SPECHT FILTRATIONS AND TENSOR SPACES FOR
THE BRAUER ALGEBRA

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Abstract. Let $m, n \in \mathbb{N}$. In this paper we study the right permutation action of the symmetric group $S_{2n}$ on the set of all the Brauer $n$-diagrams. A new basis for the free $\mathbb{Z}$-module $B_n$ spanned by these Brauer $n$-diagrams is constructed, which yields Specht filtrations for $B_n$. For any $2m$-dimensional vector space $V$ over a field of arbitrary characteristic, we give an explicit and characteristic free description of the annihilator of the $n$-tensor space $V \otimes^n$ in the Brauer algebra $B_n(-2m)$. In particular, we show that it is a $S_{2n}$-submodule of $B_n(-2m)$.

1. Introduction

Let $x$ be an indeterminate over $\mathbb{Z}$. The Brauer algebra $\mathfrak{B}_n(x)$ over $\mathbb{Z}[x]$ is a unital associative $\mathbb{Z}[x]$-algebra with generators $s_1, \ldots, s_{n-1}, e_1, \ldots, e_{n-1}$ and relations (see [16]):

- $s_i^2 = 1$, $e_i^2 = xe_i$, $e_is_i = s_ie_i$, $\forall 1 \leq i \leq n - 1$,
- $s_is_j = s_js_i$, $s_ie_j = e_js_i$, $e_ie_j = e_je_i$, $\forall 1 \leq i < j - 1 \leq n - 2$,
- $s_is_{i+1}s_i = s_{i+1}s_is_{i+1}$, $e_ie_{i+1}e_i = e_i$, $e_{i+1}e_ie_{i+1} = e_{i+1}$, $\forall 1 \leq i \leq n - 2$,
- $s_ie_{i+1}e_i = s_{i+1}e_i$, $e_{i+1}e_is_{i+1} = e_{i+1}s_i$, $\forall 1 \leq i \leq n - 2$.

$\mathfrak{B}_n(x)$ is a free $\mathbb{Z}[x]$-module with rank $(2n - 1) \cdot (2n - 3) \cdots 3 \cdot 1$. For any $\mathbb{Z}[x]$-algebra $R$ with $x$ specialized to $\delta \in R$, we define $\mathfrak{B}_n(\delta)_R := R \otimes \mathbb{Z}[x] \mathfrak{B}_n(x)$.

This algebra was first introduced by Richard Brauer (see [2]) when he studied how the $n$-tensor space $V \otimes^n$ decomposes into irreducible modules over the orthogonal group $O(V)$ or the symplectic group $Sp(V)$, where $V$ is an orthogonal vector space or a symplectic vector space. In Brauer’s original formulation, the algebra $\mathfrak{B}_n(x)$ was defined as the complex linear space with basis the set $\text{Bd}_n$ of all the Brauer $n$-diagrams, graphs on $2n$ vertices and $n$ edges with the property that

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every vertex is incident to precisely one edge. If we arrange the vertices in two rows of \( n \) each, the top and bottom rows, and label the vertices in each row of a \( n \)-diagram by the indices \( 1, 2, \ldots, n \) from left to right, then the generator \( s_i \) corresponds to the \( n \)-diagram with edges connecting vertices \( i \) (respectively, \( i + 1 \)) on the top row with \( i + 1 \) (respectively, \( i \)) on bottom row, and all other edges are vertical, connecting vertex \( k \) on the top and bottom rows for all \( k \neq i, i + 1 \). The generator \( e_i \) corresponds to the \( n \)-diagram with horizontal edges connecting vertices \( i, i + 1 \) on the top and bottom rows, and all other edges are vertical, connecting vertex \( k \) on the top and bottom rows for all \( k \neq i, i + 1 \). The multiplication of two Brauer \( n \)-diagrams is defined as follows. We compose two diagrams \( D_1, D_2 \) by identifying the bottom row of vertices in the first diagram with the top row of vertices in the second diagram. The result is a graph, with a certain number, \( n(D_1, D_2) \), of interior loops. After removing the interior loops and the identified vertices, retaining the edges and remaining vertices, we obtain a new Brauer \( n \)-diagram \( D_1 \circ D_2 \), the composite diagram. Then we define \( D_1 \cdot D_2 = x^{n(D_1, D_2)} D_1 \circ D_2 \). In general, the multiplication of two elements in \( \mathcal{B}_n(x) \) is given by the linear extension of a product defined on diagrams. For example, let \( d \) be the following Brauer 5-diagram.

![Figure 1.1](image1.png)

Let \( d' \) be the following Brauer 5-diagram.

![Figure 1.2](image2.png)

Then \( dd' \) is equal to
Note that the subalgebra of $B_n(x)$ generated by $s_1, s_2, \ldots, s_{n-1}$ is isomorphic to the group algebra of the symmetric group $\mathfrak{S}_n$ over $\mathbb{Z}[x]$.

The Brauer algebra as well as its quantization (now called Birman–Wenzl–Murakami algebra) has been studied in a number of papers, e.g., [2], [3], [4], [24], [25], [6], [32], [38], [17], [39], [16], [11]. The walled Brauer algebra (which is a variant of Brauer algebra, see [5]) is also studied in the recent preprint [10]. We are mainly interested in the Schur–Weyl duality between symplectic groups and certain specialized Brauer algebras, which we now recall. Let $K$ be an arbitrary infinite field. Let $m, n \in \mathbb{N}$. Let $V$ be a $2m$-dimensional $K$-vector space equipped with a non-degenerate skew-symmetric bilinear form $(,)$. Then (see [20], [15, Section 4]) the symplectic similitude group (respectively, the symplectic group) relative to $(,)$ is

$$GSp(V) := \left\{ g \in GL(V) \ \bigg| \ \exists d \in K \text{ with } d \neq 0, \text{ such that } (gv, gw) = d(v, w), \ \forall v, w \in V \right\}$$

(respectively,

$$Sp(V) := \left\{ g \in GL(V) \ \bigg| \ (gv, gw) = (v, w), \ \forall v, w \in V \right\}$$.

The symplectic similitude group and symplectic group $Sp(V)$ act naturally on $V$ from the left-hand side, and hence on the $n$-tensor space $V^\otimes n$. This left action on $V^\otimes n$ is centralized by certain specialized Brauer algebra, which we recall as follows. Let $\mathcal{B}_n(-2m) := \mathbb{Z} \otimes_{\mathbb{Z}[x]} \mathcal{B}_n(x)$, where $\mathbb{Z}$ is regarded as $\mathbb{Z}[x]$-algebra by specifying $x$ to $-2m$. Let $\mathcal{B}_n(-2m)_K := K \otimes_{\mathbb{Z}} \mathcal{B}_n(-2m)$, where $K$ is regarded as $\mathbb{Z}$-algebra by sending each integer $a$ to $a \cdot 1_K$. Then there is a right action of the specialized Brauer algebra $\mathcal{B}_n(-2m)_K$ on the $n$-tensor space $V^\otimes n$ which commutes with the above left action of $GSp(V)$. We recall the definition of this action as follows. Let $\delta_{ij}$ denote the value of the usual
Kronecker delta. For any $1 \leq i \leq 2m$, we set

$$i' := 2m + 1 - i.$$ 

We fix an ordered basis $\{v_1, v_2, \cdots, v_{2m}\}$ of $V$ such that

$$(v_i, v_j) = 0 = (v_i', v_{j'}), \quad (v_i, v_{j'}) = \delta_{ij} = -(v_{j'}, v_i), \quad \forall 1 \leq i, j \leq m.$$ 

For any $i, j \in \{1, 2, \cdots, 2m\}$, let

$$\epsilon_{i,j} := \begin{cases} 1 & \text{if } j = i' \text{ and } i < j, \\ -1 & \text{if } j = i' \text{ and } i > j, \\ 0 & \text{otherwise}, \end{cases}$$

The right action of $\mathfrak{B}_n(-2m)$ on $V^\otimes n$ is defined on generators by

$$(v_i \otimes \cdots \otimes v_i) s_j := -(v_i \otimes \cdots \otimes v_{i-j} \otimes v_{j+1} \otimes v_j \otimes v_{j+2} \otimes \cdots \otimes v_{n}),$$

$$(v_i \otimes \cdots \otimes v_i) e_j := \epsilon_{i,j} v_i \otimes \cdots \otimes v_{i-j} \otimes \left( \sum_{k=1}^{m} (v_{k'} \otimes v_k - v_k \otimes v_{k'}) \right) \otimes v_{j+2} \otimes \cdots \otimes v_{n}.$$ 

Let $\varphi, \psi$ be the following natural $K$-algebra homomorphisms.

$$\varphi : (\mathfrak{B}_n(-2m)_K)^{\text{op}} \to \text{End}_K(V^\otimes n),$$

$$\psi : K\text{GSp}(V) \to \text{End}_K(V^\otimes n)$$

Let $k$ be a positive integer. A composition of $k$ is a sequence of nonnegative integer $\lambda = (\lambda_1, \lambda_2, \cdots)$ with $\sum_{i\geq 1} \lambda_i = k$. A composition $\lambda = (\lambda_1, \lambda_2, \cdots)$ of $k$ is said to be a partition if $\lambda_1 \geq \lambda_2 \geq \cdots$. In this case, we write $\lambda \vdash k$. The conjugate of $\lambda$ is defined to be a partition $\lambda' = (\lambda'_1, \lambda'_2, \cdots)$, where $\lambda'_j := \#\{i | \lambda_i \geq j\}$ for $j = 1, 2, \cdots$. For any partition $\lambda = (\lambda_1, \lambda_2, \cdots)$, we use $\ell(\lambda)$ to denote the largest integer $t$ such that $\lambda_t \neq 0$.

**Lemma 1.1.** ([2], [3], [4] 1) The natural left action of $\text{GSp}(V)$ on $V^\otimes n$ commutes with the right action of $\mathfrak{B}_n(-2m)$. Moreover, if $K = \mathbb{C}$, then

$$\varphi(\mathfrak{B}_n(-2m)_\mathbb{C}) = \text{End}_{\mathcal{GSp}(V)}(V^\otimes n) = \text{End}_{\mathcal{Sp}(V)}(V^\otimes n),$$

$$\psi(\mathbb{C}\text{GSp}(V)) = \psi(\mathbb{C}\text{Sp}(V)) = \text{End}_{\mathfrak{B}_n(-2m)_\mathbb{C}}(V^\otimes n),$$

2) if $K = \mathbb{C}$ and $m \geq n$ then $\varphi$ is injective, and hence an isomorphism onto $\text{End}_{\mathcal{GSp}(V)}(V^\otimes n)$. 
3) if $K = \mathbb{C}$, then there is a decomposition of irreducible $\mathbb{C}GSp(V)$–$\mathcal{B}_n(-2m)_{\mathbb{C}}$ bimodules

$$V^\otimes n = \bigoplus_{f=0}^{[n/2]} \bigoplus_{\lambda^\dashv n-2f} \lambda \vdash n-2f \Delta(\lambda) \otimes D(\lambda'),$$

where $\Delta(\lambda)$ (respectively, $D(\lambda')$) denotes the irreducible $\mathbb{C}GSp(V)$-module (respectively, the irreducible $\mathcal{B}_n(-2m)_{\mathbb{C}}$-module) corresponding to $\lambda$ (respectively, corresponding to $\lambda'$).

By recent work of [35] and [11], part 1) and part 2) of the above lemma hold for an arbitrary infinite field. That is,

**Lemma 1.2.** ([35], [11]) Let $K$ be an arbitrary infinite field.

1) $\psi(KGSp(V)) = \text{End}_{\mathcal{B}_n(-2m)_K}(V^\otimes n)$.

2) $\varphi(\mathcal{B}_n(-2m)_K) = \text{End}_{KGSp(V)}(V^\otimes n) = \text{End}_{KSp(V)}(V^\otimes n)$, and if $m \geq n$, then $\varphi$ is also injective, and hence an isomorphism onto $\text{End}_{KSp(V)}(V^\otimes n)$.

Now there is a natural question, that is, how can one describe the kernel of the homomorphism $\varphi$. This question is closely related to invariant theory: see [7]. By the results in [11], we know that the kernel of the homomorphism $\varphi$ has a rigid structure in the sense that the dimension of $\text{Ker} \varphi$ does not depend on the choice of the infinite field $K$, and it is actually defined over $\mathbb{Z}$. Note that in the case of Schur–Weyl duality between general linear group and symmetric group ([36], [37], [7], [8]), or more generally, between the type $A$ quantum group and the type $A$ Iwahori–Hecke algebra ([26], [14]), the kernel of the similar homomorphism has already been explicitly determined in [14] in terms of Kazhdan–Lusztig basis and in [22] in terms of Murphy basis.

In this paper, we completely answer the above question by explicitly constructing an integral basis for the kernel of the homomorphism $\varphi$. Our description of $\text{Ker} \varphi$ involves a study of the permutation action of the symmetric group $S_{2n}$ on the Brauer algebra $\mathcal{B}_n(x)$. Such a permutation action was previously noted in [17]. We construct a new integral basis for this Brauer algebra, which yields integral Specht filtration of this Brauer algebra by right $S_{2n}$-modules. The kernel of $\varphi$ is just one of the $S_{2n}$-submodules appearing in this filtration. In particular, it turns out that $\text{Ker} \varphi$ is in fact a $S_{2n}$-submodule of $\mathcal{B}_n(-2m)$. The main results of this paper are presented in Theorem 2.12, Theorem 2.14 and Theorem 3.5. It would be interesting to compare the new integral
basis we obtained in this paper with the canonical basis for $\mathcal{B}_n(x)$ constructed in [17]. Finally, we remark that it might be possible to give a similar description of $\text{Ker } \varphi$ also in the orthogonal case (i.e., the case of Schur–Weyl duality between orthogonal group and certain specialized Brauer algebra). We deal with only the symplectic case in this paper because we use the main results in [11], where only the symplectic case is considered. It would also be interesting to see how the description of $\text{Ker } \varphi$ we give here can be generalized to the quantized case, i.e., the case of Schur–Weyl duality between the quantized enveloping algebra associated to the symplectic Lie algebra $\mathfrak{sp}_{2m}$ and a certain specialized Birman–Wenzl–Murakami algebra (see [9]).

