MULTIBASIC EHRHART THEORY

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ABSTRACT. In the present paper, we introduce a multibasic extension of the Ehrhart theory. We give a multibasic extension of Ehrhart polynomials and Ehrhart series. We also show that an analogue of Ehrhart reciprocity holds for multibasic Ehrhart polynomials.

INTRODUCTION

Recently, some extensions of the Ehrhart theory have been studied, for example in [3, 10, 12]. In this paper, we introduce a “multibasic” extension of the Ehrhart theory [3]. In the study of the special functions and the difference equations, the $q$-analogues of the special functions are well known. For example, the (generalized) basic hypergeometric series, the $q$-Bessel functions, the $q$-Airy functions...(see [5] for more details). These functions have a parameter $q$, which is called “the base”. If we introduce other bases $q_1, q_2, \ldots, q_N$ where $q_j \neq q_k$, we may consider the multibasic extension [6] of the original $q$-analogues. A $q$-analogue of the Ehrhart theory is given by F. Chapoton [3]. He gave the $q$-extension of the Ehrhart theory by a suitable linear form.

In the study of combinatorics, the Ehrhart theory plays an important role when we study the relationship between the convex polytopes and the integer points. We review the Ehrhart theory which was introduced by Engène Ehrhart in 1960s [4]. Let $\mathcal{P} \subset \mathbb{R}^N$ be an integral convex polytope of dimension $d$ and the set $\mathcal{P}^\circ$ its interior. If $n$ is a positive integer then we define

$$L_{\mathcal{P}}(n) = \#(n\mathcal{P} \cap \mathbb{Z}^N).$$

In other words, $L_{\mathcal{P}}(n)$ is equal to the number of the integer points in $n\mathcal{P}$ where $n\mathcal{P} = \{n\alpha | \alpha \in \mathcal{P}\}$. Ehrhart showed that the enumerative function $L_{\mathcal{P}}(n)$ is a polynomial in $n$ of degree $d$. He also gave $L_{\mathcal{P}}(0) = 1$. The polynomial $L_{\mathcal{P}}(n)$ is called the Ehrhart polynomial after his works. If we consider the Ehrhart polynomial $L_{\mathcal{P}}(n)$ where $n \in \mathbb{Z}$, we obtain the relation

$$L_{\mathcal{P}}(-n) = (-1)^d L_{\mathcal{P}^\circ}(n),$$

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which is called the Ehrhart reciprocity. The generating function of the Ehrhart polynomial \( L_P(n) \) is given by

\[
Ehr_P(t) = 1 + \sum_{n=1}^\infty L_P(n)t^n.
\]

This generating function is also called the Ehrhart series of \( P \). It is known that the Ehrhart series have the following representation by rational function

\[
Ehr_P(t) = \frac{\delta_d t^d + \delta_{d-1} t^{d-1} + \cdots + \delta_0}{(1-t)^{d+1}}
\]

where \( \delta_k \in \mathbb{Z} \), provided that \( 0 \leq k \leq d \). The vector \( \delta(P) = (\delta_0, \delta_1, \ldots, \delta_d) \in \mathbb{Z}^{d+1} \) is called the \( \delta \)-vector of \( P \). The \( \delta \)-vectors satisfy the equations \( \delta_0 = 1 \) and \( \delta_1 = \#(P \cap \mathbb{Z}^n) - (d + 1) \). A lot of significant other properties of the \( \delta \)-vectors are studied, for example in \([8, 11]\). We refer the reader to \([1, 7]\) for the detailed information about Ehrhart theory.

This paper is organized as follows. In Section 1 we introduce the concepts of the multibasic Ehrhart series and the multibasic \( \delta \)-vector, which are multibasic extensions of each one. We show that the multibasic Ehrhart series has representation as a rational function. Its numerator is a polynomial in \( t \) whose coefficients are given by the elements of \( \mathbb{Z}[q_1, q_1^{-1}, \ldots, q_N, q_N^{-1}] \), and the degree of the polynomial is (the number of vertices of \( P \)) \(-1\), at most. In Section 2 we introduce the multibasic Ehrhart polynomials for integral polytopes \( P \subset (\mathbb{R}_{\geq 0})^N \). We also give some examples of the multibasic Ehrhart polynomials for typical classes of polytopes. In Section 3 we obtain the reciprocity theorem for the multibasic Ehrhart polynomials.

