FORWARD-BACKWARD EVOLUTION EQUATIONS
AND APPLICATIONS

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Abstract. Well-posedness is studied for a special system of two-point boundary value problem for evolution equations which is called a forward-backward evolution equation (FBEE, for short). Two approaches are introduced: A decoupling method with some brief discussions, and a method of continuation with some substantial discussions. For the latter, we have introduced Lyapunov operators for FBEEs, whose existence leads to some uniform a priori estimates for the mild solutions of FBEEs, which will be sufficient for the well-posedness. For some special cases, Lyapunov operators are constructed. Also, from some given Lyapunov operators, the corresponding solvable FBEEs are identified.

1. Introduction. In this paper, we consider the following system of evolution equations:

\[
\begin{align*}
\dot{y}(t) &= Ay(t) + b(t, y(t), \psi(t)), \\
\dot{\psi}(t) &= -A^*\psi(t) - g(t, y(t), \psi(t)), \\
y(0) &= x, \quad \psi(T) = h(y(T)),
\end{align*}
\]  

(1)

where \( A : \mathcal{D}(A) \subseteq X \to X \) generates a \( C_0 \)-semigroup \( e^{A t} \) on a real separable Hilbert space \( X \) (identified with its dual \( X^* \)), with

\[
(e^{A t})^* = e^{A^* t}, \quad t \geq 0,
\]

being the adjoint semigroup generated by \( A^* \) (the adjoint operator of \( A \)), and \( b, g, \) and \( h \) being suitable maps. The above could be called a two-point boundary value problem, mimicking a similar notion for ordinary differential equations. We see that the equation for \( y(\cdot) \) is an initial value problem which should be solved forwardly, and the equation for \( \psi(\cdot) \) is a terminal value problem which should be solved backwardly. Therefore, inspired by the so-called forward-backward stochastic differential equations (FBSDEs, for short, see [17, 25, 26] for details), we prefer to call (1) a forward-backward evolution equation (FBEE, for short). In common occasions, two-point boundary value problem is related to certain eigenvalue problems, for which the well-posedness of the problem might not be the goal, instead, one might be more interested in the existence of solutions, not necessarily the uniqueness. See,

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for examples, \([3, 32, 6]\), and see also \([7]\) for some other considerations. Whereas, in this paper, we are interested in the well-posedness of \((1)\). On the other hand, our system has a special structure, involving one forward evolution equation and one backward evolution equation. Hence, we use the name FBEE to distinguish the current situation from other situations in the literature.

A pair of functions \((y(\cdot), \psi(\cdot))\) is called a strong solution of \((1)\) if these functions are differentiable almost everywhere, with the property

\[
y(t) \in \mathcal{D}(A), \quad \psi(t) \in \mathcal{D}(A^*), \quad \text{a.e. } t \in [0, T],
\]

and the equations are satisfied almost everywhere. A pair \((y(\cdot), \psi(\cdot))\) is called a mild solution (or a weak solution) to FBEE \((1)\) if the following system of integral equations are satisfied:

\[
\begin{aligned}
  y(t) &= e^{A t} x + \int_0^t e^{A(t-s)} b(s, y(s), \psi(s)) ds, \\
  \psi(t) &= e^{A^*(T-t)} h(y(T)) + \int_t^T e^{A^*(s-t)} g(s, y(s), \psi(s)) ds,
\end{aligned}
\]

\[t \in [0, T].\] (2)

Note that in the case \(A\) is bounded, \((1)\) and \((2)\) are actually equivalent, and thus, a mild solution \((y(\cdot), \psi(\cdot))\) is actually a strong solution.

Our study of the above system is mainly motivated by the study of optimal control theory. It is known that for a standard optimal control problem of an evolution equation with, say, a Bolza type cost functional, by applying the Pontryagin maximum/minimum principle, one will obtain an optimality system of the above form whose solution will be a candidate for the optimal trajectory and its adjoint \([14]\). Therefore, solvability of the above type system is important, at least for optimal control theory of evolution equations.

Roughly speaking, when \(T\) is small enough, or the Lipschitz constants of the involved functions are small enough, one can show that FBEE \((1)\) will have a unique mild solution, by means of contraction mapping theorem. On the other hand, if \((1)\) is the optimality system (obtained via Pontryagin maximum/minimum principle) of a corresponding optimal control problem for which an optimal control exists, then this FBEE admits a mild solution, which might not be unique. Further, if the corresponding optimal control has an optimal control and the optimality system admits a unique mild solution, then this solution can be used to construct the optimal control(s). Hence, under proper conditions, FBEE \((1)\) could admit a (unique) mild solution, without restriction on the length of the time horizon \(T\), and/or the size of the Lipschitz constants of the involved functions. This is actually the case if the FBEE is the optimality system of a linear-quadratic (LQ, for sort) optimal control problem satisfying proper conditions \([14]\).

In this paper, we will study the (unique) solvability of FBEE \((1)\) under some general conditions. Two approaches will be introduced: decoupling method and method of continuation. The former is inspired by the so-called invariant embedding which can be traced back to \([1, 5, 4]\). Such a method was used in the study of FBSDEs \((15, 17)\), for details). The latter is inspired by the method of continuity for elliptic partial differential equations \((see, e.g. [10])\), and FBSDEs \((11, 25, 19, 26)\). Due to the nature of FBEE \((1)\), some technical difficulties exist in applying either of these methods. We will briefly present some main idea of the decoupling method and will relatively more carefully present the method of continuation.
The rest of this paper is organized as follows. In Section 2, we will present some preliminary results, including a main motivation from optimal control theory. Linear FBEEs are carefully discussed in Section 3. In Section 4, a brief description on the decoupling method will be given. In Section 5, we will introduce the so-called Lyapunov operator which is adopted from [26] (for FBSDEs). The existence of Lyapunov operators lead to some uniform a priori estimates for the mild solutions of our FBEE. Well-posedness of FBEEs will be established in Section 6. In Section 7, we will construct some Lyapunov operators through which some well-posed FBEEs will be identified. In Section 8, we will briefly discuss some extensions of our main results. In Section 9, several illustrative examples will be presented. Finally, some concluding remarks will be made in Section 10.

2. Preliminaries. Throughout of this paper, we let $X$ be a separable real Hilbert space, with the norm $\| \cdot \|$ and the inner product $\langle \cdot , \cdot \rangle$. We identify the dual $X^*$ with $X$. The set of all bounded linear operators from $X$ to itself is denoted by $L(X)$. The set of all self-adjoint operators on $X$ is denoted by $S(X)$ and the set of all positive semi-definite operators on $X$ is denoted by $S^+(X)$. For the notational simplicity, when there is no confusion, we will not distinguish between $\lambda$ and $\lambda I$ (for any $\lambda \in \mathbb{R}$). For example, we use $\lambda - A$ to denote $\lambda I - A$. Also, if $F$ is in $S^+(X)$, we denote it by $F \geq 0$; if $F - cI \geq 0$, we simply denote it by $F \geq c$, and $F \leq c$ means $-F \geq -c$. Next, we denote $C([0,T];X) = \left\{y : [0,T] \to X \mid y(\cdot) \text{ is continuous}\right\}$, and

$$\|y(\cdot)\|_\infty = \sup_{t \in [0,T]} \|y(t)\|, \quad \forall y(\cdot) \in C([0,T];X).$$

For convenience and definiteness of our presentation, we introduce the following standing assumptions:

(H0)' $A : D(A) \subseteq X \to X$ generates a $C_0$-semigroup $e^{At}$ on a separable Hilbert space $X$.

(H0) In addition to (H0)', either

$$A^* = A, \quad (3)$$

with the spectrum $\sigma(A) \subseteq \mathbb{R}$ of $A$ satisfying

$$\sup \sigma(A) \equiv \sup \Re \sigma(A) = -\sigma_0 < 0, \quad (4)$$

or

$$A^* = -A, \quad (5)$$

for which it holds: $\sigma(A) \subseteq i\mathbb{R}$ and thus

$$\sup \Re \sigma(A) = \sigma_0 = 0. \quad (6)$$

Case (3) corresponds to the heat equation (or second order parabolic equations) with proper lower order terms and proper boundary conditions. Case (5) corresponds to the wave equation (or second order hyperbolic equations) with proper boundary conditions, without damping. Some extensions of the results presented in this paper are possible. But for the moment, we prefer not to get into the most general situations, for the simplicity of our presentation. We should keep in mind that for the case $A^* = A$, one has $\sigma_0 > 0$ and for the case $A^* = -A$, one has $\sigma_0 = 0.$
Let us now look at our main motivation of studying our FBEEs. Consider the following controlled system:

\[
\begin{align*}
\dot{y}(t) &= Ay(t) + f(t, y(t), u(t)), \quad t \in [0, T], \\
y(0) &= x,
\end{align*}
\]  
with cost functional of Bolza type:

\[
J(x; u(\cdot)) = \int_0^T f^0(s, y(s), u(s))ds + f^1(y(T)).
\]  

In the above, \( f : [0, T] \times X \times U \to X, f^0 : [0, T] \times X \times U \to \mathbb{R}, f^1 : X \to \mathbb{R} \) are suitable maps, with \( U \) being a separable metric space. We call \( x \in X \) an initial state, \( u(\cdot) \) a control, and \( y(\cdot) \) a state trajectory, respectively. Denote

\[
\mathcal{U} = \{ u : [0, T] \to U \mid u(\cdot) \text{ is measurable} \}.
\]

This is the set of all admissible controls. Under some mild conditions, for any \( x \in X \) and \( u(\cdot) \in \mathcal{U} \), state equation (7) admits a unique mild solution \( y(\cdot) \equiv y(\cdot; x, u(\cdot)) \), i.e., the solution to the following integral equation:

\[
y(t) = e^{At}x + \int_0^t e^{A(t-s)}f(s, y(s), u(s))ds, \quad t \in [0, T],
\]  
and the cost functional \( J(x; u(\cdot)) \) is well-defined. Then one can pose the following optimal control problem.

**Problem (C).** For any initial state \( x \in X \), find a \( \bar{u}(\cdot) \in \mathcal{U} \) such that

\[
J(x; \bar{u}(\cdot)) = \inf_{u(\cdot) \in \mathcal{U}} J(x; u(\cdot)).
\]  

Any \( \bar{u}(\cdot) \in \mathcal{U} \) satisfying (10) is called an optimal control, the corresponding \( \bar{y}(\cdot) \equiv y(\cdot; x, \bar{u}(\cdot)) \) is called an optimal state trajectory and \((\bar{y}(\cdot), \bar{u}(\cdot))\) is called an optimal pair.

With the above setting, we have the following standard result. To simplify the presentation, we assume that the involved maps \( f, f^0, f^1 \) have all the required measurability and smoothness. The readers are referred to [14] for details.

**Proposition 2.1. (Pontryagin’s Minimum Principle)** Let (H0)’ hold and let \((\bar{y}(\cdot), \bar{u}(\cdot))\) be an optimal pair of Problem (C). Then the following minimum condition holds:

\[
\langle \psi(t), f(t, y(t), u(t)) \rangle + f^0(t, y(t), u(t)) = \min_{u \in U} \left[ \langle \psi(t), f(t, y(t), u) \rangle + f^0(t, y(t), u) \right], \quad t \in [0, T],
\]

where \( \psi(\cdot) \) is the mild solution to the following adjoint equation:

\[
\begin{align*}
\dot{\psi}(t) &= -A^*\psi(t) - f_y(t, \bar{y}(t), \bar{u}(t))^*\psi(t) - f^0_y(t, \bar{y}(t), \bar{u}(t)), \quad t \in [0, T], \\
\psi(T) &= f^1_y(\bar{y}(T)),
\end{align*}
\]  
i.e., the following holds:

\[
\psi(t) = e^{A^*(T-t)}f^1_y(\bar{y}(T)) + \int_t^T e^{A^*(s-t)}\left[ f_y(s, \bar{y}(s), \bar{u}(s))^*\psi(s) + f^0_y(s, \bar{y}(s), \bar{u}(s)) \right]ds, \quad t \in [0, T].
\]
Note that (7) and (12) form a system with the minimum condition (11) bringing in the coupling. Suppose there exists a map \( \varphi : [0, T] \times X \times X \to U \) such that
\[
\langle \psi, f(t, y, \varphi(t, y, \psi)) \rangle + f_0(t, y, \varphi(t, y, \psi)) = \min_{u \in U} \left[ \langle \psi, f(t, y, u) \rangle + f_0(t, y, u) \right].
\]
Then we obtain the following system (dropping the bar in \( \bar{y}() \))
\[
\begin{aligned}
\dot{y}(t) &= Ay(t) + f(t, y(t), \varphi(t, y(t), \psi(t))), & & t \in [0, T], \\
\dot{\psi}(t) &= -A^*\psi(t) - f_y(t, y(t), \varphi(t, y(t), \psi(t)))^*\psi(t) \\
&\quad - f_0^y(t, y(t), \varphi(t, y(t), \psi(t))), & & t \in [0, T], \\
y(0) = x, & \quad \psi(T) = f_y^1(y(T)).
\end{aligned}
\]  
(14)

This is called the optimality system of Problem (C), which is an FBEE of form (1) with
\[
\begin{aligned}
b(t, y, \psi) &= f(t, y, \varphi(t, y, \psi)), \\
g(t, y, \psi) &= f_y(t, y, \varphi(t, y, \psi))^*\psi + f_0^y(t, y, \varphi(t, y, \psi)), \\
h(y) &= f_y^1(y).
\end{aligned}
\]
If \((y(\cdot), \psi(\cdot))\) is a mild solution of FBEE (14), then \(y(\cdot)\) will be a candidate of optimal trajectory and \(\varphi(\cdot, y(\cdot), \psi(\cdot))\) will be a candidate of optimal control.

Let us now look an interesting special case of the above. To this end, we let \(U\) also be a real Hilbert space, and
\[
\begin{aligned}
f(t, y, u) &= F(t, y) + B(t)u, & f_1^1(y) = G(y), \\
f_0(t, y, u) &= Q(t, y) + \langle S(t)y, u \rangle + \frac{1}{2}\langle R(t)u, u \rangle,
\end{aligned}
\]
for some suitable maps \(F(\cdot, \cdot), B(\cdot), G(\cdot), Q(\cdot, \cdot), S(\cdot), \text{ and } R(\cdot)\). Then the state equation becomes
\[
\begin{aligned}
\dot{y}(t) &= Ay(t) + F(t, y(t)) + B(t)u(t), & t \in [0, T], \\
y(0) = x,
\end{aligned}
\]
and the cost functional takes the form:
\[
J(x; u(\cdot)) = \int_0^T \left[ Q(y(t)) + \langle S(t)y(t), u(t) \rangle + \frac{1}{2}\langle R(t)u(t), u(t) \rangle \right] dt + G(y(T)).
\]  
(16)

Note that the right-hand side of the state equation is affine in \(u(\cdot)\) and the integrand in the cost functional is up to quadratic in \(u(\cdot)\). We therefore refer to the corresponding optimal control problem as an affine-quadratic optimal control problem (AQ problem, for short). For finite-dimensional case, general AQ problem was studied in [24]. In current case, the adjoint equation reads
\[
\begin{aligned}
\dot{\psi}(t) &= -A^*\psi(t) - F_y(t, y(t))^*\psi(t) - Q_y(t, y(t)) - S(t)^*u(t), & t \in [0, T], \\
\psi(T) &= G_y(y(T)).
\end{aligned}
\]
By the minimum condition
\[
\langle \psi(t), B(t)u(t) \rangle + \langle S(t)y(t), u(t) \rangle + \frac{1}{2}\langle R(t)u(t), u(t) \rangle
\]
\[
= \min_{u \in U} \left[ \langle \psi(t), B(t)u \rangle + \langle S(t)y(t), u \rangle + \frac{1}{2}\langle R(t)u, u \rangle \right], & t \in [0, T],
\]
with
we obtain, assuming the invertibility of $R(t)$,
\[ u(t) = -R(t)^{-1} [B(t)\psi(t) + S(t)y(t)], \quad t \in [0, T]. \]

Therefore, the corresponding optimality system reads as follows:
\[
\begin{aligned}
\dot{y}(t) &= Ay(t) + F(t, y(t)) - B(t)R(t)^{-1}S(t)y(t) - B(t)R(t)^{-1}B(t)\psi(t), \\
\dot{\psi}(t) &= -A^*\psi(t) - [Q_y(t, y(t)) - S(t)^*R(t)^{-1}S(t)y(t)] \\
&\quad - [F_y(t, y(t))^* - S(t)^*R(t)^{-1}B(t)^*]\psi(t), \\
y(0) &= x, \quad \psi(T) = G_y(y(T)).
\end{aligned}
\]

When
\[
\begin{aligned}
y \mapsto F(t, y) &\text{ is linear,} \\
y \mapsto Q(t, y) \text{, } y \mapsto G(y) &\text{ are convex,}
\end{aligned}
\]
the corresponding optimal control problem is referred to as
linear-convex problem, which was studied in [28, 29, 30]. See also [31] for some investigations on
finite-dimensional two-person zero-sum differential games of linear state equation with
non-quadratic payoff/cost functional where the convexity of $y \mapsto Q(t, y)$ and $y \mapsto G(y)$
were not assumed. Further, if
\[ F(t, y) \equiv 0, \quad Q(t, y) = \frac{1}{2}\langle Q(t)y, y \rangle, \quad G(y) = \frac{1}{2}\langle Gy, y \rangle, \]
for some $Q : [0, T] \to S(X)$ and $G \in S(X)$, the problem is reduced to a classical LQ
problem. In this case, the optimality system becomes the following linear FBEE:
\[
\begin{aligned}
\dot{y}(t) &= [A - B(t)R(t)^{-1}S(t)]y(t) - B(t)R(t)^{-1}B(t)\psi(t), \\
\dot{\psi}(t) &= -[A - B(t)R(t)^{-1}S(t)]^*\psi(t) - [Q(t) - S(t)^*R(t)^{-1}S(t)]y(t), \\
y(0) &= x, \quad \psi(T) = G_y(T).
\end{aligned}
\]

It is known that under the following conditions:
\[ R(t) \geq \delta I, \quad Q(t) - S(t)^*R(t)^{-1}S(t) \geq 0, \quad t \in [0, T], \quad G \geq 0, \]
the map $u(\cdot) \mapsto J(x; u(\cdot))$ for the current LQ problem is uniformly convex, and the
above linear FBEE (17) admits a unique mild solution $(y(\cdot), \psi(\cdot))$ ([14]).

Next, we note that under (H0)', by Hille-Yosida’s theorem, there exist $M \geq 1$
and $\omega \in \mathbb{R}$ such that
\[ \| (\lambda - A)^{-n} \| \leq \frac{M}{(\lambda - \omega)^n}, \quad \forall \lambda > \omega, \; n \geq 1, \]
and the Yosida approximation $A_\lambda$ of $A$ is well-defined:
\[ A_\lambda = \lambda A(\lambda - A)^{-1}, \quad \lambda > \omega. \]

By making a shifting and absorbing a relevant term into $b(t, y, \psi)$ (see (1)), we may
assume that $\omega = 0$ in the above. Then by [21], we may assume the following:
\[
\begin{aligned}
\|e^{A_\lambda t}\| &\leq M, \quad \forall t \geq 0, \\
\lim_{\lambda \to \infty} \|A_\lambda x - Ax\| = 0, \quad \forall x \in D(A), \\
\lim_{\lambda \to \infty} \sup_{t \in [0, T]} \|e^{A_\lambda t}x - e^{At}x\| = 0, \quad \forall x \in X.
\end{aligned}
\]

Now, let us look at (H0). It is clear that under condition (3), one has
\[ \langle Ax, x \rangle \leq -\sigma_0 \|x\|^2, \quad \forall x \in D(A), \]
\[ \text{(22)} \]
and under (5), one has
\[ \langle Ax, x \rangle = 0, \quad \forall x \in D(A). \] (23)
The following simple result is concerned with the Yosida approximation \( A_\lambda \) of \( A \), under (H0).

**Proposition 2.2.** If (3) holds, then
\[ \langle A_\lambda x, x \rangle \leq -\frac{\lambda \sigma_0}{\lambda + \sigma_0} \|x\|^2, \quad \forall x \in X, \quad \lambda > 0. \] (24)
If (5) holds, then
\[ \langle (A_\lambda + A_\lambda^*) x, x \rangle \leq 0, \quad \forall x \in X, \quad \lambda > 0. \] (25)

**Proof.** Under (3), \( A \) admits the following spectral decomposition ([8]):
\[ Ax = \int_{\sigma(A)} \mu dE_\mu x, \quad \forall x \in D(A), \] (26)
where \( \mu \mapsto E_\mu \) is the projection-valued measure associated with \( A \), and \( \sigma(A) \subseteq (-\infty, -\sigma_0] \) is the spectrum of \( A \). Consequently,
\[ A_\lambda = \lambda A(\lambda - A)^{-1} = \int_{\sigma(A)} \frac{\lambda \mu}{\lambda - \mu} dE_\mu. \]
Since the map \( \mu \mapsto \frac{\lambda \mu}{\lambda - \mu} \) is increasing on \(( -\infty, -\sigma_0] \), we have
\[ \langle A_\lambda x, x \rangle = \int_{\sigma(A)} \frac{\lambda \mu}{\lambda - \mu} d\|E_\mu x\|^2 \leq -\frac{\lambda \sigma_0}{\lambda + \sigma_0} \|x\|^2, \quad \forall x \in X. \]
Now, let (5) hold. We let \( X = X + iX \) be the complexification of \( X \), i.e.,
\[ X = \{ x + iy \mid x, y \in X \}, \]
with the following definition of addition, scalar multiplication, and inner product:
\[ (x + iy) + (\bar{x} + i\bar{y}) = (x + \bar{x}) + i(y + \bar{y}), \quad \forall x, y, \bar{x}, \bar{y} \in X, \]
\[ (\alpha + i\beta)(x + iy) = (\alpha x - \beta y) + i(\alpha y + \beta x), \quad \forall \alpha, \beta \in \mathbb{R}, \ x, y \in X, \]
\[ \langle x + iy, \bar{x} + i\bar{y} \rangle = \langle x, \bar{x} \rangle + \langle y, \bar{y} \rangle + i(\langle y, \bar{x} \rangle - \langle x, \bar{y} \rangle), \quad \forall x, \bar{x}, y, \bar{y} \in X. \]
Naturally extend \( A \) to \( A : D(A) \subseteq X \to X \) as follows
\[ \begin{cases} D(A) = D(A) + iD(A) \subseteq X, \\ A(x + iy) = Ax + iAy, \quad \forall x + iy \in D(A). \end{cases} \]
Then under (5), we have
\[ \langle A(x + iy), \bar{x} + i\bar{y} \rangle = \langle Ax + iAy, \bar{x} + i\bar{y} \rangle \]
\[ = \langle Ax, \bar{x} \rangle + \langle Ay, \bar{y} \rangle + i(\langle Ay, \bar{x} \rangle - \langle Ax, \bar{y} \rangle) \]
\[ = -\langle x, Ax \rangle - \langle y, Ay \rangle - i(y, Ax) - \langle x, Ay \rangle = -\langle x + iy, A(x + iy) \rangle, \]
which implies that
\[ A^* = -A. \]
Hence, \( A \) admits the following spectral decomposition:
\[ Az = \int_{\sigma(A)} \mu dE_\mu z, \quad \forall z \in D(A), \]
with \( \sigma(A) \subseteq i\mathbb{R} \). Consequently, for any \( \lambda > 0 \),
\[
\langle (A_\lambda + A_\lambda^*)z, z \rangle = \int_{\sigma(A)} \left[ \frac{\lambda \mu}{\lambda - \mu} - \frac{\lambda \mu}{\lambda + \mu} \right] d\|E_\mu z\|^2 \\
= -\int_{\sigma(A)} \frac{2\lambda|\mu|^2}{\lambda^2 + |\mu|^2} d\|E_\mu z\|^2 \leq 0, \quad z = x + iy \in X.
\]
Note that for any \( x \in X \), if
\[
(\lambda - A)^{-1}x = \bar{x} = \bar{x} + i\bar{y},
\]
with \( \bar{x}, \bar{y} \in X \), then
\[
x = (\lambda - A)(\bar{x} + i\bar{y}) = (\lambda - A)\bar{x} + i(\lambda - A)\bar{y}.
\]
Hence, we must have
\[
\bar{x} = (\lambda - A)^{-1}x, \quad \bar{y} = 0.
\]
Consequently,
\[
A_\lambda x = A_\lambda^* x, \quad \forall x \in X.
\]
Likewise,
\[
A_\lambda^* x = A_\lambda^* x, \quad \forall x \in X.
\]
Hence, (25) follows.

