O. G. Styrt

Orthogonality graphs of matrices over commutative rings

The paper is devoted to studying the orthogonality graph of the matrix ring over a commutative ring. It is proved that the orthogonality graph of the ring of matrices with size greater than 1 over a commutative ring with zero-divisors is connected and has diameter 3 or 4; a criterion for each value is obtained. It is also shown that each of its vertices has distance at most 2 from some scalar matrix.

Key words: associative ring with identity, commutative ring, zero-divisor, matrix ring, zero-divisor graph, orthogonality graph.

§ 1. Introduction

Researching properties of associative rings in terms of graphs of some naturally occurring algebraic binary relations takes an important place in modern mathematics. Thus, a zero-divisor graph was first defined in 1986 by Beck [1] for a commutative ring. Its vertices were all zero-divisors, and edges connected exactly all pairs of distinct elements giving zero in product. But since 1999 one uses its more convenient interpretation introduced by Anderson and Livingston in [2] via excluding the zero element of the ring from its vertex set. It is also proved in [2] that the zero-divisor graph of a commutative ring is connected and has diameter at most three; in the former treatment of the graph these statements would be trivial. A number of further papers also studies various characteristics of the zero-divisor graph: center and radius [8], concepts of planarity [4] and uniqueness of determining the ring by the graph up to an isomorphism [3, 5]. For non-commutative rings, there are several types of graphs defined by zero-divisors:
The main results for orthogonality graphs of non-commutative rings found by now concern primarily matrix rings. Thus, in the case of the basic ring being a skew field, the following properties of the orthogonality graph of the \((n \times n)\)-matrix ring are obtained: once \(n = 2\), it is disconnected and all its connected components have diameters at most 2, and, once \(n \geq 3\), it is connected and has diameter 4. These statements are proved in 2014 for a field [9] and later, in 2017 — for an arbitrary skew field [10]; they can also be easily generalized to integral domains (by reducing to the field of fractions).

In this paper, there will be the orthogonality graph of the matrix ring over a commutative ring with zero-divisors studied and the following main result proved.

**Theorem 1.1.** Let \(R\) be a commutative ring with zero-divisor set \(Z_R \neq \{0\}\). Then, for any \(n > 1\), the orthogonality graph of the ring of \((n \times n)\)-matrices over \(R\) is connected and has diameter 3 or 4, the value 3 being equivalent to the relation

\[
\forall a_0 \in Z_R \quad \exists a_1, a_2 \in R \setminus \{0\} \quad \forall i, j \in \{0, 1, 2\}, \ i \neq j:\ a_i a_j = 0, \quad (1.1)
\]

and each of its vertices has distance at most 2 from some scalar matrix.

**Theorem 1.2.** Let \(r\) be the radius of the graph under conditions of Theorem 1.1. Then

1) \(2 \leq r \leq 4\);
2) if (1.1) holds, then \(r \in \{2; 3\}\);
3) \( r = 2 \) if and only if there exists an element \( c \in R \setminus \{0\} \) such that
\[ \forall a \in Z_R \quad \text{Ann}(c) \cap \text{Ann}(a) \neq 0. \tag{1.2} \]

§ 2. Auxiliary agreements

In the paper, the following notations and agreements will be used.

1) Set-theoretical:
- While listing elements of a disordered set, figured brackets are used. As for elements of an ordered tuple, they are listed in round brackets and can be repeated.
- \( D^n := D \times \ldots \times D \) is the \( n \)-ary Cartesian power of a set \( D \).

2) General algebraic:
- All rings considered are supposed to be associative and with identity.
- \( R \) is an arbitrary ring.
- For any subset \( D \subset R \), define \( D^* := D \setminus \{0\} \). In particular, by \( R^* \) denote the subset of all nonzero (not necessarily invertible as in standard interpretation) elements of \( R \).
- An ideal in \( R \) is proper if it does not equal \( R \).
- \( M_{m \times n}(R) \) is the \( R \)-module of \((m \times n)\)-matrices over \( R \); \( M_n(R) \) is the ring \( M_{n \times n}(R) \). If in the brackets the ring is replaced with some of its subsets \( D \), then the subset of all matrices with entries from \( D \) is meant.
- \( 0_{m \times n}^n \) is the zero \((m \times n)\)-matrix; \( 0^n_n := 0 \); \( E_n \) is the identity \((n \times n)\)-matrix; \( J_r \) is the Jordan cell of size \( r \) with eigenvalue 0. If the matrix sizes are clear from the context, then the indices can be omitted.
- \( E_{kl} \) is the matrix unit \((a_{ij})\), \( a_{ij} := \delta_{ki} \delta_{lj} \).
- For a square matrix \( A \) over a commutative ring: \( \tilde{A} \) is its cofactor matrix; \( \hat{A} := (\tilde{A})^T \).
- If \( A = (a_{k_1, k_2}) \in M_{n_1 \times n_2}(R) \), \( P_i \in \{1, \ldots, n_i\}^{m_i} \) \((i = 1, 2)\), then \( A^{P_2}_{P_1} \) is the matrix \((b_{l_1, l_2}) \in M_{m_1 \times m_2}(R)\), \( b_{l_1, l_2} := a_{k_1(l_1), k_2(l_2)} \), where \( k_i(l_i) \) is the \( l_i \)-th element of \( P_i \). If numbers are repeated neither in \( P_1 \), nor in \( P_2 \), then \( A^{P_2}_{P_1} \) is the submatrix of \( A \) with row and column numbers from \( P_1 \) and \( P_2 \) respectively.

