RANDOM PARTITIONS AND THE GAMMA KERNEL

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Abstract. We study the asymptotics of certain measures on partitions (the so-called z-measures and their relatives) in two different regimes: near the diagonal of the corresponding Young diagram and in the intermediate zone between the diagonal and the edge of the Young diagram. We prove that in both cases the limit correlation functions have determinantal form with a correlation kernel which depends on two real parameters. In the first case the correlation kernel is discrete, and it has a simple expression in terms of the gamma functions. In the second case the correlation kernel is continuous and translationally invariant, and it can be a written as a ratio of two suitably scaled hyperbolic sines.

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0. Introduction

In recent years there has been a lot of interest in understanding the “random matrix type” limit behavior of different measures on partitions as the size of partitions goes to infinity. The most known result is the Baik-Deift-Johansson theorem [BDJ] that claims that the limit distribution of the (centered and scaled) first part of the random partitions distributed according to the so-called Plancherel measure is just the same as that of the largest eigenvalue of random Hermitian matrices from the Gaussian Unitary Ensemble.

The goal of this paper is to study the asymptotic behavior of the so-called z-measures and their relatives. The asymptotics of the largest parts of partitions distributed according to such measures has a representation theoretic meaning: it encodes the spectral decomposition of generalized regular representations of certain groups into irreducibles. We have computed this asymptotics in the cases of the

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infinite symmetric group and the infinite–dimensional unitary group in our previous
work, see [BO2], [BO4]. The main result of this paper is a complete description
of the limit behavior of these measures near the diagonal (smallest Frobenius co-
ordinates) and in the intermediate zone between the diagonal and the edge of the
partition (Frobenius coordinates of intermediate growth).

A more detailed description of the content of the paper follows.

The z–measures. Let $\mathcal{Y}_n$ denote the set of partitions of a natural number $n$ and
$\mathcal{Y} = \mathcal{Y}_0 \sqcup \mathcal{Y}_1 \sqcup \mathcal{Y}_2 \sqcup \ldots$ be the set of all partitions. We identify partitions and Young
diagrams. We consider a Hilbert space $H$ together with a distinguished orthonormal
basis $\{\chi_\lambda\}$ parameterized by $\lambda \in \mathcal{Y}$. The basis elements may be identified with
irreducible characters of symmetric groups of arbitrary degree. Next, we construct
a family of vectors $f_{z,\xi}$ in $H$, indexed by couples $(z,\xi) \in \mathbb{C} \times (0,1)$. Set

$$M_{z,z',\xi}(\lambda) = \frac{(f_{z,\xi},\chi_\lambda)(\chi_\lambda,f_{z',\xi})}{(f_{z,\xi},f_{z',\xi})}$$

where $z \in \mathbb{C}$, $z' \in \mathbb{C}$, $0 < \xi < 1$, and $(\cdot, \cdot)$ denotes the inner product in $H$. It
turns out that $(f_{z,\xi},f_{z',\xi}) \neq 0$, so that the above expression makes sense. Clearly,

$$\sum_{\lambda \in \mathcal{Y}} M_{z,z',\xi}(\lambda) = 1.$$

Under suitable restrictions on the parameters $z, z', \xi$ (for instance, if $z' = \bar{z}$), the
above expression for $M_{z,z',\xi}(\lambda)$ is nonnegative for any $\lambda$, so that $M_{z,z',\xi}$ is a proba-
bility measure on $\mathcal{Y}$. We call it a z–measure. This is our main object of study. An
explicit expression for $M_{z,z',\xi}(\lambda)$ is given in §1.

The measures $M_{n}^{(n)}$. Given $n$, restrict $M_{z,z',\xi}$ to $\mathcal{Y}_n \subset \mathcal{Y}$ and normalize it so
that the total mass of $\mathcal{Y}_n$ be equal to 1. Then we obtain a probability measure on
$\mathcal{Y}_n$ which turns out to be independent of $\xi$; we denote this measure by $M_{n}^{(n)}$. The
initial z–measure $M_{z,z',\xi}$ may be written as a mixture of the measures $M_{n}^{(n)}$ with
varying $n$,

$$M_{z,z',\xi} = \sum_{n=0}^{\infty} \pi(n) M_{n}^{(n)},$$

where the coefficients

$$\pi(n) = (1 - \xi)^{zz'} \frac{(zz'+1) \ldots (zz' + n - 1)}{n!} \xi^n$$

are precisely the weights of the negative binomial distribution on $\mathbb{Z}_+$ with suitable
parameters.

Frobenius coordinates. We need the Frobenius notation for Young diagrams:

$$\lambda = (p_1, \ldots, p_d \mid q_1, \ldots, q_d),$$

where $d$ is the number of diagonal boxes in $\lambda$, $p_i$ is the number of boxes in the $i$th
row to the right of the diagonal, and $q_i$ is the number of boxes in the $i$th column
below the diagonal. Note that

$$p_1 > \cdots > p_d \geq 0, \quad q_1 > \cdots > q_d \geq 0.$$

An advantage of the Frobenius notation, as compared to the conventional notation
$\lambda = (\lambda_1, \lambda_2, \ldots)$, is its obvious symmetry with respect to transposition of diagrams.
The $p_i$’s and $q_i$’s are called the Frobenius coordinates of the Young diagram $\lambda$. 

Asymptotic problems for random diagrams. Given a probability measures on Young diagrams, one may speak about random Young diagrams. A problem of interest is to study the asymptotic behavior of $M_{z,z'}^{(n)}$--random diagrams as $n \to \infty$.

In the present paper we are dealing with a different but closely related problem: the asymptotics of $M_{z,z',\xi}$--random diagrams as $\xi \nearrow 1$ (the parameters $z,z'$ remain fixed).

There is a number of different limit regimes of the asymptotics. Here we discuss three of them: one for the largest Frobenius coordinates, one for the smallest Frobenius coordinates, and one for the Frobenius coordinates of intermediate growth.

It is an interesting question how the asymptotics of $M_{z,z',\xi}$ is related to that of $M_{z,z'}^{(n)}$ in each of these regimes. For the first regime the answer is known: the limiting random point processes (i.e., measures on point configurations) are different by the multiplication by an independent random scaling factor, see [BO2, §5] and [BO1, §6]. For the third regime, the computation of [Bor1, §4.2-4.3], see also [BO1, §11], suggests that the asymptotic behavior of the p-coordinates is the same for both measures. However, this computation is rather involved, and it would be nice to have a simpler argument which would also extend to the joint asymptotics of p- and q-coordinates. No claims of this kind have been proved yet regarding the second regime, but we believe that the corresponding asymptotics of $M_{z,z',\xi}$ and $M_{z,z'}^{(n)}$ is also the same in this case.

Asymptotics of largest Frobenius coordinates [BO2]. In the first limit regime, we look at the largest Frobenius coordinates $p_1 > p_2 > \ldots$ and $q_1 > q_2 > \ldots$. These are random variables depending on $\xi$ as a parameter. As $\xi \nearrow 1$, we need to normalize them, and the suitable normalization consists in multiplying all the coordinates by $(1 - \xi)$. In the limit we obtain a couple of random infinite sequences of decreasing real numbers, which may be also interpreted as a random point configuration on the punctured line $\mathbb{R} \setminus \{0\}$, or as a random point process on $\mathbb{R} \setminus \{0\}$. This process was studied in our previous paper [BO2]. We showed that its correlation functions have determinantal form with a kernel, which we called the Whittaker kernel, because it is expressed through the classical Whittaker function.

Limit behavior of smallest Frobenius coordinates. In the second limit regime, we examine the smallest Frobenius coordinates $p_d < p_{d-1} < \ldots$ and $q_d < q_{d-1} < \ldots$. Again, these are random variables depending on $\xi$, but now no normalization is required. In the limit we obtain a couple of random infinite increasing sequences of nonnegative integers, say

$$0 \leq a_1 < a_2 < \ldots, \quad 0 \leq b_1 < b_2 < \ldots,$$

which can be conveniently interpreted as a random point configuration $X$ on the lattice $Z' := \mathbb{Z} + \frac{1}{2}$ of half-integers,

$$X = (\ldots, -b_2 - \frac{1}{2}, -b_1 - \frac{1}{2}, a_1 + \frac{1}{2}, a_2 + \frac{1}{2}, \ldots).$$

Thus we get a random point process on $Z'$, which describes the limit behavior of the random Young diagrams near the diagonal.

A different but equivalent picture of the same limit regime is obtained as follows. Set

$$X = (X \cap Z'_+) \cup (Z'_- \setminus X),$$

where $Z'_+$ and $Z'_-$ are the upper and lower half-lattices, respectively.
where
\[ \mathbb{Z}'_\pm = \{ \ldots, -\frac{5}{2}, -\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots \} \]
That is, viewing the points of \( X \) as “particles” and those of \( \mathbb{Z}' \setminus X \) as “holes”, the configuration \( X \) is formed by the particles in \( \mathbb{Z}'_+ \) and the holes in \( \mathbb{Z}'_- \).

One of the main results of the present paper is a description of both random processes on \( \mathbb{Z}' \). We show that the correlation functions for each of these two processes are given in terms of a rather simple kernel on \( \mathbb{Z}' \times \mathbb{Z}' \), which is expressed through the Euler gamma function. We call it the gamma kernel (there are two versions of the kernel which correspond to the random configurations \( X \) and \( X' \), respectively). Similarly to the Whittaker kernel, the gamma kernel depends on the parameters \( z, z' \).

The \( m \)-particle correlation function \( (m = 1, 2, \ldots) \) for the random configuration \( X \) is given by
\[
\text{Prob}\{X \supset (x_1, \ldots, x_m)\} = \det_{1 \leq i, j \leq m} [K_{\text{gamma}}(x_i, x_j \mid z, z')].
\]
Here \((x_1, \ldots, x_m)\) is an arbitrary collection of distinct points in \( \mathbb{Z}' \).

The correlation functions for the random configuration \( X \) have the same det-
minantal form, only \( K_{\text{gamma}}(x, y \mid z, z') \) is replaced by another version of the kernel, \( K_{\text{gamma}}(x, y \mid z, z') \), see Theorem 3.2 below.

**Asymptotics of intermediate Frobenius coordinates.** In the third limit regime, we consider Frobenius coordinates with intermediate growth, that is, the \( p_i \)'s and \( q_j \)'s such that
\[ 0 \ll p_i, q_j \ll \frac{1}{1-\xi} \quad \text{as } \xi \nearrow 1. \]

We show that in a suitable scaling limit, the asymptotics of the intermediate Frobenius coordinates is governed by a kernel on \( \mathbb{R} \cup \mathbb{R} \) (the union of two copies of the real line, one is for \( p \)-coordinates and the other one is for \( q \)-coordinates). We call this kernel the tail kernel. The tail kernel is translationally invariant and it is a relative of the famous sine kernel.

Notice that the tail kernel can also be obtained, via a suitable scaling limit transition, from two opposite directions: from the Whittaker kernel, see [BO1, §11], and from the gamma kernel, see §6 below.

**Remarks.** The measures \( M^{(n)}_{z, z'} \) on the finite sets \( Y_n \) first arose in Kerov–Olshanski–Vershik [KOV1], in connection with the problem of harmonic analysis on the infinite symmetric group. The asymptotics of the largest Frobenius coordinates of \( M^{(n)}_{z, z'} \)-random Young diagrams, as \( n \to \infty \), was studied in a series of our papers,
summarized in the survey [BO1]. The result provides a description of the decomposition of the so-called generalized regular representations of the infinite symmetric group on irreducible components.

The $z$-measures $M_{z,z'}$ were introduced in [BO2]. They enter a wider class of Schur measures introduced soon after by Okounkov [Ok2]. The $z$-measures initially served as a technical tool which allowed us to rederive the main results of [Bor2], [Bor3] on the limits of the measures $M_{z,z'}^{(n)}$ in a simpler way. However, the $z$-measures are also interesting in their own right.

The idea of mixing the measures $M_{z,z'}^{(n)}$ and replacing the large $n$ limit by the $\xi \rightarrow 1$ limit is similar to the idea of passing to a grand canonical ensemble in statistical mechanics. A parallelism between models of statistical mechanics and those of asymptotic combinatorics was emphasized by Vershik [V].

**Comparison with the Plancherel measure.** When the parameters $z, z'$ go to infinity, the measure $M_{z,z'}^{(n)}$ degenerates to the Plancherel measure $M^{(n)}$ on $Y_n$. Similarly, when $z, z'$ go to infinity and $\xi$ goes to $+0$, in such a way that $zz'\xi$ tends to a limit $\theta > 0$, the measure $M_{z,z',\xi}$ on $Y$ degenerates to the poissonized Plancherel measure $M_\theta$ with parameter $\theta$. The latter measure is a mixture of the measures $M^{(n)}$, where the mixing distribution on the $n$'s is the Poisson distribution (the weight of $n$ equals $e^{-\theta n}/n!$). The large $n$ limit of the measures $M^{(n)}$ can be effectively replaced by the large $\theta$ limit of the measures $M_\theta$. Due to nice properties of the Poisson distribution, both kinds of limit transition turn out to be strictly equivalent in various asymptotic regimes (see Baik-Deift-Johansson [BDJ], Borodin-Okounkov-Olshanski [BOO], Johansson [J1]).

An important difference between the Plancherel measures and the $z$-measures is that the random Plancherel diagrams have a limit form, as $n \rightarrow \infty$ or $\theta \rightarrow \infty$ (see Vershik–Kerov [VK1], [VK2], Logan-Shepp [LS]), while no such form exists for the $z$-measures. On the other hand, the statement of the asymptotic problem concerning the smallest Frobenius coordinates is the same for both kinds of measures, and the answers are formulated in similar terms: for the Plancherel measure, the role of the gamma kernel is played by the discrete sine kernel with parameter $0$ (see [BOO], especially Remark 1.8). Notice that the latter kernel is the degeneration of the gamma kernel as the parameters $z, z'$ go to infinity.

**Comparison with the measures given by the Ewens sampling formula.** The Ewens sampling formula determines a one-parameter family of probability measures on $Y_n$ for each $n = 1, 2, \ldots$:

$$ESF_t^{(n)}(\lambda) = \frac{t^{\ell(\lambda)}}{(t)_n} ESF_1^{(n)}(\lambda), \quad \lambda \in Y_n,$$

where $\ell(\lambda)$ is the number of nonzero rows in $\lambda$, $t > 0$ is the parameter, and

$$ESF_1^{(n)}(\lambda) = \frac{\text{the number of permutations in } S_n \text{ with cycle structure } \lambda}{n!}$$

There is a wide literature concerning these measures, see, e.g., the encyclopedic article Tavaré–Ewens [TE]. As shown in Kerov–Olshanski–Vershik [KO1], [KO2], both the measures $ESF_t^{(n)}$ and the measures $M_{z,z'}^{(n)}$ are involved in harmonic analysis on the infinite symmetric group, but they refer to different “levels”, the “group
level” and the “dual level”, respectively. Namely, the measures $ESF_{t}^{(n)}$ determine certain probability measures on a compactification $\mathfrak{S}$ of the infinite symmetric group, while the measures $M_{z,z'}^{(n)}$ determine the so-called spectral measures on the dual object to the infinite symmetric group. The measures on $\mathfrak{S}$ are used to build $L^{2}$ Hilbert spaces where the so-called generalized regular representations are realized, while the spectral measures govern the decomposition of those representations into irreducibles.

The large $n$ limits of the measures $ESF_{t}^{(n)}$ in various regimes were extensively studied, see, e.g., the monograph by Arratia, Barbour, and Tavaré [ABT]. At the first glance, the results look quite different as compared with our results for the measures $M_{z,z'}^{(n)}$ of a unitary matrix $U$ with respect to the measure $\lambda$. Nevertheless, it seems to us that a detailed comparison of both families of measures may be of interest since it could lead to a better understanding of the nature of probabilistic models related to partitions.

The zw–measures on signatures. By a signature of length $N$, where $N = 1, 2, \ldots$, we mean an ordered $N$–tuple of nonincreasing integers

$$\lambda = (\lambda_{1} \geq \cdots \geq \lambda_{N}), \quad \lambda_{i} \in \mathbb{Z}.$$ 

Let $SGN(N)$ be the set of all such $\lambda$’s. This is a countable set. There is a one–to–one correspondence $\lambda \mapsto \chi_{\lambda}$ between signatures $\lambda \in SGN(N)$ and irreducible characters of the compact group $U(N)$ of $N \times N$ unitary matrices. The irreducible characters $\chi_{\lambda}$ are given by the (rational) Schur functions $s_{\lambda}(u_{1}, \ldots, u_{N})$ in the eigenvalues of a unitary matrix $U \in U(N)$. Let $H_{N}$ be the Hilbert space of functions on the group $U(N)$, constant on conjugacy classes and square integrable with respect to the normalized Haar measure. The characters $\chi_{\lambda}$ form an orthonormal basis in $H_{N}$. Equivalently, $H_{N}$ can be realized as the space of symmetric functions on the torus $\mathbb{T}^{N}$ (the product of $N$ copies of the unit circle $\mathbb{T} \subset \mathbb{C}$), square integrable with respect to the measure

$$\frac{1}{N!} \prod_{1 \leq i < j \leq N} |u_{i} - u_{j}|^{2} du,$$

where $du$ is the normalized invariant measure on the torus.

We define a family $\{f_{z,w}|N\}$ of vectors in $H_{N}$, where $z, w$ are complex parameters, and we set

$$M_{z,z',w,w'}|N(\lambda) = \frac{\langle f_{z,w}|N, \chi_{\lambda}\rangle \langle \chi_{\lambda}, f_{w',z'}|N \rangle}{\langle f_{z,w}|N, f_{w',z'}|N \rangle}, \quad \lambda \in SGN(N),$$

where $(\cdot, \cdot)$ is the inner product in $H_{N}$ and $(z', w')$ is one more couple of complex numbers. An explicit expression for $M_{z,z',w,w'}|N(\lambda)$ is given in §7. Under suitable restrictions on the quadruple $(z, z', w, w')$, this expression determines, for any $N$, a probability measure on $SGN(N)$, which we call the zw-measure. For instance, the zw–measures are well defined if $z' = \bar{z}$, $w' = \bar{w}$, and $\Re(z + w) > -\frac{1}{2}$.

The zw–measures arise in the problem of harmonic analysis on the infinite–dimensional unitary group, studied in our previous papers [Ol2], [BO4].

Large $N$ limits of the zw–measures $M_{z,z',w,w'}|N$. Any signature $\lambda \in SGN(N)$ can be viewed as a couple $(\lambda^{+}, \lambda^{-})$ of Young diagrams subject to the condition $\ell(\lambda^{+}) + \ell(\lambda^{-}) \leq N$, where $\ell(\lambda^{\pm})$ stands for the number of nonzero rows in $\lambda^{\pm}$,

$$\lambda = (\lambda_{1}^{+}, \lambda_{2}^{+}, \ldots, 0, 0, \ldots, -\lambda_{2}^{-}, -\lambda_{1}^{-}).$$
A problem of interest for the zw–measures is their limit behavior as $N \to \infty$ with the parameters $z, z', w, w'$ being fixed. That is, we ask about the asymptotic distribution of the Frobenius coordinates for the random diagrams $\lambda^+, \lambda^-$. Again, one can consider (at least) three different limit regimes: the largest, smallest or intermediate coordinates, respectively.

The asymptotics of the largest Frobenius coordinates was studied in [Ol2], [BO4]. Introducing the scaling factor $1/N$ for the Frobenius coordinates of $\lambda^+$ and $\lambda^-$, we obtain in the limit 4 infinite random sequences which can be assembled in a single random point configuration living on the real axis with two punctures. We showed that this random point process is governed by a kernel, which we called the continuous hypergeometric kernel for it is expressed through the Gauss hypergeometric function. This result leads to a description of the spectral decomposition of certain unitary representations of the infinite–dimensional unitary group.

In the present paper, we are dealing with the smallest Frobenius coordinates of $\lambda^\pm$. That is, we study the limit structure of the boundary of the random shape $\lambda^\pm$ near its diagonal. Our result is that the limit correlation functions are again given by the gamma kernel. The appropriate parameters are $(z, -z')$ for $\lambda^+$ and $(-w, -w')$ for $\lambda^-$. Thus, although in the “first limit regime”, the correlation kernels obtained from the z–measures and from the zw–measures are different (the continuous hypergeometric kernel is on the next level of complexity as compared with the Whittaker kernel), the answer in the “second limit regime” is the same.

It is worth noting that the computations leading to the gamma kernel in the case of the zw–measures $M_{z, z', w, w'|N}$ are more complex than those for the z–measures $M_{z, z', \xi|N}$. Instead of the Gauss hypergeometric function $2F_1$, which is involved in the proof for the z–measures, we need to manipulate with the higher hypergeometric series — the series $3F_2$ at the unit argument.

As for the “third limit regime”, which concerns intermediate Frobenius coordinates, the answer is conjecturally given by the same tail kernel as for the z–measures. We do not prove this fact rigorously but present an argument in favor of it.

