Kleiner’s theorem for unitary representations of posets

Yurii Samoilenko
Institute of Mathematics, Tereschenkivska 3, Kyiv, Ukraine.

Kostyantyn Yusenko
Department of Mathematics, University of São Paulo, Brazil.

Abstract
A subspace representation of a poset $S = \{s_1, \ldots, s_t\}$ is given by a system $(V; V_1, \ldots, V_t)$ consisting of a vector space $V$ and its subspaces $V_i$ such that $V_i \subseteq V_j$ if $s_i \prec s_j$. For each real-valued vector $\chi = (\chi_1, \ldots, \chi_t)$ with positive components, we define a unitary $\chi$-representation of $S$ as a system $(U; U_1, \ldots, U_t)$ that consists of a unitary space $U$ and its subspaces $U_i$ such that $U_i \subseteq U_j$ if $s_i \prec s_j$ and satisfies $\chi_1 P_1 + \cdots + \chi_t P_t = 1$, in which $P_i$ is the orthogonal projection onto $U_i$.

We prove that $S$ has a finite number of unitarily nonequivalent indecomposable $\chi$-representations for each weight $\chi$ if and only if $S$ has a finite number of nonequivalent indecomposable subspace representations; that is, if and only if $S$ contains any of Kleiner’s critical posets.

Keywords: Representations of partially ordered sets, Representation-finite type, Kleiner’s theorem
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1. Introduction
Kleiner [6] described all partially ordered sets (posets) with finite number of nonequivalent indecomposable representations. We extend his description to unitary representations of posets.

*Corresponding author
Email addresses: yurii.sam@imath.kiev.ua (Yurii Samoilenko),
kay.math@gmail.com (Kostyantyn Yusenko)

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The notion of poset representations was introduced by Nazarova and Roiter [11] (see also [2, 14]). A \textit{matrix representation} of a finite poset \( S = \{s_1, \ldots, s_t\} \) over a field \( F \) is a block matrix \( \mathcal{A} = [A_1|\ldots|A_t] \) over \( F \). Two representations \( \mathcal{A} = [A_1|\ldots|A_t] \) and \( \mathcal{B} = [B_1|\ldots|B_t] \) are \textit{equivalent} if \( \mathcal{A} \) can be reduced to \( \mathcal{B} \) by elementary row transformations, elementary column transformations within \( A_i \), and additions of linear combinations of columns of \( A_i \) to columns of \( A_j \) if \( s_i \prec s_j \). The \textit{direct sum} of \( \mathcal{A} \) and \( \mathcal{B} \) is the representation

\[
\mathcal{A} \oplus \mathcal{B} := \begin{bmatrix}
A_1 & 0 & A_2 & 0 & \ldots & A_t & 0 \\
0 & B_1 & 0 & B_2 & \ldots & 0 & B_t
\end{bmatrix}.
\]

A representation is called \textit{indecomposable} if it is not equivalent to a direct sum of two representations. It is sufficient to classify only indecomposable representations since each representation is equivalent to a direct sum of indecomposable representations, uniquely determined up to isomorphism of summands.

Kleiner [6] (see also [2, Theorem 5.1] and [14, Theorem 10.1]) proved that a poset \( S \) has only a finite number of nonequivalent indecomposable representations if and only if it does not contain a full poset whose Hasse diagram is one of the form

\[
\begin{array}{cccccc}
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{array}
\]

(1)

