On Eling-Oz formula for the holographic bulk viscosity

Alex Buchel

Department of Applied Mathematics
University of Western Ontario
London, Ontario N6A 5B7, Canada
Perimeter Institute for Theoretical Physics
Waterloo, Ontario N2J 2W9, Canada

Abstract

Recently Eling and Oz [1] proposed a simple formula for the bulk viscosity of holographic plasma. They argued that the formula is valid in the high temperature (near-conformal) regime, but is expected to break down at low temperatures. We point out that the formula is in perfect agreement with the previous computations of the bulk viscosity of the cascading plasma [2, 3], as well as with the previous computations of the bulk viscosity of $\mathcal{N} = 2^*$ plasma [4, 5]. In the latter case it correctly reproduces the critical behaviour of the bulk viscosity in the vicinity of the critical point with the vanishing speed of sound.

March 2011
1 Introduction and Summary

In [1] Eling and Oz (EO) considered\(^1\) effective five-dimensional gravitational description of strongly coupled gauge theory plasma frequently arising both in phenomenological and the formal examples of the holographic gauge theory(string theory correspondence [7, 8]

\[
S_5 = \frac{1}{16 G_5 \pi} \int d^5 x \sqrt{-g} \left( R_5 - \frac{1}{2} \sum_i (\partial \phi_i)^2 - V(\phi_i) \right) + S_{gauge}. \tag{1.1}
\]

In the absence of chemical potentials for the conserved \(U(1)\) charges they proposed a simple formula for the ratio of bulk-to-shear viscosities in holographic plasma dual to (1.1):

\[
\frac{\zeta}{\eta} = \sum_i c^2_{s} T^2 \left( \frac{d \phi_i^H}{dT} \right)^2, \tag{1.2}
\]

where \(T\) is the plasma temperature (equivalently the Hawking temperature of the black brane dual to the equilibrium thermal state of the plasma),

\[
c_s^2 = \frac{d(\ln T)}{d(\ln s)}, \tag{1.3}
\]

is the speed of sound waves in plasma, and \(\phi_i^H\) are the values of the gravitational scalar fields \(\phi_i\) evaluated at the black brane horizon. The remarkable feature of the expression (1.2) is in the fact that it can be evaluated entirely from the 'horizon data'. The reason

\(^1\)See [6] for related earlier work.
why this is unexpected is that the bulk viscosity is sensitive to the microscopic scales in the theory, which in holographic correspondence are encoded in non-normalizable components of the appropriate scalar fields near the boundary\(^2\). In fact, above was precisely the reason the authors of [1] suggested that the formula (1.2) validity should be restricted to the high temperature case — in this case the scalar fields are expected not to flow a lot from the boundary to the horizon, and thus their horizon values should remain sensitive to the microscopic parameters of the boundary gauge theory. EO formula (1.2) was successfully verified [1] for a wide set of gauge theory/string theory examples, where bulk viscosity was evaluated from the sound waves attenuation coefficient [14–17]. Furthermore, it was found [1] that while (1.2) is valid in some phenomenological models of gauge/gravity correspondence [18–20], it is violated in some other models [21, 22].

In this paper we demonstrate that, at least in the context of formal examples of holographic correspondence, the range of validity of (1.2) is much wider. In particular, in section 2 we verify (1.2) for the cascading gauge theory plasma to order \(O\left(\ln^{-4} \frac{T}{\Lambda}\right)\) in the high-temperature \(T \gg \Lambda\) expansion\(^3\). Results are presented in section 2.1. In section 3 we verify (1.2) for \(\mathcal{N} = 2^*\) gauge theory plasma for all temperatures\(^4\). In particular, we demonstrate that (1.2) is valid in the vicinity of the phase transition with vanishing speed of sound [23–25]. Notice that in this latter case the finiteness of the bulk viscosity at criticality, along with the vanishing of the speed of sound at criticality, implies that the derivatives of the scalar fields must diverge at the critical point for (1.2) to be valid. Results are presented in sections 3.1 and 3.2.

It would be interesting to understand precisely why (1.2) fails for phenomenological models [20–22]. It is intriguing that these models also violate the bulk viscosity bound proposed in [4]. Thus, the derivation of (1.2) might be the first step towards proving the bulk viscosity bound in formal holographic examples.

---

\(^2\)This should be contrasted with the familiar universality of the shear viscosity to the entropy density ratio [9–12] which can be evaluated in the membrane paradigm framework [13].

\(^3\)Original computations of the bulk viscosity in this theory were done in [2,3].

