SUMS OF POWERS VIA INTEGRATION

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Abstract. Sum of powers $1^p + \cdots + n^p$, with $n, p \in \mathbb{N}$ and $n \geq 1$, can be expressed as a polynomial function of $n$ of degree $p+1$. Such representations are often called Faulhaber formulae. A simple recursive algorithm for computing coefficients of Faulhaber formulae is presented. The correctness of the algorithm is proved by giving a recurrence relation on Faulhaber formulae.

Keywords: Faulhaber formulae; recurrence relation

1. Introduction

Define $f_p(n) = 1^p + \cdots + n^p$, for $p \in \mathbb{N}, n \in \mathbb{N}^+$. One can express $f_p(n)$ as a polynomial function of $n$ of degree $p+1$. For example, $1^2 + \cdots + n^2 = \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n$. Such representations are often called Faulhaber formulae, after Johann Faulhaber (1580–1635). In this paper, we study the following problem:

For $p \in \mathbb{N}$, find the Faulhaber formula that represents $1^p + \cdots + n^p$.

Let us write $f_p(n) = a_{p+1}n^{p+1} + \cdots + a_1n + a_0$; it can be proved that $a_0 = 0$ for all $p \in \mathbb{N}$. Clearly finding the Faulhaber formula, given any $p \in \mathbb{N}$, can be reduced to finding the corresponding coefficients: $a_1, \cdots, a_{p+1}$.

A well-known relation between Faulhaber formulae and Bernoulli numbers, that is $f_p(n) = \frac{1}{p+1} \sum_{i=0}^{p+1} \binom{p+1}{i} b_i n^{p+1-i}$ with $b_i$ being the $i^{th}$ Bernoulli number (when $b_1 = \frac{1}{2}$), can be used for computing $a_1, \cdots, a_{p+1}$. This approach however requires computing Bernoulli numbers $b_0, \cdots, b_p$.

There are various algorithms in the literature for computing Bernoulli numbers. These algorithms are generally based on recurrence relations, where $b_i$ is computed using $b_0, \cdots, b_{i-1}$, e.g. see [KB67, AD09]. In this paper, we give a recurrence relation on Faulhaber formulae, which yields a direct algorithm for computing the coefficients $a_1, \cdots, a_{p+1}$.

Structure of paper. In section 2 we give a recursive algorithm for computing coefficients of Faulhaber formulae. Correctness of the algorithm is proved in section 3. Time complexity of the algorithm is also analysed in section 3.

2. Direct algorithm

We are interested in computing the coefficients of the Faulhaber formula that describes $f_p(n)$, for a given $p \in \mathbb{N}$. Write $f_p(n) = a_{p+1}n^{p+1} + \cdots + a_1n$. Let us consider a table in which rows refer to different values of $p$, and columns refer to
the powers of $n$. The element at the intersection of row $i$ and column $j$, denoted $a_{i,j}$, is meant to represent the coefficient of $n^j$ in the polynomial describing $f_i(n)$. See figure 1. Note that elements at $(i, j)$ with $j > i + 1$ are all zero.

| $i$ | $j$ $\rightarrow$ | 1 | 2 | 3 | 4 |
|-----|------------------|---|---|---|---|
| 0   | 1                | - | - | - |   |
| 1   | $\frac{1}{2}$   | $\frac{1}{2}$ | - | - |   |
| 2   | $\frac{1}{3}$   | $\frac{1}{3}$ | $\frac{1}{3}$ | - |   |
| 3   | 0                | $\frac{1}{4}$ | $\frac{1}{4}$ | $\frac{1}{4}$ | $\frac{1}{4}$ |

**Figure 1.** Intuitive description of the direct algorithm

Our goal is therefore to find the numbers in the row corresponding to $p$. We proceed inductively: First, row 0 is filled, then we fill row 1, ..., till the row numbered with $p$ is filled. The algorithm starts with placing 1 at position $(0, 1)$ in the matrix, thus filling row 0. This step reflects $f_0(n) = 1^0 + \cdots + n^0 = n$. In order to fill the $i^{th}$ row, with $i > 0$, we follow the rules below:

1. For $1 < j \leq i + 1$, let $a_{i,j} = \frac{i}{j} a_{i-1,j-1}$.
2. Next, we compute $a_{i,1}$ as $a_{i,1} = 1 - \sum_{1 < j \leq i+1} a_{i,j}$. Put differently, $a_{i,1}$ is chosen such that the sum of the numbers that appear in each row equals 1.

The procedure stops when the row corresponding to $p$ is filled. Below, it is proved that $a_{(p,j)}$, for $1 \leq j \leq p+1$, is the coefficient of $n^j$ in the polynomial that represents $f_p(n)$. Algorithm 1 implements this procedure.

Note that, since filling row $i$ only requires elements of row $i - 1$, the algorithm only stores a vector, instead of the matrix of figure 1.

**Algorithm 1 Computes coefficients of Faulhaber formulae**

Require: $p \in \mathbb{N}$

\[
\begin{align*}
a_1 & := 1 \\
\text{for } (i := 1; i \leq p; i ++) \text{ do} \\
& \quad s := 0 \\
& \quad \text{for } (j := i + 1; j > 1; j --) \text{ do} \\
& \quad \quad a_j := \frac{i}{j} a_{j-1} \\
& \quad \quad s := s + a_j \\
& \quad \text{end for} \\
& \quad a_1 := 1 - s \\
\text{end for} \\
\text{return } a_1, \cdots, a_{p+1}
\end{align*}
\]

Using the table of figure 1 and the presented algorithm, we make the following simple observations about coefficients of Faulhaber formulae.
• The coefficient of \( n^{p+1} \) in \( f_p(n) \) is \( \frac{1}{p+1} \), for any \( p \in \mathbb{N} \). This can be proved by induction: \( a_{(p, p+1)} = \frac{p}{p+1} a_{(p-1, p)} \) and \( a_{(1, 2)} = \frac{1}{2} \).

