MORITA THEORY FOR CORING EXTENSIONS AND CLEFT BICOMODULES

GABRIELLA BÖHM AND JOOST VERCRUYSSE

ABSTRACT. A Morita context is constructed for any comodule of a coring and, more generally, for an $L$-$C$ bicomodule $\Sigma$ for a coring extension $(D : L)$ of $(C : A)$. It is related to a 2-object subcategory of the category of $k$-linear functors $\mathcal{M}^C \rightarrow \mathcal{M}^D$. Strictness of the Morita context is shown to imply the Galois property of $\Sigma$ as a $C$-comodule and a Weak Structure Theorem. Sufficient conditions are found also for a Strong Structure Theorem to hold.

Cleft property of an $L$-$C$ bicomodule $\Sigma$ – implying strictness of the associated Morita context – is introduced. It is shown to be equivalent to being a Galois $C$-comodule and isomorphic to $\text{End}^C(\Sigma) \otimes_L D$, in the category of left modules for the ring $\text{End}^C(\Sigma)$ and right comodules for the coring $D$, i.e. satisfying the normal basis property.

Algebra extensions, that are cleft extensions by a Hopf algebra, a coalgebra or a Hopf algebroid, as well as cleft entwining structures (over commutative or non-commutative base rings) and cleft weak entwining structures, are shown to provide examples of cleft bicomodules.

1. Introduction

The application of Morita theory in the study of algebra extensions by a Hopf algebra (i.e. of comodule algebras for a Hopf algebra) originates in the work [21] of Cohen, Fischman and Montgomery. In that paper a Morita context has been associated to a comodule algebra of a Hopf algebra $H$, under the assumption that $H$ is a finite dimensional algebra over a field (or a Frobenius algebra over a commutative ring). The construction has been extended by Doi in [22] to arbitrary Hopf algebras $H$. Using the observation [12], that a comodule algebra of a Hopf algebra can be considered as a special instance of a base algebra of a coring with a grouplike element, in [19] a Morita context has been constructed for any coring with a grouplike element. Recall that the existence of a grouplike element in an $A$-coring $C$ is equivalent to the existence of a (left or right) $C$ comodule structure in $A$. In the paper [18] the particular $C$-comodule $A$ in [19] was replaced by any $C$-comodule $\Sigma$, which is finitely generated and projective as an $A$-module. In all of the listed (more and more general) situations, strictness of the Morita context was related to properties of the extension (or the coring) behind.

The first aim of the present paper is to generalise the construction of a Morita context further. As a first step, in Section 2 we remove the finitely generated projectivity condition in [18] and construct a Morita context for an arbitrary comodule $\Sigma$ of an $A$-coring $C$. It connects the algebra of $C$-comodule endomorphisms of $\Sigma$ and the $A$-dual algebra of $C$.

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As a next step, in Section 3 we study coring extensions \((\mathcal{D} : L)\) of \((\mathcal{C} : A)\), in the sense of \([13]\). By definition, a coring \(\mathcal{D}\) over a \(k\)-algebra \(L\) is a right extension of another coring \(\mathcal{C}\) over a \(k\)-algebra \(A\) if \(\mathcal{C}\) is a \(\mathcal{C}\)-\(\mathcal{D}\) bicomodule with the left regular \(\mathcal{C}\)-comodule structure (i.e. via the coproduct). By \([13]\) Theorem 2.6, this definition is equivalent to the existence of a \(k\)-linear functor \(U : \mathcal{M}^C \to \mathcal{M}^D\), such that the forgetful functor \(\mathcal{M}^C \to \mathcal{M}_k^L\) factors through \(U\) and the forgetful functor \(\mathcal{M}^D \to \mathcal{M}_k^L\). In this situation, to any \(L\)-\(\mathcal{C}\) bicomodule \(\Sigma\) (i.e. left \(L\)-module and right \(\mathcal{C}\)-comodule, with left \(L\)-linear \(\mathcal{C}\)-coaction) we associate a Morita context. It reduces to the Morita context in Section 2 if the coring \(\mathcal{D}\) is trivial (i.e. equal to the ground ring \(k\)).

Note that – similarly to the identification of \(k\)-algebras with \(k\)-linear categories of a single object – one can identify Morita contexts with \(k\)-linear categories of two objects (cf. \([9, Remark 3.2 (1)]\)). Namely, to a \(k\)-linear category with two objects \(a\) and \(b\), associate a Morita context of the two endomorphism algebras, \(\text{End}(a)\) and \(\text{End}(b)\). The \(k\)-modules \(\text{Hom}(a, b)\) and \(\text{Hom}(b, a)\), of morphisms between the two objects, are bimodules via the composition of morphisms. Since the restrictions of the composition of morphisms in the category to \(\text{Hom}(a, b) \otimes_k \text{Hom}(b, a)\) and to \(\text{Hom}(b, a) \otimes_k \text{Hom}(a, b)\) are balanced in the appropriate sense, both connecting homomorphisms can be constructed as their projections.

Via this identification, the Morita context in Section 3 corresponds to the full subcategory of the category of \(k\)-linear functors with two objects: the functor \(U : \mathcal{M}^C \to \mathcal{M}^D\), constructed by Brzeziński in \([13]\) for a coring extension \((\mathcal{D} : L)\) of \((\mathcal{C} : A)\) on one hand, and the functor \(\text{Hom}^C(\Sigma, -) \otimes L \mathcal{D} : \mathcal{M}^C \to \mathcal{M}^D\), given in terms of the \(L\)-\(\mathcal{C}\) bicomodule \(\Sigma\), on the other hand.

Section 3 is devoted to a study of cleft \(L\)-\(\mathcal{C}\) bicomodules. As in \([1]\) and \([19]\), our main tool is provided by Morita theory. However, as mentioned in the introduction of \([19]\), using Morita contexts associated to comodules for one coring only (say, as in Section 2), it does not seem to be possible to go beyond a study of cleft entwining structures in \([1]\) and \([19]\).

On the other hand, our language of Morita contexts for coring extensions, developed in Section 3, gives a very natural framework to introduce the cleft property of an \(L\)-\(\mathcal{C}\) bicomodule \(\Sigma\) for a coring extension \((\mathcal{D} : L)\) of \((\mathcal{C} : A)\). As a guiding example, consider a Hopf algebra \(H\) over a commutative ring \(k\) and its right \(\mathcal{C}\)-comodule algebra \(A\). Denote the coinvariants of \(A\) by \(B\). The \(k\)-module \(A \otimes_k H\) inherits a left \(A\)-module structure of \(A\) and it can be equipped with the structure of a right \(A\)-module in terms of the right \(H\)-coaction in \(A\). The \(A\)-\(A\) bimodule \(A \otimes_k H\) is an \(A\)-coring with coproduct and counit inherited from \(H\). The \(k\)-coring (coalgebra) \(H\) is a right extension of the \(A\)-coring \(A \otimes_k H\) and \(A\) is a \(k\)-\((A \otimes_k H)\) bicomodule. The Morita context, corresponding to it, consists of two subalgebras of the convolution algebra \(\text{Hom}_k(H, A)\). The two bimodules are given by the sets of right \(H\)-comodule maps \(H \to A\), w.r.t. the regular \(H\)-coaction in \(H\), \(h \mapsto h_{(1)} \otimes_k h_{(2)}\), and the twisted coaction, \(h \mapsto h_{(2)} \otimes_k S(h_{(1)})\), respectively, where \(S\) denotes the antipode of \(H\). Both connecting homomorphisms are given by projections of the restrictions of the convolution product. In light of this observation, the \(H\)-cleft property of the extension \(B \subseteq A\) is equivalent to the existence of elements \(j\) and \(\tilde{j}\) in the two bimodules of the Morita context, such that the connecting homomorphisms map \(j \otimes j\) and \(\tilde{j} \otimes j\), respectively, to the unit element of the convolution algebra. Inspired by this example, we term an \(L\)-\(\mathcal{C}\) bicomodule \(\Sigma\), for a coring extension \((\mathcal{D} : L)\) of \((\mathcal{C} : A)\), to be cleft if there exist elements \(j\) and \(\tilde{j}\) in the two bimodules of the Morita context, associated to \(\Sigma\) in Section 3 such that the connecting homomorphisms map \(j \otimes \tilde{j}\) and \(\tilde{j} \otimes j\), respectively, to the unit
element of the appropriate algebra in the Morita context. In Section 6 we show that all known examples of 'cleft extensions' – i.e. cleft algebra extensions by Hopf algebras or Hopf algebroids, cleft extensions by partial group actions, as well as cleft (weak) entwining structures (over arbitrary base) – are covered by this definition. It has to be noted that, in contrast to cleft extensions by Hopf algebras (or just coalgebras), which determine cleft entwining structures, cleft extensions by Hopf algebroids in [9] correspond to coring extensions which do not arise from any entwining structure.

As it is well known, an extension \( B \subseteq A \) of \( k \)-algebras is an \( H \)-cleft extension for a Hopf algebra \( H \) if and only if it is \( H \)-Galois with normal basis property. The Galois condition means that \( A \) is a right \( H \)-comodule algebra with coinvariants \( B \), and \( A \otimes_B A \) is isomorphic (in the category of left \( A \)-modules, right \( H \)-comodules) to \( A \otimes_k H \) via the so called canonical isomorphism. The normal basis property means that \( A \) is isomorphic to \( B \otimes_k H \) in the category of left \( B \)-modules, right \( H \)-comodules.

The notion of Galois extensions went through a wide generalisation in the past years. It was initiated by an observation by Brzeziński [12] that if \( B \subseteq A \) is an \( H \)-Galois extension then \( C := A \otimes_k H \) is a Galois \( A \)-coring. This means that \( A \) is a \( C \)-comodule with \( C \)-coinvariants \( B \), and \( C \) is isomorphic to the canonical Sweedler’s coring \( A \otimes_B A \) via the canonical isomorphism.

As a next step of generalisation [25], El Kaoutit and Gómez-Torrecillas introduced Galois comodules \( \Sigma \) for an \( A \)-coring \( C \) as right \( C \)-comodules that are finitely generated and projective as \( A \)-modules and \( C \) is isomorphic to a comatrix coring \( \text{Hom}_A(\Sigma, A) \otimes_T \Sigma \) via the canonical isomorphism, where the notation \( T := \text{End}^C(\Sigma) \) is used. The existence of such a Galois comodule \( \Sigma \) ensures that the comonad functor \( - \otimes_A C \) on the category \( \mathcal{M}_A \) of right \( A \)-modules comes from an adjunction of functors

\[
- \otimes_T \Sigma : \mathcal{M}_T \to \mathcal{M}_A \quad \text{and} \quad \text{Hom}_A(\Sigma, -) : \mathcal{M}_A \to \mathcal{M}_T.
\]

Hence, by standard Eilenberg-Moore type arguments (see e.g. [26, VI.3, Theorem 1]), the diagram

\[
\begin{array}{ccc}
\mathcal{M}_A & \xrightarrow{F^C} & \mathcal{M}_T \\
\downarrow & \swarrow \text{Hom}_A(\Sigma, -) & \downarrow \\
\mathcal{M}^C & = & \mathcal{M}_T \\
\downarrow \otimes_T \Sigma & \uparrow \otimes_T \Sigma & \downarrow \\
\mathcal{M}_A & \xrightarrow{- \otimes_A C} & \mathcal{M}_T
\end{array}
\]

is a commutative diagram in the 2-category of categories, in the sense that the inner triangle strictly commutes (here \( \mathcal{M}_T^C \) denotes the category of right \( C \)-comodules and \( F^C \) denotes the forgetful functor) and the outer one does upto the natural isomorphism

\[
\text{can}_N : \text{Hom}_A(\Sigma, N) \otimes_T \Sigma \to N \otimes_A C, \quad \phi_N \otimes x \mapsto \phi_N(x[0]) \otimes_A x[1],
\]

for any right \( A \)-module \( N \), where \( x \mapsto x[0] \otimes_A x[1] \) denotes the \( C \)-coaction in \( \Sigma \).

Finally, in [30] Wisbauer relaxed the finitely generated projectivity condition and defined Galois comodules \( \Sigma \) for an \( A \)-coring \( C \) with the requirement that the diagram (1.2) is commutative in the above sense, i.e. such that (1.3) is a natural isomorphism. Throughout the paper Galois comodules are meant in this most general sense.
As one of our main results, it is shown in Corollary 5.3 that an \( L \)-\( C \) bicomodule \( \Sigma \), for a coring extension \((D, L)\) of \((C, A)\), is cleft if and only if it is a Galois comodule for \( C \) (in the sense of \([30]\)) and isomorphic to \( T \otimes L \) as a left module for the algebra \( T := \text{End}^C(\Sigma) \) and as a comodule for the coring \( D \) (i.e. the normal basis property holds).

Let \( \Sigma \) be a Galois comodule for an \( A \)-coring \( C \) in the sense of \([30]\), and \( T := \text{End}^C(\Sigma) \). Then the Eilenberg-Moore comonad for the adjunction \((1.1)\) is naturally equivalent to the comonad \( - \otimes A \) \( C \), and the comparison functor is naturally equivalent to \( - \otimes T \) : \( \mathcal{M}_T \rightarrow \mathcal{M}_C \) (cf. diagram \((1.2)\)). Hence \( - \otimes T \) : \( \mathcal{M}_T \rightarrow \mathcal{M}_C \) is comonadic (or tripleable) if and only if the functor in the bottom row of diagram \((1.2)\) possesses a right adjoint (cf. \([15, 18.21]\)), the functor \( (1.4) \): \( \text{Hom}^C(\Sigma, -) : \mathcal{M}_C \rightarrow \mathcal{M}_T \).

In Theorem 3.6 (1) and Theorem 4.1 we show that if \( \Sigma \) is an \( L \)-\( C \) bicomodule for a coring extension \((D : L)\) of \((C : A)\), then the surjectivity of one of the connecting homomorphisms in the associated Morita context implies both the Galois property of \( \Sigma \) as a right \( C \)-comodule and the fully faithfulness of the functor \( (1.4) \) (i.e. the Weak Structure Theorem). The Strong Structure Theorem does not follow even by the strictness of the Morita context without further assumptions. In Theorem 4.6 we find sufficient conditions for it to hold. Since Morita contexts associated to cleft bicomodules are strict, these theorems – extending \([1, Proposition 4.8], [19, Theorem 4.5]\) and \([1, Theorems 4.9 and 4.10]\) – hold for them.

**Notational conventions.** All algebras of the paper are associative unital algebras over a fixed commutative ring \( k \). The multiplication in an algebra \( R \) will be denoted by \( \mu_R \) and the unit element by \( 1_R \). The algebra with the same \( k \)-module structure and opposite multiplication is denoted by \( R^{op} \). For the category of right (resp. left) \( R \)-modules we use the symbol \( \mathcal{M}_R \) (resp. \( R \mathcal{M} \)) and for its hom sets we write \( \text{Hom}_R(-, -) \) (resp. \( R \text{Hom}(-, -) \)). The forgetful functor \( \mathcal{M}_R \rightarrow \mathcal{M}_k \) will be denoted by \( G^R \).

\( R \)-rings over an algebra \( R \) are monoids in the monoidal category of \( R \)-\( R \) bimodules. An \( R \)-ring is determined by a pair, consisting of an algebra \( A \) and an algebra homomorphism (the unit of the monoid) \( \eta_A : R \rightarrow A \).

Corings (co-rings) over an algebra \( R \) are meant to be comonoids in the monoidal category of \( R \)-\( R \) bimodules. That is, an \( R \)-coring \( C \) consists of an \( R \)-\( R \) bimodule (denoted by the same symbol \( C \)), an \( R \)-\( R \) bilinear coassociative coproduct \( \Delta_C \) and an \( R \)-\( R \) bilinear counit \( \epsilon_C \) (see \([15, 17.1]\)). For the coproduct of an element \( c \in C \) we use Sweedler-Heyneman index notation, i.e. write \( \Delta_C(c) = c_{(1)} \otimes_R c_{(2)} \), without denoting summation explicitly. The category of right (resp. left) \( C \)-comodules (cf. \([15, 18.1-2]\)) will be denoted by \( \mathcal{C} \mathcal{M} \) (resp. \( \mathcal{C} \mathcal{M} \)) and its hom sets by \( \text{Hom}^C(-, -) \) (resp. \( \mathcal{C} \text{Hom}(-, -) \)). The forgetful functor \( \mathcal{C} \mathcal{M} \rightarrow \mathcal{M}_R \) will be denoted by \( F^C \). Composition of morphisms in the \( k \)-linear category \( \mathcal{C} \mathcal{M} \) equips the set \( \text{End}^C(\Sigma) \) of endomorphisms of an object \( \Sigma \) with an algebra structure. Symmetrically, also the composition of morphisms in \( \mathcal{C} \mathcal{M} \) makes the set \( \text{End}(\Lambda) \) of endomorphisms of an \( \Lambda \in \mathcal{C} \mathcal{M} \) an algebra.
2. Morita contexts associated to comodules

Generalising constructions in \[20\], \[22\], \[1\] and \[19\], Caenepeel, De Groot and Ver-crudysse in \[18\] Section 4] associated Morita contexts to comodules of an \(A\)-coring \(\mathcal{C}\). For any right \(\mathcal{C}\)-comodule \(\Sigma\), they constructed a Morita context, connecting the algebras \(\text{End}_\mathcal{C}(\Sigma)\) and the right dual \(\mathcal{C}^*\) of \(\mathcal{C}\) (cf. \(\mathcal{M}'(\Sigma)\) below). Dually, for a left \(\mathcal{C}\)-comodule \(\Lambda\) they constructed a Morita context, connecting \(\mathcal{C}\text{-End}(\Lambda)\text{op}\) and the left dual, \(\mathcal{C}^*\) (cf. \(\mathcal{M}'(\Lambda)\) below). In the case of a right \(\mathcal{C}\)-comodule \(\Sigma\), which is finitely generated and projective as right \(A\)-module, these constructions yield a Morita context connecting \(\text{End}_\mathcal{C}(\Sigma)\) and \(\mathcal{C}^*\) (cf. \(\mathcal{M}'(\Sigma^*)\) below). In the present section we give a generalisation of this last Morita context to arbitrary right \(\mathcal{C}\)-comodules \(\Sigma\). For generalities about Morita contexts, we refer to \[11\] Chapter II.3], where they are termed sets of (pre-)equivalence data.

Recall (e.g. from \[15\] 17.8 and 19.1], where however the convention of opposite multiplication is used,) that for an \(A\)-coring \(\mathcal{C}\) the \(k\)-module \(\mathcal{C}^*\) is an \(A\)-ring with multiplication

\[(ff')(c) = f'(c^{(1)} f(c^{(2)})], \quad \text{for } f, f' \in \mathcal{C}, \ c \in \mathcal{C},\]

and unit map \(A \rightarrow \mathcal{C}^*, \ a \mapsto (c \mapsto \epsilon(c)ca)\). Any right \(\mathcal{C}\)-comodule \(\Sigma\) possesses a canonical right \(\mathcal{C}^*\)-module structure with action

\[xf = x^{[0]} f(x^{[1]}), \quad \text{for } f \in \mathcal{C}, \ x \in \Sigma.\]

\(\mathcal{C}\)-comodules are \(\mathcal{C}^*\)-linear. In a symmetric way, also the right dual \(\mathcal{C}^* = \text{Hom}_A(\mathcal{C}, A)\) has an \(A\)-ring structure and there is a faithful functor \(\mathcal{C}\text{-M} \rightarrow \mathcal{C}^*\text{-M}\).

