EQUILIBRIA OF AN ANISOTROPIC NONLOCAL INTERACTION EQUATION: ANALYSIS AND NUMERICS

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Abstract. In this paper, we study the equilibria of an anisotropic, nonlocal aggregation equation with nonlinear diffusion which does not possess a gradient flow structure. Here, the anisotropy is induced by an underlying tensor field. While anisotropic forces cannot be associated with a potential in general and stationary solutions of anisotropic aggregation equations generally cannot be regarded as minimizers of an energy functional, we derive equilibrium conditions for stationary line patterns in the setting of spatially homogeneous tensor fields which can be regarded as the minimizers of a regularised energy functional depending on a scalar potential. In particular, this dimension reduction allows us to study the associated one-dimensional problem instead of the two-dimensional setting. For spatially homogeneous tensor fields, we show the existence of energy minimisers, establish Γ-convergence of the regularised energy functionals as the diffusion coefficient vanishes, and prove the convergence of minimisers of the regularised energy functional to minimisers of the non-regularised energy functional. Further, we investigate properties of stationary solutions on different domains. Finally, we prove weak convergence of a numerical scheme for the numerical solution of the anisotropic, nonlocal aggregation equation with nonlinear diffusion and any underlying tensor field, and show numerical results.

1. Introduction

The derivation, analysis and numerics of mathematical models for collective behaviour of cells, animals or humans have recently been receiving increasing attention. Based on agent-based modelling approaches, a variety of continuum models has been derived and used to describe biological aggregations such as flocks and swarms [47, 53]. Motivated by the simulation of fingerprint patterns which can be modelled as the interaction of a large number of cells [13, 43], a continuum model can be derived, given by the anisotropic aggregation equation

\[ \partial_t \rho(t,x) + \nabla \cdot [\rho(t,x) (F(\cdot, T(x)) * \rho(t, \cdot))(x)] = 0 \quad \text{in } \mathbb{R}_+ \times \mathbb{R}^2 \] (1.1)

with initial condition \( \rho|_{t=0} = \rho^0 \) in \( \mathbb{R}^2 \) for some given initial data \( \rho^0 \). Here,

\[ u_\rho(t, x) = (F(\cdot, T(x)) * \rho(t, \cdot))(x) = \int_{\mathbb{R}^2} F(x-y, T(x)) \rho(t, y) \, dy \] (1.2)

is the velocity field with \( |u_\rho(t, x)| \leq f \) for the uniform bound \( f \) of \( F \) where the term \( F(x-y, T(x)) \) denotes the force which a particle at position \( y \) exerts on a particle at position \( x \). The left-hand side of (1.1) represents the active transport of the density \( \rho \) associated to a nonlocal velocity field \( u_\rho \).

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The force $F$ depends on an underlying stress tensor field $T(x)$ at location $x$. The existence of such a tensor field $T(x)$ is motivated by experimental results for simulating fingerprints [40] and a model describing the formation of fingerprint patterns based on the interaction of so-called Merkel cells has been suggested by Kücken and Champod [43]. Due to the generality of the definition of the forces, model (1.1) can be regarded as a prototype for understanding complex phenomena in nature. Since an alignment of mass along the local stress lines is observed, we define the tensor field $T(x)$ by the directions of smallest stress at location $x$, i.e. we consider a unit vector field $s = s(x) \in \mathbb{R}^2$ and introduce a corresponding orthonormal vector field $l = l(x) \in \mathbb{R}^2$, representing the directions of largest stress. The tensor field $T(x)$ at $x$ is given by

$$T(x) := \chi s(x) \otimes s(x) + l(x) \otimes l(x) \in \mathbb{R}^{2,2}.$$  (1.3)

The parameter $\chi \in [0, 1]$ in the definition of the tensor field introduces an anisotropy in the direction $s$.

A typical aspect of aggregation models is the competition of social interactions (repulsion and attraction) between the particles which is also the focus of our research. Hence, we assume that the total force $F$ is given by

$$F(x - y, T(x)) = F_A(x - y, T(x)) + F_R(x - y).$$  (1.4)

Here, $F_R$ denotes the repulsion force that a particle at location $y$ exerts on particle at location $x$ and $F_A$ is the attraction force a particle at location $y$ exerts on particle at location $x$. The repulsion and attraction forces are of the form

$$F_R(d = d(x, y)) = f_R(|d|)d$$

and

$$F_A(d = d(x, y), T(x)) = f_A(|d|)T(x)d,$$

respectively, with radially symmetric coefficient functions $f_R$ and $f_A$, where, again, $d = d(x, y) = x - y \in \mathbb{R}^2$. An example for the force coefficients $f_R$ and $f_A$ was suggested by Kücken and Champod [43], given by

$$f_R(d) = (\alpha|d|^2 + \beta) \exp(-e_R|d|)$$  (1.5)

and

$$f_A(d) = -\gamma|d| \exp(-e_A|d|)$$  (1.6)

for nonnegative constants $\alpha$, $\beta$, $\gamma$, $e_A$ and $e_R$, and $d = (d_1, d_2) \in \mathbb{R}^2$. We assume that the total force (1.4) exhibits short-range repulsion and long-range attraction along $l$, and only repulsion along $s$, while the direction of the interaction forces is determined by the parameter $\chi \in [0, 1]$ in the definition of $T$ in (1.3). These assumptions on the force coefficients are satisfied for the parameters proposed in [31], given by

$$\alpha = 270, \quad \beta = 0.1, \quad \gamma = 10.5, \quad e_A = 95, \quad e_R = 100, \quad \chi = 0.2.$$  (1.7)

Motivated by plugging (1.3) into the definition of the total force (1.4), we consider a more general form of the total force, given by

$$F(d = d(x, y), T(x)) = f_s(|d|)(s(x) \cdot d)s(x) + f_l(|d|)(l(x) \cdot d)l(x)$$  (1.8)

for coefficient functions $f_s$ and $f_l$, where $f_s = f_{R+\chi}f_A$ and $f_l = f_{R+f_A}$ for the Kücken-Champod model.

The macroscopic model (1.1) can be regarded as the macroscopic limit of an anisotropic particle model as the number of particles $N$ goes to infinity. The $N$ interacting particles
positions \( x_j = x_j(t) \in \mathbb{R}^2, \ j = 1, \ldots, N, \) at time \( t \) satisfy
\[
\frac{dx_j}{dt} = \frac{1}{N} \sum_{k=1 \atop k \neq j}^{N} F(x_j - x_k, T(x_j)),
\]
equipped with initial data \( x_j(0) = x_j^0, \ j = 1, \ldots, N, \) for given scalars \( x_j^0, \ j = 1, \ldots, N. \) A special instance of this model has been introduced in [43] for simulating fingerprint patterns. The particle model in its general form (1.9) has been studied in [13, 22, 31]. In particular, the particles align in line patterns according to the underlying fields \( s = s(x) \) and \( l = l(x) \) [13, 22, 31]. Due to the purely repulsive forces along \( s \) and the short-range repulsive, long-range attractive forces along \( l, \) we prove for spatially homogeneous tensor fields that the stationary solution consists of line patterns along \( s. \) These stationary solutions to (1.1) can be regarded as solutions with one-dimensional support and are constant along \( s. \) For general tensor fields, we observe from the numerical simulations that line patterns can be obtained as stationary solutions.

Since our fingerprint lines do not have a one-dimensional support and, in fact, have a certain width, we widen the support of the line structures by introducing a small nonlinear diffusion on the right-hand side of (1.1), leading to the nonlocal aggregation equation with nonlinear diffusion
\[
\partial_t \rho(t,x) + \nabla \cdot [\rho(t,x)(F(\cdot, T(x)) \ast \rho(t, \cdot))(x)] = \delta \nabla \cdot (\rho(t,x) \nabla \rho(t,x)) \quad \text{in} \ \mathbb{R}_+ \times \mathbb{R}^2
\]
where \( \delta \ll 1. \) In particular, for the spatially homogeneous tensor field \( T \) with \( s = (0,1) \) and \( l = (1,0) \) straight vertical lines are obtained as stationary solutions [13, 22, 31] which can be regarded as constant solutions along the vertical axis. For solutions of this form, the diffusion term only acts perpendicular to the line patterns and not parallel. Hence, a positive diffusion coefficient \( \delta \) leads to nonlinear diffusion along the horizontal axis and we expect the widening of the vertical line profile.

1.1. Isotropic aggregation equations. While we consider anisotropic aggregation equations of the form (1.1) in this work, mainly isotropic aggregation equations [7, 9, 42, 44] of the form
\[
\rho_t + \nabla \cdot (\rho(-\nabla W \ast \rho)) = 0
\]
for a radially symmetric interaction potential \( W(d) = W(|d|) \) with \( F(d) = -\nabla W(d), \) have been studied in the literature. In particular, the study of the isotropic aggregation equations in terms of its gradient flow structure [1, 28, 29, 45, 55], the blow-up dynamics for fully attractive potentials [7, 8, 21, 27], and the rich variety of steady states [2, 3, 4, 6, 8, 15, 19, 25, 23, 33, 34, 50, 56, 57] has attracted the interest of many research groups recently. In these works, the energy
\[
\mathcal{E}(\rho) = \frac{1}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} W(x - y) \, d\rho(x) \, d\rho(y)
\]
in the \( d \)-dimensional setting plays an important role since it governs the dynamics, and its (local) minima describe the long-time asymptotics of solutions. Sharp conditions for the existence of global minimisers for a broad class of nonlocal interaction energies on the space of probability measures have been established in [51].

In terms of biological applications, nonlocal interactions on different scales [6, 32, 47] are considered for describing the interplay between short-range repulsion which prevents collisions between individuals, and long-range attraction which keeps the swarm cohesive [48, 49]. These repulsive-attractive potentials can be considered as a minimal model for pattern formation in large systems of individuals [3]. The 1D nonlocal interaction equation with a repulsive-attractive potential has been studied in [33, 34, 50] where the authors show that the behaviour of the
solution strongly depends on the regularity of the interaction potential. More precisely, the solution converges to a sum of Dirac masses for regular interaction, while it remains uniformly bounded for singular repulsive potentials. Pattern formation for repulsive-attractive potentials in multiple dimensions is studied in [9, 42, 56, 57]. It has been observed that even for quite simple repulsive-attractive potentials the energy minimizers are sensitive to the precise form of the potential and can exhibit a wide variety of patterns [41, 42, 57]. Nonlocal interaction models have been studied for specific types of repulsive-attractive potentials [2, 18, 24, 26, 27, 35]. In [2], conditions for the dimensionality of the support of local minimisers of (1.12) are obtained in terms of the repulsive strength of the potential \( W \) at the origin. Minimizers for the special class of repulsive-attractive potential which blow up approximately like the Newtonian potential at the origin have also been studied [19, 35].

