A NOTE ON QUADRATIC AND HERMITIAN GROUPS

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Abstract: In this article we deduce an analogue of Quillen’s Local-Global Principle for the elementary subgroup of the general quadratic group and the hermitian group. We show that the unstable $K_1$-groups of the hermitian groups are nilpotent by abelian. This generalizes earlier results of A. Bak, R. Hazrat, N. Vavilov and et al.

1. INTRODUCTION

The vigorous study of classical groups, more generally algebraic $K$-theory, stimulated in mid sixties in attempt to give a solution of Serre’s Problem for projective modules. This prominent theorem in commutative algebra states that finitely generated projective modules over a polynomial ring over a field are free. (cf. Faisceaux Algébriques Coherents, 1955). The beautiful book Serre’s Problem on Projective Modules by T.Y. Lam gives a comprehensive account of the mathematics surrounding Serre’s Problem. Later we see analogue of Serre’s Problem for other classical groups in the work of H. Bass, A. Suslin, V.I. Kopeiko, R. Parimala and et al. in [7], [18], [24], [20]. We discuss the following problems related to Serre’s Problem, viz. normality property of the elementary subgroup of full automorphism group, Quillen’s Local-Global Principle, stability for $K_1$-functors, and structure of unstable $K_1$-groups of classical groups and modules.

From the work of H. Bass (cf.[8]), J.S. Wilson (cf.[30]), L.N. Vaserstein (cf.[29]) and et al. it is well known that the elementary subgroup $E_n(A)$ of $GL_n(A)$ plays a crucial role in the study of Classical $K$-Theory. In [26], L.N. Vaserstein proved that for an associative ring $A$, which is finite over its center, $E_n(A)$ is a normal subgroup of $GL_n(A)$ for $n \geq 3$. Later analogue results for the symplectic and orthogonal groups were done by Kopeiko and Suslin-Kopeiko in [23] and [24] (for respective cases). The difficulties of quadratic Serre’s Problem for characteristic 2 was first noted by Bass in [7]. In fact, in many cases it is difficult to handle classical groups over fields of characteristic 2, rather than classical groups over fields of characteristic $\neq 2$. (For details see [13]). In 1969, A. Bak resolved this problem by introducing Form Rings and form parameter. We also see some results in this direction in the work of Klein, Mikhalev,
Vaserstein et al. in [16], [17], [27]. The concept of form parameter also appears in the work of K. McCrimmon which plays an important role in his classification theory of $J$organ algebras cf. [19], for details see ([14], footnote pg. 190.) and [15].

In [9], it has been shown that this normality criterion is related to the following well known Local-Global Principle introduced by A. Suslin to give a matrix theoretic proof of Serre’s conjecture on projective modules.

**Suslin’s Local-Global Principle:** Let $R$ be a commutative ring with identity and $\alpha(X) \in \text{GL}_n(R[X])$ with $\alpha(0) = I_n$. If $\alpha_m(X) \in E_n(R_m[X])$ for every maximal ideal $m \in \text{Max}(R)$, then $\alpha(X) \in E_n(R[X])$.

We see generalization of this Principle for the symplectic group in [18], and for the orthogonal group in [24]. In [9], we have shown that the question of normality of the elementary subgroup of the general linear group, symplectic and orthogonal groups, is equivalent to the above Local-Global Principle where the base ring is associative with identity and is finite over its center. In that article we have treated uniformly above three classical groups. Motivated by the work of A. Bak, R.G. Swan, L.N. Vaserstein and others in [5] the author with A. Bak and R.A. Rao has established an analogue of Suslin’s Local-Global Principle for the transvection subgroup of the automorphism group of projective, symplectic and orthogonal modules of global rank at least 1 and local rank at least 3, under the assumption that the projective module has constant local rank and that the symplectic and orthogonal modules are locally an orthogonal sum of a constant number of hyperbolic planes. In this article we deduce an analogue Local-Global Principle for the general quadratic (or unitary) and hermitian groups. We treat these two groups uniformly and give explicit proofs of those results. We have overcome many technical difficulties which comes in the hermitian case due to the elements $a_1, \ldots, a_r$ (with respect to these elements we define the hermitian groups). We assume $a_1 = 0$. The rigorous study of hermitian groups can be found in [25].

The study of stability for $K_1$-functors started in mid sixties and first appears in the work of Bass-Milnor-Serre. Then this problem was thoroughly studied by L.N. Vaserstein for the symplectic, orthogonal and unitary groups. For details cf. [26], [27], and [28]. After almost thirty years of Vaserstein’s results in 1998 this problem was revisited for the linear groups over an affine algebra by R.A. Rao and W. van der kallen in [21]. Finally, the result settled for the general quadratic and hermitian groups by A. Bak, G. Tang and V. Petrov in [4] and [3]. It has been observed that over a regular affine algebra Vaserstein’s bounds for the
stabilization can be improved for the transvection subgroup of full automorphism group of projective, and symplectic modules. But cannot be improved for the orthogonal case in general. For details cf. [10], [22].

Though the study of stability for $K_1$-functors started in mid sixties, the structure of unstable $K_1$-group first studied by A. Bak in 1991 (cf. [2]). He showed that the group $GL_n(R)/E_n(R)$ is nilpotent-by-abelian for $n \geq 3$. In [12], R. Hazrat and N. Vavilov generalized his result for Chevalley groups with irreducible root system. They have shown: Let $\Phi$ be a reduced irreducible root system of rank $\geq 2$ and $R$ be a commutative ring such that its Bass-Serre dimension $\delta(R)$ is finite. Then for any Chevalley group $G(\Phi, R)$ of type $\Phi$ over $R$ the quotient $G(\Phi, R)/E(\Phi, R)$ is nilpotent-by-abelian. In particular, $K_1(\Phi, R)$ is nilpotent of class at most $\delta(R) + 1$. They use the localization-completion method of A. Bak in [2]. In [5], the author with Bak and Rao give a uniform proof for the transvection subgroup of full automorphism group of projective, symplectic and orthogonal modules of global rank at least 1 and local rank at least 3. Our method of proof shows that for classical groups the localization part suffices. Recently, in (cf. [6]) Bak, Vavilov and Hazrat proved the relative case for the unitary and Chevalley groups. But, in my best knowledge, so far there is no result for hermitian groups. I observe that using the above Local-Global Principle, arguing as in [5], it follows that the unstable $K_1$ of hermitian group is nilpotent-by-abelian. We follow the line of Theorem 4.1 in [5].

The first part of the paper serve as an introduction. In section 3 and 4, we discuss Suslin’s Local-Global Principle and its equivalence with the normality property of the elementary subgroup of full automorphism group. Finally, in section 4 we study the nilpotent property of unstable $KH_1$.

2. Preliminaries

Let $R$ be an associative ring with identity. Recall that an involutive anti-homomorphism (involution, in short) is a homomorphism $*: R \to R$ such that $(x - y)^* = x^* - y^*$, $(xy)^* = y^*x^*$ and $(x^*)^* = x$ for any $x, y \in R$. For any left $R$-module $M$ the involution induces a left module structure to the right $R$-module $M^* =$Hom$(M, R)$ given by $(xf)v = (fv)x^*$, where $v \in M$, $x \in R$ and $f \in M^*$. Let $- : R \to R$ be an involution for which there is an element $\lambda \in R$ such that $\lambda a \lambda = \overline{a}$ for all $a \in R$. Setting $a = 1$ we obtain $\lambda \lambda = 1$. Direct computation shows that $\lambda \lambda = 1$, from which it will follow that $\lambda$ is invertible in $R$ and $\lambda^{-1} = \overline{\lambda}$. The element $\lambda$ is called a symmetry of the involution “$-$” and it is unique up to an element $c \in C(R)$, where $C(R)$ is the center.
of \( R \), such that \( \sigma = 1 \). An involution with symmetry \( \lambda \) will be called a \( \lambda \)-involution.

