Different types of the Fulde-Ferrell-Larkin-Ovchinnikov states
induced by anisotropy effects

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Abstract

The crystal structure determines both the Fermi surface and pairing symmetry of the superconducting metals. It is demonstrated in the framework of the general phenomenological approach that this is of the primary importance for the determination of the structure of the Fulde-Ferrell-Larkin-Ovchinnikov (FFLO) phase in the magnetic field. The FFLO modulation of the superconducting order parameter may be revealed in the form of the higher Landau level states or/and modulation along the magnetic field. The transition between different FFLO states could occur with the temperature variation or with the magnetic field rotation.
I. INTRODUCTION

It is well known that in type II superconductors the Abrikosov vortex state can be formed under a magnetic field. In most cases the destruction of superconductivity happens due to the orbital effect. However there can be a situation when paramagnetic effect plays an important role in destruction of superconductivity (magnetic field acting only on electron spins). In this case the non-uniform Fulde-Ferrell-Larkin-Ovchinnikov (FFLO) state [1, 2] appears in superconductors, which is characterized by the modulation of the order parameter. The structure of the FFLO phase in the real compounds may be very rich [3–12]. Interplay of orbital and paramagnetic effects has been described in the isotropic model by Gruenberg and Gunther in [3], where they calculated critical field and structure of the order parameter. It was found in [3] that the orbital effect is detrimental to the FFLO state, but still such a state can exist if the ratio of pure orbital effect \( H_{c2}^{orb}(0) \) and pure paramagnetic limit \( H_p(0) \) is larger than 1.28, i.e. the Maki parameter \( \alpha_M = \sqrt{2H_{c2}^{orb}(0)/H_p(0)} \) must be larger 1.8. Pure paramagnetic limit at \( T = 0 \) can be estimated as \( H_p(0) = \Delta_0/\sqrt{2}\mu_B \), where \( \Delta_0 \) is BCS gap at \( T = 0 \) and \( \mu_B \) is the Bohr magneton. In [3] the modulation was studied using a zero Landau level function, which holds true only for moderate Maki parameter \( \alpha_M < 9 \). It was found in [4] that for large values of Maki parameter \( \alpha_M > 9 \) the higher Landau level solutions become relevant. In this case the critical field \( H_{c2}(T) \) consists of several curves each corresponding to a different Landau level solution. The analysis of the orbital effect in the FFLO state [3, 4] were performed for the isotropic metals with s-wave type of pairing. However in [13] it was demonstrated that it readily generalized for the case of the metals with elliptic Fermi surface. In such a case the Maki parameter becomes angular dependent.

For example for the case of the quasi-2D or anisotropic 3D superconductors \( \alpha_M \) increases dramatically for the in-plane field orientation. Therefore we may expect the transitions between the usual FFLO state with zero Landau levels [3] to the state with higher Landau levels [4–7] when the magnetic field is tilted from the perpendicular orientation to the parallel one. Also crossover from the pure FFLO state to the vortex states with higher Landau levels indexes in the model of quasi-2D system has been predicted in [14]. In real compounds the deviation of the Fermi surface from the elliptic form is crucial for the adequate description
of the FFLO state as well as the type of the superconductivity pairing (e.g. $s$- or $d$-wave) [15–17]. This circumstance is related with a fact that the description of the FFLO state in the framework of Ginzburg-Landau approach needs the consideration of the higher-order derivatives of the order parameter in addition to the usual gradient terms. For example, in the case of pure paramagnetic effect the critical field and modulation vector $q$ strongly depend on anisotropy or nesting properties of the Fermi surface [18–21]. In this article we consider a realistic case with a non-elliptic Fermi surface and for the definiteness we restrict ourself to the tetragonal symmetry. For example quasi-two-dimensional superconductor CeCoIn$_5$ [9] provides favorable conditions for the formation of the FFLO state and it has a tetragonal symmetry.

A characteristic feature of the FFLO state is the existence of a tricritical point (TCP) in the field-temperature phase diagram [22]. TCP is the meeting point of three transition lines separating the normal metal, the uniform superconductivity and the FFLO state. Formation of the FFLO state near the TCP may be described by modified Ginzburg-Landau functional (MGL) [23]. Appearance of the non-uniform state is related with a change of the sign of the coefficient $g$ at the gradient term $g|\Pi_i\Psi|^2$ in the free energy density. In the standard Ginzburg-Landau theory the coefficient $g$ is positive. Here it vanishes at the TCP $(T^*, H_{c2}(T^*))$, and then becomes negative for $T < T^*$. The absolute value of $g$ grows as we move further from the TCP, for example with increasing of the magnetic field or lowering temperature. A negative $g$ means that the modulated state has a lower free energy than the uniform one. In order to obtain the modulation vector one needs to include the term with higher order derivatives in the MGL functional [23].