The z–measures on nonnegative signatures. Here we define the third family of measures, which are close relatives of the zw–measures described above. Let $\text{SGN}^+(N)$ be the subset of $\text{SGN}(N)$ formed by signatures $\lambda$ with $\lambda_N \geq 0$. We call them the nonnegative signatures. Equivalently, $\text{SGN}^+(N)$ consists of Young diagrams $\lambda$ with $\ell(\lambda) \leq N$. We fix two real parameters $a > -1$, $b > -1$. Let $H_N$ be the Hilbert space formed by symmetric functions on the $N$–dimensional cube $[-1,1]^N$, square integrable with respect to the measure

$$\prod_{1 \leq i < j \leq N} (x_i - x_j)^2 \cdot \prod_{i=1}^{N} (1 - x_i)^a (1 + x_i)^b \cdot dx_1 \ldots dx_n, \quad x_i \in [-1,1].$$

In $H_N$, we consider the orthonormal basis $\{\chi^a,b_{\lambda}\}$ formed by the (suitably normalized) multivariate Jacobi polynomials. Here the subscript $\lambda$ ranges over $\text{SGN}^+(N)$. On the other hand, we introduce a family $\{f_{z|N}\}$ of symmetric functions on the cube, depending on a complex number $z$, and we set

$$M_{z, z', a, b|N}(\lambda) = \frac{(f_{z|N}, \chi^a,b_{\lambda}) (\chi^a,b_{\lambda}, f_{z'|N})}{(f_{z|N}, f_{z'|N})}, \quad \lambda \in \text{SGN}^+(N),$$

where

$$\prod_{1 \leq i < j \leq N} (x_i - x_j)^2 \cdot \prod_{i=1}^{N} (1 - x_i)^a (1 + x_i)^b \cdot dx_1 \ldots dx_n, \quad x_i \in [-1,1].$$

In $H_N$, we consider the orthonormal basis $\{\chi^a,b_{\lambda}\}$ formed by the (suitably normalized) multivariate Jacobi polynomials. Here the subscript $\lambda$ ranges over $\text{SGN}^+(N)$. On the other hand, we introduce a family $\{f_{z|N}\}$ of symmetric functions on the cube, depending on a complex number $z$, and we set

$$M_{z, z', a, b|N}(\lambda) = \frac{(f_{z|N}, \chi^a,b_{\lambda}) (\chi^a,b_{\lambda}, f_{z'|N})}{(f_{z|N}, f_{z'|N})}, \quad \lambda \in \text{SGN}^+(N),$$

where
where \( z' \) is one more complex parameter. If \( z, z' \) satisfy certain restrictions, this gives us a probability measure on \( \text{SGN}^+(N) \), which we call the \( z \)-measure on non-negative signatures. An explicit expression is given in \( \S 8 \).

This construction is again motivated by representation theory. Specifically, for a few special values of \((a, b)\), the (suitably renormalized) multivariate Jacobi polynomials are the irreducible characters of the symplectic or orthogonal groups, or else the spherical functions on the complex Grassmannians. Then the \( z \)-measures \( M_{z, z', a, b| N} \) naturally emerge in the problem of harmonic analysis for infinite-dimensional analogs of these classical groups or for the Grassmannians.\(^1\)

For general \((a, b)\), there is no such direct representation-theoretic interpretation. Nevertheless, according to the philosophy of the modern theory of multivariate special functions (see, e.g. Heckman’s part of the book [HS]), there are good reasons to work with general parameters \((a, b)\) as well.

**Large \( N \) limit of the \( z \)-measures** \( M_{z, z', a, b| N} \). As in the case of the \( zw \)-measures, we focus on the “second limit regime”, which, in the present case, concerns the smallest Frobenius coordinates of the random diagrams \( \lambda \in \text{SGN}^+(N) \). And once again, it turns out that the limit point process is determined by the gamma kernel. The proof involves rather tedious computations with the hypergeometric series \( {}_4 F_3 \) at the unit argument.

**Asymptotics of discrete orthogonal polynomials.** The heart of the argument in the cases of the \( zw \)-measures \( M_{z, z', w, w'| N} \) and the \( z \)-measures \( M_{z, z', a, b| N} \) is a computation of the asymptotics of certain discrete orthogonal polynomials of degree \( N - 1 \) and \( N \) as \( N \to \infty \). Those are the Askey–Lesky polynomials (which generalize the classical Hahn polynomials) in the first case and the Wilson-Neretin polynomials (which generalize the classical Racah polynomials) in the second case. Even though the weight function in both cases depends on four independent parameters (except for \( N \)), the limits of the Christoffel–Darboux kernels in both cases are the same (the gamma kernel), and the result depends only on two of the four initial parameters. This suggests that the gamma kernel (and, hence, the tail kernel which is equal to its scaling limit) may play a universal role in asymptotics of general discrete orthogonal polynomials.

Recall that, as it was recently shown by Baik–Kriecherbauer–McLaughlin–Miller [BKMM, \( \S 3.1.1 \)], the discrete sine kernel is the universal microscopic limit of the Christoffel–Darboux kernels associated with generic discrete orthogonal polynomials, near a point where the macroscopic density function is continuous and takes any value strictly between 0 and 1.\(^2\)

It looks very plausible to us that the gamma kernel and the tail kernel are universal microscopic limits, in two different asymptotic regimes, of the Christoffel–Darboux kernels for generic discrete orthogonal polynomials near a point where the macroscopic density function is discontinuous, takes value 0 on one side of this point, and takes value 1 on the other side of this point. The two special cases considered in \( \S 7 \) and \( \S 8 \) below provide some evidence in support of this conjecture.

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1. This problem can be stated by analogy with the construction of [Ol2]. Notice that the \( z \)-measures \( M_{z, z', a, b| N} \) with \( a = b = 0 \) and real \( z = z' \) appeared for the first time in [Pic].

2. This means that the point configurations of the corresponding orthogonal polynomial ensemble are not empty (density 0) nor fully packed (density 1) near this point.
1. Definition of the $z$–measures

As in Macdonald [Ma] we identify partitions and Young diagrams. By $\mathcal{Y}_n$ we denote the set of partitions of a natural number $n$, or equivalently, the set of Young diagrams with $n$ boxes. By $\mathcal{Y}$ we denote the set of all Young diagrams, that is, the disjoint union of the finite sets $\mathcal{Y}_n$, where $n = 0, 1, 2, \ldots$ (by convention, $\mathcal{Y}_0$ consists of a single element, the empty diagram $\emptyset$).

Given $\lambda \in \mathcal{Y}$, let $|\lambda|$ denote the number of boxes of $\lambda$, let $\ell(\lambda)$ be the number of nonzero rows in $\lambda$, and let $\lambda'$ be the transposed diagram. For $z \in \mathbb{C}$, let

$$ (z)_\lambda = \prod_{i=1}^{\ell(\lambda)} (z - i + 1)_{\lambda_i} $$

where $(x)_k = x(x+1)\ldots(x+k-1) = \Gamma(x+k)/\Gamma(x)$ is the Pochhammer symbol.

Note that $(z)_\lambda = \prod_{(i,j) \in \lambda} (z + j - i)$ (product over the boxes of $\lambda$), which implies at once the symmetry relation

$$ (z)_\lambda = (-1)^{|\lambda|} (-z)_{\lambda'}.$$

Given $\lambda \in \mathcal{Y}, \lambda \neq \emptyset$, we denote by $\chi_\lambda$ the irreducible character of the symmetric group $S_{|\lambda|}$, indexed by $\lambda$. For $n = 1, 2, \ldots$, let $H_n$ be the space of complex functions on $S_n$, constant on conjugacy classes. We introduce an inner product in $H_n$ by the formula

$$ (f, g)_n = \frac{1}{n!} \sum_{s \in S_n} f(s)g(s). $$

The characters $\chi_\lambda$ with $\lambda \in \mathcal{Y}_n$ form an orthonormal basis in $H_n$, so that we may write

$$ H_n = \bigoplus_{\lambda \in \mathcal{Y}_n} \mathbb{C}\chi_\lambda, \quad n = 1, 2, \ldots. $$

We also agree that $H_0$ is a one–dimensional vector space with basis element denoted as $\chi_\emptyset$, $(\chi_\emptyset, \chi_\emptyset)_0 = 1$.

Given $z \in \mathbb{C}$, define a function $f_z^{(n)} \in H_n$ as follows

$$ f_z^{(n)}(s) = z^{\text{the number of cycles in } s}, \quad s \in S_n, \quad n = 1, 2, \ldots. $$

**Proposition 1.1** [KOV2, Lemma 4.1.2]. The expansion of $f_z^{(n)}$ in the basis $\{\chi_\lambda\}$ has the form

$$ f_z^{(n)} = \sum_{\lambda \in \mathcal{Y}_n} (z)_\lambda \frac{\dim \lambda}{n!} \chi_\lambda $$

where

$$ \dim \lambda = \chi_\lambda(e). $$

For the reader’s convenience we outline the proof.

**Proof.** Let $\Lambda$ be the graded algebra of symmetric functions in countably many variables, say $y_1, y_2, \ldots$, and let $\Lambda^n$ denotes the $n$th homogeneous component of $\Lambda$.
Endow $\Lambda$ with the canonical inner product ([Ma, §I.4]). Consider the characteristic map $\operatorname{ch}_n$, which is a linear isometry between $H_n$ and $\Lambda^n$ transforming the characters $\chi_\lambda$ into the Schur functions $s_\lambda$ ([Ma, §I.7]). First, we check that

$$\operatorname{ch}_n(f^{(n)}_z) = \text{the } n\text{th homogeneous component of } \prod_{i=1}^\infty (1 - y_i)^{-z}.$$ 

This reduces the claim of the proposition to the expansion

$$\prod_{i=1}^\infty (1 - y_i)^{-z} = \sum_{\lambda \in \mathbb{Y}} (z_{\lambda}) \frac{\dim \lambda}{|\lambda|!} s_\lambda,$$

which in turn can be deduced from [Ma, chapter I, (4.3)]. □

As is well known, $\dim \lambda$ coincides with the number of standard tableaux of shape $\lambda \in \mathbb{Y}$ (see [Ma, Example I.7.3]). A number of different explicit expressions are known for this quantity. For instance, for any natural $k \geq \ell(\lambda)$,

$$\dim \lambda = \det_{1 \leq i,j \leq k} \left[ \frac{1}{\Gamma(\lambda_i - i + j + 1)} \right] = \prod_{1 \leq i < j \leq k} (\lambda_i - \lambda_j + j - i) \prod_{1 \leq i \leq k} (\lambda_i + k - i)!$$

([Ma, Example I.7.6]). These formulas do not demonstrate the symmetry $\dim \lambda = \dim \lambda'$. There are two other formulas which are symmetric: the hook formula ([Ma, Example I.5.2]) and the expression in terms of Frobenius coordinates, see the beginning of §3 below.

Let us agree that $f^{(0)}_z = \chi_\emptyset \in H_0$.

**Proposition 1.2.** For any $z \in \mathbb{C}$ and any $\xi \in (0,1)$, we have

$$\sum_{n=0}^\infty \|f^{(n)}_z\|_n^2 \frac{\xi^n}{n!} < +\infty$$

so that the formal sum

$$f_{z,\xi} := \sum_{n=0}^\infty f^{(n)}_z \sqrt{\frac{\xi^n}{n!}}$$

is a well-defined element of the Hilbert space

$$H := H_0 \oplus H_1 \oplus H_2 \oplus \ldots$$

**Proof.** We will prove that

$$\|f^{(n)}_z\|_n^2 = (z\bar{z})_n = (z\bar{z})(z\bar{z}+1)\ldots(z\bar{z}+n-1).$$

Since the series

$$\sum_{n=0}^\infty \frac{(z\bar{z})_n}{n!} \xi^n$$

converges for $0 < \xi < 1$, the claim of the proposition will readily follow.

By the definition of $f^{(n)}_z$

$$\|f^{(n)}_z\|_n^2 = (f^{(n)}_z, \chi(n))_n$$

where $(n)$ is the partition $(n,0,0,\ldots)$ (the corresponding character is simply the constant function 1). Then the result follows from Proposition 1.1. □
Proposition 1.3. For any \( z, z' \in \mathbb{C} \) and any \( \xi \in (0, 1) \), we have
\[
(f_z, f_{z'}, \xi) = (1 - \xi)^{-zz'},
\]
where \((\cdot, \cdot)\) is the inner product in \( H \).

Proof. The same argument as in the proof of Proposition 1.2 gives
\[
(f_z, f_{z'}, \xi)_n = (zz')_n.
\]
Therefore,
\[
(f_z, \xi, f_{z'}, \xi) = \sum_{n=0}^{\infty} (f_z, f_{z'})_n \frac{\xi^n n!}{n!} = (1 - \xi)^{-zz'}.
\]
\[\square\]

Definition 1.4. (i) Let \( z, z' \in \mathbb{C} \) and \( \xi \in (0, 1) \). For any \( \lambda \in \mathbb{Y} \) we set
\[
M_{z, z', \xi}(\lambda) = \frac{(f_z, \chi_{\lambda}, f_{z'}, \xi)}{(f_z, f_{z'})}.
\]
Notice that the denominator is nonzero (Proposition 1.3), so that the whole expression makes sense. Since \( \{\chi_{\lambda}\}_{\lambda \in \mathbb{Y}} \) is an orthonormal basis in \( H \), we have
\[
\sum_{\lambda \in \mathbb{Y}} M_{z, z', \xi}(\lambda) = 1.
\]

From Propositions 1.2 and 1.3 we obtain an explicit expression for \( M_{z, z', \xi} \):
\[
M_{z, z', \xi}(\lambda) = (1 - \xi)^{zz'} \left( \frac{\dim \lambda}{|\lambda|!} \right)^2 \xi^{|\lambda|}.
\]

(ii) Under suitable restrictions on the triple \((z, z', \xi)\) the quantities \( M_{z, z', \xi}(\lambda) \) are nonnegative for all \( \lambda \). Then \( M_{z, z', \xi} \) is a probability measure on the countable set \( \mathbb{Y} \), which we call the \( z \)-measure on \( \mathbb{Y} \) with parameters \( z, z', \xi \). The nonnegativity property holds, for instance, if \( z' = \bar{z} \); other sufficient conditions are given in Corollary 1.9 below. The definition of the \( z \)-measures \( M_{z, z', \xi} \) was given in Borodin–Olshanski [BO2]; see also [BO3]. It is a modification of a construction due to Kerov–Olshanski–Vershik [KOV1], [KOV2]. They \( z \)-measures enter a larger class of Schur measures as defined by Okounkov [Ok2].

Example 1.5. Assume \( z = k, z' = l \), where \( k, l \) are natural numbers. Then \( (z)_{\lambda} \) vanishes unless \( \ell(\lambda) \leq k \), and likewise \( (z')_{\lambda} \) vanishes unless \( \ell(\lambda) \leq l \). If \( \ell(\lambda) \leq \min(k, l) \) then both \( (z)_{\lambda} \) and \( (z')_{\lambda} \) are strictly positive. It follows that \( M_{k, l, \xi} \) is a measure supported by diagrams \( \lambda \) with at most \( \min(k, l) \) rows. This \( z \)-measure can be obtained by the following construction.

Let \( S(\mathbb{C}^k \otimes \mathbb{C}^l) \) be the symmetric algebra of the vector space \( \mathbb{C}^k \otimes \mathbb{C}^l \). This is a graded space. Let \( A_{\xi} \) be the operator in \( S(\mathbb{C}^k \otimes \mathbb{C}^l) \) taking value \( \xi^n \) on the \( n \)th homogeneous component. On the other hand, as a bi–module over \( GL(k, \mathbb{C}) \times GL(l, \mathbb{C}) \), the space \( S(\mathbb{C}^k \otimes \mathbb{C}^l) \) is the multiplicity free direct sum of irreducible bi–modules of the form \( V_{\lambda, k} \otimes V_{\lambda, l} \), where \( \lambda \) ranges over the set of Young diagrams.
with \( \ell(\lambda) \leq \min(k, l) \) and \( V_{\lambda,k} \) denotes the irreducible polynomial \( GL(k, \mathbb{C}) \)-module indexed by \( \lambda \). Given \( \lambda \) with \( \ell(\lambda) \leq \min(k, l) \), denote by \( I_\lambda \) the projection onto the component \( V_{\lambda,k} \otimes V_{\lambda,l} \). Then we have

\[
M_{k,l,\xi}(\lambda) = \frac{\text{tr}(A_{\xi}I_\lambda)}{\text{tr} A_{\xi}}, \quad \ell(\lambda) \leq \min(k, l).
\]

A closely related interpretation is as follows. Consider the set \( \text{Mat}(k, l; \mathbb{Z}_+) \) of \( k \times l \) matrices with entries in \( \mathbb{Z}_+ \). The Robinson–Schensted–Knuth algorithm (RSK, for short) determines a projection of \( \text{Mat}(k, l; \mathbb{Z}_+) \) onto the set of Young diagram with at most \( \min(k, l) \) rows (see e.g. Sagan [Sa, Theorem 4.8.2]). Let \( M_{k,l,\xi} \) be the probability measure on \( \text{Mat}(k, l; \mathbb{Z}_+) \) defined by the condition that the matrix entries are independent random variables distributed according to the geometric distribution with parameter \( \xi \). Then the push–forward of \( M_{k,l,\xi} \) under RSK is \( M_{k,l,\xi} \).

Asymptotics of the first part of random partitions distributed according to \( M_{k,l,\xi} \), has been thoroughly studied by Johansson [J1].

**Example 1.6.** Once again, let \( k, l \) be two natural numbers, and take \( z = k, z' = -l \). Then \((z)_\lambda(z')_\lambda\) vanishes unless \( \ell(\lambda) \leq k \) and \( \ell(\lambda') \leq l \), that is, \( \lambda \) must be contained in the rectangular shape of size \( k \times l \). If this condition is satisfied then the sign of \((z)_\lambda(z')_\lambda\) equals \((-1)^{|\lambda|}\). Assume now that \( \xi < 0 \) (we temporarily abandon the restriction \( \xi \in (0, 1) \)). Then the factor \( \xi^{|\lambda|} \) in Definition 1.4 will compensate the oscillation of the sign of \((z)_\lambda(z')_\lambda\), and we again obtain a probability measure, \( M_{k,-l,\xi} \). Note that it is supported by a finite set of Young diagrams.

Both interpretations of the measure \( M_{k,l,\xi} \) given in Example 1.5 can be extended to the measure \( M_{k,-l,\xi} \), with suitable modifications. Namely, the symmetric algebra \( S(\mathbb{C}^k \otimes \mathbb{C}^l) \) is replaced by the exterior algebra \( \wedge(\mathbb{C}^k \otimes \mathbb{C}^l) \). Let \( A_{\xi}' \) be the operator in this graded space taking value \((-\xi)^n\) on the \( n \)th homogeneous component. The exterior algebra decomposes into irreducible bi–modules of the form \( V_{\lambda,k} \otimes V_{\lambda',l} \). Let \( I_\lambda \) denote the projection onto \( V_{\lambda,k} \otimes V_{\lambda',l} \). Then

\[
M_{k,-l,\xi}(\lambda) = \frac{\text{tr}(A_{\xi}'I_\lambda)}{\text{tr} A_{\xi}'}, \quad \ell(\lambda) \leq k, \quad \ell(\lambda') \leq l.
\]

Further, consider the (finite) set \( \text{Mat}(k, l; \{0, 1\}) \) of \( k \times l \) matrices with entries in \{0, 1\}. We equip \( \text{Mat}(k, l; \{0, 1\}) \) with the probability measure \( \tilde{M}_{k,-l,\xi} \) such that the matrix entries are independent and identically distributed according to

\[
\text{Prob}(0) = \frac{1}{1 + |\xi|}, \quad \text{Prob}(1) = \frac{|\xi|}{1 + |\xi|}.
\]

Instead of the Robinson–Schensted–Knuth algorithm we apply its dual version (dRSK), see [Sa, Theorem 4.8.5]. Taking the push–forward of \( M_{k,-l,\xi} \) with respect to dRSK we obtain \( M_{k,-l,\xi} \).

Asymptotics of the first part of random partitions distributed according to \( M_{k,-l,\xi} \), has been thoroughly studied by Gravner–Tracy–Widom [GTW].
Example 1.7. Let the parameters $z, z', \xi$ vary in such a way that
\[ |z| \to \infty, \quad |z'| \to \infty, \quad \xi \to 0, \quad zz'\xi \to \theta > 0. \]
Then we obtain in the limit the \textit{poissonized Plancherel measure} with parameter $\theta$,
\[ M_\theta(\lambda) = e^{-\theta} \left( \frac{\dim \lambda}{|\lambda|!} \right)^2 \theta^{|\lambda|}. \]
Asymptotics of the Plancherel measure has been studied by many authors, see e.g. \[LS\], \[VK1\], \[VK2\], \[BDJ\], \[BOO\], \[J2\], \[Ok3\] and references therein.

Proposition 1.8. Let $z, z'$ be nonzero complex numbers. The quantity $(z)_\lambda(z')_\lambda$ is nonnegative for any $\lambda \in \mathbb{Y}$ if and only if one of the following three conditions holds:

(i) The numbers $z, z'$ are not real and are conjugate to each other.

(ii) Both $z, z'$ are real and are contained in the same open interval of the form $(m, m + 1)$, where $m \in \mathbb{Z}$.

