Universal Central Extensions of the Matrix
Leibniz Superalgebras $\mathfrak{sl}(m, n, A)$

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Abstract. The universal central extensions and their extension kernels of the matrix Lie superalgebra $\mathfrak{sl}(m, n, A)$, the Steinberg Lie superalgebra $\mathfrak{sl}(m, n, A)$ in category $\text{SLeib}$ of Leibniz superalgebras are determined under a weak assumption (compared with [MP]) using the first Hochschild homology and the first cyclic homology group.

1. Introduction

Leibniz algebra was introduced by Loday [Lo1], studied by Cuvier [C] and others. Loday-Pirashvili [LP] established the concept of universal enveloping algebras of Leibniz algebras and interpreted the Leibniz (co)homology $HL_*$ (resp. $HL^*$) as a $Tor$-functor (resp. $Ext$-functor). The central extensions of Leibniz algebras and Lie superalgebras have been investigated recently (for instance, see [Lo1], [Lo2], [Gao2], [Gao3], [IK], [LH1], [LH2], [MP], etc.). Leibniz superalgebra and its cohomology were further discussed by Dzhumadil'daev in [D]. The universal central extension of the matrix Lie superalgebras over an associative algebra $A$ in category $\text{SLie}$ of Lie superalgebras was obtained in [MP] under the assumption that $m + n \geq 5$.

Theorem 1.1. [MP] If $m + n \geq 5$, the universal central extension of $\mathfrak{sl}(m, n, A)$ in category $\text{SLie}$ is $\mathfrak{sl}(m, n, A)$ with kernel $HC_1(A)$, where $HC_1(A)$ is the first cyclic homology group of $A$.

Mainly motivated by [MP], [LP], [AG], [Gao], [Gao2] and [KL], we get the universal central extension of the matrix Lie superalgebra $\mathfrak{sl}(m, n, A)$ in category $\text{SLeib}$ of Leibniz superalgebras and prove a main theorem under a weak assumption that $m + n \geq 3$:

Theorem 1.2. If $m + n \geq 3$ with $\text{char} \ K \neq 2$ if $m + n = 4$, $\text{char} \ K \neq 3$ if $m + n = 3$, the universal central extension of $\mathfrak{sl}(m, n, A)$ in category $\text{SLeib}$ is $\mathfrak{sl}(m, n, A)$ with kernel $HH_1(A)$, where $HH_1(A)$ is the first Hochschild homology group of $A$.

More precisely, we determine the universal central extension and its kernel of $\mathfrak{sl}(m, n, A)$ in category $\text{SLeib}$ in Sections 3, 4; and so does for the Steinberg Lie superalgebra $\mathfrak{sl}(m, n, A)$ in $\text{SLeib}$ as well in Section 5.

Key words and phrases. Leibniz or Steinberg superalgebras; kernel of central extension, cyclic homology.

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2. Leibniz superalgebras

Throughout this paper, \( K \) denotes a field with characteristic \( \neq 2, 3 \), \( A \) an associative unital \( K \)-algebra.

**Definition 2.1.** A Leibniz superalgebra is a \( \mathbb{Z}_2 \)-graded \( K \)-vector space \( L = L_0 \oplus L_1 \) with a \( K \)-bilinear map \([-, -]: L \times L \to L\) satisfying \([L_\alpha, L_{\alpha'}] \subseteq L_{\alpha + \alpha'}\) (\( \alpha, \alpha' \in \mathbb{Z}_2 \)) and the Leibniz identity \([[a, b], c] = [a, [b, c]] - (-1)^{|a||b|} [b, [a, c]],\) for homogenous \( a, b, c \in L \), where \(|a|\) denotes the degree of \( a \) for a homogenous element \( a \in L \).

Clearly, \( L_0 \) is a Leibniz algebra; any Lie superalgebra is a Leibniz superalgebra; any Leibniz algebra is a trivial Leibniz superalgebra. A Leibniz superalgebra is a Lie superalgebra if and only if \([a, b] + (-1)^{|a||b|} [b, a] = 0\), for homogenous \( a, b \in L \).