In this paper we study the effects of crystal (or pairing) anisotropy on the FFLO phase. Using MGL approach we introduce free energy density $\mathcal{F}$ describing tetragonal system. We examine the case of the Fermi surface close to elliptic one. Therefore $\mathcal{F}$ can be divided into isotropic and perturbative parts. We demonstrate that the higher Landau level solutions may be realized for arbitrary values of Maki parameter in contrast with isotropic model. This is a special mechanism of the higher Landau level phase formation in 3D system. Moreover depending on various type of deviation of the Fermi surface from isotropic form three possible solutions for the FFLO state can be realized: (a) maximum modulation occurs along the magnetic field with zero Landau level state, (b) both modulation and higher Landau level
state, (c) highest possible Landau level and no modulation along the field (or modulation with very small wave-vector). Moreover due to the specific form of the Fermi surface the variation of magnetic field orientation may provoke transitions between the states with different Landau levels.

The main goal of the present paper is to demonstrate that in the presence of the orbital effect and for the realistic Fermi surface the very different types of the FFLO state could be realized. In particular, if the preferred modulation direction is perpendicular to the magnetic field this can results in the formation of the higher Landau levels mixed state with no modulation along the field at all. Our approach is fully justified near the TCP and for superconductors with large Maki parameters. However qualitatively it provides the understanding of the FFLO state at all temperatures and for arbitrary strength of the orbital effect. Here we calculate the line of the second order transition from the normal to the superconducting state. For this purpose we use the quadratic over the superconducting order parameter MGL functional. To describe the properties of the FFLO state it is needed to retain the higher order terms over the superconducting order parameter. The situation is completely analogous with that of the Abrikosov vortex lattice. From the symmetry reasons it is clear that the transitions between the mixed states describing by the different Landau levels will be the first order transitions. However the appearance of the modulation along the magnetic field may occur through a continuous transition. All these interesting questions deserve further studies but they are well beyond the scope of the present article.

II. FFLO STATE IN ANISOTROPIC GINZBURG-LANDAU MODEL

The most general form of the MGL functional quadratic over $\Psi$ is

$$\mathcal{F} = \alpha |\Psi|^2 - \sum_{i=1}^{3} g_i |\Pi_i \Psi|^2 + \sum_{i=1}^{3} \gamma_i |\Pi_i^2 \Psi|^2 + \sum_{i \neq j} \varepsilon_{ij} |\Pi_i \Pi_j \Psi|^2,$$

where $\alpha(H, T) = \alpha_0(T - T_{cu}(H))$, $T_{cu}(H)$ is transition temperature into the uniform superconducting state, $\Pi_i = -i\hbar \frac{\partial}{\partial x_i} - \frac{2e}{c} A_i$ are momentum operators and $A_i$ are the components of the vector potential, further we put $\hbar = 1$. Assuming the tetragonal symmetry in the case of the pure elliptic Fermi surface we have following relation for the effective mass $m_z \neq m_x = m_y = m$. The elliptical Fermi surface could be transformed into isotropic one.
by the following scaling transformation \( z' = \sqrt{m_z/m_z} \) [13]. Components of the magnetic field also transform as \( H \rightarrow H' = \left( \frac{m_x}{m_z} H_x, \frac{m_y}{m_z} H_y, H_z \right) \). Further we suppose that actual Fermi surface deviation from the elliptical form is small and after corresponding scaling transformation the functional (1) for the tetragonal symmetry is written as

\[
\mathcal{F} = \alpha |\Psi|^2 - g \sum_{i=1}^{3} |\Pi_i \Psi|^2 + \gamma \sum_{i=1}^{3} \Pi_i^2 |\Psi|^2 + \varepsilon_z |\Pi_x^2 \Psi|^2 + \frac{\varepsilon_x}{2} \left( |\Pi_x \Pi_y \Psi|^2 + |\Pi_y \Pi_z \Psi|^2 \right) \tag{2}
\]

\[
+ \bar{\varepsilon} \left( |\Pi_x^2 \Psi|^2 + |\Pi_x \Pi_z \Psi|^2 + |\Pi_y \Pi_y \Psi|^2 + |\Pi_y \Pi_z \Psi|^2 \right).
\]

Coefficients \( g, \gamma, \varepsilon_z, \varepsilon_x, \bar{\varepsilon} \) depend on structure of the Fermi surface, but for the FFLO appearance \( g \) must be positive. The terms \(-g \sum_{i=1}^{3} |\Pi_i \Psi|^2 + \gamma \sum_{i=1}^{3} \Pi_i^2 |\Psi|^2\) describe the elliptic form of the Fermi surface (for \( s \)-wave superconductivity) and terms with coefficients \( \varepsilon_z, \varepsilon_x \) and \( \bar{\varepsilon} \) are considered as perturbation.

