COMPLEXITY OF HOMOGENEOUS SPACES
AND GROWTH OF MULTIPlicITIES

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Abstract. The complexity of a homogeneous space $G/H$ under a
reductive group $G$ is by definition the codimension of general orbits
in $G/H$ of a Borel subgroup $B \subseteq G$. We give a representation-
theoretic interpretation of this number as the exponent of growth
for multiplicities of simple $G$-modules in the spaces of sections of
homogeneous line bundles on $G/H$. For this, we show that these
multiplicities are bounded from above by the dimensions of certain
Demazure modules. This estimate for multiplicities is uniform, i.e.,
it depends not on $G/H$, but only on its complexity.

1. Introduction

Let $G$ be a connected reductive group over an algebraically closed
field $\mathbb{k}$ of characteristic 0, and $H \subseteq G$ a closed subgroup. Consider the
homogeneous space $G/H$. Choose a Borel subgroup $B \subseteq G$. By lower
semicontinuity, general $B$-orbits in $G/H$ have maximal dimension. The
minimal (=typical) codimension $c = c(G/H)$ of $B$-orbits is called the
complexity of $G/H$. (Clearly, it does not depend on the choice of $B$
since all Borel subgroups are conjugate.) By the Rosenlicht theorem
[VP, 2.3], $c(G/H)$ equals the transcendence degree of $\mathbb{k}(G/H)^B$ over $\mathbb{k}$.

This numerical invariant plays an important rôle in the geometry
of $G/H$. For instance, the class of homogeneous spaces of complexity
zero, called spherical spaces, is particularly nice [Kn1], [Bri2], [Vin]. It
includes many classical spaces, all symmetric spaces, etc. Also the equi-
variant embedding theory of $G/H$ depends crucially on its complexity,
see [LV], [Tim].

In this note, we describe $c(G/H)$ in terms of representation theory
related to $G/H$.

Let $\Lambda = \Lambda(B)$ be the weight lattice of $B$, and $\Lambda_+ \subseteq \Lambda$ be the set
of dominant weights. By $V(\lambda)$ denote the simple $G$-module of highest
weight $\lambda \in \Lambda_+$. For any rational $G$-module $M$, let $\text{mult}_\lambda M$ denote the
multiplicity of $V(\lambda)$ in $M$.
It turns out that $c(G/H)$ characterizes the growth of multiplicities in spaces of global sections of $G$-line bundles over $G/H$. Here is our main result:

**Theorem 1.** The complexity $c(G/H)$ is the minimal integer $c$ such that $\text{mult}_\lambda H^0(G/H, \mathcal{L}) = O(|\lambda|^c)$ over all $\lambda \in \Lambda_+$ and all $G$-line bundles $\mathcal{L} \to G/H$, where $| \cdot |$ is any fixed norm on the vector space spanned by $\Lambda$. Moreover, this estimate for multiplicities is uniform over all $H \subseteq G$ such that $c(G/H) = c$. If $G/H$ is quasiaffine, then it suffices to consider only $\text{mult}_\lambda \mathbb{k}[G/H]$.

A weaker version of Theorem 1 under some restrictive conditions (multiplicities in $\mathbb{k}[G/H]$ for quasiaffine $G/H$ provided that $\mathbb{k}[G/H]$ is finitely generated; no uniform estimate) appeared in [AP]. The relation between complexity and growth of multiplicities is known for quite a time, see partial results in [Pa1, 1.1], [Pa2, 2.4], [Bri2, 1.3].

We prove Theorem 1 in Section 2. The idea is to embed the “space of multiplicity” in the dual of a certain Demazure submodule in $V(\lambda)$, associated with an element of length $c$ in the Weyl group (Lemma 2).

In Section 3 we justify the term “complexity” by providing a much more precise information on multiplicities on homogeneous spaces of complexity $\leq 1$. Actually, the spherical case is well known [VK] and is included in the text only for convenience of the reader. In the case of complexity 1, a formula similar to ours for multiplicities in $\mathbb{k}[G/H]$ provided that it is a finitely generated unique factorization domain, and $G/H$ is quasiaffine, was obtained in [Pa1, 1.2], see also [Pa2, 2.4.19].

