SEMI-TOPOLOGICAL K-THEORY FOR CERTAIN PROJECTIVE VARIETIES

MIRCEA VOINEAGU

Abstract. In this paper we compute Lawson homology groups and semi-topological K-theory for certain threefolds and fourfolds. We consider smooth complex projective varieties whose zero cycles are supported on a proper subvariety. Rationally connected varieties are examples of such varieties. The computation makes use of a technique of Bloch and Srinivas, of the Bloch-Kato conjecture and of the spectral sequence relating morphic cohomology and semi-topological K-theory.

Contents

1. Introduction 1
2. Notations and Recollection 3
3. First results concerning generalized cycle maps 6
4. Cycle action on morphic cohomology 10
5. Comparing Lawson homology with singular homology 12
6. About varieties with small Chow group 20
7. The case of projective linear varieties 23
References 23

1. Introduction

Eric Friedlander and Mark Walker introduced in [15] the (singular) semi-topological K-theory of a complex projective variety $X$. This is defined by

$$K^*_{ss}(X) = π_*(Mor(X, Grass)^+)$$

where $Grass = \coprod_{n,N} Grass_n(\mathbb{P}^N)$. By $Mor(X, Grass)^+$ we denote the topological group given by the homotopy completion of the space of algebraic maps between $X$ and $Grass$.

Semi-topological K-theory lies between algebraic and topological K-theory in the sense that the natural map from the algebraic K-theory $K_* (X)$ of a variety $X$ to the connective (complex) topological K-theory $ku^*(X^{an})$ of its underlying analytic space $X^{an}$ factors through the semi-topological K-theory of $X$, i.e.

$$K_q (X) \to K^*_q (X) \to ku^{-q}(X^{an})$$

for any $q \geq 0$.

2000 Mathematics Subject Classification. 19E20, 19E15, 14F43.
In [25] Blaine Lawson introduced the (Lawson) homology groups of a projective complex variety \( X \), which are given by

\[
L_r H_n(X) = \pi_{n-2r}(\mathcal{Z}_r(X))
\]

where \( \mathcal{Z}_r(X) \) is the naive group completion of the topological monoid

\[
\mathcal{C}_r(X) = \coprod_d \mathcal{C}_{r,d}(X)
\]

with \( \mathcal{C}_{r,d}(X) \) the Chow variety of subvarieties of \( X \) of dimension \( r \) and degree \( d \).

In [14] Eric Friedlander and Blaine Lawson introduced the morphic cohomology, a cohomology theory dual to Lawson homology [13]. They defined

\[
L^r H^n(X) = \pi_{2r-n}(\mathcal{Z}^r(X))
\]

where \( \mathcal{Z}^r(X) \) is the naive group completion of the following topological monoid

\[
Mor(X, \mathcal{C}_r(\mathbb{P}^r))/Mor(X, \mathcal{C}_0(\mathbb{P}^{r-1})).
\]

Morphic cohomology groups are related to the semi-topological K-theory by means of a semi-topological spectral sequence [12] compatible with the motivic spectral sequence and Atiyah-Hirzebruch spectral sequence [12].

In this paper we study the map

\[
K^\text{sst}_i(X) \to ku^{-i}(X^{an})
\]

for various complex projective varieties \( X \).

We divide the paper in six sections. In the second section we fix the notations and recall some essential results that we need in the paper.

In the third section we study the effects of the Bloch-Kato conjecture on the kernel and cokernel of the generalized cycle maps. We give a new proof of a theorem of Bloch about the torsion of the singular cohomology of a smooth projective variety. We also study the torsion of the Borel-Moore homology of a quasi-projective smooth variety. At the end of the section we construct a birational invariant using Lawson homology.

In the fourth section we study the action of an algebraic cycle on morphic cohomology groups. Our approach is slightly different than the approach pioneered by C. Peters [29] and our results include the results of [29].

In the fifth section we start comparing Lawson homology and singular homology of smooth projective varieties with zero cycles supported on a subvariety. We essentially use the results of the previous two sections and a technique introduced by Bloch and Srinivas [4].

The main goal of this section is to study the semi-topological K-theory of our “degenerate” varieties. One of the main results of the section is the following theorem which computes semi-topological K-theory of “degenerate” threefolds.

**Theorem 1.1.** Let \( X \) be a smooth projective complex threefold such that there is a proper subvariety \( V \subset X \) with \( CH_0(X \setminus V) = 0 \). Then:

\[
K^\text{ss}t_i(X) \simeq ku^{-i}(X^{an}), i \geq 1,
\]

\[
K^\text{ss}t_0(X) \hookrightarrow ku^0(X^{an}).
\]
Moreover if $X$ is a rationally connected threefold then
\[ K_{i}^{sst}(X) \simeq ku^{-i}(X^{an}), i \geq 0. \]

This computation generalizes a result of [12] about the semi-topological K-theory of a rational threefold.

The following result describes the semi-topological K-theory of some “degenerate” fourfolds.

**Theorem 1.2.** Let $X$ be a smooth projective fourfold such that there is a proper subvariety $V \subset X$ of $\dim(V) \leq 2$ with $CH_0(X \setminus V) = 0$. Then:
\[ K_{i}^{sst}(X) \simeq ku^{-i}(X^{an}), i \geq 3, \]
\[ K_2^{sst}(X) \hookrightarrow ku^{-2}(X^{an}), \]
\[ K_{i}^{sst}(X)_\mathbb{Q} \simeq ku^{-i}(X^{an})_\mathbb{Q}, i = 1, 2, \]
\[ K_0^{sst}(X)_\mathbb{Q} \hookrightarrow ku^0(X^{an})_\mathbb{Q}. \]

We may contrast the above results with a result of H. Gillet [17] (see also C. Pedrini and C. Weibel [28]). He proved that the image of the map
\[ K_n(X) \to ku^{-n}(X) \]
is finite for any $n > 0$ and for any $X$ smooth complex projective variety.

In the sixth section of the paper we give some consequences of a theorem of Jannsen [21] and Laterveer [24] concerning a special decomposition of the diagonal for varieties with small Chow groups.

In the last section of the paper we study morphic cohomology of projective smooth linear varieties. The main idea is to use a K"unneth formula for such varieties proved by R. Joshua [22] and by B. Totaro [32]. The results in this section were proved in [12] using other tools.

This work started from a question of my advisor Eric Friedlander referring to the semi-topological K-theory of Fano varieties. I gratefully acknowledge his guidance and many valuable suggestions, in particular the suggestion that the Bloch-Kato conjecture may be helpful in the morphic cohomology context. I am also thankful to Mark Walker for carefully reading the manuscript and making useful remarks and to Jeremiah Heller for helpful discussions.

## 2. Notations and Recollection

Throughout this paper $X$ will denote a smooth projective irreducible variety over the complex numbers of dimension $d$ (unless otherwise stated). By $H^{p,q}_{et}(X)$, $L^pH^q(X)$ and $L_pH_q(X)$ we denote motivic cohomology, morphic cohomology and Lawson homology. For a field $E$ we denote $K^M_*(E)$ to be the Milnor K-theory of $E$. By $cyc^{p,q}$, respectively $cyc_{p,q}$ we denote the generalized cycle maps
\[ cyc^{p,q} : L^pH^q(X) \to H^q(X) \]
respectively
\[ cyc_{p,q} : L_pH_q(X) \to H_q(X). \]
We denote by $K^{q,n} := \text{Ker}(\text{cyc}^{q,n})$, $K_{p,q} := \text{Ker}(\text{cyc}_{p,q})$ and by $C^{q,n} := \text{Coker}(\text{cyc}^{q,n})$, $C_{q,n} = \text{Coker}(\text{cyc}_{q,n})$. For an abelian group $A$ we denote $mA := \{a \in A \mid ma = 0\}$.

If for a variety $X$ there is a proper subvariety $V \subset X$ such that $\text{CH}_0(X \setminus V) = 0$ we say as in [4] that $X$ is “degenerate” and also that its zero cycles “are supported on subvariety $V$”.

For a complex variety $X$ we denote $X^*$ a resolution of singularities for $X$.

We will start recalling the basics about the (co)niveau filtration of the singular (co)homology. Let

\[ N_k H_n(X) = \sum_{\dim(W) \leq k} \text{Im}(H_n(W) \to H_n(X)) \]

be a step in the niveau filtration of $H_n(X)$. This is an ascending filtration

\[ 0 \subset N_0 H_n(X) \subset ... \subset N_k H_n(X) \subset ... \subset H_n(X) \]

which has the property that

(1) \[ N_k H_n(X) = H_n(X) \]

for any $k \geq \min\{n, d\}$.