\(^4\)Original computations of the bulk viscosity in this theory were done in [4,5].
2 Bulk viscosity of the cascading plasma

Effective gravitational action for the cascading gauge theory plasma was derived in [2]

\[
S_5 = \frac{1}{16\pi G_5} \int_\mathcal{M}_5 \text{vol}_\mathcal{M}_5 \left\{ R_5 - \frac{40}{3} (\partial f)^2 - 20(\partial w)^2 - \frac{1}{2} (\partial \Phi)^2 - \frac{1}{4 P^2} (\partial K)^2 e^{-\Phi - 4f - 4w - P} \right\},
\]

where

\[
P = -24e^{-\frac{16}{3}f - 2w} + 4e^{-\frac{16}{3}f - 12w} + P^2 e^{\frac{8}{3}f + 4w} + \frac{1}{2} K^2 e^{-\frac{40}{3}f}.
\]

In this case (1.2) reads

\[
\left. \frac{\zeta}{\eta} \right|_{EO} = \xi_4^{4T^2} \left[ \left( \frac{d\Phi^H}{dT} \right)^2 + \frac{80}{3} \left( \frac{df^H}{dT} \right)^2 + 40 \left( \frac{dw^H}{dT} \right)^2 + \frac{1}{2 P^2} e^{-4H - 4f^H - 4w^H} \left( \frac{dK^H}{dT} \right)^2 \right].
\]

As argued in [3], it is technically challenging to study bulk viscosity of the cascading plasma for all temperatures. Instead, it was studied in [3] perturbatively (to the third sub-leading order) at high temperature. From (2.3)-(2.7) of [3]

\[
h(x) = \frac{K_*}{4 \tilde{a}_0^2} + \frac{K_*}{\tilde{a}_0^2} \sum_{n=1}^{\infty} \left\{ \left( \frac{P^2}{K_*} \right)^n \left( \xi_{2n}(x) - \frac{5}{4} \eta_{2n}(x) \right) \right\},
\]

\[
f_2(x) = \tilde{a}_0 + \frac{\tilde{a}_0}{\tilde{a}_0} \sum_{n=1}^{\infty} \left\{ \left( \frac{P^2}{K_*} \right)^n \left( -2\xi_{2n}(x) + \eta_{2n}(x) + \frac{4}{5} \lambda_{2n}(x) \right) \right\},
\]

\[
f_3(x) = \tilde{a}_0 + \frac{\tilde{a}_0}{\tilde{a}_0} \sum_{n=1}^{\infty} \left\{ \left( \frac{P^2}{K_*} \right)^n \left( -2\xi_{2n}(x) + \eta_{2n}(x) - \frac{1}{5} \lambda_{2n}(x) \right) \right\},
\]

\[
K(x) = K_* + K_* \sum_{n=1}^{\infty} \left\{ \left( \frac{P^2}{K_*} \right)^n \kappa_{2n}(x) \right\},
\]

\[
g(x) = 1 + \sum_{n=1}^{\infty} \left\{ \left( \frac{P^2}{K_*} \right)^n \zeta_{2n}(x) \right\}.
\]

where

\[
\left\{ f = \frac{1}{4} \ln h + \frac{2}{5} \ln f_3 + \frac{1}{10} \ln f_2, \quad \omega = \frac{1}{10} \ln \frac{f_3}{f_2}, \quad \Phi = \ln g \right\}.
\]

We use \( EO \) to distinguish the expressions obtained applying (1.2).
Notice that $\tilde{a}_0$ dependence disappears for the scalars entering (2.3), and the only temperature dependence enters via $K_\star = K_\star(T)$. We refer the reader to [3] for the asymptotic parametrization of various functions \{\kappa_{2n}, \zeta_{2n}, \eta_{2n}, \lambda_{2n}, \zeta_{2n}\}.

The pressure $\mathcal{P}$, the energy density $\mathcal{E}$, and the entropy density $s$ of the cascading plasma are given by [26]

\[
\frac{\mathcal{P}}{sT} = \frac{3}{7} \left( \frac{7}{12} - \tilde{a}_{2,0} \right), \quad \frac{\mathcal{E}}{sT} = \frac{3}{4} \left( 1 + \frac{4}{7} \tilde{a}_{2,0} \right),
\]

(2.10)

with (see (2.58) of [3])

\[
\hat{a}_{2,0} = \frac{7}{12} \frac{P^2}{K_\star} + \left( \frac{7}{6} \kappa_2^0 - \frac{35}{3} + \frac{7}{12} \zeta_1^2 + \frac{7}{24} \ln 2 \right) \frac{P^4}{K_\star^2} + \left( -\frac{35}{36} \ln 2 - \frac{35}{18} \kappa_2^0 + \frac{7}{48} \ln^2 2 \\
+ \frac{175}{108} + \frac{7}{24} \kappa_2^0 \ln 2 + \frac{7}{12} \kappa_2^0 \ln 2 + \frac{7}{12} \zeta_2^2 + \frac{7}{6} \kappa_3^2 - \frac{35}{36} \zeta_1^2 \right) \frac{P^6}{K_\star^3} + \left( \frac{175}{72} \ln 2 \\
+ \frac{12}{12} \zeta_3^2 + \frac{175}{54} \kappa_2^0 - \frac{35}{18} \kappa_2^0 \ln 2 - \frac{35}{36} \zeta_1^2 \ln 2 - \frac{875}{324} + \frac{175}{108} \zeta_2^2 - \frac{35}{48} \ln^2 2 \\
+ \frac{7}{96} \ln^3 2 + \frac{7}{48} \zeta_2^2 \ln^2 2 + \frac{7}{12} \kappa_2^0 \ln 2 + \frac{7}{24} \kappa_2^0 \ln^2 2 + \frac{7}{24} \zeta_2^2 \ln 2 + \frac{7}{6} \kappa_4^2 \\
- \frac{35}{36} \zeta_2^2 - \frac{35}{18} \kappa_3^2 \right) \frac{P^8}{K_\star^4} + \mathcal{O}\left( \frac{P^{10}}{K_\star^5} \right).
\]

