SEMISIMPLICITY OF THE CATEGORIES OF YETTER-DRINFE LD M O D U LES AND LONG DIMODULES

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Abstract. Let $k$ be a field, and $H$ a Hopf algebra with bijective antipode. If $H$ is commutative, noetherian, semisimple and cosemisimple, then the category $\mathcal{YD}^H$ of Yetter-Drinfeld modules is semisimple. We also prove a similar statement for the category of Long dimodules, without the assumption that $H$ is commutative.

Introduction

Let $H$ be a Hopf algebra at the same time acting and coacting on a vector space $M$. We can impose various compatibility relations between the action and coaction, leading to different notions of Hopf modules. Hopf modules are already considered by Sweedler [13], and they have to satisfy the relation

$$\rho(hm) = \Delta(h)\rho(m) = h_1m_0 \otimes h_2m_1$$

One can also require that the $H$-coaction is $H$-linear:

$$\rho(hm) = h\rho(m) = hm_0 \otimes m_1$$

A module satisfying this condition is called a Long dimodule. Long dimodules are the building stones of the Brauer-Long group, in the case where the Hopf algebra $H$ is commutative, cocommutative and faithfully projective (see [7], and [1] for a detailed discussion). Long dimodules are also connected to a non-linear equation (see [9]). Another - at first sight complicated and artificial - compatibility relation is the following:

$$h_1m_0 \otimes h_2m_1 = (h_2m)_0 \otimes (h_2m)_1h_1$$

A module that satisfies it is called a Yetter-Drinfeld module. There is a close connection between Yetter-Drinfeld modules and the Drinfeld double (see [4]): if $H$ is finitely generated projective, then the

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category of Yetter-Drinfeld modules is isomorphic to the category of modules over the Drinfeld double. Yetter-Drinfeld modules have been studied intensively by several authors over the past fifteen years, see for example [2], [6], [8], [11], this list is far from exhaustive. One of the important features is the fact that the category of Yetter-Drinfeld modules is braided monoidal. As Long dimodules, Yetter-Drinfeld modules are related to a non-linear equation, the quantum Yang-Baxter equation (see e.g. [5]). If \( H \) is commutative and cocommutative, then Yetter-Drinfeld modules coincide with Long dimodules.

In this note, we give sufficient conditions for the categories of Yetter-Drinfeld modules and Long dimodules to be semisimple (Section 3) and we study projective and injective dimension in these categories. Our main result is that the category of Yetter-Drinfeld modules is semisimple if \( H \) is a commutative, noetherian, semisimple and cosemisimple Hopf algebra over a field \( k \). The same is true for the category of Long dimodules, without the assumption that \( H \) is commutative.

For generalities on Hopf algebras, we refer the reader to [3], [10], [13]. For a detailed study of Hopf modules and their generalizations, we refer to [2].

1. Preliminary Results

Let \( k \) be a commutative ring, and \( H \) a faithfully flat Hopf algebra with bijective antipode \( S \). Unadorned \( \otimes \) and Hom will be over \( k \). We will use the Sweedler-Heyneman notation for comultiplication and coaction: for \( h \in H \), we write

\[
\Delta(h) = h_1 \otimes h_2
\]

(summation implicitly understood), and for a right \( H \)-comodule \((M, \rho_M)\) and \( m \in M \), we write

\[
\rho_M(m) = m_0 \otimes m_1
\]

\( _H M \) and \( M^H \) will be the categories of respectively left \( H \)-modules and left \( H \)-linear maps, and right \( H \)-comodules and right \( H \)-colinear maps. If \( M \) and \( N \) are right \( H \)-comodules, then we denote the \( k \)-module consisting of right \( H \)-colinear maps from \( M \) to \( N \) by \( \text{Hom}^H(M, N) \).

\[
M^{coH} = \{ m \in M \mid \rho_M(m) = m \otimes 1 \}
\]

is called the \( k \)-submodule of coinvariants of \( M \). Observe that \( H^{coH} = k \).

