A geometric approach to the generalized Noether theorem

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We provide a geometric extension of the generalized Noether theorem for scaling symmetries recently presented in [37]. Our version of the generalized Noether theorem has several positive features: it is constructed in the most natural extension of the phase space, allowing for the symmetries to be vector fields on such manifold and for the associated invariants to be first integrals of motion; it has a direct geometrical proof, paralleling the proof of the standard phase space version of Noether’s theorem; it automatically yields an inverse Noether theorem; it applies also to a large class of dissipative systems; and finally, it allows for a much larger class of symmetries than just scaling transformations which form a Lie algebra, and are thus amenable to algebraic treatments.

I. MOTIVATION AND PREVIOUS WORKS

Noether’s theorem is one of the most profound and beautiful results in mathematical physics. This is because it is very easy to state and prove (in fact, it is one of the central topics in elementary courses on mechanics) but its consequences are far-reaching, ranging from the standard conservation of energy and angular momentum in classical mechanics, to the existence of Noether charges in general relativity (one of them being black holes’ entropy [32]), up to the reduction theorems in symplectic, Poisson, and contact geometry [1].

However, it is also well-known that there exist some important transformations that act like symmetries but are not Noether symmetries, for which it has long been believed that it does not exist an associated invariant quantity. The most famous example of such symmetries is that of Kepler scalings

\[ t \rightarrow \lambda^3 t, \quad q \rightarrow \lambda^2 q, \quad S \rightarrow \lambda S, \quad \lambda = \text{const.} \]  

Indeed, it is known that this is a type of symmetry of the Kepler problem (actually, this is the symmetry underlying Kepler’s third law [36]), but it is not of Noether type, as it is clear because e.g. the action and the dynamics are rescaled by the transformation. This type of symmetries that rescale the dynamics by multiplying it by a constant term is sometimes referred to in the literature as a scaling symmetry and they are examples of the more general dynamical similarities [28].

Let us remark that scaling symmetries have profound physical consequences. For instance, they have been employed to derive generalizations of the virial theorem [12, 13, 37]; moreover, it has been recently argued that one can use such

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symmetries in order to reduce a Hamiltonian system to a purely relational description in terms of the observables of the theory, where the (equivalent) dynamics can be shown to be free of some spurious singularities such as the big bang singularity of Einstein’s equations \[28, 29\]. Therefore it is of primary importance to have a general theory of such symmetries that may help recognize and classify them, and even more so if such theory could help identify the associated conserved quantities (if any) that can then be used to perform the reduction of the system, as it is the case for the standard Noether theorem.

Recently, in \[37\] a generalized version of Noether’s theorem that applies to scaling symmetries has been proved. They proved that to any one-parameter family of transformations \( q \rightarrow q'(t') \), \( t \rightarrow t' \) that rescales the Lagrangian (and hence the action) as

\[
L(q', dq'/dt', t') = \Lambda L(q, dq/dt, t)
\]

up to a boundary term, where \( \Lambda \) is a constant, one can always associate a conserved quantity of the form

\[
Q = p \cdot \delta q - H\delta t - S(t)\delta \Lambda,
\]

where \( p = \frac{dL}{dq} \) are the conjugate momenta, \( H \) is the Hamiltonian, and \( S(t) = \int_0^t Ld\tau \) is the on-shell action, namely, the action calculated along the trajectory. For instance, in the case of Kepler scalings \[1\], the associated invariant \[3\] is

\[
Q_K = 2p \cdot q - 3tH_K - S(t),
\]

where

\[
H_K = \frac{p \cdot p}{2m} - \frac{4e}{\sqrt{q \cdot q}}.
\]

is the Kepler Hamiltonian.

One of the most intriguing aspects of such generalized Noether theorem is the fact that the invariants \[3\] are not functions on the phase space of the system, since they include also the on-shell action variable. Therefore, a natural question arising about these invariants is the following

In what sense are the invariants \[3\] first integrals of motion, that is, well-defined functions from the (possibly extended) phase space of the system to the real numbers that are constant along trajectories?

Or, more precisely,

Can we define the generalized Noether theorem geometrically in some appropriate (extension of the) phase space, so that the infinitesimal generators of scaling symmetries are actual vector fields on such manifold and their related invariants \[3\] are actual functions on the manifold?

Moreover, if the above is possible, it would be great if this geometric structure would allow for a simple proof of the generalized Noether theorem, possibly paralleling the one of the standard Noether theorem, and also provide a direct geometric relationship between the infinitesimal generators and their corresponding invariants.

An interesting approach to solve some of the above puzzles has been presented in \[17\], where the authors construct a Noether theorem for scaling symmetries from the variational principle in the vertical extension of the standard phase space. The advantages of such approach are its general scope and the fact that it effectively yields invariants associated with scaling symmetries which do not depend on computing the on-shell action. However, the relationship between the original scaling symmetry and the constructed invariant is not so clear, since it turns out that the scaling symmetry in the original configuration space is not the Noether symmetry associated with the invariant, and that another symmetry can be found to be its associated Noether symmetry. Moreover, the construction is based on performing first the variational formulation in the vertical extension of the tangent space and then projecting back down to the original manifold, which makes the derivation a bit obscure and can hamper the analysis of further questions such as e.g. whether the new symmetries obtained from this construction form a Lie algebra.

On the other hand, the presence of the term \( S(t) \) in \[3\] is remindful of contact Hamiltonian mechanics, where the contact phase space directly includes the on-shell action as an additional dynamical variable, thus serving as a natural arena in which the invariants \[3\] can be described. Indeed, a great deal of work has been devoted recently to the study of the so-called contact Hamiltonian systems, which are Hamiltonian systems on contact manifolds, the “odd-dimensional counterpart of symplectic manifolds” (see e.g. \[2, 7, 9, 15, 19, 33, 34\] for the formal theory and \[5, 8, 10, 14\] and the references therein for some applications). In this context, the standard symplectic phase
space is extended to include an additional direction which in local coordinates turns out to be precisely the on-shell action of the system, and there is a natural way in which standard symplectic Hamiltonian systems can be embedded into contact Hamiltonian systems (and vice versa). Moreover contact Hamiltonian systems can be derived from Herglotz’ variational principle \cite{20, 21, 31}, and a counterpart of the standard Noether theorem for such case has been proposed both in the variational formulation \cite{20, 21, 23} as well as in the geometric setting \cite{16, 18}.

Motivated by all the above considerations, in this work we provide a geometric extension of the generalized Noether theorem for scaling symmetries recently presented in \cite{37}. To do so, we use the geometric formulation for contact Hamiltonian systems in their extended phase space (including time). In this way we will be able to prove very easily a generalized Noether theorem and its inverse that include as a particular case the association of scaling symmetries with their related invariants found in \cite{37}. Moreover, we can prove directly that all the Noether symmetries defined in this way form a Lie algebra. Further benefits of our approach are that the definition of generalized symmetries in this context allows for even more general symmetries than just scaling symmetries of the dynamics, and that it also applies directly to a large class of dissipative systems. Finally, in order to concretely show some of the consequences of our approach, we will present some paradigmatic examples in which the general theory recovers known results and reveals new interesting features.

