On extensions of hook Weyl modules

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Abstract

We determine the integral extension groups \( \text{Ext}^1(\Delta(h), \Delta(h(k))) \) and \( \text{Ext}^k(\Delta(h), \Delta(h(k))) \), where \( \Delta(h), \Delta(h(k)) \) are the Weyl modules of the general linear group \( GL_n \) corresponding to hook partitions \( h = (a, 1^b), h(k) = (a + k, 1^{b-k}) \).

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1. Introduction

This paper concerns polynomial representations of the general linear group \( GL_n \) over the integers. For a partition \( \lambda \), let \( \Delta(\lambda) \) denote the Weyl module of \( GL_n \) of highest weight \( \lambda \). The extension groups \( \text{Ext}^i(\Delta(\lambda), \Delta(\mu)) \) play an important role in the theory. For example, the \( p \)-torsion of \( \text{Ext}^1(\Delta(\lambda), \Delta(\mu)) \) yields the Hom space between the corresponding modular Weyl modules of \( GL_n(K) \), where \( K \) is an algebraically closed field of characteristic \( p > 0 \), and the dimensions of the higher modular extensions may be obtained through torsion and restriction of integral extensions. Jantzen’s sum formula can be viewed and proved via integral extension groups [4].

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There are not many cases where explicit computations of integral extension groups between Weyl modules have been carried out. In \[2\] the \(GL_2\) case was treated and in \[5\] the \(GL_3\) case when \(\lambda\) and \(\mu\) differ by a multiple of a single root, both for \(i = 1\). In \[1\] the case \(\lambda = (1^a), \mu = (a)\) was studied and in \[15\] the situation where \(\lambda, \mu\) are hooks differing by a single root was considered. In \[14\] the case where \(\lambda, \mu\) are any partitions differing by a single root was settled.

As the modular extension groups are intimately related to the integral ones, we mention the result on neighboring Weyl modules \[11\] Part II Section 7, and the \(SL_2\) result in \[16\] for all \(i\) generalizing \[9\] and \[7\]. More can be found in \[8\].

Let \(h, h(k)\) be hooks, \(h = (a, 1^b), h(k) = (a + k, 1^b - k)\), where \(k\) is an integer such that \(1 \leq k \leq b\). It follows that \(\text{Ext}^i(\Delta(h)), \Delta(h(k))) = 0\) if \(i > k\). In this paper we determine \(\text{Ext}^1(\Delta(h)), \Delta(h(k)))\) and \(\text{Ext}^k(\Delta(h)), \Delta(h(k)))\) (Theorem 3.5 and Theorem 4.1). Our approach utilizes presentation matrices for various Ext groups that we determine from the description of generators and relations of Weyl modules of \[3\] \((i = 1)\) and from the projective resolutions of \[15\] \((i = k)\). Using these and the degree reduction theorem of Kulkarni \[13\], we identify cyclic generators of extension groups of the form \(\text{Ext}^i(\Delta(h), M), (i = 1, k)\), where \(M\) is a tensor product of a divided power and an exterior power. Computing the image of these generators under canonical maps yields the results.

2. Recollections

2.1. Notation

Let \(F\) be a free abelian group of finite rank \(n\). Fixing a basis of \(F\) yields an identification of general linear groups \(GL(F) = GL_n(\mathbb{Z})\). We will be working with homogeneous polynomial representations of \(GL_n(\mathbb{Z})\) of degree \(r\), or equivalently, with modules over the Schur algebra \(S_{\mathbb{Z}}(n, r)\) \[10\], Section 2.4. We will write \(S(n, r)\) in place of \(S_{\mathbb{Z}}(n, r)\). By \(DF = \sum_{i \geq 0} D_i F\) and \(\wedge F = \sum_{i \geq 0} \wedge^i F\) we denote the divided power algebra of \(F\) and the exterior algebra of \(F\) respectively. We will usually omit \(F\) and write \(D_i\) and \(\wedge^i\).

From \[10\] or \[2\], Proposition 2.1, we recall that for each sequence \(a_1, ..., a_n\) of non negative integers \(a_i\) that sum to \(r\), the \(S(n, r)\)-module \(D_{a_1} \otimes ... \otimes D_{a_n}\) is...
projective. Throughout this paper all tensor products are over the integers.

For a partition $\lambda$ of $r$ with at most $n$ parts, we denote by $\Delta(\lambda)$ the corresponding Weyl module for $S(n, r)$. If $\lambda = (a)$ is a partition with one part, then $\Delta(\lambda) = D_a$, and if $\lambda = (1^b)$, then $\Delta(\lambda) = \Lambda^b$. A hook $h$ is a partition of the form $h = (a, 1^b)$. The following complex of $S(n, r)$-modules (which is the dual of the usual Koszul complex) is exact

$$0 \rightarrow D_{a+b} \rightarrow \ldots \rightarrow D_{a+1} \otimes \Lambda^{b-1} \rightarrow \theta_a \rightarrow D_a \otimes \Lambda^b \rightarrow \ldots \rightarrow \Lambda^{a+b} \rightarrow 0,$$

where $\theta_a$ is the composition $D_{a+1} \otimes \Lambda^{b-1} \xrightarrow{\triangle \otimes 1} D_a \otimes D_1 \otimes \Lambda^{b-1} \xrightarrow{1 \otimes m} D_a \otimes \Lambda^b$, where $\triangle$ (respectively, $m$) is the indicated component of the comultiplication (resp., multiplication) map of the Hopf algebra $DF$ (resp., $\Lambda F$). It is well known that if $h = (a, 1^b)$ is a hook, $b \geq 1$, then $\Delta(h) \simeq \text{cok}(\theta_a) \simeq \ker(\theta_{a-1})$, so that we have the following short exact sequence

$$0 \rightarrow \Delta(h(1)) \xrightarrow{i} D_a \otimes \Lambda^b \xrightarrow{\pi_0} \Delta(h) \rightarrow 0, \quad (2.1)$$

where $h(1) = (a+1, 1^{b-1})$, the map $i$ is induced by $\theta_a$ on generators and the map $\pi_0$ is induced by the identity map on generators.

**Notation:** Throughout this paper we use the notation $h = (a, 1^b), h(k) = (a + k, 1^{b-k}), 1 \leq k \leq b, r = a + b$.

2.2. **Straightening law**

We recall the straightening law and the semi-standard basis theorem for $\Delta(h)$ (\cite[Theorem II.3.16]{[1]}). Fix an ordered basis $e_1, \ldots, e_n$ of $F$. For simplicity, we denote the element $e_i$ by $i$ and accordingly the element $e_{i_1}^{(a_1)} \ldots e_{i_t}^{(a_t)} \otimes e_{j_1} \wedge \ldots \wedge e_{j_b} \in D_a \otimes \Lambda^b$ by $i_1^{(a_1)} \ldots i_t^{(a_t)} \otimes j_1 \ldots j_b$. The image of this element under the identification $\Delta(h) \simeq \text{cok}(\theta_a)$ will be denoted by $i_1^{(a_1)} \ldots i_t^{(a_t)}|j_1 \ldots j_b$. Now suppose $i_1 < i_2 < \ldots < i_t$ and $j_1 \leq i_1$. Then in $\Delta(h)$ we have

$$i_1^{(a_1)} \ldots i_t^{(a_t)}|j_1 \ldots j_b = \begin{cases} -\sum_{s \geq 2} i_1^{(a_1+1)} \ldots i_s^{(a_s-1)} \ldots i_t^{(a_t)}|j_s j_2 \ldots j_b, & \text{if } j_1 = i_1 \\ -\sum_{s \geq 1} j_1 i_1^{(a_1)} \ldots i_s^{(a_s-1)} \ldots i_t^{(a_t)}|j_s j_2 \ldots j_b, & \text{if } j_1 < i_1. \end{cases}$$

A $\mathbb{Z}$-basis of $\Delta(h)$ is the set of all $i_1^{(a_1)} \ldots i_t^{(a_t)}|j_1 \ldots j_b$, where $a_1 + \ldots + a_t = a$, $i_1 < \ldots < i_t$ and $i_1 < j_1 < \ldots < j_b$. 

3
2.3. Resolutions of hooks

We will use the explicit finite projective resolution \( P_\ast(a,b) \) of \( \Delta(h) \),

\[
0 \to \ldots \to P_2(a,b) \xrightarrow{\theta_2(a,b)} P_1(a,b) \xrightarrow{\theta_1(a,b)} P_0(a,b)
\]

of [15], Theorem 1, which we now recall. For short we denote the tensor product \( D_{a_1} \otimes \cdots \otimes D_{a_m} \) of divided powers by \( D(a_1,\ldots,a_m) \). Let \( P_i(a,b) = \sum D(a_1,\ldots,a_{b+1-i}) \) where the sum ranges over all sequences \((a_1,\ldots,a_{b+1-i})\) of positive integers of length \( b+1-i \) such that \( a_1 + \ldots + a_{b+1-i} = a+b \) and \( a \leq a_1 \leq a+i \). The differential \( \theta_i(a,b) \) is defined by sending \( x_1 \otimes \cdots \otimes x_{b+1-i} \in D(a_1,\ldots,a_{b+1-i}) \) to

\[
\sum_{j=1}^{s} (-1)^{j+i} x_1 \otimes \cdots \otimes \Delta(x_j) \otimes \cdots \otimes x_{b+1-i} \in D(a_1,\ldots,u,v,\ldots,a_{s}),
\]

where \( s = b+1-i \) and \( \Delta(x_j) \) is the image of \( x_j \) under the two-fold diagonalization \( D(a_j) \to \sum D(u,v) \), where the sum ranges over all positive integers \( u,v \) such that \( u+v = a_j \) and \( D(a_1,\ldots,u,v,\ldots,a_{b+1-i}) \) is a summand of \( P_{i-1}(a,b) \) with \( u \) located at position \( j \). We denote by \( \triangle_{u,v} : D(a_j) \to D(u,v) \) the indicated component of the two-fold diagonalization \( D(a_j) \to \sum D(u,v) \).