Without orbital effect the momentum operators are simplified to \( \Pi_i = -i \frac{\partial}{\partial x_i} \) and the solution for the order parameter could be presented as \( \Psi = \Psi_q \exp(i\vec{q} \cdot \vec{r}) \). In the case of elliptic Fermi surface we have a degeneracy over direction of the FFLO modulation \( q \). The crystal structure effects are expressed via the terms with \( \varepsilon_z, \varepsilon_x, \bar{\varepsilon} \) and they lift this degeneracy and determine the direction of the FFLO modulation. The free energy density is written as

\[
\mathcal{F} = \sum_q \left\{ \alpha - gq^2 + \gamma q^4 + \varepsilon_z q^4 \cos^2 \theta + \varepsilon_x q^4 \sin^2 \theta \cos^2 \varphi + \bar{\varepsilon} q^4 \sin^2 \theta \cos^2 \varphi \right\} |\Psi_q|^2, \tag{3}
\]

where we have used spherical system for \( \vec{q} \): \( q_x = q \cos \varphi \sin \theta, q_y = q \sin \varphi \sin \theta, q_z = q \cos \theta \).

Due to the tetragonal symmetry the modulation in \( xy \)-plane is either parallel to \( x \) or \( y \) axis (\( \varepsilon_x > 0 \)) or along the bisector (\( \varepsilon_x < 0 \)). For definiteness we may suppose that \( \varepsilon_x > 0 \). Note that in the case \( \varepsilon_x < 0 \) the rotation of the \( xy \) axis by \( \pi/4 \) provides us the same functional (3) with renormalized coefficients \( \varepsilon_z' \) and \( \bar{\varepsilon}' \) but with \( \varepsilon_x > 0 \). The wave vector of modulation \( q \) will be in \( xy \)-plane if \( \bar{\varepsilon} > 2 \varepsilon_z, \varepsilon_x < 0 \), parallel to \( z \) axis if \( \bar{\varepsilon} > 0, \varepsilon_x > 0 \) (see Fig. 1). For the region \( \bar{\varepsilon} < 0, \bar{\varepsilon} < 2 \varepsilon_z \) the direction of modulation is at angle \( \theta = \frac{1}{2} \arccos \left( \frac{\bar{\varepsilon}}{(\bar{\varepsilon} - \varepsilon_x)} \right) \) to the \( xy \)-plane. Therefore the crystal anisotropy (or/and pairing anisotropy) lifts the degeneracy over the direction of the FFLO modulation and the whole diagram in the \((\varepsilon_z, \bar{\varepsilon})\) plane is presented in Fig. 1.
FIG. 1: Modulation $(\tilde{\varepsilon}, \varepsilon_z)$ diagram in the case of the absence of the orbital effect (pure paramagnetic limit). Areas with different patterns correspond to different orientation of the wave-vector modulation. The phase diagram does not depend on the $\varepsilon_x$ value.

III. ORBITAL EFFECT FOR THE MAGNETIC FIELD APPLIED ALONG Z AXIS

The exact solution of the linearized equation for $\Psi(\vec{r})$ is unavailable in general case if the orbital effect is taken into account. The Landau level solution with additional modulation along the field works only for the case of elliptical Fermi surface. In this case we obtain the degeneracy over modulation $q$ and Landau level $n$, when we move from the TCP. However if the anisotropy effects are taken into account they lift this degeneracy. We demonstrate that depending on the parameters of the system very different types of the FFLO state could be realized.

We begin with the case when magnetic field $H$ is applied along tetragonal $z$ axis and the gauge is chosen as $A=(yH, 0, 0)$. Following for example Ref. [7] we can express our operators $\Pi_i$ using boson operators of creation $\eta$ and annihilation $\eta^+$ as $\Pi_x = i \frac{1}{\sqrt{2} \xi_H} (\eta - \eta^+)$, $\Pi_y = \frac{1}{\sqrt{2} \xi_H} (\eta + \eta^+)$, where $\xi_H = \sqrt{\frac{eH}{2c}}$. Our goal is to find $T_c(H)$, which is the transition temperature into the FFLO state, i.e. we need to find the solution which gives the maximum
of $\alpha(H, T) = \alpha_0(T_c - T_{cu}(H))$. To do this we use the variation method [24] and look for a maximum of $\alpha$ written as

$$\alpha(H, T) = \max \left\{ \frac{\int [\alpha |\Psi|^2 - F] d^3r}{\int |\Psi|^2 d^3r} \right\}. \quad (4)$$

