Bikei Module Invariants of Unoriented Surface-Links

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Abstract

We extend our previous work from [5] on biquandle module invariants of oriented surface-links to the case of unoriented surface-links using bikei modules. The resulting infinite family of enhanced invariants proves be effective at distinguishing unoriented and especially non-orientable surface-links; in particular, we show that these invariants are more effective than the bikei homset cardinality invariant alone at distinguishing non-orientable surface-links. Moreover, as another application we note that our previous biquandle modules which do not satisfy the bikei module axioms are capable of distinguishing different choices of orientation for orientable surface-links as well as classical and virtual links.

Keywords: surface-links, marked graph diagrams, bikei, enhancements

2020 MSC: 57K12

1 Introduction

Surface-links are compact surfaces which may be knotted and linked in $\mathbb{R}^4$. Surface-links may be represented combinatorially via several diagrammatic systems such as broken surface diagrams drawn as immersions of surfaces into $\mathbb{R}^3$ with breaks to indicate crossing information, graphs representing the singular set of the embedding enhanced with relative height information known as braid charts, and ordered sequences of link diagrams representing horizontal cross-sections of the surface-link with respect to a choice of vertical direction known as movie diagrams. If we isotope the surface-link to have all of its maxima at one level, minima at another and saddle points at the same intermediate level, the cross-section at the saddle-level is a knotted 4-valent graph whose vertices represent the saddle points. Marking these to indicate the direction of the saddle encodes all the information necessary to recover the full surface-link, as shown by Lomanaco in [8]. Such a diagram is known as a marked vertex diagram, ch-diagram or marked graph diagram. The combinatorial moves on marked graph diagrams encoding ambient isotopy in $\mathbb{R}^4$ are known as the Yoshikawa moves; see e.g. [6, 7] for more.

Bikei are algebraic structures with axioms encoding the Reidemeister moves for unoriented knots and links. Bikei form a special case of biquandles on the one hand and generalize kei, also known as involutatory quandles, on the other. Bikei were introduced in [1]; our notation in this paper comes from [3].

Biquandle modules are algebraic structures associated to biquandles in which each order pair of biquandle elements determines an Alexander biquandle-style pair of operations on a commutative ring $R$ with identity. The axioms are chosen so that a biquandle module over a biquandle $X$ associates an invariant $R$-module to each biquandle homomorphism $f : B(L) \to X$ from the fundamental biquandle of an oriented link $L$ to a finite target biquandle $X$. The multiset of these modules is then an enhanced invariant whose cardinality determines the biquandle counting invariant but is in general a stronger invariant. See [4] for more.

In [5] the authors used biquandle modules to define invariants of oriented surface-links. In particular, the use of orientation is critical for the definition of these invariants. For classical knots and links this is not a problem since every classical link has at least one orientation. However, surface-links include both orientable and non-orientable cases. In this paper we adapt the biquandle module idea to the case of non-orientable surface-links by defining a notion of bikei modules, the unoriented version of biquandle modules.

*Email: yewon1129@gmail.com The first author was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education, Science and Technology (NRF-2019R1F1A1060205).
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The paper is organized as follows. In Section 2 we review the basics of surface-link theory and marked graph diagrams. In Section 3 we recall bikei and bikei colorings of surface-links. We then define the notion of bikei modules and give some examples. In Section 4 we use bikei modules to define enhanced invariants of unoriented surface-links and provide some computational illustrations and examples. We conclude in Section 5 with some questions for future research.

2 Surface-Links

We begin with a definition.

**Definition 1.** A *surface-link* is a closed 2-manifold smoothly (or piecewise linearly and locally flatly) embedded in the 4-space $R^4$. Two surface-links are *equivalent* if they are ambient isotopic. A *non-oriented surface-link* is a non-orientable surface-link or an orientable surface-link without orientation, while an *oriented surface-link* is an orientable surface-link with a fixed orientation.

We can specify non-oriented surface-links using *marked graph diagrams*, unoriented 4-valent spatial graphs with vertices marked with a small bar, in the following way. At each vertex we smooth the vertices in both ways to obtain two link diagrams connected by a surface (a cobordism between the links) with saddle points at the marked vertices.

![Marked Graph Diagrams](image)

If the links are unknots, we can cap them off with disks to obtain a closed surface-link; such marked graph diagrams are called *admissible*. Diagrams which are not admissible represent cobordisms between the links represented by the smoothed diagrams.

