Learning-to-Rank with Partitioned Preference: Fast Estimation for the Plackett-Luce Model

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Abstract

We consider the problem of listwise learning-to-rank (LTR) on data with partitioned preference, where a set of items are sliced into ordered and disjoint partitions, but the ranking of items within a partition is unknown. The Plackett-Luce (PL) model has been widely used in listwise LTR methods. However, given $N$ items with $M$ partitions, calculating the likelihood of data with partitioned preference under the PL model has a time complexity of $O(N + S!)$, where $S$ is the maximum size of the top $M - 1$ partitions. This computational challenge restrains existing PL-based listwise LTR methods to only a special case of partitioned preference, top-$K$ ranking, where the exact order of the top $K$ items is known. In this paper, we exploit a random utility model formulation of the PL model and propose an efficient approach through numerical integration for calculating the likelihood. This numerical approach reduces the aforementioned time complexity to $O(N + MS)$, which allows training deep-neural-network-based ranking models with a large output space. We demonstrate that the proposed method outperforms well-known LTR baselines and remains scalable through both simulation experiments and applications to real-world eXtreme Multi-Label (XML) classification tasks. The proposed method also achieves state-of-the-art performance on XML datasets with relatively large numbers of labels per sample.

1 Introduction

Ranking is a core problem in many information retrieval systems, such as recommender systems, search engines, and online advertising. The industry-scale ranking systems are typically applied to millions of items in a personalized way for billions of users. To meet the need of scalability and to exploit a huge amount of user feedback data, learning-to-rank (LTR) has been the most popular paradigm for building the ranking system, empowered by advanced machine learning models.

Existing LTR approaches can be categorized into three groups: pointwise [8], pairwise [5], and listwise [6] methods. The pointwise and pairwise LTR methods convert the ranking problem into regression or classification tasks on single or pairs of items respectively. The listwise LTR methods, instead, directly optimize utility functions defined on ranked lists of items. As the ranking data we observe in the real world are often presented as (partially) ordered lists of items, theoretically listwise methods are able to preserve more information of the interrelations among items in the list [14]. Empirically, the choice of the utility function plays an important role in the performance of these methods. One of the most popular utility models for listwise ranking is the Plackett-Luce (PL)

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model [24 20]. Given the utility scores of \( N \) items, \( \mathbf{w} = [w_1, w_2, \cdots, w_N]^T \), under a PL model, the probability of observing a certain permutation of the items, \( i_1, i_2, \cdots, i_N \), is defined as

\[
p((i_1, i_2, \cdots, i_N); \mathbf{w}) = \frac{\prod_{j=1}^{N} \exp(w_{i_j})}{\sum_{\pi \in \Pi} \prod_{j=1}^{N} \exp(w_{\pi_j})}.
\]  

(1)

Most existing listwise LTR methods using a PL utility model (e.g., ListMLE [29] and ListNet [6]) have the constraint that they can only be applied to data with top-\( K \) ranking, where the exact order of the top \( K \) items in a list is known. This is because evaluating the likelihood of more general partial rankings under the PL model is usually intractable. Unfortunately, in real-world applications, often times such top-\( K \) ranking is not available due to reality of data collection and noises. Indeed, it is common that we only know a set of items are preferred to another set, and there is no explicit information about the preference rankings within the sets. For example, in a page of recommended items, we usually only observe binary clicks (or a small number of ordinal ratings) as user feedback but do not know the exact order among the clicked items (or items with the same rating scale).

In this paper, we consider learning PL-based neural network ranking models from large-scale ranking data with partitioned preference [16], defined as following: given \( N \) items, partitioned preference slices the items into \( M \) disjoint partitions, where order of items within each partition are unknown while the \( M \) partitions have a global order. Partitioned preference is a strictly more general class of partial rankings compared to top-\( K \) ranking and can better characterize the ranking data with ties and noises. However, computing the exact likelihood of data with partitioned preference under the PL model requires roughly a time complexity \( O(N + S!) \), where \( S \) is the size of the largest partition among the top \( M - 1 \) partitions. This makes it computationally intractable when \( S \) is only mildly large, and this deficiency has restrained the application to large-scale real-world problems. In this paper, we overcome this computational challenge by exploiting a random utility model formulation of the PL model with Gumbel distribution [23] and converting the likelihood to numerical integral which can be evaluated in a time complexity of \( O(N + MS) \). We further investigate the numerical stability of the proposed numerical-integral approach when it is applied to gradient-based listwise LTR, and we provide a practical implementation that can successfully train neural network ranking models for items at million-level. We also discuss the connection between our proposed method and generalized rank-breaking methods [14].

We evaluate the effectiveness of the proposed method through both simulation and experiments with real-world datasets. For simulation, we show that the proposed method can better recover the ground-truth utility scores compared to baseline methods, including a method [10] that approximates the PL likelihood with a tractable lower bound. We also test the proposed method on real-world extreme multilabel (XML) classification datasets [3]. Given features of each sample, the XML classification task requires a machine learning model to tag the most relevant subset of an extremely large label set. This task is essentially an LTR problem for partitioned preference, and the performance is usually measured by ranking metrics such as Precision@k. The proposed method outperforms other popular listwise and pairwise LTR baselines on the XML datasets. The proposed method also achieves state-of-the-art performance on datasets which have a relatively large number of labels per sample.

2 Related Work

2.1 Learning-to-Rank

Our work falls in the area of LTR [18]. The goal of LTR is to build machine learning models to rank a list of items for a given context (e.g., a user) based on the feature representation of the items and the context. The choice of the ranking objective plays an important role in learning the ranking models. Existing ranking objectives can be generally categorized in to three groups: pointwise [8], pairwise [13 5], and listwise [6 29 27 7 2 28]. Listwise methods are typically designed by either optimizing smooth surrogate of ranking metrics [27] or exploiting probabilistic ranking models [6 29].

PL-based listwise LTR. The PL model has been widely used in listwise LTR methods [6 29 26]. For example, both ListNet [6] and ListMLE [29] calculate the probability of the observed ranked
list given a PL model parameterized by a scoring function over the items. ListNet learns the scoring function by optimizing the KL divergence between the model distribution and an empirical distribution based on training labels, while ListMLE maximizes the likelihood of the observed ranked list. However, most existing PL-based listwise methods cannot be applied to partitioned preference data, due to the aforementioned computational complexity of evaluating the likelihood. Our work tackles the computational challenge with a novel numerical approach.

