The Bismut-Elworthy-Li type formulae for stochastic
differential equations with jumps

Atsushi TAKEUCHI∗

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Abstract

Consider jump-type stochastic differential equations with the drift, diffusion and jump terms. Logarithmic derivatives of densities for the solution process are studied, and the Bismut-Elworthy-Li type formulae can be obtained under the uniformly elliptic condition on the coefficients of the diffusion and jump terms. Our approach is based upon the Kolmogorov backward equation by making full use of the Markovian property of the process.

Keywords: heat kernel, jump process, logarithmic derivative, Malliavin calculus.

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1 Introduction

The Malliavin calculus has played an important role in many fields, as one of powerful tools in infinite dimensional analysis. That has also given us an attractive solution to the hypoelliptic problem for the differential operator associated with a stochastic differential equation, by means of probabilistic methods. It is well known that the Hörmander condition on the coefficients of the equation, which is the condition about the Lie algebra generated by the vector fields associated with the coefficients, yields the existence of the smooth density function. See [3, 18] and references therein. Bismut [4] also studied the logarithmic derivatives of the density function with respect to the initial point of a stochastic differential equation on Riemannian manifolds. His approach is based upon the Girsanov transform on Brownian motions. The formula has a nice flavour with the precise estimate of heat kernels or large deviation principles. Elworthy and Li [8] also tackled the

∗E-mail: takeuchi@sci.osaka-cu.ac.jp
Postal Address: Department of Mathematics, Osaka City University, Sugimoto 3-3-138, Sumiyoshi-ku, Osaka 558-8585, JAPAN
same problem in more general class of stochastic differential equations on Riemannian manifolds, via the martingale methods. Nowadays, the celebrated formulae are called the Bismut-Elworthy-Li formulae after their great contributions. The logarithmic derivatives of the density function is equivalent to the Greeks computations for pay-off functions in mathematical finance. Fournié et al. [9] applied the Malliavin calculus on the Wiener space to the sensitivity analysis for asset price dynamics models. They also applied their results to the numerical computations of the Greeks.

All works stated above, paid attention to the case of the processes without any jumps. There has been a natural and non-trivial question whether a similar approach is applicable to the sensitivity analysis in case of jump processes. The interests in jump processes are recently getting more and more in mathematical finance. In the present paper, we shall study the Bismut-Elworthy-Li type formulae for jump processes, with respect to the initial point and the parameter governing the equation. There are some approaches to tackle the problem on the sensitivities: the Girsanov transform approach ([14]) for Lévy processes initiated by Bismut [3], the martingale methods ([5]) similarly to [8] in case of diffusion processes, and an application of the Malliavin calculus on the Wiener-Poisson space ([1, 6, 7]). In particular, Davis and Johansson [6], and Cass and Friz [5] studied in case of jump diffusion processes, but their approach does not take any effects from the jump term. Bally et al. [1] studied the Malliavin calculus with respect to the jump amplitudes and the jump times, and used the integration by parts formula in order to give numerical algorithms for the sensitivity computations in a model driven by Lévy processes. The goal in the present paper is to compute the logarithmic derivatives of densities, including not only the effect from the diffusion terms, but also the one from the jump terms. The results obtained in this paper corresponds to give another approach on the logarithmic derivatives of the density studied by Bismut [3] using the Girsanov transforms.

This paper is organized as follows: In Section 2, we shall prepare some notations and introduce our framework. In Section 3, the main theorems on the logarithmic derivatives for the density with respect to the initial point and the parameter, are given. Those proofs are done in Section 4. Some typical examples are given in the final section.

2 Preliminaries

At the beginning, we shall introduce some general notations. For $\alpha, \beta \in \mathbb{N}$, denote by $C^k_\mathcal{K}(\mathbb{R}^\alpha; \mathbb{R}^\beta)$ the class of $k$ times continuously differentiable, $\mathbb{R}^\beta$-valued mappings on $\mathbb{R}^\alpha$, and by $C^k_{1+b}(\mathbb{R}^\alpha; \mathbb{R}^\beta)$ the class of $C^k(\mathbb{R}^\alpha; \mathbb{R}^\beta)$-functions with bounded derivatives of all orders more than 1. The subscript $K$ of $C^k_\mathcal{K}(\mathbb{R}^\alpha; \mathbb{R}^\beta)$ indicates the compact support. Denote by $\nabla = (\nabla_1, \ldots, \nabla_d)$ the gradient operator in $\mathbb{R}^d$, by $\partial_z = (\partial_{z_1}, \ldots, \partial_{z_m})$ the one in $\mathbb{R}^m$, and by $\partial_\xi = (\partial_{\xi_1}, \ldots, \partial_{\xi_l})$ the one in $\mathbb{R}^l$. Write the identity by $I_d = (\delta_{jk} ; 1 \leq j, k \leq d) \in \mathbb{R}^d \otimes \mathbb{R}^d$. For $M \in \mathbb{R}^\alpha \otimes \mathbb{R}^\beta$, the symbol $[M]_{ij}$ indicates
the \((i, j)\)-component of \(M\). For a subset \(N \subset \mathbb{R}^d\), denote its closure by \(\overline{N}\), its boundary by \(\partial N\), and its Lebesgue measure by \(|N|\). For \(\mathbb{R}^d \otimes \mathbb{R}^m\)-valued function \(\Phi\) on \(\mathbb{R}_0^m\) and \(L \in \mathbb{R}^m \otimes \mathbb{R}^l \otimes \mathbb{R}^m\), define

\[
\{\text{div} \ z [\Phi(z)]\}_k = \sum_{i=1}^m \partial_{z_i} (\Phi_{ik}(z)), \quad \text{div} \ z [\Phi(z)] = ([\text{div} \ z [\Phi(z)]]_1, \ldots, [\text{div} \ z [\Phi(z)]]_d),
\]

\[
\{\text{Tr} [L]\}_k = \sum_{i=1}^m L_{iki}, \quad \text{Tr} [L] = ([\text{Tr} [L]]_1, \ldots, [\text{Tr} [L]]_l).
\]

Denote by \(c_i\)'s different positive finite constants.

Write \(\mathbb{R}^m_0 = \mathbb{R}^m \setminus \{0\}\), and let \(d\nu\) be a Lévy measure on \(\mathbb{R}^m_0\). Moreover, suppose that

**Assumption 1** The measure \(d\nu\) satisfies the following three conditions:

(i) for any \(p \geq 1\),

\[
\int_{\mathbb{R}^m_0} (|z| I_{|z| \leq 1} + |z|^p I_{|z| > 1}) \, d\nu < +\infty,
\]

(ii) there exists a constant \(\alpha > 0\) such that, for any \(\theta \in S^{m-1}\),

\[
\liminf_{\rho \to 0} \rho^\alpha \int_{\mathbb{R}^m_0} (|z \cdot \theta/\rho|^2 \wedge 1) \, d\nu > 0,
\]

(iii) there exists a \(C^1\)-density \(g(z)\) with respect to the Lebesgue measure on \(\mathbb{R}^m_0\) such that

\[
\lim_{|z| \to \infty} |z|^2 g(z) = 0.
\]

**Remark 2.1** Lévy processes such as tempered stable processes, inverse Gaussian processes, etc. satisfy Assumption 1.

**Remark 2.2** In order to study the existence of a smooth density for jump processes, Ishikawa and Kunita [13], and Picard [19] impose the following two conditions to the measure \(d\nu\) instead of Assumption 1:

(iv) there exists \(0 < \beta < 2\) such that

\[
\liminf_{\rho \to 0} \rho^{-\beta} \int_{|z| \leq \rho} |z|^2 \, d\nu > 0,
\]

(v) there exists a positive definite matrix \(\Xi \in \mathbb{R}^m \otimes \mathbb{R}^m\) such that, for any \(\theta \in S^{m-1}\),

\[
\liminf_{\rho \to 0} \left(\int_{|z| \leq \rho} |z|^2 \, d\nu\right)^{-1} \int_{|z| \leq \rho} |z \cdot \theta|^2 \, d\nu \geq \theta \cdot \Xi \theta.
\]
The condition (iv) is called the order condition on the measure \( dv \), and the Lévy process satisfying the condition (v) is called non-degenerate. It can be easily checked that the above conditions (iv) and (v) imply (ii) in Assumption 1. In fact,

\[
\int_{\mathbb{R}^m_0} (|z \cdot \theta|/\rho|^2 \land 1) \, dv \geq \rho^{-2} \int_{|z| \leq \rho} |z \cdot \theta|^2 \, dv \geq c_1 \rho^{-2+\theta} \cdot \Xi \theta.
\]

Let \( T > 0 \) be fixed, and \((\Omega, \mathcal{F}, \mathbb{P})\) the underlying probability space. Denote an \( m \)-dimensional Brownian motion with \( W_0 = 0 \) by \( \{W_t = (W^1_t, \ldots, W^m_t)^T; \ t \in [0, T]\} \), and by \( d\mu \) the Poisson random measure on \([0, T] \times \mathbb{R}^m_0\) with the intensity \( d\mu(t) = dt \, dv \). Let \( \mathcal{F}_t = \{\mathcal{F}_t; \ t \in [0, T]\} \) be the augmented filtration generated by \( W \) and \( d\mu \) with respect to \( \mathbb{P} \). Define \( d\tilde{\mu} = d\mu - d\mu \) and \( d\tilde{\nu} = I_{(|z| \leq 1)} d\mu + I_{(|z| > 1)} d\mu \).

Let \( a_i : \mathbb{R}^l \times \mathbb{R}^d \rightarrow \mathbb{R}^d \) \( (i = 0, 1, \ldots, m) \) and \( b : \mathbb{R}^l \times \mathbb{R}^{d} \times \mathbb{R}^m_0 \rightarrow \mathbb{R}^d \) such that

**Assumption 2** The \( \mathbb{R}^d \)-valued functions \( a_0, a_1, \ldots, a_m, b \) satisfy

(i) \( a_i (\varepsilon, \cdot) \in C^\infty_{1+b} (\mathbb{R}^d; \mathbb{R}^d) \) for each \( \varepsilon \in \mathbb{R}^l \), and \( a_i (\cdot, y) \in C^1_{1+b} (\mathbb{R}^l; \mathbb{R}^d) \) for each \( y \in \mathbb{R}^d \),

(ii) \( b (\varepsilon, \cdot) \in C^\infty_{1+b} (\mathbb{R}^{d} \times \mathbb{R}^m_0; \mathbb{R}^d) \) for each \( \varepsilon \in \mathbb{R}^l \), and \( b_z (\cdot, y) \in C^1_{1+b} (\mathbb{R}^l; \mathbb{R}^d) \) for each \((y, z) \in \mathbb{R}^d \times \mathbb{R}^m_0 \),

(iii) for each \( \varepsilon \in \mathbb{R}^l \),

\[
\lim_{|z| \rightarrow 0} b_z (\varepsilon, y) = 0, \quad \inf_{y \in \mathbb{R}^d} \inf_{z \in \mathbb{R}^m_0} \left| \det [I_d + \nabla b_z (\varepsilon, y)] \right| > 0.
\]

Write \( a = (a_1, \ldots, a_m) \). For \((\varepsilon, x) \in \mathbb{R}^l \times \mathbb{R}^d \), consider the stochastic differential equation:

\[
dx_t = a_0 (\varepsilon, x_t) \, dt + a (\varepsilon, x_t) \circ dW_t + \int_{\mathbb{R}^m_0} b_z (\varepsilon, x_t) \, d\tilde{\mu}, \quad x_0 = x.
\]

Since the coefficients satisfy the Lipschitz and linear growth conditions under Assumption 2, there exists a unique solution \( \{x_t = x_{t}^{\varepsilon, x}; \ t \in [0, T]\} \) (cf. [12]). The infinitesimal generator \( \mathcal{L}^\varepsilon \) associated with the solution process \( \{x_t; \ t \in [0, T]\} \) is given by

\[
(\mathcal{L}^\varepsilon f)(\varepsilon) = A_0^\varepsilon f (\varepsilon) + \frac{1}{2} \sum_{i=1}^{m} A_i^\varepsilon A_i^\varepsilon f (\varepsilon) + \int_{\mathbb{R}^m_0} \left( \mathbb{B}^\varepsilon f (\varepsilon) - B_z^\varepsilon f (\varepsilon, y) I_{(|z| \leq 1)} \right) \, dv
\]

for \( f \in C^2_b (\mathbb{R}^d; \mathbb{R}) \), where \( A_i^\varepsilon f (\varepsilon) = \nabla f (\varepsilon) a_i (\varepsilon, y) \), \( B_z^\varepsilon f (\varepsilon) = \nabla f (\varepsilon) b_z (\varepsilon, y) \) are vector fields, \( A_i^\varepsilon A_i^\varepsilon f (\varepsilon) = \nabla (\nabla f (\varepsilon) a_i (\varepsilon, y)) a_i (\varepsilon, y) \) and \( \mathbb{B}^\varepsilon f (\varepsilon) = f (y + b_z (\varepsilon, y)) - f (y) \). Moreover, Assumption 2 yields that the mapping \( \mathbb{R}^d \ni x \mapsto x_{t}^{\varepsilon, x} \in \mathbb{R}^d \) has a \( C^1 \)-modification for each \((t, \varepsilon) \in [0, T] \times \mathbb{R}^l \).
and its Jacobi matrix $Z_t := \nabla x_t$ satisfies the linear stochastic differential equation:

$$dZ_t = \nabla a_0(x, x_t) Z_t dt + \nabla a(x, x_t) Z_t \circ dW_t + \int_{\mathbb{R}^m} \nabla b_z(x, x_{t-}) Z_t - d\bar{\mu}, \quad Z_0 = I_d. \quad (2.2)$$

