Invariant dynamics of scalar perturbations of inflaton and gravitational fields.

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Abstract

A gauge-independent, invariant theory of linear scalar perturbations of inflation and gravitational fields has been created. This invariant theory allows one to compare gauges used in the work of other researchers and to find the unambiguous criteria to separate the physical and coordinate effects. It is shown, in particular, that the so-called longitudinal gauge, commonly used when considering inflation instability, leads to a fundamental overestimation of the effect because of non-physical perturbations of the proper time in the frame of reference specified by this gauge (see main text for the numbers). Back reaction theories employing this sort of gauge also involve coordinate effects. A comprehensive review of papers that use the longitude gauge to analyze back reaction effects and inflationary instability can be found in [1]. The invariant theory created here shows that the classical Lifshitz (1946) [2] gauge does not lead to non-physical perturbations of the proper time. This is true because in his pioneering work, Lifshitz used the synchronous frame of reference, which enables to use one and same time for the description of both the background and perturbation evolution. This gauge can therefore be used to analyze the inflation regime and the back reaction of perturbations on this regime properly. The first theory of back reaction on background of all types of perturbations (scalar, vector and tensor) based on this gauge was published in 1975 [3]. More recently it has been applied to the inflation regime [4]. The investigation of long-length perturbations, which characterize the stability of the inflationary process, and quantum fluctuations, which form the Harrison-Zel’dovich spectrum at the end of inflation, is performed in the invariant form. The invariant theory proposed allows one to examine the effect of quantum fluctuations on the inflationary stage when the periodic regime changes to an aperiodic one. Numerical examples are presented in the main text. The theory also properly describes the spectrum reconstruction during the epoch when the inflationary stage changes to the Friedman one. The final effect of this process will be investigated in future space experiments. That only the invariant theory must be used to analyze space experiments is one of the conclusions of the present work.
1 Introduction.

The present paper proposes a new approach to the two main problems of the theory of scalar perturbations in the system of inflaton and gravitational fields. These problems have been extensively discussed in the literature.

The first problem is about the dynamics of the perturbations the length of which is comparable with the Universe size at the beginning of inflation. Two aspects of the problem are: (i) the stability of the inflationary process; (ii) the back reaction of the long-length perturbations on the expansion of the Universe in the past and in the present epoch. (The latter is one of the approaches to the problem of dark energy.)

The second problem is how the Harrison–Zel’dovich spectrum was formed from the inflaton field vacuum fluctuations, the length of which is much less than the Universe size at the beginning of the inflation. As it is known, the solution of this problem is the basis for the modern theory of the large scale structure formation in the Universe.

The conventional approach to solving the problems mentioned above is based on the investigation of the scalar perturbations in the longitudinal gauge. The theory is formulated in terms of the relativistic scalar potential \( \Phi(\vec{x}, t) = \delta g_{00}(\vec{x}, t) \); all other observable values are expressed via \( \Phi(\vec{x}, t) \). The subject of the discussion, which constantly appears in the literature, is if the predictions of the physical consequences of the theory are invariant. Two point of view clashed at this point. Some people suppose, that the longitudinal gauge is physically preferred because its main object, \( \Phi(\vec{x}, t) \), is invariant itself [5]. The opposite claim is that \( \Phi(\vec{x}, t) \) describes effects in the fixed frame of reference [6, 7]; in other frames of reference the observed phenomena may look in another way.

The question under discussion is of principal value now, because in the coming years experimental researches of the relic density perturbations spectrum will be performed with the help of the space apparatus. In fact, the question is if the theory in the longitudinal gauge can be used to reconstruct the past of the Universe on the basis of the observations that will be performed. The Fourier-image of \( \Phi_k(t) \) is the functional \( \Phi_k\{H(t)\} \), and the Hubble function \( H(t) \) contains information about the inflaton field and the inflationary process itself. The theoretical reconstruction of the past is possible only in the case of \( \Phi_k\{H(t)\} \) is an invariant functional. On the other hand, in the case of the information contained in \( \Phi_k\{H(t)\} \) essentially depends on the properties of the frame of reference that is prescribed by the longitudinal gauge, the theory of relativistic scalar potential can not be used to interpret the results of the experiment.

Our claim — that the theory in the longitudinal gauge is noninvariant — is made on the basis of the proposed invariant dynamics of the scalar perturbations.

