Counting Humps in Motzkin paths

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Abstract. In this paper we study the number of humps (peaks) in Dyck, Motzkin and Schröder paths. Recently A. Regev noticed that the number of peaks in all Dyck paths of order $n$ is one half of the number of super Dyck paths of order $n$. He also computed the number of humps in Motzkin paths and found a similar relation, and asked for bijective proofs. We give a bijection and prove these results. Using this bijection we also give a new proof that the number of Dyck paths of order $n$ with $k$ peaks is the Narayana number. By double counting super Schröder paths, we also get an identity involving products of binomial coefficients.

Keywords: Dyck paths, Motzkin paths, Schröder paths, humps, peaks, Narayana number.

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1 Introduction

A Dyck path of order (semilength) $n$ is a lattice path in $\mathbb{Z} \times \mathbb{Z}$, from $(0,0)$ to $(2n,0)$, using up-steps $(1,1)$ (denoted by $U$) and down-steps $(1,-1)$ (denoted by $D$) and never going below the $x$-axis. We use $\mathcal{D}_n$ to denote the set of Dyck paths of order $n$. It is well known that $\mathcal{D}_n$ is counted by the $n$-th Catalan number (A000108 in [8])

$$C_n = \frac{1}{n+1} \binom{2n}{n}.$$ 

A peak in a Dyck path is two consecutive steps $UD$. It is also well known (see, for example, [1, 4, 11]) that the number of Dyck paths of order $n$ with $k$ peaks is the Narayana number (A001263):

$$N(n, k) = \frac{1}{n} \binom{n}{k} \binom{n}{k-1}.$$ 

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Counting Dyck paths with restriction on peaks has been studied by many authors, see for example [2, 3, 5]. Here we are interested in counting peaks in all Dyck paths of order \( n \). By summing over the above formula over \( k \) we immediately get the following result: the total number of peaks in all Dyck paths of order \( n \) is

\[
p_{dn} = \sum_{k=1}^{n} kN(n, k) = \binom{2n-1}{n}.
\]

If we allow a Dyck path to go below the \( x \)-axis, we get a super Dyck path. Let \( SD_n \) denote the set of super Dyck paths of order \( n \). By standard arguments we have

\[
s_{dn} = \#SD_n = \binom{2n}{n} = 2 \binom{2n-1}{n} = 2p_{dn}, \quad (1.1)
\]

That is, the number of super Dyck paths of order \( n \) is twice the number of peaks in all Dyck paths of order \( n \). This curious relation was first noticed by Regev [7], who also noticed that similar relation holds for Motzkin paths, which we will explain next.

A Motzkin path of order \( n \) is a lattice path in \( \mathbb{Z} \times \mathbb{Z} \), from \((0, 0)\) to \((n, 0)\), using up-steps \((1, 1)\), down-steps \((1, -1)\) and flat-steps \((1, 0)\) (denoted by \( F \)) that never goes below the \( x \)-axis. Let \( M_n \) denote all the Motzkin paths of order \( n \). The cardinality of \( M_n \) is the \( n \)-th Motzkin number \( m_n \) (A001006), which satisfies the following recurrence relation

\[
m_0 = 1, \quad m_1 = 1, \quad m_n = m_{n-1} + \sum_{i=2}^{n} m_{i-2}m_{n-i}, \quad \text{for} \quad n \geq 2,
\]

and have generating function

\[
\sum_{n \geq 0} m_n x^n = \frac{1 - x - \sqrt{1 - 2x - 3x^2}}{2x^2}.
\]

A hump in a Motzkin path is an up step followed by zero or more flat steps followed by a down step. We use \( hm_n \) to denote the total number of humps in all Motzkin paths of order \( n \). We can similarly define super Motzkin paths to be Motzkin paths that are allowed to go below the \( x \)-axis, and use \( SM_n \) to denote the set of super Motzkin paths of order \( n \). Using a recurrence relation and the WZ method [6, 12], Regev ([7]) proved that

\[
s_{mn} = \#SM_n = \sum_{j \geq 0} \binom{n}{j} \binom{n-j}{j} = 2hm_n + 1 \quad (1.2)
\]

and asked for a bijective proof of (1.1) and (1.2). The main result of this paper is such a bijective proof.