It is easy to see that $N_d H_n(X) = H_n(X)$ for any natural $n$. For $n < d$ the above equality follows from an induction argument using weak Lefshetz theorem. For $X$ smooth we know that the niveau filtration is isomorphic to the coniveau filtration of the cohomology of $X$, i.e

(2) \[ N_k H_n(X) \simeq N^{d-k} H^{2d-n}(X) \]

where we define

\[ N^k H^n(X) = \sum_{cd(W) \geq k} \text{Im}(H^n(W) \to H^n(X)). \]

From (1) and from (2) we conclude that

\[ N_{d-1} H_{2d-n}(X) \simeq N^1 H^n(X) \simeq H^n(X) \simeq H_{2d-n}(X) \]

for any $n$ such that $2d - n \leq d - 1 \iff n \geq d + 1$. We also know ([8] and [35]) the following property of the generalized cycle maps

**Proposition 2.1. ([8] and [35]) For a smooth projective variety $X$**

\[ \text{Im(cyc}^{q,n}) \subset N^{n-q} H^n(X) \]

**with equality when $n = 2q$ or $n = 2q - 1$.**

For a quasi-projective variety, Deligne [6] and Gillet-Soule [18] defined a weight filtration on the Borel-Moore homology of $U^{an}$ (denoted by $H^{BM}_*(U^{an})$). We recall the definition of this filtration. Choose a compactification $U \subseteq X$ so that $X$ is a projective complex variety and let $Y$ be the reduced complement of $U$ in $X$. Consider $\mathbb{Z} Sing_*$ () the functor taking a space $Z$ to the complex associated to the simplicial set $\text{Sing}_*Z$. We may construct two hypercovers ([18]) $X_* \to X$ and $Y_* \to Y$ such that $X_n$ and $Y_n$ are smooth projective varieties and such that there is a map $Y_* \to X_*$. 

4
which covers the embedding $Y \subset X$. Denoting $U_n = X_n \sqcup Y_{n-1}$ we may construct a bicomplex
\begin{equation}
\ldots \to \mathbb{Z} \text{Sing}_*(U_1) \to \mathbb{Z} \text{Sing}_*(U_0).\tag{3}
\end{equation}
The homology of the total complex of the bicomplex (3) gives the Borel-Moore homology \cite{18}. The weight filtration for $H_{BM}^*(U^\text{an})$ is the increasing filtration
\[\ldots \subseteq W_t H_{BM}^n(U^\text{an}) \subseteq W_{t+1} H_{BM}^n(U^\text{an}) \subseteq \ldots\]
where
\[W_t H_{BM}^n(U^\text{an}) := \text{image}(h_n(\mathbb{Z} \text{Sing}_*(U_{n+t}) \to \ldots \to \mathbb{Z} \text{Sing}_*(U_0)) \to H_{BM}^n(U^\text{an})).\]
It can be proven (\cite{18}) that
\[W_t H_{BM}^n(U^\text{an}) = 0\]
for any $t < -n$ and
\[W_t H_{BM}^n(U^\text{an}) = H_{BM}^n(U^\text{an})\]
for any $t \leq d - n$, where by $d$ we denote the dimension of the variety $U$.

The generalized cycle maps of a quasi-projective variety have the following property:

**Proposition 2.2.** (\cite{12}) For any quasi-projective complex variety $U$ the image of the canonical map
\[\text{cyc}_{t,n} : L_t H_n(U) \to H_{BM}^n(U^\text{an})\]
lies in the part of weight at most $-2t$ of Borel-Moore homology.

We will recall now the following conjecture due to Bloch and Kato.

**Theorem 2.1.** (Bloch-Kato conjecture) For any $n \geq 0$ and any field $E$ the norm residue homomorphism
\[K_n^M(E)/l \to H_{et}^n(E, \mu_m^\otimes q)\]
is an isomorphism.

This conjecture was proven by V. Voevodsky for any $m = 2^l$ and for any natural number $l > 0$ (this part is also called Milnor conjecture). The general case appears to be proven from the work of M. Rost and V. Voevodsky. A. Suslin and V. Voevodsky \cite{30} (see also Geisser-Levine \cite{16}) proved that the Bloch-Kato conjecture is equivalent to a conjecture due to Beilinson-Lichtenbaum.

**Theorem 2.2.** (Beilinson-Lichtenbaum conjecture) The map
\[H_{et}^n(X, \mathbb{Z}/m(q)) \to H_{et}^n(X, \mu_m^\otimes q)\]
is isomorphism for $n \leq q$ and injective for $n \leq q + 1$ for any $X$ smooth quasi-projective variety.

A. Suslin proposed the following characterization of morphic cohomology with integral coefficients (see \cite{12} and \cite{35}).

**Conjecture 2.1.** (Suslin’s conjecture) The map
\[L^q H^n(X, \mathbb{Z}) \to H^n(X, \mathbb{Z})\]
is isomorphism for $n \leq q$ and injective for $n \leq q + 1$ for any $X$ smooth quasi-projective variety.
We notice that the last conjecture contains a conjecture due to E. Friedlander and B. Mazur [8].

**Conjecture 2.2.** *(Friedlander-Mazur conjecture)* For any complex smooth quasi-projective variety \( X \)

\[
L^qH^n(X) = 0
\]

for any \( n < 0 \).

The Friedlander-Lawson duality theorem ([13], [10]) between morphic cohomology and Lawson homology will be used throughout the paper.

**Theorem 2.3.** *(Friedlander-Lawson [13], Friedlander [10])* For any \( X \) quasi-projective smooth complex variety of dimension \( d \)

\[
L^sH^n(X) \cong L_{d-s}H_{2d-n}(X)
\]

for any \( n \leq 2s \), \( n \in \mathbb{Z} \) and \( 0 \leq s \leq 2d \).

3. First results concerning generalized cycle maps

We start this section with some applications of the Bloch-Kato conjecture in the context of Lawson homology. The point b) in proposition 3.1 is known as Bloch’s theorem [3]. In proposition 3.2 we analyze the torsion of the Borel-Moore homology of a smooth quasi-projective variety.

**Proposition 3.1.** Let \( X \) a quasi-projective smooth variety. Then:

a) Let \( n \leq q + 1 \). Then \( K^{q,n} \) is divisible and \( C^{q,n} \) is torsion free.

b) Suppose \( X \) is projective. Then the torsion of \( H^n(X) \) is supported in codimension one for any \( n > 0 \).

c) \( L^qH^n(X) \) is uniquely divisible for \( n < 0 \) and \( L^qH^0(X) \) is torsion free (for any \( q \geq 0 \)).

**Proof.** We write the diagram of universal coefficient sequences for both cohomologies:

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & L^qH^n(X) \otimes \mathbb{Z}/m & \longrightarrow & L^qH^n(X, \mathbb{Z}/m) & \longrightarrow & mL^qH^{n+1}(X) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & H^n(X) \otimes \mathbb{Z}/m & \longrightarrow & H^n(X, \mathbb{Z}/m) & \longrightarrow & mH^{n+1}(X) & \longrightarrow & 0.
\end{array}
\]

We recalled in the second section that Bloch-Kato conjecture implies that the map

\[ H^q_{\text{et}}(X, \mathbb{Z}/m(q)) \to H^n_{\text{et}}(X, \mu_m^{\otimes q}) \]

is isomorphism for \( n \leq q \) and injective for \( n \leq q + 1 \) for any \( X \) smooth quasi-projective variety. The above map factors through the cycle map from morphic cohomology to the singular cohomology [14]. In [31] it is proved that

\[ H^q_{\text{et}}(X, \mathbb{Z}/m(q)) \cong L^qH^n(X, \mathbb{Z}/m) \]

for any allowed \( n, q \) and any complex projective variety \( X \).
In conclusion the middle vertical map from the above diagram is injective for \( n \leq q + 1 \) and isomorphism for \( n \leq q \). Using the snake lemma we conclude that we have the following exact sequence:

\[
0 \to L^q H^n(X) \otimes \mathbb{Z}/m \to H^n(X) \otimes \mathbb{Z}/m \to Ker(m L^q H^{n+1}(X) \to_m H^{n+1}(X)) \to 0
\]

for any \( n \leq q \) and that the map \( L^q H^n(X) \otimes \mathbb{Z}/m \to H^n(X) \otimes \mathbb{Z}/m \) is an injection for \( n \leq q + 1 \). Moreover we conclude that for \( n \leq q \) we have

\[
m L^q H^{n+1}(X) \to_m H^{n+1}(X)
\]
surjective. This means actually that

\[
m L^q H^n(X) \to_m H^n(X)
\]
is surjective for any \( n \leq q + 1 \) and any \( m > 1 \). It implies that

(4) \( \text{torsion}(Im(cyc^{q,n})) = \text{torsion}(H^n(X)) \)

for any \( n \leq q + 1 \). Because each image of the cycle map is included in a step of the coniveau filtration we have

\[
\text{torsion}(N^{n-q} H^n(X)) = \text{torsion}(H^n(X))
\]

for any \( n \leq q + 1 \). The only case when we conclude something non-trivial from the equality above is when \( n = q + 1 \). In this case for any \( 0 < n \leq d + 1 \) we have

\[
\text{torsion}(N^1 H^n(X)) = \text{torsion}(H^n(X))
\]

which implies our point b). Consider now the composition

\[
L^q H^n(X) \otimes \mathbb{Z}/m \to Im(cyc^{q,n}) \otimes \mathbb{Z}/m \to H^n(X) \otimes \mathbb{Z}/m.
\]

The first map is still surjective because \(- \otimes \mathbb{Z}/m\) is a right exact functor. For \( n \leq q + 1 \) the composition is injective. This implies that

\[
L^q H^n(X) \otimes \mathbb{Z}/m \simeq Im(cyc^{q,n}) \otimes \mathbb{Z}/m
\]

and that

\[
Im(cyc^{q,n}) \otimes \mathbb{Z}/m \hookrightarrow H^n(X) \otimes \mathbb{Z}/m
\]

for any \( n \leq q + 1 \) and \( m > 1 \). Consider now the following short exact sequence

\[
0 \to K^{q,n} \to L^q H^n(X) \to Im(cyc^{q,n}) \to 0.
\]

Tensoring with \( \mathbb{Z}/m \) we obtain the following exact sequence:

\[
0 \to_m K^{q,n} \to_m L^q H^n(X) \to_m Im(cyc^{q,n}) \to K^{q,n} \otimes \mathbb{Z}/m
\]

\[
\to_m L^q H^n(X) \otimes \mathbb{Z}/m \to_m Im(cyc^{q,n}) \otimes \mathbb{Z}/m \to 0.
\]

For \( n \leq q + 1 \) the map \( a_2 \) is an isomorphism and the map \( a_1 \) is a surjection. From exactness of the sequence we get

\[
K^{q,n} \otimes \mathbb{Z}/m = 0
\]

for any \( n \leq q + 1 \) and \( m > 1 \). This implies that \( K^{q,n} \) is divisible for \( n \leq q + 1 \).

