BIFURCATION SOLUTIONS OF GROSS-PITAEVSKII EQUATIONS FOR SPIN-1 BOSE-EINSTEIN CONDENSATES

DONG DENG
Department of Mathematics
Sichuan University
Chengdu, Sichuan 610064, China

RUIKUAN LIU*
College of Science
Southwest Petroleum University
Chengdu, Sichuan 610500, China

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Abstract. The main aim of this paper is to study the bifurcation solutions associated with the spinor Bose-Einstein condensates. Based on the Principle of Hamilton Dynamics and the Principle of Lagrangian Dynamics, a general pattern formation equation for the spinor Bose-Einstein condensates is established. Moreover, three kinds of critical conditions for eigenvalues are obtained under spectrum analysis and the different external confining potentials. With the change of different external potentials, the different topological structures of bifurcation solutions for the spinor Bose-Einstein condensates system are derived from steady state bifurcation theory.

1. Introduction. It is well known that phase transition is a universal phenomena of Nature. The central problem in statistical physics and in nonlinear sciences is on phase transitions. A phase transition refers to transitions of the system from one state to another, as the control parameter crosses certain critical threshold. Quantum phase transition (QPT) is an important topic in condensed matter physics. The QPT is a phase transition between different quantum states by varying some control parameters, such as magnetic field or pressure. Note that Bose-Einstein condensation (BEC) related to QPT in cold atomic gases was originally hypothesized by Bose and Einstein in 1924, although the first BEC was only produced experimentally in 1995, winning the researchers Cornell, Ketterle and Wieman a Nobel Prize [1, 9]. Since then, more quantum phase transitions were discovered including the ferromagnetic transition in 1996 [3], the experiment on ultra cold atoms in optical lattices in 2001 [13], and the experiment on crystals of $CoNb_2O_6$ by Coldea and collaborators in 2010 [7]. Other various interesting experiments related to BEC we refer to [4, 8, 13, 39] and references therein. In these experiments, a large number of (bosonic) atoms are confined to a trap and cooled to very low temperatures. Below

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* Corresponding author: Ruikuan Liu.
a critical temperature condensation of a large fraction of particles into the same one-particle state occurs.

In classical statistical physics, there is no clear definition on the QPT. However, it is worth mentioning that the QPT was considered as the phase transition in an energy conservation quantum system in [21], a general scalar QPT model was established basing on the Principle of Hamilton Dynamics (PHD) and the Principle of Lagrangian Dynamics (PLD), and this new model have been used to describing the Bose-Einstein condensate (BEC). Recently, Ma and Wang [30] provided another systematic theoretical study on quantum phase transitions associated with the Bose-Einstein condensates, the superfluidity and the superconductivity. The topological structure equations for QPT and these transitions in the physical space for the quantum systems have been obtained in [22, 30].

Note that the Gross-Pitaevskii equation (GPE) in [33] provides a relatively good description of the behavior of scalar BEC when the interaction between bosons is ignored. Also, the Hamiltonian energy for such a BEC system with $J = 1$ is derived by T.-L. Ho [14], T. Ohmi and K. Machida [31]. These two quantum systems are receiving an increasing attention from many researchers in the past several decades, see [2, 5, 6, 8, 10, 11, 16, 17, 19, 20, 21, 22, 33, 34, 35, 38, 39]. More recently, Luckins and Van Gorder [19] studied stationary and quasi-stationary solutions for the cubic-quintic Gross-Pitaevskii equation modeling Bose-Einstein condensates in one, two, and three spatial dimensions under the assumption of radial symmetry with the BEC dynamics influenced by a confining potential.

In nature, it is well-known that there are two different kinds of phase transitions including the dynamical phase transitions, and topological phase transitions (TPTs), also called the pattern formation transitions, for the details we refer to [22, 30]. The rigorous theory of dynamical phase transitions in the dissipative system goes back to the pioneering work established by Ma and Wang in [28], many interesting results related to dynamical phase transitions refer to [12, 18, 22, 23, 24, 25, 26, 27, 28, 36, 37]. For the rigorous mathematical theory associated with topological phase transitions, we refer to [22] and the series works for TPTs in [30]. To the best of our knowledge, typical examples of TPTs is originated from the pioneering work by Kosterlitz and Thouless in [15], where they identified a completely new type of phase transitions in two-dimensional systems where topological defects play a crucial role. With this work, they received 2016 Nobel prize in physics.

In order to better understand QPT related to spinor Bose-Einstein condensates (sBEC), what we are concerned in this paper is to study bifurcation solutions for the Bose-Einstein condensate system with the spinor $J = 1$. First, we establish a pattern formation equation related to sBEC with the Hamilton energy [14, 17, 31] basing on two fundamental physical principles, which are Principle of Hamilton Dynamics (PHD) and Principle of Lagrangian Dynamics (PLD) in [29]. Furthermore, we get three kinds of critical conditions related to eigenvalues basing on the spectrum analysis and three kinds of confining potentials. In addition, we derive the topological structure of bifurcation solution related to sBEC due to steady state bifurcation theory.

The rest of this article is organized as follows. Section 2 introduces two fundamental physical principles and the Lyapunov-Schmidt procedures. The pattern formation equations with the Bose-Einstein condensate system with the spinor $J = 1$ are given in section 3. The eigenvalues for the linear pattern formation equations...
and critical conditions are derived in section 4. Section 5 devotes to give the main results and also to prove them.

2. Preliminaries.

2.1. Fundamental principles of physical system. In classical mechanics, a physical motion system can be described by three dynamic principles: the Newtonian Dynamics, the Lagrange Dynamics, and the Hamiltonian Dynamics. Note that both the Lagrange Dynamics and the Hamiltonian Dynamics remain valid in the quantum physics. In the following, we firstly state the two fundamental Principles—the Principle of Lagrangian Dynamics and the Principle of Hamiltonian Dynamics.

Lemma 2.1. (Principle of Lagrangian Dynamics [29]) For a physical motion system, there are functions

\[ u = (u_1, u_2, \cdots, u_N), \]

which describe the states of this system, and there exists a functional of \( u \), given by

\[ L(u) = \int_{\Omega} \mathcal{L}(u, Du, \cdots, D^m u) dx, \]

then the state functions of this system satisfy the variational equation as follows:

\[ \delta L(u) = 0. \]

The functional \( L \) is called the Lagrange action, and \( \mathcal{L} \) is called the Lagrange density.

Lemma 2.2. (Principle of Hamilton Dynamics [29]) For any conservation physical system, there are two sets of state functions

\[ u = (u_1, \cdots, u_N), \text{ and } v = (v_1, \cdots, v_N), \]

such the energy density \( \mathcal{H} \) is a functional of \( u \) and \( v \):

\[ \mathcal{H} = \mathcal{H}(u, v, \cdots, D^m u, D^m v), \quad m \geq 0. \]

The Hamiltonian (total energy) of the system is

\[ H(u, v) = \int_{\Omega} \mathcal{H}(u, v, \cdots, D^m u, D^m v) dx, \quad \Omega \subset \mathbb{R}^3, \]

provided that the system is described by continuous fields. Moreover, the state functions \( u \) and \( v \) satisfy the equations

\[ \frac{\partial u}{\partial t} = \alpha \frac{\delta H}{\delta v}, \]
\[ \frac{\partial v}{\partial t} = -\alpha \frac{\delta H}{\delta u}, \]

where \( \alpha \) is a constant.

