DYNAMICAL MULTIFRACTAL ZETA-FUNCTIONS, MULTIFRACTAL PRESSURE AND FINE MULTIFRACTAL SPECTRA

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Abstract. We introduce multifractal pressure and dynamical multifractal zeta-functions providing precise information of a very general class of multifractal spectra, including, for example, the fine multifractal spectra of self-conformal measures and the fine multifractal spectra of ergodic Birkhoff averages of continuous functions.

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References

1. Introduction.

For a Borel measure \( \mu \) on \( \mathbb{R}^d \) and a positive number \( \alpha \), let us consider the set of those points \( x \) in \( \mathbb{R}^d \) for which the measure \( \mu(B(x, r)) \) of the ball \( B(x, r) \) with center \( x \) and radius \( r \) behaves like \( r^\alpha \) for small \( r \), i.e. the set

\[
\left\{ x \in K \left| \lim_{r \searrow 0} \frac{\log \mu(B(x, r))}{\log r} = \alpha \right. \right\}.
\] (1.1)

If the intensity of the measure \( \mu \) varies very widely, it may happen that the sets in (1.1) display a fractal-like character for a range of values of \( \alpha \). In this case it is natural to study the Hausdorff dimensions of the sets in (1.1) as \( \alpha \) varies. We therefore define the fine multifractal spectrum of \( \mu \) by

\[
f_\mu(\alpha) = \dim_H \left\{ x \in K \left| \lim_{r \searrow 0} \frac{\log \mu(B(x, r))}{\log r} = \alpha \right. \right\}.
\] (1.2)

where \( \dim_H \) denotes the Hausdorff dimension; here and below we use the following convention, namely, we define the Hausdorff of the empty set to be \( -\infty \), i.e. we put

\[
\dim_H \emptyset = -\infty.
\]
The second main ingredient in multifractal analysis is the Renyi dimensions. Renyi dimensions quantify the varying intensity of a measure by analyzing its moments at different scales. Formally, for \( q \in \mathbb{R} \), the \( q \)th Renyi dimensions \( \tau_{\mu}(q) \) of \( \mu \) is defined by

\[
\tau_{\mu}(q) = \lim_{r \to 0} \frac{\log \int K_{\mu}(B(x, r))^{q-1} \, d\mu(x)}{-\log r},
\]

provided the limit exists. One of the main problems in multifractal analysis is to understand the multifractal spectrum and the Renyi dimensions, and their relationship with each other. During the past 20 years there has been an enormous interest in computing the multifractal spectra of measures in the mathematical literature and within the last 15 years the multifractal spectra of various classes of measures in Euclidean space \( \mathbb{R}^d \) exhibiting some degree of self-similarity have been computed rigorously, see the textbooks [Fa2,Pe] and the references therein.

Dynamical zeta-functions were introduced by Artin & Mazur in the mid 1960’s [ArMa] based on an analogy with the number theoretical zeta-functions associated with a function field over a finite ring. Subsequently Ruelle [Rue1,Rue2] associated zeta-functions to certain statistical mechanical models in one dimensions. Motivated by the powerful techniques provided by the use of Artin-Mazur zeta-functions in dynamical systems and Ruelle zeta-functions in dynamical systems, Lapidus and collaborators (see the intriguing books by Lapidus & van Frankenhuysen [Lap-vF1,Lap-vF2] and the references therein) have recently introduced and pioneered to use of zeta-functions in fractal geometry. Inspired by this, within the past 2-3 years several authors have paralleled this development by introducing zeta-functions into multifractal geometry. For example, in [Bak,Ol4,Ol5] the authors introduced multifractal zeta-functions tailored to study multifractal spectra of self-conformal measures, and in [Le-VeMe,Ol4,Ol5] the authors introduced multifractal zeta-functions designed to study the multifractal Renyi dimensions of self-conformal measures. In addition, we note that Lapidus and collaborators have introduced various intriguing multifractal zeta-functions [LapRo,LapLe-VeRo]. However, the multifractal zeta-functions in [LapRo,LapLe-VeRo] serve very different purposes and are significantly different from the multifractal zeta-functions introduced in this paper and in [Bak,Le-VeMe,Ol4,Ol5].

It has been a major challenge to introduce and develop a natural and meaningful theory of multifractal zeta-functions paralleling the existing powerful theory of dynamical zeta-functions introduced and developed by Ruelle [Rue1,Rue2] and others, see for example, the surveys and books [Bal1,Bal2,ParPo1,ParPo2] and the references therein. The purpose of this paper is to propose such a theory. In particular, we introduce a family of multifractal zeta-functions motivated by the definition of Ruelle’s dynamical zeta-functions. Whereas the zeta-functions in [Bak,Ol4,Ol5] were designed to study the multifractal Renyi dimensions, the zeta-functions in this paper (and the zeta-functions in [MiOl]) are tailored to provide precise information of very general classes of multifractal spectra, including, for example, the multifractal spectra of self-conformal measures and the multifractal spectra of ergodic Birkhoff averages of continuous functions, see Section 6.

The framework developed in this paper will be formulated in the setting of self-conformal sets and self-conformal measures. For this reason we will now briefly recall the definition of self-conformal constructions.

2. The setting, Part 1: Self-conformal sets and self-conformal measures.

2.1. Notation from symbolic dynamics. We first recall the notation and terminology from symbolic dynamics that will be used in this paper Fix a positive integer \( N \). Let \( \Sigma = \{1, \ldots, N\} \) and for a positive integer \( n \), write

\[
\Sigma^n = \{1, \ldots, N\}^n,
\]

\[
\Sigma^* = \bigcup_{n \geq 1} \Sigma^n,
\]

\[
\Sigma^N = \{1, \ldots, N\}^N.
\]
i.e. \( \Sigma^n \) is the family of all strings \( i = i_1 \ldots i_n \) of length \( n \) with \( i_j \in \{1, \ldots, N\} \); \( \Sigma^* \) is the family of all finite strings \( i = i_1 \ldots i_m \) with \( m \in \mathbb{N} \) and \( i_j \in \{1, \ldots, N\} \); and \( \Sigma^N \) is the family of all infinite strings \( i = i_{12} \ldots \) with \( i_j \in \{1, \ldots, N\} \). For an infinite string \( i = i_{12} \ldots \in \Sigma^N \) and a positive integer \( n \), we will write \( i|_n = i_1 \ldots i_n \). In addition, for a positive integer \( n \) and a finite string \( i = i_1 \ldots i_n \in \Sigma^n \) with length equal to \( n \), we will write \(|i| = n\), and we let \(|i|\) denote the cylinder generated by \( i \), i.e.

\[
|i| = \{ j \in \Sigma^N \mid j|_n = i \}.
\]

Also, let \( S : \Sigma^N \to \Sigma^N \) denote the shift map, i.e.

\[
S(i_1 i_2 \ldots) = i_2 i_3 \ldots.
\]

### 2.2. Self-conformal sets and self-conformal measures.

Next, we recall the definition of self-conformal (and self-similar) sets and measures. A conformal iterated function system with probabilities is a list \( (V, X, (S_i)_{i=1,\ldots,N}) \) where

- \( V \) is an open, connected subset of \( \mathbb{R}^d \).
- \( X \) is a compact set with \( X \subseteq V \) and \( X^\circ = X \).
- \( S_i : V \to V \) is a contractive \( C^{1+\gamma} \) diffeomorphism with \( 0 < \gamma < 1 \) such that \( S_i X \subseteq X \) for all \( i \).
- The Conformality Condition: For each \( x \in V \), we have that \( (DS_i)(x) \) is a contractive similarity map, i.e. there exists \( r_i(x) \in (0,1) \) such that \(|(DS_i)(x)u - (DS_i)(x)v| = r_i(x)|u - v|\) for all \( u, v \in \mathbb{R}^d \); here \( (DS_i)(x) \) denotes the derivative of \( S_i \) at \( x \).

It follows from [Hu] that there exists a unique non-empty compact set \( K \) with \( K \subseteq X \) such that

\[
K = \bigcup_i S_i K.
\]

The set \( K \) is called the self-conformal set associated with the list \( (V, X, (S_i)_{i=1,\ldots,N}) \); in particular, if each map \( S_i \) is a contracting similarity, then the set \( K \) is called the self-similar set associated with the list \( (V, X, (S_i)_{i=1,\ldots,N}) \). In addition, if \( (p_i)_{i=1,\ldots,N} \) is a probability vector then it follows from [Hu] that there is a unique probability measure \( \mu \) with \( \text{supp} \, \mu = K \) such that

\[
\mu = \sum_i p_i \mu \circ S_i^{-1}.
\]

The measure \( \mu \) is called the self-conformal measure associated with the list \( (V, X, (S_i)_{i=1,\ldots,N}, (p_i)_{i=1,\ldots,N}) \); if each map \( S_i \) is a contracting similarity, then the measure \( \mu \) is called the self-similar measure associated with the list \( (V, X, (S_i)_{i=1,\ldots,N}, (p_i)_{i=1,\ldots,N}) \). We will frequently assume that the list \( (V, X, (S_i)_{i=1,\ldots,N}) \) satisfies the Open Set Condition defined below. Namely, the list \( (V, X, (S_i)_{i=1,\ldots,N}) \) satisfies the Open Set Condition (OSC) if there exists an open, non-empty and bounded set \( O \) with \( O \subseteq X \) and \( S_i O \subseteq O \) for all \( i \) such that \( S_i O \cap S_j O = \emptyset \) for all \( i, j \) with \( i \neq j \).

For \( i = i_1 \ldots i_n \in \Sigma^* \), we will write

\[
\begin{align*}
p_i &= p_{i_1} \cdots p_{i_n},
S_i &= S_{i_1} \cdots S_{i_n},
K_i &= S_i K.
\end{align*}
\]

Next, we define the natural projection map \( \pi : \Sigma^N \to K \) by

\[
\{ \pi(i) \} = \bigcap_n K_{i|n}
\]

for \( i \in \Sigma^N \). Finally, we define the scaling map \( \Lambda : \Sigma^N \to \mathbb{R} \) by

\[
\Lambda(i) = \log|DS_{i_1}(\pi Si)|
\]

for \( i = i_1 i_2 \ldots \in \Sigma^N \).
3. **The setting, Part 2: Pressure and dynamical zeta-functions.**

Throughout this section, and in the remaining parts of the paper, we will use the following notation. Namely, if \((a_n)\) is a sequence of complex numbers and if \(f\) is the power series defined by \(f(z) = \sum_n a_n z^n\) for \(z \in \mathbb{C}\), then we will denote the radius of convergence of \(f\) by \(\sigma_{\text{rad}}(f)\), i.e. we write

\[
\sigma_{\text{rad}}(f) = \text{"the radius of convergence of } f\text{"}.
\]

Our definitions and results are motivated by the notion of pressure from the thermodynamic formalism and the dynamical zeta-functions introduced by Ruelle [Rue1,Rue2]; see, also [Bal1,Bal2,ParPo1,ParPo2]. In addition, Bowen’s formula expressing the Hausdorff dimension of a self-conformal set in terms of the pressure (or the dynamical zeta-function) of the scaling map \(\Lambda\) in (2.5) also plays a leitmotif in our work. Because of this we now recall the definition of pressure and dynamical zeta-function, and the statement of Bowen’s formula. Let \(\varphi : \Sigma^\mathbb{N} \to \mathbb{R}\) be a continuous function. The pressure of \(\varphi\) is defined by

\[
P(\varphi) = \lim_{n \to \infty} \frac{1}{n} \log \sum_{|i|=n} \sup_{u \in [i]} \exp \sum_{k=0}^{n-1} \varphi S^k u,
\]  

(3.1)

see [Bo2] or [ParPo2]; we note that it is well-known that the limit in (3.1) exists. Also, the dynamical zeta-function of \(\varphi\) is defined by

\[
\zeta_{\text{dyn}}(\varphi; z) = \sum_n \frac{z^n}{n} \left( \sum_{|i|=n} \sup_{u \in [i]} \exp \sum_{k=0}^{n-1} \varphi S^k u \right)
\]

(3.2)

for those complex numbers \(z\) for which the series converge, see [ParPo2]. We now list two easily established and well-known properties of the pressure \(P(\varphi)\) and of the radius of convergence \(\sigma_{\text{rad}}\left(\zeta_{\text{dyn}}(\varphi; \cdot)\right)\) of the power-series \(\zeta_{\text{dyn}}(\varphi; \cdot)\). While both results are well-known and easily proved (see, for example, [Bar,Fa2]), we have decided to list them since they play an important part in the discussion of our results.

**Theorem A** (see, for example, [Bar,Fa2]). **Radius of convergence.** Fix a continuous function \(\varphi : \Sigma^\mathbb{N} \to \mathbb{R}\). Then we have

\[
- \log \sigma_{\text{rad}}\left(\zeta_{\text{dyn}}(\varphi; \cdot)\right) = P(\varphi).
\]

**Theorem B** (see, for example, [Bar,Fa2]). **Continuity and monotony properties of the pressure.** Fix a a continuous function \(\Phi : \Sigma^\mathbb{N} \to \mathbb{R}\) with \(\Phi < 0\). Then the function \(t \to P(t\Phi)\), where \(t \in \mathbb{R}\), is continuous, strictly decreasing and convex with \(\lim_{t \to -\infty} P(t\Phi) = \infty\) and \(\lim_{t \to \infty} P(t\Phi) = -\infty\). In particular, there is a unique real number \(s\) such that

\[
P(s\Phi) = 0;
\]

alternatively, \(s\) is the unique real number such that

\[
\sigma_{\text{rad}}\left(\zeta_{\text{dyn}}(s\Phi; \cdot)\right) = 1.
\]

The main importance of the pressure (for the purpose of this exposition) is that it provides a beautiful formula for the Hausdorff dimension of a self-conformal set satisfying the OSC. This result was first noted by [Bo1] (in the setting of quasi-circles) and is the content of the next result.
Theorem C (see, for example, [Bar,Fa2]). Bowen’s formula. Let $K$ be the self-conformal set defined by (2.1) and let $\Lambda : \Sigma^N \to \mathbb{R}$ be the scaling function defined by (2.5). Let $s$ be the unique real number such that

$$P(s\Lambda) = 0;$$

alternatively, $s$ is the unique real number such that

$$\sigma_{\text{rad}}(\zeta_{\text{dyn}}(s\Lambda; \cdot)) = 1.$$ 

If the OSC is satisfied, then we have

$$\dim_H K = s.$$ 

Any meaningful theory of dynamical multifractal zeta-functions is likely to produce multifractal analogues of Bowen’s equation. We will propose a framework for such a theory in Section 5. However, before doing so, we believe that it is useful to illustrate the underlying ideas in a simple setting. For this reason we will now illustrate how meaningful multifractal dynamical zeta-functions might be defined for self-conformal measures.

4. Motivation of the main results.

To illustrate the ideas behind our main definitions in a simple setting, we consider the following example involving self-conformal measures. Fix a conformal iterated function system $(V, X, (S_i)_{i=1,...,N})$ and a be a probability vector $(p_1, \ldots, p_N)$. We let $K$ denote the self-conformal set associated with the list $(V, X, (S_i)_{i=1,...,N})$, i.e. $K$ is the unique non-empty and compact subset of $\mathbb{R}^d$ satisfying (2.1), and we let $\mu$ be the self-conformal measure associated with the list $(S_1, \ldots, S_N, p_1, \ldots, p_N)$, i.e. $\mu$ is the unique Borel probability measure on $\mathbb{R}^d$ satisfying (2.2). Recall, that the multifractal spectrum of $\mu$ is defined by

$$f_{\mu}(\alpha) = \dim_H \{ x \in K \mid \lim_{r \to 0} \frac{\log \mu B(x, r)}{\log r} = \alpha \},$$

for $\alpha \in \mathbb{R}$. If the OSC is satisfied, then the multifractal spectrum $f_{\mu}$ is given by the following formula. Namely, define $\Phi : \Sigma^N \to \mathbb{R}$ by $\Phi(i) = \log p_i$, for $i = i_1i_2 \ldots \in \Sigma^N$ and let $\Lambda : \Sigma^N \to \mathbb{R}$ denote the scaling map in (2.5). Next, define $\beta : \mathbb{R} \to \mathbb{R}$ by

$$P(q\Phi + \beta(q)\Lambda) = 0;$$

alternatively, the function $\beta : \mathbb{R} \to \mathbb{R}$ is defined by

$$\sigma_{\text{rad}}(\zeta_{\text{dyn}}(q\Phi + \beta(q)\Lambda; \cdot)) = 1.$$ 

If the OSC is satisfied, then it follows from [CaMa,Pa] that

$$f_{\mu}(\alpha) = \beta^*(\alpha)$$

for all $\alpha \in \mathbb{R}$ where $\beta^*$ denotes the Legendre transform of $\beta$; recall, that if $\varphi : \mathbb{R} \to \mathbb{R}$ is a function, then the Legendre transform $\varphi^* : \mathbb{R} \to [\infty, \infty]$ of $\varphi$ is defined by $\varphi^*(x) = \inf_y (xy + \varphi(y))$.