In general case $\Psi(x, y, z) = \sum C_n \varphi_n(x, y)e^{iq_{n}z}$, but since the anisotropy effects are small we can approximate our solution only with a single Landau level function $\varphi_N$ [25], which is well known solution for the system with isotropic form of the Fermi surface. In our calculations we use the following properties of Landau functions: $\int \varphi_n \varphi_md^3r = \delta_{nm}$, $\eta \varphi_n = \sqrt{n} \varphi_{n-1}$, $\eta^+ \varphi_n = \sqrt{n+1} \varphi_{n+1}$; we also normalize $C_n$ so that $\int |\Psi|^2 d^3r = 1$. To neglect the other Landau level functions $(n \neq N)$ in our $\Psi$ representation their corresponding coefficients should be small comparing to the coefficient $C_N$. This leads to the following condition for the applicability of the single level approximation $\xi_H^{-2} \gg \frac{4}{7} \sqrt{\frac{\varepsilon}{\gamma}}$ which is determined by the Fermi surface deviation $\varepsilon$ from the elliptic form ($\varepsilon$ is of the order of average between $\tilde{\varepsilon}$, $\varepsilon_z$ and $\varepsilon_x$ and its exact value depends on magnetic field orientation). Calculating $\alpha(H, T)$ using the operators $\eta$ and $\eta^+$ we can express it in terms of $q_z$ and $\xi_n^{-2} = \xi_H^{-2}(2n + 1)$ as

$$\alpha(H, T) = \max \left\{ g \left[ q_z^2 + \xi_n^{-2} \right] - \gamma \left[ q_z^2 + \xi_n^{-2} \right]^2 \right\} - \varepsilon_z q_z^4 - \varepsilon_x \xi_n^{-4} - \varepsilon_x \xi_n^{-2} - \frac{\varepsilon_z}{8} \xi_n^{-1} - \frac{5}{8} \xi_H^{-4} \right\}. \quad (5)$$

If we consider only unperturbed part $g \left[ q_z^2 + \xi_n^{-2} \right] - \gamma \left[ q_z^2 + \xi_n^{-2} \right]^2$, put $u = q_z^2 + \xi_n^{-2}$ and take derivative with respect to $u$ we obtain the maximum point at $u_0 = g/2\gamma$. When the system is close to the TCP, $g$ is small so that $g/2\gamma < \xi_H^{-2}$. In this case we obtain the lowest Landau level $n = 0$ and no modulation along $z$ ($q_z = 0$). If we move further from the TCP and $g$ grows ($\xi_H^{-2} < g/2\gamma < 3\xi_H^{-2}$) then $n$ remains equal to 0, but $q_z$ is not. If Maki parameter is large then our approach will still be valid even far away from the TCP. In this case $g/2\gamma \geq 3\xi_H^{-2}$ and $n$ can be larger than 0. Hence, we have degeneracy over choosing of Landau level $n$ and $q_z$. But as was written earlier taking into account small perturbative terms with $\varepsilon_z, \varepsilon_x, \tilde{\varepsilon}$ we remove this degeneracy and find the maximum $\alpha(H, T)$ with a respect to $q_z = \sqrt{u_0 - \xi_n^{-2}}$. There are three possible types of solutions (combinations of $n$ and $q_z$) depending on $\varepsilon_z, \varepsilon_x, \tilde{\varepsilon}$: (a) maximum modulation $q_z = \sqrt{g/2\gamma - \xi_H^{-2}}$ and zero Landau level $n$; (b) non zero modulation...
\( q_z = \sqrt{\frac{g}{2\gamma} (\varepsilon - 4\tilde{\varepsilon}) / (8\varepsilon_z + \varepsilon - 8\tilde{\varepsilon})} \) with \( n = \left[ \frac{1}{2} (\xi_H^2 (u_0 - q_z^2) - 1) \right] \), where brackets \([\ ]\) mean that only integer part is taken; (c) highest possible Landau level \( n = \left[ \frac{g}{4\gamma\xi_H^2} - \frac{1}{2} \right] \) and near zero modulation. All these cases are shown in Fig. 2. However due to integer nature of \( n \) the modulation \( q_z \) has a very special behavior in cases b) and c), it changes abruptly every time when our solution jumps from one to another Landau level. It should be noted that in the case c) the wave-vector of modulation \( q_z \) instead of being zero oscillates with \( H \) (or with \( g \), when we move further form the TCP) due to the mismatch of \( u_0 = g/2\gamma \) and \( \xi_n^{-2} = \xi_H^{-2}(2n + 1) \). The diagram shown in Fig. 2 looks similar to that in Fig. 1 with the exception of the shift along both axis due to the presence of the \( \varepsilon_x \) coefficient. When we get zero wave-vector of modulation in Fig. 2 the same area in Fig. 1 corresponds to modulation in \( xy \)-plane. If modulation vector \( q \) along the applied magnetic field is zero then modulation could arise in the direction perpendicular to the field. When \( n \) and \( q_z \) is intermediate (not maximum and not zero, lower right part of the diagrams) FFLO modulation could be formed in both perpendicular and parallel directions to the field \( (q_\parallel^2 + q_\perp^2 = g/2\gamma - \xi_H^{-2}) \). It should be noted that we cannot achieve a smooth transition from one diagram to another by decreasing \( H \) to 0 due the condition of single Landau level approximation \( \xi_H^{-2} \gg \frac{g}{9\sqrt{\gamma}} \), despite the fact that the only difference between diagrams in Figs. 1 and 2 is shift of the intersection point due to the presence of the \( \varepsilon_x \) coefficient. The same is true for the case \( H \parallel x \) where difference will be more significant.