If \( A,B \) are \( S(n,r) \)-modules, we write \( \text{Hom}(A,B) \) and \( \text{Ext}^i(A,B) \) in place of \( \text{Hom}_{S(n,r)}(A,B) \) and \( \text{Ext}^i_{S(n,r)}(A,B) \) respectively.

We recall the recursions

\[
P_0(a,b) = D(a) \otimes P_0(1,b-1),
\]

\[
P_i(a,b) = P_{i-1}(a+1,b-1) \oplus D(a) \otimes P_i(1,b-1), \quad i > 0
\]

and that under these identifications we have the following.

\[\text{Remark 2.1. If } M \text{ is a } S(n,r)\text{-module, the differential } \text{Hom}(\theta_i(a,b), M) \text{ of the complex } \text{Hom}(P_\ast(a,b), M) \text{ looks like}\]

\[
\text{Hom}(P_{i-2}(a+1,b-1), M) \xrightarrow{\oplus} \text{Hom}(P_{i-1}(a+1,b-1), M) \oplus \text{Hom}(D(a) \otimes P_i(1,b-1), M) \text{ with } M \text{ replaced by } \text{Ext}^i_{S(n,r)}(A,B) \text{ respectively.}\]

\[\text{Hom}(D(a) \otimes P_{i-1}(1,b-1), M) \xrightarrow{\oplus} \text{Hom}(D(a) \otimes P_i(1,b-1), M)\]
where the top horizontal map is $\text{Hom}(\theta_{i-1}(a+1,b-1), M)$, the bottom one is $-\text{Hom}(1 \otimes \theta_i(1,b-1), M)$ and the restriction of the diagonal one on the summand $\text{Hom}(D(a,j,a_2,\ldots,a_m), M)$ is $\text{Hom}(\Delta_{a,j} \otimes 1 \otimes \ldots \otimes 1, M)$.

For any $S(n,r)$-module $M$ and any sequence $a_1,\ldots,a_m$ of non negative integers such that $a_1 + \ldots + a_m = r$ and $m \leq n$, we identify the $\mathbb{Z}$-module

$$\text{Hom}(D(a_1,\ldots,a_m), M)$$

with the $(a_1,\ldots,a_m)$ weight subspace of $M$ (with respect to the action of $\mathbb{Z}^n$) according to [2], eqn. (11) on p. 178. We will use such identifications freely throughout this paper.

In particular, suppose $M$ is a skew Weyl module for $S(n,r)$ (denoted be $K_{\lambda/\mu}(F)$ in [3]). Using the $\mathbb{Z}$-basis of $M$ given by the semi-standard tableaux [3], Theorem II.3.16, we see that the $\mathbb{Z}$-module $\text{Hom}(D(a_1,\ldots,a_m), M)$ may be identified with the $\mathbb{Z}$-submodule of $M$ that has basis the semi-standard tableaux of $M$ that contain the entry $i$ exactly $a_i$ times, $i = 1,\ldots,m$. We call this the semi-standard basis of $\text{Hom}(D(a_1,\ldots,a_m), M)$. (Perhaps we should remark that what we have called semi-standard tableaux are called 'co-standard' in [3], Definition II.3.2: the entries in each row are weakly increasing from left to right and the entries in each column are strictly increasing from top to bottom.)

We record here a handy computational remark. For $M$ a skew Weyl module and $T \in \text{Hom}(D(a_1,\ldots,a_m), M)$ a semi-standard basis element, let $\phi_t(T)$, $1 \leq t < m$, be the element of $\text{Hom}(D(a_1,\ldots,a_t+a_{t+1},\ldots,a_m), M)$ obtained from $T$ by replacing each occurrence of $j > t$ by $j - 1$. If $t \geq m$, let $\phi_t(T) = 0$. By extending linearly, we obtain for each degree $i$ a map of $\mathbb{Z}$-modules

$$\phi_i : \text{Hom}(P_i(a,b), M) \to \text{Hom}(P_{i+1}(a,b), M).$$

It is clear that only a finite number of these maps are nonzero. From the definition of the differential of $P_*(a,b)$, we obtain the following description for the differential of $\text{Hom}(P_*(a,b), M)$.

**Remark 2.2.** With the previous notation, $\text{Hom}(\theta_t(a,b), M) = \sum_{i \geq 1}(-1)^{t-1}\phi_t$. 

5
Let \( n \geq b + 1 \). Then \( P_*(a,b) \) is a projective resolution of \( \Delta(h) \). We claim that the \( \mathbb{Z} \)-module \( \text{Ext}^i(\Delta(h), M) \), where \( M \) is any skew Weyl \( S(n,r) \)-module, is isomorphic to the torsion submodule of the cokernel \( E^i(\Delta(h), M) \) of the map \( \text{Hom}(\theta_*(a,b), M) \). Indeed, by the argument of [6], bottom of p. 634 to the top of p. 635, we have

\[
E^i(\Delta(h), M) \simeq \text{Ext}^i(\Delta(h), M) \oplus N,
\]

where \( N \) is the image of the map \( \text{Hom}(\theta_*(a,b), M) \). (The argument given in loc. cit. is stated for \( i = 1 \) but is valid for any \( i \geq 1 \).) As a submodule of a free \( \mathbb{Z} \)-module, \( N \) is a free \( \mathbb{Z} \)-module. On the other hand, the \( \mathbb{Z} \)-module \( \text{Ext}^i(\Delta(h), M) \) is torsion for all \( i \geq 1 \) by [2], last paragraph of Section 8. Hence from the above isomorphism we obtain that the torsion submodule of \( E^i(\Delta(h), M) \) is isomorphic to \( \text{Ext}^i(\Delta(h), M) \).

2.4. The extensions \( \text{Ext}^i(\Delta(h), D_{a+k} \otimes \wedge^{b-k}) \)

We will use the following lemma several times. Let

\[
 r_k = \gcd\left( (k+1)_1, \ldots, (k+1)_k \right)
\]

and note that \( r_k = 1 \) unless \( k + 1 = p^e \), \( p \) prime, in which case \( r_k = p \).

**Lemma 2.3.** Suppose \( n \geq b + 1 \) and \( 1 \leq k < b \). Then

\[
\text{Ext}^i(\Delta(h), D_{a+k} \otimes \wedge^{b-k}) = \text{Ext}^i(\wedge^{k+1}, D_{k+1}).
\]

In particular, \( \text{Ext}^1(\Delta(h), D_{a+k} \otimes \wedge^{b-k}) = \mathbb{Z}_2 \) and \( \text{Ext}^k(\Delta(h), D_{a+k} \otimes \wedge^{b-k}) = \mathbb{Z}_{r_k} \).

**Proof.** A special case of the main result, Theorem 2, of [13] yields

\[
\text{Ext}^i(\Delta(a,1^b), D_{a+k} \otimes \wedge^{b-k}) = \text{Ext}^i(D_{a-1} \otimes \wedge^{k+1}, D_{a+k}).
\]

Applying contravariant duality [11], p. 209, and [2], Theorem 7.7, we have

\[
\text{Ext}^i(D_{a-1} \otimes \wedge^{k+1}, D_{a+k}) = \text{Ext}^i(\wedge^{a+k}, D_{k+1} \otimes \wedge^{a-1}).
\]

Again by [13], \( \text{Ext}^i(\wedge^{a+k}, D_{k+1} \otimes \wedge^{a-1}) = \text{Ext}^i(\wedge^{k+1}, D_{k+1}) \) and the first equality of the lemma follows.

We have \( \text{Ext}^1(\wedge^{k+1}, D_{k+1}) = \mathbb{Z}_2 \), by [1], Section 4, and \( \text{Ext}^k(\wedge^{k+1}, D_{k+1}) = \mathbb{Z}_{r_k} \), according to [13], eqn. (6) p. 2207.

\[\square\]
2.5. Summary of notation

For the reader’s convenience we gather here some of the notation introduced in the previous subsections that will be used often.

