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LIMIT-PERIODIC SCHRÖDINGER OPERATORS WITH LIPSCHITZ CONTINUOUS IDS

DAVID DAMANIK AND JAKE FILLMAN

Abstract. We show that there exist limit-periodic Schrödinger operators such that the associated integrated density of states is Lipschitz continuous. These operators arise in the inverse spectral theoretic KAM approach of Pöschel.

1. Introduction

The main result of this paper is the following.

Theorem 1.1. There are limit-periodic Schrödinger operators with a Lipschitz continuous integrated density of states.

The main point of this result is that it exhibits a new phenomenon. To the best of our knowledge there were no previous examples of Schrödinger operators with almost periodic potentials such that the associated integrated density of states (IDS) has such a strong regularity property. On the Hölder scale, the best previously known result states $\frac{1}{2}$-Hölder continuity of the integrated density of states under suitable assumptions; compare, for example, [2, 13]. We refer the reader also to [3, 12, 21] for other results on the Hölder regularity of the IDS for almost periodic Schrödinger operators.

The $\frac{1}{2}$-Hölder continuity results mentioned above are typically optimal in the context in which they are established, as demonstrated by the presence of square-root singularities at the endpoints of the gaps of the spectrum (see, e.g., [17]). This explains why a result like Theorem 1.1 was not only not known, but is indeed quite surprising.

Let us now make the terminology more precise. For the sake of simplicity we will work in the context of discrete one-dimensional Schrödinger operators, that is, operators of the form

$$[H\psi](n) = \psi(n + 1) + \psi(n - 1) + V(n)\psi(n)$$

acting in $\ell^2(\mathbb{Z})$ with a potential $V : \mathbb{Z} \to \mathbb{R}$. We write $H_V$ for $H$ if we want to emphasize the dependence on the potential. The potential $V$ is called
almost periodic if it is bounded and the set of its translates is relatively compact in $\ell^\infty(\mathbb{Z})$. In other words,

$$\Omega = \{V(\cdot - m) : m \in \mathbb{Z}\}^{\|\cdot\|_{\infty}}$$

is a compact subset of $\ell^\infty(\mathbb{Z})$. It turns out that $\Omega$ is a compact abelian group, where the group structure is the one obtained by continuous extension of the natural group structure (induced by $\mathbb{Z}$) on the set of translates. Thus, $\Omega$ carries a natural measure, namely normalized Haar measure, which we denote by $\mu$. Then, the shift $S_\omega = \omega(\cdot - 1)$ defines a minimal homeomorphism from $\Omega$ to itself, and the dynamical system $(\Omega, S)$ is uniquely ergodic (with $\mu$ being the unique preserved probability measure).

Note that each element $\omega$ of $\Omega$ belongs to $\ell^\infty(\mathbb{Z})$ and hence can serve as the potential of a Schrödinger operator:

$$(1.2) \quad [H_\omega \psi](n) = \psi(n + 1) + \psi(n - 1) + \omega(n)\psi(n)$$

for $\psi \in \ell^2(\mathbb{Z})$.

The integrated density of states $k : \mathbb{R} \to \mathbb{R}$ (henceforth IDS) can be introduced in many equivalent ways. Let us use the following definition, which is the most convenient for what follows later. By a standard result from the theory of ergodic Schrödinger operators [6, 7, 8], there exists a measure, denoted by $dk$ and called the density of states measure, such that for every continuous function $g : \mathbb{R} \to \mathbb{R}$ and $\mu$-almost every $\omega \in \Omega$, we have

$$(1.3) \quad \int g\, dk = \lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} \langle \delta_n, g(H_\omega)\delta_n \rangle.$$ 

Moreover, in the current setting, we know that (1.3) holds simultaneously for every continuous $g$ and every $\omega \in \Omega$ by unique ergodicity.

The distribution function $k$ of the density of states measure is the IDS associated with the family (1.2) (resp., the initial operator (1.1)). It is also a standard result from the general theory that $k$ is always continuous [6, 7, 8], and many papers have been devoted to the investigation of properties of $k$ that go beyond this basic result.