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• The coefficient of \( n^{p-2} \) in \( f_p(n) \) is zero, for any \( p \geq 3 \). This can be proved by induction: \( a_{(p, p-2)} = \frac{p}{p-2} a_{(p-1, p-3)} \) and \( a_{(3, 1)} = 0 \).

3. Recurrence relation on Faulhaber formulae

In this section we prove that algorithm 1 correctly computes the coefficients of Faulhaber formulae. For this, first, a recurrence relation on Faulhaber formulae is proved.

**Lemma 1** (Recurrence on Faulhaber formulae).

• \( f_0(n) = n \),

• for \( p > 0 \), \( f_p(n) = p \int_0^n f_{p-1}(t) \, dt + (1 - p) \int_0^1 f_{p-1}(t) \, dt) n \)

**Proof.** The first part, \( f_0(n) = 1^0 \pm \cdots + 2^0 = n \), can be proved by straightforward induction. In the following, therefore, we assume \( p > 0 \). To prove the second part, we recall the following relations (e.g. see [AS72, chapter 23]):

I \( f_{p-1}(n) = \frac{1}{p}(B_p(n+1) - B_p(0)) \), for \( p \in \mathbb{N}^+ \).

II \( \int_a^b B_i(t) \, dt = \frac{1}{i+1}(B_{i+1}(b) - B_{i+1}(a)) \).

III \( B_i(n+1) - B_i(n) = in^{i-1} \), for any \( n \in \mathbb{N}, i > 1 \).

where \( B_i(t) \) is the \( i^{th} \) Bernoulli polynomial. Using (I) and (II), we get:

\[
\int_a^b p f_{p-1}(t) \, dt \\
= \int_a^b B_p(t+1) - B_p(0) \, dt \\
= \frac{1}{p+1}(B_{p+1}(b+1) - B_{p+1}(a+1)) - B_p(0)(b-a)
\]

Therefore

\[
1 - \int_0^1 p f_{p-1}(t) \, dt \\
= 1 - \frac{1}{p+1}(B_{p+1}(2) - B_{p+1}(1)) + B_p(0) \\
= B_p(0)
\]

The last simplification step is due to (III). Note that since \( p > 0 \), we have \( p+1 > 1 \), satisfying the precondition of (III). Similarly

\[
\int_0^n p f_{p-1}(t) \, dt \\
= \frac{1}{p+1}(B_{p+1}(n+1) - B_{p+1}(1)) - nB_p(0)
\]

As a result

\[
\int_0^n p f_{p-1}(t) \, dt + (1 - \int_0^1 p f_{p-1}(t) \, dt) n \\
= \frac{1}{p+1}(B_{p+1}(n+1) - B_{p+1}(1)) \\
= \frac{1}{p+1}(B_{p+1}(n+1) - B_{p+1}(0))
\]
The last simplification step is again due to (III): $B_{p+1}(1) - B_{p+1}(0) = 0$, with $p + 1 > 1$. Finally

$$f_p(n) = \frac{1}{p+1} (B_{p+1}(n+1) - B_{p+1}(0)) = \int_0^n p f_{p-1}(t) \, dt + (1 - \int_0^1 p f_{p-1}(t) \, dt) n$$

This completes the proof. □

Now, we are ready to prove the correctness of algorithm 1.

**Theorem 1** (Correctness). Given $p \in \mathbb{N}$, algorithm 1 outputs the coefficients of the Faulhaber formula that represents $f_p(n)$.

**Proof.** Let us assume the coefficient of $n^j$ is $\alpha$ in $f_i(n)$, for some $1 < j \leq i + 1$, and the coefficient of $n^{j-1}$ in $f_{i-1}(n)$ is $\beta$. From the recurrence relation of lemma 1, we get $\alpha = \frac{j}{i} \beta$. This is simply because $\int_0^n t^k \, dt = \frac{1}{k+1} n^{k+1}$, where $k$ is a any positive rational number. This directly results in the way algorithm 1 recursively computes $a_{(i,j)}$, for $1 < j \leq i + 1$, and $i > 0$.

Now, note that $f_p(1) = 1$, for any $p \in \mathbb{N}$. Moreover, note that $f_p(1) = a_{p+1} + \cdots + a_1$. Therefore, $a_1 = 1 - (a_{p+1} + \cdots + a_2)$. This immediately results in the way algorithm 1 computes $a_{(i,1)}$, for $i > 0$. □

Below, we turn to time complexity of algorithm 1. To measure the computational complexity, we count the number of multiplication and addition (or, subtraction) operations that are performed on rational numbers. Assignments to constants, and incrementing and decrementing natural numbers (i.e. counters in the algorithm) are thus assumed to take negligible time.

**Theorem 2** (Time complexity). Time complexity of the direct algorithm is quadratic in $p$.

**Proof.** Note that the outer for loop is repeated $p$ times, and the inner for loop is repeated $1 + \cdots + p = \frac{p(p+1)}{2}$ times. It is straightforward to see that, for $p \in \mathbb{N}$, the number of addition operations is $\frac{1}{2} p(p+1) + p$, and the number of multiplication operations that are performed on rational numbers is $\frac{1}{2} p(p+1)$.

□

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