To conclude this section, let us introduce some assumptions on the coefficients of FBEE (1).

**H1** The maps \( b, g : [0, T] \times X \times X \rightarrow X \) and \( h : X \rightarrow X \) are continuous, and the map \( (y, \psi) \mapsto (b(t, y, \psi), g(t, y, \psi), h(y)) \) is locally Lipschitz.

**H2** In addition to (H1), let the map \( (y, \psi) \mapsto (b(t, y, \psi), g(t, y, \psi), h(y)) \) be uniformly Lipschitz and of uniformly linear growth, i.e., there exists a constant \( L > 0 \) such that
\[
\begin{cases}
|b(t, 0, 0)| \leq L, & t \in [0, T], \\
|b(t, y, \psi) - b(t, \bar{y}, \bar{\psi})| \leq L|y - \bar{y}| + L\|\psi - \bar{\psi}\|, & \forall t \in [0, T], y, \bar{y}, \psi, \bar{\psi} \in X, \\
|g(t, 0, 0)| \leq L, & t \in [0, T], \\
|g(t, y, \psi) - g(t, \bar{y}, \bar{\psi})| \leq L|y - \bar{y}| + L\|\psi - \bar{\psi}\|, & \forall t \in [0, T], y, \bar{y}, \psi, \bar{\psi} \in X,
\end{cases}
\]
and
\[
|h(y) - h(\bar{y})| \leq L|y - \bar{y}|, \quad \forall y, \bar{y} \in X.
\]

**H3** In addition to (H1), let the map \( (y, \psi) \mapsto (b(t, y, \psi), g(t, y, \psi), h(y)) \) be Fréchet differentiable, with continuous Fréchet derivatives.

Note that (H3) is neither stronger nor weaker than (H2), since the Fréchet derivatives \( b_x, b_\psi, g_x, g_\psi, h_y \), if they exist, are not necessarily uniformly bounded. We let \( G_1, G_2, G_3 \) be the set of all \((b, g, h)\) satisfying (H1), (H2), and (H3), respectively. Any \((b, g, h) \in G_1\) uniquely generates an FBEE (1.1) (without mentioning the well-posedness). Hence, any \((b, g, h) \in G_1\) is called the generator of an FBEE of form (1).
3. Linear FBEEs. In this section, we consider the following linear FBEE:

\[
\begin{align*}
\dot{y}(t) &= Ay(t) + B_{11}(t)y(t) + B_{12}(t)\psi(t) + b_0(t), \\
\dot{\psi}(t) &= -A^*\psi(t) - B_{21}(t)y(t) - B_{22}(t)\psi(t) - g_0(t), \\
y(0) &= x, \quad \psi(T) = Hy(T) + h_0,
\end{align*}
\tag{30}
\]

with

\[
\begin{align*}
B_{ij}(\cdot) &\in L^\infty(0, T; \mathcal{L}(X)), \quad i, j = 1, 2, \\
b_0(\cdot), g_0(\cdot) &\in L^1(0, T; X), \quad H \in \mathcal{L}(X), \quad h_0 \in X.
\end{align*}
\tag{31}
\]

The above is a special case of (1). A pair \((y(\cdot), \psi(\cdot))\) is called a mild solution to (30) if the following holds:

\[
\begin{align*}
y(t) &= e^{At}x + \int_0^t e^{A(t-s)}[B_{11}(s)y(s) + B_{12}(s)\psi(s) + b_0(s)]ds, \quad t \in [0, T], \\
\psi(t) &= e^{A^*(T-t)}[Hy(T) + h_0] + \int_t^T e^{A^*(s-t)}[B_{21}(s)y(s) + B_{22}(s)\psi(s) + g_0(s)]ds, \quad t \in [0, T].
\end{align*}
\]

Our first result is the following.

**Proposition 3.1.** Let (H0)' and (31) hold. Then FBEE (30) admits a mild solution if the following operator:

\[
\psi(\cdot) \mapsto \psi(\cdot) - \int_0^T (\Phi_{22}(T, \cdot)H\Phi_{11}(T, s)
+ \int_s^T \Phi_{22}(r, \cdot)B_{21}(r)\Phi_{11}(r, s)dr)B_{12}(s)\psi(s)ds
\]

is invertible on \(C([0, T]; X)\), where \(\Phi_{11}(\cdot, \cdot)\) and \(\Phi_{22}(\cdot, \cdot)\) are evolution operators generated by \(A + B_{11}(\cdot)\) and \(A^* + B_{22}(\cdot)\), respectively.

**Proof.** By the variation of constants formula, we have

\[
y(t) = \Phi_{11}(t, 0)x + \int_0^t \Phi_{11}(t, s)[B_{12}(s)\psi(s) + b_0(s)]ds,
\]

and

\[
\psi(t) = \Phi_{22}(T, t)[Hy(T) + h_0] + \int_t^T \Phi_{22}(s, t)[B_{21}(s)y(s) + g_0(s)]ds
= \Phi_{22}(T, t)\left[H\Phi_{11}(T, 0)x + \int_0^T \Phi_{11}(T, s)\left(B_{12}(s)\psi(s) + b_0(s)\right)ds\right] + h_0
+ \int_t^T \Phi_{22}(s, t)\left[B_{21}(s)\Phi_{11}(s, 0)x
+ \int_0^s \Phi_{11}(s, r)\left[B_{12}(r)\psi(r) + b_0(r)\right]dr\right] + g_0(s)]ds
= \int_0^T \Phi_{22}(T, t)H\Phi_{11}(T, s) + \int_0^T \Phi_{22}(r, t)B_{21}(r)\Phi_{11}(r, s)dr)B_{12}(s)\psi(s)ds
+ (\Phi_{22}(T, t)H\Phi_{11}(T, 0) + \int_t^T \Phi_{22}(s, t)B_{21}(s)\Phi_{11}(s, 0)ds)x
\]
\[
+ \int_0^T (\Phi_{22}(T, t) H \Phi_{11}(T, s) + \int_{sT}^s \Phi_{22}(r, t) B_{21}(r) \Phi_{11}(r, s) dr) b_0(s) ds \\
+ \int_t^T \Phi_{22}(s, t) g_0(s) ds + \Phi_{22}(T, t) h_0, \quad t \in [0, T].
\]

The above is a Fredholm integral equation for \( \psi(\cdot) \) of the second kind. By our assumption, it has a unique solution. Then our result follows.

Next, we consider a special case: \( A^* = -A \). For such a case, we have that
\[
A \equiv \begin{pmatrix} A & 0 \\ 0 & -A^* \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} \quad (32)
\]
generates a \( C_0 \)-group \( e^{At} \) on \( X \times X \). Hence, if we denote
\[
\mathbb{E}(t) = \begin{pmatrix} B_{11}(t) & B_{12}(t) \\ -B_{21}(t) & -B_{22}(t) \end{pmatrix}, \quad (33)
\]
then \( A + \mathbb{B}(-\cdot) \) generates an evolution operator \( \hat{\Phi}(-\cdot, \cdot) \) on \( X \times X \). The following result concerns the well-posedness of the corresponding linear FBEE.

**Proposition 3.2.** Let \( A^* = -A \) and (31) hold. Then linear FBEE (30) admits a unique mild solution \((y(\cdot), \psi(\cdot))\) for any \( h_0 \in X \) if and only if
\[
\left[ (-H, I) \hat{\Phi}(T, 0) \begin{pmatrix} 0 \\ I \end{pmatrix} \right]^{-1} \in \mathcal{L}(X \times X). \quad (34)
\]

**Proof.** Suppose (30) admits a unique mild solution. Then we have
\[
\begin{pmatrix} y(t) \\ \psi(t) \end{pmatrix} = \hat{\Phi}(t, 0) \begin{pmatrix} x \\ \psi(0) \end{pmatrix} + \int_0^t \hat{\Phi}(t, s) \begin{pmatrix} b_0(s) \\ -g_0(s) \end{pmatrix} ds, \quad t \in [0, T], \quad (35)
\]
with \( \psi(0) \) undetermined. By the condition at \( t = T \), we have
\[
h_0 = -H y(T) + \psi(T) = (-H, I) \begin{pmatrix} y(T) \\ \psi(T) \end{pmatrix} = (-H, I) \hat{\Phi}(T, 0) \begin{pmatrix} x \\ \psi(0) \end{pmatrix} + \int_0^T (-H, I) \hat{\Phi}(t, s) \begin{pmatrix} b_0(s) \\ -g_0(s) \end{pmatrix} ds
\]
\[
= (-H, I) \hat{\Phi}(T, 0) \begin{pmatrix} 0 \\ I \end{pmatrix} \psi(0) + (-H, I) \hat{\Phi}(T, 0) \begin{pmatrix} I \\ 0 \end{pmatrix} x + \int_0^T (-H, I) \hat{\Phi}(t, s) \begin{pmatrix} b_0(s) \\ -g_0(s) \end{pmatrix} ds.
\]
Hence, in order for any \( h_0 \in X \) the above uniquely determines \( \psi(0) \), we need (34). Conversely, if (34) holds, one obtains
\[
\psi(0) = \left[ (-H, I) \hat{\Phi}(T, 0) \begin{pmatrix} 0 \\ I \end{pmatrix} \right]^{-1} \left[ h_0 - (-H, I) \hat{\Phi}(T, 0) \begin{pmatrix} I \\ 0 \end{pmatrix} x \right. \left. - \int_0^T (-H, I) \hat{\Phi}(t, s) \begin{pmatrix} b_0(s) \\ -g_0(s) \end{pmatrix} ds \right]. \quad (36)
\]
From this we obtain the mild solution \((y(\cdot), \psi(\cdot))\) of FBEE (30).

We note that in principle, condition (34) is checkable, although it might be practically complicated. We also note that, in the above, the condition that \( A^* = -A \) or \( e^{At} \) is a group, plays an essential role. It seems that if \( e^{At} \) is just a semigroup,
not a group, the arguments used above will not work (since \( \Phi(\cdot, \cdot) \) in the above might not be defined).

We now return to the general linear FBEE (30) (without assuming (H0)). Suppose \((y(\cdot), \psi(\cdot))\) is a strong solution to linear FBEE (30). Inspired by the well-known invariant imbedding idea ([4, 15, 27, 16]), we suppose that the following relation holds:

\[
\psi(t) = \mathcal{P}(t)y(t) + p(t), \quad t \in [0, T],
\]

for some Fréchet differentiable functions \( \mathcal{P} : [0, T] \to \mathcal{L}(X) \) and \( p : [0, T] \to X \). Then, formally, we should have

\[
-A^*[\mathcal{P}(t)y(t) + p(t)] - B_{21}(t)y(t) - B_{22}(t)[\mathcal{P}(t)y(t) + p(t)] - g(t) = \frac{d\psi(t)}{dt}
\]

\[
= \mathcal{P}(t)y(t) + \mathcal{P}(t)(Ay(t) + B_{11}(t)y(t) + B_{12}(t)[\mathcal{P}(t)y(t) + p(t)] + b_0(t)) + \frac{dp(t)}{dt}
\]

\[
= \left( \frac{\mathcal{P}(t)}{\mathcal{P}(t)} + \mathcal{P}(t)A + \mathcal{P}(t)B_{11}(t) + B_{22}(t)\mathcal{P}(t) \right) y(t) + \mathcal{P}(t)B_{12}(t)p(t) + \mathcal{P}(t)b_0(t) + \frac{dp(t)}{dt}.
\]

Hence,

\[
0 = \left( \frac{\mathcal{P}(t)}{\mathcal{P}(t)} + \mathcal{P}(t)A + \mathcal{P}(t)B_{11}(t) + B_{22}(t)\mathcal{P}(t) \right) y(t) + \mathcal{P}(t)B_{12}(t)p(t) + \mathcal{P}(t)b_0(t) + \frac{dp(t)}{dt}.
\]

This suggests that we choose \( \mathcal{P}(\cdot) \) satisfying the following:

\[
\begin{cases}
\frac{\mathcal{P}(t)}{\mathcal{P}(t)} + \mathcal{P}(t)A + \mathcal{P}(t)B_{11}(t) + B_{22}(t)\mathcal{P}(t) + \mathcal{P}(t)B_{12}(t)p(t) + \mathcal{P}(t)b_0(t) + \frac{dp(t)}{dt} = 0, & t \in [0, T], \\
\mathcal{P}(T) = H,
\end{cases}
\]

(37)

and choose \( p(\cdot) \) satisfying

\[
\begin{cases}
\frac{dp(t)}{dt} + A^*p(t) + \left( \mathcal{P}(t)B_{12}(t) + B_{22}(t) \right)p(t) + \mathcal{P}(t)b_0(t) + g(t) = 0, & t \in [0, T], \\
p(T) = h_0.
\end{cases}
\]

(38)

Equation (37) is called a differential Riccati equation. Any \( \mathcal{P} : [0, T] \to \mathcal{L}(X) \) is called a mild solution of (37) if the following integral equation is satisfied:

\[
\mathcal{P}(t) = e^{A^*(T-t)}He^{A(T-t)} + \int_t^T e^{A^*(s-t)} \left[ \mathcal{P}(s)B_{11}(s) + B_{22}(s)\mathcal{P}(s) + \mathcal{P}(s)B_{12}(s)\mathcal{P}(s) + B_{21}(s)e^{A(s-t)}ds, \right] \quad t \in [0, T].
\]

(39)

Note that if \( A \) is bounded, (37) and (39) are equivalent. Further, recalling that \( \Phi_{11}(\cdot, \cdot) \) and \( \Phi_{22}(\cdot, \cdot) \) are the evolution operators generated by \( A + B_{11}(\cdot) \) and \( A^* + B_{22}(\cdot) \), respectively, one sees that (39) is also equivalent to the following:

\[
\mathcal{P}(t) = \Phi_{22}(T, t)H\Phi_{11}(T, t)
\]

\[
+ \int_t^T \Phi_{22}(s, t) \left[ \mathcal{P}(s)B_{12}(s)\mathcal{P}(s) + B_{21}(s) \right] \Phi_{11}(s, t) ds, \quad t \in [0, T].
\]

(40)
Having the above derivation, we now present the following result.

**Proposition 3.3.** Let (H0) and (31) hold. Let Riccati equation (37) admit a unique mild solution \( \mathbb{P} : [0, T] \to \mathcal{L}(X) \). Then linear FBEE (30) admits a unique mild solution \((y(\cdot), \psi(\cdot))\).

**Proof.** For any \( \lambda > 0 \), consider the following:

\[
\mathbb{P}_\lambda(t) = e^{A_\lambda(T-t)}He^{A_\lambda(T-t)} + \int_0^T e^{A_\lambda(s-t)}\mathcal{Q}(s)e^{A_\lambda(s-t)}ds, \quad t \in [0, T],
\]

where, with the mild solution \( \mathbb{P}(\cdot) \) of the Riccati equation (37),

\[
\mathcal{Q}(s) = \mathbb{P}(s)B_{11}(s) + B_{22}(s)\mathbb{P}(s) + \mathbb{P}(s)B_{12}(t)\mathbb{P}(s) + B_{21}(s).
\]

Clearly, \( \mathbb{P}_\lambda(\cdot) \) is uniformly bounded (by noting (21)). Moreover, for any \( x \in X \),

\[
\begin{align*}
\|\mathbb{P}_\lambda(t)x - \mathbb{P}(t)x\| & \leq \|e^{A_\lambda(T-t)}He^{A_\lambda(T-t)}x - e^{A(T-t)}He^{A(T-t)}x\| \\
& + \int_0^T \|e^{A_\lambda(s-t)}\mathcal{Q}(s)e^{A_\lambda(s-t)}x - e^{A(T-t)}\mathcal{Q}(s)e^{A(s-t)}x\|ds \\
& \leq \|e^{A_\lambda(T-t)}H[e^{A(T-t)}x - e^{A(T-t)}x]\| \\
& + \|e^{A_\lambda(T-t)}He^{A(T-t)}x - e^{A(T-t)}He^{A(T-t)}x\| \\
& + \int_0^T \left( \|e^{A_\lambda(s-t)}\mathcal{Q}(s)e^{A_\lambda(s-t)}x - e^{A(T-t)}\mathcal{Q}(s)e^{A(s-t)}x\| \right)ds \\
& \leq K\|e^{A_\lambda(T-t)}x - e^{A(T-t)}x\| + \|e^{A_\lambda(T-t)}He^{A(T-t)}x - e^{A(T-t)}He^{A(T-t)}x\| \\
& + \int_0^T \left( K\|e^{A_\lambda(s-t)}x - e^{A(s-t)}x\| \right)ds \to 0.
\end{align*}
\]

Hereafter, \( K > 0 \) represents a generic constant which can be different from line to line. Note that \( \mathbb{P}_\lambda(\cdot) \) also solves the following Lyapunov equation:

\[
\begin{cases}
\dot{\mathbb{P}}_\lambda(t) + \mathbb{P}_\lambda(t)A_\lambda + A_\lambda^T\mathbb{P}_\lambda(t) + \mathcal{Q}(t) = 0, & t \in [0, T], \\
\mathbb{P}_\lambda(T) = H.
\end{cases}
\]

(41)

Now, we let \( p_\lambda(\cdot) \) be the solution of the following:

\[
\begin{cases}
\dot{p}_\lambda(t) + A_\lambda^Tp_\lambda(t) + [\mathbb{P}_\lambda(t)B_{12}(t) + B_{22}(t)]p_\lambda(t) + \mathbb{P}_\lambda(t)b_0(t) + g_0(t) = 0, & t \in [0, T], \\
p_\lambda(T) = h_0.
\end{cases}
\]

(42)

It is clear that

\[
\|p_\lambda(\cdot)\|_\infty < \infty.
\]

We estimate

\[
\|p_\lambda(t) - p(t)\| \leq \|e^{A_\lambda(T-t)}h_0 - e^{A(T-t)}h_0\| \\
+ \int_0^T \left\| e^{A_\lambda(s-t)}\left( \mathbb{P}_\lambda(s)B_{12}(s) + B_{22}(s) \right)p_\lambda(s) \\
- e^{A(T-t)}\left( \mathbb{P}(s)B_{12}(s) + B_{22}(s) \right)p(s) \right\| ds.
\]
Then by Gronwall’s inequality, we have
\[
\|y(t) - \lambda(t)\| \leq e^{A(t-s)}\left(\mathbb{P}_\lambda(s)\varphi_0(s) + g_0(s) - e^{A^*(t-s)}\left(\mathbb{P}(s)\varphi_0(s) + g_0(s)\right)\right)ds
\]
\[
\leq e^{A^*(T-t)}\left(\mathbb{P}_\lambda(s)\varphi_0(s) + g_0(s) - e^{A^*(T-t)}\left(\mathbb{P}(s)\varphi_0(s) + g_0(s)\right)\right)ds
\]
By the convergence of \(\mathbb{P}_\lambda(\cdot) \to \mathbb{P}(\cdot)\) and \(\psi_\lambda(\cdot) \to \psi(\cdot)\), we have
\[
\lim_{\lambda \to \infty} \|y_\lambda(\cdot) - y(\cdot)\|_\infty = 0.
\]
Let us rewrite the Riccati equation as follows:

\[ -A_\lambda^*[P_\lambda(t)y_\lambda(t) + p_\lambda(t)] - B_{21}(t)y_\lambda(t) - B_{22}(t)[P(t)y_\lambda(t) + p_\lambda(t)] - g_0(t) + [P_\lambda(t) - P(t)]B_{21}(t)y_\lambda(t) + [P_\lambda(t)B_{12}(t)P_\lambda(t) - P(t)B_{12}(t)P(t)]y_\lambda(t) = -A_\lambda^*\psi_\lambda(t) - B_{21}(t)y_\lambda(t) - B_{22}(t)\psi_\lambda(t) - g_0(t) + R_\lambda(t), \]

where

\[ R_\lambda(t) = \left( [P_\lambda(t) - P(t)]B_{11}(t) + [P_\lambda(t)B_{12}(t)P_\lambda(t) - P(t)B_{12}(t)P(t)] \right)y_\lambda(t). \]

Since \( R_\lambda(t) \) is a trivial case, we see that \( \Phi(t) = \Phi(t) \) for which the above equation is linear and it always has a mild solution

\( \psi_\lambda(-) \rightarrow \psi(-) \),

and \((y(-), \psi(-))\) is a mild solution to FBBE (30).

Now, a natural question is when the Riccati equation (37) admits a mild solution. Let us rewrite the Riccati equation as follows:

\[ P(t) = \Phi_{22}(T, t)H\Phi_{11}(T, t) + \int_t^T \Phi_{22}(s, t)B_{21}(s)\Phi_{11}(s, t)ds \]

\[ + \int_t^T \Phi_{22}(s, t)P(s)B_{12}(s)P(s)\Phi_{11}(s, t)ds, \quad t \in [0, T]. \]

A trivial case is \( B_{12}(\cdot) = 0 \) for which the above equation is linear and it always has a solution, under (HO)' and (31). In general, we have the following result.

**Proposition 3.4.** Suppose (HO)' and (31) hold.

(i) Equation (43) admits at most one solution \( P(\cdot) \in C([0, T]; L(X)) \).