2.2. Lyapunov-Schmidt procedures. In order to obtain the bifurcation solutions, we need to introduce the reduction procedures, which have been established by Ma and Wang in [23].

Let \( X_1 \) and \( X \) be two Hilbert spaces, and \( X_1 \subset X \) be a dense and compact inclusion. Consider a parameter family of nonlinear operator equations

\[ L_{\lambda} u + G(u, \lambda) = 0, \quad (1) \]

where \( L_{\lambda} : X_1 \rightarrow X \) is a completely continuous field, and

\[ G(u, \lambda) = o(\|u\|) \quad (2) \]
is a $C^r$ mapping depending on the parameter $\lambda$.

Let the eigenvalues (counting multiplicity) of $L_\lambda$ be given by $\{\beta_1(\lambda), \beta_2(\lambda), \cdots\}$ with $\beta_i(\lambda) \in \mathbb{R}^k (1 \leq i \leq m)$ such that

$$
\beta_i(\lambda) \begin{cases} < 0, & \text{if } \lambda < \lambda_0, \\
= 0, & \text{if } \lambda = \lambda_0, \forall 1 \leq i \leq m, \\
> 0, & \text{if } \lambda > \lambda_0, 
\end{cases}
$$

(3)

$$
\beta_j(\lambda_0) \neq 0, \text{ for } \forall j \geq m + 1.
$$

(4)

According to the spectral decomposition theory in [23], let $\{\omega_1(\lambda), \omega_2(\lambda), \cdots\}$ be the generalized eigenvectors corresponding to $\{\beta_1(\lambda), \beta_2(\lambda), \cdots\}$, and $E_1^\lambda = \text{span}\{\omega_1(\lambda), \omega_2(\lambda), \cdots, \omega_n(\lambda)\}$, then near $\lambda = \lambda_0$ the space $X_1$ can be decomposed into the following direct sum

$$
X_1 = E_1^\lambda \oplus E_2^\lambda,
$$

(5)

where $E_2^\lambda$ is the complement of $E_1^\lambda$.

Now, we give the Lyapunov-Schmidt procedure. For any $\lambda$ near $\lambda_0$, the linear operator $L_\lambda$ can be decomposed into $L_\lambda = L_1^\lambda \oplus L_2^\lambda$ such that

$$
L_1^\lambda = L_\lambda |_{E_1^\lambda} : E_1^\lambda \rightarrow E_1^\lambda,
$$

$$
L_2^\lambda = L_\lambda |_{E_2^\lambda} : E_2^\lambda \rightarrow E_2^\lambda.
$$

Thus, near $\lambda_0$, (1) can be equivalently written as

$$
L_1^\lambda v_1 + P_1 G(v_1 + v_2, \lambda) = 0,
$$

(6)

$$
L_2^\lambda v_2 + P_2 G(v_1 + v_2, \lambda) = 0,
$$

(7)

where $P_1 : X \rightarrow E_1^\lambda$ and $P_2 : X \rightarrow E_2^\lambda$ are the canonical projections, $v = v_1 + v_2$, $v_1 \in E_1^\lambda$ and $v_2 \in E_2^\lambda$. From (4), $L_2^\lambda$ is a linear homeomorphism near $\lambda_0$. By the implicit function theorem in [23], there exists a solution $v_2 = f(v_1, \lambda)$ of (7), which is called center manifold function. Then bifurcation equation (1) degenerate to the following equation

$$
L_1^\lambda v_1 + P_1 G(v_1 + f(v_1, \lambda), \lambda) = 0.
$$

(8)

where $v_1 \in E_1^\lambda$ and $\dim E_1^\lambda = m < \infty$.

2.3. Definition of bifurcation. Finally, we shall give the definition of bifurcation for the steady state abstract equations basing on [23, 32].

Definition 2.3. We say that the equation (1) bifurcates from $(0, \lambda_0)$ a solution $(u_\lambda, \lambda) \in X \times \mathbb{R}$ if there exists a sequence of solutions $(u_n, \lambda_n)$ of (1) which satisfy

$$
\lim_{n \to \infty} \lambda_n = \lambda_0, \quad \lim_{n \to \infty} ||u_n|| = 0.
$$

3. Spinor Bose-Einstein condensates.

3.1. Invariant of Spinor condensate system. Let the wave function $\psi = (\psi_+, \psi_0, \psi_-)$ of a spinor condensate system be expressed as

$$
\psi = \zeta e^{i\varphi}, \quad \zeta = (\zeta_+, \zeta_0, \zeta_-), \quad \zeta = |\psi|.
$$

(9)

In subsection 3.1 of [30], Ma and Wang gave the physical meaning of $\zeta$ and $\varphi$: $\zeta^2$ represents the distribution density of particles in the spinor condensate system, $\frac{\hbar}{m} \nabla \varphi$ represents the velocity field of the flow of particles. Therefore, $\zeta$ and $\varphi$ describe the structure of spinor condensates in the physical space, and the equations of $\zeta$ and $\varphi$ are called pattern formation equations.
It is well known that the pattern formation equations for $\zeta$ and $\varphi$ can be derived from the field equations basing on the Lagrangian dynamic principles

$$i\hbar \frac{\partial \Psi}{\partial t} = \frac{\delta}{\delta \Psi^*} H(\Psi),$$

where $H(\Psi)$ is the Hamiltonian energy of the system, $\hbar$ is the Plank constant, $\Psi^*$ is the complex conjugation of $\Psi$.

On the one hand, we shall show the Hamiltonian system (10) is conserved, i.e.

$$\frac{d}{dt} H(t) = \delta \delta \Psi^1 H, \frac{d\Psi^1}{dt} + \delta \delta \Psi^2 H, \frac{d\Psi^2}{dt} = 0,$$

where $\langle \cdot, \cdot \rangle$ represents the inner product, $\delta$ is the variational operator.

On the other hand, we will show the number of particles is also conserved. For the Hamiltonian system (10), the number of particles $N = \langle \Psi, \Psi \rangle = \int_{\Omega} |\Psi^1|^2 + |\Psi^2|^2 \, dx$, and

$$\frac{\delta}{\delta \Psi^1} N = 2\Psi^1, \frac{\delta}{\delta \Psi^2} N = 2\Psi^2.$$

Observe that

$$\int_{\Omega} \Psi^1 \cdot \frac{\delta}{\delta \Psi^2} N \, dx = \int_{\Omega} \Psi^2 \cdot \frac{\delta}{\delta \Psi^1} N \, dx.$$

By the conservation quantity theorem in [29], it is easy to see that the number of particles is conserved.

Note that both the Hamiltonian system (10) and the number of particles are conserved. That is to say, the total Hamiltonian energy $H(t)$ and the number of particles $N$ are two invariants of spinor condensate system. Hence the solution of (10) must have the following expression:

$$\Psi = e^{-i\mu t/\hbar} \psi(x),$$

where $\psi$ is defined as (9), $\mu \in \mathbb{R}$ is the chemical potential, which is a constant and the more physical properties of chemical potential refer to [17, 22]. By (10) and (14), we can derive that

$$\mu \psi = \frac{\delta}{\delta \psi^*} H(\psi).$$

Consequently, we reduce the equation (10) to the steady state form (15), which is crucial to get the following pattern formation equations for sBEC.