Let \( SM^U_n(k) \) (\( SM^D_n(k) \)) denote the set of paths in \( SM_n \) with \( k \) peaks and the first non-flat step is \( U \), and the last non-flat step is \( U \) (\( D \)). Let \( SM^*_n \) denote all paths in \( SM_n \) whose first non-flat step is \( U \), and define

\[
\mathcal{H}M_n = \{(M, P) | M \in M_n, P \text{ is a hump of } M \}.
\]

The main result of this paper is the following:
**Theorem 1.1** There is a bijection $\Phi : \mathcal{H}M_n \to SM_n^{U*}$ such that if $(M, P) \in \mathcal{H}M_n$ and $L = \Phi(M, P)$, then there are $k$ humps in $M$ if and only if $L \in SM_n^{U}(k-1) \cup SM_n^{D}(k)$.

The outline of the paper is as follows. In Section 2 we define the bijection $\Phi$ and prove Theorem 1.1. In section 3 we apply $\Phi$ to Dyck paths and give a new proof of the Narayana numbers. In section 4 we apply $\Phi$ to Schröder paths and get an identity involving products of binomial coefficients by double counting super Schröder paths whose $F$ steps are $m$-colored.

**2 The bijection $\Phi : \mathcal{H}M_n \leftrightarrow SM_n^{U*}$**

Note that a Motzkin path $M$ of order $n$ can also be considered as a sequence $M = M_1M_2 \cdots M_n$, with $M_i \in \{U, F, D\}$, and the number of $U$’s is not less than the number of $D$’s in every subsequence $M_1M_2 \cdots M_k$ of $M$. Hence a hump in $M$ is a subsequence $P = M_iM_{i+1} \cdots M_{i+k+1}, k \geq 0$, such that $M_i = U, M_{i+1} = M_{i+2} = \cdots = M_{i+k} = F$ and $M_{i+k+1} = D$. We call the end point of step $M_i$ a hump point, and will also denoted as $P$. Similarly, if there exists $i$ such that $M_i = D, M_{i+1} = M_{i+2} = \cdots = M_{i+k} = F, k \geq 0, M_{i+k+1} = U$, then we call the subsequence $M_iM_{i+1} \cdots M_{i+k+1}$ a valley of $M$, and the end point of $M_{i+k}$ is called a valley point. The end point $(n, 0)$ of $M$ is also considered as a valley point.

Suppose $L$ is a path in $\mathbb{Z} \times \mathbb{Z}$ from $O(0, 0)$ to $N(n, 0)$, and $A$ a lattice point on $M$, we use $x_A$ and $y_A$ to denote the $x$-coordinate and $y$-coordinate of $A$, respectively. The sub-path of $L$ from point $A$ to point $B$ is denoted by $L_{AB}$. We use $\bar{L}$ to denote the lattice path obtained from $L$ by interchanging all the up-steps and down-steps in $L$, and keep the flat-steps unchanged.

Now we are ready to define the map $\Phi$ and prove Theorem 1.1.

**Proof of Theorem 1.1:**

1. The map $\Phi : \mathcal{H}M_n \to SM_n^{U*}$.

For any $(M, P) \in \mathcal{H}M_n$, we define $L = \Phi(M, P)$ by the following rules:

- Let $C$ be the leftmost valley point in $M$ such that $x_C > x_P$;
- Let $B$ be the rightmost point in $M$ such that $x_B < x_P, y_B = y_C$;
- Let $A$ be the rightmost point in $M$ such that $y_A = 0, x_A \leq x_B$;
- Set $L_0 = M_{OA}, L_1 = M_{AB}, L_2 = M_{BC}, L_3 = M_{CN}$ (Note that $L_0, L_1$ and $L_3$ may be empty);
- Define $L = \Phi(M, P) = L_0L_2\bar{L}_3\bar{L}_1$.

