ONE HIGHER DIMENSIONAL ANALOG OF JET SCHEMES

CORNELIA YUEN

ABSTRACT. We develop the theory of truncated wedge schemes, a higher dimensional analog of jet schemes. We prove some basic properties and give an irreducibility criterion for truncated wedge schemes of a locally complete intersection variety analogous to Mustaţă’s for jet schemes. We show that the reduced subscheme structure of a truncated wedge scheme of any monomial scheme is itself a monomial scheme in a natural way by explicitly describing generators of each of the minimal primes of any truncated wedge scheme of a monomial hypersurface. We give evidence that the irreducible components of the truncated wedge schemes of a reduced monomial hypersurface all have multiplicity one.

1. Introduction

A wedge is a two-dimensional analog of an arc. More precisely, for $X$ a scheme of finite type over an algebraically closed field $k$ of characteristic zero, a wedge on $X$ is a $k$-morphism $\operatorname{Spec} k[[s, t]] \to X$, which can be thought of as an “infinitesimal surface germ” on $X$.

The study of wedges was initiated by Lejeune-Jalabert in 1980 in an attack on Nash’s conjecture [6]. Her idea rested on the observation that any wedge $\operatorname{Spec} k[[s, t]] \to X$ can be precomposed with the natural map $\operatorname{Spec} k[[s]] \to \operatorname{Spec} k[[s, t]]$, dual to the map of rings sending $t \mapsto t$ and $s \mapsto t$, producing an arc called the “center” of the wedge. She showed that Nash’s conjecture (for surfaces) could be settled by an affirmative answer to the following question: Does a wedge centered at a “general” arc on a normal surface singularity lift to its minimal resolution of singularities? Later, Reguera [9] considered this problem for wedges on higher dimensional varieties, and showed that a positive answer to Nash’s question is equivalent to a positive answer to this wedge extension problem.

In this paper, we study the analogous notion to $m$-jets. Recall that an $m$-jet is a $k$-morphism $\operatorname{Spec} k[t]/(t^{m+1}) \to X$. It is known that the set of all $m$-jets of $X$ carries a natural scheme structure, called the $m$th jet scheme of $X$ and denoted by $\mathcal{J}_m(X)$.

Definition 1.1. Let $X$ be a scheme of finite type over $k$. An $m$-wedge of $X$ is a $k$-morphism

$$\operatorname{Spec} k[s, t]/(s, t)^m \to X.$$
As with $m$-jets, the collection of all $m$-wedges forms a scheme $W_m(X)$ in a natural way, called the $m^{th}$ wedge scheme of $X$. We also have the natural projection maps $\pi_i^{m-1} : W_m(X) \to W_{m-1}(X)$ and $\pi_i : W_m(X) \to X$ induced by pulling back an $m$-wedge via the natural maps $\text{Spec } k[s, t]/(s, t)^{m+1} \hookrightarrow \text{Spec } k[s, t]/(s, t)^m$ and $\text{Spec } k \hookrightarrow \text{Spec } k[s, t]/(s, t)^{m+1}$.

Analogous to the situation with arcs, the set of all wedges also forms a scheme $W_\infty(X) = \varprojlim W_m(X)$, called the wedge scheme of $X$.

We begin our study of truncated wedge schemes by showing some basic properties, including a functorial representation, base change under étale morphisms, smoothness of truncated wedge schemes of a smooth scheme, and a description of the first truncated wedge scheme in terms of the first jet scheme. We also give an irreducibility criterion for truncated wedge schemes of a locally complete intersection variety analogous to Mustaţa’s for jet schemes.

Next we conduct a detailed study of the truncated wedge schemes of monomial schemes. We show that the reduced subscheme structure of a truncated wedge scheme of any monomial scheme is itself a monomial scheme in a natural way. We give explicit generators of each of the minimal primes of any truncated wedge scheme of a monomial hypersurface, analogous to that of Goward and Smith for jet schemes. Lastly, we give evidence that the irreducible components of the truncated wedge schemes of a reduced monomial hypersurface all have multiplicity one.

2. Properties

First we discuss how the set of all $m$-wedges of $X$ forms a scheme.

Let $X$ be the affine space $\mathbb{A}^r$, then an $m$-wedge of $X$ corresponds to a $k$-algebra homomorphism

$$k[x_1, \ldots, x_r] \to k[s, t]/(s, t)^{m+1}$$

$$x_i \mapsto x_i^{(0, 0)} + \ldots + x_i^{(m, 0)} s^m + x_i^{(m-1, 1)} s^{m-1} t + \ldots + x_i^{(0, m)} t^m.$$  

So the truncated wedge scheme $W_m(\mathbb{A}^r) \cong \text{Spec } k[x_k^{(i_k, j_k)}]$ where $1 \leq k \leq r$ and $0 \leq i_k + j_k \leq m$. In other words, the scheme $W_m(\mathbb{A}^r) = \mathbb{A}^{\frac{1}{2}r(m+1)(m+2)}$.

Now if $X \subseteq \mathbb{A}^r$ is a closed subscheme, then its truncated wedge scheme $W_m(X)$ is a closed subscheme of $W_m(\mathbb{A}^r)$. Say $X = \text{Spec } k[x_1, \ldots, x_r]/(f_1, \ldots, f_d)$. Then an $m$-wedge of $X$ corresponds to a $k$-algebra homomorphism

$$\phi : k[x_1, \ldots, x_r]/(f_1, \ldots, f_d) \to k[s, t]/(s, t)^{m+1}$$

$$x_i \mapsto x_i^{(0, 0)} + \ldots + x_i^{(m, 0)} s^m + \ldots + x_i^{(0, m)} t^m,$$  

subject to the relations $\phi(f_k) = 0$. Therefore, $W_m(X)$ is defined by the ideal $W_m(X)$ whose generators $g_{ij}$ are the coefficients of $s^i t^j$ in $\phi(f_k)$ for $1 \leq k \leq d$, $0 \leq i + j \leq m$. Note that this computation commutes with localization. So this local construction of truncated wedge schemes can be patched together to give a scheme structure on the set of $m$-wedges of $X$ for any scheme $X$ of finite type over $k$. 