2. The $\mathfrak{S}_{2n}$-action on $\mathcal{B}_n(x)$

In this section, we shall first recall (cf. [17]) the right permutation action of the symmetric group $\mathfrak{S}_{2n}$ on the set $\text{Bd}_n$. Then we shall construct a new $\mathbb{Z}$-basis for the resulting right $\mathfrak{S}_{2n}$-module, which yields filtrations of $\mathcal{B}_n(x)$ by right $\mathfrak{S}_{2n}$-modules. Certain submodules occurring in this filtration will play central role in the next section.

For any involution $\sigma$ in the symmetric group $\mathfrak{S}_{2n}$, the conjugate $w^{-1}\sigma w$ of $\sigma$ by $w \in \mathfrak{S}_{2n}$ is still an involution. Therefore, we have a right action of the symmetric group $\mathfrak{S}_{2n}$ on the set of all the involutions in $\mathfrak{S}_{2n}$. Note that the set $\text{Bd}_n$ of all the Brauer $n$-diagrams can be naturally identified with the set of all the involutions in $\mathfrak{S}_{2n}$. Hence we get (cf. [17]) a right permutation action of the symmetric group $\mathfrak{S}_{2n}$ on the set $\text{Bd}_n$ of all the Brauer $n$-diagrams. We use “$\ast$” to denote this right permutation action.

We shall adopt a new labeling of the vertices in each Brauer diagram. Namely, for each Brauer $n$-diagram $D$, we shall label the vertices in the top row of $D$ by odd integers $1, 3, 5, \ldots, 2n - 1$ from left to right, and label the vertices in the bottom row of $D$ by even integers $2, 4, 6, \ldots, 2n$ from left to right. This way of labeling is more convenient when studying the permutation action from $\mathfrak{S}_{2n}$. We shall keep this way of labeling from this section until the end of Section 3, and we shall recover our original way of labeling only in Section 4. Let us look at an example. Suppose $n = 4$, $s_7 = (7, 8)$ is a transposition in $\mathfrak{S}_8$. Let $D$ be the following Brauer 4-diagram.
We first identify $D$ with following diagram with 8-vertices.

Then $D \ast s_7$ can be computed in the following way.

Finally, $D \ast s_7$ is equal to the following Brauer 4-diagram.

We use $\beta : \text{Bd}_n \cong \{w \in \mathfrak{S}_{2n} | w^2 = 1\}$ to denote the natural identification of $\text{Bd}_n$ with the set of involutions in $\mathfrak{S}_{2n}$. For any $w \in \mathfrak{S}_{2n}$ and any $D \in \text{Bd}_n$, $D \ast w = \beta^{-1}(w^{-1}\beta(D)w)$.

For any $\mathbb{Z}$-algebra $R$, we use $\mathcal{B}_{n,R}$ to denote the free $R$-module spanned by all the Brauer $n$-diagrams in $\text{Bd}_n$. Then $\mathcal{B}_{n,R}$ becomes a right $R[\mathfrak{S}_{2n}]$-module. Let $\mathcal{B}_n := \mathcal{B}_{n,\mathbb{Z}}$. Clearly, there is a canonical isomorphism $\mathcal{B}_{n,R} \cong R \otimes_{\mathbb{Z}} \mathcal{B}_n$, which is also a right $R[\mathfrak{S}_{2n}]$-module isomorphism. Taking $R = \mathbb{Z}[x]$, we deduce that the Brauer algebra
$\mathcal{B}_n(x)$ becomes a right $\mathbb{Z}[x][\mathfrak{S}_{2n}]$-module. Similarly, the specialized Brauer algebra $\mathcal{B}_n(-2m)$ becomes a right $K[\mathfrak{S}_{2n}]$-module.

For any integer $i$ with $1 \leq i \leq 2n$, we define

$$\gamma(i) := \begin{cases} i + 1, & \text{if } i \text{ is odd}, \\ i - 1, & \text{if } i \text{ is even}. \end{cases}$$

Then $\gamma$ is an involution on $\{1, 2, \cdots, 2n\}$. It is well-known that the subgroup

$$\{ w \in \mathfrak{S}_{2n} | (\gamma(a))w = \gamma(aw) \text{ for any integer } a \text{ with } 1 \leq a \leq 2n \}$$

is isomorphic to the wreath product $\mathbb{Z}_2 \wr \mathfrak{S}_n$ of $\mathbb{Z}_2$ and $\mathfrak{S}_n$, which is a Weyl group of type $B_n$ (c.f. [23]).

**Lemma 2.1.** For any $\mathbb{Z}$-algebra $R$, there is a right $R[\mathfrak{S}_{2n}]$-module isomorphism

$$\mathcal{B}_{n,R} \cong \text{Ind}_{R[\mathbb{Z}_2 \wr \mathfrak{S}_n]}^{R[\mathfrak{S}_{2n}]} 1_R,$$

where $1_R$ denotes the rank one trivial representation of $R[\mathbb{Z}_2 \wr \mathfrak{S}_n]$.

**Proof.** Let $1_{\mathfrak{S}_{2n}}$ be the element in $\text{Bd}_n$ that connects $2i - 1$ to $2i$ for each integer $i$ with $1 \leq i \leq n$. Since $\mathfrak{S}_{2n}$ acts transitively on the set of all the Brauer $n$-diagrams, it is easy to see that the map $\xi_R$ which send $1_R$ to $1_{\mathfrak{S}_{2n}}$ extends naturally to a surjective $R[\mathfrak{S}_{2n}]$-module homomorphism from $\text{Ind}_{R[\mathbb{Z}_2 \wr \mathfrak{S}_n]}^{R[\mathfrak{S}_{2n}]} 1_R$ onto $\mathcal{B}_{n,R}$.

If $R$ is a field, then we can compare the dimensions of both modules. In that case, we know that the surjection $\xi_R$ must be an injection, and hence be an isomorphism. In general, since there are natural isomorphisms

$$\text{Ind}_{R[\mathbb{Z}_2 \wr \mathfrak{S}_n]}^{R[\mathfrak{S}_{2n}]} 1_R \cong R \otimes_{\mathbb{Z}} \text{Ind}_{\mathbb{Z}[\mathfrak{S}_{2n}]}^{\mathbb{Z}[\mathfrak{S}_{2n}]} 1_{\mathbb{Z}}, \quad \mathcal{B}_{n,R} \cong R \otimes_{\mathbb{Z}} \mathcal{B}_{n,\mathbb{Z}},$$

and $\xi_R$ is naturally identified with $1_R \otimes_{\mathbb{Z}} \xi_{\mathbb{Z}}$, it suffices to show that $\xi_{\mathbb{Z}}$ is an isomorphism. Note also that the short exact sequence

$$0 \to \text{Ker} \xi_{\mathbb{Z}} \to \text{Ind}_{\mathbb{Z}[\mathfrak{S}_{2n}]}^{\mathbb{Z}[\mathfrak{S}_{2n}]} 1_{\mathbb{Z}} \to \mathcal{B}_{n,\mathbb{Z}} \to 0$$

splits as $\mathbb{Z}$-modules. It follows that $\text{Ker} \xi_R$ is canonically isomorphic to $R \otimes_{\mathbb{Z}} \text{Ker} \xi_{\mathbb{Z}}$ for any $\mathbb{Z}$-algebra $R$. Let $N := \text{Ker} \xi_{\mathbb{Z}}$. It is enough to show that $N = 0$. By [11 Proposition 3.8], we only need to show that $N_{(p)} = 0$ for each prime number $p$. Let $k_p := \mathbb{Z}/(p)$, the residue field at the prime number $p$. It is clear that $k_p \cong \mathbb{Z}_{(p)}/(p)\mathbb{Z}_{(p)}$. Note that

$$N_{(p)}/(p)N_{(p)} \cong k_p \otimes_{\mathbb{Z}_{(p)}} N_{(p)} \cong k_p \otimes_{\mathbb{Z}_{(p)}} \text{Ker} \xi_{k_p} \cong \text{Ker} \xi_{k_p} = 0.$$

Applying Nakayama’s lemma ([11 2.6]), we conclude that $N_{(p)} = 0$. This completes the proof of the lemma. \qed
For any positive integer $k$ and any composition $\mu = (\mu_1, \cdots, \mu_s)$ of $k$, the Young diagram of $\mu$ is defined to be the set $[\mu] := \{(a,b) | 1 \leq a \leq s, 1 \leq b \leq \mu_a\}$. The elements of $[\mu]$ are called nodes of $\mu$. A $\mu$-tableau $t$ is defined to be a bijective map from the Young diagram $[\mu]$ to the set $\{1, 2, \cdots, k\}$. For each integer $a$ with $1 \leq a \leq k$, we define $\text{res}_t(a) = j - i$ if $t(i,j) = a$. We denote by $t_\mu$ the $\mu$-tableau in which the numbers $1, 2, \cdots, k$ appear in order along successive rows. The row stabilizer of $t_\mu$, denoted by $S_\mu$, is the standard Young subgroup of $S_k$ corresponding to $\mu$. For example, if $k = 6, \mu = (2, 3, 1)$, then

$$
\begin{array}{cccc}
1 & 2 & & \\
3 & 4 & 5 & \\
6 & & & \\
\end{array}
$$

$S_\mu$ = the subgroup of $S_6$ generated by $\{s_1, s_3, s_4\}$.

We define

$$
x_\mu = \sum_{w \in S_\mu} w, \quad y_\mu = \sum_{w \in S_\mu} (-1)^{\ell(w)} w,
$$

where $\ell(-)$ is the length function in $S_k$. If $\mu$ is a partition of $k$, we denote by $t_\mu$ the $\mu$-tableau in which the numbers $1, 2, \cdots, k$ appear in order along successive columns. Let $w_\mu \in S_k$ be such that $t_\mu w_\mu = t_\mu$. For example, if $k = 6, \mu = (3, 2, 1)$, then

$$
\begin{array}{cccc}
1 & 2 & 3 & \\
4 & 5 & 6 & \\
& & & \\
\end{array}
$$

$t_\mu = 4 \ 5 \ \ 1 \ 4 \ 6$, $t_\mu = 2 \ 5$, $w_\mu = (2, 4)(3, 6)$.

We use $P_n$ to denote the set of all the partitions of $n$. For any partition $\mu$ of $2n$, we define the associated Specht module $S^\mu$ to be the right ideal of the group algebra $\mathbb{Z}[S_{2n}]$ generated by $y_\mu w_\mu x_\mu$. In particular, $S^{(2n)}$ is the one-dimensional trivial representation of $S_{2n}$, while $S^{(1^{2n})}$ is the one dimensional sign representation of $S_{2n}$. By [13, Theorem 3.5] and [33, 5.3], our $S^\mu$ is isomorphic to the (dual) Specht module $\overline{S}^\mu$ introduced in [33, Section 5]. For any $\mathbb{Z}$-algebra $R$, we write $S^\mu_R := R \otimes_{\mathbb{Z}} S^\mu$. Then $\{S^\mu_R \mid \mu \vdash 2n\}$ is a complete set of pairwise non-isomorphic simple $\mathbb{Q}[S_{2n}]$-modules.

For any composition $\lambda = (\lambda_1, \cdots, \lambda_s)$ of $n$, let $2\lambda := (2\lambda_1, \cdots, 2\lambda_s)$, which is a composition of $2n$. We define $2P_n := \{2\lambda \mid \lambda \in P_n\}$.

**Lemma 2.2.** There is a right $\mathbb{Q}[S_{2n}]$-modules isomorphism:

$$
\mathfrak{B}_{n, \mathbb{Q}} \cong \bigoplus_{\lambda \in 2P_n} S^\lambda_{\mathbb{Q}}.
$$

**Proof.** This follows from Lemma 2.2 and [30, Chapter VII, (2.4)]. □
Let $a$ be an integer with $0 \leq a \leq n$. Let $1 \leq i_1, \cdots, i_a, j_1, \cdots, j_a \leq 2n$ be $2a$ pairwise distinct integers. Let 

$$I := \{1, 2, \cdots, 2n\} \setminus \{i_1, \cdots, i_a, j_1, \cdots, j_a\}.$$ 

Let $\mathcal{S}_I$ be the symmetric group on the set $I$. Let $\mathcal{I} := \{1, 2, \cdots, 2n\}$ 

$\setminus \{i_1, \cdots, i_a, j_1, \cdots, j_a\}$. Let 

$$Bd_{n}(\mathcal{I}, \mathcal{J}) := \{D \in Bd_n \mid D \text{ connects } i_s \text{ with } j_s \text{ for each } 1 \leq s \leq a\}.$$ 

**Lemma 2.3.** With the notations as above, for any $w \in \mathcal{S}_I$, we have 

$$\left( \sum_{D \in Bd_{n}(\mathcal{I}, \mathcal{J})} D \right) \ast w = \sum_{D \in Bd_{n}(\mathcal{I}, \mathcal{J})} D.$$ 

**Proof.** For any $D \neq D' \in Bd_{n}(\mathcal{I}, \mathcal{J})$, it is clear that 

$$D \ast w \neq D' \ast w \in Bd_{n}(\mathcal{I}, \mathcal{J}).$$ 

Therefore, the lemma follows easily from a counting argument. \qed

**Definition 2.4.** For any non-negative even integers $a, b$ with $a + b \leq 2n$, we define 

$$\begin{align*}
Bd(a)_{(b)} & := \left\{ D \in Bd_n \mid \begin{array}{l}
\text{the vertex labeled by } i \text{ is connected with } \theta(i) \text{ whenever } \\
i \leq a \text{ or } i > a + b
\end{array} \right\}, \\
X^{(a)}_{(b)} & := \sum_{D \in Bd(a)_{(b)}} D.
\end{align*}$$

For any even integer $k$ with $0 \leq k \leq 2n$, let $X_{(k)} := X^{(0)}_{(k)}$. 

