NONUNIQUENESS OF CALABI-YAU METRICS WITH MAXIMAL VOLUME GROWTH

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Abstract. We construct a family of inequivalent Calabi-Yau metrics on $\mathbb{C}^3$ asymptotic to $\mathbb{C} \times A_2$ at infinity, in the sense that any two of these metrics cannot be related by a scaling and a biholomorphism. This provides the first example of families of Calabi-Yau metrics asymptotic to a fixed tangent cone at infinity, while keeping the underlying complex structure fixed. We propose a refinement of a conjecture of Székelyhidi [23] addressing the classification of such metrics.

1. Introduction

Since the celebrated work of Yau [25], Calabi-Yau manifolds have been studied intensively in Kähler geometry, complex algebraic geometry and physics. In the complete non-compact case, much has been known in 2 complex dimensions since the foundational works of Kronheimer [14] [15] (see for example [2] [3] [4] [21] and the references therein). In higher dimensions, Conlon-Hein [9] recently classified asymptotically conical Calabi-Yau manifolds, building on the important work of Tian-Yau [24].

In this paper, we are interested in Calabi-Yau manifolds with maximal volume growth, which include asymptotically conical manifolds. In this more general setting, the tangent cones at infinity are still Calabi-Yau cones. However, in general these cones can have non-isolated singularities. Many examples of Calabi-Yau manifolds with maximal volume growth and singular tangent cones at infinity have been constructed over the years. Biquard-Gauduchon [1] constructed hyperkähler metrics on cotangent bundles of certain hermitian symmetric spaces, whose tangent cones are realized as nilpotent orbit closures in $\text{sl}(N, \mathbb{C})$. Joyce [13] constructed QALE metrics as resolutions of $\mathbb{C}^n/\Gamma$, where the action of the discrete group $\Gamma$ is not free. This approach has been generalized by Conlon-Degeratu-Rochon [7] to admit more complicated singularities. More recently, Conlon-Rochon [10], Li [16] and Székelyhidi [22] constructed Calabi-Yau metrics on $\mathbb{C}^3$ with tangent cone given by $\mathbb{C} \times A_2$ at infinity. Here $A_2$ denotes the singular hypersurface given by $\{x_1^2 + x_2^2 + x_3^2 = 0\} \subset \mathbb{C}^3$ equipped with the flat cone metric. We remark that in [10] and [22], there are various generalizations in higher dimensions that admit tangent cones of the form $\mathbb{C} \times V$, where $V$ is a Calabi-Yau cone with an isolated singularity at the vertex.

The classification of Calabi-Yau manifolds with maximal volume growth is still largely an uncharted territory. To begin, it is expected that the tangent cones at infinity are unique, as they are affine varieties [18]. Therefore one might to try to classify Calabi-Yau manifolds asymptotic to a certain tangent cone at infinity. A recent breakthrough that fits into this picture is due to Székelyhidi [23], who showed
that the Calabi-Yau metric on $\mathbb{C}^n$ asymptotic to $\mathbb{C} \times A_1$ is unique up to scaling and biholomorphism. Their method is to compare the unknown metric to scalings of a model metric using better and better holomorphic gauges. These gauges are given by adapted sequences of bases in Donaldson-Sun theory [11] in combination with certain automorphisms of the cone at infinity. The next simplest case is to study Calabi-Yau metrics on $\mathbb{C}^3$ asymptotic to $\mathbb{C} \times A_2$ at infinity, where $A_2$ is the singular hypersurface given by $\{x_1^2 + x_2^3 + x_3 = 0\} \subset \mathbb{C}^3$. An example of such a metric has been obtained by Székelyhidi in [22].

To state our result, we recall the following setup originally considered in [22]. Consider the hypersurface $X_1 \subset \mathbb{C}^{n+1}$ given by the equation

$$z + f(x_1, \ldots, x_n) = 0,$$

where $f : \mathbb{C}^n \to \mathbb{C}$ is a polynomial, so $X_1$ is biholomorphic to $\mathbb{C}^n$. Write $x = (x_1, \ldots, x_n)$.

**Setup 1.1.** We impose the following restrictions on $f$:

- $x_i$ has weight $w_i > 0$ under the action of $t \in \mathbb{C}^*$:
  $$t \cdot x_i = t^{w_i} x_i.$$

- $f$ is homogeneous of degree $d > 1$:
  $$t \cdot f(x) = f(t \cdot x) = t^d f(x).$$

- $V_0 = f^{-1}(0) \subset \mathbb{C}^n$ has an isolated singularity at $0 \in \mathbb{C}^n$.
- $V_0$ admits a Calabi-Yau cone metric $\omega_{V_0}$ compatible with the $\mathbb{C}^*$ action.

Suppose that we are in the above setup. Let $V_1 = \{1 + f(x) = 0\} \subset \mathbb{C}^n$. Then $V_1$ admits by [8] a unique asymptotically conical Calabi-Yau metric $\omega_{V_1}$ with asymptotic cone $V_0$ (see Section 2 for the precise meaning of uniqueness).

We would like to degenerate $X_1$ to its “tangent cone at infinity”: let us define a $\mathbb{C}^*$ action on $\mathbb{C}^{n+1}$ given by $F_t(z, x) = (tz, t \cdot x)$. Then $F_t^{-1} X_1$ has the equation

$$t^{1-d} z + f(x) = 0.$$

Since $d > 1$, as $t \to \infty$, $F_t^{-1} X_1 \to X_0$, where

$$X_0 = \mathbb{C} \times V_0$$

is equipped with the Calabi-Yau cone metric $\omega_0 = \sqrt{-1} \partial \bar{\partial}|z|^2 + \omega_{V_0}$. This fits into the framework of Donaldson-Sun theory [11] (see also [17] for the case when the tangent cone at infinity is smooth but the manifold is not necessarily polarized). In [22], Székelyhidi constructed a Calabi-Yau metric on $X_1$ asymptotic to $X_0$ at infinity. From the fibration point of view, the map $z : X_1 \to \mathbb{C}$ has regular fibers biholomorphic to $V_1$, and the central fiber is given by $V_0$. Roughly speaking, the metric on $X_1$ can be seen as a perturbation of the “semi-Ricci-flat” metric which restricts to scalings of $\omega_{V_1}$ on the regular fibers and $\omega_{V_0}$ on the central fiber.

In this paper, we restrict to the case when $n = 3$. Set $f = x_1^2 + x_2^3 + y^3$, where we write $y = x_3$, so $V_0$ is the $A_2$ singularity. Recall that $V_0 \cong \mathbb{C}^3/\mathbb{Z}_3$ is equipped with the flat cone metric. The variables $z, x_1, x_2, y$ have weights $1, 3, 3, 2$, respectively, and so $d = 6$ (see Example [12] for more details). We consider the hypersurface $X_{1,6} \subset \mathbb{C}^4$ given by

$$z + by + x_1^2 + x_2^3 + y^3 = 0,$$
where $b \in \mathbb{C}$. Under the $\mathbb{C}^*$ action $F_t$, $X_{1,b}$ still degenerates to $X_0$. However, the fibration structure is different from $X_1$ considered in [22] when $b \neq 0$: there are now two singular fibers, each of which has one $A_1$ singularity. For each $b \in \mathbb{C}$, we construct Calabi-Yau metrics on $X_{1,b}$ asymptotic to $X_0$. We then distinguish these metrics using certain normalization of holomorphic functions with polynomial growth. As a consequence, we obtain the main theorem of this paper:

**Theorem 1.2.** There exists a family of Calabi-Yau metrics $\omega_b$, $b \in [0, \infty)$, on $\mathbb{C}^3$ with tangent cone $\mathbb{C} \times A_2$ at infinity. Any $\omega_b$ and $\omega_{b'}$ are related by a biholomorphism and a scaling if and only if $b = b'$.

One way to understand this phenomenon of nonuniqueness is that these metrics should correspond to different ways to smooth out the $A_2$ singularity. In particular, each $X_{1,b}$ has a distinct fibration structure, with distinct singular fiber positions and singularity types.

In Sections 2 and 3, we describe our construction of $\omega_b$ by a gluing technique similar to the one used in [22]. The main difference in our case is that the fibration is more complicated, and as a result the approximate solution is not obvious to write down. A crucial observation is that in our case, away from the singular fibers and the origin, the metric should still be modeled on either $\mathbb{C} \times V_0$ or $\mathbb{C} \times V_1$ depending on the regions. This allows us to write down an approximate solution on $X_{1,b}$ using the approximate solution on $X_1$ and the nearest point projection from $X_1$ to $X_{1,b}$ (outside large compact sets) with respect to a certain cone metric on the ambient $\mathbb{C}^4$.

In Section 4, we describe our method for distinguishing these metrics, and conclude the proof of Theorem 1.2. In particular, we generalize the application of Donaldson-Sun theory [11] as seen in [23] to construct special embeddings of Calabi-Yau metrics on $\mathbb{C}^3$ with tangent cone $\mathbb{C} \times A_2$ at infinity. We also obtain a normalization of holomorphic functions from the gluing construction in the previous sections. Our method for distinguishing these metrics is then a combination these results. At the end of this paper, we propose a refinement of a conjecture of Székelyhidi [23], and discuss preliminary results as well as some difficulties that arise in this setting.

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2. Weighted analysis on $X_1$

In this section, we explain mostly without proofs the construction of the approximate solution on $X_1$, as well as the weighted analysis in [22]. We will however give a detailed proof of Proposition 2.4 below, since a consequence of its proof is a normalization of the holomorphic functions with respect to the approximate metric (see Corollary 2.6). This will be used in Section 4.

2.1. The approximate solution. We work in Setup 1.1. Recall in [22] that there is a cone metric $\sqrt{-1} \partial \overline{\partial} R^2$ on $\mathbb{C}^n$, compatible with the $\mathbb{C}^*$ action, such that the radial function $R$, when restricting to $V_0$, is uniformly equivalent to the distance function $r$ on $V_0$. Using $\sqrt{-1} \partial \overline{\partial} r^2$, we can extend $r$ homogeneously to a function, also called $r$, on $\mathbb{C}^n$. $\sqrt{-1} \partial \overline{\partial} r^2$ defines a Kähler metric on $V_1$ (away from a large
compact set) which is asymptotic to the Calabi-Yau cone $V_0$ under the nearest point projection. By [8, Theorem 2.4] and [8, Theorem 3.1], there exists a unique complete Calabi-Yau metric $\sqrt{-1} \partial \bar{\partial} \phi$ on $V_1$ asymptotic to $\sqrt{-1} \partial \bar{\partial} r^2$. In particular, $(V_1, \sqrt{-1} \partial \bar{\partial} \phi)$ is asymptotically conical with cone $V_0$.