Next, we show that truncated wedge schemes have nice properties similar to those of jet schemes. The first example is truncated wedge schemes as representing schemes of a functor.

**Proposition 2.1.** Given a scheme $X$ of finite type over $k$, the $m^{th}$ wedge scheme $W_m(X)$ represents the functor

$$F : k\text{-Schemes} \to \text{Sets}$$

$$Z \mapsto \text{Hom}_k(Z \times_k \text{Spec } k[s, t]/(s, t)^{m+1}, X),$$

where the right hand side denotes the set of morphisms of $k$-schemes from the scheme $Z \times_k \text{Spec } k[s, t]/(s, t)^{m+1}$ to $X$.

**Proof.** Our goal is to show that the functor $F$ described in the proposition is equivalent to the functor of points of $W_m(X)$:

$$F_{W_m(X)} : k\text{-Schemes} \to \text{Sets}$$

$$Z \mapsto \text{Hom}_k(Z, W_m(X)).$$

To understand the set $\text{Hom}_k(Z \times_k \text{Spec } k[s, t]/(s, t)^{m+1}, X)$, we may assume both $X$ and $Z$ are affine [3 Proposition VI-2]. Say $X = \text{Spec } k[x_1, \ldots, x_r]/(f_1, \ldots, f_d)$ and $Z = \text{Spec } R$ for some $k$-algebra $R$. An element of $\text{Hom}_k(Z \times_k \text{Spec } k[s, t]/(s, t)^{m+1}, X)$ corresponds to a $k$-algebra map

$$\phi : k[x_1, \ldots, x_r]/(f_1, \ldots, f_d) \to R[s, t]/(s, t)^{m+1}$$

$$x_k \mapsto x_k^{(0,0)} + \ldots + x_k^{(0,m)}t^m$$

where $x_k^{(i_j,j_k)} \in R$ arbitrary, subject to the conditions $\phi(f_l) = 0$ for $l = 1, \ldots, d$. So an element of $\text{Hom}_k(Z \times_k \text{Spec } k[s, t]/(s, t)^{m+1}, X)$ corresponds to an $N$-tuple $\alpha = (\alpha_1^{(0,0)}, \ldots, \alpha_1^{(1,m-1)}, \alpha_1^{(0,m)}, \ldots, \alpha_l^{(0,0)}, \ldots, \alpha_l^{(0,m)}) \in R^N$ satisfying the equations $\phi(f_l) = 0$ for all $l$. This means that $\alpha$ is an $R$-valued point of $W_m(X)$. In other words, we have shown that

$$\text{Hom}_k(Z \times_k \text{Spec } k[s, t]/(s, t)^{m+1}, X) = \text{Hom}_k(Z, W_m(X))$$

for all $k$-schemes $Z$, and hence the scheme $W_m(X)$ represents the functor $F$. \qed

Now using this functorial point of view, we will show that truncated wedge schemes behave well under étale morphisms.

**Proposition 2.2.** If $f : X \to Y$ is an étale morphism of finite type $k$-schemes, then $W_m(X) \cong W_m(Y) \times_Y X$ for all $m$.

**Proof.** We show the equality on the level of the corresponding functor of points. That is, we show the two sets of $k$-morphisms

$$\text{Hom}_k(Z, W_m(X)) = \text{Hom}_k(Z \times_k \text{Spec } k[s, t]/(s, t)^{m+1}, X)$$
and

$$\text{Hom}_k(Z, \mathcal{W}_m(Y) \times_Y X) = \text{Hom}_k(Z, \mathcal{W}_m(Y)) \times_{\text{Hom}_k(Z, Y)} \text{Hom}_k(Z, X)$$

are the same for all $k$-schemes $Z$. Note it suffices to check the equality for all affine schemes $Z$. Fix a $k$-scheme $Z = \text{Spec } R$ and consider the commutative diagram

$$
\begin{array}{ccc}
Z & \xrightarrow{\gamma} & \text{Spec } k[s, t]/(s, t)^{m+1} \\
\alpha \downarrow & & \downarrow \beta \\
X & \xrightarrow{\delta} & Y
\end{array}
$$

where the top horizontal map is the closed embedding $\text{Spec } R \hookrightarrow \text{Spec } R[s, t]/(s, t)^{m+1}$.

Obviously, a $Z$-valued $m$-wedge $\gamma \in \text{Hom}_k(\text{Spec } R[s, t]/(s, t)^{m+1}, X)$ of $X$ induces a $Z$-valued $m$-wedge $\beta \in \text{Hom}_k(\text{Spec } R[s, t]/(s, t)^{m+1}, Y)$ and a map $\alpha \in \text{Hom}_k(\text{Spec } R, X)$ by composition. Conversely, the maps $\alpha$ and $\beta$ together induce a unique map $\gamma$ by the definition of formal étaleness [3, Definition (17.1.1)]. So we have shown

$$\text{Hom}_k(Z \times_k \text{Spec } k[s, t]/(s, t)^{m+1}, X) = \text{Hom}_k(Z \times_k \text{Spec } k[s, t]/(s, t)^{m+1}, Y) \times_{\text{Hom}_k(Z, Y)} \text{Hom}_k(Z, X),$$

or equivalently,

$$\text{Hom}_k(Z, \mathcal{W}_m(X)) = \text{Hom}_k(Z, \mathcal{W}_m(Y) \times_Y X)$$

for all $k$-schemes $Z$. That is, the schemes $\mathcal{W}_m(X)$ and $\mathcal{W}_m(Y \times_Y X)$ are isomorphic. \qed

When $X$ is a smooth scheme over $k$, we also have the result that $\mathcal{W}_m(X)$ is an affine bundle over $X$:

**Corollary 2.3.** Let $X$ be a smooth scheme over $k$ of dimension $n$. Then $\mathcal{W}_m(X)$ is locally an $\mathbb{A}^{\frac{1}{2}mn(m+3)}$-bundle over $X$. In particular, $\mathcal{W}_m(X)$ is smooth of dimension $\frac{1}{2}n(m+1)(m+2)$.

