatment.

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I. INTRODUCTION

The neutrino flux from a core-collapse supernova (SN) is a powerful tool to probe fundamental neutrino properties as well as the dynamics of the explosion [1, 2]. The diagnostic role played by neutrinos during a stellar collapse is largely associated with the signatures imprinted on the observable SN neutrino burst by flavor conversions occurring deep inside the star.

It has been understood that the paradigm of neutrino flavor transformation in supernovae [3], based primarily on the Mikheyev-Smirnov-Wolfenstein (MSW) effect with the ordinary matter [4], was incomplete. New surprising and unexpected effects have been found to be important in the region close to the neutrinosphere (see [3, 5] for recent reviews). Here the neutrino density is so high that the neutrino-neutrino interactions dominate the flavor evolution, producing collective oscillations. The most important observational consequence of ν–ν interactions is a swap of the νe and ¯νe spectra with the non-electron νx and ¯νx spectra in certain energy ranges [6, 8].

The development of the spectral swaps is strongly dependent on the original SN neutrino fluxes. In the accretion phase, i.e. t < 0.5 ms after the core-bounce, neutrino number fluxes are expected to be ordered as ϕνe > ϕνμ ≫ ϕντ [5, 11]. In such a scenario, one finds that for normal neutrino mass hierarchy (NH, Δm2atm = m23 − m21 > 0) collective oscillations do not play a significant role. For inverted hierarchy (IH, Δm2atm < 0), the end of collective oscillations is marked by a complete exchange of the e and x flavors for almost all antineutrinos. For neutrinos, the exchange happens only above a characteristic energy fixed by the lepton number conservation, giving rise to a spectral split in their energy distributions [12, 13].

The neutrino number fluxes may be significantly different at later times, i.e. during the cooling phase. In Garching simulations [11], one finds a cross-over among the different ν spectra. As a consequence, the original fluxes exhibit a different ordering Φνe > Φνμ ≫ Φντ. A study of this latter case was performed [16], finding the occurrence of unexpected multiple spectral splits for both neutrinos and antineutrinos, in normal and inverted mass hierarchies. The rich phenomenology of the spectral splits, and its dependence on the original neutrino energy spectra was further explored in the extensive study performed in [17].

Most of the collective flavor dynamics of SN neutrinos can be explained in an effective two-flavor (2ν) framework [18]. Collective oscillations are triggered by an instability in the two-flavor “H-sector” associated with the atmospheric mass-squared difference Δm2atm and the mixing angle θ13. Three-flavor (3ν) effects are due to the “L-sector,” governed by the smaller solar mass splitting Δm2sol = m23 − m21 > 0 and the mixing angle θ12. These effects have been studied for neutrino fluxes typical of the accretion phase [12, 21]. It was recently shown [22] that they are able to trigger collective flavor conversions, even if the mixing angle θ13 is exactly zero. However, apart from this initial kick, no new sizeable effect was found in the subsequent neutrino flavor evolution.

Recently, the 3ν case was studied for a scenario relevant to the cooling phase [23]. It was found that in the inverted mass hierarchy, the presence of the solar sector can “erase” the high-energy spectral splits that would have occurred in the νe and ¯νe spectra for a 2ν flavor evolution governed by the Δm2atm and θ13. Moreover, the final electron antineutrino energy spectrum exhibits a “mixed” nature, i.e. the spectral
find it worthwhile to take a closer look at the three-flavor
energy spectral features are expected to be significantly
easier to observe at neutrino detectors. In these
three-flavor effects are associated with an instability in
the $L$ sector and to the subsequent non-adiabatic flavor
evolution driven by $\Delta m^2_{sol}$. Building on this insight, we
find it worthwhile to take a closer look at the three-flavor
effects during the cooling phase, and to understand the
origin and nature of the 3$\nu$ instability.

We explain how the 3$\nu$ effects are crucially dependent
on the neutrino mass hierarchy. For normal mass
hierarchy, there is no fundamental difference between $H$
and $L$ sectors, since in this case both mass splittings
are positive and the effective in-medium mixing angles
are both small. Thus, even in a 2$\nu$ set-up one would
expect spectral splits driven by the $L$-sector parameters,
and expect them to be exactly where the splits appear
in the case of $H$-sector in normal hierarchy. However
we find that, it is not the case. In fact with typical
SN parameters, collective oscillations fail to produce
any significant conversions for the $L$-sector. This is
so, because the $L$-sector has a lower natural frequency
($\omega_L = \Delta m^2_{sol}/2E$, where $E$ is a typical SN $\nu$ energy) than
the $H$-sector and the collective interaction strength drops
at a rate much faster than it. This does not leave enough
time for the instability to grow and makes the collective
evolution non-adiabatic. Therefore, spectral splits fail to
develop. Nevertheless, the system is in an unstable
situation. Indeed, as we will show, a small perturbation
in the initial conditions is enough to develop the collective
flavor conversions, producing spectral swaps also in this
two-flavor case.

In a realistic situation, this initial perturbation for
the $L$-sector is provided by 3$\nu$ effects that couple this
sector to the $H$-sector. Oscillations in the $H$-sector are
communicated to the $L$-sector, allowing the instability to
grow much faster. The spectral swapping still remains
less adiabatic than in the $H$-sector. As a result, for
normal mass hierarchy - where both $H$ and $L$-sectors
trigger the same instability and “compete” to convert the
high-energy $\nu$ and $\bar{\nu}$ spectra - the $H$-sector wins. Three-
flavor effects do not cause a significant change. The
interesting bit happens for inverted mass hierarchy. In
this case, the $H$-sector and the $L$-sector have instabilities
in different parts of the spectrum and therefore do not
compete with each other. Instead, they “cooperate” and
act on complementary parts of the energy spectra. The
$L$-sector instability catalyzed by the $H$-sector, operates
in the high-energy region without hindrance, and causes
an additional swap, that erases the spectral split found
in the 2$\nu$ flavor evolution. The low adiabaticity in
the $L$-sector is responsible for somewhat smeared splits,
and the effect is particularly important for splits at
higher energies, especially for the antineutrinos. In the
remainder of this paper, we illustrate these aspects using
simple examples and provide a semi-analytical treatment.