Recall from \[18\] Section 4] that, for a right comodule \(\Sigma\) of an \(A\)-coring \(\mathcal{C}\) and \(T: = \text{End}_\mathcal{C}(\Sigma), \text{Hom}_\mathcal{C}(\mathcal{C}, \Sigma)\) is a \(T\)-\(\mathcal{C}^*\) bimodule with

\[(t \omega)(c) = t \circ \omega(g(c^{(1)}) c^{(2)})], \quad \text{for } t \in T, \ \omega \in \text{Hom}_\mathcal{C}(\mathcal{C}, \Sigma), \ g \in \mathcal{C}^*,\]

and \(\Sigma^* := \text{Hom}_A(\mathcal{A}, \Sigma)\) is a \(\mathcal{C}^*\text{-T}\) bimodule with

\[(g \xi t)(x) = g(\xi \circ t(x^{[0]}), x^{[1]}), \quad \text{for } t \in T, \ \xi \in \Sigma^*, \ g \in \mathcal{C}^*.\]

They constitute a Morita context

\[(2.1) \quad \mathcal{M}'(\Sigma) = \{ (T, \mathcal{C}^*, \text{Hom}_\mathcal{C}(\mathcal{C}, \Sigma), \Sigma^*, \mathcal{V}', \mathcal{V}') \},\]

with connecting homomorphisms

\[\mathcal{V}' : \Sigma^* \otimes \text{Hom}_\mathcal{C}(\mathcal{C}, \Sigma) \rightarrow \mathcal{C}^*, \quad \xi \otimes \omega \mapsto \xi \circ \omega,\]

\[\mathcal{V}' : \text{Hom}_\mathcal{C}(\mathcal{C}, \Sigma) \otimes \Sigma^* \rightarrow T, \quad \omega \otimes \xi \mapsto (x \mapsto \omega(\xi(x^{[0]}), x^{[1]})).\]

For the purposes of the present paper, however, another Morita context associated to \(\Sigma \in \mathcal{M}^\mathcal{C}\) turns out to be more useful. Define the \(k\)-module

\[(2.2) \quad Q := \{ q \in \text{Hom}_A(\Sigma, \mathcal{C}) \mid \forall x \in \Sigma, c \in \mathcal{C} \quad q(x^{[0]})(c)x^{[1]} = c^{(1)}q(x)(c^{(2)}) \} .\]

In the following lemma some properties of the \(k\)-module \(Q\) are collected, needed in order to see that \(\Sigma\) and \(Q\) are bimodules connecting the algebras \(T\) and \(\mathcal{C}^*\).

**Lemma 2.1.** For an \(A\)-coring \(\mathcal{C}\) and its right comodule \(\Sigma\), the \(k\)-module \(Q\) in \[(2.2)\] obeys the following properties.
(1) \( Q \) is isomorphic to the \( k \)-module

\[
(2.3) \quad Q' = \{ q \in A\text{Hom}(C, \Sigma^*) \mid \forall x \in \Sigma, c \in C \quad c^{(1)} q(c^{(2)})(x) = q(c)(x^{[0]}x^{[1]}) \},
\]

defined in terms of the left \( A \)-module \( \Sigma^* : = \text{Hom}_A(\Sigma, A) \);

(2) Let \( M \) be a right \( C \)-comodule. For any \( q \in Q \) and \( m \in M \), the map \( \Sigma \to M \), 
\[ x \mapsto mq(x) \] is right \( C \)-colinear, i.e. for every \( m \in M \) there is a \( k \)-linear map 
\[ Q \to \text{Hom}^C(\Sigma, M), \quad q \mapsto mq(-) ; \]

(3) \( Q \) is a \( k \)-submodule of \( \text{Hom}_C(\Sigma, ^*C) \);

(4) \( Q \) is a \( ^*C \)-\( T \) bimodule, for \( T : = \text{End}^C(\Sigma) \), with actions
\[
\begin{align*}
(fq)(x) & : = fq(x), \quad \text{for } f \in ^*C, \ q \in Q, \ x \in \Sigma \quad \text{and} \\
(gt)(x) & : = q(t(x)), \quad \text{for } q \in Q, \ t \in T, \ x \in \Sigma .
\end{align*}
\]

\textbf{Proof.} (1) The isomorphism is given by switching the arguments. That is, by the map
\[ Q \to Q', \quad q \mapsto \left( c \mapsto q(-)(c) \right) . \]

(2) Since \( ^*C \) is an \( A \)-ring and the elements of \( Q \) are right \( A \)-linear, the map \( \Sigma \to M \), 
\[ x \mapsto mq(x) \] is right \( A \)-linear, for \( q \in Q \) and a right \( C \)-comodule \( M \) and \( m \in M \). In order to see that it is also right \( C \)-colinear, use the right \( A \)-linearity of a \( C \)-coaction in the first equality and \( (2.2) \) in the second one, to conclude that, for any right \( C \)-comodule \( M, m \in M, q \in Q \) and \( x \in \Sigma \),
\[
\begin{align*}
(m^{[0]}q(x)(m^{[1]}))^{[0]} \otimes (m^{[0]}q(x)(m^{[1]}))^{[1]} & = m^{[0]} \otimes m^{[1]}q(x)(m^{[2]}) \\
& = m^{[0]} \otimes q(x^{[0]})(m^{[1]}x^{[1]}) = m^{[0]}q(x^{[0]})(m^{[1]}) \otimes x^{[1]} .
\end{align*}
\]

(3) Using the right \( A \)-linearity of \( q \in Q \) and the defining identity \( (2.2) \), one checks that, for \( x \in \Sigma, f \in ^*C \) and \( c \in C \),
\[
q(xf)(c) = q(x^{[0]}f(x^{[1]}))(c) = q(x^{[0]})(c)f(x^{[1]}) = f(q(x^{[0]})(c)x^{[1]}) = f(c^{(1)} q(x)(c^{(2)})) = (q(x)f)(c) ,
\]

where in the first equality the form of the \( ^*C \)-action in \( \Sigma \) has been used, and in the last one the multiplication law in \( ^*C \).

(4) For \( f \in ^*C \) and \( q \in Q \), \( fq \) is an element of \( \text{Hom}_A(\Sigma, ^*C) \) by the right \( A \)-linearity of \( q \) and the fact that \( ^*C \) is an \( A \)-ring. Since \( q \) is an element of \( Q \), so is \( fq \) as, for \( x \in \Sigma \) and \( c \in C \),
\[
(fq)(x^{[0]})(c)x^{[1]} = (fq(x^{[0]}))(c)x^{[1]} = q(x^{[0]})(c^{(1)} f(c^{(2)}))x^{[1]} = c^{(1)} q(x)(c^{(2)} f(c^{(3)})) = c^{(1)}(fq(x))(c^{(2)}) = c^{(1)}(fq)(x)(c^{(2)}) ,
\]

where the first and last equalities follow by the the form of the \( ^*C \)-action in \( Q \) and the second and penultimate equalities follow by the multiplication law in \( ^*C \). The third equality follows by \( (2.2) \).

Since \( q \in Q \) and \( t \in T \) are right \( A \)-linear, \( qt \) is an element of \( \text{Hom}_A(\Sigma, ^*C) \). Since \( t \in T \) is colinear and \( q \) is an element of \( Q \), it follows that \( qt \in Q \) as, for \( x \in \Sigma \) and \( c \in C \),
\[
(qt)(x^{[0]})(c)x^{[1]} = q\left(t(x^{[0]})\right)(c)t(x^{[1]}) = c^{(1)}(qt)(x)(c^{(2)}) ,
\]

where the form of the \( T \)-action in \( Q \) has been used.
It is straightforward to check that both actions are associative and unital and they commute. \[ \square \]

**Remark 2.2.** In the case when \( C \) is a locally projective left \( A \)-module, the \( k \)-module (2.2) has a particularly simple characterisation, as \( Q \equiv \text{Hom}_C(\Sigma, C) \). Indeed, by Lemma 2.1 (3), \( Q \subseteq \text{Hom}_C(\Sigma, C) \). The converse inclusion is proven as follows. Recall that local projectivity of the left \( A \)-module \( C \) means that for any finite subset \( S \subset C \), there exists a dual basis \( \{e_i\} \subset C \) and \( \{f_i\} \subset C^* \) such that \( c = \sum_i f_i(c)e_i \), for all \( c \in S \). For an element \( x \in \Sigma \), fix finite sets \( \{x_j\} \subset \Sigma \) and \( \{c_j\} \subset C \) such that

\[
\left( \sum_{j=1}^8 x_j \otimes c_j \right) = \sum_i \left( \sum_{j=1}^8 x_j \otimes f_i(c_j)e_i \right) = \sum_i \left( \sum_{j=1}^8 x_j \otimes f_i(c_j)e_i \right) = \sum_i \left( \sum_{j=1}^8 x_j \otimes f_i(c_j)e_i \right) = \sum_i \left( \sum_{j=1}^8 x_j \otimes f_i(c_j)e_i \right).
\]

where the fourth equality follows by the right \( C \)-linearity of \( q \). This shows that \( q \) belongs to the \( k \)-module \( Q \) in (2.2).

If the \( A \)-coring \( C \) is locally projective as a left \( A \)-module then the image of a right \( C \)-module \( \Sigma \) under an element \( q \in \text{Hom}_C(\Sigma, C) \) lies within the rational part \((C)_{\text{rat}}^* \) of \( C \), cf. [15, 20.1]. (Recall that the rational part of a right \( C \)-module is the biggest \( C \)-submodule that possesses a \( C \)-comodule structure.) This implies that \( Q = \text{Hom}_A(\Sigma, C) = \text{Hom}_C(\Sigma, (C)_{\text{rat}}^*) = \text{Hom}_C^*(\Sigma, C) \).

If \( C \) is a finitely generated and projective left \( A \)-module, with dual bases \( \{e_i\} \subset C \) and \( \{f_i\} \subset C^* \), then \( C \) possesses a right \( C \)-comodule structure with coaction \( f \mapsto \sum_i f_i \otimes A e_i \). In this case \( Q \) is identical to the \( k \)-module \( \text{Hom}_C^*(\Sigma, C) \equiv \text{Hom}_C(\Sigma, C) \).

In light of Lemma 2.1 there is another Morita context associated to \( \Sigma \),

\[
(2.4) \quad \mathbb{M}(\Sigma) = (T, C, \Sigma, Q, \nabla, \nu),
\]

where \( txf = t(x^{[0]}_f(x^{[1]})) \), for \( t \in T, x \in \Sigma \) and \( f \in C \), with connecting homomorphisms

\[
(2.5) \quad \nabla : Q \otimes T \rightarrow C, \quad q \otimes x \mapsto q(x),
\]

\[
(2.6) \quad \nu : C \otimes Q \rightarrow T, \quad x \otimes q \mapsto xq(-).
\]

In a symmetric way, to a left \( C \)-comodule \( \Lambda \) one can associate Morita contexts \( \mathbb{M}(\Lambda) \), connecting the algebras \( C \text{End}(\Lambda) \) and \( C \), and \( \mathbb{M}(\Lambda) \), connecting \( C \text{End}(\Lambda) \) and \( C \).

**Remark 2.3.** Note that a finitely generated and projective right \( A \)-module \( \Sigma \) is a right comodule for an \( A \)-coring \( C \) if and only if \( C^* \) is a left \( C \)-comodule. Indeed, with the help of a dual basis \( \{x_i\} \subset \Sigma \) and \( \{\xi_i\} \subset C^* \), a bijective correspondence is given between right \( C \)-coactions \( x \mapsto x^{[0]}_A x^{[1]} \) in \( \Sigma \), and left \( C \)-coactions \( \xi \mapsto \sum_i \xi(x_i^{[0]}_A)x_i^{[1]} \xi_i \) in
\(\Sigma^*\). In this case the Morita context \(\mathcal{M}(\Sigma)\) coincides with \(\mathcal{M}'(\Sigma^*)\) and \(\mathcal{M}'(\Sigma)\) coincides with \(\mathcal{M}(\Sigma^*)\).

Since any right \(\mathcal{C}\)-comodule \(\Sigma\) has also a right \(*\mathcal{C}\)-module structure, one can associate a further Morita context with it, as in [4, II.4]. Namely,

\begin{equation}
\mathcal{N}(\Sigma) = \left(\text{End}_{\mathcal{C}}(\Sigma), *\mathcal{C}, \Sigma, \text{Hom}_{\mathcal{C}}(\Sigma, *\mathcal{C}), \Delta, \delta\right)
\end{equation}

with connecting maps

\begin{align}
\Delta : \text{Hom}_{\mathcal{C}}(\Sigma, *\mathcal{C}) & \otimes_{\mathcal{C}} \text{End}_{\mathcal{C}}(\Sigma) \to *\mathcal{C}, \\
& q \otimes q \mapsto q(x) \\
\delta : \Sigma \otimes \text{Hom}_{\mathcal{C}}(\Sigma, *\mathcal{C}) & \to \text{End}_{\mathcal{C}}(\Sigma), \\
& x \otimes q \mapsto xq(-).
\end{align}

In the next generalisation of [18, Proposition 4.7] the relationship between the Morita contexts \(\mathcal{N}(\Sigma)\) in (2.7) and \(\mathcal{M}(\Sigma)\) in (2.4) is investigated.

**Proposition 2.4.** Let \(\mathcal{C}\) be an \(A\)-coring and \(\Sigma\) a right \(\mathcal{C}\)-comodule. There exists a morphism of Morita contexts \(\mathcal{M}(\Sigma) \to \mathcal{N}(\Sigma)\), which becomes an isomorphism if \(\mathcal{C}\) is locally projective as left \(A\)-module.

**Proof.** There exist inclusions \(T = \text{End}_{\mathcal{C}}(\Sigma) \subset \text{End}_{\mathcal{C}}(\Sigma)\) and, by Lemma 2.1 (3), \(Q \subset \text{Hom}_{\mathcal{C}}(\Sigma, *\mathcal{C})\). Comparing (2.5) and (2.6) with (2.8) and (2.9), it is straightforward to see that these inclusions, together with the identity maps of \(*\mathcal{C}\) and \(\Sigma\), establish a morphism of Morita contexts.

Now assume that \(\mathcal{C}\) is locally projective as left \(A\)-module. By [15, 19.2-3], for any right \(\mathcal{C}\)-comodules \(M\) and \(M'\), \(\text{Hom}_{\mathcal{C}}(M, M') = \text{Hom}_{\mathcal{C}}(M, M')\). Hence in particular \(T = \text{End}_{\mathcal{C}}(\Sigma)\). By Remark 2.2, \(Q = \text{Hom}_{\mathcal{C}}(\Sigma, *\mathcal{C})\), which completes the proof. \(\square\)

By standard Morita theory, if the connecting map \(\nabla\) in the Morita context \(\mathcal{M}(\Sigma)\) in (2.4) is surjective, then \(\Sigma\) is a finitely generated projective left \(T\)-module and a right \(*\mathcal{C}\)-generator. If \(\nabla\) is surjective then \(\Sigma\) is a finitely generated projective right \(*\mathcal{C}\)-module and a left \(T\)-generator.

**Lemma 2.5.** Let \(\mathcal{C}\) be an \(A\)-coring and \(\Sigma\) a right \(\mathcal{C}\)-comodule. Consider the Morita context \(\mathcal{M}(\Sigma) = (T, *\mathcal{C}, \Sigma, Q, v, v)\) in (2.4).

1. If the connecting map \(\nabla\) in (2.5) is surjective then \(\mathcal{C}\) is a finitely generated projective left \(A\)-module.

2. If the connecting map \(\nabla\) in (2.6) is surjective then \(\Sigma\) is a finitely generated projective right \(\mathcal{C}\)-module.

**Proof.** (1) If \(\nabla\) is surjective then there exist finite sets \(\{q_i\} \subset Q\) and \(\{x_i\} \subset \Sigma\) such that

\[\epsilon = \sum_i q_i \cdot x_i \equiv \sum_i q_i(x_i).\]

Introduce finite sets \(\{f_j\} \subset *\mathcal{C}\) and \(\{c_j\} \subset \mathcal{C}\) via the requirement that

\[\sum_j f_j \otimes c_j = \sum_i q_i(x_i^{[0]} \otimes x_i^{[1]}) = \sum_i c^{(1)} q_i(x_i)(c^{(2)}) = c^{(1)} \epsilon(c^{(2)}) = c,\]
where the second equality follows by the definition (2.2) of $Q$. Hence $C$ is a finitely generated projective left $A$-module, as stated.

(2) If $\nu$ is surjective then there exist finite sets $\{x_i\} \subset \Sigma$ and $\{q_i\} \subset Q$ such that

$$1_T = \sum_i x_i \circ q_i \equiv \sum_i x_i^0 q_i(-)(x_i^1).$$

A dual basis for the right $A$-module $\Sigma$ can be constructed introducing finite sets $\{y_j\} \subset \Sigma$ and $\{\xi_j\} \subset \Sigma^*$ via the requirement that

$$\sum_j y_j \otimes \xi_j = \sum_i x_i^0 \otimes q_i(-)(x_i^1).$$

Theorem 2.6. Let $C$ be an $A$-coring and $\Sigma$ its right comodule. Consider the Morita context $\mathbb{M}(\Sigma)$, associated to $\Sigma$ in (2.4). If the connecting map $\nu$ in (2.6) is surjective, then the functor $- \otimes_T \Sigma : \mathcal{M}_T \to \mathcal{M}_T^C$ in the bottom row of diagram (1.2) is fully faithful.

Proof. The proof consists of a verification of the bijectivity of the unit of the adjunction of functors $- \otimes_T \Sigma : \mathcal{M}_T \to \mathcal{M}_T^C$ and $\text{Hom}^C(\Sigma, -) : \mathcal{M}_T^C \to \mathcal{M}_T$, i.e. the map

$$\eta_N : N \to \text{Hom}^C(\Sigma, N \otimes_T \Sigma), \quad n \mapsto (x \mapsto n \otimes x),$$

for any right $T$-module $N$. Choose elements $\{x_i\} \subset \Sigma$ and $\{q_i\} \subset Q$ such that

$$\sum_i x_i \circ q_i = 1_T.$$ The inverse of the map (2.10) can be constructed as

$$\eta_N : \text{Hom}^C(\Sigma, N \otimes_T \Sigma) \to N, \quad \zeta_N \mapsto (N \otimes \nu)(\sum_i \zeta_N(x_i) \otimes q_i).$$

Indeed, the identity $\eta_N \circ \eta_N = N$ obviously holds true. For the other equality, use the associativity of the Morita context $\mathbb{M}(\Sigma)$ to compute, for $\zeta_N \in \text{Hom}^C(\Sigma, N \otimes_T \Sigma)$ and $x \in \Sigma$,

$$(\eta_N \circ \eta_N)(\zeta_N)(x) = (N \otimes_T \nu \otimes T \Sigma)(\sum_i \zeta_N(x_i) \otimes q_i \otimes x)$$

$$= (N \otimes_T \Sigma \otimes T \nu)(\sum_i \zeta_N(x_i) \otimes q_i \otimes x)$$

$$= \sum_i \zeta_N(x_i)(q_i \cdot x) = \zeta_N(\sum_i x_i(q_i \cdot x)) = \zeta_N(\sum_i (x_i \circ q_i)x) = \zeta_N(x),$$

where in the fourth equality we used the right $^*C$-linearity of $\zeta_N \in \text{Hom}^C(\Sigma, N \otimes_T \Sigma)$. \hfill $\square$

Proposition 2.7. Let $C$ be an $A$-coring which is finitely generated and projective as a left $A$-module and let $\Sigma$ be a right comodule. Put $T : = \text{End}^C(\Sigma)$. The following statements are equivalent.