**Proof.** Since $X$ is smooth over $k$, $X$ is covered by an open affine cover $\{U_i\}_i$ with $U_i \rightarrow V_i$ étale, for some $V_i$ open subset of $\mathbb{A}^n$. Then

$$\mathcal{W}_m(U_i) \cong \mathcal{W}_m(V_i) \times_{V_i} U_i \quad \text{(by Proposition 2.2)}$$

$$\cong (\mathcal{W}_m(\mathbb{A}^n) \times_{\mathbb{A}^n} V_i) \times_{V_i} U_i \quad \text{(since an open immersion is étale)}$$

$$\cong \mathcal{W}_m(\mathbb{A}^n) \times_{\mathbb{A}^n} U_i.$$

Now $\mathcal{W}_m(\mathbb{A}^n)$ is an $\mathbb{A}^{\frac{1}{2}nm(m+3)}$-bundle over $\mathbb{A}^n$, therefore $\mathcal{W}_m(X)$ is an $\mathbb{A}^{\frac{1}{2}nm(m+3)}$-bundle over $X$. \qed

Since a tangent vector on a scheme $X$ is simply a 1-jet by definition, the first jet scheme of $X$ is the total tangent space of $X$. Interestingly, the first wedge scheme is also related to it.

**Proposition 2.4.** Let $X$ be any scheme of finite type over $k$. Then $\mathcal{W}_1(X)$ is isomorphic to $\mathcal{J}_1(X) \times_X \mathcal{J}_1(X)$ in the category of $k$-schemes.
Proof. We show equality on the level of functors of points; that is, we show the two sets of \( k \)-morphisms

\[
\text{Hom}_k(Z, W_1(X)) = \text{Hom}_k(Z \times_k \text{Spec} k[s, t]/(s, t)^2, X)
\]

and

\[
\text{Hom}_k(Z, J_1(X) \times_X J_1(X))
\]

are the same for all \( k \)-schemes \( Z \). Again, it is sufficient to check the equality for all affine schemes \( Z \).

In the category of \( k \)-Algebras, we always have coproducts and pushout squares while products and pullback squares are much less common (for definition of pushout squares and pullback squares, see \[7\] p. 65 and p. 71 respectively). However, we do have the following pullback square:

\[
\begin{array}{ccc}
  k[s, t]/(s, t)^2 & \longrightarrow & k[s, t]/(s^2, t) \\
  \downarrow & & \downarrow \\
  k[s, t]/(s^2) & \longrightarrow & k
\end{array}
\]

where the horizontal maps are natural surjections sending \( t \) to zero, and the vertical maps are natural surjections sending \( s \) to zero. That is, the ring \( k[s, t]/(s, t)^2 \) is a product in the category of \( k \)-algebras.

We leave the checking to the reader, but caution that analogous diagrams with \( m > 2 \) in place of 2 are not also pullback squares. Now we apply the contravariant functor \( \text{Spec}(\cdot) \) to our pullback square of \( k \)-algebras, and get the commutative diagram:

\[
\begin{array}{ccc}
  \text{Spec} k[s, t]/(s, t)^2 & \longleftarrow & \text{Spec} k[s, t]/(s^2, t) \\
  \uparrow & & \uparrow \\
  \text{Spec} k[s, t]/(s^2) & \longleftarrow & \text{Spec} k
\end{array}
\]

where both the top horizontal and the left vertical maps are closed embeddings. We will show that this diagram is a pushout square in the category of \( k \)-schemes. In other words, for any \( k \)-scheme \( Y \) forming a commutative diagram

\[
\begin{array}{ccc}
  Y & \longleftarrow & \text{Spec} k[s, t]/(s, t)^2 \\
  \downarrow & & \downarrow \\
  \text{Spec} k[s, t]/(s^2) & \longleftarrow & \text{Spec} k
\end{array}
\]

there exists a unique \( k \)-morphism \( \text{Spec} k[s, t]/(s, t)^2 \to Y \) making the whole diagram commutative.

Since both the images of \( \text{Spec} k[s, t]/(s^2, t) \) and \( \text{Spec} k[s, t]/(s, t^2) \) in \( Y \) are contained in some affine subscheme \( Y_0 \subseteq Y \), we can replace \( Y \) with \( Y_0 \) and therefore assume \( Y \) is affine. Now the existence of the unique \( k \)-morphism \( \text{Spec} k[s, t]/(s, t)^2 \to Y \) is
obvious because of the pullback square of $k$-algebras mentioned earlier. So the scheme $\text{Spec} k[s, t]/(s, t)^2$ satisfies the universal property of a coproduct of $k$-schemes.

