Hadamard States and Adiabatic Vacua

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A proof is presented that in a spatially flat Robertson-Walker spacetime the adiabatic vacuum of scalar quantum field is a Hadamard state only if it is of infinite order and vice versa every Hadamard state lies in the Fock space based on an adiabatic vacuum.

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I. INTRODUCTION

It is well documented in the literature [1,2] that the so-called Hadamard condition determining the singularity structure of the two-point function $G^{(2)}(x_0, x_1)$ of a quantum field is considered to be closely related to the notion of adiabatic vacuum on a homogeneous spacetime. Both concepts are of asymptotic nature, one with respect to distance in spacetime, the other with wave number in phase space: the complementarity is convincing on a heuristical level.

Several authors [3–5] did perform explicit calculations based on both methods and derived the identical result. But this early work focused more on the question of rendering a concrete physical quantity finite than to establish a one-to-one correspondence between each term of the asymptotic series attached to either approach. Today a different emphasis seems to prevail: the ability to renormalize the stress-energy tensor as an expectation value is based on the proper restriction of the class of admissible states, not on a specific computational scheme. The crucial question remains, how to distinguish admissible states.

The Hadamard condition is certainly the most thoroughly investigated [6,7] criterion for this purpose. It allows us to separate the state-dependent information in the two-point function of the quantum field from the purely geometrical terms that implement the causal structure of the theory. In a linear quantum theory this feature of the two-point function is sufficient to establish a state as physically acceptable.

Unfortunately, complying with the Hadamard condition alone does not suffice to qualify an arbitrary two-point function as an expectation value of the product of two quantum field operators at different points in spacetime. The two-point function has to meet certain positivity requirements that result by the definition of a quantum state as an positive functional on the algebra of observables [2,8].

In practice the real and symmetric part of a quantal two-point function, which contains the singularity, is obtained from a formal integral with measure $d\mu(k)$:

$$G^{(1)}(x_0, x_1) = \Re \int d\mu(k) \, \psi^*_k(x_0) \psi_k(x_1)$$  \hspace{1cm} (1)

where the functions $\psi_k$ are normalized solutions of the linear scalar wave equation

$$[-\nabla_{\mu} \nabla^{\mu} + V(x)] \psi_k(x) = 0$$  \hspace{1cm} (2)

and serve as an orthonormal basis of the “one-particle” Hilbert space $\mathcal{H}$. This procedure ensures positivity but not consistency with the Hadamard condition. The latter calls for a specific but in general by no means obvious specification of the “one-particle” Hilbert space: the asymptotic Hadamard series has to be transcribed into an asymptotic series for the initial conditions on the functions $\psi_k$.

In this paper I will address this translation problem only in its simplest form: I assume a Robertson-Walker (RW) spacetime with flat spacelike hypersurfaces and the line-element

$$ds^2 = a^2(\eta) \left( -d\eta^2 + d\bar{x}^2 \right)$$  \hspace{1cm} (3)

or more precise: of the integral kernel defining the two-point distribution.
where $a$ is a $C^\infty$-function on an open interval containing the point $\eta_i$.

In this simplified setting there exists a constructive, physically motivated \cite{10,11} and mathematically sound \cite{12} prescription how to specify the Cauchy data for $\psi_k$. The states selected by the procedure are called adiabatic. I will show that the Hadamard condition and the adiabatic prescription are in fact equivalent concerning the class of states they admit. Moreover, the Hadamard condition is as specific about the initial conditions on $\psi_k$ as an asymptotic criterion could be. In technical terms: If $\{\psi_k\}$ is a basis suitable for the Hadamard condition, then the set $\tilde{\psi}_k$ with

$$\tilde{\psi}_k = \alpha_k \psi_k + \beta_k \psi_k^*$$

and $|\alpha_k|^2 - |\beta_k|^2 = 1$ is suitable as well if and only if $\lim_{k \to \infty} k^n \beta_k = 0$ for all $n$.

