Similarity Dimension of Fractal Curves with Multiple Generators

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Abstract

We propose a definition for the similarity dimension of fractal curves with multiple generators.

1 Introduction

Most fractal curves as the Koch curve [vK04, vK06, Man77] are defined by an initiator and one generator. While the initiator is defined by a polygonal chain or just a single line segment, the generator is defined by a polygonal chain with at least two line segments. The fractal curve is generated iteratively. In a first step all line segments of the initiator are replaced by scaled down copies of the generator with defined orientations. In the following steps, hence iterations, the line segments are replaced again accordingly. To obtain a continuous curve that cannot be differentiated anywhere, the process is repeated an infinite number of times.

In literature a large number of fractal curves can be found, but all fractal curves known to the author rely on one generator only. An exception is the FASS (space filling, self avoiding, self similar and simple) curve of the regular pentagon recently derived in [Pau21]. In detail appropriate decorations were applied on all prototiles of an appropriate cyclotomic aperiodic substitution tiling (CAST) which is also a stone inflation.

Figure 1.1: The figure shows the 7th iteration of a fractal curve with multiple generators as shown in Figure 4.1. The different line segment types are marked in green, blue and black. The initiator of the curve is line segment $L_1$. 
Aperiodic substitution tilings as described in [BG13, FGH, and references therein] and its cyclotomic variant as discussed in [Pau17, and references therein] are defined by a set of prototiles which can be expanded with a linear map - the “inflation multiplier” - and dissected into copies of prototiles in original size - the “substitution rule”. Substitution tilings allow to cover the entire Euclidean plane without gaps and overlaps with tiles of finite size or to tile a finite area with tiles of infinitesimal size. For the latter case just the expansion is omitted. Obviously in both cases similarity dimension of the tiling is \( D = 2 \). A space filling curve derived from the tiling has the same property. Since most substitution tilings rely on \( n \geq 2 \) proto tiles and substitution rules we can also describe the corresponding FASS curve by a set of \( n \) substitution rules or more precisely \( n \) generators. On closer inspection it turns out that the terms “iteration” and “substitution” as well as the terms “generator” and “substitution rule” have almost identical meanings here.

A FASS curve regardless of the number generators is just a special case of a fractal curve with similarity dimension \( D = 2 \). This raises an interesting question: How to derive the similarity dimension of a fractal curve with multiple generators in general?

In the following sections we will derive a proposal for an appropriate definition.

## 2 Perron-Frobenius Theorem

For the analysis of aperiodic substitution tilings it is common to use substitution matrices based on the substitution rules. We will apply the same principle to fractal curves with multiple generators. The substitution matrix \( M \) describes, how each of the line segments \( L_i; 1 \leq i \leq n \) is replaced by an individual polygonal chain \( G_i \) (the generator), each made of a number \( k_{i,1}, k_{i,2} \ldots k_{i,n} \) of line segments \( L_1, L_2 \ldots L_n \) scaled down by a linear map with a scaling factor \( 0 < r_i < 1; r_i \in \mathbb{R} \). The scaling factor \( r_i \) is defined as the ratio of the length of a line segment \( L_i \) to the distance between start and end point of the polygonal chain of generator \( G_i \).

As long as \( M \geq 0 \) is a primitive matrix so that \( \exists (k, M \geq 0) \Rightarrow M^k > 0 \) we can apply the Perron–Frobenius theorem as introduced in [Per07] and [Fro12]. That means the Perron–Frobenius eigenvalue of the substitution matrix \( M \) is a real positive number so that \( \lambda_{PF} \in \mathbb{R}_{>0} \) and the modulo of every other eigenvalue is smaller than \( \lambda_{PF} \). Furthermore the corresponding left and right Perron-Frobenius eigenvectors \( x_{PFL} \) and \( x_{PFR} \) also have only real positive elements.

As discussed in [BG13] and references therein the elements of a normalized right Perron-Frobenius eigenvector \( x_{PFR} \) which is defined as

\[
x_{PFR} = \frac{x_{PFR}}{\|x_{PFR}\|_1} = \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{pmatrix}
\]

so that \( \sum_{i=1}^{n} f_i = 1 \) (2.1)

describe the relative frequency of the proto tiles - or applied to our problem - the relative frequency of the different types of line segments \( L_1, L_2 \ldots L_n \) in a fractal curve. Furthermore the Perron–Frobenius eigenvalue \( \lambda_{PF} \) represents the multiplication factor applied to the total number of tiles during the substitution. As a consequence it also represents the multiplication factor applied to the total number of line segments during an iteration.

