Small rational curves on the moduli space of stable bundles

Liu Min
School of Mathematical Sciences, Fudan University, Shanghai 200433, P.R.China
E-mail: liumin@amss.ac.cn

Abstract

For a smooth projective curve $C$ with genus $g \geq 2$ and a degree 1 line bundle $L$ on $C$, let $M := SU_C(r, L)$ be the moduli space of stable vector bundles of rank $r$ over $C$ with the fixed determinant $L$. In this paper, we study the small rational curves on $M$ and estimate the codimension of the locus of the small rational curves. In particular, we determine all small rational curves when $r = 3$.

Keywords: Small rational curves; Moduli space of vector bundles over a curve.
Mathematics Subject Classification 2000: 14D20, 14H60

1 Introduction

Let $C$ be a smooth projective curve of genus $g \geq 2$ and $L$ a line bundle on $C$ of degree $d$. Let $M := SU_C(r, L)$ be the moduli space of stable vector bundles of rank $r$ and with the fixed determinant $L$. Throughout this paper, we assume that $d = 1$. In this case, $M$ is a smooth projective Fano variety with $Pic(M) = \mathbb{Z} \cdot \Theta$ and $-K_M = 2\Theta$, where $\Theta$ is an ample divisor ([8], [2]). So, as a Fano variety, the index of $M$ is 2.

For any smooth Fano variety $X$, a rational curve $l \subset X$ is called a line on $X$ if the index of $X$ equals $-K_X \cdot l$. Also we say that $l$ has degree $k$ if $-K_X \cdot l$ equals $k$ times the index of $X$. For the moduli space $M$, so defined degree of a rational curve $l \subset M$ equals $\deg(\Theta|_l)$. So, for any rational curve $\phi : \mathbb{P}^1 \rightarrow M$, we can define its degree $\deg(\phi^*\Theta)$ with respect to the ample generator of $Pic(M)$. It’s an open problem if every smooth Fano variety with picard number 1 has a line (see Problem V1.13 of [4]).

Ramanan [8] found a family of lines on $M$, but these lines lie in a proper closed subset. It has been observed that $M$ is covered by rational curves of degree $r$, which are called Hecke curves (Corollary 5.16 of [7]). Sun proved that there are no lines on $M$ except those found by Ramanan (Theorem 2 of [9]), and he also proved that rational curves of minimal degree passing through a general point of $M$ are Hecke curves (Theorem 1 of [9]), which answers a question of Jun-Muk Hwang (Question 1 of [3]) and also implies that all the rational curves of $\Theta$—degree smaller than $r$, called small rational curves ([5]), must lie in a proper closed subset. It has been proved that the locus of small rational curves consists of $r - 1$ irreducible components ([1]).

The aim of this paper is to estimate the codimension of the locus of small rational curves and to determine the small rational curves when $r = 3$. In particular, we can

\[\text{Supported in part by the National Natural Science Foundation of China (Grant No. 10871106)}\]
see that each small rational curve is defined by a vector bundle on $C \times \mathbb{P}^1$ of certain fixed extension types. We also construct all small rational curves on $M$ when $r = 3$, which will be useful to study the Chow-group of the moduli space $M$.

This paper is organized as follows. In section 2, using the degree formula of rational curves due to Sun (3), we show that, for any small rational curve, there exist a sequence fixed semi-stable bundles $V_1, \cdots, V_{n-1}$ of degree 0 and a fixed stable bundle $V_n$ of degree 1 on $C$ such that the bundles corresponding to points of the small rational curve are obtained by extensions of $V_1, \cdots, V_{n-1}$ and $V_n$. And we also prove that the locus of small rational curves is a closed subvariety of codimension at least $\min_{0 < r_1 < \cdots < r_n = r} \{\sum_{i=2}^{n} r_{i-1}(r_i - r_{i-1})(g - 2) + r_{n-1}(r_n - r_{n-1} - 1)\}$, where $0 < r_1 < \cdots < r_n = r$ runs over all positive integers $r_i$ satisfying $\sum_{i=1}^{n-1} r_i(\alpha_i - \alpha_{i+1}) < r$ for some $n$ and some integers $\alpha_1 > \cdots > \alpha_n$.

In section 3, we determine all small rational curves on $M = SU_C(3, \mathcal{L})$. Let $J_C$ be the Jacobian of $C$, and $U_C(2, 1)$ the moduli space of stable bundles on $C$ of rank 2 and degree 1. Let $\mathcal{R} \subset J_C \times U_C(2, 1)$ be the closed subvariety consisting of $([\xi], [V_2])$ satisfying $\xi \otimes \det(V_2) \cong \mathcal{L}$. We construct a projective bundle $q : \mathcal{P} \to \mathcal{R}$ such that, for any $([\xi], [V_2]) \in \mathcal{R}$, $q^{-1}(([\xi], [V_2])) \cong \mathbb{P}\text{Ext}^1(V_2, \xi)$. Let $SU_C(2, 0)$ be the open subset of stable bundles in $U_C(2, 0)$, and $J_C$ the moduli space of degree 1 line bundles on $C$. Let $\mathcal{R}' \subset SU_C(2, 0) \times J_C$ be the closed subvariety consisting of $([V_1], [\xi'])$ satisfying $\det(V_1) \otimes \xi' \cong \mathcal{L}$. By using Hecke transformation, we construct a scheme $\mathbb{P}(\mathcal{V}^*)$ parametrizing a family of stable bundles of rank 2 and degree 0, which maps onto $SU_C(2, 0)$. Let $T = (\mathbb{P}(\mathcal{V}^*)^* \times J_C) \times_{SU_C(2, 0) \times J_C} \mathcal{R}'$. Then there is a surjective morphism $\theta : T \to \mathcal{R}'$. We also construct a projective bundle $q' : \mathcal{P}' \to T$ such that, for any $([V_1], [\xi']) \in \mathcal{R}'$ and any $t \in \theta^{-1}([V_1], [\xi'])$, $q'^{-1}(t) \cong \mathbb{P}\text{Ext}^1(\xi', V_1)$. We have:

**Theorem 1.1.** There are morphisms $\Phi : \mathcal{P} \to M$ and $\Psi : \mathcal{P}' \to M$ such that any rational curve obtained by the following ways are small rational curves and any small rational curve $\phi : \mathbb{P}^1 \to M$ can be obtained by one of the following ways:

(i) it’s the image (under $\Phi$) of a rational curve of degree 2 in $\mathcal{P}$ in the fiber of $q$;

(ii) it’s the image (under $\Phi$) of a double cover of a line in $\mathcal{P}$ in the fiber of $q$;

(iii) it’s the image (under $\Phi$) of a line in $\mathcal{P}$ which is not in a fiber of $q$ and maps to a line in $U_C(2, \mathcal{L}')$ of degree 1;

(iv) it’s the image (under $\Psi$) of a line in $\mathcal{P}'$ in the fiber of $q'$.

