On a theorem of Garza regarding algebraic numbers with real conjugates

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1 Introduction. For an algebraic number $\alpha$, that is, a root of an irreducible polynomial $\phi(x)$ with integer coefficients, the absolute height of $\alpha$ is defined by

$$H(\alpha) = \frac{|c|^{1/d} \prod_{i=1}^{d} \max(1, |\alpha_i|)^{1/d}}{\prod_{i=1}^{d} (x - \alpha_i)}$$

in case $\phi(x) = c \prod_{i=1}^{d} (x - \alpha_i)$. The following lower estimate for the absolute height of $\alpha$ was recently found by J. Garza (G, Theorem 1):

**Theorem:** Let $\alpha \neq 0, \pm 1$ be an algebraic number with $r > 0$ real Galois conjugates. Then

$$H(\alpha) \geq \left( \frac{2^{1-1/R} + \sqrt{4^{1-1/R} + 4}}{2} \right)^{R/2}$$

where $R = r/d$ is the fraction of Galois conjugates $\alpha_i$ of $\alpha$ which are real.

If $R = 1$, i.e., $\alpha$ is a totally real, the bound simplifies to Schinzel’s estimate (see [S], Corollary 1’)

$$H(\alpha) \geq \left( \frac{1 + \sqrt{5}}{2} \right)^{1/2}$$

stated in loc. cit. for algebraic integers only. A short proof of Schinzel’s bound in this case was given in [HS]. In this note we show that a similar method as in [HS] together with basic properties of absolute values of number fields also leads to a new derivation of Garza’s bound.

2 Proof of Theorem. We start with an elementary estimate.

**Lemma:** For $0 < a < \frac{1}{2}$ let $f(x) = |x|^{1/2-a} |1 - x^2|^a$. Then the function $f(x)/\max(1, |x|)$ has the global maximum $M_C = 2^a$ on the complex plane and the global maximum

$$M_R = (4a)^a (1 - 2a)^{1/4 - a/2} (1 + 2a)^{-1/4 - a/2}$$

on the real axis.

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Proof of the lemma: One has \( f(x) \leq 2^a \) for \( |x| \leq 1 \) and \( f(i) = 2^a \). For \( |x| \geq 1 \) one gets \( f(x)/|x| \leq |x|^{-1/2-a}(2|x|^2)^a \leq 2^a \) proving the first statement. For the second statement, one verifies by using the first derivative and computing the boundary values that \( f(x) \) reaches the stated global maximum in the interval \([0, 1]\) at \( x_1 = \sqrt{\frac{1-2a}{1+2a}} \) and that \( f(x)/|x| \) reaches the same global maximum in the interval \([1, \infty)\) at \( x_2 = \sqrt{\frac{1+2a}{1-2a}} \).

Continuing with the notation from the lemma, one has for an algebraic integer \( \alpha \) the estimate
\[
\prod_{i=1}^{d} f(\alpha_i) = |\phi(0)|^{1/2-a} |\phi(1)| \phi(-1)|^a \geq 1.
\]
Therefore,
\[
\prod_{i=1}^{d} \max(1, |\alpha_i|) \geq M_{R}^{-r} M_{C}^{-d} \prod_{i=1}^{d} f(\alpha_i) \geq M_{R}^{-r} M_{C}^{-d}
\]
or \( H(\alpha) \geq M_{R}^{-R} M_{C}^{-R-1} \) for the height. Applying the lemma for \( a = \frac{1}{2}(1 + 4^{1/R})^{-1/2} \) gives
\[
H(\alpha) \geq (4a)^{-aR}(1 - 2a)^{(a/2 - 1/4)R} (1 + 2a)^{(a/2 + 1/4)R} 2^{a(R-1)} \]
\[
= \left( \left( 1 + \frac{4^{1/R}}{4} \right)^{a} \left( \frac{4^{1/R}}{1 + 4^{1/R}} \right)^{a(1-1/R)} \right) \frac{1 + 2a}{(1 - 4a^2)^{1/2}} \right)^{R/2} \]
\[
= \left( \left( 1 + \frac{4^{1/R}}{4^{1/2}} \right)^{1/2} \left( 1 + (1 + 4^{1/R})^{-1/2} \right) \right)^{R/2} \left( \frac{2^{1-1/R} + \sqrt{4^{1-1/R} + 4}}{2} \right)^{R/2},
\]
which finishes the proof of the theorem in the case of the algebraic integers.