(iii) One of the numbers $z, z'$ (say, $z$) is a nonzero integer while $z'$ has the same sign and, moreover, $|z'| > |z| - 1$.

Proof. Consider two cases: (1) both $z, z'$ are not integers; (2) at least one of $z, z'$ is an integer.

(1) In this case, the quantity $(z)_\lambda(z')_\lambda$ does not vanish. It is strictly positive for all $\lambda \in \mathbb{Y}$ if and only if $(z + k)(z' + k) > 0$ for any integer $k$, which is equivalent to $(z, z')$ satisfying (i) or (ii).

(2) Without loss of generality we may assume that either $z$ is an integer and $z'$ is not, or both $z, z'$ are integers and $|z| \leq |z'|$. Next, by virtue of the symmetry $(z)_\lambda(z')_\lambda = (-z)_\lambda(-z')_\lambda$, we may assume $z = m = 1, 2, \ldots$ (note that $z = 0$ is excluded by the hypothesis). Then $(z)_\lambda$ vanishes if $\ell(\lambda) > m$, and is strictly positive if $\ell(\lambda) \leq m$. Therefore, the quantity $(z)_\lambda(z')_\lambda$ is nonnegative for all $\lambda$ if and only if $(z')_\lambda \geq 0$ for all $\lambda$ with $\ell(\lambda) \leq m$, which means that $z'$ must be a real number $> m - 1$. (Note that $z' \neq m - 1$ because of the assumption $|z| \leq |z'|$.) $\square$

Corollary 1.9. Let $z, z'$ satisfy one of the conditions (i), (ii), (iii) of Proposition 1.8, and let $0 < \xi < 1$. Then the $z$–measure $M_{z, z', \xi}$ with parameters $z, z', \xi$ is well defined as a probability measure.

Notice the symmetry relation
\[ M_{z, z', \xi}(\lambda) = M_{-z, -z', \xi}(\lambda'). \]
Henceforth we assume that the parameter $\xi$ belongs to the open interval $(0, 1)$.

2. The Hypergeometric and Gamma Kernels (First Form)

Let $Z'$ denote the lattice of proper half–integers:
\[ Z' = \mathbb{Z} + \frac{1}{2} = \{ \pm \frac{1}{2}, \pm \frac{3}{2}, \pm \frac{5}{2}, \ldots \}. \]
Consider the space of all subsets $X \subset Z'$. By assigning to any $X \subset Z'$ its characteristic function we identify that space with the space $\{0, 1\}^Z$ of all doubly infinite binary sequences indexed by elements of the lattice $Z'$:
\[ (\ldots, a_{-3/2}, a_{-1/2} | a_{1/2}, a_{3/2}, \ldots), \quad a_x \in \{0, 1\} \quad \forall x \in Z'. \]
We endow the space \( \{0,1\}^Z \) with the product topology, which makes it a compact topological space.

To any diagram \( \lambda \in Y \) we assign a subset \( \lambda(\lambda) \subset Z' \),
\[
\lambda(\lambda) = \{ \lambda_i - i + \frac{1}{2} \mid i = 1, 2, \ldots \},
\]
which we identify with the corresponding binary sequence \((a_z(\lambda))_{x \in Z'}\) (so that \(a_x = 1\) if \(x\) equals \(\lambda_i - i + \frac{1}{2}\) for some \(i\), and \(a_x = 0\) otherwise). Thus, we obtain an embedding \(Y \hookrightarrow \{0,1\}^Z\). For instance, the empty diagram turns into the binary sequence \((\ldots 111 | 000 \ldots)\), and the diagram \(\lambda = (3,1) \in Y_3\) turns into the binary sequence \((\ldots 11101 | 00100 \ldots)\). The binary sequence \((a_z(\lambda))_{x \in Z'}\) has a simple geometric meaning: given \(k = 1, 2, \ldots\), the digit \(a_{z,k-1/2}\) is 1 or 0 depending on whether the \(k\)th segment of the boundary of \(\lambda\) above/below the diagonal is vertical or horizontal.

Note that image of \(Y\) is dense in \(\{0,1\}^Z\), so that \(\{0,1\}^Z\) is a compactification of the discrete space \(Y\).

**Definition 2.1.** Let \(P\) be an arbitrary probability measure on the compact space \(\{0,1\}^Z\). The \(m\)-point correlation function \((m = 1, 2, \ldots)\) of \(P\), denoted as \(\rho_m(\cdot \mid P)\), is defined on \(m\)-point subsets \(X = \{x_1, \ldots, x_m\} \subset Z'\). The value \(\rho_m(x_1, \ldots, x_m \mid P)\) at \(X\) is the probability that the random (with respect to \(P\)) set contains \(X\). Equivalently, this is the probability that the random (with respect to \(P\)) binary sequence has 1’s at the positions \(x_1, \ldots, x_m\). □

Notice that \(P\) is uniquely determined by its correlation functions. Indeed, using the inclusion/exclusion principle we can compute the \(P\)-measure of any cylinder set of the form \(\{a_x \mid a_{y_1} = \varepsilon_1, \ldots, a_{y_m} = \varepsilon_m\}\) with arbitrary \(y_1, \ldots, y_m \in Z'\) and \(\varepsilon_1, \ldots, \varepsilon_m = 0, 1\).

It turns out that the correlation functions of the \(z\)-measures can be explicitly computed.

**Theorem 2.2.** Assume that both \(z, z'\) are not integers. That is, \((z, z')\) is subject to one of the conditions (i), (ii) of Proposition 1.8, but not to the condition (iii). Let \(P_{z, z', \xi}\) be the push-forward of the \(z\)-measure \(M_{z, z', \xi}\) under the embedding \(\lambda \mapsto \lambda(\lambda)\) of the discrete space \(Y\) into the compact space \(\{0,1\}^Z\).

The correlation functions of \(P_{z, z', \xi}\), as defined in Definition 2.1, have determinantal form
\[
\rho_m(x_1, \ldots, x_m \mid P_{z, z', \xi}) = \det_{1 \leq i, j \leq m} [K(x_i, x_j \mid z, z', \xi)],
\]
where \(K(x, y \mid z, z', \xi)\) is a function on \(Z' \times Z'\) not depending on \(m\). Specifically,
\[
K(x, y \mid z, z', \xi) = \frac{P(x)Q(y) - Q(x)P(y)}{x - y},
\]
with
\[
P(x) = P(x \mid z, z', \xi) = (zz')^{1/4}e^{x/2}(1 - \xi)(z + z')^{1/2} \times \left(\frac{\Gamma(z + x + 1/2)\Gamma(z' + x + 1/2)}{\Gamma(z + 1)\Gamma(z' + x + 1)}\right)^{1/2} \cdot \frac{F(-z, -z'; x + 1/2; \xi; \xi)}{F(x + 1/2)}.
\]
\[ Q(x) = Q(x \mid z, z', \xi) = (zz')^{3/4}e^{(x+1)/2(1-\xi)(z+z')/2-1} \times \left( \frac{\Gamma(z+x+\frac{1}{2})\Gamma(z'+x+\frac{1}{2})}{\Gamma(z+1+1)} \right)^{1/2} \cdot F(-z+1,-z'+1;x+\frac{3}{2};\xi^{-1}) / \Gamma(x+\frac{3}{2}), \]

where \( F(a,b;c;w) \) stands for the Gauss hypergeometric function.

**Comments.** 1. The ratio \( F(a,b;c;w) / \Gamma(c) \) is an entire function in the parameter \( c \), see Erdelyi [Er1, 2.1.6]. Next, under our assumptions on the parameters \( z, z' \),
\[
\frac{\Gamma(z+x+\frac{1}{2})\Gamma(z'+x+\frac{1}{2})}{\Gamma(z+1+1)} > 0.
\]

This implies that \( P(x) \) and \( Q(x) \) are well defined on the whole lattice \( \mathbb{Z}' \).

2. Moreover, the expressions of the functions \( P(x), Q(x) \) are also well defined in a neighborhood of \( \mathbb{Z}' \) in \( \mathbb{C} \), and these are analytic functions. This makes it possible to define the value of ratio \( (P(x)Q(y) - Q(x)P(y))/x - y \) on the diagonal \( x = y \), by making use of the l'Hospital rule.

3. Notice that
\[
Q(x \mid z, z', \xi) = \left( \frac{zz'}{(z-1)(z'-1)} \right)^{1/4} P(x+1 \mid z-1, z'-1, \xi).
\]

4. We call \( K(x,y \mid z, z', \xi) \) the discrete hypergeometric kernel.

**Proof of Theorem 2.2.** As is shown below (Corollary 4.3), Theorem 2.2 is equivalent to Theorem 3.2, and the latter theorem was proved in Borodin–Olshanski [BO2]. On the other hand, Theorem 2.2 can be proved directly, see Okounkov [Ok1] and Borodin–Okounkov [BOk, Example 3].

Recall that the parameter \( \xi \) of the \( z \)-measure ranges over the open interval \( (0, 1) \). What happens when \( \xi \) tends to one of the endpoints 0, 1? From the definition of the \( z \)-measures it easily follows that as \( \xi \) tends to 0, the \( z \)-measure tends to the Dirac measure at \( \emptyset \), while the limit as \( \xi \) tends to 1 is the zero measure:
\[
\lim_{\xi \to 1} M_{z,z',\xi}(\lambda) = 0, \quad \forall \lambda \in \mathbb{Y}.
\]

However, the \( \xi \not\to 1 \) limit becomes nontrivial when instead of \( \mathbb{Y} \) we take its compactification \( \{0, 1\}^{2'} \supset \mathbb{Y} \).

**Theorem 2.3.** Let \( (z, z') \) and \( P_{z,z',\xi} \) be as in Theorem 2.2. As \( \xi \not\to 1 \), the measures \( P_{z,z',\xi} \) weakly converge to a probability measure \( P^{\text{gamma}}_{z,z'} \) on \( \{0, 1\}^{2'} \). The correlation functions of the limit measure have determinantal form,
\[
\rho_m(x_1, \ldots, x_m \mid P^{\text{gamma}}_{z,z'}) = \det \left[ K^{\text{gamma}}_{z,z'}(x_i, x_j \mid z, z') \right],
\]

where \( K^{\text{gamma}}(x, y \mid z, z', \xi) \) is a function on \( \mathbb{Z}' \times \mathbb{Z}' \) not depending on \( m \). Specifically,
\[
K^{\text{gamma}}(x, y \mid z, z') = \frac{\sin(\pi z) \sin(\pi z')}{\pi \sin(\pi z - z')} \times \left\{ \frac{\Gamma(z+x+\frac{1}{2})\Gamma(z'+x+\frac{1}{2})\Gamma(z+y+\frac{1}{2})\Gamma(z'+y+\frac{1}{2})}{\Gamma(z+x+\frac{3}{2})\Gamma(z'+y+\frac{3}{2})} \right\}^{-1/2} \times \frac{\Gamma(z+x+y+\frac{1}{2})}{x - y}.
\]
**Comments.**  1. The expression in the curved brackets is strictly positive because of our assumptions on the parameters $z, z'$.

2. If $z = z'$, which is only possible when $z = z' \in \mathbb{R} \setminus \mathbb{Z}$, then the above expression takes a simpler form

$$K_{\psi}(x, y \mid z) = \left( \frac{\sin(\pi z)}{\pi} \right)^2 \frac{\psi(z + x + \frac{1}{2}) - \psi(z + y + \frac{1}{2})}{x - y},$$

where $\psi(u) = \Gamma'(u)/\Gamma(u)$ is the logarithmic derivative of the $\Gamma$–function.

3. On the diagonal $x = y$ we have

$$K_{\gamma}(x, x \mid z, z') = \frac{\sin(\pi z) \sin(\pi z')}{\pi \sin(\pi (z - z'))} (\psi(z + x + \frac{1}{2}) - \psi(z' + x + \frac{1}{2}))$$

$$K_{\psi}(x, y \mid z) \bigg|_{x=y} = \left( \frac{\sin(\pi z)}{\pi} \right)^2 \psi'(z + x + \frac{1}{2}).$$

4. We call $K_{\gamma}(x, y \mid z, z')$ and $K_{\psi}(x, y \mid z)$ the gamma kernel and the psi kernel, respectively.

**Proof of Theorem 2.3.** We will show that the discrete hypergeometric kernel $K(x, y \mid z, z', \xi)$ of Theorem 2.2 has a pointwise limit as $\xi \nearrow 1$, and the result is the gamma kernel. This will imply Theorem 2.3. (Notice, however, that the functions $P(x \mid z, z', \xi)$ and $Q(x \mid z, z', \xi)$, in general, do not have limits as $\xi \nearrow 1$.)

We use the formula (see Erdelyi [Er1, 2.1.4 (17)])

$$\frac{1}{\Gamma(c)} F(a, b; c; w) = \frac{\Gamma(b - a)(-w)^{-a}}{\Gamma(b)\Gamma(c - a)} F(a, 1 - c + a; 1 - b + a; w^{-1}) + \frac{\Gamma(a - b)(-w)^{-b}}{\Gamma(a)\Gamma(c - b)} F(b, 1 - c + b; 1 - a + b; w^{-1}), \quad w \in \mathbb{C} \setminus [0, +\infty)$$

For fixed $a, b, c$ and large negative $w$, we write

$$F(a, 1-c+a; 1-b+a; w^{-1}) = 1+O(w^{-1}), \quad F(b, 1-c+b; 1-a+b; w^{-1}) = 1+O(w^{-1}),$$

which gives

$$\frac{1}{\Gamma(c)} F(a, b; c; w) = \frac{\Gamma(b - a)(-w)^{-a}}{\Gamma(b)\Gamma(c - a)} (1 + O(w^{-1})) + \frac{\Gamma(a - b)(-w)^{-b}}{\Gamma(a)\Gamma(c - b)} (1 + O(w^{-1})), \quad (2.1)$$

Specializing this simple estimate to

$$a = -z, \quad b = -z', \quad c = x + \frac{1}{2}, \quad w = \frac{\xi}{\xi - 1}$$

and to

$$a = -z + 1, \quad b = -z' + 1, \quad c = x + \frac{3}{2}, \quad w = \frac{\xi}{\xi - 1}$$
we obtain (below we denote by $\delta_1, \delta_2, \ldots$ suitable quantities of the type $1 + O(1 - \xi)$ whose precise form is unessential)

$$P(x \mid z, z', \xi) = (zz')^{1/4} \left( \Gamma(z + x + \frac{1}{2}) \Gamma(z' + x + \frac{1}{2}) \right)^{1/2} \frac{\Gamma(z - z')(1 - \xi)^{(z'-z)/2}}{\Gamma(-z') \Gamma(z + x + \frac{1}{2})} \delta_1 + \frac{\Gamma(z - z)(1 - \xi)^{(z-z)/2}}{\Gamma(-z) \Gamma(z' + x + \frac{1}{2})} \delta_2$$

$$Q(y \mid z, z', \xi) = (zz')^{3/4} \left( \Gamma(z + y + \frac{1}{2}) \Gamma(z' + y + \frac{1}{2}) \right)^{1/2} \frac{\Gamma(z - z')(1 - \xi)^{(z'-z)/2}}{\Gamma(z + 1) \Gamma(z' + 1)} \delta_3 + \frac{\Gamma(z' - z)(1 - \xi)^{(z'-z)/2}}{\Gamma(-z + 1) \Gamma(z' + y + \frac{1}{2})} \delta_4$$

Substituting these expressions into $P(x)Q(y)$ one sees that the term involving the factor $(1 - \xi)^{z'-z}$ or $(1 - \xi)^{z-z'}$ will cancel with the corresponding term in $Q(x)P(y)$, within a quantity of the form $(1 - \xi)^{\pm (z-z')}$ $O(1 - \xi)$. Such a quantity is negligible, because $|\Re(z' - z)| < 1$, as it follows from our assumptions on $z, z'$. Thus, only terms not involving the factors $(1 - \xi)^{\pm (z-z')}$ survive in $P(x)Q(y) - Q(x)P(y)$.

Writing these terms down we get

$$P(x \mid z, z', \xi) Q(y \mid z, z', \xi) - Q(x \mid z, z', \xi) P(y \mid z, z', \xi)$$

$$= \frac{\sin(\pi z) \sin(\pi z')}{\pi \sin(\pi (z - z'))} \left\{ \Gamma(z + x + \frac{1}{2}) \Gamma(z' + x + \frac{1}{2}) \Gamma(z + y + \frac{1}{2}) \Gamma(z' + y + \frac{1}{2}) \right\}^{-1/2} \times \left\{ \Gamma(z + x + \frac{1}{2}) \Gamma(z' + y + \frac{1}{2}) - \Gamma(z' + x + \frac{1}{2}) \Gamma(z + y + \frac{1}{2}) + o(1) \right\}.$$

This proves the claim of the theorem for $x \neq y$ and $z \neq z'$. To remove the restriction $x \neq y$ we remark that the above formula holds not only for $x, y$ on the lattice $Z'$ but also in a suitable neighborhood $U$ of the lattice $Z'$ in $\mathbb{C}$. Moreover, one can prove that the remaining term $o(1)$ admits a uniform bound provided that $x, y$ range over compact subsets in $U \times U$. Thus, as $\xi \to 1$, the left–hand side (which is a holomorphic function in $U \times U$, vanishing on the diagonal $x = y$) converges to the right–hand side with the remaining term (which has the same vanishing property) removed, uniformly on compact sets. This makes it possible to remove the indeterminacy on the diagonal $x = y$ using the L'Hospital rule.

Finally, to handle the case $z = z'$ we apply a similar argument of analytical continuation, using the fact that the expressions for the kernels are locally holomorphic functions in $(z, z')$. ☐

3. The hypergeometric and gamma kernels (second form)

Recall the definition of the Frobenius coordinates of a nonempty diagram $\lambda \in \mathcal{Y}$: these are the integers $p_1 > \cdots > p_d \geq 0$, $q_1 > \cdots > q_d \geq 0$, where $d$ is the number of boxes on the main diagonal of $\lambda$ and

$$p_i = \lambda_i - i, \quad q_i = \lambda'_i - i, \quad i = 1, \ldots, d.$$

Any collection of integers $p_1 > \cdots > p_d \geq 0$, $q_1 > \cdots > q_d \geq 0$ corresponds to a Young diagram. The transposition $\lambda \leftrightarrow \lambda'$ corresponds to interchanging $p_i \leftrightarrow q_i$. 

\[ \begin{array}{c}
\end{array} \]
In terms of Frobenius coordinates, the expression for the z-measure, see Definition 1.4, can be rewritten as follows

\[
M_{z,z',\xi}(\lambda) = \left(1 - \xi\right)^{zz'} \xi^{\left|\lambda\right|} (zz')^d \\
\times \prod_{i=1}^{d} \left((z + 1)p_i(z' + 1)p_i(-z + 1)q_i(-z' + 1)q_i \right) \cdot \left(\frac{\dim \lambda}{|\lambda|!}\right)^2,
\]

where

\[
\dim \lambda = \frac{\prod_{1 \leq i < j \leq d}(p_i - p_j)(q_i - q_j)}{\prod_{1 \leq i \leq d}(p_i + q_i + 1)\prod_{1 \leq i \leq d} p_i!q_i!}
\]

To any diagram \(\lambda \in \mathcal{Y}\) we assign a finite subset \(X(\lambda) \subset \mathbb{Z}'\):

\[
X(\lambda) = X_+(\lambda) \cup X_-(\lambda), \\
X_+(\lambda) = \{\bar{p}_1, \ldots, \bar{p}_d\} \subset \mathbb{Z}'_+,
\]

\[
X_-(\lambda) = \{-\bar{q}_1, \ldots, -\bar{q}_d\} \subset \mathbb{Z}'_-
\]

where

\[
\bar{p}_i = p_i + \frac{1}{2}, \quad \bar{q}_i = q_i + \frac{1}{2}, \quad i = 1, \ldots, d
\]

are the modified Frobenius coordinates of \(\lambda\) and

\[
\mathbb{Z}'_+ = \left\{\ldots, -\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \ldots\right\}, \\
\mathbb{Z}'_- = \left\{\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots\right\}
\]

By convention, \(X(\emptyset) = \emptyset\). Note that \(\lambda\) is uniquely determined by \(X(\lambda)\), so that the correspondence \(\lambda \mapsto X(\lambda)\) is an embedding of \(\mathcal{Y}\) into the space \(\{0,1\}^{\mathbb{Z}'}\).

**Proposition 3.1.** The correspondence \(\lambda \mapsto X(\lambda)\), defined above, and the correspondence \(\lambda \mapsto \bar{X}(\lambda)\), which was defined at the beginning of \(\S 2\), are related to each other as follows. For any \(\lambda \in \mathcal{Y}\),

\[
X(\lambda) = \bar{X}(\lambda) \triangle \mathbb{Z}'_+, \\
\bar{X}(\lambda) = X(\lambda) \triangle \mathbb{Z}'_-
\]

where the symbol \(\triangle\) denotes the symmetric difference of two sets.