## 2 The Locus of Small Rational Curves on Moduli Space

In [9], it was shown that any rational curve $\phi : \mathbb{P}^1 \to M$ is defined by a vector bundle $E$ on $C \times \mathbb{P}^1$ and a degree formula was discovered, which we call Sun’s degree formula. To recall it, let $X = C \times \mathbb{P}^1$, $f : X \to C$ and $\pi : X \to \mathbb{P}^1$ be the projections. On a general fiber $f^{-1}(\xi) = X_\xi$, $E$ has the form

$$E \mid_{X_\xi} = \bigoplus_{i=1}^{n} \mathcal{O}_{X_\xi}(\alpha_i)_{i}^{\oplus r_i}, \quad \alpha_1 > \cdots > \alpha_n.$$
The $n$-tuple $\alpha = (\alpha_1^{r_1}, \ldots, \alpha_n^{r_n})$ is called the generic splitting type of $E$. Tensoring $E$ by $\pi^* O_{\mathbb{P}^1}(-\alpha_i)$, we can assume without loss of generality that $\alpha_n = 0$. Any such $E$ admits a relative Harder-Narasimhan filtration

$$0 = E_0 \subset E_1 \subset \cdots \subset E_n = E$$

in which the quotient sheaves $F_i = E_i/E_{i-1}$ are torsion-free with generic splitting type $(\alpha_i^{r_i})$ respectively. Let $F_i' = F_i \otimes \pi^* O_{\mathbb{P}^1}(-\alpha_i)$, thus they have generic splitting type $(0^{r_i})$ respectively. Without risk of confusion, we denote the degree of $F_i$ (resp. $E_i$) on the general fiber of $\pi$ by $\deg(F_i)$ (resp. $\deg(E_i)$). Accordingly, $\mu(E_i)$ (resp. $\mu(E)$) denotes the slope of restriction of $E_i$ (resp. $E$) to the general fiber of $\pi$ respectively. Let $\text{rk}(E_i)$ denote the rank of $E_i$. Then we have the following so called Sun’s degree formula (see the formula (2.2) in [9])

$$\deg(\varphi^*(\Theta)) = r \sum_{i=1}^{n} c_2(F_i') + \sum_{i=1}^{n-1} (\text{drk}(E_i) - r\deg(E_i))(\alpha_i - \alpha_{i+1}). \quad (1)$$

When $\varphi : \mathbb{P}^1 \to M$ is a small rational curve, $\deg(\varphi^*(\Theta)) < r$, we have $n \geq 2$, $c_2(F_i') = 0$ and $F_i' = f^* V_i$ for some locally free sheaf $V_i$ on $C$ by Lemma 2.2 in [9]. So $F_i = f^* V_i \otimes \pi^* O_{\mathbb{P}^1}(\alpha_i)$ and $E$ can be obtained by a sequence of extensions

$$0 \to f^* V_1 \otimes \pi^* O_{\mathbb{P}^1}(\alpha_1) \to E_2 \to f^* V_2 \otimes \pi^* O_{\mathbb{P}^1}(\alpha_2) \to 0$$

$$0 \to E_2 \to E_3 \to f^* V_3 \otimes \pi^* O_{\mathbb{P}^1}(\alpha_3) \to 0$$

$$\vdots$$

$$0 \to E_{n-1} \to E_n = E \to f^* V_n \to 0.$$

When $d = 1$, we have the following elementary lemma:

**Lemma 2.1.** (1) $E_i$ are families of semi-stable bundles of degree 0 on $C$ parametrised by $\mathbb{P}^1$ for all $1 \leq i \leq n - 1$.

(2) $V_i$ are semi-stable bundles of degree 0 on $C$ for all $1 \leq i \leq n - 1$, and $V_n$ is stable of degree 1.

**Proof.** (1) Since $E$ is a family of stable bundles of slope $\mu(E) = \frac{1}{r}$, all the proper subbundles $E_i$ ($1 \leq i \leq n - 1$) have degrees $\deg(E_i) \leq 0$. On the other hand, by the degree formula, we have $\deg(E_i) \geq 0$ for all $1 \leq i \leq n - 1$. Otherwise, there is an $E_{i_0}$ of degree $\deg(E_{i_0}) \leq -1$ for some $1 \leq i_0 \leq n - 1$, and then $\deg(\varphi^*(\Theta)) \geq (\text{rk}(E_{i_0}) - r\deg(E_{i_0}))(\alpha_{i_0} - \alpha_{i_0+1}) \geq 1 + r > r$, which contradicts to $\deg(\varphi^*(\Theta)) < r$. Hence $\deg(E_i) = 0$ for all $1 \leq i \leq n - 1$.

For any proper subbundle $F$ of the restriction of $E_i$ to a fiber of $\pi$, we consider $F$ as a subbundle of the restriction of $E$ to this fiber, and then we have $\mu(F) < \mu(E) = \frac{1}{r}$ by the stability of the family $E$, so $\deg(F) \leq 0$ and $\mu(F) \leq 0 = \mu(E_i)$. This implies that $E_i$ is a family of semi-stable bundles parametrised by $\mathbb{P}^1$.

(2) It’s easy to prove that $V_i$ ($1 \leq i \leq n - 1$) is of degree 0 and that $V_n$ is of degree 1.
For $i = 1$, since $f^*V_1 \otimes \nu^*O_{P_1}(\alpha_1) = E_1$ is a family of semi-stable bundles, $V_1$ is semi-stable.

For any proper quotient sheaf $Q$ of $V_i$ ($2 \leq i \leq n - 1$), we consider $Q$ as a quotient sheaf of the restriction of $E_i$ to a general fiber, and then we have $\mu(Q) \geq \mu(E_i) = 0 = \mu(V_i)$, Hence, $V_i$ ($2 \leq i \leq n - 1$) is semi-stable of degree 0.

For any proper quotient sheaf $Q$ of $V_n$, we consider $Q$ as a proper quotient sheaf of the restriction of $E$ to a general fiber, and we have $\mu(Q) > \mu(E) = 1$, so $\deg(Q) \geq 1$ and $\mu(Q) \geq \frac{1}{\nu_1} \geq \frac{1}{\nu_1(V_n)} = \mu(V_n)$, which implies that $V_n$ is semi-stable and hence stable.

When $d = 1$, Sun’s degree formula \([1]\) for a small rational curve is

$$\deg(\phi^*(\Theta)) = \sum_{i=1}^{n-1} \nu_i(E_i)(\alpha_i - \alpha_{i+1}).$$

(2)

Now, let’s estimate the dimension and codimension of the locus of small rational curves. Let $n \leq r$ be a positive integer, and fix $n$ positive integers $r_1 < \cdots < r_{n-1} < r_n = r$ such that

$$\sum_{i=1}^{n-1} r_i(\alpha_i - \alpha_{i+1}) < r$$

(3)

for some integers $\alpha_1 > \cdots > \alpha_{n-1} > \alpha_n$.

Let $d_1 = \cdots = d_{n-1} = 0$ and $d_n = 1$. We define a subset $S_{r_1,\ldots,r_n}$ of $M = SU_C(r,\mathcal{L})$ as follows:

$$S_{r_1,\ldots,r_n} := \left\{ [W] \in M \mid \begin{array}{l}
W \text{ has a filtration } 0 = W_0 \subset W_1 \subset \cdots \subset W_n = W \\
\text{ such that } W_i/W_{i-1} \text{ is a vector bundle of rank } r_i - r_{i-1} \text{ and degree } d_i - d_{i-1} \text{ for any } 1 \leq i \leq n
\end{array} \right\}. \quad (4)$$

From above description, we know that all small rational curves lie in

$$S = \bigcup_{0 < r_1 < \cdots < r_n = r} S_{r_1,\ldots,r_n},$$

where $r_1 < \cdots < r_n$ runs over $n$ positive integers $r_i$ that satisfying inequality \((3)\) for some $n$ and some integers $\alpha_1 > \cdots > \alpha_n$.

**Proposition 2.2.** (i) $S_{r_1,\ldots,r_n}$ is Zariski-closed in $M$, and therefore can be regarded as a reduced subscheme.

