A Projective Model Structure on Pro Simplicial Sheaves, and the Relative Étale Homotopy Type

Ilan Barnea  Tomer M. Schlank

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Abstract

In this work we shall introduce a much weaker and easy to define structure then a model category, which we call a "weak fibration category". Our main theorem says that a weak fibration category can be "completed" into a full model category structure on it’s Pro category, provided the Pro category satisfies a certain two out of three property. This generalizes a theorem of Isaksen [Isa]. Applying this result to the weak fibration category of simplicial sheaves over a Grothendieck site, where the weak equivalences and the fibrations are local in the sense of Jardine [Jar], gives a new model structure on the category of pro simplicial sheaves. Using this model structure we define a pro space associated to a topos, as a result of applying a derived functor. We show that our construction lifts Artin and Mazur’s Étale homotopy type [AM], in the relevant special case. Our definition extends naturally to a relative notion, namely, a pro object associated to a map of topoi. This relative notion lifts the relative étale homotopy type that was used in [HaSc] for the study of obstructions to the existence of rational points. Thus we embed the results of [HaSc] in a suitable model structure. This relative notion also enables to generalize these homotopical obstructions from fields to general base schemas and general maps of topoi.

1 Introduction: Weak Fibration Categories

Model categories, introduced in [Qui], provides a very general context in which it is possible to set up the basic machinery of homotopy theory. However, the structure of a model category is not always available. The structure of a model category is determined by the classes of weak equivalences and fibrations (since the class of cofibrations is then determined by a left lifting property). There are situations in which there is a natural definition of weak equivalences and fibrations, however, the resulting structure is not a model category. A notable example is the category of simplicial sheaves over a Grothendieck site, where the weak equivalences and the fibrations are local, in the sense of Jardine.

In this paper we shall introduce a much weaker and easy to define structure then a model category, which we call a "weak fibration category". Our main
Theorem (Theorem 4.4) says that a weak fibration category can be "completed" into a full model category structure on its Pro category, provided the Pro category satisfies a certain two out of three property.

The notion of a weak fibration category is closely related to K. S. Brown’s notion of a "category of fibrant objects" ([Bro]), and Baues’s notion of a "fibration category" ([Bau]). These notions were introduced as a more flexible structure than a model category, in which to do abstract homotopy theory.

Moreover, we translate many useful definitions and constructions from the theory of model categories, s.t. a right Quillen functor, a simplicial structure, right properness etc., into the language of weak fibration categories. We then show that these notions give rise to the corresponding notions in the theory of model categories, when going to the pro category.

We now give the exact definition:

**Definition 1.1.** Let $\mathcal{C}$ be a category with finite limits, and let $\mathcal{M} \subseteq \mathcal{C}$ be a subcategory. We say that $\mathcal{M}$ is closed under pullbacks if whenever we have a pullback square:

$$
\begin{array}{ccc}
A & \rightarrow & B \\
\downarrow^g & & \downarrow^f \\
C & \rightarrow & D
\end{array}
$$

s.t. $f$ is in $\mathcal{M}$, then $g$ is in $\mathcal{M}$.

**Definition 1.2.** A weak fibration category is a category $\mathcal{C}$ with an additional structure of two subcategories:

$\mathcal{F}, \mathcal{W} \subseteq \mathcal{C}$

that contain all the isomorphisms, such that the following conditions are satisfied:

1. $\mathcal{C}$ has all finite limits.
2. $\mathcal{W}$ has the 2 out of 3 property.
3. The subcategories $\mathcal{F}$ and $\mathcal{F} \cap \mathcal{W}$ are closed under pullbacks.
4. Every map $A \rightarrow B$ in $\mathcal{C}$ can be factored as $A \xrightarrow{f} C \xrightarrow{g} B$, where $f$ is in $\mathcal{W}$ and $g$ is in $\mathcal{F}$. We denote this property by $Mor(\mathcal{C}) = \mathcal{F} \circ \mathcal{W}$.

The maps in $\mathcal{F}$ are called fibrations, and the maps in $\mathcal{W}$ are called weak equivalences.

**Remark 1.3.** Note that we do not require the factorizations in Definition 1.2 (4) to be functorial.

**Example 1.** Let $(\mathcal{M}, \mathcal{W}, \mathcal{C}, \mathcal{F})$ be a model category. Then $(\mathcal{M}, \mathcal{W}, \mathcal{F})$ is a weak fibration category.
A triple \((\mathcal{C}, \mathcal{W}, \text{Co}f)\) is called a weak cofibration category, if \((\mathcal{C}^{\text{op}}, \mathcal{W}^{\text{op}}, \text{Co}f^{\text{op}})\) is a weak fibration category (Definition 1.2).

**Example 2.** Let \(\mathcal{S}\) denote the category of simplicial sets, and let \(\mathcal{S}_f\) denote its full subcategory of compact objects. It is not hard to check that, with the usual notions of weak equivalence and cofibrations, \(\mathcal{S}_f\) becomes a weak cofibration category. Note however, that \(\mathcal{S}_f\) is not a model category. For example factorizations into an acyclic cofibration followed by a fibration, do not generally exist.

**Example 3.** Let \(\Gamma\) be a profinite group, and let \(\mathcal{C}\) be the category of simplicial sets with a continuous \(\Gamma\) action. Consider \(\mathcal{C}\) as a weak fibration category, where the weak equivalences and the fibrations are induced from those in simplicial sets. If \(\Gamma\) is finite, then \(\mathcal{C}\) is a model category. However, if \(\Gamma\) is infinite, it is not hard to check that if \(\mathcal{C}\) was a model category, every cofibrant object would have a free action of \(\Gamma\). But this is impossible, since all the stabilizers of this object must be of finite index (since the action of \(\Gamma\) is continuous).

**Example 4.** More generally, take \(\text{SSh}(\mathcal{C})\) to be the category of simplicial sheaves on a Grothendieck site \(\mathcal{C}\), where both weak equivalences and fibrations are local as in [Jar] (see Section 9). This is the main example of a weak fibration category we will consider in this paper.

In order to describe our main result more explicitly, we need some preliminaries from the theory of pro categories. This is explained in more detail in Section 2. Let \(\mathcal{C}\) be a category. Then there is a natural fully faithful functor \(\mathcal{C} \to \text{Pro}(\mathcal{C})\). By abuse of notation we will consider objects and morphisms of \(\mathcal{C}\), also as objects and morphisms of \(\text{Pro}(\mathcal{C})\), using this functor. If \(M\) is any class of morphisms in \(\mathcal{C}\), there is a naturally corresponding class of morphisms in \(\text{Pro}(\mathcal{C})\), called \(L^\sim M\). These are maps in \(\text{Pro}(\mathcal{C})\), that are isomorphic to a natural transformation, which is level-wise in \(M\). We can now state our main result:

**Theorem 1.4.** Let \((\mathcal{C}, \mathcal{W}, \mathcal{F})\) be a small weak fibration category. Assume that \(L^\sim \mathcal{W} \subseteq \text{Pro}(\mathcal{C})\) satisfies the 2-out-of-3 property. Then there exist a model category structure on \(\text{Pro}(\mathcal{C})\) s.t. the weak equivalences are \(L^\sim \mathcal{W}\). Moreover, this model category is fibrantly generated, with \(\mathcal{F}\) as the set of generating fibrations, and \(\mathcal{F} \cap \mathcal{W}\) as the set of generating acyclic fibrations.

In Theorem 1.4 we will give a more explicit description of the fibrations in this model structure, but this requires some more definitions.

The idea behind Theorem 1.4 is, that the main reason why \((\mathcal{C}, \mathcal{W}, \mathcal{F})\) is not a model category, is the absence of factorizations of maps \(A \to B\) in \(\mathcal{C}\), into a cofibration followed by an acyclic fibration: \(A \to C \to B\). If \(\mathcal{C}\) was a model category, such a factorization would be a (homotopy) initial object in the category of all factorizations of \(A \to B\) into a general map followed by an acyclic fibration. If \(\mathcal{C}\) is only a weak fibration category, such an initial object does not necessarily exist. In this case, we take \(\mathcal{C}\) to be the entire inverse system of such factorizations, thus resulting in a pro object. However, the category of
factorizations is not necessarily directed. An important part of the proof is to replace it with a related category that is (see Proposition 3.1).

Given a model category $\mathcal{C}$, model categories on $\text{Pro}(\mathcal{C})$ where studied by Edwards and Hastings [EH], Isaksen [Isa] and other authors. Here we obtain a model structure on $\text{Pro}(\mathcal{C})$, while assuming a weaker structure on $\mathcal{C}$ itself. In the case where $\mathcal{C}$ is a model category, our model structure is identical to the one described in [EH], [Isa].

Applying Theorem 4.4 to the weak fibration category of simplicial sheaves (see example 1 above), we get a novel model structure on the category $\text{Pro}(\text{SSh}(\mathcal{C}))$, of pro simplicial sheaves. This model structure is moreover simplicial, by Proposition 7.5, and Proposition 9.9. Since every local fibration (and in particular every level wise fibration) is a fibration in this model structure, it can be considered a projective model structure on $\text{Pro}(\text{SSh}(\mathcal{C}))$. We elaborate more on this model structure in Section 9.2.

**Theorem 1.5.** Let $\mathcal{C}$ be a small Grothendieck site, and let $\text{SSh}(\mathcal{C})$ be the category of simplicial sheaves on $\mathcal{C}$. Then there exists a simplicial model category structure on $\text{Pro}(\text{SSh}(\mathcal{C}))$ s.t. the weak equivalences are $Lw^\cong(W)$, where $W$ is the class of local fibrations in $\text{SSh}(\mathcal{C})$.

Moreover, this model category is fibrantly generated, with the local fibrations in $\text{SSh}(\mathcal{C})$ as the set of generating fibrations, and local acyclic fibrations in $\text{SSh}(\mathcal{C})$ as the set of generating acyclic fibrations.

In [Jar2], Jardine considers a different model structure on pro simplicial sheaves, with the same class of weak equivalences. This model structure can be thought of as "injective" (since every level-wise cofibration is a cofibration in this model structure).

**Remark 1.6.** Technically speaking, in order to use Theorem 4.4 we need $\text{SSh}(\mathcal{C})$ to be small, and it is not. This set theoretical issue can be solved using the language of universes, as is done for example in [LVoC]. However, a full treatment of these issues will make the paper more difficult to read, and will not have significant implication on the results and arguments. Thus we choose to ignore these set theoretical issues in this paper.

Using our new model structure on $\text{Pro}(\text{SSh}(\mathcal{C}))$, we obtain naturally a derived functor definition of the étale homotopy type defined by Artin and Mazur in [AM]. We use Propositions 10.1 and 6.3 to show that:

**Theorem 1.7.** Let $X$ locally Noetherian scheme, and let $X_{\text{ét}}$ be its étale topos. Let $\pi_0 : X_{\text{ét}} \rightarrow \text{Set}$ be the functor induced by the functor which sends a scheme to its set of connected scheme-theoretic components. Then prolongation by $\pi_0$:

$$\text{Pro}(\pi_0) : \text{Pro}(X_{\text{ét}}^\Delta^{op}) \rightarrow \text{Pro}(\text{Set}^\Delta^{op}),$$

is a left Quillen functor, relative to our projective model structures.

**Remark 1.8.** In the existing injective model structure of Jardine mentioned above, $\text{Pro}(\pi_0) : \text{Pro}(X_{\text{ét}}^\Delta^{op}) \rightarrow \text{Pro}(\text{Set}^\Delta^{op})$ is not a left Quillen functor.
Theorem 1.7 enables to make the following:

**Definition 1.9.** We define the étale topological realization of $X$, to be:

$$|X|_{\text{et}} := L \text{Pro}(\pi_0)(\ast_{X}) \in \text{Pro}(\Delta^{op}) = \text{Pro}(S),$$

Where $\ast_{X}$ is a terminal object of $X \Delta^{op}$.

The above definition of the étale topological realization, is closely related to Artin and Mazur’s étale homotopy type:

**Theorem 1.10.** Under the natural functor:

$$\text{Ho} : \text{Pro}(S) \to \text{Pro}(\text{Ho}(S)),$$

$\text{Ho}(|X|_{\text{et}})$ is isomorphic to Artin and Mazur’s étale homotopy type.

For the proof of Theorem 1.10, see Lemma 10.3.

In [Fri], Frindlander also lifted the étale homotopy type of Artin and Mazur, from an object in $\text{Pro}(\text{Ho}(S))$ to an object in $\text{Pro}(S)$. He did so by replacing the classical notion of hypercovering by the more involved one of rigid hypercovering. We achieve the same goal, but without appealing to rigid hypercomplexions.

Moreover, our definition extends naturally to a general site (see Definition 10.2). Our notion of topological realization of a topos is also closely related to those considered by Lurie ([Lur], chapter 7) and Toën-Vezzosi ([ToVe]).

The definition of topological realization also extends naturally to a relative notion. Namely, given a morphism of sites: $f : C \to D$, we give a derived functor definition of the topological realization of $f$, which is an object in $\text{Pro}(\text{SSh}(D))$ (see Definition 10.5). The non relative notion is obtained by considering the site morphism $C \to \ast$. A case of special interest is when $f$ is the morphism of étale sites induced by a scheme morphism: $X \to \text{spec}(K)$. In this case the relative topological realization lifts the notion of the relative étale homotopy type $\text{Et}_{/K}(X)$ considered in [HaSc] by Harpaz and the second author, in the context of studying rational points (in a similar way that the topological realization of the étale site of a variety $X$ lifts the étale homotopy type $\text{Et}(X)$).

We use Propositions 10.1 and 6.3, to show that:

**Theorem 1.11.** Let $X/K$ be a locally Noetherian scheme over a field $K$, and let $f^* : X_{\text{et}} \to \text{Spec}(K)_{\text{et}}$ be the induced morphism of étale topoi. Let $f_* : \text{Spec}(K)_{\text{et}} \to X_{\text{et}}$, be the left adjoint to $f^*$. Then $f_*$ has a left adjoint, denoted: $f_! : X_{\text{et}} \to \text{Spec}(K)_{\text{et}}$. Then prolongation by $f_!$:

$$\text{Pro}(f_!) : \text{Pro}(X_{\text{et}}^{\Delta^{op}}) \to \text{Pro}((\text{Spec}(K)_{\text{et}})^{\Delta^{op}}),$$

is a left Quillen functor, relative to our projective model structures.

**Remark 1.12.** Again, in the existing injective model structure of Jardine, $\text{Pro}(f_!) : \text{Pro}(X_{\text{et}}^{\Delta^{op}}) \to \text{Pro}((\text{Spec}(K)_{\text{et}})^{\Delta^{op}})$ is not a left Quillen functor.

Theorem 1.11 enables to make the following:
**Definition 1.13.** We define the relative étale topological realization of \( X/K \), to be:
\[
|X_{\text{et}}|_{\text{Spec}(K)_{\text{et}}} := \mathbb{L} \text{Pro}(f_!)(\ast_{X_{\text{et}}}) \in \text{Pro}((\text{Spec}(K)_{\text{et}})^{\Delta^{op}}),
\]
Where \( \ast_{X_{\text{et}}} \) is a terminal object of \( X_{\text{et}}^{\Delta^{op}} \).

The following theorem relates the above definition of the relative étale topological realization, with the relative étale homotopy type defined in [HaSc].

**Theorem 1.14.** Now consider the natural functor:
\[
\text{Ho} : \text{Pro}((\text{Spec}(K)_{\text{et}})^{\Delta^{op}}) \rightarrow \text{Pro}(\text{Ho}((\text{Spec}(K)_{\text{et}})^{\Delta^{op}})),
\]
Where \( \text{Ho}((\text{Spec}(K)_{\text{et}})^{\Delta^{op}}) \) is defined with respect to strict weak equivalences as in [Goë]. Then \( \text{Ho}(|X_{\text{et}}|_{\text{Spec}(K)_{\text{et}}}) \) is isomorphic to \( \text{Et}/K(X) \) as defined in [HaSc].

Following the path suggested in [HaSc], and by Ambrus Pál in [Pál], the results presented here give a model structure in which homotopy theory can be used to define and study obstructions to the existence of rational points. Further, using our results it is possible to generalize these obstructions to general base schemas and general maps of topoi. We elaborate more on this in sections 10.1 and 10.2.

A central corollary of this paper is the construction of a new "projective" model structure on pro simplicial sheaves. In [Jar2], Jardine considers a different model structure on pro simplicial sheaves, with the same class of weak equivalences. This model structure can be thought of as "injective" (Since every level-wise cofibration is a cofibration in this model structure). We will show in section 11 that the identity functors constitute a Quillen equivalence between these two model structures. As a direct consequence we get a very short and conceptual proof of Verdier’s hypercovering theorem. In the same paper, Jardine states that his main objective is to give a common framework for traditional étale homotopy theory and the homotopy theory of simplicial presheaves, for the purpose of properly comparing these theories. The model structure defined in this paper was defined with the same goal in mind, and has the extra advantage of allowing the derived functor definition for the étale homotopy type, as in Theorem 1.7 and Definition 1.9.

### 1.1 Organization of the paper

We begin in Section 2 with a brief account of the necessary background on pro-categories. In Section 3 we prove a factorization lemma (Proposition 3.1), which will be the main tool in proving the existence of our model structure. This section is the technical heart of the paper. Section 4 contains our main result (Theorem 4.4), concerning the existence of a model structure on \( \text{Pro}(\mathcal{C}) \), when \( \mathcal{C} \) is a weak fibration category. In Section 5 we point out a relation between our main factorization lemma, Proposition 5.1 and the dual of Quillen’s small object argument. In Section 6 we define a natural notion of a morphism between
weak fibration categories, which we call a weak right Quillen functor. In Section 7 we define the notion of a simplicial weak fibration category. In Section 8 we discuss the notion of homotopy in a weak fibration category. This way we can relate more directly our construction to the one defined by Artin and Mazur [AM]. These results will be used later in sections 10 and 11 to connect our theory with the more classical approaches. In Section 9 we consider our main examples, namely, the categories of simplicial sheaves and simplicial presheaves on a Grothendieck site. We show that with the notions of local weak equivalences and local fibrations, they both become pro admissible simplicial weak fibration categories. Using our main theorem we deduce the existence of induced simplicial model structures on their pro categories. In Section 10 we apply the results of the previous two sections, to give a derived functor definition of the étale homotopy type of [AM]. We also generalize this to the topological realization of a general site, as explained before. In Section 11 we compare our ”projective” model structure on pro simplicial presheaves of Section 9.2 with the ”injective” model structure on the same category, that can be deduced from [Lsa], when applied to [Jar].

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2 Preliminaries on Pro-Categories

In this section we bring a short review of the necessary background on pro-categories. Some of the definitions and lemmas given here are slightly non standard. This material can be found in [AM], [EH], and [Lsa]. Many of the ideas in this section (and paper) are influenced by Isaksen’s work on pro categories (see for example [Lsa]).

Definition 2.1. A category $I$ is called cofiltered (or directed) if the following conditions are satisfied:

1. $I$ is non-empty.

2. for every pair of objects $s, t \in I$, there exists an object $u \in I$, together with morphisms $u \to s$ and $u \to t$.

3. for every pair of morphisms $f, g : s \to t$ in $I$, there exists a morphism $h : u \to s$ in $I$, s.t. $f \circ h = g \circ h$. 

7
Thus, a poset $A$ is directed iff $A$ is non-empty, and for every $a, b \in A$, there exists $c \in A$, s.t. $c \geq a, c \geq b$.

We shall use repeatedly the following notion:

**Definition 2.2.** Let $T$ be a partially ordered set, and let $A$ be a subset of $T$. We will say that $A$ is a Reysha of $T$, if $x \in A, y \in T, y < x$, implies: $y \in A$.

**Example 5.** $T$ is a Reysha of $T$. If $t \in T$ is a maximal element, then $T \{t\}$ is a Reysha of $T$. For any $t \in T$: $R_t := T_{\leq t}$ is a Reysha of $T$.

**Definition 2.3.** Let $C$ be a category. The category $C^{\leq}$ has as objects: $\text{Ob}(C) \coprod \infty$, and the morphisms are the morphisms in $C$, together with a unique morphism: $\infty \rightarrow c$, for every $c \in C$.

In particular, if $C = \emptyset$ then $C^{\leq} = \{\infty\}$.

**Lemma 2.4.** A poset $A$ is directed iff for every finite Reysha $R \subset A$ (see Definition 2.2), there exist an element $c \in A$ such that $c \geq r$, for every $r \in R$. A category $C$ is directed iff for every finite poset $R$, and for every functor $F : R \rightarrow C$, there exist $c \in C$, together with compatible morphisms $c \rightarrow F(r)$, for every $r \in R$ (that is, a morphism $\text{Diag}(c) \rightarrow F$ in $C^R$, or equivalently we can extend the functor $F : R \rightarrow T$ to a functor $R^{\leq} \rightarrow C$).

**Proof.** Clear. $\square$

A category is called small if it has only a set of objects and a set of morphisms. A diagram in a category is called cofiltered if its indexing category is so.

**Definition 2.5.** Let $C$ be a category. The category $\text{Pro}(C)$ has as objects all small cofiltered diagrams in $C$ (of arbitrary shape). The morphisms are defined by the formula:

$$\text{Hom}_{\text{Pro}(C)}(X, Y) := \lim_{s \rightarrow t} \text{Colim}_{t} \text{Hom}_{C}(X_t, Y_s).$$

Composition of morphisms is defined in the obvious way.

Thus, if $X : I \rightarrow C, Y : J \rightarrow C$ are objects in $\text{Pro}(C)$, giving a morphism $X \rightarrow Y$ means specifying, for every $s \in J$ a morphism $X_t \rightarrow Y_s$ in $C$, for some $t \in I$. These morphisms should of course satisfy some compatibility condition. In particular, if the indexing categories are equal: $I = J$, then any natural transformation: $X \rightarrow Y$ gives rise to a morphism $X \rightarrow Y$ in $\text{Pro}(C)$.