(2.11)

The precise temperature dependence of $K_\star$ was determined in [27]

\[
\frac{K_\star}{P^2} = \frac{1}{2} \ln \left( \frac{64\pi^4}{81} \times \frac{sT}{\Lambda^4} \right).
\]

(2.12)

Using (2.12) and the expressions for the pressure and the energy density from (2.10) we find

\[
\frac{c_s^2}{\kappa_2^0} = \frac{\partial \mathcal{P}}{\partial \mathcal{E}} = \frac{1}{3} \frac{7 - 12\hat{a}_{2,0} - 6P^2 \frac{d\hat{a}_{2,0}}{dK_\star}}{7 + 4\hat{a}_{2,0} + 2P^2 \frac{d\hat{a}_{2,0}}{dK_\star}}.
\]

(2.13)

Thus, given the perturbative high temperature expansion for $\hat{a}_{2,0}$ we can evaluate from...
(2.13) the perturbative high temperature expansion for $c_s^2$

$$3c_s^2 = -1 - \frac{4}{3} P^2 + \left( \frac{10}{3} - \frac{2}{3} \ln 2 - \frac{8}{3} \kappa_2^2 + \frac{2}{3} \xi_1^2 \right) \frac{P^4}{K^2} + \left( -8 - \frac{2}{3} \xi_1^2 \ln 2 \right) + \frac{10}{3} \ln 2 - \frac{2}{3} \ln^2 2 - \frac{8}{3} \kappa_3^2 + \frac{40}{9} \xi_1^2 - \frac{4}{3} \kappa_2^2 \ln 2 + \frac{80}{9} \kappa_2^2 + \frac{4}{3} \xi_2^2 - \frac{2}{3} \ln 2 - \frac{2}{3} \xi_2^2 \ln 2 - \frac{40}{9} \kappa_2^2 \ln 2 + \frac{16}{9} \kappa_2^2 \kappa_2^2 + \frac{4}{9} \xi_2^2 \ln 2 - \frac{212}{9} \kappa_2^2 + \frac{106}{9} \xi_1^2 + \frac{46}{9} \xi_2^0 + \frac{92}{9} \kappa_3^2 - \frac{4}{3} \xi_3^2 - \frac{8}{3} \kappa_4^2 \right) \frac{P^6}{K^2} + \mathcal{O} \left( \frac{P^{10}}{K^4} \right).$$

(2.14)

Consistency of the first law of thermodynamics (which was verified in [3] with an accuracy of $\alpha 10^{-7}$) implies (see (4.5) of [3])

$$\frac{dK_s}{d\ln T} =$$

$$2 P^2 + \left( - \frac{4}{3} + 2 \xi_1^2 + 4 \kappa_2^2 + \ln 2 \right) \frac{P^4}{K_s} + \left( 4 \kappa_2^2 - \frac{8}{3} \kappa_2^2 - \frac{8}{5} (\lambda_1^0)^2 - \frac{4}{3} \xi_1^2 + 79 \right) \frac{P^6}{K^2}$$

$$- 60 (\eta_1^0)^2 + 2 \xi_2^0 + 2 \kappa_2^2 \ln 2 + \frac{1}{2} \ln^2 2 - 3 \ln 2 + \frac{1}{2} \ln 2 \xi_1^2 - 24 \xi_2^0 \right) \frac{P^8}{K^4} + \mathcal{O} \left( \frac{P^{10}}{K^4} \right).$$

(2.15)
We now have all the necessary ingredients of evaluate (2.3) to order \( \mathcal{O}\left(\frac{P^{10}}{K^*}\right) \)

