ERGODIC BEHAVIOUR OF A DOUGLAS-RACHFORD OPERATOR AWAY FROM THE ORIGIN

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Abstract. It is shown that away from the origin, the Douglas-Rachford operator with respect to a sphere and a convex set in a Hilbert space can be approximated by another operator which satisfies a weak ergodic theorem. Similar results for other projection and reflection operators are also discussed.

1. Introduction

1.1. Background. Given a set $A$ in a Hilbert space $\mathbb{H}$, denote by $P_A : \mathbb{H} \rightrightarrows \mathbb{H}$ the multi-valued projection operator, that is,

$$P_A x = \left\{ y \in A \mid \|x - y\| = \inf_{z \in A} \|x - z\| \right\},$$

where here and in what follows, $\| \cdot \|$ denotes the Hilbert norm on $\mathbb{H}$. Also, if $I : \mathbb{H} \to \mathbb{H}$ is the identity operator, denote by $R_A : \mathbb{H} \rightrightarrows \mathbb{H}$ the reflection operator, which is given by

$$R_A = 2P_A - I.$$

Given two sets $A, B \subseteq \mathbb{H}$, define the Douglas-Rachford operator by

$$T_{A,B} = \frac{I + R_B R_A}{2}. \quad (1.1)$$

Given $x \in \mathbb{H}$, let $\{x_n\}_{n=0}^{\infty} \subseteq \mathbb{H}$, be the sequence which is defined as follows,

$$x_{n+1} = T_{A,B} x_n = T_{A,B}^n x_0, \quad x_0 = x. \quad (1.2)$$

This sequence is also known as the Douglas-Rachford iteration of $x$. It was studied first in [DR56] as an algorithm for finding an intersection point of two sets. Indeed, it is not hard to check that

$$Tx = x \iff P_A x \in A \cap B, \quad (1.3)$$

and so any point $x \in A \cap B$ is a fixed point of $T_{A,B}$.

Analysing the Douglas-Rachford operator (1.1) and the iteration sequence (1.2) are well known questions with interesting applications. This question has been studied in a convex setting (that is, when both $A$ and $B$ are convex), as well as in a non-convex setting (when either $A$ or $B$ is not convex). See for example [BCL02, LM79] for the convex case and [ERT07, GE08] for the non-convex case.

In the case $A$ is convex, it is known that the projection operator $P_A$ is firmly non-expansive, that is, for every $x, y \in \mathbb{H},$

$$\|P_A x - P_A y\|^2 + \|(I - P_A)x - (I - P_A)y\|^2 \leq \|x - y\|^2.$$

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See for example [GK90, Thm. 12.2]. It then follows that the reflection operator $R_A$ is non-expansive, that is, for every $x, y \in \mathbb{H}$,
\[
\|R_Ax - R_Ay\| \leq \|x - y\|,
\]
and the Douglas-Rachford operator is firmly non-expansive. See for example [GK90, Thm. 12.1]. From the results of [Opi67], it then follows that the Douglas-Rachford iteration (1.2) is weakly convergent. In the case $\mathbb{H}$ is finite dimensional, the weak convergence implies strong (norm) convergence.

While the convex case is well understood, much less is known about the non-convex case. One of the simplest examples of a non-convex setting is the case of a sphere and a line. This case was studied in [AAB13, BS11, Ben15, Gil16]. Let
\[
\mathbb{S} = \{x \in \mathbb{H} \mid \|x\| = 1\},
\]
and for $\lambda \geq 0$,
\[
L_\lambda = \{te_1 + \lambda e_2 \in \mathbb{H} \mid t \in \mathbb{R}\},
\]
where here $\{e_1, e_2, \ldots\}$ is an orthonormal basis of $\mathbb{H}$. It was shown in [Ben15] that if $\lambda \in (0, 1)$, then for every $x \in \mathbb{H}$ with $\langle x, e_1 \rangle \neq 0$, the Douglas-Rachford iteration converges in norm to one of the two intersection points of $\mathbb{S}$ and $L_\lambda$. Here and in what follows $\langle \cdot, \cdot \rangle$ denotes the inner product on $\mathbb{H}$. Global convergence for the case $\lambda = 0$ was already proved in [BS11]. The result in [Ben15] improved previous results, which only gave local convergence. It was also shown in [BS11], that if $\langle x, e_1 \rangle = 0$ or if $\lambda \geq 1$, the Douglas-Rachford iteration is not convergent. Note that the case $\lambda \leq 0$ is completely analogous. Other non-convex cases were considered in [AABT16, HL13, Pha16].