II. A BRIEF INTRODUCTION TO SYMPLECTIC AND CONTACT HAMILTONIAN MECHANICS

In order to make this paper self-contained, we introduce here the very basic tools from symplectic and contact Hamiltonian systems that are needed in the following. We refer to \cite{7, 9, 15} for more detailed accounts and to \cite{15, 18} for the Lagrangian counterparts.

A. The symplectic case

We start with some standard definitions, that can be found in any textbook on analytical mechanics, e.g. \cite{1}.

**Definition 1.** The symplectic phase space of a mechanical system is the cotangent bundle $T^*Q$, where $Q$ is the configuration manifold of the system, endowed with the canonical symplectic form $\Omega = -d\alpha$, with $\alpha$ being the Liouville 1-form.

In local Darboux coordinates $(q^a, p_a)$, we have that $\alpha = p_a dq^a$, and thus $\Omega = dq^a \wedge dp_a$.

**Definition 2.** A symplectic Hamiltonian system is a triple $(T^*Q, \Omega, H)$, with $H : T^*Q \rightarrow \mathbb{R}$ a sufficiently regular function called the symplectic Hamiltonian.

**Definition 3.** A symplectic Hamiltonian vector field $X_H$ is defined to be the only solution to the condition

$$\iota_{X_H} \Omega = -dH. \quad (6)$$

One can directly check that in Darboux coordinates this leads to the standard Hamilton equations

$$\dot{q}^a = \frac{\partial H}{\partial p_a}, \quad \dot{p}_a = -\frac{\partial H}{\partial q^a}, \quad (7)$$

from which one can recover e.g. the Newtonian dynamics of conservative systems.

Now we are ready to define Noether symmetries and conserved quantities.

**Definition 4.** A Noether symmetry of a symplectic Hamiltonian system is a vector field $Y \in \mathfrak{X}(T^*Q)$ such that $\iota_Y \Omega = -dF$ ($Y$ is Hamiltonian for some Hamiltonian function $F$) and $\mathcal{L}_Y H = 0$ ($Y$ preserves $H$).

**Definition 5.** A conserved quantity is a function $F : T^*Q \rightarrow \mathbb{R}$ such that $\mathcal{L}_{X_H} F = 0$.

Furthermore, we note that the condition \cite{6} provides an isomorphism between the Lie algebra of Hamiltonian vector fields on $T^*Q$ with the Lie bracket and the Lie algebra of functions on $T^*Q$ with the Poisson bracket

$$\{F, G\}_P := \iota_{X_F} \iota_{X_G} \Omega. \quad (8)$$

Equipped with this isomorphism, a Noether symmetry can equivalently be expressed as a Hamiltonian vector field $X_F$ such that $\{F, H\}_P = 0$. Thus, Noether’s theorem can be proved in one line using the antisymmetry of the Poisson bracket, to obtain (see \cite{11} for more comments)
Theorem 1 (Symplectic Noether). $X_F$ is a Noether symmetry of a symplectic Hamiltonian system if and only if $F$ is a conserved quantity.

Now, in order to understand Noether’s theorem from a more general perspective, we need to introduce some general definitions.

Definition 6. A dynamical similarity of a vector field $X$ is a vector field $Y \in \mathfrak{X}(T^*Q)$ such that $[Y, X] = \Lambda X$, for some (in general non-constant) function $\Lambda$.

As a particular but important case of the above, we have

Definition 7. A dynamical symmetry of a vector field $X$ is a vector field $Y \in \mathfrak{X}(T^*Q)$ such that $[Y, X] = 0$.

It is easy to verify that any Noether symmetry is a dynamical symmetry of $X_H$; however the converse is not true, as the following proposition in the particular case $\lambda_1 = \lambda_2$ clearly shows.

Proposition 1. Let $Y \in \mathfrak{X}(T^*Q)$ be such that $\mathcal{L}_Y \Omega = \lambda_2 \Omega$ ($Y$ is a non-strictly canonical symmetry [12, 13, 28]) and $\mathcal{L}_Y H = \lambda_1 H$ ($Y$ rescales $H$). Then $[Y, X_H] = (\lambda_1 - \lambda_2) X_H$.

Proof. We have

$$
\iota_{[X_H, Y]} \Omega = \iota_{X_H} \mathcal{L}_Y \Omega - \mathcal{L}_Y \iota_{X_H} \Omega = \lambda_2 \iota_{X_H} \Omega + d \mathcal{L}_Y H = (\lambda_1 - \lambda_2) dH, \quad (9)
$$

and therefore, since $\Omega$ is non-degenerate, we conclude that $[Y, X_H] = (\lambda_1 - \lambda_2) X_H$. \square

Clearly, by the isomorphism described above, we know that we cannot associate any function of the phase space to a vector field that rescales $\Omega$ (these are not Hamiltonian), and therefore we do not have much hope to extend Noether’s theorem in order to include dynamical symmetries and similarities in the standard phase space. As a particularly relevant example for our exposition, we first note that Kepler scalings (1) can be shown to be induced by the action of the following vector field on the symplectic phase space

$$
Y_{KS} = 2q^a \partial_{q^a} - a \partial_R, \quad (10)
$$

Then we have the following result

Proposition 2. $Y_{KS}$ is a dynamical similarity of Kepler’s Hamiltonian vector field.

Proof. Let $H_K$ be the Kepler Hamiltonian [5] and $\Omega_K$ the canonical symplectic form. Then, as one can directly check, $\mathcal{L}_{Y_{KS}} H_K = -2 H_K$ and $\mathcal{L}_{Y_{KS}} \Omega_K = \Omega_K$, and therefore by Proposition 1 we have $[Y_{KS}, X_{H_K}] = -3 X_{H_K}$. \square

We remark that the result $[Y_{KS}, X_{H_K}] = -3 X_{H_K}$ appearing in the proof of Proposition 2 is expected, as it means that the dynamics after the transformation induced by $Y_{KS}$ is rescaled by $\lambda^{-3}$, with $\lambda$ being the mock parameter along the flow of $Y_{KS}$. Therefore by scaling the time variable as in (1), we would observe no difference in the trajectories, and therefore we can consider $Y_{KS}$ as a “symmetry” of the Kepler dynamics in some sense (note also that the proper scaling of the action in (1) can be recovered by dimensional analysis). However, as a consequence of Proposition 2 and the above discussion, we conclude that it would be nonsensical to look for a conserved quantity associated to $Y_{KS}$ in the standard symplectic phase space.

B. The contact case

As a further generalization of the above results, we now provide analogue definitions for the contact case and proceed to prove the “standard” Noether theorem in this case.

Definition 8. The contact phase space is the canonical contactification of the cotangent bundle, that is, the manifold $T^*Q \times \mathbb{R}$ endowed with the contact 1-form $\eta = dS - \pi^* a$, where $\pi_S : T^*Q \times \mathbb{R} \to T^*Q$ is the standard projection, and $S$ is the global coordinate on $\mathbb{R}$.

