Gently modulating opto-mechanical systems

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We introduce a framework of opto-mechanical systems that are driven with a mildly amplitude-modulated light field, but that are not subject to classical feedback or squeezed input light. We find that in such a system one can achieve large degrees of squeezing of a mechanical micromirror – signifying quantum properties of opto-mechanical systems – without the need of any feedback and control, and within parameters reasonable in experimental settings. Entanglement dynamics is shown of states following classical quasi-periodic orbits in their first moments. We discuss the complex time-dependence of the modes of a cavity-light field and a mechanical mode in phase space. Such settings give rise to certifiable quantum properties within experimental conditions feasible with present technology.

Periodically driven quantum systems exhibit a rich behavior and display non-equilibrium properties that are absent in their static counterparts. By appropriately exploiting time-periodic driving, strongly correlated Bose-Hubbard-type models can be dynamically driven to quantum phase transitions [1]. Systems can be dynamically decoupled from their environments to avoid decoherence in quantum information science [2], and quite intriguing dynamics of Rydberg atoms strongly driven by microwaves [3] can arise. It has also been noted that such time-dependent settings may give rise to entanglement dynamics in oscillating molecules [4]. A framework of such periodically driven systems is provided by the theory of linear differential equations with periodic coefficients or inhomogeneities, including Floquet’s theorem [5].

In this work, we aim at transferring such ideas to describe a new and in fact quite simple regime of opto-mechanical systems, of micromirrors as part of a Fabry-Perot cavity [6, 7, 8, 9]: So to one of the settings [10, 11, 12, 13, 14] that are the most promising candidates in the race of exploring certifiable quantum effects involving macroscopic mechanical modes. This is an instance of a regime of driving with mildly amplitude-modulated light. We find that in this regime, high degrees of squeezing below the vacuum noise level can be reached, signifying genuine quantum dynamics. More specifically, in contrast to earlier descriptions of opto-mechanical systems with a periodic time-dependence in some aspect of the description, we will not rely on classical feedback based on processing of measurement-outcomes – a promising idea in its own right in a continuous-measurement perspective [15, 16] – or resort to driving with squeezed light. Instead, we will consider the plain setting of a time-periodic amplitude modulation of an input light. The picture developed here, based in the theory of differential equations, gives rise to a framework of describing such situations. We find that large degrees of squeezing can be reached (complementing other very recent non-periodic approaches based on cavity-assisted squeezing using an additional squeezed light beam [17]). It is the practical appeal of this work that such quantum signatures can be reached without the necessity of any feedback, no driving with additional fields, and no squeezed light input (the scheme by far outperforms direct driving with a single squeezed light mode): In a nutshell, one has to simply gently shake the system in time with the right frequency to have the mechanical and optical modes rotate appropriately around each other, reminding of parametric amplification, and to so directly certify quantum features of such a system.

Time-dependent picture of system. Before we discuss the actual time-dependence of the driven system, setting the stage, we start our description with the familiar Hamiltonian of a system of a Fabry-Perot cavity of length $L$ and finesse $F$ being formed on one end by a moving micromirror,

$$H = \hbar \omega_c a \dagger a + \frac{1}{2} \hbar \omega_m (p^2 + q^2) - \hbar G_0 a \dagger a q$$

$$+ i \hbar \sum_{n = -\infty}^{\infty} (E_n e^{-i(\omega_0 + n\Omega)_t} a \dagger - E_n^* e^{i(\omega_0 + n\Omega)_t} a).$$

Here, $\omega_m$ is the frequency of the mechanical mode with quadratures $q$ and $p$ satisfying the usual commutation relations of canonical coordinates, while the bosonic operators $a$ and $a \dagger$ are associated to the cavity mode with frequency $\omega_c$ and decay rate $\kappa = \pi c/(2FL)$. $G_0 = \sqrt{\hbar/(m\omega_m)}$ is the coupling coefficient of the radiation pressure, where $m$ is the effective mass of the mode of the mirror being used. Importantly, we allow for any periodically modulated driving at this point, which can be expressed in such a Fourier series, where $\Omega = 2\pi/\tau$ and $\tau > 0$ is the modulation period. The main frequency of the driving field is $\omega_0$ while the modulation coefficients $\{ E_n \}$ are related to the power of the associated sidebands $\{ P_n \}$ by $|E_n|^2 = 2\kappa P_n/(\hbar \omega_0)$. The resulting dynamics under this Hamiltonian together with an unavoidable coupling of the mechanical mode to a thermal reservoir and cavity losses gives rise to the quantum Langevin equation in the reference frame rotating with frequency $\omega_{\text{ref}}$, $\dot{q} = \omega_{\text{ref}} p$, and