3) On zero-divisor types:
- An element \( a \in R \) is called
  - a left (resp. right) zero-divisor if there exists an element \( b \in R^* \) such that \( ab = 0 \) (resp. \( ba = 0 \));
  - a zero-divisor if it is either left or right zero-divisor;
  - a two-sided zero-divisor if it is both left and right zero-divisor.

At that,
– in a commutative ring, the concepts of all zero-divisor types are equivalent;
– zero is a two-sided zero-divisor; if there are no other zero-divisors, then \( R \) is called a ring without zero-divisors.

• An integral domain is a commutative ring without zero-divisors.

4) From general graph theory:
• All graphs considered are assumed to be undirected.
• \( \Gamma = (V, E) \) is an arbitrary graph; \( V \) and \( E \) are its vertex and edge sets respectively. In doing so, one can (usually with more convenience) define \( E \) via a symmetric binary relation on \( V \).
• Two vertices are adjacent if they are connected with an edge.
• A subgraph is a graph with vertex set \( V' \subset V \) and, unless otherwise stated, with the same binary relation restricted on \( V' \).
• A path is a sequence of vertices where any two neighbor ones are adjacent.
• The length of a path is the number of its edges.
• The distance between vertices \( v \) and \( w \) (not. \( d(v, w) \)) is the minimum of lengths of paths between them; if they do not exist, then set \( d(v, w) := +\infty \); the sign is obvious in this context and therefore will be omitted. Clearly, \( (d(v, w) = 0) \iff (v = w) \).
• The distance from a vertex \( v \) to a subset \( W \subset V \) (not. \( d(v, W) \)) is the number
  \[
  \min\{d(v, w): w \in W\}.
  \]
• \( d(v) := \sup\{d(v, w): w \in W\} \ (v \in V) \).
• The diameter of \( \Gamma \) is the number
  \[
  \text{diam}(\Gamma) := \sup\{d(v, w): v, w \in V\} = \max\{d(v): v \in V\}.
  \]
• The radius of \( \Gamma \) is the number
  \[
  \text{rad}(\Gamma) := \min\{d(v): v \in W\}. \quad (2.1)
  \]
  Clearly, \( \text{rad}(\Gamma) \leq \text{diam}(\Gamma) \leq 2 \cdot \text{rad}(\Gamma) \).

• A graph is connected if there exists a path between any two of its vertices.

**Remark.** It is easy to see that a graph with finite diameter is connected. The converse fails; an example is the set of positive integers with the neighborhood relation.

5) On special graphs in algebraic structures:
• \( O(R) \) is the orthogonality graph of the ring \( R \) (for a commutative ring it is the same as the zero-divisor graph).
• Vertices of \( O(R) \) are all nonzero two-sided zero-divisors of \( R \); the orthogonality relation \( (xy = yx = 0) \) is written as \( (x \perp y) \); \( O_R(x) \) is the set of all vertices orthogonal to \( x \).

*Possibly \( \infty \).*
§ 3. Proofs of the results

Consider an arbitrary commutative ring $R$. Denote by $\text{Ann}(a)$ ($a \in R$) the ideal \( \{ x \in R : ax = 0 \} \) and by $Z_R$ the set \( \{ a \in R : \text{Ann}(a) \neq 0 \} \) of all zero-divisors. Further, let $S$ be the ring $M_n(R)$ ($n > 1$). Via the natural ring embedding $R \to S$, $a \to aE$, identify $R$ with the subring $RE \subset S$ (and, thus, $O(R)$ — with a subgraph of the graph $O(S)$). For $A \in S$, set $I_A := \text{Ann}(\det A) \triangleleft R$.