For a Leibniz superalgebra \( L \), let \( L_{S\text{Lie}} \) be the quotient of \( L \) by the ideal generated by elements \([x, y] + (-1)^{|x||y|} [y, x],\) for homogenous \( x, y \in L \). Clearly, \( L_{S\text{Lie}} \) is a Lie superalgebra. The canonical projection \( \pi : L \to L_{S\text{Lie}} \) is universal among the maps from \( L \) to Lie superalgebras. In fact, the functor \((-)_{S\text{Lie}} : S\text{Lie} \to S\text{Leib} \) is left adjoint to \( \text{inc} : S\text{Lie} \to S\text{Leib} \). The cohomology of Leibniz superalgebras has been defined in \( [D] \). The following results are clear.

**Proposition 2.2.** \( A \) Leibniz superalgebra \( L \) admits a universal central extension \( \hat{L} \) if and only if \( L \) is perfect (i.e., \([L, L] = L\)).

**Lemma 2.3.** \( A \) Leibniz superalgebra \( L \). If \( Y \) is perfect, then there exists only one homomorphism \( h \) from \( Y \) to \( X \) such that \( f \circ h = g \).

Consider the matrix Lie superalgebra

\[
\mathfrak{gl}(m, n, A) := \left\{ X \middle| X = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \right\},
\]

where \( A, B, C, D \) are matrices in size \( m \times m, m \times n, n \times m, n \times n \) respectively with coefficients in \( A \), and \( m, n \geq 0, m + n \geq 2 \). Its supercommutator is defined as

\[
[X, Y] = XY - (-1)^{|a||b|} YX \text{ for } X \in \mathfrak{gl}(m, n, A), Y \in \mathfrak{gl}(m, n, A), \alpha, \beta \in \mathbb{Z}_2.
\]

By definition, the special linear Lie superalgebra with coefficients in \( A \) is

\[
\mathfrak{sl}(m, n, A) := \mathfrak{gl}(m, n, A)^{(1)} = \{ X \in \mathfrak{gl}(m, n, A) | \text{str} X = 0 \},
\]

where \( \text{str} X := \text{tr} A - \text{tr} D \) is the supertrace of \( X \). Note that if \( n \neq m \), the Lie superalgebra \( \mathfrak{sl}(m, n, A) \) is simple. Obviously, \( \mathfrak{sl}(m, n, A) \) has generators \( E_{ij}(a) (1 \leq i \neq j \leq m + n, a \in A) \), and subject to the relations below:

\[
\begin{align*}
[E_{ij}(a), E_{kl}(b)] &= 0, \text{ if } i \neq l, \text{ and } j \neq k; \\
[E_{ij}(a), E_{kl}(b)] &= E_{il}(ab), \text{ if } i \neq l, \text{ and } j = k; \\
[E_{ij}(a), E_{kl}(b)] &= -(1)^{\tau_{ij}\tau_{kl}} E_{kj}(ba), \text{ if } i = l, \text{ and } j \neq k,
\end{align*}
\]

where

\[
\tau_{ij} = \begin{cases} 0, & \text{if } 1 \leq i, j \leq m, \text{ or } m + 1 \leq i, j \leq m + n; \\
1, & \text{if } 1 \leq i \leq m, j > m, \text{ or } 1 \leq j \leq m, i > m.
\end{cases}
\]

For \( \tau_{ij} \), it is clear that \( \tau_{ij} = \tau_{ji} \) and

\[
(-1)^{\tau_{ij} + \tau_{jk}} = (-1)^{\tau_{ik}}. \tag{2.1}
\]

Note that \( \mathfrak{sl}(m, n, A) = \mathfrak{sl}(m, n, A)_0 \bigoplus \mathfrak{sl}(m, n, A)_1, \mathfrak{sl}(m, n, A)_\alpha = \langle E_{ij}(a), a \in A | \tau_{ij} = \alpha \rangle, \alpha \in \mathbb{Z}_2. \)
By definition, the Steinberg Lie superalgebra $\mathfrak{sl}(m, n, \mathcal{A})$ is a Lie superalgebra generated by symbols $v_{ij}(a)$, $1 \leq i \neq j \leq m + n$, $a \in \mathcal{A}$, subject to the relations:

1. $v_{ij}(k_1 a + k_2 b) = k_1 v_{ij}(a) + k_2 v_{ij}(b)$, for $a, b \in \mathcal{A}$, $k_1, k_2 \in K$;
2. $[u_{ij}(a), u_{kl}(b)] = 0$, if $i \neq l$, and $j \neq k$;
3. $[u_{ij}(a), u_{kl}(b)] = u_{il}(ab)$, if $i \neq l$, and $j = k$;
4. $[u_{ij}(a), u_{kl}(b)] = (-1)^{q_a q_b} u_{kj}(ba)$, if $i = l$, and $j \neq k$.