We assume that \(- : R \to R\) denote a \( \lambda \)-involution on \( R \). Let \( \max^\lambda(R) = \{a \in R \mid a = -\lambda a\} \) and \( \min^\lambda(R) = \{a - \lambda a \mid a \in R\} \). One checks that \( \max^\lambda(R) \) and \( \min^\lambda(R) \) are closed under the operation \( a \mapsto \overline{\lambda} a x \) for any \( x \in R \). A \( \lambda \)-form parameter on \( R \) is an additive subgroup \( \Lambda \) of \( R \) such that \( \min^\lambda(R) \subseteq \Lambda \subseteq \max^\lambda(R) \), and \( \overline{\lambda} A x \subseteq \Lambda \) for all \( x \in R \). Note that if \( \lambda = -1 \), then \( \max^\lambda(R) = R \). The symplectic case is when \( \Lambda = R \). On the other hand if \( \lambda = 1 \), then \( \min^\lambda(R) = 0 \). The orthogonal case is when \( \Lambda = 0 \).

Let \( R \) possesses a \( \lambda \)-involution \(- : a \mapsto \overline{\lambda} a \), for \( a \in R \). For a matrix \( M = (m_{ij}) \) over \( R \) we define \( \overline{\lambda} M = (\overline{\lambda} m_{ij})^t \). Let \( A_1 \) denotes the diagonal matrix \([a_1, \ldots, a_r] \) for \( a_1, \ldots, a_r \in R \), and \( A = A_1 \perp I_{n-r} \). We define the forms

\[
\psi_n^a = \begin{pmatrix} 0 & \lambda a_n \\ I_n & 0 \end{pmatrix} \quad \text{and} \quad \psi_n^h = \begin{pmatrix} A & \lambda n_n \\ I_n & 0 \end{pmatrix}.
\]

**Definition 2.1.** General Quadratic Group \( GQ_{2n}(R, \Lambda) \): The group generated by the all non-singular \( 2n \times 2n \) matrices \( \{\sigma \in GL_{2n}(R) \mid \overline{\lambda} \psi_n^a \sigma = \psi_n^a\} \), where \( \sigma = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \) (\( \alpha, \beta, \gamma, \delta \) are \( n \times n \) block matrices) and \( \overline{\lambda} \alpha, \overline{\lambda} \delta \in \Lambda \).

**Definition 2.2.** General Hermitian Group of the elements \( a_1, \ldots, a_r \) \( GH_{2n}(R, a_1, \ldots, a_r, \Lambda) \): The group generated by the all non-singular \( 2n \times 2n \) matrices \( \{\sigma \in GL_{2n}(R) \mid \overline{\lambda} \psi_n^h \sigma = \psi_n^h\} \).

A typical element in \( GQ_{2n}(R, \Lambda) \) and \( GH_{2n}(R, a_1, \ldots, a_r, \Lambda) \) is denoted by a \( 2n \times 2n \) matrix \( \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \) where \( \alpha, \beta, \gamma, \delta \) are \( n \times n \) block matrices. There is a standard embedding, \( GQ_{2n}(R, \Lambda) \to GQ_{2n+2}(R, \Lambda) \) given by

\[
\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \mapsto \begin{pmatrix} \alpha & 0 & \beta & 0 \\ 0 & 1 & 0 & 0 \\ \gamma & 0 & \delta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
\]

called the stabilization map. This allows us to identify \( GQ_{2n}(R, \Lambda) \) with a subgroup in \( GQ_{2n+2}(R, \Lambda) \). Similarly, \( GH_{2n}(R, a_1, \ldots, a_r, \Lambda) \to GH_{2n+2}(R, a_1, \ldots, a_r, \Lambda) \).

We recall the definition of the *elementary* quadratic matrices in \( GQ_{2n}(R, \Lambda) \). Let \( \rho \) be the permutation defined by \( \rho(i) = n + i \) for \( i = 1, \ldots, n \). Let \( e_i \) denote the vector with 1 in the \( i \)-th position and 0’s elsewhere. Let \( e_{ij} \) be the matrix with 1 in the \( ij \)-th position and 0’s elsewhere. We define

\[
q_{ij}(a) = I_{2n} + a e_{ij} - \overline{\lambda} e_{\rho(i)\rho(j)} \quad \text{for} \quad a \in R, \quad \text{and} \quad 1 \leq i, j \leq n, i \neq j,
\]

\[
q_{ij}(a) = I_{2n} + a e_{\rho(i)\rho(j)} - \overline{\lambda} e_{ij} \quad \text{for} \quad a \in R, \quad \text{and} \quad 1 \leq i, j \leq n,
\]
(Note that, if \( i = j \), it forces that \( a \in \Lambda \), and
\[
q_{ij}(a) = I_{2n} + ae_{\rho(i)j} - \overline{ae_{\rho(j)i}} \quad \text{for } a \in R, \quad 1 \leq i, j \leq n.
\]
(Note that, if \( i = j \), it forces that \( a \in \overline{\Lambda} \). One checks that these above matrices belong to \( GQ_{2n}(R, \Lambda) \) when \( a \in R \).

**Definition 2.3. n’th Elementary Quadratic Group** \( EQ_{2n}(R, \Lambda) \):
The subgroup generated by \( q\varepsilon_{ij}(a), qr_{ij}(a) \) and \( ql_{ij}(a) \) for \( a \in R \) and \( 1 \leq i, j \leq n \).

To define **elementary Hermitian matrices**, we need to consider the set \( C = \{ (x_1, \ldots, x_r)^t \in (R^r)^t \mid \sum_{i=1}^r x_i a_i x_i \in \min^{-\Lambda}(R) \} \) for \( a_1, \ldots, a_r \) as above. In order to overcome the technical difficulties caused by the elements \( a_1, \ldots, a_r \), we shall finely partition a typical matrix \( \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \) of \( GH_{2n}(R, a_1, \ldots, a_n, \Lambda) \) into the form
\[
\begin{pmatrix}
\alpha_{11} & \alpha_{12} & \beta_{11} & \beta_{12} \\
\alpha_{21} & \alpha_{22} & \beta_{21} & \beta_{22} \\
\gamma_{11} & \gamma_{12} & \delta_{11} & \delta_{12} \\
\gamma_{21} & \gamma_{22} & \delta_{21} & \delta_{22}
\end{pmatrix}
\]
where \( \alpha_{11}, \beta_{11}, \gamma_{11}, \delta_{11} \) are \( r \times r \) matrices, \( \alpha_{12}, \beta_{12}, \gamma_{12}, \delta_{12} \) are \( r \times (n-r) \) matrices, \( \alpha_{21}, \beta_{21}, \gamma_{21}, \delta_{21} \) are \( (n-r) \times r \) matrices, and \( \alpha_{22}, \beta_{22}, \gamma_{22}, \delta_{22} \) are \( (n-r) \times (n-r) \) matrices. By \([25], Lemma 3.4\),

(1) the columns \( \alpha_{11} - I_r, \alpha_{12}, \beta_{11}, \beta_{12}, \overline{\gamma_{11}}, \overline{\delta_{12}}, \overline{\gamma_{21}}, \overline{\delta_{21}}, \overline{\gamma_{11}} - I_r, \overline{\delta_{21}} \in C \).

One checks straightforward that the subgroup of \( GH_{2n}(R, a_1, \ldots, a_n, \Lambda) \) consisting of
\[
\begin{pmatrix}
I_r & 0 & 0 & 0 \\
0 & \alpha_{22} & 0 & \beta_{22} \\
0 & 0 & I_r & 0 \\
0 & \gamma_{22} & 0 & \delta_{22}
\end{pmatrix} \in GH_{2n}(R, a_1, \ldots, a_n)
\]
\( \cong GH_{2(n-r)}(R, a_1, \ldots, a_n, \Lambda) \).