1.2. An ergodic theorem for Lipschitz approximations of the Douglas-Rachford operator. It follows from the results of [Ben15], that the convergence of the Douglas-Rachford iteration is uniform on compact sets. See [Gil16] for the exact argument (in [Gil16] one considers a finite dimensional Hilbert space, but the case for an infinite dimensional space is similar). Define the following sets,
\[
\mathbb{H}_+ = \{x \in \mathbb{H} \mid \langle x, e_1 \rangle > 0\}, \quad \mathbb{H}_- = \{x \in \mathbb{H} \mid \langle x, e_1 \rangle < 0\}, \quad \mathbb{H}_0 = \{x \in \mathbb{H} \mid \langle x, e_1 \rangle = 0\}. \quad (1.6)
\]
It is straightforward to show that if $T = T_{S, L_\lambda}$, then $T(\mathbb{H}_+) \subseteq \mathbb{H}_+$, $T(\mathbb{H}_-) \subseteq \mathbb{H}_-$, $T(\mathbb{H}_0) \subseteq \mathbb{H}_0$. In particular, it follows that if $K \subseteq \mathbb{H}_+$ or $K \subseteq \mathbb{H}_-$ is compact, then
\[
\sup_{x, y \in K} \|T^n x - T^n y\| \underset{n \to \infty}{\to} 0. \quad (1.7)
\]
An estimate of the form (1.7) is also known as a weak ergodic theorem. This type of theorems appears in the literature of population biology. See for example [Coh79]. See also [Nus90, RZ03] for further discussion on weak ergodic theorems.

In this note, we are interested in an estimate of the form (1.7) for the Douglas-Rachford operator in a more general setting where one of the sets is the unit sphere $\mathbb{S}$ (1.4) and the other set is a convex set in $\mathbb{H}$, and the two sets have non-empty intersection (also known as the feasible case). This of course includes the case of the sphere and any affine subspace of $\mathbb{H}$. While we are unable to show an estimate of the form (1.7) for the Douglas-Rachford operator itself, what we can show is that away from the origin, the Douglas-Rachford operator can be approximated by another operator that satisfies (1.7). The main result of this note reads as follows.

**Theorem 1.1.** Assume that $C \subseteq \mathbb{H}$ is a convex set, let $\mathbb{S}$ be the unit sphere in $\mathbb{H}$ (1.4), and assume that $\mathbb{S} \cap C \neq \emptyset$. Let $T = T_{\mathbb{S}, C}$, and let $x_0 \in \mathbb{S} \cap C$. Assume also that $\alpha, \beta, \gamma \geq 0$ are such
that $\beta \in [0,1)$, $r \geq \frac{2}{1 - \beta}$, and $\alpha \leq \frac{1}{1 - \beta}$. Then there exists $G : \mathbb{H} \rightarrow \mathbb{H}$ such that
\[
\sup_{x \in B(x_0, r) \setminus B(0, 1 - \beta)} \|Gx - Tx\| \leq 2r \left( 1 - \alpha(1 - \beta) \right),
\]
and for all $n \in \mathbb{N}$,
\[
\sup_{x,y \in B(x_0, r)} \|G^n x - G^n y\| \leq 2r \alpha^n.
\]

In Theorem 1.1 and in what follows, $B(x, r)$ denotes the open ball around $x$ with radius $r$ with respect to the norm $\| \cdot \|$, while $B[x, r]$ denotes the closed ball. If we consider $T = T_{C,S}$ rather than $T_{S,C}$, Theorem 1.1 does not necessarily hold. See Remark 2.2 and Remark 3.3 below.

The proof of Theorem 1.1 is done in two steps. First, it is shown that away from the origin, the Douglas-Rachford operator satisfies a Lipschitz condition, and so using classical extension results, it can be extended to a Lipschitz map on all of $\mathbb{H}$. This is discussed in Section 2. By using further smoothing operations, it is shown that away from the origin, the Douglas-Rachford operator can be approximated by another operator which satisfies an estimate of the form (1.7). The proof of Theorem 1.1 is presented in Section 3.

In the special case where $C = L_\lambda$, as defined in (1.5), we have in fact a slightly stronger result, namely that we can construct $G$ such that $\mathbb{H}_+ \cup \mathbb{H}_0$ (alternatively, $\mathbb{H} \cup \mathbb{H}_0$) is invariant under $G$. See Remark 2.3 and Remark 3.4 below.