**Acknowledgements.** This note was written during my stay at Institut Fourier in spring 2003. I would like to thank this institution for hospitality, and M. Brion for invitation and for stimulating discussions. Thanks are also due to I. V. Arzhantsev for some helpful remarks.

**Notation.**

- The character lattice of an algebraic group $H$ is denoted by $\Lambda(H)$ and is written additively.
- By $M^H$ we denote the set of $H$-fixed points in the set $M$ acted on by a group $H$. If $M$ is a vector space, and $H$ acts linearly, then $M_\chi = M_\chi^H$ is the $H$-eigenspace of eigenweight $\chi \in \Lambda(H)$.
- Throughout the paper, $G$ is a connected reductive group, and $B \subseteq G$ a fixed Borel subgroup. The semigroup $\Lambda_+$ of dominant weights is considered relative to $B$. We fix an opposite Borel subgroup $B^-$, a maximal torus $T = B \cap B^-$, and consider the Weyl group $W$ of $G$ relative to $T$.
- By $\lambda^*$ denote the highest weight of the dual $G$-module to the simple $G$-module $V(\lambda)$ of highest weight $\lambda \in \Lambda_+$. 
2. Upper bound for multiplicities

We begin with basic facts about line bundles on homogeneous spaces. Every line bundle $\mathcal{L} \to G/H$ admits a $G$-linearization, i.e., a fiberwise linear $G$-action compatible with the projection onto the base, if we possibly replace $G$ by a finite cover [KKLY] §2. (Alternatively, one may replace $\mathcal{L}$ by its sufficiently big tensor power.) Every $G$-line bundle is isomorphic to a homogeneous bundle $\mathcal{L}(\chi) = \mathcal{L}_{G/H}(\chi) = G \times^H \mathbb{k}_\chi$, where $H$ acts on the fiber $\mathbb{k}_\chi \simeq \mathbb{k}$ via a character $\chi \in \Lambda(H)$. The bundle $\mathcal{L}(\chi)$ is trivial (regardless the $G$-linearization) iff $\chi$ is the restriction of a character of $G$.

The space of global sections $H^0(G/H, \mathcal{L}(\chi)) \simeq \mathbb{k}[G]_{-\chi}^{(H)}$ is a rational $G$-module, where $H$ acts on $G$ by right translations, and $G$ by left translations. By the Frobenius reciprocity [Jan] I.3.3–3.4, we have

$$\text{mult}_\chi H^0(G/H, \mathcal{L}(\chi)) = \dim V(\lambda^*)_{-\chi}^{(H)}.$$

In particular, $H^0(G/B, \mathcal{L}(-\lambda)) = V(\lambda^*)$ whenever $\lambda \in \Lambda_+$, and 0, otherwise (the Borel–Weil theorem).

Observe that for any rational $G$-module $M$ we have $\text{mult}_\lambda M = \dim M_{\lambda}^B$, $\forall \lambda \in \Lambda_+$. In particular,

$$\text{mult}_\lambda H^0(G/H, \mathcal{L}(\chi)) = \dim \mathbb{k}[G]_{(\lambda, -\chi)}^{(B \times H)}.$$

The nonzero spaces $\mathbb{k}[G]_{(\lambda, -\chi)}^{(B \times H)}$ represent complete linear systems of $B$-stable divisors on $G/H$, i.e., linear systems of pairwise rationally equivalent $B$-stable divisors which cannot be enlarged by adding new $B$-stable effective divisors. The respective biweights $(\lambda, \chi)$ form a subsemigroup $\Sigma = \Sigma(G/H) \subseteq \Lambda(B \times H)$. Two biweights $(\lambda, \chi), (\lambda', \chi') \in \Sigma$ determine the same linear system on $G/H$ iff $(\lambda, \chi)$ differs from $(\lambda', \chi')$ by a twist of $G$-linearization, i.e., by $(\epsilon|_B, -\epsilon|_H)$, $\epsilon \in \Lambda(G)$.