A potential $V$ is called periodic if there is a period $p \in \mathbb{Z}_+$ such that $V(\cdot - p) = V(\cdot)$; it is called limit-periodic if it lies in the $\ell^\infty$-closure of the space of periodic potentials. It is well known, and not hard to see, that every limit-periodic $V$ is almost periodic in the sense above.

Having recalled the necessary notions, it is now clear what our objective is. We wish to prove the existence of a limit-periodic $V$ such that for a suitable constant $C$, we have

$$(1.4) \quad \int \chi_I\, dk \leq C|I|$$

for any interval $I \subseteq \mathbb{R}$, where $|I|$ denotes the length of the interval $I$. 
We will make crucial use of an inverse spectral theoretic KAM approach to limit-periodic Schrödinger operators due to Pöschel [16]. The relevant details will be described in Section 2. The results from this paper allow us to estimate the left-hand side of (1.4) in terms of quantities that are variants of a geometric series and hence can be computed explicitly. The necessary calculations are provided in Section 3. With this explicit estimate in hand, the proof of Theorem 1.1 can then be concluded in Section 4.

Let us end the introduction with a few general remarks:

(1) We consider the one-dimensional case for convenience. Pöschel’s work [16] can treat limit-periodic Schrödinger operators in $\ell^2(\mathbb{Z}^d)$ for arbitrary $d \in \mathbb{Z}_+$. However, our main goal in this paper is to exhibit a new phenomenon and we have decided to do so in the simplest setting.

(2) The obstruction to an improvement of a $1/2$ Hölder continuity result for the IDS holds fairly generally in the regime of absolutely continuous spectral measures. Concretely, if an almost periodic Schrödinger operator $H_V$ has purely absolutely continuous spectral type and its spectrum $\Sigma$ is a homogeneous set in the sense of Carleson [4], then its density of states measure coincides with the equilibrium measure of $\Sigma$. Then, the IDS can be no better than $1/2$-Hölder continuous by standard product formulae for the equilibrium measure; compare [19, Eq. (6.4.2)].

(3) The known classes of almost periodic Schrödinger operators with singular continuous spectral measures typically come with a rather thin spectrum (e.g., of zero Lebesgue measure) or are dual to operators with absolutely continuous spectrum. In either scenario, one would not expect the IDS to be very regular. It would be nice to prove a general result to this effect.

(4) The above remarks suggest that a necessary condition for a very regular (e.g., Lipschitz continuous) IDS in the context of almost periodic Schrödinger operators may be the pure point nature of the spectral measures. Indeed, the examples we construct have pure point spectrum. Actually, they are even uniformly localized, which is a much stronger property that does not occur very often. Concretely, uniform localization is known to fail for the Anderson model and the almost Mathieu operator (in the localized regime) [11, Appendix C]. The question of necessity is also quite delicate, since it is known that the Maryland model, which is not almost-periodic (since it is unbounded), has Lipschitz IDS but does not necessarily exhibit localization [14, 18].

(5) Our result may be contrasted with the celebrated estimate of Wegner [20] for the Anderson model: Suppose $\{\omega(n)\}_{n \in \mathbb{Z}}$ is sequence of i.i.d. random variables in $\mathbb{R}$ whose common distribution (often called the single-site distribution) is absolutely continuous with bounded density $g$. The associated

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1Clearly, pure point spectrum is not sufficient, which may be seen by considering, e.g., the supercritical almost Mathieu operator with Diophantine frequency.
IDS is Lipschitz continuous; indeed,

$$|k(E_1) - k(E_2)| \leq C\|g\|_{\infty}|E_1 - E_2|$$

for a constant $C$. For expository presentations, see [1, Chapter 4] and [15, Section 5]. On the other hand, Lipschitz continuity may fail if the single-site distribution is too singular; this is proved explicitly in the strongly coupled Bernoulli case in [5], but as pointed out there, the argument extends to some more general cases. In any event, even in the Anderson model, Lipschitz continuity of the IDS does not always hold and needs sufficient regularity of the single-site distribution.

