Canonical superdiffusion and energy fluctuation divergence

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(Dated: May 2016)

We propose a noble physical model obtained from a Hamiltonian with periodic potential. This model is canonical, reversible and brings about chaotic superdiffusion with energy fluctuation divergence. The analytical formula of invariant density can be obtained in some parameter range. In the range it is proved that the map is Anosov diffeomorphism and the invariant measure is a SRB measure. We calculate the analytical formula of Lyapunov exponent.

PACS numbers: 45.05.+x, 46.40.Ff, 96.12.De

Introduction

Sinai-Ruelle-Bowen (SRB) measure which is a special case of Gibbs measure [1] plays an important role in dynamical system and statistical dynamics points of view. In the case of Hénon map, Logistic map or Baker’s map, it is proved that there is a SRB measure [2–4] although in former two explicit form do not given and Baker’s map does not have time-reversal symmetry.

Superdiffusion is also an important phenomenon in dynamical system and statistical dynamics.

In classical systems, there is no symplectic map in which it is proved analytically that superdiffusion occurs with measure unity. For example, in Standard map [5], it is said that superdiffusion occurs in some part of domain not in the whole domain.

In this paper, (i) new classical model whose SRB measure is given in explicit form in some parameter range is proposed. (ii) On the way to prove the existence of SRB measure, it is also proved that the map is Anosov diffeomorphism and mixing in the parameter range. (iii) It is proved that energy fluctuation diverges because action variables are in accordance with Cauchy distribution. (iv) Lyapunov exponent is obtained using these properties.

At first Hamiltonian is introduced by

\[
H(I_1, I_2, \theta_1, \theta_2) = \frac{1}{2} \left( I_1^2 + I_2^2 \right) - \varepsilon \log |\cos(\pi(\theta_1 - \theta_2))|. \tag{1}
\]

Let an interval \( I \) be as \( I = \left[ -\frac{1}{2}, \frac{1}{2} \right] \) and the map \( T \) is obtained by the Hamiltonian by leap frog method

\[
T(I_1, I_2, \theta_1, \theta_2) = \begin{pmatrix} I_1 - \varepsilon \tan(\pi(I_1 + \theta_1 - I_2 - \theta_2)) \\ I_1 + \theta_1 \mod \{-1/2, 1/2\} \\ I_2 + \varepsilon \tan(\pi(I_1 + \theta_1 - I_2 - \theta_2)) \\ I_2 + \theta_2 \mod \{-1/2, 1/2\} \end{pmatrix}. \tag{3}
\]

A two dimensional symplectic map with tangent function is also researched in [6]. Figures 1 and 2 show the behavior of the potential \( V(\theta_1, \theta_2) \). The absolute value of \( V(\theta_1, \theta_2) \) diverges in \( \{(\theta_1, \theta_2) | \cos(\pi(\theta_1 - \theta_2)) = 0\} \). This map has a conserved quantity for momentum \( C \) such that

\[ C \equiv I_1(0) + I_2(0) = \cdots = I_1(n) + I_2(n) = \cdots . \]

We can reduce degree of freedom by using \( C \) as

\[
I_2(n) = C - I_1(n), \tag{4}
\]

\[
\theta_1(n) + \theta_2(n) = \theta_1(0) + \theta_2(0) + nC,
\]

\[ |\cos(\pi(\theta_1(n) - \theta_2(n)))| \]

FIG. 1. The shape of potential \( V(\theta_1, \theta_2) = -\varepsilon \log |\cos(\pi(\theta_1 - \theta_2))| \) for \( \varepsilon = 2.0 \). The potential diverges in \( \{(\theta_1, \theta_2) | \cos(\pi(\theta_1 - \theta_2)) = 0\} \).

FIG. 2. The shape of potential \( V(\theta_1, \theta_2) = -\varepsilon \log |\cos(\pi(\theta_1 - \theta_2))| \) for \( \varepsilon = -2.0 \).
Then, we get such formulas as Equations (5) and (6). By subtracting Eq. (6) from Eq. (5), we obtain Eq. (7) Now by changing variables as

\[ \begin{align*}
  p_n & \equiv \theta_1(n-1) - \theta_2(n-1) \mod [-1/2, 1/2], \\
  q_n & \equiv \theta_1(n) - \theta_2(n) \mod [-1/2, 1/2], 
\end{align*} \]

we get another equation which is topologically conjugate with Eq. (5) as

\[ \tilde{T} : x_n \mapsto \tilde{T}_x n, x_n \in I \times I, \]

\[ \begin{pmatrix} p_{n+1} \\ q_{n+1} \end{pmatrix} = \tilde{T} \begin{pmatrix} p_n \\ q_n \end{pmatrix} = \begin{pmatrix} q_n \mod [-1/2, 1/2] \\ 2q_n - p_n - 2\varepsilon \tan(\pi q_n) \mod [-1/2, 1/2] \end{pmatrix} \] (8)