Now we define the Steinberg Leibniz superalgebra $\mathfrak{stl}(m, n, \mathcal{A})$ to be a Leibniz superalgebra with generators $v_{ij}(a)$, $1 \leq i \neq j \leq m + n$, $a \in \mathcal{A}$ and subject to the above relations (1)—(4).

Note that $\mathfrak{stl}(m, n, \mathcal{A}) = \mathfrak{stl}(m, n, \mathcal{A})_0 \oplus \mathfrak{stl}(m, n, \mathcal{A})_1$, $\mathfrak{stl}(m, n, \mathcal{A})_a = \langle v_{ij}(a) \mid a \in \mathcal{A}, \tau_{ij} = \alpha \rangle$ for $\alpha \in \mathbb{Z}_2$. Clearly, relations (3)—(4) make sense only if $m + n \geq 3$.

Define homomorphisms $\varphi$ (resp. $\psi$) of Lie (resp. Leibniz) superalgebras $\varphi \ (\text{resp. } \psi) : \mathfrak{st}(m, n, \mathcal{A})$ (resp. $\mathfrak{stl}(m, n, \mathcal{A})$) $\longrightarrow \mathfrak{sl}(m, n, \mathcal{A})$ as $\varphi(u_{ij}(a)) = E_{ij}(a)$ (resp. $\psi(v_{ij}(a)) = E_{ij}(a)$). Clearly, $\varphi, \psi$ are surjective. Moreover, $\mathfrak{st}(m, n, \mathcal{A}) = \mathfrak{st}(m, n, \mathcal{A})_{\pi}$, where $\pi : \mathfrak{st}(m, n, \mathcal{A}) \longrightarrow \mathfrak{sl}(m, n, \mathcal{A})$.

**Lemma 2.4.** The Steinberg Leibniz superalgebra $\mathfrak{stl}(m, n, \mathcal{A})$ with $m + n \geq 3$ is perfect.

### 3. Universal central extension of $\mathfrak{sl}(m, n, \mathcal{A})$ in $\mathfrak{SLie}$

**Theorem 3.1.** If $m + n \geq 3$, then $(\mathfrak{stl}(m, n, \mathcal{A}), \psi)$ is a central extension of the Leibniz superalgebra $\mathfrak{st}(m, n, \mathcal{A})$.

To begin with the proof, let us calculate $\text{Ker} \psi$ first, and then prove that $[\text{Ker} \psi, \mathfrak{stl}(m, n, \mathcal{A})] = 0$.

Denote by $P$ (resp. $Q$) the $K$-submodule of $\mathfrak{stl}(m, n, \mathcal{A})$ generated by $v_{ij}(a)$ with $i < j$ (resp. $i > j$). Clearly, we have: $P = P_0 \oplus P_1$ and $Q = Q_0 \oplus Q_1$; the restrictions of $\psi$ to $P$ and $Q$ are injective; the images of $v_{ij}(a)$ from $P$ (resp. $Q$) under $\psi$ are strictly uppertrangular (resp. lowertrangular) matrices in $\mathfrak{sl}(m, n, \mathcal{A})$.

Let $H_{ij}(a, b) := [v_{ij}(a), v_{ij}(b)]$ for $1 \leq i \neq j \leq m + n$, $a, b \in \mathcal{A}$, $H$ the submodule of $\mathfrak{stl}(m, n, \mathcal{A})$ generated by $H_{ij}(a, b), i \neq j, a, b \in \mathcal{A}$. Then the following Lemma is evident.