### 1.3. Other projection and reflection operators.

Given two sets $A, B \subseteq \mathbb{H}$, the Douglas-Rachford operator (1.1) is a special case of the following parametric family of operators. Given $s_1, s_2, s_3 \in [0,1]$, define
\[
T_{A,B}^{s_1,s_2,s_3} = s_1 I + (1 - s_1) (s_2 I + (1 - s_2) R_B) (s_3 I + (1 - s_3) R_A).
\]
(1.8)

As before, $I$ denotes the identity operator and $R_A, R_B$, denote the reflection operators on $A, B$, respectively. Note that the Douglas-Rachford operator defined in (1.1) corresponds to the case $s_1 = \frac{1}{2}, s_2 = s_3 = 0$. See [BST15] for a more detailed discussion on this family of operators. It is straightforward to show that the main result, Theorem 1.1, holds in fact for this more general family (1.8). See Remark 2.1 and Remark 3.2 below.

**Theorem 1.2.** Assume that $C \subseteq \mathbb{H}$ is a convex set, let $S$ be the unit sphere in $\mathbb{H}$ (1.4) and assume that $S \cap C \neq \emptyset$. Let $s_1, s_2, s_3 \in [0,1]$, let $T = T_{S,C}^{s_1,s_2,s_3}$, and let $x_0 \in S \cap C$. Assume also that $\alpha, \beta, r \geq 0$ are such that $\beta \in [0,1)$, and $r$ and $\alpha$ satisfy
\[
r \geq \frac{2(1 + \beta - 2(s_1 + (1 - s_1)(s_2 + s_3) + (1 - s_1)(1 - s_2)s_3)\beta)}{1 - \beta},
\]
and
\[
\alpha \leq \frac{1 + \beta - 2(s_1 + (1 - s_1)(s_2 + s_3) + (1 - s_1)(1 - s_2)s_3)\beta}{1 - \beta}.
\]

Then there exists $G : \mathbb{H} \rightarrow \mathbb{H}$ such that
\[
\sup_{x \in B(x_0, r) \setminus B(0, 1 - \beta)} \|Gx - Tx\| \leq 2r \left( 1 - \frac{\alpha(1 - \beta)}{1 + \beta - 2(s_1 + (1 - s_1)(s_2 + s_3) + (1 - s_1)(1 - s_2)s_3)\beta} \right),
\]
and for all $n \in \mathbb{N}$,
\[
\sup_{x,y \in B(x_0, r)} \|G^n x - G^n y\| \leq 2r \alpha^n.
\]
Note that choosing $s_1 = \frac{1}{2}$ and $s_2 = s_3 = 0$ in Theorem 1.2 gives Theorem 1.1. Another well known case is when $s_1 = 0$ and $s_2 = s_3 = \frac{1}{2}$, in which case we obtain
\[ T_{A,B}^{0,\frac{1}{2},\frac{1}{2}} = P_B P_A, \]
also known as the Von-Neumann operator [vN50]. Regarding the convergence of the iteration sequence $x_{n+1} = P_B P_A x_n$, $x_0 = x$, it was shown in [vN50] that if $A, B$, are both subspaces in $\mathbb{H}$, then $x_n \xrightarrow{n \to \infty} P_{A \cap B} x$ (norm convergence). It was later shown in [BB93] that if $0 \in \text{int}(A - B)$ or $A - B$ is a closed subspace, then the iteration sequence converges linearly (that is, when the rate of convergence is $ca^n$, where $c > 0$ is a constant and $\alpha \in [0, 1)$).

For the von Neumann operator, we have in fact a stronger result than Theorem 1.1, which reads as follows.

**Theorem 1.3.** Assume that $C \subseteq \mathbb{H}$ is a convex set, let $S$ be the unit sphere in $\mathbb{H}$ (1.4), and assume that $S \cap C \neq \emptyset$. Let $T = P_C P_S$, and let $x_0 \in S \cap C$. Also, assume that $\alpha, \beta, r \geq 0$ are such that $\beta \in [0, 1)$, $r \geq 2$, and $\alpha \leq \frac{1}{2 - \beta}$. Then there exists $G : \mathbb{H} \to \mathbb{H}$ such that
\[
\sup_{x \in B[x_0, r]} \| G x - T x \| \leq 2r(1 - \alpha(1 - \beta)),
\]
and
\[
\sup_{x, y \in B[x_0, r]} \| G^n x - G^n y \| \leq 2ra^n.
\]

Note that Theorem 1.3 is slightly stronger than Theorem 1.1 since we only require $r \geq 2$, rather than $r \geq \frac{1}{1 - \beta}$. Similar to the case of Theorem 1.1, we cannot change the order of the projections in Theorem 1.3. See Remark 4.1 below. Theorem 1.3 is proved in Section 4.

