Weak field equations and generalized FRW cosmology on the tangent Lorentz bundle

A Triantafyllopoulos\(^1\) and P C Stavrinos\(^2\)

\(^1\) Department of Physics, Section of Astrophysics, Astronomy and Mechanics, National and Kapodistrian University of Athens, Panepistimiopolis 15783, Athens, Greece
\(^2\) Department of Mathematics, National and Kapodistrian University of Athens, Panepistimiopolis 15784, Athens, Greece

E-mail: alktrian@phys.uoa.gr and pstavrin@math.uoa.gr

Received 6 November 2017, revised 26 February 2018
Accepted for publication 27 February 2018
Published 16 March 2018

Abstract
We study field equations for a weak anisotropic model on the tangent Lorentz bundle \(TM\) of a spacetime manifold. A geometrical extension of general relativity (GR) is considered by introducing the concept of local anisotropy, i.e. a direct dependence of geometrical quantities on observer 4−velocity. In this approach, we consider a metric on \(TM\) as the sum of an h-Riemannian metric structure and a weak anisotropic perturbation, field equations with extra terms are obtained for this model. As well, extended Raychaudhuri equations are studied in the framework of Finsler-like extensions. Canonical momentum and mass-shell equation are also generalized in relation to their GR counterparts. Quantization of the mass-shell equation leads to a generalization of the Klein−Gordon equation and dispersion relation for a scalar field. In this model the accelerated expansion of the universe can be attributed to the geometry itself. A cosmological bounce is modeled with the introduction of an anisotropic scalar field. Also, the electromagnetic field equations are directly incorporated in this framework.

Keywords: tangent Lorentz bundle, weak gravitational field, mass-shell, Raychaudhuri equation, Finsler-like gravitational equations, modified dispersion relation, cosmological bounce

1. Introduction
During the last decade there has been a considerable interest in the study of applications of Finsler geometry in different topics of Physics, such as in modified gravity theories, modern cosmology, quantum gravity etc [1–32]. It has been proposed that Finsler gravity can be used towards studying the physical phenomena in the universe.
The development of research for the evolution of the universe can be combined with a locally anisotropic structure of the Finslerian gravitational field. Finsler-gravity models allow intrinsically local anisotropies including vector variables $y_i = \frac{dx_i}{d\tau}$, $(i = 0, 1, 2, 3)$ in the framework of a tangent (Lorentz) bundle [19, 20, 25, 33]. Those approaches were elaborated as unified descriptions and modifications/generalizations of Einstein gravity theory. The $y-$dependence essentially characterizes the Finslerian gravitational field and has been combined with the concept of anisotropy and the broken Lorentz symmetry which causes the deviation from Riemannian geometry, since the latter can not explain all the gravitational effects in the universe. Therefore, the consideration of Finsler geometry as a candidate for studying gravitational theories provides that matter dynamics take place [30, 31].

In the theory of Finslerian gravitational field a peculiar velocity field is produced by the gravity of mass fluctuations which are due to anisotropic distribution and motion of particles. This can be physically described in the framework of Finsler-like geometrical structure of spacetime. Einstein Finsler-like gravity theories are considered as natural candidates for investigation of local anisotropies and the dark energy problem [1, 5, 7, 21, 34–38]. Also, extended modified gravity theories in the framework of tangent Lorentz bundles allow generalizations of the $f(R, T, \ldots)$ ones [20, 25, 39].

Finsler geometry gives a metric extension of the background metric of space-time in higher-order dimensions. Finsler gravity and cosmology models were developed in [2, 4, 5, 7, 9, 10, 15, 19, 40] extending geometrical and physical ideas and were related to quantum gravity and modified dispersion relations, broken Lorentz symmetry, nonlinear symmetries and gravitational waves [7, 13, 27, 37, 41]. Causality on a tangent bundle is induced by Lorentzian structure of the base spacetime manifold [24, 42, 43].

Spacetimes which are described by Finsler geometry allow deviation from Lorentz invariance symmetries [7, 44]. A theory which naturally describes Lorentz violation phenomena in quantum gravity while preserving Einstein’s general relativity in the background level is the standard model extension theory (SME) [14, 26, 45–48]. This theory is related with experimental investigations and observational efforts in astrophysics, cosmology and high energy physics [46, 47, 49–51]. In this framework Finsler structures for $b-$spaces were developed, giving a remarkable geometrization in the study of elementary particle theories [9, 14, 26, 46, 47]. Additionally, the ticking rate of clocks which is crucial to the magnitude redshift calculations is determined from the background metric geometry of spacetime. In a direction-dependent space-time the ticking rate depends on the direction [9, 14, 29].

Einstein–Finsler theories of gravity play an important role in the resolution of cosmological problems and generalize cosmological models. Based on such an approach we can get additional information for the gravity e.g. in connection with an electromagnetic field, inflaton, scalar field or spinor field. Particularly, the dynamics of Finsler geometry (velocity space) contributes decisively in the stability and acceleration of the universe. It is possible that this consideration can be used to analyze the implications of quantum gravity and related Lorentz violations in the early universe and in present day cosmology.

In the framework of Finsler extensions of general relativity, Raychaudhuri equations and energy conditions have also been studied [13, 52–56].

Finsler geometry includes torsions, more than one covariant derivatives and anisotropic curvatures extending the framework of field equations of general relativity and cosmology. A unified description of the Finslerian gravitation of a spacetime manifold is given by a metric function $F$, a total metric $\mathcal{G}$ on the tangent bundle of $M$, a metrical compatible connection and a nonlinear connection $N$ [57–59].
This paper is organized as follows. In the second section we present the basic geometrical structures on the tangent bundle $TM$ of a manifold $M$.

In the third section a specific type of distinguished $(d-)$metric is introduced on $TM$, which consists of a background h-Riemannian perturbed by a locally anisotropic weak field. Based on a metrical compatible $d-$connection and the corresponding field equations on the tangent Lorentz bundle, we present the field equations for our model. The extra terms of the derived field equations are connected with the anisotropic sector of geometric structure of space-time and give an interpretation for possible anisotropies of the universe.

In the fourth section we present a generalization of the definition of canonical momentum for our weak field model. Consequently, mass shell relation is also extended, and a generalization of the Klein–Gordon equation for a massive scalar field is derived. Additionally, a modified dispersion relation for the scalar field is calculated.

In the fifth section an extended FRW cosmological model on the tangent bundle is presented in which Raychaudhuri equations, energy conditions and cosmological bounce are studied. As well, an anisotropic scalar field is introduced on $TM$ and its dynamics is described in a specific case of the weak field model.

In the sixth section, Maxwell equations are generalized on the framework of the tangent Lorentz bundle, particularly for the weak field model.

Finally, in the concluding remarks, we summarize and discuss our results.

2. Preliminaries

In this section we present in brief the basic concepts of geometry on the tangent bundle of a background manifold, for more details see [57–61].

We consider a 4D spacetime manifold $M$ and its 8D tangent bundle $(TM, \pi, M)$ or for short $TM$, which around a point $p \in TM$ is equipped with local coordinates $\{X^A\} = \{x^i, y^a\}$ where $x^i$ are the local coordinates on the base manifold $M$ around $\pi(p)$ and $y^a$ are the coordinates on the fiber. The range of values for the indices is $i, j, \ldots, s = 0, \ldots, 3$ and $a, b, \ldots, f = 0, \ldots, 3$. On the tangent space $T_pTM$ an adapted basis $\{\delta_i, \dot{\partial}_a\}$ is defined, where

\begin{equation}
    x^i = x^i(x^0, \ldots, x^3)
\end{equation}

and

\begin{equation}
    y^a = \frac{\partial x^i}{\partial x^\sigma} y^\sigma
\end{equation}

where $x^i = \delta^i_a x^a$, $\delta^i_i$ is the Kronecker delta, and

\begin{equation}
    \det \left| \frac{\partial x^i}{\partial x^\sigma} \right| \neq 0.
\end{equation}

A nonlinear connection $N$ with coefficients $N^\mu_\nu(x, y)$ is defined a priori on $TM$ [59–61]. Under a local coordinate transformation, coefficients $N^\mu_\nu$ obey the following transformation rule:

\begin{equation}
    N^\mu_\nu(x, y) = \frac{\partial x^\mu}{\partial x'^\rho} \frac{\partial x^\rho}{\partial x^\nu} N^\rho_\sigma(x, y) + \frac{\partial x^\mu}{\partial x'^\rho} \frac{\partial^2 x^\rho}{\partial x'^\sigma \partial x^\nu} y^\sigma.
\end{equation}
On the tangent space $T_p TM$ an adapted to local coordinates basis or $\{\delta_i, \dot{\partial}_a\}$ is defined by the relation

$$\delta_i = \frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N_i^a \frac{\partial}{\partial y^a}$$

and

$$\dot{\partial}_a = \frac{\partial}{\partial y^a}.$$  \hfill (5)

For simplicity the adapted to local coordinates basis will hereafter be called adapted basis. The horizontal distribution or h-space of $T_p TM$ is spanned by $\delta_i$ while the vertical distribution or v-space of $T_p TM$ is spanned by $\dot{\partial}_a$. Under a local coordinate transformation, adapted basis vectors transform as:

$$\delta_i' = \frac{\partial x^i}{\partial x'^i} \delta_i, \quad \dot{\partial}_a' = \frac{\partial x^a}{\partial y^a} \dot{\partial}_a.$$  \hfill (7)

The adapted to local coordinates dual basis of the adjoint tangent space $T^*_p TM$ is $\{dx^i, \delta_y^a\}$ with the definition

$$\delta_y^a = dy^a + N_j^a dx^j.$$  \hfill (8)

For simplicity the adapted to local coordinates dual basis will hereafter be called dual adapted basis. The transformation rule for $\{dx^i, \delta_y^a\}$ is:

$$dx'^i = \frac{\partial x^i}{\partial x'^i} dx^i, \quad \delta_y'^a = \frac{\partial x^a}{\partial y^a} \delta_y^a.$$  \hfill (9)

Tensor algebra can be performed in the adapted basis in the usual way.

The bundle $TM$ is equipped with a distinguished metric ($d-$metric) $G:

$$G = g_{ij}(x, y) dx^i \otimes dx^j + h_{ab}(x, y) \delta_y^a \otimes \delta_y^b$$

where the h-metric $g_{ij}$ and v-metric $h_{ab}$ are defined to be of Lorentzian signature $(-, +, +, +)$. A tangent bundle equipped with such a metric will be called a tangent Lorentz bundle. Proper time $\tau$ is defined to be measured by the norm

$$d\tau = \sqrt{-g_{ij}(x, \dot{x}) dx^i dx^j}.$$  \hfill (11)

The distinguished connection ($d-$connection) $D$ is defined as a covariant differentiation rule that preserves h-space and v-space:

$$D_{\delta_i} \delta_j = L^a_{ij}(x,y) \delta_j, \quad D_{\dot{\partial}} \delta_j = C^i_{ja}(x,y) \delta_i,$$

$$D_{\delta_i} \dot{\partial}_b = L^a_{ib}(x,y) \dot{\partial}_b, \quad D_{\dot{\partial}} \dot{\partial}_b = C^a_{ba}(x,y) \dot{\partial}_a.$$  \hfill (12) \hfill (13)

From these the definitions for partial covariant differentiation follow as usual, e.g. for $X \in T_p TM$ we have the definitions for covariant h-derivative

$$X^A |_j \equiv \langle^{(h)} \rangle D_j X^A \equiv \delta_j X^A + L^A_{ij} X^B$$

and covariant v-derivative

$$X^A |_b \equiv \langle^{(v)} \rangle D_b X^A \equiv \partial_b X^A + C^A_{ba} X^B.$$  \hfill (14) \hfill (15)
The curvature of the nonlinear connection is defined by
\[ \Omega^a_{jk} = \frac{\delta N^a_i}{\delta x^j} - \frac{\delta N^a_i}{\delta x^j}. \] (16)

The components of the torsion tensor of the \( d \)-connection that we need are
\[ T^i_{jk} = L^i_{jk} - L^i_{kj} \quad T^a_{jk} = \Omega^a_{jk} \quad T^a_{bc} = C^a_{bc} - C^a_{cb}. \] (17)

The \( h \)-curvature tensor of the \( d \)-connection in the adapted basis and the corresponding \( h \)-Ricci tensor have respectively the components
\[ R^i_{jl} = \delta_j L^i_{lk} - \delta_k L^i_{lj} + L^i_{lk} L^l_{jk} - L^i_{lj} L^l_{kj} + C^i_{jk} \Omega^j_{kl} \] (18)
\[ R^i_{jl} = \delta_j L^i_{lk} - \delta_k L^i_{lj} + L^i_{lk} L^l_{jk} - L^i_{lj} L^l_{kj} + C^i_{jk} \Omega^j_{kl} \] (19)

while the \( v \)-curvature tensor of the \( d \)-connection in the adapted basis and the corresponding \( v \)-Ricci tensor have respectively the components
\[ S^a_{bcd} = \dot{\partial}_b C^a_{cd} - \dot{\partial}_c C^a_{bd} + C^a_{bc} C^a_{fd} - C^a_{bd} C^a_{fc} \] (20)
\[ S^a_{ab} = S^a_{abc} = \dot{\partial}_b C^a_{ac} - \dot{\partial}_a C^a_{bc} + C^a_{ab} C^a_{ec} - C^a_{ac} C^a_{eb} \] (21)

The generalized Ricci scalar curvature in the adapted basis is defined as
\[ R = g^{ij} R_{ij} + h^{ab} S_{ab} = R + S \] (22)
where
\[ R = g^{ij} R_{ij} \quad S = h^{ab} S_{ab}. \] (23)

A \( d \)-connection can be uniquely defined given that the following conditions are satisfied [57]:

- The \( d \)-connection is metric compatible
- Coefficients \( L^i_{jk}, L^a_{bk}, C^i_{jc}, C^a_{bc} \) depend solely on the quantities \( g_{ij}, h_{ab} \) and \( N^a_i \)
- Coefficients \( L^i_{jk} \) and \( C^a_{bc} \) are torsion free, i.e. \( T^i_{jk} = T^a_{bc} = 0 \).