**XML classification as a ranking problem.** We also find that the literature of XML classification is remotely related to our work, as the XML classification tasks were initially established as a reformulation of ranking problems \[1\] \[25\], the performance of which is primarily evaluated by various ranking metrics such as Precision@k or nDCG@k \[3\]. We treat the XML classification tasks as special cases of ranking with partitioned preference, where the class labels are considered items, and for each document its relevant labels form one partition and irrelevant labels form a second, lower-ranked partition. In this work, we apply the proposed method for ranking with partitioned preference to the XML classification datasets, and we find it achieving the state-of-the-art performance on datasets where the first partition, i.e., the set of relevant labels, is relatively large.

2.2 Ranking Aggregation

*Ranking aggregation* aims to integrate multiple partial or full rankings into one ranking. The multiple rankings are considered as noisy samples from underlying ground truth ranking preferences. Ranking aggregation is a broader research area that includes LTR as a subproblem. Statistical modeling is a popular approach for ranking aggregation. Various statistical models \[21\] \[20\] \[24\] are proposed to model the ranking generation process in the real world. Among them, the PL model \[20\] \[24\] is one of the most widely-used. Evaluating the likelihood of the PL model on various types of partial rankings has been widely studied \[11\] \[22\] \[17\]. However, we note that many of these studies \[22\] \[17\] are designed for ranking data without any features, and thus are not suitable for machine learning tasks. Further, many of these works \[11\] \[18\] do not scale up to large-scale partitioned-preference data. Our work differs from the ranking aggregation methods with a focus on large-scale LTR tasks. It is also worth noting that the connection between the PL model and Gumbel distribution under the framework of random utility model has been well-known due to Yellott Jr \[30\] and McFadden \[23\]. Our key insight of the proposed method also comes from this connection.

3 Approach

3.1 Problem Formulation: Learning PL Model from Partitioned Preference

We first give the formal formulation of learning a PL model from ranking data with partitioned preference. Suppose there are \(N\) different items in total and we denote the set \(\{1, \ldots , N\}\) by \([N]\). The partitioned preference is formally defined below.

**Definition 1 (Partitioned Preference \[16\] \[19\]).** A group of \(M\) disjoint partitions of \([N]\), \(S_1, S_2, \ldots , S_M\), is called a partitioned preference if (a) \(S_1 \supset \cdots \supset S_M\), where \(S_m > S_{\tilde{m}}\) indicates that any item in \(m\)-th partition has a higher rank than items in the \(\tilde{m}\)-th partition; (b) the rank of items within the same partition is unknown.

Clearly, \(\cup_{m=1}^{M} S_m = [N]\) and \(S_m \cap S_{\tilde{m}} = \emptyset\) for any \(1 \leq m \neq \tilde{m} \leq M\). We also denote the size of each partition \(S_m\) as \(n_m\), \(m = 1, \ldots , M\). Under a PL model parameterized by \(w\) as defined in Eq. (1), the probability of observing such a partitioned preference \(S_1 \supset \cdots \supset S_M\) is given by

\[
P(S_1 \supset \cdots \supset S_M; w) = \prod_{m=1}^{M-1} \left( \sum_{(i_1, \ldots , i_m) \in \sigma(S_m)} \frac{\prod_{l=1}^{n_m} \exp(w_{i_l})}{\prod_{l=1}^{n_m} \exp(w_{i_l})} \right)
\]

\[
= \prod_{\ell \in \cup_{m=1}^{M-1} S_m} \exp(w_{\ell}) \prod_{m=1}^{M-1} \left( \sum_{(i_1, \ldots , i_m) \in \sigma(S_m)} \frac{1}{\prod_{l=1}^{n_m} \exp(w_{j_l}) - \sum_{r=1}^{l-1} \exp(w_{i_r})} \right),
\]

where \(\sigma(\cdot)\) is a function that maps a set of items to the set of all possible permutations of these items, and \(R_m\) is the set of items that do not belong to the top \(m-1\) partitions, i.e., \(R_m = \cup_{r=m}^{M} S_r\).
Typically, the utility scores \( \mathbf{w} \) are themselves parameterized functions, e.g. neural networks, of the feature representation of the items and the context of ranking (e.g., a particular user). Suppose the features for item \( i \) are denoted as \( \mathbf{v}_i \) and the item-independent context features are denoted as \( x \). Then the utility score of \( i \) for a given context (e.g., user) can be written as \( u_i(x, \mathbf{v}_i; \theta) \), where \( \theta \) represents the neural network parameters. The problem of learning a PL model from data with partitioned preference can be formulated as maximizing the likelihood in Eq. (2) over \( \theta \).

However, evaluating the likelihood function naively by Eq. (2) requires a time complexity of \( O(N + n_1! + \cdots + n_{M-1}!) \), which becomes intractable as long as one partition is mildly large (e.g., \( 20! > 10^{18} \)). In Section 3.2 we propose a scalable approach that is able to compute the likelihood and its gradient without the factorial term. In the rest of the paper, we simplify the utility function \( w_i(x, \mathbf{v}_i; \theta) \) as \( w_i \) for each \( i \in [N] \) when there is no ambiguity.

3.2 Efficient Evaluation of the Likelihood and Gradients

In this subsection, we present how to efficiently learn the PL model from partitioned preference. We derive a numerical integral approach based on the random utility model formulation of the PL model with Gumbel distribution [30][23].

The random utility model formulation of PL. A random utility model assumes that, for each context, the utility of preferring the item \( i \in [N] \) is a random variable \( u_i = w_i + \epsilon_i \). In particular, \( w_i \) is the aforementioned parameterized utility function, and \( \epsilon_i \) is a random noise term that contains all the unobserved factors affecting the individual’s utility. When each \( \epsilon_i \) independently follows a standard Gumbel distribution (or equivalently, each \( u_i \) independently follows Gumbel(\( w_i \)), a Gumbel distribution with the location parameter set to \( w_i \)), we have the following fact [30], for any permutation of items \( (i_1, \cdots, i_N) \),

\[
P(u_{i_1} > u_{i_2} > \cdots > u_{i_N}) = \prod_{j=1}^N \frac{\exp(u_{i_j})}{\sum_{k=j}^N \exp(u_{i_k})}.
\]

It implies that, after sampling \( N \) independent Gumbel variables, the ordered indices returned by sorting the Gumbel variables follow the PL model. Following this result, we have developed Proposition 1 that characterizes the preference for a given context between any two disjoint partitions.