1. THE MULTIBASIC EHRHART SERIES

In the present paper we assume that each polytope is convex. Let \( a = (a_1, \ldots, a_N) \in \mathbb{Z}^N \) be an integer point. The Laurent monomial \( q^a \) is defined as

\[
q^a := q_1^{a_1} q_2^{a_2} \cdots q_N^{a_N}, \quad q^0 := 1
\]

where \( 0 := (0, 0, \ldots, 0) \). Recall that for a given rational cone or rational polytope \( S \subset \mathbb{R}^N \),

\[
\sigma_S(q) = \sigma_S(q_1, q_2, \ldots, q_N) := \sum_{a \in S \cap \mathbb{Z}^N} q^a
\]

is called the integer-point transform of \( S \). Let \( P \subset \mathbb{R}^N \) be an integral polytope of dimension \( d \). For any \( n \in \mathbb{Z}_{>0} \), let \( nP = \{n\alpha | \alpha \in P\} \). Now we define the multibasic Ehrhart series as follows:

\[
Ehr_P(t; q) = Ehr_P(t; q_1, q_2, \ldots, q_N) := 1 + \sum_{n=1}^\infty \sigma_{nP}(q)t^n.
\]

Note that this definition gives us the expansion of the classical Ehrhart series. Indeed the special case \( Ehr_P(t; 1, 1, \ldots, 1) \) is the classical Ehrhart series.
Lemma 1.1 ([1], Theorem 3.5). For any \( w_1, w_2, \ldots, w_d \in \mathbb{Z}^N \) if \( K := \{ \sum_{i=1}^d r_i w_i \mid r_i \geq 0 \} \) is a simplicial \( d \)-cone, then for \( v \in \mathbb{R}^N \), the integer-point transform of \( v + K \) is given by

\[
\sigma_{v+K}(q) = \frac{\sigma_{v+(q)}(q)}{\prod_{i=1}^d (1 - q^{w_i})},
\]

where \( \Gamma := \{ \sum_{i=1}^d r_i w_i \mid 0 \leq r_i < 1 \} \).

Proposition 1.2. Let \( \Delta \subset \mathbb{R}^N \) be an integral simplex of dimension \( d \) with vertices \( v_1, v_2, \ldots, v_{d+1} \). Then the multibasic Ehrhart series of \( \Delta \) is

\[
\text{Ehr}_\Delta(t; q) = \sum_{a \in \Gamma \cap \mathbb{Z}^{N+1}} q_1^{a_1} q_2^{a_2} \cdots q_N^{a_N} t^{a_{N+1}} \prod_{i=1}^{d+1} (1 - q^v_i t),
\]

where \( \Gamma = \{ \sum_{i=1}^{d+1} r_i (v_i, 1) \mid 0 \leq r_i < 1 \} \). More precisely the numerator of \( \text{Ehr}_\Delta(t; q) \) is a polynomial in \( t \), which has Laurent polynomial in \( N \) variables as its coefficients and the coefficients of each Laurent polynomials is either 0 or 1. Moreover the degree of this polynomial is at most \( d \).

Proof. We define a set \( \text{cone}(\Delta) \) as follows:

\[ \text{cone}(\Delta) := \left\{ \sum_{i=1}^{d+1} r_i (v_i, 1) \mid r_i \geq 0 \right\} \subset \mathbb{R}^{N+1}. \]

Note that \( \text{cone}(\Delta) \) is a simplicial \((d + 1)\)-cone, and we obtain \( n\Delta \) by considering intersection of \( \text{cone}(\Delta) \) and a hyperplane \( x_{N+1} = n \). Then we can calculate as follows by Lemma 1.1:

\[
\text{Ehr}_\Delta(t; q) = \text{Ehr}_\Delta(t; q_1, q_2, \ldots, q_N) = 1 + \sum_{n=1}^{\infty} \sigma_n \Delta(q_1, q_2, \ldots, q_N) t^n
\]

\[
= \sum_{(a_1, \ldots, a_{N+1}) \in \text{cone}(\Delta) \cap \mathbb{Z}^{N+1}} q_1^{a_1} q_2^{a_2} \cdots q_N^{a_N} t^{a_{N+1}} = \sigma_{\text{cone}(\Delta)}(q_1, q_2, \ldots, q_N, t)
\]

\[
= \sigma_{\Gamma}(q_1, q_2, \ldots, q_N, t) = \frac{\sum_{a \in \Gamma \cap \mathbb{Z}^{N+1}} q_1^{a_1} q_2^{a_2} \cdots q_N^{a_N} t^{a_{N+1}}}{\prod_{i=1}^{d+1} (1 - q^v_i t)}. \]

Since the \((N + 1)\)-th coordinate of each generator of \( \text{cone}(\Delta) \) is 1, the \((N + 1)\)-th coordinate \( a_{N+1} \) of \( a \in \Gamma \cap \mathbb{Z}^{N+1} \) is \( r_1 + \cdots + r_{d+1} \), where \( 0 \leq r_1, \ldots, r_{d+1} < 1 \). Hence since \( r_1 + \cdots + r_{d+1} \) is an integer, the degree in \( t \) of numerator is at most \( d \), as desired.