We use the symbol \( D \) instead of \( D \) for a connection satisfying the above conditions, and call it a canonical and distinguished \( d \)-connection. Metric compatibility translates into the conditions:
\[ D^k g_{ij} = 0, \quad (h) \quad D^k h_{ab} = 0, \quad (v) \quad D^c g_{ij} = 0, \quad (h) \quad D^c h_{ab} = 0. \] (24)

The coefficients of canonical and distinguished \( d \)-connection can be found in [57]:
\[ L^i_{jk} = \frac{1}{2} g^{ih} (\delta_h g_{kj} + \delta_j g_{hk} - \delta_k g_{hj}) \] (25)
\[ L^a_{bk} = \dot{\partial}_b N^a_k + \frac{1}{2} h^{ac} (\delta_k h_{bc} - h_{dk} \dot{\partial}_b N^d_k - h_{bd} \dot{\partial}_c N^d_k) \] (26)
\[ C^a_{jc} = \frac{1}{2} g^{ih} \dot{\partial}_c g_{hi} \] (27)
\[ C^i_{bc} = \frac{1}{2} h^a_{bd} \left( \partial_i h_{db} + \partial_b h_{dc} - \partial_d h_{bc} \right). \]  

A geodesic curve on \( TM \) is defined by the equation

\[ \frac{d y^a}{d \tau} + 2 G^a(x, y) = 0, \quad y^i = \frac{dx^i}{d \tau} \]  

where

\[ G^a(x, y) \equiv \frac{1}{4} \tilde{g}^{ab} \left( \frac{\partial^2 \mathcal{K}}{\partial y^b \partial x^c} y^c - \frac{\partial \mathcal{K}}{\partial x^a} \right). \]

In the above relation we defined \( \mathcal{K} \equiv g_{ab}(x, y) y^a y^b, \ g_{ab} \equiv \delta^{[a}_{\kappa} \delta^{b]}_{\lambda} g_{\kappa \lambda} \) and \( \tilde{g}_{ab} \equiv \frac{1}{2} \frac{\partial^2 \mathcal{K}}{\partial y^a \partial y^b} \). An h-vector \( \xi^i = \frac{dx^i}{d \lambda} \) represents the horizontal part of a tangent vector on \( TM \). It can be timelike, null or spacelike:

- timelike: \( g_{ij}(x, \xi) \xi^i \xi^j < 0 \)
- null: \( g_{ij}(x, \xi) \xi^i \xi^j = 0 \)
- spacelike: \( g_{ij}(x, \xi) \xi^i \xi^j > 0 \).

The curve \( x^i(\lambda) \) is timelike, null or spacelike for some value \( \lambda_0 \) of the parameter \( \lambda \) if the tangent vector \( \xi(\lambda_0) \) has the corresponding property. We see that the definition of proper time rel. (11) only makes sense for a timelike segment of a curve. Massive point particles in spacetime subject only to gravity are described by timelike geodesics, while massless ones are described by null geodesics.

The symmetric part of a \((0, 2)\) h-tensor \( A_{ij} \) is defined as

\[ A_{(ij)} = \frac{1}{2} (A_{ij} + A_{ji}) \]

and the antisymmetric part of \( A_{ij} \) is defined as

\[ A_{[ij]} = \frac{1}{2} (A_{ij} - A_{ji}) . \]

2.1. The pseudo-Finsler metric

We define a function \( F(x, y) : TM \rightarrow \mathbb{R} \) for which the following properties hold:

1. \( F \) is continuous on \( TM \) and smooth on \( \tilde{TM} \equiv TM \setminus \{0\} \) i.e. the tangent bundle minus the null structure \( \{(x, y) \in TM | F(x, y) = 0\} \).
2. \( F \) is positively homogeneous of first degree on its second argument:

\[ F(x', y^a) = k F(x', y^a), \quad k > 0. \]

3. The form

\[ f_{ab}(x, y) = \frac{1}{2} \frac{\partial^2 F^2}{\partial y^a \partial y^b} \]

defines a non-degenerate matrix:

\[ \det [f_{ab}] \neq 0. \]
A metric $f_{ab}(x, y)$ given by rel. (34) is called a pseudo-Finsler metric. From properties 2 and 3 it becomes obvious that $f_{ab}(x, y)$ is positively homogeneous of zero degree on its second argument.

Function $F(x, y)$ will generally not be smooth at the null structure of the tangent bundle $TM$, as is evident from condition 1. Thus, relations (34) and (35) are defined only in the time-like or only in the spacelike domain of $F(x, y)$. For this reason, instead of using $F(x, y)$ to define distances on the base manifold $M$, we use a pseudo-metric tensor $g_{ij}(x, y)$ homogeneous of degree zero on $y$ which is defined everywhere on $TM$. This tensor is the horizontal part of the metric on $TM$, rel. (10). Metric tensor $g_{ij}(x, y)$ can always be derived from a function $F(x, y)$ when restricted on the appropriate domain, specifically for $F(x, y) = \sqrt{|g_{ij}(x, y)y^iy^j|}$ in a timelike domain of $TM$ we get

$$g_{ij}(x, y) = -\delta^a_i \delta^b_j f_{ab}(x, y) \quad (36)$$

while for the same $F(x, y)$ in a spacelike domain of $TM$ we get

$$g_{ij}(x, y) = \delta^a_i \delta^b_j f_{ab}(x, y). \quad (37)$$

From rel. (11) we see that the proper time

$$\tau = \int_{\lambda_1}^{\lambda_2} \left( -g_{ij}(x, x') \frac{dx^i}{d\lambda} \frac{dx^j}{d\lambda} \right)^{1/2} d\lambda \quad (38)$$

where $x' = dx/d\lambda$, is independent from the choice of parametrization of the path due to homogeneity of zero degree of the h-metric on $x'$.

Geodesics equation rel. (29) in the case of a pseudo-Finsler metric takes the form

$$\frac{dy^a}{d\tau} + \gamma^a_{ij} y^i y^j = 0, \quad y^i = \frac{dx^i}{d\tau} \quad (39)$$

where the Christoffel symbols of the second kind for the h-metric are

$$\gamma^k_{ij} = \frac{1}{2} g^{kl} \left( \partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij} \right) \quad (40)$$

and $\gamma^{ka}_j = \delta^a_k \gamma^k_{ij}$.

### 3. Field equations for a weakly anisotropic model

In this section, we study weak field equations in the framework of a tangent Lorentz bundle. Previous approaches concerning a weak field limit on the tangent bundle have been studied in [62, 63].

We consider a tangent bundle $TM$ equipped with a $d-$metric

$$\mathcal{G}(x, y) = g_{ij}(x, y) dx^i \otimes dx^j + h_{ab}(x, y) \delta y^a \otimes \delta y^b \quad (41)$$

where the h-metric and v-metric can be decomposed as:

$$g_{ij}(x, y) = (h)g_{ij}(x) + \tilde{g}_{ij}(x, y) \quad (42)$$

and

$$h_{ab}(x, y) = \eta_{ab} + \tilde{h}_{ab}(x, y) \quad (43)$$
where \( (b) g_{ij}(x) \) is the background h-space metric, \( \eta_{ab} \) is the background v-space Minkowski metric, \( \tilde{g}_{ij}(x,y) \), \( \tilde{h}_{ab}(x,y) \) are weak tensorial anisotropic (y-dependent) fields with \( \det[\tilde{g}] \ll 1 \) and \( \det[\tilde{h}] \ll 1 \), \( \det[\tilde{g}], \det[\tilde{h}] \) are the determinants of \( \tilde{g}_{ij} \) and \( \tilde{h}_{ab} \) respectively. The signature convention \((- , +,+,+)\) is assumed for the individual h-space and v-space background metrics. With these definitions, rel. (41) is defined to be the sum of a background metric \( (b) G(x) \) and a perturbation \( \tilde{G}(x,y) \), where

\[
(b) G(x) = (b) g_{ij}(x) \delta x^i \otimes \delta x^j + \eta_{ab} \delta y^a \otimes \delta y^b
\]

and

\[
\tilde{G}(x,y) = \tilde{g}_{ij}(x,y) \delta x^i \otimes \delta x^j + \tilde{h}_{ab}(x,y) \delta y^a \otimes \delta y^b.
\]

The specific choice of metric on \( TM \) allows us to readily generalize Einstein’s general relativity, resulting from rel. (42) being the sum of a pseudo-Riemannian space metric (in the sense that it only depends on the position \( x \) on the base manifold \( M \)) and a weak anisotropic field. On the other hand, the v-metric of the v-space has no corresponding form in general theory of relativity. In that case, we consider the sum of a flat background and a weak anisotropic field rel. (43).

The background Christoffel symbols are

\[
(b) \gamma^r_{jk}(x) = (b) g^{ir}(x) (b) \gamma_{rjk}(x) = \frac{1}{2} (b) g^{ir}(x) [\partial_k (b) g_{jr}(x) + \partial_j (b) g_{kr}(x) - \partial_r (b) g_{jk}(x)]
\]

and the corresponding Riemann curvature tensor, Ricci tensor and Ricci scalar curvature are

\[
(b) R^r_{ijk} = \partial_j (b) \gamma^r_{ik} - \partial_k (b) \gamma^r_{ij} + (b) \gamma^m_{ik} (b) \gamma^r_{mj} - (b) \gamma^m_{ij} (b) \gamma^r_{mk} \]

\[
(b) R_{ij} = (b) R^k_{ijk} = \partial_k (b) \gamma^k_{ij} - \partial_j (b) \gamma^k_{ik} + (b) \gamma^m_{ij} (b) \gamma^k_{mk} - (b) \gamma^m_{ik} (b) \gamma^k_{mj} \]

\[
(b) R = (b) g^{ij}(x) (b) R_{ij}.
\]

From relations (42) and (43) the inverse h-metric and v-metric to first order with respect to \( \tilde{g}_{ij} \) and \( \tilde{h}_{ab} \) immediately follow:

\[
g^{ij}(x,y) = (b) g^{ij}(x) - \tilde{g}^{ij}(x,y)
\]

and

\[
h^{ab}(x,y) = \eta^{ab} - \tilde{h}^{ab}(x,y)
\]

where

\[
\tilde{g}^{ij} = (b) g^{ik} (b) g^{j}_{kl} \tilde{g}_{kl}
\]

and

\[
\tilde{h}^{ab} = \eta^{ac} \gamma^{bd} \tilde{h}_{cd}.
\]

3.1. Proper time in the weak field model

By using Taylor expansion in rel. (11), proper time can be written in the weak-field as

\[
d\tau = \sqrt{(b) g_{ij}(x) \delta x^i \delta x^j - \frac{1}{2} \left( (b) g_{ij}(x) \delta x^i \delta x^j \right)^{-1/2} \tilde{g}_{ij}(x,y) \delta x^i \delta x^j}
\]
where we kept terms up to first order with respect to \( \tilde{g}_{ij}(x, y) \). The extra term on the rhs of (54) can cause a change in the ticking rate of a clock depending on its orientation. Relation (54) is generally not an invariant quantity under the Lorentz symmetry group, as a Lorentz boost will not just transform the local frame of reference, but also change the position on the fiber, on which \( \tilde{g}_{ij}(x, y) \) depends. This is an example of Lorentz violation due to anisotropy.