The plan of our work is as follows. In Section 2,
we present our formalism for the SN neutrino flavor
evolution and set up our numerical calculations, i.e.
state our inputs for neutrino masses, mixing parameters,
original supernova neutrino energy spectra and luminosities
at late times. In Section 3, we present the 2$\nu$ results
in the $H$ and $L$ sectors, in particular showing the
lack of adiabaticity in the $L$-sector and the role
of the small perturbations in the initial conditions
to circumvent that. In Section 4, we present a complete
3$\nu$ calculation, showing that the $H$-sector catalyzes the
$L$-sector and then instabilities in both sectors develop
- in cooperation for inverted mass hierarchy, and in
competition for normal mass hierarchy. We provide an
estimates of the relative adiabaticity of the low energy
and high energy splits - explaining the mixed spectra
observed in the antineutrino sector. Finally in Section 5,
we comment on our results and conclude.

II. EQUATIONS OF MOTION AND
NUMERICAL FRAMEWORK

Mixed neutrinos are described by matrices of density
$\rho_p$ and $\bar{\rho}_p$ for each (anti)neutrino mode. The diagonal
entries are the usual occupation numbers whereas the off-
diagonal terms encode phase information. The equations
of motion (EoMs) are \cite{3, 24}

$$i\partial_t \rho_p = [H_p, \rho_p], \quad (1)$$

where the Hamiltonian is

$$H_p = \Omega_p + V + \sqrt{2}G_F \int \frac{d^3q}{(2\pi)^3} (\rho_q - \bar{\rho}_q)/(1 - \mathbf{v}_q \cdot \mathbf{v}_q), \quad (2)$$

$\mathbf{v}_p$ being the velocity. The matrix of the vacuum oscillation
frequency for neutrinos is $\Omega_p = (m^2_1, m^2_2, m^2_3)/2|p|
$ in the mass basis. For antineutrinos $\Omega_{\bar{p}} \rightarrow -\Omega_p$. The
matter effect due to the background electron density $n_e$
is represented, in the weak-interaction basis, by $V = \sqrt{2}G_F n_e \text{diag}(1,0,0)$.

In spherical symmetry the EoMs can be expressed as
a closed set of differential equations along the radial
direction \cite{25, 24}. The factor $(1 - \mathbf{v}_q \cdot \mathbf{v}_q)$ in the
Hamiltonian, implies “multi-angle” effects for neutrinos
moving on different trajectories \cite{4}. However, for realistic
supernova conditions the modifications are small, allowing
for a single-angle approximation \cite{24, 25}. We
implement this approximation by launching all neutrinos
with $45^\circ$ relative to the radial directions \cite{14, 25}.

For the numerical illustrations, we take the neutrino
mass-squared differences in vacuum to be $\Delta m^2_{sol} =
2 \times 10^{-3}$ eV$^2$ and $\Delta m^2_{atm} = 8 \times 10^{-5}$ eV$^2$, close to their
current best-fit values \cite{27}. The values of the mixing parameters
relevant for SN neutrino flavor conversions are $\sin^2 \theta_{12} \simeq 0.31$ and $\sin^2 \theta_{13} \simeq 0.04$ \cite{27}. The matter
effect in the region of collective oscillations (up to a few
100 km) can be accounted for by choosing small (matter
suppressed) mixing angles \cite{28}, which we take to be $\theta_{13} =$
\(\bar{\theta}_{12} = 10^{-3}\), and considering as effective mass-square differences in matter \(\Delta m_{\text{atm}}^2 = \Delta m_{\text{atm}}^2 \cos \theta_{13} \approx \Delta m_{\text{atm}}^2\) and \(\Delta m_{\text{sol}}^2 = \Delta m_{\text{sol}}^2 \cos 2\theta_{12} \approx 0.4 \Delta m_{\text{sol}}^2\) \cite{28,29}. We ignore possible subleading CP violating effects \cite{21} by setting \(\delta_{CP} = 0\). MSW conversions typically occur after collective effects have ceased \cite{20,30}. Their effects then factorize and can be included separately. Therefore, we neglect them in the following.

The content of a given neutrino species \(\nu_{\alpha}\) with energy \(E\) at a position \(r\) is given by

\[
\rho_{\alpha\alpha}(E,r) = \frac{F_{\nu_{\alpha}}(E,r)}{F(E,r)}, \tag{3}
\]

where \(F_{\nu_{\alpha}}\) is the energy distribution of \(\nu_{\alpha}\) and \(F\) is the sum of the energy distributions of all flavors, for neutrinos and antineutrinos respectively. For the three relevant SN \(\nu\) energy distributions at the neutrinosphere \(F_{\nu_{\alpha}}^0\), we take

\[
F_{\nu_{\alpha}}^0(E) = \Phi_{\nu_{\alpha}}^0 \varphi_{\nu_{\alpha}}(E), \tag{4}
\]