The Jacobian of Eq. (10) is given as

\[ J(n) = \begin{pmatrix} 0 & -\frac{\varepsilon}{\cos^2(\pi q_n)} \\ -1 & 2 - \frac{\varepsilon}{\cos^2(\pi q_n)} \end{pmatrix} \] (11)

The local instability condition (for at least one eigenvalue of the matrix 11, its absolute value is larger than unity) for (11) is as

\[ \varepsilon < 0, \frac{2}{\pi} < \varepsilon. \] (12)

**Theorem 1.** The uniform distribution

\[ \rho(p,q) = 1 \] (13)

is an invariant density of the map \( T_x \) on the manifold \( I \times I \).

**Proof.** We prove the proposition by showing the uniform distribution \( \rho \) is a solution of Perron-Frobenius Equation

\[ f(x,y) = \int \int_{-1/2}^{1/2} f(p,q) \delta(X - T_x(Y)) \, dpdq, \]

where \( X = (x,y) \), \( Y = (p,q) \).

When a uniform distribution satisfies Perron-Frobenius Equation, uniform distribution (13) is an invariant density for \( (I \times I, \tilde{T}_x) \). Because the values of \( \rho(x,y) \) and \( \rho(p,q) \) is equivalent, we show for any point \( X = (x,y) \) there is only one point \( (p', q') \) which satisfies \( X - T_x(p', q') = 0 \).

First, \( X \) is fixed. \( x \) is on the \( I = [-1/2, 1/2] \). Then we can determine only one \( q \in I \) which satisfies \( x - q = 0 \). Then \( x, y \) and \( q \) are fixed. Then there is only one \( p \in I \) to satisfies

\[ y - (2q - p - 2\varepsilon \tan(q)) \mod [-1/2, 1/2] = 0. \] (15)

Therefore uniform distribution is a solution of the Perron-Frobenius Equation.

When \( p \) and \( q \) distribute uniformly, \( \varepsilon \tan(\pi p) \) or \( \varepsilon \tan(\pi q) \) are in accordance with the Cauchy distribution as

\[ f(x) = \frac{1}{\pi x^2 + \varepsilon^2}. \] (16)

**Theorem 2.** The probability variables \( p \) and \( q \) are independent.

**Proof.** For any \( L^1 \) class function \( A \) and \( B \),

\[ \mathbb{E}[A(p)B(q)] = \int \int A(p)B(q) \left( \frac{1}{4} dpdq \right), \]

\[ = \int \int A(p) \left( \frac{1}{2} dp \right) \int B(q) \left( \frac{1}{2} dq \right), \]

\[ = \mathbb{E}[A(p)]\mathbb{E}[B(q)]. \]

Then, \( p \) and \( q \) are independent.

**Mixing property** Let consider a set \( A \) defined by

\[ A = \{(p,q)|^3 n \in \mathbb{Z} \text{ s.t. } q' = \pm 1/2, (p', q') = \tilde{T}_x(p, q)\}. \]

Then, define manifold \( M = (I \times I) \setminus A \).

**Definition 3.** A diffeomorphism \( f : M \to M \) where \( M \) is a closed manifold is Anosov diffeomorphism when there exists a direct sum decomposition of the tangent bundle \( T_x M \) at each point \( x \) into complementary subspace \( E_x^u, E_x^s \) such that

\[ (D_x f)E_x^u = E_x^f(x), \]

\[ (D_x f)E_x^s = E_x^f(x), \]

\[ \xi \in E_x^u \implies \| (D_x f^n)E_x^u \| \geq K \lambda^n \| \xi \|, \]

\[ \xi \in E_x^s \implies \| (D_x f^n)E_x^s \| \leq K \lambda^n \| \xi \|, \]

where \( K > 0, 0 < \lambda < 1 \) are determined by \( x \) not by \( \xi \) or \( n \).

