SUTURED MANIFOLDS AND $\ell^2$-BETTI NUMBERS

GERRIT HERRMANN

Abstract. Using the virtual fibering theorem of Agol we show that a sutured 3-manifold $(M, R_+, R_-, \gamma)$ is taut if and only if the $\ell^2$-Betti numbers of the pair $(M, R_-)$ are zero. As an application we can characterize Thurston norm minimizing surfaces in a 3-manifold $N$ with empty or toroidal boundary by the vanishing of certain $\ell^2$-Betti numbers.

1. Introduction

A sutured manifold $(M, R_+, R_-, \gamma)$ is a compact, oriented 3-manifold $M$ together with a set of disjoint oriented annuli and tori $\gamma$ on $\partial M$ which decomposes $\partial M \setminus \gamma$ into two subsurfaces $R_-$ and $R_+$. We refer to Section 2 for a precise definition. We say that a sutured manifold is balanced if $\chi(R_+) = \chi(R_-)$. Balanced sutured manifolds arise in many different contexts. For example 3-manifolds cut along non-separating surfaces can give rise to balanced sutured manifolds.

Given a surface $S$ with connected components $S_1 \cup \ldots \cup S_k$ we define its complexity to be $\chi_-(S) = \sum_{i=1}^k \max \{ -\chi(S_i), 0 \}$. A sutured manifold is called taut if $R_+$ and $R_-$ have the minimal complexity among all surfaces representing $[R_-] = [R_+] \in H_2(M, \gamma; \mathbb{Z})$.

Theorem 1.1 (Main theorem). Let $(M, R_+, R_-, \gamma)$ be a connected irreducible balanced sutured manifold. Assume that each component of $\gamma$ and $R_\pm$ is incompressible and $\pi_1(M)$ is infinite, then the following are equivalent

1. the manifold $(M, R_+, R_-, \gamma)$ is taut,
2. the $\ell^2$-Betti numbers of $(M, R_-)$ are all zero i.e. $H_2^{(2)}(M, R_-) = 0$,
3. the map $H_1^{(2)}(R_-) \to H_1^{(2)}(M)$ is a weak isomorphism.

Remark 1.2. The same statement holds true if one replaces $R_-$ with $R_+$.

By Gabai’s theory of sutured manifold decompositions we obtain the following result about Thurston norm minimizing surfaces (see Section 2 for a definition) in an irreducible 3-manifold $N$ with empty or toroidal boundary.

Theorem 1.3. Let $N$ be a connected orientable irreducible compact 3-manifold with empty or toroidal boundary and $\Sigma \hookrightarrow N$ a properly embedded decomposition surface, then the following are equivalent

1. $\Sigma$ is Thurston norm minimizing in the sense of Section 2,
2. the $\ell^2$-Betti numbers of the pair $(N \setminus \nu(\Sigma), \Sigma_-)$ are zero, where $\nu(\Sigma) \cong \Sigma \times (-1,1)$ is the interior of a tubular neighborhood $\Sigma \times [-1,1]$ and $\Sigma_-$ is given by $\Sigma \times \{-1\}$.

Remark 1.4. If one removes the assumption of $\Sigma$ being a decomposition surface, then (1) still implies (2).
As an application of Theorem 1.3 we have the following theorem first proven by Friedl and Lück with different methods [FL18].

**Theorem 1.5.** Let $N$ be a connected compact oriented irreducible 3-manifold with empty or toroidal boundary and let $\phi \in H^1(N; \mathbb{Z})$ be a primitive cohomology class. We write $N_{\ker \phi} \to N$ for the cyclic covering corresponding to $\ker \phi$. We have

$$b_1^{(2)}(N_{\ker \phi}) = x_N(\phi).$$

Here $x_N$ denotes the Thurston norm on $H^1(N; \mathbb{Z})$ (see Section 2 for a definition).

Another application of Theorem 1.1 is that for a taut sutured manifold the $\ell^2$-torsion is well defined.

**Corollary 1.6.** Let $(M, R_+, R_-, \gamma)$ be a taut sutured manifold, then the pair $(M, R_-)$ is $\ell^2$-det-acyclic and hence the $\ell^2$-torsion $\rho^{(2)}(M, R_-)$ is well defined.

We refer to [Lü02, Definition 3.91] for the definitions of $\ell^2$-det-acyclic and $\ell^2$-torsion.

**Proof of Corollary 1.6.** This follows from Theorem 1.1, Lemma 2.16, and [Sch01, Theorem 1.11].

If $M$ is a complete hyperbolic 3-manifold of finite volume, then by a result of Lück and Schick [LS99, Theorem 0.7] the $\ell^2$-torsion of $M$ is defined and one has

$$\rho^{(2)}(M) = \frac{-1}{6\pi} \cdot \text{Vol}_{3h}(M).$$

This result together with our corollary raises the following question.

**Question 1.7.** What topological and geometrical information of a taut sutured manifold $(M, R_+, R_-, \gamma)$ are contained in $\rho^{(2)}(M, R_-)$?

The author will pursue this question in a future paper with Ben-Aribi and Friedl. Theorem 1.1 can be seen as an $\ell^2$-analogue of the following theorem by Friedl and T. Kim.

**Theorem 1.8.** [FK13, Theorem 1.1] Let $(M, \gamma)$ be a connected irreducible balanced sutured manifold with $M \neq S^1 \times D^2$ and $M \neq D^3$. Then $(M, \gamma)$ is taut if and only if $H^*_a(M, R_-; \mathbb{C}^k) = 0$ for some unitary representation $\alpha: \pi_1(M) \to U(k)$.

**Outline of the content.** The paper is organised as follows. In Section 2 we review the basic definitions and introduce our notation. In Section 3 we discuss some basic properties of $\ell^2$-Betti numbers and sutured manifolds before we prove the main result in Section 4. In Section 5 we show how Theorem 1.5 follows from Theorem 1.3.

**Acknowledgements.** The author gratefully acknowledges the support provided by the SFB 1085 Higher Invariants at the University of Regensburg, funded by the DFG. Moreover, I would like to thank my advisor Stefan Friedl for his many suggestions and his helpful advice.
2. Definitions

2.1. Norm minimizing surfaces and quasi-fibers. Let $M$ be an oriented irreducible 3-manifold and $A \subset \partial M$ a subsurface. The Thurston norm is defined by

$$x_M: H_2(M, A; \mathbb{Z}) \longrightarrow \mathbb{Z}$$

$$\sigma \longmapsto \min \left\{ \chi_-(S) \mid [S] = \sigma \text{ and } S \text{ is properly embedded i.e. } \partial S = S \cap A \right\}.$$ 

This map extends to a semi norm on $H_2(M, A; \mathbb{R})$ by a result of Thurston [Th86].

Recall that an embedded surface $S \hookrightarrow M$ is called incompressible if for all choices of base points the inclusion map induces a monomorphism on the fundamental groups. We call a properly embedded surface $S$ in $(M, A)$ Thurston norm minimizing if $S$ is incompressible, $x_M([S, \partial S]) = \chi_-(S)$ and $S$ has no disk or sphere components. The condition for a Thurston norm minimizing surface to be incompressible is not very restrictive. For example if the surface doesn’t contain a torus or annulus component, then it is automatically satisfied ([AFW15, Chapter 3 C.22]).

Example 2.1. Thurston has shown that if $F \subset M$ with $\chi(F) \leq 0$ is a fiber of a fibration $M \rightarrow S^1$, then $F$ is Thurston norm minimizing in $(M, \partial M)$.

We will also make use of the following theorem due to Gabai.