**Proof.** This can be proved using a simple geometric argument, cf. Borodin–Olshanski [BO4, \S 4]. Notice that the claim is equivalent to the classical Frobenius lemma, see Macdonald [Ma, Example I.1.15 (a)]. □

In terms of binary sequences, the claim of Proposition 3.1 can be restated as follows. Let \(a \mapsto a^\circ\) denote the involutive homeomorphism of the space \(\{0,1\}^{\mathbb{Z}'}\) which applies the transposition \(0 \leftrightarrow 1\) to all digits indexed by negative semi-integers. Then we have \(X(\lambda) = (\bar{X}(\lambda))^\circ\), \(\bar{X}(\lambda) = (X(\lambda))^\circ\).

Let \(P_{z,z',\xi}\) be the push–forward of the measure \(M_{z,z',\xi}\) under the embedding \(\mathcal{Y} \hookrightarrow \{0,1\}^{\mathbb{Z}'}\) defined by the correspondence \(\lambda \mapsto X(\lambda)\). Then, by Proposition 3.1, \(P_{z,z',\xi}\) coincides with image of the measure \(\bar{P}_{z,z',\xi}\) under the involution \(a \mapsto a^\circ\). We aim to write down the correlation functions of \(P_{z,z',\xi}\).

\(^3\)This claim was exploited in Borodin–Okounkov–Olshanski [BOO, (1.2)]. In that paper, \(X(\lambda)\) and \(X(\lambda)\) were denoted as \(\mathcal{D}(\lambda)\) and \(\mathcal{F}(\lambda)\), respectively.
Theorem 3.2. Let \((z, z')\) be as in Theorem 2.2. The correlation functions of the measure \(P_{z, z', \xi}\) have determinantal form
\[
\rho_m(x_1, \ldots, x_m \mid P_{z, z', \xi}) = \det_{1 \leq i, j \leq m} [K(x_i, x_j)], \quad m = 1, 2, \ldots,
\]
where the kernel
\[
K(x, y) = K(x, y \mid z, z', \xi)
\]
on \(\mathbb{Z}' \times \mathbb{Z}'\) is defined by the following formulas depending on the sign of \(x\) and \(y\).

- For \(x > 0, y > 0\):
  \[
  \frac{P(x \mid z, z', \xi) Q(y \mid z, z', \xi) - Q(x \mid z, z', \xi) P(y \mid z, z', \xi)}{x - y}.
  \]

- For \(x > 0, y > 0\):
  \[
  \frac{P(x \mid z, z', \xi) P(-y \mid -z, -z', \xi) + Q(x \mid z, z', \xi) Q(-y \mid -z, -z', \xi)}{x - y}.
  \]

- For \(x < 0, y > 0\):
  \[
  \frac{P(-x \mid -z, -z', \xi) P(y \mid z, z', \xi) + Q(-x \mid -z, -z', \xi) Q(y \mid z, z', \xi)}{-x - y}.
  \]

- For \(x < 0, y > 0\):
  \[
  \frac{P(-x \mid -z, -z', \xi) Q(-y \mid -z, -z', \xi) - Q(-x \mid -z, -z', \xi) P(-y \mid -z, -z', \xi)}{-x - y}.
  \]

Here \(P\) and \(Q\) are the functions introduced in Theorem 2.2.

Comments. 1. Notice that \(K(y, x) = \text{sgn}(x) \text{sgn}(y) K(x, y)\).

2. The indeterminacy \(0/0\) on the diagonal \(x = y\) is removed by making use of the l'Hospital rule.

Proof. See Borodin–Olshanski [BO2, Theorem 3.3]. □

Actually, Theorem 3.3 in [BO2] contains a stronger claim (see Theorem 3.4 below). In order to state it, we introduce a kernel \(A\) on \(\mathbb{Z}'_+ \times \mathbb{Z}'_+\) by
\[
A(x, y \mid z, z', \xi) = \frac{\xi^{(x-y)/2} \sqrt{\sin(\pi z) \sin(\pi z')}}{\pi} \times \frac{\Gamma(z + x + \frac{1}{2}) \Gamma(z' + x + \frac{1}{2})}{\Gamma(x + \frac{1}{2})} \cdot \frac{\sqrt{\Gamma(z - y + \frac{1}{2}) \Gamma(-z' - y + \frac{1}{2})}}{\Gamma(-y + \frac{1}{2})}, \quad x, y \in \mathbb{Z}'_+.
\]
Proposition 3.3. The kernel $A(x, y \mid z, z', \xi)$ is of trace class, i.e., the corresponding operator $l^2(Z'_-) \to l^2(Z'_+)$ is of trace class.

Proof. First of all, note that the denominator $x - y$ does not vanish, because $x - y \geq 1$. Observe that

$$\sum_{x \in Z'_+, y \in Z'_-} |A(x, y \mid z, z', \xi)| < \infty.$$  

Indeed, in the expression for the kernel, the ratios of gamma factors have at most polynomial growth,

$$\frac{\sqrt{\Gamma(z + x + \frac{1}{2})\Gamma(z' + x + \frac{1}{2})}}{\Gamma(x + \frac{1}{2})} \sim x^{(z+z')/2}, \quad x \to +\infty$$  

$$\frac{\sqrt{\Gamma(-z - y + \frac{1}{2})\Gamma(-z' - y + \frac{1}{2})}}{\Gamma(-y + \frac{1}{2})} \sim |y|^{-(z+z')/2}, \quad y \to -\infty$$  

while $\xi^{(x-y)/2} = \xi^{(x+|y|)/2}$ has an exponential decay.

Now the claim follows from a well–known sufficient condition: an infinite matrix $A = [A_{ij}]$ is of trace class if the sum $\sum |A_{ij}|$ is finite. Here is a simple argument that justifies the sufficiency.

It is enough to show that $\|A\|_1 \leq \sum |A_{ij}|$, where $\|A\|_1$ is the trace norm. We have

$$\|A\|_1 = \sup_B |\text{tr}(AB)|,$$

where $B$ ranges over the set of all (say, finite–dimensional) matrices with $\|B\| \leq 1$, and $\|B\|$ is the ordinary norm. But

$$|\text{tr}(AB)| = \left| \sum_{i,j} A_{ij} B_{ji} \right| \leq \sum_{i,j} |A_{ij}| \cdot |B_{ji}| \leq \sum_{i,j} |A_{ij}|,$$

because $\|B\| \leq 1$ implies $|B_{ji}| \leq 1$. \qed

Next, introduce a kernel $L$ on $Z' \times Z'$ by

$$L(x, y \mid z, z', \xi) = \begin{cases} 0, & x > 0, \ y > 0 \\ A(x, y \mid z, z', \xi), & x > 0, \ y < 0 \\ -A(y, x \mid z, z', \xi), & x < 0, \ y > 0 \\ 0, & x < 0, \ y < 0. \end{cases} \quad (3.2)$$

Let $L$ be the operator in the Hilbert space $l^2(Z')$ defined by this kernel. By Proposition 3.3, $L$ is of trace class, so that $\det(1 + L)$ makes sense.

It is readily checked ([BO2, Proposition 3.1]) that

$$M_{z, z', \xi}(\lambda) = \frac{\det_{x, y \in X(\lambda)} [L(x, y \mid z, z', \xi)]}{\det(1 + L)}, \quad \lambda \in \mathbb{Y}.$$  

By a general claim (see [BO2, §2]), this implies that

$$\rho_m(x_1, \ldots, x_m \mid P_{z, z', \xi}) = \det_{1 \leq i, j \leq m} \left[ \frac{L}{1 + L}(x_i, x_j) \right].$$
Theorem 3.4. Let \( L \) be the operator in \( \ell^2(\mathbb{Z}') \) with kernel \( L(x, y \mid z, z', \xi) \). The kernel \( K(x, y \mid z, z', \xi) \) is precisely the matrix of the operator \( \frac{L}{1 + L} \).

Proof. See Borodin–Olshanski [BO2, Theorem 3.3]. \( \square \)

The next claim is a counterpart of Theorem 2.3.

Theorem 3.5. As \( \xi \nearrow 1 \), the measures \( P_{z, z', \xi} \) weakly converge to a probability measure \( P_{z, z'}^{\text{gamma}} \) on \( \{0, 1\}^{\mathbb{Z}'} \). The correlation functions of the limit measure have determinantal form,

\[
\rho_m(x_1, \ldots, x_m \mid P_{z, z'}^{\text{gamma}}) = \det_{1 \leq i, j \leq m} [K_{z, z'}^{\text{gamma}}(x_i, x_j \mid z, z')],
\]

where the kernel \( K_{z, z'}^{\text{gamma}}(x, y \mid z, z') \) on \( \mathbb{Z}' \times \mathbb{Z}' \), which is equal to the pointwise limit of the kernel \( K(x, y \mid z, z', \xi) \) as \( \xi \nearrow 1 \), is given by the following formulas depending on the signs of the arguments \( x, y \).

- For \( x > 0, y > 0 \), the kernel is given by same expression as in Theorem 2.3:

\[
\frac{\sin(\pi z) \sin(\pi z')}{\pi \sin(\pi (z - z'))} \left\{ \Gamma(z + x + \frac{1}{2}) \Gamma(z' + x + \frac{1}{2}) \Gamma(z + y + \frac{1}{2}) \Gamma(z' + y + \frac{1}{2}) \right\}^{-1/2} \\
\times \frac{\Gamma(z + x + \frac{1}{2}) \Gamma(z' + y + \frac{1}{2}) - \Gamma(z' + x + \frac{1}{2}) \Gamma(z + y + \frac{1}{2})}{x - y}
\]

- For \( x > 0, y < 0 \):

\[
\frac{\sqrt{\sin(\pi z) \sin(\pi z')}}{\pi \sin(\pi (z - z'))} \left\{ \Gamma(z + x + \frac{1}{2}) \Gamma(z' + x + \frac{1}{2}) \Gamma(-z - y + \frac{1}{2}) \Gamma(-z' - y + \frac{1}{2}) \right\}^{-1/2} \\
\times \frac{\sin(\pi z) \Gamma(z + x + \frac{1}{2}) \Gamma(-z - y + \frac{1}{2}) - \sin(\pi z') \Gamma(z' + x + \frac{1}{2}) \Gamma(-z' - y + \frac{1}{2})}{x - y}
\]

- For \( x < 0, y > 0 \):

\[
\frac{\sqrt{\sin(\pi z) \sin(\pi z')}}{\pi \sin(\pi (z - z'))} \left\{ \Gamma(-z - x + \frac{1}{2}) \Gamma(-z' - x + \frac{1}{2}) \Gamma(z + y + \frac{1}{2}) \Gamma(z' + y + \frac{1}{2}) \right\}^{-1/2} \\
\times \frac{\sin(\pi z) \Gamma(-z - x + \frac{1}{2}) \Gamma(z + y + \frac{1}{2}) - \sin(\pi z') \Gamma(-z' - x + \frac{1}{2}) \Gamma(z' + y + \frac{1}{2})}{x - y}
\]

- For \( x < 0, y < 0 \):

\[
\frac{\sin(\pi z) \sin(\pi z')}{\pi \sin(\pi (z - z'))} \left\{ \Gamma(-z - x + \frac{1}{2}) \Gamma(-z' - x + \frac{1}{2}) \Gamma(-z - y + \frac{1}{2}) \Gamma(-z' - y + \frac{1}{2}) \right\}^{-1/2} \\
\times \frac{\Gamma(-z - x + \frac{1}{2}) \Gamma(-z' - y + \frac{1}{2}) - \Gamma(-z' - x + \frac{1}{2}) \Gamma(-z - y + \frac{1}{2})}{x - y}
\]

Proof. The case \( x, y > 0 \) was proved in Theorem 2.3. This immediately implies the case \( x, y < 0 \), because of an obvious symmetry of the formulas of Theorem 3.2 (changing the signs of \( x, y \) is equivalent to changing the signs of \( z, z' \)). In the remaining two cases we argue just as in the proof of Theorem 2.3. \( \square \)
4. The relation between two forms of kernels

Our next goal is to describe a relation between the two types of the discrete hypergeometric kernel \( K(x, y \mid z, z', \xi) \) and \( K(x, y \mid z, z, \xi) \) and, similarly, between the two types of the gamma kernel \( K_{\text{gamma}}(x, y \mid z, z', \xi) \) and \( K_{\text{gamma}}(x, y \mid z, z') \).

Given an arbitrary kernel \( K(x, y) \) on \( Z' \times Z' \), we assign to it another kernel,

\[
K^\circ(x, y) = \begin{cases} 
K(x, y), & x > 0, \\
\delta_{xy} - K(x, y), & x < 0,
\end{cases}
\]

where \( \delta_{xy} \) is the Kronecker symbol. Slightly more generally, given an arbitrary map \( \varepsilon : Z' \to \mathbb{R}^* \), we set

\[
K^{\circ, \varepsilon}(x, y) = \varepsilon(x)K^\circ(x, y)\varepsilon(y)^{-1}.
\]

**Proposition 4.1.** Let \( P \) be a probability measure on \( \{0, 1\}^Z \) and \( P^{\circ} \) be its image under the involutive homeomorphism \( a \mapsto a^\circ \) of the space \( \{0, 1\}^Z \), introduced after Proposition 3.1. Assume that the correlation functions of \( P \) have determinantal form with a certain kernel \( K(x, y) \),

\[
\rho_m(x_1, \ldots, x_m \mid P) = \det_{1 \leq i,j \leq m} [K(x_i, x_j)], \quad m = 1, 2, \ldots.
\]

Then the correlation functions of the measure \( P^{\circ} \) also have a similar determinantal form, with the kernel \( K^{\circ}(x, y) \) as defined above or, equally well, with the kernel \( K^{\circ, \varepsilon}(x, y) \), where the map \( \varepsilon : Z' \to \mathbb{R}^* \) may be chosen arbitrarily,

\[
\rho_m(x_1, \ldots, x_m \mid P^{\circ}) = \det_{1 \leq i,j \leq m} [K^{\circ}(x_i, x_j)] = \det_{1 \leq i,j \leq m} [K^{\circ, \varepsilon}(x_i, x_j)], \quad m = 1, 2, \ldots.
\]

**Proof.** The factor \( \varepsilon(\cdot) \) does not affect the values of determinants in right–hand side of the above formula, so that we may take \( \varepsilon(\cdot) = 1 \). Then the result is obtained by applying the inclusion/exclusion principle, see Proposition A.8 in Borodin–Okounkov–Olshanski [BOO]. □

**Theorem 4.2.** The kernels \( K(x, y \mid z, z', \xi) \) and \( K(x, y \mid z, z', \xi) \), introduced in §2 and §3, respectively, are related to each other by the transformation \( K \mapsto K^{\circ, \varepsilon} \), where

\[
\varepsilon(x) = \begin{cases} 
1, & x \in Z'_+, \\
(-1)^k, & x = -(k + \frac{1}{2}) \in Z'_-, \quad k = 0, 1, 2, \ldots.
\end{cases}
\]

**Comments.** 1. Since the kernels in question are associated with the measures \( P_{z, z', \xi} \) and \( P_{z, z', \xi} \), which are related to each other by the involution, the claim of the proposition is not surprising, in view of Proposition 4.1. The point is the explicit form of the factor \( \varepsilon(\cdot) \).

2. The claim of the theorem generalizes Lemma 2.5 in Borodin–Okounkov–Olshanski [BOO].

Before giving a proof let us state a corollary.
Corollary 4.3. Theorem 2.3 and Theorem 3.2 are equivalent.

Proof. Indeed, this follows from Proposition 4.1 and Theorem 4.2. □

Proof of Theorem 4.2. (a) Let us check the desired relation between $K(x, y | z, z', \xi)$ and $K(x, y | z, z', \xi)$ for an arbitrary couple $x, y$ outside the diagonal $x = y \in Z'$. The classical transformation formula [Er1, 2.8(19)] implies

$$\frac{1}{\Gamma(c)} F(a, b; c; \frac{\xi}{\xi-1}) \bigg|_{c=-k} = (-1)^{k+1} \xi^{k+1} (1-\xi)^{a+b-1}(a)_{k+1}(b)_{k+1}$$

$$\times \frac{1}{\Gamma(k+2)} F(1-a, 1-b; k+2; \frac{\xi}{\xi-1})$$

for any $a, b \in \mathbb{C}$ and any $k = -1, 0, 1, 2, \ldots$

From this we derive

$$P(x | z, z', \xi) = (-1)^{x-1/2} Q(-x | -z, -z', \xi), \quad x \in Z'_-$$

$$Q(x | z, z', \xi) = (-1)^{x+1/2} P(-x | -z, -z', \xi), \quad x \in Z'_-.$$  (4.1)

This readily implies the relation in question.

(b) Consider now the case $x = y \in Z'$. We have to prove that

$$K(x, x | z, z', \xi) = 1 - K(x, x | z, z', \xi), \quad x \in Z'_-.$$  

First, we will prove that

$$\frac{d}{d\xi} K(x, x | z, z', \xi) = -\frac{d}{d\xi} K(x, x | z, z', \xi), \quad x \in Z'_-.$$  

By virtue of Proposition 4.5 below, this is equivalent to

$$P(x | z, z', \xi) Q(x | z, z', \xi) = -P(-x | -z, -z', \xi) Q(-x | -z, -z', \xi), \quad x \in Z'_-,$$

which in turn follows from formulas (4.1) above.

(c) To conclude the proof it suffices to prove that

$$\lim_{\xi \to 1} K(x, x | z, z', \xi) = \lim_{\xi \to 1} (1 - K(x, x | z, z', \xi)), \quad x \in Z'_-.$$  

By virtue of Theorem 2.3, this means

$$K_{\text{gamma}}(x, x | z, z') + K_{\text{gamma}}(-x, -x | -z, -z') = 1, \quad x \in Z'_-.$$  

Using Comment 3 to Theorem 2.3 we reduce this to

$$\frac{\sin(\pi z) \sin(\pi z')}{\pi \sin(\pi (z - z'))} \left\{ \psi(z + x + \frac{1}{2}) - \psi(z' + x + \frac{1}{2}) \right.$$  

$$\left. - \psi(-z - x + \frac{1}{2}) + \psi(-z' - x + \frac{1}{2}) \right\} = 1,$$

which is verified using a well–known relation for the $\psi$–function [Er1, 1.7.1(8)];

$$\psi(a) - \psi(1 - a) = -\pi \text{ctg}(\pi a).$$  □

The counterpart of Theorem 4.2 for the gamma kernels is
Theorem 4.4. The kernels $K_{\gamma}(x, y \mid z, z')$ and $K_{\gamma}(x, y \mid z, z')$, introduced in §2 and §3, respectively, are related to each other by the transformation $K \mapsto K_{\gamma(x, y \mid z, z')}$, where

$$
\varepsilon(x) = \begin{cases} 
1, & x \in \mathbb{Z}_+ \\
(-1)^k, & x = -(k + \frac{1}{2}) \in \mathbb{Z}_-, \quad k = 0, 1, 2, \ldots 
\end{cases}
$$

Proof. This follows from Theorem 4.2 if we pass to the limit as $\xi \to 1$. On the other hand, this can be readily checked directly, because the crucial step, the coincidence of both kernels for $x = y \in \mathbb{Z}_-$, was already verified in the proof of Theorem 4.2. $\square$

The next result, which we have just used in the proof of Theorem 4.2, is also of independent interest. It is a generalization of the differentiation formula for the discrete Bessel kernel, see Borodin–Okounkov–Olshanski [BOO, (2.11) and below].

Proposition 4.5. We have

$$
\frac{d}{d\xi} K(x, y \mid z, z', \xi) = \frac{1}{2\xi} \left( P(x \mid z, z', \xi)Q(y \mid z, z', \xi) + Q(x \mid z, z', \xi)P(y \mid z, z', \xi) \right).
$$

Proof. This can be directly verified by making use of the differentiation formulas

$$
\frac{d}{d\xi} P(x \mid z, z', \xi) = \left( \frac{x}{2\xi} - \frac{z + z'}{2(1 - \xi)} \right) P(x \mid z, z', \xi) - \frac{(zz')^{1/2}}{\xi^{1/2}(1 - \xi)} Q(x \mid z, z', \xi),
$$

$$
\frac{d}{d\xi} Q(x \mid z, z', \xi) = \left( -\frac{x}{2\xi} + \frac{z + z'}{2(1 - \xi)} \right) Q(x \mid z, z', \xi) + \frac{(zz')^{1/2}}{\xi^{1/2}(1 - \xi)} P(x \mid z, z', \xi).
$$

To check these formulas we use the following differentiation formulas for the Gauss hypergeometric function, which can be derived from [Er1, 2.8 (20), (27)]:

$$
\frac{d}{d\xi} \left( \frac{F(a, b; c; \xi)}{\Gamma(c)} \right) = -\frac{ab}{(1 - \xi)^2} \frac{F(a + 1, b + 1; c + 1; \xi)}{\Gamma(c + 1)}
$$

$$
= \frac{1}{\xi} \left( \frac{a + b - 1}{1 - \xi} + c - 1 \right) \frac{F(a, b; c; \xi)}{\Gamma(c)}. \quad \square
$$

5. The projection property

Let $H = H_+ \oplus H_-$ be a Hilbert space decomposed into a direct sum of two subspaces. According to this decomposition we will write operators in $H$ in $2 \times 2$ block form. Let $A : H_+ \to H_+$ be a bounded operator and let

$$
L = \begin{bmatrix} 0 & A \\ -A^* & 0 \end{bmatrix}.
$$

This is a bounded operator in $H$. Notice that $1 + L$ is invertible. Indeed this follows from the fact that

$$
(1 + L)^*(1 + L) = \begin{bmatrix} 1 + AA^* & 0 \\ 0 & 1 + A^*A \end{bmatrix} \geq \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.
$$
Set $K = L(1 + L)^{-1}$ and write $K$ in the block form,

$$K = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$ 

Next, set

$$K^\circ = \begin{bmatrix} a & b \\ -c & 1 - d \end{bmatrix}, \quad ^\circ K = \begin{bmatrix} 1 - a & -b \\ c & d \end{bmatrix}.$$ 

Proposition 5.1. The operators $K^\circ$ and $^\circ K$ as defined above are orthogonal projections onto the subspaces

$$H^\circ = \{ Ah_ - + h_ - \mid h_ - \in H_ - \}, \quad ^\circ H = \{ h_ + \oplus (-A^*)h_ + \mid h_ + \in H_ + \},$$

which are essentially the graphs of the operators $A$ and $-A^*$, respectively. We have $K^\circ \cdot ^\circ K = ^\circ K \cdot K^\circ = 0$ and $K^\circ + ^\circ K = 1$.

