A Comparative Study of Stochastic Volatility Models

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Abstract

The correlated stochastic volatility models constitute a natural extension of the Black and Scholes-Merton framework: here the volatility is not a constant, but a stochastic process correlated with the price log-return one. At present, several stochastic volatility models are discussed in the literature, differing in the dynamics attached to the volatility. The aim of the present work is to compare the most recent results about three popular models: the Vasicek, Heston and exponential Ornstein-Uhlenbeck models. We analyzed for each of them the theoretical results known in the literature (volatility and return distribution, higher-order moments and different-time correlations) in order to test their predictive effectiveness on the outcomes of original numerical simulations, paying particular attention to their ability to reproduce empirical statistical properties of prices. The numerical results demonstrate that these models can be implemented maintaining all their features, especially in view of financial applications like market risk management or option pricing. In order to critically compare the models, we also perform an empirical analysis of financial time series from the Italian stock market, showing the exponential Ornstein-Uhlenbeck model’s ability to capture the stylized facts of volatility and log-return probability distributions.

Key words: Econophysics; Stochastic volatility models; Numerical simulations; Stylized facts
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1 Introduction

It is well documented by many statistical studies \[1,2\] that financial markets exhibit a very complex dynamics. In parallel with this empirical research, various theoretical models, based on non-Gaussian dynamics for the time evolution of price returns, have been proposed in the literature to cope with the non-trivial stylized facts observed in real markets. Among them, the stochastic volatility models (SV) \[3,4,5,6\] have received particular attention because of their analytical tractability and parsimonious use of free parameters. Since much work has been devoted to the derivation of (semi)-analytical results, the aim of the present paper is mainly to test and critically compare the most recent theoretical results by means of detailed numerical simulations, as well as to provide a further contribution to the empirical analysis of SV models already existing in the literature.

2 Theoretical approach: the models

The idea behind SV models is that the volatility $\sigma$, a constant in the Black and Scholes-Merton framework, is itself a stochastic process. In general one can define

$$\sigma(t) \equiv f(Y(t)),$$

where $Y(t)$ is a generic driving process. Given for the underlying price $S(t)$ the well-known Geometric Brownian Motion dynamics, it is convenient to make use of the zero-mean log-return $X(t)$:

$$X(t) \equiv \ln \left[ \frac{S(t)}{S_0} \right] - \mu t + \frac{1}{2} \int_t^{t_0} \sigma^2(t') \, dt',$$

(1)

whose Stochastic Differential Equation (SDE) reads

$$dX(t) = \sigma(t) \, dW_1(t).$$

(2)

Therefore, the 2-dimensional volatility-return process can be written as

$$\begin{cases} dX(t) = f(Y(t)) \, dW_1(t) , \quad X(0) = 0 \\ dY(t) = \alpha (m - Y(t)) \, dt + g(Y(t)) \, dW_2(t) , \quad Y(0) = Y_0 \end{cases}.$$  

(3)

Namely, $Y$ is taken as a mean reverting process, i.e. the deterministic term on the r.h.s of the second equation given in (3) is responsible for reverting the expectation value of $Y$ to the asymptotic value $m$ with relaxation time $1/\alpha$. The mean-reverting character of the SV models reflects the economic idea of a “normal level” of volatility, towards which an efficient market in healthy conditions tends.
Table 1
Models of volatility.

| Authors | $f(Y)$ | $Y$ process | $Y$ Stochastic Differential Equation |
|---------|--------|-------------|-------------------------------------|
| Vasicek | $Y$    | Mean-reverting OU | $dY(t) = \alpha(m - Y(t)) \, dt + k \, dW_2$ |
| Heston  | $\sqrt{Y}$ | CIR | $dY(t) = \alpha(m - Y(t)) \, dt + k \, \sqrt{Y} \, dW_2$ |
| exp-OU  | $e^Y$  | Mean-reverting OU | $dY(t) = \alpha Y(t) \, dt + k \, dW_2$ |

The easiest way to incorporate in the models the correlation between volatility and returns, that is the familiar leverage effect, is to postulate that the two Wiener processes $W_1, W_2$ are correlated by a coefficient $\rho$

$$dW_2(t) = \rho dW_1(t) + \sqrt{1 - \rho^2} dZ(t) \quad ,$$

where $Z(t)$ is a Wiener process independent of $W_1$. For this reason, SV models are given the attribute ‘correlated’. Eq. (4) is obtained recalling that each Wiener process must satisfy $\langle dW^2 \rangle = dt$.

At present several SV models are discussed in the literature, differing in the dynamics attached to $\sigma$, namely in the choice operated for $f$ and $g$ in Eq. (3).