**Definition 2.5.** Let $\lambda = (\lambda_1, \cdots, \lambda_s)$ be a composition of $2n$ such that $\lambda_i$ is even for each $i$, we define 

$$X_{\lambda} := X^{(0)}_{(\lambda_1)}X^{(\lambda_1)}_{(\lambda_2)} \cdots X^{(\lambda_1 + \cdots + \lambda_{s-1})}_{(\lambda_s)} \in \mathcal{B}_n.$$ 

**Corollary 2.6.** Let $\lambda = (\lambda_1, \cdots, \lambda_s)$ be a composition of $2n$ such that $\lambda_i$ is even for each $i$. Then, for any $w \in \mathcal{S}_\lambda$, we have that 

$$X_{\lambda} \ast w = X_{\lambda}.$$ 

**Proof.** For any non-negative even integers $a, b$ with $a + b \leq 2n$, the set $Bd(a)_{(b)}$ is just a special case of the set $Bd_{n}(\mathcal{I}, \mathcal{J})$ we defined before. Therefore, by Lemma 2.3 for any $w \in \mathcal{S}_{\{a+1, a+2, \cdots, a+b\}}$, we have 

$$X^{(a)}_{(b)} \ast w = X^{(a)}_{(b)}.$$
Now we note that the elements $X^{(\lambda_1)}_0, X^{(\lambda_2)}_1, \cdots, X^{(\lambda_1+\cdots+\lambda_{s-1})}_{\lambda_s}$ pairwise commute with each other. Hence the corollary follows at once. □

Let $k$ be a positive integer and $\mu$ be a composition of $k$. A $\mu$-tableau $t$ is called row standard if the numbers increase along rows. We use $\text{RowStd}(\mu)$ to denote the set of all the row-standard $\mu$-tableaux. Suppose $\mu$ is a partition of $k$. Then $t$ is called column standard if the numbers increase down columns, and standard if it is both row and column standard. In this case, it is clear that both $t^\mu$ and $t_\mu$ are standard $\mu$-tableaux. We use $\text{Std}(\mu)$ to denote the set of all the standard $\mu$-tableaux.

Suppose $\mu$ is a partition of $2n$. Let $\lambda \in 2P_n$. For any $t \in \text{RowStd}(\lambda)$, let $d(t) \in S_{2n}$ be such that $t^\lambda d(t) = t$. Let $X_{\lambda,t} := X^{(\lambda)}_t$. For any $\mathbb{Z}$-algebra $R$, we define

$$\mathcal{M}_R^\lambda := R\text{-Span}\left\{ X_{\nu,t} \mid t \in \text{Std}(\nu), \lambda \leq \nu \in 2P_n \right\},$$

where “$\geq$” is the dominance order defined in [34]. We write $\mathcal{M} = \mathcal{M}_2^n$. We are interested in the module $\mathcal{M}_R^\lambda$. In the remaining part of this paper, we shall see that this module is actually a right $\mathfrak{S}_{2n}$-submodule of $\mathfrak{B}_{n,R}$, and it shares many properties with the permutation module $x^\lambda R[\mathfrak{S}_{2n}]$. In particular, it also has a Specht filtration, and it is stable under base change, i.e., $R \otimes \mathbb{Z} \mathcal{M} \cong \mathcal{M}_R$ for any $\mathbb{Z}$-algebra $R$.

For our purpose, we need to recall some results in [34] and [31] on the Specht filtrations of permutation modules over the symmetric group $\mathfrak{S}_{2n}$. Let $\lambda, \mu$ be two partitions of $2n$. A $\mu$-tableau of type $\lambda$ is a map $S : [\mu] \to \{1, 2, \cdots, 2n\}$ such that each $i$ appears exactly $\lambda_i$ times. $S$ is said to be semistandard if each row of $S$ is weakly increasing and each column of $S$ is strictly increasing. Let $T_0(\mu, \lambda)$ be the set of all the semistandard $\mu$-tableaux of type $\lambda$. Then $T_0(\mu, \lambda) \neq \emptyset$ only if $\mu \geq \lambda$.

For each standard $\mu$-tableau $s$, let $\mu(s)$ be the tableau which is obtained from $s$ by replacing each entry $i$ in $s$ by $r$ if $i$ appears in row $r$ of $t^\lambda$. Then $\mu(s)$ is a $\mu$-tableau of type $\lambda$.

For each standard $\mu$-tableau $t$ and each semistandard $\mu$-tableau $S$ of type $\lambda$, we define

$$x_{S,t} := \sum_{s \in \text{Std}(\mu), \mu(s) = S} d(s)^{-1} x_\mu d(t).$$

Then by [34] Section 7, the set

$$\left\{ x_{S,t} \mid S \in T_0(\mu, \lambda), t \in \text{Std}(\mu), \lambda \leq \mu \vdash 2n \right\}$$
form a \( \mathbb{Z} \)-basis of \( x_{\lambda} \mathbb{Z}[\mathfrak{S}_{2n}] \). Furthermore, for any \( \mathbb{Z} \)-algebra \( R \), the canonical surjective homomorphism \( R \otimes \mathbb{Z} x_{\lambda} \mathbb{Z}[\mathfrak{S}_{2n}] \to x_{\lambda} R[\mathfrak{S}_{2n}] \) is an isomorphism.

For each partition \( \mu \) of \( 2n \) and for each semistandard \( \mu \)-tableau \( S \) of type \( \lambda \), according to the results in [34, Section 7] and [31], both the following \( \mathbb{Z} \)-submodules

\[
M_{\lambda}^S := \text{Z-Span}\left\{ x_{S,s}, x_{T,t} \mid s \in \text{Std}(\mu), T \in \mathcal{T}_0(\nu, \lambda), t \in \text{Std}(\nu), \mu \triangleright \nu \vdash 2n \right\},
\]

\[
M_{\lambda,\triangleright}^S := \text{Z-Span}\left\{ x_{T,t} \mid T \in \mathcal{T}_0(\nu, \lambda), t \in \text{Std}(\nu), \mu \triangleright \nu \vdash 2n \right\},
\]

are \( \mathbb{Z}[\mathfrak{S}_{2n}] \)-submodules, and the quotient of \( M_{\lambda}^S \) by \( M_{\lambda,\triangleright}^S \) is canonically isomorphic to \( S^\mu \) so that the image of the elements \( x_{S,s} \), where \( s \in \text{Std}(\mu) \), forms the standard \( \mathbb{Z} \)-basis of \( S^\mu \). In other words, it gives rise to the Specht filtrations of \( x_{\lambda} \mathbb{Z}[\mathfrak{S}_{2n}] \), each semistandard \( \mu \)-tableau of type \( \lambda \) yields a factor which is isomorphic to \( S^\mu \) so that \( x_{\lambda} \mathbb{Z}[\mathfrak{S}_{2n}] \) has a series of factors, ordered by \( \preceq \), each isomorphic to some \( S^\mu, \mu \triangleright \lambda \), the multiplicity of \( S^\mu \) being the number of semistandard \( \mu \)-tableaux of type \( \lambda \).

We write \( \mu = (\mu_1, \mu_2, \cdots) = (a_1^{k_1}, a_2^{k_2}, \cdots) \), where \( a_1 > a_2 > \cdots \), \( k_i \in \mathbb{N} \) for each \( i \), where \( a_i^{k_i} \) means that \( a_i \) repeats \( k_i \) times. Let \( \tilde{S}_\mu \) be the subgroup of \( \mathfrak{S}_{\mu'} \) consisting of all the elements \( w \) satisfying the following condition: for any integer \( t \geq 1 \), and any integers \( i, j \) with \( i \geq 1 \), \( j \leq t \leq \sum_{s=1}^{t} k_s \), and any integers \( a, b \) with \( 1 \leq a, b \leq \sum_{s=1}^{t} k_s \),

\[
(t_\mu(i, a))w = t_\mu(j, a) \quad \text{if and only if} \quad (t_\mu(i, b))w = t_\mu(j, b).
\]

Let \( \tilde{D}_\mu \) be a complete set of right coset representatives of \( \tilde{S}_\mu \) in \( \mathfrak{S}_{\mu'} \).

**Lemma 2.8.** For any partition \( \lambda \in 2\mathcal{P}_n \), let

\[
n_{\lambda} := \prod_{i \geq 1} (\lambda_i - \lambda_{i+1} + 1), \quad h_{\lambda} := \sum_{w \in \tilde{D}_\lambda} (-1)^{\ell(w)} w.
\]

Then

\[
X_\lambda * (w_\lambda y_{\lambda'}) = n_{\lambda} (X_\lambda * (w_{\lambda} h_{\lambda})),
\]

and for any \( \mathbb{Z} \)-algebra \( R \), \( 1_R \otimes \mathbb{Z} (X_\lambda * (w_{\lambda} h_{\lambda})) \neq 0 \) in \( \mathfrak{B}_{n,R} \).

**Proof.** The condition \( \lambda \in 2\mathcal{P}_n \) implies that for any \( w \in \tilde{S}_\lambda \), \( \ell(w) \) is an even integer. The first statement of this lemma now follows from the following identity:

\[
(X_\lambda * w_{\lambda}) * \left( \sum_{w \in \tilde{S}_\lambda} w \right) = n_{\lambda} X_\lambda * w_{\lambda}.
\]
Let $d$ be the Brauer $n$-diagram in which the vertex labeled by $t_\lambda(i, 2j - 1)$ is connected with the vertex labeled by $t_\lambda(i, 2j)$ for any $1 \leq i \leq \lambda'_1, 1 \leq j \leq \lambda_i/2$. Then it is easy to see that $d$ appears with coefficient 1 in the expression of $X_\lambda^*(w_\lambda h_\lambda)$ as linear combinations of basis of Brauer $n$-diagrams. It follows that for any $\mathbb{Z}$-algebra $R$, $1 \otimes_{\mathbb{Z}} (X_\lambda^*(w_\lambda h_\lambda)) \neq 0$ in $\mathfrak{B}_{n,R}$, as required.

Following [33], we define the Jucys-Murphy operators of $\mathbb{Z}[S_{2n}]$.

\[
\begin{aligned}
L_1 &: = 0, \\
L_a &= (a - 1, a) + (a - 2, a) + \cdots + (1, a), \quad a = 2, 3, \ldots, 2n.
\end{aligned}
\]

Then for each partition $\lambda$ of $2n$, and each integer $1 \leq a \leq 2n$, we have (by [13, (3.14)])

\[
(x_\lambda w_\lambda y_\lambda') L_a = \text{res}_{\lambda}(a) (x_\lambda w_\lambda y_\lambda').
\]

For each standard $\lambda$-tableau $t$, we define

\[
\Theta_t := \prod_{i=1}^{n} \prod_{u \in \text{Std}(\lambda), \text{res}_u(i) \neq \text{res}_t(i)} \frac{L_i - \text{res}_u(i)}{\text{res}_t(i) - \text{res}_u(i)}.
\]

For each partition $\lambda \in 2\mathcal{P}_n$, by Corollary 2.6 and Frobenius reciprocity, there is a surjective right $\mathbb{Z}[S_{2n}]$-module homomorphism $\pi_\lambda$ from $x_\lambda \mathbb{Z}[S_{2n}]$ onto $X_\lambda \mathbb{Z}[S_{2n}]$ which extends the map $x_\lambda \mapsto X_\lambda$. In particular, by Lemma 2.8

\[
(X_\lambda^* w_\lambda h_\lambda) L_a = \text{res}_{\lambda}(a) (X_\lambda^* (w_\lambda h_\lambda)).
\]

**Proposition 2.9.** For any partition $\lambda \in 2\mathcal{P}_n$, we have that

\[
[X_\lambda \mathbb{Q}[S_{2n}] : S^\mu_\mathbb{Q}] = 1.
\]

**Proof.** By Lemma 2.2, we have

\[
\mathfrak{B}_{n,\mathbb{Q}} \cong \bigoplus_{\mu \in 2\mathcal{P}_n} S^\mu_\mathbb{Q}.
\]

It is well-known that each $S^\mu_\mathbb{Q}$ has a basis $\{v_t\}_{t \in \text{Std}(\mu)}$ satisfying

\[
v_t L_i = \text{res}_t(i) v_t, \quad \forall 1 \leq i \leq n.
\]

Let $\lambda$ be a fixed partition in $2\mathcal{P}_n$. Since $X_\lambda \mathbb{Q}[S_{2n}] \subseteq \mathfrak{B}_{n,\mathbb{Q}}$, we can write

\[
X_\lambda^* (w_\lambda h_\lambda) = \sum_{\mu \in 2\mathcal{P}_n} \sum_{t \in \text{Std}(\mu)} A_t v_t,
\]

where $A_t \in \mathbb{Q}$ for each $t$.

For each $\mu \in 2\mathcal{P}_n$ and each $t \in \text{Std}(\mu)$, we apply the operator $\Theta_t$ on both sides of the above identity and use Lemma 2.8 and the above
discussion. We get that $A_t \neq 0$ if and only if $\mu = \lambda$ and $t = t_\lambda$. In other words, $X_\lambda * (w_{t_\lambda}) = A_{t_\lambda}v_{t_\lambda}$ for some $0 \neq A_{t_\lambda} \in \mathbb{Q}$. This implies that the projection from $X_\lambda \mathbb{Q}[\mathfrak{S}_{2n}]$ to $S_\mathbb{Q}^\lambda$ is nonzero. Hence, $[X_\lambda \mathbb{Q}[\mathfrak{S}_{2n}]: S_\mathbb{Q}^\lambda] = 1$, as required. \hfill \Box

For each partition $\lambda \in 2P_n$, by the natural surjective $\mathbb{Z}[\mathfrak{S}_{2n}]$-module homomorphism $\pi_\lambda$ from $x_\lambda \mathbb{Z}[\mathfrak{S}_{2n}]$ onto $X_\lambda \mathbb{Z}[\mathfrak{S}_{2n}]$, we know that the elements $\pi_\lambda(x_{S,t})$, where $S \in T_0(\mu, \lambda)$, $t \in \text{Std}(\mu)$, $\lambda \subseteq \mu \vdash 2n$, spans $X_\lambda \mathbb{Z}[\mathfrak{S}_{2n}]$ as $\mathbb{Z}$-module.