Very few numerical schemes apart from particle methods have been proposed to simulate solutions of isotropic aggregation equations after blow-up. The so-called sticky particle method [21] is a convergent numerical scheme, used to obtain qualitative properties of the solution such as the finite time total collapse. While numerical results have been obtained in the one-dimensional setting [38], this method is not practical to deal with finite time blow-up and the behavior of solutions after blow-up in dimensions larger than one. Let the solution to (1.1) with initial data \( \rho^{in} \) be denoted by \( \rho \) and the solution of the particle model (1.9) with initial data \( \rho^{in,N} \) be denoted by \( \rho^N(t) = \frac{1}{N} \sum_{j=1}^{N} \delta(x - x_j(t)) \) at time \( t \geq 0 \). If \( F = -\nabla W \) for some radially symmetric potential \( W \) and the initial data satisfies \( d_W(\rho^{in}, \rho^{in,N}) \to 0 \) as \( N \to \infty \) in the Wasserstein distance \( d_W \), then

\[
\sup_{t \in [0,T]} d_W(\rho(t), \rho^N(t)) \to 0.
\]

for any given \( T > 0 \) [27]. From the theoretical viewpoint, this is a very nice result, but in practice a very large number of particles is required for numerical simulations of the particle model (1.9) to obtain a good control on the error after a long time. Nevertheless, particle simulations lead to a very good understanding of qualitative properties of solutions for aggregation equations where collisions do not happen [2, 9, 13, 31, 56, 57]. For the one-dimensional setting with a nonlinear dependency of the term \( \nabla W \ast \rho \), a finite volume scheme for simulating the behaviour after blow-up has been proposed in [39] and its convergence has been shown. Extremely accurate numerical schemes have been developed to study the blow-up profile for smooth solutions [36, 37]. An energy decreasing finite volume method for a large class of PDEs including (1.11) has been proposed in [17] and a convergence result for a finite volume scheme with general measures as initial data has been shown in [27]. In particular, this numerical scheme leads to numerical simulations of solutions in dimension greater than one.

The isotropic aggregation equation (1.11) may also be modified to include linear or nonlinear diffusion terms [16]. While a linear diffusion term can be used to describe noise at the level of interacting particles, a nonlinear diffusion term can be used to model a system of interacting particles at the continuum level, and can be expressed by a repulsive potential. To see the latter, we consider the potential \( W_\delta = W + \delta \delta_0 \) for a parameter \( \delta > 0 \) and the Dirac delta \( \delta_0 \), inducing an additional strongly localised repulsion. This corresponds to a PDE with nonlinear diffusion which is given by

\[
 \rho_t + \nabla \cdot (\rho(-\nabla W \ast \rho)) = \delta \nabla \cdot (\rho \nabla \rho).
\]

More generally, adding nonlinear diffusion in (1.11) results in the class of aggregation equations

\[
 \rho_t + \nabla \cdot (\rho(-\nabla W \ast \rho)) = \delta \nabla \cdot (\rho \nabla \rho^{m-1}) \quad (1.13)
\]
with diffusion coefficient $\delta > 0$ and a real exponent $m > 1$. Of central importance for studies of (1.13) is its gradient flow formulation [1] with respect to the energy
\[
\mathcal{E}_\delta(\rho) = \frac{1}{2} \int_{\mathbb{R}^d} \rho(W * \rho + \delta \rho^{m-1}) \, dx.
\]
(1.14)
In particular, stationary states of (1.13) are critical points of the energy (1.14). The existence of global minimisers of (1.14) has recently been studied in [5] using techniques from the calculus of variations. While radially symmetric and non-increasing global minimisers exist for $m > 2$, the case $m = 2$ is critical and yields a global minimiser only for small enough diffusion coefficients $\delta > 0$. Burger et al. [11] have shown that the threshold for $\delta$ is $\|W\|_{L_1}$ for $m = 2$. Energy considerations have also been employed in [12] to study the large time behaviour of solutions to (1.13) in one dimension. The existence of finite-size, compactly supported stationary states for the general power exponent $m > 1$ is investigated in [14]. The uniqueness/non-uniqueness criteria are determined by the parameter $m$, with the critical power being $m = 2$ [30]. In particular, the steady state is unique for a fixed mass for any attractive potential and $m \geq 2$.

1.2. Contributions. In this work, we consider the anisotropic counterparts of the isotropic aggregation equations (1.11) and (1.13) with $m = 2$ which are given by (1.1) and (1.10), respectively. No gradient flow formulation exists in this case and stationary solutions of the anisotropic aggregation equation generally cannot be regarded as minimizers of an energy functional. As a first aim of this paper, we derive equilibrium conditions for stationary solutions of (1.1) and (1.10) so that stationary line patterns can be regarded as minimisers of an energy functional which depend on a scalar potential. In particular, this dimension reduction will allow us to study the associated one-dimensional problem instead of the two-dimensional setting. Using these energy functionals, we show the existence of energy minimisers, establish $\Gamma$-convergence of a regularised energy functional with vanishing diffusion, and prove the convergence of minimisers of the regularised energy functional to minimisers of the non-regularised energy functional. The second aim of this paper is to investigate the dependence of the diffusion coefficient $\delta$ on stationary solutions numerically by considering an appropriate numerical scheme for the anisotropic interaction equation (1.10) without gradient flow structure. The numerical scheme and its analysis is based on [17, 27]. The additional diffusion in the mean-field model (1.10) results in beautiful patterns which are better than the ones obtained with the particle model since too low particle numbers may result in dotted line patterns.

This paper is organised as follows. In Section 2, we consider stationary solutions for general underlying tensor fields first, before restricting ourselves to spatially homogeneous tensor fields whose support is given by line patterns. For this case, we derive equilibrium conditions which can be reformulated as the minimisers of an energy functional. We show the existence of energy minimisers, and prove $\Gamma$-convergence of the regularised energies and the convergence of minimisers of the regularised energies to minimisers of the non-regularised energy functional as the diffusion coefficient goes to zero. Finally, we consider a numerical scheme for the anisotropic, nonlocal aggregation equation with nonlinear diffusion (1.10), prove its weak convergence as the diffusion coefficient goes to zero, and show numerical results in Section 3.

2. Stationary solutions

In this section, we study the equilibria of the nonlocal aggregation equation with nonlinear diffusion (1.10). Since most applications of (1.10) require measure-valued solutions, we consider nonnegative solutions $\rho \geq 0$ only.

The stationary solutions $\rho_\infty = \rho_\infty(x,y)$ for $(x,y) \in \mathbb{R}^2$ satisfy
\[
\nabla \cdot [\rho_\infty(F(\cdot,T(x,y)) * \rho_\infty - \delta \nabla \rho_\infty)] = 0 \quad \text{a.e. in } \mathbb{R}^2,
\]
\[
\text{implies that the argument has to be constant a.e. in } \mathbb{R}^2. \text{ Since we are interested in stationary line patterns, the stationary solution } \rho_\infty \text{ should satisfy supp } \rho_\infty \subsetneq \mathbb{R}^2 \text{ for small diffusion}.
\]
coefficients $\delta > 0$ and hence it is sufficient to require
\[
\rho_\infty (F(\cdot, T(x,y)) * \rho_\infty - \delta \nabla \rho_\infty) = 0 \quad \text{a.e. in $\mathbb{R}^2$,}
\] (2.1)
or equivalently
\[
F(\cdot, T(x,y)) * \rho_\infty = \delta \nabla \rho_\infty \quad \text{on $\text{supp}(\rho_\infty)$.}
\]

In the following, we assume that the underlying tensor field is spatially homogeneous and we study the associated stationary solutions. While anisotropic forces cannot be associated with a potential in general and stationary solutions of anisotropic aggregation equations generally cannot be regarded as minimizers of an energy functional, the idea of this section is to derive conditions for stationary solutions of (1.1) and (1.10) so that stationary line patterns can be obtained by minimising energy functionals which depend on a scalar potential. In particular, this dimension reduction will allow us to study the associated one-dimensional problem instead of the two-dimensional setting. Using these energy functionals, we show the existence of energy minimisers, establish $\Gamma$-convergence of a regularised energy functional with vanishing diffusion, and prove the convergence of minimisers of the regularised energy functional to minimisers of the non-regularised energy functional.

2.1. Notation and assumptions. The aim of this section is to derive a scalar force and its scalar potential in one variable that can be used to define the associated regularised and non-regularised energy functionals. For this, we study some properties of stationary solutions for spatially homogeneous tensor fields first.

As in [13] one can show that a steady state of (1.10) for any spatially homogeneous tensor field $\tilde{T}$ is a coordinate transform of a steady state to the mean-field equation (1.10) for the tensor field $T$ with $l = (1,0)$ and $s = (0,1)$. Due to the choice of the tensor field $T$, we restrict ourselves to vertical line patterns as steady states in the following, i.e. we consider stationary solutions which are constant along the $y$-direction. To guarantee the existence of probability measures which are constant along the $y$-direction, we consider the domain $\Omega = \mathbb{R} \times [-0.5,0.5]$ instead of $\mathbb{R}^2$ in this section. This assumption on the domain leads to stationary solutions on $\Omega$ of the form
\[
\rho_\infty (x,y) = \rho_\infty (x,0) \quad \text{for a.e. } y \in [-0.5,0.5].
\] (2.2)

Note that this assumption on the domain $\Omega$ is not restrictive and by appropriate rescaling similar results can be obtained for any domain of the form $\mathbb{R} \times [a,b]$ for any $a,b \in \mathbb{R}$ with $a < b$.

The special form (2.2) of the stationary solutions motivates the definition of the space $\mathcal{P}_c(\Omega)$ of probability measures which are constant in $y$-direction. We define the space $\mathcal{P}_c(\Omega)$ by
\[
\mathcal{P}_c(\Omega) = \left\{ \rho \in L^1_+(\Omega) : \int_{\Omega} \rho \mathrm{d}(x,y) = 1, \ \rho(x,y) = \rho(x,0) \text{ for a.e. } y \in [-0.5,0.5] \right\}.
\]
Denoting the components of $F$ by $F_x,F_y$ for $d \in \Omega$, respectively, i.e. $F(d) = (F_x(d), F_y(d))$ for $d \in \Omega$, we extend $F_x,F_y$ and $\rho_\infty$, defined on $\Omega$, periodically on $\mathbb{R}^2$ with respect to the $y$-coordinate, if required, so that the convolution integrals $F_x(\cdot, T) * \rho_\infty, F_y(\cdot, T) * \rho_\infty$ can be evaluated. Since the total force $F$ in (1.8) reduces to
\[
F(d, T) = \begin{pmatrix}
(f_x(|d|)d_1) \\
(f_y(|d|)d_2)
\end{pmatrix}
\]
for the spatially homogeneous tensor field with $l = (1,0)$ and $s = (0,1)$, we have $F_x(d) = f_1(|d|)d_1$ and $F_y(d, T) = f_s(|d|)d_2$ for $d = (d_1, d_2)$. For $\rho_\infty$ satisfying (2.2), we have $F_y(\cdot, T) * \rho_\infty = 0$ since $F_y$ is an odd function in the $y$-coordinate and $F_y(\cdot, T) * \rho_\infty$ is periodically extended along the $y$-coordinate. In particular, the second equality in (2.1) is trivial. The convolution $F_x * \rho_\infty$
is of the form
\[
F_x \ast \rho_\infty(x, y) = \int_\Omega F_x(w, z) \rho_\infty(x - w, y - z) \, d(w, z) = \int_\Omega F_x(x - w, y - z) \rho_\infty(w, z) \, d(w, z)
\]
\[
= \int_\mathbb{R} \rho_\infty(w, 0) \int_{[-0.5, 0.5]} F_x(x - w, y - z) \, dz \, dw.
\]

Since a scalar force in one variable is required for a dimension reduction, this motivates to introduce a scalar odd function \(G : \mathbb{R} \to \mathbb{R}\) defined by
\[
G(x) = \int_{[-0.5, 0.5]} F_x(x, z) \, dz = x \int_{[-0.5, 0.5]} f_1(\sqrt{x^2 + z^2}) \, dz \quad (2.3)
\]
where \(G(0) = 0\). Due to the periodic extension of \(F_x\) along the \(y\)-coordinate, we have
\[
G(x) = \int_{[-0.5, 0.5]} F_x(x, y - z) \, dz \quad \text{for any} \quad y \in [-0.5, 0.5].
\]
Hence, there exists an interaction potential \(W : \mathbb{R} \to \mathbb{R}\) which is even and such that
\[
G = -\frac{d}{dx} W. \quad (2.4)
\]

For the analysis in the following sections, we require rather relaxed conditions on the potential \(W\):

**Assumption 2.1** For the interaction potential \(W\) satisfying \((2.4)\), we require

\(\text{(A1)}\) \(W\) is even, i.e. \(W(x) = W(-x)\).

