The Dynamics of Relativistic Membranes
I: Reduction to 2-dimensional Fluid Dynamics

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Abstract
We greatly simplify the light-cone gauge description of a relativistic membrane moving in Minkowski space by performing a field-dependent change of variables which allows the explicit solution of all constraints and a Hamiltonian reduction to a $SO(1,3)$ invariant 2 + 1-dimensional theory of isentropic gas dynamics, where the pressure is inversely proportional to (minus) the mass-density. Simple expressions for the generators of the Poincaré group are given. We also find a generalized Lax pair which involves as a novel feature complex conjugation. The extension to the supersymmetric case, as well as to higher-dimensional minimal surfaces of codimension one is briefly mentioned.
Who would believe that membranes are integrable?

On the other hand, realizing that geometry so often lies at the heart of physics, it is also strange not to expect beautiful structures and very special properties of relativistic surfaces, minimally embedded in space-time.

The tribute for initiating into fruitful directions concrete studies of relativistic membranes, and higher-dimensional extended objects, should go to J. Goldstone. This ‘first phase’ of membranes - not counting earlier ideas, such as 
- led to \[ \text{(1)} \]. Five years later, starting with \[ \text{(2)} \], and the supersymmetrization of membranes (cf. \[ \text{(1)} \]), came a short blossom of activity (see for instance \[ \text{(3)} \]), bringing also into fashion \textit{sdiff} algebras, their star product deformations, and their approximations via nontrivial large \( N \) limits of matrix algebras (cf. e. g. \[ \text{(4)} \]). The interest in membranes more or less stopped, around 1990, for presumably two reasons: The fact that ‘nothing could be solved’ (not even on the classical level), together with apparent proofs of nonintegrability (\[ \text{(5)} \]; note \[ \text{(6)} \]), and arguments (based on the large \( N \) matrix model regularisation of the membrane, cf. \[ \text{(7)} \], \[ \text{(8)} \]) that the supersymmetric membrane Hamiltonian should have a continuous spectrum starting at zero (cf. \[ \text{(9)} \]).

In contrast to all these negative results, and speculations, we would like to put forward, in a series of papers devoted to different aspects of relativistic surfaces (starting with the classical dynamics of a bosonic membrane moving in four-dimensional flat Minkowski space) a more optimistic point of view, which is mainly based on two observations:

Firstly, one may explicitly solve the \textit{sdiff}-constraint which so far seemed to be untractable.

Secondly, the theory is greatly simplified by performing a time-dependent change of variables, that interchanges some of the dependent and independent variables.

Both these observations use in an essential way the infinite dimensionality of the phase space, putting further doubt on conclusions drawn from the finite-dimensional regularisation of the membrane.

That hodograph-type transformations play an important rôle in the investigation and solution of nonlinear dynamics has a long history while having become particularly clear (\[ \text{(10)} \]) in the context of \textit{higher dimensional integrability}.

The paper is organised as follows: After a brief review of light-cone gauge membrane dynamics (see \[ \text{(11)} \] or \[ \text{(12)} \] for details) we show that a change from the independent to the dependent variables allows us to solve (actually integrate) the constraint and to reduce the original system of field equations, which involves four functions, to a system involving only two functions, \( q \) and \( p \). The reduction is Hamiltonian, and \( q \) and \( p \) can be interpreted as a gas-dynamical mass density and a velocity potential, respectively. We then write down a (complex conjugated) Lax equation for the system, and finally give explicit formulae for the infinitesimal generators of the Poincaré group in this model.
1. Let $\Sigma$ be a two-dimensional surface with co-ordinates $(\phi_1, \phi_2) = \vec{\phi}$ with respect to which there is a Poisson bracket for any two smooth complex-valued functions $f$ and $g$ on $\Sigma$:

$$\{f, g\} \overset{\text{def}}{=} \frac{\partial f}{\partial \phi_1} \frac{\partial g}{\partial \phi_2} - \frac{\partial f}{\partial \phi_2} \frac{\partial g}{\partial \phi_1}. \quad (1)$$