We need a useful result, essentially due to Brion. Replacing $H$ by a conjugate, we may assume that $\dim B(eH)$ is maximal among all $B$-orbits, i.e., $\text{codim } B(eH) = c = c(G/H)$.

**Lemma 1** (cf. [Brill 2.1]). There exists a sequence of minimal parabolics $P_1, \ldots, P_c \supset B$ such that $P_c \cdots P_1(eH) = G/H$. The decomposition $w = s_1 \cdots s_c$, where $s_i \in W$ are the simple reflections corresponding to $P_i$, is reduced, and $D_w = BwB = P_1 \cdots P_c$ is a “Schubert subvariety” in $G$ of dimension $c + \dim B$.

**Proof.** If $c > 0$, then $B(eH)$ is not $G$-stable, whence it is not stabilized by some minimal parabolic $P_1 \supset B$. Since $P_1/B \simeq \mathbb{P}^1$, the natural map $P_1 \times^B B(eH) \to G/H$ is generically finite, and $\text{codim } P_1(eH) = c - 1$. Continuing in the same way, we construct a sequence of minimal parabolics $P_1, \ldots, P_c \supset B$ such that $P_c \cdots P_1(eH) = G/H$, i.e., $P_c \cdots P_1 H$ is dense in $G$. The map $P_c \cdots P_1 \times^B B(eH) \to G/H$ is generically finite, hence $\dim P_c \cdots P_1 = c + \dim B$, which yields all the remaining assertions. \qed
Remark 1. The “Schubert subvarieties” \(D_w, w \in W\), form a monoid w.r.t. the multiplication of sets in \(G\), called the \(Richardson–Springer monoid\). It is generated by the \(D_s, s \in W\) a simple reflection, and is naturally identified with \(W\), the defining relations being \(s^2 = s\) and the braid relations for \(W\). The action of the Richardson–Springer monoid on the set of \(B\)-stable subvarieties (appearing implicitly in Lemma \[\text{[Kn2]}\]) is studied in \[\text{[Kn2]}\].

Let \(v_\lambda \in V(\lambda)\) be a highest weight vector. The \(B\)-submodule \(V_w(\lambda) \subseteq V(\lambda)\) generated by \(v_w\lambda, w \in W\), is called a Demazure module. In the notation of Lemma \[\text{[Kn2]}\] we have \(V_w(\lambda) = \langle D_wv_\lambda \rangle \cong H^0(S_w, \mathcal{L}_{G/B}(-\lambda))^*\), where \(S_w = D_w/B\) is a Schubert subvariety in \(G/B\) of dimension \(c\). Indeed, the restriction map \(V(\lambda^*) = H^0(G/B, \mathcal{L}(-\lambda)) \to H^0(S_w, \mathcal{L}(-\lambda))\) is surjective \[\text{[Jan, II.14.15, e]}\], and \(D_wv_\lambda\) is the affine cone over the image of \(S_w\) under the map \(G/B \to \mathbb{P}(V(\lambda))\).

**Lemma 2.** In the notation of Lemma \[\text{[Kn2]}\] \(\text{mult}_\lambda H^0(G/H, \mathcal{L}(\chi)) \leq \dim V_w(\lambda)\).

**Proof.** We show that the pairing between \(V(\lambda^*)\) and \(V(\lambda)\) provides an embedding \(V(\lambda^*)\chi \hookrightarrow V_w(\lambda)^*\). Otherwise, if \(v^* \in V(\lambda^*)\chi\) vanishes on \(V_w(\lambda)\), then it vanishes on \(D_wv_\lambda\), i.e., \(\langle D_w^{-1}v^* \rangle = \langle Gv^* \rangle = V(\lambda^*)\) vanishes at \(v_\lambda\), a contradiction. \(\square\)

**Remark 2.** A similar idea was used in \[\text{[Pa2, 2.4.18]}\] to obtain an upper bound for multiplicities in coordinate algebras of homogeneous spaces of complexity 1.