(1) The Morita context $\mathbb{M}(\Sigma)$, associated to $\Sigma$ in (2.4), is strict;

(2) $\Sigma$ is a Galois comodule and finitely generated and projective as a right $A$-module and faithfully flat as left $T$-module;

(3) The functor $- \otimes_T \Sigma : \mathcal{M}_T \to \mathcal{M}_T^C$, in the bottom row of diagram (1.2), is an equivalence with inverse (1.4).
Proof. (1)⇒(3) If the Morita context is strict then the functors $-\otimes T\Sigma : \mathcal{M}_T \to \mathcal{M}_C$ and $-\otimes_C Q : \mathcal{M}_C \to \mathcal{M}_T$ are inverse equivalences. Since both functors $-\otimes_C Q$ and $\text{Hom}_C(\Sigma, -) : \mathcal{M}_C \to \mathcal{M}_T$ are right adjoints to $-\otimes T\Sigma$, by uniqueness of adjoint functors up to natural equivalence, both are inverses of it. Since $\mathcal{C}$ is a finitely generated and projective left $A$-module, the categories $\mathcal{M}_C$ and $\mathcal{M}_C^*$ are isomorphic (cf. [15, 19.6]). Consequently, $-\otimes T\Sigma : \mathcal{M}_T \to \mathcal{M}_C$ is an equivalence with inverse (1.4).

(3)⇒(1) Since $\mathcal{C}$ is a finitely generated and projective left $A$-module, the categories $\mathcal{M}_C$ and $\mathcal{M}_C^*$ are isomorphic, hence $-\otimes T\Sigma : \mathcal{M}_T \to \mathcal{M}_C^*$ and $\text{Hom}_C(\Sigma, -) : \mathcal{M}_C^* \to \mathcal{M}_T$ are inverse equivalences. The canonical strict Morita context, associated to this equivalence, is equal to $\mathcal{M}(\Sigma)$ in (2.4).

(2)⇒(3) This follows by [15, 18.27 (2) (a)⇒(c)].

(3)⇒(2) Since we have proven already that (3) implies (1), $\Sigma$ is a finitely generated and projective right $A$-module by Lemma 2.5 (2). Then it is a Galois comodule and a faithfully flat left $T$-module by [15, 18.27 (2) (c)⇒(a)]. □

3. Morita theory for coring extensions

Let $\mathcal{D}$ be a coring over the base $k$-algebra $L$ and $\mathcal{C}$ a coring over the $k$-algebra $A$. Assume that $\mathcal{C}$ is a $\mathcal{C}$-$\mathcal{D}$ bicomodule with the left regular $\mathcal{C}$-coaction $\Delta_C$ and some right $\mathcal{D}$-coaction $\tau_C$. By definition [15, 22.1], this means that $\tau_C$ is left $A$-linear (hence $\mathcal{C} \otimes_A \mathcal{C}$ is also a $\mathcal{D}$-comodule with coaction $\mathcal{C} \otimes_A \tau_C$) and the coproduct $\Delta_C$ is right $\mathcal{D}$-colinear. Equivalently, the coproduct $\Delta_C$ is right $L$-linear (hence $\mathcal{C} \otimes_L \mathcal{D}$ is a left $\mathcal{C}$-comodule with coaction $\Delta_C \otimes_L \mathcal{D}$) and the $\mathcal{D}$-coaction $\tau_C$ is left $\mathcal{C}$-colinear. This situation was termed by Brzeziński in [13, Definition 2.1] as $\mathcal{D}$ is a right extension of $\mathcal{C}$, for the following reason. In [13, Theorem 2.6] this definition was shown to be equivalent to the existence of a $k$-linear functor $U : \mathcal{M}_C \to \mathcal{M}_D$, making the following diagram commutative.

Indeed, using the right $\mathcal{D}$-coaction $\tau_C : c \mapsto c_{[0]} \otimes_L c_{[1]}$, for $c \in \mathcal{C}$, (note our convention to use character $\tau$ for $\mathcal{D}$-coactions and lower indices of the Sweedler type to denote the components of the coproduct and of the coactions of the coring $\mathcal{D}$) any right $\mathcal{C}$-comodule $M$ can be equipped with a right $\mathcal{D}$-comodule structure with right $L$-action

\[ ml = m^{[0]} \epsilon_C(m^{[1]} l), \quad \text{for } m \in M \text{ and } l \in L, \]

and $\mathcal{D}$-coaction

\[ \tau_M : M \to M \otimes_L \mathcal{D}, \quad m \mapsto m_{[0]} \otimes_L m_{[1]} = m^{[0]} \epsilon_C(m^{[1]}_{[0]}) \otimes_L m^{[1]}_{[1]}, \]

where $\varrho^M : m \mapsto m^{[0]} \otimes_A m^{[1]}$ denotes the $\mathcal{C}$-coaction in $M$ (note our convention to use character $\varrho$ for $\mathcal{C}$-coactions and upper indices of the Sweedler type to denote the
components of the coproduct and of the coactions of the coring \( C \)). It is straightforward to check that with this definition any right \( C \)-comodule map is \( D \)-colinear. In particular, a right \( C \)-coaction, being \( C \)-colinear by coassociativity, is \( D \)-colinear.

If \( \Sigma \) is an object in the category \( L \mathcal{M}^C \) of \( L \mathcal{C} \) bicomodules, i.e. it is a left \( L \)-module and a right \( C \)-comodule with left \( L \)-linear \( C \)-coaction, then \( T : = \text{End}^C(\Sigma) \) is an \( L \)-ring with unit map \( L \rightarrow T, l \mapsto (x \mapsto lx) \). Furthermore, \( \text{Hom}^C(\Sigma, M) \) is a right \( L \)-module for any right \( C \)-comodule \( M \), via \( (\varphi_M l)(x) : = \varphi_M(lx) \), for \( \varphi_M \in \text{Hom}^C(\Sigma, M), l \in L \) and \( x \in \Sigma \). Hence, in addition to Brzeziński’s functor \( U : \mathcal{M}^C \rightarrow \mathcal{M}^D \), we can define another \( k \)-linear functor,

\[
V : = \text{Hom}^C(\Sigma, -) \otimes \mathcal{D} : \mathcal{M}^C \rightarrow \mathcal{M}^D.
\]

Consider the opposite of the category of \( k \)-linear functors and their natural transformations. The full subcategory defined by the two objects \( U \) and \( V \) determines a Morita context

\[
(3.1) \quad (\text{Nat}(V, V)^{op}, \text{Nat}(U, U)^{op}, \text{Nat}(V, U), \text{Nat}(U, V), \blacksquare, \square),
\]

where all algebra and bimodule structures are given by the opposite composition of natural transformations and both connecting homomorphisms \( \blacksquare \) and \( \square \) are given by projections of the restrictions of the opposite composition of natural transformations.

In the following proposition an equivalent description of the Morita context (3.1), in terms of sets of (co)module maps, is given. In order to formulate it, we introduce a \( k \)-submodule of \( Q' \), (i.e. the \( k \)-module associated to an \( L \mathcal{C} \) bicomodule \( \Sigma \) via (2.3)). It is defined in terms of the \( A \)-\( L \) bimodule \( \Sigma^*: = \text{Hom}_A(\Sigma, A) \), \( (a\xi l)(x) = a\xi(lx) \), for \( l \in L, \xi \in \Sigma^*, a \in A \) and \( x \in \Sigma \), as

\[
(3.2) \quad \widetilde{Q} : = \{ q \in A\text{Hom}_L(\Sigma, \Sigma^*) \mid \forall x \in \Sigma, c \in \mathcal{C} \quad c^{(1)}q(c^{(2)})(x) = q(c)(x^{[0]})x^{[1]} \}.
\]

**Proposition 3.1.** Let \( \Sigma \) be an \( L \mathcal{C} \) bicomodule for a right coring extension \( (\mathcal{D} : L) \) of \( (\mathcal{C} : A) \). Consider the corresponding Morita context (3.1). There is a set of \( k \)-linear isomorphisms

\[
(3.3) \quad \alpha_1 : \quad _L\text{Hom}_L(\mathcal{D}, T) \xrightarrow{\cong} \text{Nat}(V, V)^{op},
(3.4) \quad \alpha_2 : \quad ^C\text{End}^D(\mathcal{C})^{op} \xrightarrow{\cong} \text{Nat}(U, U)^{op},
(3.5) \quad \alpha_3 : \quad _L\text{Hom}^D(\mathcal{D}, \Sigma) \xrightarrow{\cong} \text{Nat}(V, U),
(3.6) \quad \alpha_4 : \quad \widetilde{Q} \xrightarrow{\cong} \text{Nat}(U, V),
\]

where \( T \) denotes the algebra (and \( L \)-ring) \( \text{End}^C(\Sigma) \). Moreover, the maps (3.3)-(3.6) establish an isomorphism of the Morita context (3.1) and the Morita context

\[
(3.7) \quad \widetilde{M}(\Sigma) = (L\text{Hom}_L(\mathcal{D}, T), ^C\text{End}^D(\mathcal{C})^{op}, _L\text{Hom}^D(\mathcal{D}, \Sigma), \widetilde{Q}, \blacksquare, \circ).
\]
The algebra structures, bimodule structures and connecting homomorphisms are given by the following formulae.

\[(3.8) \quad (vv')(d) = v(d_{(1)})v'(d_{(2)})\]
\[(3.9) \quad (uu')(c) = u'(u(c))\]
\[(3.10) \quad (vp)(d) = v(d_{(1)})(p(d_{(2)}))\]
\[(3.11) \quad (pu)(d) = p(d)^0\epsilon_C(u(p(d)^1))\]
\[(3.12) \quad (qv)(c) = q(c_0)v(c_{[1]})\]
\[(3.13) \quad (uq)(c) = q(u(c))\]
\[(3.14) \quad (q \cdot p)(c) = c^{(1)}q(c_2^{[0]})p(c_{[1]}^{[1]}) = q(c_0)(p(c_{[1]}^0)p(c_{[1]}^{[1]})\]
\[(3.15) \quad (p \circ q)(d) = p(d)^0q(p(d)^1)(-),\]

for \(v, v' \in \mathcal{L}\text{Hom}_{\mathcal{L}}(\mathcal{D}, T), u, u' \in \mathcal{C}\text{End}_{\mathcal{P}}(\mathcal{C}), p \in \mathcal{L}\text{Hom}_{\mathcal{P}}(\mathcal{D}, \Sigma), q \in \tilde{Q}, d \in \mathcal{D}\) and \(c \in \mathcal{C}\).

**Proof.** To an element \(v \in \mathcal{L}\text{Hom}_{\mathcal{L}}(\mathcal{D}, T)\) associate a right \(\mathcal{D}\)-colinear map,

\[
\Phi^v_M : \text{Hom}^C(\Sigma, M) \otimes \mathcal{D} \to \text{Hom}^C(\Sigma, M) \otimes \mathcal{D}, \quad \varphi_M \otimes d \mapsto \varphi_M \circ v(d_{(1)}) \otimes d_{(2)},
\]

for any right \(\mathcal{C}\)-comodule \(M\). This defines a \(k\)-module map \(\alpha_1 : \mathcal{L}\text{Hom}_{\mathcal{L}}(\mathcal{D}, T) \to \text{Nat}(V, V)^{op}, v \mapsto \Phi^v\). The bijectivity of \(\alpha_1\) is proven by constructing the inverse \(\alpha^{-1}_1\), mapping \(\Phi \in \text{Nat}(V, V)\) to the right \(L\)-linear map

\[
\mathcal{D} \to T, \quad d \mapsto ((T \otimes \epsilon_{\mathcal{D}}) \circ \Phi_{\Sigma})(1 \otimes d).
\]

Since \(\Sigma\) is an \(L\)-\(\mathcal{C}\) bicomodule, the map \(\Sigma \to \Sigma, x \mapsto lx\) is right \(\mathcal{C}\)-colinear for any \(l \in L\). Hence \(\alpha^{-1}_1(\Phi)\) is left \(L\)-linear by the naturality of \(\Phi\). The identity \(\alpha^{-1}_1 \circ \alpha_1(v) = v, v \in \mathcal{L}\text{Hom}_{\mathcal{L}}(\mathcal{D}, T)\), is obvious. The other identity \(\alpha_1 \circ \alpha^{-1}_1(\Phi) = \Phi\), for \(\Phi \in \text{Nat}(V, V)\), is checked as follows. Since the right \(\mathcal{D}\)-coaction in \(T \otimes_L \mathcal{D}\) is given by \(T \otimes_L \Delta_{\mathcal{D}}\), it follows that, for all \(t \otimes_L d \in T \otimes_L \mathcal{D}\),

\[(3.16) \quad (T \otimes \epsilon_{\mathcal{D}})(t \otimes d) = t \epsilon_{\mathcal{D}}(d_{(1)}) \otimes d_{(2)} = t \otimes d.
\]

Using the right \(\mathcal{D}\)-colinearity of \(\Phi_{\Sigma}\) (in the second equality), (3.16) (in the third equality) and the right \(\mathcal{C}\)-colinearity of \(\varphi_M\) together with the naturality of \(\Phi\) (in the last equality), we conclude that

\[
(\alpha_1 \circ \alpha^{-1}_1(\Phi))_M(\varphi_M \otimes d) = \varphi_M \circ ((T \otimes_\mathcal{L} \epsilon_{\mathcal{D}})(\Phi_{\Sigma}(1 \otimes d_{(1)}))) \otimes d_{(2)} = \varphi_M \circ ((T \otimes_\mathcal{L} \epsilon_{\mathcal{D}})(\Phi_{\Sigma}(1 \otimes d_{[0]}))) \otimes \Phi_{\Sigma}(1 \otimes \varphi_M \otimes d)_{[1]} = ((\varphi_M \otimes \Phi_{\Sigma})(1 \otimes \varphi_M \otimes d) = \Phi_M(\varphi_M \otimes d),
\]

for \(\varphi_M \otimes L d \in \text{Hom}^C(\Sigma, M) \otimes L \mathcal{D}\).

To an element \(u \in \mathcal{C}\text{End}_{\mathcal{P}}(\mathcal{C})\) associate a map

\[
\Xi^u_M : M \to M, \quad m \mapsto m^{[0]}(\epsilon_C \circ u)(m^{[1]}),
\]

for any right \(\mathcal{C}\)-comodule \(M\). It is checked to be right \(\mathcal{D}\)-colinear using the relation between the \(\mathcal{C}\) and \(\mathcal{D}\)-comodule structures of \(M\), the right \(A\)-linearity of a \(\mathcal{C}\)-coaction,
the left $\mathcal{C}$-colinearity and the right $\mathcal{D}$-colinearity of $u$ and the $\mathcal{D}$-colinearity of the $\mathcal{C}$-coaction:

\[
(m^{[0]}(\epsilon_\mathcal{C} \circ u)(m^{[1]}))^{[0]} \otimes \left((m^{[0]}(\epsilon_\mathcal{C} \circ u)(m^{[1]}))^{[2]}\right) \otimes \left((m^{[0]}(\epsilon_\mathcal{C} \circ u)(m^{[1]}))^{[2]}\right) = m^{[0]}\epsilon_\mathcal{C}((m^{[1]}(\epsilon_\mathcal{C} \circ u)(m^{[2]}))^{[0]} \otimes \left((m^{[1]}(\epsilon_\mathcal{C} \circ u)(m^{[2]}))^{[1]}\right) = m^{[0]}\epsilon_\mathcal{C}(u(m^{[1]}))^{[0]} \otimes m^{[1]}).
\]

This implies that we have a $k$-linear map $\alpha_2 : \mathcal{C}\text{End}^D(\mathcal{C})^{\text{op}} \rightarrow \text{Nat}(U, U)^{\text{op}}$, $u \mapsto \Xi^u$. In order to prove its bijectivity, we construct the inverse $\alpha_2^{-1}$, mapping $\Xi \in \text{Nat}(U, U)$ to $\Xi_\mathcal{C} \in \text{End}^D(\mathcal{C})$. We need to prove that $\Xi_\mathcal{C}$ is left $\mathcal{C}$-linear. Note first that, for any right $A$-module $N$ and $n \in N$, the map $C \rightarrow N \otimes_A \mathcal{C}$, $c \mapsto n \otimes_A c$ is right $\mathcal{C}$-colinear (where $N \otimes_A \mathcal{C}$ is a right $\mathcal{C}$-comodule via $N \otimes_A \Delta_C$). Hence, by naturality,

\[
\Xi_{N \otimes_A \mathcal{C}} = N \otimes_A \Xi_\mathcal{C}.
\]

On the other hand, the coproduct in $\mathcal{C}$ is right $\mathcal{C}$-colinear (i.e. coassociative), hence naturality implies $\Xi_{C \otimes \mathcal{C}} \circ \Delta_C = \Delta_C \circ \Xi_\mathcal{C}$. Combining these two observations, we have the left $\mathcal{C}$-colinearity of $\Xi_\mathcal{C}$ proven. The identity $\alpha_2^{-1} \circ \alpha_2(u) = u$, for $u \in \mathcal{C}\text{End}^D(\mathcal{C})$, follows easily by the $\mathcal{C}$-colinearity of $u$. The property $\alpha_2 \circ \alpha_2^{-1}(\Xi) = \Xi$, for $\Xi \in \text{Nat}(U, U)$, follows by the commutativity of the following diagram, for any right $\mathcal{C}$-comodule $M$.

\[
\begin{array}{ccc}
M & \xrightarrow{\Xi_M} & M \\
\downarrow{\phi_M} & & \downarrow{\phi_M} \\
M \otimes \mathcal{C} & \cong_{M \otimes_A \mathcal{C}} & M \otimes \mathcal{C}
\end{array}
\]

Commutativity of this diagram follows by \[3.17\], the right $\mathcal{C}$-colinearity (i.e. coassociativity) of a right $\mathcal{C}$-coaction and the naturality of $\Xi$.

To an element $p \in _L\text{Hom}^D(\mathcal{D}, \Sigma)$ associate the right $\mathcal{D}$-colinear map,

\[
\Theta^p_M : \text{Hom}^\mathcal{C}(\Sigma, M) \otimes L \mathcal{D} \rightarrow M, \quad \varphi_M \otimes d \mapsto \varphi_M(p(d)),
\]

for any right $\mathcal{C}$-comodule $M$. It defines a $k$-map $\alpha_3 : _L\text{Hom}^D(\mathcal{D}, \Sigma) \rightarrow \text{Nat}(V, U)$, $p \mapsto \Theta^p$. We prove its bijectivity by constructing the inverse $\alpha_3^{-1}$, mapping $\Theta \in \text{Nat}(V, U)$ to the right $\mathcal{D}$-colinear map,

\[
\mathcal{D} \rightarrow \Sigma, \quad d \mapsto \Theta_\Sigma(1_T \otimes d).
\]

Its left $L$-linearity follows by the right $\mathcal{C}$-colinearity of the map $\Sigma \rightarrow \Sigma$, $x \mapsto lx$, for any $l \in L$, the naturality of $\Theta$, and the fact that $L$ is a subalgebra of $T$. The identity $\alpha_3^{-1} \circ \alpha_3(p) = p$, for $p \in _L\text{Hom}^D(\mathcal{D}, \Sigma)$ is obvious, and $\alpha_3 \circ \alpha_3^{-1}(\Theta) = \Theta$, for $\Theta \in \text{Nat}(V, U)$, follows by the naturality of $\Theta$, i.e. the identity $\varphi_M(\Theta_\Sigma(t \otimes_L d)) = \Theta_M(\varphi_M \circ t \otimes_L d)$, for any right $\mathcal{C}$-comodule $M$, $\varphi_M \in \text{Hom}^\mathcal{C}(\Sigma, M)$ and $t \otimes_L d \in T \otimes_L \mathcal{D}$.

