ALGEBRAIC MONOIDS WITH AFFINE UNIT GROUP ARE AFFINE

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ABSTRACT. In this short note we prove that any irreducible algebraic monoid whose unit group is an affine algebraic group is affine.

1. INTRODUCTION

Let $k$ be an algebraically closed field of arbitrary characteristic. An algebraic monoid is an algebraic variety $M$ with an associative product $M \times M \rightarrow M$ which is a morphism of algebraic varieties, such that there exists a neutral element $1$ for this product. In this case, it can be proved that the unit group $G(M)$ – i.e. the group of invertible elements – is an algebraic group, open in $M$ (see \cite[Thm. 1]{8} and Lemma 4).

It is easy to show that the action $(G(M) \times G(M)) \times M \rightarrow M$, $((a, b), m) \mapsto amb^{-1}$ is regular, with open orbit $G(M) \cong (G(M) \times G(M))/\Delta(G(M))$, where $\Delta(G(M))$ is the diagonal. In other words, $M$ is a $G(M)$-embedding. Moreover, it is a simple embedding, i.e. there exists an unique closed orbit, namely the center of $M$ – the minimum ideal of $M$ – (see \cite[Thm. 1]{8} and Lemma 4).

It is well known that if $G$ is a quasi-affine algebraic group, then $G$ is affine (see for example \cite[Thm. 7.5.3]{3}); in \cite[Thm. 4.4]{7}, Renner proved the analog for algebraic monoids, namely that any quasi-affine algebraic monoid is affine. In particular, if $M$ is an affine algebraic monoid, then $G(M)$ is quasi-affine and hence affine. Conversely, in \cite[Prop. 1]{8} it is proved that if $M$ is an irreducible affine embedding of a (necessarily affine) algebraic group $G$, then $M$ is an algebraic monoid of unit group $G(M) = G$. These observations lead naturally to the following conjecture (presented as an open problem by Renner in \cite{7}), communicated to the author by E.B. Vinberg:

Let $M$ be an irreducible algebraic monoid whose unit group $G(M)$ is affine, then $M$ is also affine.

A partial affirmative answer was given in \cite{8}, where it is proved that any irreducible reductive monoid – i.e. with reductive unit group – is affine, and those monoids are classified in combinatorial terms. In this note we give an affirmative answer to the above conjecture (see Theorem 2). Our methods are based on a generalization to the context of algebraic monoids of results by F. Knop, H. Kraft, D. Luna and T. Vust about line bundles over affine algebraic groups (\cite{6}). In the last section, we deal with the non-irreducible case, showing that any algebraic monoid $M$ with unit group affine and dense in $M$ is an affine algebraic variety.
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2. MAIN RESULTS

We begin this section by recalling a well known fact about algebraic monoids (see [7]).

Lemma 1. Let \( M \) be a normal irreducible algebraic monoid with affine unit group \( G \). Then \( M \) is quasi-projective.

Proof. Since \( M \) is a simple \( G \)-embedding, of closed orbit its center, we can apply Sumihiro’s theorem ([9]) and obtain a finite dimensional \((G \times G)\)-module \( V \) and a \((G \times G)\)-equivariant open immersion \( M \hookrightarrow \mathbb{P}(V) \).

Let \( X \) be an algebraic variety, and \( \pi : L \to X \) the line bundle associated to an invertible coherent sheaf on \( X \). Recall that the zero section \( \sigma_0 : X \to L \) is defined in the following way: let \( U \subset X \) be a trivializing open subset of \( L \), and identify \( \pi^{-1}(U) \cong U \times \mathbb{k} \). Then \( \sigma_0|_U : U \to U \times \mathbb{k}, \sigma_0(x) = (x, 0) \) (it is an easy exercise to prove that this is well defined, see for example [5, p. 128]). We will denote as \( L^* = L \setminus \sigma_0(X) \).

The following key result is due to Demazure and Fujita (see [1, Lemma 1.1.13], [2] or [3]).

Lemma 2. Let \( X \) be a projective normal algebraic variety, \( \mathcal{L} \) an ample invertible sheaf, and \( \pi : L \to X \) the line bundle associated to an invertible coherent sheaf on \( X \). Recall that the zero section \( \sigma_0 : X \to L \) is defined in the following way: let \( U \subset X \) be a trivializing open subset of \( L \), and identify \( \pi^{-1}(U) \cong U \times \mathbb{k} \). Then \( \sigma_0|_U : U \to U \times \mathbb{k}, \sigma_0(x) = (x, 0) \) (it is an easy exercise to prove that this is well defined, see for example [5, p. 128]). We will denote as \( L^* = L \setminus \sigma_0(X) \).

The following result is a generalization of [6, Lemmas 4.2 and 4.3].