**Remark 3.** The assertion of Lemma \[\text{[Kn2]}\] can be refined and viewed in a more geometric context as follows. Consider the natural \(B\)-equivariant proper map \(\varphi : D_w^{-1}/B \cap H \cong D_w^{-1} \times B(eH) \to G/H\), which is generically finite by construction. For \(\mathcal{L} = \mathcal{L}_{G/H}(\chi)\) we have \(\varphi^* \mathcal{L} = \mathcal{L}_{D_w^{-1}/B \cap H}(\chi|_{B \cap H})\), and \(\varphi^* : H^0(G/H, \mathcal{L}) \hookrightarrow H^0(D_w^{-1}/B \cap H, \varphi^* \mathcal{L})\) gives rise to

\[
H^0(G/H, \mathcal{L}(\chi)) \hookrightarrow H^0(D_w^{-1}/B \cap H, \varphi^* \mathcal{L}(\chi)) = \mathbb{K}[D_w^{-1}(B \times B \cap H)]_{(\chi(-\lambda))}
\]

\[
\cong \mathbb{K}[D_w]_{(B \times B \cap H)} = H^0(S_w, \mathcal{L}(-\lambda))_{(-\chi)} = V_w(\lambda)^*_{(-\chi)}
\]

I am indebted to M. Brion for this remark.

Lemma \[\text{[Kn2]}\] applies to obtaining upper bounds for multiplicities in branching to reductive subgroups, cf. \[\text{[AP, Thm. 2]}\].

**Corollary.** If \(L \subseteq G\) is a connected reductive subgroup, then

\[
\text{mult}_\mu \text{res}^G_L V(\lambda) \leq \dim V_w(\lambda^*)
\]

for any two dominant weights \(\lambda, \mu\) of \(G, L\), respectively, where \(w \in W\) is provided by Lemma \[\text{[Kn2]}\] for \(H\) equal to a Borel subgroup of \(L\). Similarly,

\[
\text{length} \text{res}^G_L V(\lambda) \leq \dim V_w(\lambda^*)
\]
where \( w \in W \) corresponds to \( H \) equal to a maximal unipotent subgroup of \( L \). (Here length is the number of simple factors in an \( L \)-module.)

**Proof.** Just note that

\[
\text{mult}_\mu \text{res}^G_L V(\lambda) = \dim V(\lambda)_{\mu}^H = \text{mult}_\chi \cdot \text{H}^0(G/H, \mathcal{L}(-\mu))
\]

in the first case, and length \( \text{res}^G_L V(\lambda) = \dim V(\lambda)^H = \text{mult}_\chi \cdot \mathbb{k}[G/H] \)

in the second case, and then apply Lemma 2. \( \square \)

**Example 1.** Let \( P \supseteq B^- \) be a parabolic subgroup with the Levi decomposition \( P = L \rtimes P_u, L \supseteq T \). By \( w_L \) denote the longest element in the Weyl group of \( L \), and consider the decomposition \( w_G = w_L w^L \). Let \( H \) be the unipotent radical of \( B^- \cap L \). Then we may take \( w = w^L \) and obtain length \( \text{res}^G_L V(\lambda) \leq \dim V_{w^L}(\lambda^*) \).

**Proof of Theorem 1.** Recall the character formula for Demazure modules [Jan, II.14.18, b)]:

\[
\text{ch}_T V_w(\lambda) = \frac{1 - e^{-\alpha_1} s_1}{1 - e^{-\alpha_1}} \ldots \frac{1 - e^{-\alpha_c} s_c}{1 - e^{-\alpha_c}} e^{\lambda}
\]

where \( e^\mu \) is the monomial in the group algebra \( \mathbb{Z}[\Lambda] \) corresponding to \( \mu \in \Lambda \), \( \alpha_i \) are the simple roots defining \( P_i \), and \( s_i \in W \) are the respective simple reflections acting on \( \mathbb{Z}[\Lambda] \) in a natural way. One easily computes

\[
1 - e^{-\alpha_i} s_i = \begin{cases} e^\mu (1 + e^{-\alpha_i} + \ldots + e^{-(\mu, \alpha_i)\alpha_i}), & (\mu, \alpha_i^\vee) \geq 0, \\ 0, & (\mu, \alpha_i^\vee) \leq -1, \\ -e^\mu (e^{\alpha_i} + e^{2\alpha_i} + \ldots + e^{1+(\mu, \alpha_i^\vee)\alpha_i}), & (\mu, \alpha_i^\vee) \leq -2, \end{cases}
\]

