COMPLEX FLAT VECTOR BUNDLES AND K-THEORY

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Abstract. The purpose of this paper is to put the real part of the Riemann–Roch–Grothendieck for complex flat vector bundles into the framework of $K$-theory in the following sense. For any given submersion $\pi : X \rightarrow B$ with closed fibers $Z$ we define an analytic index $\text{ind}^{\Lambda}_{K} : \widehat{K}(X) \rightarrow \widehat{K}(B)$ associated to twisted de Rham operators without assuming the twisting bundles to be flat in differential $K$-theory, and prove that when its restriction $\text{ind}^{\Lambda}_{R/Z} : K^{-1}(X; \mathbb{R}/\mathbb{Z}) \rightarrow K^{-1}(B; \mathbb{R}/\mathbb{Z})$ to $\mathbb{R}/\mathbb{Z} K$-theory is applied to the element associated to any given $\mathbb{Z}_2$-graded complex flat vector bundle $(F, \nabla^F)$ of virtual rank zero with a Hermitian metric, it has a canonical representative given by the element associated to the cohomology bundle of $(F, \nabla^F)$. We also prove a Riemann–Roch–Grothendieck theorem for $\text{ind}^{\Lambda}_{R/Z}$ in $\mathbb{R}/\mathbb{Z} K$-theory at the differential form level. These results give a new proof of the $\mathbb{Z}_2$-graded version of the real part of the Riemann–Roch–Grothendieck theorem for complex flat vector bundles.

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1. Introduction

Let \( \pi : X \to B \) be a submersion with closed fibers \( Z \). The Riemann–Roch–Grothendieck (RRG) theorem for complex flat vector bundles states that for any complex flat vector bundle \((F, \nabla^F)\) the following equality

\[
\text{CCS}(H(Z, F|_Z), \nabla^{H(Z, F|_Z)}) = \int_{X/B} e(T^V X) \cup \text{CCS}(F, \nabla^F) \quad (1.0.1)
\]

holds in \( H^{\text{odd}}(B; \mathbb{C}/\mathbb{Q}) \), where \( \text{CCS}(F, \nabla^F) \) is the Cheeger–Chern–Simons class of \((F, \nabla^F)\) and \((H(Z, F|_Z), \nabla^{H(Z, F|_Z)})\) is the cohomology bundle of \((F, \nabla^F)\). Bismut–Lott prove a refinement of the imaginary part of \((1.0.1)\) at the differential form level \([6, \text{Theorem } 3.23]\). Bismut proves the real part of \((1.0.1)\) under the assumption that the fibers are fiberwise orientable \([5, \text{Theorem } 3.2]\). Ma–Zhang give another proof of the imaginary part of \((1.0.1)\) and prove the real part of \((1.0.1)\) in full generality \([15, \text{Theorem } 1.1]\). In \([11, \text{Theorem } 1]\) we prove the \(\mathbb{Z}_2\)-graded version of the real part of \((1.0.1)\) at the differential form level when \(\dim(Z)\) is even.

The purpose of this paper is to put the real part of the RRG theorem for complex flat vector bundles into the framework of \(K\)-theory in the following sense. For any given submersion \( \pi : X \to B \) with closed fibers \( Z \), we define an analytic index

\[
\text{ind}_{\hat{K}}^{a, A} : \hat{K}_{\text{FL}}(X) \to \hat{K}_{\text{FL}}(B)
\]

in Freed–Lott differential \(K\)-theory \([8, \text{Definition } 2.16]\) by employing the local index theory of a suitable Dirac operator \(D^{A \otimes E}\), which is called the generalized twisted de Rham operator. Denote by

\[
\text{ind}_{\mathbb{R}/\mathbb{Z}}^{a, A} : K^{-1}_{L}(X) \to K^{-1}_{L}(B)
\]

the restriction of \(\text{ind}_{\hat{K}}^{a, A}\) to \(\mathbb{R}/\mathbb{Z} K\)-theory, where \(K^{-1}_{L}\) is a geometric model of \(\mathbb{R}/\mathbb{Z} K\)-theory defined by Lott \([13, \text{Definition } 7]\). For any \(\mathbb{Z}_2\)-graded complex flat vector bundle \((F, \nabla^F)\) over \(X\) of virtual rank zero, put a \(\mathbb{Z}_2\)-graded Hermitian metric \(g^F\) on \(F \to X\) and define a \(\mathbb{Z}_2\)-graded unitary connection \(\nabla^{F,u}\) on \(F \to X\) by \((A.1.7)\). Then

\[
\mathcal{F} := (F, g^F, \nabla^{F,u}, 0) \quad (1.0.2)
\]
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is a $\mathbb{Z}_2$-graded generator of $K^{-1}_L(X)$ [13, p.286]. Similarly, denote by

$$\mathcal{H}(Z,F|_Z) = (H(Z,F|_Z), g^{H(Z,F|_Z), \nabla^{H(Z,F|_Z)})$$

the $\mathbb{Z}_2$-graded generator of $K^{-1}_L(B)$ associated to the cohomology bundle $(H(Z,F|_Z), \nabla^{H(Z,F|_Z)})$ of $(F, \nabla^F)$.

The first main result of this paper is the following theorem.

**Theorem 1.1.** Let $\pi : X \to B$ be a submersion with closed fibers $Z$. For any $\mathbb{Z}_2$-graded generator $F$ of $K^{-1}_L(X)$ of the form (1.0.2) we have

$$\text{ind}^{a\Lambda}_{\mathbb{R}/\mathbb{Z}}(F) = \mathcal{H}(Z,F|_Z).$$

Theorem 1.1 says the analytic index $\text{ind}^{a\Lambda}_{\mathbb{R}/\mathbb{Z}}(F)$ has a canonical representative. Note that Theorem 1.1 is the analytical counterpart of a statement by Bunke (see the remark after [15, Theorem 1.2]).

The second main result of this paper is the following RRG theorem for $\text{ind}^{a\Lambda}_{\mathbb{R}/\mathbb{Z}}$ in $\mathbb{R}/\mathbb{Z} K$-theory, and we prove it at the differential form level.

**Theorem 1.2.** Let $\pi : X \to B$ be a submersion with closed fibers. Then the following diagram commutes.

$$
\begin{array}{ccc}
K^{-1}_L(X) & \xrightarrow{\text{ch}_{\mathbb{R}/\mathbb{Q}}} & H^{\text{odd}}(X; \mathbb{R}/\mathbb{Q}) \\
\downarrow \text{ind}^{a\Lambda}_{\mathbb{R}/\mathbb{Z}} & & \downarrow \int_{X/B} e(T^V X) \cup (\cdot) \\
K^{-1}_L(B) & \xrightarrow{\text{ch}_{\mathbb{R}/\mathbb{Q}}} & H^{\text{odd}}(B; \mathbb{R}/\mathbb{Q})
\end{array}
$$

Here $\text{ch}_{\mathbb{R}/\mathbb{Q}} : K^{-1}_L \to H^{\text{odd}}(\mathbb{R}/\mathbb{Q})$ is the Chern character in $\mathbb{R}/\mathbb{Z} K$-theory.

By noticing that $\text{ch}_{\mathbb{R}/\mathbb{Q}}(F) = \text{Re}(\text{CCS}(F, \nabla^F))$, Theorems 1.1 and 1.2 imply the $\mathbb{Z}_2$-graded version of the real part of (1.0.1):

$$\text{Re}(\text{CCS}(H(Z,F|_Z), \nabla^{H(Z,F|_Z)})) = \text{ch}_{\mathbb{R}/\mathbb{Q}}(\mathcal{H}(Z,F|_Z))$$

$$= \text{ch}_{\mathbb{R}/\mathbb{Q}}(\text{ind}^{a\Lambda}_{\mathbb{R}/\mathbb{Z}}(F))$$

$$= \int_{X/B} e(T^V X) \cup \text{ch}_{\mathbb{R}/\mathbb{Q}}(F)$$

$$= \int_{X/B} e(T^V X) \cup \text{Re}(\text{CCS}(F, \nabla^F)).$$

The real part of (1.0.1) can be deduced from its $\mathbb{Z}_2$-graded version (1.0.3) by following an argument by Ma–Zhang [15, p.613-614], where the following equality

$$\sum_{j=0}^{\dim(Z)} (-1)^j \text{Re}(\text{CCS}(H^j(Z,F|_Z), \nabla^{H^j(Z,F|_Z)})) = 0$$

in $H^{\text{odd}}(B; \mathbb{R}/\mathbb{Q})$ plays an important role. When $\dim(Z)$ is even, (1.0.4) is a special case of the second part of a result by Bismut [5, Theorem 3.12]. See also [11, Corollary 4] for the deduction in the case $\dim(Z)$ even.
The $\mathbb{Z}_2$-graded generators $\mathcal{F}$ of the form (1.0.2) form a subgroup of $K^{-1}_L(X)$, which is denoted by $K_{\text{flat}}(X;\mathbb{R}/\mathbb{Z})$. By Theorem 1.1, the restriction $\text{ind}^a_{\text{flat},\mathbb{R}/\mathbb{Z}} : K_{\text{flat}}(X;\mathbb{R}/\mathbb{Z}) \to K_{\text{flat}}(B;\mathbb{R}/\mathbb{Z})$ of $\text{ind}^a_{\mathbb{R}/\mathbb{Z}}$ makes sense. Then Theorem 1.2 implies the following diagram commutes.

$$
\begin{array}{ccc}
K_{\text{flat}}(X;\mathbb{R}/\mathbb{Z}) & \xrightarrow{\text{Re}(\text{CS})} & H^{\text{odd}}(X;\mathbb{R}/\mathbb{Q}) \\
\text{ind}^a_{\text{flat},\mathbb{R}/\mathbb{Z}} & \downarrow & \\
K_{\text{flat}}(B;\mathbb{R}/\mathbb{Z}) & \xrightarrow{\text{Re}(\text{CS})} & H^{\text{odd}}(B;\mathbb{R}/\mathbb{Q})
\end{array}
$$

It is in this sense that we put the real part of the RRG theorem for complex flat vector bundles into the framework of $K$-theory. It is an interesting question to see if the imaginary part of (1.0.1) can also be put into the framework of $K$-theory.

The newly defined analytic index $\text{ind}^a_{K} \hat{\mathcal{K}}$ in differential $K$-theory is related to the analytic index $\text{ind}^a_{K} : \hat{\mathcal{K}}_{\text{FL}}(X) \to \hat{\mathcal{K}}_{\text{FL}}(B)$ in differential $K$-theory defined in terms of twisted spin$^c$ Dirac operators by Freed–Lott [8, Definition 7.27] in the following way.

**Proposition 1.1.** Let $\pi : X \to B$ be a submersion with closed, oriented and spin$^c$ fibers of even dimension. For any $\mathbb{Z}_2$-graded generator $\mathcal{E}$ of $\hat{\mathcal{K}}_{\text{FL}}(X)$ we have

$$\text{ind}^a_{K} \hat{\mathcal{K}}(\mathcal{E}) = \text{ind}^a_{K} (S(T^V X)^* \otimes \mathcal{E}).$$

By Proposition 1.1 one expects (at least some of) the index theoretic results for $\text{ind}^a_{K}$ would have analogues for $\text{ind}^a_{K} \hat{\mathcal{K}}$.

In this paper we work exclusively with the local index theory without the kernel bundle assumption.

### 1.1. Motivations and method of proof

In this subsection we explain the motivations of the main results and elucidate the strategy of the proofs in more detail.

A motivation of Theorem 1.1 comes from the following theorem, which is the original purpose and a main result of this paper.

**Theorem 1.3.** Let $\pi : X \to B$ be a submersion with closed, oriented and spin$^c$ fibers $Z$ of even dimension. For every $\mathbb{Z}_2$-graded generator $\mathcal{F}$ in $K^{-1}_L(X)$ of the form (1.0.2) we have

$$\text{ind}^a_{\mathbb{R}/\mathbb{Z}} (S(T^V X)^* \otimes \mathcal{F}) = \mathcal{H}(Z, F|_Z). \tag{1.1.1}$$

Here, $\text{ind}^a_{\mathbb{R}/\mathbb{Z}} : K^{-1}_L(X) \to K^{-1}_L(B)$ is the analytic in $\mathbb{R}/\mathbb{Z} K$-theory defined in terms of twisted spin$^c$ Dirac operators by Lott [13, Definition 14].

Theorem 1.3 is the analytical counterpart of a result by Ma–Zhang [15, Theorem 1.2], where the left-hand side of (1.1.1) is replaced by the topological index $\text{ind}^t_{\mathbb{R}/\mathbb{Z}} : K^{-1}_L(X) \to K^{-1}_L(B)$ in $\mathbb{R}/\mathbb{Z} K$-theory. Thus Theorem 1.1 can be regarded as the generalization of Theorem 1.3 in the sense that...
the fibers $Z$ are not assumed to be even dimensional, oriented and spin$^c$. On the other hand, Proposition 1.1 can be regarded as a generalization of Theorems 1.1 and 1.3 in the sense that the equality
\[ \text{ind}_{R/Z}^a(F) = \mathcal{H}(Z,F|_Z) = \text{ind}_{R/Z}^a(S(T^V X)^* \otimes F) \]
in $K^{-1}_L(B)$ is lifted to $\widehat{K}_F(L)$, and it holds for any $\mathbb{Z}_2$-graded generator $E$ of $\widehat{K}_F(X)$.

The strategy of proving Theorem 1.3 is, roughly speaking, to consider the following deformation of the perturbed twisted spin$^c$ Dirac operator
\[ D_t = D^{\mathcal{S}(S \otimes F)} + tV, \]
where $t \in [0,1]$. Note that $D_0$ is the twisted spin$^c$ Dirac operator defining the left-hand side of (1.1.1), and $D_1$ is the perturbed twisted spin$^c$ Dirac operator $D^{\mathcal{S}(S \otimes F)} + V$. By [6, Propositions 3.5 and 3.7] (cf. [7, Proposition 4.12] when $B$ is a point) we have
\[ D^{\mathcal{S}(S \otimes F)} + V = D^{\mathcal{S}(S \otimes F)} + V = D^{\mathcal{S}(S \otimes F)} + V = D^{\mathcal{S}(S \otimes F)} + V, \quad (1.1.2) \]
where $D^{\mathcal{S}(S \otimes F)}$ is the twisted de Rham–Hodge operator. Since the kernel bundle $\ker(D^{\mathcal{S}(S \otimes F)}) \rightarrow B$ exists and is isomorphic to $H(Z,F|_Z) \rightarrow B$ by Hodge theory, to prove Theorem 1.3 it suffices to consider the submersion $\pi \times \text{id} : X \times [0,1] \rightarrow B \times [0,1]$ and the analog of $D_t$ twisted by a pullback bundle over $X \times [0,1]$. Part of the proof of Theorem 1.3 is similar to that of [11, Corollary 2], which says the analytic indexes in differential $K$-theory defined under and without the kernel bundle assumption are equal. However, Theorem 1.3 is not a direct consequence of [11, Corollary 2], as the definition of $\text{ind}_{R/Z}^a$ does not allow any perturbation of $D^{\mathcal{S}(S \otimes F)}$.

To prove Theorem 1.3, we must first deal with a subtlety on $\text{ind}_{R/Z}^a$, namely, that the analytic index $\text{ind}_{R/Z}^a : K^{-1}_L(X) \rightarrow K^{-1}_L(B)$ is a well defined map is a consequence of the family index theorem in $R/Z$ $K$-theory [13 Corollary 3], namely,
\[ \text{ind}_{R/Z}^a = \text{ind}_{R/Z}^a : K^{-1}_L(X) \rightarrow K^{-1}_L(B). \quad (1.1.3) \]
See the first remark in [13, p.299]. We must first prove $\text{ind}_{R/Z}^a$ is a well defined map without using (1.1.3); for otherwise Theorem 1.3 is simply a direct consequence of [13, Theorem 1.2] and (1.1.3). Since $K^{-1}_L(X)$ is a commutative subring of $\widehat{K}_F(X)$ and $\text{ind}_{R/Z}^a$ is the restriction of $\text{ind}_{K}^a$ to $R/Z K$-theory [8 (7.39)], and the fact that $\text{ind}_{K}^a$ is a well defined group homomorphism [8, (2) of Corollary 7.36] is a consequence of the family index theorem in differential $K$-theory [8, Theorem 7.35], i.e.
\[ \text{ind}_{K}^a = \text{ind}_{K}^a : \widehat{K}_F(X) \rightarrow \widehat{K}_F(B), \quad (1.1.4) \]
where $\text{ind}_{K}^a : \widehat{K}_F(X) \rightarrow \widehat{K}_F(B)$ is the topological index [8, Definition 5.34] in differential $K$-theory, we must show that $\text{ind}_{K}^a$ is a well defined group.
homomorphism (Proposition 2.4) without using (1.1.4). The variational formula [11, Proposition 1] and the properties of the Bismut–Cheeger eta form (defined without the kernel bundle assumption) established in this paper play an important role in the proof of Proposition 2.4. Among these properties, the graded additivity of the Bismut–Cheeger eta form (Propositions 2.1 and 2.3) could be of independent interests.

As a byproduct of the properties of the Bismut–Cheeger eta form established in this paper, we give a proof of the following RRG theorem for ind$^a_{R/Z}$ in $R/Z$ K-theory without the kernel bundle assumption at the differential form level.

**Theorem 1.4.** Let $\pi : X \to B$ be a submersion with closed, oriented and spin$^c$ fibers of even dimension. Then the following diagram commutes.

$$
\begin{array}{ccc}
K^{-1}_L(X) & \xrightarrow{\text{ch}_{R/Q}} & H^{\text{odd}}(X; R/Q) \\
\downarrow{\text{ind}^a_{R/Z}} & & \downarrow{\int_{X/B} \text{Todd}(T^V X) \cup (-)} \\
K^{-1}_L(B) & \xrightarrow{\text{ch}_{R/Q}} & H^{\text{odd}}(B; R/Q)
\end{array}
$$

Theorem 1.2 is a generalization of Theorem 1.4 in the same way as Theorem 1.1 is a generalization of Theorem 1.3. One purpose of presenting a new proof of Theorem 1.4 is to remedy the claim made in [10], where we present a new proof of Theorem 1.4 under the kernel bundle assumption, and claim that the general case of Theorem 1.4 can be proved in a similar way. In hindsight, the claim is unjustified, as the key ingredients of the proof ([11, Proposition 1], Propositions 2.1 and 2.3) were not available at the time of writing [10]. Another purpose is to serve as a guide for the proof of Theorem 1.2.

In the course of proving Theorem 1.3, we note that a key ingredient is the equalities in (1.1.2). The appearance of $D^{\Lambda \hat{\otimes} F}$ in (1.1.2) leads us to believe that for a given submersion $\pi : X \to B$ with closed fibers $Z$, which are not assumed to be even dimensional, oriented and spin$^c$, the generalized twisted de Rham operators $D^{A \otimes E}$ (meaning the twisting bundles $E \to X$ are not assumed to be flat) is a suitable candidate in establishing a generalization of Theorem 1.3.

Another motivation of Theorems 1.1 and 1.2 comes from [11, Theorem 1], where we prove the $Z_2$-graded version of the real part of (1.0.1) at the differential form level in the case dim$(Z)$ even by considering the perturbed twisted spin$^c$ Dirac operator $D^{S \otimes (S \otimes F) + V}$. This strategy cannot be applied to the case dim$(Z)$ odd since in this case the analytic index of $D^{S \otimes (S \otimes E)}$ defines an element in the odd topological $K$-group $K^{-1}(B)$, but not $K^0(B)$. In contrast, the analytic index of the generalized twisted de Rham operator $D^{A \otimes E}$ defines an element in $K^0(B)$ regardless of the parity of dim$(Z)$. As illustrated in (1.0.3), Theorems 1.1 and 1.2 give a more unified approach to
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the $\mathbb{Z}_2$-graded version of the real part of (1.0.1) compared to [11, Theorem 1].

We now outline the approach of proving Theorem 1.1. We first employ part of the local index theory for twisted de Rham operators due to Bismut–Lott [6, §III], but without assuming the twisting bundles to be flat, to establish a local family index theorem for generalized twisted de Rham operators $D^\Lambda \otimes E$ without the kernel bundle assumption (3.1.6):

$$d\eta^{E,\Lambda}(g^E, \nabla^E, T^HX, g^{TV^HX}, L) = \int_{X/B} e(\nabla^{TV^HX}) \wedge \text{ch}(\nabla^E) - \text{ch}(\nabla^L),$$

where $\eta^{E,\Lambda}$ is the corresponding Bismut–Cheeger eta form. The details will be given in Section 3.1. We then formulate and prove a variational formula (Proposition 3.1) and some properties (Propositions 3.2 to 3.4 and Lemma 3.1 and 3.2) of $\eta^{E,\Lambda}$. We define the analytic index

$$\text{ind}_{K}^{a,\Lambda} : \overline{K}_{FL}(X) \to \overline{K}_{FL}(B)$$

associated to $D^\Lambda \otimes E$ both under and without the kernel bundle assumption in differential $K$-theory (Definition 3.1), and show that it is a well defined group homomorphism and is independent of the choice of the “suitable” candidates for $\text{ind}_{K}^{a,\Lambda}(E)$ for any fixed $\mathbb{Z}_2$-graded generator $E$ of $\overline{K}_{FL}(X)$ (Proposition 3.5). Write

$$\text{ind}_{R/Z}^{a,\Lambda} : K^{-1}_{L}(X) \to K^{-1}_{L}(B)$$

for the restriction of $\text{ind}_{K}^{a,\Lambda}$ to $R/Z K$-theory. Then we prove Theorems 1.1 and 1.2. The techniques used to prove Theorems 1.1 and 1.2 are similar to those for Theorems 1.3 and 1.4, respectively.

Since the results for generalized twisted de Rham operators are parallel to those for twisted spin$^c$ Dirac operators and the proofs for the former are formally the same as those for the latter, we give detailed proofs of all the results for twisted spin$^c$ Dirac operators, and do the same for the results for generalized twisted de Rham operators only if there are some essential differences in the proofs. One reason of doing so is that the method of proof of the results for twisted spin$^c$ Dirac operators imply those for generalized twisted de Rham operators (in some sense), but not vice versa. We leave the other details to the interested readers.

We would like to mention the work [3] by Berthomieu. For any given submersion $\pi : X \to B$ with closed fibers $Z$, he defines, among other things, direct images on topological $K$-theory and $K$-theory of complex flat vector bundles (denoted by $K_{flat}$) using de Rham–Hodge operators $D^Z, dR$ twisted by complex vector bundles which are assumed to be flat and not flat, respectively. Thus the operator he considered is a perturbation of the generalized twisted de Rham operator $D^\Lambda \otimes E$. Moreover, he defines a map
\(\pi_1 : K^0_{\text{flat}}(X) \to K^0_{\text{flat}}(B)\) directly by \(\pi_1(F, \nabla^F) = (H(Z, F|_Z), \nabla^{H(Z,F|_Z)})\) [3, Definition 22], and prove a number of properties, for instance, the functoriality of \(\pi_1\) under composition of two submersions.