2. Pöschel's Inverse Spectral Results

In this section we recall Pöschel’s approach to the spectral analysis of limit-periodic Schrödinger operators in the large-coupling regime [16]. We focus on the inverse spectral aspect of this approach, as it is this aspect that we will employ below.

First, we define the notion of an approximation function, which is Pöschel’s mechanism for making precise the notion of having divisors that decay “not too quickly.” Given a function $\Omega : [0, \infty) \to [0, \infty)$, define

$$\Phi(t) := t^{-4} \sup_{r \geq 0} \Omega(r) e^{-tr}$$

$$\kappa_t := \left\{ \{t_j\}_{j=0}^{\infty} : t \geq t_0 \geq t_1 \geq \cdots \text{ and } \sum_{j=0}^{\infty} t_j \leq t \right\}$$

$$\Psi(t) := \inf_{\kappa_t} \prod_{j=0}^{\infty} \Phi(t_j) 2^{-j-1}$$

for $t > 0$. We say that $\Omega$ is an approximation function if $\Phi(t)$ and $\Psi(t)$ are finite for every $t > 0$. One can check that $\Omega(r) = r^\alpha$ is an approximation function for each $\alpha \geq 0$.

Now, suppose that $\mathcal{M} \subseteq \ell^\infty(\mathbb{Z})$ is a Banach subalgebra with respect to pointwise addition and pointwise multiplication (in particular, $\mathcal{M}$ is assumed to contain the constant sequence 1).

Recall the shift on $\ell^\infty(\mathbb{Z})$ is denoted by $S\omega = \omega(\cdot - 1)$. We say that $\lambda : \mathbb{Z} \to \mathbb{R}$ is a distal sequence for $\mathcal{M}$ if $(\lambda - S^k\lambda)^{-1} \in \mathcal{M}$ for each $k \neq 0$ and

$$\left\| (\lambda - S^k\lambda)^{-1} \right\|_{\infty} \leq \Omega(|k|), \quad \text{for all } k \neq 0,$$

where $\Omega$ is an approximation function.\(^2\)

**Theorem 2.1** (Pöschel, 1983 [16]). If $\lambda$ is a distal sequence for $\mathcal{M}$, then there exists $\varepsilon_0 > 0$ such that for any $0 < \varepsilon \leq \varepsilon_0$, there is a sequence $V$ so that $\lambda - V \in \mathcal{M}$ and the Schrödinger operator $H_{\varepsilon^{-1}V} = \Delta + \varepsilon^{-1}V$ is

\(^2\)Here, the inverse refers to the multiplicative inverse in the Banach algebra, where multiplication is taken to mean pointwise multiplication.
spectrally localized with eigenvalues \( \{ \varepsilon^{-1} \lambda_j : j \in \mathbb{Z} \} \). Moreover, if \( \psi_k \) is the normalized eigenvector corresponding to the eigenvalue \( \lambda_k \), then there are constants \( c > 0 \) and \( d > 1 \) such that
\[
|\psi_k(n)|^2 \leq cd^{-|k-n|}
\]
for all \( k \) and \( n \).