Consider now the following exact sequence

\[
0 \to Im(cyc^{q,n}) \to H^n(X) \to C^{q,n} \to 0.
\]
Tensoring with $\mathbb{Z}/m$ we obtain the following long exact sequence:

$$
0 \rightarrow m \text{Im}(\text{cyc}^{q,n}) \xrightarrow{a_3} m H^n(X) \rightarrow m C^{q,n} \rightarrow m \text{Im}(\text{cyc}^{q,n}) \otimes \mathbb{Z}/m \rightarrow C^{q,n} \otimes \mathbb{Z}/m \rightarrow 0.
$$

For $n \leq q + 1$ the map $a_3$ is bijective and the map $a_4$ is injective. From the exactness of the sequence we get

$$
mC^{q,n} = 0
$$

for any $n \leq q + 1$ and any $m > 1$. This implies that $C^{q,n}$ is torsion free for any $n \leq q + 1$.

Suppose now that $n < 0$. Because $0 \leq q \leq d = \dim(X)$ we have $n < q$. We have the following short exact sequence

$$
0 \rightarrow L^q H^n(X) \otimes \mathbb{Z}/m \rightarrow L^q H^n(X, \mathbb{Z}/m) \rightarrow m L^q H^{n+1}(X) \rightarrow 0.
$$

Because $L^q H^n(X, \mathbb{Z}/m) = 0$ for any $n < 0$ and for any $m > 1$ we conclude that $L^q H^n(X) \otimes \mathbb{Z}/m = 0$ for any $n < 0$, $m > 1$ (i.e. $L^q H^n(X)$ is divisible for $n < 0$) and that $mL^q H^{n+1}(X) = 0$ (i.e. $L^q H^n(X)$ is torsion free for any $n \leq 0$).

**Corollary 3.1.** Let $n \leq q + 1$. Then:

a) If there is a natural nonzero number $M$ such that $MK^{q,n} = 0$ then cyc$^{q,n}$ is injective.

b) Suppose cyc$^{q,n} \otimes \mathbb{Q}$ is surjective. Then cyc$^{q,n}$ is surjective.

**Proof.**

a) From the above proposition 3.1 we get $K^{q,n}$ divisible. This implies that for any $x \in K^{q,n}$ there is an element $y \in K^{q,n}$ such that $x = My = 0$.

b) From cyc$^{q,n} \otimes \mathbb{Q}$ surjective it follows that $C^{q,n} \otimes \mathbb{Q} = 0$. From proposition 3.1 we know that $C^{q,n}$ is torsion free. This implies that $C^{q,n} = 0$.

Point b) in proposition 3.1 has the following formulation in the quasi-projective case:

**Proposition 3.2.** Let $X$ a smooth quasi-projective variety of dimension $d$ and let $n \geq d$. Fix $s = n - d + 1$. Then

$$
torsion(W_{-2s} H_n(X)) = torsion(H_{n}^{BM}(X))
$$

where $W_{-2s} H_n(X)$ is a step in the weight filtration of the Borel-Moore homology $H_{n}^{BM}(X)$ [12].

**Proof.** From [1] we have that the groups $\text{Im}(\text{cyc}^{q,q+1})$ and $H_{q+1}^{BM}(X)$ have the same torsion for any $q$ with $0 \leq q \leq d$. Using the Friedlander-Lawson duality theorem [10] we have that

$$
tors(\text{Im}(L_{n-d+1} H_n(X) \rightarrow H_n(X))) = tors(H_n(X))
$$

for any $n \geq d$.

We recalled in the second section that the cycle map from Lawson homology to the Borel-Moore homology of a smooth quasi-projective variety factors through steps in the weight filtration [12], i.e.

$$
L_s H_n(X) \rightarrow W_{-2s} H_n(X) \hookrightarrow H_n(X)
$$

for any $0 \leq s \leq d$ and $n \geq 2s$. This implies the statement of the theorem. □
The above discussion gives us the following reformulation of the Friedlander-Mazur conjecture.

**Proposition 3.3.** Let $X$ be a smooth quasi-projective variety. Then the Friedlander-Mazur conjecture is valid for $X$ if and only if $L^q H^n(X)_\mathbb{Q} \simeq 0$ for any $n < 0$.

**Proof.** The point c) in the proposition 3.1 shows that these groups are torsion free. \hfill \Box

**Remark 3.1.** (cohomological Brauer group)

Let $X$ a smooth projective variety of dimension $d$ with
\[ H^2_{\text{Zar}}(X, O_X) = H^1_{\text{Zar}}(X, O_X) = 0. \]
We know (see for example [5]) that, in these conditions, the cohomological Brauer group of $X$ has the following characterization
\[ Br(X) \simeq \text{tors}(H^3(X)). \]

Suslin’s conjecture predicts that the cycle map
\[ L^q H^n(X) \to H^n(X) \]
is an isomorphism for any $n \leq q$ and a monomorphism for $n = q + 1$. Let’s assume Suslin’s conjecture. We obtain that
\[ \text{tors}(L^3 H^3(X)) \simeq \text{tors}(L^4 H^3(X)) \simeq \ldots \simeq \text{tors}(H^3(X)). \]

But [4] shows that
\[ \text{tors}(\text{Im}(\text{cyc}^{2,3})) = \text{tors}(H^3(X)) \]
and because Suslin’s conjecture gives us that the cycle map $\text{cyc}^{2,3}$ is injective we obtain that
\[ \text{tors}(L^2 H^3(X)) \simeq \text{tors}(L^3 H^3(X)) \simeq \ldots \simeq \text{tors}(H^3(X)) \]
giving a characterization of the cohomological Brauer group of $X$ by means of morphic cohomology. We will show in sections five, six and seven that Suslin’s conjecture can be verified for certain projective varieties.

A natural question to ask is whether $\text{tors}(L^2 H^3)$ is a birational invariant in general as Suslin’s conjecture predicts. We will prove below that this is indeed the case. We will use a blow-up formula for Lawson homology proved by Hu [19] and the fact that birational maps between projective smooth varieties factor as a composition of blow-ups with centers of codimension greater than two [11].

**Proposition 3.4.** $\text{tors}(L^2 H^3)$ is a birational invariant.

That is for any $X, X'$ birational equivalent smooth projective varieties we have:
\[ \text{tors}(L^2 H^3(X)) \simeq \text{tors}(L^2 H^3(X')). \]

**Proof.** Let $X_Y \to X$ a blow up of a smooth center $Y$ of codimension greater or equal with 2. From [19] we know that
\[ L_{n-2} H_{2n-3}(X_Y) = \oplus_{1 \leq j \leq r-1} L_{n-2-j} H_{2n-3-2j}(Y) \oplus L_{n-2} H_{2n-3}(X) \]
where by $r$ we denote the codimension of $Y$ in $X$. 

9
It suffices to show that $\text{tors}(L_{n-2-j}H_{2n-3-2j}(Y)) = 0$ for any $1 \leq j \leq r - 1$. We notice that
\[ \dim Y - 1 \leq n - 2 - j \leq n - 3 \]
and
\[ 2\dim(Y) - 1 \leq 2n - 3 - 2j \leq 2n - 5. \]
If $n - 2 - j \geq \dim(Y)$ it is obvious that $\text{tors}(L_{n-2-j}H_{2n-3-2j}(Y)) = 0$.

We also have $\text{tors}(L_{\dim Y - 1}H_{2\dim Y - 1}(Y)) \simeq \text{tors}(H_{2\dim Y - 1}(Y))$.

From the universal coefficient sequence we obtain $\text{tors}(H_{2\dim Y - 1}(Y)) = \text{tors}(H_0(Y)) = 0$.

We can conclude now that $\text{tors}(L^2H^3(X)) \simeq \text{tors}(L^2H^3(X'))$ for any $X, X'$ birational equivalent smooth projective varieties. □

4. Cycle action on morphic cohomology

Let $\alpha$ be a dimension $d = \dim(X)$ irreducible algebraic cycle in $X \times X$ with the support contained in $V \times W$, where $V \subset X$ and $W \subset X$ are irreducible subvarieties. Denote $v = \dim(V)$ and $w = \dim(W)$. Consider the resolutions of singularities $i : V^* \rightarrow V$ and $j : W^* \rightarrow W$. Denote $\alpha^* \in CH_d(V^* \times W^*)$ an element such that $(i \times j)_* \alpha^* = \alpha$.

