Overparameterized (robust) models from computational constraints

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Abstract

Overparameterized models with millions of parameters have been hugely successful. In this work, we ask: can the need for large models be, at least in part, due to the computational limitations of the learner? Additionally, we ask, is this situation exacerbated for robust learning? We show that this indeed could be the case. We show learning tasks for which computationally bounded learners need significantly more model parameters than what information-theoretic learners need. Furthermore, we show that even more model parameters could be necessary for robust learning. In particular, for computationally bounded learners, we extend the recent result of Bubeck and Sellke [NeurIPS’2021] which shows that robust models might need more parameters, to the computational regime and show that bounded learners could provably need an even larger number of parameters. Then, we address the following related question: can we hope to remedy the situation for robust computationally bounded learning by restricting adversaries to also be computationally bounded for sake of obtaining models with fewer parameters? Here again, we show that this could be possible. Specifically, building on the work of Garg, Jha, Mahloujifar, and Mahmoody [ALT’2020], we demonstrate a learning task that can be learned efficiently and robustly against a computationally bounded attacker, while to be robust against an information-theoretic attacker requires the learner to utilize significantly more parameters.

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1 Introduction

The fundamental theory of learning theory characterizes the set of hypothesis functions that are learnable (e.g., to have finite VC dimension) and further shows that the simple “empirical risk minimization” can always succeed. In particular, in the above (realizable) setting, it is enough to output any \( f \in H \) that is consistent with the training set. This means that if \( H \) is a “simple” class, then at least information theoretically, there is no need to have a more complex model than what the hypothesis class \( H \) requires. More specifically, if all \( h \in H \) can be represented with \( \ell \) number of bits (also referred to as parameters), then the learner, (again information-theoretically) does not need more parameters in the output model \( f \) than what \( h \) needs.

What could cause overparameterized models? In recent years, deep neural nets with millions or even billions of parameters\(^1\) [Wortsman et al., 2022, Dai et al., 2021, Yu et al., 2022] have emerged as one of the most powerful models for very basic tasks such as image classification. A magic of DNNs is that they generalize without falling into the classical theories mentioned above, and hence they are the subject of an active line of work aiming to understand how DNNs generalize Arora et al. [2019], Allen-Zhu et al. [2019b,a], Novak et al. [2018], Neyshabur et al. [2014], Kawaguchi et al. [2017], Zhang et al. [2021], Arora et al. [2018b] and the various benefits of overparametrized regimes Xu et al. [2018], Chang et al. [2020], Arora et al. [2018a], Du et al. [2018], Lee et al. [2019]. In fact, the number of parameters in such models is so large that it is enough to memorize (and fit) to a large number of random labels [Zhang et al., 2021]. This leads us to our first main question, in which we investigate the potential cause for having large models:

Are there any learning tasks for which computationally bounded learners need to utilize significantly more model parameters than needed information-theoretically?

One should be cautious in how to formulate the question above. That is because, many simple (information-theoretically learnable) tasks are believed to be computationally hard to learn (e.g., learning parity with noise [Pietrzak, 2012]). In that case, one can interpret this as saying that the efficient learner requires infinite number of parameters, as a way of saying that the learning is not possible at all! However, as explained above, we are interested in understanding the reason behind needing a large number of model parameters when learning is possible.

Could robustness also be a cause for overparameterization? A highly sought-after property of machine learning models is their robustness to the so-called adversarial examples [Goodfellow et al., 2014]. Here we would like to find a model \( f \) such that \( f(x) = f(x') \) holds with high probability when \( x \leftarrow D \) is an honestly sampled instance and \( x' \approx x \) is a close instance that is perhaps minimally (yet adversarially) perturbed.\(^2\) A recent work of Bubeck and Sellke [2021] showed that having large model parameters could be due to the robustness of the model. Here, we are asking whether such phenomenon can have a computational variant that perhaps leads to needing even more parameters when the robust learner is running in polynomial time.

%\begin{align} % Are there any learning tasks for which computationally bounded robust learning comes at the cost of having even more model parameters than non-robust learning? \end{align}%

\(^1\) See https://paperswithcode.com/sota/image-classification-on-Imagenet for the size of the most successful models for image classification of Imagenet.

\(^2\) The closeness here could mean that a human cannot tell the difference between the two images \( x, x' \).
In fact, it was shown by Bubeck et al. [2019] and Degwaker et al. [Degwekar et al., 2019] that computational limitations of the learner could be the reason behind the vulnerability of models to adversarial examples. In this work, we ask whether the phenomenon investigated by the prior works is also crucial when the model size is considered.

**Can computational intractability of adversary help?** We ask whether natural restrictions on the adversary can help reduce model sizes. In particular, we consider the restriction to the class of polynomially bounded adversaries. In fact, when it comes to robust learning, the computationally bounded aspect could be imposed both on the learner as well as the adversary.

*Are there any learning tasks for which robust learning can be done with fewer model parameters when dealing with polynomial-time adversaries?*

Previously, it was shown that indeed working with computationally bounded adversaries can help achieving robust models [Mahloujifar and Mahmoody, 2019, Bubeck et al., 2019, Garg et al., 2020b]. Hence, we are asking a similar question in the context of model parameters and whether adversary’s limitation can help having smaller models.

### 1.1 Our results

In summary, we prove that under the computational assumption that one-way functions exist, the answer to all three of our main questions above is positive. Indeed, we show that the computational efficiency of the learner could be the cause of having overparametrized models. Furthermore, the computational efficiency of the adversary could reduce the size of the models. In particular, we prove the following theorem, in which a learning task is parameterized by $\lambda$, a hypothesis class $\mathcal{H}$ and a class of input distributions $\mathcal{D}$ (see Section 3.1 for formal definitions).

**Theorem 1** (Main results, informal). *If one-way functions exist, then for arbitrary polynomials $n < \alpha < \beta$ (e.g., $n = \lambda^{0.1}, \alpha = \lambda^5, \beta = \lambda^{10}$) over $\lambda$ the following hold.*

- **Part 1:** There is a learning task parameterized by $\lambda$ and a robustness bound (to limit how much an adversary can perturb the inputs) such that:

  - The instance size is $\Theta(n)$. (That is, the length of the input is $\Theta(n)$.)
  - There is a robust learner that uses $\Theta(\lambda)$ parameters.
  - Any polynomial-time learner needs $\Theta(\alpha)$ parameters to learn the task.
  - Any polynomial-time learner needs $\Theta(\beta)$ parameters to robustly learn the task.

- **Part 2:** There is a learning task parameterized by $\lambda$ and a robustness bound (to limit how much an adversary can perturb the inputs) such that:

  - The instance size is $\Theta(n)$.
  - When the adversary that generates the adversarial examples runs in polynomial time, there is an (efficient) learner that outputs a model with $O(1)$ parameters that robustly predicts the output labels with small error.
  - Against information-theoretic (computationally unbounded) adversaries that do the instance-perturbations, no learner can produce a model with $< \Theta(\alpha)$ parameters that later robustly make the predictions.

In Section 2, we present the high-level ideas behind the proofs of the two parts of Theorem 1. The formal constructions and proofs can be found in Section 4 and Section 5, respectively.
**Takeaway.** Here we put our work in perspective. As discussed above, prior works have considered the effect of computational efficiency (for both the learner and the attacker) on the robustness of the model. Informally, these works have shown that requiring a learner to be efficient hinders robustness, while requiring an attacker to be efficient helps achieve robustness. Our work studies the effect of computational efficiency as well but focuses on the number of parameters of the model. In spirit, we have shown a similar phenomenon. Namely, requiring a learner to be efficient increases the size of the model, while requiring an attacker to be efficient helps reduce the size of the model. Our results can be summarized as follows.

- In the non-robust case: requiring the learner to be efficient increases the size of the model.
- In the robust case:
  - Requiring the learner to be efficient increases the size of the model. This holds for both efficient and inefficient attackers.
  - Restricting to only computational efficient attackers reduces the size of the model. This holds for both efficient and inefficient learners.

**Limitations, Implications, and Open Question.** Our work shows that the phenomenon of having larger models due to computational efficiency could provably happen in certain scenarios. It does not imply, however, that this holds for all learning problems. It is a fascinating open question whether similar phenomena also happen to real-world problems. We note that this is not particular to our work, but common to most prior works in the theory of learning showing “separation” results [Degwekar et al., 2019, Mahloujifar and Mahmoody, 2019, Garg et al., 2020b].

Our results provides an explanation on why small but representative classes (e.g. 2 layers neural networks) of functions do not obtain same (robust) accuracy as larger models. This phenomenon that cannot be solely explained based on representation power of the function class might be due to computational limitations of the learning algorithm.

Finally, we note that our theorem demonstrates the separation by using the simplest setting of binary output. One can extend it to more sophisticated settings of any finite output, particularly real numbers with bounded precision. However, our work does not consider real numbers with infinite precision. In such cases, it is unclear what computational efficiency means as the learner needs to take an infinitely long input.

2 Technical overview

2.1 Efficient learner vs. information-theoretic learner

In this section, we explain the ideas behind the proof of Part 1 of Theorem 1. The formal construction and the theorems of this result are deferred to Section 4.

Consider the problem of learning an inner product function \( \text{IP}_P \) defined as \( \text{IP}_P(x) = \langle x, P \rangle \), where the inner product is done in \( \mathbb{GF}_2 \). Our first observation is that to learn \( \text{IP}_P \) with a small error where \( P \) is uniformly random, the number of model parameters that the learner employs must be as (almost) as large as the length of \( P \). Intuitively, one can argue it as follows. Let \( Z \) denote the parameters in the model that the learner outputs. Suppose that \( Z \) is shorter than \( P \). Then \( P \) must still be unpredictable given \( Z \).\(^3\) By a standard result in randomness extraction, one can argue that,

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\(^3\)That is, with high probability over the choice of \( Z = z \), the conditional distribution \( P|Z = z \) has high min-entropy. More formally, this unpredictability is measured by the average-case min-entropy (Definition 13).
for a uniform $x$, 
\[(Z, x, \langle x, P \rangle) \approx (Z, x, U_{\{0,1\}}).\]

That is, the label $\langle x, P \rangle$ looks information-theoretically uniform to the classifier who holds $Z$ and $x$. Therefore, the classifier can only output the correct label with (the trivial) probability $\approx 1/2$.

The conclusion is that learning all linear functions need a learner that outputs as many parameters as the function’s description is. However, even an efficient learner can perform the learning just as well as an information-theoretic one (e.g., using the Gaussian elimination). We now show how to modify this task to make a big difference between an efficient learner and an information-theoretic learner in terms of the number of parameters that they output.