The main result of our investigation is the strict proof of the following statement. The invariant information about the dynamics of the scalar perturbations is selected from the equations of the linear gravitational instability theory by the identical mathematical transformations without making use of any gauge condition at any stage of the calculations. Our theory is formulated as a closed system of equations for the invariant \( J_k \) of metric perturbations, invariant function \( \chi_{\text{kinv}} \) of the perturbations of the inflaton field, its derivative \( \dot{\chi}_{\text{kinv}} \) and energy density perturbations \( \delta \varepsilon_{\text{inv}} \) (see Eqs. (3), (4), (5)). The theory is such that the invariance of the physical values follows from their mathematical definitions, and the invariance of the equations does from the way to obtain
them without any gauge.

For one of the simplest model of inflaton field we perform the numerical analysis of the evolution of the invariant physical values and compare the results with those obtained in the longitudinal gauge theory. The general qualitative properties — the power-law instability of long–length perturbations and the formation of the Harrison–Zel’dovich spectrum — are the same in the two approaches, but the numerical values are different, the perturbations in the longitudinal gauge theory being several times greater. We find out that the reason of the quantitative differences is the strong perturbations of the proper time in frame of reference specified by the longitudinal gauge.

Our point of view is that the perturbations of the proper time can not lead to the large scale structure formation and thus the invariant theory must be used to interpret the space experiments. Among the theories formulated in the fixed gauges, the synchronous gauge theory proposed by Lifshitz is an adequate one, because it this gauge the evolution of the background and perturbations is analyzed in one and the same time. This physical requirement was formulated in our papers devoted to the description of the back reaction of the perturbations on the background.

2 The equations and the numerical results.

As it was mentioned in the Introduction the object of our theory is invariant $J_k$ constructed from the components of the perturbed metric

$$
J_k = \left( \frac{\Phi_k}{a} - ik_\alpha \sigma_\alpha k^\beta \lambda_k \right),
$$

and the invariant functions for the perturbation of the inflaton field and energy density

$$
\chi_k^{inv} = \delta \phi_k - \dot{\phi} \delta \tau_k(t),
\dot{\chi}_k^{inv} = \delta \dot{\phi}_k - \ddot{\phi} \delta \tau_k(t) - \frac{1}{2} \dot{\phi} \delta \Phi_k,
$$

is the perturbations of the proper time. In the expressions for the observed values the function $\delta \tau_k(t)$ describes the effects that are related to the noninertial motion of the frame of reference with respect to the background. So, these effects are due to the influence of the scalar perturbations on the clock run in the frame of reference which the scalar perturbations are studied in.

The identical transformations of the equations of the gravitational instability theory without making use of any gauge lead to the following equation for the invariant metric
\[ J_k + \left( 3H + 2 \frac{H}{H} - \frac{\dot{H}}{H} \right) \dot{J}_k + \left( \frac{k^2}{a^2} + \frac{6H - 2 \dot{H}}{H^2} - \frac{\ddot{H}}{H} + \frac{\dot{H}^2}{H^2} - 3 \frac{H \ddot{H}}{H} \right) J_k = 0 \] (3)

This equation is valid for any potential \( U(\varphi) \); a fixed potential makes function \( H(t) \) and its derivatives also fixed. The relations which connect the invariant characteristics of the gravitational and inflaton field perturbations are

\[ \chi_k \text{inv} = -\frac{1}{3\dot{\varphi}} \left( H J_k + \dot{H} + \frac{H}{J} \int_0^t J_k dt \right), \]
\[ \dot{\chi}_k \text{inv} = -\frac{H}{3\dot{\varphi}} \left[ \dot{J}_k + \left( \frac{2H}{H} - \frac{H^2}{2H} \right) J_k + \frac{\dot{H}}{2H} \int_0^t J_k dt \right]. \] (4)

In particular, these equations are used to specify the initial conditions for invariant and its derivative via those for the perturbations of inflaton field and its derivative.

For the physical and cosmological applications of the theory the relative energy density perturbations are of particular interest

\[ \frac{\Delta \varepsilon_k \text{inv}}{\varepsilon} = -\frac{1}{9H} \left[ \dot{J}_k + \left( \frac{2H}{H} - 3H - \frac{H}{H} \right) J_k \right] + \frac{\dot{H}}{3H} \int_0^t J_k dt. \] (5)

We will compare our results with those in the longitudinal gauge. The relativistic scalar potential satisfies the equation

\[ \ddot{\Phi}_k + \left( H - \frac{\ddot{H}}{H} \right) \dot{\Phi}_k + \left( \frac{k^2}{a^2} + 2\dot{H} - \frac{H \ddot{H}}{H} \right) \Phi_k = 0 \] (6)