**Lemma 4.** When the condition (12) is satisfied, the map \( \tilde{T}_x \) on \( M \) is an Anosov diffeomorphism.

According to [7, 8], for any normalized two dimensional vector

\[ \mathbf{a} = (a_1(n), a_2(n)), \]

consider cones \( L^+ \) and \( L^- \) such that

\[ L^+ = \{(a_1(n), a_2(n)); \| J(n)\mathbf{a} \| > \| \mathbf{a} \| \}, \]

\[ L^- = \{(a_1(n), a_2(n)); \| J(n)\mathbf{a} \| < \| \mathbf{a} \| \}. \]

The goal is to prove

\[ J(n)a \in L^+(\tilde{T}_x x_n), \mathcal{V}a \in L^+(x_n), \]

\[ J(n-1)^{-1}a \in L^-(\tilde{T}_{x_n}^{-1} x_n), \mathcal{V}a \in L^-(x_n). \] (21)
\[ \theta_1(n+1) - 2\theta_1(n) + \theta_1(n-1) = -\varepsilon \tan(\pi(\theta_1(n) - \theta_2(n))), \]
\[ \theta_2(n+1) - 2\theta_2(n) + \theta_2(n-1) = \varepsilon \tan(\pi(\theta_1(n) - \theta_2(n))). \]
\[ [\theta_1(n+1) - \theta_2(n+1)] - 2 [\theta_1(n) - \theta_2(n)] + [\theta_1(n-1) - \theta_2(n-1)] = -2\varepsilon \tan[\pi(\theta_1(n) - \theta_2(n))]. \]