3.2. Connection coefficients and curvature of the model

We use in the following a canonical and distinguished \( d- \) connection \( \mathcal{D} \), rel. (25)–(28). We additionally consider the connection to be Cartan-type \[57, 58\], so we have

\[
N^a_i = L^a_{(b)} y^b
\]  
(55)

\[
C^a_{bc} y^b = 0.
\]  
(56)

In the following we calculate the connection coefficients for the weakly anisotropic metric defined in relations (41)–(43).

We define a weak nonlinear connection \( \tilde{N}^a_i \) on \( TM \) homogeneous of degree 1 on \( y \) in accordance with \[60\], and demand that it is of the same order with \( \tilde{g}_{ij} \) and \( \tilde{h}_{ab} \). By using Euler’s theorem on homogeneous functions we get:

\[
y^b \tilde{\partial}_b \tilde{N}^a_i = \tilde{N}^a_i.
\]  
(57)

From relations (55) and (26) we get to first order with respect to \( \tilde{h}_{ab} \):

\[
\tilde{N}^a_i = y^b \tilde{\partial}_b \tilde{h}^a_i - y_a \eta^{bc} \tilde{\partial}_b \tilde{N}^b_i.
\]  

Contracting with \( y_a \) gives:

\[
y_b \left( \tilde{N}^a_i - \frac{1}{2} y_a \tilde{\partial}_b \tilde{h}_{ab} \right) = 0
\]

and since this holds for arbitrary \( y \) we deduce:

\[
\tilde{N}^a_i = \frac{1}{2} y_a \tilde{\partial}_b \tilde{h}_{ab}.
\]  
(58)

Since \( \tilde{N}^a_i(x, y) \) is homogeneous of degree 1 on \( y \), from rel. (58) it is deduced that \( \tilde{h}_{ab}(x, y) \) is homogeneous of degree zero and by extension \( h_{ab}(x, y) = \eta_{ab} + \tilde{h}_{ab}(x, y) \) is also homogeneous of degree zero.

Now we calculate to first order with respect to \( \tilde{h}_{ab} \) the curvature coefficients of the nonlinear connection from relations (16) and (58) and we find:

\[
\Omega^a_{jk} = 0.
\]  
(59)

The connection coefficients defined in rel. (25)–(28) give to first order with respect to \( \tilde{g}_{ij} \) and \( \tilde{h}_{ab} \):

\[
L^j_{ik} = \gamma^j_{ik} = (b) \gamma^j_{ik} - \tilde{g}^{ij} \left( b \right) \gamma_{rjk} + \frac{1}{2} (b) \tilde{g}^{ij} \left( \partial_r \tilde{g}_{kr} + \partial_k \tilde{g}_{rj} - \partial_r \tilde{g}_{jk} \right)
\]  
(60)

\[
L^a_{bi} = \frac{1}{2} \tilde{\partial}_b \tilde{h}^a_i
\]  
(61)
The h-Ricci curvature coefficients and the h-Ricci scalar are given in appendix B. From relations (21) and (63) we get to first order with respect to \( \tilde{g}_{ij} \) and \( \tilde{h}_{ab} \) the v-components of the Ricci tensor:

\[
S_{ab} = \frac{1}{2} \left( \partial^c \partial_a \tilde{h}_{bc} + \partial^c \partial_b \tilde{h}_{ac} - \left( \frac{\partial}{\nu} \right) \tilde{h}_{ab} - \partial_a \partial_b \tilde{h} \right)
\]

(64)

where \( \tilde{h} \equiv \tilde{h}_c^c \) is the trace of the perturbation on the v-metric and \( \left( \frac{\partial}{\nu} \right) \equiv \partial^c \partial_c \) is the d’Alembertian of the background v-space. From relations (23) and (64) we get the v-space scalar curvature to first order with respect to \( \tilde{h}_{ab} \):

\[
\mathcal{S} = \partial_a \partial_b \tilde{h}^{ab} - \left( \frac{\partial}{\nu} \right) \tilde{h}.
\]

(65)

From this relation we immediately get the condition for the vanishing of the v-space curvature \( S \) to first order with respect to \( \tilde{h}_{ab} \) as

\[
\partial_a \partial_b \tilde{h}^{ab} = \left( \frac{\partial}{\nu} \right) \tilde{h}.
\]

(66)

### 3.3. Field equations

Field equations on the tangent bundle for a distinguished connection with coefficients given in relations (25)–(28) are derived in appendix A using the variational principle, resulting in relations (A.23) and (A.24), which in our case are written as:

\[
R_{(ij)} - \frac{1}{2} (R + S)g_{ij} = \kappa \frac{1}{T_{ij}}
\]

(67)

\[
S_{ab} - \frac{1}{2} (R + S)h_{ab} = \kappa \frac{2}{T_{ab}}
\]

(68)

where

\[
T_{ij}^1 = - \frac{2}{\sqrt{\det[G]}} \frac{\delta \left( \sqrt{\det[G]} L_M \right)}{\delta g^{ij}}
\]

(69)

\[
T_{ab}^2 = - \frac{2}{\sqrt{\det[G]}} \frac{\delta \left( \sqrt{\det[G]} L_M \right)}{\delta h^{ab}}
\]

(70)

are the coefficients of the generalized energy-momentum tensor on \( TM \), \( \det[G] \) is the metric tensor’s determinant and \( L_M \) denotes the Lagrangian density of the matter fields.

We will make some comments in order to give some physical interpretation to the energy momentum v-tensor, rel. (70), which is an object with no equivalent in Riemannian gravity. Lorentz violations produce anisotropies in the space and the matter sector [7, 63, 64]. These act as a source of anisotropy and can contribute to the energy-momentum tensors of the horizontal and vertical space \( T_{ij} \) and \( T_{ab} \). Energy-momentum tensor \( T_{ab} \) contains more information
of anisotropy which is produced from the metric $h_{ab}$ including additional internal degrees of freedom.

From relations (67), (B.1), (B.2) and (65) we get the h-space field equations as:

$$ (b)_{ij} R_{ij} - \frac{1}{2} (b)_{ij} g_{ij} R + A_{ij} = \kappa T_{ij}. $$

(71)

From relations (68), (64), (B.2) and (65) we get the v-space field equations as:

$$ -\frac{1}{2} \eta_{ab} (b)_{ij} R + B_{ab} = \kappa T_{ab} $$

(72)

where $A_{ij}$ and $B_{ab}$ are the terms of the h-space and v-space field equations which are linear in $\tilde{g}_{ij}$ and $\tilde{h}_{ab}$, see appendix C.

4. Mass shell and dispersion relation

In this section we study the dynamics of a massive point particle and we compare it with the procedure of [41].

4.1. Generalized mass shell equation

We consider a Lagrangian $L$ homogeneous of degree one on $y^j$ and the action

$$ S = \int L \, d\tau. $$

(73)

The Lagrangian $L$ is defined as

$$ L = -m \left( -g_{ij}(x, \dot{y}) y^i y^j \right)^{1/2}, \quad y^j = \frac{dx^j}{d\tau} $$

(74)

where $m$ is the rest mass of the point particle and $\tau$ is the proper-time, rel. (54). We limit ourselves to an h-metric that can be decomposed according to rel. (42) and substituting to rel. (74) we get to first order with respect to $\tilde{g}_{ij}$:

$$ L = -m \left( L_R - \frac{1}{2} L_R^{-1} \tilde{g}_{ij}(x, \dot{y}) y^i y^j \right) $$

(75)

where the Riemannian norm $L_R$ is defined as

$$ L_R = \left( (b)_{ij} g_{ij}(x, \dot{y}) y^i y^j \right)^{1/2}. $$

(76)

The canonical four-momentum is

$$ p_i = \frac{\partial L}{\partial \dot{y}^i}. $$

(77)

By using the Lagrangian rel. (75) and setting $c = 1$ we get

$$ p_i = m \left[ y_i + \frac{1}{2} \left( \dot{h}_{ij} \right) y^j y^j \right] $$

(78)

see appendix D for more details.
This relation can be used to calculate the generalized mass shell equation for our framework:

\[ p'p_i = m^2 \left[-1 + y' \left( \hat{\partial}_i \tilde{g}_{ij} \right) y^j \right]. \]  

(79)

Rel. (78) gives to zeroth order \( p_i = my_i \), so rel. (79) can be equivalently written to first order with respect \( \tilde{g}_{ij} \) as:

\[ p'p_i = -m^2 + \frac{p'_i}{m} \left( \hat{\partial}_i \tilde{g}_{ij} \right) p^j. \]  

(80)

4.2. Dispersion relation

We know from Riemannian geometry that we can choose a coordinate system so that locally we get \( \tilde{g}_{ij} = \eta_{ij} = \text{diag}(-1, 1, 1, 1) \) and the coefficients \( \frac{\partial^2}{\partial x^i \partial x^j} \) vanish. We call this a local inertial frame. We work on such a frame, and follow the usual method for quantization of physical quantities by replacing them with operators, so in position space we get for the position operator:

\[ x_i \rightarrow x_i = x_i \]  

and for the momentum operator:

\[ p_j \rightarrow p_j = -i\hbar \partial_j \Rightarrow p_i p_j = -\hbar^2 \Box = -\hbar^2 \eta^{ij} \partial_i \partial_j \]  

(81)

where we denoted quantum operators with boldface.

Perturbation \( \tilde{g}_{ij}(x,p) \) depends on both position and momentum, so upon quantization there is an ambiguity regarding the ordering of the operators, as \( x' \) and \( p_i \) do not commute. We will not attempt to treat this ambiguity in the present work, instead we will restrict ourselves to the case where the metric perturbation \( \tilde{g}_{ij} \) is only \( y \)-dependent. This way only the momentum operator appears upon quantization, so we do not need to worry about ordering.

This approach gives a generalized equation for a scalar (spin-0) boson field \( \phi(x) \):

\[ \Box \phi = \frac{m^2}{\hbar^2} \phi - \frac{i\hbar}{m} \phi^0 \left( \hat{\partial}_i \tilde{g}_{ij} \right) \partial_i \partial_j \phi \]  

(82)

where \( \phi^0 \) indicates a spatial index \( (\alpha, \beta, \ldots = 1, 2, 3) \). This is a generalization of the Klein–Gordon equation from the standard model of particle physics, and for \( \tilde{g}_{ij} = 0 \) we get \( \Box \phi = \frac{m^2}{\hbar^2} \phi \), which is the well-known Klein–Gordon equation. We see that the extra terms in (82) are due to spacetime anisotropy.