**Proposition 1.** Given a PL model parameterized by \( \mathbf{w} \), for any \( A, B \subseteq [N] \) and \( A \cap B = \emptyset \), the probability of \( A \succ B \) is given by

\[
P(A \succ B; \mathbf{w}) = P(\min_{a \in A} g_{w_a} \succ \max_{b \in B} g_{w_b}) = \int_{u=0}^1 \prod_{a \in A} (1 - u^{\exp(w_a - w_B)}) du,
\]

where \( g_w \) denotes a random variable following Gumbel(\( w \)) and \( w_B = \log \sum_{b \in B} \exp(w_b) \).

Kool et al. [15] have shown a weaker version of Proposition 1 where \( A \cup B = [N] \). Here we extend it to the case where \( A \cup B \subset [N] \), whose proof is given in Appendix A. In particular, the second equality in Eq. (3) provides an efficient way of computing the likelihood of the preference for a given context between two disjoint partitions.

**Compute the likelihood and gradients by numerical integral.** Following the random utility model formulation of PL, we can compute the log-likelihood of the preference for a given context among \( M \) partitions and its gradient efficiently by one-dimensional numerical integration. For the likelihood function, it directly follows from Proposition 1 that

\[
P(S_1 \succ \cdots \succ S_M; \mathbf{w}) = \prod_{m=1}^{M-1} P(S_m \succ R_{m+1}; \mathbf{w}) = \prod_{m=1}^{M-1} \int_{u=0}^1 \prod_{i \in S_m} (1 - u^{\exp(w_i - w_{R_{m+1}})}) du,
\]

where \( w_{R_{m+1}} = \log \sum_{j \in R_{m+1}} \exp(w_j) \). Therefore, we can compute the log-likelihood function through \( M-1 \) one-dimensional numerical integration. The gradients of the log-likelihood w.r.t. \( \mathbf{w} \)
while partitioned preference is a general class of partial rankings for a set of items, it is not able
We discuss the potential application of the proposed numerical approach to
An arbitrary pairwise preference of
within each integrand can all be done in parallel. 2) The number of intervals
To the best of our knowledge, there is no scalable listwise method that is able to learn industry-scale
Computation costs. Suppose the number of numerical integration intervals is set as $T$. Evaluating
the likelihood and its gradients w.r.t. $u$ has time complexity $O(N + T(N - n_M))$, which is linear
in $N$. We further highlight several advantages of the proposed numerical approach. 1) The whole
computation is highly parallelizable: the computation of the $T$ integrands and the product over $S_m$
within each integrand can all be done in parallel. 2) The number of intervals $T$ can be adjusted to
control the trade-off between computation cost and accuracy. 3) In large-scale ranking data, we often
have $N \approx n_M \gg \sum_{m=1}^{M-1} n_m$, thus $T(N - n_M)$ will be negligible for large $N$.
As for the memory cost, a sequential implementation requires $O(N)$ and a parallel implementation
requires at most $O(MN + T(N - n_M))$, which is also linear in $N$ as $M$ is usually small. More
detailed analyses of computation cost are provided in Appendix A.2.
Numerical stability. As our ultimate goal is to train deep neural network based ranking models, it is
important to ensure the numerical integrations of both the likelihood in Eq. (4) and its gradients
in Eq. (5) are numerically stable. There are a few sources of numerical instability in a plain
implementation directly following Eq. (4) and Eq. (5). For instance, the double exponential terms
$u^{exp(w_i - w_{R_{m+1}})}$, or the product of small terms over $S_m$. Fortunately, we are able to overcome the
instability by some numerical tricks. We provide an detailed analysis and solution in Appendix A.3.

3.3 Improving the Computational Efficiency of Generalized Rank-Breaking Methods
We discuss the potential application of the proposed numerical approach to generalized rank-breaking
methods [14] as a final remark of this section.

While partitioned preference is a general class of partial rankings for a set of items, it is not able
to represent the class of all possible partial rankings, which is also known as arbitrary pairwise preference [19][17]. It is challenging to learn arbitrary pairwise preference using a listwise method.
To the best of our knowledge, there is no scalable listwise method that is able to learn industry-scale
PL-based ranking models. Pointwise and pairwise methods are able to deal with any types of partial
rankings at the expense of lower statistical efficiency. Generalized rank-breaking methods [14] are
recently proposed to better trade-off the computational and statistical efficiency in LTR.
An arbitrary pairwise preference of $N$ items can be represented as a directed acyclic graph (DAG) of
$N$ nodes, where each node is an item and each directed edge represents the preference over a pair
of items. The generalized rank-breaking methods first apply a graph algorithm to extract a maximal
ordered partition of $[N]$, $S_1 \succ S_2, \ldots, S_M$: a group of $M$ disjoint partitions of $[N]$ with largest
possible $M$, such that the item preference in the $M$ partitions is consistent with that of the DAG. One
difference between data with partitioned preference and data with arbitrary pairwise preference is that
the maximal ordered partition is not unique for the latter, as the maximal ordered partition does not
preserve all relationships in the DAG. With the extracted partitions, we can maximize the likelihood
of these partitions under a PL model to learn the model parameters. Khretan and Ob [14] propose to
calculate the likelihood as shown in Eq. (2), which has a time complexity involves factorials of the
partition sizes. To overcome this challenge for learning large-scale data, existing methods need to
approximate the likelihood by dropping the top $M - 1$ partitions with large sizes. In contrast, the
proposed numerical approach in this paper can be directly applied to the likelihood evaluation step of
generalized rank-breaking methods to significantly improve the computational efficiency.
4 Experiments

In this section, we report empirical results on both synthetic and real-world datasets. We compare the proposed method, denoted as PL-Partition, with two groups of baseline methods that can be applied to large-scale partitioned preference data.

First, we consider two softmax-based listwise methods: PL-LB [10] and AttRank [2]. PL-LB optimizes a lower bound of the likelihood of partitioned preference under the PL model. In particular, for each \( m = 1, \cdots, M-1 \), the term \( P(S_m \succ R_{m+1} ; w) \) in Eq. (4) is replaced by its lower bound \( n_m! \prod_{i \in S_m} \exp(w_i) / \sum_{j \in S_m \cup R_{m+1}} \exp(w_j) \). AttRank optimizes the cross-entropy between the softmax outputs and an empirical probability based on the item relevance given by training labels.

For the second group of baselines, we consider two popular pairwise methods: RankNet [5] and RankSVM [13]. RankNet optimizes a logistic surrogate loss on each pair of items that can be compared. RankSVM optimizes a hinge surrogate loss on each pair of items that can be compared.