The above argument can be extended to arbitrary algebraic numbers \( \alpha \) by using some basic algebraic number theory and properties of the absolute height (cf. \([I]\) for the case of Schinzel’s result).

Let \( k = \mathcal{Q}(\alpha) \). For a place \( \nu \) of \( k \) we denote by \( |.|_\nu \) the corresponding normalized absolute value of \( k \), so that \( \prod_{\nu} |\beta|_\nu = 1 \) for a non-zero algebraic number \( \beta \) in \( k \). Then the absolute height of \( \beta \) equals \( H(\beta) = \prod_{\nu} \max(1, |\beta|_\nu) \). With \( a \leq 1/2 \) as above, we have the estimate
\[
1 = \prod_{\nu} |\alpha - \alpha|^{-a}_{\nu} = \prod_{\nu \not= \infty} |\alpha - \alpha|^{-a}_{\nu} \cdot \prod_{\nu \in \infty} |\alpha - \alpha|^{-a}_{\nu}
\]
\[
\leq \prod_{\nu \not= \infty} |\alpha - \alpha|^{-a}_{\nu} \prod_{\nu \in \infty} \max(1, |\alpha|_\nu)^a \max(1, |\alpha^{-1}|_\nu)^a
\]
\[
\leq \prod_{\nu \not= \infty} \frac{(|\alpha_{\nu} - \alpha_{\nu}^{-1}|_\nu)^{d_{\nu}/d}}{(\max(1, |\alpha_{\nu}|)^{1/2} \max(1, |\alpha_{\nu}^{-1}|)^{1/2})^{d_{\nu}/d}} \prod_{\nu} \max(1, |\alpha|_\nu)^{1/2} \max(1, |\alpha^{-1}|_\nu)^{1/2}
\]
where \( d_{\nu} = [k_{\nu}:\mathbb{R}] \) and \( \alpha_{\nu} \) is the image of \( \alpha \) under some Galois automorphism of the Galois closure of \( k \) such that \( |\alpha|_\nu = |\alpha_{\nu}|^{d_{\nu}/d} = |\alpha_{i}|^{d_{\nu}/d} \) for some \( i \) so that one factor for
each pair \{\alpha_i, \bar{\alpha}_i\} appears in the product over the archimedean places. Since \( g(x) = |x - x^{-1}|^a / (\max(1, |x|)^{1/2} \max(1, |x^{-1}|)^{1/2}) \) is symmetric under \( x \mapsto x^{-1} \) we can assume \(|x| \geq 1\) where \( g(x) = f(x) / \max(1, |x|) \). By applying the lemma we get now the estimate

\[
1 \leq M_R^R M_C^{1-R} \cdot H(\alpha)^{1/2} H(\alpha^{-1})^{1/2}
\]

and the result follows as before by using \( H(\alpha) = H(\alpha^{-1}) \).

3 Remarks. 1. Under all functions \( \tilde{f}(x) = |x|^u |1 - x^2|^v \), the chosen \( f(x) \) gives the best estimate for \( H(\alpha) \).

2. For \( R = 1 \) the bound for \( H(\alpha) \) is optimal. One may ask if this is also the case for other values of \( R \), although it follows from the proof that there cannot exist an \( \alpha \) actually reaching the bound.

3. The main difference to Garza’s proof is that we replace a sequence of inequalities in \([G]\) with the estimate of the lemma, allowing a particular elementary proof for algebraic integers.

References

[G] J. Garza, On the height of algebraic numbers with real conjugates, Acta Arith. 128 (2007), 385–389.

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