\(\text{(A2)}\) \(W\) is continuously differentiable.

\(\text{(A3)}\) \(W\) is locally integrable on \(\Omega\).

\(\text{(A4)}\) \(W(x) \to 0\) as \(|x| \to \infty\).

\(\text{(A5)}\) There exist \(\delta > 0\) and a measure \(\bar{\rho} \in P_c(\Omega)\) such that \(\mathcal{E}_\delta(\bar{\rho}) \leq 0\).

\(\text{(A6)}\) There exists some \(x_W > 0\) such that
\[
W(x) \leq 0 \quad \text{for} \quad 0 \leq |x| \leq x_W \quad \text{and} \quad W(x) < 0 \quad \text{for some} \quad x \in (0, x_W). \quad (2.5)
\]

Using the potential \(W\), we define the energy functional
\[
\mathcal{E}(\rho_\infty) := \frac{1}{2} \int_\Omega \rho_\infty(W \ast \rho_\infty) \, d(x, y) \quad (2.6)
\]
where \(W \ast \rho_\infty(x, y)\) is regarded as the convolution with respect to the first coordinate, i.e.
\[
W \ast \rho_\infty(x, y) = \int_\mathbb{R} W(x - w) \rho_\infty(w, y) \, dw, \quad (2.7)
\]
which is constant with respect to the second coordinate. The regularisation of the energy \(\mathcal{E}\) is defined as
\[
\mathcal{E}_\delta(\rho_\infty) := \frac{1}{2} \int_\Omega \rho_\infty(W \ast \rho_\infty + \delta \rho_\infty) \, d(x, y) \quad (2.8)
\]
on \(P_c(\Omega)\).

**Remark 2.2** Note that assumptions \((\text{A1}), (\text{A2}), (\text{A3}), (\text{A4})\) are rather relaxed conditions and allow us to consider a rather general class of interaction potentials, including the one that can be derived from \(G\) based on \(F_x\) in the Kücken-Champod model. In particular, the interaction potential \(W(x)\) satisfies \(W(0) = 0\) and is bounded. Besides, the energy \(\mathcal{E} : P_c(\Omega) \to \mathbb{R}\) in \((2.8)\) is weakly lower semi-continuous with respect to weak convergence of measures. Assumption \((\text{A5})\) is required for establishing the existence of minimisers of the energy \(\mathcal{E}_\delta\) in \((2.8)\). In particular, it follows from \((\text{A5})\) that there exists a measure \(\bar{\rho} \in P_c(\Omega)\) such that
where the convolution $\rho(\delta) \leq 0$ for all $0 \leq \delta \leq \delta$. Assumption (A5) also implies that there exists $x \in (0, x_W)$ such that $W(x) < 0$.

Assumption (A6) is motivated by the form of the force $F$ in (1.4) which exhibits short-range repulsion and long-range attraction forces along $l$. Hence, there exists a constant $d_a > 0$ such that

$$(f_A + f_R)(|d|) \leq 0 \text{ for } |d| > d_a \quad \text{and} \quad (f_A + f_R)(|d|) > 0 \text{ for } 0 \leq |d| < d_a.$$  

A slightly stronger condition is given by the existence of some $x_G > 0$ such that

$$G(x) \geq 0 \text{ for } 0 \leq x \leq x_G, \quad (2.9)$$

where $G$ is defined in (2.3). Then, (A6) follows from (2.4). Note that condition (2.5) in (A6) is necessary for (A5) for $\delta > 0$ and sufficient for (A5) for $\delta \geq 0$.

**Remark 2.3** Assumption (A5) is not restrictive which is shown by the following examples for $\tilde{\rho} \in \mathcal{P}_c(\Omega)$ which satisfies (A5), provided (A6) holds. We consider $\tilde{\rho} = \frac{1}{Q_W} \chi_{Q_W}$ where $Q_W = [-x_W/2, x_W/2] \times [-0.5, 0.5]$. The non-regularised energy $E$ in (2.6) is clearly negative and for $\delta > 0$ sufficiently small, assumption (A5) is satisfied, provided (A6) holds. More generally, $\tilde{\rho} = \frac{1}{Q_W} \chi_{Q_W}$ satisfies (A5) for any $\tilde{x} \in (0, x_{W,\max})$ where $x_{W,\max} = \sup \left\{ \tilde{x} > 0 : \int_0^{\tilde{x}} W(s) \, ds \leq 0 \right\} > x_W$ and $Q_{W,\delta} = [-\tilde{x}/2, \tilde{x}/2] \times [-0.5, 0.5]$, provided (A6) holds.

Another example for measures satisfying (A5) are mollified delta distributions. Note that $\tilde{\rho}(x, y) = \delta(x) \in \mathcal{P}_c(\Omega)$ satisfies (A5) for $\tilde{E}$ since

$$\int_\Omega \rho_\infty(W * \rho_\infty) \, d(x, y) = W(0) = 0.$$  

Further note that for the one-dimensional heat kernel

$$\phi(x) = \frac{1}{\sqrt{4\pi}} \exp \left(-\frac{|x|^2}{4} \right)$$

we consider the rescaled kernel

$$\phi_\varepsilon(x, y) = \frac{1}{\sqrt{\varepsilon}} \phi \left( \frac{x}{\sqrt{\varepsilon}} \right).$$

Due to property (2.5) of $W$ we can choose $\varepsilon > 0$ and $\delta > 0$ sufficiently small such that $E_\delta(\phi_\varepsilon) \leq 0$.

The above examples show that for $\rho_\infty$ with compact, connected support (A5) is satisfied. Similarly, for any $\delta > 0$, the first term of the energy functional $E_\delta$ in (2.8) is negative provided the support of $\rho_\infty$ is sufficiently small in the $x$-direction and (A6) holds. Hence, the parameter $\delta > 0$ can be chosen sufficiently small so that (A5) is satisfied.

Note that Assumption (A5) is equivalent to the existence of minimizers of $E_\delta$ and is also equivalent to $E_\delta$ not being $H$-stable [15, 51].

**2.2. Equilibrium conditions.** Using the interaction potential $W$, the condition for equilibria in (2.1) can be reformulated as

$$\rho_\infty \partial_x (W * \rho_\infty + \delta \rho_\infty) = 0 \quad \text{a.e. in } \Omega \quad (2.10)$$

where the convolution $W * \rho_\infty$ is given by (2.7). Hence, we require

$$W * \rho_\infty + \delta \rho_\infty = C \quad \text{in each connected component of } \text{supp}(\rho_\infty) \quad (2.11)$$
for some constant $C \in \mathbb{R}$ since we are only interested in minimisers of the interaction energy $\mathcal{E}_\delta$. Note that we obtain by multiplying (2.11) by $\rho_\infty$ and integrating over $\text{supp}(\rho_\infty)$

$$\int\int_{\text{supp}(\rho_\infty)} \rho_\infty(x, y) W * \rho_\infty(x, y) \, d(x, y) + \delta \int\int_{\text{supp}(\rho_\infty)} \rho_\infty^2(x, y) \, d(x, y) = C,$$

where the unit mass of $\rho_\infty$ was used. In particular, this shows that $C = C(\delta) \in \mathbb{R}$ is uniquely determined and the integral equation (2.11) may be expressed in the equivalent fixed point form

$$\rho_\infty(x, y) = \frac{(C - W * \rho_\infty)_+}{(C - W * \rho_\infty)_+ \, d(x, y)}.$$

(2.12)

Clearly, the fixed point form is consistent with (2.2) and the dependence of $\rho_\infty$ on $\delta$ follows from $C = C(\delta)$.

It has been shown in [11] for non-trivial stationary states for purely repulsive potentials in the set $L^2(\mathbb{R}^d) \cap \mathcal{P}(\mathbb{R}^d)$ with $d \geq 1$ that minimisers are sufficient for solving the equilibrium conditions. A similar result can be shown in our setting of more general potentials and stationary states in the space $L^2(\Omega) \cap \mathcal{P}_c(\Omega)$ whose elements satisfy (2.2) in addition. In particular, a minimiser of the energy functional (2.8) is sufficient for solving (2.10).

**Proposition 2.4 (Stationary solutions via energy minimisation)** Let $\rho_\infty \in L^2(\Omega)$ be a minimiser of the energy functional (2.8) on $\mathcal{P}_c(\Omega)$ which is of the form (2.2). Then, $\rho_\infty$ satisfies (2.1).

### 2.3. Existence and convergence of minimisers

Motivated by Proposition 2.4, we consider the energy functionals $\mathcal{E}$ and $\mathcal{E}_\delta$, defined in (2.6) and (2.8). For the existence and convergence of minimisers, we have to verify that an energy minimising sequence is precompact in the sense of weak convergence of measures, and prove a $\Gamma$-convergence result. For this, we use Lions’ concentration-compactness lemma for probability measures [46], [52, Section 4.3] and reformulate it to our setting.

**Lemma 2.5 (Concentration-compactness lemma for measures)** Let $\{\rho_n\}_{n \in \mathbb{N}} \subset \mathcal{P}_c(\Omega)$. Then, there exists a subsequence $\{\rho_{n_k}\}_{k \in \mathbb{N}}$ satisfying one of the three following possibilities:

1. **(tightness up to transition)** There exists $z_k \in \Omega$ such that for all $\varepsilon > 0$ there exists $R > 0$ satisfying
   $$\int_{B_R(z_k) \cap \Omega} d\rho_{n_k}(x, y) \geq 1 - \varepsilon \quad \text{for all } k;$$

2. **(vanishing)**
   $$\limsup_{k \to \infty} \sup_{z \in \Omega} \int_{B_R(z) \cap \Omega} d\rho_{n_k}(x, y) = 0 \quad \text{for all } R > 0;$$

3. **(dichotomy)** There exists $\alpha \in (0, 1)$ such that for all $\varepsilon > 0$ there exists $R > 0$ and a sequence $\{z_k\}_{k \in \mathbb{N}} \subset \Omega$ with the following property:
   Given any $R' > R$ there are nonnegative measures $\rho^1_k$ and $\rho^2_k$ such that
   $$0 \leq \rho^1_k + \rho^2_k \leq \rho_{n_k},$$
   $$\text{supp}(\rho^1_k) \subset B_R(z_k) \cap \Omega,$$
   $$\text{supp}(\rho^2_k) \subset \Omega \setminus B_R(z_k),$$
   $$\limsup_{k \to \infty} \left( \alpha - \int_{\Omega} d\rho^1_k(x, y) + (1 - \alpha) - \int_{\Omega} d\rho^2_k(x, y) \right) \leq \varepsilon.$$
For proving the existence of minimisers of the energy functional (2.8), one can use the direct method of the calculus of variations and Lemma 2.5 to eliminate the cases ‘vanishing’ and ‘dichotomy’ of an energy minimising sequence. The proof of the existence of minimisers of the regularised energy $\mathcal{E}_\delta$ in (2.8) is very similar to the one for the non-regularised energy $\mathcal{E}$, provided in [51, Theorem 3.2]:

**Proposition 2.6 (Existence of minimisers)** Suppose $W$ satisfies assumptions (A1), (A2), (A3) and (A4). Then, the regularised energy $\mathcal{E}_\delta$ in (2.8) has a global minimiser in $\mathcal{P}_c(\Omega)$ if and only if it satisfies (A5). The non-regularised energy $\mathcal{E}$ in (2.6) has a global minimiser in $\mathcal{P}_c(\Omega)$ if and only if (A5) is satisfied for $\mathcal{E}$.

