Statistical description of ideal gas at Planck scale with strong quantum gravity measurement

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A R T I C L E   I N F O

Keywords:
Generalized Uncertainty Principle
Maximal lengths
Thermodynamics of ideal gas
Logarithmic corrections

A B S T R A C T

It has been recently introduced a set of noncommutative algebra that describes the space at the Planck scale [J. Phys. A: Math. Theor. 53, 115303 (2020)]. The interesting significant result we found is that the generalized uncertainty principle induced a maximal length of quantum gravity which revealed strong quantum gravitational effects at this scale [J. Phys. A: Math. Theor. 55, 105303 (2022)]. Based on these works, we study the thermodynamic quantities within the canonical ensembles of an ideal gas made up of \(N\) indistinguishable particles at this scale. A comparison with the results obtained in the context of minimal length scenarios and in Reissner-Nordström’s black hole indicates that the maximal length in this theory induces logarithmic corrections of deformed parameter which are consequences of strong quantum gravitational effects.

1. Introduction

The Generalized Uncertainty Principle (GUP) with minimal length has emerged as a path of finding a consistent quantum formulation to the theory of gravity [3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15]. This length scale is extremely small that its experimental search lies beyond any accessible energies in the laboratory. To circumvent the high energy requirement to probe this length scale, we have recently proposed a position deformed Heisenberg algebra that induces GUP with maximal length of quantum gravity [1]. This maximal length revealed properties similar to the classical gravity and allows probing quantum gravitational effects with low energies and with high probability densities [2].

In continuation of this work [2], we review in one hand the dynamics of free particle at the frontier of the Planck scale. We show that the spectrum of this system is weakly proportional to the ordinary one of quantum mechanics without quantum gravity perturbations. The states of this particle exhibit property similar to the Gaussian states of the standard quantum mechanics which are consequence of quantum gravitational fluctuations. In the order hand, with the spectrum of this system at hand, we investigate the statistics of the canonical ensemble of an ideal gas made of this free particle at this scale. Since the quantum gravity is strongly measured at this scale, the thermodynamic quantities induce for this description logarithmic corrections \(\ln r\). This situation perfectly fits with the quantum correction of a minimal scalar field yield UV-divergent contributions to the entropy at the extremal limit of Reissner-Nordström’s black hole [16, 17]. Comparing these results with those of minimal length measurements [18, 19, 20, 21, 22, 23] show that, the effects of both type of measurements are fundamentally different. In fact, the minimal length formalism shifts quadratically the thermodynamic quantities at the order of the deformed parameter \(\beta_{GUP}\). Thus, at the extremal limit \(\beta_{GUP} \rightarrow 0\), one recovers the ordinary quantities at this scale while in this framework by tending our deformed parameter \(r\) to zero, these thermodynamic quantities diverge. We come out from these observations that the minimal length formalism induces weak quantum gravity localization while the maximal length induces strong quantum gravity localization at the Planck scale.

This paper is outlined as follows. In the next section, we review the GUP with maximal length uncertainty [1]. Section 3, is devoted to the study of a non-relativistic free particle at the Planck scale. We give the spectrum of this system by solving analytically the Schrödinger equation. In section 4, we deduce from this spectrum a statistical description of an ideal gas in canonical ensemble. The conclusion is given in section 5.

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https://doi.org/10.1016/j.heliyon.2022.e10564
Received 5 June 2022; Received in revised form 27 July 2022; Accepted 2 September 2022

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2. GUP with maximal length uncertainty

Let $H_s = L^2(\mathbb{R}, dx)$ be the Hilbert space of square integrable functions. The arbitrary operators $\hat{x}$ and $\hat{p}$ that act on this space are defined by

$$\hat{x} = \hat{x}, \quad \hat{p} = (1 - r \hat{x} + r^2 \hat{x}^2)\hat{p},$$

(1)

satisfy the following relation [1, 2]

$$[\hat{x}, \hat{p}] = i\hbar (1 - r \hat{x} + r^2 \hat{x}^2),$$

(2)

where the parameter $r \in (0, 1)$ is the generalized uncertainty principle parameter related to quantum gravitational effects at the Planck scale [3, 4, 5] and the Hermitian operators $\hat{x}$ and $\hat{p}$ satisfy the ordinary Heisenberg uncertainty algebra

$$[\hat{x}, \hat{p}] = i\hbar.$$

(3)

From the representation (1), one can interpret $\hat{x}$ and $\hat{p}$ as the set of operators at low energies which has the standard representation in position space whereas the operators $\hat{X}, \hat{P}$ as the set of operators at high energies in the generalized representation. Furthermore, the position deformed algebra (2) is consistent with the recently proposed by Perivolaropoulos [24].

