Matrix quantum groups as matrix product operator representations of Lie groups

Romain Couvreur, Laurens Lootens, and Frank Verstraete

1Department of Physics and Astronomy, Ghent University, Krijgslaan 281, 9000 Gent, Belgium

We demonstrate that the matrix quantum group $SL_q(2)$ gives rise to nontrivial matrix product operator representations of the Lie group $SL(2)$, providing an explicit characterization of the nontrivial global $SU(2)$ symmetry of the XXZ model with periodic boundary conditions. The matrix product operators are non-injective and their set is closed under multiplication. This allows to calculate the fusion tensors acting on the virtual or quantum degrees of freedom and to obtain the recoupling coefficients, which satisfy a type of pentagon relation. We argue that the combination of this data with the well known $q$-deformed Clebsch-Gordan coefficients and 6j-symbols is consistent with a description of this quantum group in terms of bimodule categories.

Symmetries are a keystone in the description of topological phases of matter, conformal field theories and integrable systems. Recently, much attention has been paid to the study of categorical symmetries, also called topological defect lines or matrix product operator (MPO) symmetries, as these symmetries reveal important non-local properties in 2d spin systems \cite{VWZ2020} and anyonic spin chains \cite{RSS1999,AA2007,ARS2007,RR2007}. Moreover, those MPO symmetries \cite{LR2021,LR2022,LR2022b} were found to be particularly useful to characterise topological sectors and anyons in the framework of projected entangled pair states (PEPS)\cite{LE2008}. In a recent paper, MPO symmetries were shown to be governed by bimodule categories \cite{LOR2022}, where the consistency equations associated to have MPO symmetries can be understood as a set of coupled pentagon equations with generalised F-symbols \cite{LOR2022a}. This picture has enabled a lattice understanding of the Fuchs-Runkel-Schweigert construction of rational conformal field theory \cite{FIT2018}, and provides new insights into dualities in 1D quantum models \cite{LOR2022b}.

So far, this approach has focused on discrete MPO symmetries, either based on finite groups or fusion categories. In this paper, we extend this construction to the case of continuous groups, by taking advantage of the quantum group deformation of the XXZ chain \cite{RS1991}. The presence of the quantum adjective is reflected in the fact that the MPOs have a bond dimension larger than 1, providing the necessary non-commutative structure. These quantum groups and their relations to the algebraic Bethe ansatz are well known to describe symmetries of the XXZ chain \cite{QZ2002}, and its implications on the spectrum of the underlying spin chain are particularly interesting \cite{RZ2004}. While we will construct global MPOs with periodic boundary conditions which form a group representation of $SL(2)$, the local tensors form non-simple/non-injective representations of the matrix quantum group $SL_q(2)$. Those MPOs can be interpreted as the Lie group analogues of the representations built for the Lie algebra $\mathfrak{sl}_2$ \cite{QZ2002}, in which generators of the loop algebra $\mathfrak{sl}_2$ were obtained by taking a particular limit of powers of the quantum group generators of $U_q(\mathfrak{sl}_2)$ \cite{QZ2002}.

We restrict to the case where $q$ is an odd root of unity, $q^d = 1$ with $d$ odd. We show that under this condition, the product of two MPOs follows the $SL(2)$ multiplication rule. By taking derivatives of our group representations around the identity element, we recover representations of the $SU(2)$ generators as sums of $d$-body operators similar but not equal to the ones in \cite{QZ2002}. We will discuss how this construction via quantum groups fits within the bimodule category formalism of \cite{LOR2022a} and provide analytical expressions for the various associators of a bimodule category. This work makes an explicit connection between the quantum group as a deformed Lie algebra and its group-like elements, analogous to the exponential map from the Lie algebra to the Lie group. We briefly discuss the case of $q$ as an even root of unity where our MPOs are no longer representations of a group, but rather of an algebra.

Quantum groups and RTT relations:

Matrix quantum groups can be defined starting from the so-called RTT relations \cite{DR1992,SV2001,GS2018}. For the spin $1/2$ case, this leads to the matrix

$$T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

with entries that are themselves non-commuting operators satisfying the relations

$$ab = q^{-1}ba, \quad ac = q^{-1}ca, \quad bd = q^{-1}db, \quad cd = q^{-1}dc,$$

$$bc = cb, \quad ad - da = -(q-q^{-1})bc$$

together with the following unitarity constraint:

$$ad - q^{-1}bc = 1.$$  \hspace{1cm} (3)

The matrix $T$ can be interpreted as the defining tensor of a uniform matrix product operator with periodic boundary conditions, in which the physical level is 2-dimensional and the virtual level consists of the operator space. MPOs satisfying these constraints commute

* laurens.lootens@ugent.be
with the transfer matrix of the 6-vertex model or equivalently with the XXZ Hamiltonian at $\Delta = (q + q^{-1})/2$.

