The gravitational chiral anomaly of spin-1/2 field in the presence of twisted boundary conditions for ordinary field theory.

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Abstract

We calculate the chiral anomaly in the neighbourhood of the fixed point space $\mathcal{M}_h$ which is constructed by the group action of a discrete symmetry $h$ on a compact manifold $\mathcal{M}$. The Feynman diagrams approach for the corresponding supersymmetric quantum mechanical system with twisted boundary conditions is used. The result we derive in this way agrees with the generalization of the ordinary index theorem (the G-index theorem) on the spin complex.

1 Introduction

Anomalies of quantum field theories in the path integral approach can be interpreted, according to Fujikawa [1], as a symptom of the impossibility of defining an invariant measure in the functional integral which determines the graviton theory under investigation. Therefore symmetries of the classical action cease to be conserved at the quantum level. For a massless fermion $\psi$, in even-dimensions, the action of the field theory is given by:

$$\mathcal{L} = -\left(\det e^\alpha_{\mu}\right) \bar{\psi} e^\mu_\alpha(x) \gamma^\alpha \nabla_\mu \psi$$

where $\nabla_\mu \psi = \partial_\mu \psi + \frac{i}{4} \omega_{\mu ab}(e) \gamma^a \gamma^b \psi$, with $\omega_{\mu ab}$ the spin connection. Chiral anomaly is associated with the infinitesimal local transformation $\psi \rightarrow \psi + \alpha(x) \gamma_5 \psi$ where $\alpha$ is a complex space-time dependent function. Fujikawa found that for a spin-$1/2$ loop the chiral anomaly is given by the regulated Jacobian $\lim_{\beta \rightarrow 0} \text{Tr} \left[ \gamma_5 \left( -\frac{\beta}{\hbar} \mathcal{D} \mathcal{D} \right) \right]$, and $\mathcal{D} \mathcal{D}$ is the regulator.

The computation of anomalies using Fujikawa’s scheme was soon realized to be very cumbersome when trying to evaluate traces in $n$-dimensions involving products of $\gamma$ matrices. A new procedure based on one-dimensional quantum mechanics was then proposed by Alvarez-Gaumé and Witten [4]. According to this method the operators $\gamma_5$, $\nabla_\mu$, $x^\mu$, $\gamma^\mu$ were represented by operators of a corresponding quantum mechanical model, and by turning these operator expressions into path integrals, one finds that anomalies of quantum field theories can be written in terms of Feynman diagrams for certain sigma models on the worldline.

In the case of chiral anomalies the partition function receives contributions from periodic bosonic coordinates and the projection operator $(-1)^F$ replaces the antiperiodic boundary conditions for the fermions with periodic ones. It is worth noting that chiral anomalies due to their topological nature are insensitive to the method used for calculating them in contrast to the trace anomalies which are not.

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In this article we introduce twisted boundary conditions for bosons and fermions induced by automorphic maps of the base manifold. For convenience we consider $Z_N$ as the discrete isometry group. The twisted chiral anomaly will be given by the trace:

$$\text{An}(\text{chiral}) = \lim_{\beta \to 0} Tr \left( (-1)^F \hat{P} e^{-\frac{\beta}{2} H} \right)$$

where $(-1)^F$ is the $4n$ analogue of $\gamma_5$, $F$ is the one-dimensional fermion number, $\mathcal{P}$ is a projection operator which selects those states of the total Hilbert space that remain invariant under the group action and $H$ is the $N = 1$ supersymmetric Hamiltonian. The paths of the points that belong to the group invariant subspace dominate in the functional integral. Away from the fixed point space the corresponding contributions are exponentially suppressed.

The extra contribution to the chiral anomaly stems from the vacuum expectation value of a vertex that couples the complex zero modes of periodic fermions with the complex bosonic quantum fluctuations. Writing then the chiral anomaly as a double sum of the relevant twisted worldline graphs all satisfying the same boundary conditions, we recover the standard result of the literature [3, 4, 5].