While one may argue that (4.1) and (4.3) provide a pressure formula for the multifractal spectrum $f_{\mu}(\alpha)$ of a self-conformal measure (or, alternatively, that (4.2) and (4.3) provide a zeta-function formula for the multifractal spectrum $f_{\mu}(\alpha)$ of a self-conformal measure), this formula can hardly be said to be in the spirit of Bowen’s formula. Adopting this viewpoint, for a given $\alpha \in \mathbb{R}$, it is natural to attempt to introduce dynamical multifractal zeta-functions $\zeta_{\text{dyn-con}}^\alpha$ of self-conformal measures tailored, for example, to see the multifractal decomposition sets

$$\left\{ x \in K \mid \lim_{r \to 0} \frac{\log \mu B(x, r)}{\log r} = \alpha \right\}$$
more directly and, as a result of this, hopefully lead to a better conceptual understanding of the pressure formula (4.3). More precisely, and motivated by Bowen’s formula, for each $\alpha \in \mathbb{R}$ it seems natural to expect that any dynamically meaningful multifractal zeta-function $\zeta_{\alpha}^{\text{dyn-con}}$ should have the following property: there is a unique real number $\nearrow(\alpha)$ such that

$$\sigma_{\text{rad}}\left(\zeta_{\alpha}^{\text{dyn-con}}(\nearrow(\alpha) \Lambda; \cdot)\right) = 1,$$

and the number $\nearrow(\alpha)$ equals the multifractal spectrum $f_{\mu}(\alpha)$, i.e.

$$f_{\mu}(\alpha) = \nearrow(\alpha).$$

Since $f_{\mu}(\alpha)$ measures the size of the set of points $x$ for which $\lim_{r \to 0} \frac{\log \mu(B(x,r))}{\log r} = \alpha$ and since $\left(\frac{\log p_i}{\log \text{diam} K_i}\right)$ has the same form as $\frac{\log \mu(B(x,r))}{\log r}$, it is natural to define the dynamical self-conformal multifractal zeta-function $\zeta_{\alpha}^{\text{dyn-con}}(\varphi; \cdot)$ of a continuous function $\varphi : \Sigma^N \to \mathbb{R}$ by

$$\zeta_{\alpha}^{\text{dyn-con}}(\varphi; z) = \sum_n \frac{z^n}{n} \left( \sum_{|i| = n} \sup_{u \in [i]} \exp \left( \sum_{k=0}^{n-1} \varphi S^k u \right) \right),$$

for those complex numbers $z$ for which the series converges. The main difference between the classical dynamical zeta-function (3.2) and its proposed multifractal counterpart (4.4) is that in (4.4) we only sum over those strings $i$ with $|i| = n$ that are multifractally relevant. An easy and straightforward calculation, which we present in Observation 4.1 below, shows that if there is a unique real number $\nearrow(\alpha)$ such that

$$\sigma_{\text{rad}}\left(\zeta_{\alpha}^{\text{dyn-con}}(\nearrow(\alpha) \Lambda; \cdot)\right) = 1,$$

then this number is less than $f_{\mu}(\alpha)$, i.e.

$$\nearrow(\alpha) \leq f_{\mu}(\alpha).$$

(4.5)

Observation 4.1. Let $\mu$ be the self-conformal measure defined by (2.2) and let $\Lambda : \Sigma^N \to \mathbb{R}$ be the scaling function defined by (2.5). For $\alpha, t \in \mathbb{R}$, let $\zeta_{\alpha}^{\text{dyn-con}}(t\Lambda; \cdot)$ be defined by (4.4). If there is a unique real number $\nearrow(\alpha)$ such that

$$\sigma_{\text{rad}}\left(\zeta_{\alpha}^{\text{dyn-con}}(\nearrow(\alpha) \Lambda; \cdot)\right) = 1,$$

then

$$\nearrow(\alpha) = \inf \left\{ t \in \mathbb{R} \left| \sigma_{\text{rad}}(\zeta_{\alpha}^{\text{dyn-con}}(t\Lambda)) \geq 1 \right. \right\}$$

and this number is less than $f_{\mu}(\alpha)$, i.e.

$$\nearrow(\alpha) \leq f_{\mu}(\alpha).$$

(4.5)

Proof

Indeed, if $\alpha \notin -\beta'(\mathbb{R})$. then it is well-known that that for all $i \in \Sigma^*$, we have $\frac{\log p_i}{\log \text{diam} K_i} \neq \alpha$ (see, for example, [Pa]) This implies that if $\alpha \notin -\beta'(\mathbb{R})$, then the sum

$$\sum_{|i| = n} \sup_{u \in [i]} \exp \left( \sum_{k=0}^{n-1} t\Lambda S^k u \right)$$

(4.6)
is the empty sum and therefore equal to 0 for all \( n \) and all \( t \), whence \( \zeta_\alpha^{\text{dyn-con}}(t; z) = \sum_{n} z^n = 0 \) for all \( z \) and all \( t \). It follows immediately this that if \( \alpha \not\in (-\beta, 0] \), then \( \sigma_\alpha^{\text{dyn-con}}(t; \cdot) = \infty \) for all \( t \), whence \( \rho(\alpha) = \inf \{ t \in \mathbb{R} : \sigma_\alpha^{\text{dyn-con}}(t; \cdot) \geq 1 \} = -\infty \), and inequality (4.5) is therefore trivially satisfied. On the other hand, if \( \alpha \in (-\beta, 0] \), then it follows from [CaMa,Fa1,Pa] that there we can find a (unique) \( q \in \mathbb{R} \) with \( f_\alpha(q) = \alpha q + \beta(q) \). It is also well-known, see, for example, [Bar,Fa2], that there is a constant \( c > 0 \) such that for all positive integers \( n \) and all \( i \) with \( |i| = n \) and all \( u \in [i] \), we have \( C \leq |DS_t(\pi S^n u)| \leq c \). This clearly implies that there is a constant \( C \) such that for all positive integers \( n \) and all \( i \) with \( |i| = n \) and all \( u \in [i] \), we have \( \alpha q \log |DS_t(\pi S^n u)| \leq C + \alpha q \log \text{diam } K_i \). Since also \( \sum_{k=1}^{n-1} \Lambda S^k u = \log |DS_t(\pi S^n u)| \) for all positive integers \( n \) and all \( i \) with \( |i| = n \) and all \( u \in [i] \), we therefore conclude that

\[
\sum_{|i|=n} \sup_{u \in [i]} \exp \left( \sum_{k=0}^{n-1} f_\alpha(q) \Lambda S^k u \right)
\]

\[
= \sum_{|i|=n} \sup_{u \in [i]} \exp \left( \sum_{k=0}^{n-1} (\alpha q + \beta(q)) \Lambda S^k u \right)
\]

\[
= \sum_{|i|=n} \sup_{u \in [i]} \exp \left( \alpha q \sum_{k=0}^{n-1} \Lambda S^k u + \sum_{k=0}^{n-1} \beta(q) \Lambda S^k u \right)
\]

\[
\leq \sum_{|i|=n} \sup_{u \in [i]} \exp \left( C + \alpha q \log \text{diam } K_i + \sum_{k=0}^{n-1} \beta(q) \Lambda S^k u \right)
\]

\[
= \sum_{|i|=n} \sup_{u \in [i]} \exp \left( C + q \log p_i + \sum_{k=0}^{n-1} \beta(q) \Lambda S^k u \right)
\]

\[
= \sum_{|i|=n} \sup_{u \in [i]} \exp \left( C + q \Phi S^k u + \sum_{k=0}^{n-1} \beta(q) \Lambda S^k u \right)
\]

\[
eq C \sum_{|i|=n} \sup_{u \in [i]} \exp \left( \sum_{k=0}^{n-1} (q \Phi + \beta(q) \Lambda) S^k u \right),
\]
whence

\[
\sum_n \frac{z^n}{n} \sum_{|i|=n} \sup_{u \in [i]} \exp \left( \sum_{k=0}^{n-1} f_\mu(\alpha) S^k u \right) 
\leq e^C \sum_n \frac{z^n}{n} \sum_{|i|=n} \sup_{u \in [i]} \exp \left( \sum_{k=0}^{n-1} (q\Phi + \beta(q)\Lambda) S^k u \right),
\]

for all complex numbers \( z \). We immediately conclude from this that \( 1 = \sigma_{rad}(\zeta_{\alpha}^{\text{dyn-con}}(f_\mu(\alpha)\Lambda; \cdot)) \leq \sigma_{rad}(\zeta_{\alpha}^{\text{dyn-con}}(f_\mu(\alpha)\Lambda; \cdot)) \), whence \( \langle \alpha \rangle = \inf\{t \in \mathbb{R} | \sigma_{rad}(\zeta_{\alpha}^{\text{dyn-con}}(t\Lambda)) \geq 1\} \leq f_\mu(\alpha) \). This proves (4.5).

However, it is also clear that we, in general, do not have equality in (4.5). This is the content of the next observation.

**Observation 4.2.** Let \( \mu \) be the self-conformal measure defined by (2.2) and let \( \Lambda : \Sigma^N \to \mathbb{R} \) be the scaling function defined by (2.5). For \( \alpha, t \in \mathbb{R} \), let \( \zeta_{\alpha}^{\text{dyn-con}}(t\Lambda; \cdot) \) be defined by (4.4). If there is a unique real number \( \langle \alpha \rangle \) such that

\[
\sigma_{rad}(\zeta_{\alpha}^{\text{dyn-con}}(\langle \alpha \rangle \Lambda; \cdot)) = 1,
\]

then

\[
\langle \alpha \rangle = -\infty < 0 < f_\mu(\alpha)
\]

for all except at most countably many \( \alpha \in -\beta'(\mathbb{R}) \).

(4.7)

**Proof**

Indeed, the set \( \{ \log p_i \mid i \in \Sigma^* \} \) is clearly countable (because \( \Sigma^* \) is countable) and if \( \alpha \in \mathbb{R} \setminus \{ \frac{\log p_i}{\log \text{diam } K_1} \mid i \in \Sigma^* \} \), then the sum \( \sum_{|i|=n} \frac{\log p_i}{\log \text{diam } K_1} \sup_{u \in [i]} \exp \left( \sum_{k=0}^{n-1} t A S^k u \right) \) is the empty sum and therefore equal to 0 for all \( n \) and all \( t \). It follows from this, using an argument similar to the reasoning following (4.6), that \( \langle \alpha \rangle = \inf\{t \in \mathbb{R} | \sigma_{rad}(\zeta_{\alpha}^{\text{dyn-con}}(t\Lambda)) \geq 1\} = -\infty \). Since it also follows from [CaMa,Fa1,Pa] that \( f_\mu(\alpha) > 0 \) for all \( \alpha \in -\beta'(\mathbb{R}) \), we therefore conclude that \( \langle \alpha \rangle = -\infty < 0 < f_\mu(\alpha) \) for all except at most countably many \( \alpha \in -\beta'(\mathbb{R}) \).

It follows from the above discussion that while the definition of \( \zeta_{\alpha}^{\text{dyn-con}}(s) \) is “natural”, it is not does not encode sufficient information allowing us to recover the multifractal spectrum \( f_\mu(\alpha) \). The reason for the strict inequality in (4.7) is, of course, clear: even though there are no strings \( i \in \Sigma^* \) for which the ratio \( \frac{\log p_i}{\log \text{diam } K_1} \) equals \( \alpha \) if \( \alpha \in -\beta'(\mathbb{R}) \setminus \{ \frac{\log p_i}{\log \text{diam } K_1} \mid i \in \Sigma^* \} \), there are nevertheless many sequences \( \{i_n\}_n \) of strings \( i_n \in \Sigma^* \) for which the sequence of ratios \( \left( \frac{\log p_i}{\log \text{diam } K_1} \right)_n \) converges to \( \alpha \). In order to capture this, it is necessary to ensure that those strings \( i \) for which the ratio \( \frac{\log p_i}{\log \text{diam } K_1} \) is “close” to \( \alpha \) are also included in the series defining the multifractal zeta-function. For this reason, we modify the definition of \( \zeta_{\alpha}^{\text{dyn-con}} \) and introduce a self-conformal multifractal zeta-function obtained by replacing the original small “target” set \( \{\alpha\} \) by a larger “target” set \( I \) (for example, we may choose the enlarged “target” set \( I \) to be a non-degenerate interval centered at \( \alpha \)).
idea precise we proceed as follows. For a closed interval $I$, we define the self-conformal multifractal zeta-function $\zeta^\text{dyn-con}_I(\varphi; z)$ of a continuous function $\varphi : \Sigma^N \to \mathbb{R}$ by

$$\zeta^\text{dyn-con}_I(\varphi; z) = \sum_n z^n \left( \sum_{|I| = n} \sup_{u \in [I]} \exp \left( \sum_{k=0}^{n-1} \varphi S^k u \right) \right),$$

(4.8)

for those complex numbers $z$ for which the series converges. Observe that if $I = \{\alpha\}$, then

$$\zeta^\text{dyn-con}_I(\varphi; z) = \zeta^\text{dyn-con}_\alpha(\varphi; z).$$

We can now proceed in two equally natural ways. Either, we can consider a family of enlarged “target” sets shrinking to the original main “target” $\{\alpha\}$; this approach will be referred to as the shrinking target approach. Or, alternatively, we can keep the enlarged “target” set fixed and regard this as our original main “target”; this approach will be referred to as the fixed target approach. We now discuss these approaches in more detail.

(1) The shrinking target approach. For a given (small) “target” $\{\alpha\}$, we consider the following family $\{[\alpha - r, \alpha + r]\}_{r > 0}$ of enlarged “target” sets $[\alpha - r, \alpha + r]$ shrinking to the original main “target” $\{\alpha\}$ as $r \to 0$, and attempt to relate the limiting behaviour of the radius of convergence of $\zeta_{[\alpha-r,\alpha+r]}^\text{dyn-con}(t\Lambda; \cdot)$ as $r \to 0$ to the multifractal spectrum $f_\mu(\alpha)$ at $\alpha$. The next result, which is an application of one of our main results (see Theorem 6.1), shows that the multifractal zeta-functions $\zeta_{[\alpha-r,\alpha+r]}^\text{dyn-con}(t\Lambda; \cdot)$ encode sufficient information allowing us to recover the multifractal spectra $f_\mu(\alpha)$ by letting $r \to 0$.

**Theorem 4.1. Shrinking targets.** Let $\mu$ be the self-conformal measure defined by (2.2) and let $\Lambda : \Sigma^N \to \mathbb{R}$ be the scaling function defined by (2.5). For $\alpha \in \mathbb{R}$, $r > 0$ and $t \in \mathbb{R}$, let $\zeta_{[\alpha-r,\alpha+r]}^\text{dyn-con}(t\Lambda; \cdot)$ be defined by (4.8).

1. There is a unique real number $\sigma(\alpha)$ such that
   $$\lim_{r \to 0} \sigma(\alpha) = 1.$$

2. If the OSC is satisfied, then we have
   $$\sigma(\alpha) = f_\mu(\alpha).$$

**Proof**

This result is a special case of Theorem 6.1. \hfill \Box

We note that Theorem 4.1 has a very clear resemblance to Bowen’s formula in Theorem C.

(2) The fixed target approach Alternatively, we can keep the enlarged “target” set $I$ fixed and attempt to relate the radius of convergence of the multifractal zeta-function $\zeta^\text{dyn-con}_I(t\Lambda; \cdot)$ associated with the enlarger “target” set $I$ to the values of the multifractal spectrum $f_\mu(\alpha)$ for $\alpha \in I$. Of course, inequality (4.7) shows that if the “target” set $I$ is “too small”, then this is not possible. However, if the enlarger “target” set $I$ satisfies a mild non-degeneracy condition, namely condition (4.9), guaranteeing that $I$ is sufficiently “big”, then the next result, which is also an application of one of our main results (see Theorem 6.1), shows that this is possible. More precisely the result shows that if the enlarger “target” set $I$ satisfies condition (4.9), then the multifractal zeta-function $\zeta^\text{dyn-con}_I(t\Lambda; \cdot)$ associated with the enlarger “target” set $I$ encode sufficient information allowing us to recover the suprema $\sup_{\alpha \in I} f_\mu(\alpha)$ of the multifractal spectrum $f_\mu(\alpha)$ for $\alpha \in I$. 
Theorem 4.2. Fixed targets. Let \( \mu \) be the self-conformal measure defined by (2.2) and let \( \Lambda : \Sigma^N \to \mathbb{R} \) be the scaling function defined by (2.5). For a closed interval \( I \) and \( t \in \mathbb{R} \), let \( \zeta^\text{dyn-con}_I(t \Lambda; \cdot) \) be defined by (4.8). Assume that
\( \circ I \cap (-\beta'(\mathbb{R})) \neq \emptyset \) (4.9)
(where \( \circ \) denotes the interior of \( I \)).

(1) There is a unique real number \( \mathcal{F}(I) \) such that
\( \sigma_{\text{rad}}(\zeta^\text{dyn-con}_I(\mathcal{F}(I) \Lambda; \cdot)) = 1 \),

(2) If the OSC is satisfied, we have
\( \mathcal{F}(I) = \sup_{\alpha \in I} f_\mu(\alpha) \).

Proof
This result is a special case of Theorem 6.1. \( \square \)

As with Theorem 4.1, we also note that Theorem 4.2 has a very clear resemblance to Bowen’s formula in Theorem C.

We emphasise that Theorem 4.1 and Theorem 4.2 are presented in order to motivate this work and are special cases of the substantially more general and abstract theory of dynamical multifractal zeta-function developed in this paper.

The next section, i.e. Section 5, describes the general framework developed in this paper and lists our main results. In Section 6 we will discuss a number of examples, including, mixed and non-mixed multifractal spectra of self-conformal measures, and multifractal spectra of Birkhoff ergodic averages.

5. Statements of the main results.

We also denote the family of Borel probability measures on \( \Sigma^N \) and the family of shift invariant Borel probability measures on \( \Sigma^N \) by \( P(\Sigma^N) \) and \( P_S(\Sigma^N) \), respectively, i.e. we write
\[
P(\Sigma^N) = \left\{ \mu \mid \mu \text{ is a Borel probability measures on } \Sigma^N \right\},
\]
\[
P_S(\Sigma^N) = \left\{ \mu \mid \mu \text{ is a shift invariant Borel probability measures on } \Sigma^N \right\};
\]
we will always equip \( P(\Sigma^N) \) and \( P_S(\Sigma^N) \) with the weak topology. Fix a metric space \( X \) and a continuous map \( U : P(\Sigma^N) \to X \). For a positive integer \( n \), let \( L_n : \Sigma^N \to P(\Sigma^N) \) be defined by
\[
L_n \iota = \frac{1}{n} \sum_{k=0}^{n-1} \delta_{\mathcal{S}_{ki}}.
\]
(5.1)

We can now define the multifractal pressure and zeta-function associated with the space \( X \) and the map \( U \),

**Definition.** The multifractal pressure \( P^U_C(\varphi) \) and \( \overline{P}^U_C(\varphi) \) associated with the space \( X \) and the map \( U \). Let \( \varphi : \Sigma^N \to \mathbb{R} \) be a continuous map. For \( C \subseteq X \), we define the lower and upper multifractal pressure of \( \varphi \) associated with the space \( X \) and the map \( U \) and by
\[
P^U_C(\varphi) = \liminf_n \frac{1}{n} \log \sum_{|i|=n} \sup_{u \in [i]} \exp \sum_{k=0}^{n-1} \varphi \mathcal{S}_k u,
\]
\[
\overline{P}^U_C(\varphi) = \limsup_n \frac{1}{n} \log \sum_{|i|=n} \sup_{u \in [i]} \exp \sum_{k=0}^{n-1} \varphi \mathcal{S}_k u.
\]
If \( P_C^U(\varphi) \) and \( \overline{P}_C^U(\varphi) \) coincide, then we write \( P_C^U(\varphi) \) for their common value, i.e. we write \( P_C^U(\varphi) = \overline{P}_C^U(\varphi) = P_{\overline{C}}^U(\varphi) \).