Proof. The latter equality is immediate from the definition of $K^\circ$ and $^\circ K$.

Obviously, $H^\circ$ and $^\circ H$ are closed subspaces, orthogonal to each other. Moreover, as is well known, their sum is the whole $H$. (Indeed, it suffices to check that any $f \in H_+$ can be written as a sum of vectors from $H^\circ$ and $^\circ H$. This means that $h_ - - A^*h_ + = 0$, $Ah_ - + h_ + = f$,

which is reduced to $(1 + AA^*)h_ + = f$. But the latter equation is solvable, because $1 + AA^*$ is invertible.)

Next, one can directly verify that

$$a = (1 + AA^*)^{-1}AA^* = AA^*(1 + AA^*)^{-1}$$

$$b = (1 + AA^*)^{-1}A = A(1 + A^*A)^{-1}$$

$$c = -(1 + A^*A)^{-1}A^* = -A^*(1 + AA^*)^{-1}$$

$$d = (1 + A^*A)^{-1}A^*A = A^*A(1 + A^*A)^{-1}.$$ 

Using these explicit expressions for the blocks $a, b, c, d$ one can readily check that the operator $K^\circ$ is the identity on $H^\circ$ and zero on $^\circ H$. Similarly, the operator $^\circ K$ is the identity on $^\circ H$ and zero on $H^\circ$. This concludes the proof. \square

Remark 5.2. The claim of Proposition 5.1 remains true under weaker assumptions. Namely, $A$ may be an unbounded, closed operator with dense domain. \square

Theorem 5.3. The discrete hypergeometric kernel $K(x, y \mid z, z', \xi)$ on $\mathbb{Z}' \times \mathbb{Z}'$, as defined in \S 2, is a projection kernel. That is, it corresponds to an orthogonal projection operator in the Hilbert space $\ell^2(\mathbb{Z}')$.

Proof. Take $H = \ell^2(\mathbb{Z}')$, $H_+ = \ell^2(\mathbb{Z}'_+)$, $H_ - = \ell^2(\mathbb{Z}'_ -)$, and let $K$ be the operator in $H$ defined by the kernel $K(x, y \mid z, z', \xi)$. By Theorem 3.4, $K = L(1 + L)^{-1}$, where $L$ has the form $\begin{bmatrix} 0 & A \\ -A^* & 0 \end{bmatrix}$ with a certain bounded operator $A$ (recall that the kernel $A(x, y)$ is real, so that the adjoint operator $A^*$ is given by the transposed kernel). Let $\tilde{K}$ be the operator given by the kernel $\tilde{K}(x, y \mid z, z', \xi)$. By Theorem 4.2, $\tilde{K} = \varepsilon K^\circ \varepsilon^{-1}$, where $\varepsilon$ is a diagonal matrix with $\pm 1$'s on the diagonal. By Proposition
5.1, the operator \( K^{0} \) is an orthoprojection. Therefore, \( \mathbf{K} \) is an orthoprojection, too. \( \square \)

We would like to prove a similar claim for the gamma kernel \( K_{\text{gamma}}^{\xi}(x, y \mid z, z') \). By Theorem 2.3, it is the pointwise limit (as \( \xi \searrow 1 \)) of the hypergeometric kernels \( K(x, y \mid z, z', \xi) \), which are projection kernels by virtue of Theorem 5.3. That is, the operator defined by the gamma kernel is a weak limit of orthoprojections. However, the projection property is not stable under limit transitions in the weak operator topology. Indeed, one can obtain any selfadjoint operator with norm \( \leq 1 \) as a weak limit of orthoprojections in an infinite-dimensional Hilbert space. It would be nice to strengthen Theorem 2.3 by proving that the kernels (or rather the corresponding operators) actually converge in the \emph{strong} operator topology: this would suffice to conclude that the limit kernel inherits the projection property. However, to derive the strong convergence directly from the formulas, as we have done for the weak convergence, does not seem to be easy.

Below we present a simple argument, which proves the strong convergence in a roundabout way, under an additional restriction on the parameters \( z, z' \). The idea is to prove an analog of Theorem 3.4. To do this we verify the strong convergence directly from the formulas, as we have done for the weak convergence, does not seem to be easy.

Below we present a simple argument, which proves the strong convergence in a roundabout way, under an additional restriction on the parameters \( z, z' \). The idea is to prove an analog of Theorem 3.4. To do this we verify the strong convergence directly from the formulas, as we have done for the weak convergence, does not seem to be easy.

Below we present a simple argument, which proves the strong convergence in a roundabout way, under an additional restriction on the parameters \( z, z' \). The idea is to prove an analog of Theorem 3.4. To do this we verify the strong convergence directly from the formulas, as we have done for the weak convergence, does not seem to be easy.

Below we present a simple argument, which proves the strong convergence in a roundabout way, under an additional restriction on the parameters \( z, z' \). The idea is to prove an analog of Theorem 3.4. To do this we verify the strong convergence directly from the formulas, as we have done for the weak convergence, does not seem to be easy.

Below we present a simple argument, which proves the strong convergence in a roundabout way, under an additional restriction on the parameters \( z, z' \). The idea is to prove an analog of Theorem 3.4. To do this we verify the strong convergence directly from the formulas, as we have done for the weak convergence, does not seem to be easy.
By virtue of (3.1), in order to prove that this quantity is finite, it suffices to prove that
\[
\sup_{f,g} \left( \sum_{x \in \mathbb{Z}_+} \sum_{y \in \mathbb{Z}_+} \left( \frac{x}{|y|} \right)^{\gamma + \gamma'} \cdot \frac{1}{x + |y|} \cdot |f(x)| \cdot |g(y)| \right) < +\infty.
\]

It is convenient to rewrite this as
\[
\sup_{f,g} \left( \sum_{x \in \mathbb{Z}_+} \sum_{y \in \mathbb{Z}_+} \left( \frac{x}{y} \right)^{\gamma + \gamma'} \cdot \frac{1}{x + y} \cdot |f(x)| \cdot |g(y)| \right) < +\infty.
\]

Here we assume that both \( f \) and \( g \) range over the unit ball of \( \ell^2(\mathbb{Z}_+) \).

Next, we may replace the sums over \( \mathbb{Z}_+ \) by the integrals over \( \mathbb{R}_+ \) with respect to Lebesgue measure (assuming that \( f \) and \( g \) range over the unit ball of \( L^2(\mathbb{R}_+, dx) \)). Indeed, this will only strengthen the claim. The resulting claim is equivalent to the boundedness of the operator in \( L^2(\mathbb{R}_+, dx) \) with the kernel
\[
\left( \frac{x}{y} \right)^{\gamma + \gamma'} \cdot \frac{1}{x + y}.
\]

It is not hard to show that this integral operator is bounded if and only if \( |z + z'| < 1 \), see Olshanski [Ol1]. □

Similarly to (3.2), using the kernel \( A(x, y \mid z, z') \) we construct another kernel on \( \mathbb{Z} \times \mathbb{Z} \) by
\[
L(x, y \mid z, z') = \begin{cases} 
0, & x > 0, y > 0 \\
A(x, y \mid z, z'), & x > 0, y < 0 \\
-A(y, x \mid z, z'), & x < 0, y > 0 \\
0, & x < 0, y < 0.
\end{cases}
\]

The next result is the counterpart of Theorem 3.4.

**Theorem 5.5.** Let \( L : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z}) \) be the operator with kernel \( L(x, y \mid z, z') \). Assume \( |z + z'| < 1 \). Then the kernel \( K_{\text{gamma}}(x, y \mid z, z') \) is precisely the matrix of the operator \( L \).

**Proof.** Let \( L_\xi \) denote the operator with kernel \( L(x, y \mid z, z', \xi) \). We claim that \( L_\xi (1 + L_\xi)^{-1} \) strongly converges to \( L(1 + L)^{-1} \). To check this we use a standard argument. Write the formal identity
\[
(1 + L_\xi)^{-1} - (1 + L)^{-1} = (1 + L_\xi)^{-1}(L - L_\xi)(1 + L)^{-1}.
\]

Since \( (1 + L_\xi)(1 + L_\xi) \geq 1 \) (see the beginning of the section), we have \( \|(1 + L_\xi)^{-1}\| \leq 1 \). Next, the operators \( L_\xi \) are uniformly bounded and \( L_\xi \to L \) strongly: this follows from Proposition 5.4 (here we use the assumption \( |z + z'| < 1 \)). Therefore, the product in the right–hand side strongly converges to 0.

Since the kernel of \( L_\xi (1 + L_\xi)^{-1} \) is \( K(x, y \mid z, z', \xi) \) (Theorem 3.4), the latter kernel strongly converges to the kernel of \( L(1 + L)^{-1} \). On the other hand, we already know (Theorem 3.5) that \( K(x, y \mid z, z', \xi) \) pointwise converges to \( K_{\text{gamma}}(x, y \mid z, z') \). We conclude that \( K_{\text{gamma}}(x, y \mid z, z') \) is the kernel of \( L(1 + L)^{-1} \). □
Theorem 5.6. Assume $|z + z'| < 1$. The gamma kernel $K^\gamma(x, y \mid z, z')$ on $\mathbb{Z}' \times \mathbb{Z}'$, as defined in §2, is a projection kernel.

Proof. We argue precisely as in the proof of Theorem 5.3, with reference to Theorem 4.4 instead of Theorem 4.2. □

We conjecture that the claim of Theorem 5.6 holds without the restriction $|z + z'| < 1$.

6. The tail kernel

Let us study the asymptotics of the process $P^\gamma(x, y \mid z, z')$ near $+\infty$. In order to find a suitable scaling, let us look at the first correlation function (also called the density function). It is given by the value of the correlation kernel on the diagonal, which was written down in Comment 3 to Theorem 2.3 in terms of the psi function $\psi(x)$. Near $+\infty$, the psi function behaves as follows ([Er1, 1.19(7)])

$$\psi(x) = \log x - \frac{1}{2x} + O(x^{-2}).$$

Substituting this into the expression for $K^\gamma(x, x \mid z, z')$ we see that the density function of the process $P^\gamma$ behaves as

$$\frac{(z - z')\sin(\pi z)\sin(\pi z')}{\pi \sin(\pi(z - z'))} \cdot x^{-1}, \quad x \to +\infty. \quad (6.1)$$

This suggests that the scaling should have the form $x = e^{s_0 + s}$, where $s_0 \to +\infty$, because then in the coordinate $s$, the density function will be asymptotically constant. Notice that in the limit transition, the lattice turns into the real line.

All statements of this section are made under the assumption that $(z, z')$ satisfy one of the conditions (i), (ii) of Proposition 1.8.

Proposition 6.1. In the scaling limit $x = e^{s_0 + s}$, where $s_0 \to +\infty$, the correlation functions of $P^\gamma(x, x \mid z, z')$ converge, and the limit functions have determinantal form with the kernel

$$K^\text{tail}(s, t \mid z, z') = \frac{\sin(\pi z)\sin(\pi z')}{\pi \sin(\pi(z - z'))} \cdot \frac{\sinh\left(\frac{1}{2}(z - z')(s - t)\right)}{\sinh\left(\frac{1}{2}(s - t)\right)}, \quad s, t \in \mathbb{R}.$$

Proof. It suffices to examine the limit behavior of the correlation kernel $K^\gamma(x, x \mid z, z')$. Suppose that if we are given a correlation kernel $K(x, y)$ on a state space with reference measure $dx$ then, under a transformation of the state space, we have to look at the transformation of the expression $\sqrt{dxdy} K(x, y)$, rather than of $K(x, y)$ itself. In our situation, $x = \exp(s_0 + s)$, $y = \exp(s_0 + t)$, so that $\sqrt{dxdy} = \sqrt{xy}\sqrt{dsdt}$. Using the well-known asymptotics of the ratio of gamma functions ([Er1, 1.18 (4)]) and the explicit expression of the kernel in question we find that the limit

$$\lim_{s_0 \to +\infty} \left\{ \sqrt{xy} K^\gamma(x, y \mid z, z') \bigg|_{x=\exp(s_0+s), y=\exp(s_0+t)} \right\}$$

exists and equals $K^\text{tail}(s, t \mid z, z')$. □

We call $K^\text{tail}(s, t \mid z, z')$ the tail kernel with parameters $z, z'$. It determines a translationally invariant point process on $\mathbb{R}$. The tail kernel was obtained via a double limit transition: first, from the discrete hypergeometric kernel to the gamma kernel, and next, from the gamma kernel to the tail kernel. The same result can be obtained in one step, as the following proposition shows.
Proposition 6.2. Consider the discrete hypergeometric kernel $K(x, y | z, z', \xi)$ on the lattice $\mathbb{Z}'$, and make the change of variables $x = e^{s_0 + t}$, $y = e^{s_0 + t'}$. Let $s_0 \to +\infty$ and $\xi \nearrow 1$. Moreover, assume that $(1 - \xi)^{-1}$ grows faster than $e^{s_0}$; namely,
\[
e^{s_0} = O\left((1 - \xi)^{-\varepsilon}\right)
\]
where $\varepsilon > 0$ is small enough (it suffices to assume that $\varepsilon$ is smaller than $1 - |\Re(z - z')|$; we recall that $|\Re(z - z')| < 1$, see Proposition 1.8). Then the scaling limit of the kernel $K(x, y | z, z', \xi)$ is the tail kernel.

Proof. First, we slightly revise the proof of Theorem 2.3. Specifically, we cannot apply the trivial estimate (2.1), because the parameter $c = x + \frac{1}{2}$ is no longer constant. Instead of this we use the Euler integral representation of the Gauss hypergeometric function in the form
\[
F(a, 1 - c + a; 1 - b + a; w^{-1}) = \Gamma(1 - b + a) \int_0^1 \frac{u^{a-1} (1 - u)^{-b}}{\Gamma(a) \Gamma(1 - b)} (1 - uw^{-1})^{c+a-1} du
\]
(see [Er1, 2.1.3 (10)]), where, as in Theorem 2.3, $(a, b, c; w)$ is either
\[
(-z, -z', x + \frac{1}{2}; \frac{c}{\xi - 1}) \quad \text{or} \quad (-z + 1, -z' + 1, x + \frac{3}{2}; \frac{c}{\xi - 1}).
\]
(Notice that in [Er1] the Euler integral representation is given under restrictions on the parameters. However, these restrictions are inessential, because, in our notation, the expression $\frac{u^{a-1} (1 - u)^{-b}}{\Gamma(a) \Gamma(1 - b)}$ makes sense as a distribution supported by $[0, 1]$, for any complex $a, b$.)

By our hypothesis, $(c + a - 1)w^{-1} = O\left((1 - \xi)^{1-\varepsilon}\right)$, whence
\[
(1 - uw^{-1})^{c+a-1} = 1 + O\left((1 - \xi)^{1-\varepsilon}\right)
\]
uniformly in $u \in [0, 1]$. This gives
\[
F(a, 1 - c + a; 1 - b + a; w^{-1}) = 1 + O\left((1 - \xi)^{1-\varepsilon}\right)
\]
and likewise
\[
F(b, 1 - c + b; 1 - a + b; w^{-1}) = 1 + O\left((1 - \xi)^{1-\varepsilon}\right).
\]

Then we may continue the argument as in the proof of Theorem 2.3 and use the same simple estimate for ratios of gamma functions as in the proof of Proposition 6.1. □

The gamma kernel and the discrete hypergeometric kernel in the second form (which corresponds to looking at the Frobenius coordinates of Young diagrams) also have tail limits.

The density function of $P_{z, z'}^{\gamma}$ has the asymptotics
\[
\frac{(z - z') \sin(\pi z) \sin(\pi z')}{\pi \sin(\pi(z - z'))} \cdot |x|^{-1}, \quad x \to \pm \infty,
\]
which immediately follows from the asymptotics (6.1) of the density function of $P_{z, z'}$. Indeed, $P_{z, z'}^{\gamma}$ and $P_{z, z'}^{\gamma}$ coincide on $\mathbb{Z}_+^4$, and the change of sign transformation of $P_{z, z'}^{\gamma}$ is equivalent to changing the signs of the parameters $z, z'$. Thus, it makes sense to consider the scaling limit of $P_{z, z'}^{\gamma}$ at both plus and minus infinity.
Proposition 6.3. In the scaling limit \( x = \pm e^{s_0+s} \), where \( s_0 \to +\infty \), the correlation functions of \( F_{\gamma\text{amma}} \) converge, and the limit functions have determinantal form with the kernel given by

\[
\frac{\sin(\pi z) \sin(\pi z')}{\pi \sin(\pi(z-z'))} \frac{\sinh(\frac{1}{2}(z-z')(s-t))}{\sinh(\frac{1}{2}(s-t))}
\]

- For \( x = e^{s_0+s}, y = e^{s_0+t} \), the limit is the same as in Proposition 6.1:

\[
\frac{\sin(\pi z) \sin(\pi z')}{\pi \sin(\pi(z-z'))} \frac{\sinh(\frac{1}{2}(z-z')(s-t))}{\sinh(\frac{1}{2}(s-t))}
\]

- For \( x = e^{s_0+s}, y = -e^{s_0+t} \):

\[
\sqrt{\sin(\pi z) \sin(\pi z')} \cdot \frac{\sin(\pi z) e^{\frac{1}{2}(z-z')(s-t)} - \sin(\pi z') e^{-\frac{1}{2}(z-z')(s-t)}}{2 \cosh(\frac{1}{2}(s-t))}
\]

- For \( x = -e^{s_0+s}, y = e^{s_0+t} \) the kernel is the same as for \( x = e^{s_0+s}, y = e^{s_0+t} \) (the first case above).

We denote the resulting tail kernel in the second form by \( K_{\text{tail}}(s, t \mid z, z') \). It defines a determinantal point process on \( \mathbb{R} \times \mathbb{R} \) which is invariant under simultaneous translations \( (s, t) \to (s + \Delta, t + \Delta), \Delta \in \mathbb{R} \). This kernel appeared for the first time in [Ol1, Proposition 4.1], see also [Bo1, Theorem VII].

Proof of Proposition 6.3. The formulas for \( K_{\text{tail}}(s, t \mid z, z') \) are readily obtained from those for \( K_{\gamma\text{amma}}(x, y \mid z, z') \), see Theorem 3.5, using the standard asymptotics of ratios of gamma–functions, see [Er1, 1.18 (4)]. One also has to keep in mind the transformation of differentials explained in the proof of Proposition 6.1. □

The statement of Proposition 6.2 also carries over.

Proposition 6.4. In the scaling limit \( x = \pm e^{s_0+s} \), where \( s_0 \to +\infty \) with

\[
e^{s_0} = O((1 - \xi)^{-\varepsilon}), \quad 0 < \varepsilon < 1 - |\Re(z - z')|,
\]

the discrete hypergeometric kernel \( K(x, y \mid z, z', \xi) \) converges, as \( \xi \nearrow 1 \), to the tail kernel \( K_{\text{tail}}(s, t \mid z, z') \).

The proof goes along the same lines as that of Proposition 6.2, and we omit it.

Proposition 6.3 and 6.4 prove the convergence of the correlation kernels. In many situations it is simpler to establish the corresponding convergence of \( L \)-kernels (as usual, \( L = K(1-K)^{-1} \), where \( K \) is a correlation kernel). We have already used the convergence of \( L \)-kernels, see Theorems 5.5 and 5.6. The following statement shows that the convergence of the discrete hypergeometric kernel and the gamma kernel to the tail kernel can be seen on the level of the corresponding \( L \)-kernels.