Theorem 2.2. [Ga83, Corollary 6.13] Let $p: N \rightarrow M$ be a finite cover of a connected compact orientable 3-manifold $M$ and $S$ a Thurston norm minimizing surface in $(M, \partial M)$. Then $p^{-1}(S)$ is a Thurston norm minimizing surface in $(N, \partial N)$.

Definition 2.3. We call a properly embedded surface $S$ in $(M, \partial M)$ a quasi-fiber if the following two conditions are satisfied.

1. The surface $S$ is Thurston norm minimizing,
2. and there exists a fibration over the circle with fiber $F$ such that

$$x_M([F]) + x_M([S]) = x_M([F] + [S]).$$ 

Remark 2.4. A surface is a quasi-fiber if and only if it is Thurston norm minimizing and the corresponding class lies in the boundary of a fiber cone in $H_2(M, \partial M; \mathbb{R})$ [Th86].

2.2. Sutured manifolds. If $M$ is an oriented manifold then we endow $\partial M$ with the orientation coming from the outwards-pointing normal vectors. A sutured manifold $(M, R_+, R_-, \gamma)$ is a compact oriented 3-manifold with a decomposition of its boundary

$$\partial M = R_+ \cup \gamma \cup -R_-,$$

into oriented submanifolds such that

1. $\gamma$ is a collection of disjoint embedded annuli or tori,
2. $R_+ \cap R_- = \emptyset$,
Lemma 2.6. Let \( A \) be an annulus component of \( \gamma \), then \( R_- \cap A \) is a boundary component of \( A \) and of \( R_- \), and similarly for \( R_+ \cap A \). Furthermore, \([R_+ \cap A] = [R_- \cap A] \in H_1(A; \mathbb{Z})\) where we endow \( R_+ \cap A \) with the orientation coming from the boundary of the oriented manifold \( R_+ \).

We call a sutured manifold \( M \) taut, if \( M \) is irreducible and \( R_+ \) and \( R_- \) are Thurston norm minimizing viewed as properly embedded surfaces in \((M, \gamma)\). We call a sutured manifold balanced if \( \chi_- (R_+) = \chi_- (R_-) \).

Remark 2.5. In our definition of a taut sutured manifold we demand \( R_\pm \) to be incompressible, which differs from the convention most other authors choose. Our convention just rules out notorious counterexamples in the case that \( M = S^1 \times D^2 \).

An example of a balanced taut sutured manifold is given by the product sutured manifold
\[
(R \times [-1, 1], R \times 1, R \times -1, \partial R \times [-1, 1]),
\]
where \( R \) is a surface with \( R \not\approx S^2 \). Another example is given by an irreducible 3-manifold \( N \) with empty or toroidal boundary where the sutured manifold structure is given by \( \gamma = \partial N \).

2.3. Sutured manifold decomposition. In [Ga83] Gabai introduced the notation of a sutured manifold decomposition which we now recall. Let \((S, \partial S)\) be a properly embedded oriented surface in a sutured manifold \((M, R_+, R_-)\). We call \( S \) a decomposition surface if \( S \) is transversal to \( R_\pm \) and for every connected component \( c \in S \cap \gamma \) one of the following holds.

1. \( c \) is a properly embedded non-separating arc,
2. \( c \) is a simple closed curve in an annulus component of \( \gamma \) which is homologous to \([R_- \cap A] \in H_1(A; \mathbb{Z})\),
3. \( c \) is a homotopically non-trivial curve in a torus component \( T \) of \( \gamma \) and if \( c' \) is another curve in \( S \cap T \), then \( c \) and \( c' \) are homologous in \( T \).

Given a decomposition surface \( S \) we define the sutured decomposition along \( S \) by
\[
(M, R_-, R_+, \gamma) \overset{S}{\sim} (M', R'_-, R'_+, \gamma')
\]
where
\[
M' = M \setminus S \times (-1, 1),
\]
\[
\gamma' = (\gamma \cap M') \cup \nu(S'_+ \cap R_-) \cup \overline{\nu(S'_- \cap R_+)},
\]
\[
R'_+ = ((R_+ \cap M') \cup S'_+) \setminus \text{int } \gamma',
\]
\[
R'_- = ((R_- \cap M') \cup S'_-) \setminus \text{int } \gamma'.
\]

Here \( S'_+ \) (resp. \( S'_- \)) is the outward-pointing (resp. inward-pointing) part of \( S \times \{-1, 1\} \cap M' \) (See Figure 1). In the rest of the paper we suppress \( R_\pm \) from the notation and abbreviate \((M, R_+, R_-, \gamma)\) to \((M, \gamma)\) assuming that \( R_\pm \) is clear from the context. We make use of the following elementary lemma rather frequently.

Lemma 2.6. Let \((N, \emptyset, \emptyset, \partial N)\) be a sutured manifold (i.e. \( N \) has empty or toroidal boundary) and \( S \) a Thurston norm minimizing decomposition surface in \( N \), then \( N' \) defined by \( N \overset{S}{\sim} N' \) is a taut sutured manifold.
We also need the following lemma from the theory of sutured manifold decomposition due to Gabai.

**Lemma 2.7.** Let $N$ be connected irreducible oriented closed 3-manifold with empty or torodial boundary. Let $S$ and $F$ be Thurston norm minimizing decomposition surfaces with $\chi_N([S]) + \chi_M([F]) = \chi_N([F] + [S])$. We assume that $S$ and $F$ are in general position such that the number of components of $S \cap F$ is minimal. Denote by $N'$ the sutured manifold obtained by $N \xrightarrow{S} N'$, then $F' := F \cap N'$ is a decomposition surface for $N'$. Moreover, $N'$ and $N''$ are taut sutured manifolds, where $N''$ is given by $N' \xrightarrow{F'} N''$. One also has a commutative diagram of taut sutured manifold decompositions

\[
\begin{array}{ccc}
N & \xrightarrow{F} & N' \\
\downarrow{F \oplus S} & & \downarrow{F'} \\
N \setminus \nu(F) & & N' \\
\downarrow{S} & & \downarrow{C} \\
N' & & N''
\end{array}
\]

where $F \oplus S$ is the oriented cut and paste sum (see Figure 3), $S' = S \cap (N \setminus \nu(F))$, and $C = C_1, \ldots, C_n$ is a disjoint union of annuli and disks.

The main idea in the proof of that lemma is the following. The assumptions on $S$ and $F$ ensure that $F \oplus S$ is Thurston norm minimizing. Note that $M$ defined by $N \xrightarrow{F \oplus S} M$ is a taut sutured manifold, because $F \oplus S$ is Thurston norm minimizing. The same is true for $N'$. Now one can obtain $N''$ from $M$ by a decomposition surface only consisting of annuli or disks (see for example [FK14, Lemma 3.4]). Then $N''$ is taut by [Ga83, Lemma 3.12]. We refer to [Ga83, Section 3] for more details.
Later we will need the double $DM(\gamma)$ of sutured manifold $(M, \gamma)$. It is defined by

$$DM(\gamma) := M \sqcup_{R_\pm} M.$$  

This is a 3-manifold, which is closed or has toroidal boundary. It is also a sutured manifold with the sutured structure given by $(DM(\gamma), \emptyset, \emptyset, \partial DM(\gamma))$. Note that $R_+ \cup R_-$ is a decomposition surface in $DM(\gamma)$ and if one decomposes $DM(\gamma)$ along $R_+ \cup R_-$ one obtains two copies of $M$ as sutured manifolds. With this construction one can prove the following result which is analog to Theorem 2.2.