The word pro-object refers to objects of pro-categories. A simple pro-object is one indexed by the category with one object and one (identity) map. Note that for any category $C$, $\text{Pro}(C)$ contains $C$ as the full subcategory spanned by the simple objects.

Note that if $X : I \rightarrow C, Y : J \rightarrow C$ are objects in $\text{Pro}(C)$, giving a morphism $X \rightarrow Y$ means giving morphisms $X \rightarrow Y_s$ for every $s \in J$, compatible relative to morphisms in $J$, where $Y_s$ is regarded as a simple object in $\text{Pro}(C)$. 

8
If $P$ is a partially ordered set, then we may view $P$ as a category which has a single morphism $u \to v$ iff $u \geq v$. Note that this convention is opposite from the one used by some authors.

**Definition 2.6.** A cofinite poset is a poset $T$ s.t. for every $x \in T$ the set $R_x := \{ z \in T | z \leq x \}$ is finite.

Cofinite directed sets and natural transformations are extremely important in the study of pro categories, largely because of the following lemma:

**Lemma 2.7.** Let $C$ be a category, and let $F$ be any morphism in $\text{Pro}(C)$. Then $F$ is isomorphic, in the category $\text{Ar}(\text{Pro}(C))$ of arrows in $\text{Pro}(C)$, to a morphism that comes from a natural transformation s.t. the indexing category is a cofinite directed set.

**Proof.** Combine [AM], Appendix 3.2, and [EH], Theorem 2.1.6. □

**Definition 2.8.** Let $C$ be a category with finite limits, $M \subseteq \text{Mor}(C)$ a class of morphisms in $C$, $I$ a small category, and $F : X \to Y$ a morphism in $C^I$. Then $F$ will be called:

1. A levelwise $M$-map, if for every $i \in I$: the morphism $X_i \to Y_i$ is in $M$. We will denote this by $F \in Lw(M)$.

2. A special $M$-map, if the following holds:

   (a) The indexing category $I$ is a cofinite poset (see Definition 2.6).

   (b) The natural map $X_t \to Y_t \times \lim_{s \leq t} Y_s \lim_{s \leq t} X_s$ is in $M$, for every $t \in I$.

We will denote this by $F \in Sp(M)$.

**Definition 2.9.** Let $C$ be a category with finite limits, and $M \subseteq \text{Mor}(C)$ a class of morphisms in $C$. Denote by:

1. $R(M)$ the class of morphisms in $C$ that are retracts of morphisms in $M$.

2. $\perp M$ the class of morphisms in $C$ having the left lifting property w.r.t. any morphism in $M$.

3. $M \perp$ the class of morphisms in $C$ having the right lifting property w.r.t. any morphism in $M$.

4. $Lw^=(M)$ the class of morphisms in $\text{Pro}(C)$, that are isomorphic to a morphism that comes from a natural transformation which is a level-wise $M$-map.

5. $Sp^=(M)$ the class of morphisms in $\text{Pro}(C)$, that are isomorphic to a morphism that comes from a natural transformation which is a special $M$-map.
Lemma 2.10. Let $M$ be any class of morphisms in $\mathcal{C}$. Then

$$R(Lw^\sim(M)) = Lw^\sim(M).$$

Proof. Appears in Iskasen ([IsaS], Proposition 12.1).

Lemma 2.11. Let $M$ be any class of morphisms in $\mathcal{C}$. Then:

$$(R(M))^\bot = M^\bot, \quad ^\bot(R(M)) = ^\bot M,$$

$$R(M^\bot) = M^\bot, \quad R(^\bot M) = ^\bot M.$$ 

Proof. Easy diagram chase.

2.1 The Special Levelwise Lemma

In this subsection we bring a proof of Proposition 2.12, which will be used throughout the paper. We assume that $\mathcal{C}$ is a category with finite limits, and $\mathcal{M} \subseteq \mathcal{C}$ is a subcategory that is closed under pullbacks, and contains all the isomorphisms. Let $*$ denote a final object in $\mathcal{C}$.

Proposition 2.12. Let $\mathcal{C}$ be a category with finite limits, and $\mathcal{M} \subseteq \mathcal{C}$ a subcategory that is closed under pullbacks, and contains all the isomorphisms. Let $F : X \to Y$ be a natural transformation between diagrams in $\mathcal{C}$, which is a special $\mathcal{M}$-map. Then $F$ is a levelwise $\mathcal{M}$-map.

Proposition 2.12 appears in [FaIs] (see Lemma 2.3, 5.14), but without a full proof. Since we will use some of the tools also later in the paper, we chose to bring a detailed proof here.

Let $\text{Ar}(\mathcal{C})$ denote the category of arrows in $\mathcal{C}$. We define $\mathcal{M}^{\text{ar}} \subseteq \text{Mor}(\text{Ar}(\mathcal{C}))$, to be the class of morphisms represented by squares:

\[
\begin{array}{ccc}
A & \longrightarrow & B \\
\downarrow & & \downarrow \\
C & \longrightarrow & D,
\end{array}
\]

such that the natural map: $A \to B \times_D C$ is in $\mathcal{M}$.

Lemma 2.13. $\mathcal{M}^{\text{ar}} \subseteq \text{Ar}(\mathcal{C})$ is a subcategory that is closed under pullbacks, and contains all the isomorphisms.

Proof. The fact that $\mathcal{M}^{\text{ar}}$ contains all the isomorphisms is clear. To prove that $\mathcal{M}^{\text{ar}}$ is closed under composition consider the diagram:

\[
\begin{array}{ccc}
A_2 & \longrightarrow & B_2 \\
\downarrow & & \downarrow \\
A_1 & \longrightarrow & B_1 \\
\downarrow & & \downarrow \\
A_0 & \longrightarrow & B_0.
\end{array}
\]
We need to prove that if:

\[ A_1 \to A_0 \times B_0 B_1, \quad A_2 \to A_1 \times B_1 B_2 \]

are in \( \mathcal{M} \), then:

\[ A_2 \to A_0 \times B_0 B_2 \]

is in \( \mathcal{M} \).

Consider the pullback diagram:

\[
\begin{array}{c}
A_1 \times B_1 B_2 \\
\downarrow \\
A_1 = A_1 \times B_1 B_1 \\
\downarrow \\
A_0 \times B_0 B_1
\end{array}
\]

Now we get that \( A_2 \to A_0 \times B_0 B_2 \) is just the composition:

\[ A_2 \xrightarrow{\mathcal{M}} A_1 \times B_1 B_2 \xrightarrow{\mathcal{M}} A_0 \times B_0 B_2. \]

To prove that \( \mathcal{M}_{ar} \) is closed under pullback, consider a map in \( \mathcal{M}_{ar} \):

\[
\begin{array}{c}
A_1 \\
\downarrow \\
A_0
\end{array}
\xrightarrow{B_1}
\begin{array}{c}
B_1 \\
\downarrow \\
B_0
\end{array}
\]

that is, a square in \( \mathcal{C} \), such that \( A_1 \to A_0 \times B_0 B_1 \) is in \( \mathcal{M} \). We would like to pullback this map along the map:

\[
\begin{array}{c}
A_2 \\
\downarrow \\
A_0
\end{array}
\xrightarrow{B_2}
\begin{array}{c}
B_2 \\
\downarrow \\
B_0
\end{array}
\]

and show that the map represented by the square:

\[
\begin{array}{c}
A_2 \times A_0 A_1 \\
\downarrow \\
A_2 \times B_2 B_2
\end{array}
\xrightarrow{\mathcal{M}}
\begin{array}{c}
B_2 \times A_0 B_1 \\
\downarrow \\
B_2
\end{array}
\]

is in \( \mathcal{M}_{ar} \), i.e. that the map:

\[ A_2 \times A_0 A_1 \to A_2 \times B_2 (B_2 \times B_0 B_1) = A_2 \times B_0 B_1, \]

is in \( \mathcal{M} \). Now the claim is proved by considering the pull back square:

\[
\begin{array}{c}
A_2 \times A_0 A_1 \\
\downarrow \\
A_1 = A_0 \times A_0 A_1 \xrightarrow{\mathcal{M}} A_0 \times B_0 B_1.
\end{array}
\]

\[ \square \]
**Definition 2.14.** We say that an object \( X \in \mathcal{C} \) is an \( \mathcal{M} \)-object if the morphism \( X \to * \) is in \( \mathcal{M} \).

**Lemma 2.15.** An object \( A \to B \in \text{Ar}(\mathcal{C}) \) is an \( \mathcal{M}^{op} \)-object iff it is a morphism in \( \mathcal{M} \).

*Proof.* Clear. \( \square \)

**Definition 2.16.** Let \( T \) be a cofinite poset. We call a diagram \( X : T \to \mathcal{C} \), Special \( \mathcal{M} \), if for every \( t \in T \) the map:

\[
X_t = \lim_{s \leq t} X_s \to \lim_{s < t} X_s
\]

is in \( \mathcal{M} \) (or in other words if the natural transformation \( X \to * \) is special \( \mathcal{M} \) (see Definition 2.8).

**Lemma 2.17.** Let \( T \) be a finite partially ordered set, \( X : T \to \mathcal{C} \) a special \( \mathcal{M} \)-diagram, and \( t \in T \) a maximal element. Then the map:

\[
\lim_{s \in T} X_s \to \lim_{s \in T \setminus \{t\}} X_s
\]

is in \( \mathcal{M} \)

*Proof.* consider the pullback square:

\[
\begin{array}{ccc}
\lim_{s \in T} X_s & \to & \lim_{s \in T \setminus \{t\}} X_s \\
\downarrow & & \downarrow \\
\lim_{s \leq t} X_s & \to & \lim_{s < t} X_s.
\end{array}
\]

\( \square \)

**Lemma 2.18.** Let \( T \) be a cofinite partially ordered set, \( X : T \to \mathcal{C} \) a special \( \mathcal{M} \) diagram in \( \mathcal{C} \), and \( A \) a Reysha of \( T \) (see Definition 2.2). then \( X|_A : A \to \mathcal{C} \) is a special \( \mathcal{M} \) diagram.

*Proof.* Clear. \( \square \)

**Lemma 2.19.** Let \( T \) be a finite partially ordered set, \( X : T \to \mathcal{C} \) a special \( \mathcal{M} \)-diagram, and \( A \subseteq B \subseteq T \) any two Reysha’s of \( T \) (see Definition 2.2). Then the map:

\[
\lim_{s \in B} X_t \to \lim_{s \in A} X_t
\]

is in \( \mathcal{M} \).
Proof. We prove the lemma by induction on the size of $B$. The base of the induction ($B = \emptyset$) is clear. Now assume that the lemma holds for $|B| < n$ ($n \geq 1$). Let us prove the lemma for $|B| = n$. If $A = B$ the statement is clear. Otherwise, choose a maximal element $x \in B \setminus A$. We can decompose the map:

$$\lim_{s \in B} X_s \to \lim_{s \in A} X_s$$

into

$$\lim_{s \in B} X_s \to \lim_{s \in B \setminus \{x\}} X_s \to \lim_{s \in A} X_s.$$

The first map belongs to $\mathcal{M}$ by Lemmas 2.17, 2.18, and the second map belongs to $\mathcal{M}$ by the induction hypothesis. Since $\mathcal{M}$ is closed under composition, we have the desired result.

Corollary 2.20. Let $T$ be a finite partially ordered set, and $X : T \to C$ a special $\mathcal{M}$-diagram. Then for every $t \in T$, $X_t$ is an $\mathcal{M}$-object.

Proof. Apply Lemma 2.19 with $B = T \leq t$, $A = \emptyset$.

Proof of Proposition 2.12: Let $T$ be a cofinite directed set, and let $F : X \to Y$ be a morphism in $C_T$, which is a special $\mathcal{M}$-map. Let $I$ denote the category with two objects $0, 1$, and a unique morphism $0 \to 1$. Then $F$ can be regarded as a functor $F : I \to C_T$, or equivalently, as a functor $F : T \to C^I = \text{Ar}(C)$. It is straightforward to check that $F$ (in the first picture) is a special $\mathcal{M}$-map iff $F$ (in the second picture) is a special $\mathcal{M}^{\text{ar}}$-diagram. It follows from Corollary 2.20 (applied to the Reysa $T_{\leq t}$ of $T$, see Lemma 2.18), that for every $t \in T$, $F_t \in \text{Ar}(C)$ is an $\mathcal{M}^{\text{ar}}$-object. It now follows from 2.18 that for every $t \in T$, $F_t : X_t \to Y_t$ is in $\mathcal{M}$.

Corollary 2.21. Let $C$ be a category with finite limits, and $\mathcal{M} \subseteq C$ a subcategory that is closed under pullbacks, and contains all the isomorphisms. Then: $\text{Sp}^{\text{eq}}(\mathcal{M}) \subseteq \text{Lw}^{\text{eq}}(\mathcal{M})$.

2.2 A lifting Lemma

This subsection is devoted to proving the following lemma, which is the main motivation behind Definition 2.9 (5):

Lemma 2.22. Let $M \subseteq \text{Mor}(C)$ be any class of morphisms in $C$. Then: $\perp \text{Sp}^{\text{eq}}(M) = \perp M$.

Remark 2.23. The idea of the proof of Lemma 2.22 appears in [Isa] (see the proof of Lemma 4.11).

Proof. Since $M \subseteq \text{Sp}^{\text{eq}}(M)$, it is clear that $\perp \text{Sp}^{\text{eq}}(M) \subseteq \perp M$. It remains to show that $\perp \text{Sp}^{\text{eq}}(M) \supseteq \perp M$. Let $g \perp M$ and $f \in \text{Sp}^{\text{eq}}(M)$. We need to show that $g \perp f$. Without loss of generality we may assume that $f$ comes from a natural transformation $X \to Y$ with the following properties:
1. The indexing category is a cofinite directed set: \( T \).

2. The natural map \( X_t \to Y_t \times \lim_{s < t} Y_s \lim_{s < t} X_s \) is in \( M \) for every \( t \in T \).

We need to construct a lift in the following diagram:

\[
\begin{array}{ccc}
A & \rightarrow & \{X_t\} \\
\downarrow g & & \downarrow f \\
B & \rightarrow & \{Y_t\}.
\end{array}
\]

Giving a morphism \( B \to \{X_t\} \) means giving morphisms \( B \to X_t \) for every \( t \in T \), compatible relative to morphisms in \( T \), where \( X_t \) is regarded as a simple object in \( \text{Pro}(C) \). Thus, it is enough to construct compatible lifts \( B \to X_t \), in the diagrams:

\[
\begin{array}{ccc}
A & \rightarrow & X_t \\
\downarrow g & & \downarrow f_t \\
B & \rightarrow & Y_t
\end{array}
\]

for every \( t \in T \).

We will do this by induction on \( t \). If \( t \) is a minimal element of \( T \), then such a lift exists since \( g \in \perp M \), and

\[
X_t \to Y_t \times \lim_{s < t} Y_s \lim_{s < t} X_s = Y_t
\]

is in \( M \). Suppose that we have constructed compatible lifts \( B \to X_s \), for every \( s < t \). Let us construct a compatible lift \( B \to X_t \).

We will do this in two stages. First, the compatible lifts \( B \to X_s \), for \( s < t \), available by the induction hypothesis, gather together to form a lift:

\[
\begin{array}{ccc}
A & \rightarrow & \lim_{s < t} X_s \\
\downarrow g & & \downarrow f \\
B & \rightarrow & \lim_{s < t} Y_s
\end{array}
\]

and the diagram

\[
\begin{array}{ccc}
B & \rightarrow & Y_t \\
\downarrow & & \downarrow \\
\lim_{s < t} X_s & \rightarrow & \lim_{s < t} Y_s
\end{array}
\]

obviously commutes (since the morphisms \( B \to Y_t \) are compatible). Thus we get a lift

\[
\begin{array}{ccc}
A & \rightarrow & Y_t \times \lim_{s < t} Y_s \lim_{s < t} X_s \\
\downarrow g & & \downarrow \\
B & \rightarrow & Y_t.
\end{array}
\]
The second stage is to choose any lift in the square:

\[
\begin{array}{ccc}
A & \rightarrow & X_t \\
\downarrow g & & \downarrow \\
B & \rightarrow & Y_t \times_{\lim_{s<t} Y_s} \lim_{s<t} X_s
\end{array}
\]

which exists since \( g \in \bot M \), and \( X_t \rightarrow Y_t \times_{\lim_{s<t} Y_s} \lim_{s<t} X_s \) is in \( M \). In particular we get that

\[
\begin{array}{ccc}
B & \rightarrow & X_t \\
\downarrow & & \downarrow \\
& \rightarrow & \lim_{s<t} X_s
\end{array}
\]

which shows that the lift \( B \rightarrow X_t \) is compatible.

\[ \square \]

3 Factorization of Maps

In this section we prove the following proposition about factorization of maps, which will be our main tool in proving later the existence of the desired model structure.

**Proposition 3.1.** Let \( C \) be a small category that has finite limits, \( M \subseteq C \) a subcategory that is closed under pullbacks, and \( N \subseteq \text{Mor}(C) \) an arbitrary class of morphisms, such that \( M \circ N = \text{Mor}(C) \). Then every map \( f : X \rightarrow Y \) in \( \text{Pro}(C) \) can be functorially factored as: \( X \xrightarrow{g} H \xrightarrow{h} Y \), where \( h \) is in \( \text{Sp}^\infty(M) \), and \( g \) is in \( \text{Lw}^\infty(N) \cap \bot M \) (see Definition 2.9).

**Remark 3.2.**

1. There is a strong connection between Proposition 3.1 and a dual version of Quillen’s small object argument. For more details see Section 5.

2. Notice, that while we did not require that every morphism in \( C \) can be **functorially** factored as a morphism in \( N \) followed by a morphism in \( M \), we do get a functorial factorization in Proposition 3.1. The reason is that in constructing the factorization in Proposition 3.1 we go over all possible factorizations available in \( C \).

The proof of Proposition 3.1 will occupy the rest of this section.

With out loss of generality we may assume that \( f : \{X_t\}_{t \in T} \rightarrow \{Y_t\}_{t \in T} \) is a natural transformation, i.e. a map in the functor category \( C^T \), for a directed category \( T \) (see Lemma 2.7).

The proof will consist of finding a cofinite directed set \( A_f \), together with a cofinal functor \( p : A_f \rightarrow T \), and a factorization \( p^* X \xrightarrow{g} H_f \xrightarrow{h} p^* Y \), of
$p^*f : p^*X \to p^*Y$, in the category $C^{A_f}$, s.t. $h$ is a special $\mathcal{M}$ map, and $g$ is a levelwise $N$ map, that belongs to $^\perp \mathcal{M}$ as a map in $\text{Pro}(\mathcal{C})$ (see Definition 2.8). Then $X \cong p^*X, Y \cong p^*Y$ in $\text{Pro}(\mathcal{C})$, and we get the desired factorization.

**Remark 3.3.** Notice, that the above construction will give us something slightly stronger then the conclusion of the theorem. Namely, we get that $f$ is isomorphic to a natural transformation, that can be factored, as a natural transformation, into a levelwise $N$ map, that belongs to $^\perp \mathcal{M}$ as a map in $\text{Pro}(\mathcal{C})$, followed by a special $\mathcal{M}$ map. This conclusion is stronger since we get both the levelwise condition and the special condition with the same indexing category. We will not use this stronger claim in this paper, but we will use it in a future paper.

Instead of giving the construction of $A_f$ now, we shall first describe a general method for finding a factorization that has some slightly weaker properties. This more general framework will hopefully illuminate the intuition behind the construction.

To be more precise we shall start with the problem of factoring $f : X \to Y$ to $X \xrightarrow{g} H_f \xrightarrow{h} Y$ in $\text{Pro}(\mathcal{C})$, where $g \in Lw^=(N) \cap ^\perp \mathcal{M}$, $h \in Lw^=(M)$. We will do so by finding a directed category $A_f$, a cofinal functor $A_f \to T$ and a factorization $p^*X \xrightarrow{\overline{g}} H_f \xrightarrow{\overline{h}} p^*Y$, of $p^*f : p^*X \to p^*Y$, in the category $C^{A_f}$, s.t. $h$ is a levelwise $\mathcal{M}$ map, and $g$ is a levelwise $N$ map, that belongs to $^\perp \mathcal{M}$ as a map in $\text{Pro}(\mathcal{C})$.

Note that since we do not require specialness, there is no a-priori reason to assume that $A_f$ is a cofinite directed set, and $A_f$ can be any directed category.

We want to find such an $A_f$ that induces some natural "universal" factorization, into a levelwise $N$ map followed by a levelwise $\mathcal{M}$ map. A natural choice is to take the category $A_f$ to be the category $\overline{\mathcal{F}}_f$, whose objects are all pairs $(t, X_t \xrightarrow{g} H \xrightarrow{h} Y_t)$ such that $t \in T, h \circ g = f, g \in N, h \in \mathcal{M}$. A morphism:

$$(t, X_t \to H \to Y_t) \to (t', X_{t'} \to H' \to Y_{t'})$$

in $\overline{\mathcal{F}}_f$, is given by a morphism $t \to t'$ in $T$, together with a commutative diagram of the form:

$$\begin{array}{ccc}
X_t & \overset{N}{\longrightarrow} & H \\
\downarrow & & \downarrow
\end{array} \begin{array}{ccc}
\mathcal{M} & \longrightarrow & Y_t \\
\downarrow & & \downarrow
\end{array} \begin{array}{ccc}
X_{t'} & \overset{N}{\longrightarrow} & H' \\
\downarrow & & \downarrow
\end{array} \begin{array}{ccc}
\mathcal{M} & \longrightarrow & Y_{t'}
\end{array}$$

s.t. the left and right vertical maps are induced by the given morphism $t \to t'$. Note that there is a natural functor: $p : \overline{\mathcal{F}}_f \to T$. We define the functor $H_f : \overline{\mathcal{F}}_f \to \mathcal{C}$ to be the one sending $(t, X_t \to H \to Y_t)$ to $H$, and take $p^*X \xrightarrow{\overline{g}} H_f \xrightarrow{\overline{h}} p^*Y$ to be the obvious factorization of: $p^*f : p^*X \to p^*Y$ in the functor category $C^{\overline{\mathcal{F}}_f}$.