- $h = (a, 1^b)$ and $h(k) = (a+k, 1^{b-k})$: hooks, where $1 \leq k \leq b$ and $r = a+b$, (subsection 2.1).
- $i_1^{(a_1)}...i_t^{(a_t)}j_1...j_b$: the image in $\Delta(h)$, where $a = a_1 + ... + a_t$, of the element $i_1^{(a_1)}...i_t^{(a_t)} \otimes j_1...j_b \in D_a \otimes \wedge^b$ under the isomorphism $cok(\theta_a) \simeq \Delta(h)$, (subsection 2.2).
- $P^\bullet(a,b)$ and $\theta^\bullet(a,b)$: projective resolution of $\Delta(h)$ and the differential of $P^\bullet(a,b)$ respectively, (subsection 2.3).
- $E^i(\Delta(h), M)$: the cokernel of the map $\text{Hom}(\theta_i(a,b), M)$, (subsection 2.3).
- $\phi_i$: a summand of the differential $\text{Hom}(\theta_i(a,b), M)$ (defined before Remark 2.2).

3. $\text{Ext}^1(\Delta(h), \Delta(h(k)))$

In this section we determine $\text{Ext}^1(\Delta(h), \Delta(h(k)))$ for $k > 1$. The case $k = 1$ was computed in [13], Theorem 6.

3.1. Matrices $e^{(1)}(a,b,M)$ and a generator

Let $M = D_{a+k} \otimes \wedge^{b-k}$. For the semi-standard basis $B$ of the domain $\text{Hom}(D(h), M) = \text{Hom}(D(a,1,...,1), M)$ of the map $\text{Hom}(\theta_1(a,b), M)$ we have $B = B_0 \cup B_1$, where $B_0$ (resp., $B_1$) consists of those elements of $B$ that have no (resp., exactly one) appearance of 1 in the $\wedge^{b-k}$ part. We consider the usual lexicographic ordering on $B$ and note that every element of $B_0$ is less than every element of $B_1$. We have

$$|B| = \binom{k+1}{k+1}, \ |B_0| = \binom{k}{k}, \ |B_1| = \binom{b}{k+1}.$$

Likewise, for the semi-standard bases $B^1$, $B^2$, ..., $B^b$ of $\text{Hom}(D(a+1,1,...,1), M)$, $\text{Hom}(D(a,2,...,1), M)$, ..., $\text{Hom}(D(a,1,...,2), M)$ respectively, we have $B^t =$
$B_0^i \cup B_1^i$, where $B_0^i$ (resp. $B_1^i$) consists of those elements of $B^i$ that have no (resp. exactly one) appearance of 1 in the $\wedge^{b-k}$ part, and thus

$$B' = B_0^1 \cup B_1^1 \cup \ldots \cup B_0^b \cup B_1^b$$

is a basis of the codomain of the map $\text{Hom}(\theta_1(a, b), M)$. We order each set $B^i$ lexicographically and declare that every element of $B^i$ is less than every element of $B^{i+1}$. For each $i$ we have

$$|B^i| = \binom{b}{k}, |B_0^i| = \binom{b-1}{k-1}, |B_1^i| = \binom{b-1}{k}.$$ 

Consider the matrix $e^{(1)}(a, b, M) \in M_{b^k} \times (b^{b+1}) \left( \mathbb{Z} \right)$ of the map $\text{Hom}(\theta_1(a, b), M)$, with respect to the previous orderings, partitioned into $b$ row blocks according to $B' = B^1 \cup \ldots B^b$.

In the next Lemma, the missing entries of any matrix are assumed to be equal to 0.

**Lemma 3.1.** Let $p = \binom{b-1}{k-1}, q = \binom{b-1}{k}$. The matrix $e^{(1)}(a, b, M)$ has the following properties.

1. The first block is $\{ A(a, b; k) \mid B(a, b; k) \}$, where

$$A(a, b; k) = \text{diag}(a + 1, \ldots, a + 1, 1, \ldots, 1) \in M_{\binom{b}{k}} \left( \mathbb{Z} \right),$$

$$B(a, b; k) = \left( aI_q \right) \in M_{\binom{b}{k}} \times \binom{b}{k} \left( \mathbb{Z} \right).$$

2. The $t$-th block, $t > 1$, is of the form $\left( \begin{array}{c} C_t \\ D_t \end{array} \right)$, where $C_t \in M_{p \times \binom{b}{k}} \left( \mathbb{Z} \right)$, $D_t \in M_{q \times \binom{b}{k+1}} \left( \mathbb{Z} \right)$.

3. The sum of the elements in any row of the $t$-th block, where $t \geq 2$, is $(-1)^{t-1}2$.

4. The last row is of the form $(0 \ldots 0 \pm 2)$.

**Proof.** (1) The set $B_0$ consists of all $T = 1^{(a)}i_1 \ldots i_k \otimes j_1 \ldots j_{b-k} \in B$ such that

$$i_1 < \ldots < i_k, \quad j_1 < \ldots < j_{b-k}$$

$$\{i_1, \ldots, i_k\} \cup \{j_1, \ldots, j_{b-k}\} = \{2, \ldots, b + 1\}$$

$$\{i_1, \ldots, i_k\} \cap \{j_1, \ldots, j_{b-k}\} = \emptyset.$$
The definition of $\phi_1$ yields
\[
\phi_1(T) = \begin{cases} 
(a + 1)1^{(a+1)}i'_2 \cdots i'_k \otimes j'_1 \cdots j'_{b-k}, & \text{if } i_1 = 2 \\
1^{(a)}i'_2 \cdots i'_k \otimes 1j'_2 \cdots j'_{b-k}, & \text{if } i_1 \neq 2 
\end{cases},
\]
where $i' = i - 1$. From this it easily follows that the matrix of the restriction of $\phi_1$ on the subgroup of $\text{Hom}(D, M)$ spanned by $B_0$ is $A(a, b; k)$.

The set $B_1$ consists of all $S = 1^{(a-1)}i_1 \cdots i_{k+1} \otimes j_1 \cdots j_{b-k} \in B$ such that
\[
i_1 < \cdots < i_{k+1}, \quad j_1 < \cdots < j_{b-k}, \quad j_1 = 1
\]
\[
\{i_1, \ldots, i_{k+1}\} \cup \{j_1, \ldots, j_{b-k}\} = \{1, \ldots, b+1\}
\]
\[
\{i_1, \ldots, i_{k+1}\} \cap \{j_1, \ldots, j_{b-k}\} = \emptyset.
\]
Then
\[
\phi_1(S) = \begin{cases} 
a1^{(a-1)}i'_2 \cdots i'_k \otimes 1j'_2 \cdots j'_{b-k}, & \text{if } j_2 \neq 2 \\
0, & \text{if } j_2 = 2 
\end{cases},
\]
From this it easily follows that the matrix of the restriction of $\phi_1$ on the subgroup of $\text{Hom}(D, M)$ spanned by $B_1$ is $B(a, b; k)$.

(2) If $T = 1^{(a)}i_1 \cdots i_k \otimes j_1 \cdots j_{b-k} \in B_0$, then $j_1 \geq 2$, and thus for $t \geq 2$, $\phi_t(T)$ is a multiple of an element of $B_t^l$ that does not contain a 1 in the $\wedge^{b-k}$ part. Hence $\phi_t(T) \in \text{span}B_t^l$. If $S = 1^{(a-1)}i_1 \cdots i_{k+1} \otimes j_1 \cdots j_{b-k} \in B_1$, then $j_1 = 1$ and thus $\phi_t(T) \in \text{span}B_t^l$ for all $t$.

(3) Suppose $T' \in B^2$. Since $t > 1$, $t$ appears exactly twice in $T'$.

Case 1. Suppose $T'$ is of the form $T' = xt(2)y \otimes z$. Since each element of $B$ has weight $(a, 1, \ldots, 1)$, from the definition of $\phi_t$, it follows that there is a unique $T \in B$ such that $\phi_t(T) = cT'$, $c \neq 0$, namely $T = xt(t + 1)y_1 \otimes z_1$, where $y_1$ and $z_1$ are obtained from $y$ and $z$ respectively by replacing each $i > t$ by $i + 1$. We have $\phi_t(T) = 2T'$.

Case 2. Suppose $T'$ is of the form $T' = xty \otimes ztw$. Since each element of $B$ has weight $(a, 1, \ldots, 1)$, from the definition of $\phi_t$, it follows that there are exactly two $T_1, T_2 \in B$ such that $\phi_t(T_i) = c_i T'$, $c_i \neq 0$, namely
\[
T_1 = x(t + 1)y_1 \otimes ztw_1, \quad T_2 = xty_1 \otimes x(t + 1)w_1,
\]
where $y_1$ and $w_1$ are obtained from $y$ and $w$ respectively by replacing each $i > t$ by $i + 1$. We have $\phi_t(T_1) = \phi_t(T_2) = T'$.

(4) This follows from case 1 of (3) since the greatest element in $B^k$ is $T' = 1^{(a-1)}(b - k + 1) \cdots (b - 1) \otimes \Delta^{(b)} \otimes 12 \cdots (b - k)$. \hfill $\Box$

Let $1 \leq k < b$ and $M = D_{a+k} \otimes \wedge^{b-k}$. From Lemma 2.3 we have $\text{Ext}^1_1(\Delta(h), M) = \mathbb{Z}_2$. We will determine a generator of this Ext group.