**Lemma 3.2.** Every element $X \in \mathfrak{stl}(m, n, \mathcal{A})$ with $m + n \geq 3$ can be uniquely written in the form

\[ X = p + h + q, \quad \text{where } p \in P, \ h \in H, \ q \in Q. \]  

(3.1)

**Lemma 3.3.** For $m + n \geq 3$, $\text{Ker} \psi \subseteq H$.

**Proof.** Let $X = p + h + q \in \text{Ker} \psi$, where $p \in P$, $h \in H$, $q \in Q$. Then $0 = \psi(X) = \psi(p) + \psi(h) + \psi(q)$. By Lemma 3.2, we have $\psi(p) = \psi(q) = 0$, so $p = q = 0$, that is, $X = h \in H$. \hfill $\square$

**Lemma 3.4.** For $m + n \geq 3$, $[\text{Ker} \psi, \mathfrak{stl}(m, n, \mathcal{A})] = 0$.

**Proof.** By Lemma 3.3, any $t \in \text{Ker} \psi$ is expressible by $H_{kl}(b, c)$’s. By definition,

\[ [v_{ij}(a), H_{kl}(b, c)] \in P + Q. \]

So we have $[v_{ij}(a), t] = p + q$, where $p \in P, q \in Q$. Thus $\psi(p + q) = \psi([v_{ij}(a), t]) = [\psi(v_{ij}(a)), \psi(t)] = 0$ since $\psi(t) = 0$. By injectivity of the restriction of $\psi$ to $P + Q$, we get $p + q = 0$. So $[\text{Ker} \psi, \mathfrak{stl}(m, n, \mathcal{A})] = 0$. \hfill $\square$

Therefore, we complete the proof of Theorem 3.1.

We proceed to show that the central extension $(\mathfrak{stl}(m, n, \mathcal{A}), \psi)$ is universal.
THEOREM 3.5. Let $(W, \phi)$ denote a central extension of Leibniz superalgebra $\mathfrak{sl}(m, n, A)$, and $m+n \geq 3$ with $\text{char} K \neq 2$ if $m+n = 4$, $\text{char} K \neq 3$ if $m+n = 3$. Then there exists a unique homomorphism $\rho : \mathfrak{sl}(m, n, A) \to W$ such that $\phi \circ \rho = \psi$.

PROOF. For $m+n = 3$ and $\text{char} K \neq 3$: one can use the same method in the proof of Theorem 5.18 in [AG] to prove $\mathfrak{sl}(m, n, A)$ is centrally closed. The differences here are that we need set $M = v_{13}(A) \oplus v_{11}(A) \oplus v_{23}(A) \oplus v_{32}(A)$, $D = \text{ad}(v_{12}(1) + v_{21}(1))$, and $D|_{M}$ is diagonalizable with eigenvalues $\pm 1$.

For $m+n \geq 4$: since $\phi : W \to \mathfrak{sl}(m, n, A)$ is surjective, for any generator $E_{ij}(a) \in \mathfrak{sl}(m, n, A)$, we choose $e_{ij}(a) \in \phi^{-1}(E_{ij}(a))$. Then the commutator $[e_{ij}(a), e_{kl}(b)]$ does not depend on the choice of representatives of $\phi^{-1}(E_{ij}(a))$ and $\phi^{-1}(E_{kl}(b))$. Moreover, for any $j \neq k$ and $i \neq l$, and $a, b \in A$, $\phi([e_{ij}(a), e_{kl}(b)]) = [\phi(e_{ij}(a)), \phi(e_{kl}(b))] = 0$. So $[e_{ij}(a), e_{kl}(b)] \in \text{Ker} \phi$.