Next, we apply the covariant functor $Z \times_k -$ to this pushout square of $k$-schemes, and one can check that we have a pushout square:

$$
\begin{array}{ccc}
Z \times_k \text{Spec} k[s, t]/(s, t)^2 & \rightarrow & Z \times_k \text{Spec} k[s]/(s^2) \\
\downarrow & & \downarrow \\
Z \times_k \text{Spec} k[t]/(t^2) & \rightarrow & Z \times_k \text{Spec} k
\end{array}
$$

Finally, applying the contravariant and left exact functor $\text{Hom}_k(-, X)$, we have a pullback square in the category of sets:

$$
\begin{array}{ccc}
\text{Hom}_k(Z \times_k \text{Spec} k[s, t]/(s, t)^2, X) & \rightarrow & \text{Hom}_k(Z \times_k \text{Spec} k[s]/(s^2), X) \\
\downarrow & & \downarrow \\
\text{Hom}_k(Z \times_k \text{Spec} k[t]/(t^2), X) & \rightarrow & \text{Hom}_k(Z, X)
\end{array}
$$

On the other hand, we also have an obvious pullback square in the category of $k$-Schemes:

$$
\begin{array}{ccc}
\mathcal{J}_1(X) \times_X \mathcal{J}_1(X) & \rightarrow & \mathcal{J}_1(X) \\
\downarrow & & \downarrow \\
\mathcal{J}_1(X) & \rightarrow & X
\end{array}
$$

We apply the covariant and left exact functor $\text{Hom}_k(Z, -)$ and obtain again a pullback square in the category of sets:

$$
\begin{array}{ccc}
\text{Hom}_k(Z, \mathcal{J}_1(X) \times_X \mathcal{J}_1(X)) & \rightarrow & \text{Hom}_k(Z, \mathcal{J}_1(X)) \\
\downarrow & & \downarrow \\
\text{Hom}_k(Z, \mathcal{J}_1(X)) & \rightarrow & \text{Hom}_k(Z, X)
\end{array}
$$

By the uniqueness of fiber products, we have

$$\text{Hom}_k(Z \times \text{Spec} k[s, t]/(s, t)^2, X) = \text{Hom}_k(Z, \mathcal{J}_1(X) \times_X \mathcal{J}_1(X))$$

for all (affine) $k$-schemes $Z$. Therefore, $\mathcal{W}_1(X) \cong \mathcal{J}_1(X) \times_X \mathcal{J}_1(X)$. \hfill \Box

3. TRUNCATED WEDGE SCHEMES OF LOCAL COMPLETE INTERSECTIONS

In the case of a locally complete intersection variety, we give an irreducibility criterion for its truncated wedge schemes similar to that of Mustaţă for jet schemes [8, Proposition 1.4].

**Theorem 3.1.** Let $X$ be locally a complete intersection variety of dimension $n$. Then the scheme $\mathcal{W}_m(X)$ is pure dimensional if and only if

$$\dim \mathcal{W}_m(X) = \frac{1}{2} n(m + 1)(m + 2),$$
and in this case \( W_m(X) \) is locally a complete intersection. Similarly, \( W_m(X) \) is irreducible if and only if
\[
\dim \pi_m^{-1}(X^{\text{sing}}) < \frac{1}{2} n(m+1)(m+2),
\]
where \( \pi_m : W_m(X) \to X \) are the natural projections.

**Proof.** We have a decomposition
\[
W_m(X) = \pi_m^{-1}(X^{\text{sing}}) \cup \pi_m^{-1}(X^{\text{reg}})
\]
and in general \( \pi_m^{-1}(X^{\text{reg}}) \) is an irreducible component of \( W_m(X) \) of dimension \( \frac{1}{2} n(m+1)(m+2) \) by Proposition 2.3. So the “only if” part of both assertions is obvious and holds without the local complete intersection hypothesis.

Suppose that \( \dim W_m(X) = \frac{1}{2} n(m+1)(m+2) \). Working locally, we may assume that \( X \subseteq \mathbb{A}^N \) and \( X \) is defined by \( N-n \) equations. Notice that each defining equation of \( X \) gives rise to \( \frac{1}{2} (m+1)(m+2) \) defining equations of \( W_m(X) \). So \( W_m(X) \subseteq W_n(\mathbb{A}^N) = \mathbb{A}^{\frac{1}{2} N(n+1)(m+2)} \) is defined by \( \frac{1}{2} (N-n)(m+1)(m+2) \) equations. Then by Krull’s principal ideal theorem [2, Theorem 10.2], every irreducible component of \( W_m(X) \) has dimension at least \( \frac{1}{2} n(m+1)(m+2) \). Thus \( \dim W_m(X) = \frac{1}{2} n(m+1)(m+2) \) implies that \( W_m(X) \) is pure dimensional and a local complete intersection.

Now if \( \dim \pi_m^{-1}(X^{\text{sing}}) < \frac{1}{2} n(m+1)(m+2) \), the decomposition (1) yields \( \dim W_m(X) = \dim \pi_m^{-1}(X^{\text{reg}}) = \frac{1}{2} n(m+1)(m+2) \). So by what we just proved, the scheme \( W_m(X) \) is pure dimensional. Therefore, our assumption on \( \dim \pi_m^{-1}(X^{\text{sing}}) \) tells us that \( \pi_m^{-1}(X^{\text{sing}}) \) contributes no components to \( W_m(X) \). Hence, \( \pi_m^{-1}(X^{\text{sing}}) \) is contained in the closure of \( \pi_m^{-1}(X^{\text{reg}}) \), and \( W_m(X) \) is irreducible. \( \square \)

4. TRUNCATED WEDGE SCHEMES OF MONOMIAL SCHEMES

In this section, we analyze the scheme structure of truncated wedge schemes of monomial schemes. Like jet schemes, truncated wedge schemes of monomial schemes are not themselves monomial but our calculations will show that their reduced subschemes are. The following theorem is analogous to that of Goward and Smith for jet schemes [1, Theorem 3.1].

**Theorem 4.1.** Let \( X \subseteq \mathbb{A}^n \) be a monomial scheme in coordinates \( x_1, \ldots, x_n \). Then \( \sqrt{W_m(X)} \) is a square-free monomial ideal in the coordinates \( x_k^{(i_k,j_k)} \) where \( 1 \leq k \leq n \) and \( 0 \leq i_k + j_k \leq m \). The generators of \( \sqrt{W_m(X)} \) can be described as follows: for each minimal monomial generator of the defining ideal of \( X \), say \( x_1^{a_1} \cdots x_n^{a_n} \) after relabeling, the monomials
\[
\sqrt{x_1^{(i_1,j_1)}x_2^{(i_2,j_2)} \cdots x_1^{(i_k,j_k)}x_2^{(i_{k+1},j_{k+1})} \cdots x_2^{(i_{k+2},j_{k+2})} \cdots x_n^{(i_{k+r},j_{k+r})}}
\]
with \( \sum (i_k + j_k) \leq m \) are monomial generators of \( \sqrt{W_m(X)} \). The collection of all such monomials as we range through the minimal monomial generators of the defining ideal of \( X \) is a generating set for \( \sqrt{W_m(X)} \).