\[
\frac{\zeta}{\eta}_{|_{EO}} = \frac{8}{9} \frac{P^2}{K_*} + \left( -\frac{76}{27} + \frac{16}{9} \zeta_1^2 + \frac{32}{9} \kappa_2^2 + \frac{8}{9} \ln 2 - \frac{8}{9} \zeta_1^2 + \frac{8}{3} \eta_{1,h} + \frac{16}{45} \lambda_{1,h}\right) \frac{P^4}{K^2} \\
+ \left( 12 + \frac{32}{9} \kappa_2^2 \ln 2 + \frac{16}{9} \ln 2 \zeta_1^2 + \frac{32}{9} \kappa_2^2 \zeta_1^2 - \frac{8}{3} \eta_{1,h} \ln 2 - \frac{16}{45} \lambda_{1,h} - \frac{32}{9} \eta_{1,h} \kappa_2^2 \right) \frac{P^6}{K^4} \\
+ \frac{16}{3} \eta_{1,h} \zeta_1^2 + \frac{32}{3} \eta_{1,h} \kappa_2^2 + \frac{32}{45} \lambda_{1,h} \zeta_1^2 + \frac{64}{45} \lambda_{1,h} \zeta_1^2 - \frac{8}{3} \eta_{1,h} \ln 2 + \frac{8}{3} \eta_{1,h} \ln 2 \\
+ \frac{16}{45} \lambda_{1,h} \ln 2 - \frac{16}{9} \zeta_1^2 + \frac{32}{45} \eta_{1,h} \lambda_{1,h} - 8 \zeta_1^2 - 16 \kappa_2^0 + \frac{32}{9} \zeta_1^2 - \frac{56}{9} \eta_{1,h} - \frac{64}{45} \lambda_{1,h} \\
- \frac{16}{9} \kappa_2^0 - \frac{64}{3} \zeta_2^0 + \frac{16}{45} \lambda_{2,h} - \frac{8}{3} \zeta_2^0 + \frac{8}{3} \zeta_2^0 + \frac{8}{3} \eta_{2,h} - \frac{164}{27} \ln 2 + \frac{8}{9} (\zeta_{1,h}^2) \\
+ \frac{2}{3} (\ln 2)^2 + \frac{32}{9} (\kappa_2^0)^2 + \frac{8}{9} (\zeta_1^2)^2 - \frac{112}{3} (\eta_{1,h}^2) - \frac{296}{225} (\lambda_{1,h}^2) \right) \frac{P^8}{K^6} \\
+ \mathcal{V}_4 \frac{P^8}{K^6} + \mathcal{O}\left(\frac{P^{10}}{K^8}\right),
\]

(2.16)

where the coefficient \( \mathcal{V}_4 \) is given in Appendix.

Eq. (2.16) should be compared with the expression for the bulk viscosity derived in [3] (see (4.14) in [3])

\[
\frac{\zeta}{\eta} = \frac{8}{9} \frac{P^2}{K_*} + \frac{4}{3} \left( \beta_{2,2} \frac{P^4}{K_*^2} + \beta_{2,3} \frac{P^6}{K_*^4} + \beta_{2,4} \frac{P^8}{K_*^6} \right) + \mathcal{O}\left(\frac{P^{10}}{K^8}\right).
\]

(2.17)

For convenience we collect in Table 1 numerical values for all the coefficients entering (2.16) and (2.17) as computed in [3].

### 2.1 Comparison of (1.2) with bulk viscosity of the cascading plasma

Using (2.16) and (2.17) and the data from Table 1 we find:

\[
\frac{\zeta}{\eta}_{|_{EO}} = \frac{8}{9} \frac{P^2}{K_*} + 1.7634445167 \frac{P^4}{K_*^2} - 2.263612386 \frac{P^6}{K_*^4} + 3.026509767 \frac{P^8}{K_*^6} + \mathcal{O}\left(\frac{P^{10}}{K_*^8}\right),
\]

\[
\frac{\zeta}{\eta} = \frac{8}{9} \frac{P^2}{K_*} + 1.763444983 \frac{P^4}{K_*^2} - 2.2636127883 \frac{P^6}{K_*^4} + 3.026511143 \frac{P^8}{K_*^6} + \mathcal{O}\left(\frac{P^{10}}{K_*^8}\right).
\]

(2.18)

### 3 Bulk viscosity of \( \mathcal{N} = 2^* \) plasma

Effective gravitational action for \( \mathcal{N} = 2^* \) gauge theory plasma was obtained in [28]

\[
S = \frac{1}{4\pi G_5} \int_{M_5} d\xi^5 \sqrt{-g} \left[ \frac{1}{4} R - 3(\partial\alpha)^2 - (\partial\chi)^2 - \mathcal{P} \right],
\]

(3.1)
Table 1: Coefficients entering (2.16) and (2.17).

| n  | 1       | 2       | 3       | 4       |
|----|---------|---------|---------|---------|
| $\kappa^{2,0}_n$ | 0.73675974 | -0.62226255 | -0.03784377 |       |
| $\eta^{4,0}_n$ | -0.01717287 | 0.00534036 | -0.01064222 |       |
| $\lambda^{3,0}_n$ | -0.87235794 | -1.11562943 | 1.39008636 |       |
| $\zeta^{2,0}_n$ | -0.15342641 | 0.62226267 | -0.32514260 |       |
| $\kappa^0_{n,h}$ | 0.62226259 | -0.42061461 | 0.00816831 |       |
| $\xi^0_{n,h}$ | -0.07981931 | 0.01661150 | -0.00920379 |       |
| $\xi^0_{n,h}$ | 0.01919989 | -0.05277626 | 0.01385333 |       |
| $\eta^0_{n,h}$ | -0.14891337 | -0.21809464 | 0.00213345 |       |
| $\lambda^0_{n,h}$ | 0.16806881 | -0.14619173 | 0.01639579 |       |
| $\zeta^0_{n,h}$ | -0.41123352 | 0.33024116 | -0.07445122 |       |
| $\beta^2_{2,n}$ | 0.13225837 | -1.69770959 | 2.26988336 |       |

where the potential\(^6\)

$$\mathcal{P} = \frac{1}{16} \left[ \frac{1}{3} \left( \frac{\partial W}{\partial \alpha} \right)^2 + \left( \frac{\partial W}{\partial \chi} \right)^2 \right] - \frac{1}{3} W^2$$

(3.2)

is a function of $\alpha$ and $\chi$, and is determined by the superpotential

$$W = -e^{-2\alpha} - \frac{1}{2} e^{4\alpha} \cosh(2\chi).$$

(3.3)

In this case (1.2) reads:

$$\frac{\zeta}{\eta} \bigg|_{EO} = c_s^4 T^2 \left[ 24 \left( \frac{d\alpha^H}{dT} \right)^2 + 8 \left( \frac{d\chi^H}{dT} \right)^2 \right].$$

(3.4)

In what follows we present results only for $\mathcal{N} = 2^*$ gauge theory plasma at vanishing fermionic mass\(^7\) [23]. The latter case corresponds to setting $\chi \equiv 0$.