Let $\{U_t; t \in [0, T]\}$ be the solution to the linear stochastic differential equation: $U_0 = I_d$ and

$$dU_t = -U_t \nabla a_0(x, x_t) dt - U_t \nabla a(x, x_t) \circ dW_t - \int_{\mathbb{R}^m} U_{t-} \left[(I_d + \nabla b_z) \nabla b_z \right](x, x_{t-}) d\bar{\mu}$$

$$\quad + \int_{|t| \leq 1} U_t \left[(I_d + \nabla b_z) \nabla b_z \right](x, x_t) d\mu. \quad (2.3)$$

Then, $Z_t U_t = U_t Z_t = I_d$ holds for each $t \in [0, T]$ by the Itô product formula. Under Assumption 1 on the measure $d\nu$, and Assumption 2 on the coefficients, the upper estimate

$$\mathbb{E} \left[ \sup_{t \in [0, T]} \left( |x_t|^p + \|Z_t\|^p + \|U_t\|^p \right) \right] \leq c_{2,p,\varepsilon,T} (1 + |x|^p)$$

holds for any $p > 1$. See [10]. Moreover, we have

**Proposition 2.1** For each $t \in [0, T]$, the mapping $\mathbb{R}^l \ni \varepsilon \mapsto x_t \in \mathbb{R}^d$ has a $C^1$-modification, and the derivative $H_t := \partial_x x_t$ satisfies the equations: $H_0 = 0 \in \mathbb{R}^l \otimes \mathbb{R}^d$, and

$$dH_t = \nabla a_0(x, x_t) H_t dt + \nabla a(x, x_t) H_t \circ dW_t + \int_{\mathbb{R}^m} \nabla b_z(x, x_{t-}) H_{t-} d\bar{\mu}$$

$$\quad + \partial_x a_0(x, x_t) dt + \partial_x a(x, x_t) \circ dW_t + \int_{\mathbb{R}^m} \partial_x b_z(x, x_{t-}) d\bar{\mu}. \quad (2.4)$$

Moreover, it holds that, for any $p > 1$ and any compact subset $K$ in $\mathbb{R}^l$,

$$\sup_{x \in K} \mathbb{E} \left[ \sup_{t \in [0, T]} \|H_t\|^p \right] \leq c_{3,p,T}.$$

**Proof.** We shall write $x_t^\varepsilon = x_t^\delta$, in order to emphasize the dependence on $\varepsilon \in \mathbb{R}^l$ throughout the proof. Let $(\varepsilon, \delta) \in \mathbb{R}^l \times \mathbb{R}^l$. Since

$$x_t^\varepsilon - x_t^\delta = \int_0^t \left\{a_0(x, x_t^\varepsilon) - a_0(x, x_t^\delta)\right\} ds + \int_0^t \left\{a(x, x_t^\varepsilon) - a(x, x_t^\delta)\right\} \circ dW_s$$

$$\quad + \int_0^t \int_{\mathbb{R}^m} \left\{b_z(x, x_{t-}^\varepsilon) - b_z(x, x_{t-}^\delta)\right\} d\bar{\mu},$$

we can get

$$\mathbb{E} \left[ \sup_{t \leq T} |x_t^\varepsilon - x_t^\delta|^p \right] \leq c_{4,p,\varepsilon,T} |\varepsilon - \delta|^p.$$
for any $p > 1$, from Assumption 1 and 2. Thus, the Kolmogorov continuity criterion tells us that the mapping $\mathbb{R}^l \ni \varepsilon \mapsto x_t^\varepsilon \in \mathbb{R}^d$ has a continuous modification for each $t \geq 0$ and $x \in \mathbb{R}^d$.

Next, we shall study the differentiability of $x_t^\varepsilon$ in $\varepsilon \in \mathbb{R}^l$. Let $0 \neq \xi, \zeta \in \mathbb{R}$, and $e_k \in \mathbb{R}^l$ the $k$-th unit vector. Since

$$
(x_t^{\varepsilon+\xi e_k} - x_t^{\varepsilon}) / \xi = \int_0^t \left\{ a_0 \left( \varepsilon + \xi e_k, x_s^{\varepsilon+\xi e_k} \right) - a_0 \left( \varepsilon, x_s^{\varepsilon} \right) \right\} / \xi \, ds
$$

$$
+ \int_0^t \left\{ a \left( \varepsilon + \xi e_k, x_s^{\varepsilon+\xi e_k} \right) - a \left( \varepsilon, x_s^{\varepsilon} \right) \right\} / \xi \circ dW_s
$$

$$
+ \int_0^t \int_{\mathbb{R}^m} b_z \left( \varepsilon + \xi e_k, x_s^{\varepsilon+\xi e_k} \right) - b_z \left( \varepsilon, x_s^{\varepsilon} \right) \right\} / \xi \, d\mu,
$$

we can get the upper estimate

$$
\mathbb{E} \left[ \sup_{t \leq T} \left| \frac{(x_t^{\varepsilon+\xi e_k} - x_t^{\varepsilon})}{\xi} - \left( \frac{x_t^{\varepsilon+\xi e_k} - x_t^{\varepsilon}}{\xi} \right) \right|^p \right] \leq c_{5,p,x,T,e,k} |\xi - \zeta|^p
$$

for any $p > 1$. Hence, the mapping $\mathbb{R}^l \ni \varepsilon \mapsto x_t^\varepsilon \in \mathbb{R}^d$ has a $C^1$-modification with respect to the parameter $\varepsilon \in \mathbb{R}^l$ for each $t \geq 0$ and $x \in \mathbb{R}^d$, via the Kolmogorov continuity criterion, again.

Furthermore, Assumption 2 enables us to justify that the derivative $\partial_\varepsilon x_t^\varepsilon$ satisfies the equation (2.4). It is an easy work to check the upper estimate of $\partial_\varepsilon x_t$ in the assertion. □

**Corollary 2.1** The derivative $H_t = \partial_\varepsilon x_t$ can be computed as follows:

$$
H_t = Z_t \int_0^t U_s \left( \partial_\varepsilon a_0 (\varepsilon, x_s) - \int_{|i| \leq 1} \left( (I_d + \nabla b_z)^{-1} \nabla b_z \partial_\varepsilon b_z \right) (\varepsilon, x_s) \, dv \right) \, ds
$$

$$
+ Z_t \int_0^t U_s \partial_\varepsilon a (\varepsilon, x_s) \circ dW_s + Z_t \int_0^t \int_{\mathbb{R}^m} U_{s-} \left( (I_d + \nabla b_z)^{-1} \partial_\varepsilon b_z \right) (\varepsilon, x_{s-}) \, d\mu
$$

(2.5)

$$
=: Z_t \int_0^t f^\varepsilon_0 (s) \, ds + Z_t \int_0^t f^\varepsilon (s) \circ dW_s + Z_t \int_0^t \int_{\mathbb{R}^m} h^\varepsilon (s) \, d\mu.
$$

**Proof.** Obvious by Proposition 2.1 and the Itô product formula. □

### 3 Main theorems

Let us present an assumption on the coefficients of the equation (2.1), which is crucial for discussions in what follows.
Assumption 3 There exist constants $c_{6,ε}, c_{7,ε} > 0$ such that
\[
\sum_{i=1}^{m} |ξ \cdot a_i(ε, y)|^2 \geq c_{6,ε} |ξ|^2, \quad \sum_{i=1}^{m} |ξ \cdot \partial_ε b_i(ε, y)|^2 \geq c_{7,ε} |ξ|^2
\]
for any $(y, ξ) \in \mathbb{R}^d \times \mathbb{R}^d$ and $z \in \mathbb{R}^m$.

Define the $\mathbb{R}^d$-valued function $\tilde{b}$ by $\tilde{b}_i(ε, y) = \left[(I_d + \nabla b_i)^{-1} \partial_ε b_i(ε, y)\right](ε, y) z$. Then, we shall introduce the well-known criterion on the existence of the smooth density.

**Proposition 3.1 (cf. [15, 16])** If there exist constants $c_{8,ε} > 0$ and $γ > 0$ such that
\[
\inf_{y \in \mathbb{R}^d} \inf_{ξ \in S^{d-1}} \left\{ \sum_{i=1}^{m} |a_i(ε, y) \cdot ξ/ρ|^2 + \int_{\mathbb{R}_0^m} \left[|\tilde{b}_i(ε, y) \cdot ξ/ρ|^2 \wedge 1\right] dv \right\} \geq c_{8,ε} ρ^{-γ} (\text{3.1})
\]
for $0 < ρ < 1$, then the probability law of the random variable $x_T = x^{(ε)}_T$ has a density $p_T(ε, x, y)$ with respect to the Lebesgue measure on $\mathbb{R}^d$ such that $p_T(ε, x, y)$ is smooth in $y \in \mathbb{R}^d$.

**Remark 3.1** It can be easily checked that Assumption 1, 2 and 3 imply the condition (3.1) in Proposition 3.1. In fact, since
\[
1 \leq \left|\left[(I_d + \nabla b_i)^{-1}\right]^*(ε, y) ξ\right| \left|\left[I_d + \nabla b_i\right](ε, y) ξ\right| \leq c_{9,ε} \left|\left[(I_d + \nabla b_i)^{-1}\right]^*(ε, y) ξ\right|
\]
for $ξ \in S^{d-1}$, we see that
\[
\left|\left[(I_d + \nabla b_i)^{-1} \partial_ε b_i(ε, y) \right] (ε, y) ξ/ρ\right|^2 \geq c_{10,ε} \left|\left[(I_d + \nabla b_i)^{-1}\right]^*(ε, y) ξ/ρ\right|^2 \geq c_{11,ε} ρ^{-2}
\]
under Assumption 2 and 3. Then we have
\[
\int_{\mathbb{R}_0^m} \left[|\tilde{b}_i(ε, y) \cdot ξ/ρ|^2 \wedge 1\right] dv = \int_{\mathbb{R}_0^m} \left[\left|\left[(I_d + \nabla b_i)^{-1} \partial_ε b_i(ε, y) \right] (ε, y) ξ/ρ\right|^2 \wedge 1\right] dv
\]
\[
\geq c_{12,ε} \inf_{|θ|=1} \int_{\mathbb{R}_0^m} \left|\left[z \cdot θ/ρ\right|^2 \wedge 1\right] dv
\]
\[
\geq c_{13,ε} ρ^{-α}
\]
from Assumption 1 (ii), for sufficiently small $0 < ρ < 1$. \hfill \Box

We are now in a position to present main results. To avoid lengthy expressions, let us prepare some auxiliary notations. Define
\[
v_s(ε, z) = \left[(\partial_ε b) (I_d + \nabla b)\right](ε, x_s) Z_s |z|^2,
\]
Theorem 1 Let \( \varphi \) be in \( C^2_k \left( B \R^d ; \R \right) \). Then, it holds that

\[
\nabla_s \left( \mathbb{E} [ \varphi (x_T) ] \right) = \mathbb{E} \left[ \varphi (x_T) \left\{ \frac{L_{0,T} - J_{0,T}}{A_{0,T}} + \frac{K_{0,T}}{(A_{0,T})^2} \right\} \right] \quad \left( =: \mathbb{E} \left[ \varphi (x_T) \Gamma^{(1)}_T \right] \right). \tag{3.2}
\]

Next, we shall study the sensitivity in \( \varepsilon \in \R^d \). For the sake of simplicity on notations, define

\[
L^\varepsilon_t = \int_0^t dW^\varepsilon_s a (\varepsilon, x_s)^{-1} Z_s f^\varepsilon_0 (s), \quad G^\varepsilon_t = \int_0^t f^\varepsilon(s) \circ dW_s,
\]

\[
R^\varepsilon_t = \frac{1}{t} \int_0^t dW^\varepsilon_s a (\varepsilon, x_s)^{-1} Z_s G^\varepsilon_t, \quad Q^\varepsilon_t = \frac{1}{t} \int_0^t \text{Tr} \left[ a (\varepsilon, x_s)^{-1} Z_s D_s G^\varepsilon_t \right] d s,
\]

\[
\tilde{v}_s (\varepsilon, z) = \left( (\partial_z b_z)^{-1} (I_d + \nabla b_z) \right) (\varepsilon, x_s) \left( Z_s h^\varepsilon_s (s) \right), \quad F^\varepsilon_t = \int_0^t \int_{\R^m} \frac{\partial_s \left[ g (z) \tilde{v}_s (\varepsilon, z) \right]}{g (z)} d \tilde{\mu},
\]

where \( f_0^\varepsilon(s), f^\varepsilon(s) \) and \( h^\varepsilon_s(s) \) are given in Corollary 2.1, and \( \{D_s ; s \in [0, T] \} \) is the Malliavin derivative operator.