On $\mathbb{C}^{n+1}$, define $\rho^2 = |z|^2 + R^2$. This gives a cone metric on $\mathbb{C}^{n+1}$ compatible with the $\mathbb{C}^\ast$ action.

Let $\gamma_1(s)$ be a cutoff function satisfying

$$
\gamma_1(s) = \begin{cases} 1 & \text{if } s > 2 \\
0 & \text{if } s < 1.
\end{cases}
$$

and let $\gamma_2 = 1 - \gamma_1$. Define the approximate solution, at least for $\rho > P$ for sufficiently large $P > 0$, by

$$
\omega = \partial \bar{\partial} \left( |z|^2 + \gamma_1(R\rho^{-\alpha})\psi^2 + \gamma_2(R\rho^{-\alpha})|z|^{2/d}\psi(z^{-1/d} \cdot x) \right),
$$

where $\alpha \in (1/d, 1)$ is to be chosen later. Writing $\psi = \phi - r^2$, we can rewrite $\omega$ as

$$
\omega = \partial \bar{\partial} \left( |z|^2 + r^2 + \gamma_2(R\rho^{-\alpha})|z|^{2/d}\psi(z^{-1/d} \cdot x) \right).
$$

So the potential of $\omega$ grows like $\rho^2$. In particular if $\omega$ is positive definite on $\rho > P$, then we can replace $\omega$ by a metric on $X_1$ that agrees with $\omega$ on $\rho > 2P$.

The following shows that for large enough $P$, $\omega$ defines a Kähler metric, and the Ricci potential has good enough decay.

**Proposition 2.1.** Fix $\alpha \in (1/d, 1)$. The form $\omega$ defines a Kähler metric on the subset of $X_1$ where $\rho > P$, for sufficiently large $P$. For suitable constants $\kappa, C_1 > 0$ and weight $\delta < 2/d$, the Ricci potential $h$ of $\omega$ satisfies, for large $\rho$,

$$
|\nabla^i h|_\omega < \begin{cases} C_1 \rho^{\delta - 2 - i} & \text{if } R > \kappa \rho \\
C_1 \rho^{\delta - 2 - i} & \text{if } R \in (\kappa^{-1} \rho^{1/d}, \kappa \rho) \\
C_1 \rho^{2/d - 1/d} & \text{if } R < \kappa^{-1} \rho^{1/d}.
\end{cases}
$$

If in addition $d > 3$ and $\alpha$ is chosen close to 1, then we can even choose $\delta < 0$, i.e. in this case $h$ decays faster than quadratically away from the singular rays.

Since $\omega$ defines a Kähler metric on $X_1 \cap \{ \rho > P \}$, one can modify the Kähler potential so that the new metric is defined on $X_1$ and coincides with $\omega$ on $X_1 \cap \{ \rho > 2P \}$, say. This can be done for example using the “regularized maximum” as described in [23, p.2659]. We fix a modification of $\omega$ and still call it $\omega$ in the following.

### 2.2. Weighted spaces and tangent cones.

We turn to the definition of weighted spaces. The definition will account for model geometries in different regions on $X_1$, as illustrated in the previous proposition. Recall that we want to perturb the approximate solution $\omega$ to a Calabi-Yau metric on the set $\{ \rho > A \}$ for sufficiently large $A$. To proceed, we fix a large $P < A$ such that on $\{ \rho < 2P \}$ we use the usual $C^{k, \alpha}$ norm. When $\rho > P$ we define the weighted spaces in terms of the radial distance $\rho$ and the distance to the singular rays $R$. Define the smooth function

$$
w = \begin{cases} 1 & \text{if } R > 2\kappa \rho \\
R/(\kappa \rho) & \text{if } R \in (\kappa^{-1} \rho^{1/d}, \kappa \rho) \\
\kappa^{-2} \rho^{1/d - 1} & \text{if } R < \frac{1}{2} \kappa^{-1} \rho^{1/d}
\end{cases}
$$
The three regions in the definition are “away from singular rays”, “gluing region” and “near singular rays” in order. Define the Hölder seminorm as
\[
[T]_{0,\gamma} = \sup_{\rho(z) > P} \rho(z)^\gamma w(z)^\gamma \sup_{z' \neq z, z' \in B(z, c)} \frac{|T(z) - T(z')|}{d(z, z')^\gamma}.
\]
Here \(c\) is chosen so that \(B(z, c)\) has bounded geometry and is geodesically convex.

We use parallel transport along a geodesic to compare \(T(z)\) and \(T(z')\). We can now define the weighted spaces
\[
\|f\|_{C^{k,\alpha}_{\rho,\tau}} = \|f\|_{C^{k,\alpha}(\rho<2P)} + \sum_{j=0}^k \sup_{\rho > P} \rho^{-\delta+j} w^{-\tau+j} |\nabla^j f| + [\rho^{-\delta+k} w^{-\tau+k} |\nabla^k f|]_{0,\alpha}.
\]
Alternatively, if we replace \(\rho\) by a smoothing of \(\max\{1, \rho\}\), then we can express these weighted norms with respect to the metric \(\rho^{-2} w^{-2} \omega\):
\[
\|f\|_{C^{k,\alpha}_{\rho,\tau}} = \|\rho^{-\delta} w^{-\tau} f\|_{C^{k,\alpha}_{\rho^{-2} w^{-2} \omega}}.
\]
Using these norms we can define \(C^{k,\alpha}_{\rho,\tau}(X_1, \omega)\). Since we will invert the Laplacian only on \(\rho \geq A\) for \(A\) sufficiently large, for \(f\) defined on \(\rho \geq A\) we define the norms
\[
\|f\|_{C^{k,\alpha}_{\rho,\tau}(\rho^{-1}[A,\infty))} = \inf_{\hat{f}} \|f\|_{C^{k,\alpha}_{\rho,\tau}(X_1, \omega)},
\]
where the infimum is among all extensions \(\hat{f}\) of \(f\) on \(X_1\).

We record without proof some basic properties of the weighted norms:

**Proposition 2.2.** The weighted norms we just defined enjoy the following properties:

- **If** \(f \in C^{k,\alpha}_{a,b}\) and \(g \in C^{k,\alpha}_{c,d}\), then \(\|fg\|_{C^{k,\alpha}_{a+c+b+d}} \leq \|f\|_{C^{k,\alpha}_{a,b}} \leq \|g\|_{C^{k,\alpha}_{c,d}}\).
- **If** \(a < c\), then \(\|f\|_{C^{k,\alpha}_{a,b}} \geq \|f\|_{C^{k,\alpha}_{c,b}}\), and consequently \(C^{k,\alpha}_{a,b} \subset C^{k,\alpha}_{c,b}\). This is because \(\rho > P > 1\).
- **If** \(b < d\), then \(\|f\|_{C^{k,\alpha}_{a,b}} \leq \|f\|_{C^{k,\alpha}_{c,d}}\), and consequently \(C^{k,\alpha}_{a,b} \supset C^{k,\alpha}_{a,d}\). This is because \(w \leq 1\).

We can now use the weighted spaces to compare the geometry of \(X_1\) with model spaces in different regions. Write \(g, g_0\) for the Riemannian metrics of \(\omega, \omega_0\), respectively (recall that \(\omega_0\) is the cone metric on \(X_0\)). First we consider the region
\[
U = \{\rho > A, R > \Lambda \rho^{1/d}\} \cap X_1,
\]
for large \(A, \Lambda\), and let
\[
G : U \rightarrow X_0
\]
be the nearest point projection with respect to the cone metric \(\partial \tilde{\partial} (|z|^2 + R^2)\) on \(\mathbb{C}^{n+1}\). Note that we have
\[
G(z, x) = (z, x')
\]
where \(x'\) is the nearest point projection of \(x \in \mathbb{C}^n\) with respect to the cone metric \(\partial \tilde{\partial} R^2\) on \(\mathbb{C}^n\).
Proposition 2.3. Given any $\epsilon > 0$ we can choose $\Lambda > \Lambda(\epsilon)$, and $A > A(\epsilon)$ sufficiently large so that on $\mathcal{U}$ we have

$$|\nabla^i(G^*g_0 - g)|_g < \epsilon w^{-i} \rho^{-i}.$$  

for $i \leq k + 1$. In particular, in terms of weighted spaces we have

$$\|G^*g_0 - g\|_{C^{k,\alpha}_{0,0}} < \epsilon.$$  

Next we consider the region where $\rho > A$ but $R < \Lambda(\rho)$, i.e. we are close to the singular ray. Fix $z_0 \in \mathbb{C}$ and a large constant $B > 0$. Define

$$\mathcal{V} = \{|z - z_0| < B, R < \Lambda(\rho), \rho > A\} \cap X_1.$$  

We will use regions in the form of $\mathcal{V}$ to cover the neighborhood of the singular ray. We change the coordinates as follows:

$$\hat{x} = z_0^{-1/d} x, \quad \hat{z} = z_0^{-1/d} (z - z_0).$$  

Define $\hat{R} = |z_0|^{-1/d} R$, and let $\hat{\xi} = \max\{1, \hat{R}\}$. Then $(\hat{z}, \hat{x})$ satisfies the equation

$$\hat{z}^{1/d} + 1 + f(\hat{x}) = 0,$$  

and $|\hat{z}| < B, |\hat{R}| < CA$ for some fixed constant $C$ (since $|z| \sim \rho$). In terms of the new coordinates, we define the map

$$H : \mathcal{V} \to \mathbb{C} \times V_1$$  

by $H(\hat{z}, \hat{x}) = (\hat{z}, \hat{x}')$, where $\hat{x}'$ is the nearest point projection of $\hat{x}$ onto $V_1$ with respect to the ambient cone metric.

Proposition 2.4. Given $\epsilon, \Lambda > 0$, if $A > A(\epsilon, \Lambda, B)$, then we have

$$|\nabla^i(H^*g_{\mathbb{C} \times V_1} - |z_0|^{-2/d} g)|_{|z_0|^{-2/d} g} < \epsilon \hat{\xi}^{-3}$$  

for $i \leq k + 1$. In terms of weighted spaces we have

$$\||z_0|^{2/d} H^*g_{\mathbb{C} \times V_1} - g\|_{C^{k,\alpha}_{0,0}} < \epsilon.$$  

From the above two propositions we have the following:

Proposition 2.5. Let $\epsilon > 0$. If $D$ is sufficiently large, then there are $(Dc)$-Gromov-Hausdorff approximations between the annular regions

$$X_1^D = (X_1, \omega) \cap \{D^{1/2} < \rho < D\}$$  

and

$$X_0^D = (X_0, \omega_0) \cap \{D^{1/2} < \rho < D\}$$  

Recall that $X_0 = \mathbb{C} \times V_0$ is equipped with the product metric $\omega_0 = \partial\bar{\partial}(|z|^2 + r^2)$. Consequently, the tangent cone of $(X_1, \omega)$ at infinity is $(X_0, \omega_0)$.