Suppose that a \( k \)-vector space \( M \) is at the same time a left \( H \)-module and a right \( H \)-comodule. Recall that \( M \) is called a left-right Yetter-Drinfeld module if

\[
h_1 m_0 \otimes h_2 m_1 = (h_2 m)_0 \otimes (h_2 m)_1 h_1
\]
or, equivalently,
\[ \rho(hm) = h_2m_0 \otimes h_3m_1S^{-1}(h_1) \]
for all \( m \in M \) and \( h \in H \). \( M \) is called a left-right Long dimodule if
\[ \rho(hm) = hm_0 \otimes m_1 \]
for all \( m \in M \) and \( h \in H \). If \( H \) is commutative and cocommutative, then a Long dimodule is the same as a Yetter-Drinfeld module. \( H \text{YD}^H \) and \( H \text{L}^H \) will be the categories of respectively Yetter-Drinfeld modules and Long dimodules, and \( H \)-linear \( H \)-colinear maps. The \( k \)-module consisting of all \( H \)-linear \( H \)-colinear maps between two Yetter-Drinfeld modules or two Long dimodules \( M \) and \( N \) will be denoted by \( H \text{Hom}^H(M, N) \). If \( H \) is finitely generated and projective, then the category \( H \text{YD}^H \) is isomorphic to the category \( D(H)\text{M} \), where \( D(H) \) is the Drinfeld double of \( H \), and \( H \text{L}^H \) is isomorphic to \( H \otimes H^* \text{M} \).

The functors
\[ (-)^{coH} : H \text{YD}^H \to \text{M} \]
and \( (-)^{coH} : H \text{L}^H \to \text{M} \)
are exact if
\[ (-)^{coH} : \text{M}^H \to \text{M} \]
is exact. This is the case if \( H \) is cosemisimple and \( k \) is a field.

**Lemma 1.1.**

1. Let \( M \) and \( N \) be objects of \( H \text{YD}^H \). Then \( M \otimes N \) is an object of \( H \text{YD}^H \); the \( H \)-action and \( H \)-coaction are given by
\[ h(m \otimes n) = h_1m \otimes h_2n \quad \text{and} \quad \rho(m \otimes n) = m_0 \otimes n_0 \otimes m_1n_1 \]
2. Let \( M \) and \( N \) be objects of \( H \text{L}^H \). Then \( M \otimes N \) is an object of \( H \text{L}^H \); the \( H \)-action and \( H \)-coaction are given by
\[ h(m \otimes n) = h_1m \otimes h_2n \quad \text{and} \quad \rho(m \otimes n) = m_0 \otimes n_0 \otimes m_1n_1 \]
3. For any \( H \)-comodule \( N \), \( H \otimes N \) is an object of \( H \text{YD}^H \) via the following structures
\[ h(h' \otimes n) = hh' \otimes n \quad \text{and} \quad \rho(h \otimes n) = h_2 \otimes n_0 \otimes h_3n_1S^{-1}(h_1) \]
4. For any \( H \)-comodule \( N \), \( H \otimes N \) is an object of \( H \text{L}^H \) via the following structures
\[ h(h' \otimes n) = hh' \otimes n \quad \text{and} \quad \rho(h \otimes n) = h \otimes n_0 \otimes n_1 \]

**Proof.** This result is well-known, and the proof is a straightforward computation. It may be found in \[1\], p. 440], \[1\], Prop. 12.1.2], \[2\], Prop. 123], and \[2\], Sec. 7.2].\[\square\]
Lemma 1.2.  (1) Let $M$ and $N$ be in $\mathcal{H}YD^H$. If $H$ is commutative, then $M \otimes_H N$ is an object of $\mathcal{H}YD^H$. The $H$-action and $H$-coaction are given by

$$h(m \otimes n) = hm \otimes n = m \otimes hn$$

and

$$\rho_{M \otimes_H N}(m \otimes n) = m_0 \otimes n_0 \otimes n_1m_1$$

(2) Let $H$ be commutative. Let $M$ and $N$ be in $\mathcal{H}YD^H$ with $M$ finitely generated projective in $\mathcal{H}M$. Then

(a) $\mathcal{H}Hom(M, N) \in \mathcal{H}M^H$ and

$$\mathcal{H}Hom^H(M, N) = \mathcal{H}Hom(M, N)^{coH}$$

The coaction is defined by

$$\rho(f) = f_0 \otimes f_1 \in \mathcal{H}Hom(M, N) \otimes H$$

if and only if

$$f_0(m) \otimes f_1 = f(m_0)_0 \otimes f(m_0)_1S(m_1)$$

for all $m \in M$.

(b) $\mathcal{H}Hom(M, N) \in \mathcal{H}YD^H$; the $H$-action is defined by $(hf)(m) = hf(m)$.