To calculate the dispersion relation for the particle, we set

\[ p_i = \hbar k_i = \hbar \left(-\omega_k \vec{k} \right) \]  

(83)

in accordance with the procedure of [41]. From rel. (83) and applying equation (80) we get the dispersion relation:

\[ \omega_k^2 = \left| \vec{k} \right|^2 + \frac{m^2}{\hbar^2} - \frac{\hbar}{m} \left[ \omega_k^2 \left( \omega_k \partial_0 + k_\alpha \partial^\alpha \right) \tilde{g}^{00} - 2\omega_k k_\beta \left( \omega_k \partial_0 + k_\alpha \partial^\alpha \right) \tilde{g}^{0\beta} + \omega_k k_\alpha \left( \omega_k \partial_0 + k_\alpha \partial^\alpha \right) \tilde{g}^{\beta\gamma} \right] \]  

(84)

where a greek letter indicates a spatial index \( (\alpha, \beta, \ldots = 1, 2, 3) \). This is a generalization of flat isotropic spacetime dispersion relation \( \omega_k^2 = \left| \vec{k} \right|^2 + m^2 / \hbar^2 \). We see that the extra terms on the rhs of (84) are due to spacetime anisotropy.
In general relativity, the quantity \( \frac{\partial p_0}{\partial p_\beta} \) is interpreted as the group velocity \( v_{gr} \) of the wave-form and is equal to \(-\frac{y_\beta}{y_0} \). We will study whether this equality holds in our weak field framework. We differentiate rel. (80) with respect to \( p_\beta \) and after straightforward calculations we get

\[
\frac{\partial p_0}{\partial p_\beta} = -\frac{p_\beta^2}{p_0} - \frac{p_\beta^2}{2(p_0)^2} \frac{\partial \tilde{g}_{ij}}{\partial p_0} p^i p^j + \frac{1}{2p_0^2} \frac{\partial \tilde{g}_{ij}}{\partial p_\beta} p^i p^j. \tag{85}
\]

By using relation (78) we get the following relations keeping terms up to first order:

\[
\frac{\partial \tilde{g}_{kl}}{\partial p_i} = \eta^{ijm} - \frac{1}{\dot{y}} \frac{\partial \tilde{g}_{kl}}{\partial y^j} \dot{y}^m. \tag{86}
\]

\[
\frac{p_\beta}{p_0} = \frac{y_\beta}{y_0} + \frac{1}{2y_0} \frac{\partial \tilde{g}_{ij}}{\partial y^j} \dot{y}^i \dot{y}^m \frac{\partial \tilde{g}_{kl}}{\partial y^m} \dot{y}^k. \tag{87}
\]

Putting together relations (85)–(87) we arrive at the equation:

\[
\frac{\partial p_0}{\partial p_\beta} = -\frac{y_\beta}{y_0}. \tag{88}
\]

In conclusion, using homogeneity condition for the Lagrangian (75) of our generalized space, we arrive at an extended dispersion relation (84) which satisfies the group velocity equation (88). We observe that the method we developed is consistent with the one presented in [41].

5. Generalized FRW cosmology of the model

Cosmological evolution of the universe is described by the well-known spatially flat FRW metric. The dynamics of this metric is determined by the Friedmann equations. In order that these equations agree with the accelerated expansion of the universe suggested by various observational data, one has to assume the existence of some exotic matter field usually called dark energy. Some studies in order to explain the accelerated expansion, using only geometrical structures, involve modified theories of gravity, e.g. Finsler–Randers cosmology [31, 38].

In this section, we make an effort to model this acceleration using the extra structure provided by the tangent bundle. We do that by introducing a metric structure, where the horizontal part is isotropic and the generalized dynamics of the metric comes from the vertical part of the metric. For this, we consider a tangent bundle equipped with the metric tensor

\[
G = -dt \otimes dt + a(t)^2 \delta_{\alpha \lambda} dx^\alpha \otimes dx^\lambda + h_{ab}(x,y) \frac{d\gamma^a}{\dot{y}^b} \otimes \frac{d\gamma^b}{\dot{y}^b}. \tag{89}
\]

The h-metric in rel. (89),

\[
g_{ij} = \text{diag} \left( -1, a^2(t), a^2(t), a^2(t) \right) \tag{90}
\]

is the spatially flat FRW metric that depends only on the position on the base manifold. We study the case of an holonomic basis, i.e. the curvature coefficients of the nonlinear connection defined in rel. (16) are set to zero. Connection structure on this space is defined by coefficients given in rel. (25)–(28).
5.1. Field equations

Connection coefficients from rel. (25) for the metric given in rel. (89) are reduced to

\[ L^i_{jk} = \frac{1}{2} g^{ih} \left( \partial_k g_{bj} + \partial_j g_{bk} - \partial_h g_{bj} \right). \]  

(91)

These are just the Christoffel symbols for the h-metric \( g_{ij} \). The Ricci curvature tensor components from rel. (19) read:

\[ R_{ij} = \partial_k L^k_{ij} - \partial_j L^k_{ik} + L^m_{ij} L^k_{mk} - L^m_{ik} L^k_{mj}. \]  

(92)

Due to the fact that \( L^i_{jk} \) is the Christoffel connection, those are identified as the components of the Ricci tensor used in GR (General theory of Relativity), with well-known components for our choice of metric.

A simple cosmological model occurs by considering the energy and momentum described by an ideal fluid

\[ T^{ij}_{\text{matter}} = (\rho + P) u_i u_j + P g_{ij} \]  

(93)

where \( \rho(x) \) is the spacetime-dependent (isotropic) energy density, \( P(x) \) is the spacetime-dependent (isotropic) pressure and \( u \) is the four-velocity field of the fluid. Field equations from rel. (67) are written as

\[ R_{ij} - \frac{1}{2} g_{ij} R - \frac{1}{2} g_{ij} S = \kappa T^{ij}_{\text{matter}} \]  

(94)

which reduce to ordinary differential equations for the scale factor \( a(t) \) in the usual manner

\[ \left( \frac{\dot{a}}{a} \right)^2 = \frac{\kappa}{3} \rho - \frac{1}{6} S \]  

(95)

\[ \frac{\ddot{a}}{a} = -\frac{\kappa}{6} (\rho + 3P) - \frac{1}{6} S \]  

(96)

where a dot denotes differentiation with respect to coordinate time. Relations (95) and (96) are the generalized Friedmann equations for this model. By considering these equations it follows that there is an equivalence with the equations of classical FRW model in their form (for \( S = -2\Lambda \)) but their dynamical evolution is different. In classical FRW model, homogeneous and isotropic constant \( \Lambda \) is added ad hoc, while in our case, \( S \)-curvature is a dynamical anisotropic cosmological parameter which emerges from the additional degrees of freedom of the geometrical structure and plays the role of an effective cosmological constant.

At the present stage of cosmological evolution, \( S \)-curvature’s value should be very small, so it could be described by a weak field framework. From rel. (65) we see that the dynamics of \( S \) is connected with the weak field \( \tilde{h}^{ab}(x, y) \) which is determined by the field equation (72).

The matter source \( T_{ab} \) must be chosen such that \( S \)-curvature’s present value agrees with current cosmological observations.

It is known that in general relativity, the gravitational field is described by the metric tensor \( g_{ij}(x) \). In our model, the gravitational field is described by two metric tensors \( g_{ij}(x) \) and \( h_{ab}(x, y) \) of which the dynamical evolution is connected. Consequently, the v-metric gives more degrees of freedom to the model and generalizes anisotropically the cosmological evolution (rel. (95) and (96)).
5.2. Raychaudhuri equation of the model

Given a four velocity field

\[ u'(x) = \frac{dx^i}{d\tau} \]  \hspace{1cm} (97)

and setting \( y^i(x) = \delta^i_au'(x) \), we get from rel. (29) the geodesics equation:

\[ \frac{du^i}{d\tau} + 2G^i(x, u) = 0 \]  \hspace{1cm} (98)

with \( \tau \) being the proper time parameter defined in rel. (11). The tangent vector to geodesics equation, rel. (98), is given by the semispray field [58, 59]:

\[ Y(x, u) = u^i \frac{\partial}{\partial x^i} - 2G^i(x, u) \frac{\partial}{\partial u_i}. \]  \hspace{1cm} (99)

By using rel. (5), equivalently on the adapted basis we get:

\[ Y(x, u) = u^i \delta^i_0 + (u^i N^a_i(x, u) - 2G^i(x, u)) \frac{\partial}{\partial u_a}. \]  \hspace{1cm} (100)

In the framework of Finslerian extensions, Raychaudhuri equations and energy conditions have been studied in [13, 52–54]. Raychaudhuri equation was developed on the tangent bundle of a spacetime and has been derived for a timelike congruence in [13]. Adapting a tangent field \( Y \) on a timelike geodesic congruence, the h-space Raychaudhuri equation gives:

\[ \dot{\theta} = -\frac{1}{3} \theta^2 - \sigma^2 + \omega^2 - R_{ij}u^iu^j + u^i T_{ij}^{(h)} \dot{D}^k_i Y^k + u^i \Omega_{ja}^{(v)} \frac{\partial}{\partial u^a}. \]  \hspace{1cm} (101)

where \( \theta \equiv \mathcal{D}^a_i Y^k \) is the expansion, \( \sigma_{ij} \equiv \mathcal{D}^a_i Y^j - \frac{1}{2} \theta g_{ij} \) is the shear and \( \omega_{ij} \equiv \mathcal{D}^a_i \dot{Y}^j \) is the vorticity of the congruence. We use the definitions \( \sigma^2 \equiv \sigma_{ij} \sigma^{ij} \), \( \omega^2 \equiv \omega_{ij} \omega^{ij} \) and the projection tensor is \( g_{ij} + u^i u^j \). A dot denotes differentiation with respect to \( \tau \).

For the case where \( T_{\mu}^\mu = 0 \) and because \( Y^k \) is a function of position \( x \) only, equation (101) gives:

\[ \dot{\theta} = -\frac{1}{3} \theta^2 - \sigma^2 - R_{ij}u^iu^j. \]  \hspace{1cm} (102)

Field equations given in rel. (68) can be manipulated to the form:

\[ R_{ij} = \kappa \left( T_{ij} - \frac{1}{2} g_{ij} T \right) = \frac{1}{2} g_{ij} S. \]  \hspace{1cm} (103)

On the other hand, strong energy conditions for matter are defined as:

\[ T_{\mu}^\nu u^\nu u^\mu \geq \frac{1}{2} g_{ij} u^i u^j T \]  \hspace{1cm} (104)

which applied to rel. (103) give

\[ R_{ij}u^iu^j \geq \frac{1}{2} S \]  \hspace{1cm} (105)

where we used the normalization condition \( u^i u^i = -1 \). From the last relation and because of \( \sigma^2 \geq 0 \), rel. (102) gives:

\[ \dot{\theta} \leq -\frac{1}{3} \theta^2 - \frac{1}{2} S. \]  \hspace{1cm} (106)
From relations (96) and (106) we notice that v-space scalar curvature $S$ can allow an increasing expansion $\theta$. The geometrical interpretation of $\theta = \frac{1}{2} Y^k \nabla_k Y$ describes the deviation of neighbouring geodesics in a congruence along the direction of $Y$. The proper time derivative of $\theta$ gives a measure of the relative acceleration of nearby test particles freely falling along the geodesics congruence. The inequality above sets an upper limit for the proper time derivative of $\theta$, which would necessarily be non-positive in the case of $S=0$ (as is the case of Riemannian Geometry) so nearby test particles would not accelerate relative to each other following just the geometry of spacetime. In our case, the upper limit also depends on $S$, so an acceleration of nearby test particles is possible.

5.3. Energy conditions and cosmological bounce

Ordinary matter fields, i.e. cold dark matter and radiation, obey certain energy conditions. In standard FRW cosmology, those matter fields are described as ideal fluids and are characterized by spatially homogeneous energy density $\rho$ and spatially homogeneous and isotropic pressure $P$ [65]. In this case, the weak, null and strong energy conditions, hereafter WEC, NEC and SEC respectively, are given as:

- **WEC:** $\rho \geq 0$, $\rho + P \geq 0$
- **NEC:** $\rho + P \geq 0$
- **SEC:** $\rho + P \geq 0$, $\rho + 3P \geq 0$

from which, for the field equations given in relations (95) and (96), we get

- **WEC for generalized FRW:** $(\dot{a}^2) \geq -\frac{1}{6} S$
- **NEC for generalized FRW:** $(\dot{a}^2) \geq \left(\frac{\kappa}{\rho}\right)$
- **SEC for generalized FRW:** $(\dot{a}^2) \geq \left(\frac{\kappa}{\rho}\right)$ and $(\ddot{a}^2) \leq -\frac{1}{6} S$.

A cosmological bounce on an FRW universe occurs when the conditions $\dot{a}(t_0) = 0$, $\ddot{a}(t_0) > 0$ are met for a coordinate time $t_0$. Studies of cosmological bounce in modified gravity have been made e.g. in [53]. Subtracting rel. (95) from rel. (96) and applying the bounce conditions gives

$$\rho + P < 0$$

provided that $a > 0$ and $\kappa > 0$. It is apparent that a cosmological bounce for this model requires the violation of all the aforementioned conditions.

5.4. Scalar field

Various scalar field models are used in cosmology in order to describe the accelerating expansion during the inflationary period of the universe. There have been studies of scalar field models in the framework of generalized geometry e.g. in [33, 66].