**Remark 2.7** Let $\delta > 0$ be given. To see the necessity of assumption (A5) for the existence of minimisers, assume that $\mathcal{E}_\delta(\rho) > 0$ for all $\rho \in \mathcal{P}_c(\Omega)$. We consider a sequence of measures which ‘vanishes’ in the sense of Lemma 2.5(2). Let

$$
\rho(x, y) = \chi_{Q_1}(x, y),
$$

where $Q_n$ denotes the rectangle $[-0.5n, 0.5n] \times [-0.5, 0.5]$ for $n \geq 1$, and $\chi_{Q_n}$ denotes the characteristic function of $Q_n$. We consider the sequence

$$
\rho_n(x, y) = \frac{1}{n} \rho \left( \frac{x}{n}, y \right)
$$

for $n \geq 1$. Then, $\rho_n \in \mathcal{P}_c(\Omega)$ and

$$
0 < \mathcal{E}_\delta(\rho_n) = \frac{1}{n^2} \iint_{Q_n} \int_{[-0.5n, 0.5n]} W(x - w) \, dw \, d(x, y) + \frac{\delta}{n^2} \iint_{Q_n} d(x, y) = \frac{1}{n^2} \iint_{Q_n} \iint_{Q_n} W(x - w) \, d(w, z) \, d(x, y) + \frac{\delta}{n} \leq \frac{1}{n^2} \iint_{Q_n} \iint_{Q_n} |W(w)| \, d(w, z) \, d(x, y) + \frac{\delta}{n} \leq \frac{1}{n} \left( \iint_{Q_n} |W(x)| \, d(x, y) + \iint_{Q_{2n} \setminus Q_n} |W(x)| \, d(x, y) + \delta \right) \leq \frac{C(R)}{n} + 2 \sup_{|x| \geq R} |W(x)| + \frac{\delta}{n}
$$

for any $R > 0$ where

$$
C(R) := \iint_{Q_n} |W(x)| \, d(x, y).
$$

Due to (A4) we have $\sup_{|x| \geq R} |W(x)| \to 0$ as $R \to 0$, implying that for any $\varepsilon > 0$ we can choose $R$ so that $2 \sup_{|x| \geq R} |W(x)| < \varepsilon$. Then, we can choose $n$ large enough so that $\frac{C(R)+\delta}{n} < \frac{\varepsilon}{2}$ holds. Hence, $\lim_{n \to \infty} \mathcal{E}_\delta(\rho_n) = 0$, implying

$$
\inf_{\rho \in \mathcal{P}_c(\Omega)} \mathcal{E}_\delta(\rho) = 0.
$$

Since $\mathcal{E}_\delta(\rho) > 0$ for all $\rho \in \mathcal{P}_c(\Omega)$, $\mathcal{E}_\delta$ does not have a minimiser in $\mathcal{P}_c(\Omega)$.

**Theorem 2.8** (Γ-convergence of regularised energies) Suppose that $W$ satisfies (A1), (A2), (A3) and (A4). The sequence of regularised energies $\{\mathcal{E}_\delta\}_{\delta > 0}$ Γ-converges to the energy $\mathcal{E}$ with respect to the weak convergence of measures. That is,
(Limsup) For any $\rho \in \mathcal{P}_c(\Omega)$ there exists a sequence $\{\rho_\delta\}_{\delta > 0} \subset \mathcal{P}_c(\Omega)$ such that $\rho_\delta$ converges weakly to $\rho$ as $\delta \to 0$ and
\[
\limsup_{\delta \to 0} \mathcal{E}_\delta(\rho_\delta) \leq \mathcal{E}(\rho).
\]

Proof. Step 1 (Liminf): Since $W$ is lower semi-continuous and bounded from below, the weak lower semi-continuity of the first term in the energy functional $\mathcal{E}_\delta$ in (2.8) follows from the Portmanteau Theorem [54, Theorem 1.3.4], i.e.
\[
\liminf_{\delta \to 0} \frac{1}{2} \iint_{\Omega} \rho_\delta(W * \rho_\delta) \, d(x, y) \geq \frac{1}{2} \iint_{\Omega} \rho(W * \rho) \, d(x, y).
\]
Together with
\[
\liminf_{\delta \to 0} \frac{\delta}{2} \iint_{\Omega} \rho_\delta^2 \, d(x, y) \geq 0,
\]
the liminf inequality immediately follows.

Step 2 (Limsup): Let $\mu \in \mathcal{P}_c(\Omega)$ be given, let
\[
\phi(x) = \frac{1}{\sqrt{4\pi}} \exp\left(-\frac{|x|^2}{4}\right)
\]
denote the one-dimensional heat kernel and define
\[
\phi_\delta(x) = \frac{1}{\sqrt{\delta}} \phi\left(\frac{x}{\sqrt{\delta}}\right).
\]
Note that $\phi \in C^\infty(\Omega)$, $\phi(x, y) = \phi(-x, y)$ for all $(x, y) \in \Omega$, $\phi$ is constant in $y$, and
\[
\iint_{\Omega} \phi \, d(x, y) = 1.
\]
In particular, $|\phi_\delta| \leq C_\phi \sqrt{\delta}$ where $C_\phi$ denotes the bound of $\phi$. We define the measure $\rho_\delta := \phi_\delta * \rho$ which converges weakly to $\rho$ in $\mathcal{P}_c(\Omega)$. Note that
\[
\delta \iint_{\Omega} \rho_\delta^2 \, d(x, y) \leq C_\phi \sqrt{\delta} \iint_{\Omega} \rho_\delta \, d(x, y) = C_\phi \sqrt{\delta} \to 0 \quad \text{as } \delta \to 0.
\]
Due to the continuity of $W$, the term $- \iint_{\Omega} \rho(W * \rho) \, d(x, y)$ is weakly lower semi-continuous and
\[
\limsup_{\delta \to 0} \frac{1}{2} \iint_{\Omega} \rho_\delta(W * \rho_\delta) \, d(x, y) \leq \frac{1}{2} \iint_{\Omega} \rho(W * \rho) \, d(x, y),
\]
resulting in the limsup inequality. \hfill \square

Theorem 2.9 (Convergence of minimisers) Suppose that $W$ satisfies (A1), (A2), (A3) and (A4). For any $\delta > 0$ sufficiently small, suppose that $\mathcal{E}_\delta$ satisfies (A5) and let $\rho_\delta \in \mathcal{P}_c(\Omega)$ be a minimiser of the energy $\mathcal{E}_\delta$ in (2.8) for all $0 < \delta \leq \delta$. Then, there exists $\rho \in \mathcal{P}_c(\Omega)$ such that, up to a subsequence and translations, $\rho_\delta$ converges weakly to $\rho$ as $\delta \to 0$, and $\rho$ minimises the energy $\mathcal{E}$ over $\mathcal{P}_c(\Omega)$. 
Proof. Let \( \{\rho_k\}_{\delta>0} \subset \mathcal{P}_c(\Omega) \) be a sequence of minimisers of \( \mathcal{E}_\delta \). For \( \delta > 0 \) sufficiently small, we may assume that \( \mathcal{E}_\delta(\rho_k) \leq 0 \) for all \( 0 < \delta \leq \delta \) since \( \rho_k \) minimises \( \mathcal{E}_\delta \). As in [51, Theorem 3.2] one can eliminate the cases ‘vanishing’ and ‘dichotomy’ in Lemma 2.5, implying that there exists a subsequence \( \{\rho_{k_j}\}_{j\in\mathbb{N}} \) satisfying ‘tightness up to translation’, i.e. there exists \( z_k \in \Omega \) such that for all \( \varepsilon > 0 \) there exists \( R > 0 \) satisfying

\[
\int_{B_R(z_k)\cap\Omega} d\rho_{k_j}(x, y) \geq 1 - \varepsilon \quad \text{for all } k.
\]

We define \( \tilde{\rho}_{k_j} := \rho_{k_j}(\cdot - z_k) \) and hence \( \{\tilde{\rho}_{k_j}\}_{j\in\mathbb{N}} \) is tight. Since \( \mathcal{E}_{\delta_k}(\rho_{\delta_k}) = \mathcal{E}_{\delta_k}[\tilde{\rho}_{\delta_k}] \), \( \{\tilde{\rho}_{\delta_k}\}_{k\in\mathbb{N}} \) is also a sequence of minimisers of \( \mathcal{E}_{\delta_k} \) and by Prokhorov’s Theorem (cf. [10, Theorem 4.1]) there exists a further subsequence \( \{\tilde{\rho}_{\delta_{k_j}}\}_{j\in\mathbb{N}} \), not relabelled, such that \( \tilde{\rho}_{\delta_{k_j}} \) converges weakly to some measure \( \rho \in \mathcal{P}_c(\Omega) \) as \( k \to \infty \).

For showing that the measure \( \rho \) minimises the energy functional \( \mathcal{E} \), we consider an arbitrary measure \( \mu \in \mathcal{P}_c(\Omega) \). By the limsup inequality in Theorem 2.8, there exists a sequence \( \{\mu_{\delta_k}\}_{k\in\mathbb{N}} \) which converges weakly to \( \mu \) as \( k \to \infty \) such that

\[
\limsup_{k\to\infty} \mathcal{E}_{\delta_k}(\mu_{\delta_k}) \leq \mathcal{E}(\mu).
\]

Together with the liminf inequality in Theorem 2.8, this yields

\[
\lim_{k\to\infty} \mathcal{E}_{\delta_k}(\mu_{\delta_k}) = \mathcal{E}(\mu).
\]

Since the sequence of measures \( \tilde{\rho}_{\delta_{k_j}} \) is a minimising sequence of \( \mathcal{E}_{\delta_k} \) which converges weakly to \( \rho \), we obtain, again by the liminf inequality,

\[
\mathcal{E}(\rho) \leq \liminf_{k\to\infty} \mathcal{E}_{\delta_k}(\tilde{\rho}_{\delta_k}) \leq \liminf_{k\to\infty} \mathcal{E}_{\delta_k}(\mu_{\delta_k}) = \mathcal{E}(\mu).
\]

\( \square \)

Note that each local minimiser \( \rho_\infty \) of \( \mathcal{E}_\delta \) is a steady state and satisfies the equilibrium condition (2.10). To see this, note that the Euler-Lagrange conditions for minimisers [20, Proposition 2.4] state that for each connected component \( A_i \) of \( \text{supp}(\rho_\infty) \) there exists \( C_i \in \mathbb{R} \) such that

\[
W * \rho_\infty + \delta \rho_\infty = C_i \quad \text{a.e. on } A_i,
\]

\[
W * \rho_\infty + \delta \rho_\infty \geq C_i \quad \text{a.e. on } \mathbb{R}.
\]

Since \( \text{supp}(\rho_\infty) \) is connected in our setting, see Theorem 2.12 below, this implies that (2.11) is fulfilled, implying that \( \rho_\infty \) is of the form (2.12). In particular, \( \partial_x \rho \) is well-defined and condition (2.10) holds.