(ii) There is a scheme $\Sigma$ such that $C \times \Sigma$ carries a rank $r$ vector bundle $\mathcal{E}$ and a filtration

$$0 \subset \mathcal{E}_1 \subset \cdots \subset \mathcal{E}_{n-1} \subset \mathcal{E}_n = \mathcal{E}$$

such that $\mathcal{E}_i/\mathcal{E}_{i-1}$ is a vector bundle of rank $r_i - r_{i-1}$, and for any $\sigma \in \Sigma$, the restriction of $\mathcal{E}_i/\mathcal{E}_{i-1}$ to $C \times \{\sigma\}$ has degree $d_i - d_{i-1}$. 

\[ \Box \]
Proof. By the construction of the moduli space, $M$ can be regarded as a good quotient of a subscheme $R$ of a quotient scheme by a reductive group $G$. There is a vector bundle $\mathbb{E}'$ on $C \times R$ whose restriction to $C \times \{q\}$ is the bundle represented by the image of $q$ in $M$. $\mathbb{E}'$ extends to a coherent sheaf (denoted by the same symbol) on the closure $C \times \overline{R}$.

Now we will define $\{Q_i$ and $\mathbb{E}'_i$ on $C \times Q_i\}_{n \geq i \geq 1}$ by descent induction.

Set $Q_n = \overline{R}$ and $\mathbb{E}'_n = \mathbb{E}'$.

For any $n \geq i$ ($i > 1$), we assume that $Q_i$ and $\mathbb{E}'_i$ on $C \times Q_i$ have been constructed. Let $Q_{i-1}$ be the closed subscheme of relative quotient scheme $\text{Quot}_{C \times Q_i/Q_i}(\mathbb{E}'_i)$ parametrizing quotients which restrict to vector bundles of rank $r_i - r_{i-1}$ and degree $d_i - d_{i-1}$ on each $C' \times \{q\}$. It's known that there is a universal quotient bundle on $C \times Q_i \times Q_{i-1} = C \times Q_{i-1}$

$$\mathbb{E}'_i \otimes \mathcal{O}_{C \times Q_{i-1}} \rightarrow \mathcal{V}_i \rightarrow 0.$$ 

Let $\mathbb{E}'_{i-1} = \text{Ker}(\mathbb{E}'_i \otimes \mathcal{O}_{C \times Q_{i-1}} \rightarrow \mathcal{V}_i)$. The intersection $\widetilde{S}$ of $R$ with the image of the composition of projections $Q_1 \rightarrow Q_2 \rightarrow \cdots \rightarrow Q_{n-1} \rightarrow Q_n = \overline{R}$ is closed and $G$-invariant. Therefore the image of $\widetilde{S}$ in $M$, which is $S_{r_1, \ldots, r_n}$, is closed.

Set $\Sigma = R \times \overline{R}Q_1$ and $\mathbb{E}_i$ to the pullback of $\mathbb{E}'_i$ to $C \times \Sigma = (C \times Q_i) \times \Sigma$. Then it is easily seen that these have the required properties. 

Let $W$ be a vector bundle corresponding to a general point $[W] \in M$ of a component of $S_{r_1, \ldots, r_n}$. Then $W$ has a filtration

$$0 = W_0 \subset W_1 \subset W_2 \subset \cdots \subset W_{n-1} \subset W_n = W$$

such that $V_i := W_i/W_{i-1}$ is a vector bundle of rank $r_i - r_{i-1}$ and degree $d_i - d_{i-1}$ for any $1 \leq i \leq n$. Let

$$a_n := \dim U_C(r_n - r_{n-1}, 1) + \text{ext}^1(V_n, W_{n-1}) - 1,$$

and

$$a_i := \dim U_C(r_i - r_{i-1}, 0) + \text{ext}^1(V_i, W_{i-1}) - 1 \quad \text{for any} \quad 1 < i < n.$$ 

Then

$$\dim S_{r_1, \ldots, r_n} \leq \dim U_C(r_1, 0) + a_2 + \cdots + a_n - g. \quad (5)$$

Since $\mu(V_n) > \mu(W_{n-1})$, we have $\text{Hom}(V_n, W_{n-1}) = 0$ and

$$\text{ext}^1(V_n, W_{n-1}) = -\chi(V_n^* \otimes W_{n-1}) = (r_n - r_{n-1})r_{n-1}(g - 1) + r_{n-1}.$$ 

Hence

$$a_n = r_n(r_n - r_{n-1})(g - 1) + r_{n-1}.$$ 

Lemma 2.3. If $V$ is a semi-stable vector bundle, then $h^1(V) \leq \text{rk}(V) \cdot g$ provided that $\text{rk}(V) + \deg(V) \geq 0$. 

\[5\]
Proof. If $\deg(V) < 0$, then $h^0(V) = 0$ and by the Riemann-Roch, we have
\[ h^1(V) = -\chi(V) = -(\deg(V) + \text{rk}(V) \cdot (1-g)) = \text{rk}(V) \cdot g - (\text{rk}(V) + \deg(V)) \leq \text{rk}(V) \cdot g. \]

For $\deg(V) \geq 0$, we can assume that the lemma holds for $V(-m)$ for some positive integer $m$ by induction. Therefore,
\[ h^1(V) \leq h^1(V(-m)) \leq \text{rk}(V(-m)) \cdot g = \text{rk}(V) \cdot g \]

since $H^1(V/V(-m)) = 0$.

From this lemma, for $2 \leq i \leq n - 1$ we have
\[ a_i \leq r_i(r_i - r_{i-1})(g-1) + r_{i-1}(r_i - r_{i-1}), \]

and
\[ \dim U_C(r_{i-1}, 0) + a_i \leq r_{i-1}^2(g-1) + 1 + r_i(r_i - r_{i-1})(g-1) + r_{i-1}(r_i - r_{i-1}) = \dim U_C(r_i, 0) - r_{i-1}(r_i - r_{i-1})(g-2). \]

Hence
\[
\dim S_{r_1, \ldots, r_n} \leq \dim U_C(r_1, 0) + a_2 + \cdots + a_n - g \\
\leq \cdots \leq \dim U_C(r_{n-1}, 0) - \sum_{i=2}^{n-1} r_{i-1}(r_i - r_{i-1})(g-2) + a_n - g \\
= (r_n^2 - 1)(g-1) - \left( \sum_{i=2}^{n} r_{i-1}(r_i - r_{i-1})(g-2) + r_{n-1}(r_n - r_{n-1} - 1) \right).
\]

Therefore
\[
\text{Codim } S_{r_1, \ldots, r_n} \geq \sum_{i=2}^{n} r_{i-1}(r_i - r_{i-1})(g-2) + r_{n-1}(r_n - r_{n-1} - 1). \tag{6}
\]

**Theorem 2.4.** Any small rational curve in $M$ lies in a closed subset
\[ S = \bigcup_{0 < r_1 < \cdots < r_n = r} S_{r_1, \ldots, r_n} \]
of codimension at least
\[ \min_{0 < r_1 < \cdots < r_n = r} \left\{ \sum_{i=2}^{n} r_{i-1}(r_i - r_{i-1})(g-2) + r_{n-1}(r_n - r_{n-1} - 1) \right\}. \]

where $0 < r_1 < \cdots < r_n = r$ runs over $n$ positive integers $r_i$ that satisfying
\[ \sum_{i=1}^{n-1} r_i(\alpha_i - \alpha_{i+1}) < r \] for some $n$ and some integers $\alpha_1 > \cdots > \alpha_{n-1} > \alpha_n$. 