Notice that if $q = 1$, the quantum group is no longer deformed and the $SL(2)$ symmetry is restored. We consider the case where $q^d = 1$ with $d > 1$ an odd integer, for which finite-dimensional $d \times d$ matrix solutions exist to the above equations. Considering the permutation matrix $\sigma$ and the diagonal matrix $I_q$

$$\sigma = \begin{pmatrix} 0 & 1 & 0 & \cdots \\ 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad I_q = \begin{pmatrix} q & & & \\ & \ddots & & \\ & & q & \cdots \\ & & & q^{d-1} \end{pmatrix}, \quad (4)$$

the following parameterisation provides solutions to (2)

$$a = xI_q, \quad b = y\sigma, \quad c = z\sigma, \quad d = a^{-1} + ba^{-1}c = \frac{1}{x}I_{q^{-1}} + \frac{y}{x}I_{q^{-1}}\sigma, \quad (5)$$

with $x, y$ and $z$ free complex parameters. This parameterisation turns $T$ into a four-index tensor that we shall denote as the set of $d \times d$ matrices $T^{ij}$, where $i, j = 1, 2$ denote the 2-dimensional physical indices and the dependence on $x, y, z$ is implicit. These tensors allow us to define the 3-parameter family of matrix product operators (MPOs) with periodic boundary conditions on $L$ sites as

$$\mathcal{T}(x, y, z) = \sum_{\{i\} \{j\} = 1}^2 \text{Tr} \left( T^{i_1j_1} \cdots T^{i_Lj_L} \right) |i_1 \cdots i_L\rangle \langle j_1 \cdots j_L|$$

$$= \begin{array}{ccc}
& & \\
& & \\
\mathcal{T} & & \mathcal{T} \\
\mathcal{T} & & \\
& & \\
& & \\
\end{array} \quad (6)$$

where from now on we use the diagrammatic notation to depict tensor contractions. Let us consider the following transformations on the virtual degrees of freedom:

$$T(x, y, z)^{ij} \rightarrow \sigma T(x, y, z)^{ij} \sigma^{-1} = T(qx, y, z)^{ij},$$

$$T(x, y, z)^{ij} \rightarrow I_q T(x, y, z)^{ij} I_q^{-1} = T(x, qy, qz)^{ij}. \quad (7)$$

This implies that the parameterization of the MPOs $\mathcal{T}$ in terms of $(x, y, z)$ is not one-to-one, since among the $d^3$ MPOs $\mathcal{T}(q^i x, q^j y, q^k z)$ with $i, j, k = 1, \ldots, d$, only $d$ are different. Therefore, without loss of generality, we change our parameterization to $(\tilde{x}, \tilde{y}, \tilde{z}) \equiv (x^d, y^d, z^d)$ and define

$$D_k(\tilde{x}, \tilde{y}, \tilde{z})^{ij} \equiv T(\tilde{x}^{1/d}, \tilde{y}^{1/d}, \tilde{q}^{k\tilde{z}^{1/d}})^{ij},$$

$$D_k(\tilde{x}, \tilde{y}, \tilde{z}) \equiv T(\tilde{x}^{1/d}, \tilde{y}^{1/d}, \tilde{q}^{k\tilde{z}^{1/d}}), \quad (8)$$

where the choice of roots is arbitrary, but has to be fixed.

**Fusion of two MPOs:** The multiplication of two such MPOs turns out to form a closed MPO algebra [20] with structure constants that are independent of the number of sites. Indeed, by a cumbersome but straightforward calculation (see supplemental material), it can be shown that

$$\mathcal{D}_{k_1}(g_1)\mathcal{D}_{k_2}(g_2) = \sum_{k_{12}=0}^{d-1} \mathcal{D}_{k_{12}}(g_{12}) \quad (9)$$

where $g_i \equiv (\tilde{x}_i, \tilde{y}_i, \tilde{z}_i)$ labeling elements of the Lie group $SL(2)$, $\tilde{x}_i, \tilde{y}_i, \tilde{z}_i = 1$ and

$$\left( \begin{array}{c} \tilde{x}_{12} \\ \tilde{y}_{12} \\ \tilde{z}_{12} \end{array} \right) = \left( \begin{array}{c} \tilde{x}_1 \\ \tilde{y}_1 \\ \tilde{z}_1 \end{array} \right) \left( \begin{array}{c} \tilde{x}_2 \\ \tilde{y}_2 \\ \tilde{z}_2 \end{array} \right). \quad (10)$$

Hence $g_{12} = g_1 \cdot g_2$ with the product of the Lie group $SL(2)$, meaning we indeed recover the $SL(2)$ group structure starting from the matrix quantum group structure by taking the $d^6$ power! As we will show later, this is related to the fact that the corresponding raising operator in this representation of the Lie algebra of $sL_2$ flips d spins. Defining non-injective MPO tensors $D(g)^{ij}$ of bond dimension $d^2$ and their corresponding MPOs $\mathcal{D}(g)$ as

$$\mathcal{D}(g)^{ij} = \bigoplus_{k=1}^d D_k(g)^{ij}, \quad \mathcal{D}(g) = \frac{1}{d^2} \sum_{k=1}^d D_k(g), \quad (11)$$

we find that the MPOs form a faithful representation of $SL(2)$:

$$\mathcal{D}(g_1)\mathcal{D}(g_2) = \mathcal{D}(g_{12}). \quad (12)$$

The normalization factor $d^2$ is required due to the fact that $\mathcal{D}(g_{12})$ can be obtained by fusing any pair of blocks in $\mathcal{D}(g_1)$ and $\mathcal{D}(g_2)$.