The first three kinds of generators are taken for the most part from \( GQ_{2(n-r)}(R, \Lambda) \), which is embedded as above as a subgroup of \( GH_{2n} \) and the last two kinds are motivated by the result \([1]\) concerning the column of a matrix in \( GH_{2n} \). We define
\[
h\varepsilon_{ij}(a) = I_{2n} + ae_{\rho(i)j} - \overline{ae_{\rho(j)i}} \quad \text{for } a \in R, \quad r + 1 \leq i \leq n, 1 \leq j \leq n, i \neq j,
\]
\[
hr_{ij}(a) = I_{2n} + ae_{\rho(j)i} - \overline{ae_{\rho(j)i}} \quad \text{for } a \in R, \quad r + 1 \leq i \leq n, 1 \leq j \leq n,
\]
(Note that, if \( i = j \), it forces that \( a \in \Lambda \)) and
\[
hl_{ij}(a) = I_{2n} + ae_{\rho(i)j} - \overline{ae_{\rho(j)i}} \quad \text{for } a \in R, \quad 1 \leq i, j \leq n.
\]
(Note that, if \( i = j \), it forces that \( a \in \overline{\Lambda} \)). It follows from \([2]\) that these matrices belong to \( GH_{2n}(R, a_1, \ldots, a_n, \Lambda) \) when \( a \in R \).
For \( \zeta \in (x_1, \ldots, x_r)^t \in C \), let \( \zeta_f \in R \) such that \( \zeta_f + \lambda \overline{\zeta}_f = \sum_{i=1}^{r} \bar{x}_i a_i x_i \).

(The element \( \zeta_f \) is not unique in general). We define

\[
hm_i(\zeta) = \begin{pmatrix} I_r & \alpha_{12} & 0 & 0 \\ 0 & I_{n-r} & 0 & 0 \\ 0 & -\bar{\alpha}_{12} & I_r & 0 \\ 0 & \gamma_{22} & -f_{12} & I_{n-r} \end{pmatrix}
\]

for \( \zeta \in C \) and \( r + 1 \leq i \leq n \)

a \( 2n \times 2n \) matrix, where \( \alpha_{12} \) is the \( r \times (n-r) \) matrix with \( h\overline{m}_i(\zeta) e_{i-r} = \zeta \)

and all other column’s zero, and \( \gamma_{22} \) is the \( (n-r) \times (n-r) \) matrix with \( \gamma_{22} e_{i-r}, i-r = \zeta_f \) and 0 elsewhere.

As above, we define

\[
hr_i(\zeta) = \begin{pmatrix} I_r & 0 & 0 & \beta_{12} \\ 0 & I_{n-r} & -\bar{\beta}_{12} & 0 \\ 0 & 0 & I_r & 0 \\ 0 & 0 & 0 & I_{n-r} \end{pmatrix}
\]

for \( \zeta \in C \) and \( r + 1 \leq i \leq n \)

a \( 2n \times 2n \) matrix, where \( \beta_{12} \) is the \( r \times (n-r) \) matrix with \( hr_i(\zeta) e_{i-r} = \zeta \)

and all other column’s zero, and \( \beta_{22} \) is the \( (n-r) \times (n-r) \) matrix with \( \beta_{22} e_{i-r}, i-r = \zeta_f \) and 0 elsewhere.

**Definition 2.4.** Each of the above matrices is called an elementary hermitian matrix for the elements \( a_1, \ldots, a_r \).

Note that if \( \eta = e_{pq}(a) \) is an elementary generator in \( GL_v(R) \), then the matrix \( (I_{n-x} \perp \eta \perp I_{n-x} \perp \eta^{-1}) \in h\overline{e}_{ij}(a) \). It has been shown in [25, §5] that each of the above matrices is in \( \text{GH}_{2n}(R, a_1, \ldots, a_r, \Lambda) \).

**Definition 2.5. n’th Elementary Hermitian Group** \( \text{EH}_{2n}(R, a_1, \ldots, a_r, \Lambda) \): The group generated by \( h\overline{e}_{ij}(a) \), \( hr_{ij}(a) \), \( hl_{ij}(a) \), \( h\overline{m}_i(\zeta) \) and \( hr_i(\zeta) \) for \( a \in R \) and \( 1 \leq i, j \leq n \).

**Blanket Assumption:** We assume that \( 2n \geq 6 \) and \( n > r \) while dealing with the hermitian case. We consider only isotropic vectors of \( \Lambda \). We do not want to put any restriction on the elements of \( C \). Therefore we assume that \( a_i \in \text{min}^\lambda(R) \) for \( i = 1, \ldots, r \), as in that case \( C = R^r \). When dealing with the Hermitian case we always assume \( a_1 = 0 \).

**Notation 2.6.** In the sequel \( M(2n, R) \) will denote the set of all \( 2n \times 2n \) matrices. \( G(2n, R, \Lambda) \) will denote either the quadratic group \( \text{GQ}_{2n}(R, \Lambda) \) or the hermitian group \( \text{GH}_{2n}(R, a_1, \ldots, a_r, \Lambda) \). \( S(2n, R, \Lambda) \) will denote respective subgroups \( \text{SQ}_{2n}(R, \Lambda) \) or \( \text{SH}_{2n}(R, a_1, \ldots, a_r, \Lambda) \) with matrices of determinant 1, in the case when \( R \) will be commutative. \( E(2n, R, \Lambda) \) will denote the corresponding elementary subgroups \( \text{EQ}_{2n}(R, \Lambda) \) and \( \text{EH}_{2n}(R, a_1, \ldots, a_r, \Lambda) \). To treat uniformly we denote the generators of \( E(2n, R, \Lambda) \) by \( gc_{ij}(a) \) for \( a \in R \). Let \( \Lambda[X] \) denote the \( \lambda \)-form parameter on \( R[X] \) induced from \( (R, \Lambda) \), and let \( \Lambda_s \) denote the \( \lambda \)-form parameter on \( R_s \) induced from \( (R, \Lambda) \).

For any column vector \( v \in R^{2n} \) we denote \( \bar{v}_q = \overline{v}^t \psi^q_i \) and \( \bar{v}_h = \overline{v}^t \psi^h_i \).
Definition 2.7. We define the map $M : R^{2n} \times R^{2n} \to M(2n, R)$ and the inner product $\langle \cdot, \cdot \rangle$ as follows:

$$M(v, w) = v. \tilde{w} - \tilde{v}w$$

when $G(2n, R, \Lambda) = G(2n, R)$,

$$\langle v, w \rangle = \tilde{v}w$$

when $G(2n, R, \Lambda) = G(2n, R, \Lambda)$. We recall following very well known facts:

Lemma 2.8. (cf. [1], [25]) The group $E(2n, R, \Lambda)$ is perfect for $n \geq 3$ in the quadratic case and for $n \geq r + 3$ in the hermitian case, i.e.

$$[E(2n, R, \Lambda), E(2n, R, \Lambda)] = E(2n, R, \Lambda).$$

Lemma 2.9. (Splitting property): For all $x, y \in R$

$$g_{ij}(x + y) = g_{ij}(x)g_{ij}(y).$$

Lemma 2.10. Let $G$ be a group, and $a_i, b_i \in G$, for $i = 1, \ldots, n$. Then for $r_i = \prod_{j=1}^{i} a_j$ we have

$$\prod_{i=1}^{n} a_i b_i = \prod_{i=1}^{n} r_i b_i r_i^{-1} \prod_{i=1}^{n} a_i.$$

Notation 2.11. By $G(2n, R[X], \Lambda[X], (X))$ we mean the group of all invertible matrices over $R[X]$ which are $I_n$ modulo $(X)$.

