Random Walks through the Ensemble: Linking Spectral Statistics with Wavefunction
Correlations in Disordered Metals

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We use a random walk in the ensemble of impurity configurations to generate a Brownian motion model for energy levels in disordered conductors. Treating arc-length along the random walk as fictitious time, the resulting Langevin equation relates spectral statistics to eigenfunction correlations. Solving this equation at energy scales large compared with the mean level spacing, we obtain the spectral form factor, and its parametric dependence.

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Statistical properties of the spectra of finite quantum systems have been a focus for research in three successive contexts: nuclear physics, the semiclassical limit of quantum mechanics, and studies of mesoscopic conductors. A unifying idea is that the energies of individual eigenstates are frequently neither calculable nor interesting: instead, the concern should be with eigenvalue correlations, which typically are independent of many details of the Hamiltonian.

Disordered mesoscopic conductors bring to this field both new behaviour and a natural ensemble for a statistical description - the ensemble of impurity configurations. For weak disorder, new behaviour arises because there can be a broad window in time, and hence a corresponding energy interval, between the scale, \( t_{\text{el}} \), for electron scattering from impurities and that for diffusion across the system, \( t_{\text{erg}} \sim L^2/D \) (where \( L \) and \( D \) are the system size and diffusion constant). And at the mobility edge, specific, critical spectral statistics are expected. The ensemble average provides the departure point for established perturbative and non-perturbative calculations of spectral correlations in disordered metals.

In this paper we describe an alternative approach, in which the energy level distribution is averaged along a random walk through the ensemble. There are several precedents for study of levels as a function of position in the space of Hamiltonians. Most recently, a number of authors, in particular Szafer, Simons and Altshuler, have investigated parametric statistics: level correlations between different points on a smooth path in this space. Earlier, in connection with the semiclassical limit, Pechukas used motion along such a path, with coordinate \( \lambda \), to generate a dynamics for the one-dimensional gas formed by levels on the energy axis. And originally, in the context of random matrix theory (RMT), Dyson employed a random walk through the matrix ensemble as the foundation for Brownian dynamics of levels, with arc-length, \( \tau \), being the fictitious time. Pechukas' and Dyson's ideas are linked, as shown schematically in Fig. 1, by the usual relation between the end-to-end distance and the length of a random walk, \( \lambda^2 \sim \tau \).

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In previous work on level dynamics, eigenfunction correlations have played no role, either (in the semiclassical limit) by assumption, or (for random matrices) by construction, RMT having no preferred basis. In contrast, for disordered metals, the basis of position states is singled out: it is in this basis that the impurity potential is a diagonal operator. Further, specific eigenfunction correlations must be present if, for example, wavepackets spread diffusively. We show here that these correlations result in a novel Brownian level dynamics, from which we obtain the features of spectral correlations particular to mesoscopic conductors.

Our results are expressed in terms of the quantum return probability, \( p(t) \), which is required as input in this approach. Consider the time-evolution of a wavepacket that initially occupies a small volume \( V_0 \): \( p(t) \) is the probability density to remain in this volume at the time \( t \). For weak disorder, the wavepacket spreads diffusively, and \( p(t) \) is known from semiclassical arguments; at the metal-insulator transition, scaling ideas relate \( p(t) \) to fractal properties of eigenstates.

We shall be interested in the two-level correlation function (TLCF) and the spectral form factor. Let \( E_n(\tau) \) be the energy levels of a one-parameter family of Hamiltonians, \( H(\tau) \). The density of states per unit volume (DoS)
is $\rho(E, \tau) = L^{-d} \sum_{n} \delta(E - E_n(\tau))$, and the mean level spacing is $\Delta = (\rho L^d)^{-1}$, with $\rho \equiv \rho(E, \tau)$. The TLF is

$$R(E, \tau) = \rho^{-2} \langle \rho(E, \tau) \rho(0,0) \rangle - 1,$$ 

(1)

and the spectral form factor is

$$K(t, \tau) = \frac{1}{2\pi \hbar} \int_{-\infty}^{\infty} e^{-iE_t/\hbar} R(E, \tau) dE.$$ 