Example 1.3. Let \( \Delta = [a, b] \) be a 1-simplex where \( a, b \in \mathbb{Z} \) and \( a < b \). Then we obtain

\[
\text{Ehr}_\Delta(t; q) = \frac{1 + \sum_{k=a+1}^{b} q^k t}{(1 - q^2 t)(1 - q^3 t)}. \]
since \( \Gamma \cap \mathbb{Z}^2 = \{(0,0),(a+1,1),(a+2,1),\ldots,(b-1,1)\} \).

**Example 1.4** (The standard \( d \)-simplex). Let \( e_i \ (1 \leq i \leq d+1) \) be unit vectors in \( \mathbb{R}^{d+1} \) and let \( \Delta := \text{conv}\{e_1,e_2,\ldots,e_{d+1}\} \). Then the multibasic Ehrhart series of \( \Delta \) is given as

\[
\text{Ehr}_\Delta(t;q) = \frac{1}{\prod_{i=1}^{d+1}(1-q_it)}.
\]

since we have \( \Gamma \cap \mathbb{Z}^{d+2} = \{0\} \) by

\[
\Gamma = \left\{ \sum_{i=1}^{d+1} r_ie_i, 1 \right\} \left| 0 \leq r_i < 1 \right\} = \left\{ (r_1, r_2, \ldots, r_{d+1}, \sum_{i=1}^{d+1} r_i) \left| 0 \leq r_i < 1 \right\} \right.
\]

**Remark 1.5.**

\[
\sigma_{\text{cone}(P)}(q_1,q_2,\ldots,q_N,q_{N+1})
\]

is the integer-point transform of the cone over the integral polytope \( P \subset \mathbb{R}^N \). Then we have

\[
\sigma_{\text{cone}(P)}(q_1,q_2,\ldots,q_N,t) = \text{Ehr}_P(t;q)
\]

and

\[
\sigma_{\text{cone}(P)}(1,1,\ldots,1,t) = \text{Ehr}_P(t)
\]

by the same idea of the proof of Proposition 1.2. Moreover we have

\[
\sigma_{\text{cone}(P)}(q^{a_1}, q^{a_2}, \ldots, q^{a_N}, t) = \text{Ehr}_P(\lambda(t,q))
\]

where \( \text{Ehr}_P(\lambda(t,q)) \) is the \( q \)-Ehrhart series and \( \lambda = (a_1, a_2, \ldots, a_N) \) is a linear form satisfying positivity and genericity (See [3]).

Namely, the classical Ehrhart series, the \( q \)-Ehrhart series and the multibasic Ehrhart series can be obtained from \( \sigma_{\text{cone}(P)}(q_1,q_2,\ldots,q_N,q_{N+1}) \).

**Theorem 1.6.** Let \( P \subset \mathbb{R}^N \) be an integral polytope of dimension \( d \) and let \( v_1,\ldots,v_m \) its vertices. Then the multibasic Ehrhart series of \( P \) can be displayed as

\[
\text{Ehr}_P(t;q) = \frac{\delta_{m-1}t^{m-1} + \delta_{m-2}t^{m-2} + \cdots + \delta_1 t + \delta_0}{\prod_{i=1}^{m}(1-q^iv_it)}
\]

\[
\delta_k \in \mathbb{Z}[q, q^{-1}]=\mathbb{Z}[q_1, q_1^{-1}, \ldots, q_N, q_N^{-1}], \ 0 \leq k \leq m-1.
\]

Namely, the numerator of the multibasic Ehrhart series is a polynomial in \( t \), which has Laurent polynomials in \( N \) variables with coefficients in \( \mathbb{Z} \) as its coefficients. In particular the degree of the numerator of \( \text{Ehr}_P(t;q) \) is at most \( m-1 \).