II. HADAMARD SERIES AND ROBERTSON-WALKER SPACETIME

The precise statement of the Hadamard condition is intricate in its mathematical detail \cite{7}. I use the familiar yet heuristic formulation in the tradition of DeWitt and Brehme \cite{13,14}: There exits an asymptotic expansion of the symmetric two-point function

$$G^{(1)}(x_0, x_1) = \langle \Psi | \{ \hat{\phi}(x_0), \hat{\phi}(x_1) \} | \Psi \rangle \approx \frac{\Delta^{1/2}}{8\pi^2} \left( \frac{2}{\sigma} + v \ln \sigma + w \right)$$

near the coincidence limit $x_1 \to x_0$. The biscalar $\sigma(x_0, x_1)$ denotes half of the square of the geodesic interval between $x_0$ and $x_1$; $\Delta(x_0, x_1)$ is the biscalar form of the Van-Vleck-Morette determinant. The remaining symbols $v$ and $w$ represent biscalar functions with smooth behaviour when $x_1$ approaches $x_0$. It is well known that geometry and $V$ alone define $v(x_0, x_1)$ by recursion relations \cite{13,15}, whereas $w(x_0, x_1)$ contains the information about the quantum state.

The Hadamard condition is important in part because no reference to coordinates or peculiar symmetries of the spacetime has to be made. However, in order to compare this condition with the notion of an adiabatic vacuum in a RW spacetime, we have to reformulate the Hadamard series in local coordinates:\footnote{For technical reasons \cite{6,8} this local condition has to be supplemented by a global one \cite{2}: The two-point function $G^{(1)}(x_0, x_1)$ is finite whenever $x_1 \neq x_0$.}

$$G^{(1)}_{\text{RW}}(x_0, x_1) = G^{(1)}_{\text{RW}}(\eta_0, \eta_1, r) \approx \frac{1}{8\pi^2 a_0 a_1} \left( \frac{4\epsilon}{\lambda^2} + V \ln \lambda^2 + W \right).$$

Here we use the abbreviations

$$\tau := \eta_1 - \eta_0, \quad r := |\vec{x}_1 - \vec{x}_0|,$$

$$\lambda := \sqrt{|r^2 - \tau^2|}, \quad \epsilon := \text{sign} \left( r^2 - \tau^2 \right).$$

I assume further that the quantum state itself is homogenous and isotropic.
and \( a_j = a(\eta_j) \). The correspondence between \( v \) and \( V \) or \( w \) and \( W \) is then given by

\[
v(\eta_0, \eta_1, r) = \Delta^{-1/2} \frac{4}{a_0 a_1} V, \tag{7a}
\]

\[
w(\eta_0, \eta_1, r) = \Delta^{-1/2} \left\{ \frac{4}{a_0 a_1 \epsilon \lambda^2} - 2 \frac{\Delta^{1/2}}{\sigma} + \frac{V}{a_0 a_1} \ln \frac{\lambda^2}{|\sigma|} + \frac{W}{a_0 a_1} \right\}. \tag{7b}
\]

The difference \( \sqrt{\Delta/(2\sigma)} - 1/(a_0 a_1 \epsilon \lambda^2) \) is a smooth function of the coordinates due to the formula \( \nabla_\mu \nabla^\mu (\sqrt{\Delta/\sigma}) = (\nabla_\mu \nabla^\mu \sqrt{\Delta})/\sigma \).

### III. ADIABATIC MODES

#### A. Definition and Existence

If we exploit the symmetry of the RW spacetime, we can express the formal integral from Eq.(1) as

\[
G_{RW}(\eta_0, \eta_1, r) = \frac{1}{a_0 a_1 \pi^2} \int_0^\infty \frac{\sin kr}{kr} \Re \{\chi_k(\eta_1) \chi_k^*(\eta_0)\} k^2 dk. \tag{8}
\]

The modes \( \chi_k(\eta) \) satisfy the following set of equations:

\[
\chi_k''(\eta) + \Omega_k^2(\eta) \chi_k = 0, \tag{9a}
\]

\[
\Omega_k^2 = k^2 + M^2(\eta), \tag{9b}
\]

\[
M^2(\eta) = a^2 \left( \sqrt{v - \frac{a''}{a^3}} \right) \tag{9c}
\]

and the normalization constraint is

\[
\chi_k(\eta) \chi_k'(\eta)^* - \chi_k(\eta)^* \chi_k'(\eta) = i, \tag{9d}
\]

where the prime indicates differentiation with respect to \( \eta \).