Let $\{x\}, \{p\} \in H = H_x \otimes H_p$ be complete unit basis vectors respectively defined in $\{|x\rangle\} \in H_x$ and $\{|p\rangle\} \in H_p$. The action of the operators (1) on these vectors reads as follows

$$\hat{X}|x\rangle = x|x\rangle \quad \text{and} \quad \hat{P}|x\rangle = -i\hbar D_x|x\rangle, \quad x \in \mathbb{R},$$

(4)

$$\hat{X}|p\rangle = -i\hbar \hat{p}|p\rangle \quad \text{and} \quad \hat{P}|p\rangle = (1 + r \hat{x} + r^2 \hat{x}^2)\hat{p}|p\rangle, \quad p \in \mathbb{R},$$

(5)

where $D_x = (1 - r \hat{x} + r^2 \hat{x}^2)\hat{p}$ is a deformed derivation. Let us consider an arbitrary vector $\psi(x) \in H_x$ that projects this vector on the unit basis vectors $\{|x\rangle\}$ and $\{|p\rangle\}$ generates the functions $\psi(x)$ and $\psi(p)$. Therefore, the above equations are rewritten as follows

$$\hat{X}\psi(x) = x\psi(x) \quad \text{and} \quad \hat{P}\psi(x) = -i\hbar D_x\psi(x),$$

(6)

$$\hat{X}\psi(p) = i\hbar \psi(p) \quad \text{and} \quad \hat{P}\psi(p) = (1 + r \hat{x} + r^2 \hat{x}^2)\psi(p).$$

(7)

An interesting feature can be observed from the commutation relation (2) through the following uncertainty relation:

$$\Delta X \Delta P \geq \frac{\hbar}{2} \left(1 - \tau (\hat{X}) + r^2 (\hat{X}^2)\right),$$

(8)

where $\langle \hat{X} \rangle$ and $\langle \hat{X}^2 \rangle$ are the expectation values of the operators $\hat{X}$ and $\hat{X}^2$ respectively for any space representations. Using the relation $\langle \hat{X}^2 \rangle = X^2 + \langle \hat{X} \rangle^2$, equation (8) can be rewritten as a second order equation for $\Delta X$. The solutions for $\Delta X$ are as follows

$$\Delta X = \frac{\Delta P}{\hbar r} \pm \sqrt{\frac{\Delta P^2}{\hbar^2 r^2} - \frac{\langle \hat{X} \rangle}{\tau} \left(\tau \langle \hat{X} \rangle - 1\right) - \frac{1}{r^2}}.$$  

(9)

The reality of solutions gives the following minimum value of $\Delta P$

$$\Delta P = \hbar r \sqrt{1 - \frac{r}{\tau} (\hat{X}) + r^2 (\hat{X}^2)}. $$

(10)

Therefore, these equations lead to the absolute minimal uncertainty $\Delta P_{\min}$ in P-direction and to the absolute maximal uncertainty $\Delta X_{\max}$ in $X$-direction for $\langle \hat{X} \rangle = 0$ such that

$$\Delta P_{\min} = \hbar r, \quad \Delta X_{\max} = l_{\max} = \frac{1}{r}.$$  

(11)

It is well-known that [3], the existence of minimal uncertainty raised the question of singularity of the space i.e. space is inevitably bounded by minimal quantity beyond which any further localization of particle is not possible. In the present situation, the minimal momentum $\Delta P_{\min}$ leads to an absolute minimal representation of $\psi(p)$ and a maximal representation of wave function $\psi(x)$ i.e., $\psi(x) \in H = H_x$. Furthermore, the existence of the minimal momentum $\Delta P_{\min}$ induces a non-symmetry of the operator $\hat{P}$ such as

$$\hat{P}^\dagger = \hat{P} + i\tau (1 - 2r \hat{X}) \implies \hat{P}^\dagger \neq \hat{P}. $$