We remark that we can always find such a cycle up to a moving in the rational equivalence class of $\alpha$ [7]. The cycle $\alpha$ gives the following action
\[ \alpha_* : L^mH^l(X) \rightarrow L_{d-m}H_{2d-l}(X) \]
with $d = \dim(X)$ and
\[ \alpha_*(x) = pr_{2*}(pr_{1*}i^*(x) \cap \alpha^*) \]
The above map depends only on the algebraic equivalence class of $\alpha$. A similar action in the context of Lawson homology was also considered by C. Peters in [29] (see also [14]).

The above action commutes with the similarly defined action of the algebraic cycle $\alpha$ on the singular cohomology. This can be seen from the fact that the cycle maps from morphic cohomology (resp. Lawson homology) to singular cohomology.
(resp. singular homology) are natural [11] and commute with cap product with an algebraic cycle [11].

The above discussion is summarized in the following sequence of the commutative diagrams (the horizontal maps are given by the decomposition of the actions of the algebraic cycle $\alpha$ and the vertical maps are given by the cycle maps)

\[
\begin{align*}
L^{d-m}H^{2d-l}(X) & \xrightarrow{i^*} L^{d-m}H^{2d-l}(V^*) \xrightarrow{pr_{V^*}^*} L^{d-m}H^{2d-l}(V^* \times W^*) \xrightarrow{\cap \alpha^*} \\
H^{2d-l}(X) & \xrightarrow{i^*} H^{2d-l}(V^*) \xrightarrow{pr_{V^*}^*} H^{2d-l}(V^* \times W^*) \xrightarrow{\cap \alpha^*} \\
& \xrightarrow{\cap \alpha^*} L_mH_l(V^* \times W^*) \xrightarrow{(pr_{W^*})^*} L_mH_l(W^*) \xrightarrow{j_*} L_mH_l(X) \\
& \xrightarrow{\cap \alpha^*} H_l(V^* \times W^*) \xrightarrow{(pr_{W^*})^*} H_l(W^*) \xrightarrow{j_*} H_l(X)
\end{align*}
\]

(5)

for any $0 \leq m \leq d$ and any $l \geq 2m$.

The map $c_2$ is an isomorphism for any $m \leq d - v$, where $v = \text{dim}(V)$. To see this we divide it in several cases depending on the value of $2d - l$. If $2d - l > 2v \geq 0$ then

\[L^{d-m}H^{2d-l}(V^*) = H^{2d-l}(V^*) = 0\]

by [13]. If $0 \leq 2d - l \leq 2v$ then we can consider the following morphic cohomology group $L^vH^{2d-l}(V^*)$ which is isomorphic with $H^{2d-l}(V^*)$ from the Poincare duality and the Dold-Thom theorem (see [11]). At the same time the composition of s-maps

\[L^vH^{2d-l}(V^*) \to L^{d-m}H^{2d-l}(V^*)\]

is an isomorphism in this range and commutes with the cycle maps [13]. This implies that

\[L^{d-m}H^{2d-l}(V^*) \simeq H^{2d-l}(V^*).\]

Consider now $2d - l < 0$. Then by the Friedlander-Lawson duality theorem and the isomorphism of s-maps in this range we obtain

\[0 = L^vH^{2d-l}(V^*) = L^{d-m}H^{2d-l}(V^*).\]

The map $c_3$ is an isomorphism for any $m \geq w$. In the case $m = w$ we have

\[L_mH_{2m}(W^*) \simeq H_{2m}(W^*)\]

because $W^*$ is irreducible. For $m > w$ we obviously have $L_mH_l(W^*) = H_l(W^*) = 0$ since $l \geq 2m > 2w$.

The above discussion proves the following proposition:

**Proposition 4.1.** Let $\alpha$ be an irreducible algebraic cycle in $CH^d(X \times X)$ with the support contained in $V \times W$, where $V \subset X$ and $W \subset X$ are irreducible subvarieties of dimension $v$, respectively $w$. The action of the cycle $\alpha$ on the kernel and the cokernel of the map $c_1$ is zero for $m \geq w = \text{dim}(W)$ or for $m \leq d - v = \text{codim}(V)$.
Proof. From the above discussion we conclude that if $m \geq w$ then $c_3$ is an isomorphism and that if $m \leq d - v$ then $c_2$ is an isomorphism. These imply the conclusion of our proposition. □

Corollary 4.1. Suppose $\alpha$ as in proposition 4.1 and suppose $\dim(X) = \dim(V) + \dim(W)$. Then the action of the cycle $\alpha$ on the kernel and on the cokernel of the map $c_1$ is zero for any $0 \leq m \leq d$.

Proof. Direct consequence of proposition 4.1. □

Remark 4.1. We remark that to study the action of a cycle $\alpha = \sum n_i \alpha_i \in CH^d(X \times X)$ with $\text{supp}(\alpha) \subset V \times W$ it is enough to study the action of each irreducible cycle $\alpha_i$. It is obvious that $\text{supp}(\alpha_i) \subset \text{supp}(\alpha) \subset V \times W$

and that because $\text{supp}(\alpha_i)$ is irreducible there are $V_i \subset V, W_i \subset W$ irreducible components such that $\text{supp}(\alpha_i) \subset V_i \times W_i$

By using the Friedlander-Lawson duality theorem [13] we will identify the cycle map $c_1$ with the cycle map $L_m H_\ast(X) \to H_\ast(X)$.

This will identify the action of the diagonal cycle with the identity map.

Convention 4.1. From now on by “the action of $\alpha$ is zero for $m$ in some certain range” we will understand that the action of the cycle $\alpha$ on the kernel and cokernel of the cycle map $L_m H_\ast \to H_\ast$ is zero for $m$ in the respective range.

5. Comparing Lawson homology with singular homology

In this section we study the cycle maps $\text{cyc}_q^n : L^q H^n(X) \to H^n(X)$ for $X$ smooth projective complex variety with the property that its zero cycles are supported on a proper subvariety. We prove that these cycle maps behave nicely for threefolds and fourfolds with this property (being most of the time injective or bijective). We expect that there are cycle maps $\text{cyc}_q^n$ totally non-trivial for varieties $X$ of large dimension with zero cycles supported on a subvariety. As a support for our expectation is a theorem of A. Albano, and A. Collino [2] proving that for a generic smooth cubic hypersurface $X \subset \mathbb{P}^8$ the Griffiths group $Griff^4(X) \otimes \mathbb{Q}$ is infinitely generated.

We start the section by recalling a result of E. Friedlander. He proved [9] that for any smooth connected complex projective variety $X$ of dimension $d$ we have

$L_{d-1} H_{2d-2}(X) \to H_{2d-2}(X),$

$L_{d-1} H_{2d-1}(X) \simeq H_{2d-1}(X),$

$L_{d-1} H_{2d}(X) \simeq H_{2d}(X) \simeq \mathbb{Z}$

and that $L_{d-1} H_k(X) = 0$ for any $k > 2d$. 

12
We recall that Bloch and Srinivas [4] proved that if a smooth projective variety \( X \) has its zero cycles supported on a non-necessary irreducible subvariety \( V \), i.e. \( CH_0(X \setminus V) = 0 \), then the diagonal cycle decomposes as
\[
N\Delta = \alpha + \beta
\]
for some natural nonzero number \( N \) and some cycles \( \alpha, \beta \in CH^d(X \times X) \) with the support of \( \alpha \) included in \( V \times X \) and the support of \( \beta \) included in \( X \times D \), where \( D \) is a divisor of \( X \). We will also use the transpose of this decomposition, i.e
\[
N\Delta = \alpha^t + \beta^t
\]
where \( \alpha^t, \beta^t \in CH^d(X \times X) \) are supported on \( X \times V \), respectively \( D \times X \).

The following theorem computes \( K_{sst} \) for “degenerate” threefolds.

**Theorem 5.1.** Let \( X \) be a smooth projective complex threefold such that there is a proper subvariety \( V \subset X \) with \( CH_0(X \setminus V) = 0 \). Then:
\[
K_{sst}^i(X) \simeq ku^{-i}(X^{an}), i \geq 1,
\]
\[
K_{sst}^0(X) \hookrightarrow ku^0(X^{an}).
\]
Moreover if \( X \) is a rationally connected threefold then
\[
K_{sst}^i(X) \simeq ku^{-i}(X^{an})
\]
for any \( i \geq 0 \).

Some examples which fulfill the conditions above are: rationally connected threefolds (e.g. smooth Fano threefolds [23]), Kummer threefolds [4], certain quotient varieties such
\[
(X \times E)/(\mathbb{Z}/2)
\]
with \( X \) a K3 covering of an Enriques surface and \( E \) an elliptic curve [4].

The above result on rationally connected threefolds generalizes the same result on rational threefolds proved in [12] with other tools.

**Proof.** The proof of the above theorem is based on the spectral sequence relating morphic cohomology and semi-topological K-theory [12] and on the following two propositions which compute the Lawson homology groups of a threefold \( X \) with zero cycles supported on a subvariety.