Proof. 1) It is clear that $M \otimes_H N$ is an $H$-module. An easy verification shows that the $H$-coaction is well-defined on the tensor product over $H$ and that the necessary associativity and counit properties are satisfied, so that $M \otimes_H N$ is also an $H$-comodule. $M \otimes_H N$ is a Yetter-Drinfeld module, since we have for every $h \in H$ that

$$\rho_{M \otimes_H N}(hm \otimes n) = (hm)_0 \otimes n_0 \otimes n_1(hm)_1$$

$$= h_2m_0 \otimes n_0 \otimes n_1h_3m_1S^{-1}(h_1)$$

$$= h_2(m_0 \otimes n_0) \otimes h_3n_1m_1S^{-1}(h_1)$$

$$= h_2(m \otimes n)_0 \otimes h_3(m \otimes n)_1S^{-1}(h_1)$$

2a) Let us define a map

$$\pi : \text{Hom}(M, N) \rightarrow \text{Hom}(M, N \otimes H)$$

by

$$\pi(f)(m) = f(m_0)_0 \otimes f(m_0)_1S(m_1)$$

Let $f$ be $H$-linear. Using the commutativity of $H$, we obtain

$$\pi(f)(hm) = f((hm)_0)_0 \otimes f((hm)_0)_1S((hm)_1)$$

$$= (h_2f(m_0))_0 \otimes (h_2f(m_0))_1S(h_3m_1S^{-1}(h_1))$$

$$= h_2f(m_0)_0 \otimes h_4f(m_0)_1S^{-1}(h_2)h_1S(m_1)S(h_5)$$

$$= hf(m_0)_0 \otimes f(m_0)_1S(m_1) = h\pi(f)(m)$$
so \( \pi(f) \) is \( H \)-linear, and \( \pi \) restricts to a map
\[
\pi : \mathcal{H}\text{Hom}(M, N) \to \mathcal{H}\text{Hom}(M, N \otimes H)
\]
Now \( H \) is finitely generated and projective as an \( H \)-module, so we have a natural isomorphism \( \mathcal{H}\text{Hom}(M, N \otimes H) \cong \mathcal{H}\text{Hom}(M, N) \otimes H \), and we obtain a map
\[
\pi : \mathcal{H}\text{Hom}(M, N) \to \mathcal{H}\text{Hom}(M, N) \otimes H
\]
with \( \pi(f) = f_0 \otimes f_1 \) if and only if
\[
(\pi(f))(m) = f_0(m) \otimes f_1 = f(m_0)_0 \otimes f(m_0)_1 S(m_1)
\]
It is straightforward to show that \( \pi \) makes \( \mathcal{H}\text{Hom}(M, N) \) a right \( H \)-comodule. Now take \( f \in \mathcal{H}\text{Hom}^H(M, N) \) and \( m \in M \). Then
\[
\pi(f)(m) = f_0(m) \otimes f_1 = f(m_0)_0 \otimes f(m_0)_1 S(m_1) = f(m_0) \otimes m_1 S(m_2) = f(m) \otimes 1 = (f \otimes 1)(m)
\]
so \( f \) is coinvariant. Conversely, take \( f \in \mathcal{H}\text{Hom}(M, N)^{\text{co}H} \). Then for every \( m \in M \)
\[
f(m_0)_0 \otimes f(m_0)_1 S(m_1) = f_0(m) \otimes f_1 = f(m) \otimes 1
\]
and
\[
f(m_0)_0 \otimes f(m_0)_1 S(m_1) m_2 = f(m_0) \otimes m_1
\]
and it follows that
\[
\rho_N(f(m)) = \rho_N(f(m_0)) \varepsilon(m_1) = f(m_0) \otimes m_1
\]
and \( f \) is \( H \)-colinear.