This is slightly different from Proposition 9 in [22]. Since the above result is crucial for obtaining the asymptotic behavior of the distance function of $\omega$, we give a detailed proof here.
Proof of Proposition 2.4. Given \( \epsilon > 0 \), the goal is to construct a \((D\epsilon)\)-Gromov-Hausdorff approximation \( G : X^D_0 \to X^D_0 \). Let \( \Lambda > 0 \). Write \( S_\Lambda = \{ z < \Lambda o^{1/d} \} \). Recall that \( S_\Lambda \) denotes a region that is close to the singular ray of \( X_0 \). Then we can decompose \( X^D_0 \) into \( X^D_0 \setminus S_\Lambda \) and \( X^D_0 \cap S_{2\Lambda} \).

First we work on \( X^D_0 \setminus S_\Lambda \). Recall from Proposition 2.3 that once \( \Lambda \) is sufficiently large, the nearest point projection \( G : X^D_0 \setminus S_\Lambda \to X^D_0 \) is a diffeomorphism onto its image, and the error in the metric is \( |g - G^*g_0|_g < \epsilon \). Let \( x_1, x_2 \in X^D_0 \setminus S_\Lambda \), and let \( \gamma \) be a curve in \( X^D_0 \setminus S_\Lambda \) connecting \( x_1 \) and \( x_2 \). Then the error in the length is given by

\[
(2.1) \quad |\text{length}_g(\gamma) - \text{length}_{g_0}(\gamma)| \leq \text{length}_{g_0}(\gamma)\epsilon.
\]

It follows that

\[
d_{X^D_0}(x_1, x_2) \leq d_{X^D_0}(G(x_1), G(x_2))(1 + \epsilon) \\
\leq d_{X^D_0}(G(x_1), G(x_2)) + 2D\epsilon.
\]

The second inequality uses the fact that \( X_0 \) is a cone. To get the reverse inequality, we can use \((2.4)\) again and get

\[
(1 - \epsilon) d_{X^D_0}(G(x_1), G(x_2)) \leq \text{length}_g(\gamma).
\]

However, we cannot yet take the infimum of the right hand side among all curves connecting \( x_1 \) and \( x_2 \), as the minimal geodesic connecting \( x_1 \) and \( x_2 \) may pass through \( X^D_0 \cap S_\Lambda \). To \( d_{X^D_0}(x_1, x_2) \) is not too much smaller than the right hand side, we turn to the study on \( X^D_0 \cap S_{2\Lambda} \).

On \( X^D_0 \cap S_{2\Lambda} \), we define \( G : X^D_0 \to X^D_0 \) by the projection

\[
\text{pr}_1 : X^D_0 \subset C \times \mathbb{C}^n \to C \subset X_0
\]

onto the singular ray of \( X_0 \). Proposition 2.4 says that there is a map \( H : X^D_0 \cap S_{2\Lambda} \to C \times V_1 \) with \( \text{pr}_1 \circ H = \text{pr}_1 \) such that \( |g - H^*g_{C \times V_1}|_g \leq \epsilon \). Consequently, for \( x_1, x_2 \) in this region, any curve \( \gamma \) connecting \( x_1 \) and \( x_2 \) satisfies

\[
\text{length}_{g_0}(\gamma) \geq (1 - \epsilon)\text{length}_{C}(G \circ \gamma) \geq (1 - \epsilon)d_{C}(G(x_1), G(x_2)).
\]

To take the infimum of the left hand side, note that the shortest curve connecting \( x_1 \) and \( x_2 \) in \( X^D_0 \) will remain in the region \( S_{2\Lambda} \), since on the “annular region” \( S_{2\Lambda} \setminus S_\Lambda \) the metric can be made arbitrarily close to the cone metric \( \omega_0 \) by letting \( \Lambda \) and \( D \) be sufficiently large. So we have

\[
d_{X^D_0}(x_1, x_2) \geq d_{C}(G(x_1), G(x_2)) - 2D\epsilon.
\]

To get the reverse inequality, we write \( H(x_1) = (z_1, p_1), H(x_2) = (z_2, p_2) \) with \( z_i \in C \) and \( p_i \in V_1 \). From the error in the metric we get

\[
d_{X^D_0}(x_1, x_2) \leq d_{C \times V_1}(H(x_1), H(x_2))(1 + \epsilon) \\
\leq (d_C(z_1, z_2) + d_{V_1}(p_1, p_2))(1 + \epsilon) \\
\leq (d_C(z_1, z_2) + d_{V_1}(o, p_1)) + d_{V_1}(o, p_2)(1 + \epsilon).
\]

Here the second inequality follows from the Pythagorean theorem, and \( o \) is a fixed point in \( V_1 \). Since \( d_{V_1}(o, \cdot) \) is equivalent to \( R \), we can estimate

\[
(2.2) \quad d_{V_1}(o, p_1) \leq CR \leq C\Lambda D^{1/d - 1}D \ll D\epsilon
\]
by choosing $D$ sufficiently large. We conclude that

\begin{equation}
(2.3) \quad d_{X^D_p}(x_1, x_2) \leq d_C(z_1, z_2) + 2D\epsilon.
\end{equation}

We now come back to the region $X^D_p \cap S^\Lambda$. Again let $x_1, x_2 \in X^D_p \cap S^\Lambda$. Let $\gamma$ be the shortest curve in $X^D_p$ connecting $x_1$ and $x_2$. Let $x'_1$ be the first point of $\gamma$ entering the region $S^\Lambda$ and let $x'_2$ be the last point exiting $S^\Lambda$. If $\gamma_1$ is the shortest curve connecting $x_1, x'_1$, then

\[ d_{X^D_p}(x_1, x'_1) = \text{length}_{\gamma}(\gamma_1) \geq d_{X^D_p}(G(x_1), G(x'_1)) - D\epsilon \]

by (2.1). The similar inequality holds for $d_{X^D_p}(x_2, x'_2)$. We then have

\[
\begin{align*}
&d_{X^D_p}(x_1, x_2) = d_{X^D_p}(x_1, x'_1) + d_{X^D_p}(x_2, x'_2) + d_{X^D_p}(x'_1, x'_2) \\
&\quad \geq (d_{X^D_p}(G(x_1), G(x'_1)) - D\epsilon) \\
&\quad + (d_{X^D_p}(G(x_2), G(x'_2)) - D\epsilon) \\
&\quad + (d_{X^D_p}(G(x'_1), G(x'_2)) - 2D\epsilon) \\
&\quad \geq d_{X^D_p}(G(x_1), G(x_2)) - 4D\epsilon
\end{align*}
\]

using the triangle inequality and (2.3).

Finally, $G(X^D_p)$ is clearly $(D\epsilon)$-dense away from the singular ray. That $G(X^D_p)$ is $(D\epsilon)$-dense near the singular ray follows from the estimate (2.2). To get the inverse Gromov-Hausdorff approximation, away from the singular ray we can use \(d_{X^D_p}(x, x') = \frac{1}{2}(d_{X^D_p}(x, x') + d_{X^D_p}(x', x))\) and near the singular ray we can first map it to \((z, o)\), where \(o \in V^1\) is a fixed point, and then map \((z, o)\) into \(X^D_p\) using \(H^{-1}\).

\[ \square \]

The following corollary will be useful in Section 4.

**Corollary 2.6.** Let $d$ denote the distance function of \((X_1, \omega)\) and let $o \in X_1$ be a fixed point. Then $d(o, \cdot)$ is uniformly equivalent to $\rho$. Moreover, we have

\[ \lim_{\rho(x) \to \infty} \frac{d(o, x)^2}{|z|^2 + r^2} = 1. \]

**Proof.** Write $\tilde{\rho}^2 = |z|^2 + r^2$. Assume for now that $o \in X_1$ is the origin, and let $x \in X_1$, which we will let $D = \rho(x) \to \infty$. First we note that by concatenating larger and larger annuli of the form $(2^i, 2^{i+1})$, Proposition 2.3 implies that the function $d(o, \cdot)$ is equivalent to $\rho$. Since $\rho$ and $\tilde{\rho}$ are homogeneous of degree 2, they are equivalent, too.

Let $x' \in X_1$ be on the minimal geodesic connecting $o$ and $x$ such that $\rho(x') = D^{1/2}$. By Proposition 2.3 for any $\epsilon > 0$ we have for sufficiently large $D$,

\begin{equation}
(2.4) \quad d(o, x') + d_{X^0}(G(x'), G(x)) - D\epsilon < d(o, x) = d(o, x') + d(x', x) \\
\quad < d(o, x') + d_{X^0}(G(x'), G(x)) + D\epsilon,
\end{equation}

where $G$ is the $(D\epsilon)$-Gromov-Hausdorff approximation given in Proposition 2.4.

Recall that away from the singular ray, $G$ is given by the nearest point projection with respect to the cone metric $\partial \theta \rho^2$ on $C^n$, and near the singular ray we have $|z| \sim \rho$ and $G$ is given by the projection onto the singular ray. It follows that $\rho(Gx) \sim \rho(x) = D$, and so $\tilde{\rho}(x) \sim D$. As $D \to \infty$, the distance of $x$ and $G(x)$ with
respect to the scaled down cone metric $D^{-2} \partial \bar{\partial} \rho^2$ converges to 0. It follows that

\[(2.5) \quad \frac{\tilde{\rho}(G(x))}{\tilde{\rho}(x)} \to 1\]

as $D \to \infty$.

Dividing the inequality (2.4) by $\tilde{\rho}(x)$, we estimate the terms as follows:

\[\frac{d(o, x')}{\tilde{\rho}(x)} \sim \frac{D^{1/2}}{D} = D^{-1/2},\]
\[\frac{d_{X_0}(G(x'), G(x))}{\tilde{\rho}(x)} \to 1,\]

as $D \to \infty$. Here the second estimate follows from the cosine law of the cone metric on $X_0$ and (2.5). Letting $D \to \infty$ we get the desired result. For arbitrary fixed point $o \in X_1$ the same result follows by an application of the triangle inequality. □

Finally, we recall the technical heart of [22], the invertibility of the Laplacian in weighted spaces:

**Proposition 2.7.** Suppose that we choose $\tau \in (4 - 2n, 0)$ (recall that $n$ is the complex dimension of $X_1$) and $\delta$ avoids a discrete set of indicial roots. For sufficiently large $A > 0$ the Laplacian

\[\Delta : C^{2, \alpha}_{\delta, \tau}(\rho^{-1}[A, \infty), \omega) \to C^{0, \alpha}_{\delta - 2, \tau - 2}(\rho^{-1}[A, \infty), \omega)\]

is surjective with inverse bounded independently of $A$.