**Bimodule categories and pentagon equations:** In this section, we discuss how the MPOs $\mathcal{D}(g)$ fit in the context of the bimodule category language of [11], although we have to add a disclaimer that a rigorous categorical description for our construction is lacking at present. Following [11], the various consistency conditions on MPO algebras are equivalent to the pentagon equations of a $(\mathcal{C}, \mathcal{D})$-bimodule category $\mathcal{M}$. In this framework, the fusion category $\mathcal{C}$ describes the virtual labels of the MPO algebra and the fusion category $\mathcal{D}$ describes the physical labels of the MPO algebra. The bimodule category $\mathcal{M}$ then provides the necessary data, in the form of a set of $F$-symbols $\{0F, 1F, 2F, 3F, 4F\}$ to explicitly construct the MPO tensors and the relevant fusion tensors.

Using the notation of [11], the $2F$-symbol can be identified with the components of the non-injective tensor made from the $d$ blocks corresponding to a tuple
$g = (\tilde{x}, \tilde{y}, \tilde{z})$ parameterising an $SL_q(2)$ matrix:

$$(2F^{x}_{s})_{\alpha, jn'i} = n \begin{array}{c}
\alpha \\
g \\
n'\end{array} j_i$$

Here, $\alpha$ labels a representation of $SL_q(2)$; in the categorical language, we tentatively label the $\alpha$ labels as objects in the tensor category $D = \text{Rep}(SL_q(2))$, and the labels $g$ as objects in another tensor category $\mathcal{C}$. For the case of $\alpha = 1/2$, the $2F$ symbol is given by the components of $D(\gamma)^{ij}$, and we will see that the higher spin representations can be generated from the spin 1/2 representation. As mentioned above, these MPO tensors are not injective, and indeed they have a nontrivial presentation. As mentioned above, these MPO tensors are not injective, and indeed they have a nontrivial center;

$$\forall i, j, k : I_k D(\gamma)^{ij} = D(\gamma)^{ij} I_k = D_k(\gamma)^{ij},$$

where $I_k$ are $d^2 \times d^2$ orthogonal projector matrices in the center that project onto block $k$, where they act as the identity. From the MPO tensors, one can compute fusion tensors that locally decompose the product of two MPOs:

$$\begin{array}{cccc}
g_{12} & g_1 & \alpha \\
g_2 & \end{array} = \begin{array}{cccc}
g_{12} & \alpha \\
g_{23} & \end{array} \begin{array}{cccc}
g_1 & g_2 \\
g_3 & \end{array}$$

the components of which read

$$(1F^{x}_{g_{12}})_{g_{12}, k_{12}}^{n, n_1 n_2} = n_{12} \begin{array}{cccc}
g_{12} & g_1 & \alpha \\
g_2 & \end{array} = \begin{array}{cccc}
g_{12} & \alpha \\
g_2 & \end{array} \begin{array}{cccc}
g_1 & n_1 \\
g_2 & n_2 \end{array}$$

Equation (15) is known as the zipper condition, and when written out in components corresponds the pentagon equation of the form $1F^{2F} = 2F^{1F}$, expressing compatibility between $1F$ and $2F$. The multiplicity label $k$ is understood as a tuple $k = (k_1, k_2)$ with $k_i = 1, \ldots, d$, specifying the block in the virtual space labelled by $g_i$, as before. Graphically, we can depict this as

$$\begin{array}{cccc}
g_{12} & g_1 & \alpha \\
g_2 & \end{array} = \delta_{k_1, k_1'} \delta_{k_2, k_2'} \begin{array}{cccc}
g_{12} & g_1 \\
g_2 & \end{array}$$

An analytical expression for the $1F$ tensors is provided in the supplemental material.

Using the fusion tensors and their inverses, we can define the following $d^2 \times d^2$ matrix:

$$(F^{g_{12}}_{g_{12}})_{g_{12}, kl}^{g_{23}, mn} = g_{123} \begin{array}{cccc}
g_{123} & g_1 & \alpha \\
g_{23} & \end{array} = \begin{array}{cccc}
g_{123} & \alpha \\
g_{23} & \end{array} \begin{array}{cccc}
g_1 & n_1 \\
g_2 & n_2 \end{array}$$

Using the zipper condition for the fusion tensors and their inverses, we find that this matrix must commute with the MPO tensors $D_{g_{123}}^{ij}$. The matrix $(0F^{g_{12}}_{g_{12}})_{g_{12}, kl}^{g_{23}, mn}$ is therefore in the center of the MPO tensors, and so must be of the form

$$(F^{g_{12}}_{g_{12}})_{g_{12}, kl}^{g_{23}, mn} = \sum_{k_{123}} \left(0F^{g_{12}}_{g_{12}}\right)_{g_{12}, kl}^{g_{23}, mn} I_{k_{123}},$$

where the $0F_k$-symbols are now just complex scalars. They satisfy

$$g_{123} \begin{array}{cccc}
g_1 & g_2 & 0 \\
g_3 & \end{array} = \sum_{mn} \left(0F^{g_{12}}_{g_{12}}\right)_{g_{12}, kl}^{g_{23}, mn} g_{123} \begin{array}{cccc}
g_1 & g_2 & 0 \\
g_3 & \end{array} \begin{array}{cccc}
g_1 & g_2 & 0 \\
g_3 & \end{array}$$

which when written out in components gives a set of $d$ pentagon equations of the form $1F^{1F} = 0F_k 1F^{1F}$ for $k = 1, \ldots, d$, expressing compatibility between $1F$ and $0F_k$. The $0F_k$-symbols express the associativity of the fusion of three MPOs at the level of the local tensors.

