Perfect dominating sets in the Cartesian products of prime cycles.

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Abstract

We study the structure of a minimum dominating set of $C_{2n+1}^n$, the Cartesian product of $n$ copies of the cycle of size $2n+1$, where $2n+1$ is a prime.

Keywords: Perfect Lee codes; dominating sets; defining sets.

1 Introduction

Let $G$ and $H$ be two graphs. The Cartesian product of $G$ and $H$ is a graph with vertices $\{(x,y) : x \in G, y \in H\}$ where $(x,y) \sim (x',y')$ if and only if $x = x'$ and $y \sim y'$, or $x \sim x'$ and $y = y'$. Let $G^n$ denote the Cartesian product of $n$ copies of $G$. This article deals with $C_{2n+1}^n$ where $C_{2n+1}$ is the cycle of size $p := 2n+1$ and $p$ is a prime.

For our purpose, it is more convenient to view the vertices of $C_{2n+1}^n$ as the elements of the group $G := \mathbb{Z}_{2n+1}^n$. Then $x \sim y$ if and only if $x - y = \pm e_i$ for some $i \in [n]$, where $e_i = (0,\ldots,1,\ldots,0)$ is the unit vector with 1 at the $i$th coordinate. In other words, $C_{2n+1}^n$ is the Cayley graph $\Gamma(\mathbb{Z}_{2n+1}^n, \mathcal{U})$ over the group $\mathbb{Z}_{2n+1}^n$ with the set of generators $\mathcal{U} = \{\pm e_1, \ldots, \pm e_n\}$. From this point on, to emphasis the group structure of the graph we will use the Cayley graph notation $\Gamma(\mathbb{Z}_{2n+1}^n, \mathcal{U})$ instead of the Cartesian product notation of $C_{2n+1}^n$.

Let $u$ and $v$ be two vertices of a graph $G$. We say that $u$ dominates $v$ if $u = v$ or $u \sim v$. A subset $S$ of the vertices of $G$ is called a dominating set, if every vertex of $G$ is dominated by at least one vertex of $S$. A dominating set is perfect, if no vertex is dominated by more than one vertex.
Theorem 1  Let $G$ be a graph with $m$ vertices. Every function $f : V(G) \to \mathbb{C}$ can be viewed as a vector $\vec{f} \in \mathbb{C}^m$. Let $A$ denote the adjacency matrix of $G$. Note that $f : V(G) \to \{0,1\}$ is the characteristic function of a perfect dominating set if and only if $(A + I) \vec{f} = \vec{1}$.

We are interested in perfect dominating sets of $\Gamma(\mathbb{Z}^n_{2n+1}, \mathcal{U})$. Note that for an $r$-regular graph $G = (V,E)$ a dominating set is perfect if and only if it is of size $|V|/(r+1)$. Since $\Gamma(\mathbb{Z}^n_{2n+1}, \mathcal{U})$ is $2n$-regular and has $(2n+1)^n$ vertices, a dominating set is perfect if and only if it is of size $(2n+1)^{n-1}$.

Fix an arbitrary $(\epsilon_1, \ldots, \epsilon_{n-1}) \in \{-1,1\}^{n-1}$, and a $k \in \{0, \ldots, 2n\}$. The set

$$\{(x_1, \ldots, x_{n-1}, k + \sum_{i=1}^{n-1} \epsilon_i (i+1)x_i) : x_i \in \mathbb{Z}^n_{2n+1} \forall i \in [n-1]\}$$

forms a perfect dominating set in $\Gamma(\mathbb{Z}^n_{2n+1}, \mathcal{U})$, where the additions are in $\mathbb{Z}^n_{2n+1}$. To see this consider $y = (y_1, \ldots, y_n) \in \mathbb{Z}^n_{2n+1}$. Let $t = k + \sum_{i=1}^{n-1} \epsilon_i (i+1)y_i$, and $\Delta = t - y_n \pmod{2n+1}$ so that $|\Delta| \leq n$. If $\Delta \in \{-1,0,1\}$ then $y$ is dominated by $(y_1, \ldots, y_{n-1}, t)$. If $\Delta \not\in \{-1,0,1\}$, then with the notation $j := |\Delta| - 1$, $y$ is adjacent to $(y_1, \ldots, y_{j-1}, y_j - \epsilon_j \times \text{sgn}(\Delta), y_{j+1}, \ldots, y_n)$, which can easily be seen that is in the considered set.