**Computational perspective.** Now, suppose $P$ is computationally pseudorandom [Goldwasser and Micali, 1984] rather than being truly random.\(^4\) That is, let $f : \{0,1\}^\lambda \rightarrow \{0,1\}^\beta$ be a pseudorandom generator (Definition 5) and $P$ is distributed as $f(U_{\lambda})$, where $U_{\lambda}$ denotes the uniform distribution over $\lambda$ bits. Since (1) an efficient learner cannot distinguish $P \leftarrow f(U_{\lambda})$ from $P \leftarrow U_{\alpha}$ and (2) any learner who tries to learn $\text{IP}_P$ with $P \leftarrow U_{\alpha}$ needs $\alpha$ parameters, it can be proved that an efficient learner who tries to learn $\text{IP}_P$ with $P \leftarrow f(U_{\lambda})$ must also uses $\alpha$ parameters. However, an information-theoretic learner needs only $\lambda$ parameters to learn $\text{IP}_P$ with $P \leftarrow f(U_{\lambda})$ as it can essentially find the seed $s$ such that $P = f(s)$ and output the seed $s$. To conclude, to learn $\text{IP}_P$ for $P \leftarrow f(U_{\lambda})$, an information-theoretic learner only needs a few (i.e., $\lambda$) parameters and an efficient learner needs a lot of (i.e., $\alpha$) parameters. We emphasize that for a pseudorandom generator, its output length $\alpha$ could have an arbitrarily large polynomial dependence on its input length $\lambda$. For instance, it could be $\alpha = \lambda^{10}$.

**Robustness.** We now explain the ideas behind Theorem 22 in which we study the role of model robustness in the size of the model parameters. We now suppose the instance is sampled according to the distribution $D_Q$ (parametrized by a string $Q$), which is defined by 
\[
\text{Enc}(U) + Q.
\]

Here, $U$ is the uniform distribution (over the right number of bits), $\text{Enc}(U)$ is an error-correcting encoding of $U$, and the addition is coordinate-wise field addition. Moreover, the label for this instance is the inner product $\langle U, P \rangle$ for some vector $P$. Observe that if the learner learns $Q$, it can always find the correct label on a perturbed instance due to the error-correcting property of $\text{Enc}(U)$. We argue that, in order to robustly learn this task for a uniformly random $Q$, the number of parameters in the model that the learner outputs must be almost as large as the length of $Q$. Intuitively, the argument is as follows. Let $Z$ be the model that the learner outputs. Since $|Z| < |Q|$, then $Q$ must still be unpredictable given $Z$. In this case, we prove that 
\[
(Z, \text{Enc}(U) + Q + \rho) \approx (Z, U').
\]

Here, $\rho$ stands for the noise that the adversary adds to the instance. In words, the classifier holding the parameter $Z$ cannot distinguish the perturbed instance $\text{Enc}(U) + Q + \rho$ from a uniformly random string $U'$. Since $U$ is information-theoretically hidden to the classifier, it can only output the correct label $\langle U, P \rangle$ with probability $\approx 1/2$.

Next, to explore the difference between an efficient and information-theoretic learner, we consider the case where $Q$ is pseudorandomly distributed, i.e. $Q \leftarrow f(U_{\lambda})$ for some $f : \{0,1\}^\lambda \rightarrow \{0,1\}^\beta$.

\(^4\)A pseudorandom string is merely indistinguishable from a random one in eyes’ of computationally bounded distinguishers.
Again, since (1) an efficient learner cannot distinguish $Q \leftarrow U_{\beta}$ from $Q \leftarrow f(U_{\lambda})$ and (2) any learner needs at least $\approx |Q|$ parameters to learn the task with $Q \leftarrow U_{\beta}$, an efficient learner must also need at least $\approx |Q|$ parameters to learn the task. On the other hand, an information-theoretic learner could again find the seed $s$ such that $Q = f(s)$ and output the seed $s$ as the parameter. To conclude, an information-theoretic learner requires few (i.e., $\lambda$) parameters to robustly learn the task and an efficient learner needs a lot of (i.e., $\approx \beta$) parameters to robustly learn the task. (Again, $\beta$ could have an arbitrary polynomial dependence on $\lambda$.)

Making instances small. The learning tasks we considered above suffer from one drawback: the size of the instance is very large, or at least is related to the number of parameters of the model. Here we ask: is it possible to have a small instance size while an efficient learner still needs to output a very large model? We answer this question positively. In particular, for any $n = \text{poly}(\lambda)$ (e.g., $n = \lambda^{0.1}$), we construct a learning problem where the instance size is $\Theta(n)$ and the efficient learner still needs $\alpha$ (resp. $\beta$) parameters to learn (resp. robustly learn) the task. Our construction uses the “sampler” by Vadhan [2003]. Informally, a sampler $\text{samp}$ (see Theorem 12) needs a small seed $u$ and outputs a subset of $\{1,2,\ldots,\alpha\}$ of size $n$. The sampler comes with the guarantee that if $P$ is a source with high entropy, $P|_{\text{samp}(u)}$ also has high enough entropy. To illustrate the usage of the sampler, consider learning this new inner product function $IP_{\text{samp}}$ defined as

$$IP_{\text{samp}}(u,x) = \langle x, P|_{\text{samp}(u)} \rangle.$$  

For uniformly random $P$, one can similarly argue that a model must output at least $|P|$ parameters to learn the task. Let $Z$ denote the parameters in the model that the learner outputs. If $|Z| < |P|$, then $P$ contains high entropy conditioned on $Z$. By the property of the sampler, it must hold that $P|_{\text{samp}(u)}$ contains high entropy conditioned on $Z$ and $u$. Consequently,

$$\langle Z, u, x, IP_{\text{samp}}(u,x) \rangle \approx \langle Z, u, x, U_{\{0,1\}} \rangle.$$  

Namely, the classifier who sees the parameter $Z$ and the instance $(u,x)$ can only predict the label with probability $\approx 1/2$. Observe that the size of the instance $(u,x)$ is roughly $n$, while $P$ could have length $\alpha \gg n$. The use of the sampler in the robust learning case is similar to the non-robust case above. We refer the readers to Section 4 for more details.

2.2 Efficient adversary vs. information-theoretic adversary

In this section, we explain the ideas behind the proof of Part 2 of Theorem 1. The formal construction and the theorems of this result is deferred to Section 5.

We now explore whether the computational efficiency of the adversary could affect the number of parameters required to robustly learn a task. Previous work Garg et al. [2020a] considered the difference between an efficient adversary and an information-theoretic one in terms of whether a successful attack can be launched. We first recall their construction. The learning instance is sampled from the distribution

$$[\text{vk}], b, \text{Sign}(sk, b).$$  

Here, $(\text{vk}, sk)$ is the verification key and signing key pair of a signature scheme (Definition 7); $[\text{vk}]$ is an error-correcting encoding of $\text{vk}$, which ensures that $[\text{vk}]$ can always be recovered after perturbation by the adversary; $b$ is a uniform random bit and the label of the instance is simply $b$.

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5As the seed $u$ is very small.
6Every instance samples a fresh pair of verification key and signing key $(\text{vk}, sk)$. 

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The signature scheme ensures that an efficient adversary cannot forge a valid signature and, hence, any efficient adversary that perturbs \((b, \text{Sign}(sk, b))\) will result in an invalid message/signature pair. A classifier could then detect such perturbations and output a special symbol \(\bot\) indicating that perturbation is detected. On the other hand, an information-theoretic adversary could launch a successful attack by forging a valid signature \((1 − b, \text{Sign}(sk, 1 − b))\). Therefore, it could perturb the instance to be \([vk], 1 − b, \text{Sign}(sk, 1 − b)\) and, hence, flipping the label of the output.

**First idea.** We want to construct a learning problem such that the learner needs few parameters against an efficient adversary, but a lot of parameters against an information-theoretic adversary. Our first idea is to add another way of recovering \(b\) in the above learning problem. Consider the learning problem where the instance is sampled from distribution \(D_P\) defined as

\[
[vk], b, \text{Sign}(sk, b), [b + \langle u, P \rangle], [u].
\]

Here, \(u\) is uniformly distributed. Observe that if the learner learns \(P\), it can always recover \(b\) correctly from \([b + \langle u, P \rangle]\) and \([u]\) by invoking error-correction decoding. However, if the number of parameters in the model that the learner outputs have \(< |P|\) parameters, then \([b + \langle u, P \rangle]\) and \([u]\) information-theoretically hides \(b\). Again, this is because \(P\) is unpredictable given the parameter \(Z\)

\[Z, u, \langle u, P \rangle \approx Z, u, U_{\{0,1\}}.\]

Consequently, an information-theoretic adversary could again launch the attack that replace \(b, \text{Sign}(sk, b)\) with \(1 − b, \text{Sign}(sk, 1 − b)\) and successfully flipping the label. Therefore, a learner must employ \(|P|\) parameters to robustly learn the task against information-theoretic adversaries.

**Second idea.** In the above learning task, a learner employing \(|P|\) parameters can always recover the correct label against information-theoretic adversaries. However, against efficient adversaries, a learner with fewer parameters may not always recover the correct label but will sometimes output the special symbol \(\bot\) indicating that tampering is detected. Could we twist it to ensure that a learner with fewer parameters can also always recover the correct label against efficient adversaries? We positively answer this question by using list-decodable code (Definition 9). Intuitively, list-decoding ensures that given an erroneous codeword, the decoding algorithm will find the list of all messages whose encoding is close enough to the erroneous codeword.

Our new learning task has instances drawing from the distribution \(D_P\) defined as

\[
[vk], \text{LEnc}(b, \text{Sign}(sk, b)), [b + \langle u, P \rangle], [u].
\]

Here, \(\text{LEnc}\) is the encoding algorithm of the list-decoding code. The main idea is that: after the perturbation on \(\text{LEnc}(b, \text{Sign}(sk, b))\), the original message/signature pair \(b, \text{Sign}(sk, b)\) will always appear in the list output by the decoding algorithm. Now, against an efficient adversary, \(b, \text{Sign}(sk, b)\) must be the only valid message/signature pair in the list. Otherwise, this adversary breaks the unforgeability of the signature scheme. Against an information-theoretic adversary, however, it could introduce \((1 − b, \text{Sign}(sk, 1 − b))\) into the list recovered by the list-decoding algorithm. Consequently, the learner cannot tell if the correct label is \(b\) or \(1 − b\). Consequently, for this learning task, against information-theoretic adversaries, one still needs \(|P|\) parameters. Against efficient adversaries, one needs only a few parameters and can always recover the correct label.

\(^7\)That is, with high probability over the choice of \(Z = z\), the conditional distribution \(P|(Z = z)\) has high min-entropy. More formally, this unpredictability is measured by the average-case min-entropy (Definition 13).
Making instances small. Again, we have this unsatisfying feature that the instance has the same size as the model. We resolve this issue using the sampler in a similar way. We refer the readers to Section 5 for more details.