In this section, we consider an anisotropic scalar field $\phi(x, y)$ on the tangent Lorentz bundle with a Lagrangian density

$$L_\phi = -\frac{1}{2} D_A \phi D^A \phi - V(\phi)$$

where $D_A \phi = (h)_{\phi} (h)_{\phi} = (v)_{\phi} (v)_{\phi}$. This is a direct generalization of the definition of a scalar field in regular 4D spacetime. For a $d-$metric rel. (10) this density is written as:
\[
\mathcal{L}_\phi = -\frac{1}{2} g^{ij} \delta_i \phi \delta_j \phi - \frac{1}{2} h^{ab} \partial_a \phi \partial_b \phi - V(\phi).
\]  

(109)

Definitions in relations (69) and (70) give:

\[
\begin{align}
\frac{1}{\gamma} T_{ij}^{(\phi)} &= \delta_i \phi \delta_j \phi - g_{ij} \left( \frac{1}{2} g^{kl} \delta_k \phi \delta_l \phi + \frac{1}{2} h^{ab} \partial_a \phi \partial_b \phi + V(\phi) \right) \\
\frac{1}{\gamma} T_{ab}^{(\phi)} &= \partial_a \phi \partial_b \phi - h_{ab} \left( \frac{1}{2} g^{ij} \delta_i \phi \delta_j \phi + \frac{1}{2} h^{cd} \partial_c \phi \partial_d \phi + V(\phi) \right).
\end{align}
\]

(110)

(111)

Equations of motion for \( \phi \) on the tangent bundle are given by extremalizing the action

\[
S = \int \sqrt{\text{det}(g)} \mathcal{L}_\phi \, d^8x
\]

for variations of \( \phi \) or equivalently by the generalized Euler–Lagrange equations:

\[
\frac{\partial \mathcal{L}_\phi}{\partial \phi} - D\Lambda \frac{\partial \mathcal{L}_\phi}{\partial (D\Lambda \phi)} = 0.
\]

(112)

These give:

\[
\Box \phi + \Box \phi - V' = 0
\]

(113)

where we denoted

\[
\Box \equiv g^{ij} \Box_i \Box_j, \quad \Box \equiv h^{ab} \Box_a \Box_b
\]

(114)

the prime standing for differentiation with respect to \( \phi \). Equation (113) is another generalization of the Klein–Gordon equation of the standard model of particle physics (provided that \( V(\phi) = m^2 \phi^2/2h^2 \)), along with rel. (82).

If we consider the specific case where the metric takes the form (89), the connection coefficients (25) reduce to the Christoffel coefficients for FRW metric. In this case, we take all the quantities on the bundle to be spatially homogeneous functions, so the functions \( \phi(t, y) \), \( N^r_\lambda(t, y) \) depend on \( t \) and \( y \). From (114) we get:

\[
\Box \phi = -\ddot{\phi} - 3H\dot{\phi} - \left( N^a_b \dot{\partial}_a N^a_0 - \dot{N}^a_0 \right) \partial_a \phi + 2N^a_b \dot{\partial}_a \phi + \frac{1}{a^2} \delta^{\lambda \mu} N^a_\lambda N^b_\mu \dot{\partial}_a \dot{\partial}_b \phi - 2N^a_b \dot{\partial}_a \dot{\partial}_b \phi + 3HN^a \dot{\partial}_a \phi.
\]

(115)

Equation (113) then gives:

\[
\ddot{\phi} + 3H + V' = \Box \phi - \left( N^a_b \dot{\partial}_a N^a_0 - \dot{N}^a_0 \right) \partial_a \phi + 2N^a_b \dot{\partial}_a \phi + \frac{1}{a^2} \delta^{\lambda \mu} N^a_\lambda N^b_\mu \dot{\partial}_a \dot{\partial}_b \phi - 2N^a_b \dot{\partial}_a \dot{\partial}_b \phi + 3HN^a \dot{\partial}_a \phi
\]

(116)

where \( H = \dot{a}/a \) is the Hubble parameter. For a v-metric given in rel. (43) and a Cartan-type connection, nonlinear connection components are given by rel. (58). Then rel. (116) to first order with respect to \( h_{ab} \) becomes:

\[
\ddot{\phi} + 3H + V' = \eta^{ab} \dot{\partial}_a \dot{\partial}_b \phi - \eta^{ab} \eta^{cd} \left( \dot{\partial}_a \dot{\partial}_b + \dot{\partial}_b \dot{\partial}_a - \dot{\partial}_d \dot{\partial}_d \right) + \frac{1}{2} \dot{\partial}_a \dot{\partial}_b \dot{\partial}_a \phi + y_b \dot{\partial}_b \dot{\partial}_a \phi + \frac{3}{2} H N^a \dot{\partial}_a \phi.
\]

(117)

We can model the scalar field as an ideal fluid by comparing relations (110) and (94). We get

\[
\delta_i \phi = \sqrt{p_\phi + P_\phi} U_i
\]

(118)
and
\[ P_\phi = -\frac{1}{2} g^{kl} \delta_k \phi \delta_l \phi -\frac{1}{2} h^{ab} \hat{\partial}_a \phi \hat{\partial}_b \phi - V(\phi). \] (119)

By combining the above two equations and solving for \( \rho_\phi \) we find
\[ \rho_\phi = -\frac{1}{2} g^{kl} \delta_k \phi \delta_l \phi + \frac{1}{2} h^{ab} \hat{\partial}_a \phi \hat{\partial}_b \phi + V(\phi). \] (120)

For the case of the generalized FRW metric rel. (89) and for spatially homogeneous functions \( \phi \) and \( N_i^a \), energy density and pressure for the scalar fluid become:
\[ \rho_\phi = \frac{1}{2} \dot{\phi}^2 + \frac{1}{2} N_i^a N_i^b \hat{\partial}_a \phi \hat{\partial}_b \phi - \dot{\phi} N_i^a \hat{\partial}_a \phi + \frac{1}{2} h^{ab} \hat{\partial}_a \phi \hat{\partial}_b \phi + V(\phi) \] (121)

and
\[ P_\phi = \frac{1}{2} \dot{\phi}^2 - \frac{1}{2} y_b \hat{\partial}^{ab} \phi \hat{\partial}_a \phi - \dot{\phi} N_i^a \hat{\partial}_a \phi - \frac{1}{2} h^{ab} \hat{\partial}_a \phi \hat{\partial}_b \phi - V(\phi). \] (122)

For the case of the weak-field v-metric given in rel. (43), energy density and pressure in relations (121) and (122) are to first order with respect to \( \tilde{h}_{ab} \) calculated as:
\[ \rho_\phi = \frac{1}{2} \dot{\phi}^2 - \frac{1}{2} y_b \hat{\partial}^{ab} \phi \hat{\partial}_a \phi + \frac{1}{2} h^{ab} \hat{\partial}_a \phi \hat{\partial}_b \phi + V(\phi) \] (123)
\[ P_\phi = \frac{1}{2} \dot{\phi}^2 - \frac{1}{2} y_b \hat{\partial}^{ab} \phi \hat{\partial}_a \phi - \frac{1}{2} h^{ab} \hat{\partial}_a \phi \hat{\partial}_b \phi - V(\phi). \] (124)

Scalar field provides a viable model for a cosmological bounce. Going back to the bounce condition in rel. (107), relations (121) and (122) give:
\[ \rho_\phi + P_\phi < 0 \iff \dot{\phi}^2 + N_i^a N_i^b \hat{\partial}_a \phi \hat{\partial}_b \phi - 2\dot{\phi} N_i^a \hat{\partial}_a \phi < 0. \] (125)

For the weak-field case and given relations (123) and (124) we get the necessary condition for a bouncing universe:
\[ \rho_\phi + P_\phi < 0 \Rightarrow y_b \hat{\partial}^{ab} \phi \hat{\partial}_a \phi > 0. \] (126)

We remark that the scalar field would not be able to provide a viable model for cosmological bounce in ordinary GR. To satisfy the bounce conditions in the weak field it is necessary to have a nonzero \( \tilde{h}_{ab} \) and also a scalar field with nonzero directional derivative \( \hat{\partial}_a \phi \), as we can see from rel. (126).

6. Electromagnetic field tensor

Previous studies that incorporate the electromagnetic field tensor and the associated Maxwell equations in the framework of the metric tangent bundle have been made, for example in [2, 10, 36, 67, 68]. In the present work we study the electromagnetic field in the framework of tangent bundle’s geometry. In this approach we assume that the vector potential \( A \) is horizontal and isotropic:
\[ A = A_i(x) \, dx^i. \] (127)

Isotropy here means that \( A \) depends only on position coordinates \( x^i \). Taking into account [2] page 277, we consider symmetric connection coefficients \( L_i^k \) and an isotropic \( A_i(x) \) and we get
a generalized description of the electromagnetic field tensor $F_{ij} = \partial_j A_i - \partial_i A_j$ on the tangent bundle:

$$\tilde{F}_{ij} = (h) D_j A_i(x) - (h) D_i A_j(x) = \delta_j A_i(x) - \delta_i A_j(x) - L^0_j A_k(x) + L^0_i A_k(x)$$

(128)

or

$$\tilde{F}_{ij} = \partial_j A_i(x) - \partial_i A_j(x) = F_{ij}$$

(129)

Since

$$\dot{\partial}_a A_i(x) = 0.$$  

(130)

From relations (129) and (130) we get

$$\frac{1}{2} \tilde{F}_{ij}(x) \, dx^i \wedge dx^j = \partial_j A_i(x) \, dx^i \wedge dx^j$$

$$= \delta_j A_i(x) \, dx^i \wedge dx^j.$$  

(131)

This is equivalent to the relation

$$\tilde{F} = dA$$

(132)

where

$$\tilde{F} = \frac{1}{2} \tilde{F}_{ij}(x) \, dx^i \wedge dx^j.$$  

(133)

From rel. (132) we get

$$d\tilde{F} = 0.$$  

(134)

On the adapted basis, equation (134) gives

$$\frac{1}{2} \delta_k \tilde{F}_{ij}(x) \, dx^k \wedge dx^i \wedge dx^j$$

(135)

or

$$\partial_k \tilde{F}_{ij} = 0$$

(136)

where the fact that the field tensor depends only on $x$ was used.

For the second part of Maxwell’s equations, we consider the straightforward generalization:

$$\tilde{D}^k F_{ij} = J^i$$

(137)

where $J = J^i(x, y) \delta_i$ is the electromagnetic 4–current density. For a canonical $d$–connection we get:

$$\tilde{D}^k F_{ij} = \partial_k \tilde{F}_{ij} + L^k_j \tilde{F}^l_j + L^k_i \tilde{F}^l_i$$

$$= \partial_k \tilde{F}_{ij}(x) + \frac{1}{2} g^{kl}(x, y) (\delta_k g_{ij}(x, y) + \delta_i g_{kj}(x, y) - \delta_j g_{ki}(x, y)) \tilde{F}^{kl}(x)$$

$$= \partial_k \tilde{F}_{ij}(x) + \frac{1}{2} g^{kl}(x, y) \delta_i g_{kj}(x, y) \tilde{F}^{kl}(x) = J^i(x, y).$$

(138)
This is equivalent to the relation

\[ d \star \tilde{F} = \star J \]  

(139)

where \( \star \) is the Hodge duality operator of the horizontal subspace. To prove this, we first need the derivative of h-metric’s \( g_{ij} \) determinant with respect to \( x^j \). For that we use the identity for a square matrix \( M \):

\[ \ln(\det M) = \text{tr}(\ln M) \]

and find

\[ \delta_j \text{det}[g] = \text{det}[g] g^{kl} \delta_j g_{kl}. \]

(140)

Levi-Civita tensor for the h-space is

\[ \epsilon_{ijkl} = \sqrt{-\det[g]} \tilde{\tau}_{ijkl} \]

(141)

where \( \tilde{\tau}_{ijkl} \) is the Levi-Civita symbol of the h-space, the convention \( \epsilon_{0123} = 1 \) is followed.

Relation (139) in the adapted basis is written

\[ \frac{1}{4} \delta_p (\epsilon_{ijkl} \tilde{F}^{ij}) \, dx^p \wedge dx^k \wedge dx^l = \frac{1}{3!} \epsilon_{ijkl} dx^j \wedge dx^k \wedge dx^l. \]

(142)

Using relation (141) we get:

\[ \frac{1}{4} \epsilon_{ijkl} \delta_p \left( \sqrt{-\det[g]} \tilde{F}^{ij} \right) \, dx^p \wedge dx^k \wedge dx^l = \frac{1}{3!} \sqrt{-\det[g]} J_i \epsilon_{ijkl} dx^j \wedge dx^k \wedge dx^l. \]

(143)

After straightforward calculations this gives

\[ \delta_j \left( \sqrt{-\det[g]} \tilde{F}^{ij} \right) = \sqrt{-\det[g]} J^j. \]

(144)

Finally, using rel. (140) we get the equation:

\[ \partial_j \tilde{F}^{ij}(x) + \frac{1}{2} g^{H}(x,y) \delta_j g_{kl}(x,y) \tilde{F}^{ij}(x) = J^j(x,y). \]

(145)

This concludes the proof that relations (137) and (139) are equivalent.