2b) Clearly \( \mathcal{H}\text{Hom}(M, N) \) is an \( H \)-module and, by a), it is an \( H \)-comodule. On the other hand, we have
\[
((hf)_0 \otimes (hf)_1)(m) = ((hf)(m_0))_0 \otimes ((hf)(m_0))_1 S(m_1)
\]
\[
= (hf(m_0))_0 \otimes (hf(m_0))_1 S(m_1)
\]
\[
= h_2 f(m_0)_0 \otimes h_3 f(m_0)_1 S^{-1}(h_1) S(m_1)
\]
\[
= h_2 f(m_0)_0 \otimes h_3 f(m_0)_1 S(m_1) s^{-1}(h_1)
\]
\[
= h_2 f_0(m) \otimes h_3 f_1 S^{-1}(h_1)
\]
\[
= (h_2 f_0 \otimes h_3 f_1 S^{-1}(h_1))(m)
\]
so \( \mathcal{H}\text{Hom}(M, N) \in \mathcal{H}\mathcal{Y}\mathcal{D}^H \).

\[\square\]

Remark 1.3. The results in Lemma \[\ref{lem:1.2}\] remain true after we replace \( \mathcal{H}\mathcal{Y}\mathcal{D}^H \) by \( \mathcal{H}\mathcal{L}^H \). The \( H \)-coaction on \( M \otimes_H N \) is given by
\[
\rho_{M \otimes_H N}(m \otimes n) = m_0 \otimes n_0 \otimes m_1 n_1
\]
The \( H \)-coaction on \( \mathcal{H}\text{Hom}(M, N) \) is also defined by \( \Box \). Part 2a) of Lemma \[\ref{lem:1.2}\] then also holds if \( H \) is noncommutative.
Lemma 1.4. Let $V$ be a $k$-module and $N$ an $H$-module.

(1) $H\text{Hom}(H \otimes V, N)$ and $\text{Hom}(V, N)$ are isomorphic as $k$-modules.

(2) If $V$ is projective as $k$-module, then $H \otimes V$ is projective in $H\mathcal{M}$.

Proof. 1) is well-known: the $k$-isomorphism $\Phi: H\text{Hom}(H \otimes V, N) \to \text{Hom}(V, N)$ is defined by $\Phi(f)(v) = f(1 \otimes v)$.

2) follows immediately from (1). □

Let $V$ be an $H$-comodule which is finitely generated and projective as a $k$-module. By Lemmas 1.1 and 1.4, $H \otimes V$ is an object in $H\mathcal{YD}^H$ and in $H\mathcal{L}^H$, and is finitely generated projective as an $H$-module. So if $N$ is an object of $H\mathcal{YD}^H$ and if $H$ is commutative, then, by Lemma 1.2, $H\text{Hom}(H \otimes V, N)$ is an object in $H\mathcal{YD}^H$. If $N$ is an object of $H\mathcal{L}^H$, then by Remark 1.3, $H\text{Hom}(H \otimes V, N)$ is an object of $\mathcal{M}^H$; if furthermore $H$ is commutative, then $H\text{Hom}(H \otimes V, N)$ is an object of $H\mathcal{L}^H$.

Lemma 1.5. Let $H$ be commutative and $N \in H\mathcal{YD}^H$.

(1) If $V$ is an $H$-comodule which is finitely generated and projective as a $k$-module, then the $H$-comodules $H\text{Hom}(H \otimes V, N)$ and $\text{Hom}(V, N)$ are isomorphic.

(2) Let $k$ be a field and $V$ a finite-dimensional $H$-comodule that is projective as an $H$-comodule. Then $H \otimes V$ is a projective object of $H\mathcal{YD}^H$.

Proof. 1) Consider the canonical $k$-isomorphism $\phi: H\text{Hom}(H \otimes V, N) \to \text{Hom}(V, N)$, $\phi(f)(v) = f(1 \otimes v)$.

$\phi$ is $H$-colinear since

$$\phi(f)_0(v) \otimes \phi(f)_1 = (\phi(f)(v_0))_0 \otimes (\phi(f)(v_0))_1 S(v_1)$$

$$= f(1 \otimes v_0) \otimes f(1 \otimes v_0)_1 S(v_1)$$

$$= f((1 \otimes v)_0 \otimes f((1 \otimes v)_0)_1 S((1 \otimes v)_1)$$

$$= f_0(1 \otimes v) \otimes f_1$$

$$= (\phi(f_0))(v) \otimes f_1$$

2) By 1) and Lemma 1.2, we have

$$H\text{Hom}^H(H \otimes V, N) \cong H\text{Hom}(H \otimes V, N)^{coH}$$

$$\cong \text{Hom}(V, N)^{coH} \cong \text{Hom}^H(V, N)$$

□

Lemma 1.5 also holds with $H\mathcal{YD}^H$ replaced by $H\mathcal{L}^H$, and without the assumption that $H$ is commutative.
Proposition 1.6. Let \( k \) be a field. An object \( M \) of \( _H \mathcal{YD}^H \) or \( _H \mathcal{L}^H \) is finitely generated as an \( H \)-module if and only if there exists a finite dimensional \( H \)-comodule \( V \) and an \( H \)-linear \( H \)-colinear epimorphism \( \pi : H \otimes V \to M \).