$$\dot{p} = -\omega_m q - \gamma_m p + G_0 a \dagger a + \xi,$$

$$\dot{a} = -\kappa (i \Delta_0) a + i G_0 a q + \sum_{n = -\infty}^{\infty} E_n e^{-i\omega_0 t} + \sqrt{2\kappa a^\dagger a},$$

where $\Delta_0 = \omega_c - \omega_0$ is the cavity detuning. $\gamma_m$ is here an effective damping rate related to the oscillator quality factor $Q$ by $\gamma_m = \omega_m/Q$. The mechanical ($\xi$) and the optical ($a^\dagger a$) noise operators have zero mean values and are characterized by their auto correlation functions which, in the Markovian
approximation, are
\[ \langle \xi(t)\xi(t') + \xi(t')\xi(t) \rangle / 2 = \gamma_m(2n + 1)\delta(t - t') \] (3)
and
\[ \langle \tilde{n}_m(t)\tilde{n}_m(t') \rangle = \delta(t - t'), \]
where \( \tilde{n} = [\exp(\frac{\hbar\omega_m}{k_BT}) - 1]^{-1} \) is the mean thermal phonon number. Here, we have implicitly assumed that such an effective damping model holds \([18]\), which is a reasonable assumption in a wide range of parameters including the current experimental regime.

Semiclassical phase space orbits. Our strategy of a solution will be as follows: we will first investigate the classical phase space orbits of the first moments of quadratures. We then consider the quantum fluctuations around the asymptotic quasi-periodic orbits, by implementing the usual linearization of the Heisenberg equations of motion \([11, 12]\) (excluding the very weak driving regime). Exploiting results from the theory of linear differential equations with periodic coefficients, we can then proceed to describe the dynamics of fluctuations and find an analytical solution for the second moments.

If we average the Langevin equations \([2]\), assuming \( \langle a_1^\dagger a_1 \rangle \simeq |\langle a_1 \rangle|^2 \), \( \langle a_2a_2^\dagger \rangle \simeq \langle a_2 \rangle\langle q_2 \rangle \) (true in the semi-classical driving regime we are interested in), we have a nonlinear differential equation for the first moments. Far away from instabilities and multi-stabilities, a power series ansatz in the coupling \( G_0 \)
\[ \langle O \rangle(t) = \sum_{j=0}^{\infty} \sum_{n=-\infty}^{\infty} O_{j,n} e^{i\Omega t} G_0^n, \] justifying the Fourier expansion
\[ \langle O \rangle(t) = \sum_{j=0}^{\infty} \sum_{n=-\infty}^{\infty} O_{j,n} e^{i\Omega t} G_0^n, \] (4)
Substituting this in Eq. \([2]\), we find the following recursive formulae for the time independent coefficients \( O_{j,n} \),
\[ q_{n,0} = p_{n,0} = 0, \]
\[ a_{n,0} = E_{-n}/(\kappa + i(\Delta_0 + n\Omega)), \] (6)
corresponding to the zero coupling limit \( G_0 = 0 \), while for each \( j \geq 1 \), we have
\[ q_{n,j} = \omega_m \sum_{k=0}^{j-1} \sum_{m=-\infty}^{\infty} a_{m,k} a_{n+m,j-k-1} / \omega_m^2 - n\Omega^2 + i\gamma_m n\Omega, \]
and
\[ p_{n,j} = i\Omega n \overline{q}_{n,j}, a_{n,0} = -i \int \sum_{k=0}^{\infty} a_{m,k} q_{n-m,j-k-1} / \omega_m + \sum_{k=0}^{\infty} \kappa + i(\Delta_0 + n\Omega), \] (7)
Within the typical parameter space, considering only the first terms in the double expansion \([4]\), corresponding to the first side bands, leads to a good analytical approximation of the classical periodic orbits, see Fig. 1. On physical grounds, this is expected to be a good approximation, since \( G_0 \ll \omega_m \), and because high side-bands (of frequency \( n\Omega \)) fall outside the cavity bandwidth, \( n\Omega > 2\kappa \). What is more, the decay behavior of \( E_n \) related to the smoothness of the drive inherits a good approximation in terms of few sidebands.