(12)

In order to guarantee the symmetry of this operator, we arbitrary restrict the study from the infinite dimensional Hilbert space $H$ into its bounded dense domain $D_x = L^2(\mathbb{R}, dx)$ in such away that, for $\tau \to 0$, one recovers the entire space $H$. This restriction perfectly fits with the work of Nozari and Etemad in momentum space [25]. The deformed completeness relation (13) becomes

$$\langle \psi |\hat{x}\rangle = \frac{dx}{1 - r \hat{x} + r^2 \hat{x}^2} \langle \psi, \langle \psi |\hat{x}\rangle = \mathbb{I}. $$

(13)

Consequently, the scalar product between two states $|\Psi\rangle, |\Phi\rangle \in D_x$ and the orthogonality of eigenstates becomes

$$\langle \Psi |\Phi\rangle = \int_{-L_{\max}}^{+L_{\max}} \frac{dx}{1 - r |x| + r^2 |x|^2} \Psi^*(x)\Phi(x).$$

(14)

$$\langle x |\alpha\rangle = (1 - r |x| + r^2 |x|^2)\delta(x - \alpha^\prime).$$

(15)

Now let us consider this operator $\hat{P}$ in its closed interval

$$D_x(\hat{P}) = \{\psi, -i\hbar D_x\psi \in L^2(-l_{\max}, +l_{\max})\}, \quad \psi(-l_{\max}) = 0 = \psi(+l_{\max})$$

(16)

and its adjoint domain defined by

$$D_x(\hat{P}^\dagger) = \{\phi, -i\hbar D_x\phi \in L^2(-l_{\max}, +l_{\max})\}.$$  

(17)

Thus, we may write $D_x(\hat{P}) \subset D_x(\hat{P}^\dagger)$, which means that the domain of $\hat{P}$ is a proper subset of the domain of its adjoint $\hat{P}^\dagger$. To show the symmetry of the operator $\hat{P}$, we consider a functional $F(\phi, \psi)$ defined by

$$F(\phi, \psi) := \langle \phi |\hat{P}\psi\rangle - \langle \hat{P}^\dagger \phi |\psi\rangle.$$  

(18)

Using the relation (13) and a straightforward computation of this functional, we have

$$F(\phi, \psi) = \int_{-l_{\max}}^{+l_{\max}} \frac{dx}{1 - r |x| + r^2 |x|^2} \left[\psi^*(x) (-i\hbar D_x\psi(x)) - (-i\hbar D_x\psi(x))^\dagger \psi(x)\right]$$

$$= -i\hbar \int_{-l_{\max}}^{+l_{\max}} d(x) \psi^*(x)\psi(x) = -i\hbar \int_{-l_{\max}}^{+l_{\max}} \psi^*(x)\psi(x) |l_{\max}x^2 - |l_{\max}^2x^2|.$$  

(19)

Since $\psi(\pm l_{\max}) = 0$, and $\phi(x)$ can reach any arbitrary value at the boundaries. This leads to the vanishing of $F(\phi, \psi)$ i.e., $F(\phi, \psi) = 0$. Consequently, the operator $\hat{P}$ is symmetric in $D(\hat{P})$ such that

$$\langle \phi |\hat{P}\psi\rangle = \langle \hat{P}^\dagger \phi |\psi\rangle \implies \hat{P} = \hat{P}^\dagger.$$  

(20)

With these equations at hand, we are now in a position to determine the eigensystems of a particle in an infinite square well potential at the Planck scale and to deduce later from this spectrum its thermodynamic properties.

3. Spectrum of a free particle Hamiltonian

We consider the Hamiltonian of a free particle confined in an infinite square well potential at the Planck scale i.e., $-l_{\max} < x < l_{\max}$ defined by

$$\hat{H}(\hat{p}, \hat{x}) = \frac{1}{2m} \left[1 - r \hat{x} + r^2 \hat{x}^2\right]^2.$$  

(21)