3 Small Rational Curves on $M = SU_C(3, \mathcal{L})$ with $\deg \mathcal{L} = 1$

In this section, we want to determine all small rational curves on $M = SU_C(3, \mathcal{L})$ with $\deg \mathcal{L} = 1$.

In this special case, Sun’s degree formula (1) becomes

$$\deg(\phi^*(\Theta)) = 3 \sum_{i=1}^{n} c_2(F'_i) + \sum_{i=1}^{n-1} (\rk E_i - 3\deg(E_i))(\alpha_i - \alpha_{i+1}).$$

For $M = SU_C(3, \mathcal{L})$, the degree of a small rational curve is either 1 or 2 with respect to $\Theta$. When $\deg(\phi^*(\Theta)) = 1$, $\phi : \mathbb{P}^1 \to M$ is a line, which has been studied in [9] and [5]. Now we consider the case $\deg(\phi^*(\Theta)) = 2$, and in the rest of this paper we say small rational curves only for this case.

From section 2 and the degree formula (2) of a small rational curve

$$2 = \deg(\phi^*(\Theta)) = \sum_{i=1}^{n-1} \rk E_i(\alpha_i - \alpha_{i+1}),$$

any rational curve $\phi : \mathbb{P}^1 \to M$ is determined by a vector bundle $E$ on $X = C \times \mathbb{P}^1$ satisfying one of the following conditions:

(A) The vector bundle $E$ fits in an exact sequence

$$0 \to f^*V_1 \bigotimes \pi^*\mathcal{O}_{\mathbb{P}^1}(2) \to E \to f^*V_2 \to 0,$$

where $V_1, V_2$ are of rank 1, 2 and degrees 0, 1 respectively.

(B) The vector bundle $E$ fits in an exact sequence

$$0 \to f^*V_1 \bigotimes \pi^*\mathcal{O}_{\mathbb{P}^1}(1) \to E \to f^*V_2 \to 0,$$

where $V_1, V_2$ are of rank 2, 1 and degrees 0, 1 respectively.

3.1 The constructions of small rational curves

Let $U_C(2, 1)$ be the moduli space of stable vector bundles of rank 2 and degree 1. It’s known that $U_C(2, 1)$ is a smooth projective variety and there is a universal vector bundle $\mathcal{V}$ on $C \times U_C(2, 1)$. Let $J_C$ be the Jacobian of the curve $C$ and $\mathcal{L}$ be a Poincare bundle on $C \times J_C$. Consider the morphism

$$J_C \times U_C(2, 1) \xrightarrow{\cdot \times \det(\cdot)} J_C \times J_C^1 \xrightarrow{(\cdot) \otimes (\cdot)} J_C^1$$

and let $\mathcal{R}$ be its fiber at $[\mathcal{L}] \in J_C^1$. We still use $\mathcal{V}$ (resp. $\mathcal{L}$) to denote the pullback on $C \times \mathcal{R}$ by the projection $C \times \mathcal{R} \to C \times U_C(2, 1)$ (resp. $C \times \mathcal{R} \to C \times J_C$). Let $p : C \times \mathcal{R} \to \mathcal{R}$ and $\mathcal{G} = R^1p_*(\mathcal{V}^\vee \bigotimes \mathcal{L})$ which is a vector bundle of rank $2g - 1$. Let $q : \mathcal{P} = \mathbb{P}(\mathcal{G}) \to \mathcal{R}$ be the projection bundle parametrizing 1-dimensional
subspaces of \( G_t \) (\( t \in \mathbb{R} \)) and \( f : C \times \mathcal{P} \rightarrow C \) and \( \pi : C \times \mathcal{P} \rightarrow \mathcal{P} \) be the projections. Then there is a universal extension

\[
0 \rightarrow (\text{Id}_C \times q)^* \mathcal{L} \otimes \pi^* \mathcal{O}_\mathcal{P}(1) \rightarrow \mathcal{E} \rightarrow (\text{Id}_C \times q)^* \mathcal{V} \rightarrow 0
\]

(7)
on \( C \times \mathcal{P} \) such that for any point \( x = ([\xi], [V], [e]) \in \mathcal{P} \), where \( [\xi] \in J_C \), \( [V] \in U_C(2, 1) \) with \( \xi \otimes \det(V) \cong \mathcal{L} \) and \( [e] \in H^1(C, V^\vee \otimes \xi) \) being a line through the origin, the bundle \( \mathcal{E}|_{C \times \{x\}} \) is the isomorphic class of vector bundles \( E \) given by extensions

\[
0 \rightarrow \xi \rightarrow E \rightarrow V \rightarrow 0
\]

that defined by vectors on the line \( [e] \subset H^1(C, V^\vee \otimes \xi) \).

By [Lemma 3.1 in [9]], the vector bundle \( E \) given by the universal extension (7) on \( C \times \mathcal{P} \) defines a morphism

\[
\Phi : \mathcal{P} \rightarrow SU_C(3, \mathcal{L}) = M.
\]

Lemma 3.1. Let \( \alpha : \mathbb{P}^1 \rightarrow \mathcal{P} \) be a morphism satisfying \( \alpha^* \mathcal{O}_\mathcal{P}(1) \cong \mathcal{O}_{\mathbb{P}^1}(2) \), then \( \alpha : \mathbb{P}^1 \rightarrow \mathcal{P} \) is either a rational curve of degree 2 in \( \mathcal{P} \) or a double cover of a line in \( \mathcal{P} \).

Proof. Let \( Y := \alpha(\mathbb{P}^1) \) be the subvariety of \( \mathcal{P} \) with the reduced structure and \( \rho : \tilde{Y} \rightarrow Y \) be its normalization. Then \( \tilde{Y} \cong \mathbb{P}^1 \) and there is a morphism \( \alpha' : \mathbb{P}^1 \rightarrow \tilde{Y} \) such that \( \alpha = \rho \circ \alpha' \). Thus

\[
2 = \deg(\alpha^* \mathcal{O}_\mathcal{P}(1)) = \deg(\alpha') \cdot \deg(\rho^*(\mathcal{O}_\mathcal{P}(1)|_Y)),
\]

and then we have

\[
\deg(\alpha') = 1 \quad \text{and} \quad \deg(\rho^*(\mathcal{O}_\mathcal{P}(1)|_Y)) = 2
\]

or

\[
\deg(\alpha') = 2 \quad \text{and} \quad \deg(\rho^*(\mathcal{O}_\mathcal{P}(1)|_Y)) = 1.
\]

The first case implies that \( Y = \alpha(\mathbb{P}^1) \) is a rational curve of degree 2 in \( \mathcal{P} \) and \( \alpha : \mathbb{P}^1 \rightarrow \alpha(\mathbb{P}^1) \) is the normalization of \( \alpha(\mathbb{P}^1) \). The second case implies that \( Y = \alpha(\mathbb{P}^1) \) is a line in \( \mathcal{P} \) and \( \alpha' : \mathbb{P}^1 \rightarrow \tilde{Y} \) is a degree 2 morphism, i.e., \( \alpha : \mathbb{P}^1 \rightarrow \alpha(\mathbb{P}^1) \) is a double cover of a line in \( \mathcal{P} \).