Quantum fluctuations around the classical orbits. We will now turn to the actual quantum dynamics taking first moments into account separately when writing any operator as \( O(t) = \langle O \rangle(t) + \delta O(t) \). The frame will hence be provided by the motion of the first moments. In this reference frame, as long as \( |\langle a_1 \rangle| \gg 1 \), the usual linearization approximation to \([4]\) can be implemented. In what follows, we will also use the quadratures \( \delta x = (\delta a + \delta a^\dagger)/\sqrt{2} \) and \( \delta y = -i(\delta a - \delta a^\dagger)/\sqrt{2} \), and the analogous input noise quadratures \( x_1, y_1 \). For the vector of all quadratures we will write \( u = (\delta q, \delta p, \delta x, \delta y)^T \), with \( n = (0, \xi, \sqrt{2\kappa} x_m^*, \sqrt{2\kappa} y_m^*)^T \) being the noise vector \([11, 13]\). Then the time-dependent inhomogeneous equations of motion arise as \( \ddot{u}(t) = A(t) u(t) + n(t) \),
\[ A(t) = \begin{bmatrix} 0 & \omega_m & 0 & 0 \\ -\gamma_m & G_{x_1}(t) & G_{y_1}(t) & G_{y_1}(t) \\ -G_{x_1}(t) & 0 & -\kappa & \Delta(t) \\ G_{x_1}(t) & 0 & -\Delta(t) & -\kappa \end{bmatrix}, \] (8)
where the real \( A(t) \) contains the time-modulated coupling matrices and the detuning as \( G(t) = G_{x_1}(t) + iG_{y_1}(t) \),
\[ G(t) = \sqrt{2}(\alpha(t)G_0, \Delta(t) = \Delta_0 - G_0(q(t)). \] (9)
From now on we will consider quasi-periodic orbits only, so the long-time dynamics following the initial one, where the first moments follow a motion that is \( \tau \)-periodic. Then, \( A \) is \( \tau \)-periodic, and hence
\[ A(t) = A(t + \tau) = \sum_{n=-\infty}^{\infty} A_n e^{i\Omega nt}. \] (10)
In turn, if all eigenvalues of \( A(\cdot) \) having negative real parts for all \( t \in [0, \tau] \) is a sufficient condition for stability. From the Markovian assumption \([3]\), we have
\[ \langle n_1(t)n_2(t') + n_2(t')n_1(t) \rangle / 2 = \delta(t - t')D_{i,j}, \] (11)
where $D = \text{diag}(0, \gamma_m, 2\bar{n}+1, \kappa, \kappa)$. The formal solution of Eq. (8) is

$$u(t) = U(t, t_0)u(t_0) + \int_{t_0}^{t} U(t, s)n(s)ds,$$  

where $U(t, t_0)$ is the principal matrix solution of the homogeneous system satisfying $U(t, t_0) = A(t)U(t, t_0)$ and $U(t_0, t_0) = 1$. From Eqs. (8, 12), we have as an equation of motion of the covariance matrix (CM) of the two modes

$$\dot{V}(t) = A(t)V(t) + V(t)A^T(t) + D.$$  

Here, the CM $V(.)$ is the $4 \times 4$ matrix with components $V_{i,j} = \langle u_i u_j + u_j u_i \rangle/2$, collecting the second moments of the quadratures. This is again an inhomogeneous differential equation for the second moments which can readily be solved using quadrature methods, providing numerical solutions that will be used to test and justify analytical approximate results using quadrature methods, providing numerical solutions that can be reliably be solved using experimental data. Moreover, now the coefficients and not the inhomogeneity are $\tau$-periodic, $A(t) = A(t + \tau)$. Again, we can invoke results from the theory of linear differential equations to Eq. (13) [5]: We find that in the long time limit, the CM is periodic and can be written as

$$V(t) = \sum_n V_n e^{i n \Omega t}.$$  

An analytical solution for $V(.)$, is most convenient in the Fourier domain, $\tilde{f}(\omega) = \int_{-\infty}^{+\infty} e^{-i \omega t} f(t) dt$, giving rise to

$$-i\omega \tilde{u}(\omega) + \sum_{n=-\infty}^{\infty} A_n \tilde{u}(\omega - n \Omega) = -\tilde{n}(\omega).$$  