**Example 1.** The marked graph diagram

![Unknots](image)

represents an unknotted real projective plane. Smoothing the marked vertex in both ways and connecting yields a cobordism

![Cobordism](image)

between two unknots with a single crossing each; these can be capped off with Whitney’s umbrellas to complete the projective plane.
The moves on marked graph diagrams encoding ambient isotopy of the represented surface are known as 
*Yoshikawa moves*:

See [6, 7] for more about these moves.

## 3 Bikei

In this section we recall *bikei* and the biquandle counting invariant for surface-links, including non-orientable surface-links.

**Definition 2.** Let $X$ be a set. A *bikei structure* on $X$ consists of two binary operations $\ast, \ast'$ on $X$ such that

(i) For all $x \in X$, we have $x x = x x$,

(ii) For all $x, y \in X$ we have

\[
\begin{align*}
(x \ast y) \ast y &= x \\
(x \ast y) \ast y &= x \\
x \ast (y \ast x) &= x y \\
x \ast (y \ast x) &= x y
\end{align*}
\]

and

(iii) For all $x, y, z \in X$, the *exchange laws* are satisfied:

\[
\begin{align*}
(x \ast y) \ast (z \ast y) &= (x \ast z) \ast (y \ast z), \\
(x \ast y) \ast (z \ast y) &= (x \ast z) \ast (y \ast z), \text{ and} \\
(x \ast y) \ast (z \ast y) &= (x \ast z) \ast (y \ast z).
\end{align*}
\]

A bikei in which $x \ast y = x$ for all $x, y$ is called a *kei* or *involutory quandle*.

**Example 2.** Let $G$ be a group. The operations

\[
x \ast y = y x^{-1} \
\]

define a bikei structure on $X$ known as a *Takasaki kei*, *cyclic kei* or *dihedral quandle* depending on context.
Example 3. Let $R$ be a commutative ring with identity and let $X$ be an $R$-module with $t, r, s \in R$ such that $t^2 = r^2 = 1$, $s(t + r) = 0$ and $r = t + s$. Then $X$ is a biikei known as an Alexander biikei under the operations $x \ast y = tx + sy$ and $x \bar{\ast} y = rx$.

Example 4. We can specify a biikei structure on a set $\{1, 2, \ldots, n\}$ with pair of operation tables (or more succinctly as a block matrix of indices). For example the smallest non-trivial biikei has underlying set $X = \{1, 2\}$ with operations given by

\[
\begin{array}{c|cc}
\ast & 1 & 2 \\
\hline
1 & 2 & 2 \\
2 & 1 & 1 \\
\end{array}
\quad \begin{array}{c|cc}
\bar{\ast} & 1 & 2 \\
\hline
1 & 2 & 2 \\
2 & 1 & 1 \\
\end{array}
\]

or more compactly by the block matrix

\[
\begin{bmatrix}
2 & 2 & 2 & 2 \\
1 & 1 & 1 & 1 \\
\end{bmatrix}.
\]

Example 5. Let $L$ be an unoriented surface-link represented by a marked graph diagram $D$. The fundamental biikei of $L$ has a presentation with a generator for each semiarc and a crossing relation at each crossing, with semiarc meeting at a marked vertex having the same generator as shown.

The elements of the fundamental biikei are equivalence classes of biikei words in these generators (i.e. obtained recursively as $w \ast z$ or $w \bar{\ast} z$ where generators are words) modulo the equivalence relation generated by the biikei axioms and the crossing relations.

The biikei axioms are chosen precisely so that Yoshikawa moves (indeed, the first three moves, known as the Reidemeister moves in classical knot theory, are sufficient) on diagrams induce Tietze moves on the presented fundamental biikei. See \[3\] for more; here we show that Yoshikawa moves $\Omega_4$ through $\Omega_8$ do not impose any additional conditions.