Now we will prove that $L \in SM_n^{U*}$. According to the above definition, $L_0$ and $L_2$ are both Motzkin paths, therefore $\#U = \#D$ in $L_0$ and $L_2$. And for $L_1$, we have $\#U - \#D = y_B - y_A = y_B = y_C$, for $L_3$, $\#U - \#D = -y_C$. Therefore the total number of $U$’s is as much as that of $D$’s in $L$. Thus $L$ is a super Motzkin path of order $n$. Moreover, the first non-flat step in $L$ must be in $L_0$ (when $L_0$ is not empty) or in $L_2$ (when $L_0$ is empty), and $L_0, L_2$ are
both Motzkin paths, hence the first step leaving the $x$-axis must be a $U$. Therefore we proved that $L = \Phi(M, P) \in S\mathcal{M}^U_n$.

(2) The inverse of $\Phi$. 

For any $L \in S\mathcal{M}^U_n$, we define $\Psi$ by the following rules:

- Let $B$ be the leftmost point such that $y_B = 0$, and $L$ goes below the $x$-axis after $B$. (If such a point does not exist, then set $B = N$);
- Let $A$ be the rightmost point in $L$ such that $x_A < x_B, y_A = 0$;
- Let $C$ be the rightmost point in $L$ such that $x_C \geq x_B$, and $\forall G, x_G \geq x_B$ implies that $y_C \geq y_G$;
- Let $P$ be the rightmost hump point in $L$ such that $x_P < x_B$;
- Set $L_0 = L_{OA}, L_1 = L_{AB}, L_2 = L_{BC}, L_3 = L_{CN}$ (Note that $L_0, L_2$ and $L_3$ may be empty);
- Set $M = L_0\overrightarrow{L_3}L_1\overrightarrow{L_2}$, and $\Psi(L) = (M, P)$.

Now we prove that $\Psi = \Phi^{-1}$. Since $C$ is the highest point in $L_3$, and $\overrightarrow{L_3}$ and $L_3$ are symmetric with respect to the line $y = y_C$, $C$ is mapped to the lowest point in $\overrightarrow{L_3}$. Moreover, $L_0$ and $L_1$ are both Motzkin paths, then $L_0\overrightarrow{L_3}L_1$ does not go below the $x$-axis, and the $y$-coordinate of the end point of $L_0\overrightarrow{L_3}L_1$ is $y_C$. In $\overrightarrow{L_2}$, the end point is the lowest point, and the start point of $\overrightarrow{L_2}$ is $y_C$ higher than the end point. So $M = L_0\overrightarrow{L_3}L_1\overrightarrow{L_2}$ ends on the $x$-axis and never goes below it, i.e., $M \in \mathcal{M}_n$. Thus $\Psi(L) \in \mathcal{H}\mathcal{M}_n$, and it is not hard to see that $\Psi = \Phi^{-1}$.

(3) There are $k$ humps in $M$ if and only if $\Phi(M, P) \in S\mathcal{M}^{UD}_n(k) \cup S\mathcal{M}^{UU}_n(k - 1)$.

Since $\Phi(M) = L_0L_2\overrightarrow{L_3}L_1 = L$, the number of humps changes only in sub-paths $\overrightarrow{L_3}$ and $\overrightarrow{L_1}$ when $M$ is converted to $L$. If the last step of $L_1$ is $U$, then the last step in $\overrightarrow{L_1}$ becomes $D$. The number of humps in $L_1$ is the same as the number of humps in $\overrightarrow{L_1}$, and the number of humps in $\overrightarrow{L_3}$ is 1 less than the number of humps in $L_3$. The last step in $\overrightarrow{L_3}$ is $U$ step, so concatenating $\overrightarrow{L_1}$ with $\overrightarrow{L_3}$ yields a new hump. Therefore the total number of humps in $L$ is the same as in $M$. Thus we have $\Phi(M, P) \in S\mathcal{M}^{UD}_n(k)$.

If the last step in $L_1$ is $D$, then the last step in $\overrightarrow{L_1}$ is $U$. The number of humps in $\overrightarrow{L_1}$ is 1 less than the number of humps in $L_1$, and the humps in $\overrightarrow{L_3}$ is 1 less than the number of humps in $L_3$. Moreover, the last step in $\overrightarrow{L_3}$ is $U$, so concatenating $\overrightarrow{L_1}$ with $\overrightarrow{L_3}$ yields a new hump. Therefore the total number of humps in $L$ is 1 less than the number humps in $M$. Thus we have $\Phi(M, P) \in S\mathcal{M}^{UU}_n(k - 1)$.