2 The group action and the Hilbert space

We consider a compact Riemannian manifold $\mathcal{M}$ (which might be curved with nontrivial metric) of even dimension $\text{dim} \mathcal{M} = 4n$ with coordinates described by the real bosonic fields $x^\mu(\tau)$. In the supersymmetric quantum mechanics we also have Majorana fermions living on the sections of the pullback of the tangent bundle of the tangent space $T\mathcal{M}$ with $\text{dim} T\mathcal{M} = 4n$. When our base manifold has a discrete isometry $h \in G$ then there is a submanifold $\mathcal{M}_h$ (possibly of non-vanishing dimension) that remains invariant under the corresponding group action. We may also assume that the manifold $\mathcal{M}$ is Kähler equipped with a complex structure $\mathcal{J}$ preserved under the action of $h$:

$$h_* \circ \mathcal{J} = \mathcal{J} \circ h_*$$

where the asterisk represents push-forward of the vectors. If the normal bundle $N_{\mathcal{M}}(\mathcal{M}_h)$ decomposes into flat complex bundles then the real dimension at a point $p$ of $\mathcal{M}_h$ is:

$$\text{dim} \mathcal{M}_h|_p = m = 4n - 2s.$$  

The above constraint implies that $\mathcal{M}_h$ is of even dimension. The specific choice for the dimension of the base manifold is implied by the existence of the untwisted chiral anomaly only in $4n$-dimensions. Note that we may use for $\mathcal{M}_h$ the same $\mathcal{J}$ as that used for $\mathcal{M}$.

Let $h$ be an orientation preserving map which generates the $Z_N$ transformation realised by the rotation:

$$\begin{pmatrix} x^{2i}(\tau) \\ x^{2i-1}(\tau) \end{pmatrix} \rightarrow \begin{pmatrix} \cos \theta_i & -\sin \theta_i \\ \sin \theta_i & \cos \theta_i \end{pmatrix} \begin{pmatrix} x^{2i}(\tau) \\ x^{2i-1}(\tau) \end{pmatrix}$$

where $i = 1, \ldots, \text{dim} N_{\mathcal{M}}(\mathcal{M}_h)/2$, $\theta_i = \frac{2\pi}{N} k_i$, $k_i = 0, \ldots, N - 1$ and $U_i$ satisfies $U_i^N = 1$. Since $U_i$ is an orthogonal matrix it can be diagonalized by a unitary matrix $M_i$:

$$M_i U_i M_i^\dagger = U_{i,\text{diag}}$$
U_{i,\text{diag}} = \begin{pmatrix} g^{k_i} & 0 \\ 0 & g^{-k_i} \end{pmatrix} \quad (7)

with $g = e^{2i\pi/N}$. In this way the eigenvalues of $U_i$ fall into two subsets $\{g^{k_i}\}$ and $\{g^{-k_i}\}$.

Thus, adopting complex notation for the bosons on the normal bundle we impose the following twisted boundary conditions:

$$\begin{pmatrix} X^i \\ \bar{X}^i \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} x^{2i} + ix^{2i-1} \\ x^{2i} - ix^{2i-1} \end{pmatrix} \rightarrow \begin{pmatrix} g^{k_i} & 0 \\ 0 & g^{-k_i} \end{pmatrix} \begin{pmatrix} X^i \\ \bar{X}^i \end{pmatrix} \quad (8)$$

and similar twisted conditions hold for the fermions on the tangent space. Thus the base manifold coordinates split into the following set:

$$x^\mu = (x^l, X^i, \bar{X}^i) \quad (9)$$

where $x^l, l = 1, \ldots, m$ describe the fixed point manifold. These coordinates can be decomposed into background trajectories obeying the boundary condition $x^l_b(\tau = -1) = x^l_b(\tau = 0)$ and a quantum fluctuating part with $q^l(\tau = -1) = q^l(\tau = 0) = 0$. On the other hand the $X^i, \bar{X}^i, i, \bar{i} = 1, \ldots, (4n - m)/2$ describe the manifold normal to $\mathcal{M}_g$ and the appearance of twisted boundary conditions does not allow the existence of a classical part but permits only a quantum part (denoted by $Q^i, \bar{Q}^\bar{i}$ respectively).

