Brownian motion with darning (BMD in abbreviation) is introduced and studied in [4] and [5, Chapter 7]. Roughly speaking, BMD travels across the “darning area” at infinite speed, while it behaves like a regular BM outside of this area. In this paper we show that starting from a single point in its state space, BMD is the weak limit of a family of continuous-time simple random walks on square lattices with diminishing mesh sizes. From any vertex in their state spaces, the approximating random walks jump to its nearest neighbors with equal probability after an exponential holding time.

AMS 2010 Mathematics Subject Classification: Primary 60J27, 60J35; Secondary 31C25, 60J65.

Keywords and phrases: Space of varying dimension, Brownian motion, random walk, Dirichlet forms, heat kernel estimates, isoperimetric inequality, Nash-type inequality, tightness, Skorokhod space.

1 Introduction

Brownian motion with darning has been introduced and discussed in [4] and [5, Chapter 7]. Its definition can be found in, e.g., [4, Definition 1.1]. In this paper, let $K \subset \mathbb{R}^d$ be a compact connected subset with Lipschitz-continuous boundary. At every $x \in \partial K$, $K$ satisfies the “cone condition” (see, e.g., [3, Proposition 1.22]), it is thus clear that every point on $\partial K$ is regular for $K$ in the sense that $\mathbb{P}^x[\sigma_K = 0] = 1$. This allows us to define BMD by identifying $K$ as a singleton $a^*$ and equipping $E := (\mathbb{R}^d \setminus K) \cup \{a^*\}$ with the topology induced from $\mathbb{R}^d$ (see, e.g., [4, §1.1]). In other words, the distribution of the process on $\mathbb{R}^d \setminus K$ is the same as regular Brownian motion on $\mathbb{R}^d$, but the “darning area” $K$ offers zero resistance to the process. Diffusion processes with darning can be nicely characterized via Dirichlet forms and have been studied with depth in recent literatures, for example, [4, 5, 7]. In particular, we equip $E$ with a measure $m$ which is the same as the Lebesgue measure on $\mathbb{R}^d \setminus K$, and does not charge $a^*$, Then the BMD on $E$ described above can be characterized by the following Dirichlet form on $L^2(E, m(dx))$:

$$D(\mathcal{E}) = \left\{ f : f \in W^{1,2}(\mathbb{R}^d \setminus K), f \text{ is continuous on } E \right\}$$

$$\mathcal{E}(f, g) = \frac{1}{2} \int_E \nabla f(x) \cdot \nabla g(x) m(dx).$$

(1.1)

Recently, it was studied in [12, 13] how BMD can be approximated by continuous-time random walks on square lattices. However, the discussion in [12, 13] was only limited to the toy
model of “Brownian motion with varying dimension” where roughly speaking, the state space is \( \mathbb{R}^2 \cup \mathbb{R}_+ \) with a disc on \( \mathbb{R}^2 \) identified with a singleton, i.e., the “darning point. In this paper, using the method in [13], we describe how BMD on \( \mathbb{R}^d \) with a darning area \( K \) satisfying Lipschitz boundary condition can be approximated by random walks on square lattices. The results in this paper provide an intuition for the behavior of BMD upon hitting the “darning point”, and how it is affected by the geometric properties (or intuitively, the “shape”) of the boundary of the darning area.

We now introduce the state spaces of the approximating random walks. For every \( j \in \mathbb{N} \), let \( K_j := K \cap 2^{-j} \mathbb{Z}^d \). We identify all the vertices of \( 2^{-j} \mathbb{Z}^d \) that are contained in the compact set \( K \) as a singleton \( a_j^* \). Let \( E^j := (2^{-j} \mathbb{Z}^d \cap (\mathbb{R}^d \setminus K)) \cup \{a_j^*\} \).

Recall that in general, a graph \( G \) can be written as “\( G = \{G_v, G_e\} \)”, where \( G_v \) is its collection of vertices, and \( G_e \) is its collection of edges. Given any two vertices in \( a, b \in G \), if there is an unoriented edge with endpoints \( a \) and \( b \), we say \( a \) and \( b \) are adjacent to each other in \( G \), written “\( a \leftrightarrow b \) in \( G \)”. One can always assume that given two vertices \( a, b \) on a graph, there is at most one such unoriented edge connecting these two points (otherwise edges with same endpoints can be removed and replaced with one single edge). This unoriented edge is denoted by \( e_{ab} \) or \( e_{ba} \) (\( e_{ab} \) and \( e_{ba} \) are viewed as the same element in \( G_e \)). In this paper, for notational convenience, we denote by \( G_j := \{2^{-j} \mathbb{Z}^d \cup \{a_j^*\}, V_j\} \), where \( V_j \) is the collection of the edges of \( 2^{-j} \mathbb{Z}^d \).

Next we introduce the graph structure on \( E^j \). Denote by \( G^j = \{G^j_v, G^j_e\} \) the graph where \( G^j_v = E^j \) is the collection of vertices and \( G^j_e \) is the collection of unoriented edges over \( E^j \) defined as follows:

\[
G^j_v := \{e_{xy} : \exists x, y \in 2^{-j} \mathbb{Z}^d \cap (\mathbb{R}^d \setminus K), |x - y| = 2^{-j}, e_{xy} \in V_j, e_{xy} \cap K = \emptyset\}
\]

\[
\cup \{e_{xa_j^*} : x \in 2^{-j} \mathbb{Z}^d \cap (\mathbb{R}^d \setminus K), \exists \text{ at least one edge } e_{xy} \in V_j \text{ such that } e_{xy} \cap K \neq \emptyset\}. \tag{1.2}
\]

Note that \( G^j = \{G^j_v, G^j_e\} \) is a connected graph. We emphasize that given any \( x \in G^j_v, x \neq a_j^* \), there is at most one unoriented edge in \( G^j_e \) connecting \( x \) and \( a_j^* \). Denote by \( v_j(x) = \#\{e_{xy} \in G^j_e\} \), i.e., the number of vertices in \( G^j_v \) adjacent to \( x \).

In order to give definition to the approximating random walks for BMD, for every \( j \geq 1 \), we equip \( E^j \) with the measure:

\[
m_j(x) := \frac{2^{-jd}}{2d} \cdot v_j(x), \quad x \in E^j. \tag{1.3}
\]

Consider the following Dirichlet form on \( L^2(E^j, m_j) \):

\[
\begin{aligned}
\mathcal{D}(\mathcal{E}^j) &= L^2(E^j, m_j) \\
\mathcal{E}^j(f, f) &= \frac{2^{-(d-2)j}}{4d} \sum_{e_{xy}, e_{xy} \in G^j_e} (f(x) - f(y))^2,
\end{aligned}
\tag{1.4}
\]

where \( e^o_{xy} \) denotes an oriented edge from \( x \) to \( y \). In other words, given any pair of adjacent vertices \( x, y \in G^j_v \), the edge with endpoints \( x \) and \( y \) is represented twice in the sum: \( e^o_{xy} \) and \( e^o_{yx} \). One can verify that \( (\mathcal{E}^j, \mathcal{D}(\mathcal{E}^j)) \) is a regular symmetric Dirichlet form on \( L^2(E^j, m_j) \), therefore there is symmetric strong Markov process associated with it. We denote this process by \( X^j \).

In [2] we show that each \( X^j \) is a continuous-time random walk whose trajectories of \( X^j \) stay at each vertex of \( E^j \) for an exponentially distributed holding time with parameter \( 2^{-2j} \) before jumping to one of its neighbors with equal probability. The main result of this paper is that the distributions of \( \{X^j, j \geq 1\} \) converge weakly to the BMD characterized by \([11]\).
The rest of this paper is organized as follows. In §2 we first describe the behavior of $X^j$ by showing their roadmaps. Then we give a brief review on isoperimetric inequalities for weighted graphs, especially the isoperimetric inequalities for $\mathbb{Z}^d$ equipped with equal weights. Using these results, in Proposition 2.7 we prove an isoperimetric inequality for $X^j$. With the isoperimetric inequality obtained in §2 in §3 we derive a Nash-type inequality for the family of random walks \{X^j, j \geq 1\}, from which we establish heat kernel upper bounds, first on-diagonal then off-diagonal, for the entire family of \{X^j, j \geq 1\}. In §4 we use the well-known criterion of tightness presented in [9, Chapter VI, Proposition 3.21] to prove the tightness of \{X^j, j \geq 1\}. The tightness criterion is verified in Propositions 4.7-4.8 which are proved using the heat kernel.

In this paper we follow the convention that in the statements of the theorems or propositions, the capital letters $C_1, C_2, \cdots$ or $N_1, N_2, \cdots$ denote positive constants or positive integers, whereas in their proofs, the lower letters $c, c_1, \cdots$ or $n_1, n_2, \cdots$ denote positive constants or positive integers whose exact value is unimportant and may change from line to line.

2 Preliminaries

2.1 Roadmap of the approximating random walks

Suppose $E$ is a locally compact separable metric space and \{Q(x,dy)\} is a probability kernel on $(E, \mathcal{B}(E))$ with $Q(x,\{x\}) = 0$ for every $x \in E$. Given a constant $\lambda > 0$, we can construct a pure jump Markov process $X$ as follows: Starting from $x_0 \in E$, $X$ remains at $x_0$ for an exponentially distributed holding time $T_1$ with parameter $\lambda(x_0)$ (i.e., $E[T_1] = 1/\lambda(x_0)$), then it jumps to some $x_1 \in E$ according to distribution $Q(x_0,dy)$; it remains at $x_1$ for another exponentially distributed holding time $T_2$ also with parameter $\lambda(x_1)$ before jumping to $x_2$ according to distribution $Q(x_1,dy)$. $T_2$ is independent of $T_1$. $X$ then continues. The probability kernel $Q(x,dy)$ is called the roadmap, i.e., the one-step distribution of $X$, and the $\lambda(x)$ is its speed function. If there is a $\sigma$-finite measure $m_0$ on $E$ with $\text{supp}[m_0] = E$ such that

$$Q(x,dy)m_0(dx) = Q(y,dx)m_0(dy), \quad (2.1)$$

$m_0$ is called a symmetrizing measure of the roadmap $Q$. The following theorem is a restatement of [5, Theorem 2.2.2].

**Theorem 2.1 [5].** Given a speed function $\lambda > 0$. Suppose (2.1) holds, then the reversible pure jump process $X$ described above can be characterized by the following Dirichlet form $(\mathcal{E}, \mathfrak{H})$ on $L^2(E, m(x))$ where the underlying reference measure is $m(dx) = \lambda(x)^{-1}m_0(dx)$ and

$$\begin{cases}
\mathfrak{H} = L^2(E, m(x)), \\
\mathcal{E}(f,g) = \frac{1}{2} \int_{E \times E} (f(x) - f(y))(g(x) - g(y))Q(x,dy)m_0(dx).
\end{cases} \quad (2.2)$$

With the theorem above, we present the following proposition which states that at every vertex of $E^j$, $X^j$ holds for an exponential amount of time with mean $2^{-2j}$ before jumping to each of its nearest neighbors with equal probability.