Theorem 2 Let \( \varphi \in C^2_k \left( B \R^d ; \R \right) \). Then, it holds that

\[
\partial_\varepsilon ( \mathbb{E} [ \varphi (x_T) ] ) = \mathbb{E} [ \varphi (x_T) \{ L^\varepsilon_t + R^\varepsilon_t - Q^\varepsilon_t - J^\varepsilon_t \} ] \quad \left( =: \mathbb{E} \left[ \varphi (x_T) \Gamma^{(2)}_T \right] \right). \tag{3.3}
\]

Finally, we next study the second order derivative in \( x \in \R^d \). To keep the presentation as concise as possible, write \( \bar{T} = T/2 \), and define

\[
F^1_{ijk}(\varepsilon, t) = \sum_{\beta=1}^d \left\{ \nabla_{x_j} \left[ a (\varepsilon, x_i)^{-1} \right]_{\beta} Z^\beta_t + \left[ a (\varepsilon, x_i)^{-1} \right]_{\beta} \nabla_{x_j} Z^\beta_t \right\},
\]

\[
F^2_{ijk}(\varepsilon, t, z) = - \sum_{\beta=1}^d \left[ (\partial_z b_z (\varepsilon, x_i))^{-1} \right]_{\beta} \nabla_{x_i} \left[ ((I_d + \nabla b_z (\varepsilon, x_i)) Z_t)_{\beta j} \right] |z|^2,
\]

\[
F^3_{ijk}(\varepsilon, t, z) = \sum_{\beta=1}^d \left[ (\partial_z b_z (\varepsilon, x_i))^{-1} \right]_{\beta} \nabla_{x_i} \left( (\partial_z b_z (\varepsilon, x_i)) v_i (\varepsilon, z) \right) Z^\beta_t.
\]
for $1 \leq i \leq m$ and $1 \leq j,k \leq d$. Moreover, write $F^1(e,t) = (F^1_{ijk}(e,t))_{1 \leq i \leq m, 1 \leq j,k \leq d}$, and $F^\sigma(e,t,z) = (F^\sigma_{ijk}(e,t,z))_{1 \leq i \leq m, 1 \leq j,k \leq d}$ for $\sigma = 2, 3$. Then we have

**Theorem 3** Let $\varphi$ be in $C^2_K(\mathbb{R}^d; \mathbb{R})$. Then, the equality

$$
\nabla_x \nabla_x (\mathbb{E} [\varphi(x_T)])
= \mathbb{E} \left[ \varphi(x_T) \left\{ \left( \Gamma^{(i)*}_{T,T} + \frac{K_{i,T}^*}{A_{i,T} A_0.T} \right) \Gamma^{(i)}_{0,T} + \frac{K_{i,T}^* K_{0,T}}{A_{i,T} (A_0.T)^2} - \sum_{\sigma=2}^3 \int_0^T \int_{\mathbb{R}_n} \frac{\text{div}_z [F^\sigma(e,s,z)]}{A_{0,T} + |z|^2} d\mu \right\} \right]
+ \frac{1}{A_{0,T}} \left( \int_0^T F^1(e,s) dW_x + \sum_{\sigma=2}^3 \int_0^T \int_{\mathbb{R}_n} \text{div}_z [F^\sigma(e,s,z)] d\mu \right) \right]

(=: \mathbb{E} [\varphi(x_T) \Gamma^{(3)}_T])
$$

holds, where $\Gamma^{(i)}_{r,t} = (L_{r,t} - J_{r,t}) / A_{r,t} + K_{r,t} / (A_{r,t})^2$ for $0 \leq \tau \leq t \leq T$.

**Remark 3.2** For $0 \leq \tau < t \leq T$, define $N^{A}_{r,t} = \int_{r,t} \int_{\mathbb{R}_n} (e^{-|\lambda|^2} - 1) d\mu$. Then, $\mathbb{E} [A_{r,t}^{-p}] < +\infty$ holds for any $p > 1$, since the condition (ii) in Assumption 3 on the measure $d\nu$ yields that

$$
\mathbb{E} [A_{r,t}^{-p}] = \frac{1}{\Gamma(p)} \int_0^{+\infty} \lambda^{p-1} \mathbb{E} \left[ \exp \left( -\lambda A_{r,t} \right) \right] d\lambda
= \frac{1}{\Gamma(p)} \int_0^{+\infty} \lambda^{p-1} \mathbb{E} \left[ \exp \left( -\lambda A_{r,t} - N^{A}_{r,t} \right) \right] e^{N^{A}_{r,t}} d\lambda
\leq c_{14,T} \int_0^{+\infty} \lambda^{p-1} \exp \left\{ - (t-\tau) \lambda - (t-\tau) \int_{\mathbb{R}_n} (|\lambda|^2 \wedge 1) d\nu \right\} d\lambda
\leq c_{14,T} \int_0^{+\infty} \lambda^{p-1} \exp \left\{ - (t-\tau) \lambda - c_{15} (t-\tau)^{\sigma/2} \right\} d\lambda
< +\infty.
$$

Denote by $\mathcal{U}$ the family of bounded domains and their complements in $\mathbb{R}^d$. Define the class $\mathfrak{F}$ of $\mathbb{R}$-valued functions by

$$
\mathfrak{F} = \left\{ f = \sum_{k=1}^n a_k f_k I_{A_k} : n \in \mathbb{N}, a_k \in \mathbb{R}, |f_k(y)| \leq c_{16,k} (1 + |y|), A_k \in \mathcal{U} \right\}.
$$

**Corollary 3.1** Let $\varphi \in \mathfrak{F}$, and $\Gamma^{(i)}_T$ $(i = 1, 2, 3)$ be random variables defined in Theorem 1, 2 and 3. Then, the following equalities hold.

$$
\nabla_x (\mathbb{E} [\varphi(x_T)]) = \mathbb{E} [\varphi(x_T) \Gamma^{(3)}_T],
$$

9
\[ \partial_v (\mathbb{E} [\varphi (x_T)]) = \mathbb{E} [\varphi (x_T) \Gamma^{(2)}_T], \]
\[ \nabla_x \nabla_x (\mathbb{E} [\varphi (x_T)]) = \mathbb{E} [\varphi (x_T) \Gamma^{(3)}_T]. \]

**Remark 3.3** The class \( \mathcal{F} \) is smaller than that of measurable functions \( \varphi \) satisfying \( \mathbb{E} [\varphi (x_T)^2] < +\infty \). But, the class \( \mathcal{F} \) is rich enough from a practical point of view in mathematical finance, because various payoff functions for asset price dynamics such as call options, put options, digital options, and so on, are included in \( \mathcal{F} \).

**Remark 3.4** Consider the case where the \( \mathbb{R}^d \otimes \mathbb{R}^d \)-valued function \( a (\epsilon, y) a (\epsilon, y)^* \) is uniformly elliptic in \( y \in \mathbb{R}^d \), while the function \( \partial_\epsilon b_\epsilon (\epsilon, y) \partial_\epsilon b_\epsilon (\epsilon, y)^* \) is *not always* uniformly elliptic in \( y \in \mathbb{R}^d \) and \( z \in \mathbb{R}^m_0 \). Although Assumption 3 is not satisfied, this case can be also discussed in our position by ignoring any jump effects. Then, the sensitivity formulae are given as follows (cf. [5]):

\[ \Gamma^{(1)}_T = \frac{L_{0,T}}{T}, \quad \Gamma^{(2)}_T = L^F_T + R^F_T - \mathcal{Q}^c_T, \quad \Gamma^{(3)}_T = \frac{L^s_{0,T} L_{0,T}}{T^2} + \frac{1}{T} \int_0^T F^1 (\epsilon, s) dW_s. \]

Moreover, remark that Assumption 1 on the measure \( dv \) is not necessary. In case of \( b_\epsilon (\epsilon, y) \equiv 0 \), these are exactly the Bismut-Elworthy-Li formulae. See [4] and [8].

**Remark 3.5** Consider the case where the \( \mathbb{R}^d \otimes \mathbb{R}^d \)-valued function \( \partial_\epsilon b_\epsilon (\epsilon, y) \partial_\epsilon b_\epsilon (\epsilon, y)^* \) is uniformly elliptic in \( y \in \mathbb{R}^d \) and \( z \in \mathbb{R}^m_0 \), while the function \( a (\epsilon, y) a (\epsilon, y)^* \) is *not always* uniformly elliptic in \( y \in \mathbb{R}^d \). Although Assumption 3 is not satisfied, this case can be also discussed in our position by ignoring any diffusion effects. Then, the process can be of pure-jump type and of infinite-activity type, and Assumption 1 on \( dv \) is essential. The sensitivity formulae are

\[ \Gamma^{(1)}_T = - \frac{V_{0,T}}{A_{0,T}} + \frac{K_{0,T}}{(A_{0,T})^2}, \quad \Gamma^{(2)}_T = -J^F_T, \]
\[ \Gamma^{(3)}_T = \left\{ - \frac{V^*_{0,T}}{A_{0,T}} + \frac{K^*_{0,T}}{(A_{0,T})^2} \right\} \left( - \frac{V_{0,T}}{A_{0,T}} + \frac{K_{0,T}}{(A_{0,T})^2} \right) + \frac{K^*_{0,T} K_{0,T}}{A_{0,T}^2 (A_{0,T})^3} \]
\[ - \sum_{\sigma = 2}^3 \int_0^T \int_{\mathbb{R}^m_0} \int_{\mathbb{R}^m_0} \text{div}_z \left[ F^\sigma (\epsilon, s, z) \right] d\mu \right. \]
\[ \left. + \frac{1}{A_{0,T}} \sum_{\sigma = 2}^3 \int_0^T \int_{\mathbb{R}^m_0} \text{div}_z \left[ F^\sigma (\epsilon, s, z) \right] d\mu, \]

where \( A_{r,T} = \int_{r}^{T} |z|^2 \left. d\mu. \)

**Remark 3.6** Bismut [3] obtained the integration by parts formula for jump processes via the Girsanov transform, and studied the existence of smooth densities. Then, it is crucial to study the invertibility on the non-negative definite, symmetric matrices valued random variable, which is called the Malliavin covariance matrix. *The Hörmander type condition* on the coefficients of the
equation (2.1), instead of Assumption 3, enables us to check the invertibility of the Malliavin covariance matrix (cf. [15] and [16]). Then, it would be possible to compute the concrete representations as stated in Corollary 3.1 in the hypoelliptic situation via a similar manner to the one in the uniformly elliptic situation, which will be studied elsewhere (cf. [20]). □

4 Proofs

We shall devote this section to prove our main results. For \( t \in [0, T] \) and \( \varphi \in C^2_K (\mathbb{R}^d; \mathbb{R}) \), define

\[
u(t, x) = \mathbb{E} [\varphi (x_T) | x_0 = x].
\]

Then, it holds that \( u \in C^{1,2}_b ([0, T] \times \mathbb{R}^d; \mathbb{R}) \), \( \lim_{t \to T} u(t, x) = \varphi(x) \), and \((\partial_t + L^e) u = 0 \) (cf. [11]).

The following lemma can be regarded as the martingale representation on \( \varphi(x_T) \), and plays a crucial role in what follows.

