Wavelet Electrodynamics II:
Atomic Composition of Electromagnetic Waves

(Appeared in Applied and Computational Harmonic Analysis 1, 246–260, 1994)

Gerald Kaiser
Department of Mathematical Sciences
University of Massachusetts at Lowell
Lowell, MA 01854, USA
e-mail: kaiserg@woods.uml.edu

November 15, 1993

Abstract

The representation of solutions of Maxwell’s equations as superpositions of scalar wavelets with vector coefficients developed earlier is generalized to wavelets with polarization, which are matrix-valued. The construction proceeds in four stages: (1) A Hilbert space \( \mathcal{H} \) of solutions is considered, based on a conformally invariant inner product. (2) The analytic-signal transform is used to extend solutions from real space-time to a complex space-time domain \( \mathcal{T} \). The evaluation maps \( \mathcal{E}_z \), which send any solution \( F = B + iE \) to the values \( \tilde{F}(z) \) of its extension at points \( z \in \mathcal{T} \), are bounded linear maps on \( \mathcal{H} \). Their adjoints \( \Psi_z \equiv \mathcal{E}_z^* \) are the electromagnetic wavelets. (3) The eight real parameters \( z = x + iy \in \mathcal{T} \) are given a complete physical interpretation: \( x = (x, t) \in \mathbb{R}^4 \) is interpreted as a space-time point about which \( \Psi_z \) is focused. The imaginary space-time vector \( y = (y, s) \) is time-like, i.e., \( |y| < |s| \). The sign of \( s \) is interpreted as the helicity of the wavelet, while \( |s| \) is its scale. The 3-vector \( v \equiv y/s \) is the velocity of its center. Thus wavelets parameterized by the set of Euclidean points \( E = \{ (x, is) \} \) (real space and imaginary time) have stationary centers, and wavelets with \( y \neq 0 \) are Doppler-shifted versions of ones with stationary centers. All the wavelets can be obtained from a single “mother” by conformal transformations. (4) A resolution of unity is established in \( \mathcal{H} \), giving a representation of solutions as “atomic compositions” of wavelets parameterized by \( z \in E \). This yields a constructive method for generating solutions with initial data specified locally in space and by scale. Other representations, employing wavelets with moving centers, are obtained by applying conformal transformations to the stationary representation. This could be useful in the analysis of electromagnetic waves reflected or emitted by moving objects, such as radar signals.
1. Introduction

In this paper we further develop the wavelet formulation of classical electrodynamics begun in Kaiser [12-14]. There, it was shown that electromagnetic waves (solutions of Maxwell’s equations) can be expressed as linear superpositions of spherical wavelets uniquely adapted to these equations. However, the wavelets in Refs. [12-14] were scalar-valued solutions of the wave equation rather than (vector-valued) solutions of Maxwell’s equations. (Their coefficients were vector-valued, thus providing for polarization.) Although the scalar wavelets sufficed for the reconstruction of known electromagnetic waves, they could not be used for the construction of new ones according to given local data, precisely because of their scalar nature, which ignored the polarization degrees of freedom. (Their reproducing kernel was not the projection operator to the solution space.) The wavelets constructed here remedy this deficiency, since they are true electromagnetic fields, parameterized by a complete set of physically relevant variables: Their point and time of localization, the velocity of their center, their scale and their helicity.

This work is part of a general program whose main objective is to extend the elementary physical fields from real to complex space-time and interpret the imaginary parts of the space-time variables as control parameters for the wave number and frequency contents of the fields being analyzed. This approach has so far given a fully relativistic phase-space description of a variety of free field theories: Klein-Gordon, Dirac, and now Maxwell. Furthermore, the regularity resulting from the analyticity of the fields in the complex space-time domain promises to help resolve some of the fundamental difficulties plaguing these theories, related to their reliance on the concept of precise geometrical points with no substance or structure. For the extended fields, points in complex space-time have a natural interpretation as moving extended objects in real space-time which, in turn, act as elementary building blocks or “atoms” for the fields. In the case of field theories with positive mass (Klein-Gordon, Dirac), these atoms are relativistic coherent states, i.e., Gabor-like wavelets whose windows undergo scaling (Lorentz contractions) under Lorentz transformations. In the case of massless field theories such as electrodynamics, the atoms are space-time-scale wavelets transforming covariantly under the conformal group. In all cases, the atoms are uniquely determined by the field theory through covariance and analyticity.

The main ideas of the above program were reported in [16], where some of the results of the author’s thesis (Phase-Space Approach to Relativistic Quantum Mechanics, Univ. of Toronto, 1977) were summarized. A key tool for extending general fields to complex space-time, the analytic-signal transform, was developed in [17] and further investigated in [11]. Some speculation on the application of these methods to electrodynamics was advanced in Kaiser and Streater [15] for the much simpler case of a two-dimensional space-time.

In Section 2, Maxwell’s equations are solved by Fourier methods from a viewpoint in which the concepts of helicity and polarization become very clear. A Hilbert space
$\mathcal{H}$ of solutions is constructed which was proved by L. Gross [8] to carry a unitary representation of the full invariance group of the equations, namely the conformal group of space-time. In Section 3 we review the analytic-signal transform, which extends any function $f(x)$ from $\mathbb{R}^n$ to $\mathbb{C}^n$. In general, the extended function $\tilde{f}(z)$ is not analytic (there may not exist any analytic extension). But when the Fourier transform of $f$ is supported on a double convex cone (as it is, for example, when $f$ represents a free relativistic field such as an electromagnetic wave), then $\tilde{f}(z)$ is analytic in a certain tube domain $\mathcal{T}$ in $\mathbb{C}^n$. In Section 4 we show that the analytic-signal transform, when applied to electromagnetic fields, uniquely determines a set of electromagnetic wavelets. A resolution of unity is derived which allows solutions to be expressed as superpositions of wavelets. In Section 5 we compute the reproducing kernel defined by the wavelets, which in turn gives the wavelets explicitly. In Section 6 we show how arbitrary solutions in $\mathcal{H}$ can be constructed from wavelets, with initial data specified locally and by scale. In Section 7, the wavelets are given a complete physical and geometric interpretation. In Section 8 we describe some generalizations, and a possible application.

2. The Fourier Representation of Free Electromagnetic Fields

An electromagnetic wave in free space (without sources or boundaries) is described by a pair of vector fields depending on the space-time variables $x = (x, x_0)$ (where $x$ is the position and $x_0$ is the time), namely the electric field $E(x)$ and the magnetic field $B(x)$. These are subject to Maxwell’s equations,

$$\nabla \times E + \partial_0 B = 0, \quad \nabla \cdot E = 0,$$

$$\nabla \times B - \partial_0 E = 0, \quad \nabla \cdot B = 0,$$

(1)

where $\nabla$ is the gradient with respect to the space variables and $\partial_0$ is the time derivative. We have set the speed of light $c = 1$ for convenience. For the present discussion, $E$ and $B$ may be assumed to be tempered distributions, so that (2.1) holds weakly. Later, the fields will be required to belong to a certain Hilbert space. Note that the equations are symmetric under the linear mapping defined by $J : E \mapsto B$, $B \mapsto -E$, and that $J^2$ is minus the identity map. Such a mapping on a vector space is called a complex structure, by analogy with multiplication by $i$ in the complex plane. The combinations $B \pm iE$ diagonalize $J$, since $J(B \pm iE) = \pm i(B \pm iE)$. They each map Maxwell’s equations to a form in which the concepts of helicity and polarization become very simple. It will suffice to consider only $F \equiv B + iE$, since the other combination is equivalent. Eqs. (2.1) then become

$$\partial_0 F = i \nabla \times F, \quad \nabla \cdot F = 0.$$

(2)

Note that the first of these equations is an evolution equation (initial-value problem), while the second is a constraint on the initial values. Taking the divergence of the first equation shows that the constraint is conserved by time evolution. Note also that it is
the factor $i$ in (2.2) (i.e., the complex structure!) which couples the dynamics of the electric field $E$ to those of the magnetic field $B$. Eq. (2.2) implies

$$-\partial_0^2 F = \nabla \times (\nabla \times F) = \nabla (\nabla \cdot F) - \nabla^2 F = -\nabla^2 F$$  

(3)
in Cartesian coordinates, hence the components of $F$ become decoupled and each satisfies the wave equation

$$\Box F \equiv (-\partial_0^2 + \nabla^2) F = 0.$$  

(4)

Since $F$ is a tempered distribution, it has a Fourier expansion

$$F(x) = (2\pi)^{-4} \int_{\mathbb{R}^4} d^4 p e^{ip \cdot x} \hat{F}(p),$$  

(5)

where $p = (p, p_0) \in \mathbb{R}^4$ with $p \in \mathbb{R}^3$ as the spatial wave vector and $p_0$ as the frequency. We use the Lorentz-invariant inner product $p \cdot x \equiv p_0 x_0 - p \cdot x$. The wave equation (2.4) implies that $p^2 \hat{F}(p) = 0$, where $p^2 \equiv p \cdot p = p_0^2 - |p|^2$. If the distribution $\hat{F}$ has no essential support at the origin $p = 0$ (i.e., no term proportional to $\delta(p)$), it must be supported on the nipped light cone

$$C = \{(p, p_0) \in \mathbb{R}^4 \setminus \{0\} : p_0^2 = |p|^2\} = C_+ \cup C_-,$$

(6)

where $\pm p_0 = |p| > 0$ in $C_\pm$. Hence $\hat{F}$ has the form $\hat{F}(p) = 2\pi \delta(p^2) f(p)$, where $f(p)$ is a (vector-valued) distribution on $C$ or, equivalently, the pair of distributions on $\mathbb{R}^3$ given by $f_{\pm}(p) \equiv f(p, \pm |p|)$. For the moment, we assume that $f_{\pm}$ are (vector-valued) Schwartz test functions. Later the class of $f$’s will be enlarged by introducing an inner product and completing it to a Hilbert space, subject to a restriction related to our having “nipped” the light cone, which amounts, roughly, to $f(0) = 0$. Letting $\omega \equiv |p|$, we have

$$\delta(p^2) = \delta((p_0 - \omega)(p_0 + \omega)) = \frac{\delta(p_0 - \omega) + \delta(p_0 + \omega)}{2|p_0|}.$$  

(7)