An equivalent definition of poset representations can be given in terms of subspaces. A \textit{subspace representation} of \( S = \{s_1, \ldots, s_t\} \) is a tuple \( \mathcal{V} = (V; V_1, \ldots, V_t) \), in which \( V \) is a vector space over \( F \) and \( V_1, \ldots, V_t \) are its subspaces such that \( V_i \subseteq V_j \) if \( s_i \prec s_j \) (i.e., each representation is a homomorphism from \( S \) to the poset of all subspaces of \( V \)). Two subspace representations \( \mathcal{V} = (V; V_1, \ldots, V_t) \) and \( \mathcal{W} = (W; W_1, \ldots, W_t) \) are \textit{equivalent} if there exists a linear bijection \( g : V \to W \) such that \( g(V_i) = W_i \) for all \( i \). For each subspace representation \( \mathcal{V} = (V; V_1, \ldots, V_t) \), one can construct a matrix representation \( \mathcal{A} = [A_1|\ldots|A_t] \) in such a way that (i) for each \( i \) the
columns of all $A_j$ with $s_j \preceq s_i$ generate the subspace $V_i$; and (ii) two subspace representations are equivalent if and only if the corresponding matrix representations are equivalent; see [14, Chapter 3].

From now on, all representations that we consider are over the field $\mathbb{C}$ of complex numbers. By a unitary representation of dimension $d$, we mean a subspace representation $U = (U; U_1, \ldots, U_t)$ in which $U$ is a unitary space of dimension $d$. Two unitary representations $U = (U; U_1, \ldots, U_t)$ and $V = (V; V_1, \ldots, V_t)$ of a poset $S$ are unitarily equivalent if there exists a unitary bijection $\varphi : U \to V$ such that $\varphi(U_i) = V_i$ for all $i$. The orthogonal sum of unitary representations $U$ and $V$ is the unitary representation

$$U \perp V := (U \perp V; U_1 \perp V_1, \ldots, U_t \perp V_t),$$

in which $U \perp V$ denotes the orthogonal sum of $U$ and $V$. A unitary representation is called orthogonally indecomposable if it is not equivalent to an orthogonal sum of two unitary representations.

Note that the problem of classifying unitary representations is hopeless even for the poset $S = \{s_1, s_2, s_3 \mid s_1 \prec s_2\}$ since by [10, Theorem 4] it contains the problem of classifying an operator on a unitary space, and hence it contains the problem of classifying any system of operators on unitary spaces [10, 13]. The classification becomes possible for a broader class of posets if we impose additional conditions on unitary representations.

We denote the orthogonal projection onto a subspace $M \subset U$ by $P_M$ and the set of positive real numbers by $\mathbb{R}_+$. We say that a unitary representation $U = (U; U_1, \ldots, U_t)$ is a representation of weight $\chi = (\chi_1, \ldots, \chi_t) \in \mathbb{R}_+^t$ (or $\chi$-representation) if

$$\chi_1 P_{U_1} + \cdots + \chi_t P_{U_t} = 1;$$

such relations appear in many areas of mathematics, see for example [1, 7, 9, 16, 17] and references therein.

Our goal is to prove that Kleiner’s theorem holds for $\chi$-representations too:

**Theorem 1.** The following conditions are equivalent for each finite poset $S$ with $t$ elements:

(i) For each $\chi \in \mathbb{R}_+^t$, $S$ has only a finite number of indecomposable unitarily nonequivalent $\chi$-representations;

(ii) For each $\chi \in \mathbb{R}_+^t$ and $d \in \mathbb{N}$, $S$ has only a finite number of indecomposable unitarily nonequivalent $\chi$-representations of dimension $d$;
(iii) \( S \) does not contain a full poset whose Hasse diagram is one of the form (I).

2. Preliminaries

In what follows we suppose that the elements of a poset \( S \) are numbered from 1 to \(|S|\). A poset is called primitive and is denoted by \((t_1,\ldots,t_s)\) if it is the disjoint (cardinal) sum of linearly ordered sets of orders \( t_i \). The diagrams (I) and corresponding posets are called critical. The poset which corresponds to the last diagram in the list (I) is denoted by \((N,4)\). To simplify the notation we denote a subspace representation \((V;V_1,\ldots,V_t)\) of \( S \) by \((V;V_i)_{i\in S}\). The similar notation will be used for unitary representations and weights.