3 Preliminaries

For a distribution $D$, $x \leftarrow D$ denotes that $x$ is sampled according to $D$. We use $U_S$ for the uniform distribution over $S$, and we use $U_n$ to represent $U_{\{0,1\}^n}$. For a set $S$, $s \leftarrow S$ means $s \leftarrow U_S$. For any two distributions $X$ and $Y$ over a finite universe $\Omega$, their statistical distance is defined as

$$\text{SD}(X,Y) := \frac{1}{2} \sum_{\omega \in \Omega} |\text{Pr}[X = \omega] - \text{Pr}[Y = \omega]|.$$

We sometimes write $X \approx_\varepsilon Y$ to denote $\text{SD}(X,Y) \leq \varepsilon$. For a vector $x = (x_1, \ldots, x_n) \in \mathbb{F}^n$ and a subset $S = \{i_1, \ldots, i_t\} \subseteq \{1, 2, \ldots, n\}$, $x_S$ denotes $(x_{i_1}, \ldots, x_{i_t})$. For any two vectors $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$, their Hamming distance is defined as $\text{HD}(x,y) = |\{i \in \{1, 2, \ldots, n\} : x_i \neq y_i\}|$.

For a set $S$ and an integer $0 \leq t \leq |S|$, $\binom{|S|}{t}$ represents the set of subsets of $S$ of size $t$. $I$ stands for the indicator function. The base for all logarithms in this paper is 2.

3.1 Definitions related to learning and attacks

In this subsection, we present notions and definitions that are related to learning.

We use $\mathcal{X}$ to denote the inputs and $\mathcal{Y} = \{0,1\}$ to denote the outputs or the labels. By $\mathcal{H}$ we denote a hypothesis class of functions from $\mathcal{X}$ to $\mathcal{Y}$. We use $D$ to denote a distribution over $\mathcal{X} \times \mathcal{Y}$. A learning algorithm, takes a set $S \in (\mathcal{X} \times \mathcal{Y})^*$ and a parameter $\lambda$ and outputs a function $f$ that is supposed to predict a fresh sample from the same distribution that has generated the set $S$. The parameter $\lambda$ is supposed to capture the complexity of the problem, e.g., by allowing the inputs (and the running times) to grow. (E.g., this could be the VC dimension, but not necessarily so.) In particular, we assume that the members of the sets $\mathcal{X}_\lambda, \mathcal{Y}_\lambda$ can be represented with $\text{poly}(\lambda)$ bits. A proper learning for a hypothesis class $\mathcal{H}$ outputs $f \in \mathcal{H}$, while an improper learner is allowed to output arbitrary functions. By default, we work with improper learning as we do not particularly focus on whether the learning algorithms output a function from the hypothesis class. For a set $S \subseteq \mathcal{X}$ and a function $h: \mathcal{X} \to \mathcal{Y}$, by $S^h$ we denote the labeled set $\{(x, h(x)) \mid x \in S\}$. For a distribution $D$ and an oracle-aided algorithm $A$, by $A^D$ we denote giving $A$ access to a $D$ sampler.

Definition 2 (Learning problems and learners). We use $\mathcal{F} = \{F_\lambda = (\mathcal{X}_\lambda, \mathcal{Y}_\lambda, D_\lambda, \mathcal{H}_\lambda)\}_{\lambda \in \mathbb{N}}$ to denote a family of learning problems where each $\mathcal{H}_\lambda$ is a set of hypothesis functions mapping $\mathcal{X}_\lambda$ to $\mathcal{Y}_\lambda$ and $D_\lambda$ is a set of distributions supported on $\mathcal{X}_\lambda$.

- For function $\varepsilon(\cdot, \cdot)$, we say $L$ $\varepsilon$-learns $\mathcal{F}$ if
  $$\forall \lambda \in \mathbb{N}, D_\lambda \in D_\lambda, h \in \mathcal{H}_\lambda, n \in \mathbb{N}; \quad \mathbf{E}_{S \leftarrow D_\lambda^*; f \leftarrow L(S^h, \lambda)} \left[\text{Risk}(h, D_\lambda, f)\right] \leq \varepsilon(\lambda, n),$$
  where $\text{Risk}(h, D_\lambda, f) = \text{Pr}_{x \leftarrow D_\lambda}[h(x) \neq f(x)]$.

- $L$ outputs models with at most $p(\cdot)$ bits of parameters if
  $$\forall \lambda \in \mathbb{N}; \quad \log \left(\left|\text{Supp}(L(\cdot, \lambda))\right|\right) \leq p(\lambda).$$
• $L$ is an $\varepsilon$-robust learner against (all) adversaries of budget $r$ w.r.t. distance metric $d$ if

$$\forall \lambda \in \mathbb{N}, D_\lambda \in \mathcal{D}_\lambda, h \in \mathcal{H}_\lambda, n \in \mathbb{N}; \quad \mathbb{E}_{S \leftarrow D_\lambda^n; f \leftarrow L(S^h, \lambda)}[\text{Risk}_{d,r}(h, D_\lambda, f)] \leq \varepsilon(\lambda, n).$$

where

$$\text{Risk}_{d,r}(h, D_\lambda, f) = \Pr_{x \leftarrow D_\lambda} [\max_x \{f(x') \neq h(x)\} \land (d(x, x') \leq r)].$$

• $L$ runs in polynomial time if there is a polynomial $\text{poly}(\cdot)$ such that for all $\lambda \in \mathbb{N}$ and $S \in (\mathcal{X}_\lambda, \mathcal{Y}_\lambda)^*$, the running time of $L(S, \lambda)$ is bounded by $\text{poly}(|S| \cdot \lambda)$.

• $L$ is an $\varepsilon$-robust learner against polynomial-time adversaries of budget $r$ w.r.t distance $d$, if for any family of $\text{poly}(\lambda)$-time (oracle aided) adversaries $\mathcal{A} = \{A^{(j)}_\lambda\}_{\lambda \in \mathbb{N}}$ we have

$$\forall \lambda \in \mathbb{N}, D_\lambda \in \mathcal{D}_\lambda, h \in \mathcal{H}_\lambda, n \in \mathbb{N}; \quad \mathbb{E}_{S \leftarrow D_\lambda^n; A \leftarrow L(S^h, \lambda)}[\text{Risk}_{d,r,A}(h, D_\lambda, f)] \leq \varepsilon(\lambda, n),$$

where

$$\text{Risk}_{d,r,A}(h, D_\lambda, f) = \Pr_{x \leftarrow D_\lambda, x' \leftarrow A^{(j)}_\lambda(x, h(x), f)} [\max_x \{f(x') \neq f(x)\} \land (d(x, x') \leq r)].$$

Our proof also relies on the following theorem from Bubeck et al. [2019].

**Theorem 3** (Implied by Theorem 3.1 of Bubeck et al. [2019]). Let $\{C_\lambda\}_\lambda$ be a finite family of classifiers. Suppose the learning problem $\mathcal{F} = \{\mathcal{F}_\lambda\}_\lambda$ and a learner $L$ satisfy the following. For all $D_\lambda \in \mathcal{D}_\lambda$ and $h \in \mathcal{H}_\lambda$, and sample $S \leftarrow D_\lambda^n$, $L(S^h)$ always outputs a classifier $f \in C_\lambda$ such that $\text{Risk}_{d,r}(h, D_\lambda, f) = 0$ (i.e., $f$ robustly fits $h$ perfectly). Then, $L$ will $\delta$-robust learn $\mathcal{F}$ with sample complexity $\log |C_\lambda| / \delta$.

We emphasize that in the theorem above, one might pick a different set of classifiers merely for sake of computational efficiency of the learner $L$. Namely, it might be possible to information-theoretically learn a hypothesis class robustly (e.g., by a robust variant of empirical risk minimization when), but an efficient learner might choose to output its classifiers from a larger set such that it can efficiently find a member of that class.

### 3.2 Cryptographic primitives

**Definition 4** (Negligible function). A function $\varepsilon(\lambda)$ is said to be negligible, denoted by $\varepsilon(\lambda) = \text{negl}(\lambda)$, if for all polynomial $\text{poly}(\lambda)$, it holds that $\varepsilon(\lambda) < 1/\text{poly}(\lambda)$ for all sufficiently large $\lambda$.

**Definition 5** (Pseudorandom generator). Suppose $\alpha > \lambda$. A function $f: \{0,1\}^\lambda \rightarrow \{0,1\}^\alpha$ is called a pseudorandom generator (PRG) if for all polynomial-time probabilistic algorithm $A$, it holds that

$$\left| \Pr_{x \leftarrow \{0,1\}^\lambda} [A(f(x)) = 1] - \Pr_{y \leftarrow \{0,1\}^\alpha} [A(y) = 1] \right| = \text{negl}(\lambda).$$

The ratio $\alpha/\lambda$ is refer to as the (multiplicative) stretch of the PRG.

**Theorem 6** (Håstad et al. [1999]). Pseudorandom generators (with arbitrary polynomial stretch) can be constructed from any one-way functions.\(^8\)

\(^8\)A one-way function is a function that is easy to compute, but hard to invert (see Definition 32).
Definition 7 (Signature scheme). A signature scheme consists of three algorithms (Gen, Sign, Verify).

- \((vk, sk) \leftarrow \text{Gen}(1^\lambda)\): on input the security parameter \(1^\lambda\), the randomized algorithm \(\text{Gen}\) outputs a (public) verification key \(vk\) and a (private) signing key \(sk\).
- \(\sigma = \text{Sign}(sk, m)\): on input the signing key \(sk\) and a message \(m\), \(\text{Sign}\) outputs a signature \(\sigma\).
- \(b = \text{Verify}(vk, m, \sigma)\): on input a verification key \(vk\), a message \(m\), and a signature \(\sigma\), \(\text{Verify}\) outputs a bit \(b\), and \(b = 1\) indicates that the signature is accepted.

We require a (secure) signature scheme to satisfy the following properties.

- **Correctness.** For every message \(m\), it holds that
  \[
  \Pr\left[ \text{Verify}(vk, m, \sigma) = 1 : (vk, sk) \leftarrow \text{Gen}(1^\lambda), \sigma = \text{Sign}(sk, m) \right] = 1.
  \]

- **Weak unforgeability.** For any PPT algorithm \(A\) and message \(m\), it holds that
  \[
  \Pr\left[ m' \neq m \text{ and } \text{Verify}(vk, m', \sigma') = 1 : (vk, sk) \leftarrow \text{Gen}(1^\lambda), \sigma = \text{Sign}(sk, m), (m', \sigma') \leftarrow A(vk, m, \sigma) \right] = \text{negl}(\lambda).
  \]

Theorem 8 (Naor and Yung [1989], Rompel [1990]). Signature schemes can be constructed from any one-way function.

3.3 Coding theory

Let \(F\) be a finite field. A linear code \(H\) with block length \(n\) and dimension \(k\) is a \(k\)-dimensional subspace of the vector space \(F^n\). The generator matrix \(G \in F^{k \times n}\) maps every message \(m \in F^k\) to its encoding, namely, the message \(m\) is encoded as \(m \cdot G\). The distance \(d\) of the code \(H\) is the minimum distance between any two codewords. That is, \(d = \min_{(x,y) \in H \land (x \neq y)} \text{HD}(x, y)\). The rate of the codeword is defined as \(R = k/n\).