\[
\begin{align*}
\Omega_4 : & \quad x+y \quad y\bar{x}x \quad y\bar{x}x & \iff & \quad x+y \quad y\bar{x}x \\
\Omega'_4 : & \quad x\bar{y}x \quad y\bar{x}x \quad y\bar{x}x & \iff & \quad x\bar{y}x \quad y\bar{x}x \\
\Omega_5 : & \quad x\ast y \quad y\bar{x}x \quad y\bar{x}x & \iff & \quad x\ast y \quad y\bar{x}x \\
\Omega_6 : & \quad x\bar{y}\bar{x} \quad x\bar{y}x \quad x\bar{y}x & \iff & \quad x\bar{y}\bar{x} \quad x\bar{y}x \\
\Omega_7 : & \quad x\bar{y}\bar{x} \quad x\bar{y}x \quad x\bar{y}x & \iff & \quad x\bar{y}\bar{x} \quad x\bar{y}x \\
\Omega_8 : & \quad x\bar{y}\bar{x} \quad x\bar{y}x \quad x\bar{y}x & \iff & \quad x\bar{y}\bar{x} \quad x\bar{y}x \\
\end{align*}
\]

It follows that the fundamental biikei is an invariant of surface-links. However, directly comparing objects specified by presentations is difficult; thus, we want to indirectly compare these fundamental biikei by comparing their homsets onto a choice of finite biikei.

Definition 3. A map $f : X \to Y$ between biikei is a biikei homomorphism if for all $x, x' \in X$ we have $f(x \ast x') = f(x) \ast f(x')$ and $f(x \bar{\ast} x') = f(x) \bar{\ast} f(x')$. 

4
Definition 4. Let $L$ be an unoriented surface-link and $X$ a bikei. A bikei coloring of a diagram of $L$ by $X$ is an assignment of elements in $X$ to the semiarcs of $L$ such that the crossing relations are satisfied at every crossing.

A bikei coloring uniquely determines and is determined by a bikei homomorphism $f : B(L) \to X$. Thus a marked graph diagram gives us a way to combinatorially (and visually) represent elements of the homset $\text{Hom}(B, X)$ analogously to the way choosing a basis for a vector space lets us concretely represent homsets in the vector space category. In particular, the cardinality of the homset is an integer-valued invariant of unoriented surface-links known as the bikei counting invariant, denoted $\Phi^X_Z(L) = |\text{Hom}(B(L), X)|$.

Example 6. The unknotted projective plane $2_1^{-1}$ with diagram

has no colorings by the bikei $X_1 = \{1, 2\}$ with operations

\[
\begin{array}{c|cccc}
\ast & 1 & 2 & \tau & 1 & 2 \\
\hline
1 & 2 & 2 & 1 & 2 \\
2 & 1 & 1 & 1 & 1 \\
\end{array}
\]

since at the marked vertex all four colors must be equal but at the crossing colors must change from 1 to 2 or 2 to 1; hence, we have $\Phi^X_{Z_1}(2_1^{-1}) = 0$. On the other hand, the same diagram has two colorings by the other bikei structure $X_2$ on the set of two elements,

\[
\begin{array}{c|cccc}
\ast & 1 & 2 & \tau & 1 & 2 \\
\hline
1 & 1 & 1 & 1 & 1 \\
2 & 2 & 2 & 2 & 2 \\
\end{array}
\]

and we have $\Phi^X_{Z_2}(2_1^{-1}) = 2$.

Example 7. Let $X$ be the bikei structure on the $\{1, 2, 3, 4\}$ specified by the operation tables

\[
\begin{array}{c|cccc}
\ast & 1 & 2 & 3 & 4 \\
\hline
1 & 3 & 1 & 3 & 1 \\
2 & 3 & 1 & 3 & 1 \\
3 & 1 & 3 & 1 & 3 \\
4 & 4 & 2 & 2 & 2 \\
\end{array}
\]

The non-orientable surface-links $8_1^{-1,-1}$ and $9_1^{1,-2}$ are distinguished from each other by their counting invariants with respect to $X$, with $\Phi^X_{Z_2}(8_1^{-1,-1}) = 0 \neq 4 = \Phi^X_{Z_2}(9_1^{1,-2}).$
4 Bikei Module Enhancements

We begin this section with a generalization of a definition from [5].

**Definition 5.** Let $X$ be a bikei and $R$ a commutative ring with identity. A bikei module is an assignment of elements $t_{x,y}, r_{x,y}$ and $s_{x,y}$ in $R$ to each pair of elements in $X$ such that for all $x, y, z \in X$ we have

\[
\begin{align*}
 t_{x,y} + r_{x,y} & = 1 & (i) \\
 r_{x,y} & = 1 & (ii) \\
 (t_{x,y} + r_{x,y}) s_{x,y} & = 0 & (iii) \\
 t_{x,x} s_{y,y} & = r_{x,x} & (i.i) \\
 r_{y,y} & = r_{x,x} & (i.ii) \\
 r_{x,x} & = s_{x,x} & (i.iii) \\
 s_{x,x} & = r_{x,x} & (i.iv) \\
 s_{x,x} & = s_{x,x} & (i.v) \\
 (t_{x,x} + s_{x,x}) s_{y,y} & = 0 & (ii) \\
 \end{align*}
\]

We can specify a bikei module using a block matrix $[T|S|R]$ whose blocks have $t_{j,k}$, $s_{j,k}$ and $r_{j,k}$ respectively in row $j$ column $k$.