**Proposition 5.1.** Let \( X \) be a smooth projective complex threefold such that there is a proper subvariety \( V \subset X \) with \( CH_0(X \setminus V) = 0 \) and \( \dim(V) \leq 1 \). Then:

a) \( L_1H_2(X) \hookrightarrow H_2(X) \) is injective and a rational isomorphism.
b) \( L_1H_3(X) \simeq H_3(X) \).
c) \( L_2H_4(X) \simeq L_1H_4(X) \simeq H_4(X) \).
d) \( L_2H_5(X) \simeq L_1H_5(X) \simeq H_5(X) \).
e) \( L_3H_6(X) \simeq L_2H_6(X) \simeq L_1H_6(X) \simeq H_6(X) \).
f) \( L_kH_\ast(X) = 0 \) for any \( n \geq 7 \) and any \( k \geq 0 \).

In particular any such threefold fulfills Suslin’s conjecture.
Moreover if \( X \) is rationally connected threefold then
\[
L_\ast H_\ast(X) = H_\ast(X)
\]
for all possible indices.

Proof. We proof the case \( \text{dim}(V) = 1 \), the other case being similar.

Consider the above decomposition
\[
N \Delta = \alpha^t + \beta^t
\]
with \( \alpha^t \) supported on \( X \times V \) and \( \beta^t \) supported on \( D \times X \), with \( D \) a divisor in \( X \). Remark 4.1 shows that it is enough to consider the case when \( V \) and \( D \) are irreducible. We recall that we denote by \( D^* \), respectively \( V^* \), the resolution of singularities of \( D \), respectively \( V \).

Proposition 4.1 gives us that the action of \( \beta^t \) is zero on \( \text{Ker}(L^m H_\ast(X) \rightarrow H_\ast(X)) \) for \( m \leq \text{codim}(D) = 1 \) and the action of \( \alpha^t \) on the same kernel is zero for \( m \geq v = \text{dim}(V) = 1 \). This implies that for \( m = 1 \) we have
\[
N(\text{Ker}(L_1 H_k(X) \rightarrow H_k(X))) = 0
\]
for \( 2 \leq k \leq 6 \) and that
\[
NL_1 H_k(X) = 0
\]
for \( k \geq 7 \). Proposition 3.1 implies that \( L_1 H_k(X) = 0 \) for any \( k \geq 7 \) and Corollary 3.1 implies that the cycle map \( L_1 H_k(X) \rightarrow H_k(X) \) is injective for any \( 3 \leq k \leq 6 \).

Let \( x \in H_k(X) \simeq H^{6-k}(X) \) with \( 3 \leq k \leq 6 \). Then
\[
\beta^t_s x \in \text{Im}(L_1 H_k(X) \rightarrow H_k(X))
\]
because \( L^2 H^{6-k}(D^*) \simeq H^{6-k}(D^*) \) (see diagram (5)).

We have \( \alpha^t_s x = 0 \) because the action of \( \alpha^t \) on \( H_k(X) \) factors through \( H_k(V^*) \), \( \text{dim}(V) = 1 \) and \( k \geq 3 \). This implies that
\[
Nx = \beta^t_s x \in \text{Im}(L_1 H_k(X) \rightarrow H_k(X))
\]
for any \( 3 \leq k \leq 6 \). Because \( x \in H_k(X) \) is arbitrary we conclude that the rational cycle map
\[
L_1 H_k(X) \otimes \mathbb{Q} \rightarrow H_k(X) \otimes \mathbb{Q}
\]
is surjective for any \( 3 \leq k \leq 6 \). Corollary 3.1 shows that the cycle map \( L_1 H_k(X) \rightarrow H_k(X) \) is surjective for any \( 3 \leq k \leq 6 \).

For \( k = 2 \) we use a result of Bloch-Srinivas. They prove that for varieties as in our hypothesis algebraic equivalence and homological equivalence coincides for codimension 2 cycles [4]. This means that the cycle map
\[
L_1 H_2(X) \rightarrow H_2(X)
\]
is injective.

Consider now the decomposition
\[
N \Delta = \alpha + \beta
\]
with \( \alpha \) supported on \( V \times X \) and \( \beta \) supported on \( X \times D \). Proposition 4.1 gives that the action of \( \alpha \) is zero on \( \text{Ker}(L^m H_s(X) \rightarrow H_s(X)) \) for \( m \leq d - v = \text{codim}(V) = 2 \) and that the action of \( \beta \) is zero on \( \text{Ker}(L^m H_s(X) \rightarrow H_s(X)) \) for \( m \geq \text{dim}(D) = 2 \). This implies that for \( m = 2 \) we have
\[
N(\text{Ker}(L_2 H_s(X) \rightarrow H_s(X))) = 0
\]
for $4 \leq k \leq 6$ and that
$$NL_2H_k(X) = 0$$
for $k \geq 7$. Proposition 3.1 implies that $L_2H_k(X) = 0$ for any $k \geq 7$ and Corollary 3.1 implies that the cycle map $L_2H_k(X) \to H_k(X)$ is injective for any $4 \leq k \leq 6$.

Let $x \in H_k(X)$ with $4 \leq k \leq 6$. Then
$$\alpha_*(x) \in \text{Im}(L_2H_k(X) \to H_k(X))$$
because $H^{6-k}(V^*) \simeq H^{6-k}(V^*)$ (see diagram (5)). The action of $\beta$ on $H_k(X)$ is zero for $5 \leq k \leq 6$ because this action factors through $H_k(D^*)$ and $\dim(D^*) = 2$. If $k = 4$ then
$$\beta_*(x) \in \text{Im}(L_2H_4(X) \to H_4(X))$$
because $L_2H_4(D^*) \simeq H_4(D^*)$ (see diagram (5)). Because
$$N(x) = \alpha_*(x) + \beta_*(x)$$
we conclude that for $4 \leq k \leq 6$ the cycle maps
$$L_2H_k(X) \otimes \mathbb{Q} \to H_k(X) \otimes \mathbb{Q}$$
are surjective. Applying Corollary 3.1 we conclude that these surjections are with integer coefficients.

Let $X$ be a rationally connected threefold. Then $H^4(X, \mathbb{C}) = H^{2,2}(X)$ because $h^{1,3} = h^{3,1} = h^{2,0} = 0$. This implies that
$$H_2(X) \simeq H^4(X) \simeq H^{2,2}(X, \mathbb{Z})$$
where we denoted $H^{2,2}(X, \mathbb{Z}) := \{ \eta \in H^4(X) \text{ such that } \text{coef}_*(\eta) \in H^{2,2}(X) \text{ with } \text{coef}_*: H^4(X, \mathbb{C}) \to H^4(X, \mathbb{C}) \text{ being the coefficient map} \}.$$

C. Voisin proved the following theorem:

**Theorem 5.2.** (Voisin [33])

The Hodge conjecture with integral coefficients is valid for any smooth uniruled threefold.

This theorem implies that the cycle map $L_1H_2(X) \to H_2(X)$ is surjective for $X$ any smooth rationally connected threefold. □

**Proposition 5.2.** Let $X$ be a smooth projective complex threefold such that there is a proper subvariety $V \subset X$ with $\text{CH}_0(X \setminus V) = 0$ and $\dim(V) = 2$. Then:

a) $L_1H_2(X) \subset H_2(X)$.
b) $L_1H_3(X) \simeq H_3(X)$.
c) $L_2H_4(X) \subset L_1H_4(X) \simeq H_4(X)$.
d) $L_2H_5(X) \simeq L_1H_5(X) \simeq H_5(X)$.
e) $L_3H_6(X) \simeq L_2H_6(X) \simeq L_1H_6(X) \simeq H_6(X)$.
f) $L_kH_n(X) = 0$ for any $n \geq 7$ and any $k \geq 0$.

In particular any such threefold fulfills Suslin’s conjecture.

**Proof.** Consider the decomposition
$$N\Delta = \alpha + \beta$$
with $\alpha$ and $\beta$ being supported on $V \times X$, respectively $X \times D$. It is enough to consider the case when $V$ and $D$ are irreducible (see remark 4.1). The action of $\alpha$ is zero for $m \leq d - v = \text{codim}(V) = 1$ and the action of $\beta$ is zero for $m \geq \text{dim}(D) = 2$ (see Convention 4.1).

Suppose now $m = 1$. Because $D^*$ is a surface we have for $l \geq 3$

$$L_1H_l(D^*) \simeq H_l(D^*).$$

This implies that the action of $\beta$ is zero for $m = 1$ and $l \geq 3$ (see Diagram 5). Because we already know that the action of $\alpha$ is zero for $m = 1$ it implies that

$$L_1H_l(X) \simeq H_l(X)$$

for any $l \geq 3$. Applying corollary 3.1 we obtain

$$L_1H_l(X) \simeq H_l(X)$$

for any $l \geq 3$.

Suppose now $m = 2$. Because $V^*$ is a surface we get

$$L^1H^{6-l}(V^*) \simeq H^{6-l}(V^*)$$

for any $l \geq 5$ which means that for these indexes the action of $\alpha$ is zero. As we know that for $m = 2$ the action of $\beta$ is zero we can conclude as above that

$$L_2H_l(X) \simeq H_l(X)$$

for any $l \geq 5$.

The injectivity with integer coefficients in a) and c) follows from the theorem of Bloch-Srinivas \[4\] used in the proposition 5.1 and from the fact that for divisors algebraic equivalence coincides with homological equivalence.

Using now the spectral sequence argument from [12], theorem 6.1 and our propositions 5.1 and 5.2 we can conclude our theorem 5.1.