**Proposition 2.2.** For every $j \geq 1$, $X^j$ has constant speed function $\lambda_j = 2^{2j}$ and a roadmap

$$Q_j(x,dy) = \sum_{z \in E^j, z \leftrightarrow x \text{ in } G^j} q_j(x,z)\delta_{\{z\}}(dy),$$

where $q_j(x,y) = v_j(x)^{-1}$, for all $x, y \in E^j$. 

3
Proof. Define a measure $m_j^0(x) := \lambda_j m_j(x) = 2^{-(d-2\beta)} v_j(x)/(2d)$. The conclusion follows immediately from (1.4) and Theorem 2.2.

\[ \square \]

2.2 Isoperimetric inequalities for weighted graphs

In this section we summarize some results about isoperimetric inequalities for weighted graphs in [1]. In general, let $\Gamma$ be a locally finite connected graph, and let the collection of vertices of $\Gamma$ be denoted by $\mathbb{V}$. If two vertices $x, y \in \mathbb{V}$ are adjacent to each other, then the unoriented edge connecting $x$ and $y$ is assigned a unique weight $\mu_{xy} > 0$. Set $\mu_{xy} = 0$ if $x$ and $y$ are not adjacent in $\Gamma$. Denote by $\mu := \{\mu_{xy} : x, y \text{ connected in } \Gamma\}$ the assignment of the weights on all the unoriented edges. $(\Gamma, \mu)$ is called a locally finite connected weighted graph. A weighted graph $(\Gamma, \mu)$ can be equipped with the following intrinsic measure $\nu$ on $\mathbb{V}$:

\[ \nu(x) := \sum_{y \in \mathbb{V}, y \sim x} \mu_{xy}, \quad x \in \mathbb{V}. \]  

(2.3)

Given two sets of vertices $A, B$ in $\mathbb{V}$, we define

\[ \mu_\Gamma(A, B) := \sum_{x \in A} \sum_{y \in B} \mu_{xy}. \]  

(2.4)

The following definition of isoperimetric inequality is taken from [1, Definition 3.1].

Definition 2.3. For $\alpha \in [1, \infty)$, we say that a weighted graph $(\Gamma, \mu)$ satisfies $\alpha$-isoperimetric inequality $(I_{\alpha})$ if there exists $C_0 > 0$ such that

\[ \frac{\mu_\Gamma(A, \mathbb{V} \setminus A)}{\nu(A)^{1-1/\alpha}} \geq C_0, \quad \text{for every finite non-empty } A \subset \mathbb{V}. \]

The following proposition follows from the combination of [1, Theorem 3.7, Lemma 3.9, Theorem 3.14] and the proofs therein. It gives the relationship between Nash-type inequalities and isoperimetric inequalities for weighted graphs.

Proposition 2.4 ([1]). Let $(\Gamma, \mu)$ be a locally finite connected weighted graph satisfying $\alpha$-isoperimetric inequality with constant $C_0$. Let $\nu$ be the measure defined in (2.3). Then $(\Gamma, \mu)$ satisfies the following Nash-type inequality:

\[ \frac{1}{2} \sum_{x \in \mathbb{V}} \sum_{y \in \mathbb{V}, y \sim x} (f(x) - f(y))^2 \mu_{xy} \geq 4^{-(2+\alpha/2)} C_0^2 \|f\|_{L^2(\nu)}^2 \|f\|_{L^1(\nu)}^{-4/\alpha}, \quad f \in L^1(\nu) \cap L^2(\nu) \]

The next proposition follows immediately from [1, Theorem 3.26]. As a notation in [1], given a weighted graph $(\Gamma, \mu)$ with collection of vertices $\mathbb{V}$. We denote the counting measure times $2^{-jd}$ on $2^{-j} \mathbb{Z}^d$ by $\mu_j$, which can be viewed as the measure “$\nu$” in (2.3) corresponding to weighted $2^{-j} \mathbb{Z}^d$ with all edges weighing $2^{-jd}/2d$.

Proposition 2.5 ([1]). For $j \in \mathbb{N}$, let all edges of $2^{-j} \mathbb{Z}^d$ be assigned with a weight $2^{-jd}/2d$. There exists a constant $C_1 > 0$ independent of $j$ such that for any finite subset $A$ of $2^{-j} \mathbb{Z}^d$,

\[ \mu_{2^{-j}\mathbb{Z}^d}(A, 2^{-j}\mathbb{Z}^d \setminus A) \geq C_1 \cdot 2^{-j} \mu_j(A)^{(d-1)/d}. \]  

(2.5)

Before establishing the isoperimetric inequality for $X_j$, we need the following proposition which will be used throughout this article.
Proposition 2.6. There exist $C_2 > 0$ and $N_0 \in \mathbb{N}$ only depending on the darning region $K$ such that for all $j \geq N_0$,
\[ v_j(a_j^*) \leq C_2 \cdot 2^{j(d-1)}. \] (2.6)

Proof. In the following we denote the $d$- and $(d-1)$-dimensional Lebesgue measures by $m(d)$ and $m(d-1)$, respectively. For any two distinct $x, y \in E^j$ both adjacent to $a_j^*$, the two Euclidean balls $B_{1\|}(x, \frac{2^{-j}}{2})$ and $B_{1\|}(y, \frac{2^{-j}}{2})$ are disjoint. Also, for any $x \leftrightarrow a_j^*$, the Euclidean ball $B_{1\|}(x, \frac{2^{-j}}{2})$ must be contained in the set \{ $x \in \mathbb{R}^d : d_{1\|}(x, \partial K) \leq 2 \cdot 2^{-j}$ \}. Therefore,
\[ v_j(a_j^*) \cdot m(d) \left( B_{1\|} \left( x, \frac{2^{-j}}{2} \right) \right) \leq m(d) \left( \left\{ x \in \mathbb{R}^d : d_{1\|}(x, \partial K) \leq 2 \cdot 2^{-j} \right\} \right). \] (2.7)

Since $K$ has Lipschitz-continuous boundary in $\mathbb{R}^d$, $\partial K$ is $(d-1)$-dimensional in the sense of both topological and Minkowski box dimension. This means that there exists $j_0 \in \mathbb{N}$ and a constant $c > 0$ such that for all $j \geq j_0$, $\partial K$ can be covered by $c \cdot 2^{j(d-1)}$ many boxes with side length $2^{-j}$. This further implies that set \{ $x \in \mathbb{R}^d : d_{1\|}(x, \partial K) \leq 2 \cdot 2^{-j}$ \} can be covered by boxes with the same centers but side length $16 \cdot 2^{-j}$, which further implies that for some $c > 0$,
\[ m(d) \left( \left\{ x \in \mathbb{R}^d : d_{1\|}(x, \partial K) \leq 2 \cdot 2^{-j} \right\} \right) \leq c \cdot 2^{j(d-1)} \cdot 2^{-jd} = c \cdot 2^{-j}, \quad j \geq j_0. \]

The conclusion thus follows on account of (2.7) and the fact that $m(d)(B(x, 2^{-j}/2)) = s(d-1) \cdot 2^{-(j+1)d}$ for all $x \in \mathbb{R}^d$, where $s(d-1) > 0$ is a constant equal to the $(d-1)$-dimensional surface measure. \hfill \qed

Recall the graph structure on $E^j$ defined in (1.2). In the next proposition we establish an isoperimetric inequality for $X^j$ on the weighted graph $E^j$, where all the edges in $G_t^j$ are equipped with an equal weight of $2^{jd/(2d)}$.

Proposition 2.7. For every $j \in \mathbb{N}$, let all edges of $E^j$ be equipped with an equal weight $2^{-j/(2d)}$, which is consistent with the definition of $m_j$ in the sense that
\[ m_j(x) = \frac{2^{-jd}}{2d} \cdot \# \{ e_{xy} \in G^j \}. \]

For the $N_0$ specified in Proposition 2.6, there exist an integer $N_1 \geq N_0$ and a constant $C_2 > 0$ independent of $j$ such that for all $j \geq N_1$,
\[ \mu_{E^j}(A, E^j \setminus A) \geq 2^{-j} C_2 m_j(A)^{(d-1)/d}, \quad \text{for any finite set } A \subset E^j. \] (2.8)

Proof. Let $A$ be any finite subset of $E^j$. We denote by $K^j := 2^{-jd}\mathbb{Z}^d \cap K$. Recall that we denote by $G_j := \{ 2^{-jd}\mathbb{Z}^d, \mathcal{V}_j \}$, where $\mathcal{V}_j$ is the collection of the edges of $2^{-jd}\mathbb{Z}^d$. Also recall that we use "$e_{xy}$" to denote an edge connecting $x$ and $y$, including these two endpoints. In the following we establish (2.8) by dividing our discussion into two cases depending on whether $a_j^*$ is in $A$ or not.

Case (i). $a_j^* \notin A$. Thus $a_j^* \in E^j \setminus A$ and $A \subset 2^{-jd}\mathbb{Z}^d$. In view of the definition of the graph
structure $G'$ in [1,2],

$$
\mu_{E^j}(A, E^j \setminus A) = \frac{2^{-jd}}{2d} \sum_{x \in A} \# \left\{ y \in E^j \setminus A : y \leftrightarrow x \text{ in } G' \right\}
= \frac{2 - jd}{2d} \sum_{x \in A} \# \left\{ y \in (E^j \setminus (A \cup \{a^*_j\})) : y \leftrightarrow x \right\} + \frac{2^{-jd}}{2d} \# \left\{ x \in A : x \leftrightarrow a^*_j \right\}
= \frac{2^{-jd}}{2d} \sum_{x \in A} \# \left\{ y \in (2^{-j} \mathbb{Z}^d \setminus A) : y \leftrightarrow x \text{ in } 2^{-j} \mathbb{Z}^d, e_{xy} \cap K = \emptyset \right\}
+ \frac{2^{-jd}}{2d} \# \left\{ x \in A : \exists e_{xy} \in V_j \text{ such that } e_{xy} \cap K \neq \emptyset \right\}
\geq \frac{2^{-jd}}{2d} \sum_{x \in A} \# \left\{ y \in (2^{-j} \mathbb{Z}^d \setminus A) : \exists e_{xy} \in V_j \text{ such that } e_{xy} \cap K = \emptyset \right\}
+ \frac{2^{-jd}}{2d} \cdot \frac{1}{2d} \sum_{x \in A} \# \left\{ y \in 2^{-j} \mathbb{Z}^d : \exists e_{xy} \in V_j \text{ such that } e_{xy} \cap K \neq \emptyset \right\}
\geq \frac{2^{-jd}}{4d^2} \sum_{x \in A} \# \left\{ y \in 2^{-j} \mathbb{Z}^d \setminus A : y \leftrightarrow x \text{ in } 2^{-j} \mathbb{Z}^d \right\},
$$

where the first inequality above is due to the fact that for every $x \in A$ such that there is at least one edge in $V_j$ with an endpoint $x$ intersecting $K$, there are at most $2d$ many such edges with different other endpoint $y$. Now in view of (2.5) and the definition of $\mu_j$ and $\mu_{2^{-j} \mathbb{Z}^d}$ earlier in this section, we have

$$
\mu_{E^j}(A, E^j \setminus A) \geq \frac{2^{-jd}}{4d^2} \sum_{x \in A} \# \left\{ y \in 2^{-j} \mathbb{Z}^d \setminus A : y \leftrightarrow x \text{ in } 2^{-j} \mathbb{Z}^d \right\}
\geq \frac{1}{2d} \cdot \mu_{2^{-j} \mathbb{Z}^d}(A, 2^{-j} \mathbb{Z}^d) \geq \frac{C_1}{2d} \cdot 2^{-j} \mu_j(A)^{(d-1)/d} = \frac{C_1}{2d} \cdot 2^{-j} m_j(A)^{(d-1)/d}, \quad (2.9)
$$

which the $C_1$ above is the same as in (2.3). This establishes the desired inequality for the current case. Before continuing to the other case, we note that by the definition of Lipschitz-continuity, $\mu_j(K_j)$ is bounded from below by a positive constant for sufficiently large $j$. Thus there exists an integer $j_1 \geq N_0$ and a constant $c_1 > 0$ such that

$$
\mu_j(K_j) \geq c_1, \quad \text{for all } j \geq j_1. \quad (2.10)
$$

Now in view of Proposition 2.6, there exists an integer $j_2 \geq N_0$ such that

$$
\mu_j(K_j) \geq c_1 \geq \frac{2^{-jd}}{2d} \cdot C_2 \cdot 2^{j(d-1)} \geq m_j(a^*_j), \quad \text{for all } j \geq j_2. \quad (2.11)
$$