**Reed-Solomon code.** In this work, we will mainly use the Reed-Solomon (RS) code. For the RS code, every message \(m = (m_1, \ldots, m_k) \in F^k\) is parsed as a degree \(k - 1\) polynomial

\[
  f(x) = m_1 + m_2 \cdot x + \cdots + m_k \cdot x^{k-1}
\]

and the encoding is simply \((f(1), f(2), \ldots, f(n)) \in F^n\).

Definition 9 (List-decodable code). A code \(H \subseteq F^n\) is said to be \((p, L)\)-list-decodable if for any string \(x \in F^n\), there are \(\leq L\) messages whose encoding \(c\) satisfies \(\text{HD}(x, c) \leq p \cdot n\). We say the code \(H\) is efficiently \((p, L)\)-list-decodable if there is an efficient algorithm that finds all such messages.

Theorem 10 (Guruswami and Sudan [1998]). For any constant \(R > 0\), the Reed-Solomon code with constant rate \(R\) and block length \(n\) is efficiently \((1 - \sqrt{R}, \text{poly}(n))\)-list-decodable.

\(^9\)We consider a rather weak notion of unforgeability. Here, we require that the adversary cannot forge a signature when he is given only one valid pair of message and signature. This weaker security notion already suffices for our purposes. In the stronger notion, the adversary is allowed to pick \(m\) based on the given \(vk\)
3.4 Randomness extraction

Definition 11 (Min-entropy). The min-entropy of a distribution $X$ over a finite universe $\Omega$ is defined as

$$H_{\infty}(X) := -\log \left( \max_{\omega \in \Omega} \Pr[X = \omega] \right).$$

We need the following result about the sampler. A sampler (for a target set of coordinates \{1, 2, \ldots, n\}) is a deterministic mapping that only takes randomness as input and outputs a subset of \{1, 2, \ldots, n\}. Below, we let \( \binom{n}{t} = \{S \mid |S| = t, S \subseteq [n] \} \).

Theorem 12 (Lemma 6.2 and Lemma 8.4 of Vadhan [2004]). For any 0 < $\kappa_1, \kappa_2 < 1$ and any $n, t \in \mathbb{N}$, there exists a deterministic sampler $\text{samp}: \{0, 1\}^t \rightarrow \binom{n}{t}$ such that the following hold.

- If $X$ is a random variable over \{0, 1\}^n with min-entropy $\geq \mu \cdot n$, there exists a random variable $Y$ over \{0, 1\}^t with min-entropy $\geq (\mu - \kappa_1) \cdot t$ and

$$\text{SD}((U_r, X_{\text{samp}(U_r)}), (U_r, Y)) \leq \exp(-\Theta(n \kappa_1)) + \exp(-n^{\kappa_2}),$$

where the two $U_r$ in the first joint distribution refer to the same sample.

- Furthermore, $r = \Theta(n^{\kappa_2})$.

This theorem by Vadhan [2004] states that one could use a small amount of randomness to sub-sample from a distribution $X$ with the guarantee that $X_{\text{samp}(U_r)}$ is close to another distribution $Y$ that preserves the same min-entropy rate as $X$.

In certain cases, some information regarding $X$ is learned (e.g., through training). Let us denote this learned information as a random variable $Z$. Note that $X$ and $Z$ are two correlated distributions. In order to denote the min-entropy of $X$ conditioned on the learned information $Z$, we need the following notion (and lemma) introduced by Dodis et. al. Dodis et al. [2008].

Definition 13 (Average-case min-entropy Dodis et al. [2008]). For two correlated distributions $X$ and $Z$, the average-case min-entropy is defined as

$$H_{\infty}(X|Z) = -\log \left( \mathbb{E}_{z \sim Z} \left[ \max_{x} \Pr[X=x|Z=z] \right] \right).$$

Lemma 14 (Dodis et al. [2008]). We have the following two lemmas regarding the average-case min-entropy.

- If the support set of $Z$ has size at most $2^m$, we have

$$H_{\infty}(X|Z) \geq H_{\infty}(X) - m.$$

- It holds that

$$\Pr_{z \sim Z}[H_{\infty}(X|Z=z) \geq H_{\infty}(X|Z) - \log(1/\varepsilon)] \geq 1 - \varepsilon.$$  

Intuitively, the above lemma states that: if a model denoted by the random variable $Z$ is not too large, and that model captures the information revealed about a variable $X$, since the support set of $Z$ has small size, then the average-case min-entropy $H_{\infty}(X|Z)$ is large. Furthermore, for most $z$, the min-entropy of $H_{\infty}(X|Z=z)$ is almost as large as $H_{\infty}(X|Z)$.

We now recall a tool that, roughly speaking, states that if $X$ is a distribution over \{0, 1\}^n that contains some min-entropy, then the inner product (over $\mathbb{F}_2$) between $X$ and a random vector $Y$ is a uniformly random bit, even conditioned on most of $Y$. This is a special case of the celebrated leftover hash lemma Håstad et al. [1999]. We summarize this result as the following theorem. For completeness, a proof can be found in Appendix B.1.
Theorem 15 (Inner product is a good randomness extractor). For all distribution $X$ over $\{0, 1\}^n$ such that $H_\infty(X) \geq 2 \cdot \log(1/\varepsilon)$, it holds that

$$(U_n, (X, U_n)) \approx_\varepsilon U_{n+1},$$

where the two $U_n$ refer to the same sample.

Next, we need the following notion and results from Fourier analysis.

Definition 16 (Small-bias distribution Naor and Naor [1993]). Let $F_{2^\ell}$ be the finite field of order $2^\ell$. For a distribution $X$ over $F_{2^\ell}^n$, the bias of $X$ with respect to a vector $y \in F_{2^\ell}^n$ is defined as

$$\text{bias}(X, \alpha) := \left| \mathbb{E}_{x \leftarrow X} \left[ (-1)^{\text{Tr}((x, \alpha))} \right] \right|,$$

where $\text{Tr}: F_{2^\ell} \to F_2$ denote the trace map defined as $\text{Tr}(y) = y + y^2 + y^2 + \cdots + y^{2^{\ell-1}}$. The distribution $X$ is said to be $\varepsilon$-small-biased if for all non-zero vector $\alpha \in \{0, 1\}^n$, it holds that

$$\text{bias}(X, \alpha) \leq \varepsilon.$$

Note that the trace map maps elements from $F_{2^\ell}$ to $F_2$, where exactly half of the field elements maps to 1 and the other half to 0. Consequently, if $(X, \alpha)$ is a uniform distribution over $F_{2^\ell}$, then bias$(X, \alpha) = 0$. In Appendix B.2, prove the theorem below.

Theorem 17 (Small-biased Masking Lemma Dodis and Smith [2005]). Let $X$ and $Y$ be distributions over $F_{2^\ell}^n$. If $H_\infty(X) \geq k$ and $Y$ is $\varepsilon$-small-biased, it holds that

$$\text{SD} (X + Y, U_{\{0, 1\}^n}) \leq 2^{n\ell-k-1} \cdot \varepsilon.$$

Finally, we observe the following property about the noisy RS code. A proof can be found in Appendix B.3.

Theorem 18 (Noisy RS code is small-biased). Let $H$ be a RS code over $F_{2^\ell}$ with block length $n$ and rate $R$. For all integer $s \leq n$, consider the following distribution

$$D = \begin{cases} \quad \text{c} \leftarrow H, \\ \quad \text{Sample a random } S \subseteq \{1, 2, \ldots, n\} \text{ such that } |S| = s, \\ \quad \forall i \in S, \text{ replace } c_i \text{ with a random field element} \\ \quad \text{Output } c \end{cases}.$$ 

It holds that $D$ is $(1 - R)^s$-small-biased.

4 Efficient learning could cause more model parameters

In this section, we formally prove the first part of Theorem 1, which is the separation result between the number of parameters needed by unbounded v.s. bounded learners.

Our construction and theorems are formally stated as follows.

Construction 19 (Parameter-heavy models under efficient learning). Given the parameter $n < \lambda < \alpha < \beta$, we construct the following learning problem.\footnote{All the other parameters are implicitly defined by these parameters.} We rely on the following building blocks.
• Let \( f_1 : \{0,1\}^\lambda \rightarrow \{0,1\}^\alpha \) and \( f_2 : \{0,1\}^\lambda \rightarrow \{0,1\}^\beta \) be PRGs (Definition 5).

• Let \( \text{Enc} \) be a RS encoding with dimension \( k \), block length \( n \), and rate \( R = k/n \). The rate is chosen to be any constant \(< 1/3 \) and \( k \) is defined by \( R \) and \( n \). This RS code is over the field \( \mathbb{F}_{2^\ell} \) for some \( \ell = \Theta(\log \lambda) \).

• Let \( \text{samp}_1 : \{0,1\}^\alpha \rightarrow \{1,\ldots,\alpha\} \) and \( \text{samp}_2 : \{0,1\}^\beta \rightarrow \{1,\ldots,\beta\} \) be samplers. We obtain these samplers by invoking Theorem 12 with sufficiently small \( \kappa_1 \) and \( \kappa_2 \). (For instance, setting \( \kappa_1 = \Theta(1/\log \lambda) \) and \( \kappa_2 \) to be any small constant suffices.) Note that \( \kappa_1, \kappa_2 \) define \( r_1, r_2 \).

• For any binary string \( v \), we use \([v] \) for an arbitrary error-correcting encoding of \( v \) (over the field \( \mathbb{F}_{2^\ell} \) ) such that \([v] \) can correct \( > (1-R)n/2 \) errors. This can always be done by encoding \( v \) using RS code with a suitable (depending on the dimension of \( v \) ) rate. Looking forward, we shall consider an adversary that may perturb \( \leq (1-R)n/2 \) symbols. Therefore, when a string \( v \) is encoded as \([v] \) and the adversary perturb it to be \([\tilde{v}] \), it will always be error-corrected and decoded back to \( v \).

We now construct the following learning task \( \mathcal{F}_\lambda = (\mathcal{X}_\lambda, \mathcal{Y}_\lambda, \mathcal{D}_\lambda, \mathcal{H}_\lambda) \).

• \( \mathcal{X}_\lambda \) implicitly defined by the distribution \( \mathcal{D}_\lambda \).

• \( \mathcal{Y}_\lambda = \{0,1\} \).

• \( \mathcal{D}_\lambda \) consists of distributions \( D_s \) for \( s \in \{0,1\}^\lambda \), where \( D_s \) is the following:

\[
D_s = \left( [u_1], [u_2], m, \text{Enc}(m) + \left( f_2(s) \big|_{\text{samp}_2(u_2)} \right) \right),
\]

where

- \( u_1 \) and \( u_2 \) are sampled uniformly at random from \( \{0,1\}^{r_1} \) and \( \{0,1\}^{r_2} \), respectively.
- \( m \) is sampled uniformly at random from \( \mathbb{F}_{2^\ell}^k \).
- \( f_2(s) \big|_{\text{samp}_2(u_2)} \) is interpreted as a vector over \( \mathbb{F}_{2^\ell} \) and \( \text{Enc}(m) + \left( f_2(s) \big|_{\text{samp}_2(u_2)} \right) \) is coordinate-wise addition over \( \mathbb{F}_{2^\ell} \).