Recalling the standard lore or orbifold theories [6], each $k_i$-sector (one for each conjugacy class in the group) has its own mode expansion and Hilbert space $\mathcal{H}_{k_i}$. The total Hilbert space is the direct sum of $\mathcal{H}_{k_i}$:

$$\hat{\mathcal{H}}_{\text{tot}} = \bigoplus_{k_i=1}^{N} \mathcal{H}_{k_i}. \quad (10)$$

The physical Hilbert space $\mathcal{H}_{\text{phys}}^i$ consists of states that remain invariant under the action of all $\hat{h}_{k_i}$:

$$\hat{\mathcal{H}}_{\text{phys}}^i = \mathcal{P}^i \mathcal{H}^i \quad (11)$$

with the projection operator defined by:

$$\hat{\mathcal{P}}^i = \frac{1}{N} \sum_{k_i=1}^{N} \hat{h}_{k_i}. \quad (12)$$

Notice that $\mathcal{P}^i$ is indeed a projection operator since $(\hat{\mathcal{P}}^i)^2 = \hat{\mathcal{P}}^i$ and $\hat{h}_{k_i} \hat{\mathcal{P}}^i = \hat{\mathcal{P}}^i$.

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1The explicit form of $M_i$ is: $M_i = \begin{pmatrix} \alpha & i\alpha \\ i\alpha^* & \alpha^* \end{pmatrix}$ with the condition that $||\alpha|| = \frac{1}{\sqrt{2}}$.

2In the non-Abelian case the twisted sectors should be labeled by the conjugacy classes $C_i$ of the group G. The projection operator is replaced by $\hat{\mathcal{P}} = \sum_i \frac{1}{|N_i|} \sum g \in N_i g \zeta(h)$ where the centralizer $N_i$ is defined by $N_i = \{g|h \in C_i : [g,h] = 0\}$ and the action of any $g \in N_i$ on $\zeta(h)$ is defined by $g \zeta(h) = (g\zeta(h), l_1 l_2^{-1} \zeta(l_1 h l_2^{-1}), \ldots)$. 
3 Interactions and propagators

In the path integral approach and applying the background field formalism \[8\] to the untwisted chiral anomaly one obtains \[8\]:

$$An(chiral) = \left(\frac{i}{2\pi}\right)^{n/2} \int_{-1}^{0} d\tau \prod_{i=1}^{n} dx_i \sqrt{g(x)} \prod_{\alpha=1}^{n} d\psi_{1,bg}^{\alpha} < e^{-\frac{i}{\hbar^2} S_{int}} >$$  \hspace{1cm} (13)

where \( S_{int} = \int_{0}^{\infty} L_{int}(\tau) d\tau \) and

$$L_{int}(\tau) = \left[ g_{\mu\nu}(x_0 + q) - g_{\mu\nu}(x_0) \right] \left( \dot{q}^{\mu}(\tau) \dot{q}^{\nu}(\tau) + b^{\mu}(\tau) c^{\nu}(\tau) + \alpha^{\mu}(\tau) \alpha^{\nu}(\tau) \right)$$

$$+ \dot{q}^{\mu}(\tau) \omega_{\mu\alpha\beta}(x_0 + q) \psi_{1,bg}^{\alpha} \psi_{1,bg}^{\beta}$$

$$- \left( \frac{\beta \hbar}{2} \right)^2 g^{\mu\nu}(x_0 + q) \left( \Gamma_{\mu\alpha^\beta}(x_0 + q) \Gamma_{\nu\rho\sigma}(x_0 + q) - \frac{1}{2} \omega_{\mu\alpha\beta}(x_0 + q) \omega_{\nu\rho\sigma}(x_0 + q) \right)$$  \hspace{1cm} (14)