(ii) Suppose in addition that

\[ B_{22}(t) = B_{11}(t)^*, \quad t \in [0, T], \]

and

\[ H \in S^+(X), \quad -B_{12}(\cdot), B_{21}(\cdot) \in L^\infty(0, T; S^+(X)). \]

Then Riccati equation (43) admits a unique solution \( P(\cdot) \in C([0, T]; S^+(X)). \)

**Proof.** (i) Suppose \( P(\cdot), \tilde{P}(\cdot) \in C([0, T]; L(X)) \) are two solutions to (43). Then \( \tilde{P}(\cdot) = P(\cdot) - \tilde{P}(\cdot) \) satisfies the following:

\[ \tilde{P}(t) = \int_t^T \Phi_{22}(s, t) \left[ P(s)B_{12}(s)P(s) - \tilde{P}(s)B_{12}(s)\tilde{P}(s) \right] \Phi_{11}(s, t)ds \]

\[ = \int_t^T \Phi_{22}(s, t) \left[ \tilde{P}(s)B_{12}(s)P(s) + \tilde{P}(s)B_{12}(s)\tilde{P}(s) \right] \Phi_{11}(s, t)ds. \]

Here, we note that \( P(\cdot) = \tilde{P}(\cdot) + \tilde{P}(\cdot) \). Hence

\[ P(s)B_{12}(s)P(s) - \tilde{P}(s)B_{12}(s)\tilde{P}(s) \]

\[ = \left[ \tilde{P}(s) + \tilde{P}(s) \right] B_{12}(s)P(s) - \tilde{P}(s)B_{12}(s) \left[ P(s) - \tilde{P}(s) \right] \]
and (37) can be written as
\[ \tilde{P}(t)B_{12}(s)\tilde{P}(s) + \tilde{P}(s)B_{12}(s)\tilde{P}(s). \]

Then by Gronwall’s inequality, we obtain that
\[ \tilde{P}(\cdot) = P(\cdot). \]

(ii) Under our conditions, we have
\[ A^* + B_{22}(\cdot) = [A + B_{11}(\cdot)]^*. \]

Hence,
\[ \Phi_{22}(\cdot, \cdot) = \Phi_{11}(\cdot, \cdot)^*, \]

and (37) can be written as
\[
\begin{cases}
\dot{P}(t) + P(t)[A + B_{11}(t) + B_{12}(t)P(t)] + [A + B_{11}(t) + B_{12}(t)P(t)]^*P(t) \\
+ P(t)[-B_{12}(t)P(t) + B_{21}(t)] = 0, & t \in [0, T], \\
P(T) = H, &
\end{cases}
\]

(46)

We now let
\[ P_0(t) = 0, \quad t \in [0, T], \]

and let \( P_{n+1}(\cdot) \) be the mild solution of the following Lyapunov equation:
\[
\begin{cases}
P_{n+1}(t) + P_{n+1}(t)[A + B_{11}(t) + B_{12}(t)P_n(t)] \\
+ [A + B_{11}(t) + B_{12}(t)P_n(t)]^*P_{n+1}(t) \\
+ P_n(t)[-B_{12}(t)P_n(t) + B_{21}(t)] = 0, & t \in [0, T], \\
P_{n+1}(T) = H, &
\end{cases}
\]

(47)

Observe the following:
\[
0 = \dot{P}_{n+1}(t) - \dot{P}_n(t) + [P_{n+1}(t) - P_n(t)] [A + B_{11}(t)] \\
+ [A + B_{11}(t)]^*[P_{n+1}(t) - P_n(t)] + P_{n+1}(t)B_{12}(t)P_n(t) \\
+ P_n(t)B_{12}(t)P_{n+1}(t) - P_n(t)B_{12}(t)P_{n-1}(t) - \dot{P}_n(t)B_{12}(t)P_{n}(t) \\
- \dot{P}_n(t)B_{12}(t)P_{n}(t) - \dot{P}_n(t)B_{12}(t)P_{n-1}(t) \\
= \dot{P}_{n+1}(t) - \dot{P}_n(t) + [P_{n+1}(t) - P_n(t)] [A + B_{11}(t) + B_{12}(t)P_n(t)] \\
+ [A + B_{11}(t) + B_{12}(t)P_n(t)]^*[P_{n+1}(t) - P_n(t)] \\
+ [P_n(t) - P_{n-1}(t)]B_{12}(t)\begin{bmatrix}P_n(t) - P_{n-1}(t) \end{bmatrix}.
\]

This implies that
\[ P_{n+1}(t) \leq P_n(t), \quad t \in [0, T], \quad n \geq 1. \]

On the other hand, from (47), one has
\[ P_n(t) \geq 0, \quad \forall t \in [0, T], \quad n \geq 1. \]

Hence, by \([22, 18]\) for any \( t \in [T, T], \) there exists a \( P(t) \in S^+(X) \) such that
\[
\lim_{n \to \infty} \|P_n(t)x - P(t)x\| = 0, \quad \forall x \in X.
\]

(48)

Note that for any \( x \in X, \)
\[
P_{n+1}(t)x = \Phi_{11}(T, t)^*H\Phi_{11}(T, t)x + \int_t^T \Phi_{11}(s, t)^*B_{21}(s)\Phi_{11}(s, t)xds \\
+ \int_t^T \Phi_{11}(s, t)^*P_{n+1}(s)B_{12}(s)P_n(s) + P_n(s)B_{12}(s)P_{n+1}(s) \\
- \dot{P}_n(s)B_{12}(s)P_n(s)\Phi_{11}(s, t)xds, \quad t \in [0, T].
\]
Thus, making use of (48), we obtain that $\mathbb{P}(\cdot)$ is a mild solution to (43). \hfill $\square$

Let us look at linear FBEE (17) resulting from linear-quadratic optimal control problem. We rewrite (17) below:

$$
\begin{align*}
\dot{y}(t) &= Ay(t) - B(t)R(t)^{-1}S(t)y(t) - B(t)R(t)^{-1}B(t)^*\psi(t), \\
\psi(t) &= -A^*\psi(t) - [Q(t) - S(t)^*R(t)^{-1}S(t)]y(t) + S(t)^*R(t)^{-1}B(t)^*\psi(t), \\
y(0) &= x, \quad \psi(T) = G\psi(T).
\end{align*}
$$

(49)

Note that

$$
R(t) \geq \delta I, \quad \forall t \in [0, T],
$$

which leads to the existence of $R(t)^{-1}$. Hence,

$$
\begin{align*}
B_{11}(t) &= -B(t)R(t)^{-1}S(t), & B_{12}(t) &= -B(t)R(t)^{-1}B(t)^*, \\
B_{21}(t) &= [Q(t) - S(t)^*R(t)^{-1}S(t)], & B_{22}(t) &= -S(t)^*R(t)^{-1}B(t)^*.
\end{align*}
$$

Then in the case that

$$
G \geq 0, \quad Q(t) - S(t)^*R(t)^{-1}S(t) \geq 0, \quad t \in [0, T],
$$

the corresponding Riccati equation admits a unique solution and linear FBEE (17) admits a solution.

4. Decoupling method — A brief description. We note that in the previous section, the essence of the approach by means of Riccati equation is to use the ansatz

$$
\psi(t) = \mathbb{P}(t)y(t) + p(t), \quad t \in [0, T],
$$

and then determine the maps $\mathbb{P}(\cdot)$ and $p(\cdot)$ to get a solution to the original FBEE. Inspired by this, we have the following result for nonlinear cases.

**Proposition 4.1.** Let (H0)' and (H2) hold. Let $\mathbb{K} : [0, T] \times X \to X$ be Fréchet differentiable satisfying the following:

$$
\begin{align*}
\mathbb{K}_t(t, y) + \mathbb{K}_y(t, y) \left[ Ay + b(t, y, \mathbb{K}(t, y)) \right] + A^*\mathbb{K}(t, y) + g(t, y, \mathbb{K}(t, y)) &= 0, \\
\mathbb{K}(T, y) &= h(y), \quad y \in X.
\end{align*}
$$

(50)

Let $y(\cdot)$ be a classical solution to the following:

$$
\begin{align*}
\dot{y}(t) &= Ay(t) + b(t, y(t), \mathbb{K}(t, y(t))), \quad t \in [0, T], \\
y(0) &= x,
\end{align*}
$$

(51)

and

$$
\psi(t) = \mathbb{K}(t, y(t)), \quad t \in [0, T].
$$

(52)

Then $(y(\cdot), \psi(\cdot))$ is a strong solution of FBEE (1).

**Proof.** Note that as a part of requirement for $\mathbb{K}(\cdot, \cdot)$ being a solution to (50), one has

$$
\mathbb{K}(t, y) \in D(A^*), \quad \forall (t, y) \in [0, T] \times X.
$$

By (52), we have

$$
\begin{align*}
\dot{\psi}(t) &= \mathbb{K}_t(t, y(t)) + \mathbb{K}_y(t, y(t)) \left[ Ay(t) + b(t, y(t), \mathbb{K}(t, y(t)) \right] \\
&= -A^*\mathbb{K}(t, y(t)) - g(t, y(t), \mathbb{K}(t, y(t)) = -A^*\psi(t) - g(t, y(t), \psi(t)),
\end{align*}
$$

and

$$
\psi(T) = \mathbb{K}(T, y(T)) = h(y(T)).
$$
Hence, our claim follows.

It is seen that thanks to the map $\mathbb{K}(\cdot, \cdot)$, the original FBEE (1) is decoupled into (51) and (52). Because of this, we introduce the following notion:

**Definition 4.2.** A map $\mathbb{K} : [0, T] \times X \to X$ is called a **decoupling field** of FBEE (1) if it is a solution to (50).

Now the natural question is when one can solve equation (50). The linear case has been treated in the previous section. To look at the nonlinear case, let us further assume that $A^* = A$ and it has a sequence of eigenvalues

$$0 > -\sigma_0 = \lambda_1 \geq \lambda_2 \geq \lambda_3 \cdots,$$

with the corresponding eigenvectors $\{\zeta_n\}_{n \geq 1}$ which form an orthonormal basis for $X$. Then

$$\mathbb{K}(t, y) = \sum_{n=1}^{\infty} \langle \mathbb{K}(t, y), \zeta_n \rangle \zeta_n = \sum_{n=1}^{\infty} k^n(t, y) \zeta_n.$$ 

Hence,

$$0 = K_y(t, y) + K_y(t, y) [A y + b(t, y, \mathbb{K}(t, y))] + A^* \mathbb{K}(t, y) + g(t, y, \mathbb{K}(t, y))$$

$$= \sum_{n=1}^{\infty} \left( k^n(t, y) + \langle k^n(t, y), A y + b(t, y, \mathbb{K}(t, y)) \rangle + \lambda_n k^n(t, y) \right)$$

$$+ \langle g(t, y, \mathbb{K}(t, y)), \zeta_n \rangle \zeta_n.$$

Therefore, we obtain a coupled system of countably many equations:

$$\begin{cases}
    k^n(t, y) + \langle k^n(t, y), A y + b(t, y, \mathbb{K}(t, y)) \rangle + \lambda_n k^n(t, y) \\
    + \langle g(t, y, \mathbb{K}(t, y)), \zeta_n \rangle = 0, \quad (t, y) \in [0, T] \times X, \\
    k^n(T, y) = \langle h(y), \zeta_n \rangle, \quad y \in X.
\end{cases}$$

Let us look at a special case. Suppose

$$\mathbb{K}(t, y) = k^1(t, y) \zeta_1, \quad (t, y) \in [0, T] \times X, \quad \zeta_1 \in D(A^*).$$

Then

$$\begin{cases}
    k^1(t, y) + \langle k^1(t, y), A y + b(t, y, k^1(t, y) \zeta_1) \rangle + \lambda_1 k^1(t, y) \\
    + \langle g(t, y, k^1(t, y) \zeta_1), \zeta_1 \rangle = 0, \quad (t, y) \in [0, T] \times X, \\
    k^1(T, y) = h^1(y), \quad y \in X,
\end{cases}$$

and

$$\langle g(t, y, k^n(t, y) \zeta_1), \zeta_n \rangle = 0, \quad n > 1.$$  \hspace{1cm} \text{(55)}

To guarantee (55), we assume that

$$g(t, y, \text{span} \{\zeta_1\}) \subseteq \text{span} \{\zeta_1\}.$$  \hspace{1cm} \text{(56)}

We note that (54) is a first order Hamilton-Jacobi equation in the Hilbert space $X$, involving an unbounded operator $A$. Therefore, it is possible to study the existence of viscosity solution of it. When the viscosity solution has certain regularity, one might be able to obtain a decoupling field $\mathbb{K}(t, y) = k^1(t, y) \zeta_1$ for our FBEE. Apparently, this is merely a very special case for the general FBEEs, and it already looks complicated. Hence, there is a very long way to go in this direction to establish a satisfactory theory (for nonlinear FBEEs). We hope to report some further results in this direction in our future publications.
5. **Lyapunov operators and a priori estimates.** We now look at the solvability of FBEES by another method, called method of continuity. We first look at the following linear FBE:

\[
\begin{aligned}
\dot{y}(t) &= Ay(t) + B_{11}(t)y(t) + B_{12}(t)\psi(t) + b_0(t), & t \in [0, T], \\
\dot{\psi}(t) &= -A^*\psi(t) - B_{21}(t)y(t) - B_{22}(t)\psi(t) - g_0(t), & t \in [0, T], \\
y(0) &\text{ and } \psi(T) \text{ are given,}
\end{aligned}
\]

with $B_{ij} : [0, T] \to L(X)$. Let

\[
A = \begin{pmatrix} A & 0 \\ 0 & -A^* \end{pmatrix}, \quad B(t) = \begin{pmatrix} B_{11}(t) & B_{12}(t) \\ -B_{21}(t) & -B_{22}(t) \end{pmatrix},
\]

Then FBE (57) can be written as

\[
\begin{aligned}
\begin{pmatrix} \dot{y}(t) \\ \dot{\psi}(t) \end{pmatrix} &= \begin{pmatrix} A + B(t) \\ -A^* \end{pmatrix} \begin{pmatrix} y(t) \\ \psi(t) \end{pmatrix} + \begin{pmatrix} b_0(t) \\ -g_0(t) \end{pmatrix}, & t \in [0, T], \\
y(0) &\text{ and } \psi(T) \text{ are given.}
\end{aligned}
\]

We introduce the following Lyapunov differential equation for operator-valued function $\Pi(\cdot)$:

\[
\ddot{\Pi}(t) + \Pi(t)[A - M(t)] + [A - M(t)]^*\Pi(t) + \mathcal{Q}(t) = 0, \quad t \in [0, T],
\]

where

\[
\Pi(t) = \begin{pmatrix} P(t) \\ \Gamma(t) \end{pmatrix}, \quad M(t) = \begin{pmatrix} M(t) & 0 \\ 0 & -M(t)^* \end{pmatrix}, \quad \mathcal{Q}(t) = \begin{pmatrix} Q_0(t) & \Theta(t)^* \\ \Theta(t) & Q_0(t) \end{pmatrix},
\]

with $M, M, \Theta : [0, T] \to L(X)$ and $Q_0, Q_0 : [0, T] \to S(X)$ to be properly chosen later. We may equivalently write (60) as follows:

\[
\dot{P}(t) + P(t)[A - M(t)] + [A^* - M(t)^*]P(t) + Q_0(t) = 0,
\]

\[
\dot{\Gamma}(t) + \Gamma(t)[A - M(t)] - [A - M(t)]^*\Gamma(t) + \Theta(t) = 0,
\]

and

\[
\dot{\Gamma}(t)^* - \Gamma(t)^*[A^* - M(t)^*] + [A^* - M(t)]^*\Gamma(t)^* + \Theta(t)^* = 0.
\]

Let us first look at (61) and (62). Operator-valued functions $P(\cdot)$ and $\dot{P}(\cdot)$ are mild solutions to (61) and (62), respectively, if the following hold:

\[
P(t) = e^{A^*(t-t)}P(T)e^{-A(t-t)} - \int_t^T e^{A^*(s-t)}[P(s)M(s) + M(s)^*P(s) - Q_0(s)]e^{A(s-t)}ds, \quad t \in [0, T],
\]

and

\[
\dot{P}(t) = e^{At}P(0)e^{A^*t} - \int_0^t e^{A(t-s)}[\dot{P}(s)\dot{M}(s)^* + \dot{M}(s)\dot{P}(s) + \dot{Q}_0(s)]e^{A(t-s)}ds, \quad t \in [0, T].
\]

We use the above definition simply because when $A$ is bounded, (61) is equivalent to (65), and (62) is equivalent to (66). Further, if we let $\Phi(\cdot, \cdot)$ and $\dot{\Phi}(\cdot, \cdot)$ be the
evolution operators generated by $A - M(\cdot)$ and $A - \bar{M}(\cdot)$, respectively, then $P(\cdot)$ and $\bar{P}(\cdot)$ admit the following representations:

$$P(t) = \Phi(T, t)^* P(T) \Phi(T, t) + \int_t^T \Phi(s, t)^* Q_0(s) \Phi(s, t) ds, \quad t \in [0, T],$$

and

$$\bar{P}(t) = \bar{\Phi}(t, 0) \bar{P}(0) \bar{\Phi}(t, 0)^* - \int_0^t \bar{\Phi}(t, s) \bar{Q}_0(s) \bar{\Phi}(t, s)^* ds, \quad t \in [0, T].$$

Thus, if

$$P(T), -\bar{P}(0), Q_0(t), \bar{Q}_0(t) \geq 0, \quad t \in [0, T],$$

then

$$P(t) \geq 0, \quad \bar{P}(t) \leq 0, \quad t \in [0, T].$$

Now, let us look at (63) and (64), which are equivalent. We assume that (H0) holds. Therefore, we have two cases to discuss.

**Case 1.** Let (3) hold. In this case, since $A$ is dissipative, the appearance of the term $\Gamma(t) A - A \Gamma(t)$ makes (63) and (64) difficult to solve in general. To overcome this, we require that for all $t \in [0, T], \Gamma(t) : D(A) \to D(A)$ and

$$\Gamma(t) A x = A \Gamma(t) x, \quad t \in [0, T], \quad x \in D(A).$$

Then both (63) and (64) are reduced to the following:

$$\Gamma(t) - \Gamma(t) M(t) + M(t) \Gamma(t) + \Theta(t) = 0,$$

which admits a unique solution as long as, say, $M(\cdot), \bar{M}(\cdot)$ and $\Theta(\cdot)$ are bounded and $\Gamma(T) \in L(X)$ is given. Actually, if $\Psi(\cdot, \cdot)$ and $\bar{\Psi}(\cdot, \cdot)$ are evolution operators generated by $-M(\cdot)$ and $\bar{M}(\cdot)$, respectively, then

$$\Gamma(t) = \bar{\Psi}(T, t) \Gamma(T) \Psi(T, t) + \int_t^T \bar{\Psi}(s, t) \Theta(s) \Psi(s, t) ds, \quad t \in [0, T].$$

Note that under (3), $A$ admits a spectral decomposition

$$A = \int_{\sigma(A)} \mu dE_\mu,$$

with $\sigma(A) \subseteq (-\infty, -\sigma_0]$ being the spectral of $A$, if

$$\begin{cases}
\Gamma(T) = \int_{\sigma(A)} \gamma(\mu) dE_\mu, \\
\Theta(t) = \int_{\sigma(A)} \theta(t, \mu) dE_\mu, \\
M(t) = \int_{\sigma(A)} m(t, \mu) dE_\mu, \\
\bar{M}(t) = \int_{\sigma(A)} \bar{m}(t, \mu) dE_\mu,
\end{cases}$$

for some suitable maps $\gamma : \sigma(A) \to \mathbb{R}$ and $\theta, m, \bar{m} : [0, T] \times \sigma(A) \to \mathbb{R}$, then

$$\Gamma(t) = \int_{\sigma(A)} \left( e^{\int_t^T [\bar{m}(s, \mu) - m(s, \mu)] ds} \gamma(\mu) + \int_t^T e^{\int_s^T [\bar{m}(\tau, \mu) - m(\tau, \mu)] d\tau} \theta(\tau, \mu) d\tau \right) dE_\mu.$$

Hence, (70) holds in this case. In particular, if

$$\begin{cases}
\gamma(\mu) = \gamma, \\
\theta(t, \mu) = \theta(t), \\
m(t, \mu) = m(t), \\
\bar{m}(t, \mu) = \bar{m}(t),
\end{cases} \quad (t, \mu) \in [0, T] \times \sigma(A),$$

we have

$$\Gamma(t) = \left( e^{\int_t^T [\bar{m}(s) - m(s)] ds} \gamma + \int_t^T e^{\int_s^T [\bar{m}(\tau) - m(\tau)] d\tau} \theta(\tau) d\tau \right) I."
Also, as a special case of (73), if
\[
\begin{cases}
\Gamma(t) = \gamma(t)A_{\lambda}, & \Theta(t) = \theta(t)A_{\lambda}, \\
M(t) = m(t)A_{\lambda}, & \bar{M}(t) = \bar{m}(t)A_{\lambda},
\end{cases} \quad t \in [0, T],
\]
for some suitable scalar functions \(\gamma(\cdot), \theta(\cdot), m(\cdot), \bar{m}(\cdot)\), then
\[
\Gamma(t) = \left( e^{\int_{t}^{T} [\bar{m}(s) - m(s)] ds} \gamma(\cdot) + \int_{t}^{T} e^{\int_{s}^{T} [\bar{m}(s) - m(s)] ds} \theta(\tau) d\tau \right) A_{\lambda},
\]
for which (70) will also hold.

**Case 2.** Let (5) hold. Then \(e^{At}\) is a group. Consequently, \(e^{-At}\) is well-defined. Similar to the case of \(P(\cdot)\), a map \(\Gamma(\cdot)\) is called a mild solution to equation (63), if the following holds:
\[
\Gamma(t) = e^{-A(T-t)} \Gamma(T)e^{A(T-t)} + \int_{t}^{T} e^{-A(s-t)} [M(s)\Gamma(s) - \Gamma(s)M(s) + \Theta(s)] e^{A(s-t)} ds, \quad t \in [0, T].
\]
By recalling the evolution operators \(\Phi(\cdot, \cdot)\) and \(\bar{\Phi}(\cdot, \cdot)\) generated by \(A - M(\cdot)\) and \(A - \bar{M}(\cdot)\), (noting that in the current case, \(\bar{\Phi}(s, t)^{-1}\) exists) we have
\[
\Gamma(t) = \bar{\Phi}(T, t)^{-1} \Gamma(T) \Phi(T, t) + \int_{t}^{T} \bar{\Phi}(s, t)^{-1} \Theta(s) \Phi(s, t) ds, \quad t \in [0, T].
\]
We point out that in the current case, (70) is not needed. However, if (73) holds and we are working in a complex Hilbert space, we will still have (70) and \(\Gamma(\cdot)\) can also be given by (72).

In what follows, when we say a mild solution \(\Pi(\cdot)\) of (60), we mean that \(P(\cdot)\) and \(\bar{P}(\cdot)\) are given by (65) and (66), respectively, and \(\Gamma(\cdot)\) is defined by (72) such that (70) holds for the case \(A = A^*\) and \(\Gamma(\cdot)\) is defined by (77) for the case \(A^* = -A\) (since we prefer to stay with a real Hilbert space).

The following is the main result of this section and it will play an important role below.