where \(\Phi_{\nu_{\alpha}}^0 = L_{\nu_{\alpha}}/(E_{\nu_{\alpha}})\) is the number flux, defined in terms of the neutrino luminosity \(L_{\nu_{\alpha}}\) and the neutrino average energy \(\langle E_{\nu_{\alpha}} \rangle\). \(\varphi_{\nu_{\alpha}}(E)\) is the normalized neutrino spectrum \((\int dE \varphi_{\nu_{\alpha}} = 1)\), parametrized as \cite{11}:

\[
\varphi_{\nu_{\alpha}}(E) = \frac{\beta^\beta}{\Gamma(\beta)} E^{\beta-1} e^{-\beta E/\langle E_{\nu_{\alpha}} \rangle}, \tag{5}
\]

where \(\beta\) is a spectral parameter, and \(\Gamma(\beta)\) is the Euler gamma function. The values of the parameters are model dependent \cite{9,10}. For our numerical illustrations, we choose

\[
(\langle E_{\nu_{\alpha}} \rangle, \langle E_{\bar{\nu}_{\alpha}} \rangle, \langle E_{\bar{\nu}_{\beta}} \rangle) = (12, 15, 18) \text{ MeV}, \tag{6}
\]

and \(\beta = 4\), from the admissible parameter ranges \cite{11}. We take ratios of the fluxes at late-times to be \cite{10}

\[
\Phi_{\nu_{\alpha}}^0 : \Phi_{\nu_{\beta}}^0 : \Phi_{\nu_{\tau}}^0 = 0.85 : 0.75 : 1.0, \tag{7}
\]

where we have assumed equal \(\nu\) and \(\bar{\nu}\) neutrino and antineutrino initial fluxes.

The strength of the neutrino-neutrino interactions can be parametrized as \cite{22}

\[
\mu_0 = \sqrt{2} G_F |F_{\nu_{\alpha}}^0 - F_{\nu_{\alpha}}^0|, \tag{8}
\]

where the fluxes are taken at the neutrinosphere radius \(R = 10\) km. Numerically, we assume \(\mu_0 = 7 \times 10^5 \text{ km}^{-1}\). When formally deriving the single-angle approximation, one explicitly obtains that the radial dependence of the neutrino-neutrino interaction strength can be written as \cite{26}

\[
\mu(r) = \frac{\mu_0 R^2}{r^2} C_r, \tag{9}
\]

where the \(r^{-2}\) scaling comes from the geometrical flux dilution, and the collinearity factor

\[
C_r = \left[ \frac{1 - \sqrt{1 - (R/r)^2}}{(R/r)^2} \right]^2 - 1
\]

\[
= \frac{1}{2} \left( \frac{R}{r} \right)^2 \text{ for } r \rightarrow \infty, \tag{10}
\]

arises from the \((1 - \cos \theta)\) structure of the neutrino-neutrino interaction. The asymptotic behavior for large \(r\) agrees with what one obtains by considering that all neutrinos are launched at 45° to the radial-direction \cite{19}. The known decline of the neutrino-neutrino interaction strength, \(\mu(r) \sim r^{-4}\) for \(r \gg R\), is evident.

### III. MULTIPLE SPECTRAL SPLITS IN A TWO-FLAVOR SCENARIO

#### A. \(2\nu\) H-system

We start our investigation with the flavor evolution in the \(H\)-sector characterized by \((\Delta m_{\text{atm}}^2, \theta_{13})\). As explained in \cite{16}, spectral swaps can develop around energies \(E = E_c\) of the original neutrino spectra (except at \(E = 0\)), where spectra of different flavors cross each other, i.e. at the crossing points

\[
F_{\nu_{\alpha}}(E_c) - F_{\nu_{\alpha}}(E_c) = 0, \tag{11}
\]

for neutrinos and antineutrinos, respectively. A given crossing point is unstable if

\[
d(F_{\nu_{\alpha}} - F_{\nu_{\beta}})/dE < 0 \text{ for } \text{IH}, \tag{12}
\]

\[
d(F_{\nu_{\beta}} - F_{\nu_{\gamma}})/dE > 0 \text{ for } \text{NH},
\]

and analogously for antineutrinos.

In Fig. 1, we show the results of the flavor evolution for neutrinos (left panel) and antineutrinos (right panel) in inverted mass hierarchy. In the upper panels we show the initial and final \(\nu_e\) and \(\nu_x\) energy spectra, while in the lower panels we show the electron neutrino survival probability \(P_{\nu_e}\). Following the instability conditions stated above, one finds that in IH the spectral swap would develop for neutrinos around \(E_c \approx 13\) MeV and for antineutrinos around \(E_c \approx 9\) MeV. The development of a spectral swap in the middle of the energy spectra produces two splits in the final \(\nu\) spectra. In particular, in the \(\nu_e\) final spectrum the swapped region is between \(5\) MeV and \(23\) MeV, while for \(\bar{\nu}_e\) is between \(3\) MeV and \(17\) MeV.

In Fig. 2, the corresponding results for normal hierarchy are shown. In this case the only unstable crossing point is at \(E_c \rightarrow \infty\) in the tail of the energy spectra, therefore the resulting \(\nu\) and \(\bar{\nu}\) spectra exhibit only a high-energy swap and a single split each. In this case, the split is at \(E \approx 23\) MeV for \(\nu_e\), and at \(E \approx 17\) MeV for \(\bar{\nu}_e\).

