Cosmic Acceleration from Modified Gravity:

Lessons from Worked Examples

Wayne Hu
II Workshop Challenges
Campos do Jordao, December 2009
Outline

- Three Regimes of Modified Gravity
  - **Horizon** scale - acceleration dominated, conservation regime
  - **Intermediate** scale - modified Newtonian, scalar tensor regime
  - **Small** scale - general relativistic, non-linearly suppressed regime

- Collaborators
  - Alexander Belikov
  - Dragan Huterer
  - Hiro Oyaizu
  - Hiranya Peiris
  - Yong-Seon Song†
  - Sheng Wang†
  - Wenjuan Fang
  - Michael Mortonson
  - Fabian Schmidt
  - Iggy Sawicki†
  - Amol Upadhye†
  - Alexey Vikhlinin
  - Marcos Lima†

  ← Brazilian connection
Charting Out the Expansion

- **Standard candle**: apparent brightness of objects with a fixed luminosity to judge distance
- **Standard ruler**: apparent (angular) separation of objects with a fixed physical separation to judge distance

Supernovae
1998 Discovery

Sound waves
CMB+Galaxies
Incomplete Geometry or Energy?

- General relativity says \textit{Gravity} = \textit{Geometry}

- And \textit{Geometry} = \textit{Matter-Energy}

- Could the \textit{missing energy} required by \textit{acceleration} be an \textit{incomplete} description of how \textit{matter determines geometry}?
Dynamical Tests of Acceleration

- Dark energy slows growth of structure in highly predictive way

- Deviation significantly $>2\%$ rules out $\Lambda$ with or without curvature

- Excess $>2\%$ rules out quintessence with or without curvature and early dark energy [as does $>2\%$ excess in $H_0$]
Dynamical Tests of Acceleration

- Dark energy slows growth of structure in highly predictive way

Mortonson, Hu, Huterer (2009)
Parameterizing Acceleration

- Cosmic acceleration, like the cosmological constant, can either be viewed as arising from
  - Missing, or dark energy, with \( w \equiv \frac{\bar{p}}{\bar{\rho}} < -\frac{1}{3} \)
  - Modification of gravity on large scales

\[
G_{\mu\nu} = 8\pi G \left( T^M_{\mu\nu} + T^\text{DE}_{\mu\nu} \right)
\]
\[
F(g_{\mu\nu}) + G_{\mu\nu} = 8\pi G T^M_{\mu\nu}
\]

- Proof of principle models for both exist: quintessence, k-essence; DGP braneworld acceleration, \( f(R) \) modified action
- Compelling models for either explanation lacking
- Study models as illustrative toy models whose features can be generalized
Post-Friedmann Parameterization

• Assume:
  Gravity is a metric theory
  Stress-energy tensor covariantly conserved
  Linear metric deviations around FRW

\[ ds^2 = -(1 + 2\Psi)dt^2 + a^2(1 + 2\Phi)dx^2 \]

\[ g = \frac{\Phi + \Psi}{\Phi - \Psi} \]

• Dark energy equivalence

\[ F(g_{\mu\nu}) + G_{\mu\nu} = 8\pi G T^M_{\mu\nu} \]
\[ -F(g_{\mu\nu}) = 8\pi G T^{DE}_{\mu\nu} \]

with \( \nabla^{\mu}T^{DE}_{\mu\nu} = 0 \), \( g \) as effective anisotropic stress, leaving 1 dof

• Differences in the dynamics that \( g \) specifies or closure relation
Three Regimes

- Three regimes with different dynamics
- Examples $f(R)$ and DGP braneworld acceleration
- Parameterized Post-Friedmann description
- Non-linear regime return to General Relativity / Newtonian dynamics

Diagram:

- General Relativistic Non-Linear Regime
- Scalar-Tensor Regime
- Conserved-Curvature Regime

Key:
- $r_*$: halos, galaxy
- $r_c$: large scale structure
- $r$: CMB
Three Regimes

- **Metric:** \( ds^2 = -(1 + 2\Psi)dt^2 + a^2(1 + 2\Phi)dx^2 \)