**Lemma 4.1** For \( \varphi \in C^2_K (\mathbb{R}^d; \mathbb{R}) \), it holds that

\[
\varphi(x_T) = \mathbb{E} [\varphi(x_T)] + \int_0^T \nabla u(s, x_s) a(\varepsilon, x_s) dW_s + \int_0^T \int_{\mathbb{R}^d} \mathcal{Q}^e_u(s, x_{s-}) d\tilde{\mu}.
\]

**Proof.** Let \( t \in [0, T] \). Since \( u \in C^{1,2}_b ([0, T] \times \mathbb{R}^d; \mathbb{R}) \) and \((\partial_t + L^e) u = 0 \), the Itô formula yields

\[
u(t, x_t) = \nu(0, x) + \int_0^t \nabla u(s, x_s) a(\varepsilon, x_s) dW_s + \int_0^t \int_{\mathbb{R}^d} \mathcal{Q}^e_u(s, x_{s-}) d\tilde{\mu}.
\]

Since \( \varphi \in C^2_K (\mathbb{R}^d; \mathbb{R}) \), it holds that \( u(t, x_t) = \mathbb{E} [\varphi(x_T) | F_T] \to \mathbb{E} [\varphi(x_T) | F_T] = \varphi(x_T) \) as \( t \to T \) (cf. [12]). It can be easily checked from Assumption 2 that stochastic integrals in the right hand side of the equality (4.2) converge to the ones in (4.1) as \( t \to T \), respectively. □

Taking the differential of (4.1) in Lemma 4.1, we have

**Lemma 4.2** For \( 1 \leq k \leq d \) and \( \varphi \in C^2_K (\mathbb{R}^d; \mathbb{R}) \), it holds that

\[
\nabla_{x_k} (\varphi(x_T)) = \mathbb{E} [\nabla_{x_k} (\varphi(x_T))] + \int_0^T \nabla_{x_k} (\nabla u(s, x_s) a(\varepsilon, x_s)) dW_s
\]
\[+ \int_0^T \int_{\mathbb{R}^d} \nabla_{x_k} (\mathcal{Q}^e_u(s, x_{s-})) d\tilde{\mu}.
\]

(4.3)
Proof. We shall write \( x_t = x_t^\epsilon \), in order to emphasize the dependence on \( x \in \mathbb{R}^d \) throughout the proof. Taking the derivative of the equality (4.1) in Lemma 4.1, we have
\[
\nabla_{x_t} (\varphi (x_T^\epsilon)) = \nabla_{x_t} \left( \mathbb{E} [\varphi (x_T^\epsilon)] \right) + \nabla_{x_t} \left( \int_0^T \nabla u (s, x_s^\epsilon) a (\epsilon, x_s^\epsilon) dW_s \right) \\
+ \nabla_{x_t} \left( \int_0^T \int_{\mathbb{R}^n} \mathcal{B}_z^\epsilon u (s, x_s^\epsilon) d\mu \right).
\]

Let \( 0 < \delta < 1 \), and \( e_k \in \mathbb{R}^d \) be the \( k \)-th unit vector. Then, we have
\[
\left| \mathbb{E} \left[ \frac{\varphi (x_T^{\epsilon+\delta e_k}) - \varphi (x_T^\epsilon)}{\delta} \right] - \mathbb{E} [\nabla_{x_t} (\varphi (x_T^\epsilon))] \right| \leq \int_0^1 \mathbb{E} \left[ \left| \nabla_{x_t} \varphi (x_T^\epsilon) \right| d\sigma, \right.
\]
which tends to 0 as \( \delta \searrow 0 \), because \( \varphi \in C^2_K (\mathbb{R}^d; \mathbb{R}) \) and \( Z_t \in L^2 (\Omega, \mathbb{P}) \). Hence, we get
\[
\nabla_{x_t} (\mathbb{E} [\varphi (x_T^\epsilon)]) = \mathbb{E} [\nabla_{x_t} (\varphi (x_T^\epsilon))].
\]

On the other hand, since \( u \in C^{1,2}_b ([0, T) \times \mathbb{R}^d; \mathbb{R}) \), Assumption 2 and \( x_t, Z_t \in L^p (\Omega, \mathbb{P}) \) for any \( p > 1 \), we have
\[
\mathbb{E} \left[ \left| \int_0^T \left( \nabla u (s, x_s^\epsilon + \delta e_k) a (\epsilon, x_s^\epsilon + \delta e_k) - \nabla u (s, x_s^\epsilon) a (\epsilon, x_s^\epsilon) \right) \right| dW_s \right] \\
\leq \int_0^T \mathbb{E} \left[ \left| \nabla_{x_t} \left( \nabla u (s, x_s^\epsilon + \sigma e_k) a (\epsilon, x_s^\epsilon + \sigma e_k) - \nabla_{x_t} (\nabla u (s, x_s^\epsilon) a (\epsilon, x_s^\epsilon)) \right) \right| ds \right]
\]
which tends to 0 as \( \delta \searrow 0 \). Thus, we get
\[
\nabla_{x_t} \left( \int_0^T \nabla u (s, x_s^\epsilon) a (\epsilon, x_s^\epsilon) dW_s \right) = \int_0^T \nabla_{x_t} (\nabla u (s, x_s^\epsilon) a (\epsilon, x_s^\epsilon)) dW_s.
\]

Similarly to the above, it holds that
\[
\nabla_{x_t} \left( \int_0^T \int_{\mathbb{R}^n} \mathcal{B}_z^\epsilon u (s, x_s^\epsilon) d\mu \right) = \int_0^T \int_{\mathbb{R}^n} \nabla_{x_t} (\mathcal{B}_z^\epsilon u (s, x_s^\epsilon)) d\mu.
\]

Lemma 4.3 For \( \varphi \in C^2_K (\mathbb{R}^d; \mathbb{R}) \), it holds that
\[
\mathbb{E} \left[ \varphi (x_T) \int_0^T \int_{\mathbb{R}^n} |z|^2 d\mu \right] = \mathbb{E} \left[ \int_0^T \int_{\mathbb{R}^n} u (t, x_t + b (\epsilon, x_t)) |z|^2 d\mu \right],
\]

□
\[ \mathbb{E} \left[ \phi(x_T) \int_0^T \int_{\mathbb{R}^m} h^e_\varepsilon(t) \, d\mu \right] = \mathbb{E} \left[ \int_0^T \int_{\mathbb{R}^m} u(t, x_t + b_\varepsilon(\varepsilon, x_t)) \, h^e_\varepsilon(t) \, d\mu \right]. \]

**Proof.** We shall only prove the second assertion, because the first assertion can be proved in a similar manner. Write \( M^e_T = \int_0^T \int_{\mathbb{R}^m} h^e_\varepsilon(t) \, d\mu, \quad M^e_T = \int_0^T \int_{\mathbb{R}^m} h^e_\varepsilon(t) \, d\hat{\mu}. \) Remark that

\[
\mathbb{E} \left[ \int_0^T \nabla u(t, x_t) \, a(\varepsilon, x_t) \, dW_t, M^e_T \right] = \mathbb{E} \left[ \int_0^T \sum_{i=1}^m \nabla u(t, x_t) \, a_i(\varepsilon, x_t) \, M^e_t \, dW_t \right] + \mathbb{E} \left[ \int_0^T \left( \int_0^t \nabla u(s, x_s) \, a(\varepsilon, x_s) \, dW_s \right) \, dM^e_t \right] = \mathbb{E} \left[ \int_0^T \left( \int_0^t \nabla u(s, x_s) \, a(\varepsilon, x_s) \, dW_s \right) \, d\hat{M}^e_t \right],
\]

and

\[
\mathbb{E} \left[ \int_0^T \int_{\mathbb{R}^m} \mathcal{B}^e_\varepsilon u(t, x_{t-}) \, d\hat{\mu}, M^e_T \right] = \mathbb{E} \left[ \int_0^T \int_{\mathbb{R}^m} \mathcal{B}^e_\varepsilon u(t, x_{t-}) \, h^e_\varepsilon(t) \, d\mu \right] + \mathbb{E} \left[ \int_0^T \left( \int_0^t \mathcal{B}^e_\varepsilon u(s, x_{t-}) \, d\mu \right) \, dM^e_t \right] + \mathbb{E} \left[ \int_0^T \left( \int_0^t \mathcal{B}^e_\varepsilon u(s, x_{t-}) \, d\hat{\mu} \right) \, d\hat{M}^e_t \right] = \mathbb{E} \left[ \int_0^T \int_{\mathbb{R}^m} \mathcal{B}^e_\varepsilon u(t, x_{t-}) \, h^e_\varepsilon(t) \, d\hat{\mu} \right] + \mathbb{E} \left[ \int_0^T \left( \int_0^t \mathcal{B}^e_\varepsilon u(s, x_{t-}) \, d\hat{\mu} \right) \, d\hat{M}^e_t \right]
\]

from the Itô formula. Since \( u \in C^{1,2}_p([0, T) \times \mathbb{R}^d; \mathbb{R}), \quad U_\varepsilon \in L^p(\Omega, \mathbb{P}) \) for any \( p > 1, \) and

\[
\mathbb{E} \left[ \left\{ \int_0^T \nabla u(s, x_s) \, a(\varepsilon, x_s) \, dW_s + \int_0^T \int_{\mathbb{R}^m} \mathcal{B}^e_\varepsilon u(s, x_{s-}) \, d\hat{\mu} \right\} \, h^e_\varepsilon(t) \right] = 0 \in \mathbb{R}^t \otimes \mathbb{R}^d,
\]

the equality (4.1) in Lemma 4.1 enables us to see that

\[
\mathbb{E} \left[ \phi(x_T) \, M^e_T \right] = \mathbb{E} \left[ \phi(x_T) \right] \mathbb{E} \left[ \hat{M}^e_T \right] + \mathbb{E} \left[ \int_0^T \nabla u(t, x_t) \, a(\varepsilon, x_t) \, dW_t, M^e_T \right] + \mathbb{E} \left[ \int_0^T \int_{\mathbb{R}^m} \mathcal{B}^e_\varepsilon u(t, x_{t-}) \, d\hat{\mu}, M^e_T \right] + \mathbb{E} \left[ \int_0^T \int_{\mathbb{R}^m} \mathcal{B}^e_\varepsilon u(t, x_{t-}) \, h^e_\varepsilon(t) \, d\hat{\mu} \right]
\]

\[
= \mathbb{E} \left[ \phi(x_T) \right] \mathbb{E} \left[ \hat{M}^e_T \right] + \mathbb{E} \left[ \int_0^T \nabla u(t, x_t) \, a(\varepsilon, x_t) \, dW_t, M^e_T \right] + \mathbb{E} \left[ \int_0^T \int_{\mathbb{R}^m} \mathcal{B}^e_\varepsilon u(t, x_{t-}) \, d\hat{\mu}, M^e_T \right] + \mathbb{E} \left[ \int_0^T \int_{\mathbb{R}^m} \mathcal{B}^e_\varepsilon u(t, x_{t-}) \, h^e_\varepsilon(t) \, d\hat{\mu} \right]
\]

\[
- \mathbb{E} \left[ \int_0^T \left( \int_0^t \nabla u(s, x_s) \, a(\varepsilon, x_s) \, dW_s \right) \, d\hat{M}^e_t \right] - \mathbb{E} \left[ \int_0^T \left( \int_0^t \int_{\mathbb{R}^m} \mathcal{B}^e_\varepsilon u(s, x_{s-}) \, d\hat{\mu} \right) \, d\hat{M}^e_t \right].
\]
4.1 Proofs of Theorem 1 and 2

We shall reveal each term in Theorem 1 and 2 in what follows.

**Lemma 4.4** Let \( \varphi \in C^2_K(\mathbb{R}^d; \mathbb{R}) \). Then, it holds that

\[
\mathbb{E}\left[\nabla_x (\varphi (x_T))\right] = \mathbb{E}\left[\varphi (x_T) L_{0,T}\right], \\
\mathbb{E}\left[\nabla_x (\varphi (x_T)) \int_0^T f_0^e (t) \, dt\right] = \mathbb{E}\left[\varphi (x_T) L^e_T\right].
\]

**Proof.** We shall only prove the second assertion, because the first one can be done in a similar manner. Since Lemma 4.1 tells us that the process \( \{u(t, x_t) ; t \in [0, T]\} \) is \((\mathcal{F}_t)\)-martingale, so is \( \{\nabla_x (u(t, x_t)) ; t \in [0, T]\} \), similarly to the proof of Lemma 4.2. Then, for \( t < \tau < T \), we have

\[
\mathbb{E}\left[\nabla_x (u(t, x_t)) f_0^e (t) \right] = \mathbb{E}\left[\nabla_x (u(\tau, x_\tau)) f_0^e (t) \right].
\]

Hence, taking the limit as \( \tau \to T \) yields that \( \mathbb{E}\left[\nabla_x (u(t, x_t)) f_0^e (t) \right] = \mathbb{E}\left[\nabla_x (\varphi (x_T)) f_0^e (t) \right] \), because

\[
\nabla_x (u(\tau, x_\tau)) = \nabla_x (u(0, x)) + \int_0^\tau \nabla_x (\nabla u(s, x_s) a(\epsilon, x_s)) \, dW_s + \int_0^\tau \int_{\mathbb{R}^n} \nabla_x \left( \nabla^c u(s, x_s, \cdot) \right) \, d\mu
\]

\[
\to \nabla_x (u(0, x)) + \int_0^T \nabla_x (\nabla u(s, x_s) a(\epsilon, x_s)) \, dW_s + \int_0^T \int_{\mathbb{R}^n} \nabla_x \left( \nabla^c u(s, x_s, \cdot) \right) \, d\mu
\]

\[
= \nabla_x (\varphi (x_T)).
\]

Therefore, the Fubini theorem and Lemma 4.1 yield that

\[
\mathbb{E}\left[\nabla_x (\varphi (x_T)) \int_0^T f_0^e (t) \, dt\right] = \mathbb{E}\left[\int_0^T \nabla_x (u(t, x_t)) f_0^e (t) \, dt\right] = \mathbb{E}\left[\int_0^T \nabla u(t, x_t) a(\epsilon, x_t) \, dW_t L^e_T\right] = \mathbb{E}\left[\varphi (x_T) L^e_T\right],
\]

because of Assumption 3 and \( U_t \in L^2(\Omega, \mathbb{P}) \).