A subspace representation \( \mathcal{V} = (V;V_i)_{i\in S} \) is called schurian if all its endomorphisms are trivial; that is, the ring \( \text{End}(\mathcal{V}) := \{ g \in M_{\dim V}(\mathbb{F}) \mid g(V_i) \subseteq V_i, \ i \in S \} \) is isomorphic to \( \mathbb{F} \). Any schurian representation is indecomposable.

Any unitary representation \( \mathcal{U} = (U;U_i)_{i\in P} \) can be viewed as a subspace representation; the forgetful map is denoted by \( F \). If \( \mathcal{U} \) is an indecomposable \( \chi \)-representation, then \( F(\mathcal{U}) \) is schurian (see [9, Theorem 1]).

Lemma 2. Let \( P_i, Q_i \in M_n(\mathbb{C}), \ i = 1,\ldots,m \) be orthogonal projections such that
\[
\chi_1P_1 + \cdots + \chi_mP_m = \chi_1Q_1 + \cdots + \chi_mQ_m
\]
for \((\chi_1,\ldots,\chi_m)\) with positive real \( \chi_i \). Let there exist a diagonal matrix \( D = \text{diag}(r_1,\ldots,r_n) \) with positive components such that \( P_iDQ_i = DQ_i \) for all \( i \). Then \( r_1 = \cdots = r_n \) and \( P_i = Q_i \) for all \( i \).

Proof. Write \( P_i = [p_{i,k,l}^{(i)}], \ Q_i = [q_{i,k,l}^{(i)}], \ P_iDQ_i = [t_{i,k,l}^{(i)}] \), where \( t_{i,k,l}^{(i)} = \sum_{j=1}^n r_j p_{k,j}^{(i)} q_{l,j}^{(i)} \). Without losing generality, we may assume that \( r_1 = \max\{r_1,\ldots,r_n\} \). Since \( D \sum_{i=1}^m \chi_iQ_i = \sum_{i=1}^m \chi_iP_iDQ_i \), we have
\[
\sum_{i=1}^m \chi_iq_{1,1}^{(i)} = \sum_{i=1}^m \chi_i \sum_{j=1}^n r_j p_{1,j}^{(i)} q_{1,j}^{(i)} \leq r_1 \sum_{i=1}^m \chi_i \sum_{j=1}^n p_{1,j}^{(i)} q_{1,j}^{(i)}.
\]

For
\[
x := [\sqrt{\chi_1P_{1,1}^{(1)}}, \ldots, \sqrt{\chi_1P_{1,n}^{(1)}}, \ldots, \sqrt{\chi_mP_{1,1}^{(m)}}, \ldots, \sqrt{\chi_mP_{1,n}^{(m)}}]^T \in \mathbb{C}^{nm},
\]
\[
y := [\sqrt{\chi_1q_{1,1}^{(1)}}, \ldots, \sqrt{\chi_1q_{1,n}^{(1)}}, \ldots, \sqrt{\chi_mq_{1,1}^{(m)}}, \ldots, \sqrt{\chi_mq_{1,n}^{(m)}}]^T \in \mathbb{C}^{nm},
\]

(without \( P_iDQ_i = DQ_i \).
Lemma 2 ensures that

\[ \psi \vdash \chi. \]

Then

\[ \psi \vdash \chi. \]

Hence \((\psi \vdash \chi\psi)\). Let \(g \vdash \chi\psi\). We prove the converse statement.

Let \(g \vdash \chi\psi\). If there exists an invertible \(F\) such that \(F\psi = \chi\psi\), then \(\psi = \chi\psi\). Otherwise, if \(\psi \neq \chi\psi\), then \(g\psi \neq \chi\psi\).

Proof. If \(\U = (U; U_i)_{i \in S}\) and \(\U' = (U'; U'_i)_{i \in S}\) are unitarily equivalent if and only if the corresponding subspaces representations \(F(\U)\) and \(F(\U')\) are equivalent.