The idea of the proof is to cover $X_1$ (outside a big compact set) by the open subset $U$ and open subsets of types $V$ near the singular rays. On each such open set, the model Laplacian is invertible with respect to the corresponding model weighted space. Then one construct a parametrix by patching local inverses together using cutoff functions.

### 3. Construction of new Calabi-Yau metrics

We now turn to constructing a new family of Calabi-Yau metrics on $\mathbb{C}^3$, building on the results in the previous section. Similar to the construction of Calabi-Yau metrics on $X_1$ as in [22], we will consider the following family of hypersurfaces

\[X_{1,b} = \{z + by + x_1^2 + x_2^2 + y^3 = 0\} \subset \mathbb{C}^4,\]

where $b \in \mathbb{C}$. More generally, we could consider

\[X_{a,b} = \{az + by + x_1^2 + x_2^2 + y^3 = 0\} \subset \mathbb{C}^4,\]

where $a \neq 0 \in \mathbb{C}$ and $b \in \mathbb{C}$. The effect of $a$ can be taken care of by rescaling the metric. So we will assume $a = 1$. Later in Section 4 we will give a detailed explanation why the following construction of Calabi-Yau metrics on $X_{a,b}$ would possibly give all the Calabi-Yau metrics on $\mathbb{C}^3$ with tangent cone $\mathbb{C} \times A_2$ at infinity.

Let $\Omega = dx_1 \wedge dx_2 \wedge dy$ be the holomorphic volume form on $X_{1,b}$. The rest of the section is dedicated to proving the following:
Theorem 3.1. There exists a complete Kähler metric \( \omega_{1,b} ^3 \) on \( X_{1,b} \) such that
\[
\omega_{1,b} ^3 = \sqrt{-1} \Omega \wedge \overline{\Omega},
\]
and that the tangent cone at the infinity given by \( C \times A_2 \).

Let
\[
\Phi = |z|^2 + \gamma_1 (R \rho^{-\alpha}) r^2 + \gamma_2 (R \rho^{-\alpha}) |z|^{2/d} \phi(z^{-1/d} \cdot (x,y))
\]
be the Kähler potential of the approximate solution on \( X_{1,b} \) constructed in the previous section. The strategy is to use the nearest point projection \( G : X_1 \cap \{ \rho > A \} \to X_{1,b} \cap \{ \rho > A \} \) with respect to the ambient cone metric \( \partial \bar{\partial} \rho^2 \) for large enough \( A > 0 \), to pull back the volume form \( \sqrt{-1} \Omega \wedge \overline{\Omega} \) as well as the complex structure \( J \) on \( X_{1,b} \), and solve
\[
(\sqrt{-1} \partial_b \bar{\partial}_b (\Phi + u)) ^3 = \sqrt{-1} \Omega_b \wedge \overline{\Omega}_b.
\]

Here \( \partial_b \) and \( \bar{\partial}_b \) are the partial differentials with respect to the complex structure \( J_b = G_\ast (J(G^{-1})_a) \), and \( \Omega_b = G_\ast \Omega \) is the pullback of the holomorphic volume form. Once this is done, we push forward this metric using \( G \) to \( X_{1,b} \) and obtain a Calabi-Yau metric outside a large compact subset. Then we extend it to a Kähler metric on \( X_{1,b} \) which is Ricci-flat outside a large compact subset. We can then apply Hein’s version of the Tian-Yau perturbation theorem [12] to perturb it again to a genuine Calabi-Yau metric on \( X_{1,b} \).

Remark 3.2. One could try to write down an explicit approximate solution on \( X_{1,b} \) without relying on the nearest point projection, and apply the techniques in the previous section directly on \( X_{1,b} \), but then an issue is that the fibration is non-trivial away from the singular fibers. This potentially would make the analysis harder. We use the nearest point projection because near the singular ray and far from the singular fibers, we are still comparing the geometry of \( X_{1,b} \) to the geometry of \( C \times V_1 \). See the proof of Proposition 3.3 below.

The nearest point projection \( G : X_1 \to X_{1,b} \) is only defined outside compact subsets containing the origin \( 0 \in C^4 \), as the cone metric \( \partial \bar{\partial} \rho^2 \) is singular at \( 0 \) (and also singular along the singular rays \( C \subset C^4 \)). Recall that scaling down the metric amounts to making the coordinate change \( z \to D^{-1} z, x \to D^{-1} \cdot x \). One might be tempted to conclude that the error going from \( X_1 \) to \( X_{1,b} \) is \( O(b \rho^{-4}) \) by comparing the defining equations. If this were true, then we may apply Hein’s perturbation theorem directly to perturb the Calabi-Yau metric \( \omega \) on \( X_1 \) to a (pullback of) Calabi-Yau metric on \( X_{1,b} \). Unfortunately this is not the case, as both \( X_1 \) and \( X_{1,b} \) converges to \( X_0 \), whose singular set is complex one-dimensional. To get meaningful \( C^{k,\alpha} \) bounds of the errors introduced by the nearest point projection, we need to apply the region analysis in Proposition 2.1 in the previous section, comparing the geometry in each region to those of different model spaces.

3.1. Decay of the Ricci potential. Let us write \( \omega_b = \sqrt{-1} \partial_b \bar{\partial}_b \Phi \) as the approximate solution. As mentioned above, we want to solve (3.1) on \( X_1 \cap \{ \rho > A \} \) for large enough \( A \). To solve for \( u \), we want to ensure that the Ricci potential
\[
h = \log \frac{\omega_b ^3}{\sqrt{-1} \Omega_b \wedge \overline{\Omega}_b}
\]
has fast enough decay in order to apply the technical results discussed in the previous section. We have the following generalization of Proposition 2.1 in our \(\mathbb{C} \times A_2\) case.

**Proposition 3.3.** Fix \(\alpha \in (1/d, 1)\). The form \(\omega_b\) defines a Kähler metric with respect to the deformed complex structure \(J_b\) on the \(X_1 \cap \{\rho > P\}\), for sufficiently large \(P\) (depending on \(b\)). For suitable constants \(\kappa, C_1 > 0\) and weight \(\delta < 2/d\), the Ricci potential \(h\) of \(\omega_b\) with respect to \(G^*(\sqrt{-1}\Omega_b \wedge \overline{\Omega}_b)\) and the error in the complex structure satisfy, for large \(\rho\),

\[
|\nabla^i h|_\omega, |\nabla^i (\omega_b - \omega)|, |\nabla^i (J_b - J)|_\omega < \max\{1, b\} \begin{cases} C_1 \rho^{\delta-2-i} & \text{if } R > \kappa \rho \\ C_1 \rho^{\delta} R^{-2-i} & \text{if } R \in (\kappa^{-1} \rho^{1/d}, \kappa \rho) \\ C_1 \rho^{\delta-2-i/d} & \text{if } R < \kappa^{-1} \rho^{1/d}. \end{cases}
\]

In fact, since \(d = 6\), we can choose \(\delta \in [-1/3, 1/3]\). In terms of the weighted spaces defined in the previous section, we have that

\[
\|h\|_{c^{\kappa,\alpha}, \omega}, \|\omega_b - \omega\|_{c^{\kappa,\alpha}, \omega}, \|J_b - J\|_{c^{\kappa,\alpha}, \omega} \leq C_k \max\{1, b\}
\]

for a uniform constant \(C_k > 0\).

**Proof.** The proof is very similar to Proposition 2.1 before. The main difference is that in this case the complex structure as well as the holomorphic volume form are deformed. As a result the Ricci potential is given by

\[
h = \log \frac{\omega_b^3}{\sqrt{-1}\Omega_b \wedge \overline{\Omega}_b} = \log \frac{\omega^3}{\sqrt{-1}\Omega \wedge \overline{\Omega}} + \log \frac{\omega_b^3}{\omega^3} + \log \frac{\sqrt{-1}\Omega_b \wedge \overline{\Omega}_b}{\sqrt{-1}\Omega \wedge \overline{\Omega}}.
\]

Here we recall that \(\Omega\) is the holomorphic volume form on \(X_1\). Thus we will have additional errors introduced by the change in the complex structure as well as the change in the volume form. For the metric, we can estimate the error by

\[
\omega_b - \omega = d(J_b - J)d\Phi.
\]

Since \(\Phi\) has growth rate 2, it follows that the error in the metric is dominated by the error in the change of the complex structure. We perform the region analysis as in the proof of Proposition 2.1.

**Region I:** Suppose \(R > \kappa \rho\) and \(\rho \in (D/2, 2D)\) for some large \(D\). Since \(R > (\kappa/2)D\), we are uniformly away from the singular rays. We study the scaled metric \(D^{-2}\omega\) in terms of the rescaled coordinates \(\tilde{z} = D^{-1}z\), \(\tilde{x} = D^{-1} \cdot x\). The equation of \(X_1\) becomes

\[
D^{1-d} \tilde{z} + f(\tilde{x}) = 0,
\]

and the equation of \(X_{1, b}\) becomes

\[
D^{1-d} \tilde{z} + bD^{2-d} \tilde{y} + f(\tilde{x}) = 0.
\]

Thus the extra error is of order \(bD^{2-d}\). We can choose any \(\delta\) such that \(\delta - 2 > 2 - d\). Since \(d = 6\), we can make \(\delta < 0\).

**Region II:** Suppose now that \(R \in (K/2, 2K)\) for some \(K < \kappa \rho\), \(K/2 > 2\rho^\alpha\) and \(\rho \in (D/2, 2D)\). In this case \(\rho\) is comparable to \(|z|\). We assume that for some fixed \(z_0\) we have \(|z - z_0| < K\). We now scale the metric by \(K\), and define

\[
\tilde{z} = K^{-1}(z - z_0), \quad \tilde{x} = K^{-1} \cdot x, \quad \tilde{r} = K^{-1}r.
\]
The equation of \( X \) is
\[
K^{-d}(K\tilde{z} + z_0) + f(\tilde{x}) = 0,
\]
while the equation of \( X_{1,b} \) is
\[
K^{-d}(K\tilde{z} + z_0) + bK^{2-d}\tilde{y} + f(\tilde{x}) = 0.
\]
Since \(|\tilde{y}| \sim 1\), thus the extra error in the Ricci potential is of order \( bK^{2-d} \). Since \( d = 6 \) and \( K > 4\rho^\alpha \), we have
\[
bK^{4-d}K^{-2} < bCD(4-d)\alpha K^{-2}
\]
for a constant \( C \). We can choose \( \delta < 0 \) such that \((4-d)\alpha < \delta \). If \( \alpha \) is close to 1 then we can choose \( \delta = -1 \).