- **Superhorizon regime:** \( \zeta = \text{const.}, \ g(a) = (\Phi + \Psi)/(\Phi - \Psi) \)

- **Linear regime - closure ↔ “smooth” dark energy density:**

\[
\nabla^2 (\Phi - \Psi)/2 = -4\pi G a^2 \Delta \rho
\]

\( G \) can be promoted to \( G(a) \), \( G(a, k) \) but for scalar degrees of freedom conformal invariance requires \( G = G_N \) and

- **Non-linear regime:**

\[
\nabla^2 (\Phi - \Psi)/2 = -4\pi G a^2 \Delta \rho
\]

\[
\nabla^2 \Psi = 4\pi G a^2 \Delta \rho + \frac{1}{2} \nabla^2 \phi
\]

with non-linearity in the field equation

\[
\nabla^2 \phi = g_{\text{lin}}(a) a^2 (8\pi G \Delta \rho - N[\phi])
\]
Worked Examples
Modified Action $f(R)$ Model

- $R$: Ricci scalar or “curvature”
- $f(R)$: modified action (Starobinsky 1980; Carroll et al 2004)

$$S = \int d^4x \sqrt{-g} \left[ \frac{R + f(R)}{16\pi G} + \mathcal{L}_m \right]$$

- $f_R \equiv df/\,dR$: additional propagating scalar degree of freedom (metric variation)
- $f_{RR} \equiv d^2f/\,dR^2$: Compton wavelength of $f_R$ squared, inverse mass squared
- $B$: Compton wavelength of $f_R$ squared in units of the Hubble length

$$B \equiv \frac{f_{RR}}{1 + f_R R'} \frac{H}{H'}$$

- $' \equiv d/d\ln a$: scale factor as time coordinate
**Modified Einstein Equation**

- In the **Jordan frame**, gravity becomes 4th order but matter remains minimally coupled and separately conserved

\[
G_{\alpha\beta} + f_R R_{\alpha\beta} - \left( \frac{f}{2} - \Box f_R \right) g_{\alpha\beta} - \nabla_\alpha \nabla_\beta f_R = 8\pi G T_{\alpha\beta}
\]

- Trace can be interpreted as a scalar field equation for \( f_R \) with a density-dependent effective potential (\( p = 0 \))

\[
3\Box f_R + f_R R - 2f = R - 8\pi G \rho
\]

- For small deviations, \( |f_R| \ll 1 \) and \( |f / R| \ll 1 \),

\[
\Box f_R \approx \frac{1}{3} (R - 8\pi G \rho)
\]

the field is sourced by the deviation from GR relation between curvature and density and has a mass

\[
m_{f_R}^2 \approx \frac{1}{3} \frac{\partial R}{\partial f_R} = \frac{1}{3 f_{RR}}
\]
DGP Braneworld Acceleration

- **Braneworld acceleration** (Dvali, Gabadadze & Porrati 2000)

\[ S = \int d^5x \sqrt{-g} \left[ \frac{(5)R}{2\kappa^2} + \delta(\chi) \left( \frac{(4)R}{2\mu^2} + \mathcal{L}_m \right) \right] \]

with crossover scale \( r_c = \kappa^2/2\mu^2 \)

- Influence of bulk through **Weyl tensor anisotropy** - solve **master equation** in bulk (Deffayet 2001)

- Matter still **minimally coupled** and conserved

- Exhibits the 3 regimes of modified gravity
  - **Weyl tensor anisotropy** dominated conserved curvature regime \( r > r_c \) (Sawicki, Song, Hu 2006; Cardoso et al 2007)
  - **Brane bending** scalar tensor regime \( r_* < r < r_c \) (Lue, Soccimarro, Starkman 2004; Koyama & Maartens 2006)
  - **Strong coupling** General Relativistic regime \( r < r_* = (r_c^2r_g)^{1/3} \) \( r_g = 2GM \) (Dvali 2006)
DGP Field Equations