The following lemma taken from [1, Lemma 2.1] allows us to reduce the proof of Theorem 4.1 to the hypersurface case.
Lemma 4.2. If $I$ and $J$ are monomial ideals in a polynomial ring, then $\sqrt{I+J} = \sqrt{I} + \sqrt{J}$.

4.1. Monomial hypersurface case.

Theorem 4.3. Let $X = \text{Spec } k[x_1, \ldots, x_n]/(x_1^{a_1} \cdots x_r^{a_r})$ be a monomial hypersurface. Then

i. The minimal primes of $W_m(X)$ are precisely the minimal members of the set of prime ideals

\[ (x_k^{(i_k,j_k)} : 0 \leq i_k + j_k < t_k, 1 \leq k \leq r) \]

where $0 \leq t_k \leq m + 1$ and $\sum a_k t_k \geq m + 1$. (Here, we use the convention that the value $t_k = 0$ means that the variable $x_k$ does not appear at all.)

ii. The radical $\sqrt{W_m(X)}$ is the monomial ideal generated by the monomials

\[ \sqrt{x_1^{(i_1,j_1)} x_2^{(i_2,j_2)} \cdots x_r^{(i_r,j_r)}} \]

where $\sum (i_k + j_k) \leq m$.

Before we prove this result, we first need to describe the generators of $W_m(X)$. Recall that the polynomials defining the truncated wedge scheme $W_m(X)$ are the coefficients of $s^t t^j$ in the product

\[ \prod_{k=1}^r (x_k^{(0,0)} + x_k^{(1,0)} s + x_k^{(0,1)} t + \ldots + x_k^{(m,0)} s^m + x_k^{(m-1,1)} s^{m-1} t + \ldots + x_k^{(0,m)} t^m)^{a_k} \]

Therefore, $W_m(X)$ has generators of the form

\[ g_{ij} = \sum x_1^{(i_1,j_1)} \cdots x_1^{(i_1,j_1)} x_2^{(i_2,j_2)} \cdots x_2^{(i_2,j_2)} \cdots x_r^{(i_r,j_r)} \]

where $0 \leq i + j \leq m$, $0 \leq i_k, j_k \leq m$, $\sum i_k = i$ and $\sum j_k = j$.

Proof of Theorem 4.3. We will first show that part (i) follows from part (ii). Recall that the radical of any ideal is equal to the intersection of its minimal primes. So by part (ii), it is clear that $\sqrt{W_m(X)}$ is a monomial ideal. Next, observe that the monomials

\[ x_1^{(i_1,j_1)} x_2^{(i_2,j_2)} \cdots x_1^{(i_1,j_1)} x_2^{(i_2,j_2)} \cdots x_r^{(i_r,j_r)} \]

with $\sum (i_k + j_k) \leq m$ are precisely the terms of the polynomials $g_{ij}$ as $i + j$ ranges from 0 to $m$. Since every monomial ideal must contain all terms of its generators, it follows that the monomials in (i) and therefore the monomials listed in part (ii) are all contained in $\sqrt{W_m(X)}$. On the other hand, since these monomials are all square-free, they generate a radical ideal containing $W_m(X)$. Thus, this is the smallest radical ideal containing $W_m(X)$, and hence must be $\sqrt{W_m(X)}$ exactly.

To prove part (ii), we induce on $m$. The ideal $W_0(X)$ is $\left( (x_1^{(0,0)})^{a_1} \cdots (x_r^{(0,0)})^{a_r} \right)$ and its minimal primes are obviously $(x_i^{(0,0)})$ for $1 \leq i \leq r$. For the inductive step, let $Q$
be a prime containing $W_m(X)$. Since $W_{m-1}(X) \subset W_m(X)$, the prime $Q$ contains one of the minimal primes $P$ of $W_{m-1}(X)$. So by induction,

$$P = (x_k^{(i_k,j_k)} : 0 \leq i_k + j_k < t_k, 1 \leq k \leq r)$$

for some $0 \leq t_k \leq m$ and $\sum a_k t_k \geq m$. Fix the indices $t_1, \ldots, t_r$ corresponding to $P$. We want to show $Q$ contains a “full layer” of variables; that is, a set of elements

$$\{x_n^{(i_n,t_n-n)} : 0 \leq i_n \leq t_n\}$$

for some $0 \leq t_n \leq r$.

Consider the set $S$ of primes lying between $P$ and $Q$, and containing several “initial partial layers”. In other words, consider primes contained in $Q$ and of the form

$$P + (x_k^{(i_k,t_k-i_k)} : 0 \leq i_k < c_k, 1 \leq k \leq r)$$

for some $0 \leq c_k \leq t_k$.

Note that the set $S$ is nonempty because $P$ is a member (take $c_k$ to be zero for all $k$). Now choose a maximal element $\mathcal{P}$ in the set $S$ ($\mathcal{P}$ exists by the Noetherian property), and consider the monomial

$$\alpha = (x_1^{(c_1,t_1-c_1)})^{a_1} \cdots (x_r^{(c_r,t_r-c_r)})^{a_r}.$$ Clearly, $\alpha \notin \mathcal{P}$ and $\alpha$ is a monomial term of the generator $g_{c,m-c}$ where $c = \sum_{k=1}^r a_k c_k$.