Theorem 3. Two \(\chi\)-representations \(\U = (U; U_i)_{i \in S}\) and \(\U' = (U'; U'_i)_{i \in S}\) are unitarily equivalent if and only if the corresponding subspaces representations \(F(\U)\) and \(F(\U')\) are equivalent.

Proof. If \(\U\) is unitarily equivalent to \(\U'\), then \(F(\U)\) is equivalent to \(F(\U')\).

Let us prove the converse statement. \(F(\U)\) is equivalent to \(F(\U')\) if and only if there exists an invertible \(g : U \rightarrow U'\) such that

\[ g^{-1}P_{U_i}gP_{U_i} = P_{U_i}, \quad gP_{U_i}g^{-1}P_{U'_i} = P_{U'_i}, \quad i \in S. \]

Let \(g = \varphi \psi D \psi^*\) be the polar decomposition of \(g\), where \(\varphi : U \rightarrow U'\) and \(\psi : U \rightarrow U'\) are unitary maps and \(D\) is a positively defined diagonal operator. Then

\[ (\psi D^{-1} \psi^* \varphi)P_{U'_i}(\varphi \psi D \psi^*)P_{U_i} = P_{U_i}, \quad i \in S. \]

Hence \((\psi^* \varphi P_{U'_i} \varphi \psi)D(\psi^* P_{U'_i} \psi) = D(\psi^* P_{U'_i} \psi)\) for all \(i\). Since \(\U\) and \(\U'\) are \(\chi\)-representations,

\[ \sum_{i \in S} \chi_i(\psi^* \varphi P_{U'_i} \varphi \psi) = \chi_0 I, \quad \sum_{i \in S} \chi_i(\psi^* P_{U'_i} \psi) = \chi_0 I. \]

Lemma 2 ensures that \(\psi^* \varphi P_{U'_i} \varphi \psi = \psi^* P_{U_i} \psi\) for all \(i\). Therefore, \(\varphi^* P_{U'_i} \varphi = P_{U_i}\) for all \(i\), and so \(\U\) is unitarily equivalent to \(\U'\). 

\(\square\)
Remark 4. By similar argumentation, one can show that $\chi$-representation $U$ is orthogonally indecomposable if and only $F(U)$ is indecomposable. The connection between usual and orthoscalar representations of quivers was established [9, Theorem 1] in the same way as in Theorem 3.

A representation $V = (V; V_i)_{i \in S}$ of weight $\chi = (\chi_i)_{i \in S}$ is called $\chi$-stable if $\sum_{i \in S} \chi_i \dim V_i = \dim V$ and
\[
\sum_{i \in S} \chi_i \dim (V_i \cap M) < \dim M
\]
for any proper subspace $0 \neq M \subset V$.

**Lemma 5.** If $U = (U; U_i)_{i \in S}$ is an indecomposable $\chi$-representation, then $F(U)$ is $\chi$-stable.

**Proof.** Equating the traces of both sides in (2), we obtain $\sum_{i \in S} \chi_i \dim U_i = \dim U$. If $M$ is any proper subspace of $U$, then $\sum_{i \in S} \chi_i P_U P_M = P_M$. Equating the traces of both sides in the last equality, we get
\[
\sum_{i \in S} \chi_i \text{tr}(P_{U_i} P_M) = \dim M.
\]
By [4, Theorem 2], $\text{tr}(P_{M_1 \cap M_2}) \leq \text{tr}(P_{M_1} P_{M_2})$ for each two subspaces $M_1$ and $M_2$, and so
\[
\sum_{i \in S} \chi_i \text{tr}(P_{U_i \cap M}) \leq \sum_{i \in S} \chi_i \text{tr}(P_{U_i} P_M) = \dim M.
\]
It remains to prove that the last inequality is strict. Indeed, assume that $\text{tr}(P_{U_i \cap M}) = \text{tr}(P_{U_i} P_M)$ for all $i$. Then each $P_{U_i}$ commutes with $P_M$. Hence the subspace $M$ is invariant with respect to the projections $P_{U_i}$ and the representation $U$ is decomposable. This contradicts the assumption. \qed

The converse statement to Lemma 5 also holds: if a representation $V = (V; V_i)_{i \in S}$ is $\chi$-stable, then one can choose a scalar product in $V$ in such a way that $V$ becomes a $\chi$-representation; see [3, Theorem 3.5]. Using results from [3, 7, 16], one can prove the following theorem.