1.2. Outline. The paper is organized as follows. In Sections 2 and 3 we prove the main results of this paper. Section 2 is devoted to the results related to twisted spin\(^c\) Dirac operators, and Section 3 is devoted to the results related to generalized twisted de Rham operators. Since the background material is scattered in the literature, to make the paper as self-contained as possible we have collected the material in an Appendix.

1.3. Notations and conventions. In this paper \(X\) and \(B\) are closed manifolds and \(I\) is the closed interval \([0, 1]\). Given a manifold \(X\), define \(\bar{X} = X \times I\). Given \(t \in I\), define a map \(i_{X,t} : X \to \bar{X}\) by \(i_{X,t}(x) = (x, t)\). Denote by \(p_X : \bar{X} \to X\) the standard projection map. For \(k \geq 0\), denote by \(\Omega^k_{Q}(X, \mathbb{C})\) the set of all \(\mathbb{C}\)-valued closed \(k\)-forms on \(X\) with periods in \(Q\), and write \(\Omega^k_{Q}(X)\) for \(\Omega^k_{Q}(X, \mathbb{R})\).

Let \(E \to X\) be a complex vector bundle. If \(E \to X\) is \(\mathbb{Z}_2\)-graded, denote by \(E^{\text{op}} \to X\) the \(\mathbb{Z}_2\)-graded complex vector bundle whose \(\mathbb{Z}_2\)-grading is the opposite of \(E \to X\). We will also use the notation \(\text{op}\) for other \(\mathbb{Z}_2\)-graded objects. Given another complex vector bundle \(F \to X\), denote by \(E \otimes F \to X\) the \(\mathbb{Z}_2\)-graded tensor product if either one is ungraded, and by \(E \hat{\otimes} F \to X\) if both are \(\mathbb{Z}_2\)-graded.

A triple \((E, g^E, \nabla^E)\) consisting of a complex vector bundle with a Hermitian metric and a unitary connection is said to be \(\mathbb{Z}_2\)-graded if \(E \to X\) is \(\mathbb{Z}_2\)-graded, and \(g^E\) and \(\nabla^E\) preserve the \(\mathbb{Z}_2\)-grading (which will also be called \(\mathbb{Z}_2\)-graded). A \(\mathbb{Z}_2\)-graded triple \((E, g^E, \nabla^E)\) is said to be balanced if \(E^+ = E^-\), \(g^{E,+} = g^{E,-}\), and \(\nabla^{E,+} = \nabla^{E,-}\).

Let \(\pi : M \to B\) a smooth fiber bundle with compact fibers of dimension \(n\) which satisfies certain orientability assumptions. Then

\[
\pi^* \alpha \wedge \beta = \alpha \wedge \left( \int_{M/B} \beta \right)
\]

for any \(\alpha \in \Omega^*_{\hat{B}}(B)\) and \(\beta \in \Omega^*_{\hat{M}}(M)\). If \(M\) has nonempty boundary, then Stokes’ theorem for integration along the fibers [9, Problem 4 (p.311)] states that for any \(\omega \in \Omega^k_{\hat{B}}(M)\) we have

\[
(-1)^{k-n+1} \int_{\partial M/B} i^* \omega = \int_{M/B} d^B M \omega - d^B \int_{M/B} \omega, \tag{1.3.1}
\]

where \(i : \partial M \rightarrow M\) is the inclusion map.

Choose and fix \(a \in (0, 1)\). Let \(\alpha : [0, \infty) \to I\) be a smooth function that satisfies \(\alpha(t) = 0\) for all \(t \leq a\) and \(\alpha(t) = 1\) for all \(t \geq 1\).

For any differential forms \(\omega\) and \(\eta\), we write \(\omega \equiv \eta\) if \(\omega\) and \(\eta\) differ by an exact form.
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2. The main results for twisted spin$^c$ Dirac operators

In this section we prove the main results for twisted spin$^c$ Dirac operators. In Section 2.1 we establish some properties of the Bismut–Cheeger eta form. We then give a proof that $\text{ind}^E_K$ is a well defined group homomorphism in Section 2.2 and prove Theorem 1.3 in Section 2.3. In Section 2.4 we give a proof of Theorem 1.4.

We refer to Appendices A.2 and A.3 for a quick summary on the local index family theorem for twisted spin$^c$ Dirac operators without the kernel bundle assumption, especially for the meaning of a $\mathbb{Z}_2$-graded triple $(L, g^L, \nabla^L)$ where $L \to B$ satisfies the MF property for $D^{S\otimes F}$, and to Appendix A.5 for the analytic indexes in differential and $\mathbb{R}/\mathbb{Z}$ $K$-theory. We refer the readers to [2] and [8] for the details.

2.1. Some properties of the Bismut–Cheeger eta form

The following proposition says the Bismut–Cheeger eta form is additive with respect to ungraded direct sums.

**Proposition 2.1.** Let $\pi : X \to B$ be a submersion with closed, oriented and spin$^c$ fibers of even dimension, and $(E, g^E, \nabla^E)$ and $(F, g^F, \nabla^F)$ are two triples. Choose and fix a spin$^c$ structure on $T^VX \to X$, and put geometric data on $\pi : X \to B$ as in Section A.2. If $(L_E, g_L^E, \nabla_L^E)$ and $(L_F, g_L^F, \nabla_L^F)$ are $\mathbb{Z}_2$-graded triples so that $L_E \to B$ and $L_F \to B$ satisfy the MF properties for $D^{S\otimes F}$ and $D^{S\otimes F}$, respectively, then $(L_E \oplus L_F, g_L^E \oplus g_L^F, \nabla_L^E \oplus \nabla_L^F)$ is a $\mathbb{Z}_2$-graded triple so that $L_E \oplus L_F \to B$ satisfies the MF property for $D^{S\otimes (E\oplus F)}$, and

\[
\eta^{E\otimes F}_G(g^F \oplus g^E, \nabla^F \oplus \nabla^E, T^H X, g^{TVX}, L_E \oplus L_F)
\equiv \eta^F(g^F, \nabla^F, T^H X, g^{TVX}, L_E) + \eta^E(g^E, \nabla^E, T^H X, g^{TVX}, L_F).
\]

**Proof.** We first show that $L_E \oplus L_F \to B$ satisfies the MF property for $D^{S\otimes (E\oplus F)}$. By the definitions of $L_E \to B$ and $L_F \to B$ there exist complementary closed subbundles $K^+_E \to B$ and $K^+_F \to B$ of $(\pi_E)^* \to B$ and $(\pi_F)^* \to B$ such that

\[
(\pi_E)^+ = K^+_E \oplus L^+_E, \quad (\pi_E)^- = K^-_E \oplus L^-_E,
\]

\[
(\pi_F)^+ = K^+_F \oplus L^+_F, \quad (\pi_F)^- = K^-_F \oplus L^-_F.
\]

(2.1.1)
$D^S \otimes E$ and $D^S \otimes F$ are of the form

$$D^S \otimes E = \begin{pmatrix} a_E & 0 \\ 0 & d_E \end{pmatrix}, \quad D^S \otimes F = \begin{pmatrix} a_F & 0 \\ 0 & d_F \end{pmatrix}$$

(2.1.2)

with respect to (2.1.1), and $a_E : K^+_E \to K^-_E$ and $a_F : K^+_F \to K^-_F$ are isomorphisms, respectively.

From (2.1.1) we have

$$
\begin{align*}
(p_*(E \otimes F))^+ &= (p_*(E))^+ \oplus (p_*(F))^+ = (K^+_E \oplus K^+_F) \oplus (L^+_F \oplus L^+_F), \\
(p_*(E \otimes F))^- &= (p_*(E))^- \oplus (p_*(F))^- = (K^-_E \oplus K^-_F) \oplus (L^-_F \oplus L^-_F).
\end{align*}
$$

(2.1.3)

Note that $D^S \otimes (E \otimes F) = D^S \otimes E \oplus D^S \otimes F$. With respect to (2.1.3), $D^S \otimes (E \otimes F)$ is given by

$$D^S \otimes (E \otimes F) = \begin{pmatrix} a_E & 0 & 0 & 0 \\ 0 & a_F & 0 & 0 \\ 0 & 0 & d_E & 0 \\ 0 & 0 & 0 & d_F \end{pmatrix}$$

(2.1.4)

(2.1.4) shows that $D^S \otimes (E \otimes F)|_{K^+_E \oplus K^+_F} = D^S \otimes E|_{K^+_E} \oplus D^S \otimes F|_{K^+_F}$, which is an isomorphism. Thus $L_E \otimes L_F \to B$ satisfies the MF property for $D^S \otimes (E \otimes F)$.

We show that the Bismut superconnection $\widehat{B}^{E \otimes F}$ is additive in the following sense:

$$\widehat{B}^{E \otimes F} = \widehat{B}^E \oplus \widehat{B}^F.$$  

(2.1.5)

First note that

$$\pi_*(E \otimes F) = \pi_*(E \otimes F) \oplus (L_E \otimes L_F)^{op} = (\pi_*(E \otimes L_E)^{op}) \oplus (\pi_*(F \otimes L_F)^{op}) = \pi_*E \oplus \pi_*F.$$

Since

$$D^S \otimes (E \otimes F) = D^S \otimes E \oplus D^S \otimes F,$$

it follows that $D^S \otimes (E \otimes F)(a) = D^S \otimes E(a) \oplus D^S \otimes F(a)$ for any $a \in \mathbb{C}$. Thus

$$D^S \otimes (E \otimes F)(a) = D^S \otimes E(a) \oplus D^S \otimes F(a)^*$$

$$= (D^S \otimes E(a) \oplus D^S \otimes F(a))^*$$

Thus (2.1.5) holds.

Let $t, T \in (0, \infty)$ satisfy $t < T$. By (2.1.5) and (A.1.5) we have

$$CS(\widehat{B}^{E \otimes F}_t, \widehat{B}^{E \otimes F}_T) \equiv CS(\widehat{B}^E \oplus \widehat{B}^F_t, \widehat{B}^E \oplus \widehat{B}^F_T) \equiv CS(\widehat{B}^E_t, \widehat{B}^E_T) + CS(\widehat{B}^F_t, \widehat{B}^F_T).$$

(2.1.6)
By letting $t \to 0$ and $T \to \infty$ in (2.1.6), the result follows. □

**Lemma 2.1.** Let $\pi : X \to B$ be a submersion with closed, oriented and spin$^c$ fibers of even dimension, and $(E^+, g^+, \nabla^+)$ and $(E^-, g^-, \nabla^-)$ are two triples. Choose and fix a spin$^c$ structure on $T^V X \to X$, and put geometric data on $\pi : X \to B$ as in Section A.2. Let $L_{E^+} \to B$ and $L_{E^-} \to B$ be $\mathbb{Z}_2$-graded complex vector bundles satisfying the MF property for $D^{S\otimes E^+}$ and $D^{S\otimes E^-}$, respectively. Define a $\mathbb{Z}_2$-graded triple $(E, g^E, \nabla^E)$ by $E = E^+ \oplus E^-$, $g^E = g^+ \oplus g^-$ and $\nabla^E = \nabla^+ \oplus \nabla^-$. Denote by $D^{S\otimes E}$ the corresponding twisted spin$^c$ Dirac operator. Then $L_{E^+} \oplus L_{E^-} \to B$ is a $\mathbb{Z}_2$-graded complex vector bundle satisfying the MF property for $D^{S\otimes E}$.

**Proof.** Since $L_{E^+} \to B$ and $L_{E^-} \to B$ satisfy the MF property for $D^{S\otimes E^+}$ and $D^{S\otimes E^-}$, there exist complementary closed subbundles $K_{E^+}^p \to B$ and $K_{E^-}^p \to B$ of $(\pi_* E^+)^\pm \to B$ and $(\pi_* E^-)^\pm \to B$ such that

\[
\begin{align*}
(\pi_* E^+)^+ &= K_{E^+}^+ \oplus L_{E^+}^-, \\
(\pi_* E^+)^- &= K_{E^-}^- \oplus L_{E^+}^+, \\
(\pi_* E^-)^+ &= K_{E^+}^\pm \oplus L_{E^-}^+, \\
(\pi_* E^-)^- &= K_{E^-}^\pm \oplus L_{E^-}^-,
\end{align*}
\]

(2.1.7)

$D^{S\otimes E^+}$ and $D^{S\otimes E^-}$ are of the form

\[
D^{S\otimes E^+}_+ = \begin{pmatrix} a_{E^+} & 0 \\ 0 & a_{E^-} \end{pmatrix}, \\
D^{S\otimes E^-}_+ = \begin{pmatrix} a_{E^-} & 0 \\ 0 & a_{E^-} \end{pmatrix}
\]

with respect to (2.1.7), and $a_{E^+} : K_{E^+}^\pm \to K_{E^-}^\pm$ and $a_{E^-} : K_{E^+}^\pm \to K_{E^-}^\pm$ are isomorphisms, respectively.

From (2.1.7) we have

\[
\begin{align*}
(\pi_* E)^+ &= (\pi_* E^+)^+ \oplus (\pi_* E^-)^- = K_{E^+}^+ \oplus K_{E^-}^- \oplus L_{E^+}^+, \oplus L_{E^-}^-, \\
(\pi_* E)^- &= (\pi_* E^-)^+ \oplus (\pi_* E^-)^+ = K_{E^-}^+ \oplus K_{E^-}^- \oplus L_{E^+}^-, \oplus L_{E^-}^+.
\end{align*}
\]

(2.1.8)

In other words, $\pi_* E = K_{E^+} \oplus K_{E^-} \oplus L_{E^+} \oplus L_{E^-}$. With respect to (2.1.8), $D^{S\otimes E}_+$ is given by

\[
D^{S\otimes E}_+ = \begin{pmatrix} a_{E^+} & 0 & 0 & 0 \\ 0 & a_{E^-} & 0 & 0 \\ 0 & 0 & a_{E^-} & 0 \\ 0 & 0 & 0 & a_{E^-} \end{pmatrix},
\]

where $D^{S\otimes E}_- = (D^{S\otimes E}_+)^\ast = \begin{pmatrix} a_{E^-} & 0 \\ 0 & a_{E^-} \end{pmatrix}$. Note that $D^{S\otimes E}_+(K_{E^+} \oplus K_{E^-}) = a_{E^+} \oplus a_{E^-}$, which is an isomorphism. Thus $L_{E^+} \oplus L_{E^-} \to B$ satisfies the MF property for $D^{S\otimes E}$. □

**Remark 2.1.** Let $V$ be a $\mathbb{Z}_2$-graded vector space. Since $\text{End}(V)^+ = \text{End}(V^+) \oplus \text{End}(V^-)$, it follows that

\[
\text{End}(V^{\text{op}^+}) = \text{End}(V^{\text{op}^+}) \oplus \text{End}(V^{\text{op}^-}) = \text{End}(V^-) \oplus \text{End}(V^+).
\]

Thus if $T \in \text{End}(V^{\text{op}^+})$, then $T \in \text{End}(V)^+$ and

\[
\text{str}_{V^{\text{op}}} (T) = \text{tr}(T|_{V^{\text{op}^+}}) - \text{tr}(T|_{V^{\text{op}^-}}) = \text{tr}(T|_{V^-}) - \text{tr}(T|_{V^+}) = -\text{str}_V (T).
\]
The following lemma says switching the $\mathbb{Z}_2$-grading of a $\mathbb{Z}_2$-graded triple induces a minus sign in the corresponding Bismut–Cheeger eta form.

**Lemma 2.2.** Let $\pi : X \to B$ be a submersion with closed, oriented and spin$^c$ fibers of even dimension, and $(E, g^E, \nabla^E)$ a $\mathbb{Z}_2$-graded triple. Choose and fix a spin$^c$ structure on $T^V X \to X$, and put geometric data on $\pi : X \to B$ as in Section A.2. If $(L, g^L, \nabla^L)$ is a $\mathbb{Z}_2$-graded triple so that $L \to B$ satisfies the MF property for $D^{S \otimes E}$, then $(L^{\text{op}}, g^{L^{\text{op}}}, \nabla^{L^{\text{op}}})$ is a $\mathbb{Z}_2$-graded triple so that $L^{\text{op}} \to B$ satisfies the MF property for $D^{S \otimes E^{\text{op}}}$, and

$$
\tilde{\eta}^{E^{\text{op}}}(g^{E^{\text{op}}}, \nabla^{E^{\text{op}}}, T^H X, g^{T^V X}, L^{\text{op}}) \equiv -\tilde{\eta}^E(g^E, \nabla^E, T^H X, g^{T^V X}, L). \quad (2.1.9)
$$

**Proof.** We first show that $L^{\text{op}} \to B$ satisfies the MF property for $D^{S \otimes E^{\text{op}}}$. Since $L \to B$ satisfies the MF property for $D^{S \otimes E}$, there exist complementary closed subbundles $K^\pm \to B$ of $(\pi_*(E))^\pm \to B$ such that (A.3.7) holds, $D^{S \otimes E}_+$ is block diagonal as a map with respect to (A.3.7) and $D^{S \otimes E}_{K^+}$ is an isomorphism.

Since the even and the odd part of $(\pi^* T^V X) \otimes E \to X$ are given by

$$(S(\pi^* T^V X) \otimes E)^+ = S(\pi^* T^V X)^+ \otimes E^+ \oplus S(\pi^* T^V X)^- \otimes E^-,$$

$$(S(\pi^* T^V X) \otimes E)^- = S(\pi^* T^V X)^+ \otimes E^- \oplus S(\pi^* T^V X)^- \otimes E^+,$$

respectively, it follows that $(S \otimes E)^\pm = (S \otimes E)^\mp$. Thus $(\pi_*(E^{\text{op}}))^\pm = (\pi_*(E))^\mp$. Since $\pi_* E = K \oplus L$, it follows that $\pi_*(E^{\text{op}}) = K^{\text{op}} \oplus L^{\text{op}}$, i.e.

$$(\pi_*(E^{\text{op}}))^+ = K^- \oplus L^-, \quad (\pi_*(E^{\text{op}}))^- = K^+ \oplus L^+.$$ \quad (2.1.10)

Moreover, we have $D^{S \otimes E^{\text{op}}}_+ = D^{S \otimes E}_- = (D^{S \otimes E})^*$. Thus $D^{S \otimes E^{\text{op}}}_+$ is also block diagonal as a map with respect to (2.1.10), and $D^{S \otimes E^{\text{op}}}_{K^{\text{op}}^+}$ is the inverse of $D^{S \otimes E}_{K^+}$, which is an isomorphism. Thus $L^{\text{op}} \to B$ satisfies the MF property for $D^{S \otimes E^{\text{op}}}$.

It is obvious that $g^{L^{\text{op}}} = P^{L^{\text{op}}} g^{\pi_* (E^{\text{op}})}$ and $\nabla^{L^{\text{op}}} = P^{L^{\text{op}}} \nabla^{\pi_*(E^{\text{op}}),H}$. By applying Remark 2.1 to $\frac{d}{dt} e^{-\frac{1}{2} t \tilde{\eta}(\xi)}$, we have (2.1.9). \hfill $\square$

The following proposition says the Bismut–Cheeger eta form of a balanced $\mathbb{Z}_2$-graded triple vanishes.

**Proposition 2.2.** Let $\pi : X \to B$ be a submersion with closed, oriented and spin$^c$ fibers of even dimension, and $(V, g^V, \nabla^V)$ a balanced $\mathbb{Z}_2$-graded triple over $X$. Choose and fix a spin$^c$ structure on $T^V X \to X$, and put geometric data on $\pi : X \to B$ as in Section A.2. If $(L_+, g^{L_+}, \nabla^{L_+})$ is a $\mathbb{Z}_2$-graded triple so that $L_+ \to B$ satisfies the MF property for $D^{S \otimes V^+}$, then there exists a balanced $\mathbb{Z}_2$-graded triple $(L, g^L, \nabla^L)$ such that $L \to B$ satisfies the MF property for $D^{S \otimes V}$ and

$$
\tilde{\eta}^V(g^V, \nabla^V, T^H X, g^{T^V X}, L) \equiv 0.
$$
Proof. Since \((V^+, g^{V+,+}, \nabla^{V,+}) = (V^-, g^{V,-}, \nabla^{V,-})\), \((L_+, g^{L+}, \nabla^{L_+})\) is a \(\mathbb{Z}_2\)-graded triple so that \(L_+ \rightarrow B\) also satisfies the MF property for \(D^S \otimes V^-\). Define a \(\mathbb{Z}_2\)-graded triple \((L, g^L, \nabla^L)\) by
\[
L = L_+ \oplus L_+^{\text{op}}, \quad g^L = g^{L_+} \oplus g^{L_+^{\text{op}}}, \quad \nabla^L = \nabla^{L_+} \oplus \nabla^{L_+^{\text{op}}}.
\] (2.1.11)
By Lemma 2.1, \(L \rightarrow B\) satisfies the MF property for \(D^S \otimes V^\oplus\). Since
\[
L^+ = L_+^+ \oplus L_+^- \quad \text{and} \quad L^- = L_+^+ \oplus L_+^-,
\]
and similarly for \(g^L\) and \(\nabla^L\), the \(\mathbb{Z}_2\)-graded triple \((L, g^L, \nabla^L)\) is balanced. By Lemma 2.2 and the definitions of \((V, g^V, \nabla^V)\) and \((L, g^L, \nabla^L)\), we have
\[
\hat{\eta}^V (g^V, \nabla^V, T^V X, g^{TV X}, L) \equiv -\hat{\eta}^V (g^{V,\text{op}}, \nabla^{V,\text{op}}, T^V X, g^{TV X}, L^{\text{op}})
\]
\[
\equiv -\hat{\eta}^V (g^V, \nabla^V, T^V X, g^{TV X}, L).
\]
Thus \(\hat{\eta}^V (g^V, \nabla^V, T^V X, g^{TV X}, L) \equiv 0\). \(\square\)

The following proposition says the Bismut–Cheeger eta form is \(\mathbb{Z}_2\)-graded additive with respect to \(\mathbb{Z}_2\)-graded triples.

**Proposition 2.3.** Let \(\pi : X \rightarrow B\) be a submersion with closed, oriented and spin\(^c\) fibers of even dimension, and \((E^+, g^+, \nabla^+)\) and \((E^-, g^-, \nabla^-)\) are two triples. Choose and fix a spin\(^c\) structure on \(T^V X \rightarrow X\), and put geometric data on \(\pi : X \rightarrow B\) as in Section A.2. Let \((L_{E^+}, g^{L_{E^+}}, \nabla^{L_{E^+}})\) and \((L_{E^-}, g^{L_{E^-}}, \nabla^{L_{E^-}})\) be \(\mathbb{Z}_2\)-graded triples so that \(L_{E^+} \rightarrow B\) and \(L_{E^-} \rightarrow B\) satisfy the MF property for \(D^S \otimes E^+\) and \(D^S \otimes E^-\), respectively. Define a \(\mathbb{Z}_2\)-graded triple \((E, g^E, \nabla^E)\) by \(E = E^+ \oplus E^-\), \(g^E = g^+ \oplus g^-\) and \(\nabla^E = \nabla^+ \oplus \nabla^-\). Then
\[
\hat{\eta}^E (g^E, \nabla^E, T^E X, g^{TE X}, L_{E^+} \oplus L_{E^-}^{\text{op}}) \equiv \hat{\eta}^E (g^+, \nabla^+, T^E X, g^{TE X}, L_{E^+}) - \hat{\eta}^E (g^-, \nabla^-, T^E X, g^{TE X}, L_{E^-}).
\]

**Proof.** Since \(L_{E^+} \oplus L_{E^-}^{\text{op}} \rightarrow B\) satisfies the MF property for \(D^S \otimes E\) by Lemma 2.1, the Bismut–Cheeger eta form \(\hat{\eta}^E (g^E, \nabla^E, T^E X, g^{TE X}, L_{E^+} \oplus L_{E^-}^{\text{op}})\) is well defined.