• \( \mathcal{H}_\lambda \) consists of all functions \( h_s : \mathcal{X} \rightarrow \mathcal{Y} = \{0,1\} \) for all \( s \in \{0,1\}^\lambda \), where \( h_s \) is:

\[
h_s(x) = \left( m, \left( f_1(s) \big|_{\text{samp}_1(u_1)} \right) \right),
\]

and the inner product is over \( \mathbb{F}_2 \) and \( m \) is interpreted as a string \( \in \mathbb{F}_{2^\ell}^k \) in the natural way.

• **Adversary.** The entire input \( x = ( [u_1], [u_2], m, \text{Enc}(m) + \left( f_2(s) \big|_{\text{samp}_2(u_2)} \right) ) \) is interpreted as a vector over \( \mathbb{F}_{2^\ell} \) and we consider an adversary that may perturb \( \leq (1-R)n/2 \) symbols. That is, the adversary has a budget of \( (1-R)n/2 \) in Hamming distance over \( \mathbb{F}_{2^\ell} \).

**Theorem 20.** An information-theoretic learner can (robustly) \( \varepsilon \)-learn the task of Construction 19 with parameter size \( 2\lambda \) and sample complexity \( \Theta(\varepsilon^2) \). Moreover, an efficient learner can (robustly) \( \varepsilon \)-learn this task with parameter size \( \alpha + \beta \) and sample complexity \( \Theta(\frac{\alpha + \beta}{\varepsilon}) \).

**Theorem 21.** Any efficient learner that outputs models with \( \leq \alpha/2 \) parameters cannot \( \varepsilon \)-learn \( F_\lambda \) of Construction 19 for \( \varepsilon < 1/3 \).
Theorem 22. There exists some constant $c$ such that the following holds. In the presence of an adversary that may perturb $(1 - R)n/2$ symbols, any efficient learner for the task of Construction 19 that outputs a model with $c \cdot \beta / \log \lambda$ parameters cannot $\varepsilon$-robustly learn $F_{\lambda}$ for $\varepsilon < 1/3$.

First, observe that instance size is approximately $\Theta((k + n) \cdot \ell) = \Theta(n \cdot \log \lambda)$ as the size of the sampler inputs $u_1$ and $u_2$ are sufficiently small. Our theorems prove the following. First, in Theorem 20 we establish the efficient (robust) learnability of the task of Construction 19, where the efficient-learner variant requires more parameters. Then, in Theorem 21 we establish the lower bound on the number of parameters needed by an efficient learner. Finally, in Theorem 22 we establish the lower bound on the number of parameters needed by efficient robust learners.

In the rest of this section, we prove these theorems.

4.1 Proof of Theorem 20

Since the learning task of Theorem 20 has a finite hypothesis class, its learnability follows from the classical result of learning finite classes [Shalev-Shwartz and Ben-David, 2014]. Moreover, this can be done efficiently as this is a linear task. When it comes to learning robust functions, one can also use the result of Bubeck et al. [2019] for robustly learning finite classes. The learner of Bubeck et al. [2019] simply uses the empirical-risk minimization, however this is done with respect to the robust empirical risk. This learner is not always polynomial-time, even if the (regular) risk minimization can be done efficiently. However, we would like to find robust learners also efficiently. The formal proof follows.

Proof of Theorem 20. Consider the set of functions $f_{s, s'} : \mathcal{X}_{\lambda} \rightarrow \mathcal{Y}_{\lambda}$ for all $s, s' \in \{0, 1\}^{\lambda}$ as follows.

1. On input $x = ([u_1], [u_2], \bar{m}, \bar{d})$, it invokes the error-correcting decoding algorithm on $[u_1]$ and $[u_2]$ to find $u_1$ and $u_2$.
2. It uses $\bar{d} + f_2(s')_{\text{samp}(u_2)}$ to get an encoding $\bar{c}$ of $m$.
3. It invokes the error-correcting decoding on $\bar{c}$ to get $m$.
4. It outputs $\langle m, f_1(s)_{\text{samp}(u_1)} \rangle$.

One of the function $f_{s, s'}$ will (perfectly) robustly fit the distribution since all the encodings $[u_1], [u_2], \text{Enc}(m)$ tolerates $(1 - R)n/2$ perturbation. Since, there are $2^{2\lambda}$ such functions, by Theorem 3, we conclude that an information-theoretic learner can $\varepsilon$-learn this task with $2\lambda$ parameters and sample complexity $\Theta(\lambda/\varepsilon)$.

Note that an efficient learner might not be able to find such a function $f_{s, s'}$ as it requires inverting a pseudorandom generator. However, an efficient learner can still learn using more samples and parameters as follows. Consider the set of functions $f_{P, Q} : \mathcal{X}_{\lambda} \rightarrow \mathcal{Y}_{\lambda}$ for all $P \in \{0, 1\}^{\alpha}$ and $Q \in \{0, 1\}^{\beta}$ as follows.

1. On input $x = ([u_1], [u_2], \bar{m}, \bar{d})$, it invokes the error-correcting decoding algorithm on $[u_1]$ and $[u_2]$ to find $u_1$ and $u_2$.
2. It uses $\bar{d} + Q_{\text{samp}(u_2)}$ to get an encoding $\bar{c}$ of $m$.
3. It invokes the error-correcting decoding on $\bar{c}$ to get $m$.

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11 Results for infinite classes could be found in subsequent works Montasser et al. [2019].
4. It outputs $\langle m, P_{\text{samp}(u_1)} \rangle$.

Similarly, one of the function $f_{P,Q}$ will (perfectly) robustly fit the distribution since all the encodings $[u_1],[u_2],\text{Enc}(m)$ tolerates $(1-R)n/2$ perturbation. Finding $f_{P,Q}$ that fits all the samples only requires linear operations and, hence, is efficiently learnable. As there are $2^{\alpha+\beta}$ such functions, again by Theorem 3, we conclude that an efficient learner can $\varepsilon$-learn this task with $\alpha+\beta$ parameters and sample complexity $\Theta((\alpha+\beta)/\varepsilon)$. To apply Theorem 3, we simply pretend that the hypothesis set is the larger set of $2^{\alpha+\beta}$ such functions, in which case our learner finds one of these $2^{\alpha+\beta}$ functions that perfectly matches with the training set with zero robust empirical risk. \hfill $\square$

4.2 Proof of Theorem 21

Proof. We start by defining another learning problem $F'_\lambda$. This learning problem is identical to $F_\lambda$ for $\mathcal{X}_\lambda$, $\mathcal{Y}_\lambda$, and $\mathcal{D}_\lambda$. However, $\mathcal{H}'_\lambda$ consists of all functions $h_P$ for all $P \in \{0,1\}^\alpha$, such that

$$h_P(x) = \langle m, \left(P_{\text{samp}(u_1)}\right) \rangle.$$ 

On a high level, our proof consists of two claims.

Claim 23. Fix any distribution $D_{s'} \in \mathcal{D}_\lambda$. We consider a random hypothesis function $h_s$ and $h_P$, where $s \leftarrow \{0,1\}^\lambda$ and $P \leftarrow \{0,1\}^\alpha$. It holds that

$$\mathbb{E}_{S \leftarrow D^n_{s'}; f \leftarrow L(S^{h_s},\lambda)} \left[Risk(h_s, D_\lambda, f)\right] \approx_{\text{negl}(\lambda)} \mathbb{E}_{S \leftarrow D^n_{s'}, f \leftarrow L(S^{h_P},\lambda)} \left[Risk(h_P, D_\lambda, f)\right].$$

Claim 24. For any learner $L$ (with an arbitrary sample complexity) with $\leq \alpha/2$ parameters, we have

$$\mathbb{E}_{S \leftarrow D^n_{s'}; f \leftarrow L(S^{h_P},\lambda)} \left[Risk(h_P, D_\lambda, f)\right] > 3/8.$$ 

Note that, if both claims are correct, the theorem statement is true.

We first show Claim 23. Observe that, given the string $P$, one can compute the function $h_P(x)$ efficiently. Now, given a string $P$, which is either a pseudorandom string (i.e., $P \leftarrow f_1(U_\lambda)$) or a truly random string (i.e., $P \leftarrow \{0,1\}^\alpha$). Consider the following distinguisher

$$\left\{ S \leftarrow D^n_{s'}, f \leftarrow L(S^{h_P},\lambda), x \leftarrow D_{s'}, \text{Output } \mathbb{I}(h_P(x) = f(x)) \right\}.$$ 

If $P$ is pseudorandom, the probability that the distinguisher outputing 1 is

$$\mathbb{E}_{S \leftarrow D^n_{s'}; f \leftarrow L(S^{h_s},\lambda)} \left[Risk(h_s, D_\lambda, f)\right];$$

if $P$ is truly random, the probability that the distinguisher outputs 1 is

$$\mathbb{E}_{S \leftarrow D^n_{s'}; f \leftarrow L(S^{h_P},\lambda)} \left[Risk(h_P, D_\lambda, f)\right].$$

Therefore, if Claim 23 does not hold, we break the pseudorandom property of the PRG.

It remains to prove Claim 24.

Since $P$ is sampled uniformly at random, we have $H_\infty(P) = \alpha$. Let $Z$ denote the random variable $L(\cdot,\lambda)$, i.e., the model learned by the learner. Since the learner’s output model employs $\leq \alpha/2$ parameters, we have $\text{Supp}(Z) \leq 2^{\alpha/2}$. And by Lemma 14, we must have

$$\tilde{H}_\infty(P|Z) \geq \alpha/2.$$
Now, let us define the set\(^{12}\)
\[
\text{Good} = \{ z \in \text{Supp}(Z) : H_\infty(P|Z = z) \leq \alpha/4 \}.
\]

Lemma 14 implies that
\[
\Pr[Z \in \text{Good}] \leq 2^{-\alpha/4} = \text{negl}(\lambda).
\]

In the rest of the analysis, we conditioned on the event that \(Z \notin \text{Good}\), which means \(H_\infty(P|Z = z) > \alpha/4\). Now, let \(x = ([u_1], [u_2], m, \text{Enc}(m) + (f_2(s')_{\text{samp}_2(u_2)}))\) be the test sample. Since \(P\) has min-entropy rate \(> 1/4\), the property of the sampler (Theorem 12) guarantees that there exists a distribution \(D\) such that
\[
H_\infty(D) \geq \left( \frac{1}{4} - \kappa_1 \right) \cdot k^\lambda > \frac{1}{5} \cdot k^\lambda
\]
and
\[
\text{SD} \left( (u_1, D), (u_1, P|_{\text{samp}_1(u_1)}) \right) \leq \exp(-\Theta(\alpha\kappa_1)) + \exp(-n^{\kappa_2}) = \text{negl}(\lambda).
\]