To an element $q \in \hat{Q}$ associate the right $\mathcal{D}$-colinear map,

\[
\Omega^q_M : M \rightarrow \text{Hom}^\mathcal{C}(\Sigma, M) \otimes L \mathcal{D}, \quad m \mapsto m^{[0]} q(m^{[1]} r) (-) \otimes m^{[1]} r
\]

\[
= m^{[0]} q(m^{[1]} r) (-) \otimes m^{[1]} r.
\]
for any right $C$-comodule $M$. The two forms of $\Omega^q_M$ are equal by the right $D$-colinearity of the $C$-coaction. It is a well-defined map by the $A$-$L$ bilinearity of $q$ and Lemma 2.1 (2). The association $q \mapsto \Omega^q$ defines a $k$-map $\alpha_4 : \widetilde{Q} \to \text{Nat}(U, V)$. We prove its bijectivity by constructing the inverse $\alpha_4^{-1}$, mapping $\Omega \in \text{Nat}(U, V)$ to the right $L$ linear map

$$C \rightarrow \Sigma^*, \quad c \mapsto \epsilon_C((\text{Hom}^C(\Sigma, C) \otimes \epsilon_D)(\Omega_C(c))(-)).$$

By the right $C$-colinearity of the map $C \rightarrow N \otimes_A C$, $c \mapsto n \otimes_A c$, for any right $A$-module $N$ and $n \in N$, and the naturality of $\Omega$, the map $\Omega_{N \otimes A C}$ can be written as a composite of $N \otimes_A \Omega_C : N \otimes_A C \rightarrow N \otimes_A \text{Hom}^C(\Sigma, C) \otimes_L D$ and the obvious map $N \otimes_A \text{Hom}^C(\Sigma, C) \otimes_L D \rightarrow \text{Hom}^C(\Sigma, N \otimes_A C) \otimes_L D$. Applying this fact to the case $N = A$, we conclude on the left $A$-linearity of $\Omega_C$, hence of $\alpha_4^{-1}(\Omega)$. Consider the following commutative diagram.

The upper left square is commutative by the right $C$-colinearity (i.e. coassociativity) of the coproduct in $C$ and the naturality of $\Omega$. The lower left triangle is commutative by the previous observation that $\Omega_{N \otimes A C}$ factors through $N \otimes_A \Omega_C$, for any right $A$-module $N$. The squares in the middle column are commutative by the isomorphism of $k$-modules $\text{Hom}^C(\Sigma, N \otimes_A C) \cong \text{Hom}_A(\Sigma, N)$, for any right $A$-module $N$, cf. [15 18.10]. The upper line in the diagram gives an equivalent expression for $\alpha_4^{-1}(\Omega)$. Comparing the incoming arrows from above and below in $\text{Hom}_A(\Sigma, C)$ on the outer right, we conclude that $\alpha_4^{-1}(\Omega)$ is an element of $\widetilde{Q}$.

The identity $\alpha_4^{-1} \circ \alpha_4(q) = q$, for $q \in \widetilde{Q}$, follows by the right $A$-linearity of $\epsilon_C$ and the left $A$-linearity of $q$. In order to prove the converse property, $\alpha_4 \circ \alpha_4^{-1}(\Omega) = \Omega$, for $\Omega \in \text{Nat}(U, V)$, consider the following diagram in $\mathcal{M}^L$, for any $M \in \mathcal{M}^C$. 

![Diagram](attachment:image.png)
The commutativity of the upper quadrangle follows by the naturality of $\Omega_M$. The lower quadrangle commutes by the right $D$-comodule of $\rho_M$ and the right lower quadrangle does by the right $D$-
comodule of $\Omega_C$ and the fact that $\epsilon_D$ is the counit of $D$. The property that $\Omega_{M \otimes A C}$ factors through $M \otimes \Omega_C$ implies that the triangle commutes as well. Evaluating the upper and lower incoming arrow in $\Hom_C(\Sigma, M \otimes_A C) \otimes_L D$, on an element $m \in M$, one finds

$$(\rho_M \circ -) \circ \Omega_M(m) = m_{[0]} \otimes A (\Hom_C(\Sigma, C) \otimes_A \epsilon_D(\Omega_C(m_{[1]}))) \otimes_L m_{[1]}.$$  

Application of $(M \otimes_A \epsilon_C) \circ - \otimes_L D$ to both sides of this equation yields the required identity $\Omega_M = \alpha_4 \circ \alpha_4^{-1}(\Omega)_M$ in $\Hom_C(\Sigma, M) \otimes_L D$.

The proof is completed by showing that the maps $\alpha_1$, $\alpha_2$, $\alpha_3$ and $\alpha_4$ define a morphism of Morita contexts. Indeed, it is straightforward to check that, for $v, v' \in L\Hom_L(D, T)$, $u, u' \in \mathcal{C} end_P(D)$, $p \in L\Hom_P(D, \Sigma)$ and $q \in \tilde{Q}$,

$$\alpha_1(v) \circ \alpha_1(v') = \alpha_1(v) \circ \alpha_1(v') = \alpha_2(u) \circ \alpha_2(u') = \alpha_2(u) \circ \alpha_2(u') = \alpha_3(v p) \circ \alpha_3(v p) = \alpha_3(v p) \circ \alpha_3(v p) = \alpha_4(q) \circ \alpha_4(q) = \alpha_4(q) \circ \alpha_4(q) = \alpha_1(p \circ q).$$

This ends the proof. □

**Remark 3.2.** The commutative base ring $k$ can be considered as a trivial ($k$-) coring, which is a right extension of any $A$-coring $C$. A right $C$-comodule $\Sigma$ can be viewed as a $k$-$C$ bicomodule for the right coring extension $k$ of $C$, hence there is an associated Morita context $\tilde{M}(\Sigma) = (\Hom_k(k, T), \mathcal{C} end(C)^{op}, \Hom_k(k, \Sigma), \tilde{Q} \equiv Q', \bullet \circ )$, as in (3.7). Obviously, $\Hom_k(k, T) \cong T$ and $\Hom_k(k, \Sigma) \cong \Sigma$. By a hom-tensor relation for comodules [15, 18.11], $\mathcal{C} end(C)^{op} \cong \mathcal{C}$. By Lemma 2.1 (1), $Q \cong Q' \equiv \tilde{Q}$. Composing these isomorphisms with the Morita maps in (3.7), one obtains the structure maps of the Morita context (2.4). Hence the Morita context $\tilde{M}(\Sigma)$, associated to $\Sigma$ as a $k$-$C$ bicomodule, coincides with $M(\Sigma)$, associated to a right $C$-comodule $\Sigma$ in (2.4).

**Lemma 3.3.** Let the $L$-coring $D$ be a right extension of the $A$-coring $C$ and let $\Sigma$ be an object in $L\mathcal{M}_C$. Consider the Morita context $\tilde{M}(\Sigma)$ in (3.7). If there exist finite sets of elements $\{j_\ell\} \subset \mathcal{L} Hom_P(D, \Sigma)$ and $\{\tilde{j}_\ell\} \subset Q$ such that $\sum_\ell \tilde{j}_\ell \bullet j_\ell = C$ (i.e. connecting map $\bullet$ in (3.14) is surjective), then

1. the identity $\sum_\ell \tilde{j}_\ell(c_{[0]})(j_\ell(c_{[1]})) = \epsilon_C(c)$ holds, for all $c \in C$;
2. the identity $\sum_\ell m_{[0]}^{[0]} \tilde{j}_\ell(m_{[0]}^{[1]})(j_\ell(m_{[1]})) = m$ holds, for any right $C$-comodule $M$ and $m \in M$.

**Proof.** (1) This follows by applying $\epsilon_C$ to (3.14).

(2) By the $D$-comodularity of a right $C$-coaction in a right $C$-comodule $M$, for any $m \in M$,

$$\sum_\ell m_{[0]}^{[0]} \tilde{j}_\ell(m_{[0]}^{[1]})(j_\ell(m_{[1]})) = \sum_\ell m_{[0]}^{[0]} \tilde{j}_\ell(m_{[1]}^{[1]})(j_\ell(m_{[1]})) = m^{[0]} \epsilon_C(m_{[1]}) = m,$$

where we used part (1) in the second equation. □

**Proposition 3.4.** Let the $L$-coring $D$ be a right extension of the $A$-coring $C$. Take $\Sigma \in L\mathcal{M}_C$ and consider the associated Morita context $\tilde{M}(\Sigma)$ in (3.7). If both the map $\bullet$ in (3.14) and the counit $\epsilon_C$ are surjective then $\Sigma$ is a generator of right $A$-modules.
Proof. Choose elements \( \{j_\ell\} \subset \text{LHom}^D(D, \Sigma) \) and \( \{\tilde{g}_\ell\} \subset \tilde{Q} \) such that \( \sum_\ell \tilde{g}_\ell \cdot j_\ell = C \), and an element \( c \in C \) such that \( \epsilon_C(c) = 1_A \). Fix finite sets \( \{x_i\} \subset \Sigma \) and \( \{\xi_i\} \subset \Sigma^* \) such that
\[
\sum_i \xi_i \otimes x_i = \sum_\ell \tilde{g}_\ell(c[0]) \otimes j_\ell(c[1]).
\]
Then \( \sum_i \xi_i(x_i) = \epsilon_C(c) = 1_A \), by Lemma 3.3 (1), which proves the claim. \( \square \)

A lemma by Beck (cf. a dual version of [3, 3.3 Proposition 3]) states that if \( F : A \to B \) is a comonadic functor and \( \lambda \) is a split epimorphism in the category \( A \), such that \( F(\lambda) \) has a kernel in \( B \), then also \( \lambda \) has a kernel in \( A \) and \( F \) preserves this kernel. Since for an \( A \)-coring \( C \) the forgetful functor \( \mathcal{M}^C \to \mathcal{M}_A \) is obviously comonadic, the next lemma follows by this general result. Still, for the convenience of the reader we include a complete proof, which is a simplified version of Beck’s arguments (as in our case the target category \( \mathcal{M}_A \) is abelian).

**Lemma 3.5.** The following statements about right comodules \( M \) and \( N \) for an \( A \)-coring \( C \) are equivalent.

(i) There exists an index set \( \{\ell\} \) of finite order \( s \) and collections of morphisms \( \{\kappa_\ell\} \subset \text{Hom}^C(M, N) \) and \( \{\lambda_\ell\} \subset \text{Hom}^C(N, M) \), indexed by \( \ell = 1 \ldots s \), such that
\[
\sum_\ell \lambda_\ell \circ \kappa_\ell = M;
\]

(ii) \( M \) is a direct summand of the direct sum \( N^s \) as a right \( C \)-comodule.

**Proof.** In terms of \( \{\kappa_\ell\} \) and \( \{\lambda_\ell\} \) as in (i), construct a map \( \kappa : M \to N^s \) defined by \( \kappa(m) = (\kappa_\ell(m))_\ell \). Clearly, \( \kappa \) has a left inverse, \( \lambda : N^s \to M \), \( \lambda((n_\ell)_\ell) = \sum_\ell \lambda_\ell(n_\ell) \). Recall that the category of comodules of any \( A \)-coring \( C \) has direct sums that do coincide with the direct sums in \( \mathcal{M}_A \) (see e.g. [29, 3.3]), so \( \lambda \) and \( \kappa \) are morphisms in \( \mathcal{M}^C \). In particular, \( \lambda \) and \( \kappa \) are right \( A \)-linear, so there is a split exact sequence in the abelian category of right \( A \)-modules,
\[
0 \longrightarrow M' \overset{\nu}{\longrightarrow} N^s \overset{\lambda}{\longrightarrow} M \longrightarrow 0,
\]
where \( (M', \nu) \) is the kernel of \( \lambda \) in \( \mathcal{M}_A \). The proof is completed by showing that \( M' \) is a right \( C \)-comodule and \( \nu \) is a \( C \)-colinear section. Denoting a right \( A \)-linear retraction of \( \nu \) by \( \varpi \), introduce a right \( A \)-module map
\[
\varphi^{M'} : = (\varpi \otimes \mathcal{A}) \circ \varphi^N \circ \nu : M' \to M' \otimes \mathcal{A}.
\]
Since the \( A \)-module maps satisfy \( \nu \circ \varpi + \kappa \circ \lambda = N^s \), the colinearity of \( \kappa \) and \( \lambda \) implies that \( \nu \circ \varpi \) is \( \mathcal{C} \)-colinear. Hence
\[
(\nu \otimes \mathcal{A}) \circ \varphi^{M'} = (\nu \circ \varpi \otimes \mathcal{A}) \circ \varphi^N \circ \nu = \varphi^N \circ \nu.
\]
Composing (3.18) by \( \varpi \) on both sides and using the colinearity of \( \nu \circ \varpi \), one obtains also \( \varphi^{M'} \circ \varpi = (\varpi \otimes \mathcal{A}) \circ \varphi^N \). Furthermore, using (3.18) (in the first, second and fourth equalities) and the coassociativity of \( \varphi^N \) (in the third equality), one checks that
\[
(\nu \otimes \mathcal{A} \otimes \mathcal{A}) \circ (\varphi^{M'} \otimes \mathcal{A}) \circ \varphi^{M'} = (\varphi^N \circ \nu \otimes \mathcal{A}) \circ \varphi^{M'} = (\varphi^N \otimes \mathcal{A} \otimes \mathcal{A}) \circ (\nu \otimes \mathcal{A}) \circ \varphi^{M'}
\]
\[
= (N^s \otimes \Delta_C) \circ \varphi^N \circ \nu = (N^s \otimes \Delta_C) \circ (\nu \otimes \mathcal{A}) \circ \varphi^{M'}
\]
\[
= (\nu \otimes \mathcal{A} \otimes \mathcal{A}) \circ (M' \otimes \Delta_C) \circ \varphi^{M'}.
\]
Since $\nu \otimes_A C \otimes_A C$ is a (split) monomorphism, this implies the coassociativity of $\varrho^M$.
Finally, by the counitality of $\varrho^{N*}$,
$$(M' \otimes \epsilon_C) \circ \varrho^M = (M' \otimes \epsilon_C) \circ (\varpi \otimes A) \circ \varrho^{N*} \circ \nu = \varpi \circ \nu = M',$$
which finishes the proof of the implication $(i) \Rightarrow (ii)$. The converse implication is obvious. \hfill \Box

Since $L$-$C$ bicomodules, for an algebra $L$ and an $A$-coring $C$, can be identified with right comodules for the $L^{op} \otimes_k A$-coring $L^{op} \otimes_k C$, Lemma 3.5 applies also to $L$-$C$ bicomodules.

**Theorem 3.6.** Let the $L$-coring $D$ be a right extension of the $A$-coring $C$. Take $\Sigma \in L \mathcal{M}^C$ and consider the Morita context $\mathcal{M}(\Sigma)$ in (3.7). In particular, put $T := \text{End}^C(\Sigma)$.

1. The map $\ddagger$ in (3.14) is surjective if and only if $\Sigma$ is a Galois $C$-comodule and $\Sigma$ is a direct summand of a direct sum $T \otimes_L D)^s$ as $T$-$D$-bicomodule, for an appropriate finite integer $s$.

2. The Morita context $\mathcal{M}(\Sigma)$ is strict if and only if $\Sigma$ is a Galois $C$-comodule, $\Sigma$ is a direct summand of $(T \otimes_L D)^s$ and $T \otimes_L D$ is a direct summand of $\Sigma^s$, both as $T$-$D$-bicomodules, where $s$ and $z$ are finite integers.

**Proof.** (1) By surjectivity of $\ddagger$, there exist elements $\{j_\ell\} \subset \text{Hom}^D(D, \Sigma)$ and $\{\bar{j}_\ell\} \subset \bar{Q}$ (cf. (3.2)) such that $\sum_\ell \bar{j}_\ell \bullet j_\ell = C$. In terms of the isomorphisms $\alpha_3$ and $\alpha_4$ in Proposition 3.1, we denote $\alpha_3(j_\ell) = J_\ell$ and $\alpha_4(\bar{j}_\ell) = \bar{J}_\ell$, for all values of $\ell$.

First we check that the surjectivity of $\ddagger$ implies the Galois property of the right $C$-comodule $\Sigma$. To this end, we construct the inverse of the canonical natural transformation (1.3). For any right $A$-module $N$, put

$$\Upsilon_N : N \otimes_A C \rightarrow \text{Hom}_A(\Sigma, N) \otimes_T \Sigma, \quad n \otimes_A c \mapsto \sum_\ell n \bar{j}_\ell(c_{[0]})(- \otimes_T j_\ell(c_{[1]})).$$

Since $T$ is an $L$-ring, $\Upsilon_N$ is a well defined map. It is natural in $N$, being a sum of composites of natural morphisms,

$$N \otimes_A C \xrightarrow{(\bar{j}_\ell)n \otimes_A c} \text{Hom}^C(\Sigma, N) \otimes A \otimes C \xrightarrow{\text{Hom}_A(\Sigma, N) \otimes A} \text{Hom}_A(\Sigma, N) \otimes T \otimes_L D \xrightarrow{\text{Hom}_A(\Sigma, N) \otimes T(J_\ell)c} \text{Hom}_A(\Sigma, N) \otimes_T \Sigma.$$

We claim that $\Upsilon$ yields the inverse of the canonical natural transformation (1.3). Indeed, we find that, for $n \otimes_A c \in N \otimes_A C$,

$$\text{can}_N(\Upsilon_N(n \otimes_A c)) = \sum_\ell n \otimes_A \bar{j}_\ell(c_{[0]})(j_\ell(c_{[1]}))^{[0]}j_\ell(c_{[1]})^{[1]}$$

$$= n \otimes_A \sum_\ell (\bar{j}_\ell \bullet j_\ell)(c) = n \otimes_A c.$$

On the other hand, for $\phi_N \otimes_T x \in \text{Hom}_A(\Sigma, N) \otimes_T \Sigma$,

$$\Upsilon_N(\text{can}_N(\phi_N \otimes_T x)) = \sum_\ell \phi_N(x_{[0]}^{[0]}\bar{j}_\ell(x_{[0]}^{[1]}))(-) \otimes_T j_\ell(x_{[1]}^{[1]}))$$

$$= \sum_\ell \phi_N(x_{[0]}^{[0]}\bar{j}_\ell(x_{[0]}^{[1]}))(-) \otimes_T j_\ell(x_{[1]}^{[1]}),$$
where in the second equality we used the right $D$-colinearity of the $C$-coaction in $\Sigma$ and the right $A$-linearity of $\phi_N$. Using Lemma 2.1 (2) at the right $C$-comodule $\Sigma$, we conclude that $x[0]^0 \tilde{\eta}(x[1])(-) \in T$ for all $x \in \Sigma$ and any value of the index $\ell$. Hence

$$\sum_\ell \phi_N(x[0]^0 \tilde{\eta}(x[1])(-)) \otimes \tilde{\eta}(x[1]) = \phi_N \otimes \sum_\ell x[0]^0 \tilde{\eta}(x[1])(-) \otimes x[1] = \phi_N \otimes x,$$

where the last equality follows by Lemma 3.3 (2), applied to the right $C$-comodule $\Sigma$. This proves that if the Morita map $\bullet$ is surjective then $\Sigma$ is a Galois $C$-comodule.