Lemma 2.12. The group $G(2n, R[X], \Lambda[X], (X)) \cap E(2n, R[X], \Lambda[X])$ is generated by the elements of the type $\varepsilon g_{ij}(Xh(X))\varepsilon^{-1}$, where $\varepsilon \in E(2n, R, \Lambda), h(X) \in R[X].$

3. Suslin’s Local-Global Principle

In his remarkable thesis (cf. [1]) Bak showed that for a form ring $(R, \Lambda)$ the elementary subgroup $E(2n, R, \Lambda)$ is perfect for $n \geq 3$ and hence a normal subgroup of $G(2n, R, \Lambda)$. As we have noted earlier that this question is related to Suslin’s Local-Global Principle for the elementary subgroup. In [25], Tang has shown that for $n \geq r + 3$ the elementary hermitian group $E(2n, R, a_1, \ldots, a_r, \Lambda)$ is perfect for $n \geq r + 3$ and hence a normal subgroup of $G(2n, R, a_1, \ldots, a_r, \Lambda)$. In this section we deduce analogue Local-Global Principle for the elementary subgroup of the general quadratic and hermitian groups when $R$ is finite over its center. We use this result in §5 to prove the nilpotent property of unstable $KH_1$. Furthermore, we show that if $R$ is finite over its center the normality of elementary subgroup is equivalent to Local-Global Principle. This generalizes our result in [9].

The following is the key Lemma, and it tells us the reason why we need to assume that the size of the matrix is at least 6. In [9] proof is given for the same result for the linear group. Arguing in similar manner by using identities of commutator laws result follows in the unitary and hermitian cases. A list of commutator laws for elementary generators is stated in
Lemma 3.1. For \( m > 0 \), there are \( h_t(X, Y, Z) \in R[X, Y, Z] \) such that
\[
ge_{pq}(Z)ge_{ij}(X^{2m}Y)ge_{pq}(-Z) = \frac{k}{t}ge_{pq}(X^m h_t(X, Y, Z)).
\]

Corollary 3.2. If \( \varepsilon = \varepsilon_1 \varepsilon_2 \cdots \varepsilon_r \), where each \( \varepsilon_j \) is an elementary generator, then there are \( h_t(X, Y) \in R[X, Y] \) such that
\[
ge_{pq}(X^{2^r m}Y)^{e^{-1}} = \prod_{t=1}^k ge_{pq}(X^m h_t(X, Y)).
\]

Lemma 3.3. Let \((R, \Lambda)\) be a form ring and \( v \in E(2n, R, \Lambda)e_{2n} \). Let \( w \in R^{2n} \) be a column vector such that \( \langle v, w \rangle = 0 \). Then \( I_{2n} + M(v, w) \in E(2n, R, \lambda) \).

Proof. Let \( v = \varepsilon e_{2n} \), where \( \varepsilon \in E(2n, R, \Lambda) \). Then it follows that \( I_{2n} + M(v, w) = \varepsilon (I_{2n} + M(e_{2n}, w_1))^{e^{-1}} \), where \( w_1 = \varepsilon^{n-1}w \). Since \( \langle e_{2n}, w_1 \rangle = \langle v, w \rangle = 0 \), we get \( w_1' = (w_1, \ldots, w_{1n-1}, 0, w_{1n+1}, \ldots, w_{12n}) \). Therefore, as \( \lambda X = X \lambda = 1 \),
\[
I_{2n} + M(v, w) = \left\{ \begin{array}{l}
eg q_{1n}(-\overline{\varepsilon}^{n+1}) q_{1n}(-\overline{\varepsilon}^{n+1}) q_{1n}^{e^{-1}} \\
\prod_{1 \leq j \leq n} \varepsilon h_{1n}(-\overline{\varepsilon}^{n+1}) h_{1n}(-\overline{\varepsilon}^{n+1}) h_{1n}^{e^{-1}} \\
\prod_{1 \leq k \leq r} \varepsilon h_{1n}(-\overline{\varepsilon}^{n+1}) h_{1n}(-\overline{\varepsilon}^{n+1}) h_{1n}^{e^{-1}} \\
\prod_{1 \leq j \leq n} \varepsilon h_{1n}(-\overline{\varepsilon}^{n+1}) h_{1n}(-\overline{\varepsilon}^{n+1}) h_{1n}^{e^{-1}} \end{array} \right.
\]
(in the quadratic and hermitian cases respectively), where \( \overline{\varepsilon}^{n+k} = (w_1n-k, 0, \ldots, 0) \). Hence the result follows.

Note that the above implication is true for any associative ring with identity. From now onwards we assume that \( R \) is finite over its center \( C(R) \).

Lemma 3.4. (Dilation Lemma) Let \( \alpha(X) \in G(2n, R[X], \Lambda[X]) \), with \( \alpha(0) = I_{2n} \). If \( \alpha_s(X) \in E(2n, R_s[X], \Lambda_s[X]) \) for some non-nilpotent \( s \in R \), then \( \alpha(bX) \in E(2n, R[X], \Lambda[X]) \) for \( b \in s' C(R) \), \( l \gg 0 \).

(Actually, we mean there exists some \( \beta(X) \in E(2n, R[X], \Lambda[X]) \) such that \( \beta(0) = I_{2n} \) and \( \beta_s(X) = \alpha(bX) \). But, since there is no ambiguity, for simplicity we are using the notation \( \alpha(bX) \) instead of \( \beta_s(X) \)).

Proof. We prove the Lemma in 2 steps.

Step 1. First we prove the following:

If \( \alpha(X) = I_{2n} + X^d M(v, w) \), where \( v \in E(2n, R, \Lambda)e_{2n} \) and \( \langle v, w \rangle = 0 \), then \( \alpha(X) \in E(2n, R[X], \Lambda[X]) \) and it can be written as a product decomposition of the form \( \prod_{i=1}^k \ge_{pq}(X \alpha_i(X)) \) for \( d \gg 0 \).

Replacing \( R \) by \( R[X] \) in Lemma 3.3, we get that \( I_n + X \alpha(X) \in E(n, R[X], \Lambda[X]) \). Let \( v = \varepsilon e_1 \), where \( \varepsilon \in E(n, R) \). Since \( \lambda X = X \lambda = 1 \),
as in the proof of Lemma 3.3 we can write

\[
I_{2n} + M(v, w) = \begin{cases} 
\prod_{1 \leq j \leq n} \varepsilon q_{i_n}(-x \overline{x}, \overline{w}_{n+1}) q_{e_{j_n}}(-x \overline{x}, \overline{w}_j) d_{n+1}^\varepsilon \\
\prod_{1 \leq k \leq r} e h_{i_n}(-x \overline{x}, \overline{w}_{n+1}) h e_{j_n}(-x \overline{x}, \overline{w}_j) h m_n(-x \overline{x}, \overline{w}_k) d_{n+1}^\varepsilon \end{cases}
\]

(in the quadratic and hermitian cases respectively), where \( \overline{w}_{1+n+k} = (w_{1+n+k}, 0, \ldots, 0) \).

Now we split the proof into following two cases.

Case I: \( \varepsilon \) is an elementary generators of the type \( g e_{pq}(x) \), \( x \in R \). First applying the homomorphism \( X \mapsto X^2 \) and then applying Lemma 3.1 over \( R[X] \) we get \( I_n + X^2 M(v, w) = \prod_{t=1}^{k} \left( \prod_{j=1}^{l} g e_{p(j,t)}(X h_{j(t)}(X)) \right) \), where \( h_{j(t)}(X) \in R[X] \). Now the result follows for \( d \gg 0 \).

Case II: \( \varepsilon \) is a product of elementary generators of the type \( g e_{pq}(x) \). Let \( \mu(\varepsilon) = r \). First applying the homomorphism \( X \mapsto X^2 \) and then applying the Corollary 3.2 we see that the result is true for \( d \gg 0 \).