Hence (2.5) becomes

$$F(x) = (2\pi)^{-3} \int_{\mathbb{R}^3} \frac{d^3 p}{2|p_0|} e^{-i p \cdot x} \left[ e^{i \omega t} f_+(p) + e^{-i \omega t} f_-(p) \right]$$

$$= \int_{C} d\tilde{p} \ e^{i p \cdot x} f(p),$$  

(8)

where $d\tilde{p} \equiv (2\pi)^{-3} d^3 p / 2|p_0|$ is a Lorentz-invariant measure on $C$. In order for (2.8) to give a solution of (2.2), $f(p)$ must further satisfy the algebraic conditions

$$i p_0 f(p) = p \times f(p), \quad p \cdot f(p) = 0.$$  

(9)
for all \( p \in C \), and the first of these equations suffices since it implies the second. Let \( \mathbf{v}(p) = p/p_0 \), so that \( p \in C \) if and only if \( |\mathbf{v}| = 1 \). Define the operator \( \Gamma \equiv \Gamma(p) \) on arbitrary functions \( \mathbf{g} : C \to \mathbb{C}^3 \) by

\[
(\Gamma \mathbf{g})(p) \equiv -i \mathbf{v}(p) \times \mathbf{g}(p), \quad p \in C.
\]

(10)

\( \Gamma(p) \) is represented by the Hermitian matrix

\[
\Gamma(p) = i \begin{bmatrix} 0 & v_3 & -v_2 \\ -v_3 & 0 & v_1 \\ v_2 & -v_1 & 0 \end{bmatrix} = \frac{1}{p_0} \begin{bmatrix} 0 & p_3 & -p_2 \\ -p_3 & 0 & p_1 \\ p_2 & -p_1 & 0 \end{bmatrix},
\]

(11)

with matrix elements \( \Gamma_{mn}(p) = ip_0^{-1} \sum_{k=1}^3 \varepsilon_{mnk} p_k \), where \( \varepsilon_{mnk} \) is the totally antisymmetric tensor with \( \varepsilon_{123} = 1 \). In terms of \( \Gamma(p) \), (2.9) becomes

\[
\Gamma \mathbf{f}(p) = \mathbf{f}(p).
\]

(12)

Now for any \( \mathbf{g} : C \to \mathbb{C}^3 \),

\[
\Gamma^2 \mathbf{g} = -\mathbf{v} \times (\mathbf{v} \times \mathbf{g}) = \mathbf{g} - (\mathbf{v} \cdot \mathbf{g}) \mathbf{v},
\]

(13)

so \( \Gamma(p)^2 \) is the orthogonal projection to the subspace of \( \mathbb{C}^3 \) orthogonal to \( \mathbf{v}(p) \), and it follows that

\[
\Gamma^3 \mathbf{g} = \Gamma \mathbf{g}.
\]

(14)

The eigenvalues of \( \Gamma(p) \), for each \( p \in C \), are therefore 1, 0 and -1, and (2.12) states that \( \mathbf{f}(p) \) is an eigenvector with eigenvalue 1. Since \( \Gamma(p)^* = -\Gamma(p) \), (2.12) implies that \( \Gamma \mathbf{\bar{f}} = -\mathbf{f} \). A similar operator was defined and studied in much more detail by Moses [19] in connection with fluid mechanics as well as electrodynamics.

Consider a single component of (2.8), i.e., the plane-wave solution

\[
\mathbf{F}_p(x) \equiv e^{ip \cdot x} \mathbf{f}(p) = \mathbf{B}_p(x) + i \mathbf{E}_p(x),
\]

(15)

with arbitrary but fixed \( p \in C \) and \( \mathbf{f}(p) \neq \mathbf{0} \). The electric and magnetic fields are obtained by taking the real and imaginary parts. Now \( \Gamma(p) \mathbf{f}(p) = \mathbf{f}(p) \) and \( \Gamma(p) \mathbf{\bar{f}}(p) = -\mathbf{\bar{f}}(p) \) imply

\[
\Gamma(p) \mathbf{F}_p(x) = \mathbf{F}_p(x), \quad \Gamma(p) \mathbf{\bar{F}}_p(x) = -\mathbf{\bar{F}}_p(x).
\]

(16)

Since \( \Gamma(p)^* = \Gamma(p) \), these eigenvectors of \( \Gamma(p) \) with eigenvalues 1 and -1 must be orthogonal: \( \mathbf{F}_p(x)^* \mathbf{F}_p(x) = \mathbf{F}_p(x) \cdot \mathbf{F}_p(x) = 0 \), where the asterisk denotes the Hermitian transpose. Taking real and imaginary parts, we get

\[
|\mathbf{B}_p(x)|^2 = |\mathbf{E}_p(x)|^2, \quad \mathbf{B}_p(x) \cdot \mathbf{E}_p(x) = 0.
\]

(17)
The first equation shows that neither $\mathbf{B}_p(x)$ nor $\mathbf{E}_p(x)$ can vanish at any $x$ (since $f(p) \neq 0$). Furthermore, (2.16) implies that $\mathbf{p} \times \mathbf{E}_p(x) = p_0 \mathbf{B}_p(x)$. Thus, for any $x$, \(\{\mathbf{p}, \mathbf{E}_p(x), \mathbf{B}_p(x)\}\) is a right-handed orthogonal basis if $p_0 > 0$ (i.e., $p \in C_+$) and a left-handed orthogonal basis if $p_0 < 0$ ($p \in C_-$). Taking the real and imaginary parts of (2.15) and using $f(p) = \mathbf{B}_p(0) + i \mathbf{E}_p(0)$, we have

\[
\begin{align*}
\mathbf{B}_p(x) &= \cos(p \cdot x) \mathbf{B}_p(0) - \sin(p \cdot x) \mathbf{E}_p(0), \\
\mathbf{E}_p(x) &= \cos(p \cdot x) \mathbf{E}_p(0) + \sin(p \cdot x) \mathbf{B}_p(0).
\end{align*}
\]

An observer at any fixed location $\mathbf{x} \in \mathbb{R}^3$ sees these fields rotating, as a function of time, in the plane orthogonal to $\mathbf{p}$. If $p \in C_+$, the rotation is that of a right-handed corkscrew, or helix, moving in the direction of $\mathbf{p}$, whereas if $p \in C_-$, it is that of a left-handed corkscrew. Hence $\mathbf{F}_p(x)$ is said to have positive helicity if $p \in C_+$ and negative helicity if $p \in C_-$.

A general solution of the form (2.8) has positive helicity if $f(p)$ is supported in $C_+$ and negative helicity if $f(p)$ is supported in $C_-$. Other states of polarization, such as linear or elliptic, are obtained by mixing positive and negative helicities. The significance of the complex combination $\mathbf{F}(x) = \mathbf{B}(x) + i \mathbf{E}(x)$ therefore seems to be that in Fourier space, the sign of the frequency $p_0$ gives the helicity of the solution! (Usually in signal analysis, the sign of the frequency is not given any physical interpretation, and negative frequency is regarded as a convenient mathematical artifact.) In other words, the combination $\mathbf{B} + i \mathbf{E}$ “polarizes” the helicity, with positive and negative helicity states being represented in $C_+$ and $C_-$, respectively. Had we used the opposite combination $\mathbf{B} - i \mathbf{E}$, $C_+$ and $C_-$ would have parameterized the plane-wave solutions with opposite helicities. Nothing new seems to be gained by considering this alternative. (In fact, Maxwell’s equations are invariant under the continuous group of duality rotations, of which the complex structure $J$ mapping $\mathbf{E}$ to $\mathbf{B}$ and $\mathbf{B}$ to $-\mathbf{E}$ is a special case. In the complexified solution space, the combinations $\mathbf{B} \pm i \mathbf{E}$ form invariant subspaces with respect to the duality rotations. That gives the choice of $\mathbf{B} + i \mathbf{E}$ an interpretation in terms of group representation theory.)

In order to eliminate the constraint, we now proceed as follows: Let

\[
\Pi(p) \equiv \frac{1}{2} \left[ \Gamma(p) + \Gamma^2(p) \right].
\]

Explicitly,

\[
\Pi(p) = \frac{1}{2p_0^2} \begin{bmatrix}
  p_0^2 - p_1^2 & -p_1 p_2 + i p_0 p_3 & -p_1 p_3 - i p_0 p_2 \\
  -p_1 p_2 - i p_0 p_3 & p_0^2 - p_2^2 & -p_2 p_3 + i p_0 p_1 \\
  -p_1 p_3 + i p_0 p_2 & -p_2 p_3 - i p_0 p_1 & p_0^2 - p_3^2
\end{bmatrix}.
\]

The established properties $\Gamma^* = \Gamma = \Gamma^3$ imply that $\Pi^* = \Pi = \Pi^2$ and $\Gamma \Pi = \Pi$, which proves that $\Pi(p)$ is the orthogonal projection to eigenvectors of $\Gamma(p)$ with eigenvalue
1. Thus, to satisfy the constraint (2.12), we need only replace the constrained function \( f(p) \) in (2.8) by \( \Pi(p)f(p) \), where now \( f(p) \) is unconstrained:

\[
F(x) = \int_C d\tilde{p} \ e^{ip \cdot x} \Pi(p) f(p).
\]

Consequently, the mapping \( f \mapsto F \) is not one-to-one since \( \Pi \) is a projection operator. In fact, \( f \) is closely related to the potentials for \( F \), which consist of a real 3-vector potential \( \mathbf{A}(x) \) and a real scalar potential \( A_0(x) \) such that

\[
\mathbf{B} = \nabla \times \mathbf{A}, \quad \mathbf{E} = -\partial_0 \mathbf{A} - \nabla A_0.
\]

The combination \((\mathbf{A}(x), A_0(x))\) is called a “4-vector potential” for the field. We can assume without loss of generality that the potential satisfies the Lorentz condition

\[
\nabla \cdot \mathbf{A} + \partial_0 A_0 = 0 \quad (\text{Jackson [10]}).
\]

Since \( \mathbf{A} \) and \( A_0 \) also satisfy the wave equation (2.4), they have Fourier representations similar to (2.8):

\[
\mathbf{A}(x) = \int_C d\tilde{p} \ e^{ip \cdot x} a(p), \quad A_0(x) = \int_C d\tilde{p} \ e^{ip \cdot x} a_0(p).
\]

The Lorentz condition means that \( p_0 a_0(p) = \mathbf{p} \cdot a(p) \), or \( a_0(p) = \mathbf{v} \cdot a(p) \), so \( a_0 \) is determined by \( a \). Eqs. (2.22) will be satisfied provided that the Fourier representatives \( e(p) \), \( b(p) \) of \( \mathbf{E}, \mathbf{B} \) satisfy

\[
b = -i \mathbf{p} \times a = p_0 \Gamma a, \quad e = -ip_0 a + i \mathbf{p} a_0 = -ip_0 \Gamma^2 a.
\]

Hence \( F = \mathbf{B} + i \mathbf{E} \) is represented in Fourier space by

\[
b(p) + i e(p) = p_0 [\Gamma(p) + \Gamma(p)^2] a(p) = 2p_0 \Pi(p) a(p).
\]

This shows that we can interpret the unconstrained function \( f(p) \) in (2.21) as being directly related to the 3-vector potential by

\[
f(p) = 2p_0 a(p),
\]

modulo terms annihilated by \( \Pi(p) \), which correspond to eigenvalues \(-1\) and \(0\) of \( \Gamma(p) \). Seen in this light, the non-uniqueness of \( f \) in (2.21) is an expression of gauge freedom in the \( \mathbf{B} + i \mathbf{E} \) representation, as seen from Fourier space. In the space-time domain, \((\mathbf{B}, \mathbf{E})\) are the components of a 2-form \( F \) in \( \mathbb{R}^4 \) and \((\mathbf{A}, A_0)\) are the components of a 1-form \( \mathbf{A} \). Then Eqs. (2.1) become \( dF = 0 \) and \( \delta F = 0 \) (where \( \delta \) is the divergence with respect to the Lorentzian inner product), Eqs. (2.22) become unified as \( F = dA \), the Lorentz condition reads \( \delta A = 0 \) and the gauge freedom corresponds to the invariance of \( F \) under \( A \to A + d\chi \), where \( \chi(x) \) is a scalar solution of the wave equation.