**Proposition 2.10.** For any two partitions $\lambda, \mu$ of $2n$, and for any $S \in T_0(\mu, \lambda)$, we have that $\pi_\lambda(M_S^\lambda) \subseteq M^\lambda$. In particular, $X_\lambda \mathbb{Z}[\mathfrak{S}_{2n}] \subseteq M^\lambda$.

**Proof.** We first prove a weak version of the claim in this proposition. That is, for any two partitions $\lambda, \mu$ of $2n$, and for any $S \in T_0(\mu, \lambda)$,

$$\pi_\lambda(M_S^\lambda) \subseteq M^\lambda_{\mathbb{Q}}.$$  

We consider the dominance order "$\preceq$" and make induction on $\lambda$. We start with the partition $(2n)$, which is the unique maximal partition of $2n$ with respect to "$\preceq$". In this case, $x_{(2n)} \mathbb{Z}[\mathfrak{S}_{2n}] = \mathbb{Z}x_{(2n)}$, and $X_{(2n)} \mathbb{Z}[\mathfrak{S}_{2n}] = \mathbb{Z}X_{(2n)}$, it is easy to see the claim in this proposition is true for $\lambda = (2n)$.

Now let $\lambda \vdash (2n)$ be a partition of $2n$. Assume that for any partition $\nu$ of $2n$ satisfying $\nu \triangleright \lambda$, the claim in this proposition is true. We now prove the claim for the partition $\lambda$.

Let $\mu \triangleright \lambda$ be a partition of $2n$ with $T_0(\mu, \lambda) \neq \emptyset$. We consider again the dominance order "$\preceq$" and make induction on $\mu$. Since $T_0((2n), \lambda)$ contains a unique element $S_*$, $\text{Std}((2n)) = \{t^{(2n)}\}$ and

$$\pi_\lambda(x_{S_*, t^{(2n)}}) = \pi_\lambda(x_{(2n)}) = X_{(2n)} \in M^\lambda.$$  

So in this case the claim of this proposition is still true.

Now let $\mu \triangleright \lambda$ be a partition of $2n$ with $T_0(\mu, \lambda) \neq \emptyset$ and $\mu \vdash (2n)$. Assume that for any partition $\nu$ of $2n$ satisfying $T_0(\nu, \lambda) \neq \emptyset$ and $\nu \triangleright \mu$, and for any $S \in T_0(\nu, \lambda)$,

$$\pi_\lambda(M_S^\lambda) \subseteq M^\lambda_{\mathbb{Q}}.$$  

Let $S \in T_0(\mu, \lambda)$. The homomorphism $\pi_\lambda$ induces a surjective map from $M_S^\lambda/M_{S, \triangleright}^\lambda$ onto

$$\left(\pi_\lambda(M_S^\lambda)\right)/\left(\pi_\lambda(M_{S, \triangleright}^\lambda)\right).$$  

Hence it also induces a surjective map $\tilde{\pi}_\lambda$ from

$$\left(\mathbb{Q} \otimes_{\mathbb{Z}} M_S^\lambda\right)/\left(\mathbb{Q} \otimes_{\mathbb{Z}} M_{S, \triangleright}^\lambda\right) \cong \mathbb{Q} \otimes_{\mathbb{Z}} \left(M_S^\lambda/M_{S, \triangleright}^\lambda\right) \cong S_\mathbb{Q}^\mu.$$
onto

\[ \mathbb{Q} \otimes_{\mathbb{Z}} \left( \pi_\lambda(M^\lambda_S)/\pi_\lambda(M^\lambda_{S_{\mu}},) \right). \]

Since \( S^\mu_Q \) is irreducible, the above map is either a zero map or an isomorphism. If it is a zero map, then (by induction hypothesis)

\[ \pi_\lambda(M^\lambda_S) \subseteq \pi_\lambda(M^\lambda_{S_{\mu}},) \mathbb{Q} \subseteq \mathcal{M}^\lambda_Q. \]

It remains to consider the case where \( \tilde{\pi}_\lambda \) is an isomorphism. In particular,

\[ \mathbb{Q} \otimes_{\mathbb{Z}} \left( \pi_\lambda(M^\lambda_S)/\pi_\lambda(M^\lambda_{S_{\mu}},) \right) \cong S^\mu_Q. \]

Applying Lemma 2.2, we know that \( \mu \in 2\mathcal{P}_n \).

On the other hand, the homomorphism \( \pi_\mu \) also induces a surjective map from \( x_\mu \mathbb{Z}[\mathfrak{S}_{2n}]/M^\mu_{S_{0,\mu}} \) onto

\[ \left( \pi_\mu(x_\mu \mathbb{Z}[\mathfrak{S}_{2n}]) \right)/\left( \pi_\mu(M^\mu_{S_{0,\mu}}) \right) = X_\mu \mathbb{Z}[\mathfrak{S}_{2n}]/\left( \pi_\mu(M^\mu_{S_{0,\mu}}) \right), \]

where \( S_0 \) is the unique semistandard \( \mu \)-tableau in \( \mathcal{T}_0(\mu, \mu) \). Hence it also induces a surjective map \( \tilde{\pi}_\mu \) from

\[ \left( \mathbb{Q} \otimes_{\mathbb{Z}} x_\mu \mathbb{Z}[\mathfrak{S}_{2n}]/(\mathbb{Q} \otimes_{\mathbb{Z}} M^\mu_{S_{0,\mu}}) \right) \cong \mathbb{Q} \otimes_{\mathbb{Z}} \left( x_\mu \mathbb{Z}[\mathfrak{S}_{2n}]/M^\mu_{S_{0,\mu}} \right) \cong S^\mu_Q \]
ton to

\[ \mathbb{Q} \otimes_{\mathbb{Z}} \left( X_\mu \mathbb{Z}[\mathfrak{S}_{2n}]/\pi_\mu(M^\mu_{S_{0,\mu}}) \right) \cong \left( X_\mu \mathbb{Q}[\mathfrak{S}_{2n}] \right)/\left( \mathbb{Q} \otimes_{\mathbb{Z}} \pi_\mu(M^\mu_{S_{0,\mu}}) \right). \]

By the Specht filtration of \( M^\mu_Q \), we know that \( S^\mu_Q \) does not occur as composition factor in \( \mathbb{Q} \otimes_{\mathbb{Z}} M^\mu_{S_{0,\mu}} \). Hence \( S^\mu_Q \) does not occur as composition factor in \( \mathbb{Q} \otimes_{\mathbb{Z}} \pi_\mu(M^\mu_{S_{0,\mu}}) \). By Proposition 2.9, \( S^\mu_Q \) occurs as composition factor with multiplicity one in \( X_\mu \mathbb{Q}[\mathfrak{S}_{2n}] \). Therefore, \( X_\mu \mathbb{Q}[\mathfrak{S}_{2n}] \neq \mathbb{Q} \otimes_{\mathbb{Z}} \pi_\mu(M^\mu_{S_{0,\mu}}) \). It follows that \( \tilde{\pi}_\mu \) must be an isomorphism. Hence

\[ \mathbb{Q} \otimes_{\mathbb{Z}} \left( X_\mu \mathbb{Z}[\mathfrak{S}_{2n}]/\pi_\mu(M^\mu_{S_{0,\mu}}) \right) \cong S^\mu_Q. \]

We write \( A = \pi_\lambda(M^\lambda_S), B = x_\mu \mathbb{Z}[\mathfrak{S}_{2n}] \). Since \( S^\mu_Q \) appears only once in \( \mathfrak{B}_{n,Q} \), it follows that \( S^\mu_Q \) must occur as composition factor in the module

\[ (\mathbb{Q} \otimes_{\mathbb{Z}} A) \cap (\mathbb{Q} \otimes_{\mathbb{Z}} B) = \mathbb{Q} \otimes_{\mathbb{Z}} (A \cap B). \]

Hence \( S^\mu_Q \) can not occur as composition factor in the module

\[ (\mathbb{Q} \otimes_{\mathbb{Z}} A)/(\mathbb{Q} \otimes_{\mathbb{Z}} (A \cap B)) \cong \mathbb{Q} \otimes_{\mathbb{Z}} (A/A \cap B). \]

Therefore, the image of the canonical projection \( \mathbb{Q} \otimes_{\mathbb{Z}} A \to \mathbb{Q} \otimes_{\mathbb{Z}} (A/A \cap B) \) must be contained in the image of \( \mathbb{Q} \otimes_{\mathbb{Z}} \pi_\lambda(M^\lambda_{S_{\mu}}) \). However,
by induction hypothesis, both \( \pi_\lambda(M^\lambda_{2^n}) \) and \( B \) are contained in the \( \mathbb{Q} \)-span of \( \left\{ X_{\alpha,u} \mid u \in \text{Std}(\alpha), \lambda \leq \alpha \in 2\mathcal{P}_n \right\} \). It follows that

\[
\pi_\lambda(M^\lambda_{2^n}) \subseteq M^\lambda_{2^n},
\]

as required.

Now we begin to prove \( \pi_\lambda(M^\lambda_{\lambda S}) \subseteq M^\lambda_{\lambda Q} \). Suppose that

\[
\pi_\lambda(M^\lambda_{\lambda S}) \not\subseteq M^\lambda_{\lambda Q}.
\]

Then (by the \( \mathbb{Z} \)-freeness of \( \mathcal{B}_n \)) there exist an element \( x \in M^\lambda_{\lambda S} \), some integers \( a, a_u \), and a prime divisor \( p \in \mathbb{N} \) of \( a \), such that

\[
a\pi_\lambda(x) = \sum_{\lambda \leq \alpha \in 2\mathcal{P}_n} \sum_{u \in \text{Std}(\alpha)} a_u X_\alpha * d(u),
\]

and

\[
\Sigma_p := \{ \alpha \in 2\mathcal{P}_n \mid \lambda \leq \alpha, p \not| a_u, \text{ for some } u \in \text{Std}(\alpha) \} \neq \emptyset.
\]

We take an \( \alpha \in \Sigma_p \) such that \( \alpha \) is minimal with respect to “\( \leq \)”. Then we take an \( u \in \text{Std}(\alpha) \) such that \( p \not| a_u \) and \( \ell(d(u)) \) is maximal among the elements in the set \( \{ u \in \text{Std}(\alpha) \mid p \not| a_u \} \). Let \( \sigma_u \) be the unique element in \( \mathcal{S}_{2n} \) such that \( d(u)\sigma_u = w_\alpha \) and \( \ell(w_\alpha) = \ell(d(u)) + \ell(\sigma_u) \).

We consider the finite field \( \mathbb{F}_p \) as a \( \mathbb{Z} \)-algebra. By [12, (4.1)], we know that for any partitions \( \beta, \gamma \) of \( 2n \), and element \( w \in \mathcal{S}_{2n} \),

\[
x_{\beta} w_{\gamma} \neq 0 \text{ only if } \gamma \geq \beta; \quad x_{\beta} w_{\gamma} \neq 0 \text{ only if } w \in \mathcal{S}_{\gamma} w_{\beta}.
\]

Hence by Lemma 2.8 and the homomorphism \( \pi_\lambda \),

\[
X_{\beta} * (w_{\gamma}) \neq 0 \text{ only if } \gamma \geq \beta; \quad X_{\beta} * (w_{\gamma}) \neq 0 \text{ only if } w \in \mathcal{S}_{\gamma} w_{\beta}.
\]

Using Lemma 2.8 again, we get

\[
0 = 1_{\mathbb{F}_p} \otimes_{\mathbb{Z}} \left( a\pi_\lambda(x) * (\sigma_u h_\alpha) \right) = 1_{\mathbb{F}_p} \otimes_{\mathbb{Z}} \left( a_u X_\alpha * (w_\alpha h_\alpha) \right) \neq 0,
\]

which is a contradiction. This prove that \( \pi_\lambda(M^\lambda_{2^n}) \subseteq M^\lambda_{2^n} \).

\[ \square \]

**Corollary 2.11.** For any partition \( \lambda \in 2\mathcal{P}_n \) and any \( \mathbb{Z} \)-algebra \( R \), \( M^\lambda_{\lambda R} \) is a right \( \mathcal{S}_{2n} \)-submodule of \( \mathcal{B}_{n,R} \).

**Proof.** This follows directly from Proposition 2.10. \( \square \)

**Theorem 2.12.** For any partition \( \lambda \in 2\mathcal{P}_n \) and any \( \mathbb{Z} \)-algebra \( R \), the canonical map \( R \otimes_{\mathbb{Z}} M^\lambda \rightarrow M^\lambda_R \) is an isomorphism, and the set

\[
\left\{ X_{\nu,t} \mid t \in \text{Std}(\nu), \lambda \leq \nu \in 2\mathcal{P}_n \right\}
\]

form an \( R \)-basis of \( M^\lambda_R \). In particular, the set

\[
\left\{ X_{\lambda,t} \mid t \in \text{Std}(\lambda), \lambda \in 2\mathcal{P}_n \right\}
\]
form an $R$-basis of $\mathcal{B}_{n,R}$.

Proof. We take $\lambda = (2^n)$, then $X_\lambda \mathbb{Z}[S_{2n}] = \mathcal{B}_n$. It is well-known that $\mathcal{B}_{n,R} \cong R \otimes \mathbb{Z} \mathcal{B}_n$ for any $\mathbb{Z}$-algebra $R$. Applying Proposition 2.10 and counting the dimension, we get that for any $\mathbb{Z}$-algebra $R$ which is a field, the set

$$\left\{ X_\lambda, t \mid t \in \text{Std}(\lambda), \lambda \in \mathcal{P}_n \right\}$$

must form an $R$-basis of $\mathcal{B}_{n,R}$. Since each basis element $X_\lambda, t$ is integrally defined and $\mathcal{B}_{n,R} \cong R \otimes \mathbb{Z} \mathcal{B}_n$ for any $\mathbb{Z}$-algebra $R$, it follows that the above set is still an $R$-basis of $\mathcal{B}_{n,R}$ for any $\mathbb{Z}$-algebra $R$.