**Region III:** Suppose \( R \in (K/2, 2K), K \in (\rho^\alpha, 2\rho^\alpha) \) and \( \rho \in (D/2, 2D) \). Thus \(|z| \) is comparable to \( D \). We are in the gluing region. We scale as in Region II. The equation of \( X \) becomes
\[
K^{-d}(K\tilde{z} + z_0) + f(\tilde{x}) = 0,
\]
and the equation of \( X_{1,b} \) becomes
\[
K^{-d}(K\tilde{z} + z_0) + bK^{2-d}\tilde{y} + f(\tilde{x}) = 0.
\]
The extra error in the Ricci potential is then again of order \( bK^{2-d} \). Since \( K \sim D^\alpha \), we can estimate it as follows:
\[
bK^{4-d}K^{-2} < bCD(4-d)\alpha K^{-2}.
\]
So here we can choose \( 0 > \delta > (4-d)\alpha \).

**Region IV:** Suppose now that \( R \in (K/2, 2K), K \in (\kappa\rho^{1/d}, \rho^\alpha/2), \) and \( \rho \in (D/2, 2D) \). Then we have \(|z| \sim D \). We scale in the same way as in Regions II, III. The equation of \( X \) is
\[
K^{-d}(K\tilde{z} + z_0) + f(\tilde{x}) = 0,
\]
and we are comparing \( X \) to \( C \times V_{K^{-d}z_0} \), given by the equation
\[
K^{-d}z_0 + f(\tilde{x}) = 0.
\]
On the other hand the error going from \( X_{1,b} \) to \( X \) is still of order \( bK^{2-d} \). Since \( K > \kappa\rho^{1/d} \), we get
\[
K^{4-d}K^{-2} \leq bCD^{4/d-1}K^{-2}.
\]
Since \( 4/d - 1 < 0 \), we can choose \( \delta < 0 \).

**Region V:** Suppose that \( R < 2\kappa^{-1}\rho^{1/d}, \rho \in (D/2, 2D) \). Then \(|z| \) is comparable to \( D \). Fix \( z_0 \) and let \( z \) be very close \( z_0 \). We scale by \(|z_0|^{1/d} \):
\[
\tilde{z} = z_0^{-1/d}(z - z_0), \quad \tilde{x} = z_0^{-1/d}x, \quad \tilde{r} = |z|^{-1/d}r.
\]
So we have \(|\tilde{z}|, |\tilde{r}| < C \). We are near the singular rays. So we compare \( X \):
\[
z_0^{1/d-1}z + 1 + f(\tilde{x}) = 0
\]
with \( C \times V_1 \):
\[
1 + f(\tilde{x}) = 0.
\]
On the other hand, the equation of \( X_{1,b} \) becomes
\[
z_0^{1/d-1}z + 1 + bz_0^{2/d-1}\tilde{y} + f(\tilde{x}) = 0.
\]
So the extra error in this case is $b|z_0|^{(2-d)(1/d)} \leq bCD^{2/d-1} \leq bCD^{5/2-d}$, where we choose $0 > \delta \geq 4/d - 1$. □

As indicated in the proof, the decay rate of the error introduced by the nearest point projection is slower than quadratic in the region close to the singular rays, so we cannot apply Heins’s perturbation theorem directly. But as the proposition concludes, we still have good decay rates that allow us to improve upon using the contraction mapping principle as in [22]. We first need to take care of the fact that in our case, the Laplacian is also perturbed:

**Lemma 3.4.** Let $\tau \in (-2, 0)$, and let $\delta$ avoid a discrete set of indicial roots. The Laplacian $\Delta_b$ with respect to the metric defined by $\omega_b$ is a map from $C^{2,\alpha}_b(\rho^{-1}[A, \infty))$ to $C^{0,\alpha}_{\delta-2,\tau-2}(\rho^{-1}[A, \infty))$ with bounded right inverse when $A$ is sufficiently large.

**Proof.** Let $P : C^{0,\alpha}_{\delta-2,\tau-2}(\rho^{-1}[A, \infty)) \to C^{2,\alpha}_b(\rho^{-1}[A, \infty))$ be the right inverse for $\Delta$. For $u \in C^{k,\alpha}_{\delta,\tau}$, by direct computation we have

$$\|\Delta_b u - \Delta u\|_{C^{0,\alpha}_{\delta-2,\tau-2}} \leq \|\nabla (g_b - g) \ast \nabla u\|_{C^{0,\alpha}_{\delta-2,\tau-2}} + \|(g_b - g) \ast \nabla^2 u\|_{C^{0,\alpha}_{\delta-2,\tau-2}}.$$ 

Using the properties of the weighted norms in Proposition 2.2, we have

$$\|\nabla (g_b - g) \ast \nabla u\|_{C^{0,\alpha}_{\delta-2,\tau-2}} \leq C\|\nabla (g_b - g)\|_{C^{0,\alpha}_{\delta-2,\tau}} \|\nabla u\|_{C^{0,\alpha}_{\delta-2,\tau-2}} \leq C\|g_b - g\|_{C^{1,\alpha}_{\delta-2,\tau}} \|u\|_{C^{2,\alpha}_{\delta-2,\tau-2}}.$$ 

Similarly,

$$\|(g_b - g) \ast \nabla^2 u\|_{C^{0,\alpha}_{\delta-2,\tau-2}} \leq C\|g_b - g\|_{C^{1,\alpha}_{\delta-2,\tau}} \|\nabla^2 u\|_{C^{0,\alpha}_{\delta-2,\tau}} \leq C\|g_b - g\|_{C^{1,\alpha}_{\delta-2,\tau}} \|u\|_{C^{2,\alpha}_{\delta-2,\tau-2}}.$$ 

It follows that

$$\|\Delta_b u - \Delta u\|_{C^{0,\alpha}_{\delta-2,\tau-2}} \leq C\|g_b - g\|_{C^{2,\alpha}_{\delta-2,\tau-2}} \|u\|_{C^{2,\alpha}_{\delta,\tau}}$$

for a uniform constant $C > 0$. By Proposition 3.3, $\|g_b - g\|_{C^{2,\alpha}_{\delta-2}(\rho^{-1}[A, \infty))}$ can be made arbitrarily small once $A \gg 1$. It follows that

$$\|u - \Delta_b Pu\|_{C^{0,\alpha}_{\delta-2,\tau-2}} \leq \|\Delta u Pu - \Delta Pu\|_{C^{0,\alpha}_{\delta-2,\tau-2}} \leq C\|g_b - g\|_{C^{2,\alpha}_{\delta-2,\tau-2}} \ll \|u\|_{C^{2,\alpha}_{\delta,\tau}}.$$ 

It follows that $\Delta_b$ admits a bounded right inverse. □

### 3.2. Perturbing to genuine solution.

We use the approximate solution $\omega$ on $X_1$ and the weighted spaces defined in the previous section. Recall that our goal is to first solve (3.1) on $X_1 \cap \{ \rho > A \}$ for large $A$. Define

$$\mathcal{B} = \{ u \in C^{2,\alpha}_{\delta,\tau} : \|u\|_{C^{2,\alpha}_{\delta,\tau}} < \varepsilon_0 \},$$
where $\tau$ is now chosen to be close to 0 and $\epsilon_0$ is sufficiently small such that $\omega + \partial\bar{\partial}u$ has the same tangent cone at infinity as $\omega$. Consider the following operator

$$F : B \to C_{\delta-2, \tau-2}^{0, \alpha}(\rho^{-1}[A, \infty))$$

$$u \mapsto \log \left( \frac{(\bar{\omega} + \sqrt{-1}\partial\bar{\partial}u)^3}{\sqrt{-1}\Omega_b \wedge \Omega_b} \right)_{\rho^{-1}[A, \infty]},$$

and write

$$F(u) = F(0) + \Delta_b u + Q(u),$$

where $Q$ is the nonlinear part of $F$. Here $F(0) = h$ is given by the Ricci potential defined above. The goal is to find $u \in B$ such that $F(u) = 0$, or equivalently

$$(3.2) \quad \Delta_b u = -F(0) - Q(u).$$

Let $P$ be the right inverse for $\Delta_b$ in Lemma 3.4. Define the map

$$N(u) = P(-F(u) - Q(u)).$$

Then finding a solution to (3.2) is the same as finding a fixed point of $N$. Note that we have a uniform bound for $P$ independent of sufficiently large $A$. Thus we can enlarge $A$ when needed. From an explicit formula for $Q$ (e.g. expand $\log \det(I + A) - \text{tr}A$ using eigenvalues for $A$), we see that if

$$\|\partial_b \bar{\partial}_b u\|_{C_{0, 0}^{\alpha}} \ll 1,$$

then we have the estimate

$$\|Q(u) - Q(v)\|_{C_{\delta-2, \tau-2}^{0, \alpha}} \leq C(\|\partial_b \bar{\partial}_b u\|_{C_{0, 0}^{\alpha}} + \|\partial_b \bar{\partial}_b v\|_{C_{0, 0}^{\alpha}})\|\partial_b \bar{\partial}_b (u - v)\|_{C_{\delta-2, \tau-2}^{0, \alpha}}.$$

To estimate $\|\partial_b \bar{\partial}_b u\|_{C_{0, 0}^{\alpha}}$ in terms of the norm of $u$, we have

$$\|\partial_b \bar{\partial}_b u\|_{C_{0, 0}^{\alpha}} \leq \|\sqrt{-1}\partial\bar{\partial}u\|_{C_{0, 0}^{\alpha}} + \|\partial_b \bar{\partial}_b - \sqrt{-1}\partial\bar{\partial}\|_{C_{0, 0}^{\alpha}}\|u\|_{C_{2, 2}^{\alpha}}$$

$$\leq C(1 + \|b - J\|_{C_{2, 2}^{\alpha}})\|u\|_{C_{2, 2}^{\alpha}}$$

by Proposition 3.3. Since we have

$$\rho^\delta w^\tau \leq C\rho^{\delta-2+(\tau-2)(1/d-1)}\rho^\tau w^2,$$

which implies

$$\|u\|_{C_{2, 2}^{\alpha}} \leq C\|u\|_{C_{2, \tau}^{\alpha}},$$

by choosing $\epsilon_0 < C\max\{1, b\}^{-1}$ for a uniform constant $C > 0$ we have

$$\|N(u) - N(v)\|_{C_{\delta-2, \tau-2}^{\alpha}} < \frac{1}{2}\|u - v\|_{C_{\delta-2, \tau-2}^{\alpha}}$$

for $u, v \in B$; i.e. $N$ is a contraction mapping. It remains to ensure that $N$ maps $B$ into $B$. First we note that by the estimates of Proposition 3.3 we have $F(0) \in C_{\delta-2, \tau-2}^{0, \alpha}$ for some $\delta' < \delta$ (increase $\delta$ if necessary) sufficiently close to $\delta$. It follows that