3.2. Pattern formation equations for sBEC. 1. Ho-Ohmi-Machida energy for optical potential well spinor systems.

Atoms of such a dilute gas system are binding together by an electric potential. The Hamiltonian energy for such a BEC system with $J = 1$ is derived by T.-L. Ho [14], T. Ohmi and K. Machida [31] in the following form, which is $SO(3)$ spinor invariant:

$$\mathcal{H} = \int_{\Omega} \left[ \frac{\hbar^2}{2m} |\nabla \Psi|^2 + V|\Psi|^2 + \frac{g_1}{2} |\Psi|^4 + \frac{g_2}{2} |\Psi^1 \hat{F} \Psi|^2 \right] \, dx,$$
respectively, $\hbar$ is the Planck constant, $\hat{F}$ is given by as follows

$$F_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad F_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad F_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$ 

Hence $|\Psi^\dagger \hat{F} \Psi|^2$ in (16) can be written as

$$|\Psi^\dagger \hat{F} \Psi|^2 = |\Psi^\dagger F_1 \Psi|^2 + |\Psi^\dagger F_2 \Psi|^2 + |\Psi^\dagger F_3 \Psi|^2,$$

and

$$g_1 = \frac{4\pi \hbar^2 a_0 + 2a_1}{3}, \quad g_2 = \frac{4\pi \hbar^2 a_1 - a_0}{3}. \quad (17)$$

Here $a_0$, $a_1$ are the lengths of s-wave scattering for $J = 0$ and $J = 1$ respectively, and

$$g_2 \begin{cases} < 0 & \text{for ferromagnetic,} \\ > 0 & \text{for anti-ferromagnetic.} \end{cases} \quad (18)$$

It is easy to check that $g_1 > 0$, $g_1 + g_2 > 0$.

2. Spinor Bose-Einstein Condensate Model

From (10) and (16), we can get the following pattern formation equations

$$i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m} \Delta \Psi + V \Psi + g_1 |\Psi|^2 \Psi + g_2 (\Psi^\dagger \hat{F} \Psi) \cdot (F \Psi), \quad (19)$$

where $\Psi^\dagger$ is the conjugate of $\Psi$, $g_1$ and $g_2$ are shown as (17). Owing to the quantum system is energy conservation, the wave function should take the following form basing on the equivalent form (14) in subsection 3.1

$$\Psi = e^{-i\mu t/\hbar} \Phi = e^{-i\mu t/\hbar} (\varphi_+, \varphi_0, \varphi_-), \quad (20)$$

where $\mu$ is the chemical potential.

For simplicity, we take $\Phi = (\varphi_+, \varphi_0, \varphi_-)$ as a real-valued function. Then, together (19) with (20), we can derive the final pattern formation equations for sBEC as follows

\[
\begin{cases}
\frac{\hbar^2}{2m} \Delta \varphi_+ + (\alpha + \mu) \varphi_+ - g_1 |\Phi|^2 \varphi_+ - g_2 \varphi_- \varphi_0^2 = 0, \\
\frac{\hbar^2}{2m} \Delta \varphi_0 + (\beta + \mu) \varphi_0 - g_1 |\Phi|^2 \varphi_0 - 2g_2 \varphi_+ \varphi_- \varphi_0 = 0, \\
\frac{\hbar^2}{2m} \Delta \varphi_- + (\gamma + \mu) \varphi_- - g_1 |\Phi|^2 \varphi_- - g_2 \varphi_+ \varphi_0^2 = 0,
\end{cases}
\]

(21)

where $|\Phi|^2 = \varphi_+^2 + \varphi_0^2 + \varphi_-^2$, $g_1$ and $g_2$ are defined by (17).
In this paper, we focus on the final pattern formation equations (21) with Dirichlet boundary condition
\[ \varphi_+|_{\partial \Omega} = 0, \quad \varphi_0|_{\partial \Omega} = 0, \quad \varphi_-|_{\partial \Omega} = 0. \] (22)
In the following, we mainly study the bifurcation solutions for the problem (21) to (22).

4. Abstract forms and critical conditions.

4.1. Abstract operator forms. Now, we define the following function spaces:
\[ H_1 = H^2(\Omega, \mathbb{R}^3) \cap H^1_0(\Omega, \mathbb{R}^3), \]
\[ H = L^2(\Omega, \mathbb{R}^3). \]
Setting \( \Phi = (\varphi_+, \varphi_0, \varphi_-)^T \). Let the operator \( A_\lambda : H_1 \to H \) be defined by
\[
A\Phi = \begin{pmatrix}
-\frac{\hbar^2}{2m} \Delta & 0 & 0 \\
0 & -\frac{\hbar^2}{2m} \Delta & 0 \\
0 & 0 & -\frac{\hbar^2}{2m} \Delta
\end{pmatrix}
\begin{pmatrix}
\varphi_+ \\
\varphi_0 \\
\varphi_-
\end{pmatrix}. \tag{23}
\]
Let \( B \) be the linear operator represented by
\[
B\Phi = \begin{pmatrix}
\alpha + \mu & 0 & 0 \\
0 & \beta + \mu & 0 \\
0 & 0 & \gamma + \mu
\end{pmatrix}
\begin{pmatrix}
\varphi_+ \\
\varphi_0 \\
\varphi_-
\end{pmatrix}. \tag{24}
\]
Then the linear terms of the equations (21) are equivalent to the following form
\[
L_\lambda \Phi = (A + B) \Phi. \tag{25}
\]
Obviously, \( L_\lambda : H_1 \to H \) is a parameterized linear completely continuous field depending continuously on \( \alpha, \beta, \gamma \).

Next, let \( G : H_1 \to H \) be
\[
G(\Phi) = \begin{pmatrix}
G_1(\Phi) \\
G_2(\Phi) \\
G_3(\Phi)
\end{pmatrix} = \begin{pmatrix}
-g_1|\Phi|^2 \varphi_+ - g_2 \varphi_- \varphi_0 - g_2 (\varphi_+^2 + \varphi_0^2 - \varphi_-^2) \varphi_+ \\
-g_1|\Phi|^2 \varphi_0 - 2g_2 \varphi_+ \varphi_- \varphi_0 - g_2 (\varphi_+^2 + \varphi_0^2) \varphi_0 \\
-g_1|\Phi|^2 \varphi_- - g_2 \varphi_+ \varphi_0^2 - g_2 (\varphi_+^2 + \varphi_0^2 - \varphi_-^2) \varphi_-
\end{pmatrix}. \tag{26}
\]
It is not difficult to see that (27) represents the nonlinear terms of the equations (21).

Hence, we conclude the equivalently abstract formulation of the problem (21)–(22) as follows
\[
L_\lambda \Phi + G(\Phi) = 0. \tag{28}
\]
4.2. Eigenvalues and critical conditions.