**Theorem 6.** An indecomposable unitary representation $U$ is a $\chi$-representation if and only if the corresponding subspace representation $F(U)$ is $\chi$-stable.
3. Proof of Theorem 1

The implication (i) ⇒ (ii) is trivial.

(iii) ⇒ (i). Assume that the Hasse diagram of $S$ does not contain any of critical diagrams (1). If $S$ has an infinite number of indecomposable unitarily nonequivalent $\chi$-representations for some weight $\chi$, then by Theorem 3 it has an infinite number of nonequivalent indecomposable subspace representations. By Kleiner’s theorem, $S$ contains a critical diagram; a contradiction.

(ii) ⇒ (iii). We say that a poset $S$ is unitary representation-infinite if there exist $d \in \mathbb{N}$ and $\chi^S \in R_+^{[S]}$ such that $S$ has an infinite number of indecomposable unitarily nonequivalent $\chi^S$-representations of dimension $d$. Our aim is to prove that critical posets are unitary representation-infinite.

One can show that critical primitive posets are unitary representation-infinite using [1, 8, 12]. Namely, there exists a correspondence between the $\chi$-representations of a given poset $S$ and the representations of a certain *-algebra $A_{\Gamma,\omega}$ associated with a star-shaped graph $\Gamma$, which is determined by the Hasse diagram of $S$, and the parameter $\omega$ is determined by the weight $\chi$. If $\Gamma$ is an extended Dynkin graph (which corresponds to some primitive critical $S$), then one can choose the parameter $\omega$ such that $A_{\Gamma,\omega}$ has an infinite number of unitarily nonequivalent irreducible representations. The complete description of such representations was given in [1, 8, 12] (see also Remark 11). But we use another method that handles both primitive and non-primitive cases.

Denote by $e^{(n)}_i$ (or $e_i$ if no confusion can arise) the $n$-dimensional vector in which the $i$-th coordinate is 1 and the others are 0. Denote by $e_{i_1\ldots i_k}$ the vector $e_{i_1} + \cdots + e_{i_k}$ and by $\langle x_1,\ldots,x_m \rangle$ the vector space spanned by $x_1,\ldots,x_m \in \mathbb{C}^n$.

For each critical poset $S$, we define a family of its subspace representations $\mathcal{V}_\lambda(S)$ that depend on a complex parameter $\lambda \in \mathbb{C}$.

- If $S = (1,1,1,1)$, then $\mathcal{V}_\lambda(S)$ consists of the space $\mathbb{C}^2$ and its subspaces
  \[ \langle e_1 \rangle, \quad \langle e_2 \rangle, \quad \langle e_1 + e_2 \rangle, \quad \langle e_1 + \lambda e_2 \rangle \]

- If $S = (2,2,2)$, then $\mathcal{V}_\lambda(S)$ consists of the space $\mathbb{C}^3$ and its subspaces
  \[ \langle e_{123}, e_1 + \lambda e_3 \rangle, \quad \langle e_1, e_2 \rangle, \quad \langle e_2, e_3 \rangle \]
  \[ \langle e_{123} \rangle, \quad \langle e_1 \rangle, \quad \langle e_3 \rangle \]
• If $S = (1, 3, 3)$, then $\mathcal{V}_\lambda(S)$ consists of the space $\mathbb{C}^4$ and its subspaces

$$
\begin{align*}
\langle e_1, e_4, e_2 + \lambda e_3 \rangle &\quad \langle e_1, e_2, e_3 \rangle \\
\langle e_1, e_4 \rangle &\quad \langle e_2, e_3 \rangle \\
\langle e_{123}, e_{24} \rangle &\quad \langle e_4 \rangle
\end{align*}
$$