Associated to the contact 1-form $\eta$, there is a unique vector field, called the Reeb vector field, defined by the conditions $\iota_R d\eta = 0$ and $\iota_R \eta = 1$. In local Darboux coordinates $(q^a, p_a, S)$, we have that $\eta = dS - p_a dq^a$ and $R = \partial / \partial S$.

Definition 9. A contact Hamiltonian system is a triple $(T^*Q \times \mathbb{R}, \eta, h)$, with $h : T^*Q \times \mathbb{R} \to \mathbb{R}$ a sufficiently regular function called the contact Hamiltonian.
Definition 10. A contact Hamiltonian vector field $X_h$ is defined to be the only solution to the conditions
\[ \iota_{X_h} d\eta = dh - R(h)\eta \quad \text{and} \quad \iota_{X_h} \eta = -h. \] (11)

One can directly check that in Darboux coordinates this leads to the contact Hamiltonian equations
\[ \dot{q}^a = \frac{\partial h}{\partial p_a} \quad \dot{p}_a = -\frac{\partial h}{\partial q^a} - p_a \frac{\partial h}{\partial S} \quad \dot{S} = p_a \frac{\partial h}{\partial p_a} - h, \] (12)
from which one can recover the standard Hamilton equations (7) for $q^a$ and $p_a$ whenever $h$ does not depend on $S$. Note also that from the last equation in (12), the new variable $S$ is the action of the system evaluated along the trajectories of the dynamics, that is the on-shell action. Moreover, the contact Hamiltonian equations (12) can be used to model mechanical systems with different types of dissipative terms (see e.g. [6, 7, 18]).

Now we are ready to define Noether symmetries and the analogue of conserved quantities in the contact case. First we point out that the condition (11) provides a (global) isomorphism between the Lie algebra of contact vector fields on $T^*Q \times \mathbb{R}$ with the Lie bracket and the Lie algebra of functions on $T^*Q \times \mathbb{R}$ with the Jacobi bracket
\[ \{F, G\}_J := \iota_{[X_F, X_G]} \eta. \] (13)

Then, starting from the symplectic analogue and from (13), we have the following natural definitions.

Definition 11. A Noether symmetry of a contact Hamiltonian system is a contact Hamiltonian vector field $X_F \in \mathfrak{X}(T^*Q \times \mathbb{R})$ such that $\{F, h\}_J = 0$.

Moreover, we can generalize the concept of conserved quantities to the case of dissipative systems, which is the general case for contact systems, as follows.

Definition 12. A dissipated quantity of a contact Hamiltonian system is a function $F : T^*Q \times \mathbb{R} \to \mathbb{R}$ that is dissipated at the same rate as the contact Hamiltonian, i.e. $\mathcal{L}_{X_h} F = -R(h)F$ (recall that $\mathcal{L}_{X_h} h = -R(h)h$).

Then, the contact version of Noether’s theorem goes as follows [16, 18]:

Theorem 2 (Contact Noether). $X_F$ is a Noether symmetry of a contact Hamiltonian system if and only if $F = -\iota_{X_F} \eta$ is a dissipated quantity.

Proof. The proof directly follows from the fact that
\[ \mathcal{L}_{X_h} F = -\mathcal{L}_{X_h} (\iota_{X_F} \eta) = -\iota_{X_F} \mathcal{L}_{X_h} \eta - \iota_{\mathcal{L}_{X_h} X_F} \eta = R(h)\iota_{X_F} \eta - \iota_{[X_h, X_F]} \eta = -R(h)F - \iota_{[X_h, X_F]} \eta. \]

Theorem 2 has been proved previously in this form in [16, 18], while similar results for the non-dissipative case were already elucidated in [3]. Here we discuss a slight difference between our approach and theirs, mostly related to the definition of symmetries. In [18] a more restrictive definition of symmetry has been considered, that is, $Y$ such that $[Y, X_h] = 0$. This has been called an infinitesimal dynamical symmetry in [18] and Noether’s theorem (therein called dissipation theorem) has been proved for such symmetries, analogously to the proof presented here. However, the condition for $Y$ to be a Noether symmetry (Definition 11) is more general (it allows for $[X_F, X_h] \neq 0$), and therefore some symmetries may escape the more restrictive definition in [18]. On the other hand, in [16] the most general definition of symmetry has been considered that can lead to Theorem 2. Indeed, from the proof of Theorem 2 one finds out that the necessary and sufficient condition for the existence of a dissipated quantity associated to a vector field $Y$ is to have $\iota_{[X_h, Y]} \eta = 0$. This is indeed the definition of symmetry considered in [16], where it has been referred to as a dynamical symmetry. However, in such case the definition is too weak, and this hampers the 1:1 correspondence between dissipated quantities and symmetries which constitutes part of the beauty of the standard Noether theorem. Note also that the term “dynamical symmetry” employed in such reference may be confusing with respect to Definition 7. Therefore we will refer to such symmetries as generalized dynamical symmetries in the following discussion. For these reasons we prefer to consider a different concept of symmetry, Noether’s symmetry, as stated in Definition 11. Indeed, for a given dissipated quantity $F$, our definition selects the associated symmetry in the class of generalized dynamical symmetries corresponding to $F$ that coincides exactly with the contact Hamiltonian vector field associated with $F$ (cf. [16, Remark 3]).

Moreover, the following important Corollary of Theorem 2 has been observed already both in [16] and in [18].
Corollary 1. Given two dissipated quantities $F_1$ and $F_2$, their ratio $F_1/F_2$, whenever it is well-defined, is a conserved quantity (note also that the contact Hamiltonian is, by definition, a dissipated quantity).

At this point we are ready to focus on the comparison between Theorem 1 and Theorem 2 and on discussing the role of dynamical similarities (including e.g. Kepler scalings) in the contact case.

To begin, a direct comparison between the symplectic and the contact versions of Noether’s theorem leads to the following observation: in the case where $h$ does not depend on $S$, both $h$ itself and any other dissipated quantity are in fact conserved quantities, and therefore we get a generalization of the symplectic version of Noether’s theorem, given that now both the symmetries and their associated invariants can depend explicitly on $S$, similarly to 3. However, it is also clear that the dissipated (or conserved) quantities associated to Noether’s symmetries in the contact case are functions of positions, momenta and the action only, and thus 3 cannot be recovered, since in general they are functions of the time variable too. To further illustrate this point, we proceed to show that, as in the symplectic case, Kepler scalings (11) are not Noether symmetries according to any of the definitions discussed above. To do so, first we note that Kepler scalings are induced by the action of the following vector field on the contact phase space

$$Y^c_{KS} = 2q^a \partial_{q^a} - p_a \partial_{p_a} + S \partial_S.$$  \hspace{1cm} (14)

Then we have the following result, whose proof is exactly analogous to that of the corresponding result in the symplectic case and can be also obtained by a direct calculation

Proposition 3. $Y^c_{KS}$ is a non-trivial dynamical similarity of Kepler’s contact Hamiltonian vector field. In particular $[Y^c_{KS}, X^K_h] = -3X^c_h$, where $X^K_h$ is the contact Hamiltonian vector field associated to Kepler’s Hamiltonian $K$.