Next we prove that $\Sigma$ is a direct summand of $\left(T \otimes_L D\right)^s$ as $T$-$D$-bicomodule, where $s$ is the cardinality of the index set $\{\ell\}$ above. For any value of $\ell$, put

$$(3.19) \quad \kappa_\ell : = (\tilde{\eta}_\ell)_\Sigma : \Sigma \to T \otimes_L D, \quad x \mapsto x[0]^0 \tilde{\eta}(x[1])(-) \otimes x[1], \quad \text{and} \quad \kappa_\ell : = (\eta_\ell)_\Sigma : \Sigma \otimes_L C \to \Sigma, \quad t \otimes d \mapsto (\eta_\ell)(d).$$

Since the left $L$-action in $\Sigma$ is right $C$-colinear by assumption, and any $C$-colinear map is $D$-colinear, both the right $C$, and $D$-coactions in $\Sigma$ are left $L$-linear. This way $\kappa_\ell$, being a composite of left $L$-linear right $D$-colinear maps, is left $L$-linear and right $D$-colinear. The map $\tilde{\kappa}_\ell$ is obviously left $L$-linear and it is right $D$-colinear by the colinearity of $\bar{\eta}_\ell$ and any $t \in T$. It follows by Lemma 3.3 (2) that $\sum_\ell \tilde{\kappa}_\ell \circ \kappa_\ell(x) = x$. By Lemma 3.5 this implies that $\Sigma$ is a direct summand of $\left(T \otimes_L D\right)^s$ as $T$-$D$-bicomodule, where $s$ is the cardinality of the index set $\{\ell\}$.

In order to prove the converse statement, we make use of Lemma 3.3 again. Since $\Sigma$ is a direct summand of $\left(T \otimes_L D\right)^s$ as $T$-$D$-bicomodule, there exist maps $\kappa_\ell \in \tau_{\Hom}(\Sigma, T \otimes_L D)$ and $\tilde{\kappa}_\ell \in \tau_{\Hom}(T \otimes_L D, \Sigma)$, for $\ell = 1 \ldots s$, such that $\sum_\ell \tilde{\kappa}_\ell \circ \kappa_\ell = \Sigma$. We can define maps $\bar{\eta}_\ell$ and $\tilde{\eta}_\ell$ as follows. For any value of $\ell$, put

$$(3.20) \quad \bar{\eta}_\ell : D \to \Sigma, \quad d \mapsto \tilde{\eta}_\ell(1_T \otimes d).$$

By the colinearity of $\tilde{\kappa}_\ell$, $\bar{\eta}_\ell$ is right $D$-colinear. Since $\tilde{\kappa}_\ell$ is left $T$-linear and $L$ is a subalgebra of $T$, $\tilde{\eta}_\ell$ is also left $L$-linear. Furthermore, $\Sigma$ is a Galois $C$-comodule by assumption, hence we can set

$$(3.21) \quad \tilde{\eta}_\ell : = [\Sigma^* \otimes (T \otimes L) \circ \kappa_\ell] \circ \text{can}_A^{-1} : C \to \Sigma^*,$$

for $\ell = 1 \ldots s$. The map $\text{can}_A$ is left $A$-linear and right $C$-colinear. Hence it is also right $D$-colinear, so, in particular, right $L$-linear. Since this way $\tilde{\eta}_\ell$ is a composite of left $A$-linear, right $L$-linear maps, it is left $A$-linear and right $L$-linear. Let us prove that $\tilde{\eta}_\ell \in \bar{Q}$. Consider the following commutative diagram.

Here $f_c$ denotes a right $A$-linear morphism from $A$ to $C$, defined for an element $c \in C$ as $f_c(a) = ca$. The lower square on the left hand side commutes because of the naturality
of \( \text{can}^{-1} \). By the explicit form (1.3) of \( \text{can} \), for \( \xi \otimes_{T} x \in \Sigma^{*} \otimes_{T} \Sigma \),
\[
\text{can}_{C} \circ \left( \left( - \otimes C \right) \circ \rho_{T}^{*} \otimes \Sigma \right)(\xi \otimes x) = \xi(x^{[0]})x^{[1]} \otimes x^{[2]} = \Delta_{C} \circ \text{can}_{A}(\xi \otimes x).
\]

Hence also the upper square on the left hand side commutes. The commutativity of the squares on the right hand side is obvious. Both the upper and lower horizontal lines express the map \( \tilde{j}_{\ell} \). In terms of the map \( f_{\ell} \) introduced above, \( \Delta_{C}(c) = (f_{\ell(1)} \otimes A_{C})(c(2)) \).

Hence the commutativity of the diagram implies, for all \( c \in C \) and \( x \in \Sigma \),
\[
\begin{align*}
\tilde{j}_{\ell}(c)(x^{[0]})x^{[1]} & = \left( \left[ \text{Hom}_{A}(\Sigma, C) \otimes_{T} (T \otimes \epsilon_{D}) \circ \kappa_{\ell} \right] \circ \text{can}_{A}^{-1} \circ \Delta_{C}(c) \right)(x) \\
& = \left( \left[ \text{Hom}_{A}(\Sigma, C) \otimes_{T} (T \otimes \epsilon_{D}) \circ \kappa_{\ell} \right] \circ \text{can}_{A}^{-1} \circ (f_{\ell(1)} \otimes A_{C})(c(2)) \right)(x) \\
& = c(1) \tilde{j}_{\ell}(c(2))(x).
\end{align*}
\]

That is, \( \tilde{j}_{\ell} \in \tilde{Q} \), for all values of the index \( \ell \). The surjectivity of \( \left( C \text{End}^{D}(C) \right) \)-\( C \text{End}^{D}(C) \) bilinear map \( \circ \) is proven by showing \( \sum_{\ell} \tilde{j}_{\ell} \circ j_{\ell} = C \). Use the right \( D \)-colinearity of \( \text{can}_{A}^{-1} \) (in the second equality), the right \( D \)-colinearity of \( \kappa_{\ell} \) (in the third one) and the left \( T \)-linearity of \( \bar{\kappa}_{\ell} \) (in the penultimate one) to compute the composite of the right \( D \)-coaction \( \tau_{C} \) on \( C \) with \( \sum_{\ell} \tilde{j}_{\ell} \otimes_{L} j_{\ell} \). It yields
\[
\begin{align*}
\sum_{\ell} (\tilde{j}_{\ell} \otimes j_{\ell}) \circ \tau_{C} & = \sum_{\ell} \left\{ [\Sigma^{*} \otimes_{T} (T \otimes \epsilon_{D}) \circ \kappa_{\ell}] \otimes j_{\ell} \right\} \circ (\text{can}_{A}^{-1} \otimes D) \circ \tau_{C} \\
& = \sum_{\ell} \left\{ [\Sigma^{*} \otimes_{T} (T \otimes \epsilon_{D}) \circ \kappa_{\ell}] \otimes j_{\ell} \otimes \tau_{\Sigma} \right\} \circ \text{can}_{A}^{-1} \\
& = \sum_{\ell} [\Sigma^{*} \otimes_{T} (T \otimes \epsilon_{D}) \otimes \kappa_{\ell}] \otimes (T \otimes \Delta_{D}) \circ \kappa_{\ell} \circ \text{can}_{A}^{-1} \\
& = \sum_{\ell} [\Sigma^{*} \otimes_{T} (T \otimes j_{\ell}) \circ \kappa_{\ell}] \circ \text{can}_{A}^{-1} \\
& = [\Sigma^{*} \otimes_{T} \sum_{\ell} \bar{\kappa}_{\ell} \circ \kappa_{\ell}] \circ \text{can}_{A}^{-1} = \text{can}_{A}^{-1}.
\end{align*}
\]

Note that the evaluation map \( \Sigma^{*} \otimes_{T} \Sigma \rightarrow A \), \( \xi \otimes_{T} x \mapsto \xi(x) \) is equal to \( \epsilon_{C} \circ \text{can}_{A} \).

Hence equation (3.22) implies
\[
\begin{align*}
\sum_{\ell} \tilde{j}_{\ell} \circ j_{\ell} & = \sum_{\ell} (\epsilon_{C} \circ \text{can}_{A} \otimes A_{C}) \circ (\Sigma^{*} \otimes_{T} \rho_{T}) \circ (\tilde{j}_{\ell} \otimes j_{\ell}) \circ \tau_{C} \\
& = (\epsilon_{C} \circ \text{can}_{A} \otimes A_{C}) \circ (\Sigma^{*} \otimes_{T} \rho_{T}) \circ \text{can}_{A}^{-1} = (\epsilon_{C} \otimes A_{C}) \circ \Delta_{C} = C,
\end{align*}
\]

where the third equality follows by the right \( C \)-colinearity of \( \text{can}_{A} \).

(2) Suppose that the Morita context (3.7) is strict. In view of part (1), we have to prove only that \( T \otimes_{L} D \) is a direct summand of \( \Sigma^{z} \), for some integer \( z \). By the surjectivity of \( \phi \), there exist elements \( \{ h_{i} \} \subset_{L} \text{Hom}^{D}(D, \Sigma) \) and \( \{ \tilde{h}_{i} \} \subset \tilde{Q} \) such that \( \sum_{i}(h_{i} \circ \tilde{h}_{i})(d) = \epsilon_{D}(d)1_{T} \), for \( d \in D \). Similarly to (3.19), for any value of \( i \), we define left \( L \)-linear and right \( D \)-colinear morphisms,
\[
\begin{align*}
\lambda_{i} & : \Sigma \rightarrow T \otimes_{L} D, \quad x \mapsto x^{[0]}_{\Sigma} \tilde{h}_{i}(x^{[1]}_{\Sigma})(- \otimes x^{[1]}_{\Sigma}) \quad \text{and} \\
\tilde{\lambda}_{i} & : T \otimes_{L} D \rightarrow \Sigma, \quad t \otimes d \mapsto t(h_{i}(d)).
\end{align*}
\]
They satisfy, for any \( t \otimes_L d \in T \otimes_L \mathcal{D} \),
\[
\sum_i \lambda_i(\tilde{h}_i(t \otimes d)) = \sum_i t(h_i(d))_0[0] \tilde{h}_i(t(h_i(d))_0[1])(- \otimes t(h_i(d))_1)
\]
\[
= \sum_i t(h_i(d))(0) \tilde{h}_i(h_i(d))(1)(- \otimes d(2))
\]
\[
= \sum_i t((h_i \circ \tilde{h}_i)(d(1)) \otimes d(2)) = t\epsilon_D(d(1)) \otimes d(2) = t \otimes d,
\]
where the second equality follows by \( \mathcal{C} \)- and \( \mathcal{D} \)-colinearity of \( t \in T, \mathcal{D} \)-colinearity of \( h_i \), and right \( \mathcal{A} \)-linearity of \( t \in T \). By Lemma 3.5 we conclude that \( T \otimes_L \mathcal{D} \) is a direct summand of \( \Sigma^z \), where \( z \) is the cardinality of the index set \( \{i\} \).

Finally, we show that if \( \Sigma \) is a Galois \( \mathcal{C} \)-comodule and \( T \otimes_L \mathcal{D} \) is a direct summand of \( \Sigma^z \), then \( \diamond \) is surjective. Let \( \{\lambda_i\} \subset \tau\text{Hom}^P(T \otimes_L \mathcal{D}, \Sigma) \) and \( \{\lambda_i\} \subset \tau\text{Hom}^P(\Sigma, T \otimes_L \mathcal{D}) \) be sets of morphisms, such that \( \sum_i \lambda_i \circ \tilde{h}_i = T \otimes_L \mathcal{D} \). Repeating the arguments in part (1) (cf. (3.20) and (3.21)), we define maps \( h_i \in \tau\text{Hom}^P(\mathcal{D}, \Sigma) \) and \( \tilde{h}_i \in \tilde{Q} \) as
\[
h_i : = \tilde{\lambda}_i(1_T \otimes -) \quad \text{and} \quad \tilde{h}_i : = [\Sigma^* \otimes (T \otimes \epsilon_D) \circ \lambda_i \circ \text{can}_A^{-1}].
\]

For any \( x \in \Sigma \), the association \( a \mapsto xa \) defines a right \( \mathcal{A} \)-module map \( A \to \Sigma \). By naturality of the canonical maps (3.3), for \( x \in \Sigma, c \in \mathcal{C} \), and any value of the index \( i \),
\[
(3.23) \quad x\tilde{h}_i(c)(-) = x\{[\text{End}_A(\Sigma, A) \otimes (T \otimes \epsilon_D) \circ \lambda_i \circ \text{can}_A^{-1}(c)](-)
\]
\[
= [\text{End}_A(\Sigma) \otimes (T \otimes \epsilon_D) \circ \lambda_i \circ \text{can}_A^{-1}(x \otimes c)].
\]

By (1.3), \( \text{can}_\Sigma(\Sigma \otimes_T x) = x[0] \otimes_A x[1] \), for \( x \in \Sigma \). Hence (3.23) implies the following equality of right \( \mathcal{A} \)-linear endomorphisms of \( \Sigma \).
\[
x[0] \tilde{h}_i(x[1])(-) = (T \otimes \epsilon_D) \circ \lambda_i(x),
\]
for all \( x \in \Sigma \) and any value of the index \( i \). Then it follows that, for \( d \in \mathcal{D} \),
\[
\sum_i (h_i \circ \tilde{h}_i)(d) = \sum_i h_i(d)[0] \tilde{h}_i(h_i(d))[1](-) = \sum_i (T \otimes \epsilon_D)(\lambda_i(h_i(d)))
\]
\[
= (T \otimes \epsilon_D)(\sum_i (\lambda_i \circ \tilde{\lambda}_i)(1_T \otimes d)) = \epsilon_D(d)1_T.
\]

This proves that \( \sum_i h_i \circ \tilde{h}_i \) is the unit element of the algebra \( L\text{Hom}_L(\mathcal{D}, T) \), and thus the surjectivity of the \( (L\text{Hom}_L(\mathcal{D}, T)) \)-bilinear map \( \diamond \).

**Remark 3.7.** It follows by the proof of Theorem 3.6 that the finite number \( s \) in the theorem can be chosen equal to the cardinality of the sets \( \{j_i\} \subset L\text{Hom}^P(\mathcal{D}, \Sigma) \) and \( \{j_i\} \subset \tilde{Q} \), such that \( \sum_i j_i \cdot j_i = \mathcal{C} \). Similarly, the number \( z \) can be chosen equal to the cardinality of the sets \( \{h_i\} \subset L\text{Hom}^P(\Sigma, T \otimes_L \mathcal{D}) \) and \( \{\tilde{h}_i\} \subset \tilde{Q} \), such that \( \sum_i h_i \circ \tilde{h}_i = \epsilon_D(-)1_T \).

**Proposition 3.8.** Let the \( L \)-coring \( \mathcal{D} \) be a right extension of the \( A \)-coring \( \mathcal{C} \) and \( \Sigma \in L\mathcal{M}^C \). Consider the associated Morita context \( \tilde{\mathcal{M}}(\Sigma) \) in (3.7). If the connecting map \( \circ \) in (3.14) is surjective, then \( \Sigma \) is a \( \mathcal{D} \)-equivariantly \( L \)-relative projective left module of the algebra \( T = \text{End}^C(\Sigma) \), i.e. the left action
\[
(3.24) \quad T \otimes_L \Sigma \to \Sigma, \quad t \otimes x \mapsto t(x)
\]
is a coretraction in $\mathcal{T}\mathcal{M}^D$.

Proof. A retraction of the map (3.24) can be constructed in terms of the elements $\{\tilde{j}_\ell\} \subset \tilde{\mathcal{Q}}$ and $\{j_\ell\} \subset \mathbb{L}\text{Hom}(\mathcal{D}, \Sigma)$, satisfying $\sum_\ell \tilde{j}_\ell \cdot j_\ell = \mathcal{C}$, as

$$\sigma : \Sigma \rightarrow T \otimes \Sigma, \quad x \mapsto \sum_\ell x^{[0]} \tilde{j}_\ell(x^{[1]}[0])(- \otimes j_\ell(x^{[1]}[1])) \equiv \sum_\ell x^{[0]} \tilde{j}_\ell(x^{[1]}[1])(- \otimes j_\ell(x^{[1]})).$$

It is a well defined map by Lemma 2.1 (2). Being a composite of right $\mathcal{D}$-colinear maps, it is $\mathcal{D}$-colinear. Its $T$-linearity follows by the fact that any $t \in T$ is $\mathcal{C}$-colinear and hence, in particular $\mathcal{D}$-colinear and $A$-linear. That $\sigma$ is a retraction of the map (3.24) is a direct consequence of Lemma 3.3 (2). □

4. Weak and strong structure theorems

As an $\mathbb{L}$-$\mathcal{C}$ bicomodule $\Sigma$, for a right coring extension $(\mathcal{D} : \mathbb{L})$ of $(\mathcal{C} : A)$, is in particular a right $\mathcal{C}$-comodule, it determines an adjunction of functors

$$(4.1) \quad - \otimes_T \Sigma : \mathcal{M}_T \rightarrow \mathcal{M}^\mathcal{C},$$

from the category of right modules for the algebra $T = \text{End}^\mathcal{C}(\Sigma)$ to the category of right comodules for the coring $\mathcal{C}$, as in the bottom row of diagram (1.2), and

$$(4.2) \quad \text{Hom}^\mathcal{C}(\Sigma, -) : \mathcal{M}^\mathcal{C} \rightarrow \mathcal{M}_T.$$

In the present section we study the ‘descent theory’ of coring extensions, i.e. investigate in what situations the functor (4.2) is fully faithful or an equivalence with inverse (4.1).

Theorem 4.1 (Weak Structure Theorem). Let the $\mathbb{L}$-coring $\mathcal{D}$ be a right extension of the $\mathbb{L}$-coring $\mathcal{C}$. Take $\Sigma \in \mathbb{L}\mathcal{M}^\mathcal{C}$ and consider the associated Morita context $\widehat{\mathbb{M}}(\Sigma)$ in (3.7). If the map $\bullet$ in (3.14) is surjective then the functor (4.2) is fully faithful.

Proof. The property, that the functor (4.2) is fully faithful, is equivalent to the bijec-
tivity of the counit of the adjunction,

$$(4.3) \quad \varepsilon_M : \text{Hom}^\mathcal{C}(\Sigma, M) \otimes_T \Sigma \rightarrow M, \quad \varphi_M \otimes x \mapsto \varphi_M(x),$$

for $T = \text{End}^\mathcal{C}(\Sigma)$ and any right $\mathcal{C}$-comodule $M$. Note that the restriction of the map $(M \otimes_A \varepsilon_C) \circ \text{can}_M$ to $\text{Hom}^\mathcal{C}(\Sigma, M) \otimes_T \Sigma$ is equal to $\varepsilon_M$. Furthermore, by the right $\mathcal{C}$-colinearity of $\varepsilon_M$, we have $\varrho^M \circ \varepsilon_M = (\varepsilon_M \otimes_A \mathcal{C}) \circ (\text{Hom}^\mathcal{C}(\Sigma, M) \otimes_T \varrho^\Sigma)$, which equals the restriction of $\text{can}_M$. By Theorem 3.6 (1), the surjectivity of $\bullet$ implies that $\Sigma$ is a Galois $\mathcal{C}$-comodule. Taking the explicit form of $\text{can}^{-1}$ in the proof of Theorem 3.6 into account, we have

$$\text{can}_M^{-1} \circ \varrho^M(m) = \sum_\ell m^{[0]}[0] \tilde{j}_\ell(m^{[1]}[0])(- \otimes j_\ell(m^{[1]}[1])), \quad \text{for } m \in M,$$

which is an element of $\text{Hom}^\mathcal{C}(\Sigma, M) \otimes_T \Sigma$, by Lemma 2.1 (2). In light of these observations, the inverse of $\varepsilon_M$ can be constructed as

$$\text{can}_M^{-1} \circ \varrho^M : M \rightarrow \text{Hom}^\mathcal{C}(\Sigma, M) \otimes_T \Sigma.$$

□
Corollary 4.2. Let the $L$-coring $\mathcal{D}$ be a right extension of the $A$-coring $\mathcal{C}$ and $\Sigma$ an $L$-$\mathcal{C}$ bicomodule. Consider the Morita contexts $\bar{\mathcal{M}}(\Sigma)$, associated to $\Sigma$ in (3.7), and $\bar{\mathcal{M}}(\Sigma)$, associated to $\Sigma$ as a $\mathcal{C}$-comodule in (2.4). Suppose that the map $\triangleleft$, given in (3.14), is surjective. Then the connecting map $\triangledown$, given in (2.5), is surjective if and only if $\mathcal{C}$ is a finitely generated and projective left $A$-module.