First note that we regard the triple \((E \oplus E^-, g^E \oplus g^-, \nabla^E \oplus \nabla^-)\) to be ungraded, i.e. we do not take the involution defining the \(\mathbb{Z}_2\)-grading of \(E \oplus E^- \rightarrow B\) into the account, no matter \(E^- \rightarrow B\) is regarded as \(\mathbb{Z}_2\)-graded or not. By Proposition 2.1
\[
((L_{E^+} \oplus L_{E^-}^{\text{op}}) \oplus L_{E^-}, (g^{L_{E^+}} \oplus g^{L_{E^-}}, \nabla^{L_{E^+}} \oplus \nabla^{L_{E^-}}) \oplus (g^{L_{E^-}} \oplus g^{L_{E^-}}, \nabla^{L_{E^-}} \oplus \nabla^{L_{E^-}}))
\] (2.1.12)
is a \(\mathbb{Z}_2\)-graded triple so that \((L_{E^+} \oplus L_{E^-}^{\text{op}}) \oplus L_{E^-} \rightarrow B\) satisfies the MF property for \(D^S \otimes (E \oplus E^-)\).

Define a balanced \(\mathbb{Z}_2\)-graded triple \((\mathcal{E}, g^{\mathcal{E}}, \nabla^{\mathcal{E}})\), where \(\mathcal{E}^\pm = E^- \oplus g^{\mathcal{E}}, \mathcal{E}^\pm = g^- \oplus \nabla^{\mathcal{E}}\). Similarly, we regard the triple \((E^+ \oplus \mathcal{E}, g^+ \oplus g^{\mathcal{E}}, \nabla^+ \oplus \nabla^{\mathcal{E}})\) to be ungraded. By Proposition 2.2 there exists a balanced \(\mathbb{Z}_2\)-graded triple
(L, g_L, \nabla L), where L = L_E \oplus L_{\text{op}}, and similarly for g_L and \nabla L, such that L \to B satisfies the MF property for D^{S \otimes E} and

\eta (g_L, \nabla L, TH X, g_T VX) \equiv 0.

(2.1.13)

By Proposition 2.1

(L_E, + L_{\text{op}}, g_{L_E} \oplus g_L, \nabla L_E \oplus \nabla L)

(2.1.14)

is a Z_2-graded triple so that L \oplus L_{\text{op}} \to B satisfies the MF property for D^{S \otimes (E \oplus \text{op})}.

Since L_E \oplus L_{\text{op}} \oplus L_{E^*} = L_E \oplus L_{\text{op}} as Z_2-graded vector bundles, by applying [11] Proposition 1 to the Z_2-graded triples (2.3.13) and (2.3.14), we have

\eta (g_E \oplus g_{-}, \nabla E \oplus \nabla_{-}, TH X, g_T VX, E, L_E \oplus L_{E^*} \oplus L_{E^*})

- \eta (g_{+} \oplus g_{\text{op}}, \nabla_{+} \oplus \nabla_{\text{op}}, TH X, g_T VX, E, L_{E^*} \oplus L_{\text{op}})

\equiv \int_{X/B} \text{Todd}(\nabla T VX) \wedge \text{CS}(\nabla E \oplus \nabla_{-}, \nabla_{+} \oplus \nabla_{\text{op}}) - \text{CS}(\nabla L_{E^*} \oplus \nabla L_{E^*} \oplus \nabla L_{E^*} \oplus \nabla_{L})

(2.1.15)

Since \nabla E \oplus \nabla_{-} = \nabla_{+} \oplus \nabla_{-} \oplus \nabla_{=} \oplus \nabla_{\text{op}} and

\nabla_{L_E^*} \oplus \nabla_{L_{E^*} \cdot \text{op}} \oplus \nabla_{L_E} = \nabla_{L_E^*} \oplus \nabla_{L_E} \oplus \nabla_{L_{E^*} \cdot \text{op}} = \nabla_{L_E^*} \oplus \nabla_{L},

the Chern–Simons forms in (2.1.15) vanish in \frac{\Omega^{\text{odd}}(B)}{\text{Im}(d)}. Thus (2.1.15) becomes

\eta (g_E \oplus g_{-}, \nabla E \oplus \nabla_{-}, TH X, g_T VX, E, L_E \oplus L_{E^*} \oplus L_{E^*})

\equiv \eta (g_{+} \oplus g_{\text{op}}, \nabla_{+} \oplus \nabla_{\text{op}}, TH X, g_T VX, E, L_{E^*} \oplus L_{\text{op}}).

(2.1.16)

By applying Proposition 2.1 to both sides of (2.1.16), we have

\eta (g_E, \nabla E, TH X, g_T VX, E, L_E \oplus L_{E^*} \oplus L_{E^*}) + \eta (g_{-}, \nabla_{-}, TH X, g_T VX, L_E \oplus L_{E^*})

\equiv \eta (g_{+}, \nabla_{+}, TH X, g_T VX, L_{E^*} \oplus \nabla_{\text{op}}) + \eta (g_{\text{op}}, \nabla_{\text{op}}, TH X, g_T VX, L_{E^*} \oplus L_{\text{op}}).

By (2.1.12), the result follows.

2.2. The analytic index in differential K-theory. We now prove the analytic index ind^A_K : \overline{K}(X) \to \overline{K}(B) is a well defined group homomorphism without using (1.1.4).

Proposition 2.4. Let \pi : X \to B be a submersion with closed, oriented and spin^c fibers of even dimension. Choose and fix a spin^c structure on T^V X \to X, and put geometric data on \pi : X \to B as in Section A.2. The differential analytic index

\text{ind}^A_K : \overline{K}(X) \to \overline{K}(B)

associated to twisted spin^c Dirac operators is a well defined group homomorphism.
Proof: Let $\mathcal{E}_0$ and $\mathcal{E}_1$ be $\mathbb{Z}_2$-graded generators of $\overline{K}_{FL}(X)$. Fix $k \in \{0, 1\}$. Denote by $\mathcal{D}^{S\otimes E_k}$ the corresponding twisted spin$^c$ Dirac operator. Let $(L_k, g^{L_k}, \nabla^{L_k})$ be a $\mathbb{Z}_2$-graded triple so that $L_k \rightarrow B$ satisfies the MF property for $\mathcal{D}^{S\otimes E_k}$. By (A.5.6) we have

$$\text{ind}^a_B(\mathcal{E}_k; L_k) = \left( L_k \cdot g^{L_k}, \nabla^{L_k}, \int_{X/B} \text{Todd}(\nabla^{T^*X}) \wedge \omega_k+\nabla^{E_k}(g^{E_k}, \nabla^{E_k}, T^*X, g^{T^*X}, L_k) \right).$$

We first show that $\text{ind}^B_k: \overline{K}_{FL}(X) \rightarrow \overline{K}_{FL}(B)$ is a well defined map. Suppose $\mathcal{E}_0 = \mathcal{E}_1$. Then there exist balanced $\mathbb{Z}_2$-graded triples $(V_0, g^{V_0}, \nabla^{V_0})$ and $(V_1, g^{V_1}, \nabla^{V_1})$ such that (A.5.1) and (A.5.2) hold. Let $\eta^{V_0}(g^{V_0}, \nabla^{V_0}, \nabla^{V_0})$ be a $\mathbb{Z}_2$-graded triple so that $L_{V_0} \rightarrow B$ satisfies the MF property for $\mathcal{D}^{S\otimes V_0}$. By Proposition 2.2, $L_{V_0} \rightarrow B$ in the balanced $\mathbb{Z}_2$-graded triple $(L_{V_0}, g^{V_0}, \nabla^{V_0})$ of the form (2.1.11) satisfies the MF property for $\mathcal{D}^{S\otimes V_0}$, and

$$\eta^{V_0}(g^{V_0}, \nabla^{V_0}, T^*X, g^{T^*X}, L_{V_0}) \equiv 0. \quad (2.2.1)$$

By Proposition 2.1, $(L_k \oplus L_{V_0} \cdot g^{L_k}, \nabla^{L_k}, \nabla^{V_0})$ is a $\mathbb{Z}_2$-graded triple so that $L_k \oplus L_{V_0} \rightarrow B$ satisfies the MF property for $\mathcal{D}^{S\otimes (E_k \oplus V_0)}$. By applying [11] Proposition 1] to the $\mathbb{Z}_2$-graded triples

$$(E_0 \oplus V_0, g^{E_0} \oplus g^{V_0}, \nabla^{E_0} \oplus \nabla^{V_0}) \quad \text{and} \quad (E_0 \oplus V_1, g^{E_1} \oplus g^{V_1}, \nabla^{E_1} \oplus \nabla^{V_1})$$

there exist balanced $\mathbb{Z}_2$-graded triples $(W_0, g^{W_0}, \nabla^{W_0})$ and $(W_1, g^{W_1}, \nabla^{W_1})$ such that

$$L_0 \oplus L_{V_0} \oplus W_0 \equiv L_1 \oplus L_{V_1} \oplus W_1 \quad (2.2.2)$$

as $\mathbb{Z}_2$-graded complex vector bundles, and

$$\eta^{E_1 \oplus V_0}(g^{E_0} \oplus g^{V_0}, \nabla^{E_0} \oplus \nabla^{V_0}, L_0 \oplus L_{V_0}) - \overline{\eta}^{E_0 \oplus V_0}(g^{E_0} \oplus g^{V_0}, \nabla^{E_0} \oplus \nabla^{V_0}, L_0 \oplus L_{V_0}) \equiv \int_{X/B} \text{Todd}(\nabla^{T^*X}) \wedge \text{CS}(\nabla^{E_0} \oplus \nabla^{V_0}, \nabla^{E_0} \oplus \nabla^{V_0}) - \text{CS}(\nabla^{L_0} \oplus \nabla^{M_0} \oplus \nabla^{W_0}, \nabla^{L_1} \oplus \nabla^{M_1} \oplus \nabla^{W_1}) \quad (2.2.3)$$

By writing $M_k = L_k \oplus W_k$, and similarly for $g^{M_0}$ and $\nabla^{M_0}$, we see that $(M_k, g^{M_k}, \nabla^{M_k})$ is a balanced $\mathbb{Z}_2$-graded triple. By (A.5.2), Proposition 2.1 and (2.2.1), (2.2.3) is reduced to

$$\overline{\eta}^{E_1}(g^{E_1}, \nabla^{E_1}, T^*X, g^{T^*X}, L_0) = \text{Todd}(\nabla^{T^*X}) \wedge (\omega_0 - \omega_1) - \text{CS}(\nabla^{L_0} \oplus \nabla^{M_0} \oplus \nabla^{W_0}, \nabla^{L_1} \oplus \nabla^{M_1}),$$

which is equivalent to

$$\text{CS}(\nabla^{L_0} \oplus \nabla^{M_0} \oplus \nabla^{M_1}) \equiv \overline{\eta}^{E_0}(g^{E_0}, \nabla^{E_0}, T^*X, g^{T^*X}, L_0) + \int_{X/B} \text{Todd}(\nabla^{T^*X}) \wedge \omega_0$$

$$- \left( \overline{\eta}^{E_1}(g^{E_1}, \nabla^{E_1}, T^*X, g^{T^*X}, L_1) + \int_{X/B} \text{Todd}(\nabla^{T^*X}) \wedge \omega_1 \right).$$
Thus \( \text{ind}^a_K(\mathcal{E}_0;L_0) = \text{ind}^a_K(\mathcal{E}_1;L_1) \).

It follows immediately from Proposition 2.1 that
\[
\text{ind}^a_K(\mathcal{E}_0 + \mathcal{E}_1;L_0 \oplus L_1) = \text{ind}^a_K(\mathcal{E}_0;L_0) + \text{ind}^a_K(\mathcal{E}_1;L_1).
\]

Thus \( \text{ind}^a_K \) is a group homomorphism. \( \square \)

An immediate consequence of Proposition 2.4 is the following corollary.

**Corollary 2.1.** Let \( \pi: X \to B \) be a submersion with closed, oriented and spin\(^c\) fibers of even dimension. Choose and fix a spin\(^c\) structure on \( T^V X \to X \), and put geometric data on \( \pi: X \to B \) as in Section A.2. The \( \mathbb{R}/\mathbb{Z} \) analytic index
\[
\text{ind}^a_{\mathbb{R}/\mathbb{Z}}: K^{-1}_L(X) \to K^{-1}_L(B)
\]
associated to twisted spin\(^c\) Dirac operators is a well defined group homomorphism.

By [11, Corollary 1] we can drop the notation \( L \) in \( \text{ind}^a_K \) and in \( \text{ind}^a_{\mathbb{R}/\mathbb{Z}} \).

### 2.3. A proof of Theorem 1.3

Let \( F \to X \) be a \( \mathbb{Z}_2 \)-graded complex flat vector bundle with \( \mathbb{Z}_2 \)-graded flat connection \( \nabla^F \) of virtual rank zero (i.e. \( \text{rank}(F^+) = \text{rank}(F^-) \)). Let \( g^F \) be a \( \mathbb{Z}_2 \)-graded Hermitian metric on \( F \to X \). Then the unitary connection \( \nabla^{F,u} \) on \( F \to X \) defined by (A.1.7) is \( \mathbb{Z}_2 \)-graded. Write \( g^F = g^+ \oplus g^- \) and \( \nabla^F = \nabla^+ \oplus \nabla^- \). Note that \( \nabla^{F,u,\pm} = \nabla^{F,u} \). Since \( \text{ch}(\nabla^{F,u}) = \text{ch}(\nabla^{F,u}_+) - \text{ch}(\nabla^{F,u}_-) = \text{rank}(F^+) - \text{rank}(F^-) = 0 \) by (A.1.8), it follows that
\[
\mathcal{F} = (F, g^F, \nabla^{F,u}, 0)
\]
is a \( \mathbb{Z}_2 \)-graded generator of \( K^{-1}_L(X) \).

Let \( \pi: X \to B \) be a submersion with closed fibers \( Z \). Put geometric data on \( \pi: X \to B \) as in Section A.2. Define a quadruplet \( \mathcal{H}(Z,F|_Z) \) by
\[
\mathcal{H}(Z,F|_Z) := (H(Z,F|_Z), g^{H(Z,F|_Z)}, \nabla^{H(Z,F|_Z),u}, 0).
\]

Since the \( \mathbb{Z}_2 \)-grading of \( H(Z,F|_Z) \to B \) is given by
\[
H(Z,F|_Z)^+ = H^{\text{even}}(Z,F^+|_Z) \oplus H^{\text{odd}}(Z,F^-|_Z),
\]
\[
H(Z,F|_Z)^- = H^{\text{even}}(Z,F^-|_Z) \oplus H^{\text{odd}}(Z,F^+|_Z),
\]
it follows that
\[
\text{ch}(\nabla^{H(Z,F|_Z),u}) = \text{ch}(\nabla^{H(Z,F|_Z)^+,u}) - \text{ch}(\nabla^{H(Z,F|_Z)^-,u})
\]
\[
= \text{rank}(H(Z,F^+|_Z) - \text{rank}(H(Z,F^-|_Z))
\]
\[
= \text{rank}(H(Z,F^+|_Z)) - \text{rank}(H(Z,F^-|_Z))
\]
\[
= \chi(Z) \text{rank}(F^+) - \chi(Z) \text{rank}(F^-) = 0.
\]

Thus \( \mathcal{H}(Z,F|_Z) \) is a \( \mathbb{Z}_2 \)-graded generator of \( K^{-1}_L(B) \).

We now prove Theorem 1.3.
Theorem 2.1. Let \( \pi : X \to B \) be a submersion with closed, oriented and spin\(^c\) fibers \( Z \) of even dimension \( n \). Choose and fix a spin\(^c\) structure on \( T^V X \to X \), and put geometric data on \( \pi : X \to B \) as in Section A.2. For any \( \mathbb{Z}_2 \)-graded generator \( \mathcal{F} \) of \( K_L^{-1}(X) \) of the form (2.3.1) we have

\[
\text{ind}^a_{\mathbb{R}/Z}(S(T^V X)^* \otimes \mathcal{F}) = \mathcal{H}(Z, F|_Z),
\]

where \( S(T^V X)^* \) is given by (A.5.7).

Proof. Let \( \mathcal{F} \) have the form (2.3.1). Define \( \mathbb{Z}_2 \)-graded generator \( p^*_X \mathcal{F} = (\mathcal{F}, g^\mathcal{F}, \nabla^\mathcal{F}, u, 0) \) of \( K_L^{-1}(X) \) by \( \mathcal{F} := p^*_X \mathcal{F}, g^\mathcal{F} = p^*_X g^\mathcal{F} \) and \( \nabla^\mathcal{F}, u = p^*_X \nabla^\mathcal{F}, u \).

Pull the geometric data on \( \pi : X \to B \) back to \( \tilde{\pi} : \tilde{X} \to \tilde{B} \), where \( \tilde{\pi} = \pi \times \text{id}_t \). The chosen spin\(^c\) structure on \( T^V X \to X \) induces a spin\(^c\) structure on \( T^V \tilde{X} \to \tilde{X} \). Let \( \nabla^{S(T^V \tilde{X}) \otimes \mathcal{F}}, u \) be the tensor product of \( \nabla^{S(T^V \tilde{X})} \) and \( \nabla^\mathcal{F}, u \). By (A,5.3),

\[
S(T^V \tilde{X})^* \otimes p^*_X \mathcal{F} = (S(T^V \tilde{X})^* \otimes \mathcal{F}, g^{S(T^V \tilde{X})^*} \otimes g^\mathcal{F}, \nabla^{S(T^V \tilde{X}) \otimes \mathcal{F}}, u, 0)
\]

is a \( \mathbb{Z}_2 \)-graded generator of \( K_L^{-1}(\tilde{X}) \).

Let \( \{\tilde{e}_1, \ldots, \tilde{e}_n\} \) be a local orthonormal frame for \( T^V \tilde{X} \to \tilde{X} \). Denote by \( D^{S \otimes (S \otimes \mathcal{F})} \) the twisted spin\(^c\) Dirac operator. Consider the following operator

\[
D = D^{S \otimes (S \otimes \mathcal{F})} + t\tilde{V},
\]

where \( t \in I \) and \( \tilde{V} \) is given by (A.4.6) adapted to the current setting. Note that \( D \) is an odd self-adjoint operator acting on \( \Gamma(\tilde{X}, S(T^V \tilde{X}) \otimes (S(T^V \tilde{X})^* \otimes \mathcal{F})) \).

Let \( (\mathcal{L}, g^\mathcal{L}, \nabla^\mathcal{L}) \) be a \( \mathbb{Z}_2 \)-graded triple so that \( \mathcal{L} \to \tilde{B} \) satisfies the MF property for \( D \), i.e. there exist complementary closed subbundles \( \mathcal{K}^\pm \to \tilde{B} \) of \( (\pi_*(S(T^V \tilde{X})^* \otimes \mathcal{F}))^\pm \to \tilde{B} \) such that

\[
(\pi_*(S(T^V \tilde{X})^* \otimes \mathcal{F}))^+ = \mathcal{K}^+ \oplus \mathcal{L}^+,
\]

\[
(\pi_*(S(T^V \tilde{X})^* \otimes \mathcal{F}))^- = \mathcal{K}^- \oplus \mathcal{L}^-,
\]

\( D^\pm \) is block diagonal as a map with respect to (2.3.4) and \( D_+|_{\mathcal{K}^+} : \mathcal{K}^+ \to \mathcal{K}^- \) is an isomorphism. Let \( \mathcal{K} = \mathcal{K}^+ \oplus \mathcal{K}^- \). Note that

\[
i_{B,j}^i \pi_*(S(T^V \tilde{X})^* \otimes \mathcal{F}) \equiv \pi_*(S(T^V X)^* \otimes \mathcal{F})
\]

as \( \mathbb{Z}_2 \)-graded complex vector bundles over \( B \) for \( j \in \{0, 1\} \). Write \( L_j \to B \) for \( i_{B,j}^i \mathcal{L} \to B \) and \( K_j \to B \) for \( i_{B,j}^i \mathcal{K} \to B \). Note that \( K_0 \equiv K_1 \) and

\[
L_0 \equiv L_1
\]

as \( \mathbb{Z}_2 \)-graded complex vector bundles. Write \( g^{L_j} = i_{B,j}^i g^\mathcal{L} \) and \( \nabla^{L_j} = i_{B,j}^i \nabla^\mathcal{L} \).