Recall that \(x = ([u_1], [u_2], m, \text{Enc}(m) + (f_2(s')_{\text{samp}_2(u_2)}))\) and \(y = h_P(x) = \langle m, (P|_{\text{samp}_1(u_1)})\rangle\). Consequently,
\[
\text{SD} \left( ((x, z), y), (\langle x, z \rangle, U_{(0,1)}) \right) \\
= \text{SD} \left( ((u_1, m, z), y), (\langle u_1, m, z \rangle, U_{(0,1)}) \right) \quad \text{(as } u_2 \text{ is independent of } y) \\
\leq \text{SD} \left( ((u_1, m, z), \langle m, D \rangle), (\langle u_1, m, z \rangle, U_{(0,1)}) \right) + \text{negl}(\lambda) \\
= \text{SD}(P|_{\text{samp}_1(u_1)}|Z = z, D) \leq \text{negl}(\lambda)
\]
(\text{Theorem 15 and } H_\infty(D) > \frac{1}{5} \cdot k^\lambda)

Therefore, in the learner’s view \((x, z), y\) is statistically \(\text{negl}(\lambda)\)-close to uniform. Therefore,
\[
\mathbb{E}_{S \leftarrow D^\lambda_{\lambda}; f \leftarrow L(S^{h_P}, \lambda)} [\text{Risk}(h_P, D_{\lambda}, f)] \\
\geq \Pr_{S \leftarrow D^\lambda_{\lambda}; f \leftarrow L(S^{h_P}, \lambda)} [z \in \text{Good}] \\
\quad + \Pr_{S \leftarrow D^\lambda_{\lambda}; f \leftarrow L(S^{h_P}, \lambda)} [z \notin \text{Good}] \cdot \mathbb{E}_{S \leftarrow D^\lambda_{\lambda}; f \leftarrow L(S^{h_P}, \lambda)} [\text{Risk}(h_P, D_{\lambda}, f)|z \notin \text{Good}] \\
\geq \text{negl}(\lambda) + (1 - \text{negl}(\lambda)) \cdot \left( \frac{1}{2} - \text{negl}(\lambda) \right) > 3/8.
\]

This shows Claim 24 and completes the proof of the theorem. \(\square\)

### 4.3 Proof of Theorem 22

**Proof.** The high-level structure of the proof is similar to the proof of Theorem 21. We consider a new learning problem \(F_{\lambda}^*\) that has the same \(X_{\lambda}, Y_{\lambda},\) and \(H_{\lambda}\). However, \(D_{\lambda}\) consists of all distribution \(D_Q\) for all \(Q \in \{0, 1\}^D\), where the distribution \(D_Q\) is
\[
D_Q = \left( [u_1], [u_2], m, \text{Enc}(m) + (Q|_{\text{samp}_2(u_2)}) \right).
\]

The proof consists of two claims.

\(^{12}\)This set is called “good” as it is good for the learner.
Claim 25. Fix a hypothesis $h_{s'} \in \mathcal{H}_s$. We consider a random distribution over $\mathcal{D}_s$ and $\mathcal{D}'_s$. That is, $D_s$ and $D_Q$ are sampled with $s \sim \{0,1\}^\lambda$ and $Q \sim \{0,1\}^\beta$. It holds that

$$\mathbb{E}_{S \sim D_s^n, f \sim L(S^{h_{s'}}, \lambda)} \left[ \text{Risk}(h_{s'}, D_s, f) \right] \approx_{\negl(\lambda)} \mathbb{E}_{S \sim D'_s, f \sim L(S^{h_{s'}}, \lambda)} \left[ \text{Risk}(h_{s'}, D_Q, f) \right].$$

Claim 26. For any learner $L$ (with an arbitrary sample complexity) with $\leq c \cdot \beta / \log \lambda$ parameters, it holds that

$$\mathbb{E}_{S \sim D_s^n, f \sim L(S^{h_{s'}}, \lambda)} \left[ \text{Risk}(h_{s'}, D_s, f) \right] > 3/8.$$

Note that these two claims prove the theorem. To see Claim 25, observe that given a string $Q$, we can sample efficiently from $D_Q$. Analogous to the proof of Claim 23, if Claim 25 does not hold, we may break the pseudorandom property of the PRG using this (efficient) learner $L$.

It remains to prove Claim 26.

Let $Z$ denote the random variable $L(\cdot, \lambda)$, i.e., the parameter of the learner. Since $Q$ is uniformly random, we have

$$H_\infty(Q|Z) \geq (1 - c/\log \lambda)\beta.$$

Let us define the set

$$\text{Good} = \{ z \in \text{Supp}(Z) : H_\infty(Q|Z = z) \leq (1 - 2c/\log \lambda)\beta \}.$$

Lemma 14 implies that

$$\Pr[Z \in \text{Good}] \leq 2^{-c\beta/\log \lambda} = \negl(\lambda).$$

In the rest of the analysis, we conditioned on the event that $Z \not\in \text{Good}$, which means $H_\infty(Q|Z = z) > (1 - 2c/\log \lambda)\beta$.

Now, we consider the following adversary $A$ that perturbs $(1 - R)n/2$ symbols. Given a test instance $(x, y)$, where

$$x = ([u_1], [u_2], m, \text{Enc}(m) + (Q|_{\text{samp}_2(u_2)})),$$

the adversary will do the following.

- Replace $m$ with a uniformly random string. This costs a budget of $Rn$.
- Samples a random subset $T \subseteq \{1, 2, \ldots, n\}$ of size $(1 - 3R)n/2$. It adds noises to $\text{Enc}(x) + (Q|_{\text{samp}_2(u_2)})$ at precisely those indices from $S$. This costs a budget of $(1 - 3R)n/2$.

For simplicity, let us denote the distribution of this noise by $\rho$. That is, $\rho$ is a distribution over $\mathbb{F}_2^n$ such that it is 0 everywhere except for a random subset $T$ and for those $i \in T$, $\rho_i$ is uniformly random.

We now argue that the perturbed instance is statistically $\negl(\lambda)$-close to the distribution

$$\left( [u_1], [u_2], U_{k \cdot t}, U_{n \cdot t} \right).$$

It suffices to prove that $\text{Enc}(m) + (Q|_{\text{samp}_2(u_2)}) + \rho$ is close to the uniform distribution. By Theorem 18, $\text{Enc}(m) + \rho$ is close to $\text{Enc}(m) + (1 - R)^{\frac{(1 - 3R)n}{2}}$-small-biased.

Furthermore, since $Q$ has min-entropy rate $> (1 - 2c/\log \lambda)$, the property of the sampler (Theorem 12) guarantees that there exists a distribution $D$ such that

$$H_\infty(D) \geq (1 - 2c/\log \lambda - \kappa_1) \cdot n\ell > (1 - 3c/\log \lambda) \cdot n\ell.$$
and
\[ \text{SD}((u_2, D), (u_2, Q|\text{samp}_2(u_2))) \leq \exp(-\Theta(\beta \kappa_1)) + \exp(-n^{\kappa_2}) = \text{negl}(\lambda). \] (1)

Finally, by Theorem 17, we have \((\text{Enc}(m) + \rho) + D\) is
\[ 2^{\frac{\log \lambda}{10}} \cdot (1 - R)^{\frac{1 - 3R}{2}} \]
close to the uniform distribution. Observe that as long as
\[ c < \frac{\log \lambda}{3\ell} \cdot (1 - 3R) \log \frac{1}{1 - R} = \Theta(1), \]
the closeness is negligible in \(\lambda\). Overall,
\[ \text{SD}((\text{Enc}(m) + Q|\text{samp}_2(u_2) + \rho, U_n) \]
\[ \leq \text{SD}((\text{Enc}(m) + Q|\text{samp}_2(u_2) + \rho, \text{Enc}(m) + D + \rho) + \text{SD}(\text{Enc}(m) + D + \rho, U_n)) \]
\[ \leq \text{negl}(\lambda) + \text{negl}(\lambda) = \text{negl}(\lambda). \]

Therefore, given a test instance \(x\) and the perturbed input \(x'\), we have
\[ \text{SD}(((x', z), y), ((x', z), U_{\{0,1\}})) \]
\[ \leq \text{SD}(((u_1, u_2, U_{\kappa_1}, U_{\kappa_2}), y), ((u_1, u_2, U_{\kappa_1}, U_{\kappa_2}), U_{\{0,1\}})) + \text{negl}(\lambda) \]
\[ = \text{SD}(((u_1, m, f_1(s')|\text{samp}_1(u_1))), (u_1, U_{\{0,1\}})) + \text{negl}(\lambda) \]
\[ = \text{negl}(\lambda). \]

Hence, when \(z \notin \text{Good}\), given a perturbed test input \(x'\), the correct label \(y\) is information-theoretically unpredicatable from the learner with \(\text{negl}(\lambda)\) advantage.

Putting everything together, we have
\[ \text{E}_{S \leftarrow D^{|S|}_Q; f \leftarrow L(S^h, \nu, \lambda)} [\text{Risk}(h_{s'}, D_Q, f)] \]
\[ \geq \text{Pr}_{S \leftarrow D^{|S|}_Q; f \leftarrow L(S^h, \nu, \lambda)} [z \in \text{Good}] \]
\[ + \text{Pr}_{S \leftarrow D^{|S|}_Q; f \leftarrow L(S^h, \nu, \lambda)} [z \notin \text{Good}] \cdot \text{E}_{S \leftarrow D^{|S|}_Q; f \leftarrow L(S^h, \nu, \lambda)} [\text{Risk}(h_{s'}, D_Q, f)|z \notin \text{Good}] \]
\[ \geq \text{negl}(\lambda) + (1 - \text{negl}(\lambda)) \cdot \left(\frac{1}{2} - \text{negl}(\lambda)\right) > 3/8. \]

This completes the proof of the claim and the entire theorem. \(\square\)

5 Computationally robust learning could cause fewer parameters

In this section, we formally prove Part 2 of Theorem 1. Our construction and theorems are formally stated as follows.

Construction 27 (Learning task for bounded/unbounded attackers). Given the parameters \(n < \lambda < \alpha\), we construct the following learning problem.\(^{13}\) We use the following tools.

\(^{13}\) All the other parameters are implicitly defined by these parameters.
(Gen, Sign, Verify) be a signature scheme (see Definition 7).

Let \( \text{LEnc} \) be a RS encoding with dimension \( k \), block length \( n \), and rate \( R = k/n \). We pick the rate \( R \) to be any constant \(< 1/4 \) and \( k \) is defined by \( R \) and \( n \). This RS code is over the field \( \mathbb{F}_{2^\ell} \) for some \( \ell = \Theta(\log \lambda) \).

Let \( \text{samp} : \{0,1\}^r \rightarrow \left(\{0,1\}^n\right)^{\alpha} \) be samplers. (We obtain these samplers by invoking Theorem 12 with sufficiently small \( \kappa_1 \) and \( \kappa_2 \). For instance, setting \( \kappa_1 = \Theta(1/\log \lambda) \) and \( \kappa_2 \) to be any small constant suffices.)