The $0F_k$-symbols have to satisfy coupled pentagon equations, derived from the two equivalent ways of recoupling the fusion tensors that arise when looking at the product of four MPOs:

Using equations (17) and (20), and defining the shorthand notation $(\omega_{g_{1234}})_{kl}^{pq} = (0F^{g_{12}}_{g_{12}})_{g_{12}, kl}^{g_{23}, mn}$, this leads to the pentagon equations

$$\sum_{r s} (\omega_{g_{1234}})_{r s} \omega_{g_{1234}}^{t q} = \sum_{r s t q} (\omega_{g_{1234}})_{r s} \omega_{g_{1234}}^{t q} (\omega_{g_{234}})_{t q}^{n p} \begin{array}{cccc}
g_1 & g_2 & 0 \\
g_3 & \end{array} \begin{array}{cccc}
g_1 & g_2 & 0 \\
g_3 & \end{array}$$

These pentagon equations should relate to some underlying tensor category, but besides noting their similarity to the three-cocycle conditions characterizing the third cohomology of a group, we do not have a concrete categorical foundation for this equation at present.

So far, we have only explicitly constructed the spin 1/2 representation. To generate higher spin representations, we contract two spin 1/2 MPO tensors along the virtual degrees of freedom. If we decompose this product by acting with generalized Clebsch-Gordan coefficients on the physical degrees of freedom, the blocks of
this decomposition correspond to higher spin representations of $SL_q(2)$. These generalized Clebsch-Gordan coefficients satisfy the pulling-through condition

$$
\begin{array}{c}
\begin{array}{cc}
\alpha \\
\beta \\
\gamma
\end{array}
\end{array}
\begin{array}{c}
g
\end{array}
\begin{array}{c}
\alpha \\
\beta \\
\gamma
\end{array}
= \begin{array}{c}
\begin{array}{cc}
\alpha \\
\beta \\
\gamma
\end{array}
\end{array}
\begin{array}{c}
g
\end{array}
\begin{array}{c}
\begin{array}{cc}
\alpha \\
\beta \\
\gamma
\end{array}
\end{array}
\end{array}
\tag{23}
$$

where $\alpha, \beta, \gamma \in \text{Rep}(SL_q(2))$, and their components give the $^3F$ symbols of the bimodule category:

$$
(3F^\alpha_{\beta\gamma})_{ij}^{1k} = \begin{array}{c}
\begin{array}{cc}
i \\
\beta \\
k
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{cc}
\alpha \\
\gamma
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{cc}
j \\
\nu
\end{array}
\end{array}
\tag{24}
$$

These Clebsch-Gordan coefficients have been described in the literature and formulas exist for general $q$ (see e.g. [21] and references therein), but one has to be careful at roots of unity. Just as an illustration, let us consider the fusion of a spin 1 and a spin 1/2 representations. For $d$ odd such that $d > 3$, the representation theory is exactly the same as in $SL_2$

$$
V_1 \otimes V_{1/2} = V_{1/2} \oplus V_{3/2}
\tag{25}
$$

whereas for $d = 3$, we can no longer decompose this product as a sum of irreps but we obtain

$$
V_1 \otimes V_{1/2} = \mathcal{P}_{1,1}
\tag{26}
$$

following the convention of [22]. The module $\mathcal{P}_{1,1}$ is reducible but contains Jordan blocks. These features are important since, in the continuum limit of the spin chains, they are related to the logarithmic structure of the conformal field theory and are generically found in non-unitary systems. In order to get consistent F-symbols and pentagon equations, we define the Clebsch-Gordan coefficients as the transformations that bring the product of two representations into Jordan normal form, which differ from the ones obtained by taking a limit of the generic formula.

Finally, these Clebsch-Gordan coefficients or $^3F$ symbols are recoupled by the $6j$ or $^4F$ symbols, encoding the associativity of recoupling:

$$
(4F_{\delta\mu\nu}^{\alpha\beta\gamma})_{\mu}^{\nu} = \begin{array}{c}
\begin{array}{cc}
\delta \\
\mu \\
\nu
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{cc}
\alpha \\
\beta \\
\gamma
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{cc}
\delta \\
\nu
\end{array}
\end{array}
\tag{27}
$$

where the tensor on the left is exactly proportional to the identity on the physical space.

**Symmetries of vertex models:** The MPOs are topological symmetries of vertex models. Indeed a large class of lattice models can be built from the $^3F$ tensors (Clebsch-Gordan coefficients). For example the Temperley-Lieb generator $e_i$ on site $i$ in the integrable 6-vertex model (or XXZ chain) is

$$
e_i = \begin{array}{c}
\begin{array}{cc}
1/2 \\
0
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{cc}
1/2 \\
i
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{cc}
i+1 \\
1/2
\end{array}
\end{array}
\tag{28}
$$

and hence commutes with the MPOs due to the pulling through condition (23). The $^2F$ tensors describe topological symmetries of either the XXZ Hamiltonian or the 6-vertex transfer matrix. Higher-spin Hamiltonians and transfer matrices also have MPO-symmetries. Starting from (23), the Clebsch-Gordan coefficients can be used to create higher representations of the symmetries that satisfy the same pentagon-like equations.