The perturbations of the inflaton field and energy density are related to \( \Phi_k \) as

\[ \chi_k = \frac{1}{\dot{\varphi}} \left( \ddot{\Phi}_k + H \dot{\Phi}_k \right), \quad \frac{\Delta \varepsilon_k}{\varepsilon} = -\left( \frac{k^2}{3H^2a^2} \Phi_k + 1 \right) \Phi_k - \frac{\dot{\Phi}_k}{H} \] (7)

The comparison of (3), (4), (5) and (6), (7) shows that the mathematical structure of invariant theory and that of the theory in the longitudinal gauge are essentially different. This is not surprisingly, because the invariant theory by definition exclude the effects coming from the perturbations of the proper time. Contrary, in the frame of reference specified by the longitudinal gauge, the observed phenomena strongly depend on the properties of the frame of reference itself, because the scalar relativistic potential \( \Phi_k \) describes not only the physical perturbations but also the perturbations of the proper time.

We perform a formal mathematical analysis and show that (i) the invariant formulation of the theory is compatible with any gauge; the set of the gauge only fixes the expressions for \( \chi_k - \chi_k \text{inv}, \dot{\chi}_k - \dot{\chi}_k \text{inv}, \Delta \varepsilon - \Delta \varepsilon \text{inv} \); (ii) the invariant effect can be separated from the results
obtained in the longitudinal gauge theory. In this gauge both invariant and noninvariant parts of energy density perturbations are expressed via $\Phi_k$. The invariant part exactly coincide with (3), the function $J_k$, which satisfies equation (3), being now defined as $J_k(l.g.) = -(\Phi_k/H) + \Phi_k/3$. The noninvariant part, which is absent, for example, in synchronous frame of reference, is the longitudinal gauge is

$$
\delta \varepsilon_{k, \text{noninv}} = \varepsilon \delta \tau_k = H \dot{H} \left( \text{const} + \frac{1}{a} \int_0^t J_k(l.g.) \, dt - \int_0^t J_k(l.g.) \, dt \right)
$$

Unfortunately, such separation of effects can not be performed in the longitudinal gauge theory as it is; it is necessary to know beforehand, what effects are invariant. We think, that is the reason why the physical effects were not separated from the coordinate ones in previous works and so the total effect were taken into account in physical and cosmological applications of the theory.

We perform the numerical comparison of the invariant effects with the total effects obtained in the longitudinal gauge. The comparison is made in the framework of the simplest model of inflation with the inflaton potential $U(\varphi) = m^2 \varphi^2/2$, where $m = 0.15$ in the system of units $\hbar = c = 8\pi G = 1$. In the model under consideration the slow-rolling regime holds form $t = 0$ till $t_{\text{inf}} = 125$, e-folding parameter being equal to 70.

On Figs.1,2 the relative values of inflaton field perturbations and energy density perturbations are shown in the long–length–wave limit $k \ll a_0 H_0$. The normalization is made with respect to the corresponding values in the initial moment of time. The effect of the power-low instability holds both in the invariant theory and in the longitudinal gauge theory. One can see from Figs. 1a, 2a, the differences in the grows of the perturbations take place from the very beginning of the inflation. By the end of the inflation, as one can see form Figs. 1b, 2b, the differences in the predictions of the theories achieve 4-5 times, the instability being stronger in the longitudinal gauge theory. The physical reason of such differences is that the measurements of the background and the perturbations are nonsynchronized in the frame of reference specified by the longitudinal gauge. The background physical parameters are measured in some later moment of time, when their values have decreased during the inflation. The result that the longitudinal gauge theory yields the overestimated values for the long-length–wave perturbations was also obtained in [7] in the framework of the theory in the synchronous gauge.

The results of the invariant theory answer the question about the stability of inflation in the slow-rolling regime. For the simplest model discussed here, Figs. 1b, 2b show that the relative perturbation of the inflaton field increases by the end of inflation in 25 times, the relative perturbation of the energy density does in 15 times. It is known, the end of the inflation is the most important epoch for the formation of the observed properties of the Universe, so we need to discuss the influence of the perturbations on this regime. The initial long–length–wave perturbations of the inflaton field $\chi(0)/\varphi(0)$, generally speaking, is a random value the nature of which must be clarified in quantum geometrodynamics; the value $\chi(0)/\varphi(0) \sim 0.1$ seems to be physically reasonable. In this case the long–length–wave perturbations generate the additional stochastic effects comparable with the effects taken into account in the background solution. In particular, the evolution of the system in the end of the inflation must be strongly influenced by the back reaction of the perturbations
on the background. The invariant theory of the back reaction of the scalar perturbations on
the background is the subject of another paper [4]. We would like to emphasize, such back
reaction plays an important role in forming the intermediate stage, when the inflationary
epoch changes to the Friedman one. It is this intermediate stage that the final properties of
the Universe which are the subject of coming cosmic experiments are formed.