We have mentioned that the torsion subgroup of the abelian group $E^1_1(\Delta(h), M)$ with presentation matrix $e^{(1)}(a, b, M)$ is isomorphic to $\text{Ext}^1_1(\Delta(h), M)$. We denote by $\pi$ the natural projection $\pi : \text{Hom}(P(a, b), M) \to E^1_1(\Delta(h), M)$.
Lemma 3.2. Let \( h = (a, 1^b) \), \( 1 \leq k < b \) and \( M = D_{a+k} \otimes \wedge^{b-k} \). A cyclic generator of the abelian group \( \text{Ext}^1(\Delta(h), M) \) is \( \pi(g_k) \), where

\[
g_k = \binom{a+1}{2} \sum_{T \in B_0^k} T + a \sum_{T \in B_1^k} T + \sum_{i=2}^{b} (-1)^{i-1} \left( a \sum_{T \in B_0^i} T + \sum_{T \in B_i^1} T \right).
\]

Proof. Let \( E_i \) be the \( i \)-th column of \( e^{(1)}(a, b, M) \) and let \( p = \binom{b-1}{k-1} \), \( q = \binom{b-1}{k} \). Consider the \( e^{(1)}(a, b, M) \) partitioned into \( b \) blocks each consisting of \( p+q = \binom{b}{k} \) consecutive rows. From Lemma 3.1 it follows that

\[
a(E_1 + \ldots + E_{(k)}) + E_{(k)+1} + \ldots + E_{(k+1)} =
\frac{a(a+1)}{p}, \ldots, \frac{a(a+1)}{q}, \frac{2a}{p}, \ldots, \frac{-2a}{q}, \ldots, \frac{-2a}{p}, \ldots, \frac{-2a}{q}.
\]

Hence in the cokernel \( E^1(\Delta(h), M) \) of the differential \( \text{Hom}(\theta_1(a, b), M) \) we have \( 2\pi(g_k) = 0 \). This shows that \( \pi(g_k) \in \text{Ext}^1(\Delta(h), M) \). We have \( \pi(g_k) \neq 0 \), since otherwise the integer matrix-column \( \frac{1}{2}X \), where \( X \) is the right hand side of (3.1), would be a \( \mathbb{Z} \)-linear combination of columns of \( e^{(1)}(a, b, M) \). This is not possible because the last entry of \( \frac{1}{2}X \) is \( \pm 1 \) while all entries of the last row of \( e^{(1)}(a, b, M) \), \( k > 0 \), are even according to Lemma 3.1 (4). Since \( \text{Ext}^1(\Delta(h), M) = \mathbb{Z}_2 \), it follows that \( \pi(g_k) \) generates \( \text{Ext}^1(\Delta(h), M) \).

3.2. Proof for \( \text{Ext}^1(\Delta(h), \Delta(h(k))) \).

We determine \( \text{Ext}^1(\Delta(h), \Delta(h(k))), k \geq 2 \) in this subsection. The main computation is done in the next two lemmas the first of which takes care of the case \( k = 2 \).

We will use the following notation. If \( f : M \to N \) is a map of \( S(n, r) \)-modules, we have in the usual way various induced maps

\[
\text{Hom}(P_i(a, b), M) \to \text{Hom}(P_i(a, b), N),
\]

\[
E^i(\Delta(h), M) \to E^i(\Delta(h), N),
\]

\[
\text{Ext}^i(\Delta(h), M) \to \text{Ext}^i(\Delta(h), N)
\]

of abelian groups which will all will be denoted by \( f^* \). For an integer \( m \) let \( \epsilon_m \) be the remainder of the division of \( m \) by 2.
In the statement of the next Lemma, we recall that \( \text{Ext}^1(\Delta(h), D_{a+1} \otimes \wedge^{b-1}) = \mathbb{Z}_2 \) according to Lemma 2.3. Also from \( \text{[15]} \), Theorem 6, we have that \( \text{Ext}^1(\Delta(h), \Delta(h(1))) \) is a cyclic group.

**Lemma 3.3.** Let \( h = (a, b) \) and \( h(1) = (a + 1, b-1), b \geq 2 \). The map

\[
\text{Ext}^1(\Delta(h), D_{a+1} \otimes \wedge^{b-1}) \xrightarrow{\pi_0} \text{Ext}^1(\Delta(h), \Delta(h(1)))
\]

induced by \( D_{a+1} \otimes \wedge^{b-1} \xrightarrow{\pi_0} \Delta(h(1)) \) is multiplication by the integer \( \frac{(a+\epsilon_b-1)(a+b)}{2} \).

**Proof.** The matrix of the map \( \text{Hom}(P_0(a, b), \Delta(h(1))) \to \text{Hom}(P_1(a, b), \Delta(h(1))) \) with respect to the lexicographic order of semi-standard bases is the following \( b \times b \) matrix according to \( \text{[15]} \), p. 2211,

\[
\begin{pmatrix}
(a + 1 & -1 & 1 & \ldots & (1)^{b-1} \\
-1 & -1 & 0 & \ldots & 0 \\
0 & 1 & 1 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & (1)^{b-1} & (1)^{b-1}
\end{pmatrix}
\]

Let

\[
S_1 = 1^{(a+1)}|23\ldots b, S_2 = 1^{(a)}|2|23\ldots b, \ldots, S_b = 1^{(a)}|b|23\ldots b
\]

be the semi-standard basis of \( \text{Hom}(P_1(a, b), \Delta(h(1))) \). From the above matrix it follows that for each \( i \) there is an integer \( m_i \) such that \( \pi(S_i) = m_i \pi(S_1) \), and hence a cyclic generator of \( E^1(\Delta(h), \Delta(h(1))) \) is \( \pi(S_1) \). Moreover by adding the even numbered columns 2, 4, \ldots, \( 2\left[\frac{b}{2}\right] \) we have the relation

\[
\sum_{i=2}^{b} (-1)^{i-1} \pi(S_i) = \left[\frac{b}{2}\right] \pi(S_1)
\]

where \( \left[\frac{b}{2}\right] \) is the largest integer less than or equal to \( \frac{b}{2} \).

Using the notation established at the beginning of subsection 3.1, we have \( B_0^i = \{ T_i \} \) and \( B_1^i = \{ T_2, \ldots, T_b \} \), where \( T_1 = 1^{(a+1)} \otimes 23\ldots b \) and \( T_j = 1^{(a)}|j|23\ldots j\ldots b \), \( j = 2, \ldots, b \), and it is understood that \( j \) means that \( j \) is omitted. Now \( \pi_0(T_1) = S_1 \). Also

\[
\pi_0(T_j) = 1^{(a)}|j|23\ldots j\ldots b = -1^{(a+1)}|j|2\ldots j\ldots b = (-1)^{j-1}S_1,
\]

where in the second equality the straightening law was used. Thus

\[
\pi_0 \left( \sum_{j=2}^{b} \left[\frac{X}{2}\right] B_0^j + aB_1^j \right) = \left( \left[\frac{a+1}{2}\right] + a(\epsilon_2 - 1) \right) S_1,
\]

where \( X = \sum_{T \in X} T \), if \( X \) is one of the sets \( B_0^j, B_1^j \). A similar computation for \( j = 2, \ldots, b \) yields

\[
\pi_0 \left( B_0^j + B_1^j \right) = (a + \epsilon_b - 1) S_j,
\]

11
and therefore in $E^1(\Delta(h, \Delta(h(1))))$ we have

$$\pi^*_0(\pi(g_1)) = \left(\frac{a+b+1}{2} + a(\epsilon_b - 1)\right)\pi(S_1) + (a + \epsilon_b - 1) \sum_{i=2}^{b} (-1)^{i-1} \pi(S_i)$$
$$= \left(\frac{a+b+1}{2} + a(\epsilon_b - 1) + (a + \epsilon_b - 1)\frac{b}{2}\right)\pi(S_1).$$

It is easy to verify that $\left(\frac{a+b+1}{2} + a(\epsilon_b - 1) + (a + \epsilon_b - 1)\frac{b}{2}\right) = \frac{(a+\epsilon_b-1)(a+b)}{2}$. We have shown that the map $\Ext^1(\Delta(h), D_{a+1} \otimes \land^{b-1}) \xrightarrow{\pi^*_0} \Ext^1(\Delta(h), \Delta(h(1)))$ is multiplication by this integer. Since $g_1$ is a generator of $\Ext^1(\Delta(h), D_{a+1} \otimes \land^{b-1})$ (Lemma 3.2) which is the torsion submodule of $E^1(\Delta(h), D_{a+1} \otimes \land^{b-1})$, the result follows.

Since multiplication in the divided power algebra is commutative, we will often denote a semi-standard basis element of the form $1^{(a)}i_1...i_s t^{(2)}i_{s+1}...i_{k-2} \otimes j_1...j_{b-k}$ by $1^{(a)}t^{(2)}i_1...i_{k-2} \otimes j_1...j_{b-k}$ and likewise for $1^{(a)}i_1...i_s t^{(2)}i_{s+1}...i_{k-1} \otimes j_1...j_{b-k-1}$.

The Ext groups appearing in the next Lemma are both equal to $\mathbb{Z}_2$ by Lemma 2.3. We want to identify a particular map between these.