• If $S = (1, 2, 5)$, then $\mathcal{V}_\lambda(S)$ consists of the space $\mathbb{C}^6$ and its subspaces

$$
\begin{align*}
\langle e_1, e_2, e_3, e_4, e_5 + \lambda e_6 \rangle &\quad \langle e_1, e_2, e_3, e_4 \rangle \\
\langle e_1, e_2, e_3, e_4 \rangle &\quad \langle e_2, e_3, e_4 \rangle \\
\langle e_1, e_2, e_5, e_6 \rangle &\quad \langle e_3, e_4 \rangle \\
\langle e_{123}, e_{245}, e_{16} \rangle &\quad \langle e_5, e_6 \rangle \\
\langle e_5, e_6 \rangle &\quad \langle e_4 \rangle
\end{align*}
$$

• If $S = (N, 4)$, then $\mathcal{V}_\lambda(S)$ consists of the space $\mathbb{C}^5$ and its subspaces

$$
\begin{align*}
\langle e_1, e_2, e_3, e_4 \rangle &\quad \langle e_2, e_3, e_4 \rangle \\
\langle e_2, e_3, e_4 \rangle &\quad \langle e_3, e_4 \rangle \\
\langle e_{235}, e_{134}, e_5, e_3 + \lambda e_4 \rangle &\quad \langle e_1, e_2, e_5 \rangle \\
\langle e_{235}, e_{134} \rangle &\quad \langle e_5 \rangle \\
\langle e_5 \rangle &\quad \langle e_4 \rangle
\end{align*}
$$

Denote by $V_\lambda^S$ the only subspace from $\mathcal{V}_\lambda(S)$ that depends on the parameter $\lambda$ and denote by $a$ the element from $S$ that corresponds to $V_\lambda^S$. Deleting $V_\lambda^S$ from $\mathcal{V}_\lambda(S)$, we obtain the subspace representation $\mathcal{V}(S_a) = (V^S_i; V^S_i)_{i \in S_a}$ of primitive poset $S_a := S \setminus \{a\}$.

**Proposition 7.** $\mathcal{V}_\lambda(S)$ is not equivalent to $\mathcal{V}_\mu(S)$ if $\lambda \neq \mu$ for each critical poset $S$. All subspace representation $\mathcal{V}(S_a)$ are schurian.
Proof. This proposition is proved by straightforward computations. □

Let $S$ be a critical poset. The poset $S_a$ is primitive and does not contain any of the critical posets, its subspace representation $V(S_a)$ is schurian. By [3, Proposition 3.1], there exists a weight which we denote by $\chi^a$, such that $V(S_a)$ is $\chi^a$-stable. Write

$$R := \min \left\{ \dim M - \sum_{i \in S_a} \chi^a_i \dim (V^S_i \cap M) \mid M \text{ is a proper subspace of } V^S \right\}.$$ 

The subspace representation $V(S_a)$ is $\chi^a$-stable, hence $R > 0$. Let $\varepsilon$ be such that $R > \varepsilon > 0$. Write $T := 1 + (R - \varepsilon) \dim V^S_\lambda (\dim V^S)^{-1}$ and

$$\chi^S = (\chi^i)^{i \in S}, \quad \chi^S_i := \begin{cases} \chi^a_i \cdot T^{-1}, & \text{if } i \in S_a, \\ (R - \varepsilon) \cdot T^{-1}, & \text{if } i = a. \end{cases}$$