**Pöschel’s Example: A Limit-Periodic Distal Sequence.** Let \( \mathcal{P}_n \) denote the set of sequences in \( \ell^\infty(\mathbb{Z}) \) having period \( 2^n \), and \( \mathcal{P} = \bigcup_n \mathcal{P}_n \); the space
\[
\mathcal{L} = \mathcal{P}
\]
is a Banach algebra and a subspace of the space of all limit-periodic sequences. We may construct a distal sequence for \( \mathcal{L} \) as follows. For \( j \in \mathbb{Z}_+ \), define the set \( A_j \) by
\[
A_j := \begin{cases} 
\bigcup_{N \in \mathbb{Z}} [N \cdot 2^j, N \cdot 2^j + 2^{j-1}], & j \text{ even;} \\
\bigcup_{N \in \mathbb{Z}} [N \cdot 2^j + 2^{j-1}, (N+1) \cdot 2^j], & j \text{ odd.}
\end{cases}
\]
For example, \( A_1 = 2\mathbb{Z} + 1 \), the collection of odd integers. Let \( a_j = \chi_{A_j} \) denote the characteristic function of \( A_j \). Then, the sequence
\[
\lambda_n = \sum_{j=1}^{\infty} a_j(n) 2^{-j}
\]
belongs to \( \mathcal{L} \); moreover, the inequality
\[
\left\| (\lambda - 5^k \lambda)^{-1} \right\| \leq 16|k|
\]
for \( k \neq 0 \) means that \( \lambda \) is distal.

In order to prove Theorem 1.1, we will need to characterize the integers for which \( \lambda_n \) belongs to dyadic intervals of the form
\[
I_{m,j} := \left[ \frac{j}{2^m}, \frac{j+1}{2^m} \right).
\]

**Lemma 2.2.** For any \( m \in \mathbb{Z}_+ \) and any integer \( 0 \leq j < 2^m \), there is an integer \( \ell = \ell(j, m) \) so that
\[
\lambda_k \in I_{m,j} \iff k \in \ell + 2^m \mathbb{Z}.
\]

**Proof.** Given \( k \in \mathbb{Z} \), let us first note that the sequence \( \{a_j(k)\}_{j=1}^{\infty} \) cannot terminate in an infinite string of ones. Concretely, suppose \( k \geq 0 \) and choose \( r \) even and large enough that \( 2^r > k \). Then, for every odd \( s > r \), one has \( k \in [0, 2^{s-1}) \), whence \( k \notin A_s \). The argument for \( k < 0 \) is similar.

Given \( j \) and \( m \), let \( j = \sum_{i=1}^{m} b_i 2^{i-1} \) with \( b_i \in \{0, 1\} \). Then, since the sequence \( \{a_i(k)\} \) cannot terminate in an infinite string of 1’s, it follows that \( \lambda_k \in I_{m,j} \) if and only if \( a_i(k) = b_i \) for every \( 1 \leq i \leq m \). By [16, Lemma 2.1], there is a unique \( \ell \in [0, 2^m) \) such that
\[
(a_1(\ell), \ldots, a_m(\ell)) = (b_1, \ldots, b_m).
\]
By the definition of the \(a_j\), \((a_1, \ldots, a_m)\) is a \(2^m\)-periodic map from \(\mathbb{Z}\) to \(\{0, 1\}^m\), which concludes the proof of the lemma. \(\square\)

3. Fun and Games with Geometric Series

In this section we prove some basic statements about quantities related to geometric series. These results will allow us in the next section to estimate the weight assigned by the density of states measure of the operators in question to suitable intervals.

**Lemma 3.1.** Let \(d > 1\), \(\Delta > 0\). Then, for all \(x \in \mathbb{R}\),

\[
\sum_{j \in \mathbb{Z}} d^{-|x-j\Delta|} = \frac{d^{-s} + d^{-(\Delta-s)}}{1 - d^{-\Delta}},
\]

where \(s := \text{dist}(x, \Delta \mathbb{Z})\).

**Proof.** Let \(g(x)\) denote the left-hand side of (3.1), and note that \(g(x + \Delta) \equiv g(x)\), so it suffices to consider \(x \in [0, \Delta)\); for such \(x\), one has

\[
\begin{cases} 
  x - j\Delta \geq 0 \text{ whenever } j \leq 0 \\
  x - j\Delta \leq 0 \text{ whenever } j \geq 1.
\end{cases}
\]

Then,

\[
g(x) = \sum_{j=1}^{\infty} d^{-(j\Delta-x)} + \sum_{j=-\infty}^{0} d^{-(x-j\Delta)} = \frac{d^{-(\Delta-x)}}{1 - d^{-\Delta}} + \frac{d^{-x}}{1 - d^{-\Delta}} = \frac{d^{-s} + d^{-(\Delta-s)}}{1 - d^{-\Delta}}.
\]