**Claim 4.4.** All monomial terms of $g_{c,m-c}$ except $\alpha$ belong to the prime $\mathcal{P}$.

**Proof.** Let $\beta$ be a monomial term of $g_{c,m-c}$. Then

$$\beta = x_1^{(i_1,j_1)} \cdots x_1^{(i_{a_1+1},j_{a_1+1})} \cdots x_2^{(i_{a_1+2},j_{a_1+2})} \cdots x_r^{(i_{a_1+\cdots+a_r},j_{a_1+\cdots+a_r})}$$

where $\sum i_t = c$ and $\sum j_t = m - c$. Suppose $\beta \notin \mathcal{P}$, and so in particular, $\beta \notin P$. Then the description of $P$ implies that every sum of superscripts $i_t + j_t$ attached to each $x_k$ is at least $t_k$ for all $k = 1, \ldots, r$. So

$$m = \sum (i_t + j_t) \geq \sum a_k t_k \geq m$$

implies that $i_t + j_t = t_k$ for all $k$ and $l$. If $i_t \geq c_k$ for all $k$ and $l$, then

$$c = \sum i_t \geq \sum a_k c_k = c$$

implies that $i_t = c_k$ for all $k, l$. So

$$\beta = (x_1^{(c_1,t_1-c_1)})^{a_1} \cdots (x_r^{(c_r,t_r-c_r)})^{a_r} = \alpha.$$ Thus, the only monomial term of $g_{c,m-c}$ not in $\mathcal{P}$ is $\alpha$. This completes the proof of Claim 4.4.

Since $g_{c,m-c} \in W_m(X) \subset Q$, Claim 4.4 gives $\alpha \in Q$ and so $x_n^{(c_n,t_n-c_n)} \in Q$ for some $n \in \{1, \ldots, r\}$ by primality of $Q$. Fix such an $n$. If $c_n < t_n$, then $\mathcal{P} + (x_n^{(c_n,t_n-c_n)})$ is an element of the set $S$, violating maximality of $\mathcal{P}$. Therefore, $c_n = t_n$, which means $Q$ contains the ideal

$$P' := P + (x_n^{(i_n,t_n-i_n)} : 0 \leq i_n \leq t_n),$$

a complete layer, as desired.

We are left to check that $W_m(X)$ is contained in the ideal $P'$. Since $W_{m-1}(X) \subset P$, the only generators of $W_m(X)$ not already in $P$ are $g_{a,m-a}$ for $0 \leq a \leq m$. By a similar
argument as in the proof of Claim 4.4, we see that a term of \( g_{a,m-a} \) not in \( P \) must have every sum of superscripts \( i_k + j_l \) attached to each \( x_k \) exactly \( t_k \). Obviously, this term is in the ideal \( (x^{(i_n,t_n-t_n)}_n) : 0 \leq i_n \leq t_n \). Since every prime containing \( W_m(X) \) contains a prime of the form \( P' \), the minimal primes of \( W_m(X) \) are precisely those described in (2).

\[ \square \]

In the special case when \( X \) is a reduced monomial hypersurface, we get a slightly sharper result:

**Corollary 4.5.** Let \( X = \text{Spec} k[x_1, \ldots, x_n]/(x_1 \cdots x_r) \) be a reduced hypersurface monomial scheme. Then the minimal primes of \( W_m(X) \) are exactly the primes of the form

\[ P = (x_k^{(i_k,j_k)} : 0 \leq i_k + j_k < t_k, 1 \leq k \leq r) \]

where \( 0 \leq t_k \leq m + 1 \) and \( \sum t_k = m + 1 \). (Again, \( t_k = 0 \) means the variable \( x_k \) does not appear.)

**Remark 4.6.** This result tells us that truncated wedge schemes of a monomial scheme are not in general pure dimensional. For example, take \( r = 3 \) and \( m = 2 \) in Corollary 4.5. Then \( W_2(X) \) has a minimal prime of height 6 (for example, take \( t_1 = 3 \) and \( t_2 = t_3 = 0 \), a minimal prime of height 4 (take \( t_1 = 2, t_2 = 1, t_3 = 0 \)), and a minimal prime of height 3 (take \( t_1 = t_2 = t_3 = 1 \)).

**4.2. Alternate point of view.** In this section, we present another approach towards Theorem 4.3. We will examine \( W_m(X) \) first as a set; that is, we will look at the reduced subscheme of \( W_m(X) \) and then describe the irreducible components of \( W_m(X) \). For \( X = \text{Spec} k[x_1, \ldots, x_n]/(x_1^{a_1}\cdots x_r^{a_r}) \), a point in \( W_m(X) \) corresponds to an \( N \)-tuple \( (x_i^{(i_l,j_l)}) \in k^N, N = \frac{r}{2}r(m+1)(m+2) \), satisfying

\[ \prod_{l=1}^{r} \left( \sum_{s,t} x_l^{(i_l,j_l)} s^{i_l} t^{j_l} \right)^{a_l} \in (s,t)^{m+1}. \]

We write \( \text{ord}(f) \) for the smallest positive integer \( n \) such that \( f \in (s,t)^n \), and note that \( \text{ord}(fg) = \text{ord}(f) + \text{ord}(g) \). Then writing \( p_l \) for \( \sum x_l^{(i_l,j_l)} s^{i_l} t^{j_l} \), the condition that

\[ \prod_{l=1}^{r} p_l^{a_l} \in (s,t)^{m+1} \]

means \( \text{ord}(\prod p_l^{a_l}) \geq m + 1 \), or equivalently, \( \sum a_l \text{ord}(p_l) \geq m + 1 \). Now let

\[ t_l := \text{ord}(p_l) = \min\{c : x_l^{(i_l,j_l)} = 0 \text{ for all } i_l + j_l < c\}. \]