Obviously, this Hamiltonian is Hermitian $\hat{H} = \hat{H}^\dagger$. The time-independent Schrödinger equation is given by [2]
The solution of the equation (22) is given by
\[
\psi_k(x) = A \exp \left( \frac{2k}{\tau} \sqrt{\frac{\tau}{3}} \left[ \arctan \left( \frac{2x - 1}{\sqrt{3}} \right) + \frac{\pi}{6} \right] \right),
\]  
where \( k = \frac{\sqrt{2m} x}{\hbar} \) and \( A \) is a constant. Then by normalization, \( \langle \psi_k | \psi_k \rangle = 1 \) and using the deformed completeness relation (13), we find that
\[
A = \sqrt{\frac{\tau \sqrt{3}}{\pi}}.
\]  
Substituting this equation (24) into the equation (23), we have
\[
\psi_k(x) = \sqrt{\frac{\tau \sqrt{3}}{\pi}} \exp \left( \frac{2k}{\tau} \sqrt{\frac{\tau}{3}} \left[ \arctan \left( \frac{2x - 1}{\sqrt{3}} \right) + \frac{\pi}{6} \right] \right).
\]

Based on the references [3, 25], the scalar product of the formal eigenstates is given by
\[
\langle \psi_k | \psi_{k'} \rangle = \frac{\tau \sqrt{3}}{\pi (k - k')} \sin \left( \frac{\pi (k - k')}{\tau \sqrt{3}} \right).
\]

This relation shows that, the normalized eigenstates (25) are no longer orthogonal. However, if one tends \( (k - k') \rightarrow \infty \), these states become orthogonal
\[
\lim_{(k-k') \rightarrow \infty} \langle \psi_k | \psi_{k'} \rangle = 0.
\]  
These properties show that, the states \(| \psi_k \rangle \) are essentially Gaussians centered at \( (k - k') \rightarrow 0 \) (see Fig. 1). This observation indicates primordial fluctuations at this scale and these fluctuations increase with the quantum gravitational effects. They can be assimilated to the coherent states of harmonic oscillator [26] which are known as states that mediate a smooth transition between the quantum and classical worlds. This transition is manifested by the saturation of the Heisenberg uncertainty principle \( \Delta_x \Delta_p = \frac{\hbar}{2} \). In comparison with coherent states, the states \(| \psi_k \rangle \) strongly saturate the GUP \( \Delta_x \Delta_p = \hbar \) at the Planck scale and could be used to describe the transition states between the quantum world and unknown world for which the physical descriptions are out of reach.

Since \( \psi_k \in D_{\psi} \), i.e., it vanishes at the boundaries \( \psi_k(-l_{\text{max}}) = 0 = \psi_k(+l_{\text{max}}) \), the above wave functions (25) for \( \psi(-l_{\text{max}}) = 0 \) becomes
\[
\psi_k(x) = \sqrt{\frac{\tau \sqrt{3}}{\pi}} \sin \left( \frac{2k}{\tau} \sqrt{\frac{\tau}{3}} \left[ \arctan \left( \frac{2x - 1}{\sqrt{3}} \right) + \frac{\pi}{6} \right] \right).
\]  

The quantization follows from the boundary condition \( \psi(l_{\text{max}}) = 0 \) and leads to the equation
\[
\frac{2k}{\tau} \sqrt{\frac{\tau}{3}} \left[ \arctan \left( \frac{2\tau l_{\text{max}} - 1}{\sqrt{3}} \right) + \frac{\pi}{6} \right] = n\pi \quad \text{with} \quad n \in \mathbb{N}.
\]

This equation becomes
\[
k = \frac{n\pi \sqrt{3}}{2 \arctan \left( \frac{2\tau l_{\text{max}} - 1}{\sqrt{3}} \right) + \frac{\pi}{6}}.
\]

Replacing \( l_{\text{max}} = \frac{l}{\tau} \) is the equation (31), we have
\[
E_n = \frac{27\hbar^2 \pi^2 \tau^2}{8m} \quad \text{with} \quad \tau \in (0, 1).
\]

In term of the maximal length of quantum gravity, we have
\[
E_n = \frac{27\hbar^2 \pi^2 \tau^2}{8m} = 0.6606 \tau^2.
\]

From this result, if we assume that the maximal length \( l_{\text{max}} \) is the order of the ordinate length \( L \) of square well potential in the basic quantum mechanics i.e. \( l_{\text{max}} = L \), the spectrum of this system is expressed as follows
\[
E_n = 0.6606 \Rightarrow E_n < \varepsilon_n,
\]

where \( \varepsilon_n = \frac{h^2 \pi^2 \tau^2}{8m} \) is the spectrum of a free particle in an infinite square well potential of the basic quantum mechanics with the fundamental energy \( \varepsilon_1 = \frac{h^2 \pi^2 \tau^2}{8m} \). This result shows that the strong gravity measurement induces weak transition energies. This indicates that, the quantum gravity induces a more pronounced contraction of energy levels which, consequently implies the decrease of energy band structures [2].