**Step 2.** Let \( \alpha_s(X) = \prod_k g e_{i(k)}(h_k(X)) \), where \( h_k(X) \in R_s[X] \). So, \( \alpha_s(X T^d) = \prod_k g e_{i(k)}(h_k(X T^d)) \). Choose \( d \geq 2^r \) and \( r = \mu(\alpha(X)) \).

Since \( \alpha(0) = I_n \), as in Lemma 2.12 we get

\[
\alpha_s(X T^d) = \prod_k e_k g e_{i(k)}(X T^d \lambda_k(X T^d)) e_k^{-1} = \prod_k (I_n + X T^d \lambda_k(X T^d)) e_k M(e_{i(k)}, e_{\sigma(j)(k)}) e_k^{-1},
\]

for \( \lambda_k(X T^d) \in R_s[X, T] \), \( e_k \in E(n, R_s, \Lambda_s) \). Let \( v_k = e_k e_{i(k)} \). Then taking \( w_k(X, T) = X \lambda_k(X T^d) e_k e_{\sigma(j)(k)} \) otherwise we get \( \langle v_k, w_k(X, T) \rangle = 0 \) (without loss of generality we are assuming \( \sigma(i) \neq j \)). Applying result in Step I over the polynomial ring \( (R_s[X])[T] \) we get \( I_n + T^d M(v_k, w_k(X, T)) \in E(n, R_s[X, T], \Lambda_s[X, T]) \), and can be expressed as a product of the form \( \prod g e_{p(k)q(k)}(T h_k(X, T)) \), where \( h_k(X, T) \in R_s[X, T] \). Let \( l \) be the maximum of the powers occurring in the denominators of \( h_k(X, T) \) for all \( k \). Now applying the homomorphism \( T \mapsto s^m T \) for \( m \geq l \) we get \( \alpha(b X T^d) \in E(n, R[X, T], \Lambda[X, T]) \) for some \( b \in (s^l) C(R) \). Finally, putting \( T = 1 \) we get the required result.

**Theorem 3.5. (Local-Global Principle)** If \( \alpha(X) \in G(2n, R[X], \Lambda[X]) \), \( \alpha(0) = I_n \) and \( \alpha_m(X) \in E(n, R_m[X], \Lambda_m[X]) \) for every maximal ideal \( m \in \text{Max}(C(R)) \), then \( \alpha(X) \in E(2n, R[X], \Lambda[X]) \). (Note that \( R_m \) denotes \( S^{-1} R \), where \( S = C(R) - m \).)

**Proof.** Since \( \alpha_m(X) \in E(2n, R_m[X], \Lambda_m[X]) \) for all \( m \in \text{Max}(C(R)) \), for each \( m \) there exists \( s \in C(R) - m \) such that \( \alpha_s(X) \in E(n, R_s[X], \Lambda_s[X]) \).
Let \( \theta(X,T) = \alpha_s(X+T)\alpha_s(T)^{-1} \). Then
\[
\theta(X,T) \in E(2n, (R_s[T])[X], \Lambda_s[T][X])
\]
and \( \theta(0,T) = I_n \). Then by Dilation Lemma, applied with base ring \( R[T] \),
\[
\theta(bX,T) \in E(2n, R[X,T], \Lambda[X,T]) \text{ for some } b \in (s^l)C(R), l \gg 0. \quad (A)
\]
Let \( b_1, b_2, \ldots, b_r \in C(R) \) be such that (A) holds and \( b_1 + \cdots + b_r = 1 \). Then \( \theta(b_iX,T) \in E(2n, R[X,T], \Lambda[X,T]) \) and hence \( \prod_{i=1}^r \theta(b_iX,T) \in E(2n, R[X,T], \Lambda[X,T]) \). But,
\[
\alpha(X) = \left( \prod_{i=1}^{r-1} \theta(b_iX,T) \right)_{T=bi+1X+\cdots+b_rX} \theta(b_rX,0).
\]
Since \( \alpha(0) = I_n \), \( \alpha(X) \in E(2n, R[X,T], \Lambda[X,T]) \). \( \square \)

4. Equivalence of Normality and Local-Global

Next we show that if \( R \) is a commutative ring with identity and \( A \) is an associative \( R \)-algebra such that \( A \) is finite as a left \( R \)-module, then the normality criterion of elementary subgroup is equivalent to Suslin’s Local-Global Principle for above two classical groups.

One of the crucial ingredients in the proof of the above theorem is the following result which states that the group \( E \) acts transitively on unimodular vectors. The precise statement of the fact is the following:

**Theorem 4.1.** Let \( R \) be a semilocal ring (not necessarily commutative) with involution and \( v = (v_1, \ldots, v_{2n})^t \) be a unimodular and istropic vector in \( R^{2n} \). Then \( v \in E(2n, R)e_{2n} \) for \( n \geq 2 \). i.e. \( E(2n, R) \) acts transitively on \( \text{Um}_{2n}(R) \).

Let us first recall some known facts before we give a proof of the theorem.

**Definition 4.2.** An associative ring \( R \) is said to be **semilocal** if \( R/\text{rad}(R) \) is artinian semisimple.

We recall the following three lemmas.

**Lemma 4.3.** (H. Bass) Let \( A \) be an associative \( R \)-algebra such that \( A \) is finite as a left \( R \)-module and \( R \) be a commutative local ring with identity. Then \( A \) is semilocal.

**Proof.** Since \( R \) is local, \( R/\text{rad}(R) \) is a division ring by definition. That implies \( A/\text{rad}(A) \) is a finite module over the division ring \( R/\text{rad}(R) \) and hence is a finitely generated vector space. Thus \( A/\text{rad}(A) \) artinian as \( R/\text{rad}(R) \) module and hence \( A/\text{rad}(A) \) artinian as \( A/\text{rad}(A) \) module, so it is an artinian ring.

It is known that an artin ring is semisimple if its radical is trivial. Thus \( A/\text{rad}(A) \) is semisimple, as \( \text{rad}(A/\text{rad}(A)) = 0 \). Hence \( A/\text{rad}(A) \) artinian semisimple. Therefore, \( A \) is semilocal by definition. \( \square \)
Lemma 4.4. (H. Bass) ([8], Lemma 4.3.26) Let $R$ be a semilocal ring (may not be commutative), and let $I$ be a left ideal of $R$. Let $a$ in $R$ be such that $Ra + I = R$. Then the coset $a + I = \{a + x \mid x \in I\}$ contains a unit of $R$.

Proof. We give a proof due to R.G. Swan. We can factor out the radical and assume that $R$ is semisimple artinian. Let $I = (Ra \cap I) \oplus I'$. Replacing $I$ by $I'$ we can assume that $R = Ra \oplus I$. Let $f : R \to Ra$ by $r \mapsto ra$ for $r \in R$. Therefore, we get an split exact sequence $0 \to J \to R \xrightarrow{f} Ra \to 0$, for some ideal $J$ in $R$ which gives us a map $g : R \to J$ such that $R \xrightarrow{(f,g)} Ra \oplus J$ is an isomorphism. Since $Ra \oplus J \cong R \cong Ra \oplus I$ cancellation (using Jordon-Hölder or Krull-Schmidt) shows that $J \cong I$.

If $h : R \cong J \cong I$, then $R \xrightarrow{(f,g)} Ra \oplus I \cong R$ is an isomorphism sending $1$ to $(a,i)$ to $a + i$, where $i = h(1)$. Hence it follows that $a + i$ is a unit. □

Lemma 4.5. Let $R$ be a semisimple artinian ring and $I$ be a left ideal of $R$. Let $J = Ra + I$. Write $J = Re$, where $e$ is an idempotent (possible since $J$ is projective. For detail cf. [11] Theorem 4.2.7). Then there is an element $i \in I$ such that $a + i = ue$, where $u$ is a unit in $R$.