For reasons that will be explained later, we also consider the subcategory $\mathcal{F}_f \subseteq \overline{\mathcal{F}}_f$ containing all the objects, that contains only morphisms as above, s.t. the induced map $H \to H' \times_{Y_{t'}} Y_t$ is in $\mathcal{M}$. Note that again, there is a
natural functor: \( p : \mathcal{F}_f \to T \), and an obvious factorization \( p^*X \xrightarrow{\alpha} H_f \xrightarrow{h} p^*Y \), of \( p^*f : p^*X \to p^*Y \) in \( C^{\mathcal{F}_f} \).

One immediate problem is that \( \mathcal{F}_f, \mathcal{F}_f \) are not necessarily directed. One axiom is satisfied, namely, for every pair of objects, there is an object that dominates both. However not every pair of parallel morphism can be equalized (see Definition 2.1).

We solve this problem by finding a directed category \( A_f \), together with a functor \( F : A_f \to \mathcal{F}_f \), that captures all the information in \( \mathcal{F}_f \) relevant for us. Ideally, we would like \( F \) to be cofinal. However, this is not possible, because of the following lemma:

**Lemma 3.4.** Let \( A \) be a directed category, \( D \) any category, and \( F : A \to D \) a cofinal functor. Then \( D \) is directed.

**Proof.** By [EH], Theorem 2.1.6, we may assume that \( A \) is a directed poset. By [Hir] section 14.2, for every \( c \in D \), the over category \( F/c \) is nonempty and connected.

Let \( c, d \in D \), \( F/c, F/d \) are non empty, so there exist \( q, p \in A \), and morphisms in \( D \) of the form:

\[
F(q) \to d, F(p) \to c.
\]

\( A \) is directed, so there exist \( r \in A \) s.t. \( r \geq p, q \). Then \( F(r) \in D \), and we have morphisms in \( D \) of the form:

\[
F(r) \to F(q) \to d, F(r) \to F(p) \to c.
\]

Let \( f, g : c \to d \) be two parallel morphisms in \( D \). \( F/c \) is nonempty, so there exist \( p \in A \), and a morphism in \( D \) of the form: \( h : F(p) \to c \). Then \( gh, fh \in F/d \), and \( F/d \) is connected, so there exist elements in \( A \) of the form:

\[
p \leq p_1 \geq p_2 \leq \ldots p_n \geq p,
\]

that connect \( gh, fh : F(p) \to d \) in the over category \( F/d \). \( A \) is directed, so there exist \( q \in A \), s.t. \( q \geq p, p_1, \ldots, p_n \). It follows that we have a commutative diagram in \( D \) of the form:

\[
\begin{array}{ccc}
F(p) & \xrightarrow{l_1} & F(q) & \xrightarrow{l_2} & F(p) \\
\downarrow{gh} & & \downarrow{fh} & & \\
\downarrow{d} & & & & \\
\end{array}
\]

But, \( l_1 = l_2 = l \), since \( A \) is a poset. Define: \( t := hl : F(q) \to c \). then:

\[
ft = fhl = ghl = gt.
\]

\( \Box \)

We thus demand that \( F \) satisfies a different property, which we now define:
Definition 3.5. A functor $F : J \to I$ is called \textit{pre cofinal}, if for every morphism in $I$ of the form $f : i \to F(j)$, there exist a morphism $g : j' \to j$ in $J$, s.t. $F(g)$ factors through $f$:

\[
\begin{array}{ccc}
F(j') & \xrightarrow{F(g)} & F(j) \\
\downarrow{i} & & \downarrow{f} \\
\end{array}
\]

We will prove the following:

Lemma 3.6. Let $I$ be a directed category, and let $F : I \to \mathcal{F}_f$ (or $F : I \to \mathcal{F}_{\bar{f}}$) be a pre cofinal functor. Then we have a cofinal functor: $pF : I \to T$, and an obvious factorization:

\[
\begin{array}{ccc}
(pF)^*X & \xrightarrow{gf} & F^*H_f \xrightarrow{hf} (pF)^*Y, \\
\end{array}
\]

of $(pF)^*f : (pF)^*X \to (pF)^*Y$, in $\mathcal{C}$, s.t. $h_F$ is a levelwise $\mathcal{M}$ map, and $g_F$ is a levelwise $\mathcal{N}$ map. It thus remains to show that $F \cong p^*X, Y \cong p^*Y$ in $\text{Pro}(\mathcal{C})$, and we get a factorization:

\[
\begin{array}{ccc}
X & \xrightarrow{g} & H_f \xrightarrow{h} Y, \\
\end{array}
\]

of $f : X \to Y$ in $\text{Pro}(\mathcal{C})$, s.t. $h$ is in $\text{Lw}^\Xi(\mathcal{M})$, and $g$ is in $\text{Lw}^\Xi(\mathcal{N}) \cap \perp \mathcal{M}$

Proof. We prove the Lemma for $\mathcal{F}_f$, and the proof for $\mathcal{F}_{\bar{f}}$ is identical.

Clearly we have a factorization:

\[
\begin{array}{ccc}
(pF)^*X & \xrightarrow{gf} & F^*H_f \xrightarrow{hf} (pF)^*Y, \\
\end{array}
\]

of $(pF)^*f : (pF)^*X \to (pF)^*Y$, in $\mathcal{C}$, s.t. $h_F$ is a levelwise $\mathcal{M}$ map, and $g_F$ is a levelwise $\mathcal{N}$ map. It thus remains to show that $pF : I \to T$ is cofinal, and $g_F \in \perp \text{Sp}(\mathcal{M})$, as a map in $\text{Pro}(\mathcal{C})$.

In order to proceed we need to prove the following Lemma:

Lemma 3.7. Suppose we have a commutative diagram in $\mathcal{C}$ of the form:

\[
\begin{array}{ccc}
X & \xrightarrow{f} & C \\
\downarrow{i} & & \downarrow{M} \\
Y & \xrightarrow{N} & D, \\
\end{array}
\]

Then we can embed this diagram in a bigger commutative diagram of the form:

\[
\begin{array}{ccc}
X & \xrightarrow{f} & C \\
\downarrow{i} & & \downarrow{M} \\
Y' & \xrightarrow{N} & D, \\
\end{array}
\]

s.t. the induced map $Y' \to Y \times_D C$ is in $\mathcal{M}$. 

18
Remark 3.8. Notice the resemblance of Lemma 3.7 to [Bro] I 2, Lemma 1.

Proof. Consider the diagram:

\[
\begin{array}{c}
X \downarrow \downarrow \rightarrow C \\
Y \times_D C \downarrow \downarrow \rightarrow C \\
Y \downarrow \downarrow \rightarrow D.
\end{array}
\]

Since \( \mathcal{M} \circ N = Mor(C) \), we can factor the map \( X \to Y \times_D C \), and obtain:

\[
\begin{array}{c}
X \downarrow \downarrow \rightarrow C \\
N \downarrow \downarrow \rightarrow H' \downarrow \downarrow \rightarrow Y' \\
Y' \downarrow \downarrow \rightarrow C \\
Y \downarrow \downarrow \rightarrow D.
\end{array}
\]

and since \( \mathcal{M} \) is closed under composition, we get the desired result.

Lemma 3.9. The functor: \( p : \mathcal{F}_f \to T \) is pre cofinal.

Proof. Let \( (t, X_t \to H \to Y_t) \) be an object of \( \mathcal{F}_f \), and let \( t' \to t \) be a morphism in \( T \). It is enough to show that there exist a morphism in \( \mathcal{F}_f \) of the form:

\[
\begin{array}{c}
X_{t'} \downarrow \downarrow \rightarrow H' \downarrow \downarrow \rightarrow Y_{t'} \\
X_t \downarrow \downarrow \rightarrow H \downarrow \downarrow \rightarrow Y_t,
\end{array}
\]

s.t. the left and right vertical maps are induced by the given morphism \( t' \to t \).

We have a commutative diagram of the form:

\[
\begin{array}{c}
X_{t'} \downarrow \downarrow \rightarrow Y_{t'} \\
X_t \downarrow \downarrow \rightarrow Y_t.
\end{array}
\]

Thus, we can apply Lemma 3.7 with: \( X := X_{t'}, Y := Y_{t'}, C := H, D := Y_t \), and get the desired result.
Lemma 3.10. The composition of pre cofinal functors is pre cofinal.

Proof. Let $G: K \to J$, $F: J \to I$ be pre cofinal functors. Let $f: i \to F(G(k))$ be a morphism in $I$. $F$ is pre cofinal, so there exist a morphism $g: j' \to G(k)$ in $J$, s.t. $f \circ h = F(g)$. $G$ is pre cofinal, so there exist a morphism $h: k' \to k$ in $K$, s.t. $g \circ l = G(h)$. It follows that:

$$f \circ h \circ F(l) = F(g) \circ F(l) = F(g \circ l) = F(G(h)).$$

$\square$

Lemma 3.11. Let $F: J \to I$ be a functor between directed categories. Then $F$ is pre cofinal iff $F$ is (left) cofinal.

Proof. .

$(\Rightarrow)$ By [Hir] section 14.2, we need to show that for every $i \in I$, the over category $F_{ji}$ is nonempty and connected. Let $i \in I$.

Let $J$ be nonempty, so we can choose $j \in J$. $F(j), i \in I$, and $I$ is directed, so there exist morphisms in $I$ of the form: $f: i \to i, g: j' \to F(j)$. $F$ is pre cofinal, so there exist a morphism $h: j' \to j$ in $J$, s.t. $F(h) = g \circ k$. In particular: $f \circ k : F(j') \to i$, is an object in $F_{ji}$.

Let $f_1 : F(j_1) \to i, f_2 : F(j_2) \to i$ be two objects in $F_{ji}$. Since $J$ is directed, there exist morphisms in $J$ of the form: $g_1 : j_3 \to j_1, g_2 : j_3 \to j_2$. Then: $f_1 F(g_1), f_2 F(g_2) : F(j_3) \to i$, are two parallel morphisms in $I$. Since $I$ is directed, there exist a morphism: $h : i' \to F(j_3)$ in $I$, s.t. $f_1 F(g_1) h = f_2 F(g_2) h$. Since $F$ is pre cofinal, there exist a morphism $k : j_4 \to j_3$ in $J$, s.t. $F(k) = h l$. It follows that:

$$f_1 F(g_1 k) = f_1 F(g_1) F(k) = f_1 F(g_1) h l = f_2 F(g_2) h l = f_2 F(g_2) F(k) = f_2 F(g_2 k).$$

Thus, we have morphisms in $F_{ji}$:

$$
\begin{array}{c}
\xymatrix{ F(j_1) & F(j_4) & F(j_2) \\
\downarrow f_1 & \downarrow i & \downarrow f_2 \\
F(g_1 k) & F(g_2 k) & \\
}\end{array}
$$

$(\Leftarrow)$ Let $f : i \to F(j)$ be a morphism in $I$. $F$ is left cofinal, so the category $F_{ji}$ is nonempty. Let $h : F(j') \to i$ be an object in $F_{ji}$. Then $f h, id_{F(j)}$, are two objects in the over category $F_{F(j)}$. $F$ is left cofinal, so the category $F_{F(j)}$ is connected. It follows that $f h, id_{F(j)}$ can be connected by a zig zag of morphisms $F_{F(j)}$. Since $J$ is directed, we can find morphisms $g : j'' \to j', k : j'' \to j$ in $J$, s.t. $f h F(g) = F(k)$.

$\square$

Corollary 3.12. The functor $pF : I \to T$ is cofinal.

Proof. This follows from the fact that $F : I \to F_f$ is pre cofinal, and from Lemmas: 3.10, 3.11 and 3.11.
It remains to show that $g_{F} \in \perp M$, as a map in $\text{Pro}(C)$. Consider the following diagram:

$$
\begin{array}{c}
\{X_{p(F(a))}\}_{a \in I} \\
\downarrow g_{F} \\
\{H_{F(a)}\}_{a \in I} \\
\downarrow \\
\end{array} \longrightarrow 
\begin{array}{c}
C \\
\downarrow M \\
D. \\
\end{array}
$$

We need to show the existence of a lift in the above square. It follows from the definition of morphisms in $\text{Pro}(C)$, that there exist some $a_{0} \in I$, such that the above square factors as:

$$
\begin{array}{c}
\{X_{p(F(a))}\}_{a \in I} \\
\downarrow g_{F} \\
\{H_{F(a)}\}_{a \in I} \\
\downarrow \\
\end{array} \longrightarrow 
\begin{array}{c}
X_{p(F(a_{0}))} \\
\downarrow g_{F(a_{0})} \\
H_{F(a_{0})} \\
\downarrow \\
\end{array} \longrightarrow 
\begin{array}{c}
C \\
\downarrow M \\
D. \\
\end{array}
$$

In order to finish the proof, it is enough to find a morphism: $a_{0}' \to a_{0}$ in $I$, s.t. in the following diagram we can add a dotted line:

$$
\begin{array}{c}
X_{p(F(a_{0}))} \\
\downarrow g_{F(a_{0})} \\
H_{F(a_{0})} \\
\downarrow \\
\end{array} \longrightarrow 
\begin{array}{c}
C \\
\downarrow M \\
D. \\
\end{array}
$$

By Lemma 3.7 we have a commutative diagram in $C$ of the form:

$$
\begin{array}{c}
X_{p(F(a_{0}))} \\
\downarrow g_{F(a_{0})} \\
H_{F(a_{0})} \\
\downarrow \\
\end{array} \longrightarrow 
\begin{array}{c}
C \\
\downarrow M \\
D. \\
\end{array}
$$

We thus have a morphism in $\mathcal{F}_{f}$, of the form:

$$
\begin{array}{c}
X_{p(F(a_{0}))} \\
\downarrow g_{F(a_{0})} \\
\downarrow \\
\end{array} \longrightarrow 
\begin{array}{c}
Z \\
\downarrow M \\
Y_{p(F(a_{0}))} \\
\downarrow \\
\end{array}.
$$

$F : A \to \mathcal{F}_{f}$ is pre cofinal, so there exist a morphism: $a_{0}' \to a_{0}$ in $I$, s.t. the induced morphism:

$$
\begin{array}{c}
X_{p(F(a_{0}'))} \\
\downarrow g_{F(a_{0}')} \\
\downarrow \\
\end{array} \longrightarrow 
\begin{array}{c}
H_{F(a_{0}')} \\
\downarrow M \\
Y_{p(F(a_{0}'))} \\
\downarrow \\
\end{array}.
$$
factors as:

\[
\begin{array}{ccc}
X_{p(F(a'_0))} & \xrightarrow{g_{F(a'_0)}} & H_{F(a'_0)} & \xrightarrow{h_{F(a'_0)}} & Y_{p(F(a'_0))} \\
\downarrow & & \downarrow & & \downarrow \\
X_{p(F(a_0))} & \xrightarrow{N} & Z & \xrightarrow{\mathcal{M}} & Y_{p(F(a_0))} \\
\downarrow & & \downarrow & & \downarrow \\
X_{p(F(a_0))} & \xrightarrow{g_{F(a_0)}} & H_{F(a_0)} & \xrightarrow{h_{F(a_0)}} & Y_{p(F(a_0))}.
\end{array}
\]

Composing the morphisms \(H_{F(a'_0)} \to Z\) and \(Z \to C\), we get the desired lift. \(\square\)

**Remark 3.13.** The reason for considering also the category \(\mathcal{F}_f\), is that we do not know how to find a pre cofinal functor \(F : I \to \mathcal{F}_f\), s.t. \(I\) is a directed category.

**Lemma 3.14.** Let \(A\) be any category, and let \(p : A \to T\) be a functor. Then a factorization: \(p^*X \xrightarrow{g} H \xrightarrow{h} p^*Y\), of \(p^*f : p^*X \to p^*Y\), in the category \(\mathcal{C}^A\), s.t. \(h\) is a levelwise \(\mathcal{M}\) map and \(g\) is a levelwise \(\mathcal{N}\) map, gives rise in a natural way to a functor: \(q : A \to \mathcal{F}_{\mathcal{F}_f}\), such that the composition \(A \xrightarrow{2} \mathcal{F}_{\mathcal{F}_f} \to T\) is \(p : A \to T\).

If \(A\) is a cofinite poset, and \(h\) is a special \(\mathcal{M}\) map, then one can replace \(\mathcal{F}_{\mathcal{F}_f}\) with \(\mathcal{F}_f\) above.

**Proof.** The only non trivial claim is that if \(A\) is a cofinite poset, and \(h\) is a special \(\mathcal{M}\) map, we can replace \(\mathcal{F}_{\mathcal{F}_f}\) with \(\mathcal{F}_f\).

Let \(a < b\) be elements of \(A\). then the morphism \(q(b) \to q(a)\), in \(\mathcal{F}_{\mathcal{F}_f}\), is given by the commutative diagram:

\[
\begin{array}{ccc}
X_{p(b)} & \xrightarrow{g_{b}} & H(b) & \xrightarrow{h_{b}} & Y_{p(b)} \\
\downarrow & & \downarrow & & \downarrow \\
X_{p(a)} & \xrightarrow{g_{a}} & H(a) & \xrightarrow{h_{a}} & Y_{p(a)}.
\end{array}
\]

We need to show, that the induced map: \(H(b) \to H(a) \times_{Y_{p(a)}} Y_{p(b)}\), is in \(\mathcal{M}\).

Since the map \(h : H \to p^*Y\) in \(\mathcal{C}^A\), is a special \(\mathcal{M}\) map, it follows from Lemma 2.19 when applied to the Reishas \(R_b\) and \(R_a\) of \(A\), that the map:

\[
\lim_{R_b} H \to \lim_{R_a} H \times_{\lim_{R_{op}} Y} \lim_{R_b} Y \circ p,
\]

is in \(\mathcal{M}\), or in other words, that the map: \(H(b) \to H(a) \times_{Y_{p(a)}} Y_{p(b)}\), is in \(\mathcal{M}\). \(\square\)

Now, to prove Proposition 3.1, we need to find a cofinite directed set \(A_f\), together with a functor \(p : A_f \to T\), and a factorization \(p^*X \xrightarrow{g} H \xrightarrow{h} p^*Y\), of \(p^*f : p^*X \to p^*Y\), in the category \(\mathcal{C}^{A_f}\), s.t. \(h\) is a special \(\mathcal{M}\) map, \(g\) is a levelwise \(\mathcal{N}\) map, and the induced functor \(A_f \to \mathcal{F}_f\) is pre cofinal.
We shall define $A_f, p : A_f \to T$ and the factorization: $p^*X \overset{g}{\to} H \overset{h}{\to} p^*Y$, inductively.

We start with defining $A^n_0 := \emptyset$, and $p : A^n_0 \to T$, $p^*X \overset{g}{\to} H \overset{h}{\to} p^*Y$, in the only possible way.

Now, suppose we have defined an $n$-level cofinite poset $A^n_\emptyset$, a functor $p^n : A^n_\emptyset \to T$ and a factorization of $p^n \ast f$, denoted: $p^n \ast X \overset{g}{\to} H \overset{h}{\to} p^n \ast Y$, s.t. $g$ is levelwise $N$ and $h$ is special $M$.

We define $B^n_{j+1}$ to be the set of all tuples $(R, p : R \overset{\emptyset}{\to} T, p^*X \overset{g}{\to} H \overset{h}{\to} p^*Y)$, such that $R$ is a finite Reysha in $A^n_\emptyset$ (see Definition 2.2), $p : R \overset{\emptyset}{\to} T$ is a functor such that $p|R = p^n|R$ and $p^*X \overset{g}{\to} H \overset{h}{\to} p^*Y$ is a factorization of $p^n f$ in $C^{R^\emptyset}$, such that $g$ is levelwise $N$, $h$ is special $M$ and when restricted to $C^R$, this factorization is the same as $p^n \ast X \overset{g}{\to} H \overset{h}{\to} p^n \ast Y$ restricted to $C^R$.

As a set, we define: $A^n_{j+1} := A^n_\emptyset \coprod B^n_{j+1}$. For $c \in A^n_j$, we set $c < (R, p : R \overset{\emptyset}{\to} T, p^*X \overset{g}{\to} H \overset{h}{\to} p^*Y)$ iff $c \in R$. Thus we have defined an $n+1$-level cofinite poset: $A^{n+1}_n$. We now define $p^{n+1} : A^{n+1}_n \to T$ by $p^{n+1}|A^n_j = p^n$ and $p^{n+1}((R, p : R \overset{\emptyset}{\to} T, p^*X \overset{g}{\to} H \overset{h}{\to} p^*Y)) = p(\infty)$, where $\infty \in R^\emptyset$ is the initial object. It is clear the factorization $p^n \ast X \overset{g}{\to} H \overset{h}{\to} p^n \ast Y$, extends naturally to a factorization $p^{n+1} \ast X \overset{g}{\to} H \overset{h}{\to} p^{n+1} \ast Y$ s.t. $g$ is levelwise $N$ and $h$ is special $M$.

Notice that $A^n_1 = B^n_{1} = Ob(F_{\emptyset})$, and the map: $p^1 : A^n_1 = Ob(F_{\emptyset}) \to T$ is the natural projection.

Now we define $A_f = \cup A^n_f$.