In the orthonormal light-cone gauge, the dynamics of a bosonic membrane $x^\mu : \mathbb{R} \times \Sigma \to \mathbb{R}^{(1, 3)}$ in four-dimensional Minkowski space is described by the two transverse co-ordinate functions $(x_1, x_2) = \vec{x}$ and their conjugate momenta $(p_1, p_2) = \vec{p}$, whereas one of the light-cone co-ordinates, $\zeta$, becomes cyclic (such that its conjugate momentum $-\eta$ is conserved). $\eta$ times the Hamiltonian governing the dynamics of the membrane is then given by

$$H = \frac{1}{2} \int_\Sigma d\phi_1 d\phi_2 (p_1^2 + p_2^2 + \{x_1, x_2\}^2), \quad (2)$$

provided the constraint

$$K(\vec{\phi}) \overset{\text{def}}{=} \{p_1, x_1\}(\vec{\phi}) + \{p_2, x_2\}(\vec{\phi}) = 0 \quad (3)$$

holds which reflects the remainder of the diffeomorphism invariance of the original relativistic action (that was taken to be proportional to the three-dimensional volume swept out in Minkowski space). Using the usual dynamical Poisson bracket $\{\ldots, \}$ on the classical fields,

$$\{x_i(\vec{\phi}), p_j(\vec{\phi}')\} = \delta_{ij} \delta^{(2)}(\vec{\phi} - \vec{\phi}') \quad , \quad (4)$$

one gets the following equations of motion from the Hamiltonian (H):

$$\dot{x}_1 = p_1, \quad (5)$$

$$\dot{x}_2 = p_2, \quad (6)$$

$$\dot{p}_1 = \{x_1, x_2\}, \quad (7)$$

$$\dot{p}_2 = -\{x_1, x_2\}, \quad (8)$$

which imply

$$\{x_1, x_2\} = \{p_1, x_2\} + \{x_1, p_2\} \quad . \quad (9)$$

2. In order to obtain an unconstrained description of the membrane we make the following change of independent variables:

$$(\phi_0 \overset{\text{def}}{=} t, \phi_1, \phi_2) \mapsto (x_0 = t, x_1(t, \vec{\phi}), x_2(t, \vec{\phi})) \quad , \quad (10)$$
with the Jacobian
\[
\left( \frac{\partial \vec{x}}{\partial \vec{\phi}} \right) = \begin{pmatrix} 1 & 0 & 0 \\ \dot{x}_1 & \frac{\partial x_1}{\partial \phi_1} & \frac{\partial x_1}{\partial \phi_2} \\ \dot{x}_2 & \frac{\partial x_2}{\partial \phi_1} & \frac{\partial x_2}{\partial \phi_2} \end{pmatrix}
\] (11)
and its Jacobi determinant
\[
J(\vec{\phi}) \overset{\text{def}}{=} \det \left( \frac{\partial \vec{x}}{\partial \vec{\phi}} \right)(\vec{\phi}) = \{x_1, x_2\}(\vec{\phi})
\] . (12)

Along with this transformation (assumed to be nonsingular) we use the notation \( f(\vec{\phi}) = \hat{f}(\vec{x}) \) for any function \( f \) on \( \Sigma \) to indicate the transformation.

This transformation is tailored to simplify Poisson brackets with the fields \( x_i \) (\( i = 1, 2; \epsilon_{12} = -\epsilon_{21} = 1 \)):
\[
\{f, x_i\}(\vec{\phi}) = -\epsilon_{ij} \frac{\partial \hat{f}}{\partial x_j}(\vec{x}) \hat{J}(\vec{x})
\] . (13)