Definition. The dynamical multifractal zeta-function \( \zeta^\text{dyn, \, U}_C(\varphi; \cdot) \) associated with the space \( X \) and the map \( U \). Let \( \varphi : \Sigma^N \to \mathbb{R} \) be a continuous map. For \( C \subseteq X \), we define the dynamical multifractal zeta-function \( \zeta^\text{dyn, \, U}_C(\varphi; \cdot) \) associated with the space \( X \) and the map \( U \) by

\[
\zeta^\text{dyn, \, U}_C(\varphi; z) = \sum_n \frac{z^n}{n} \left( \sum_{|i|=n} \sup_{U L_n[i] \subseteq C} \exp \left( \sum_{k=0}^{n-1} \varphi S^k u \right) \right)
\]

for those complex numbers \( z \) for which the series converges.

Remark. It is clear that if \( C = X \), then the multifractal “constraint” \( UL_n[i] \subseteq C \) is vacuously satisfied, and the multifractal pressure and dynamical multifractal zeta-function reduce to the usual pressure and the usual dynamical zeta-function, i.e.

\[
P_X^U(\varphi) = \overline{P}_X^U(\varphi) = P(\varphi)
\]

and

\[
\zeta^\text{dyn, \, U}_X(\varphi; \cdot) = \zeta^\text{dyn}(\varphi; \cdot).
\]

Before developing the theory of the multifractal pressure and the multifractal zeta-functions further we make the following two simple observations. Firstly, we note (see Proposition 5.1) that the expected relationship between the multifractal pressure and the radius of convergence of the multifractal zeta-function holds. Secondly, we would expect any dynamically meaningful theory of dynamical multifractal zeta-functions to lead to multifractal Bowen formulas. For this to hold, we must, at the very least, ensure that there are unique solutions to the relevant multifractal Bowen equations. i.e. we must ensure that there are unique real numbers \( \hat{f}(C) \) and \( \overline{f}(C) \) solving the following equations, namely,

\[
\limsup_{r \to 0} \overline{P}^U_{B(C, r)}(\hat{f}(C) \Phi) = 0,
\]

\[
\overline{P}^U_{C}(\overline{f}(C) \Phi) = 0,
\]

That there are unique numbers \( \hat{f}(C) \) and \( \overline{f}(C) \) satisfying (5.2) is our second simple observation (see Proposition 5.2).

Proposition 5.1. Radius of convergence. Let \( X \) be a metric space and let \( U : \mathcal{P}(\Sigma^N) \to X \) be continuous with respect to the weak topology. Let \( C \subseteq X \) be a subset of \( X \). Fix a continuous function \( \varphi : \Sigma^N \to \mathbb{R} \). We have

\[
-\log \sigma_{\text{rad}}(\zeta_C^\text{dyn, \, U}(\varphi; \cdot)) = \overline{P}_C^U(\varphi).
\]

Proof: This follows immediately from the fact that if \( (a_n) \) is a sequence of complex numbers and \( f(z) = \sum_n a_n z^n \), then \( \sigma_{\text{rad}}(f) = \frac{1}{\limsup_n |a_n|^2} \).

\( \square \)
Proposition 5.2. Continuity and monotonicity of the multifractal pressure. Let $X$ be a metric space and let $U : \mathcal{P}(\Sigma^N) \to X$ be continuous with respect to the weak topology. Let $C \subseteq X$ be a subset of $X$. Fix a continuous map $\Phi : \Sigma^N \to \mathbb{R}$ with $\Phi < 0$. Let $C$ be a subset of $X$. Then the functions $t \to \limsup_{r \to 0} P^U_{B(C,r)}(t\Phi)$ and $t \to P^U_C(t\Phi)$, where $t \in \mathbb{R}$, are continuous, strictly decreasing and convex with $\lim_{t \to -\infty} \limsup_{r \to 0} P^U_{B(C,r)}(t\Phi) = \infty$ and $\lim_{t \to \infty} \limsup_{r \to 0} P^U_{B(C,r)}(t\Phi) = -\infty$, and $\lim_{t \to -\infty} \limsup_{r \to 0} P^U_C(t\Phi) = \infty$ and $\lim_{t \to \infty} \limsup_{r \to 0} P^U_C(t\Phi) = -\infty$. In particular, there are unique real numbers $\overline{f}(C)$ and $\underline{f}(C)$ such that

$$\limsup_{r \to 0} P^U_{B(C,r)}(\overline{f}(C) \Phi) = 0,$$

$$P^U_C(\overline{f}(C) \Phi) = 0;$$

$\underline{f}(C)$ and $\overline{f}(C)$ are the unique real numbers such that

$$\limsup_{r \to 0} \sigma_{\text{rad}}(\zeta^U_{B(C,r)}(\overline{f}(C) \Phi; \cdot)) = 1,$$

$$\sigma_{\text{rad}}(\zeta^U_C(\overline{f}(C) \Phi; \cdot)) = 1.$$

Proof

This is not difficult to prove and for sake of brevity we have decided to omit the proof. \qed

We can now state our main results. The results are divided into two parts paralleling the discussion in Section 4.2. The first part (consisting of Theorem 5.3 and Corollary 5.4) presents our results in the shrinking target setting, and the second part consisting of Theorem 5.5 and Corollary 5.6) presents our results in the fixed target setting. More precisely, in the shrinking target setting, Theorem 5.3 provide a variational principle for the multifractal pressure and Corollary 5.4 provide a variational principle for the solution $\overline{f}(C)$ to the multifractal Bowen equation (5.3).

Theorem 5.3. The shrinking target variational principle for the multifractal pressure. Let $X$ be a metric space and let $U : \mathcal{P}(\Sigma^N) \to X$ be continuous with respect to the weak topology. Let $C \subseteq X$ be a subset of $X$. Fix a continuous function $\varphi : \Sigma^N \to \mathbb{R}$.

1. We have

$$\lim_{r \to 0} P^U_{B(C,r)}(\varphi) = P^U_{B(C,r)}(\varphi) = \sup_{\mu \in \mathcal{P}_F(\Sigma^N)} \left( h(\mu) + \int \varphi \, d\mu \right).$$

2. We have

$$\lim_{r \to 0} -\log \sigma_{\text{rad}}(\zeta^\text{dyn, U}_{B(C,r)}(\varphi; \cdot)) = \sup_{\mu \in \mathcal{P}_F(\Sigma^N)} \left( h(\mu) + \int \varphi \, d\mu \right).$$

Theorem 5.3 is proved in Section 8.
Corollary 5.4. The shrinking target multifractal Bowen equation. Let $X$ be a metric space and let $U : \mathcal{P}(\Sigma^N) \to X$ be continuous with respect to the weak topology. Let $C \subseteq X$ be a subset of $X$. Fix a continuous function $\Phi : \Sigma^N \to \mathbb{R}$ with $\Phi < 0$ and let $\hat{f}(C)$ be the unique real number such that

$$\limsup_{r \to 0} \mathcal{P}^{\hat{f}}_{B(C,r)}(\hat{f}(C) \Phi) = 0;$$

alternatively, $\hat{f}(C)$ is the unique real number such that

$$\limsup_{r \to 0} \sigma_{rad}(\zeta^{U}_{B(C,r)}(\hat{f}(C) \Phi; \cdot)) = 1.$$ 

Then

$$\hat{f}(C) = \sup_{\mu \in \mathcal{P}(\Sigma^N)} \frac{h(\mu)}{\int \Phi \, d\mu}.$$

Proof

It follows from Theorem 5.3 and the definition of $\hat{f}(C)$ that

$$\sup_{\mu \in \mathcal{P}(\Sigma^N)} \left( h(\mu) + \hat{f}(C) \int \Phi \, d\mu \right) \leq \limsup_{r \to 0} \mathcal{P}^{\hat{f}}_{B(C,r)}(\hat{f}(C) \Phi) = 0. \quad (5.5)$$

The desired formula for $\hat{f}(C)$ follows easily from (5.5). $\square$
Of course, if the set $C$ is “too small”, then it follows from the discussion in Section 4.2 that we, in general, cannot expect any meaningful results in the fixed target setting. However, if the set $C$ satisfies a non-degeneracy condition guaranteeing that it is not “too small” (namely condition (5.6) below), then meaningful results can be obtained in the fixed target setting. This is the contents of Theorem 2.5 and Corollary 2.6 below. Indeed, Theorem 2.5 and Corollary 2.6 provide variational principles for the multifractal pressure and for the solution $\bar{f}(C)$ to the multifractal Bowen equation (5.4) in the fixed target setting.

**Theorem 5.5. The fixed target variational principle for the multifractal pressure.** Let $X$ be a normed vector space. Let $\Gamma : \mathcal{P}(\Sigma^N) \to X$ be continuous and affine and let $\Delta : \mathcal{P}(\Sigma^N) \to \mathbb{R}$ be continuous and affine with $\Delta(\mu) \neq 0$ for all $\mu \in \mathcal{P}(\Sigma^N)$. Define $U : \mathcal{P}(\Sigma^N) \to X$ by $U = \frac{1}{\Delta}$. Let $C$ be a closed and convex subset of $X$ and assume that $^0 C \cap U(\mathcal{P}_S(\Sigma^N)) \neq \emptyset$.

(1) We have

$$P^U_C(\varphi) = \sup_{\mu \in \mathcal{P}_S(\Sigma^N)} \left( h(\mu) + \int \varphi \, d\mu \right) = \sup_{\mu \in \mathcal{P}_S(\Sigma^N)} \left( h(\mu) + \int \varphi \, d\mu \right).$$

(2) We have

$$-\log \sigma_{rad}(\zeta^{\mu_U}(\varphi; \cdot)) = \sup_{\mu \in \mathcal{P}_S(\Sigma^N)} \left( h(\mu) + \int \varphi \, d\mu \right) = \sup_{\mu \in \mathcal{P}_S(\Sigma^N)} \left( h(\mu) + \int \varphi \, d\mu \right).$$

Theorem 5.5 is proved in Section 9.

**Corollary 5.6. The fixed target multifractal Bowen equation.** Let $X$ be a normed vector space. Let $\Gamma : \mathcal{P}(\Sigma^N) \to X$ be continuous and affine and let $\Delta : \mathcal{P}(\Sigma^N) \to \mathbb{R}$ be continuous and affine with $\Delta(\mu) \neq 0$ for all $\mu \in \mathcal{P}(\Sigma^N)$. Define $U : \mathcal{P}(\Sigma^N) \to X$ by $U = \frac{1}{\Delta}$. Let $C$ be a closed and convex subset of $X$ and assume that $^0 C \cap U(\mathcal{P}_S(\Sigma^N)) \neq \emptyset$.

Let $\Phi : \Sigma^N \to \mathbb{R}$ be continuous with $\Phi < 0$. Let $\bar{f}(C)$ be the unique real number such that

$$P^U_C(\bar{f}(C) \Phi) = 0;$$

alternatively, $\bar{f}(C)$ is the unique real number such that

$$\sigma_{rad}(\zeta^{\mu_U}(\bar{f}(C) \Phi; \cdot)) = 1.$$

Then

$$\bar{f}(C) = \sup_{\mu \in \mathcal{P}_S(\Sigma^N)} \frac{h(\mu)}{\int \Phi \, d\mu}.$$

Proof

The proof is similar to the proof of Corollary 5.4 using Theorem 5.5 and the definition of $\bar{f}(C)$.

In the next section we will show that in many cases, the solutions $\bar{f}(C)$ and $\bar{f}(C)$ to the multifractal Bowen equations (5.3) and (5.4) coincide with the usual multifractal spectra.
6. Applications: multifractal spectra of measures and multifractal spectra of ergodic Birkhoff averages

We will now consider several of applications of Theorem 5.3 and Theorem 5.5 to multifractal spectra of measures and ergodic averages. In particular, we consider the following examples:

- Section 6.1: Multifractal spectra of self-conformal measures.
- Section 6.2: Mixed multifractal spectra of self-conformal measures.
- Section 6.3: Multifractal spectra of ergodic Birkhoff averages.

6.1. Multifractal spectra of self-conformal measures. Fix a conformal iterated function system \((V, X, (S_i)_{i=1,\ldots,N})\) and a probability vector \((p_1, \ldots, p_N)\). We let \(K\) denote the self-conformal set defined by (2.1), and we let \(\mu\) denote the self-conformal measure defined by (2.2). We also recall that the Hausdorff multifractal spectrum \(f_\mu(\alpha)\) of \(\mu\) is defined by

\[
\dim_H \left\{ x \in K \left| \lim_{r \to 0} \frac{\log \mu B(x,r)}{\log r} = \alpha \right. \right\},
\]

for \(\alpha \in \mathbb{R}\), and that the multifractal spectrum \(f_\mu(\alpha)\) can be computed as follows, see, for example, [ArPa,CaMa,Pa]. Define \(\Phi, \Lambda : \Sigma^N \to \mathbb{R}\) by \(\Phi(i) = \log p_i\) and let \(\lambda : \Sigma^N \to \mathbb{R}\) denote the scaling map defined in (2.5). Finally, let \(\beta(q)\) be the unique real number such that

\[
P(\beta(q)\Lambda + q\Phi) = 0; \quad (6.1)
\]

alternatively, the function \(\beta : \mathbb{R} \to \mathbb{R}\) is defined by

\[
\sigma_{\text{rad}}(\zeta_{\text{dyn}}(q\Phi + \beta(q)\Lambda; \cdot)) = 1. \quad (6.2)
\]

The multifractal spectrum \(f_\mu(\alpha)\) can now be computed as follows. If the OSC is satisfied, then it follows from [ArPa,CaMa,Pa] that

\[
f_\mu(\alpha) = \beta^*(\alpha); \quad (6.3)
\]

recall, that if if \(\varphi : \mathbb{R} \to \mathbb{R}\) is a function, then the Legendre transform \(\varphi^* : \mathbb{R} \to [\infty, \infty]\) of \(\varphi\) is defined by \(\varphi^*(x) = \inf_y (xy + \varphi(y))\).

Of course, in general, the limit \(\lim_{r \to 0} \frac{\log \mu B(x,r)}{\log r}\) may not exist. Indeed, recently Barreira & Schmeling [BaSc] (see also Olsen & Winter [OlWi1,OlWi2], Xiao, Wu & Gao [XiWuGa] and Moran [Mo]) have shown that the set of divergence points, i.e. the set of points \(x\) for which the limit \(\lim_{r \to 0} \frac{\log \mu B(x,r)}{\log r}\) does not exist, typically is highly “visible” and “observable”, namely it has full Hausdorff dimension. More precisely, it follows from [BaSc] that if the OSC is satisfied and \(t\) denotes the Hausdorff dimension of \(K\), then

\[
\left\{ x \in K \left| \text{the expression } \frac{\log \mu B(x,r)}{\log r} \text{ diverges as } r \to 0 \right. \right\} = \emptyset
\]

provided \(\mu\) is proportional to the \(t\)-dimensional Hausdorff measure restricted to \(K\), and

\[
\dim_H \left\{ x \in K \left| \text{the expression } \frac{\log \mu B(x,r)}{\log r} \text{ diverges as } r \to 0 \right. \right\} = \dim_H K
\]

provided \(\mu\) is not proportional to the \(t\)-dimensional Hausdorff measure restricted to \(K\). This suggests that the set of divergence points has a surprising rich and complex fractal structure, and in order to explore this more carefully Olsen & Winter [OlWi1,OlWi2] introduced various generalised multifractal spectra functions designed to “see” different sets of divergence points. In order to define
these spectra we introduce the following notation. If $M$ is a metric space and $\varphi : (0, \infty) \to M$ is a function, then we write $\text{acc}_{r \searrow 0} f(r)$ for the set of accumulation points of $f$ as $r \searrow 0$, i.e.

$$\text{acc}_{r \searrow 0} \varphi(r) = \left\{ x \in M \mid x \text{ is an accumulation point of } \varphi \text{ as } r \searrow 0 \right\}.$$ 

In [OlWi1] Olsen & Winter introduced and investigated the generalised Hausdorff multifractal spectrum $F_\mu$ of $\mu$ defined by

$$F_\mu(C) = \dim_H \left\{ x \in K \mid \frac{\log \mu B(x, r)}{\log r} \subseteq C \right\}$$

for $C \subseteq \mathbb{R}$. Note that the generalised spectrum is a genuine extension of the traditional multifractal spectrum $f_\mu(\alpha)$, namely if $C = \{\alpha\}$ is a singleton consisting of the point $\alpha$, then clearly $F_\mu(C) = f_\mu(\alpha)$. There is a natural divergence point analogue of Theorem A. Indeed, the following divergence point analogue of Theorem A was first obtained by Moran [Mo] and Olsen & Winter [OlWi1], and later in a less restrictive setting by Li, Wu & Xiong [LiWuXi] (see also [Ca,Vo] for earlier but related results in a slightly different setting).