It is clear that by taking the limit on all the $p^n$ we obtain a functor $p : A_f \to T$, and a factorization $p^*X \overset{g}{\to} H \overset{h}{\to} p^*Y$ of $p^*X \overset{h'}{\to} p^*Y$, s.t. $g$ is levelwise $N$ and $h$ is special $M$.

So, to conclude, we need only prove two things:

1. $A_f$ is directed.

2. The induced functor $g : A_f \to F_\emptyset$ is pre cofinal.

While proving that $A_f$ is directed, we shall use the following fact.

**Lemma 3.15.** Let $R$ be a finite poset, and let $f : X \to Y$ be a map in $C^{R^\emptyset}$.

Let $X|_R \overset{g}{\to} H \overset{h}{\to} Y|_R$ be a factorization of $f|_R$, such that $g$ is levelwise $N$ and $h$ is special $M$. Then all the factorizations of $f$ of the form $X \overset{g'}{\to} H' \overset{h'}{\to} Y$, such that $g'$ is levelwise $N$, $h'$ is special $M$ and $H'|_R = H, g'|_R = g, h'|_R = h$, are in natural 1-1 correspondence with all factorizations of the map $X(\infty) \to \lim H \times_{\lim Y} Y(\infty)$ of the form $X(\infty) \overset{g''}{\to} H'(\infty) \overset{h''}{\to} \lim H \times_{\lim Y} Y(\infty)$, s.t. $g'' \in N$ and $h'' \in M$ (in particular there always exists one, since $M \circ N = Mor(C)$).

**Proof.** To define a factorizations of $f$ of the form $X \overset{g'}{\to} H' \overset{h'}{\to} Y$ as above, we need to define:
1. An object: \( H'(\infty) \in \mathcal{C} \).

2. Compatible morphisms: \( H'(\infty) \to H(r) \), for every \( r \in R \) (or in other words, a morphism: \( H'(\infty) \to \lim_{R} H \)).

3. A factorization \( X(\infty) \xrightarrow{g_{\infty}} H'(\infty) \xrightarrow{h_{\infty}} Y(\infty) \), of \( f_{\infty} : X(\infty) \to Y(\infty) \), s.t:

   (a) The resulting \( g' : X \to H' \), \( h' : H' \to Y \) are natural transformations (we only need to check that the following diagram commutes:

   \[
   \begin{array}{ccc}
   X(\infty) & \xrightarrow{g_{\infty}} & H'(\infty) \xrightarrow{h_{\infty}} Y(\infty) \\
   \downarrow & & \downarrow \\
   \lim_{R} X & \xrightarrow{\lim_{R} g} & \lim_{R} H \xrightarrow{\lim_{R} h} \lim_{R} Y.
   \end{array}
   \]

   (b) \( g' : X \to H' \) is levelwise \( N \) (we only need to check that \( g'_{\infty} \in N \)).

   (c) \( h' : H' \to Y \) is special \( M \) (we only need to check the special condition on \( \infty \in R^{\sqcup} \)).

From this the lemma follows easily. \( \square \)

**Lemma 3.16.** \( A_f \) is directed.

**Proof.** To prove that \( A_f \) is directed we need to show that for every finite reysha \( R \subset A_f \), there exist an element \( c \in A_f \), such that \( c \geq r \) for every \( r \in R \) (see Lemma 2.4). Indeed let \( R \subset A_f \) be a finite reysha, since \( R \) is finite there exist some \( n \in \mathbb{N} \) such that \( R \subset A_{f}^{n} \). We can take \( c \) to be any element in \( B_{f}^{n+1} \) of the form \( (R, p : R_{\sqcup} \to T, p^* X \xrightarrow{g} H \xrightarrow{h} p^* Y) \). To show that such an element exists, note that since \( T \) is directed we can extend the functor \( p : R \to T \) to a functor \( p : R_{\sqcup} \to T \) (see Lemma 2.4). Now, the existence of the suitable factorization \( p^* X \xrightarrow{g} H \xrightarrow{h} p^* Y \), follows from Lemma 3.15. \( \square \)

To complete the proof we need to show that the induced functor \( q : A_f \to F_f \) is pre-cofinal.

**Lemma 3.17.** The functor: \( q : A_f \to F_f \) is pre-cofinal.

**Proof.** Let \( c \in A_f \), and let \( d : i \to q(c) \) be a map in \( F_f \). There exist a unique \( n \geq 1 \), s.t. \( c \in A_{f}^{n} \setminus A_{f}^{n-1} = B_{f}^{n} \). We can write \( c \) as \( c = (R, p : R_{\sqcup} \to T, p^* X \xrightarrow{g} H \xrightarrow{h} p^* Y) \), where \( R \) is a finite reysha in \( A_{f}^{n-1} \). Then we have: \( q(c) = (p(\infty), X_{p(\infty)} \xrightarrow{g} H(\infty) \xrightarrow{h} Y_{p(\infty)}) \), where \( \infty \in R^{\sqcup} \) is the initial object.
Note that $R_c := \{ a \in A^f | c \geq a \} \subseteq A^f$ is a naturally isomorphic to $R^{\leq}$. Let

$$
\begin{array}{c}
X_t \xrightarrow{g} H \xrightarrow{h} Y_t \\
\downarrow \hspace{1cm} \downarrow \hspace{1cm} \downarrow \\
X_{p(q(c))} \xrightarrow{g_{q(c)}} H_{q(c)} \xrightarrow{h_{q(c)}} Y_{p(q(c))},
\end{array}
$$

be the morphism $d$ in $\mathcal{F}_f$.

Now it is enough to find $c' \in B^{n+1}_f$ such that $c' > c$, $q(c') = (t, X_t \xrightarrow{g} h \xrightarrow{h} Y_t)$, and the induced map: $q(c') \to q(c)$ is exactly $d$.

We shall take $c' := (R_c, p' : R^{\leq}_c \to T, p^*X \xrightarrow{d} H' \xrightarrow{h} p'^*Y)$, where:

$$p'|_{R_c} = p'|_{R^{\leq}} = p|_{R^{\leq}}, p'(\infty') = t,$$

where $\infty' \in R^{\leq}_c$ is the initial object.

To define the functor $p' : R^{\leq}_c \to T$, it remains to define a morphism: $p'(\infty') = t \to p(q(c))$, and we take this morphism to be the one given by $d : i \to q(c)$. We extend the factorization $p^*X \xrightarrow{d} H \xrightarrow{h} p'^*Y$, to a factorization $p^*X \xrightarrow{d} H' \xrightarrow{h} p'^*Y$, using the morphism $d : i \to q(c)$. To show that $c' \in B^{n+1}_f$, it remains to check that $H' \xrightarrow{h} p'^*Y$ is a special $\mathcal{M}$ map in $C^{R^{\leq}_c}$. We only need to check the special condition on $\infty' \in R^{\leq}_c$. This just says that the induced map $k : h \to H_{q(c)} \times_{Y_{(q(c))}} Y_t$ belongs to $\mathcal{M}$. But this follows from the fact that $d$ is a morphism in $\mathcal{F}_f$.

Now it is clear that: $c' > c$, $q(c') = (t, X_t \xrightarrow{g} h \xrightarrow{h} Y_t)$, and the induced map: $q(c') \to q(c)$ is exactly $d$.  

\[
\square
\]

Remark 3.18. 

1. We were not able to find intrinsic conditions on a category $\mathcal{D}$, weaker than directedness, that will imply the existence of a pre cofinal functor: $\mathcal{P} \to \mathcal{D}$, s.t. $\mathcal{P}$ is a cofinite directed set.

2. Our construction produces a pre cofinal functor: $q : A^f \to \mathcal{F}_f$, s.t. the induced map: $q^*H \to (pq)^*Y$, in the category $C^{A^f}$, is a special $\mathcal{M}$ map. We do not know if for every pre cofinal functor $q : A \to \mathcal{F}_f$, s.t. $A$ is directed, the induced map: $q^*H \to (pq)^*Y$, will be in $Sp(\mathcal{M})$.

4 The Model Structure on $Pro(\mathcal{C})$

In this section we show how to construct a model category out of a weak fibration category. Namely, given a weak fibration category $(\mathcal{C}, \mathcal{W}, \mathcal{F})$, that satisfies an extra condition (which we call "admissibility"), we shall construct a model structure on $Pro(\mathcal{C})$. The weak equivalences in this model structure will be maps that are isomorphic to a natural transformation, that is a levelwise $\mathcal{W}$
map. The cofibrations will be maps that have the left lifting property with respect to $F \cap W$ (considered as maps in Pro($C$)). The fibrations will be maps that are retracts of natural transformations, that are special $F$ maps. We begin with some definitions.

**Definition 4.1.** A relative category is a pair: $(C, W)$, consisting of a category $C$, and a subcategory $W \subseteq C$, that contains all the isomorphisms, and satisfies the 2 out of 3 property. $W$ is called the subcategory of weak equivalences.

**Remark 4.2.** Any weak fibration category, is naturally a relative category, when ignoring the fibrations.

**Definition 4.3.** A relative category $(C, W)$ will be called pro admissible (resp. ind admissible), if $L w \sim = (W) \subseteq \text{Pro}(C)$ (resp. $L w \sim = (W) \subseteq \text{Ind}(C)$) satisfies the 2-out-of-3 property.

**Theorem 4.4.** Let $(C, W, F)$ be a small pro admissible weak fibration category. Then there exist a model category structure on Pro($C$) s.t:

1. The weak equivalences are $W := L w \sim = (W)$.
2. The fibrations are $F := R(\text{Sp}^\sim (F))$.
3. The cofibrations are $C := \perp \text{Sp}^\sim (F \cap W) = \perp (F \cap W)$.

Moreover, this model category is fibrantly generated, with set of generating fibrations $F$ and set of generating acyclic fibrations $F \cap W$.

**Remark 4.5.** The definition of a model category that we refer to in Theorem 4.4, is the one used in [Hov] or [Hir]. In particular, we require functorial factorizations. Notice, that we did not require functorial factorizations in the definition of a weak fibration category. We do get functorial factorizations in Pro($C$), by Proposition 3.4 (see the remark following the proposition). Note also that we only required the existence of finite limits in the definition of a weak fibration category, while in Pro($C$) we do get the existence of arbitrary limits and colimits.

**Proof.** The proof will consist of a sequence of lemmas, verifying the different axioms of a model structure.

**Lemma 4.6.** The category Pro($C$) is complete and cocomplete.

**Proof.** By definition the category $C$ has finite limits. It follows that $C^{op}$ has finite colimits. By the results of [AR], the category $\text{Ind}(C^{op})$ is locally presentable, and in particular complete and cocomplete. It follows that $\text{Ind}(C^{op})^{op} = \text{Pro}(C)$ is also complete and cocomplete.

**Lemma 4.7.** The classes $W, F$ and $C$ contain all isomorphisms.

**Proof.** Clear.
Lemma 4.8. The classes $W$, $F$, and $C$ are closed under retracts.

Proof. This is trivial for $F$, and follows from Lemma 2.10 for $W$, and from Lemma 2.11 for $C$. □

Lemma 4.9. We have $(C \cap W) \perp F$.

Proof. We need to show that $C \cap W \subseteq \frac{1}{\times}R(\operatorname{Sp}^\leq(F))$. But $\frac{1}{\times}R(\operatorname{Sp}^\leq(F)) = \frac{1}{\times}Sp^\leq(F) = \frac{1}{\times}F$, by Lemma 2.11 and Lemma 2.22. Thus, it is enough to show that there exist a lift in every square of the form:

$$
\begin{array}{ccc}
X & \longrightarrow & A \\
\downarrow_{\phantom{\times}W} & & \downarrow_{\phantom{\times}F} \\
Y & \longrightarrow & B.
\end{array}
$$

Without loss of generality, we may assume that $X \rightarrow Y$ is a natural transformation, which is is a level-wise $W$-map. Thus we have a diagram of the form:

$$
\begin{array}{ccc}
\{X_t\}_{t \in T} & \longrightarrow & A \\
\downarrow_{\phantom{\times}C} & & \downarrow_{\phantom{\times}F} \\
\{Y_t\}_{t \in T} & \longrightarrow & B.
\end{array}
$$

By the definition of morphisms in $\operatorname{Pro}(C)$, there exist $t \in T$, such that the above square factors as:

$$
\begin{array}{ccc}
\{X_t\}_{t \in T} & \longrightarrow & X_t & \longrightarrow & A \\
\downarrow_{\phantom{\times}C} & & \downarrow_{\phantom{\times}W} & & \downarrow_{\phantom{\times}F} \\
\{Y_t\}_{t \in T} & \longrightarrow & Y_t & \longrightarrow & B.
\end{array}
$$

By taking the fiber product we get the following diagram:

$$
\begin{array}{ccc}
\{X_t\}_{t \in T} & \longrightarrow & X_t & \longrightarrow & A \\
\downarrow_{\phantom{\times}C} & & \downarrow_{\phantom{\times}W} & & \downarrow_{\phantom{\times}F} \\
\{Y_t\}_{t \in T} & \longrightarrow & Y_t & \longrightarrow & B.
\end{array}
$$
Now factor the map $X_t \to Y_t \times_B A$ into a map in $W$, followed by a map in $F$:

We compose and get:

where the map $H \to Y_t$ belongs to $W$, because $W$ has the 2 out of 3 property. But now we have a lift in the left bottom square, by definition of $C$.  

**Lemma 4.10.** We have: 

$$F \cap W = R(Sp^\simeq(F \cap W)).$$

**Proof.** By Corollary 2.21 we have:

$$Sp^\simeq(F \cap W) \subseteq Lw^\simeq(F \cap W) \subseteq Lw^\simeq(W) = W.$$ 

We also have:

$$Sp^\simeq(F \cap W) \subseteq Sp^\simeq(F) \subseteq R(Sp^\simeq(F)) = F.$$ 

Thus, by Lemma 4.8 We have:

$$R(Sp^\simeq(F \cap W)) \subseteq F \cap W.$$ 

Now let $h : A \to B \in F \cap W$. We can apply Proposition 3.1 for $N = Mor(C), M = F \cap W$, and get a factorization of $h$ as:

$$A \xrightarrow{g \in C} C \xrightarrow{f \in Sp^\simeq(F \cap W)} B$$
Since $f \in Sp^\simeq (F \cap W) \subseteq W$, and $W$ has the 2 out of 3 property, we have that: $g \in C \cap W$. It follows from Lemma 4.9 that we have a lift in the following square:

Thus we get:

and $h$ is indeed a retract of $f \in Sp^\simeq (F \cap W)$.

**Corollary 4.11.** We have:

$$C = \bot (F \cap W).$$

**Proof.** By Lemma 4.10 we have:

$$C = \bot (Sp^\simeq (F \cap W)) = \bot (R(Sp^\simeq (F \cap W))) = \bot (F \cap W).$$

**Lemma 4.12.** Every map $X \to Y$ in $Pro(C)$ can be functorially factored as:

$$X \xleftarrow{C} Z \xrightarrow{F \cap W} Y.$$

**Proof.** We can apply Proposition 3.1 for $N = Mor(C), M = F \cap W$, and get a functorial factorization of the form:

But $C = \bot Sp^\simeq (F \cap W)$, and by Lemma 4.10 we have: $Sp^\simeq (F \cap W) \subseteq F \cap W$.

**Lemma 4.13.** Every map $X \to Y$ in $Pro(C)$ can be functorially factored as:

$$X \xleftarrow{C \cap W} Z \xrightarrow{F} Y.$$

**Proof.** We can apply Proposition 3.1 for $N = W, M = F$, and get a functorial factorization of the form:

But $Sp^\simeq (F) \subseteq F, W = Lw^\simeq (W)$, and $\bot Sp^\simeq (F) \subseteq \bot Sp^\simeq (F \cap W) = C$. 

Lemma 4.14. We have:
\[ F = (C \cap W)^\perp. \]

Proof. By Lemma 4.9 we have:
\[ F \subseteq (C \cap W)^\perp. \]
Now let \( h : A \to B \in (C \cap W)^\perp \). By Lemma 4.13 we can factor \( h \) as:
\[ A \xrightarrow{\text{\( g \in C \cap W \)}} C \xrightarrow{\text{\( f \in F \)}} B. \]

We get the commutative diagram:
\[
\begin{array}{ccc}
A & \xrightarrow{k} & C \\
\downarrow{g} & & \downarrow{f} \\
C & \xrightarrow{h} & B
\end{array}
\]
where the existence of \( k \) is clear. Rearranging, we get:
\[
\begin{array}{ccc}
A & \xrightarrow{g} & C \\
\downarrow{h} & & \downarrow{f} \\
B & \xrightarrow{k} & A
\end{array}
\]
and we see that \( h \) is a retract of \( f \in F \). But \( F \) is closed under retracts, so \( h \in F \).

Corollary 4.15. The classes \( W, C \) and \( F \) are closed under composition.

Proof. For \( W \) this follows from the 2 out of 3 property, for \( C \) it is clear from the definition, and for \( F \) it follows from Lemma 4.14.

The 2 out of 3 property for \( W \) holds by assumption. Thus, all the axioms for a model structure have been verified.

It remains to show that Pro\((C)\) is fibrantly generated, with set of generating fibrations \( \mathcal{F} \) and set of generating acyclic fibrations \( \mathcal{F} \cap W \). The category \( C \) has finite limits, so \( C^{\text{op}} \) has finite colimits. By the results of [AR], the category \( \text{Ind}(C^{\text{op}}) \) is locally presentable, and in particular every object of \( \text{Ind}(C^{\text{op}}) \) is small. It follows that every object of \( \text{Ind}(C^{\text{op}})^{\text{op}} = \text{Pro}(C) \) is cosmall. By [Hov] Definition 2.1.17, it remains to show that:
\[ C = ^\perp (\mathcal{F} \cap W), (C \cap W) = ^\perp \mathcal{F}. \]
The first equality holds by definition, and the second follows from Lemmas 2.11 and 2.22:
\[ (C \cap W) = ^\perp \mathcal{F} = ^\perp R(Sp^\perp(\mathcal{F})) = ^\perp Sp^\perp(\mathcal{F}) = ^\perp \mathcal{F}. \]
Remark 4.16. In this remark we compare Theorem 4.4 with the main result in Isaksen’s paper [Isa].

1. Theorem 4.4 was proved by Isaksen, for the case that the weak fibration category \((C, W, F)\) comes from a model category \((C, Cof, W, F)\). He also shows that in this case the cofibrations in \(\text{Pro}(C)\) are given by: \(C = Lw^\sim(Cof)\). At first glance it might seem that the result we obtained here is strictly stronger than Isaksen’s, as the conclusion of his theorem is shown to hold under much weaker assumptions. However, Isaksen’s theorem holds also for large model categories, whereas our theorem assumes that the weak fibration category is small.

2. The approach taken by Isaksen is to begin with a model category \(C\), and to work hard to define a model category structure also on \(\text{Pro}(C)\). As we see here, the latter may exist without the former. Namely, \(C\) can be a weak fibration category, that is not a model category, while on \(\text{Pro}(C)\) there will still be an induced model structure. The main reason for this phenomenon is that the absence of an initial factorization in \(C\), can be solved when working in \(\text{Pro}(C)\), by simply ”running over” all possible factorizations (See the introduction, and the proof of Proposition 3.1).

3. Our theorem can be applied also to a large weak fibration category if we pass to the next universe, i.e. if we allow also for large indexing diagrams in \(\text{Pro}(C)\). In other words, since our main factorization theorem is based on running over all possible factorizations in \(C\), the size of the resulting indexing category will necessarily be the same as the size of \(C\). This also has to do with the fact that our factorization theorem can be viewed as a form of the cosmall object argument, that demands the existence of a set of generating maps, see Section 3.

4. Sometimes it is possible to deal with the above set theoretical issues in a different way. When in the (large) class of all possible factorizations, there always exists a cofinal (small) subset, one can consider only those factorization, and obtain a small indexing category. In this case there will be an induced model structure on the pro category, in the usual sense. This is possible, for example, for simplicial sheaves on a small Grothendieck site (See Section 3). However, in this case one obtains a model category without functorial factorizations. The ability to restrict to a small cofinal system is explained for example in the proof of Theorem 3.4.1 in [ToVe1].

Remark 4.17. In [Isa] section 3 it is shown that if \(\mathcal{M}\) is a proper model category, and \(W\) is the class of weak equivalences in \(\mathcal{M}\), then \(Lw^\sim(W) \subseteq \text{Pro}(\mathcal{M})\) satisfies the 2 out of 3 property. It follows that \((\mathcal{M}, W)\) is pro admissible.

5 Relation to the cosmall object argument

The main technical tool in proving the existence of our model structure (Theorem 4.4), is the factorization Lemma (Proposition 3.1). Classically, when
proving the existence of model structures, factorizations are usually proven by means of Quillen’s small object argument. In this section we discuss the close relationship between the two approaches.

The following is based on [Isa] section 5.

**Definition 5.1.** Let $D$ be a category with all small limits, $M \subseteq \text{Mor}(D)$ a class of morphisms in $D$, and $\lambda$ an ordinal. A $\lambda$-tower in $D$, relative to $M$, is a diagram $X : \lambda \to D$, s.t. for all limit ordinals $t < \lambda$, the natural map $X_t \to \lim_{s \leq t} X_s$ is an isomorphism, and for all non limit ordinals $t < \lambda$, the map $X_t \to X_{t-1}$ is in $M$. The (transfinite) composition of the $\lambda$-tower $X$ is defined to be the natural map $\lim_\lambda X \to X(0)$.

**Definition 5.2.** Let $D$ be a category with all small limits, and $M \subseteq \text{Mor}(D)$ a class of morphisms in $D$. A relative $M$-cocell complex, is a transfinite composition of pullbacks of elements of $M$. That is, $f : A \to B$ is a relative $M$-cocell complex if there exist an ordinal $\lambda$, and a $\lambda$-tower in $D$, relative to pullbacks of maps in $M$, s.t. $f$ is isomorphic to the composition of the $\lambda$-tower $X$. We denote the collection of all relative $M$-cocell complexes by $\text{cocell}(M)$.