**Maximum Landau level and residual modulation.** Due to the fact that \( n \) is integer the wave-vector of modulation \( q_z \) may not be equal exactly to zero but to some value less than \( \sqrt{u_0 - \xi_n^{-2}} \) when \( n \) is maximum. To calculate this value we maximize \( \alpha(H, T) \) from the Eq. (5) again but this time \( \xi_n^{-2} = \xi_H^{-2}(2n + 1) \) will be treated as constant. We obtain \( q_z^2 = \max \left( 0, \frac{u_0 - \xi_n^{-2} - (\tilde{\varepsilon}/2\gamma)\xi_n^{-2}}{1 + (\varepsilon_x/\gamma)} \right) \) and the general solution for residual modulation \( q_z \) will oscillate with \( H \) or with \( g \) (absolute value of \( g \) is increasing when we are moving away from the TCP). Wave vector of modulation is zero when \( \xi_n^{-2} \) is close to \( u_0 = g/2\gamma \), then is start to increase linearly with \( g \) until it drops to zero again when the solution ”jumps” to another Landau level (Fig. 3). With the increasing effect of anisotropy, the area of zero wave-vector modulation widens, and in the limiting case it will cover all parameter range. Similar results were obtained for isotropic case at \( \alpha_M > 9 \) at low temperature [4].
FIG. 2: Modulation diagram in the case when the magnetic field applied along \( z \) axis. There are 3 areas on the diagram corresponding to 3 types of the solution for modulation vector \( q_z \) and Landau level \( n \). Modulation direction is always parallel to the applied field and \( \varepsilon_x \) here is chosen equal to \( \gamma \).

FIG. 3: Dependence of residual modulation on the parameter \( g \) (normalized to \( \gamma \xi_H^{-2} \)). The wave vector of modulation \( q \) is shown here by the solid line and measured in units of \( \xi_H^{-1} \). The values of Landau level \( n \) is shown by the dashed line. The parameter \( \bar{\varepsilon} \) is chosen here as 0.2\( \gamma \).
IV. THE CASE OF THE MAGNETIC FIELD APPLIED ALONG X AXIS

For magnetic field applied along the \(x\) axis we choose \( \vec{A} \) as \((0, 0, yH)\), the order parameter \( \Psi(x, y, z) = \varphi_N(y)e^{iq_xx} \) where \( q_x \) is modulation along the \(x\) axis, and reintroduce creation and annihilation operators as \( \eta = \frac{\xi_H}{\sqrt{2}}(\Pi_y - i\Pi_z) \) and \( \eta^+ = \frac{\xi_H}{\sqrt{2}}(\Pi_y + i\Pi_z) \). Repeating the same calculations as for the \(H||z\) case we have

\[
\alpha(H, T) = \max \left\{ g \left[ q_x^2 + \xi_n^{-2} \right] - \gamma \left[ q_x^2 + \xi_n^{-2} \right]^2 \right. \\
-\frac{3}{8} \varepsilon_x^{-4} - \frac{1}{2} \varepsilon_x q_x^2 \xi_n^{-2} - \frac{1}{2} \varepsilon_x \xi_n^{-4} - \frac{3}{8} \xi_H^{-4} - \frac{5}{8} \xi_H^{-4} \left. \right\}.
\]

Again we put \( u = q_x^2 + \xi_n^{-2} \) and find that maximum point \( u_0 = g/2\gamma \) is the same for unperturbed part. The degeneracies over \( q_x \) and \( n \) are removed in a similar way by taking into account the perturbative terms \( \varepsilon_z, \varepsilon_x, \varepsilon_H \). Diagram for maximums of \( \alpha(H, T) \) in the case with magnetic field applied along \(x\) axis is shown in Fig. 4. Three main areas of the diagrams are similar to the ones shown in Fig. 2: (a) maximum modulation \( q_x = \sqrt{g/2\gamma - \xi_H^{-2}} \) and zero Landau level \( n \); (b) non zero modulation \( q_x = \sqrt{g/2\gamma (3\varepsilon_z - \varepsilon_H - 2\varepsilon_x)/(3\varepsilon_z - 3\varepsilon_H - 4\varepsilon_x)} \) with \( n = \left[ \frac{1}{2} (\xi_H^2 (u_0 - q_x^2) - 1) \right] \); (c) highest possible Landau level \( n = \left[ \frac{1}{2} \xi_H^2 - \frac{1}{2} \right] \) and residual modulation described earlier. Diagrams shown in Figs. 4 and 1 have some similarities with a respect to the conditions for the different types of solutions. On both diagrams the FFLO modulation along \(x\) (or \(y\)) axis corresponds to the upper-right quarter and there is no modulation in \(xy\) plane in the left quarters of the diagrams.