Remark 5.1. Bloch-Srinivas \[4\] proved that a Kummer threefold has its zero cycles supported on a subvariety of dimension two. These Kummer threefolds show that the injectivity in the point a) of proposition 5.2 is the best we can get because a Kummer threefold has $h^{2,0} \neq 0$.

Remark 5.2. For $X$ and $V$ as in the theorem 5.1 and $\text{dim}(V) \leq 1$ the injection

$$K_0^{\text{sst}}(X) \hookrightarrow ku^0(X^{\text{an}})$$

is moreover a rational isomorphism. For $\text{dim}(V) = 2$ this rational isomorphism is not longer true as we can see from the example given in remark 5.1.

Convention 5.1. Remark 4.1 shows that it is enough to study the action of a irreducible cycle. In the rest of the paper, without reducing the generality, we will understand that a decomposition of the form

$$N \Delta = \alpha + \beta$$

with $\alpha$ supported on $V \times X$ and $\beta$ supported on $X \times D$ has $V$ and $D$ irreducible varieties.
The next theorem computes $K^{sst}$ for certain “degenerate” fourfolds.

**Theorem 5.3.** Let $X$ be a smooth projective fourfold such that there is a proper subvariety $V \subset X$ of $\dim(V) \leq 2$ with $CH_0(X \setminus V) = 0$. Then:

- $K_i^{sst}(X) \simeq ku^{-i}(X^{an})$, $i \geq 3$,
- $K_2^{sst}(X) \hookrightarrow ku^{-2}(X^{an})$,
- $K_i^{sst}(X)_Q \simeq ku^{-i}(X^{an})_Q$, $i = 1, 2$,
- $K_0^{sst}(X)_Q \hookrightarrow ku^0(X^{an})_Q$.

Some examples of varieties which fulfill the conditions of the theorem are: rationally connected fourfolds, certain quotient varieties as in [4] etc.

**Proof.** The proof is similar to the proof of the theorem about $K^{sst}$ for degenerate threefolds. It is a corollary of the spectral sequence relating morphic cohomology groups and $K^{sst}$ and of the computation of some Lawson groups.

**Proposition 5.3.** Let $X$ be a smooth projective fourfold such that there is a proper subvariety $V \subset X$ of $\dim(V) \leq 1$ with $CH_0(X \setminus V) = 0$. Then:

- $L_1 H_2(X)_Q \simeq H_2(X)_Q$.
- $L_1 H_3(X)_Q \simeq H_3(X)_Q$.
- $L_2 H_4(X) \hookrightarrow L_1 H_4(X) \simeq H_4(X)$.
- $L_2 H_5(X) \simeq L_1 H_5(X) \simeq H_5(X)$.
- $L_3 H_6(X) \simeq L_2 H_6(X) \simeq L_1 H_6(X) \simeq H_6(X)$.
- $L_3 H_7(X) \simeq L_2 H_7(X) \simeq L_1 H_7(X) \simeq H_7(X)$.
- $L_4 H_8(X) \simeq L_3 H_8(X) \simeq L_2 H_8(X) \simeq L_1 H_8(X) \simeq H_8(X)$.
- $L_k H_n(X) = 0$ for any $n \geq 9$ and any $k \geq 0$.

In particular any such fourfold fulfills Suslin’s conjecture.

**Proof.** Consider the decomposition

$$N\Delta = \alpha + \beta$$

with $\alpha$ supported on $V \times X$ and $\beta$ supported on $X \times D$. The action of $\alpha$ is zero for $m \leq 3$ and the action of $\beta$ is zero for $m \geq 3$ (see Convention 4.1). This implies that

$$L_3 H_3(X) \otimes Q \simeq H_3(X) \otimes Q$$

and because of Corollary 3.1 we obtain

$$L_3 H_3(X) \simeq H_3(X).$$

Because $D^*$ is a smooth threefold we have that the cycle map

$$L_2 H_l(D^*) \to H_l(D^*)$$

is an isomorphism for $l \geq 5$ and a monomorphism for $l = 4$. This implies that the action of $\beta$ on the kernel and cokernel of the cycle map

$$L_2 H_l(X) \to H_l(X)$$

is zero for $l \geq 5$. Because we already know that the action of $\alpha$ is zero for $m = 2$ we conclude using Corollary 3.1 that

$$L_2 H_l(X) \simeq H_l(X)$$
for any \( l \geq 5 \). The injection from the point c) comes from the fact that for such varieties algebraic equivalence and homological equivalence coincide on codimension 2 cycles \[4\].

Consider now the decomposition

\[ N \Delta = \alpha^t + \beta^t \]

with \( \alpha^t \) supported on \( X \times V \) and \( \beta \) supported on \( D \times X \). The action of \( \alpha^t \) is zero for \( m \geq 1 \) and the action of \( \beta^t \) is zero for \( m \leq 1 \) (see Convention \[4.1\]). This implies that

\[ L_1 H^*_s(X) \otimes \mathbb{Q} \simeq H^*_s(X) \otimes \mathbb{Q} \]

and from Corollary \[3.1\] we obtain

\[ L_1 H_l(X) \simeq H_l(X) \]

for any \( l \geq 4 \).

**Proposition 5.4.** Let \( X \) be a smooth projective fourfold such that there is a proper subvariety \( V \subset X \) of \( \dim(V) = 2 \) with \( CH_0(X \setminus V) = 0 \). Then:

\begin{enumerate}
  \item \( L_1 H_2(X)_0 \hookrightarrow H_2(X)_0 \).
  \item \( L_1 H_3(X)_0 \iso H_2(X)_0 \).
  \item \( L_2 H_4(X) \hookrightarrow L_1 H_4(X) \simeq H_4(X) \).
  \item \( L_2 H_5(X) \simeq L_1 H_5(X) \simeq H_5(X) \).
  \item \( L_3 H_6(X) \iso L_2 H_6(X) \iso L_1 H_6(X) \simeq H_6(X) \).
  \item \( L_3 H_7(X) \iso L_2 H_7(X) \iso L_1 H_7(X) \simeq H_7(X) \).
  \item \( L_4 H_8(X) \iso L_3 H_8(X) \iso L_2 H_8(X) \iso L_1 H_8(X) \simeq H_8(X) \).
  \item \( L_k H_n(X) = 0 \) for any \( n \geq 9 \) and any \( k \geq 0 \).
\end{enumerate}

In particular any such fourfold fulfills Suslin’s conjecture.

**Proof.** Consider the decomposition

\[ N \Delta = \alpha + \beta \]

with \( \alpha \) supported on \( V \times X \) and \( \beta \) supported on \( X \times D \). The action of \( \alpha \) is zero for \( m \leq 2 \) and the action of \( \beta \) is zero for \( m \geq 3 \) (see Convention \[4.1\]).

Because \( D^* \) is a smooth threefold we have that the cycle map

\[ L_2 H_l(D^*) \to H_l(D^*) \]

is an isomorphism for \( l \geq 5 \) and a monomorphism for \( l = 4 \). This implies that the action of \( \beta \) on the kernel and cokernel of the cycle map

\[ L_2 H_l(X) \to H_l(X) \]

is zero for \( l \geq 5 \). Because we already know that the action of \( \alpha \) is zero for \( m = 2 \) we conclude using corollary \[3.1\] that

\[ L_2 H_l(X) \simeq H_l(X) \]

for any \( l \geq 5 \). The injection from the point c) comes from the fact that for such varieties algebraic equivalence and homological equivalence coincide on codimension 2 cycles \[4\].
Consider the action of $\alpha$ on the kernel and the cokernel of the cycle maps

$$L_3H_l(X) \rightarrow H_l(X).$$

This action factors through $L^1H^{8-l}(V^*) \simeq L_1H_{l-4}(V^*)$ (see diagram 3). Because $V^*$ is a surface we have that the cycle map

$$L_1H_{l-4}(V^*) \simeq H_{l-4}(V^*)$$

for any $l \geq 7$ and injective for $l = 6$. Because the action of $\beta$ is zero for $m = 3$ we have that

$$L_3H_l(X) \otimes \mathbb{Q} \simeq H_l(X) \otimes \mathbb{Q}$$

for any $l \geq 7$. From Corollary 3.1 we conclude that

$$L_3H_l(X) \simeq H_l(X)$$

for any $l \geq 7$. The injectivity in point e) comes from the fact that on divisors algebraic equivalence coincides with homological equivalence.

Consider now the decomposition

$$N\Delta = \alpha^t + \beta^t$$

with $\alpha^t$ and $\beta^t$ being supported on $X \times V$, respectively $D \times X$. The action of $\alpha^t$ is zero for $m \geq 2$ and the action of $\beta^t$ is zero for $m \leq 1$. Because $V^*$ is a surface, we have that

$$L_1H_l(V^*) \rightarrow H_l(V^*)$$

is an isomorphism for any $l \geq 3$ and a monomorphism for $l = 2$. This implies that

$$L_1H_l(X) \otimes \mathbb{Q} \rightarrow H_l(X) \otimes \mathbb{Q}$$

is an isomorphism for any $l \geq 3$ and a monomorphism for $l = 2$. Using Corollary 3.1 we can conclude that

$$L_1H_l(X) \simeq H_l(X)$$

for any $l \geq 4$.

Using now the spectral sequence argument from [12], theorem 6.1 and our Propositions 5.3 and 5.4 we can conclude our Theorem 5.3.