Proof. If there exist a finite dimensional \( H \)-comodule \( V \) and an epimorphism of \( H \)-modules \( \pi : H \otimes V \to M \), then \( H \otimes V \) is finitely generated as an \( H \)-module and \( M \) is a quotient of \( H \otimes V \) in \( _H \mathcal{M} \), so \( M \) is finitely generated in \( _H \mathcal{M} \).

Suppose that \( M \) is finitely generated as an \( H \)-module, with generators \( \{m_1, \ldots, m_n\} \). By [3, 5.1.1], there exists a finite dimensional \( H \)-subcomodule \( V \) of \( M \) containing \( \{m_1, \ldots, m_n\} \) and the \( k \)-linear map \( \pi : H \otimes V \to M, \pi(h \otimes v) = hv \) is an \( H \)-linear \( H \)-colinear epimorphism. \( \square \)

Let \( H^* \) be the linear dual of \( H \). If \( M \) and \( N \) are \( H \)-comodules, then \( \text{Hom}_k(M, N) \) is a left \( H^* \)-module, with \( H^* \)-action

\[
(h^* f)(m) = h^* (f(m_0)S(m_1))f(m_0)
\]

(adapt the proof of [12, Proposition 1.1]).

Lemma 1.7. Let \( H \) be commutative. For \( M, N \in _H \mathcal{YD}^H \), \( _H \text{Hom}(M, N) \) is a left \( H^* \)-submodule of \( \text{Hom}_k(M, N) \).

Proof. For all \( \alpha \in H^* \), \( f \in \text{Hom}_H(M, N) \), \( h \in H \) and \( m \in M \), we have

\[
(\alpha f)(hm) = \alpha \left( f((hm)_0)S((hm)_1) \right) f((hm)_0)
= \alpha \left( f(h_2m_0)_1S(h_3m_1S^{-1}(h_1)) \right) f(h_2m_0)
= \alpha \left( (h_2(f(m_0)))_1h_1S(m_1)S(h_3) \right) (h_2f(m_0))_0
= \alpha \left( h_4f(m_0)_1S^{-1}(h_2)h_1S(m_1)S(h_5) \right) h_3f(m_0)_0
= \alpha \left( f(m_0)_1S(m_1) \right) hf(m_0)_0
= h(\alpha f)(m)
\]

and it follows that \( \alpha f \) is \( H \)-linear. Observe that we used the commutativity of \( H \). \( \square \)

Recall that a left \( H^* \)-module \( M \) is called rational if there exists a right \( H \)-coaction on \( M \) inducing the left \( H^* \)-action.

Proposition 1.8. Let \( H \) be a commutative Hopf algebra over a field \( k \). If \( M, N \in _H \mathcal{YD}^H \) with \( M \) finitely generated as \( H \)-module, then \( _H \text{Hom}(M, N) \in _H \mathcal{YD}^H \).
Proof. By Proposition 1.6, there exist a finite dimensional $H$-subcomodule $V$ of $M$ and an $H$-linear $H$-colinear epimorphism $\pi : H \otimes V \to M$. So we obtain an injective $k$-linear map

$$H\text{Hom}(\pi, N) : H\text{Hom}(M, N) \to H\text{Hom}(H \otimes V, N)$$

For all $\alpha \in H^*$, $f \in H\text{Hom}(M, N)$, $h \in H$ and $v \in V$, we have $\pi(h \otimes v) = hv$, $(1 \otimes v)_0 \otimes (1 \otimes v)_1 = 1 \otimes v_0 \otimes v_1$ and

$$(\alpha f) \circ \pi(1 \otimes v) = (\alpha f)(v) = \alpha(f(v)_1 S(v_1))f(v)_0$$

$$= \alpha(f(\pi(1 \otimes v)_0))_1 S(v_1)f(\pi(1 \otimes v)_0)_0$$

$$= \alpha(f(\pi(1 \otimes v)_0))(1 \otimes v)$$

This relation and the fact that $(\alpha f) \circ \pi$ and $\alpha(f \circ \pi)$ are $H$-linear imply that $((\alpha f) \circ \pi)(h \otimes v) = (\alpha(f \circ \pi))(h \otimes v)$, and it follows that the map $H\text{Hom}(\pi, N)$ is $H^*$-linear. By Lemma 1.2, $H\text{Hom}(H \otimes V, N)$ is an $H$-comodule, and therefore a rational $H^*$-module. It follows that $H\text{Hom}(M, N)$ is a rational $H^*$-module, being an $H^*$-submodule of the rational $H^*$-module $H\text{Hom}(H \otimes V, N)$. This shows that $H\text{Hom}(M, N)$ is an $H$-comodule. By Lemma 1.2, $H\text{Hom}(M, N) \in H\text{YD}^H$. \hfill $\Box$