As pointed out in [23], and further explored in [24, 25], strongly coupled $\mathcal{N} = 2^*$ gauge theory plasma undergoes a second-order phase transition at

$$\delta \equiv \left( \frac{m_b}{T} \right)^2 = \delta_c = 5.4098(6).$$

(3.5)

\(^6\)We set the five-dimensional gauged supergravity coupling to one. This corresponds to setting the radius $L$ of the five-dimensional sphere in the undeformed metric to 2.

\(^7\)Although we did not verify this fact explicitly, based on the agreement for $m_f = 0$, we do not expect discrepancy between (3.4) and the direct computation of the bulk viscosity for $m_f \neq 0$.  

8
Figure 1: (Colour online) Horizon value of the scalar field $\rho_0 = e^{\alpha H}$ as the function of $\delta \equiv (m_b T)^2$. Data points are blue; the solid red line is the best fit of the data with order 15 polynomial in $\delta$. The dashed vertical green line represents the critical point of the theory $\delta = \delta_c$.

At the critical point the speed of sound vanishes. We discuss separately the cases $\delta < \delta_c$, and $|\delta_c - \delta| \ll 1$.

We use asymptotic parametrization of the scalar function, the speed of sound, and the bulk viscosity as in [5]

$$\rho \equiv \ln \alpha, \quad c_s = \frac{\beta_1}{\sqrt{3}}, \quad \frac{\zeta}{\eta} = \frac{4}{3}(\beta_2 - 1).$$

(3.6)

3.1 Comparison of (1.2) with $N = 2^*$ bulk viscosity away from criticality: $\delta < \delta_c$

We rewrite (3.4) assuming functional dependence $\rho^H \equiv \ln \alpha^H = \rho_0(\delta)$

$$\left. \frac{\zeta}{\eta} \right|_{EO} = \frac{32\delta^2 \beta_1^4}{3\rho_0^2} \left( \frac{d\rho_0}{d\delta} \right)^2.$$  

(3.7)

From eq. (2.11) of [5],

$$\delta = \frac{24\pi^2}{\sqrt{2}} \rho_{11} e^{\delta_0}.$$  

(3.8)

Thus, given the data sets $\{\rho_{11}, \rho_{10}, \rho_0, a_0, a_1, \beta_1, \beta_2\}$ obtained in [4,5] we can reconstruct the functional dependence $\rho_0(\delta)$. The result of such reconstruction is presented in
Figure 2: (Colour online) Comparison of the EO prediction for $N = 2^*$ plasma bulk viscosity with the explicit computations from the quasinormal modes [4]. The dashed vertical green line represents the critical point of the theory $\delta = \delta_c$.

Figure 1. Data points are blue — there are altogether 320 points. The solid red line is the best fit to the data with the polynomial

$$\rho_0^{fit} = \sum_{i=0}^{15} \mathcal{R}_i \delta^i.$$  \hspace{1cm} (3.9)

Having an (approximate) analytic expression for $\rho_0 \approx \rho_0^{fit}(\delta)$, we can compute the prediction for the bulk viscosity of $N = 2^*$ plasma from (1.2)

$$\left. \frac{\zeta}{\eta} \right|_{EO} \approx \frac{32\delta^2\beta_4^4}{3\rho_0^2} \left( \frac{d\rho_0^{fit}}{d\delta} \right)^2.$$ \hspace{1cm} (3.10)

We find that the numerical agreement between (3.10) and the original result (3.6) is excellent — Figure 2 presents the functional dependence of

$$\left( \frac{\zeta/\eta}{\zeta/\eta - 1} \right) \text{ vs } \delta.$$ \hspace{1cm} (3.11)

Again, the vertical green dashed line represents the critical point of $N = 2^*$ plasma. The agreement is relatively worse for small values of $\delta$ and in the vicinity of the critical point. In the former case, this is caused by numerical errors for the evaluation of the bulk viscosity (the larger errors are induced due to small amplitude profiles of the gravitational scalar field $\alpha$), and in the latter case the relatively worse agreement is
Figure 3: (Colour online) Horizon value of the scalar field \( \rho_0 = e^{\alpha H} \) and \( \delta \equiv (\frac{m_b}{T})^2 \) as a function of \( \beta \equiv c_s^2 \) at criticality. Data points are blue; the solid red lines are the best fit of the data with polynomials in \( \beta \) — see (3.14).

caused by the choice of the 'fit' (3.9) (in the vicinity of the critical point \( \frac{d\rho_0}{d\delta} \) is expected to diverge as \( \beta_1^{-2} \)). As was already pointed out in [1], the agreement between (3.7) and (3.6) can be established analytically to order \( O(\delta) \). In the next subsection, with a suitable parametrization of the horizon value of the scalar field at criticality, we drastically improve the agreement between (3.7) and (3.6) for \( \delta \) close to \( \delta_c \).