It is known that repeatedly multiplying a vector \( x \geq 0 \) by a primitive matrix \( M \) cause the result to converge to the Perron-Frobenius eigenvector of the matrix. That means regardless of the initial setting the relative frequencies of the types of lines segments of the fractal curves will converge with an increasing number of iterations to the relative frequencies described by \( x_{PFR} \).
The first three iterations (substitutions) of the Koch curve as described in [vK04, vK06, Man77]. The triangles mark the orientation of the line segments and are not part of the curve itself.

The initiator of the curve is line segment $L_1$. The scaling factor is given by $r_1 = \frac{1}{3}$. All angles are multiples of $60° = \frac{\pi}{3}$ rad. The similarity dimension is $D = \frac{\log(4)}{\log(3)} \approx 1.26186$.

### 3 Similarity Dimension

During the research of fractals and fractal structures different types of fractal dimensions were defined by well known mathematicians like F. Hausdorff, H. Minkowsky, B. B. Mandelbrot and many more. For strictly self similar structures as discussed in this article it is sufficient to focus on the similarity dimension.

H.-O. Peitgen et. al. defined strict self similarity in [PJS+91] as follows: “If the figure can be decomposed into parts which are exact replicas of the whole, then the figure is called strictly self-similar. Every part of a strictly self-similar structure contains an exact replica of the whole.”

For the purpose of this paper we assume that strict self similarity also applies to a set of figures which can be decomposed into a set of parts which are exact replicas of elements in the set of wholes.

The similarity dimension is defined as follows:

\[ 1 = nr^D \]  
(3.1)

Where $n \in \mathbb{N}$ is the number of copies, $0 < r < 1$; $r \in \mathbb{R}$ is the scaling factor and $D > 0$; $D \in \mathbb{R}$ is the similarity dimension. This equation works for many fractal curves such as the Koch curve as shown in Figure 3.1. Its generator is defined by a polygonal chain of four line segments of unit length scaled down by $r_1 = \frac{1}{3}$.

B. B. Mandelbrot proposed in [Man77] a generalization for cases where the $n$ line segments of the generator have different lengths and so different scaling factors $0 < r_i < 1$; $r \in \mathbb{R}$; $1 \leq i \leq n$.

\[ 1 = \sum_{i=1}^{n} r_i^D \]  
(3.2)

A corresponding example is shown in Figure 3.2.
Figure 3.2: The first five iterations (substitutions) of a fractal curve based on one generator with 3 line segments with two different lengths. The triangles mark the orientation of the line segments and are not part of the curve itself.

The initiator of the curve is line segment $L_1$ as shown on the upper left.

The generator which is equivalent to the first iteration is shown in the upper right.

The scaling factors are given by $r_1 = r_2 = 2$, $r_3 = \frac{1}{\sqrt{2}}$. All angles are right angles. The similarity dimension is $D \simeq 1.52361$.

This equation can also be written as:

$$1 = n \sum_{i=1}^{n} \frac{1}{n} r_i^D$$  \hspace{1cm} (3.3)

Here $n$ is the factor which increases the number of line segments in each iteration. The line segments with different scaling factors $r_i$ have all the same relative frequency $\frac{1}{n}$. To describe the case where the line segments have individual relative frequencies we replace $n$ with $\lambda_{PF}$ and $\frac{1}{n}$ with $f_i$ as introduced in Section 2 and derive the proposed definition for the similarity dimension of fractal curves with multiple generators:

$$1 = \lambda_{PF} \sum_{i=1}^{n} f_i r_i^D$$  \hspace{1cm} (3.4)
4 The Example

In the following part we discuss an example, in detail the curve shown in Figure 1.1 and Figure 4.1. The curve is defined by a set of three generators or substitution rules \( G_1, G_2 \) and \( G_3 \) which are assigned to a set of three lines \( L_1, L_2 \) and \( L_3 \) with lengths \( l_1, l_2 \) and \( l_3 \). The generators \( G_i \) describe how a line segment \( L_i \) can be replaced by a polygonal chain made of \( k_{1,i}, k_{2,i} \) and \( k_{3,i} \) line segments \( L_1, L_2 \) and \( L_3 \) scaled down by \( r_i \). As a consequence each generator \( G_i \) and each type of line \( L_i \) is also assigned to a corresponding inflation multiplier \( r_i \).