- DGP field equations

\[ G_{\mu \nu} = 4 r_c^2 f_{\mu \nu} - E_{\mu \nu} \]

where \( f_{\mu \nu} \) is a tensor quadratic in the 4-dimensional Einstein and energy-momentum tensors

\[ f_{\mu \nu} \equiv \frac{1}{12} A A_{\mu \nu} - \frac{1}{4} A^\alpha_{\mu} A_{\nu \alpha} + \frac{1}{8} g_{\mu \nu} \left( A_{\alpha \beta} A^{\alpha \beta} - \frac{A^2}{3} \right) \]

\[ A_{\mu \nu} \equiv G_{\mu \nu} - \mu^2 T_{\mu \nu} \]

and \( E_{\mu \nu} \) is the bulk Weyl tensor

- Background metric yields the modified Friedmann equation

\[ H^2 \mp \frac{H}{r_c} = \frac{\mu^2 \rho}{3} \]

- For perturbations, involves solving metric perturbations in the bulk through the “master equation”
$f(R)$ Expansion History
Modified Friedmann Equation

- Expansion history parameterization: *Friedmann equation* becomes

\[
H^2 - f_R(\dot{H}H' + H^2) + \frac{1}{6} f + H^2 f_{RR}R' = \frac{8\pi G \rho}{3}
\]

- Reverse engineering $f(R)$ from the expansion history: for any desired $H$, solve a 2nd order diffeq to find $f(R)$

- Allows a *family* of $f(R)$ models, parameterized in terms of the Compton wavelength parameter $B$
Modified Friedmann Equation

- Expansion history parameterization: Friedmann equation becomes

\[ H^2 - f(R)(HH' + H^2) + \frac{1}{6}f + H^2 f_{RR}R' = \frac{8\pi G \rho}{3} \]

- Reverse engineering \( f(R) \) from the expansion history: for any desired \( H \), solve a 2nd order diffeq to find \( f(R) \)

- Allows a family of \( f(R) \) models, parameterized in terms of the Compton wavelength parameter \( B \)

- Formally includes models where \( B < 0 \), such as \( f(R) = -\mu^4/R \), leading to confusion as to whether such models provide viable expansion histories

- Answer: no these have short-time scale tachyonic instabilities at high curvature and limit as \( B \to 0 \) from below is not GR

- \( B > 0 \) family has very different implications for structure formation but with identical distance-redshift relations
Expansion History Family of $f(R)$

- Each expansion history, matched by dark energy model $[w(z), \Omega_{\text{DE}}, H_0]$ corresponds to a family of $f(R)$ models due to its 4th order nature.
- Parameterized by $B \propto f_{RR} = \frac{d^2f}{dR^2}$ evaluated at $z=0$.

Song, Hu & Sawicki (2006)
Expansion History Family of $f(R)$

- Each expansion history, matched by dark energy model $[\omega(z), \Omega_{DE}, H_0]$ corresponds to a family of $f(R)$ models due to its 4th order nature.
- Parameterized by $B \propto f_{RR} = \frac{d^2f}{dR^2}$ evaluated at $z=0$.

Song, Hu & Sawicki (2006)
Expansion History Family of $f(R)$

- Each expansion history, matched by dark energy model $[w(z), \Omega_{DE}, H_0]$ corresponds to a family of $f(R)$ models due to its 4th order nature.
- Parameterized by $B \propto f_{RR} = \frac{d^2f}{dR^2}$ evaluated at $z=0$.

Song, Hu & Sawicki (2006)
Expansion History Family of $f(R)$

- Each expansion history, matched by dark energy model $[w(z), \Omega_{DE}, H_0]$ corresponds to a family of $f(R)$ models due to its 4th order nature.
- Parameterized by $B \propto f_{RR} = \frac{d^2f}{dR^2}$ evaluated at $z=0$.