In this paper, we decided to take into consideration the Vasicek, Heston and exponential Ornstein-Uhlenbeck (exp-OU) models, three of the most popular within the family of SV models. They are listed in Tab. 1. It’s worth mentioning that for each of them the volatility distribution can be obtained analytically, since its SDE features a single Wiener process. The same does not hold for the return distribution, which has to be worked out starting from the Fokker-Planck equation for the two-dimensional volatility-return process of Eq. (3).

In Tab. 2 we summarize the basic theoretical characteristics of each model, as obtained in [7,8,9]. The main feature in the analysis of SV models is that the predicted log-return probability distribution functions (PDF) can be expressed only by means of their characteristic functions $\varphi_X(\omega, t)$, defined as

$$\varphi_X(\omega, t) = \int d\sigma_0 \varphi_X(\omega, t|\sigma_0)p_{st}(\sigma_0)$$

$$\quad = \int d\sigma_0 \int dx \, e^{i\omega x} p_X(x, t|\sigma_0)p_{st}(\sigma_0) \quad .$$

The first integral is obtained under the important hypothesis of considering the initial/final volatility at a normal level. Among the models considered, Eq. (5) can be inverted analytically only in the case of the exp-OU model, leading to a closed-form expression for the return distribution [9].

Moreover, a good SV model must account not only for the volatility and return PDFs, but also must give realistic predictions for leverage effect and volatility autocorrelation observed in financial markets. These can be attained
Table 2: Theoretical features of SV models.

|                        | Vasicek          | Heston           | exp-OU            |
|------------------------|------------------|------------------|-------------------|
| Volatility PDF         | Normal           | Gamma            | Log-normal        |
| Log-return PDF         | $\varphi_X(\omega, t)$ | $\varphi_X(\omega, t)$ | $\varphi_X(\omega, t)$ |
|                        | non-inv          | non-inv          | invertible        |
| $\mathcal{L}(\tau)$   | $\rho e^{-\alpha\tau} H(\tau)$ | ?               | $\rho e^{-k^2\tau} H(\tau)$ |
| $\mathcal{C}(\tau)$   | $\sim e^{-\alpha\tau}$ | $e^{-\alpha\tau}$ | $\frac{\exp[4\beta e^{-\alpha\tau}]-1}{3e^{4\beta}-1}$ |
|                        | 1 time scale     | 1 time scale     | 2 time scales     |

by defining the respective statistical coefficients

$$
\mathcal{L}(\tau) \doteq \frac{E[\sigma(t+\tau)^2dX(t)]}{E[\sigma(t)^2]^2} \approx \frac{E[dX(t+\tau)^2dX(t)]}{E[dX(t)^2]^2},
$$

(6)

as in Ref. [10], and the analogous

$$
\mathcal{C}(\tau) \doteq \frac{\langle \sigma(t)^2\sigma(t+\tau)^2 \rangle - \langle \sigma(t)^2 \rangle^2}{\text{Var} \langle \sigma(t)^2 \rangle} \\
\approx \frac{\langle dX(t)^2dX(t+\tau)^2 \rangle - \langle dX(t)^2 \rangle^2}{\langle dX(t)^4 \rangle - \langle dX(t)^2 \rangle^2}.
$$

(7)

The correlations at different times between the Wiener processes allow to compute the expressions appearing in Tab. 2, where $\beta = k^2/2\alpha$. It is worth mentioning that in the case of the leverage coefficient, $\mathcal{L}(\tau)$ is null for $\tau < 0$ ($H$ is the Heaviside step function), thus respecting the observed facts. As for the volatility autocorrelation it must be noticed that the exp-OU model yields, in a wholly natural way, two main time constants, thus resulting the most realistic of the three.

3 Numerical results

The theoretical results were tested \textit{ab initio} by means of original numerical simulations of the models, whose SDEs were discretized following a standard Euler-Maruyama method. As a rule of thumb, to have a sufficiently accurate simulation of the return-volatility paths the time step $\Delta t$ must be significantly shorter than the mean-reversion time $1/\alpha$. The model parameters used in the simulated SDEs were taken to assume values comparable with those appearing in the literature.

The produced paths were used to generate Monte Carlo populations whose distributions (or distribution parameters) were graphically compared with the
analytical results regarding the volatility and log-return processes. We tested
the time evolution of the mean and variance values of the volatility distribu-
tions, the shape of the same distributions for several time instants, and the
log-return distributions at different times. The simulated paths were evolved
over time intervals ranging from a few days to about one financial year. The
most delicate step in the procedure was indeed the mere graphical contrast,
since it requested to invert the analytical characteristic functions of log-return
distributions in the Vasicek and Heston models. Such operation was done
with the help of the built-in Fast Fourier Transform functions of ROOT and
MATHEMATICA®. The shape of the leverage and volatility autocorrelation
coefficients versus the time delay τ (see Eqs. 6 and 7) was also obtained. The
expectation values appearing in the correlation functions were calculated on
a single, very long simulated return series, corresponding to \(\sim\)100 years of
trading, as widely used in the financial practice. The entire analysis and more
technical details can be found in Ref. [11].