Finally, we have the following important no-go result.

Proposition 4. Any non-trivial dynamical similarity of a contact Hamiltonian vector field is not a generalized dynamical symmetry. In particular they are not Noether symmetries.

Proof. By definition of a non-trivial dynamical similarity, we have $[Y, X_h] = \Lambda X_h$, with $\Lambda \neq 0$. Therefore $\iota_{[Y, X_h]} \eta = \Lambda \iota_{X_h} \eta = -\Lambda h \neq 0$.

We conclude that scaling symmetries cannot be Noether symmetries in the contact phase space, as it was the case in the symplectic phase space. In the next section we show how to extend the contact version of Noether’s theorem in order to include such symmetries.

III. THE GENERALIZED NOETHER THEOREM

In this section we extend all the previous results to the case of generalized symmetries defined on the extended contact phase space and their associated dissipated (conserved) quantities. We will see that this is at the same time an extension of the symplectic and contact versions of Noether’s theorems, and of the generalized Noether theorem for scaling symmetries proved in 37.

As usual, we start with the necessary definitions. In order to consider contact Hamiltonian systems with an explicit time dependence, we follow 7 and define the following.

Definition 13. We call the extended contact phase space the manifold $T^* Q \times \mathbb{R} \times \mathbb{R}$, endowed with the 1-form $\eta^E = \pi^*_t \eta + h dt$, where $\pi_t : T^* Q \times \mathbb{R} \times \mathbb{R} \to T^* Q \times \mathbb{R}$ is the projection and $t$ is the coordinate on $\mathbb{R}$.

Note that this extension is not canonical, in the sense that it depends on the system at hand through $h$, and that $(T^* Q \times \mathbb{R} \times \mathbb{R}, \eta^E)$ in general is a pre-symplectic manifold 32. Indeed, $d\eta^E$ is non-degenerate (hence symplectic) if and only if $\partial h/\partial S \neq 0$.

Moreover, we define time-dependent contact Hamiltonian vector fields as follows.

Definition 14. A time-dependent contact Hamiltonian vector field $X^t_h$ is the solution to the conditions

$$\iota_{X^t_h} d\eta^E = -R(h)\eta^E \quad \text{and} \quad \iota_{X^t_h} \eta^E = 0,$$  \hspace{1cm} (15)

where $R(h) = \partial h/\partial S$. We point out that this definition is different from that of a pre-symplectic system, as the right hand side of the first condition in (15) is not a closed 1-form 32. Note that in this case $X^t_h$ is not uniquely fixed by the two conditions in (15), as one has the freedom to choose a global function $f : T^* Q \times \mathbb{R} \times \mathbb{R} \to \mathbb{R} - \{0\}$ that
Proof. Let $X^t_k$, that is, $X^t_k$ is uniquely defined up to (nowhere-vanishing) rescalings. For instance, fixing $f = 1$, in local adapted coordinates one recovers the contact Hamiltonian equations for time-dependent Hamiltonians

$$q^a = \frac{\partial h}{\partial p_a}, \quad \dot{p}_a = -\frac{\partial h}{\partial q^a} - p_a \frac{\partial h}{\partial S}, \quad \dot{S} = p_a \frac{\partial h}{\partial p_a} - h, \quad t = 1,$$  

from which we infer that changing the function $f$ amounts to reparametrizing the dynamics.

To avoid clutter of notation and without loss of generality, we will also assume from now on that $f = 1$, that is, $X^t_k = X^t_k + \partial_t$, where $X^t_k$ is the contact Hamiltonian vector field associated with $h$. All the results can be easily generalized by including the factor $f$ back into the corresponding equations.

Now we are ready to study symmetries and associated dissipated (conserved) quantities in this extended phase space.

**Definition 15.** We call $Y \in \mathfrak{X}(T^*Q \times \mathbb{R} \times \mathbb{R})$ a generalized Noether symmetry of a time-dependent contact Hamiltonian system if $\mathcal{L}_Y \eta^E = \lambda \eta^E$ for some $\lambda \in C^\infty(T^*Q \times \mathbb{R} \times \mathbb{R})$.

One can check that this definition includes the previous definitions of Noether symmetries of symplectic and contact Hamiltonian systems (cf. Theorem 4 below). However, in this case we allow also for the symmetry to have a non-zero time component, and therefore such symmetries naturally allow for time rescalings.

We have the following important property.

**Proposition 5.** Generalized Noether symmetries form a Lie algebra with the Lie bracket.

*Proof.* Let $\mathcal{L}_{Y_1} \eta^E = \lambda_1 \eta^E$ and $\mathcal{L}_{Y_2} \eta^E = \lambda_2 \eta^E$. Then

$$\mathcal{L}_{[Y_1, Y_2]} \eta^E = \mathcal{L}_{Y_1} \mathcal{L}_{Y_2} \eta^E - \mathcal{L}_{Y_2} \mathcal{L}_{Y_1} \eta^E = (Y_1(\lambda_2) - Y_2(\lambda_1)) \eta^E.$$  

We can also define dissipated quantities in analogy with the standard contact case.

**Definition 16.** A dissipated quantity in the extended contact phase space is a function $F : T^*Q \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ that satisfies

$$\mathcal{L}_{X^t_k} F = -R(h) F.$$  

\begin{equation}
\label{dissipation_equation}
\mathcal{L}_{X^t_k} F = -R(h) F.
\end{equation}

In the following we will refer to condition \[17\] as the dissipation equation.

Note that the Hamiltonian is not necessarily a dissipated quantity itself. Indeed, as it happens in the symplectic case where $H$ is not conserved whenever it depends explicitly on $t$, here $h$ is not a dissipated quantity whenever it depends explicitly on $t$.

Next, we have the following Lemma.

**Lemma 1.** Let $Y$ be a generalized Noether symmetry. Then $\iota_{[Y, X^t_k]} \eta^E = 0$.

*Proof.*

$$\iota_{[Y, X^t_k]} \eta^E = [\mathcal{L}_Y, \iota_{X^t_k}] \eta^E = \mathcal{L}_Y \iota_{X^t_k} \eta^E - \iota_{X^t_k} \mathcal{L}_Y \eta^E = -\iota_{X^t_k} \eta^E = 0.$$  

Now we are ready to prove the generalized Noether Theorem.

**Theorem 3** (Generalized Noether Theorem). Let $Y$ be a generalized Noether symmetry of a time-dependent contact Hamiltonian system. Then $F = -\iota_Y \eta^E$ is a dissipated quantity.