V. MAGNETIC FIELD APPLIED IN XY PLANE

If a magnetic field \(H\) is applied in the \(xy\)-plane (\(\beta\) is the angle between \(\vec{H}\) and \(x\) axis), then it is convenient to rotate \(x, y\) axis around \(z\) by angle \(\beta\) to reduce the problem to the case \(H||x\). Under this rotation the terms with coefficients \(g, \gamma, \varepsilon_z\) remain unchanged and the rest parts are transformed according to rules \(x' = x \cos \beta + y \sin \beta, y' = y \cos \beta - x \sin \beta\).
There are three areas on the diagram corresponding to different types of the solution for modulation vector $q_x$ and Landau level $n$. Modulation direction is always parallel to the applied field. The position of the intersection point is determined by the coefficient $\varepsilon_x$.

\[
\Pi_x = \Pi'_x \cos \beta - \Pi'_y \sin \beta \tag{7}
\]
\[
\Pi_y = \Pi'_x \sin \beta + \Pi'_y \cos \beta \tag{8}
\]
\[
\Pi_z = \Pi'_z = -i \frac{\partial}{\partial z} - \frac{2e}{c} y' H \tag{9}
\]

The operators $\eta$ and $\eta^+$ are expressed using new $\Pi'_x$, $\Pi'_x$ and $\Pi'_z$ as before in the case $H \parallel x$. Due to the symmetry of the problem only $\varepsilon_x$ term in Eq. (2) acquires the dependence on $\beta$ in the final expression for $\alpha(H, T)$

\[
\alpha(H, T) = \max \left\{ g \left[ q_x'^2 + \xi_n^{-2} \right] - \gamma \left[ q_x'^2 + \xi_n^{-2} \right]^2 - \varepsilon_x \frac{3}{8} \xi_n^{-4} - \frac{1}{2} \varepsilon_x q_x'^2 \xi_n^{-2} - \varepsilon_x \frac{\sin^2 2\beta}{4} \left( q_x'^4 - 3q_x'^2 \xi_n^{-2} + \frac{3}{8} \xi_n^{-4} + \frac{3}{8} \xi_n^{-4} \right) - \frac{1}{2} \varepsilon_x q_x'^2 \xi_n^{-2} - \varepsilon_x \frac{1}{8} \xi_n^{-4} - \varepsilon_x \frac{3}{8} \xi_n^{-4} - \frac{5}{8} \xi_n^{-1} \right\}, \tag{10}
\]

where $q_x'$ is modulation along the new $x'$ axis parallel to the magnetic field $H$. Directly from the $\varepsilon_x$ term it can be concluded that the angles $\beta = 0, \pm \frac{\pi}{2}, \pi$ will lead to the old results, when

FIG. 4: Modulation diagram ($\overline{\varepsilon}, \varepsilon_z$) in the case when the magnetic field is applied along $x$ axis.
magnetic field is applied along $x$ or $y$ axis. The main results for the maximum of $\alpha(H, T)$ will be similar to the case $H \parallel x$, with the only exception that separation lines on phase diagram are changed to three lines: $\tilde{\varepsilon} = -3\varepsilon_z + \frac{5}{2}\sin^2 2\beta \varepsilon_x$, $\tilde{\varepsilon} = 3\varepsilon_z + \left(\frac{15}{4}\sin^2 2\beta - 2\right) \varepsilon_x$ and $\tilde{\varepsilon} = \left(\frac{5}{4}\sin^2 2\beta - 1\right) \varepsilon_x$. However this change only affects the initial shift of the diagram from the center. For example in Fig. 5 the case $\beta = \frac{\pi}{2}(n + \frac{1}{2})$ is shown and the intersection point has shifted to the opposite quarter of the graph. For general values of $\beta$, the intersection point is situated at $(\varepsilon_z, \tilde{\varepsilon}) = \left((-\frac{5}{12}\sin^2 2\beta + \frac{1}{3}) \varepsilon_x, (\frac{5}{2}\sin^2 2\beta - 1) \varepsilon_x\right)$ on a line segment connecting the two intersection points for $\beta = \frac{\pi}{2} n$ and $\beta = \frac{\pi}{2}(n + \frac{1}{2})$. On phase diagram like in previous cases we have 3 possible solutions:

1. $q_x^2 = u_0 \left(3\varepsilon_z - \tilde{\varepsilon} - 2\varepsilon_x + \frac{15}{4}\sin^2 2\beta \varepsilon_x\right) / \left(3\tilde{\varepsilon} - 3\varepsilon_z - 4\varepsilon_x + \frac{35}{4}\sin^2 2\beta \varepsilon_x\right)$, $n = \left[\frac{1}{2} \left(\xi^2_H (u_0 - q_x^2) - 1\right)\right]$, when $\tilde{\varepsilon} < \min \left(3\varepsilon_z - 2\varepsilon_x + \frac{15}{4}\sin^2 2\beta \varepsilon_x, \varepsilon_x (\frac{5}{2}\sin^2 2\beta \varepsilon_x - 1)\right)$.