The graph $O(R)$ is connected and has diameter at most $3$ (see Theorem 2.3 in [2, § 2]). Besides, if $R$ is a skew body, then

1) once $n = 2$, the graph $O(S)$ is disconnected and all its connected components have diameters $\leq 2$;
2) once $n \geq 3$, the graph $O(S)$ is connected and has diameter 4.

These results are obtained in [9, § 4] for fields (Lemma 4.1 and Theorem 4.5 respectively), and in [10, § 2] are generalized to arbitrary skew-fields (Lemma 2.2 and Theorem 2.1 respectively). They are also shifted to integral domains (by reducing to the field of fractions).

**Theorem 3.1.** For any matrix $A \in S$ and proper ideal $I \triangleleft R$ containing $\det A$, there exists a matrix $B \in S \setminus (M_n(I))$ such that $AB, BA \in M_n(I)$.

□ For $m \in \mathbb{N}$, set $Q_m := \{1, \ldots, m\}$ and $P_m := \{1, \ldots, m\} \in \mathbb{N}^m$.

Consider all triples $(k, P', P'')$ ($k \geq 0$, $P', P'' \in (Q_n)^k$) satisfying the relation $\det(A_{P''}^{P'}) \notin \notin I$. For each of them, numbers are repeated neither in $P_1$, nor in $P_2$, and, by condition, $k < n$. Besides, at least one of such triples exists: for $k := 0$ and empty tuples $P', P''$, the corresponding $(0 \times 0)$-matrix has determinant $1 \notin I$. Hence, we can fix one of these triples with the largest possible $k$, and then $0 \leq k < n$, $m := k + 1 \in Q_n$.

**Case 1.** $P' = P'' = P_k$.

By construction, $\det(A_{P_k}^{P_k}) \notin I$. Further, set $C := A_{P_m}^{P_m} \in M_m(R)$,

$$B := \begin{pmatrix} \tilde{C} \\ 0_{m-m} \\ 0_{n-m} \end{pmatrix} \in S.$$

Then $b_{m,m} = \det(A_{P_k}^{P_k}) \notin I$ implying $B \notin M_n(I)$. Show that $AB, A^T B^T \in M_n(I)$, i.e. that, for any $p, q \in Q_n$, the matrix entries $(AB)_{p,q}$ and $(A^T B^T)_{p,q}$ belong to $I$. Assume that $p \in Q_n$ and $q \in Q_m$ (otherwise $(AB)_{p,q} = (A^T B^T)_{p,q} = 0$). Let $P \in (Q_n)^m$ be the tuple obtained form $P_m$ by changing the $q$-th element with $p$. Due to maximality of $k$ and the inequality $m > k$, we have $\det(A_{P_m}^{P_m}), \det(A_{P_m}^{P_m}) \in I$,

\[
(AB)_{p,q} = \sum_{i \in Q_n} (a_{p,i} b_{i,q}) = \sum_{i \in Q_m} (a_{p,i} (\tilde{C})_{i,q}) = \sum_{i \in Q_m} (a_{p,i} (\tilde{C})_{q,i}) = \det(A_{P_m}^{P_m}) \in I;
\]

\[
(A^T B^T)_{p,q} = \sum_{i \in Q_n} ((A^T)_{p,i} (B^T)_{i,q}) = \sum_{i \in Q_m} (a_{i,p} (\tilde{C})_{i,q}) = \det(A_{P_m}^{P_m}) \in I.
\]
Thereby, it is proved that $AB, (BA)^T = A^T B^T \in M_n(I)$ implying $BA \in M_n(I)$.

**Case 2.** $P', P'' \in (Q_n)^k$ are arbitrary tuples.

In each of the tuples $P'$ and $P''$ all numbers are distinct. Hence, via suitable permutations of rows and columns, one can obtain from $A$ a matrix $A_0$ satisfying Case 1 with the same $k$. By proved above, there exists a matrix $B_0 \in S \setminus (M_n(I))$ such that $A_0 B_0, B_0 A_0 \in M_n(I)$. At that, there exist monomial (therefore, invertible) matrices $C_1, C_2 \in S$ such that $A = C_1 A_0 C_2^{-1}$. Hence, via suitable permutations of rows (resp. columns), and, consequently, $B := C_2 B_0 C_1^{-1} \in S \setminus (M_n(I)), AB = C_1 (A_0 B_0) C_2^{-1} \in M_n(I), BA = C_2 (B_0 A_0) C_1^{-1} \in M_n(I)$. \hfill ■

**Corollary 3.1.** If $A \in S$ and $c \in I_A^*$, then, in the subset $(cS)^* \subset S$, there exists an element orthogonal to $A$.