Song, Hu & Sawicki (2006)
Instability at High Curvature

- Tachyonic instability for negative mass squared $B<0$ makes high curvature regime increasingly unstable: high density $\neq$ high curvature
- Linear metric perturbations immediately drop the expansion history to low curvature solution

Sawicki & Hu (2007)
Engineering $f(R)$ Models

- Mimic $\Lambda$CDM at high redshift
- **Accelerate** the expansion at low redshift without a cosmological constant
- Sufficient **freedom** to vary expansion history within observationally allowed range
- **Contain** the phenomenology of $\Lambda$CDM in both cosmology and solar system tests as a **limiting case** for the purposes of constraining small deviations
- Suggests

  \[
  f(R) \propto \frac{R^n}{R^n + \text{const.}}
  \]

  such that modifications **vanish** as $R \to 0$ and go to a **constant** as $R \to \infty$
Form of $f(R)$ Models

- Transition from zero to constant across an adjustable curvature scale
- Slope $n$ controls the rapidity of transition, field amplitude $f_{R0}$ position
- Background curvature stops declining during acceleration epoch and thereafter behaves like cosmological constant

Hu & Sawicki (2007)
Expansion History

- Effective equation of state $w_{\text{eff}}$ scales with field amplitude $f_{R0}$
- Crosses the phantom divide at a redshift that decreases with $n$
- Signature of degrees of freedom in dark energy beyond standard kinetic and potential energy of k-essence or quintessence or modified gravity

Hu & Sawicki (2007)
DGP Expansion History
DGP Expansion History

- Matching the DGP expansion history to a dark energy model with the same expansion history
- Effective equation of state $w(z)$ [\(w_0 \sim -0.85, \ w_a \sim 0.35\)]
DGP Expansion History

- Crossover scale $r_c$ fit to SN relative distance from $z=0$: $H_0D_A$
DGP Normal Branch

- On the normal branch, expansion does not self-accelerate and dark energy in the form of a brane tension or scalar field necessary.

\[
H^2 + \frac{H}{r_c} = \frac{\mu^2}{3} (\rho_m + \rho_{DE})
\]

- Gravity is still modified as in the self-accelerated branch (but with attractive forces).

- Ghost free in the quantum theory.

- Can choose \( \rho_{DE} \) to match any desired expansion history including flat \( \Lambda \)CDM.

\[
H^2 \equiv \frac{\mu^2}{3} (\rho_m + \rho_\Lambda) \rightarrow \rho_{DE}
\]

- Separate out geometrical and dynamical tests of acceleration.
Conserved Curvature Regime
Curvature Conservation

- On superhorizon scales, energy momentum conservation and expansion history constrain the evolution of metric fluctuations (Bertschinger 2006)

- For adiabatic perturbations in a flat universe, conservation of comoving curvature applies $\zeta' = 0$ where $' \equiv d/d \ln a$ (Bardeen 1980)

- Gauge transformation to Newtonian gauge

$$ds^2 = -(1 + 2\Psi)dt^2 + a^2(1 + 2\Phi)dx^2$$

yields (Hu & Eisenstein 1999)

$$\Phi'' - \Psi' - \frac{H''}{H'}\Phi' - \left(\frac{H'}{H} - \frac{H''}{H'}\right)\Psi = 0$$

- Modified gravity theory supplies the closure relationship $\Phi = -\gamma(\ln a)\Psi$ between and expansion history $H = \dot{a}/a$ supplies rest.
Two Gravitational Potentials

- **Newtonian potential**: $\Psi = \delta g_{00}/2g_{00}$ which non-relativistic particles feel

- **Space curvature**: $\Phi = \delta g_{ii}/2g_{ii}$

- **Combination** $\Phi_\neq = (\Phi - \Psi)/2$ for gravitational lensing + redshift

Caldwell et al (2007); Amendola et al (2007); Sawicki & Hu (2007); Amin et al (2007); Jain & Zhang (2007)
Deviation Parameter

- Express the 4th order nature of equations as a deviation parameter

\[ \Phi'' - \Psi' - \frac{H''}{H'} \Phi' - \left( \frac{H'}{H} - \frac{H''}{H'} \right) \Psi = \left( \frac{k}{aH} \right)^2 B \epsilon \]

- Einstein equation become a second order equation for \( \epsilon \)

- In high redshift, high curvature \( R \) limit this is

\[ \epsilon'' + \left( \frac{7}{2} + 4 \frac{B'}{B} \right) \epsilon' + \frac{2}{B} \epsilon = \frac{1}{B} \times \text{metric sources} \]

\[ B = \frac{f_{RR}}{1 + f_R} R' \frac{H}{H'} \]