4.2.1. Eigenvalues and Eigenvectors. Consider the following linearized eigenvalue equations of (28)

\[
\begin{aligned}
&\frac{\hbar^2}{2m} \Delta \varphi_+ + (\alpha + \mu) \varphi_+ = \eta \varphi_+, \\
&\frac{\hbar^2}{2m} \Delta \varphi_0 + (\beta + \mu) \varphi_0 = \eta \varphi_0, \\
&\frac{\hbar^2}{2m} \Delta \varphi_- + (\gamma + \mu) \varphi_- = \eta \varphi_-, \\
&\varphi_+|_{\partial\Omega} = \varphi_0|_{\partial\Omega} = \varphi_-|_{\partial\Omega} = 0.
\end{aligned}
\]

(29)

Let \( \lambda_k \) and \( e_k \) be the k-th eigenvalue and eigenvector of the Laplacian with the Dirichlet condition:

\[
\begin{aligned}
-\Delta e_k &= \lambda_k e_k, \\
e_k|_{\partial\Omega} &= 0.
\end{aligned}
\]

(30)

Let the matrix \( M_k \) be defined by

\[
M_k = \begin{pmatrix}
-\frac{\hbar^2}{2m} \lambda_k + \mu + \alpha & 0 & 0 \\
0 & -\frac{\hbar^2}{2m} \lambda_k + \mu + \beta & 0 \\
0 & 0 & -\frac{\hbar^2}{2m} \lambda_k + \mu + \gamma
\end{pmatrix}, \quad k \in \mathbb{N}^+.
\]

(31)

It is clear that \( M_k \) is the coefficient matrix of (29) and the eigenvalues of \( L_\lambda \) have the following expressions

\[
\begin{aligned}
\eta_{k+} &= -\frac{\hbar^2}{2m} \lambda_k + \mu + \alpha, \\
\eta_{k0} &= -\frac{\hbar^2}{2m} \lambda_k + \mu + \beta, \\
\eta_{k-} &= -\frac{\hbar^2}{2m} \lambda_k + \mu + \gamma,
\end{aligned}
\]

(32)

where \( \mu \) is the chemical potential, \( \alpha, \beta, \) and \( \gamma \) are control parameters. Then, the corresponding eigenvectors of the eigenvalues \( \eta_{k+}, \eta_{k0}, \eta_{k-} \) can be expressed as

\[
\begin{aligned}
w_{k+} &= \begin{pmatrix} e_k \\ 0 \\ 0 \end{pmatrix}, \\
w_{k0} &= \begin{pmatrix} 0 \\ e_k \\ 0 \end{pmatrix}, \\
w_{k-} &= \begin{pmatrix} 0 \\ 0 \\ e_k \end{pmatrix}.
\end{aligned}
\]

(33)

Let \( L_\lambda^* \) be the conjugate operator of \( L_\lambda \). Clearly, the eigenvalues of \( L_\lambda^* \) are \( \eta_{k+}, \eta_{k0}, \eta_{k-} \), which are the same to the eigenvalues of \( L_\lambda \). Then, the corresponding eigenvalues should be expressed as

\[
\begin{aligned}
w_{k+}^* &= w_{k+}, \\
w_{k0}^* &= w_{k0}, \\
w_{k-}^* &= w_{k0},
\end{aligned}
\]

(34)

which also satisfy

\[
\begin{aligned}
\langle w_{k+}, w_{k+}^* \rangle &= 1, \\
\langle w_{k0}, w_{k0}^* \rangle &= 1, \\
\langle w_{k-}, w_{k-}^* \rangle &= 1, \\
\langle w_{k+}, w_{m+}^* \rangle &= 0, \\
\langle w_{k0}, w_{m0}^* \rangle &= 0, \\
\langle w_{k-}, w_{m-}^* \rangle &= 0 (k \neq m),
\end{aligned}
\]

(35)
4.2.2. Critical conditions. Now, we consider the three kinds of critical conditions according to different control parameters.

Then, we introduce the critical numbers

$$\alpha_n^c = \frac{\hbar^2}{2m} \lambda_n - \mu, \quad n \in \mathbb{N}^+, \quad (36)$$

where $\lambda_n$ is given by (30).

**Case 1.** $\alpha = \beta = \gamma > 0$.

Note that $\eta_k = \eta_{k+} = \eta_{k0} = \eta_{k-} = -\frac{\hbar^2}{2m} \lambda_k + \mu + \alpha$, which implies that the bounding potential acting on $\psi_+, \psi_0, \psi_-$ are the same, i.e. the external potential field is well-distributed. Obviously, the following lemma holds ture.

**Lemma 4.1.** If $\alpha = \beta = \gamma > 0$ and $\alpha$ is the control parameter, then there must exists a positive integer $n$ such that

$$\eta_n = \begin{cases} < 0, & \alpha < \alpha_n^c, \\ = 0, & \alpha = \alpha_n^c, \\ > 0, & \alpha > \alpha_n^c, \end{cases} \quad (37)$$

and

$$\eta_k(\alpha_n^c) = -\frac{\hbar^2}{2m} \lambda_k + \frac{\hbar^2}{2m} \lambda_n \neq 0, \quad \text{for any } k \neq n.$$  

**Case 2.** $\alpha = \beta \neq \gamma$ ($\gamma > 0$).

It is easy to see that $\eta_{k+} = \eta_{k0} = -\frac{\hbar}{2m} + \mu + \alpha, \eta_{k-} = -\frac{\hbar}{2m} + \mu + \gamma$, which indicates that the external potential field is not well-distributed. By simple calculation, we can get the following lemma.

**Lemma 4.2.** Let $\alpha = \beta \neq \gamma$. If $\alpha$ is the control parameter, then there must exists a positive integer $n$ such that

$$\eta_{n+} = \eta_{n0} = \begin{cases} < 0, & \alpha < \alpha_n^c, \\ = 0, & \alpha = \alpha_n^c, \\ > 0, & \alpha > \alpha_n^c, \end{cases} \quad (38)$$

and

$$\eta_{k+}(\alpha_n^c) = \eta_{k0}(\alpha_n^c) \neq 0, \quad \text{for any } k \neq n.$$

and

$$\eta_{k-} \neq 0, \quad \text{for any } k \in \mathbb{N},$$

provided that $\gamma$ is a given constant.

**Case 3.** $\alpha \neq \beta \neq \gamma$.

Notice that $\eta_{k+}, \eta_{k0}$ and $\eta_{k-}$ are shown as (32). It is clear that $\alpha \neq \beta \neq \gamma$ indicates the external potential field is nonuniform. Analogously, we can obtain the following lemma.

**Lemma 4.3.** If $\alpha \neq \beta \neq \gamma$ and $\alpha$ is the control parameter, then there must exists a positive integer $n$ such that

$$\eta_{n+} = \begin{cases} < 0, & \alpha < \alpha_n^c, \\ = 0, & \alpha = \alpha_n^c, \\ > 0, & \alpha > \alpha_n^c, \end{cases} \quad (39)$$
and
\[ \eta_k (\alpha_n^c) \neq 0, \text{ for any } k \neq n, \]
and
\[ \eta_k \neq 0, \eta_k \neq 0, \text{ for any } k \in \mathbb{N}, \]
provided that \( \beta \) and \( \gamma \) are given constants.

5. Main results and proofs. In this section, we give the main theorems and also prove them. Hereafter, we always assume that the eigenvalue \( \eta_k^+ \), \( \eta_k^0 \) and \( \eta_k^- \) in (32) are simple.

For convenience, we denote
\[ \eta_n = \eta_{n^+} = \eta_{n^0} = \eta_{n^-} = -\frac{h^2}{2m} \lambda_n + \mu + \alpha, \]
where \( g_1 \) and \( g_2 \) are defined by (17), \( \lambda_n \) and \( e_n \) satisfy (30).

**Theorem 5.1.** Assume that \( \alpha = \beta = \gamma > 0 \), \( \alpha \) is the control parameter, and the \( k \)-th eigenvalue of Laplacian operator with multiplicity 1. Then the system (21) with (22) has a series of bifurcation points \( (\alpha_{k_n}, 0) \), where
\[ \alpha_{k_n} = \frac{h^2}{2m} \lambda_n - \mu(n = 1, 2, \cdots). \]
Furthermore, the following assertions hold true:

1. If \( \alpha < \alpha_{k_n} \), then the system (21) with (22) has no bifurcation solutions; If \( \alpha > \alpha_{k_n} \), then the system (21) with (22) bifurcates infinity non-trivial solutions at \( (\alpha_{k_n}, 0) \).