An immediate consequence is a simplification of the constraint equation (3):
\[
K(\vec{\phi}) = (\vec{\nabla} \times \hat{p})(\vec{x}) \hat{J}(\vec{x}) = 0
\] (14)
where \( \vec{\nabla} \times \hat{p} \overset{\text{def}}{=} \partial_1 \hat{p}_2 - \partial_2 \hat{p}_1 \). The solution of eqn (14) is simply a gradient of some function \( p \) depending on the \( \vec{x} \)-variables:
\[
\hat{p} = \vec{\nabla} p
\] (15)
Actually, we had first solved the constraint in the original co-ordinates:
\[
p_1 = \frac{\{P, x_2\}}{\{x_1, x_2\}}
\] (16)
\[
p_2 = -\frac{\{P, x_1\}}{\{x_1, x_2\}}
\] , (17)
the function \( p \) being equal to \( \hat{P} \). The verification that this indeed satisfies the original constraint (3) is simple if one uses the fact that in two dimensions every Poisson bracket remains a Poisson bracket when multiplied by an arbitrary smooth function. Note again that this uses in an essential way the infinite dimensionality of the phase space and does not seem to have any immediate analogue in the large \( N \) matrix model regularisation of the membrane.

Now the equations of motion can be reduced: Using (11) one observes that the ‘old’ time derivative becomes a substantial (or material) derivative known from fluid dynamics:
\[
\frac{\partial}{\partial \phi_0} = \left( \vec{\nabla} \cdot \hat{p} \right) \cdot \vec{\nabla} + \frac{\partial}{\partial x_0}
\] . (18)
The r. h. s. of eqs (7, 8) becomes an ordinary first derivative (of $\hat{J}^2$) in the $\vec{x}$-co-ordinates. Now denoting by ($\cdot$) the differentiation with respect to $t = x_0$ and defining

$$q(\vec{x}) \overset{\text{def}}{=} \frac{1}{f(\vec{x})} = \det(\frac{\partial \vec{x}}{\partial \vec{y}})(\vec{x})$$

(19)

we get

$$\dot{q} = -q(\vec{\nabla})^2 p - \vec{\nabla} p \cdot \vec{\nabla} q$$

(20)

$$\dot{p} = \frac{1}{2}(\frac{1}{q^2} - (\vec{\nabla} p)^2) + c(t)$$

(21)

where $c(t)$ arises as an integration ‘constant’ (for the moment, we will take it to be zero). What about eqs (5, 6) which were used to obtain eqn (19)? They are equivalent to the equations

$$\left(\frac{\partial}{\partial x_0} + (\vec{\nabla} p) \cdot \vec{\nabla}\right) \varphi_i = 0$$

(22)

Taking the gradient w. r. t. $\vec{x}$ of eqn (21) we get a pair of eqs that are well-known in fluid dynamics:

$$\dot{q} + \vec{\nabla} \cdot (q \vec{\nabla} p) = 0$$

(23)

$$q \vec{\nabla} p + q(\vec{\nabla} p \cdot \vec{\nabla}) \vec{\nabla} p + \vec{\nabla}(\frac{1}{q}) = 0$$

(24)

Viewing $q$ as a mass density and $\vec{v} = \vec{\nabla} p$ as a velocity field the first equation is nothing but the continuity equation whereas the second one is the Euler equation of irrotational ($\vec{\nabla} \times \vec{v} = 0$) isentropic 2-dimensional gas dynamics with a pressure $-\frac{1}{q}$. The function $p$ plays the rôle of a velocity potential.

Another feature of the reduced eqs (20, 21) is that they are again Hamiltonian: treating $q$ and $p$ as canonical variables, i. e.

$$\{q(\vec{x}), p(\vec{y})\} = \delta^{(2)}(\vec{x} - \vec{y})$$

(25)

and expressing the old Hamiltonian (2) in the reduced variables

$$\hat{H} = \frac{1}{2} \int d^2 x \ q(\vec{x})((\vec{\nabla} p(\vec{x}))^2 + \frac{1}{q(\vec{x})^2})$$

(26)

the above equations of motion can be deduced from $\hat{H}$ in the canonical way, a fact which does not seem to be mentioned in textbooks on gas dynamics.