Case (ii). $a^*_j \in A$. In this case $E^j \setminus A = 2^{-j} \mathbb{Z}^d \setminus A$. Recall that we let $K_j = K \cap 2^{-j} \mathbb{Z}^d$. Thus for
all $j \geq j_2$ given in (2.11),
\[
\begin{align*}
\mu_{E^j}(A, E^j \setminus A) \\
= \frac{2^{-jd}}{2d} \sum_{x \in A} \# \{ y \in E^j \setminus A : y \leftrightarrow x \text{ in } G^j \} \\
= \frac{2^{-jd}}{2d} \sum_{x \in A, x \neq a_j^*} \# \{ y \in E^j \setminus A : y \leftrightarrow x \text{ in } G^j \} + \frac{2^{-jd}}{2d} \# \{ y \in E^j \setminus A : y \leftrightarrow a_j^* \} \\
\geq \frac{2^{-jd}}{2d} \sum_{x \in A, x \neq a_j^*} \# \{ y \in (2^{-j} \mathbb{Z}^d \setminus A), \exists e_{xy} \in \mathcal{V}_j : e_{xy} \cap K = \emptyset \} \\
+ \frac{2^{-jd}}{4d^2} \sum_{x \in 2^{-j} \mathbb{Z}^d} \# \{ y \in (2^{-j} \mathbb{Z}^d \setminus A), \exists e_{xy} \in \mathcal{V}_j : e_{xy} \cap K \neq \emptyset \} \\
\geq \frac{2^{-jd}}{2d} \sum_{x \in A \setminus \{a_j^*\}} \# \{ y \in (2^{-j} \mathbb{Z}^d \setminus A), \exists e_{xy} \in \mathcal{V}_j : e_{xy} \cap K = \emptyset \} \\
+ \frac{2^{-jd}}{4d^2} \sum_{x \in A \setminus \{a_j^*\}} \# \{ y \in (2^{-j} \mathbb{Z}^d \setminus A), \exists e_{xy} \in \mathcal{V}_j \} \\
+ \frac{2^{-jd}}{4d^2} \sum_{x \in K_j} \# \{ y \in (2^{-j} \mathbb{Z}^d \setminus A) : e_{xy} \in \mathcal{V}_j \} \\
\geq \frac{2^{-jd}}{4d^2} \sum_{x \in (A \setminus \{a_j^*\}) \cup K_j} \# \{ y \in 2^{-j} \mathbb{Z}^d \setminus A : y \leftrightarrow x \text{ in } 2^{-j} \mathbb{Z}^d \},
\end{align*}
\]
where the first inequality is due to the fact that for every $y \in 2^{-j} \mathbb{Z}^d \setminus A$ such that there is at least one edge in $\mathcal{V}_j$ with an endpoint $y$ intersecting $K$, there are at most $2d$ many such edges in $\mathcal{V}_j$ with different other endpoint $x$. The proof is complete in view of (2.9) and (2.12).

3 Nash-type inequality and heat kernel upper bound for random walks on lattices with darning

In this section, using the isoperimetric inequality obtained in Proposition 2.7, we establish first a Nash-type inequality and then heat kernel upper bound for $X_j$.

**Proposition 3.1.** For every $j \in \mathbb{N}$, let $(P_t^j)_{t \geq 0}$ be the transition semigroup of $X^j$ with respect to $m_j$. There exists a constant $C > 0$ independent of $j$ such that for all $j \geq N_1$ specified in Proposition 2.7,
\[
\|P_t^j\|_{1 \rightarrow \infty} \leq \frac{C}{td^{d/2}}, \quad \forall t \in (0, +\infty].
\]
Proof. It follows from (2.8) and Proposition 2.4 that
\[
\frac{1}{2} \sum_{x \in E^j} \sum_{y \in E^j, y \neq x} (f(x) - f(y))^2 \frac{2^{-jd}}{2d} \geq C_2^j \cdot 2^{-2j} \cdot 4^{-2-\frac{d}{2}} \|f\|_{L^2(m_j)}^{2+\frac{4}{d}} \|f\|_{L^1(m_j)}^{\frac{4}{d}}, \quad f \in L^1(m_j) \cap L^2(m_j).
\]
In view of the definition of $E^j$, this implies that
\[
E^j(f, f) \geq C_2^j \cdot 4^{-2-\frac{d}{2}} \|f\|_{L^2(m_j)}^{2+\frac{4}{d}} \|f\|_{L^1(m_j)}^{\frac{4}{d}}, \quad f \in L^1(m_j) \cap L^2(m_j). \tag{3.2}
\]
The conclusion now follows from [3, Theorem 2.9].

Now for every $j \in \mathbb{N}$, we define a metric $d_j(\cdot, \cdot)$ on $E^j$ as follows:
\[
d_j(x, y) := 2^{-j} \times \text{smallest number of edges between } x \text{ and } y \text{ in } G^j. \tag{3.3}
\]
With the above on-diagonal heat kernel estimate, using the standard Davies’s method, we next derive an off-diagonal heat kernel upper bound estimate for $X^j$. Since there is a Nash-type inequality holds for each $X^j$, the family of transition density function of $(P_t^j)_{t \geq 0}$ with respect to $m_j$ exists for every $j \in \mathbb{N}$. We denote this by $\{p_j(t, x, y), t > 0, x, y \in E^j\}$.

**Proposition 3.2.** For every $j \geq 1$, fix a sequence of $\{\alpha_j\}_{j \geq 1}$ satisfying $\alpha_j \leq 2^j - 1$. There exists $C_4 > 0$ independent of $j$, such that for all $j \geq N_1$ specified in Proposition 2.7,
\[
p_j(t, x, y) \leq \frac{C_4}{t^{d/2}} \exp\left(-\alpha_j d_j(x, y) + 4t\alpha_j^2\right), \quad 0 < t < \infty, \; x, y \in E^j. \tag{3.4}
\]

Proof. We prove this result using [3, Corollary 3.28]. For each $j$, we set
\[
\widehat{F}^j := \{h + c : h \in D(E^j), \; h \text{ bounded, and } c \in \mathbb{R}\}.
\]
It is known that the regular symmetric Dirichlet form $(E^j, D(E^j))$ is associated with the following energy measure $\Gamma^j$:
\[
E^j(u, u) = \int_{E^j} \Gamma^j(u, u), \quad u \in \widehat{F}^j.
\]
Now we define $\widehat{F}^j_\infty$ as a subset of $\psi \in \widehat{F}^j$ satisfying the following conditions:

(i) Both $e^{-2\psi} \Gamma^j(e^\psi, e^\psi)$ and $e^{2\psi} \Gamma^j(e^{-\psi}, e^{-\psi})$ as measures are absolutely continuous with respect to $m_j$ on $E^j$.

(ii) Furthermore,
\[
\Gamma^j(\psi) := \left(\left\| \frac{de^{-2\psi}}{dm_j} \Gamma^j(e^\psi, e^\psi) \right\|_{\infty} \vee \left\| \frac{de^{2\psi}}{dm_j} \Gamma^j(e^{-\psi}, e^{-\psi}) \right\|_{\infty} \right)^{1/2} < \infty. \tag{3.5}
\]
For a fixed constant $\alpha_j \leq 2^j - 1$, we denote by
\[
\psi_{j,n}(x) := \alpha_j \cdot (d_j(x, a_j^*) \wedge n). \tag{3.6}
\]
In order to apply [3, Corollary 3.28], we need to verify that $\psi_{j,n} \in \widehat{F}^j_\infty$ for every $n$. Notice that $\psi_{j,n}$ is a constant outside of a bounded domain of $E^j$, therefore it is in $\widehat{F}^j$. We first note that
\[
|1 - e^x| \leq |2x|, \quad \text{for } -\frac{1}{2} < x < \frac{1}{2}. \tag{3.7}
\]
We now first verify conditions (i) and (ii) above. Viewing $e^{-2\psi_j,n} \Gamma^j(e^{\psi_j,n}, e^{\psi_j,n})$ as a measure on $E^j$, given any subset $A \subset E^j$, we have

\[
e^{-2\psi_j,n} \Gamma^j(e^{\psi_j,n}, e^{\psi_j,n})(A) = \frac{2^{-(d-2)j}}{4d} \sum_{x \in E^j \cap A} e^{-2\psi_j,n}(x) \left[ \sum_{y+x \in E^j} \left( e^{\psi_j,n}(y) - e^{\psi_j,n}(x) \right)^2 \right]
\]

\[
\leq \frac{2^{-(d-2)j}}{4d} \sum_{x \in E^j \cap A} \sum_{y+x \in E^j} \left( 1 - e^{\alpha_j(d_j(y,a_j^*) \land n - d_j(x,a_j^*) \land n)} \right)^2
\]

\[
= \frac{2^{-(d-2)j}}{4d} \sum_{x \in E^j \cap A} \sum_{y \neq a_j^*} \left( 1 - e^{\alpha_j(d_j(y,a_j^*) \land n - d_j(x,a_j^*) \land n)} \right)^2
\]

\[
+ \frac{2^{-(d-2)j}}{4d} \cdot 1_{\{a_j^* \in A\}} \cdot \sum_{y+a_j^*} \left( 1 - e^{\alpha_j(d_j(y,a_j^*) \land n)} \right)^2
\]

\[\tag{3.7}\]

\[
\leq \frac{2^{-(d-2)j}}{4d} \cdot \# \{ x \in E^j \cap A, x \neq a_j^* \} \cdot 2d \cdot (2 \cdot \alpha_j \cdot 2^{-j})^2
\]

\[+ \frac{2^{-(d-2)j}}{4d} \cdot 1_{\{a_j^* \in A\}} \cdot v_j(a_j^*) \cdot (2 \cdot \alpha_j \cdot 2^{-j})^2
\]

\[
\leq 2 \cdot 2^{-jd} \cdot \# \{ x \in E^j \cap A, x \neq a_j^* \} \cdot \alpha_j^2 + \frac{2^{-jd}}{d} \cdot 1_{\{a_j^* \in A\}} \cdot v_j(a_j^*) \cdot \alpha_j^2
\]