As usual, the proof is a one-liner:

*Proof.*

$$\mathcal{L}_{X^t_k} F = -\mathcal{L}_{X^t_k} (\iota_Y \eta^E) = -\iota_Y \mathcal{L}_{X^t_k} \eta^E - \iota_{[X^t_k, Y]} \eta^E = R(h) \iota_Y \eta^E - \iota_{[X^t_k, Y]} \eta^E = -R(h) F - \iota_{[X^t_k, Y]} \eta^E.$$  

□
We remark that the dissipated quantity associated with a generalized Noether symmetry is obtained exactly in the same way as in Theorem 2 with $\eta$ replaced by $\eta^E$. This replacement is fundamental in order to obtain dissipated (conserved) quantities associated to scaling symmetries of the type obtained in 37 (cf. Equation 43).

Clearly, we could have proved Theorem 3 even under the more general definition of symmetry $\iota_{[X_h, Y]}(\eta^E) = 0$. However, in such case we would lose the possibility to associate to any invariant a somewhat “unique” symmetry, as we now show.

**Theorem 4** (Inverse generalized Noether Theorem). Let $F$ be a dissipated quantity. Then

$$Y_F = X_F + Y^t X^t_h$$

(18)

is the most general form of its associated generalized Noether symmetry, where $X_F$ is the contact Hamiltonian vector field associated to $F$ (see Definition 17) and $Y^t$ is a free function.

**Proof.** Given $F$, we wish to find the corresponding $Y$ that satisfies $\mathcal{L}_Y \eta^E = \lambda \eta^E$ ($Y$ generalized Noether symmetry) and $\iota_Y \eta^E = -F$, (the associated dissipated quantity is $F$). Using these two conditions and Cartan’s identity, we get immediately $\iota_Y d\eta^E = dF + \lambda \eta^E$, which, when written in Darboux coordinates is equivalent to the following conditions

$$\lambda = -Y^t \frac{\partial h}{\partial S} - \frac{\partial F}{\partial S}$$

(19)

and

$$Y^a = Y^t \frac{\partial h}{\partial p_a} + \frac{\partial F}{\partial p_a}$$

(20)

$$Y_a = -Y^t \left[ \frac{\partial h}{\partial q^a} + p_a \frac{\partial h}{\partial S} \right] - \frac{\partial F}{\partial q^a} - p_a \frac{\partial F}{\partial S}$$

(21)

$$Y^a \frac{\partial h}{\partial q^a} + Y_a \frac{\partial h}{\partial p_a} + Y^S \frac{\partial h}{\partial S} = -\left( Y^t \frac{\partial h}{\partial S} + \frac{\partial F}{\partial S} \right) h + \frac{\partial F}{\partial t}.$$  

(22)

Now from $\iota_Y \eta^E = -F$ and the above conditions, we get the additional requirement

$$Y^S = Y^t \left( p_a \frac{\partial h}{\partial p_a} - h \right) + p_a \frac{\partial F}{\partial p_a} - F.$$  

(23)

From (20), (21) and (23) we get that $Y = X_F + Y^t (X^t_h + \partial_t) = X_F + Y^t X^t_h$, while equation (22) can be rewritten after some algebra as $X^t_h (F) = -\frac{\partial h}{\partial S} F$, that is, the dissipation equation 17.

Theorems 3 and 4 are the main contributions of this work. Let us make some comments about these results: first, we get a generalization of the contact version of Noether’s theorem, as now the symmetries and the associated invariants can depend on $t$ explicitly. Additionally, if $h$ does not depend on $S$ and $t$, then we get a generalization of the symplectic version of Noether’s theorem, given that the symmetries and the associated invariants now can depend on $S$ and $t$ explicitly. This will be the case of e.g. Kepler scalings. Secondly, as anticipated, in the most general case $\mathcal{L}_X h = -R(h) h + \partial h$, and therefore whenever $h$ depends explicitly on $t$, it is not a dissipated quantity itself. This implies that whenever $h$ depends on $t$ we lose the fact that $F/h$ is a conserved quantity, as it happened in the contact version of Noether theorem. However, whenever $h$ does not depend on $t$ we have that $F/h$ is still a conserved quantity, even when $F$ depends on $t$. Moreover, we still have the following analogue of Corollary 1.

**Corollary 2.** Given two dissipated quantities $F$ and $G$, the function $F/G$, whenever defined, is a conserved quantity.

Finally, note that for $Y_F$ as in (18), we have $\iota_{Y_F} \eta^E = -F$. Moreover observe that, since $\iota_{Y^t, X^t_h} \eta^E = 0$ and $\mathcal{L}_{Y^t, X^t_h} \eta^E = -Y^t R(h) \eta^E$, we can always add a term of the form $Y^t X^t_h$ to any given generalized Noether symmetry and obtain another generalized Noether symmetry corresponding to the same invariant. Therefore such “gauge freedom” in the fixing of the symmetry is unavoidable. We say that the correspondence between symmetries and invariants is 1:1 up to the mentioned gauge freedom.

A fundamental question for our discussion at this point is the relationship between generalized Noether symmetries and the dynamical similarities of $X^t_h$. This is the content of the next two propositions. We start with a lemma.

**Lemma 2.** Let $A \in \mathfrak{X}(T^* Q \times \mathbb{R} \times \mathbb{R})$ be such that

i) $A \in \ker \eta^E$
\[ \iota_{A}d\eta^{E} = f_{A}\eta^{E}, \text{ for some function } f_{A}. \]

Then \( A = A^{t}X_{\lambda}^{t} \), with \( f_{A} = -\left( \frac{\partial h}{\partial q^{a}} \right)A^{t} \).

**Proof.** We have \( A \in \ker \eta^{E} \implies A^{S} = p_{a}A^{a} - hA^{t} \). Then we can directly compute that

\[
\iota_{A}d\eta^{E} = f_{A}\eta^{E} \implies A^{a} = A^{t}\frac{\partial h}{\partial p_{a}} \\
A_{a} = -A^{t}\left( \frac{\partial h}{\partial q^{a}} + p_{a}\frac{\partial h}{\partial S} \right) \\
f_{A} = -\frac{\partial h}{\partial S}A^{t} \\
f_{A}h = A^{a}\frac{\partial h}{\partial q^{a}} + A_{a}\frac{\partial h}{\partial p_{a}} + A^{S}\frac{\partial h}{\partial S}.
\]

Using (26) we can rewrite the other equations as

\[ A^{a} = A^{t}\frac{\partial h}{\partial p_{a}}, \quad (28) \]
\[ A_{a} = -A^{t}\left( \frac{\partial h}{\partial q^{a}} + p_{a}\frac{\partial h}{\partial S} \right), \quad (29) \]
\[ A^{S} = A^{t}\left( p_{a}\frac{\partial h}{\partial p_{a}} - h \right), \quad (30) \]

meaning that \( A = A^{t}X_{\lambda} + A^{t}\partial_{t} = A^{t}X_{\lambda}^{t} \), as claimed. \( \square \)

Now we can prove the following important implication, which states that generalized Noether symmetries are a subset of the dynamical similarities of \( X_{\lambda}^{t} \).