$$\|F(0)\|_{C_{\delta-2, \tau-2}^{0, \alpha}(\rho^{-1}[A, \infty))} < CA^{\delta'-\delta}.$$
Combining the estimates above, we have that if \( u \in \mathcal{B} \), then
\[
\|N(u)\|_{C^{2,\alpha}_k} \leq \|N(0)\|_{C^{2,\alpha}_k} + \|N(u) - N(v)\|_{C^{2,\alpha}_k} \\
\leq \|F(0)\|_{C^{0,\alpha}_{\delta,-2}(\rho^{-1}[A,\infty])} + \frac{1}{2}\|u\|_{C^{2,\alpha}_k} \\
\leq \max\{1, b\}CA^{\delta-\delta} + \frac{\epsilon_0}{2}.
\]
We see that to make \( N \) maps into \( \mathcal{B} \), we need to remove larger and larger compact subsets as \( b \) gets larger. In sum we can make \( N \) a contraction mapping by choosing \( A \) sufficiently large (depending on \( b \)). Thus there exists \( u \in C^{k,\alpha}_{\delta,\tau} (\rho^{-1}[A,\infty]) \) with \( \|u\|_{C^{2,\alpha}_k} < \epsilon_0 \) such that
\[
(\omega_b + \sqrt{-1}\partial\bar{\partial}u)^3 = \sqrt{-1}\Omega_b \wedge \Omega_b,
\]
on \( \rho^{-1}[A,\infty] \). Pushing forward to \( X_{1,b} \), we have
\[
(\partial\bar{\partial}((\Phi + u) \circ G^{-1}))^3 = \sqrt{-1}\Omega \wedge \Omega
\]
on \( \rho^{-1}[A,\infty] \). We can modify the Kähler potential so that it defines a Kähler potential \( \hat{\Phi}_b \) on \( X_{1,b} \) such that \( \partial\bar{\partial}\hat{\Phi}_b \) agrees with \( \partial\bar{\partial}((\Phi + u) \circ G^{-1}) \) on \( \rho^{-1}[2A,\infty] \).

Set \( \hat{\omega}_b = \partial\bar{\partial}\hat{\Phi}_b \).

We now apply Hein’s version \(^{12}\) of the Tian-Yau perturbation. Recall that \((X_1 \cap \rho^{-1}[A,\infty], \omega)\) is covered by sets of type \( U \) and type \( V \) as in Proposition 2.3 and Proposition 2.4, respectively. Pulling back using the nearest point projection, it follows that the same holds for \((X_{1,b} \cap \rho^{-1}[2A,\infty], \hat{\omega}_b)\). We can rescale accordingly to compare in each region to the model geometries \( X_0 \) and \( C \times V_1 \). The rescaled coordinates then give the desired \( C^{3,\alpha} \) coordinates. For the compact part we simply cover it with a finite number of coordinate balls. This shows that \((X_{1,b}, \hat{\omega}_b)\) admits a \( C^{3,\alpha} \) quasi-atlas. That \( \hat{\omega}_b \) is \( \text{SOB}(6) \) follows from that \( \hat{\omega}_b \) is Ricci-flat outside a compact subset and the tangent cone at infinity is \( X_0 \), which together imply maximal volume growth by Colding’s volume convergence \(^{[3]}\). We can then apply \(^{12}\) Proposition 4.1 to perturb \( \hat{\omega} \) to a genuine Calabi-Yau metric \( \omega_{1,b} = \partial\bar{\partial}\check{\varphi}_{1,b} \) on \( X_{1,b} \). When \( b = 0 \), this recovers the Calabi-Yau metric constructed on \( X_1 = X_{1,0} \) in \(^{22}\).

A few notes about this construction are in order. First, this construction should generalize to construct families of Calabi-Yau metrics asymptotic to \( C \times A_k, k \geq 3 \), including the ones constructed in \(^{22}\). One could consider hypersurfaces in \( \mathbb{C}^4 \) given by \( az + b_1y + b_2y^2 + \ldots + b_{k-2}y^{k-1} + x_1^2 + x_2^2 + y^{k+1} = 0 \), with \( a \neq 0 \in \mathbb{C} \) and \( b_i \in \mathbb{C} \). Second, what we know about these metrics \( \omega_{1,b} \) for now is that they are unique up to subquadratic perturbation of the Kähler potential by \(^{[5]}\) Theorem 1.3). So a small perturbation of the initial data or the choice of the right inverse of the Laplacian does not affect the resulting metric. It is not clear at this moment whether \( \omega_{1,b} \) and \( \omega_{1,b'} \) are related by an automorphism of \( \mathbb{C}^3 \) up to scaling, because the construction involves nearest point projections which are not even holomorphic to begin with. To distinguish them we need to exploit the explicit nature of the asymptotics.

More generally, we would like to know if the gluing construction above gives all the Calabi-Yau metrics on \( \mathbb{C}^3 \) with tangent cone \( \mathbb{C} \times A_2 \) at infinity. We will discuss some preliminary results in the next section.
4. Distinguishing the metrics

In this section, we conclude the proof of Theorem 1.2 and discuss some preliminary results about Conjecture 4.7 below.

Uniqueness results in singular perturbation problems are usually hard to obtain, and very few results in the Calabi-Yau setting are known. We would like to follow a similar strategy in [23] to study the classification problem in our case. For this we first compute subquadratic harmonic functions on the cone $C \times A_2$.

4.1. Subquadratic harmonic functions on cones. We first recall the following characterization of subquadratic harmonic functions of Calabi-Yau cones $C(Y)$:

**Lemma 4.1.** Suppose $C(Y)$ is a metric tangent cone of a non-collapsed Gromov-Hausdorff limit of Kähler-Einstein manifolds. Let $r$ denote the radial coordinate so that $r \partial_r$ is the homothetic vector field. Let $J$ denote the complex structure. Suppose $u$ is a harmonic function on $C(Y)$. Then we have the following:

1. If $u$ is $s$-homogeneous ($r \partial_r u = su$) with $s < 2$, then $u$ is pluriharmonic.
2. If $u$ is $2$-homogeneous harmonic, then $u = u_1 + u_2$, where $u_1$ is pluriharmonic, and $u_2$ is $J(r \partial_r)$-invariant.
3. The space of real holomorphic vector fields that commute with $r \partial_r$ can be written as $p \oplus Jp$, where $p$ is spanned by $r \partial_r$ and vector fields of the form $\nabla_u$, where $u$ is a $J(r \partial_r)$-invariant harmonic function homogeneous of degree 2. $Jp$ consists of real holomorphic Killing vector fields.

For a proof, see [5, Lemma 3.1] and the references therein. We apply this lemma to systematically calculate subquadratic harmonic functions on $C(Y)$. First we note that since $C(Y)$ is an affine variety, Lemma 4.1 (1) and (2) imply that many of these subquadratic harmonic functions are given by the real part of subquadratic holomorphic functions. We are more interested in quadratic harmonic functions and turn to real holomorphic vector fields. We note that $p$ has another characterization:

$$p = \{ V : V \text{ is real holomorphic with linear growth and } JV(r^2) = 0 \}.$$  

Since $C(Y)$ is an affine variety, it is useful to find $W = V - iJV$ first and then take the real part of $W$. We follow this approach and calculate a few examples relevant to this paper.

**Example 4.1.** Let us consider $C(Y) = C \times A_1$, defined as the hypersurface $\{ x_1^2 + \ldots + x_n^2 = 0 \} \subset C \times C^n$, $n \geq 3$. $C(Y)$ is equipped with a Ricci-flat Kähler metric

$$\omega_0 = \sqrt{-1} \partial \bar{\partial} (|z|^2 + |x|^{2\frac{n-2}{n-1}},$$

where the coordinate $z$ has weight 1 and the coordinates $x_i$ have weight $(n-1)/(n-2)$. Any (complex) holomorphic vector field $W$ in $(p \oplus Jp) \otimes C$ is given by

$$W = b(z \partial_z + a_{ij} x_i \partial_{x_j}),$$

where the coefficients $b$ and $a_{ij}$ are such that $W(x_1^2 + \ldots + x_n^2) = 0$ and $\text{Im} W(r^2) = 0$. From these two equations, we get that $b$ and $\lambda$ are real, and that $a_{ij} = \sqrt{-1} b_{ij} + \lambda \delta_{ij}$, where $(b_{ij}) \in \mathfrak{o}(n, \mathbb{R})$. Write $W_1 = b(z \partial_z + \lambda x_i \partial_{x_i})$, and $W_2 = \sqrt{-1} b_{ij} x_i \partial_{x_j}$. Note that $\text{Re} W_2(r^2)$ does not contain the $|z|^2$ term, so in particular it is not proportional...
to $r^2$. It follows that $\text{Re}W_2(r^2)$ is a harmonic function. It remains to look at $W_1$. For $\text{Re}W_1(r^2)$ to be a harmonic function, we need

$$\Delta \text{Re}W_1(r^2) = \Delta \left( b|x|^2 + \frac{n-2}{n-1}|x|^{2\frac{n-2}{n-1}} \right) = 2b + 2\lambda(n-2) = 0,$$

and so $W_1 = (n-2)z\partial_z - x_i\partial_{x_i}$. The corresponding harmonic functions are

$$u_1 = W_1(r^2) = (n-2)|z|^2 - \frac{n-2}{n-1}|x|^{2\frac{n-2}{n-1}},$$

$$u_2 = W_2(r^2) = \sqrt{-1} \frac{n-2}{n-1}|x|^{2\frac{n-2}{n-1}} b_{ij} x_i x_j.$$

In [23], the same result is obtained using Fourier transform in the $C$-direction.

**Example 4.2.** Let $A_2$ denote the singularity

$$\{x_1^2 + x_2^2 + y^3 = 0\} \subset C^3.$$

Then $A_2$ is isomorphic to $C^2/Z_3$ via the map

$$C^2 \to C^3$$

$$(z_1, z_2) \mapsto \left( \frac{z_1^3 + z_2^3}{2}, \frac{z_1^3 - z_2^3}{2\sqrt{-1}}, \zeta z_1 z_2 \right),$$

where $\zeta$ is a cubic root of $-1$. The holomorphic volume form is given by

$$\Omega = \frac{dx_1 \wedge dx_2}{3y^2}.$$

Pulling $\Omega$ back to $C^2$ gives a constant multiple of $dz_1 \wedge dz_2$. The standard flat metric on $C^2$ thus gives the correct Calabi-Yau cone metric on $A_2$. The potential $r^2$ on $A_2$, using the ambient coordinates $x_1, x_2$ and $y$, is given by

$$r^2 = \left( |x_1|^2 + |x_2|^2 + \sqrt{(|x_1|^2 + |x_2|^2)^2 - |y|^6} \right)^{1/3}$$

$$+ \left( |x_1|^2 + |x_2|^2 - \sqrt{(|x_1|^2 + |x_2|^2)^2 - |y|^6} \right)^{1/3}.$$

This can be seen by solving a cubic equation. The complexified radial vector field on $C^2$, $z_i \partial_{z_i}$, pushes forward to

$$3x_1 \partial_{x_1} + 3x_2 \partial_{x_2} + 2y \partial_y.$$

So $x_1, x_2$ have weight 3 and $y$ has weight 2. Alternatively, the weights can be read off from the complex Monge-Ampère equation.