Recall that $m_j(x) = \frac{2^{-jd}}{2^d} \cdot v_j(x)$. We conclude that for some $c > 0$ independent of $j$, it holds

\[
\frac{\left\| de^{-2\psi_j,n} \Gamma^j(e^{\psi_j,n}, e^{\psi_j,n}) \right\|}{dm_j} \leq \sqrt{2} \alpha_j, \quad \text{for all } j \geq 1. \tag{3.8}
\]

Similarly, it can be computed that

\[
e^{2\psi_j,n} \Gamma^j(e^{-\psi_j,n}, e^{-\psi_j,n})(A)
\]

\[
\leq \frac{2^{-(d-2)j}}{4d} \sum_{x \in E^j \cap A} \sum_{y+x \in E^j} \left( 1 - e^{\alpha_j(d_j(x,a_j^*) \land n - d_j(y,a_j^*) \land n)} \right)^2
\]

\[
+ \frac{2^{-(d-2)j}}{4d} \cdot 1_{\{a_j^* \in A\}} \cdot \sum_{y+a_j^*} \left( 1 - e^{-\alpha_j(d_j(y,a_j^*) \land n)} \right)^2
\]

\[\tag{3.7}\]

\[
\leq \frac{2^{-(d-2)j}}{4d} \cdot \# \{ x \in E^j \cap A, x \neq a_j^* \} \cdot (2d) \cdot (2 \cdot \alpha_j \cdot 2^{-j})^2
\]

\[+ \frac{2^{-(d-2)j}}{4d} \cdot 1_{\{a_j^* \in A\}} \cdot v_j(a_j^*) \cdot (2 \cdot \alpha_j \cdot 2^{-j})^2
\]

\[
\leq 2 \cdot 2^{-jd} \cdot \# \{ x \in E^j \cap A, x \neq a_j^* \} \cdot \alpha_j^2 + \frac{2^{-jd}}{d} \cdot 1_{\{a_j^* \in A\}} \cdot v_j(a_j^*) \cdot \alpha_j^2
\]

Similar to (3.8), this shows that

\[
\frac{\left\| de^{2\psi_j,n} \Gamma^j(e^{\psi_j,n}, e^{\psi_j,n}) \right\|}{dm_j} \leq \sqrt{2} \alpha_j, \quad \text{for all } j \geq 1,
\]
Proposition 4.1 (Chapter VI, Theorem 3.21 in [11]). In particular, given any $T > 1$, with $\min_{x} x$, proof.

Proof. To prove this, in Proposition 3.2, given any $T > 0$, there exists $C_6 > 0$ such that

$$p_j(t, x, y) \leq \frac{C_6}{t^{d/2}} \left( e^{-d_j(x,y)/2t^2} + e^{-2d_j(x,y)/t^4} \right), \text{ for all } (t, x, y) \in (0, T) \times E^j \times E^j.$$

The desired result follows from first taking $x = x_0$ and $y = y_0$, then simplying the right hand side above by dividing it into two cases: $d_j(x_0, y_0) \geq 32t_0 \cdot 2^{j-1}$ and $d_j(x_0, y_0) \leq 32t_0 \cdot 2^{j-1}$.  

4 Tightness of the approximating random walks

The next proposition taken from [11] is a well-known criterion for tightness for càdlàg processes. As a standard notation, given a metric $d(\cdot, \cdot)$, we denote by

$$w_d(x, \theta, T) := \inf_{\{t_i\}_{1 \leq i \leq n} \in \Pi} \max_{1 \leq t \leq n} \sup_{s \in [t, t_{n-1}]} d(x(s), x(t)),$$

where $\Pi$ is the collection of all possible partitions of the form $0 = t_0 < t_1 < \cdots < t_{n-1} < T \leq t_n$ with $\min_{1 \leq i \leq n} (t_i - t_{i-1}) \geq \theta$ and $n \geq 1$.

Proposition 4.1 (Chapter VI, Theorem 3.21 in [11]). Let $\{Y_k, \mathbb{P}^y\}_{k \geq 1}$ be a sequence of càdlàg processes on state space $E$. Given $y \in E$, the laws of $\{Y_k, \mathbb{P}^y\}_{k \geq 1}$ are tight in the Skorokhod space $D([0,T], E, \rho)$ if and only if

(i). For any $T > 0$, $\delta > 0$, there exist $K_1 \in \mathbb{N}$ and $M > 0$ such that for all $k \geq K_1$,

$$\mathbb{P}^y \left[ \sup_{t \in [0,T]} |Y^k_t| > M \right] < \delta.$$  \hspace{1cm} (4.1)

(ii). For any $T > 0$, $\delta_1, \delta_2 > 0$, there exist $\delta_3 > 0$ and $K_2 > 0$ such that for all $k \geq K_2$,

$$\mathbb{P}^y \left[ w_{\rho} \left( Y^k, \delta_3, T \right) > \delta_1 \right] < \delta_2.$$  \hspace{1cm} (4.2)
In this section, we use the heat kernel upper bounds obtained in Corollary 3.3 to verify the two conditions in Proposition 2.7. We begin with the following two lemmas that can be proved in the same manner as those in [13, Lemma 4.2, Proposition 4.3]. In particular, Lemma 4.2 is a standard result due to the strong Markov property of $X^j$, and Lemma 4.3 results from the fact that $X^j$ making jumps at a rate of $2^{-2j}$, for each $j \geq 1$. We skip the proofs of these two lemmas.

**Lemma 4.2.** Given any $T, M > 0$, for any sufficiently large $j \in \mathbb{N}$ such that $2^{-j} < T$, it holds for all $x \in E^j$ that

$$
\mathbb{P}^x \left[ \sup_{t \in [0, T]} |X^j_t|_\rho \geq M \right] \leq \mathbb{P}^x \left[ \sup_{t \in [0, 8^{-j}]} |X^j_t|_\rho \geq M \right] + \mathbb{P}^x \left[ |X^j_T|_\rho \geq \frac{M}{2} \right] + \mathbb{P}^x \left[ T - 8^{-j} \leq \tau_M \leq T, |X^j_T|_\rho \leq \frac{M}{2} \right] + \mathbb{P}^x \left[ 8^{-j} \leq \tau_M \leq T - 8^{-j}, |X^j_T|_\rho \leq \frac{M}{2} \right],
$$

where $\tau_M := \inf \{ t > 0 : |X^j_t|_\rho \geq M \}$.

**Lemma 4.3.** For any $\delta > 0$, any $T > 0$, there exists $M_1 > 0$ such that for all $j \geq 1$:

$$
\sup_{y \in E^j} \mathbb{P}^y \left[ \sup_{t \in [0, 8^{-j}]} \rho \left( X^j_0, X^j_t \right) \geq M_1 \right] < \delta.
$$

Given any $r > 0$, we denote the boundary of a “cube” in $\mathbb{R}^d$ centered at the origin with side length $2r$ by

$$
S_r := \left\{ (x_1, \ldots, x_d) \in \mathbb{R}^d : \max_{1 \leq i \leq d} |x_i| = r \right\}. \quad (4.3)
$$

For the remaining of this paper, we fix a starting point $x_0 \in \cap_{j \geq 1} E^j$ and a $k_0 \in \mathbb{N}$, such that both $K$ and $x_0$ are contained in the set

$$
\left\{ (x_1, \ldots, x_d) \in \mathbb{R}^d : \max_{1 \leq i \leq d} |x_i| \leq k_0 \right\}. \quad (4.4)
$$

By elementary geometry, it can be told that for all $r \geq 2k_0, j \geq 1$,

$$
\# \left( S_r \cap E^j \right) \leq (2d) \cdot (2r \cdot 2^{j-1}). \quad (4.5)
$$

When $k \geq 2k_0$, for all $j \geq 1$, the definition of $d_j$ implies that

$$
d_j(x_0, y) \geq \rho(x_0, y) \geq \frac{k}{2}, \quad \text{for } y \in S_k \cap E^j. \quad (4.6)
$$

**Lemma 4.4.** Fix $x_0 \in \cap_{j \geq 1} E^j$ and $T > 0$. For any $\delta > 0$, there exists $M_2 > 0$ such that for all $j \geq N_1$ specified in Proposition 2.7 such that $e^{-j} < T$:

$$
\sup_{8^{-j} \leq t \leq T} \mathbb{P}^{x_0} \left[ d_j(X^j_t, x_0) \geq M_2 \right] < \delta.
$$

**Proof.** We use Proposition 3.3 to prove this. We first note that the sequence of metrics $\{d_j\}_{j \geq 1}$ is non-increasing in $j$. Therefore, for the $k_0$ specified in (4.3), one can choose $M > 2k_0$ sufficiently large such that

$$
d_j(a^*_j, x_0) \leq M, \quad \text{for all } j \geq 1. \quad (4.7)
$$
For \( M \) satisfying (4.7), in view of the definition of \( m_j \) and the heat kernel upper bound in Corollary 3.3, there exists some \( c_1 > 0 \) independent of \( j \) such that

\[
\mathbb{P}^{x_0} \left[ d_j(X_t^j, x_0) \geq M \right] \leq \sum_{d_j(y, x_0) \geq M} \frac{c_1}{t^{d/2}} \left( e^{-\frac{d_j(x_0,y)^2}{64t}} + e^{-\frac{2j d_j(x_0,y)}{4}} \right) m_j(dy).
\]

To give an upper bound for each of the two terms on the right hand side above, we first compute for a generic \( k \geq 0 \),

\[
\begin{aligned}
\sum_{l=k \cdot 2^j}^{\infty} \sum_{y \in E \cap S_{2k_0 + l \cdot 2^{-j}}} & \frac{1}{t^{d/2}} 2^{-jd} \cdot e^{-\frac{d_j(x_0,y)^2}{64t}} \\
\leq & \sum_{l=k \cdot 2^j}^{\infty} \sum_{y \in E \cap S_{2k_0 + l \cdot 2^{-j}}} \frac{1}{t^{d/2}} 2^{-jd} \cdot e^{-\frac{(2k_0 + l \cdot 2^{-j})^2}{256t}} \cdot (2d) \cdot (4k_0 + 2l \cdot 2^{-j})^{d-1} \cdot 2^{j(d-1)} \\
\leq & \sum_{l=k \cdot 2^j}^{\infty} \frac{2d \cdot 2^{-j}}{t^{d/2}} \cdot (4k_0 + 2l \cdot 2^{-j})^{d-1} \cdot e^{-\frac{(2k_0 + l \cdot 2^{-j})^2}{512t}} \\
\leq & \sum_{l=k \cdot 2^j}^{\infty} 2^{-j} \cdot (4k_0 + 2l \cdot 2^{-j})^{d-1} \cdot e^{-\frac{(2k_0 + l \cdot 2^{-j})^2}{512t}} \left( \sup_{0 < t \leq T} \frac{1}{t^{d/2}} e^{-\frac{(2k_0)^2}{512t}} \right) \\
\leq & c_2 \sum_{l=k \cdot 2^j}^{\infty} 2^{-j} \cdot (4k_0 + 2l \cdot 2^{-j})^{d-1} \cdot e^{-\frac{(2k_0 + l \cdot 2^{-j})^2}{512t}} \\
\leq & c_2 \sum_{u=k}^{\infty} \sum_{l=2^{j-u}} \cdot 2^{-j} \cdot (4k_0 + 2l \cdot 2^{-j})^{d-1} \cdot e^{-\frac{(2k_0 + u)^2}{512t}} \\
\leq & c_2 \sum_{u=k}^{\infty} (4k_0 + 2u + 2)^{d-1} \cdot e^{-\frac{(2k_0 + u)^2}{512t}} \\
\leq & c_2 \cdot \left( \sup_{x \geq 2k_0} (2x + 2)^{d-1} e^{-\frac{x^2}{512t}} \right) \sum_{u=k}^{\infty} e^{-\frac{(2k_0 + u)^2}{512t}} \leq c_2 \sum_{u=k}^{\infty} \sum_{u=k}^{\infty} e^{-\frac{(2k_0 + u)^2}{512t}}. \quad (4.9)
\end{aligned}
\]