(2)

We obtain $K(t, \tau)$ for times shorter than the Heisenberg time, $t_{\text{He}} \approx \hbar/\Delta$, in terms of $p(t)$. We defer discussion of the parametric dependence until the end of this paper. The non-parametric form factor, $K(t) \equiv K(t, \tau = 0)$, is

$$K(t) = \frac{1}{1 + (\pi \rho)^{-1} \int_{t_0}^{t_0 + t} p(t') dt'} (\Delta/2\pi \hbar)^2 L^d.$$ 

(3)

In the diffusive regime, $t_{\text{el}} \lesssim t \lesssim t_{\text{erg}}$, $p(t)$ at leading order reduces to the classical return probability for random walks, multiplied by a symmetry factor $2/\beta$, where $\beta = 1, 2$ or 4 is the usual index [14]:

$$p_0(t) = \frac{2}{\beta(4\pi D t)^{d/2}}.$$ 

(4)

There exist quantum (weak-localization) corrections to Eq. (3), which are smaller by a factor $g_0^{-1}$, where $g_0 \ll 1$ is the dimensionless conductance at scale of the elastic mean free path, $\ell$. Noting that in the ballistic regime, $t \lesssim t_{\text{el}}$, $p(t)$ saturates at $p_0(t_{\text{el}}) \sim 1/\ell^d$, one sees that the second term in the denominator of Eq. (3) also gives weak localization corrections: it is of order $(t_{\text{el}} \Delta/\hbar)(L/\ell)^d \sim g_0^{-1}$ for $d \geq 2$, and of order $g_0^{-1} \ln(t/t_{\text{el}})$ for $d = 2$. Neglecting these corrections in both the numerator and denominator of Eq. (3), it reduces to

$$K_0(t) = (\Delta/2\pi \hbar)^2 L^d |p_0(t)|.$$ 

(5)

To leading order, this expression is also valid in the ergodic regime, $t_{\text{erg}} \lesssim t \lesssim t_{\text{el}}$, where the classical return probability saturates at $(2/\beta)L^d$ so that the second term in the denominator of Eq. (3) is of order $t/t_{\text{el}} \ll 1$ and may be neglected.

Equation (3) coincides with the result obtained in Ref. [15], using the diagonal approximation in semiclassical periodic-orbit theory. The Fourier transform of this expression corresponds to the TLF obtained originally by Altshuler and Shklovskii from the lowest order of the diagrammatic expansion [1]. In the diffusive regime, $R(s,0) \sim A_d s^{d/2-2}$, and in the ergodic regime, $R(s,0) \sim -1/s^{2}$ [16], where $s = E/\Delta$, and $A_d$ is a numerical coefficient which is zero for $d = 2$ [17].

There are indications that our approach is useful beyond the diagonal approximation. We have checked Eq. (3) in the diffusive regime by direct diagrammatic expansion of both sides in powers of $g_0^{-1}$, calculating the first two non-trivial orders for $d = 2$. This is the most interesting case because, for $d = 2$, $p_0(t) \propto 1/t$, so that $K(t)$ is constant and $R(s,0) = 0$ in the diagonal approximation. We find [13] that Eq. (3) is exact up to order $g_0^{-2}$. For $\beta = 1, 4$ this means that both the leading ($g_0^{-1}$) contribution to $R(s,0)$ [17] and the $g_0^{-2}$ correction are exact. In the unitary case ($\beta = 2$) the first order terms in the numerator and denominator of Eq. (3) cancel, in agreement with the known absence of corrections to $K_0(t)$ at this order, and it is the $g_0^{-2}$ correction which governs $R(s,0)$; again it is given exactly by Eq. (3). For $d > 2$, first order corrections to the diagonal approximation are small by an additional factor, $(t_{\text{el}}/t)^{d/2-1}$; but Eq. (3) reproduces their form but not the numerical coefficient. On the other hand, at the mobility edge in $d > 2$, Eq. (3) is consistent with the results of independent calculations, as we discuss at the end of the paper.