For distinct $i, j, k, l$, let $[e_{ik}(a), e_{kj}(b)] = e_{ij}(ab) + C_{ij}^{k}(a, b)$, $[e_{kj}(b), e_{ik}(a)] = -(-1)^{r_{jk}r_{il}}e_{ij}(ab) + D_{ij}^{k}(a, b)$, where $C_{ij}^{k}(a, b), D_{ij}^{k}(a, b) \in \text{Ker} \phi$. Take $l \notin \{i, j, k\}$, then
\[
[e_{ik}(a), e_{kj}(bc)] = [e_{ik}(a), [e_{kl}(b), e_{lj}(c)]] = [e_{ik}(a), e_{kl}(b), e_{lj}(c)]
\]
\[
= [[e_{ik}(a), e_{kl}(b)], e_{lj}(c)] + (-1)^{r_{jk}r_{il}}[e_{ij}(b), [e_{ik}(a), e_{lj}(c)]]
\]
\[
= [e_{il}(ab), e_{lj}(c)],
\]
i.e.,
\[
[e_{ik}(a), e_{kj}(bc)] = [e_{il}(ab), e_{lj}(c)].
\]

In particular,
\[
[e_{ik}(a), e_{kj}(c)] = [e_{il}(ab), e_{lj}(c)].
\]

It follows that $C_{ij}^{k}(a, c) = C_{ij}^{k}(a, c)$, which shows that $C_{ij}^{k}$ is independent of the choice of $k$. Setting $C_{ij}(a, b) = C_{ij}^{k}(a, b)$, we have
\[
[e_{ik}(a), e_{kj}(b)] = e_{ij}(ab) + C_{ij}(a, b),
\]
where $C_{ij}(a, b) \in \text{Ker} \phi$. Taking $a = 1$, we have
\[
[e_{ik}(1), e_{kj}(b)] = e_{ij}(b) + C_{ij}(1, b).
\]

Set $w_{ij}(a) = e_{ij}(a) + \text{Ker} \phi = [e_{ik}(1), e_{kj}(a)] + \text{Ker} \phi$. We shall show that for all $i \neq j, a \in A, w_{ij}(a)$’s satisfy relations (1)—(4) in the definition of Steinberg Leibniz superalgebra, i.e.,
\[
\begin{align*}
(5) & \quad w_{ij}(k_{1}a + k_{2}b) = k_{1}w_{ij}(a) + k_{2}w_{ij}(b), \quad \text{for} \quad a, b \in A, k_{1}, k_{2} \in K; \\
(6) & \quad [w_{ij}(a), w_{kl}(b)] = 0, \quad \text{if} \quad i \neq l, j \neq k; \\
(7) & \quad [w_{ij}(a), w_{kl}(b)] = w_{il}(ab), \quad \text{if} \quad i \neq l, j = k; \\
(8) & \quad [w_{ij}(a), w_{kl}(b)] = -(1)^{r_{ij}r_{kl}}w_{kj}(ba), \quad \text{if} \quad i = l, j \neq k.
\end{align*}
\]

If $m + n \geq 5$, the proof of (5)—(8) is essentially the same as that in [KL] (or [Gao2, MP]).

If $m + n = 4$, we only prove (6) for the case when $\{i, j, l, k\} = \{1, 2, 3, 4\}$,
\[
[w_{ij}(a), w_{kl}(bc)] = [w_{ij}(a), [w_{kl}(b), w_{il}(c)]]
\]
\[
= [[[w_{ij}(a), w_{kl}(b)], w_{il}(c)] = -(-1)^{r_{ij}r_{ik}}[w_{kj}(ba), w_{il}(c)],
\]
\[
[w_{ij}(a), w_{kl}(bc)] = [w_{ij}(a), [w_{kl}(b), w_{jl}(c)]]
\]
\[
= (-1)^{r_{ij}r_{kl}}[w_{kj}(b), [w_{ij}(a), w_{jl}(c)]] = (-1)^{r_{ij}r_{kl}}[w_{kj}(b), w_{il}(ac)].
\]

So we have
\[
[w_{ij}(a), w_{kl}(bc)] = -(-1)^{r_{ij}r_{kl}}[w_{kj}(ba), w_{il}(c)] = (-1)^{r_{ij}r_{kl}}[w_{kj}(b), w_{il}(ac)].
\]

Taking $a = b = 1$ in (3.6), we have
\[
[w_{ij}(1), w_{kl}(c)] = -[w_{kj}(1), w_{il}(c)] = [w_{kj}(1), w_{il}(c)] = 0,
\]

(3.7)
unless \( \tau_{ij} = 1 \) (noting \((-1)^{\tau_{ij}} \tau_{ik} + (-1)^{\tau_{ij}} \tau_{kj} = (-1)^{\tau_{ij}} \tau_{ik}(1 + (-1)^{\tau_{ij}})) \).