**Some comments about spin-1/2:** Let us consider the case of the periodic spin-1/2 XXZ chain. Choosing $(\tilde{x}, \tilde{y}, \tilde{z}) = (1, 0, 0)$, the identity MPO on $L = 2M$ sites is diagonal and the coefficient of a state of magnetisation is

$$
\sum_{k=0}^{d-1} q^{kS_z} = \begin{cases} 
1 & \text{if } S_z = 0 \mod p \\
0 & \text{else}
\end{cases}
\tag{29}
$$

thus the MPOs will only be non-zero for sector with a magnetisation that is a multiple of $d$. We define the following operators

$$
J_i = (\partial_i D(g))|_{g=(1,0,0)}, \quad i = 1, 2, 3
\tag{30}
$$

and their commutation relations can be easily calculated. For example, using (10), we have

$$
[J_1, J_2] = \partial_2 \partial_\bar{y} (D(\tilde{x}, \tilde{y}, g) - D(\tilde{x}, \tilde{y}, 0)) |_{\tilde{x}=1, \tilde{y}=0} = 2 \partial_\bar{y} (\tilde{y} \partial_\bar{y} \mathcal{O}(1, \tilde{y}, 0)) |_{\tilde{y}=0} = 2 J_2.
\tag{31}
$$

In a similar fashion, we get the two following relations

$$
[J_1, J_3] = -2 J_3, \quad [J_2, J_3] = J_1.
\tag{32}
$$

We now can define

$$
J_+ = \frac{1}{\sqrt{2}} J_2, \quad J_- = \frac{1}{\sqrt{2}} J_3, \quad J_z = \frac{1}{2} J_1
\tag{33}
$$

that satisfy the usual $sl_2$ commutation relations. As a pedagogical example, let us consider the case $d = 3$. The $sl_2$ operator $J_+$ (33) reads

$$
J_+ = \frac{1}{\sqrt{2}} \sum_{m=1}^{3} \sum_{1 \leq i_1 < i_2 < i_3 \leq L} (q^{m \sigma_3})^{\otimes (i_1-1)} \otimes (q^{m \sigma_3})^{\otimes (i_2-i_1-1)} \otimes (q^{m \sigma_3})^{\otimes (L-i_3-1)} \otimes (q^{m \sigma_3})^{\otimes (L-i_3-1)}
\tag{34}
$$
The operators are in general built from $d$ raising or lowering operators and we recover a similar expression as found in [18], but here there is an extra sum over the index $m$. However our MPOs are not simply the exponential map applied to these operators. In the case of open boundary conditions it is possible, by carefully choosing the boundaries operators, to recover the usual $U_q(\mathfrak{sl}_2)$ generators [16] by keeping only one creation or annihilation operator.

The even case $d = 2n$: The case of even $d \geq 4$ is more involved. In particular we find that (10) does not hold. First, still using the MPO definition (5), the parameterization that we use is $g = (\tilde{x}, \tilde{y}, \tilde{z}) = (x^n, y^n, z^n)$. This change of power is needed and illustrated in the appendix (50) when calculating the fusion of two MPOs. For $d$ odd, the product of two injective MPOs resulted in a sum that was grouped as a single non-injective MPO. This is no longer possible and the fusion leads to a sum over two non-injective MPOs corresponding to different elements $g_3$ and $g_4$.

$$D(g_1)D(g_2) = 2n^2 (D(g_3) + D(g_4)).$$

As a consequence our set of MPOs no longer forms a group but an algebra. In the following we briefly discuss the case $n > 1$. The element $g_3$ is given by the following matrix multiplication

$$\begin{pmatrix} \tilde{x}_1 \tilde{y}_1 \\ \tilde{z}_1 \end{pmatrix} \begin{pmatrix} \tilde{x}_2 \tilde{y}_2 \\ \tilde{z}_2 \end{pmatrix}$$

whereas $g_4$ is given by the rule

$$\begin{pmatrix} \tilde{x}_1 \tilde{y}_1 \\ \tilde{z}_1 \end{pmatrix} \sigma_z \begin{pmatrix} \tilde{x}_2 \tilde{y}_2 \\ \tilde{z}_2 \end{pmatrix} \sigma_z.$$

Because of the similarity transformations that can be performed on the virtual space, the signs of $x^n$ or $y^n$ and $z^n$ simultaneously, can be reversed. Thus these elements must be identified. Our matrix space is then $SU(2)$ with an equivalence relation identifying the four elements in the set

$$\{(\tilde{x}, \tilde{y}, \tilde{z}), (-1)^i \tilde{x}, (-1)^j \tilde{y}, (-1)^k \tilde{z} \}, i, j, k = 0, 1\}.$$ (38)

This identification is consistent with the two multiplication rules and the conjugation by $\sigma_z$ in (37).