**Proposition 2.8.** Let $(M, R_+, R_-, \gamma)$ be a sutured manifold. Let $p: N \to M$ be a finite cover of $M$, then the preimage of $R_+ \cup R_-$ and $\gamma$ under $p$ induces a sutured structure on $N$. If in addition $M$ is taut and $\gamma$ is incompressible, then the induced structure on $N$ is taut as well.

**Proof.** The first assertion of the proposition is clear. Hence we only show that if $M$ is taut and $\gamma$ is incompressible, then $N$ is taut. Under these two assumptions one has by a result of Gabai ([Ga83, Lemma 3.7]) that $R_- \cup R_+$ is Thurston norm minimizing in $H_2(DM(\gamma), \partial DM(\gamma); \mathbb{Z})$. The proposition follows from Theorem 2.2 Lemma 2.6 and the fact that a finite cover of an irreducible 3-manifold is again irreducible [MSY82, Theorem 3].

2.4. **Incompressible decomposition surfaces in a product sutured manifold.** We quickly recall a result of Waldhausen [Wa68], which will be essential in the proof of Lemma 3.6. Let $F$ be a connected surface possibly with boundary and $F \not= S^2$, then we endow $F \times [-1, 1]$ with the product sutured manifold structure $(F, F_+, F_-, \partial F \times I)$, where $F_+ = F \times \{1\}$ and $F_- = F \times \{-1\}$. We denote by $p: F \times [-1, 1] \to F$ the canonical projection. A properly embedded surface $S$ in $F \times [-1, 1]$ is called horizontal if $p|_S$ is a homeomorphism onto its image.

**Remark 2.9.** A horizontal surface $S$ is by definition homeomorphic to a subsurface of $F$, so that we can view it as an embedding $S \to F \times [-1, 1], x \mapsto (x, h(x))$. Therefore $S$ is isotopic to a subsurface of $F_-$ (resp. $F_+$) by pushing (resp. lifting) the interval factor.
Given a connected incompressible decomposition surface \( S \) in \( F \times [-1, 1] \) there are evidently two possibilities:

1. \( S \) intersects \( F_+ \) and \( F_- \),
2. \( S \cap F_+ = \emptyset \) or \( S \cap F_- = \emptyset \).

Waldhausen [Wa68, Proposition 3.1] showed that in the second case if \( S \cap F_+ = \emptyset \) (resp. \( S \cap F_- = \emptyset \)), then \( S \) is ambient isotopic to a horizontal surface via an ambient isotopy fixing \( F_\pm \).

Remark 2.10. The second case also includes the possibility that \( S \) intersects neither \( F_+ \) nor \( F_- \). In this situation \( S \) is ambient isotopic to \( F \times \{ t \} \) for \( t \in (-1, 1) \).

Later we need the following lemma, which easily follows from Waldhausen’s result.

Lemma 2.11. Let \( N \not\cong S^1 \times S^2 \) be a connected 3-manifold which fibers over \( S^1 \) with fiber \( F \). We fix an embedding of \( F \) and an identification \( N \setminus \nu(F) \cong F \times [-1, 1] \). Let \( \Sigma \) be an incompressible not necessarily connected surface in \( N \) such that \( F \cap \Sigma \) has minimal number of connected components compared to all other embeddings in the isotopy class of \( \Sigma \). Then one of the following holds:

1. \( \Sigma \) consists of parallel copies of surfaces, all isotopic to \( F \),
2. every component of \( \Sigma' := \Sigma \cap N \setminus \nu(F) \) intersects \( F_+ \) and \( F_- \).

Proof. A standard argument using the irreducibility of \( N \) and our hypothesis on \( F \cap \Sigma \) shows, that \( F \cap \Sigma \) is incompressible and hence \( \Sigma' \) is incompressible in \( F \times [-1, 1] \). If \( \Sigma' \) does not intersect \( F_+ \cup F_- \), then by Remark 2.10 every component of \( \Sigma' \) is ambient isotopic to \( F \times \{ t \} \) for some \( t \in (-1, 1) \). Therefore \( \Sigma \) consist of parallel copies of \( F \). Now let \( C \) be connected component of \( \Sigma' \), which intersects \( F_- \) at least once but does not intersect \( F_+ \). Then there is an ambient isotopy fixing \( F_+ \), which makes \( C \) into a horizontal surface. Since this isotopy fixes \( F_\pm \), this isotopy extends to an isotopy of \( \Sigma \) in \( N \). If we further assume that \( C \) is an innermost among such a connected component, then one can use the isotopy from Remark 2.9 to remove the intersection component which corresponds to \( C \cap F_- \). But this contradicts our assumptions on \( F \cap \Sigma \) and hence \( C \) has to intersect \( F_+ \). The same argument with the roles of \( F_+ \) and \( F_- \) interchanged proves the lemma.

2.5. \( \ell^2 \)-Betti numbers. Here we introduce \( \ell^2 \)-Betti numbers and discuss some of their basic properties. For more details on \( \ell^2 \)-Betti numbers and proofs of the facts stated here we refer to [Li02, Chapter 1].

Let \( G \) be a group. We endow \( \mathbb{C}[G] \) with the pre-Hilbert space structure for which \( G \) is an orthonormal basis and denote by \( \ell^2(G) \) the closure of \( \mathbb{C}[G] \) with respect to the norm induced from the scalar product. Multiplication with elements in \( G \) induces an isometric left \( G \)-action on \( \ell^2(G) \). We define the group von Neumann algebra \( \mathcal{N}(G) \) to be the set of all bounded linear operators from \( \ell^2(G) \) to itself which commute with this left action.
**Definition 2.12.** Let \( \hat{X} \) be a \( CW \)-complex on which a group \( G \) acts co-compactly, freely and cellularly. Then \( C^\ell_2 (\hat{X}; \mathcal{N}(G)) \) is a finite free \( \mathbb{Z}[G] \)-module and we can consider the chain complex

\[
C^\ell_2 (\hat{X}; \mathcal{N}(G)) := \ell^2 (G) \otimes_{\mathbb{Z}[G]} C^\ell_2 (\hat{X})
\]

with the \( \ell^2 \)-homology defined by

\[
H^\ell_i (\hat{X}; \mathcal{N}(G)) := \ker (\text{Id} \otimes \partial_i) / \overline{\text{im} (\text{Id} \otimes \partial_{i+1})},
\]

where \( \overline{\text{im} (\text{Id} \otimes \partial_{i-1})} \) denotes the closure of \( \text{im} (\text{Id} \otimes \partial_{i-1}) \) with respect to the Hilbert space structure on \( \ell^2 (G) \). The \( \ell^2 \)-Betti numbers are then given by

\[
b^\ell_i (\hat{X}; \mathcal{N}(G)) = \dim_{\mathcal{N}(G)} H^\ell_i (\hat{X}; \mathcal{N}(G)),
\]

where \( \dim_{\mathcal{N}(G)} \) denotes the von Neumann dimension \cite[Definition 1.10]{[Lü02]}. Moreover, we define the \( \ell^2 \)-Euler characteristic by

\[
\chi^{\ell} (\hat{X}; \mathcal{N}(G)) := \sum_{i \in \mathbb{N}_0} (-1)^i \cdot b_i^{\ell} (\hat{X}; \mathcal{N}(G)).
\]

**Remark 2.13.** These numbers only depend on the homotopy type of \( \hat{X}/G \) and are especially independent of the choice of \( CW \)-structure.