From this example, we realize that in the two mass hierarchies the instabilities occur around different and well-separated crossing points. This leads to spectral swaps that occur in different energy ranges for the two mass hierarchies. In fact, these ranges are non-overlapping and almost complementary, i.e. the high-energy ends of the swaps in IH are the low-energy ends.
neutrinos\(F_n\)(a.u.)
e, init\(x, init\)e, fin\(x, fin\)antineutrinos

FIG. 1: Flavor evolution in 2\(\nu\) H-sector for neutrinos (left panels) and antineutrinos (right panels) in inverted mass hierarchy. Upper panels: the \(\nu_e\) (red) and \(\nu_x\) (blue) energy-spectra, before (dashed curves) and after (continuous curves) the collective oscillations. Lower panels: survival probability \(P_{ee}\) for electron (anti)neutrinos. The grey-regions represent the range in which the spectral swap occurs.

of the swaps in NH. This implies that if we take the \(\nu\) spectra swapped by the conversions in IH, as an input for conversions in NH, these latter would swap also the high-energy spectrum of the electron (anti)neutrinos, giving the impression that the high-energy split has been “erased.” This observation will play an important role in our understanding of the full three-flavor evolution.

B. 2\(\nu\) L-system

We now consider the L-system. Since \(\Delta m^2_{sol} > 0\), we expect a behavior similar to that of the H-system in normal hierarchy. However, we find that in this case no flavor conversion occurs. This has to be attributed to two reasons - insufficient growth of the instability, and lack of adiabaticity. The strength of the instability is given by the off-diagonal components in the density matrix. For a simple pendular system with energy \(E\), the time-period for growth of off-diagonal components is a few times the pendular time-period \(\tau_{\text{pend}}\)\[8, 16\]

\[
\tau_{\text{pend}} \approx \sqrt{\frac{2E}{\Delta m^2_{sol}\mu}}. \tag{13}
\]

which scales logarithmically with the small in-medium mixing angle. This time period is roughly \(\sqrt{\Delta m^2_{atm}/\Delta m^2_{sol}} \approx 8\) times larger for the L-sector, causing the instability to develop relatively slowly. The slow growth is further exacerbated by a relatively fast decrease in collective neutrino interaction strength \(\mu\).

During the spectral swapping phenomenon, the spectrum near the crossing acts like an inverted pendulum\[16\]. The swap sweeps through the spectrum on each side of the crossing, and the modes at the edge of the swap precess at an average oscillation frequency \(\kappa\)\[16\]. Adiabaticity requires\[2\]

\[
|\frac{d\ln \mu}{dr}| < \kappa. \tag{14}
\]

Since the collective neutrino interaction strength \(\mu\) goes as \(r^{-4}\), the rate at which neutrino refractive effects are decreasing is \(|d\ln \mu/dr| \approx 1/50 \text{ km}^{-1}\) at \(r \approx 200 \text{ km}\), approximately where the swapping takes place. One finds \(\kappa \approx \Delta m^2_{sol}/2E \approx 1/400 \text{ km}^{-1}\) for a typical energy \(E \approx 32 \text{ MeV}\) in the region of the swap, so the adiabaticity condition is not met.

Nevertheless, the L-system has the same instability as the H-system in NH, around the crossing point \(E_c \to \infty\) of the (anti)neutrino energy spectra. Indeed, if we

FIG. 2: Flavor evolution in the 2\(\nu\) H-sector for neutrinos (left panels) and antineutrinos (right panels) in normal mass hierarchy. Upper panels: the \(\nu_e\) (red) and \(\nu_x\) (blue) energy-spectra, before (dashed curves) and after (continuous curves) the collective oscillations. Lower panels: survival probability \(P_{ee}\) for electron (anti)neutrinos. The grey-regions represent the range in which the spectral swap occurs.
sufficient to take off-diagonal seeds in the density matrix $|\rho^0_{\alpha\beta}| = \epsilon \times (\rho^0_{e\alpha} + \rho^0_{e\beta})$, (15) with $\epsilon \geq 10^{-5}$, to produce significant flavor conversions. However, relative to the $H$-sector in the normal hierarchy, the swap is less sharp. This is naturally expected because the adiabaticity does not change significantly by taking an initial perturbation.

In Fig. 3, we show the effects of these artificially triggered flavor conversions on the electron neutrinos (left panels) and antineutrinos (right panels). Upper panels show the energy spectrum, while the lower panels display the neutrino survival probability. We find that it is sufficient to take off-diagonal seeds in the density matrix

$$|\rho^0_{\alpha\beta}| = \epsilon \times (\rho^0_{e\alpha} + \rho^0_{e\beta}),$$

with $\epsilon \geq 10^{-5}$, to produce significant flavor conversions. However, relative to the $H$-sector in the normal hierarchy, the swap is less sharp. This is naturally expected because the adiabaticity does not change significantly by taking an initial perturbation.

In Fig. 4, we show the $|\rho_{\alpha\beta}|$ component of the $\nu$ density matrix for $E = 32$ MeV for three cases: (i) the $H$-system with NH, (ii) the $L$-system with $\epsilon = 10^{-6}$, and (iii) the $L$-system seeded with $\epsilon = 10^{-4}$. We realize that the growth of the off-diagonal component is almost absent for the $L$-system with $\epsilon = 10^{-6}$. Even when there are adequately large off-diagonal elements, as in the case of $\epsilon = 10^{-4}$, the growth is relatively slow compared to the $H$-system, so that significant flavor changes can start only at large radius ($r \gtrsim 300$ km). Consequently, due to the violation of adiabaticity, collective flavor conversions do not have enough time to develop complete splits before the effects of the neutrino-neutrino interactions become negligible.

