On a functional equation characterizing linear similarities

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Abstract. The aim of this paper is to give an answer to a question posed by Alsina, Sikorska and Tomás. Namely, we show that, under suitable assumptions, a function \( f : X \to Y \) from a normed space \( X \) into a normed space \( Y \), satisfying the functional equation

\[
\frac{f\left(y - \frac{\rho'_\pm(x, y)}{\|x\|^2} x\right)}{\|f(x)\|^2} = \frac{f(y) - \frac{\rho'_\pm(f(x), f(y))}{\|f(x)\|^2} f(x)}{\|x\|^2}, \quad x, y \in X
\]

has to be a linear similarity (scalar multiple of a linear isometry).

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1. Introduction

Let \((X, \|\cdot\|)\) be a real normed space. We define norm derivatives \( \rho'_\pm : X \times X \to \mathbb{R} \) by \( \rho'_\pm(x, y) := \|x\| \cdot \lim_{t \to 0^\pm} \frac{\|x + ty\| - \|x\|}{t} \). The convexity of the norm yields that \( \rho'_+ \) and \( \rho'_- \) are well-defined. Now we define \( \rho'_+ \)-orthogonality: \( x \perp \rho'_+ y \iff \rho'_+(x, y) = 0 \). The following properties can be found, e.g., in \([1,2]\).

\(\text{(nd1)}\) \(\forall x, y \in X \forall \alpha \in \mathbb{R} \) \( \rho'_\pm(x, \alpha x + y) = \alpha \|x\|^2 + \rho'_\pm(x, y) \);

\(\text{(nd2)}\) \(\forall x, y \in X \forall \alpha \geq 0 \) \( \rho'_\pm(\alpha x, y) = \alpha \rho'_\pm(x, y) = \rho'_\pm(x, \alpha y) \);

\(\text{(nd3)}\) \(\forall x, y \in X \forall \alpha < 0 \) \( \rho'_\pm(\alpha x, y) = \alpha \rho'_\pm(x, y) = \rho'_\pm(x, \alpha y) \);

\(\text{(nd4)}\) \(\forall x, y \in X \) \( |\rho'_\pm(x, y)| \leq \|x\| \cdot \|y\| \), \( \rho'_\pm(x, x) = \|x\|^2 \), \( \rho'_-(x, y) \leq \rho'_+(x, y) \);

\(\text{(nd5)}\) \(\forall x, y, z \in X \) \( \rho'_+(x, y + z) \leq \rho'_+(x, y) + \rho'_+(x, z) \).

A normed space \( X \) is said to be smooth if for every \( x \in X \setminus \{0\} \) there is a unique supporting functional at \( x \), i.e., a unique functional \( x^* \in X^* \) such that \( \|x^*\| = 1 \) and \( x^*(x) = \|x\| \). Moreover, we may state this definition in an equivalent form, namely: \( X \) is smooth \( \iff \rho'_+ = \rho'_- \iff \forall x \in X \rho'_+(x, \cdot) \) is linear. If \( X \) is smooth, then the following condition holds (see \([1]\)):
can compute the height vector from \( y \) linearly independent vectors \( x \) and \( y \), \( h = \lambda x - y \).

In a real inner product space \((X, \langle \cdot, \cdot \rangle)\), given the triangle determined by two linearly independent vectors \( x, y \) and the zero vector (i.e., \( \triangle \{x, y, 0\} \)), one can compute the height vector from \( y \) to the side \( x \) and orthogonal to \( x \) using the formula \( h(x, y) := y - \frac{x \langle y, x \rangle}{\|x\|^2} \). Then \( x \perp h(x, y) \). The same might be done for normed spaces using the function \( \rho'_+ \) as a generalization of an inner product.

In this case we consider the height function \( h(x, y) := y - \frac{\rho'(x, y)}{\|x\|^2} x \).