Lemma 3. Let \( G \) be an affine algebraic group and \( X \) a projective normal \( G \)-embedding, \( \pi : L \to X \) be a \((G \times G)\)-linearized line bundle, and consider the commutative diagram:
If we denote as \( H = (i^*L)^* = (i^*L) \setminus \sigma_0(G) \), then \( H \cong G \times \k^* \) and \( L^* \) is an \( H \)-embedding.

**Proof.** Since \( L \) is \((G \times G)\)-linearized, then \( G \) acts on the fibre \( \pi^{-1}(1) \cong \k \) as the diagonal \( \Delta(G) \subset G \times G \), i.e. \( g \cdot l = (g, g) \cdot l \) for all \( l \in \pi^{-1}(1) \). Hence, \( G \) acts by multiplication with a character \( \lambda : G \to \k^* \). Moreover, for all \( g \in G \) and \( l \in \pi^{-1}(g) \) we have that
\[
(a, g^{-1}ag) \cdot l = ((1, g^{-1})(a, a)(a, g)) \cdot l = (1, g^{-1})(\lambda(a)(1, g) \cdot l) = \lambda(a)l.
\]

Extend \( \lambda : G \to \k^* \) to \( G \times G \) by \( \tilde{\lambda}(g, g') = \lambda(g) \), and change the linearization by considering \((a, b) \star l = \tilde{\lambda}^{-1}(a, b) \cdot l = \lambda^{-1}(a, b) \cdot l \). Then \( L \) is trivial over \( G \), and \( H = (i^*L)^* \cong G \times \k^* \) is an algebraic group such that \( H 	imes H \) acts on \( L^* \) by \((l, s)(g, g') \cdot l = st^{-1}(g, g') \star l\); it is clear that \( H \) is an open orbit for this action. \( \square \)

**Theorem 1.** Let \( M \) be a normal irreducible algebraic monoid with unit group an affine algebraic group \( G \). Then there exists a \((G \times G)\)-linearized line bundle \( \pi : N \to M \), such that \( N^* \) is an affine algebraic monoid with unit group \( H = \pi^{-1}_N(G) \cong G \times \k^* \). Moreover, \( \pi : N^* \to M \) is a morphism of algebraic monoids, and is the quotient of \( N^* \) by \( \pi^{-1}(1) \cong \k^* \).

**Proof.** By Sumihiro’s theorem, there exists an open \((G \times G)\)-equivariant immersion \( \varphi : M \hookrightarrow X \), where \( X \) is a projective \( G \)-embedding. It is clear that we can suppose that \( X \) is normal, and that there exists a very ample invertible \((G \times G)\)-linearizable sheaf \( \mathcal{L} \) on \( X \) (see [12] Proposition 2.4). Let \( \pi : L \to X \) be the line bundle associated to the dual of \( \mathcal{L} \), and \( N = \varphi^*(M) \) its restriction to \( M \).

By Lemma [12] \( N \) is a \( H = \pi^{-1}_N(G) \)-embedding. Let \( \mu_1 : H \times L^* \to L^*, \mu_1(h, l) = (h, 1) \cdot l, \mu_2 : L^* \times H \to L^*, \mu_2(l, h) = (1, h^{-1}) \cdot l \). Since both \( \mu_1 \) and \( \mu_2 \) coincide with the product on \( H \) when restricted to \( H \times H = (H \times L^*) \cap (L^* \times H) \), they induce a morphism \( \mu : U = H \times L^* \cap L^* \times H \to L^* \). Since \( L^* = \text{Spec} R(X, L) \setminus \{0\} \) is a quasi-affine normal variety by Lemma [2] and clearly \( \text{codim}(L^* \times L^*) \setminus U \geq 2 \), we can extend the morphism \( \mu \) to a morphism \( \mu : L^* \times L^* \to \text{Spec} R(X, L) \).

It suffices to prove that \( \mu(N^* \times N^*) \subseteq N^* \). Indeed, if this is the case then \( \mu \) is an associative product in \( N^* \), since it is associative on the open subset \( H \times H \subset N^* \times N^* \times N^* \). Then \( N^* \) is a quasi-affine algebraic monoid, and it follows from the result of Renner cited at the introduction (see [14] Thm. 4.4) that \( N^* \) is an affine algebraic monoid.