An adiabatic mode of \( n \)-th order, \( \chi_k^n \), is a specific kind of mode, defined by

\[
\chi_k^n(\eta) = \frac{\exp \left( -i \int_{\eta_i}^\eta \frac{n}{W_k} d\eta \right)}{\sqrt{2W_k}} \zeta_k^n(\eta), \tag{10a}
\]

with the intial conditions \( \zeta_k^n(\eta_i) = 1 \) and \( \zeta_k^n'(\eta_i) = 0 \), and the iteration scheme

\[
W_k^2 = \Omega_k^2, \tag{10b}
\]

\[
W_k^{n+2} = \Omega_k^2 - \mathcal{A} \left[ W_k^n \right] = \Omega_k^2 - \frac{1}{2} \left( \frac{W_k^n}{W_k} - \frac{3}{2} \left( \frac{W_k^n}{W_k} \right)^2 \right). \tag{10c}
\]
The definition is feasible only if \( \dot{W}_k > 0 \) holds throughout the dynamical evolution. But if \( |\partial^n_{\eta_i} \eta_i M^2| \) is bounded for all natural numbers \( m \), then \( \dot{W}_k > 0 \) is true beyond a certain threshold \( k_\infty(n) \) for the wave number \( k \).

Analyzing the asymptotic behaviour of \( \dot{W}_k \) as done in a recent paper by Lüders and Roberts [12] one can infer [16] that for all \( k \geq k_\infty(n) \):

1. \( \dot{W}_k / k \) is represented by a convergent power series in \( k^{-2} \),

2. \( O \left( 1 - \left( \frac{n}{\zeta_k} \right) \right) = O \left( k^{-2(n+1)} \right) \) and

3. the expansion

\[
\frac{n}{n} \chi_k (\eta) = \frac{e^{-ik(\eta-\eta_i)}}{\sqrt{2k}} \left( t_0 (\eta) + \frac{t_1 (\eta)}{k} + \ldots + \frac{t_{2n} (\eta)}{k^{2n}} + O \left( k^{-2(n+1)} \right) \right)
\]

is valid.

As a consequence the mode product can be expanded

\[
\frac{n}{n} \chi_k (\eta_1) \chi_k^* (\eta_0) = \exp \left( -i \int_{\eta_0}^{n} W_k d\eta \right) \frac{\zeta_k (\eta_0) \zeta_k (\eta_1)}{2 \sqrt{W_k (\eta_1) W_k (\eta_0)}} \exp \left( -ik \phi \right) \left( A_0 + \frac{A_1}{k} + \ldots + \frac{A_{2n}}{k^{2n}} + O \left( k^{-2(n+1)} \right) \right)
\]

with

\[
A_m (\eta_1, \eta_0) = \sum_{n=0}^{m} t_n (\eta_1) t_{m-n}^* (\eta_0).
\]

The coefficients \( A_m \) are independent of \( \eta_i \); they inherit from the expansion of \( \dot{W}_k \) the symmetries

\[
\Im \{ A_{2m} \} = \Re \{ A_{2m+1} \} = 0, \quad A_{2m} (\eta_0, \eta_1) - A_{2m} (\eta_1, \eta_0) = A_{2m+1} (\eta_0, \eta_1) + A_{2m+1} (\eta_1, \eta_0) = 0.
\]

In the appendix we list the first four coefficients \( A_m \).

The motivation for introducing adiabatic modes is to control the ultraviolet behaviour of the modes; the immediate application of adiabatic modes is the calculation of the ultraviolet contribution

\[
S [\chi_k] = \frac{1}{a_0 a_1 \pi^2} \int_{k_\infty}^{\infty} \frac{\sin kr}{kr} \Re \{ \chi_k (\eta_1) \chi_k^* (\eta_0) \} k^2 dk
\]

to the two-point function when we assume that \( \chi_k \) has the same asymptotic behaviour as \( \dot{\chi}_k \) in any order \( n \).
B. Integration

Substituting the asymptotic expansion \( \exp(-ik\tau)/(2k) \sum_{m=0}^{\infty} A_m k^{-m} \) for \( \chi_k \chi_k^* \), we obtain

\[
S[\chi_k] \simeq \frac{1}{2\pi^2 a_1 a_0 r} \Re \int_{k_-}^{\infty} e^{-ik\tau} \sin kr \sum_{m=0}^{\infty} A_m k^{-m} \, dk. \tag{15}
\]