\( \forall i = 1, \ldots, c, \forall \mu \in \Lambda \), where \( \alpha_i^\vee \) is the respective simple coroot. It is then easy to deduce that \( \dim V_w(\lambda) = O(|\lambda|^c) \), and Lemma 2 yields the desired estimate in Theorem 1.

Alternatively, observing that \( S_w \) is a projective variety of dimension \( c \), one may deduce that \( \dim \text{H}^0(S_w, \mathcal{L}(-\lambda)) \) grows no faster than \( |\lambda|^c \) as follows.

Without loss of generality we may assume \( G \) to be semisimple and simply connected. Let \( \omega_1, \ldots, \omega_l \) be the fundamental weights of \( G \). Put \( X = \mathbb{P}_{S_w} (\mathcal{L}(\omega_1) \oplus \cdots \oplus \mathcal{L}(\omega_l)), \) a projective space bundle over \( S_w \) with fiber \( \mathbb{P}^{l-1} \). Let \( \mathcal{O}_X(1) \) be the antiautotological line bundle over \( X \), and \( \pi : X \to S_w \) the projection map. Then

\[
\pi_* \mathcal{O}_X(k) = \bigoplus_{k_1 + \cdots + k_l = k, \ k_i \geq 0} \mathcal{L}(-k_1 \omega_1 - \cdots - k_l \omega_l)
\]

\( R^i \pi_* \mathcal{O}_X(k) = 0, \ \forall i > 0 \)

Now \( R_w = \bigoplus_{k \geq 0} H^0(X, \mathcal{O}(k)) = \bigoplus_{\lambda \in \Lambda^+} H^0(S_w, \mathcal{L}(-\lambda)) \) is a \( \Lambda \)-graded algebra, and the quotient field of \( R_w \) is a monogenic transcendental extension of \( \mathbb{k}(X) \), hence it has transcendence degree \( n = c + l \). Furthermore, \( R_w \) is finitely generated. Indeed, \( R_w \) is a quotient of the
$G$-algebra $R = \bigoplus_{\lambda \in \Lambda_+} H^0(G/B, \mathcal{L}(-\lambda))$. The multihomogeneous components of $R$ are the simple $G$-modules $V(\lambda^*)$, hence $R$ is generated by its components with $\lambda = \omega_1, \ldots, \omega_l$. (The spectrum of $R_w$ is the so-called multicone over the Schubert variety $S_w$, studied in [KR]. For instance, this multicone has rational singularities.) Finally, a general property of finitely generated multigraded algebras implies that $\dim H^0(S_w, \mathcal{L}(-\lambda)) = O(|\lambda|^n \cdot \text{rk} \Lambda) = O(|\lambda|^c)$. This estimate is uniform in $H$, because there are finitely many choices for $w$. It remains to show that the exponent $c$ cannot be made smaller.

Let $f_1, \ldots, f_c$ be a transcendence base of $k(G/H)^B$. There exist a $G$-line bundle $\mathcal{L}$ and $B$-eigenvectors $\sigma_0, \ldots, \sigma_c \in H^0(G/H, \mathcal{L})$ of the same weight $\lambda$ such that $f_i = \sigma_i/\sigma_0$, $\forall i = 1, \ldots, c$. (Indeed, $\mathcal{L}$ and $\sigma_0$ may be determined by a sufficiently big $B$-stable effective divisor majorizing the poles of all $f_i$.)

Consider the graded algebra $R = \bigoplus_{n \geq 0} R_n$, $R_n = H^0(G/H, \mathcal{L}^\otimes n)_{n\lambda}^B$. Clearly, $\sigma_0, \ldots, \sigma_c$ are algebraically independent in $R$, whence

$$\text{mult}_{n\lambda} H^0(G/H, \mathcal{L}^\otimes n) = \dim R_n \geq \binom{n+c}{c} \sim n^c$$

This proves our claim.