2. The bifurcation solutions set \( \Sigma \) can be expressed as
\[ \Sigma = \{ E^2, S^1, 10 \text{ singular points} \}, \]
where \( E^2 \) is the part of 2-dimensional sphere, \( S^1 \) is the ellipse.

**Proof.** According to the assumptions, it is easy to see that the Lemma 4.1 holds. Based on the spectral theory of linear completely continuous field [23, 28], the spaces \( H \) and \( H_1 \) can be decomposed into the following form
\[ H_1 = E_1 + E_2, \]
where
\[ E_1 = \text{span}\{ w_{n^+}, w_{n^0}, w_{n^-} \}, \quad E_2 = \text{span}\{ w_{k^+}, w_{k^0}, w_{k^-}, k \neq n \}. \]

Let \( \Phi = \Phi_1 + \Phi_2 \). Then \( \Phi_1 \) and \( \Phi_2 \) have the following expressions
\[ \Phi_1 = x_{n^+} w_{n^+} + x_{n^0} w_{n^0} + x_{n^-} w_{n^-} = \begin{pmatrix} x_{n^+} e_n \\ x_{n^0} e_n \\ x_{n^-} e_n \end{pmatrix} = \begin{pmatrix} \varphi^+ \\ \varphi^0 \\ \varphi^- \end{pmatrix}, \]
\[ \Phi_2 = \sum_{k \neq n} (x_{k^+} w_{k^+} + x_{k^0} w_{k^0} + x_{k^-} w_{k^-}) = \begin{pmatrix} \sum_{k \neq n} x_{k^+} e_k \\ \sum_{k \neq n} x_{k^0} e_k \\ \sum_{k \neq n} x_{k^-} e_k \end{pmatrix}. \]
Inserting (42) and (43) into (28), and making inner product with $w_{n+}^*, w_{n0}^*, w_{n-}^*$ respectively, we conclude the following reduction equations

\begin{align}
\eta_{n+} x_{n+} + \int_{\Omega} G_1(\Phi_1 + \Phi_2) e_n dx &= 0, \\
\eta_{n0} x_{n0} + \int_{\Omega} G_2(\Phi_1 + \Phi_2) e_n dx &= 0, \\
\eta_{n-} x_{n-} + \int_{\Omega} G_3(\Phi_1 + \Phi_2) e_n dx &= 0.
\end{align}

(44)

Direct calculation shows that $\Phi_2$ is the higher order term of $x_{n+}, x_{n0}$ and $x_{n-}$, so we can omit it. Consider the following equations

\begin{align}
\eta_{n+} x_{n+} + \int_{\Omega} G_1(\Phi_1) e_n dx &= 0, \\
\eta_{n0} x_{n0} + \int_{\Omega} G_2(\Phi_1) e_n dx &= 0, \\
\eta_{n-} x_{n-} + \int_{\Omega} G_3(\Phi_1) e_n dx &= 0.
\end{align}

(45)

Let $|\Phi|^2 = \varphi_+^2 + \varphi_0^2 + \varphi_-^2 = (x_{n+}^2 + x_{n0}^2 + x_{n-}^2) e_n^2$. Then, from (27), we can obtain

\begin{align*}
G_1(\Phi_1) &= -(g_1 + g_2) x_{n+}^3 e_n^3 - (g_1 + g_2) x_{n+} x_{n0}^2 e_n^3 \\
&\quad + (g_2 - g_1) x_{n+} x_{n-}^2 e_n^3 - g_2 x_n x_{n0}^2 e_n^3, \\
G_2(\Phi_1) &= -(g_1 + g_2) x_{n+}^2 x_{n0} e_n^3 - g_1 x_{n0}^3 e_n^3 - (g_1 + g_2) x_{n+} x_{n-} e_n^3 \\
&\quad + g_2 x_{n-} x_{n0} e_n^3 - 2 g_2 x_{n+} x_{n0} x_{n-} e_n^3, \\
G_3(\Phi_1) &= (g_2 - g_1) x_{n+} x_{n-} e_n^3 - (g_1 + g_2) x_{n+} x_{n-} e_n^3 \\
&\quad + (g_2 - g_1) x_{n+} x_{n0} e_n^3 - g_2 x_{n+} x_{n0} e_n^3.
\end{align*}

It follow from (46) that (45) can be rewritten as

\begin{equation}
\begin{aligned}
\eta_{n+} x_{n+} &= \int_{\Omega} e_n^4 dx [(g_1 + g_2) x_{n+}^3 + (g_1 + g_2) x_{n+} x_{n0}^2 \\
&\quad - (g_2 - g_1) x_{n+} x_{n-}^2 + g_2 x_{n+} x_{n0}^2] + o(3) = 0, \\
\eta_{n0} x_{n0} &= \int_{\Omega} e_n^4 dx [(g_1 + g_2) x_{n+}^3 x_{n0} + g_1 x_{n0}^3 \\
&\quad + (g_1 + g_2) x_{n+} x_{n0} - 2 g_2 x_{n+} x_{n0} x_{n-}] + o(3) = 0, \\
\eta_{n-} x_{n-} &= \int_{\Omega} e_n^4 dx [(g_1 + g_2) x_{n0}^3 x_{n-} \\
&\quad - (g_2 - g_1) x_{n+}^2 x_{n-} + (g_1 + g_2) x_{n-}^3 + g_2 x_{n+} x_{n0}^2] + o(3) = 0.
\end{aligned}
\end{equation}

(47)

For convenience, we denote $x = x_{n+}, y = x_{n0}, z = x_{n-}, D = B - A$, then (47) can be expressed as

\begin{align}
\eta_n x - C x^3 - C x y^2 + D x z^2 - B y^2 z &= 0, \\
\eta_n y - C x^2 y - A y^3 - C y z^2 - 2 B x y z &= 0, \\
\eta_n z + D x^2 z - C y^2 z - C z^3 - B x y^2 &= 0,
\end{align}

(48)(49)(50)

where $\eta_n, A, B$ and $C$ are given by (40).

It is easy to check that (49) is equivalent to

\[ y(\eta_n - C x^2 - A y^2 - C z^2 - 2 B x z) = 0, \]

which implies that

\[ y = 0 \quad \text{or} \quad y^2 = \frac{1}{A}(\eta_n - C x^2 - A z^2 - 2 B x z). \]
(i) If \( y = 0 \), then we derive from (48) and (50) that

\[
\begin{align*}
\eta_n x - Cx^3 + Dxz^2 &= 0, \\
\eta_n z + Dx^2 z - Cz^3 &= 0.
\end{align*}
\]

(51)

It is readily to check that

\[
\begin{align*}
x = 0, \quad z &= 0, \\
x = 0, \quad z^2 &= \frac{\eta_n}{C},
\end{align*}
\]

or

\[
\begin{align*}
z = 0, \quad x &= 0, \\
z = 0, \quad x^2 &= \frac{\eta_n}{C},
\end{align*}
\]