2.4. Properties of stationary solutions. Note that the odd function \( G \), defined by \( G(x) = \int_{[-0.5,0.5]} F_x(x, z) \, dz \) in (2.3), is nonnegative for \( x \geq 0 \) for the force \( F_x \) in the Kücken-Champod model, see [31] or Section 1 for the precise definition of the force coefficients. Since \( G = -\frac{d}{dx^2} W \), we can make stronger assumptions on \( G \) and \( W \) than in (2.9) and (2.5), respectively, and we assume in this subsection that

\[
W'(x) = G(x) \geq 0 \quad \text{for all } x \geq 0
\]

and

\[
W(x) \leq 0 \quad \text{for all } |x| \geq 0.
\]

In particular, the assumptions on the potential \( W \) for the one-dimensional results in [11] are satisfied and the results also hold for the stationary states \( \rho_\infty \) satisfying (2.2). We obtain:

Corollary 2.10 Let \( \delta > 0 \) be given.

- If \( \delta \geq \|W\|_{L^1} \), there exists no stationary solution \( \rho_\infty \) in \( L^2 \cap \mathcal{P}_c(\Omega) \) of the form (2.2) which satisfies (2.1).
• If $\delta < \|W\|_{L^1}$, there exists a minimiser $\rho_\infty \in L^2(\Omega) \cap \mathcal{P}_c(\Omega)$ of the energy functional (2.8) which is symmetric in $x$, non-increasing on $x \geq 0$, and of the form (2.2).

To relate the cases $\delta < \|W\|_{L^1}$ and $\delta \geq \|W\|_{L^1}$ to assumption (A5) note that

$$-\iint_{\Omega} \rho_\infty W * \rho_\infty \, dx, dy \leq \|W\|_{L^1} \iint_{\Omega} \rho_\infty^2 \, dx, dy$$

by Young’s convolution inequality and property (2.14) of $W$, implying

$$\mathcal{E}_5(\rho_\infty) = \frac{1}{2} \iint_{\Omega} \rho_\infty(W * \rho_\infty + \delta \rho_\infty) \, dx, dy \geq \frac{\delta - \|W\|_{L^1}}{2} \iint_{\Omega} \rho_\infty^2 \, dx, dy$$

and hence, a necessary condition for (A5) is given by $\delta \leq \|W\|_{L^1}$.

Due to conditions (2.13) and (2.14), properties of the stationary solution of the one-dimensional case in [11] can also be extended to our setting:

**Proposition 2.11** For any given $L > 0$ there exists a unique symmetric function $\rho_L \in C^2([-L, L] \times [-0.5, 0.5])$ with unit mass, $\rho_L(x, y) = \rho_L(0, 0)$ for all $y \in [-0.5, 0.5]$, and $\partial_x \rho_L(x, y) \leq 0$ for $x \geq 0, y \in [-0.5, 0.5]$, such that $\rho_L$ solves (2.11) for some $\delta = \delta(L) > 0$ where $C = 2\mathcal{E}_5(\rho_L)$ in (2.11). Such a function $\rho_L$ also satisfies $\partial_x^2 \rho_L(0, y) < 0$ for all $y \in [-0.5, 0.5]$. Moreover, $\delta(L)$ is the largest eigenvalue of the compact operator

$$W_L[\rho_L](x) := \left( \int_0^L \rho_L(w, 0) \left( W(x-w) + W(x+w) - W(L-w) - W(L+w) \right) \, dw \right)$$

on the Banach space

$$\mathcal{Y}_L := \{ \rho_L \in C([-0.5, 0.5]) : \rho_L(0, y) = 0 \text{ for all } y \in [-0.5, 0.5] \}.$$

The simple eigenvalue $\delta(L)$ is uniquely determined as a function of $L$ with the following properties:

1. $\delta(L)$ is continuous and strictly increasing with respect to $L$,
2. $\lim_{L \to +\delta} \delta(L) = \|W\|_{L^1}$,
3. $\delta(0) = 0$.

**Theorem 2.12** Let $\delta < \|W\|_{L^1}$. Then, there exists a unique $\rho_L \in L^2(\Omega) \cap \mathcal{P}_c(\Omega)$ with unit mass and zero centre of mass such that (2.10) is satisfied. Moreover,

- $\rho_L$ is symmetric in $x$ and monotonically decreasing on $x > 0$,
- $\rho_L \in C^2(\text{supp}(\rho_L))$,
- $\text{supp}(\rho_L)$ is a bounded, connected set in $\Omega$,
- $\rho_L$ has a global maximum at $x = 0$, and $\partial_x^2 \rho_L(0, y) < 0$ for all $y \in [-0.5, 0.5]$,
- $\rho_L$ is the global minimiser of the energy $\mathcal{E}_\delta$ in (2.8).

### 2.5. Stationary solutions on the torus.

To compare the analytical results to the numerical simulations, we consider the two-dimensional unit torus $\mathbb{T}^2$, or equivalently, the unit square $[-0.5, 0.5]^2$ with periodic boundary conditions as the domain in this section. For minimisers $\rho_5$ of the energy functional $\mathcal{E}_5$ in (2.8), we require $\rho_\infty(x, y) = \rho_\infty(x, 0)$ for all $y \in [-0.5, 0.5]$ with zero centre of mass. Note that the uniform distribution on $[-0.5, 0.5]^2$ also satisfies these conditions.

In contrast to steady states on $\Omega = \mathbb{R} \times [-0.5, 0.5]$ in Theorem 2.12, steady states on the unit torus may not have connected support and may be composed of finitely many stripes of equal width and equal distances between each other. To see this, let us consider minimisers of the non-regularised energy $\mathcal{E}$ in (2.6), and suppose that we have an odd number $n$ of stripes first.
Let
\[ \rho_\infty(x, y) = \frac{1}{n} \sum_{k=1}^{n} \delta_{x_k}(x) \]  
(2.15)
for \( x_1, \ldots, x_n \in (-0.5, 0.5) \) with \( x_1 < \ldots < x_n \). We introduce the general velocity field \( V \in C^1([-0.5, 0.5]) \) such that \( V(x_j) = v_k \) for some given \( v_1, \ldots, v_n \in \mathbb{R} \). Let \( u(x, y, s) \) be a local solution to the Cauchy problem
\[ \partial_s u + \partial_x(uV) = 0, \quad u(x, y, 0) = \rho_\infty(x, y). \]
The evolution of the energy \( E \) in (2.6) along \( u \) at time \( s = 0 \) is given by
\[ \frac{d}{ds} E(u(x, y, s)) \bigg|_{s=0} = \int_{[-0.5,0.5]^2} (W*u) \partial_x u \, dx \, dy = \int_{[-0.5,0.5]^2} \rho_\infty V(W' * \rho_\infty) \, dx \, dy \]
since \( \rho_\infty(-0.5, y) = \rho_\infty(0.5, y) = 0 \) for all \( y \in [-0.5, 0.5] \). Here, appropriate periodic extensions of \( W' \) and \( \rho_\infty \) are considered in the convolution integral. Note that
\[ \int_{[-0.5,0.5]^2} \rho_\infty V \partial_x (W * \rho_\infty) \, dx \, dy = \sum_{k=1}^{n} v_k \sum_{j=1}^{n} W'(x_k - x_j), \]
and we require \( \frac{d}{ds} E(u(x, y, s)) \bigg|_{s=0} = 0 \) for minimisers of \( E \) for any velocity field \( V \), implying
\[ \sum_{j=1}^{n} W'(x_k - x_j) = \sum_{j=1}^{n} W'(x_k - x_j) = 0, \]
(2.16)
since \( W'(0) = G(0) = 0 \). Condition (2.16) is satisfied for equidistant points \( x_1, \ldots, x_n \) with
\[ x_k = \frac{k}{n} - \frac{n + 1}{2n}, \quad k = 1, \ldots, n, \]
(2.17)
since \( W'(d) = -W'(-d) \), and for general potentials it is unlikely that (2.16) is satisfied if (2.17) is not fulfilled. In particular, a minimiser \( \rho_\infty \) of \( E \) of the form (2.15) with zero centre of mass satisfying (2.16) and consisting of an odd number \( n \) of parallel lines consists of \( n \) equidistant lines at locations \( x_k \) in (2.17). The single straight vertical line with zero centre of mass is included in the property of locations \( x_k \) in (2.17).

For an even number \( n \) of lines, we can proceed in a similar way as above. Condition (2.16) implies that for minimisers \( \rho_\infty \) of \( E \) consisting of an even number of lines the property \( W'(-0.5) = W'(0.5) = 0 \) is required in addition to equidistant lines at locations \( x_k \) in (2.17). Note that \( W'(-0.5) = W'(0.5) = 0 \) is equivalent to \( f_l(0.5) = 0 \) for the force coefficient \( f_l \) in the definition of the force \( F_{x_k}(d) = f_l(|d|)d \).

More generally, for minimizers of \( E \) we require the measure \( \rho_\infty(x, y) = \frac{1}{n} \sum_{k=1}^{n} \delta_{x_k}(x) \) for \( n \in \mathbb{N} \) arbitrary to be a periodic function of period \( \frac{1}{n} \). This motivates to consider measures \( \rho_\infty \) which are periodic of period \( \frac{1}{n} \) in \( x \) for some \( n \in \mathbb{N} \), constant in \( y \), and whose support \( \text{supp}(\rho_\infty) \) is not connected, i.e. \( \text{supp}(\rho_\infty) \) consists of \( n \) connected components \( M_k, k = 1, \ldots, n \), with
\[ M_k = M_j + \frac{k - j}{n}, \quad j, k \in \{1, \ldots, n\}. \]
(2.18)
We further assume that \( \rho_\infty \) is symmetric in \( x \) on \( M_k \) for \( k = 1, \ldots, n \). Note that for measures with zero centre of mass, we can assume without loss of generality that \( \rho_\infty(-0.5) = \rho_\infty(0.5) = 0 \).
For $\delta > 0$ and $\rho_\infty \in L^2([-0.5, 0.5]^2)$, we may also consider the regularised energy $\mathcal{E}_\delta$ in (2.8). For the evolution of the energy $\mathcal{E}_\delta$, we obtain

$$\frac{d}{ds} \mathcal{E}_\delta(u(x, y, s))\bigg|_{s=0} = \int_{[-0.5,0.5]^2} (W \ast u + \delta u) \partial_x u \, dx \, dy \bigg|_{s=0}$$

$$= \int_{[-0.5,0.5]^2} \rho_\infty V \partial_x (W \ast \rho_\infty + \delta \rho) \, dx \, dy.$$ 

For any velocity field $V \in C^1([-0.5,0.5])$ which is constant on each connected component of $\text{supp}(\rho_\infty)$ with $v_k \in \mathbb{R}$ such that $V(x) = v_k$ for all $(x, y) \in M_k$ for $k = 1, \ldots, n$, we have

$$\delta \int_{[-0.5,0.5]^2} \rho_\infty V \partial_x \rho_\infty \, dx \, dy = \frac{\delta}{2} \sum_{k=1}^n v_k \int_{M_k} \partial_x \rho_\infty^2 \, dx \, dy = 0$$

and due to the periodicity of $\rho_\infty$ we obtain

$$\int_{[-0.5,0.5]^2} \rho_\infty V \partial_x (W \ast \rho_\infty) \, dx \, dy$$

$$= \sum_{k=1}^n v_k \int_{M_k} \rho_\infty (W' \ast \rho_\infty) \, dx \, dy$$

$$= \sum_{k=1}^n v_k \int_{M_k} \rho_\infty (x, y) \sum_{j=1}^n \int_{M_j} W'(x-w)\rho_\infty(w, z) \, dz \, dw \, dx \, dy.$$ 