Proof. Since $R = J + R(1 - e) = Ra + I + R(1 - e)$, using Lemma 4.4 we can find a unit $u = a + i + x(1 - e)$ in $R$ for some $x \in R$. Since $a + i \in Re$, it follows that $ue = a + i$. □

Corollary 4.6. Let $R$ be a semisimple artinian ring and $(a_1, \ldots, a_n)^t$ be a column vector over $R$, where $n \geq 2$. Let $\Sigma a_i = Re$, where $e$ is an idempotent. Then there exists $\varepsilon \in E_n(R)$ such that $\varepsilon(a_1, \ldots, a_n)^t = (0, \ldots, 0, e)^t$.

Proof. By Lemma 4.5 we can write $ue = \Sigma a_i = a_n$, where $u$ is a unit. Therefore, applying an elementary transformation we can assume that $a_n = e$. Multiplying from the left by $(I_{n-2} \perp u \perp u^{-1})$ we can make $a_n = e$. Since all $a_i$ are left multiple of $e$, further elementary transformations reduce our vector to the required form. □

The following observation will be needed to do the case $2n = 4$.

Lemma 4.7. Let $R$ be a semisimple artinian ring and $e$ be an idempotent. Let $f = 1 - e$, and $b$ be an element of $R$. If $fRb \subseteq Re$, then we have $b \in Re$.

Proof. Since $R$ is a product of simple rings, it will suffice to do the case in which $R$ is simple. If $e = 1$, we are done. Otherwise $RfR$ is a non-zero two sided ideal, and hence $RfR = R$. Since $Rb = RfRb \subseteq Re$, we have $b \in Re$. □
Lemma 4.8. Let $R$ be a semisimple artinian ring and let $- : R \to R$ be a $\lambda$-involution on $R$. Let $(x y)^t$ be a unimodular element of $R^{2n}$, where $2n \geq 4$. Then there exists an element $\varepsilon \in E(2n, R)$ such that $\varepsilon (x y)^t = (x' y')^t$, where $x'$ is a unit in $R$.

Proof. Let $x = (x_1, \ldots, x_r)^t$ and $b = (y_1, \ldots, y_r)^t$. We claim that there exists $\varepsilon \in E(2n, R)$ such that $\varepsilon (x y)^t = (x' y')^t$, where $x'$ is a unit in $R$. Among all $(x' y')^t$ of this form, choose one for which the ideal $I = \Sigma Rx_i$ is maximal. Replacing the original $(x y)^t$ by $(x' y')^t$ we can assume that $I = \Sigma R x_i$ is maximal among such ideals. Write $I = Re$, where $e$ is an idempotent in $R$. By Corollary 4.6 we can find an element $\eta \in E_n(R)$ such that $\eta x = (0, 0, \ldots, e)^t$. Hence we assume that $x = (0, 0, \ldots, e)^t$.

We claim that $y_i \in Re$ for all $i \geq 1$.

First we consider the case $2n \geq 6$. Assume $y_1 \notin I$, but $y_i \in I$ for all $i \geq 2$. If we apply $q_{1n}(1)$ in the quadratic case then this replaces $y_n$ to $y_n - y_1$ but not changes $e$ and $y_1$. On the other hand for the hermitian case we do not have the generator $q_{1n}(1)$. But if we apply $hm_{n}(1, \ldots, 1)$, then it changes $y_2$ but does not changes $e$ and $b_1$. Therefore, in both the cases we can therefore assume that some $y_i$ with $i > 1$ is not in $I$. (Here recall that we have put no restriction on $C$, i.e. for us $C = R^\sigma$). Apply $qr_{ii}(1)$ with $2 \leq i \leq n$ in the quadratic case. This changes $x_i = 0$ (for $i > 1$) to $y_i$ while $x_n = e$ is preserved. The ideal generated by the entries of $x$ now contains $Re + Ry_i$, which is larger than $I$, a contradiction, as $I$ is maximal. In the hermitian case if we apply suitable $hr_{i}(1, \ldots, 1)$ then also we see that the ideal generated by the entries of $x$ now contains $Re + Ry_i$, hence a contradiction.

If $2n = 4$, we can argue as follows. Let $f = 1 - e$. Let us assume that $y_i \notin I$ as above. Then by Lemma 4.7 it will follow that we can find some $s \in R$ such that $fsy_1 \neq Re$. First consider the quadratic case. Applying $qr_{21}(fs)$ replaces $x_2 = e$ by $c = e + fsb_1$. As $ec = e$, $I = Re \subset Rc$. Also, $fc = fsb_1 \in Rc$ but $fc \notin I$. Hence $I \not\subseteq Rc$, a contradiction. We can get the similar contradiction for $y_2$ by applying $qr_{22}(fs)$. In the hermitian case, apply $hr_{1}(1)$ to get the contradiction for $y_1$. Now note that in this $r = 1$ as we have assume $r < n$. Hence we can apply $qr_{22}(fs)$ to get the contradiction.

Since all $y_i$ lie in $Re$, the left ideal generated by the all entries of $(x y)^t$ is $Re$, but as this column vector is unimodular $Re = R$, and therefore $e = 1$.

Proof of Theorem 4.1. Let $J$ be the Jacobson radical of $R$. Since the left and the right Jacobson radical are same, $J$ is stable under the involution which therefore passes to $R/J$. Let $\varepsilon$ be as in Lemma 4.8 for the image $(x' y')^t$ of $(x y)^t$. By lifting $\varepsilon$ from $R/J$ to $R$ and applying it to
\((x\ y)^t\) we reduce to the case where \(x_n\) is a unit in \(R\). Let \(\alpha = x_n \perp x_n^{-1}\). Then applying \((I_{n-2} \perp \alpha \perp I_{n-2} \perp \alpha^{-1})\) we can assume that \(x_n = 1\).

Next applying \(\Pi_{i=1}^n ql_{ni}(-y_i)\) and \(\Pi_{i=1}^n hl_{ni}(-y_i)\) in the respective cases we get \(y_1 = \cdots = y_{n-1} = 0\). As isotropic vector remains isotropic under elementary quadratic (hermitian) transformation, we have \(y_n + \lambda \gamma_n = 0\), hence \(ql_{11}(\lambda \gamma_n)\) and \(hl_{11}(\lambda \gamma_n)\) are defined and applying it reduces \(y_n\) to 0 in both the cases. Now we want to make \(x_i = 0\) for \(i = 1, \ldots, n\). In the quadratic case it can be done by applying \(\Pi_{i=1}^n h\varepsilon_{in}(-x_i)\). Note that this transformation does not affect any \(y_i\)'s, as \(y_i = 0\). In the hermitian case we can make \(x_{r+1} = \cdots = x_n = 0\) as before applying \(\Pi_{i=r+1}^n q\varepsilon_{in}(-x_i)\).

To make \(x_1 = \cdots = x_r = 0\) we have to recall that the set \(C = R^r\), i.e. there is no restriction on the set \(C\). Hence \(hr_n(-x_1, \ldots, -x_r)\) is defined and applying it we get \(x_1 = \cdots = x_r = 0\). Also note that other \(x_i\)'s and \(y_i\)'s remain unchanged. Finally, applying \(hl_{mn}(1)\) and then \(hr_{mn}(-1)\) we get the required vector \((0, \ldots, 0, 1)\). This completes the proof. 

\[\square\]

**Theorem 4.9.** Let \(R\) be a commutative ring with identity and \(A\) an associative \(R\)-algebra such that \(A\) is finite as a left \(R\)-module. Then the following are equivalent for \(n \geq 3\) in the quadratic case and \(n \geq r + 3\) in the hermitian case:

1. **(Normality)** \(E(2n, A, \Lambda)\) is a normal subgroup of \(G(2n, A, \Lambda)\).
2. **(L-G Principle)** If \(\alpha(X) \in G(2n, A[X], \Lambda[X]), \alpha(0) = I_n\) and \(\alpha_m(X) \in E(n, A_m[X], A_m[X])\)

for every maximal ideal \(m \in \text{Max}(R)\), then

\[\alpha(X) \in E(2n, A[X], \Lambda[X]).\]

(Note that \(A_m\) denotes \(S^{-1}A, \) where \(S = R - m\).)