Hence, there are 8 bifurcation solutions

\[
\begin{align*}
\tilde{\Phi}_1^\pm &= \pm \sqrt{\frac{\eta_n}{C}} w_{n+} + o(|\eta_n|^{\frac{1}{2}}), \\
\tilde{\Phi}_2^\pm &= \pm \sqrt{\frac{\eta_n}{C}} w_{n-} + o(|\eta_n|^{\frac{1}{2}}), \\
\tilde{\Phi}_3^\pm &= \pm \sqrt{\frac{\eta_n}{C}} w_{n+} + \sqrt{\frac{\eta_n}{C}} w_{n-} + o(|\eta_n|^{\frac{1}{2}}), \\
\tilde{\Phi}_4^\pm &= \pm \sqrt{\frac{\eta_n}{C}} w_{n+} - \sqrt{\frac{\eta_n}{C}} w_{n-} + o(|\eta_n|^{\frac{1}{2}}).
\end{align*}
\]

(ii) If \( y \neq 0 \), then we can obtain the following equivalent equation by inserting \( y^2 = \frac{1}{A}(\eta_n - Cx^2 - Az^2 - 2Bxz) \) into (48) or (50),

\[(x + z)(-\eta_n + C(x + z)^2) = 0.
\]

Hence, we have

\[
\begin{align*}
x &= z = 0, \\
y &= \pm \sqrt{\frac{\eta_n}{A}}
\end{align*}
\]

(52)

or

\[
\begin{align*}
x &= -z, \\
2Ax^2 + Ay^2 &= \eta_n \quad (xyz \neq 0),
\end{align*}
\]

(53)

or

\[
\begin{align*}
x^2 + y^2 + z^2 &= \frac{\eta_n}{C}, \\
xz > 0, \quad y \neq 0.
\end{align*}
\]

(54)

Note that (52) implies the other two bifurcation solutions, (53) indicates the bifurcation solutions are \( S^1 \), and it follows from (54) that the bifurcation solutions are the part of 2-dimensional sphere \( S^2 \), which complete the proof.

\[
\text{Figure 1. The graph of the part of } S^2.
\]
Remark 1. If we take the bifurcation solutions $\Phi$ as
$$\Phi = x w_{n+} + y w_{n0} + z w_{n-} + o(\eta_n^{1/2}),$$
then the coefficients $x, y, z$ satisfy
$$\begin{cases} 
  x^2 + y^2 + z^2 = \frac{\eta_n}{C}, \\
  xz > 0, \quad y \neq 0,
\end{cases}$$
which implies the topological structure $E^2$ of bifurcation solutions is the part of 2-dimensional sphere, and is given by Figure 1. Also, the coefficients $x, y, z$ satisfy
$$\begin{cases} 
  x = -z, \\
  2Ax^2 + Ay^2 = \eta_n (xyz \neq 0),
\end{cases}$$
which indicates the topological structure $S^1$ is the ellipse shown in Figure 2. In addition, the 10 singular points can be expressed as follows
$$\begin{align*}
  \Phi^+_1 &= \pm \sqrt{\frac{\eta_n}{C}} w_{n+} + o(\eta_n^{1/2}), \\
  \Phi^+_2 &= \pm \sqrt{\frac{\eta_n}{A}} w_{n0} + o(\eta_n^{1/2}), \\
  \Phi^+_3 &= \pm \sqrt{\frac{\eta_n}{C}} w_{n-} + o(\eta_n^{1/2}), \\
  \Phi^+_4 &= \pm \sqrt{\frac{\eta_n}{C}} w_{n+} + \sqrt{\frac{\eta_n}{C}} w_{n-} + o(\eta_n^{1/2}), \\
  \Phi^+_5 &= \pm \sqrt{\frac{\eta_n}{C}} w_{n+} - \sqrt{\frac{\eta_n}{C}} w_{n-} + o(\eta_n^{1/2}),
\end{align*}$$
where $\alpha_n^c$ is defined as (36), $w_{n+}$, $w_{n0}$, $w_{n-}$ are given by (33), and $o(\cdot)$ is the higher-order term.

Remark 2. Suppose that $\alpha = \beta = \gamma > 0$. If we regard $\beta$ or $\gamma$ as the control parameter and the k-th eigenvalue of Laplacian operator with multiplicity 1, then we will get the similar results.

Theorem 5.2. Assume that $\alpha = \beta \neq \gamma$, $\alpha$ is the control parameter, $\gamma$ is a given positive constant, and the k-th eigenvalue of Laplacian operator with multiplicity 1. Then the system (21) with (22) has a series of bifurcation points $(\alpha_n^{c}, 0)$, where $\alpha_n^{c} = \frac{\hbar^2}{2m} \lambda_n - \mu(n = 1, 2, \cdots)$. Furthermore, the following assertions hold true:
(1) If $\alpha < \alpha_n^c$, then the system (21) with (22) has no bifurcation solutions; If $\alpha > \alpha_n^c$, then the system (21) with (22) bifurcates four non-trivial solutions at $(\alpha_n^c, 0)$.

(2) The four bifurcation solutions can be expressed as

$$\Phi_1^\pm = \pm \sqrt{\frac{\eta_n}{C}} w_{n_+} + o(|\eta_n|^{1/2}),$$

$$\Phi_2^\pm = \pm \sqrt{\frac{\eta_n}{A}} w_{n_0} + o(|\eta_n|^{1/2}),$$

where $\alpha_n^c$ is defined as (36), $w_{n_+}$, $w_{n_0}$ are given by (33), and $o(\cdot)$ is the higher-order term.

Proof. According to the assumptions, it is clear that the Lemma 4.2 holds. Analogously, the spaces $H$ and $H_1$ can be decomposed into the following form

$$H_1 = E_1 \oplus E_2, \quad H = E_1 \oplus E_2,$$

where

$$E_1 = \text{span}\{w_{n_+}, w_{n_0}\}, \quad E_2 = \text{span}\{w_{k_+}, \forall k \in \mathbb{N}; w_{k_0}, \forall k \neq n\}.$$

Let

$$\tilde{\Phi} = \Phi_1 + \Phi_2, \quad \Phi_1 \in E_1, \quad \Phi_2 \in E_2, \quad (55)$$

where

$$\Phi_1 = x_{n_+} w_{n_+} + x_{n_0} w_{n_0},$$

$$\Phi_2 = \sum_{k \neq n} (x_{k_+} w_{k_+} + x_{k_0} w_{k_0}) + \sum_{k \in \mathbb{N}} z_{k_-} w_{k_-}.$$

Inserting (55) into (28), and making inner product with $w_{n_+}^*, w_{n_0}^*$ respectively, we deduce that

$$\begin{cases}
\eta_{n_+} x_{n_+} + \int_\Omega G_1(\Phi_1 + \Phi_2; \lambda) e_n dx = 0, \\
\eta_{n_0} x_{n_0} + \int_\Omega G_2(\Phi_1 + \Phi_2; \lambda) e_n dx = 0.
\end{cases} \quad (56)$$

Furthermore, we need to calculate $\Phi_2$. Taking inner product (28) with $w_{k_+}^*(k \neq n)$, $w_{k_0}^*(k \neq n)$ and $w_{k_-}^*(k \in \mathbb{N})$ respectively, we can get that

$$\begin{cases}
\eta_{k_+} x_{n_+} + \int_\Omega G_1(x_{n_+} w_{n_+} + x_{n_0} w_{n_0}) e_n dx = 0, \forall k \neq n, \\
\eta_{k_0} x_{n_0} + \int_\Omega G_2(x_{n_+} w_{n_+} + x_{n_0} w_{n_0}) e_n dx = 0, \forall k \neq n, \\
\eta_{k_-} x_{k_-} + \int_\Omega G_3(x_{n_+} w_{n_+} + x_{n_-} w_{n_-}) e_n dx = 0, \forall k \in \mathbb{N}.
\end{cases} \quad (57)$$