**Proposition 5.1.** Let \((y(\cdot), \psi(\cdot))\) be a mild solution of linear FBEE (57) and \(\Pi(\cdot)\) be a mild solution of Lyapunov differential equation (60). Then
\[
\langle \Pi(T) \begin{pmatrix} y(T) \\ \psi(T) \end{pmatrix}, \begin{pmatrix} y(T) \\ \psi(T) \end{pmatrix} \rangle - \langle \Pi(0) \begin{pmatrix} y(0) \\ \psi(0) \end{pmatrix}, \begin{pmatrix} y(0) \\ \psi(0) \end{pmatrix} \rangle
= \int_{0}^{T} \left[ \begin{pmatrix} \langle \Pi(t)[\mathbb{B}(t) + M(t)] + [\mathbb{B}(t) + M(t)]^* \Pi(t) - Q(t) \rangle y(t) \\ \psi(t) \end{pmatrix}, \begin{pmatrix} y(t) \\ \psi(t) \end{pmatrix} \right] dt + 2\langle \Pi(t) \begin{pmatrix} b_0(t) \\ -g_0(t) \end{pmatrix}, \begin{pmatrix} y(t) \\ \psi(t) \end{pmatrix} \rangle dt.
\]

**Proof.** For \(\lambda > 0\), let \(\Phi_{\lambda}(\cdot, \cdot)\) and \(\bar{\Phi}_{\lambda}(\cdot, \cdot)\) be the evolution operators generated by \(A_{\lambda} - M(\cdot)\) and \(A_{\lambda} - \bar{M}(\cdot)\), respectively. Define
\[
P_{\lambda}(t) = \Phi_{\lambda}(T, t)^* P(T) \Phi_{\lambda}(T, t) + \int_{t}^{T} \Phi_{\lambda}(s, t)^* Q_0(s) \Phi_{\lambda}(s, t) ds, \quad t \in [0, T],
\]
and
\[
P_{\lambda}(t) = \bar{\Phi}_{\lambda}(t, 0)^* \bar{P}(0) \bar{\Phi}_{\lambda}(t, 0) - \int_{0}^{t} \bar{\Phi}_{\lambda}(t, s) Q_0(s) \bar{\Phi}_{\lambda}(t, s)^* ds, \quad t \in [0, T].
\]
For the case $A^* = A$, we have (72) with (70) which leads to
\[ \Gamma(t)A\lambda = A\lambda \Gamma(t), \quad t \in [0, T]. \]
For the case $A^* = -A$, we define
\[ \Gamma(t) = \Phi(t)^{-1}\Gamma(T)\Phi(t) + \int_t^T \Phi(s)\Theta(s)\Phi(s)ds, \quad t \in [0, T]. \quad (81) \]
A direct computation shows that
\[ \Pi(t) + \Pi(t)[A\lambda - M(t)] + [A\lambda - M(t)]^*\Pi(t) + Q(t) = 0, \]
where
\[ A\lambda = \begin{pmatrix} A\lambda & 0 \\ 0 & -A\lambda \end{pmatrix}, \quad Q(t) = \begin{pmatrix} Q_0(t) & \Theta(t)^* \\ \Theta(t) & Q_0(t) \end{pmatrix}, \]
and
\[ \Pi(t) = \begin{pmatrix} P_\lambda(t) & \Gamma_\lambda(t)^* \\ \Gamma_\lambda(t) & P_\lambda(t) \end{pmatrix}, \]
with $\Gamma_\lambda(\cdot) = \Gamma(\cdot)$ for the case $A^* = A$. At the same time, we let
\[ \begin{cases} \begin{aligned} \frac{\partial y_\lambda}{\partial t} + g_\lambda &= [A\lambda + B(t)]y_\lambda + b_0(t) + q_\lambda(t), \\ y_\lambda(0) &= y(0), \quad \psi_\lambda(T) = \psi(T). \end{aligned} \end{cases} \]
Then
\[ \langle \Pi_\lambda(T) \begin{pmatrix} y_\lambda(T) \\ \psi_\lambda(T) \end{pmatrix}, \begin{pmatrix} y_\lambda(T) \\ \psi_\lambda(T) \end{pmatrix} \rangle - \langle \Pi_\lambda(0) \begin{pmatrix} y_\lambda(0) \\ \psi_\lambda(0) \end{pmatrix}, \begin{pmatrix} y_\lambda(0) \\ \psi_\lambda(0) \end{pmatrix} \rangle + 2\langle \Pi_\lambda(T) \begin{pmatrix} b_0(t) \\ -g_\lambda(t) \end{pmatrix}, \begin{pmatrix} y_\lambda(t) \\ \psi_\lambda(t) \end{pmatrix} \rangle dt \]
\[ = \int_0^T \left[ \langle [\Pi_\lambda(t)[B(t) + M(t)] + [A\lambda + B(t)]\Pi_\lambda(t) + Q(t)] \begin{pmatrix} y_\lambda(t) \\ \psi_\lambda(t) \end{pmatrix}, \begin{pmatrix} y_\lambda(t) \\ \psi_\lambda(t) \end{pmatrix} \rangle \right] dt. \]
Passing to the limit, we obtain our result. \qed

Next, we let
\[ G_0 = \left\{ (b_0, g_0, h_0) \mid b_0(\cdot), g_0(\cdot) \in L^2(0, T; X), \quad h_0 \in X \right\}. \]
For any $x \in X$ and $(b, g, h) \in G_1$, $(b_0, g_0, h_0) \in G_0$, and $\rho \in [0, 1]$, consider the following FBEE:
\[ \begin{cases} \begin{aligned} \dot{y}(t) &= Ay(t) + \rho b(t, y(t), \psi(t)) + b_0(t), \\ \dot{\psi}(t) &= -A^* \psi(t) - \rho q(t, y(t), \psi(t)) - g_0(t), \\ y(T) &= x, \quad \psi(T) = \rho h(y(T)) + h_0. \end{aligned} \end{cases} \quad t \in [0, T], \quad (82) \]
It is easy to see that for $\rho = 0$, (82) is a trivial decoupled FBEE which admits a unique mild solution, and for $\rho = 1$, (82) is essentially the same as (although it looks a little more general than) FBEE (1). We will show that under certain conditions, there exists an absolute constant $\varepsilon > 0$ such that when (82) is (uniquely) solvable for
some \( \rho \in [0, 1) \), it must be (uniquely) solvable for (82) with \( \rho \) replaced by \((\rho + \varepsilon) \wedge 1\). Then by repeating the same argument, we obtain the (unique) solvability of (1) over \([0, T]\). Such an argument is called a method of continuation (see [25]). In doing so, the key is to establish an a priori estimate for the mild solutions to (82), uniform in \( \rho \in [0, 1] \). To this end, we need to make some preparations.

For any \( \lambda > 0 \), we introduce the following approximate system of FBEE (82):

\[
\begin{align*}
\dot{y}_\lambda(t) &= A\lambda y_\lambda(t) + \rho b(t, y_\lambda(t), \psi_\lambda(t)) + b_0(t), \\
\dot{\psi}_\lambda(t) &= -A\lambda \psi_\lambda(t) - \rho g(t, y_\lambda(t), \psi_\lambda(t)) - g_0(t), \\
y_\lambda(0) &= x, \quad \psi_\lambda(T) = \rho h(y_\lambda(T)) + h_0.
\end{align*}
\]  

(83)

Suppose for the initial condition \( x \in X \), the generator \((b, g, h) \in \mathcal{G}_4 \) and \((b_0, g_0, h_0) \in \mathcal{G}_0 \). FBEE (83) admits a solution \((y_\lambda(\cdot), \psi_\lambda(\cdot))\). Also, let \((\hat{y}_\lambda(\cdot), \hat{\psi}_\lambda(\cdot))\) be a solution of (83) with \((b, g, h) \in \mathcal{G}_1\), \((b_0, g_0, h_0) \in \mathcal{G}_0\), and \(x\) respectively replaced by \((\hat{b}, \hat{g}, \hat{h}) \in \mathcal{G}_1\), \((\hat{b}_0, \hat{g}_0, \hat{h}_0) \in \mathcal{G}_0\), and \(\hat{x} \in X\). Define

\[
\hat{y}(\cdot) = \hat{y}_\lambda(\cdot) - y_\lambda(\cdot), \quad \hat{\psi}(\cdot) = \hat{\psi}_\lambda(\cdot) - \psi_\lambda(\cdot).
\]  

(84)

Denote

\[
\begin{align*}
\hat{b}(t) &= \int_0^1 b(t, y_\lambda(t), \psi_\lambda(t)) + \alpha \hat{y}(t), \psi_\lambda(t))d\alpha, \\
\hat{h}(t) &= \int_0^1 h(t, y_\lambda(t), \psi_\lambda(t)) + \alpha \hat{\psi}(t))d\alpha,
\end{align*}
\]

and set

\[
\begin{align*}
\delta b(t) &= \hat{b}(t, y_\lambda(t), \psi_\lambda(t)) - b(t, y_\lambda(t), \psi_\lambda(t)), \quad \delta h(t) = \hat{h}(t, y_\lambda(t), \psi_\lambda(t)) - h(t, y_\lambda(t), \psi_\lambda(t)), \\
\delta g(t) &= g(t, y_\lambda(t), \psi_\lambda(t)) - g(t, \hat{y}(t), \hat{\psi}(t)), \quad \delta g_0(t) = \hat{g}_0(t) - \hat{g}_0(t), \\
\delta h_0 &= \hat{h}_0 - h_0, \quad \hat{x} = \bar{x} - x.
\end{align*}
\]  

(86)

Then \((\hat{y}(\cdot), \hat{\psi}(\cdot))\) satisfies

\[
\begin{align*}
\dot{\hat{y}}(t) &= A\lambda \hat{y}(t) + \rho \hat{b}(t) \hat{y}(t) + \rho \hat{h}(t) \hat{\psi}(t) + \rho \delta b(t) + \delta h_0(t), \\
\dot{\hat{\psi}}(t) &= -A\lambda \hat{\psi}(t) - \rho \hat{g}(t) \hat{y}(t) - \rho \hat{g}(t) \hat{\psi}(t) - \rho \delta g(t) - \delta g_0(t), \\
\hat{y}(0) &= \bar{x}, \quad \hat{\psi}(T) = \rho \hat{h}(\hat{y}(T)) + \delta h + \delta h_0.
\end{align*}
\]  

(87)

For the above linear FBEE, we have the following result.

**Proposition 5.2.** Let \((b, g, h) \in \mathcal{G}_3\) and

\[
\begin{align*}
L_{b\psi}(t) &= \sup_{(y, \psi) \in X \times X} \left[ \max \sigma \left( \frac{b(t, y, \psi) + b(t, \psi, y)}{2} \right) \right]^{+}, \quad t \in [0, T], \\
L_{g\psi}(t) &= \sup_{(y, \psi) \in X \times X} \left[ \max \sigma \left( \frac{g(t, y, \psi) + g(t, \psi, y)}{2} \right) \right]^{+}, \quad t \in [0, T].
\end{align*}
\]  

(88)
where \( \sigma(\Lambda) \) is the spectrum of the operator \( \Lambda \in \mathcal{L}(X) \). Let \((y^0(\cdot), \psi^0(\cdot))\) be a solution to \(\text{FBEE} (83)\), and \((\tilde{y}^0(\cdot), \tilde{\psi}^0(\cdot))\) be a solution of \((83)\) corresponding to \((b, \tilde{g}, \tilde{h}) \in \mathcal{G}_1, (b_0, g_0, h_0) \in \mathcal{G}_0 \) and \( \tilde{x} \in X \). Then

\[
\| \tilde{y}(\cdot) \|_\infty \leq \rho \int_0^T e^{\rho \int_0^t L_{b_0}(\tau) d\tau} \| \tilde{b}_0(s) \tilde{\psi}(s) \| ds \\
+ K \left[ \| \tilde{x} \| + \int_0^T \left( \| \delta b(s) \| + \| \delta b_0(s) \| \right) ds \right], 
\]

and

\[
\| \tilde{\psi}(\cdot) \|_\infty \leq \rho \left[ e^{\rho \int_0^T L_{g_0}(\tau) d\tau} \| \tilde{g}_0(T) \| + \int_0^T \left( e^{\rho \int_0^\tau L_{g_0}(\tau) d\tau} \| \tilde{g}_0(s) \tilde{\psi}(s) \| ds \right) \\
+ K \left[ \| \tilde{h} \| + \| \delta h_0 \| + \int_0^T \left( \| \delta g(s) \| + \| \delta g_0(s) \| \right) ds \right].
\]

The proof is straightforward and for reader’s convenience, a proof is presented in the appendix.

We note that in the above proposition, it is only assumed that \((b, g, h) \in \mathcal{G}_3\) (the set of all generators satisfying \((H3)\)). Therefore, the Fréchet derivatives \( b, b_0, \psi, \tilde{\psi} \), and so on are not necessarily bounded. However, it is still possible that

\[
\int_0^T \left( \| \tilde{b}_0(s) \tilde{\psi}(s) \| + \| \tilde{g}_0(s) \tilde{\psi}(s) \| \right) ds < \infty.
\]

On the other hand, in the case \((b, g, h) \in \mathcal{G}_4\), we have

\[
\| \tilde{y}(\cdot) \|_\infty \leq \rho \int_0^T e^{\rho \int_0^t L_{b_0}(\tau) d\tau} \| \tilde{b}_0(s) \tilde{\psi}(s) \| ds \\
+ K \left[ \| \tilde{x} \| + \int_0^T \left( \| \delta b(s) \| + \| \delta b_0(s) \| \right) ds \right] \\
\leq \rho \left[ \int_0^T e^{\rho \int_0^\tau L_{b_0}(\tau) d\tau} \| \tilde{b}_0(s) \tilde{\psi}(s) \| ds \right] \| \tilde{\psi}(\cdot) \|_\infty \\
+ K \left[ \| \tilde{x} \| + \int_0^T \left( \| \delta b(s) \| + \| \delta b_0(s) \| \right) ds \right],
\]

and

\[
\| \tilde{\psi}(\cdot) \|_\infty \leq \rho \left[ e^{\rho \int_0^T L_{g_0}(\tau) d\tau} \| \tilde{g}_0(T) \| + \int_0^T \left( e^{\rho \int_0^\tau L_{g_0}(\tau) d\tau} \| \tilde{g}_0(s) \tilde{\psi}(s) \| ds \right) \\
+ K \left[ \| \tilde{h} \| + \| \delta h_0 \| + \int_0^T \left( \| \delta g(s) \| + \| \delta g_0(s) \| \right) ds \right] \\
\leq \rho^2 \left[ e^{\rho \int_0^T L_{g_0}(\tau) d\tau} \| \tilde{g}_0(T) \| + \int_0^T e^{\rho \int_0^\tau L_{g_0}(\tau) d\tau} \| \tilde{g}_0(s) \tilde{\psi}(s) \| ds \right] \\
\cdot \left[ \int_0^T e^{\rho \int_0^\tau L_{y_0}(\tau) d\tau} \| \tilde{b}_0(s) \| ds \right] \| \tilde{\psi}(\cdot) \|_\infty \\
+ K \left[ \| \tilde{x} \| + \| \delta h \| + \| \delta h_0 \| + \int_0^T \left( \| \delta b(s) \| + \| \delta b_0(s) \| + \| \delta g(s) \| + \| \delta g_0(s) \| \right) ds \right].
\]
Hence, when the following holds:

$$
\rho^2 \left[ e^{\rho T} L_{\psi}(\tau) \|h_\psi\| + \int_0^T e^{\rho \int_0^\tau L_{\psi}(\rho) d\tau} \|g_\psi(s)\| ds \right] \cdot \left[ \int_0^T e^{\rho \int_0^\tau L_{\psi}(\rho) d\tau} \|g_\psi(s)\| ds \right] < 1,
$$

FBEE (83) admits a unique mild solution \((y_\rho^\lambda(\cdot), \psi_\rho^\lambda(\cdot))\), by means of contraction mapping theorem. It is not hard to see that condition (91) holds when one of the following holds:

- The parameter \(\rho = 0\), this is a trivial case, for which the FBEE is linear and decoupled.
- The time duration \(T\) is small enough.
- The coupling is weak enough in the sense that the Lipschitz constant of \(b(t, y, \psi)\) with respect to \(\psi\) (the bound of \(b_\psi(\cdot)\)), and/or the Lipschitz constants of \(g(t, y, \psi)\) and \(h(y)\) with respect to \(y\) (the bounds of \(g_y(\cdot)\) and \(h_y(\cdot)\)) are small enough. An extreme case is that \(b(t, y, \psi)\) is independent of \(\psi\), or \(g(t, y, \psi)\) and \(h(y)\) are independent of \(y\), which corresponds to the decoupled case.

From Proposition 5.2, we see that due to the coupling, in general, one can only obtain an estimate of \(\hat{\psi}(\cdot)\) in terms of \(\hat{\psi}(\cdot)\), and an estimate of \(\hat{\psi}(\cdot)\) in terms of \(\hat{y}(\cdot)\). In order to obtain an a priori estimate on the whole \((\hat{y}(\cdot), \hat{\psi}(\cdot))\), we need either have an estimate for

$$
\int_0^T \|\hat{b}_\psi(s)\hat{\psi}(s)\|^2 ds
$$

independent of \(\hat{y}(\cdot)\), or have an estimate for

$$
\|\hat{h}_\psi(T)\|^2 + \int_0^T \|\hat{g}_\psi(s)\hat{\psi}(s)\|^2 ds
$$

independent of \(\hat{\psi}(\cdot)\). We now search conditions under which this is possible. To this end, we introduce the following notions.

**Definition 5.3.** A continuous function \(\Pi(\cdot) \equiv \begin{pmatrix} P(\cdot) & \Gamma(\cdot)^* \\ \Gamma(\cdot) & P(\cdot) \end{pmatrix} : [0, T] \to \mathcal{S}(X \times X)\) is called a type (I) Lyapunov operator of the generator \((b, g, h) \in \mathcal{G}_3\) if there exist \(Q : [0, T] \to \mathcal{S}(X \times X)\) and \(M : [0, T] \to \mathcal{L}(X \times X)\) with

$$
Q(t) = \begin{pmatrix} Q_0(t) & \Theta(t)^* \\ \Theta(t) & Q_0(t) \end{pmatrix}, \quad M(t) = \begin{pmatrix} M(t) & 0 \\ 0 & -\bar{M}(t)^* \end{pmatrix}, \quad t \in [0, T],
$$

such that \(\Pi(\cdot)\) is a mild solution to the Lyapunov differential equation

$$
\dot{\Pi}(t) + \Pi(t)[A - M(t)] + [A - M(t)]^*\Pi(t) + Q(t) = 0, \quad t \in [0, T].
$$

(92)
and for some constants $\mu, K > 0$, the following are satisfied:

\[
\begin{cases}
\Pi(0) + \begin{pmatrix} -K & 0 \\ 0 & 0 \end{pmatrix} \leq 0, \\
\Pi(T) + \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} \rho h_y(y) & 0 \\ 0 & \rho h_y(y) \end{pmatrix} \geq 0, \\
\Pi(t) \leq 0, \\
\Pi(t) \leq 0,
\end{cases}
\]

(93)

where

\[
\mathbb{H}^\rho(t, \Pi(t), y, \psi) - Q(t) + \mu \begin{pmatrix} g_y(t, y, \psi) & g_y(t, y, \psi) \\ 0 & 0 \end{pmatrix} \Pi(t) \leq 0,
\]

\[
\forall(t, y, \psi) \in [0, T] \times X \times X, \rho \in [0, 1],
\]

and

\[
\mathbb{H}^\rho(t, \Pi(t), y, \psi) = \rho \left[ \Pi \mathbb{B}(t, y, \psi) + \mathbb{B}(t, y, \psi) \Pi \right] + \Pi \mathbb{M}(t) + \mathbb{M}(t) \Pi,
\]

(94)

If (93) is replaced by the following:

\[
\begin{cases}
\Pi(0) + \begin{pmatrix} -K & 0 \\ 0 & 0 \end{pmatrix} \leq 0, \\
\Pi(T) + \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} \rho h_y(y) & 0 \\ 0 & \rho h_y(y) \end{pmatrix} \geq 0, \\
\Pi(t) \leq 0, \\
\Pi(t) \leq 0,
\end{cases}
\]

(95)

then $\Pi(\cdot)$ is called a type (II) Lyapunov operator of $(b, g, h)$.

If $\Pi(\cdot)$ is either a type (I) or Type (II) Lyapunov operator of $(b, g, h)$, we simply call it a Lyapunov operator of $(b, g, h)$.

The existence of a Lyapunov operator gives some kind of compatibility of the coefficients in FBEE (1), which will guarantee the well-posedness of the FBEE. We will carefully discuss properties and existence of Lyapunov operators a little later. First, we present the following result gives the (uniform) stability of mild solutions to (82) when the generator $(b, g, h) \in \mathcal{G}_3$ admits a Lyapunov operator.

**Proposition 5.4.** Let $(b, g, h) \in \mathcal{G}_3$ admit a Lyapunov operator $\Pi(\cdot)$ of either type (I) or (II). For any $\rho \in [0, 1]$, and $x \in X$, let $(y^\rho(\cdot), \psi^\rho(\cdot))$ be a mild solution of FBEE (82) with some $(b_0(\cdot), g_0(\cdot), h_0) \in \mathcal{G}_0$, and let $(\tilde{y}^\rho(\cdot), \tilde{\psi}^\rho(\cdot))$ be a mild solutions of FBEE (82) corresponding to another generator $(b, g, h) \in \mathcal{G}_1$, and some
\[(\bar{b}_0(\cdot), \bar{g}_0(\cdot), \bar{h}_0) \in \mathcal{G}_0, \bar{x} \in X. \text{ Then} \]

\[
\|\bar{y}^\rho(\cdot) - y^\rho(\cdot)\|_\infty + \|\bar{\psi}^\rho(\cdot) - \psi^\rho(\cdot)\|_\infty \\
\leq K \left\{ \|\bar{x} - x\|^2 + \|\bar{h}_0 - h_0\|^2 + \|\bar{h}(\bar{y}^\rho(T)) - h(y^\rho(T))\|^2 \right. \\
+ \int_0^T \left( \|\bar{b}(s, \bar{y}^\rho(s), \bar{\psi}^\rho(s)) - b(t, \bar{y}^\rho(t), \bar{\psi}^\rho(s))\|^2 + \|\bar{b}_0(s) - b_0(s)\|^2 \\
+ \|\bar{g}(s, \bar{y}^\rho(s), \bar{\psi}^\rho(s)) - g(s, \bar{y}^\rho(s), \bar{\psi}^\rho(s))\|^2 + \|\bar{g}_0(s) - g_0(s)\|^2 \right) ds \right\}, \tag{96}
\]

uniformly in \(\rho \in [0, 1]\). In particular, if \((\bar{b}, \bar{g}, \bar{h}) = (b, g, h)\), then

\[
\|\bar{y}^\rho(\cdot) - y^\rho(\cdot)\|_\infty^2 + \|\bar{\psi}^\rho(\cdot) - \psi^\rho(\cdot)\|_\infty^2 \\
\leq K \left\{ \|\bar{x} - x\|^2 + \|\bar{h}_0 - h_0\|^2 + \int_0^T \left( \|\bar{b}_0(s) - b_0(s)\|^2 + \|\bar{g}_0(t) - g_0(s)\|^2 \right) ds \right\}, \tag{97}
\]

uniformly in \(\rho \in [0, 1]\).

**Proof.** Recall notations in (84)–(86) and noting Proposition 5.1, we have (suppressing \(s\) when it has no ambiguity)

\[
\langle \begin{pmatrix} I & \rho(\bar{h})^* \end{pmatrix} \Pi(T) \begin{pmatrix} I & 0 \\ \rho\bar{h} \end{pmatrix} \begin{pmatrix} \hat{\bar{y}}(T) \\ \hat{x} \end{pmatrix} \begin{pmatrix} \hat{\bar{y}}(T) \\ \hat{x} \end{pmatrix} \rangle \\
- \langle \Pi(0) \begin{pmatrix} \hat{x} \\ \hat{x} \end{pmatrix} \begin{pmatrix} \hat{x} \\ \hat{x} \end{pmatrix} \rangle \\
= \langle \Pi(T) \begin{pmatrix} \hat{\bar{y}}(T) \\ \rho\bar{h} \hat{\bar{y}}(T) + \rho\delta h + \delta h_0 \end{pmatrix} \begin{pmatrix} \hat{x} \\ \hat{x} \end{pmatrix} \begin{pmatrix} \hat{x} \\ \hat{x} \end{pmatrix} \rangle \\
- \langle \Pi(0) \begin{pmatrix} \hat{x} \\ \hat{x} \end{pmatrix} \begin{pmatrix} \hat{x} \\ \hat{x} \end{pmatrix} \rangle \\
= \langle \Pi(T) \begin{pmatrix} \hat{\bar{y}}(T) \\ \hat{\psi}(T) \end{pmatrix} \begin{pmatrix} \hat{\bar{y}}(T) \\ \hat{\psi}(T) \end{pmatrix} \rangle \\
- \langle \Pi(0) \begin{pmatrix} \hat{x} \\ \hat{x} \end{pmatrix} \begin{pmatrix} \hat{x} \\ \hat{x} \end{pmatrix} \rangle \\
= \int_0^T \left\{ \langle \begin{pmatrix} \hat{\bar{y}^\rho} - \mathcal{Q} \\ \hat{\psi} \end{pmatrix} \begin{pmatrix} \hat{\bar{y}} \\ \hat{\psi} \end{pmatrix} \rangle + 2\langle \begin{pmatrix} \hat{\bar{y}} \\ \hat{\psi} \end{pmatrix} \begin{pmatrix} \rho\delta b + \delta b_0 \\ \rho\delta g + \delta g_0 \end{pmatrix} \rangle \right\} ds \\
= \int_0^T \langle \begin{pmatrix} \hat{\bar{y}^\rho} - \mathcal{Q} \\ \Pi \end{pmatrix} \begin{pmatrix} \hat{\bar{y}} \\ \rho\delta b + \delta b_0 \\ \rho\delta g + \delta g_0 \end{pmatrix} \begin{pmatrix} \hat{\bar{y}} \\ \rho\delta b + \delta b_0 \\ \rho\delta g + \delta g_0 \end{pmatrix} \rangle ds,
\]

where

\[
\mathcal{H}^\rho(t, \Pi, \psi) = \rho \left[ \Pi \bar{\mathcal{B}} + \bar{\mathcal{B}}^* \Pi \right] + \Pi \mathcal{M}(t) + \mathcal{M}(t)^* \Pi,
\]

and

\[
\bar{\mathcal{B}} = \begin{pmatrix} \bar{b}_y \\ -\bar{g}_y \end{pmatrix},
\]

with \(\bar{b}_y\), etc. given by (85). Consequently, in the case that \(\Pi(\cdot)\) is a type (I) Lyapunov operator of \((b, g, h)\), we have
Hence,

\[
\langle \Pi(T) \left( \frac{\hat{g}(T)}{\rho h_y(T) + \rho h + \delta h_0} \right), \left( \frac{\hat{g}(T)}{\rho h_y(T) + \rho h + \delta h_0} \right) \rangle - \langle \Pi(0) \left( \frac{\hat{x}}{\hat{\psi}(0)} \right), \left( \frac{\hat{x}}{\hat{\psi}(0)} \right) \rangle \geq \mu \| \hat{h}_y(T) \|^2 - K \left( \| \hat{x} \|^2 + \| \delta h \|^2 + \| \delta h_0 \|^2 \right),
\]

and

\[
\int_0^T \left( \left( \frac{\hat{g}(T)}{\rho h_y(T) + \rho h + \delta h_0} \right), \left( \frac{\hat{g}(T)}{\rho h_y(T) + \rho h + \delta h_0} \right) \right) ds \leq \int_0^T \left( - \mu \| \hat{g}_y(s) \hat{g}(s) \|^2 + K \| \rho \delta b(s) + \delta b_0(s) \|^2 + K \| \rho \delta g(s) + \delta g_0(s) \|^2 \right) ds.
\]

Hence,

\[
\| \hat{h}_y(T) \|^2 + \int_0^T \| \hat{g}_y(s) \hat{g}(s) \|^2 ds \leq K \left( \| \hat{x} \|^2 + \| \delta h \|^2 + \| \delta h_0 \|^2 \right)
\]

\[
+ \int_0^T \left( \| \delta b(s) \|^2 + \| \delta b_0(s) \|^2 + \| \delta g(s) \|^2 + \| \delta g_0(s) \|^2 \right) ds.
\]

Combining the above with (89) and (90), we obtain (96).