If $A_n \neq 0$, corresponding to no-modulation, we are in the usual regime where the spectra are centered around $\pm \omega_m$ for the mechanical oscillator and around $\pm \Delta$ for the optical mode. The modulation introduces sidebands shifted by $\pm n \Omega$. If the modulation is weak, only the first two sidebands at $\pm \Omega$ significantly contribute. For strong modulation also further sidebands play a role: Disregarding higher sidebands means truncating the summation to $\pm \bar{n}$, (valid if $\omega \pm n \Omega \approx 0$). Then Eq. (15) can be written as $\tilde{A}(\omega) = \tilde{n}(\omega)$, where $\tilde{A}(\omega) = \begin{pmatrix} \tilde{A}(-2\Omega) & \ldots & \tilde{A}(-\Omega) & \tilde{A}(0) & \tilde{A}(\Omega) & \ldots & \tilde{A}(2\Omega) \end{pmatrix}$ and $\tilde{n}(\omega)$ are $4 \times (2\bar{n} + 1)$ vectors, while, in terms of 4 blocks,

$$\tilde{A}(\omega) = \begin{bmatrix} B_{-N} & A_{-1} & A_{-2} & \cdots & A_{-2N} \\ A_1 & B_{-(N-1)} & A_{-1} & \cdots & \vdots \\ A_2 & A_1 & B_{-(N-2)} & \cdots & \vdots \\ \vdots & \vdots & \ddots & \cdots & \vdots \\ A_{2N} & \cdots & A_1 & B_N \end{bmatrix} \quad \text{where } B_k = A_0 - i(\omega + k\Omega).$$

From the correlation properties of the noise vector $n(.)$, we have that

$$\phi_{x,j}(\omega, \omega') = \langle \tilde{n}_i(\omega)\tilde{n}_j^*(\omega') + \tilde{n}_j(\omega')\tilde{n}_i^*(\omega) \rangle/2 = \sum_{n=-2N}^{2N} \delta(\omega - \omega' - n\Omega)D_n,$$  

where $D_0 = \text{diag}(D, D, \ldots, D)$, then $D_1$ is the matrix that has $D$ on all first right off diagonal blocks, $D_2$ on the second off diagonals, with $D_n$ analogously defined, and $D_{-n} = D_n^T$. We now define the two frequency correlation function as $\tilde{V}_{ij}(\omega, \omega') = \langle \tilde{n}_i(\omega)\tilde{n}_j^*(\omega') + \tilde{n}_j(\omega')\tilde{n}_i^*(\omega) \rangle/2$. We have $\tilde{V}(\omega, \omega') = \tilde{A}^{-1}(\omega)\phi(\omega, \omega')[\tilde{A}^{-1}(\omega')]^T$. We are interested only on the central $4 \times 4$ block of $\tilde{V}$ which we call $\tilde{V}(\omega, \omega') = \{\tilde{V}(\omega, \omega')\}_{ij}$. From $\phi(\omega, \omega')$, we find

$$\tilde{V}(\omega, \omega') = \sum_{n=-2N}^{2N} \tilde{V}_n(\omega)\delta(\omega - \omega' - n\Omega),$$  

where $\tilde{V}_n(\omega) = [\tilde{A}^{-1}(\omega)D_n[\tilde{A}^{-1}(\omega - n\Omega)]]_{ij}$. This means that the driving modulation correlates different frequencies, but only if they are separated by a multiple of the modulation frequency $\Omega$. By inverse Fourier transforms we recover the time periodic expression for the CM, where the all the components $V_n$ are given by the integral of their noise spectra, i.e.,

$$V_n = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \tilde{V}_n(\omega)d\omega.$$  