Remark 5.3. As smooth cubic fourfolds show, the injectivity

$$K_0^{sst}(X)_\mathbb{Q} \hookrightarrow ku^0(X^{an})_\mathbb{Q}$$

is the best we can obtain.

The following proposition was previously known in the case of generic cubic hypersurfaces [20] which are known to be rationally connected.

**Proposition 5.5.** Let $X$ be a projective smooth variety of dimension $d \geq 3$ and suppose that there is a subvariety $V \subset X$ of $\dim(V) \leq 2$ with $CH_0(X \setminus V) = 0$. Then

$$N^1H^d(X) = H^d(X).$$
Proof. Without restricting the generality we may suppose \( \dim(V) = 2 \). We consider the following decomposition of the diagonal

\[ N\Delta = \alpha^t + \beta^t \]

with \( \alpha^t \) supported on \( X \times V \) and \( \beta^t \) supported on \( D \times X \). As before we obtain that the action of \( \alpha^t \) is zero for \( m \geq 2 \) and the action of \( \beta^t \) is zero for \( m \leq 1 \) (see Convention 4.1). Denote \( V^* \) a desingularization of \( V \). Because \( V^* \) is a surface we have

\[ L_1H_l(V^*) \otimes \mathbb{Q} \cong H_l(V^*) \otimes \mathbb{Q} \]

for any \( l \geq 3 \). This implies that

\[ L_1H_l(X) \mathbb{Q} \cong H_l(X) \mathbb{Q} \]

for any \( l \geq 3 \). From Corollary 3.1 we get

\[ L_1H_d(X) \simeq H_d(X) \]

and because the image of this cycle map is included in \( N_{d-1}H_d(X) = N^1H^d(X) \) we obtain our conclusion. \( \square \)

6. About varieties with small Chow group

It is proved by Jannsen [21] and Laterveer [24] that if the following cycle maps are injective

\[ CH_k(X) \otimes \mathbb{Q} \hookrightarrow H_{2k}(X) \otimes \mathbb{Q} \]

for any \( 0 \leq k \leq r \) then we have the following decomposition of the diagonal

(6) \[ N\Delta = \alpha_0 + \alpha_1 + \alpha_2 + \ldots + \alpha_r + \beta \]

where \( \alpha_i \) are supported on \( V_i \times W_{d-i} \) and \( \beta \) is supported on \( X \times \Gamma^{r+1} \) and \( N \) is a nonzero natural number (the lower indices represent the dimension of the subvariety and the upper indices represent the codimension of the subvariety). We denote by \( d \) the dimension of the variety \( X \).

We say that \( X \) has small Chow group of rank \( r \) if and only if the first \( r \) cycle maps are injective ([7], [24]). The following theorem and corollary are extensions of the main results of C. Peters [29].

**Theorem 6.1.** Let \( X \) be a smooth projective variety with small Chow group of rank \( r \). Then there is a natural nonzero number \( N \) such that

a) \( NK_{s,s} = 0 \) for any \( s \in \{0, 1, \ldots, r+1\} \) \( (29) \).

b) \( NK^{s,s} = 0 \) for any \( s \in \{0, 1, \ldots, r+2\} \).

**Proof.** Because \( X \) has small Chow group of rank \( r \), the diagonal cycle decomposes as in (6). Because the cycles \( \alpha_i \) are supported on \( V_i \times W_{d-i} \) with \( \dim(V_i) + \dim(W_{d-i}) = \dim(X) \) we know from Corollary 4.1 that the action of \( \alpha_i \) is zero for any \( m \) (see Convention 4.1). This implies that

\[ N\Delta_s = \beta_s \]

on the kernel and the cokernel of the cycle map

\[ L_mH_l(X) \rightarrow H_l(X). \]
But the action of $\beta$ is zero for $m \geq n - r - 1$ because it factors through $L_m H_l(\Gamma^{r+1})$ (see Diagram 5). We know [9] that the cycle map

$$L_{n-r-2} H_s(\Gamma^{r+1}) \to H_s(\Gamma^{r+1})$$

is injective (where $\Gamma^{r+1} \to \Gamma^{r+1}$ is a resolution of singularities). This implies that the action of $\beta$ on the kernel of the cycle map

$$L_m H_l(X) \to H_l(X)$$

is zero for any $m \geq n - r - 2$. This means that

$$N(Ker(L^m H^*(X) \to H^*(X))) = 0$$

for $0 \leq m \leq r + 2$

Point a) was proved in [29]. \hfill \square

**Corollary 6.1.** Let $X$ be a smooth projective variety such that rational equivalence coincides with homological equivalence in $CH_s(X) \otimes \mathbb{Q}$ in degrees less or equal than $r$. Then the algebraic equivalence coincides with homological equivalence in $CH_s(X) \otimes \mathbb{Q}$ in degrees less or equal than $r+1$ ([29]) and in degrees greater or equal than $n-r-2$.

**Remark 6.1.** It is conjectured [27] that for $X$ a smooth complete intersection in $\mathbb{P}^n$ of multi-degree $d_1 \geq d_2 \geq \ldots \geq d_s$ we have

$$CH_l(X) \otimes \mathbb{Q} \simeq \mathbb{Q}$$

for any $l \leq k - 1$ where $k = \lfloor \frac{n - \sum_{i=2}^s d_i}{d_1} \rfloor$, the integer part of the rational number.

In particular this would imply that $X$ has small Chow group of dimension $k - 1$.

Supposing this conjecture and using Theorem 6.1 we conclude that in our case we have

$$Griff^r(X) \otimes \mathbb{Q} = 0$$

for any $r \geq n - k$ and $r \leq k + 1$ and moreover

$$K_{r,s}^r \otimes \mathbb{Q} = 0$$

for the same range of indexes.

J. Lewis proved the statements for the Griffiths groups of a generic hypersurface in [26] (without using the above mentioned conjecture).

It is known [27] that the conjecture from Remark 6.1 is valid for generic cubic fivefold and sixfold. The next proposition studies the Lawson homology groups of such cubics.

**Proposition 6.1.** Let $X$ be a smooth generic cubic of dimension $d = 5$ or 6. Then Suslin’s conjecture is valid for $X$.

**Proof.** In [27] it is proved that a generic smooth cubic of dimension $d \geq 5$ has

$$CH_0(X) \simeq CH_1(X) \simeq \mathbb{Z}.$$ 

This implies that there is a decomposition

$$N\Delta = \alpha_0 + \alpha_1 + \beta$$
with $\alpha_i$ supported on $V_i \times W_{d-i}$ and $\beta$ supported on $X \times \Gamma_{d-2}$, cycles of codimension $d$ in $X \times X$. As in Proposition 6.1 we get the equality $N\Delta_* = \beta_*$ on the kernel and the cokernel of the cycle map $L_m H_*(X) \to H_*(X)$ for any $m$.

Suppose $d = 5$. Because the action of $\beta$ is zero for $m \geq 3$ and the action of $\beta'$ is zero for $m \leq 5 - 3 = 2$ we obtain that the Lawson homology of a generic smooth cubic fivefold is isomorphic with singular homology up to torsion, i.e

$$L_* H_*(X) \otimes \mathbb{Q} \simeq H_*(X) \otimes \mathbb{Q}.$$ 

Using Corollary 3.1 we obtain Suslin’s conjecture for generic smooth cubic fivefold.

Suppose now $d = 6$. Then the action of $\beta$ is zero for $m \geq 4 = \text{dim}(\Gamma_{d-2})$ and the action of $\beta'$ is zero for $m \leq 6 - 4 = 2 = \text{codim}(\Gamma_{d-2})$. We remark that the action of $\alpha_i$ is still zero for any $m \geq 1$. As above, we conclude that for any generic smooth cubic sixfold

$$L_m H_*(X) \otimes \mathbb{Q} \simeq H_*(X) \otimes \mathbb{Q}$$

for any $m \geq 4$ and any $m \leq 2$. The action of $\beta$ on $L_3 H_1(X)$ factors through $L_3 H_1(\Gamma_4) \to H_1(\Gamma_4)$ which is an isomorphism for any $l \geq 7$ and a monomorphism for $l = 6$. It implies that the cycle map $L_3 H_0(X) \otimes \mathbb{Q} \to H_0(X) \otimes \mathbb{Q}$ is injective and that $L_3 H_1(X) \otimes \mathbb{Q} \simeq H_1(X) \otimes \mathbb{Q}$ for any $l \geq 7$. Using now Corollary 3.1 we conclude that Suslin’s conjecture holds for any generic smooth cubic sixfold. □

**Theorem 6.2.** Let $X$ be a smooth projective variety. If $CH^*_{\mathbb{Q}}(X) \simeq H^*_{\mathbb{Q}}(X)$ then $L_* H_*(X)_{\mathbb{Q}} \simeq H_*(X)_{\mathbb{Q}}$.

In particular Suslin’s conjecture is valid for such $X$.