The next problem is the theory of quantum fluctuations of the inflaton field, which are
short–length–wave ones in the beginning of the inflation. As it is known, the theory of these
fluctuations is, in fact, the theory of relic density perturbations, the evolution of which have
lead to the formation of the large scale structure of the Universe. It is necessary to differ
three stages of relic perturbations formation: (i) the stage of quantum oscillations, on which
$k > aH$; (ii) the state of aperiodic increase from the moment of the oscillation termination
till the end of inflation; (iii) the stage of spectrum reconstruction (metric preheating [8])
on which the final formation of the relic perturbation spectrum takes place. The quantum
theory of fluctuations is able to describe the evolution of fluctuations on all the three stages.
On the first and the second stages the analysis can be performed on the given background,
which correspond to the slow-rolling regime. On the third stage, as it was mentioned above,
the back reaction of the perturbations must be taken into account.

Below we’ll describe the results of investigation on the first and the second stages. Gen-
erally speaking, there exists a method of numerical estimations of relic perturbation level
and spectrum at the end of inflation, which do not use the gravitational instability theory
[9]. Notice at once, the results of the proposed invariant theory of quantum fluctuation are
in quantitative agreement with this estimations. However, the invariant perturbation the-
ory is more regular method, because it allow to describe successively all the three stages of
spectrum formation.

Now let us pass to the quantum perturbation theory as it is. First of all, notice, there
is a formal problem in the theory: what function is the object of quantization. The pose
of this problem is induced by our understanding that the coordinate effects that can be
eliminated by the appropriate choice of a classical frame of reference must not to be subject
to quantization. In the longitudinal gauge theory the discussion of this problem is again
reduced to the question whether the description in terms of the relativistic potential is
invariant or not. One of the achievements of our invariant theory is that this question is
uniquely solved. Only the invariant metric function which is uniquely connected with the
invariant characteristics of the inflaton field is the subject of quantization. For the short–
length–wave fluctuations $k/a_0H_0 \gg 1$ the normalization of quantum operator is easy to
find:

$$\hat{J}_k \approx \frac{3\dot{\Phi}}{H} \hat{\Psi}_k, \quad \hat{\Psi}_k \approx \frac{1}{\sqrt{2ka}} \left[ \hat{c}_k \exp \int_0^t \frac{k}{a} dt + \hat{c}_k^+ \exp \left( -\int_0^t \frac{k}{a} dt \right) \right], \quad \frac{ka}{H} \gg 1 \quad (8)$$

where $\hat{c}_k$ and $\hat{c}_k^+$ are annihilation and creation operators in quantum field theory. Notice,
expressions (8) are used only to specify the initial conditions; the quantum dynamics itself
is described by the exact operator equation (8). The subject of calculations is the value of
energy density fluctuations averaged over the Heisenberg vacuum specified in the beginning
of the inflation.

$$\left| \frac{\delta \varepsilon_{k_{\text{inv}}}}{\varepsilon} \right| \equiv \sqrt{\langle 0 | \left( \frac{\delta \varepsilon_{k_{\text{inv}}}}{\varepsilon} \right)^2 | 0 \rangle} \tag{9}$$

In order to calculate the value (9) one can solve equation (3) under $|c_k| = |c_{-k}^+| = 1/2$. The averaging over the phases of the complex numbers $c_k$, $c_{-k}$ and taking the absolute value corresponds to the procedure of quantum averaging.

In the longitudinal gauge theory we can set the subject of quantization only by the formal agreement without strict proof. The quantum relativistic potential is

$$\dot{\Phi}_k \approx \dot{\varphi} \hat{\Psi}_k \approx -\frac{i \hat{\varphi}}{\sqrt{2k^{3/2}}} \left[ c_k \exp \int_0^t \frac{k \, dt}{a} - c_{-k}^+ \exp \left( -\int_0^t \frac{k \, dt}{a} \right) \right] \tag{10}$$

We’ll pass to the comparison of the results. For quantum fluctuations of the energy density there is a region of oscillations, which is the more wide, the less the initial length of the wave is. In the model under consideration, $k \sim 10^6$ corresponds to the large scale structure of the modern Universe. In order to demonstrate the physical and mathematical contents of the theory, we use $k = 50$, for which the detailed computer calculations are easy to done. The evolution of quantum fluctuations on the region of oscillations is shown on Fig. 3a for the invariant theory and Fig. 3b for the longitudinal gauge theory. One can see, there is no difference in the predictions of the two theories in this region. This conclusion is physically clear, because the effect of clocks nonsynchronization in the longitudinal gauge theory is very small in comparison with quick change of the amplitude of quantum fluctuations, which are dictated by the proper dynamics of this fluctuations. However, the quantum fluctuations inevitable goes to the aperiodic evolution stage, which is shown on Fig. 3c. Here the strong quantitative difference between the two theories is evinced.