### IV. MULTIPLE SPECTRAL SPLITS IN A THREE-FLAVOR SCENARIO

#### A. Speed-up of the $\Delta m^2_{sol}$-instability

Equipped with the insights of the previous section, we are ready to analyze the behavior of the flavor conversions in the complete three-flavor scenario. We work in the rotated basis $(\nu_e, \nu_x, \nu_y) = R^T(\theta_{23}) (\nu_e, \nu_{\mu}, \nu_\tau)$ [18]. This is equivalent to take $\theta_{23} = 0$ in the neutrino mixing matrix (which makes no difference to $\nu_e$ and $\bar{\nu}_e$ evolution, if $\mu$ and $\tau$ flavors are treated identically). The vacuum Hamiltonian is written as [18]

$$\Omega(E) = \frac{\Delta m^2_{\text{sol}}}{2E} \left( \begin{array}{ccc} s^2_{13} & 0 & c_{13}s_{13} \\ 0 & 0 & 0 \\ c_{13}s_{13} & 0 & c^2_{13} \end{array} \right) + \frac{\Delta m^2_{\text{atm}}}{2E} \left( \begin{array}{ccc} c^2_{13}s^2_{12} & c_{12}c_{13}s_{12} & -c_{13}s^2_{12}s_{13} \\ c_{12}c_{13}s_{12} & c^2_{12} & -c_{12}s_{12}s_{13} \\ -c_{13}s^2_{12}s_{13} & -c_{12}s_{12}s_{13} & s^2_{12}s^2_{13} \end{array} \right),$$

(16)

where $c_{ij} = \cos \bar{\theta}_{ij}$ and $s_{ij} = \sin \bar{\theta}_{ij}$. In the limit of $\theta_{13} = 0$, the $\nu_e - \nu_\mu$ ($H$) and the $\nu_e - \nu_\tau$ ($L$) sectors are completely decoupled.

When the neutrino-neutrino interaction is strong, all the density matrix mode $\rho_\nu$ stay pinned to each other.

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1 The speed-up of flavor instabilities under the effects of very small seeds in the initial conditions was already pointed out in [31].
exhibiting collective flavor conversions. Now we review the factorization of the $H$ sector from other sub-leading contributions, as shown in [18]. For each energy mode, we rewrite our density matrix and Hamiltonian as

$$\rho = \rho^{(0)} + \rho^{(1)},$$  \hspace{1cm} (17)

$$H = H^{(0)} + H^{(1)},$$  \hspace{1cm} (18)

where the superscript $(0)$ refers to off-components in the $L$-sector and all diagonal components, while $(1)$ to all others, namely

$$\rho^{(0)} = \begin{pmatrix} \rho_{ee} & \rho_{ex} & 0 \\ \rho_{ex}^* & \rho_{xx} & 0 \\ 0 & 0 & \rho_{yy} \end{pmatrix},$$  \hspace{1cm} (19)

and

$$\rho^{(1)} = \begin{pmatrix} 0 & 0 & \rho_{ey} \\ 0 & 0 & \rho_{xy} \\ \rho_{ey}^* & \rho_{xy}^* & 0 \end{pmatrix}.$$  \hspace{1cm} (20)

Analogous expressions hold for $H^{(0)}$ and $H^{(1)}$.

Note from Eq. (18) that $H^{(0)}$, which contains the $e - x$ block, causes oscillations due to $\Delta m^2_{sol}$, while $H^{(1)}$, containing the $e - y$ off-diagonal terms, gives $\Delta m^2_{atm}$-driven oscillations. Putting this decomposition into the equations of motion, one finds [18]

$$i\dot{\rho}^{(0)} = [H^{(0)}, \rho^{(0)}] + [H^{(1)}, \rho^{(1)}],$$  \hspace{1cm} (21)

$$i\dot{\rho}^{(1)} = [H^{(1)}, \rho^{(0)}] + [H^{(0)}, \rho^{(1)}].$$  \hspace{1cm} (22)

Analogous equations hold for the antineutrinos. Note the interesting structure of the EoMs, that is an outcome of the commutation relations - $\dot{\rho}^{(0)}$ depends only on commutators $[H^{(0)}, \rho^{(0)}]$ and $[H^{(1)}, \rho^{(1)}]$, while $\dot{\rho}^{(1)}$ depends only on the cross-terms [18].

In a pure $2\nu$ $L$-system evolution, $H^{(1)}$ and $\rho^{(1)}$ are zero. Thus Eq. (21) has only the first term on r.h.s., and Eq. (22) is irrelevant. Once off-diagonal components of $\rho^{(0)}$, driven by $H^{(0)}$, develop asymmetrically in neutrinos and antineutrinos, the instability grows under the action of $H^{(0)}$ (proportional to $\Delta m^2_{sol}$), i.e. the pendular timescale is $\tau_{\text{pend}} \sim 1/\sqrt{\omega L \mu}$. The addition of off-diagonal elements to $\rho^{(0)}$ kick-starts the process.

In a complete $3\nu$ system, $H^{(1)}$ is non-zero and thus produces the off-diagonal components in $\rho^{(0)}$ and $\rho^{(1)}$ more quickly. Not only do initial off-diagonal terms get generated in the $L$-sector, but also those terms grow at a much faster rate. The growth is speeded up by the second term in the equation of motion Eq. (21), i.e. by $H^{(1)}$ which induces oscillations $\Delta m^2_{atm}$-dependent at the leading order. Therefore, it leads to a growth of the instability at almost the same rate as for the $H$-system, and much faster than an isolated $L$-system.