By the $R$-linear independence of the elements in this set and Corollary 2.11 we also get that, for any partition $\lambda \in \mathcal{P}_n$, the set

$$\left\{ X_{\nu, t} \mid t \in \text{Std}(\nu), \lambda \prec \nu \in \mathcal{P}_n \right\}$$

must form an $R$-basis of $\mathcal{M}^{\lambda}_{R}$. Therefore, for any $\mathbb{Z}$-algebra $R$, the canonical map $R \otimes \mathcal{M}^{\lambda}_R \to \mathcal{M}^{\lambda}_{R}$ is an isomorphism. □

Remark 2.13. Note that for any partition $\lambda \in \mathcal{P}_n$, $X_\lambda \mathbb{Z}[S_{2n}] \subseteq \mathcal{M}^{\lambda}$. But $X_\lambda \mathbb{Z}[S_{2n}]$ is not necessarily equal to $\mathcal{M}^{\lambda}$ in general. For example, one sees easily that

$$X_{(6,2)} \notin X_{(4,4)} \mathbb{Z}[S_8].$$

In fact, if this is not the case, then we can write

$$X_{(6,2)} + \sum_i a_i X_{(4,4)} \ast w_i = \sum_i b_j X_{(4,4)} \ast w'_j,$$

for some positive integers $a_i, b_j$ and some elements $w_i, w'_j \in S_{2n}$. However, if we express both sides into linear combinations of Brauer 4-diagrams and count the number of terms, we find that this is impossible (as the equation $15 + 9a = 9b$ has no solutions in $\mathbb{Z}$).

Theorem 2.14. For any partition $\lambda \in \mathcal{P}_n$ and any $\mathbb{Z}$-algebra $R$, we define

$$\mathcal{M}^{\lambda}_{R} := R \text{-Span}\left\{ X_{\nu, t} \mid t \in \text{Std}(\nu), \lambda \prec \nu \in \mathcal{P}_n \right\}.$$

Then $\mathcal{M}^{\lambda}_{R}$ is a right $R[S_{2n}]$-submodule of $\mathcal{M}^{\lambda}_R$, and there is a $R[S_{2n}]$-module isomorphism

$$\mathcal{M}^{\lambda}_R / \mathcal{M}^{\lambda}_{R} \cong S^\lambda_R.$$
Proof. It suffices to consider the case where $R = \mathbb{Z}$. We first show that

$$M^\lambda_Q \cong \bigoplus_{\lambda \leq \mu \in 2P_n} S^\mu_Q, \quad M^{>\lambda}_Q \cong \bigoplus_{\lambda < \mu \in 2P_n} S^\mu_Q.$$ 

For each $\mu \in 2P_n$, we use $\rho^\lambda_\mu$ to denote the composition of the embedding $M^\lambda_Q \hookrightarrow \mathcal{B}_{n,Q}$ and the projection $\mathcal{B}_{n,Q} \twoheadrightarrow S^\mu_Q$. Suppose that $\rho^\lambda_\mu \neq 0$. Then $\rho^\lambda_\mu$ must be a surjection. We claim that $\mu \triangleright= \lambda$. In fact, if $\mu \ntriangleright= \lambda$, then for any $\lambda \leq \nu \in 2P_n$, $\mu \ntriangleright= \nu$, and $x_\nu \mathbb{Z}[S_{2\lambda}] w_\mu x_\mu w_\mu y_\nu = 0$, hence $X_{\nu,1} \ast (w_\mu x_\mu w_\mu y_\nu) = 0$ for any $t \in \text{Std} (\nu)$. It follows that $M^\lambda_Q (w_\mu x_\mu w_\mu y_\nu ) = 0$. Therefore, $S^\mu_Q (w_\mu x_\mu w_\mu y_\nu) = 0$. On the other hand, since $S^\mu_Q \cong x_\mu w_\mu y_\mu Q[S_{2\lambda}]$, and by [28] Lemma 5.7,

$$x_\mu w_\mu y_\mu (w_\mu x_\mu w_\mu y_\nu) = \left( \prod_{(i,j) \in [\mu]} h_{i,j}^\mu \right) x_\mu w_\mu y_\nu \neq 0,$$

where $h_{i,j}^\mu$ is the $(i,j)$-hook length in $[\mu]$, we get a contradiction. Therefore, $\rho^\lambda_\mu \neq 0$ must imply that $\mu \triangleright= \lambda$. Now counting the dimensions, we deduce that $M^\lambda_Q \cong \bigoplus_{\lambda \leq \mu \in 2P_n} S^\mu_Q$. In a similar way, we can prove that $M^{>\lambda}_Q \cong \bigoplus_{\lambda < \mu \in 2P_n} S^\mu_Q$. It follows that $M^\lambda_Q / M^{>\lambda}_Q \cong S^\lambda_Q$.

We now consider the natural map from $x_\lambda \mathbb{Z}[S_{2\lambda}]$ onto $M^\lambda_Q / M^{>\lambda}_Q \cong S^\lambda_Q$. Since $Q \otimes_{\mathbb{Z}} M^\lambda_\mathbb{S}_{0,\triangleright}$ does not contain $S^\lambda_Q$ as a composition factor, it follows that (by Proposition 2.10) the image of $M^\lambda_\mathbb{S}_{0,\triangleright}$ must be contained in $M^{>\lambda}_Q$. Therefore we get a surjective map from $S^\lambda_Q$ onto $M^\lambda_Q / M^{>\lambda}_Q \cong S^\lambda_Q$. This map sends the standard basis of $S^\lambda_Q$ to the canonical basis of $M^\lambda_Q / M^{>\lambda}_Q$. So it must be injective as well, as required. \hfill \Box

3. The $n$-tensor space $V^\otimes n$

In this section, we shall use the results obtained in Section 2 and in [11] to give an explicit and characteristic free description of the annihilator of the $n$-tensor space $V^\otimes n$ in the Brauer algebra $\mathcal{B}_n(-2m)$.

Let $K$ be an arbitrary infinite field. Let $m, n \in \mathbb{N}$. Let $V$ be a $2m$-dimensional symplectic $K$-vector space. Let $Sp (V)$ be the corresponding symplectic group, acting naturally on $V$, and hence on the $n$-tensor space $V^\otimes n$ from the left-hand side. As we mentioned in the introduction, this left action on $V^\otimes n$ is centralized by the specialized Brauer algebra $\mathcal{B}_n(-2m)_K := K \otimes_{\mathbb{Z}} \mathcal{B}_n(-2m)$, where $K$ is regarded as $\mathbb{Z}$-algebra by sending each integer $a$ to $a \cdot 1_K$. The Brauer algebra
$B_n(-2m)_K$ acts on the $n$-tensor space $V^\otimes n$ from the right-hand side. Let $\varphi$ be the natural $K$-algebra homomorphism

$$\varphi : (B_n(-2m)_K)^{\text{op}} \to \text{End}_K(V^\otimes n).$$

The following is one of the main results in [11], which generalize earlier results in [2], [3], [4] for the case when $K = \mathbb{C}$.

**Lemma 3.1.** ([11, (1.2)]) Let $K$ be an arbitrary infinite field. Then

$$\varphi(B_n(-2m)_K) = \text{End}_{KSp}(V^\otimes n),$$

and if $m \geq n$, then $\varphi$ is also injective, and hence an isomorphism.

By the discussion in [11, Section 3], $V^\otimes n$ is a tilting module over $KSp(V)$. By [14, (4.4)], the dimension of $\text{End}_{KSp(V)}(V^\otimes n)$ is independent of the choice of the infinite field $K$. Therefore, the dimension of $\text{Ker} \varphi := \{ y \in B_n(-2m)_K \mid \varphi(y) = 0 \}$ is also independent of the choice of the infinite field $K$. That is, the dimension of the annihilator of the $n$-tensor space $V^\otimes n$ in the Brauer algebra $B_n(-2m)_K$ is independent of the choice of the infinite field $K$.

**Lemma 3.2.** With the notations as above, we have that

$$\dim(\text{Ker} \varphi) = \sum_{\lambda \in 2P_n, \lambda_1 \geq 2m} \dim S^\lambda.$$

**Proof.** By Lemma 2.2 and Lemma 3.1 it suffices to consider the case where $K = \mathbb{C}$ and to show that

$$\dim \text{End}_{KSp_{2n}(V)}(V^\otimes n) = \sum_{\lambda \in 2P_n, \lambda_1 \leq 2m} \dim S^\lambda.$$

Note that $\dim S^\lambda = \dim S^{\lambda'}$, and

$$\text{End}_{KSp_{2n}(V)}(V^\otimes n) \cong \left((V^\otimes n) \otimes (V^\otimes n)^*\right)^{Sp(V)} \cong (V^\otimes 2n)^{Sp(V)}.$$

Therefore, it suffices to show that

$$\dim(V^\otimes 2n)^{Sp(V)} = \sum_{\lambda \in 2P_n, \lambda_1 \leq 2m} \dim S^{\lambda'}.$$

By the well-known Schur-Weyl duality between the general linear group $GL(V)$ and the symmetric group $\mathfrak{S}_{2n}$ on the $2n$-tensor space $V^\otimes 2n$, we know that there is a ($GL(V), \mathfrak{S}_{2n}$)-bimodules decomposition

$$V^\otimes 2n \cong \bigoplus_{\lambda \in 2P_n, \lambda \leq 2n, \ell(\lambda) \leq 2m} \tilde{\Delta}_\lambda \otimes S^\lambda,$$
where \( \widetilde{\Lambda} \) denotes the irreducible Weyl module with highest weight \( \lambda \) over \( GL(V) \). Here we identify \( \lambda \) with \( \lambda_1 \varepsilon_1 + \cdots + \lambda_{2n} \varepsilon_{2n} \), \( \varepsilon_1, \cdots, \varepsilon_{2n} \) are the fundamental dominant weights of \( GL(V) \). It follows that

\[
\dim(V^{\otimes 2n})_{Sp(V)} = \sum_{\lambda \in 2P_n, \lambda_1 \leq 2m} \dim((\widetilde{\Lambda} \downarrow_{Sp(V)})_{Sp(V)}) \dim(S^\lambda).
\]

By the branching law (see [29, Proposition 2.5.1]) from \( GL(V) \) to \( Sp(V) \), we know that

\[
\dim((\widetilde{\Lambda} \downarrow_{Sp(V)})_{Sp(V)}) = 1
\]

if \( \lambda' \in 2P_n \), and 0 otherwise. This proves that

\[
\dim(V^{\otimes 2n})_{Sp(V)} = \sum_{\lambda \in 2P_n, \lambda_1 \leq 2m} \dim S^\lambda,
\]

as required. \( \square \)

Let \( a, b \) be two integers such that \( 0 \leq a, b \leq n \) and \( a + b \) is even. Let

\[
I^\text{odd}_a := \{1, 3, 5, \cdots, 2a - 1\}, \quad I^\text{even}_b := \{2, 4, 6, \cdots, 2b\}.
\]

Let \( k := \max\{2a, 2b\} \). If \( k = 2a \), we define \( Bd_n(a,b) \) to be the set of all the Brauer \( n \)-diagrams \( D \) such that for each integer \( s \in \{1, 2, \cdots, 2b, 2b + 1, 2b + 3, \cdots, 2a - 1\} \), \( D \) connects the vertex labeled by \( s \) with the vertex labeled by \( t \) for some integer \( t \in \{1, 2, \cdots, 2b, 2b + 1, 2b + 3, \cdots, 2a - 1\} \setminus \{s\} \); and for each integer \( s \) with \( a + 1 \leq s \leq n \), \( D \) connects the vertex labeled by \( 2s - 1 \) with the vertex labeled by \( 2s \); and for each integer \( s \) with \( 1 \leq s \leq (a - b) / 2 \), \( D \) connects the vertex labeled by \( 2b + 4s - 2 \) with the vertex labeled by \( 2b + 4s \). If \( k = 2b \), we define \( Bd_n(a,b) \) to be the set of all the Brauer \( n \)-diagrams \( D \) such that for each integer \( s \in \{1, 2, \cdots, 2a, 2a + 2, 2a + 4, \cdots, 2b\} \), \( D \) connects the vertex labeled by \( s \) with the vertex labeled by \( t \) for some integer \( t \in \{1, 2, \cdots, 2a, 2a + 2, 2a + 4, \cdots, 2b\} \setminus \{s\} \); and for each integer \( b + 1 \leq s \leq n \), \( D \) connects the vertex labeled by \( 2s - 1 \) with the vertex labeled by \( 2s \); and for each integer \( 1 \leq s \leq (b - a) / 2 \), \( D \) connects the vertex labeled by \( 2a + 4s - 3 \) with the vertex labeled by \( 2a + 4s - 1 \).

**Lemma 3.3.** Let \( a, b \) be two integers such that \( 0 \leq a, b \leq n \) and \( a + b \) is even. Suppose that \( a + b \geq 2m + 2 \), then

\[
\sum_{D \in Bd_n(a,b)} D \in \text{Ker} \varphi.
\]
The proof of Lemma 3.3 is somewhat complicated and will be given in Section 4.

Given any two subsets \( A^{(1)} \subseteq \{1, 3, \cdots, 2n-1\} \), \( A^{(2)} \subseteq \{2, 4, \cdots, 2n\} \) with \( |A^{(1)}| + |A^{(2)}| \) is even, we set \( 2n_0 = |A^{(1)}| + |A^{(2)}| \), and

\[
\{a_1, a_2, \cdots, a_{2n-2n_0}\} := \{1, 2, \cdots, 2n\} \setminus (A^{(1)} \cup A^{(2)}).
\]

Let \( (i_1, j_1, i_2, j_2, \cdots, i_{n-n_0}, j_{n-n_0}) \) be a fixed permutation of \( \{a_1, a_2, \cdots, a_{2n-2n_0}\} \).