2. Lipschitz behaviour of the Douglas-Rachford operator

Given two Banach spaces $(X, \| \cdot \|_X)$ and $(Y, \| \cdot \|_Y)$, a set $D \subseteq X$, and a map $f : D \to Y$, define the Lipschitz constant of $f$ to be
\[
\| f \|_{\text{lip}} = \sup_{x, y \in D \atop x \neq y} \frac{\| f(x) - f(y) \|_Y}{\| x - y \|_X}.
\]
A map $f : X \to Y$ is said to be Lipschitz if $\| f \|_{\text{lip}} < \infty$. Note that if $C \subseteq \mathbb{H}$, then $T_{S,C}$ is not necessarily Lipschitz on $\mathbb{H}$, since $P_S = x/\| \cdot \|_\mathbb{H}$, which is not Lipschitz. However, it is shown below that if $C \subseteq \mathbb{H}$ is convex, the Douglas-Rachford operator can be ‘smoothed’ in a neighbourhood of the origin such that the smoothed operator satisfies a Lipschitz condition.

**Theorem 2.1.** Assume that $C \subseteq \mathbb{H}$ is a convex set, and let $S$ be the unit sphere in $\mathbb{H}$ (1.4). Let $T = T_{S,C}$, and let $\beta \in [0, 1)$. Then there exists $F : \mathbb{H} \to \mathbb{H}$ such that
\[
F \big|_{\mathbb{H} \setminus B(0, 1 - \beta)} = T,
\]
and
\[
\| F \|_{\text{lip}} \leq \frac{1}{1 - \beta}.
\]

We begin with the following proposition.

**Proposition 2.1.** Assume that $x, y \in \mathbb{H} \setminus B(0, 1 - b)$. Then
\[
\| R_S x - R_S y \| \leq \frac{1 + \beta}{1 - \beta} \| x - y \|.
\]
Proof. Recall that
\[ R_S x = 2P_S x - x = \left( \frac{2}{\|x\|} - 1 \right)x. \]

Hence,
\[
\|R_S x - R_S y\|^2 = \|R_S x\|^2 + \|R_S y\|^2 - 2\langle R_S x, R_S y \rangle
\]
\[ = (2 - \|x\|)^2 + (2 - \|y\|)^2 - 2 \left( \frac{2}{\|x\|} - 1 \right) \left( \frac{2}{\|y\|} - 1 \right) \langle x, y \rangle \]
\[ = 4 - 4\|x\| + \|x\|^2 + 4 - 4\|y\| + \|y\|^2 - 2 \left( \frac{4}{\|x\|\|y\|} - \frac{2}{\|x\|} - \frac{2}{\|y\|} \right) \langle x, y \rangle - 2\langle x, y \rangle \]
\[ = \|x - y\|^2 + 4(2 - \|x\| - \|y\|) \left( 1 - \frac{\langle x, y \rangle}{\|x\|\|y\|} \right). \tag{2.1} \]

Now, since \( \|x\|\|y\| \leq \frac{\|x\|^2 + \|y\|^2}{2} \) for all \( x, y \in \mathbb{H} \), if \( x, y \in \mathbb{H} \setminus B(0, 1 - \beta) \), then
\[
\|x\|\|y\| - \langle x, y \rangle \leq \frac{\|x\|^2 + \|y\|^2}{2} - \langle x, y \rangle \]
\[
\leq \frac{\|x\|\|y\|}{(1 - \beta)^2} \left( \frac{\|x\|^2 + \|y\|^2}{2} - \langle x, y \rangle \right) \]
\[ = \frac{\|x\|\|y\|}{2(1 - \beta)^2} (\|x\|^2 + \|y\|^2 - 2\langle x, y \rangle) \]
\[ = \frac{\|x\|\|y\|}{2(1 - \beta)^2} \|x - y\|^2, \]

where in (*) we used the fact that
\[ \frac{\|x\|^2 + \|y\|^2}{2} - \langle x, y \rangle \geq \|x\|\|y\| - \langle x, y \rangle \geq 0, \]
and the fact that \( \|x\| \geq 1 - \beta \) and \( \|y\| \geq 1 - \beta \). Therefore, if \( x, y \in \mathbb{H} \setminus B(0, 1 - \beta) \), then
\[ 1 - \frac{\langle x, y \rangle}{\|x\|\|y\|} \leq \frac{1}{2(1 - \beta)^2} \|x - y\|^2. \tag{2.2} \]

Plugging (2.2) into (2.1), it follows that if \( x, y \in \mathbb{H} \setminus B(0, 1 - \beta) \), then \( 2 - \|x\| - \|y\| \leq 2\beta \), and so
\[
\|R_S x - R_S y\|^2 \leq \left( 1 + 4(2 - \|x\| - \|y\|) \right) \frac{1}{2(1 - \beta)^2} \|x - y\|^2 \]
\[ \leq \left( 1 + \frac{4\beta}{(1 - \beta)^2} \right) \|x - y\|^2 \]
\[ = \frac{(1 + \beta)^2}{(1 - \beta)^2} \|x - y\|^2. \]