The expectation value \( <> \) means that all quantum fields must be contracted using the appropriate propagators. In the expression \[10\] \((b^{\mu}, c^{\nu})\) and \(\alpha^{\mu}\) is a set of anticommuting and commuting Lee-Yang ghosts respectively \[10\]. When we take the limit \( \beta \to 0 \) only one-loop graphs survive and the corresponding interaction is:

$$L_{int}(\tau) = \frac{1}{2} \dot{q}^{\mu}(\tau) \dot{q}^{\nu}(\tau) R_{\mu\nu ab}(\omega(x_0)) \psi_{1,bg}^{a} \psi_{1,bg}^{b}$$  \hspace{1cm} (15)

where \( R_{\mu\nu ab}(\omega(x_0)) = 2\partial_{[\mu}\omega_{\nu]ab}(x_0) \) in a frame with \( \omega_{\mu\nu}(x_0) = 0 \).

In our case the total interaction Langrangian density decomposes into an untwisted and a twisted part:

$$L_{int} = L_U + L_T$$

$$= \frac{1}{4\beta} \left( \psi_{1,bg}^{a} \psi_{1,bg}^{b} R_{ablk}(\omega) q^l q^k + \frac{1}{2} \psi_{1,bg}^{i} \psi_{1,bg}^{j} R_{ijmn}(\omega) Q^m Q^n \right).$$  \hspace{1cm} (16)

The first term corresponds to the familiar interaction \[10\] on the base manifold while the second describes the interaction on the normal bundle.

The propagator for bosons that move along the fibres \( R^2 \) of the normal bundle is:

$$< Q^i(\tau) Q^j(\sigma) >_{k_i} = -\beta \hbar g^{ij} \Delta_{k_i}^b (\tau - \sigma)$$  \hspace{1cm} (17)

where \( g^{ij} \) is the metric on the normal bundle and \( \Delta_{k_i}^b \) is the Green’s function given by:

$$\Delta_{k_i}^b (\tau - \sigma) = \sum_{m} \frac{\phi_m(\tau) \phi_m^*(\sigma)}{\lambda_m}.$$  \hspace{1cm} (18)

The \( \lambda_m \)'s are the eigenvalues of the kinetic operator \( d^2/d\tau^2 \), \( \Delta_{k_i}^b \) satisfies:

$$\frac{\partial^2}{\partial \tau^2} \Delta_{k_i}^b (\tau - \sigma) = \delta(\tau - \sigma)$$  \hspace{1cm} (19)

\[3\]In principle one could compute an infinite set of Feynman diagrams with a spin-\( \frac{1}{2} \) field circulating in a loop and external gravitons couple at the vertices of the fermion loop. The advantage of the background formalism is encapsulated in the fact that it takes the contributions of all vertices at once.

\[4\]The significance of such ghosts is implied by the integration over the momenta \( p_\mu(\tau) \) in the transition from phase space to configuration space. A factor \( \sqrt{g} \) is then produced and by exponentiating it, introducing the Lee-Yang ghosts, we are led to a perfectly regular term in the action.
and in addition, it is subjected to the boundary condition:

\[ \Delta^{b}_{k_i}(\tau, -1) = e^{i\theta_i} \Delta^{b}_{k_i}(\tau, 0). \]  

(20)

on the time interval \([-1, 0]\). One then finds the solution:

\[ \Delta^{b}_{k_i}(\tau - \sigma) = \sum_{n=-\infty}^{+\infty} \frac{e^{i(2\pi n + \theta_i)(\tau - \sigma)}}{i(2\pi n + \theta_i)^2}. \]

(21)

from which by setting \(k_i = 0\) we recover the center-of-mass Green’s function [7]:

\[ \Delta^{b}_{cm}(\tau - \sigma) = -\sum_{n=-\infty}^{+\infty} \frac{\tau^{2\pi n (\tau - \sigma)}{(2\pi n)^2}}{2} = \frac{1}{2}(\tau - \sigma) \epsilon(\tau - \sigma) - \frac{1}{2}(\tau - \sigma)^2 - \frac{1}{12}. \]

(22)

The fermionic propagator is:

\[ <\Psi^i(\tau)\Psi^j(\sigma) >_{k_i} = g^{ij} \Delta^{f}_{k_i}(\tau - \sigma) \]