As an example, Figure 1 shows a Motzkin path $M \in \mathcal{M}_{41}$ with a circled hump point $P$, and Figure 2 shows a super Motzkin path $L \in S\mathcal{M}^*_n = \Phi(M, P)$.

From Theorem 1.1 we can easily get the following result.
Corollary 2.2 For all \( n \geq 0 \), we have
\[
sm_n = 2hm_n + 1,
\]
(2.1)
and
\[
hm_n = \frac{1}{2} \left( \sum_{j \geq 0} \binom{n}{j} \binom{n-j}{j} - 1 \right).
\]
(2.2)

Proof. Equation (2.1) follows immediately from Theorem 1.1. To prove (2.2) we count super Motzkin paths with \( j U \) steps. We can first choose the \( j U \) steps among the total \( n \) steps, then choose \( j \) steps as \( D \) steps among the remaining \( n-j \) steps. Thus we have
\[
sm_n = \sum_{j \geq 0} \binom{n}{j} \binom{n-j}{j}.
\]
Combine with equation (2.1) we get equation (2.2).

3 Counting peaks in Dyck paths and the Narayana numbers

Note that when restricted to Dyck paths, \( \Phi \) is a bijection between super Dyck paths and peaks in Dyck paths. Therefore we have the following result.

Corollary 3.3 For all \( n \geq 0 \), we have
\[
sd_n = 2pd_n,
\]
and
\[
pd_n = \binom{2n-1}{n}.
\]
Moreover, from the bijection Φ we can easily get a new proof for the Narayana numbers. To this end we need the following lemma.

**Lemma 3.4** Let $SD_n^{UD}(k)$ ($SD_n^{UU}(k)$) denote the set of super Dyck paths of order $n$ with $k$ peaks whose first step is $U$ and last step is $D$ ($U$), then we have

\[
\#SD_n^{UD}(k) = \binom{n-1}{k-1}^2, \tag{3.1}
\]

\[
\#SD_n^{UU}(k) = \binom{n-1}{k}^{n-1}, \tag{3.2}
\]

and the number of super Dyck paths with $k$ peaks of order $n$ is $\binom{n}{k}^2$.

**Proof.** Each $L \in SD_n^{UD}(k)$ can be written uniquely as a word $L = U^{x_1}D^{y_1}U^{x_2}D^{y_2} \cdots U^{x_k}D^{y_k}$, such that

\[
\begin{cases}
x_1 + x_2 + \cdots + x_k = n, & x_1, x_2, \ldots, x_k \geq 1, \\
y_1 + y_2 + \cdots + y_k = n, & y_1, y_2, \ldots, y_k \geq 1.
\end{cases}
\]

The number of solutions for the $x_i$’s and for the $y_i$’s both equal to $\binom{n-k+k-1}{k-1} = \binom{n-1}{k-1}$. Hence equation (3.1) is proved.

Each $L' \in SD_n^{UU}(k)$ can be written uniquely as a word $L' = U^{x_1}D^{y_1}U^{x_2}D^{y_2} \cdots U^{x_k}D^{y_k}U^{x_k+1}$, such that

\[
\begin{cases}
x_1 + x_2 + \cdots + x_k + x_{k+1} = n, & x_1, x_2, \ldots, x_{k+1} \geq 1 \\
y_1 + y_2 + \cdots + y_k = n, & y_1, y_2, \ldots, y_k \geq 1
\end{cases}
\]

There are $\binom{n-k+k+1-1}{k}$ solutions for the $x_i$’s and $\binom{n-1}{k-1}$ solutions for the $y_i$’s. Hence equation (3.2) is proved.