Alsina et al. \cite{Alsina} investigated functions \( f : X \to X \) that transform the height of the triangle with sides \( x, y, x - y \) into the corresponding height of the triangle determined by sides \( f(x), f(y), f(x) - f(y) \), i.e. \( f(\triangle \{x, y, 0\}) = \triangle \{f(x), f(y), 0\} \). Namely, they studied the condition \( f(h(x, y)) = h(f(x), f(y)) \), which leads to the functional equation \( f \left( y - \frac{\rho'(x, y)}{\|x\|^2} x \right) = f(y) - \frac{\rho'(f(x), f(y))}{\|f(x)\|^2} f(x) \).

In particular, Alsina et al. \cite{Alsina} obtained the following result.

**Theorem 1.** \cite[p. 102, Theorem 3.7.2]{Alsina} If \( X \) is a real normed linear space and \( f : X \to X \) is a continuous function, then \( f \) is a solution of

\[
f \left( y - \frac{\rho'(x, y)}{\|x\|^2} x \right) = f(y) - \frac{\rho'(f(x), f(y))}{\|f(x)\|^2} f(x), \quad x, y \in X
\]

and vanishes only at zero if and only if, \( f \) is a linear similarity.

At the end of their book \cite[p. 178, Open problem 6]{Alsina} Alsina, Sikorska and Tomás put the following problem.

*Open problem* Solve the functional equation

\[
f \left( y - \frac{\rho'(x, y)}{\|x\|^2} x \right) = f(y) - \frac{\rho'(f(x), f(y))}{\|f(x)\|^2} f(x), \quad x, y \in X, \quad (1)
\]

where \( f : X \to X \) is injective and \( f(x) \neq 0 \) whenever \( x \neq 0 \).

The aim of this paper is to present a partial solution of the above open problem. In particular, we will prove that the assumption of the continuity of \( f \) is redundant in some circumstances. Moreover, it is not necessary to assume that \( f \) is injective.

### 2. Results

Throughout this section we will work with real normed spaces of dimensions not less than 2. We will consider the norm derivatives in various spaces \((X, Y)\); however, we will use one common symbol \( \rho'_+ \) for them. We will prove that \( f : X \to Y \) is a solution of \((1)\) if and only if it is a linear similarity (scalar multiple of a linear isometry). This assertion, however, can be obtained under the assumption of the smoothness of \( X \). But, unlike Theorem 1, it will not be assumed that a function \( f \) is continuous.
Lemma 2. Let $X, Y$ be normed spaces, let $f : X \to Y$ satisfy (1). Then $f(0) = 0$.

Proof. By (1) we get $f(0) = f \left( y - \frac{\rho_+(y,y)}{\|y\|^2} y \right) = f(y) - \frac{\rho_+(f(y),f(y))}{\|f(y)\|^2} f(y) = f(y) - f(y) = 0$. □

Now we prove the first main result of this paper.

Theorem 3. Let $X, Y$ be normed spaces and let $f : X \to Y$ satisfy (1). Suppose that $z \neq 0 \Rightarrow f(z) \neq 0$. Then $f$ is additive.

Proof. First we will prove that $f$ preserves the linear independence of two vectors. Suppose that $f(y) = \alpha f(x)$ and $x \neq 0$. Then

$$f \left( y - \frac{\rho_+(x,y)}{\|x\|^2} x \right) \overset{(1)}{=} \alpha f(x) - \frac{\rho_+(f(x),\alpha f(x))}{\|f(x)\|^2} f(x) \overset{(nd1)}{=} \alpha f(x) - \frac{\rho_+(f(x),f(x))}{\|f(x)\|^2} f(x) \overset{(nd2)}{=} 0.$$

From the assumption (i.e. $f(z) = 0 \Rightarrow z = 0$) we have that $y - \frac{\rho_+(x,y)}{\|x\|^2} x = 0$, hence the vectors $x, y$ are linearly dependent. So, we have proved that $f$ preserves the linear independence of two vectors.