For any binary string \( v \), we use \([v]\) for an arbitrary error-correcting encoding of \( v \) (over the field \( \mathbb{F}_{2^\ell} \)) such that \([v]\) can correct \((1 - \sqrt{R})n \) errors. This can always be done by encoding \( v \) using RS code with a suitable (depending on the dimension of \( v \)) rate. Looking forward, we shall consider an adversary that may perturb \( (1 - \sqrt{R})n \) symbols. Therefore, when a string \( v \) is encoded as \([v]\) and the adversary perturbs it to be \([\tilde{v}]\), it will always be error-corrected and decoded back to \( v \).

We now construct the following learning task \( F_\lambda = (\mathcal{X}_\lambda, \mathcal{Y}_\lambda, \mathcal{D}_\lambda, \mathcal{H}_\lambda) \).

- \( \mathcal{X}_\lambda \) is \( \{0,1\}^N \) for some \( N \) that is implicitly defined by \( \mathcal{D}_\lambda \).
- \( \mathcal{Y}_\lambda \) is \( \{0,1\} \).
- The distribution \( \mathcal{D}_s \) consists of all distribution \( D_s \) for \( s \in \{0,1\}^\alpha \), where
  \[
  D_s = \left( [u, [v]], [vk], \text{LEnc}(b, \text{Sign}(sk, b)), [b + \langle v, s \rangle_{\text{samp}(u)}] \right).
  \]

Here,

- \( u \) is sampled uniformly at random from \( \{0,1\}^\tau \) and \( v \) is sampled uniformly at random from \( \{0,1\}^n \).
- \( (vk, sk) \leftarrow \text{Gen}(1^\lambda) \) are sampled from the signature scheme.
- \( b \) is sampled uniformly at random from \( \{0,1\} \). \((b, \text{Sign}(sk, b))\) is interpreted as a vector in \( \mathbb{F}_{2^\ell}^k \) in the natural way.
- \( h_\lambda \) consists of one single function \( h \). On input \( x = ([u, [v]], \text{LEnc}(b, \text{Sign}(sk, b)), [b + \langle v, s \rangle_{\text{samp}(u)}]) \), \( h(x) \) simply decodes \( \text{LEnc}(b, \text{Sign}(sk, b)) \) and output \( b \).

**Adversary.** The entire input \(([u, [v]], [vk], \text{LEnc}(b, \text{Sign}(sk, b)), [b + \langle v, s \rangle_{\text{samp}(u)}])\) is interpreted as a vector over \( \mathbb{F}_{2^\ell} \) and we consider an adversary that may perturb \( (1 - \sqrt{R})n \) symbols. That is, the adversary has a budget of \((1 - \sqrt{R})n \) for Hamming distance over \( \mathbb{F}_{2^\ell} \).

**Theorem 28.** For the learning task of Construction 27, there is an efficient learner (with \( 0 \) sample complexity) that outputs a model with no parameter and \( \text{negl}(\lambda) \)-robustly learns \( F_\lambda \) against computationally-bounded adversaries of budget \((1 - \sqrt{R})n \).

**Theorem 29.** For computationally unbounded adversaries, any information-theoretic learner with \( \alpha/2 \) parameters cannot \( \varepsilon \)-robustly learn \( F_\lambda \) for \( \varepsilon < 1/3 \) for the learning task of Construction 27.

We note that the instance size is (approximately) \( \Theta(n \cdot \ell) = \Theta(n \cdot \log \lambda) \). We shall prove two properties of this construction. In Theorem 28, we establish the upper bound of learnability with few parameters under efficient (polynomial-time) attacks. Later, in Theorem 29, we establish the lower bound of the number of parameters when the attacker is unbounded.

In the rest of this section, we formally prove these theorems.
5.1 Proof of Theorem 28

Proof. The learner is defined as follows. On input a perturbed instance \( x' = ([u], [v], [vk], \tilde{c}, \tilde{d}) \), it does the following:

1. Invoke the error-correction algorithm to recover \( vk \).
2. Invoke the list-decoding algorithm on \( \tilde{c} \) to find a list of message/signature \((b_i, \sigma_i)\) pairs.
3. Run the verifier to find any valid message/signature pair \((b^*, \sigma^*)\) and output \( b^* \). If no such pair exists, output a random bit, and if there are more than one such pair, pick one arbitrarily.

Observe that the learner can always recover the correct \( vk \) since the encoding \([vk]\) tolerates \((1 - \sqrt{R})n\) errors.

Next, suppose the original instance is \(([u], [v], [vk], \text{LEnc}(b, \text{Sign}(sk, b)))\). Then, \((b, \text{Sign}(sk, b))\) is always in the list of message/signature pairs output by the list-decoding algorithm. This is due to that \( \text{LEnc}(b, \text{Sign}(sk, b)) \) is \((1 - \sqrt{R})n\)-close to the perturbed encoding \( \tilde{c} \) and the list decoding algorithm outputs all such messages whose encoding is \((1 - \sqrt{R})n\)-close to the perturbed one.

Finally, fix any distribution \( D_s \). It must hold that, with \( 1 - \text{negl}(\lambda) \) probability, there does not exist a valid message/signature pair where the message is \( 1 - b \). If this does not hold, one may utilize this learning adversary \( A \) to break the unforgeability of the signature scheme as follows: on input the verification key \( vk \) and a valid message/signature \((b, \text{Sign}(sk, b))\), the signature adversary samples the test instance and feed it to the adversary \( A \), obtaining a perturbed instance \( x' = ([\tilde{u}], [\tilde{v}], [vk], \tilde{c}, \tilde{d}) \).\(^{14}\) The signature adversary uses the same procedure as the efficient learner to recover a list of message/signature pairs. If there is a valid message/signature pair with message \( 1 - b \), clearly, the signature adversary breaks the unforgeability of the signature scheme. Since the signature scheme is \( \text{negl}(\lambda) \)-secure, it must hold that, with \( 1 - \text{negl}(\lambda) \) probability, there does not exist a valid message/signature pair where the message is \( 1 - b \).

Consequently, this efficient learner outputs the correct label \( b \) with \( 1 - \text{negl}(\lambda) \) probability. Thus, for all efficient adversary \( A \),

\[
\mathbb{E}_{f \leftarrow L(\emptyset, \lambda)} [\text{Risk}_{d,r}(h, D_s, f)] = \text{negl}(\lambda),
\]

and this finishes the proof. \( \square \)

5.2 Proof of Theorem 29

Proof. We sample \( s \) uniformly at random from \( \{0, 1\}^\alpha \) and prove that

\[
\mathbb{E}_{s \leftarrow D_s^*; f \leftarrow L(\mathbb{S}^b, \lambda)} [\text{Risk}_{d,r}(h, D_s, f)] > 1/3.
\]

The proof is similar to the proof of Theorem 21.

Let \( Z \) denote \( L(\cdot, \lambda) \), i.e., the parameters of the model output by the learner. Given a test instance \( x = ([u], [v], [vk], \text{LEnc}(b, \text{Sign}(sk, b)), [b + \langle v, s|_{\text{samp}(u)} \rangle]) \), we first prove the following claim.

Claim 30. With overwhelming probability over \( Z \),

\[
\left( Z, ([u], [v], [vk], \text{LEnc}(b, \text{Sign}(sk, b)), [b + \langle v, s|_{\text{samp}(u)} \rangle]) \right) \approx_{\text{negl}(\lambda)} \left( Z, ([u], [v], [vk], \text{LEnc}(b, \text{Sign}(sk, b)), [U_{\{0,1\}}]) \right).
\]

\(^{14}\)Note that the adversary can efficiently sample from \( D_s \) as the \((vk, sk)\) pairs for every instance are independent.
That is, the learner cannot distinguish the two distributions given \( Z \).

First, we have \( H_\infty(s|Z) \geq \alpha/2 \). Define the set

\[
\text{Good} = \{ z : H_\infty(s|Z = z) \leq \alpha/4 \}.
\]

By Lemma 14, \( \Pr[Z \in \text{Good}] \leq 2^{-\alpha/4} = \negl(\lambda) \). For the rest of the analysis, we conditioned on \( Z \notin \text{Good} \). Since \( H_\infty(s|Z = z) \geq \alpha/4 \), by the property of the sampler (Theorem 12), there exists a distribution \( D \) such that

\[
H_\infty(D) \geq \left( \frac{1}{4} - \kappa_1 \right) \alpha > \frac{1}{5}\alpha
\]

and

\[
SD \left( (u, s_{\text{samp}}(u)), (u, D) \right) \leq \exp(-\Theta(\beta\kappa_1)) + \exp(-n^{\kappa_2}) = \negl(\lambda). \tag{2}
\]

Finally, by Theorem 15, we have

\[
SD \left( (v, \langle v, s_{\text{samp}}(u) \rangle), (v, U_{\{0,1\}}) \right) \\
\leq SD \left( (v, \langle v, s_{\text{samp}}(u) \rangle), (v, \langle v, D \rangle) \right) + SD \left( (v, \langle v, D \rangle), (v, U_{\{0,1\}}) \right) \quad \text{(Triangle inequality)} \\
\leq \negl(\lambda) + \negl(\lambda). \quad \text{(Equation 2 and Theorem 15)}
\]

This completes the proof of Claim 30.

Now, consider the following adversary \( A \) that perturbs \( n/2 \) symbols and does the following. (Observe that \( n/2 < (1 - \sqrt{R})n \) for \( R < 1/4 \) and, hence, the adversary is within budget.)

1. \( A \) decodes \( L\text{Enc}(b, \text{Sign}(sk, b)) \) to find \( b \). Let \( \tilde{b} = 1 - b \). It forges a valid signature \( \sigma = \text{Sign}(sk, \tilde{b}) \) and encode it \( L\text{Enc}(\tilde{b}, \sigma) \).

2. Now, for a random subset \( T \subseteq \{1, 2, \ldots, n\} \) of size \( |T| = n/2 \), \( A \) replaces \( L\text{Enc}(b, \text{Sign}(sk, b)) \) with \( L\text{Enc}(\tilde{b}, \sigma) \) on those \( i \in T \). Then, after perturbation, the string \( L\text{Enc}(b, \text{Sign}(sk, b)) \) becomes a random string that has Hamming distance exactly \( n/2 \) from both \( (0, \text{Sign}(sk, 0)) \) and \( (1, \text{Sign}(sk, 1)) \). Let us call this distribution \( X \). Note that \( X \) is independent of \( b \).