**Proof.** Since the set \( \text{cone}(P) \) is a pointed cone, \( \text{cone}(P) \) can be triangulated into simplicial cones such that each simplicial cone does not have any new generator. We remark that the intersection of simplicial cones is a simplicial cone. We can obtain the integer-point transform of each simplicial cone by Lemma 1.1. By the principle of inclusion-exclusion, we obtain the integer-point transform of \( \text{cone}(P) \) as
an alternating sum of the integer-point transforms of those simplicial cones. Thus the integer-point transform of cone(\(P\)) is given by

\[
\sigma_{\text{cone}(P)}(q_1, q_2, \ldots, q_{N+1}) = \frac{f}{\prod_{i=1}^{m}(1 - q^i q_{N+1})}
\]

where \(f \in \mathbb{Z}[q_1, q_1^{-1}, \ldots, q_{N+1}, q_{N+1}^{-1}]\). By Proposition 1.2, the highest exponent of \(q_{N+1}\) that appear in the numerator of the integer-point transform of each \((d+1)\)-dimensional simplicial cone is at most \(d\). Thus the highest exponent of \(q_{N+1}\) that appear in \(f\) is at most \(d + \{m - (d + 1)\} = d - 1 + \{m - (d - 1 + 1)\} = m - 1\). Since \(\sigma_{\text{cone}(P)}(q_1, q_2, \ldots, q_n, t) = \text{Ehr}_P(t; q)\), we obtain the conclusion. \(\square\)

We call

\[
\delta_q(P) := (\delta_0, \delta_1, \ldots, \delta_{m-1}), \quad \delta_k \in \mathbb{Z}[q, q^{-1}]
\]

the multibasic \(\delta\)-vector of \(P\), where \(\sum_{k=0}^{m-1} \delta_k t^k\) is the numerator of \(\text{Ehr}_P(t; q)\) that we acquire in Theorem 1.6.

**Corollary 1.7.** Let \(P \subset \mathbb{R}^N\) be an integral polytope and \(v_1, v_2, \ldots, v_m\) its vertices. Then for the multibasic \(\delta\)-vector of \(P\), the following properties hold.

\[
\delta_0 = 1, \quad \delta_1 = \sigma_P(q) - \sum_{i=1}^{m} q^{v_i}
\]

**Corollary 1.8.** Let \(P \subset \mathbb{R}^N\) be an integral polytope and let \(P' = P + v, v \in \mathbb{Z}^N\). Then we have

\[
\text{Ehr}_{P'}(t; q) = \text{Ehr}_P(q^t; q), \quad \delta_q(P') = (\delta_0, \delta_1 q^v, \delta_2 q^{2v}, \ldots, \delta_{m-1} q^{(m-1)v}), \quad \delta_k \in \mathbb{Z}[q, q^{-1}].
\]

**Proof.** Since we have \(\sigma_{nP'}(q) = \sigma_{nP}(q) \cdot q^{nv}\) for all \(n \in \mathbb{Z}_{>0}\), it then follows that

\[
\text{Ehr}_{P'}(t; q) = 1 + \sum_{n=1}^{\infty} \sigma_{nP'}(q)t^n = 1 + \sum_{n=1}^{\infty} \sigma_{nP}(q) \cdot q^{nv}t^n = 1 + \sum_{n=1}^{\infty} \sigma_{nP}(q)(q^t)^n = \text{Ehr}_P(q^t; q).
\]

Hence the numerator of \(\text{Ehr}_{P'}(t; q)\) is \(\sum_{k=0}^{m-1} \delta_k(q^v t)^k = \sum_{k=0}^{m-1} (\delta_k q^{kv}) t^k\), as desired. \(\square\)

### 2. Multibasic Ehrhart Polynomials

In this section, we show the existence of a multibasic Ehrhart polynomial. The notation

\[
[n]_q := \frac{1 - q^n}{1 - q}, \quad n \in \mathbb{Z}
\]

is a \(q\)-integer \([3]\). We review the key relation as follows:

\[
(1 + qx - x)|_{x=[n]_q} = 1 + q \frac{1 - q^n}{1 - q} - \frac{1 - q^n}{1 - q} = q^n.
\]
Then we have the following relation immediately:

\[
q^n = \prod_{k=1}^{N} (1 + q_k x_k - x_k) \bigg|_{x_k = [n]_{q_k}}
\]

where \( n = (n, n, \ldots, n) \).