Since $W'$ is an odd function, we have

$$\int_{M_k} \rho_\infty (x, y) \int_{M_k} W'(x-w)\rho_\infty(w, z) \, dz \, dw \, dx \, dy = 0$$

and

$$\int_{M_k} \rho_\infty (x, y) \int_{M_{k+j}} W'(x-w)\rho_\infty(w, z) \, dz \, dw \, dx \, dy$$

$$= -\int_{M_k} \rho_\infty (x, y) \int_{M_{k-j}} W'(x-w)\rho_\infty(w, z) \, dz \, dw \, dx \, dy$$

due to the symmetry of $\rho_\infty$ in $x$ on each $M_k$ and the translation property (2.18) of two connected components of $\text{supp}(\rho_\infty)$. Under the above assumptions, this implies that

$$\int_{[-0.5,0.5]^2} \rho_\infty V \partial_x (W \ast \rho_\infty) \, dx \, dy = 0$$

for $n$ odd, while for $n$ even, we have to require in addition that $W'(-0.5) = W'(0.5) = 0$, i.e. $f_i(0.5) = 0$, as before. Note that for general potentials $W$, it is unlikely that minimizers $\rho_\infty$ of $\mathcal{E}_\delta$ for $\delta \geq 0$ exist which do not satisfy the conditions that $\rho_\infty$ is symmetric in $x$ on $M_k$, and that all connected components $M_k$ of $\text{supp}(\rho_\infty)$ are of equal size, equidistant, and given by the translation property (2.18). In particular, this shows that the energy functionals $\mathcal{E}_\delta$ and $\mathcal{E}$ for probability measures defined on the torus $\mathbb{T}^2$ may have multiple local minimisers due to the dependence on $n$. The support of these minimisers may not be connected and may consist of a finite number of connected components of equal size, satisfying the translation property (2.18). Besides, symmetry in $x$ on each connected component $M_k$ is required for minimisers, implying the periodicity of minimisers in $x$. 

EQUILIBRIA OF AN ANISOTROPIC NONLOCAL INTERACTION EQUATION: ANALYSIS AND NUMERICS15
3. Numerical scheme and its convergence

3.1. Numerical methods. For the numerical simulations, we consider the positivity-preserving finite-volume method for nonlinear equations with gradient structure proposed in [17] for isotropic interaction equations (1.11). We consider the domain $\mathbb{R}^2$ and extend the scheme [17] to the anisotropic interaction equations with or without diffusion in (1.10) or (1.1), respectively. This is achieved by replacing $-\nabla W$ by $F(\cdot, T)$, requiring additional care in calculating the term $(F(\cdot, T(x, y)) \ast \rho(t, \cdot))(x, y)$ for $(x, y) \in \mathbb{R}^2$ efficiently.

In two spatial dimensions, we consider a Cartesian grid, given by $x_i = i \Delta x$ and $y_j = j \Delta y$ for $i, j \in \mathbb{Z}$. Let $C_{ij}$ denote the cell of the spatial discretisation $C_{ij} = [x_i, x_{i+1}) \times [y_j, y_{j+1})$, and let the time discretisation be given by $t_n = \sum_{i=0}^{n-1} \Delta t_i$ for $n \in \mathbb{N}$ with time steps $\Delta t_i$. Let $\rho_{ij}^n$ denote the approximation of the solution $\rho(t_n, x_i, y_j)$ to the anisotropic nonlocal interaction equation with diffusion (1.10) with initial condition $\rho(t_0) = \rho^0$ in $\mathbb{R}^2$ for a given probability measure $\rho^0$. Note that (1.10) can be rewritten as

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho u_\rho) = \delta \nabla \cdot (\rho \nabla \rho)$$

where $u_\rho$ is defined in (1.2) with

$$|u_\rho(t, x, y)| \leq f$$

for the uniform bound $f$ of $F$. Assuming that $\rho^0 \in \mathcal{P}_2(\mathbb{R}^2)$ where $\mathcal{P}_2(\mathbb{R}^2)$ denotes the space of probability measures with finite second order moment, we define its discretisation

$$\rho_{ij}^0 = \frac{1}{\Delta x \Delta y} \int_{C_{ij}} \rho^0 d(x, y) \geq 0$$

(3.1)

for $(i, j) \in \mathbb{Z}^2$. Since $\rho^0$ is a probability measure, the total mass of the system is $\sum_{i,j} \rho_{ij}^0 \Delta x \Delta y = 1$ initially. Given an approximating sequence $\{\rho_{ij}^n\}_{i,j}$ at time $t_n$, we consider the scheme

$$\rho_{ij}^{n+1} = \rho_{ij}^n - \frac{\Delta t^n}{\Delta x} \left( (u_x)^{n+1/2, j}_{i+1/2, j} \rho_{i+1/2, j}^n - (u_x)^{n-1/2, j}_{i-1/2, j} \rho_{i-1/2, j}^n \right)$$

$$- \frac{\Delta t^n}{\Delta y} \left( (u_y)^{n+1/2, j}_{i, j+1/2} \rho_{i, j+1/2}^n - (u_y)^{n-1/2, j}_{i, j-1/2} \rho_{i, j-1/2}^n \right)$$

$$+ \frac{\Delta t^n}{2 \Delta x} f \left( \rho_{i+1/2, j}^n - 2 \rho_{i, j}^n + \rho_{i-1/2, j}^n \right)$$

$$+ \frac{\Delta t^n}{2 \Delta y} f \left( \rho_{i, j+1/2}^n - 2 \rho_{i, j}^n + \rho_{i, j-1/2}^n \right)$$

$$+ \frac{\delta \Delta t^n}{2(\Delta x)^2} \left( (\rho_{i+1, j}^n)^2 - 2(\rho_{i, j}^n)^2 + (\rho_{i-1, j}^n)^2 \right) + \frac{\delta \Delta t^n}{2(\Delta y)^2} \left( (\rho_{i, j+1}^n)^2 - 2(\rho_{i, j}^n)^2 + (\rho_{i, j-1}^n)^2 \right)$$

(3.2)

for the uniform bound $f$ of the force $F$ and parameter $\delta > 0$. Here, we use the notation

$$\rho_{i+1/2, j} = \frac{\rho_{ij} + \rho_{i+1, j}}{2}, \quad \rho_{i, j+1/2} = \frac{\rho_{ij} + \rho_{i, j+1}}{2},$$

$$(u_x)^{i+1/2, j}_{i+1/2, j} = \frac{(u_x)^{i+1/2, j}_{i+1, j} + (u_x)^{i+1/2, j}_{i, j+1}}{2}, \quad (u_y)^{i+1/2, j}_{i+1/2, j} = \frac{(u_y)^{i+1/2, j}_{i+1, j} + (u_y)^{i+1/2, j}_{i, j+1}}{2},$$

where the macroscopic velocity is defined by

$$(u_x)^{i, j} = \frac{1}{\Delta x \Delta y} \sum_{k,l} \rho_{kl}(F_x)^{kl}_{ij}, \quad (u_y)^{i, j} = \frac{1}{\Delta x \Delta y} \sum_{k,l} \rho_{kl}(F_y)^{kl}_{ij}$$

(3.3)
with
\[
(F_x)^{kl}_{ij} = \int_{C_{kl}} \left( \iint_{C_{ij}} F_x(x - x', y - y', T(x, y)) \, d(x, y) \right) \, d(x', y'),
\]
and
\[
(F_y)^{kl}_{ij} = \int_{C_{kl}} \left( \iint_{C_{ij}} F_y(x - x', y - y', T(x, y)) \, d(x, y) \right) \, d(x', y')
\]
for the components \(F_x, F_y\) of \(F\) with \(F = (F_x, F_y)\). A change of variable also yields
\[
(u_x)^{i+1/2}_{j} = \frac{1}{\Delta x \Delta y} \sum_{k,l} \rho_{k+l+1/2}(F_x)^{kl}_{ij}, \quad (u_y)^{i+1/2}_{j} = \frac{1}{\Delta x \Delta y} \sum_{k,l} \rho_{k+l+1/2}(F_y)^{kl}_{ij}.
\]

Note that \((F_x)^{kl}_{ij}\) and \((F_y)^{kl}_{ij}\) can be determined explicitly in the numerical simulations instead of evaluating the integrals, and can also be precomputed for making the computation of the discretised velocity fields more efficient. Further note that the last line of the numerical scheme (3.2) can be regarded as a discretisation of the nonlinear diffusion \(\delta \nabla \cdot (\rho \nabla \rho) = \frac{\delta}{2} (\partial_x^2 \rho^2 + \partial_y^2 \rho^2)\).

### 3.2. Properties of the scheme: conservation of mass, positivity, convergence.

In [27], the convergence of a finite volume method is shown for general measure solutions of the (isotropic) aggregation equation with mildly singular potentials. In this section, we establish a CFL condition for the numerical scheme (3.2) for the anisotropic aggregation equation (1.10) and prove its weak convergence.

**Lemma 3.1** Let \(\rho^n \in \mathcal{P}_2(\mathbb{R}^2)\) and define \(\rho^0_{ij}\) by (3.1). The conservation of mass is satisfied for all \(n\), i.e.
\[
\sum_{i,j \in \mathbb{Z}} \rho^n_{ij} \Delta x \Delta y = \sum_{i,j \in \mathbb{Z}} \rho^0_{ij} \Delta x \Delta y = 1.
\]
For spatially homogeneous tensor fields, conservation of the centre of mass also holds, i.e.
\[
\sum_{i,j \in \mathbb{Z}} x_i \rho^n_{ij} = \sum_{i,j \in \mathbb{Z}} x_i \rho^0_{ij}, \quad \sum_{i,j \in \mathbb{Z}} y_i \rho^n_{ij} = \sum_{i,j \in \mathbb{Z}} y_i \rho^0_{ij}.
\]

**Proof.** The conservation of mass is directly obtained by summing over \(i\) and \(j\) in (3.2), and noting that \(\sum_{i,j \in \mathbb{Z}} \rho^n_{ij} \Delta x \Delta y = 1\). The conservation of the centre of mass follows from a discrete integration by parts and the fact that \((F_x)^{kl}_{ij} = -(F_x)^{ij}_{kl}\) for spatially homogeneous tensor fields. \(\square\)

For proving the convergence of the numerical scheme, a CFL condition is required:

**Lemma 3.2** Let \(\rho^n \in \mathcal{P}_2(\mathbb{R}^2)\) and define \(\rho^0_{ij}\) by (3.1). Suppose that the force \(F\) is bounded by \(f\) and, given the spatial discretisation \(\Delta x, \Delta y\), assume that the \(n\)th time step \(\Delta t^n\) satisfies
\[
\left( 2f \left( \frac{1}{\Delta x} + \frac{1}{\Delta y} \right) + \delta r_n \left( \frac{1}{(\Delta x)^2} + \frac{1}{(\Delta y)^2} \right) \right) \Delta t^n \leq 1 \tag{3.4}
\]
where
\[
r_n = \sup_{ij} \rho^n_{ij}.
\]
Then the sequences defined in (3.2)–(3.3) satisfy
\[
\rho^n_{ij} \geq 0, \quad |(u_x)^n_{ij}| \leq f, \quad |(u_y)^n_{ij}| \leq f.
\]
for all $i$, $j$ and $n$. In particular, there exists a constant $r > 0$ such that

$$\sup_{n,i,j} \rho^0_{ij} \leq r. \quad (3.5)$$

**Proof.** By the definition of the velocity (3.3) and the uniform bound $f$ of the force $F$ we obtain

$$|(u_x)_{ij}^n| \leq \Delta x \Delta y f \sum_{k,l} \rho^0_{kl} = f, \quad |(u_y)_{ij}^n| \leq f \quad (3.6)$$

for all $i, j, n$.