Fix two linearly independent vectors $a, b \in X$. Then we have

$$f(b) - \frac{\rho_+(f(a),f(b))}{\|f(a)\|^2} f(a) \overset{(1)}{=} f \left( b - \frac{\rho_+(a,b)}{\|a\|^2} a \right) \overset{(nd1)}{=} f \left( a + b - \frac{\rho_+(a,a+b)}{\|a\|^2} a \right) \overset{(1)}{=} f(a+b) - \frac{\rho_+(f(a),f(a+b))}{\|f(a)\|^2} f(a).$$

It follows from the above equalities that

$$f(a + b) = f(b) + \left( \frac{\rho_+(f(a),f(a+b))}{\|f(a)\|^2} - \frac{\rho_+(f(a),f(b))}{\|f(a)\|^2} \right) f(a). \quad (2)$$

Putting $b, a$ in place of $a, b$, respectively, in the above equality we get

$$f(a + b) = f(a) + \left( \frac{\rho_+(f(b),f(b+a))}{\|f(b)\|^2} - \frac{\rho_+(f(b),f(a))}{\|f(b)\|^2} \right) f(b). \quad (3)$$

We know that $f(a), f(b)$ are linearly independent. Thus, combining (2) and (3), we immediately get $\frac{\rho_+(f(a),f(a+b))}{\|f(a)\|^2} - \frac{\rho_+(f(a),f(b))}{\|f(a)\|^2} = 1$. Now equality (2) becomes $f(a + b) = f(b) + 1 \cdot f(a)$. To sum up, it has been shown that

$$a, b \text{ are linearly independent } \Rightarrow f(a + b) = f(a) + f(b). \quad (4)$$

Now let $x$ and $y$ be linearly dependent. We may assume that $x \neq 0 \neq y$. We consider two cases. Assume first that $y = \gamma x$ for some $\gamma \in \mathbb{R} \setminus \{-1\}$. There are linearly independent vectors $a, b \in X$ such that $a + b = x$. Then
\[ f(x+y) = f(x+\gamma x) = f(a+b+\gamma x) \quad (4) \]
\[ f(a)+f(b)+f(\gamma x) \quad (4) \]
\[ f(a)+f(b)+f(\gamma x) \quad (4) \]
\[ f(a+b)+f(\gamma x) = f(x)+f(\gamma x). \]

To sum up, it has been shown that
\[ x \in X \setminus \{0\}, \gamma \in \mathbb{R} \setminus \{-1\} \Rightarrow f(x+\gamma x) = f(x)+f(\gamma x). \quad (5) \]

Now assume \( y = -x \). We have
\[ f(x) = f(2x + (-\frac{1}{2}) 2x) \quad (5) \]
\[ f(2x) + f(-x) = f(x) + f(-x). \]

It follows from the above equalities that \( 0 = f(x) + f(-x). \)

By Lemma 2 we already know that \( f(0) = 0. \)

Therefore \( f(x+y) = f(x+(-x)) = f(0) = 0 = f(x)+f(-x) = f(x)+f(y). \) So, we have the additivity of \( f \) on the whole space \( X \).

\[ \square \]

Lemma 4. Let \( X,Y \) be normed spaces and let \( f: X \to Y \) satisfy (1). Then \( f \) preserves \( \rho_+ \)-orthogonality.

Proof. Assume that \( x \perp_{\rho_+} y, \) i.e., \( \rho'_+(x,y) = 0. \) We assume that \( f(x) \neq 0 \) (if \( f(x) = 0, \) then \( f(x) \perp_{\rho_+} f(y) \)). Notice that \( f(y) = f \left( y - \frac{\rho'_+(x,y)}{\|x\|^2} x \right) \quad (1) \]
\[ f(y) = f \left( y - \frac{\rho'_+(f(x),f(y))}{\|f(x)\|^2} f(x) \right), \]

hence \( \rho'_+(f(x),f(y)) = 0. \) This gives \( \rho'_+(f(x),f(y)) = 0. \)

Hence \( f(x) \perp_{\rho_+} f(y). \) Thus, in fact, \( f \) preserves \( \rho_+ \)-orthogonality.