For any vertex \( v \) of a rational polytope \( P \subset \mathbb{R}^N \), the vertex cone \( K_v \) is defined by

\[
K_v := \{ v + r(x - v) \mid x \in P, r \in \mathbb{R}_{\geq 0} \}.
\]

Let \( v \) be a vertex of \( P \) and set

\[
C_i := (-v_i) + K_{v_i} = \{ r(x - v_i) \mid x \in P, r \in \mathbb{R}_{\geq 0} \}.
\]

**Lemma 2.1** (Brion’s theorem, [2]). Let \( P \subset \mathbb{R}^N \) be a rational polytope of dimension \( d \). Then

\[
\sigma_{P}(q) = \sum_{v \text{ a vertex of } P} \sigma_{K_v}(q).
\]

**Theorem 2.2.** Let \( P \subset (\mathbb{R}_{\geq 0})^N \) be an integral polytope of dimension \( d \). Then there exists a polynomial

\[
L_P(x_1, x_2, \ldots, x_N) \in \mathbb{Q}(q)[x_1, x_2, \ldots, x_N]
\]

such that

\[
L_P([n]_{q_1}, [n]_{q_2}, \ldots, [n]_{q_N}) = \sigma_{nP}(q), \quad \forall n \in \mathbb{Z}_{>0}.
\]

In particular, the degree of \( L_P(x_1, x_2, \ldots, x_N) \) coincides with \( \max \{ \sum_{k=1}^{N} v_{ik} | 1 \leq i \leq m \} \) where \( v_i = (v_{i1}, v_{i2}, \ldots, v_{iN}), 1 \leq i \leq m \) are vertices of \( P \).

**Proof.** By Lemma 2.1 we have

\[
\sigma_{nP}(q) = \sum_{i=1}^{m} \sigma_{K_{nv_i}}(q),
\]

since \( K_{nv_i} \) can be obtained as a parallel translation of \( C_i \), then we have \( \sigma_{K_{nv_i}}(q) = \sigma_{C_i}(q) \cdot q^{nv_i} \). Therefore we obtain

\[
\sum_{i=1}^{m} \sigma_{K_{nv_i}}(q) = \sum_{i=1}^{m} \sigma_{C_i}(q) \cdot q^{nv_i}.
\]

By using (1), we have

\[
q^{nv_i} = \prod_{k=1}^{N} (1 + q_k x_k - x_k)^{v_{ik}} \bigg|_{x_k = [n]_{q_k}}
\]

Thus, we have

\[
L_P(x_1, x_2, \ldots, x_N) = \sum_{i=1}^{m} \sigma_{C_i}(q) \prod_{k=1}^{N} (1 + q_k x_k - x_k)^{v_{ik}}
\]

Our claim on the degree of \( L_P(x_1, x_2, \ldots, x_N) \) follows immediately since \( \sigma_{C_i}(q) \in \mathbb{Q}(q) \).
We call the polynomial $L_P(x_1, x_2, \ldots, x_N)$ the multibasic Ehrhart polynomial of $P$.

**Corollary 2.3.** Let $P \subset (\mathbb{R}_{\geq 0})^N$ be an integral polytope, $P' = P + v \subset (\mathbb{R}_{\geq 0})^N$ and $v = (v_1, v_2, \ldots, v_N) \in \mathbb{Z}^N$. Then we have

$$L_{P'}(x_1, x_2, \ldots, x_N) = L_P(x_1, x_2, \ldots, x_N) \prod_{k=1}^{N} (1 + q_k x_k - x_k)^{v_k}.$$ 

**Proof.** By Theorem 2.2 we have

$$L_{P'}(x_1, x_2, \ldots, x_N) = \sum_{i=1}^{m} \sigma_{C_i}(q) \prod_{k=1}^{N} (1 + q_k x_k - x_k)^{v_k + v_k}$$

$$= L_P(x_1, x_2, \ldots, x_N) \prod_{k=1}^{N} (1 + q_k x_k - x_k)^{v_k}. \hfill \square$$

**Lemma 2.4.** The sum of each integer-point transform of $C_i$ is equals to one, namely,

$$\sum_{i=1}^{m} \sigma_{C_i}(q) = 1.$$ 

**Proof.** Let $Q$ be the image of $P$ by a dilation and translation and $\varepsilon_1, \ldots, \varepsilon_m \in \mathbb{R}^N$ its vertices, that is

$$Q := rP + \alpha = \text{conv} (\{\varepsilon_1, \ldots, \varepsilon_m\}), \ r, \alpha \in \mathbb{R}^N.$$ 

Then we can choose $\varepsilon_1, \ldots, \varepsilon_m$ in such a way that:

- the origin of $\mathbb{R}^N$ is a unique integer point in $Q$,
  $$Q \cap \mathbb{Z}^N = \{0\};$$
  - the integer points contained in each vertex cone of $Q$ are precisely the integer points contained in the corresponding vertex cone $C_i$,
  $$K_{\varepsilon_i} \cap \mathbb{Z}^N = C_i \cap \mathbb{Z}^N, \ \text{for all } i = 1, \ldots, m.$$