2. $q_x^2 \approx 0$ (near zero residual modulation), maximum $n$ in this case will be equal to $\left[\frac{g}{4\gamma\xi^2_H} - \frac{1}{2}\right]$. This corresponds to $-3\varepsilon_z + \frac{5}{4}\sin^2 2\beta \varepsilon_x > \tilde{\varepsilon} > 3\varepsilon_z - 2\varepsilon_x + \frac{15}{4}\sin^2 2\beta \varepsilon_x$.

3. $q_x^2 = \sqrt{\frac{g}{2\gamma} - \xi^2_H}$ - maximum modulation along the magnetic field with $n = 0$. This corresponds to $\tilde{\varepsilon} > \max \left(\varepsilon_x (\frac{5}{2}\sin^2 2\beta \varepsilon_x - 1), -3\varepsilon_z + \frac{5}{4}\sin^2 2\beta \varepsilon_x\right)$.

Using $q_x^2$ corresponding to the maximum of $\alpha(H, T)$ we find $T_c(H)$. When the parameters of our system (the actual values of $\tilde{\varepsilon}, \varepsilon_z, \varepsilon_x$ coefficients) correspond to the point in $(\tilde{\varepsilon}, \varepsilon_z)$ plane situated near one of the separation lines in Figs. 4 or 5 then the magnetic field rotation can lead to transition between the corresponding two phases. The simplest case with $\tilde{\varepsilon} = 0$, $\varepsilon_z = 0$ is shown in the Figs. 6 and 7. For positive $\varepsilon_x$ the transition is between states $(q = 0, n = \max)$ and $(q > 0, n > 0)$; for negative $\varepsilon_x$ the two states are $(q = \max, n = 0)$ and $(q > 0, n > 0)$. Integer nature of Landau level $n$ manifests itself in the state $(q > 0, n > 0)$, when the FFLO modulation could change several Landau levels while magnetic field rotates in a given region. In this case $T_c$ line consists of several curves each corresponding to a different Landau level solution. If switching between the different solutions does not occur then general $T_{cu}$ dependence will be reduced to the simple sinusoidal form with period $\pi/4$. 

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FIG. 5: Modulation diagram in the case when the magnetic field applied in xy-plane. Solid separation lines correspond to the case of $\beta = \pi/4$ and dashed lines to the case of $\beta = 0$. Patterned area corresponds to the region where the type of the solution for $q$ and $n$ could be changed during rotation along $z$ axis.

FIG. 6: Transition temperature dependence on the angle $\beta$ of the magnetic field in $xy$-plane. For illustration we have chosen $\varepsilon_x = \bar{\varepsilon} = 0$, $\varepsilon_x = 0.1\gamma$, $g$ (normalized to $\gamma\xi_H^{-2}$) equal to 100. There are switching between two types of the solution: $(q > 0, n > 0)$ and $(q = 0, n = \text{max})$. 
FIG. 7: Transition temperature dependence on the angle $\beta$ of the magnetic field in $xy$-plane. For illustration we have chosen $\varepsilon_z = \varepsilon = 0$, $\varepsilon_x = -0.1\gamma$, $g$ (normalized to $\gamma\xi_H^{-2}$) equal to 40. There are switching between two types of the solution: $(q > 0, n > 0)$ and $(q = \text{max}, n = 0)$. The inset shows zoom of the region near switching point. $T_c$ line consists of several curves each corresponding to a different Landau level $n$ ($n = 0, 1$ and $2$ in this case).

VI. CUBIC SYMMETRY

In the case of cubic symmetry $\varepsilon_z$ is equal to 0 and $\varepsilon = \varepsilon_x = \varepsilon$. In the absence of the orbital effect the direction of modulation will be along one of the axis if $\varepsilon > 0$, and along one of the main diagonals if $\varepsilon < 0$. In the presence of orbital effect when magnetic field is applied along one of the cubic axis the type of solution for maximum $\alpha(H, T)$ depends only on the sign of $\varepsilon$. If $\varepsilon < 0$ then $q$ is equal to $\sqrt{\frac{2}{3}u_0}$ and $n = \left[\frac{2}{7}\xi_H^2u_0 - \frac{1}{2}\right]$. For $\varepsilon > 0$ there will be the choice between highest and zero modulations. If $u_0 > 7\xi_H^{-2}$ then we obtain the maximum modulation along the field $q = \sqrt{u_0 - \xi_H^{-2}}$ and Landau level $n$ will be zero. But when $u_0 < 7\xi_H^{-2}$ it is more favorable to have zero (or some residual due to the mismatch) modulation $q$ and highest Landau level $n = \left[\frac{1}{2}\xi_H^2u_0 - \frac{1}{2}\right]$. In the latter case when the modulation along the field is absent ($q \approx 0$) it turns out that the maximum Landau level can not be higher than 3. In the case with magnetic field applied along the one of the diagonals we have the same situation but with the opposite sign of $\varepsilon$. Note that in all these
cases the momentum and Landau level do not depend on \( \varepsilon \) at the first approximation.