Even in the case when \( \tau_{ij} = 1 \), one can choose a \( k \not\in \{i, j\} \) such that \( \tau_{kj} \neq 1 \) due to (2.1), then use (3.7) again to obtain

\[
[w_{ij}(1), w_{kl}(c)] = \pm [w_{kj}(1), w_{id}(c)] = 0, \tag{3.8}
\]

for all \( \{i, j, l, k\} = \{1, 2, 3, 4\} \).

Taking \( b = 1 \) in (3.6), together with (3.8), we obtain (6).

The others are also essentially the same as those in the case of \( m + n \geq 5 \).

Now define a homomorphism \( \rho : \mathfrak{sl}(m, n, A) \to W \) by \( \rho(v_{ij}(a)) = w_{ij}(a) \). From the above, we see that this mapping is well-defined, and \( \phi \circ \rho = \psi \). The uniqueness of the mapping \( \rho \) follows from Lemmas 2.2, 2.3.

**Remark 3.6.** In (3.7), we used the restriction \( \text{char } K \neq 2 \). From the proof of Theorem 5.18 in [AG], we must restrict \( \text{char } K \neq 3 \) if \( m + n = 3 \) (also see [Gao1]).

### 4. Kernel of universal central extension \( \mathfrak{sl}(m, n, A, \psi) \)

In treating with the Leibniz superalgebras’ case, in order to calculate the kernel of the universal central extension of \( \mathfrak{sl}(m, n, A) \), we have to introduce a so-called modified Hochschild homology. By calculation, the modified Hochschild boundary must be the \( K \)-linear map \( d_n : A^{\otimes (n+1)} \to A^{\otimes n} \) defined by the formula (with the last summand different from the usual definition of the Hochschild boundary, see [Lo2]):

\[
d_n(a_0 \otimes a_1 \otimes \cdots \otimes a_n) = \sum_{i=0}^{n-1} (-1)^i a_0 \otimes a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n - a_1 \otimes \cdots \otimes a_{n-1} \otimes a_n a_0,
\]

which still can be checked to satisfy \( d_n \circ d_{n+1} = 0 \). Thus one can consider the \( n \)-th homology group \( HH_n(A) = \ker d_n/\text{Im } d_{n+1} \).

**Remark 4.1.** The last summand in the definition (cf. [Lo2]) of the usual Hochschild boundary \( d_n \) is \((-1)^n a_n a_0 \otimes a_1 \otimes \cdots \otimes a_{n-1} \).

Let \( M_{m+n}(A) \) be the \( K \)-algebra of \( (m+n) \times (m+n) \)-matrices written in the block form

\[
\begin{pmatrix}
A & B \\
C & D
\end{pmatrix},
\]

where \( A, B, C, D \) are matrices in size \( m \times m, m \times n, n \times m, n \times n \) respectively with coefficients in \( A \), and \( m, n \geq 0, m+n \geq 2 \).

**Theorem 4.2.** The kernel \( T(m,n) \) of universal central extension \( \mathfrak{sl}(m,n,A), \psi \) of \( \mathfrak{sl}(m,n,A) \) in \( \textbf{SLeib} \) under the assumption \( m + n \geq 3 \) (with \( \text{char } K \neq 2 \) if \( m+n=4 \), \( \text{char } K \neq 3 \) if \( m+n=3 \)) is isomorphic to \( HH_1(A) \).

**Proof.** The proofs are analogous to those in [KL] (also see [Gao2], [MP]), except for some properties of \( H_{ij}(a,b) \) and the homomorphism \( \theta : \mathfrak{sl}(m,n,A) \to A \otimes A/\text{Im } d_2 \). In our case, \( H_{ij} \)'s satisfy the following properties:

1) \( H_{ij}(ab,c) = H_{ik}(a,bc) + (-1)^{\tau_{ik}} H_{kj}(b,ca) \);
2) \( H_{ij}(1,a) = (-1)^{\tau_{ij}} H_{ji}(1,a) \), for any distinct \( i,j \) (we also set \( h(a,b) = H_{1i}(a,b) - H_{1i}(1,ba) \), which is independent of \( i(\neq 1) \)).
\[ \theta \text{ is defined by } \theta((x,y)) = \sum_{i,j} \psi(x)_{ij} \otimes \psi(y)_{ji} = \text{str}_2(\psi(x) \otimes \psi(y)) \],

where \( \text{str}_2 : M_{m+n}(A) \otimes M_{m+n}(A) \to A \otimes A \) is given by

\[
\text{str}_2\left(\begin{array}{cc} A_1 & B_1 \\ C_1 & D_1 \end{array}\right) \otimes \left(\begin{array}{cc} A_2 & B_2 \\ C_2 & D_2 \end{array}\right)
\]

\[ = \text{tr}_2\left(\begin{array}{cc} A_1 & 0 \\ 0 & 0 \end{array}\right) \otimes \left(\begin{array}{cc} A_2 & B_2 \\ 0 & 0 \end{array}\right) + \left(\begin{array}{cc} 0 & B_1 \\ 0 & 0 \end{array}\right) \otimes \left(\begin{array}{cc} 0 & 0 \\ C_2 & D_2 \end{array}\right)
\]

\[ - \text{tr}_2\left(\begin{array}{cc} 0 & B_2 \\ C_1 & 0 \end{array}\right) \otimes \left(\begin{array}{cc} 0 & 0 \\ 0 & D_1 \end{array}\right) + \left(\begin{array}{cc} 0 & 0 \\ 0 & D_2 \end{array}\right) \otimes \left(\begin{array}{cc} 0 & 0 \\ 0 & 0 \end{array}\right)\),

and \( \text{tr}_2 \) was defined in [KL] by \( \text{tr}_2(P \otimes Q) = \sum_{1 \leq i,j \leq 1} p_{ij} \otimes q_{ij} \), for \( P, Q \in M_1(A) \).

**Corollary 4.3.** ([LB3]) When \( A \) is commutative, \( HH_1(A) \cong \Omega^1_{\text{Lie}} \), i.e.,

\[ 0 \to \Omega^1_{\text{Lie}} \to \text{st}(m,n,A) \to \text{st}(m,n,A) \to 0 \]

is the universal central extension of \( \text{st}(m,n,A) \) in \( \text{SLeib} \) under the assumption:

\[ m + n \geq 3 \text{ with char } K \neq 2 \text{ if } m + n = 4, \text{ or char } K \neq 3 \text{ if } m + n = 3. \]

**Remark 4.4.** If take \( n = 0 \) in \( \text{st}(m,n,A) \), we recover the same results in [LP], [Gao2] for Leibniz algebras. The stable case (characteristic 0) is due to Cuvier-Loday theorem (see [Lo2]).

**5. Universal central extension of \( \text{st}(m,n,A) \) in \( \text{SLeib} \)**

In what follows, it is necessary for us to point out an important relationship between our modified Hochschild homology (only applicable to the Leibniz superalgebras’ case) and the cyclic homology.

Consider the complex of the \( K \)-modules \( C_\ast(A) \), where \( C_0(A) = A \) and for \( n \geq 1 \) the module \( C_n(A) \) is the factor module of the \( K \)-module \( A^\otimes(n+1) \) by the \( K \)-submodule generated by elements \( a_0 \otimes \cdots a_n + (-1)^a_1 \otimes \cdots a_n \otimes a_0, a_i \in A, i = 0, \cdots, n \). The cyclic boundary induced by the modified Hochschild boundary (also denoted \( d_n \)) is exactly the one induced by the usual Hochschild boundary. Thus we consider the essentially same \( n \)-th cyclic homology group \( HC_n(A) = \text{Ker } d_n/\text{Im } d_{n+1} \) as usual in [Lo2]. Moreover, there is a natural projection \( p : HH_\ast(A) \to HC_\ast(A) \) induced by the projection \( A^\otimes(n+1) \to C_n(A) \).