Hence,
\[
\|R_S x - R_S y\| \leq \frac{1 + \beta}{1 - \beta} \|x - y\|, \]

and this completes the proof. \( \square \)
Another tool which is needed in the proof of Theorem 2.1 is the following theorem, known as Kirszbraun’s Theorem. See for example [BL00, GK90]. Given a set $D \subseteq \mathbb{H}$, let $\text{conv}(D)$ denote its closed convex hull, where the convex hull is given by

$$\text{conv}(D) = \left\{ \sum_{i=1}^{n} t_i x_i \mid x_i \in D, \ t_i \geq 0, \ 1 \leq i \leq n, \ \sum_{i=1}^{n} t_i = 1, \ n \in \mathbb{N} \right\}.$$ 

Kirszbraun’s theorem reads as follows.

**Theorem 2.2.** Assume that $D_1, D_2 \subseteq \mathbb{H}$. Assume that $f : D_1 \to D_2$ is Lipschitz. Then there exists $F : \mathbb{H} \to \text{conv}(D_2)$ such that $F|_{D_1} = f$ and $\|F\|_{\text{lip}} = \|f\|_{\text{lip}}$.

We are now in a position to prove Theorem 2.1

**Proof of Theorem 2.1.** Since $C$ is convex, it follows that $R_C$ is non-expansive. Let $x, y \in \mathbb{H} \setminus B(0, 1 - \beta)$. Then

$$\|Tx - Ty\| = \left\| \left( \frac{I + R_C R_S}{2} \right) x - \left( \frac{I + R_C R_S}{2} \right) y \right\|$$

$$= \left\| \frac{x - y}{2} + \frac{R_C R_S x - R_C R_S y}{2} \right\|$$

$$\leq \frac{1}{2} \|x - y\| + \frac{1}{2} \|R_C R_S x - R_C R_S y\|$$

$$\leq \frac{1}{2} \|x - y\| + \frac{1}{2} \|R_S x - R_S y\|$$

$$\leq \frac{1}{2} \|x - y\| + \frac{1 + \beta}{2(1 - \beta)} \|x - y\|$$

$$= \frac{\|x - y\|}{1 - \beta},$$

where in (*) we used the fact that $C$ is convex and thus $R_C$ is non-expansive, and in (**) we used Proposition 2.1. Applying Theorem 2.2 to $T$ on the sets $D_1 = \mathbb{H} \setminus B(0, 1 - \beta)$ and $D_2 = \mathbb{H}$ completes the proof.

**Remark 2.1.** Note that if $T = T_{s_1, s_2, s_3}^{s_{1, s_2, s_3}}$ is as defined in (1.8), then in particular,

$$T = (s_1 + (1 - s_1)(s_2 + s_3)) I + (1 - s_1)s_2(1 - s_3) R_S + (1 - s_1)(1 - s_2)s_3 R_C + (1 - s_1)s_2s_3 R_C R_S.$$

Note also that

$$(s_1 + (1 - s_1)(s_2 + s_3)) + (1 - s_1)s_2(1 - s_3) + (1 - s_1)(1 - s_2)s_3 + (1 - s_1)s_2s_3 = 1.$$

Hence, if $C$ is convex, then since both $I$ and $R_C$ are non-expansive, using Proposition 2.1, for every $x, y \in \mathbb{H} \setminus B(0, 1 - \beta)$,

$$\|Tx - Ty\| \leq (s_1 + (1 - s_1)(s_2 + s_3) + (1 - s_1)(1 - s_2)s_3) \|x - y\|$$

$$+ ((1 - s_1)s_2(1 - s_3) + (1 - s_1)s_2s_3) \frac{1 + \beta}{1 - \beta} \|x - y\|$$

$$= \frac{1 + \beta - 2(s_1 + (1 - s_1)(s_2 + s_3) + (1 - s_1)(1 - s_2)s_3) \beta}{1 - \beta} \|x - y\|.$$  

(2.3)

Thus, repeating the proof of Theorem 2.1, we obtain a similar result, but now the Lipschitz constant is the one given in (2.3).
Remark 2.2. Even if $C \subseteq \mathbb{H}$ is convex, the map $x \mapsto R_SR_Cx$ need not satisfy a Lipschitz condition, since $R_C$ might be arbitrarily close to 0 (indeed, it might even not be defined). Thus, in general, Theorem 2.1 does not hold for the operator $T = T_{C,S}$.