3.2 Comparison of (1.2) with \( \mathcal{N} = 2^* \) bulk viscosity at criticality \( |\delta_c - \delta| \ll 1 \)

An important prediction of [4, 5] is that the bulk viscosity of \( \mathcal{N} = 2^* \) plasma remains finite at criticality, where \( c_s^2 = 0 \). The EO expression (3.7), if correct, implies that the derivative of the scalar field \( \rho_0 \) with respect to \( \delta \) must diverge as \( c_s^{-2} \) at criticality. Thus \( \delta \) parametrization of \( \rho_0 \) is not very useful. Instead, in the vicinity of the critical point we rewrite (3.7) as a function of

\[
\beta \equiv c_s^2 \equiv \frac{\beta_1^2}{3}. \tag{3.12}
\]

We find

\[
\frac{\zeta}{\eta} \bigg|_{EO} = \frac{96\delta^2}{\rho_0^2} \left( \frac{d\rho_0}{d\beta} \right)^2 \left( \beta \frac{d\beta}{d\delta} \right)^2,
\tag{3.13}
\]

where now we understand \( \rho_0 = \rho_0(\beta) \) and \( \delta = \delta(\beta) \). These two functions can easily be reconstructed from the data sets \( \{\rho_{11}, \rho_{10}, \rho_0, a_0, a_1, \beta_1, \beta_2\} \) obtained in [4, 5] — see Figure 3. Blue dots represent the data points, the solid red lines are the fits to the
\[
\left( \frac{\zeta/\eta|_{EO}}{\zeta/\eta} - 1 \right)
\]

Figure 4: (Colour online) Comparison of the EO prediction for \(\mathcal{N} = 2^*\) plasma bulk viscosity with the explicit computations from the quasinormal modes [4,5] in the vicinity of the critical point, i.e., for \(\beta = 0\).

Data:

\[
\begin{align*}
\delta^{fit} &= 5.40987 - 1.14476 \beta^2 - 4.64346 \beta^3 - 14.55312 \beta^4, \\
\rho_0^{fit} &= 0.83305 + 0.09281 \beta + 0.17683 \beta^2 + 0.36473 \beta^3.
\end{align*}
\]  

(3.14)

We can now compute the prediction for the bulk viscosity of \(\mathcal{N} = 2^*\) plasma from (1.2)

\[
\frac{\zeta}{\eta}|_{EO} = \frac{96 \delta^2}{\rho_0^{fit}} \left( \frac{d\rho_0^{fit}}{d\beta} \right)^2 \left( \beta \frac{d\beta}{d\delta^{fit}} \right)^2.
\]  

(3.15)

Figure 4 presents the functional dependence of \(\left( \frac{\zeta/\eta|_{EO}}{\zeta/\eta} - 1 \right)\) vs \(\beta\), in the vicinity of the critical point of \(\mathcal{N} = 2^*\) plasma.

Acknowledgments

I would like to thank Yaron Oz for valuable correspondence. Research at Perimeter Institute is supported by the Government of Canada through Industry Canada and by the Province of Ontario through the Ministry of Research & Innovation. AB gratefully acknowledges further support by an NSERC Discovery grant and support through the Early Researcher Award program by the Province of Ontario.
A  Coefficient $\mathcal{V}_4$

\[
\mathcal{V}_4 = \frac{3080}{81} \xi_{1,0}^2 + \frac{6160}{81} \kappa_{2,0}^2 - \frac{404}{27} \xi_{0,0}^2 + \frac{244}{9} \eta_{0,0}^2 + \frac{808}{135} \lambda_{1,0}^2 + \frac{64}{9} \kappa_{0,0}^2 + \frac{608}{9} \zeta_{0,2}^2 - \frac{64}{45} \lambda_{0,2}^2
\]

\[
+ \frac{32}{9} \xi_{2,0}^2 - \frac{80}{9} \xi_{2,0}^2 + \frac{16}{9} \eta_{0,2} - \frac{32}{9} \kappa_{3,0}^2 - \frac{32}{9} \zeta_{3,0}^2 + \frac{8}{3} \eta_{3,0} + \frac{16}{45} \zeta_{3,0}^2 - \frac{704}{225} \xi_{1,0}^2 - \frac{16}{9} \xi_{1,0}^2 - \frac{448}{3} \eta_{0,2}^2 - \frac{224}{3} \zeta_{1,0}^2
\]

\[
+ 32 \kappa_{4,0}^2 + \frac{16}{9} \xi_{3,0}^2 - \frac{32}{45} \xi_{1,0}^2 \eta_{1,0}^2 + \frac{64}{45} \eta_{1,0}^2 \lambda_{1,0}^2 \xi_{1,0}^2 + \frac{128}{45} \eta_{1,0}^2 \lambda_{1,0}^2 \kappa_{2,0}^2 - \frac{16}{9} \xi_{1,0}^2 \eta_{1,0}^2 \zeta_{1,0}^2
\]