We observe that equations (134) and (139) of electromagnetism in our space are equivalent in form with the Riemannian ones which is a result of our initial assumptions. Of course, the dynamics of the fields is not equivalent in the two geometries. The extended geometrical structure of the tangent bundle can result to an extension of physical predictions of the theory.

In our approach we have considered the 4-current density \( J_i(x,y) \) to depend on \( y \) in order to be consistent with the anisotropic geometric structure which is obvious in rel. (138) and (145). This can remove possible inconsistencies that could be inherited from the assumption of an isotropic field on an anisotropic background.

Another point of view that will give us isotropic field equations for electromagnetism and resolve any possible inconsistencies can be considered by introducing the action

\[ S_{EM} = \int_M d^4x \left( \int_{\mathcal{T},M} d^4y \sqrt{|\det[G]|} \mathcal{L}_{EM} \right) \]

(146)

and perform the integration over the fiber before extremalizing over a subset of the base manifold \( M \).
7. Concluding remarks

In this work, we studied a weak-field model on the tangent bundle, which provides an insight in local anisotropy of modified Einstein gravity. Field equations rel. (71) extend Einstein’s equations of GR with extra terms and a generalized energy-momentum tensor. The extra terms are connected with the anisotropic part of the geometry and can be interpreted as a possible anisotropy of the universe.

We derived the mass shell equation for a locally flat background h-metric, which is a generalization of the known \( p^i p_i = -m^2 \), rel. (79) and (80). An extension of the Klein–Gordon equation was given and a dispersion relation for the scalar field’s modes was derived as well, rel. (84).

A profound result of modified gravity theory on \( TM \) is given in rel. (95), (96), where the simple case of an isotropic FRW h-metric is considered. Field equations resemble the Friedmann equations, with an extra term being the scalar curvature \( S \) of v-space. This extra term can give rise to an accelerating expansion of space. From a cosmological point of view, this can give an insight on a possible physical interpretation of v-metric \( h_{ab} \).

On the other hand, it turns out that this generalized FRW cosmology cannot describe a bouncing universe, at least not for an ideal fluid matter field obeying basic energy conditions. We introduced a scalar field model and we derived the condition which can describe a bouncing cosmology, violating several energy conditions in the process. As well, a generalized form of Raychaudhuri equation was given in section 5.2 for the present model. In this approach, extra anisotropic terms can determine the accelerating/decelerating expansion of the universe.

Finally, we incorporated the electromagnetic field equations in the tangent bundle framework, taking the 4-potential to be isotropic, and we resulted in equations similar to those of regular Riemannian models of gravity. Relations (134) and (139) remain invariant in form under the introduction of local anisotropy at the description of gravity.

A further investigation of the consideration presented in this work regarding the electromagnetic field is required. This can be studied in the near future.

In the present work, the weak field equations provide an infrastructure for the study of gravitational waves and cosmological perturbations on the tangent Lorentz bundle. This will be a task for a future project.

Acknowledgments

The authors would like to express their thanks to the unknown referees for their valuable comments on the text. We also thank Dr S Basilakos for discussions on the manuscript.

Appendix A. Field equations on the tangent bundle

In the following we consider a tangent Lorentz bundle equipped with a nonlinear connection, a \( d \)-metric, rel. (10), and a canonical and distinguished \( d \)-connection, rel. (25)–(28).

The action of the fields in an arbitrary closed subset \( A \) of \( TM \) is

\[
S_{TM} = \frac{1}{2\kappa} S_G + S_M = \frac{1}{2\kappa} \int_A \sqrt{\left| \det \left[ \mathcal{G} \right] \right|} \mathcal{L}_G d^8 \mathcal{X} + \int_A \sqrt{\left| \det \left[ \mathcal{G} \right] \right|} \mathcal{L}_M d^8 \mathcal{X} \quad (A.1)
\]

where
\[ S_G = \int_A d^8x \sqrt{|\det [g]|} R = \int_A d^8x \sqrt{|\det [\mathcal{G}]|} (R + S) \] (A.2)

is the gravitational part of the action and
\[ S_M = \int_A d^8x \sqrt{|\det [\mathcal{G}]|} \mathcal{L}_M \] (A.3)

is the action of the matter fields, while \( \det [\mathcal{G}] \) is the \( d \)-metric’s determinant. This is a direct generalization of the Einstein–Hilbert action. Constant \( \kappa \) will be specified by the limit of this framework where general relativity is obtained. The volume element on \( TM \) is defined by
\[ d^8x = dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 \wedge dy_0 \wedge dy^1 \wedge dy^2 \wedge dy^3. \] (A.4)

We observe that the terms in \( \delta y^a \) involving \( dx^i \), rel. (8), will drop out in the exterior product with \( dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 \), so we can equivalently write (A.4) as:
\[ d^8x = dx^0 \wedge dx^1 \wedge dx^2 \wedge \delta y^0 \wedge \delta y^1 \wedge \delta y^2 \wedge \delta y^3. \] (A.5)

The independent fields of the underlying geometry are \( g_{ij}(x,y), h_{ab}(x,y) \) and \( N_i^a(x,y) \). We will derive equations relating these fields to the matter fields by extremalizing the action rel. (A.1) with respect to variations \( \delta g_{ij}, \delta h_{ab} \) and \( \delta N_i^a \) which vanish at the boundary \( \partial A \). The variation of the curvature coefficients \( \Omega^a_i, \) rel. (16), is given by
\[ \delta \Omega_i^a = \delta \bar{g}_{ij} N_j^a - \left( \partial_i \bar{N}_j^a \right) \bar{g}_{jk} \delta N_j^k + \left( \partial_i \bar{N}_j^a \right) \bar{g}_{jk} \delta N_j^k. \] (A.6)

The variations of connection coefficients \( L_{jk}^i \) and \( C_{ia}^k \) are
\[ \delta L_{jk}^i = -\frac{1}{2} \left( g^m_{(j} D_k \delta g_{i) m} + g^m_{km} D_j \delta g_{i m} - g^m_{km} D_l \delta g_{i m} \right) \] (A.7)
\[ - (C_{ia}^k \delta N_j^a - g^i_{jk} \delta g_{km} \delta N_j^m). \]
\[ \delta C_{ia}^k = \frac{1}{2} C_{ija} \delta g^{jk} - \frac{1}{2} C_{ia}^{mb} \delta g_{mb} - \frac{1}{2} g^i_{im} D_a \delta g^{mk}. \] (A.8)

The variation of the h-Ricci tensor \( R_j^i \) is given by
\[ \delta R_j^i = D_k \left( \delta L_{jk}^i + C_{ia}^k \delta N_j^a \right) - D_j \left( \delta L_{ik}^j + C_{ia}^k \delta N_i^a \right) + \mathcal{D}_a \left( \frac{1}{2} \Omega_b^m \delta g_{im} \delta g^{mk} \right) \]
\[ + \left( D_k C_{ia}^k + C_{ia}^k L_{jk}^k - C_{ia}^k \left( \partial_i \bar{N}_j^a \right) - \left( \partial_i \bar{N}_j^a \right) \bar{g}_{jk} \right) \delta N_j^a \]
\[ - \left( D_k C_{ia}^k + C_{ia}^k L_{jk}^a - C_{ia}^k \left( \partial_i \bar{N}_j^k \right) - \left( \partial_i \bar{N}_j^k \right) \bar{g}_{jk} \right) \delta N_j^a \]
\[ - \frac{1}{2} \left( \delta \bar{g}_{ij} \delta \bar{g}_{im} - \Omega^a_{ij} \delta \bar{g}_{im} \right) \delta g^{jk} \] (A.9)

and the variation of the h-Ricci scalar tensor \( R \) is given by
\[ \delta R = \left( g^{jk} \partial_i \bar{L}_{ij}^k - g^{ij} \partial_i \bar{L}_{jk}^j \right) \delta N_j^a + \left( \Omega^a_{ij} C_{ja}^k + R_{ij} \right) \delta g^{jk} + \mathcal{D}_a Z^0 \] (A.10)

where \( \mathcal{D}_a \equiv \delta_a^b D_b + \delta_a^b D_a \) and \( Z^0 = (Z^0, 0) \), with

\[ Z^0 = \left( \frac{1}{2} \Omega_b^m \delta g_{im} \delta g^{mk}, 0 \right). \]
\[ Z^k \equiv g^{ij} \delta L^k_{ij} - g^{ik} \delta L^j_{ij}. \]

The variation of v-Ricci tensor \( S_{ab} \) and v-Ricci scalar \( S \) takes the form
\[
\delta S_{ab} = \mathcal{D}_c \delta C_{ab}^c - \mathcal{D}_b \delta C_{ac}^c
\]
\[
\delta S = S_{ab} \delta h^{ab} + \mathcal{D}_b K^b
\] (A.11)

where \( K^b = (0, K^c) \), with
\[
K^c \equiv h^{ab} \delta C_{ab}^c - h^{ac} \delta C_{cb}^c.
\] (A.12)

Metric tensor, rel. (10), is represented in the adapted basis as a block diagonal matrix. From a well known theorem regarding such matrices we get
\[
\det [G] = \det [g] \det [h]
\] (A.13)

where \( \det [g] \) and \( \det [h] \) are the determinants of the h-metric and v-metric respectively. As far as the variation of the metric determinant’s square root is concerned, we find
\[
\delta \sqrt{\det [G]} = \delta \left[ \sqrt{\det [g]} \sqrt{\det [h]} \right]
\] (A.14)
\[
= \frac{1}{2 \sqrt{-\det [g]\sqrt{-\det [h]}}} \left( \det [g] \delta \det [h] + \det [h] \delta \det [g] \right)
\] (A.15)
\[
= -\frac{1}{2} \sqrt{\det [G]} \left( g_{ij} \delta g^{ij} + h_{ab} \delta h^{ab} \right).
\] (A.16)

From relations (A.10), (A.11) and (A.16) we get the variation of the geometrical part of the action:
\[
\delta S_G = \int_A d^8 X \sqrt{\det [G]} \left[ -\frac{1}{2} \left( g_{ij} \delta g^{ij} + h_{ab} \delta h^{ab} \right) (R + S) 
+ \left( g^{ik} \partial_k L^j_{ij} - g^{ij} \partial_k L^k_{ij} \right) \delta N^a_k
+ \left( \Omega^a_{c} C^c_{ja} + R_{ij} \right) \delta g^{ij} + S_{ab} \delta h^{ab} + \mathcal{D}_b \left( Z^b + K^b \right) \right].
\] (A.17)

From Stokes theorem we get:
\[
\int_A d^8 X \mathcal{D}_B \left( \sqrt{\det [G]} \left( Z + K \right)^B \right) = \oint_{\partial A} d^7 X \sqrt{\det [\mathcal{G}]} \left( n_B (Z + K)^B \right) = 0
\] (A.18)

where \( \det [\mathcal{G}] \) is the determinant of the metric \( \mathcal{G} \) of the boundary space \( \partial A \). We have assumed the vanishing of the boundary term.