- \( R \to \infty, B \to 0 \) and for \( B < 0 \) short time-scale tachyonic instability appears making previous models not cosmologically viable

\[ f(R) = -M^{2+2n} / R^n \]
Linear Theory for $f(R)$

- In $f(R)$ model, “superhorizon” behavior persists until Compton wavelength smaller than fluctuation wavelength $B^{1/2}(k/aH) < 1$
- Once Compton wavelength becomes larger than fluctuation

$$B^{1/2}(k/aH) > 1$$

perturbations are in scalar-tensor regime described by $\gamma = 1/2$. 
Linear Theory for $f(R)$

- In $f(R)$ model, “superhorizon” behavior persists until Compton wavelength smaller than fluctuation wavelength $B^{1/2}(k/aH) < 1$
- Once Compton wavelength becomes larger than fluctuation

$$B^{1/2}(k/aH) > 1$$

perturbations are in scalar-tensor regime described by $\gamma = 1/2$.
- Small scale density growth enhanced and

$$8\pi G\rho > R$$

low curvature regime with order unity deviations from GR
- Transitions in the non-linear regime where the Compton wavelength can shrink via chameleon mechanism
- Given $k_{NL}/aH \gg 1$, even very small $f_R$ have scalar-tensor regime
PPF $f(R)$ Description

- **Metric and matter evolution well-matched by PPF description**
- **Standard GR tools apply (CAMB), self-consistent, gauge invar.**

Hu & Sawicki (2007); Hu (2008)
Integrated Sachs-Wolfe Effect

- CMB photons transit gravitational potentials of large-scale structure
- If potential decays during transit, gravitational blueshift of infall not cancelled by gravitational redshift of exit
- Spatial curvature of gravitational potential leads to additional effect $\Delta T/T = -\Delta (\Phi - \Psi)$
Integrated Sachs-Wolfe Effect

- CMB photons transit gravitational potentials of large-scale structure
- If potential decays during transit, gravitational blueshift of infall not cancelled by gravitational redshift of exit
- Spatial curvature of gravitational potential leads to additional effect $\Delta T/T = -\Delta(\Phi - \Psi)$
Reduction of potential decay can eliminate the ISW effect at the quadrupole for $B_0 \sim 3/2$

In conjunction with a change in the initial power spectrum can also bring the total quadrupole closer in ensemble average to the observed quadrupole.

Song, Hu & Sawicki (2006)
**ISW Quadrupole**

- **Reduction of large angle anisotropy** for $B_0 \sim 1$ for same expansion history and distances as $\Lambda$CDM
- **Well-tested** small scale anisotropy unchanged

---

Song, Hu & Sawicki (2006)
ISW-Galaxy Correlation

- Decaying potential: galaxy positions \textit{correlated} with CMB
- Growing potential: galaxy positions \textit{anticorrelated} with CMB
- Observations indicate \textit{correlation}
Galaxy-ISW Anti-Correlation

- Large Compton wavelength $B^{1/2}$ creates potential growth which can anti-correlate galaxies and the CMB
- In tension with detections of positive correlations across a range of redshifts
DGP Horizon Scales

- **Metric and matter evolution well-matched by PPF description**
- **Standard GR tools apply (CAMB), self-consistent, gauge invar.**

Hu & Sawicki (2007); Hu (2008)
DGP CMB Large-Angle Excess

- Extra dimension **modify** gravity on large scales
- 4D universe **bending** into extra dimension alters gravitational redshifts in **cosmic microwave background**
CMB in DGP

- Adding cut off as an epicycle can fix distances, ISW problem
- Suppresses polarization in violation of EE data - cannot save DGP!

Fang et al (2008)
CMB in DGP

- Adding **cut off** as an epicycle can fix distances, ISW problem
- Suppresses polarizations in violation of EE data - cannot save DGP!