(23)

and \(\Delta^{f}_{k_i}\) is given by:

\[ \Delta^{f}_{k_i}(\tau - \sigma) = \sum_{n=-\infty}^{+\infty} \frac{e^{i(2\pi n + \theta_i)(\tau - \sigma)}}{i(2\pi n + \theta_i)}. \]

(24)

### 4 Twisted chiral anomaly

The untwisted chiral anomaly, in which both bosonic and fermionic fields are periodic, can be written equivalently as in [7]:

\[ An(chiral) = \left( -\frac{i}{2\pi} \right)^{n/2} \int_{-1}^{0} \prod_{i=1}^{n} dx_{0}^{i} \sqrt{g(x_{0})} \prod_{a=1}^{n} d\psi^{a}_{1,0} \exp \left[ \frac{1}{2} \text{Tr} \ln \left( \frac{R_{\mu\nu}/4}{\sinh(R_{\mu\nu}/4)} \right) \right]. \]

(25)

At first sight the expression (25) is obscured since for example \(R_{4}\) could equally mean \(\text{tr}(R_{4})\) or \(\text{tr}(R_{2})\text{tr}(R_{2})\). The recipe one has to apply is to write down the series for the logarithm, replace \(R_{m} \rightarrow \text{tr}(R_{m})\) everywhere, and only then one should take the exponential.

Using the interaction Lagrangian density (16) and the propagators (17), (23) one evaluates the total integrated chiral anomaly of the relevant worldline graphs to be proportional to:

\[ <e^{-\frac{1}{2}S_{int}} > = e^{\frac{1}{2} \text{Tr} \ln \left( \frac{R_{k_i}/4}{\sinh(R_{k_i}/4)} \right)} \times \exp \left[ \frac{1}{N} \sum_{k_{i}=1}^{N-1} \sum_{l=1}^{\infty} \left( -\frac{1}{\beta h} \right)^{l} \frac{(l-1)!}{l!} 2^{l-1} \text{tr} \left( \frac{R_{m}/8}{\beta h} \right)^{l} (-\beta h)^{l} I_{k_{i},l} \right] \]

(26)

where

\[ I_{k_{i},l} = \int_{-1}^{0} d\tau_{1} \int_{-1}^{0} d\tau_{2} \cdots \int_{-1}^{0} d\tau_{l} \Delta^{\star}_{k_{i}}(\tau_{1} - \tau_{2}) \Delta^{\star}_{k_{i}}(\tau_{2} - \tau_{3}) \cdots \Delta^{\star}_{k_{i}}(\tau_{l} - \tau_{1}) \]

(27)

and all bosons are twisted in the same way with \(\Delta^{\star}_{k_{i}}(\tau - \sigma) = \frac{\partial}{\partial \sigma} \Delta_{k_{i}}(\tau - \sigma)\). The combinatorial factors \((l-1)!\) and \(2^{l-1}\) stand for the ways one can contract \(l\) vertices and two \(q\) fields at \(l\) vertices respectively. The multiple integral can be evaluated using successively:

\[ \int_{-1}^{0} \Delta^{\star}_{k_{i}}(\tau_{1} - \tau_{2}) \Delta^{\star}_{k_{i}}(\tau_{2} - \tau_{3}) d\tau_{2} = \Delta_{k_{i}}(\tau_{1} - \tau_{3}) \]

(28)
\[
\int_{-1}^{0} \Delta_{\beta}(\tau_1 - \tau_3) \Delta_{\beta}^*(\tau_3 - \tau_4) \, d\tau_3 = \left( \frac{\partial}{\partial \tau_4} \right)^{-1} \Delta_{\beta}^*(\tau_1 - \tau_4) \tag{29}
\]

and the final integral reads:

\[
I_{k,i} = \int_{-1}^{0} \int_{-1}^{0} d\tau_1 d\tau_l \left( \frac{\partial}{\partial \tau_l} \right)^{-(l-3)} \Delta_{\beta}(\tau_1 - \tau_l) \Delta_{\beta}^*(\tau_l - \tau_1)
\]