In the final line we used that \(s = x\) for \(x \in [0, \Delta/2)\) and \(s = \Delta - x\) for \(x \in [\Delta/2, \Delta)\). Thus, the right-hand side of (3.1) and \(g\) coincide for \(x \in [0, \Delta)\). Since the right-hand side of (3.1) is clearly \(\Delta\)-periodic in \(x\), we are done. \(\square\)

**Corollary 3.2.** For \(d > 1\), \(m \in \mathbb{Z}_+, \ell \in \mathbb{Z}\), we have

\[
\lim_{N \to \infty} \frac{1}{N} \sum_{j \in \mathbb{Z}} \sum_{n=0}^{N-1} d^{-|n-\ell-j2^m|} = 2^{-m} \cdot \frac{1 + d^{-1}}{1 - d^{-1}}.
\]

**Proof.** For \(n \in \mathbb{Z}\), Lemma 3.1 yields

\[
\phi(n) := \sum_{j \in \mathbb{Z}} d^{-|n-\ell-j2^m|} = \frac{d^{-s} + d^{-(2^m-s)}}{1 - d^{-2^m}},
\]
where \( s = \text{dist}(n - \ell, 2^m \mathbb{Z}) \). As observed in Lemma 3.1, one has \( \phi(n + 2^m) \equiv \phi(n) \) and hence the limit on the left-hand side of (3.2) exists and satisfies

\[
\lim_{N \to \infty} \frac{1}{N} \sum_{j \in \mathbb{Z}} \sum_{n=0}^{N-1} d^{-|n - \ell - j 2^m|} = 2^{-m} \sum_{n=0}^{2^m-1} \phi(n). \tag{3.3}
\]

As \( n \) ranges from 0 to \( 2^m - 1 \), \( s \) attains all values in \((0, 2^m) \cap \mathbb{Z}\) twice and attains the values 0 and \( 2^m - 1 \) once each; note this means that \( 2^m - s \) attains each value in \([2^m - 1, 2^m] \cap \mathbb{Z}\) twice except for the endpoints. Thus, we get

\[
\sum_{n=0}^{2^m-1} \phi(n) = \frac{1}{1 - d^{-2m}} \left( \left( 2 \sum_{i=0}^{2^m} d^{-i} \right) - 1 - d^{-2m} \right)
= \frac{1}{1 - d^{-2m}} \left( \frac{2(1 - d^{-2^m - 1})}{1 - d^{-1}} - (1 + d^{-2m}) \right)
= \frac{1 + d^{-1}}{1 - d^{-1}}.
\]

In view of (3.3), we are done. \( \square \)

4. PROOF OF THEOREM 1.1

In this section we prove Theorem 1.1. In the inverse spectral theoretic Pöschel approach, we will choose the initial distal sequence so that for \( \varepsilon \) small enough, the resulting limit-periodic operator satisfies the desired estimate (1.4) with a suitable constant \( C \) after rescaling the energy to account for the \( \varepsilon \) factor. We will see that the primary object that determines the size of \( C \) is the uniform exponential decay rate of the eigenfunctions.

Since it is important for our purposes, let us briefly note that (in the present setting) (1.3) holds for all \( \omega \in \Omega \) with \( g \) replaced by \( \chi_I \); that is,

\[
\int \chi_I \, dk = \lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} \langle \delta_n, \chi_I(H_\omega) \delta_n \rangle.
\]