**Theorem D [LiWuXi,Mo,OlWi1].** Let $\mu$ be the self-conformal measure defined by (2.2). Let $C$ be a closed subset of $\mathbb{R}$. If the OSC is satisfied, then we have

$$F_\mu(C) = \sup_{\alpha \in C} \beta^*(\alpha).$$

As a first application of Theorem 5.3, Corollary 5.4, Theorem 5.5 and Corollary 5.6 we obtain a dynamical multifractal zeta-function with an associated Bowen equation whose solution equals the generalised multifractal spectrum $F_\mu(C)$ of a self-conformal measure $\mu$. This is the content of the next theorem.

**Theorem 6.1.** Dynamical multifractal zeta-function for multifractal spectra of self-conformal measures. Let $(p_1, \ldots, p_N)$ be a probability vector, and let $\mu$ denote the self-conformal measure associated with the list $(V, X, (S_i)_{i=1,\ldots,N}, (p_i)_{i=1,\ldots,N})$, i.e. $\mu$ is the unique probability measure such that $\mu = \sum_i p_i \mu \circ S_i^{-1}$.

For $C \subseteq \mathbb{R}$ and an continuous function $\varphi : \Sigma^\mathbb{N} \to \mathbb{R}$, we define the dynamical self-conformal multifractal zeta-function by

$$\zeta^{\text{dyn-con}}_C(\varphi; z) = \sum_n \frac{z^n}{n} \left( \sum_{|i|=n} \sup_{u \in [i]} \exp \left( \frac{n-1}{\log \text{diam } \Lambda_i} \sum_{k=0}^{n-1} \varphi S^k u \right) \right).$$

Let $\Lambda$ be defined by (2.5) and let $\beta$ be defined by (6.1) (or, alternatively, by (6.2)).

1. Assume that $C \subseteq \mathbb{R}$ is closed.

1.1 There is a unique real number $\mathcal{A}(C)$ such that

$$\lim_{r \searrow 0} \sigma_{\text{rad}} \left( \zeta^{\text{dyn-con}}_{B(C,r)}(\mathcal{A}; \cdot) \right) = 1.$$

It $\alpha \in \mathbb{R}$ and $C = \{\alpha\}$, then we will write $\mathcal{A}(\alpha) = \mathcal{A}(C)$.

1.2 We have

$$\mathcal{A}(C) = \sup_{\alpha \in C} \beta^*(\alpha).$$
If the OSC is satisfied, then we have
\[
\mathcal{F}(C) = F_\mu(C) = \dim_H \left\{ x \in K \left| \frac{\log \mu(B(x,r))}{\log r} \right|_{acc} \to 0 \subseteq C \right\}.
\]

In particular, if the OSC is satisfied and \(\alpha \in \mathbb{R}\), then we have
\[
\mathcal{F}(C) = f_\mu(\alpha) = \dim_H \left\{ x \in K \left| \lim_{r \to 0} \frac{\log \mu(B(x,r))}{\log r} = \alpha \right\}.
\]

(2) Assume that \(C \subseteq \mathbb{R}\) is a closed interval with \(C \cap (-\beta'(\mathbb{R})) \neq \emptyset\).

(2.1) There is a unique real number \(\mathcal{F}(C)\) such that
\[
\sigma_{rad}(\zeta_{\mathcal{F}(C)}^{\text{dyn-con}}(\mathcal{F}(C) \Lambda ; \cdot)) = 1.
\]

(2.2) We have
\[
\mathcal{F}(C) = \sup_{\alpha \in C} \beta^*(\alpha).
\]

(2.3) If the OSC is satisfied then
\[
\mathcal{F}(C) = F_\mu(C) = \dim_H \left\{ x \in K \left| \frac{\log \mu(B(x,r))}{\log r} \right|_{acc} \to 0 \subseteq C \right\}.
\]

Proof
This follows immediately from the more general Theorem 6.2 in Section 6.2 by putting \(M = 1\).

6.2. Mixed multifractal spectra of self-conformal measures. Recently mixed (or simultaneous) multifractal spectra have generated an enormous interest in the mathematical literature, see [BaSa,Mo,Ol2,Ol3]. Indeed, previous results (for example, (6.3) and Theorem D) only considered the scaling behaviour of a single measure. Mixed multifractal analysis investigates the simultaneous scaling behaviour of finitely many measures. Mixed multifractal analysis thus combines local characteristics which depend simultaneously on various different aspects of the underlying dynamical system, and provides the basis for a significantly better understanding of the underlying dynamics. We will now make these ideas precise. For \(m = 1, \ldots, M\), let \((p_{m,1}, \ldots, p_{m,N})\) be a probability vector, and let \(\mu_m\) denote the self-conformal measure associated with the list \((V, X, (S_i)_{i=1,\ldots,N}, (p_{m,i})_{i=1,\ldots,N})\), i.e. \(\mu_m\) is the unique probability measure such that
\[
\mu_m = \sum_i p_{m,i} \mu_m \circ S_i^{-1}.
\]

(6.4)

The mixed multifractal spectrum \(f_\mu\) of the list \(\mu = (\mu_1, \ldots, \mu_M)\) is defined by
\[
f_\mu(\alpha) = \dim_H \left\{ x \in K \left| \lim_{r \to 0} \left( \frac{\log \mu_1(B(x,r))}{\log r}, \ldots, \frac{\log \mu_M(B(x,r))}{\log r} \right) = \alpha \right\}
\]
for \(\alpha \in \mathbb{R}^M\). Of course, it is also possible to define generalised mixed multifractal spectra designed to “see” different sets of divergence points. Namely, we define the generalised mixed Hausdorff multifractal spectrum \(F_\mu\) of the list \(\mu = (\mu_1, \ldots, \mu_M)\) by
\[
F_\mu(C) = \dim_H \left\{ x \in K \left| \frac{\log \mu_1(B(x,r))}{\log r}, \ldots, \frac{\log \mu_M(B(x,r))}{\log r} \right|_{acc} \subseteq C \right\}
\]
for \( C \subseteq \mathbb{R}^M \). Again we note that the generalised mixed multifractal spectrum is a genuine extensions of the traditional mixed multifractal spectrum \( F_\mu(\alpha) \), namely, if \( C = \{\alpha\} \) is a singleton consisting of the point \( \alpha \), then clearly \( F_\mu(C) = f_\mu(\alpha) \). Assuming the OSC, the generalised mixed multifractal spectrum \( F_\mu(C) \) can be computed [Mo,Ol2]. In order to state the result from [Mo,Ol2], we introduce the following definitions. Define \( \Lambda, \Phi : \Sigma^N \rightarrow \mathbb{R} \) for \( m = 1, \ldots, M \) by \( \Lambda(i) = \log |DS_i(\pi S_i)| \) and \( \Phi_m(i) = \log p_{m,i} \) for \( i = i_{1,2} \in \Sigma^N \), and write \( \Phi = (\Phi_1, \ldots, \Phi_M) \). For \( x, y \in \mathbb{R}^M \), we let \( \langle x | y \rangle \) denote the usual inner product of \( x \) and \( y \), and define \( \beta : \mathbb{R}^M \rightarrow \mathbb{R} \) by

\[
0 = P(\beta(q) \Lambda + \langle q | \Phi \rangle);
\]

alternatively, the function \( \beta : \mathbb{R}^M \rightarrow \mathbb{R} \) is defined by

\[
\sigma_{rad}(\zeta_{dyn}^n(\langle q | \Phi \rangle + \beta(q) \Lambda; :)) = 1.
\]

Finally, if \( \varphi : \mathbb{R}^M \rightarrow \mathbb{R} \) is a function, we define the Legendre transform \( \varphi^* : \mathbb{R}^M \rightarrow [-\infty, \infty] \) of \( \varphi \) by

\[
\varphi^*(x) = \inf_y (\langle x | y \rangle + \varphi(y)).
\]

The generalised mixed multifractal spectra \( f_\mu \) and \( F_\mu \) are now given by the following theorem.

**Theorem E [Mo,Ol2].** Let \( \mu_1, \ldots, \mu_M \) be defined by (4.1) and let \( C \subseteq \mathbb{R}^M \) be a closed set. Put \( \mu = (\mu_1, \ldots, \mu_M) \). If the OSC is satisfied, then we have

\[
F_\mu(C) = \sup_{\alpha \in C} \beta^*(\alpha).
\]

In particular, if the OSC is satisfied and \( \alpha \in \mathbb{R}^M \), then we have

\[
f_\mu(\alpha) = \beta^*(\alpha).
\]

As a second application of Theorem 5.3, Corollary 5.4, Theorem 5.5 and Corollary 5.6 we obtain a dynamical multifractal zeta-function with an associated Bowen equation whose solution equals the generalised mixed multifractal spectrum \( F_\mu(C) \) of a list \( \mu \) of self-conformal measures. This is the content of the next theorem.

**Theorem 6.2.** Multifractal zeta-functions for mixed multifractal spectra of self-conformal measures. For \( m = 1, \ldots, M \), let \( (p_{m,1}, \ldots, p_{m,N}) \) be a probability vector, and let \( \mu_m \) denote the self-conformal measure associated with the list \( (V, X, (S_i)_{i=1,\ldots,N}, (p_{m,i})_{i=1,\ldots,N}) \), i.e. \( \mu_m \) is the unique probability measure such that \( \mu_m = \sum p_{m,i} \mu_m \circ S_i^{-1} \).

For \( C \subseteq \mathbb{R}^M \) and an continuous function \( \varphi : \Sigma^N \rightarrow \mathbb{R} \), we define the dynamical self-conformal multifractal zeta-function by

\[
\zeta_{\text{dyn-con}}(\varphi; z) = \sum_n \frac{z^n}{n} \left( \sum_{|i|=n} \sup_{u \in [i]} \exp \left( \frac{\log p_{1,i}}{\log \delta_{\mu_1}(S_i)} + \cdots + \frac{\log p_{M,i}}{\log \delta_{\mu_M}(S_i)} \right) \right).
\]

Let \( \Lambda \) be defined by (2.5) and let \( \beta \) be defined by (6.5) (or, alternatively, by (6.6)).

1. Assume that \( C \subseteq \mathbb{R}^M \) is closed.
2. (1.1) There is a unique real number \( \lambda(C) \) such that

\[
\lim_{r \rightarrow 0} \sigma_{rad} \left( \zeta_{\text{dyn-con}}^{(C,r)}(\lambda(C) \Lambda; :) \right) = 1.
\]
It $\mathbf{a} \in \mathbb{R}^M$ and $\mathcal{C} = \{\mathbf{a}\}$, then we will write $\mathcal{F}(\mathbf{a}) = \mathcal{F}(\mathcal{C})$.

(2.1) We have

$$\mathcal{F}(\mathcal{C}) = \sup_{\mathbf{a} \in \mathcal{C}} \beta^*(\mathbf{a}).$$

(2.2) We have

$$\mathcal{F}(\mathcal{C}) = F_{\mu}(\mathcal{C}) = \dim_{\mathcal{H}} \left\{ x \in K \left| \lim_{r \searrow 0} \left( \frac{\mu_1(B(x,r))}{\log r} , \ldots , \frac{\mu_M(B(x,r))}{\log r} \right) \right. \right\} \subseteq \mathcal{C}. \right.$$\)

(2.3) If the OSC is satisfied then

$$\mathcal{F}(\mathcal{C}) = F_{\mu}(\mathcal{C}) = \dim_{\mathcal{H}} \left\{ x \in K \left| \lim_{r \searrow 0} \left( \frac{\mu_1(B(x,r))}{\log r} , \ldots , \frac{\mu_M(B(x,r))}{\log r} \right) \right. \right\} \subseteq \mathcal{C}. \right.$$\)

Assume that $C \subseteq \mathbb{R}^M$ is closed and convex with $C \cap (-\nabla \beta(\mathbb{R}^M)) \neq \emptyset$.

(2.1) There is a unique real number $\mathcal{F}(\mathcal{C})$ such that

$$\sigma_{\text{rad}} \left( \zeta_{\mathcal{C}}^{\text{dyn-con}} \left( \mathcal{F}(\mathcal{C}) ; \cdot \right) \right) = 1.$$\)

(2.2) We have

$$\mathcal{F}(\mathcal{C}) = \sup_{\mathbf{a} \in \mathcal{C}} \beta^*(\mathbf{a}).$$

(2.3) If the OSC is satisfied then

$$\mathcal{F}(\mathcal{C}) = F_{\mu}(\mathcal{C}) = \dim_{\mathcal{H}} \left\{ x \in K \left| \lim_{r \searrow 0} \left( \frac{\mu_1(B(x,r))}{\log r} , \ldots , \frac{\mu_M(B(x,r))}{\log r} \right) \right. \right\} \subseteq \mathcal{C}. \right.$$\)

We will now prove Theorem 6.2. Recall that the function $\Lambda : \Sigma^N \to \mathbb{R}$ is defined by $\Lambda(i) = \log |DS_i(\pi S)|$ for $i = i_1 i_2 \ldots \in \Sigma^N$. Also, recall that $\Phi = (\Phi_1 , \ldots , \Phi_M)$ where $\Phi_m : \Sigma^N \to \mathbb{R}$ is defined by $\Phi_m(i) = \log p_{m,i}$ for $i = i_1 i_2 \ldots \in \Sigma^N$. We now introduce the following definitions. For $\mu \in \mathcal{P}(\Sigma^N)$, write $f \Phi d\mu = (\int \Phi_1 d\mu , \ldots , \int \Phi_M d\mu)$, and define $\Gamma : \mathcal{P}(\Sigma^N) \to \mathbb{R}^M$ and $\Delta : \mathcal{P}(\Sigma^N) \to \mathbb{R}$ by

$$\Gamma(\mu) = \int \Phi d\mu , \quad \Delta(\mu) = \int \Lambda d\mu.$$\)

Observe that the maps $\Gamma$ and $\Delta$ are affine and continuous. Finally, define $U : \mathcal{P}(\Sigma^N) \to \mathbb{R}^M$ by $U = \frac{\Gamma}{\Delta}, \quad i.e.$

$$U_{\mu} = \frac{\Gamma(\mu)}{\Delta(\mu)} = \frac{\int \Phi d\mu}{\int \Lambda d\mu} , \quad (6.7)$$\)

and note that if $i \in \Sigma^*$, then

$$U_{L_{[i][i]}[i]} = \left\{ \left( \frac{\log p_{1,i}}{\log |DS_i(\pi u)|} , \ldots , \frac{\log p_{M,i}}{\log |DS_i(\pi u)|} \right) \left| u \in \Sigma^N \right. \right\} .$$\)

It therefore follows that

$$\zeta_{\mathcal{C}}^{\text{dyn,U}} (\varphi ; z) = \sum_{n} \frac{z^n}{n} \left\{ \sum_{|i| = n} \sup_{u \in \Sigma^n} \exp \left( \sum_{k=0}^{n-1} \varphi S^k u \right) \right\} , \quad (6.8)$$\)

In order to prove Theorem 6.2, we first prove that radii of convergence of the zeta-functions $\zeta_{\mathcal{C}}^{\text{dyn,U}} (\varphi ; \cdot)$ and $\zeta_{\mathcal{C}}^{\text{dyn-con}} (\varphi ; \cdot)$ are comparable; this is the context of Proposition 6.5. However, in order to prove Proposition 6.5 we first prove two small auxiliary results, namely, Proposition 6.3 and Proposition 6.4.
Proposition 6.3. Let $U$ be defined by (6.7). Let $\Lambda$ be defined by (2.5) and let $\beta$ be defined by (6.5). Let $C \subseteq \mathbb{R}^M$ be a closed set and let $t$ be the unique real number such that

$$\sigma_{\text{rad}}(\zeta_C^{\text{dyn},U}(t\Lambda;\cdot)) = 1.$$ 

Then

$$\sup_{\mu \in \mathcal{P}_S(\mathbb{R}^H)} \left( h(\mu) + t \int \Lambda \, d\mu \right) = 0,$$

and we have

$$t = \sup_{\mu \in \mathcal{P}_S(\mathbb{R}^H)} \frac{h(\mu)}{\int \Lambda \, d\mu} = \sup_{\alpha \in C} \beta^*(\alpha).$$

Proof

We first note that it follows immediately from Theorem 5.5 that

$$\sup_{\mu \in \mathcal{P}_S(\mathbb{R}^H)} \left( h(\mu) + t \int \Lambda \, d\mu \right) = \mathcal{P}_C^{\epsilon}(t\Lambda) = -\log \sigma_{\text{rad}}(\zeta_C^{\text{dyn},U}(t\Lambda;\cdot)) = 0.$$ 

It therefore suffices to prove the following three inequalities, namely

$$\sup_{\alpha \in C} \beta^*(\alpha) \leq \sup_{\mu \in \mathcal{P}_S(\mathbb{R}^H)} \frac{h(\mu)}{\int \Lambda \, d\mu},$$

(6.9)

$$\sup_{\mu \in \mathcal{P}_S(\mathbb{R}^H)} - \frac{h(\mu)}{\int \Lambda \, d\mu} \leq t,$$

(6.10)

$$t \leq \sup_{\alpha \in C} \beta^*(\alpha).$$

(6.11)

Proof of (6.9). For $s \in \mathbb{R}$ and $q \in \mathbb{R}^M$, let $\mu_{s,q}$ denote the Gibbs state of $s\Lambda + \langle q|\Phi \rangle$. We now prove the following three claims.

Claim 1. For all $q$, we have $\frac{\int \Phi \, d\mu_{\beta(q),q}}{\int \Lambda \, d\mu_{\beta(q),q}} = -\nabla \beta(q)$.

Proof of Claim 1. Define $F : \mathbb{R} \times \mathbb{R}^M \to \mathbb{R}$ by $F(s, q) = P(s\Lambda + \langle q|\Phi \rangle)$ for $s \in \mathbb{R}$ and $q \in \mathbb{R}^M$. It follows from [Rue1] that $F$ is real analytic with

$$\nabla F(s, q) = \left( \int \Lambda \, d\mu_{s,q}, \int \Phi \, d\mu_{s,q} \right).$$

(6.12)

Next, since $0 = F(\beta(q), q)$ for all $q$, it follows from (6.12) and an application of the chain rule that

$$0 = \int \Lambda \, d\mu_{\beta(q),q} \nabla \beta(q) + \int \Phi \, d\mu_{\beta(q),q}$$

for all $q$. This clearly implies that $-\nabla \beta(q) = \frac{\int \Phi \, d\mu_{\beta(q),q}}{\int \Lambda \, d\mu_{\beta(q),q}}$ for all $q$. This completes the proof of Claim 1.