**Proposition 8.** The subspace representations $V_\lambda(S)$ are $\chi^S$-stable for all $\lambda$ and $S$.

**Proof.** Note that

$$\sum_{i \in S} \chi^S_i \dim V^S_i = T^{-1} \sum_{i \in S} \chi^a_i \dim V^S_i + \chi^S_a \dim V^S_\lambda = T^{-1} \dim V^S + (1 - T^{-1}) \dim V^S = \dim V^S.$$ 

Let $M$ be any proper subspace of $V^S$. Then

$$\sum_{i \in S} \chi^S_i \dim (V^S_i \cap M) = T^{-1} \sum_{i \in S} \chi^a_i \dim (V^S_i \cap M) + \chi^S_a \dim (V^S_\lambda \cap M) \leq T^{-1} (\dim M (1 - R) + (R - \varepsilon) \dim (V^S_\lambda \cap M))$$

$$< T^{-1} \dim M (1 - \varepsilon) < \dim M.$$ 

Hence $V_\lambda(S)$ is $\chi^S$-stable. □

**Proposition 9.** Critical posets are unitary representation-infinite.

**Proof.** By Proposition 7 and Proposition 8 any critical poset $S$ has an infinite number of nonequivalent $\chi^S$-stable subspace representations. By Theorem 6, $S$ has an infinite number of indecomposable unitarily nonequivalent $\chi^S$-representations. □
Proposition 10. If a poset $S$ contains a critical poset (as a full subposet), then $S$ is unitary representation-infinite.

Proof. Suppose that $S$ contains a critical poset $S_c$. By Proposition 9, there exists a weight $\chi^c$ such that $S_c$ has an infinite number of indecomposable unitarily nonequivalent $\chi^c$-representations of dimension $d$. Define the following subset of $S$:

$$S_{\text{max}} := \{a \in S \mid b \prec a \text{ for some } b \in S_c\}.$$  

For each $\chi^c$-representation $U = (U; U_i)_{i \in S_c}$, define the unitary representation $U' = (U; U'_i)_{i \in S}$ of $S$ as follows:

$$U'_i := \begin{cases} 0, & \text{if } i \notin S_{\text{max}} \cup S_c, \\ U_i, & \text{if } i \in S_c, \\ U, & \text{if } i \in S_{\text{max}}. \end{cases}$$

It is easy to check that $U'$ is $\chi'$-representation, in which $\chi' = (\chi'_i)_{i \in S}$ is defined by

$$\chi'_i := \begin{cases} \chi^c_i \cdot (1 + |S_{\text{max}}|)^{-1}, & \text{if } i \in S_c, \\ (1 + |S_{\text{max}}|)^{-1}, & \text{otherwise}. \end{cases}$$

Hence $S$ is unitary representation-infinite. \hfill \Box

The implication (ii) $\Rightarrow$ (iii) follows from Proposition 10. This finishes the proof of Theorem 1.

Remark 11. Define the following weights:

$$\begin{align*}
\chi^{(1,1,1,1)} &= \frac{1}{2}(1,1,1,1), \\
\chi^{(2,2,2)} &= \frac{1}{3}(1,1,1,1,1,1), \\
\chi^{(1,3,3)} &= \frac{1}{4}(2,1,1,1,1,1,1), \\
\chi^{(1,2,5)} &= \frac{1}{6}(3,2,2,1,1,1,1,1), \\
\chi^{(N,4)} &= \frac{1}{5}(2,1,1,2,1,1,1,1). 
\end{align*}$$
Each weight $\chi^S$ obtained from the minimal imaginary root of the quadratic form related to a critical poset $S$. We checked (describing all possible subdimension vectors) that the representations $V_\lambda(S)$ are $\chi^S$-stable for any $\lambda \in \mathbb{C} \setminus \{0, 1\}$. Hence they give rise to an infinite family of nonequivalent $\chi^S$-representations. For primitive $S$ one can obtain the precise description of projections for such representations using the results from [1, 8, 12]. The description in the case $S = (N, 4)$ is unknown.

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