Proof. In the case of (21), the condition that \( \mathbf{a} \in L^+(x_n) \) is expressed by
\[ \mathbf{a} \in L^+(x_n) \iff \left\{ \begin{array}{l} a_1(n) > 1 - \frac{\pi\varepsilon}{\cos^2(\pi q_n)}, \quad \varepsilon > 2 \pi \\ a_2(n) < 1 - \frac{\pi\varepsilon}{\cos^2(\pi q_n)}, \quad \varepsilon < 0. \end{array} \right. \] (23)
Let define \( \mathbf{a}' = J(n)\mathbf{a} = (a_1(n+1), a_2(n+1)) \). The condition that \( \mathbf{a}' \in L^+(x_{n+1}) \) is expressed by
\[ J(n)\mathbf{a} \in L^+(\tilde{T}_\varepsilon x_n), \quad \iff \left\{ \begin{array}{l} a_1(n+1) > 1 - \frac{\pi\varepsilon}{\cos^2(\pi q_{n+1})}, \quad \varepsilon > 2 \pi \\ a_2(n+1) < 1 - \frac{\pi\varepsilon}{\cos^2(\pi q_{n+1})}, \quad \varepsilon < 0. \end{array} \right. \] (24)
By substituting \( a_1(n+1) = a_2(n) \) and \( a_2(n+1) = -a_1(n) + 2 \left( 1 - \frac{\pi\varepsilon}{\cos^2(\pi q_n)} \right) a_2(n) \),
\[ \frac{a_1(n+1)}{a_2(n+1)} = \frac{a_2(n)}{a_2(n)} = 1 \]
Then, when the condition (23) is satisfied, it holds that
\[ 1 - \frac{\pi\varepsilon}{\cos^2(\pi q_k)} < \frac{a_1(k)}{a_2(k)} < 0, \quad \forall k \geq n + 1, \varepsilon > \frac{2}{\pi}, \]
\[ 0 < \frac{a_1(k)}{a_2(k)} < 1 - \frac{\pi\varepsilon}{\cos^2(\pi q_k)}, \quad \forall k \geq n + 1, \varepsilon < 0. \] (25)
Therefore the condition (21) holds. Then consider a subset \( LL^+(x_n) \) of \( L^+(x_n) \) defined by
\[ LL^+(x_n) = \left\{ (a_1(n), a_2(n)) \left| 1 - \frac{\pi\varepsilon}{\cos^2(\pi q_n)} < \frac{a_1(n)}{a_2(n)} < 0 \right. \right\} \]
when \( \varepsilon > 2 \pi \) and
\[ LL^+(x_n) = \left\{ (a_1(n), a_2(n)) \left| 0 < \frac{a_1(n)}{a_2(n)} < 1 - \frac{\pi\varepsilon}{\cos^2(\pi q_n)} \right. \right\} \]
when \( \varepsilon < 0 \). It holds that
\[ J(n)\mathbf{a} \in LL^+(\tilde{T}_\varepsilon x_n), \quad \forall \mathbf{a} \in LL^+(x_n). \] (27)
In the case of (22), it holds that
\[ \mathbf{a} \in L^-(x_n) \iff \left\{ \begin{array}{l} a_1(n) < 1 - \frac{\pi\varepsilon}{\cos^2(\pi q_n)}, \quad \varepsilon > 2 \pi \\ a_2(n) > 1 - \frac{\pi\varepsilon}{\cos^2(\pi q_n)}, \quad \varepsilon < 0. \end{array} \right. \] (28)
Then,
\[ J(n)^{-1} \mathbf{a} \in L^-(\tilde{T}_\varepsilon^{-1} x_n), \quad \iff \left\{ \begin{array}{l} a_1(n-1) < 1 - \frac{\pi\varepsilon}{\cos^2(\pi q_{n-1})}, \quad \varepsilon > 2 \pi \\ a_2(n-1) > 1 - \frac{\pi\varepsilon}{\cos^2(\pi q_{n-1})}, \quad \varepsilon < 0. \end{array} \right. \] (29)
By substituting \( a_1(n-1) = 2 \left( 1 - \frac{\pi\varepsilon}{\cos^2(\pi q_{n-1})} \right) a_1(n) - a_2(n) \) and \( a_2(n-1) = a_1(n) \), one can see condition (29) holds when condition (28) holds.
Then, by considering a orbit \( \{x_n\}_{n=-\infty}^{\infty} \) there exists a cone \( L^-(x_{\infty}) \) and \( l^-(x_n) \) such that
\[ L^-(x_n) \supset J^{-1}(n)L^-(x_{n+1}), \]
\[ \supset J^{-1}(n)J^{-1}(n+1)L^-(x_{n+2}), \quad \ldots \]
\[ \supset l^-(x_n), \]
\[ l^-(x_n) \equiv \left( \prod_{k=n}^{\infty} J^{-1}(k) \right) L^-(x_{\infty}). \]
Then, one can choose any eigenvector spaces \( E^u_x \) and \( E^s_x \) from \( LL^+(x_n) \) and \( l^-(x) \) each other by
\[ E^u_{x_n} \subset LL^+(x_n), \quad E^s_{x_n} \subset l^-(x_n). \] (30)
It is established that \( \left( D_{x_n}\tilde{T}_\varepsilon \right) E^u_{x_n} \subset LL^+(x_{n+1}) \) and \( \left( D_{x_n}\tilde{T}_\varepsilon \right) E^s_{x_n} \subset L^-(x_{n+1}) \). Then, one can define \( E^u_{x_n+1}, \quad E^s_{x_n+1} \) by
\[ E^u_{x_n+1} = \left( D_{x_n}\tilde{T}_\varepsilon \right) E^u_{x_n}, \quad E^s_{x_n+1} = \left( D_{x_n}\tilde{T}_\varepsilon \right) E^s_{x_n}. \] (31)
\( E^u_{x_n} \) and \( E^s_{x_n} \) are independent by their definition so that, it holds that
\[ T_{x_n} M = E^u_{x_n} \oplus E^s_{x_n}. \]
Next let’s determine \( K, \lambda \). Stretching rate \( \sigma \) is defined by
\[ \sigma(x_n, \mathbf{a}) = \frac{\|J(n)\mathbf{a}\|^2}{\|\mathbf{a}\|^2}, \]
\[ = \frac{a_1^2 + a_2^2 - 4a_1a_2 \left( 1 - \frac{\pi\varepsilon}{\cos^2(\pi q_n)} \right)}{a_1^2 + a_2^2}, \]
\[ = 1 - 4 \frac{a_1a_2 \left( 1 - \frac{\pi\varepsilon}{\cos^2(\pi q_n)} \right)}{a_1^2 + a_2^2} + 4 \left( 1 - \frac{\pi\varepsilon}{\cos^2(\pi q_n)} \right)^2 \frac{a_1^2 + a_2^2}{a_1^2 + a_2^2}, \]
\[ = 1 + 4 \left( 1 - \frac{\pi\varepsilon}{\cos^2(\pi q_n)} \right) \left( \frac{1}{\sin^2 \phi - \sin \phi \cos \phi} \right), \]
where \( \sin^2 \phi = \frac{a_1}{a_1^2 + a_2^2}, \sin \phi \cos \phi = \frac{a_1 a_2}{a_1^2 + a_2^2}, \) \(-\pi < \phi \leq \pi\). Then, it holds that if \( a \in LL^+(x_n), \)
\[
\left(1 - \frac{\pi \varepsilon}{\cos^2(\pi q_n)}\right) \sin^2 \phi - \sin \phi \cos \phi \left\{ \begin{array}{ll}
< 0, & \varepsilon > \frac{\pi}{2}, \\
> 0, & \varepsilon < 0,
\end{array} \right. \tag{32}
\]
Let define \( \alpha_n = \left(1 - \frac{\pi \varepsilon}{\cos^2(\pi q_n)}\right) \), and
\[
g(\phi) \equiv \alpha_n \sin^2 \phi - \sin \phi \cos \phi,
g'(\phi) = \alpha_n \sin(2\phi) - \cos(2\phi),
= \sin(2\phi) (\alpha_n - \cot(2\phi)) .
\]