Since

$$x_{n_+} w_{n_+} + x_{n_0} w_{n_0} = \begin{pmatrix} x_{n_+} e_n \\ x_{n_0} e_n \\ 0 \end{pmatrix},$$

which implies that

$$\varphi_+ = x_{n_+} e_n, \quad \varphi_0 = x_{n_0} e_n, \quad \varphi_- = 0.$$

Then

$$|\Phi|^2 = \varphi_+^2 + \varphi_0^2 + \varphi_-^2 = x_{n_+}^2 e_n^2 + x_{n_0}^2 e_n^2.$$ 

By (27), we find that

$$\begin{align}
G_1(x_{n_+} w_{n_+} + x_{n_0} w_{n_0}) &= -(g_1 + g_2)x_{n_+}^3 e_n^3 - (g_1 + g_2) x_{n_0} x_{n_+} e_n^3, \\
G_2(x_{n_+} w_{n_+} + x_{n_0} w_{n_0}) &= -(g_1 + g_2)x_{n_+}^2 x_{n_0} e_n^3 - g_1 x_{n_0}^3 e_n^3, \\
G_3(x_{n_+} w_{n_+} + x_{n_0} w_{n_0}) &= -g_2 x_{n_+}^2 x_{n_0} e_n^3.
\end{align} \quad (58)$$
Together (57) with (58), we derive that

\[ \eta_{k_+} x_{k_+} - \left[ (g_1 + g_2)x_{n_+}^3 - (g_1 + g_2)x_{n_+} x_{n_0}^2 \right] \int_{\Omega} e_n^3 e_k dx = 0, \]

\[ \eta_{k_0} x_{k_0} - \left[ (g_1 + g_2)x_{n_+} x_{n_0}^2 + g_1 x_{n_0}^3 \right] \int_{\Omega} e_n^3 e_k dx = 0, \]

\[ \eta_{k_-} x_{k_-} - g_2 x_{n_+} x_{n_0}^2 \int_{\Omega} e_n^3 e_k dx = 0, \]

which imply

\[ x_{k_+} = \left( \frac{(g_1 + g_2)x_{n_+}^3}{\eta_{k_+}} + \frac{(g_1 + g_2)x_{n_+} x_{n_0}^2}{\eta_{k_+}} \right) \int_{\Omega} e_n^3 e_k dx, \forall \ k \neq n, \]

\[ x_{k_0} = \left( \frac{(g_1 + g_2)x_{n_+} x_{n_0}^2}{\eta_{k_0}} + \frac{g_1 x_{n_0}^3}{\eta_{k_0}} \right) \int_{\Omega} e_n^3 e_k dx, \forall \ k \neq n \]

\[ z_{k_-} = \frac{g_2 \int_{\Omega} e_n^3 e_k dx}{\eta_{k_-}} x_{n_+} x_{n_0}^2, \forall \ k \in \mathbb{N}. \]

For simplicity, we denote

\[ A_k = \frac{g_1 + g_2 \int_{\Omega} e_n^3 e_k dx}{\eta_{k_+}}, \quad B_k = \frac{g_1 \int_{\Omega} e_n^3 e_k dx}{\eta_{k_0}}, \quad C_k = \frac{g_2 \int_{\Omega} e_n^3 e_k dx}{\eta_{k_-}}, \]

\[ \phi_1 = \sum_{k \neq n} A_k e_k, \quad \phi_2 = \sum_{k \neq n} B_k e_k, \quad \phi_3 = \sum_{k \in \mathbb{N}} C_k e_k. \]

Since \( \eta_{k_0} = \eta_{k_+} \), (60) can be rewritten as

\[ x_{k_+} = A_k x_{n_+}^3 + A_k x_{n_+} x_{n_0}^2, \]

\[ x_{k_0} = A_k x_{n_+} x_{n_0} + B_k x_{n_0}^3, \]

\[ z_{k_-} = C_k x_{n_+} x_{n_0}^2. \]

Thus,

\[ \Phi_2 = \sum_{k \neq n} x_{k_+} w_{k_+} + \sum_{k \neq n} x_{k_0} w_{k_0} + \sum_{k \neq n} z_{k_-} w_{k_-} \]

\[ = \left( \sum_{k \neq n} A_k e_k x_{n_+}^3 + \sum_{k \neq n} A_k e_k x_{n_+} x_{n_0}^2 \right) \left( \frac{\phi_1 x_{n_+}^3 + \phi_1 x_{n_+} x_{n_0}^2}{\phi_1 x_{n_+} x_{n_0} + \phi_2 x_{n_0}^3} \right) \]

\[ = \left( \sum_{k \neq n} A_k e_k x_{n_+} x_{n_0} + \sum_{k \neq k_0} B_k e_k x_{n_0}^3 \right) \left( \frac{\phi_3 x_{n_+} x_{n_0}^2}{\phi_3 x_{n_+} x_{n_0} + \phi_2 x_{n_0}^3} \right). \]

Note that

\[ \Phi_1 + \Phi_2 = \begin{pmatrix} x_{n_+} e_n + \phi_1 x_{n_+}^3 + \phi_1 x_{n_+} x_{n_0}^2 \\ x_{n_0} e_n + \phi_1 x_{n_0}^2 x_{n_0} + \phi_2 x_{n_0}^3 \\ \phi_3 x_{n_+} x_{n_0}^2 \end{pmatrix}. \]

Clearly, \( \Phi_2 \) is the higher-order of \( x_{n_+} \) and \( x_{n_0} \), so we can omit it. Hence, (56) can be rewritten as

\[ \left\{ \begin{array}{l} \eta_{n_+} x_{n_+} + \int_{\Omega} G_1(\Phi_1) e_n dx + o(3) = 0, \\
\eta_{n_0} x_{n_0} + \int_{\Omega} G_2(\Phi_1) e_n dx + o(3) = 0, \end{array} \right. \]
which is equivalent to
\[ \eta_{n+} x_{n+} - [(g_1 + g_2)x_{n+}^3 + (g_1 + g_2)x_{n+}^2x_n] \int_\Omega \xi_n^4 dx + o(3) = 0, \]
\[ \eta_{n0} x_{n0} - [(g_1 + g_2)x_{n+}^2x_{n0} - g_1x_{n0}^3] \int_\Omega \xi_n^4 dx + o(3) = 0. \]  

Obviously, (65) is equivalent to
\[
\begin{align*}
\eta_{n+} x_{n+} - Cx_{n+}^3 - Cx_{n0} x_{n+}^2 &= 0, \\
\eta_{n0} x_{n0} - Cx_{n0} x_{n0} - Ax_{n0}^3 &= 0,
\end{align*}
\]
where \( A \) and \( C \) are defined as (40). Taking \( \eta_{n+} = \eta_{n0} = \eta_n \), (66) can be rewritten as
\[
\begin{align*}
x_{n+}(\eta_n - Cx_{n+}^2 - Cx_{n0}^2) &= 0, \\
x_{n0}(\eta_n - Cx_{n0}^2 - Ax_{n0}^3) &= 0.
\end{align*}
\]

(1) If \( x_{n+} = 0 \), then \( x_{n0} = 0 \) or \( x_{n0}^2 = \eta_n A \).

(2) If \( x_{n0} = 0 \), then \( x_{n+} = \pm \sqrt{\frac{\eta_n}{C}} \).