Outlook: In this paper, we have shown how nontrivial MPO-representations of a group can arise from $q$-deformed algebras. The next logical step is to consider the physical implications of such symmetries in the context of tensor network and lattice models. In particular, we can construct the tube algebra [8] which are needed to describe topological sectors and/or twisted boundary conditions. The even case shall also be considered properly in a future work and is especially important since many lattice models fall into this category. In particular, it would be important to connect this work with the topological symmetries in the RSOS picture, models in part dual to the XXZ chain. A very interesting question is how to deal with the case that $q$ is not a root of unity; one could speculate that algebraic structures as encountered in the context of tropical geometry emerge [23]. An additional direct extension of this work is to generalise the procedure to different classical groups (and their associated quantum groups) and supersymmetric ones.

Acknowledgements: We are grateful to Jacob Bridgeman, Jacopo De Nardis and Joris Van der Jeugt for helpful comments and interesting discussions. This work has received funding from the Research Foundation Flanders via grant nr. G087918N and G0E1820N. RC and LL are supported by a fellowship from the Research Foundation Flanders (FWO).

[1] D. Aasen, R. S. Mong, and P. Fendley, Journal of Physics A: Mathematical and Theoretical 49, 354001 (2016).
[2] R. Vanhove, M. Bal, D. J. Williamson, N. Bultinck, J. Haegeman, and F. Verstraete, Physical review letters 121, 177203 (2018).
[3] C. Gils, E. Ardonne, S. Trebst, D. A. Huse, A. W. Ludwig, M. Troyer, and Z. Wang, Physical Review B 87, 235120 (2013).
[4] M. Buican and A. Gromov, Communications in Mathematical Physics 356, 1017 (2017).
[5] J. Belletête, A. M. Gainutdinov, J. L. Jacobsen, H. Saleur, and T. S. Tavares, (2020), arXiv:2003.11293 [math-ph].
[6] B. Firvu, V. Murg, J. I. Cirac, and F. Verstraete, New Journal of Physics 12, 025012 (2010).
[7] N. Schuch, I. Cirac, and D. Perez-Garcia, Annals of Physics 325, 2153 (2010).
[8] N. Bultinck, M. Mariën, D. J. Williamson, M. B. Şahinoglu, J. Haegeman, and F. Verstraete, Annals of physics 378, 183 (2017).
[9] J. I. Cirac, D. Perez-Garcia, N. Schuch, and F. Verstraete, Reviews of Modern Physics 93, 045003 (2021).
[10] P. Etingof, S. Gelaki, D. Nikshych, and V. Ostrik, Tensor categories, Vol. 205 (American Mathematical Soc., 2016).
[11] L. Lootens, J. Fuchs, J. Haegeman, C. Schweigert, and F. Verstraete, SciPost Phys. 10, 53 (2021).
[12] R. Vanhove, L. Lootens, H.-H. Tu, and F. Verstraete, “Topological aspects of the critical three-state potts model,” (2021), arXiv:2107.11177 [math-ph].
[13] L. Lootens, C. Delcamp, G. Ortiz, and F. Verstraete, “Category-theoretic recipe for dualities in one-dimensional quantum lattice models,” (2021), arXiv:2112.09091 [quant-ph].
[14] S. L. Woronowicz, Communications in Mathematical Physics 111, 613 (1987).
[15] L. D. Faddeev, N. Y. Reshetikhin, and L. A. Takhtajan, in Algebraic analysis (Elsevier, 1988) pp. 129–139.
[16] V. Pasquier and H. Saleur, Nuclear Physics B 330, 523 (1990).
[17] R. Vasseur, J. L. Jacobsen, and H. Saleur, Nuclear Physics B 851, 314 (2011).
[18] T. Deguchi, K. Fabricius, and B. M. McCoy, Journal of Statistical Physics 102, 701 (2001).
[19] L. C. Biedenharn and M. A. Lohe, Quantum Group Symmetry and Q-Tensor Algebras (World Scientific, 1995).
[20] N. Bultinck, M. Marien, D. Williamson, M. Sahinoglu, J. Haegeman, and F. Verstraete, Annals of Physics 378, 183 (2017).
[21] E. Ardonne and J. Slingerland, Journal of Physics A: Mathematical and Theoretical 43, 395205 (2010).
[22] P. Bushlanov, A. Gainutdinov, and I. Tipunin, Nuclear Physics B 862, 232 (2012).
[23] D. Maclagan and B. Sturmfels, Introduction to tropical geometry, Vol. 161 (American Mathematical Society, 2021).
SUPPLEMENTARY MATERIAL

We consider the fusion of two simple MPOs given by (6) on the virtual space. We are going to show that we can find a change a basis such that (9) is explicit together with the fusion rule (10). We write

\[ A = a_1 \otimes a_2 + b_1 \otimes c_2 \quad B = a_1 \otimes b_2 + b_1 \otimes d_2 \]
\[ C = c_1 \otimes a_2 + d_1 \otimes c_2 \quad D = c_1 \otimes b_2 + d_1 \otimes d_2 \]  

(39)

the fused operators for each physical spin state. (2). We will now show that the four fused operators can also be written as in (5).