That \( \mathcal{L} \to \tilde{B} \) satisfies the MF property for \( D \) implies

\[
L_0 \to B \text{ satisfies the MF property for } D^{S \otimes (S \otimes \mathcal{F})}, \text{ and}
\]

\[
L_1 \to B \text{ satisfies the MF property for } D^{S \otimes (S \otimes \mathcal{F})} + V.
\]
By applying [11, Proposition 1] to $D$, $(L_0, g^{L_0}, \nabla^{L_0})$ and $(L_1, g^{L_1}, \nabla^{L_1})$, and by noting (2.3.5), we have

$\tilde{\eta}^{S(T^V X)^* \otimes F} (g^{S(T^V X)^*} \otimes g^F, \nabla^{S(T^V X)^* \otimes F}, T^H_X, g^{T^V X}, L_1)$

$- \tilde{\eta}^{S(T^V X)^* \otimes F} (g^{S(T^V X)^*} \otimes g^F, \nabla^{S(T^V X)^* \otimes F}, T^H_X, g^{T^V X}, L_0)$

$(2.3.8)$

$\equiv -\text{CS}(\nabla^{L_0}, \nabla^{L_1})$,

where the Bismut–Cheeger eta form

$\tilde{\eta}^{S(T^V X)^* \otimes F} (g^{S(T^V X)^*} \otimes g^F, \nabla^{S(T^V X)^* \otimes F}, T^H_X, g^{T^V X}, L_1)$

in (2.3.8) is associated to the Bismut superconnection $\tilde{\mathcal{D}}^{S(T^V X)^* \otimes F}$ with

$\tilde{\mathcal{D}}^{S(T^V X)^* \otimes F} = \mathcal{D}^{S(T^V X)^* \otimes F}(1) = \begin{pmatrix} D^{S \otimes (S \otimes F)} + V \quad i^- \\ p^+ \quad 0 \end{pmatrix}$

(cf. (A.3.9)). By (2.3.6) and [11, Corollary 1], we have

$\text{ind}_{\mathbb{Z}/2}^Z(S(T^V X)^* \otimes F)$

$= \begin{pmatrix} L_0, g^{L_0}, \nabla^{L_0}, \tilde{\eta}^{S(T^V X)^* \otimes F} (g^{S(T^V X)^*} \otimes g^F, \nabla^{S(T^V X)^* \otimes F}, T^H_X, g^{T^V X}, L_0) \end{pmatrix}$

$(2.3.9)$

where $\nabla^{S(T^V X)^* \otimes F}$ is the tensor product of $\nabla^{S(T^V X)^*}$ and $\nabla^{F, u}$. By (2.3.9) and (2.3.2), to prove (2.3.3) it suffices to show that there exist balanced $\mathbb{Z}_2$-graded triples $(W_0, g^{W_0}, \nabla^{W_0})$ and $(W_1, g^{W_1}, \nabla^{W_1})$ such that

$L_0 \oplus W_0 \cong H(Z, F|_Z) \oplus W_1$ \hspace{1cm} (2.3.10)

as $\mathbb{Z}_2$-graded complex vector bundles, and

$\tilde{\eta}^{S(T^V X)^* \otimes F} (g^{S(T^V X)^*} \otimes g^F, \nabla^{S(T^V X)^* \otimes F}, T^H_X, g^{T^V X}, L_0)$

$\equiv \text{CS}(\nabla^{L_0} \oplus \nabla^{W_0}, \nabla^{H(Z, F|_Z)}, T^H_X, g^{T^V X}, L_0)$

$(2.3.11)$

Since

$S(T^V X) \otimes S(T^V X)^* \cong S_0(T^V X) \otimes S_0(T^V X)^* \cong \Lambda(T^V X)^*$

$(2.3.12)$

we have $\pi_1^A F \cong \pi_1(S(T^V X)^* \otimes F)$. Under the isomorphism (2.3.12), the Clifford multiplication on $S(T^V X) \to X$ corresponds to that on $\Lambda(T^V X)^* \to X$ [2, Proposition 3.5]. Let $\nabla^{S(T^V X) \otimes S(T^V X)^*}$ be the tensor product of $\nabla^{S(T^V X)}$ and $\nabla^{S(T^V X)^*}$. Since

$\nabla^{S(T^V X) \otimes S(T^V X)^*} = \Lambda^A(T^V X)^*$ \hspace{1cm} (2.3.13)

the tensor product of $\nabla^{S(T^V X)}$ and $\nabla^{S(T^V X)^* \otimes F, u}$ is equal to that of $\nabla^{\Lambda(T^V X)^*}$ and $\nabla^{F, u}$. Thus $D^{S \otimes (S \otimes F)} = D^{A \otimes F}$. By (A.4.5), we have

$D^{S \otimes (S \otimes F)} + V = D^{A \otimes F} + V = D^{Z, dR}$ \hspace{1cm} (2.3.14)

where $V$ is given by (A.4.6). It follows from (2.3.14) and (A.4.3) that $H(Z, F|_Z) \to B$ satisfies the MF property for $D^{S \otimes (S \otimes F)} + V$. Note that $g^{H(Z, F|_Z)}$ is the
projected $\mathbb{Z}_2$-graded Hermitian metric on $H(Z,F|_Z) \to B$. By (A.3.4) and (A.4.1), (2.3.15) implies that
\[
\nabla^{\pi_0(S^r(T^V X)\otimes F),u} = \nabla^{\pi_1 F,u}.
\] (2.3.15)
It follows from (A.4.4) that $\nabla^{H(Z,F|_Z),u}$ is the projected $\mathbb{Z}_2$-graded unitary connection on $H(Z,F|_Z) \to B$. By applying Proposition 1 to $B^S(S^\ast \otimes F) \oplus V, (L_1, g^{L_1}, \nabla^{L_1})$ and $(H(Z,F|_Z), g^{H(Z,F|_Z)}, \nabla^{H(Z,F|_Z),u})$, there exist balanced $\mathbb{Z}_2$-graded triples $(W_0, g^{W_0}, \nabla^{W_0})$ and $(W_1, g^{W_1}, \nabla^{W_1})$ such that
\[
L_1 \oplus W_0 \cong H(Z,F|_Z) \oplus W_1
\] (2.3.16)
as $\mathbb{Z}_2$-graded complex vector bundles, and
\[
\eta(S^r(T^V X)\otimes F) \oplus g^F, \nabla^{S^r(T^V X)\otimes F,u}, T^H X, g^{T^V X}, H(Z,F|_Z))
\]
\[
- \eta(S^r(T^V X)\otimes F) \oplus g^F, \nabla^{S^r(T^V X)\otimes F,u}, T^H X, g^{T^V X}, L_1)
\] (2.3.17)
\[
\equiv -\text{CS}(\nabla^{L_1} \oplus W_0, \nabla^{H(Z,F|_Z),u} \oplus \nabla^{W_1}).
\]
By (2.3.5), (2.3.16) becomes
\[
L_0 \oplus W_0 \cong H(Z,F|_Z) \oplus W_1,
\]
which is (2.3.10). Moreover, by adding (2.3.8) to (2.3.17), we have
\[
\eta(S^r(T^V X)\otimes F) \oplus g^F, \nabla^{S^r(T^V X)\otimes F,u}, T^H X, g^{T^V X}, H(Z,F|_Z))
\]
\[
- \eta(S^r(T^V X)\otimes F) \oplus g^F, \nabla^{S^r(T^V X)\otimes F,u}, T^H X, g^{T^V X}, L_0)
\] (2.3.18)
\[
\equiv -\text{CS}(\nabla^{L_0} \oplus W_0, \nabla^{H(Z,F|_Z),u} \oplus \nabla^{W_1}).
\]
By (A.1.5) and (A.1.4), we have
\[
-\text{CS}(\nabla^{L_0} \oplus W_0, \nabla^{H(Z,F|_Z),u} \oplus \nabla^{W_1})
\]
\[
\equiv -\text{CS}(\nabla^{L_0} \oplus W_1) - \text{CS}(\nabla^{W_0} \oplus \nabla^{W_1}) - \text{CS}(\nabla^{L_1} \oplus W_0, \nabla^{H(Z,F|_Z),u} \oplus \nabla^{W_1})
\] (2.3.19)
\[
\equiv -\text{CS}(\nabla^{L_0} \oplus W_0, \nabla^{H(Z,F|_Z),u} \oplus \nabla^{W_1})
\]
\[
\equiv -\text{CS}(\nabla^{L_0} \oplus W_0, \nabla^{H(Z,F|_Z),u} \oplus \nabla^{W_1}).
\]
By (2.3.19), (2.3.18) becomes
\[
\eta(S^r(T^V X)\otimes F) \oplus g^F, \nabla^{S^r(T^V X)\otimes F,u}, T^H X, g^{T^V X}, H(Z,F|_Z))
\]
\[
- \eta(S^r(T^V X)\otimes F) \oplus g^F, \nabla^{S^r(T^V X)\otimes F,u}, T^H X, g^{T^V X}, L_0)
\] (2.3.20)
\[
\equiv -\text{CS}(\nabla^{L_0} \oplus W_0, \nabla^{H(Z,F|_Z),u} \oplus \nabla^{W_1}).
\]
By (2.3.20), to prove (2.3.11) it suffices to show that
\[
\eta(S^r(T^V X)\otimes F) \oplus g^F, \nabla^{S^r(T^V X)\otimes F,u}, T^H X, g^{T^V X}, H(Z,F|_Z)) \equiv 0.
\] (2.3.21)
To prove (2.3.21), denoted by $\mathcal{B}^{S^r(T^V X)\otimes F}$ the Bismut superconnection with $B^{S^r(T^V X)\otimes F}_{[0]} = B^S(S^\ast \otimes F) + V$ (cf. (A.3.5)). By (2.3.14) and (2.3.15), we have $\mathcal{B}^{S^r(T^V X)\otimes F} = \mathcal{B}^{dR}$. Denote by $\eta^{S^r(T^V X)\otimes F}$ the Bismut–Cheeger eta form
associated to $D^S(T^VX)\hat{\otimes} F$. By (2.3.14) and (A.4.3), the definition of $g^{H(Z,F|Z)}$, (A.4.4) and (2.3.15), we have

$$(H(Z,F|Z), g^{H(Z,F|Z)}, \nabla H(Z,F|Z), u) \equiv (\ker(D^S(\hat{\otimes} F) + V), g^\chi(\ker(D^S(\hat{\otimes} F) + V), \nabla(\ker(D^S(\hat{\otimes} F) + V))$$

as $\mathbb{Z}_2$-graded triples. Thus

$$\eta^{S(T^VX)\hat{\otimes} F}(g^{S(T^VX)} \otimes g^F, \nabla S(T^VX)\hat{\otimes} F, u, T^H X, g^{T^V X}, H(Z,F|Z))$$

$$\equiv \eta^{S(T^VX)\hat{\otimes} F}(g^{S(T^VX)} \otimes g^F, \nabla S(T^VX)\hat{\otimes} F, u, T^H X, g^{T^V X}, \ker(D^S(\hat{\otimes} F) + V))$$

$$\equiv \eta^{\text{MF}} = 0,$$

where the second equality follows from [11] (3.35) and the fourth equality follows from (A.4.7).

2.4. The RRG theorem for $\text{ind}^\mathbb{Z}_{\mathbb{R}}_{\mathbb{Z}}$ in $\mathbb{R}/\mathbb{Z}$ $K$-theory. We now prove Theorem 1.4.

**Theorem 2.2.** Let $\pi : X \to B$ be a submersion with closed, oriented and spin$^c$ fibers of even dimension. Choose and fix a spin$^c$ structure on $T^V X \to X$, and put geometric data on $\pi : X \to B$ as in Section 1.2. Then the following diagram commutes.

$$\begin{align*}
K_L^{-1}(X) & \xrightarrow{\text{ch}_{\mathbb{R}/\mathbb{Q}}} H^{\text{odd}}(X; \mathbb{R}/\mathbb{Q}) \\
\text{ind}_{\mathbb{R}/\mathbb{Z}} & \downarrow \\
K_L^{-1}(B) & \xrightarrow{\text{ch}_{\mathbb{R}/\mathbb{Q}}} H^{\text{odd}}(B; \mathbb{R}/\mathbb{Q})
\end{align*}$$

(2.4.1)

**Proof.** Let $E = (E, g^E, \nabla^E, \omega)$ be a $\mathbb{Z}_2$-graded generator of $K_L^{-1}(X)$. Denote by $D^S g^{E^+}$ and $D^S g^E$ the twisted spin$^c$ Dirac operators associated to $(E^+, g^{E^+, \nabla^{E^+}})$ and $(E^-, g^{E^-, \nabla^{E^-}})$. Let $(L^+_{E^+}, g_{L^+_{E^+}}, \nabla_{L^+_{E^+}})$ and $(L^-_{E^-}, g_{L^-_{E^-}}, \nabla_{L^-_{E^-}})$ be $\mathbb{Z}_2$-graded triples so that $L^+_{E^+} \to B$ and $L^-_{E^-} \to B$ satisfy the MF property for $D^S g^{E^+}$ and $D^S g^E$, respectively.

Denote by $D^S g^E$ the twisted spin$^c$ Dirac operator associated to $(E, g^E, \nabla^E)$. By Lemma 2.1 $(L^+_{E^+} \oplus L^+_{E^+}, g_{L^+_{E^+}} \oplus g_{L^+_{E^+} \cdot \text{op}}, \nabla_{L^+_{E^+}} \oplus \nabla_{L^+_{E^+} \cdot \text{op}})$ is a $\mathbb{Z}_2$-graded triple so that $L^+_{E^+} \oplus L^+_{E^+} \to B$ satisfies the MF property for $D^S g^E$. By [11] Corollary 1], the $\mathbb{R}/\mathbb{Z}$ analytic index of $E$ is given by

$$\text{ind}_{\mathbb{R}/\mathbb{Z}}^\mathbb{Z}(E) = \left(\int_{X/B} \text{Todd}(\nabla^{T^V X}) \wedge \omega \right. + \left. \eta^{E}(g^E, \nabla^E, T^H X, g^{T^V X}, L^+_{E^+} \oplus L^+_{E^+})\right)$$
By the definition of $\mathcal{E}$ and (A.3.4), there exists a $k_1 \in \mathbb{N}$ such that $k_1E^+ \equiv k_1E^-$. On the other hand, since

$$
\text{ch}(\nabla^{L^E_+}) - \text{ch}(\nabla^{L^E_-}) = \text{ch}(\nabla^{L^E_+} \oplus \nabla^{L^E_-}^{\text{op}})
= -d\left( \int_{X/B} \text{Todd}(\nabla^{T^VX}) \wedge \omega + \tilde{\eta}^E (g^E, \nabla^E, T^H X, g^{T^VX}, L_{E^+} \oplus L_{E^-}^{\text{op}}) \right),
$$

(2.4.2)

there exists a $k_2 \in \mathbb{N}$ such that $k_2L_{E^+} \equiv k_2L_{E^-}$. Moreover, since

$$
\text{ch}(\nabla^{L^E_+} \oplus L^{\text{op}}_{E^+}) - \text{ch}(\nabla^{L^E_+} \oplus L^{\text{op}}_{E^-}) = \text{ch}(\nabla^{L^E_+,+} \oplus \nabla^{L^E_-,+}) - \text{ch}(\nabla^{L^E_+,+} \oplus \nabla^{L^E_-,+})
= \text{ch}(\nabla^{L^E_+,+} - \nabla^{L^E_-,+} + \nabla^{L^E_+,+} - \nabla^{L^E_-,+})
= \text{ch}(\nabla^{L^E_+} - \nabla^{L^E_-}),
$$

it follows from (2.4.2) that there exists a $k_3 \in \mathbb{N}$ such that

$$
k_3(L_{E^+} \oplus L^{\text{op}}_{E^+})^+ = k_3(L_{E^-} \oplus L^{\text{op}}_{E^-})^-.
$$

Let $k$ be the least common multiple of $k_1, k_2$ and $k_3$. Note that such a $k$ satisfies

$$
kE^+ \equiv kE^-, \quad kL_{E^+} \equiv kL_{E^-}, \quad k(L_{E^+} \oplus L^{\text{op}}_{E^+})^+ \equiv k(L_{E^-} \oplus L^{\text{op}}_{E^-})^-.
$$

(2.4.3)

By the left-most isomorphism in (2.4.3), a differential form representative of

$$
\int_{X/B} \text{Todd}(T^V X) \cup \text{ch}_{\mathbb{R}/\mathbb{Q}}(\mathcal{E})
$$

is

$$
\int_{X/B} \text{Todd}(\nabla^{T^VX}) \wedge \left( \frac{1}{k} \text{CS}(k\nabla^{E,-}, k\nabla^{E,+}) + \omega \right),
$$

(2.4.4)

and by the right-most isomorphism in (2.4.3), a differential form representative of $\text{ch}_{\mathbb{R}/\mathbb{Q}}(\text{ind}_{\mathbb{R}/\mathbb{Z}}^(E)(\mathcal{E}))$ is

$$
\frac{1}{k} \text{CS}(k(\nabla^{L^E_+} \oplus \nabla^{L^E_-,+}), k(\nabla^{L^E_+} \oplus \nabla^{L^E_-,+})) + \int_{X/B} \text{Todd}(\nabla^{T^VX}) \wedge \omega
$$

$$
+ \tilde{\eta}^E (g^E, \nabla^E, T^H X, g^{T^VX}, L_{E^+} \oplus L_{E^-}^{\text{op}}).
$$

(2.4.5)

By (2.4.4) and (2.4.5), to show diagram (2.4.1) commutes it suffices to prove that

$$
k\tilde{\eta}^E (g^E, \nabla^E, T^H X, g^{T^VX}, L_{E^+} \oplus L_{E^-}^{\text{op}}) = \int_{X/B} \text{Todd}(\nabla^{T^VX}) \wedge \text{CS}(k\nabla^{E,-}, k\nabla^{E,+})
$$

$$
- \text{CS}(k(\nabla^{L^E_+} \oplus \nabla^{L^E_-,+}), k(\nabla^{L^E_+} \oplus \nabla^{L^E_-,+}))
$$

(2.4.6)

up to closed odd forms on $B$ with periods in $\mathbb{Q}$. 
By applying [11] Proposition 1 to \((kE^+, kg^{E,+}, k\nabla^{E,+})\) and \((kE^-, kg^{E,-}, k\nabla^{E,-})\), and by noting the middle isomorphism in (2.4.3), we have
\[
\eta^{kE^+}(kg^{E,+}, k\nabla^{E,+}, T^H X, g^{T^V X}, kL_{E^+}) - \eta^{kE^-}(kg^{E,-}, k\nabla^{E,-}, T^H X, g^{T^V X}, kL_{E^-}) \\
\equiv \int_{X/B} \text{Todd}(\nabla^{T^V X}) \wedge \text{CS}(k\nabla^{E,-}, k\nabla^{E,+}) - \text{CS}(k\nabla^{L_{E^-}}, k\nabla^{L_{E^+}}).
\] (2.4.7)

By Propositions [2.3] and [2.1], we have
\[
\eta^{kE^+}(kg^{E,+}, k\nabla^{E,+}, T^H X, g^{T^V X}, kL_{E^+}) - \eta^{kE^-}(kg^{E,-}, k\nabla^{E,-}, T^H X, g^{T^V X}, kL_{E^-}) \\
\equiv \eta^{kE^-}(kg^{E,-}, k\nabla^{E,-}, T^H X, g^{T^V X}, k(L_{E^-} \oplus L^\text{op}_{E^-})) \\
\equiv \eta^{kE^+}(g^{E^+}, \nabla^{E^+}, T^H X, g^{T^V X}, L_{E^+} \oplus L^\text{op}_{E^+}).
\]

On the other hand, by (A.1.6), (A.1.3) and (A.1.5), we have
\[
\text{CS}(k\nabla^{L_{E^-}}, k\nabla^{L_{E^+}}) \equiv \text{CS}(k\nabla^{L_{E^-}+}, k\nabla^{L_{E^-}}, k\nabla^{L_{E^+}}, k\nabla^{L_{E^+}+}) \\
\equiv \text{CS}(k\nabla^{L_{E^-}+}, k\nabla^{L_{E^+}}, k\nabla^{L_{E^-}}, k\nabla^{L_{E^+}+}) \\
\equiv \text{CS}(k\nabla^{L_{E^-}}, k\nabla^{L_{E^+}}, k\nabla^{L_{E^-}+}, k\nabla^{L_{E^+}}) \\
\equiv \text{CS}(k\nabla^{L_{E^+}}, k\nabla^{L_{E^-}}, k\nabla^{L_{E^-}+}, k\nabla^{L_{E^+}}).
\]

Thus (2.4.7) reduces to (2.4.6). \(\square\)

Note that (2.4.6) is the refinement of Theorem [2.2] at the differential form level.

3. The main results for generalized twisted de Rham operators

In this section we prove the main results for generalized twisted de Rham operators. In Section 3.1 we state part of the local index theory for generalized twisted de Rham operators by Bismut–Lott [6, §III] without assuming the twisting bundles to be flat. In Section 3.2 we prove a variational formula and state some properties of the corresponding Bismut–Cheeger eta form. In Section 3.3 we define the analytic index associated to generalized twisted de Rham operators in differential K-theory and show that it is a well-defined group homomorphism. In Section 3.4 we prove Theorem [1.1]. Finally, in Section 3.5 we prove Theorem [1.2].

We refer to Appendices A.2 and A.4 for a quick summary on the local index family theorem for the de Rham operators twisted by complex flat vector bundles. We refer to [6] for the details.

3.1. Local index theory for generalized twisted de Rham operator. Let \(\pi : X \to B\) be a submersion with closed fibers \(Z\) of dimension \(n\). Put geometric data on \(\pi : X \to B\) as in Section A.2. We consider the complexified exterior bundle \(\Lambda(T^V X)^* \to X\) to be \(\mathbb{Z}_2\)-graded, where the \(\mathbb{Z}_2\)-grading is given by even and odd forms.
Let \((E,g^E,\nabla^E)\) be a triple. Denote by \(g^{\Lambda(T^V X)^*}\) and \(\nabla^{\Lambda(T^V X)^*}\) the extensions of \(g^{T^V X}\) and \(\nabla^{T^V X}\) to \(\Lambda(T^V X)^* \to X\), respectively. Then \(g^{\Lambda(T^V X)^* \otimes E} := g^{\Lambda(T^V X)^*} \otimes g^E\) is a \(\mathbb{Z}_2\)-graded Hermitian metric and
\[
\nabla^{\Lambda(T^V X)^* \otimes E} = \nabla^{\Lambda(T^V X)^*} \otimes \text{id}_{\Gamma(X,E)} + \text{id}_{\Gamma(X,\Lambda(T^V X)^*)} \otimes \nabla^E
\]
is a \(\mathbb{Z}_2\)-graded unitary connection on \(\Lambda(T^V X)^* \otimes E \to X\), respectively. Define the generalized twisted de Rham operator \(D^{\Lambda \otimes E}\) by
\[
D^{\Lambda \otimes E} = \sum_{k=1}^n c(e_k) \nabla^{\Lambda(T^V X)^* \otimes E}_{e_k}.
\]
It acts on \(\Gamma(X,\Lambda(T^V X)^* \otimes E)\) and is odd self-adjoint, i.e. \(D^{\Lambda \otimes E} = (D^{\Lambda \otimes E})^*\).

Define an infinite rank \(\mathbb{Z}_2\)-graded complex vector bundle \(\pi^*_s E \to B\) whose fiber over \(b \in B\) is
\[
(\pi^*_s E)_b = \Gamma(Z_b, (\Lambda(T^V X)^* \otimes E)|Z_b).
\]
Note that \(\Omega(X, E) \equiv \Omega(B, \pi^*_s E)\). Define an \(L^2\)-metric on \(\pi^*_s E \to B\) by
\[
g^{\pi^*_s E}(s_1, s_2)(b) = \int_{Z_b} g^{\Lambda(T^V X)^* \otimes E}(s_1, s_2) d\text{vol}(Z).
\]
Define a connection on \(\pi^*_s E \to B\) by
\[
\nabla^{\pi^*_s E}_{U} s = \nabla^{\Lambda(T^V X)^* \otimes E}_{U} s,
\]
where \(s \in \Gamma(B, \pi^*_s E)\) and \(U \in \Gamma(B, TB)\). Note that \(\nabla^{\pi^*_s E}\) preserves the \(\mathbb{Z}_2\)-grading of \(\pi^*_s E \to B\). The connection on \(\pi^*_s E \to B\) defined by
\[
\nabla^{\pi^*_s E, U} := \nabla^{\pi^*_s E} + \frac{1}{2} k,
\]
where \(k\) is given by \([\mathbf{A.2.1}]\), is \(\mathbb{Z}_2\)-graded and unitary with respect to \(g^{\pi^*_s E}\).