Hereafter, with the spectrum (32) of this system at hand we are able to study in detail the statistical properties of this system at the Planck scale.

4. Statistical descriptions

In this section we study the thermodynamical properties of canonical ensemble of ideal gas by means of the results obtained in the previous section. We consider \( N \) noninteracting particles of the above system at temperature \( T \). The corresponding partition function [18, 19, 20, 21, 22, 23, 27, 28, 29, 30, 31] is given by

Fig. 1. Variation of \( \langle \psi_k | \psi_k \rangle \) versus \( k \cdot k' \) with \( h = 1 \).
\[ Z_N = \frac{Z^N}{N!} = \frac{1}{\tau^N N!} \left( \frac{8\pi m}{27\hbar^2} \right)^{N/2}, \]

where \( \beta = \frac{1}{k_B T} \), \( k_B \) is the Boltzmann constant and

\[ Z = \int_0^\infty dx \exp \left[ -\frac{27\hbar^2 x^2}{8m} \right] = \frac{1}{\tau} \left( \frac{8\pi m}{27\hbar^2} \right)^{1/2}. \]

The internal energy \( E \) is given by

\[ E = -\left( \frac{\partial \ln Z_N}{\partial \beta} \right)_N = \frac{1}{2} N k_B T. \]

This equation shows that the presence of a deformed parameter \( \tau \) does not influence the internal energy of the canonical ideal gas [28]. Hence the heat capacity remains unchanged

\[ C_V = \left( \frac{\partial E}{\partial T} \right)_N = \frac{1}{2} N k_B. \]

Another important quantity is the Helmholtz free energy,

\[ A = -k_B T \ln Z_N = a + N k_B T \ln \tau, \]

where

\[ a = -k_B T \ln \left[ \frac{1}{N!} \left( \frac{8\pi m}{27\hbar^2} \right)^{N/2} \right]. \]

is the ordinary Helmholtz free energy. This shows that the ordinary Helmholtz free energy is corrected by a logarithm of the deformed parameter such that \( \Delta A = N k_B T \ln \tau \).

Now, the entropy of the system can be derived from equation (39) as,

\[ S = -\left( \frac{\partial A}{\partial T} \right)_N = s - N k_B \ln \tau, \]

where the first term

\[ s = -\left( \frac{\partial a}{\partial T} \right)_N = k_B \ln \left[ \frac{N}{2} - \left( \frac{8\pi m}{27\hbar^2} \right)^{N/2} \right], \]

is the ordinary entropy and the second term represents the correction to the ordinary entropy \( \Delta S = -N k_B T \ln \tau \). Note that this logarithmic correction to the ordinary entropy is closed to the quantum corrections at the extremal limit of Reissner-Nordström’s black hole [16, 17]. According to several works in this frame, the entropy logarithmic correction is due to large quantum fluctuations at this scale. This situation is consistent with our previous works [1, 2] since the quantum gravity measurement is important at this scale.

Finally, the generalized chemical potential, \( M \) is given by

\[ M = \left( \frac{\partial A}{\partial N} \right)_T = \mu + k_B T \ln \tau, \]

where

\[ \mu = \left( \frac{\partial a}{\partial N} \right)_T = -k_B T \ln \left[ \frac{1}{N} \left( \frac{8\pi m}{27\hbar^2} \right)^{1/2} \right]. \]

stands for the chemical potential in the undeformed case \( M = k_B T \ln \tau \) is the induced logarithmic chemical correction. In summary of the above logarithmic corrections due to the strong gravity at this scale, we have:

\[ \Delta A = N k_B T \ln \tau, \quad \Delta S = -N k_B \ln \tau, \quad \Delta M = k_B T \ln \tau. \]