Squeezing and entanglement modulation. We will now see that the mild amplitude-modulated driving in the cooling regime is exactly the tool that we need in order to arrive at strong degrees of squeezing, in the absence of feedback or squeezed light. We will apply the previous general theory to setting of an optomechanical system that is experimentally feasible with present technology. In fact, all values that we assume have been achieved already and reported on in publications with the exception of assuming a relatively good mechanical $Q$-factor. The reasonable set of experimental parameters [9] that we assume is $L = 255\text{mm}$, $F = 1.4 \times 10^4$, $\omega_m = 2\pi \text{MHz}$, $Q = 10^9$, $m = 150\text{ng}$, $T = 0.1\text{K}$. Then we consider the – in the meantime well known – self-cooling regime [7] in which a cavity eigen-mode is driven with a red detuned laser $\Delta \Omega \simeq \omega_m$ (with wavelength $\lambda = 1064\text{nm}$), but we also add a small sinusoidal modulation to the input amplitude with a frequency $\Omega = 2\omega_m$, so twice the mechanical frequency. To be more precise we choose the power of the carrier component equal to $P_0 = 10\text{mW}$, and the power of the two modulation sidebands equal to $P_{\pm 1} = 2\text{mW}$.

We approximate the asymptotic classical mean values in Eq. (14) by truncating the series only to the first terms with indexes $j = 0, \ldots, 3$ and $n = -1, 0, 1$, giving rise to a good approximation. Fig. 1 shows that, after less than 50 modulation periods, the first moments reach quasi-periodic orbits which are well approximated by our analytical results.
FIG. 2: (a) Variance of the mirror position and (b) light-mirror entanglement $E_N$ as functions of time. In both (a) and (b) the non-modulated driving regime (blue), the modulated driving regime (green) and the numerical solutions (black dashed/dotted) are plotted. (a) also shows the standard quantum limit (red dashed) at $1/2$, the minimum eigenvalue of the mirror covariance matrix (black dashed) and its analytical estimation [20] in the RWA (orange).

In order to calculate the variances of the quantum fluctuations around the classical orbits, we truncate the sum in Eq. (15) to $N = 2$ and we apply all the previous theory to find the covariance matrix $V$. In Fig. 2 we compare two regimes: with or without ($P_{\pm 1} = 0$) modulation (computed analytically and numerically). We see that the modulation of the driving field causes the emergence of significant true quantum squeezing below the Heisenberg limit of the mechanical oscillator state and also the interesting phenomenon of light-mirror entanglement oscillations. This dynamics reminds of the effect of parametric amplification [13] [16], as if the spring constant of the mechanical motion was varied in time with just twice the frequency of the mechanical motion, leading to the squeezing of the mechanical mode. For related ideas of reservoir engineering, making use of bichromatic microwave coupling to a charge qubit of nano-mechanical oscillators, see Refs. [23]. Here, it is a more complicated joint dynamics of the cavity field and the mechanical mode – where the dynamics of the first and the second moments can be separated – which for large times yet yields a similar effect. Indeed, this squeezing can directly be measured when considering the output power spectrum, following Ref. [20], and no additional laser light is needed for the readout, giving hence rise to a relatively simple certification of the squeezing. Entanglement here refer to genuine quantum correlations between the mirror and the field mode, as being quantified by the logarithmic negativity defined as $E_N(\rho) = \log \| \rho^F \|_1$, essentially the trace-norm of the partial transpose, which is a proper entanglement measure [21] [22]. The minimum eigenvalue of the mirror covariance matrix – the logarithm thereof typically referred to as single mode squeezing parameter – is almost constant and this means that the state is always squeezed but that the squeezing direction continuously rotates in phase space with the same period of the modulation. Calling this rotating squeezed quadrature $\delta x_R$, a rough estimate of its variance can be calculated in the rotating-wave approximation (RWA), compare, e.g., Ref. [24],

$$\langle \delta x_R^2 \rangle = \frac{1}{2} \bar{n} - \frac{2\kappa(G_0 - G_{-1})(G_0\bar{n} + G_{-1}(\bar{n} + 1))}{\gamma_m + 2\kappa(G_0 - G_{-1}^2 + 2\gamma_m \kappa)}$$

with $\{G_n\}$ being defined as $G(t) = \sum_{n=-\infty}^{\infty} G_n e^{i\omega t}$.