Figure 1. Comparison between simulated return PDFs (dots) and theoretical results
(lines). Left panel: Heston model simulated on a 20-day time horizon. Right panel:
exp-OU model simulated at various time horizons vs the analytical expression of [9]

In Fig. [1][2] we show some examples of the analysis carried out: all the numer-
ical outcomes agree very well with the predictions of Tab. [2] for each model,
showing in the same time that these are correct and, conversely, that the
models can be effectively simulated even with a quite simple strategy. These
conclusions hold for both log-return PDFs (Fig. [1]) and the different-time cor-
relation functions between log-return and volatility, i.e. the leverage effect,
and between volatility and itself (Fig. [2] left and right, respectively).

4 Empirical analysis

We also performed an empirical analysis of financial data in order to crit-
ically compare the models and to establish whether they can be successful
in predicting the stylized facts of real markets. The time series used, freely
Figure 2. Comparison between simulated different-time correlations for exp-OU model and their analytical forms. Left panel: leverage effect analytical expression compared with the numerical leverage function. Right panel: volatility autocorrelation function compared with the corresponding analytical form.

downloaded from Yahoo Web Site[1], are collections of daily closing prices of the Italian assets Bulgari SpA, Brembo and Fiat SpA from January 2000 to May 2007. Here, we present in detail the analysis of Fiat SpA data; the entire study performed on Italian shares can be found in Ref. [12].

Fig. 3 shows a comparison between the theoretical distributions displayed in Tab. 2 and the empirical daily volatility for Fiat SpA, evaluated by means of the proxy described in Ref. [2] as absolute daily returns. It’s worth mentioning that a similar analysis is also performed in Ref. [13]. The parameter values of the fitted curves are obtained according to a multidimensional minimization procedure based on the maximum likelihood approach. From Fig. 3 the best agreement between empirical data and the theory clearly emerges for the Log-Normal distribution predicted in the exp-OU framework, whereas the Normal and the Gamma densities tend to underestimate the large distribution’s tail, as already remarked in Ref. [2].

Figure 3. Fit to empirical daily volatility PDF of Fiat SpA.

1 http://finance.yahoo.com/
In the light of this result we propose a comparison between historical log-return probability density and the analytical formula derived by Masoliver and Perelló (MP) in the exp-OU framework [9]. Figure 4 (left panel) compares daily log-returns with the MP theory, together with the Normal and the fat-tailed Student-$t$ PDFs [11][14]. To solve the optimization problem and find the best parameter values required for the fit, we implement a numerical algorithm based on the MINUIT program of the CERN library. In particular, for the MP function we perform a multidimensional fit over four free parameters, finding out values in good agreement with those quoted in the literature [9]. From Fig. 4 (left panel) it emerges that the Student-$t$ and MP curves are in good agreement with the empirical return distribution for both the central body and the tails of the histogram, while the Normal distribution fails to reproduce the data. Moreover, it’s also quite evident that Student-$t$ better captures the extreme events of the distribution, due to its strong leptokurtic nature.

Figure 4. Comparison between empirical returns and the MP theory [9] for Fiat SpA. Left panel: fit of daily returns (histogram) on MP curve in comparison with the Normal and the Student-$t$ PDFs. Right panel: returns (dots) for different time horizons, shifted each other by one decade, vs the MP prediction (line).

In the right panel of Fig. 4 we show the log-return distributions for several time horizons (points) in comparison with the MP theory (solid line). It’s worth mentioning that all the theoretical curves appearing in the figure are generated by changing in the analytical formula only the value of the temporal parameter according to the time lag under analysis, while for the other parameters we use the same values evaluated from the fit of daily data. In this way, we can directly compare empirical data and theoretical predictions: the overall agreement is very good. The success of the MP theory emerges also noting that when the time lag increases, the left tail of the empirical distributions becomes fatter and the analytical solution tends to increase the absolute value of its skewness, becoming more asymmetric and similar to the data shape.
5 Conclusions

For all of the considered SV models we proved an almost perfect convergence between theoretical results and numerical simulations, of particular interest in view of financial applications like risk management and option pricing. Among them, exp-OU has been found to be the most successful in predicting the empirical volatility distribution. The theoretical analysis of the model yields also a double time scale in the volatility autocorrelation, a well-known fact in real financial data. The model also fits the log-return distribution quite well, even if a better agreement with the PDF tails for infra-week data would require to substitute the Wiener noise in the model SDEs with a non-Gaussian one, therefore leaving the Black-Scholes-Merton framework.

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