On the other hand, in the case that \( \Pi(\cdot) \) is a type (II) Lyapunov operator for \( (b, g, h) \), we have

\[
\langle \Pi(T) \left( \frac{\hat{g}(T)}{\rho h_y(T) + \rho h + \delta h_0} \right), \left( \frac{\hat{g}(T)}{\rho h_y(T) + \rho h + \delta h_0} \right) \rangle - \langle \Pi(0) \left( \frac{\hat{x}}{\hat{\psi}(0)} \right), \left( \frac{\hat{x}}{\hat{\psi}(0)} \right) \rangle \geq -K \left( \| \hat{x} \|^2 + \| \delta h \|^2 + \| \delta h_0 \|^2 \right),
\]

and

\[
\int_0^T \left( \left( \frac{\hat{g}(T)}{\rho h_y(T) + \rho h + \delta h_0} \right), \left( \frac{\hat{g}(T)}{\rho h_y(T) + \rho h + \delta h_0} \right) \right) ds \leq \int_0^T \left( - \mu \| \hat{b}_y(s) \hat{g}(s) \|^2 + K \| \rho \delta b(s) + \delta b_0(s) \|^2 + K \| \rho \delta g(s) + \delta g_0(s) \|^2 \right) ds.
\]

Hence,

\[
\int_0^T \| \hat{b}_y(s) \hat{g}(s) \|^2 ds \leq K \left( \| \hat{x} \|^2 + \| \delta h \|^2 + \| \delta h_0 \|^2 \right)
\]

\[
+ \int_0^T \left( \| \delta b(s) \|^2 + \| \delta b_0(s) \|^2 + \| \delta g(s) \|^2 + \| \delta g_0(s) \|^2 \right) ds.
\]

Then, combining the above with (89) and (90), we again obtain (96).
6. Well-posedness of FBEEs via Lyapunov operators. We now state and prove the following theorem concerning the well-posedness of FBEE (1).

**Theorem 6.1.** Let \((b, g, h) \in \mathcal{G}_2 \cap \mathcal{G}_3\) admit a type (I) or (II) Lyapunov operator \(\Pi(\cdot)\). Then FBEE (1) admits a unique mild solution \((y(\cdot), \psi(\cdot))\). Moreover, the following estimate holds:

\[
\|y(\cdot)\|_\infty + \|\psi(\cdot)\|_\infty \leq K \left[ \|x\| + \|h(0)\| + \int_0^T \left( \|b(s, 0, 0)\| + \|g(s, 0, 0)\| \right) ds \right].
\] (98)

Further, if \((\tilde{y}(\cdot), \tilde{\psi}(\cdot))\) is a mild solution of FBEE (1) corresponding to \((\tilde{b}, \tilde{g}, \tilde{h}) \in \mathcal{G}_2 \cap \mathcal{G}_3\), then the following stability estimate holds:

\[
\|\tilde{y}(\cdot) - y(\cdot)\|_\infty + \|\tilde{\psi}(\cdot) - \psi(\cdot)\|_\infty \leq K \left\{ \|\tilde{x} - x\| + \|\tilde{h}(\tilde{y}(T)) - h(y(T))\|
\right.
\]
\[+ \int_0^T \left( \|\tilde{b}(s, \tilde{y}(s), \tilde{\psi}(s)) - b(s, y(s), \psi(s))\|
\right.
\]
\[\left. + \|\tilde{g}(s, \tilde{y}(s)) - g(s, y(s), \psi(\cdot))\| \right) ds \}.
\] (99)

**Proof.** Let \((b_0, g_0, h_0) \in \mathcal{G}_0\). Let \(\rho \in [0, 1]\). Suppose the following (coupled) FBEE admits a unique mild solution \((y(\cdot), \psi(\cdot))\):

\[
\begin{aligned}
\dot{y}_\rho(t) &= Ay_\rho(t) + \rho b(t, y_\rho(t), \psi_\rho(t)) + b_0(t), \\
\dot{\psi}_\rho(t) &= -A^* \psi_\rho(t) + \rho g(t, y_\rho(t), \psi_\rho(t)) + g_0(t), \\
y_\rho(0) &= x, \\
\psi(T) &= \rho h(y(T)) + h_0,
\end{aligned}
\] (100)

and the following estimate holds:

\[
\|y(\cdot)\|_\infty + \|\psi(\cdot)\|_\infty \leq K \left\{ \|x\| + \|h(0)\| + \int_0^T \left( \|b(s, 0, 0)\| + \|g(s, 0, 0)\| \right) ds \right\}.
\] (101)

Now, let \(\varepsilon > 0\) such that \(\rho + \varepsilon \in [0, 1]\). Consider the following coupled FBEE:

\[
\begin{aligned}
\dot{y}^{\rho+\varepsilon}(t) &= Ay^{\rho+\varepsilon}(t) + (\rho + \varepsilon) b(t, y^{\rho+\varepsilon}(t), \psi^{\rho+\varepsilon}(t)) + b_0(t), \\
\dot{\psi}^{\rho+\varepsilon}(t) &= -A^* \psi^{\rho+\varepsilon}(t) + (\rho + \varepsilon) g(t, y^{\rho+\varepsilon}(t), \psi^{\rho+\varepsilon}(t)) + g_0(t), \\
y^{\rho+\varepsilon}(0) &= x, \\
\psi^{\rho+\varepsilon}(T) &= (\rho + \varepsilon) h(y^{\rho+\varepsilon}(T)) + h_0,
\end{aligned}
\] (102)

To obtain the (unique) solvability of the above problem, we introduce the following sequence of problems:

\[
\begin{aligned}
y^{\rho+\varepsilon,0}(\cdot) &= \psi^{\rho+\varepsilon,0}(\cdot) = 0, \\
y^{\rho+\varepsilon,k+1}(t) &= Ay^{\rho+\varepsilon,k+1}(t) + \rho b(t, y^{\rho+\varepsilon,k+1}(t), \psi^{\rho+\varepsilon,k+1}(t)) \\
&\quad + \varepsilon b(t, y^{\rho+\varepsilon,k+1}(t), \psi^{\rho+\varepsilon,k+1}(t)) + b_0(t), \\
\dot{\psi}^{\rho+\varepsilon,k+1}(t) &= -A^* \psi^{\rho+\varepsilon,k+1}(t) + \rho g(t, y^{\rho+\varepsilon,k+1}(t), \psi^{\rho+\varepsilon,k+1}(t)) \\
&\quad + \varepsilon g(t, y^{\rho+\varepsilon,k+1}(t), \psi^{\rho+\varepsilon,k+1}(t)) + g_0(t), \\
y^{\rho+\varepsilon,k+1}(0) &= x, \\
\psi^{\rho+\varepsilon,k+1}(T) &= \rho h(y^{\rho+\varepsilon,k+1}(T)) + \varepsilon h(y^{\rho+\varepsilon,k+1}(T)) + h_0.
\end{aligned}
\] (103)

By our assumption, inductively, for each \(k \geq 0\), as long as \((y^{\rho+\varepsilon,k}(\cdot), \psi^{\rho+\varepsilon,k}(\cdot)) \in C([t, T]; X \times X)\), the above FBEE admits a unique mild solution \((y^{\rho+\varepsilon,k+1}(\cdot),\)
with $\psi \in C([t, T]; X \times X)$. Further,

$$\|y^{\rho+\varepsilon,k+1}(\cdot)\|_\infty + \|\psi^{\rho+\varepsilon,k+1}(\cdot)\|_\infty$$

$$\leq K \left\{ \|x\| + \|h(0)\| + \|h(y^{\rho+\varepsilon,k+1}(T)) + h_0\| 
+ \int_0^T \left( \|b(s, 0, 0)\| + \|b(s, y^{\rho+\varepsilon,k}(s), \psi^{\rho+\varepsilon,k}(s)) + b_0(s)\| \right) ds 
+ \int_0^T \left( \|g(s, 0, 0)\| + \|g(s, y^{\rho+\varepsilon,k}(s), \psi^{\rho+\varepsilon,k}(s)) + g_0(t)\| \right) ds \right\}$$

$$\leq K \varepsilon \left( \|y^{\rho+\varepsilon,k}(\cdot)\|_\infty + \|\psi^{\rho+\varepsilon,k}(\cdot)\|_\infty \right)$$

$$+ K \left\{ \|x\| + \|b(0)\| + \|h_0\| + \int_0^T \left( \|b(s, 0, 0)\| + \|b_0(s)\| \right) ds 
+ \|g(s, 0, 0)\| + \|g_0(s)\| \right) ds \right\}.$$

Now, since $(b, g, h)$ admits a type (I) or type (II) Lyapunov operator $P(\cdot)$, by Proposition 5.4, we obtain

$$\|y^{\rho+\varepsilon,k+1}(\cdot) - y^{\rho+\varepsilon,k}(\cdot)\|_\infty + \|\psi^{\rho+\varepsilon,k+1}(\cdot) - \psi^{\rho+\varepsilon,k}(\cdot)\|_\infty$$

$$\leq \varepsilon K \left\{ \|h(y^{\rho+\varepsilon,k}(T)) - h(y^{\rho+\varepsilon,k-1}(T))\| 
+ \int_0^T \left[ \|b(s, y^{\rho+\varepsilon,k}(s), \psi^{\rho+\varepsilon,k}(s)) - b(s, y^{\rho+\varepsilon,k-1}(s), \psi^{\rho+\varepsilon,k-1}(s))\| 
+ \|g(s, y^{\rho+\varepsilon,k}(s), \psi^{\rho+\varepsilon,k}(s)) - g(s, y^{\rho+\varepsilon,k-1}(s), \psi^{\rho+\varepsilon,k-1}(s))\| \right] ds \right\}$$

$$\leq K_0 \left( \|y^{\rho+\varepsilon,k}(\cdot) - y^{\rho+\varepsilon,k-1}(\cdot)\|_\infty + \|\psi^{\rho+\varepsilon,k}(\cdot) - \psi^{\rho+\varepsilon,k-1}(\cdot)\|_\infty \right).$$

Here, $K_0 > 0$ is an absolute constant (independent of $k \geq 1$). Thus, taking $\varepsilon > 0$ small enough so that $\varepsilon K_0 \leq \frac{1}{2}$, we obtain

$$\lim_{k \to \infty} \left( \|y^{\rho+\varepsilon,k}(\cdot) - y^{\rho+\varepsilon}(\cdot)\|_\infty + \|\psi^{\rho+\varepsilon,k}(\cdot) - \psi^{\rho+\varepsilon}(\cdot)\|_\infty \right) = 0,$$

with

$$\left\{ \begin{array}{l}
y^{\rho+\varepsilon}(\cdot) = \sum_{k=1}^\infty \left[ y^{\rho+\varepsilon,k}(\cdot) - y^{\rho+\varepsilon,k-1}(\cdot) \right], \\
\psi^{\rho+\varepsilon}(\cdot) = \sum_{k=1}^\infty \left[ \psi^{\rho+\varepsilon,k}(\cdot) - \psi^{\rho+\varepsilon,k-1}(\cdot) \right].
\end{array} \right.$$

which is the unique mild solution of FBEE (102). Further, let $k \to \infty$ in (104), we obtain

$$\|y^{\rho+\varepsilon}\|_\infty + \|\psi^{\rho+\varepsilon}\|_\infty \leq K \varepsilon \left( \|y^{\rho+\varepsilon}(\cdot)\|_\infty + \|\psi^{\rho+\varepsilon}(\cdot)\|_\infty \right)$$
Let \( \text{Theorem 7.1.} \) be practically more convenient to use than the definition.

The posedness of corresponding FBEEs. First of all, we prove the following result which will construct some Lyapunov operators, through which we obtain well-poseness of FBEE:

\[
\begin{align*}
\text{Then, } \bar{y}(0) &= x, \quad \psi(T) = h(y(T)) + b_0,
\end{align*}
\]

with the mild solution \((\bar{y}(\cdot), \psi(\cdot))\) satisfying

\[
\|y(\cdot)\| + \|\psi(\cdot)\| \leq K \left\{ \|x\| + \|h(0)\| + \int_0^T \left( \|b(s, 0, 0)\| + \|b_0(s)\| \right) ds \right\}.
\]

Continuing the above procedure, we obtain the solvability of the following coupled FBEE:

\[
\begin{align*}
\dot{y}(t) &= Ay(t) + b(t, y(t), \psi(t)) + b_0(t), \\
\dot{\psi}(t) &= -A^*\psi(t) - g(t, y(t), \psi(t)) - g_0(t), \\
y(0) &= x, \\
\psi(0) &= x,
\end{align*}
\]

Thus, in particular, by taking \((b_0, g_0, h_0) = 0\), we obtain the solvability of FBEE (1.1) with estimate

\[
\|y(\cdot)\| + \|\psi(\cdot)\| \leq K \left\{ \|x\| + \|h(0)\| + \int_0^T \left( \|b(s, 0, 0)\| + \|b_0(s)\| \right) ds \right\}.
\]

Now, let \((\bar{y}(\cdot), \bar{\psi}(\cdot))\) be a mild solution to (1.1) corresponding to \((\bar{b}, \bar{g}, \bar{h}) \in \mathcal{G}_2 \cap \mathcal{G}_3\). Then, \((\bar{y}(\cdot) - y(\cdot), \bar{\psi}(\cdot) - \psi(\cdot))\) satisfies a linear FBE with the generator admitting a type (I) or type (II) Lyapunov operator \(\Pi(\cdot)\), the same as that for the generator \((b, g, h)\). Hence, applying (105), we obtain the following stability estimate:

\[
\|\bar{y}(\cdot) - y(\cdot)\| + \|\bar{\psi}(\cdot) - \psi(\cdot)\| \leq K \left\{ \|x - \bar{x}\| + \|\bar{h}(\bar{y}(T)) - h(\bar{y}(T))\| \\
+ \int_0^T \left( \|\bar{b}(s, \bar{g}(s), \bar{\psi}(s)) - b(s, \bar{y}(s), \bar{\psi}(s))\| \\
+ \|\bar{g}(s, \bar{y}(s), \bar{\psi}(s)) - g(s, \bar{y}(s), \bar{\psi}(s))\| \right) ds \right\}.
\]

This proves the theorem.

7. Construction of Lyapunov operators and solvable FBEEs. In this section, we will construct some Lyapunov operators, through which we obtain well-posedness of corresponding FBEEs. First of all, we prove the following result which is practically more convenient to use than the definition.

**Theorem 7.1.** Let \((H0)\) hold and let \((b, g, h) \in \mathcal{G}_2 \cap \mathcal{G}_3\). Let \(\Pi(\cdot) \equiv \begin{pmatrix} P(\cdot) & \Gamma(\cdot)^* \\ \Gamma(\cdot) & P(\cdot)^* \end{pmatrix}\) be a mild solution to linear Lyapunov differential equation (60) for some

\[
M(\cdot) \equiv \begin{pmatrix} M(\cdot) & 0 \\ 0 & -M(\cdot)^* \end{pmatrix}, \quad Q(\cdot) \equiv \begin{pmatrix} Q_0(\cdot) & \Theta(\cdot)^* \\ \Theta(\cdot) & Q_0(\cdot)^* \end{pmatrix}.
\]

\[
+K \left\{ \|x\| + \|h(0)\| + \int_0^T \left( \|b(s, 0, 0)\| + \|b_0(s)\| \right) ds \right\}.
\]

Note that the constant \(K\) in front of \(\varepsilon\) above is universal. Then choose an \(\varepsilon > 0\) satisfying \(K\varepsilon \leq \frac{1}{2}\), so that the first term on the right hand side can be absorbed into the left hand, leading to the following:

\[
\|y^\rho + \varepsilon(\cdot)\| + \|\psi^\rho + \varepsilon(\cdot)\| \leq K \left\{ \|x\| + \|h(0)\| + \int_0^T \left( \|b(s, 0, 0)\| + \|b_0(s)\| \right) ds \right\}.
\]
Then $\Pi(\cdot)$ is both a Lyapunov operator of types (I) and (II) for $(b, g, h)$ if the following hold:
\begin{equation}
\bar{P}(t) \leq -\delta, \quad P(T) \geq \delta, \quad (108)
\end{equation}
\begin{equation}
P(T) + h_y(y)^* \Gamma(T) + \Gamma(T)^* h_y(y) + h_y(y)^* \bar{P}(T) h_y(y) \geq \delta, \quad \forall y \in X, \quad (109)
\end{equation}
and
\begin{equation}
\begin{cases}
\Pi(t) \bar{M}(t) + \bar{M}(t)^* \Pi(t) - Q(t) \leq -\delta, & t \in [0, T], \\
\Pi(t) \mathcal{B}(t, y, \psi) + \mathcal{B}(t, y, \psi)^* \Pi(t) + \Pi(t) \mathcal{M}(t) + \bar{M}(t)^* \Pi(t) - Q(t) \leq -\delta, & (t, y, \psi) \in [0, T] \times X \times X.
\end{cases} \quad (110)
\end{equation}
for some $\delta > 0$, with
\begin{equation}
\mathcal{B}(t, y, \psi) = \begin{pmatrix} b_y(t, y, \psi) & b_{\psi}(t, y, \psi) \\ -g_y(t, y, \psi) & -g_{\psi}(t, y, \psi) \end{pmatrix}, \quad (t, y, \psi) \in [0, T] \times X \times X.
\end{equation}

Note that $\delta > 0$ appears in (108)–(110) does not have to be the same. But, we can always make them the same by shrinking $\delta$ if necessary.

**Proof.** First of all, in order $\Pi(\cdot)$ to be a type (I) Lyapunov operator of the generator $(b, g, h)$, one needs (93). Hence, at $t = 0$, one needs
\begin{equation}
0 \geq \Pi(0) + \begin{pmatrix} -K & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} P(0) - K & \Gamma(0)^* \\ \Gamma(0) & P(0) \end{pmatrix}, \quad (111)
\end{equation}
for some $K > 0$, which will be ensured by the following:
\begin{equation}
\bar{P}(0) \leq -\delta, \quad (112)
\end{equation}
for some $\delta > 0$. Next, at $t = T$, one needs
\begin{equation}
0 \leq \begin{pmatrix} I & \rho h_y(y)^* \\ 0 & I \end{pmatrix} \Pi(T) \begin{pmatrix} I & 0 \\ \rho h_y(y) & I \end{pmatrix} + \begin{pmatrix} -\mu h_y(y)^* h_y(y) & 0 \\ 0 & K \end{pmatrix}
\end{equation}
\begin{equation}
= \rho^2 \begin{pmatrix} 0 & h_y(y)^* \\ 0 & 0 \end{pmatrix} \Pi(T) \begin{pmatrix} 0 & h_y(y) \\ 0 & 0 \end{pmatrix}
\end{equation}
\begin{equation}
+ \rho \begin{pmatrix} 0 & h_y(y)^* \\ 0 & 0 \end{pmatrix} \Pi(T) + \Pi(T) \begin{pmatrix} 0 & 0 \\ h_y(y) & 0 \end{pmatrix}
\end{equation}
\begin{equation}
\begin{pmatrix} 0 & h_y(y)^* h_y(y) \\ 0 & K \end{pmatrix}, \quad \forall y \in X, \quad \rho \in [0, 1], \quad (112)
\end{equation}
for some $\mu, K > 0$. If we are able to show the following (which will be done below)
\begin{equation}
\bar{P}(T) \leq 0, \quad (113)
\end{equation}
then
\begin{equation}
\begin{pmatrix} 0 & h_y(y)^* \\ 0 & 0 \end{pmatrix} \Pi(T) \begin{pmatrix} 0 & h_y(y) \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} h_y(y)^* \bar{P}(T) h_y(y) & 0 \\ 0 & 0 \end{pmatrix} \leq 0. \quad (114)
\end{equation}
Hence, (112) is true if and only if it is true for $\rho = 0, 1$, i.e.,
\begin{equation}
0 \leq \Pi(T) + \begin{pmatrix} -\mu h_y(y)^* h_y(y) & 0 \\ 0 & K \end{pmatrix} = \begin{pmatrix} P(T) - \mu h_y(y)^* h_y(y) & \Gamma(T)^* \\ \Gamma(T) & \bar{P}(T) + K \end{pmatrix}, \quad (115)
\end{equation}
and

\[
0 \leq \begin{pmatrix}
0 & h_y(y)^* \\
0 & h_y(y)
\end{pmatrix} \Pi(T) \begin{pmatrix}
0 & 0 \\
0 & 0
\end{pmatrix} + \left[ \begin{pmatrix}
0 & h_y(y)^* \\
0 & 0
\end{pmatrix} \Pi(T) + \Pi(T) \begin{pmatrix}
0 & 0 \\
0 & h_y(y)
\end{pmatrix} \right]
\]

\[
+ \Pi(T) + \begin{pmatrix}
-\mu h_y(y)^* h_y(y) & 0 \\
0 & K
\end{pmatrix}
\]

\[
= \begin{pmatrix}
P(T) + h_y(y)^* \Gamma(T) + \Gamma(T)^* h_y(y) + h_y(y)^* \bar{P}(T) h_y(y) - \mu h_y(y)^* h_y(y) \\
\Gamma(T)^* + h_y(y)^* \bar{P}(T)
\end{pmatrix}
\]

By choosing \( \mu > 0 \) small enough and \( K > 0 \) large enough, we see that the above is implied by the following:

\[
\begin{pmatrix}
P(T) & \Gamma(T)^* \\
\Gamma(T) & \bar{P}(T) + K
\end{pmatrix} \geq \delta,
\]

for some \( \delta > 0 \). This will further be implied by

\[
P(T) \geq \delta, \quad P(T) + h_y(y)^* \Gamma(T) + \Gamma(T)^* h_y(y) + h_y(y)^* \bar{P}(T) h_y(y) \geq \delta,
\]

for some \( \delta > 0 \). Hence, to summarize, at \( t = 0, T \), it suffices to have (108)–(109).