**Proof.** Our condition on $X$ gives us the following decomposition of the diagonal

$$N\Delta = \alpha_0 + \alpha_1 + ... + \alpha_d$$

where each $\alpha_i$ is supported on $V_i \times W_{d-i}$. We remark that $\text{dim}(V) + \text{dim}(W_{d-i}) = \text{dim}X = d$. Using Corollary 4.1 we can conclude that

$$Nx = N\Delta_*(x) = 0$$

for any $x$ in the kernel and in the cokernel of the cycle map

$$L_* H_*(X) \to H_*(X).$$

This implies that

$$L_m H_1(X) \otimes \mathbb{Q} \simeq H_1(X) \otimes \mathbb{Q}.$$ 

We conclude now Suslin’s conjecture for $X$ by using Corollary 3.1 □

We notice that the same techniques used in the Theorem 6.1 and in Theorem 6.2 give us the following proposition (which was already known (see [34])).

**Proposition 6.2.** Let $X$ be a smooth projective complex variety such that the cycle class maps

$$cl : CH^l(X)_{\mathbb{Q}} \to H_{2l}(X, \mathbb{Q})$$

are injective for $l \leq k$. Then $H^{p,q}(X) = 0$ for

a) $p \neq q$, $p + q$ even and $q \leq k$.

b) $|p - q| > 1$, $p + q$ odd and $q \leq k$.
Proof. The vanishing of the above Hodge numbers come from the equalities in the coniveau filtration generated by the decomposition of the diagonal and from the fact that coniveau filtration with complex coefficients is included in the Hodge filtration.

7. The case of projective linear varieties

Totaro [32] and Jannsen [21] gave the definition of a linear variety (see also [22]).

Definition. A complex variety is called 0-linear if it is either empty set or isomorphic to any affine space \( \mathbb{A}^n_\mathbb{C} \). Let \( n > 0 \). A complex variety \( Z \) is \( n \)-linear if there is a triple \((U, X, Y)\) of complex varieties so that \( Y \subset X \) is a closed immersion with \( U \) its complement; \( Y \) and one of the varieties \( U \) or \( X \) is \((n-1)\)-linear and \( Z \) is the other member in \( U, X \). We say that \( Z \) is linear if it is \( n \)-linear for some \( n \geq 0 \).

Among examples of linear varieties are toric varieties, flag varieties ([32],[21],[22]). R. Joshua [22] and B. Totaro [32] proved the following Künneth formula for projective linear varieties:

**Theorem 7.1.** ([32],[22]) Let \( X \) a projective smooth linear variety of dimension \( d \). Then
\[
CH^*(X) \otimes CH^*(X) \simeq CH^*(X \times X).
\]

In particular there is a decomposition of the diagonal cycle \( \Delta \in CH^d(X \times X) \) of the form
\[
\Delta = \sum \alpha_i \times \beta_i
\]
with \( \alpha_i, \beta_i \in CH^*(X) \) algebraic cycles with \( \dim(\alpha_i) + \dim(\beta_i) = d \).

Using Corollary [1.1] and Theorem [7.1] we can conclude that the action of \( \Delta \) is zero on the kernel and the cokernel of the cycle map \( L_m H_s(X) \to H_s(X) \). This implies the following proposition:

**Proposition 7.1.** Let \( X \) a projective smooth linear variety. Then
\[
L_s H_s(X) \simeq H_s(X).
\]
In particular we have
\[
K^s_{ist}(X) \simeq ku^{-i}(X^{an})
\]
for any \( i \geq 0 \).

The above proposition was first proved in [12] by other methods.

**References**

1. Abramovich, D. and Karu, K. and Matsuki, K. and Włodarczyk, J., *Torification and factorization of birational maps*, J. Amer. Math. Soc. 15 (2002), no. 3, 531–572 (electronic).
2. Albano, A. and Collino, A., *On the Griffiths group of the cubic sevenfold*, Math. Ann. 299 (1994), no. 4, 715–726.
3. Bloch, S., *Lectures on algebraic cycles*, Duke University Mathematics Series, IV, Duke University Mathematics Department, Durham, N.C., 1980.
4. Bloch, S. and Srinivas, V., *Remarks on correspondences and algebraic cycles*, Amer. J. Math. 105 (1983), no. 5, 1235–1253.
5. Danilov, V. I., *Cohomology of algebraic varieties*, Algebraic geometry, II, Encyclopaedia Math. Sci., vol. 35, Springer, Berlin, 1996, pp. 1–125, 255–262.
6. Deligne, P., *Théorie de Hodge. III*, Inst. Hautes Études Sci. Publ. Math. (1974), no. 44, 5–77.
7. Esnault, H. and Levine, M., *Surjectivity of cycle maps*, Astérisque (1993), no. 218, 203–226, Journées de Géométrie Algébrique d’Orsay (Orsay, 1992).
8. Friedlander, E. M. and Mazur, B., *Correspondence homomorphisms for singular varieties*, Ann. Inst. Fourier (Grenoble) 44 (1994), no. 3, 703–727.
9. Friedlander, Eric M., *Algebraic cycles, Chow varieties, and Lawson homology*, Compositio Math. 77 (1991), no. 1, 55–93.
10. Friedlander, Eric M., *Algebraic cocycles on normal, quasi-projective varieties*, Compositio Math. 110 (1998), no. 2, 127–162.
11. Friedlander, Eric M., *Bloch-Ogus properties for topological cycle theory*, Ann. Sci. École Norm. Sup. (4) 33 (2000), no. 1, 57–79.
12. Friedlander, Eric M. and Haesemeyer, C. and Walker, Mark E., *Techniques, computations, and conjectures for semi-topological K-theory*, Math. Ann. 330 (2004), no. 4, 759–807.
13. Friedlander, Eric M. and Lawson, H. Blaine, *Duality relating spaces of algebraic cocycles and cycles*, Topology 36 (1997), no. 2, 533–565.
14. Friedlander, Eric M. and Lawson, H. Blaine, *A theory of algebraic cocycles*, Ann. of Math. (2) 136 (1992), no. 2, 361–428.
15. Friedlander, Eric M. and Walker, Mark E., *Semi-topological K-theory using function complexes*, Topology 41 (2002), no. 3, 591–644.
16. Geisser, T. and Levine, M., *The Bloch-Kato conjecture and a theorem of Suslin-Voevodsky*, J. Reine Angew. Math. 530 (2001), 55–103.
17. Gillet, H., *On the K-theory of surfaces with multiple curves and a conjecture of Bloch*, Duke Math. J. 51 (1994), no. 1, 195–233.
18. Gillet, H. and Soulé, C., *Descent, motives and K-theory*, J. Reine Angew. Math. 478 (1996), 127–176.
19. Hu, W., *Some birational invariants defined by Lawson homology*, Preprint, arXiv:math.AG/0511722.
20. Izadi, E., *A Prym construction for the cohomology of a cubic hypersurface*, Proc. London Math. Soc. (3) 79 (1999), no. 3, 535–568.
21. Jannsen, U., *Mixed motives and algebraic K-theory*, Lecture Notes in Mathematics, vol. 1400, Springer-Verlag, Berlin, 1990, With appendices by S. Bloch and C. Schoen.
22. Joshua, R., *Algebraic K-theory and higher Chow groups of linear varieties*, Math. Proc. Cambridge Philos. Soc. 130 (2001), no. 1, 37–60.
23. Kollár, J. and Miyaoka, Y. and Mori, S., *Rationally connected varieties*, J. Algebraic Geom. 1 (1992), no. 3, 429–448.
24. Laterveer, R., *Algebraic varieties with small Chow groups*, J. Math. Kyoto Univ. 38 (1998), no. 4, 673–694.
25. Lawson, Jr., H. Blaine, *Algebraic cycles and homotopy theory*, Ann. of Math. (2) 129 (1989), no. 2, 253–291.
26. Lewis, J. D., *Cylinder homomorphisms and Chow groups*, Math. Nachr. 160 (1993), 205–221.
27. Paranjape, K. H., *Cohomological and cycle-theoretic connectivity*, Ann. of Math. (2) 139 (1994), no. 3, 641–660.
28. Pedrini, C. and Weibel, C., *The higher K-theory of complex varieties*, K-Theory 21 (2000), no. 4, 367–385, Special issues dedicated to Daniel Quillen on the occasion of his sixtieth birthday, Part V.
29. Peters, C., *Lawson homology for varieties with small Chow groups and the induced filtration on the Griffiths groups*, Math. Z. 234 (2000), no. 2, 209–223.
30. Suslin, A. and Voevodsky, V., *Singular homology of abstract algebraic varieties*, Invent. Math. 123 (1996), no. 1, 61–94.
31. Bloch-Kato conjecture and motivic cohomology with finite coefficients, The arithmetic and geometry of algebraic cycles (Banff, AB, 1998), NATO Sci. Ser. C Math. Phys. Sci., vol. 548, Kluwer Acad. Publ., Dordrecht, 2000, pp. 117–189.

32. Totaro, B., Chow groups, Chow cohomology and linear varieties, Preprint, 1995.

33. Voisin, C., On integral Hodge classes on uniruled or Calabi-Yau threefolds, Preprint, arXiv:math.AG/0412279.

34. Hodge theory and complex algebraic geometry. II, Cambridge Studies in Advanced Mathematics, vol. 77, Cambridge University Press, Cambridge, 2003, Translated from the French by Leila Schneps.

35. Walker, Mark E., The morphic Abel-Jacobi map, Preprint, May 9, 2005, K-theory Preprint Archives, http://www.math.uiuc.edu/K-theory/0740/.

Department of Mathematics, Northwestern University, Evanston, IL 60208
E-mail address: mircea@math.northwestern.edu