Conclusions and outlook. In this work we have introduced a framework of describing periodically amplitude-modulated optomechanical systems. Interestingly, such a surprisingly simple setting feasible with present technology [9] leads to a setting showing high degrees of mechanical squeezing, with no feedback or additional fields needed. We hope that such ideas contribute to experimental studies finally certifying first quantum mechanical effects in macroscopic mechanical systems, constituting quite an intriguing perspective.

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values of $2|V^{1/2}iS^{1/2}|$, where $S$ is the $4 \times 4$-matrix with $S_{1,2} = S_{4,3} = 1$, $S_{2,1} = S_{3,4} = -1$ and zero otherwise.

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**APPENDICES**

In this appendix, we will summarize a number of additional statements useful for the main text.

**Differential equations**

In this appendix, we will consider several issues related to solutions of inhomogeneous first-order differential equations that are used in the main text. Most of the presented material is standard material from the theory of time-dependent periodic linear systems. At the end, yet, we will consider a bound to the error made when assuming a periodic solution that is tailored to the specific situation studied above. Consider a linear first-order system,

$$\dot{x}(t) = B(t)x(t),$$

where $B(t)$ being some complex square matrix with entries dependent on $t \geq t_0$ with some $t_0 > 0$. Then the linear first-order system has a unique solution for $x(t_0) = x_0$ for all times $t > t_0$. The principal matrix solution is the solution of

$$P(t, t_0) = B(t)P(t, t_0),$$

with $P(t, t_0) = 1$. Floquet’s theorem now states the following:

Lemma 1 (Floquet’s theorem) If $B(\cdot)$ is periodic, $B(t) = B(t + \tau)$ for some $\tau > 0$ for all $t \geq t_0$, then the principal matrix solution has the form

$$P(t, t_0) = X(t, t_0)e^{(t-t_0)Y(t_0)},$$

where the matrices $X(\cdot, \cdot)$ and $Y(\cdot)$ are $\tau$-periodic in all their arguments and $X(t_0, t_0) = 1$.

So if one factors out an exponential, the remainder is periodic in time. The eigenvalues of the monodromy matrix

$$M(t_0) = P(t_0 + \tau, t_0)$$

are then referred to as Floquet multipliers, and the eigenvalues of $Y(t_0)$ are known as Floquet exponents. Floquet exponents are $t_0$ independent, in fact the matrix $Y(t_0)$ is similar to $Y(t'_0)$ for any $t'_0$. If all Floquet exponents have a negative real part, the system is asymptotically stable.

Lemma 2 (Solution to inhomogeneous problem) The solution to the inhomogeneous system

$$\dot{x}(t) = B(t)x(t) + g(t),$$

with initial condition $x(t_0) = x_0$ is given by

$$x(t) = P(t, t_0)x_0 + \int_{t_0}^t ds P(t, s)g(s).$$

We will now show that under simple conditions that, when both $B(\cdot)$ and $g(\cdot)$ are $\tau$-periodic, we will asymptotically arrive at a solution with the same time period. Let us define the interval $I = [0, \tau]$. In order to prepare that statement, we will use the following bound.

Lemma 3 (Bound from Floquet exponents) In the above notation, if the system is stable and if $t - t_0 > 1$, then

$$\max_{u \in I} \|e^{(t-t_0+u)Y(t_0-u)}\| \leq c n(t - t_0 + \tau)^{-n} e^{\lambda(t-t_0)},$$

where

$$c = \max_{u \in I} \|W(u)\| \|W^{-1}(u)\|,$$

where $W(u)$ is a similarity transformation that brings $Y(u)$ to a Jordan normal form.