**Diagonalising A**

Let us start by diagonalising A, its action on the tensor product of the virtual space is

\[ A|i, j\rangle = x_1 x_2 q^{i+j-2}|i, j\rangle + y_1 z_2|i - 1, j - 1\rangle. \]  

(40)

There are \(d\) trivial invariant subspaces \(V_k, k = 1, \ldots, d\), defined as

\[ V_k = \{|i, i+k), i = 1, ..., d\}, \]  

(41)

that we can use to diagonalise A. All indices, \(i\) and \(k\) are taken modulo \(d\). We start with a vector \(v_k\)

\[ v_k^X = \sum_{i=1}^{d} \alpha_{i,k}^X |i, i+k\rangle \]  

(42)

such that \(A v_k^X = X v_k^X\). Acting with \(A\), we find

\[ A v_k^X = \sum_{i=1}^{d} (x_1 x_2 q^{2i+k-2} \alpha_{i,k}^X + y_1 z_2 \alpha_{i+1,k}^X) |i, i+k\rangle \]  

(43)

An eigenvector \(X\) can be found using the recursion relation on the \(\alpha\) coefficients

\[ \frac{\alpha_{i+1,k}^X}{\alpha_{i,k}^X} = \frac{X - x_1 x_2 q^{2i+k-2}}{y_1 z_2} \]  

(44)

thus we have the consistency equation for \(X\)

\[ (y_1 z_2)^d = \prod_{i=1}^{d} (X - x_1 x_2 q^{2i+k}). \]  

(45)

Expanding the product on the left-hand side

\[ (y_1 z_2)^d = X^d + X^{d-1}(-x_1 x_2 q^k) \sum_{i} q^{2i} + X^{d-2}(-x_1 x_2 q^k)^2 \sum_{i_1 < i_2} q^{2i_1+2i_2} \]
\[ + \ldots + (-x_1 x_2 q^k)^d \sum_{i_1 < \ldots < i_d} q^{2i_1+\ldots+2i_d}. \]  

(46)

(47)

together with the identity

\[ \sum_{i_1 < \ldots < i_m} q^{i_1+\ldots+i_m} = \begin{cases} 0 & \text{if } q^m \neq 1 \\ 2(-1)^{m+1} & \text{if } q^m = 1 \end{cases} \]  

(48)

we find, for \(d\) odd, that only the first and last term in the expansion are non zero. We get simply

\[ X^d = (x_1 x_2)^d + (y_1 z_2)^d. \]  

(49)

If \(d\) is even \((d = 2n)\) then the middle term in the expansion is also non zero, we get three terms that can be factorized in the following form

\( (X^n - (x_1 x_2)^n + (y_1 z_2)^n)(X^n - (x_1 x_2)^n - (y_1 z_2)^n) = 0. \)  

(50)
The eigenvalues are independent of \( k \). Each sector \( V_k \) contains an eigenvector associated to one of the \( d \) possible values of \( X \) and the eigenvectors can be calculated directly from the recursion relation on the \( \alpha_{1,k}^X \). If we take as a convention \( \alpha_{1,k}^X = 1 \) then

\[
\alpha_{i,k}^X = (y_1 z_2)^{-i-1} \prod_{p=2}^{i} (X - x_1 x_2 q^{2p+k-4})
\]  

(51)

Writing \( B \) and \( C \) as permutation matrix

In an eigenspace of \( A \) associated to an eigenvalue \( X \) we need to find vectors \( v^{X,W} \) such that

\[
A.v^{X,W} = X.v^{X,W}, \quad B.v^{X,W} = W C.v^{X,W}, \quad \text{with} \quad v^{X,W} = \sum_{k=0}^{d-1} \beta_{k}^{X,W} v_k^X
\]  

(52)

since we want to have a form where \( B \) and \( C \) are proportional to the same permutation matrix. We first calculate the action of \( B \) and \( C \) on our basis vectors \( v_k^X \)

\[
C.v_k^X = \frac{z_2}{x_1} v_{k-1}^{q^{-1}X} + \frac{z_1}{x_1 y_1 z_2} X q^{-1}(X - x_1 x_2 q^{k-1}) v_{k+1}^{q^{-1}X}
\]  

(53)

\[
B.v_k^X = \frac{y_2}{x_2} X q^{-k} v_{k-1}^{q^{-1}X} + \frac{1}{x_2 z_2} q^{-k}(X - x_1 x_2 q^{k-1}) v_{k+1}^{q^{-1}X}.
\]  

(54)

Action on \( v^{X,\delta} \) then gives

\[
B.v^{X,W} = \sum_{k=0}^{d-1} \left( \frac{\beta_{k+1}^{X,W} z_2}{x_2} X q^{-k-1} - \frac{z_2}{x_1} \right) v_k^{q^{-1}X}
\]  

(55)

\[
C.v^{X,W} = \sum_{k=0}^{d-1} \left( \frac{\beta_{k}^{X,W} z_2}{x_1} + \frac{\beta_{k-1}^{X,W}}{z_1 x_1 y_1 z_2} q^{-1}(X - x_1 x_2 q^{k-1}) \right) v_k^{q^{-1}X}
\]  

(56)

which leads, using (52), to the following relation

\[
\beta_{k+1}^{X,W} \left( \frac{y_2}{x_2} X q^{-k-1} - \frac{z_2}{x_1} \right) = \frac{\beta_{k-1}^{X,W}}{z_2} X q^{-k-1} \left( \frac{z_1}{x_1 y_1 z_2} X q^{-1} - \frac{q^{-k}}{x_2} \right)
\]  