Remark 3.2. Let \( \alpha : \mathbb{P}^1 \rightarrow \mathcal{P} \) be a morphism satisfying \( \alpha^* \mathcal{O}_\mathcal{P}(1) \cong \mathcal{O}_{\mathbb{P}^1}(p) \), where \( p \) is a prime number. Then, similar to the proof of the above lemma, we can prove that \( \alpha : \mathbb{P}^1 \rightarrow \mathcal{P} \) is either a rational curve of degree \( p \) in \( \mathcal{P} \) or a \( p \)-fold cover of a line in \( \mathcal{P} \). In particular, when \( p = 3 \), we call \( \alpha : \mathbb{P}^1 \rightarrow \mathcal{P} \) a triple cover of a line in \( \mathcal{P} \).

Proposition 3.3. (1) For any rational curve of degree 2 in the fiber of \( q : \mathcal{P} \rightarrow \mathcal{R} \), its image is a small rational curve on \( M \).

(2) For any double cover of a line in the fiber of \( q : \mathcal{P} \rightarrow \mathcal{R} \), its image is a small rational curve on \( M \).
\textbf{Proof.} Let $\alpha : \mathbb{P}^1 \to \mathcal{P}$ be either a rational curve of degree 2 in the fibre of $q$ or a double cover of a line in the fibre of $q$, we always have $\alpha^*\mathcal{O}_\mathcal{P}(1) \cong \mathcal{O}_{\mathbb{P}^1}(2)$. Let $q(\alpha(\mathbb{P}^1)) = ([\xi], [V]) \in \mathcal{R}$ and $E = (Id_C \times \alpha)^*\mathcal{E}$. Then the morphism

$$\Phi \circ \alpha : \mathbb{P}^1 \longrightarrow M$$

is defined by $E$, which fits in an exact sequence

$$0 \longrightarrow f^*\xi \otimes \pi^*\mathcal{O}_\mathbb{P^1}(2) \longrightarrow E \longrightarrow f^*V \longrightarrow 0.$$ 

Thus $c_2(E) = 2$ and $c_1(E)^2 = 4$. Then the degree of $\deg\Phi^*(\Theta)|_{\mathbb{P}^1}$ equals

$$\frac{1}{2}\Delta(E) = 3 \times c_2(E) - \frac{1}{2}(3 - 1)c_1(E)^2 = 6 - 4 = 2,$$

which implies that $\Phi(\alpha(\mathbb{P}^1))$ is a small rational curve in $M = SU_C(3, \mathcal{L})$ of degree 2 with respect to $\Theta$. \hfill \Box

\textbf{Proposition 3.4.} Let $\mathbb{P}^1 \subset \mathcal{P}$ be a line, i.e., $\mathcal{O}_\mathcal{P}(1)|_{\mathbb{P}^1} \cong \mathcal{O}_{\mathbb{P}^1}(1)$, which is not in any fiber of $q$. Then $\Phi|_{\mathbb{P}^1} : \mathbb{P}^1 \longrightarrow M$ is a small rational curve if and only if $p_2 \circ q(\mathbb{P}^1)$ is a line in $SU_C(2, \mathcal{L}')$ for some line bundle $\mathcal{L}'$ of degree 1, where $p_2 : \mathcal{R} \to U_C(2, 1)$ is the projection.

\textit{Proof.} Let $p_1 : \mathcal{R} \to J_C$ be the projection. Since $J_C$ is an abelian variety, $p_1 \circ q : \mathbb{P}^1 \to J_C$ is a constant morphism and let $p_1 \circ q(\mathbb{P}^1) = [\mathcal{L}_1] \in J_C$. Then $p_2 \circ q|_{\mathbb{P}^1} : \mathbb{P}^1 \to U_C(2, 1)$ factors through $SU_C(2, \mathcal{L} \otimes \mathcal{L}_1^{-1})$ and it is defined by a vector bundle $E' = (id_C \times p_2 \circ q \circ \alpha)^*\mathcal{V}$ on $C \times \mathbb{P}^1$, where $\alpha : \mathbb{P}^1 \to \mathcal{P}$ denotes the immersion. Thus $\Phi|_{\mathbb{P}^1} : \mathbb{P}^1 \longrightarrow M$ is a rational curve defined by

$$(id_C \times \alpha)^*(0) \longrightarrow (id_C \times q)^*\mathcal{L} \otimes \pi^*\mathcal{O}_\mathcal{P}(1) \longrightarrow \mathcal{E} \longrightarrow (id_C \times q)^*\mathcal{V} \longrightarrow 0),$$

which is isomorphic to

$$0 \longrightarrow f^*\mathcal{L}_1 \otimes \pi^*\mathcal{O}_{\mathbb{P}^1}(1) \longrightarrow (id_C \times \alpha)^*\mathcal{E} := E \longrightarrow E' \longrightarrow 0. \quad (8)$$

If $\Phi|_{\mathbb{P}^1} : \mathbb{P}^1 \longrightarrow M$ is a small rational curve, we have known that $E$ satisfies either condition (A) or condition (B). If $E$ satisfy (A), then $\mathbb{P}^1 \subset \mathcal{P}$ must be in a fiber of $q$. Thus $E$ satisfies (B), and then $E$ fits in an exact sequence

$$0 \longrightarrow f^*V_1 \otimes \pi^*\mathcal{O}_{\mathbb{P}^1}(1) \longrightarrow E \longrightarrow f^*V_2 \longrightarrow 0,$$

where $V_1, V_2$ are vector bundles of rank 2, 1 and degrees 0, 1 respectively.

Since $H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-1)) = 0$, and by Künneth formula, we have

$$H^0(C \times \mathbb{P}^1, f^*(\mathcal{L}_1^{-1} \otimes V_2) \otimes \pi^*\mathcal{O}_{\mathbb{P}^1}(-1)) = H^0(C, \mathcal{L}_1^{-1} \otimes V_2) \otimes H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-1)) = 0.$$ 

Thus

$$\text{Hom}(f^*\mathcal{L}_1 \otimes \pi^*\mathcal{O}_{\mathbb{P}^1}(1), f^*V_2) = H^0(C \times \mathbb{P}^1, f^*(\mathcal{L}_1^{-1} \otimes V_2) \otimes \pi^*\mathcal{O}_{\mathbb{P}^1}(-1)) = 0.$$
Then there’s an induced injective morphism $f^*L_1 \otimes \pi^*O_{\mathbb{P}^1}(1) \to f^*V_1 \otimes \pi^*O_{\mathbb{P}^1}(1)$ and a morphism $\beta : E' \to f^*V_2$ satisfying a commutative diagram

$$
\begin{array}{cccccc}
0 & \longrightarrow & f^*L_1 \otimes \pi^*O_{\mathbb{P}^1}(1) & \longrightarrow & E & \longrightarrow & E' & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow \beta & & \\
0 & \longrightarrow & f^*V_1 \otimes \pi^*O_{\mathbb{P}^1}(1) & \longrightarrow & E & \longrightarrow & f^*V_2 & \longrightarrow & 0.
\end{array}
$$

By the snake lemma, $\beta$ is surjective and $\ker(\beta) = f^*(V_1/L_1) \otimes \pi^*O_{\mathbb{P}^1}(1)$. This implies that $p_2 \circ q|_{\mathbb{P}^1} : \mathbb{P}^1 \to SU_C(2, L \otimes L_1^{-1})$ is a line in $SU_C(2, L \otimes L_1^{-1})$.