**VII. DISCUSSION**

We investigated the influence of the crystal structure effects on the FFLO state based on the modified Ginzburg-Landau approach. We analyzed the possible solutions for the FFLO modulation vector and relevant Landau level functions. We have used the single-level approximation, but we believe that qualitatively our results would remain valid even if we take into account the general multi-level representation of the order parameter. For illustration we have restricted ourself to the tetragonal symmetry because most promising material for FFLO realization CeCoIn\(_5\) has namely this type of the symmetry. Our results can be easily generalized to any symmetry as long as deviation of the Fermi surface from the elliptic form can be treated as a perturbation. In the opposite case the single-level Landau function solution will be transformed into a series of higher level functions. Also this will lead to the broadening of the \( q = 0 \) region shown in Fig. 3, which means that for a wide range of parameters in such a case there will be no more modulation along the field. The form of the Fermi surface determines the direction of the FFLO modulation in the pure paramagnetic limit. We see that in the presence of the orbital effect the system tries in some way to reproduce this optimal directions of the FFLO modulation by varying the Landau level index \( n \) and wave-vector of the modulation along the field.

The higher Landau level solutions has been predicted for the FFLO phase in 2D superconductors in tilted magnetic field [5–7], in 3D \( d \)-wave and quasi-2D \( s \)- and \( d \)-wave superconductors [11, 27], and in 3D isotropic superconductors at low temperature provided the Maki parameter is large [4]. Here we have demonstrated that for certain field orientations such states naturally appear in real 3D compounds in a whole region of the FFLO phase existence (without any restriction to the value of Maki parameter). This behavior is related with crystal structure and/or pairing symmetry effects. The isotropic models used so far to describe FFLO state fail to predict these different types of the scenarios of the FFLO transitions. Indeed following the isotropic (or quasi-isotropic) model the transition to the FFLO state with the increase of the magnetic field always occurs via the modulation appearance along the field direction. On the contrary in the present paper we predict the FFLO transition as a formation of the higher Landau level states. The vortex state that
corresponds to these higher Landau level solutions have a rather complicated structure due to the competition between two length scales, the average distance between vortices and the FFLO period [12, 26, 28]. Recently in [29] the very special vortex phases with spatial line nodes forming a variety of 3D spatial configurations has been predicted. Therefore we may expect that the mixed state in the FFLO superconductor may be very different from the usual Abrikosov lattice, provided that the higher Landau level solutions are realized. The experimentally verified consequences of these scenario of the FFLO transition are the first order transitions between the states with different Landau level solutions (namely between \( n = 0 \) and \( n = 1 \)), accompanying by the strong change of the vortex lattice structure. The standard experimental techniques of the vortex lattice observation (including the neutron scattering) could be used to detect these transformations.

It is commonly believed that the FFLO state in CeCoIn\(_5\) corresponds to the state with the modulation along the magnetic field for both field orientations: along the tetragonal axis and in the basal plane. However, comparing the \((\xi, \xi_z)\) diagrams (Figs. 2 and 4) we see that the situation when we have a zero Landau level solution for this two field orientations is improbable. In CeCoIn\(_5\) the crystal structure effects are rather important – for example in [30] the vortex lattice reorientation transition have been reported as well as in-plane anisotropy of the upper critical field [31]. In such a case we can expect that for one of these field orientations the Landau level solution with \( n \geq 1 \) may be realized. Note that such a possibility in connection with the FFLO state in CeCoIn\(_5\) has been discussed in [32]. If the crystal structure effects are large enough for the Landau level solutions with \( n \geq 1 \) the modulation along the field may be absent. Very recently [33] the modulated antiferromagnetic ordering has been reported in the low temperature superconducting phase of CeCoIn\(_5\) at the magnetic field in the basal plane. The antiferromagnetic ordering plays for the FFLO state the role of the crystal structure effect favoring the orientation of the FFLO modulation wavevector along the antiferromagnetic one [13]. The texture in the superconducting order parameter revealed by NMR experiments looks different for different field orientations [9] as well as the anomaly in the local magnetic inductor measurements [34]. This may indicate on the different types of the FFLO state for different field orientations. Presumably for the field orientation in the basal plane there are no FFLO modulation along the field.
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