It now follows that for any \( \delta > 0 \), there exists \( k_1 \in \mathbb{N} \) such that for all \( k \geq k_1 \),

\[
\begin{aligned}
\sum_{l=k \cdot 2^j}^{\infty} \sum_{y \in E \cap S_{2k_0 + l \cdot 2^{-j}}} \frac{1}{t^{d/2}} 2^{-jd} \cdot e^{-\frac{d_j(x_0,y)^2}{32t}} & \leq \sum_{l=k}^{\infty} \sum_{l=k}^{\infty} e^{-\frac{(2k_0 + l)^2}{512t}} \leq \frac{\delta}{2}, \quad (4.10)
\end{aligned}
\]
For the second term on the right hand side of (4.8), similarly we have for any generic \( k \geq 0 \),

\[
\sum_{l=k-2}^{\infty} \sum_{y \in E^l \cap S_{2k_0+l-2-j}} \frac{1}{l^{d/2}} e^{-\frac{2^j d_j(x_0, y)}{4}} \cdot 2^{-jd} \leq \sum_{l=k-2}^{\infty} \frac{1}{l^{d/2}} e^{-\frac{2^j (2k_0 + l - 2 - j)}{8}} (2d) \cdot (4k_0 + 2l \cdot 2^{-j})^{d-1} \cdot 2^{-j} \quad (t \geq 8^{-j}) \]

\[
\leq 2d \sum_{l=k-2}^{\infty} 2^{-j} (4k_0 + 2l \cdot 2^{-j})^{d-1} e^{-\frac{2^j (2k_0 + l - 2 - j)}{16}} \left( \sup_{j \geq 1} 8^j d e^{-\frac{2^j}{16}} \right) \leq c_3 \sum_{l=k-2}^{\infty} 2^{-j} (4k_0 + 2l \cdot 2^{-j})^{d-1} e^{-\frac{2^j (2k_0 + l - 2 - j)}{16}} \leq c_3 \sum_{l=k-2}^{\infty} (4k_0 + 2u + 2)^{d-1} e^{-\frac{2^j (2k_0 + u)}{16}} \leq c_3 \left( \sum_{u=k}^{\infty} (2x + 2)^{d-1} e^{-\frac{u}{16}} \right) \sum_{u=k}^{\infty} e^{-\frac{2^j (2k_0 + u)}{16}} \leq c_1 \sum_{u=k}^{\infty} e^{-\frac{2k_0 + u}{16}}. \quad (4.11)
\]

Therefore, for any \( \delta > 0 \), there exists \( k_2 \in \mathbb{N} \) such that for all \( k \geq k_2 \),

\[
\sum_{l=k-2}^{\infty} \sum_{y \in E^l \cap S_{2k_0+l-2-j}} \frac{1}{l^{d/2}} e^{-\frac{2^j d_j(x_0, y)}{4}} \cdot 2^{-jd} \leq c_1 \sum_{u=k}^{\infty} e^{-\frac{2k_0 + u}{16}} < \delta_2. \quad (4.12)
\]

Finally, we note that for any \( j \geq 1 \), \( E^j \subset \bigcup_{l=0}^{\infty} S_{l, 2^{-j}} \). Thus for the \( k_1, k_2 \) specified in (4.10) and (4.12), there exists \( M > 0 \) sufficiently large such that

\[
\{ y \in E^j : d_j(x_0, y) \geq M \} \subset \bigcup_{l=(k_1 \cup k_2)-2^j} (E^j \cap S_{2k_0+l-2^{-j}}) \quad \text{for all } j \geq 1. \quad (4.13)
\]

This combining with (4.8), (4.10), and (4.12) implies that given \( \delta > 0 \), for such chosen \( M > 0 \),

\[
\mathbb{P}^{x_0} \left[ d_j(X^j_t, x_0) \geq M \right] < \delta, \quad \text{for all } j \geq 1. \quad (4.14)
\]

\[ \Box \]

Lemma 4.5. For any fixed \( T > 0 \), \( \delta > 0 \), there exist \( M_3 > 0 \) and an integer \( N_2 \geq N_1 \) satisfying \( T > 2 \cdot 8^{-N_2} \), such that for all \( j \geq N_2 \) and \( M > M_3 \), it holds that

\[
\mathbb{P}^y \left[ |X^j_t|_\rho \leq \frac{M}{2} \right] < \delta, \quad \text{for all } |y|_\rho > M, t \in [8^{-j}, T - 8^{-j}].
\]

Proof. For a generic \( M > 0 \), given \( |y|_\rho > M \) and \( t \in [8^{-j}, T - 8^{-j}] \),

\[
\mathbb{P}^y \left[ |X^j_t|_\rho \leq \frac{M}{2} \right] = \sum_{\rho(x, a^*_j) \leq M/2} p_j(t, y, x)m_j(x) + p_j(t, y, a^*_j)m_j(a^*_j) \quad (4.15)
\]

13
In view of Proposition 2.6 and Corollary 3.3, since \(d_j(y, a^*_j) \geq \rho(y, a^*_j) \geq M \) and \( t \in [8^{-j}, T - 8^{-j}] \), there exists \( c_1 > 0 \) such that

\[
p_j(t, y, a^*_j)m_j(a^*_j) \leq \frac{c_1 2^{-j}}{t^{d/2}} \left( e^{-\frac{M^2}{8\pi t}} + e^{-\frac{2jM}{8\pi t}} \right) \leq c_1 \left( \frac{1}{t^{d/2}} e^{-\frac{M^2}{8\pi t}} + \frac{2^{-j}}{8^{-jd/2}} e^{-\frac{2jM}{8\pi t}} \right). \tag{4.16}\]

Given any \( \delta > 0 \), there exists \( c_2 > 0 \) such that when \( M \geq c_2 \),

\[
\sup_{0 < t \leq T} \frac{c_1}{t^{d/2}} e^{-\frac{M^2}{8\pi t}} < \frac{\delta}{4}.
\]

For the second term on the right hand side of (4.16), given any \( \delta > 0 \), there exist \( k_1 \in \mathbb{N} \) and \( c_3 > 0 \) such that for all \( j \geq k_1 \) and all \( M > c_3 \) specified above,

\[
\frac{c_1 2^{-j}}{8^{-jd/2}} e^{-\frac{2jM}{8\pi t}} = c_1 (2^j)^{-j} e^{-j \frac{M^2}{8\pi t}} < \frac{\delta}{4}.
\]

Thus we have showed that given any \( \delta > 0 \), there exist \( c_4 > 0 \) and \( k_2 \in \mathbb{N} \) such that for all \( M > c_4 \) and all \( j \geq k_1 \),

\[
p_j(t, y, a^*_j)m_j(a^*_j) < \frac{\delta}{2}. \tag{4.17}\]

For the other term on the right hand side of (4.15), first we note that since \( K \subset \{ (x_1, \ldots, x_d) \in \mathbb{R}^d : \max_{1 \leq i \leq d} |x_i| \leq k_0 \} \),

\[
\left\{ x \in E^j : \rho(x, a^*_j) \leq \frac{M}{2} \right\} \subset \left\{ (x_1, \ldots, x_d) \in \mathbb{R}^d : \max_{1 \leq i \leq d} |x_i| \leq k_0 + 2M \right\}.
\]

This implies that there exists some \( c_5 > 0 \) such that for \( M \geq 2k_0 \),

\[
\# \left\{ x \in E^j : \rho(x, a^*_j) \leq \frac{M}{2} \right\} \leq c_5 \cdot M^d \cdot 2^j d. \tag{4.18}\]

Again since \( d_j(y, x) \geq \rho(y, x) \geq M \) and \( t \in [8^{-j}, T - 8^{-j}] \), by Corollary 3.3,

\[
\sum_{\rho(x, a^*_j) \leq M/2 \atop x \neq a^*_j} p_j(t, y, x)m_j(x) \leq \sum_{\rho(x, a^*_j) \leq M/2 \atop x \neq a^*_j} c_6 \left( \frac{1}{t^{d/2}} e^{-\frac{M^2}{8\pi t}} + \frac{1}{8^{-jd/2}} e^{-\frac{2jM}{8\pi t}} \right) 2^{-jd} \tag{4.18} \]

\[
\leq c_6 \cdot M^d \cdot 2^j d \left( \frac{1}{t^{d/2}} e^{-\frac{M^2}{8\pi t}} + \frac{1}{8^{-jd/2}} e^{-\frac{2jM}{8\pi t}} \right) 2^{-jd} \]

\[
(M \geq 2k_0) \leq c_6 M^d e^{-\frac{M^2}{16\pi t}} \left( \sup_{0 < t \leq T} e^{-\frac{4M^2}{8\pi t}} \right) + c_6 \cdot M^d \cdot 2^{3jd/2} e^{-\frac{3jM}{8\pi t}}
\]

\[
\leq c_6 M^d e^{-\frac{M^2}{16\pi t}} + c_6 \cdot M^d \cdot 2^{3jd/2} e^{-\frac{3jM}{8\pi t}}. \tag{4.19}\]

Given any \( \delta > 0 \), for the first term on the right hand side of (4.19), there exists \( c_7 \geq 2k_0 \) such that for all \( M \geq c_7 \), \( c_6 \cdot M^d \cdot e^{-\frac{M^2}{16\pi t}} < \delta /4 \). For the second term on the right hand side of (4.15), we first denote by \( c_8 := \sup_{M \geq c_7} M^d e^{-M/16} \). Thus for all \( M \geq c_7 \),

\[
c_6 \cdot M^d \cdot 2^{3jd/2} e^{-\frac{3jM}{8\pi t}} \leq c_6 \cdot M^d \cdot 2^{3jd/2} e^{-\frac{3jM}{16}} \cdot e^{-\frac{M}{16}} \leq c_6 c_8 \cdot 2^{3jd/2} e^{-\frac{3jM}{16}}. \tag{4.20}\]
From the last display above, one can tell that there exists \( k_3 \in \mathbb{N} \) such that for all \( j \geq k_3 \) and all \( M \geq c_1 \),

\[
c_6 \cdot M^d \cdot 2^{3d} \cdot 2^{3j} \cdot e^{-\frac{\delta^2 d}{8}} < \frac{\delta}{4}.
\]

(4.21)