Conversely, if $p_2 \circ q|_{\mathbb{P}^1}$ is a line in $SU_C(2, L \otimes L_1^{-1})$, then by the result of lines in [9] and [1], there are line bundles $L_2, L_3$ on $C$, and an exact sequence

$$0 \longrightarrow f^*L_2 \otimes \pi^*O_{\mathbb{P}^1}(1) \longrightarrow E' \longrightarrow f^*L_3 \longrightarrow 0 \quad (9)$$

on $C \times \mathbb{P}^1$ with $L_1 \otimes L_2 \otimes L_3 = L$. Then by [8] and [9], we have $c_2(E) = 2, c_1(E)^2 = 4$ and $\Phi(\mathbb{P}^1)$ is a small rational curve in $M = SU_C(3, L)$ of degree 2 with respect to $\Theta$. \[\square\]

Note that the a small rational curve in Proposition 3.3 is defined by a vector bundle satisfying (A), and a small rational curve in Proposition 3.4 is defined by a vector bundle satisfying (B) but $V_1$ is not stable. Now we will construct the small rational curves which are defined by vector bundles satisfying (B) and at the same time $V_1$ is stable. Unlike the above way to obtain small rational curves, there is no universal family of vector bundles on $C$ parametrised by $SU_C(2, 0)$. Now we use the Hecke transformation ([9], [1]) to construct a family of stable vector bundles parametrised by a scheme which maps onto $SU_C(2, 0)$.

Let $\mathcal{V} \longrightarrow C \times U_C(2, 1)$ be a universal family of vector bundles of rank 2 and degree 1, and $\pi_\mathcal{V} : \mathbb{P}(\mathcal{V}) \longrightarrow C \times U_C(2, 1)$ be the associated projective bundle. Then we can construct a family $K(\mathcal{V})$ of vector bundles of rank 2 and degree 0 on $C$ parametrised by $\mathbb{P}(\mathcal{V})$ fits in an exact sequence of sheaves on $C \times \mathbb{P}(\mathcal{V})$

$$0 \longrightarrow K(\mathcal{V})^\vee \longrightarrow p^*_\mathcal{V}(\mathcal{V}) \pi^*_\mathcal{V} \mathcal{V} \longrightarrow p^*_\mathcal{V}(\mathcal{V}) \tau \otimes O(\mathcal{V})^\vee \longrightarrow 0,$$

where $\tau$ is the tautological bundle on $\mathbb{P}(\mathcal{V})$, and $\mathbb{P}(\mathcal{V})$ is considered as a divisor in $C \times \mathbb{P}(\mathcal{V})$ by the inclusion $(p_C \circ \pi_\mathcal{V}, Id_{\mathcal{V}(\mathcal{V})})$.

To define $K(\mathcal{V})$ we now construct a surjective homomorphism $\beta_\mathcal{V} : p^*_\mathcal{V}(\mathcal{V}) \pi^*_\mathcal{V} \mathcal{V} \longrightarrow p^*_\mathcal{V}(\mathcal{V}) \tau \otimes O(\mathcal{V})^\vee$ or, what is the same, an element of $H^0(C \times \mathbb{P}(\mathcal{V}), Hom(p^*_\mathcal{V}(\mathcal{V}) \pi^*_\mathcal{V} \mathcal{V}, p^*_\mathcal{V}(\mathcal{V}) \tau \otimes O(\mathcal{V})^\vee))$,

which is mapped isomorphically by $(p_C \circ \pi_\mathcal{V}, Id_{\mathcal{V}(\mathcal{V})})^*$ to

$$H^0(\mathbb{P}(\mathcal{V}), Hom(p^*_\mathcal{V}(\mathcal{V}) \pi^*_\mathcal{V} \mathcal{V}, p^*_\mathcal{V}(\mathcal{V}) \tau \otimes O(\mathcal{V})^\vee)),
$$

$$H^0(\mathbb{P}(\mathcal{V}), (p_C \circ \pi_\mathcal{V}, Id_{\mathcal{V}(\mathcal{V})})^* Hom(p^*_\mathcal{V}(\mathcal{V}) \pi^*_\mathcal{V} \mathcal{V}, p^*_\mathcal{V}(\mathcal{V}) \tau \otimes O(\mathcal{V})^\vee))
\approx H^0(\mathbb{P}(\mathcal{V}), Hom((p_C \circ \pi_\mathcal{V}, Id_{\mathcal{V}(\mathcal{V})})^* p^*_\mathcal{V}(\mathcal{V}) \pi^*_\mathcal{V} \mathcal{V}, (p_C \circ \pi_\mathcal{V}, Id_{\mathcal{V}(\mathcal{V})})^*(p^*_\mathcal{V}(\mathcal{V}) \tau \otimes O(\mathcal{V})^\vee)))
\approx H^0(\mathbb{P}(\mathcal{V}), Hom(\pi^*_\mathcal{V} \mathcal{V}, \tau)).
For the associated projective bundle $\pi_V : \mathbb{P}(V) \to C \times U_C(2, 1)$, there is a canonical surjective homomorphism $\alpha_V : \pi^*_V \to \tau$. We define $\beta_V$ by setting $(p_C \circ \pi_V, Id_{\mathbb{P}(V)})^* \beta_V = \alpha_V$. It is easy to see that the homomorphism $\beta_V$ thus defined is surjective, and the kernel of this homomorphism is locally free, which defines a vector bundle $H(V)$. Let $K(V)$ be its dual. Then $K(V)$ is a family of semi-stable vector bundles of rank 2 and degree 0 on $C$, and $K(V)$ induces a surjective morphism

$$\theta : \mathbb{P}(V^*) \to U_C(2, 0).$$

Let $\mathbb{P}(V^*) := \theta^{-1}(SU_C(2, 0))$, where $SU_C(2, 0)$ denotes the open set of stable bundles in $U_C(2, 0)$. Then $K(V)|_{C \times \mathbb{P}(V^*)}$ is a family of stable vector bundles parametrised by $\mathbb{P}(V^*)$.

Now consider the morphism

$$g : SU_C(2, 0) \times J_C^1 \cong (\mathbb{P}(V) \times J_C^1) \to C \times J_C^1 \to J_C^1 \otimes (\mathbb{P}(V^*) \times J_C^1).$$

Let $R'$ be its fiber of at $[\mathcal{L}] \in J_C^1$ and $T$ be the fiber of $(\theta|_{\mathbb{P}(V^*)} \times Id_{J_C^1}) \circ g$ at $[\mathcal{L}] \in J_C^1$, then $T = (\mathbb{P}(V^*) \times J_C^1)_{SU_C(2, 0) \times J_C^1} \to R'$. We still use $K(V)$, $\mathcal{Q}^1$ to denote the pullback on $C \times T$ by the projections $C \times T \to C \times \mathbb{P}(V^*)$ and $C \times T \to C \times J_C^1$ respectively, where $\mathcal{Q}_t^1$ is a Poincaré bundle on $C \times J_C^1$. Let $p : C \times T \to T$ and $F = \mathcal{F} = T \times \mathbb{P}(\mathcal{O}_T(2))$. Then $F$ is a vector bundle of rank $2g$. Let $q' : \mathcal{P}' := \mathbb{P}(\mathcal{F}) \to T$ be the projective bundle parametrizing 1-dimensional subspaces of $\mathcal{F}_t$ ($t \in T$) and $f : C \times \mathcal{P}' \to C$ and $\pi : C \times \mathcal{P}' \to \mathcal{P}'$ be the projections. Then there is a universal extension

$$0 \to (Id_C \times q')^*K(V) \otimes \pi^*\mathcal{O}_{\mathcal{P}'}(1) \to \mathcal{E}' \to (Id_C \times q')^*\mathcal{Q}^1 \to 0 \quad (10)$$

on $C \times \mathcal{P}'$ such that for any point $x = ([0 \to V_1^\prime \to V \to \mathcal{O}_p \to 0], [\xi], [\eta]) \in \mathcal{P}'$ with det$(V_1^\prime) \otimes \xi = \mathcal{L}$, $0 \to V_1^\prime \to V \to \mathcal{O}_p \to 0$ represents a point in $\mathbb{P}(V^*)$ and $[\xi] \subset H^1(C, \mathcal{E}^\vee \otimes V_1)$ being a line through the origin, the bundle $\mathcal{E}'|_{C \times \{x\}}$ is the isomorphic class of vector bundle $E'$ given by extensions

$$0 \to V_1 \to E' \to \xi \to 0$$

that defined by vectors on the line $[\xi] \subset H^1(C, \mathcal{E}^\vee \otimes V_1)$.