Finally, if $G/H$ is quasiaffine, then there even exist $\sigma_0, \ldots, \sigma_c \in k[G/H]$ with the same properties. □

Remark 4. For a given $H$, the estimate of the multiplicity by $\dim V_w(\lambda)$ may be not sharp. However, it is natural to ask whether it is a sharp uniform estimate over all homogeneous spaces with given complexity. More precisely, we formulate the following

Question. Given an element $w \in W$ of length $c$, does there exist a subgroup $H \subseteq G$ such that $\text{mult}_\lambda H^0(G/H, \mathcal{L}(\chi)) = \dim V_w(\lambda)$ for sufficiently general $(\lambda, \chi) \in \Sigma(G/H)$?

Example 2. In the notation of Example 1, put $H = P_u$. Then always $\chi = 0$, so that $H^0(G/H, \mathcal{L}(\chi)) = k[G/H]$. We may take $w = w_L$. Then $V(\lambda^*)^H$ is a simple $L$-module of lowest weight $-\lambda$, and $V_w(\lambda)$ is the dual $L$-module of highest weight $\lambda$. It follows that $\text{mult}_\lambda k[G/H] = \dim V_w(\lambda)$.

3. Case of small complexity

Homogeneous spaces of complexity $\leq 1$ are distinguished among all homogeneous spaces by their nice behaviour. For instance, they have a well-developed equivariant embedding theory [LV, 8–9], [Tim, 2–5]. There are also more explicit formulæ for multiplicities in this case.

Theorem 2. In the above notation,

(1) If $c(G/H) = 0$, then $\text{mult}_\lambda H^0(G/H, \mathcal{L}(\chi)) = 1$, $\forall (\lambda, \chi) \in \Sigma$. 

(2) If \( c(G/H) = 1 \), then there exists a pair \((\lambda_0, \chi_0) \in \Sigma\), unique up to a shift by \((\varepsilon_B, -\varepsilon_H)\), \( \varepsilon \in \Lambda(G) \), such that

\[
\text{mult}_\lambda H^0(G/H, \mathcal{L}(\chi)) = n + 1
\]

where \( n \) is the maximal integer such that \((\lambda, \chi) - n(\lambda_0, \chi_0) \in \Sigma(G/H)\).

Proof. The assertion is well known in the case \( c = 0 \), and we prove it just to keep the exposition self-contained. Assuming the contrary yields two non-proportional \( B \)-eigenvectors \( \sigma_0, \sigma_1 \in H^0(G/H, \mathcal{L}(\chi)) \) of the same weight \( \lambda \). Hence \( f = \sigma_1/\sigma_0 \in \mathbb{k}(G/H)^B \), \( f \neq \text{const} \), a contradiction.

In the case \( c = 1 \), we have \( \mathbb{k}(G/H)^B \simeq \mathbb{k}(\mathbb{P}^1) \) by the Lüroth theorem. Consider the respective rational map \( \pi : G/H \dashrightarrow \mathbb{P}^1 \), whose general fibers are (the closures of) general \( B \)-orbits. By a standard argument, \( \pi \) is given by two \( B \)-eigenvectors \( \sigma_0, \sigma_1 \in H^0(G/H, \mathcal{L}(\chi_0)) \) of the same weight \( \lambda_0 \) for a certain \((\lambda_0, \chi_0) \in \Sigma \). Moreover, \( \sigma_0, \sigma_1 \) are algebraically independent, and each \( f \in \mathbb{k}(G/H)^B \) can be represented as a homogeneous rational fraction in \( \sigma_0, \sigma_1 \) of degree 0.