\[
= \frac{(-1)^l}{(2\pi)^l} \sum_{n=-\infty}^{+\infty} \frac{1}{(n + \alpha_{k_l})^{l}} \tag{30}
\]

where \(\alpha_{k_l} = \frac{k_l}{N}\). If we compare this result with the one derived from the untwisted sector \([7]\) we observe that \(I_{k,i}\) is no longer zero for odd values of \(l\). To demonstrate this consider the case \(I_{k,i,1}\) then:

\[
I_{k,i,1} = \int_{-1}^{0} \Delta_{\beta}^*(0) \, d\tau_1 = \frac{1}{2i\pi} \sum_{n=-\infty}^{+\infty} \frac{1}{(n + \alpha_{k_l})}
\]

\[
= \frac{1}{2i\pi} [f(1 - \alpha_{k_l}) - f(\alpha_{k_l})] = \frac{1}{2i} \cot(\pi \alpha_{k_l}) \tag{31}
\]

where \(f(z) = \frac{\Gamma'(z)}{\Gamma(z)}\) and \(\Gamma(z)\) is the well known gamma function (see Appendix for details). Consider now a closed q-loop with an odd number of vertices \((-\frac{1}{4\pi}R_{\mu\nu}(x_0)\psi_{1,0}^{\mu}\psi_{1,0}^{\nu} = -\frac{1}{4\pi}R_{\mu\nu}(x_0)\)
and perform all the contractions of the related quantum bosonic fields (two at each vertex namely \(g^\mu(\tau), \bar{g}^\nu(\tau)\)). There always be one vertex in the end for which upon contracting the attached bosonic fields (at a given time) will give, apart from a numerical factor, a result \(\propto g^\mu R_{\mu\nu}R^{2k} = 0\). Only parity violating amplitudes are anomalous and thus the twisted chiral anomaly receives contributions exclusively from even values of \(l\). Therefore the anomaly is proportional to:

\[
< A_{twisted} > = \frac{1}{N} \exp \left[ \sum_{k_{i=1}}^{N-1} \sum_{l=1}^{+\infty} \left( \frac{1}{4} \right)^{2l} 2^{2l-1} \frac{(2l-1)!}{(2l)!} Tr \left( \frac{R_{mn}}{2} \right)^{2l} I_{k_l,2l} \right]
\]

\[
= \exp \left[ \frac{1}{2} \ln \left( \frac{1}{N^2} \right) + \frac{1}{2} Tr \left( \sum_{k_{i=1}}^{N-1} \sum_{l=1}^{+\infty} \frac{z^{2l}}{2l} \sum_{n=-\infty}^{+\infty} \frac{1}{(n + \alpha_{k_l})^{2l}} \right) \right] \tag{32}
\]

where \(z = \frac{R_{mn}}{8\pi}\). Performing first the summation over \(l\), then over \(n\) and making use of the identity (see Appendix for the proof):

\[
\left( 1 \pm \frac{z}{\alpha_i} \right) \prod_{n=1}^{+\infty} \left( 1 \pm \frac{z}{n + \alpha_{k_i}} \right) \left( 1 \mp \frac{z}{n - \alpha_{k_i}} \right) = \frac{\sin(\pi (z \pm \alpha_{k_i}))}{\sin(\pi \alpha_{k_i})} \tag{33}
\]

yields:

\[
< A_{twisted} > = \exp \left[ \frac{1}{2} Tr \ln \left( \frac{1}{N^2} \prod_{k_{i=1}}^{N-1} \left( \frac{\sin(\pi \alpha_{k_i}) \sin(\pi \alpha_{k_i}^*)}{i \sin(\frac{\alpha_{k_i}^*}{2}) i \sin(\frac{\alpha_{k_i}}{2})} \right) \right) \right]
\]
\begin{align*}
\exp \left[ \frac{1}{2} Tr \ln \prod_{k_i=1}^{N-1} \left( \frac{1}{2i \sin(\frac{\alpha_{k_i}}{2})} \frac{1}{2i \sin(\frac{\alpha_{k_i}^*}{2})} \right) \right] \\
= \exp \left[ Tr \ln \prod_{k_i=1}^{N-1} \left( \frac{1}{2i \sin(\frac{\alpha_{k_i}}{2})} \right) \right]
\end{align*}