Note that at the extremal limit of the space \( (\tau \to 0) \), the above thermodynamic quantities diverge such as

\[ \lim_{\tau \to 0} \Delta A = \infty \Rightarrow \lim_{\tau \to 0} A = \infty, \]

\[ \lim_{\tau \to 0} \Delta S = \infty \Rightarrow \lim_{\tau \to 0} S = \infty, \]

\[ \lim_{\tau \to 0} \Delta M = \infty \Rightarrow \lim_{\tau \to 0} M = \infty. \]

Comparing these results with the analogous obtained in the minimal length scenarios [18, 19, 20, 21, 22, 23] show that, the effect of the maximal length in this framework and that of the minimal length are fundamentally different. In fact, the minimal length induces a weak quantum gravity localization while the maximal length in this context induces a strong quantum gravity localization. Recently, numerically experimentations have been addressed to examine the effect of the minimal length on the exceptional points in optomechanical sensors [32]. It has been shown that, the weak effect of quantum gravity induced by the minimal length can be sensed via pushing the system towards a second-order exceptional point, where the spectra of the non-Hermitian system exhibit non-analytic and even discontinuous behavior exceptional point, where the spectra of the non-Hermitian system exhibit non-analytic and even discontinuous behavior. In agreement with this result [32], the weak quantum gravity measurement shifts quadratically the thermodynamic quantities and at the extremal limit of deformed parameter \( \rho_{GUP} (\rho_{GUP} \to 0) [18, 19, 20, 21, 22, 23] \), one recovers the ordinary quantities. This concept of weak quantum gravity measurement can also be observed in general relativity where the weak curvature of the space-time fabric can be approximated as flat. However in the present work, the quantum gravity measurement induces logarithmic corrections.

Moreover, to show another way the strong quantum gravity measurement within the space (39), we reduce this algebra through first order approximation of the deformed parameter \( \tau \). We obtain

\[ [\hat{X}, \hat{P}] = i\hbar(1-\tau, \hat{X}), \]

where \( \tau_r \) is the reduced or the first order parameter. In this way, the algebra (49) indicates a space that measures weak quantum gravitational effects and can be connected to the \( q \)-deformed Heisenberg algebra frequently used in 1D to describe position-dependent mass particles [33, 34, 35, 36, 37]. The spectrum of free particle in a square well potential of length \( L \) is given by [33, 34, 35]

\[ E_n = \frac{n^2 \pi^2 \hbar^2}{2m L^2} \]

\[ \tau_r L < 1. \]

The partition function associated to the spectrum (50) is given by

\[ Z_N = \left( \frac{2m}{\pi \hbar^2} \right)^{N/2} \left( \frac{r^2}{\ln^2(1-\tau_r L)} \right)^{-N/2}. \]

Repeating the same thermodynamic computations, we obtain the similar results as in [28] where the ordinary thermodynamic quantities as the Helmholtz free energy \( a \), the entropy \( s \) and the chemical potential \( u \) shift quadratic corrections of the deformed parameter \( \tau_r \), except the internal energy which is invariant under this quantum gravitational effects

\[ A = -k_B T \ln \left[ \frac{2m}{\pi \hbar^2} \right]^{N/2} - k_B T \ln \left[ \frac{r^2}{\ln^2(1-\tau_r L)} \right]^{N/2}. \]

\[ S = \frac{1}{2} N k_B \ln \left[ \frac{2m}{\pi \hbar^2} \right]^{N/2} + k_B \ln \left[ \frac{r^2}{\ln^2(1-\tau_r L)} \right]^{N/2}. \]

\[ M = -k_B T \ln \left[ \frac{2m}{\pi \hbar^2} \right]^{1/2} - k_B T \ln \left[ \frac{r^2}{\ln^2(1-\tau_r L)} \right]^{1/2}. \]

\[ E = \frac{1}{2} N k_B T \quad \text{and} \quad C_V = \frac{1}{2} N k_B. \]
and

\[ \lim_{r \to 0} A = a = -k_B T \ln \left( \frac{2mL^2}{\pi \hbar^2} \right)^{1/2} \]  
\[ \lim_{r \to 0} S = s = \frac{1}{2} N k_B \ln \left( \frac{2mL^2}{\pi \hbar^2} \right)^{1/2} \]  
\[ \lim_{r \to 0} M = u = -k_B T \ln \left( \frac{2mL^2}{\pi \hbar^2} \right)^{1/2} \].