Remark 1.9. 1) Lemma 1.7 is still true if we replace $H\text{YD}^H$ by $H\text{L}^H$, without the assumption that $H$ is commutative.

2) We have the following Long dimodule version of Proposition 1.8: for a (not necessarily commutative) Hopf algebra over a field $k$, and $M, N \in H\text{L}^H$, with $M$ finitely generated as an $H$-module, $H\text{Hom}(M, N) \in M^H$.

2. Projective and injective dimension in the category of Yetter-Drinfeld modules

Lemma 2.1. Let $H$ be commutative, and $M, N, P \in H\text{YD}^H$, with $N$ finitely generated projective as an $H$-module.

1) We have a $k$-isomorphism

$$H\text{Hom}^H(M, H\text{Hom}(N, P)) \cong H\text{Hom}^H(M \otimes_H N, P)$$

2) The functor

$$H\text{Hom}(N, -) : H\text{YD}^H \to H\text{YD}^H$$

preserves injective objects.

Proof. 1) We have a natural isomorphism

$$\phi : H\text{Hom}(M, H\text{Hom}(N, P)) \to H\text{Hom}(M \otimes_H N, P)$$
given by $\phi(f)(m \otimes n) = f(m)(n)$. We will show that $\phi$ restricts to an isomorphism between $\text{Hom}_H(M, \text{Hom}(N, P))$ and $\text{Hom}_H(M \otimes_H N, P)$. Take $f \in \text{Hom}_H(M, \text{Hom}(N, P))$ and $\phi(f) = g$. Then $f$ is $H$-colinear if and only if
\[
 f(m_0) \otimes m_1 = f(m_0) \otimes f(m_1)
\]
for all $m \in M$. Using (1), we find that this is equivalent to
\[
 f(m_0)(n) \otimes m_1 = f(m_0)(n) \otimes f(m)(n_0) = f(m)(n_0) \otimes f(m)(n_1) S(n_1)
\]
for all $m \in M$ and $n \in N$, or
\[
 g(m_0 \otimes n) \otimes m_1 = g(m \otimes n_0) \otimes g(m \otimes n_0) S(n_1)
\]
which is equivalent to
\[
 g(m_0 \otimes n_0) \otimes m_1 n_1 = g(m \otimes n_0) \otimes g(m \otimes n) n_1
\]
and this equation means that $g$ is $H$-colinear.

2) If $I$ is an injective object of $\mathcal{YD}_H^H$, then the functor
\[
 \text{Hom}_H(-, I) : \mathcal{YD}_H^H \to k \mathcal{M}
\]
is exact. On the other hand, $N$ is $H$-projective, hence the functor
\[
 (-) \otimes_H N : \mathcal{YD}_H^H \to \mathcal{YD}_H^H
\]
is exact, and it follows from (1) that
\[
 \text{Hom}_H(-, \text{Hom}(N, I)) : \mathcal{YD}_H^H \to k \mathcal{M}
\]
is exact. \qed

If $k$ is a field, then the category of Yetter-Drinfeld modules $\mathcal{YD}_H^H$ is Grothendieck, and every object has an injective resolution. For every Yetter-Drinfeld module $M$, we can define the right derived functors $\text{Ext}_H^H(M, -)$ of the covariant left exact functor
\[
 \text{Hom}_H(M, -) : \mathcal{YD}_H^H \to k \mathcal{M}
\]

**Proposition 2.2.** Let $H$ be a commutative Hopf algebra over a field $k$, and $M, N, P \in \mathcal{YD}_H^H$ with $N$ finitely generated projective as an $H$-module. Then
\[
 \text{Ext}_H^H(M, \text{Hom}(N, P)) \cong \text{Ext}_H^H(M \otimes_H N, P)
\]

*Proof.* By the first part of Lemma 2.1, the functors
\[
 \text{Hom}_H(M, \text{Hom}(N, -)) \quad \text{and} \quad \text{Hom}_H(M \otimes_H N, -)
\]
coincide on $\mathcal{YD}_H^H$. By the projectivity and the finiteness assumptions on $N$, the $\text{Hom}(N, -)$ is an exact endofunctor of $\mathcal{YD}_H^H$. By the second part of Lemma 2.1, it preserves the injective objects of $\mathcal{YD}_H^H$. \qed
Thus the functor $H\text{Hom}(N, -)$ preserves injective resolutions in $HYD^H$.