Let
\[
i := (i_1, i_2, \cdots, i_{n-n_0}), \quad j := (j_1, j_2, \cdots, j_{n-n_0}).
\]

We define \( \text{Bd}_{n}^{(1)}(A^{(1)}, A^{(2)}) \) to be the set of all the Brauer \( n \)-diagrams \( D \) such that for each integer \( s \in A^{(1)} \cup A^{(2)} \), \( D \) connects the vertex labeled by \( s \) with a vertex labeled by \( t \) for some integer \( t \in (A^{(1)} \cup A^{(2)}) \setminus \{s\} \), and for each integer \( s \) with \( 1 \leq s \leq n - n_0 \), \( D \) connects the vertex labeled by \( i_s \) with the vertex labeled by \( j_s \). Note that the set \( \text{Bd}_{n}(a, b) \) we defined before is a special case of the set \( \text{Bd}_{n}^{(1)}(A^{(1)}, A^{(2)}) \) we defined here.

**Corollary 3.4.** With the notations as above and suppose that \( 2n_0 = |A^{(1)}| + |A^{(2)}| \geq 2m + 2 \), then we have

\[
\sum_{D \in \text{Bd}_{n}^{(1)}(A^{(1)}, A^{(2)})} D \in \text{Ker} \varphi.
\]

**Proof.** Let \( n_1 = |A^{(1)}|, n_2 = |A^{(2)}| \). If \( n_1 \geq n_2 \), then for any Brauer diagram \( D \in \text{Bd}_{n}^{(1)}(A^{(1)}, A^{(2)}) \), there exist at least \( \frac{n_1 - n_2}{2} \) bottom horizontal edges between the vertices labeled by the integers in the following set

\[
\{2, 4, 6, \cdots, 2n\} \setminus A^{(2)}.
\]

As a result, we deduce that there exist elements \( \sigma_{A^{(1)}} \in \mathfrak{S}_{\{1, 3, \cdots, 2n-1\}} \), \( \sigma_{A^{(2)}} \in \mathfrak{S}_{\{2, 4, \cdots, 2n\}} \) and a Brauer diagram \( D_1 \), such that

(1) for any integer \( a \) with \( 1 \leq a \leq n_1 \), \( D_1 \) connects the vertex labeled by \( 2a - 1 \) with the vertex labeled by \( 2a \).

(2) \n
\[
\sigma_{A^{(1)}} \left( \sum_{D \in \text{Bd}_{n}^{(1)}(A^{(1)}, A^{(2)})} D \right) \sigma_{A^{(2)}} = D_1 \left( \sum_{D \in \text{Bd}_{n}(|A^{(1)}|, |A^{(2)}|)} D \right).
\]

In this case, since both \( \varphi(\sigma_{A^{(1)})} \) and \( \varphi(\sigma_{A^{(2)})} \) are invertible, it follows directly from Lemma 3.3 that \( \sum_{D \in \text{Bd}_{n}^{(1)}(A^{(1)}, A^{(2)})} D \in \text{Ker} \varphi. \)
If \( n_1 \leq n_2 \), then for any Brauer diagram \( D \in \text{Bd}_{i,j}^{n_1}(A^{(1)}, A^{(2)}) \), there exist at least \( \frac{n_2-n_1}{2} \) top horizontal edges between the vertices labeled by the integers in the following set
\[
\{1, 3, 5, \ldots, 2n - 1\} \setminus A^{(1)}.
\]
As a result, we deduce that there exist elements \( \sigma_{A^{(1)}} \in \mathfrak{S}_{(1,3,\ldots,2n-1)} \), \( \sigma_{A^{(2)}} \in \mathfrak{S}_{(2,4,\ldots,2n)} \) and a Brauer diagram \( D_2 \), such that

(3) for any integer \( a \) with \( 1 \leq a \leq n_2 \), \( D_2 \) connects the vertex labeled by \( 2a-1 \) with the vertex labeled by \( 2a \).

(4) \[
\sigma_{A^{(1)}} \left( \sum_{D \in \text{Bd}_{i,j}^{n_1}(A^{(1)}, A^{(2)})} D \right) \sigma_{A^{(2)}} = D_2 \left( \sum_{D \in \text{Bd}_{i,j}^{n_2}(A^{(1)}, A^{(2)})} D \right).
\]

By the same argument as before, we deduce that \( \sum_{D \in \text{Bd}_{i,j}^{n_1}(A^{(1)}, A^{(2)})} D \in \text{Ker} \varphi \) in this case. This completes the proof of the corollary. \( \square \)

The following is the main result of this section, which gives an explicit and characteristic free description of the annihilator of the \( n \)-tensor space \( V \otimes^n \) in the Brauer algebra \( \mathfrak{B}_n(-2m) \).

**Theorem 3.5.** With the notations as in Lemma 3.1 and Lemma 3.2, we have that
\[
\text{Ker} \varphi = \mathcal{M}_K^{(2m+2,2n-m-1)},
\]
where \( (2m+2,2n-m-1) : = (2m+2, \underbrace{2, \ldots, 2}_{n-m-1 \text{ copies}}) \), \( \mathcal{M}_K^{(2m+2,2n-m-1)} \) is the right \( K[\mathfrak{S}_{2n}] \)-module associated to \( (2m+2,2n-m-1) \) as defined in Section 2. In particular, \( \text{Ker} \varphi \) is a \( \mathfrak{S}_{2n} \)-submodule.

**Proof.** It is easy to see that for any partition \( \mu \in \mathcal{P}_n \), \( \mu \trianglerighteq (2m+2,2n-m-1) \) if and only if \( \mu_1 > 2m \). Therefore,
\[
\dim \mathcal{M}_K^{(2m+2,2n-m-1)} = \sum_{\lambda \in \mathcal{P}_n \setminus \mathcal{P}_{2n}, \lambda_1 > 2m} \dim S^\lambda.
\]
Applying Lemma 3.1 and Lemma 3.2 we see that to prove this theorem, it suffices to show that \( \mathcal{M}_K^{(2m+2,2n-m-1)} \subseteq \text{Ker} \varphi \). Equivalently, it suffices to show that for any partition \( \lambda = (\lambda_1, \ldots, \lambda_s) \in \mathcal{P}_n \) satisfying \( \lambda_1 > 2m \), and any \( w \in \mathfrak{S}_{2n} \), \( \varphi(X_\lambda \ast w) = 0 \).

By the definition of the element \( X_\lambda \), the action “\( \ast \)” and the multiplication rule of Brauer diagrams, we deduce that
\[
X_\lambda \ast w = \sum_{ij(D)} \sum_{D \in \text{Bd}_{i,j}^{n_1}(A^{(1)}, A^{(2)})} D,
\]
where

\[ A^{(1)} := \left\{ (i)w \mid i = 1, 2, 3, \cdots, \lambda_1 \right\} \cap \left\{ 1, 3, 5, \cdots, 2n - 1 \right\}, \]

\[ A^{(2)} := \left\{ (i)w \mid i = 1, 2, 3, \cdots, \lambda_1 \right\} \cap \left\{ 2, 4, 6, \cdots, 2n \right\}, \]

and \( |A^{(1)}| + |A^{(2)}| = 2n_0 = \lambda_1 \), and \( i := (i_1, i_2, \cdots, i_{n-n_0}) \), \( j := (j_1, j_2, \cdots, j_{n-n_0}) \) such that \((i_1, j_1, i_2, j_2, \cdots, i_{n-n_0}, j_{n-n_0}) \) is a permutation of the integers in \( \{1, 2, \cdots, 2n \} \setminus (A^{(1)} \cup A^{(2)}) \).

We now apply Corollary 3.4. It follows immediately that \( \varphi(X_\lambda \ast w) = 0 \) as required. This completes the proof of the theorem. \( \square \)

Remark 3.6. Let \( V_Z \) be a free \( Z \)-module with basis \( \{v_1, v_2, \cdots, v_{2m}\} \).

For any \( Z \)-algebra \( R \), we define \( V_R := R \otimes_Z V_Z \). The same formulae (see Section 1) define an action of the algebra \( \mathfrak{B}_n(-2m) \) on \( V^n_Z \), and hence an action of \( \mathfrak{B}_n(-2m)_R \) on \( V^n_R \). Let \( \mathcal{S}^S_R(m,n) \) (see [11, Section 2] and [35]) be the symplectic Schur algebra over \( R \). If \( R \) is a field, then \( \mathcal{S}^S_R(m,n) \) is a quasi-hereditary algebras over \( R \), and \( V^n_R \) is a tilting module over \( \mathcal{S}^S_R(m,n) \). Applying [14, (4.4)], we know that, for any \( Z \)-algebra \( R \), there is a canonical isomorphism

\[ R \otimes_Z \text{End}_{\mathcal{S}^S_R(m,n)}(V^n_Z) \cong \text{End}_{\mathcal{S}^S_R(m,n)}(V^n_R). \]

Note that

\[ \varphi(\mathfrak{B}_n(-2m)_R) \subseteq \text{End}_{\mathcal{S}^S_R(m,n)}(V^n_R) \]

By the main result in [11], we know that the above inclusion \( \subseteq \) can be replaced by \( = \) when \( R = K \) is an infinite field \( K \). In fact, this is always true for any \( Z \)-algebra \( R \) (by using some localization argument in commutative algebras). As a consequence, our Theorem 3.5 is also always true if we replace the infinite field \( K \) by any \( Z \)-algebra \( R \).

4. Proof of Lemma 3.3

We shall first fix some notations and convention. Note that the element \( \sum_{D \in \text{Bd}_n(a,b)} D \) in Lemma 3.3 actually lies in \( \mathfrak{B}_n \), we can choose to work inside the Brauer algebra \( \mathfrak{B}_n(-2m)_C \) in this section. Furthermore, throughout this section, we shall recover our original way of labeling of vertices in each Brauer \( n \)-diagram. That is, the vertices in each row of a Brauer \( n \)-diagram will be labeled by the indices \( 1, 2, \cdots, n \) from left to right. This way of labeling is more convenient when we need to express each Brauer diagram in terms of the standard
generators \(s_i, e_i\) for \(1 \leq i \leq n - 1\) and to consider the action of Brauer diagrams on the \(n\)-tensor space \(V^\otimes n\).

Let \(f\) be an integer with \(0 \leq f \leq \lfloor n/2 \rfloor\), where \(\lfloor n/2 \rfloor\) is the largest non-negative integer not bigger than \(n/2\). Let \(\nu = \nu_f := ((2^f), (n - 2f))\), where \((2^f) := (2, 2, \cdots, 2)\) and \((n - 2f)\) are considered as partitions of \(2f\) and \(n - 2f\) respectively. In general, a bipartition of \(n\) is a pair \((\lambda^{(1)}, \lambda^{(2)})\) of partitions of numbers \(n_1\) and \(n_2\) with \(n_1 + n_2 = n\). The notions of Young diagram, bitableaux, etc., carry over easily. Let \(t^\nu\) be the standard \(\nu\)-bitableau in which the numbers \(1, 2, \cdots, n\) appear in order along successive rows of the first component tableau, and then in order along successive rows of the second component tableau. We define

\[
\mathcal{D}_\nu := \left\{ d \in \mathcal{S}_n \middle| (t^{(1)}, t^{(2)}) = t^\nu d \text{ is row standard and the first column of } t^{(1)} \text{ is an increasing sequence when read from top to bottom} \right\}.
\]

For each partition \(\lambda\) of \(n - 2f\), we denote by \(\text{Std}_{2f}(\lambda)\) the set of all the standard \(\lambda\)-tableaux with entries in \(\{2f + 1, \cdots, n\}\). The initial tableau \(t^\lambda\) in this case has the numbers \(2f + 1, \cdots, n\) in order along successive rows. Again, for each \(t \in \text{Std}_{2f}(\lambda)\), let \(d(t)\) be the unique element in \(\mathcal{S}_{\{2f + 1, \cdots, n\}} \subseteq \mathcal{S}_n\) with \(d^\nu d(t) = t\). Let \(\sigma \in \mathcal{S}_{\{2f + 1, \cdots, n\}\}}\) and \(d_1, d_2 \in \mathcal{D}_\nu\). Then \(d_1^{-1}e_1e_3\cdots e_{2f-1}\sigma d_2\) corresponds to the Brauer \(n\)-diagram where the top horizontal edges connect \((2i - 1)d_1\) and \((2i)d_1\), the bottom horizontal edges connect \((2i - 1)d_2\) and \((2i)d_2\), for \(i = 1, 2, \cdots, f\), and the vertical edges connect \((j)d_1\) with \((j)d_2\) for \(j = 2f + 1, 2f + 2, \cdots, n\).

**Lemma 4.1.** ([4 Corollary 3.3]) With the above notations, the set

\[
\left\{ d_1^{-1}e_1e_3\cdots e_{2f-1}\sigma d_2 \middle| 0 \leq f \leq \lfloor n/2 \rfloor, \sigma \in \mathcal{S}_{\{2f + 1, \cdots, n\}}, d_1, d_2 \in \mathcal{D}_\nu, \right\}
\]

is a basis of Brauer algebra \(B_n(x)_R\), which coincides with the natural basis given by Brauer \(n\)-diagrams.

Given an element \(d_1^{-1}e_1e_3\cdots e_{2f-1}\sigma d_2\) as above, let \(D\) be its representing Brauer \(n\)-diagram. Let \(v_i := v_{i_1} \otimes v_{i_2} \otimes \cdots \otimes v_{i_n}\) be a simple \(n\)-tensor in \(V^\otimes n\).