Now, we look at \( t \in (0, T) \). One needs

\[
\begin{pmatrix}
\mathbb{P}(t, y, \psi) - \mathbb{Q}(t) + \mu \left( g_y(t, y, \psi)^* g_y(t, y, \psi) 0 \\
0 & 0
\right) \Pi(t)
\end{pmatrix} \leq 0,
\]

for some \( \mu, K > 0 \). The left-hand side is affine in \( \rho \). Hence, the above is true for all \( \rho \in [0, 1] \) if and only if it is true for \( \rho = 0, 1 \), i.e.,

\[
\begin{pmatrix}
\Pi(t) \mathcal{M}(t) + \mathcal{M}(t)^* \Pi(t) - \mathbb{Q}(t) + \mu \left( g_y(t, y, \psi)^* g_y(t, y, \psi) 0 \\
0 & 0
\right) \Pi(t)
\end{pmatrix} \leq 0,
\]

\[
\begin{pmatrix}
\Pi(t) \mathcal{B}(t, y, \psi) + \mathcal{B}(t, y, \psi)^* \Pi(t) + \mathcal{M}(t) \Pi(t) - \mathcal{R}(t) + \mu \left( g_y(t, y, \psi)^* g_y(t, y, \psi) 0 \\
0 & 0
\right) \Pi(t)
\end{pmatrix} \leq 0,
\]

for some \( \mu, K > 0 \). These are implied by the following: (by letting \( \mu > 0 \) small enough)

\[
\begin{pmatrix}
\Pi(t) \mathcal{M}(t) + \mathcal{M}(t)^* \Pi(t) - \mathbb{Q}(t) \\
\Pi(t)
\end{pmatrix} \leq -\delta,
\]

and

\[
\begin{pmatrix}
\Pi(t) \mathcal{B}(t, y, \psi) + \mathcal{B}(t, y, \psi)^* \Pi(t) + \Pi(t) \mathcal{M}(t) + \mathcal{M}(t)^* \Pi(t) - \mathbb{Q}(t) \\
\Pi(t)
\end{pmatrix} \leq -\delta,
\]

for some \( \delta > 0 \). It is clear that (115)–(116) hold for some large \( K > 0 \) if (110) holds.

Further, we note that the first condition in (110) implies

\[
\begin{pmatrix}
-P(t) M(t) - M(t)^* P(t) + Q_0(t) \geq \delta, \\
\bar{P}(t) \bar{M}(t) + \bar{M}(t) \bar{P}(t) + \bar{Q}_0(t) \geq \delta
\end{pmatrix}
\]
Hence, for any $t \in [t, T]$, using $P(T) \geq \delta$ and $\bar{P}(0) \leq -\delta$, we obtain
\begin{equation}
P(t) = e^{A(t-T)} P(T) e^{A(T-t)} + \int_t^T e^{A(s-t)} [-P(s)M(s) - M(s)^*P(s) + Q_0(s)] e^{A(t-s)} ds \geq 0,
\end{equation}
and
\begin{equation}
\bar{P}(t) = e^{At} \bar{P}(0) e^{A^*t} - \int_0^t e^{A(t-s)} [\bar{P}(s) \bar{M}(s)^* + \bar{M}(s) \bar{P}(s) + \bar{Q}_0(s)] e^{A^*(t-s)} ds \leq 0.
\end{equation}
In particular, (113) holds. This proves that under (108)–(110), $\Pi(\cdot)$ is a type (I) Lyapunov operator for the generator $(b, g, h)$.

We can similarly prove that under (108)–(110), $\Pi(\cdot)$ is also a type (II) Lyapunov operator for the generator $(b, g, h)$.

\begin{flushright}$\Box$
\end{flushright}

\textbf{Corollary 7.2.} Let (H0) hold and let $(b, g, h) \in G_2 \cap G_3$. Let $\Pi(\cdot) \equiv \begin{pmatrix} P(\cdot) & \Gamma(\cdot)^* \\ \Gamma(\cdot) & P(\cdot) \end{pmatrix}$ be a mild solution to (60) for some $\mathbb{M}(\cdot)$ and $\mathbb{Q}(\cdot)$ of form (107) such that (108)–(109) hold. Suppose the following hold:
\begin{equation}
\begin{aligned}
&\Pi(t)M(t) + M(t)^*\Pi(t) - Q(t) \leq -\delta - \varepsilon, \quad t \in [0, T], \\
&\Pi(t)B(t, y, \psi) + B(t, y, \psi)^*\Pi(t) \leq \varepsilon, \quad (t, y, \psi) \in (0, T) \times X \times X,
\end{aligned}
\end{equation}
for some $\delta, \varepsilon > 0$. Then $\Pi(\cdot)$ is both a Lyapunov operator of types (I) and (II) for $(b, g, h)$. In particular, this is the case if the following holds:
\begin{equation}
\begin{aligned}
&Q(t) \geq \delta + \varepsilon, \quad t \in [0, T], \\
&\Pi(t)B(t, y, \psi) + B(t, y, \psi)^*\Pi(t) \leq \varepsilon, \quad (t, y, \psi) \in (0, T) \times X \times X,
\end{aligned}
\end{equation}
for some $\delta, \varepsilon > 0$.

\textbf{Proof.} It is clear that (119) implies (110). In particular, by letting $\mathbb{M}(\cdot) = 0$, we see that (120) implies (110). \hfill \Box

We note that condition (119) is more convenient to check than (110). Next, we look at some concrete special cases of Theorem 7.1, which will be more practically useful. We first present the following result.

\textbf{Lemma 7.3.} Let (H0) hold and $p_1, \bar{p}_0, q_0, \bar{q}_0, \theta, \gamma \in \mathbb{R}$ and $m, \bar{m} > -\sigma_0$. Let
\begin{equation}
M(t) = mI, \quad \bar{M}(t) = \bar{m}I, \\
Q_0(t) = q_0I, \quad \bar{Q}_0(t) = \bar{q}_0I, \quad \Theta(t) = \theta I.
\end{equation}
The mild solution $\Pi(\cdot)$ of (60) satisfying
\begin{equation}
P(T) = p_1 I, \quad \bar{P}(0) = -\bar{p}_0 I, \quad \Gamma(T) = \gamma I,
\end{equation}
is given by the following: In the case $A^* = A$, for any $t \in [0, T],$
\begin{equation}
\Pi(t) = \begin{pmatrix}
\gamma e^{(m-m)(T-t)} + \theta e^{(m-m)(T-t) - 1} \\
\gamma e^{(m-m)(T-t) + \theta e^{(m-m)(T-t) - 1}} \\
\end{pmatrix} T \\
\begin{pmatrix}
p_1 e^{2(A-m)(T-t)} + \theta e^{2(A-m)(T-t)} - \bar{p}_0 e^{2(A-m)(T-t)} \\
q_0 e^{2(A-m)(T-t)} - \bar{q}_0 e^{2(A-m)(T-t)} - 1 \\
\end{pmatrix} T + 1/2 \begin{pmatrix}
q_0(A-m)^{-1} e^{2(A-m)(T-t)} - 1 \\
0 \\
\end{pmatrix} T.
\end{equation}
and in the case $A^* = -A$, for $t \in [0, T]$,
\[
\Pi(t) = \begin{cases}
\left[p_1 e^{-2m(T-t)} + \frac{q_0}{2m} \left(1 - e^{-2m(T-t)}\right)\right] I \\
\left[\gamma e^{(m-m)(T-t)} + \theta m (m-m)(T-t) I\right] I
\end{cases}
\]
(124)

In the above, the following convention is adopted:
\[
\frac{1 - e^{-\alpha \beta}}{\alpha} \equiv \beta, \quad \text{if } \alpha = 0.
\]
(125)

In particular, if
\[
\bar{m} = m,
\]
then in the case $A^* = A$, for $m \in \mathbb{R}(A)$ and $t \in [0, T]$,
\[
\Pi(t) = \begin{cases}
\left[p_1 e^{2(A-m)(T-t)} \left[\gamma + \theta(T-t)\right] I \right] I \\
\left[\gamma + \theta(T-t)\right] I \end{cases}
\]
(127)

and in the case $A^* = -A$, for $t \in [0, T]$,
\[
\Pi(t) = \begin{cases}
\left[p_1 e^{-2m(T-t)} + \frac{q_0}{2m} \left(1 - e^{-2m(T-t)}\right)\right] I \\
\left[\gamma + \theta(T-t)\right] I \end{cases}
\]
(128)

Further, if $m = \bar{m} = 0$, then, for $A^* = A$, $t \in [0, T]$,
\[
\Pi(t) = \begin{cases}
\left(p_1 + \frac{q_0}{2} A^{-1}\right) e^{2A(T-t)} + \frac{q_0}{2} A^{-1} \left[\gamma + \theta(T-t)\right] I \\
\left[\gamma + \theta(T-t)\right] I \end{cases}
\]
(129)

and for $A^* = -A$, $t \in [0, T]$,
\[
\Pi(t) = \begin{cases}
\left[p_1 + q_0 (T-t)\right] I \\
\left[\gamma + \theta(T-t)\right] I \end{cases}
\]
(130)

Proof. With the choice of (121), we have
\[
P(t) = e^{(A-m)^*(T-t)} P(T) e^{(A-m)(T-t)}
\]
\[
+ \int_t^T e^{(A-m)^*(s-t)} Q_s(s) e^{(A-m)(s-t)} ds
\]
(131)

and
\[
\tilde{P}(t) = e^{(A-\bar{m})t} P(0) e^{(A-\bar{m})^* t} - \int_0^t e^{(A-\bar{m})(t-s)} Q_0(s) e^{(A-\bar{m})^*(t-s)} ds
\]
(132)

Also, in the current case, $\Gamma(\cdot)$ satisfies
\[
\dot{\Gamma}(t) + (\bar{m} - m) \Gamma(t) + \theta I = 0,
\]
which leads to (with \( \Gamma(T) = \gamma I \))

\[
\Gamma(t) = \left[ \gamma e^{(\bar{m}-m)(T-t)} + \theta \int_t^T e^{(\bar{m}-m)(s-t)} ds \right] I
\]

\[
= \left[ \gamma e^{(\bar{m}-m)(T-t)} + \theta e^{(\bar{m}-m)(T-t)} - \frac{1}{\bar{m} - m} \right] I = \Gamma(t)^*, \quad t \in [0, T].
\] (133)

When \( \bar{m} - m = 0 \), the above is understood as

\[
\Gamma(t) = \left[ \gamma + \theta(T-t) \right] I, \quad t \in [0, T].
\] (134)

We now look at two cases.

In the case \( A^* = A \), (131) and (132) become

\[
P(t) = p_1 e^{2(A-m)(T-t)} + \frac{q_0}{2} (A - m)^{-1} \left[ e^{2(A-m)(T-t)} - I \right], \quad t \in [0, T],
\] (135)

and

\[
\bar{P}(t) = -\bar{p}_0 e^{2(A-\bar{m})t} - \frac{\bar{q}_0}{2} (A - \bar{m})^{-1} \left[ e^{2(A-\bar{m})t} - I \right], \quad t \in [0, T].
\] (136)

In the case \( A^* = -A \), (131) and (132) become

\[
P(t) = \left( p_1 e^{-2m(T-t)} + q_0 \int_t^T e^{-2m(s-t)} ds \right) I
\]

\[
= \left( p_1 e^{-2m(T-t)} + \frac{q_0}{2m} (1 - e^{-2m(T-t)}) \right) I, \quad t \in [0, T],
\] (137)

with the above understood as follows when \( m = 0 \),

\[
P(t) = \left( p_1 + q_0(T-t) \right) I, \quad t \in [0, T],
\] (138)

and

\[
\bar{P}(t) = -\left( \bar{p}_0 e^{-2\bar{m}t} + \bar{q}_0 \int_0^t e^{-2\bar{m}(t-s)} ds \right) I
\]

\[
= -\left( \bar{p}_0 e^{-2\bar{m}t} + \frac{\bar{q}_0}{2\bar{m}} (1 - e^{-2\bar{m}t}) \right) I, \quad t \in [0, T],
\] (139)

with the above understood as follows when \( \bar{m} = 0 \),

\[
\bar{P}(t) = -\left( \bar{p}_0 + \bar{q}_0 t \right) I, \quad t \in [0, T].
\] (140)

The rest conclusions are clear. \( \square \)

Combining Theorem 7.1 or Corollary 7.2 with Lemma 7.3, we can present many concrete cases for which the corresponding FBEEs are well-posed. For the simplicity of presentation, we only consider below the case that (126) holds. First, we present a simple lemma.

**Lemma 7.4.** Let

\[
f(\kappa) = \alpha e^{-\kappa} + \beta \frac{1 - e^{-\kappa}}{\kappa}, \quad \kappa > 0,
\]

with \( \alpha, \beta > 0 \). Then \( \kappa \to f(\kappa) \) is decreasing on \( (0, \infty) \) and

\[
0 = \lim_{\kappa \to \infty} f(\kappa) = \inf_{\kappa > 0} f(\kappa) < \sup_{\kappa > 0} f(\kappa) = \lim_{\kappa \to 0} f(\kappa) = \alpha + \beta.
\]

**Proof.** We note that

\[
f'(\kappa) = -\alpha e^{-\kappa} + \beta \frac{e^{-\kappa} - (1 - e^{-\kappa})}{\kappa} = -\left( \frac{\alpha}{e^\kappa} + \frac{e^\kappa - 1 - \kappa}{\kappa^2 e^\kappa} \right) < 0.
\]
Then our conclusion follows immediately.

**Theorem 7.5.** Let (H0) hold and let \((b, g, h) \in \mathcal{G}_2 \cap \mathcal{G}_3\). Suppose there are constants \(p_1, \bar{p}_0, q_0, \bar{q}_0, \delta, \varepsilon > 0, m > -\sigma_0\), and \(\gamma, \theta \in \mathbb{R}\) such that

\[
p_1 I + \gamma [h_y(y) + h_y(y)^*] - \left[\bar{p}_0 e^{-2(\sigma_0+m)(T-t)} + \frac{\bar{q}_0(1 - e^{-2(\sigma_0+m)(T-t)})}{2(\sigma_0 + m)}\right]h_y(y)^*h_y(y) \geq \delta,\tag{141}
\]

\[
\left(\frac{q_0 e^{-2(\sigma_0+m)(T-t)}}{\sigma_0+m} - 2m p_1 e^{-2(\sigma_0+m)(T-t)} - \bar{q}_0 e^{-2(\sigma_0+m)t} + \frac{\bar{p}_0 e^{-2(\sigma_0+m)t}}{\sigma_0+m} - 2m \bar{p}_0 e^{-2(\sigma_0+m)t}\right) \geq \tilde{\delta} + \varepsilon, \quad \forall t \in [0, T].\tag{142}
\]

Then the corresponding FBBE is well-posed if one of the following holds:

(i) In the case \(A^* = A\), it holds that

\[
\begin{align*}
&\left(p_1 e^{2(A-m)(T-t)} \begin{pmatrix} 0 & 0 \\ 0 & -\bar{p}_0 e^{2(A-m)t} \end{pmatrix}\right) \mathbb{B}(t, y, \psi) + \mathbb{B}(t, y, \psi)^* \left(p_1 e^{2(A-m)(T-t)} \begin{pmatrix} 0 & 0 \\ 0 & -\bar{p}_0 e^{2(A-m)t} \end{pmatrix}\right) \\
&+ \left[\gamma + \theta(T-t)\right] \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \mathbb{B}(t, y, \psi) + \mathbb{B}(t, y, \psi)^* \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \right. \\
&\left. + \frac{1}{2} \left( \begin{pmatrix} q_0 (A-m)^{-1} & 0 \\ 0 & -\bar{q}_0 (A-m)^{-1} \end{pmatrix} e^{2(A-m)(T-t) - I} \right) \mathbb{B}(t, y, \psi) + \mathbb{B}(t, y, \psi)^* \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \right] \leq \varepsilon.\tag{143}
\end{align*}
\]

(ii) In the case \(A^* = -A\), it holds that

\[
\begin{align*}
&\left(p_1 e^{-2m(T-t)} \begin{pmatrix} 0 & 0 \\ 0 & -\bar{p}_0 e^{-2mt} \end{pmatrix}\right) \mathbb{B}(t, y, \psi) + \mathbb{B}(t, y, \psi)^* \left(p_1 e^{-2m(T-t)} \begin{pmatrix} 0 & 0 \\ 0 & -\bar{p}_0 e^{-2mt} \end{pmatrix}\right) \\
&+ \left[\gamma + \theta(T-t)\right] \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \mathbb{B}(t, y, \psi) + \mathbb{B}(t, y, \psi)^* \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \right. \\
&\left. + \left( q_0 \begin{pmatrix} 1 - e^{-2m(T-t)} \ & 0 \\ 0 & 1 - e^{-2mt} \end{pmatrix} \right) \mathbb{B}(t, y, \psi) + \mathbb{B}(t, y, \psi)^* \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \right] \leq \varepsilon.\tag{144}
\end{align*}
\]

Proof. (i) We consider the case \(A^* = A\). First of all, by taking

\[
P(T) = p_1 I > 0, \quad \bar{P}(0) = -\bar{p}_0 I < 0,\tag{145}
\]

we see that (108) holds. Next, to get (109), we look at the following (recalling (139)):

\[
P(T) + h_y(y)^* \Gamma(T) + \Gamma(T)^* h_y(y) + h_y(y)^* \bar{P}(T) h_y(y) = p_1 I + \gamma [h_y(y) + h_y(y)^*] - h_y(y)^* \left(\bar{p}_0 e^{2(A-m)T} + \frac{\bar{q}_0}{2} (A-m)^{-1} \left[e^{2(A-m)T} - I\right]\right) h_y(y).
\]
Let us estimate the quadratic term in \( h_p(y) \) on the right hand side of the above.
To this end, we observe the following: For any \( \tau > \sigma \) (recall \( m > -\sigma_0 \)),

\[
\bar{p}_0 e^{-2(\tau+\sigma)T} + \tilde{q}_0 T \left( 1 - e^{-2(\tau+\sigma)T} \right) \equiv \bar{p}_0 e^{-\kappa} + \frac{\tilde{q}_0 T (1 - e^{-\kappa})}{\kappa} \equiv f(\kappa),
\]

with \( \kappa = 2(\tau + \sigma)T \geq 2(\sigma_0 + m)T > 0 \). By Lemma 7.4, we have

\[
\sup_{\kappa \geq 2(\sigma_0 + m)T} f(\kappa) = f(2(\sigma_0 + m)T) = \bar{p}_0 e^{-2(\sigma_0 + m)T} + \frac{\tilde{q}_0 (1 - e^{-2(\sigma_0 + m)T})}{2(\sigma_0 + m)}.
\]

By the spectral decomposition of \( A \), making use of (141), one has

\[
P(T) + h_y(y) \Gamma(T) + \Gamma(T)^* h_y(y) + h_y(y)^* \bar{P}(T) h_y(y)
= p_1 I + \gamma [h_y(y) + h_y(y)^*]
\geq p_1 I + \gamma [h_y(y) + h_y(y)^*]
- \left[ \bar{p}_0 e^{-2(\sigma_0 + m)T} + \frac{\tilde{q}_0 (1 - e^{-2(\sigma_0 + m)T})}{2(\sigma_0 + m)} \right] h_y(y)^* h_y(y) \geq \delta,
\]

which gives (109). Next, note that (in the case \( \bar{m} = m \))

\[
- \Pi(t) \bar{M}(t) - \bar{M}(t)^* \Pi(t) + Q(t) = \begin{pmatrix} -2mP(t) + q_0 I & 0 \\ 0 & 2mP(t) + \tilde{q}_0 I \end{pmatrix} + \frac{\delta I}{2mP(t) + \tilde{q}_0 I}.
\]

We now look at the following (noting (135) and (136)):

\[
-2mP(t) + q_0 I = -2mp_1 e^{2(A-m)(T-t)} - m \sigma_0 (A - m)^{-1} \left[ e^{2(A-m)(T-t)} - I \right] + q_0 I
= -2mp_1 e^{2(A-m)(T-t)} + q_0 \left[ A - m e^{2(A-m)(T-t)} \right] (A - m)^{-1},
\]

and

\[
2mP(t) + q_0 I = -2mp_0 e^{2(A-m)t} - m \tilde{q}_0 (A - m)^{-1} \left[ e^{2(A-m)t} - I \right] + \tilde{q}_0 I
= -2mp_0 e^{2(A-m)t} + \tilde{q}_0 \left[ A - m e^{2(A-m)t} \right] (A - m)^{-1}.
\]

Similar to the above, for any \( \tau > \sigma \) and \( m > -\sigma_0 \), we have \( (\tau + m > 0) \)

\[
-2mp_1 e^{-2(\tau+m)(T-t)} + \frac{q_0 (\tau + m e^{-2(\tau+m)(T-t)})}{\tau + m}
= -2mp_1 e^{-2(\tau+m)(T-t)} + q_0 + \frac{2mq_0 (T-t) (e^{-2(\tau+m)(T-t)} - 1)}{2(\tau + m)(T-t)}
= q_0 - \left[ 2mp_1 e^{-\kappa} - \frac{2mq_0 (T-t) (1 - e^{-\kappa})}{\kappa} \right] \equiv q_0 - f(\kappa),
\]

with \( \kappa = 2(\tau + m)(T-t) > 2(\sigma_0 + m)(T-t) \). By Lemma 7.4 again, we have

\[
-2mp_1 e^{-2(\tau+m)(T-t)} + \frac{q_0 (\tau + m e^{-2(\tau+m)(T-t)})}{\tau + m}
\geq q_0 - \sup_{\kappa \geq 2(\sigma_0 + m)(T-t)} f(\kappa) = q_0 - f(2(\sigma_0 + m)(T-t))
= q_0 - 2mp_1 e^{-2(\sigma_0 + m)(T-t)} + \frac{2mq_0 (e^{-2(\sigma_0 + m)(T-t)} - 1)}{2(\sigma_0 + m)}
\]
Consequently, using the spectral decomposition of $A$, we have

$$-2mP(t) + q_0 I \geq \begin{pmatrix} \sigma_0 + me^{-2(\sigma_0 + m)(T-t)} & -2m p_1 e^{-2(\sigma_0 + m)(T-t)} \\ \theta I & -2m \bar{p}_0 e^{-2(\sigma_0 + m)t} \end{pmatrix},$$

$$2m \bar{P}(t) + \bar{q}_0 I \geq \begin{pmatrix} \sigma_0 + me^{-2(\sigma_0 + m)(T-t)} & -2m p_1 e^{-2(\sigma_0 + m)(T-t)} \\ \theta I & -2m \bar{p}_0 e^{-2(\sigma_0 + m)t} \end{pmatrix},$$

for some $\delta > 0$, provided (142) holds. Then by (143), together with the representation of $\Pi(\cdot)$ from Lemma 7.3, we have

$$\Pi(t) B(t, y, \psi) + B(t, y, \psi)^* \Pi(t) \leq \varepsilon.$$

Hence, Corollary 7.2 applies.

(ii) We now consider the case $A^* = -A$. Again, we still have (108) by (141).