The von Neumann dimension behaves very similarly to the ordinary dimension. For example it is additive under weakly short exact sequences. With this property one easily shows that \( \ell^2 \)-Euler characteristic and Euler characteristic are related by

\[
\chi^{\ell} (\hat{X}; \mathcal{N}(G)) = \chi (\hat{X}/G).
\]  

One of the main features of \( \ell^2 \)-Betti numbers is given by the next proposition.

**Proposition 2.14.** Let \( \hat{X} \) be a \( CW \)-complex on which a group \( G \) acts co-compactly, freely and cellularly. Let \( H \triangleleft G \) be a finite index subgroup. Then

\[
b^\ell_i (\hat{X}; \mathcal{N}(H)) = [G : H] \cdot b^\ell_i (\hat{X}; \mathcal{N}(G)).
\]

In this paper we deal with two special cases of groups acting on \( CW \)-complexes which therefore obtain a shortened notation. If \( X \) is a finite connected \( CW \)-complex with fundamental group \( \pi \), we denote by the corresponding universal cover \( \tilde{X} \to X \) and then we define the \( \ell^2 \)-homology of \( X \) by:

\[
H^\ell_i (X) := H^\ell_i (\tilde{X}; \mathcal{N}(\pi)).
\]

We write \( b^\ell_i (X) = \dim_{\mathcal{N}(\pi)} H^\ell_i (X) \) and in the case that \( X = X_1 \sqcup \ldots \sqcup X_n \) has several connected components we set \( b^\ell_i (X) := \sum_{k=1}^n b^\ell_i (X_k) \).

Moreover, if \( Y \subset X \) is a subcomplex, then we set \( \tilde{Y} := p^{-1} (Y) \) and define

\[
H^\ell_i (Y \subset X) := H^\ell_i (\tilde{Y}; \mathcal{N}(\pi)), \quad b^\ell_i (Y \subset X) := \dim_{\mathcal{N}(\pi)} H^\ell_i (\tilde{Y}; \mathcal{N}(\pi)),
\]

\[
H^\ell_i (X, Y) := H^\ell_i (\tilde{X}, \tilde{Y}; \mathcal{N}(\pi)), \quad b^\ell_i (X, Y) := \dim_{\mathcal{N}(\pi)} H^\ell_i (\tilde{X}, \tilde{Y}; \mathcal{N}(\pi)).
\]

With this pullback of coefficients one has Mayer-Vietoris sequences and the long exact sequence associated to a pair. **Proposition 2.14** and Equation (1) holds equally in the relative case. Moreover, if the inclusion \( Y \to X \) induces a
monomorphism on the fundamental group for any choice of base-point, then one has by the induction principle [L" u02, Theorem 1.35(10)]:
\[ b_i^2(Y \subset X) = b_i^2(Y). \]
If \( \phi: \pi \to \mathbb{Z} \) is an epimorphism, we denote by \( \hat{X} \) the covering corresponding to \( \ker \phi \) and introduce the notation
\[ H_i^\phi(X) := H_i^\phi(\hat{X}; N(\mathbb{Z})). \]
Let \( \langle t \rangle \cong \mathbb{Z} \cong \pi/\ker \phi \) denote a generator, then by [L" u02, Lemma 1.34] one has
\[ \dim_{N(\mathbb{Z})} H_i^\phi(X) = \dim_{C(t)} H_i(X; C(t)^\phi), \]
where \( H_i(X; C(t)^\phi) \) denotes the homology of the chain complex \( C(t) \otimes_{\mathbb{Z}[t^\pm 1]} C_{CW}^* \hat{X} \).

2.6. **Approximation of \( \ell^2 \)-Betti numbers.** In this paragraph we recall the L"uck-Schick approximation result of \( \ell^2 \)-Betti numbers. In order to state the theorem and that it applies in our situation we need some preliminaries.

**Definition 2.15.** Let \( G \) be the smallest class of groups which contains the trivial group and is closed under the following processes:

1. If \( H < \pi \) is a normal subgroup such that \( H \in G \) and \( \pi/H \) is amenable then \( \pi \in G \).
2. If \( \pi \) is the direct limit of a directed system of groups \( \pi_i \in G \), then \( \pi \in G \).
3. If \( \pi \) is the inverse limit of a directed system of groups \( \pi_i \in G \), then \( \pi \in G \).
4. The class \( G \) is closed under taking subgroups.

The precise definition of an amenable group doesn’t play a role for this article. We only need that finite groups are amenable. This is sufficient to prove the following lemma.

**Lemma 2.16.** Every fundamental group of a compact 3-manifold lies in \( G \).

**Proof.** By fact \([1]\) every finite group lies in \( G \). Then by fact \([3]\) the profinite completion of a group lies in \( G \). Since residually finite groups are subgroups of their profinite completion we have by fact \([4]\) that all residually finite groups are in \( G \). So the lemma follows from the fact that all 3-manifold groups are residually finite [AFW15, Chapter 3 C.29]. \( \square \)

We are now able to state the approximation result of Schick which extended earlier results by L"uck [L" u94].

**Theorem 2.17.** [Sch01, Theorem 1.14] Let \( X \) be a CW-complex with a free cellular and co-compact action of a group \( G \) in the class \( G \). Let \( G = G_1 \supset G_2 \supset \ldots \) be a nested sequence of normal subgroups such that \( \cap_{i \in \mathbb{N}} G_i = \{e\} \). Denote by \( X_i = X/G_i \) and by \( \Gamma_i = G/G_i \) the corresponding quotients. Further assume that \( \Gamma_i \in G \) for all \( i \in \mathbb{N} \), then one has for all \( p \in \mathbb{Z} \)
\[ \lim_{i \to \infty} b_i^p(X_i; N(\Gamma_i)) = b_i^p(X; N(G)). \]
Note that the action of each \( \Gamma_i \) on \( X_i \) is co-compact, free and cellular.
3. Basics of $\ell^2$-Betti numbers of sutured manifolds

In this section we prove the equivalence of statements (2) and (3) in Theorem 1.1. Moreover, we prove a vanishing criteria for the $\ell^2$-homology of a cyclic covering of a sutured manifold.

As mentioned in the beginning we state every result only for the pair $(M, R_-)$, but by Poincaré Lefschetz duality (see theorem below) all results hold equally for the pair $(M, R_+)$.

**Theorem 3.1.** [Lück 02, Theorem 1.35(3)] Let $X$ be a compact and connected $n$-manifold together with a decomposition $\partial X = Y_1 \cup Y_2$, where $Y_1$ and $Y_2$ are compact $(n-1)$-dimensional submanifolds of $\partial X$ with $\partial Y_1 = \partial Y_2$. Then

$$b^{(2)}_{n-i}(X, Y_1) = b^{(2)}_i(X, Y_2).$$

**Remark 3.2.** Lück only discusses the case $Y_1 = \partial X$ but the same proof he gives works also in the above setting (compare [Br93, Chapter 5.9 Exercise 3]).

We also use this duality to obtain a general result about $\ell^2$-Betti numbers of balanced sutured manifolds.

**Proposition 3.3.** Let $M$ be a balanced sutured manifold with infinite fundamental group, then $b^{(2)}_1(M, R_-) = b^{(2)}_2(M, R_-)$, and $b^{(2)}_1(M, R_-) = 0$ implies for all $i \in \mathbb{N}$

$$b^{(2)}_i(M, R_-) = 0.$$

**Proof.** One has $b^{(2)}_0(M, R_\pm) = 0$, because $\pi_1(M)$ is infinite [Lück 02, Theorem 1.35(8)] and by Poincaré Lefschetz duality (Theorem 3.1) $b^{(2)}_3(M, R_\pm) = 0$, too. Since $M$ is balanced, we have

$$\chi(R_-) = \frac{\chi(R_+ \cup \gamma \cup R_-)}{2} = \frac{\chi(\partial M)}{2} = \chi(M)$$

and hence $\chi(M, R_-) = 0$. By the relation between $\ell^2$-Euler characteristic and Euler characteristic (see Equation (1)) we obtain

$$\chi^{(2)}(\tilde{M}, \tilde{R}_-; N(\pi_1(M))) = \chi(M, R_-) = 0$$

and thus $b^{(2)}_1(M, R_-) = b^{(2)}_2(M, R_-)$.