Any (complex) holomorphic vector field of linear growth is given by

$$W = a_{ij} x_i \partial_{x_j} + by \partial_y,$$

where the coefficients $a_{ij}$ and $b$ are chosen so that $W(x_1^2 + x_2^2 + y^3) = 0$ and $\text{Im}W(r^2) = 0$. It follows that

$$W = by \partial_y + \sqrt{-1} b_{ij} x_i \partial_{x_j} + cx_i \partial_{x_i},$$

where $b, c$ are real with $3b + 2c = 0$ and $b_{ij}$ is real and skew-symmetric. Thus $W_1 = \sqrt{-1} b_{ij} x_i \partial_{x_j}$, and $W_2 = \frac{1}{2} y \partial_y + \frac{1}{2} x_i \partial_{x_i}$. $W_2$ is the (complexified) radial
vector. The space of homogeneous \((Jr\partial_r)\)-invariant quadratic growth harmonic functions on \(A_2\) is generated by

\[
u_1 = W_1(r^2) = \frac{1}{3} \sqrt{-1b_{ij}} \frac{r^2}{\sqrt{(|x_1|^2 + |x_2|^2)^2 - |y|6}} x_i\bar{x}_j,
\]

**Example 4.3.** We now assume that our cone is \(C \times A_2\). Following the calculations in the previous examples, it is easily seen that the space of \((Jr\partial_r)\)-invariant homogeneous harmonic functions with quadratic growth on

\[C \times A_2 = \{x_1^2 + x_2^2 + y^3 = 0\} \subset C^4 = C \times C^3\]

is generated by

\[
u_1 = \frac{1}{3} \sqrt{-1b_{ij}} \frac{r^2}{\sqrt{(|x_1|^2 + |x_2|^2)^2 - |y|6}} x_i\bar{x}_j,
\nu_2 = 2|z|^2 - r^2,
\]

where \(u_2\) corresponds to the vector \(W_2 = z\partial_z - \frac{1}{2}(2y\partial_y + 3x_i\partial_x_i)\).

Let us consider

\[V = \text{Re}(z\partial_z + \frac{1}{3}y\partial_y + \frac{1}{2}x_i\partial_x_i),\]

Then \(L_V\Omega = n\beta\Omega\), and

\[V(|z|^2 + r^2) - \beta(|z|^2 + r^2) = \frac{5}{18} u_2,\]

where \(\beta = \frac{4}{9}\).

\(V\) generates biholomorphisms

\[\Phi_t(z, x_1, x_2, y) = (e^{t/2}z, e^{t/4}x_1, e^{t/4}x_2, e^{t/6}y),\]

where \(t \in C\).

Let us recall the notion \(X_0 = C \times A_2\) and \(X_{1,b}\) in the previous sections. The automorphisms \(\Phi_t\) fix \(X_0\), and move \(X_{1,b}\):

\[\Phi_t(X_{1,b}) = X_{1,e^{t/3}b}.\]

Thus the only \(X_{1,b}\) that is fixed by \(\Phi_t\) is \(X_{1,0}\).

The effect of the automorphism \(\Phi_t\) on the cone metric and the holomorphic volume form on \(X_0\) is seen as

\[\Phi_t^*(|z|^2 + r^2) = e^{t/2}|z|^2 + e^{t/6}r^2,\]

\[\Phi_t^*(dx_1 \wedge dx_2 \wedge dy) = e^{2t/3}dx_1 \wedge dx_2 \wedge dy.\]

So

\[e^{-4t/9}\Phi_t^*|z|^2 + r^2 = e^{5t/9}|z|^2 + e^{-5t/18}r^2\]

defines a Calabi-Yau cone metric on \(X_0\) with the same volume form as that of \(|z|^2 + r^2\). Taking Taylor expansion, we have

\[e^{-4t/9}\Phi_t^*|z|^2 + r^2 = (|z|^2 + r^2) + \frac{5}{18} u_2 t + O(t^2).\]

It follows that up to first order, perturbing the cone metric by \(u_2\) corresponds to applying the automorphism \(\Phi_t\) and rescaling. As in [23], the reason we want
to consider $V$ in place of $\text{Re} W_2$ is that $W_2$ does not fix any of $X_{1,b}$. Since the automorphisms $\Phi$ fix the hypersurface $X_1 = X_{1,0}$, we can still prove a result similar to [23] (see Proposition 4.1 below).

4.2. Donaldson-Sun theory. We now apply Donaldson-Sun theory [11] to construct sequences of special embeddings of $C^3$ into $C^4$ using holomorphic functions with polynomial growth. The following is similar to [23, Proposition 3.1]:

**Proposition 4.2.** Suppose $X = C^3$ is equipped with a Calabi-Yau metric $\omega$ with $C \times A_2$ as tangent cone at infinity. Then there exists a sequence of holomorphic embeddings $F_i : X \to C^4$ with the following properties:

1. On the ball $B_1$, the map $F_i$ gives a $\Psi(i^{-1})$-Gromov-Hausdorff approximation to the embedding $B(0,1) \to C^4$, where $B(0,1)$ is the unit ball in $C \times A_2$.
2. The image $F_i(X)$ is given by the equation
   \[ a_i z + b_i y + x_1^2 + x_2^2 + y^3 = 0, \]
   for some $a_i > 0, b_i \geq 0$. Either all $b_i = 0$ or all $b_i \neq 0$.
3. There exists a point $o \in X$ such that $F_i(o) = 0$ for all $i$.
4. The volume form $\omega^3$ satisfies
   \[ 2^{-6i} \omega^3 = F_i^* (\sqrt{-\Omega} \wedge \overline{\Omega}), \]
   where $\Omega = a_i^{-1} dx_1 \wedge dx_2 \wedge dy$ is the holomorphic volume form on $X_{a_i,b_i} = F_i(X)$.
5. $a_i/a_{i+1} \to 2^5$ and $b_i/b_{i+1} = 2^{3/2}(a_i/a_{i+1})^{1/2} \to 2^4$ (when $b_i \neq 0$) as $i \to \infty$. Furthermore, the number $b = b_i a_i^{-1/2} 2^{3i/2}$ is independent of $i$ and independent of the sequence.

We call any sequence of embeddings satisfying the above properties a sequence of special embeddings.

**Proof.** The proof follows a similar strategy of [23 Proposition 3.1]. Let $x_1, x_2, y, z$ be holomorphic functions on the cone $X_0 = C \times A_2$ with weight 3, 3, 2, 1, correspondingly. Recall that the defining equation is given by $x_1^2 + x_2^2 + y^3 = 0$. Let $F_i = (x_1^i, x_2^i, y^i, z^i)$ be the sequence of holomorphic embeddings of $X$ into $C^4$, where the components have weights 3, 3, 2, 1 respectively, such that over the balls $B(p_i, 1) = B(p, 2^i)$ scaled down to unit size, $F_i$ converge in the Gromov-Hausdorff sense to $F = (x_1, x_2, y, z)$, the embedding of $X_0$ to $C^4$. Such a sequence of embeddings can be obtained using adapted sequences of bases for holomorphic functions (see [11] Proposition 3.26]). By comparing dimensions of the corresponding spaces of holomorphic functions with polynomial growth, we see that $x_1^i, x_2^i, y^i, z^i$ must satisfy a polynomial equation, each term of which has weight at most 6. For notational simplicity we suppress the index $i$ in the discussion below. By making a change of variables that does not change the weights of the variables (i.e. completing the squares to kill off the terms $x_1, x_2, x_1 x_2$ and a shift in $y$ by a scalar to kill off the $y^2$ term), the equation reduces to

\[ x_1^2 + x_2^2 + y^3 + f(z)y + g(z) = 0, \]

where $f(z)$ is a polynomial of degree at most 4 and $g(z)$ is a polynomial of degree at most 6. We claim that $f(z)$ can only be a constant and that $g(z)$ can only be linear. Otherwise, by putting suitable weights to $x_1, x_2, y, z$, we may assume that the variety degenerates to one of the following singular hypersurfaces:
\[
\begin{align*}
\bullet & \quad x_1^2 + x_2^2 + y^3 + z^k y = 0, \\
\bullet & \quad x_1^2 + x_2^2 + y^3 + z^l = 0, \\
\bullet & \quad x_1^2 + x_2^2 + y^3 + a z^2 y + b z^3 = 0, \quad a, b \neq 0,
\end{align*}
\]

where \(1 \leq k \leq 4\) and \(2 \leq l \leq 6\) (\(l = 1\) is biholomorphic to \(\mathbb{C}^3\)).

In the first two cases, the Milnor number of each isolated singularity is positive.

By Milnor’s fibration theorem \[19\], the smoothing has nontrivial topology. In fact, it is homotopy equivalent to a bouquet of spheres, where the number of spheres is given by the Milnor number. Therefore it cannot be homeomorphic to \(\mathbb{C}^3\). In the third case, if \(27b^2 + 4a^3 \neq 0\) then we again have an isolated singularity (it is the three-dimensional \(A_2\)). If \(27b^2 + 4a^3 = 0\), then we have an isolated line singularity of the form \(x_1^2 + x_2^2 + v w^2 = 0\) after a change of variables. In this case the Milnor fiber is still homotopy equivalent to a bouquet of spheres \[20\]. It follows that \(f(z)\) can only be a constant, and \(g(z)\) must be linear. For now we conclude that the image of \(F_i\) in \(\mathbb{C}^4\) is given by

\[
e_i + a_i z + b_i y + x_1^2 + x_2^2 + y^3 = 0,
\]

where \(e_i, a_i\) and \(b_i\) are complex numbers. To kill off the constant term, we make a change of variables \(z \rightarrow z + a_i^{-1} e_i\). We need to ensure that \(a_i^{-1} e_i \rightarrow 0\) as \(i \rightarrow 0\). As pointed out in \[23\] Lemma 5, if we have two sets of \(s\)-holomorphic functions \((z, x_1, x_2, y)\) and \((\tilde{z}, \tilde{x}_1, \tilde{x}_2, \tilde{y})\) on \(X\) with weights \((1, 3, 3, 2)\) such that

\[
a z + b y + x_1^2 + x_2^2 + y^3 = 0
\]

and

\[
c \tilde{z} + d \tilde{y} + \tilde{x}_1^2 + \tilde{x}_2^2 + \tilde{y}^3 = 0,
\]

then using the fact that both sets of \(s\)-holomorphic functions generate the space of \(s\)-holomorphic functions with growth rates \(\leq 6\), we see that \(\tilde{z} = k_1 z, \tilde{y} = k_2 y\) and \((\tilde{x}_1, \tilde{x}_2) = A(x_1, x_2)\) for some scalars \(k_1, k_2\) and an invertible matrix \(A\) with \(A^T A = k_3^2 \text{Id}\). We may assume \(A = k_3 \text{Id}\) for some scalar \(k_3\). From this it follows that the two sets of \(s\)-holomorphic functions have a common zero \(o \in X\). Since \(F_i\) converges to the standard embedding \(F : B(0, 1) \rightarrow \mathbb{C}^4\) of the cone, it follows that \(F_i(o) \rightarrow 0 \in \mathbb{C}^4\). This implies that \(a_i^{-1} e_i \rightarrow 0\). Thus we can absorb this small constant term to \(z\). We make a stop and conclude what we got so far:

- A sequence of embeddings \(F_i = (x_1^i, x_2^i, y^i, z^i)\) of \(X\) into \(\mathbb{C}^4\) such that \(F_i\) converges to \(F\) over \(B(p_i, 1) \rightarrow B(0, 1)\).
- The image \(F_i(X)\) is given by the equation

\[
a_i z + b_i y + x_1^2 + x_2^2 + y^3 = 0,
\]

with \(a_i, b_i \rightarrow 0\) as \(i \rightarrow 0\).
- There exists \(o \in X\) such that \(F_i(o) = 0\).