To see this, let \( \varepsilon > 0 \) and choose continuous functions \( f \leq \chi_I \leq g \) so that

\[
\int (\chi_I - f) \, dk < \varepsilon \quad \text{and} \quad \int (g - \chi_I) \, dk < \varepsilon;
\]

note that this already uses continuity of \( dk \). Then, for \( N \) large enough,

\[
\frac{1}{N} \sum_{n=0}^{N-1} \langle \delta_n, \chi_I(H_\omega) \delta_n \rangle 
\leq \frac{1}{N} \sum_{n=0}^{N-1} \langle \delta_n, g(H_\omega) \delta_n \rangle
< \int g \, dk + \varepsilon
< \int \chi_I \, dk + 2\varepsilon.
\]
Similarly, approximating $\chi_I$ with $f$ instead of $g$, one gets
\[ \frac{1}{N} \sum_{n=0}^{N-1} \langle \delta_n, \chi_I(H)\delta_n \rangle > \int \chi_I \, dk - 2\varepsilon \]
for all sufficiently large $N$; thus, (4.1) follows.

**Remark 4.1.** Alternatively, one may start with the statement that, for every $I$, (4.1) holds for $\mu$-almost every $\omega$, which follows readily from Birkhoff’s ergodic theorem. Picking an $\omega$ for which (4.1) holds, we may use the extension of Pöschel’s work to all $\omega \in \Omega$ by Damanik and Gan [9, 10] and proceed in a similar fashion as below, but with $H$ replaced by $H_\omega$.

**Proof of Theorem 1.1.** Let $\lambda$ denote the distal sequence defined in Section 2 and $\varepsilon > 0$ sufficiently small. Then, as discussed in Section 2, there is a limit-periodic potential $V$ so that $H := \Delta + \varepsilon^{-1}V$ has eigenvalues $\{\varepsilon^{-1}\lambda_k\}_{k \in \mathbb{Z}}$ and a complete set of eigenfunctions $\{\psi_k\}$ so that $\varepsilon H \psi_k = \lambda_k \psi_k$ and
\[ |\psi_k(n)|^2 \leq cd^{-|n-k|} \]
for constants $c > 0$ and $d > 1$.

As already mentioned, our goal is to prove (1.4) with a suitable constant $C$. It clearly suffices to do this for every (rescaled) dyadic interval $I$ of the form
\[ I = \varepsilon^{-1}I_{m,j}, \]
where $I_{m,j} = [j2^{-m}, (j + 1)2^{-m})$ as in (2.1). For convenience, we define $E_k = \varepsilon^{-1}\lambda_k$ so that $E_k \in I$ if and only if $\lambda_k \in I_{m,j}$. Thus, by Lemma 2.2, there exists $\ell = \ell(I)$ so that
\[ E_k \in I \iff k \in \{\ell + j2^m : j \in \mathbb{Z}\}. \]
Recall that by (4.1), we have
\[ \int \chi_I \, dk = \lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} \langle \delta_n, \chi_I(H)\delta_n \rangle. \]

Next note that
\[ \langle \delta_n, \chi_I(H)\delta_n \rangle = \sum_{E_k \in I} \langle \psi_k, \delta_n \rangle \langle \delta_n, \psi_k \rangle \]
\[ = \sum_{E_k \in I} |\psi_k(\delta_n)|^2. \]

Thus, we have
\[ \int \chi_I \, dk = \lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} \sum_{E_k \in I} |\psi_k(n)|^2. \]
\[
\leq \limsup_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} \sum_{j \in \mathbb{Z}} c \cdot d^{n - j 2^m} = c \cdot 2^{-m} \cdot \frac{1 + d^{-1}}{1 - d^{-1}} = c \cdot \varepsilon \cdot \frac{1 + d^{-1}}{1 - d^{-1}} \cdot |I|.
\]

Here we used (4.5) and (4.6) in the first step, (4.2) and (4.4) in the second step, (3.2) in the third step, and (4.3) in the fourth step.

This proves (1.4) with the constant \(c \cdot \varepsilon \cdot \frac{1 + d^{-1}}{1 - d^{-1}}\) for every rescaled dyadic interval \(I\) of length \(\varepsilon^{-1}2^{-m}\), which in turn implies (1.4) with the same constant for every interval \(I\), concluding the proof. \(\square\)

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