This gives us already the equivalence of (2) and (3) in the main result (Theorem 1.1):

**Corollary 3.4.** Let $(M, \gamma)$ be an irreducible balanced sutured manifold. Assume that $\gamma$ is incompressible and $\pi_1(M)$ is infinite. Then $b^{(2)}_*(M, R_-) = 0$ if and only if $H^{(2)}_1(R_- \subset M) \to H^{(2)}_1(M)$ is a monomorphism.

**Proof.** We look at the long exact sequence in $\ell^2$-homology of the pair $(M, R_-)$. Note that $H^{(2)}_2(M) = 0$, because $M$ is irreducible and $\pi_1(M)$ is infinite [Lück 02, Theorem 4.1]. Therefore the sequence becomes

$$0 \to H^{(2)}_1(M, R_-) \to H^{(2)}_1(R_- \subset M) \to H^{(2)}_1(M) \to \ldots .$$

Now the corollary follows from Proposition 3.3 and the fact that the von Neumann dimension is zero if and only if the module is zero [Lück 02, Theorem 1.12(1)].
We end this section with two technical lemmas, which are the cornerstone for the proof of Theorem 4.7. The first of these two lemmas is a vanishing result of certain $\ell^2$-Betti numbers and the second lemma gives a sufficient criteria two apply the first one.

**Lemma 3.5.** Let $(M, \gamma)$ be a connected sutured manifold and let $\phi \in H^1(M; \mathbb{Z})$ be non-trivial. If there is a decomposition surface $S$ such that the class $[S] \in H_2(M, \partial M; \mathbb{Z})$ is Poincaré dual to $\phi$ and $M \cong_{S} M'$ results in a product sutured manifold $M'$, then

$$H^{\phi,i}(M, R_{-}) = 0.$$ 

Before giving the proof it is worth discussing a simple case which motivates the proof. Namely, if $N$ is sutured manifold with empty or toroidal boundary where the sutured manifold structure is given by $\gamma = \partial N$. In this case one has $R_{-} = \emptyset$. Moreover, if $S$ is a decomposition surface $S$ such that the decomposition $N \cong_{S} N'$ results in a product sutured manifold, then $S$ is a fiber of a fibration of $N$ over $S^1$. The cover corresponding to $\ker(\phi)$ is homeomorphic to $S \times \mathbb{R}$ and hence $H_*(N; \mathbb{Z}(t)^{\phi}) = H_*(S \times \mathbb{R}; \mathbb{Z}) \otimes_{\mathbb{Z}[t^{\pm}]} \mathbb{C}(t) = 0$.

**Proof of Lemma 3.5** We have two canonical embeddings of $S$ into the boundary of $M'$ which we denote by $i_{\pm}: S \to M'$. Moreover, we denote by $S_{\pm}$ the images $i_{\pm}(S)$.

Since $\phi$ is the Poincaré dual of $S$, the cyclic cover $p: \hat{M} \to M$ corresponding to $\ker \phi$ can be described by

$$\hat{M} = (M' \times \mathbb{Z})/ \sim,$$

where $(S_{-}, i)$ is glued to $(S_{+}, i + 1)$ in the obvious way (see Figure 3). The deck transformation group acts on the $\mathbb{Z}$-factor. We denote by $t$ the generator of this action i.e. $t \cdot (x, i) = (x, i + 1)$ for all $x \in M'$.

We decompose $\hat{M}$ into the subsets $\{M' \times \{i\}\}_{i \in \mathbb{Z}}$ and we set $\partial_- S = S \cap R_-$. Given a CW-complex $X$ we abbreviate $C_{*}^{CW}(X; \mathbb{Z})$ and $H_*(X; \mathbb{Z})$ to $C_{*}(X)$ and $H_*(X)$ so that we can express the Mayer–Vietoris sequence in cellular homology with respect to the above decomposition in the following way:

$$0 \to C_*(S, \partial_- S) \otimes_{\mathbb{Z}} \mathbb{Z}[t^\pm] \xrightarrow{i_{-} - i_{+}} C_*(M', M' \cap R_-) \otimes_{\mathbb{Z}} \mathbb{Z}[t^\pm] \to C_*(\hat{M}, \hat{R}_-) \to 0.$$ 

This yields a long exact sequence in homology

$$\ldots \to H_i(S, \partial_- S) \otimes_{\mathbb{Z}} \mathbb{Z}[t^\pm] \xrightarrow{i_{-} - i_{+}} H_i(M', M' \cap R_-) \otimes_{\mathbb{Z}} \mathbb{Z}[t^\pm] \to H_i(\hat{M}, \hat{R}_-) \to \ldots$$

Next we show that the inclusion $i_{-}: (S, \partial_- S) \to (M', M' \cap R_-)$ induces an isomorphism on homology.

By assumption $M'$ is a product sutured manifold i.e. $M' = R'_{-} \times I$ with $R'_{-} = (M' \cap R_-) \cup S_{-}$. Therefore, we get by homotopy invariance

$$H_*(M', M' \cap R_-) = H_*(R'_{-} \times I, M' \cap R_-) \cong H_*(R'_{-}, M' \cap R_-)$$

and by excision

$$H_*(R'_{-}, M' \cap R_-) = H_*(S, \partial_- S).$$
We refer to Figure 4 for an illustration of this argument.

We now continue with the rest of the proof. Because $H_*(M', M' \cap R_-) \cong H_*(S, \partial_- S)$ is free abelian it makes sense to talk about the determinant. And since $i_-$ induces an isomorphism on homology we have $\det_{\mathbb{Z}}(i_-) \neq 0$. This of course implies that $\det_{\mathbb{Z}[t]}(i_- - t \cdot i_+) \neq 0$. Therefore the map $i_- - t \cdot i_+$ is invertible over $\mathbb{C}(t)$. Also note that we have $H_*(\overline{M}, \partial_- R_-; \mathbb{Z}) = H_*(M, R_-; \mathbb{Z}[t^\pm]^{\phi})$. 

**Figure 3.** This is a schematic picture one dimension reduced. It illustrates the cyclic cover $\hat{M}$ of $M$ corresponding to the kernel of $\phi$, where $\phi$ is Poincaré dual to an embedded surface $S$.

**Figure 4.** A schematic picture one dimension reduced. The sutured decomposition $M \overset{S}{\leadsto} M'$ results in a product sutured manifold. The dashed lines show the $[0, 1]$-factor of the product. By homotopy invariance and excision one has an isomorphism $H_*(S, \partial_- S) \overset{i_-}{\to} H_*(M', M' \cap R_-)$. 

We refer to Figure 4 for an illustration of this argument.
SUTURED MANIFOLDS AND $\ell^2$-BETTI NUMBERS

Figure 5. A schematic picture of how the loop $c$ in the proof of Proposition 4.6 is constructed. We see the product surface $F \times [-1, 1]$. The grey rectangle represents $\Sigma \cap F \times [-1, 1]$. The points $x_i$ and $x'_i$ are identified by the monodromy in $N$. The concatenation of the paths gives a loop in $N$.