**Proof.** We have proved the Lemma \[3.3\] for any form ring with identity. In particular, suppose \(E(2n, A, \Lambda)\) is a normal subgroup of \(G(2n, A, \Lambda)\).

Let \(\alpha = I_n + M(v, w)\), where \(v = Ae_1\), and \(A \in G(n, A, \Lambda)\). Then we can write \(\alpha = A(I_n + M(e_1, w_1))A^{-1}\), where \(w_1 = A^{-1}w\). Hence it is enough to show that \(I_n + M(e_1, w_1) \in E(n, R)\). Now arguing as in the proof of Lemma \[3.3\] we get the result. Now in section \[5\] we have proved Local-Global Principle as a consequence of Lemma \[3.3\]. Hence the implication follows.

To prove the converse we need \(A\) to be finite as \(R\)-module, where \(R\) is a commutative ring with identity (i.e. a ring with trivial involution).

Let \(\alpha \in E(2n, A, \Lambda)\) and \(\beta \in G(2n, A, \Lambda)\). Then \(\alpha = \Pi g_{ij}(x), x \in A\). Hence, \(\beta \alpha \beta^{-1} = \Pi (I_{2n} + x\beta M(x_1, x_2)\beta^{-1})\), where \(x_1\) and \(x_2\) are suitably chosen standard basis vectors. Now let \(v = \beta e_i\) and \(w = x\beta e_j\). Then we get \(\beta \alpha \beta^{-1} = \Pi (I_{2n} + M(v, w))\), where \(v \in \text{Um}_{2n}(A)\) and \(\langle v, w \rangle = 0\). We show that each \((I_{2n} + M(v, w)) \in E(2n, A, \Lambda)\).
Let $\gamma(X) = I_{2n} + XM(v, w)$. Then $\gamma(0) = I_{2n}$. By Lemma 5.1, it follows that $S^{-1}A$ is a semilocal ring, where $S = R - m, m \in \text{Max}(R)$. Since $v \in U_{2n}(A)$, using Theorem 4.1 we get $v \in E(2n, S^{-1}A, S^{-1}A)e_1$, hence $Xv \in E(2n, S^{-1}A[X], S^{-1}A[X])e_1$. Therefore, applying Lemma 5.3 over $S^{-1}(A[X], \Lambda[X])$ it follows that

$$\gamma_m(X) \in E(2n, S^{-1}A[X], S^{-1}A[X]).$$

Now applying Theorem 5.3 it follows that $\gamma(X) \in E(2n, A[X], \Lambda[X])$. Finally, putting $X = 1$ we get the result.

\[\square\]

5. Nilpotent property for $K_1$ of Hermitian groups

We devote this section to discuss the study of nilpotent property of unstable $K_1$-groups. The literature in this direction can be seen in the work of A. Bak, N. Vavilov and R. Hazarat and that we have already discussed in the Introduction. Throughout this section we assume $R$ is a commutative ring with identity, i.e. we are considering trivial involution and $n \geq r + 3$. Following is the statement of the theorem.

**Theorem 5.1.** The quotient group $SH_{2n}(R,a_1,\ldots,a_r)_{\text{EH}_{2n}(R,a_1,\ldots,a_r)}$ is nilpotent for $n \geq r + 3$. The class of nilpotency is at the most $\max(1, d + 3 - n)$, where $d = \dim(R)$.

The proof follows by omitting the proof of Theorem 4.1 in [5].

**Lemma 5.2.** Let $\beta \in SH(2n, R, \Lambda)$, with $\beta \equiv I_n$ modulo $I$, where $I$ is an ideal contained in the Jacobson radical $J(R)$ of $R$. Then there exists $\theta \in EH_{2n}(R, a_1, \ldots, a_r)$ such that $\beta \theta = \text{the diagonal matrix } [d_1, d_2, \ldots, d_{2n}]$, where each $d_i$ is a unit in $R$ with $d_i \equiv 1$ modulo $I$, and $\theta$ a product of elementary generators with each congruent to identity modulo $I$.

**Proof.** The diagonal elements are units. Let $\beta = (\beta_{ij})$, where $d_i = \beta_{ii} = 1 + s_i$ with $s_i \in I \subset J(R)$ for $i = 1, \ldots, 2n$, and $\beta_{ij} \in I \subset J(R)$ for $i \neq j$. First we make all the $(2n, j)$-th, and $(i, 2n)$-th entries zero for $i = 2, \ldots, n, j = 2, \ldots, n$. Then repeating the above process we can reduce the size of $\beta$. Since we are considering trivial involution, we take

$$\alpha = \prod_{j=1}^n h_{n+1,j}(-\beta_{2n,j}d_{2n,j}^{-1}) \prod_{n+1 \leq j \leq 2n-1} h_{n,j}(b_{nj}d_{nj}^{-1}) \prod_{r+1 \leq j \leq 2n-1} h_{r+1,n}(b_{rn}d_{rn}^{-1}),$$

where $j = i - r$ and $\beta_{ij} = (0, \ldots, 0, \beta_{2nj})$, and

$$\gamma = \prod_{r+1 \leq j \leq 2n-1} h_{r+1,n}(a_{r+j}d_{2n,j}^{-1})h_{n,r}(\eta),$$

where $a_t = 0$ for $t > r$, and $\eta = (\beta_{12n}d_{2n,1}^{-1}, \beta_{22n}d_{2n,2}^{-1}, \ldots, \beta_{n2n}d_{2n,n}^{-1})$. Then the last column and last row of $\gamma \beta \alpha$ become $(0, \ldots, 0, d_{2n})^t$, where $d_{2n}$ is a unit in $R$ and $d_{2n} \equiv 1$ modulo $I$. Repeating the process we can modify $\beta$ to the required form.

\[\square\]
Lemma 5.3. Let \((R, \Lambda)\) be a commutative form ring, i.e. with trivial involution and \(s\) a non-nilpotent element in \(R\). Let \(D\) denote the diagonal matrix \([d_1, \ldots, d_{2n}]\), where \(d_i \equiv 1 \mod (s^l)\) for \(l \geq 2\). Then
\[
[ge_{ij} \left( \frac{a}{s} X \right), D] \subset \text{EH}_{2n}(R[X], a_1, \ldots, a_r) \cap \text{SH}_{2n}((s^{l-1})R, a_1, \ldots, a_r).
\]

Proof. Let \(d = d_i d_j^{-1}\). Then using a list of commutator laws for elementary generators is stated in [23], pg. 237-239) for the hermitian group, it follows that
\[
[ge_{ij} \left( \frac{a}{s} X \right), D] = ge_{ij} \left( \frac{a}{s} X \right) ge_{ij} \left( -\frac{a}{s} d X \right).
\]
Since \(d_id_j \equiv 1 \mod (s^l)\) for \(l \geq 2\), we can write \(d = 1 + s^m \lambda\) for some \(m > 2\) and \(\lambda \in R\). Hence
\[
ge_{ij} \left( \frac{a}{s} X \right) ge_{ij} \left( -\frac{a}{s} d X \right) = ge_{ij} \left( \frac{a}{s} X \right) ge_{ij} \left( -\frac{a}{s} s^m \lambda X \right) = ge_{ij} \left( -\frac{a}{s} s^m \lambda X \right) \in \text{EH}_{2n}(R[X], a_1, \ldots, a_r) \cap \text{SH}_{2n}((s^{m-1})R, a_1, \ldots, a_r).
\]

Lemma 5.4. Let \((R, \Lambda)\) be as above, \(s \in R\) a non-nilpotent element in \(R\) and \(a \in R\). Then for \(l \geq 2\)
\[
[ge_{ij} \left( \frac{a}{s} X \right), \text{SH}_{2n}(s^l R, a_1, \ldots, a_r)] \subset \text{EH}_{2n}(R[X], a_1, \ldots, a_r).
\]
More generally, \([\varepsilon(X), \text{SH}_{2n}(s^l R[X], a_1, \ldots, a_r)] \subset \text{EH}_{2n}(R[X], a_1, \ldots, a_r)\) for \(l > 0\) and \(\varepsilon(X) \in \text{EH}_{2n}(R_X[X], a_1, \ldots, a_r)\).