Then a point in \( W_m(X) \) corresponds to an \( N \)-tuple \( (x_l^{(i_l,j_l)}) \in k^N \) satisfying \( x_l^{(i_l,j_l)} = 0 \) for \( 1 \leq l \leq r, 0 \leq i_l + j_l < t_l \), and \( \sum a_l t_l \geq m + 1 \). Thus the vanishing set of the
generators of $W_m(X)$ is
\[ V(g_{ij}) = \bigcup_{\sum a_{ilt} \geq m+1} \bigcap (x^{(i_l,j_l)}_t : 0 \leq i_l + j_l < t_l, 1 \leq l \leq r) \]
\[ = V\left( \bigcap_{\sum a_{ilt} \geq m+1} (x^{(i_l,j_l)}_t : 0 \leq i_l + j_l < t_l, 1 \leq l \leq r) \right). \]

Since each ideal $(x^{(i_l,j_l)}_t : 0 \leq i_l + j_l < t_l, 1 \leq l \leq r)$ is prime, their intersection is a radical ideal. Therefore, the radical
\[ \sqrt{W_m(X)} = \bigcap_{\sum a_{ilt} \geq m+1} (x^{(i_l,j_l)}_t : 0 \leq i_l + j_l < t_l, 1 \leq l \leq r) \]
is a monomial ideal. Since the right hand side of (5) is a primary decomposition of $\sqrt{W_m(X)}$, the minimal primes of $W_m(X)$ are precisely the minimal members of the collection of primes
\[ \left\{ (x^{(i_k,j_k)}_t : 0 \leq i_k + j_k < t_k, \sum a_k t_k \geq m+1) \right\}. \]

4.3. Multiplicity. Although the defining ideal and its minimal primes of a truncated wedge scheme of a reduced monomial hypersurface $X$ are very similar to those of a jet scheme, there are important differences. From Remark 4.6 we see that the truncated wedge schemes of $X$ are not pure dimensional, while the jet schemes are [4, Theorem 2.2]. Multiplicities behave quite differently too. It was shown in [10] that the multiplicity of the jet schemes $J_m(X)$ along a component is a multinomial coefficient, depending on the component. But Macaulay calculations suggest that the multiplicity of $W_m(X)$ along any irreducible component is always one.

Conjecture 4.7. If $X = \text{Spec } k[x_1, \ldots, x_n]/(x_1 \cdots x_r)$ is a reduced monomial hypersurface, then the multiplicity of $W_m(X)$ along any irreducible component is one.

Remark 4.8. This conjecture does not imply the schemes $W_m(X)$ are reduced because they may have embedded components. For example, a Macaulay calculation shows that for the scheme $X = \text{Spec } k[x,y]/(xy)$, the ideal $W_1(X)$ has three minimal primes
\[ (x_0, y_0), \ (x_0, x_{01}, x_{10}), \ (y_0, y_{01}, y_{10}) \]
and one non-minimal associated prime $(x_0, y_0, x_{01} y_{10} - x_{10} y_{01})$.

We will prove Conjecture 4.7 in the case when $r = 2$ or $3$. We will show that a certain subset of the generators of $W_m(X)$ generate the maximal ideal of the local ring along the corresponding irreducible component.

Theorem 4.9. For $X = \text{Spec } k[x, y, z_1, \ldots, z_n]/(xy)$, the multiplicity of $W_m(X)$ along any irreducible component is one.

Proof. Fix a minimal prime $P$ of $W_m(X)$, that is,
\[ P = (x_{ij}, y_{kl} : 0 \leq i + j < t_1, 0 \leq k + l < t_2) \]
Proof. Consider a minimal prime

\[ P = (x_{ij}, y_{kl}, z_{pq} : 0 \leq i + j < t_1, 0 \leq k + l < t_2, 0 \leq p + q < t_3) \]

for some \( 0 \leq t_1, t_2 \leq m + 1 \) such that \( t_1 + t_2 = m + 1 \). For each \( x_{ij} \in P \), pick the generator \( g_{i,j+t_2} \), and for each \( y_{kl} \in P \), pick the generator \( g_{k+t_1,l} \). These are generators of \( W_m(X) \) because \( 0 \leq i + j < t_1, 0 \leq k + l < t_2 \) and \( t_1 + t_2 = m + 1 \) imply that \( 0 \leq i + (j + t_2), (k + t_1) + l \leq m \). Also, note that \( g_{i,j+t_2} \neq g_{k+t_1,l} \) since \( i < t_1 \).

We want to show that this subset of generators form a regular system of parameters in \( R_P \) where \( R \) is the coordinate ring of \( \mathcal{W}_m(\text{Spec} \ k[x, y, z_1, \ldots, z_n]) \). Indeed, we will show that the images of these elements in \( PR_P/P^2R_P \) form a basis for the cotangent space \( P/P^2 \) of the regular local ring \( R_P \).

Let \( M \) be the matrix whose rows express the elements \( y_{ab} \) as \( R_P/PR_P \)-linear combination of the basis \( x_{ij}, y_{kl} \) of the vector space \( P/P^2 \). Then from the expression (3), we can check that the matrix \( M \) has the form

\[
\begin{pmatrix}
  x_{00} & x_{10} & \cdots & x_{0,t_1-1} & y_{00} & y_{10} & \cdots & y_{0,t_2-1} \\
  y_{0,t_2} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
  y_{1,t_2} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
  \vdots & \ddots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
  y_{0,t_1+t_2-1} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
  y_{t_1,0} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
  y_{t_1+1,0} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
  \vdots & \ddots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
  y_{t_1,t_2-1} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
  \end{pmatrix}
\]

where \( \ast \)'s are of the form \( x_{ab} \) where \( (a,b) \neq (t_1,0) \) or \( y_{ab} \) where \( (a,b) \neq (0,t_2) \).