Claim 2. For all $q$, we have $-\frac{h(\mu_{\beta(q),q})}{\int \Lambda \, d\mu_{\beta(q),q}} \geq \beta^*(-\nabla \beta(q))$.

Proof of Claim 2. Since $\mu_{\beta(q),q}$ is a Gibbs state of $\beta(q)\Lambda + \langle q|\Phi \rangle$ and $P(\beta(q)\Lambda + \langle q|\Phi \rangle) = 0$, we deduce that $0 = P(\beta(q)\Lambda + \langle q|\Phi \rangle) = h(\mu_{\beta(q),q}) + \int (\beta(q)\Lambda + \langle q|\Phi \rangle) \, d\mu_{\beta(q),q} = h(\mu_{\beta(q),q}) + \beta(q) \int \Lambda \, d\mu_{\beta(q),q} + \langle q|\Phi \, d\mu_{\beta(q),q} \rangle$. Hence

$$-\frac{h(\mu_{\beta(q),q})}{\int \Lambda \, d\mu_{\beta(q),q}} = \beta(q) + \frac{\langle q|\Phi \, d\mu_{\beta(q),q} \rangle}{\int \Lambda \, d\mu_{\beta(q),q}} = \beta(q) + \frac{\langle q|\Phi \, d\mu_{\beta(q),q} \rangle}{\int \Lambda \, d\mu_{\beta(q),q}}.$$  

(6.13)
Combining Claim 1 and (6.13) now yields

\[-h(\mu_\beta(q), q) \int \Lambda d\mu_\beta(q, q) = \beta(q) + (q | - \nabla \beta(q)) \geq \inf_r (\beta(r) + (r | - \nabla \beta(q))) = \beta^*(-\nabla \beta(q))\]

for all \(q\). This completes the proof of Claim 2.

**Claim 3.** For all \(\alpha \in \mathbb{R}^M\), we have \(\beta^*(\alpha) \leq \sup_{\mu \in \mathcal{P}_S(\Sigma^N)} \frac{h(\mu)}{\int \Lambda d\mu}\).

**Proof of Claim 3.** If \(\beta^*(\alpha) = -\infty\), then the statement is clear. Hence, we may assume that \(\beta^*(\alpha) > -\infty\). In this case it follows from the convexity of \(\beta\) that there is a point \(q_{\alpha} \in \mathbb{R}^M\) such that \(\alpha = -\nabla \beta(q_{\alpha})\), see [Ro]. It therefore follows from Claim 1 that the measure \(\mu_{\beta(q_{\alpha}), q_{\alpha}}\) satisfies

\[U_{\mu_{\beta(q_{\alpha}), q_{\alpha}}} = \int \Phi d\mu_{\beta(q_{\alpha}), q_{\alpha}} = -\nabla \beta(q_{\alpha}) = \alpha,\]

whence, using Claim 2,

\[\sup_{\mu \in \mathcal{P}_S(\Sigma^N)} \frac{h(\mu)}{\int \Lambda d\mu} \geq \sup_{U_{\mu} = \alpha} \frac{h(\mu_{\beta(q_{\alpha}), q_{\alpha}})}{\int \Lambda d\mu_{\beta(q_{\alpha}), q_{\alpha}}} = \beta^*(-\nabla \beta(q_{\alpha})) = \beta^*(\alpha)\]

This completes the proof of Claim 3.

We can now prove the required inequality. Indeed, it follows immediately from Claim 3 that

\[\sup_{\alpha \in C} \beta^*(\alpha) \leq \sup_{\alpha \in C} \sup_{\mu \in \mathcal{P}_S(\Sigma^N)} \frac{h(\mu)}{\int \Lambda d\mu} \leq \sup_{U_{\mu} \in C} \frac{h(\mu)}{\int \Lambda d\mu}.\]

This completes the proof of (6.9).

**Proof of (6.10).** Fix \(\mu \in \mathcal{P}_S(\Sigma^N)\) with \(U_{\mu} \in C\). It follows from the definition of \(t\) that \(h(\mu) + t \int \Lambda d\mu \leq 0\), and since \(\Lambda < 0\), we therefore conclude that \(- \frac{h(\mu)}{\int \Lambda d\mu} \leq t\). Taking supremum over all \(\mu \in \mathcal{P}_S(\Sigma^N)\) with \(U_{\mu} \in C\) in this inequality now gives the desired result. This completes the proof of (6.10).

**Proof of (6.11).** Fix \(\mu \in \mathcal{P}_S(\Sigma^N)\) with \(U_{\mu} \in C\). Next, let \(q \in \mathbb{R}^M\). It now follows from the definition of \(t\) and \(\beta(q)\) that We now have

\[\sup_{\nu \in \mathcal{P}_S(\Sigma^N)} \left( h(\nu) + t \int \Lambda d\nu \right) = 0 = P(\beta(q)\Lambda + (q|\Phi)) \tag{6.14}\]

Also, using the variational principle (see [Wa]) we conclude that \(P(\beta(q)\Lambda + (q|\Phi)) = \sup_{\nu \in \mathcal{P}_S(\Sigma^N)} (h(\nu) + \int (\beta(q)\Lambda + (q|\Phi)) d\nu) \geq h(\mu) + \beta(q)\int \Lambda d\mu + (q|\Phi d\mu)\). We deduce from this and (6.14) that

\[\sup_{\nu \in \mathcal{P}_S(\Sigma^N)} \left( h(\nu) + t \int \Lambda d\nu \right) \geq h(\mu) + \beta(q)\int \Lambda d\mu + \left(q \int \Phi d\mu\right). \tag{6.15}\]

Next, observe that \(U_{\mu} = \frac{\Phi d\mu}{\int \Lambda d\mu}\), whence \(\int \Phi d\mu = \int \Lambda d\mu U_{\mu}\), and it therefore follows from (6.15) that

\[\sup_{\nu \in \mathcal{P}_S(\Sigma^N)} \left( h(\nu) + t \int \Lambda d\nu \right) \geq h(\mu) + \beta(q)\int \Lambda d\mu + \left(q \int \Lambda d\mu U_{\mu}\right)\]

\[= h(\mu) + (\beta(q) + (q|U_{\mu})) \int \Lambda d\mu. \tag{6.16}\]
Taking supremum over all $\mathbf{q}$ in (6.16) and using the fact that $\Lambda < 0$ now gives
\[
\sup_{\nu \in \mathcal{P}_S(\Sigma^N)} \left( h(\nu) + t \int \Lambda \, d\nu \right) \geq \sup_{\mathbf{q}} \left( h(\mu) + (\beta(\mathbf{q}) + (\mathbf{q}|U\mu)) \int \Lambda \, d\mu \right)
\]
\[
= h(\mu) + \inf_{\mathbf{q}} \left( \beta(\mathbf{q}) + (\mathbf{q}|U\mu) \right) \int \Lambda \, d\mu
\]
\[
= h(\mu) + \beta^*(U\mu) \int \Lambda \, d\mu.
\]
(6.17)

By assumption $U\mu \in C$, whence $\beta^*(U\mu) \leq \sup_{\mathbf{a} \in C} \beta^*(\mathbf{a})$. It follows from this and the inequality $\Lambda < 0$ that $\beta^*(U\mu) \int \Lambda \, d\mu \geq (\sup_{\mathbf{a} \in C} \beta^*(\mathbf{a})) \int \Lambda \, d\mu$, and we therefore conclude from (6.17) that
\[
\sup_{\nu \in \mathcal{P}_S(\Sigma^N)} \left( h(\nu) + t \int \Lambda \, d\nu \right) \geq h(\mu) + \left( \sup_{\mathbf{a} \in C} \beta^*(\mathbf{a}) \right) \int \Lambda \, d\mu.
\]
(6.18)

Finally, taking supremum over all $\mu \in \mathcal{P}_S(\Sigma^N)$ with $U\mu \in C$ in (6.18) yields
\[
\sup_{\nu \in \mathcal{P}_S(\Sigma^N)} \left( h(\nu) + t \int \Lambda \, d\nu \right) \geq \sup_{\mu \in \mathcal{P}_S(\Sigma^N)} \left( h(\mu) + \left( \sup_{\mathbf{a} \in C} \beta^*(\mathbf{a}) \right) \int \Lambda \, d\mu \right).
\]
(6.19)

Since $\Lambda < 0$, we now deduce from inequality (6.19) that $t \leq \sup_{\mathbf{a} \in C} \beta^*(\mathbf{a})$. This completes the proof of (6.11). \qed

**Proposition 6.4.** Let $U$ be defined by (6.7). Let $\Lambda$ be defined by (2.5) and let $\beta$ be defined by (6.5). Then $-\nabla \beta(\mathbb{R}^M) \subseteq U(\mathcal{P}_S(\Sigma^N))$.

**Proof**
This follows from Claim 1 in the proof of Proposition 6.3. \qed

**Proposition 6.5.** Let $U$ be defined by (6.7). Fix a continuous function $\varphi : \Sigma^N \to \mathbb{R}$.

1. There is a sequence $(\Delta_n)_n$ with $\Delta_n > 0$ and $\Delta_n \to 0$ such that for all closed subsets $W$ of $\mathbb{R}^M$ and for all $n \in \mathbb{N}$, $i \in \Sigma^n$ and $\mathbf{u} \in \Sigma^N$, we have
\[
\text{dist} \left( \left( \frac{\log p_{1,i}}{\log |DS_i(\pi \mathbf{u})|}, \ldots, \frac{\log p_{M,i}}{\log |DS_i(\pi \mathbf{u})|} \right), W \right)
\]
\[
\quad \leq \text{dist} \left( \left( \frac{\log p_{1,i}}{\log \text{diam } K_i}, \ldots, \frac{\log p_{M,i}}{\log \text{diam } K_i} \right), W \right) + \Delta_n,
\]
(6.20)

\[
\text{dist} \left( \left( \frac{\log p_{1,i}}{\log \text{diam } K_i}, \ldots, \frac{\log p_{M,i}}{\log \text{diam } K_i} \right), W \right)
\]
\[
\quad \leq \text{dist} \left( \left( \frac{\log p_{1,i}}{\log |DS_i(\pi \mathbf{u})|}, \ldots, \frac{\log p_{M,i}}{\log |DS_i(\pi \mathbf{u})|} \right), W \right) + \Delta_n.
\]
(6.21)

2. Let $W$ be a closed subset of $\mathbb{R}^M$. For all $r > 0$, we have
\[
\sigma_{\text{rad}}(\varsigma_{\text{dyn,}U}^{\text{con}}(\varphi; \cdot)) \leq \sigma_{\text{rad}}(\varsigma_{\text{dyn,}W}^{\text{con}}(\varphi; \cdot)),
\]
(6.22)

\[
\sigma_{\text{rad}}(\varsigma_{\text{dyn,}U}^{\text{con}}(\varphi; \cdot)) \leq \sigma_{\text{rad}}(\varsigma_{\text{dyn,}U}^{\text{U}}(\varphi; \cdot)).
\]
(6.23)
(3) Assume that $C \subseteq \mathbb{R}^M$ is closed. Then we have
\[
\lim_{r \to 0} \sigma_{\text{rad}}(s_{B(C,r)}(\varphi)) = \lim_{r \to 0} \sigma_{\text{rad}}(s_{B(C,r)}(\psi))
\]

(4) Assume that $C \subseteq \mathbb{R}^M$ is closed and convex with $C \cap (-\nabla \beta(\mathbb{R}^M)) \neq \emptyset$. Then we have
\[
\sigma_{\text{rad}}(s_C^{\text{dyn},\text{con}}(\varphi)) = \sigma_{\text{rad}}(s_C^{\text{dyn},U}(\varphi))
\]

Proof
(1) It is well-known and follows from the Principle of Bounded Distortion (see, for example, [Bar,Fa2]) that there is a constant $c > 0$ such that for all integers $n$ and all $i$ with $|i| = n$ and all $u, v \in [i]$, we have $\frac{1}{c} \leq \frac{DS_I(\pi^0 u)}{|DS_I(\pi^0 u)|} \leq c$ and $\frac{1}{c} \leq \frac{|DS_I(\pi^0 u)|}{|DS_I(\pi^0 v)|} \leq c$. It is not difficult to see that the desired result follows from this.

(2) Fix $r > 0$. Let $(\Delta_n)_n$ be the sequence from (1). Since $\Delta_n \to 0$, we can find a positive integer $N_r$ such that if $n \geq N_r$, then $\Delta_n < r$. Consequently, using (6.21) in Part (1), for all $n \geq N_r$, we have
\[
\sum_{|i| = n} \sup_{u \in [i]} \exp \sum_{k=0}^{n-1} \varphi S^k u \leq \sum_{|i| = n} \sup_{u \in [i]} \exp \sum_{k=0}^{n-1} \varphi S^k u
\]

A similar argument using (6.20) in Part 1 shows that
\[
\sum_{|i| = n} \sup_{u \in [i]} \exp \sum_{k=0}^{n-1} \varphi S^k u \leq \sum_{|i| = n} \sup_{u \in [i]} \exp \sum_{k=0}^{n-1} \varphi S^k u.
\]

The desired results follow immediately from inequalities (6.24) and (6.25).

(3) This result follows easily from Part (2).
(4) "\(\geq\)" It follows from (6.22), Theorem 5.3 and Theorem 5.5 that
\[
- \log \sigma_{rad}(\zeta_C^{\text{dyn-con}}(\varphi; \cdot)) \leq \liminf_{r \searrow 0} - \log \sigma_{rad}(\zeta_{B(I(C,r)}^{\text{dyn,U}}(\varphi; \cdot)) \quad \text{[by (6.22)]}
\]
\[
= \sup_{\mu \in \mathcal{P}_S(\Sigma^N)} \left( h(\mu) + \int \varphi \, d\mu \right) \quad \text{[by Theorem 5.3]}
\]
\[
= - \log \sigma_{rad}(\zeta_C^{\text{dyn,U}}(\varphi; \cdot)). \quad \text{[by Theorem 5.5]}
\]
It follows from this inequality that \(\sigma_{rad}(\zeta_C^{\text{dyn-con}}(\varphi; \cdot)) \geq \sigma_{rad}(\zeta_C^{\text{dyn,U}}(\varphi; \cdot))\).

"\(\leq\)" For \(\varepsilon > 0\), write \(I(C,\varepsilon) = \{x \in C \mid \text{dist}(x, \partial C) \geq \varepsilon\}\).

Next, fix \(\varepsilon > 0\) and note that if \(r > 0\) with \(2r < \varepsilon\), then it follows from (6.23) applied to \(W = B(I(C,\varepsilon),r)\) that
\[
- \log \sigma_{rad}(\zeta_{B(I(C,\varepsilon),r)}^{\text{dyn-con}}(\varphi; \cdot)) \geq - \log \sigma_{rad}(\zeta_{B(I(C,\varepsilon),r)}^{\text{dyn,U}}(\varphi; \cdot)) . \quad \text{(6.26)}
\]
However, for \(r > 0\) with \(2r < \varepsilon\) it follows from the convexity of \(C\) that \(B(B(I(C,\varepsilon),r),r) \subseteq B(I(C,\varepsilon),2r) \subseteq C\), whence \(\sigma_{rad}(\zeta_C^{\text{dyn-con}}(\varphi; \cdot)) \leq \sigma_{rad}(\zeta_{B(I(C,\varepsilon),r)}^{\text{dyn-con}}(\varphi; \cdot))\), and so \(- \log \sigma_{rad}(\zeta_C^{\text{dyn-con}}(\varphi; \cdot)) \geq - \log \sigma_{rad}(\zeta_{B(I(C,\varepsilon),r)}^{\text{dyn-con}}(\varphi; \cdot))\). We conclude from this and (6.26) that if \(r > 0\) with \(2r < \varepsilon\), then
\[
- \log \sigma_{rad}(\zeta_C^{\text{dyn-con}}(\varphi; \cdot)) \geq - \log \sigma_{rad}(\zeta_{B(I(C,\varepsilon),r)}^{\text{dyn,U}}(\varphi; \cdot)). \quad \text{(6.27)}
\]
Next, since \(I(C,\varepsilon)\) is closed, it follows from (6.27) and Theorem 5.3 that if \(\varepsilon > 0\), then
\[
- \log \sigma_{rad}(\zeta_C^{\text{dyn-con}}(\varphi; \cdot)) \geq \limsup_{r \searrow 0} - \log \sigma_{rad}(\zeta_{B(I(C,\varepsilon),r)}^{\text{dyn,U}}(\varphi; \cdot))
\]
\[
= \sup_{\mu \in \mathcal{P}_S(\Sigma^N)} \left( h(\mu) + \int \varphi \, d\mu \right) . \quad \text{(6.27)}
\]
Taking supremum over all \(\varepsilon > 0\) in (6.27) gives
\[
- \log \sigma_{rad}(\zeta_C^{\text{dyn-con}}(\varphi; \cdot)) \geq \sup_{\varepsilon > 0} \sup_{\mu \in \mathcal{P}_S(\Sigma^N)} \left( h(\mu) + \int \varphi \, d\mu \right)
\]
\[
= \sup_{\mu \in \mathcal{P}_S(\Sigma^N)} \left( h(\mu) + \int \varphi \, d\mu \right) \quad \text{[since } \bigcup_{\varepsilon > 0} I(C,\varepsilon) = \overset{\circ}{C}] \]
\[
= \sup_{\mu \in \mathcal{P}_S(\Sigma^N)} \left( h(\mu) + \int \varphi \, d\mu \right). \quad \text{(6.28)}
\]

Now note that it follows from Proposition 6.4 that \(-\nabla \beta(\mathbb{R}^M) \subseteq U(\mathcal{P}_S(\Sigma^N))\). Since \(\overset{\circ}{C} \cup (-\nabla \beta(\mathbb{R}^M)) \neq \emptyset\), we therefore deduce that \(\overset{\circ}{C} \cup U(\mathcal{P}_S(\Sigma^N)) \neq \emptyset\), and an application of Theorem 5.5 now gives
\[
\sup_{\mu \in \mathcal{P}_S(\Sigma^N)} \left( h(\mu) + \int \varphi \, d\mu \right) = - \log \sigma_{rad}(\zeta_C^{\text{dyn,U}}(\varphi; \cdot)). \quad \text{(6.29)}
\]
Finally, combining (6.28) and (6.29) yields $-\log \sigma_{rad}(\zeta^\text{dyn-con}_C(\varphi; \cdot)) \geq -\log \sigma_{rad}(\zeta^\text{dyn-U}_C(\varphi; \cdot))$. It follows from this inequality that $\sigma_{rad}(\zeta^\text{dyn-con}_C(\varphi; \cdot)) \leq \sigma_{rad}(\zeta^\text{dyn-U}_C(\varphi; \cdot))$. \qed

We can now prove Theorem 6.2.