Suppose \(D^{\Lambda \otimes E}\) satisfies the kernel bundle assumption, i.e. the family of complex vector spaces \(\{\ker(D^{\Lambda \otimes E})_{b \in B}\}\) form a vector bundle. In this case we write \(\ker(D^{\Lambda \otimes E}) \to B\) for the resulting complex vector bundle, which is \(\mathbb{Z}_2\)-graded. Note that the family of complex vector spaces \(\{\text{Im}(D^{\Lambda \otimes E})_{b \in B}\}\) form an infinite rank \(\mathbb{Z}_2\)-graded subbundle of \(\pi^*_s E \to B\), and is denoted by \(\text{Im}(D^{\Lambda \otimes E}) \to B\). Their \(\mathbb{Z}_2\)-gradings are given by
\[
\ker(D^{\Lambda \otimes E})^\pm = \ker(D^{\Lambda \otimes E})\quad \text{and} \quad \text{Im}(D^{\Lambda \otimes E})^\pm = \text{Im}(D^{\Lambda \otimes E}).
\]
Note that the direct sum decompositions
\[
(\pi^*_s E)^+ = \text{Im}(D^{\Lambda \otimes E})^+ \oplus \ker(D^{\Lambda \otimes E})^+, \quad (\pi^*_s E)^- = \text{Im}(D^{\Lambda \otimes E})^- \oplus \ker(D^{\Lambda \otimes E})^-
\]
are orthogonal. Denote by \(p_{\ker(D^{\Lambda \otimes E})} : \pi^*_s E \to \ker(D^{\Lambda \otimes E})\) the orthogonal projection. Then
\[
g^{\ker(D^{\Lambda \otimes E})} := p_{\ker(D^{\Lambda \otimes E})} g^{\pi^*_s E}, \quad \nabla^{\ker(D^{\Lambda \otimes E})} := p_{\ker(D^{\Lambda \otimes E})} \nabla^{\pi^*_s E, U}.
\]
are a $\mathbb{Z}_2$-graded Hermitian metric and a $\mathbb{Z}_2$-graded unitary connection on \( \ker(D^{A \otimes E}) \to B \), respectively.

Define a Bismut superconnection on \( \pi_*^A E \to B \) by

\[
\mathbb{B}^{E,A} = D^{A \otimes E} + \nabla \pi_*^A E,u - \frac{c(T)}{4},
\]

where \( T \) is given by \( \text{A.2.2} \). The rescaled Bismut superconnection is given by

\[
\mathbb{B}^{E,A}_t = \sqrt{t}D^{A \otimes E} + \nabla \pi_*^A E,u - \frac{c(T)}{4\sqrt{t}}.
\]

By essentially the same argument in the proof of [6, Theorem 3.15], we have

\[
\lim_{t \to 0} \text{ch}(\mathbb{B}^{E,A}_t) = \int_{X/B} e(\nabla^{TVX}) \wedge \text{ch}(\nabla^E).
\]

(3.1.2)

On the other hand, by [2, Theorem 9.19] we have

\[
\lim_{t \to 0} \text{ch}(\mathbb{B}^{E,A}_t) = \text{ch}(\nabla^{ker(D^{A \otimes E})}).
\]

(3.1.3)

Note that (3.1.2) and (3.1.3) hold regardless of the parity of \( n \).

The Bismut–Cheeger eta form associated to \( \mathbb{B}^{E,A} \) is defined to be

\[
\tilde{\eta}^{E,A}(g^E, \nabla^E, T^H X, g^{TVX}) = \int_0^\infty \text{str} \left( \frac{d\mathbb{B}^{E,A}_t}{dt} e^{-\frac{t}{2m}(\mathbb{B}^{E,A}_t)^2} \right) dt.
\]

By (3.1.2) and (3.1.3), the local FIT for \( D^{A \otimes E} \) under the kernel bundle assumption is

\[
d\tilde{\eta}^{E,A}(g^E, \nabla^E, T^H X, g^{TVX}) = \begin{cases} 
\int_{X/B} e(\nabla^{TVX}) \wedge \text{ch}(\nabla^E) - \text{ch}(\nabla^{ker(D^{A \otimes E})}), & \text{if } n \text{ is even} \\
-\text{ch}(\nabla^{ker(D^{A \otimes E})}), & \text{if } n \text{ is odd}
\end{cases}
\]

(3.1.2)

In the general case, i.e. \( D^{A \otimes E} \) does not satisfy the kernel bundle assumption, let \( L \to B \) be a \( \mathbb{Z}_2 \)-graded complex vector bundle satisfying the MF property for \( D^{A \otimes E} \), i.e. there exist complementary closed subbundles \( K^\pm \to B \) of \( (\pi_*^A E)^\pm \to B \) such that

\[
(\pi_*^A E)^+ = K^+ \oplus L^+, \quad (\pi_*^A E)^- = K^- \oplus L^-.
\]

(3.1.4)

\( D_+^{A \otimes E} : (\pi_*^A E)^+ \to (\pi_*^A E)^- \) is block diagonal as a map with respect to (3.1.4), and \( D^{A \otimes E}_{K^+} : K^+ \to K^- \) is a smooth bundle isomorphism. Define

\[
\tilde{\pi}_*^A E = \pi_*^A E \oplus L^\text{op}.
\]

Let \( i^- : L^- \to (\pi_*^A E)^- \) be the inclusion map and \( p^+ : (\pi_*^A E)^+ \to L^+ \) the projection map. For any \( \alpha \in \mathbb{C} \), define a map \( \widetilde{D}^E : (\pi_*^A E)^+ \to (\tilde{\pi}_*^A E)^- \) by

\[
\widetilde{D}^E(\alpha) = \begin{pmatrix} D_+^{A \otimes E} & \alpha i^- \\ 0 & 0 \end{pmatrix}.
\]

Note that \( \widetilde{D}^E(\alpha) \) is invertible for any \( \alpha \neq 0 \).
Let $P^L : \pi^A_\ast E \to L$ be the projection map with respect to (3.1.4). Then
\[ g^L = P^L g^{\pi^A_\ast E}, \quad \nabla^L = P^L \nabla^{\pi^A_\ast E, u} \]
are a $\mathbb{Z}_2$-graded Hermitian metric and a $\mathbb{Z}_2$-graded unitary connection on $L \to B$, respectively. Henceforth, whenever $(L, g^L, \nabla^L)$ is a $\mathbb{Z}_2$-graded triple and $L \to B$ satisfies the MF property for $D^{A\otimes E}$, $g^L$ and $\nabla^L$ are obtained as above unless otherwise specified.

Define a $\mathbb{Z}_2$-graded unitary connection on $\pi^{\pi^A_\ast E} \to B$ by
\[ \nabla^{\pi^{\pi^A_\ast E}, u} = \nabla^{\pi^{\pi^A_\ast E, u}} \oplus \nabla^{L, \text{op}}. \]
Define a Bismut superconnection on $\pi^{\pi^A_\ast E} \to B$ by
\[ \hat{B}^{E,A} = \hat{B}^E (1) + \nabla^{\pi^{\pi^A_\ast E}, u} - \frac{c(T)}{4}. \] (3.1.5)
The rescaled Bismut superconnection is given by
\[ \hat{B}^{E,A}_t = \sqrt{t} \hat{B}^E (\alpha(t)) + \nabla^{\pi^{\pi^A_\ast E}, u} - \frac{c(T)}{4 \sqrt{t}}, \]
where $\alpha : [0, \infty) \to [0, 1]$ is the smooth function below (3.1.1). Since $\hat{B}^E (\alpha(t))$ is invertible for any $t \geq 1$, it follows that $\lim \text{ch}(\hat{B}^{E,A}_t) = 0$. By (3.1.2), we have
\[ \lim_{t \to 0} \text{ch}(\hat{B}^{E,A}_t) = \lim_{t \to 0} \text{ch}(\hat{B}^{E,A}) - \text{ch}(\nabla^L) = \int_{X/B} e(\nabla^{TVX}) \wedge \text{ch}(\nabla^E) - \text{ch}(\nabla^L). \]

The Bismut–Cheeger eta form $\tilde{\eta}^{E,A}$ associated to $\hat{B}^{E,A}$ is defined to be
\[ \tilde{\eta}^{E,A}(g^E, \nabla^E, T^H X, g^{TVX}, L) = \int_0^\infty \text{str} \left( \frac{d \hat{B}^{E,A}_t}{dt} e^{-\frac{1}{\sqrt{t}} (\hat{B}^{E,A}_t)^2} \right) dt. \]

The local FIT for $D^{A\otimes E}$ is given by
\[ d\tilde{\eta}^{E,A}(g^E, \nabla^E, T^H X, g^{TVX}, L) = \begin{cases} \int_{X/B} e(\nabla^{TVX}) \wedge \text{ch}(\nabla^E) - \text{ch}(\nabla^L), & \text{if } n \text{ is even} \\ -\text{ch}(\nabla^L), & \text{if } n \text{ is odd} \end{cases}. \] (3.1.6)

3.2. A variational formula and some properties of the Bismut–Cheeger eta form. The following proposition is a variational formula of the Bismut–Cheeger eta form $\tilde{\eta}^{E,A}$.

**Proposition 3.1.** Let $\pi : X \to B$ be a submersion with closed fibers of dimension $n$. Fix $k \in [0, 1]$. Let $(E, S_k^E, \nabla_k^E)$ be a triple. Put geometric data $(T^H_k X, g_k^{TVX})$ on $\pi : X \to B$ as in Section 3.2, and denote by $D^{A\otimes E}_k$ the generalized twisted de Rham operator. Let $(L_k, g^{L_k}, \nabla^{L_k})$ be a $\mathbb{Z}_2$-graded triple so
that $L_k \to B$ satisfies the MF property for $D^{\Lambda \otimes E}_k$. Then there exist balanced $Z_2$-graded triples $(W_0, g^W_0, \nabla^W_0)$ and $(W_1, g^W_1, \nabla^W_1)$ such that

$$L_0 \oplus W_0 \cong L_1 \oplus W_1$$

as $Z_2$-graded complex vector bundles, and

$$\tilde{\eta}^{E, \Lambda}(g^E_1, \nabla^E_1, T_1^H X, g^T^V X, L_1) - \tilde{\eta}^{E, \Lambda}(g^E_0, \nabla^E_0, T_0^H X, g^T^V X_0, L_0)$$

$$\equiv \int_{X/B} \overline{\mathcal{C}}(\nabla^V_0, \nabla^V_1) \wedge \text{ch}(\nabla^E_0) + \int_{X/B} e(\nabla^T_1) \wedge \text{CS}(\nabla^E_0, \nabla^E_1)$$

(3.2.1)

$$- \text{CS}(\nabla^L_0 \oplus \nabla^W_0, \nabla^L_1 \oplus \nabla^W_1)$$

if $n$ is even, and

$$\tilde{\eta}^{E, \Lambda}(g^E_1, \nabla^E_1, T_1^H X, g^T^V X, L_1) - \tilde{\eta}^{E, \Lambda}(g^E_0, \nabla^E_0, T_0^H X, g^T^V X, L_0)$$

$$\equiv - \text{CS}(\nabla^L_0 \oplus \nabla^W_0, \nabla^L_1 \oplus \nabla^W_1)$$

(3.2.2)

if $n$ is odd.

One can prove Proposition 3.1 along the lines of [11, Proposition 1] with the help of the analogue of [11, Lemma 1], which says the Bismut–Cheeger eta form is stable under perturbation of split quadruples. Since the proof of [11, Proposition 1] is quite involved, instead of proving Proposition 3.1 in full generality, we prove a special case, where $g^E_k$ and $\nabla^E_1$ are pulled back from a triple $(E, g^E_1, \nabla^E_1)$ over $\tilde{X}$, and the $Z_2$-graded triples $(L_k, g^L_k, \nabla^L_k)$ are pulled back from a $Z_2$-graded triple $(L, g^L, \nabla^L)$ over $\tilde{B}$ so that $L \to \tilde{B}$ satisfies the MF property for $D^{\Lambda \otimes E}$.

**Proof.** Since the space of the splitting map is affine, there exists a smooth path of horizontal distributions $\{T^H_i X \to X\}_{t \in I}$ joining $T^H_0 X \to X$ and $T^H_1 X \to X$. Let $(E, g^E_0, \nabla^E_0)$ be a triple over $\tilde{X}$. Since $i_{X,0}^* \nabla^E_1 \cong i_{X,1}^* \nabla^E_1$, we write $E = i_{X,0}^* \nabla^E_1$. Fix $k \in \{0, 1\}$. Write $g^E_k = i_{X,k}^* \nabla^E_1$ and $\nabla^E_k = i_{X,k}^* \nabla^E_1$. Then there exists a smooth path $(g^E_i, \nabla^E_i, T^H_i X, g^T^V X_i)_{t \in I}$ joining $(g^E_0, \nabla^E_0, T^H_0 X, g^T^V X_0)$ and $(g^E_1, \nabla^E_1, T^H_1 X, g^T^V X_1)$. Define a new path $\beta$ by

$$\beta(t) = \left\{ \begin{array}{ll}
(g^E_0, \nabla^E_0, T^{H_t} X, g^T^V X_t), & \text{for } t \in [0, \frac{1}{2}] \\
(g^E_{2t-1}, \nabla^E_{2t-1}, T^{H_{2t-1}} X, g^T^V X_{2t-1}), & \text{for } t \in \left[ \frac{1}{2}, 1 \right] \end{array} \right.$$ 

Note that $\beta(k) = (g^E_k, \nabla^E_k, T^H_k X, g^T^V X_k)$. The smooth path $\beta$ defines $T^H \tilde{X} \to \tilde{X}$ and $g^T^V \tilde{X}$ on $\tilde{\pi} : \tilde{X} \to \tilde{B}$. Denote by $D^{\Lambda \otimes E}$ the generalized twisted de Rham operator defined in terms of $(E, g^E_0, \nabla^E_0)$, and by $D^{\Lambda \otimes E}_k$ the generalized twisted de Rham operator defined in terms of $(E, g^E_k, \nabla^E_k)$. 


Let \((\mathcal{L}, g^L, \nabla^L)\) be a \(\mathbb{Z}_2\)-graded triple so that \(\mathcal{L} \to B\) satisfies the MF property for \(D^{\Lambda \otimes E}\). By \((3.1.6)\) we have

\[
d\eta^E,^L (g^E, g^L, T^H, g^TV) = \begin{cases} 
\int_{X/B} e(\nabla^T V) \wedge \text{ch}(\nabla^E) - \text{ch}(\nabla^L), & \text{if } n \text{ is even} \\
-\text{ch}(\nabla^L), & \text{if } n \text{ is odd}
\end{cases}
\]

Write \(L_k = i_{B,k}^* L, g^{L_k} = i_{B,k}^* g^L\) and \(\nabla^{L_k} = i_{B,k}^* \nabla^L\). Note that \(L_0 \cong L_1\) as \(\mathbb{Z}_2\)-graded complex vector bundles. By integrating the left-hand side of \((3.2.3)\) along the fibers of \(p_B : \tilde{B} \to B\) and by \((1.3.1)\), we have

\[
- \int \bar{B}/B (g^E, g^L, T^H, g^TV) = \int_{\tilde{B}/B} d\eta^E,^L (g^E, g^L, T^H, g^TV) - d \int_{\tilde{B}/B} \eta^E,^L (g^E, g^L, T^H, g^TV)
\]

\[(3.2.3)\]

Now suppose \(n\) is even. Since \(\int_{X/B} \int_{\tilde{X}/X} = \int_{\tilde{X}/X} \int_{\tilde{B}/B} \circ \int_{\tilde{X}/B} \) (see, for example, [9] Problem 3(ii), p.311]), it follows from \((3.2.3)\) that \((3.2.4)\) becomes

\[
\eta^E,^L (g^E_1, g^E_1, T^H, g^TV, L_1) - \eta^E,^L (g^E_0, g^E_0, T^H, g^TV, L_0)
\]

\[
\equiv - \int_{\tilde{B}/B} \left( \int_{\tilde{X}/\tilde{B}} e(\nabla^T V) \wedge \text{ch}(\nabla^E) - \text{ch}(\nabla^L) \right)
\]

\[
\equiv \int_{\tilde{X}/\tilde{X}} \left( \int_{\tilde{X}/\tilde{X}} -e(\nabla^T V) \wedge \text{ch}(\nabla^E) \right) - \text{CS}(\nabla^{L_0}, \nabla^{L_1})
\]

\[
\equiv \int_{X/B} e(\nabla^T V) \wedge \text{ch}(\nabla^E) + \int_{X/B} e(\nabla^T V) \wedge \text{CS}(\nabla^{E_0}, \nabla^{E_1}) = \text{CS}(\nabla^{L_0}, \nabla^{L_1})
\]

where the last equality follows from the definition of the smooth path \(\beta\).

Now suppose \(n\) is odd. By \((3.2.3), (3.2.4)\) becomes

\[
\eta^E,^L (g^E_1, g^E_1, T^H, g^TV, L_1) - \eta^E,^L (g^E_0, g^E_0, T^H, g^TV, L_0)
\]

\[
\equiv - \int_{\tilde{B}/B} -\text{ch}(\nabla^L) = -\text{CS}(\nabla^{L_0}, \nabla^{L_1}).
\]

The following results correspond to those in Section 2.1. We state it for the sake of completeness, and omit the proofs since they are formally the same as those in Section 2.1.

**Proposition 3.2.** Let \(\pi : X \to B\) be a submersion with closed fibers, and \((E, g^E, \nabla^E)\) and \((F, g^F, \nabla^F)\) are two triples. Put geometric data on \(\pi : X \to B\) as in Section A.2. If \((L_E, g^{L_E}, \nabla^{L_E})\) and \((L_F, g^{L_F}, \nabla^{L_F})\) are \(\mathbb{Z}_2\)-graded triples so that \(L_E \to B\) and \(L_F \to B\) satisfy the MF properties for \(D^{\Lambda \otimes E}\) and \(D^{\Lambda \otimes F}\), respectively, then...
respectively, then \((L_E \oplus L_F, g^L_E \oplus g^L_F, \nabla^L_E \oplus \nabla^L_F)\) is a \(Z_2\)-graded triple so that \(L_E \oplus L_F \to B\) satisfies the MF property for \(D^{\Lambda \otimes (E \oplus F)}\), and
\[
\tilde{\eta}^{E \oplus F, \Lambda}(g^E \oplus g^F, \nabla^E \oplus \nabla^F, T^H X, g^{TV, X}, L_E \oplus L_F) \\
\equiv \tilde{\eta}^{E, \Lambda}(g^E, \nabla^E, T^H X, g^{TV, X}, L_E) + \tilde{\eta}^{F, \Lambda}(g^F, \nabla^F, T^H X, g^{TV, X}, L_F).
\]

**Lemma 3.1.** Let \(\pi : X \to B\) be a submersion with closed fibers, and \((E^+, g^+, \nabla^+\) and \((E^-, g^-, \nabla^-\) are two triples. Put geometric data on \(\pi : X \to B\) as in Section \(A.2\). Let \(L_E^+ \to B\) and \(L_E^- \to B\) be \(Z_2\)-graded complex vector bundles satisfying the MF property for \(D^{\Lambda \otimes E^+}\) and \(D^{\Lambda \otimes E^-}\), respectively. Define a \(Z_2\)-graded triple \((E, g^E, \nabla^E)\) by \(E = E^+ \oplus E^-, g^E = g^+ \oplus g^-\) and \(\nabla^E = \nabla^+ \oplus \nabla^-\), and denote by \(D^\Lambda \otimes E\) the corresponding generalized twisted de Rham operator. Then \(L_E^+ \oplus L_E^- \to B\) is a \(Z_2\)-graded complex vector bundle satisfying the MF property for \(D^\Lambda \otimes E\).

**Lemma 3.2.** Let \(\pi : X \to B\) be a submersion with closed fibers, and \((E, g^E, \nabla^E)\) a \(Z_2\)-graded triple. Put geometric data on \(\pi : X \to B\) as in Section \(A.2\). If \((L, g^L, \nabla^L)\) is a \(Z_2\)-graded triple so that \(L \to B\) satisfies the MF property for \(D^\Lambda \otimes E\), then \((L^\text{op}, g^{L \text{op}}, \nabla^{L \text{op}})\) is a \(Z_2\)-graded triple so that \(L^\text{op} \to B\) satisfies the MF property for \(D^\Lambda \otimes E^\text{op}\), and
\[
\tilde{\eta}^{E \oplus F, \Lambda}(g^E, \nabla^E, T^H X, g^{TV, X}, L^\text{op}) \equiv \tilde{\eta}^{E, \Lambda}(g^E, \nabla^E, T^H X, g^{TV, X}, L).
\]

**Proposition 3.3.** Let \(\pi : X \to B\) be a submersion with closed fibers, and \((V, g^V, \nabla^V)\) a balanced \(Z_2\)-graded triple over \(X\). Put geometric data on \(\pi : X \to B\) as in Section \(A.2\). If \((L_+, g^{L_+}, \nabla^{L_+})\) is a \(Z_2\)-graded triple so that \(L_+ \to B\) satisfies the MF property for \(D^\Lambda \otimes V^+\), then there exists a balanced \(Z_2\)-graded triple \((L, g^L, \nabla^L)\) such that \(L \to B\) satisfies the MF property for \(D^\Lambda \otimes V\), and
\[
\tilde{\eta}^{V, \Lambda}(g^V, \nabla^V, T^H X, g^{TV, X}, L) \equiv 0.
\]

By Propositions 3.1, 3.2 and 3.3 and Lemmas 3.1 and 3.2 one can prove the following proposition along the lines of Proposition 2.3.

**Proposition 3.4.** Let \(\pi : X \to B\) be a submersion with closed fibers, and \((E^+, g^+, \nabla^+)\) and \((E^-, g^-, \nabla^-\) are two triples. Put geometric data on \(\pi : X \to B\) as in Section \(A.2\). Let \((L_{E^+}, g^{L_{E^+}}, \nabla^{L_{E^+}})\) and \((L_{E^-}, g^{L_{E^-}}, \nabla^{L_{E^-}})\) be \(Z_2\)-graded triples so that \(L_{E^+} \to B\) and \(L_{E^-} \to B\) satisfy the MF property for \(D^\Lambda \otimes E^+\) and \(D^\Lambda \otimes E^-\), respectively. Define \((E, g^E, \nabla^E)\) a \(Z_2\)-graded triple by \(E = E^+ \oplus E^-\), \(g^E = g^+ \oplus g^-\) and \(\nabla^E = \nabla^+ \oplus \nabla^-\). Then
\[
\tilde{\eta}^{E, \Lambda}(g^E, \nabla^E, T^H X, g^{TV, X}, L_{E^+} \oplus L_{E^-}^\text{op}) \\
\equiv \tilde{\eta}^{E^+, \Lambda}(g^+, \nabla^+, T^H X, g^{TV, X}, L_{E^+}) - \tilde{\eta}^{E^-, \Lambda}(g^-, \nabla^-, T^H X, g^{TV, X}, L_{E^-}).
\]

3.3. **An analytic index in differential K-theory.** We define the analytic index in differential K-theory associated to generalized twisted de Rham operators.
Definition 3.1. Let $\pi: X \to B$ be a submersion with closed fibers of dimension $n$ and $E$ a $\mathbb{Z}_2$-graded generator of $\hat{K}_{FL}(X)$. Put geometric data on $\pi: X \to B$ as in Section A.2. If $D^{A\otimes E}$ satisfies the kernel bundle assumption, define

$$\text{ind}^a_{\hat{K}}(E) = \left( \ker(D^{A\otimes E}), g_{\ker(D^{A\otimes E})}, \nabla_{\ker(D^{A\otimes E})}, \int_{X/B} e(\nabla^T V_X) \wedge \omega + \tilde{\eta}^{E,A}(g^E, \nabla^E, T^H X, g^T V_X) \right).$$

In the general case, i.e. $D^{A\otimes E}$ does not satisfy the kernel bundle assumption, define

$$\text{ind}^a_{\hat{K}}(E; \mathcal{L}) = \left( \mathcal{L}, g_{\mathcal{L}}, \nabla_{\mathcal{L}}, \int_{X/B} e(\nabla^T V_X) \wedge \omega + \tilde{\eta}^{E,A}(g^E, \nabla^E, T^H X, g^T V_X, \mathcal{L}) \right).$$

(3.3.1)

If $n$ is odd, the term $\int_{X/B} e(\nabla^T V_X) \wedge \omega$ vanishes.