By condition, $I := \text{Ann}(c) \trianglelefteq R$ is a proper ideal containing $\det A$. According to Theorem 3.1, there exists a matrix $B \in S \setminus (M_n(I))$ such that $AB, BA \in M_n(I)$. Thus, $C := cB \neq 0$ and $c(AB) = c(BA) = 0$, i.e. $C \in (cS)^*$ and $AC = CA = 0$. \hfill ■

**Lemma 3.1.** For any $A \in S$, the following conditions are equivalent:

1) $\det A \in Z_R$;
2) $I_A \neq 0$;
3) in $S^*$, there exists an element orthogonal to $A$;
4) $A$ is a two-sided zero-divisor;
5) $A$ is a zero-divisor.

The implications 1) $\iff$ 2) and 3) $\implies$ 4) $\implies$ 5) obviously follow from definitions, and the implication 2) $\implies$ 3) from Corollary 3.1.

Prove the implication 5) $\implies$ 1). Suppose that, without loss of generality, $A$ is a left zero-divisor, i.e. that $AB = 0$ for some $B \in S^*$. Then $\bar{A} A = (\det A) E$ implying $(\det A) B = \bar{A} AB = 0$. It remains to use non-triviality of $B$. \hfill ■

**Corollary 3.2.** All zero-divisors in $S$ are two-sided.

Let $Z_S \subset S$ be the subset of all elements $A \in S$ satisfying each of the equivalent conditions 1)–5) of Lemma 3.1, i.e. the set of all zero-divisors of the ring $S$. Then the vertex set of the graph $O(S)$ is $Z_S^*$.

Further, we will assume that $Z_R^* \neq \emptyset$.

**Statement 3.1.** If $I \trianglelefteq R$ and $I \neq 0$, then $Z_R \cap I \neq \{0\}$.

Suppose that $Z_R \cap I = \{0\}$. There exist elements $b \in I^*$ and $c \in Z_R^*$; then $bc \in Z_R \cap I = \{0\}$. So, $bc = 0 \neq c$ that implies $b \in Z_R \cap I^* = \emptyset$, a contradiction. \hfill ■

**Lemma 3.2.** If, for a subset $D \subset S$, the ideal $I := \bigcap_{A \in D} I_A \trianglelefteq R$ is nonzero, then there exist elements $b \in Z_R^*$ and $C_A \in S^*$, $A \in D$, such that $b E \perp C_A \perp A$ ($A \in D$).
According to Statement 3.1, the ideal \( I \) contains an element \( c \in Z_R \). Then \( bc = 0 \) where \( b \in Z_R \). Further, for any \( A \in D \), we have \( c \in I_A \) and, by Corollary 3.1, there exist an element \( C_A \in (cS)^* \) orthogonal to \( A \); at that, \( bC_A \in bcS = 0 \), \( bE \perp C_A \).

\[ \square \]

**Corollary 3.3.**

1) For any \( A \in Z_S^* \), we have \( d(A, O(R)) \leq 2 \).
2) If \( A_1, A_2 \in Z_S^* \) and \( I_{A_1} \cap I_{A_2} \neq 0 \), then \( d(A_1, A_2) \leq 4 \).

\[ \square \]

**Lemma 3.3.** If \( A_i \in Z_S^* \), \( c_i \in I_{A_i}^* \) \((i = 1, 2)\) and \( c_1 c_2 = 0 \), then \( d(A_1, A_2) \leq 3 \).

\[ \square \]

By Corollary 3.1, for each \( i = 1, 2 \), there exists an element \( C_i \in (c_i S)^* \) such that \( C_i \perp A_i \). In this case, \( C_1 C_2, C_2 C_1 \in c_1 c_2 S = 0 \), \( C_1 \perp C_2 \).

\[ \square \]

**Definition.** We will say that an ideal \( I \subset R \) does not have zero-divisors if \( I^* I^* \neq 0 \), i.e. if the ring \( I \) does not have zero-divisors.

**Lemma 3.4.** If \( A_1, A_2 \in Z_S^* \) and \( d(A_1, A_2) > 3 \), then \( I_{A_i} \) \((i = 1, 2)\) is the same ideal without zero-divisors.

\[ \square \]

According to Lemma 3.3, \( I_{A_1}^* I_{A_2}^* \neq 0 \). It remains to prove that \( I_{A_1} = I_{A_2} \).