From the point of view of phase space reduction these features are quite natural: the left hand side of the constraint equation (3) can be viewed as a momentum map for the $SDiff(\Sigma)$ (i. e. group of area preserving diffeomorphisms on $\Sigma$) action on the original fields $\vec{x}$. Therefore we did a phase space reduction.
at momentum level zero. One could also consider a reduction at a nonzero momentum level thus introducing a fixed degree of vorticity in the reduced phase space living on additional co-adjoint orbits of $SDiff(\Sigma)$. It is interesting that this analysis does not depend on the fact that $\Sigma$ is two-dimensional (we shall give a more detailed differential geometric account thereof elsewhere).

When extending the analysis to the supersymmetric case, one finds (with $\psi(\vec{x})$ a complex anti-commuting field)

$$\tilde{H}_s = \frac{1}{2} \int d^2x \left[ q(\vec{\nabla}p)^2 + ip(\vec{\nabla}^2 \psi + \bar{\psi} \tilde{\nabla}^2 \psi) \right. $$

$$+ \frac{1}{q} \left( 1 + \frac{\psi \bar{\psi}}{4} \left( \partial \bar{\psi} \partial \psi - \partial \psi \partial \bar{\psi} \right) + i(\psi \bar{\partial} \bar{\psi} + \bar{\psi} \partial \bar{\psi}) \right) \right].$$

(27)

Its derivation and canonical structure (which is more complicated than in the purely bosonic case) will be given in a separate paper.

Note that the connection between eqn (27) and eqn (26) has, as a simple consequence, the solution of constant pressure irrotational gas dynamics

$$\tilde{H}_0 = \frac{1}{2} \int d^2x \ q(\vec{x})(\vec{\nabla}p(\vec{x}))^2$$

(28)

in terms of 'free' fields $(x_1(\vec{\varphi}), x_2(\vec{\varphi}))$ only subjected to the constraint (3): Just solve the equations in the old variables, i. e. $\vec{x}_i = 0$ for $i = 1, 2$ and $\{x_1, \dot{x}_1\} + \{x_2, \dot{x}_2\} = 0$, trivially giving

$$x_i(t, \vec{\varphi}) = x_i(0, \vec{\varphi}) + t \dot{x}_i(0, \vec{\varphi})$$

(29)

$$\{x_1(0, \cdot), \dot{x}_1(0, \cdot)\} + \{x_2(0, \cdot), \dot{x}_2(0, \cdot)\} = 0.$$  

(30)

For any choice of the initial conditions eqn (29) when inverted to functions $\varphi_i(t, \vec{x})$ will give $q(t, \vec{x})$ as the Jacobi determinant $det(\partial \varphi_i/\partial x_j)$. This is a nice (infinite dimensional) example of the projection method of Olshanetsky and Perelomov (cf. [13]).

Finally note that the one-dimensional analogue of the Hamiltonian (26) can presumably be solved by quite a variety of different methods such as direct linearization, infinite charge algebra, collective field theory or matrix model limit.

3. Going back to the membrane Hamiltonian (26) observe that the equations of motion take a remarkably simple form - exhibiting the special nature of the $\frac{1}{q^*}$-potential - when using the linear differential operator

$$D \overset{\text{def}}{=} q \left( \frac{\partial}{\partial t} + (\vec{\nabla}p) \cdot \vec{\nabla} \right)$$

(31)
and complex co-ordinates $z = \frac{1}{2}(x_1 + ix_2)$, $\partial = \partial_1 - i\partial_2$, $\bar{\partial} = \partial_1 + i\partial_2$, namely:

\begin{align*}
D\left(\frac{1}{q}\right) &= \partial\bar{\partial}p \\
D(\partial p) &= \partial\left(\frac{1}{q}\right) \\
D(\bar{\partial} p) &= \bar{\partial}\left(\frac{1}{q}\right).
\end{align*}