We next turn to the derivation of our results. Let the fictitious time $\tau$ parameterize a Brownian path through an ensemble of disordered conductors, so that

$$H(\tau) = H_0 + \int_{0}^{\tau} d\tau' V(\tau',r),$$

(6a)

and

$$H_0 = -\frac{\hbar^2}{2m} \nabla^2 + U(r).$$

(6b)

We take $U(r)$ and $V(\tau, r)$ to be Gaussian distributed with zero average and

$$\langle U(r)U'(r') \rangle = \frac{\langle V(\tau, r)V'(\tau', r') \rangle = v^2 L^d \delta(\tau - \tau') \delta(r - r').}$$

(7)

We use perturbation theory to second order to calculate the change of $E_n(\tau)$ in response to $V(\tau, r)\delta\tau$:

$$\delta E_n = E_{nn} + \sum_{n \neq m} \frac{V_{nm}V_{mn}}{E_n - E_m},$$

$$V_{nn} = \int_{\tau}^{\tau + \delta \tau} d\tau' \langle n|V(\tau', r)|m \rangle,$$

(8a)

where $\langle r | n \rangle \equiv \psi_n(\tau, r)$ are the corresponding eigenfunctions of $H(\tau')$. We average over $V$ noting that

$$\langle V_{nn'}V_{nm} \rangle = \langle V_{nn'}V_{mm} \rangle = v^2 \int d\tau c_{nm}(\tau),$$

(8b)

$$c_{nm}(\tau) = L^d \int d^d r |\psi_n(\tau, r)|^2 |\psi_m(\tau, r)|^2.$$
\[
\langle \xi_n(\tau)\xi_m(\tau') \rangle = v^2 \delta(\tau - \tau') c_{nm}(\tau) .
\] (9b)

Both the spectrum of the random force and the drift term are expressed in terms of \( c_{nm}(\tau) \), the eigenstate correlation function, which we later show to be proportional to the return probability \( p(t) \).

The appearance of the eigenstate correlation function in Eqs. (8) is the essential difference between our Brownian level dynamics for disordered conductors, and that of Dyson for random matrices. The analogues to Eqs. (8) in Dyson’s work [11] have the basis-independent form

\[
\langle V_{nm}V_{mn} \rangle = 2/\beta ; \quad \langle V_{nn}V_{mm} \rangle = \delta_{nn} .
\] (10)

In contrast to Eqs. (8), these display two simplifying features: they are independent of the eigenvectors, and they lead to diagonal random force correlations. As a result, the limiting level distribution in RMT can be calculated exactly, proceeding via a Fokker-Planck formulation [1].

For disordered conductors, the Langevin equation for energy levels is not closed, since Eq. (8) involves \( c_{nm}(\tau) \), and the random forces have off-diagonal correlations. In consequence, approximations are required, which are most transparent within the Langevin description.

Two central assumptions are necessary in order to make progress. We believe that they are reasonable provided one considers only behaviour at energy scales much larger than the mean level spacing. First, we replace \( c_{nm}(\tau) \) by its average over the ensemble of \( H_0 \), which, within the energy window of interest, is a function only of the energy difference, \( E = E_n - E_m \):

\[
\langle c_{n,n+i}(\tau) \rangle \equiv c(E) .
\] (11)

Thus we neglect correlations between fluctuations in \( c_{nm}(\tau) \) and those in \( \rho(E) \) [12]. Second, we linearize Eqs. (8) in the deviation of level separation \( E_n - E_m \) from its mean value \( (n - m)\Delta \),

\[
E_n(\tau) = n\Delta + \varepsilon_n(\tau) .
\]