There are many results in the direction of constructing perfect dominating sets in the Cartesian product of cycles (see [3] and its references). However the authors are unaware of any result in the direction of characterizing the structure of perfect dominating sets. We consider the simplest case $\Gamma(\mathbb{Z}^n_{2n+1}, \mathcal{U})$ where $2n+1$ is a prime. Even in this simple case we are unable to characterize all the perfect dominating sets. However we prove the following theorem in this direction.

**Theorem 1** Let $2n+1$ be a prime and $S \subseteq \Gamma(\mathbb{Z}^n_{2n+1}, \mathcal{U})$ be a perfect dominating set. Then for every $(x_1, \ldots, x_n) \in \mathbb{Z}^n_{2n+1}$ and every $i \in [n]$,

$$|S \cap \{(y_1, \ldots, y_n) : y_j = x_j \forall j \neq i\}| = 1.$$ 

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$$|S \cap \{(y_1, \ldots, y_n) : y_j = x_j \forall j \neq i\}| = 1.$$ 

**Theorem 1** says that when $2n+1$ is a prime, every parallel-axis line contains exactly one point from every perfect dominating set of $\Gamma(\mathbb{Z}^n_{2n+1}, \mathcal{U})$. It is easy to construct examples to show that the condition of $2n+1$ being a prime is necessary [4].

Let $\mathcal{F}$ be a family of sets. For $S \in \mathcal{F}$, a set $D \subseteq S$ is called a defining set for $(S, \mathcal{F})$ (or for $S$ when there is no ambiguity), if and only if $S$ is the only superset of $D$ in $\mathcal{F}$. The size of the minimum defining set for $(S, \mathcal{F})$ is called its defining number. Defining sets are studied for various families of $\mathcal{F}$ (See [3] for a survey on the topic). Let $\mathcal{F}$ be the family of all minimum dominating sets of $\Gamma(\mathbb{Z}^n_{2n+1}, \mathcal{U})$. Note that since $\Gamma(\mathbb{Z}^n_{2n+1}, \mathcal{U})$ is regular and contains at least one perfect dominating...
set, a set $S \subseteq V(G)$ is a minimum dominating set if and only if it is a perfect dominating set. In [2] Chartrand et al. studied the size of defining sets of $\mathcal{F}$ for $n = 2$. Based on this case they conjectured that the smallest defining set over all minimum dominating sets of $\Gamma(\mathbb{Z}_{2n+1}^n, \mathcal{U})$ is of size exactly $n$. As it is noticed by Richard Bean [private communication], the conjecture fails for $n = 3$, as in this case there are perfect dominating sets with defining number 2 (See Remark [3]). So far there is no nontrivial bound known for the defining numbers of minimum dominating sets of $\Gamma(\mathbb{Z}_{2n+1}^n, \mathcal{U})$. We prove the following theorem.

**Theorem 2** Let $2n + 1$ be a prime and $\mathcal{F}$ be the family of all minimum dominating sets of $\Gamma(\mathbb{Z}_{2n+1}^n, \mathcal{U})$. Every $S \in \mathcal{F}$ has a defining set of size at most $n!2^n$.

The proof of Theorem [1] uses Fourier analysis on finite Abelian groups. In Section 2 we review Fourier analysis on $\mathbb{Z}_p^n$. Section 3 is devoted to the proof of Theorem [2] Section 4 contains further discussions about the defining sets of minimum dominating sets of $\Gamma(\mathbb{Z}_{2n+1}^n, \mathcal{U})$.

### 2 Background

In this section we introduce some notations and review Fourier analysis on $G = \mathbb{Z}_p^n$. For a nice and more detailed, but yet brief introduction we refer the reader to [1]. See also [6] for a more comprehensive reference.

Aside from its group structure we will also think of $G$ as a measure space with the uniform (product) measure, which we denote by $\mu$. For any function $f : G \to \mathbb{C}$, let

$$\int_G f(x)dx = \frac{1}{|G|} \sum_{x \in G} f(x).$$

The inner product between two functions $f$ and $g$ is $\langle f, g \rangle = \int_G f(x)\overline{g(x)}dx$. Let

$$\omega = e^{2\pi i/p},$$

where $i$ is the imaginary number. For any $x \in G$, let $\chi_x : G \to \mathbb{C}$ be defined as

$$\chi_x(y) = \omega^{\sum_{i=1}^n x_iy_i}.$$

It is easy to see that these functions form an orthonormal basis. So every function $f : G \to \mathbb{C}$ has a unique expansion of the form $f = \sum \hat{f}(x)\chi_x$, where $\hat{f}(x) = \langle f, \chi_x \rangle$ is a complex number.
3 Proof of Theorem 1