where \( \tilde{\alpha}_{k_i} = 2\pi \alpha_{k_i} + \frac{iR_{m \bar{n}}}{4} \), \( \tilde{\alpha}_{k_i}^* \) is the complex conjugate of \( \tilde{\alpha}_{k_i} \) and

\[
\prod_{k_i=1}^{N-1} \sin(\pi \alpha_{k_i}) = \frac{N}{2^{N-1}}.
\]

The last equality in (34) can be justified from the fact that:

\[
sin(\pi \alpha_{k_i} + z) = \begin{cases} 
\sin(\pi \alpha_{k_i} + 2 - z) & \text{for } k_i + 2 < k_i \\
\sin(\pi \alpha_{k_i} - z) & \text{for } k_i + 2 = N \\
\sin(\pi \alpha_{k_i} - 2 - z) & \text{for } k_i + 2 > k_i
\end{cases}
\]

This expression is in agreement with the one found by Eguchi [3] and Witten [4, 5].

\section{Conclusions}

We have extended the Feynman diagrams formalism developed by the authors [7, 8] to calculate chiral anomalies when twisted boundary conditions are present. The starting point was the expression that incorporates all the q-loop contributions to the chiral anomaly in which all bosons were subjected to identical boundary conditions.

A complete and meticulous examination of this problem would require steps similar to those presented in [7, 8] but this task is postponed for a future investigation. It is believed our result to be changed only by a numerical factor which for the untwisted case was found to be \( \left( \frac{-i}{2\pi} \right)^{2n} \).

\section{Appendix}

We define the generalised zeta function by the equation:

\[
\zeta(s, a) = \sum_{n=0}^{+\infty} \frac{1}{(a + n)^s}
\]

where \( a \) is a constant. The contribution \( I_{k_1,1} \) can be rewritten as:

\[
I_{k_1,1} = \zeta(1, \alpha_{k_1}) - \zeta(1, 1 - \alpha_{k_1}).
\]

The zeta function has the following asymptotic form when \( s \to 1 \):

\[
\lim_{s \to 1} \left[ \zeta(s, \alpha_{k_1}) - \frac{1}{s - 1} \right] = -\frac{\Gamma'(\alpha_{k_1})}{\Gamma(\alpha_{k_1})}.
\]

\footnote{In the untwisted case it was derived from \( J = (-i)^{n/2} \prod_{n=1}^{N} (\Psi^a + \Psi^b) \) which is the definition of the \( \gamma_5 \) analogue for the corresponding supersymmetric quantum mechanical model, the doubling of the Majorana fermions, the transition element of the bosonic action, the integration over the fermionic coherent states and the rescaling of the fermionic quantum and background fields.}
Also from $\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}$ by differentiation with respect to $z$ we get:

$$\frac{\Gamma'(z)}{\Gamma(z)} - \frac{\Gamma'(1-z)}{\Gamma(1-z)} = -\pi \cot(\pi z). \quad (40)$$

Combining (39) and (40) we arrive at (31).

The expression (33) is easily proved with the help of:

$$\prod_{n=1}^{+\infty} \left[ 1 - \left( \frac{z}{n} \right)^2 \right] = \frac{\sin(\pi z)}{\pi z}. \quad (41)$$

One then has:

$$\left( 1 \pm \frac{z}{\alpha_{ki}} \right) \prod_{n=1}^{+\infty} \left( 1 \pm \frac{z}{n + \alpha_{ki}} \right) \left( 1 \mp \frac{z}{n - \alpha_{ki}} \right) = \left( 1 \pm \frac{z}{\alpha_{ki}} \right) \prod_{n=1}^{+\infty} \left[ 1 - \left( \frac{z}{n} - \alpha_{ki} \right)^2 \right] = \frac{\sin \pi (z \pm \alpha_{ki})}{\sin(\pi \alpha_{ki})}. \quad (42)$$

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