**Proposition 6.** \( Y \) generalized Noether symmetry \( \implies [Y, X_{\lambda}^{t}] = \lambda X_{\lambda}^{t}. \)

**Proof.** To prove the result, we will prove that \( [Y, X_{\lambda}^{t}] \in \ker \eta^{E} \) and that \( \iota_{[Y, X_{\lambda}^{t}]d\eta^{E}} = f_{\eta^{E}} \) for some function \( f \), and then we use Lemma 2. We start with

\[ \lambda_{Y}\eta^{E} = \lambda\eta^{E} \implies \iota_{Y}d\eta^{E} - dF = \lambda\eta^{E}, \quad (31) \]

On the other hand,

\[ \iota_{[Y, X_{\lambda}^{t}]\eta^{E}} = \mathcal{L}_{Y}X_{\lambda}^{t}\eta^{E} - \mathcal{L}_{Y}X_{\lambda}^{t}\eta^{E} - \iota_{X_{\lambda}^{t}}\iota_{Y}d\eta^{E} = -\iota_{X_{\lambda}^{t}}dF + \iota_{Y}t_{X_{\lambda}^{t}}d\eta^{E} = \lambda_{X_{\lambda}^{t}}\eta^{E} = 0, \quad (32) \]

and therefore \([X_{\lambda}^{t}, Y] \in \ker \eta^{E}\). Finally, a direct computation shows that

\[ \iota_{[Y, X_{\lambda}^{t}]d\eta^{E}} = \mathcal{L}_{[Y, X_{\lambda}^{t}]\eta^{E}} = \mathcal{L}_{Y}\mathcal{L}_{X_{\lambda}^{t}}\eta^{E} - \mathcal{L}_{X_{\lambda}^{t}}\mathcal{L}_{Y}\eta^{E} = -\left[ \mathcal{L}_{Y}R(h) + \mathcal{L}_{X_{\lambda}^{t}}\lambda \right]\eta^{E}, \quad (33) \]

and therefore by Lemma 2 we conclude. \( \square \)

The following example illustrates that in general the inclusion defined by Proposition 6 is strict, i.e., generalized Noether symmetries are a proper subset of dynamical similarities.

**Example 1.** Let \( X_{K}^{t} \) be the contact Hamiltonian vector field associated with the Kepler Hamiltonian \( \mathcal{H}_{K} \) in the extended contact phase space and consider the vector field \( Y = \mathcal{H}_{K}\partial_{S} \). One can directly check that \([Y, X_{K}^{t}] = 0 \), and thus \( Y \) is a (trivial) dynamical similarity. However, it is not a generalized Noether symmetry, as \( \mathcal{L}_{Y}\eta^{E} \neq \lambda\eta^{E} \) for any function \( \lambda \).

Despite the above remark, we have the next proposition, which is an immediate corollary of Theorem 4 and provides in some sense the inverse of Proposition 6 as it guarantees that for any dynamical similarity there exists an associated dissipated (conserved) quantity, and thus a related generalized Noether symmetry.

**Proposition 7.** Let \( Y \) be such that \([Y, X_{\lambda}^{t}] = \lambda X_{\lambda}^{t} \). Then \( F := -\iota_{Y}\eta^{E} \) is a dissipated quantity and there exists a generalized Noether symmetry associated with \( F \).
Proof. Since \([Y, X^\lambda_b] = \lambda X^\lambda_b\), then \(\iota_{[Y, X^\lambda]} T^E = 0\) and thus from the proof of Theorem 3 we know that \(T\) is a dissipated quantity. Hence, from Theorem 4 it follows that there is a generalized Noether symmetry associated to \(T\).

At this point we have reached our aim, that is, we have found the appropriate extension of the phase space and the appropriate definition of (generalized) Noether symmetries in such space such that we can guarantee that to any dynamical similarity of \(X^\lambda\) we can associate a dissipated (conserved) quantity via the generalized Noether theorem. This will be the case of e.g. Kepler scalings, as we will prove shortly.

### IV. SCALINGS AS GENERALIZED NOETHER SYMMETRIES

To keep the discussion as general as possible, let us consider a contact Hamiltonian of the form

\[
h = \frac{1}{2m} \mathbf{p} \cdot \mathbf{p} + f(t)V(\mathbf{q}) + g(S),
\]

where

\[
V(\xi) = \xi^k V(\mathbf{q}), \quad g(\xi) = \xi^\kappa g(S),
\]

i.e., \(V(\mathbf{q})\) and \(g(S)\) are homogeneous functions of degree \(k\) and \(\kappa\) respectively. The functions \(f(t)\) and \(g(S)\) are so far arbitrary functions of time \(t\) and the variable \(S\).

Consider now a general scaling transformation

\[
q' = \zeta^\alpha q, \quad p' = \zeta^\beta p, \quad S' = \zeta^\gamma S, \quad t' = \zeta^\sigma t,
\]

where \(\zeta \in \mathbb{R}^>0\) is the scaling parameter. The corresponding infinitesimal transformation is of the form

\[
\delta q = \left(\frac{\partial}{\partial \zeta} q'\right)_{\zeta=1} = \alpha q, \quad \delta p = \left(\frac{\partial}{\partial \zeta} p'\right)_{\zeta=1} = \beta p, \quad \delta S = \left(\frac{\partial}{\partial \zeta} S'\right)_{\zeta=1} = \gamma S, \quad \delta t = \left(\frac{\partial}{\partial \zeta} t'\right)_{\zeta=1} = \sigma t,
\]

and thus the associated generator

\[
Y = Y^a \partial_{q^a} + Y_a \partial_{p_a} + Y^S \partial_S + Y^t \partial_t
\]

has components

\[
Y^a = \alpha q^a, \quad Y_a = \beta p_a, \quad Y^S = \gamma S, \quad Y^t = \sigma t.
\]

We now insert these components in (18) and obtain the following equations

\[
\alpha q^a = \frac{\partial F}{\partial p_a} + Y^t p_a, \quad \beta p_a = -\frac{\partial F}{\partial q^a} - p_a \frac{\partial F}{\partial S} - Y^t \left(f(t) \frac{\partial V}{\partial q^a} + p_a \frac{dg}{dS}\right),
\]

\[
\gamma S = p_a \frac{\partial F}{\partial p_a} - F + Y^t \left(\frac{1}{2m} \mathbf{p} \cdot \mathbf{p} - f(t) V(\mathbf{q}) - g(S)\right),
\]

\[
\sigma t = Y^t.
\]

In order to find the solution, we proceed as follows. First, we replace equation (42) into (39), (40) and (41). This yields the new set of equations

\[
\alpha q^a = \frac{\partial F}{\partial p_a} + \sigma t p_a, \quad \beta p_a = -\frac{\partial F}{\partial q^a} - p_a \frac{\partial F}{\partial S} - \sigma t \left(f(t) \frac{\partial V}{\partial q^a} + p_a \frac{dg}{dS}\right),
\]

\[
\gamma S = p_a \frac{\partial F}{\partial p_a} - F + \sigma t \left(\frac{1}{2m} \mathbf{p} \cdot \mathbf{p} - f(t) V(\mathbf{q}) - g(S)\right).
\]
Then, we solve for \( \frac{\partial F}{\partial p_a} \) in equation (43) and obtain
\[
\frac{\partial F}{\partial p_a} = \alpha q^a - \sigma t \frac{p_a}{m},
\] (46)
which can now be inserted in (45) to obtain the associated dissipated quantity
\[
F = \alpha q \cdot p - \sigma t h(q, p, S, t) - \gamma S,
\] (47)
which should now be compared to (3). The third step is to insert this result into equation (44), which leads to the following condition
\[
(\beta - \gamma + \alpha) + \sigma t \frac{dg}{dS} = 0.
\] (48)
As \( g(S) \) is a time independent function, there are only two possibilities to find a solution of (48): (i) \( \frac{dg}{dS} = 0 \), which implies \( g(S) = g_0 = \text{constant} \), or (ii) \( \sigma = 0 \). Let us analyze each case separately.