Let \( \vec{0} = (0, \ldots, 0) \), \( \vec{1} = (1, \ldots, 1) \), and \( e_i = (0, \ldots, 1, \ldots, 0) \), the unit vector with 1 at the \( i \)-th coordinate. Let \( p = 2n + 1 \) be a prime and \( S \) be a perfect dominating set in \( G \), and let \( f \) be the characteristic function of \( S \), i.e. \( f(x) = 1 \) if \( x \in S \) and \( f(x) = 0 \) otherwise. Let

\[
D = \{ \pm e_1, \pm e_2, \ldots, \pm e_n \},
\]

be the set of unit vectors and their negations. For every \( \tau \in D \) define

\[
f_\tau(x) = f(x + \tau).
\]

Let \( g = f + \sum_{\tau \in D} f_\tau \).

We have

\[
g = \left( \sum_{y \in G} \hat{f}(y) \chi_y \right) + \sum_{\tau \in D} \sum_{y \in G} \hat{f}_\tau(y) \chi_y = \sum_{y \in G} \hat{f}(y) \left( \sum_{\tau \in D \cup \{0\}} \chi_y(\tau) \right) \chi_y. \tag{2}
\]

Since \( f \) is the characteristic function of a perfect dominating set, we have \( g(x) = 1 \), for every \( x \in G \). So \( g = \chi_\vec{0} \). By uniqueness of Fourier expansion, for every \( y \neq \vec{0} \),

\[
0 = \hat{g}(y) = \hat{f}(y) \sum_{\tau \in D \cup \{0\}} \chi_y(\tau) = \hat{f}(y) \left( 1 + \sum_{i=1}^{n} \omega^{y_i} + \sum_{i=1}^{n} \omega^{-y_i} \right). \tag{3}
\]

Now we turn to the key step of the proof. Since \( 2n + 1 \) is a prime, \( \hat{f}(y) \neq 0 \), we have

\[
\{y_1, \ldots, y_n\} \cup \{-y_1, \ldots, -y_n\} = \{1, \ldots, 2n\}. \tag{4}
\]

Denote the set of all \( y \) satisfying \( \{4\} \) by \( \mathcal{Y} \). For \( 1 \leq i \leq n \), let

\[
D_i = \{ke_i: 0 \leq k \leq 2n\}.
\]

Define \( g_i = \sum_{\tau \in D_i} f_\tau \). Similar to \( \{2\} \), we get

\[
g_i = \sum_{y \in \mathcal{Y}} \hat{f}(y) \left( \sum_{\tau \in D_i} \chi_y(\tau) \right) \chi_y.
\]
When \( y \in Y \), since \( y_i \neq 0 \), we have
\[
\sum_{\tau \in D_i} \chi_y(\tau) = \sum_{k=0}^{2n} \omega^{ky_i} = 0.
\]
When \( y \notin Y \) and \( y \neq \vec{0} \), \( \hat{f}(y) = 0 \). So
\[
g_i = \left( \hat{f}(0) \sum_{\tau \in D_i} \chi_{\vec{0}}(\tau) \right) \chi_{\vec{0}} = 1. \tag{5}
\]
Note that \( g_i(x) \) counts the number of elements in \( S \cap \{(y_1, \ldots, y_n) : y_j = x_j \ \forall j \neq i\} \).
This completes the proof.

**Remark 2** The above proof can be translated to the language of linear algebra (However in the linear algebra language the key observation (4) becomes less obvious). Indeed, let \( m = (2n + 1)^n \) denote the number of vertices. From Remark 1 we know that \( f : \mathbb{Z}_{2n+1}^n \rightarrow \mathbb{C} \) is the characteristic function of a perfect dominating set if and only if \((A + I)\vec{f} = \vec{1}\), where \( A \) is the adjacency matrix of \( \Gamma(\mathbb{Z}_{2n+1}^n, \mathcal{U}) \). The reader may notice that in the proof of Theorem 1, \( \vec{g} = (A + I)\vec{f} \), and thus (2) shows that \( \chi_y \) is a family of orthonormal eigenvectors of \( A + I \). Moreover, among these eigenvectors, the ones that correspond to the 0 eigenvalue are exactly \( \chi_y \) with \( y \in Y \). Hence the rank of \( A + I \) is \( m - |Y| = (2n + 1)^n - n!2^n \). We will use this fact in the proof of Theorem 2.