For proving the nonnegativity of the scheme (3.2), note that we can rewrite (3.2) as

$$\rho_{ij}^{n+1} = \rho_{ij}^n \left(1 - \frac{\Delta t}{\Delta x} \left(\frac{(u_x)_{j+1/2}^n - (u_x)_{j-1/2}^n + 2f}{2}\right) - \frac{\Delta t}{\Delta y} \left(\frac{(u_y)_{i+1/2}^n - (u_y)_{i-1/2}^n + 2f}{2}\right) - \delta \frac{\Delta t}{\Delta x} (\rho_{ij}^n - \delta \Delta t^2 \rho_{ij}^n)ight) + \rho_{i+1,j}^n \frac{\Delta t}{\Delta x} \left(f - (u_x)_{i+1/2}^n\right) + \rho_{i-1,j}^n \frac{\Delta t}{\Delta x} \left(f + (u_x)_{i-1/2}^n\right) + \rho_{i,j+1}^n \frac{\Delta t}{\Delta y} \left(f - (u_y)_{i+1/2}^n\right) + \rho_{i,j-1}^n \frac{\Delta t}{\Delta y} \left(f + (u_y)_{i-1/2}^n\right) + \delta \frac{\Delta t}{\Delta x} \left((\rho_{i+1,j}^n)^2 + (\rho_{i-1,j}^n)^2\right) + \delta \frac{\Delta t}{\Delta y} \left((\rho_{i,j+1}^n)^2 + (\rho_{i,j-1}^n)^2\right). \quad (3.7)$$

We show the nonnegativity of $\rho_{ij}^n$ by induction on $n$. For $n \in \mathbb{N}$ given, we assume that $\rho_{ij}^n \geq 0$ for all $i, j \in \mathbb{Z}$. Note that due to condition (3.4), all coefficients in (3.7) of $\rho_{ij}^0$, $\rho_{i+1,j}^n$, $\rho_{i-1,j}^n$, $\rho_{i,j+1}^n$ and $\rho_{i,j-1}^n$ are nonnegative, and the terms in the last line are also nonnegative. By induction, we deduce $\rho_{ij}^{n+1} \geq 0$ for all $i, j \in \mathbb{Z}$.

Since $\rho_{ij}^0 \geq 0$, the conservation of mass implies the uniform boundedness of $\rho_{ij}^n$, i.e. there exists a constant $r > 0$ such that (3.5) is satisfied.

Next, we consider the convergence of the scheme in a weak topology. Let $\mathcal{M}_{loc}(\mathbb{R}^d)$ denote the space of local Borel measures on $\mathbb{R}^d$. For $\rho \in \mathcal{M}_{loc}(\mathbb{R}^d)$, we denote the total variation of $\rho$ by $|\rho|(\mathbb{R}^d)$ and we denote the space of measures in $\mathcal{M}_{loc}(\mathbb{R}^d)$ with finite total variation by $\mathcal{M}_b(\mathbb{R}^d)$. The space of measures $\mathcal{M}_b(\mathbb{R}^d)$ is always endowed with the weak topology $\sigma(\mathcal{M}_b, C_0)$.

Let the characteristic function on some set $[t_n, t_{n+1}] \times C_{ij} \subset \mathbb{R}^d \times \mathbb{R}^d$ be denoted by $\chi_{[t_n, t_{n+1}] \times C_{ij}}$. For $\Delta = \max\{\Delta x, \Delta y\}$, we define the reconstruction of the discretisation by

$$\rho_\Delta(t, x, y) = \sum_{n \in \mathbb{N}} \sum_{i \in \mathbb{Z}} \sum_{j \in \mathbb{Z}} \rho_{ij}^n \chi_{[t_n, t_{n+1}] \times C_{ij}}(t, x, y),$$

where the boundedness of $\rho_\Delta$ independent of $\Delta$ follows from Lemma 3.2. Using the definition $u_{ij}^n = ((u_x)_{ij}^n, (u_y)_{ij}^n)$ in (3.3), we obtain

$$u_{ij}^n = \frac{1}{\Delta x \Delta y} \int_{C_{ij}} F(\cdot, T(x, y)) \ast \rho_\Delta(t_n, \cdot)(x, y) d(x, y)$$

and

$$u_\Delta(t, x, y) = \sum_{n \in \mathbb{N}} \sum_{i \in \mathbb{Z}} \sum_{j \in \mathbb{Z}} u_{ij}^n \chi_{[t_n, t_{n+1}] \times C_{ij}}(t, x, y).$$

**Theorem 3.3** Suppose that the continuous force $F$ is bounded by $f$ and that the tensor field $T$ is continuous. We consider $\rho^n \in \mathcal{P}_2(\mathbb{R}^2)$ and define $\rho_{ij}^0$ by (3.1). Let $S > 0$ be fixed, and suppose that the discretisation in time and space satisfies (3.4) where $\rho_{ij}^n$ are obtained from (3.2) for $\rho_{ij}^0$ given. Then, the discretisation $\rho_\Delta$ converges weakly in $\mathcal{M}_b([0, S] \times \mathbb{R}^2)$ towards the solution $\rho$ of
(1.1) as $\Delta = \max\{\Delta x, \Delta y\}$ and $\delta$ go to 0 where for each $\Delta$, the sequence of time steps $\{\Delta t^n\}$ satisfies (3.4).

**Proof.** Lemma 3.1 implies the nonnegativity of $\rho_{ij}^n$ provided condition (3.4) holds. By the conservation of mass, we have that the sequence $\{\rho_{\Delta}\}_{\Delta > 0}$ of nonnegative bounded measures satisfies $|\rho_{\Delta}(t)|(\mathbb{R}^2) = 1$ for all $t \in [0, S]$. Hence, there exists a subsequence, still denoted by $\{\rho_{\Delta}\}_{\Delta > 0}$, which converges to $\rho$ in the weak topology as $\Delta = \{\Delta x, \Delta y\}$ goes to 0 where for each $\Delta$, the sequence of time steps $\{\Delta t^n\}$ satisfies (3.4). Hence,

$$
\int_{0}^{S} \int_{\mathbb{R}^2} \phi(t, x, y) \rho_{\Delta}(t, x, y) \, d(x, y) \, dt \to \int_{0}^{S} \int_{\mathbb{R}^2} \phi(t, x, y) \rho(t, x, y) \, d(x, y) \, dt
$$

for all $\phi \in C_0([0, S] \times \mathbb{R}^2)$.

Let $\Delta x, \Delta y$ and $\Delta = \max\{\Delta x, \Delta y\}$ be given. For $S > 0$ given, note that $N_S \in \mathbb{N}_{>0}$ and $\Delta t^{N_S-1} > 0$ can be chosen such that $S = t_{N_S} = \sum_{n=0}^{N_S-1} \Delta t^n$ and condition (3.4) are satisfied. We set $\Delta t = \min_n \Delta t^n$ and choose $N$ such that $S = N \Delta t$. Let $\mathcal{D}([0, S] \times \mathbb{R}^2)$ denote the space of smooth, compactly supported test functions on $[0, S] \times \mathbb{R}^2$ and for $s_n = n \Delta t$ consider

$$
\phi_{ij}^n = \int_{s_n}^{s_n+1} \int_{C_{ij}} \phi(t, x, y) \, d(x, y) \, dt.
$$

Note that $\rho_{\Delta}(s_{n+1}, x_i, y_j) - \rho_{\Delta}(s_n, x_i, y_j) \in \{0, \rho_{ij}^{\sigma(n)+1} - \rho_{ij}^{\sigma(n)}\}$ for $\sigma(n) \in \{0, \ldots, N_S - 1\}$. Here, $\sigma(n)$ is an increasing function defined iteratively with $\sigma(0) = 0$ and $\sigma(n + 1) = \sigma(n)$ if $\rho_{\Delta}(s_{n+1}, x_i, y_j) = \rho_{\Delta}(s_n, x_i, y_j)$ and $\sigma(n + 1) = \sigma(n) + 1$ if $\rho_{\Delta}(s_{n+1}, x_i, y_j) = \rho_{\Delta}(s_n, x_i, y_j) = \rho_{ij}^{\sigma(n)+1} - \rho_{ij}^{\sigma(n)}$. We define $\tilde{\sigma}_{ij}^{n+1} = \phi_{ij}^n$ if $\rho_{\Delta}(s_{n+1}, x_i, y_j) = \rho_{\Delta}(s_n, x_i, y_j) = \rho_{ij}^{\sigma(n)+1} - \rho_{ij}^{\sigma(n)}$. In particular, we have

$$
\frac{1}{\Delta t} \sum_{n=0}^{N_S-1} \sum_{i,j \in \mathbb{Z}} \left( \rho_{\Delta}(s_{n+1}, x_i, y_j) - \rho_{\Delta}(s_n, x_i, y_j) \right) \tilde{\phi}_{ij}^n = \frac{1}{\Delta t} \sum_{n=0}^{N_S-1} \sum_{i,j \in \mathbb{Z}} \left( \rho_{ij}^{\sigma(n)+1} - \rho_{ij}^{\sigma(n)} \right) \tilde{\phi}_{ij}^n
$$

$$
= - \sum_{n=0}^{N_S-1} \sum_{i,j \in \mathbb{Z}} \rho_{ij}^{\sigma(n)} \tilde{\phi}_{ij}^{n+1} = - \sum_{n=0}^{N_S-1} \sum_{i,j \in \mathbb{Z}} \rho_{ij}^{\sigma(n)} \tilde{\phi}_{ij}^{n} - \sum_{n=0}^{N_S-1} \sum_{i,j \in \mathbb{Z}} \rho_{ij}^{\sigma(n)} \phi_{ij}^{n+1} - \phi_{ij}^{n} \Delta t
$$

$$
= - \sum_{n=0}^{N_S-1} \int_{s_n}^{s_n+1} \int_{\mathbb{R}^2} \rho_{\Delta}(t, x, y) \phi(t, x, y) - \phi(t - \Delta t, x, y) \frac{\Delta t}{\Delta t} \, d(x, y) \, dt
$$

$$
\to - \int_{0}^{S} \int_{\mathbb{R}^2} \rho(t, x, y) \partial_t \phi(t, x, y) \, d(x, y) \, dt
$$

as $\Delta$ goes to 0, where the limit integral follows from $\phi(t, x, y) - \phi(t - \Delta t, x, y) = \partial_t \phi(t, x, y) \Delta t + \mathcal{O}((\Delta t)^2)$, the weak convergence of $\rho_{\Delta}$ to $\rho$ and the boundedness of the measure $\rho_{\Delta}$ with a bound not depending on the mesh. Note that for