Then Eq. (9b) becomes

\[
\frac{d\varepsilon_n(\tau)}{d\tau} = -v^2 \sum_{|l| \neq 0} (\varepsilon_n - \varepsilon_l) f(l\Delta) + \xi_n(\tau) (12)
\]

where \( f(E) \equiv d\varphi(E)/dE \) with \( \varphi(E) = c(E)/E \). This is diagonalized by Fourier transform, using as coordinates

\[
\mathcal{E}(t, \tau) = \frac{\Delta}{2\pi h} \sum_n \varepsilon_n(\tau) e^{-i\Delta nt/h} ,
\] (13a)

which, for \( 0 < t \ll h/\Delta \), and \( \varepsilon_n t/h \) are essentially Fourier components of the DoS:

\[
\mathcal{E}(t, \tau) \simeq \frac{i}{2\pi \rho t} \int_{-\infty}^{\infty} \rho(E, \tau) e^{-iEt/h} dE .
\] (13b)

In these variables, Eqs. (8) become

\[
\frac{d\mathcal{E}(t, \tau)}{d\tau} = -\frac{2\pi hv^2}{\Delta} F(t) \mathcal{E}(t, \tau) + \Xi(t, \tau) ,
\]

\[
\langle \Xi(t, \tau) \Xi(t', \tau') \rangle = v^2 \delta(t-t') \delta(\tau - \tau') C(t) ,
\] (14a)

where \( C(t), F(t) \) and \( \Xi(t, \tau) \) are the Fourier transforms of \( c(E), f(E) \) and \( \xi_n(\tau) \), defined as in Eqs. (13). For example,

\[
C(t) = \frac{\Delta}{2\pi h} \sum_l c(l\Delta) e^{-i\Delta lt/h} \simeq \int_{-\infty}^{\infty} c(E) e^{-iEt/h} \frac{dE}{2\pi h} .
\]

It follows from the definition of \( f(E) \), after Eq. (12), that \( F(t) \) is related to \( C(t) \). Simple manipulations lead to

\[
F(t) = \left| \frac{t}{\hbar} \right| + \frac{|t|}{\hbar^2} \int_0^t C(t') dt' ,
\]

where we have used the fact that \( F(t) \) is an even function of \( t \) to fix the constant of integration. Finally, we note that, since \( c(E) = 1 \) for \( |E| \gtrsim \hbar / t_{\text{el}} \), \( C(t) \) contains a \( \delta \)-function at \( t = 0 \), so that

\[
F(t) = \frac{|t|}{2\hbar^2} + \frac{|t|}{\hbar^2} \int_0^{|t|} C(t') dt' .
\] (15)

Now we solve Eqs. (14), fixing at this point the units of \( \tau \) by setting \( v^2 = \Delta/2\pi h \):

\[
\langle \mathcal{E}(t, \tau + \tau') \mathcal{E}(t', \tau') \rangle = \delta(t+t') \frac{C(t)\Delta}{4\pi h F(t)} e^{-F(t)|\tau|} .
\]

As \( \mathcal{E}(t, \tau) \) is the Fourier transform of \( \rho(E, \tau) \), we immediately obtain the spectral form factor:

\[
K(t, \tau) = \left( \frac{t}{\hbar} \right)^2 \frac{C(t)\Delta}{4\pi h F(t)} e^{-F(t)|\tau|} .
\] (16)

Note that the form factor is expressed in terms of eigenfunction correlations via the interplay of the restoring force, \( F \), and the noise correlator, \( C \). It is the latter which is responsible for the difference between spectral statistics in the ergodic and diffusive regimes.

We relate \( F \) and \( C \) to the return probability of a diffusing electron, by considering a wavepacket made from the eigenstates of \( H(\tau) \) and concentrated initially near the origin, in a volume of size \( \ell^3 \):

\[
\Psi(r, t) = A \sum_n \psi_n(0)^* \psi_n(r) e^{-iE_n t/h} .
\]

Here the summation is limited to \( N \) levels with energies \( E_n \lesssim \hbar / t_{\text{el}} \) so that \( N \sim \hbar / t_{\text{el}} \Delta \), and the normalization constant is \( A^2 = L^2 N^{-1} \). The ensemble-averaged return probability, \( p(t) = \langle |\Psi(0, t)|^2 \rangle \), is given for \( t > 0 \) by

\[
p(t) = A^2 \sum_{nm} c_{nm} e^{-i(E_n - E_m) t/h} = 2\pi h \rho C(t) ,
\] (17)

where we have again (as in the solution of the linearized Langevin equation) neglected correlations between the fluctuations of \( \xi_n(\tau) \) and \( c_{nm}(\tau) \).