## 4 Proof of Theorem 2

As it is observed in Remark 1 every perfect dominating set of \( \Gamma(\mathbb{Z}_{2n+1}^n, \mathcal{U}) \) corresponds to a zero-one vector \( \vec{f} \in \mathbb{C}^m \) that satisfies \((A + I)\vec{f} = \vec{1}\). Let
\[
V := \text{span}\{\vec{f} : f \in \mathcal{F}\}.
\]
Trivially
\[
dim V \leq 1 + (m - \text{rank}(A + I)) = 1 + n!2^n.
\]
Also for a subset \( D \) of vertices of \( \Gamma(\mathbb{Z}_{2n+1}^n, \mathcal{U}) \), define
\[
V_D := \text{span}\{\vec{f} : f \in \mathcal{F} \text{ and } \forall x \in D, f(x) = 1\},
\]
Note that \( V = V_\emptyset \).

To prove Theorem 2 we start from \( D = \emptyset \). At every step, if \( D \) does not extend uniquely to \( S \), then there exists a vertex \( v \in S \) such that \( \dim V_{D \cup \{v\}} < \dim V_D \); we add \( v \) to \( D \). Since \( \dim V_\emptyset \leq 1 + n!2^n \), we can obtain a set \( D \) of size at most \( n!2^n \) such that the dimension of \( V_D \) is at most 1. This completes the proof as there is at most one non-zero, zero-one vector in a vector space of dimension 1.
5 Future directions

We ask the following question:

**Question 1** For a prime $2n + 1$, are there examples of perfect dominating sets in $\Gamma(Z_{2n+1}, U)$ that are not of the form (1)?

If the answer to Question 1 turns out to be negative, then we can improve the bound of Theorem 2:

**Proposition 1** Let $p = 2n + 1$ be a prime, and let $T$ denote the set of perfect dominating sets of the form (1). Every $(S, T)$ where $S \in T$ has a defining set of size $1 + \lceil \frac{n-1}{\log_2 p} \rceil$.

**Proof.** Suppose that $S \in T$. Then $S$ is of the form:

$$\{(x_1, \ldots, x_{n-1}, k + \sum_{i=1}^{n-1} \epsilon_i(i + 1)x_i) : x_i \in \mathbb{Z}_p \forall i \in [n-1]\}.$$  

Let $m = \lceil \log_2 p \rceil$. We will use the easy fact that for any $c \in \mathbb{Z}_p$, the equation $\sum_{i=0}^{m-1} \epsilon_i 2^i =_p c$ has at most one solution $(\epsilon_0, \epsilon_1, \ldots, \epsilon_{m-1}) \in \{-1, +1\}^m$. For $i, j \geq 0$, define $\alpha_{i,j} \in \mathbb{Z}_p$ to be the solution to $(i + j + 1)\alpha_{i,j} =_p 2^j$.

Let $u = (0, 0, \ldots, 0, b)$ be the unique vertex in $S$ with the first $n - 1$ coordinates equal to 0, and for every $1 \leq i \leq n - 1$ consider the unique vector $u_i = (0, \ldots, 0, \alpha_{i,0}, \alpha_{i,1}, \ldots, \alpha_{i,k_i}, 0, \ldots, 0, b_i) \in S$, where $\alpha_{i,0}$ is in the $i$th coordinate and $k_i = \min(m-1, n-i-1)$. We claim that the set $D = \{u, u_0, u_m, \ldots, u_m(\lceil \frac{n-1}{m} \rceil -1)\}$ is a defining set for $(S, T)$. Since $S$ is of form (1), clearly $k = b$, and for every $0 \leq i \leq \lceil \frac{n-1}{m} \rceil - 1$, we have:

$$b_{mi} - b = \sum_{j=0}^{k_{mi}} \epsilon_{mi+j}(mi + j + 1)\alpha_{mi,j} = \sum_{j=0}^{k_{mi}} \epsilon_{mi+j}2^j.$$  

The above equation has only one solution $(\epsilon_{mi}, \epsilon_{mi+1}, \ldots, \epsilon_{mi+k_{mi}}) \in \{-1, +1\}^{k_{mi}+1}$. Considering this for all $u_{mi} \in D$ determines $(\epsilon_1, \epsilon_2, \ldots, \epsilon_{n-1})$. Thus the set $D$ is a defining set for $(S, T)$.  

**Remark 3** For $n = 2, 3$ the answer to Question 1 is negative. Thus when $n = 3$, Proposition 1 implies that there is a defining set of size 2 for a perfect dominating set. This disproves the conjecture of [2] which is already observed by Richard Bean [private communication].
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