Here,

$$\lambda = \max_j \text{re}(\lambda_j),$$

where $\lambda_j$ are the Floquet exponents. The norm $\|\|\|$ is the norm

$$\|A\| = \sup \frac{\|Ax\|}{\|x\|},$$

induced by the usual Euclidean vector norm $\|\|$. So up to a constant, the convergence is essentially governed by the largest real part of the Floquet exponents. To show this bound, note that

$$e^{(t-t_0+u)Y(t_0-u)} = W^{-1}(t_0 - u) \left( \oplus_j M_j(\delta)e^{\lambda_j} \right) W(t_0 - u),$$

where

$$M_j(\delta) = \begin{bmatrix} 1 & \delta & \frac{\delta^2}{2!} & \cdots & \frac{\delta^{(n_j-1)}}{(n_j-1)!} \\ 0 & 1 & \delta & \frac{\delta^2}{2!} & \cdots & \frac{\delta^{(n_j-1)}}{(n_j-1)!} \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots \\ 0 & 0 & \cdots & 1 \end{bmatrix}, \quad \delta = t - t_0 + u,$$

and $n_j$ is the dimension of the Jordan block associated with the Floquet exponent $\lambda_j$, giving rise to the above expression [27], by acknowledging that

$$\|M_j(\delta)\| \leq n_j \delta^{n_j-1} \leq n \delta^{n-1}$$

for all $j$, by bounding the $\|\|$ norm from above by the $\|\|$ norm, and since $\delta > 1$. We can now turn back to our original problem of the periodicity of the asymptotic solution.
Theorem 1 (Asymptotic periodicity) If the system is stable and both \(B(.)\) and \(g(.)\) are \(t\)-periodic (including the case of constant \(B\) or \(g\)), then the solution \(x(.)\) of the inhomogeneous problem is asymptotically also \(t\)-periodic, with known bounds, in time exponentially suppressed, on the error made.

From [26], with the new integration variable \(u = s - \tau\), one can write for the solution of the inhomogeneous system

\[
x(t + \tau) = P(t + \tau, t_0)x_0 + \int_{t_0-\tau}^{t} du \, P(t + \tau, u + \tau)g(u + \tau)
\]

which is bounded from above by

\[
\tau m \max_{v \in I} \|g(v)\| \max_{v \in I} \|e^{(s-t_0+\tau)Y(s-v)}\|,
\]

where

\[
m = \max_{(t',t) \in I^2} \|X(t, t')\|.
\]

Finally, by using Lemma 3, we have

\[
\|x(t + \tau) - x(t)\| \leq e^{\lambda s(t-t_0)} m c n (t - t_0 + \tau)^{n-1} \times \left(2\|x_0\| + \tau \max_{v \in I} \|g(v)\|\).
\]

Now it is clear that, in the long time limit \((t - t_0 \to \infty)\), the first factor \(e^{\lambda s(t-t_0)}\) exponentially suppresses the RHS of Eq. (39), proving the theorem.

Squeezing in the rotating wave approximation

Suppose that we are in the particular resonance condition such that

\[
\Delta(t) = \omega_m + \sum_{n \neq 0} \Delta_n e^{in\Omega t}
\]

and \(\Omega = 2\omega_m\). Then, in the limit of \(\omega_m \gg |G(t)|, \kappa\), a simple analytical expression for the covariance matrix can be obtained in the rotating wave approximation. We observe that we have two arbitrary degrees of freedom that we can choose to simplify the calculation: The global phase of the driving laser and the initial time corresponding to \(t = 0\). If we expand the coupling coefficient \(G(t) = \sum_{n=-\infty}^{\infty} G_n e^{in\Omega t}\), then we can assume without loss of generality assume that \(G_0, G_{-1} \in \mathbb{R}\).

Now we move to an interaction picture introducing the slowly varying bosonic operators,

\[
a_s = a_s e^{i\omega_m t}, \quad b_s = b_s e^{i\omega_m t}.
\]

In this reference frame, if we neglect terms rotating at frequency \(2\omega_m\), we obtain an equation analogous to Eq. (13):

\[
\dot{V}_s(t) = A_s V_s + V_s A_s^T + D_s
\]

where,

\[
A_s = \frac{1}{2} \begin{bmatrix}
-\gamma_m & 0 & 0 & G_{-1} - G_0 \\
0 & -\gamma_m & G_{-1} + G_0 & 0 \\
G_{-1} + G_0 & 0 & -2\kappa & 0 \\
0 & 0 & -2\kappa & -2\kappa
\end{bmatrix},
\]

and

\[
D = \text{diag} (\gamma(n + 1/2), \gamma(n + 1/2), \kappa, \kappa).
\]

We observe that, in the RWA, only the coefficients \(G_0\) and \(G_{-1}\) matter. They correspond to the cooling and the heating sidebands of the input driving laser.