Now we use the above sequence, the fact that $\mathbb{C}(t)$ is flat over $\mathbb{Z}[t^\pm]$ and that $i_- - t \cdot i_+$ is invertible over $\mathbb{C}(t)$ to obtain $H_\ast(M, R_-; \mathbb{C}(t)^\phi) = 0$. Then Equation (3) yields $H^{\phi, (2)}_\ast(M, R_-) = 0$. □

Lemma 3.6. Let $(N, \emptyset, \emptyset, \emptyset)$ be a taut sutured manifold and let $\Sigma$ be a quasi-fiber in $N$ which is not the fiber of a fibration. Then the sutured manifold $M$ obtained by $N \leadsto^\Sigma M$ contains a decomposition surface $S$ such that $M \leadsto^S M'$ results in a product sutured manifold $M'$ and the class $[S] \in H_2(M, \partial M; \mathbb{Z})$ is non-trivial.

Proof. Since $\Sigma$ is a quasi-fiber there is a fibration of $N$ over $S^1$ with fiber $F$ and $\chi_N([F]) + \chi_N([\Sigma]) = \chi_N([F] + [\Sigma])$. We can assume that $F$ and $\Sigma$ are in general position such that the number of components of $\Sigma \cap F$ is minimal compared to all surfaces isotopic to $\Sigma$ and $F$.

We define $F' := F \cap M = F \setminus \nu(F \cap \Sigma)$ and make the following claim.

Claim. The sutured decomposition $(M, \Sigma_+, \Sigma_-, \gamma) \leadsto^{F'} (M', R'_+, R'_-, \gamma')$ results in a product sutured manifold.

We set $\Sigma' = \Sigma \cap (N \setminus \nu(F))$. By Lemma 2.7 we have that $M'$ is a taut sutured manifold and by the commutativity of the diagram in Lemma 2.7 we have that $N \leadsto_F N \setminus \nu(F) \leadsto^{\Sigma'} M'$. Moreover, $N \setminus \nu(F) \cong F \times [-1, 1]$ is a product, since $F$ is a fiber of a fibration. The taut sutured manifold decomposition of a product sutured manifold is again a product sutured manifold ([Ga83, Remark 4.9(4)]) and hence $M'$ is a product.

It remains to show, that $[F'] \in H_2(M, \partial M)$ is non-trivial. This follows directly from the next claim.

Claim. There is a closed curve $c$ in $N$, which doesn’t intersect $\Sigma$ but has a positive intersection number with $F$. 

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The curve $c$ is constructed as follows (see Figure 5). We choose a point $x_0 \in F_+ \setminus \Sigma'$ and a path $p_0$ not intersecting $\Sigma'$ to $F_-$. Such a path always exists by Lemma 2.11 and the assumption that $\Sigma$ is not a fiber of a fibration. The monodromy sends the endpoint of this path to a new point $x_1'$ on $F_+$ maybe in a different connected component of $M'$. We repeat this process to obtain another path $p_1$ connecting $x_1'$ with another point $x_2 \in F_-$, which is sent to $x_2' \in F_+$ via the monodromy. Since there are only finitely many connected components of $M'$ we can after several iterations of this process join $x_n$ with $x_0$ in $F_+$ by a path $p$ not intersecting $\Sigma'$. All these paths patched together give a closed loop in $N$. This loop does not intersect $\Sigma$ but gives a positive intersection number with $F$.

Since $c$ does not intersect $\Sigma$ it is a loop in $M$ and since it has positive intersection number with $F$, the class $[F'] \in H_2(M, \partial M)$ is non-trivial. 

\section{Proof of the Main Theorem}

\subsection{$\ell^2$-acyclic implies taut}

The basic idea is that $\ell^2b^i(M, R_-)$ is an upper bound for how far a sutured manifold $(M, \gamma)$ is away from being taut.

\begin{lemma}[Half lives, half dies] Let $W$ be a compact connected $(2k + 1)$-dimensional manifold, then
\[
\dim_{N(\pi_1(W))} \ker \left( i_*: H^2_k(\partial W \subset W) \to H^2_k(W) \right) = \frac{1}{2} \cdot \ell^2k^2(\partial W \subset W).
\]
\end{lemma}

\begin{proof}

We write $G = \pi_1(W)$ and $i_*: H^2_k(\partial W) \to H^2_k(W)$. Because of the additivity of the von Neumann dimension we get
\[
\ell^2k^2(\partial W \subset W) = \dim_{N(G)} \ker(i_*) + \dim_{N(G)} \overline{\im(i_*)}.
\]
Therefore it is sufficient to prove that $\dim_{N(G)} \ker(i_*) = \dim_{N(G)} \overline{\im(i_*)}$. To show this one considers the long exact sequence in homology associated to the pair $(\partial W, W)$. This sequence can be decomposed into long exact sequences
\[
\cdots \to H^2_{k+1}(W, \partial W) \to \ker(i_*) \to 0 \to \cdots \\
\downarrow \quad \downarrow \quad \downarrow \\
\cdots \to H^2_k(W) \to H^2_k(\partial W \subset W) \to H^2_k(W) \to \cdots \\
\downarrow \quad \downarrow \\
\cdots \to 0 \to \overline{\im(i_*)} \to H^2_k(W) \to \cdots.
\]

One has $\ell^2k^2_{k+1+i}(W, \partial W) = \ell^2k^2_i(W)$ and $\ell^2k^2_k(\partial W \subset W) = \ell^2k^2_{k-1}(\partial W \subset W)$ by Poincaré duality. So we see that the Euler characteristic of the upper and the lower exact sequence contains the same summands except for $\dim_{N(G)} \ker(i_*)$ and $\dim_{N(G)} \overline{\im(i_*)}$. But both sequences have zero Euler characteristic, because they are exact. Hence $\dim_{N(G)} \ker(i_*) = \dim_{N(G)} \overline{\im(i_*)}$. \hfill \Box

\begin{lemma}

Suppose $(M, \gamma)$ is a connected irreducible sutured manifold with infinite fundamental group and $\gamma$ is incompressible. Then there exists a Thurston norm minimizing representative $S \subset M \setminus (R_+ \cup R_-)$ of $[R_-] \in H_2(M, \gamma; \mathbb{Z})$ such that $M$ cut along $S$ is the union of two disjoint (not necessarily connected) compact manifolds $M_\pm$ with $R_\pm \subset \partial M_\pm$.

\end{lemma}
Proof. Let $T$ be a properly embedded surface homologous to $[R_-]$, with $\chi_-(T) = x_M([R_-])$ and the intersection of $T$ with $R_{\pm}$ is empty. We will show that a subsurface of $T$ has the desired properties.

First we find a subsurface $S$ of $T$ which is Thurston norm minimizing. By the definition of Thurston norm minimizing we have to show that there is a subsurface $S \subset T$ which is homologous to $T$, which has the same complexity, does not have sphere or disk components, and is incompressible.

Since $M$ is irreducible every sphere component of $T$ is null homologous and we can omit it. Now let $D \subset T$ be a disk component. From the definition one has $\partial D \subset \gamma$. Therefore we have to consider two cases. The first case that $\partial D$ is homotopically trivial in $\gamma$. In this case we can close the circle with another disk in $\gamma$ and obtain a sphere. By the assumption $M$ is irreducible and therefore this sphere bounds a 3-ball. Hence the disk $D$ is homologically trivial in $H_2(M, \gamma; \mathbb{Z})$. Therefore $[T] = [T \setminus D]$ and we define $S := T \setminus D$. Note that $S$ has the same complexity as $T$. The other case that $\partial D$ is non trivial in $\gamma$ can not occur because $\gamma$ is incompressible. That $S$ is indeed incompressible is a consequence of the loop theorem and the fact that it has minimal complexity. See also [AFW15, Chapter 3 C.22] or [Ca07, Lemma 5.7].