**Example 8.** Using our python code, we found 512 bikei module structures on the bikei

\[
\begin{array}{c|cc}
 x & 1 & 2 \\
 1 & 2 & 2 \\
 2 & 1 & 1 \\
\end{array}
\quad
\begin{array}{c|cc}
 x & 1 & 2 \\
 1 & 2 & 2 \\
 2 & 1 & 1 \\
\end{array}
\]

with coefficients in $R = \mathbb{Z}_8$, including for instance

\[
\begin{bmatrix}
 3 & 7 & 4 & 0 & 7 & 5 \\
 7 & 3 & 0 & 4 & 5 & 7 \\
\end{bmatrix}
\]

The motivation is as follows: Given an element $f$ of the bikei homset represented by a bikei coloring of a marked graph diagram, we would like to define an invariant $R$-module by modifying the Alexander bikei operations to have coefficients which are functions of the bikei colors at each crossing. Assigning a module generator to each semiarc, we can picture these generators as “beads.”

![Diagram](image)

We then define the module $M_f$ as the kernel of the coefficient matrix of the homogeneous determined by the system of linear equations including

\[
\begin{align*}
 0 & = t_{x,y}a + s_{x,y}b - d \\
 0 & = r_{x,y}b - c \\
\end{align*}
\]
at each crossing. The elements of $M_f$ can be represented as bead colorings of the $X$-colored diagrams, i.e. assignments of elements of $R$ to each semiarc in the $X$-colored diagram satisfying the above-pictured condition at every crossing. The bikei module axioms are the conditions on the coefficients coming from the bikei-colored Yoshikawa moves (again, the first three, i.e. the Reidemeister moves, are sufficient) together with the 180-degree rotational symmetry required by the lack of orientation.

A choice of bikei module over a bikei space with coefficients in $R$ associates an invariant $R$-module to each $X$-coloring of a diagram $D$ of an unoriented surface-link $L$. More precisely, Yoshikawa moves induce $R$-module isomorphisms on these modules, and consequently the multiset of such $R$-modules over the homset $\text{Hom} (BK(L), X)$ is an invariant of unoriented surface-links.

We illustrate the computation of the invariant with an example.

**Example 9.** Let $X = \{1, 2\}$ have the trivial bikei structure, i.e. the structure specified by the operation tables

$$
\begin{array}{c|cc}
\ast & 1 & 2 \\
\hline
1 & 1 & 1 \\
2 & 2 & 2 \\
\end{array}
\quad
\begin{array}{c|cc}
\ast & 1 & 2 \\
\hline
1 & 1 & 1 \\
2 & 2 & 2 \\
\end{array}
$$

and let $R = \mathbb{Z}_5$; then $X$ has $R$-module structures including

$$
\begin{bmatrix}
1 & 4 \\
1 & 4 \\
\end{bmatrix}
\quad
\begin{bmatrix}
0 & 2 & 1 & 1 \\
0 & 4 & 4 \\
\end{bmatrix}.
$$

To compute the invariant for the unknotted projective plane $2_1^{-1}$, we note that there are two $X$-colorings of a marked graph diagram representing $2_1^{-1}$ as shown:

These then give us the systems of bead coloring equations

$$
a = r_{11}a \quad & a = r_{22}a \\
\frac{a}{1} = t_{11}a + s_{11}a \quad & \frac{a}{2} = t_{22}a + s_{22}a
$$

which become

$$
a = 1a \quad & a = 4a \\
\frac{a}{1} = 1a + 0a \quad & \frac{a}{2} = 1a + 0a
$$

so the associated $R$-modules are respectively the kernels of the matrices $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, and the invariant value is the multiset $\{(\mathbb{Z}_5)^0, (\mathbb{Z}_5)^1\}$.