Let $C$ be a small category with finite limits. As we explained in the proof of Lemma 4.6, the category $\text{Pro}(C)$ has all small limits.

**Proposition 5.3.** For any class of morphisms $M \subseteq \text{Mor}(C)$, we have: $\text{Sp}^\simeq(M) \subseteq \text{cocell}(M)$, in $\text{Pro}(C)$.

**Proof.** [Isa], Proposition 5.2.

In [Isa], Isaksen conjectures a partial converse to Proposition 5.3. Namely, that for any class of morphisms $M \subseteq \text{Mor}(C)$, we have: $R(\text{cocell}(M)) \subseteq R(\text{Sp}^\simeq(M))$, in $\text{Pro}(C)$. This conjecture fails as stated, as the following counterexample demonstrates. Take $C$ to be the category:

\[
\begin{array}{ccc}
a & \to & b \\
\downarrow & & \downarrow \\
c & \to & d, \\
\end{array}
\]

where the square is commutative, and take $M$ to consist only of the unique map $b \to d$. It is easy to verify, that there is a natural equivalence of categories $\text{Pro}(C) \simeq C$, and under this equivalence, $R(\text{Sp}^\simeq(M))$ is just $M$. Thus $R(\text{Sp}^\simeq(M))$ is clearly not closed under pullbaks, in contrary to $R(\text{cocell}(M))$.

However, using Theorem 4.4, we can prove Isaksen’s conjecture in the case where $M$ is a subcategory, that is closed under pullbaks.

**Proposition 5.4.** Let $M \subseteq C$ be a subcategory that is closed under pullbacks and contains all the isomorphisms. Then $R(\text{cocell}(M)) \subseteq R(\text{Sp}^\simeq(M))$. 

32
Proof. Since $\mathcal{M} \subseteq R(Sp^\otimes(\mathcal{M}))$, it is enough to show that class $R(Sp^\otimes(\mathcal{M})) \subseteq Mor(\text{Pro}(\mathcal{C}))$ is closed under pullbacks and transfinite compositions (see Definition 6.1).

It is easy to see that $(\mathcal{C}, C, M)$ is a small weak fibration category. Moreover, $Lw^\otimes(\mathcal{C}) = Mor(\text{Pro}(\mathcal{C}))$ by Lemma 2.7, so $(\mathcal{C}, C, M)$ is clearly pro admissible. Thus, it follows from Theorem 4.4 that there exist a model category structure on Pro($\mathcal{C}$) s.t. the fibrations are precisely $F := R(Sp^\otimes(F))$. In particular it follows that $R(Sp^\otimes(F)) = (\mathcal{C} \cap W)^\perp$, and thus $R(Sp^\otimes(M))$ is closed under pullbacks and transfinite compositions by well known arguments (see for example [Lur] A.1.1).

Corollary 5.5. Let $\mathcal{M} \subseteq C$ be a subcategory that is closed under pullbacks and contains all the isomorphisms. Then $R(\text{cocell}(\mathcal{M})) = R(Sp^\otimes(\mathcal{M}))$.

Proof. Combine Propositions 5.3 and 5.4.

Corollary 5.6. Let $M \subseteq Mor(\mathcal{C})$ be any class of morphisms. Then every map $f : X \to Y$ in $\text{Pro}(\mathcal{C})$, can be functorially factored as $X \xrightarrow{h} H \xrightarrow{g} Y$, where $g$ is in $\text{cocell}(M)$, and $h$ is in $\perp M$.

Proof. Let $\mathcal{M}$ denote the smallest subcategory of $\mathcal{C}$ that is closed under pullbacks and contains all the isomorphisms, that also contains $M$. Since $\text{cocell}(M), (\perp M)^\perp$ are closed under pull-back and transfinite composition, we have:

1. $\text{cocell}(M) = \text{cocell}(\mathcal{M})$.
2. $\perp M = \perp \mathcal{M}$.

Thus the corollary follows by combining Propositions 5.1, 5.3 and Lemma 2.22.

Remark 5.7. Note that every simple object in Pro($\mathcal{C}$) is $\omega$-cosmall. Thus Corollary 5.6, can be viewed as a special case of Quillen’s cosmall object argument.

We now present an application of Proposition 5.4. Let $\mathcal{S}_f$ denote the category of simplicial sets with finitely many non-degenerate simplices. Let $\mathcal{A}$ denote the smallest subcategory of $\mathcal{S}_f$, that contains all the isomorphisms and is closed under push outs, that also contains all the horn inclusions $\Lambda^n_i \to \Delta^n$. In other words, if $H$ denotes the set of horn inclusions, then maps in $\mathcal{A}$ are just finite relative $H$-cell complexes in $\mathcal{S}_f$. That is, maps that can be obtained as a finite composition of push outs of horn inclusions, starting from an arbitrary object in $\mathcal{S}_f$. Clearly, every map in $\mathcal{A}$ is a trivial cofibration in $\mathcal{S}_f$.

Lemma 5.8. Every trivial cofibration in $\mathcal{S}_f$, is a retract of a map in $\mathcal{A}$.

Proof. Let $f : A \to B$ be a trivial cofibration in $\mathcal{S}_f$. By the results of [Hov] 2.1, $f$ belongs to $R(\text{cell}(H)) = R(\text{cell}(\mathcal{A}))$, as a map in Ind($\mathcal{S}_f$) $\simeq S$. By the dual of Proposition 5.4 $f$ also belongs to $R(coSp^\otimes(\mathcal{A}))$. Thus, there exist $h \in coSp^\otimes(\mathcal{A})$, s.t $f$ is a retract of $h$. Without loss of generality we may assume
that \( h : \{X_t\}_{t \in T} \to \{Y_t\}_{t \in T} \) is a natural transformation, which is a cospecial \( \mathcal{A} \)-map. We have the following retract diagram:

\[
\begin{array}{ccc}
A & \xrightarrow{f} & \{X_t\} & \xrightarrow{f} & A \\
\downarrow{h} & & \downarrow{h} & & \downarrow{h} \\
B & \xrightarrow{f} & \{Y_t\} & \xrightarrow{f} & B.
\end{array}
\]

It follows from the definition of morphisms in \( \text{Ind} (S_f) \), that there exist \( t_0 \in T \), such that the above diagram can be factored as:

\[
\begin{array}{ccc}
A & \xrightarrow{f} & X_{t_0} & \xrightarrow{f} & A \\
\downarrow{h_{t_0}} & & \downarrow{h} & & \downarrow{h} \\
B & \xrightarrow{f} & Y_{t_0} & \xrightarrow{f} & B.
\end{array}
\]

It follows that \( f \) is a retract of \( h_{t_0} \), in \( S_f \). But by the dual of Lemma 2.12 \( h \) is a levelwise \( \mathcal{A} \)-map. In particular \( h_{t_0} \) belongs to \( \mathcal{A} \), and we get the desired result.

\[\square\]

6 Weak Right Quillen Functors

In this section we will discuss a natural notion of a morphism between weak fibration categories, which we call a weak right Quillen functor. We then prove, that a weak right Quillen functor between pro admissible weak fibration categories, gives rise to a right Quillen functor between the corresponding model structures on the pro categories.

**Definition 6.1.** Let \( F : \mathcal{D} \to \mathcal{C} \) be a functor, between two weak fibration categories. Then \( F \) will be called a weak right Quillen functor if \( F \) commutes with finite limits, and preserves fibrations and trivial fibrations.

**Remark 6.2.** If \( F : \mathcal{D} \to \mathcal{C} \) is a weak right Quillen functor between model categories, then \( F \) is not necessarily a right Quillen functor, since \( F \) is only assumed to commute with finite limits, not arbitrary limits.

The main fact we want to prove about weak right Quillen functors is the following:

**Proposition 6.3.** Let \( F : \mathcal{D} \to \mathcal{C} \) be a weak right Quillen functor between two pro admissible weak fibration categories. Then prolongation of \( F \) induces a Quillen adjunction of the form:

\[
L_F : \text{Pro}(\mathcal{C}) \rightleftarrows \text{Pro}(\mathcal{D}) : \text{Pro}(F),
\]

relative to the model structures defined in Theorem 4.4. Further, if \( F \) has a left adjoint \( G : \mathcal{C} \to \mathcal{D} \), we have:

\[
L_F \cong \text{Pro}(G).
\]
Proof. It is a classical fact that a functor from a category with finite limits to \( \text{Set} \), that commutes with finite limits, is pro representable (that is, representable by a pro object). It follows that \( \text{Pro}(F) : \text{Pro}(\mathcal{D}) \to \text{Pro}(\mathcal{C}) \) has a left adjoint: \( L_F \).

It thus remains to show that \( L_F \) preserves cofibrations and trivial cofibrations.

Let \( f : X \to Y \) be a cofibration in \( \text{Pro}(\mathcal{C}) \). Then by definition \( f \in \perp (\mathcal{F}_C \cap \mathcal{W}_C) \). We need to show that \( L_F(f) \in \perp (\mathcal{F}_D \cap \mathcal{W}_D) \). Consider the following commutative square in \( \text{Pro}(\mathcal{D}) \):

\[
\begin{array}{ccc}
L_F(X) & \longrightarrow & A \\
\downarrow & & \downarrow \text{g} \in \mathcal{F}_D \cap \mathcal{W}_D \\
L_F(Y) & \longrightarrow & B.
\end{array}
\]

We need to show that the above square has a lift. by adjointness, it is enough to show that the following square in \( \text{Pro}(\mathcal{C}) \) has a lift:

\[
\begin{array}{ccc}
X & \longrightarrow & F(A) \\
\downarrow & & \downarrow F(g) \\
Y & \longrightarrow & F(B).
\end{array}
\]

But \( F : \mathcal{D} \to \mathcal{C} \) is a weak right Quillen functor, so \( F(g) \in \mathcal{F}_C \cap \mathcal{W}_C \), and thus such a lift exists by assumption.

Let \( f : X \to Y \) be a trivial cofibration in \( \text{Pro}(\mathcal{C}) \). Since \( \text{Pro}(\mathcal{C}) \) is fibrantly generated, with set of generating fibrations \( \mathcal{F}_C \), it follows that: \( f \in \perp \mathcal{F}_C \), and we proceed as before.

Now assume that \( F \) has a left adjoint \( G \). We need to prove that \( L_F \cong \text{Pro}(G) \). By uniqueness of adjoint functors, it is enough to show that \( \text{Pro}(G) \) is left adjoint to \( \text{Pro}(F) \). Let \( c = \{c_i\}_{i \in I} \in \text{Pro}(\mathcal{C}), \) and \( d = \{d_j\}_{j \in J} \in \text{Pro}(\mathcal{D}) \). We have:

\[
\text{Hom}_{\text{Pro}(\mathcal{D})}(\text{Pro}(G)(c), d) = \lim_{\text{colim}_{i \in I}} \text{Hom}(G(c_i), d_j) = \lim_{\text{colim}_{j \in J}} \text{Hom}_C(c_i, F(d_j)) = \text{Hom}_{\text{Pro}(\mathcal{C})}(c, \text{Pro}(F)(d)).
\]

7 Simplicial Weak Fibrations Categories

In this section we define the notion of a simplicial weak fibration category. We then prove that a pro admissible simplicial weak fibration category, gives rise to a simplicial model structure on it’s pro category.

Recall that \( S_f \) denotes the category of simplicial sets with finitely many non-degenerate simplices.
Definition 7.1. Let $\mathcal{C}$ be a category with finite limits. Then $\mathcal{C}$ is said to be weakly cotensored (closed) over $S_f$ if there exist a functor: $\text{Map}(-,-) : S_f^{op} \times \mathcal{C} \to \mathcal{C}$, s.t:

1. The functor $\text{Map}(-,-) : S_f^{op} \times \mathcal{C} \to \mathcal{C}$ commutes with finite limits (or equivalently, for every $X \in \mathcal{C}$, and $K \in S_f$, the functors: $\text{Map}(K,-) : \mathcal{C} \to \mathcal{C}$ and $\text{Map}(-,X) : S_f^{op} \to \mathcal{C}$, both commute with finite limits).

2. There exist coherent natural isomorphisms:

$$\text{Map}(L, \text{Map}(K,X)) \cong \text{Map}(K \times L, X),$$

$$\text{Map}(\Delta_0, X) \cong X,$$

for $X \in \mathcal{C}$; $K, L \in S_f$.

We denote: $X^L := \text{Map}(L,X)$, for $X \in \mathcal{C}$ and $L \in S_f$.

Remark 7.2. Compare Definition 7.1 to [GJ] II. Lemma 2.4.

Example 6. Suppose $\mathcal{C}$ is a category that has finite limits. $s\mathcal{C} = \mathcal{C} \Delta^{op}$ is naturally weakly cotensored over $S_f$.

If $\mathcal{C}$ is weakly cotensored over $S_f$, then $\mathcal{C}$ is naturally enriched over $S$, by defining for every $X, Y \in \mathcal{C}$:

$$\text{Map}_C(X,Y)_n := \text{Hom}_C(X,Y^{\Delta^n}).$$

Proposition 7.3. Let $\mathcal{C}$ be a category, weakly cotensored over $S_f$. Then $\text{Pro}(\mathcal{C})$ is naturally enriched, tensored and cotensored over $S$.

Proof. Let $K \in S_f$, the functor $\text{Map}(K,-) : \mathcal{C} \to \mathcal{C}$ commutes with finite limits. It follows that $\text{Pro}(\text{Map}(K,-)) : \text{Pro}(\mathcal{C}) \to \text{Pro}(\mathcal{C})$ has a left adjoint: $(-) \otimes K$:

$$(-) \otimes K : \text{Pro}(\mathcal{C}) \rightleftarrows \text{Pro}(\text{Map}(K,-))$$

More concretely, let $X : I \to \mathcal{C}$ be an object in $\text{Pro}(\mathcal{C})$. Then $X \otimes K \in \text{Pro}(\mathcal{C})$ is isomorphic (defined) to the Grothendieck construction of the functor: $\text{Hom}_C(X,(-)^K) : \mathcal{C} \to \text{Set}$. Since this functor commutes with finite limits, this is well defined.

Note that if $(-)^K : \mathcal{C} \to \mathcal{C}$ has a left adjoint, $(-) \otimes K : \mathcal{C} \to \mathcal{C}$, then $X \otimes K : I \to \mathcal{C}$ is naturally isomorphic to $i \mapsto X(i) \otimes K$, in $\text{Pro}(\mathcal{C})$.

We define: $X^K : I \to \mathcal{C} \in \text{Pro}(\mathcal{C})$, by $(X^K)(i) := X(i)^K \in \mathcal{C}$.

Now let $X, Y \in \text{Pro}(\mathcal{C})$, and $K \in S$. We define:

1. $\text{Map}_{\text{Pro}(\mathcal{C})}(X,Y) \in S$ by:

$$\text{Map}_{\text{Pro}(\mathcal{C})}(X,Y)_n := \text{Hom}_{\text{Pro}(\mathcal{C})}(X \otimes \Delta^n, Y) = \text{Hom}_{\text{Pro}(\mathcal{C})}(X, Y^{\Delta^n}),$$

or in other words:

$$\text{Map}_{\text{Pro}(\mathcal{C})}(X,Y) := \lim_{s} \colim_{t} \text{Map}_{\mathcal{C}}(X_t, Y_s).$$
2. $X \otimes K \in \text{Pro}(\mathcal{C})$ by:

$$X \otimes K := \text{colim}_{K_f \subseteq K, K_f \in S_f} X \otimes K_f.$$ 

3. $X^K \in \text{Pro}(\mathcal{C})$ by:

$$X^K := \text{lim}_{K_f \subseteq K, K_f \in S_f} X^{K_f}.$$ 

By Proposition 16.1 in [IsaS], it suffices to check the axioms only for finite simplicial sets.

Most of the axioms are obvious; we verify only the non-trivial ones here. Let $X, Y \in \text{Pro}(\mathcal{C})$, and let $K \in S_f$. We use the fact that $\text{Hom}_S(K, -)$ commutes with filtered colimits, because $K$ is finite. It follows by direct calculation that:

$$\text{Map}_{\text{Pro}(\mathcal{C})}(X \otimes K, Y) \cong \text{Map}_{\text{Pro}(\mathcal{C})}(X, Y^K) \cong \text{Map}_S(K, \text{Map}_{\text{Pro}(\mathcal{C})}(X, Y)).$$

Definition 7.4. Let $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ be a weak fibration category which is also weakly cotensored over $S_f$. We say that $\mathcal{C}$, with this structure, is a simplicial weak fibration category, if for every cofibration $i : K \to L$ in $S_f$, and every fibration $p : X \to Y$ in $\mathcal{C}$, the induced map:

$$X^L \to X^K \times_{Y^K} Y^L$$

is a fibration, which is acyclic if either $i$ or $p$ is.

Let $\mathcal{C}$ be a simplicial weak fibration category. Then for every $K \in S_f$ and every fibrant $X \in \mathcal{C}$, the functors $\text{Map}(K, -) : \mathcal{C} \to \mathcal{C}$ and $\text{Map}(-, X) : S_f^{op} \to \mathcal{C}$, are weak right Quillen functors.

Proposition 7.5. Let $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ be a pro admissible simplicial weak fibration category. Then $\text{Pro}(\mathcal{C})$, with the model structure described in Theorem 4.4, is naturally a simplicial model category.

Proof. By Proposition 7.3, $\text{Pro}(\mathcal{C})$ is naturally enriched, tensored and cotensored over $S$. By Theorem 4.4, $\text{Pro}(\mathcal{C})$ has a model structure. It remains to check Quillen’s Axiom SM7. By Proposition 16.1 in [IsaS], it suffices to check this axiom only for finite simplicial sets.

Let $i : K \to L$ be a cofibration in $S_f$, and let $j : A \to B$ be a cofibration in $\text{Pro}(\mathcal{C})$. We need to show that the natural map:

$$A \otimes L \coprod_{A \otimes K} B \otimes K \to B \otimes L$$

is a cofibration, that is acyclic if either $i$ or $j$ is. In order to check this we use the criterion that:

$$\mathcal{C} = \mathcal{R}(S_{\mathcal{F}}^= (\mathcal{F} \cap \mathcal{W})) = \mathcal{R}((\mathcal{F} \cap \mathcal{W})),$$
\[ \mathcal{C} \cap \mathcal{W} = \perp R(Sp^{\mathcal{C}}(\mathcal{F})) = \perp \mathcal{F} , \]

in Pro(\mathcal{C}). Thus, given a fibration \( p : X \to Y \) in \( \mathcal{C} \), it is enough to show that the following diagram:

\[
\begin{array}{ccc}
A \otimes L & \coprod_{A \otimes K} & B \otimes K \\
\downarrow & & \downarrow p \\
B \otimes L & \to & Y
\end{array}
\]

has a lift, if at least one of the maps \( i, j \) or \( p \) is acyclic. By adjunction this is equivalent to showing that the diagram:

\[
\begin{array}{ccc}
A & \to & X^L \\
\downarrow & & \downarrow \\
B & \to & X^K \times_{Y^K} Y^L
\end{array}
\]

has a lift, if at least one of the maps \( i, j \) or \( p \) is acyclic. But this follows immediately, from the definition of a simplicial weak fibration category, and the criterion above.

\[ \Box \]

8 Homotopies in Weak Fibration Categories and Comparison with Artin and Mazur’s Construction

Let \( \mathcal{C} \) be an admissible weak fibration category. Then according to Theorem 4.4 there is an induced model structure on Pro(\( \mathcal{C} \)). In particular, every map \( f : X \to Y \) in Pro(\( \mathcal{C} \)) can be functorially factored as: \( X \xrightarrow{C} Z \xrightarrow{F \cap \mathcal{W}} Y \), and as \( X \xrightarrow{C \cap \mathcal{W}} Z \xrightarrow{F} Y \). The existence of both factorizations was proven in section 4 using Proposition 3.1 which takes as data two classes of maps in \( \mathcal{C} \): \( N \) and \( M \).

To achieve the first factorization we took \( N = Mor(\mathcal{C}), M = \mathcal{F} \cap \mathcal{W}, \) and to get the second factorization we took \( N = \mathcal{W}, M = \mathcal{F} \) (See Lemmas 4.13, 4.12).

In the proof of Proposition 3.1 we assumed the map \( f : X \to Y \) was given as a natural transformation in \( CT \), for some directed category \( T \). We started with the intuition of taking the middle pro object in the factorization of \( f \) to be indexed by the categories \( \mathcal{F}_f \) or \( \mathcal{F}_f \), whose objects are all factorizations in \( \mathcal{C} \) of the form \( X_t \xrightarrow{N} H \xrightarrow{M} Y_t \), where \( t \in T \).

The main problem was that \( \mathcal{F}_f \) or \( \mathcal{F}_f \), where not necessarily directed. We addressed the non-directness of \( \mathcal{F}_f \) by constructing a pre-cofinal functor \( A_f \to \mathcal{F}_f \), for a suitable directed poset \( A_f \).

Suppose \( \mathcal{C} \) is the weak fibration category of simplicial sheaves on a Grothendieck site (See section 9), \( f : \phi \to * \) is the unique map from the initial object to the terminal object in Pro(\( \mathcal{C} \)), and we are interested in factoring \( f \) as \( \phi \xrightarrow{C} Z \xrightarrow{F \cap \mathcal{W}} \)
Then the corresponding category $\overline{F}_f$ is the category of hypercoverings in $C$, studied by Artin and Mazur in [AM]. They also encountered the problem of the non-directness of $\overline{F}_f$, but they address it differently. They consider the homotopy category of the category of hypercoverings, which turns out to be directed. Thus they are working in $\text{Pro}(\text{Ho}(C))$ instead of $\text{Pro}(C)$. It turns out that if the weak fibration category $C$ is simplicial, the homotopy categories of $\overline{F}_f$ and $F_f$ will always be directed.