**Proof of Theorem 6.2**

(1.1) and (2.1): The statements in Part (1.1) and Part (2.1) of Theorem 6.2 follow immediately from Proposition 5.2 and Proposition 6.5.\(\Box\)

(1.2) and (2.2): The statements in Part (1.2) and Part (2.2) of Theorem 6.2 follow immediately from Proposition 5.2 and Proposition 6.5.

(1.3) and (2.3): The statements in Part (1.3) and Part (2.3) of Theorem 6.2 follow immediately from Proposition 6.3.

6.3. Multifractal spectra of ergodic Birkhoff averages. We first fix $\gamma \in (0, 1)$ and define the metric $d_\gamma$ on $\Sigma^N$ as follows. For $i,j \in \Sigma^N$ with $i \neq j$, we will write $i \wedge j$ for the longest common prefix of $i$ and $j$ (i.e. $i \wedge j = u$ where $u$ is the unique element in $\Sigma^*$ for which there are $k,l \in \Sigma^N$ with $k = k_1k_2 \ldots$ and $l = l_1l_2 \ldots$ such that $k_1 \neq l_1$, $i = uk$ and $j = ul$). The metric $d_\gamma$ is now defined by

$$d_\gamma(i,j) = \left\{ \begin{array}{ll} 0 & \text{if } i = j; \\
\gamma^{\|i\wedge j\|} & \text{if } i \neq j,
\end{array} \right.$$ 

for $i,j \in \Sigma^N$; throughout this section, we equip $\Sigma^N$ with the metric $d_\gamma$ and continuity and Lipschitz properties of functions $f : \Sigma^N \to \mathbb{R}$ from $\Sigma^N$ to $\mathbb{R}$ will always refer to the metric $d_\gamma$. Multifractal analysis of Birkhoff averages has received significant interest during the past 10 years, see, for example, [BaMe, FaFe, FaFeWu, FeLaWu, Oli, Ol3, OlWi2]. The multifractal spectrum $F^\text{erg}_f$ of ergodic Birkhoff averages of a continuous function $f : \Sigma^N \to \mathbb{R}$ is defined by

$$F^\text{erg}_f(\alpha) = \dim_H \pi \left\{ i \in \Sigma^N \left| \lim_{n} \frac{1}{n} \sum_{k=0}^{n-1} f(S^k i) = \alpha \right. \right\}$$

for $\alpha \in \mathbb{R}$. One of the main problems in multifractal analysis of Birkhoff averages is the detailed study of the multifractal spectrum $F^\text{erg}_f$. For example, Theorem D below is proved in different settings and at various levels of generality in [FaFe, FaFeWu, FeLaWu, Oli, Ol3, OlWi2].

**Theorem F** [FaFe, FaFeWu, FeLaWu, Oli, Ol3, OlWi2]. Let $f : \Sigma^N \to \mathbb{R}$ be a Lipschitz function. Let $C \subseteq \mathbb{R}$ be a closed subset of $\mathbb{R}$. If the OSC is satisfied, then

$$\dim_H \pi \left\{ i \in \Sigma^N \left| \lim_{n} \frac{1}{n} \sum_{k=0}^{n-1} f(S^k i) \subseteq C \right. \right\} = \sup_{\mu \in \mathcal{P}(\Sigma^N)} \frac{-h(\mu)}{\int f \mu} - \frac{h(\mu)}{\int \Lambda d\mu}.$$ 

In particular, if the OSC is satisfied and $\alpha \in \mathbb{R}$, then we have

$$\dim_H \pi \left\{ i \in \Sigma^N \left| \lim_{n} \frac{1}{n} \sum_{k=0}^{n-1} f(S^k i) = \alpha \right. \right\} = \sup_{\mu \in \mathcal{P}(\Sigma^N)} \frac{-h(\mu)}{\int f \mu} - \frac{h(\mu)}{\int \Lambda d\mu}.$$ 

As a third application of Theorem 2.1 we obtain a zeta-function whose abscissa of convergence equals the multifractal spectrum $F^\text{erg}_f$ of ergodic Birkhoff averages of a Lipschitz function $f$. This is the content of the next theorem.
Theorem 6.6. Multifractal zeta-function for multifractal spectra of ergodic Birkhoff averages. Let $f : \Sigma^N \to \mathbb{R}$ be a Lipschitz function. For $C \subseteq \mathbb{R}$ and an continuous function $\varphi : \Sigma^N \to \mathbb{R}$, we define the dynamical ergodic multifractal zeta-function by

$$\zeta_{\text{dyn}, C, f} (\varphi; z) = \sum_n z^n \left( \sum_{|i| = n} \sup_{u \in [i]} \exp \left( \sum_{k=0}^{n-1} f(S^k u) \right) \right),$$

where we write $\bar{\imath} = \bar{i} \ldots$ for $i \in \Sigma^*$. Let $\Lambda : \Sigma^N \to \mathbb{R}$ be defined by (2.5). Assume that $C \subseteq \mathbb{R}$ is closed.

1. There is a unique real number $\varphi(C)$ such that

$$\lim_{r \to 0} \sigma_{\text{rad}} \left( \zeta_{B(r), f} (\varphi(C) \Lambda; \cdot) \right) = 1.$$

2. We have

$$\varphi(C) = \sup_{\alpha \in C} \sup_{\mu \in \mathcal{P}(\Sigma^N)} \int f \, d\mu = \sup_{\alpha \in C} \frac{h(\mu)}{\int f \, d\mu}.$$

3. If the OSC is satisfied, then we have

$$\varphi(C) = \dim_H \pi \left( \left\{ i \in \Sigma^N \mid f(S^k i) \subseteq C \right\} \right).$$

In particular, if the OSC is satisfied and $\alpha \in \mathbb{R}$, then we have

$$\varphi(\alpha) = \dim_H \pi \left( \left\{ i \in \Sigma^N \mid \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} f(S^k i) = \alpha \right\} \right).$$

We will now prove Theorem 6.6. Recall, that the function $\Lambda : \Sigma^N \to \mathbb{R}$ is defined by $\Lambda(i) = \log |D\pi S_i (\pi \bar{i} \bar{\imath})|$ for $i = i_1 i_2 \ldots \in \Sigma^*$. Define $U : \mathcal{P}(\Sigma^N) \to \mathbb{R}$ by

$$U \mu = \int f \, d\mu.$$ 

and note that if $i \in \Sigma^*$, then

$$UL_{|i|}[i] = \left\{ \frac{1}{|i|} \sum_{k=0}^{n-1} f(S^k (i u)) \mid u \in \Sigma^N \right\}.$$ 

It therefore follows that

$$\zeta_{\text{dyn}, U} (\varphi; z) = \sum_n z^n \left( \sum_{|i| = n} \sup_{u \in [i]} \exp \left( \sum_{k=0}^{n-1} f(S^k (i u)) \right) \right),$$

In order to prove Theorem 6.6, we first prove the following auxiliary result.
Proposition 6.7. Let $U$ be defined by (6.30). Fix a continuous function $\varphi : \Sigma^N \to \mathbb{R}$.

1. There is a sequence $(\Delta_n)_n$ with $\Delta_n > 0$ for all $n$ and $\Delta_n \to 0$ such that for all closed subsets $C$ of $\mathbb{R}$ and for all $n \in \mathbb{N}$, $i \in \Sigma^n$ and $u \in \Sigma^N$, we have

$$\text{dist} \left( \frac{1}{n} \sum_{k=0}^{n-1} f(S^k(iu)) \right), C \right) \leq \text{dist} \left( \frac{1}{n} \sum_{k=0}^{n-1} f(S^k(i)) \right), C \right) + \Delta_n,$$

recall, that for $i \in \Sigma^*$, we write $i = \cdots i i$. 

2. Let $C$ be a closed subset of $\mathbb{R}^M$. We have

$$\lim_{r \searrow 0} \sigma_{\text{rad}}(\zeta_{\text{dyn-erg}}_{B(C,r),f}(\varphi; \cdot)) = \lim_{r \searrow 0} \sigma_{\text{rad}}(\zeta_{\text{dyn,U}}_{B(C,r)}(\varphi; \cdot)).$$

Proof

(1) Let $\text{Lip}(f)$ denote the Lipschitz constant of $f$. It is clear that for all $n \in \mathbb{N}$, $i \in \Sigma^n$ and $u \in \Sigma^N$, we have

$$\left| \frac{1}{n} \sum_{k=0}^{n-1} f(S^k(i)) - \frac{1}{n} \sum_{k=0}^{n-1} f(S^k(iu)) \right| \leq \frac{1}{n} \sum_{k=0}^{n-1} |f(S^k(i)) - f(S^k(iu))|$$

$$\leq \text{Lip}(f) \frac{1}{n} \sum_{k=0}^{n-1} d_{\gamma}(S^k(i), S^k(iu))$$

$$\leq \text{Lip}(f) \frac{1}{n} \sum_{k=0}^{n-1} \gamma^k$$

$$\leq \text{Lip}(f) \frac{1}{n(1 - \gamma)}.$$  \hfill (6.32)

It is not difficult to see that the desired result follows from (6.32).

(2) This statement follows from Part (1) by an argument very similar to the proofs of Part (2) and Part (3) in Proposition 6.5, and the proof is therefore omitted. \hfill \Box

We can now prove Theorem 6.6.

Proof of Theorem 6.6

(1) This statement follows immediately from Proposition 5.2 and Proposition 6.7.

(2) This statement follows immediately from Part (1) using Corollary 5.4. \hfill \Box

(3) This statement follows immediately from Part (2) using Theorem F. \hfill \Box

7. Proofs. Preliminary results: the modified multifractal pressure

In this section we introduce our main technical tool, namely, the modified multifractal pressure; see definition (7.2) below. The two main results in this section are Theorem 7.3 providing a variational principle for the modified multifractal pressure and Theorem 7.5 showing that the multifractal pressure and the modified multifractal pressure are (almost) comparable. Both Theorem 7.3 and Theorem 7.5 play major roles in the in the proof of Theorem 5.3 in Section 8 and in the proof of Theorem 5.5 in Section 9.

We first define the modified multifractal pressure. We start by introducing some notation. If $i \in \Sigma^*$, then we define $\bar{i} \in \Sigma^N$ by $\bar{i} = ii \ldots$. We also define $M_n : \Sigma^N \to P_S(\Sigma^N)$ by

$$M_n i = L_n \left( \bar{i} | \bar{i} \right) = \frac{1}{n} \sum_{k=0}^{n-1} \delta_{S^k(i)}(\bar{i} | \bar{i} \right).$$  \hfill (7.1)
for \( i \in \Sigma^N \); recall, that the map \( L_n : \Sigma^N \rightarrow \mathcal{P}(\Sigma^N) \) is defined in (5.1). Furthermore, note that if \( i \in \Sigma^N \), then \( M_n i \) is shift invariant, i.e. \( M_n \) maps \( \Sigma^N \) into \( \mathcal{P}(\Sigma^N) \) as claimed. Next, let \( P \) denote the probability measure on \( \Sigma^N \) given by

\[
P = \frac{1}{N} \sum_{i=1}^{N} \delta_i.
\]

For a continuous function \( \varphi : \Sigma^N \rightarrow \mathbb{R} \), we define \( F_{\varphi} : \mathcal{P}(\Sigma^N) \rightarrow \mathbb{R} \) by

\[
F_{\varphi}(\mu) = \int \varphi \, d\mu.
\]

Observe that since \( \varphi \) is bounded, i.e. \( \| \varphi \|_\infty < \infty \), we conclude that \( \| F_{\varphi} \|_\infty < \infty \). Next, for a positive integer \( n \), define probability measures \( P_n, Q_{\varphi,n} \in \mathcal{P}(\mathcal{P}(\Sigma^N)) \) by

\[
P_n = P \circ M_n^{-1},
\]

\[
Q_{\varphi,n}(E) = \frac{\int_E \exp(nF_{\varphi}) \, dP_n}{\int \exp(nF_{\varphi}) \, dP_n} \quad \text{for Borel subsets } E \text{ of } \mathcal{P}(\Sigma^N).
\]

Finally, we define modified multifractal pressures as follows. Namely, for \( C \subseteq X \), we define the modified lower and upper multifractal pressure of \( \varphi \) associated with the space \( X \) and the map \( U \) and by

\[
Q^L_{\varphi}(C) = \limsup_{n} \frac{1}{n} \log \sum_{|i|=n} \sup_{u \in [i]} \exp \sum_{k=0}^{n-1} \varphi S^k u,
\]

\[
Q^U_{\varphi}(C) = \liminf_{n} \frac{1}{n} \log \sum_{|i|=n} \sup_{u \in [i]} \exp \sum_{k=0}^{n-1} \varphi S^k u.
\]

We now turn towards the proof of the first main result in this section, namely, Theorem 7.3 providing a variational principle for the modified multifractal pressure. The proof of Theorem 7.3 is based on large deviation theory. In particular, we need Varadhan’s [Va] large deviation theorem (Theorem 7.1.(i) below), and a non-trivial application of this (namely Theorem 7.1.(ii) below) providing first order asymptotics of certain “Boltzmann distributions”. However, we begin with a definition.

**Definition.** Let \( X \) be a complete separable metric space and let \( (P_n)_n \) be a sequence of probability measures on \( X \). Let \( (a_n)_n \) be a sequence of positive numbers with \( a_n \rightarrow \infty \) and let \( I : X \rightarrow [0, \infty] \) be a lower semicontinuous function with compact level sets. The sequence \( (P_n)_n \) is said to have the large deviation property with constants \( (a_n)_n \) and rate function \( I \) if the following two conditions hold:

(i) For each closed subset \( K \) of \( X \), we have

\[
\limsup_{n} \frac{1}{a_n} \log P_n(K) \leq - \inf_{x \in K} I(x);
\]

(ii) For each open subset \( G \) of \( X \), we have

\[
\liminf_{n} \frac{1}{a_n} \log P_n(G) \geq - \inf_{x \in G} I(x).
\]
Theorem 7.1. Let $X$ be a complete separable metric space and let $(P_n)_n$ be a sequence of probability measures on $X$. Assume that the sequence $(P_n)_n$ has the large deviation property with constants $(a_n)_n$ and rate function $I$. Let $F : X \to \mathbb{R}$ be a continuous function satisfying the following two conditions:

(i) For all $n$, we have
$$\int \exp(a_n F) \, dP_n < \infty.$$ 

(ii) We have
$$\lim_{M \to \infty} \limsup_n \frac{1}{a_n} \log \int_{\{M \leq F\}} \exp(a_n F) \, dP_n = -\infty.$$ 

(Observe that the Conditions (i)–(ii) are satisfied if $F$ is bounded.) Then the following statements hold.

(1) We have
$$\lim_n \frac{1}{a_n} \log \int \exp(a_n F) \, dP_n = -\inf_{x \in X} (I(x) - F(x)).$$

(2) For each $n$ define a probability measure $Q_n$ on $X$ by
$$Q_n(E) = \frac{\int_E \exp(a_n F) \, dP_n}{\int \exp(a_n F) \, dP_n}.$$ 

Then the sequence $(Q_n)_n$ has the large deviation property with constants $(a_n)_n$ and rate function $(I - F) - \inf_{x \in X} (I(x) - F(x)).$

Proof
Statement (1) follows from [El, Theorem II.7.1] or [DeZe, Theorem 4.3.1], and statement (2) follows from [El, Theorem II.7.2].

Before stating and proving Theorem 7.3, we establish the following auxiliary result.