4.1 Simulation

We conduct experiments on synthetic data which is generated from a PL model. The goal of this simulation study is two-fold: 1) we investigate how accurate the proposed method can recover the ground truth utility scores of a PL model; 2) we empirically compare the computation costs of different methods over data with different scales.

**Synthetic data generated from a PL model.** We first generate a categorical probability simplex \( p \in \Delta^{N-1} \) as the ground truth utility scores for \( N \) items following \( p = \text{softmax}(q) \) and \( q_i \sim \text{uniform}(0, \log N) \), \( 1 \leq i \leq N \). Then we draw \( n \) samples of full ranking from a PL model parameterized by \( p \). Finally, we randomly split the full rankings into \( M \) partitions and remove the order within each partition to get the partitioned preference. We note that the synthetic data is stateless (i.e., there is no feature for each sample), as this simulation focuses on the estimation of the PL utility scores \( p \) rather than the relationship between \( p \) and sample features. Further, in large-scale real-world applications, we often only observe the order of limited items per sample. For example, a user can only consume a limited number of recommended items. To respect this pattern, we restrict the total number of items in top \( M - 1 \) partitions to be at most 500 regardless of \( N \). We fix \( M = 4 \) and generate data with varying \( (N, n) \) and random seeds.

**Experiment setups.** As the synthetic data is stateless, we only need to train \( N \) free parameters, with each parameter corresponding to an item. We use the proposed method and baseline methods to respectively train the parameters using stochastic gradient descent with early-stopping. We use AdaGrad optimizer with initial learning rate of 0.1 for all methods. We report the mean squared error (MSE) between the softmax of these free parameters and the PL utility scores \( p \) as a measure of how accurately different methods recover \( p \).

**MSE of the estimated PL utility scores.** Figure 1 shows the MSE of the estimated PL utility scores by different methods over various \( N \) and \( n \). To better compare results across data with different numbers of items \( N \), we further include an oracle reference method PL-TopK, which has access to the full ranking of the items in the top \( M - 1 \) partitions and optimizes the corresponding PL likelihood. First, as expected, the proposed PL-Partition method best recovers the ground truth utility scores in terms of MSE on all data configurations, as it numerically approximates the PL likelihood of the partitioned preference data. However, it is worth noting that PL-LB, while also trying to approximate the PL likelihood with a lower bound, performs even worse than pairwise methods when \( N \) is small, which indicates the existing lower bound method is not sufficient to take the full advantage of the PL model.

**Computation cost.** Figure 2 shows the time and memory costs of different methods over various \( N \). The results are obtained using a single Nvidia V100 GPU. We report the total time of running 1000 steps of stochastic gradient descent with batch size 20. We also report the peak CUDA memory. The costs of both time and memory for the pairwise methods grow faster than those for the listwise methods as \( N \) increases. The two listwise baseline methods, AttRank and PL-LB, have similar memory cost. PL-LB has a larger running time due to the calculation of multiple partition functions. The proposed PL-Partition method has an overhead cost due to the numerical integration. However, we
observe that this overhead cost is amortized as $N$ increases. When $N = 10^5$, the computational cost of PL-Partition becomes close to that of PL-LB. Overall, this benchmark empirically demonstrates that the proposed method is scalable for large-scale applications.

4.2 Real-World Datasets

Experiment setups. We also verify the effectiveness of the proposed method on 4 real-world XML datasets [3]: Delicious-1K (D-1K), Eurlex-4K (E-4K), Wiki10-31K (W-31K), and Delicious-200K (D-200K). The trailing number in the name of each dataset indicates the number of classes in the dataset. Following many existing works [25, 4] and the official instructions of XML classification repository [3], we evaluate different methods with 4 types of ranking metrics, Precision@k, Propensity-Scored Precision@k, nDCG@k, and Propensity-Scored nDCG@k. We observe that nDCG-based metrics show similar trends compared to their Precision-based counterparts. Due to space limit, we leave the results of nDCG metrics in Appendix A.5.

We first compare the proposed PL-Partition method with the 4 baseline methods. Note that PL-LB and AttRank collapse into exactly the same method on XML classification as the number of partitions is 2. So we only report the results of PL-LB. For each method, we train a neural network model with the same architecture, 2-layer fully connected network with ReLU activations and hidden size of 256. We train the neural networks with stochastic gradient descent using the ADAM optimizer. The batch size is fixed to 128. We use the official train-test split of each dataset and further split the training set into training and validation (9:1 for D-200K and 3:1 for other datasets). We tune the learning rate by line-search from $\{10^{-4}, 10^{-3}, 10^{-2}\}$ and apply early-stopping in training, based on validation sets.

We also compare PL-Partition with two state-of-the-art embedding-based XML classifiers, SLEEC [4] and LEML [31], which are listed on the XML repository leaderboard [3]. SLEEC

\footnote{In practice, both pairwise and listwise methods can be made more scalable by negative sampling.}

\footnote{The average number of labels per sample (shown in the brackets after the dataset names): D-1K (19.03), E-4K (5.31), W-31K (18.64), D-200K (75.54). See Appendix A.5 for more summary statistics of these datasets.}
Table 1: Precision@k and propensity-scored precision@k on the real-world XML datasets. Due to space limit, PL-Partition, RankNet, and RankSVM are respectively renamed as PL-P, R-Net, and R-SVM. **Bold numbers** indicate the best performance.