**Lemma 3.5.** Let $V_1, V_2$ are vector bundles of ranks $r_1$, $r_2$ and degrees 0, 1 respectively. Let $0 \to V_1 \to V \to V_2 \to 0$ be a non-trivial extension.

(i) If $V_1$ and $V_2$ are stable, then $V$ is stable;

(ii) if $V$ is stable, then $V_1$ and $V_2$ are semistable.

**Proof.** (i) Let $V' \subset V$ be a proper subbundle and $V'_2 \subset V_2$ be its image. Then we have a subbundle $V_1' \subset V_1$ such that $V'$ fits in an exact sequence

$$0 \to V_1' \to V' \to V_2' \to 0.$$ 

If $V'_2 \neq V_2$, then $\deg(V'_2) \leq 0$, and we always have $\deg(V'_2) \leq 0$. Hence $\mu(V') = \frac{\deg(V_2') + \deg(V'_2)}{\rk(V')} \leq 0 < \mu(V)$. Thus we assume that $V'_2 = V_2$, then $V_1' \neq V_1$ and $\deg(V'_2) < 0$. Hence, $\mu(V') = \frac{\deg(V'_2) + 1}{\rk(V')} \leq 0 < \mu(V)$.

(ii) It’s easy to check.
By the lemma above, $\mathcal{E}'$ is a family of stable vector bundles of rank 3 and with fixed determinant $\mathcal{L}$ on $C$ parametrised by $\mathcal{P}'$. Then $\mathcal{E}'$ defines a morphism

$$\Psi : \mathcal{P}' := \mathbb{P}(\mathcal{F}) \longrightarrow SU_C(3, \mathcal{L}) = M.$$ 

**Proposition 3.6.** For any line $\mathbb{P}^1 \subset \mathcal{P}'$ in the fiber of $q'$, its image in $M$ is a small rational curve.

**Proof.** Let $q'(\mathbb{P}^1) = ([0 \to V_1^\vee \to V \to \mathcal{O}_p \to 0], [\xi]) \in \mathcal{T}$ and $E' = \mathcal{E}'|_{C \times \mathbb{P}^1}$. Then the morphism

$$\Psi|_{\mathbb{P}^1} : \mathbb{P}^1 \longrightarrow M$$

is defined by $E'$, which fits in an exact sequence

$$0 \longrightarrow f^*V_1 \otimes \pi^*\mathcal{O}_{\mathbb{P}^1}(1) \longrightarrow E' \longrightarrow f^*\xi \longrightarrow 0.$$ 

Thus $c_2(E') = 2$ and $c_1(E')^2 = 4$. Then the degree of $\Psi|_{\mathbb{P}^1}$ is equal to

$$\deg \Psi^*(\Theta)|_{\mathbb{P}^1} = \frac{1}{2} \Delta(E') = 3c_2(E') - c_1(E')^2 = 2,$$

which implies that $\Psi(\mathbb{P}^1)$ is a small rational curve in $M = SU_C(3, \mathcal{L})$ of $\Theta$-degree 2 and $\Psi|_{\mathbb{P}^1}$ is its normalization. \hfill $\Box$

**Theorem 3.7.** There exist small rational curves on the moduli space $M$. Moreover, any rational curve obtained by the following ways are small rational curves and any small rational curve $\phi : \mathbb{P}^1 \longrightarrow M$ can be obtained by one of the following ways:

(i) it’s the image (under $\Phi$) of a rational curve of degree 2 in $\mathcal{P}$ in the fiber of $q$;

(ii) it’s the image (under $\Phi$) of a double cover of a line in $\mathcal{P}$ in the fiber of $q$;

(iii) it’s the image (under $\Phi$) of a line in $\mathcal{P}$ which is not in a fiber of $q$ and maps to a line in $U_C(2, \mathcal{L}')$ for some line bundle $\mathcal{L}'$ of degree 1;

(iv) it’s the image (under $\Psi$) of a line in $\mathcal{P}'$ in the fiber of $q'$. 

**Proof.** By Propositions 3.3, 3.4 and 3.6 rational curves obtained from (i), (ii), (iii) and (iv) are small rational curves.

If $\phi : \mathbb{P}^1 \longrightarrow M$ is a small rational curve, then it’s defined by a vector bundle $E$ on $C \times \mathbb{P}^1$ satisfied either condition (A) or condition (B). If $E$ satisfy (A), then $\phi : \mathbb{P}^1 \longrightarrow M$ is the image (under $\Phi$) of either a rational curve of degree 2 in $\mathcal{P}$ in the fiber of $q$ at $([V_1], [V_2]) \in \mathcal{R}$ or a double cover of a line in $\mathcal{P}$ in the fiber of $q$ at $([V_1], [V_2]) \in \mathcal{R}$.

Now we assume that $E$ satisfies (B), i.e, small rational curve $\phi : \mathbb{P}^1 \longrightarrow M$ is defined by a vector bundle $E$ on $C \times \mathbb{P}^1$, which fits in an exact sequence

$$0 \longrightarrow f^*V_1 \otimes \pi^*\mathcal{O}_{\mathbb{P}^1}(1) \longrightarrow E \longrightarrow f^*V_2 \longrightarrow 0,$$

where $V_1, V_2$ are vector bundles of ranks 2, 1 and degrees 0, 1 respectively.

If $V_1$ is semi-stable but not stable, then there is a line sub-bundle $\mathcal{L}_1$ of $V_1$ of degree 0, and then $f^*\mathcal{L}_1 \otimes \pi^*\mathcal{O}_{\mathbb{P}^1}(1)$ is a sub-bundle of $E$. Hence $\phi : \mathbb{P}^1 \longrightarrow M$ factors as $\mathbb{P}^1 \rightarrow \mathcal{P} \rightarrow M$ where $\mathbb{P}^1 \rightarrow \mathcal{P}$ is a rational curve of degree 1, and $p_1 \circ q$
maps it to a point \([L_1] \in J_C\). Let \(E' := E/(f^*L_1 \otimes \pi^*\mathcal{O}_{\mathbb{P}^1}(1))\). Then \(E'\) fits in an exact sequence

\[
0 \rightarrow f^*L_2 \otimes \pi^*\mathcal{O}_{\mathbb{P}^1}(1) \rightarrow E' \rightarrow f^*V_2 \rightarrow 0
\]

with \(L_1 \otimes L_2 \otimes V_2 = L\). This implies that \(p_2 \circ q(\mathbb{P}^1)\) is a line in \(U_C(2, L \otimes L_{-1})\).