Next, for the current case, recalling (136),

$$P(T) + h_y(y)^* \Gamma(T) + \Gamma(T^*) h_y(y) + h_y(y)^* \bar{P}(T) h_y(y) = p_1 I + \gamma [h_y(y) + h_y(y)^*]$$

$$= \frac{p_1}{2m} \left( 2m - e^{-2m(T-t)} \right) h_y(y)^* h_y(y) \geq \delta.$$
and

$$2m\bar{P}(t) + \bar{q}_0 I = -2m\left(\bar{p}_0 e^{2mt} + \frac{\bar{q}_0}{2m}(1 - e^{-2mt})\right) I + \bar{q}_0 I$$

$$= -2m\bar{p}_0 e^{-2mt} + \bar{q}_0 e^{-2mt} = (\bar{q}_0 - 2m\bar{p}_0)e^{-2mt},$$

Hence,

$$-\Pi(t)M(t) - M(t)^\ast \Pi(t) + Q(t)$$

$$= \left((\bar{q}_0 - 2m\bar{p}_0)e^{-2mt}I\right) \left(\theta I\right) \left(\bar{q}_0 - 2m\bar{p}_0)e^{-2mt}I\right) \geq \delta, \quad \forall t \in [0, T],$$

for some $\delta > 0$, provided

$$\left((\bar{q}_0 - 2m\bar{p}_0)e^{-2mt}I\right) \left(\theta I\right) \left(\bar{q}_0 - 2m\bar{p}_0)e^{-2mt}I\right) > 0, \quad \forall t \in [0, T],$$

which is implied by (142) with $\sigma_0 = 0$. The rest proof is obvious. \qed

**Corollary 7.6.** Let (H0) hold, and let $(b, g, h) \in \mathcal{G}_2 \cap \mathcal{G}_3$. Let $p_1, \bar{p}_0, \bar{q}_0, \bar{\delta}, \bar{\delta}, \bar{\varepsilon} > 0$, $\gamma \in \mathbb{R}$ such that

$$p_1 + \gamma[h_y(y) + h_y(y)^\ast] - (\bar{p}_0 + \bar{q}_0)\bar{h}_y(y)^\ast \bar{h}_y(y) \geq \delta, \quad (149)$$

and

$$\left(q_0[1 + 2\sigma_0(T - t)] + 2\sigma_0 p_1\right) \left(\theta \bar{q}_0[1 + 2\sigma_0 t] + 2\sigma_0\bar{p}_0\right) \geq \bar{\delta} + \bar{\varepsilon}, \quad t \in [0, T]. \quad (150)$$

Then the corresponding FBEE is well-posed if one of the following holds:

(i) For the case $A^\ast = A$, the following holds:

$$\begin{pmatrix} p_1 e^{2(A + \sigma_0)(T - t)} & 0 & 0 \\
0 & -\bar{p}_0 e^{2(A + \sigma_0)t} & 0 \\
0 & 0 & -\bar{q}_0 e^{2(A + \sigma_0)t} \end{pmatrix} \mathbb{B}(t, y, \psi)$$

$$+ \mathbb{B}(t, y, \psi)^\ast \begin{pmatrix} p_1 e^{2(A + \sigma_0)(T - t)} & 0 & 0 \\
0 & -\bar{p}_0 e^{2(A + \sigma_0)t} & 0 \\
0 & 0 & -\bar{q}_0 e^{2(A + \sigma_0)t} \end{pmatrix} \mathbb{B}(t, y, \psi)^\ast \begin{pmatrix} 0 & I & 0 \\
I & 0 & 0 \\
0 & 0 & I \end{pmatrix} \left[\gamma + \theta(T - t)\right]$$

$$+ \left[\begin{pmatrix} q_0(T - t)\eta(A + \sigma_0)(T - t) \\
0 & -\bar{q}_0 t\eta((A + \sigma_0)t) \\
0 & -\bar{q}_0 t\eta((A + \sigma_0)t) \end{pmatrix} \mathbb{B}(t, y, \psi)^\ast \begin{pmatrix} q_0(T - t)\eta(A + \sigma_0)(T - t) \\
0 & -\bar{q}_0 t\eta((A + \sigma_0)t) \\
0 & -\bar{q}_0 t\eta((A + \sigma_0)t) \end{pmatrix} \leq \bar{\varepsilon}, \quad (151)$$

where

$$\eta(\kappa) = \begin{cases} e^\kappa - 1, & \kappa \neq 0, \\
1, & \kappa = 0. \end{cases} \quad (152)$$

(ii) For the case $A^\ast = -A$, the following holds:

$$\begin{pmatrix} [p_1 + q_0(T - t)]I & [\gamma + \theta(T - t)]I \\
[\gamma + \theta(T - t)]I & -[\bar{p}_0 + \bar{q}_0 t]I \end{pmatrix} \mathbb{B}(t, y, \psi)$$

$$+ \mathbb{B}(t, y, \psi)^\ast \begin{pmatrix} [p_1 + q_0(T - t)]I & [\gamma + \theta(T - t)]I \\
[\gamma + \theta(T - t)]I & -[\bar{p}_0 + \bar{q}_0 t]I \end{pmatrix} \leq \bar{\varepsilon}. \quad (153)$$

**Proof.** By letting $m \to -\sigma_0$ in (141) and (142), we have (149)–(150). Thus, when (149)–(150) hold, for $m$ sufficiently close to $-\sigma_0$, (141)–(142) hold.
By the definition of $\eta(\cdot)$, we have that
\begin{equation*}
\frac{e^{2(\mu-m)(T-t)}-1}{2(\mu-m)} = (T-t)\eta(2(\mu-m)(T-t)),
\end{equation*}
and
\begin{equation*}
\lim_{m \to -\sigma_0} \frac{1}{2} (A - m)^{-1} [e^{A-m}(T-t) - I] = (T-t) \int_{\sigma(A)} \eta((\mu + \sigma_0)(T-t)) dE_\mu.
\end{equation*}

Then sending $m \to -\sigma_0$ in (143), we obtain (151).

Now, for the case $A^* = A$, by sending $m \to 0$, we obtain (153).

Although the conditions stated in Theorem 7.5 and Corollary 7.6 still look lengthy, they are practically checkable. To illustrate this, let us look at an interesting situation covered.

**Corollary 7.7.** Let (H0) hold and $(b, g, h) \in G_2 \cap G_3$. Let
\begin{equation}
h_y(y) + h_y(y)^* \geq 0, \quad \forall y \in X. \tag{154}
\end{equation}

Let
\begin{equation}
\begin{aligned}
B_{11}(t, y, \psi) &= B_{22}(t, y, \psi)^*, \\
B_{12}(t, y, \psi) &= B_{12}'(t, y, \psi) \leq 0, \quad (t, y, \psi) \in [0, T] \times X \times X, \\
B_{21}(t, y, \psi) &= B_{21}'(t, y, \psi)^* \geq \delta.
\end{aligned} \tag{155}
\end{equation}

for some $\delta > 0$, and
\begin{equation}
\bar{p}_0(t)B_{22} + B_{22}'\bar{p}_0(t) \leq 0, \quad \forall t \in [0, T], \tag{156}
\end{equation}
where
\begin{equation}
\bar{p}_0(t) = \begin{cases}
\bar{p}_0 e^{2(A+\sigma_0)t} + \bar{q}_0 t \eta((A + \sigma_0)t), & \text{if } A^* = A, \\
\bar{p}_0 + \bar{q}_0 t, & \text{if } A^* = -A, \tag{157}
\end{cases}
\end{equation}
with $p_1, q_0, \bar{p}_0, \bar{q}_0 > 0$ and $\eta(\cdot)$ is defined by (152). Then the FBEE generated by $(b, g, h)$ is well-posed.

**Proof.** First of all, by (154), and the boundedness of $h_y(\cdot)$ (since $(b, g, h) \in G_2 \cap G_3$), we can find $p_1 > 0$ large so that (149) holds, and $\gamma > 0$ is allowed to be arbitrarily large. Also, by letting $\theta = 0$, we see that (150) holds, as long as $q_0, \bar{q}_0 > 0$. Next, we define
\begin{equation}
p_1(t) = \begin{cases}
p_1 e^{2(A+\sigma_0)(T-t)} + q_0 (T-t) \eta((A + \sigma_0)(T-t)), & \text{if } A^* = A, \\
[p_1 + q_0 (T-t)]t, & \text{if } A^* = -A. \tag{158}
\end{cases}
\end{equation}

Then according to Corollary 7.6, the FBEE is well-posed if the following holds:
\begin{equation}
\varepsilon \geq \left( p_1(t) \gamma I \right) \begin{pmatrix} B_{11} & B_{12} \\ B_{11}' & -B_{22} \end{pmatrix} \left( \begin{pmatrix} p_1(t) \gamma I \\ \gamma I \end{pmatrix} ^T - \bar{p}_0(t) \right) + \left( \begin{pmatrix} B_{11} & -B_{21} \\ B_{12} & -\bar{p}_0(t) \end{pmatrix} \right) \left( p_1(t) \gamma I \right) \begin{pmatrix} p_1(t) \gamma I \\ \gamma I \end{pmatrix} \tag{159}
\end{equation}
\begin{equation}
\begin{aligned}
&= \left( p_1(t)B_{11} + \gamma B_{11} + \bar{p}_0(t)B_{21} \right) \left( \begin{pmatrix} p_1(t) \gamma I \\ \gamma I \end{pmatrix} \right) + B_{12} \left( \begin{pmatrix} p_1(t) \gamma I \\ \gamma I \end{pmatrix} \right) \\
&= \left( p_1(t)B_{11} + B_{11}' + \gamma B_{11} + \bar{p}_0(t)B_{21} \right) \left( \begin{pmatrix} p_1(t) \gamma I \\ \gamma I \end{pmatrix} \right) + B_{12} \left( \begin{pmatrix} p_1(t) \gamma I \\ \gamma I \end{pmatrix} \right) \\
&= \left( p_1(t)B_{11} + B_{11}' + \gamma B_{11} + \bar{p}_0(t)B_{21} \right) \left( \begin{pmatrix} p_1(t) \gamma I \\ \gamma I \end{pmatrix} \right) + B_{12} \left( \begin{pmatrix} p_1(t) \gamma I \\ \gamma I \end{pmatrix} \right).
\end{aligned} \tag{159}
\end{equation}
This is equivalent to the following:

\[
\begin{pmatrix}
\varepsilon + \gamma |B_{21} + B_{21}^*| - |p_1(t)B_{11} + B_{11}^*p_1(t)| \\
\varepsilon - \gamma |B_{12} + B_{12}^*| - |\tilde{p}_0(t)B_{22} + B_{22}^*\tilde{p}_0(t)| \\
-|B_{12}p_1(t) + \tilde{p}_0(t)B_{21}| \\
\end{pmatrix} \geq 0,
\]

which is implied by

\[
\gamma |B_{21} + B_{21}^*| - |p_1(t)B_{11} + B_{11}^*p_1(t)| > 0,
\]

and

\[
\varepsilon - \gamma |B_{12} + B_{12}^*| - |\tilde{p}_0(t)B_{22} + B_{22}^*\tilde{p}_0(t)| \\
-|B_{12}p_1(t) + \tilde{p}_0(t)B_{21}| \left(\gamma |B_{21} + B_{21}^*| - |p_1(t)B_{11} + B_{11}^*p_1(t)|\right)^{-1} \\
\geq 0.
\]

Note that

\[
\left\| \left(\gamma |B_{21} + B_{21}^*| - |p_1(t)B_{11} + B_{11}^*p_1(t)|\right)^{-1} \right\| \\
\leq \frac{1}{\gamma} \left\| (B_{21} + B_{21}^*)^{-1} \right\| \left\| \left(1 - \frac{1}{\gamma} (B_{21} + B_{21}^*)^{-1} |p_1(t)B_{11} + B_{11}^*p_1(t)|\right)^{-1} \right\| \\
\leq \frac{1}{\gamma} \left\| (B_{21} + B_{21}^*)^{-1} \right\| \frac{1}{1 - \frac{1}{\gamma} \left\| (B_{21} + B_{21}^*)^{-1} |p_1(t)B_{11} + B_{11}^*p_1(t)|\right\|} \\
= \gamma - \left\| (B_{21} + B_{21}^*)^{-1} |p_1(t)B_{11} + B_{11}^*p_1(t)|\right\|.
\]

Therefore, it suffices to have (note (155)–(156))

\[
\varepsilon \geq \frac{|B_{12}p_1(t) + \tilde{p}_0(t)B_{21}|^2 \left\| (B_{21} + B_{21}^*)^{-1} \right\|^2}{\gamma - \left\| (B_{21} + B_{21}^*)^{-1} |p_1(t)B_{11} + B_{11}^*p_1(t)|\right\|},
\]

which can be achieved by letting \(\gamma > 0\) sufficiently large. Then our conclusion follows.

Let us look at some more cases.

**Corollary 7.8.** Let (H0) hold. Suppose \((b, g, h) \in G_2 \cap G_3\) such that

\[
h_y(y) + h_y(y)^* \geq 0, \quad \forall y \in X,
\]

and

\[
\begin{pmatrix}
0 & I \\
I & 0
\end{pmatrix} B(t, y, \psi) + B(t, y, \psi)^* \begin{pmatrix}
0 & I \\
I & 0
\end{pmatrix} \leq -\delta, \quad \forall (t, y, \psi) \in [0, T] \times X \times X,
\]

for some \(\delta > 0\). Then the corresponding FBEE is well-posed.

**Proof.** First, by letting \(p_1, \tilde{p}_0, q_0, \tilde{q}_0 > 0\), with \(q_0, \tilde{q}_0 > 0\) suitably large, we will have (150). Then letting \(p_1 > 0\) large, we will have (149). Next, by noting

\[
0 \leq \eta(\kappa) \leq 1, \quad \forall \kappa \leq 0,
\]

we see that under the condition \((b, g, h) \in G_4\), either \(A^* = A\) or \(A^* = -A\), we always have the boundedness of all the terms involved in the left-hand sides of (151) and (153), respectively. Hence, under condition (160), we can find \(\gamma > 0\) large enough so that (151) and (153) holds, respectively. Due to the condition (159), by letting \(\gamma > 0\) large, (149) will not be affected. Then Corollary 7.6 applies to get the well-posedness of the corresponding FBEE. \(\square\)
Note that (160) is equivalent to the following:
\[
\begin{align*}
&\left( -[g_y(t, y, \psi) + g_y(t, y, \psi)]^* b_y(t, y, \psi)^* - g_y(t, y, \psi) \right) \\
&\left( b_y(t, y, \psi) - g_y(t, y, \psi)^* \right) \leq -\delta,
\end{align*}
\]
\[
\forall (t, y, \psi) \in [0, T] \times X \times X.
\]
This is further equivalent to the uniform monotonicity of the following map
\[
\left( \begin{array}{c}
y(t, \psi) \\
\psi(t, \psi)
\end{array} \right) \mapsto \left( \begin{array}{c}
g(t, y, \psi) \\
-b(t, y, \psi)
\end{array} \right),
\]
in the sense that for some \( \delta > 0 \),
\[
\begin{align*}
\left( \begin{array}{c}
g(t, y, \psi) - g(t, \bar{y}, \bar{\psi}) \\
-b(t, y, \psi) + b(t, \bar{y}, \bar{\psi})
\end{array} \right),
\left( \begin{array}{c}
y - \bar{y} \\
\psi - \bar{\psi}
\end{array} \right) \geq \delta(||y - \bar{y}||^2 + ||\psi - \bar{\psi}||^2),
\end{align*}
\]
\[
\forall t \in [0, T], y, \bar{y}, \psi, \bar{\psi} \in X.
\]
It is possible to cook up many other cases from Theorem 7.5 and/or Corollary 7.6, for which the corresponding FBEEs are well-posed. Let us list some of them here.

**Corollary 7.9.** Let (H0) hold and \((b, g, h) \in \mathcal{G}_2 \cap \mathcal{G}_3\). Then the corresponding FBEE is well-posed if one of the following holds:

(i) For some \( \delta, \varepsilon > 0 \),
\[
I + h_y(y) + h_y(y)^* \geq \delta, \quad \forall y \in X.
\]
In the case \( A^* = A \), for all \((t, y, \psi) \in [0, T] \times X \times X \),
\[
\begin{pmatrix}
e^{2(A+\sigma_0)(T-t)} & I \\
I & 0
\end{pmatrix} B(t, y, \psi) + \begin{pmatrix} I \\
I
\end{pmatrix} \leq 0,
\]
and in the case \( A^* = -A \), for all \((t, y, \psi) \in [0, T] \times X \times X \),
\[
\begin{pmatrix}
I & I \\
I & 0
\end{pmatrix} B(t, y, \psi) + \begin{pmatrix} I \\
I
\end{pmatrix} \leq 0.
\]

(ii) For some \( \delta, \varepsilon > 0 \),
\[
I - h_y(y)^* h_y(y) \geq \delta, \quad \forall y \in X.
\]
In the case \( A^* = A \), for all \((t, y, \psi) \in [0, T] \times X \times X \),
\[
\begin{pmatrix}
e^{2(A+\sigma_0)(T-t)} & 0 \\
0 & e^{2(A+\sigma_0)t}
\end{pmatrix} B(t, y, \psi) \\
+ \begin{pmatrix} I \\
I
\end{pmatrix} \leq 0,
\]
and in the case \( A^* = -A \), for all \((t, y, \psi) \in [0, T] \times X \times X \),
\[
B(t, y, \psi) + \begin{pmatrix} I \\
I
\end{pmatrix} \leq 0. \quad (161)
\]

(iii) For some \( \delta, \varepsilon > 0 \),
\[
I + h_y(y) + h_y(y)^* - h_y(y)^* h_y(y) \geq \delta, \quad \forall y \in X.
\]
In the case \( A^* = A \), for all \((t, y, \psi) \in [0, T] \times X \times X \),
\[
\begin{pmatrix}
e^{2(A+\sigma_0)(T-t)} & I \\
I & e^{2(A+\sigma_0)t}
\end{pmatrix} B(t, y, \psi) \\
+ \begin{pmatrix} I \\
I
\end{pmatrix} \leq 0, \quad (162)
\]
\[
\begin{pmatrix}
e^{2(A+\sigma_0)(T-t)} & I \\
I & e^{2(A+\sigma_0)t}
\end{pmatrix} B(t, y, \psi) \\
+ \begin{pmatrix} I \\
I
\end{pmatrix} \leq 0. \quad (163)
\]
and in the case $A^* = -A$, for all $(t, y, \psi) \in [0, T] \times X \times X$,

$$\begin{pmatrix} I & I \\ I & I \end{pmatrix} B(t, y, \psi) + B(t, y, \psi)^* \begin{pmatrix} I & I \\ I & I \end{pmatrix} \leq 0.$$  \hspace{1cm} (164)

**Proof.** (i) We take $p_1 = \gamma > 0$ large enough and take $\tilde{p}_0, q_0, \tilde{q}_0 > 0$ small enough, $\theta = 0$. Then we may apply Corollary 7.6 to get our claim.

(ii) and (iii) can be proved similarly. \hfill \square

Inspired by the above result, it is easy for us to prove many other results of similar nature. We prefer not to get into exhausting details.

8. **More general cases.** In this section, we will briefly consider some more general cases.

First of all, we consider the case $(b, g, h) \in G_2$, i.e., the generator $(b, g, h)$ only satisfies (H2), and might not be Fréchet differentiable in $(y, \psi)$. Such a situation happens in many optimal control problems. To study such a case, let us recall some results from [20].

Let $f : X \to X$ be Lipschitz continuous and $\tilde{y} \in X$. For any linear subspace $L \subseteq X$, we define $L$-Gâteaux-Jacobian $D_L f(\tilde{y}) \in \mathcal{L}(L; X)$ by the following (if the limit exists):

$$D_L f(\tilde{y})(x) = f'(\tilde{y}; x) = \lim_{t \to 0} \frac{f(\tilde{y} + tx) - f(\tilde{y})}{t}, \quad \forall x \in X.$$  

The set of all points $\bar{y} \in X$ for which $D_L f(\bar{y})$ exist is denoted by $\Omega_L(f)$. Next, we let

$$\partial_L f(\bar{y}) = \bigcap_{\delta > 0} \left\{ D_L f(y) \mid y \in \Omega_L(f), \|y - \bar{y}\| \leq \delta \right\}$$

and define the generalized Jacobian of $f(\cdot)$ at $\bar{y}$ by the following:

$$\partial f(\bar{y}) = \left\{ \Psi \in \mathcal{L}(X; X) \mid \Psi|_L \in \partial_L f(\bar{y}), \forall L \text{ subspace of } X \right\}.$$  

For any $y, z \in X$, define $y \otimes z : \mathcal{L}(X) \to \mathbb{R}$ by

$$(y \otimes z)(\Psi) = \langle \Psi(y), z \rangle, \quad \forall \Psi \in \mathcal{L}(X).$$

Then $y \otimes z \in \mathcal{L}(\mathcal{L}(X); \mathbb{R})$. Let

$$X \otimes X = \text{span} \left\{ y \otimes z \mid y, z \in X \right\} \subseteq \mathcal{L}(X)^*.$$  

The weak topology induced by $X \otimes X$ on $\mathcal{L}(X)$ is called the weak*-operator-topology, denoted by $\beta(X)$. The following can be found in [20].

**Proposition 8.1.** If $f : X \to X$ is Lipschitz near $\bar{y}$, then $\partial f(\bar{y})$ is non-empty, bounded, and $\beta(X)$-compact.

More interestingly, we have the following mean-value theorem (see [20], Theorem 4.4).

**Proposition 8.2.** Let $f : X \to X$ be locally Lipschitz. Then for any $y, \bar{y} \in X$,

$$f(y) - f(\bar{y}) \in \left[ \bigcap_{\lambda \in [0, 1]} \partial f(\bar{y} + \lambda(y - \bar{y})) \right] (y - \bar{y}).$$

With the above preparation, we now consider the case that $(b, g, h) \in G_2$. Naturally, we need only to define

$$B(s, y, \psi) = \begin{pmatrix} b_y & b_\psi \\ -g_y & -g_\psi \end{pmatrix},$$
with
\[ b_y \in \partial_y b(t, y, \psi), \quad b_\psi \in \partial_\psi b(t, y, \psi), \quad g_y \in \partial_y g(t, y, \psi), \quad g_\psi \in \partial_\psi g(t, y, \psi). \]

Then, all the results from previous sections for \((b, g, h) \in \mathcal{G}_3\) can be carried over properly to the case \((b, g, h) \in \mathcal{G}_2\).

Next, we consider \((b, g, h) \in \mathcal{G}_3\), i.e., the generator \((b, g, h)\) may be not globally Lipschitz with respect to \(y\) and/or \(\psi\), or equivalently, the Fréchet derivative of \((y, \psi) \mapsto (b(t, y, \psi), g(t, y, \psi), h(y))\) might be not bounded. In such cases, a priori uniform boundedness of the mild solution \((y(\cdot), \psi(\cdot))\) could play an essential role. Let us indicate one such a case. To this end, we introduce the following

\[ (H3)' \text{ In addition to (H3), there is a non-decreasing function } f : [0, \infty) \to [0, \infty) \text{ such that} \]

\[
\|b_y(t, y, \psi)\| + \|b_\psi(t, y, \psi)\| + \|g_y(t, y, \psi)\| + \|g_\psi(t, y, \psi)\| + \|h_y(y)\| \leq f(\|y\| + \|\psi\|), \quad \forall (t, y, \psi) \in [0, T] \times X \times X.
\]

Moreover,
\[
\begin{aligned}
\langle b(t, y, \psi), y \rangle &\leq L(1 + \|y\|^2), \\
\langle g(t, y, \psi), \psi \rangle &\leq L(1 + \|\psi\|^2),
\end{aligned} \quad \forall (t, y, \psi) \in [0, T] \times X \times X.
\]

Under \((H3)'\), if \((y_\lambda^{\omega}(\cdot), \psi_\lambda^{\omega}(\cdot))\) is a solution to (83), then
\[
\|y_\lambda^{\omega}(t)\|^2 = \|x\|^2 + 2\int_0^t \langle y_\lambda^{\omega}(s), A\lambda y_\lambda^{\omega}(s) + \rho b(s, y_\lambda^{\omega}(s), \psi_\lambda^{\omega}(s)) + b_0(s) \rangle ds
\]
\[
\leq \|x\|^2 + 2L\int_0^t \left(1 + \|y_\lambda^{\omega}(s)\|^2 + \|y_\lambda^{\omega}(s)\| \|b_0(s)\|\right) ds.
\]

Then by Gronwall’s inequality, we have
\[
\|y_\lambda^{\omega}(\cdot)\|_{\infty} \leq K \left(1 + \|x\| + \int_0^T \|b_0(r)\| dr\right).
\]

Similarly,
\[
\|\psi_\lambda^{\omega}(t)\|^2 = \|\psi_\lambda^{\omega}(T)\|^2 - 2\int_t^T \langle \psi_\lambda^{\omega}(s), -A\lambda^* \psi_\lambda^{\omega}(s) - \rho g(s, y_\lambda^{\omega}(s), \psi_\lambda^{\omega}(s)) - g_0(s) \rangle ds
\]
\[
\leq \|\psi_\lambda^{\omega}(T)\|^2 + 2L\int_t^T \left(1 + \|\psi_\lambda^{\omega}(s)\|^2 + \|\psi_\lambda^{\omega}(s)\| \|g_0(s)\|\right) ds.
\]

Hence, it follows from Gronwall’s inequality that
\[
\|\psi_\lambda^{\omega}(\cdot)\|_{\infty} \leq K \left(1 + \|\psi_\lambda^{\omega}(T)\| + \int_0^T \|g_0(s)\| ds\right)
\]
\[
\leq K \left(1 + \|h(0)\| + f(\|y_\lambda^{\omega}(T)\|) \|y_\lambda^{\omega}(T)\| + \|h_0\| + \int_0^T \|g_0(s)\| ds\right) \leq K.
\]

Consequently, the relevant proofs will go through as if \((H4)\) is assumed.