Extremization of the action \( S_{TM} \) with respect to \( g_{ij}, h_{ab} \) and \( N^a_i \) gives:
\[
\int_A d^8 X \sqrt{\det [G]} \left[ \frac{1}{2\kappa} \left[ R_{ij} - \frac{1}{2} (R + S) g_{ij} + \Omega^a_{c} C^c_{ja} - \kappa T^a_i \right] \right] \delta g^{ij}
+ \int_A d^8 X \sqrt{\det [G]} \left[ \frac{1}{2\kappa} \left[ S_{ab} - \frac{1}{2} (R + S) h_{ab} - \kappa T_{ab} \right] \right] \delta h^{ab}
+ \int_A d^8 X \sqrt{\det [G]} \left[ \frac{1}{2\kappa} \left( g^{ik} \partial_k L^j_{ij} - g^{ij} \partial_k L^k_{ij} - \kappa T^a_i \right) \right] \delta N^a_k = 0
\] (A.19)
where the energy momentum tensor coefficients are defined as

\[ T^1_{ij} \equiv - \frac{2}{\sqrt{\operatorname{det}[G]}} \frac{\delta \left( \sqrt{\operatorname{det}[G]} L_M \right)}{\delta g^{ij}} \]  

(A.20)

\[ T^2_{ab} \equiv - \frac{2}{\sqrt{\operatorname{det}[G]}} \frac{\delta \left( \sqrt{\operatorname{det}[G]} L_M \right)}{\delta h^{ab}} \]  

(A.21)

\[ \frac{3}{2} T^k_a \equiv - \frac{2}{\sqrt{\operatorname{det}[G]}} \frac{\delta \left( \sqrt{\operatorname{det}[G]} L_M \right)}{\delta N^a_k} \]  

(A.22)

From rel. (A.19) we get the field equations

\[ R_{(ij)} - \frac{1}{2} (R + S) g_{ij} + \Omega^a_{ki} c^{jk}_a = \kappa T^1_{ij} \]  

(A.23)

\[ S_{ab} - \frac{1}{2} (R + S) h_{ab} = \kappa T^2_{ab} \]  

(A.24)

\[ g^{ik} \partial_a L^i_j - g^{il} \partial_a L^k_l = \kappa \frac{3}{2} T^k_a. \]  

(A.25)

Here we have presented a more general approach than the one we follow at the other sections. Specifically, we have assumed \( g_{ij}, h_{ab} \) and \( N^a_i \) to be independent dynamic fields on the tangent bundle. However, at the rest of our work we consider the nonlinear connection \( N^a_i \) as an a priori defined structure on the tangent bundle, so field equation (A.25) cannot be considered valid.

A.1. Determination of constant \( \kappa \)

We consider now the limit where \( N^a_i \) goes to zero, \( h_{ab} \) goes to \( \eta_{ab} = \text{diag}(-1, 1, 1, 1) \) and differentiation with respect to \( y \) of any quantity defined on \( TM \) goes to zero. The adapted basis \( \{ \delta, \dot{\delta} \} \) and its dual \( \{ dx^i, dy^a \} \) defined in relations (5)–(8) reduce then to \( \{ \partial, \dot{\partial} \} \) and \( \{ dx^i, dy^a \} \) respectively. Curvature coefficients \( \Omega^a_b_i, \) rel. (16), vanish in this limit. The metric on the tangent bundle, rel. (10), becomes:

\[ G = g_{ij}(x) dx^i \otimes dx^j + \eta_{ab} dy^a \otimes dy^b. \]  

(A.26)

The connection coefficients \( L^i_{jk} \) given in rel. (25) are then

\[ L^i_{jk} = \frac{1}{2} g^{ik} \left( \partial_k g_{ij} - \partial_j g_{ik} + \partial_h g_{hk} \right). \]  

(A.27)

We observe that \( L^i_{jk} = \gamma^i_{jk} \) in the GR limit. The rest of the connection coefficients rel. (26)–(28) vanish. Moreover, the quantities \( R_{ij} \) and \( R \) defined in relations (19) and (23) are given in this limit as:

\[ R_{ij} = \partial_h \gamma^k_{ij} - \partial_j \gamma^k_{ih} + \gamma^m_{ij} \gamma^k_{mk} - \gamma^m_{ik} \gamma^k_{mj} \]  

(A.28)

\[ R = g^{ij} R_{ij}. \]  

(A.29)
These are identified as the Ricci tensor and Ricci scalar of Riemannian geometry for the metric $g_{ij}$ as can be seen by the respective definitions, while $S_{ab}$ and $S$ vanish. Using rel. (A.13), the determinant of the metric of rel. (A.26) is given as:

$$\det [\mathcal{G}] = \det [g] \det [\eta] = - \det [g]$$

so rel. (A.20) becomes

$$\frac{1}{T_{ij}} = - \frac{2}{\sqrt{-\det [g]}} \frac{\delta}{\delta g^{ij}} \left( \frac{\sqrt{-\det [g]} \mathcal{L}_M}{\eta} \right).$$

This is the definition of the energy momentum tensor of GR. Field equations rel. (A.23) reduce to

$$R_{ij} - \frac{1}{2} R g_{ij} = \kappa T_{ij}.$$

Given the relations (A.28), (A.29) and (A.31), we identify these as the Einstein field equations for the metric $g_{ij}(x)$.

Geodesics equation (39) reduces to

$$\frac{dy^a}{d\tau} + \gamma^a_{ij}(x) y^i y^j = 0, \quad y^i = \frac{dx^i}{d\tau}.$$  

This is the geodesics equation defined in GR for a metric tensor $g_{ij}(x)$.

From relations (A.31)–(A.33) we deduce that in this limit we get ordinary general relativity for a spacetime equipped with metric tensor $g_{ij}(x)$. Constant $\kappa$ is thus specified as

$$\kappa = 8\pi G/c^4$$

where $G$ is the gravitational constant.

Field equation (A.24) in this limit become

$$-\frac{1}{2} R \eta_{ab} = \kappa T_{ab}.$$

These equations restrict the way matter fields can depend upon the v-metric. In this sense, they do not affect the dynamics of $g_{ij}(x)$.

**Appendix B. Horizontal space Ricci curvature coefficients and Ricci scalar**

The components of the h-Ricci tensor are obtained from relations (19) and (60) as:

$$R_{ij} = \left( \begin{array}{c} \text{ob} \end{array} \right) R_{ij} + \left( \begin{array}{c} \text{ob} \end{array} \right) R^{\eta}_{ij} + \left( \begin{array}{c} \text{ob} \end{array} \right) \gamma^r_{im} \left( \begin{array}{c} \text{ob} \end{array} \right) \gamma^r_{ij} - \left( \begin{array}{c} \text{ob} \end{array} \right) \gamma^r_{im} \left( \begin{array}{c} \text{ob} \end{array} \right) \gamma^r_{ij} - \left( \begin{array}{c} \text{ob} \end{array} \right) \gamma^r_{ij} \frac{\delta}{\delta x^r} \tilde{g}_r^k$$

$$+ \left( \begin{array}{c} \text{ob} \end{array} \right) g^{kr} \left( \begin{array}{c} \text{ob} \end{array} \right) \left( \partial_p \tilde{g}_{qr} - \frac{1}{2} \partial_r \tilde{g}_{pq} \right) - \left( \begin{array}{c} \text{ob} \end{array} \right) g^{kr} \frac{\delta}{\delta x^r} \tilde{g}_r^k$$

$$+ \left( \begin{array}{c} \text{ob} \end{array} \right) g^{kr} \left( \partial_p \tilde{g}_{kr} - \frac{1}{2} \partial_k \tilde{g}_{pr} \right) - \left( \begin{array}{c} \text{ob} \end{array} \right) g^{kr} \frac{\delta}{\delta x^r} \tilde{g}_r^k.$$

Using this result, the h-Ricci scalar curvature from rel. (23) to first order with respect to $\tilde{g}_{ij}$ and $\tilde{h}_{ab}$ equals:
\[ R = (b) R - \tilde{g}^i (b) R^i_j + (b) \tilde{g}^i (b) R^i_{k\dot{j}} + (b) \gamma^m_{k\dot{j}} (b) \gamma^r_{\dot{i}m} - (b) \gamma^m_{\dot{i}m} (b) \gamma^r_{\dot{i}m} - (b) \gamma^r_{\dot{i}j}(b) \tilde{g}^k_r \]

\[ + (b) g^{i} \left\{ \left( \partial_k (b) g^{jk} \right) \left( \partial_j \tilde{g}^k_r - \frac{1}{2} \partial_j \tilde{g}^k \right) \right. - \left( \partial_k (b) g^{jk} \right) \partial_j \tilde{g}^k_r \]

\[ + (b) g^{kr} \left[ \partial_k \partial_j \tilde{g}^r_{ij} - \frac{1}{2} \partial_k \partial_j \tilde{g}^k_r + \frac{1}{2} (b) \gamma^m_{ij} \partial_m \tilde{g}^k_r \right] \]

\[ - (b) \gamma^m_{ij} \left( \partial_i \tilde{g}^m_{ij} + \partial_m \tilde{g}^m_{ij} - \partial_i \tilde{g}^m_{mj} \right) + (b) \gamma^m_{\dot{i}m} \left( \partial_j \tilde{g}^m_{ij} - \frac{1}{2} \partial_j \tilde{g}^m \right) \right\}. \quad \text{(B.2)} \]

**Appendix C. Coefficients \( A_{ij} \) and \( B_{ab} \)**

The quantities \( A_{ij} \) and \( B_{ab} \) which appear in relations (71) and (72) respectively, are the terms which are of first order in \( \tilde{g}_{ij} \) and \( \tilde{h}_{ab} \) and are defined as follows:

\[ A_{ij} = - \frac{1}{2} g^{ij} (b) R + \frac{1}{2} g^{ij} (b) R_{pq} (b) R_{pq} - (b) \gamma^m_{\dot{i}m} (b) \gamma^r_{\dot{i}m} + (b) \gamma^r_{\dot{i}j}(b) \tilde{g}^k_r \]

\[ + \delta_{ij} \left( (b) R^i_j + (b) \gamma^m_{\dot{i}m} (b) \gamma^r_{\dot{i}m} \right) \tilde{g} + (b) \gamma^m_{\dot{i}m} (b) \gamma^r_{\dot{i}m} \tilde{g} + (b) \gamma^r_{\dot{i}j}(b) \tilde{g}^k_r \]

\[ + (b) g^{kr} \left[ \partial_k \partial_j \tilde{g}^r_{ij} - \frac{1}{2} \partial_k \partial_j \tilde{g}^k_r + \frac{1}{2} (b) \gamma^m_{ij} \partial_m \tilde{g}^k_r \right] \]

\[ - (b) \gamma^m_{ij} \left( \partial_i \tilde{g}^m_{ij} + \partial_m \tilde{g}^m_{ij} - \partial_i \tilde{g}^m_{mj} \right) + (b) \gamma^m_{\dot{i}m} \left( \partial_j \tilde{g}^m_{ij} - \frac{1}{2} \partial_j \tilde{g}^m \right) \right\}. \quad \text{(C.1)} \]

\[ B_{ab} = - \frac{1}{2} \left( \partial^i (b) \partial_i \tilde{h}_{bc} + \partial^i (b) \partial_i \tilde{h}_{bc} - \partial^i (b) \partial_i \tilde{h} \right) - \frac{1}{2} \tilde{h}_{ab} (b) R \]

\[ + \frac{1}{2} \tilde{h}_{ab} \tilde{g}^i (b) R^i_j + \frac{1}{2} \tilde{h}_{ab} \tilde{g}^{ij} (b) R^i_{k\dot{j}} + (b) \gamma^m_{k\dot{j}} (b) \gamma^r_{\dot{i}m} - (b) \gamma^m_{\dot{i}m} (b) \gamma^r_{\dot{i}m} - (b) \gamma^m_{ij}(b) \tilde{g}^k_r \]

\[ - \frac{1}{2} \tilde{h}_{ab} \tilde{g}^{ij} \left\{ \left( \partial_k (b) g^{jk} \right) \left( \partial_j \tilde{g}^k_r - \frac{1}{2} \partial_j \tilde{g}^k \right) \right. - \left( \partial_k (b) g^{jk} \right) \partial_j \tilde{g}^k_r \]

\[ + (b) g^{kr} \left[ \partial_k \partial_j \tilde{g}^r_{ij} - \frac{1}{2} \partial_k \partial_j \tilde{g}^k_r + \frac{1}{2} (b) \gamma^m_{ij} \partial_m \tilde{g}^k_r \right] \]

\[ - (b) \gamma^m_{ij} \left( \partial_i \tilde{g}^m_{ij} + \partial_m \tilde{g}^m_{ij} - \partial_i \tilde{g}^m_{mj} \right) + (b) \gamma^m_{\dot{i}m} \left( \partial_j \tilde{g}^m_{ij} - \frac{1}{2} \partial_j \tilde{g}^m \right) \right\}. \quad \text{(C.2)} \]
Appendix D. Canonical momentum for the massive particle in a weakly anisotropic metric space

We consider a tangent bundle equipped with the metric tensor defined in rel. (41)–(43). The trajectory in spacetime for the massive particle of section 4.1 is described by the Lagrangian rel. (75):

\[ L = -m \left( -y'_i y_i \right)^{1/2} = -m \left( L_R - \frac{1}{2} L_R^{-1} \tilde{g}_{ij}(x, y) y_i y_j \right) \]

where \( y'_i = g_{ij} y'^j \) and in the second equality only terms up to first order with respect to \( \tilde{g}_{ij} \) are kept. The Riemannian norm \( L_R \) is defined in rel. (76) as:

\[ L_R = \left( -g_{ij}(x) y^i y^j \right)^{1/2}. \]

The canonical momentum is found as

\[ p_i = \dot{y}_i = mL_R^{-1} \left[ y_i + \frac{1}{2} \left( \partial_y \tilde{g}_{ij} + L_R^{-1} \tilde{g}_{ij} y_i y^j \right) y^j \right]. \tag{D.1} \]

We normalize the fiber coordinates as \( y_i y^i = -1 \). We find to first order with respect to \( \tilde{g}_{ij} \):

\[ -y_i y^i = L_R - \frac{1}{2} L_R^{-1} \tilde{g}_{ij} y_i y^j = 1 \]

\[ \Rightarrow L_R^2 - L_R - \frac{1}{2} \tilde{g}_{ij} y_i y^j = 0. \tag{D.2} \]

Solving the quadratic equation with respect to \( L_R \) gives to first order with respect to the perturbation:

\[ L_R = \frac{1}{2} \left( 1 \pm \sqrt{1 + 2 \tilde{g}_{ij} y_i y^j} \right) = 1 + \frac{1}{2} \tilde{g}_{ij} y_i y^j \]  \tag{D.3} \]

where the second solution, namely \( L_R = -\frac{1}{2} \tilde{g}_{ij} y_i y^j \) is rejected as it gives a vanishing Riemannian norm in zero order.