The following lemma is the application of the integration by parts formula on the Wiener space. 

\[
= \mathbb{E}\left[\int_0^T \nabla u(t, x_t) a(\epsilon, x_t) \, dW_t L^e_T\right] = \mathbb{E}\left[\varphi (x_T) L^e_T\right],
\]
**Lemma 4.5** Let \( \varphi \in C^2_K \left( \mathbb{R}^d ; \mathbb{R} \right) \). Then, it holds that

\[
\mathbb{E} \left[ \nabla_x (\varphi (x_T)) \int_0^T f^\varepsilon (t) \circ dW_t \right] = \mathbb{E} \left[ \varphi (x_T) \left( R^\varepsilon_T - Q^\varepsilon_T \right) \right].
\]

**Proof.** Since \( D_s \varphi (x_T) = \nabla \varphi (x_T) Z_T U_s a (\varepsilon, x_s) \) for \( s \in [0, T] \) from the chain rule on the operator \( D \), the integration by parts formula implies that

\[
\mathbb{E} \left[ \nabla_x (\varphi (x_T)) G_T \right] = \mathbb{E} \left[ \frac{1}{T} \int_0^T D_s \varphi (x_T) a (\varepsilon, x_s)^{-1} Z_s G_T \, ds \right]
\]

\[
= \mathbb{E} \left[ \varphi (x_T) \frac{1}{T} D^* (a (\varepsilon, x)^{-1} Z G_T) \right],
\]

where \( D^* \) is the Skorokhod integral operator. Remark that \( G_T \in \mathcal{D}_\infty \left( \mathbb{R}^l \otimes \mathbb{R}^d \right) \) from Assumption 2 (cf. [18]). Then, we see that

\[
\begin{align*}
D^* (a (\varepsilon, x)^{-1} Z G_T) &= D^* (a (\varepsilon, x)^{-1} Z) G_T - \int_0^T \text{Tr} \left[ a (\varepsilon, x_s)^{-1} Z_s D_s G_T \right] \, ds \\
&= \left\{ \int_0^T (dW_s)^* a (\varepsilon, x_s)^{-1} Z_s \right\} G_T - \int_0^T \text{Tr} \left[ a (\varepsilon, x_s)^{-1} Z_s D_s G_T \right] \, ds \\
&= T R^\varepsilon_T - T Q^\varepsilon_T
\end{align*}
\]

from Proposition I-1.3.3 in [18]. \( \square \)

**Lemma 4.6** Let \( \varphi \in C^2_K \left( \mathbb{R}^d ; \mathbb{R} \right) \). Then, it holds that

\[
\begin{align*}
\mathbb{E} \left[ \nabla_x (\varphi (x_T)) \int_0^T \int_{\mathbb{R}_0^m} |z|^2 \, d\mu \right] &= -\mathbb{E} \left[ \varphi (x_T) J_{0,T} \right], \\
\mathbb{E} \left[ \nabla_x (\varphi (x_T)) \int_0^T \int_{\mathbb{R}_0^m} h^\varepsilon (t) \, d\mu \right] &= -\mathbb{E} \left[ \varphi (x_T) J^\varepsilon_T \right].
\end{align*}
\]

**Proof.** We shall prove the second assertion only, because the first assertion can be obtained in a similar manner. Recall \( M^\varepsilon_T = \int_0^T \int_{\mathbb{R}_0^m} h^\varepsilon (t) \, d\mu \) and \( \tilde{M}^\varepsilon_T = \int_0^T \int_{\mathbb{R}_0^m} h^\varepsilon (t) \, d\tilde{\mu} \). Lemma 4.1 implies that

\[
\mathbb{E} \left[ \int_0^T \int_{\mathbb{R}_0^m} \nabla_x \left( h^\varepsilon (t) \right) \, d\tilde{\mu} \right] = \mathbb{E} \left[ \int_0^T \int_{\mathbb{R}_0^m} \nabla_x \left( h^\varepsilon (t) \right) \, d\tilde{\mu} \right]
\]

\[
= \mathbb{E} \left[ \varphi (x_T) \int_0^T \int_{\mathbb{R}_0^m} \nabla_x \left( h^\varepsilon (t) \right) \, d\tilde{\mu} \right].
\]
On the other hand, it holds that
\[
\mathbb{E} \left[ \int_0^T \int_{\mathbb{R}^m_0} u(t, x_t) \nabla_x \left( h^\varepsilon_x(t) \right) d\hat{\mu} \right] = \mathbb{E} \left[ \varphi(x_T) \int_0^T \int_{\mathbb{R}^m_0} \nabla_x \left( h^\varepsilon_x(t) \right) d\hat{\mu} \right]
\]
from the Fubini theorem. Thus, we have
\[
\mathbb{E} \left[ \int_0^T \int_{\mathbb{R}^m_0} u(t, x_t + b_\varepsilon(\varepsilon, x_t)) \nabla_x \left( h^\varepsilon_x(t) \right) d\hat{\mu} \right] = \mathbb{E} \left[ \varphi(x_T) \int_0^T \int_{\mathbb{R}^m_0} \nabla_x \left( h^\varepsilon_x(t) \right) d\mu \right] = \mathbb{E} \left[ \varphi(x_T) \nabla_x M^\varepsilon_T \right].
\]
Here, the second equality can be justified, similarly to the proof of Lemma 4.2. Furthermore, multiplying \( J^\varepsilon_T \) by the equality (4.1) in Lemma 4.1, it holds that
\[
\mathbb{E} \left[ \varphi(x_T) J^\varepsilon_T \right] = \mathbb{E} \left[ \int_0^T \int_{\mathbb{R}^m_0} \mathcal{B}^x u(s, x_s) \text{div}_x \left[ g(z) \tilde{v}_s(\varepsilon, z) \right] dz ds \right]
\]
\[
= -\mathbb{E} \left[ \int_0^T \int_{\mathbb{R}^m_0} \partial_\varepsilon \left( u(s, x_s + b_\varepsilon(\varepsilon, x_s)) \right) \tilde{v}_s(\varepsilon, z) d\hat{\mu} \right]
\]
\[
= -\mathbb{E} \left[ \int_0^T \int_{\mathbb{R}^m_0} \nabla_x \left( u(s, x_s + b_\varepsilon(\varepsilon, x_s)) \right) h^\varepsilon_x(s) d\hat{\mu} \right].
\]
Here we have used the integration by parts for the second equality, from the condition (iii) in Assumption 1. Therefore, Assumption 2 and \( \varphi \in C^2_K(\mathbb{R}^d; \mathbb{R}) \) enables us to obtain that
\[
\mathbb{E} \left[ \nabla_x \left( \varphi(x_T) \right) M^\varepsilon_T \right] = \mathbb{E} \left[ \nabla_x \left( \varphi(x_T) M^\varepsilon_T \right) \right] - \mathbb{E} \left[ \varphi(x_T) \nabla_x M^\varepsilon_T \right]
\]
\[
= \nabla_x \left( \mathbb{E} \left[ \int_0^T \int_{\mathbb{R}^m_0} u(t, x_t + b_\varepsilon(\varepsilon, x_t)) h^\varepsilon_x(t) d\hat{\mu} \right] \right)
\]
\[
- \mathbb{E} \left[ \int_0^T \int_{\mathbb{R}^m_0} u(t, x_t + b_\varepsilon(\varepsilon, x_t)) \nabla_x \left( h^\varepsilon_x(t) \right) d\hat{\mu} \right]
\]
\[
= \mathbb{E} \left[ \int_0^T \int_{\mathbb{R}^m_0} \nabla_x \left( u(s, x_s + b_\varepsilon(\varepsilon, x_s)) \right) h^\varepsilon_x(s) d\hat{\mu} \right]
\]
\[
= -\mathbb{E} \left[ \varphi(x_T) J^\varepsilon_T \right].
\]
\[\square\]

**Corollary 4.1** For \( \varphi \in C^2_K(\mathbb{R}^d; \mathbb{R}) \), it holds that
\[
\mathbb{E} \left[ \nabla_x \left( \varphi(x_T) \right) A_{0,T} \right] = \mathbb{E} \left[ \varphi(x_T) \left( L_{0,T} - J_{0,T} \right) \right]. \tag{4.4}
\]
Corollary 4.1. The Fubini theorem leads to is a martingale measure. See [17] for details.

In a similar manner to Corollary 4.1, we can get

\[
\int_0^T \left( e^{-\lambda t} - 1 \right) d\tilde{\mu}.
\]

Define a new probability measure \( \mathbb{P}^A \) via the Girsanov transform

\[
\frac{d\mathbb{P}^A}{d\mathbb{P}} \bigg|_{\mathcal{F}_T} = \exp \left\{ - \int_0^T \int_{\mathbb{R}^m} \lambda |z|^2 \, d\mu - N^4_{0, T} \right\},
\]

and denote by \( \mathbb{E}^A[ \cdot ] \) the expectation with respect to the measure \( \mathbb{P}^A \). Then, under the measure \( \mathbb{P}^A \),

\( d\mu \) is the Poisson random measure with the intensity \( d\tilde{\mu}_A := \exp \left\{ -\lambda |z|^2 \right\} d\tilde{\mu} \), and \( d\tilde{\mu}_A := d\mu - d\tilde{\mu}_A \) is a martingale measure. See [17] for details.

We shall rewrite the equation (2.1) as follows:

\[
dx_t = \tilde{a}_0 (\varepsilon, x_t) \, dt + a(\varepsilon, x_t) \circ dW_t + \int_{\mathbb{R}^m} b_\varepsilon (\varepsilon, x_t) \, d\tilde{\mu}_A,
\]

where

\[
\tilde{a}_0 (\varepsilon, y) = a_0 (\varepsilon, y) + \int_{|z| \leq 1} b_\varepsilon (\varepsilon, y) \left( e^{-\lambda |z|^2} - 1 \right) \, dv,
\]

\( d\tilde{\mu}_A = \mathbb{1}_{|z| \leq 1} d\tilde{\mu}_A + \mathbb{1}_{|z| > 1} d\mu. \)

In a similar manner to Corollary 4.1, we can get

\[
\nabla_s \left( \mathbb{E}^A \left[ \varphi (x_T) A_{0, T} \right] \right) = \mathbb{E}^A \left[ \varphi (x_T) \left( L_{0, T} - J^{(4)}_{0, T} \right) \right],
\]

where

\[
J^{(4)}_{r, t} = \int_r^T \int_{\mathbb{R}^m} \text{div}_z \left[ \frac{e^{-\lambda |z|^2} g (z) v_s (\varepsilon, z)}{e^{-\lambda |z|^2} g (z)} \right] d\tilde{\mu}_A
\]

for \( 0 \leq r \leq t \leq T \). The Fubini theorem yields that

\[
\int_0^\infty e^{-\lambda T + N^4_{0, T}} \mathbb{E}^A \left[ \varphi (x_T) L_{0, T} \right] \, d\lambda = \mathbb{E} \left[ \left\{ \int_0^\infty e^{-\lambda A_{0, T}} \, d\lambda \right\} \varphi (x_T) L_{0, T} \right] = \mathbb{E} \left[ \varphi (x_T) \frac{L_{0, T}}{A_{0, T}} \right].
\]

Since \( d\tilde{\mu}_A = d\tilde{\mu} + \left( 1 - e^{-\lambda |z|^2} \right) d\tilde{\mu} \) and

\[
\text{div}_z \left[ \frac{e^{-\lambda |z|^2} g (z) v_s (\varepsilon, z)}{e^{-\lambda |z|^2} g (z)} \right] = -2\lambda z^* v_s (\varepsilon, z) + \frac{\text{div}_z \left[ g (z) v_s (\varepsilon, z) \right]}{g (z)},
\]

Proof of Theorem 1. Our goal is to get rid of \( A_{0, T} \) from the left hand side of the equality in Corollary 4.1. The Fubini theorem leads to

\[
\mathbb{E} \left[ \varphi (x_T) \right] = \mathbb{E} \left[ \varphi (x_T) A_{0, T} \right] = \int_0^\infty \mathbb{E} \left[ \varphi (x_T) \exp \left\{ -\lambda A_{0, T} - N^4_{0, T} \right\} \right] e^{N^4_{0, T}} d\lambda,
\]

where \( N^4_{r, t} = \int_r^t \int_{\mathbb{R}^m} (e^{-\lambda |z|^2} - 1) \, d\tilde{\mu}. \)
we have \( J_{r,t} = J_{r,t} - \lambda K_{r,t} \) from the condition (iii) in Assumption 1. Hence, we can get

\[
\int_0^\infty e^{-\lambda t} \mathbb{E} \left[ \varphi(x_T) J_{0,T}^{(4)} \right] d\lambda = \mathbb{E} \left[ \int_0^\infty e^{-\lambda t} \varphi(x_T) (J_{0,T} - \lambda K_{0,T}) d\lambda \right] = \mathbb{E} \left( \varphi(x_T) \frac{J_{0,T}}{A_{0,T}} \right) - \mathbb{E} \left( \varphi(x_T) \frac{K_{0,T}}{(A_{0,T})^2} \right)
\]

from the Fubini theorem. The proof of Theorem 1 is complete.

\[ \square \]

**Proof of Theorem 2.** By summing up the equalities in Lemma 4.4, 4.5 and 4.6, the assertion of Theorem 2 holds.

\[ \square \]

### 4.2 Proof of Theorem 3

We shall reveal each terms in Theorem 3. Concerning the continuous part, it holds that

**Lemma 4.7** For \( 1 \leq j, k \leq d \), and \( \varphi \in C^2_K \left( \mathbb{R}^d ; \mathbb{R} \right) \), it holds that

\[
\mathcal{T} \nabla_{x_j} \nabla_{x_k} \left( \mathbb{E} \left[ \varphi(x_T) \right] \right) = \mathbb{E} \left[ \varphi(x_T) \left( \mathcal{T}^{(1),j} L^k_{r,0,T} + \int_0^T \sum_{i=1}^m F_{ijk} \varphi(x_s) dW^i_s \right) \right].
\]