Maxwell’s equations are invariant under a large group of space-time transformations. Such transformations produce new solutions from known ones by acting on the
underlying space-time variables (possibly with a multiplier to rotate or scale the vector fields). Some trivial examples are space and time translations: Obviously, a translated version of a solution is again a solution, since the equations have constant coefficients. Similarly, a rotated version of a solution is a solution. A less obvious example is Lorentz transformations, which are interpreted as transforming to a uniformly moving reference frame in space-time. (In fact, it was in the study of the Lorentz invariance of Maxwell’s equations that the Special Theory of Relativity originated; see Einstein et al. [6].) The scale transformations \( x \rightarrow ax, \ a \neq 0 \), also map solutions to solutions, since Maxwell’s equations are homogeneous in the space-time variables. Finally, the equations are invariant under “special conformal transformations” (Bateman [2], Cunningham [4]), which can be interpreted as transforming to a uniformly accelerating reference frame (Page [20]; Hill [9]). Altogether, these transformations form a 15-dimensional Lie group called the conformal group, which is locally isomorphic to \( SU(2, 2) \) and is here denoted by \( C \).

To construct the machinery of wavelet analysis, we introduce a Hilbert space structure on the solutions. It is important to choose the inner product to be invariant under the largest possible group of symmetries, since this allows the largest set of solutions in \( \mathcal{H} \) to be generated by unitary transformations from any one known solution. (In quantum mechanics, invariance of the inner product is also an expression of the fundamental invariance of the laws of nature with respect to the symmetries in question.) Let \( f(p) \) satisfy (2.12), and let \( a(p) \) be a vector potential for \( f \) satisfying the Lorentz condition, so that the scalar potential is determined by \( a_0(p) = \mathbf{v} \cdot a(p) \). By (2.25),

\[
|f(p)|^2 = 4p_0^2 |\Pi(p) a(p)|^2 = 4p_0^2 \overline{a(p)} \cdot \Pi(p) a(p) = 2p_0^2 a(p) \cdot \Gamma(p) a(p) + 2p_0^2 \overline{a(p)} \cdot \Gamma(p)^2 a(p). \tag{27}
\]

The first term is

\[-2ip_0^2 \overline{a(p)} \cdot (\mathbf{v} \times a(p)) = 2ip_0^2 \mathbf{v} \cdot (\overline{a(p)} \times a(p)), \tag{28}\]

which cancels its counterpart with \( p \rightarrow -p \) on account of the reality condition \( \overline{a(-p)} = a(p) \). Thus

\[
\int_C \frac{d\tilde{p}}{p_0^2} |f(p)|^2 = 2 \int_C d\tilde{p} \overline{a(p)} \cdot [a(p) - \mathbf{v} (\mathbf{v} \cdot a(p))] = 2 \int_C d\tilde{p} \left[ |a(p)|^2 - |a_0(p)|^2 \right]. \tag{29}
\]

The integrand in the last expression is the negative of the Lorentz-square of the 4-potential \((a(p), a_0(p))\). Consequently, the integral can be shown to be invariant under
Lorentz transformations. (Note that \(|a|^2 - |a_0|^2 \geq 0\), vanishing only when \(a(p)\) is a multiple of \(p\), in which case \(f = 0\). This corresponds to “longitudinal polarization.”) Hence (2.29) defines a norm on solutions which is invariant under Lorentz transformations as well as space-time translations. In fact, the norm (2.29) is uniquely determined, up to a constant factor, by the requirement that it be so invariant. Moreover, Gross [8] has shown it to be invariant under the full conformal group \(C\). Again we eliminate the constraint by replacing \(f(p)\) with \(\Pi(p)f(p)\). Thus, let \(\mathcal{H}\) be the set of all solutions \(F(x)\) defined by (2.21) with \(f : C \rightarrow \mathbb{C}^3\) square-integrable in the sense that

\[
\|F\|^2 = \int_C \frac{d^3p}{p_0^2} |\Pi(p)f(p)|^2 = (2\pi)^{-3} \int_C \frac{d^3p}{2|p|^3} |\Pi(p)f(p)|^2 < \infty. \tag{30}
\]

\(\mathcal{H}\) is a Hilbert space under the inner product obtained by applying the polarization identity to (2.30) and using \((\Pi f)\ast \Pi g = f\ast \Pi \ast \Pi g = f\ast \Pi g\):

\[
\langle F, G \rangle = \int_C \frac{d^3p}{p_0^2} f(p)\ast \Pi(p) g(p). \tag{31}
\]

\(\mathcal{H}\) will be our main arena for developing the wavelet analysis and synthesis of solutions. Note that when (2.12) holds and \(f_\pm(p) \equiv f(p, \pm|p|)\) are Schwartz test function as we assumed earlier, then

\[
f(0) = f_\pm(0) = 0 \tag{32}
\]

must hold in order that (2.30) be satisfied. However, now that we have our Hilbert structure, we complete to the larger class of all (generalized) functions \(f\) satisfying (2.30).

To show the invariance of (2.30) under conformal transformations, Gross derived an equivalent norm expressed directly in terms of the values of the fields in space at any particular time \(x_0 = t\):

\[
\|F\|_{Gross}^2 \equiv \frac{1}{\pi^2} \int_{R^4} \frac{d^3x \, d^3y}{|x - y|^2} F(x, t)\ast F(y, t). \tag{33}
\]

The right-hand side is independent of \(t\) due to the invariance of Maxwell’s equations under time translations (which is, in turn, related to the conservation of energy). A disadvantage of the expression (2.33) is that it is non-local, since it uses the values of the field simultaneously at the space points \(x\) and \(y\). In fact, it is known that no local expression for the inner product can exist in terms of the field values in (real) space-time \(R^4\) (Bargmann and Wigner [1]). In Section 4, we derive an alternate expression for the inner product directly in terms of the values of the electromagnetic fields, extended analytically to complex space-time. This expression is “local” in the space-scale domain.
(rather than in space alone). But first we must introduce the tool which implements the extension to complex space-time.

3. The Analytic-Signal Transform

Given a vector function \( \mathbf{F} : \mathbb{R}^n \to \mathbb{C}^m \), we define its analytic-signal transform as the function \( \tilde{\mathbf{F}} : \mathbb{C}^n \to \mathbb{C}^m \) given by the following line integral in \( \mathbb{R}^n \):

\[
\tilde{\mathbf{F}}(x + iy) = \frac{1}{\pi i} \int_{-\infty}^\infty \frac{d\tau}{\tau - i} \mathbf{F}(x + \tau y).
\] (3.1)

This transform was introduced and studied in Kaiser [11, 17], where it was shown to be related to the Fourier transform \( \hat{\mathbf{F}}(p) \) of \( \mathbf{F}(x) \) by

\[
\tilde{\mathbf{F}}(x + iy) = (2\pi)^{-n} \int_{\mathbb{R}^n} d^n p 2\theta(p \cdot y) e^{ip \cdot (x + iy)} \hat{\mathbf{F}}(p).
\] (3.2)

Here \( \theta \) is the unit step function, defined by \( \theta(u) = 0 \) if \( u < 0 \), \( \theta(0) = \frac{1}{2} \), \( \theta(u) = 1 \) if \( u > 0 \). For (3.2) to make sense, it suffices that \( \hat{\mathbf{F}}(p) \) be absolutely integrable, since \( |\theta(p \cdot y) e^{ip \cdot (x + iy)}| \leq 1 \). For concreteness, we assume for the time being that \( \hat{\mathbf{F}} \in L^1(\mathbb{R}^4) \) and use (3.2) to define \( \tilde{\mathbf{F}} \), viewing (3.1) as “motivation.” (A study of (3.1) in the context of distribution theory is currently being undertaken by T. Takiguchi [24].) Note that setting \( y = 0 \) on the right gives the inverse Fourier transform of \( \hat{\mathbf{F}}(p) \), so that formally we have \( \tilde{\mathbf{F}}(x) = \mathbf{F}(x) \) and \( \tilde{\mathbf{F}} \) is an extension of \( \mathbf{F} \) from \( \mathbb{R}^n \) to \( \mathbb{C}^n \). (This is made more precise below, in (3.7).) Of course, this extension is usually not analytic, since in general there exists no analytic extension. However, when \( \mathbb{R}^n \) is space-time and \( \mathbf{F} \) represents a free physical field such as an electromagnetic field \( (m = 3) \), a Klein-Gordon field \( (m = 1) \) or a Dirac field \( (m = 4) \), then \( \tilde{\mathbf{F}}(p) \equiv \tilde{\mathbf{F}}(p, p_0) \) vanishes outside the solid light cone

\[
V \equiv \{(p, p_0) \in \mathbb{R}^4 : p^2 \equiv p_0^2 - |p|^2 \geq 0, \ p_0 \neq 0\} = V_+ \cup V_-,
\] (3.3)

where \( \pm p_0 \geq |p| > 0 \) in \( V_\pm \). (In the electromagnetic case, for example, \( \tilde{\mathbf{F}} \) is supported on the boundary \( C \) of \( V \) as a consequence of the wave equation (2.4).) Hence the integral in (3.2) extends only over \( V \). Formally, the obstacle to the analyticity of \( \tilde{\mathbf{F}}(z) \) in (3.2) is the factor \( \theta(p \cdot y) \) (which is necessary, generally, to ensure that the integral converges in the region of Fourier space where \( e^{-p \cdot y} > 1 \)). However, when \( \tilde{\mathbf{F}} \) is supported in \( V \), that obstacle can be removed as follows: Suppose \( y \) is such that \( p \cdot y > 0 \) for all \( p \in V_+ \) and \( p \cdot y > 0 \) for all \( p \in V_- \). (This means that the hyperplane Lorentz-orthogonal to \( y \) separates \( V_+ \) and \( V_- \)!) Then \( \theta(p \cdot y) = 1 \) for all \( p \in V_+ \) and \( \theta(p \cdot y) = 0 \) for all \( p \in V_- \), hence the integral now extends only over \( V_+ \), and the obstructing factor is identically \( =1 \) in that cone. Furthermore, the extra factor \( e^{-p \cdot y} \) coming from the analytic continuation of the Fourier kernel provides exponential damping, which leads to the analyticity of \( \tilde{\mathbf{F}} \).
at \((x + iy)\), for all \(x\). Similarly, if \(y\) is such that \(p \cdot y > 0\) for all \(p \in V_+\) and \(p \cdot y < 0\) for all \(p \in V_-\), then \(\hat{F}\) is again analytic at \(x + iy\), for all \(x\). The above sets of imaginary space-time points \(y\) are, by definition, the dual cones \(V'_\pm\) of \(V_\pm\) (Stein and Weiss [23]),

\[
V'_\pm \equiv \{y = (y, y_0) \in \mathbb{R}^4 : p \cdot y \equiv p_0 y_0 - p \cdot y > 0 \text{ for all } p \in V_\pm\}. \tag{3.4}
\]