For ease of comparison, we can define a polynomial invariant from this homset by summing a formal variable $u$ raised to the number of elements of each module in the case that $R$ is finite (that is, the number of bead colorings of each $X$-colored diagram) or raised to the rank of the module in case $R$ is infinite. More formally, we have:

**Definition 6.** Let $X$ be a finite bikei, $L$ an unoriented surface-link represented by a marked graph diagram $D$, $R$ a commutative ring with identity and $M$ a choice of $X$-module with coefficients in $R$. For each homset
element \( f : B\mathcal{K}(L) \to X \), let \( D_f \) be the corresponding \( X \)-coloring of \( D \), and let \( M_f \) be the \( R \)-module of bead colorings of \( D_f \) with \( M \) coefficients. We then define the bikei module enhanced polynomial of \( L \) to be

\[
\Phi^M_X(L) = \left\{ \begin{array}{l}
\sum_{f \in \text{Hom}(B\mathcal{K}(L),X)} u^{|M_f|} |R| \in \mathbb{N} \\
\sum_{f \in \text{Hom}(B\mathcal{K}(L),X)} u^{\text{Rank}(M_f)} \text{ otherwise}
\end{array} \right.
\]

**Theorem 1.** The polynomial \( \Phi^M_X(L) \) is an invariant of surface-links.

**Proof.** The conditions defining bikei modules are the conditions required for invariance of \( M_f \) under a generating set of unoriented Yoshikawa moves.

**Example 10.** Let \( X = \{1, 2, 3, 4\} \) have bikei structure defined by the operation tables

\[
\begin{array}{cccc}
* & 1 & 2 & 3 \\
1 & 1 & 1 & 2 \\
2 & 2 & 2 & 1 \\
3 & 4 & 4 & 3 \\
4 & 3 & 3 & 4
\end{array}
\quad
\begin{array}{cccc}
\bar{*} & 1 & 2 & 3 \\
1 & 1 & 1 & 2 \\
2 & 2 & 2 & 1 \\
3 & 3 & 3 & 3 \\
4 & 4 & 4 & 4
\end{array}
\]

and let \( R = \mathbb{Z}_3 \). Then the matrix of \( t, s, r \) coefficients

\[
\begin{bmatrix}
2 & 2 & 1 & 1 & 2 & 2 & 0 & 0 & 1 & 1 & 1 & 1 \\
2 & 2 & 1 & 1 & 2 & 2 & 0 & 0 & 1 & 1 & 1 & 1 \\
2 & 1 & 2 & 2 & 0 & 0 & 0 & 0 & 1 & 1 & 2 & 1 \\
1 & 2 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 2 & 1
\end{bmatrix}
\]

defines a bikei module over \( X \) with \( R \) coefficients.

The non-orientable surface-links \( 8_{1}^{-1,-1} \) and \( 9_{1}^{1,-2} \) both have bikei homsets of cardinality 8, so the links are not distinguished by the counting invariant alone with respect to this bikei. However, the bikei module enhanced invariants with respect to the listed bikei module are

\[
\Phi^M_X(8_{1}^{-1,-1}) = 5u + 2u^3 + u^9
\]

and

\[
\Phi^M_X(9_{1}^{1,-2}) = 2u + 4u^3 + 2u^9;
\]

hence the surface-links are distinguished by the enhanced invariant. In particular this example shows that the enhanced invariant is not determined by the size of the homset and hence is a proper enhancement.

**Example 11.** We computed the bikei module enhanced invariant for the non-orientable surface-links with \( ch \)-index up to 10 as listed in [9] with respect to the bikei and bikei module in example 10; the results are in the table.

| \( \Phi^M_X(L) \) | \( L \) |
|-----------------|-----------------|
| \( 3u + u^3 \)  | \( 2_{1}^{-1}, 10_{1}^{0,-2}, 10_{1}^{-1,-1}, 10_{2}^{-1,-1} \) |
| \( 5u + 2u^3 + u^9 \) | \( 8_{1}^{-1,-1}, 10_{1}^{-2,-2} \) |
| \( 2u + 4u^3 + 2u^9 \) | \( 9_{1}^{1,-2} \) |
| \( 6u + 4u^3 + 2u^9 \) | \( \tau_{0}^{1,-2} \) |