**Lemma 4.2.** With the notations as above, we have that

\[
v_i D = (-1)^{\ell(d_1^{-1}\sigma d_2)}(v_i \circ D),
\]

where \(v_i \circ D\) can be described as follows:

1) If \((j)d_1^{-1}\sigma d_2 = (k)\) for \(j \in \{(2f + 1)d_1, (2f + 2)d_1, \cdots, (n)d_1\}\), then the \(k\)th position of \(v_i \circ D\) is \(v_{i_j}\).
2) For each $1 \leq j \leq f$, the $((2j - 1)d_2, (2j)d_2)$th position of $v_\perp \circ D$ is the following sum:
\[
\epsilon_{i(2j-1)d_1} \cdot \epsilon_{(2j)d_1} \sum_{k=1}^{m} (v_{k'} \otimes v_{k} - v_{k} \otimes v_{k'}). \]

**Remark 4.3.** Intuitively, the action of the Brauer $n$-diagram $D$ on $v_\perp$ can be thought as follows. Let $(a_1, b_1), \cdots, (a_f, b_f)$ be the set of all the horizontal edges in the top row of $D$, where $a_s \prec b_s$ for each $s$ and $a_1 < a_2 < \cdots < a_f$. Let $(c_1, d_1), \cdots, (c_f, d_f)$ be the set of all the horizontal edges in the bottom row of $D$, where $c_s \prec d_s$ for each $s$ and $c_1 < c_2 < \cdots < c_f$. Then for each $1 \leq j \leq f$, the $(c_j, d_j)$th position of $v_\perp \circ D$ is the following sum:
\[
\epsilon_{i_{a_j}, i_{b_j}} \sum_{k=1}^{m} (v_{k'} \otimes v_{k} - v_{k} \otimes v_{k'}). \]

We list those vertices in the top row of $D$ which are not connected with horizontal edges from left to right as $i_{k_{2f+1}}, i_{k_{2f+2}}, \cdots, i_{k_n}$. Then, for each integer $j$ with $2f + 1 \leq j \leq n$, the $(j\sigma d_2)$th position of $v_\perp \circ D$ is $v_{i_{k_j}}$.

We fix an arbitrary element $d_2 \in \mathcal{D}_\nu$. We define
\[
\text{Bd}^{(f)}(n; d_2) := \{ d_1^{-1} e_1 e_3 \cdots e_{2f-1} \sigma d_2 \mid d_1 \in \mathcal{D}_\nu, \sigma \in S_{\{2f+1, 2f+2, \ldots, n\}} \}. \]

Note that $\text{Bd}^{(f)}(n; d_2)$ consists of all the Brauer $n$-diagrams whose bottom horizontal edges are
\[
((1)d_2, (2)d_2), ((3)d_2, (4)d_2), \cdots, ((2f-1)d_2, (2f)d_2). \]

**Lemma 4.4.** Let $f$ be an integer with $0 \leq f \leq [n/2]$. Let $d_2 \in \mathcal{D}_f$. Then for any $\sigma \in S_n$,
\[
\sigma \left( \sum_{D \in \text{Bd}^{(f)}(n; d_2)} D \right) = \sum_{D \in \text{Bd}^{(f)}(n; d_2)} D. \]

**Proof.** It suffices to show that for each integer $1 \leq i < n$,
\[
(4.5) \quad s_i \left( \sum_{D \in \text{Bd}^{(f)}(n; d_2)} D \right) = \sum_{D \in \text{Bd}^{(f)}(n; d_2)} D. \]

In fact, for $D, D' \in \text{Bd}^{(f)}(n; d_2)$ with $D \neq D'$, it is clear that $s_i D \neq s_i D'$, and both $s_i D$ and $s_i D'$ are still lie in $\text{Bd}^{(f)}(n; d_2)$. Now counting the number of Brauer $n$-diagrams occurring in both sides, we prove (4.5) and hence also prove the lemma. \qed
Similarly, we fix an arbitrary element $d_1 \in \mathcal{D}_\nu$ and define
\[
\text{Bd}^{(f)}(d_1; n) := \{ d_1^{-1}e_1e_3 \cdots e_{2f-1}\sigma d_2 \mid d_2 \in \mathcal{D}_\nu, \sigma \in \mathcal{S}_{\{2f+1, 2f+2, \ldots, n\}} \}.
\]
Then \(\text{Bd}^{(f)}(d_1; n)\) consists of all the Brauer \(n\)-diagrams whose top horizontal edges are
\[
((1)d_1, (2)d_1), ((3)d_1, (4)d_1), \ldots, ((2f-1)d_1, (2f)d_1).
\]
The following result can be proved in the same way as Lemma 4.4.

**Lemma 4.6.** Let \(f\) be an integer with \(0 \leq f \leq [n/2]\). Let \(d_1 \in \mathcal{D}_f\). Then for any \(\sigma \in \mathcal{S}_n\),
\[
\left( \sum_{D \in \text{Bd}^{(f)}(d_1; n)} D \right) \sigma = \sum_{D \in \text{Bd}^{(f)}(d_1; n)} D.
\]

Let \(i = (i_1, i_2, \ldots, i_n)\), where \(1 \leq i_j \leq 2m\) for each \(j\). An ordered pair \((s, t)\) \((1 \leq s < t \leq n)\) is called a *symplectic pair* in \(i\) if \(i_s = i'_t\). Two ordered pairs \((s, t)\) and \((u, v)\) are called disjoint if \(\{s, t\} \cap \{u, v\} = \emptyset\).

We define the *symplectic length* \(\ell_s(i)\) to be the maximal number of disjoint symplectic pairs \((s, t)\) in \(i\). We now consider a special case of Lemma 3.3.

**Proposition 4.7.** We have that
\[
\sum_{D \in \text{Bd}_n(m+1, m+1)} D \in \text{Ker} \varphi.
\]

**Proof.** By the above discussion and the definition of \(\text{Bd}_n(m+1, m+1)\), any Brauer diagram \(d \in \text{Bd}_n(m+1, m+1)\) only acts on the first \(m+1\) components of any simple \(n\)-tensor \(v_{i_1} \otimes v_{i_2} \otimes \cdots \otimes v_{i_n} \in V^\otimes n\). Therefore, to show that \(\sum_{D \in \text{Bd}_n(m+1, m+1)} D \in \text{Ker} \varphi\), we can assume without loss of generality that \(n = m + 1\). Note that
\[
\sum_{D \in \text{Bd}_n(m+1, m+1)} D = \sum_{0 \leq f \leq [n/2]} \sum_{d_2 \in \mathcal{D}_f} \sum_{D \in \text{Bd}^{(f)}(d_2)} D.
\]

Suppose that \(\sum_{D \in \text{Bd}_n(m+1, m+1)} D \notin \text{Ker} \varphi\). Then there exists a simple \(n\)-tensor \(v_i = v_{i_1} \otimes v_{i_2} \otimes \cdots \otimes v_{i_n} \in V^\otimes n\), such that
\[
\sum_{D \in \text{Bd}_n(m+1, m+1)} D \neq 0.
\]
Let \(i := (i_1, \ldots, i_n)\). Suppose that \(\ell_s(i) = f\) for some integer \(0 \leq f \leq [n/2]\).
By Lemma 4.4, we have that for any $\sigma \in \mathfrak{S}_n$,
\[ \sum_{D \in \text{Bd}_n(m+1,m+1)} v_D s_{(j,k)} D = \sum_{D \in \text{Bd}_n(m+1,m+1)} v_D D = 0, \]
where $s_{(j,k)}$ denotes the transposition $(j, k)$ in $\mathfrak{S}_n$, and we have used the fact that the length of $s_{(j,k)}$ is an odd integer. Therefore, we can deduce that $i_1, i_2, \ldots, i_n$ are pairwise distinct. Hence, $i_{2s-1} = i'_{2s} < i_{2s}$ for each integer $1 \leq s \leq f$, $i_1 < i_3 < \cdots < i_{2f-1}$ and $i_{2f+1} < i_{2f+2} < \cdots < i_n$.

By Lemma 4.2, we get that
\[ (4.8) \quad v_1 \sum_{0 \leq g \leq f} \sum_{d_1 \in \mathfrak{D}_g} \sum_{D \in \text{Bd}(g)(d_1;n)} D \neq 0. \]

By Lemma 4.6, it is easy to see that for any $w \in \mathfrak{S}_n$,
\[ v_1 \sum_{0 \leq g \leq f} \sum_{d_1 \in \mathfrak{D}_g} \sum_{D \in \text{Bd}(g)(d_1;n)} Dw = v_1 \sum_{0 \leq g \leq f} \sum_{d_1 \in \mathfrak{D}_g} \sum_{D \in \text{Bd}(g)(d_1;n)} D. \]

In particular, by the same argument as before, any simple $n$-tensor $v_{b_1} \otimes \cdots \otimes v_{b_n}$ with $b_j = b_k$ for some integers $j \neq k$ can not appear with nonzero coefficient in the expansion of the left-hand side of (4.8). Therefore if a simple $n$-tensor $v_D$ appears with nonzero coefficient in the expansion of the left-hand side of (4.8), then $b_1, \ldots, b_n$ must be pairwise distinct. Applying Lemma 4.4 and Lemma 4.6 again, we can further assume (without loss of generality) that there exists such a $v_D$ which appears with nonzero coefficient in the expansion of the left-hand side of (4.8), such that $b_1, \ldots, b_n$ are pairwise distinct, $\ell_s(b) = f$, $b_{2s-1} = b'_{2s} < b_{2s}$ for each integer $1 \leq s \leq f$, $b_1 < b_3 < \cdots < b_{2f-1}$, and there exists integer $0 \leq r \leq f$ such that $b_t = i_t$ for each integer $2r + 1 \leq t \leq n$, and $\{b_1, b_2, \ldots, b_{2r}\} \cap \{i_1, i_2, \ldots, i_{2r}\} = \emptyset$.

Let $g$ be an integer with $0 \leq g \leq f$, $d_1 \in \mathfrak{D}_g$, $D \in \text{Bd}(g)(d_1;n)$, where
\[ D = d_1^{-1} e_1 e_3 \cdots e_{2g-1} \sigma d_2, \quad \sigma \in \mathfrak{S}_{\{2g+1,2g+2,\ldots,n\}}, \quad d_2 \in \mathfrak{D}_g. \]

We claim that $v_D$ appears with nonzero coefficient in the expansion of $v_1 D$ if and only if
1) $g \geq r$, $\sigma = 1$, and
2) the horizontal edges in the top row of $D$ are of the form

$$(1, 2), (3, 4), \cdots, (2r - 1, 2r), (2a_1 - 1, 2a_1), (2a_2 - 1, 2a_2), \cdots,$$

$$(2a_{g-r} - 1, 2a_{g-r}),$$

where $a_1, \cdots, a_{g-r}$ are some integers satisfying $r + 1 \leq a_1 < a_2 < \cdots < a_{g-r} \leq f$, and

3) the horizontal edges in the bottom row of $D$ is the same as those in the top row of $D$, i.e., $d_2 = d_1$.

In fact, for any Brauer diagram $D$ satisfying the above conditions 1), 2), 3), by Remark 4.3, $v_b$ does appear with nonzero coefficient in the expansion of $v_i D$, the coefficient is $(-1)^g$; on the other hand, suppose that $v_b$ appears with nonzero coefficient in the expansion of $v_i D$. By our assumption on $i$ and $b$, it is easy to see that the tensor factor $v_{b_{2s-1}} \otimes v_{b_{2s}}$ with $1 \leq s \leq r$ can only be produced through the action of $e_{2t-1}$ for some $1 \leq t \leq g$. This implies that $g \geq r$. For each integer $j$ with $2g + 1 \leq j \leq n$, by Remark 4.3, the action of $D$ on $v_b$ move the vector in the $jd_1$th position of $v_b$ (i.e., $v_{j\sigma d_1}$) to the $(j\sigma d_2)$th position. By our assumption on $i$, $b$, again, we deduce that $jd_1 = j\sigma d_2$. But by the definition of $\mathcal{D}_g$,

$$(2g + 1)d_1 < (2g + 2)d_1 < \cdots < (n)d_1,$$

$$(2g + 1)d_2 < (2g + 2)d_2 < \cdots < (n)d_2.$$ It follows that $\sigma = 1$, and $jd_1 = j\sigma d_2$ for any $2g + 1 \leq j \leq n$. Now the remaining statements of our claim follows easily from the fact that $\sigma = 1$, our assumption on $i$ and $b$ as well as Remark 4.3.

Therefore, the coefficient of $v_b$ in the expansion of

$$\sum_{d_1 \in \mathcal{D}_g} \sum_{D \in Bd^{(g)}(d_1:n)} v_i D$$

is equal to

$$(-1)^g \binom{f - r}{g - r}.$$ Hence the coefficient of $v_b$ in the left-hand side of (4.8) is

$$\sum_{r \leq g \leq f} (-1)^g \binom{f - r}{g - r} = 0,$$

a contradiction. This completes the proof of the proposition. \hfill \Box

Next we consider a more general situation than Proposition 4.7, which is still a special case of Lemma 3.3.
Proposition 4.9. Let \(a, b\) be two integers such that \(0 \leq a, b \leq n\) and \(a + b\) is even. Suppose that \(a + b \geq 2m + 2\) and \(a \geq b\), then
\[
\sum_{D \in \text{Bd}_n(a, b)} D \in \text{Ker} \varphi.
\]

Proof. By the assumption that \(a \geq b\) and the definition of \(\text{Bd}_n(a, b)\), any Brauer diagram \(d \in \text{Bd}_n(a, b)\) only acts on the first \(a\) components of any simple \(n\)-tensor in \(V^\otimes n\). Therefore, we can assume without loss of generality that \(n = a\). Also, because of Proposition 4.7, we can assume that \(n = a > b\). In particular, \(n = a \geq m + 2, n + b \geq 2m + 2\). Suppose that \(\sum_{D \in \text{Bd}_n(a, b)} D \notin \text{Ker} \varphi\). Then there exists a simple \(n\)-tensor \(v_i \in V^\otimes n\) such that
\[
(4.10) \quad v_i \sum_{D \in \text{Bd}_n(a, b)} D \neq 0.
\]
Applying Lemma 4.4, we know that for any \(\sigma \in S_n\),
\[
(4.11) \quad \sigma \sum_{D \in \text{Bd}_n(a, b)} D = \sum_{D \in \text{Bd}_n(a, b)} D.
\]
Therefore, by the same argument as before, we deduce that \(i_1, \cdots, i_n\) are pairwise distinct.