\(V'_+\) and \(V'_-\) are the open future light cone and the open past light cone in space-time (as opposed to Fourier space, where \(V_\pm\) live). The union \(V' \equiv V'_+ \cup V'_-\) will be called the dual cone of \(V = V_+ \cup V_-\). Explicitly,

\[
V'_\pm = \{(y, y_0) : \pm y_0 > |y|\}, \quad V' = \{y \in \mathbb{R}^4 : y^2 \equiv y_0^2 - |y|^2 > 0\}. \tag{3.5}
\]

The argument used above to motivate the definition of \(V'_\pm\) can be made precise, leading to the conclusion that \(\hat{F}(z)\) is indeed analytic in

\[
\mathcal{T} \equiv \{z = x + iy \in \mathbb{C}^4 : y \in V'\} = \mathcal{T}_+ \cup \mathcal{T}_- , \tag{3.6}
\]

where \(\mathcal{T}_\pm\) is the set of \(z\)'s with \(y \in V'_\pm\). The fact that \(V\) and \(V'\) are almost identical is due to our arbitrary choice \(c = 1\) for the speed of light. \(V'\) is actually “reciprocal” to \(V\): As \(c\) increases, \(V\) narrows and \(V'\) widens.

A general function \(F(x)\) with \(\text{supp} \hat{F}(p) \subset V\) therefore becomes “polarized” when extended to \(\mathcal{T}\): The positive-frequency part \((p \in V_+)\) determines \(\hat{F}(z)\) in \(\mathcal{T}_+\), and the negative-frequency part determines it in \(\mathcal{T}_-\). The positive and negative frequency components mix on the common boundary \(\mathbb{R}^4\) of \(\mathcal{T}_+\) and \(\mathcal{T}_-\). (The boundaries of \(\mathcal{T}_\pm\) are seven-dimensional, but their intersection is \(\mathbb{R}^4\).) If only real vector functions \(F\) are considered, then the positive and negative frequency parts are “coupled” by the reality condition \(\overline{\hat{F}(-p)} = \hat{F}(p)\), with the corresponding relation \(\overline{\hat{F}(\bar{z})} = \hat{F}(z)\) in \(\mathcal{T}\). If \(F\) is allowed to be complex-valued, its positive- and negative-frequency parts become independent. In the case of electrodynamics, we saw in the last section that they correspond to the positive-helicity and negative-helicity parts of an electromagnetic wave in the \(B + iE\) representation. The separation of helicities into \(C_+\) and \(C_-\) in Fourier space is translated, by the analytic-signal transform, to their separation into \(\mathcal{T}_+\) and \(\mathcal{T}_-\).

From a mathematical point of view, it suffices for \(\hat{F}(p)\) to be supported in any double cone of the form \(V_+ \cup V_-\), where \(V_\pm\) are convex cones intersecting only at their common vertex. Then \(\hat{F}\) is analytic at \(x + iy\) whenever the hyperplane orthogonal to \(y\) separates \(V_+\) and \(V_-\), which again means that \(y\) belongs to the dual \(V'\) of \(V\), defined as in (3.5). The name “analytic-signal transform” derives from the fact that when \(n = m = 1\) and \(f\) is real-valued, then \(\hat{f}(x + iy)\) coincides with the “analytic signal” of \(f\) for \(y > 0\), as first defined by D. Gabor [7] in his famous paper on communication theory. In fact, if \(n = 1\), then \(\hat{f}(x + iy)\) is simultaneously the analytic extension of the positive-frequency part of \(f\) to the complex upper half-plane and of the negative-frequency part of \(f\) to the lower half-plane. (These two half-planes now play the roles of \(\mathcal{T}_+\) and \(\mathcal{T}_-\).)
As already mentioned, $\tilde{F}(x) = F(x)$ formally, i.e., by setting $y = 0$ in (3.2). More precisely, $F$ is the boundary value of $F$ in the sense that for any $y \in \mathbb{R}^n \backslash \{0\},$

$$\lim_{\varepsilon \to 0^+} \left[ \tilde{F}(x + i\varepsilon y) + \tilde{F}(x - i\varepsilon y) \right] = 2F(x) \quad \text{a.e.} \quad (3.7)$$

On the other hand, the “jump” of $\tilde{F}$ across $\mathbb{R}^n$ is

$$\lim_{\varepsilon \to 0^+} \left[ \tilde{F}(x + i\varepsilon y) - \tilde{F}(x - i\varepsilon y) \right] = 2iH_y F(x) \quad \text{a.e.,} \quad (3.8)$$

where

$$H_y F(x) \equiv \frac{1}{\pi} \text{PV} \int_{-\infty}^{\infty} \frac{d\tau}{\tau} F(x - \tau y) \quad (3.9)$$

is the multidimensional Hilbert transform of $F$ in the direction of $y \neq 0$ (Stein [22]) and $\text{PV}$ denotes the Cauchy principal value.

4. The Electromagnetic Wavelets $\Psi_z$

We are now ready to pursue our main theme, the construction of the electromagnetic wavelets and their resolution of unity. (For general background on wavelet theory, the reader may consult Chui [3], Daubechies [5], Kaiser [18] and the references therein.)

Consider the extension of the electromagnetic field $F(x)$ to the tube domain $T$ defined in (3.5) and (3.6). Combining (2.21) and (3.2), we obtain

$$\tilde{F}(x + iy) = \int_C dp \, 2\theta(p \cdot y) e^{ip(x+iy)} \Pi(p) f(p). \quad (4.1)$$

As earlier, assume that $f_\pm(p) \equiv f(p, \pm|p|)$ are vector-valued Schwartz test functions, to begin with. Then $\tilde{F}$ is analytic in $T$. Fix an arbitrary $z = x + iy \in T$ (i.e., $y^2 \equiv y_0^2 - |y|^2 > 0$) and consider the linear operator $E_z : \mathcal{H} \to \mathbb{C}^3$ defined by $E_z F = \tilde{F}(z)$. This is an evaluation map which, when applied to the field $F$, gives the value of its extension at the complex space-time point $z$. Because of the analyticity of $\tilde{F}$, $E_z$ turns out to be bounded, as will be seen later. (It becomes unbounded as $y^2 \to 0$.) We now define the electromagnetic wavelets as the adjoint operators $\Psi_z = E_z^* : \mathbb{C}^3 \to \mathcal{H}$. To find these explicitly, choose any orthonormal basis $u_1, u_2, u_3$ of $\mathbb{C}^3$ and let $\Psi_{z,k} \equiv \Psi_z u_k \in \mathcal{H}, k = 1, 2, 3$. This gives three solutions of Maxwell’s equations, all of which will be wavelets “at” $z$. $\Psi_z$ is a matrix-valued solution of maxwell’s equations, obtained by putting the three (column) vector solutions $\Psi_{z,k}$ together. It will be convenient to use the following “star notation,” introduced in Kaiser [17]: For any $F \in \mathcal{H}$, let $F^* : \mathcal{H} \to \mathbb{C}$ denote the linear functional obtained by taking inner products with $F$:

$$F^* G \equiv \langle F, G \rangle, \quad G \in \mathcal{H}. \quad (4.2)$$

12
\( \mathbf{F}^* \) is not to be confused with the Hermitian transpose \( \mathbf{F}(x)^* \) of \( \mathbf{F}(x) \in \mathbb{C}^3 \). Then the \( k \)-th component of \( \tilde{\mathbf{F}}(z) \) with respect to the basis \( \{ \mathbf{u}_k \} \) is

\[
\tilde{F}_k(z) \equiv u_k^* \tilde{F}(z) = u_k^* \mathcal{E}_x \mathbf{F} = u_k^* \Psi_z^* \mathbf{F} = (\Psi_z \mathbf{u}_k)^* \mathbf{F} = \langle \Psi_{z,k}, \mathbf{F} \rangle. \tag{4.3}
\]

By (4.1),

\[
u_k^* \tilde{F}(z) = \int_C \frac{d\tilde{p}}{p_0^2} 2p_0^2 \theta(p \cdot y) e^{ip \cdot z} \mathbf{u}_k^* \Pi(p) f(p), \tag{4.4}
\]

which shows that \( \Psi_{z,k} \) is given in the Fourier domain by

\[
\Psi_{z,k}(p) = 2p_0^2 \theta(p \cdot y) e^{-ip \cdot \bar{z}} \Pi(p) \mathbf{u}_k. \tag{4.5}
\]

Note that each \( \Psi_{z,k}(p) \) satisfies the constraint since \( \mathbf{\Gamma}(p) \Pi(p) = \Pi(p) \). The matrix-valued wavelet \( \Psi_z \) in the Fourier domain is therefore

\[
\Psi_z(p) = 2p_0^2 \theta(p \cdot y) e^{-ip \cdot \bar{z}} \Pi(p). \tag{4.6}
\]

In the space-time domain we have (using \( \Pi(p) \psi_z(p) = \psi_z(p) \))

\[
\Psi_z(x') \equiv \int_C d\tilde{p} \ e^{ip \cdot x'} \Psi_z(p) = \int_C d\tilde{p} \ 2p_0^2 \theta(p \cdot y) e^{ip \cdot (x' - \bar{z})} \Pi(p). \tag{4.7}
\]