Note the well-known exact sequence

\[ 0 \to HC_1(A) \to C_1(A)/\text{Im } d_2 \xrightarrow{d_2} A, \]

as well as the Connes operator \( B \) (see [Lo2]), which is a \( K \)-linear map \( A^\otimes(n+1) \to A^\otimes(n+2) \) by

\[ B(a_0 \otimes \cdots a_n) = \sum_{i=0}^n (-1)^n a_i (1 \otimes a_i \otimes \cdots a_n \otimes a_0 \otimes \cdots a_{i-1}) \]

\[ + (-1)^{n+i} (a_i \otimes \cdots a_n \otimes a_0 \otimes \cdots a_{i-1} \otimes 1), \]

such that \( Bd_n + d_{n+1}B = 0 \). Using these objects, one can prove the following theorem.

**Theorem 5.1.** ([MP]) The universal central extension of \( \text{st}(m,n,A) \) in category \( \text{SLeib} \) is \( \text{st}(m,n,A) \) with kernel \( HC_1(A) \) under the assumption: \( m + n \geq 5 \).

**Remark 5.2.** Using the same methods in the proofs of Sections 3 & 4, we can show that Theorem 5.1 still holds under a weak assumption below.

**Theorem 5.3.** The universal central extension of \( \text{st}(m,n,A) \) in category \( \text{SLeib} \) is \( \text{st}(m,n,A) \) with kernel \( HC_1(A) \) under the assumption: \( m+n \geq 3 \) with char \( K \neq 2 \) if \( m+n = 4 \), char \( K \neq 3 \) if \( m+n = 3 \).
Now we give the main theorem of this section.

**Theorem 5.4.** Steinberg Leibniz superalgebra \((\mathfrak{st}(m,n,A),\pi)\) is the universal central extension of \(\mathfrak{sl}(m,n,A)\) in category \(\mathbf{SLie}\) with the kernel isomorphic to \(\text{Im}\ B\) under the assumption: \(m+n \geq 3\) with \(\text{char} \ K \neq 2\) if \(m+n = 4\), or \(\text{char} \ K \neq 3\) if \(m+n = 3\).

**Proof.** The proving idea is similar to that in [LP].

Let \(\mathfrak{g} = \mathfrak{st}(m,n,A)\), then \(\mathfrak{g}_{\mathbf{SLie}} = \mathfrak{sl}(m,n,A)\). From [MP] or Theorem 5.3, we see that \(\mathfrak{g}_{\mathbf{SLie}}\) is the universal central extension of \(\mathfrak{sl}(m,n,A)\) in \(\mathbf{SLie}\) for \(m+n \geq 3\). By Theorem 3.5, Theorem 4.2 and Theorem 5.3, we have the following exact diagram. Moreover, it is clear that this diagram is commutative.

\[
\begin{array}{ccccccc}
\text{HC}_0(A) & \downarrow B \\
0 \rightarrow & H H_1(A) & \rightarrow & \mathfrak{st}(m,n,A) & \rightarrow & \mathfrak{sl}(m,n,A) & \rightarrow 0 \\
\downarrow p & \downarrow \pi & & \rightarrow & & || & \\
0 \rightarrow & HC_1(A) & \rightarrow & \mathfrak{st}(m,n,A) & \rightarrow & \mathfrak{sl}(m,n,A) & \rightarrow 0 \\
\downarrow & \downarrow & & & & & \\
0 & 0 & & & & &
\end{array}
\]

Then we have \(\text{Ker} \ (\pi) \cong \text{Ker} \ p \cong \text{Im} \ B\).

Hence \(0 \rightarrow \text{Im} \ B \rightarrow \mathfrak{st}(m,n,A) \rightarrow \mathfrak{sl}(m,n,A) \rightarrow 0\) is a central extension in \(\mathbf{SLie}\). Moreover, for \(m+n \geq 3\), this is a universal extension by Theorem 5.3 and Theorem 4.2. Therefore,

\(HL_2(\mathfrak{st}(m,n,A)) \cong \text{Im} \ B\).

\(\square\)

**Remark 5.5.** Taking \(n = 0\) in Theorem 5.4, we obtain that (4.6) in [LP] also holds for all \(m \geq 3\) with \(\text{char} \ K \neq 2\) if \(m = 4\), \(\text{char} \ K \neq 3\) if \(m = 3\) (see [Gao2]).

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