Proof. If the connecting map $\triangledown$ in (2.5) is surjective then $\mathcal{C}$ is a finitely generated projective left $A$-module by Lemma 2.5 (1). Conversely, if $\mathcal{C}$ is a finitely generated projective left $A$-module then the connecting map (2.5) is equal to the counit (4.3) for the right $\mathcal{C}$-comodule $^*\mathcal{C}$ (cf. Remark 2.2). Then it is an isomorphism by Theorem 4.1.

Recall from Theorem 2.6 that a sufficient condition for the functor (4.1) to be fully faithful is the surjectivity of the map (2.6). Motivated by this result, in what follows we look for conditions under which the map (2.6) is surjective.

Proposition 4.3. Let the $L$-coring $\mathcal{D}$ be a right extension of the $A$-coring $\mathcal{C}$ and take $\Sigma \in L\mathcal{M}^\mathcal{C}$. Consider the Morita contexts $\bar{\mathcal{M}}(\Sigma)$, associated to the $L$-$\mathcal{C}$ bicomodule $\Sigma$ in (3.7), and $\bar{\mathcal{M}}(\Sigma)$, associated to $\Sigma$ as a $\mathcal{C}$-comodule in (2.4). In particular, denote $T = \text{End}^\mathcal{C}(\Sigma)$. If the connecting map $\circlearrowleft$, given in (3.15), is surjective and there exist elements $\{v_j\} \subset L\text{Hom}_L(\mathcal{D}, T)$ and $\{d_j\} \subset \mathcal{D}$, such that $\sum_j v_j(d_j) = 1_T$, then also the connecting map $\triangledown$, given in (2.6), is surjective (hence $\Sigma$ is a finitely generated and projective right $A$-module by Lemma 2.5 (2)).

Proof. By Lemma 2.1 (1), $\tilde{Q}$ can be viewed as a $k$-submodule of $Q \cong Q'$ in (2.2). Hence (identifying $q \in \tilde{Q}$ with the corresponding element of $Q$), it follows by the explicit forms of the maps $\circlearrowleft$ and $\triangledown$ that $(p \circlearrowleft q)(d) = p(d) \circlearrowleft q$, for $p \in L\text{Hom}_L(\mathcal{D}, \Sigma)$, $q \in \tilde{Q}$ and $d \in \mathcal{D}$. Let $\{h_i\} \subset L\text{Hom}_L(\mathcal{D}, \Sigma)$ and $\{\tilde{h}_i\} \subset Q$ be sets of morphisms such that $\sum_i h_i \circlearrowleft \tilde{h}_i = \epsilon_\mathcal{D}(-)1_T$. Then

$$1_T = \sum_j v_j(d_j) = \sum_j v_j(d_{j(1)})\epsilon_\mathcal{D}(d_{j(2)}) = \sum_j v_j(d_{j(1)})\left(\sum_i (h_i \circlearrowleft \tilde{h}_i)(d_{j(2)})\right)$$

$$= \sum_j v_j(d_{j(1)})\left(\sum_i (h_i(d_{j(2)}) \circlearrowleft \tilde{h}_i)\right) = \sum_i \left(\sum_j v_j h_i(d_{j(2)})\right) \circlearrowleft \tilde{h}_i$$

where the penultimate equality follows by the left $T$-linearity of $\triangledown$ and the last one follows by (3.10). Since $\triangledown$ is $T$-$T$ bilinear, this proves its surjectivity.

By standard Morita theory, Proposition 4.3 implies the following.

Corollary 4.4. Under the assumptions (and using the notations) of Proposition 4.3, $\Sigma$ is a generator of left $T$-modules. That is, $T$ is a direct summand of a direct sum $\Sigma^*$, as a left $T$-module, where $z$ is the cardinality of the sets $\{h_i\} \subset L\text{Hom}_L(\mathcal{D}, \Sigma)$ and $\{\tilde{h}_i\} \subset Q$, such that $\sum_i h_i \circlearrowleft \tilde{h}_i = \epsilon_\mathcal{D}(-)1_T$.

Remark 4.5. The assumption in Proposition 4.3 about the existence of elements $\{v_j\} \subset L\text{Hom}_L(\mathcal{D}, T)$ and $\{d_j\} \subset \mathcal{D}$, such that $\sum_j v_j(d_j) = 1_T$, holds in various situations, studied in connection with cleft entwining structures in [19] Theorem 4.5] and [1] Theorems 4.9 and 4.10].
(1) If the counit $\epsilon_D$ of $\mathcal{D}$ is surjective, then there exists $d \in \mathcal{D}$ such that $\epsilon_D(d) = 1_L$. Putting $v : \mathcal{D} \to T$, $d' \mapsto \epsilon_D(d')1_T$, we have $v(d) = 1_T$.

(2) If $\mathcal{D}$ contains a grouplike element then it is mapped by $\epsilon_D$ to $1_L$, by definition. Hence $\epsilon_D$ is surjective, being $L$-$L$ bilinear, so the considerations in part (1) apply.

(3) If $\mathcal{D}$ is faithfully flat as a left, or as a right $L$-module then $\epsilon_D$ is surjective, since $\epsilon_D \otimes_L \mathcal{D}$ and $\mathcal{D} \otimes \epsilon_D$ are epimorphisms, split by the coproduct $\Delta_{\mathcal{D}}$. Hence this is an example of the situation in part (1) as well.

**Theorem 4.6** (Strong Structure Theorem). Let the $L$-coing $\mathcal{D}$ be a right extension of the $A$-coing $\mathcal{C}$ and $\Sigma$ an $L$-$\mathcal{C}$ bicomodule. Denote $T : = \text{End}^C(\Sigma)$. If the associated Morita context \((3.7)\) is strict and there exist elements $\{v_i\} \subset \mathcal{L}\text{Hom}_L(\mathcal{D}, T)$ and $\{d_j\} \subset \mathcal{D}$, such that $\sum_j v_j(d_j) = 1_T$, then the functors \((4.1)\) and \((4.2)\) are inverse equivalences.

**Proof.** This is an immediate consequence of Theorem 4.1, Theorem 2.6 and Proposition 4.3. \(\square\)

Note that under the hypothesis of Theorem 4.6 $\Sigma$ is a finitely generated and projective right $A$-module (cf. Lemma 2.5 (2)), hence a Galois comodule in the sense of \(23\) (cf. Theorem 3.6). On the other hand, under the assumptions in Theorem 4.6 $\Sigma$ is not necessarily flat as a left $T$-module (equivalently, $\mathcal{C}$ is not necessarily flat as a left $A$-module). This way Theorem 4.6 covers cases which are not treated by the Galois Comodule Structure Theorem \[15\] 18.27]. This will be clear by the observation in Section 6 that the Fundamental Theorem of Hopf modules (for arbitrary Hopf algebras or Hopf algebroids) is a particular instance of Theorem 4.6.

5. **Cleft bicomodules**

The results in Section 3 and Section 4 allow for an application to the main case of interest, when there exist ‘invertible’ elements in the Morita context, associated to a bicomodule of a coring extension, in the following sense. Let $\mathcal{D}$ be an $L$-coing which is a right extension of an $A$-coing $\mathcal{C}$. For an $L$-$\mathcal{C}$ bicomodule $\Sigma$ consider the associated Morita context $\tilde{M}(\Sigma) = (\mathcal{L}\text{Hom}_L(\mathcal{D}, T), \mathcal{C}\text{End}^D(\mathcal{C})^\text{op}, \mathcal{L}\text{Hom}^D(\mathcal{D}, \Sigma), \tilde{Q}, \bullet, \diamond) \text{ in } (3.7)$, where $T = \text{End}^C(\Sigma)$ and $\tilde{Q}$ is the $k$-module \((3.2)\).

**Definition 5.1.** An object $\Sigma$ of $\mathcal{L}\mathcal{M}^C$ is called a weak cleft bicomodule for the right coring extension $(\mathcal{D} : L)$ of $(\mathcal{C} : A)$ provided there exist elements $j \in \mathcal{L}\text{Hom}^D(\mathcal{D}, \Sigma)$ and $\tilde{j} \in \tilde{Q}$ such that $\tilde{j} \mathcal{D} j = \mathcal{C}$.

An object $\Sigma$ of $\mathcal{L}\mathcal{M}^C$ is called a cleft bicomodule for the right coring extension $(\mathcal{D} : L)$ of $(\mathcal{C} : A)$ provided there exist elements $j \in \mathcal{L}\text{Hom}^D(\mathcal{D}, \Sigma)$ and $\tilde{j} \in \tilde{Q}$ such that $\tilde{j} \mathcal{D} j = \mathcal{C}$ and, in addition, $j \Diamond \tilde{j} = \epsilon_D(-)1_T$.

Note that if $\Sigma \in \mathcal{L}\mathcal{M}^C$ is a (weak) cleft bicomodule for a right coring extension $(\mathcal{D} : L)$ of $(\mathcal{C} : A)$, with morphisms $j \in \mathcal{L}\text{Hom}^D(\mathcal{D}, \Sigma)$ and $\tilde{j} \in \tilde{Q}$ as in Definition 5.1, then, by Proposition 3.1 the natural transformation $\alpha_\mathcal{D}(\tilde{j})$ in Proposition 3.1 is a (left) inverse of $\alpha_\mathcal{D}(j)$ there.

In standard Hopf Galois theory, cleft extensions are Galois extensions with additional ‘normal basis property’. In order to derive a similar result for coring extensions, we impose the following definition.
Definition 5.2. An $L$-$C$ bicomodule $\Sigma$ for a right coring extension $(D : L)$ of $(C : A)$ is said to obey the weak normal basis property if it is isomorphic to a direct summand of $T \otimes_L D$, as a $T$-$D$ bicomodule, for $T = \text{End}_C(\Sigma)$.

An $L$-$C$ bicomodule $\Sigma$ for a right coring extension $(D : L)$ of $(C : A)$ is said to obey the normal basis property if it is isomorphic to $T \otimes_L D$, as a $T$-$D$ bicomodule.

Note that the normal basis property of an $L$-$C$ bicomodule $\Sigma$, for a right coring extension $(D : L)$ of $(C : A)$, implies the isomorphism of the cotensor product $\Sigma \square_D W$ to $T \otimes_L W$, as a left $T$: $= \text{End}_C(\Sigma)$-module, for any left $D$-comodule $W$. This way, properties (like projectivity or freeness) of the left $L$-module $W$ are inherited by the associated left $T$-module $\Sigma \square_D W$.

It is immediately clear from the definition that the Morita context [3.7], associated to a cleft bicomodule of a coring extension, is strict. Hence the proof of Theorem 3.6 and Remark 3.7 lead to the following relation between cleft bicomodules and Galois comodules which satisfy the normal basis property.

Corollary 5.3. (1) $\Sigma \in L\mathcal{M}^C$ is a weak cleft bicomodule for the right coring extension $(D : L)$ of $(C : A)$ if and only if $\Sigma$ is a Galois $C$-comodule and satisfies the weak normal basis property.

(2) $\Sigma \in L\mathcal{M}^C$ is a cleft bicomodule for the right coring extension $(D : L)$ of $(C : A)$ if and only if $\Sigma$ is a Galois $C$-comodule and satisfies the normal basis property.

Corollary 4.4 has the following consequence.

Corollary 5.4. Let the $L$-coring $D$ be a right extension of the $A$-coring $C$ and $\Sigma \in L\mathcal{M}^C$ a cleft bicomodule. Put $T_\Sigma := \text{End}_C(\Sigma)$. If there exist elements $\{v_j\} \subset L\text{Hom}_L(D, T)$ and $\{d_j\} \subset D$ such that $\sum_j v_j(d_j) = 1_T$ then $\Sigma$ contains the left regular $T$-module as a direct summand.

Theorems 4.1 and 4.6 imply the following structure theorems.

Corollary 5.5. For the functors (4.1) and (4.2), associated to a cleft $L$-$C$ bicomodule $\Sigma$ of a right coring extension $(D : L)$ of $(C : A)$, the following assertions hold.

(1) (Weak Structure Theorem) The functor (4.2) is fully faithful.

(2) (Strong Structure Theorem) The functors (4.1) and (4.2) are inverse equivalences provided that there exist elements $\{v_j\} \subset L\text{Hom}_L(D, T)$ and $\{d_j\} \subset D$, such that $\sum_j v_j(d_j) = 1_T$, where the notation $T = \text{End}_C(\Sigma)$ is used.

6. Examples

6.1. Cleft entwining structures. An entwining structure [14] consists of a $k$-algebra $A$, a $k$-coalgebra $D$ and a $k$-linear map $\psi : D \otimes_k A \to A \otimes_k D$, satisfying

(6.1) $\psi \circ (D \otimes_k \mu_A) = (\mu_A \otimes_k D) \circ (A \otimes_k \psi) \circ (\psi \otimes_k A)$

(6.2) $\psi \circ (D \otimes_k 1_A) = 1_A \otimes_k D$

(6.3) $(A \otimes_k \Delta_D) \circ \psi = (\psi \otimes_k D) \circ (D \otimes_k \psi) \circ (\Delta_D \otimes_k A)$

(6.4) $(A \otimes_k \epsilon_D) \circ \psi = \epsilon_D \otimes_k A$.

The index notation $\psi(d \otimes_k a) = a \psi \otimes_k d^\psi$ will be used, where implicit summation is understood. To an entwining structure $(A, D, \psi)$ one can associate an $A$-coring $C := A \otimes_k D$ as follows ([12, Proposition 2.2]). The left $A$-module structure is given.
by left multiplication in the first tensorand and the right $A$-module structure is given by $(a \otimes_k d)a' = aa'_\psi \otimes_k d^\psi$, for $d \in D$ and $a, a' \in A$. The coproduct is
\[
\Delta_C : A \otimes C \rightarrow C \otimes C \simeq A \otimes C \otimes C,
\]
\[
a \otimes d \mapsto (a \otimes d_{(1)}) \otimes (1_A \otimes d_{(2)}) \simeq a \otimes d_{(1)} \otimes d_{(2)},
\]
and the counit is $\epsilon_C : A \otimes_k \epsilon_D$. Clearly, $C$ is a $C$-$D$ bicomodule with the left regular $C$-coaction $\Delta_C$ and right $D$-coaction $\tau_C : = A \otimes_k \delta_D$. That is, $D$ is a right extension of $C$. This implies that any right $C$-comodule possesses a right $D$-comodule structure. What is more, right $C$ comodules (also called entwined modules) are those right $D$-comodules $M$ that are right $A$-modules as well and the compatibility condition
\[
(ma)_{[0]} \otimes (ma)_{[1]} = m_{[0]}a_\psi \otimes m_{[1]}\psi
\]
holds, for any $m \in M$ and $a \in A$.

An entwining structure $(A, D, \psi)$ has been termed cleft in [1] Definition 4.6 if $A$ (with the right regular $A$-module structure) is an entwined module and there exists a convolution invertible right $D$-colinear map $\lambda : D \rightarrow A$. (In the paper [19] a cleft entwining structure is meant in the more restrictive sense that in addition $D$ possesses a grouplike element $x$ and the right $D$-coaction in $A$ is of the form $a \mapsto \psi(x \otimes_k a)$. Note that in this case $\text{End}^D(A) \simeq \text{Hom}^D(k, A)$, i.e. the $C$, and $D$-coinvariants of $A$ coincide.)

**Proposition 6.1.** An entwining structure $(A, D, \psi)$ is cleft if and only if $A$ (with the right regular $A$-module structure) is a cleft bicomodule for the coring extension $D$ of $C : = A \otimes_k D$.

**Proof.** Let us assume first that $(A, D, \psi)$ is a cleft entwining structure. In this case $A$ is an entwined module, i.e. a right $C$-comodule, by assumption. Let $\lambda : D \rightarrow A$ be a right $D$-colinear map, with convolution inverse $\tilde{\lambda}$. Put $\tilde{\lambda} : = \lambda$ and $\tilde{\lambda} : C \rightarrow A, a \otimes_k d \mapsto a\tilde{\lambda}(d)$.

We need to prove that $\tilde{\lambda}$ is an element of the appropriate bimodule (3.2) in the Morita context $\tilde{M}(A)$, associated to $A$ as in (3.7), that is, of
\[
\tilde{Q} \simeq \{q \in \text{AHom}(C, A) | \forall d \in D, a \in A : \psi(d_{(1)} \otimes q(1_A \otimes d_{(2)})a) = q(1_A \otimes d)a_{[0]} \otimes a_{[1]} \},
\]
(which is equal to $\text{cHom}(C, A)$, cf. Remark 2.3). Note that, by [1] Lemma 4.7 1], the identity
\[
\tilde{\lambda}(d)_{[0]} \otimes 1_A_{[1]} = \tilde{\lambda}(d_{(2)})\psi \otimes d_{(1)}\psi
\]
holds true, for any $d \in D$. Using the assumption that $A$ is an entwined module (in the second equality), (6.6) (in the third one), and property (6.1) of entwining structures (in the fourth one), one checks that, for $d \in D$ and $a \in A,$
\[
\tilde{\lambda}(d_{(1)} \otimes \tilde{\lambda}(d_{(2)})a) = \psi(d_{(1)} \otimes \tilde{\lambda}(d_{(2)})a),
\]
that is, $\tilde{\lambda} \in \tilde{Q}$. By the assumption that $\tilde{\lambda}$ is left convolution inverse of $\lambda$, for $a \in A$ and $d \in D,$
\[
(\tilde{\lambda} \bullet j)(a \otimes d) = a\tilde{\lambda}(d_{(1)})\lambda(d_{(2)})_{[0]} \otimes \lambda(d_{(2)})_{[1]} = a\tilde{\lambda}(d_{(1)})\lambda(d_{(2)}) \otimes d_{(3)} = a\epsilon_D(d_{(1)}) \otimes d_{(2)} = a \otimes d,
\]
where the second equality follows by the colinearity of \( \lambda \). Similarly, since \( \tilde{\lambda} \) is also right convolution inverse of \( \lambda \), for \( d \in D \),

\[
(j \circ \tilde{j})(d) = \lambda(d)|_0 \tilde{\lambda}(\lambda(d)|_1) = \lambda(d|_1)\tilde{\lambda}(d|_2) = \epsilon_D(d)1_A.
\]

This proves that \( A \) is a cleft bicomodule, as stated.