Suppose that \( I_{A_1} \neq I_{A_2} \). Without loss of generality, assume that there exists an element \( c \in I_{A_1} \setminus I_{A_2} \). Setting \( a := \det A_2 \), we have \( I_{A_2} = \text{Ann}(a) \), \( b := ca \in I_{A_1}^* \), and \( bA_2 = ca I_{A_2} = 0 \) implying \( bI_{A_2}^* \subset \{0\} \cap (I_{A_1}^* I_{A_2}^*) = \emptyset \), \( I_{A_2} = \emptyset \), \( I_{A_2} = 0 \), a contradiction.

**Theorem 3.2.** The graph \( O(S) \) is connected and has diameter at most 4.

\[ \square \]

Suppose that there exist elements \( A_1, A_2 \in Z_S^* \) satisfying the inequality \( d(A_1, A_2) > 4 \). By Lemma 3.4, \( 0 \neq I_{A_1} = I_{A_2} = I_{A_1} \cap I_{A_2} \) that contradicts with Corollary 3.3.

**Theorem 3.3.** We have \( \text{diam}(O(S)) \geq 3 \), the strict inequality being equivalent to the existence of an ideal \( \text{Ann}(a) \subset R \) \((a \in Z_R)\) without zero-divisors.

\[ \square \]

Similarly with examples from [9, 10] giving lower estimates of the diameter, for an arbitrary \( a \in Z_R \), set \( I := \text{Ann}(a) \subset R \) and \( A := J_n + ae_{n1} \in S \). Note that

- \( A, A^T \in Z_S^* \), \( O_S(A) = I^* E_{1n}, O_S(A^T) = I^* E_{n1}; \)
- \( a_{12} = 1 \neq a_{21}, (AA^T)_{11} = 1 \) and \( O_S(A) \cap O_S(A^T) = \emptyset \), that implies \( d(A, A^T) \geq 3 \);
- if \( I^* I^* \neq 0 \), then \( (O_S(A))_1(O_S(A^T)) = (I^* I^*)_{11} \neq 0 \) and, hence, \( d(A, A^T) \geq 4 \).

Due to mentioned above, \( \text{diam}(O(S)) \geq 3 \), the strict inequality following from the existence of an ideal \( \text{Ann}(a) \subset R \) \((a \in Z_R)\) without zero-divisors. Conversely, in the case of the strict inequality, by Lemma 3.4, for some elements \( A \in Z_S \) and \( a := \det A \in Z_R \), the ideal \( I_A = \text{Ann}(a) \subset R \) does not have zero-divisors.

Now the main Theorem 1.1 follows from Theorems 3.2 and 3.3, and Corollary 3.3. It implies (see (2.1)) the statements 1) and 2) of Theorem 1.2. Let us prove 3).

\[ \square \]

In general, without identity.
Suppose that $\text{rad}(O(S)) = 2$. There exist elements $C \in Z_S^*$, $c \in R^*$ and $k, l \in Q_n$ such that $d(C, A) \leq 2$ ($A \in Z_S^*$) and $c_{kl} = c$. Further, there exists a permutation $\sigma \in S_n$ such that $m := \sigma(k) \neq l$.

Let $a \in Z_R$ be an arbitrary element.

Set $I := \text{Ann}(a) < R$ and $A := \left( \sum_{i \neq k} E_{i, \sigma(i)} \right) + aE_{km} \in S$. Note that

- $A \in Z_S^*$, $O_S(A) = I^*E_{mk}$;
- $A \neq C$ (otherwise $a_{kl} = c \neq 0$, $m = l$);
- $(m, k) \neq (k, l)$ (otherwise $m = k = l$), that implies $C \notin O_S(A)$.

Thus, $d(C, A) = 2$, so, there exists an element $B \in Z_S^*$ orthogonal to $C$ and $A$. We have $B = bE_{mk}$ where $b \in I^*$. Meanwhile, $BC = 0$, $0 = (BC)_{ml} = bc$, $b \in \text{Ann}(c) \cap I^*$.

Due to arbitrariness of $a \in Z_R$, the element $c \in R^*$ satisfies (1.2).

Conversely, assume that (1.2) holds for some $c \in R^*$. Show that the element $C := cE \in I^*S$ satisfies, for each $A \in Z_S^*$, the inequality $d(C, A) \leq 2$.

Let $A \in Z_S^*$ be an arbitrary element. Then $\det A \in Z_R$, and, by (1.2), there exists an element $b \in I_A$ such that $cb = 0$. Further, according to Corollary 3.1, there exists an element $B \in (bS)^*$ orthogonal to $A$; in this case, $cB \in cbS = 0$, $C \in Z_S^*$, $C \perp B \perp A$, $d(C, A) \leq 2$.

So, Theorem 1.2 is completely proved.

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