By the two conditions and Lemma 2.1 we obtain

$$1 = \sigma_Q(q) = \sum_{v \text{ a vertex of } Q} \sigma_{K_v}(q) = \sum_{i=1}^{m} \sigma_{K_{\varepsilon_i}}(q) = \sum_{i=1}^{m} \sigma_{C_i}(q). \hfill \square$$

**Corollary 2.5.** Let $P \subset \mathbb{R}^N$ be an integral polytope. Then the constant part of the multibasic Ehrhart polynomial $L_P(x_1, x_2, \ldots, x_N)$ is equals to one.
Proof. By substituting \( x_i = 0 \) to \( L_P(x_1, x_2, \ldots, x_N) \), we have

\[
L_P(0, 0, \ldots, 0) = \sum_{i=1}^{m} \sigma_{C_i}(q).
\]

By Lemma 2.4, we obtain the conclusion. □

Next, we give examples of the multibasic Ehrhart polynomials by Lemma 2.4. We can compute \( L_P(x_1, x_2, \ldots, x_N) \) for the classes of polytopes.

**Example 2.6.** We fix the sets as follow:

1. The \( d \)-simplex of \( \mathbb{R}^{d+1} \) is given by \( \Delta := \text{conv}(\{e_1, e_2, \ldots, e_{d+1}\}) \),

2. The \( d \)-simplex of \( \mathbb{R}^d \) is given by \( \Delta' := \text{conv}(\{0, e_1, e_2, \ldots, e_d\}) \),

3. The unit \( d \)-cube is given by \( \Box := \text{conv}(\{(x_1, x_2, \ldots, x_d) \mid x_i = 0 \text{ or } 1, 1 \leq i \leq d\}) \).

Then each multibasic Ehrhart polynomial is given by:

1. \( L_{\Delta}(x_1, x_2, \ldots, x_{d+1}) = \sum_{i=1}^{d+1} q_i^d (q_i - 1) \prod_{1 \leq j \leq d+1, j \neq i} (q_i - q_j) x_i + 1 \),

2. \( L_{\Delta'}(x_1, x_2, \ldots, x_d) = \sum_{i=1}^{d} q_i^d \prod_{1 \leq j \leq d, j \neq i} (q_i - q_j) x_i + 1 \),

3. \( L_{\Box}(x_1, x_2, \ldots, x_d) = \prod_{1 \leq i \leq d} (q_i x_i + 1) \).

Proof. We consider each case.

1. The \( d \)-simplex \( \Delta \subset \mathbb{R}^{d+1} \) case.

For any \( i \) (provided that \( 1 \leq i \leq d+1 \) ), the generator of each vertex cone \( K_{e_i} \) is given by \( \{e_j - e_i \mid 1 \leq j \leq d+1, j \neq i\} \). Then we have

\[
\Gamma_i := \left\{ \sum_{1 \leq j \leq d+1, j \neq i} r_j (e_j - e_i) \right\}_{0 \leq r_j < 1}
\]

\[
= \left\{ \left( r_1, r_2, \ldots, r_{i-1}, - \sum_{1 \leq j \leq d+1, j \neq i} r_j, r_{i+1}, \ldots, r_{d+1} \right) \right\}_{0 \leq r_j < 1}.
\]

Namely, we have \( \Gamma_i \cap \mathbb{Z}^{d+1} = \{0\} \) for any \( i \). By Lemma 1.1,

\[
\sigma_{C_i}(q) = \sigma_{(-e_i) + K_{e_i}}(q) = \frac{1}{\prod_{1 \leq j \leq d+1, j \neq i} (1 - q_i^{-1} q_j)}.
\]
Combining Lemma 2.4 and Theorem 2.2, we obtain

\[
L_{\Delta}(x_1, x_2, \ldots, x_{d+1}) = \sum_{i=1}^{d+1} \sigma_{C_i}(q)(1 + q_i x_i)
\]

\[
= \sum_{i=1}^{d+1} \sigma_{C_i}(q)(q_i - 1)x_i + \sum_{i=1}^{d+1} \sigma_{C_i}(q)
\]

\[
= \sum_{i=1}^{d+1} \frac{q_i - 1}{\prod_{\substack{1 \leq j \leq d+1 \backslash j \neq i}} (1 - q_i^{-1} q_j)} x_i + 1
\]

\[
= \sum_{i=1}^{d+1} \frac{q_i^d(q_i - 1)}{\prod_{\substack{1 \leq j \leq d+1 \backslash j \neq i}} (q_i - q_j)} x_i + 1.
\]

(2) The \(d\)-simplex \(\Delta' \subset \mathbb{R}^d\) case.