**Lemma 3.4.** Let $h = (a, 1^b)$ and $1 < k < b$. The map

$$\Ext^1(\Delta(h), D_{a+k} \otimes \land^{b-k}) \xrightarrow{\theta^*} \Ext^1(\Delta(h), D_{a+k-1} \otimes \land^{b-k+1})$$

induced by $D_{a+k} \otimes \land^{b-k} \xrightarrow{\theta} D_{a+k-1} \otimes \land^{b-k+1}$ is multiplication by $a + \epsilon_{b-k+1} - 1$.

**Proof.** Consider the semi-standard basis $B' = B_0^1 \cup B_1^1 \cup \ldots \cup B_0^b \cup B_1^b$, with the notation as in the beginning of subsection 3.1, of the codomain of the map $\Hom(\theta_1(a, b), D_{a+k} \otimes \land^{b-k})$. Likewise we denote by $C' = C_0^1 \cup C_1^1 \cup \ldots \cup C_0^b \cup C_1^b$ the semi-standard basis of the codomain of the map $\Hom(\theta_1(a, b), D_{a+k-1} \otimes \land^{b-k+1})$.

If $X$ is any one of the sets $B_0^t, B_1^t, C_0^t, C_1^t$, we let $\overline{X} = \sum_{T \in X} T$.

We claim that for each $t = 1, \ldots, b$,

$$\theta^*(\overline{B_0^t}) = \epsilon_{b-k+1} \overline{C_0^t} + \overline{C_1^t}, \quad (3.2)$$
$$\theta^*(\overline{B_1^t}) = -\epsilon_{b-k} \overline{C_1^t}, \quad (3.3)$$

Indeed, let $t > 1$. We note that $B_0^t$ consists of all $1^{(a)}t^{(2)}i_1...i_{k-2} \otimes j_1...j_{b-k}$ such that

$$i_1 < \ldots < i_{k-2}, \quad j_1 < \ldots < j_{b-k},$$
$$\{i_1, \ldots, i_{k-2}\} \cap \{j_1, \ldots, j_{b-k}\} = \emptyset, \quad \{i_1, \ldots, i_{k-2}\} \cup \{j_1, \ldots, j_{b-k}\} = \{2, \ldots, \hat{t}, \ldots, b\}.$$
and of all \(1^{(a)}t_{i_1...i_{k-1} \otimes j_1...j_{b-k-1}}\) such that

\[i_1 < ... < i_{k-1}, \ j_1 < ... < t < ... < j_{b-k-1},\]

\(\{i_1,...,i_{k-1}\} \cap \{j_1,...,j_{b-k-1}\} = \emptyset, \ \{i_1,...,i_{k-1}\} \cup \{j_1,...,j_{b-k-1}\} = \{2,...,\tilde{t},...,b\}\)

The definition of \(\theta^*\) on the above elements yields

\[
\theta^*(1^{(a)}t^{(2)}_{i_1...i_{k-2} \otimes j_1...j_{b-k}}) = 1^{(a-1)}t^{(2)}_{i_1...i_{k-2} \otimes j_1...j_{b-k}} + 1^{(a)}t_{i_1...i_{k-2} \otimes tj_1...j_{b-k}}
+ \sum_{u=1}^{k-2}1^{(a)}t^{(2)}_{i_1...\hat{i}_u...i_{k-2} \otimes i_u j_1...j_{b-k-1}),}
\]

(3.4)

\[
\theta^*(1^{(a)}t_{i_1...i_{k-1} \otimes j_1...l...j_{b-k-1-1}}) = 1^{(a-1)}t_{i_1...i_{k-1} \otimes j_1...l...j_{b-k-1}} + \sum_{u=1}^{k-1}1^{(a)}t_{i_1...\hat{i}_u...i_{k-1} \otimes i_u j_1...l...j_{b-k-1-1})),
\]

(3.5)

where \(\hat{i}_u\) means that \(i_u\) is omitted. We note that each term in the right hand side of equations (3.4) and (3.5) is of the form \(\pm S\), where \(S \in C^i\). Moreover, the terms in the right hand side of (3.4) are distinct and those in the right hand side of (3.5) are distinct. Now let \(S \in C^t = C^t_0 \cup C^t_1\).

(1) Let \(S \in C^t_0\).

(a) Suppose \(S = 1^{(a)}t_{u_1...u_{k-2} \otimes v_1...v_{b-k-1}}\). From (3.4) and (3.5), it follows that the elements \(T_i \in B^t\) such that \(S\) appears with nonzero coefficient in \(\theta^*(T_i)\) are

\[T_0 = 1^{(a)}t^{(2)}_{u_1...u_{k-2} \otimes v_1...v_{b-k}},\]
\[T_i = 1^{(a)}t_{u_1...u_{k-2}v_i \otimes v_1...\hat{v}_i...l...v_{b-k-1}}, \ i = 1, ..., b-k.\]

Moreover by straightening the \(\wedge^{b-k+1}\) part, the coefficient of \(S\) in \(\theta^*(T_0)\) is \((-1)^s, \ s = \#\{i : v_i < t\}\), and the coefficient of \(S\) in \(\theta^*(T_i)\) is

\[
\begin{cases}
(-1)^{i-1}, & \text{if } i \leq s \\
(-1)^i, & \text{if } i \geq s + 1.
\end{cases}
\]

Therefore, by summing over \(B^t_0\) be see that the coefficient of \(S\) in \(\theta^*(B^t_0)\) is

\[
\sum_{i=1}^{b-k+1}(-1)^{i-1} = \epsilon_{b-k+1}.
\]

(b) Suppose \(S = 1^{(a)}t^{(2)}_{u_1...u_{k-3} \otimes v_1...v_{b-k-1+1}}\). From (3.4) and (3.5) it follows that the elements \(T_i \in B^t\) such that \(S\) appears with nonzero coefficient in \(\theta^*(T_i)\) are

\[T_i = 1^{(a)}t^{(2)}_{u_1...u_{k-3}v_i \otimes v_1...\hat{v}_i...l...v_{b-k}}, \ i = 1, ..., b-k.\]
Moreover by straightening the $\Lambda^{b-k+1}$ part, the coefficient of $S$ in $\theta^*(T_0)$ is $(-1)^{i-1}$. Thus summing over $B_0^i$, the coefficient of $S$ in $\theta^*(B_0^i)$ is $\sum_{i=1}^{b-k+1} (-1)^{i-1} = \epsilon_{b-k+1}$.

(2) Let $S \in C'_1$.

(a) Suppose $S = 1^{(a-1)}t^{(2)}u_1...u_{k-2} \otimes 1v_1...v_{b-k}$. From (3.4) and (3.5), it follows that there is a unique $T \in B_0^i$ such that $S$ appears with nonzero coefficient in $\theta^*(T)$, namely $T = 1^{(a)}t^{(2)}u_1...u_{k-2} \otimes v_1...v_{b-k}$, and the coefficient is 1.

(b) Suppose $S = 1^{(a-1)}tu_1...u_{k-1} \otimes 1v_1...v_{b-k-1}$. From (3.4) and (3.5), it follows that there is a unique $T \in B_0^i$ such that $S$ appears with nonzero coefficient in $\theta^*(T)$, namely $T = 1^{(a)}tu_1...u_{k-1} \otimes v_1...v_{b-k-1}$, and the coefficient is 1.

From the cases (1) and (2), equation (3.2) follows for $t > 1$. The proof of (3.3), $t > 1$, is similar (and a bit shorter) and omitted. Finally, the proof of (3.2) and (3.3) for $t = 1$ is similar (and a bit simpler) and omitted.

We now prove the statement of the Lemma.

Case 1. Suppose $b - k + 1$ is even. By substituting (3.2) and (3.3) we obtain

$$\theta^*(g_k) = \binom{a}{2}C_1^1 + (a - 1) \sum_{i=2}^{b} (-1)^{i-1} C_0^i$$

and thus

$$\theta^*(g_k) - (a - 1)g_{k-1} = -\binom{a}{2} \left( (a + 1)C_0^1 + C_1^1 + 2 \sum_{i=2}^{b} (-1)^{i-1} C_0^i \right).$$

But $\pi((a+1)C_0^1 + C_1^1 + 2 \sum_{i=2}^{b} (-1)^{i-1} C_0^i) = 0$ in $E^1(\Delta(h), D_{0+k} \otimes \Lambda^{b-k+1})$ because this is the relation coming from adding the first $\binom{b}{k-1}$ columns of the matrix $e^{(1)}(a, b, D_{a+k} \otimes \Lambda^{b-k+1})$ according to Lemma 3.1 (1)-(3). Thus in this case the map $\theta^*$ is multiplication by $a - 1$.

Case 2. Suppose $b - k + 1$ is odd. By substituting (3.2) and (3.3) we obtain

$$\theta^*(g_k) = \left( a + \frac{1}{2} \right) \left( C_0^1 + C_1^1 \right) + a \sum_{i=2}^{b} (-1)^{i-1} (C_0^i + C_1^i)$$

and using this we have

$$\theta^*(g_k) - ag_{k-1} = -\binom{a}{2} \left( (a + 1)C_0^1 + C_1^1 + 2 \sum_{i=2}^{b} (-1)^{i-1} C_0^i \right).$$

Thus in this case the map $\theta^*$ is multiplication by $a$. \qed
Theorem 3.5. Let $h = (a, 1^b)$ and $h(k) = (a + k, 1^{b-k})$, where $2 \leq k \leq b$. If $n \geq b + 1$, then $\text{Ext}^{1}(\Delta(h), \Delta(h(k))) = 0$ unless $a + b + k$ is odd in which case $\text{Ext}^{1}(\Delta(h), \Delta(h(k))) = \mathbb{Z}_2$.