|          | PL-P | PL-LB | R-Net | R-SVM | PL-P | PL-LB | R-Net | R-SVM |
|----------|------|-------|-------|-------|------|-------|-------|-------|
| D-1K     | 66.72| 66.12 | 64.11 | 61.95 | 33.22| 30.36 | 32.02 | 31.03 |
| P@1      | 66.72| 64.11 | 61.95 |       | 33.22| 30.36 | 32.02 | 31.03 |
|            | 61.30| 60.13 | 58.46 | 57.05 | 34.23| 31.69 | 32.76 | 31.61 |
|            | 56.63| 54.72 | 53.60 | 52.74 | 35.02| 31.82 | 32.85 | 32.20 |
|            | 47.61| 45.76 | 45.62 | 44.57 | 35.26| 31.91 | 33.27 | 32.71 |
| E-4K     | 78.12| 66.46 | 77.57 | 76.46 | 41.93| 34.55 | 42.71 | 42.71 |
| P@1      | 78.12| 66.46 | 77.57 |       | 41.93| 34.55 | 42.71 | 42.71 |
| P@3      | 62.81| 51.58 | **63.05**| 61.88 | 43.46| 34.37 | 45.55 | **45.69**|
| P@5      | 51.75| 41.35 | **52.55**| 50.95 | 45.70| 34.48 | **48.61**| 47.43 |
| P@10     | 33.79| 26.99 | 34.36 | 32.56 | 57.13| 42.50 | **60.83**| 57.50 |
| W-31K    | 85.97| 80.73 | 82.35 | 80.88 | 13.01| 12.90 | 12.55 | 12.55 |
| P@1      | 85.97| 80.73 | 82.35 |       | 13.01| 12.90 | 12.55 | 12.55 |
| P@3      | 63.01| 44.88 | 56.96 | 50.81 | 14.03| 13.18 | 12.20 | 12.20 |
| P@5      | 51.75| 41.35 | 52.55 | 50.95 | 14.03| 13.18 | 12.20 | 12.20 |
| P@10     | 33.79| 26.99 | 34.36 | 32.56 | 15.70| 14.90 | 13.04 | 13.04 |
| D-200K   | 47.58| 40.38 | 41.93 | 41.41 | 8.72 | 7.79  | 7.06  | 7.13  |
| P@1      | 47.58| 40.38 | 41.93 |       | 8.72 | 7.79  | 7.06  | 7.13  |
| P@3      | 42.09| 37.40 | 38.92 | 38.46 | 9.19 | 8.17  | 8.13  | 8.13  |
| P@5      | 39.23| 35.65 | 36.94 | 36.67 | 9.82 | 8.46  | 8.93  | 8.92  |
| P@10     | 35.11| 32.40 | 33.66 | 33.44 | 11.01| 10.20 | 10.21 | 10.21 |

Table 2: Comparisons between the proposed PL-Partition and SLEEC and LEML. **Bold numbers** indicate the best performance. The results of SLEEC and LEML are from the XML repository [3].

|          | PL-Partition | SLEEC | LEML | PL-Partition | SLEEC | LEML |
|----------|--------------|-------|------|--------------|-------|------|
| D-1K     | 66.72 | **67.59** | 65.67 | 33.22 | 32.11 | 30.73 |
| P@1      | 66.72 | **67.59** | 65.67 | 33.22 | 32.11 | 30.73 |
| P@3      | 61.30 | **61.38** | 60.55 | 34.23 | 33.21 | 32.43 |
| P@5      | 56.63 | **56.56** | 52.80 | 35.02 | 33.83 | 33.26 |
| E-4K     | 78.12 | **79.26** | 63.40 | 41.93 | 34.25 | 28.10 |
| P@1      | 78.12 | **79.26** | 63.40 | 41.93 | 34.25 | 28.10 |
| P@3      | 62.81 | **64.30** | 50.35 | 43.46 | 39.83 | 27.20 |
| P@5      | 51.75 | **52.33** | 41.28 | 45.70 | 42.76 | 29.09 |
| W-31K    | 85.97 | 85.88 | 73.47 | 13.01 | 11.14 | 9.41  |
| P@1      | 85.97 | 85.88 | 73.47 | 13.01 | 11.14 | 9.41  |
| P@3      | 73.07 | 72.98 | 62.43 | 13.46 | 11.86 | 10.07 |
| P@5      | 63.01 | 62.70 | 54.35 | 14.03 | 12.40 | 10.55 |
| D-200K   | 47.58 | **47.85** | 40.73 | 8.72 | 7.17  | 6.06  |
| P@1      | 47.58 | **47.85** | 40.73 | 8.72 | 7.17  | 6.06  |
| P@3      | 42.09 | **42.21** | 37.71 | 9.19 | 8.16  | 7.24  |
| P@5      | 39.23 | **39.43** | 35.84 | 9.82 | 8.96  | 8.10  |

and LEML share similar model architectures with our setup, i.e., 2-layer neural networks, but use different training objectives: SLEEC uses a nearest-neighbor loss; LEML uses a least-square loss.

**Results.** As can be seen in Table 1, the proposed PL-Partition method significantly outperforms the softmax-based listwise method PL-LB on all datasets, indicating the importance of optimizing the proper utility function for listwise methods. PL-Partition also outperforms the pairwise methods RankSVM and RankNet on D-1K, W-31K, and D-200K, where the number of labels per sample is relatively large. When the number of labels per sample is relatively small, breaking the labels into pairwise comparisons leads to little loss of information, and pairwise methods perform well (E-4K).

Table 2 shows the comparison between PL-Partition and embedding-based XML classifiers SLEEC and LEML. SLEEC is better than LEML on all metrics. PL-Partition achieves similar performance to SLEEC on Precision@k and significantly outperforms the baselines on Propensity-Scored Precision@k. The propensity-scored metrics are believed to be less biased towards the head items. Thus the results indicate PL-Partition has better performance than SLEEC for the torso or tail items.

**Discussions.** We note for the task of XML classification, tree-based methods [25, 12] sometimes outperform the embedding-based methods. The focus of this paper is to develop scalable listwise LTR methods for learning neural network ranking models from partitioned preference, instead of methods tailored for XML classification. Therefore we restrain our comparison to embedding methods only, whose model architectures are similar as the 2-layer neural networks in our experiment setup. We also note that SLEEC outperforms tree-based methods for D-200K on the XML repository leaderboard [3]. This indicates PL-Partition achieves state-of-the-art performance on D-200K, where the top partition size is relatively large. Guo et al. [9] recently showed that, with advanced regularization techniques, embedding-based methods trained by RankSVM or PL-LB can be significantly improved to surpass
state-of-the-art tree-based methods on most XML datasets. It seems an interesting future direction to apply such regularization techniques on our proposed LTR objective.

5 Conclusion

In this paper, we study the problem of learning neural network ranking models with a Plackett-Luce-based listwise LTR method from data with partitioned preferences. We overcome the computational challenge of calculating the likelihood of partitioned preferences under the PL model by proposing an efficient numerical integration approach. The key insight of this approach comes from the random utility model formulation of Plackett-Luce with Gumbel distribution. Our experiments on both synthetic data and real-world data show that the proposed method is both more effective and scalable compared to popular existing LTR methods.

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A Appendix

A.1 Proof of Proposition

Proof. We first show \( P(A \succ B; w) = P(\min_{a \in A} g_{w_a} > \max_{b \in B} g_{w_b}) \). If \( A \cup B = [N] \), then the event of \( A \succ B \) is equivalent to the event of \( \min_{a \in A} g_{w_a} > \max_{b \in B} g_{w_b} \) so this equality holds true. Otherwise, assume there is a \( c \in [N] \) but \( c \notin A \cup B \).