In this section we will relate the solution in [AM] to the one described in this paper. For this we first need to discuss the notion of homotopy in a simplicial weak fibration category. Note that our approach is slightly different from that of [Bro], and is more influenced from [AM]. We shall then be able to show that after going to $\text{Pro}(\text{Ho}(C))$ our solution and the one described in [AM] are naturally isomorphic. This will allow us to obtain a simpler model for the cofibrant replacements of objects (after going to $\text{Pro}(\text{Ho}(C))$). These results will be used later in sections 10 and 11.

Remark 8.1. When working with a category $C$ with both a notion of weak-equivalences and a simplicial structure, one should distinguish between the "homotopy category" $\text{Ho}(C)$, which is what one gets when formally inverting the weak-equivalences, and $\pi C$ which is what one gets when taking connected components of mapping spaces. Those two notions are usually not the same. For example, when $C$ is a simplicial model category, it is only guaranteed that

$$\text{Map}_{\text{Ho}(C)}(A, B) = \text{Map}_{\pi C}(A, B),$$

when $A$ is cofibrant and $B$ is fibrant.

Artin and Mazur are working in [AM] with fibrant-cofibrant objects, and are thus not faced with this distinction. Here we will work with $\pi C$. This will suffice since in our case, as in every reasonable one, the natural functor $C \to \text{Ho}(C)$ factors through $\pi C$.

Let $(C, W, F)$ be a simplicial weak fibration category. Recall that an object $C \in C$ is called fibrant or contractible, if the unique map $C \to *$ is a fibration or a weak equivalence respectively.

Let $X \in C$. Consider the following diagram in $S_f$:

$$\Delta^{[0]} \coprod \Delta^{[1]} \to \Delta^1 \to \Delta^0.$$ 

Applying $X(-)$ to this diagram, we obtain the following diagram in $C$:

$$X \to X^{\Delta^1} \xrightarrow{(\pi_0, \pi_1)} X \times X.$$ 

The composition of the above morphisms is the diagonal map. Note, that if $X \in C$ is fibrant, the first map is a weak equivalence, and the second map is a fibration. We denote $I := \Delta^1$.

The following definition is motivated by [AM], pg. 102.

Definition 8.2. Let $f, g : C \to D$ be two maps in $C$. A strict homotopy from $f$ to $g$ is a one simplex $H \in \text{Map}_C(C, D)_1$ going from $f$ to $g$ (or equivalently, a
map: $H : C \to D^I$, s.t. $\pi_0H = f$ and $\pi_1H = g$. We say $f$ is \textit{strictly homotopic} to $g$, if there exist a strict homotopy from $f$ to $g$. $f, g$ will be called \textit{homotopic}, denoted $f \sim g$, if they belong to the same connected component of $\text{Map}_\mathcal{C}(C, D)$ (or equivalently, if they can be related by a chain of strict homotopies).

Clearly, the homotopy relation is an equivalence relation on $\mathcal{C}(C, D)$, for every $C, D \in \mathcal{C}$. We denote by $\pi \mathcal{C}$ the category defined by:

1. $\text{Ob}(\pi \mathcal{C}) = \text{Ob}(\mathcal{C})$.
2. For every $C, D \in \text{Ob}(\mathcal{C})$: $\pi \mathcal{C}(C, D) := \pi_0\text{Map}_\mathcal{C}(C, D) = C(C, D) / \sim$.
3. Composition and identities in $\pi \mathcal{C}$ are induced from those in $\mathcal{C}$.

Let $f : X \to Y$ be a map in $\text{Pro}(\mathcal{C})$, given as a natural transformation in $\mathcal{C}^T$, for some directed category $T$. Recall that Proposition \ref{prop} takes as data two classes of maps in $\mathcal{C}$: $N$ and $M$. In the proof, we have defined two categories $\mathcal{F}_f = \mathcal{F}_f(N, M)$ and $\overline{\mathcal{F}}_f = \overline{\mathcal{F}}_f(N, M)$. Motivated by the introduction to this section, we would like have a notion of homotopy between maps in the following categories:

$$
\mathcal{F}_f(W, F), \overline{\mathcal{F}}_f(W, F), \mathcal{F}_f(C, F \cap W), \overline{\mathcal{F}}_f(C, F \cap W).
$$

All these categories are subcategories of the category $\mathcal{F}_f(C, C)$. We will turn $\mathcal{F}_f(C, C)$ into an $S$ enriched category, and thus all the above subcategories will also inherit an $S$ enriched structure. This will define the desired notion of homotopy.

Notice, that $(\mathcal{C}, \mathcal{C}, \mathcal{F} \cap W)$ is also an admissible simplicial weak fibration category (see the proof of Theorem \ref{thm}). It follows that the latter two categories are a special case of the former. Thus we will only be interested in the categories:

$$
\mathcal{F}_f := \mathcal{F}_f(W, F), \overline{\mathcal{F}}_f := \overline{\mathcal{F}}_f(W, F).
$$

We will now turn $\mathcal{F}_f(C, C)$ into an $S$ enriched category.

First assume that $T = \ast$ is the trivial category, or in other words that $X$ and $Y$ are simple objects. In this case $\mathcal{F}_f(C, C)$ is not only simplicially enriched, but also a simplicial weak fibration category, as we will now explain. It is not hard to check that the category $\mathcal{F}_f(C, C)$ has finite limits. We define, for every $B = (X \to H \to Y) \in \mathcal{F}_f(C, C)$ and every $K \in S_f$:

$$
B^K := (X \to Y \times_{Y^K} H^K \to Y) \in \mathcal{F}_f(C, C).
$$

It is not hard to check that all the axioms for a category, weakly cotensored over $S_f$ are satisfied. Let:

$$
\begin{array}{ccc}
H_1 & \to & H_2 \\
\downarrow k & & \downarrow \\
X & \to & Y
\end{array}
$$
be a morphism in $\mathcal{F}_f(C, C)$. We define it to be a fibration or a weak equivalence, if $k$ is so in $C$. It is not hard to check that we have turned $\mathcal{F}_f(C, C)$ into a simplicial weak fibration category. Notice that $(X \xrightarrow{=} X \xrightarrow{f} Y)$ is an initial object, and $(X \xrightarrow{f} Y) \xrightarrow{=} (X \xrightarrow{H} Y)$ is a terminal object in $\mathcal{F}_f(C, C)$. Thus, an object $(X \xrightarrow{H} Y) \in \mathcal{F}_f(C, C)$ is fibrant iff $H$ is a fibration in $C$.

We now return to the case where $T$ is a general directed category. In this case $\mathcal{F}_f(C, C)$ does not have finite limits in general. However, we can still define a functor: $\text{Map}(-, -) = (-)^{(-)} : \mathcal{S}^{op} \times \mathcal{F}_f(C, C) \to \mathcal{F}_f(C, C)$ by:

$$B^K := (t, X_t \to Y_t \times Y^K t H \to Y_t) \in \mathcal{F}_f(C, C),$$

for every $B = (t, X_t \to H \to Y_t) \in \mathcal{F}_f(C, C)$ and every $K \in \mathcal{S}_f$. We can use this functor to turn $\mathcal{F}_f(C, C)$ into a category enriched over $\mathcal{S}$, by defining for every $A, B \in \mathcal{F}_f(C, C)$:

$$\text{Map}_{\mathcal{F}_f(C, C)}(A, B)_n := \text{Hom}_{\mathcal{F}_f(C, C)}(A, B^\Delta^n).$$

Let $B = (t, X_t \xrightarrow{g} H \xrightarrow{h} Y_t)$ be an object in $\mathcal{F}_f(C, C)$. Then the factorization of the diagonal map:

$$B \to B^I \xrightarrow{(\pi_0, \pi_1)} B \times B,$$

described above, takes on the form:

![Diagram]( bağlı taraflı bağlantı)

Note, that if $B \in \mathcal{F}_f(C, F)$, then the first vertical map is a weak equivalence and the second vertical map is a fibration.

Let $k, l : B' = (t', X_{t'} \to H' \to Y_{t'}) \to B = (t, X_t \to H \to Y_t)$ be two maps in $\mathcal{F}_f(C, C)$. A strict homotopy from $k$ to $l$ is a map $\mathcal{H} : B' \to B^I$, s.t. $\pi_0 \circ \mathcal{H} = k$ and $\pi_1 \circ \mathcal{H} = l$:

![Diagram]( bağlı taraflı bağlantı)
In particular, if $k$ is (strictly) homotopic to $l$, then $k, l$ induce the same morphism: $\phi : t' \to t$, in $T$. Thus, given $k, l$, the information in $\mathcal{H}$ is defined entirely by the map $H' \to H^{\phi}$ in $\mathcal{C}$.

It follows that the functor $H_f : \mathcal{F}_f(\mathcal{C}, \mathcal{C}) \to \mathcal{C}$, given on objects by:

$$(t, X_t \to H \to Y_t) \to H,$$

commutes with the cotensor action. That is, there exist a coherent natural transformation:

$$H_f((t, X_t \to H \to Y_t)^K) \to (H_f(t, X_t \to H \to Y_t))^K$$

(between two functors $\mathcal{S}_f^\text{op} \times \mathcal{F}_f(\mathcal{C}, \mathcal{C}) \to \mathcal{C}$), given by the projection:

$$Y_t \times_Y H^K \to H^K.$$

It follows that $H_f$ is naturally an $\mathcal{S}$ enriched functor. Thus, there is a naturally induced functor: $\pi \mathcal{F}_f(\mathcal{C}, \mathcal{C}) \to \pi \mathcal{C}$. Restricting this functor to the subcategories $\pi \mathcal{F}_f$ and $\pi \mathcal{F}_f$, we get the following functors:

$$\pi \mathcal{H}_f : \pi \mathcal{F}_f \to \pi \mathcal{C},$$

$$\pi \mathcal{H}_f : \pi \mathcal{F}_f \to \pi \mathcal{C}.$$

We now want to prove the following:

**Proposition 8.3.** Let $H_f^A : A_f \to \mathcal{C}$ be the middle object in the factorization of $f$ constructed in Proposition 3.1 (for the case $N = \mathcal{W}, \mathcal{M} = \mathcal{F}$). then:

1. The categories $\pi \mathcal{F}_f$ and $\pi \mathcal{F}_f$ are directed.
2. The composition: $A_f \to \mathcal{F}_f \to \pi \mathcal{F}_f$, is cofinal.
3. The natural inclusion $\pi \mathcal{F}_f \to \pi \mathcal{F}_f$, is cofinal.
4. The image of the object $H_f^A \in \text{Pro}(\mathcal{C})$ under the natural map $\text{Pro}(\mathcal{C}) \to \text{Pro}(\pi \mathcal{C})$, is isomorphic to $\pi H_f$.

**Proof.** We prove 1 for $\pi \mathcal{F}_f$, and the proof in the other case is similar. Let $B_1 = (t_1, X_{t_1} \to H_1 \to Y_{t_1}), B_2 = (t_2, X_{t_2} \to H_2 \to Y_{t_2})$ be two objects in $\pi \mathcal{F}_f$. We need to find an element $B_0 = (t_0, X_{t_0} \to H_0 \to Y_{t_0})$ in $\pi \mathcal{F}_f$, with maps $B_0 \to B_1$ and $B_0 \to B_2$. This can be done using Lemma 3.15 applied on the discrete poset of two elements $R = \{1, 2\}$, and Lemma 2.19.

Let $[k], [l] : B_1 \to B_2$ be two parallel morphisms in $\pi \mathcal{F}_f$. We denote: $B_1 := (t_1, X_{t_1} \to H_1 \to Y_{t_1})$ and $B_2 := (t_2, X_{t_2} \to H_2 \to Y_{t_2})$. Then $k, l$ induce morphisms $p(k), p(l) : t_1 \to t_2$ in $T$. There exist a morphism in $T$ of the form $g : t_1' \to t_1$ s.t. $p(k) \circ g = p(l) \circ g$. Using Lemma 3.15 applied on the poset of two elements $R = \{1', 1\}$, s.t. $1' > 1$, there exist a morphism in $\pi \mathcal{F}_f$ of the form $m : B_{1'} \to B_1$ s.t. $k \circ m, l \circ m : B_{1'} \to B_2$ induce the same morphism in $T$. Thus we can assume that $k, l$ induce the same morphism: $t_1 \to t_2$, in $T$. 

42
Consider the objects $B_2 \times B_2 = (t_2, X_{t_2} \to H_2 \times Y_{t_2} \to Y_{t_2})$ and $B_1^l := (t_2, X_{t_2} \to H_2^{f_{Y_{t_2}}} \to Y_{t_2})$, in $F_f(C, C)$. Note that we have a natural map $B_1^l \to B_2 \times B_2$ in $F_f(C, C)$. Since $k, l$ induce the same morphism: $t_1 \to t_2$ in $T$, we also have an induced map: $B_1^l \to B_2 \times B_2$. To complete the proof of 1, we need to find an object $B_0 := (t_0, X_{t_0} \to H_0 \to Y_{t_0})$ in $F_f$, that can fit into a commutative diagram in $F_f(C, C)$ of the form:

![Diagram]

s.t. the map: $s$ is in $F_f$ (since then $H$ will be a strict homotopy from $ls$ to $ks$).

Define $t_0 := t_1$, and the map $t_0 \to t_1$ to be the identity. Consider the commutative square:

![Diagram]

Factor the induced map $X_{t_0} \to H_2^{f_{Y_{t_2}}} \times_{H_2 \times Y_{t_2}} H_2$ as:

$X_{t_0} \xrightarrow{\psi} H_0 \xrightarrow{\varphi} H_2^{f_{Y_{t_2}}} \times_{H_2 \times Y_{t_2}} H_2 \xrightarrow{\varphi} H_1 \to Y_{t_0}$,

to get the desired $B_0$. $H_2^{f_{Y_{t_2}}} \times_{H_2 \times Y_{t_2}} H_2 \to H_1$ is in $F$ since $H_2 \to Y_{t_2}$ is in $F$. Since $H_0 \to H_1$ is in $F$, it follows that $s$ is in $F_f$. This proves 1 (Notice that this construction is like applying Lemma 3.15 to the diagram:

![Diagram]

Remark 8.4. 1 says that the $S$ enriched categories $F_f$ and $F_f$ are directed “up to homotopy”. Actually, by the same method of proof, it can be shown that these $S$ enriched categories are directed “up to all higher homotopies”, in the following sense:

Let $B_1, B_2 \in F_f$, and let $n \geq 1$. Then for every simplicial map $\partial \Delta^n \to \text{Map}_{F_f}(B_1, B_2)$, there exist a morphism $s : B_0 \to B_1$ in $F_f$, s.t. in the diagram:

![Diagram]
there exist a dotted arrow, making the diagram commutative (and similarly for $\mathcal{F}_f$).

**Lemma 8.5.** The natural functors:

$$
\mathcal{F}_f \to \pi\mathcal{F}_f,
$$

$$
\mathcal{F}_f \to \pi\mathcal{F}_f
$$

are pre-cofinal (Definition 3.5).

**Proof.** By definition these functors are onto, on both objects and morphisms, and are thus clearly pre-cofinal.

We have by Lemma 8.4 that the functor $A_f \to \mathcal{F}_f$ is pre-cofinal, and by Lemma 3.5 that the functor $\mathcal{F}_f \to \pi\mathcal{F}_f$ is pre-cofinal. Thus by Lemma 3.10 the functor $A_f \to \pi\mathcal{F}_f$ is pre-cofinal. But now, since $A_f, \pi\mathcal{F}_f$ are both directed (1 and Lemma 3.16), we get by Lemma 3.11 that the functor $A_f \to \pi\mathcal{F}_f$ is cofinal. This proves 2.

We now prove 3. Let $B \in \pi\mathcal{F}_f$. By [Hir] section 14.2, it is enough to show that the over category $i/B$ is nonempty and connected. It is nonempty since it contains $[\text{id}_B]$. Let $[f] \in \pi\mathcal{F}_f(B_1, B)$ and $[g] \in \pi\mathcal{F}_f(B_2, B)$. It is enough to show that there exist $[l] \in \pi\mathcal{F}_f(B_0, B_1)$ and $[k] \in \pi\mathcal{F}_f(B_0, B_2)$, s.t. $[f][l] = [g][k]$.

We denote: $B := (t, X_t \to H \to Y_t), B_1 := (t_1, X_{t_1} \to H_1 \to Y_{t_1})$ and $B_2 := (t_2, X_{t_2} \to H_2 \to Y_{t_2})$. Since $T$ is directed, there exist $t_3 \in T$ that fits into a commutative diagram:

$$
\begin{array}{c}
\Delta^n \\
\downarrow \\
\partial\Delta^n
\end{array}
\xymatrix{
\partial\Delta^n \ar[r] \ar[d] & \text{Map}_{\mathcal{F}_f}(B_1, B_2) \ar[r]^{\delta^*} & \text{Map}_{\mathcal{F}_f}(B_0, B_2) \ar[d] \\
\Delta^n & & \text{Map}_{\mathcal{F}_f}(B_0, B_2)
\end{array}
$$

Using Lemma 3.15 applied on the poset of two elements $R = \{3, 1\}$, s.t. $3 > 1$, we get an object $B_{t_1}$ in $\pi\mathcal{F}_f$, and a morphism in $\pi\mathcal{F}_f$ of the form $m : B_{t_1} \to B_1$.

Using Lemma 3.15 applied on the poset of two elements $R = \{3, 2\}$, s.t. $3 > 2$, we get an object $B_2$ in $\pi\mathcal{F}_f$, and a morphism in $\pi\mathcal{F}_f$ of the form $m : B_{t_2} \to B_2$. Thus we can assume that $t_1 = t_2 = t'$, and $f, g$ induce the same morphism: $t' \to t$, in $T$.

Consider the objects $B \times B = (t, X_t \to H \times Y_t, H \to Y_t)$, $B^l := (t, X_{t_l} \to H_{t_l} \to Y_{t_l})$ and $B_1 \times B_2 = (t', X_{t'} \to H_{t'} \times Y_{t'}, H_{t'} \to Y_{t'})$, in $\mathcal{F}_f(C, C)$. Note that
we have a natural map $B^I \rightarrow B \times B$ in $\mathcal{F}_f(\mathcal{C}, \mathcal{C})$. Since $f, g$ induce the same morphism: $t' \rightarrow t$ in $T$, we also have an induced map: $B_1 \times B_2 \xrightarrow{f \times g} B \times B$.

To complete the proof we need to find an object $B_0 := (t_0, X_{t_0} \rightarrow H \rightarrow Y_{t_0})$ in $\mathcal{F}_f$, that can fit into a commutative diagram in $\mathcal{F}_f(\mathcal{C}, \mathcal{C})$ of the form:

$$
\begin{array}{c}
B_0 \\
\downarrow (l, k) \\
B_1 \times B_2 \\
\downarrow f \times g \\
B \times B
\end{array}
$$

s.t. the map: $(l, k)$ is in $\mathcal{F}_f$ (since then $H$ will be a strict homotopy from $fl$ to $gk$).

Define $t_0 := t'$, and the map $t_0 \rightarrow t'$ to be the identity. Consider the commutative square:

$$
\begin{array}{ccc}
X_{t_0} & \xrightarrow{H^I} & H^I_{Y_i} \\
\downarrow & & \downarrow \\
H_1 \times_{Y_i} H_2 & \xrightarrow{f \times g} & H \times_{Y_i} H
\end{array}
$$

Factor the induced map $X_{t_0} \rightarrow H^I_{Y_i} \times_{(H \times Y_i, H)} (H_1 \times_{Y_i} H_2)$ as:

$$
X_{t_0} \xrightarrow{W} H_0 \xrightarrow{\varphi} H^I_{Y_i} \times_{(H \times Y_i, H)} (H_1 \times_{Y_i} H_2) \xrightarrow{\varphi} H_1 \times_{Y_i} H_2 \xrightarrow{\varphi} Y_{t_0},
$$

to get the desired $B_0, H^I_{Y_i} \times_{(H \times Y_i, H)} (H_1 \times_{Y_i} H_2) \rightarrow H_1 \times_{Y_i} H_2$ is in $\mathcal{F}$ since $H \rightarrow Y_i$ is in $\mathcal{F}$. Since $H_0 \rightarrow H_1 \times_{Y_i} H_2$ is in $\mathcal{F}$, it follows that $(l, k)$ is in $\mathcal{F}_f$.

This proves 3.

4 follows from the commutativity of the following diagram:

$$
\begin{array}{ccc}
A_f & \xrightarrow{H_f} & \mathcal{C} \\
\downarrow \pi \mathcal{F}_f & & \downarrow \pi \mathcal{F}_f \\
\pi \mathcal{F}_f & \xrightarrow{\pi H_f} & \pi \mathcal{C}
\end{array}
$$

We now specialize to the case of computing a cofibrant replacement in $\text{Pro}(\mathcal{C})$, for the terminal object $* \in \mathcal{C}$. Assume that there exist an initial object $\phi \in \mathcal{C}$. Then such a cofibrant replacement can be achieved by factoring the unique map $\phi \rightarrow *$ in $\text{Pro}(\mathcal{C})$ as $\phi \xrightarrow{C} H^A \xrightarrow{FC \cap W} *$. Such a factorization is given by Proposition 3.1 taking $N = \text{Mor}(\mathcal{C}), M = \mathcal{F} \cap W$, as explained in the beginning of this section. Let $A$ denote the indexing category of the pro object $H^A$ in the above factorization.