In Fig. 5, we show the diagonal components of the neutrino density matrix $\rho_{ee}, \rho_{xx}, \rho_{yy}$ (upper panel) and the off-diagonal $\rho_{ex}, \rho_{xy}$ (lower panel) for a given energy mode with $E = 32$ MeV as a function of $r$ in inverted mass hierarchy. The initial behavior is qualitatively similar to the one in normal hierarchy (not shown). The pendular oscillations of $\rho_{ee}$ begin to develop at $r \simeq 35$ km for the $e - x$ sector, and proceed as pure $2\nu$ transitions till $r \simeq 50$ km. Up to this point, $\rho_{ex}$ has not evolved significantly. All off-diagonal components increase rapidly, but the $\rho_{ex}$ starts to develop only after $\rho_{ey}$, and saturates at $r \simeq 60$ km, when $e - x$ conversions start. Note that, $\rho_{ex}$ grows faster than a $2\nu$ $L$-system (shown in Fig. 4), as predicted. The $\theta_{13}$-coupling between the $L$ and $H$ sectors induces three-flavor effects in the neutrino conversions: The initial kick, associated with $\Delta m^2_{atm}$, is necessary to trigger the instability in the $L$-system.

**B. Inverted mass hierarchy**

In Fig. 6, we show the complete development of the $\rho_{ee}, \rho_{xx}$ and $\rho_{yy}$ components of the density matrix in inverted mass hierarchy for neutrinos (left panels) and antineutrinos (right panels) for four representative energy modes. We observe that once the conversions have been started, the different energy modes for the three neutrino species oscillate in phase, confirming the collective behavior of the flavor conversions. However, the final fate of $\rho_{ee}, \rho_{xx}$ and $\rho_{yy}$ depends on their energy.
As a result of the three-flavor effects, the $\nu_e$ mixes with both $\nu_x$ and $\nu_y$. Therefore, the $\nu_e$ flavor conversions can be described in terms of the combined effects of the $L$ and $H$ two-neutrino systems. We find that the effects of the $H$-sector on the $\nu_e - \nu_x$ conversions saturate before the ones of the $L$-sector, as expected by the hierarchy between the two mass splittings. As we have discussed in Section III-A, $\Delta m^2_{\text{atm}} < 0$ and $\Delta m^2_{\text{sol}} > 0$ are expected to process complementary parts of the neutrino energy spectra. Therefore, their effects do not interfere in the same energy range. In particular, in the part of the neutrino energy spectra unstable under the effects of $\Delta m^2_{\text{atm}}$, the $\rho_{xx}$ and $\rho_{yy}$ would swap, while $\rho_{xx}$, which has been perturbed from its stable equilibrium configuration, would relax to it again. Conversely, in the part of the neutrino energy spectra unstable under the effects of $\Delta m^2_{\text{sol}}$, the $\rho_{ee}$ and $\rho_{xx}$ swap, while $\rho_{yy}$ comes back to its initial value.

In the three-flavor space, the neutrino ensemble behaves like a pendulum. Once it is perturbed from its initial configuration, it would evolve toward its stable equilibrium position which may be different for different energy modes. In inverted mass hierarchy, the highest energy modes relax to the $x$ state, while the intermediate ones to the $y$ state, while lowest energy modes remain in the $e$ flavor.

For the modes represented in Fig. 6, we see that for $E = 2.5$ MeV none of the three neutrino species is affected by significant flavor changes. For $E = 7.5$ MeV the flavor conversions produce a swap between $\rho_{ee}$ and $\rho_{yy}$, while $\rho_{xx}$ comes back to its original value. At 23 MeV, we are in the transition region between $e - y$ and $e - x$ conversions, therefore $\rho_{yy}$ and $\rho_{xx}$ are both partially swapped into $\rho_{ee}$. Finally, at $E = 40$ MeV, $\rho_{ee}$ and $\rho_{xx}$ exchange their initial values, while $\rho_{yy}$ returns to its original value. Therefore, the behavior of the diagonal components of the density matrix at different energies would produce a single split in the $\nu_e$ energy spectrum, since all the $\nu_e$ modes at sufficiently high energy would swap with either $\nu_y$ or $\nu_x$. Therefore, the high-energy spectral split at $E \simeq 23$ MeV observed in inverted mass hierarchy in the $2\nu$ evolution of Sec. III-A is washed-out by the three-flavor effects. In the antineutrinos (right panels), we find an analogous behavior. At $E = 2.5$ MeV $\rho_{ee}$, $\rho_{xx}$ and $\rho_{yy}$ are essentially unchanged. At $E = 13$ MeV the flavor conversions produce a swap between $\rho_{ee}$ and $\rho_{yy}$, while $\rho_{xx}$ comes back to its original value. For $E = 23$ MeV, $\rho_{yy}$ returns to its original value, while $\rho_{ee}$ and $\rho_{xx}$ tend to exchange their initial values. However, as we will discuss later, due to the less...
adiabatic behavior around the splitting region, the $\tilde{\nu}_e$-$\tilde{\nu}_x$ swap is not complete. Finally, for $E = 40$ MeV $\rho_{ee}$ and $\rho_{xx}$ completely convert into each other and $\rho_{yy}$ is stable.

In the upper panels of Fig. 7, we show the neutrino (left panels) and antineutrino (right panels) spectra before and after the complete 3$\nu$ flavor conversions in inverted hierarchy. In the lower panel we show the corresponding $P_{ee} = P(\nu_e \to \nu_e)$, $P_{xx} = P(\nu_x \to \nu_x)$, and $P_{yy} = P(\nu_y \to \nu_y)$ probabilities, which can be defined approximately as the off-diagonal elements of the density matrix are small at the end of the collective evolution. Namely, these three probabilities read

$$P(\nu_e \to \nu_\alpha) = \frac{\rho_{e\alpha}^f - \rho_{e\alpha}^0}{\rho_{e\alpha}^f - \rho_{e\alpha}^0}, \quad (23)$$

where 0 and $f$ indicate the initial and final values of $\rho$, respectively. We first consider the neutrino flavor evolution.