We introduce a few notations to assist the proof. For any \( D \subseteq [N] \), let \( G(D) = \{ g_i \ | \ i \in D \} \). Further let \( \Omega(A \succ B; D) \) be the set of all possible permutations of \( D \) that are consistent with the partial ranking \( A \succ B \), i.e.,

\[
\Omega(A \succ B; D) = \{ (i_1, \cdots, i_N) \in \sigma(D) \mid k < l, \forall i_k \in A, i_l \in B \}.
\]

Then we can write the LHS as

\[
P(A \succ B; w) = \sum_{(i_1, \cdots, i_N) \in \Omega(A \succ B; [N])} P(g_{w_{i_1}} > g_{w_{i_2}} > \cdots > g_{w_{i_N}})
\]

\[
= \sum_{(i_1, \cdots, i_N) \in \Omega(A \succ B; [N])} \prod_{i=1}^N \int_{\mathbb{R}^N} \mathbb{I}[g_{w_{i_1}} > g_{w_{i_2}} > \cdots > g_{w_{i_N}}] \mathcal{G}([N])
\]

\[
= \prod_{i=1}^N \int_{\mathbb{R}^N} \sum_{(i_1, \cdots, i_N) \in \Omega(A \succ B; [N])} \mathbb{I}[g_{w_{i_1}} > g_{w_{i_2}} > \cdots > g_{w_{i_N}}] \mathcal{G}([N])
\]

\[
= \prod_{i=1}^N \int_{\mathbb{R}^N} \sum_{k=1}^N \sum_{(j_1, \cdots, j_{N-1}) \in \Omega(A \succ B; [N] \setminus \{c\})} \mathbb{I}[g_{w_{j_1}} > \cdots > g_{w_{j_{k-1}}} > g_{w_c} > g_{w_{j_k}} > \cdots > g_{w_{j_{N-1}}}]
\]

\[
= \prod_{i=1}^N \int_{\mathbb{R}^N} \sum_{k=1}^N \sum_{(j_1, \cdots, j_{N-1}) \in \Omega(A \succ B; [N] \setminus \{c\})} \mathbb{I}[g_{w_{j_1}} > \cdots > g_{w_{j_{N-1}}}]
\]

\[
\cdot \int_{g_{w_c}} \sum_{k=1}^N \mathbb{I}[g_{w_{j_k-1}} > g_{w_c} > g_{w_{j_k}}], \tag{6}
\]

where in the last three qualities, we define \( g_{w_{j_k}} \equiv +\infty \) and \( g_{w_{j_{N-1}}} \equiv -\infty \) to ease the notation. We slightly abused the notation \( g_{w_i} \) by using it to refer both the Gumbel random variables in the first line and the corresponding integral variables in the following lines. We have also omitted the integral variables and the probability densities \( df(g_{w_i}) \) in the derivation. The last equality utilizes the fact that all the Gumbel variables are independent.

Note that, in Eq. (6), given \( g_{w_{j_1}} > \cdots > g_{w_{j_{N-1}}} \), \( \sum_{k=1}^N \mathbb{I}[g_{w_{j_k-1}} > g_{w_c} > g_{w_{j_k}}] \equiv 1 \) regardless the choice of \( (j_1, \cdots, j_{N-1}) \). Therefore,

\[
\prod_{i=1}^N \int_{g_{w_c}} \sum_{k=1}^N \mathbb{I}[g_{w_{j_k-1}} > g_{w_c} > g_{w_{j_k}}] \equiv 1,
\]

11
and

\[ P(A \succ B; w) = \int \cdots \int_{\Omega(A \succ B; \{\{\}])} g(\{\} \setminus \{\}) \sum_{\{j_1, \ldots, j_{N-1}\} \in \Omega(A \succ B; \{\{\}])} \mathbb{I}[g_{w_{j_1}} > \cdots > g_{w_{j_{N-1}}}]. \tag{7} \]

By applying Eq. (7) to all the items that do not belong to \( A \cup B \), we get

\[ P(A \succ B; w) = \int \cdots \int_{\Omega(A \cup B; \{\})} g(\{\}) \sum_{\{j_{A|+;B}, \ldots, j_{A|+;B}\} \in \Omega(A \succ B; A \cup B)} \mathbb{I}[g_{w_{j_1}} > \cdots > g_{w_{j_{A|+;B}}}]. \tag{8} \]

And note that this reduces to a situation equivalent to the case \( A \cup B = [N] \). Therefore we have shown \( P(A \succ B; w) = P(\min_{a \in A} g_{w_a} > \max_{b \in B} g_{w_b}) \).

The proof for \( P(\min_{a \in A} g_{w_a} > \max_{b \in B} g_{w_b}) = \int_{u=0}^{1} \prod_{a \in A} (1 - u^{exp(w_i - w_{R_{m+1}})}) du \) remains the same no matter if \( A \cup B = [N] \) or not, as the Gumbel variables are independent. Therefore we refer the reader to the Appendix B of Kool et al. [15] for the proof. \( \square \)

### A.2 Analysis of Computation Costs

Recall the likelihood is

\[ P(S_1 \succ \cdots \succ S_M; w) = \prod_{m=1}^{M-1} P(S_m \succ R_{m+1}; w) = \prod_{m=1}^{M-1} \int_{u=0}^{1} \prod_{i \in S_m} \left(1 - u^{exp(w_i - w_{R_{m+1}})}\right) du, \]

where \( w_{R_{m+1}} = \log \sum_{j \in R_{m+1}} exp(w_j) \). For partition \( m = 1, \cdots, M - 1 \), calculating \( w_{R_{m+1}} \) requires \( O(N) \) time complexity but only needs to be calculated once. Suppose we set the number of numerical integration intervals as \( T \). Calculating the integrand \( \prod_{i \in S_m} (1 - u^{exp(w_i - w_{R_{m+1}})}) \)

given \( w_{R_{m+1}} \) requires time complexity \( O(n_m) \). So the numerical integration for partition \( m \) requires \( O(N + Tn_m) \) time complexity. The time complexity for evaluating the likelihood in total is \( O(N + T(N - n_M)) \).

### A.3 Numerical Stability

In this section, we study the numerical stability of the likelihood in Eq. (4) and its gradients in Eq. (5).

For each individual integrations in Eq. (4) indexed by \( m = 1, \cdots, M - 1 \), the two major sources of numerical instability are:

1. \( u^{exp(w_i - w_{R_{m+1}})} \), for \( i \in S_m \), has a form of double exponential and may suffer from numerical errors if \( w_i - w_{R_{m+1}} \) deviates much from 0;
2. \( \prod_{i \in S_m} (1 - u^{exp(w_i - w_{R_{m+1}})}) \) is a product of small numbers and may suffer from numerical errors.