Recall the $\mathcal{S}$ enriched categories $\mathcal{F}_f$ and $\mathcal{F}_f^\mathcal{S}$ constructed above, for $N = \text{Mor}(\mathcal{C}), M = \mathcal{F} \cap W$ and $f : \phi \rightarrow *$. $\mathcal{F}_f^\mathcal{S}$ is just the full (simplicial) subcategory.
of $C$ spanned by the fibrant and contractible objects. We thus denote $\mathcal{C}_{fw} := \mathcal{F}_f$. $\mathcal{F}_f$ is the (simplicial) subcategory of $C$ spanned by the fibrant and contractible objects, and acyclic fibrations between them. We thus denote $\mathcal{C}_{fw} := \mathcal{F}_f$. By Proposition 8.3 we have that:

1. The categories $\pi \mathcal{C}_{fw}$ and $\pi \mathcal{C}_{fw}$ are directed.
2. The composition: $A \to \mathcal{C}_{fw} \to \pi \mathcal{C}_{fw}$, is cofinal.
3. The natural inclusion: $\pi \mathcal{C}_{fw} \to \pi \mathcal{C}_{fw}$, is cofinal.

As we will see in subsequent sections, many important pro objects can be extracted from $H^A : A \to C$. This is done via functors from $C$ to other categories. Given such a functor $F : C \to D$, we can compose it with $H^A$, and get a pro object in $D$:

$$A \xrightarrow{H^A} C \xrightarrow{F} D.$$ 

This pro object can be thought of as a left derived functor of $F$, evaluated at the terminal object. Suppose now that $F$ is a homotopy functor, that is, it factors through $\pi C$:

$$F : C \to \pi C \to D.$$ 

In this case the pro object $F \circ H^A : A \to D$ factors through $\pi \mathcal{C}_{fw}$:

Since the natural functor: $A \to \pi \mathcal{C}_{fw}$ is left cofinal, it follows that the pro object $F \circ H^A : A \to D$ is isomorphic, in $\text{Pro}(D)$, to a pro object indexed by $\pi \mathcal{C}_{fw}$. Thus $F \circ H^A$ admits a much simpler representation in this case. Note, however, that the cofibrant replacement itself: $H^A : A \to C$, does not factor through $\pi \mathcal{C}_{fw}$, and thus is not isomorphic to any functor $\pi \mathcal{C}_{fw} \to C$.

9 Simplicial Presheafs as a Weak Fibration Category

Let $C = (C, \tau)$ be a small Grothendieck site, and let $SPS(C) := \mathcal{S}^{\text{simp}}$ denote the category of simplicial presheafs on $C$. In [Jar], Jardine defines the notions of combinatorial weak equivalences and local fibrations in $SPS(C)$. In the same paper Jardine defines a model structure on $SPS(C)$. However the local fibrations are not the fibrations in this model structure. Jardine (in [Jar]) proves almost all that is needed to show that combinatorial weak equivalences and local
fibrations give rise to a simplicial weak fibration category structure on $SPS(C)$ (without considering this notion directly). In this section we complete the proof of this fact, and also review some of the definitions and proofs presented in [Jar], for the sake of completeness. We follow the common convention in the field, and call Jardine’s combinatorial weak equivalences local weak equivalences (see [DuIs], [Jar1]).

Let $X$ be a simplicial set, and let $x \in X_0$. For every $1 \leq m$, we have:

$$\pi_m(X, x) = \pi_m(|X|, x),$$

i.e. the corresponding homotopy group of the realization $|X|$, at the point $x$. The set of path components of $X$: $\pi_0(X) := \pi_0(|X|)$, has a combinatorial description: $\pi_0(X)$ is the coequalizer of: $X_1 \rightrightarrows X_0$.

A simplicial set map $f: X \to Y$ is called a weak equivalence, if:

1. The function: $\pi_0(X) \to \pi_0(Y)$ is a bijection.
2. The induced map: $\pi_m(X, x) \to \pi_m(Y, f(x))$ is an isomorphism, for every $x \in X_0$, and every $1 \leq m$.

There is a base point free way to describe this. For every $1 \leq m$, define the set:

$$\pi_m(X) := \bigsqcup_{x \in X_0} \pi_m(X, x).$$

There is a canonical map: $\pi_m(X) \to X_0$, which is a group object in the category: $Set_{/X_0}$. This group object is abelian if $n \geq 2$. Any simplicial set map $f: X \to Y$, induces a commutative diagram:

$$\xymatrix{ \pi_m(X) \ar[r] & \pi_m(Y) \\ X_0 \ar[u] \ar[r] & Y_0. \ar[u] }$$

The map $f$ is a weak equivalence if:

1. The function $\pi_0(X) \to \pi_0(Y)$ is a bijection.
2. The diagram above is a pullback diagram for every $1 \leq m$.

The above constructions are all functorial. Thus, given a simplicial presheaf $X \in SPS(C)$, we can define a presheaf: $\pi_0(X)$, and presheaf maps: $\pi_m(X) \to X_0$, for every $m \geq 1$. Any simplicial presheaf map $f: X \to Y$, induces a presheaf morphism: $\pi_0(X) \to \pi_0(Y)$, and a commutative diagram of presheaves as above. For every $m \geq 0$, write $\widetilde{\pi_m}X$ for the sheaf associated to the presheaf $\pi_mX$. Now we can give the definition of a local weak equivalence. A map $f: X \to Y$ of simplicial presheaves is called a local weak equivalence, if:

1. The map $\widetilde{\pi_0}X \to \widetilde{\pi_0}Y$ is an isomorphism of sheaves.
2. The diagram:

\[
\begin{array}{ccc}
\pi_m(X) & \rightarrow & \pi_m(Y) \\
\downarrow & & \downarrow \\
\tilde{X}_0 & \rightarrow & Y_0
\end{array}
\]

is a pullback diagram in \( \text{Sh}(\mathcal{C}) \), for every \( 1 \leq m \).

We denote the class of local weak equivalences in \( \text{SPS}(\mathcal{C}) \), by \( \mathcal{W} \). Note that every levelwise weak equivalence in \( \text{SPS}(\mathcal{C}) \), is a local weak equivalence, since the two conditions are satisfied at the presheaf level, and hence also at the sheaf level (since sheafification commutes with pull backs).

**Definition 9.1.** Let \( f : A \rightarrow B \) be a map of simplicial sets, and let \( g : X \rightarrow Y \) be a map in \( \text{SPS}(\mathcal{C}) \). We say that \( g \) has the *local* right lifting property with respect to \( f \), if for every \( U \in \mathcal{C} \), and every square of the form:

\[
\begin{array}{ccc}
A & \rightarrow & X(U) \\
\downarrow & & \downarrow \\
B & \rightarrow & Y(U)
\end{array}
\]

there exist a covering sieve \( R \) of \( U \), such that for every \( \phi : V \rightarrow U \) in \( R \), there is a lift:

\[
\begin{array}{ccc}
A & \rightarrow & X(U) & \rightarrow & X(V) \\
\downarrow & & \downarrow & & \downarrow \\
B & \rightarrow & Y(U) & \rightarrow & Y(V)
\end{array}
\]

In this case we shall denote:

\[ f \perp_l g \]

Let \( M \) be a class of maps of simplicial sets. We denote by \( M \perp_l \) the class of all morphisms in \( \text{SPS}(\mathcal{C}) \), having the local right lifting property with respect to every map in \( M \).

**Definition 9.2.** Let \( f : X \rightarrow Y \) be a map in \( \text{SPS}(\mathcal{C}) \).

1. Let \( \text{Cof} \) denote the subcategory of cofibrations (inclusions) in \( S_f \).
2. Let \( \text{CW} \) denote the subcategory of acyclic cofibrations in \( S_f \).
3. We say that \( f \) is a *local acyclic fibration*, if \( f \in \text{Cof} \perp_l \). We denote: \( \mathcal{F}_{\text{W}} := \text{Cof} \perp_l \).
4. We say that \( f \) is a *local fibration*, if \( f \in \text{CW} \perp_l \). We denote: \( \mathcal{F} := \text{CW} \perp_l \).

**Remark 9.3.**
1. Note that every level wise (acyclic) fibration in \( SPS(\mathcal{C}) \) is also a local (acyclic) fibration. In fact, we have the usual lifting property and not just the local one.

2. By considering the map \( \phi \to \Delta^0 \) in \( \text{Cof} \), we see that every local acyclic fibration in \( SPS(\mathcal{C}) \), is a local epimorphism in dimension 0.

**Lemma 9.4.** Let \( f : X \to Y \) be a map in \( SPS(\mathcal{C}) \). Then:

1. \( f \) is a local acyclic fibration iff \( f \) has the local right lifting property with respect to all inclusions of the form \( \partial \Delta^n \to \Delta^n \) \((n \geq 0)\).

2. \( f \) is a local fibration iff \( f \) has the local right lifting property with respect to all inclusions of the form \( \Lambda^n_k \to \Delta^n \) \((n \geq 0, 0 \leq k \leq n)\) \((This \ is \ the \ definition \ of \ a \ local \ fibration \ given \ in \ [Jar].)\)

**Proof.**

1. This is explained in [Jar] (see the remark after Corollary 1.5).

2. This follows from Lemma 5.8 and [Jar] Corollary 1.4.

**Lemma 9.5.** We have:

\[ \mathcal{F}W = \mathcal{F} \cap \mathcal{W}. \]

**Proof.** See [DuIs] Proposition 7.2 (based heavily on [Jar]).

**Proposition 9.6.** \((SPS(\mathcal{C}), \mathcal{W}, \mathcal{F})\) is a weak fibration category.

**Proof.** \( SPS(\mathcal{C}) \) has all limits and colimits, and they are computed object wise. Since \( \mathcal{F} \) is defined by a local lifting property, it is easy to see that it is a subcategory, that contains all the isomorphisms, and is closed under pullbacks. The same is true for \( \mathcal{F} \cap \mathcal{W} \), by Lemma 9.5. The fact that \( \mathcal{W} \) has the 2 out of 3 property, and contains all the isomorphisms, is also clear. Thus it remains to show the existence of factorizations. Consider a functorial factorization to an acyclic cofibration followed by a fibration, in the standard model structure on simplicial sets. Given a map \( f : X \to Y \) in \( SPS(\mathcal{C}) \), we can apply this functorial factorization levelwise, and obtain a factorization of \( f \) in \( SPS(\mathcal{C}) \):

\[ X \to Z \to Y, \]

where \( X \to Z \) is a levelwise weak equivalence, and thus in \( \mathcal{W} \), and \( Z \to Y \) is a levelwise fibration, and thus in \( \mathcal{F} \).

Note that the weak fibration category \((SPS(\mathcal{C}), \mathcal{W}, \mathcal{F})\), is naturally enriched over \( S \). For a simplicial presheaf \( X \in SPS(\mathcal{C}) \), and a finite simplicial set \( K \in S_f \), we define \( K \otimes X, X^K \in SPS(\mathcal{C}) \) levelwise. This makes \( SPS(\mathcal{C}) \) tensored and cotensored over \( S_f \). The following two Lemmas are based partly on Corollary 7.4 in [DuIs].
**Lemma 9.7.** Let $f : X \to Y$ be a map in $SPS(C)$. Then the following are equivalent:

1. $f$ is a local acyclic fibration.
2. For every map $K \to L$ in $Cof$, the induced map:
   \[ X^L \to Y^L \times_{Y^K} X^K \]
   is a local acyclic fibration.
3. For every map $K \to L$ in $Cof$, the induced map:
   \[ X^L \to Y^L \times_{Y^K} X^K \]
   is a local epimorphism in dimension 0.

**Proof.**

(1) $\Rightarrow$ (2) Let $K \to L$ be a map in $Cof$. We need to show that $X^L \to Y^L \times_{Y^K} X^K$ has the local right lifting property with respect to every map $T \to S$ in $Cof$ (9.2). By adjointness, we need to check that $X \to Y$ has the local right lifting property with respect to all maps of the form:

\[ j : L \times T \coprod_{K \times T} K \times S \to L \times S. \]

Since $\mathcal{S}$ is a simplicial model category, $j$ is a cofibration in $\mathcal{S}_f$. Therefore the result follows from the definition of a local acyclic fibration (9.2).

(2) $\Rightarrow$ (3) Obvious (see the remark following definition 9.2).

(3) $\Rightarrow$ (1) Let $i : K \to L$ be a map in $Cof$, and let $U \in \mathcal{C}$. Consider a square of the form:

\[
\begin{array}{ccc}
K & \xrightarrow{k} & X(U) \\
\downarrow{i} & & \downarrow{f_U} \\
L & \xrightarrow{l} & Y(U).
\end{array}
\]

by assumption, the induced map: $(X^L)_0 \to (Y^L \times_{Y^K} X^K)_0$ is a local epimorphism. But $(l,k) \in (Y(U)^L \times_{Y(U)^K} X(U)^K)_0$. Thus, there exist a covering sieve $R$ of $U$, such that for every $\phi : V \to U$ in $R$, there exist $g \in (X(V)^L)_0$ that maps to the restriction of $(l,k)$ to $V$, or in other words, there exist a lift $g$ in the diagram:

\[
\begin{array}{ccc}
K & \xrightarrow{k} & X(U) & \to & X(V) \\
\downarrow{i} & & \downarrow{} & & \downarrow{} \\
L & \xrightarrow{l} & Y(U) & \to & Y(V).
\end{array}
\]

\[ \square \]
Lemma 9.8. Let $f : X \to Y$ be a map in $SPS(C)$. Then the following are equivalent:

1. $f$ is a local fibration.
2. For every map $K \to L$ in $CW$, the induced map:
   $$X^L \to Y^L \times_{Y^K} X^K$$
   is a local acyclic fibration.
3. For every map $K \to L$ in $CW$, the induced map:
   $$X^L \to Y^L \times_{Y^K} X^K$$
   is a a local epimorphisms in dimension 0.

Proof. Similar to Lemma 9.7.

Proposition 9.9. The weak fibration category $(SPS(C), W, F)$ is simplicial (7.4).

Proof. Let $i : K \to L$ be a cofibration in $\mathcal{S}_f$, and let $p : X \to Y$ be a fibration in $SPS(C)$. We need to show that the induced map:

$$X^L \to Y^L \times_{Y^K} X^K$$

is a local fibration, that is acyclic if either $i$ or $p$ is.

Let us show that $X^L \to Y^L \times_{Y^K} X^K$ has the local right lifting property with respect to every map $T \to S$ in $CW$ (9.2). By adjointness, we need to check that $X \to Y$ has the local right lifting property with respect to all maps of the form:

$$j : L \times T \prod_{K \times T} K \times S \to L \times S.$$ 

Since $\mathcal{S}$ is a simplicial model category, $j$ is an acyclic cofibration in $\mathcal{S}_f$. Therefore the result follows from the definition of a local fibration (9.2).

Suppose that $i$ is acyclic. Then $X^L \to Y^L \times_{Y^K} X^K$ is acyclic by Lemma 9.8.

Suppose that $p$ is acyclic. Then $X^L \to Y^L \times_{Y^K} X^K$ is acyclic by Lemma 9.7.

9.1 Simplicial sheafs as a weak fibration category

Let $SSH(C) := Sh(C)^\Delta^{op}$ denote the category of simplicial sheafs on $C$. Note that $SSH(C)$ is just the full subcategory of $SPS(C)$ spanned by the objects that satisfy the (usual) sheaf condition, since limits in $SPS(C)$ are calculated levelwise. It is a classical fact (see for example [Jar]) that there is a functor $L : PSh(C) \to PSh(C)$, such that $L^2$ is left adjoint to the inclusion $i : Sh(C) \to PSh(C)$. $L^2$ is called the sheafification functor. We can take these functors dimension-wise, and obtain a functor $L : SPS(C) \to SPS(C)$, and an adjunction:

$$L^2 : SPS(C) \cong SSH(C) : i.$$
Definition 9.10. We say that a map in $SSh(\mathcal{C})$ is a local weak equivalence (resp. local fibration) if it is a local weak equivalence (resp. local fibration) as a map in $SPS(\mathcal{C})$.

By abuse of notation we denote the class of local weak equivalences (resp. local fibrations) in $SSh(\mathcal{C})$ also by $W$ (resp. $\mathcal{F}$).

Proposition 9.11. $(SSh(\mathcal{C}), W, \mathcal{F})$ is a weak fibration category.

Proof. $Sh(\mathcal{C})$ is a topos, and thus has all limits and colimits. It follows that $SSh(\mathcal{C}) = Sh(\mathcal{C})^{\Delta^{op}}$ also has all limits and colimits, and they are computed level wise. Since the inclusion $i : SSh(\mathcal{C}) \hookrightarrow SPS(\mathcal{C})$ has a left adjoint, it commutes with pullbacks. It is thus easy to see that $\mathcal{F}, \mathcal{F} \cap W$ are subcategories, that contain all the isomorphisms, and are closed under pullbacks. The fact that $W$ has the 2 out of 3 property, and contains all the isomorphisms is also clear. Thus it remains to show the existence of factorizations. Let $f : X \to Y$ be a map in $SSh(\mathcal{C})$. We already proved that in $SPS(\mathcal{C})$ we have a factorization: $X \xrightarrow{W} Z \xrightarrow{\mathcal{F}} Y$. Now consider the commutative diagram:

\[
\begin{array}{ccc}
X & \xrightarrow{W} & Z \\
\downarrow{\cong} & & \downarrow{\cong} \\
L^2(X) & \xrightarrow{f} & L^2(Y)
\end{array}
\]

By [Jar] Lemma 1.6, the middle vertical map is in $\mathcal{F}W$, and by [Jar] Corollary 1.8, the map $L^2(Z) \to L^2(Y)$ is in $\mathcal{F}$. Thus we get that $f$ is in $W$, and $g$ is in $\mathcal{F}$. \hfill \Box

The category $SSh(\mathcal{C})$, inherits an $S$ enriched structure as a full subcategory of $SPS(\mathcal{C})$. For a simplicial sheaf $X \in SSh(\mathcal{C})$, and a finite simplicial set $K \in S_f$, we can define $K \otimes X, X^K$ as in $SPS(\mathcal{C})$, and then take sheafification. This makes $SSh(\mathcal{C})$ tensored and cotensored over $S_f$. It is not hard to check that this structure turns $(SSh(\mathcal{C}), W, \mathcal{F})$ into a simplicial weak fibration category. Furthermore, Lemmas 9.8 and 9.7 remain valid, if we replace $SPS(\mathcal{C})$ by $SSh(\mathcal{C})$.

9.2 The new model structures

As shown in [Jar], Theorems 2,5, there exist proper model category structures on the categories $SPS(\mathcal{C}), SSh(\mathcal{C})$, in which the weak equivalences are the local weak equivalences. Thus, as relative categories, $(SPS(\mathcal{C}), W), (SSh(\mathcal{C}), W)$ are pro admissible (see Remark 4.17). We have shown in this section that $SPS(\mathcal{C}), SSh(\mathcal{C})$ can also be given (other) weak fibration structures, with the same class of weak equivalences. It follows from Theorem 4.4 that there are induced model category structures on $Pro(SPS(\mathcal{C})), Pro(SSh(\mathcal{C}))$.

Consider the inclusion functor:

\[
i : SSh(\mathcal{C}) \hookrightarrow SPS(\mathcal{C}).\]
Since $i$ has a left adjoint ($L^2$) it commutes with finite limits, and it clearly preserves local fibrations and local acyclic fibrations. Thus $i$ is a weak right Quillen functor, and it induces a Quillen adjunction (6.3):

$$\text{Pro}(L^2) : \text{Pro}(SPS(C)) \leftrightarrows \text{Pro}(SSh(C)) : \text{Pro}(i).$$

We claim that this Quillen adjunction is a Quillen equivalence. This follows easily from the fact that both $L^2$ and $i$ preserve local weak equivalences, and the unit and counit of the adjunction $L^2 \dashv i$ are also weak equivalences (see [Jar] Lemma 1.6).

Consider the sheafification functor:

$$L^2 : SPS(C) \rightarrow SSh(C).$$

It is a well known fact that $L^2$ commutes with finite limits (see for example [Jar]). By [Jar] Corollary 1.8, $L^2$ preserves local fibrations, and by [Jar] Lemma 1.6, $L^2$ preserves local acyclic fibrations. Thus $L^2$ is a weak right Quillen functor, and it induces a Quillen adjunction (6.3):

$$L_{L^2} : \text{Pro}(SSh(C)) \leftrightarrows \text{Pro}(SPS(C)) : \text{Pro}(L^2).$$

This Quillen adjunction can also be shown to be a Quillen equivalence.

### 10 The Étale Homotopy Type as a Derived Functor

Given an algebraic variety $X$, Artin and Mazur defined in [AM] the notion of the étale homotopy type of $X$, by applying the connected components functor to the hypercoverings in the étale site of $X$. This gives rise to an object in the category $\text{Pro}(\text{Ho}(S))$, where $S$ is the category of simplicial sets. Artin and Mazur’s construction can be easily generalized to any locally connected site $C$. However, for many applications it is essential to lift Artin and Mazur’s construction from $\text{Pro}(\text{Ho}(S))$ to $\text{Pro}(S)$. This was achieved by Frindlander in [Fri], by replacing hypercoverings with rigid hypercoverings. In this section we shall give an alternative solution, by using the model structure described in Section 9. This new approach will give a nice description of the étale homotopy type as the result of applying a derived functor, and will also have the advantage of working with usual hypercoverings rather than the more involved rigid hypercoverings (see Definition 10.2 and Lemma 10.3 below). Another advantage is that our construction works over any site.

**Proposition 10.1.** Let $T = Sh(C)$, $S = Sh(D)$ be two topoi, and let:

$$f^* : S \rightleftarrows T : f_*,$$

be a geometric morphism. Then $f^*$ induce a weak right Quillen functor:

$$f^* : S^{\Delta^{op}} \rightarrow T^{\Delta^{op}},$$

53
relative to the local weak fibration structure on simplicial sheafs, described in Section 9.