Proposition 3.5. Let $\pi: X \to B$ be a submersion with closed fibers of dimension $n$ and $E$ a $\mathbb{Z}_2$-graded generator of $\hat{K}_{FL}(X)$. Put geometric data on $\pi: X \to B$ as in Section A.2.

(a) The analytic index $\text{ind}^a_{\hat{K}}: \hat{K}_{FL}(X) \to \hat{K}_{FL}(B)$ is a well defined group homomorphism.

(b) If $(L_0, g^{L_0}, \nabla^{L_0})$ and $(L_1, g^{L_1}, \nabla^{L_1})$ are $\mathbb{Z}_2$-graded triples so that $L_0 \to B$ and $L_1 \to B$ satisfy the MF property for $D^{A\otimes E}$, then

$$\text{ind}^a_{\hat{K}}(E; L_0) = \text{ind}^a_{\hat{K}}(E; L_1).$$

(c) Suppose $D^{A\otimes E}$ satisfies the kernel bundle assumption. If $(L, g^L, \nabla^L)$ is a $\mathbb{Z}_2$-graded triple so that $L \to B$ satisfies the MF property for $D^{A\otimes E}$, then

$$\text{ind}^a_{\hat{K}}(E; L) = \text{ind}^a_{\hat{K}}(E).$$

Note that (a) of Proposition 3.5 can be proved along the lines of Proposition 2.5 by using Propositions 3.1, 3.2 and Lemma 3.1. Also, (b) of Proposition 3.5 is an immediate consequence of Proposition 3.1 and it can be proved along the lines of [11, Corollary 1]. Similarly, (c) of Proposition 3.5 can be proved along the lines of [11, Corollary 2]. We still give a proof of (c) since part of the arguments will be used to prove Theorem 3.1.

Proof. Since $\ker(D^{A\otimes E}) \to B$ exists, it satisfies the MF property for $D^{A\otimes E}$. By (b) of Proposition 3.5 we have $\text{ind}^a_{\hat{K}}(E; \mathcal{L}) = \text{ind}^a_{\hat{K}}(E; \ker(D^{A\otimes E}))$. Thus it suffices to show that

$$\text{ind}^a_{\hat{K}}(E; \ker(D^{A\otimes E})) = \text{ind}^a_{\hat{K}}(E).$$

(3.3.2)
To show (3.3.2), it suffices to show that

\[ \tilde{\eta}^{E,A}(g^E, \nabla^E, THX, g^{TVX}, \ker(\mathcal{D}^{\Lambda \otimes \hat{E}})) \equiv \tilde{\eta}^{E,A}(g^E, \nabla^E, THX, g^{TVX}) \]  

(3.3.3)

regardless of the parity of \( n \). Recall that the Bismut–Cheeger eta form on the left-hand side of (3.3.3) is associated to the Bismut superconnection \( \overline{B}^{E,A} \) (cf. (3.1.5)) with

\[ \overline{B}^{E,A}_{[1]} = \nabla^{E,h} = \nabla^{E,h} \oplus \nabla^{\ker(\mathcal{D}^{\Lambda \otimes \hat{E}}),op}. \]

Choose and fix \( t, T \in (0, \infty) \) that satisfy \( T \geq 1 \) and \( t < a \). Since \( \overline{B}^{E,A}_T = B^{E,A}_t \oplus \nabla^{\ker(\mathcal{D}^{\Lambda \otimes \hat{E}}),op} \), it follows from (A.1.5), (A.1.3) and (A.1.4) that

\[
\begin{align*}
\text{CS}(\overline{B}^{E,A}_T, \overline{B}^{E,A}_t) &= \text{CS}(B^{E,A}_T, B^{E,A}_t) \\
&\equiv \text{CS}(B^{E,A}_T, B^{E,A}_t) - \text{CS}(B^{E,A}_T, B^{E,A}_t) - \text{CS}(\nabla^{\ker(\mathcal{D}^{\Lambda \otimes \hat{E}}),op}, B^{E,A}_t) \\
&\equiv \text{CS}(B^{E,A}_T, B^{E,A}_t) + \text{CS}(\nabla^{\ker(\mathcal{D}^{\Lambda \otimes \hat{E}}),op}, B^{E,A}_t) \\
&\equiv \text{CS}(B^{E,A}_T, B^{E,A}_t) + \text{CS}(\nabla^{\ker(\mathcal{D}^{\Lambda \otimes \hat{E}}),op}, B^{E,A}_t).
\end{align*}
\]

By letting \( T \to \infty \) and \( t \to 0 \) in above, we have

\[ \tilde{\eta}^{E,A}(g^E, \nabla^E, THX, g^{TVX}, \ker(\mathcal{D}^{\Lambda \otimes \hat{E}})) - \tilde{\eta}^{E,A}(g^E, \nabla^E, THX, g^{TVX}) \]

\[ \equiv \lim_{T \to \infty} \text{CS}(\overline{B}^{E,A}_T, B^{E,A}_t) \oplus \nabla^{\ker(\mathcal{D}^{\Lambda \otimes \hat{E}}),op}. \]

By the estimates in [2] §9.3 we have

\[ \lim_{T \to \infty} \text{CS}(\overline{B}^{E,A}_T, B^{E,A}_t) = 0. \]

Thus (3.3.3) holds. □

Let \( \pi : X \to B \) be a submersion with closed fibers. For a given \( \mathbb{Z}_2 \)-graded generator \( \mathcal{E} \) of \( K^{-1}_L(X) \), define \( \ind_{\mathbb{R}/\mathbb{Z}}^{E,A}(\mathcal{E}) \) by the right-hand side of (3.3.1). The \( \mathbb{Z}_2 \)-graded version of (3.1.6) guarantees that \( \ind_{\mathbb{R}/\mathbb{Z}}^{E,A}(\mathcal{E}) \) is a \( \mathbb{Z}_2 \)-graded generator of \( K^{-1}_L(B) \). Proposition 3.5 applies to \( \ind_{\mathbb{R}/\mathbb{Z}}^{E,A} \) as well.

We now prove Proposition 1.1

**Proposition 3.6.** Let \( \pi : X \to B \) be a submersion with closed, oriented and spin' fibers of even dimension. Choose and fix a spin' structure on \( TVX \to X \), and put geometric data on \( \pi : X \to B \) as in Section A.2. For any \( \mathbb{Z}_2 \)-graded generator \( \mathcal{E} \) of \( K_{FL}(X) \) we have

\[ \ind_{\mathbb{R}}^{E,A}(\mathcal{E}) = \ind_{\mathbb{R}}^{E,A}(S(TVX)^\ast \otimes \mathcal{E}), \]

(3.3.4)

where \( S(TVX)^\ast \) is given by (A.5.7).
Proof: Write $\mathcal{E} = (E, g^E, \nabla^E, \omega)$. By [A.5.3] we have

$$S(T^V X)^* \otimes E = (S(T^V X)^r \otimes E, g^E, \nabla^E, \nabla^S(T^V X)^r \otimes E, \text{ch}(\nabla^S(T^V X)^r) \wedge \omega),$$

where $\nabla^S(T^V X)^r \otimes E$ is the tensor product of $\nabla^S(T^V X)^r$ and $\nabla^E$. By [A.5.6], $\text{ind}_{K}^a(S(T^V X)^* \otimes E)$ is given by

$$\text{ind}_{K}^a(S(T^V X)^* \otimes E) = \left( H, g^H, \nabla^H, \int_{X/B} \text{Todd}(\nabla^T^V X) \wedge \omega, \right.$$ 

$$+ \tilde{\eta}^S(T^V X)^r \otimes E (g^S(T^V X)^r \otimes g^E, \nabla^S(T^V X)^r \otimes E, T^H X, g^{T^V X}, H) \bigg),$$

(3.3.5)

where $(H, g^H, \nabla^H)$ is a $\mathbb{Z}_2$-graded triple so that $H \to X$ satisfies the MF property for $D^S \otimes \otimes E$.

By (2.3.12), we have

$$S(T^V X) \otimes S(T^V X)^* \otimes E \cong \Lambda(T^V X)^* \otimes E$$

(3.3.6)
as $\mathbb{Z}_2$-graded complex vector bundles. Under the isomorphism $[3.3.6]$, the Clifford multiplication on $S(T^V X) \to X$ corresponds to that on $\Lambda(T^V X)^r \to X$. By (2.3.13), the tensor product of $\nabla^S(T^V X)$ and $\nabla^S(T^V X)^r$ is equal to that of $\nabla^\Lambda(T^V X)^r$ and $\nabla^E$. Thus

$$D^\Lambda \otimes E = D^S \otimes \otimes E.$$ (3.3.7)

Since (3.3.6) implies

$$\pi^\Lambda_* E \equiv \pi_* (S(T^V X)^* \otimes E)$$

(3.3.8)
as $\mathbb{Z}_2$-graded complex vector bundles, it follows that $H \to B$ also satisfies the MF property for $D^\Lambda \otimes E$. By (b) of Proposition [3.5], $\text{ind}_{K}^a(\mathcal{E})$ is given by

$$\text{ind}_{K}^a(\mathcal{E}) = \left( H, g^H, \nabla^H, \int_{X/B} e(\nabla^T^V X) \wedge \omega + \tilde{\eta}^E, g^E, \Lambda(T^V X)^r \otimes g^E, T^H X, \nabla^T^V X, H) \bigg).$$

(3.3.9)

By [3.3.5] and [3.3.6], to prove (3.3.4), it suffices to show that

$$\int_{X/B} e(\nabla^T^V X) \wedge \omega + \tilde{\eta}^E, g^E, \Lambda(T^V X)^r \otimes g^E, T^H X, \nabla^T^V X, H) \equiv \int_{X/B} \text{Todd}(\nabla^T^V X) \wedge \omega \wedge H, \nabla^S(T^V X)^r \otimes E, T^H X, g^{T^V X}, H).$$

(3.3.10)

Under the isomorphism (3.3.8), we have $g^\pi^\Lambda_* E = g^\pi_* (S(T^V X)^* \otimes E)$ and $\nabla^\pi^\Lambda_* E, u = \nabla^\pi_* (S(T^V X)^* \otimes E), u$ (cf. (2.3.15)). Thus

$$\nabla^\pi_{(S(T^V X)^* \otimes E), u} = \nabla^\pi_* (S(T^V X)^* \otimes E), u \otimes \nabla^H, op = \nabla^\pi^\Lambda_* E, u \otimes \nabla^H, op = \nabla^\pi^\Lambda^E, u.$$ (3.3.11)

Thus (3.3.7) and (3.3.11) show that $\nabla^E, \Lambda = \nabla^S(T^V X)^r \otimes E$, which implies

$$\tilde{\eta}^S(T^V X)^r \otimes E (g^S(T^V X)^r \otimes g^E, \nabla^S(T^V X)^r \otimes E, T^H X, g^{T^V X}, H) \equiv \tilde{\eta}^E, \Lambda (g^E, \nabla^E, T^H X, g^{T^V X}, H).$$
On the other hand, by (A.3.3) and [1, (8.30)] we have

$$\text{Todd}(\nabla T^V X) \wedge \text{ch}(\nabla S(T^V X))^\vee) = A(\nabla T^V X) \wedge e^{\frac{1}{2}c_1(\nabla^i)} \wedge \text{ch}(\nabla S(T^V X)) \wedge e^{-\frac{1}{2}c_1(\nabla^i)}$$

$$= A(\nabla T^V X) \wedge \frac{e(\nabla T^V X)}{A(\nabla T^V X)}$$

$$= e(\nabla T^V X).$$

Thus (3.3.10) holds. \(\square\)

### 3.4. A proof of Theorem 1.1

We now prove Theorem 1.1.

**Theorem 3.1.** Let \(\pi : X \to B\) be a submersion with closed fibers \(Z\) of dimension \(n\). Put geometric data on \(\pi : X \to B\) as in Section A.2. For any \(\mathbb{Z}_2\)-graded generator \(F\) of \(K_L^{-1}(X)\) of the form (2.3.1) we have

$$\text{ind}_{B/Z}^e(F) = \mathcal{H}(Z,F|_Z).$$

**Proof.** Let \(F\) have the form (2.3.1). Define a \(\mathbb{Z}_2\)-graded generator \(p_X^* F = (\mathcal{F}, g, \nabla F, 0)\) of \(K_L^{-1}(X)\) by \(\mathcal{F} := p_X^* F, g = p_X^* g^F\) and \(\nabla F = p_X^* \nabla^F\).

Pull the geometric data on \(\pi : X \to B\) back to \(\pi : \tilde{X} \to \tilde{B}\). Let \(\nabla^\Lambda(T^V \tilde{X}) \otimes \tilde{F}\) be the tensor product of \(\nabla^\Lambda(T^V \tilde{X})\) and \(\nabla \bar{F}\). Let \(\tilde{e}_1, \ldots, \tilde{e}_n\) be a local orthonormal frame for \(T^V \tilde{X} \to \tilde{X}\). Denote by \(D^\Lambda \tilde{\phi}\) the generalized twisted de Rham operator. Let

$$D = D^\Lambda \tilde{\phi} + i \tilde{V},$$

where \(t \in I\) and \(\tilde{V}\) is given by (A.4.6) adapted to the current setting. Note that \(D\) is an odd self-adjoint operator acting on \(\Gamma(\tilde{X}, \Lambda(T^V \tilde{X})^* \otimes \tilde{F})\).

Let \((\mathcal{L}, g, \nabla \bar{F})\) be a \(\mathbb{Z}_2\)-graded triple so that \(\mathcal{L} \to \tilde{B}\) satisfies the MF property for \(D\), i.e. there exist complementary closed subbundles \(K^\pm \to \tilde{B}\) of \((\pi^* \mathcal{F})^\pm \to \tilde{B}\) such that

\[
\begin{align*}
(p^* \mathcal{F})^+ &= K^+ \oplus L^+, \\
(p^* \mathcal{F})^- &= K^- \oplus L^-,
\end{align*}
\]

\(D^+\) is block diagonal as a map with respect to (3.4.2), and \(D^+_a : K^+ \to K^-\) is an isomorphism. Write \(K \to \tilde{B}\) for the \(\mathbb{Z}_2\)-graded complex vector bundle defined by \(\tilde{K} = K^+ \oplus K^-\). Note that

$$i^*_B \alpha_i \mathcal{F} = \pi^*_B F$$

as \(\mathbb{Z}_2\)-graded complex vector bundles over \(B\) for \(j \in \{0, 1\}\). Write \(L_j \to B\) for \(i^*_B \mathcal{L} \to B\) and \(K_j \to B\) for \(i^*_B \mathcal{K} \to B\). Note that \(K_0 \cong K_1\) and

$$L_0 \cong L_1$$

as \(\mathbb{Z}_2\)-graded complex vector bundles. Write \(g^{L_j}\) for \(i^*_B g^L\) and \(\nabla^{L_j}\) for \(i^*_B \nabla^L\).
That \( \mathcal{L} \rightarrow \widetilde{B} \) satisfies the MF property for \( D \) implies
\[
L_0 \rightarrow B \text{ satisfies the MF property for } D^{\Lambda \otimes F}, \quad \text{and} \quad (3.4.4)
\]
\[
L_1 \rightarrow B \text{ satisfies the MF property for } D^{\Lambda \otimes F} + V. \quad \text{(3.4.5)}
\]

By applying Proposition 3.1 to \( D, (L_0, g^{L_0}, \nabla^{L_0}) \) and \( (L_1, g^{L_1}, \nabla^{L_1}) \), and by noting (3.4.3), we have
\[
\widetilde{\eta}^{F,A}(g^F, \nabla^{F,u}, T^H X, g^{T^V X}, L_1) - \widetilde{\eta}^{F,A}(g^F, \nabla^{F,u}, T^H X, g^{T^V X}, L_0) \equiv - \text{CS}(\nabla^{L_0}, \nabla^{L_1}) \quad \text{(3.4.6)}
\]

regardless of the parity of \( n \). Note that the Bismut–Cheeger eta form
\[
\widetilde{\eta}^{F,A}(g^F, \nabla^{F,u}, T^H X, g^{T^V X}, L_1)
\]
in (3.4.6) is associated to the Bismut superconnection \( \widetilde{B}^{F,A} \) with
\[
\widetilde{B}^{F,A}_{[0]} = D^{F}(1) = \begin{pmatrix} D^{\Lambda \otimes F} + V & i^- \\ p^+ & 0 \end{pmatrix}
\]
(cf. (3.1.5)). By (3.4.4) and (b) of Proposition 3.5 we have
\[
\text{ind}_{\mathbb{R}/Z}(\mathcal{F}) = \left( L_0, g^{L_0}, \nabla^{L_0}, \widetilde{\eta}^{F,A}(g^F, \nabla^{F,u}, T^H X, g^{T^V X}, L_0) \right). \quad \text{(3.4.7)}
\]

By (3.4.7) and (2.3.2), to prove (3.4.1) it suffices to show that there exist balanced \( \mathbb{Z}_2 \)-graded triples \( (W_0, g^{W_0}, \nabla^{W_0}) \) and \( (W_1, g^{W_1}, \nabla^{W_1}) \) such that
\[
L_0 \oplus W_0 \cong H(Z, F|_Z) \oplus W_1 \quad \text{(3.4.8)}
\]
as \( \mathbb{Z}_2 \)-graded complex vector bundles, and
\[
\eta^{F,A}(g^F, \nabla^{F,u}, T^H X, g^{T^V X}, L_0) \equiv \text{CS}(\nabla^{L_0} \oplus \nabla^{W_0}, \nabla^{H(Z,F|_Z),u} \oplus \nabla^{W_1}). \quad \text{(3.4.9)}
\]

It follows from (A.4.5) and (A.4.3) that the \( \mathbb{Z}_2 \)-graded complex vector bundle \( H(Z, F|_Z) \rightarrow B \) also satisfies the MF property for \( D^{\Lambda \otimes F} + V \). Note that \( g^{H(Z,F|_Z)} \) is the projected \( \mathbb{Z}_2 \)-graded Hermitian metric on \( H(Z, F|_Z) \rightarrow B \), and \( \nabla^{H(Z,F|_Z),u} \) is the projected \( \mathbb{Z}_2 \)-graded unitary connection on \( H(Z, F|_Z) \rightarrow B \) by (A.4.4). By applying Proposition 3.1 to \( D^{\Lambda \otimes F} + V, (L_1, g^{L_1}, \nabla^{L_1}) \) and \( (H(Z, F|_Z), g^{H(Z,F|_Z)}, \nabla^{H(Z,F|_Z),u}) \) there exist balanced \( \mathbb{Z}_2 \)-graded triples \( (W_0, g^{W_0}, \nabla^{W_0}) \) and \( (W_1, g^{W_1}, \nabla^{W_1}) \) such that
\[
L_1 \oplus W_0 \cong H(Z, F|_Z) \oplus W_1 \quad \text{(3.4.10)}
\]
as \( \mathbb{Z}_2 \)-graded complex vector bundles, and
\[
\eta^{F,A}(g^F, \nabla^{F,u}, T^H X, g^{T^V X}, (H(Z, F|_Z))) - \eta^{F,A}(g^F, \nabla^{F,u}, T^H X, g^{T^V X}, L_1) \equiv - \text{CS}(\nabla^{L_1} \oplus \nabla^{W_0}, \nabla^{H(Z,F|_Z),u} \oplus \nabla^{W_1}) \quad \text{(3.4.11)}
\]
regardless of the parity of \( n \). By (3.4.3), (3.4.10) becomes
\[
L_0 \oplus W_0 \cong H(Z, F|_Z) \oplus W_1,
\]
which is (3.4.8). By adding (3.4.6) to (3.4.11), we have
\[ \tilde{\eta}^{F,A}(g^F, \nabla^{F,u}, T^H X, g^{T^V X}, H(Z,F|_Z)) - \tilde{\eta}^{F,A}(g^F, \nabla^{F,u}, T^H X, g^{T^V X}, L_0) \equiv -\mathrm{CS}(V_{L_0}, \nabla^{L_1}) - \mathrm{CS}(V_{L_1} \oplus V_{W_0}, \nabla^{H(Z,F|_Z),u} \oplus \nabla^{W_1}). \] (3.4.12)

By (2.3.19), (3.4.12) becomes
\[ \tilde{\eta}^{F,A}(g^F, \nabla^{F,u}, T^H X, g^{T^V X}, H(Z,F|_Z)) - \tilde{\eta}^{F,A}(g^F, \nabla^{F,u}, T^H X, g^{T^V X}, L_0) \equiv -\mathrm{CS}(V_{L_0} \oplus V_{W_0}, \nabla^{H(Z,F|_Z),u} \oplus \nabla^{W_1}). \]

Thus, to prove (3.4.9) it suffices to show that
\[ \tilde{\eta}^{F,A}(g^F, \nabla^{F,u}, T^H X, g^{T^V X}, H(Z,F|_Z)) \equiv 0. \] (3.4.13)

To prove (3.4.13), denote by B^{F,A} the Bismut superconnection with B^{F,A}_{[0]} = D^{\Lambda \otimes F} + V (cf. (3.1.1)). By (A.4.5) we have B^{F,A} = D^{dR}. Denote by \eta^{F,A} the Bismut–Cheeger eta form associated to B^{F,A}. By (A.4.3) and (A.4.5), the definition of \hat{g}(H(Z,F|_Z), V\ker(D^{\Lambda \otimes F} + V)) as \mathbb{Z}_2-graded triples. Thus
\[ \tilde{\eta}^{F,A}(g^F, \nabla^{F,u}, T^H X, g^{T^V X}, H(Z,F|_Z)) \] \[ \equiv \eta^{F,A}(g^F, \nabla^{F,u}, T^H X, g^{T^V X}, \ker(D^{\Lambda \otimes F} + V)) \] \[ \equiv \eta^{dR} = 0, \]
where the second equality follows from (3.3.3) and the forth equality follows from (A.4.7). \( \square \)

3.5. The RRG theorem for \( \text{ind}^{a\Lambda}_{\mathbb{R}/\mathbb{Z}} \) in \( \mathbb{R}/\mathbb{Z} \) K-theory. We now prove Theorem 1.2

**Theorem 3.2.** Let \( \pi : X \to B \) be a submersion with closed fibers \( Z \) of dimension \( n \). By putting geometric data on \( \pi : X \to B \) as in Section A.2, the following diagram commutes
\[ \begin{array}{ccc}
K^{-1}_{-1}(X) & \xrightarrow{\text{ch}_{\mathbb{R}/\mathbb{Q}}} & H^{\text{odd}}(X; \mathbb{R}/\mathbb{Q}) \\
\text{ind}^{a\Lambda}_{\mathbb{R}/\mathbb{Z}} & & \\
K^{-1}_{-1}(B) & \xrightarrow{\text{ch}_{\mathbb{R}/\mathbb{Q}}} & H^{\text{odd}}(B; \mathbb{R}/\mathbb{Q})
\end{array} \] (3.5.1)

**Proof.** Let \( E = (E, g^E, \nabla^E, \omega) \) be a \( \mathbb{Z}_2 \)-graded generator of \( K^{-1}_{-1}(X) \). Denote by \( D^{\Lambda \otimes E^+} \) and \( D^{\Lambda \otimes E^-} \) the generalized twisted de Rham operator associated to \( (E^+, g^+, \nabla^+) \) and \( (E^-, g^-, \nabla^-) \). Let \( (L_{E^+}, g^{L_{E^+}}, \nabla^{L_{E^+}}) \) and \( (L_{E^-}, g^{L_{E^-}}, \nabla^{L_{E^-}}) \) be
$\mathbb{Z}_2$-graded triples so that $L_{E^+} \to B$ and $L_{E^-} \to B$ satisfy the MF property for $D^{\Lambda \otimes \mathbb{E}}$ and $D^{\Lambda \otimes \mathbb{E}^*}$, respectively.