Proof. Applying $\text{Hom}(\Delta(h), -)$ to the short exact sequence (2.1) for $h(k-1)$ in place of $h$ yields the exact sequence

$$0 \to \text{Ext}^{1}(\Delta(h), \Delta(h(k))) \xrightarrow{i_*} \text{Ext}^{1}(\Delta(h), D_{a+k-1} \otimes \wedge^{b-k+1})$$

because $\text{Hom}(\Delta(h), \Delta(h(k-1))) = 0$ as $\mathbb{Q} \otimes \Delta(h)$ and $\mathbb{Q} \otimes \Delta(h(k-1))$ are distinct irreducible representations of $GL_n(\mathbb{Q})$.

First let $k = 2$. From the above exact sequence, Lemma 2.3 and [12], Theorem 6, it follows that $\text{Ext}^{1}(\Delta(h), \Delta(h(2)))$ is the kernel of the map $\mathbb{Z}_2 \to \mathbb{Z}_{a+b}$ which is multiplication by the integer $\frac{(a+1)(a+b)}{2}$ according to Lemma 3.3. Thus the result follows.

Suppose $k \geq 3$. Then $\text{Ext}^{1}(\Delta(h), \Delta(h(k-1)))$ injects in $\text{Ext}^{1}(\Delta(h), D_{a+k-2} \otimes \wedge^{b-k+2})$ and thus $\text{Ext}^{1}(\Delta(h), \Delta(h(k)))$ is the kernel of the composite map

$$\psi : \text{Ext}^{1}(\Delta(h), D_{a+k-1} \otimes \wedge^{b-k+1}) \to \text{Ext}^{1}(\Delta(h), D_{a+k-2} \otimes \wedge^{b-k+2}).$$

This map is induced by $D_{a+k-1} \otimes \wedge^{b-k+1} \xrightarrow{\theta} D_{a+k-2} \otimes \wedge^{b-k+2}$. According to Lemma 3.4, $\psi : \mathbb{Z}_2 \to \mathbb{Z}_2$ is by multiplication by $a + b - k - 1$. Hence the result follows. \qed

4. $\text{Ext}^{k}(\Delta(h), \Delta(h(k)))$

Let $h = (a, 1^b)$ and $h(k) = (a + k, 1^{b-k})$, where we assume throughout this section that $1 \leq k \leq b$. We will prove the following result.

Theorem 4.1. If $n \geq b + 1$, then $\text{Ext}^{k}(\Delta(h), \Delta(h(k))) = \mathbb{Z}_{d_k}$, where $d_k = \gcd((a+1^b), (a+2^b), \ldots, (a+k^b))$.

This result is known in the special cases $a = 1$, $b = k$ [1], Section 4, and any $a, b = k$ [15], eqn. (6) p. 2207.

According to the following Remark, the above Ext group is the highest possible nonzero Ext group between the hooks $h$ and $h(k)$. We thank both H. H Andersen and the referee for pointing out an error in a previous version of this paper concerning the proof of the Remark and for suggesting the proof that follows.

Remark. Let $n \geq b + 1$. If $i > k$, then $\text{Ext}^{i}(\Delta(h), \Delta(h(k))) = 0$. 

Proof. We use induction on $k$. For $k = 0$ the result follows from the general fact that no Weyl module has non trivial self extension, see [10], B.4. Remark. Applying $\text{Hom}(\Delta(h), -)$ to the short exact sequence

$$0 \to \Delta(h(k + 1)) \to \Delta(a + k) \otimes \Lambda(b - k) \to \Delta(h(k)) \to 0$$

we obtain the exact sequence

$$\text{Ext}^i(\Delta(h), \Delta(h(k))) \to \text{Ext}^{i+1}(\Delta(h), \Delta(h(k + 1))) \to \text{Ext}^{i+1}(\Delta(h), \Delta(a + k) \otimes \Lambda^{b-k}).$$

The term on the left is zero by induction. By Lemma 2.3, the term on the right is

$$\text{Ext}^{i+1}(\Delta(h), \Delta(a + k) \otimes \Lambda^{b-k}) = \text{Ext}^{i+1}(\Lambda^{k+1}, \Delta(k + 1))$$

which is zero by induction since $i + 1 > k$. Hence the middle term is zero. \qed

Recall the following notation from Section 2.3. $E^i(\Delta(h), M)$ is the cokernel of the differential $\text{Hom}(\theta_i(a, b), M)$ of the complex $\text{Hom}(P_*(a, b), M)$, where $M$ is a skew Weyl module. The torsion part of this abelian group is isomorphic to $\text{Ext}^i(\Delta(h), M)$. Let $\pi$ the natural projection $\pi : \text{Hom}(P_*(a, b), M) \to E^i(\Delta(h), M)$ and $e^{(i)}(a, b, M)$ the matrix of the map $\text{Hom}(\theta_i(a, b), M)$ with respect to orderings to be specified.

We order lexicographically the semi-standard basis of $\text{Hom}(D(a_1, ..., a_m), \Delta(h(k)))$. Now if $(a_1, ..., a_m)$ is greater than $(b_1, ..., b_{m'})$, where $m, m' \leq n$, in the usual lexicographic ordering of sequences, we declare that each element of the semi-standard basis of $\text{Hom}(D(a_1, ..., a_m), \Delta(h(k)))$ is less than each element of the semi-standard basis of $\text{Hom}(D(b_1, ..., b_{m'}), \Delta(h(k)))$. With respect to the above orderings, Remark 2.1 yields the following, where the missing entries in the bottom left part of the matrix are equal to zero.

**Lemma 4.2.** For $i > 1$ and $b > 1$, $e^{(i)}(a, b, \Delta(h(k))) = \left( \begin{array}{c|c|c} & (a + 1, b - 1, \Delta(h(k))) \\ \hline A & \star & \star \\
\end{array} \right)$, where $A = e^{(i-1)}(a + 1, b - 1, \Delta(h(k)))$.

**4.1. A generator of $\text{Ext}^k(\Delta(h), D_{a+k} \otimes \Lambda^{b-k})$**

In this subsection we will identify a generator of $\text{Ext}^k(\Delta(h), D_{a+k} \otimes \Lambda^{b-k})$. We assume throughout that $1 \leq k \leq b$. 

16
Let $\Gamma_k \in \text{Hom}(P_k(a, b), D_{a+k} \otimes \wedge^{b-k})$,

$$\Gamma_k = \binom{a+k}{k+1} \Delta_1 - \Delta_2 + \cdots + (-1)^{b-k} \Delta_q,$$

(4.1)

where $q = b - k + 1$ and

$$\Delta_1 = 1^{(a+k)} \otimes 2 \ldots q \in \text{Hom}(D(a + k, 1, \ldots, 1), D_{a+k} \otimes \wedge^{b-k})$$

$$\Delta_i = 1^{(a-1)i(k+1)} \otimes 1 \ldots \widehat{i} \ldots q$$

$$\in \text{Hom}(D(a, 1, \ldots, k + 1, \ldots, 1), D_{a+k} \otimes \wedge^{b-k}),$$

$i = 2, \ldots, q$, where $k + 1$ is located at the $i$-th position. Consider the natural projection $\pi : \text{Hom}(P_k(a, b), D_{a+k} \otimes \wedge^{b-k}) \to E^k(\Delta(h), D_{a+k} \otimes \wedge^{b-k})$.

**Lemma 4.3.** If $k + 1 = p^r$, $p$ prime, then $\pi(\Gamma_k) \neq 0$.

**Proof.** Suppose $\Gamma_k$ is equal to a linear combination of the columns of $A$, where $A = e^{(k)}(a, b, D_{a+k} \otimes \wedge^{b-k})$. Then the coefficient $(-1)^{q-1}$ of $\Delta_q$, $q = b - k + 1$, is a linear combination of the entries of the last row of $A$, as $\Delta_q$ is the last basis element in $\text{Hom}(P_k(a, b), D_{a+k} \otimes \wedge^{b-k})$ with respect to the lexicographic order. We claim that the set of nonzero elements in the last row of $A$ is $\{(-1)^{q-1}(k+1), \ldots, (-1)^{q-1}(k+1)\}$.