Combining (4.21) with the previous discussion regarding the first term on the right hand side of (4.15), it has been shown that there exist \( c_7 > 0 \) and \( k_3 \in \mathbb{N} \) such that for all \( M \geq c_7 \) and all \( j \geq k_3 \)

\[
\sum_{\substack{\rho(x,a^j) \leq M/2 \\
x \neq a^j}} p_j(t,y,x) m_j(x) < \frac{\delta}{4} + \frac{\delta}{4} = \frac{\delta}{2}.
\]

(4.22)

Finally, combining (4.15), (4.17), and (4.22), it has been shown that for all \( M \geq c_4 \lor c_7 \) and all \( j \geq k_2 \lor k_3 \),

\[
\mathbb{P}^y \left[ \left| X^j_t \right|_\rho \leq \frac{M}{2} \right] < \delta, \quad \text{for all } |y|_\rho > M, \; t \in [8^{-j}, T - 8^{-j}].
\]

Lemma 4.6. Fix \( x_0 \in \bigcap_{j \geq 1} E^j \). For every \( T > 0 \) and every \( \delta > 0 \), for the \( N_2 \) and \( M_3 \) given in Lemma 4.5, it holds for all \( j \geq N_2 \) and all \( M > M_3 \) that

\[
\mathbb{P}^{x_0} \left[ 8^{-j} \leq \tau_M \leq T - 8^{-j}, \left| X^j_T \right|_\rho \leq \frac{M}{2} \right] < \delta.
\]

Proof.

\[
\begin{align*}
\mathbb{P}^{x_0} & \left[ 8^{-j} \leq \tau_M \leq T - 8^{-j}, \left| X^j_T \right|_\rho \leq \frac{M}{2} \right] \\
& = \int_{8^{-j}}^{T - 8^{-j}} \mathbb{E}^{x_0} \left[ \mathbb{P}^{X^j_{T-s}} \left[ \left| X^j_{T-s} \right|_\rho \leq \frac{M}{2} \right] ; \tau_M \in ds \right] \\
& \leq \int_{8^{-j}}^{T - 8^{-j}} \mathbb{E}^{x_0} \left[ \sup_{y \geq |y|_\rho M} \mathbb{P}^{y} \left[ \left| X^j_T \right|_\rho \leq \frac{M}{2} \right] ; \tau_M \in ds \right] \\
& \leq \sup_{y \geq |y|_\rho M} \mathbb{P}^{y} \left[ \left| X^j_T \right|_\rho \leq \frac{M}{2} \right].
\end{align*}
\]

(4.23)

The conclusion then follows from Lemma 4.5.

\[ \square \]

Proposition 4.7. Given \( x_0 \in \bigcap_{j \geq 1} E^j \), given any \( T > 0 \), \( \delta > 0 \), there exist \( M_4 > 0 \) and an integer \( N_3 \geq N_1 \), such that for all \( j \geq N_3 \),

\[
\mathbb{P}^{x_0} \left[ \sup_{t \in [0, T]} \left| X^j_t \right|_\rho > M_4 \right] < \delta.
\]

(4.24)

Proof. On account of Lemma 4.2, it suffices to show that given any \( \delta > 0 \), \( T > 0 \), there exist \( c_1 > 0 \) and \( n_1 \in \mathbb{N} \) with \( 8^{-n_1} < T/2 \), such that for all \( M > c_1 \) and all \( j \geq n_1 \), the following hold:

(i) \( \mathbb{P}^{x_0} \left[ \sup_{t \in [0, 8^{-j}]} \left| X^j_t \right|_\rho \geq M \right] < \delta; \)
(ii) \( \mathbb{P}^x_0 \left[ X^j_T \right| \geq \frac{M}{2} \right] < \delta. \)

(iii) \( \mathbb{P}^x_0 \left[ T - 8^{-k} \leq \tau_M \leq T, \left| X^j_T \right| \leq \frac{M}{2} \right] < \delta; \)

(iv) \( \mathbb{P}^x_0 \left[ 8^{-j} \leq \tau_M \leq T - 8^{-j}, \left| X^j_T \right| \leq \frac{M}{2} \right] < \delta. \)

We claim that all (i)-(iv) are satisfied for all \( M \geq |x_0|_\rho + 2 \cdot \max_{1 \leq i \leq 4} M_i \) and \( j \geq \max_{1 \leq i \leq 3} N_i \), where the \( M_i \)'s and the \( N_i \)'s are given in Lemmas 4.3 and 4.4. Actually it is evident that (i) and (ii) hold on account of Lemma 4.3 and 4.4, respectively, together with triangle inequality for distances. (iv) holds in view of Lemma 4.6. To justify (iii), by the same argument as that for \([13\text{ (4.18)}]\) we have

\[
\mathbb{P}^x_0 \left[ T - 8^{-j} \leq \tau_M \leq T, \left| X^j_T \right| \leq \frac{M}{2} \right] \leq \sup_{|y| \geq M} \mathbb{P}^y \left[ \sup_{t \in [0, 8^{-j}]} \rho \left( X^j_t, X^j_0 \right) \geq \frac{M}{2} \right]. \tag{4.25}
\]

Thus (iii) follows from Lemma 4.5.

As a standard notation, given a metric \( d(\cdot, \cdot) \), we denote by

\[
w_d(x, \theta, T) := \inf_{\{t_i\}_{i=1}^n \in \Pi} \max_{1 \leq i \leq n} \sup_{s, t \in [t_{i-1}, t_i]} d(x(s), x(t)),
\]

where \( \Pi \) is the collection of all possible partitions of the form \( 0 = t_0 < t_1 < \cdots < t_{n-1} < T \leq t_n \) with \( \min_{1 \leq i \leq n} (t_i - t_{i-1}) \geq \theta \) and \( n \geq 1 \).

**Proposition 4.8.** Fix any \( x_0 \in \cap_{j \geq 1} E^j \). For any \( T > 0, \delta_1, \delta_2 > 0 \), there exist \( \delta_3 > 0 \) and an integer \( N_4 \geq N_1 \) such that for all \( j \geq N_4 \),

\[
\mathbb{P}^x_0 \left[ w_{\rho} \left( X^j, \delta_3, T \right) > \delta_1 \right] < \delta_2, \tag{4.26}
\]

where

\[
w_{\rho}(x, \delta_3, T) := \inf_{\{t_i\}} \max_{i} \sup_{s, t \in [t_{i-1}, t_i]} \rho(x(s), x(t)),
\]

where \( \{t_i\} \) ranges over all possible partitions of the form \( 0 = t_0 < t_1 < \cdots < t_{n-1} < T \leq t_n \) with \( \min_{1 \leq i \leq n} (t_i - t_{i-1}) \geq \delta_3 \) and \( n \geq 1 \).

**Proof.** Using the same argument as that for \([13\text{ (4.52)-(4.54)}]\), one can get for \( j \in \mathbb{N} \) and any \( \delta_1, \delta_3 > 0 \),

\[
\mathbb{P}^x_0 \left[ w_{\rho} \left( X^j, \delta_3, T \right) > \delta_1 \right] \leq 2 \left( \frac{T}{\delta_3} + 1 \right) \sup_{y \in E \atop 0 \leq s \leq \delta_3} \mathbb{P}^y \left[ \rho \left( X^j_0, X^j_s \right) \geq \frac{\delta_1}{4} \right]. \tag{4.27}
\]

For the right hand side of (4.27), we further have that for \( \delta_3 > 0 \) and \( j \in \mathbb{N} \) satisfying \( 8^{-j} < \delta_3 \),

\[
\sup_{0 < s \leq \delta_3} \mathbb{P}^y \left[ \rho \left( X^j_0, X^j_s \right) \geq \frac{\delta_1}{4} \right] \leq \sup_{0 < s \leq 8^{-j}} \mathbb{P}^y \left[ \rho \left( X^j_0, X^j_s \right) \geq \frac{\delta_1}{4} \right] + \sup_{8^{-j} \leq s \leq \delta_3} \mathbb{P}^y \left[ \rho \left( X^j_0, X^j_s \right) \geq \frac{\delta_1}{4} \right]. \tag{4.28}
\]
We first handle the second term on the right hand side above. For any \( \delta_3 > 0 \), \( t > 0 \), \( j \in \mathbb{N} \) such that \( 8^{-j} < t < \delta_3 \), for any \( y \in E_j \), there exists some \( c_1 > 0 \) such that
\[
\Pr[ \rho(y, X_t^j) \geq \frac{\delta_1}{4} ] \leq \Pr[ d_j(y, X_t^j) \geq \frac{\delta_1}{4} ]
\]
\[
\leq \sum_{x \in E_j, d_j(x, y) \geq \frac{\delta_1}{4}, x \neq a_j^*} \frac{c_1}{t d^{j/2}} \left( e^{-\frac{d_j(x, y)^2}{64t}} + e^{-\frac{2^j d_j(x, y)}{4}} \right) m_j(x)
\]
\[+ \frac{c_1}{t d^{j/2}} \left( e^{-\frac{\delta_1^2}{256t}} + e^{-\frac{2^j \delta_1}{16}} \right) m_j(a_j^*). \tag{4.29}
\]

For the first term on the right hand side of (4.29), we first note that for a given \( y \in E_j \) we have
\[
\left\{ x \in E_j : d_j(x, y) \geq \frac{\delta_1}{4}, x \neq a_j^* \right\} \subset \bigcup_{k=1}^{\infty} \left\{ x \in E_j : d_j(x, y) \leq \frac{\delta_1}{4} + k, x \neq a_j^* \right\}.
\]
In view of the definition of \( k_0 \) in (4.4), the diameter of \( K \) under Euclidean distance is at most \( 2k_0 \). Thus for all \( j \geq 1 \) and all \( x, y \in E_j \), it holds
\[d_j(x, y) \geq \rho(x, y), \quad \rho(x, y) + 2k_0 \geq |x - y|,
\]
Therefore,
\[
\left\{ x \in E_j : d_j(x, y) \leq \frac{\delta_1}{4} + k, x \neq a_j^* \right\} \subset \left\{ x \in 2^{-j} \mathbb{Z}^d : |x - y| \leq \frac{\delta_1}{4} + k + 2k_0 \right\}.
\]
It then follows that for any \( k \in \mathbb{N} \),
\[
\# \left\{ x \in E_j : d_j(x, y) \leq \frac{\delta_1}{4} + k, x \neq a_j^* \right\} \leq \left( \frac{\delta_1}{4} + k + 2k_0 \right)^{2d}. \tag{4.30}
\]
Now for the first term on the right hand side of (4.29), we have
\[
\sum_{x \in E_j, d_j(x, y) \geq \frac{\delta_1}{4}, x \neq a_j^*} \frac{c_1}{t d^{j/2}} \left( e^{-\frac{d_j(x, y)^2}{64t}} + e^{-\frac{2^j d_j(x, y)}{4}} \right) m_j(x)
\]
\[\leq \sum_{k=0}^{\infty} \sum_{x \in E_j, d_j(x, y) \leq \frac{\delta_1}{4} + k} \frac{c_1}{t d^{j/2}} \left( e^{-\frac{d_j(x, y)^2}{64t}} + e^{-\frac{2^j d_j(x, y)}{4}} \right) m_j(x)
\]
\[\quad \leq \sum_{k=0}^{\infty} \frac{c_1}{t d^{j/2}} \left( e^{-\frac{(\delta_1 + k)^2}{256t}} + e^{-\frac{2^j (\delta_1 + k)}{16}} \right) \cdot (\delta_1 + k + 2k_0)^{2d} 2^j 2^{jd} 2^{-jd} \tag{4.30}
\]
\[\leq \sum_{k=0}^{\infty} \frac{c_1(\delta_1 + k + 2k_0)^{2d}}{t d^{j/2}} e^{-\frac{(\delta_1 + k)^2}{256t}} + \sum_{k=0}^{\infty} \frac{c_1(\delta_1 + k + 2k_0)^{2d}}{t d^{j/2}} e^{-\frac{2^j (\delta_1 + k)}{16}}. \tag{4.31}
\]
Now, for the first term on the right hand side of (4.31), for any \( 8^{-j} < t < \delta_3 \leq T \),
\[
\sum_{k=0}^{\infty} \frac{c_1(\delta_1 + k + 2k_0)^{2d}}{t d^{j/2}} e^{-\frac{(\delta_1 + k)^2}{256t}} \leq e^{-\frac{\delta_3^2}{512t_3}} \left( \sup_{t \in (0, T]} \frac{c_1}{t d^{j/2}} e^{-\frac{\delta_3^2}{1024t}} \right) \sum_{k=0}^{\infty} (\delta_1 + k + 2k_0)^{2d} e^{-\frac{(\delta_1 + k)^2}{1024t}}
\]
\[\leq c_2(\delta_1, T, k_0) e^{-\frac{\delta_3^2}{512t_3}}. \tag{4.32}
\]
For the second term on the right hand side of (4.31), noticing that $t \geq 8^{-j}$,