Denote by $D^{\Lambda \otimes \mathbb{E}}$ the generalized twisted de Rham operator associated to $(E, g^E, \nabla^E)$. By Lemma 3.1 $(L_{E^+} \oplus L_{E^-}^{\text{op}}, g^{L_{E^+}} \oplus g^{L_{E^-}^{\text{op}}}, \nabla^{L_{E^+}} \oplus \nabla^{L_{E^-}^{\text{op}}})$ is a $\mathbb{Z}_2$-graded triple so that $L_{E^+} \oplus L_{E^-}^{\text{op}} \to B$ satisfies the MF property for $D^{\Lambda \otimes \mathbb{E}}$. By (b) of Proposition 3.5 we have

$$\text{ind}^{\varphi, \Lambda}_{\mathbb{R}/\mathbb{Z}}(\mathcal{E}) = \left( L_{E^+} \oplus L_{E^-}^{\text{op}}, g^{L_{E^+}} \oplus g^{L_{E^-}^{\text{op}}}, \nabla^{L_{E^+}} \oplus \nabla^{L_{E^-}^{\text{op}}}, \int_{X/B} e(\nabla^{TV} X) \wedge k \text{CS}(k \nabla^{E^-}, k \nabla^{E^+} + \omega) \right).$$

As in the proof of Theorem 2.2 there exists a $k \in \mathbb{N}$ such that

$$kE^+ \cong kE^-, \quad kL_{E^+} \cong kL_{E^-}, \quad k(L_{E^+} \oplus L_{E^-}^{\text{op}})^+ \cong k(L_{E^+} \oplus L_{E^-}^{\text{op}}) -.$$ (3.5.2)

By the left-most isomorphism in (3.5.2), a differential form representative of

$$\int_{X/B} e(T^V X) \cup \text{ch}_{\mathbb{R}/\mathbb{Q}}(\mathcal{E})$$

is

$$\int_{X/B} e(\nabla^{TV} X) \wedge \left( \frac{1}{k} \text{CS}(k \nabla^{E^-}, k \nabla^{E^+} + \omega) \right).$$ (3.5.3)

and by the right-most isomorphism in (3.5.2), a differential form representative of $\text{ch}_{\mathbb{R}/\mathbb{Q}}(\text{ind}^{\varphi, \Lambda}_{\mathbb{R}/\mathbb{Z}}(\mathcal{E}))$ is

$$\frac{1}{k} \text{CS}(k(\nabla^{L_{E^+}^{-}} \oplus \nabla^{L_{E^-}^{+}}), k(\nabla^{L_{E^+}^{+}} \oplus \nabla^{L_{E^-}^{-}})) + \int_{X/B} e(\nabla^{TV} X) \wedge \omega.$$ (3.5.4)

By (3.5.3) and (3.5.4), to show diagram (3.5.1) commutes it suffices to prove that

$$k \tilde{\eta}^{E, \Lambda}(g^E, \nabla^E, T^H X, g^{TV} X, L_{E^+} \oplus L_{E^-}^{\text{op}}) = \int_{X/B} e(\nabla^{TV} X) \wedge \text{CS}(k \nabla^{E^-}, k \nabla^{E^+}) - \text{CS}(k(\nabla^{L_{E^+}^{-}} \oplus \nabla^{L_{E^-}^{+}}), k(\nabla^{L_{E^+}^{+}} \oplus \nabla^{L_{E^-}^{-}})).$$ (3.5.5)

up to closed odd forms on $B$ with periods in $\mathbb{Q}$.

By applying Proposition 3.1 to $(kE^+, k g^{E^+}, k \nabla^{E^+})$ and $(kE^-, k g^{E^-}, k \nabla^{E^-})$, and by noting the middle isomorphism in (3.5.2), we have

$$\tilde{\eta}^{kE^+, \Lambda}(k g^{E^+}, k \nabla^{E^+}, T^H X, g^{TV} X, k L_{E^+}) - \tilde{\eta}^{kE^-, \Lambda}(k g^{E^-}, k \nabla^{E^-}, T^H X, g^{TV} X, k L_{E^-})$$

$$\equiv \int_{X/B} e(\nabla^{TV} X) \wedge \text{CS}(k \nabla^{E^-}, k \nabla^{E^+}) - \text{CS}(k \nabla^{L_{E^+}^{+}}, k \nabla^{L_{E^-}^{-}}).$$ (3.5.6)
By Propositions 3.4 and 3.2 we have

\[ \eta^{kE,A}(g^E, k\nabla^E, T^H_X, g^{TV_X}, kL^E) - \eta^{kE,A}(g^E, k\nabla^E, T^H_X, g^{TV_X}, kL_E^+ - kL_E^-) \equiv \eta^{kE,A}(g^E, k\nabla^E, T^H_X, g^{TV_X}, k(L_E^+ \oplus L_E^-)) \]

\[ \equiv k\eta^{E,A}(g^E, \nabla^E, T^H_X, g^{TV_X}, L_E^+ \oplus L_E^-). \]

On the other hand, as in the proof Theorem 2.2, we have

\[ \text{CS}(k\nabla^L_E, k\nabla^L_E) \equiv \text{CS}(k\nabla^L_E, - + k\nabla^L_E, + - k\nabla^L_E, - + k\nabla^L_E). \]

Thus (3.5.6) reduces to (3.5.5). \[ \square \]

Note that (3.5.5) is the refinement of Theorem 3.2 at the differential form level.

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Appendix

In this Appendix we review all the necessary background material. In Section A.1 we recall the definitions of the Chern character form, the Chern–Simons form, the $\hat{A}$-genus form, the Euler form, and the Cheeger–Chern–Simons class for complex flat vector bundles, and fix sign convention. In Section A.2 we fix the notations of the geometric data that are put on a submersion. In Sections A.3 and A.4 we review the statements of the local index theory for twisted spin$^c$ Dirac operators and twisted de Rham operators, respectively. In Section A.5 we recall Freed–Lott differential $K$-theory and its analytic index, and regard $\mathbb{R}/\mathbb{Z} K$-theory and its analytic index as special cases.

A.1. Some primary and secondary characteristic forms. Let $E \to X$ be a $\mathbb{Z}_2$-graded complex vector bundle with a superconnection $\mathcal{A}^E$. The Chern character form of $\mathcal{A}^E$ is defined by
\[
\text{ch}(\mathcal{A}^E) = \text{str}(e^{-\frac{1}{2\pi i}(\mathcal{A}^E)^2}) \in \Omega^{\text{even}}(X; \mathbb{C}).
\]
Given two superconnections $\mathcal{A}_0^E$ and $\mathcal{A}_1^E$ on $E \to X$, let $\mathcal{A}_t^E$ be a smooth curve of superconnections joining $\mathcal{A}_0^E$ and $\mathcal{A}_1^E$. Define a $\mathbb{Z}_2$-graded complex vector bundle $\mathcal{E} \to \tilde{X}$ by $\mathcal{E} = p^*XE$. Then
\[
\mathcal{A}^E = \mathcal{A}_t^E + dt \wedge \frac{\partial}{\partial t}
\]
is a superconnection on $\mathcal{E} \to \tilde{X}$. The Chern–Simons form $\text{CS}(\mathcal{A}_0^E, \mathcal{A}_1^E) \in \Omega^{\text{odd}}(X; \mathbb{C})/\text{Im}(d)$ is defined to be
\[
\text{CS}(\mathcal{A}_0^E, \mathcal{A}_1^E) = -\int_{\tilde{X}/X} \text{ch}(\mathcal{A}^E). \tag{A.1.1}
\]
Note that $\text{CS}(\mathcal{A}_0^E, \mathcal{A}_1^E)$ is independent of the choice of $\mathcal{A}_t^E$ joining $\mathcal{A}_0^E$ and $\mathcal{A}_1^E$ [14, Theorem B.5.4], and satisfies
\[
d \text{CS}(\mathcal{A}_0^E, \mathcal{A}_1^E) = \text{ch}(\mathcal{A}_1^E) - \text{ch}(\mathcal{A}_0^E).
\]
In the case that $E \to X$ is equipped with $\mathbb{Z}_2$-graded Hermitian metrics $g_0$ and $g_1$, and $\mathcal{A}_k^E$ is unitary with respect to $g_k$ for $k \in \{0, 1\}$, we refer to [14, (B.5.21)] for the corresponding construction of $\mathcal{A}^E$.

Another equivalent definition of the Chern–Simons form is given by
\[
\text{CS}(\mathcal{A}_0^E, \mathcal{A}_1^E) = -\frac{1}{2\pi i} \int_0^1 \text{str}\left(\frac{d\mathcal{A}_s^E}{ds} e^{-\frac{1}{2\pi i}(\mathcal{A}_s^E)^2}\right) ds. \tag{A.1.2}
\]
The choices of 0 and 1 are immaterial. If $t < T$ are two fixed positive real numbers, then one can replace 0 by $t$ and 1 by $T$ in (A.1.2).
It follows from (A.1.1) that the Chern–Simons form satisfies the following properties:

\[ \text{CS}(A^E_1, A^E_0) = - \text{CS}(A^E_0, A^E_1), \]  
\[ \text{CS}(A^E_1, A^E_0) = \text{CS}(A^E_1, A^E_2) + \text{CS}(A^E_2, A^E_0), \]  
\[ \text{CS}(A^E_1 \oplus A^E_0, A^E_0 \oplus A^E_0) = \text{CS}(A^E_1, A^E_0) + \text{CS}(A^E_1, A^E_0), \]  
\[ \text{CS}(\nabla^E_1, \nabla^E_0) = \text{CS}(\nabla^E_1, \nabla^E_0) - \text{CS}(\nabla^E_1, \nabla^E_0), \]

where \( \nabla^E_1 \) and \( \nabla^E_0 \) are \( \mathbb{Z}_2 \)-graded connections on \( E \to X \).

Let \( H \to X \) be a real vector bundle of rank \( k \) with a Euclidean metric \( g^H \) and a Euclidean connection \( \nabla^H \). Denote by \( R^H \) the curvature of \( \nabla^H \). The \( \hat{A} \)-genus form of \( \nabla^H \) is defined to be

\[ \hat{A}(\nabla^H) = \sqrt{\det \left( \frac{-1}{4\pi R^H} \sinh \left( -\frac{1}{4\pi R^H} \right) \right)} \in \Omega^4_{\mathbb{Q}}(X), \]

Denote by \( o(H) \to X \) the orientation bundle of \( H \to X \). It is a flat real line bundle, which is trivial if and only if \( H \to X \) is oriented. Let \( \text{Pf} \) be the Pfaffian. The Euler form of \( \nabla^H \) is defined to be

\[ e(\nabla^H) = \begin{cases} 
\text{Pf} \left( \frac{R^H}{2\pi} \right), & \text{if } k \text{ is even} \\
0, & \text{if } k \text{ is odd}
\end{cases} \]

Note that \( e(\nabla^H) \in \Omega^k(X, o(H)) \). Write \( e(H) = [e(\nabla^H)] \in H^k(X; o(H)) \) for the Euler class of \( H \to X \). If \( \nabla^H_0 \) and \( \nabla^H_1 \) are two Euclidean connections on \( H \to X \), one can define a transgression form \( \tilde{e}(\nabla^H_0, \nabla^H_1) \in \frac{\Omega^{k-1}(X)}{\text{Im}(d)} \) by

\[ \tilde{e}(\nabla^H_0, \nabla^H_1) = - \int_{\tilde{X}/X} e(\nabla^H), \]

where \( \nabla^H \) is a Euclidean connection on \( \mathcal{H} \to \tilde{X} \) given by \[14, (B.5.21)]\]. Note that

\[ d\tilde{e}(\nabla^H_0, \nabla^H_1) = e(\nabla^H_1) - e(\nabla^H_0). \]

Let \( F \to X \) be a complex flat vector bundle with flat connection \( \nabla^F \). Put a Hermitian metric \( g^F \) on \( F \to X \). Define

\[ \omega(F, g^F) := (g^F)^{-1}(\nabla^F g^F) \in \Omega^1(X, \text{End}(F)). \]

The connection on \( F \to X \) defined by

\[ \nabla^{F,u} = \nabla^F + \frac{1}{2} \omega(F, g^F) \]  
(A.1.7)

is unitary with respect to \( g^F \) and has curvature \( (\nabla^{F,u})^2 = \frac{1}{4} \omega(F, g^F)^2 \). Since \( \text{tr}(\omega(F, g^F)^{2k}) = 0 \) for any \( k \in \mathbb{N} \), it follows that

\[ \text{ch}(\nabla^{F,u}) = \text{rank}(F). \]  
(A.1.8)
Remark A.1. Let \((E, g^E, \nabla^E)\) and \((F, g^F, \nabla^F)\) be two triples which satisfy \(E \cong F\). Let \(f : E \to F\) be a smooth bundle isomorphism. By [12, Theorem 8.8 of Chapter 1] one can always assume \(f\) to be an isometric isomorphism, i.e. \(g^E = f^* g^F\). Thus \(f^* \nabla^F\) is a unitary connection on \(E \to X\).

Henceforth, under such a situation, we will suppress the isometric isomorphism \(f\), (in particular, this convention applies to the connections in the Chern–Simons forms), and regard \(g^E\) and \(\nabla^F\) as a Hermitian metric and a unitary connection on \(E \to X\) if no confusion arises. This convention applies to \(\mathbb{Z}_2\)-graded triples as well.

Let \(F \to X\) be a \(\mathbb{Z}_2\)-graded complex flat vector bundle with \(\mathbb{Z}_2\)-graded flat connection \(\nabla^F = \nabla^+ + \nabla^-\) of virtual rank zero. Since \(\text{ch}(\nabla^+) - \text{ch}(\nabla^-) = 0\), there exists a \(k \in \mathbb{N}\) such that \(kF^+ \cong kF^-\) (see, for example, [11, Remark 1]). Put a \(\mathbb{Z}_2\)-graded Hermitian metric \(g^F = g^+ \oplus g^-\) on \(F \to X\). Define unitary connections \(\nabla^{\pm, u}\) on \(F^\pm \to X\) with respect to \(g^\pm\) by (A.1.7). The real part of the Cheeger–Chern–Simons class of \((F, \nabla^F)\) is given by

\[
\text{Re}(\text{CCS}(F, \nabla^F)) = \left[ \frac{1}{k} \text{CS}(k \nabla^{+, u}, k \nabla^{+, u}) \right] \mod \mathbb{Q}. \quad (A.1.9)
\]

A.2. Putting geometric data on a submersion. Let \(\pi : X \to B\) be a submersion with closed fibers \(Z\) of dimension \(n\). Denote by \(T^V X \to X\) the vertical tangent bundle. Let \(T^H X \to X\) be a horizontal distribution for \(\pi : X \to B\), i.e. \(TX = T^V X \oplus T^H X\). Denote by \(p^{TVX} : TX \to T^V X\) the projection map. Put a metric \(g^{TVX}\) on \(T^V X \to X\) and a Riemannian metric \(g^{TB}\) on \(T^B \to B\), respectively. Define a metric \(g^{TX}\) on \(TX \to X\) by

\[
g^{TX} = g^{TVX} \oplus \pi^* g^{TB}.
\]

Denote by \(\nabla^{TX}\) and \(\nabla^{TB}\) the Levi-Civita connections on \(TX \to X\) and \(TB \to X\) associated to \(g^{TX}\) and \(g^{TB}\), respectively. Then \(\nabla^{TVX} := p^{TVX} \nabla^{TX}\) is a Euclidean connection on \(T^V X \to X\) with respect to \(g^{TVX}\).

Define a connection \(\nabla^{TX}\) on \(TX \to X\) by

\[
\nabla^{TX} = \nabla^{TVX} \oplus \pi^* \nabla^{TB}.
\]

Then \(S := \nabla^{TX} - \nabla^{TX} \in \Omega^1(X, \text{End}(TX))\). By [3, Theorem 1.9] the \((3, 0)\) tensor \(g^{TX}(S(\cdot), \cdot)\) depends only on \((T^H X, g^{TVX})\). Let \(\{e_1, \ldots, e_n\}\) be a local orthonormal frame for \(T^V X \to X\). For any \(U \in \Gamma(B, TB)\), denote by \(U^H \in \Gamma(X, T^H X)\) its horizontal lift. Define a horizontal one-form \(k\) on \(X\) by

\[
k(U^H) = - \sum_{k=1}^n g^{TX}(S(e_k) e_k, U^H). \quad (A.2.1)
\]

For any two \(U, V \in \Gamma(B, TB)\),

\[
T(U, V) := -p^{TVX}[U^H, V^H]. \quad (A.2.2)
\]
is a horizontal two-form with values in $T^V X$, and is called the curvature of $\pi : X \to B$.

Denote by $d \text{vol}(Z)$ the Riemannian volume element of the fiber $Z$, which is a section of $\Lambda^n(T^V X)^* \otimes o(T^V X) \to X$.

A.3. **Local index theory for twisted spin$^c$ Dirac operator.** Let $\pi : X \to B$ be a submersion with closed, oriented and spin$^c$ fibers $Z$ of even dimension $n$. Choose and fix a spin$^c$ structure on $T^V X \to X$, and put geometric data on $\pi : X \to B$ as in Section A.2. This fixes a complex line bundle $L \to X$ satisfying $w_2(T^V X) = c_1(L) \mod 2$. Let $g^L$ be a Hermitian metric and $\nabla^L$ a unitary connection on $L \to X$. The spinor bundle $S(T^V X) \to X$ associated to the chosen spin$^c$ structure of $T^V X \to X$ is a complex vector bundle over $X$ given by

$$S(T^V X) = S_0(T^V X) \otimes L^\frac{1}{2},$$

where $S_0(T^V X)$ is the spinor bundle for the locally existing spin structure of $T^V X \to X$, and $L^\frac{1}{2}$ is the locally existing square root of $L \to X$. Since $n$ is even, $S(T^V X) \to X$ is $\mathbb{Z}_2$-graded.

Note that $g^{T^V X}$ and $\nabla^{T^V X}$ lift uniquely to a Hermitian metric $g^{S_0(T^V X)}$ and a unitary connection $\nabla^{S_0(T^V X)}$ on the locally existing spinor bundle $S_0(T^V X)$. On the other hand, $g^L$ and $\nabla^L$ induce a Hermitian metric $g^{L^\frac{1}{2}}$ and a unitary connection $\nabla^{L^\frac{1}{2}}$ on the locally existing complex line bundle $L^\frac{1}{2}$. Define a $\mathbb{Z}_2$-graded Hermitian metric and a $\mathbb{Z}_2$-graded unitary connection on $S(T^V X) \to X$ by

$$g^{S(T^V X)} := g^{S_0(T^V X)} \otimes g^{L^\frac{1}{2}},$$

$$\nabla^{S(T^V X)} := \nabla^{S_0(T^V X)} \otimes \text{id} + \text{id} \otimes \nabla^{L^\frac{1}{2}},$$

respectively. The first Chern form of $\nabla^L$ is defined to be $c_1(\nabla^L) = -\frac{1}{2\pi i} \text{tr}((\nabla^L)^2)$. The Todd form of $\nabla^{T^V X}$ is defined to be

$$\text{Todd}(\nabla^{T^V X}) = \overline{A}(\nabla^{T^V X}) \wedge e^{\frac{1}{2\pi i}(\nabla^L)}.$$

The triples $(E,g^E,\nabla^E)$ and $(S(T^V X),g^{S(T^V X)},\nabla^{S(T^V X)})$ induce the $\mathbb{Z}_2$-graded triple $(S(T^V X) \otimes E,g^{S(T^V X) \otimes E},\nabla^{S(T^V X) \otimes E})$, where $g^{S(T^V X) \otimes E}$ and $\nabla^{S(T^V X) \otimes E}$ are the tensor products of $g^{S(T^V X)}$ and $g^E$, and of $\nabla^{S(T^V X)}$ and $\nabla^E$, respectively. Define the twisted spin$^c$ Dirac operator $D^{S \otimes E}$ by

$$D^{S \otimes E} = \sum_{k=1}^n c(c_k)\nabla_{c_k}^{S(T^V X) \otimes E},$$

where $c$ is the Clifford multiplication. It acts on $\Gamma(X,S(T^V X) \otimes E)$ and is odd self-adjoint.
Define an infinite-rank $\mathbb{Z}_2$-graded complex vector bundle $\pi_*E \to B$ whose fiber over $b \in B$ is given by

$$(\pi_*E)_b = \Gamma(Z_b, (S(T^V X) \otimes E)|_{Z_b}).$$

The space of sections of $\pi_*E \to B$ is defined to be

$$\Gamma(B, \pi_*E) := \Gamma(X, S(T^V X) \otimes E).$$

Define an $L^2$-metric $g^\pi_*E$ on $\pi_*E \to B$ by

$$g^\pi_*E(s_1, s_2)(b) = \int_{Z_b} g^{S(T^V X) \otimes E}(s_1, s_2) d\text{vol}(Z).$$

Define a connection on $\pi_*E \to B$ by

$$\nabla_{U} \pi_*E_s := \nabla_{S(T^V X) \otimes E}U_s,$$

where $s \in \Gamma(B, \pi_*E)$ and $U \in \Gamma(B, TB)$. Then the connection on $\pi_*E \to B$ defined by

$$\nabla_{U} \pi_*E, u := \nabla_{U} \pi_*E + \frac{1}{2}k,$$

where $k$ is given by (A.2.1), is $\mathbb{Z}_2$-graded and unitary with respect to $g^\pi_*E$.