Indeed, let $i \in \{1, \ldots, q\}$, $\lambda = (\lambda_1, \ldots, \lambda_{q+1})$, where $\lambda_1 \in \{0, \ldots, k - 1\}$, $\lambda_j \geq 1$ for $j \in \{2, \ldots, q + 1\}$, $\sum_{j=1}^{q+1} \lambda_j = b$, and consider a semi-standard basis element $T \in \text{Hom}(D(a + \lambda_1, \lambda_2, \ldots, \lambda_{q+1}), D_{a+k} \otimes \wedge^{b-k})$, such that

$$\phi_i(T) = c\Delta_q, \quad c \in \mathbb{Z} - \{0\}.$$

By Remark 2.2, the left hand side has weight

$$\left(1^{a+\lambda_1}, 2^{\lambda_2}, \ldots, (\lambda_1+\lambda_{q+1}), \ldots, q^{\lambda_{q+1}}\right),$$

which must be equal to the weight $(1^a, 2, \ldots, q-1, q^{k+1})$ of the right hand side. So $\lambda_1 = 0$ and if $1 < i \leq q - 1$ then $\lambda_i + \lambda_{i+1} = 1$ which contradicts the hypothesis $\lambda_i \geq 1$ for $i \in \{2, \ldots, q+1\}$. This implies that $i = q$, $\lambda_1 = 0$, $\lambda_j = 1$ for $j \in \{2, \ldots, q-1\}$ and $\lambda_q + \lambda_{q+1} = k + 1$. Hence $T = 1^{(a-1)}q^{\lambda_q} (q + 1)^{\lambda_{q+1}} \otimes 1 \ldots (q - 1)$, where $\lambda_q + \lambda_{q+1} = k + 1$. For such a $T$ we have $(-1)^{q-1} \phi_q(T) = (-1)^{q-1}(k+1)\Delta_q$, which proves the claim.

It follows that $\text{gcd}\left(\binom{k+1}{1}, \dotsc, \binom{k+1}{k+1}\right) = 1$ contradicting the assumption $k + 1 = p^r$, $p$ prime. Hence $\pi(\Gamma_k) \neq 0$. \qed

Let $q = b - k + 1$. Define $T_{1,j} \in \text{Hom}(D(a + k - 1, 1, \ldots, 1), D_{a+k} \otimes \wedge^{b-k})$,

$j = 2, \ldots, q + 1,$

$$T_{1,j} = 1^{(a+k-1)}j \otimes 2 \ldots \widehat{j} \ldots (q + 1),$$
and $T_{i,j} \in \text{Hom}(D(a,1,\ldots,k,1), D_{a+k} \otimes \wedge^{b-k})$, where $k$ is at the $i$th position, $i = 2, \ldots, q$, $j = i+1, \ldots, q+1$,

$$T_{i,j} = 1^{(a-1)}i^{(k)}j \otimes 1 \ldots \hat{i} \ldots \hat{j}(q+1).$$

Let

$$A = \binom{a+k-1}{k} \sum_{j=2}^{q+1} (-1)^j T_{1,j} + \sum_{i=2,j>i}^{q,q+1} (-1)^{j-i-1} T_{i,j}$$

and consider $\phi(A)$, where $\phi$ is the differential $\phi = \text{Hom}(\theta_k(a,b), D_{a+k} \otimes \wedge^{b-k})$.

**Lemma 4.4.** We have $\phi(A) = (k+1)\Gamma_k$. Moreover, if $k+1 = p^r$, $p$ prime, then $\pi(\Gamma_k)$ is a a generator of $\text{Ext}^k(\Delta(h), D_{a+k} \otimes \wedge^{b-k})$.

**Proof.** For the first statement, it suffices to show that

$$\phi_1(A) = \binom{a+k-1}{k} (a+k) \Delta_1$$

and $\phi_t(A) = (k+1) \Delta_t$ for $t = 2, \ldots, q$, since $\binom{a+k-1}{k} (a+k) = (k+1) \binom{a+k}{k+1}$. Using Remark 2.2 an immediate calculation in each case shows the following.

For $i = 1$:

- Then $\phi_1(T_{1,2}) = (a+k) \Delta_1$.
- $\phi_1(T_{2,j}) = \binom{a+k-1}{k} \phi_1(T_{1,j})$ for all $j \geq 3$.
- $\phi_1(T_{ij}) = 0$ for all $j > i$.

Upon substituting,

$$\phi_1(A) = \binom{a+k-1}{k} (a+k) \Delta_1 + \sum_{j=2}^{q+1} (-1)^j \phi_1(T_{1,j}) + \sum_{j=3}^{q+1} (-1)^{j-3} \phi_1(T_{2,j})$$

$$= \binom{a+k-1}{k} (a+k) \Delta_1.$$

Similarly, for $t > 1$, an immediate calculation in each case yields the following.

- $\phi_t(T_{i,t}) = \phi_t(T_{i,t+1})$ and $\phi_t(T_{i,j}) = 0$ if $j \neq t, t+1$.
- $\phi_t(T_{i,t+1}) = (k+1) \Delta_t$ and $\phi_t(T_{i,j}) = \phi_t(T_{i+1,j})$ if $j \geq t + 2$.
- $\phi_t(T_{ij}) = 0$ for all $j > i$.

Upon substituting,

$$\phi_t(A) = \binom{a+k-1}{k} \left( (-1)^t \phi_t(T_{i,t}) + (-1)^{t+1} \phi_t(T_{i,t+1}) \right)$$

$$+ \sum_{i=2}^{t-1} \left( (-1)^t \phi_t(T_{i,t}) + (-1)^{t+1} \phi_t(T_{i,t+1}) \right)$$

$$+ (a+k) \Delta_t + \sum_{j=t+2}^{q+1} \left( (-1)^{j-t-1} \phi_t(T_{i,j}) + (-1)^{j-t} \phi_t(T_{i,j}) \right)$$

$$= (a+k) \Delta_t.$$

Let $k+1 = p^r$, $p$ prime. By Lemma 4.3 and the first part of the present Lemma, $\pi(\Gamma_k)$ is a nonzero torsion element of the abelian group $E^k(\Delta(h), D_{a+k} \otimes \wedge^{b-k})$. Thus it is a nonzero element of $\text{Ext}^k(\Delta(h), D_{a+k} \otimes \wedge^{b-k})$ which according to Lemma 2.3 is $\mathbb{Z}_p$. Hence it is a generator of $\text{Ext}^k(\Delta(h), D_{a+k} \otimes \wedge^{b-k})$.  \(\square\)
4.2. Relations and a generator of $E^k(\Delta(h), \Delta(h(k)))$

Let $q = b - k + 1$ and define $\delta_1 \in \text{Hom}(D(a + k, 1, \ldots, 1), \Delta(h(k)))$ and $\delta_{i,j} \in \text{Hom}(D(a + k - j, 1, \ldots, j + 1, \ldots, 1), \Delta(h(k)))$, where $j + 1$ is located at the $i$-th position, by

$$\delta_1 = 1^{(a+k)} | 2 \ldots q,$$
$$\delta_{i,j} = 1^{(a+k-j)_{i(j)}} | 2 \ldots q, i = 2, \ldots, q, j = 0, \ldots, k,$$

where it is understood that for $j = 0$ we have $\delta_{i,0} = \delta_1$.

**Lemma 4.5.** In $E^k(\Delta(h), \Delta(h(k)))$ the following relations hold.

$$\pi(\delta_{i,j}) = (a+k+i-j)\pi(\delta_1), i = 2, \ldots, q-1, j = 0, \ldots, k.$$  \hspace{1cm} (4.2)

**Proof.** Let $S_{i,j} \in \text{Hom}(P_{k-1}(a,b), \Delta(h(k))), i = 1, \ldots, q$,

$$S_{i,j} = 1^{(a+k-j)(i+1)(j)} | 2 \ldots (i+1) \ldots (q+1)$$

and consider the differential in degree $k$

$$\sum_{t \geq 1} (-1)^{t-1} \phi_t : \text{Hom}(P_{k-1}(a,b), \Delta(h(k))) \to \text{Hom}(P_k(a,b), \Delta(h(k)))$$

of the complex $\text{Hom}(P_*(a,b), \Delta(h(k)))$. An immediate calculation in each case using Remark 2.2 (and the straightening law for the first equality in the case $i > 1$) yields the following.

$$i = 1 : \phi_1(S_{1,j}) = (a+k)\delta_1, \phi_2(S_{1,j}) = \delta_{2,j}, \phi_j(S_{1,j}) = 0, j > 2.$$  

$$i > 1 : \phi_1(S_{i,j}) = (-1)^{i-1}\delta_{i,j-1}, \phi_t(S_{i,j}) = 0, t \in \{2, \ldots, i-1\}, \phi_i(S_{i,j}) = \delta_{i,j}, \phi_{i+1}(S_{i,j}) = \delta_{i+1,j}, \phi_t(S_{i,j}) = 0, t \geq i + 2.$$  \hspace{1cm} (4.3)

In $E^k(a,b, \Delta(h(k)))$, we have the relations

$$\sum_{t=1}^q (-1)^{t-1}\pi(\phi_t(S_{i,j})) = 0, i = 1, \ldots, q.$$  

Substituting the above for $i = 1$ yields

$$\pi(\delta_{2,j}) = (a+k)\pi(\delta_1), j = 0, \ldots, k$$  \hspace{1cm} (4.2)

and substituting the above for $i \geq 2$ yields

$$\pi(\delta_{i+1,j}) = \pi(\delta_{i,j}) + \pi(\delta_{i,j-1}), i = 2, \ldots, q-1, j = 1, \ldots, k.$$  \hspace{1cm} (4.3)

The equation of the Lemma follows by induction on $i$ using (4.2) and (4.3). \qed
Lemma 4.6. $E^k(\Delta(h), \Delta(h(k)))$ is a cyclic group generated by $\pi(\delta_1)$.