Proposition 6.5. In the scaling limits of Propositions 6.3, 6.4, the kernels \( L(x, y \mid z, z') \) defined at the end of §5, and \( L(x, y \mid z, z', \xi) \) defined by (3.2) converge to a
kernel on $\mathbb{R} \times \mathbb{R}$ given by

- For $x = e^{s_0 + t}$, $y = e^{s_0 + t}$, the kernel is identically equal to 0.
- For $x = e^{s_0 + t}$, $y = -e^{s_0 + t}$:
  \[ \frac{\sin(\pi z) \sin(\pi z')}{\pi} \cdot \frac{e^{\frac{1}{2}(z + z')(s - t)}}{2 \cosh(\frac{1}{2}(s - t))} \]
- For $x = -e^{s_0 + t}$, $y = e^{s_0 + t}$:
  \[ -\frac{\sin(\pi z) \sin(\pi z')}{\pi} \cdot \frac{e^{\frac{1}{2}(z + z')(s - t)}}{2 \cosh(\frac{1}{2}(s - t))} \]
- For $x = -e^{s_0 + t}$, $y = -e^{s_0 + t}$, the kernel is also identically equal to 0.

Proof. Direct computation. $\square$

Let us denote the kernel defined in Proposition 6.5 by $L_{\text{tail}}(s, t)$. It is easy to see that it defines a bounded operator in $L^2(\mathbb{R} \cup \mathbb{R})$ if and only if $|z + z'| < 1$. Similarly to Theorems 3.4 and 5.5, we have the following claim.

**Proposition 6.6** [Ol1, Proposition 4.2]. If $|z + z'| < 1$ then

\[ K_{\text{tail}} = \frac{L_{\text{tail}}}{1 + L_{\text{tail}}}. \]

**Proof.** Let us identify $L^2(\mathbb{R} \cup \mathbb{R})$ with $L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$. Then we may interpret integral operators in this Hilbert space as $2 \times 2$ matrix–valued integral operators on $\mathbb{R}$. Thus, we may write

\[ L_{\text{tail}} = \begin{bmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{bmatrix}, \quad K_{\text{tail}} = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix}, \]

where all the blocks are integral operators in $L^2(\mathbb{R})$,

\[ L_{ij} = L_{ij}(s, t), \quad K_{ij} = K_{ij}(s, t), \quad i, j = 1, 2. \]

Actually, these integral operators are translationally invariant, so that we may write

\[ L_{ij}(s, t) = L_{ij}(s - t), \quad K_{ij}(s, t) = K_{ij}(s - t), \quad i, j = 1, 2. \]

Given a function $f(s)$ on $\mathbb{R}$, let $\hat{f}(u)$ denote its Fourier transform,

\[ \hat{f}(u) = \int_{-\infty}^{+\infty} e^{ius} f(s) ds. \]

The Fourier transform is an isometry between $L^2(\mathbb{R}, ds)$ and $L^2(\mathbb{R}, du)$. By virtue of the translation invariance, the Fourier images of $L_{\text{tail}}$ and $K_{\text{tail}}$ are operators of multiplication by $2 \times 2$ matrix–valued functions in $u$:

\[ \hat{L}_{\text{tail}}(u) = \begin{bmatrix} \hat{L}_{11}(u) & \hat{L}_{12}(u) \\ \hat{L}_{21}(u) & \hat{L}_{22}(u) \end{bmatrix}, \quad \hat{K}_{\text{tail}}(u) = \begin{bmatrix} \hat{K}_{11}(u) & \hat{K}_{12}(u) \\ \hat{K}_{21}(u) & \hat{K}_{22}(u) \end{bmatrix}. \]
From the explicit expressions for $L$ and $K$ (see Propositions 6.3 and 6.5) it follows that their Fourier images have the form

$$
\hat{L}_{\text{tail}}(u) = \begin{bmatrix} 0 & c(u) \\ -c(u) & 0 \end{bmatrix}, \quad \hat{K}_{\text{tail}}(u) = \begin{bmatrix} a(u) & b(u) \\ -b(u) & a(u) \end{bmatrix},
$$

where

$$
c(u) = \left\{ \frac{\sin(\pi z) \sin(\pi z')}{\pi} \cdot \frac{e^{\frac{1}{2}(z+z')s}}{2 \cosh(\frac{s}{2})} \right\}_{s \to u},
$$

$$
a(u) = \left\{ \frac{\sin(\pi z) \sin(\pi z')}{\pi \sin(\pi (z - z'))} \cdot \frac{\sinh\left(\frac{1}{2}(z - z')s\right)}{2 \sinh(\frac{s}{2})} \right\}_{s \to u},
$$

$$
b(u) = \left\{ \frac{\sqrt{\sin(\pi z) \sin(\pi z')}}{\pi} \cdot \frac{\sin(\pi z) e^{\frac{1}{2}(z-z')s} - \sin(\pi z') e^{-\frac{1}{2}(z-z')s}}{2 \cosh(\frac{s}{2})} \right\}_{s \to u}.
$$

The required Fourier images can be evaluated from the tables, see formulas 1.9(14) and 3.2(15) in Erdelyi [Er2],

$$
\left\{ \frac{\sinh\left(\frac{1}{2}(z - z')s\right)}{\sinh(\frac{s}{2})} \right\}_{s \to u} = \frac{2\pi \sin(\pi (z - z'))}{\cos(2\pi i u) + \cos(\pi (z - z'))},
$$

$$
\left\{ \frac{e^{\frac{1}{2}(z\pm z')s}}{2 \cosh(\frac{s}{2})} \right\}_{s \to u} = \frac{\pi}{\cos(\pi i u - \frac{1}{2}\pi (z \pm z'))}.
$$

From these formulas we get explicit expressions

$$
c(u) = \frac{\sqrt{\sin(\pi z) \sin(\pi z')}}{\cos(\pi i u - \frac{1}{2}\pi (z + z'))},
$$

$$
a(u) = \frac{2\sin(\pi z) \sin(\pi z')}{\cos(2\pi i u) + \cos(\pi (z - z'))},
$$

$$
b(u) = 2\sqrt{\sin(\pi z) \sin(\pi z')} \frac{\cos(\pi i u + \frac{1}{2}\pi (z + z'))}{\cos(2\pi i u) + \cos(\pi (z - z'))}.
$$

Now, the claim of the proposition is equivalent to the relations

$$
a(u) = \frac{c(u)c(u)}{1 + c(u)c(u)}, \quad b(u) = \frac{c(u)}{1 + c(u)c(u)},
$$

which are checked directly from the above expressions. □

7. ZW-measures on signatures

In this section, we replace the set $\mathcal{Y}$ of Young diagrams by the set $\text{SGN}(N)$ of signatures of length $N$. Here $N = 1, 2, \ldots$, and a signature $\lambda \in \text{SGN}(N)$ is an $N$–tuple of weakly decreasing integers,

$$\lambda = (\lambda_1, \ldots, \lambda_N), \quad \lambda_1 \geq \cdots \geq \lambda_N.$$
We will describe a family of probability measures on the sets SGN(N) (for more detail, see Olshanski [Ol2] and Borodin–Olshanski [BO4]). Then we will study the behavior of the measures as $N \to \infty$, there the limit transition is similar to the “second regime” considered in §2 above. We show that the final result is again described in terms of the gamma kernel.

Our probability measures on SGN(N) depend on 4 complex parameters $z, z', w, w'$ and have the form

$$M_{z,z',w,w'}|N(\lambda) = (\text{const}_N)^{-1} \cdot M'_{z,z',w,w'}|N(\lambda)$$

where

$$M'_{z,z',w,w'}|N(\lambda) = \prod_{i=1}^{N} \left( \frac{1}{\Gamma(z - \lambda_i + i) \Gamma(z' - \lambda_i + i)} \times \frac{1}{\Gamma(w + N + 1 + \lambda_i - i) \Gamma(w' + N + 1 + \lambda_i - i)} \right) \cdot (\text{Dim}_N(\lambda))^2,$$

$$\text{Dim}_N(\lambda) = \prod_{1 \leq i < j \leq N} \frac{\lambda_i - \lambda_j + j - i}{j - i},$$

and

$$\text{const}_N = \sum_{\lambda \in \text{SGN}(N)} M'_{z,z',w,w'}|N(\lambda)$$

is the normalizing constant depending on $z, z', w, w', N$. Under suitable conditions on the quadruple $(z, z', w, w')$, the measures $M_{z,z',w,w'}|N(\lambda)$ are well defined for all $N$. That is, the weights $M'_{z,z',w,w'}|N(\lambda)$ are nonnegative and their sum over $\lambda \in \text{SGN}(N)$ is finite. A criterion for that to happen and for all the weights $M_{z,z',w,w'}|N(\lambda)$ to be strictly positive is provided below. See [Ol2, §7] for detailed explanations.

The measures $M_{z,z',w,w'}|N$ can be obtained by a construction which is quite similar to that described in §1, with the finite symmetric group $S_n$ replaced by the compact group $U(N)$ of unitary $N \times N$ matrices. Let $\mu_N$ be the normalized Haar measure on $U(N)$, and let $H_N$ be the Hilbert space of square integrable functions on $U(N)$ (with respect to $\mu_N$), constant on conjugacy classes. In $H_N$, there is a distinguished orthonormal basis formed by the irreducible characters $\chi_\lambda$ of the group $U(N)$. Here $\lambda$ ranges over $\text{SGN}(N)$.

Let $\mathbb{T}$ be the unit circle in $\mathbb{C}$ and $T^N$ be the product of $N$ copies of $\mathbb{T}$ (the $N$–dimensional torus). Given a unitary matrix $U \in U(N)$, we assign to it the unordered $N$–tuple of its eigenvalues, $(u_1, \ldots, u_N)$. Any element of $H_N$ can be viewed as a function in $(u_1, \ldots, u_N)$, that is, as a symmetric function on the torus $T^N$. In particular, the irreducible characters $\chi_\lambda(U)$ are the (rational) Schur functions $s_\lambda(u_1, \ldots, u_N)$,

$$s_\lambda(u_1, \ldots, u_N) = \frac{\det_{1 \leq i,j \leq N}^{\lambda_j + N - j} u_i^{j}}{\det_{1 \leq i,j \leq N}^{N - j} u_i^{j}}.$$
The whole Hilbert space $H_N$ can be identified with the Hilbert space of symmetric functions on $\mathbb{T}^N$, square integrable with respect to the measure

$$
\tilde{\mu}_N(du) = \frac{1}{N!} \prod_{1 \leq i < j \leq N} |u_i - u_j|^2 \prod_{i=1}^{N} du_i,
$$

which is the push–forward of $\mu_N$ under the correspondence $U \mapsto (u_1, \ldots, u_N)$. Here $du_i$ is the normalized invariant measure on the $i$th copy of $\mathbb{T}$.

Given two complex numbers $z, w$, we define a symmetric function on $\mathbb{T}^N$ by

$$
f_{z,w|N}(u) = \prod_{i=1}^{N} (1 + u_i)^z (1 + \bar{u}_i)^w.
$$

If $\Re(z + w) > -\frac{1}{2}$ then $f_{z,w|N}$ belongs to the space $H_N$. Let $(z', w')$ be another couple of complex numbers with $\Re(z' + w') > -\frac{1}{2}$. We set

$$
M_{z,z',w,w'|N}(\lambda) = \frac{(f_{z,w|N}, \chi_\lambda)(\lambda, f_{w', z'|N})}{(f_{z,w|N}, f'_{w', z'|N})}, \quad \lambda \in \text{SGN}(N),
$$

where $(\cdot, \cdot)$ is the inner product in $H_N$. It turns out that this definition leads us to the explicit formula given above. Notice that $\text{Dim}_N \lambda$ is equal to the value of the character $\chi_\lambda$ at $1 \in U(N)$ (equivalently, to the value of the Schur function $s_\lambda$ at $(1, \ldots, 1) \in \mathbb{T}^N$).

Similarly to the identification of the Young diagrams with points in $\{0, 1\}^{Z'}$ described at the beginning of §2, we identify a signature $\lambda = (\lambda_1, \ldots, \lambda_N)$ with a binary sequence $X(\lambda) = (\ldots, a_{-3/2}, a_{-1/2} | a_{1/2}, a_{3/2} \ldots)$ by

$$
a_j = \begin{cases} 
1, & \text{if } j \in \{\lambda_i - i + \frac{1}{2} | i = 1, \ldots, N\}, \\
0, & \text{otherwise}.
\end{cases}
$$

Note that this identification establishes a one-to-one correspondence between $\text{SGN}(N)$ and the elements from $\{0, 1\}^{Z'}$ with exactly $N$ 1’s.

In what follows we will assume that neither of the parameters $z, z', w, w'$ is an integer. This is always the case if we require the weights of all signatures to be nonzero. Further, the condition of $M_{z,z',w,w'|N}(\lambda)$ of being positive for all $\lambda \in \text{SGN}(N)$ is equivalent to both pairs $(z, z')$ and $(w, w')$ satisfying one of the conditions (i) and (ii) of Proposition 1.8, and the convergence of the series $
 \sum_{\lambda \in \text{SGN}(N)} M_{z,z',w,w'|N}(\lambda)$ is equivalent to the inequality

$$
\Re(z + z' + w + w') > -1,
$$

see [Ol2] for proofs. We also assume these conditions to be satisfied.

The following statement is an analog of Theorem 2.2.

**Theorem 7.1** ([BO4, Theorem 7.1]). Let $P_{z,z',w,w'|N}$ be the push–forward of the measure $M_{z,z',w,w'|N}$ under the embedding $\lambda \mapsto X(\lambda)$ of $\text{SGN}(N)$ into $\{0, 1\}^{Z'}$ defined above. Then its correlations functions have determinantal form

$$
\rho_m(x_1, \ldots, x_m | P_{z,z',w,w'|N}) = \det_{1 \leq i, j \leq m} [K(x_i, x_j | z, z', w, w' | N)],
$$

$$
m = 1, 2, \ldots, \quad x_1, \ldots, x_m \in \mathbb{Z'},
$$
Theorem 7.2. The measures $\mathbb{P}_{z, z', w, w'} | N$ weakly converge, as $N \to \infty$, to the probability measure $\mathbb{P}^{\gamma}$ defined in Theorem 2.3.4

Proof. Similarly to the proof of Theorem 2.3, we will show that the kernel $K(x, y \mid z, z', w, w' \mid N)$ has a pointwise limit as $N \to \infty$ which equals the gamma kernel. We will assume that $x \neq y$, $z \neq z'$, and $\Sigma \neq 0$. The convergence is easily extended to these sets by analytic continuation, as is explained at the end of the proof of Theorem 2.3.

We will use the following transformation formula for $\mathcal{F}_2$ with the unit argument, see Bailey [Ba, 3.2(2)]:

$$3\mathcal{F}_2 \left[ \begin{array}{c} a, b, c \\ e, f \end{array} \mid 1 \right] = \frac{\Gamma(a) \Gamma(b) \cdots}{\Gamma(c) \Gamma(d) \cdots}$$

Notation. The symbol $\mathcal{F}_2$ above stands for the higher hypergeometric series of type (3,2), see e.g. [Er1, chapter 4] and [Ba]. We also use the notation

$$_3\mathcal{F}_2 \left[ \begin{array}{c} -N, z + w', z' + w' \\ \Sigma, x + w' + \frac{1}{2} \end{array} \mid 1 \right]$$

where the correlation kernel is given by

$$\begin{align*}
K(x, y \mid z, z', w, w' \mid N) &= \frac{1}{\varphi_{N-1}} \frac{p_N(x)p_{N-1}(y) - p_{N-1}(x)p_N(y)}{x - y}\sqrt{f(x)f(y)}, \\
p_N(x) &= \frac{\Gamma(x + w' + N + \frac{1}{2})}{\Gamma(x + w' + \frac{1}{2})^3} \mathcal{F}_2 \left[ \begin{array}{c} -N, z + w', z' + w' \\ \Sigma, x + w' + \frac{1}{2} \end{array} \mid 1 \right], \\
p_{N-1}(x) &= \frac{\Gamma(x + w' + N + \frac{1}{2})}{\Gamma(x + w' + \frac{1}{2} + 1)^3} \mathcal{F}_2 \left[ \begin{array}{c} -N + 1, z + w' + 1, z' + w' + 1 \\ \Sigma + 2, x + w' + \frac{1}{2} + 1 \end{array} \mid 1 \right], \\
\varphi_{N-1} &= \frac{\Gamma(\Sigma + N + 1, z + w + 1, z' + w + 1, z' + w' + 1)}{\Gamma(\Sigma + N + 1, z + w + 1, z + w' + 1, z' + w + 1)}, \\
f(x) &= \frac{1}{\Gamma(z - x + \frac{1}{2}) \Gamma(z' - x + \frac{1}{2}) \Gamma(w + x + N + \frac{1}{2}) \Gamma(w' + x + N + \frac{1}{2})}.
\end{align*}$$

with $\Sigma = z + z' + w + w'$.

Comments. 1. The functions $p_{N-1}$ and $p_N$ are monic (i.e., the highest coefficient is equal to 1) orthogonal polynomials on $\mathbb{Z}'$ of degree $(N - 1)$ and $N$, corresponding to the weight function $f(x)$, and $\varphi_{N-1} = \|p_{N-1}\|^2_{f(\mathbb{Z}, f)}$. The determinantal structure of the correlation functions with the kernel expressed through orthogonal polynomials as above is a standard fact from Random Matrix Theory. Up to the factor $\sqrt{f(x)f(y)}$, the kernel is the Christoffel–Darboux kernel for the orthogonal polynomials with weight $f(x)$.

2. If $\Sigma = 0$ then the formula for $p_N$ above does not make sense because it involves a hypergeometric function with a zero lower index. However, the kernel itself admits an analytic continuation to the set $\Sigma = 0$, see [BO4, (7.3)].
If $a \to -\infty$, and $b, c, e, f$ are fixed, the $3F_2$’s in the right-hand side are equal to $1 + O(|a|^{-1})$, and using

$$\frac{\Gamma(1 - a)}{\Gamma(1 + b - a)} = (-a)^{-b}(1 + O(|a|^{-1})), \quad \frac{\Gamma(1 - a)}{\Gamma(1 + c - a)} = (-a)^{-c}(1 + O(|a|^{-1})),\]$$

we obtain

$$3F_2\left[\begin{array}{c} \frac{a, b, c}{e, f} \mid 1 \end{array}\right] = (-a)^{-b}f\left[\begin{array}{c} e, f, c - b \mid e - b, f - b, c \end{array}\right](1 + O(|a|^{-1}))$$

$$+ (-a)^{-c}f\left[\begin{array}{c} e, f, b - c \mid e - c, f - c, b \end{array}\right](1 + O(|a|^{-1})).$$

Applying this estimate to $p_N$ and $p_{N-1}$ with $a = -N$ and $-N + 1$, respectively, we get

$$p_N(x) = N^{-z - w'}\Gamma\left[\sum, z' - z \mid z' + w, -z + x + \frac{1}{2}, z' + w'\right](1 + O(\frac{1}{N}))$$

$$+ \text{a similar expression with } z \text{ and } z' \text{ interchanged } \cdot \Gamma(x + w' + N + \frac{1}{2}).$$

$$p_{N-1}(x) = N^{-z - w'-1}\Gamma\left[\sum + 2, z' - z \mid z' + w + 1, -z + x + \frac{1}{2}, z' + w' + 1\right](1 + O(\frac{1}{N}))$$

$$+ \text{a similar expression with } z \text{ and } z' \text{ interchanged } \cdot \Gamma(x + w' + N + \frac{1}{2}).$$

As we substitute these formulas into the expression $p_N(x)p_{N-1}(y) - p_{N-1}(x)p_N(y)$, we see that the part coming from $O(\frac{1}{N})$ is equal to

$$\Gamma(x + w' + N + \frac{1}{2})\Gamma(y + w' + N + \frac{1}{2})\gamma_{-z - z' - 2w' - 1}, o(1),$$

due to the fact that $|\Re(z - z')| < 1$ and $N\gamma_{-z - z'}(\frac{1}{N}) = o(1)$ as $N \to \infty$. Furthermore, four of the remaining eight terms cancel out, and we get (using the relation $\Gamma(s + 1) = s\Gamma(s)$ a few times)

$$p_N(x)p_{N-1}(y) - p_{N-1}(x)p_N(y) = \Gamma(x + w' + N + \frac{1}{2})\Gamma(y + w' + N + \frac{1}{2})$$

$$\times N^{-z - z' - 2w' - 1}\Gamma\left[\sum, \sum + 2, z' - z, z - z' \mid z + w, z' + w, z + w', z' + w'\right]$$

$$\times \left(\frac{1}{\Gamma(-z + x + \frac{1}{2})\Gamma(-z' + y + \frac{1}{2})}\left(\frac{1}{(z + w)(z + w') - \frac{1}{(z' + w')(z' + w')}}\right)^{\frac{1}{\Gamma(-z' + x + \frac{1}{2})\Gamma(-z + y + \frac{1}{2})}(\frac{1}{(z' + w')(z' + w') - \frac{1}{(z + w)(z + w')}) + o(1)}\right).$$

Simplifying

$$\frac{1}{(z + w)(z + w')} - \frac{1}{(z' + w')(z' + w')} = \frac{(z' - z)\Sigma}{(z + w)(z + w')(z + w')(z' + w')}.$$
and using the formula for \( h_{N-1} \), we see that
\[
\frac{p_N(x)p_{N-1}(y) - p_{N-1}(x)p_N(y)}{h_{N-1}(x-y)} = \frac{\Gamma(\Sigma + N + 1)}{\Gamma(N)} \cdot ((z'-z)\Gamma(z'-z)\Gamma(z-z')) \\
\times \Gamma(x+w' + N + \frac{1}{2}) \Gamma(y+w' + N + \frac{1}{2}) N^{-z'-z'-2w'-1}(1 + o(1)) \\
\times \left( \frac{1}{\Gamma(-z + x + \frac{1}{2})\Gamma(-z'+y + \frac{1}{2})} - \frac{1}{\Gamma(-z' + x + \frac{1}{2})\Gamma(-z + y + \frac{1}{2})} \right) \frac{1}{x-y}. 
\]

Since \( \Gamma(\Sigma + N + 1)/\Gamma(N) \sim N^{\Sigma+1} \) and \((z'-z)\Gamma(z'-z)\Gamma(z-z') = \pi/\sin(\pi(z-z'))\), we see that the above expression equals
\[
\frac{\pi}{\sin(\pi(z-z'))} \Gamma\left(x+w' + N + \frac{1}{2}\right) \Gamma\left(y+w' + N + \frac{1}{2}\right) N^{w-w'}(1 + o(1)) \\
\times \left( \frac{1}{\Gamma(-z + x + \frac{1}{2})\Gamma(-z'+y + \frac{1}{2})} - \frac{1}{\Gamma(-z' + x + \frac{1}{2})\Gamma(-z + y + \frac{1}{2})} \right) \frac{1}{x-y}. 
\]

It remains to multiply this expression by \( \sqrt{f(x)f(y)} \) and take the limit \( N \to \infty \). The weight function \( f(x) \) consists of four gamma-factors, two of which do not depend on \( N \) while the two others do. Taking the factors in \( \sqrt{f(x)f(y)} \) which are independent on \( N \), we obtain
\[
\frac{1}{\sqrt{\Gamma(z-x+\frac{1}{2})\Gamma(z'-x+\frac{1}{2})\Gamma(z-y+\frac{1}{2})\Gamma(z'-y+\frac{1}{2})}} \\
\times \left( \frac{1}{\Gamma(-z+x+\frac{1}{2})\Gamma(-z'+y+\frac{1}{2})} - \frac{1}{\Gamma(-z'+x+\frac{1}{2})\Gamma(-z+y+\frac{1}{2})} \right) \frac{1}{\sin(\pi z)\sin(\pi z')} \\
= \frac{\pi^2}{\sin(\pi z)\sin(\pi z')} \times (\Gamma(-z'+x+\frac{1}{2})\Gamma(-z+y+\frac{1}{2}) - \Gamma(-z+x+\frac{1}{2})\Gamma(-z+y+\frac{1}{2})). 
\]

Here we used the fact that due to our restrictions on \((z, z')\), the product \( \sin(\pi z)\sin(\pi z') \) is always positive, so we can pull it out of the square root.