Conversely, assume that \( A \) is a cleft bicomodule for the coring extension \( D \) of \( C \). Then it is, in particular, an entwined module. Let \( j \in \text{Hom}^D(D, A) \) and \( \tilde{j} \in \tilde{Q} \) be elements of the bimodules in the Morita context \( \tilde{M}(A) \), associated to \( A \) as in (3.7), such that \( j \cdot \tilde{j} = C \) and \( j \circ \tilde{j} = 1_A \epsilon_D(\cdot) \). Then \( \lambda: = j: D \to A \) is right \( D \)-colinear and \( \tilde{\lambda}: d \mapsto \tilde{j}(1_A \otimes_k d) \) is its convolution inverse.

\[\Box\]

6.2. Cleft extensions of algebras by a coalgebra. Let \( D \) be a coalgebra over \( k \) and \( A \) a \( k \)-algebra and a right \( D \)-comodule. In \( \square \) \( A \) has been termed a \( D \)-cleft extension of the subalgebra

\[ T: = \{ t \in A \mid \forall a \in A \quad (ta)|_0 \otimes (ta)|_1 = ta|_0 \otimes_{} a|_1 \}, \]

if it is a \( D \)-Galois extension, i.e. the canonical map

\[
(6.9) \quad A \otimes_{} T \to A \otimes_{} D, \quad a \otimes_{} a' \mapsto a a'|_0 \otimes_{} a'|_1
\]

is bijective, and there exists a convolution invertible right \( D \)-colinear map \( \lambda: D \to A \).

Recall that for any \( D \)-Galois extension \( A \) of \( T \) there exists a (unique) entwining structure \( (A, D, \psi) \) such that \( A \) is an entwined module (cf. \( \square \) 34.6)). On the other hand, the canonical map (6.9) is bijective for any cleft entwining structure \( (A, D, \psi) \) by \( \square \) Proposition 4.8 1]. This means that cleft extensions of algebras by a coalgebra are in one-to-one correspondence with cleft entwining structures. Combining this observation with Proposition 6.1, we conclude that \( A \) is a \( D \)-cleft extension of \( T \) if and only if \( A \) is a cleft bicomodule for the coring extension \( D \) of \( C: = A \otimes_k D \).

6.3. Cleft extensions of algebras by a Hopf algebra. Let \( D \) be a Hopf algebra over \( k \) and \( A \) a right comodule algebra. The algebra \( A \) and the coalgebra underlying \( D \) are entwined by the map

\[ \psi : D \otimes_{} k A \to A \otimes_{} D, \quad d \otimes_{} a \mapsto a|_0 \otimes_{} da|_1. \]

Since \( 1_D \) is a grouplike element in \( D \), \( 1_A \otimes_{} k 1_D \) is a grouplike element in the \( A \)-coring \( C: = A \otimes_k D \), associated to the entwining structure \( (A, D, \psi) \). Hence \( A \) is an entwined module.

\( A \) is called a \( D \)-cleft extension of its \( D \)-coinvariant subalgebra if and only if there exists a convolution invertible right \( D \)-colinear map \( \lambda: D \to A \) (see e.g. \( \square \) Definition 7.2.1]), i.e. if and only if \( (A, D, \psi) \) is a cleft entwining structure. (Note that this way a cleft extension of algebras by a Hopf algebra is a cleft extension by the underlying coalgebra.) By Proposition 6.1 this is equivalent to \( A \) being a cleft bicomodule for the coring extension \( D \) of \( C: = A \otimes_k D \).

6.4. Cleft weak entwining structures. A weak entwining structure \( \square \) consists of a \( k \)-algebra \( A \), a \( k \)-coalgebra \( D \) and a \( k \)-linear map \( \psi : D \otimes_{} k A \to A \otimes_{} k D \), such that the compatibility conditions (6.1) and (6.3) hold true, while (6.2) and (6.4) are replaced by

\[
(6.10) \quad \psi \circ (D \otimes_{} 1_A) = (e \otimes_{} D) \circ \Delta_D, \quad \text{and}
\]

\[
(6.11) \quad (A \otimes_{} \epsilon_D) \circ \psi = \mu_A \circ (e \otimes_{} A),
\]
respectively, where $e_0 : = (A \otimes_k \epsilon_D) \circ \psi \circ (D \otimes_k 1_A) : D \to A$.

To a weak entwining structure $(A, D, \psi)$ one can associate an $A$-coring $C_0 = \{ a_1 A_\psi \otimes_k d^\psi \}_{a_1 \in A, d \in D}$ (cf. [12] Proposition 2.3 or [15], 37.4]). The left $A$-module structure is given by left multiplication in the first tensorand, and the right $A$-module structure is given by $(a_1 A_\psi \otimes_k d^\psi) a' = a a'_\psi \otimes_k d^\psi$. The coproduct is given by the restriction of $A \otimes_k \Delta_D$, i.e. by

$$\Delta_C : C \to C \otimes C,$$

$$a_1 A_\psi \otimes_k d^\psi \mapsto (a_1 A_\psi \otimes_k d^\psi_{(1)}) \otimes (1_A \otimes_k d^\psi_{(2)}) = (a_1 A_\psi \otimes_k d^\psi) \otimes (1_A \otimes_k d^\psi)'.$$

The counit is given by the restriction of $A \otimes_k \epsilon_D$, i.e. by

$$\epsilon_C : C \to A, \quad a_1 A_\psi \otimes_k d^\psi \mapsto a_1 A_\psi \epsilon_D(d^\psi) = ae(d).$$

$C$ is a $C$-$D$ bicomodule with the left regular $C$-coaction $\Delta_C$ and right $D$-coaction, given by the restriction of $A \otimes_k \Delta_D$, i.e.

$$\tau_C : C \to C \otimes D, \quad a_1 A_\psi \otimes_k d^\psi \mapsto (a_1 A_\psi \otimes_k d^\psi_{(1)}) \otimes d^\psi_{(2)} = (a_1 A_\psi \otimes_k d^\psi_{(1)}) \otimes d^\psi_{(2)}.$$

where the equality of the two forms of $\tau_C$ follows by (6.10) and the coassociativity of $\Delta_D$. That is, $D$ is a right extension of $C$. Right $C$-comodules are called weak entwined modules and they can be characterised as right $D$-comodules $M$, that are right $A$-modules as well such that the compatibility condition (6.5) holds true.

By [2], Definition 1.9, a weak entwining structure $(A, D, \psi)$ is cleft if $A$ (with the right regular $A$-module structure) is a weak entwined module and there exists a right $D$-colinear map $\lambda : D \to A$ and a $k$-linear map $\bar{\lambda} : D \to A$, satisfying (6.6) and

$$1_{A_\psi} \bar{\lambda}(d^\psi) = \bar{\lambda}(d) \quad \text{and} \quad \bar{\lambda}(d_{(1)})\lambda(d_{(2)}) = e(d), \quad \text{for } d \in D.$$

(The first condition in (6.13) can be read as a convenient normalisation. Indeed, if there exists $\bar{\lambda} \in \text{Hom}_k(D, A)$, satisfying (6.6) and the second condition in (6.13), then it can be replaced by the (non-zero) map $d \mapsto 1_{A_\psi} \bar{\lambda}(d^\psi)$.)

**Proposition 6.2.** A weak entwining structure $(A, D, \psi)$ is cleft if and only if $A$ (with the right regular $A$-module structure) is weak left bicomodule for the right coring extension $D$ of the $A$-coring $C$, associated to the weak entwining structure $(A, D, \psi)$.

**Proof.** Let us assume first that $(A, D, \psi)$ is a left cleft entwining structure. We construct elements $j$ and $\bar{j}$ in the bimodules of the Morita context (3.7), associated to $A$, such that $\bar{j} \ast j = C$. Put $j : = \lambda$ and

$$\bar{j} : C \to A, \quad a_1 A_\psi \otimes_k d^\psi \mapsto a_1 A_\psi \bar{\lambda}(d^\psi) = a \bar{\lambda}(d),$$

where $\lambda : D \to A$ is a right $D$-colinear map and $\bar{\lambda} : D \to A$ is a $k$-linear map, satisfying (6.6) and (6.13). Analogously to (6.7) and (6.8), assumption (6.6) implies that $\bar{j}$ is left $C$-colinear, i.e. an element of

$$\bar{Q} \simeq \{ q \in A \text{Hom}(C, A) \mid \forall d \in D, a \in A \}
\psi(d_{(1)} \otimes a) = q(1_A \otimes d_{(2)}) a [0] \otimes a [1] \},$$

and (6.13) implies $\bar{j} \circ j = C$.

Conversely, assume that $A$ is a weak cleft bicomodule, i.e. there exist elements $j \in \text{Hom}_D(D, A)$ and $\bar{j} \in \bar{Q}$ in the bimodules of the Morita context $\bar{M}(A)$, associated to $A$ in (3.7), such that $\bar{j} \ast j = C$. The map $\lambda : = j : D \to A$ is right $D$-colinear.
Together with the map $\lambda : d \mapsto \overline{j}(1_{A\psi} \otimes_k d^\psi)$, for $d \in \mathcal{D}$, they satisfy (6.13) and (6.6). Indeed, the first condition in (6.13) follows by the left $A$-linearity of $\overline{j}$ and (6.1). The second one follows by $\overline{j} \circ j = C$ as, for $d \in \mathcal{D}$,

$$
e(d) = 1_{A\psi} e_\mathcal{D}(d^\psi) = (A \otimes e_\mathcal{D})((\overline{j} \circ j)(1_{A\psi} \otimes_k d^\psi))$$

$$= \overline{j}(1_{A\psi} \otimes_k d(1)) j(d(2)) = \overline{\lambda}(d(1)) \lambda(d(2)),$$

where the third equality follows by the forms (3.14) of the map $\bullet$ and (6.12) of the $\mathcal{D}$-coaction in $\mathcal{C}$. Condition (6.6) is easily seen to follow by the assumption that $\overline{j}$ is an element of the bimodule $Q$.

Note that the first condition in (6.13) and (6.6), imposed on a weak entwining structure in [13], gain an explanation by Proposition 6.2. They mean that $\overline{\lambda} \in \text{Hom}_k(\mathcal{D}, A)$ corresponds to an element of $\overline{\mathcal{Q}} \subseteq A\text{Hom}(\mathcal{C}, A)$ in Proposition 6.2 via the isomorphism

$$A\text{Hom}(\mathcal{C}, A) \cong \{ \nu \in \text{Hom}_k(\mathcal{D}, A) \mid \forall d \in \mathcal{D} \ 1_{A\psi} \nu(d^\psi) = \nu(d) \}.$$  

6.5. **Cleft extensions by partial group actions.** Extending the definition of (idempotent) partial actions of finite groups on commutative algebras in [17] and [18], Caenepeel and De Groot introduced in [17] idempotent partial actions of finite groups $G$ on arbitrary algebras $A$, as follows. An idempotent partial $G$-action on $A$ consists of a collection $\{e_\sigma\}_{\sigma \in G}$ of central idempotents in $A$ and a collection $\{\alpha_\sigma : Ae_{\sigma^{-1}} \to Ae_\sigma\}_{\sigma \in G}$ of isomorphisms of ideals, satisfying the conditions

$$A_1 = A \quad \text{and} \quad \alpha_1 = A,$$

$$\alpha_\sigma (\alpha_\tau (ae_{\tau^{-1}}) e_{\sigma^{-1}}) = \alpha_{\sigma \tau} (ae_{\tau^{-1}} e_{\sigma^{-1}}) e_{\sigma}, \quad \text{for } \sigma, \tau \in G, \ a \in A.$$  

They constructed an $A$-coring for such a partial action, $\{e_\sigma, \alpha_\sigma\}_{\sigma \in G}$ of $G$ on $A$, as a $k$-module $\mathcal{C} : = \bigoplus_{\sigma \in G} Ae_\sigma$ with $A$-$A$ bimodule structure

$$a_1(\alpha_\sigma a_2) = a_1 a_2 \alpha_\sigma (a_2 e_{\sigma^{-1}}) \nu_\sigma,$$

for $a_1, a_2, \ a \in A$, and elements $\nu_\sigma$ of $\mathcal{C}$, taking the value $e_\sigma$ in the component $\sigma$ and 0 everywhere else, for $\sigma \in G$. The coproduct and the counit are inherited from the coalgebra $k(G)$, the $k$-dual of the group algebra. Explicitly,

$$\Delta_\mathcal{C}(\alpha_\sigma) = \sum_{\tau \in G} \alpha_\sigma \otimes_k \nu_{\tau^{-1} \sigma}$$

and

$$\epsilon_\mathcal{C}(\alpha_\sigma) = a_0 \delta_{\sigma,1}, \quad \text{for } a_0 \in \mathcal{C}.$$  

Note that the coalgebra $(k$-coring) $k(G)$ is a right extension of the $A$-coring $\mathcal{C}$. That is, there exists a left $\mathcal{C}$-colinear right $k(G)$-coaction in $\mathcal{C}$,

$$\tau_\mathcal{C} : \mathcal{C} \to \mathcal{C} \otimes_k k(G), \quad a \nu_\sigma \mapsto \sum_{\tau \in G} \alpha_\sigma \otimes_k u_{\tau^{-1} \sigma},$$

where $\{u_\sigma\}_{\sigma \in G}$ is the $k$-basis for $k(G)$, dual to the basis $\{\sigma\}_{\sigma \in G}$ of the group algebra. Since $\mathcal{C}$ possesses a grouplike element,

$$\sum_{\sigma \in G} \nu_\sigma,$$

(cf. [17, Lemma 2.3]), $A$ possesses a right $\mathcal{C}$-comodule structure (and hence a right $k(G)$-comodule structure).
By a plausible definition we call A a \textit{cleft extension} of its \(G\)-invariant subalgebra \(\{ a \in A \mid \forall \sigma \in G \quad \alpha_{\sigma}(ae_{\sigma^{-1}}) = ae_{\sigma} \}\) if there exists a convolution invertible right colinear map from the right regular \(k(G)\)-comodule to \(A\).

\textbf{Proposition 6.3.} Let \(G\) be a finite group with an idempotent partial action \(\{e_{\sigma}, \alpha_{\sigma}\}_{\sigma \in G}\) on an algebra \(A\). Let \(C\) be the associated \(A\)-coring. Then \(A\) is a cleft extension of its \(G\)-invariant subalgebra if and only if \(A\) is a cleft bicomodule for the coring extension \(k(G)\) of \(C\).

\textit{Proof.} The proof is surprisingly similar to that of Proposition 6.1.

Assume first that \(A\) is a cleft bicomodule, that is, there exist elements \(\tilde{j} \in \tilde{Q}\) and \(j \in \text{Hom}^{k(G)}(k(G), A)\) in the bimodules of the Morita context \(\mathcal{M}(A)\), associated to \(A\) as in (3.7), such that \(\tilde{j} \cdot j = \mathcal{C}\) and \(j \circ \tilde{j} = 1_{Ae_{k(G)}}(-)\). Put \(\lambda := j : k(G) \to A\). It is right colinear. We claim that it is also convolution invertible.

Using the notations, introduced earlier in this section, the \(k(G)\)-coaction in \(A\), determined by the grouplike element (6.14), comes out as

\[\tau_A : A \to A \otimes k(G), \quad a \mapsto \sum_{\sigma \in G} \alpha_{\sigma}(ae_{\sigma^{-1}}) \otimes u_{\sigma}.\]

Then, since \(k(G)\) is a free \(k\)-module of finite rank, the colinearity of \(\lambda\) means that

\[(6.15) \quad \alpha_{\tau}(\lambda(u_{\sigma})e_{\tau^{-1}}) = \lambda(u_{\sigma^{-1}}), \quad \text{for } \sigma, \tau \in G.\]

Condition (6.15) implies, in particular, that \(\lambda(u_{\sigma})e_{\tau} = \lambda(u_{\sigma})\), for any \(\sigma, \tau \in G\). (Hence there exist no non-trivial \(k(G)\)-comodule maps \(k(G) \to A\) if the ideals \(\{ Ae_{\sigma}\}_{\sigma \in G}\) have no non-trivial intersection.)

Now put \(\tilde{\lambda}(u_{\sigma}) : = \tilde{j}(u_{\sigma})\), for \(\sigma \in G\). By \(\tilde{j} \cdot j = \mathcal{C}\), \(\tilde{\lambda}\) is left convolution inverse of \(\lambda\). Similarly, it follows by \(j \circ \tilde{j} = 1_{Ae_{k(G)}}(-)\) that

\[\sum_{\tau \in G} \alpha_{\tau}(\lambda(u_{\sigma})e_{\tau^{-1}}) \tilde{\lambda}(u_{\tau}) = \delta_{\sigma, 1} 1_{A} \quad \text{for } \sigma \in G.\]

Using the colinearity of \(\lambda\), i.e., the identity (6.15), we conclude that \(\tilde{\lambda}\) is also right convolution inverse of \(\lambda\).

Conversely, assume that there exists a right \(k(G)\)-comodule map \(\lambda : k(G) \to A\) with convolution inverse \(\tilde{\lambda}\). We construct elements \(\tilde{j} \in \text{Hom}^{k(G)}(k(G), A)\) and \(\tilde{j} \in \tilde{Q}\) such that \(\tilde{j} \cdot j = \mathcal{C}\) and \(j \circ \tilde{j} = 1_{Ae_{k(G)}}(-)\). Put \(j : = \lambda : k(G) \to A\). Since \(\tilde{\lambda}\) is convolution inverse of a right \(k(G)\)-comodule map \(\lambda\), its range is in the intersection of the ideals \(\{ Ae_{\sigma}\}_{\sigma \in G}\). Hence we can put

\[\tilde{j} : \mathcal{C} \to A, \quad av_{\sigma} \mapsto a\tilde{\lambda}(u_{\sigma}).\]

The conditions \(\tilde{j} \cdot j = \mathcal{C}\) and \(j \circ \tilde{j} = 1_{Ae_{k(G)}}(-)\) follow easily by the assumptions that \(\tilde{\lambda}\) is left, and right convolution inverse of \(\lambda\), respectively, and the colinearity condition (6.15). Furthermore, using that \(\tilde{\lambda}\) is left convolution inverse of \(\lambda\) (in the second equality), the colinearity condition (6.15) (in the third one) and the assumption that
" is left convolution inverse of \( \lambda \) (in the last one), we deduce that
\[
\alpha_\tau(\bar{\lambda}(u_{\tau^{-1}})ae_{\tau^{-1}}) = \sum_{\omega \in G} \delta_{\omega,\tau} \alpha_\tau(\bar{\lambda}(u_{\omega^{-1}})ae_{\tau^{-1}})
\]
\[
= \sum_{\omega,\mu \in G} \bar{\lambda}(u_{\mu})\lambda(u_{\mu^{-1}\omega^{-1}})\alpha_\tau(\bar{\lambda}(u_{\omega^{-1}})ae_{\tau^{-1}})
\]
\[
= \sum_{\omega,\mu \in G} \bar{\lambda}(u_{\mu})\alpha_\tau(\lambda(u_{\mu^{-1}\omega})e_{\tau^{-1}})\alpha_\tau(\bar{\lambda}(u_{\omega^{-1}})ae_{\tau^{-1}})
\]
\[
= \sum_{\omega,\mu \in G} \bar{\lambda}(u_{\mu})\alpha_\tau(\lambda(u_{\mu^{-1}\omega})e_{\tau^{-1}})\alpha_\tau(\bar{\lambda}(u_{\omega^{-1}})ae_{\tau^{-1}}) = \bar{\lambda}(u_{\sigma})\alpha_\tau(\bar{\lambda}(u_{\omega^{-1}})ae_{\tau^{-1}}),
\]
for \( a \in A \) and \( \sigma, \tau \in G \). Using the forms of the coproduct \( \Delta_C \) in \( C \) and the \( C \)-coaction (determined by the grouplike element \( (6.14) \)) in \( A \), it is straightforward to check that this is equivalent to the property that \( j \) is an element of the bimodule \( \tilde{Q} \), associated to \( A \) as in \( [3,2] \).