Proof. Induction on $k$, the case $k=1$ owing to the first paragraph of the proof of Lemma 3.3.

Let $T \in \text{Hom}(P_k(a, b), \Delta(h(k)))$ be a semi-standard tableau of weight $\lambda = (\lambda_1, ..., \lambda_q)$, $q = b - k + 1$. If $\lambda_1 > a$ then by induction and Lemma 4.2 (since $h(k) = a + 1 + (k - 1), 1^{b-1-(k-1)}$, $\pi(T)$ is a multiple of $\pi(\delta_1)$).

Suppose $\lambda_1 = a$ in which case $T = 1^{(a)}(2^{(\lambda_2-1)} \ldots q^{(\lambda_q-1)})2 \ldots q$. (In this notation it is understood that if $\lambda_j = 1$, then the term $j^{(\lambda_j-1)}$ is omitted.) We will show that there are semi-standard tableaux $T_i \in \text{Hom}(P_k(a, b), \Delta(h(k)))$ and $a_i \in \mathbb{Z}$ such that $\pi(T) = \sum_j a_i \pi(T_j)$ and $T_j < T$ for every $j$ in the lexicographic ordering of semi-standard tableaux of $\text{Hom}(P_k(a, b), \Delta(h(k)))$. Since this set is finite and $\delta_1$ is the least element, we obtain by induction on the ordering that $\pi(T) = a \pi(\delta_1)$, $a \in \mathbb{Z}$.

There exists an $i \geq 2$ such that $\lambda_i \geq 2$ because $k > 0$. Let $m$ be the largest such $i$ and let

$$S = 1^{(a)}2^{(\lambda_2-1)} \ldots m^{(\lambda_m-1)}|2 \ldots m \ldots (q+1),$$

which is a semi-standard tableau in $\text{Hom}(P_{k-1}(a, b), \Delta(h(k)))$. Then for $t \in \{1, \ldots, q\}$, straightforward calculations yield the following, where we assume that $m \geq 3$.

1. $\phi_1(S) = (-1)^{m-2}a^{\lambda_2-1}1^{(a+\lambda_2-1)}2^{(\lambda_3-1)} \ldots (m-1)^{\lambda(m-2)}|2 \ldots q.$
2. $\phi_1(S) = 0$ if $m \geq 4$, $t = 2, \ldots, m-2$.
3. $\phi_{m-1}(S) = (-1)^m(\lambda_{m-1}+\lambda_m-2)1^{(a+\lambda_2)}2^{(\lambda_3-1)} \ldots (m-1)^{\lambda_m+\lambda_m-2}|2 \ldots q.$
4. $\phi_m(S) = (-1)^{m-1}T$.
5. $\phi_T(S) = 0$, if $j \geq m + 1$.

The tableaux in the right-hand sides of equations (1) and (3) are semi-standard and less than $T$ in our ordering since $\lambda_2 > 0$, and the coefficient of $T$ in the right hand side of (4) is $\pm 1$. Hence the desired result for $m \geq 3$ follows from $\sum_{t=1}^q (-1)^{t-1} \pi \phi_t(S) = 0$.

Let $m = 2$. Then similarly, $0 = \sum_{i=1}^q (-1)^{t-1} \pi \phi_t(S) = (a^{\lambda_2-1}) \pi(\delta_1) - \pi(T)$ and the result follows. \hfill \Box

4.3. Proof of Theorem 4.1

We prove Theorem 4.1 by induction on $k$, the case $k = 1$ owing to 15.

Theorem 6. Applying $\text{Hom}(\Delta(h), -)$ to the short exact sequence

$$0 \rightarrow \Delta(h(k+1)) \overset{\delta_k}{\rightarrow} D_{a+k} \otimes \Lambda^{b-k} \overset{\pi_{h_b}}{\rightarrow} \Delta(h(k)) \rightarrow 0$$

20
yields the exact sequence

\[ \cdots \to \text{Ext}^k(\Delta(h), D_{a+k} \otimes \wedge^{b-k}) \xrightarrow{\pi_k^*} \text{Ext}^k(\Delta(h), \Delta(h(k))) \to \text{Ext}^{k+1}(\Delta(h), \Delta(h(k+1))) \to 0 \]

(4.4)

because from Lemma 2.3, \( \text{Ext}^{k+1}(\Delta(h), D_{a+k} \otimes \wedge^{b-k}) = \text{Ext}^{k+1}(\wedge^{k+1}, D_{k+1}) = 0 \) as the length of the projective resolution \( P_k(1, k) \) of \( \wedge^{k+1} \) is less than \( k+1 \).

By Lemma 2.3 we have \( \text{Ext}^k(\Delta(h), D_{a+k} \otimes \wedge^{b-k}) = Z_{r_k} \).

If \( \text{Ext}^k(\Delta(h), \Delta(h(k))) = 0 \), then by induction \( d_k = 1 \). Since \( d_k | d_{k+1} \), we have \( d_{k+1} = 1 \). Moreover, \( \text{Ext}^{k+1}(\Delta(h), \Delta(h(k+1))) = 0 \) by (4.4) and hence the result holds in this case.

We may assume that \( \text{Ext}^k(\Delta(h), \Delta(h(k))) \neq 0 \). Since \( E^k(\Delta(h), \Delta(h(k))) \) is a cyclic \( \mathbb{Z} \)-module according to Lemma 4.6, and its torsion subgroup is nonzero, we have \( E^k(\Delta(h), \Delta(h(k))) = \text{Ext}^k(\Delta(h), \Delta(h(k))) \).

**Case 1**: Let \( k+1 = p^e \), \( p \) prime. By Lemma 4.4, \( \pi(\Gamma_k) \) is a generator of \( \text{Ext}^k(\Delta(h), D_{a+k} \otimes \wedge^{b-k}) \). We compute its image under the map \( \pi_k^* \) of (4.4). With the notation established at the beginning of subsections 4.1 and 4.2 and using the straightening law and Lemma 4.5 we have

\[
\pi_k^*(\pi(\Delta_i)) = \pi(1^{(a-1)}i^{(k+1)}|1\ldots\hat{i}\ldots q) = (-1)^{i+1}\pi(\delta_{i,k})
\]

\[ = (-1)^{i+1}(\frac{a+k+i-2}{k})\pi(\delta_{1,k}). \]

By substituting in (4.1) and using the binomial coefficient identity

\[
\binom{a+k}{k+1} + \sum_{i=2}^{q} \binom{a+k+i-2}{k} = \binom{a+b}{k+1},
\]

we obtain \( \pi_k^*(\Gamma_k) = \binom{a+b}{k+1}\pi(\delta_{1,k}) \). Since, by Lemma 4.6, \( \pi(\delta_{1,k}) \) is a generator of \( \text{Ext}^k(\Delta(h), \Delta(h(k))) \), we obtain from (4.4) and the induction hypothesis that \( \text{Ext}^{k+1}(\Delta(h), \Delta(h(k+1))) = \mathbb{Z}_{d_{k+1}} \).

**Case 2**: Suppose \( k+1 \) is divisible by two distinct primes, whence according to Lemma 2.3, \( \text{Ext}^k(\Delta(h), D_{a+k} \otimes \wedge^{b-k}) = 0 \). From (4.4) it suffices to show that \( d_k = d_{k+1} \).

By Theorem 1 of \([12]\), \( d_k = \frac{a+b}{l_k} \), where \( l_k = \text{lcm}(1^{\eta_1}, 2^{\eta_2}, \ldots, k^{\eta_k}) \) and \( \eta_i = 1 \) if \( i | a+b \), and \( \eta_i = 0 \) otherwise. If \( k+1 \nmid a+b \), then \( \eta_{k+1} = 0 \) and hence \( l_k = l_{k+1} \).
If $k + 1 | a + b$, then, since $k+1$ is divisible by two distinct primes, every prime power factor of $k+1$ is less than $k+1$. Hence $l_{k+1} = l_k$. □

Let $K$ be an infinite field of characteristic $p > 0$, $S_K(n,r)$ the Schur algebra for $GL_n(K)$ and $\Delta_K(\lambda)$ the Weyl module for $S_K(n,r)$ corresponding to a partition $\lambda$ of $r$ with at most $n$ parts. Then $S_K(n,r) = K \otimes S(n,r)$ and $\Delta_K(\lambda) = K \otimes \Delta(\lambda)$. From this and the universal coefficient theorem [2], Theorem 5.3, our results yield the following.

**Corollary 4.7.** Let $K$ be an infinite field of characteristic $p > 0$ and $n \geq b + 1$.

1. Let $2 \leq k \leq b$. Then $\text{Hom}_{S_K(n,r)}(\Delta_K(h), \Delta_K(h(k))) = 0$, unless $p = 2$ and $a + b + k$ is odd, in which case $\text{Hom}_{S_K(n,r)}(\Delta_K(h), \Delta_K(h(k))) = K$.

2. Let $1 \leq k \leq b$. Then $\text{Ext}^k_{S_K(n,r)}(\Delta_K(h), \Delta_K(h(k))) = 0$, unless $p | \binom{a+b}{i}, i = 1, \ldots, k$, in which case $\text{Ext}^k_{S_K(n,r)}(\Delta_K(h), \Delta_K(h(k))) = K$.

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23