We remark that the generator of the vertex cone \(K_0\) is \(\{e_1, e_2, \ldots, e_d\}\) and the generator of each vertex cone \(K_{e_i}\) is given by \(\{0 - e_i\} \cup \{e_j - e_i \mid 1 \leq j \leq d+1, j \neq i\}\). Then we have

\[
\Gamma_0 := \left\{ \sum_{j=1}^{d} r_j e_j \left| 0 \leq r_j < 1 \right. \right\} = \left\{ (r_1, r_2, \ldots, r_d) \mid 0 \leq r_j < 1 \right\},
\]

\[
\Gamma_i := \left\{ r_0(0 - e_i) + \sum_{\substack{1 \leq j \leq d \backslash j \neq i}} r_j (e_j - e_i) \left| 0 \leq r_0, r_j < 1 \right. \right\}
\]

\[
= \left\{ (r_1, r_2, \ldots, r_{i-1}, -\sum_{\substack{0 \leq j \leq d \backslash j \neq i}} r_j, r_{i+1}, \ldots, r_d) \mid 0 \leq r_0, r_j < 1 \right\}.
\]

Therefore, we have \(\Gamma_0 \cap \mathbb{Z}^d = \Gamma_i \cap \mathbb{Z}^d = \{0\}\), provided that \(1 \leq i \leq d\). By Lemma 1.1, we obtain

\[
\sigma_{C_0}(q) = \sigma_{K_0}(q) = \frac{1}{\prod_{1 \leq j \leq d} (1 - q_j)},
\]

\[
\sigma_{C_i}(q) = \sigma_{(-e_i) + K_{e_i}}(q) = \frac{1}{(1 - q_i^{-1}) \prod_{\substack{1 \leq j \leq d \backslash j \neq i}} (1 - q_i^{-1} q_j)}.
\]
By Lemma 2.4 and Theorem 2.2, we have

\[ L_\Delta(x_1, x_2, \ldots, x_d) = \sigma_{C_0}(q) + \sum_{i=1}^{d} \sigma_{C_i}(q)(1 + q_i x_i - x_i) \]

\[ = \sum_{i=1}^{d} \sigma_{C_i}(q)(q_i - 1)x_i + \sum_{i=0}^{d} \sigma_{C_i}(q) \]

\[ = \sum_{i=1}^{d} \frac{q_i - 1}{(1 - q_i^{-1}) \prod_{1 \leq j \leq d, j \neq i} (1 - q_i^{-1} q_j)} x_i + 1 \]

\[ = \sum_{i=1}^{d} \prod_{1 \leq j \leq d, j \neq i} (q_i - q_j) x_i + 1. \]

(3) The unit \( d \)-cube \( \square \subset \mathbb{R}^d \) case.

If we consider the special case \( x_i = [n]_{q_i} \), we obtain

\[ L_\square([n]_{q_1}, [n]_{q_2}, \ldots, [n]_{q_d}) = \prod_{1 \leq i \leq d} (1 + q_1 + q_1^2 + \cdots + q_1^n) \]

\[ = \sum_{a \in n \square \cap \mathbb{Z}^d} q^a = \sigma_{n \square}(q). \]

\[ \square \]

### 3. Multibasic Ehrhart reciprocity

Let \( \mathcal{P}^\circ \) be the interior of \( \mathcal{P} \). We define the multibasic Ehrhart series for the interior of the polytope \( \mathcal{P} \subset \mathbb{R}^N \) of dimension \( d \) as follows:

\[ \text{Ehr}_{\mathcal{P}^\circ}(t; q) := \sum_{n=1}^\infty \sigma_{n \mathcal{P}^\circ}(q)t^n. \]

We consider a multibasic analogue of the Ehrhart reciprocity.

**Lemma 3.1** (Stanley’s reciprocity theorem, [9]). Let \( \mathcal{K} \subset \mathbb{R}^N \) be a rational \( d \)-cone with the origin as apex. Then

\[ \sigma_{\mathcal{K}} \left( \frac{1}{q_1}, \frac{1}{q_2}, \ldots, \frac{1}{q_N} \right) = (-1)^d \sigma_{\mathcal{K}^\circ}(q_1, q_2, \ldots, q_N). \]

We also prepare the following lemma to study the reciprocity.