**Example 12.** Let \( X = \{1, 2, 3, 4\} \) have bikei structure defined by the operation tables

\[
\begin{array}{cccc}
* & 1 & 2 & 3 \\
1 & 1 & 1 & 2 \\
2 & 2 & 2 & 1 \\
3 & 4 & 4 & 3 \\
4 & 3 & 3 & 4
\end{array}
\quad
\begin{array}{cccc}
\bar{*} & 1 & 2 & 3 \\
1 & 1 & 1 & 2 \\
2 & 2 & 2 & 1 \\
3 & 3 & 3 & 3 \\
4 & 4 & 4 & 4
\end{array}
\]
and let $R = \mathbb{Z}_3$. Then the matrix of $t, s, r$ coefficients
\[
\begin{bmatrix}
1 & 2 & 1 & 1 & 1 & 0 & 0 & 0 & 2 & 2 & 2 \\
2 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 2 & 2 & 2 \\
2 & 1 & 2 & 2 & 0 & 0 & 0 & 0 & 1 & 1 & 2 \\
1 & 2 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 2 \\
\end{bmatrix}
\]
defines a bikei module over $X$ with $R$ coefficients. We computed the bikei module enhanced invariant for the orientable and non-orientable surface-links with $ch$-index up to 10 as listed in [9]; the results are in the table.

| $\Phi^M_X(L)$ | $L_{ori}$ | $\Phi^M_X(L)$ | $L_{non-ori}$ |
|----------------|-----------|----------------|---------------|
| $4u^3$         | $2_1, 10_1, 10_3$ | $3u + u^3$ | $2_1^{-1}, 10_1^{1,-2}, 10_2^{1,-2}, 10_1^{-1,1}$ |
| $2u^3 + 10u^9$ | $6_{1,1}$ | $u + 6u^3 + u^9$ | $8_1^{-1,-1}, 10_1^{1,-2}$ |
| $2u^3 + 2u^9$  | $8_{1,2}, 9_{1,1}, 10_2, 10_2^{1,1}$ | $6u^3 + 2u^9$ | $g_{1}^{1,2}$ |
| $2u^3 + 6u^9$  | $10_{1,1}^{1,1}$ | $10u^3 + 2u^9$ | $7_1^{0,-2}$ |
| $2u^3 + 14u^9$ | $9_{1,0}^{1,1}$ | $6u^3 + 10u^9$ | $10_{1,1}^{1,0}$ |
| $2u + 4u^3 + 10u^9$ | $9_{2,0}^{1,1}$ | $10_2^{1,0}$ | $10_{1,0}^{1,0}$. |
| $4u^3 + 4u^9$  | $9_{1,1}^{2,1}$ | $18u^9 + 30u^{27}$ | $10_{1,0}^{2,0}$ |

**Remark 1.** In [4], non-orientable surface-links could not be distinguished using ideal coset invariants, so these examples show that the method of this paper represents an improvement over the methods of [4].

**Remark 2.** Though we have developed bikei modules as invariants of surface-links, the invariants they define also apply to unoriented classical and virtual knots and links, which may be regarded as trivial cobordisms between two copies of the classical or virtual knot or link. In particular, every bikei module is a bicrossed module as defined in [5] satisfying some extra conditions; hence the set of bikei modules of a bikei $X$ is a subset of the set of bicrossed modules of $X$. The bicrossed modules over $X$ which are not bikei modules are those that may be sensitive to orientation; thus, to find invariants which can potentially distinguish knots from their reverses, we can use bicrossed module invariants using bicrossed modules which are not bikei modules.

## 5 Questions and Future Directions

We have considered only bikei modules over finite rings of the form $\mathbb{Z}_n$ for simplicity of computation using python. Bikei modules over infinite rings, especially polynomial rings, are of great interest as generalizations of the Alexander invariant for these non-orientable surface-links. Our method of exhaustive computer search over finite rings obviously does not work for these cases, so developing other methods of finding bikei modules is of definite interest.

What is the relationship between bikei modules and other bikei invariants? Do bikei modules have an interpretation as bikei extensions analogous to abelian extensions of quandles by cocycles as in [2]?

A more computational question: we searched many examples of bikei modules over finite rings but could not distinguish the surface-links $8_1^{1,-1}$ and $10_1^{2,-2}$. We conjecture that this is a consequence of our computational limitation to small cardinality bikei and rings, and ask what is the smallest bikei $X$ and bikei module $M$ over a finite ring $R$ such that

$$
\Phi^M_X(8_1^{1,-1}) \neq \Phi^M_X(10_1^{2,-2}).
$$
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