In the following corollary, $H\text{pdim}^H(-)$ and $H\text{injdim}^H(-)$ denote respectively the projective and injective dimension in the category $HYD^H$.

**Corollary 2.3.** Let $H$ be a commutative Hopf algebra over a field $k$, and $M, N, P \in HYD^H$ with $N$ finitely generated projective as an $H$-module. Then

1. $H\text{pdim}^H(M \otimes_H N) \leq H\text{pdim}^H(M)$.
2. $H\text{injdim}^H(H\text{Hom}(N, P)) \leq H\text{injdim}^H(P)$.

**Remarks 2.4.**
1) Let $H$ be semisimple. Then the projectivity assumption in Lemma 2.1, Proposition 2.2 and Corollary 2.3 is no longer needed.
2) If $k$ is a field, then $HL^H$ is a Grothendieck category with enough injective objects, and every Long dimodule has an injective resolution. For every $M \in HL^H$, we can then define the right derived functors $H\text{Ext}^H(M, -)$ of the covariant left exact functor

$$H\text{Hom}^H(M, -) : HL^H \rightarrow kM$$

All the results of this Section remain valid for $HL^H$. If $H$ is semisimple, then the projectivity assumptions are not needed.

### 3. Semisimplicity of the category of Yetter-Drinfeld modules

Throughout this Section, $k$ will be a field, and $H$ a commutative Hopf algebra. Recall that $M \in HYD^H$ is called simple if it has no proper subobjects; a direct sum of simples is called semisimple. If every $M \in HYD^H$ is semisimple, then we call the category $HYD^H$ semisimple. We say that $HYD^H$ satisfies condition $(\dagger)$ if the following holds:

- if $M \in HYD^H$ is finitely generated as a left $H$-module, then $H\text{Hom}(M, -) : HYD^H \rightarrow HYD^H$ is exact.
- By Proposition 1.8, $H\text{Hom}(M, N) \in HYD^H$ if $H$ is commutative and $M$ is finitely generated as an $H$-module. Also observe that $HYD^H$ satisfies condition $(\dagger)$ if $H$ is semisimple.

**Proposition 3.1.** Let $H$ be commutative. Assume that $HYD^H$ satisfies condition $(\dagger)$ and that the functor

$$(-)^{coH} : HYD^H \rightarrow M$$

is exact. If $M \in HYD^H$ is finitely generated as an $H$-module, then $M$ is a projective object in $HYD^H$. 
Proof. We know that
\[ H\Hom^H(M, -) \cong H\Hom(M, -)^{\text{co}H} \]
so \( H\Hom^H(M, -) \) is exact since it is isomorphic to the composition of two exact functors. \( \square \)

**Corollary 3.2.** With the same assumptions as in Proposition 3.1, and with \( H \) noetherian, we have that every object \( M \in \mathcal{YD}^H \) which is finitely generated as an \( H \)-module is a direct sum in \( \mathcal{YD}^H \) of a family of simple subobjects that are finitely generated as \( H \)-modules.

Proof. Let \( N \) be a subobject of \( M \) in \( \mathcal{YD}^H \). Then \( M/N \) is finitely generated as an \( H \)-module and we have an exact sequence
\[ 0 \to N \to M \to M/N \to 0 \]
in \( \mathcal{YD}^H \). \( N \) is finitely generated as \( H \)-module, since \( H \) is noetherian, so it follows from Proposition 3.1 that \( M/N \) and \( N \) are projective in \( \mathcal{YD}^H \), hence the sequence \( (2) \) splits in \( \mathcal{YD}^H \). \( \square \)