Defining the following functional $l$ of the fields

\begin{equation}
l = \frac{e^{-i\theta}}{q} + e^{i\theta} \partial p
\end{equation}

with $e^{i\theta}$ a spectral parameter (reflecting the explicit $U(1)$-invariance of the model) the equation

\begin{equation}
Dl = \partial\bar{l}
\end{equation}

encodes the membrane-dynamics. Since the integral $\int d^2\varphi$ over the $\varphi$-variables becomes an integral $\int d^2 x \ q(\vec{x})$ (with density $q$) under the transformation (10) it is clear that if

\begin{equation}
\int d^2 x \ D(C(q, p, \partial p, \bar{\partial} p, ...)) = 0
\end{equation}

for some functional $C$ of the fields $q$ and $p$, the integral over the $\varphi$-variables of the corresponding functional in the original variables will be a constant of motion. Moreover, it is easy to check that eqns (35) and (36) can be retransformed as follows:

\begin{equation}
L \overset{\text{def}}{=} e^{i\theta}\{x_1, x_2\} + e^{i\theta}(p_1 - i p_2)
\end{equation}

satisfies the ‘Lax like’ equation

\begin{equation}
\dot{L} = \{L, M\}
\end{equation}

(where the function $M$ is given by $M = ie^{-i\theta}(x_1 - ix_2)$) provided the constraint holds. The original Hamiltonian (2) is simply given by

\begin{equation}
H = \frac{1}{2} \int d^2 x \ \bar{L}L = \frac{1}{2} \int d^2 x \ \bar{l}l.
\end{equation}

Note that the $\theta$-dependent cross terms automatically vanish upon integration due to the constraint eqn (3).

4. Let us now comment on the 4-dimensional Poincaré invariance of the membrane: Consider the following functionals of the fields $q$ and $p$ (where
\[ P_+ \overset{\text{def}}{=} \eta \]  
\[ P_- \overset{\text{def}}{=} \frac{\dot{H}}{\eta} = \frac{1}{2\eta} \int d^2x \; q((\vec{\nabla} p)^2 + \frac{1}{q^2}) \]  
\[ P_i \overset{\text{def}}{=} \int d^2x \; q \partial_i p \]  
\[ J_{12} \overset{\text{def}}{=} \int d^2x \; q(x_1 \partial_2 p - x_2 \partial_1 p) \]  
\[ J_{i+} \overset{\text{def}}{=} \eta \int d^2x \; qx_i \]  
\[ J_{i-} \overset{\text{def}}{=} -\eta \zeta_0 \]  
\[ J_{i+} \overset{\text{def}}{=} \frac{1}{2\eta} \int d^2x \; qx_i((\vec{\nabla} p)^2 + \frac{1}{q^2}) - \frac{1}{2\eta} \int d^2x \; q \partial_i (p^2) \]  
\[ \ldots \ldots \ldots \ldots \]  
Here, \( \zeta_0 \) is the integral over the light-cone co-ordinate \( \zeta \) of the membrane. Simply using the canonical Poisson brackets
\[ \{ \{ q(\vec{x}), p(\vec{x}') \} \} = \delta^{(2)}(\vec{x} - \vec{x}') \]
\[ \{ \zeta_0, \eta \} = -1 \]
one may convince oneself that the \( J \)'s above constitute a basis of the Lie algebra \( so(1, 3) \) of the Lorentz group, and that the \( P \)'s commute among themselves and with the \( J \)'s in the correct way. In particular \( (i = 1, 2) \):
\[ \{ J_{-i}, P_+ \} = \delta_{ij}P_- \]  
\[ \{ J_{+i}, P_- \} = P_i \]  
\[ \{ J_{-i}, P_- \} = 0 \]  
\[ \{ J_{1-}, J_{2-} \} = 0 \]  
\[ \{ J_{i+}, J_{k-} \} = \delta_{ik}J_{+k} - J_{ik} \]  
The relativistic mass of the membrane
\[ M^2 = 2P_+P_- - \vec{P}^2 = \int d^2x \; q((\vec{\nabla} p)^2 + \frac{1}{q^2}) - P_{1}^2 - P_{2}^2 \]
Poisson-commutes with all the ten generators given in eqs. (41) - (47).

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