**Proof.** Define \( \ell_s(\varepsilon) = a(\varepsilon, x_s)^{-1} Z_s \). Multiplying \( L^k_{r,0,T} \) by (4.3) in Lemma 4.2, we have

\[
\mathbb{E} \left[ (\nabla_{x_j} \varphi(x_T)) L^k_{r,0,T} \right] = \mathbb{E} \left[ \int_0^T \sum_{i=1}^m \nabla_{x_j} \left[ \nabla u(s, x_s) a_i(\varepsilon, x_s) \right] \ell_s^{il}(\varepsilon) ds \right]
\]

\[
= \mathbb{E} \left[ \int_0^T \sum_{i=1}^m \nabla_{x_j} \left[ \nabla u(s, x_s) \nabla_{x_i} (\varphi(x_T)) \ell_s^{il}(\varepsilon) ds \right] \right]
\]

\[
= \mathbb{E} \left[ \int_0^T \nabla_{x_j} \nabla_{x_k} (u(s, x_s)) ds \right] - \mathbb{E} \left[ \int_0^T \sum_{i=1}^m \nabla_{x_j} (u(s, x_s)) \nabla_{x_k} \left[ a(\varepsilon, x_s)^{-1} \right] ds \right]
\]

\[
- \mathbb{E} \left[ \int_0^T \sum_{i=1}^m \nabla_{x_j} (u(s, x_s)) a_i(\varepsilon, x_s) \nabla_{x_k} \left[ a(\varepsilon, x_s)^{-1} \right] ds \right],
\]

because \( \sum_{i=1}^m \nabla_{x_i} a_i(\varepsilon) \left[ a(\varepsilon)^{-1} \right] = - \sum_{i=1}^m a_i(\varepsilon) \nabla_{x_i} \left[ a(\varepsilon)^{-1} \right] \). Similarly to the proof of Lemma 4.2, we can get \( \mathbb{E} \left[ \nabla_{x_j} \nabla_{x_k} (u(s, x_s)) \right] = \mathbb{E} \left[ \nabla_{x_j} \nabla_{x_k} (u(t, x_s)) \right] \) for \( 0 \leq s \leq t \leq T \). Since \( \varphi \in C^2_K \left( \mathbb{R}^d ; \mathbb{R} \right) \),
taking the limit as $t \nearrow T$ leads to

$$\mathbb{E} \left[ \nabla_{x_j} \nabla_{x_k} (u(s, x_s)) \right] = \mathbb{E} \left[ \nabla_{x_j} \nabla_{x_k} (\varphi(x_T)) \right] = \nabla_{x_j} \nabla_{x_k} (\mathbb{E} \left[ \varphi(x_T) \right]) .$$

Thus, we see that

$$T \nabla_{x_j} \nabla_{x_k} (\mathbb{E} \left[ \varphi(x_T) \right]) = \mathbb{E} \left[ \nabla_{x_j} (\varphi(x_T)) \right] L_{0,T}^k + \mathbb{E} \left[ \int_0^T \sum_{\beta=1}^d \nabla_{\beta} u(s, x_s) \nabla_{x_k} Z_{s}^{\beta} d\bar{s} \right]$$

$$+ \mathbb{E} \left[ \int_0^T \sum_{i=1}^m \sum_{\beta=1}^d \nabla u(s, x_s) a_i(\varepsilon, x_s) \nabla_{x_j} \left[ a(\varepsilon, x_s)^{-1} \right]_{i\beta} Z_{s}^{\beta} d\bar{s} \right].$$

(4.6)

Denote by $\{P_t; t \in [0, T]\}$ the $(C_0)$-semigroup associated with the process $\{x_t; t \in [0, T]\}$. We shall replace $T$ and $\varphi$ in the equality (4.6) by $\tilde{T}$ and $P_{\tilde{T}} \varphi$, respectively. Then, it holds that

$$\tilde{T} \nabla_{x_j} \nabla_{x_k} (\mathbb{E} \left[ P_{\tilde{T}} \varphi(x_T) \right]) = \tilde{T} \nabla_{x_j} \nabla_{x_k} \left( \mathbb{E} \left[ \varphi(x_T) \big| \mathcal{F}_{\tilde{T}} \right] \right) = \tilde{T} \nabla_{x_j} \nabla_{x_k} (\mathbb{E} \left[ \varphi(x_T) \right]).$$

In a similar manner to Theorem 1, the first term of the right hand side of (4.6) is equal to

$$\mathbb{E} \left[ \nabla_{x_j} (P_{\tilde{T}} \varphi(x_T)) \right] L_{0,\tilde{T}}^k = \mathbb{E} \left[ \nabla_{x_j} \left( \mathbb{E} \left[ \varphi(x_T) \big| \mathcal{F}_{\tilde{T}} \right] \right) \right] L_{0,\tilde{T}}^k = \mathbb{E} \left[ \varphi(x_T) \Gamma_{T,\tilde{T}}^{1,j} L_{0,\tilde{T}}^k \right].$$

Define $\tilde{u}(t, x) = \mathbb{E} \left[ P_{\tilde{T}} \varphi(x_T) \big| x_0 = x \right]$ for $t \in [0, \tilde{T}]$ and $x \in \mathbb{R}^d$. Replacing $T$ and $\varphi$ in Lemma 4.1 by $\tilde{T}$ and $P_{\tilde{T}} \varphi$, respectively, we have

$$P_{\tilde{T}} \varphi(x_T) = \mathbb{E} \left[ P_{\tilde{T}} \varphi(x_T) \right] + \int_0^{\tilde{T}} \nabla \tilde{u}(s, x_s) a(s, x_s) dW_s + \int_0^{\tilde{T}} \int_{\mathbb{R}^d} \mathbb{E} \tilde{u}(s, x_{s-}) d\tilde{\mu}.$$ (4.7)

Multiplying $\int_0^{\tilde{T}} \sum_{i=1}^m \sum_{\beta=1}^d \left[ a(\varepsilon, x_s)^{-1} \right]_{i\beta} \nabla_{x_k} Z_{s}^{\beta} dW_s^i$ by the equality (4.7), we have

$$\mathbb{E} \left[ \varphi(x_T) \int_0^{\tilde{T}} \sum_{i=1}^m \sum_{\beta=1}^d \left[ a(\varepsilon, x_s)^{-1} \right]_{i\beta} \nabla_{x_k} Z_{s}^{\beta} dW_s^i \right]$$

$$= \mathbb{E} \left[ P_{\tilde{T}} \varphi(x_T) \int_0^{\tilde{T}} \sum_{i=1}^m \sum_{\beta=1}^d \left[ a(\varepsilon, x_s)^{-1} \right]_{i\beta} \nabla_{x_k} Z_{s}^{\beta} dW_s^i \right]$$

$$= \mathbb{E} \left[ \int_0^{\tilde{T}} \sum_{\beta=1}^d \nabla_{\beta} \tilde{u}(s, x_s) \nabla_{x_k} Z_{s}^{\beta} d\bar{s} \right].$$
Multiplying $\int_0^T \sum_{i=1}^m \sum_{\beta=1}^d \nabla_{x_j} \left[ a(\epsilon, x_s)^{-1} \right]_{ij} Z^{\delta k}_s dW^j_s$ by the equality (4.7), we have

$$
\mathbb{E} \left[ \varphi(x_T) \int_0^T \sum_{i=1}^m \sum_{\beta=1}^d \nabla_{x_j} \left[ a(\epsilon, x_s)^{-1} \right]_{ij} Z^{\delta k}_s dW^j_s \right]
= \mathbb{E} \left[ P_{\tilde{T}} \varphi(x_T) \int_0^\tilde{T} \sum_{i=1}^m \sum_{\beta=1}^d \nabla_{x_j} \left[ a(\epsilon, x_s)^{-1} \right]_{ij} Z^{\delta k}_s dW^j_s \right]
= \mathbb{E} \left[ \int_0^\tilde{T} \sum_{i=1}^m \sum_{\beta=1}^d \nabla \tilde{u}(s, x_s) a_i(x_s) \nabla_{x_j} \left[ a(\epsilon, x_s)^{-1} \right]_{ij} Z^{\delta k}_s ds \right].
$$

Therefore we can get

$$
\tilde{T} \nabla_{x_j} \nabla_{x_i} \mathbb{E}[\varphi(x_T)] = \mathbb{E} \left[ \varphi(x_T) \Gamma^{(1),j}_{\tilde{T},\tilde{T}} + \mathbb{E} \left[ \varphi(x_T) \int_0^\tilde{T} \sum_{i=1}^m \sum_{\beta=1}^d \left[ a(\epsilon, x_s)^{-1} \right]_{ij} \nabla_{x_i} Z^{\delta j}_s dW^j_s \right] \right.
+ \mathbb{E} \left[ \varphi(x_T) \int_0^\tilde{T} \sum_{i=1}^m \sum_{\beta=1}^d \nabla_{x_j} \left[ a(\epsilon, x_s)^{-1} \right]_{ij} Z^{\delta k}_s dW^j_s \right]
= \mathbb{E} \left[ \varphi(x_T) \left\{ \Gamma^{(1),j}_{\tilde{T},\tilde{T}} + \int_0^\tilde{T} \sum_{i=1}^m F_{ijk} \left( \epsilon, s \right) dW^j_s \right\} \right].
$$

Concerning the jump part, it holds that

**Lemma 4.8** For $1 \leq j, k \leq d$, and $\varphi \in C^2_K(\mathbb{R}^d; \mathbb{R})$, it holds that

$$
\nabla_{x_j} \nabla_{x_i} \left( \mathbb{E} \left[ \varphi(x_T) \int_0^\tilde{T} \int_{\mathbb{R}^n_+} |z|^2 d\mu \right] \right)
= \mathbb{E} \left[ \varphi(x_T) \left\{ -\Gamma^{(1),j}_{\tilde{T},\tilde{T}} + \int_0^\tilde{T} \int_{\mathbb{R}^n_+} \sum_{\sigma=2}^3 \left[ div \left[ F^{\sigma} \left( \epsilon, s, z \right) \right] \right]_{jk} d\mu \right\} \right].
$$

**Proof.** Since

$$
\nabla_{x_j} \nabla_{x_i} \left( u(s, x_s + b_\epsilon(x_s)) \right)
= \nabla_{x_i} \left( \sum_{\beta=1}^d \nabla_{x_\beta} u(s, x_s + b_\epsilon(x_s)) \left[ (I_d + \nabla b_\epsilon(x_s)) Z_s \right]_{\beta j} \right)
= \sum_{\beta, \delta=1}^d \nabla_{x_i} \nabla_{x_\beta} u(s, x_s + b_\epsilon(x_s)) \left[ (I_d + \nabla b_\epsilon(x_s)) Z_s \right]_{\beta j} \left[ (I_d + \nabla b_\epsilon(x_s)) Z_s \right]_{\delta k}
$$
multiplying \( J_{0,T}^k \) by the equality (4.3) in Lemma 4.2 yields that

\[
\mathbb{E} \left[ \nabla_{x_j} (\varphi (x_T)) \right] = \mathbb{E} \left[ \int_0^T \int_{\mathbb{R}^m} \nabla_{x_j} (\mathbb{E} [ u(s, x_s) | \mathcal{F}_s]) d\sigma d\tau \right]
\]

where Assumption 1 is used for the second equality.