The inverse \( L_R^{-1} \) of rel. (D.3), as well as \( L_R^{-2} \), are easily found to be

\[ L_R^{-1} = 1 - \frac{1}{2} \tilde{g}_{ij} y^j, \quad L_R^{-2} = 1 - \tilde{g}_{ij} y^j. \tag{D.4} \]

Putting together relations (D.1) and (D.4) and keeping terms up to first order with respect to \( \tilde{g}_{ij} \) we get the canonical momentum given in rel. (78):

\[ p_i = m \left[ y_i + \frac{1}{2} \left( \partial_y \tilde{g}_{ij} \right) y^j y^j \right]. \tag{D.5} \]

ORCID iDs

P C Stavrinos  https://orcid.org/0000-0002-1187-8017
References

[1] Stavrinos P C 2004 Congruences of fluids in a Finslerian anisotropic space-time Int. J. Theor. Phys. 44 245–54
[2] Stavrinos P C and Diakogiannis F I 2004 Finslerian structure of anisotropic gravitational field Gravit. Cosmol. Russ. Gravit. Soc. 10 269–78
[3] Stavrinos P C and Ikeda S 2006 Connection considerations of gravitational field in Finsler spaces Int. J. Theor. Phys. 45 743–9
[4] Gibbons G W, Gomis J and Pope C N 2007 General very special relativity is Finsler geometry Phys. Rev. D 76 081701
[5] Stavrinos P C, Koulentesis A P and Stathakopoulos M 2008 Friedman-like Robertson–Walker model in generalized metric space-time with weak anisotropy Gen. Relativ. Gravit. 40 1403
[6] Stavrinos P C 2009 Gravitational and cosmological considerations based on the Finsler and Lagrange metric structures Nonlinear Anal. 71 e1380–92
[7] Koulentesis A P, Stathakopoulos M and Stavrinos P C 2010 General very special relativity in Finsler cosmology Phys. Rev. D 79 104011
[8] Vacaru S I 2010 New classes of off-diagonal cosmological solutions in Einstein gravity Int. J. Theor. Phys. 49 2753
[9] Kostelecký V A 2011 Riemann–Finsler geometry and Lorentz-violating kinematics Phys. Lett. B 701 137–43
[10] Pfeifer C and Wohlfarth M N R 2011 Causal structure and electrodynamics on Finsler spacetimes Phys. Rev. D 84 044039
[11] Skakala J and Visser M 2011 Bi-metric pseudo-Finslerian spacetimes J. Geom. Phys. 61 1396–400
[12] Vacaru S I 2011 On general solutions of Einstein equations Int. J. Geom. Methods Mod. Phys. 8 9
[13] Stavrinos P C 2012 Weak gravitational field in Finsler–Randers space and Raychaudhuri equation Gen. Relativ. Gravit. 44 3029–45
[14] Kostelecký V A, Russell N and Tso R 2012 Bipartite Riemann–Finsler geometry and Lorentz violation Phys. Lett. B 716 470–4
[15] Vacaru S I 2012 Principles of Einstein–Finsler gravity and perspectives in modern cosmology Int. J. Mod. Phys. 21 1250072
[16] Vacaru S I 2012 Modified dispersion relations in Hořava–Lifshitz gravity and Finsler brane models Gen. Relativ. Gravit. 44 1015–42
[17] Torrome R G, Piccione P and Vitório H 2012 On Fermat’s principle for causal curves in time oriented Finsler spacetimes J. Math. Phys. 53 123511
[18] Koulentesis A P, Stathakopoulos M and Stavrinos P C 2012 Covariant kinematics and gravitational bounce in Finsler space-times Phys. Rev. D 86 124025
[19] Pfeifer C and Wohlfarth M N R 2012 Finsler geometric extension of Einstein gravity Phys. Rev. D 85 064009
[20] Stavrinos P C and Vacaru S I 2013 Cyclic and ekpyrotic universes in modified Finsler oscillating gravity on tangent Lorentz bundles Class. Quantum Grav. 30 5
[21] Baslakos S and Stavrinos P C 2013 Cosmological equivalence between the Finsler–Randers space-time and the DGP gravity model Phys. Rev. D 87 043506
[22] Chang Z, Li M H and Wang S 2013 Finsler geometric perspective on the bulk flow in the universe Phys. Lett. B 723 257–60
[23] Lin L K 2013 A spherical symmetrical spacetime solution in Finsler gravity Int. J. Theor. Phys. 53 1271–5
[24] Minguzzi E 2014 Light cones in Finsler spacetime Commun. Math. Phys. 334 1529–51
[25] Stavrinos P C, Vacaru O and Vacaru S I 2014 Modified Einstein and Finsler like theories on tangent Lorentz bundles Int. J. Mod. Phys. D 23 1450094
[26] Foster J and Lehner R 2015 Classical-physics applications for Finsler b space Phys. Lett. B 746 164–70
[27] Fuster A and Pabst C 2016 Finsler pp-waves Phys. Rev. D 94 104072
[28] Voicu N 2017 Volume forms for time oriented Finsler spacetimes J. Geom. Phys. 112 85–94
[29] Hohmann M and Pfeiffer C 2017 Geodesics and the magnitude-redshift relation on cosmologically symmetric Finsler spacetimes Phys. Rev. D 95 104021
[30] Silva J E G, Maluf R V and Almeida C A S 2017 A nonlinear dynamics for the scalar field in Randers spacetime Phys. Lett. B 766 263–7
[31] Papagiannopoulos G, Basilakos S, Paliathanasis A, Savvidou S and Stavrinos P C 2017 Finsler–Randers cosmology: dynamical analysis and growth of matter perturbations Class. Quantum Grav. 34 225008
[32] Wang D and Meng X-H 2017 Weighing neutrinos in Finslerian cosmological models Phys. Rev. D 96 023538
[33] Stavrinos P C and Ikeda S 1999 Some connections and variational principle to the Finslerian scalar-tensor theory of gravitation Rep. Math. Phys. 44 221–30
[34] Chang Z and Li X 2008 Modified Newton’s gravity in Finsler space as a possible alternative to dark matter hypothesis Phys. Lett. B 668 453–6
[35] Chang Z and Li X 2009 Modified Friedmann model in Randers–Finsler space of approximate Berwald type as a possible alternative to dark energy hypothesis Phys. Lett. B 676 173–6
[36] Voicu N 2011 On the fundamental equations of electromagnetism in Finslerian spacetimes Prog. Electromagn. Res. 113 83
[37] Vacaru S I 2011 Finsler branes and quantum gravity phenomenology with Lorentz symmetry violations Class. Quantum Grav. 28 215991
[38] Basilakos S, Kouretsis A P, Saridakis E N and Stavrinos P C 2013 Resembling dark energy and modified gravity with Finsler–Randers cosmology Phys. Rev. D 88 123510
[39] Vacaru S I 2013 Decoupling of field equations in Einstein and modified gravity J. Phys.: Conf. Ser. 453 012021
[40] Caponio E and Stancarone G 2016 Standard static Finsler spacetimes
[41] Kostelecký V A and Russel N 2010 Classical kinematics for Lorentz violation Phys. Lett. B 693 443–7
[42] Ishikawa H 1981 Note on Finslerian relativity J. Math. Phys. 22 995
[43] Caponio E, Javaloyes M A and Sánchez M 2011 On the interplay between Lorentzian causality and Finsler metrics of Randers type Rev. Mat. Iberoamericana 27 919–52
[44] Kouretsis A P, Stathakopoulos M and Stavrinos P C 2010 Imperfect fluids, Lorentz violations and Finsler cosmology Phys. Rev. D 82 064035
[45] Bogoslovsky G Yu 1977 A special-relativistic theory of the locally anisotropic space-time Il Nuovo Cimento B 40 116–34
[46] Colladay D and Kostelecký V A 1997 CPT violation and the standard model Phys. Rev. D 55 6760
[47] Colladay D and Kostelecký V A 1998 Lorentz-violating extension of the standard model Phys. Rev. D 58 116002
[48] Cohen A G and Glashow S L 2006 Very special relativity Phys. Rev. Lett. 97 021601
[49] Kostelecký V A and Russel N 1999 CPT and Lorentz tests in hydrogen and antihydrogen Phys. Rev. Lett. 82 2254
[50] Bogoslovsky G Yu 2007 Some physical displays of the space anisotropy relevant to the feasibility of its being detected at a laboratory (arXiv:0706.2621)
[51] Kostelecký V A and Russel N 2011 Data tables for Lorentz and CPT violation Rev. Mod. Phys. 83 11
[52] Minguzzi E 2015 Raychaudhuri equation and singularity theorems in Finsler spacetimes Class. Quantum Grav. 32 185008
[53] Singh T, Chaubey R and Singh A 2015 Bounce conditions for FRW models in modified gravity theories Eur. Phys. J. Plus 130 31
[54] Stavrinos P C and Alexiou M 2018 Raychaudhuri equation in the Finsler–Randers spacetime and generalized scalar-tensor theories Int. J. Geom. Methods Mod. Phys. 15 1850039
[55] Mohseni M and Fathi M 2016 Focusing of world-lines in Weyl gravity Eur. Phys. J. Plus 131 21
[56] Thompson R T and Fathi M 2017 Covariant kinematics of light in media and a generalized Raychaudhuri equation Phys. Rev. D 96 105006
[57] Miron R, Watanabe S and Ikeda S 1987 Some connections on tangent bundle and their applications to general relativity Tensor NS 46 8–22
[58] Vacaru S, Stavrinos P, Gaburov E and Gonja D 2006 Clifford and Riemann–Finsler Structures in Geometric Mechanics and Gravity (Bucharest: Geometry Balkan Press)
[59] Bucataru I and Miron R 2007 Finsler–Lagrange Geometry (Bucharest: Editura Academiei Romane)
[60] Kandatu A 1966 Tangent bundle of a manifold with a non-linear connection Kodai Math. Sem. Rep. 18 259–70
[61] Crasmareanu M 2012 Nonlinear connections for conformal gauge theories on path-spaces and duality Publicationes Math. Debrecen 81 167–77
[62] Balan V and Stavrinos P C 2001 Weak gravitational fields in generalized metric spaces Geom. Balkan Press Proc. 6 27–37
[63] Balan V and Stavrinos P C 2004 Weak linearized gravitational models based on Finslerian \((\alpha, \beta)\) metrics Geom. Balkan Press Proc. 10 39–52
[64] Kostelecký V A 2004 Gravity, Lorentz violation, and the standard model Phys. Rev. D 69 105009
[65] Carroll S M 2004 Spacetime and Geometry: an Introduction to General Relativity (San Francisco, CA: Addison-Wesley)
[66] Stavrinos P C and Ikeda S 2000 Variational principle to the generalized scalar U tensor theory of gravitation II. Rev Bull. Calcutta Math. Soc. 8 1–2
[67] Ikeda S 1990 Some remarks on the Lagrangian theory of electromagnetism Tensor NS 49 204–8
[68] Kouretsis A P 2014 Cosmic magnetization in curved and Lorentz violating space-times Eur. Phys. J. C 74 2879