Now that we have the wavelets, we want to make them into a “basis” that can be used to decompose and compose arbitrary solutions. This will be accomplished by constructing a “resolution of unity” in terms of the wavelets. To this end, we derive an expression for the inner product in \( \mathcal{H} \) directly in terms of the values \( \tilde{\mathbf{F}}(z) \) of the extended fields. To begin with, it will suffice to consider the values of \( \tilde{\mathbf{F}} \) only at Euclidean space-time points, i.e., at points with an imaginary time coordinate \( z_0 = is \) and real space coordinates \( z = \mathbf{x} \). In order for \( z \) to belong to \( \mathcal{T} \), it is only necessary to have \( s \neq 0 \). We denote the set of all such points by \( E \). The name “Euclidean” stems from the fact that at such points, the negative of the indefinite Lorentzian metric restricts to the positive-definite Euclidean metric on \( E \): \( -z^2 = -(is)^2 + |\mathbf{x}|^2 = s^2 + |\mathbf{x}|^2 \). Later, \( \mathbf{x} \) will be interpreted as the center of the wavelets \( \Psi_{z,k} \), and \( s \) as their helicity and scale. Using (4.1) and letting \( \omega \equiv |p| = |p_0| \), we have

\[
\tilde{\mathbf{F}}(\mathbf{x}, is) = \int_C d\tilde{p} \ 2\theta(p_0 s) e^{-p_0 s - i\mathbf{p} \cdot \mathbf{x}} \Pi(p) f(p)
\]

\[
= 2 \int_{\mathbb{R}^3} dp \ e^{-i\mathbf{p} \cdot \mathbf{x}} \left[ \theta(\omega s) e^{-\omega s} \Pi(\mathbf{p}, \omega) f(\mathbf{p}, \omega) + \theta(-\omega s) e^{\omega s} \Pi(\mathbf{p}, -\omega) f(\mathbf{p}, -\omega) \right]
\]

\[
= \left[ \omega^{-1} \theta(\omega s) e^{-\omega s} \Pi(\mathbf{p}, \omega) f(\mathbf{p}, \omega) + \omega^{-1} \theta(-\omega s) e^{\omega s} \Pi(\mathbf{p}, -\omega) f(\mathbf{p}, -\omega) \right]^\wedge(\mathbf{x}), \tag{4.8}
\]
where $\vee$ denotes the inverse Fourier transform with respect to $p$. Hence by Plancherel’s formula,

$$\int_{\mathbb{R}^3} d^3x \left| \hat{F}(x, is) \right|^2 = \int_{\mathbb{R}^3} \frac{d^3p}{(2\pi)^3 \omega^2} \left[ \theta(\omega s) e^{-2\omega s} |\Pi(p, \omega) f(p, \omega)|^2 + \theta(-\omega s) e^{2\omega s} |\Pi(p, -\omega) f(p, -\omega)|^2 \right],$$

where we used $\theta(u)^2 = \theta(u)$ and $\theta(u) \theta(-u) = 0$ for $u \neq 0$. Thus

$$\int_E d^3x ds |\hat{F}(x, is)|^2 = \int_{\mathbb{R}^3} \frac{d^3p}{2(2\pi)^3 \omega^3} \left[ |\Pi(p, \omega) f(p, \omega)|^2 + |\Pi(p, -\omega) f(p, -\omega)|^2 \right]$$

$$= \int_C \frac{dp}{p_0^2} |\Pi(p) f(p)|^2 = \int_C \frac{dp}{p_0^2} f(p)^* \Pi(p) f(p) = \|F\|^2,$$

since $\Pi^* \Pi = \Pi^2 = \Pi$. Let $\tilde{H}$ be the set of all analytic-signal transforms $\tilde{F}$ of solutions $F \in H$. For $\tilde{F}, \tilde{G} \in \tilde{H}$, write

$$\langle \tilde{F}, \tilde{G} \rangle = \int_E d^3x ds \tilde{F}(z)^* \tilde{G}(z).$$

Then (4.10) leads immediately to the following result.

**Theorem 1.** $\tilde{H}$ is a Hilbert space under the inner product (4.11), and the map $F \mapsto \tilde{F}$ is unitary from $H$ onto $\tilde{H}$.

**Proof:** By the polarization identity, (4.10) implies

$$\langle \tilde{F}, \tilde{G} \rangle = \langle F, G \rangle,$$

so the map is an isometry. It is obviously surjective, by the definition of $\tilde{H}$. $\blacksquare$

With the “star notation” introduced earlier, the Hermitian transpose $\tilde{F}(z)^* : \mathbb{C}^3 \to \mathbb{C}$ of the ‘column vector’ $\tilde{F}(z) \in \mathbb{C}^3$ is the composition

$$\tilde{F}(z)^* = (\Psi_z^* F)^* = F^* \Psi_z,$$

where $F^* : H \to \mathbb{C}$ denotes the linear functional (4.2). Hence the integrand in (4.11) is

$$\tilde{F}(z)^* \tilde{G}(z) = (\Psi_z^* F)^* \Psi_z^* G = F^* \Psi_z \Psi_z^* G,$$

where $\Psi_z \Psi_z^* : H \to H$ is the composition of $\Psi_z^*$ and $\Psi_z$, and (4.11) reads
\[ \int_E d^3x \, ds \, F^* \Psi_z \Psi_z^* G = F^* G, \quad F, G \in \mathcal{H}. \]  
(4.15)

**Theorem 2.**

(a) The wavelets \( \Psi_z \) with \( z \in E \) give the following resolution of the identity \( I \) in \( \mathcal{H} \):

\[ \int_E d^3x \, ds \, \Psi_z \Psi_z^* = I, \]  
(4.16)

where the equality holds in the weak topology of \( \mathcal{H} \), i.e., (4.15) is satisfied.

(b) Every solution \( F \in \mathcal{H} \) can be written as a superposition of the wavelets \( \Psi_z \) with \( z = (x, is) \in E \), according to

\[ F = \int_E d^3x \, ds \, \Psi_{x,is} \Psi_{x,is}^* F = \int_E d^3x \, ds \, \Psi_{x,is} \tilde{F}(x, is), \]  
(4.17)

i.e.,

\[ F(x') = \int_E d^3x \, ds \, \Psi_{x,is}(x') \tilde{F}(x, is) \quad \text{a.e.} \]  
(4.18)

(4.17) holds weakly in \( \mathcal{H} \) (i.e., the inner products of both sides with any member of \( \mathcal{H} \) are equal). However, for the extended fields, we have

\[ \tilde{F}(z') = \Psi_{z'}^* F = \int_E d^3x \, ds \, \Psi_{z'}^* \Psi_{x,is} \tilde{F}(x, is) \]  
(4.19)

pointwise for all \( z' \in T \).

**Proof:** Only the pointwise convergence in (4.19) remains to be shown. This follows from the boundedness of \( \Psi_{z'} \), which will be proved in Section 5.

The pointwise equality fails, in general, for the boundary values \( F(x) \) because the evaluation maps (or, equivalently, their adjoints \( \Psi_z \)) become unbounded as \( y \to 0 \). This will be seen in the next section.

The opposite composition \( \Psi^*_z, \Psi_z : \mathbb{C}^3 \to \mathbb{C}^3 \) is a matrix-valued function on \( T \times T \):

\[ K(z', \bar{z}) \equiv \Psi^*_z \Psi_z = \int_C \frac{d\vec{p}}{p_0^2} 4p_0^4 \theta(p \cdot y) \theta(p \cdot y) e^{i p \cdot (z' - \bar{z})} \Pi(p)^2 \]  
(4.20)
Eq. (4.19) shows that \( K(z' | \tilde{z}) \) is a reproducing kernel for the Hilbert space \( \tilde{\mathcal{H}} \); see Kaiser [11] for background and references. The boundary value of \( K(z' | \tilde{z}) \) as \( y' \to 0 \) is, according to (3.7) and (4.7), given by

\[
K(x' | \tilde{z}) = \frac{1}{2} \lim_{\varepsilon \to 0^+} \left[ K(x' + i\varepsilon y' | \tilde{z}) + K(x' - i\varepsilon y' | \tilde{z}) \right] = \Psi_z(x').
\]  

(4.21)

Hence, to find the wavelets explicitly, we must compute their reproducing kernel. This is done in the next section.

The meaning of the index \( k \) in \( \Psi_{z,k} \) deserves to be examined. Since \( \Pi(p) \) is the orthogonal projection to the eigenspace of \( \Gamma(p) \) with the nondegenerate eigenvalue 1, all the columns (as well as the rows) of \( \Pi(p) \) are all multiples of one another. But the coefficients are \( p \)-dependent, and the algebraic linear dependence in Fourier space translates to a differential equation in space-time, relating the different wavelets \( \Psi_{z,k} \). For the columns, this differential equation is just Maxwell’s vector equation (2.2). (Recall that the scalar equation is then implied by the wave equation.) Since \( \Pi(p) \) is Hermitian, the same argument goes for the rows. Explicitly,

\[
\Gamma(p)\psi_z(p) = \psi_z(p) = \psi_z(p)\Gamma(p).
\]

(4.22)

When multiplied through by \( p_0 \) and transformed to space-time, these read

\[
\nabla' \times \Psi_z(x') = -i\partial'_0 \Psi_z(x') = \Psi_z(x') \times \hat{\nabla'},
\]

(4.23)

where \( \partial'_0 \) denotes the partial with respect to \( x'_0 \), \( \nabla' \) the gradient with respect to \( x' \), and \( \hat{\nabla'} \) indicates that \( \nabla' \) acts to the left, i.e., on the column index. This states that not only the columns, but also the rows of \( \Psi_z \) are solutions of Maxwell’s equations. The three wavelets \( \Psi_{z,k} \) are thus coupled. Note also that since \( \Psi_z(x') = \Psi_{z-x'}(0) \), Eq. (4.23) can be rewritten as

\[
\nabla \times \Psi_z = -i\partial_0 \Psi_z = \Psi_z \times \hat{\nabla},
\]

(4.24)

where \( \partial_0 \) and \( \nabla \) are the corresponding operators with respect to the labels \( x_0 = \text{Re} z_0 \) and \( x = \text{Re} z \).

We will see in Section 7 that the reconstruction of \( \mathbf{F}(x') \) from \( \tilde{\mathbf{F}}(x, is) \) can be obtained by a much simpler method than (4.18), using only a single scalar wavelet \( \Psi_{x,is}(x') \) instead of the matrix wavelet \( \Psi_{x,is}(x') \) (or three vector wavelets \( \Psi_{x,is,k}(x') \)). However, that presumes that we already know \( \tilde{\mathbf{F}}(x, is) \), and without this knowledge the reconstruction becomes meaningless, since no new solutions can be obtained this way. The use of matrix wavelets will be necessary in order to give a generalization of (4.17)–(4.19), where \( \tilde{\mathbf{F}}(x, is) \) can be replaced with an unconstrained coefficient function. In other words, we need matrix wavelets in space-time for exactly the same reason that
\( \Pi(p) \) was needed in Fourier space (Eq. (2.21)): To eliminate the constraints in the coefficient function.