As for the gamma-factors in \( \sqrt{f(x)f(y)} \) that do depend on \( N \), we get
\[
\frac{\Gamma\left(x+w' + N + \frac{1}{2}\right) \Gamma\left(y+w' + N + \frac{1}{2}\right) N^{w-w'}}{\sqrt{\Gamma(w+x+N+\frac{1}{2})\Gamma(w'+x+N+\frac{1}{2})\Gamma(w+y+N+\frac{1}{2})\Gamma(w'+y+N+\frac{1}{2})}} \\
= 1 + O\left(\frac{1}{N}\right). 
\]

Thus, gathering all pieces together, we see that as \( N \to \infty \) we have the estimate
\[
K(x, y \mid z, z', w, w') \sim \frac{p_N(x)p_{N-1}(y) - p_{N-1}(x)p_N(y)}{h_{N-1}(x-y)} \sqrt{f(x)f(y)} \\
= K^{\text{Gamma}}(x, y \mid -z, -z') \cdot (1 + o(1)). \quad \boxed{} 
\]
Remark 7.3. Observe that the measure $M_{z,z',w,w'}[N]$ has the following symmetry property:

$$M_{z,z',w,w'}[N](\lambda_1, \ldots, \lambda_N) = M_{w,w',z,z'}[N](\lambda_N, \ldots, \lambda_1).$$

Hence, the measure $\mathbb{P}_{z,z',w,w'}[N]$ on $\{0,1\}^Z$ is invariant with respect to the simultaneous switch $(z, z') \leftrightarrow (w, w')$ of the parameters and the involution

$$\{0,1\}^Z \to \{0,1\}^Z', \quad (a_j)_{j \in \mathbb{Z}} \mapsto (\tilde{a}_j = a_{-N-j})_{j \in \mathbb{Z}}.$$

This means that Theorem 7.2 also implies the following claim: Embed $\text{SGN}(N)$ into $\{0,1\}^Z = \{(\ldots, a_{-3/2}, a_{-1/2} | a_{1/2}, a_{3/2}, \ldots)\}$ by

$$a_j = \begin{cases} 1, & \text{if } j \in \{-\lambda_i - (N + 1 - i) + 1/2 | i = 1, \ldots, N\}, \\ 0, & \text{otherwise}. \end{cases}$$

Then the push–forwards of the measures $M_{z,z',w,w'}[N]$ under these embeddings weakly converge to $\mathbb{P}_{\gamma}^\alpha \otimes \mathbb{P}_{-\gamma}^\alpha$ as $N \to \infty$.

Remark 7.4. It is natural to ask whether the two limit transitions, the one of Theorem 7.2 and the one described in Remark 7.3 above, lead to asymptotically independent random point processes. The answer turns out to be positive, and the exact statement is as follows.

Consider an embedding of $\text{SGN}(N)$ into $\{0,1\}^Z \times \{0,1\}^Z'$ defined by using the map $\lambda \mapsto \overline{\lambda}(\lambda)$ (described just before Theorem 7.1) on the first coordinate, and using the map described in Remark 7.3 on the second coordinate. Then the push–forwards of the measures $M_{z,z',w,w'}[N]$ under these embeddings converge, as $N \to \infty$, to the product measure $\mathbb{P}_{\gamma}^\alpha \otimes \mathbb{P}_{-\gamma}^\alpha$.

The proof follows from the fact that the correlation kernel $K(x, y | z, z', w, w' | N)$ tends to zero as $N \to \infty$ if one of the arguments $(x, y)$ is in a finite neighborhood of 0, while the other one is in a finite neighborhood of $-N$. To prove such an estimate one uses the symmetry of the polynomials $p_N(x)$ and $p_{N-1}(x)$ with respect to $(z, z', w, w', x) \leftrightarrow (w, w', z, z', N-x)$ (which follows from the obvious symmetry of the weight function $f(x)$), and the same estimate of the $3F_2$ series as was used in the proof of Theorem 7.2 above.

Similarly to the discrete hypergeometric kernel, the kernel $K(x, y | z, z', w, w' | N)$ of Theorem 7.1 also has a second form. This form corresponds to a representation of the signatures $\lambda \in \text{SGN}(N)$ through Frobenius coordinates. Given a signature $\lambda \in \text{SGN}(N)$ we view it as a pair of Young diagrams $(\lambda^+, \lambda^-)$: one consists of positive $\lambda_i$'s and the other one consists of minus negative $\lambda_i$'s, zeros can go in either of the two:

$$\lambda = (\lambda_1^+, \lambda_2^+, \ldots, -\lambda_2^-, -\lambda_1^-).$$

Write the diagrams $\lambda^+$ and $\lambda^-$ through their Frobenius coordinates:

$$\lambda^\pm = (p_1^+, \ldots, p_{d^+}^+, | q_1^+, \ldots, q_{d^+}^+).$$

Now we associate to the signature $\lambda$ a finite subset $X(\lambda) \subset \mathbb{Z}'$ (or, equivalently, an element in $\{0,1\}^{Z'}$) as follows:

$$X(\lambda) = \{p_1^+ + \frac{i}{2} \} \cup \{-q_i^+ - \frac{1}{2}\} \cup \{-p_j^+ - N - \frac{1}{2}\} \cup \{q_j^+ - N + \frac{1}{2}\},$$

where $i = 1, \ldots, d^+$ and $j = 1, \ldots, d^-$. Then we have the following analog of Proposition 3.1.
Proposition 7.5 ([BO4, Proposition 4.1]). For any \( \lambda \in \text{SGN}(N) \), the two finite subsets \( X(\lambda) \) and \( X'(\lambda) \) are related by

\[
X(\lambda) = X(\lambda) \triangle \{ -\frac{1}{2}, -\frac{3}{2}, \ldots, -N + \frac{1}{2} \}, \quad X'(\lambda) = X(\lambda) \triangle \{ -\frac{1}{2}, -\frac{3}{2}, \ldots, -N + \frac{1}{2} \},
\]

where \( \triangle \) denotes the symmetric difference of two sets.

Note that the notation in [BO4] is slightly different, all points are shifted to the right by \( N/2 \) comparing to our notation here.

Theorem 8.7 of [BO4] proves that the push–forward of the measure \( M_{z,z',w,w'}[N] \) under the map \( \lambda \mapsto X'(\lambda) \) has determinantal correlation functions and gives explicit formulas for the kernel. Let us denote the correlation kernel by \( K(x, y \mid z, z', w, w' \mid N) \). (Once again, this kernel is different from that in [BO4] by the shift \( x \mapsto x + \frac{N}{2} \).)

The two correlation kernels, \( K(x, y \mid z, z', w, w' \mid N) \) of [BO4, Theorem 7.1] used in Theorems 7.1, 7.2 above, and \( K(x, y \mid z, z', w, w' \mid N) \) are related by a simple transform similar to that of Theorems 4.2 and 4.4. This fact is explained in [BO4, Theorem 5.10] in a fairly general framework. Together with Theorems 4.4 and 7.2, this implies that \( K(x, y \mid z, z', w, w' \mid N) \) converges to \( K_{\text{gamma}}(x, y \mid z, z') \) as \( N \to \infty \).

Note that, similarly to Theorem 5.3, the kernel \( K(x, y \mid z, z', w, w' \mid N) \) represents an orthogonal projection operator by the very definition; its range is the \( N \)-dimensional space \( \text{Span}\{ \sqrt{f(x)}, x \sqrt{f(x)}, \ldots, x^{N-1} \sqrt{f(x)} \} \).

Furthermore, [BO4, Theorem 8.7] shows that \( K(x, y \mid z, z', w, w' \mid N) \) has a rather simple \( L \)-kernel, \( L = K(1 - K)^{-1} \), presented in [BO4, §6]. (This is an analog of Theorem 3.4 above.) Using the explicit form of this \( L \)-kernel, it is not hard to show that a natural analog of Proposition 6.5 holds true. However, this is not enough to ensure the convergence of the correlation kernels for all admissible values of parameters (the reason being the unboundedness of the limit \( L \)-kernels for \( |z + z'| \geq 1 \)). Hence, it is of interest to evaluate the asymptotic behavior of the correlation kernel directly, so we state this as

Problem 7.6 (cf. Proposition 6.4). Show that in the scaling limit \( x = \pm e^{s_0 + \varepsilon} \), where \( s_0 \to +\infty \) with

\[
e^{s_0} = O(N^\varepsilon), \quad 0 < \varepsilon < 1 - |\Re(z - z')|,
\]

the kernel \( K(x, y \mid z, z', w, w' \mid N) \) converges to the tail kernel \( K_{\text{tail}}(s, t \mid -z, -z') \).

Due to the symmetry explained in Remark 7.3, solving this problem will also imply the convergence of \( K(x, y \mid z, z', w, w' \mid N) \) in the scaling limit \( x = -N^\varepsilon e^{s_0 + \varepsilon} \) to the tail kernel \( K_{\text{tail}}(s, t \mid -w, -w') \).

8. \( \mathbb{Z} \)-measures on nonnegative signatures

In this section, we deal with the subset \( \text{SGN}^+(N) \subset \text{SGN}(N) \) formed by the signatures \( \lambda \in \text{SGN}(N) \) with \( \lambda_N \geq 0 \). Elements of \( \text{SGN}^+(N) \) may be called nonnegative signatures of length \( N \). We will consider a family of probability measures on \( \text{SGN}^+(N) \) depending on parameters \( z, z', a, b \), where \( (z, z') \) is a couple of complex numbers satisfying suitable conditions, and \( a, b \) are real numbers such that \( a > -1, b > -1 \). It is convenient to denote

\[
\varepsilon = \frac{a + b + 1}{2}.
\]
We set

\[ M_{z,z',a,b|N}(\lambda) = (\text{const}_N)^{-1} \cdot M'_{z,z',a,b|N}(\lambda), \quad \lambda \in \text{SGN}^+(N), \]

where

\[
M'_{z,z',a,b|N}(\lambda) = \prod_{i=1}^{N} \left( \frac{(N+\varepsilon+\lambda_i-i)\Gamma(N+2\varepsilon+\lambda_i-i)\Gamma(N+a+1+\lambda_i-i)}{\Gamma(N+b+1+\lambda_i-i)\Gamma(N+1+\lambda_i-i)} \right) \\
\times \frac{1}{\Gamma(z-\lambda_i+i)\Gamma(z'+\lambda_i-i+\varepsilon)\Gamma(z+2N+2\varepsilon+\lambda_i-i)\Gamma(z'+2N+2\varepsilon+\lambda_i-i)} \\
\times \prod_{1 \leq i < j \leq N} \left( (N+\lambda_i-i+\varepsilon)^2 - (N+\lambda_j-j+\varepsilon)^2 \right)^2
\]

and

\[
\text{const}_N = \sum_{\lambda \in \text{SGN}^+(N)} M'_{z,z',a,b|N}(\lambda).
\]

One sufficient condition ensuring the existence of the probability measures for all \( N \) is \( z' = \bar{z}, \Re z > -\frac{b+1}{2} \).

Once again, the above formula can be obtained following the same general scheme. As the Hilbert space \( H_N \) we now take the space of symmetric functions on the \( N \)-dimensional cube \([-1,1]^N\], square integrable with respect to the measure

\[
\bar{\mu}_N(dx) = \prod_{1 \leq i < j \leq N} (x_i - x_j)^2 \cdot \prod_{i=1}^{N} (1 - x_i)^a(1 + x_i)^b \cdot dx_1 \ldots dx_n.
\]

A distinguished orthogonal basis in \( H_N \) is formed by the multivariate Jacobi polynomials

\[
P^{a,b}_{\lambda|N}(x_1, \ldots, x_N) = \frac{\det_{1 \leq i,j \leq N} [P^{a,b}_{\lambda+i+N-j}(x_i)]}{\det_{1 \leq i,j \leq N} [P^{a,b}_{N-j}(x_i)]},
\]

where \( P^{a,b}_m(y) \) are the classical Jacobi polynomials, orthogonal on the segment \(-1 \leq y \leq 1\) with the weight function \((1 - y)^a(1 + y)^b\). Let us set

\[
\chi^{a,b}_\lambda(x_1, \ldots, x_N) = \frac{P^{a,b}_{\lambda|N}(x_1, \ldots, x_N)}{\|P^{a,b}_{\lambda|N}\|}, \quad \lambda \in \text{SGN}^+(N),
\]

where \( \| \cdot \| \) stands for the norm in \( H_N \). The normalized polynomials \( \chi^{a,b}_\lambda \) form an orthonormal basis in \( H_N \).

We define

\[
f_{z|N}(x_1, \ldots, x_N) = \prod_{i=1}^{N} (1 + x_i)^z.
\]

Then the formula for the measure is obtained from the expression

\[
M_{z,z',a,b|N}(\lambda) = \frac{(f_{z|N} \cdot \chi^{a,b}_\lambda)(\chi^{a,b}_\lambda \cdot f_{z'|N})}{(f_{z|N}, f_{z'|N})}, \quad \lambda \in \text{SGN}^+(N).
\]
Notice that for special values of \((a, b)\), the multivariate Jacobi polynomials \(P_{A[N]}^{a,b}\), suitably renormalized, can be interpreted as the irreducible characters of the compact classical groups \(O(2N + 1)\), \(Sp(2N)\), \(O(2N)\), or as indecomposable spherical functions on the complex Grassmannians \(U(2N + k)/U(N + k) \times U(N)\). See, e.g., Okounkov–Olshanski [OkOl], Berezin–Karpelevich [BK].

Let us take the same embedding \(\text{SGN}(N)\) into \(\{0, 1\}^{\mathbb{Z}}\) (which is identified with subsets of \(\mathbb{Z}'\)) as we took in §7: \(\lambda \mapsto \tilde{X}(\lambda) = \{\lambda_i - i + \frac{1}{2}\}_{i=1}^{N}\). Denote by \(\mathcal{P}_{z', a, b}^{N}\) the push–forward of the measure \(M_{z', a, b}^{N}\) under this embedding. Standard tools of Random Matrix Theory provide us with the following claim, cf. Theorem 2.2 and Theorem 7.1.

**Proposition 8.1.** The correlation functions of the measure \(\mathcal{P}_{z', a, b}^{N}\) have determinantal form

\[
\rho_m(x_1, \ldots, x_m \mid \mathcal{P}_{z', a, b}^{N}) = \det_{1 \leq i, j \leq m} [K(x_i, x_j \mid z, z', a, b \mid N)],
\]

where \(m = 1, 2, \ldots\), \(x_1, \ldots, x_m \in \mathbb{Z}'\),

where the correlation kernel has the form

\[
K(x, y \mid z, z', a, b \mid N) = \frac{q_N(\hat{x}^2)q_{N-1}(\hat{y}^2) - q_{N-1}(\hat{x}^2)q_N(\hat{y}^2)}{h_{N-1} \cdot (\hat{x}^2 - \hat{y}^2)} \sqrt{g(x)g(y)},
\]

where \(\hat{x} = N + x + \frac{\varepsilon}{2}, \hat{y} = N + y + \varepsilon - \frac{1}{2}\), \(\{q_l\}_{l \geq 0}\) are monic polynomials, \(\deg q_l = l\), satisfying

\[
\sum_{x \in \mathbb{Z}'} q_k(\hat{x}^2) q_l(\hat{x}^2) g(x) = h_k \delta_{kl}
\]

with

\[
g(x) = (N + \varepsilon + x - \frac{1}{2}) \Gamma \left[ N + 2\varepsilon + x - \frac{1}{2}, N + a + x + \frac{1}{2} \right] \\
\times \Gamma \left[ z - x + \frac{1}{2}, z' - x + \frac{1}{2}, z + 2N + 2\varepsilon + x - \frac{1}{2}, z' + 2N + 2\varepsilon + x - \frac{1}{2} \right].
\]

Note that the weight function \(g(x)\) vanishes when \(x \leq -(N + \frac{1}{2})\) due to \(\Gamma(N + x + \frac{1}{2})\) in the denominator. Also note that \(g(x)\) has a polynomial asymptotics as \(x \to +\infty\), namely

\[
g(x) \sim x^{1-4N-2b-2(z+z')}, \quad x \to +\infty.
\]

We will make the assumption that the 4N moment of \(g(x)\) is finite, which will guarantee the existence of \(q_l\) up to \(l = N\). This means that \(z + z' > 1 - b\). The measure \(\mathcal{P}_{z', a, b}^{N}\) exists under a milder condition of finiteness of the \(4(N-1)\) moment of \(g(x)\), and our results can be extended to this wider domain of parameters by analytic continuation. However, we will not provide a detailed argument in this paper.

The weight function \(g(x)\) generalizes that associated with the classical Racah polynomials, see e.g. [KS]. Namely, if we assume that \(g(x)\) vanishes if \(x\) is greater than some fixed number, which may be achieved by requiring one of the parameters
are orthogonal with respect to the weight

\[ \text{Theorem 2.3.} \]

Theorem 2.3.

The exceptional sets by analytic continuation as explained at the end of the proof of \([Ner, \text{Proposition 8.2 (2)}]\).

These statements hold whenever the corresponding series are convergent.

\[ \text{Proof.} \]

Taking arbitrary complex numbers \(a_1, a_2, a_3, a_4\) and \(\alpha\), consider the weight function

\[ w(t | a_1, a_2, a_3, a_4; \alpha) = \frac{\alpha + t}{\prod_{j=1}^{4} \Gamma(a_j + \alpha + t)\Gamma(a_j - \alpha - t)}, \quad t \in \mathbb{Z}. \]

**Proposition 8.2 ([Ner, §3.4]).** The polynomials

\[
Q_n((t + \alpha)^2) = \Gamma \left[ \begin{array}{c} 2 - a_1 - a_2, 2 - a_1 - a_4, 2 - a_1 - a_4 + 1 + 4n \\ 2 - a_1 - a_4, 2 - a_1 - a_4 - 1 + 4n \\ \end{array} \right] \\
\times \frac{\sin(2\pi(2n + 3 - a_1 - a_4 - a_4))}{2\pi \sin(\pi(a_1 + a_2 + a_3 + a_4))}
\]

are orthogonal with respect to the weight \(w(t)\), and

\[
H_n = \sum_{t \in \mathbb{Z}} Q_n^2((t + \alpha)^2) = \frac{\sin(2\pi(2n + 3 - a_1 - a_4 - a_4))}{2\pi \sin(\pi(a_1 + a_2 + a_3 + a_4))}
\]

\[
\times \frac{n! \prod_{i,j=1}^{4} \Gamma(2 - a_i - a_j + n)}{(3 - a_1 - a_2 - a_3 - a_4 + n)\Gamma(3 - a_1 - a_2 - a_3 - a_4 + n)}. \]

These statements hold whenever the corresponding series are convergent.