It remains to show that $S$ separates the manifold $M$ into at least two disjoint parts. We have an intersection form:

$$H_2(M, \gamma; \mathbb{Z}) \times H_1(M, R_+ \cup R_-; \mathbb{Z}) \rightarrow \mathbb{Z}$$

Let $p$ be a path from $R_+$ to $R_-$, then the intersection number of $[p]$ and $[R_-]$ is equal to 1. So every surface homologous to $[R_-]$ has to intersect $p$ at least once and therefore separates $R_-$ and $R_+$. □

Lemma 4.3. With the notation of the previous lemma and the additional assumption that $R_-$ is incompressible one has

$$\frac{1}{2}(\chi(S) - \chi(R_+)) \leq b_2(M, R_-).$$

Proof. Applying Mayer-Vietoris on $U := R_- \cup S$ and $V = \gamma'$ for the boundary $\partial M_- = R_+ \cup \gamma' \cup S$ we get an weak isomorphism

$$H_1^2(\partial M_- \subset M) \cong H_1^2(R_- \subset M) \oplus H_1^2(S \subset M).$$

Here we used that $\gamma' \subset \gamma$ is incompressible in $M$. We set $G = \pi_1(M)$ and will consider all $\ell^2$-homology with the coefficient system coming from $M$. Therefore, we drop “$\subset M$” from the notation. From Lemma 4.1 applied to the boundary of $M_-$, we get

$$\frac{1}{2}(b_1^2(R_-) + b_1^2(S)) = \dim_{\mathcal{N}(G)} \ker \left( H_1^2(\partial M_-) \rightarrow H_1^2(M_-) \right).$$

By the standard inequality

$$\dim_{\mathcal{N}(G)} \ker(i: A \oplus B \rightarrow C) \leq \dim_{\mathcal{N}(G)} \ker(i: A \rightarrow C) + \dim_{\mathcal{N}(G)} B$$
applied to $H_1^{(2)}(\partial M_-) = H_1^{(2)}(R_-) \oplus H_1^{(2)}(S) \to H_1^{(2)}(M_-)$ we obtain further
\[
\frac{1}{2}(b_1^{(2)}(R_-) + b_1^{(2)}(S)) = \dim_{\mathbb{N}(G)} \ker \left( H_1^{(2)}(\partial M_-) \to H_1^{(2)}(M_-) \right)
\leq \dim_{\mathbb{N}(G)} \ker \left( H_1^{(2)}(R_-) \to H_1^{(2)}(M_-) \right) + b_1^{(2)}(S)
\leq \dim_{\mathbb{N}(G)} \ker \left( H_1^{(2)}(R_-) \to H_1^{(2)}(M) \right) + b_1^{(2)}(S)
\leq \dim_{\mathbb{N}(G)} \im \left( H_2^{(2)}(M, R_-) \to H_1^{(2)}(R_-) \right) + b_1^{(2)}(S)
\leq b_2^{(2)}(M, R_-) + b_1^{(2)}(S).
\]

Recall that we dropped “$\subset M$” from the notation but by assumption $R_-$ and $S$ are incompressible and hence $b_1^{(2)}(R_- \subset M) = b_1^{(2)}(R_-)$ and $b_1^{(2)}(S \subset M) = b_1^{(2)}(S)$. For surfaces with infinite fundamental group one has $-\chi(S) = b_1^{(2)}(S)$ and $-\chi(R_-) = b_1^{(2)}(R_-)$, which finishes the proof.

Now the direction $(2) \Rightarrow (1)$ of the main theorem follows easily.

**Corollary 4.4.** Let $(M, \gamma)$ be an irreducible connected balanced sutured manifold with infinite fundamental group and such that $\gamma$ and $R_-$ are incompressible. If $b_2^{(2)}(M, R_-) = 0$ then $M$ is taut.

**Proof.** Let $S$ be a surface obtained from Lemma 4.2. By construction of $S$ one has $-\chi(S) = x_M([R_-])$ and hence $\chi(S) - \chi(R_-) \geq 0$. By assumption we have $b_2^{(2)}(M, R_-) = 0$ and Lemma 4.3 now implies that
\[
0 \leq \chi(S) - \chi(R_-) = -x_M([R_-]) - \chi(R_-) \leq 0.
\]

Therefore we get $x_M([R_-]) = -\chi(R_-)$ and since $(M, \gamma)$ was balanced we obtain $x_M([R_-]) = -\chi(R_+)$, too. But this is the definition of a taut sutured manifold. \qed

### 4.2. Taut implies $\ell^2$-acyclic.

We are now going to show that if $(M, \gamma)$ is taut then $H_2^{(2)}(M, R_-)$ is zero. Since $\ell^2$-Betti numbers are multiplicative under finite covers (Proposition 2.14) and a finite cover of a taut sutured manifold is again taut (Proposition 2.8) the above statement is true if and only if it is true for a finite cover. The proof consists of three steps.

1. There exists a finite cover $\hat{M} \to M$ and a decomposition surface $S \subset \hat{M}$ such that $\hat{M} \xrightarrow{\phi} \hat{M}'$ is a product sutured manifold.

2. Denote by $\phi \in H_1(\hat{M}, \hat{M})$ the Poincaré dual of $S$, then $H_1^{\phi(2)}(\hat{M}, R_-) = 0$.

3. We use the second step and Schick’s approximation result to conclude $H_2^{(2)}(M, R_-) = 0$.

For the first step we need the virtual fibering theorem due to Agol. Notice that the fundamental group of an irreducible 3-manifold with non-empty boundary is virtually RFRS [AFW15, Corollary 4.8.7], which was proved by Przytycki and Wise [PW17]. We don’t need the precise definition of RFRS, we only need that the following theorem holds in our situation.

**Theorem 4.5.** [Ag08, Theorem 5.1] Let $M$ be an irreducible 3-manifold with empty or toroidal boundary. Assume $\pi_1(M)$ is infinite and virtually RFRS. If
If $\Sigma$ is a Thurston norm minimizing surface, then there is a finite sheeted cover $p: N \to M$ such that $p^{-1}(\Sigma)$ is a quasi-fiber.

The next proposition is implicit in the article of Agol [Ag08]. We will show how it follows from Theorem 4.5.

**Proposition 4.6.** Let $(M, \gamma)$ be a connected taut sutured manifold with infinite fundamental group and such that $\gamma$ is incompressible. Then there exists a connected finite cover $(\hat{M}, \hat{\gamma})$ and a decomposition surface $S$ in $\hat{M}$, such that $\hat{M} \xrightarrow{\hat{S}} \hat{M}'$ is a product sutured manifold and $[S] \in H_2(\hat{M}, \partial \hat{M})$ is non-trivial.

**Proof.** If $(M, \gamma)$ is a product sutured manifold then one can take the annulus obtained from a homologically non trivial curve in $R_- \times I$. Therefore we will in the following assume that $(M, \gamma)$ is a taut sutured manifold which is not a product. Recall the construction of the double $DM(\gamma)$ of a sutured manifold $M$:

$$DM(\gamma) := M \sqcup_{R_\pm} M.$$ 

This is a 3-manifold, which is closed or has toroidal boundary. By our assumptions $\gamma$ incompressible and $(M, \gamma)$ is taut. Hence $R_- \cup R_+$ is Thurston norm minimizing in $DM(\gamma)$ ([Ga83, Lemma 3.7]).