5. The Reproducing Kernel

In order to obtain detailed information on the wavelets, we compute the reproducing kernel (4.20) explicitly. Note, first of all, that if \( y' \cdot y < 0 \) (i.e., \( z' \in T_+ \) and \( z \in T_- \) or \( z' \in T_- \) and \( z \in T_+ \)), then \( K(z' | z) = 0 \) since \( p \cdot y' \) and \( p \cdot y \) have opposite signs for all \( p \in C \). Hence it suffices to compute the kernel for \( z' \) and \( z \) in the same half of \( T \). Furthermore, \( \Pi(-p) = \Pi(p) \) since \( \Gamma(-p) = \Gamma(p) \). Hence, letting \( z' \to \bar{z}' \) and \( z \to \bar{z} \) in (4.20) gives

\[
K(z' | z) = 4 \int_C d\bar{p} \ p_0^2 \theta(-p \cdot y') \theta(-p \cdot y) e^{ip(z'-z)} \Pi(p) \\
= 4 \int_C d\bar{p} \ p_0^2 \theta(p \cdot y') \theta(p \cdot y) e^{ip(z-\bar{z}')} \Pi(p) = K(z | \bar{z'}),
\]

where the last equality is obtained by letting \( p \to -p \). Thus it suffices to compute the kernel for \( z', \bar{z} \in T_+ \). In this case,

\[
K(z' | \bar{z}) = 4 \int_{C_+} d\bar{p} \ p_0^2 e^{ip(z'-\bar{z})} \Pi(p) \equiv L(z' - \bar{z})
\]

is analytic in \( w \equiv z' - \bar{z} \in T_+ \). It can be shown that \( L(iy) \) with \( y \in V_+ \) uniquely determines \( L(w) \) for all \( w \in T_+ \) by analytic continuation, hence it suffices to compute only \( L(iy) \) for \( y \in V'_+ \). Now the matrix elements of \( 2p_0^2 \Pi(p) \) are given by (2.20):

\[
2p_0^2 \Pi_{mn}(p) = \delta_{mn}p_0^2 - p_m p_n + i \sum_{k=1}^3 \varepsilon_{mnk} p_0 p_k.
\]

To compute \( L(iy) \), it is useful to write the coordinates of \( y \) in contravariant form: \( y^0 = y_0, \ y^m = -y_m \ (m = 1, 2, 3) \), so that \( p \cdot y = \sum_{\mu=0}^3 p_\mu y^\mu \). Letting \( \partial_\mu \) denote the partial derivative with respect to \( y^\mu \) for \( \mu = 0, 1, 2, 3 \), we have

\[
2 \int_{C_+} d\bar{p} \ p_\mu p_\nu e^{-p \cdot y} = \partial_\mu \partial_\nu S(y) \equiv S_{\mu\nu}(y), \quad \mu, \nu = 0, 1, 2, 3,
\]

where

\[
S(y) \equiv 2 \int_{C_+} d\bar{p} \ e^{-p \cdot y}, \quad y \in V'_+.
\]

Thus (5.3) and (5.4) give the matrix elements of \( L(iy) \) as
\[ L_{mn}(iy) = \delta_{mn}S_{00}(y) - S_{mn}(y) + i \sum_{k=1}^{3} \varepsilon_{mnk} S_{0k}(y), \quad m, n = 1, 2, 3. \] (5.6)

It only remains to compute \( S(y) \). For this, we use the fact that \( S(y) \) is invariant under Lorentz transformations, since \( p \cdot y \) and \( d\vec{p} \) are invariant and \( C_+ \) is a homogeneous space for the proper Lorentz group. Since \( y \in V_+ \), there exists a Lorentz transformation mapping \( y \) to \((0, \lambda)\), where \( \lambda(y) \equiv (y_0^2 - |y|^2)^{1/2} > 0 \). The invariance of \( S \) implies that \( S(y) = S(0, \lambda) \). Letting \( \omega = |\vec{p}| \) again, we thus have

\[ S(y) = 2 \int_{\mathbb{R}^3} \frac{d^3p}{16\pi^3|\vec{p}|} e^{-\lambda|\vec{p}|} = \frac{1}{2\pi^2} \int_0^\infty \omega d\omega e^{-\omega\lambda} = \frac{1}{2\pi^2\lambda^2}. \] (5.7)

Taking partials with respect to \( y^\mu \) and \( y^\nu \) gives

\[ S_{\mu\nu}(y) = \frac{4y_\mu y_\nu - g_{\mu\nu}\lambda^2}{\pi^2\lambda^6}, \quad \mu, \nu = 0, 1, 2, 3, \] (5.8)

where \( g_{\mu\nu} = \text{diag}(1, -1, -1, -1) \) is the Lorentz metric. It follows that

\[ L_{mn}(iy) = \frac{2}{\pi^2\lambda^6} \left[ \delta_{mn}(y_0^2 + y_1^2 + y_2^2 + y_3^2) - 2y_m y_n + 2i \sum_{k=1}^{3} \varepsilon_{mnk} y_0 y_k \right]. \] (5.9)

To compute \( L(w) \) for \( w \in \mathcal{T}_+ \), we need only replace \( y \) with \(-iw\). This gives

\[ L_{mn}(w) = \frac{2}{\pi^2w^6} \left[ \delta_{mn}(w_0^2 + w_1^2 + w_2^2 + w_3^2) - 2w_m w_n + 2i \sum_{k=1}^{3} \varepsilon_{mnk} w_0 w_k \right], \] (5.10)

where \( w^6 \equiv (w \cdot w)^3 \). The full kernel is obtained by setting \( w = z' - \bar{z} \) and multiplying by \( \theta(y' \cdot y) \), which ensures that it vanishes when \( z' \) and \( z \) are in opposite halves of \( \mathcal{T} \):

\[ K(z' | \bar{z}) = \theta(y' \cdot y) L(z' - \bar{z}), \quad z', \bar{z} \in \mathcal{T}. \] (5.11)

In Section 4 we stated that due to the analyticity of \( \bar{F}(z) \), the evaluation maps \( \mathcal{E}_z \) (and with them, the wavelets \( \Psi_z = \mathcal{E}_z^* \)) are bounded, and that they become unbounded as \( z = x + iy \) approaches the boundary of \( \mathcal{T} \), i.e., \( y^2 \to 0 \). This can now be verified by examining \( K(z | \bar{z}) = \Psi_z^* \Psi_z \). By (5.11),

\[ K(z | \bar{z}) = \theta(y^2) L(2iy) = L(2iy) \] (5.12)

for all \( z \in \mathcal{T} \), since \( y^2 > 0 \) in \( V' \). Eq. (5.9) shows that \( \Psi_z^* \Psi_z \) is indeed bounded when \( z \in \mathcal{T} \) diverges as \( y^2 \to 0 \). For example, if \( y = (0, s) \) (which can always be arranged by applying a Lorentz transformation), then
\begin{equation}
\Psi_z^* \Psi_z = \frac{1}{8\pi^2 s^4} I,
\end{equation}

where \( I \) is the identity matrix in \( \mathbb{C}^3 \).

6. Atomic Composition of Electromagnetic Waves

The reproducing kernel computed in the last section can be used to construct electromagnetic waves according to local specifications, rather than merely to reconstruct known solutions from their analytic-signal transforms on \( E \). This is especially interesting because the Fourier method for constructing solutions (Section 2) uses plane waves and is therefore completely unsuitable to deal with questions involving local properties of the fields. It will be shown in Section 7 that the wavelets \( \Psi_{x+iy}(x') \) are localized solutions of Maxwell’s equations, at the “initial” time \( x'_0 = x_0 \). Hence we call the composition of waves from wavelets “atomic.”

Suppose \( \tilde{F} \) is the analytic-signal transform of a solution \( F \in \mathcal{H} \) of Maxwell’s equations. Then according to (4.10),

\begin{equation}
\int_E d^3x \, ds \, |\tilde{F}(x, is)|^2 = ||F||^2 < \infty.
\end{equation}

Let \( \mathcal{L}^2(E) \) be the set of all measurable functions \( \Phi : E \to \mathbb{C}^3 \) for which the above integral converges. \( \mathcal{L}^2(E) \) is a Hilbert space under the obvious inner product, obtained from (6.1) by polarization. (In fact, we could identify \( E \) with \( \mathbb{R}^4 \) and \( \mathcal{L}^2(E) \) with \( L^2(\mathbb{R}^4) \) since the set \( \mathbb{R}^4 \setminus E = \{(x,0) : x \in \mathbb{R}^3\} \) has zero measure in \( \mathbb{R}^4 \). But this could cause confusion between the Euclidean region \( E \) and real spacetime \( \mathbb{R}^4 \).) Define the map \( R_E : \mathcal{H} \to \mathcal{L}^2(E) \) by

\begin{equation}
(R_E F)(x, is) \equiv \Psi^*_{x,is} F = \tilde{F}(x, is).
\end{equation}

That is, \( R_E F \) is the restriction \( \tilde{F} \mid_E \) to \( E \) of the analytic-signal transform \( \tilde{F} \) of \( F \). Then (6.1) implies that the range \( \mathcal{W} \) of \( R_E \) is a closed subspace of \( \mathcal{L}^2(E) \), and \( R_E \) maps \( \mathcal{H} \) isometrically onto \( \mathcal{W} \). (In the Physics literature, an operator which transforms fields in real space-time to their counterparts in Euclidean space-time is called a Wick rotation.)

The following theorem characterizes the range of \( R_E \) and gives the adjoint \( R_E^* \).
Theorem 3.
(a) The range of \( R_E \) is the set \( \mathcal{W} \) of all \( \Phi \in \mathcal{L}^2(E) \) satisfying the “consistency condition”

\[
\Phi(z') = \int_E d^3x \, ds \, K(z' \mid x, -is) \, \Phi(x, is), \tag{6.3}
\]
pointwise in \( z' \in E \).
(b) The adjoint operator \( R^*_E : \mathcal{L}^2(E) \rightarrow \mathcal{H} \) is given by

\[
R^*_E \Phi = \int_E d^3x \, ds \, \Psi_{x,is} \Phi(x, is), \tag{6.4}
\]
where the integral converges weakly in \( \mathcal{H} \).