Note that the polynomials \(Q_n\) are not monic, the highest coefficient \(k_n\) of \(Q_n\) is equal to

\[ k_n = (n + 3 - a_1 - a_2 - a_3 - a_4)n! = \frac{\Gamma(2n + 3 - a_1 - a_2 - a_3 - a_4)}{\Gamma(n + 3 - a_1 - a_2 - a_3 - a_4)}. \]

In what follows we will use Proposition 8.2 to evaluate the correlation kernel from Proposition 8.1 in terms of hypergeometric functions in order to prove the following result, cf. Theorems 2.3, 7.2.

**Theorem 8.3.** The measures \(P_{z, -z'}^{a, b | N}\) weakly converge, as \(N \to \infty\), to the probability measure \(P_{z, -z'}^{\text{gamma}}\) on \(\{0, 1\}^Z\) defined in Theorem 2.3.

**Proof.** The argument resembles those in the proofs of Theorem 2.3 and 7.1 but it is more technically involved. Once again, we will compute the pointwise asymptotics of the correlation kernel and show that it converges to the gamma kernel. The argument below requires that \(x \neq y\) and \(z \neq z'\). The result is extended to these exceptional sets by analytic continuation as explained at the end of the proof of Theorem 2.3.

First of all, applying the identities

\[
\Gamma(N + 2\varepsilon + x - \frac{1}{2}) = \frac{\pi}{\sin(\pi(N + 2\varepsilon + x - \frac{1}{2}))\Gamma(-N - 2\varepsilon - x + \frac{1}{2})}, \]

\[
\Gamma(N + a + x + \frac{1}{2}) = \frac{\pi}{\sin(\pi(N + a + x + \frac{1}{2}))\Gamma(-N - a - x + \frac{1}{2})}, \]

z, \(z'\) to be an integer, then \(\{q_i\}\) are exactly the (normalized) Racah polynomials. However, it is not immediately obvious how to generalize the Racah polynomials to the nonintegral values of \(z\) and \(z'\).

Fortunately, the orthogonal polynomials that we need were recently computed by Neretin [Ner]. Actually, Neretin considers even more general situations when the lattice is infinite at both plus and minus infinity. Let us state his result.

Take arbitrary complex numbers \(a_1, a_2, a_3, a_4\) and \(\alpha\), consider the weight function

\[ w(t | a_1, a_2, a_3, a_4; \alpha) = \frac{\alpha + t}{\prod_{j=1}^{4} \Gamma(a_j + \alpha + t)\Gamma(a_j - \alpha - t)}, \quad t \in \mathbb{Z}. \]
we observe that the weight function \( g(x) \) of Proposition 8.1 is proportional to the weight function \( w(t \mid a_1, a_2, a_3, a_4; \alpha) \) with the following identification of parameters:

\[
t = N + x - \frac{1}{2}, \quad \alpha = \varepsilon, \quad a_1 = 1 - \varepsilon, \quad a_2 = b + 1 - \varepsilon, \quad a_3 = z + N + \varepsilon, \quad a_4 = z' + N + \varepsilon.
\]

Hence, using this identification and the notation \( \hat{K} = \hat{K}(x, y \mid z, z', a, b \mid N) \), we may rewrite the correlation kernel \( k_{N-1} \) in the form

\[
\frac{k_{N-1}}{k_N H_{N-1}} \frac{Q_N(\hat{z}^2)Q_{N-1}(\hat{y}^2) - Q_{N-1}(\hat{z}^2)Q_N(\hat{y}^2)}{(\hat{z}^2 - \hat{y}^2)} \sqrt{w(\hat{z} - \varepsilon)w(\hat{y} - \varepsilon)}.
\]

It is the asymptotics of this expression that we are going to compute.

Our next goal is to transform the \( {}_4F_3 \) hypergeometric functions that enter the formulas for \( Q_{N-1} \) and \( Q_N \) into a form suitable for the limit transition \( N \to \infty \). We will do this in two steps.

First, we use the formula [Ba, 7.2(1)]:

\[
{}_4F_3 \left[ \begin{array}{l} X, Y, Z, -n \\ U, V, W \end{array} \right] \mid 1 = \Gamma \left[ \begin{array}{l} V - Z + n, W - Z + n, V, W \\ V - Z, W - Z, V + n, W + n \end{array} \right]
\times {}_4F_3 \left[ \begin{array}{l} U - X, U - Y, Z, -n \\ 1 - V + Z - n, 1 - W + Z - n, U \end{array} \right] \mid 1
\]

which holds if the \( {}_4F_3 \) series are terminating \((n = 1, 2, \ldots)\) and Saalschützian, that is, the sum of upper indices is greater than the sum of the lower indices by one: \( U + V + W = X + Y + Z - n + 1 \).

Applying this formula to \( Q_N \) with \( n = N \) and

\[
X = N + 3 - a_1 - a_2 - a_3 - a_4 = 1 - (z + z' + N + b),
\]

\[
Y = 1 - a_1 - t - \alpha = -N - x + \frac{1}{2},
\]

\[
Z = 1 - a_1 + t + \alpha = N + 2\varepsilon + x - \frac{1}{2} = N + a + b + x + \frac{1}{2}
\]

\[
U = 2 - a_1 - a_2 = a + 1,
\]

\[
V = 2 - a_1 - a_3 = 1 - z - N, \quad W = 2 - a_1 - a_4 = 1 - z' - N,
\]

we obtain

\[
Q_N(\hat{z}^2) = \Gamma \left[ \begin{array}{l} 2N + z + a + b + x + \frac{1}{2}, 2N + z' + a + b + x + \frac{1}{2}, N + a + 1 \\ N + z + a + b + x + \frac{1}{2}, N + z' + a + b + x + \frac{1}{2}, a + 1 \end{array} \right]
\times {}_4F_3 \left[ \begin{array}{l} N + z + z' + a + b, N + a + x + \frac{1}{2}, N + a + b + x + \frac{1}{2}, -N \\ N + z + a + b + x + \frac{1}{2}, N + z' + a + b + x + \frac{1}{2}, a + 1 \end{array} \right] \mid 1.
\]

Similarly,

\[
Q_{N-1}(\hat{z}^2) = \Gamma \left[ \begin{array}{l} 2N + z + a + b + x + \frac{1}{2}, 2N + z' + a + b + x + \frac{1}{2}, N + a \\ N + z + a + b + x + \frac{1}{2}, N + z' + a + b + x + \frac{1}{2} + 1, a + 1 \end{array} \right]
\times {}_4F_3 \left[ \begin{array}{l} N + z + z' + a + b + 1, N + a + x + \frac{1}{2}, N + a + b + x + \frac{1}{2}, -N + 1 \\ N + z + a + b + x + \frac{1}{2} + 1, N + z' + a + b + x + \frac{1}{2} + 1, a + 1 \end{array} \right] \mid 1.
\]
The second transformation formula for $\,_4F_3$ that we are about to use looks as follows:

$$\begin{align*}
_4F_3 \left[ \begin{array}{c} X, Y, Z, -n \\ U, V, W \end{array} \right] & = \Gamma \left[ \begin{array}{c} 1 + X - U, 1 + Y - U, 1 + Z - U, 1 - n - U, V, V - W \\ V - X, V - Y, V - Z, V + N, 1 - U, 1 - U + W \end{array} \right] \\
\times _4F_3 \left[ \begin{array}{c} W - X, W - Y, W - Z, W + n \\ 1 - U + W, 1 - V + W, W \end{array} \right] + \text{a similar expression with } V \text{ and } W \text{ interchanged}.
\end{align*}$$

This formula also holds for a terminating Saalschützian $\,_4F_3$ series, and it can be obtained by successful applications of [Ba, 7.1(1)] and [Ba, 7.5(3)].

For $Q_N$, we take $n = N$ and

$$\begin{align*}
X &= N + z + z' + a + b, \quad Y = N + a + x + \frac{1}{2}, \quad Z = N + a + b + x + \frac{1}{2}, \\
U &= a + 1, \quad V = N + z + a + b + x + \frac{1}{2}, \quad W = N + z' + a + b + x + \frac{1}{2}.
\end{align*}$$

Then the transformation formula yields

$$\begin{align*}
Q_N(\vec{z}^2) &= \Gamma \left[ \begin{array}{c} 2N + z + a + b + x + \frac{1}{2}, 2N + z' + a + b + x + \frac{1}{2}, N + a + 1 \\ N + z + a + b + x + \frac{1}{2}, N + z' + a + b + x + \frac{1}{2}, a + 1 \end{array} \right] \\
\times \Gamma \left[ \begin{array}{c} N + z + z' + b, N + x + \frac{1}{2}, N + b + x + \frac{1}{2}, -N - a, N + z + a + b + x + \frac{1}{2}, z - z' \\ -z' + x + \frac{1}{2}, z + b, z, 2N + z + a + b + x + \frac{1}{2}, -a, N + z' + b + x + \frac{1}{2} \end{array} \right] \\
\times _4F_3 \left[ \begin{array}{c} -z + x + \frac{1}{2}, z' + b, z', 2N + z' + a + b + x + \frac{1}{2}, 1 - z + z', N + z' + a + b + x + \frac{1}{2} \\ 1 \end{array} \right] + \text{a similar expression with } z \text{ and } z' \text{ interchanged}.
\end{align*}$$

We now see that four gamma–factors cancel out, and also

$$\Gamma \left[ \begin{array}{c} N + a + 1, -N - a \\ a + 1, -a \end{array} \right] = (-1)^N.$$

Further, we observe that the $\,_4F_3$ factors are of the form $1 + O\left(\frac{1}{N}\right)$ as $N \to \infty$. Indeed, this is true about any

$$\begin{align*}
_4F_3 \left[ \begin{array}{c} X, Y, Z, 2N + T \\ N + U, V, N + W \end{array} \right] \end{align*}$$

with finite $X, Y, Z, T, U, V, W; \, V \neq 0, -1, -2, \ldots$, as follows from the series representation of $\,_4F_3$.

Taking this into account we obtain

$$\begin{align*}
Q_N(\vec{z}^2) &= (-1)^N \Gamma(N + z + z' + b)\Gamma(N + x + \frac{1}{2})\Gamma(N + b + x + \frac{1}{2}) \\
\times \left( \Gamma \left[ \begin{array}{c} 2N + z' + a + b + x + \frac{1}{2}, z - z' \\ -z' + x + \frac{1}{2}, z + b, z, N + z' + a + b + x + \frac{1}{2}, N + z' + b + x + \frac{1}{2} \end{array} \right] (1 + O\left(\frac{1}{N}\right)) \\
+ \text{a similar expression with } z \text{ and } z' \text{ interchanged} \right).
\end{align*}$$
Similarly,
\[
Q_{N-1}(\tilde{x}^2) = (-1)^{N-1}\Gamma(N + z + z' + b + 1)\Gamma(N + x + \frac{1}{2})\Gamma(N + b + x + \frac{1}{2})
\times \left( \Gamma \left[ \begin{array}{c} 2N + z' + a + b + x + \frac{1}{2}, z - z' \\
- z' + x + \frac{1}{2}, z + b + 1, z + 1, N + z' + a + b + x + \frac{3}{2}, N + z' + b + x + \frac{3}{2} \end{array} \right] 
(1 + O(\frac{1}{N})) + \text{a similar expression with } z \text{ and } z' \text{ interchanged} \right).
\]

Using a number of times the asymptotic relation \(\Gamma(M+c)/\Gamma(M) = M^c(1+O(\frac{1}{M}))\), \(M \to +\infty\), \(c\) is fixed, we simplify the above expressions to get
\[
\frac{(-1)^N Q_N(\tilde{x}^2)}{\sqrt{\Gamma(2N + z + a + b + x + \frac{1}{2})\Gamma(2N + z' + a + b + x + \frac{1}{2})}} = \Gamma(N + z + z' + b)
\times \left( \Gamma \left[ \begin{array}{c} z - z' \\
- z' + x + \frac{1}{2}, z + b, z \end{array} \right] (2N)^{\frac{z - z'}{2}} N^{-2z' - a - b}(1 + O(\frac{1}{N})) 
+ \text{a similar expression with } z \text{ and } z' \text{ interchanged} \right)
\]
and
\[
\frac{(-1)^{N-1} Q_{N-1}(\tilde{x}^2)}{\sqrt{\Gamma(2N + z + a + b + x + \frac{1}{2})\Gamma(2N + z' + a + b + x + \frac{1}{2})}} = \Gamma(N + z + z' + b + 1)
\times \left( \Gamma \left[ \begin{array}{c} z - z' \\
- z' + x + \frac{1}{2}, z + b + 1, z + 1 \end{array} \right] (2N)^{\frac{z - z'}{2}} N^{-2z' - a - b - 2}(1 + O(\frac{1}{N})) 
+ \text{a similar expression with } z \text{ and } z' \text{ interchanged} \right).
\]

Now as we compute \(Q_N(\tilde{x}^2)Q_{N-1}(\tilde{y}^2) - Q_{N-1}(\tilde{x}^2)Q_N(\tilde{y}^2)\) normalized by the square root of the product of four gamma functions as above, we observe that the terms involving nonzero powers of \((2N)\) cancel out leaving a remainder of the form
\[
\Gamma(N + z + z' + b)\Gamma(N + z + z' + b + 1)N^{-2(z + z' + a + b + 1)} \cdot o(1),
\]
where we used the fact that \(N^{\pm(z - z')}O(\frac{1}{N}) = o(1)\). Hence, the whole expression equals
\[
-N^{-2(z + z' + a + b + 1)} \Gamma \left[ \begin{array}{c} N + z + z' + b, N + z + z' + b + 1, z - z', z' - z \\
z + 1, z' + 1, z + b + 1, z' + b + 1 \end{array} \right] 
\times \left( \frac{z + b - z'(z' + b)}{\Gamma(-z' + x + \frac{1}{2})\Gamma(-z + y + \frac{1}{2})} + \frac{z'(z' + b) - z(z + b)}{\Gamma(-z' + y + \frac{1}{2})\Gamma(-z' + y + \frac{1}{2})} + o(1) \right).
\]

Simplifying, we obtain
\[
N^{-2(z + z' + a + b + 1)} \Gamma \left[ \begin{array}{c} N + z + z' + b, N + z + z' + b + 1 \\
z + 1, z' + 1, z + b + 1, z' + b + 1 \end{array} \right] (1 + o(1)) 
\times \frac{\pi(z + z' + b)}{\sin(\pi(z - z'))} \left( \frac{1}{\Gamma(-z' + x + \frac{1}{2})\Gamma(-z + y + \frac{1}{2})} - \frac{1}{\Gamma(-z + x + \frac{1}{2})\Gamma(-z' + y + \frac{1}{2})} \right).
\]
To complete the computation of the asymptotics of the correlation kernel, it remains to take care of the factors

\[ \frac{k_{N-1}}{k_N} \sqrt{w(\bar{x} - \varepsilon)w(\bar{y} - \varepsilon)} \frac{1}{H_N(\bar{x}^2 - \bar{y}^2)}. \]

We see that

\[ \frac{k_{N-1}}{k_N} = \Gamma \left[ \begin{array}{c} 2N + 1 - a_1 - a_2 - a_3 - a_4, N + 3 - a_1 - a_2 - a_3 - a_4 \\ N + 2 - a_1 - a_2 - a_3 - a_4, 2N + 3 - a_1 - a_2 - a_3 - a_4 \end{array} \right] \]

\[ = \Gamma \left[ \begin{array}{c} -1 - z - z' - b, 1 - z - z' - b - N \\ -z - z' - b - N, 1 - z - z' - b \end{array} \right] = -(z + z' + b)(z + z' + b + 1)N(1 + O(1)) \]

and (assuming \( x \neq y \))

\[ \frac{1}{\bar{x}^2 - \bar{y}^2} = \frac{1}{(N + x + \varepsilon - \frac{1}{2})^2 - (N + y + \varepsilon - \frac{1}{2})^2} = \frac{1}{2N(x - y + o(1))}. \]

Further, let us consider the factor \( \sqrt{w(\bar{x} - \varepsilon)w(\bar{y} - \varepsilon)} \). Observe that out of the eight gamma-functions that enter the expression

\[ w(\bar{x} - \varepsilon) = \frac{N + 2\varepsilon + x - \frac{1}{2}}{\Gamma(-N - a - b - x + \frac{1}{2})\Gamma(-a - N - x + \frac{1}{2})\Gamma(N + b + x + \frac{1}{2})} \]

\[ \times \frac{1}{\Gamma(z - x + \frac{1}{2})\Gamma(z' - x + \frac{1}{2})\Gamma(2N + z + a + b + x + \frac{1}{2})\Gamma(2N + z' + a + b + x + \frac{1}{2})} \]

we have already used the last two to normalize \( Q_N \) and \( Q_{N-1} \) above. The remaining contribution of \( \sqrt{w(\bar{x} - \varepsilon)w(\bar{y} - \varepsilon)} \) equals

\[ \frac{|\sin(\pi(a + b))\sin(\pi a)|}{\pi^2} \frac{N^{2a+1}(1 + O(\frac{1}{N}))}{\sqrt{\Gamma(z - x + \frac{1}{2})\Gamma(z' - x + \frac{1}{2})\Gamma(z - y + \frac{1}{2})\Gamma(z' - y + \frac{1}{2})}}. \]

Finally, using the formula of Proposition 8.2, we obtain, using the periodicity of sine several times,

\[ H_{N-1} = \pm \frac{\sin(\pi z)\sin(\pi z')\sin(\pi(a + b))\sin(\pi a)\sin(\pi(z + z' + a + b))}{2\pi^4 \sin(\pi(z + z' + b))(z - z' - b - 1)} \]

\[ \times \frac{\sin(\pi(z + b))\sin(\pi(z' + b))}{\pi^2} \Gamma \left[ \begin{array}{c} -z, -z', N + a, -N - z - z' - a - b, -z - b, -z' - b \\ -N - z - z' - b \end{array} \right]. \]

Simplifying and using the fact that \( H_{N-1} \) must be positive, we obtain

\[ H_{N-1} = \frac{|\sin(\pi(a + b))\sin(\pi a)|}{2\pi^2} \Gamma \left[ \begin{array}{c} N, N + a \\ 1 + z, 1 + z', 1 + z + b, 1 + z' + b \end{array} \right] N^{-a}(1 + O(\frac{1}{N})). \]
Gathering all the pieces together, we obtain that the correlation kernel \( K(x, y \mid z, z', a, b \mid N) \), up to the factor \((1 + o(1))\), is equal to

\[
N^{-2(z+z'+a+b+1)} \Gamma \left[ \frac{N + z + z' + b, N + z + z' + b + 1}{z + 1, z' + 1, z + b + 1, z' + b + 1} \right]
\times \frac{\pi (z + z' + b)}{\sin(\pi(z - z'))} \left( \frac{1}{\Gamma(-z' + x + \frac{1}{2}) \Gamma(-z + y + \frac{1}{2})} - \frac{1}{\Gamma(-z + x + \frac{1}{2}) \Gamma(-z' + y + \frac{1}{2})} \right)
\times (-1)^{z + z' + b} (z + z' + b + 1) \cdot \frac{1}{2N(x - y)}
\times \frac{\sin(\pi(a + b)) \sin(\pi a)}{\pi^2} \frac{N^{2a+1}}{\sqrt{\Gamma(z - x + \frac{1}{2}) \Gamma(z' - x + \frac{1}{2}) \Gamma(z - y + \frac{1}{2}) \Gamma(z' - y + \frac{1}{2})}}
\times \frac{\left| \sin(\pi(a + b)) \sin(\pi a) \right|}{2\pi^2} \frac{N, N + a}{\left[ 1 + z, 1 + z', 1 + z + b, 1 + z' + b \right]} N^{-a}
\right]
\right) N^{-a} \right]^{1-1}
\right]
\right]
which, thanks to the asymptotic relation

\[
\Gamma \left[ \frac{N + z + z' + b, N + z + z' + b + 1}{N, N + a} \right] = N^{2(z+z'+b)+1-a} (1 + O(\frac{1}{N}))
\]
is readily seen to be asymptotically equal to \( K^{\text{gamma}}(x, y \mid z, -z') \). \( \square \)

Similarly to the discrete hypergeometric kernel and the \( 3F2 \) kernel of §7, the correlation kernel \( K(x, y \mid z, z', a, b \mid N) \) also has a second form \( K(x, y \mid z, z', a, b \mid N) \) related to representing signatures in terms of the Frobenius coordinates. Using [BO4, Theorem 5.10], it is easy to show that Theorem 8.3 proved above also implies the convergence of the second form \( K(x, y \mid z, z', a, b \mid N) \) to the second form of the gamma kernel \( K^{\text{gamma}}(x, y \mid z, -z') \).

One can also compute the \( L \)-kernel, \( L = K(1 - K)^{-1} \), and consider the tail scaling limit of the correlation kernels and the \( L \)-kernel, but we will postpone the discussion of these issues until a later publication.

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