\[
- \frac{32}{3} \xi_{0,2}^2 \eta_{1,0}^2 \kappa_{2,0}^2 - \frac{32}{45} \eta_{1,0}^2 \lambda_{1,0}^2 \kappa_{2,0}^2 - \frac{32}{9} \xi_{1,0}^2 \kappa_{2,0}^2 \zeta_{2,0}^2 + \frac{32}{3} \eta_{1,0}^2 \kappa_{2,0}^2 \zeta_{2,0}^2
\]

\[
+ \frac{64}{45} \lambda_{1,0}^2 \kappa_{2,0}^2 \xi_{1,0}^2 - \frac{272}{9} \eta_{2,0}^2 \xi_{2,0}^2 - \frac{32}{3} \xi_{1,0}^2 \eta_{1,0}^2 + \frac{64}{45} \xi_{1,0}^2 \lambda_{1,0}^2 \xi_{1,0}^2 - \frac{512}{27} \xi_{1,0}^2 \kappa_{2,0}^2 - \frac{176}{9} \eta_{1,0}^2 \xi_{1,0}^2
\]

\[
- \frac{352}{9} \eta_{1,0}^2 \kappa_{2,0}^2 - \frac{512}{135} \lambda_{0,1}^2 \xi_{1,0}^2 - \frac{1024}{135} \lambda_{0,1}^2 \kappa_{2,0}^2 + \frac{256}{27} \eta_{1,0}^2 \kappa_{2,0}^2 - \frac{128}{45} \eta_{1,0}^2 \lambda_{1,0}^2
\]

\[
+ \frac{704}{75} (\theta_{1,0}^2 \lambda_{1,0}^2 \xi_{1,0}^2 + \frac{8}{9} \eta_{1,0}^2 (\lambda_{1,0}^2 \xi_{1,0}^2)^2 + \frac{16}{9} \xi_{1,0}^2 \eta_{0,2}^2 - \frac{448}{135} (\eta_{1,0}^2 \kappa_{1,0}^2 \xi_{1,0}^2 - \frac{224}{3} (\xi_{1,0}^2 \eta_{1,0}^2)^2
\]

\[
+ \frac{296}{27} \xi_{1,0}^2 \lambda_{0,1}^2 \xi_{1,0}^2 - \frac{8}{3} \xi_{1,0}^2 \eta_{0,2}^2 - \frac{16}{9} \xi_{1,0}^2 \lambda_{0,2}^2 \xi_{1,0}^2 + \frac{8}{3} \eta_{1,0}^2 \xi_{0,2}^2 + \frac{8}{135} \eta_{1,0}^2 (\lambda_{1,0}^2 \xi_{1,0}^2)^2 - \frac{16}{45} \lambda_{1,0}^2 \xi_{1,0}^2
\]

\[
+ \frac{16}{45} \xi_{0,2}^2 \kappa_{2,0}^2 - \frac{272}{9} \eta_{2,0}^2 \xi_{2,0}^2 - \frac{32}{3} \xi_{1,0}^2 \eta_{1,0}^2 + \frac{64}{45} \xi_{1,0}^2 \lambda_{1,0}^2 \xi_{1,0}^2 - \frac{512}{27} \xi_{1,0}^2 \kappa_{2,0}^2 - \frac{176}{9} \eta_{1,0}^2 \xi_{1,0}^2
\]

\[
- \frac{352}{9} \eta_{1,0}^2 \kappa_{2,0}^2 - \frac{512}{135} \lambda_{0,1}^2 \xi_{1,0}^2 - \frac{1024}{135} \lambda_{0,1}^2 \kappa_{2,0}^2 + \frac{256}{27} \eta_{1,0}^2 \kappa_{2,0}^2 - \frac{128}{45} \eta_{1,0}^2 \lambda_{1,0}^2
\]

\[
+ \frac{704}{75} (\theta_{1,0}^2 \lambda_{1,0}^2 \xi_{1,0}^2 + \frac{8}{9} \eta_{1,0}^2 (\lambda_{1,0}^2 \xi_{1,0}^2)^2 + \frac{16}{9} \xi_{1,0}^2 \eta_{0,2}^2 - \frac{448}{135} (\eta_{1,0}^2 \kappa_{1,0}^2 \xi_{1,0}^2 - \frac{224}{3} (\xi_{1,0}^2 \eta_{1,0}^2)^2
\]

\[
+ \frac{296}{27} \xi_{1,0}^2 \lambda_{0,1}^2 \xi_{1,0}^2 - \frac{8}{3} \xi_{1,0}^2 \eta_{0,2}^2 - \frac{16}{9} \xi_{1,0}^2 \lambda_{0,2}^2 \xi_{1,0}^2 + \frac{8}{3} \eta_{1,0}^2 \xi_{0,2}^2 + \frac{8}{135} \eta_{1,0}^2 (\lambda_{1,0}^2 \xi_{1,0}^2)^2 - \frac{16}{45} \lambda_{1,0}^2 \xi_{1,0}^2
\]