Take \( M \in \mathcal{YD}^H \) and \( V \) a right \( H \)-subcomodule of \( M \). We will set
\[ HV = \{ \sum_{i \in I} a_i v_i \mid a_i \in H, v_i \in V, \text{ where } I \text{ is a finite set} \} \]
\( HV \) is a subobject of \( M \) in \( \mathcal{YD}^H \); the \( H \)-action and \( H \)-coaction on \( HV \) are given by
\[ h(\sum_{i \in I} a_i v_i) = \sum_{i \in I} h a_i v_i \]
\[ \rho(\sum_{i \in I} a_i v_i) = \sum_{i \in I} (a_i)_2(v_i)_0 \otimes (a_i)_3(v_i)_1 S^{-1}((a_i)_1) \]

**Corollary 3.3.** Let \( H \) be commutative and noetherian. Assume that \( \mathcal{YD}^H \) satisfies condition (†), and that the functor \((-)^{\text{co}H}\) from \( \mathcal{YD}^H \) to \( \mathcal{M} \) is exact. Then \( M \in \mathcal{YD}^H \) is a direct sum in \( \mathcal{YD}^H \) of a family of simple subobjects that are finitely generated as \( H \)-modules. Therefore \( M \) is a semisimple object in \( \mathcal{YD}^H \) and \( \mathcal{YD}^H \) is a semisimple category.

Proof. Every \( m \in M \) is contained in a finite-dimensional \( H \)-subcomodule \( V_m \) of \( M \), see e.g. [3, 5.1.1]. Then \( HV_m \) is finitely generated as \( H \)-module, and, by Corollary 3.2, each \( HV_m \) is a direct sum of a family of simple subobjects of \( HV_m \) (and of \( M \)) in \( \mathcal{YD}^H \), which are finitely generated as an \( H \)-module. Consequently each \( m \in M \) is contained in a simple object which is finitely generated as an \( H \)-module, so \( M \) is a sum of simple objects finitely generated as an \( H \)-module. The sum is a direct sum since the intersection of two simple objects is trivial. \( \square \)
Corollary 3.4. Let $H$ be commutative, noetherian (in particular: finite dimensional), semisimple and cosemisimple. Then each $M \in H\mathcal{YD}^H$ is a direct sum in $H\mathcal{YD}^H$ of a family of simple subobjects of $M$ finitely generated as $H$-modules. Hence $M$ is semisimple in $H\mathcal{YD}^H$ and $H\mathcal{YD}^H$ is a semisimple category.

Proof. The cosemisimplicity of $H$ implies that the functor $(-)^{coH} : \mathcal{M}^H \to \mathcal{M}$ is exact, and, a fortiori $(-)^{coH} : H\mathcal{YD}^H \to \mathcal{M}$ is exact. □

Take $M, N \in H\mathcal{L}^H$, with $M$ finitely generated as an $H$-module. By Proposition 1.8 and Remark 1.9, $H\text{Hom}(M, N) \in \mathcal{M}^H$, and we can study the semisimplicity of $H\mathcal{L}^H$. We will say that $H\mathcal{L}^H$ satisfies condition (†) if the functor $H\text{Hom}(M, -) : H\mathcal{L}^H \to \mathcal{M}^H$ is exact for every $H$-finitely generated $M \in H\mathcal{L}^H$. The previous results of this Section then remain true after we replace the category of Yetter-Drinfeld modules by Long dimodules, and without the assumption that $H$ is commutative. We state the results without proof.

Proposition 3.5. Assume that $H\mathcal{L}^H$ satisfies condition (†) and that the functor $(-)^{coH} : \mathcal{M}^H \to \mathcal{M}$ is exact. Then every $H$-finitely generated $M \in H\mathcal{L}^H$ is a projective object in $H\mathcal{L}^H$.

Corollary 3.6. Let $H$ be left noetherian, and assume that the conditions of Proposition 3.3 are satisfied. Then every $H$-finitely generated $M \in H\mathcal{L}^H$ is a direct sum in $H\mathcal{L}^H$ of a family of simple subobjects of $M$ that are finitely generated as $H$-modules. $H\mathcal{L}^H$ is a semisimple category.

Corollary 3.7. Let $H$ be left noetherian (in particular: finite dimensional), semisimple and cosemisimple. Then each $M \in H\mathcal{L}^H$ is a direct sum in $H\mathcal{L}^H$ of a family of simple subobjects of $M$ that are finitely generated as $H$-modules. Hence $M \in H\mathcal{L}^H$ is semisimple and $H\mathcal{L}^H$ is a semisimple category.
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