Therefore, after the perturbation, the perturbed instance is statistically close to

\[
([u], [v], [vk], X, [U_{\{0,1\}}]),
\]

which is independent of \( b \). Hence, the learner’s output will not agree with \( b \) with probability \( \geq 1/2 - \negl(\lambda) \). Putting everything together, we have

\[
\begin{align*}
\mathbb{E}_{S \leftarrow D^h_{n}; f \leftarrow L(S^h, \lambda)} \left[ \text{Risk}_{d, f}(h, D, f) \right] \\
\geq \Pr_{S \leftarrow D^h_{n}; f \leftarrow L(S^h, \lambda)} \left[ z \in \text{Good} \right] \\
+ \Pr_{S \leftarrow D^h_{n}; f \leftarrow L(S^h, \lambda)} \left[ z \notin \text{Good} \right] \cdot \mathbb{E}_{S \leftarrow D^h_{n}; f \leftarrow L(S^h, \lambda)} \left[ \text{Risk}_{d, f}(h, D, f) \right] | z \notin \text{Good} \\
\geq \negl(\lambda) + (1 - \negl(\lambda)) \cdot \left( \frac{1}{2} - \negl(\lambda) \right) > 1/3.
\end{align*}
\]

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A Additional preliminaries

A.1 Cryptographic primitives

Definition 31 (Computational indistinguishability). We say two ensembles of distributions $X = \{X_\lambda\}_{\lambda \in \mathbb{N}}$ and $Y = \{Y_\lambda\}_{\lambda \in \mathbb{N}}$ are computationally indistinguishable if for any probabilistic polynomial-time (PPT) algorithm $A$, it holds that

$$\left| \Pr_{x \leftarrow X_\lambda} [A(x) = 1] - \Pr_{y \leftarrow Y_\lambda} [A(y) = 1] \right| = \text{negl}(\lambda).$$

Definition 32 (One-way function). An ensemble of functions $\{f_\lambda : \{0,1\}^\lambda \rightarrow \{0,1\}^\lambda\}_\lambda$ is called a one-way function if for all probabilistic polynomial-time probabilistic algorithm $A$, it holds that

$$\Pr \left[ x \leftarrow \{0,1\}^\lambda, y = f_\lambda(x), x' \leftarrow A(1^\lambda, y) : f_\lambda(x') = y \right] = \text{negl}(\lambda).$$

A.2 Coding theory

Fact 33. The following facts hold about the Reed-Solomon code.

- The distance of the Reed-Solomon code is $d = n - k + 1$. Moreover, the decoding is possible efficiently: there is a PPT algorithm that maps any erroneous codeword that contains up to $\leq (n-k)/2$ errors to the nearest correct (unique) codeword. In other words, one can efficiently correct up to $\frac{1-R}{2}$ fraction of errors, where $R$ is the code’s rate.
The encoding of a random message is \( k \)-wise independent. That is, for all subset \( S \subseteq \{1, 2, \ldots, n\} \) such that \(|S| \leq k\), the following distribution

\[
\begin{cases}
  m \leftarrow \mathbb{F}^k, & c = m \cdot G \\
  \text{Output } c_S
\end{cases}
\]

is uniform over \( \mathbb{F}^{|S|} \). This follows from the fact that any \( \leq k \) columns of the generator matrix of the RS code is full-rank.

B Missing Proofs

B.1 Proof of Theorem 15

The theorem follows from the following derivation.

\[
\begin{align*}
\text{SD} \left( Y, \langle X, Y \rangle, (Y, U_{\{0,1\}}) \right) \\
= \mathbb{E}_{y \leftarrow Y} \text{SD} \left( \langle X, y \rangle, U_{\{0,1\}} \right) \\
= \frac{1}{2} \cdot \mathbb{E}_{y \leftarrow Y} \left[ \left| \Pr \left[ \langle X, y \rangle = 0 \right] - \Pr \left[ \langle X, y \rangle = 1 \right] \right| \right] \\
\leq \frac{1}{2} \cdot \sqrt{\mathbb{E}_{y \leftarrow Y} \left[ \left( \Pr \left[ \langle X, y \rangle = 0 \right] - \Pr \left[ \langle X, y \rangle = 1 \right] \right)^2 \right]} \quad \text{(Jensen’s inequality)} \\
= \frac{1}{2} \cdot \sqrt{\mathbb{E}_{y \leftarrow Y} \left[ \Pr \left[ \langle X, y \rangle = 0 \right] - \Pr \left[ \langle X, y \rangle \neq \langle X', y \rangle \right] \right]} \\
= \frac{1}{2} \cdot \sqrt{\mathbb{E}_{y \leftarrow Y} \left[ \Pr \left[ \langle X - X', y \rangle = 0 \right] - \Pr \left[ \langle X - X', y \rangle = 1 \right] \right]} \\
= \frac{1}{2} \cdot \sqrt{\mathbb{E}_{y \leftarrow Y} \left[ \frac{1}{2} \right]} \quad \text{(when } x \neq x'\text{, the inner term is always 0)} \\
= \frac{1}{2} \cdot \sqrt{\frac{1}{2} \cdot \sum_{\omega} \left( \Pr \left[ X = \omega \right] \right)^2} \\
\leq \frac{1}{2} \cdot \sqrt{\frac{1}{2} \cdot \sum_{\omega} \Pr \left[ X = \omega \right] \cdot 2^{-H_{\infty}(X)}} \quad \text{(By definition of min-entropy)} \\
= \frac{1}{2} \cdot \sqrt{2^{-H_{\infty}(X)-1}} \leq \varepsilon
\end{align*}
\]

B.2 Proof of Theorem 17

Dodis and Smith [2005] proved this theorem for \( \mathbb{F}_2 \). We are simply revising their proof for the field \( \mathbb{F}_{2^\ell} \). Within this proof, we shall use \( \mathbb{F} \) for \( \mathbb{F}_{2^\ell} \). We need the following claims.

Claim 34 (Parseval’s identity). \( \sum_{\alpha} \text{bias}(X, \alpha)^2 = |\mathbb{F}|^n \cdot \sum_{\omega} \left( \Pr \left[ X = \omega \right] \right)^2 \).

Proof. Observe that

\[ \sum_{\alpha} \text{bias}(X, \alpha)^2 \]
= \sum_\alpha \left( \mathbb{E}_{x \leftarrow X} \left[ (-1)^{\text{Tr}(x,\alpha)} \right] \right)^2 \\
= \sum_\alpha \mathbb{E}_{x,x' \leftarrow X} \left[ (-1)^{\text{Tr}(x,\alpha)} \cdot (-1)^{\text{Tr}(x',\alpha)} \right] \\
= \sum_\alpha \mathbb{E}_{x,x' \leftarrow X} \left[ (-1)^{\text{Tr}(x+x',\alpha)} \right] \quad \text{(Since the trace map is additive)} \\
= \mathbb{E}_{x,x' \leftarrow X} \left[ \sum_\alpha (-1)^{\text{Tr}(x+x',\alpha)} \right] \\
= |F|^n \Pr_{x,x' \leftarrow X} [x=x'] . \\

Here, we use the fact that, when \( x \neq x' \), the inner term is 0 as the trace map maps half of the field to 0 and the other half to 1.

Note that the last line is exactly equal to

\[ |F|^n \cdot \sum_\omega (\Pr [X = \omega])^2. \]

Claim 35 (Bias of Convolution is product of bias). \( \text{bias}(X + Y, \alpha) = \text{bias}(X, \alpha) \cdot \text{bias}(Y, \alpha) \).

Proof. Observe that

\[
\text{bias}(X + Y, \alpha) \\
= \sum_\omega \Pr [X + Y = \omega] \cdot (-1)^{\text{Tr}(\omega,\alpha)} \\
= \sum_\omega \sum_{\omega'} \Pr [X = \omega'] \Pr [Y = \omega - \omega'] \cdot (-1)^{\text{Tr}(\omega,\alpha)} \\
= \sum_{\omega''} \sum_\omega \Pr [X = \omega] \Pr [Y = \omega''] \cdot (-1)^{\text{Tr}(\omega' + \omega'', \alpha)} \\
= \left( \sum_{\omega'} \Pr [X = \omega'] \cdot (-1)^{\text{Tr}(\omega',\alpha)} \right) \cdot \left( \sum_{\omega''} \Pr [Y = \omega''] \cdot (-1)^{\text{Tr}(\omega'',\alpha)} \right) \\
= \text{bias}(X, \alpha) \cdot \text{bias}(Y, \alpha). \]

Given these two claims, we prove the theorem as follows.

\[ \text{SD} (X + Y, U_F^n) \]
\[ = \frac{1}{2} \cdot \sum_\omega |\Pr [X + Y = \omega] - \Pr [U_F^n = \omega]| \]
\[ \leq \frac{1}{2} \cdot \sqrt{|F|^n \cdot \sum_\omega (\Pr [X + Y = \omega] - \Pr [U_F^n = \omega])^2} \quad \text{(Cauchy-Schwartz)} \]
\[ = \frac{1}{2} \cdot \sqrt{\sum_\alpha (\text{bias}(X + Y, \alpha) - \text{bias}(U_F^n, \alpha))^2} \quad \text{(Parseval)} \]
\[ = \frac{1}{2} \cdot \sqrt{\sum_{\alpha \neq 0^n} \text{bias}(X + Y, \alpha)^2} \quad \text{(Since bias}(U_F^n, \alpha) = 0 \text{ for all } \alpha \neq 0^n.) \]
\[
\frac{1}{2} \sqrt{\sum_{\alpha \neq 0^n} \text{bias}(X, \alpha)^2 \cdot \text{bias}(Y, \alpha)^2} \quad \text{(By Claim 35)}
\]

\[
\leq \frac{\varepsilon}{2} \sqrt{\sum_{\alpha \neq 0^n} \text{bias}(X, \alpha)^2} \quad \text{(Since Y is small-biased)}
\]

\[
\leq \frac{\varepsilon}{2} \sqrt{\left| F \right|^n \sum_{\omega} \left( \Pr [X = \omega] \right)^2} \quad \text{(Parseval)}
\]

\[
= \frac{\varepsilon}{2} \sqrt{\left| F \right|^n \sum_{\omega} \Pr [X = \omega] \cdot 2^{-H_\infty(X)}} \quad \text{(Definition of min-entropy)}
\]

\[
= \frac{\varepsilon}{2} \sqrt{\left| F \right|^n \cdot 2^{-H_\infty(X)}}
\]

\[
= 2^{\frac{\varepsilon}{2} - 1} \cdot \varepsilon
\]

B.3 Proof of Theorem 18

We divide all possible linear tests \( \alpha \) into two cases.

- **Small linear tests are fooled by RS code.** We say that \( \alpha \) is a small linear test if \(|\{i : \alpha_i \neq 0\}| \leq Rn\). By Fact 33, a random codeword projects onto any \( \leq Rn \) coordinates is always a uniform distribution. Hence, \( \langle D, \alpha \rangle \) is always uniform. Consequently, \( \text{bias}(D, \alpha) = 0 \) for all small linear test.

- **Large linear tests are fooled by the noise.** Suppose \( \alpha \) is such that \(|T| > Rn\), where \( T = \{i : \alpha_i \neq 0\} \). Observe that \( S \) is a random subset of size \( s \) and \( T \) is a fixed set of size \( > Rn \). Clearly, \( S \cap T = \emptyset \) happens with probability \( \leq (1 - R)^s \). Now, conditioned on the event that \( S \cap T \neq \emptyset \), we again have \( \langle D, \alpha \rangle \) is a uniform distribution (because of the random noise). Consequently, for large \( \alpha \), we have \( \text{bias}(D, \alpha) \leq (1 - R)^s \).

Therefore, for all possible \( \alpha \), \( \text{bias}(D, \alpha) \) is small. Hence, the theorem follows.