Now to show that the matrix \( M \) is invertible, note that its determinant is a polynomial in the variables \( x_{ab} \) and \( y_{cd} \) for \( a + b \geq t_1 \) and \( c + d \geq t_2 \), interpreted as an element in the field

\[ R_P/PR_P \cong k(x_{ab}, y_{cd} : t_1 \leq a + b \leq m, t_2 \leq c + d \leq m). \]

This polynomial is not the zero polynomial because plugging in the values \( x_{t_1,0} = y_{0,t_2} = 1 \) and all other \( x_{ab} \) and \( y_{cd} \) to be zero, we get the nonzero value 1. So the matrix \( M \) is invertible, which means the elements \( y_{0,t_2}, \ldots, y_{t_1,t_2-1} \) form a basis for the \( R_P/PR_P \)-vector space \( P/P^2 \). Therefore, the generators \( g_{0,t_2}, \ldots, g_{t_1,t_2-1} \) form a regular system of parameters in \( R_P \). Thus, \( W_m(X)R_P = (g_{0,t_2}, \ldots, g_{t_1,t_2-1})R_P = PR_P \), and so \( \ell(R_P/W_m(X)) = 1 \).

Theorem 4.10. For \( X = \text{Spec} \ k[x, y, z, w_1, \ldots, w_n]/(xyz) \), the multiplicity of \( W_m(X) \) along any irreducible component is one.

Proof. Consider a minimal prime

\[ P = (x_{ij}, y_{kl}, z_{pq} : 0 \leq i + j < t_1, 0 \leq k + l < t_2, 0 \leq p + q < t_3) \]

for some \( 0 \leq t_r \leq m + 1 \) and \( \sum t_r = m + 1 \). Without loss of generality, we may assume \( t_1 \leq t_2 \leq t_3 \). For each \( x_{ij} \in P \), pick the generator \( g_{i,j+t_2} \). For each \( y_{kl} \in P \), pick the generator \( g_{k+t_1,l} \). For each \( z_{pq} \in P \), pick the generator \( g_{p+q+t_2} \). Notice that these are generators of \( W_m(X) \), since \( 0 \leq i + j < t_1, 0 \leq k + l < t_2, 0 \leq p + q < t_3 \) and
\[ \sum t_r = m + 1 \] imply that \[ 0 \leq i + (j + t_2 + t_3), (k + t_1 + t_3) + l, (p + t_1) + (q + t_2) \leq m. \] Also, they are distinct because \( i < t_1 \) and \( l < t_2 \).

We want to show that the images of these generators in \( PR_P/P^2R_P \) form a basis for the cotangent space \( P/P^2 \) of the regular local ring \( R_P \), where \( R \) is the coordinate ring of the scheme \( W_m(\text{Spec } k[x, y, z, w_1, \ldots, w_n]) \). As in the previous proof, let \( M \) be the matrix whose rows express the elements \( y_{ab} \) as \( R_P/PR_P \)-linear combination of the basis \( x_{ij}, y_{kl} \) and \( z_{pq} \) of the vector space \( P/P^2 \). We will show that \( M \) is invertible.

From the expression (3), one can check that the matrix \( M \) has the form

\[
\begin{pmatrix}
  h_1 + \star & \star & \star \\
  \vdots & \cdots & \vdots \\
  \star & \star & \star \\
  \star & h_2 + \star & \star \\
  \star & \star & \star \\
  \star & \star & \star \\
  \star & \star & \star \\
  \end{pmatrix}
\]

where \( h_1 = \frac{y_{0,2}z_{0,3}}{x_{0,2}z_{0,3}}, \) \( h_2 = \frac{x_{0,2}z_{0,3}}{x_{0,2}y_{0,2}}, \) and \( h_3 = \frac{x_{0,2}y_{0,2}}{z_{0,3}} \). The symbol \( \star \) represents a polynomial in the variables \( x_{ab} \) where \( (a, b) \neq (t_1, 0) \), \( y_{cd} \) where \( (c, d) \neq (0, t_2) \), and \( z_{ef} \) where \( (e, f) \neq (0, t_3) \). The symbol \( \bigcirc \) represents an arbitrary polynomial in \( R_P/PR_P \). Interpreting the entries of this matrix as elements of the field

\[ \mathbb{P} = k(x_{ab}, y_{cd}, z_{ef} : t_1 \leq a + b \leq m, t_2 \leq c + d \leq m, t_3 \leq e + f \leq m), \]

we see that to check the determinant of \( M \) is nonzero, it suffices to check it has nonzero value for some choice of values for \( x_{ab}, y_{cd} \) and \( z_{ef} \). Now setting \( x_{t_1,0} = y_{0,t_2} = z_{0,t_3} = z_{t_3,0} = 1 \) and all the others to be zero, we get

\[
\begin{pmatrix}
  x_{ij} & \bigcirc & 0 & \cdots & 0 & \cdots & 0 \\
  \vdots & \cdots & \vdots & \cdots & \vdots & \cdots & \vdots \\
  1 & \bigcirc & 0 & \cdots & 0 & \cdots & 0 \\
  \vdots & \cdots & \vdots & \cdots & \vdots & \cdots & \vdots \\
  \end{pmatrix}
\]

where the \( \bigcirc \)'s represent arbitrary elements of \( k \).

By reversing the order of \( x_{ij} \) and therefore the order of \( g_{ij} = x_{ij} + \star \), we obtain a lower-triangular matrix with 1 on the diagonal. So the above matrix has determinant 1, and
therefore the matrix $M$ is invertible. Thus, the set
\[ \{ g_{i,j+t_2+t_3}, g_{k+t_1+t_3,l}, g_{p+t_1,q+t_2} : 0 \leq i + j < t_1, 0 \leq k + l < t_2, 0 \leq p + q < t_3 \} \]
is a regular system of parameters in $R_P$, and hence $\ell(R_P/W_m(X)) = 1$. \hfill \Box

**Remark 4.11.** In the case when $r = 4$, Conjecture 4.7 was checked up to $m = 5$ by Macaulay 2.

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SUNY POTSDAM, DEPARTMENT OF MATHEMATICS, 44 PIERREPONT AVENUE, POTSDAM, NY 13676, USA

*E-mail address:* yuenco@potsdam.edu