We are able to perform the integration on any term of the sum if we tacitly regard \( \tau \) shifted into the complex plane: \( \tau \to \tau + i\delta \) and \( \delta > 0 \). This operation specifies how \( G^{(1)}(x_0, x_1) \) has to be understood as an integral kernel of a distribution [7]. The integration yields

\[
S[\chi_k] \simeq \frac{1}{8\pi^2 a_0 a_1} \left\{ \frac{4\epsilon}{\lambda^2} + U \frac{1}{r} \ln \left| \frac{r + \tau}{r - \tau} \right| + \tilde{V} \ln \lambda^2 + \tilde{W} \right\} \tag{16a}
\]

with

\[
\tilde{V}(\eta_0, \eta_1) = 2 \sum_{n=0}^{\infty} \sum_{m=0}^{n} \left( \frac{2n+1}{2m} \times \frac{(-1)^n}{(2n+1)!} A_{2n+2} \right)
- \left( \frac{2n+1}{2m+1} \times \frac{(-1)^n}{(2m+2)!} i\tau A_{2n+3} \right) r^{2(n-m)\tau} 2m, \tag{16b}
\]

\[
U(\eta_0, \eta_1) = 2 \sum_{n=0}^{\infty} \sum_{m=0}^{n} \left( \frac{2n+1}{2m} \times \frac{(-1)^n}{(2n+1)!} \tau A_{2n+2} \right)
- \left( \frac{2n}{2m} \times \frac{(-1)^n}{(2n)!} iA_{2n+1} \right) r^{2(n-m)\tau} 2m \tag{16c}
\]

and \( \tilde{W} \) is a smooth function of the coordinates.

IV. COMPARISON

The singularity structure of \( S[\chi_k] \) corresponds to the Hadamard series of Eqs.(6) only if every term in the series \( U \) vanishes everywhere.

The integral \( S[\chi_k] \) represents a solution to the wave equation; in order to cancel the logarithmic terms, \( U/r \) and \( \tilde{V} \) have to be asymptotic solutions in their own right. Therefore the equation

\[
\left[ \partial^2_{\eta_1 \eta_1} - \partial^2_{rr} + M^2 \right] U = 0 \tag{17}
\]

holds and \( \lim_{r_0 \to \eta_1} U = \lim_{r_0 \to \eta_1} \partial_{\eta_1} U = 0 \) is a necessary and sufficient condition that \( U \) vanishes everywhere [17]. It is obvious that \( U \) is odd in \( \tau \); it remains to prove is that \( \partial_{\eta_1} U \) vanishes on the the hypersurface \( \tau = 0 \) as well.

We adopt a bracket notation \([A]\) to indicate \( \lim_{r_0 \to \eta_1} A \) and derive from the above formula for \( U \)
\[ [\partial_n U] = 2 \sum_{n=0}^{\infty} (-1)^{n+1} ([A_{2n+2}] + i [\partial_n A_{2n+1}]) \frac{r^{2n}}{(2n)!}. \] (18)

We show that \([A_{2n+2}] = [\partial_n A_{2n+1}] / i\) using the normalization constraint Eq.(9d): it follows from

\[
\left[ \chi_k (\eta_1) \partial_{\eta_0} \chi_k^* (\eta_0) \right] - \left[ \partial_{\eta_1} \chi_k (\eta_1) \chi_k^* (\eta_0) \right] = i
\] (19a)

that

\[
\Im \left[ \partial_{\eta_1} \chi_k (\eta_1) \chi_k^* (\eta_0) \right] = -\frac{1}{2},
\] (19b)

The imaginary part of the asymptotic series yields using the symmetries of Eq.(13)

\[
\Im \left[ \partial_{\eta_1} \chi_k (\eta_1) \chi_k^* (\eta_0) \right] \cong \Im \left[ \partial_{\eta_1} (\eta_1) \chi_k (\eta_0) \right] = -\frac{1}{2}
\]

Both results, Eqs.(19) and (20), are compatible for an arbitrary \(k\) only if the required condition, Eq.(18), is met for all \(A_m\)-coefficients.