The theoretical predictions for the parameter of the Harrison–Zel’dovich spectrum are of the most interest. This parameter is defined as follows

$$\Delta = \left( \frac{\langle 0 | \left( \frac{\delta \varepsilon_{k_{\text{inv}}}}{\varepsilon} \right)^2 | 0 \rangle}{2\pi^2} \right)^{1/2} \tag{11}$$

The results of calculation of $\Delta$ without the averaging over the initial phases of quantum fluctuations are shown on Fig. 4. As it was expected, in the both theories we have a flat spectrum. However, the numerical values in the longitudinal gauge theory are strongly high than those in the invariant theory. As it was noticed earlier, the difference is generated by the effect of the clock unsynchronization on the stage of the aperiodic evolution.

So, before the interpretation of the results of the cosmic experiments, we must clarify, what information is in the results of the measurements. If we are sure that we are able to separate the invariant information, then the invariant theory is needed for its theoretical interpretation.

Appendix.

In this appendix we’ll show how the invariant equation (3) can be obtained.
The perturbed metric (1) is substituted to the perturbed Einstein equations $\delta R^k_i - \frac{1}{2} \delta^k_i R = \delta T^k_i$, where $\delta T^k_i$ is energy–momentum tensor of inflanton field, the potential being arbitrary. From (0) and (0) Einstein’s equations the perturbations of inflanton field are expressed via perturbations of metric, this allows to write the (0) Einstein equation down in the form of the two following equations for three variables

$$\ddot{N}_k + 3H\dot{N}_k - \frac{k^2}{3a^2}(N_k + 3\Phi_k) - \dot{L}_k - 3HL_k = 0 \quad (12)$$

$$L_k = -\frac{1}{3H} \left[ \ddot{N}_k - \left( 3H + \frac{\ddot{H}}{H} \right) \dot{N}_k + \frac{k^2}{a^2}N_k - 3H\Phi_k - 3 \left( 2\dot{H} - \frac{H\ddot{H}}{H} \right) \Phi_k \right] = 0 \quad (13)$$

Here

$$N_k = \mu_k + \lambda_k, \quad L_k = \dot{\mu}_k + 2ik\sigma_k.$$  

Equations (12), (13) show that all gauges are to be presented in the form of the additional constraints for $N_k$, $L_k$, $\Phi_k$. In particular, the synchronous gauge is $\Phi_k = 0$, the longitudinal gauge is $L_k = N_k$. It is of principal importance, however, equations (12), (13) allows to separate the invariant information without making use of any gauge. Substituting (13) to (12) lead to the equation in which the variables $N_k$, $\Phi_k$ are present only in the invariant combination $J_k = (N_k/H) - 3\Phi_k$, which is a linear superposition of two Bardeen’s invariant. The equation obtained is the desired equation of invariant dynamics (3) that was written in Section 2.

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Fig. 1a: The normalized perturbation of long-length wave inflaton field in the beginning of the inflation

- 1 - invariant theory
- 2 - longitudinal gauge

Fig. 1b: The normalized perturbation of long-length wave inflaton field up to the end of inflation

- 1 - invariant theory
- 2 - longitudinal gauge
Fig. 2a: The normalized long-length wave perturbation of energy density in the beginning of the inflation

1 - invariant theory
2 - longitudinal gauge

Fig. 2b: The normalized long-length wave perturbation of energy density up to the end of inflation

1 - invariant theory
2 - longitudinal gauge
Fig. 3a. Quantum fluctuations of energy density: $k=50$.
The epoch of oscillations in the invariant theory.

Fig. 3b. Quantum fluctuations of energy density: $k=50$.
The epoch of oscillations in the longitudinal theory.
Fig. 3c. Quantum fluctuations of energy density: $k=50$.
The epoch of the aperiodic evolution $t>4$.

Fig. 4. The spectrum of density perturbations formed by the evolution of quantum fluctuations by the end of inflation.

1 - invariant theory
2 - longitudinal gauge