![Graph showing CMB data in DGP](image)

Fang et al (2008)
Linear Scalar Tensor Regime
Post-Newtonian Large Scale Structure

- Testing gravity with the **large scale structure** of the universe

---

**Density Field**
Galaxy Clustering

**Velocity Field**
Redshift Space Distortion
Three Regimes

- Three regimes with different dynamics
- Examples $f(R)$ and DGP braneworld acceleration
- Parameterized Post-Friedmann description

- Non-linear regime return to General Relativity / Newtonian dynamics

General Relativistic Non-Linear Regime
Scalar-Tensor Regime
Conserved-Curvature Regime

$halos, galaxy$  $large scale structure$  $CMB$
Three Regimes

- **Metric:** \( ds^2 = -(1 + 2\Psi)dt^2 + a^2(1 + 2\Phi)dx^2 \)

- **Superhorizon regime:** \( \zeta = \text{const.}, \ g(a) = (\Phi + \Psi)/(\Phi - \Psi) \)

- **Linear regime - closure ↔ “smooth” dark energy density:**

\[
\nabla^2(\Phi - \Psi)/2 = -4\pi Ga^2 \Delta \rho
\]

\( G \) can be promoted to \( G(a), G(a, k) \) but for scalar degrees of freedom conformal invariance requires \( G = G_N \) and

- **Non-linear regime:**

\[
\nabla^2(\Phi - \Psi)/2 = -4\pi Ga^2 \Delta \rho
\]

\[
\nabla^2 \Psi = 4\pi Ga^2 \Delta \rho + \frac{1}{2} \nabla^2 \phi
\]

with non-linearity in the field equation

\[
\nabla^2 \phi = g_{\text{lin}}(a)a^2 (8\pi G \Delta \rho - N[\phi])
\]
 Linear real space power spectrum enhanced on small scales
- Degeneracy with galaxy bias and lack of non-linear predictions leave constraints from shape of power spectrum

![Graph of Linear Power Spectrum](image)

- $P_L(k)$ (Mpc/$h$)$^3$
- $k (h$/Mpc)
- $B_0 = 0, 0.001, 0.01, 0.1, 1$
- $0 (~\Lambda$CDM)
Power Spectrum Data

- **Linear** power spectrum enhancement fits SDSS LRG data better than $\Lambda$CDM but

![Graph showing power spectrum data]
Power Spectrum Data

- **Linear** power spectrum enhancement fits SDSS LRG data better than $\Lambda$CDM but
- **Shape** expected to be altered by non-linearities

Song, Peiris & Hu (2007)
Current Constraints

- Likelihood analysis of SDSS LRG $P(k)$, WMAP $C_l$, SNIa $d_L$
- Degeneracy between non-linearity and $f(R)$ enhancement allows whole range of Compton wavelengths from infinitesimal to horizon sized
- Requires cosmological simulation of $f(R)$ to predict non-linearities

Song, Peiris & Hu (2007)
Redshift Space Distortion

- Relationship between velocity and density field given by continuity with modified growth rate ($f_v = \frac{d\ln D}{d\ln a}$)
- Redshift space power spectrum further distorted by Kaiser effect
PPF DGP Description

- Metric and matter evolution well-matched by PPF description
- Standard GR tools apply (CAMB), self-consistent, gauge invar.

Hu & Sawicki (2007); Hu (2008)
DGP Power Spectrum

- Constant suppression in the linear regime for self-acceleration

Lue, Scoccimarro, Starkman (2004); Hu & Sawicki (2007)
Non-Linear GR Regime
Three Regimes

- Three regimes with different dynamics
- Examples $f(R)$ and DGP braneworld acceleration
- Parameterized Post-Friedmann description
- Non-linear regime return to General Relativity / Newtonian dynamics

General Relativistic Non-Linear Regime
Scalar-Tensor Regime
Conserved-Curvature Regime

$r_*$
$r_c$

halos, galaxy
large scale structure
CMB
Three Regimes

- Fully worked $f(R)$ and DGP examples show 3 regimes
- **Superhorizon** regime: $\zeta = \text{const.}, g(a)$
- **Linear** regime - closure condition - analogue of “smooth” dark energy density:

  \[
  \frac{\nabla^2 (\Phi - \Psi)}{2} = -4\pi G a^2 \Delta \rho
  \]

  \[
  g(a, x) \leftrightarrow g(a, k)
  \]