(1) Case of \( \varepsilon > \frac{\pi}{2} \).
Considering \( \alpha_n < \frac{\alpha_{1(n)}}{\alpha_{2(n)}} = \cot \phi_n < 0 \), the range of \( \phi_n \) is expressed by
\[
\frac{\pi}{2} < \phi_n < \psi_n, -\frac{\pi}{2} < \phi_n < \psi_n - \pi, \tag{33}
\]
where \( \cot \psi_n = \alpha_n \). Since \( g'(\phi) \) is positive in this range, It becomes
\[
g(\phi_n) < g(\psi_n) = \frac{2\alpha_n}{1 + \alpha_n^2} < 0,
\]
\[
\sigma(x_n, \alpha) < 1 + 4\alpha_n \cdot \frac{2\alpha_n}{1 + \alpha_n^2} < 1 + \frac{16(1 - \pi \varepsilon)^2}{1 + (1 - \pi \varepsilon)^2}.
\]
Then by defining \( K = 1 \) and \( \lambda = \sqrt{1 + \frac{16(1 - \pi \varepsilon)^2}{1 + (1 - \pi \varepsilon)^2}} \), condition (19) is satisfied.

(II) Case of \( \varepsilon < 0 \).
Considering \( 0 < \cot \phi_n < \alpha_n \), the range of \( \phi_n \) is expressed by
\[
\psi_n < \phi_n < \frac{\pi}{2}, \psi_n - \pi < \phi_n < -\frac{\pi}{2} . \tag{34}
\]
Since \( g'(\phi) \) is positive in this range, It becomes
\[
g(\phi_n) > g(\psi_n) = \frac{2\alpha_n}{1 + \alpha_n^2} > 0,
\]
\[
\sigma(x_n, \alpha) > 1 + 4\alpha_n \cdot \frac{2\alpha_n}{1 + \alpha_n^2} > 1 + \frac{16(1 - \pi \varepsilon)^2}{1 + (1 - \pi \varepsilon)^2} .
\]
Then by defining \( K = 1 \) and \( \lambda = \sqrt{1 + \frac{16(1 - \pi \varepsilon)^2}{1 + (1 - \pi \varepsilon)^2}} \), condition (19) is satisfied.

For a symplectic map, since the shrinking rate is a inverse of stretching rate, condition (20) is also holds. Therefore, the map \( T_\varepsilon \) on \( M \) is an Anosov diffeomorphism.

According to [8], Anosov diffeomorphism is K-system. Therefore the theorem below holds.

**Theorem 5.** When the condition (12) is satisfied, dynamical system \((M, T_\varepsilon, \frac{1}{2} dpdq)\) has the mixing property.

According to [1], if \((M, T_\varepsilon)\) is a Anosov diffeomorphism, \( M \) is also Axiom A attractor. Since Lebesgue measure \( dpdq \) is preserved by \( T_\varepsilon \) on \( M \), and dynamical system \((M, T_\varepsilon, dpdq)\) is ergodic for \( \varepsilon < 0, \frac{\pi}{2} < \varepsilon \). Therefore, Lebesgue measure is a unique SRB measure.