6.6. Cleft entwining structures over arbitrary base. An entwining structure over an algebra \( L \) consists of an \( L \)-ring \( A \), an \( L \)-coring \( D \) and an \( L-L \) bilinear map \( \psi : D \otimes_L A \to A \otimes_L D \), satisfying conditions \( (6.1)-(6.4) \), with the only modification that \( k \)-module tensor products are replaced by \( L \)-module tensor products. Just as in the case of commutative base rings, \( C : = A \otimes_L D \) possesses an \( A \)-coring structure (cf. \([5]\) Example 4.5) such that \( D \) is a right extension of \( C \).

Recall that, for an \( L \)-ring \( A \) and an \( L \)-coring \( D \), the set of bimodule maps \( \text{Hom}_L(D, A) \) is an algebra with the convolution product \( (fg)(d) = f(d_{(1)})g(d_{(2)}) \) and unit \( 1_A \).

In complete analogy with Proposition \( 6.4 \) one proves the following.

**Proposition 6.4.** Let \( (A, D, \psi) \) be an entwining structure over an algebra \( L \) and let \( C : = A \otimes_L D \) be the associated \( A \)-coring. A (with the right regular \( A \)-module structure) is a cleft \( L \)-C bicomodule for the coring extension \( D \) of \( C \) if and only if the following assertions hold.

(a) \( A \) (with the right regular \( A \)-module structure) is an entwined module, i.e. it is a right \( D \)-comodule such that the compatibility condition
\[
(aa')_{[0]} \otimes (aa')_{[1]} = a_{[0]}a'_{[1]} \otimes a_{[1]} \psi
\]
holds true, for all \( a, a' \in A \);
(b) The right \( D \)-coaction \( a \mapsto a_{[0]} \otimes_L a_{[1]} \) in \( A \) is left \( L \)-linear;
(c) There exists a convolution invertible morphism \( \lambda \in \text{Hom}^D(D, A) \subseteq \text{Hom}_L(D, A) \).

If conditions (a)-(c) in Proposition \( 6.4 \) hold, we call the \( L \)-entwining structure \( (A, D, \psi) \) cleft.

Note that if the coring \( D \) possesses a grouplike element \( x \), then the left \( L \)-linearity of the \( D \)-coaction \( a \mapsto \psi(x \otimes_L a) \) in \( A \) is equivalent to the element \( x \) to be central in the \( L-L \) bimodule \( D \).

6.7. Cleft extensions of algebras by a Hopf algebroid. A Hopf algebroid \( \mathcal{H} \) consists of a left bialgebroid structure \( \mathcal{H}_L \), over a base algebra \( L \), and a right bialgebroid structure \( \mathcal{H}_R \), over a base algebra \( R \), on the same total algebra \( H \), and a \( k \)-linear map \( S : H \to H \), called the antipode, relating the two bialgebroid structures \([10],[6]\). For
a Hopf algebroid \( \mathcal{H} \), we denote by \( \gamma_L \) and \( \pi_L \) (resp. \( \gamma_R \) and \( \pi_R \)) the coproduct and the counit of the constituent bialgebroid \( \mathcal{H}_L \) (resp. \( \mathcal{H}_R \)).

The category of right comodules for the right bialgebroid \( \mathcal{H}_R \) is a monoidal category such that the forgetful functor to the bimodule category \( _R \mathcal{M}_R \) is strict monoidal. Right \( \mathcal{H}_R \)-comodule algebras are defined as monoids in the category of right \( \mathcal{H}_R \)-comodules, hence they are in particular \( R \)-rings \[28\]. A right \( \mathcal{H}_R \)-comodule algebra \( A \) determines an entwining structure over \( R \) with \( R \)-ring \( A \) and \( R \)-coring \( (H, \gamma_R, \pi_R) \), underlying the bialgebroid \( \mathcal{H}_R \) (cf. \[7\] (3.17)). Hence there exists a corresponding \( A \)-coring \( \mathcal{C} := A \otimes_R H \), with coproduct \( A \otimes_R \gamma_R \), inherited from \( \mathcal{H}_R \). By the definition of a Hopf algebroid, the coproduct \( \gamma_R \) in \( \mathcal{H}_R \) is (left and right) \( \mathcal{H}_L \)-colinear, hence \( \mathcal{C} \) possesses a \( \mathcal{C} \)-\( \mathcal{H}_L \) bicomodule structure with the left regular \( \mathcal{C} \)-coaction and right \( \mathcal{H}_L \)-coaction \( A \otimes_R \gamma_L \). That is, the \( L \)-coring \( (H, \gamma_L, \pi_L) \), underlying the bialgebroid \( \mathcal{H}_L \), is a right extension of the \( A \)-coring \( \mathcal{C} = A \otimes_R H \). Note that this coring extension does not correspond to any entwining structure.

Under the additional assumption that the right \( \mathcal{H}_R \)-comodule algebra \( A \) is also an \( L \)-ring with left \( L \)-linear \( \mathcal{H}_R \)-coaction, one can associate to it a Morita context like in \[9\] Remark 3.2 (1)]. It is formulated in terms of the two convolution products, \((f, g) \mapsto \mu_A \circ (f \otimes g) \circ \gamma_L\), for \( f \in \text{Hom}_L(H, A) \) and \( g \in \text{lHom}(H, A) \) on one hand, and \((f', g') \mapsto \mu_A \circ (f' \otimes g') \circ \gamma_R\), for \( f' \in \text{Hom}_R(H, A) \) and \( g' \in \text{rHom}(H, A) \) on the other hand. These two convolution products define the convolution algebras \( \text{lHom}_L(H, A) \) and \( \text{rHom}_R(H, A) \), respectively, and also their bimodules \( \text{lHom}_R(H, A) \) and \( \text{rHom}_L(H, A) \). The connecting homomorphisms of the Morita context are defined as projections of the appropriate convolution product. Note that, under the assumption imposed on \( A \), it is in particular an \( L \)-\( \mathcal{C} \) bicomodule. The precise relation of the Morita context described above to the one associated to \( A \) as in \( \text{(3.7)} \), is formulated in the following lemma.

**Lemma 6.5.** Let \( \mathcal{H} = (\mathcal{H}_L, \mathcal{H}_R, S) \) be a Hopf algebroid and \( A \) a right \( \mathcal{H}_R \)-comodule algebra. Denote the associated \( A \)-coring \( A \otimes_R H \) by \( \mathcal{C} \). Assume that \( A \) is also an \( L \)-ring and its \( \mathcal{H}_R \)-coaction is left \( L \)-linear. Then the Morita context \( \overline{\mathcal{M}}(A) \), associated to the \( L \)-\( \mathcal{C} \) bicomodule \( A \) as in \( \text{(3.7)} \), is isomorphic to a sub-Morita context of \[16\] \( \text{(6.16)} \) \( (\text{lHom}_L(H, A) , \text{rHom}_R(H, A) , \text{lHom}_R(H, A) , \text{rHom}_L(H, A) , \bullet , \phi) \),

where the algebra and bimodule structures are given by the respective convolution product and the connecting homomorphisms \( \bullet \) and \( \phi \) are defined as projections of the convolution products.

*Proof.* The endomorphism algebra \( T = \text{End}^\mathcal{C}(A) \) can be identified with the subalgebra of \( \mathcal{C} \) (equivalently, \( \mathcal{H}_R \) -coinvariants in \( A \)), via \( T \ni t \mapsto t(1_A) \) cf. \[15\] 28.4. This injection \( T \hookrightarrow A \) of \( L \)-rings defines an injection of convolution algebras

\[ \iota_1 : \text{lHom}_L(H, T) \hookrightarrow \text{lHom}_L(H, A). \]

The algebra of left \( \mathcal{C} \)-colinear right \( \mathcal{H}_L \)-colinear endomorphisms of \( A \otimes_R H \) can be injected into the other convolution algebra \( \text{rHom}_R(H, A) \) via the map

\[ \iota_2 : \text{cEnd}^{\mathcal{H}_L}(\mathcal{C}) \hookrightarrow \text{rHom}_R(H, A), \quad u \mapsto (A \otimes \pi_R) \circ u \circ (1_A \otimes h) \]

Right \( \mathcal{H}_L \)-colinear maps are right \( R \)-linear by \[9\] Theorem 2.1], so we have an obvious injection

\[ \iota_3 : \text{lHom}^{\mathcal{H}_L}(H, A) \hookrightarrow \text{lHom}_R(H, A). \]
Finally, by standard hom-tensor relations, we have an inclusion
\[ \iota_4 : \tilde{Q} \hookrightarrow \text{AHom}_L(A \otimes_R H, A) \cong \text{RHom}_L(H, A). \]
It is left to the reader as an easy exercise to check that the four injections constructed define a morphism of Morita contexts. \( \square \)

In [8, Example 3.11] a right \( \mathcal{H}_R \)-comodule algebra \( A \) (with \( \mathcal{R} \)-ring structure \( \eta_R : \mathcal{R} \to A \)) for a Hopf algebroid \( \mathcal{H} = (\mathcal{H}_L, \mathcal{H}_R, S) \) has been called an \( \mathcal{H} \)-cleft extension of its \( \mathcal{H}_R \)-coinvariant subalgebra \( B \) if the following conditions are satisfied.

(a) \( A \) is an \( \mathcal{L} \)-ring (with unit morphism \( \eta_L : \mathcal{L} \to A \)) and \( B \) is an \( \mathcal{L} \)-subring of \( A \);
(b) There exists a left \( \mathcal{L} \)-linear right \( \mathcal{H}_R \)-colinear map \( \lambda : H \to A \) which is invertible in the Morita context \( (6.16) \), i.e. for which there exists a left \( \mathcal{L} \)-linear right \( \mathcal{L} \)-linear map \( \bar{\lambda} : H \to A \) such that
\[ \lambda \hat{\otimes} \bar{\lambda} \equiv \mu_A \circ (\lambda \otimes \bar{\lambda}) \circ \gamma_R = \eta_L \circ \pi_L \quad \text{and} \quad \bar{\lambda} \hat{\otimes} \lambda \equiv \mu_A \circ (\bar{\lambda} \otimes \lambda) \circ \gamma_L = \eta_R \circ \pi_R. \]

For more details on cleft extensions by Hopf algebroids, in particular for their characterisation as crossed products, we refer to [9]. This definition, cited from [8], can be reformulated using the terminology of the present paper as follows. The proof is a simple generalisation of the one of Proposition 6.1 using [9] Lemmas 3.6 and 3.7.

**Proposition 6.6.** Let \( \mathcal{H} = (\mathcal{H}_L, \mathcal{H}_R, S) \) be a Hopf algebroid and \( A \) a right \( \mathcal{H}_R \)-comodule algebra. Let \( \mathcal{C} \) be the associated \( \mathcal{A} \)-coring \( A \otimes_R \mathcal{H} \), with coproduct inherited from \( \mathcal{H}_R \). \( A \) is an \( \mathcal{H} \)-cleft extension of its \( \mathcal{H}_R \)-coinvariant subalgebra if and only if the following hold.

(a) \( A \) is an \( \mathcal{L} \)-ring;
(b) \( A \) (with the left \( \mathcal{L} \)-module structure in (a), the right regular \( \mathcal{A} \)-module structure and the given \( \mathcal{H}_R \)-comodule structure) is a cleft \( \mathcal{L} \)-\( \mathcal{C} \) bicomodule for the coring extension \( \mathcal{H}_L \) of \( \mathcal{C} \).

It should be emphasized that cleft extensions of algebras by a Hopf algebroid are *not* examples of the kind discussed in Section 6.6. As explained, one can associate an \( \mathcal{R} \)-entwining structure – with \( \mathcal{R} \)-ring \( A \) and \( \mathcal{R} \)-coring underlying \( \mathcal{H}_R \) – to a right \( \mathcal{H}_R \)-comodule algebra \( A \), for a Hopf algebroid \( \mathcal{H} = (\mathcal{H}_L, \mathcal{H}_R, S) \). It isn’t true, however, that this \( \mathcal{R} \)-entwining structure was cleft for an \( \mathcal{H} \)-cleft extension. The right \( \mathcal{H}_R \)-coaction in \( A \) is determined by the grouplike element \( 1_H \) (cf. last paragraph in Section 6.6), which is *not* central in the \( R \)-\( R \) bimodule \( \mathcal{H}_R \). Hence the right \( \mathcal{H}_R \)-coaction in \( A \) is *not* left \( \mathcal{R} \)-linear in the sense of a cleft \( \mathcal{R} \)-entwining structure (i.e. of Proposition 6.4 (b)).

As a particular example of an \( \mathcal{H} \)-cleft extension, consider the right regular \( \mathcal{H}_R \)-comodule algebra for a Hopf algebroid \( \mathcal{H} = (\mathcal{H}_L, \mathcal{H}_R, S) \). It is an \( \mathcal{L} \)-ring via the source map of \( \mathcal{H}_L \). The coinvariant subalgebra is the image of the base algebra \( \mathcal{R} \) under the target map of \( \mathcal{H}_R \), which coincides with the image of the base algebra \( \mathcal{L} \) under the source map of \( \mathcal{H}_L \). The (left \( \mathcal{L} \)-linear right \( \mathcal{H}_R \)-colinear) identity map \( H \to H \) possesses an inverse in the associated Morita context \( (6.16) \), the antipode. Hence \( \mathcal{R}^{\text{op}} \subseteq H \) is an \( \mathcal{H} \)-cleft extension. That is, by Proposition 6.6, \( H \) is a cleft \( \mathcal{L}-(H \otimes_R H) \) bicomodule. In light of this observation, the Fundamental Theorem of Hopf modules for a Hopf algebroid [6, Theorem 4.2] is a special instance of Corollary 5.5 (2) in the current paper (note the existence of a grouplike element \( 1_H \) in \( \mathcal{H}_L \),...
cf. Remark 4.5 (1) and (2)). This explains why the Fundamental Theorem of Hopf modules can be proven without assuming $H$ to be a faithfully flat $R$-module (unlike the Galois Coring Structure Theorem [13, 28.19]).

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References

[1] J. Abuhlail, *Morita contexts for corings and equivalences*, in: ‘Hopf algebras in non-commutative geometry and physics’. S. Caenepeel and F. Van Oystaeyen (eds.), Marcel Dekker 2005, pp. 1–29.
[2] J.N. Alonso Álvarez, J.M. Fernández Vilaboa, R. González Rodríguez and A.B. Rodríguez Raposo, *Weak $C$-cleft extensions, weak entwining structures and weak Hopf algebras*, J. Algebra 284 (2005), 679–704.
[3] M. Barr and C. Wells, *Toposes, Triples and Theories*. Available at http://www.cwru.edu/artsci/math/wells/pub/qtt.html.
[4] H. Bass, *Algebraic K-theory*. Benjamin, New York, 1968.
[5] G. Böhm, *Internalbialgebroids, entwining structures and corings*, in: ‘Algebraic structures and their representations’ J. A. de la Peña, E. Vallejo and N. Atakishiyev (eds.), AMS Contemp. Math. 376 (2005), 207–226.
[6] G. Böhm, *Integral Theory for Hopf algebroids*, Algebra Rep. Theory 4 (2005), 563–599.
[7] G. Böhm, *Galois theory for Hopf algebroids*, Ann. Univ. Ferrara - Sez. VII - Sc. Mat. 51 (2005), 233–262.
[8] G. Böhm and T. Brzeziński, *Strong connections and the relative Chern-Galois character for corings*, Int. Res. Math. Notices 2005:42 (2005), 2579–2625.
[9] G. Böhm and T. Brzeziński, *Cleft extensions of Hopf algebroids*, Preprint arXiv:math.RA/0510253.
[10] G. Böhm and K. Szlachányi, *Hopf algebroids with bijective antipodes: axioms, integrals and duals*, J. Algebra 274 (2004), 585-617.
[11] T. Brzeziński, *On modules associated to coalgebra Galois extensions*, J. Algebra 215 (1999), 290–317.
[12] T. Brzeziński, *The structure of corings. Induction functors, Maschke-type theorem, and Frobenius and Galois-type properties*, Alg. Rep. Theory 5 (2002), 389–410.
[13] T. Brzeziński, *A note on coring extensions*, Ann. Univ. Ferrara - Sez. VII - Sc. Mat. 51 (2005), 15–27.
[14] T. Brzeziński and S. Majid, *Coalgebra bundles*, Comm. Math. Phys. 191 (1998), 467–492.
[15] T. Brzeziński and R. Wisbauer, *Corings and Comodules*. Cambridge University Press, Cambridge, 2003.
[16] S. Caenepeel and E. De Groot, *Modules over weak entwining structures*, AMS Contemp. Math. 267 (2000), 31–54.
[17] S. Caenepeel and E. De Groot, *Galois corings applied to partial Galois theory*, in: ‘Proceedings of the International Conference on Mathematics and Applications, ICMA 2004’ S.L. Kalla and M.M. Chawla (eds.), Kuwait University, Kuwait 2005.
[18] S. Caenepeel, E. De Groot and J. Vercruysse, *Galois theory for comatrix corings: descent theory, Morita theory, Frobenius and separability properties*, Preprint arXiv:math.RA/0406436, to appear in Trans. Amer. Math. Soc.

[19] S. Caenepeel, J. Vercruysse and S. Wang, *Morita theory for corings and cleft entwining structures*, J. Algebra 276 (2004), 210–235.

[20] S. Chase and M.E. Sweedler, *Hopf algebras and Galois theory*, Lect. Notes in Math. 97, Springer 1969.

[21] M. Cohen, D. Fischman and S. Montgomery, *Hopf Galois extensions, smash products, and Morita equivalence*, J. Algebra 133 (1990), 351–372.

[22] Y. Doi, *Generalised smash products and Morita contexts for arbitrary Hopf algebras*, in: ‘Advances in Hopf Algebras’ J. Bergen and S. Montgomery (eds.), Marcel Dekker 1994.

[23] M. Dokuchaev and R. Exel, *Associativity of crossed products by partial actions, enveloping actions and partial representations*, Trans. Amer. Math. Soc. 357 (2005), 1931–1952.

[24] M. Dokuchaev, M. Ferrero and A. Pacques, *Partial actions and Galois theory*, Preprint 2004, to appear in J. Pure Appl. Algebra.

[25] L. El Kaoutit and J. Gómez-Torrecillas, *Comatrix corings: Galois corings, Descent Theory, and a Structure Theorem for Cosemisimple corings*, Math. Z. 244 (2003), 887–906.

[26] S. Mac Lane, *Categories for the working mathematician*. Second Edition, Springer 1998.

[27] S. Montgomery, *Hopf algebras and their actions on rings*. AMS, Providence, 1993.

[28] P. Schauenburg, *Bialgebras over noncommutative rings, and a structure theorem for Hopf bimodules*, Applied Categorical Structures 6 (1998), 193-222.

[29] R. Wisbauer, *On the category of comodules over corings*, in ‘Mathematics and mathematics education (Bethlehem, 2000)’, World Sci. Publishing, River Edge, NJ, 2002, 325–336.

[30] R. Wisbauer, *On Galois comodules*, Preprint arXiv:math.RA/0408251 v2, to appear in Comm. Algebra.