Remark 2.3. In the case $C = L_\lambda$, as defined in (1.5), if $T = T_{S,L_\lambda}$, then $\mathbb{H}_+, \mathbb{H}_-, \mathbb{H}_0$ as defined in (1.6) are all invariant under $T$. Hence, by applying Theorem 2.2 with $D_1 = (\mathbb{H}_+ \cup \mathbb{H}_0) \setminus B(0, 1 - \beta)$ (resp. $(\mathbb{H}_- \cup \mathbb{H}_0) \setminus B(0, 1 - \beta)$) and $D_2 = \mathbb{H}_+ \cup \mathbb{H}_0$ (resp. $\mathbb{H}_- \cup \mathbb{H}_0$), it follows that in Theorem 2.1 we can choose $F : \mathbb{H}_+ \cup \mathbb{H}_0 \to \mathbb{H}_+ \cup \mathbb{H}_0$ (resp. $F : \mathbb{H}_- \cup \mathbb{H}_0 \to \mathbb{H}_- \cup \mathbb{H}_0$). Note that we cannot choose $F : \mathbb{H}_+ \to \mathbb{H}_+$ or $F : \mathbb{H}_- \to \mathbb{H}_-$ as these are not closed sets.

3. Proof of Theorem 1.1

Given a set $D \subseteq \mathbb{H}$, define

$$\text{diam}(D) = \sup_{x,y \in D} \|x - y\|.$$  

The next proposition shows that on a bounded convex set, we can ‘smooth’ Lipschitz maps, so that the smoothed map satisfies an estimate of the form (1.7). The smoothing operation is similar to the one which appeared in [RZ03].

Proposition 3.1. Assume that $D \subseteq \mathbb{H}$ is bounded and convex, and let $F : D \to D$ be a Lipschitz map. Then for every $\alpha \leq \|F\|_{\text{lip}}$ there exists a map $G : D \to D$ such that

$$\|Gx - Fx\| \leq \left(1 - \frac{\alpha}{\|F\|_{\text{lip}}}\right) \text{diam}(D),$$

and for all $n \in \mathbb{N}$,

$$\sup_{x,y \in D} \|G^n x - G^n y\| \leq \alpha^n \text{diam}(D).$$

In particular, if $\alpha \in [0, 1)$,

$$\sup_{x,y \in D} \|G^n x - G^n y\| \xrightarrow{n \to \infty} 0.$$

Proof. Let $\theta \in D$ and $\gamma \in [0, 1]$. Define

$$Gx = (1 - \gamma)Fx + \gamma \theta.$$  

Then since $D$ is convex, it follows that $G(D) \subseteq D$, and

$$\sup_{x \in D} \|Gx - Fx\| = \sup_{x \in D} \gamma \|Fx - \theta\| \leq \gamma \text{diam}(D).$$

Also,

$$\|G\|_{\text{lip}} = (1 - \gamma)\|F\|_{\text{lip}}.$$  

Choosing $\gamma = 1 - \frac{\alpha}{\|F\|_{\text{lip}}} \in [0, 1]$ and using the fact that $\|G^n\|_{\text{lip}} \leq \|G\|_{\text{lip}}^n$ completes the proof. □

We are now in a position to prove Theorem 1.1.

Proof of Theorem 1.1. Since $x_0 \in S \cap C$, we have $Tx_0 = x_0$, see (1.3). Let $F : \mathbb{H} \to \mathbb{H}$ be the map obtained from Theorem 2.1. Let $x \in B[x_0, r]$. If $x \notin B[0, 1]$ then $R_S = R_g$, where

$$\mathbb{B} = \{x \in \mathbb{H} \mid \|x\| \leq 1\},$$

which is convex. Thus, in this case, $R_C, R_S$ and therefore $T$ are all non-expansive, and so

$$\|Fx - Fx_0\| = \|Tx - Tx_0\| \leq \|x - x_0\| \leq r.$$
If \( x \in B[0,1] \), then by Theorem 2.1,
\[
\| F x - F x_0 \| \leq \frac{\| x - x_0 \|}{1 - \beta} \leq \frac{2}{1 - \beta}.
\]
Therefore, if \( r \geq \frac{2}{1 - \beta} \), then
\[
F(B[x_0, r]) \subseteq B[x_0, r].
\]
Now, \( \text{diam}(B[x_0, r]) = 2r \). Applying Proposition 3.1 to the function \( F \) on the domain \( D = B[x_0, r] \), it follows that for every \( \alpha \leq \frac{1}{1 - \beta} \), there exists \( G : \mathbb{H} \rightarrow \mathbb{H} \) which satisfies \( G(B[x_0, r]) \subseteq B[x_0, r] \), and such that
\[
\sup_{x \in B[x_0, r]} \| Gx - Fx \| \leq 2r (1 - \alpha (1 - \beta)),
\]
and
\[
\sup_{x, y \in B[x_0, r]} \| G^n x - G^n y \| \leq 2r \alpha^n.
\]
Since
\[
\sup_{x \in B[x_0, r] \setminus B(0,1-\beta)} \| Gx - Tx \| \leq \sup_{x \in B[x_0, r]} \| Gx - Fx \|,
\]
the proof is complete. \( \square \)