(3) If \( x_{n+} \neq 0, x_{n0} \neq 0, \eta_n - Cx_{n+}^2 - Cx_{n0}^2 = 0 \) and \( \eta_n - Cx_{n0}^2 - Ax_{n0}^3 = 0 \), then by the similar analysis to (2), we can get the same conclusion of (2). Therefore, we obtain the following nontrivial coefficients of the bifurcation solutions
\[
\left(0, \pm \sqrt{\frac{\eta_n}{A}}, 0\right), \quad \left(\pm \sqrt{\frac{\eta_n}{C}}, 0, 0\right),
\]
which complete the proof.

**Remark 3.** Suppose that \( \alpha = \gamma \neq \beta \), or \( \beta = \gamma \neq \alpha \). If we regard \( \alpha \) or \( \beta \) as the control parameter and the \( k \)-th eigenvalue of Laplacian operator with multiplicity 1, then we will obtain the similar results.

**Theorem 5.3.** Assume that \( \alpha \neq \beta \neq \gamma \), \( \alpha \) is the control parameter, \( \beta, \gamma \) are two given positive constants, and the \( k \)-th eigenvalue of Laplacian operator with multiplicity 1. Then the system (21) with (22) has a series of bifurcation points \((\alpha_n^\epsilon, 0)\), where \( \alpha_n^\epsilon = \frac{\mu^2}{2m} \lambda_n - \mu(n = 1, 2, \cdots) \). Furthermore, the following assertions hold true:

(1) If \( \alpha < \alpha_n^{\epsilon} \), then the system (21) with (22) has no bifurcation solutions; If \( \alpha > \alpha_n^{\epsilon} \), then the system (21) with (22) bifurcates two non-trivial solutions at \((\alpha_n^{\epsilon}, 0)\).

(2) The two bifurcation solutions can be expressed as
\[
\Phi^\pm = \pm \sqrt{\frac{\eta_n}{C}} w_{n+} + o(|\eta_n|^{\frac{1}{2}}),
\]
where \( \alpha_n^{\epsilon} \) is defined as (36), \( w_{n+} \) is given by (33), and \( o(\cdot) \) is the higher-order term.

**Proof.** According to the assumptions, we can easy to see that the Lemma 4.3 holds. Analogously, the spaces \( \mathcal{H} \) and \( \mathcal{H}_1 \) can be decomposed into \( \mathcal{H}_1 = E_1 \oplus E_2, \mathcal{H} = E_1 \oplus \overline{E}_2 \), where
\[
E_1 = \text{span}\{w_{n+}\}, \quad E_2 = \text{span}\{w_{k+}, \forall k \neq n; \ w_{k0}, w_{k-} \ \forall k \in \mathbb{N}\}.
\]
Let \( \Phi = \Phi_1 + \Phi_2, \, \Phi_1 \in E_1, \, \Phi_2 \in E_2 \) (68)

Then \( \Phi_1 \) and \( \Phi_2 \) have the following expressions:

\[
\Phi_1 = x_{n_+} w_{n_+} \in E_1,
\]

\[
\Phi_2 = \sum_{k \neq n} x_{k_+} w_{k_+} + \sum_{k \in \mathbb{N}} y_{k_0} w_{k_0} + \sum_{k \in \mathbb{N}} z_{k_-} w_{k_-},
\]

Inserting (68) into (28) and making inner product with \( w_{n_+}^* \), we obtain the following reduction equation

\[
\eta_{n_+} x_{n_+} + \langle G(\Phi(x)), w_{n_+}^* \rangle = 0.
\]

Furthermore, making inner product with \( w_{n_+}^* (k \neq n) \), \( w_{k_0}^* (k \in \mathbb{N}) \), \( w_{k_-}^* (k \in \mathbb{N}) \) on both sides of (28) to get

\[
\begin{align*}
\eta_{n_+} x_{n_+} + \int_{\Omega} G_1(\Phi, \lambda) e_k dx &= 0, \quad \forall \, k \neq n, \\
\eta_{k_0} y_{k_0} + \int_{\Omega} G_2(\Phi, \lambda) e_k dx &= 0, \quad \forall \, k \in \mathbb{N}, \\
\eta_{k_-} z_{k_-} + \int_{\Omega} G_3(\Phi, \lambda) e_k dx &= 0, \quad \forall \, k \in \mathbb{N},
\end{align*}
\]

and

\[
\begin{align*}
\eta_{k_+} x_{k_+} + \int_{\Omega} G_1(\eta_{k_+} x_{n_+}, \lambda) e_k dx &= 0, \quad (71) \\
\eta_{k_0} y_{k_0} + \int_{\Omega} G_2(\eta_{k_+} x_{n_+}, \lambda) e_k dx &= 0, \quad (72) \\
\eta_{k_-} z_{k_-} + \int_{\Omega} G_3(\eta_{k_+} x_{n_+}, \lambda) e_k dx &= 0. \quad (73)
\end{align*}
\]

Since \( x_{n_+} w_{n_+} = \begin{pmatrix} x_{n_+} e_n \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \varphi_+ \\ \varphi_0 \\ \varphi_- \end{pmatrix} \). Then \( |\Phi|^2 = \varphi_+^2 + \varphi_0^2 + \varphi_-^2 = x_{n_+}^2 e_n^2 \).

Meanwhile, it follows from (27) that

\[
\begin{align*}
G_1(x_{n_+} w_{n_+}, \lambda) &= -(g_1 + g_2) e_n^2 x_{n_+}^3, \\
G_2(x_{n_+} w_{n_+}, \lambda) &= 0, \\
G_3(x_{n_+} w_{n_+}, \lambda) &= 0.
\end{align*}
\]

Combining (71) and (74), we have

\[
\eta_{k_+} x_{k_+} - (g_1 + g_2) \int_{\Omega} e_n^3 e_k dx x_{n_+}^3 = 0, \quad (75)
\]

which implies

\[
x_{k_+} = \frac{(g_1 + g_2) \int_{\Omega} e_n^3 e_k dx}{\eta_{k_+}} x_{n_+}^3. \quad (76)
\]

Since

\[
\Phi_2 = \sum_{k \neq n} x_{k_+} w_{k_+} = \begin{pmatrix} \sum_{k \neq n} x_{k_+} e_k \\ 0 \\ 0 \end{pmatrix}. \quad (77)
\]

Note that \( \Phi = x_{n_+} w_{n_+} + \Phi_2 \). Therefore, it follows from (77) that

\[
\int_{\Omega} G_1(\Phi, \lambda) e_n dx = -(g_1 + g_2) \int_{\Omega} e_n^4 dx x_{n_+}^3 + o(3). \quad (78)
\]
Combining (69) and (78), we get the reduction equation as follows
\[
\eta^n x_n + x_n^3 - (g_1 + g_2) \int_\Omega e^{4n} dx + o(3) = 0.
\]
(79)

It follows (79) that
\[
x_n^2 = \frac{\eta^n}{C},
\]
i.e. the nontrivial coefficients of the bifurcation solutions have the following expressions
\[
(\pm \sqrt{\frac{\eta^n}{C}}, 0, 0).
\]
which complete the proof.

**Remark 4.** Suppose that \(\alpha \neq \gamma \neq \beta\). If we regard \(\beta\) or \(\gamma\) as the control parameter and the k-th eigenvalue of Laplacian operator with multiplicity 1, then we will obtain the similar results.

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E-mail address: dd0328a@163.com
E-mail address: liuruikuan2008@163.com