From (3.1) and (3.2) we have that the number of super Dyck paths with $k$ peaks of order $n$ is

\[
\binom{n-1}{k-1}^2 + \binom{n-1}{k}^{n-1} + 2 \binom{n-1}{k} \binom{n-1}{k-1} = \binom{n}{k}^2.
\]

**Corollary 3.5** The number of Dyck paths of order $n$ with $k$ peaks is:

\[
N(n, k) = \frac{1}{n} \binom{n}{k} \binom{n}{k-1}.
\]

**Proof.** From theorem 1.1 we know that each Dyck path of order $n$ with $k$ peaks is mapped to $k$ super Dyck paths, and each of the $k$ super Dyck paths is either in $SD_n^{UU}(k-1)$ or in $SD_n^{UD}(k)$. Therefore we have $kN(n, k) = \#SD_n^{UU}(k-1) + \#SD_n^{UD}(k)$. From Proposition 3.4 we can conclude that

\[
N(n, k) = \frac{1}{k} \left( \binom{n-1}{k-1}^2 + \binom{n-1}{k-2} \binom{n-1}{k-1} \right) = \frac{1}{n} \binom{n}{k} \binom{n}{k-1}.
\]

Bijective proof of this result can also be found in [11, Exercise 6.36(a)].
4 Humps in Schröder paths

In this section we count the number of humps in a third kind of lattice paths: Schröder paths. A *Schröder path* of order \( n \) is a lattice path in \( \mathbb{Z} \times \mathbb{Z} \), from \((0,0)\) to \((n,n)\), using up-steps \((0,1)\), down-steps \((1,0)\) and flat-steps \((1,1)\) (denoted by \(U, D, F\), respectively) and never going below the line \( y = x \). Note that Schröder paths are different from rotating Motzkin paths 45 degrees counterclockwise, since the \( F \) steps in these two kinds of paths are different. However, the bijection \( \Phi \) still works when counting humps in Schröder paths. Let \( ss_n \) denote the number of super Schröder paths of order \( n \), and \( hs_n \) denote the number of humps in all Schröder paths of order \( n \). We have the following result.

**Corollary 4.6** For all \( n \geq 0 \), we have

\[
ss_n = 2hs_n + 1, \tag{4.1}
\]

and

\[
hs_n = \frac{1}{2} \sum_{k=0}^{n} \binom{n+k}{2k} \binom{2k}{k} - 1. \tag{4.2}
\]

**Proof.** Apply the bijection \( \Phi \) to Schröder paths we immediately get (4.1). Next we will count \( ss_n \). Let \( L \) be a super Schröder path of order \( n \) with \( k \) humps, then there are \( k \) \( U \) steps, \( k \) \( D \) steps, and \( n - k \) \( F \) steps in \( L \). We can first choose a super Dyck path of order \( k \) and then “insert” \( n - k \) \( F \) steps to get \( L \). There are \( \binom{2k}{k} \) ways to choose a super Dyck paths, and \( \binom{n-k+2k+1-1}{2k} = \binom{n+k}{2k} \) ways for the insertion. Therefore we have

\[
ss_n = \sum_{k=0}^{n} \binom{n+k}{2k} \binom{2k}{k}.
\]

From the above formula and (4.1) we get (4.2). \( \blacksquare \)

The above proof inspired us to get the following identity, which is listed as an exercise in [9, Exercise 3(g) of Chapter 1].

**Corollary 4.7** For all \( n \geq 0 \), we have

\[
\sum_{k=0}^{n} \binom{n}{k}^2 (m+1)^k = \sum_{k=0}^{n} \binom{n+k}{2k} \binom{2k}{k} m^{n-k}. \tag{4.3}
\]

**Proof.** We will first prove (4.3) \( m = 1 \). From the proof of Corollary 4.6 we know that the right hand side of (4.3) is the number of super Schröder paths of order \( n \) when \( m = 1 \). Now we count \( ss_n \) with a different method to obtain the left hand. Let \( L \) be a super Dyck path of order \( n \) with \( k \) peaks, for each peak of \( L \), we can either keep it invariant or change it into a \( F \) step to we get two super Schröder paths. Hence each \( L \) is mapped to \( 2^k \) super Schröder paths, thus the left hand side of (4.3) when \( m = 1 \) also equals \( ss_n \). Therefore we proved (4.3) for \( m = 1 \).

For general \( m \) we count the number of super Schröder paths in which the \( F \) steps are \( m \)-colored. Now every super Dyck path with \( k \) peaks is mapped to \((m+1)^k\) colored super Schröder
paths. So the total number of such path is \( \sum_{k=0}^{n} \binom{n}{k}^2 (m + 1)^k \). On the other hand, from the proof of Theorem 4.6 we know that the right hand side of (4.3) also counts the number of such paths, hence we proved (4.3).

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