  $G$ can be promoted to $G(a)$ but conformal invariance relates fluctuations to field fluctuation that is small

- **Non-linear** regime:

  \[
  \frac{\nabla^2 (\Phi - \Psi)}{2} = -4\pi G a^2 \Delta \rho
  \]

  \[
  \nabla^2 \Psi = 4\pi G a^2 \Delta \rho - \frac{1}{2} \nabla^2 \phi
  \]
Nonlinear Interaction

Non-linearity in the field equation

$$\nabla^2 \phi = g_{\text{lin}}(a)a^2 (8\pi G \Delta \rho - N[\phi])$$

recovers linear theory if $N[\phi] \to 0$

- For $f(R)$, $\phi = f_R$ and

$$N[\phi] = \delta R(\phi)$$

a non-linear function of the field

Linked to gravitational potential

- For DGP, $\phi$ is the brane-bending mode and

$$N[\phi] = \frac{r^2_c}{a^4} \left[ (\nabla^2 \phi)^2 - (\nabla_i \nabla_j \phi)^2 \right]$$

a non-linear function of second derivatives of the field

Linked to density fluctuation
Non-Linear Chameleon

- For $f(R)$ the field equation

$$\nabla^2 f_R \approx \frac{1}{3} (\delta R(f_R) - 8\pi G \delta \rho)$$

is the non-linear equation that returns general relativity

- High curvature implies short Compton wavelength and suppressed deviations but requires a change in the field from the background value $\delta R(f_R)$

- Change in field is generated by density perturbations just like gravitational potential so that the chameleon appears only if

$$\Delta f_R \leq \frac{2}{3} \Phi,$$

else required field gradients too large despite $\delta R = 8\pi G \delta \rho$ being the local minimum of effective potential
Non-Linear Dynamics

• Supplement that with the modified Poisson equation

\[ \nabla^2 \Psi = \frac{16 \pi G}{3} \delta \rho - \frac{1}{6} \delta R(f_R) \]

• Matter evolution given metric unchanged: usual motion of matter in a gravitational potential \( \Psi \)

• Prescription for \( N \)-body code

• Particle Mesh (PM) for the Poisson equation

• Field equation is a non-linear Poisson equation: relaxation method for \( f_R \)

• Initial conditions set to GR at high redshift
Environment Dependent Force

- **Chameleon suppresses extra force (scalar field) in high density, deep potential regions**

  - density: $\max[\ln(1+\delta)]$
  - potential: $\min[\Psi']$
  - field: $\min[f_R/f_{R0}]$

Oyaizu, Lima, Hu (2008)
Environment Dependent Force

- For large background field, gradients in the scalar prevent the chameleon from appearing

Oyaizu, Lima, Hu (2008)
512³ PM-relaxation code resolves the chameleon transition to GR: greatly reduced non-linear effect
• Artificially turning off the chameleon mechanism restores much of enhancement

\[ \frac{P(k)}{P_{GR}(k)} = 1 \]

Oyaizu, Lima, Hu (2008)
N-body Power Spectrum

- Models where the chameleon absent today (large field models) show residual effects from a high redshift chameleon

\[
P(k) / P_{GR}(k) - 1
\]

\[|f_{R0}| = 10^{-4}\]

\[|f_{R0}| = 10^{-6}\]

Oyaizu, Lima, Hu (2008)
Halo Model

- Model *density field* as (linearly) clustered *NFW halos* with mass function
Halo Model

- Associate luminous observables with dark halos
Mass Function

- Enhanced **abundance** of rare dark matter halos (clusters) with extra force

Schmidt, Lima, Oyaizu, Hu (2008)
Halo Bias

- Halos at a fixed mass less rare and less highly biased

Schmidt, Lima, Oyaizu, Hu (2008)
Halo Mass Correlation

- Enhanced forces vs lower bias

Schmidt, Lima, Oyaizu, Hu (2008)
Halo Model

- Power spectrum trends also consistent with halos and modified collapse

Schmidt, Lima, Oyaizu, Hu (2008)
Nonlinear Interaction

Non-linearity in the field equation

\[ \nabla^2 \phi = g_{\text{lin}}(a) a^2 (8\pi G \Delta \rho - N[\phi]) \]

recovers linear theory if \( N[\phi] \to 0 \)