The Bismut superconnection on $\pi_*E \to B$ is defined to be

$$\mathbb{B}^E = D^{S \otimes E} + \nabla_{\pi_*E, u} - \frac{c(T)}{4},$$

(A.3.5)

where $T$ is given by (A.2.2). The rescaled Bismut superconnection is given by

$$\mathbb{B}^E_t = \sqrt{t}D^{S \otimes E} + \nabla_{\pi_*E, u} - \frac{c(T)}{4\sqrt{t}}.$$  

By [2, Theorem 10.23] we have

$$\lim_{t \to 0} \text{ch}(\mathbb{B}^E_t) = \int_{X/B} \text{Todd}(\nabla_{T^V X}) \wedge \text{ch}(\mathbb{B}^E).$$

(A.3.6)

The Bismut–Cheeger eta form associated to $\mathbb{B}^E$ is defined to be

$$\tilde{\eta}^E(g^E, \nabla^E, T^HX, g^{T^V X}) = \int_{0}^{\infty} \text{str} \left( \frac{dB^E_t}{dt} e^{-\frac{1}{4}t \text{Im}(\mathbb{B}^E)^2} \right) dt.$$  

Miščenko–Fomenko [16] (see also [8, Lemma 7.13]) prove that there exist finite rank subbundles $L^\pm \to B$ and complementary closed subbundles $K^\pm \to B$ of $(\pi_*E)^\pm \to B$ such that

$$(\pi_*E)^+ = K^+ \oplus L^+,$$

$$(\pi_*E)^- = K^- \oplus L^-,$$

(A.3.7)

$D_{+}^{S \otimes E}: (\pi_*E)^+ \to (\pi_*E)^-$ is block diagonal as a map with respect to (A.3.7), and $D_{+}^{S \otimes E}|_{K^+}: K^+ \to K^-$ is an isomorphism.

Given $L^\pm \to B$ satisfying the above conditions, we say the $\mathbb{Z}_2$-graded complex vector bundle $L \to B$, defined by $L = L^+ \oplus L^-$, satisfies the MF
property for $D^S \otimes E$. If $L \to B$ satisfies the MF property for $D^S \otimes E$, then the analytic index of $[E] \in K(X)$ is defined to be

$$\text{ind}^a([E]) = [L^+] - [L^-] \in K(B).$$

It is proved in [16, p.96-97] that the definition of $\text{ind}^a([E])$ does not depend on the choice of $L \to B$ satisfying the MF property for $D^S \otimes E$.

Let $P^L : \pi_* E \to L$ be the projection map with respect to (A.3.7). Then

$$g^L := P^L g^{\pi_* E}, \quad \nabla^L := P^L \nabla^{\pi_* E,u},$$

are a $\mathbb{Z}_2$-graded Hermitian metric and a $\mathbb{Z}_2$-graded unitary connection on $L \to B$, respectively. Henceforth, whenever $(L, g^L, \nabla^L)$ is a $\mathbb{Z}_2$-graded triple and $L \to B$ satisfies the MF property for $D^S \otimes E$, $g^L$ and $\nabla^L$ are obtained as above unless otherwise specified.

Given an $L \to B$ satisfying the MF property for $D^S \otimes E$, define an infinite rank $\mathbb{Z}_2$-graded complex vector bundle $\tilde{\pi}_* E \to B$ by

$$(A.3.8) \quad \tilde{\pi}_* E := \pi_* E \oplus L^{\text{op}}.$$

Let $i^- : L^- \to (\pi_* E)^{-}$ be the inclusion map and $p^+ : (\pi_* E)^+ \to L^+$ the projection map with respect to (A.3.7). Let $\alpha \in \mathbb{C}$. With respect to (A.3.8), define a map $\tilde{D}^E_+ (\alpha) : (\tilde{\pi}_* E)^+ \to (\tilde{\pi}_* E)^-$ by

$$\tilde{D}^E_+ (\alpha) = \begin{pmatrix} D^S \otimes E + \alpha i^- & \alpha p^- \\ 0 & 0 \end{pmatrix}.$$

$\tilde{D}^E_+ (\alpha)$ is invertible for all $\alpha \neq 0$ [8, Lemma 7.20]. Define a map $\tilde{D}^E (\alpha) : \tilde{\pi}_* E \to \tilde{\pi}_* E$ by

$$\tilde{D}^E (\alpha) := \begin{pmatrix} 0 & (\tilde{D}^E_+ (\alpha))^* \\ \tilde{D}^E_+ (\alpha) & 0 \end{pmatrix}.$$ 

Define a $\mathbb{Z}_2$-graded unitary connection on $\tilde{\pi}_* E \to B$ by

$$\nabla^{\tilde{\pi}_* E,u} := \nabla^{\pi_* E,u} \oplus \nabla^{L^{\text{op}}}.$$

Define a Bismut superconnection on $\tilde{\pi}_* E \to B$ by

$$\tilde{B}^E = \tilde{D}^E (1) + \nabla^{\tilde{\pi}_* E,u} - \frac{c(T)}{4} (A.3.9)$$

The rescaled Bismut superconnection is given by

$$\tilde{B}^E_t = \sqrt{t} \tilde{D}^E (\alpha (t)) + \sqrt{t} \nabla^{\tilde{\pi}_* E,u} - \frac{c(T)}{4 \sqrt{t}},$$

where $\alpha : [0, \infty) \to [0, 1]$ is the smooth function below (1.3.1). Since $\tilde{D}^E (\alpha (t))$ is invertible for $t \geq 1$, we have

$$\lim_{t \to \infty} \text{ch} (\tilde{B}^E_t) = 0. \quad (A.3.10)$$

On the other hand, for $t \leq a$, $\tilde{B}^E_t$ decouples, i.e.

$$\tilde{B}^E_t = \left( \sqrt{t} D^S \otimes E + \nabla^{\pi_* E,u} - \frac{c(T)}{4 \sqrt{t}} \right) \oplus \nabla^{L^{\text{op}}} = B^E_t \oplus \nabla^{L^{\text{op}}}.$$
By (A.3.6) we have
\[
\lim_{t \to 0} \text{ch}(\mathcal{B}_t^E) = \lim_{t \to 0} \text{ch}(\mathcal{B}_t^E) + \text{ch}(\nabla^{L,0})
\]
\[
= \int_{X/B} \text{Todd}(\nabla^{T^V X}) \wedge \text{ch}(\nabla^E) - \text{ch}(\nabla^L).
\]  \hfill (A.3.11)

The Bismut–Cheeger eta form associated to $\mathcal{B}^E$ is defined to be
\[
\tilde{\eta}^E(g^E, \nabla^E, T^H X, g^{T^V X}, L) = \int_0^\infty \text{str} \left( \frac{d}{dt} e^{-\frac{t}{\text{str}(\mathcal{B}^E)}} \right) dt.
\]

By (A.3.10) and (A.3.11) the local FIT for $D^S \otimes E$ is
\[
d\tilde{\eta}^E(g^E, \nabla^E, T^H X, g^{T^V X}, L) = \int_{X/B} \text{Todd}(\nabla^{T^V X}) \wedge \text{ch}(\nabla^E) - \text{ch}(\nabla^L).
\]  \hfill (A.3.12)

### A.4. Local index theory for twisted de Rham operator

Let $\pi : X \to B$ be a submersion with closed fibers $Z$ of dimension $n$. Put geometric data on $\pi : X \to B$ as in Section A.2. The complexified exterior bundle $\Lambda(T^V X)^* \to X$ is a Clifford module with Clifford multiplication $c(Y) = \epsilon(Y) - i(Y)$, where $Y \in \Gamma(X, T^V X)$, $\epsilon$ is the exterior multiplication and $i$ is the interior multiplication. Here $T^V X \to X$ is identified with $(T^V X)^* \to X$ via $g^{T^V X}$.

Let $F \to X$ be a complex flat vector bundle with flat connection $\nabla^F$. Put a Hermitian metric $g^F$ on $F \to X$. Define the connection $\nabla^{F,u}$ on $F \to X$ by (A.1.7). Denote by $g^{\Lambda(T^V X)^*}$ and $\nabla^{\Lambda(T^V X)^*}$ the extensions of $g^{T^V X}$ and $\nabla^{T^V X}$ to $\Lambda(T^V X)^* \to X$, respectively. Let $g^{\Lambda(T^V X)^* \otimes F} = g^{\Lambda(T^V X)^*} \otimes g^F$ and
\[
\nabla^{\Lambda(T^V X)^* \otimes F,u} = \nabla^{\Lambda(T^V X)^*} \otimes \text{id}_{\Gamma(X, F)} + \text{id}_{\Gamma(X, \Lambda(T^V X)^*)} \otimes \nabla^{F,u},
\]
respectively. Define the twisted de Rham operator $D^{\Lambda \otimes F}$ by
\[
D^{\Lambda \otimes F} = \sum_{k=1}^n c(e_k) \nabla^{\Lambda(T^V X)^* \otimes F,u}.
\]

It acts on $\Gamma(X, \Lambda(T^V X)^* \otimes F)$ and is odd self-adjoint.

Define an infinite rank $\mathbb{Z}$-graded complex vector bundle $\pi^A F \to B$ whose fiber over $b \in B$ is given by
\[
(\pi^A F)_b := \Gamma(Z_b, (\Lambda(T^V X)^* \otimes F)|_{Z_b}).
\]

Note that $\Omega(X, F) \equiv \Omega(B, \pi^A F)$. Define an $L^2$-metric on $\pi^A F \to B$ by
\[
g^{\pi^A F}(s_1, s_2)(b) = \int_{Z_b} g^{\Lambda(T^V X)^* \otimes F}(s_1, s_2) d\text{vol}(Z).
\]

Define a connection on $\pi^A F \to B$ by
\[
\nabla^U_{\pi^A F} s = \nabla^U_{\pi^A F,u} s,
\]
where \( s \in \Gamma(B, \pi_*^A F) \) and \( U \in \Gamma(B, TB) \). Note that \( \nabla_{\pi_*^A F} \) preserves the \( \mathbb{Z} \)-grading of \( \pi_*^A F \to B \). The connection on \( \pi_*^A F \to B \) defined by

\[
\nabla_{\pi_*^A F, u} := \nabla_{\pi_*^A F} + \frac{1}{2} k,
\]

where \( k \) is given by (A.4.1), is unitary with respect to \( g_{\pi_*^A F} \).

Denote by \( d^Z \) the fiberwise de Rham operator coupled with \( \nabla^F \) acting on \( \pi_*^A F \to B \). The connection on \( \pi_*^A F \to B \) defined by

\[
\nabla_{U}^{\pi_*^A F} s := \mathcal{L}_{UH} s,
\]

where \( \mathcal{L}_{UH} : \Gamma(X, \Lambda(T^V X)^* \otimes F) \to \Gamma(X, \Lambda(T^V X)^* \otimes F) \) is the Lie derivative, is \( \mathbb{Z} \)-graded.

The exterior differential \( d^X : \Omega(X,F) \to \Omega(X,F) \) coupled with \( \nabla^F \) can be regarded as a flat superconnection on \( \pi_*^A F \to B \) of total degree 1 and can be decomposed as

\[
d^X = d^Z + \nabla_{\pi_*^A F} + i_T,
\]

where \( T \) is given by (A.2.2). Consider \( d^Z \) as an element in \( \Gamma(B, \text{Hom}((\pi_*^A F)^*,(\pi_*^A F)^{**1})) \).

For each \( b \in B \),

\[
0 \longrightarrow (\pi_*^A F)^0 \xrightarrow{d^Z_b} (\pi_*^A F)^1 \xrightarrow{d^Z_b} \cdots \xrightarrow{d^Z_b} (\pi_*^A F)^n \xrightarrow{d^Z_b} 0
\]

is a cochain complex. Denote by \( H^k(Z_b,F|Z_b) \) the associated \( k \)-th cohomology group and define \( H(Z_b,F|Z_b) := \bigoplus_{k=0}^n H^k(Z_b,F|Z_b) \). Define a \( \mathbb{Z} \)-graded complex vector bundle \( H(Z,F|Z) \to B \) whose fiber over \( b \in B \) is given by \( H(Z,F|Z)_b := H(Z_b,F|Z_b) \). Denote by \( \psi : \ker(d^Z) \to H(Z,F|Z) \) the quotient map. For \( s \in \Gamma(B, H^k(Z,F|Z)) \), let \( e \in \Gamma(B, (\pi_*^A F)^k \cap \ker(d^Z)) \) satisfy \( \psi(e) = s \). The connection on \( H(Z,F|Z) \to B \) defined by

\[
\nabla_{U}^{H(Z,F|Z)} s = \psi(\nabla_{U}^{\pi_*^A F} e)
\]

is a well defined \( \mathbb{Z} \)-graded flat connection.

Denote by \( d^{Z_*} \) the formal adjoint of \( d^Z \) with respect to \( g_{\pi_*^A F} \). Define the twisted de Rham–Hodge operator by

\[
D^Z, \text{dR} = d^Z + d^{Z_*}.
\]

It acts on \( \Gamma(X, \Lambda(T^V X)^* \otimes F) \) and is odd self-adjoint. Denote by \( (\nabla_{\pi_*^A F}^*)^* \) the formal adjoint of \( \nabla_{\pi_*^A F}^* \) with respect to \( g_{\pi_*^A F} \). The connection on \( \pi_*^A F \to B \) defined by

\[
\nabla_{\pi_*^A F, u} := \frac{1}{2} (\nabla_{\pi_*^A F} + (\nabla_{\pi_*^A F}^*)^*)
\]

is unitary with respect to \( g_{\pi_*^A F} \). By [6, Propositions 3.5 and 3.7] we have

\[
\nabla_{\pi_*^A F, u} = \nabla_{\pi_*^A F, u}.
\]
By Hodge theory we have
\[ H(Z_b, F|_{Z_b}) \cong \ker(D^{Z, dR}). \]

Define a \( Z \)-graded complex vector bundle \( \ker(D^{Z, dR}) \to B \) whose fiber over \( b \in B \) is given by \( \ker(D^{Z, dR})_b := \ker(D^Z, dR) \). Then \( \ker(D^Z, dR) \to B \) is a finite rank subbundle of \( \pi^\Lambda F \to B \) and
\[ H(Z, F|_Z) \cong \ker(D^{Z, dR}) \]  

as \( Z \)-graded complex vector bundles. Let \( \rho_{\ker(D^{Z, dR})} : \pi^\Lambda F \to \ker(D^{Z, dR}) \) be the orthogonal projection. Define a \( Z \)-graded Hermitian metric on \( \ker(D^{Z, dR}) \to B \) by \( g_{\ker(D^{Z, dR})} = \rho_{\ker(D^{Z, dR})} g_{\pi^\Lambda F} \). Denote by \( g^{H(Z, F|_Z)} \) the \( Z \)-graded Hermitian metric on \( H(Z, F|_Z) \to B \) obtained by pulling back \( g_{\ker(D^{Z, dR})} \) via \( A.4.3 \).

The connection on \( H(Z, F|_Z) \to B \) defined by
\[ \nabla^{H(Z, F|_Z), u} = \frac{1}{2} \left( \nabla^{H(Z, F|_Z)} + (\nabla^{H(Z, F|_Z)})^* \right) \]
is unitary with respect to \( g^{H(Z, F|_Z)} \). Since \( \nabla^{H(Z, F|_Z), u} = \rho_{\ker(D^{Z, dR})} \nabla^{\pi^\Lambda F, u} \) by \[6\] Proposition 3.14, it follows from \( A.4.2 \) that
\[ \nabla^{H(Z, F|_Z), u} = \rho_{\ker(D^{Z, dR})} \nabla^{\pi^\Lambda F, u}. \]  

Set \( \overline{c}(Y) = \varepsilon(Y) + i(Y) \). By \[6\] Propositions 3.5 and 3.7 we have
\[ D^{Z, dR} = D^\Lambda F + V, \]  

where
\[ V = -\frac{1}{2} \sum_{k=1}^n \overline{c}(e_k) \omega(F, g^F)(e_k). \]  

Note that \( V \) is odd self-adjoint and anti-commutes with the \( c(X) \)'s.

Define a Bismut superconnection on \( \pi^\Lambda F \to B \) by
\[ B_{dR} := D^{Z, dR} + \nabla^{\pi^\Lambda F, u} - \frac{c(T)}{4}. \]

The rescaled Bismut superconnection is given by
\[ B_{dR}^t := \sqrt{t} D^{Z, dR} + \nabla^{\pi^\Lambda F, u} - \frac{c(T)}{4\sqrt{t}}. \]

The Bismut–Cheeger eta form associated to \( B_{dR} \) is defined by
\[ \tilde{\eta}_{dR} := \int_0^\infty \text{str} \left( \frac{d^n B_{dR}^t}{dt} e^{-\frac{1}{2t}(\rho^d_{dR})^2} \right) dt. \]

By \[5\] Theorem 3.7 (see also \[17\] Proposition 2.3) we have
\[ \tilde{\eta}_{dR} = 0. \]  

(A.4.7)
A.5. **Differential K-theory and its analytic index.** The differential $K$-group $\widetilde{K}_{\text{FL}}(X)$ of $X$ can be described in terms of $\mathbb{Z}_2$-graded generators, which have the form $E = (E, g^E, \nabla^E, \omega)$, where $(E, g^E, \nabla^E)$ is a $\mathbb{Z}_2$-graded triple and $\omega \in \Omega^\text{odd}(X)$. Two $\mathbb{Z}_2$-graded generators $E_0$ and $E_1$ are equal in $\widetilde{K}_{\text{FL}}(X)$ if and only if there exist balanced $\mathbb{Z}_2$-graded triples $(V_0, g^{V_0}, \nabla^{V_0})$ and $(V_1, g^{V_1}, \nabla^{V_1})$ such that

\[
E_0 \oplus V_0 \cong E_1 \oplus V_1 \quad \text{as } \mathbb{Z}_2\text{-graded complex vector bundles} \quad (A.5.1)
\]

and

\[
\omega_0 - \omega_1 \equiv \text{CS}(\nabla^{E_0} \oplus \nabla^{V_0}, \nabla^{E_1} \oplus \nabla^{V_1}). \quad (A.5.2)
\]

Note that $\widetilde{K}_{\text{FL}}(X)$ is a commutative ring, whose ring structure is given by

\[
(E, g^E, \nabla^E, \omega_E) \otimes (F, g^F, \nabla^F, \omega_F) = (E \otimes F, g^E \otimes g^F, \nabla^{E \otimes F}, \omega_E \wedge \chi(\nabla^E) + \chi(\nabla^F) \wedge \omega_F + \omega_E \wedge d\omega_F), \quad (A.5.3)
\]

where $\nabla^{E \otimes F}$ is the tensor product of $\nabla^E$ and $\nabla^F$.

Define a map $\chi_E : \widetilde{K}_{\text{FL}}(X) \to \Omega^\text{even}_Q(X)$ by $\chi_E(\mathcal{E}) = \chi(\nabla^E) + d\omega$. The kernel of $\chi_E$ is the $\mathbb{R}/\mathbb{Z}$ $K$-group $K^{-1}_L(X)$. Thus a $\mathbb{Z}_2$-graded generator of $K^{-1}_L(X)$ is a $\mathbb{Z}_2$-graded generator $E$ of $\widetilde{K}_{\text{FL}}(X)$ satisfying $\chi(\nabla^E) = -d\omega$. The latter condition, which is equivalent to

\[
\chi(\nabla^{E+}) - \chi(\nabla^{E-}) = -d\omega, \quad (A.5.4)
\]

implies $\text{rank}(E^+) = \text{rank}(E^-)$.

The $\mathbb{R}/\mathbb{Q}$ Chern character $\chi_{\mathbb{R}/\mathbb{Q}} : K^{-1}_L(X) \to H^\text{odd}(X; \mathbb{R}/\mathbb{Q})$ is defined as follows. Let $\mathcal{E}$ be a $\mathbb{Z}_2$-graded generator in $K^{-1}_L(X)$. By (A.5.4), there exists a $k \in \mathbb{N}$ such that $kE^+ \equiv kE^-$. Then the odd form $\frac{1}{k} \text{CS}(k\nabla^{E-}, k\nabla^{E+}) + \omega$ is closed. Define $\chi_{\mathbb{R}/\mathbb{Q}}(\mathcal{E})$ by

\[
\chi_{\mathbb{R}/\mathbb{Q}}(\mathcal{E}) = \left[ \frac{1}{k} \text{CS}(k\nabla^{E-}, k\nabla^{E+}) + \omega \right] \mod \mathbb{Q}. \quad (A.5.5)
\]

Note that $\chi_{\mathbb{R}/\mathbb{Q}}(\mathcal{E})$ is independent of the choice of $k$ [13, p.289].

Let $\pi : X \to B$ be a submersion with closed, oriented and spin$^c$ fibers of even dimension and $\mathcal{E}$ a $\mathbb{Z}_2$-graded generator in $\widetilde{K}_{\text{FL}}(X)$. Choose and fix a spin$^c$ structure on $T^V X \to X$, and put geometric data on $\pi : X \to B$ as in Section A.2. Let $(L, g^L, \nabla^L)$ be a $\mathbb{Z}_2$-graded triple so that $L \to B$ satisfies the MF property for $D^{\text{spin}c \otimes E}$. Define $\text{ind}^a_K(\mathcal{E}; L)$ by

\[
\text{ind}^a_K(\mathcal{E}; L) = \left( L, g^L, \nabla^L, \int_{X/B} \text{Todd}(\nabla^{T^V X}) \wedge \omega + \eta^E(g^E, \nabla^E, T^H X, g^{T^V X}, L) \right). \quad (A.5.6)
\]

For a $\mathbb{Z}_2$-graded generator $\mathcal{E}$ of $K^{-1}_L(X)$, we still define $\text{ind}^a_{\mathbb{R}/\mathbb{Z}}(\mathcal{E})$ by the right-hand side of (A.5.6). The $\mathbb{Z}_2$-graded version of (A.3.12) guarantees that $\text{ind}^a_{\mathbb{R}/\mathbb{Z}}(\mathcal{E})$ is a $\mathbb{Z}_2$-graded generator of $K^{-1}_L(B)$. 

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Note: The text has been reformatted for clarity and readability, with mathematical expressions and formulas accurately transcribed and aligned with the surrounding text. The document represents a complex subject, possibly dealing with advanced topics in differential geometry and algebraic topology, focusing on $K$-theory and analytic indices.
Denote by $S(T^V X)^* \to X$ the dual of the spinor bundle $S(T^V X) \to X$. Then

$$S(T^V X)^* := (S(T^V X)^*, g^{S(T^V X)^*}, \nabla^{S(T^V X)^*}, 0),$$

(A.5.7)

where $g^{S(T^V X)^*}$ and $\nabla^{S(T^V X)^*}$ are dual to (A.3.1) and (A.3.2), respectively, is a $\mathbb{Z}_2$-graded generator of $\hat{K}_{FL}(X)$. For any $\mathbb{Z}_2$-graded generator $E$ of $K_1^{-1}(X)$, one can check that the product $S(T^V X)^* \otimes E$ given by (A.5.3) defines a $\mathbb{Z}_2$-graded generator in $K_1^{-1}(X)$. 