Proof. First fix \((i, j)\) for \(i \neq j\).

Let \(\alpha(X) = [g_{ij} \left( \frac{a}{s} X \right), \beta]\) for some \(\beta \in \text{SH}_{2n}(s^l R, a_1, \ldots, a_r)\). Since \(s\) is in \(J(R)\), the diagonal entries of \(\beta\) are unipotent. As \(l \geq 2\), it follows that \(\alpha(X) \in \text{SH}_{2n}(R[X], a_1, \ldots, a_r)\). Since \(\text{EH}_{2n}(R[X], a_1, \ldots, a_r)\) is a normal subgroup of \(\text{SH}_{2n}(R[X], a_1, \ldots, a_r)\) for \(n \geq r + 3\) (cf. [23], Theorem 4.2), we get
\[
\alpha(X) \in \text{EH}_{2n}(R_X[X], a_1, \ldots, a_r).
\]

Let \(B = 1 + sR\). We show that \(\alpha_B(X) \in \text{EH}_{2n}(R_B[X], a_1, \ldots, a_r)\). Since \(s \in J(R_B, \Lambda_B)\), it follows from Lemma 5.2 that we can decompose \(\beta_B = \varepsilon_1 \cdots \varepsilon_l D\), where \(\varepsilon_i = ge_{p_i q_i} \left( s^l \lambda_i \right) \in \text{EH}_{2n}(R_B(a_1, \ldots, a_r); \lambda_i \in R_B\) and \(D = \text{the diagonal matrix } [d_1, \ldots, d_{2n}]\) with \(d_i\) is a unit in \(R\) and \(d_i \equiv 1 \mod (s^l)\) for \(l \geq 2; i = 1, \ldots, 2n\). If \(t = 1\), then using the commutator laws for elementary generators is stated in [23], pg. 237-239) it follows from Lemma 5.3 that \(\alpha_B(X) \in \text{EH}_{2n}(R_B[X], a_1, \ldots, a_r)\). Suppose \(t > 1\). Then
\[
\alpha_B(X) = [ge_{ij} \left( \frac{a}{s} X \right), \varepsilon_1 \cdots \varepsilon_l D] = [ge_{ij} \left( \frac{a}{s} X \right), \varepsilon_1 \left[ ge_{ij} \left( \frac{a}{s} X \right), \varepsilon_2 \cdots \varepsilon_l D \right] \varepsilon_1^{-1}
\]
and by induction each term is in \(\text{EH}_{2n}(R_B[X], a_1, \ldots, a_r)\), hence
\[
\alpha_B(X) \in \text{EH}_{2n}(R_B[X], a_1, \ldots, a_r).
\]
Since \(\alpha(0) = I_{n}\), by the Local-Global Principle for the hermitian groups (Theorem 3.5) it follows that \(\alpha(X) \in \text{EH}_{2n}(R[X], a_1, \ldots, a_r)\).
Corollary 5.5. Let \( R \) be as above, \( s \in R \) be a non-nilpotent element in \( R \) and \( a \in R \). Then for \( l \geq 2 \)
\[
\left[ \epsilon, \text{SH}_{2n}(s^l R, a_1, \ldots, a_r) \right] \subset \text{EH}_{2n}(R, a_1, \ldots, a_r).
\]
More generally, \( \left[ \epsilon, \text{SH}_{2n}(s^l R, a_1, \ldots, a_r) \right] \subset \text{EH}_{2n}(R, a_1, \ldots, a_r) \) for \( l \gg 0 \) and \( \epsilon \in \text{EH}_{2n}(R^n, a_1, \ldots, a_r) \).

Proof of Theorem 5.1. Recall
Let \( G \) be a group. Define \( Z^0 = H, Z^1 = [G, G] \) and \( Z^i = [G, Z^{i-1}] \). Then \( G \) is said to be nilpotent if \( Z^r = \{e\} \) for some \( r > 0 \), where \( e \) denotes the identity element of \( H \).

Since the map \( \text{EH}_{2n}(R, a_1, \ldots, a_r) \to \text{EH}_{2n}(R/I, \pi_1, \ldots, \pi_r) \) is surjective we may and do assume that \( R \) is a reduced ring. Note that if \( t \geq d + 3 \), then the group \( \text{SH}_{2n}(R, a_1, \ldots, a_r)/\text{EH}_{2n}(R, a_1, \ldots, a_r) = \text{KH}_1(R, a_1, \ldots, a_r) \), which is abelian and hence nilpotent. So we consider the case \( t \leq d + 3 \). Let us first fix a \( t \). We prove the theorem by induction on \( d = \dim R \). Let \( G = \text{SH}_{2n}(R, a_1, \ldots, a_r)/\text{EH}_{2n}(R, a_1, \ldots, a_r) \). Let \( m = d + 3 - t \) and \( \alpha = [\beta, \gamma] \) for some \( \beta \in G \) and \( \gamma \in Z^{m-1} \). Clearly, the result is true for \( d = 0 \). Let \( \tilde{\beta} \) be the pre-image of \( \beta \) under the map \( \text{SH}_{2n}(R, a_1, \ldots, a_r) \to \text{SH}_{2n}(R, a_1, \ldots, a_r)/\text{EH}_{2n}(R, a_1, \ldots, a_r) \). If \( R \) is local then arguing as Lemma 5.2 is follows that \( \text{EH}_{2n} = \text{SH}_{2n} \), hence we can choose a non-zero-divisor \( s \in R \) such that \( \tilde{\beta}_s \in \text{EH}_{2n}(R_s, a_1, \ldots, a_r) \).

Consider \( \overline{G} \), where \( \overline{G} \) denote reduction modulo \( s^l \) for some \( l \gg 0 \). By the induction hypothesis \( \overline{\gamma} = \{1\} \) in \( \overline{\text{SH}}_{2n} \). Since \( \text{EH}_{2n} \) is a normal subgroup of \( \text{SH}_{2n} \) for \( n \geq r + 3 \), by modifying \( \gamma \) we may assume that \( \overline{\gamma} \in \text{SH}_{2n}(R, s^l R, a_1, \ldots, a_r) \), where \( \tilde{\gamma} \) is the pre image of \( \gamma \) in \( \text{SH}_{2n}(R, a_1, \ldots, a_r) \). Now by Corollary 5.5 it follows that \( \overline{[\beta, \gamma]} \in \text{EH}_{2n}(R, a_1, \ldots, a_r) \). Hence \( \alpha = \{1\} \) in \( \overline{G} \).

Remark 5.6. In (\[9\], Theorem 3.1) it has been proved that the question of normality of the elementary subgroup and the Local-Global Principle are equivalent for the elementary subgroups of the linear, symplectic and orthogonal groups over an almost commutative ring with identity. There is a gap in the proof of the statement (3) \( \Rightarrow \) (2) of Theorem 3.1 in (\[9\]) (for an almost commutative ring). The fact that over a non-commutative semilocal ring the elementary subgroups of the classical groups acts transitively on the set of unimodular and istropic \( \langle v, v \rangle = 0 \) vectors of length \( n \geq 2 \) in the linear case and \( n = 2r \geq 4 \) in the non-linear cases has been used in the proof, but it is not mentioned anywhere in the article. This was pointed by Professor R.G. Swan and he provided us a proof for the above result.

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