**Remark 3.1.** Note that by Proposition 3.1, the choice of \( G \) in Theorem 1.1 depends on \( \alpha \) and on the centre point \( x_0 \). \( \diamond \)

**Remark 3.2.** If we consider now the operator \( T = T^{s_1, s_2, s_3}_C \) as defined in (1.8), then repeating the proof of Theorem 1.1 but now using Remark 2.1, we obtain Theorem 1.2. Note that the conditions on \( \alpha \) and \( r \) that we need are \( r \geq 2\| F \|_{\text{lip}} \) and \( \alpha \leq \| F \|_{\text{lip}} \), where \( F \) is the function obtained in Theorem 2.1 (applied now to the operator \( T \)). These are exactly the conditions that appear in Theorem 1.2. \( \diamond \)

**Remark 3.3.** Since, by Remark 2.2, Theorem 2.1 does not necessarily hold if we let \( T = T_{C,S} \), the same is true for Theorem 1.1. \( \diamond \)

**Remark 3.4.** In the case of the sphere and the line, \( C = L_\lambda, \lambda \in [0,1] \) as defined in (1.5), it follows from Remark 2.3 that we can choose \( G : \mathbb{H}_+ \cup \mathbb{H}_0 \rightarrow \mathbb{H}_+ \cup \mathbb{H}_0 \) such that
\[
\sup_{x \in B[x_0, r] \setminus B(0,1-\beta)} \| Gx - Tx \| \leq 2r (1 - \alpha (1 - \beta)),
\]
and for all \( n \in \mathbb{N} \),
\[
\sup_{x, y \in B[x_0, r]} \| G^n x - G^n y \| \leq 2r \alpha^n.
\]
If we replace \( \mathbb{H}_+ \) by \( \mathbb{H}_- \) we obtain a similar result. \( \diamond \)

### 4. Proof of Theorem 1.3

We begin with the following proposition, which shows that the projection operator on the sphere, \( P_3 \), satisfies a Lipschitz condition away from the origin.
**Proposition 4.1.** For every \( x, y \in \mathbb{H} \setminus \{0\}, \)

\[
\left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\| \leq \max \left\{ \frac{1}{\|x\|}, \frac{1}{\|y\|} \right\} \|x - y\|.
\]

In particular, if \( \beta \in [0, 1) \), \( x, y \in \mathbb{H} \setminus B(0, 1 - \beta) \), and \( S \) is the unit sphere in \( \mathbb{H} \) \((1.4)\),

\[
\|P_S x - P_S y\| \leq \frac{\|x - y\|}{1 - \beta}.
\]

**Proof.** Assume without loss of generality that \( \|x\| \leq \|y\| \). Then

\[
\frac{1}{\|x\|^2} \|x - y\|^2 \leq \left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\|^2 = \frac{\|y\|^2}{\|x\|^2} - 2 \langle x, y \rangle \left( \frac{1}{\|x\|^2} - \frac{1}{\|x\| \|y\|} \right) - 1
\]

\[
\geq \frac{\|y\|^2}{\|x\|^2} - 2 \|x\| \|y\| \left( \frac{1}{\|x\|^2} - \frac{1}{\|x\| \|y\|} \right) - 1 = \|y\| \|x\| + 1 = \left( \frac{\|y\|}{\|x\|} - 1 \right)^2 \geq 0,
\]

where in (*) we used the fact that \( \langle x, y \rangle \leq \|x\| \|y\| \) and the fact that \( \frac{1}{\|x\|^2} - \frac{1}{\|x\| \|y\|} \geq 0 \) (since \( \|x\| \leq \|y\| \)). Thus,

\[
\left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\| \leq \frac{1}{\|x\|} \|x - y\|,
\]

which completes the proof of the first statement. The second statement follows as \( P_S x = x/\|x\| \) for all \( x \in \mathbb{H} \setminus \{0\} \). \(\square\)

We are now in a position to prove Theorem 1.3.