As we have already discussed, the effects of the L and H systems process complementary parts of the neutrino energy spectra. Indeed, as we can see from the conversion probabilities, in the energy range where $P_{ex} = 1$, we have $P_{xy} = 0$ and vice versa. In particular, the solar mass splitting $\Delta m^2_{sol}$ induces an electron survival probability $P_{ee} = 0$ at high-energies, erasing the H-sector high-energy split at $E \simeq 23$ MeV, as was pointed out in Ref. [23]. Thus, the final electron neutrino spectrum reads

$$F_{\nu_e}^f = P_{ee} F_{\nu_e}^0 + (1 - P_{ee}) F_{\nu_x}^0 \approx \begin{cases} F_{\nu_e}^0 & \text{for } E \lesssim 5 \text{ MeV} \\ F_{\nu_x}^0 & \text{for } E \gtrsim 5 \text{ MeV} \end{cases}, \quad (24)$$

Conversely, the behavior of the $y$ and $x$ spectra is the same as that of the non-electron species in 2$\nu$ flavor evolution in IH and NH, respectively, i.e.

$$F_{\nu_y}^f = P_{xy} F_{\nu_e}^0 + (1 - P_{xy}) F_{\nu_y}^0 \approx \begin{cases} F_{\nu_e}^0 & \text{for } E \lesssim 5 \text{ MeV} \\ F_{\nu_x}^0 & \text{for } 5 \text{ MeV} \lesssim E \lesssim 23 \text{ MeV} \\ F_{\nu_y}^0 & \text{for } E \gtrsim 23 \text{ MeV} \end{cases}, \quad (25)$$

and

$$F_{\nu_x}^f = P_{xx} F_{\nu_e}^0 + (1 - P_{xx}) F_{\nu_x}^0 \approx \begin{cases} F_{\nu_e}^0 & \text{for } E \lesssim 23 \text{ MeV} \\ F_{\nu_x}^0 & \text{for } E \gtrsim 23 \text{ MeV} \end{cases}. \quad (26)$$

Indeed, the $\nu_e$ and $\nu_x$ spectra will be affected respectively only by the L or the H sector, since $x - y$ conversions are strongly suppressed as evident from the vacuum Hamiltonian in Eq. (14). One should note that while the high-energy spectral split is no longer present in the $\nu_e$ final energy spectrum, $\nu_y$ and $\nu_x$ final spectra still present this high-energy feature. Therefore, MSW effects in supernova, vacuum mixing and Earth effects that occur later can make the high-energy split disappear in the observable electron neutrino spectra at Earth.

Moving on to the antineutrinos, we realize that the $e-x$ swap is not sharp, due to the imperfect adiabaticity in the L sector. This implies that the swap of $\tilde{\nu}_e$ and $\tilde{\nu}_x$ spectra at intermediate energies (14 MeV $\lesssim E \lesssim$ 40 MeV) is not complete and this explains the "mixed" $\tilde{\nu}_e$ spectrum, observed also in the numerical simulations in Ref. [22]. This effect has to be attributed to the low adiabaticity of the $\Delta m^2_{sol}$-induced conversions, which is particularly visible for antineutrinos. The adiabaticity of a spectral split depends on the condition in Eq. (14). The $\tilde{\nu}_e$ and $\tilde{\nu}_x$ spectra are closer to each other than the corresponding neutrino spectra. This implies that in the high-energy region, where the swap takes place,

$$|F_{\tilde{\nu}_e}(E) - F_{\tilde{\nu}_x}(E)| \ll |F_{\nu_e}(E) - F_{\nu_x}(E)|. \quad (27)$$

In the final phases of the swapping dynamics, the neutrino and antineutrino spectra evolve quite independently, and the precession frequencies of the two blocks are not governed by a common $\mu$, but by individual $\mu$'s proportional to the flux differences in Eq. (27). They behave essentially as two uncoupled oscillators because the neutrino-neutrino interaction $\mu$ is now smaller than the frequency difference of the two blocks. One can see this clearly in the numerical simulations of Ref. [16], shown at Ref. [32]. The frequency $\tau^{-1}_\text{pend} \approx \sqrt{2/\mu}$ is lower for antineutrinos [16], and thus adiabaticity tends to be broken more severely for antineutrinos.

C. Normal mass hierarchy

In Fig. 8, we show the complete development of the $\rho_{ee}$, $\rho_{xx}$ and $\rho_{yy}$ components of the density matrix in normal hierarchy for neutrinos (left panels) and for antineutrinos (right panels) for different energy modes. For both neutrinos and antineutrinos, the swapping dynamics occurs mainly between the $\rho_{ee}$ and the $\rho_{yy}$, while $\rho_{xx}$ experiences only nutations around its initial equilibrium value, before relaxing completely to it when the flavor conversions are saturated. Therefore, the flavor evolution is close to the 2$\nu$ $H$-case in normal hierarchy discussed in Sec. III-A. Indeed, both $\Delta m^2_{atm}$ and $\Delta m^2_{sol}$ are positive. Therefore, the L-sector would behave as a replica of the H-sector, but with a smaller mass splitting. In this condition, both H and L sectors process the same regions of the electron neutrino energy spectra. However, the hierarchy between the two mass splittings produces the dominance of the $\nu_e - \nu_y$ swaps, while conversion effects in the $\nu_e - \nu_x$ sector remain inhibited by adiabaticity violation. The only region where the $L$-instability can compete with $H$ is very close to the split. In particular, for low-energy neutrino modes ($E = 2.5$, 18 MeV in Fig. 8) the $\rho_{ee}$ comes back to its initial value, while at higher energies ($E = 23$, 40 MeV) exchanges its initial value with $\rho_{yy}$. For the antineutrinos, $\rho_{ee}$ remains at its initial value for the modes at $E = 2.5$ MeV.
At $E = 18$ MeV, we are around the splitting region where the adiabaticity in the $H$-system is more severely violated, and the $e$-$y$ conversions are not complete. Under these conditions, the $L$ instability can also play a role, producing a weak swap in $\rho_{xx}$. Finally, at $E = 40$ MeV, conversions again occur only between $e$ and $y$ states, producing a complete swap of the initial $\rho_{ee}$ and $\rho_{yy}$ values.