A. Case (i): Homogeneous non-dissipative time-dependent systems

In this case \( g(S) = \text{constant} \) and this constant can be taken to be zero without loss of generality. Therefore, the Hamiltonian function (44) takes the following expression
\[
h = \frac{1}{2m} p \cdot p + f(t)V(q).
\] (49)
Recall that since in this case \( h \) does not depend on \( S \), then any dissipated quantity is in fact a conserved one.

We now return to (48) and notice that it can be used to fix \( \beta \) in terms of \( \alpha \) and \( \gamma \)
\[
\beta = \gamma - \alpha.
\] (50)
Now, for \( F \) as in (47) and \( h \) as above, we can rewrite the dissipation equation (17) as
\[
\left( \alpha - \frac{\gamma}{2} - \frac{\sigma}{2} \right) \frac{p \cdot p}{m} + (\gamma - \sigma)f(t)V(q) - \alpha f(t)q \cdot \nabla V - \sigma t \dot{f}(t)V = 0.
\] (51)
Clearly, the coefficient of the \( p \cdot p \) term must be zero, hence
\[
\alpha = \frac{\gamma}{2} + \frac{\sigma}{2}.
\] (52)
On the other hand, the homogeneity of the potential \( V(q) \) yields the following relation
\[
q \cdot \nabla V(q) = kV(q),
\] (53)
which is used to replace \( q \cdot \nabla V(q) \) in (51). As a result we obtain the condition
\[
f(t) \left[ \gamma \left( 1 - \frac{k}{2} \right) - \sigma \left( 1 + \frac{k}{2} \right) \right] = \sigma t \dot{f}(t).
\] (54)
We can now consider three cases:

Case (1). If \( \sigma = 0 \), then (54) gives
\[
\gamma \left( 1 - \frac{k}{2} \right) = 0.
\] (55)
which admits two solutions: \( \gamma = 0 \) and \( k = 2 \). The first option yields a trivial solution, \( F = 0 \), and the second solution gives a conserved quantity \( F \) of the form
\[
F_0(q, p, S) = q \cdot p - 2S.
\] (56)
Case (2). If \( \dot{f}(t) = 0 \), then (54) results in
\[
\gamma \left( 1 - \frac{k}{2} \right) - \sigma \left( 1 + \frac{k}{2} \right) = 0,
\] (57)
which admits three possible solutions:
2.1) If $k = 2$, then $\sigma = 0$, and this recovers the second solution of Case (1), where $F_0$ was defined.

2.2) If $k = -2$, then $\gamma = 0$, and the conserved quantity takes the form

$$F_1(q, p, t) = q \cdot p - 2t h(q, p, t).$$

(58)

Note that in this particular case the dependence on $S$ in the conserved quantity disappears. This is because in this case scaling transformations are canonical symmetries [22].

2.3) If $k \neq \pm 2$, then $\gamma = \frac{(2+k)}{(2-k)} \sigma$. In this case, the conserved quantity is

$$F_2(q, p, S, t) = 2 - \frac{k}{2} q \cdot p - S - \frac{(2+k)}{(2-k)} S.$$  

(59)

Note that this case contains the Kepler system with Hamiltonian [5]. Given that in such case the potential is homogeneous of degree $k = -1$, the conserved quantity takes the form

$$F(K) = 2q \cdot p - \frac{tH_K}{3} - \frac{(2+k)}{3} S.$$  

(60)

and $F$ is of the form

$$F(q, p, S, t) = (\Lambda + 1) q \cdot p - 2H_K t - 2\Lambda S.$$  

(65)

B. Homogeneous dissipative time-dependent systems

If $\frac{dg}{dS} \neq 0$, then $\sigma = 0$ in (48) and this yields a condition for $\beta$ as given in (50). Moreover, the Hamiltonian in this case is still of the form (34) but the expression for the function $F$ is now given as

$$F = \alpha q \cdot p - \gamma S.$$  

(66)

Inserting this expression into the dissipation equation (17), we obtain

$$\left(\alpha - \frac{2}{2}\right) \frac{P \cdot P}{m} - 2 \alpha f(t) kV(q) + \gamma f(t) V(q) + \gamma g(S) = 0,$$  

(67)
where the homogeneity condition of the potential \( \kappa \) was used. As before, the coefficient of \( \mathbf{p} \cdot \mathbf{p} \) has to be zero, which means
\[
\alpha = \frac{\gamma}{2},
\] (68)
and now (67) takes the form
\[
\gamma f(t)k V(q) \left(1 - \frac{k}{2}\right) + \gamma g(S)(1 - \kappa) = 0,
\] (69)
where \( \kappa \) is the homogeneity degree of \( g(S) \).

Again, there are three cases to be considered to obtain solutions of this algebraic equation. The case in which \( \gamma = 0 \) gives the trivial solution, \( F = 0 \). The other two cases, (i) \( k = 0 \) and \( \kappa = 1 \) and (ii) \( k = 2 \) and \( \kappa = 1 \) yield the same dissipated function \( F \) which turns out to be \( F_0 \) (cf. equation (50)). Remarkably, in both cases the dissipative term \( g(S) \) is forced to be of the form \( g(S) = g_0S \).

In summary, a Hamiltonian of the type (44) admits a scaling symmetry (36) only in cases where \( k = 0 \) or \( k = 2 \). We conclude by remarking that for the case of the harmonic oscillator, where \( k = 2 \), the dissipated function (56) has been already found in [7] by a direct calculation.

V. HARMONIC TYPE POTENTIALS WITH LINEAR DISSIPATION

Clearly, Theorems 3 and 4 do not apply only to scaling symmetries (36). To illustrate this point, let us conclude by considering one-dimensional systems with Hamiltonians of the type
\[
h = \frac{1}{2m} p^2 + \frac{m}{2} f(t)q^2 + g_0S,
\] (70)
where \( f(t) \) is an arbitrary function of time \( t \) and \( g_0 \) is an arbitrary real parameter. These systems can model e.g. a one-dimensional harmonic oscillator with a linear dissipation and a time-dependent factor in the potential. We recall that in this case, due to the explicit time dependence, the Hamiltonian \( h \) is not a dissipated quantity.