Proof. $f^*: S \to T$ preserves finite limits by definition of a geometric morphism, so $f^*: S^{\Delta^{op}} \to T^{\Delta^{op}}$ also preserves finite limits. Further, since $f^*: S \to T$ preserves local epimorphisms, it follows from Lemmas 9.8 and 9.7 that $f^*: S^{\Delta^{op}} \to T^{\Delta^{op}}$ preserves local fibrations and local acyclic fibrations.

Definition 10.2. Let $T$ be a topos. Consider the unique geometric morphism:

$\Gamma^*: Set \rightleftarrows T : \Gamma_!$.

$\Gamma_!$ is the global sections functor, and $\Gamma^*$ is the constant sheaf functor. By Propositions 10.1 and 6.3, we have a Quillen adjunction:

$L_{\Gamma^*}: \text{Pro}(T^{\Delta^{op}}) \rightleftarrows \text{Pro}((Set)^{\Delta^{op}}): \text{Pro}(\Gamma^*)$.

We define the **topological realization** of $T$, to be:

$|T| := \text{L} L_{\Gamma^*}(*_T) \in \text{Pro}((Set)^{\Delta^{op}}) = \text{Pro}((\mathcal{S}))$

Where $*_T$ is a terminal object of $T^{\Delta^{op}}$.

If $\mathcal{C}$ is a small Grothendieck site, we define the **topological realization** of $\mathcal{C}$, to be: $|\mathcal{C}| := |\text{Sh}(\mathcal{C})|$.

A case of special interest is when $T$ is locally connected, i.e when $\Gamma^*: Set \to T$ has a left adjoint $\Gamma_!: T \to Set$. In geometric situations, the functor $\Gamma_!$ is induced by the functor which sends a scheme to its set of connected scheme-theoretic components. Thus we shall denote $\pi_0 := \Gamma_!$. By Proposition 6.3, we get, when $T$ is locally connected:

$L_{\Gamma^*} \cong \text{Pro}(\pi_0)$.

It follows that:

$|T| = \text{L} \text{Pro}(\pi_0)(*_T)$.

This formula allow us to give quite a concrete description of $|T|$. Recall that in order to compute a left derived functor, one should apply the original functor to a cofibrant replacement. Thus, we should apply $\text{Pro}(\pi_0)$ to a cofibrant replacement of $*_T$. In section 8, we gave a description of such a cofibrant replacement, as a functor

$H^A: A_T \to T^{\Delta^{op}}$.

**Example 7.** Let $\mathcal{C}$ be a small category, equipped with the trivial topology. Then $T = \text{Sh}(\mathcal{C}) = (\mathcal{C})^{\text{op}}$ is the category of functors $(\mathcal{C})^{\text{op}} \to Set$. $\Gamma^* = \Delta: Set \to (\mathcal{C})^{\text{op}}$ is the diagonal functor. $T$ is locally connected, since $\Delta$ has a left adjoint, which is the colimit functor: $\text{colim} = \Gamma_! = \pi_0: (\mathcal{C})^{\text{op}} \to Set$. By definition:

$|\mathcal{C}| = |T| = \text{L} \text{Pro}(\text{colim})(*_T)$.  

54
The above defined weak fibration structure on \( T^{\Delta_{op}} = \mathcal{S}^{\Delta_{op}} \), is just the projective model structure on \( \mathcal{S}^{\Delta_{op}} \). Let \( E(C^{op}) \to * \) be a cofibrant replacement to the terminal object in the projective structure on \( \mathcal{S}^{\Delta_{op}} \). By [Isa] we have that the cofibrations in the projective model structure on \( \text{Pro}(\mathcal{S}^{\Delta_{op}}) \) are just \( Lw_{\sim}(\text{Cof}) \), where \( \text{Cof} \) are the cofibrations in \( \mathcal{S}^{\Delta_{op}} \). Thus, \( E(C^{op}) \to * \) is also a cofibrant replacement to the terminal object in the projective structure on \( \text{Pro}(\mathcal{S}^{\Delta_{op}}) \), and we get that:

\[
|C| = |T| = \text{L Pro}(\text{colim}(\ast_T)) = \text{Pro}(\text{colim}(E(C^{op}))) = \colim_{C^{op}} E(C^{op}) = \text{hocolim}_{C^{op}} \ast \simeq N(C^{op}) \simeq N(C).
\]

**Lemma 10.3.** Let \( X \) be a locally notherian scheme, and \( X_{\text{ét}} \) its étale topos. Consider the natural functor:

\[
\text{Ho} : \text{Pro}(\mathcal{S}) \to \text{Pro}(\text{Ho}(\mathcal{S})).
\]

Then \( \text{Ho}(|X_{\text{ét}}|) \) is isomorphic to the étale homotopy type of \( X \), defined in [AM].

**Proof.** Define \( T := X_{\text{ét}} \). First note that \( T \) is locally connected, so the discussion above applies. Following the notation of Section 8, let \( \pi T^{\Delta_{op}} f_w \) denote the full sub \( \mathcal{S} \)-category of \( T^{\Delta_{op}} \) spanned by the locally fibrant locally contractible objects, and let \( \pi T^{\Delta_{op}} f_w \), denote its homotopy category. As we have shown in Section 8 \( \pi T^{\Delta_{op}} f_w \) is directed, and we have a left cofinal functor: \( A_T \to \pi T^{\Delta_{op}} f_w \).

We have a commutative diagram:

\[
\begin{array}{ccc}
A_T & \xrightarrow{H^A} & T^{\Delta_{op}} f_w \\
\downarrow & & \downarrow \\
\pi T^{\Delta_{op}} f_w & \xrightarrow{\gamma} & H\mathcal{O}(\mathcal{S}).
\end{array}
\]

It follows that the pro object \( \gamma|T| = \gamma \pi_0 H^A : A_T \to \text{Ho}(\mathcal{S}) \) factors through \( \pi T^{\Delta_{op}} f_w \). Since \( A_T \to \pi T^{\Delta_{op}} f_w \) is left cofinal, it follows that the pro object \( \gamma|T| : A_T \to \text{Ho}(\mathcal{S}) \) is isomorphic, in \( \text{Pro}(\text{Ho}(\mathcal{S})) \), to a pro object: \( \pi T^{\Delta_{op}} f_w \to \text{Ho}(\mathcal{S}) \).

In order to get Artin and Mazur’s construction, we should restrict this pro object only to hypercoverings, i.e. to those locally fibrant locally contractible simplicial sheaves, which are levelwise a coproduct of representables in the étale site of \( X \). However, since the hypercoverings are cofinal among all the locally fibrant locally contractible simplicial sheaves (Lemma 2.2 in [Jar3]), The resulting object in \( \text{Pro}(\text{Ho}(\mathcal{S})) \) is isomorphic.

**Remark 10.4.** As we have mentioned, Artin and Mazur’s construction can be generalized to any locally connected topos \( T \). Lemma 10.3 remains valid also in this more general situation, and the proof is exactly the same.
10.1 The relative homotopy type

The notion of a relative étale homotopy type was considered in [HaSc] as a useful construction for the study of rational points. However, similarly to Artin and Mazur étale homotopy type, the relative étale homotopy type was not given within a suitable model category. In this section we lift this construction in a suitable way.

Definition 10.5. Let \( T = Sh(C), S = Sh(D) \) be two topoi, and let:

\[
f^* : S \xrightarrow{\simeq} T : f_*
\]

be a geometric morphism. By Propositions \( \text{[10.1]} \) and \( \text{[6.3]} \) we have a Quillen adjunction:

\[
L f_* : \text{Pro}(T^{\Delta^{op}}) \xrightarrow{\simeq} \text{Pro}(S^{\Delta^{op}}) : \text{Pro}(f^*)
\]

We define the relative topological realization of \( T \) over \( S \), to be:

\[
|T|_S := LLf_*(\ast_T) \in \text{Pro}(S^{\Delta^{op}}),
\]

Where \( \ast_T \) is a terminal object of \( T^{\Delta^{op}} \).

If the above geometric morphism corresponds to the morphism of sites: \( \phi : C \to D \), we also define the relative topological realization of \( C \) over \( D \), to be:

\[
|C|_D := |T|_S.
\]

As in the case \( S = \text{Set} \), if the geometric morphism \( f^* : S \xrightarrow{\simeq} T : f_* \) is essential, i.e if \( f^* : S \to T \) has a left adjoint \( f_! : T \to S \), we have:

\[
L f_* \cong \text{Pro}(f_!),
\]

and this allows us to give very a concrete description of \( |T|_S \). There is also an analogue of Lemma \( \text{[10.3]} \) if we are only interested in the image of \( |T|_S \) in \( \text{Pro}(\text{Ho}(S^{\Delta^{op}})) \).

Since this construction is functorial, we get a functor:

\[
| \bullet |_S : \text{Topoi}/S \to \text{Pro}(S^{\Delta^{op}}).
\]

It is easy to verify, that for every topos \( S \) we have \( |S|_S \simeq \ast_S \). Thus, by the functoriality of \( | \bullet |_S \), we have a map:

\[
h : T(S) \to [\ast_S, |T|_S]_{\text{Pro}(S^{\Delta^{op}})},
\]

Where \( T(S) \) is the set of geometric morphisms \( s_* : S \to T \), which are sections of the map \( f_* : T \to S \). The codomain of \( h \) above has an obstruction theory and a Bousfield-Kan type spectral sequence. Thus the map \( h \) can be used to study sections of maps of topoi. This will be discussed in more detail in a future paper.

56
10.2 Rational points

The work presented in this paper originated from the motivation of finding a suitable model structure in which the general machinery of abstract homotopy theory can be used to define and study obstructions to the existence of rational points. Such obstructions were studied without a framework of a model structure by Y. Harpaz and the second author in [HaSc] and by Ambrus Pál in [Pal].

In [HaSc], Harpaz and the second author defined a notion of a relative étale homotopy type of a variety $X/K$ over a field $K$. This construction was then used to study rational $K$-points on $X$, by using some notion of homotopy fixed points.

However, similar to the construction of Artin and Mazur in [AM], the construction in [HaSc] is homotopical rather than topological, namely, it gives an object in $\text{Pro}(\text{Ho}((\text{Spec}K)_{\text{ét}}^{\Delta^{op}}))$ rather than $\text{Pro}((\text{Spec}K)_{\text{ét}}^{\Delta^{op}})$. Furthermore, the above notions are defined by ad-hock constructions, and are not given conceptual definitions in a suitable model category. The construction of the relative topological realization presented here, gives an object $\text{Pro}((\text{Spec}K)_{\text{ét}}^{\Delta^{op}})$, and allows us to define the above notions using the language of model categories.

Indeed, given a field $K$ and a $K$-variety $X/K$, we can define:

$$\text{Top}_K(X) := |X_{\text{ét}}|_{\text{Spec}K} \in \text{Pro}((\text{Spec}K)_{\text{ét}}^{\Delta^{op}}),$$

and we get a map:

$$h : X(K) \to [*, \text{Top}_K(X)]_{\text{Pro}((\text{Spec}K)_{\text{ét}}^{\Delta^{op}})}.$$ 

This map is closely related to the map: $h : X(K) \to X(hK)$, presented in [HaSc] and [Pal], and can be used to study rational points.

Furthermore, having a “topological” object and a model structure, allows one to use the general machinery of model categories in order to give simpler and more conceptual proofs to the results in [HaSc]. This also enables to generalize the homotopy obstruction theory of [HaSc], from fields to arbitrary base schemas. This approach will be discussed in a future paper.

Remark 10.6. Note that in [AM], Artin and Mazur work in some localization of $\text{Pro}(\text{Ho}(\text{Set}^{\Delta^{op}}))$ (namely, the $\xi$-localization). This localization also has a model theoretic counterpart, as a localization of our model structure on pro-simplicial sheaves. This will also be described in detail in a future paper.

11 Comparison with the Isaksen-Jardine Model Structure

In this section we compare our “projective” model structure on pro simplicial presheaves of section 9.2 with the “injective” model structure on the same category, that can be deduced from [Isa], when applied to [Jar]. Namely, we show that the identity functors constitute a Quillen equivalence between these two
model structures. As a direct consequence we get a very short and conceptual proof of Verdier’s hypercovering theorem.

Let $\mathcal{C}$ be a small Grothendieck site. We use the notation of section 9. As shown in [Jar], there exist a model structure on the category $SPS(\mathcal{C})$, in which the cofibrations are the levelwise cofibrations, and the weak equivalences are the local weak equivalences. Furthermore, this model structure is proper (see [Jar1], Theorem 2). It follows from Theorem 4.4 and Remark 4.17 (or Theorem 4.15 in [Isa]), that there exist a model category structure on $\text{Pro}(SPS(\mathcal{C}))$, which we will denote by $\text{Pro}(SPS(\mathcal{C}))_I$, s.t:

1. The weak equivalences are $W_I := Lw^\approx(W)$.
2. The fibrations are $F_I := R(Sp^\approx(F_J))$, where $F_J$ is the class of fibrations in the Jardine structure on $SPS(\mathcal{C})$.
3. The cofibrations are $C_I := \perp Sp^\approx(F_J \cap W) = \perp(F_J \cap W)$.

This model structure on $\text{Pro}(SPS(\mathcal{C}))$ was considered by Jardine in [Jar2]. We call $\text{Pro}(SPS(\mathcal{C}))_I$ the injective model structure on $\text{Pro}(SPS(\mathcal{C}))$, since every levelwise cofibration in $SPS(\mathcal{C})$ is a cofibration in $\text{Pro}(SPS(\mathcal{C}))_I$ (between simple objects).

As we have shown in section 9 there exist a weak fibration structure on the category $SPS(\mathcal{C})$, in which the fibrations are the local fibrations, and the weak equivalences are the local weak equivalences (see Proposition 9.6). Furthermore, this weak fibration structure is pro admissible (see 9.2). It follows from Theorem 4.4 that there exist a model category structure on $\text{Pro}(SPS(\mathcal{C}))$, which we will denote by $\text{Pro}(SPS(\mathcal{C}))_P$, s.t:

1. The weak equivalences are $W_P := Lw^\approx(W)$.
2. The fibrations are $F_P := R(Sp^\approx(F))$.
3. The cofibrations are $C_P := \perp Sp^\approx(F \cap W) = \perp(F \cap W)$.

We call $\text{Pro}(SPS(\mathcal{C}))_P$ the projective model structure on $\text{Pro}(SPS(\mathcal{C}))$, since every local fibration in $SPS(\mathcal{C})$ is a fibration in $\text{Pro}(SPS(\mathcal{C}))_P$ (between simple objects).

Let $f$ be a fibration in the Jardine model structure on $SPS(\mathcal{C})$. Since the Jardine model structure is a left Bousfield localization of the injective model structure on $SPS(\mathcal{C})$ (see [Lur] section A.3.3), $f$ is also a fibration in the injective model structure on $SPS(\mathcal{C})$. It follows that $f$ is a levelwise fibration in $SPS(\mathcal{C})$, and in particular a local fibration in $SPS(\mathcal{C})$. Thus: $F_J \subseteq F$. It follows that:

$$C_P = \perp(F \cap W) \subseteq \perp(F_J \cap W) = C_I.$$ 

From this inclusion we conclude trivially that:

$$id : \text{Pro}(SPS(\mathcal{C}))_P \rightleftarrows \text{Pro}(SPS(\mathcal{C}))_I : id$$

58
is a Quillen equivalence between the projective and injective model structures on \( \text{Pro}(SPS(C)) \).

Since the Jardine model structure is a left Bousfield localization of the injective model structure on \( SPS(C) \), and the injective model structure on \( SPS(C) \) is simplicial (see \( \text{Lur} \) section A.3.3), we conclude that the Jardine model structure is also simplicial. By Proposition 4.17 (or by \( \text{Isa} \) Theorem 4.17), it follows that the injective model structure on \( \text{Pro}(SPS(C)) \) is simplicial.

The weak fibration category structure on \( SPS(C) \), defined in section 9, is simplicial, as explained there (see Proposition 9.9). It follows from Proposition 7.5, that the projective model structure on \( \text{Pro}(SPS(C)) \) is also simplicial.

We conclude this section with an illustration concerning Verdier’s hypercovering theorem \( \text{[AM]} \). Let \( F : C^{op} \to Ab \) be a presheaf of abelian groups on \( C \), and let \( n \geq 0 \). There is a well known functor \( K(-, n) : Ab \to S \), from abelian groups into Kan simplicial sets (see \( \text{GJ} \)). Composing this functor with \( F \) we get a simplicial presheaf: \( K(F, n) : C^{op} \to S \). As explained in \( \text{Jar} \) (see also \( \text{Pro} \)), there is a natural isomorphism:

\[
H^n(C, \tilde{F}) \cong [*, K(F, n)],
\]

where:

1. \( H^n(C, \tilde{F}) \) is the \( n \)'th sheaf cohomology group of the site \( C \), with coefficients in the (sheaf associated to the) presheaf \( F \).

2. \([*, K(F, n)]\) denotes maps in the homotopy category of the Jardine model structure on \( SPS(C) \), between the constant simplicial presheaf with value \(*\), and the presheaf \( K(F, n) \).

Since \( SPS(C) \), in the Jardine model structure, and \( \text{Pro}(SPS(C)) \), in both the projective and injective model structures, are simplicial model categories, we have:

\[
H^n(C, \tilde{F}) \cong [*, K(F, n)] \cong \pi_0(\text{Map}_{L(\text{SPS}(C))}(*, K(F, n))) \cong \pi_0(\text{Map}_{SPS(C)}(*, K(F, n))) \cong \pi_0(\text{Map}_{\text{Pro}(SPS(C))}(*, K(F, n))) \cong \pi_0(\text{Map}_{\text{Pro}(\text{SPS}(C))}(*, K(F, n))) \cong \pi_0(\text{Map}_{\text{Pro}(\text{SPS}(C))}(*, K(F, n))) \cong \pi_0(\text{Map}_{\text{Pro}(\text{SPS}(C))}(*, K(F, n)))
\]

where:

1. \( L \) denotes the Dwyer Kan localization functor, from relative categories to simplicial categories.

2. The superscripts \( c, f \) denote cofibrant and fibrant replacements respectively, in the relevant model categories.

In Section \( \text{[8]} \) we gave a description of a cofibrant replacement for \(*\) in \( \text{Pro}(SPS(C))_P \), as a functor: \( H^A : A \to SPS(C) \). Using this description (same
notation) and the fact that $\pi_0 : \mathcal{S} \to \text{Set}$ commutes with filtered colimits, we get:

$$H^n(C, \tilde{F}) \cong \pi_0(\text{Map}_{\text{Pr}(\mathcal{SP}(\mathcal{C}))}(\mathcal{H}^A, K(F, n))) \cong \pi_0(\text{colim}_{a \in \mathcal{A}^{op}} \text{Map}_{\mathcal{SP}(\mathcal{C})}(\mathcal{H}^A(a), K(F, n)))$$

$$\cong \text{colim}_{a \in \mathcal{A}^{op}} \pi_0(\text{Map}_{\mathcal{SP}(\mathcal{C})}(\mathcal{H}^A(a), K(F, n)))$$

Consider the functor:

$$G := \pi_0(\text{Map}_{\mathcal{SP}(\mathcal{C})}(-, K(F, n)) : \mathcal{SP}(\mathcal{C})^{op} \to \text{Set.}$$

We have a commutative diagram:

$$\begin{array}{ccc}
\mathcal{A}^{op} & \xrightarrow{\mathcal{H}^A} & \mathcal{SP}(\mathcal{C})^{op} \xrightarrow{\pi_0(\text{Map}_{\mathcal{SP}(\mathcal{C})}(\mathcal{H}^A, K(F, n)))} \mathcal{S} \\
\downarrow & & \downarrow \pi_0 \\
\pi \mathcal{SP}(\mathcal{C})^{op} & \xrightarrow{\pi \mathcal{H}} & \pi \mathcal{SP}(\mathcal{C})^{op} \xrightarrow{\pi_0} \text{Set.}
\end{array}$$

It follows that the ind object $G \circ (\mathcal{H}^A)^{op} : \mathcal{A}^{op} \to \text{Set}$ factors through $\pi \mathcal{SP}(\mathcal{C})^{op}$:

$$G \circ (\mathcal{H}^A)^{op} : \mathcal{A}^{op} \to \pi \mathcal{SP}(\mathcal{C})^{op} \to \text{Set.}$$

As we have shown in Section \ref{cofinal}, $\mathcal{A}^{op} \to \pi \mathcal{SP}(\mathcal{C})^{op}$ is right cofinal. We thus get that:

$$H^n(C, \tilde{F}) \cong \text{colim}_{a \in \mathcal{A}^{op}} \pi_0(\text{Map}_{\mathcal{SP}(\mathcal{C})}(\mathcal{H}^A(a), K(F, n)))$$

$$\cong \text{colim}_{U \in \pi \mathcal{SP}(\mathcal{C})^{op}} \pi_0(\text{Map}_{\mathcal{SP}(\mathcal{C})}(U, K(F, n))) \cong \text{colim}_{U \in \pi \mathcal{SP}(\mathcal{C})^{op}} H^n_{\text{Cech}}(C, F, U),$$

where the last isomorphism is a classical observation. This is exactly Verdier’s theorem, saying that the sheaf cohomology of a site can be computed as the colimit over all hypercoverings in the site, of the Čech cohomologies.

Remark 11.1. In order to get Verdier’s theorem, we should restrict this last colimit only to hypercoverings, i.e. to those locally fibrant locally contractible simplicial presheaves, which are levelwise representable in the étale site of $X$. However, since the hypercoverings are cofinal among all the locally fibrant locally contractible simplicial presheaves (Lemma 2.2 in \cite{Jar3}), The resulting colimit is isomorphic.
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Department of Mathematics, Hebrew University, Jerusalem

E-mail address: ilan.barnea770@gmail.com

Department of Mathematics, Hebrew University, Jerusalem

E-mail address: tomer.schlank@gmail.com