We can use the virtual fibering Theorem 4.5 to obtain a finite cover $p: W \to DM(\gamma)$ such that the surface $p^{-1}(R_- \cup R_+)$ is a quasi-fiber. We write $\Sigma = p^{-1}(R_- \cup R_+)$. The taut sutured manifold $W'$ given by $W \xrightarrow{\Sigma} W'$ is by construction a finite cover of $M$. Therefore by Lemma 3.6 applied to $\Sigma$ and $W$ we see that a connected component $\hat{M} \subset W'$ is the desired finite cover of $M$. \qed

Now we can finally prove the rest of our main result.

**Theorem 4.7.** If $M$ is a connected taut sutured manifold with infinite fundamental group and $\gamma$ is incompressible, then $b_*^{(2)}(M, R_-) = 0$.

**Proof.** Let $M$ be a taut sutured manifold. Recall that for any finite cover $\hat{M} \to M$ we have

$$[\hat{M} : M] \cdot b_*^{(2)}(M, R_-) = b_*^{(2)}(\hat{M}, \hat{R_-})$$

and if $M$ is taut then $\hat{M}$ is taut as well. Therefore it is sufficient to prove the theorem for a suitable finite cover. By this observation together with Proposition 4.6 we will assume that $(M, \gamma)$ admits a decomposition surface $S$ such that $M'$ defined by $M \xrightarrow{S} M'$ is a product sutured manifold and $[S] \in H_2(M, \partial M)$ is non-trivial. If we denote by $\phi \in H^1(M; \mathbb{Z})$ the Poincaré dual of $S$ then by Lemma 3.5 we have

$$H_1^{\phi, (2)}(M, R_-) = 0.$$ 

Since the fundamental group of $M$ is residually finite we obtain a nested sequence $\pi_1(M) = \pi \supset \pi_1 \supset \pi_2 \ldots$ of normal subgroups such that $\pi/\pi_i$ is finite and $\bigcap_{i \in I} \pi_i = \{e\}$. Denote by $p_i: M_i \to M$ the corresponding finite cover. Denote by $S_i := p_i^{-1}(S)$ the pre-image of the surface $S$ and by $\phi_i = p_i^*(\phi)$ the pull back of $\phi$. Obviously, $S_i$ is the Poincaré dual of $\phi_i$. Furthermore, $M_i \xrightarrow{S_i} M'_i$ results in a
product sutured manifold since $M'_i$ is a cover of $M'$ and $M'$ is a product sutured manifold. Hence we can apply Lemma 3.5 again to obtain

$$H^{\phi_i}(2)(M_i, R_{i-}) = 0.$$  

Denote by $G_i = \ker(\phi) \cap \pi_i$ the kernel of $\phi_i$. We see that $G_i$ is normal in $\pi_i$. Moreover, by the third isomorphism Theorem we have

$$\frac{(\pi/G_i)}{(\pi_i/G_i)} \cong \pi/\pi_i$$  

and hence $(\pi/G_i) \cap (\pi_i/G_i)$ is finite index. This yields

$$0 = \dim_{\mathbb{Z}} H^{\phi_i}(2)(M_i, R_i) = b^{(2)}_1(\tilde{M}/G_i, \tilde{R}_-/G_i; \mathcal{N}(\pi_i/G_i)) = [\pi : \pi_i] \cdot b^{(2)}_1(\tilde{M}/G_i, \tilde{R}_-/G_i; \mathcal{N}(\pi/G_i))$$

and in particular

$$b^{(2)}_1(\tilde{M}/G_i, \tilde{R}_-/G_i; \mathcal{N}(\pi/G_i)) = 0.$$  

The groups $\pi/G_i$ are by construction virtually cyclic and hence lie in the class $\mathcal{G}$. We can now apply Schick’s approximation Theorem 2.17 to the nested and cofinal sequence of normal subgroups $\pi_1(M) \supset G_1 \supset G_2 \ldots$ and obtain

$$b^{(2)}_1(M, R_-) = \lim_{i \to \infty} b^{(2)}_1(\tilde{M}/G_i, \tilde{R}_-/G_i; \mathcal{N}(\pi/G_i)) = 0.$$

□

5. Applications

In this section we prove Theorem 1.5

**Theorem 1.5.** Let $N$ be a connected compact oriented irreducible 3-manifold with empty or toroidal boundary and let $\phi \in H^1(N; \mathbb{Z})$ be a primitive cohomology class. We write $N_{\ker \phi} \to N$ for the cyclic covering corresponding to $\ker \phi$. We have

$$b^{(2)}_1(N_{\ker \phi}) = x_N(\phi).$$

Note that $N_{\ker \phi}$ is not a finite CW-complex. We refer to Chapter 6 of Lück’s monograph [Lü02] for a definition of $\ell^2$-betti numbers in this context. In the proof we are only using that the von Neumann dimension behaves well with respect to colimits of modules.

**Proof of Theorem 1.5.** Let $G = \ker \phi$ and $\Sigma$ be a Thurston norm minimizing decomposition surface Poincaré dual to $\phi$. We write $M = N \setminus \nu(\Sigma)$ and construct inductively

$$X_0 = M,$$

$$X_n = M \bigsqcup_{\Sigma_- = \Sigma_+} X_{n-1} \bigsqcup_{\Sigma_- = \Sigma_+} M.$$  

By abuse of notation we write $\Sigma$ for $\Sigma_-$ in $X_0$. Since $\phi$ is primitive we have $\lim_{n \in \mathbb{N}} X_n = N_{\ker \phi}$. We refer to Figure 6 for an illustration of the situation.

Also note that all inclusions $\Sigma \to X_1 \to X_n$ and $X_n \to N_{\phi}$ are $\pi_1$-injective and we can use the induction principle stated in Equation (2).
By the excision isomorphism we have
\[ b^{(2)}(X_n, X_{n-1}; \mathcal{N}(G)) = b^{(2)}((M, \Sigma_+) \cup (M, \Sigma_-); \mathcal{N}(G)) \]
and hence by the main theorem
\[ b^{(2)}(X_n, X_{n-1}; \mathcal{N}(G)) = b^{(2)}(X_n, X_{n-1}) = 0. \]
We can consider the triple \((X_n, X_{n-1}, \Sigma)\) and its associated long exact sequence in homology. It follows inductively that \(b^{(2)}(X_n, \Sigma; \mathcal{N}(G)) = 0\).

We also have the isomorphisms (see the PhD-thesis for more details [He19]):
\[ \lim_{n \to \infty} H^{(2)}_*(X_n, \Sigma; \mathcal{N}(G)) \cong H^{(2)}_*(\lim_{n \to \infty} X_n, \Sigma; \mathcal{N}(G)) = H^{(2)}_*(N_{\ker \phi}, \Sigma; \mathcal{N}(G)). \]
Then by cofinality of the von Neumann dimension ([Lü02, Theorem 6.13]) we have
\[ b^{(2)}_*(N_{\ker \phi}, \Sigma) = \lim_{n \to \infty} b^{(2)}_*(X_n, \Sigma; \mathcal{N}(G)) = 0. \]
We look at the long exact sequence in homology associated to the pair \((N_{\ker \phi}, \Sigma)\) and conclude from the additivity of the von Neumann dimension that
\[ b^{(2)}_1(N_{\ker \phi}) = b^{(2)}_1(\Sigma) = -\chi(\Sigma) = x_N(\phi). \]

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Fakultät für Mathematik, Universität Regensburg, Germany,

www.gerrit-herrmann.de

E-mail address: gerrit.herrmann@mathematik.uni-regensburg.de