**Lemma 3.2.** Let \( \mathcal{P} \subset \mathbb{R}^N \) be an integral polytope. For any \( n \in \mathbb{Z}_{>0} \), we have

\[ \sum_{n \leq 0} L_\mathcal{P}([n]_{q_1}, [n]_{q_2}, \ldots, [n]_{q_N}) t^n + \sum_{n \geq 1} L_\mathcal{P}([n]_{q_1}, [n]_{q_2}, \ldots, [n]_{q_N}) t^n = 0. \]
Proof. Let \( \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m \) be the vertices of \( \mathcal{P} \). We recall from Theorem 2.2 that there exists a relation
\[
L_{\mathcal{P}}([n]_{q_1}, [n]_{q_2}, \ldots, [n]_{q_N}) = \sum_{i=1}^{m} \sigma_{c_i}(q) \cdot q^{n_{v_i}}.
\]
Then we have
\[
\sum_{n \leq 0} L_{\mathcal{P}}([n]_{q_1}, [n]_{q_2}, \ldots, [n]_{q_N}) t^n + \sum_{n \geq 1} L_{\mathcal{P}}([n]_{q_1}, [n]_{q_2}, \ldots, [n]_{q_N}) t^n
\]
\[
= \sum_{n \leq 0} \left\{ \sum_{i=1}^{m} \sigma_{c_i}(q) \cdot q^{n_{v_i}} \right\} t^n + \sum_{n \geq 1} \left\{ \sum_{i=1}^{m} \sigma_{c_i}(q) \cdot q^{n_{v_i}} \right\} t^n
\]
\[
= \sum_{i=1}^{m} \sigma_{c_i}(q) \left\{ \sum_{n \geq 0} (q^{n_{v_i} t^{-1}})^n \right\} + \sum_{i=1}^{m} \sigma_{c_i}(q) \left\{ \sum_{n \geq 1} (q^{n_{v_i}})^n \right\} = 0.
\]
Therefore, we obtain the conclusion. \( \square \)

**Theorem 3.3.** Let \( \mathcal{P} \subset \mathbb{R}^N \) be an integral polytope of dimension \( d \). For any \( n \in \mathbb{Z}_{>0} \), we have
\[
L_{\mathcal{P}}([n]_{q_1}, [-n]_{q_2}, \ldots, [-n]_{q_N}) = (-1)^d \sigma_{n_{\mathcal{P}^c}} \left( \frac{1}{q} \right).
\]

Proof. By applying Lemma 3.1 to the cone over \( \mathcal{P} \),
\[
\sigma_{\text{cone}(\mathcal{P})} \left( \frac{1}{q_1}, \frac{1}{q_2}, \ldots, \frac{1}{q_N}, \frac{1}{q_{N+1}} \right) = (-1)^{d+1} \sigma_{(\text{cone}(\mathcal{P}))^c} (q_1, q_2, \ldots, q_N, q_{N+1}).
\]
We consider the special case \( q_{N+1} = t \). By Remark 1.5
\[
\text{Ehr}_{\mathcal{P}} \left( \frac{1}{t}, \frac{1}{q} \right) = (-1)^{d+1} \text{Ehr}_{\mathcal{P}^c}(t; q).
\]
Then we have
\[
1 + \sum_{n=1}^{\infty} \sigma_{n_{\mathcal{P}}} \left( \frac{1}{q} \right) \left( \frac{1}{t} \right)^n = (-1)^{d+1} \sum_{n=1}^{\infty} \sigma_{n_{\mathcal{P}^c}}(q) t^n.
\]
By Theorem 2.2 we have
\[
1 + \sum_{n=1}^{\infty} \sigma_{n_{\mathcal{P}}} \left( \frac{1}{q} \right) \left( \frac{1}{t} \right)^n = \sum_{n \leq 0} L_{\mathcal{P}}([-n]_{1/q_1}, [-n]_{1/q_2}, \ldots, [-n]_{1/q_N}) \cdot t^n.
\]
We also obtain
\[
\sum_{n \leq 0} L_{\mathcal{P}}([-n]_{1/q_1}, \ldots, [-n]_{1/q_N}) \cdot t^n = - \sum_{n \geq 1} L_{\mathcal{P}}([-n]_{1/q_1}, \ldots, [-n]_{1/q_N}) \cdot t^n.
\]
by Lemma 3.2 Therefore,
\[
\sum_{n \geq 1} L_{\mathcal{P}}([-n]_{1/q_1}, [-n]_{1/q_2}, \ldots, [-n]_{1/q_N}) \cdot t^n = (-1)^d \sum_{n \geq 1} \sigma_{n_{\mathcal{P}^c}}(q) t^n.
\]
Finally, we acquire the conclusion. \( \square \)
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