If \(V_1\) is stable, then the small rational curve \(\phi : \mathbb{P}^1 \rightarrow M\) factors through \(\mathbb{P}\operatorname{Ext}^1(V_2, V_1)\). Hence the small rational curve \(\phi : \mathbb{P}^1 \rightarrow M\) can factors through the fiber \(q'^{-1}(t)\), which means that it’s the image (under \(\Psi\)) of a line in \(\mathcal{P}'\) in the fiber of \(q'\) at \(t \in T\).

**Remark 3.8.** When a small rational curve \(\phi : \mathbb{P}^1 \rightarrow M\) is defined by a vector bundle \(E\) on \(C \times \mathbb{P}^1\) satisfied condition (B) and moreover \(V_1\) is stable, then \(\phi\) can be obtained by the fourth way in the above theorem, but the way is not unique. Indeed, it can factors through every fiber of \(q'\) over \(\theta^{-1}([V_1], [V_2])\).

### 3.2 Remarks on lines and the locus of small rational curves

For the lines on the moduli space \(M\), we have the following result:

**Theorem 3.9.** There exist lines on the moduli space \(M\). For any line \(l \subseteq M\), there is a line \(\mathbb{P}^1 \subseteq \mathcal{P}\) in the fibre of \(q\) such that \(\Phi(\mathbb{P}^1) = l\).

**Proof.** It’s a special case of Theorem 3.4 of [9] and Theorem 2.7 of [5].

Recall that any small rational curve \(\phi : \mathbb{P}^1 \rightarrow M\) is defined by a vector bundle \(E\) on \(C \times \mathbb{P}^1\) satisfied either condition (A) or condition (B), which implies that the image of \(\phi\) lies in \(S_{1,2}\) or \(S_{2,1}\), which are defined as in section 2. And for any point \([V] \in S_{1,2}\), i.e., there is a sub line bundle \(V_1 \subseteq V\) of degree 0 and \(V/V_2\) is a vector bundle. Let \(\xi = ([V_1], [V/V_1]) \in \mathcal{R}\), and let \(\mathcal{C}\) be a rational curve of degree 2 in \(\mathcal{P}\xi\) passing through \([0 \rightarrow V_1 \rightarrow V \rightarrow V/V_1 \rightarrow 0]\), then the image of \(\mathcal{C}\) in \(M\) is a small rational curve passing through \([V]\). Similarly, for any point in \(S_{2,1}\), there also exists a small rational curve passing through it. Hence

**Proposition 3.10.** All the small rational curve lie in and cover \(S_{1,2} \cup S_{2,1}\), which is a closed subset with codimension at least \(2g - 3\).

Similarly, we have

**Proposition 3.11.** All the lines lie in and cover \(S_{1,2}\), which is a closed subset with codimension at least \(2g - 3\).

### 3.3 Remark on Hecke curves and their limits

From [9], we have known that, a minimal rational curve in \(M = SU_{C}(3, L)\) passing through a generic point is a Hecke curve, which has degree 3 with respect to \(\Theta\). In
this subsection, we will study the limits of Hecke curves, i.e., the rational curves in $M$ has degree 3 with respect to $\Theta$.

From Sun’s degree formula (1), we have

$$3 = \deg \phi^*(\Theta) = 3 \sum_{i=1}^{n} c_2(F'_i) + \sum_{i=1}^{n-1} (\text{rk} E_i - 3\deg(E_i))(\alpha_i - \alpha_{i+1}),$$

the vector bundle $E$ on $C \times \mathbb{P}^1$ which defines the rational curve $\phi : \mathbb{P}^1 \to M$ with degree 3, must satisfy one of the following cases:

**Case 1:** $n = 1$ and $c_2(E) = 1$;

**Case 2:** $\phi : \mathbb{P}^1 \to M$ is defined by a vector bundle $E$ on $C \times \mathbb{P}^1$ satisfying

$$0 \to f^*V_1 \otimes \pi^* \mathcal{O}_{\mathbb{P}^1}(3) \to E \to f^*V_2 \to 0,$$

where $V_1, V_2$ are of rank 1, 2 and degrees 0, 1 respectively.

**Case 3:** $\phi : \mathbb{P}^1 \to M$ is defined by a vector bundle $E$ on $C \times \mathbb{P}^1$ satisfying

$$0 \to f^*V_1 \otimes \pi^* \mathcal{O}_{\mathbb{P}^1}(2) \to E_2 \to f^*V_2 \otimes \pi^* \mathcal{O}_{\mathbb{P}^1}(1) \to 0,$$

$$0 \to E_2 \to E \to f^*V_3 \to 0,$$

where $V_1, V_2, V_3$ are line bundles of degrees 0, 0 and 1 respectively.

If $E$ satisfy case 1, $\phi : \mathbb{P}^1 \to M$ is a Hecke curve. Similar as the discussion of small rational curves on $M$, we have

**Theorem 3.12.** If a rational curve of degree 3 (with respect to $\Theta$) is not a Hecke curve, it can be obtained from one of the following ways:

(i) it’s the image (under $\Phi$) of a degree 3 rational curve in $\mathcal{P}$ in the fibre of $q$;

(ii) it’s the image (under $\Phi$) of a triple cover of a line in $\mathcal{P}$ in the fibre of $q$;

(iii) it’s the image (under $\Phi$) of a degree 2 rational curve in $\mathcal{P}$, which is not in a fibre of $q$ and maps to a line in $\mathcal{U}_C(2, \mathcal{L}')$ for some line bundle $\mathcal{L}'$ of degree 1;

(iv) it’s the image (under $\Phi$) of a double cover of a line in $\mathcal{P}$, which is not in a fibre of $q$ and maps to a line in $\mathcal{U}_C(2, \mathcal{L}')$ for some line bundle $\mathcal{L}'$ of degree 1.

Moreover, any rational curve coming from one of above four ways is a rational curve of degree 3 (with respect to $\Theta$) which is not a Hecke curve.

**Acknowledgments**

The author would like to thank her supervisor Professor Xiaotao Sun for the helpful suggestions in the preparation of this paper.

**References**

[1] I. Choe: Loci of rational curves of small degree on the moduli space of vector bundles, Bull. Korean Math. Soc. 48 (2011), no.2, 377-386.
[2] J.-M. Drezet and M.S. Narasimhan: Groupe de Picard des varits de modules de fibrés semistables sur les courbes algébriques, Invent. Math. 97 (1989), 53-94.

[3] J.M. Hwang: Hecke curves on the moduli space of vector bundles over an algebraic curve, Proceedings of the Symposium Algebraic Geometry in East Asia, Kyoto (2001), 155-164.

[4] J. Kollár: Rational curves on algebraic varieties. Ergebnisse der Mathematik und ihrer Grenzgebiete 3. Folge. Band. 32, Springer-Verlag Berlin-Heidelberg, 1999.

[5] N. Mok and X. Sun: Remarks on lines and minimal rational curves, Sciences in China Series A: Mathematics 52 (2009), no.6, 1-16.

[6] M.S. Narasimhan and S. Ramanan: Deformations of the moduli space of vector bundles over an algebraic curve, Ann. of Math. 101, 391-417 (1975).

[7] M.S. Narasimhan and S. Ramanan: Geometry of Hecke cycles I. In: C.P. Ramanujam attribute. Springer Verlag, 1978, pp. 291-345.

[8] S. Ramanan: The moduli spaces of vector bundles over an algebraic curve, Math. Annalen 200 (1973), 69-84.

[9] X. Sun: Minimal rational curves on the moduli spaces of stable bundles, Math. Ann. 331 (2005), 925-937.