- For \( f(R) \), \( \phi = f_R \) and

\[ N[\phi] = \delta R(\phi) \]

a non-linear function of the field

Linked to gravitational potential

- For DGP, \( \phi \) is the brane-bending mode and

\[ N[\phi] = \frac{r_c^2}{a^4} \left[ (\nabla^2 \phi)^2 - (\nabla_i \nabla_j \phi)^2 \right] \]

a non-linear function of second derivatives of the field

Linked to density fluctuation
DGP N-Body

- DGP nonlinear derivative interaction solved by relaxation revealing the Vainshtein mechanism

Newtonian Potential  Brane Bending Mode

Schmidt (2009); Chan & Scoccimarro (2009)
Vainshtein Suppression

- Modification to gravitational potential saturates at the Vainshtein radius

Lue, Scoccimarro, Starkman (2004); Schmidt, Hu, Lima (2009)
DGP Halo Modeling

- Mass function, halo bias, power spectrum well modelled by spherical collapse and halo model, especially the non-linear Vainshtein mechanism

Schmidt, Hu, Lima (2009)
Apparent Equivalence Principle Violation

- Self-field of a “test mass” can saturate an external field (for $f(R)$ in the gradient, for DGP in the second derivatives)

Hui, Nicolis, Stubbs (2009); Hu (2009)
Parameterizing Non-Linear Regime

- No (field) superposition principle
  - apparent “equivalence principle” violation - internal vs external linear theory not necessarily recovered by averaging

- Deviations are environment dependent $[\phi]$ not scale dependent $[g(a, k)]$ - massive dark matter halos show suppressed modifications

- PPF: parameterize as a halo model through the 1 halo term
  - simple form: some rms density scaling
  - general form: suppression is some monotonic function of parent halo mass describing interpolation in abundance, profile etc. parameterization of general form can be motivated by spherical collapse calculations given a specific non-linear mechanism
Solar System Tests
Solar Profile

- Density profile of Sun is not a constant density sphere - interior photosphere, chromosphere, corona
- Density drops by ~25 orders of magnitude - does curvature follow?

Hu & Sawicki (2007)
Field Solution

- Field solution smoothly relaxes from exterior value to high curvature interior value $f_R \sim 0$, minimizing potential + kinetic

- Juncture is where thin-shell criterion is satisfied $|\Delta f_R| \sim \Delta \Phi$

Hu & Sawicki (2007)
Solar Curvature

- Curvature drops suddenly as field moves slightly from zero
- Enters into low curvature regime where $R < 8\pi G \rho$

Hu & Sawicki (2007)
Solar System Constraint

- Cassini constraint on PPN $|\gamma-1|<2.3\times10^{-5}$
- Easily satisfied if galactic field is at potential minimum $|f_{Rg}|<4.9\times10^{-11}$
- Allows even order unity cosmological fields

![Graph showing the variation of $|\gamma-1|$ with $rr_\odot$ for different values of $|f_{R0}|$.](image.png)

Hu & Sawicki (2007)
Summary

- Lessons from the $f(R)$ and DGP worked examples – 3 regimes:
  - large scales: conservation determined
  - intermediate scales: scalar-tensor
  - small scales: GR in high density regions, modified in low

- **Large** scales: expansion history and metric ratio
  \[ g = \frac{(\Phi + \Psi)}{(\Phi - \Psi)} \] through curvature conservation

- **Intermediate** scales: scalar tensor modified Newtonian regime, $g$ and Poisson equation

- **Small** scales: nonlinear interaction of modification field makes $g$ depend on local environment (not scale) - density or potential - suppressing deviations

- $N$-body (PM-relaxation) simulations show halo model framework can describe observables in the nonlinear regime