To derive the symmetries of this system let us write the dissipation equation (17) explicitly
\[
\frac{p}{m} \frac{\partial F}{\partial q} - \left( mf(t)q + \lambda p \right) \frac{\partial F}{\partial p} + \left( \frac{1}{2m} p^2 - \frac{m}{2} f(t)q^2 - g_0S \right) \frac{\partial F}{\partial S} + \frac{\partial F}{\partial t} + g_0F = 0.
\] (71)

Let us look for generalized Noether symmetries of the Hamiltonian (70) for two particular cases: (i) the non-dissipative Hamiltonians, where \( g_0 = 0 \), and (ii) linear dissipative term, where \( g_0 \neq 0 \). Each of these cases will be treated using the following ansatz for \( F \):
\[
F(q,p,S,t) = A(q,S,t)p^2 + B(q,S,t)p + C(q,S,t).
\] (72)

In the first case, when \( g_0 = 0 \), the ansatz (72) yields a solution of the form
\[
F(q,p,S,t) = C_0 F_0(q,p,S) + C_1 F_{LR}(q,p,t),
\] (73)
where \( F_0 \) was defined in (50) and \( F_{LR} \) is given by
\[
F_{LR}(q,p,t) = \rho(t)^2 \frac{\rho_0}{\rho(t)^2} q^2.
\] (74)

Note that, since the dissipation equation is linear, we have obtained a linear combination of two independent solutions, \( F_0 \) and \( F_{LR} \). Moreover, \( F_{LR} \) is the well-known Lewis-Riesenfeld invariant [24, 25], and \( \rho(t) \) has to satisfy the auxiliary Ermakov equation
\[
\dot{\rho}(t) + f(t)\rho(t) = \frac{\rho_0}{\rho(t)^2},
\] (75)
where \( \rho_0 \) is an arbitrary real constant. It is not surprising that we have obtained the Lewis-Riesenfeld invariant as a generalized Noether invariant, as it can be shown to be a standard Noether invariant [20]. Moreover, we emphasize that both \( F_0 \) and \( F_{LR} \) are conserved quantities in this case.
To generalize the above discussion, let us consider now the case where \( g_0 \neq 0 \) and use the same ansatz. A direct calculation shows that in this case the solution of the dissipation equation (71) takes the form

\[
F(q,p,S,t) = C_0 F_0(q,p,S) + C_1 F_{GLR}(q,p,t) + C_2 F_{EM}(q,p,t)
\]

where \( F_{GLR} \) and \( F_{EM} \) are defined as

\[
F_{GLR}(q,p,t) = a^2(t) \frac{L^2}{2m} + \frac{1}{2} \frac{1}{2} \left( g_0 a^2(t) - 2a(t) \dot{a}(t) \right) q p + \frac{\dot{a}^2(t) - g_0 a(t) \dot{a}(t) + \frac{g_0^2 a^2(t)}{4} + \frac{a_0^3}{a^2(t)} \left( 1 + \frac{3 a_0 g_0^2}{4} \right) m q^2}{2},
\]

\[
F_{EM}(q,p,t) = b(t) p + m \dot{b}(t) q,
\]

and the auxiliary functions \( a(t) \) and \( b(t) \) satisfy the equations

\[
\dot{a}(t) + f(t) a(t) - \frac{g_0^2}{4} a(t) = \frac{a_0^3}{a^2(t)} \left( 1 + \frac{3}{4} a_0 g_0^2 \right),
\]

\[
\dot{b}(t) + g_0 \dot{b}(t) + f(t) b(t) = 0,
\]

where \( a_0 \) is an arbitrary real constant.

The function \( F_{GLR} \) can be considered as a generalization of the Lewis-Riesenfeld invariant \( F_{LR} \) to the case where the system has a further linear dissipative term. This can be checked by taking \( g_0 = 0 \) in (77) and (79) and observing that they reduce to (74) and (75) respectively. Contrary to the previous case where \( g_0 = 0 \), these dissipated quantities are now not conserved. However, according to Corollary \( \frac{1}{2} \), their quotient is a conserved quantity.

To conclude this section, let us write the generalized Noether symmetry associated with the generalized Lewis-Riesenfeld dissipated quantity \( F_{GLR} \), which reads

\[
X_{F_{GLR}} = \left[ \frac{a(t) p}{m} + \frac{1}{2} \left( g_0 a(t) - \dot{a}(t) \right) q \partial_q + \frac{1}{2} \left( g_0 a(t) - \dot{a}(t) \right) p + m \left( f(t) a(t) - \frac{g_0}{2} \dot{a}(t) + \frac{\dot{a}(t)}{2} \right) q \partial_p + \frac{a(t) p^2}{2m} - \frac{m}{2} \left( f(t) a(t) - \frac{g_0}{2} \dot{a}(t) + \frac{\dot{a}(t)}{2} \right) q \partial_S \right],
\]

where we are considering \( Y^t = 0 \) in (18).

VI. CONCLUSIONS AND FUTURE WORK

We have proved a geometric extension of the generalized Noether theorem for scaling symmetries recently put forward in (27). Our construction stems from the observation that the invariants associated with scaling symmetries, cf. (4), in general include an explicit dependence on the on-shell action of the system and on the time variable, and therefore we argued that the extended contact phase space is the appropriate minimal geometric setting to include such invariants and their related symmetries. Indeed, by carefully constructing the extended contact phase space and the related Hamiltonian dynamics, we have shown that a sensible definition of Noether symmetries exists such that an extension of the generalized Noether theorem for scaling symmetries that applies to all dynamical similarities and also to some dissipative systems can be immediately found (Theorem 14), together with its inverse statement (Theorem 14).

As we have argued, these theorems have several positive features: in the first place, their proofs are very simple and are natural generalizations of their counterparts in the standard symplectic and contact cases; moreover, they apply equivalently to conservative systems and to those dissipative systems that admit a description in terms of time-dependent contact Hamiltonian systems; by construction, in this space the thus-obtained conserved or dissipated quantities are actual functions on the manifold and do not contain spurious terms involving integrals over the dynamics; finally, one can directly show that the generalized Noether symmetries thus derived form a Lie algebra, and therefore they are amenable to be treated in algebraic terms, in the lines of e.g. (27).

We hope that the analysis initiated in this work will be helpful to clarify the origin and structure of scaling symmetries and their related invariants, by putting them on the same footing as other standard Noether symmetries. As further developments, we will address the comparison of our results with the “Eisenhart-Duval lift” of mechanical systems employed in (31), with the “vertical extension” of the tangent space presented in (17), with the reduction in the pre-symplectic setting described in (27), and also with the “unit-free approach” to Hamiltonian mechanics introduced.
in [32], and we will provide a deeper analysis of the Lie-algebraic structure of the generalized Noether symmetries for various systems of interest in physics, e.g. in cosmology [28][29].
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