Proof: If \( \Phi \in \mathcal{W} \), then \( \Phi(x, is) = \tilde{F}(x, is) \) for some \( F \in \mathcal{H} \), and (6.3) reduces to (4.19), which holds pointwise in \( z' \in E \). On the other hand, given a function \( \Phi \in \mathcal{L}^2(E) \) which satisfies (6.3), let \( F \) denote the right–hand side of (6.4). Then for any \( G \in \mathcal{H} \),

\[
G^* F = \int_E d^3x \, ds \, \tilde{G}(x, is)^* \Phi(x, is) = \langle R_E G, \Phi \rangle_{L^2}, \tag{6.5}
\]
where we have used \( G^* \Psi_{x,is} = (\Psi_{x,is}^* G)^* = \tilde{G}(x, is)^* \). Hence the integral in (6.4) converges weakly in \( \mathcal{H} \). The transform of \( F \) under \( R_E \) is

\[
(R_E F)(z') = \Psi_{z'}^*, F = \int_E d^3x \, ds \, \Psi_{z'}^* \Psi_{x,is} \Phi(x, is)
\]

\[
= \int_E d^3x \, ds \, K(z' \mid x, -is) \, \Phi(x, is) = \Phi(z'), \tag{6.6}
\]
by (6.3). Hence \( \Phi \in \mathcal{W} \) as claimed, proving (a). Eq. (6.5) states that \( \langle G, F \rangle_{\mathcal{H}} = \langle R_E G, \Phi \rangle_{L^2} \). That shows that \( F = R^*_E \Phi \), proving (b).

Eq. (6.4) constructs a solution \( R^*_E \Phi \in \mathcal{H} \) from a coefficient function \( \Phi \in \mathcal{L}^2(E) \). When \( \Phi \) is actually the transform \( R_E F \) of a solution \( F \in \mathcal{H} \), then \( \Phi(x, is) = \Psi_{x,is}^* F \) and

\[
R^*_E \Phi = \int_E d^3x \, ds \, \Psi_{x,is} \Psi_{x,is}^* F = F, \tag{6.7}
\]
by (4.17). Thus \( R^*_E R_E = I \), the identity in \( \mathcal{H} \). (This is equivalent to (6.1).) We now examine the opposite composition.

Theorem 4. The orthogonal projection to \( \mathcal{W} \) in \( \mathcal{L}^2(E) \) is the composition \( P \equiv R_E R^*_E : \mathcal{L}^2(E) \rightarrow \mathcal{L}^2(E) \), which is given by

\[
(P \Phi)(z') \equiv \int_E d^3x \, ds \, K(z' \mid x, -is) \, \Phi(x, is). \tag{6.8}
\]
Proof: By (6.4),

\[ (R_E R_E^* \Phi)(z') \equiv \Psi_{z'}^* R_E \Phi = \int_E d^3x \, ds \, \Psi_{z'}^* \Psi_{x,is} \Phi(x,is) = (P \Phi)(z'), \quad \text{(6.9)} \]

since \( \Psi_{z'}^* \Psi_{x,is} = K(z',x,-is) \). Hence \( R_E R_E^* = P \). This also shows that \( P^* = P \). Furthermore, \( R_E^* R_E = I \) implies that \( P^2 = P \), hence \( P \) is indeed the orthogonal projection to its range. It only remains to show that the range of \( P \) is \( \mathcal{W} \). If \( \Phi = R_E F \in \mathcal{W} \), then \( R_E R_E^* \Phi = R_E R_E^* R_E R_E F = R_E F = \Phi \). Conversely, any function in the range of \( P \) has the form \( \Phi = R_E R_E^* \Theta \) for some \( \Theta \in L^2(E) \), hence \( \Phi = R_E F \) where \( F = R_E^* \Theta \in \mathcal{H} \). \( \blacksquare \)

When the coefficient function \( \Phi \) in (6.4) is the transform of an actual solution, then \( R_E^* \) reconstructs that solution. However, this process does not appear to be too interesting, since we must have a complete knowledge of \( F \) to compute \( \Phi(x,is) \). For example, to compute \( \tilde{F}(x,is) \) by (3.1), we must know \( F(x,t) \) for all \( x \) and all \( t \). Hence, no “initial-value problem” is solved by (6.4) when applied to \( \Phi \in \mathcal{W} \). However, the option of applying (6.4) to arbitrary \( \Phi \in L^2(E) \) is a very attractive one, since it is guaranteed to produce a solution without any assumptions on \( \Phi \) other than square-integrability. It is appropriate to call \( R_E^* \) the construction operator associated with the resolution of unity (4.16). It can be used to construct solutions in \( \mathcal{H} \) from unconstrained functions \( \Phi \in L^2(E) \). In fact, it is interesting to compare the wavelet construction formula

\[ F(x') = \int_E d^3x \, ds \, \Psi_{x,is}(x') \Phi(x,is) \quad \text{(6.10)} \]

directly with its Fourier counterpart (2.21):

\[ F(x') = \int_C dp \, e^{ip \cdot x'} \Pi(p) f(p). \quad \text{(6.11)} \]

In both cases, the coefficient functions \( \Phi \) and \( f \) are unconstrained (except for the respective square-integrability requirements). The building blocks in (6.10) are the matrix-valued wavelets parameterized by \( E \), whereas those in (6.11) are the matrix-valued plane-wave solutions \( e^{ip \cdot x'} \Pi(p) \) parameterized by \( C \).

7. Interpretation of the Wavelet Parameters

Our goal in this section is twofold: (a) Reduce the wavelets \( \Psi_z \) to a sufficiently simple form that they can actually be visualized, and (b) use the ensuing picture to give a complete physical and geometric interpretation of the eight complex space-time parameters \( z \in T \) labeling \( \Psi_z \). That the wavelets can be visualized at all is quite remarkable,
since $\Psi_z(x')$ is a complex matrix-valued function of $x' \in \mathbb{R}^4$ and $z \in \mathcal{T}$. However, the symmetries of Maxwell’s equations can be used to reduce the number of effective variables one by one, until all that remains is a single complex-valued function of two real variables, whose real and imaginary parts can be graphed separately.

We begin by showing that the parameters $z \in \mathcal{T}$ can be eliminated entirely. Recall that $\Psi_z(x')$ is the boundary value of the reproducing kernel, according to (5.11) and (3.7):

$$
\Psi_z(x') = \frac{1}{2} \lim_{\varepsilon \to 0} \left[ K(x' + i\varepsilon y' \mid \bar{z}) + K(x' - i\varepsilon y' \mid \bar{z}) \right]
$$

$$
= \frac{1}{2} \lim_{\varepsilon \to 0} \left[ \theta(y' \cdot y) L(x' + i\varepsilon y' - \bar{z}) + \theta(-y' \cdot y) L(x' - i\varepsilon y' - \bar{z}) \right]
$$

$$
= \frac{1}{2} \left[ \theta(y' \cdot y) + \theta(-y' \cdot y) \right] L(x' - \bar{z}) = \frac{1}{2} L(x' - \bar{z}).
$$

Hence

$$
\Psi_{x+iy}(x') = \frac{1}{2} L(x' - x + iy) = \Psi_{iy}(x' - x),
$$

and $\Psi_{x+iy}$ is a translated version of $\Psi_{iy}$. It therefore suffices to examine only $\Psi_{iy}$ with $y \in V'$. Eq. (4.7), combined with $\Pi(-p) = \Pi(p)$, shows that $\Psi_z(x')^* = \Psi_{\bar{z}}(x')$, hence it suffices to look only at $y \in V'_+$. To reduce the number of parameters still further, we use the fact that Maxwell’s equations are invariant under Lorentz transformations, and this invariance implies certain transformation properties for the wavelets. The covariance of the wavelets under the Lorentz group and, more generally, under the conformal group, will be studied in detail elsewhere. Here we remark only that Lorentz transformations relate all the wavelets with equal values of $y^2$, hence it suffices to study only $\Psi_{iy}$ with $y = (0, s)$ and $s > 0$. The physical significance of this will be discussed below. Finally, note that $\Gamma(ap) = \Gamma(p)$ for any $a > 0$, since $v(ap) \equiv ap/p_0 = v(p)$. Hence $\Pi(ap) = \Pi(p)$, and (4.7) implies that

$$
\Psi_{0,is}(x') = s^{-4} \Psi_{0,i}(x'/s).
$$

Thus all the wavelets $\Psi_z, z \in \mathcal{T}$, can be obtained by space-time translations, Lorentz transformations and scalings from the single “mother wavelet”

$$
\Psi(x) \equiv \Psi_{0,i}(x) = 2 \int_{C_+} d\tilde{\tau} \ p_0^2 e^{-p_0} e^{ip \cdot x} \Pi(p).
$$

(Of course, any one of the $\Psi_z$’s can equally be chosen as the mother!) In particular, the wavelets parameterized by $(x, is) \in E$ are
\[
\Psi_{x,i,s}(x', t') = s^{-4} \Psi \left( \frac{x' - x}{s}, \frac{t'}{s} \right).
\] (7.5)

Let \([\Psi(x, t)]_{mn}\) denote the matrix elements of \(\Psi(x, t)\). By (5.10), with \(w_0 = t+i\), \(w = x\) and \(r = |x|\), we have

\[
[\Psi(x, t)]_{mn} = \frac{1}{\pi^2} \frac{\delta_{mn} [(t+i)^2 + r^2] - 2x_m x_n + 2i(t+i) \sum_{k=1}^{3} \varepsilon_{mnk} x_k}{[(t+i)^2 - r^2]^3}.
\] (7.6)

This is still a complex matrix-valued function in \(\mathbb{R}^4\), hence impossible to visualize directly. We now eliminate the polarization degrees of freedom. Returning to the Fourier representation of solutions, note that if \(f(p)\) already satisfies the constraint (2.12), then \(\Pi(p) f(p) = f(p)\) and (4.10) reduces to

\[
\int_E d^3x \, ds \, |\tilde{F}(x, is)|^2 = \int_C d\tilde{\rho} \frac{|f(p)|^2}{p_0^2} = \|F\|^2.
\] (7.7)

Define the scalar wavelets by

\[
\Psi_z(x') \equiv \int_C d\tilde{\rho} \, 2p_0^2 \theta(p \cdot y) e^{-p \cdot (x' - \tilde{z})}
\] (7.8)

and the corresponding scalar kernel \(K : \mathcal{T} \times \mathcal{T} \rightarrow \mathbb{C}\) by

\[
K(z' | \tilde{z}) = \int_C d\tilde{\rho} \, 4p_0^2 \theta(p \cdot y') \theta(p \cdot y) e^{ip \cdot (z' - \tilde{z})}.
\] (7.9)

Then (7.7), with essentially the same argument as in the proof of Theorem 1, now gives the relations

\[
\tilde{F}(z') = \int_E d^3x \, ds \, K(z' | x, -is) \tilde{F}(x, is) \quad \text{pointwise in } z' \in \mathcal{T},
\]

\[
F(x') = \int_E d^3x \, ds \, \Psi_{x, is}(x') \tilde{F}(x, is) \quad \text{a.e.}
\] (7.10)

\[
F = \int_E d^3x \, ds \, \Psi_{x, is} \tilde{F}(x, is) \quad \text{weakly in } \mathcal{H}.
\]

The first equation states that \(K(z' | \tilde{z})\) is still a reproducing kernel on the range \(\mathcal{W}\) of \(R_E : \mathcal{H} \rightarrow \mathcal{L}^2(E)\). The second and third equations state that an arbitrary solution \(F \in \mathcal{H}\) can be represented as a superposition of the scalar wavelets, with \(\tilde{F} = R_E F\) as a (vector) coefficient function. Thus, when dealing with coefficient functions in the range \(\mathcal{W}\) of \(R_E\), it is unnecessary to use the matrix-valued wavelets. The main advantage of
the latter (and a very important one) is that they can be used even when the coefficient function \( \Phi(x, is) \) does not belong to the range \( W \) of \( R_E \), since they project \( \Phi \) to \( W \).