Replace \( T \) by \( \tilde{T} \) and \( \varphi \) by \( P_T \varphi \), respectively. In a similar manner to Theorem 1, we have

\[
\mathbb{E} \left[ \nabla_{x_j} (P_T \varphi (x_T)) \right] = \mathbb{E} \left[ \nabla_{x_j} \left( \mathbb{E} \left[ \varphi (x_T) \mid \mathcal{F}_s \right] \right) \right] = \mathbb{E} \left[ \varphi (x_T) \Gamma_{\tilde{T}, T}^{i,j} J_{0,T}^k \right].
\]

Since \( \tilde{u} (t, x) = \mathbb{E} \left[ (P_T \varphi) (x_T) \right] x_0 = x \) for \( t \in [0, \tilde{T}] \) and \( x \in \mathbb{R}^d \), we see that

\[
\mathbb{E} \left[ \int_0^{\tilde{T}} \int_{\mathbb{R}^m} \nabla_{x_j} \nabla_{x_k} (\tilde{u}(s, x_s + b_\varepsilon (e, x_s))) |z|^2 d\mu \right]
\]
\[
= \nabla_{x_j} \nabla_{x_k} \left( \mathbb{E} \left[ \int_0^T \int_{\mathbb{R}^m_0} \tilde{u}(s, x_s + b_z(e, x_s)) |z|^2 \, d\mu \right] \right)
= \nabla_{x_j} \nabla_{x_k} \left( \mathbb{E} \left[ P_T \varphi(x_T) \int_0^T \int_{\mathbb{R}^m_0} |z|^2 \, d\mu \right] \right)
= \nabla_{x_j} \nabla_{x_k} \left( \mathbb{E} \left[ \varphi(x_T) \int_0^T \int_{\mathbb{R}^m_0} |z|^2 \, d\mu \right] \right)
\]
from $\varphi \in C^2_R \left( \mathbb{R}^d ; \mathbb{R} \right)$ and Lemma 4.3. On the other hand, Assumption 1 implies that

\[
\mathbb{E} \left[ \int_0^T \int_{\mathbb{R}^m_0} \sum_{\beta=1}^d \nabla_{\beta} \tilde{u}(s, x_s + b_z(e, x_s)) \nabla_{x_i} \left( (I_d + \nabla b_z(e, x_s)) Z_s \right) |z|^2 \, d\mu \right]
= -\mathbb{E} \left[ \int_0^T \int_{\mathbb{R}^m_0} \sum_{i=1}^m \sum_{\beta=1}^d \nabla_{\beta} \tilde{u}(s, x_s + b_z(e, x_s)) \nabla_{x_j} \partial_{z_i} \left( b_{\zeta}^\varphi(e, x_s) \right) F_{ijk}^3(e, s, z) \, d\mu \right]
= \mathbb{E} \left[ P_T \varphi(x_T) \int_0^T \int_{\mathbb{R}^m_0} \{ \text{div}_z [F^3(e, s, z)] \} \, d\mu \right]
= \mathbb{E} \left[ \varphi(x_T) \int_0^T \int_{\mathbb{R}^m_0} \{ \text{div}_z [F^3(e, s, z)] \} \, d\mu \right]
\]
from (4.7) in the proof of Lemma 4.7. Similarly, we have

\[
-\mathbb{E} \left[ \int_0^T \int_{\mathbb{R}^m_0} \sum_{i=1}^m \sum_{\beta=1}^d \nabla_{\beta} \tilde{u}(s, x_s + b_z(e, x_s)) \nabla_{x_j} \partial_{z_i} \left( b_{\zeta}^\varphi(e, x_s) \right) v_{ik}^\varphi(e, z) \, d\mu \right]
= -\mathbb{E} \left[ \int_0^T \int_{\mathbb{R}^m_0} \sum_{i=1}^m \sum_{\beta=1}^d \nabla_{\beta} \tilde{u}(s, x_s + b_z(e, x_s)) \nabla_{x_j} \partial_{z_i} \left( b_{\zeta}^\varphi(e, x_s) \right) F_{ijk}^3(e, s, z) \, d\mu \right]
= \mathbb{E} \left[ \varphi(x_T) \int_0^T \int_{\mathbb{R}^m_0} \{ \text{div}_z [F^3(e, s, z)] \} \, d\mu \right]
= \mathbb{E} \left[ P_T \varphi(x_T) \int_0^T \int_{\mathbb{R}^m_0} \{ \text{div}_z [F^3(e, s, z)] \} \, d\mu \right]
\]
from Assumption 1 and (4.7) in the proof of Lemma 4.7. The proof of Lemma 4.8 is complete. □
Corollary 4.2 For $1 \leq j, k \leq d$ and $\varphi \in C^2_k\left(\mathbb{R}^d ; \mathbb{R}\right)$, it holds that

$$
\nabla_{x_j} \nabla_{x_k} \left( \mathbb{E} \left[ \varphi(x_T) A_{0,T} \right] \right) = \mathbb{E} \left[ \varphi(x_T) \Gamma^{(1),(1)}_{\tau,T} \left( L^k_{0,T} - J^k_{0,T} \right) \right] + \mathbb{E} \left[ \varphi(x_T) \left\{ \int_0^{\tau} F^1(\varepsilon, s) \, dW_s + \sum_{\sigma \geq 2} \int_0^{\tau} \int_{\mathbb{R}^n} \text{div}_z [F^\sigma(\varepsilon, s, z)] \, d\tilde{\mu}_s \right\} \right]_{jk}.
$$

Proof of Theorem 3. Our goal is to remove $A_{0,T}$ in the left hand side of the equality in Corollary 4.2. In order to do that, we shall adopt the same strategy as in Theorem 1. Define a new probability measure $\tilde{\mathbb{P}}^d$ by

$$
d\tilde{\mathbb{P}}^d = \exp \left\{ -\lambda A_{0,T} - N^d_{0,T} \right\}
$$

via the Girsanov transform, and $\tilde{\mathbb{P}}^d [\cdot]$ is the expectation with respect to the measure $\tilde{\mathbb{P}}^d$, where $N^d_{0,T} = \int_{0}^{T} \left( e^{-|z|^2} - 1 \right) \, d\tilde{\mu}$. As stated in the proof of Theorem 1, we have

$$
\mathbb{E} \left[ \varphi(x_T) \right] = \mathbb{E} \left[ \varphi(x_T) \frac{A_{0,T}}{A_{0,T}} \right] = \int_0^\infty \tilde{\mathbb{P}}^d \left[ \varphi(x_T) A_{0,T} \right] e^{N^d_{0,T}} \, d\lambda,
$$

and, under the measure $\tilde{\mathbb{P}}^d$, $d\mu$ is the Poisson random measure with intensity $d\tilde{\mu}_\lambda := e^{-|z|^2} \, d\tilde{\mu}$. Moreover, $d\tilde{\mu}_\lambda := d\mu - d\tilde{\mu}_\lambda$ is a martingale measure (cf. [17]).

Define $\Gamma^{(1),(4)}_{\tau,T} = \left( L_{\tau,T} - J^{(4)}_{\tau,T} \right) / A_{\tau,T} + K_{\tau,T} / (A_{\tau,T})^2$. Applying Corollary 4.2 with respect to $\tilde{\mathbb{P}}^d [\cdot]$, we have

$$
\nabla_{x_j} \nabla_{x_k} \left( \mathbb{E} \left[ \varphi(x_T) \right] \right) = \int_0^\infty e^{N^d_{0,T}} \nabla_{x_j} \nabla_{x_k} \left( \tilde{\mathbb{P}}^d \left[ \varphi(x_T) A_{0,T} \right] \right) \, d\lambda
$$

$$
= \int_0^\infty e^{N^d_{0,T}} \left\{ \tilde{\mathbb{P}}^d \left[ \varphi(x_T) \Gamma^{(1),(1)}_{\tau,T} \left( L^k_{0,T} - J^k_{0,T} \right) \right] + \tilde{\mathbb{P}}^d \left[ \varphi(x_T) \left\{ \int_0^{\tau} F^1(\varepsilon, s) \, dW_s + \sum_{\sigma \geq 2} \int_0^{\tau} \int_{\mathbb{R}^n} \text{div}_z [F^\sigma(\varepsilon, s, z)] \, d\tilde{\mu}_s \right\} \right] \right\} \, d\lambda
$$

$$
=: I_1 + I_2,
$$

where $J^{(4)}_{\tau,T} = \int_{\mathbb{R}^n} \left\{ \text{div}_z \left[ e^{-|z|^2} g(z) \, v_s(\varepsilon, z) \right] \right\} \, d\tilde{\mu}_\lambda$ for $0 \leq \tau \leq t \leq T$. Since $d\tilde{\mu}_\lambda = d\mu + \left( 1 - e^{-|z|^2} \right) d\tilde{\mu}$ and $J^{(4)}_{\tau,T} = J_{\tau,T} - \lambda K_{\tau,T}$ as seen in the proof of Theorem 1, we have

$$
\Gamma^{(1),(4)}_{\tau,T} = \frac{L_{\tau,T} - J_{\tau,T}}{A_{\tau,T}} + \frac{K_{\tau,T}}{(A_{\tau,T})^2} + \lambda \frac{K_{\tau,T}}{A_{\tau,T}}.
$$
Thus, it holds that

\[
I_1 = \mathbb{E} \left[ \varphi(x_T) \int_0^\infty e^{-\lambda A_0,\tilde{T}} \left( \Gamma_{\tilde{T}}^{(1),j} + \lambda \frac{K_{\tilde{T}}^j}{A_{\tilde{T},T}} \right) \left( L_{0,\tilde{T}}^k - J_{0,\tilde{T}}^k + \lambda K_{0,\tilde{T}}^k \right) d\lambda \right]
\]

\[
= \mathbb{E} \left[ \varphi(x_T) \int_0^\infty e^{-\lambda A_0,\tilde{T}} \left( \Gamma_{\tilde{T}}^{(1),j} \left( L_{0,\tilde{T}}^k - J_{0,\tilde{T}}^k + \lambda K_{0,\tilde{T}}^k \right) + \lambda \frac{K_{\tilde{T}}^j}{A_{\tilde{T},T}} \left( L_{0,\tilde{T}}^k - J_{0,\tilde{T}}^k \right) + \lambda^2 \frac{K_{\tilde{T}}^j}{A_{\tilde{T},T}} K_{0,\tilde{T}}^k \right) d\lambda \right]
\]

\[
= \mathbb{E} \left[ \varphi(x_T) \left\{ \Gamma_{\tilde{T}}^{(1),j} \Gamma_{\tilde{T}}^{(1),k} + \frac{K_{\tilde{T}}^j}{A_{\tilde{T},T}} \frac{L_{0,\tilde{T}}^k - J_{0,\tilde{T}}^k}{(A_{0,\tilde{T}})^2} + 2 \frac{K_{\tilde{T}}^j}{A_{\tilde{T},T}} \frac{K_{0,\tilde{T}}^k}{(A_{0,\tilde{T}})^3} \right\} \right]
\]

from the Fubini theorem. Similarly, we have

\[
I_2 = \mathbb{E} \left[ \varphi(x_T) \int_0^\infty e^{-\lambda A_0,\tilde{T}} \left\{ \int_0^\tilde{T} F^1(\epsilon, s) dW_s + \int_0^\tilde{T} \int_{\mathbb{R}} \sum_{\sigma=2}^3 \text{div}_\epsilon [F^\sigma(\epsilon, s, z)] d\tilde{\mu}_\sigma \right\} d\lambda \right]
\]

\[
= \mathbb{E} \left[ \varphi(x_T) \frac{1}{A_{0,\tilde{T}}} \left\{ \int_0^\tilde{T} F^1(\epsilon, s) dW_s + \int_0^\tilde{T} \int_{\mathbb{R}} \sum_{\sigma=2}^3 \text{div}_\epsilon [F^\sigma(\epsilon, s, z)] d\mu \right\} \right]
\]

\[
- \mathbb{E} \left[ \varphi(x_T) \left\{ \sum_{\sigma=2}^3 \int_{\mathbb{R}} \frac{\text{div}_\epsilon [F^\sigma(\epsilon, s, z)]}{A_{0,\tilde{T}} + |z|^2} d\tilde{\mu}_\sigma \right\} \right].
\]

The proof of Theorem 3 is complete. \qed

### 4.3 Proof of Corollary 3.1

For \( \varphi \in C^2_K(\mathbb{R}^d; \mathbb{R}) \), all sensitivity formulae are the direct consequences of Theorem 1, 2 and 3. The strategy to remove the regularity conditions, and to extend to the class \( \tilde{\mathcal{X}} \), is almost parallel to the one studied in [14].

First, we shall extend from \( C^2_K(\mathbb{R}^d; \mathbb{R}) \) to \( C_K(\mathbb{R}^d; \mathbb{R}) \). Since \( \varphi \in C_K(\mathbb{R}^d; \mathbb{R}) \) can be approximated uniformly and boundedly by a sequence \( \{\varphi_n; n \in \mathbb{N}\} \), we see that, for each compact set \( H \subset \mathbb{R}^d \),

\[
\left| \mathbb{E} \left[ \varphi(x_T) \right] - \mathbb{E} \left[ \varphi_n(x_T) \right] \right| \leq \|\varphi - \varphi_n\|_\infty,
\]

\[
\sup_{x \in H} \left| \nabla_x (\mathbb{E} \left[ \varphi_n(x_T) \right]) - \mathbb{E} \left[ \varphi(x_T) \Gamma_T^{(1)} \right] \right| \leq \sup_{x \in H} \mathbb{E} \left[ \left| \Gamma_T^{(1)} \right|^2 \right]^{1/2} \|\varphi_n - \varphi\|_\infty,
\]

which tends to 0 as \( n \to +\infty \). Thus, the sensitivity formula \( \nabla_x (\mathbb{E} \left[ \varphi(x_T) \right]) = \mathbb{E} \left[ \varphi(x_T) \Gamma_T^{(1)} \right] \) holds for \( \varphi \in C_K(\mathbb{R}^d; \mathbb{R}) \).