Substituting for \( C(t) \) in Eq. (16), we find:
\[ K(t, \tau) = \left( \frac{\Delta}{2\pi\hbar} \right)^2 L^d \frac{|t| p(t)}{M(t)} \exp \left[ -\frac{M(t)}{2\hbar^2} |t\tau| \right], \quad (18a) \]
\[ M(t) = 1 + \frac{1}{\pi\hbar \rho} \int_0^t p(t') dt'. \quad (18b) \]

These expressions, which are the central result of this work, relate spectral statistics to the return probability and thus, via Eq. (17), to the wavefunction correlations [3]. For the non-parametric problem, Eq. (18a) reduces to Eq. (3), and the ensuing behaviour in the ergodic and diffusive regimes has been discussed following Eq. (4).

Parametric correlations in these regimes, from Eqs. (5), are as follows. At leading order we ignore weak-localization contributions to \( p(t) \) and \( M(t) \), approximating them by \( p_0(t) \) and 1, respectively. Then, for \( t \ll t_H \),
\[ K(t, \tau) = K_0(t) e^{-|t\tau|/2\hbar^2}. \quad (19) \]

Its Fourier transform gives the parametric TLCF in both the ergodic regime (where it reproduces the envelope of the TLCF found in Refs. [8,12]) and the diffusive regime (where it is consistent with the results of diagrammatic calculations). For \( E = 0 \) we obtain
\[ R(0, \lambda) = \frac{2}{\beta} \left\{ \begin{array}{ll} \lambda^{-4}, & \text{ergodic regime} \\ B_d \lambda^{-d}/2 \lambda^{-4}, & \text{diffusive regime} \end{array} \right. \quad (20) \]
where \( \lambda^2 \equiv (\pi\tau/\hbar\Delta) \) and \( B_d \) is a constant which is non-zero in \( d = 2 \), so that the leading approximation is sufficient for the parametric TLCF in \( d = 2 \), in contrast to the non-parametric TLCF, \( R(E, \tau = 0) \).

Finally, we turn to spectral correlations for a system at the metal-insulator transition [6,21]. At the critical point, power-law decay of the return probability is expected for \( t_{el} \ll t \ll t_{eq} \sim t_H \), with the power related to a multifractal dimension of wavefunctions [20]:
\[ p(t) \propto t^{-1+\eta/d}, \quad (21) \]
where \( 0 \leq \eta \leq d - d_2 < d \), and \( d_2 \) is the dimension associated with \(|\psi|^4\). In consequence, for the same range of \( t \), \( K(t) \) is constant, a result obtained previously from a diagrammatic analysis [4]. As in \( d = 2 \), the behaviour of \( R(E, \tau = 0) \) at the mobility edge is, in fact determined [3], not by this limiting behaviour for \( K(t) \), but by the way in which the limit is approached as \( t/t_H \to 0 \); we defer further discussion to a future publication. Parameteric correlations at the mobility edge have not previously been investigated. We find \( K(t, \tau) \propto K_0 \exp(-a\tau|t|^{1+\eta/d}) \), where \( a \) is a constant, giving
\[ R(0, \lambda) \propto \lambda^{-2/\eta+2/\eta}. \quad (22) \]

Thus the multifractal properties of critical eigenstates, via the exponent \( \eta \), are reflected in parametric spectral statistics at the mobility edge.

In summary, we have developed a simple and transparent approach for calculating spectral correlations in disordered metals. By averaging along a random walk through the ensemble of impurity configurations, we have expressed level statistics in all regimes in terms of the quantum return probability for a spreading wavepacket.

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