Kool et al. [15] have proposed to use a change of variable trick to alleviate the first issue, and convert the numerical integration to the logarithm scale to alleviate the second issue. And these numerical tricks can be applied to stabilize the individual integrations in Eq. (4).

For the gradients in Eq. (5), we need to take more care in addition to the two tricks above. We first expand the gradients of the log-likelihood w.r.t. \( w \) in Eq. (5) below.

\[ \nabla_{w} \log P(S_1 \succ \cdots \succ S_M; w) = \sum_{m=1}^{M-1} \nabla_{w} \log P(S_m \succ R_{m+1}; w) \]

\[ = \sum_{m=1}^{M-1} \frac{1}{P(S_m \succ R_{m+1}; w)} \nabla_{w} P(S_m \succ R_{m+1}; w) \]

\[ = \sum_{m=1}^{M-1} \frac{1}{P(S_m \succ R_{m+1}; w)} \nabla_{w} \int_{u=0}^{1} \prod_{i \in S_m} (1 - u^{exp(w_i - w_{R_{m+1}})}) du. \tag{9} \]
Further note that
\[
\nabla_w \prod_{i \in S_m} (1 - u^{\exp(w_i - w_{R_{m+1}})})
\]
\[
= \sum_{i \in S_m} \left[ \prod_{j \in S_m \setminus \{i\}} (1 - u^{\exp(w_j - w_{R_{m+1}})}) \right] \left[ - \nabla_w u^{\exp(w_i - w_{R_{m+1}})} \right]
\]
\[
= - \left[ \prod_{j \in S_m} (1 - u^{\exp(w_j - w_{R_{m+1}})}) \right] \sum_{i \in S_m} \frac{u^{\exp(w_i - w_{R_{m+1}})} \log u}{1 - u^{\exp(w_i - w_{R_{m+1}})}} \nabla_w u^{\exp(w_i - w_{R_{m+1}})}.
\]  \hspace{1cm} (10)

Plug Eq. (10) into the gradients Eq. (9), we have
\[
\nabla_w \log P(S_1 \succ \cdots \succ S_M; w)
\]
\[
= - \sum_{m=1}^{M-1} \frac{1}{P(S_m \succ R_{m+1}; w)} \int_{u=0}^{1} \left[ \prod_{j \in S_m} (1 - u^{\exp(w_j - w_{R_{m+1}})}) \right] \frac{u^{\exp(w_i - w_{R_{m+1}})} \log u}{1 - u^{\exp(w_i - w_{R_{m+1}})}} \nabla_w u^{\exp(w_i - w_{R_{m+1}})} du
\]
\[
= - \sum_{m=1}^{M-1} \frac{1}{P(S_m \succ R_{m+1}; w)} \sum_{i \in S_m} \nabla_w u^{\exp(w_i - w_{R_{m+1}})}
\]
\[
\cdot \int_{u=0}^{1} \left[ \prod_{j \in S_m} (1 - u^{\exp(w_j - w_{R_{m+1}})}) \right] \frac{u^{\exp(w_i - w_{R_{m+1}})} \log u}{1 - u^{\exp(w_i - w_{R_{m+1}})}} du
\]  \hspace{1cm} (11)

Now we apply the aforementioned change of variable trick to stabilize \( u^{\exp(w_j - w_{R_{m+1}})} \), for \( j \in S_m \).
Let \( v = u^{\exp(-c - w_{R_{m+1}})} \), where \( c \) is a constant to be determined. The integration in Eq. (11) for each \( m = 1, \cdots, M - 1 \) can be re-written as
\[
\int_{u=0}^{1} \prod_{j \in S_m} (1 - u^{\exp(w_j - w_{R_{m+1}})}) \frac{u^{\exp(w_i - w_{R_{m+1}})} \log u}{1 - u^{\exp(w_i - w_{R_{m+1}})}} du
\]
\[
= \exp(2c + 2w_{R_{m+1}}) \int_{v=0}^{1} \prod_{j \in S_m} (1 - v^{\exp(w_j + c)}) \frac{v^{\exp(w_i + w_{R_{m+1}} + 2c) - 1} \log v}{1 - v^{\exp(w_i + c)}} dv
\]  \hspace{1cm} (12)

The double exponential term \( u^{\exp(w_j - w_{R_{m+1}})} \) is replaced by \( v^{\exp(w_j + c)} \). If \( c \) can be set to make most \( w_j + c \) not far from 0, then we can reduce the numerical errors of \( v^{\exp(w_j + c)} \).

We note that, however, \( \frac{v^{\exp(w_i + w_{R_{m+1}} + 2c) - 1} \log v}{1 - v^{\exp(w_i + c)}} \) in each integrand is not well-defined at \( v = 0 \) and \( v = 1 \). So we investigate the numerical errors of the integration near 0 and 1.

When \( v \) is close to 1, everything in the integrand is bounded except for \( \frac{\log v}{1 - v^{\exp(w_i + c)}} \). By L’Hospital’s rule, we have
\[
\lim_{v \to 1} \frac{\log v}{1 - v^{\exp(w_i + c)}} = \lim_{v \to 1} \frac{1/v}{1 - v^{\exp(w_i + c)} v^{\exp(w_i + c) - 1}}
\]
\[
= - \frac{1}{\exp(w_i + c)}.
\]

For any \( \epsilon > 0 \), by continuity, \( \exists M < \infty \), such that
\[
\left| \frac{\log v}{1 - v^{\exp(w_i + c)}} \right| < M
\]
for any $v \in [1 - \epsilon, 1]$. Therefore,

\[
\left| \int_{1-\epsilon}^{1} \prod_{j \in S_m} (1 - v^{\exp(w_j + c)}) \frac{v^{\exp(w_i + w_{R_{m+1}} + 2c)} - 1 \log v}{1 - v^{\exp(w_i + c)}} \, dv \right|
\]

\[
\leq \int_{1-\epsilon}^{1} \prod_{j \in S_m} (1 - v^{\exp(w_j + c)}) \frac{v^{\exp(w_i + w_{R_{m+1}} + 2c)} - 1 \log v}{1 - v^{\exp(w_i + c)}} \, dv
\]

\[
\leq M \int_{1-\epsilon}^{1} \prod_{j \in S_m} (1 - v^{\exp(w_j + c)}) \, dv
\]

\[
\leq M \prod_{j \in S_m} \exp(w_j + c) \epsilon^{|S_m|+1},
\]

where for the last second inequality we have used the fact that $(1 - x)^a \geq 1 - ax$ when $x \to 0$. Therefore, we do not need to worry much about numerical error when $v$ is close to 1.