\[
+ \frac{16}{45} \xi_{0,2}^2 \kappa_{2,0}^2 - \frac{272}{9} \eta_{2,0}^2 \xi_{2,0}^2 - \frac{32}{3} \xi_{1,0}^2 \eta_{1,0}^2 + \frac{64}{45} \xi_{1,0}^2 \lambda_{1,0}^2 \xi_{1,0}^2 - \frac{512}{27} \xi_{1,0}^2 \kappa_{2,0}^2 - \frac{176}{9} \eta_{1,0}^2 \xi_{1,0}^2
\]

\[
- \frac{352}{9} \eta_{1,0}^2 \kappa_{2,0}^2 - \frac{512}{135} \lambda_{0,1}^2 \xi_{1,0}^2 - \frac{1024}{135} \lambda_{0,1}^2 \kappa_{2,0}^2 + \frac{256}{27} \eta_{1,0}^2 \kappa_{2,0}^2 - \frac{128}{45} \eta_{1,0}^2 \lambda_{1,0}^2
\]
References

[1] C. Eling and Y. Oz, “A Novel Formula for Bulk Viscosity from the Null Horizon Focusing Equation,” arXiv:1103.1657 [hep-th].

[2] A. Buchel, Phys. Rev. D 72, 106002 (2005) [arXiv:hep-th/0509083].

[3] A. Buchel, Nucl. Phys. B 820, 385 (2009) [arXiv:0903.3605 [hep-th]].

[4] A. Buchel, Phys. Lett. B 663, 286 (2008) [arXiv:0708.3459 [hep-th]].

[5] A. Buchel and C. Pagnutti, Nucl. Phys. B 816, 62 (2009) [arXiv:0812.3623 [hep-th]].

[6] M. Fujita, JHEP 0810, 031 (2008) [arXiv:0712.2289 [hep-th]].

[7] J. M. Maldacena, Adv. Theor. Math. Phys. 2, 231 (1998) [Int. J. Theor. Phys. 38, 1113 (1999)] [arXiv:hep-th/9711200].

[8] O. Aharony, S. S. Gubser, J. M. Maldacena, H. Ooguri and Y. Oz, Phys. Rept. 323, 183 (2000) [arXiv:hep-th/9905111].

[9] A. Buchel and J. T. Liu, Phys. Rev. Lett. 93, 090602 (2004) [arXiv:hep-th/0311175].

[10] P. Kovtun, D. T. Son and A. O. Starinets, Phys. Rev. Lett. 94, 111601 (2005) [arXiv:hep-th/0405231].

[11] A. Buchel, Phys. Lett. B 609, 392 (2005) [arXiv:hep-th/0408095].

[12] P. Benincasa, A. Buchel and R. Naryshkin, Phys. Lett. B 645, 309 (2007) [arXiv:hep-th/0610145].

[13] P. Kovtun, D. T. Son and A. O. Starinets, JHEP 0310, 064 (2003) [arXiv:hep-th/0309213].

[14] J. Mas and J. Tarrio, JHEP 0705, 036 (2007) [arXiv:hep-th/0703093].

[15] P. Benincasa and A. Buchel, Phys. Lett. B 640, 108 (2006) [arXiv:hep-th/0605076].
[16] P. Benincasa, A. Buchel and A. O. Starinets, Nucl. Phys. B 733, 160 (2006) [arXiv:hep-th/0507026].

[17] A. Buchel, Nucl. Phys. B 841, 59 (2010) [arXiv:1005.0819 [hep-th]].

[18] I. Kanitscheider and K. Skenderis, JHEP 0904, 062 (2009) [arXiv:0901.1487 [hep-th]].

[19] A. Yarom, JHEP 1004, 024 (2010) [arXiv:0912.2100 [hep-th]].

[20] U. Gursoy, E. Kiritsis, G. Michalogiorgakis and F. Nitti, JHEP 0912, 056 (2009) [arXiv:0906.1890 [hep-ph]].

[21] S. S. Gubser, S. S. Pufu and F. D. Rocha, JHEP 0808, 085 (2008) [arXiv:0806.0407 [hep-th]].

[22] S. S. Gubser and A. Nellore, Phys. Rev. D 78, 086007 (2008) [arXiv:0804.0434 [hep-th]].

[23] A. Buchel, S. Deakin, P. Kerner and J. T. Liu, Nucl. Phys. B 784, 72 (2007) [arXiv:hep-th/0701142].

[24] A. Buchel and C. Pagnutti, Nucl. Phys. B 834, 222 (2010) [arXiv:0912.3212 [hep-th]].

[25] A. Buchel and C. Pagnutti, Phys. Rev. D 83, 046004 (2011) [arXiv:1010.3359 [hep-th]].

[26] O. Aharony, A. Buchel and A. Yarom, Phys. Rev. D 72, 066003 (2005) [arXiv:hep-th/0506002].

[27] O. Aharony, A. Buchel and P. Kerner, Phys. Rev. D 76, 086005 (2007) [arXiv:0706.1768 [hep-th]].

[28] K. Pilch and N. P. Warner, Nucl. Phys. B 594, 209 (2001) [arXiv:hep-th/0004063].