$$\sum_{k=0}^{\infty} c_1(\delta_1 + k + 2k_0)^{2d} \frac{t^{d/2}}{t^{d/2}} e^{-\frac{\delta_1}{16}} \leq c_1 \left( (2^j)^{3d/2} e^{-\frac{2^j \delta_1}{32}} \right) \sum_{k=0}^{\infty} (\delta_1 + k + 2k_0)^{2d} e^{-\frac{\delta_1}{16}}$$

$$\leq \left( (2^j)^{3d/2} e^{-\frac{2^j \delta_1}{32}} \right) \cdot c_3(\delta_1, k_0). \quad (4.33)$$

Now we deal with the second term on the right hand side of (4.29). For any $t < T$ where $\delta_3 > 0$ is a generic constant,

$$\frac{c_1}{t^{d/2}} \left( e^{-\frac{\delta_1^2}{256T}} + e^{-\frac{\delta_1}{16}} \right) m_j(a_j^*) \leq \frac{c_4}{t^{d/2}} \left( e^{-\frac{\delta_1^2}{256T}} + e^{-\frac{\delta_1}{16}} \right) 2^{-j}$$

$$\leq c_4 \cdot e^{-\frac{\delta_1^2}{512T^2}} \left( \sup_{t \in (0,T]} \frac{1}{t^{d/2}} e^{-\frac{\delta_1^2}{256T}} \right) + c_4 \cdot (2^j)^{3d/2} e^{-\frac{2^j \delta_1}{16}}$$

$$\leq c_5(\delta_1, T) e^{-\frac{\delta_1^2}{512T^2}} + c_4 \cdot (2^j)^{3d/2} e^{-\frac{2^j \delta_1}{16}}. \quad (4.34)$$

From here, first we replace the two terms on the right hand side of (4.31) with (4.32) and (4.33), then plug what is obtained as well as (4.34) into the right hand side of (4.29). Consolidating the common terms gives us that for any pair of $\delta_3 > 0$, $j \in \mathbb{N}$ such that $8^{-j} < \delta_3$,

$$\sup_{t \in [8^{-j}, \delta_3]} \sup_{y \in E^j} \mathbb{P}^y \left[ \rho(y, X_t^j) \geq \frac{\delta_1}{4} \right] \leq c_6(\delta_1, T, k_0) e^{-\frac{\delta_1^2}{512T^2}} + c_7(\delta_1, k_0) \left( (2^j)^{3d/2} e^{-\frac{2^j \delta_1}{16}} \right). \quad (4.35)$$

From the above, given any pair of $\delta_1, \delta_2 > 0$, we can first choose $\delta_3 > 0$ sufficiently small such that

$$c_6(\delta_1, T, k_0) e^{-\frac{\delta_1^2}{512T^2}} < \frac{\delta_2 \delta_3}{4(T + 2 \delta_3)}. \quad (4.36)$$

With this $\delta_3$ chosen above, then we choose $n_1 > 0$ satisfying $8^{-n_1} < \delta_3$ such that for all $j \geq n_1$,

$$c_7(\delta_1, k_0) \left( (2^j)^{3d/2} e^{-\frac{2^j \delta_1}{16}} \right) < \frac{\delta_2 \delta_3}{4(T + 2 \delta_3)}.$$

Thus given any pair of $\delta_1, \delta_2 > 0$, there exists $\delta_3 > 0$ and $n_1 \in \mathbb{N}$ with $8^{-n_1} < \delta_3$ such that for all $j \geq n_1$,

$$\sup_{y \in E^j \cap \delta_{\delta_1}, \delta_{\delta_1} \leq t \leq \delta_1} \sup_{t \in [8^{-j}, \delta_3]} \mathbb{P}^y \left[ \rho \left( X_0^j, X_t^j \right) \geq \frac{\delta_1}{4} \right] < \frac{\delta_2 \delta_3}{2(T + 2 \delta_3)}. \quad (4.37)$$

The same argument as that for (4.50) yields that given any $\delta_1, \delta_2 > 0$ and the $\delta_3 > 0$ therefore selected in (4.36), there exists $n_2$ such that for all $j \geq n_2$,

$$\sup_{y \in E^j \cap \delta_2, \delta_2 \leq t \leq \delta_1} \sup_{t \in [8^{-j}, \delta_3]} \mathbb{P}^y \left[ \rho \left( X_0^j, X_t^j \right) \geq \frac{\delta_1}{4} \right] < \frac{\delta_2 \delta_3}{2(T + 2 \delta_3)}. \quad (4.38)$$
Finally, plugging (4.37) and (4.38) into the right hand side of (4.27) yields that given $\delta_1, \delta_2 > 0$, there exist $\delta_3 > 0$ and $n_1, n_2 \in \mathbb{N}$ such that for all $j \geq n_1 \vee n_2$,

$$\mathbb{P}^{x_0} \left[ w_{\rho} (X^j, \delta_3, T) > \delta_1 \right] \leq 2 \left( \left\lceil \frac{T}{\delta_3} \right\rceil + 1 \right) \cdot \frac{\delta_2 \delta_3}{T + 2 \delta_3} \leq 2 \cdot \frac{T + 2 \delta_3}{\delta_3} \cdot \frac{\delta_2 \delta_3}{T + 2 \delta_3} < 2\delta_2.$$

\[ \square \]

**Theorem 4.9.** Fix $x_0 \in \bigcap_{j \geq 1} E^j$. For every $T > 0$, the laws of $\{X^j, \mathbb{P}^{x_0}, j \geq 1\}$ are $C$-tight in the Skorokhod space $\mathcal{D}([0, T], E, \rho)$ equipped with the Skorokhod topology.

**Proof.** This follows immediately from [10, Chapter VI, Proposition 3.21]. In view of Propositions 4.7-4.8 \[ \square \]

**Remark 4.10.** By the same proof as that to Theorem 4.9, it can be shown that given any $T > 0$, the laws of $\{X^j, \mathbb{P}^{0,j}, j \geq 1\}$ are $C$-tight in the Skorokhod space $\mathcal{D}([0, T], E, \rho)$ equipped with the Skorokhod topology.

Before proving the next lemma, we define the following class of functions:

$$\mathcal{G} := \{ f \in C^3_c(\mathbb{R}^d), f = \text{constant on } K \}. \quad (4.39)$$

For $f \in \mathcal{G}$, we define

$$\mathcal{L}^j f(x) := 2^{2j} \sum_{y \leftrightarrow x \text{ in } E^j} (f(y) - f(x)) \frac{1}{v_j(x)}, \quad \text{for } x \in E^j_0. \quad (4.40)$$

We also set

$$S^j := \{ x \in E^j : x \neq a^*_j, v_j(x) = 2d \text{ in } G^j \}, \quad (4.41)$$

where $G^j$ has been defined in [10]

**Lemma 4.11.** For every $\delta > 0$ and every $f \in \mathcal{G}$, there exists some $n_{\delta, f} \in \mathbb{N}$, such that for all $j \geq n_{\delta, f}$:

(i) $m_j(E^j \setminus S^j) < \delta$;

(ii) As $j \to \infty$, $\mathcal{L}^j f$ converges uniformly to

$$\mathcal{L} f(x) := \frac{1}{2d} \Delta f(x) + O(1) 2^{-j} \quad \text{on } S^{n_{\delta, f}}. \quad (4.42)$$

Also, there exists some constant $C_7 > 0$ independent of $j$ such that for all $j \geq 1$,

$$\mathcal{L}^j f(x) \leq C_7, \quad \text{for all } x \in E^j.$$

**Proof.** To claim (i), we notice that $\{E^j \setminus S^j\} \subset \{x = a^*_j \text{ or } x \leftrightarrow a^*_j\}$. Thus by Proposition 2.6, there exists $c_1 > 0$ such that for all $j \geq j_0$ specified in Proposition 2.6

$$m_j(E^j \setminus S^j) \leq m_j(\{x = a^*_j \text{ or } x \leftrightarrow a^*_j\}) \leq m_j(a^*_j) + v_j(a^*_j) \cdot 2^{-jd} \leq c_1 \cdot 2^{-j}.$$ 

Using Taylor’s expansion we have for any $f \in \mathcal{G}$ and any $j \geq j_0$,

$$\tilde{\mathcal{L}} f(x) = 2^{2j} \sum_{y \leftrightarrow x} \left[ \sum_{i=1}^d \frac{\partial f(x)}{\partial x_i} (y_i - x_i) + \frac{1}{2} \sum_{i,l=1}^d \frac{\partial^2 f(x)}{\partial x_i \partial x_l} (y_i - x_i)(y_l - x_l) + O(1) |y - x|^3 \right] \frac{1}{v_j(x)}.$$
Hence
\[ \mathcal{L}^j f(x) = \frac{1}{2d} \Delta f(x) + O(1)2^{-j}, \quad \text{for } x \in S^j. \] (4.43)

Since \( \{S^j\}_{j \geq 1} \) is an increasing sequence of sets in \( j \), both (i) and (ii) have been justified. To justify the last claim, we first note that by (4.43) and the fact that \( f \in C^3_c(\mathbb{R}^d) \), there exists \( c_2 > 0 \) independent of \( j \) such that
\[ \mathcal{L}^j f(x) \leq c_2, \quad \text{for all } x \in \bigcup_{j \geq 1} S^j. \] (4.44)