Comparing these results with those obtained by Bensalem and Bouaziz [28], we finally confirm that the concept of maximal length induces a maximal localization/measurement of quantum gravity and this localization increases with the order of the deformed parameter \( \epsilon \).

5. Conclusion

In this paper, we studied the effects of maximal measurement of quantum gravity on the thermodynamics of ideal gas in the canonical ensemble of \( N \) noninteracting particles at the Planck scale. The interesting physical result we found is that, at the Planck scale with strong quantum gravity localization, the ordinary thermodynamic parameters as the Helmholtz free energy \( s \), the entropy \( S \), and the chemical potential \( \mu \) shift logarithm corrections of the deformed parameter \( \epsilon \), except the internal energy which is invariant under this strong gravitational effects.

The logarithm corrections induced by these quantities are consistent with the ones obtained at the extremal limit of Reissner-Nordström’s black hole due to the high effectiveness of quantum fluctuations [16, 17]. To confirm these results, we compared our results with those of the minimal length formalism available in the literature [18, 19, 20, 21, 22, 23]. On this basis, it has been shown that both scenarios are fundamentally different. In fact, the minimal length formalism shifts quadratically the thermodynamic quantities at the order of the deformed parameter \( \beta_{UV} \). So, at the extremal limit i.e. \( \beta_{UV} \to 0 \) one recovers the ordinary quantities while in this framework by tending \( \epsilon \to 0 \), the modified thermodynamic quantities diverge because of the logarithm corrections. With respect to these remarks, we can state that the minimal length formalism [18, 19, 20, 21, 22, 23] induced weak gravitation localization while the emergence of maximal length in this algebra induced strong quantum gravitation measurement at the Planck scale. This result comes to confirm our recent result [2] where we have shown that this concept of maximal length has similarities with the classical properties of gravity.

Declarations

Author contribution statement

Latévi M. Lawson: Conceived and designed the experiments; Performed the experiments; Analyzed and interpreted the data; Contributed reagents, materials, analysis tools or data; Wrote the paper.

Declaration of interests statement

The authors declare no conflict of interest.

Data availability statement

No data was used for the research described in the article.

Funding statement

This research did not receive any specific grant from funding agencies in the public, commercial, or not-for-profit sectors.

Additional information

No additional information is available for this paper.

Acknowledgements

I would like to thank the referees for giving such constructive comments which considerably improved the quality of the paper. The author acknowledges support from DAAD (German Academic Exchange Service) under the DAAD postdoctoral in region grant.