In order to prove that \( \mu(N^* \times N^*) \subseteq N^* \), consider \( u, v \in N^* \). There exists an affine open subset \( V \subseteq M \) such that \( \pi(u) \pi(v) \in V \) and \( \pi^{-1}(V) \cong V \times \k^* \). Let \( m : M \times M \to M \) be the product and consider \( W = m^{-1}(V) \subset M \times M; \) let \( W' = \pi^{-1}(W) \cap (H \times N^*) \). Then \( W' \) is an open subset of \( \pi^{-1}(W) \), with complement of codimension greater than 2, such that \( \mu(W') \subset \pi^{-1}(V) \). Since \( N^* \) and hence \( \pi^{-1}(W) \) are normal, it follows that \( \mu|_{W'} : W' \to \pi^{-1}(V) \) extends to a
morphism $\tilde{\mu}: \pi^{-1}(W) \to \pi^{-1}(V)$. Since both $\mu$ and $\tilde{\mu}$ are continuous functions and $\mu |_{W'} = \tilde{\mu} |_{W'}$, it follows that $\mu |_{\pi^{-1}(W)} = \tilde{\mu}$; in particular, $\mu(u, v) \in \pi^{-1}(V) \subset N^*$. By construction, the map $\pi: N^* \to M$ is a morphism of algebraic monoids, with central kernel $\pi^{-1}(1) \cong \mathbb{k}^*$. Hence, $M$ is a quotient of $N^*$ by $\mathbb{k}^*$.

**Theorem 2.** Let $G$ be an affine algebraic group and $M$ an algebraic monoid with unit group $G$, affine algebraic group. Then $M$ is affine.

**Proof.** We can assume without loss of generality that $M$ is normal (see for example [8, Lemma 1]) Applying Theorem 1 we deduce that $M$ is the quotient of an affine algebraic variety by an algebraic torus, and hence it is affine. \hfill $\square$

### 3. The non-irreducible case

Let $G$ be a non-connected affine algebraic group, and assume that $M$ is an algebraic monoid with unit group $G$. In order to obtain a better control of the geometry of $M$ it is natural to impose the density condition $G = M$, as the following examples show:

**Examples**

1. Let $S$ be an arbitrary algebraic variety and $s_0 \in S$. Then $m: S \times S \to S$, $m(s, t) = s_0$ is an associative product. If $M$ is an arbitrary algebraic monoid, then the products on $M$ and $S$ extend to a product $\mu$ on $M \cup S$ (disjoint union) by $\mu(a, s) = s$ for all $a \in M$ and $s \in S$. Then $M \cup S$ is an algebraic monoid, of unit group $G(M)$ and zero $s_0$.

2. Assume now that $M$ has a zero $0$, and consider the equivalence relationship on $M \cup S$ induced by $0 \sim s_0$. Then $\mu$ induces a product $\tilde{\mu}$ on $N = (M \cup S)/\sim$, in such a way that $N$ is an algebraic monoid with unit group $G(M)$. Observe that $N$ can be realized as the closed subvariety $N \cong (M \times \{s_0\}) \cup \{0\} \times S \subset M \times S$.

The following lemma is an easy generalization of [8, Thm. 1], where the case of irreducible algebraic monoids is treated, hence we omit the proof.

**Lemma 4.** Let $M$ be an algebraic monoid of unit group $G$, dense in $M$. Then $M$ is a simple $G$-embedding, with unique closed orbit the center of $M$. \hfill $\square$

The following theorem generalizes [8, Prop. 2], where it is proved that if an algebraic monoid verifies the density condition $G(M) = M$, then any two irreducible components are isomorphic.

**Theorem 3.** Let $M$ be an algebraic monoid with affine dense unit group $G$ and center $Y$. Let $G = \bigcup_{i=1}^n G_i$, where $1 \in G_1$, be the decomposition in irreducible components of $G$; then $M = \bigcup_{i=1}^n M_i$. If we set $M_i = G_i \cdot Y$, $i = 1, \ldots, n$, then $M_i$ is an affine algebraic monoid of unit group $G_i$, and $M_i \cong M_1$ as an algebraic variety for all $i = 1, \ldots, n$. In particular, $M$ is an affine algebraic variety.

Moreover, $M_i \cap Y \subset G_i \cdot Y_1 = Y_1 \cdot G_i$, where $Y_1$ denotes the center of $M_1$.

**Proof.** Consider $\mu: M \times M \to M$, the product on $M$; then for all $j = 1, \ldots, n$, $M_1 \cdot M_j = \mu(M_1 \times M_j)$ is irreducible and contains $1 \cdot M_j = M_j$. Hence $M_1 \cdot M_j = M_j$; in particular, $M_1$ is an algebraic monoid with unit group $G_1$ and it follows from Theorem 2 that $M_1$ is an affine algebraic variety. Moreover, if $g_i \in G_i$, then
$\ell_{g_i} : M_1 \rightarrow M_i$, $\ell_{g_i}(m) = g_im$ is an isomorphism with inverse $\ell_{g_i}^{-1}$, and thus $M$ is an affine algebraic variety.

In order to prove the last assertion, observe that $G_1 \cdot Y_1 \subset M_1 \cdot Y_1 \subset M_i \cap Y$, and that since $G_1$ is normal in $G$, it follows that $G_i \cdot Y_1 = Y_1 \cdot G_i$. □

**Corollary 1.** Let $M$ be an algebraic monoid with zero $0 \in M$ satisfying the density condition. Then every irreducible component of $M$ contains $0$. □

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