The stability condition for the differential Eq. (43) is \(G_1^2 - G_0^2 \leq 2\gamma_m \kappa\), which is always satisfied if the cooling process is predominant with respect to the heating. Differently from Eqs. (13), Eq. (43) has constant coefficients and therefore, in the stable regime, \(V_s(t)\) reaches the an asymptotic constant value \(V_s = \lim_{t \to \infty} V_s(t)\).

The matrix \(V_s\) can be calculated by imposing the derivative in (43) equal to zero and solving the remaining linear system. We report only the mirror variances: \((V_s)_{1,1} = f_{-}, (V_s)_{2,2} = f_{+}\) and \((V_s)_{1,2} = 0\), where

\[
f_{\pm} = \frac{1}{2} \left\{ 1 + \eta \mp \frac{2\kappa(G_0 \pm G_{-1})[G_0 \bar{n} + G_{-1}(\bar{n} + 1)]}{(\gamma_m + 2\kappa)(G_0^2 - G_{-1}^2 + 2\gamma_m \kappa)} \right\}.
\]

Three particular limits. If \(G_{-1} = 0\), which corresponds to a red detuned driving laser without modulation, we observe the usual cooling of the mirror. For \(G_{-1} = G_0\), we recover the QND measurement setting studied in Ref. [15], where a
symmetric driving with opposite detuning couples the light only with a quadrature of the mirror. Indeed, the variance $V_{s11}$ is unaffected by the driving while the other is increased due to radiation pressure noise. Finally if $G_{-1} < G_0$, we observe that the mirror can reach a steady state which is both cooled and squeezed without the need of any feedback and control.

Squeezed environments versus modulation

In recent work [17] it has been shown that it is possible to squeeze the mechanical mode by using a constant driving field with $\Delta = \omega_m$ and an additional squeezed vacuum beam with frequency resonant with cavity $\omega_s = \omega_c$. In the RWA formulation used in the previous section, this corresponds to a coefficient matrix $A'$ having the same structure as in Eq. (44) but with $G_0 = G'$ and $G_{-1} = 0$, while

$$D' = \text{diag}(\gamma(n + 1/2), \gamma(n + 1/2), s\kappa, s^{-1}\kappa),$$

(47)

where $s$ is a parameter quantifying the squeezing of the environment. If we solve the system for the mirror variances, instead of Eq. 46 we get

$$f'_\pm = \frac{1}{2} + n - \frac{\kappa G^2(2n + 1 - s_{\pm}^1)}{(\gamma_m + 2\kappa)(G^2 + 2\gamma_m\kappa)},$$

(48)

Now we show that, within the validity of the RWA, for what concerns the squeezing of the mirror, this approach is equivalent to our scheme in which the cavity is driven by a single amplitude modulated laser and no squeezing is required. In fact, by choosing the modulation parameters such that

$$G_0 = G(s^{-1/2} + s^{1/2})/2, \quad G_1 = G(s^{-1/2} - s^{1/2})/2,$$

(49)

(50)

the previous formula given in (46) reduces exactly to Eq. 48. This means that the two approaches are formally equivalent as far as the squeezing is concerned and the choice between one or the other depends only on technical factors.

Finally, we may ask if a good mechanical squeezing is possible by just driving the cavity with a single squeezed non-modulated field with $\Delta = \omega_m$. Differently from the proposal of Ref. [17], in this case the squeezed input noise is not resonant with the cavity mode but rotates with a frequency $\omega_m$. This means that, in the RWA, the squeezing is averaged to zero and the optical environment looks like an effective thermal bath. Therefore, the only convenient choice is to use an additional squeezed beam which is resonant with cavity as in Ref. [17]. Only in this way the squeezing direction of the environment does not rotate and can have a significant effect on the system dynamics.