We denote by \( c_K := f|_K \). Given any \( x \leftrightarrow a_j^* \) in \( E^j \), by the definition of \( G^j \) in (1.2), there must exist a point \( a \in K \) such that \( |x - a| \leq 2^{-j} \). Since \( f \in C^3_c(\mathbb{R}^d) \) and is constant on \( K \), the first order derivatives of \( f \) vanish on \( K \). By Taylor expansion, there exists some constant \( c_3 > 0 \) only depending on \( f \) but not \( j \) such that
\[ (f(x) - c_K) = |f(x) - f(a)| \leq \sum_{i=1}^d \left| \frac{\partial f}{\partial x_i}(a) \right| |x - a| + c_2 \cdot |x - a|^2 \leq c_3 \cdot 2^{-2j}. \] (4.45)

Thus by the definition of \( \mathcal{L}^j \) and Proposition 2.6
\[ \mathcal{L}^j f(a_j^*) = 2^{2j} \sum_{x \leftrightarrow a_j^*} (f(x) - c_K) \cdot v_j(a_j^*)^{-1} \leq 2^{2j} \cdot c_3 \cdot 2^{-2j} \leq c_3. \] (4.46)

To bound \( \mathcal{L}^j f(x) \) for \( x \leftrightarrow a_j^* \), we first note in this case,
\[ \mathcal{L}^j f(x) = 2^{2j} \sum_{y \leftrightarrow x} (f(y) - f(x)) \cdot v_j(x)^{-1} \leq 2^{2j} \max_{x \leftrightarrow a_j^*} \max_{y \leftrightarrow x} |f(y) - f(x)|. \] (4.47)

For each \( y \) in the second “max” in (4.47), notice that there exists \( a \in K \) such that \( |y - a| \leq 2 \cdot 2^{-j} \). Thus by similar reasoning for (4.45), \( |f(y) - c_K| \leq c_3 \cdot (2 \cdot 2^{-j})^2 \). Since for all \( x \) in the first “max” in (4.47), \( |f(x) - c_K| \leq c_3 \cdot 2^{-2j} \), by triangle inequality it follows that for some \( c_4 > 0 \) independent of \( j \), for all \( j \geq 1 \), \( \mathcal{L}^j f(x) \leq c_4 \) for all \( x \) such that \( x \leftrightarrow a_j^* \). This together with (4.46) shows
\[ \mathcal{L}^j f(x) \leq c_3 \vee c_4, \quad \text{for all } x \in \bigcup_{j \geq 1} (E^j \setminus S^j). \]

This combined with (4.44) proves the last claim of this lemma. The proof is complete. \( \square \)

The following lemma is used in the proof of the main result: Theorem 4.13.

**Lemma 4.12.** Fix \( T > 0 \). Given any \( \delta > 0 \), there exist \( C_8 > 0 \) and an integer \( N_\delta \geq N_1 \), such that for all \( j \geq N_\delta \),
\[ \sup_{t \in [(2^j \delta)^{-2/d}, T]} \mathbb{P}^{x_0} \left[ X_t^j \notin S^j \right] \leq C_8 \delta. \]

**Proof.** Given any \( \delta > 0 \), choose \( j_\delta \in \mathbb{N} \) large enough such that \( (2^{j_\delta} \delta)^{-2/d} < T \). For any \( t \in [(2^j \delta)^{-2/d}, T] \) with \( j \geq j_\delta \), by Corollary 3.3 and Proposition 2.6, there exists \( c_1, c_2 > 0 \) such
that
\[
\mathbb{P}^{x_0} \left[ X^j_t \notin S^j \right] \leq \sum_{y \notin S^j} \frac{c_1}{l^d/2} m_j(y) \\
\leq \sum_{x \rightarrow a_j^*} \left( \frac{c_1}{l^d/2} \cdot m_j(x) \right) + \frac{c_1}{l^d/2} \cdot m_j(a_j^*)
\]
(Proposition 2.6) 
\[
(2 \cdot 2^{(d-1)} \cdot c_1 l^{-d/2} \cdot 2^{-jd} + c_1 l^{-d/2} \cdot c_2 \cdot 2^{(d-1)} \cdot 2^{-jd})
\]
This proves the desired result by letting \( N_\delta \) be the selected \( j_\delta \).

**Theorem 4.13.** Given \( x_0 \in \cap_{j \geq 1} E^j \), \( \{X^j, \mathbb{P}^{x_0}, j \geq 1\} \) converges weakly to the BMD described in (1.1) starting from \( x_0 \).

**Proof.** Since the laws of \( \{X^j\}_{j \geq 1} \) are C-tight in \( D([0,T], E, \rho) \), any sequence has a weakly convergent subsequence supported on the set of continuous paths. Denote by \( \{X^{j_l} : l \geq 1\} \) any such weakly convergent subsequence, and denote by \( Y \) its weak limit which is continuous. By Skorokhod representation theorem (see, e.g., [9, Chapter 3, Theorem 1.8]), we may assume that \( \{X^{j_l}, l \geq 1\} \) as well as \( Y \) are defined on a common probability space \( (\Omega, \mathcal{F}, \mathbb{P}) \), so that \( \{X^{j_l}, l \geq 1\} \) converges almost surely to \( Y \) in the Skorokhod topology.

For every \( t \in [0,T] \), we set \( \mathcal{M}^{j_l} := \sigma(X_s^{j_l}, s \leq t) \) and \( \mathcal{M}_t := \sigma(Y_s, s \leq t) \). It is obvious that \( \mathcal{M}_t \subset \sigma(\mathcal{M}^{j_l} : j \geq 1) \). By the same argument as that for [13] Theorem 5.3 with the use of Lemma 5.2 in [13] being replaced with Lemma [1.12], it can be shown that \( (Y, \mathbb{P}^{x_0}) \) is indeed a solution to the \( \mathcal{D}([0,T], E, \rho) \) martingale problem \( (\mathcal{L}, \mathcal{G}) \) with respect to the filtration \( \{\mathcal{M}_t\}_{t \geq 0} \).

Next we claim that the BMD associated with the Dirichlet form described by (1.1) is a strong Feller process. To see this, we denote by \( \{G_\alpha^a\}_{\alpha > 0} \) the resolvent operators of \( X \), and denote by \( \{G_\alpha^{E\setminus\{a^*\}}\}_{\alpha > 0} \) the part process of \( X \) killed hitting \( a^* \), which has the same distribution as regular Brownian motion on \( \mathbb{R}^d \) killed upon hitting \( K \). By strong Markov property, for \( x \in E \), for every bounded measurable function \( f : E \rightarrow \mathbb{R} \),
\[
G_\alpha f(x) = G_\alpha^{E\setminus\{a^*\}} f(a^*) + \int_0^\infty e^{-\alpha s} \mathbb{P}^{x_0}[\sigma_{\{a^*\}} \in ds]
\]
(4.48)

In the right hand side above, the map \( x \mapsto \mathbb{P}^{x_0}[\sigma_{\{a^*\}} \in ds] \) is continuous because every point on the boundary of \( K \subset \mathbb{R}^d \) is regular for \( K \). Therefore we conclude that \( X \) is a strong Feller process because \( G_\alpha^{E}(b\mathcal{B}(E)) \subset C(E) \). In view of [11, §1.5], the infinitesimal generator of \( X \) can be described by \( (\mathcal{L}, \mathcal{D}(\mathcal{L})) \), where \( u \in \mathcal{F} \) is in \( \mathcal{D}(\mathcal{L}) \) if there exists \( f \in L^2(\mathbb{E}) \) such that
\[
\frac{1}{2} \int_{E\setminus\{a^*\}} \nabla u(x) \nabla v(x) \, dm = \int_{E\setminus\{a^*\}} f(x) \cdot v(x) \, dm, \quad \text{for all } v \in \mathcal{F}.
\]
(4.49)
It also holds that \( \mathcal{L}u = f = \frac{1}{2} \Delta u \) for \( u \in \mathcal{D}(\mathcal{L}) \). It then is clear that the bp-closure of the graph of \( (\mathcal{L}, \mathcal{D}(\mathcal{L})) \) is contained in the bp-closure of the graph of \( (\mathcal{L}, \mathcal{G}) \). By [2] Lemma 3.4.18, any solution to the martingale problem \( (\mathcal{L}, \mathcal{G}) \) must be a solution to the martingale problem \( (\mathcal{L}, \mathcal{G}) \). Since \( X \) is a strong Feller process, the martingale problem \( (\mathcal{L}, \mathcal{D}(\mathcal{L})) \) must be well-posed with its unique solution being \( X \). Therefore the martingale problem \( (\mathcal{L}, \mathcal{G}) \) must be well-posed with its unique solution being \( X \). This means that \( X \) is the sequential limit of any weakly convergent subsequence of \( \{X^j\}_{j \geq 1} \), the proof is complete.

**Remark 4.14.** In view of Remark 4.10, the same argument can show that \( \{X^j, \mathbb{P}^{a^*}, j \geq 1\} \) converges weakly to the BMD given by (1.1) starting from \( a^* \).
References

[1] M. T. Barlow, *Random walks and heat kernels on graphs*. Cambridge University Press, 2017.

[2] A. Bovier, *Markov Processes*. Available at https://wt.iam.uni-bonn.de/fileadmin/WT/Inhalt/people/An.pdf.

[3] E. A. Carlen, S. Kusuoka and D. W. Stroock, Upper bounds for symmetric Markov transition functions, *Ann. Inst. H. Poincaré. Probab. Statist.* 23 (1987) 245-287.

[4] Z.-Q. Chen, *Brownian motion with darning*. Available at https://sites.math.washington.edu/~zchen/BMD_lecture.pdf.

[5] Z.-Q. Chen and M. Fukushima, *Symmetric Markov Processes, Time Change and Boundary Theory*. Princeton University Press, 2011.

[6] Z.-Q. Chen, P. J. Fitzsimmons, K. Kuwae and T.-S. Zhang, Stochastic calculus for symmetric Markov processes. *Ann. Probab.* 36 (2008), 931-970.

[7] Z.-Q. Chen and S. Lou, Brownian motion on some spaces with varying dimension. *Ann. Probab.* 47 (2019), 213-269.

[8] K. L. Chung and Z. Zhao, *From Brownian motion to Schrödinger’s equation*. Springer, 2001.

[9] S. N. Ethier and T. G. Kurtz, *Markov Processes: Characterization and Convergence*. Wiley, New York, 1986.

[10] M. Fukushima, Y. Oshima, and M. Takeda, *Dirichlet Forms and Symmetric Markov Processes*, Second revised and extended edition. de Gruyter Studies in Mathematics, vol. 19, Walter de Gruyter & Co., Berlin, 2011.

[11] J. Jacob and A. N. Shiryaev, *Limit Theorems for Stochastic Processes*. Springer, Berlin, 1987.

[12] S. Lou, Discrete approximate to Brownian motion with varying dimension in bounded domains. To appear in *Kyoto J. Math*. Available at https://arxiv.org/pdf/2007.01933.pdf.

[13] S. Lou, Discrete approximate to Brownian motion with varying dimension in unbounded domains. Available at https://arxiv.org/pdf/2110.12716.pdf.

[14] F. Markus, *Markov processes and martingale problems*. Available at https://www.math.unipd.it/~fischer/Didattica/MarkovMP.pdf.

Shuwen Lou
Department of Mathematics and Statistics, Loyola University Chicago, Chicago, IL 60660, USA
E-mail: slou1@luc.edu