In Fig. 9, we show the neutrinos (left panels) and antineutrino (right panels) spectra before and after the complete $3\nu$ flavor conversions in NH. In the lower panels we represent the corresponding $P_{ee}$, $P_{ex}$ and $P_{ey}$ probabilities. Once more, we realize that the flavor evolution can be mostly described in terms of two-flavor $\nu_e - \nu_y$ transitions, while the role of $\nu_e - \nu_x$ conversions is subleading. The spectral splitting features are remarkably similar to the ones found for the $2\nu$ $H$-system in normal hierarchy (Sec. III-A). In particular, the $\nu_e$ spectrum presents a single split at $E \simeq 23$ MeV and $\bar{\nu}_e$ spectrum around $E \simeq 17$ MeV. Only close to the splitting regions, we find subleading effects associated with $\nu_e - \nu_x$ conversions.

V. CONCLUSIONS

Collective neutrino flavor conversions in supernovae, associated with neutrino-neutrino interactions, have been recognized to induce peculiar spectral swaps among the different neutrino species. The development of these features is associated with instabilities in the flavor space. In particular, these instabilities would develop around the crossing points of the original SN neutrino fluxes. Then, the neutrino mass hierarchy determines if a crossing point is unstable under the effects of the collective oscillations. A particularly intriguing case is the one in which the original SN neutrino fluxes exhibit an ordering with $\Phi_{\nu_x} \sim \Phi_{\nu_e} \sim \Phi_{\bar{\nu}_e}$, possible during the cooling phase. In this case, the two-flavor study realized by [16], found the occurrence of multiple spectral splits for both neutrinos and antineutrinos, depending on the neutrino mass hierarchy. A recent numerical exploration of this case performed in [23], has found that when three-flavor effects are taken into account in inverted mass hierarchy, the high-energy spectral swaps observed in the $2\nu$ evolution are erased by effects related to $\Delta m^2_{\text{sol}}$.

Motivated by this intriguing result, in our paper we have performed a detailed study of the three-flavor
effects in the collective oscillations for supernova neutrino spectra typical of the cooling phase. We have found that the effects of $\Delta m^2_{\odot}$ in the three-flavor evolution are important only in inverted mass hierarchy. In this case, the presence of $\Delta m^2_{\odot}$ gives rise to instabilities in regions of the neutrino energy spectra that were stable under the two-flavor evolution governed by $\Delta m^2_{\text{atm}}$ and $\delta_{13}$. Therefore, the combinations of these two different instabilities would produce a wash-out of the high-energy splitting features in the $\nu_e$ and $\bar{\nu}_e$ spectra. Conversely, in normal mass hierarchy the three-flavor instabilities and the two-flavor one act in the same regions of the neutrino energy spectrum, leading only to minor departures from the two-flavor evolution. Essentially, the system behaves like a pendulum in $3\nu$ flavor space. It can topple towards either the $\nu_y$ state or the $\nu_x$ state. In inverted hierarchy, when the $H$ and $L$ instabilities are in different regions of energy, the pendulum topples towards $\nu_y$ for the $H$-instability, and towards $\nu_x$ for the $L$-instability. In normal hierarchy, when the instabilities are in the same region of energy, the pendulum topples towards $\nu_y$, as the $L$-instability is relatively non-adiabatic. As a consequence, in inverted mass hierarchy the electron (anti)neutrino spectrum at the end of the collective oscillations would present only a very low-energy ($E \lesssim 5$ MeV) splitting feature, being completely swapped to the original non-electron spectra at higher energies.

We wish to emphasize that the high-energy splitting features may survive in the observable electron (anti)neutrino spectrum at Earth, even in inverted hierarchy. Indeed, MSW matter effects in SN, vacuum mixing and Earth effects would further mix the $\nu_x$ and $\nu_y$ spectra. The non-electron spectrum at the end of the collective oscillations, still contains high-energy splitting features, since for the non-electron species the collective flavor conversions have occurred essentially as in the two-flavor case. Therefore, especially for neutrinos, which have sharper spectral swaps, the electron neutrino signal at Earth could still present observable splitting features at high energy. Also, due to the lower neutrino luminosity at sufficiently late times, the initial collective interaction strength $\mu_0$ can be somewhat lower than is assumed in Ref.[23] and this work. We found that in certain regions of the spectral parameter space, reducing $\mu_0$ by a factor of 10 makes the three-flavor effect disappear due to a stronger adiabaticity violation in the $L$ sector. In principle, this effect could produce interesting signatures in the time evolution of the SN neutrino signal.

In conclusion, the non-linear equations that govern the flavor evolution of neutrinos emitted during a stellar collapse are a continuous source of surprises and new effects. During this last year, dramatic changes have occurred in the picture consolidated after the initial exploration of collective supernova neutrino oscillations. The discovery of this new three-flavor effect is the most recent of these changes. After our study, it appears that its impact on the collective neutrino flavor conversions is conceptually and quantitatively well under control.

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