$$
\tilde{\phi}_{ij}^n = \int_{t_n}^{t_{n+1}} \int_{C_{ij}} \phi(t, x, y) \, d(x, y) \, dt
$$
we have
\[
\sum_{n=0}^{N_S-1} \sum_{i,j} \frac{1}{2\Delta x} (\rho_\Delta(t_n, x_{i+1,j+1}) - 2\rho_\Delta(t_n, x_{i,j}) + \rho_\Delta(t_n, x_{i-1,j+1})) \phi^n_{i,j}
\]
\[
= \int_0^S \int \frac{\rho_\Delta(t, x) + 2\phi(t, x, y) + \phi(t, x - \Delta x, y)}{2\Delta x} \text{d}(x, y) dt \to 0
\]
as \Delta \to 0 since \(|\phi(t, x + \Delta x, y) - 2\phi(t, x, y) + \phi(t, x - \Delta x, y)| \leq \|\partial_{xx}\phi\|_\infty (\Delta x)^2\). Due to the boundedness of the force \(F(\cdot, T(x))\), we can show in a similar way as in [27] that
\[
\sum_{n=0}^{N_S-1} \sum_{i,j} \frac{1}{\Delta x} \left( (u_{in})_{i+1/2,j}^n - (u_{in})_{i-1/2,j}^n \right) \phi^n_{i,j}
\]
\[
\to - \int_0^S \int \partial_x \phi(t, x, y) (F_x(\cdot, T(x, y)) \ast \rho(t, \cdot))(x, y) \rho(t, x, y) \text{d}(x, y) dt
\]
as \Delta \to 0 by the continuity of \(F = (F_x, F_y)\) and \(T\) where \(F_x\) denotes the first component of the force \(F\). Further note that we have
\[
\delta \sum_{n=0}^{N_S-1} \sum_{i,j} \frac{1}{2(\Delta x)^2} \left( (\rho_{i+1,j}^n)^2 - 2(\rho_{ij}^n)^2 + (\rho_{i-1,j}^n)^2 \right) \phi^n_{i,j}
\]
\[
= \delta \sum_{n=0}^{N_S-1} \sum_{i,j} \frac{1}{2(\Delta x)^2} \left( \rho_{ij}^n \phi_{i+1,j}^n - 2\phi_{ij}^n + \phi_{i-1,j}^n \phi_{ij}^n \right)
\]
\[
\leq \frac{1}{2} \delta \|\partial_{xx}\phi\|_\infty \int_0^S \int \left( \rho_\Delta(t, x, y) \right)^2 \text{d}(x, y) dt.
\]
The boundedness of \(\rho_\Delta\), independent of \(\Delta\), guarantees that the right-hand side goes to 0 as \(\Delta\) goes to 0.

Multiplying (3.2) by \(\phi_{ij}^n\), summing over \(n, i, j\), and taking the limits \(\delta\) and \(\Delta\) to 0, we obtain
\[
\int_0^S \int \left[ \partial_t \phi(t, x, y) + \nabla \phi(t, x, y) \cdot (F(\cdot, T(x, y)) \ast \rho(t, \cdot))(x, y) \right] \rho(t, x, y) \text{d}(x, y) dt = 0
\]
in the limit, i.e. \(\rho\) is a solution in the sense of distributions of the anisotropic aggregation equation (1.1).

4. Numerical results

In this section, we show simulation results for solving the anisotropic aggregation equation with nonlinear diffusion (1.10) numerically using the numerical scheme (3.2). For the numerical simulations, we consider the force coefficients \(f_s\) and \(f_l\) in (1.8) with \(f_s = f_R + \chi f_A\) and \(f_l = f_R + f_A\) as suggested in [31], where \(f_R\) and \(f_A\) are defined in (1.5) and (1.6). To be consistent with the work of Kücken and Champod [43], we assume that the total force (1.8) defined via the tensor field \(T(x, y) := \chi s(x, y) \otimes s(x, y) + l(x, y) \otimes l(x, y)\) in (1.3) exhibits short-range repulsion and long-range attraction along \(l\) and repulsion along \(s\). In the following, we consider the force coefficients \(f_R\) and \(f_A\) with the parameter values in (1.7). The computational domain is given by \([-0.5, 0.5]^2\) with periodic boundary conditions.
4.1. Spatially homogeneous tensor fields. In this section, we show stationary solutions to the anisotropic interaction equation (1.10), obtained with the numerical scheme (3.2), for the spatially homogeneous tensor field $T$ with $s = (0, 1)$ and $l = (1, 0)$, cf. Figures 1–4. Note that the stationary solutions for the tensor field $T$ are constant in $y$-direction in all these figures.

The stationary solution to (1.10), obtained with the numerical scheme (3.2) for different values of the diffusion coefficient $\delta$, is shown in Figure 1. Here, we consider uniformly distributed initial data on a disc of radius $R = 0.05$ with centre $(0, 0)$ on the computational domain $[-0.5, 0.5]^2$, where the spatial discretisation is given by a grid of size 50 in each spatial direction, and the time step is chosen according to the CFL condition (3.4). Due to the choice of initial data, this leads to a single straight vertical line as stationary solution, provided $\delta$ is chosen sufficiently small. As expected, an increase in $\delta$ leads to the widening of the single straight vertical line which is stable for sufficiently small values of $\delta$. For larger values of $\delta$, e.g. $\delta = 5 \times 10^{-7}$, the uniform distribution is obtained as stationary solution.

![Figure 1](image)

*Figure 1. Stationary solution to the anisotropic interaction equation (1.10), obtained with the numerical scheme (3.2) on a grid of size 50 in each spatial direction and different diffusion coefficients $\delta$ for the spatially homogeneous tensor field with $s = (0, 1)$ and $l = (1, 0)$ and uniformly distributed initial data on a disc on the computational domain $[-0.5, 0.5]^2$."

In Figure 2, we investigate the role of the grid size on the stationary solution by considering grid sizes of 50, 100 and 200 in each spatial direction for the diffusion parameter $\delta = 10^{-10}$ and uniformly distributed initial data on a disc. Clearly, the stationary solution is given by a step function in the $x$-coordinate. Finer grids lead to step functions with more steps and smaller step heights compared to the grid size of 50 where only one step occurs.

![Figure 2](image)

*Figure 2. Stationary solution to the anisotropic interaction equation (1.10), obtained with the numerical scheme (3.2) on grids of sizes 50, 100 and 200 in each spatial direction for the diffusion coefficient $\delta = 10^{-10}$ for the spatially homogeneous tensor field with $s = (0, 1)$ and $l = (1, 0)$ and uniformly distributed initial data on a disc on the computational domain $[-0.5, 0.5]^2$."

The stationary solution for grid sizes of 100 and 200 in each spatial direction and uniformly distributed initial data on the computational domain $[-0.5, 0.5]^2$ is shown in Figure 3, and is given by equidistant, parallel vertical line patterns. Note that we obtain the same number of parallel lines for the different grid sizes.

![Figure 3](image1.png)

**Figure 3.** Stationary solution to the anisotropic interaction equation (1.10), obtained with the numerical scheme (3.2) on grids of sizes 100 and 200 in each spatial direction for the diffusion coefficient $\delta = 10^{-10}$ for the spatially homogeneous tensor field with $s = (0, 1) \text{ and } l = (1, 0)$ and uniformly distributed initial data on the computational domain $[-0.5, 0.5]^2$.

In Figure 4, we show the stationary solution for uniformly distributed initial data on the computational domain $[-0.5, 0.5]^2$ for different diffusion coefficients $\delta$. Note that as $\delta$ increases, the stable line patterns become wider and this may result in a decrease in the number of parallel lines. If $\delta$ is larger than a certain threshold, e.g. $\delta = 5 \cdot 10^{-9}$, the parallel line patterns are no longer stable and the stationary solution is given by the uniform distribution on the computational domain $[-0.5, 0.5]^2$. The plot of the cross-section of the stationary solution for diffusion coefficient $\delta = 10^{-9}$ is shown in Figure 5. Note that the solution is finite and has no blow-up.

![Figure 4](image2.png)

**Figure 4.** Stationary solution to the anisotropic interaction equation (1.10), obtained with the numerical scheme (3.2) on a grid of size 200 in each spatial direction and different diffusion coefficients $\delta$ for the spatially homogeneous tensor field with $s = (0, 1) \text{ and } l = (1, 0)$ and uniformly distributed initial data on the computational domain $[-0.5, 0.5]^2$. 
4.2. Spatially inhomogeneous tensor fields. In this section, we consider stationary solutions to the anisotropic interaction equation (1.10), obtained with the numerical scheme (3.2), for different spatially inhomogeneous tensor fields.

In Figure 6, we consider fingerprint images in Figures 6(A) and 6(D), use these fingerprint images to construct the vector field $s = s(x, y)$ in Figures 6(B) and 6(E), and show the resulting stationary solutions for the diffusion coefficient $\delta = 10^{-10}$ and uniformly distributed initial data on a grid of size 50 in each spatial direction in Figures 6(C) and 6(F), respectively. For the construction of the tensor field we firstly proceed as in [31], and then we rescale the tensor field appropriately to the given grid size.

In Figure 7, we consider the tensor field in Figure 6(B) of part of a fingerprint, and show the numerical solution at different iterations of the numerical scheme (3.2) on a grid of size 50 in each spatial direction for the diffusion coefficient $\delta = 10^{-10}$ and uniformly distributed initial data on the computational domain $[-0.5, 0.5]^2$. Note that the resulting numerical solution is close to being stationary.

Similarly as in Figure 4 for spatially homogeneous tensor fields, we show the stationary solution for different diffusion coefficients $\delta$ in Figure 8. For the numerical results in Figure 8, the spatially inhomogeneous tensor field in Figure 6(B) and a grid of size 50 in each spatial direction are considered. As $\delta$ increases, the line patterns become wider, provided the diffusion coefficient $\delta$ is below a certain threshold. If $\delta > 0$ is above this threshold, e.g. for $\delta = 10^{-9}$, the uniform distribution is obtained as stationary solution. Note that this threshold is smaller than the one in Figure 4 for spatially homogeneous tensor fields.

Motivated by the simulation results in [31], we consider different rescalings of the forces in Figure 9 to vary the distances between the fingerprint lines, i.e. we consider $F(\eta d(x, y), T(x))$ where $\eta > 0$ is the rescaling factor. As before, we consider the diffusion coefficient $\delta = 10^{-10}$ on a grid of size 50 in each spatial direction and uniformly distributed initial data on $[-0.5, 0.5]^2$. For $\eta = 1$ we recover the same stationary solution as in Figure 6(C), while the distances between the fingerprint lines become larger for $\eta \in (0, 1)$ and smaller for $\eta > 1$. Note that the resulting patterns for the mean-field model (1.10) are better for $\eta > 1$ than for the associated particle model, see [31, Figure 24], since only dotted lines are possible for particle simulations with $N = 2400$ and higher particle numbers result in very long simulation times.

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Figure 6. Stationary solution to the anisotropic interaction equation (1.10), obtained with the numerical scheme (3.2) on a grid of size 50 in each spatial direction and diffusion coefficient $\delta = 10^{-10}$ for different spatially inhomogeneous tensor fields from real fingerprint images and uniformly distributed initial data on the computational domain $[-0.5, 0.5]^2$.

Figure 7. Numerical solution to the anisotropic interaction equation (1.10) after $n$ iterations, obtained with the numerical scheme (3.2) on a grid of size 50 in each spatial direction with diffusion coefficient $\delta = 10^{-10}$ for the spatially inhomogeneous tensor field of part of a fingerprint and uniformly distributed initial data on the computational domain $[-0.5, 0.5]^2$.
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Figure 8. Stationary solution to the anisotropic interaction equation (1.10), obtained with the numerical scheme (3.2) on a grid of size 50 in each spatial direction for different values of the diffusion coefficient $\delta$ for a given spatially inhomogeneous tensor field and uniformly distributed initial data on the computational domain $[-0.5, 0.5]^2$.

Figure 9. Stationary solution to the anisotropic interaction equation (1.10), obtained with the numerical scheme (3.2) on a grid of size 50 in each spatial direction, diffusion coefficient $\delta = 10^{-10}$ and different force rescalings $\eta$ for a given spatially inhomogeneous tensor field and uniformly distributed initial data on the computational domain $[-0.5, 0.5]^2$.

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