**Proof of Theorem 1.3.** Note first that if \( r \geq 2 \), then since \( x_0 \in \mathbb{S} \), \( B[0, 1] \subseteq B[x_0, r] \). Therefore, \( P_S(B[x_0, r] \setminus \{0\}) = \mathbb{S} \). Now, since \( C \) is convex, \( P_C \) is non-expansive, and so for all \( x \in B[x_0, r] \setminus \{0\} \),

\[
\|P_C P_S x - P_C P_S x_0\| \leq \|P_S x - P_S x_0\| \leq 2 \leq r.
\]  \((4.1)\)

Therefore,

\[
P_C P_S(B[x_0, r] \setminus \{0\}) \subseteq B[x_0, r]
\] \((4.2)\)

In particular, it follows that

\[
P_C P_S((1 - \beta)\mathbb{S}) \subseteq B[x_0, r],
\]

where

\[
(1 - \beta)\mathbb{S} = \{ x \in \mathbb{H} \mid \|x\| = 1 - \beta \}.
\]

Thus, by Theorem 2.2, there exists \( F : \mathbb{H} \to B[x_0, r] \) such that \( \|F\|_{\text{lip}} = \frac{1}{1 - \beta} \) and \( F|_{(1 - \beta)\mathbb{S}} = P_S P_C \).

Define \( F_1 : \mathbb{H} \to \mathbb{H} \),

\[
F_1 x = \begin{cases} 
F x & x \in B[0, 1 - \beta], \\
P_C P_S x & x \in \mathbb{H} \setminus B(0, 1 - \beta).
\end{cases}
\] \((4.3)\)

If \( x, y \in B[0, 1 - \beta] \) or \( x, y \in \mathbb{H} \setminus B(0, 1 - \beta) \) then since \( \|F\|_{\text{lip}} = \frac{1}{1 - \beta} \) and by Proposition 4.1,

\[
\|F_1 x - F_1 y\| \leq \frac{\|x - y\|}{1 - \beta}.
\] \((4.4)\)
If, without loss of generality, \( x \in B[0, 1 - \beta] \) and \( y \in \mathbb{H} \setminus B(0, 1 - \beta) \), then there exists \( t \in [0, 1] \) such that \( \|tx + (1 - t)y\| = 1 - \beta \). Thus,
\[
\|F_1x - F_1y\| \leq \|F_1x - F_1(tx + (1 - t)y)\| + \|F_1(tx + (1 - t)y) - F_1y\| \\
\overset{(4.3)}{=} \|Fx - F(tx + (1 - t)y)\| + \|P_CP_3(tx + (1 - t)y) - P_CP_3y\| \\
\leq (1 - t)\frac{\|x - y\|}{1 - \beta} + t\frac{\|x - y\|}{1 - \beta} \\
= \frac{\|x - y\|}{1 - \beta},
\]
where in (*) we used the fact that \( \|F\|_{\text{lip}} = \frac{1}{1 - \beta} \) and Proposition 4.1. Combining (4.4) and (4.5), it follows that \( \|F_1\|_{\text{lip}} = \frac{1}{1 - \beta} \). Now, if \( r \geq 2 \),
\[
F_1(B[0, 1 - \beta]) = F(B[0, 1 - \beta]) \subseteq B[x_0, r],
\]
and
\[
F_1(B[x_0, r] \setminus B(0, 1 - \beta)) = P_CP_3(B[x_0, r] \setminus B(0, 1 - \beta)) \subseteq B[x_0, r],
\]
where in (*) we used the fact that \( F(\mathbb{H}) \subseteq B[x_0, r] \). Altogether,
\[
F_1(B[x_0, r]) \subseteq B[x_0, r],
\]
and \( \|F_1\|_{\text{lip}} = \frac{1}{1 - \beta} \). Applying Proposition 3.1 to \( F_1 \) on the domain \( B[x_0, r] \) completes the proof. \( \square \)

**Remark 4.1.** Note that we cannot change the order of projections in Theorem 1.3. Indeed, it is possible that \( P_Cx = 0 \) for some \( x \in \mathbb{H} \), and then \( P_CLy \) is not defined. Even if \( \|P_Cx\| > 0 \), \( \|P_CLy\| > 0 \), then by Proposition 4.1,
\[
\|P_CLy - P_CLy\| \leq \max\left\{\frac{1}{\|P_Cx\|}, \frac{1}{\|P_CLy\|}\right\} \|P_Cx - P_CLy\| \leq \max\left\{\frac{1}{\|P_Cx\|}, \frac{1}{\|P_CLy\|}\right\} \|x - y\|,
\]
but \( \max\left\{\frac{1}{\|P_Cx\|}, \frac{1}{\|P_CLy\|}\right\} \) can be very large. Thus, we do not obtain an estimate similar to (4.1). \( \diamond \)

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