**Cauchy distribution** Action variable \( I_1(n) \) can be expressed by
\[
I_1(n) = I_1(0) - \varepsilon \sum_{k=1}^{n} \tan \left[ \pi \{ \theta_1(k) - \theta_2(k) \} \right] . \tag{35}
\]
A probability variable \( s_k = \{-\varepsilon \tan \left[ \pi \{ \theta_1(k) - \theta_2(k) \} \right] \} \) is according to the Cauchy distribution whose scale parameter is \( |\varepsilon| \) for \( \varepsilon < 0, \frac{\pi}{2} < \varepsilon \). That is \( \{s_k\} \) is stationary and strongly mixing. Then according to [11], \( \{I_1(n)\} \) is in accordance with a stable distribution. Figure 3 shows log-log plot of the distribution \( f(x) \) of \( \{I_1(100)\} \) at \( \varepsilon = 0.65 \) obtained by numerical experiment and fitted function \( g(x) \). The number of initial points \( N \) is \( N = 10^6 \). When \( \{I_1(100)\} \) are in accordance with Cauchy distribution whose scale parameter is \( \alpha \), the probability variables \( \{x = (I_1(100) - \mu)/\sqrt{\sigma}\} \) are considered to be in accordance with \( g(x) \) defined by
\[
g(x) = \frac{1}{\pi} \frac{a\sqrt{\sigma}}{(\sqrt{\sigma}x)^2 + a^2}, \tag{36}
\]
where \( \mu \) and \( \sqrt{\sigma} \) are an average and a variance respectively obtained from finite number of probability variables \( \{I_1(100)\} \). By fitting \( g(x) \) to the data using least squares method, fitted parameter \( \hat{\alpha} \) is obtained as \( \hat{\alpha} \approx 68.0 \approx 0.5 \times 10^2 \). This result shows \( I_1 \) is in accordance with Cauchy distribution, so that the true average and variance of \( I_1 \) do not exist.

The fluctuation of Energy In not integrable system, there is a trade-off relation between the conservation of Energy and the that of symplecticity [10]. Then, this symplectic map, the energy cannot be conserved and fluctuates. Especially in this map since the distribution with \( I_1 \) and \( I_2 \) are in accordance with Cauchy distribution, their variances \( \sigma(I_{1,2}) \) diverge. Then the kinetic energy and a total one also diverge. The Figure 4 shows the behavior of the fluctuation of energy.
FIG. 4. The time behavior of total Energy $H(I_1, I_2, \theta_1, \theta_2)$. It fluctuates and its variance diverges. The initial condition is $(I_1, I_2, \theta_1, \theta_2) = (1.41, 1.51 + \frac{\pi}{2}, 0.1, -1.4)$.

**Superdiffusion** Considering that $I_1$ is in accordance with Stable distribution and referring the Figure 3, superdiffusion occurs. Figure 5 shows the log-log plot with times evolution of Mean Square Displacement (MSD) for $I_1$ at $\varepsilon = 0.66 > \frac{2}{\pi}$ and there occurs superdiffusion. The inclination of the fitting line is about 1.80 > 1.

**Lyapunov exponent** Since the Lebesgue measure is unique SRB measure for $\varepsilon < 0$, $\frac{2}{\pi} < \varepsilon$, KS entropy $h(\tilde{T}_\varepsilon)$ is expressed as

$$h(\tilde{T}_\varepsilon) = \int \int \log |\det (D\tilde{T}_\varepsilon|_{C^\alpha})| dpdq, \quad (37)$$

$$= \int \int \log |\gamma| dpdq, \quad (38)$$

where $\gamma$ is a eigenvalue of Jacobian $J(x)$ whose absolute value is larger than unity.

$$\gamma = \begin{cases} \gamma_-, & \text{when } \varepsilon > \frac{2}{\pi}, \\ \gamma_+, & \text{when } \varepsilon < 0, \end{cases} \quad (39)$$

$$\gamma^\pm = 1 - \frac{\pi \varepsilon}{\cos^2(\pi q)} \pm \sqrt{\left(1 - \frac{\pi \varepsilon}{\cos^2(\pi q)}\right)^2 - 1}. \quad (40)$$

$h(\tilde{T}_\varepsilon)$ is equivalent to a positive Lyapunov exponent. Then, in $|\varepsilon| \gg 1$, the Lyapunov exponent can be expressed by

$$\lambda(\varepsilon) = \log \left|2 \left(1 - 2 \varepsilon \pm \sqrt{4 \varepsilon (\varepsilon - 1)}\right)\right|. \quad (41)$$

Figure 6 shows the comparison between the numerical result and analytical formula of Lyapunov exponent. The numerical result is consistent with analytic formula in $|\varepsilon| \gg 1$.

**Conclusion** We proposed canonical deterministic model with superdiffusion. We showed the condition of superdiffusion. The analytical formula of invariant density is obtained and mixing property is shown when this condition satisfied. We also calculate Lyapunov exponent and it is consistent with a numerical experiment. The action variables are in accordance with the Cauchy distribution whose scale parameter is $|\varepsilon|$ in $\varepsilon < 0$, $\frac{2}{\pi} < \varepsilon$. Therefore, this model has energy fluctuation divergence.

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