For concrete PDEs, there are some other ways to obtain uniform boundedness of the (weak) solutions to the system. We will see some of such below.
9. **Several illustrative examples.** In this section, we look at several examples.

**Example 9.1. (Linear-Convex Optimal Control Problem)** Consider an optimal control problem with a linear state equation:

\[
\begin{aligned}
\dot{y}(t) &= Ay(t) + Bu(t), \\
y(0) &= x,
\end{aligned}
\]

and with the cost functional

\[
J(x; u(\cdot)) = \int_0^T \left( Q(y(t)) + \frac{1}{2} \langle Ru(t), u(t) \rangle \right) ds + G(y(T)),
\]

where \( y \mapsto Q(y) \) and \( y \mapsto G(y) \) are \( C^2 \) and convex. Then Pontryagin minimum principle leads to the optimality system:

\[
\begin{aligned}
\dot{y}(t) &= Ay(t) - BR^{-1}B^*\psi(t), \\
\dot{\psi}(t) &= -A^*\psi(t) - Qy(t), \\
y(0) &= x, \\
\psi(T) &= Gy(T).
\end{aligned}
\]

In this case, we have

\[
\begin{aligned}
b(t, y, \psi) &= -BR^{-1}B^*\psi, \\
g(t, y, \psi) &= Qy(y), \\
h(y) &= Gy(y).
\end{aligned}
\]

Thus,

\[
\begin{aligned}
b_y(t, y, \psi) &= 0, \\
b_{\psi}(t, y, \psi) &= -BR^{-1}B^*, \\
g_y(t, y, \psi) &= Q_{yy}(y), \\
g_{\psi}(t, y, \psi) &= 0, \\
h_y(y) &= G_{yy}(y).
\end{aligned}
\]

Then

\[
\mathbb{B}(t, y, \psi) = \begin{pmatrix} 0 & -BR^{-1}B^* \\ -Q_{yy}(y) & 0 \end{pmatrix},
\]

Hence, under conditions

\[
R \geq \delta, \quad M \geq G_{yy}(y) \geq 0, \quad M \geq Q_{yy}(y) \geq \delta, \quad \forall y \in X,
\]

for some \( M, \delta > 0 \), all the conditions of Corollary 7.7 hold, and the FBEE (167) admits a unique mild solution. A further special case is the following:

\[
Q(y) = \frac{1}{2} \langle Qy, y \rangle, \quad G(y) = \frac{1}{2} \langle Gy, y \rangle,
\]

for some \( Q, G \in S^+(X) \). In this case, the FBEE can be written as

\[
\begin{aligned}
\dot{y}(t) &= Ay(t) - BR^{-1}B^*\psi(t), \\
\dot{\psi}(t) &= -A^*\psi(t) - Qy(t), \\
y(0) &= x, \\
\psi(T) &= Gy(T).
\end{aligned}
\]

Hence, according to the above, when

\[
R \geq \delta, \quad G \geq 0, \quad Q \geq \delta,
\]

for some \( \delta > 0 \), the FBEE (168) admits a unique mild solution.

**Example 9.2. (AQ Problem)** For the simplicity of presentation, we let \( S(\cdot) = 0 \), and assume that all the involved functions are time-independent. Then the
optimality system reads

\[
\begin{align*}
\dot{y}(t) &= Ay(t) + F(y(t)) - BR^{-1}B^*\psi(t), \\
\dot{\psi}(t) &= -A^*\psi(t) - Q_y(y(t)) - F_y(y(t))^*\psi(t), \\
y(0) &= x, \quad \psi(T) = G_y(y(T)),
\end{align*}
\]

(169)

with \(A^* = A\) or \(A^* = -A\). Thus,

\[
\begin{align*}
&\begin{cases}
  b(t, y, \psi) = F(y) - BR^{-1}B^*\psi, \\
g(t, y, \psi) = Q_y(y) + F_y(y)^*\psi, \\
h(y) = G_y(y).
\end{cases}
\end{align*}
\]

(170)

Let \(\{\xi_n\}_{n \geq 1}\) be an orthonormal basis of \(X\), under which we may let

\[
F(y) = \sum_{n=1}^{\infty} \langle F(y), \xi_n \rangle \xi_n = \sum_{n=1}^{\infty} f^n(y)\xi_n.
\]

Then

\[
F_y(y)z = \lim_{\delta \to 0} \frac{F(y + \delta z) - F(y)}{\delta} = \sum_{n=1}^{\infty} \langle f^n(y), z \rangle \xi_n = \sum_{n=1}^{\infty} [\xi_n \otimes f^n(y)]z.
\]

Thus,

\[
F_y(y) = \sum_{n=1}^{\infty} \xi_n \otimes f^n(y),
\]

and

\[
F_y(y)^*\psi = \sum_{n=1}^{\infty} [f^n(y) \otimes \xi_n]\psi = \sum_{n=1}^{\infty} f^n(y)\langle \xi_n, \psi \rangle.
\]

Hence,

\[
[F_y(y)^*\psi]_y = \sum_{n=1}^{\infty} f^n_{yy}(y)\langle \xi_n, \psi \rangle \in \mathbb{S}(X), \quad \forall y \in X.
\]

Consequently,

\[
\begin{align*}
&\begin{cases}
b_y(s, y, \psi) = F_y(y), \\
g \psi(s, y, \psi) = -BR^{-1}B^*, \\
g_y(s, y, \psi) = Q_{yy}(y) + [F_y(y)^*\psi]_y, \\
h_y(y) = G_{yy}(y).
\end{cases}
\end{align*}
\]

Then

\[
\mathbb{B}(s, y, \psi) = \begin{pmatrix} F_y(y) & -BR^{-1}B^* \\ -Q_{yy}(y) - [F_y(y)^*\psi]_y & -F(y)^* \end{pmatrix} \equiv \begin{pmatrix} B_{11} & B_{12} \\ -B_{21} & -B_{22} \end{pmatrix}
\]

From this, we can calculate

\[
\begin{align*}
&\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \mathbb{B}(s, y, \psi) + \mathbb{B}(s, y, \psi)^* \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \\
&= \begin{pmatrix} -[g_y(s, y, \psi) + g_y(s, y, \psi)^*] & b_y(s, y, \psi)^* - g \psi(s, y, \psi)^* \\
b_y(s, \psi)^* - g \psi(s, y, \psi)^* & b_y(s, y, \psi)^* + b_y(s, y, \psi)^* \end{pmatrix} \\
&= -2 \begin{pmatrix} Q_{yy}(y) + [F_y(y)^*\psi]_y & 0 \\ 0 & BR^{-1}B^* \end{pmatrix}
\end{align*}
\]
Next, we note that if \( \psi(\cdot) \) is a mild solution to the backward evolution equation in (169), we have

\[
\left\| \psi(t) \right\|^2 = \left\| \psi(T) \right\|^2 + 2 \int_t^T \left( [A + F_y(y(s))] \psi(s) + Q_y(y(s)) , \psi(s) \right) ds
\]

\[
\leq \left\| G_y(y(T)) \right\|^2 + 2 \int_t^T \left\| Q_y(y(s)) \right\| \left\| \psi(s) \right\| ds
\]

\[
\leq \left\| G_y(\cdot) \right\|_{\infty}^2 + 2 \int_t^T \left\| Q_y(\cdot) \right\|_{\infty} \left\| \psi(s) \right\| ds \equiv \varphi(t).
\]

Then

\[
\dot{\varphi}(t) = -2 \left\| Q_y(\cdot) \right\|_{\infty} \left\| \psi(t) \right\| \geq -2 \left\| Q_y(\cdot) \right\|_{\infty} \sqrt{\varphi(t)},
\]

which leads to

\[
\left( \sqrt{\varphi(t)} \right)' \geq -\left\| Q_y(\cdot) \right\|_{\infty}.
\]

Then

\[
\left\| \psi(t) \right\| = \sqrt{\varphi(t)} = \sqrt{\varphi(T)} - \int_t^T \left( \sqrt{\varphi'(s)} \right) ds
\]

\[
\leq \left\| G_y(\cdot) \right\|_{\infty} + \left\| Q_y(\cdot) \right\|_{\infty} T, \quad \forall t \in [0, T].
\]

Hence,

\[
\left\| F_y(\cdot)^* \psi \right\| = \left\| F_y(\cdot) \right\| \left\| \psi \right\| \leq \left\| F_y(\cdot) \right\| \left( \left\| G_y(\cdot) \right\|_{\infty} + \left\| Q_y(\cdot) \right\|_{\infty} T \right),
\]

\[
\forall \left\| \psi \right\| \leq \left( \left\| G_y(\cdot) \right\|_{\infty} + \left\| Q_y(\cdot) \right\|_{\infty} T \right).
\]

Consequently, if we assume

\[
\begin{aligned}
&G_{yy}(y) \geq 0, \quad \forall y \in X, \quad BR^{-1}B^* \geq \delta, \\
&Q_{yy}(y) \geq \left\| F_y(\cdot) \right\| \left( \left\| G_y(\cdot) \right\|_{\infty} + \left\| Q_y(\cdot) \right\|_{\infty} T \right) + \delta, \quad \forall y \in X,
\end{aligned}
\]

for some \( \delta > 0 \), then (169) admits a unique mild solution, by Corollary 7.8.

**Example 9.3. (Optimal Control of a Parabolic PDE).** We now consider an optimal control problem for a parabolic equation. Such a problem was studied in [23]. The controlled state equation reads:

\[
\begin{aligned}
y_t &= \Delta y - (\lambda + u) y + f, \quad \text{in } (0, T) \times \Omega, \\
y|_{\partial \Omega} &= 0, \\
y(0, x) &= y_0(x), \quad x \in \Omega,
\end{aligned}
\]

(171)

where \( y(t, x) \) is the state and \( u(t, x) \) is the control, and \( \Omega \subseteq \mathbb{R}^n \) is a bounded domain with smooth boundary \( \partial \Omega \). The cost functional is the following:

\[
J(u(\cdot)) = \frac{1}{2} \int_0^T \int_{\Omega} \left( L|y - y_d|^2 + Nu^2 \right) dx dt + \frac{1}{2} \int_{\Omega} M|y(T, x) - z(x)|^2 dx.
\]

(172)

We assume that

\[
\begin{aligned}
f(t, x) &\geq 0, \quad y_d(t, x) \leq 0, \quad (t, x) \in (0, T) \times \Omega, \\
y_0(x) &\geq 0, \quad z(x) \leq 0, \quad x \in \Omega.
\end{aligned}
\]
According to [23], optimal control exists and the optimality system reads:

\[
\begin{align*}
&y_t = \Delta y - \lambda y - \frac{1}{N} \psi y^2 + f, \quad \text{in } (0,T) \times \Omega, \\
&\psi_t = -\Delta \psi + \lambda \psi + \frac{1}{N} \psi \psi^2 - L(y - y_d), \quad \text{in } (0,T) \times \Omega, \\
&y|_{\partial \Omega} = \psi|_{\partial \Omega} = 0, \\
&y(0,x) = y_0(x), \quad \psi(T,x) = M(y(T,x) - z(x)), \quad x \in \Omega,
\end{align*}
\]

Then we have

\[
\begin{align*}
&b(s,y,\psi) = -\lambda y - \frac{1}{N} \psi y^2 + f, \\
&g(s,y,\psi) = -\lambda \psi - \frac{1}{N} \psi \psi^2 + L(y - y_d).
\end{align*}
\]

Hence,

\[
\begin{align*}
&b_y = -\lambda - \frac{2}{N} \psi \psi, \quad b_\psi = -\frac{1}{N} \psi^2, \\
&g_y = L - \frac{1}{N} \psi^2, \quad g_\psi = -\lambda - \frac{2}{N} \psi \psi = b_y.
\end{align*}
\]

Then

\[
B(t,y,\psi) = \begin{pmatrix}
-\lambda - \frac{2}{N} \psi \psi & -\frac{1}{N} \psi^2 \\
L + \frac{1}{N} \psi^2 & \lambda + \frac{2}{N} \psi \psi
\end{pmatrix}.
\]

Thus,

\[
\begin{pmatrix}
0 & I \\
I & 0
\end{pmatrix} B(t,y,\psi) + B(t,y,\psi) \begin{pmatrix}
0 & I \\
I & 0
\end{pmatrix} = 2 \begin{pmatrix}
-L + \frac{1}{N} \psi^2 & 0 \\
0 & -\frac{1}{N} \psi^2
\end{pmatrix} \leq 0,
\]

provided \( \psi \) is bounded (which was shown in [23]) and \( N \) is large enough.

10. **Concluding remarks.** We have discussed the well-posedness of FBEEs which is mainly motivated by the optimality systems of optimal control problems for infinite dimensional evolution equations. We have presented some basic results from two approaches: the decoupling method and the method of continuity. It is seen that the theory is far from mature and many challenging questions are left open. Here is a partial list of these:

- In the direction of decoupling method, it is widely open that how one can construct decoupling field, through solving a PDE in Hilbert space.
- In the direction of method of continuity, more careful analysis is need to make the stated condition easier to use.
- More general generators \( A \) other than \( A^* = A \) and \( A^* = -A \). Also, taking into account of PDEs, the generator \( (b,g,h) \) might be unbounded (involving differential operators).
- The existence of Lyapunov operators, together with other relevant conditions, is a sufficient condition for the well-posedness of FBEEs, and it seems that such a condition is not necessary in general. However, we believe that for some case, such a condition is very close to a necessary condition. Further exploration is desired.
- It is expected that a general FBEE could have more than one solutions. How one could explore in that direction also remains widely open.

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Appendix. Proof of Proposition 5.2. First, we have

\[
\|\hat{y}(s)\|^2 = \|\hat{x}\|^2 + \int_t^s \left( 2\langle \hat{y}(r), A_{\lambda}\hat{y}(r) + \rho \hat{b}(r) \hat{y}(r) + \rho \hat{b}_\psi(s) \hat{\psi}(s) + \delta b(r) + \delta b_0(r) \rangle \right) dr
\]

\[
\leq \|\hat{x}\|^2 + \int_t^s \left[ \left( A_{\lambda} + A_{\lambda}^* + \rho \hat{b}(r) + \hat{\psi}(s) \right) \hat{\psi}(s) + \rho \delta b(r) + \delta b_0(r) \right] dr
\]

\[
\leq \|\hat{x}\|^2 + 2 \int_t^s \left[ \rho L_{by}(r) \hat{\psi}(r) \right] dr \equiv \varphi(s)^2.
\]

Then

\[
\varphi(s) \hat{\psi}(s) = \rho L_{by}(s) \|\hat{y}(s)\|^2 + \|\hat{y}(s)\| \|\rho \hat{b}_\psi(s) \hat{\psi}(s) + \rho \delta b(s) + \delta b_0(s)\|
\]

\[
\leq \rho L_{by}(s) \varphi(s)^2 + \varphi(s) \|\rho \hat{b}_\psi(s) \hat{\psi}(s) + \rho \delta b(s) + \delta b_0(s)\|
\]

\[
\equiv a_2(s) \varphi(s)^2 + a_1(s) \varphi(s).
\]

Consequently,

\[
\hat{\psi}(s) \leq a_2(s) \varphi(s) + a_1(s).
\]

Hence, by Gronwall’s inequality,

\[
\|\hat{y}(s)\| \leq \varphi(s) \leq e^{\int_t^s a_2(\tau) d\tau} \|\hat{x}\| + \int_t^s e^{\int_t^s a_2(\tau) d\tau} a_1(\tau) d\tau
\]

\[
= e^{\rho \int_t^s L_{by}(\tau) d\tau} \|\hat{x}\| + e^{\rho \int_t^s L_{by}(\tau) d\tau} \|\rho \hat{b}_\psi(s) \hat{\psi}(s) + \rho \delta b(r) + \delta b_0(r)\|
\]

\[
\leq \rho \int_t^s e^{\rho \int_t^s L_{by}(\tau) d\tau} \|\rho \hat{b}_\psi(s) \hat{\psi}(s) + \rho \delta b(r) + \delta b_0(r)\| d\tau + K \left[ \|\hat{x}\| + \int_t^s \left( \|\delta b(r)\| + \|\delta b_0(r)\| \right) d\tau \right].
\]

This proves (89). We now prove (90). One has

\[
\|\hat{\psi}(T)\|^2 - \|\hat{\psi}(s)\|^2
\]

\[
= \int_s^T 2 \langle \hat{\psi}(r), -A_{\lambda} \hat{\psi}(r) - \rho \hat{g}_y(r) \hat{y}(r) - \rho \hat{g}_\psi(s) \hat{\psi}(s) - \rho \delta g(r) - \delta g_0(r) \rangle dr
\]

\[
\geq -2 \int_s^T \left( \langle \hat{g}_y(s) + \hat{g}_\psi(s) \rangle \hat{\psi}(s), \hat{\psi}(s) \right) + \langle \hat{\psi}(s), \hat{g}_y(s) \hat{y}(s) + \rho \delta g(s) + \delta g_0(s) \rangle ds
\]

\[
\geq 2 \int_s^T \left\{ - \rho L_{gy}(r) \|\hat{\psi}(r)\|^2 - \|\hat{\psi}(r)\| \rho \hat{g}_y(r) \hat{y}(r) + \rho \hat{g}(r) + \delta g_0(r) \right\} ds,
\]

which leads to

\[
\|\hat{\psi}(s)\|^2 \leq \|\hat{\psi}(T)\|^2 + 2 \int_s^T \left\{ \rho L_{gy}(r) \|\hat{\psi}(r)\|^2 + \|\hat{\psi}(r)\| \rho \hat{g}_y(r) \hat{y}(r) + \rho \hat{g}(r) + \delta g_0(r) \right\} dr \equiv \varphi(s)^2.
\]
Then
\[ \varphi(s)\dot{\varphi}(s) = -\rho L_{g\psi}(r)\|\dot{\varphi}(s)\|^2 + \|\ddot{\varphi}(s)\| \rho g(s) + \rho \dot{g}(s) + \eta g_0(s) \|
\geq -\rho L_{g\psi}(r)\varphi(s)^2 + \|\rho \ddot{\varphi}(s)\| \rho g(s) + \rho \dot{g}(s) + \eta g_0(s) \| \varphi(s)
\equiv -a_2(s)\varphi(s)^2 - a_1(s)\varphi(s).
\]
Thus,
\[ \dot{\varphi}(s) + a_2(s)\varphi(s) \geq -a_1(s).
\]
Also,
\[ \left( e^{-\int_s^T a_2(\tau)d\tau} \varphi(s) \right)' \geq -a_1(s)e^{-\int_s^T a_2(\tau)d\tau} \]
\[ \varphi(T) - e^{-\int_s^T a_2(\tau)d\tau} \varphi(s) \geq - \int_s^T a_1(r)e^{-\int_s^T a_2(\tau)d\tau} dr.
\]
Hence,
\[ \|\dot{\varphi}(s)\| \leq \varphi(s) \leq e^{\int_s^T a_2(\tau)d\tau} \varphi(T) + \int_s^T a_1(r)e^{\int_s^T a_2(\tau)d\tau} dr
\leq e^{\int_s^T L_{g\psi}(r)d\tau} \|\dot{\varphi}(s)\| + \rho \dot{h} + \delta h_0
\]
\[ + \int_s^T e^{\int_s^T L_{g\psi}(r)d\tau} \|\rho g_0(\tau)\| \|\ddot{\varphi}(s)\| + \rho \delta g(\tau) + \eta g_0(\tau) \| d\tau
\leq e^{\int_s^T L_{g\psi}(r)d\tau} \|\dot{\varphi}(s)\| + \int_s^T e^{\int_s^T L_{g\psi}(r)d\tau} \|\ddot{\varphi}(s)\| + \|\dot{\varphi}(s)\| + \|\ddot{\varphi}(s)\| d\tau
\]
\[ + K \left[ \|\dot{h}\| + \|\delta h_0\| + \int_s^T \left( \|\delta g(\tau)\| + \|\delta g_0(\tau)\| \right) d\tau \right].
\]
This completes the proof. □

Note that if we let \((y_0^c(\cdot), \psi_0^c(\cdot))\) be the mild solution of the following:
\[
\begin{align*}
\dot{y}_0^c(s) &= Ay_0^c(s) + \rho b(s,0,0) + b_0(s), \\
\dot{\psi}_0^c(s) &= -A^\ast \psi_0^c(s) - \rho g(s,0,0) - g_0(s), \\
y_0^c(t) &= x, \quad \psi_0^c(T) = \rho h(0) + h_0.
\end{align*}
\]
Then
\[ y_0^c(s) = e^{A(s-t)}x + \int_t^s e^{A(s-r)}[\rho b(r,0,0) + b_0(r)]dr.
\]
Hence,
\[ \|y_0^c(\cdot)\| \leq \|x\| + \int_t^T \|\rho b(r,0,0) + b_0(r)\| dr.
\]
Also,
\[ \psi_0^c(s) = e^{A^\ast(T-s)}\psi_0^c(T) + \int_s^T e^{A^\ast(r-s)}[\rho g(r,0,0) + g_0(r)]dr.
\]
Hence,
\[ \|\psi_0^c(\cdot)\| \leq \|\rho h(0) + h_0\| + \int_t^T \|\rho g(r,0,0) + g_0(r)\| dr.
\]
Now, taking
\[
\begin{align*}
\bar{b}(s,y,\psi) &= b(s,0,0), \quad \bar{b}_0(s) = \bar{b}_0(s), \\
\bar{g}(s,y,\psi) &= g(s,0,0), \quad \bar{g}_0(s) = \bar{g}_0(s), \\
\bar{h}(y) &= h(0), \quad \bar{h}_0 = h_0.
\end{align*}
\]
Then
\[
\begin{aligned}
\delta b(s) &= b(s, 0, 0) - b(s, y_0'(s), \psi_0'(s)), & \quad \delta b_0(s) = \bar{b}_0(s) - b_0(s) = 0, \\
\delta g(s) &= g(s, 0, 0) - g(s, y_0'(s), \psi_0'(s)), & \quad \delta g_0(s) = \bar{g}_0(s) - g_0(s) = 0, \\
\delta h &= h(0) - h(y_0'(T)), & \quad \delta h_0 = \bar{h}_0 - h_0 = 0, \\
\end{aligned}
\]
(175)
\[\|\hat{y}(\cdot)\|_{\infty} \leq \rho \int_t^T e^{\rho \int_t^r L_{b_y}(\tau) d\tau} \|\bar{b}_y(r)\hat{y}(r)\| dr. \] (176)
and
\[\|\hat{\psi}(\cdot)\|_{\infty} \leq \rho \left[ e^{\rho \int_t^T L_{b_y}(\tau) d\tau} \|\bar{h}_y\hat{y}(T)\| + \int_t^T e^{\rho \int_t^r L_{b_y}(\tau) d\tau} \|\bar{g}_y(r)\hat{y}(r)\| dr \right]. \] (177)

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