Theorem 7.2. Let $X$ be a metric space and let $U : \mathcal{P}(\Sigma^\mathbb{N}) \to X$ be continuous with respect to the weak topology. Let $C \subseteq X$ be a subset of $X$. Fix a continuous function $\varphi : \Sigma^\mathbb{N} \to \mathbb{R}$. Then there is a constant $c$ such that for all positive integers $n$, we have

$$\sum_{|k|=n} \sup_{\mathbf{u} \in [k]} \exp \sum_{k=0}^{n-1} \varphi S^k \mathbf{u} \leq c \, N^n \, Q_{\varphi,n} \left( \{ U \in C \} \right) \int \exp(n \varphi) \, dP_n,$$

and

$$\sum_{|k|=n} \sup_{\mathbf{u} \in [k]} \exp \sum_{k=0}^{n-1} \varphi S^k \mathbf{u} \geq \frac{1}{c} \, N^n \, Q_{\varphi,n} \left( \{ U \in C \} \right) \int \exp(n \varphi) \, dP_n.$$

Proof
For each positive integer $n$ and each $i$ with $|i| = n$, we write $s_i = \sup_{\mathbf{u} \in [i]} \exp \sum_{k=0}^{n-1} \varphi S^k \mathbf{u}$ for sake of brevity. Let $C$ be a subset of $X$. For each positive integer $n$, we clearly have

$$\int_{\{ j \in \Sigma^n \mid U_M \subseteq C \}} s_{ij} \, dP(i)$$

$$= \sum_{|k|=n} \left[ \int_{U_M \subseteq C} s_{ij} \, dP(i) \right] + \sum_{|k|=n} \left[ \int_{\mathbf{u} \in [k] \cap \{ j \in \Sigma^n \mid U_M \subseteq C \}} s_{ij} \, dP(i) \right].$$
Now observe that if \( k \in \Sigma^* \) with \(|k| = n\) and \( k \cap \{j \in \Sigma^n | UM_n[j][n] \subseteq C\} \neq \emptyset \), then there is \( u \in [k] \cap \{j \in \Sigma^n | UM_n[j][n] \subseteq C\} \). Since \( u \in [k] \), we conclude that \( u = kv \) for some \( v \in \Sigma^n \). Next, since also \( kv = u \in \{j \in \Sigma^n | UM_n[j][n] \subseteq C\} \), we conclude that \( UM_n[k] = UM_n[(kv)[n] = UM_n[u][n] \subseteq C \). This shows that

\[
\sum_{|k| = n} s_k P \left( [k] \cap \left\{ j \in \Sigma^n \mid UM_n[j][n] \subseteq C \right\} \right).
\]

Combining (7.3) and (7.4) gives

\[
\int \sum_{|k| = n} s_k P \left( [k] \cap \left\{ j \in \Sigma^n \mid UM_n[j][n] \subseteq C \right\} \right) dP(i)
\]

\[
= \sum_{|k| = n} s_k P \left( [k] \cap \left\{ j \in \Sigma^n \mid UM_n[j][n] \subseteq C \right\} \right).
\]

However, if \( k \in \Sigma^* \) with \(|k| = n \) and \( UM_n[k] \subseteq C \), then it is clear that \( [k] \subseteq \{j \in \Sigma^n | UM_n[j][n] \subseteq C\} \), whence \( [k] \cap \{j \in \Sigma^n | UM_n[j][n] \subseteq C\} = [k] \). This and (7.5) now imply that

\[
\int \sum_{|k| = n} s_k P \left( [k] \cap \left\{ j \in \Sigma^n \mid UM_n[j][n] \subseteq C \right\} \right) dP(i)
\]

\[
= \sum_{|k| = n} s_k P \left( [k] \cap \left\{ j \in \Sigma^n \mid UM_n[j][n] \subseteq C \right\} \right).
\]
whence
\[ \sum_{|k|=n} s_k = N^n \int_{\{j \in \Sigma^n \mid U_{M_n} [j] \subseteq C\}} s_{|n|} dP(i). \]  
(7.6)

It follows from the Principle of Bounded Distortion (see, for example, [Bar,Fa2]) that there is a constant \( c > 0 \) such that if \( n \in \mathbb{N}, \ i \in \Sigma^n \) and \( u, v \in [i] \), then \( \frac{1}{c} \leq \frac{\exp \sum_{k=0}^{n-1} \varphi^{k} u}{\exp \sum_{k=0}^{n-1} \varphi^{k} v} \leq c \). In particular, this implies that for all \( n \in \mathbb{N} \) and for all \( i \in \Sigma^n \), we have
\[ \frac{1}{c} \exp \sum_{k=0}^{n-1} \varphi^{k} i \leq s_i \leq c \exp \sum_{k=0}^{n-1} \varphi^{k} i. \]  
(7.7)

**Claim 1.** For all positive integers \( n \), we have
\[ \sum_{|k|=n} s_k \leq c N^n \int_{\{j \in \Sigma^n \mid U_{M_n} [j] \subseteq C\}} \exp (nF_\varphi (M_n i)) dP(i), \]  
(7.8)

\[ \sum_{|k|=n} s_k \geq \frac{1}{c} N^n \int_{\{j \in \Sigma^n \mid U_{M_n} [j] \subseteq C\}} \exp (nF_\varphi (M_n i)) dP(i). \]  
(7.9)

**Proof of Claim 1.** It follows from (7.6) and (7.7) that if \( n \) is a positive integer, then we have
\[ \sum_{|k|=n} s_k \leq c N^n \int_{\{j \in \Sigma^n \mid U_{M_n} [j] \subseteq C\}} \exp \left( \sum_{k=0}^{n-1} \varphi^k \left( \frac{1}{|n} \right) \right) dP(i) \]
\[ = c N^n \int_{\{j \in \Sigma^n \mid U_{M_n} [j] \subseteq C\}} \exp \left( n \int \varphi d(M_n i) \right) dP(i) \]
\[ = c N^n \int_{\{j \in \Sigma^n \mid U_{M_n} [j] \subseteq C\}} \exp (nF_\varphi (M_n i)) dP(i). \]

This proves inequality (7.8). Inequality (7.9) is proved similarly. This completes the proof of Claim 1.

**Claim 2.** For all positive integers \( n \), we have \( \{j \in \Sigma^n \mid U_{M_n} [j] \subseteq C\} = \{j \in \Sigma^n \mid U_{M_n} [j] \subseteq C\} \).

**Proof of Claim 2.** Indeed, it is clear that \( \{j \in \Sigma^n \mid U_{M_n} [j] \subseteq C\} \subseteq \{j \in \Sigma^n \mid U_{M_n} [j] \subseteq C\} \). We will now prove the reverse inclusion. We therefore fix \( j \in \Sigma^n \) with \( U_{M_n} [j] \subseteq C \). We must now prove that \( U_{M_n} [j] \subseteq C \). In order to do this, we let \( u \in [j] \). Since \( u \in [j] \), we conclude that \( u \in [j] \), whence \( U_{M_n} u = U_{M_n} (\bar{u}) = U_{M_n} (\bar{j} \bar{u}) = U_{M_n} j \subseteq C \). This completes the proof of Claim 2.

For all positive integers \( n \), we now deduce from Claim 1 and Claim 2 that
\[ \sum_{|k|=n} s_k \leq c N^n \int_{\{j \in \Sigma^n \mid U_{M_n} [j] \subseteq C\}} \exp (nF_\varphi (M_n i)) dP(i) \]
\[ = c N^n \int_{\{U_{M_n} \in C\}} \exp (nF_\varphi (M_n i)) dP(i). \]
\[ c N^n \int_{\{U \in C\}} \exp(nF_\varphi) \, dP_n \]
\[ = c N^n \int_{\{U \in C\}} \exp(nF_\varphi) \, dP_n. \]

Similarly, we prove that for all positive integers \( n \), we have
\[ \sum_{|k| = n} s_k \geq \frac{1}{c} N^n Q_{\varphi,n}(\{U \in C\}) \int \exp(nF_\varphi) \, dP_n. \]

This completes the proof of Theorem 7.2. \( \square \)

We can now state and prove the first main result in this section, namely, Theorem 7.3.

**Theorem 7.3.** The variational principle for the modified multifractal pressure.. Let \( X \) be a metric space and let \( U : \mathcal{P}(\Sigma^N) \to X \) be continuous with respect to the weak topology. Let \( C \subseteq X \) be a subset of \( X \). Fix a continuous function \( \varphi : \Sigma^N \to \mathbb{R} \).

1. If \( G \) is an open subset of \( X \), then
   \[ Q^U_G(\varphi) \geq \sup_{\mu \in \mathcal{P}_S(\Sigma^N)} \left( h(\mu) + \int \varphi \, d\mu \right). \]
   
2. If \( K \) is a closed subset of \( X \), then
   \[ Q^U_K(\varphi) \leq \sup_{\mu \in \mathcal{P}_S(\Sigma^N)} \left( h(\mu) + \int \varphi \, d\mu \right). \]

**Proof**

We introduce the simplified notation from the proof of Theorem 7.2, i.e. for each positive integer \( n \) and each \( i \) with \( |i| = n \), we write \( s_i = \sup_{u \in [i]} \exp \sum_{k=0}^{n-1} \varphi^k u \). First note that it follows immediately from Theorem 7.2 that
\[ \liminf_n \frac{1}{n} \log \sum_{|i| = n} s_i \geq \log N + \limsup_n \frac{1}{n} \log Q_{\varphi,n}(\{U \in G\}) \]
\[ + \limsup_n \frac{1}{n} \log \int \exp(nF_\varphi) \, dP_n, \]
\[ \limsup_n \frac{1}{n} \log \sum_{|i| = n} s_i \leq \log N + \limsup_n \frac{1}{n} \log Q_{\varphi,n}(\{U \in K\}) \]
\[ + \limsup_n \frac{1}{n} \log \int \exp(nF_\varphi) \, dP_n. \]

Next, we observe that it follows from \([E1]\) that the sequence \((P_n = P \circ M_n^{-1})_n \subseteq \mathcal{P}(\mathcal{P}_S(\Sigma^N))\) has the large deviation property with respect to the sequence \((n)_n\) and rate function \( I : \mathcal{P}_S(\Sigma^N) \to \mathbb{R} \) given by \( I(\mu) = \log N - h(\mu) \). We therefore conclude from Part (1) of Theorem 7.1 that
\[ \lim_n \frac{1}{n} \log \int \exp(nF_\varphi) \, dP_n = - \inf_{\nu \in \mathcal{P}_S(\Sigma^N)} (I(\nu) - F_\varphi(\nu)). \]

Also, since the sequence \((P_n = P \circ M_n^{-1})_n \subseteq \mathcal{P}(\mathcal{P}_S(\Sigma^n))\) has the large deviation property with respect to the sequence \((n)_n\) and rate function \(I: \mathcal{P}_S(\Sigma^n) \to \mathbb{R}\) given by \(I(\mu) = \log N - h(\mu)\), we conclude from Part (2) of Theorem 7.1 that the sequence \((Q_{\varphi,n})_n\) has the large deviation property with respect to the sequence \((n)_n\) and rate function \((I - F_{\varphi}) - \inf_{\nu \in \mathcal{P}_S(\Sigma^n)} (I(\nu) - F_{\varphi}(\nu))\). As the set \(\{U \in G\} = U^{-1}G\) is open and the set \(\{U \in K\} = U^{-1}K\) is closed, it therefore follows from the large deviation property that

\[
\limsup_n \frac{1}{n} \log Q_{\varphi,n} \left( \{U \in G\} \right) \leq \inf_{\mu \in \mathcal{P}_S(\Sigma^n)} \left( (I(\mu) - F_{\varphi}(\mu)) - \inf_{\nu \in \mathcal{P}_S(\Sigma^n)} (I(\nu) - F_{\varphi}(\nu)) \right),
\]

\[
\limsup_n \frac{1}{n} \log Q_{\varphi,n} \left( \{U \in K\} \right) \geq -\inf_{\mu \in \mathcal{P}_S(\Sigma^n)} \left( (I(\mu) - F_{\varphi}(\mu)) - \inf_{\nu \in \mathcal{P}_S(\Sigma^n)} (I(\nu) - F_{\varphi}(\nu)) \right). \tag{7.14}
\]

Combining (7.12), (7.13) and (7.14) now yields

\[
\limsup_n \frac{1}{n} \log \sum_{|i|=n} s_i \geq \log N + \limsup_n \frac{1}{n} \log Q_{\varphi,n} \left( \{U \in G\} \right)
\]

\[
+ \limsup_n \frac{1}{n} \log \int \exp \left( nF_{\varphi} \right) dP_n
\]

\[
\geq \log N - \inf_{\mu \in \mathcal{P}_S(\Sigma^n)} \left( (I(\mu) - F_{\varphi}(\mu)) - \inf_{\nu \in \mathcal{P}_S(\Sigma^n)} (I(\nu) - F_{\varphi}(\nu)) \right)
\]

\[
= \log N + \sup_{\mu \in \mathcal{P}_S(\Sigma^n)} (F_{\varphi}(\mu) - I(\mu))
\]

\[
= \sup_{\mu \in \mathcal{P}_S(\Sigma^n)} \left( \int \varphi \, d\mu + h(\mu) \right).
\]

This completes the proof of inequality (7.10). Inequality (7.11) is proved similarly. \(\square\)

We now turn towards the second main result in this section, namely, Theorem 7.5 showing that the multifractal pressure and the modified multifractal pressure are (almost) comparable. We first prove a small auxiliary lemma.

**Lemma 7.4.** Let \((X, d)\) be a metric space and let \(U: \mathcal{P}(\Sigma^n) \to X\) be continuous with respect to the weak topology. Let \(C\) be a subset of \(X\) and \(r > 0\).

1. For all \(n\), we have

\[
\left\{ u \in \Sigma^n \left| UL_u[u] \subseteq C \right. \right\} \subseteq \left\{ u \in \Sigma^n \left| UM_u[u] \subseteq C \right. \right\}.
\]

2. There is a positive integer \(N_r\) such that if \(n \geq N_r\), \(u \in \Sigma^n\) and \(k, l \in \Sigma^n\), then we have \(d\left( UL_n(uk), UL_n(ul) \right) \leq r\).
(3) There is a positive integer $N_r$ such that if $n \geq N_r$, then we have
\[
\left\{ u \in \Sigma^n \mid UM_n[u] \subseteq C \right\} \subseteq \left\{ u \in \Sigma^n \mid UL_u[u] \subseteq B(C, r) \right\}.
\]

**Proof**

(1) This statement follows immediately from the fact that if $u \in \Sigma^n$, then $M_n[u] = \{ L_n[u] \}$.

(2) Fix $\gamma \in (0, 1)$ and let $d_\gamma$ denote the metric on $\Sigma^N$ introduced in Section 6.3. For a function $f : \Sigma^N \to \mathbb{R}$, we let $\text{Lip}(f)$ denote the Lipschitz constant of $f$ with respect to the metric $d_\gamma$, i.e.,
\[
\text{Lip}(f) = \sup_{i,j \in \Sigma^N, i \neq j} \frac{|f(i) - f(j)|}{d_\gamma(i, j)}.
\]

and we therefore conclude from (7.15) that $\delta > 0$ such that all measures $\mu, \nu \in \mathcal{P}(\Sigma^N)$ satisfy the following implication:
\[
L(\mu, \nu) < \delta \implies d(U\mu, U\nu) < r.
\]

Next, choose a positive integer $N_r$ such that
\[
\frac{1}{N_r(1 - \gamma)} < \delta.
\]

If $n \geq N_r$, $u \in \Sigma^n$ and $k, l \in \Sigma^N$, then it follows from (7.16) that
\[
L(L_n(uk), L_n(ul)) = \sup_{f: \Sigma^N \to \mathbb{R}, \text{Lip}(f) \leq 1} \left\{ f d(L_n(uk)) - f d(L_n(ul)) \right\}
\]
\[
= \sup_{f: \Sigma^N \to \mathbb{R}, \text{Lip}(f) \leq 1} \left| \frac{1}{n} \sum_{i=0}^{n-1} f(S^i(uk)) - \frac{1}{n} \sum_{i=0}^{n-1} f(S^i(ul)) \right|
\]
\[
\leq \sup_{f: \Sigma^N \to \mathbb{R}, \text{Lip}(f) \leq 1} \frac{1}{n} \sum_{i=0}^{n-1} |f(S^i(uk)) - f(S^i(ul))|
\]
\[
\leq \frac{1}{n} \sum_{i=0}^{n-1} d_{\Sigma^N}(S^i(uk), S^i(ul))
\]
\[
= \frac{1}{n} \sum_{i=0}^{n-1} \frac{1}{\mathcal{N}[S^i(uk) \wedge S^i(ul)]}
\]
\[
\leq \frac{1}{N_r} \sum_{i=0}^{n-1} \frac{1}{N^{n-i}}
\]
\[
< \frac{1}{N_r(1 - \gamma)}
\]
\[
< \delta,
\]
and we therefore conclude from (7.15) that $d(U L_n(uk), U L_n(ul)) < r$.

(3) It follows from Part (2) that there is a positive integer $N_r$ such that if $n \geq N_r$, $u \in \Sigma^n$ and $k, l \in \Sigma^N$, then $d(U L_n(uk), U L_n(ul)) < r$. We now claim that if $n \geq N_r$, then
\[
\left\{ u \in \Sigma^n \mid UM_n[u] \subseteq C \right\} \subseteq \left\{ u \in \Sigma^n \mid UL_u[u] \subseteq B(C, r) \right\}.
\]
In order to prove this inclusion, we fix $n \geq N_r$ and $u \in \sum^n$ with $UM_n[u] \subseteq C$. We must now prove that $UL_n[u] \subseteq B(C,r)$. Fix $i \in [u]$. Since $i \in [u]$, we can now find a (unique) $k \in \sum^m$ such that $i = uk$, whence

$$\text{dist} (UL_n i, C) \leq d(UL_n i, UL_n u) + \text{dist} (UL_n u, C) = d(UL_n (uk), UL_n (u\overline{u})) + \text{dist} (UL_n \overline{u}, C).$$

(7.17)

However, since $n \geq N_r$ and $u \in \sum^n$, we conclude that $d(UL_n (uk), UL_n (u\overline{u})) < r$. Also, $UL_n \overline{u} = UM_n \overline{u} \in UM_n [u] \subseteq C$, whence $\text{dist}(UL_n \overline{u}, C) = 0$. It therefore follows from (7.17) that

$$\text{dist} (UL_n i, C) = d(UL_n (uk), UL_n (u\overline{u})) + \text{dist} (UL_n \overline{u}, C) < r.$$  

This completes the proof. □

We can now state and prove the second main result in this section, namely, Theorem 7.5.

**Theorem 7.5.** Let $X$ be a metric space and let $U : \mathcal{P}(\sum^m) \to X$ be continuous with respect to the weak topology. Let $C \subseteq X$ be a subset of $X$ and $r > 0$. Fix a continuous function $\varphi : \sum^m \to \mathbb{R}$. Then we have

$$P^l_U (\varphi) \leq \Omega^l_U (\varphi) \leq P^l_{B(C,r)} (\varphi),$$

$$P_C (\varphi) \leq \Omega^l_C (\varphi) \leq P^l_{B(C,r)} (\varphi).$$

**Proof**

This follows immediately from Lemma 7.4. □

8. **Proof of Theorem 5.3**

The purpose of this section is to prove Theorem 5.3.