Now we prove the second main result of this paper.

Theorem 5. Let \( X,Y,f \) be as in Theorem 3. Suppose that \( X \) is smooth. Then \( f \) is homogeneous.

Proof. Fix \( y \) in \( X \setminus \{0\}. \) We know that \( \dim \text{span}\{y\} = 1, \) so, it is best to think of \( f|_{\text{span}\{y\}} : \text{span}\{y\} \to Y \) as a function \( f : \mathbb{R} \to Y. \)

Now we can prove that \( f|_{\text{span}\{y\}} \) is homogeneous. Since we already know that \( f \) is additive, it suffices to show that \( \|f|_{\text{span}\{y\}}(\cdot)\| \) is bounded below on the segment \( \{\gamma y : \gamma \in [1,2]\}. \) Let \( \beta \in (0,1]. \) Applying (nd6), there exists a \( w \in X \setminus \{0\} \) such that \( y \perp_{\rho_+} w \) and \( y+w \perp_{\rho_+} \beta y - w. \) It follows from Lemma 4 that \( f(y) \perp_{\rho_+} f(w). \)

Therefore,
\[
\|f(y)\|^2 = \rho'_+(f(y),f(y)) + 0 = \rho'_+(f(y),f(y)) + \rho'_+(f(y),f(w)) \quad (nd1) = \rho'_+(f(y),f(y) + f(w)) \quad (nd4) \leq \|f(y)\| \cdot \|f(y) + f(w)\|, \]
\[
\text{and dividing by } \|f(y)\|, \text{ we obtain } \|f(y)\| \leq \|f(y) + f(w)\|. \]

But since also \( y+w \perp_{\rho_+} \beta y - w, \) we conclude that \( f(y+w) \perp_{\rho_+} f(\beta y - w), \) and by the additivity of \( f \) we have \( f(y) + f(w) \perp_{\rho_+} f(\beta y) - f(w). \) In the same manner we can prove
\[
\|f(y) + f(w)\| \leq \|f(y) + f(w) + f(\beta y) - f(w)\|. \]

Therefore \( \|f(y) + f(w)\| \leq \|f(y) + f(\beta y\|). \) From this we deduce that
\[
\|f(y)\| \leq \|f(y) + f(w)\| \leq \|f(y) + f(\beta y)\| = \|f(y + \beta y)\|. \]

Thus we have proved:
\[
\beta \in (0,1] \Rightarrow \|f(y)\| \leq \|f((1+\beta)y)\|.
\]
Observe that the above condition implies that \( \|f|_{\text{span}\{y\}}(\cdot)\| \) is bounded below on the segment \( \{\gamma y : \gamma \in [1, 2]\} \). The proof of Theorem 5 is complete.

We can combine the results of Theorems 3 and 5 and Lemma 4 to obtain the third main result. Finally, we can solve (1) completely.

**Theorem 6.** Let \( X, Y \) be real normed spaces. Suppose that \( X \) is smooth. Assume that \( f: X \to Y \) is nonzero, suppose that \( z \neq 0 \Rightarrow f(z) \neq 0 \). Then, the following conditions are equivalent:

1. \( f \) satisfies (1),
2. \( f \) is linear and \( \exists \gamma > 0 \forall x \in X \|f(x)\| = \gamma \|x\| \).

**Proof.** We prove (a) \( \Rightarrow \) (b). It follows from Theorems 3, 5 that \( f \) is linear. According to Lemma 4, \( f \) preserves \( \rho \)-orthogonality. The class of linear mappings preserving \( \rho \)-orthogonality coincides with the class of linear similarities (cf. [3, Theorem 5]). The proof of this implication is complete. The converse implication has a trivial verification.

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