When $v$ is close to 0, suppose $\prod_{j \in S_m} (1 - v^{\exp(w_j + c)}) / (1 - v^{\exp(w_i + c)}) \leq M$. Then

\[
\left| \int_{0}^{\epsilon} \prod_{j \in S_m} (1 - v^{\exp(w_j + c)}) \frac{v^{\exp(w_i + w_{R_{m+1}} + 2c)} - 1 \log v}{1 - v^{\exp(w_i + c)}} \, dv \right|
\]

\[
\leq \int_{0}^{\epsilon} \prod_{j \in S_m} (1 - v^{\exp(w_j + c)}) \frac{v^{\exp(w_i + w_{R_{m+1}} + 2c)} - 1 \log v}{1 - v^{\exp(w_i + c)}} \, dv
\]

\[
\leq M \int_{0}^{\epsilon} -v^{\exp(w_i + w_{R_{m+1}} + 2c)} - 1 \log v \, dv.
\]

Also, it is easy to verify that

\[
\int_{0}^{\epsilon} e^a \log v = \left( \frac{1}{a+1} \log \epsilon - \frac{1}{(a+1)^2} \right) e^{a+1}.
\]

Let $a = \exp(w_i + w_{R_{m+1}} + 2c) - 1$, then

\[
\left| \int_{0}^{\epsilon} \prod_{j \in S_m} (1 - v^{\exp(w_j + c)}) \frac{v^{\exp(w_i + w_{R_{m+1}} + 2c)} - 1 \log v}{1 - v^{\exp(w_i + c)}} \, dv \right|
\]

\[
\leq M \left( \frac{1}{(a+1)^2} - \frac{1}{a+1} \log \epsilon \right) e^{a+1}.
\]

Further by exploiting the inequality $1 - \frac{1}{x} < \log x$, we have

\[
\left| \int_{0}^{\epsilon} \prod_{j \in S_m} (1 - v^{\exp(w_j + c)}) \frac{v^{\exp(w_i + w_{R_{m+1}} + 2c)} - 1 \log v}{1 - v^{\exp(w_i + c)}} \, dv \right|
\]

\[
\leq M \left( \frac{1}{(a+1)^2} - \frac{1}{a+1} + \frac{1}{a+1} \epsilon \right) e^{a+1}
\]

\[
\leq M e^a.
\]

The numerical error can be well-controlled as long as $a > 0 \Rightarrow \exp(w_i + w_{R_{m+1}} + 2c) > 1 \Rightarrow w_i + w_{R_{m+1}} + 2c > 0$.

In practice, we found that there is a wide range of $c$ that can give reasonable numerical stability.
Figure 3: MSE of the estimated PL parameters vs various numbers of items $N$ and number of samples $n$. Both x-axis and y-axis are in the logarithmic scale with base 10. The results are averaged over 5 different random seeds and error bars indicate the standard error of the mean.

Table 3: Summary Statistics of the XML Classification Datasets

| Dataset  | #Feature | #Label | #Train | #Test | Avg. #Sample per Label | Avg. #Label per Sample |
|----------|----------|--------|--------|-------|------------------------|------------------------|
| D-1K     | 500      | 983    | 12920  | 3185  | 311.61                 | 19.03                  |
| E-4K     | 5000     | 3993   | 15539  | 3809  | 25.73                  | 5.31                   |
| W-31K    | 101938   | 30938  | 14146  | 6616  | 8.52                   | 18.64                  |
| D-200K   | 782585   | 205443 | 196606 | 100095| 72.29                  | 75.54                  |

A.4 Supplemental Details for Simulation

We also provide additional simulation results with $N = 100, 1000, 10000$ in Figure 3. The trend is similar as what has been shown in Figure 1 in Section 4.1.

A.5 Supplemental Details for Experiments on XML Datasets

We provide the summary statistics of the 4 XML classification datasets in Table 4. We also provide the results of the nDCG-based metrics in Table 4. The nDCG-based metrics are highly correlated with their Precision-based counterparts.

Table 4: nDCG@k and propensity-scored nDCG@k on the real-world XML datasets. Due to space limit, PL-Partition, RankNet, and RankSVM are respectively renamed as PL-P, R-Net, and R-SVM. **Bold numbers** indicate the best performance.

| Dataset  | PL-P @1 | PL-LB @1 | R-Net @3 | R-SVM @5 | PL-P @10 | PL-LB @10 | R-Net @3 | R-SVM @5 | PL-P @10 | PL-LB @10 | R-Net @3 | R-SVM @5 |
|----------|---------|----------|----------|----------|----------|-----------|----------|----------|----------|-----------|----------|----------|
| D-1K     | 66.72   | 66.12    | 64.11    | 61.95    | **33.22**| 30.36     | 32.02    | 31.03    | **37.58**| 34.55     | 35.77    | 34.87    |
| E-4K     | 59.25   | 57.51    | 56.29    | 55.01    | **37.58**| 34.55     | 35.77    | 34.87    | **37.58**| 34.55     | 35.77    | 34.87    |
| W-31K    | **52.48**| 50.70    | 50.23    | 49.08    | **39.48**| 36.09     | 37.43    | 36.52    | **39.48**| 36.09     | 37.43    | 36.52    |
| D-200K   | 78.12   | 66.46    | 71.57    | 67.46    | **42.81**| 34.55     | 42.71    | 42.71    | **42.81**| 34.55     | 42.71    | 42.71    |

| Dataset  | PL-P @1 | PL-LB @1 | R-Net @3 | R-SVM @5 | PL-P @10 | PL-LB @10 | R-Net @3 | R-SVM @5 | PL-P @10 | PL-LB @10 | R-Net @3 | R-SVM @5 |
|----------|---------|----------|----------|----------|----------|-----------|----------|----------|----------|-----------|----------|----------|
| E-4K     | 60.60   | 55.16    | 66.56    | 65.32    | **47.61**| 38.18     | 49.83    | 49.87    | **52.98**| 39.19     | 52.50    | 52.50    |
| W-31K    | 59.66   | 48.71    | **59.86**| 58.61    | **50.51**| 39.19     | **52.98**| 52.50    | **52.98**| 39.19     | 52.50    | 52.50    |
| D-200K   | **57.34**| 46.78    | 57.78    | 56.35    | **50.76**| 39.19     | **53.33**| 52.72    | **50.76**| 39.19     | 52.72    | 52.72    |