(57)

Taking a product over the index \( k \) leads to the following consistency equation for \( W \):

\[
W^d = \frac{x_1^d}{x_2^d} \times \frac{y_1^d + y_2^d (y_1^d z_1^d + y_2^d z_2^d)}{z_2^d + z_1^d (y_1^d z_1^d + y_2^d z_2^d)}
\]  

(58)

Acting with \( B \) and \( C \) on \( v^{X,W} \), one can show that

\[
B^d.v^{X,W} = Y^d.v^{X,W}, \quad \text{with} \quad Y^d = \frac{y_2^d}{x_2^d} (x_1^d x_2^d + y_1^d z_1^d) + \frac{y_1^d}{x_2^d}
\]  

(59)

and

\[
C^d.v^{X,W} = Z^d.v^{X,W}, \quad \text{with} \quad Z^d = \frac{z_1^d}{x_1^d} (x_1^d x_2^d + y_1^d z_1^d) + \frac{z_2^d}{x_1^d}
\]  

(60)

and we can check that indeed \( W^d = Y^d/Z^d \). To get the right form, we also need to a basis where \( B \) and \( C \) verify

\[
B.v^{X,W} = Y.v^{X,W} \quad \text{and} \quad C.v^{X,W} = Z.v^{X,W}
\]  

(61)

which is simpler to implement through the condition

\[
B^2.v^{X,W} = Y^2.v^{X,W} \quad \text{and} \quad C^2.v^{X,W} = Z^2.v^{X,W}
\]  

(62)
because the recursion formula (67) relates $\beta^X_{k+2}$ to $\beta^X_k$. After a direct calculation, we find the following relation must be enforced

$$
Y^2\beta_0^{-2}X,W = \frac{\beta^X_0}{x_2^2z_2^2} \left( y_2^2z_2^2X^2q^{-4} + q^{-1}(X - x_1x_2)(X - x_1x_2q^{-2}) + y_2z_2q^{-2}X(q + q^{-1})(X - x_1x_2) \right). \tag{63}
$$

We are left with $d$ free parameters, corresponding to the $d$ coefficients $\left\{ \beta^X_{0,q^kW} \right\}_{k=0,...,d-1}$ that can be adjusted to normalise each block.

Fusion rules

In the end, we find that there are $d$ blocks, each parametrised by a root of $W^d$ obtained from (58) where the $d$ vector one block are given by the $d$ roots of $X^d$ from equation (49). Equivalently, choosing the root of $\delta^d$ is equivalent to either choose the root of $Y^d$ or $Z^d$. We thus obtain $d$ distinct blocks, in agreement with the similarity transformations mentioned in the main text. Using equations (59), (59) and (60) we observe that

$$
\begin{pmatrix}
X^d & Y^d \\
Z^d & T
\end{pmatrix} = \begin{pmatrix}
x_1^d & y_1^d \\
z_1^d & t_1
\end{pmatrix} \cdot \begin{pmatrix}
x_2^d & y_2^d \\
z_2^d & t_2
\end{pmatrix}
$$

with $T, t_1, t_2$ chosen such that the determinant of each matrix is one. A this point, we need to chose $X, Y$ and $Z$ in a chosen quadrant as mentioned in the main text. Defining $g_3 = (\tilde{X}, \tilde{Y}, \tilde{Z}) = (X, Y, Z)$ leads to equation (10) in the main text.

$^3F$ definition

To summarise and using the notation of the main text, we have

$$
(_3F^2)_{g_1,g_2}^{s,(s_1,l_2),(s_1,l_1)}_{g_3,(k_1,k_2)_{(s_3,l_3)}} = \delta_{g_1,g_2,g_3} \delta_{s_1,k_1} \delta_{l_2,k_2} \beta^q X, q^3 W^q \alpha_{s_1, s_2 - s_3} \tag{65}
$$

where, comparing with the main text formula, we have the composite indices $n_i = (s_i, l_i)$, and $k = (k_1, k_2)$. The $\alpha$ and $\beta$ coefficients are given by

$$
\alpha^X_{s+1, p} = \frac{X - x_1x_2q^{(s+p-2)}}{y_1z_2} \alpha^X_{s, p}, \quad \alpha^X_{0, p} = 1 \tag{66}
$$

$$
\beta^X_{p+1} \left( \frac{y_2}{x_2} X q^{-p-1} - \frac{z_2}{x_1} W \right) = \frac{\beta^X_{p-1}}{z_2} (X - x_1x_2q^{p-1}) \left( \frac{z_1}{x_1y_1} W - \frac{q-p}{x_2} \right) \tag{67}
$$

together with the consistency equation between $\beta_0$ coefficients

$$
Y^2\beta_0^{q^{-2}X,W} = \frac{\beta^X_0}{x_2^2z_2^2} \left( y_2^2z_2^2X^2q^{-4} + q^{-1}(X - x_1x_2)(X - x_1x_2q^{-2}) + y_2z_2q^{-2}X(q + q^{-1})(X - x_1x_2) \right). \tag{68}
$$