References

[1] L. Lawson, Minimal and maximal lengths from position-dependent noncommutativity, J. Phys. A, Math. Theor. 53 (2020) 115303.
[2] L. Lawson, Position-dependent mass in strong quantum gravitational background fields, J. Phys. A, Math. Theor. 55 (2022) 105303.
[3] A. Kempf, G. Mangano, R. Mann, Hilbert space representation of the minimal length uncertainty relation, Phys. Rev. D 52 (1995) 1108.
[4] A. Kempf, Noncommutative geometric regularization, Phys. Rev. B 54 (1997) 5174.
[5] A. Kempf, G. Mangano, Minimal length uncertainty relation and ultraviolet regularization, Phys. Rev. D 55 (1997) 7909.
[6] A. Kempf, Maximal localization in the presence of minimal uncertainties in positions and in momentum, Phys. Rev. B 54 (1997) 5174.
[7] A. Kempf, Non-pointlike particle in harmonic oscillators, J. Phys. A, Math. Gen. 30 (1997) 2093.
[8] A. Kempf, 2000 unsharp degrees of freedom and the generating of symmetries, Phys. Rev. D 54 (1997) 5174.
[9] L. Lawson, I. Nonkané, K. Sologa, The damped harmonic oscillator at the classical limit of the Snyder-de Sitter space, J. Math. Res. 13 (2021) 2.
[10] A. Fring, L. Gouba, F. Scholtz, Strings from position-dependent noncommutativity, J. Phys. A, Math. Theor. 43 (2010) 345401.
[11] L. Lawson, L. Gouba, G. Avorsou, Two-dimensional noncommutative gravitational quantum well, J. Phys. A, Math. Theor. 50 (2017) 475202.
[12] F. Scardigli, Generalized uncertainty principle in quantum gravity from micro-black hole gedanken experiment, Phys. Lett. B 45 (1999) 39–44.
[13] P. Pedram, A higher order GUP with minimal length uncertainty and maximal momentum, Phys. Lett. B 714 (2012) 317–323.
[14] K. Nozari, A. Etemadi, Minimal length, maximal momentum, and Hilbert space representation of quantum mechanics, Phys. Rev. B 85 (2012) 104029.
[15] A. Tawfik, A. Diab, A review of the generalized uncertainty principle, Rep. Prog. Phys. 78 (2015) 126001.
[16] S.N. Solodukhin, On “Non-Geometric” contribution to the entropy of black hole due to quantum corrections, Phys. Rev. D 51 (1995) 618.
[17] R. Mann, S. Solodukhin, Universality of quantum entropy for extreme black holes, Nucl. Phys. B 523 (1998) 293.
[18] M. Abbasyan-Moqal, P. Pedram, The minimal length and quantum partition functions, J. Stat. Mech. Theory Exp. 2014 (2014).
[19] P. Wang, H. Yang, X. Zhang, Quantum gravity effects on statistics and compact star configurations, J. High Energy Phys. 08 (2010) 043.
[20] T. Fityo, Statistical physics in deformed spaces with minimal length, Phys. Lett. A 372 (2008) 5872–5877.
[21] M. Mirtorabi, S. Miraboutalebi, A. Masoudi, L. Farhang Mattin, Quantum gravity modifications of the relativistic ideal gas thermodynamics, Phys. A, Stat. Mech. Appl. 506 (2018) 602–612.
[22] K. Nozari, V. Hosseinazadeh, M. Gorgi, High temperature dimensional reduction in Snyder space, Phys. Lett. B 750 (2015) 218–224.
[23] M. Abbasyan-Moqal, P. Pedram, Generalized uncertainty principle and thermo-statistics: a semiclassical approach, Int. J. Theor. Phys. 55 (2016) 1953–1961.
[24] L. Perivolaropoulos, Cosmological horizons, uncertainty principle, and maximum length quantum mechanics, Phys. Rev. D 95 (2017) 103523.
[25] K. Nozari, A. Etemadi, Minimal length, maximal momentum and Hilbert space representation of quantum mechanics, Phys. Rev. D 85 (2012) 104029.
[26] J.P. Gazeau, Coherent States in Quantum Physics, Wiley-Vch Verlag GmbH Co. KgaA, 2009.
[27] S. Bensalem, D. Bouaziz, Statistical description of an ideal gas in maximum length quantum mechanics, Phys. A, Stat. Mech. Appl. 523 (2019) 583–592.
[28] S. Bensalem, D. Bouaziz, On the thermodynamics of relativistic ideal gases in the presence of a maximal length, Phys. Lett. A 384 (2020) 126911.
[29] R. Patiria, Statistical Mechanics, 2nd edition, Butterworth-Heinemann, Oxford, 1996.
[30] S. Miraboutalebi, L. Mattin, Thermodynamics of canonical ensemble of an ideal gas in presence of Planck-scale effects, Can. J. Phys. 93 (2014) 574–579.
[31] S. Dan, P. Majumdar, R. Bhaduri, General logarithmic corrections to black hole entropy, Class. Quantum Gravity 19 (2002) 2355.
[32] Dianzhao Cui, T. Li, Jianging Li, Xuedi Yu, Detecting deformed commutators with exceptional points in optomechanical sensors, New J. Phys. 23 (2021) 123037.
[33] R. Costa Filho, M. Almeida, G. Farias, J. Andrade Jr., Displacement operator for quantum systems with position-dependent mass, Phys. Rev. A 84 (2011) 050102.
[34] B. da Costa, E. Borges, Generalized space and linear momentum operators in quantum mechanics, J. Math. Phys. 55 (2014) 062105.
[35] S. Habib Mazharimousavi, Revisiting the displacement operator for quantum systems with position-dependent mass, Phys. Rev. A 85 (2012) 034102.

[36] G. Bruno da Costa, Ernesto P. Borges, A position-dependent mass harmonic oscillator and deformed space, J. Math. Phys. 59 (2018) 042101.
[37] M. Rego-Monteiro, F. Nobre, Classical field theory for a non-Hermitian Schrodinger equation with position-dependent masses, Phys. Rev. A 88 (2013) 032105.