Let \(f := (n - b)/2\). We define
\[
\Sigma_f = \left\{ (a, b) := ((a_1, b_1), \cdots, (a_f, b_f)) \mid 1 \leq a_1 < \cdots < a_f \leq n, \quad a_i < b_i \text{ for each } 1 \leq i \leq f \right\}.
\]
For each \((a, b) \in \Sigma_f\), we define \(\text{Bd}_{a,b}(n, b)\) to be the set of all the Brauer \(n\)-diagrams in \(\text{Bd}_n(a, b)\) whose rightmost \(f\) horizontal edges in the top row are exactly those pairs in \((a, b)\). It is clear that
\[
\text{Bd}_n(a, b) = \bigsqcup_{(a,b) \in \Sigma_f} \text{Bd}_{a,b}(n, b).
\]
Therefore,
\[
(4.12) \quad v_i \sum_{D \in \text{Bd}_{a,b}(n, b)} D \neq 0,
\]
for some \((a, b) \in \Sigma_f\). Henceforth, we fix such an \((a, b)\). We list the integers in the set \(\{k| a_1 \leq k \leq n \} \setminus \{a_i, b_i| 1 \leq i \leq f\}\) as \(t_1, t_2, \cdots, t_{b+1-a_1}\) such that \(t_1 < t_2 < \cdots < t_{b+1-a_1}\). We define
\[
I_{a,b} := \left\{ \bar{j} := (j_1, \cdots, j_{b+1-a_1}) \mid 1 \leq j_1 < \cdots < j_{b+1-a_1} \leq b \right\}.
\]
For each element \(\sigma \in S_{b+1-a_1}\), we define \(\text{Bd}_{a,b}^{\sigma,j}(n, b)\) to be the set of all the Brauer \(n\)-diagrams in \(\text{Bd}_{a,b}(n, b)\) whose vertical edges connects
the vertex labeled by \( t_s \) in the top row with the vertex labeled by \( j_s \sigma \) in the bottom row for each integer \( s \) with \( 1 \leq s \leq b+1-a_1 \). It is clear that

\[
\mathrm{Bd}_{a,b}(n,b) = \bigcup_{\sigma \in \mathcal{S}_{b+1-a_1}} \mathrm{Bd}_{a,b}^{\sigma j}(n,b).
\]

Now let \( k := (k_1, \cdots, k_n) \) be such that \( v_k \) appears with nonzero coefficient in the expansion of (4.12). Using the same argument in the proof of Lemma 4.6, one can show that for any \( \sigma \in \mathcal{S}_b \)

\[
\left( \sum_{D \in \mathrm{Bd}_{a,n}(n,b)} D \right) \sigma = \sum_{D \in \mathrm{Bd}_{a,n}(n,b)} D.
\]

As a result, we deduce that \( k_1, \cdots, k_b \) are pairwise distinct. Furthermore, we can assume (if necessary, we replace \( v_i \) by \( v_i \sigma \) for some \( \sigma \in \mathcal{S}_b \)) that

\[
k_{b-s+1} = i_{b+2-a_1-s}, \quad \text{for} \quad s = 1, 2, \cdots, b+1-a_1.
\]

It follows that \( v_k \) appears with nonzero coefficient in the expansion of (4.13)

\[
v_k \sum_{D \in \mathrm{Bd}_{a,n}^{(0)}(n,b)} D,
\]

where \( \mathrm{Bd}_{a,b}^{(0)} := \mathrm{Bd}_{a,b}^{1,(a_1,a_1+1,\cdots,b)} \). We divide the remaining proof into two cases:

**Case 1.** \( \ell_s(i) = (n - b)/2 \). Then it follows from (4.13) and our assumption on \( \sigma \) that \( \{i_1, \cdots, i_{a_1-1}\} = \{k_1, \cdots, k_{a_1-1}\} \), and the vectors \( v_{i_1}, \cdots, v_{i_{a_1-1}} \) lie in a subspace with dimension

\[
\leq m - \frac{n-b}{2} - (b+1-a_1) \leq a_1 - 2.
\]

Hence \( i_s = i_t \) for some \( 1 \leq s \neq t \leq a_1 - 1 \). But (4.13) implies that

\[
(v_{i_1} \otimes \cdots \otimes v_{i_{a_1-1}}) \sum_{\sigma \in \mathcal{S}_{a_1-1}} \sigma \neq 0,
\]
a contradiction, as required.

**Case 2.** \( \ell_s(i) > (n - b)/2 \). Let \( W \) be the symplectic subspace generated by \( v_{k_1}, \cdots, v_{k_{a_1-1}} \). Then (4.13) implies that \( \dim W \leq 2(a_1 - 2) \). Hence \( \tilde{m} := \dim W/2 \leq a_1 - 2 \). Note that if \( i_s \notin W \) for some \( 1 \leq s \leq a_1 - 1 \), then we must have that \( i_s = i_t \) for some integer \( t \) with \( 1 \leq t \leq a_1 - 1 \). Furthermore, in this case, if we replace the tensor factors \( v_{i_1}, v_{i_t} \) in \( v_k \) by \( v_{k_1}, v_{k_t} \) respectively, then the coefficient of \( v_k \) in (4.13) changes at most one sign. Therefore, (4.13) implies that there
exists a simple $(a_1 - 1)$-tensor $\tilde{v}$ in $W^{\otimes a_1 - 1}$ such that $v_{k_1} \otimes \cdots \otimes v_{k_{a_1 - 1}}$ appears with non-zero coefficient in

$$\tilde{v} \sum_{D \in \text{Bd}_{a_1 - 1}(\tilde{m}, \tilde{m})} D,$$

where the above element $D$ is understood as element in the Brauer algebra $\mathfrak{B}_n(-2\tilde{m})$, acting on the $(a_1 - 1)$-tensor space $W^{\otimes a_1 - 1}$. This is impossible by Proposition 4.7. Hence we complete the proof of this proposition. \hfill $\square$

Finally, thanks to Proposition 4.9, to complete the proof of Lemma 3.3, we only need to prove the following proposition.

**Proposition 4.14.** Let $a, b$ be two integers such that $0 \leq a, b \leq n$ and $a + b$ is even. Suppose that $a + b \geq 2m + 2$ and $b \geq a$, then

$$\sum_{D \in \text{Bd}_n(a, b)} D \notin \text{Ker } \varphi.$$

**Proof.** By the assumption that $b \geq a$ and the definition of $\text{Bd}_n(a, b)$, any Brauer diagram $d \in \text{Bd}_n(a, b)$ only acts on the first $b$ components of any simple $n$-tensor in $V^{\otimes n}$. Therefore, we can assume without loss of generality that $n = b$. Also, because of Proposition 4.7, we can assume that $n = b > a$. In particular, $n = b \geq m + 2, n + a \geq 2m + 2$.

Suppose that $\sum_{D \in \text{Bd}_n(a, n)} D \notin \text{Ker } \varphi$. Then there exists a simple $n$-tensor $v_1 \in V^{\otimes n}$ such that

$$v_1 \sum_{D \in \text{Bd}_n(a, n)} D \neq 0. \tag{4.15}$$

It follows that $i_{a + 2s - 1} = i'_{a + 2s}$ for each integer $s$ with $1 \leq s \leq (n - a)/2$. In particular, $\ell_s(\tilde{d}) \geq (n - a)/2$. Using the same argument in the proof of Lemma 4.6, one can show that for any $\sigma \in \mathfrak{S}_a$,

$$\sigma \left( \sum_{D \in \text{Bd}_n(a, n)} D \right) = \sum_{D \in \text{Bd}_n(a, n)} D.$$

As a result, we deduce that $i_1, \ldots, i_a$ are pairwise different.

Let $f := (n - a)/2$. We define

$$\Sigma_f = \left\{ (a, b) : (a_1, b_1), \ldots, (a_f, b_f) \left| \begin{array}{c} 1 \leq a_1 < \cdots < a_f \leq n, \\ a_i < b_i \text{ for each } 1 \leq i \leq f \end{array} \right\} \right. \cup \left\{ (a, b) : (a_1, b_1), \ldots, (a_f, b_f) \left| \begin{array}{c} 1 \leq a_1 < \cdots < a_f \leq n, \\ a_i < b_i \text{ for each } 1 \leq i \leq f \end{array} \right\} \right.

$$

For each $(a, b) \in \Sigma_f$, we define $\text{Bd}_{a, b}(a, n)$ to be the set of all the Brauer $n$-diagrams in $\text{Bd}_n(a, n)$ whose rightmost $f$ horizontal edges in
the bottom row are exactly those pairs in \((a, b)\). It is clear that
\[
\text{Bd}_n(a, n) = \bigsqcup_{(a, b) \in \Sigma_f} \text{Bd}_{a, b}(a, n).
\]

Therefore,
\[
(4.16) \quad v_i \sum_{D \in \text{Bd}_{a, b}(a, n)} D \neq 0,
\]
for some \((a, b) \in \Sigma_f\). Henceforth, we fix such an \((a, b)\). We list the integers in the set \(\{k|a_1 \leq k \leq n\}\) \(\setminus \{a_i, b_i|1 \leq i \leq f\}\) as \(t_1, t_2, \ldots, t_{a+1-a_1}\) such that \(t_1 < t_2 < \cdots < t_{a+1-a_1}\). We define
\[
I_{a, b} := \left\{ j := (j_1, \ldots, j_{a+1-a_1}) \middle| 1 \leq j_1 < \cdots < j_{a+1-a_1} \leq a \right\}.
\]

For each element \(\sigma \in \mathfrak{S}_{a+1-a_1}\), we define \(\text{Bd}_{a, b}^{\sigma, j}(a, n)\) to be the set of all the Brauer \(n\)-diagrams in \(\text{Bd}_{a, b}(a, n)\) whose vertical edges connects the vertex labeled by \(t_s\) in the bottom row with the vertex labeled by \(j_s\sigma\) in the top row for each integer \(s\) with \(1 \leq s \leq a+1-a_1\). It is clear that
\[
\text{Bd}_{(a, b)}(a, n) = \bigsqcup_{\sigma \in \mathfrak{S}_{a+1-a_1}} \text{Bd}_{a, b}^{\sigma, j}(a, n).
\]

Now let \(k := (k_1, \cdots, k_n)\) be such that \(v_k\) appears with nonzero coefficient in the expansion of \((4.12)\). Applying Lemma 4.6 we know that for any \(\sigma \in \mathfrak{S}_n\),
\[
(4.17) \quad \left( \sum_{D \in \text{Bd}_{a, b}(a, n)} D \right) \sigma = \sum_{D \in \text{Bd}_{a, b}(a, n)} D.
\]

As a result, we deduce that \(k_1, \cdots, k_n\) are pairwise distinct. Furthermore, we can assume (if necessary, we replace \(v_k\) by \(v_k\sigma\) for some \(\sigma \in \mathfrak{S}_n\)) that
\[
i_{a-s+1} = k_{a+2-a_1-s} \quad \text{for} \quad s = 1, 2, \cdots, a+1-a_1.
\]

It follows that \(v_k\) appears with nonzero coefficient in the expansion of
\[
(4.18) \quad v_k \sum_{D \in \text{Bd}_{a, b}^{(0)}(a, n)} D,
\]
where \(\text{Bd}_{a, b}^{(0)} := \text{Bd}_{a, b}^{1, (a_1, a_1+1, \cdots, a)}\). We divide the remaining proof into two cases:
Case 1. \( \ell_s(i) = (n - a)/2 \). Then it follows from (4.18) and our results on \( i \) and \( k \) that \( \{i_1, \ldots, i_{a-1}\} = \{k_1, \ldots, k_{a-1}\} \), and the vectors \( v_{i_1}, \ldots, v_{i_{a-1}} \) lie in a subspace with dimension
\[
\leq m - \frac{n - a}{2} - (a + 1 - a_1) \leq a_1 - 2.
\]
Hence \( i_s = i_t \) for some \( 1 \leq s \neq t \leq a_1 - 1 \). But (4.18) implies that
\[
(v_{i_1} \otimes \cdots \otimes v_{i_{a-1}}) \sum_{\sigma \in \mathfrak{S}_{a_1-1}} \sigma \neq 0,
\]
a contradiction, as required.

Case 2. \( \ell_s(i) > (n - a)/2 \). Let \( W \) be the symplectic subspace generated by \( v_{k_1}, \ldots, v_{k_{a_1-1}} \). Note that \( \ell_s(k) = \ell_s(i) \). It follows that \( \dim W \leq 2(a_1 - 2) \). Hence \( \tilde{m} := (\dim W)/2 \leq a_1 - 2 \). Note that if \( i_s \not\in W \) for some \( 1 \leq s \leq a_1 - 1 \), then we must have that \( i_s = i_t \) for some integer \( t \) with \( 1 \leq t \leq a_1 - 1 \). Furthermore, in this case, if we replace the tensor factors \( v_{i_s}, v_i \) in \( v_k \) by \( v_{k_1}, v'_{k_1} \) respectively, then the coefficient of \( v_{k_2} \) in (4.18) changes at most one sign. Therefore, (4.18) implies that there exists a simple \( (a_1 - 1) \)-tensor \( \tilde{v} \) in \( W^{\otimes a_1 - 1} \) such that \( v_{k_1} \otimes \cdots \otimes v_{k_{a_1-1}} \) appears with non-zero coefficient in
\[
\tilde{v} \sum_{D \in \mathfrak{B}_{a_1-1}(\tilde{m}, \tilde{m})} D,
\]
where the above element \( D \) is understood as element in the Brauer algebra \( \mathfrak{B}_n(-2\tilde{m}) \), acting on the \( (a_1 - 1) \)-tensor space \( W^{\otimes a_1 - 1} \). This is impossible by Proposition 4.7. Hence we complete the proof of this proposition. \( \square \)

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