The scalar wavelets and kernel were introduced and studied in Kaiser [12, 13]. They cannot, of course, be solutions of Maxwell’s equations precisely because they are scalars. But they do satisfy the wave equation, since every component of \( \Psi_z \) does so. To see their relation to the corresponding matrix quantities, note that \( \Pi(p) \) is a projection operator of rank 1, hence \( \text{Trace} \, \Pi(p) = 1 \). Taking the trace on both sides of Eqs. (4.7) and (4.20) therefore gives

\[
\Psi_z(x') = \text{Trace} \, \Psi_z(x'), \quad K(z' | \bar{z}) = \text{Trace} \, K(z' | \bar{z}).
\]  

(7.11)

Taking the trace amounts, roughly, to averaging over polarizations. The trace of the mother wavelet \( \Psi \) is

\[
\text{Trace} \, \Psi(x) = \frac{1}{\pi^2} \frac{3(t+i)^2 + r^2}{[(t+i)^2 - r^2]^3} \equiv \Psi(r,t), \quad r \equiv |x|.
\]  

(7.12)

Because it is spherically symmetric, \( \Psi(r,t) \) can be easily plotted. Its real and imaginary parts are shown in Figs. 1 and 2. These figures confirm that \( \Psi \) is a spherical wave converging towards the origin as \(-\infty < t < 0\), becoming localized in a sphere of radius \( \sqrt{3} \) around the origin at \( t = 0 \), and then diverging away from the origin as \( 0 < t < \infty \). Figure 3 shows \( \Psi(r,0) \), which is real. Even though \( \Psi(r,0) \) does not have compact support (it decays as \( r^{-4} \)), it is seen to be very well localized in \( |x| \leq \sqrt{3} \).

Now that we have a reasonable interpretation of \( \Psi_{x+iy} \) with \( y = (0,s) \), let us return to interpret the wavelets with \( y \neq 0 \). Suppose \( y = (y,s) \in V'_+ \), and let \( v = y/s \). Then \( y \in V'_+ \) implies \( |v| < 1 \). (We have chosen units of length and time in which the speed of light \( c = 1 \); for general units, \( |v| < c \).) Hence we can perform a Lorentz transformation to a reference frame moving with velocity \(-v\) relative to the original frame. In the new frame, \( y \) has coordinates \( y' = (0, \sqrt{s^2 - |y|^2}) \), hence our wavelet has a stationary center. Returning to the original frame, we conclude that \( \Psi_{x+iy} \) is a wave whose center is moving with the uniform velocity \( v = y/s \). It is a Doppler-shifted version of the stationary wavelet with \( y' = 0 \). Thus each of the eight real parameters \( z = (x,t) + i(y,s) \in T \) has a physical and geometric significance: \( x \) and \( t \) give the location and time at which \( \Psi_z \) is localized; \( v = y/s \) gives the velocity of its center; \( |s| \) gives its scale (width at time \( t \)), and the sign of \( s \) gives its helicity. Since all these parameters, as well as the wavelets which they label, were a direct consequence of the extension of the electromagnetic field to complex space-time, it would appear that \( T \) is a natural arena in which to study electrodynamics.
8. Moving and Accelerating Wavelet Representations

The construction of the electromagnetic wavelets has been completely unique, in the following sense: (a) The inner product (2.31) on solutions is uniquely determined, up to a constant factor, by the requirement that it be Lorentz-invariant. (b) The analytic extensions of the positive- and negative-frequency parts of \( \mathbf{F} \) to \( \mathcal{T}_+ \) and \( \mathcal{T}_- \), respectively, are certainly unique, hence so is \( \tilde{\mathbf{F}}(z) \). (c) The evaluation maps \( \mathcal{E}_z \mathbf{F} = \tilde{\mathbf{F}}(z) \) are unique, hence so are their adjoint \( \Psi_z \equiv \mathcal{E}_z^* \). On the other hand, the choice of the Euclidean space-time region \( E \) as a parameter space for expanding solutions is rather arbitrary. \( E \) may be regarded as the group of space translations and scalings, acting on real space-time by 
\[
g_{sx} x' = sx' + (x, 0).
\]
As such, it is a subgroup of the conformal group \( \mathcal{C} \), which consists of space-time translations, scalings, space rotations, Lorentz transformations and special conformal transformations. \( E \) is invariant under space rotations but not under time translations, Lorentz transformations or special conformal transformations. This non-invariance can be exploited by applying any of the latter transformations to the resolution of unity (4.16) and using the transformation properties of the wavelets. The general idea is that when \( g \in \mathcal{C} \) is applied to (4.16), then another such resolution of unity is obtained in which \( E \) is replaced by its image \( gE \) under \( g \). If \( gE = E \), nothing new results. If \( g \) is a time translation, then the wavelets parameterized by \( gE \) are all localized at some time \( t \neq 0 \) rather than \( t = 0 \). If \( g \) is a Lorentz transformation, then all wavelets with \( z \in gE \) have centers which move with a uniform non-zero velocity rather than being stationary. Finally, if \( g \) is a special conformal transformation, then \( gE \) is a curved submanifold of \( \mathcal{T} \) and the wavelets parameterized by \( gE \) have centers with varying velocities. This is consistent with results obtained by Page [20] and Hill [9], who showed that special conformal transformations can be interpreted as mapping to an \textit{accelerating} reference frame.

As a possible application, consider an electromagnetic pulse reflected or emitted by a moving object. After the reflection time, and far away from boundaries, the pulse may be approximated by a solution of Maxwell’s equations in free space, hence it can be analyzed as in Section 4 using wavelets with stationary centers. However, the analysis is likely to be more efficient (i.e., have fewer significant coefficients) if it is made in the reference frame in which the reflecting object is at rest, with the reflection time as the initial time of localization. From the viewpoint of the receiver, this means that a representation with “co-moving” wavelets should be used instead of one with stationary centers. The details will be presented elsewhere.

I thank R. F. Streater for his hospitality at King’s College, where we had some helpful discussions concerning the helicity of the electromagnetic wavelets.
References

[1] V. Bargmann and E. P. Wigner, Group-theoretical discussion of relativistic wave equations, Proc. Natl. Acad. Sci. U. S. 34 (1948) 211-233.
[2] H. Bateman, The transformation of the electrodynamical equations, Proc. London Math. Soc. 8 (1910) 223-264.
[3] C. K. Chui, An Introduction to Wavelets, Academic Press, 1992.
[4] E. Cunningham, The principle of relativity in electrodynamics and an extension thereof, Proc. London Math. Soc. 8 (1910) 77-98.
[5] I. Daubechies, Ten Lectures on Wavelets, SIAM, 1992.
[6] A. Einstein, H. A. Lorentz, H. Weyl and H. Minkowski, The Principle of Relativity, Dover, 1923.
[7] D. Gabor, Theory of communication, Proc. IEE (London), Ser. 3, 93 (1946) 429-457.
[8] L. Gross, Norm invariance of mass-zero equations under the conformal group, J. Math. Phys. 5 (1964) 687-695.
[9] E. L. Hill, On accelerated coordinate systems in classical and relativistic mechanics, Phys. Rev. 67 (1945) 358-363; On the kinematics of uniformly accelerated motions and classical electromagnetic theory, *ibid.* 72 (1947) 143-149; The definition of moving coordinate systems in relativistic theories, *ibid.* 84 (1951) 1165-1168.
[10] J. D. Jackson, Classical Electrodynamics, Wiley, 1975.
[11] G. Kaiser, Quantum Physics, Relativity, and Complex spacetime: Towards a New Synthesis, North-Holland, Amsterdam, 1990.
[12] G. Kaiser, Wavelet electrodynamics, Physics Letters A 168 (1992) 28-34.
[13] G. Kaiser, in Progress in Wavelet Analysis and Applications, Y. Meyer and S. Roques, eds., Editions Frontières, 1993.
[14] G. Kaiser, Space-time-scale analysis of electromagnetic waves, in Proc. of IEEE-SP Internat. Symp. on Time-Frequency and Time-Scale Analysis, Victoria 1992.
[15] G. Kaiser and R. F. Streater, Windowed Radon transforms, analytic signals and the wave equation, in Wavelets—A Tutorial in Theory and Applications, C. K. Chui, ed., Academic Press, New York, 1992.
[16] G. Kaiser, Phase-space approach to relativistic quantum mechanics, Part I: Coherent-state representation of the Poincaré group, J. Math. Phys. 18 (1977) 952-959; part II: Geometrical Aspects, *ibid.* 19 (1978) 502-507.
[17] G. Kaiser, Quantized fields in complex space-time, Ann. Phys. 173 (1987) 338-354.
[18] G. Kaiser, A Friendly Guide to Wavelets, Birkhäuser, Boston, 1994.
[19] H. E. Moses, Eigenfunctions of the curl operator, rotationally invariant Helmholtz theorem, and applications to electromagnetic theory and fluid mechanics, SIAM J. Appl. Math. 21 (1971) 114-144.
[20] L. Page, A new relativity, Phys. Rev. 49 (1936) 254-268.

[21] W. Rühl, Distributions on Minkowski space and their connection with analytic representations of the conformal group, Commun. math. Phys. 27 (1972) 53-86.

[22] E. Stein, *Singular Integrals and Differentiability Properties of Functions*, Princeton University Press, 1970.

[23] E. Stein and G. Weiss, *Fourier Analysis on Euclidean Spaces*, Princeton University Press, 1971.

[24] T. Takiguchi, “The windowed Radon transform for distributions,” Univ. of Tokyo preprint, 1994.