We still need to conclude (4) and (5) in the statement of the proposition. Pulling back the volume form using \(F_i\), we have

\[
F_i^\ast((\sqrt{-1}\Omega \wedge \overline{\Omega})) = |g_i|^2 \omega^3
\]

for some nowhere vanishing polynomial growth holomorphic function \(g_i\) on \(X\). Therefore \(g_i\) must be a constant (recall that \(X\) is biholomorphic to \(\mathbb{C}^3\)). By Colding’s volume convergence,

\[
2^{-6i} \int_{B(p, 2^{2i})} \omega^3 \rightarrow \int_{B(0, 1)} F^\ast((\sqrt{-1}\Omega \wedge \overline{\Omega}))
\]
it follows that \(2^{6i} |g_i|^2 \to 1\) as \(i \to 0\). Scaling \(z\) by a factor \(\Psi(i^{-1})\)-close to 1, we may assume \(|g_i|^2 = 2^{-6i}\). Finally, the image of \(F_i\) and \(F_{i+1}\) are given by
\[
a_iz + b_iy + x_1^2 + x_2^2 + y^2 = 0
\]
and
\[
a_{i+1}z + b_{i+1}y + x_1^2 + x_2^2 + y^2 = 0,
\]
respectively. Using the argument finding \(o\) such that \(F_i(o) = 0\) above, we see that the coefficients of these equations satisfy
\[
\frac{a_i}{k_1a_{i+1}} = \frac{b_i}{k_2b_{i+1}} = 1 = \frac{1}{k_3^2},
\]
where \(k_i\) are such that \(z^{i+1} = k_1z^i, y^{i+1} = k_2y^i, (x_1^{i+1}, x_2^{i+1}) = k_3(x_1^i, x_2^i)\). By the definition of adapted sequences of bases ([11, Proposition 3.26]), we have \(k_1 \to 2^{-1}, k_2 \to 2^{-2}, k_3 \to 2^{-3}\) as \(i \to \infty\). The negative of the powers of 2 here are the respective growth rates of the functions. From these, along with the relation given by the volume forms
\[
2^{6i}F_i^*(\sqrt{-1}\Omega \wedge \overline{\Omega}) = 2^{6(i+1)}F_{i+1}^*(\sqrt{-1}\Omega \wedge \overline{\Omega}),
\]
we deduce the limits of \(a_i/a_{i+1}\) and \(b_i/b_{i+1}\). The same method shows that \(b\) is independent of the sequence constructed here. Finally, to make \(a_i > 0\) we simply rotate the \(z\) variable. To make \(b_i \geq 0\), we compose \(F_i\) with the following linear automorphism of \(\mathbb{C}^4\):
\[
G_i(z, x_1, x_2, y) = (e^{t_i/2}z, e^{t_i/4}x_1, e^{t_i/4}x_2, e^{t_i/6}y)
\]
(this is \(\Phi_i\) in Example 4.3) for some suitable \(e^{t_i} \in S^1\). Note that \(G_i\) preserves the volume form. \(\square\)

From the proposition we immediately have the following:

**Corollary 4.3.** Suppose \(\omega, \omega'\) are two isometric Calabi-Yau metrics on \(\mathbb{C}^3\) with tangent cone \(\mathbb{C} \times A_2\) at infinity. Then \(b = b'\) in Proposition 3.2.

Note that Corollary 4.3 does not imply that the metrics we constructed in Theorem 3.1 are distinct in our sense. Actually, if we apply an automorphism and also a scaling to a metric in Corollary 4.3, then its invariant \(b\) scales correspondingly.

When \(b = 0\), we have the following uniqueness result:

**Proposition 4.4.** Let \(X\) be a Calabi-Yau manifold biholomorphic to \(\mathbb{C}^3\) with tangent cone \(\mathbb{C} \times A_2\) at infinity. If \(b = 0\) in Proposition 4.2, then up to scaling, \(X\) is isometric to \(X_{1,0}\) equipped with the Calabi-Yau metric \(\omega_{1,0}\) in Theorem 3.1.

**Proof.** The proof is very similar to the \(\mathbb{C} \times A_1\) case in [23], modulo the special embeddings established in Proposition 4.2 and the computations of quadratic harmonic functions and the corresponding vector fields and automorphisms on \(\mathbb{C} \times A_2\) that are supplemented in Example 4.3. Note that the key reason we can follow the proof in [23] is that all the vector fields and automorphisms associated to quadratic harmonic functions of \(\mathbb{C} \times A_2\) actually fix \(X_{1,0}\). \(\square\)

We now turn to distinguishing the metrics in Theorem 3.1. For this, we need the following explicit asymptotic information of the metrics that we have constructed:
In particular we have
\[ F \] 
Proposition 4.5. Let \( \omega_{1,b} \) be the Calabi-Yau metric on \( X_{1,b} \) constructed in Theorem 3.1 and let \( d \) be the distance function with respect to \( \omega_{1,b} \). Then we have
\[ \lim_{\rho \to \infty} \frac{d(0,(z,x))^2}{|z|^2 + r^2} = 1, \]
where \( 0 \in X_{1,b} \subset \mathbb{C}^4 \).

Proof. Since the metric \( \omega_{1,b} \) is a small perturbation in weighted spaces of the approximate solution \( \omega \) on \( X_1 \), this follows directly from Corollary 2.6.

Together with the above proposition, we can follow the idea of the proof of Proposition 1.2 to distinguish the model metrics \( \omega_{1,b} \) on \( X_{1,b} \):

Proposition 4.6. There exist a biholomorphism \( F : X_{1,b} \to X_{1,b'} \) and a scaling \( c > 0 \) such that \( F^*\omega_{1,b'} = c^2\omega_{1,b} \) if and only if \( b = b' \).

Proof. Let \( (z,x_1,x_2,y) \) and \( (z',x_1',x_2',y') \) be the coordinate functions on \( X_{1,b} \) and \( X_{1,b'} \), respectively. Since \( F^*\omega_{1,b'} = c^2\omega_{1,b} \), the set of functions \( (z' \circ F, x_1' \circ F, x_2' \circ F, y' \circ F) \) has the same set of growth rates that of \( (z, x_1, x_2, y) \). By comparing the equations we necessarily have \( z' \circ F = a_1z, y' \circ F = a_2y \) and \( x_1' \circ F = a_3x_1 \) (say) for some \( a_i \neq 0 \in \mathbb{C} \), and
\[ \frac{1}{a_1} = \frac{b}{b'a_2} = \frac{1}{a_2} = \frac{1}{a_3}. \]
In particular we have \( F(0) = 0 \) and \( F(x) \to \infty \) as \( \rho(x) \to \infty \). By comparing the volume forms we have
\[ |a_3|^4|a_2|^2 = c^6. \]
On the other hand, using the assumption and the fact that \( r \) is homogeneous, we get
\[ F^* \left( \frac{d(0,(z',x'))^2}{|z'|^2 + r^2} \right) = \frac{c^2d(0,(z,x))^2}{|a_1|^2(|z|^2 + r^2)}. \]
Taking limit of both sides of the above equation as \( \rho \to \infty \) and using Proposition 4.5 we conclude that \( c = |a_1| \). Combining these we see that \( c = 1 \) and \( b = b' \).

Proof of Theorem 1.2. This is a combination of Theorem 3.1 and Proposition 4.6.

Based on these elementary observations, we state the following refinement of a conjecture of Székelyhidi [23]:

Conjecture 4.7. The space of Calabi-Yau metrics on \( \mathbb{C}^3 \) with tangent cone \( \mathbb{C} \times A_2 \) at infinity, up to biholomorphism and scaling, is parametrized by \( \mathbb{C}/S^1 \cong \mathbb{R}_{\geq 0} \).

Difficulties arise when one tries to generalize the decay estimate approach in [23] to prove Conjecture 4.7. An initial technical issue is that the linear automorphisms \( \Phi_t : \mathbb{C}^4 \to \mathbb{C}^4 \) in Example 4.5 which correspond to the quadratic harmonic function \( 2|z|^2 - r^2 \) on the cone \( \mathbb{C} \times A_2 \), do not preserve the hypersurface \( X_{1,b} \). It is therefore crucial to understand how the metrics \( \omega_{1,b} \), or their potentials \( \varphi_b \), change with respect to the parameter \( b \). In the terminology in [5] Section 3, we expect that for a given \( b \), the metrics \( \Phi_t^*\omega_{1,b+\epsilon/3} \) on \( X_{1,b} \) for \( |t| \ll 1 \) form a family of model metrics parametrized by small quadratic harmonic functions on the cone \( \mathbb{C} \times A_2 \). Another difficulty, which seems more substantial, is due to the parameter space...
[0, ∞) being non-compact. In the sequence of special embeddings, if $a_i$ deviates largely from $2^{-5i}$, then $b_i$ will deviate even more from $2^{-4i}$ as the decay rate of $b_i$ is slower than $a_i$. To follow a similar argument as seen in [23], we need to have some kind of uniform control of the family of spaces $(X_{1,b_i}, \omega_{1,b_i})$ as $b \to \infty$, possibly with suitable rescalings. To overcome these difficulties, a finer gluing construction might be needed in order to understand the metric behavior in the compact region. Alternatively, one could also try to establish a priori estimates for the complex Monge-Ampère equation in the maximal volume growth setting. We leave these to future work.

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