The curve as shown in Figure 1.1 and Figure 4.1 has the following substitution matrix:

\[
M = \begin{pmatrix}
  k_{1,1} & k_{1,2} & k_{1,3} \\
  k_{2,1} & k_{2,2} & k_{2,3} \\
  k_{3,1} & k_{3,2} & k_{3,3}
\end{pmatrix} = \begin{pmatrix}
  1 & 2 & 0 \\
  1 & 0 & 2 \\
  1 & 0 & 0
\end{pmatrix}
\]

(4.1)

The columns correspond to the generators \( G_1, G_2 \) and \( G_3 \) while the rows are assigned to the number of line segments \( L_1, L_2 \) and \( L_3 \). Obviously the substitution matrix \( M \) is primitive because \( M^3 \) has only positive entries greater zero:

\[
M^3 = \begin{pmatrix}
  9 & 6 & 4 \\
  5 & 6 & 4 \\
  3 & 2 & 4
\end{pmatrix} > 0
\]

(4.2)

The normalized right Perron-Frobenius eigenvector \( x_{PF}^{*} \) and the Perron-Frobenius eigenvalue \( \lambda_{PF} \) are calculated with numerical methods:

\[
\lambda_{PF} \simeq 2.46750
\]

(4.3)

\[
x_{PF}^{*} = \begin{pmatrix}
  f_1 \\
  f_2 \\
  f_3
\end{pmatrix} \simeq \begin{pmatrix}
  0.46750 \\
  0.34303 \\
  0.18946
\end{pmatrix}
\]

(4.4)

Together with Equation (3.4) and the scaling factors \( r_1, r_2, r_3 \) in Figure 4.1 we have:

\[
1 \simeq 2.46750 \left( 0.46750 (0, 618034)^D + 0.34303 (0, 381966)^D + 0.18946 (0, 618034)^D \right)
\]

(4.5)

The similarity dimension \( D \) of the curve in Figure 1.1 and Figure 4.1 is calculated with a numerical method:

\[
D \simeq 1.47814
\]

(4.6)

5 Summary

Based on B. B. Mandelbrot’s equation for the similarity dimension of fractal curves with one generator and line segments with different lengths and scaling factors and the tool sets for the analysis of aperiodic substitution tilings such as the Perron Frobenius theorem, we derived a proposal for a definition for the similarity dimension of fractal curves with multiple generators.
Figure 4.1: The first row shows the three initiators, the line segments $L_1$ (green), $L_2$ (blue) and $L_3$ (black) with individual orientations. The triangles mark the orientation of the line segments and are not part of the curve itself.

The second row shows the three corresponding generators $G_1$ (green), $G_2$ (blue) and $G_3$ (black) which are equivalent to the first iterations. Each of the generators is defined by a polygonal chain, made of the line segments $L_1$ (green), $L_2$ (blue) and $L_3$ (black) with orientations, scaled down by $r_i$, so that the distance between start and end point of $G_i$ is equal to the length of the assigned line segment $L_i$.

The other rows show the result of repeated iterations hence substitutions.

The relative lengths of line segments $L_1$, $L_2$ and $L_3$ are given by $l_1/l_2 = l_1/l_3 = \varphi = \sin(2\pi/5)/\sin(\pi/5) \simeq 1.618034$.

The scaling factors $r_1$, $r_2$ and $r_3$ are given by $r_1 = r_3 = 1/\varphi = \varphi - 1 \simeq 0.618034$ and $r_2 = r_1^2 = 1/\varphi^2 = 2 - \varphi \simeq 0.381966$.

All angles are multiples of $36^\circ = \pi/5 \text{ rad}$.

The similarity dimension is $D \simeq 1.47814$ as derived in Section 4.
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