**Lemma 8.1.** Let $X$ be a metric space and let $F : X \to \mathbb{R}$ be an upper semi-continuous function. Let $K_1, K_2, \ldots \subseteq X$ be non-empty compact subsets of $X$ with $K_1 \supseteq K_2 \supseteq \ldots$. Then

$$\inf_{n} \sup_{x \in K_n} F(x) = \sup_{x \in \bigcap_{n} K_n} F(x).$$

**Proof**

First note that it is clear that $\inf_{n} \sup_{x \in K_n} F(x) \geq \sup_{x \in \bigcap_{n} K_n} F(x)$. We will now prove the reverse inequality, namely, $\inf_{n} \sup_{x \in K_n} F(x) \leq \sup_{x \in \bigcap_{n} K_n} F(x)$. Let $\varepsilon > 0$. For each $n$, we can choose $x_n \in K_n$ such that $F(x_n) \geq \sup_{x \in K_n} F(x) - \varepsilon$. Next, since $K_n$ is compact for all $n$ and $K_1 \supseteq K_2 \supseteq \ldots$, we can find a subsequence $(x_{n_k})_k$ and a point $x_0 \in \bigcap_{n} K_n$ such that $x_{n_k} \to x_0$. Also, since $K_{n_1} \supseteq K_{n_2} \supseteq \ldots$, we conclude that $\sup_{x \in K_{n_1}} F(x) \geq \sup_{x \in K_{n_2}} F(x) \geq \ldots$, whence $\inf_{k} \sup_{x \in K_{n_k}} F(x) = \lim_{k} \sup_{x \in K_{n_k}} F(x)$. This implies that $\inf_{n} \sup_{x \in K_n} F(x) \leq \inf_{k} \sup_{x \in K_{n_k}} F(x) = \lim_{k} \sup_{x \in K_{n_k}} F(x) \leq \lim_{k} \sup_{x \in K_{n_k}} F(x_{n_k}) + \varepsilon$. However, since $x_{n_k} \to x_0$, we deduce from the upper semi-continuity of the function $F$, that $\lim_{k} \sup_{x \in K_{n_k}} F(x_{n_k}) \leq F(x_0)$. Consequently $\inf_{n} \sup_{x \in K_n} F(x) \leq \lim_{k} \sup_{x \in K_{n_k}} F(x_{n_k}) + \varepsilon \leq F(x_0) + \varepsilon \leq \sup_{x \in \bigcap_{n} K_n} F(x) + \varepsilon$. Finally, letting $\varepsilon \searrow 0$ gives the desired result. □

We can now prove Theorem 5.3.
Proof of Theorem 5.3
(1) We must prove the following two inequalities, namely,

\[
\sup_{\mu \in \mathcal{P}(\Sigma^N)} \left( h(\mu) + \int \varphi \, d\mu \right) \leq \inf_{r > 0} \frac{P_{B(C,r)}^U}{U_{\mu \in \mathcal{C}}} \left( \varphi \right),
\]

(8.1)

\[
\inf_{r > 0} \frac{P_{B(C,r)}^U}{U_{\mu \in \mathcal{C}}} \left( \varphi \right) \leq \sup_{\mu \in \mathcal{P}(\Sigma^N)} \left( h(\mu) + \int \varphi \, d\mu \right).
\]

(8.2)

Proof of (8.1). Since \( B(C,r) \) is open with \( \mathcal{C} \subseteq B(C,r) \), we conclude from Theorem 7.3 that

\[
\sup_{\mu \in \mathcal{P}(\Sigma^N)} \left( h(\mu) + \int \varphi \, d\mu \right) \leq \sup_{\mu \in \mathcal{P}(\Sigma^N)} \left( h(\mu) + \int \varphi \, d\mu \right) \leq \frac{Q_{B(C,r)}^U}{U_{\mu \in \mathcal{C}}} \left( \varphi \right).
\]

(8.3)

Taking infimum over all \( r > 0 \) in (8.3) gives

\[
\sup_{\mu \in \mathcal{P}(\Sigma^N)} \left( h(\mu) + \int \varphi \, d\mu \right) \leq \inf_{r > 0} \frac{Q_{B(C,r)}^U}{U_{\mu \in \mathcal{C}}} \left( \varphi \right).
\]

(8.4)

Next, we note that it follows from Theorem 7.5 that \( \frac{Q_{B(C,r)}^U}{U_{\mu \in \mathcal{C}}} \left( \varphi \right) \leq \frac{P_{B(C,r)}^U}{U_{B(C,r)}} \left( \varphi \right) \). Combining this inequality with and (8.4) and using the fact that \( B(B(C,r), r) \subseteq B(C, 2r) \), we now conclude that

\[
\sup_{\mu \in \mathcal{P}(\Sigma^N)} \left( h(\mu) + \int \varphi \, d\mu \right) \leq \inf_{r > 0} \frac{Q_{B(C,r)}^U}{U_{\mu \in \mathcal{C}}} \left( \varphi \right) \leq \inf_{r > 0} \frac{P_{B(C,2r)}^U}{U_{\mu \in \mathcal{C}}} \left( \varphi \right) \leq \inf_{s > 0} \frac{P_{B(C,s)}^U}{U_{\mu \in \mathcal{C}}} \left( \varphi \right).
\]

This completes the proof of inequality (8.1).

Proof of (8.2). Since \( \overline{B(C,r)} \) is closed, we conclude from Theorem 7.3 that

\[
\inf_{r > 0} \frac{P_{B(C,r)}^U}{U_{\mu \in \mathcal{C}}} \left( \varphi \right) \leq \inf_{r > 0} \frac{P_{B(C,r)}^U}{U_{\mu \in \mathcal{C}}} \left( \varphi \right) \leq \inf_{r > 0} \sup_{\mu \in \mathcal{P}(\Sigma^N)} \left( h(\mu) + \int \varphi \, d\mu \right).
\]

Letting \( U_S : \mathcal{P}(\Sigma^N) \to X \) denote the restriction of \( U \) to \( \mathcal{P}(\Sigma^N) \), the above inequality can be written as

\[
\inf_{r > 0} \frac{P_{B(C,r)}^U}{U_{\mu \in \mathcal{C}}} \left( \varphi \right) \leq \inf_{r > 0} \sup_{\mu \in U_S^{-1}(B(C,r))} \left( h(\mu) + \int \varphi \, d\mu \right) = \inf_{n} \sup_{\mu \in U_S^{-1}(B(C,r))} \left( h(\mu) + \int \varphi \, d\mu \right).
\]

(8.5)
Next, note that since \( \overline{B(C, \frac{1}{n})} \) is closed and \( U_S \) is continuous, the set \( U_S^{-1}\overline{B(C, \frac{1}{n})} \) is a closed subset of \( \mathcal{P}_S(\Sigma^n) \). As \( \mathcal{P}_S(\Sigma^n) \) is compact, we therefore deduce that \( U_S^{-1}\overline{B(C, \frac{1}{n})} \) is compact. Also, note that it follows from [Wa] that the entropy map \( h : \mathcal{P}_S(\Sigma^n) \rightarrow \mathbb{R} \) is upper semi-continuous. We conclude from this that the map \( F : \mathcal{P}_S(\Sigma^n) \rightarrow \mathbb{R} \) defined by \( F(\mu) = h(\mu) + \int \varphi \, d\mu \) is upper semi-continuous. Finally, since the sets \( K_n = U_S^{-1}\overline{B(C, \frac{1}{n})} \) are compact with \( K_1 \supseteq K_2 \supseteq K_3 \supseteq \ldots \) and \( F \) is upper semi-continuous, we deduce from Lemma 8.1 that

\[
\inf_{n} \sup_{\mu \in U_S^{-1}\overline{B(C, \frac{1}{n})}} \left( h(\mu) + \int \varphi \, d\mu \right) = \inf_{\mu \in K_n} F(\mu) = \sup_{\mu \in K_n} F(\mu) = \sup_{\mu \in \bigcap_n U_S^{-1}B(C, \frac{1}{n})} \left( h(\mu) + \int \varphi \, d\mu \right). \tag{8.6}
\]

Observe that \( \bigcap_n U_S^{-1}B(C, \frac{1}{n}) \subseteq U_S^{-1}(\bigcap_n \overline{B(C, \frac{1}{n})}) = U_S^{-1}C \), whence

\[
\sup_{\mu \in \bigcap_n U_S^{-1}B(C, \frac{1}{n})} \left( h(\mu) + \int \varphi \, d\mu \right) \leq \sup_{\mu \in \bigcap_n U_S^{-1}C} \left( h(\mu) + \int \varphi \, d\mu \right) = \sup_{\mu \in \mathcal{P}_S(\Sigma^n)} \left( h(\mu) + \int \varphi \, d\mu \right). \tag{8.7}
\]

Finally, combining (8.5), (8.6) and (8.7) gives inequality (8.2).

(2) This part follows immediately from Part (1) and Proposition 5.1. \( \square \)

9. PROOF OF THEOREM 5.5

The purpose of this section is to prove Theorem 5.5. We first prove two small lemmas.

**Lemma 9.1.** Let \( \Delta : \mathcal{P}(\Sigma^n) \rightarrow \mathbb{R} \) be continuous with \( \Delta(\mu) \neq 0 \) for all \( \mu \in \mathcal{P}(\Sigma^n) \). The either \( \Delta < 0 \) or \( \Delta > 0 \).

**Proof**
Assume, in order to reach a contradiction, that there are \( \mu_-, \mu_+ \in \mathcal{P}(\Sigma^n) \) such that \( \Delta(\mu_-) < 0 \) and \( \Delta(\mu_+) > 0 \). For \( t \in [0, 1] \), let \( \mu_t = t \mu_- + (1 - t) \mu_+ \in \mathcal{P}(\Sigma^n) \) and define \( f : [0, 1] \rightarrow \mathbb{R} \) by \( f(t) = \Delta(\mu_t) \). The function \( f \) is clearly continuous with \( f(0) = \Delta(\mu_-) > 0 \) and \( f(1) = \Delta(\mu_+) < 0 \), and we therefore conclude from the intermediate value theorem that there is a number \( t_0 \in (0, 1) \) such that \( \Delta(\mu_{t_0}) = f(t_0) = 0 \). However, this clearly contradicts the fact that \( \Delta(\mu) \neq 0 \) for all \( \mu \in \mathcal{P}(\Sigma^n) \). \( \square \)

**Lemma 9.2.** Let \( X \) be a normed vector space. Let \( \Gamma : \mathcal{P}(\Sigma^n) \rightarrow X \) be continuous and affine and let \( \Delta : \mathcal{P}(\Sigma^n) \rightarrow \mathbb{R} \) be continuous and affine with \( \Delta(\mu) \neq 0 \) for all \( \mu \in \mathcal{P}(\Sigma^n) \). Define \( U : \mathcal{P}(\Sigma^n) \rightarrow X \) by \( U = \frac{1}{X} \). Let \( C \) be a closed and convex subset of \( X \) and assume that

\[
C \cap U\left( \mathcal{P}(\Sigma^n) \right) \neq \emptyset.
\]

Then

\[
\sup_{\mu \in \mathcal{P}(\Sigma^n)} \left( h(\mu) + \int \varphi \, d\mu \right) = \sup_{\mu \in \mathcal{P}(\Sigma^n)} \left( h(\mu) + \int \varphi \, d\mu \right) \quad \text{for all} \quad U \mu \in C.
\]
Proof
For brevity define $F : \mathcal{P}_S(\Sigma^N) \rightarrow \mathbb{R}$ by $F(\mu) = h(\mu) + \int \varphi \, d\mu$. It clearly suffices to show that
\[
\sup_{\mu \in \mathcal{P}_S(\Sigma^N)} F(\mu) \leq \sup_{\mu \in \mathcal{P}_S(\Sigma^N)} F(\mu).
\]
(9.1)

We will now prove inequality (9.1). Write $s = \sup_{\mu \in \mathcal{P}_S(\Sigma^N), \nu \in C} F(\mu)$. Fix $\varepsilon > 0$. It follows from the definition of $s$ that we can choose $\lambda \in \mathcal{P}_S(\Sigma^N)$ with $U \lambda \in C$ and $F(\lambda) > s - \varepsilon$. Also, since $\tilde{C} \cap U\left\{ \mathcal{P}_S(\Sigma^N) \right\} \neq \emptyset$, we can find $\nu \in \mathcal{P}_S(\Sigma^N)$, with $U \nu \in \tilde{C}$. For $t \in (0,1)$ we now define $\gamma_t \in \mathcal{P}_S(\Sigma^N)$ by $\gamma_t = t\nu + (1-t)\lambda$. Next, we prove the following three claims.

Claim 1. For all $t \in (0,1)$, we have $U \gamma_t \in \tilde{C}$.

Proof of Claim 1. Fix $t \in (0,1)$. Write $a = \frac{t \Delta(\nu)}{\Delta(\nu) + (1-t)\Delta(\lambda)}$ and $b = \frac{(1-t)\Delta(\lambda)}{\Delta(\nu) + (1-t)\Delta(\lambda)}$. We now make a few observations. We first observe that it follows from Lemma 9.1 that either $\Delta < 0$ or $\Delta > 0$. This clearly implies that $a, b \in (0,1)$. Next, we note that $U \gamma_t = \frac{\Gamma(\nu + (1-t)\lambda)}{\Gamma(\nu) + (1-t)\Gamma(\lambda)} = \frac{t \Delta(\nu)}{\Delta(\nu) + (1-t)\Delta(\lambda)} \nu + \frac{(1-t)\Delta(\lambda)}{\Delta(\nu) + (1-t)\Delta(\lambda)} \lambda = aU \nu + bU \lambda$. We can now prove that $U \gamma_t \in \tilde{C}$. Indeed, since $a, b \in (0,1)$ with $a + b = 1$ and $U \lambda \in C$ and $U \nu \in \tilde{C}$, we conclude from [Co, p. 102, Proposition 1.11] that $U \gamma_t = aU \nu + bU \lambda \in \tilde{C}$. This completes the proof of Claim 1.

Claim 2. There is $t_0 \in (0,1)$ such that $F(\gamma_{t_0}) > s - \varepsilon$.

Proof of Claim 2. Since the entropy function $h : \mathcal{P}_S(\Sigma^N)$ is affine (see [Wa]), we conclude that $F$ is affine, and so $F(\gamma_t) = F(\nu + (1-t)\lambda) = tF(\nu) + (1-t)F(\lambda) \rightarrow F(\lambda) > s - \varepsilon$. This implies that there is $t_0 \in (0,1)$ with $F(\gamma_{t_0}) > s - \varepsilon$. This completes the proof of Claim 2.

Claim 3. There is $\pi \in \mathcal{P}_S(\Sigma^N)$ with $U \pi \in \tilde{C}$ such that $F(\pi) > s - \varepsilon$.

Proof of Claim 3. It follows from Claim 2 that there is $t_0 \in (0,1)$ such that $F(\gamma_{t_0}) > s - \varepsilon$ and Claim 1 implies that $U \gamma_{t_0} \in C$. We now put $\pi = \gamma_{t_0}$. This completes the proof of Claim 3.

We can now prove inequality (9.1). It follows from Claim 3 that there is $\pi \in \mathcal{P}_S(\Sigma^N)$ with $U \pi \in \tilde{C}$ such that $F(\pi) > s - \varepsilon$, whence $s - \varepsilon < F(\pi) \leq \sup_{\mu \in \mathcal{P}_S(\Sigma^N), \nu \in C} F(\mu)$. Finally, letting $\varepsilon \searrow 0$ gives $s \leq \sup_{\mu \in \mathcal{P}_S(\Sigma^N), \nu \in C} F(\mu)$. \hfill $\square$

We can now prove Theorem 5.5.

Proof of Theorem 5.5
In view of Lemma 9.2, it suffices to prove the following two inequalities, namely,
\[
\sup_{\mu \in \mathcal{P}_S(\Sigma^N)} \left( h(\mu) + \int \varphi \, d\mu \right) \leq P^U_C(\varphi), \tag{9.2}
\]

\[
\sup_{\mu \in \mathcal{P}_S(\Sigma^N)} \left( h(\mu) + \int \varphi \, d\mu \right) \leq P^U_C(\varphi). \tag{9.3}
\]

Proof of inequality (9.2). For $r > 0$, let $G_r = \{x \in C \mid \text{dist}(x, X \setminus C) > r\}$, and note that $G_r$ is open with $B(G_r, \rho) \subseteq C$ for all $0 < \rho < r$. We therefore conclude from Theorem 7.3 and Theorem
Taking supremum over all $r > 0$ in (9.4) yields
\[
\mathcal{P}_{C}^{U}(\varphi) \geq \sup_{r>0} \sup_{\mu \in \mathcal{P}_{S}(\Sigma^{N}) \cup \mathcal{C}} \left( h(\mu) + \int \varphi \, d\mu \right). \tag{9.5}
\]

However, it is easily seen that $\bigcup_{r>0} G_{r} \supseteq C$, whence $\bigcup_{r>0} U_{S}^{-1} G_{r} = U_{S}^{-1}(\bigcup_{r>0} G_{r}) \supseteq U_{S}^{-1} C$. We conclude from this inclusion and inequality (9.5) that
\[
\mathcal{P}_{C}^{U}(\varphi) \geq \sup_{\mu \in U_{S}^{-1} C} \left( h(\mu) + \int \varphi \, d\mu \right) = \sup_{\mu \in \mathcal{P}_{S}(\Sigma^{N}) \cup \mathcal{C}} \left( h(\mu) + \int \varphi \, d\mu \right). \tag{9.5}
\]

This proves inequality (9.2).

Proof of inequality (9.3). Since $C$ is closed we immediately conclude from Theorem 7.3 and Theorem 7.5 that
\[
\mathcal{P}_{C}^{U}(\varphi) \leq Q_{C}^{U}(\varphi) \leq \sup_{\mu \in \mathcal{P}_{S}(\Sigma^{N}) \cup \mathcal{C}} \left( h(\mu) + \int \varphi \, d\mu \right). \tag{9.5}
\]

This proves inequality (9.3). \hfill \Box

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