V. CONCLUSIONS

We have established the anticipated equivalence between the Hadamard vacuum and the adiabatic vacuum of infinite order without evaluating explicitly any expansion coefficients. Instead the asymptotic quality of the adiabatic iteration combined with symmetries inferred from the normalization constraint for the modes suffices to prove that after integration the Hadamard series dressed in coordinates will be matched term by term. It is further obvious from Eqs.(16) and Sec. IV that an adiabatic vacuum of \(n\)-th order leaves the difference \(G^{(1)}[\chi_k] - S[k^2] a^2\) a \(C^{2n-1}\) function instead of a \(C^\infty\) one.

Yet the distance between both concepts so intimately intertwined is far from being trivial. Solutions to the wave equation that asymptotically resemble Minkowskian plane waves may be constructed by means of Riemann normal coordinates in a normal neighbourhood in an arbitrary smooth spacetime, but imposing the normalization constraint as an integral over a Cauchy hypersurface, impedes our ability to turn these solutions easily into a set of modes with the desired asymptotic quality. In a homogeneous spacetime the constraint collapses to a mere wronskian determinant and there is not necessity to handle any issue global to the hypersurface: this facilitated our comparison between the Hadamard expansion and the adiabatic vacuum.

The global issue is hidden in the Hadamard condition by supplementing the raw asymptotic expansion with the global restriction (cf. Footnote 4) that the two-point functions is finite whenever their arguments are separate. In the Robertson-Walker spacetime there is
no need for such a global caveat: elementary estimates assure that the integral Eq. (8) if converging in the sense of a distribution as outlined in subsection III B, while $0 < |\tau|, r < \epsilon$ and $\epsilon > 0$, it also converges if $r$ is extended over the interval $r > 0$, and, by Cauchy evolution, if $\tau$ is spread out. So Kay’s conjecture [8,11] that the global addition to the Hadamard condition is superfluous provided $G^{(1)}(x, x')$ is derived from an actual state, is at least true in a spacetime as symmetric as the assumed RW spacetime.

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COEFFICIENTS OF THE ADIABATIC EXPANSIONS

\[ A_0 = 1, \quad A_1 = -\frac{i}{2} \int_{\eta_0}^{\eta_1} M^2 d\bar{\eta}, \]

\[ A_2 = -\frac{1}{4} \left( M^2 (\eta_0) + M^2 (\eta_1) \right) - \frac{1}{8} \left[ \int_{\eta_0}^{\eta_1} M^2 d\bar{\eta} \right]^2, \]

\[ A_3 = \frac{i}{8} \left\{ \left( M^2 (\eta_0) + M^2 (\eta_1) \right) \int_{\eta_0}^{\eta_1} M^2 d\bar{\eta} + \int_{\eta_0}^{\eta_1} \left( M^2 \right)'' + M^4 d\bar{\eta} \right. \]

\[ + \left. \frac{1}{6} \left[ \int_{\eta_0}^{\eta_1} M^2 d\bar{\eta} \right]^3 \right\}, \]

\[ A_4 = \frac{1}{16} \left\{ \frac{5}{2} \left( M^4 (\eta_1) + M^4 (\eta_0) + \frac{2}{5} M^2 (\eta_1) M^2 (\eta_0) \right) \right. \]

\[ + \left( M^2 (\eta_1) \right)'' + \left( M^2 (\eta_0) \right)'' + \left[ \left( M^2 (\eta_1) \right)' - \left( M^2 (\eta_0) \right)' \right] \int_{\eta_0}^{\eta_1} M^2 d\bar{\eta} \right. \]

\[ + \frac{1}{2} \left( M^2 (\eta_1) + M^2 (\eta_0) \right) \left[ \int_{\eta_0}^{\eta_1} M^2 d\bar{\eta} \right]^2 + \int_{\eta_0}^{\eta_1} M^2 d\bar{\eta} \int_{\eta_0}^{\eta_1} M^4 d\bar{\eta} \]

\[ + \left. \frac{1}{24} \left[ \int_{\eta_0}^{\eta_1} M^2 d\bar{\eta} \right]^4 \right\}. \quad (21) \]
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