Second, we shall extend to the class \( C_b(\mathbb{R}^d; \mathbb{R}) \) of bounded continuous functions. Let \( \sigma \in \)
(0, 1) be fixed, and write \( N(y; \delta) = \{ \tilde{y} \in \mathbb{R}^d : |\tilde{y} - y| < \delta \} \) for \( y \in \mathbb{R}^d \) and \( \delta > 0 \). For \( \varphi \in C_b \left( \mathbb{R}^d ; \mathbb{R} \right) \), we can find the sequence \( \{ \varphi_n ; n \in \mathbb{N} \} \) of continuous functions defined by

\[
\varphi_n(x) = \begin{cases} 
\varphi(x), & \text{if } x \in N(0; n - \sigma), \\
0, & \text{if } x \in N(0; n + \sigma)^c,
\end{cases}
\]

and \( \varphi_n(x) \in [0, \varphi(x)] \) for each \( x \in \left( N(0; n - \sigma) \right)^c \cap N(0; n + \sigma) \), where \([0, -1]\) should be understood as \([-1, 0]\). Clearly, \( \varphi_n \in C_K \left( \mathbb{R}^d ; \mathbb{R} \right) \), and \( \sup_{n \in \mathbb{N}} \| \varphi_n \|_\infty = \| \varphi \|_\infty \). The dominated convergence theorem leads to

\[
|\mathbb{E} [\varphi(x_T)] - \mathbb{E} [\varphi_n(x_T)]| \to 0
\]
as \( n \to +\infty \). On the other hand, since

\[
\mathbb{E} [\| \varphi_n(x_T) - \varphi(x_T) \|^2] \leq \mathbb{E} [\| \varphi(x_T) \|^2 I_{\{x_T > n - \sigma\}}] \leq \frac{\| \varphi \|^2_\infty}{(n - \sigma)^2} \mathbb{E} [x_T^2] \to 0
\]
as \( n \to +\infty \), we have

\[
\sup_{x \in H} \left| \nabla_x (\mathbb{E} [\varphi_n(x_T)]) - \mathbb{E} \left[ \varphi(x_T) \Gamma_T^{(1)} \right] \right| \leq \sup_{x \in H} \mathbb{E} \left[ \| \Gamma_T^{(1)} \|^2 \right]^{1/2} \sup_{x \in H} \mathbb{E} \left[ \| \varphi_n(x_T) - \varphi(x_T) \|^2 \right]^{1/2},
\]

which tends to 0 as \( n \to +\infty \). Hence, we can obtain the sensitivity formula \( \nabla_x (\mathbb{E} [\varphi(x_T)]) = \mathbb{E} \left[ \varphi(x_T) \Gamma_T^{(1)} \right] \) for \( \varphi \in C_b \left( \mathbb{R}^d ; \mathbb{R} \right) \).

Thirdly, we shall extend to the class of finite linear combinations of indicator functions, which leads us to extend to the class \( \mathcal{F} \) immediately, via the standard truncation argument. It is sufficient to consider the case \( \varphi = I_U \) for a subset \( U \) in \( \mathbb{R}^d \). Then, we can find a sequence \( \{ \varphi_n ; n \in \mathbb{N} \} \) of continuous functions such that

\[
\varphi_n(x) = \begin{cases} 
\varphi(x), & \text{if } x \in U_-, \\
0, & \text{if } x \in U_+^c,
\end{cases}
\]

and \( \varphi_n(x) \in [0, \varphi(x)] \) for \( x \in U_-^c \cap U_+ \), where

\[
U_+ = \left\{ y \in \mathbb{R}^d : |y - \tilde{y}| < \frac{1}{n} \left( \tilde{y} \in \partial U \right) \right\} \cup U, \quad U_- = \left\{ y \in \mathbb{R}^d : |y - \tilde{y}| < \frac{1}{n} \left( \tilde{y} \in \partial U \right) \right\} \cap U.
\]

Clearly, \( \varphi_n \in C_b \left( \mathbb{R}^d ; \mathbb{R} \right) \), and \( \sup_{n \in \mathbb{N}} \| \varphi_n \|_\infty \leq 1 \). The dominated convergence theorem implies that

\[
|\mathbb{E} [\varphi(x_T)] - \mathbb{E} [\varphi_n(x_T)]| \to 0
\]
as \( n \to +\infty \). On the other hand, since there exists a smooth density \( p_T(\varepsilon, x, y) \) for the random variable \( x_T \) with respect to the Lebesgue measure on \( \mathbb{R}^d \) as stated in Proposition 3.1, we have
\[
\sup_{\varepsilon \in H} \mathbb{E} \left[ |\varphi_n(x_T) - \varphi(x_T)|^2 \right] = \sup_{\varepsilon \in H} \mathbb{E} \left[ |\varphi_n(x_T) - \varphi(x_T)|^2 ; x_T \in U_-^c \cap U_+ \right] \\
\leq 4 \left| U_-^c \cap U_+ \right| \sup_{\varepsilon \in H} \sup_{y \in U_-^c \cap U_+} p_T(\varepsilon, x, y),
\]
which tends to 0 as \( n \to +\infty \), because of \( \left| U_-^c \cap U_+ \right| \to 0 \). Hence, we have
\[
\sup_{\varepsilon \in H} \left| \nabla_x \left( \mathbb{E} \left[ \varphi_n(x_T) \right] \right) - \mathbb{E} \left[ \varphi(x_T) \Gamma_T^{(1)} \right] \right| \leq \sup_{\varepsilon \in H} \mathbb{E} \left[ |\Gamma_T^{(1)}|^2 \right]^{1/2} \sup_{\varepsilon \in H} \mathbb{E} \left[ |\varphi_n(x_T) - \varphi(x_T)|^2 \right]^{1/2},
\]
which tends to 0 as \( n \to +\infty \). Therefore, we can conclude that the sensitivity formula
\[
\nabla_x \left( \mathbb{E} \left[ \varphi(x_T) \right] \right) = \mathbb{E} \left[ \varphi(x_T) \Gamma_T^{(1)} \right]
\]
holds for \( \varphi \in \mathfrak{F} \).

The regularity condition on the function \( \varphi \) in Theorem 2 and 3 can be relaxed to the class \( \mathfrak{F} \) in a similar manner. \( \square \)

## 5 Examples

### Example 1 (Lévy processes)

Let \( m = d = 1 \), and \((x, \gamma, \sigma_1, \sigma_2) \in \mathbb{R}^4 \). Consider the \( \mathbb{R} \)-valued process \( \{x_t ; t \in [0, T]\} \) given by
\[
x_t = x + \gamma t + \sigma_1 W_t + \sigma_2 \int_0^t \int_{\mathbb{R}_0} z \, d\tilde{\mu}.
\]

Consider the case of \( \sigma_1 \neq 0 \) and \( \sigma_2 \neq 0 \). Since
\[
A_{\tau, t} = (t - \tau) + \int_\tau^t \int_{\mathbb{R}_0} |z|^2 \, d\mu, \quad L_{\tau, t} = \frac{W_t - W_\tau}{\sigma_1}, \quad J_{\tau, t} = \int_\tau^t \int_{\mathbb{R}_0} \left\{ \frac{g(z) |z|^2}{\sigma_2 g(z)} \right\}' \, d\tilde{\mu},
\]
\[
K_{\tau, t} = \int_\tau^t \int_{\mathbb{R}_0} \frac{z}{\sigma_2} \, d\mu, \quad L_T^\gamma = \frac{W_T}{\sigma_1}, \quad R_T^{\sigma_1} - Q_T^{\sigma_1} = \frac{W_T^2 - T}{\sigma_1 T},
\]
\[
L_T^{\sigma_2} = -\frac{W_T}{\sigma_1} \int_{|z| \leq 1} \, dz, \quad J_T^{\sigma_2} = \int_0^T \int_{\mathbb{R}_0} \left\{ \frac{g(z) |z|^2}{\sigma_2 g(z)} \right\}' \, d\tilde{\mu},
\]
we have
\[
\Gamma^{(1)}_T = \frac{L_0 + J_0}{A_0} + \frac{K_0}{(A_0)^2}, \quad \Gamma^{(2,\gamma)}_T = L_T, \quad \Gamma^{(2,\sigma_1)}_T = R^{\sigma_1}_T - Q^{\sigma_1}_T, \quad \Gamma^{(2,\sigma_2)}_T = L_T^{\sigma_2} + J_T^{\sigma_2},
\]
\[
\Gamma^{(3)}_T = \left\{ \Gamma^{(1)}_T + \frac{K_T}{A_T A_0} \right\} \Gamma^{(1)}_T + \frac{K_T}{A_T (A_0)^2},
\]

As stated in Remark 3.4 and 3.5, the case of either \( \sigma_1 \neq 0 \) or \( \sigma_2 \neq 0 \) is also in our position. In the case of \( \sigma_1 \neq 0 \), since

\[
L_{\tau,\gamma} = \frac{W_t - W_T}{\sigma_1}, \quad L_T = \frac{W_T - T}{\sigma_1}, \quad R^{\sigma_1}_T = \frac{W_T^2 - T}{\sigma_1}, \quad L^{\sigma_2}_T = -\frac{W_T}{\sigma_1} \int_{|z| \leq 1} z d\nu,
\]

we have

\[
\Gamma^{(1)}_T = \frac{L_0}{T}, \quad \Gamma^{(2,\gamma)}_T = L_T, \quad \Gamma^{(2,\sigma_1)}_T = R^{\sigma_1}_T - Q^{\sigma_1}_T, \quad \Gamma^{(2,\sigma_2)}_T = L^{\sigma_2}_T, \quad \Gamma^{(3)}_T = \frac{L_0}{T} \frac{L_{\tau,\gamma}}{T^2}.
\]

In the case of \( \sigma_2 \neq 0 \), since

\[
J_{\tau,\gamma} = \int_{\tau}^{T} \int_{\mathbb{R}} \frac{g_\gamma(|z|^2)^{\gamma}}{\sigma_2 g_\nu(z)} d\mu, \quad K_{\tau,\gamma} = \int_{\tau}^{T} \int_{\mathbb{R}} \frac{z}{\sigma_2} g(z) d\mu, \quad J^{\sigma_2}_T = \int_{\tau}^{T} \int_{\mathbb{R}} \frac{g_\gamma(|z|^2)^{\gamma}}{\sigma_2 g(z)} d\mu,
\]

we have

\[
\Gamma^{(1)}_T = -\frac{V_0}{A_0} + \frac{K_0}{(A_0)^2}, \quad \Gamma^{(2,\sigma_1)}_T = J^{\sigma_2}_T, \quad \Gamma^{(3)}_T = \left\{ \Gamma^{(1)}_T + \frac{K_T}{A_T A_0} \right\} \Gamma^{(1)}_T + \frac{K_T}{A_T (A_0)^2},
\]

where \( A_{\tau,\gamma} = \int_{\tau}^{T} \int_{\mathbb{R}} |z|^2 d\mu. \)

**Example 2 (geometric Lévy processes)** Let \( m = d = 1, (\gamma, \sigma_1, \sigma_2) \in \mathbb{R}^3 \), and \( \{X_t ; t \in [0, T]\} \) the \( \mathbb{R} \)-valued Lévy process represented as follows:

\[
X_t = \gamma t + \sigma_1 W_t + \sigma_2 \int_0^t \int_{\mathbb{R}} z d\mu.
\]

Let \( x > 0 \), and \( \{x_t ; t \in [0, T]\} \) the \( \mathbb{R} \)-valued process defined by \( x_t = xe^{X_t} \), which is called the geometric Lévy process. Let \( \varphi \in \mathbb{R} \) be bounded. Since

\[
\nabla_x \left( \mathbb{E} \left[ \varphi \left( xe^{X_t} \right) \right] \right) = \nabla_x \left( \mathbb{E} \left[ \varphi \left( xe^{X_t} \right) e^{X_t} \right] \right) = \frac{1}{x} \nabla_x \left( \mathbb{E} \left[ \varphi \left( e^{X+X_t} \right) \right] \right) \Big|_{X=\log x},
\]

\[
\partial_x \left( \mathbb{E} \left[ \varphi \left( x T \right) \right] \right) = \mathbb{E} \left[ \varphi \left( x T \right) x T \right] = \partial_x \left( \mathbb{E} \left[ (\varphi \circ \psi)(X + X_T) \right] \right) \Big|_{X=\log x},
\]

27
\begin{align*}
\partial_{\sigma_1} \left( \mathbb{E} \left[ \varphi \left( x_T \right) \right] \right) &= \mathbb{E} \left[ \varphi' \left( x_T \right) \right] \left. \mathbb{E} \left[ (\varphi \circ \psi) (X + x_T) \right] \right|_{X = \log x}, \\
\partial_{\sigma_2} \left( \mathbb{E} \left[ \varphi \left( x_T \right) \right] \right) &= \mathbb{E} \left[ \varphi' \left( x_T \right) \right] \left. \mathbb{E} \left[ (\varphi \circ \psi) (X + x_T) \right] \right|_{X = \log x}, \\
\nabla^2_x \left( \mathbb{E} \left[ \varphi \left( x e^{X_t} \right) \right] \right) &= \mathbb{E} \left[ \varphi'' \left( x e^{X_t} \right) e^{2X_t} \right] \left. \left\{ \nabla^2_X \left( \mathbb{E} \left[ \varphi \left( e^{X_t + X} \right) \right] \right) - \nabla_X \left( \mathbb{E} \left[ \varphi \left( e^{X_t + X} \right) \right] \right) \right\} \right|_{X = \log x},
\end{align*}

we can calculate the corresponding weights \( \tilde{\Gamma}_{(1)}^{(1)}, \tilde{\Gamma}_{(2,\gamma)}^{(2,\gamma)}, \tilde{\Gamma}_{(2,\sigma_1)}^{(2,\sigma_1)}, \tilde{\Gamma}_{(2,\sigma_2)}^{(2,\sigma_2)} \text{ and } \tilde{\Gamma}_{(3)}^{(3)} \) by using the result in Example 1. \( \square \)

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