Now put \((\mu, \tau) = (\lambda, \chi) - n(\lambda_0, \chi_0) \), fix \( \sigma_\mu \in H^0(G/H, \mathcal{L}(\tau))|_{\mu}^{(B)} \), and take any \( \sigma_\lambda \in H^0(G/H, \mathcal{L}(\chi))|_{\lambda}^{(B)} \). Then \( f = \sigma_\lambda/\sigma_\mu \in \mathbb{k}(G/H)^B \), whence \( f = F_1/F_0 \) for some \( m \)-forms \( F_0, F_1 \) in \( \sigma_0, \sigma_1 \). We may assume the fraction to be reduced and decompose \( F_1 = L_1 \cdots L_m, F_0 = M_1 \cdots M_m \), as products of linear forms, with all \( L_i \) distinct from all \( M_j \).

Then \( \sigma_\lambda M_1 \cdots M_m = \sigma_\mu \sigma_\mu^0 L_1 \cdots L_m \).

Being fibers of \( \pi \), the divisors of \( \sigma_0, L_i, M_j \) on \( G/H \) either coincide or have no common components. By the maximality of \( n \), the divisor of \( \sigma_\mu \) does not majorize any one of \( M_j \). Therefore \( M_1 = \cdots = M_m = \sigma_0 \), \( m \leq n \), and \( \sigma_\lambda/\sigma_\mu \) is an \( n \)-form in \( \sigma_0, \sigma_1 \). The assertion follows.

Example 3. Let \( G = \text{SL}_3, H = T \) (the diagonal torus), \( B \) be the upper-triangular subgroup. The space \( G/H \) can be regarded as the space of ordered triangles in \( \mathbb{P}^2 \), i.e., the set of all ordered triples \( p = (p_1, p_2, p_3) \in \mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2 \) such that \( p_1, p_2, p_3 \) do not lie on one line.

Let \( \ell_i \subset \mathbb{P}^2 \) denote the line joining \( p_j \) and \( p_k \), where \((i, j, k)\) is a cyclic permutation of \((1, 2, 3)\). By \( p_0 \) denote the \( B \)-fixed point in \( \mathbb{P}^2 \), and by \( \ell_0 \) the \( B \)-stable line.

There are the following \( B \)-stable prime divisors on \( G/H \):

\[
D_i = \{ p \mid p_i \in \ell_0 \} = \text{div} \ g_{3i}, \quad \lambda_i = \omega_2, \ \chi_i = -\varepsilon_i
\]

\[
D'_i = \{ p \mid p_0 \in \ell_i \} = \text{div} \ \Delta_i, \quad \Delta_i = \begin{vmatrix} g_{2j} & g_{2k} \\ g_{3j} & g_{3k} \end{vmatrix}, \quad \lambda_i' = \omega_1, \ \chi_i' = \varepsilon_i
\]

\[
D_t = B \cdot p(t) = \text{div} (g_{32} \Delta_2 + t g_{33} \Delta_3), \quad \lambda_0 = \omega_1 + \omega_2, \ \chi_0 = 0
\]

where \( i = 1, 2, 3, t \in \mathbb{P}^1 \setminus \{0, 1, \infty\} \), and the vertices of the triangle \( p(t) \) are: \( p_1(t) = (0 : 0 : 1), p_2(t) = (0 : 1 : 1), p_3(t) = (1 : t : 1) \).
Here \( g_{ij} \) are matrix entries of \( g \in G \), and \( H \)-semiinvariant polynomials in \( g_{ij} \) are regarded as sections of \( G \)-line bundles on \( G/H \). We also indicate their biweights \((\lambda, \chi) \in \Sigma\), denoting by \( \omega_i \) the fundamental weights, and by \( \varepsilon_i \) the diagonal matrix entries of \( H \). Observe that
\[
g_{31} \Delta_1 + g_{32} \Delta_2 + g_{33} \Delta_3 = 0.
\]

It follows that \( c(\text{SL}_3/T) = 1 \). Now it is an easy combinatorial exercise to deduce from Theorem 2 that \( \mu_\lambda H^0(\text{SL}_3/T, \mathcal{L}(\chi)) = n + 1 \), where
\[
n = \frac{k_1 + k_2}{2} - \frac{1}{6} \sum_{i=1}^{3} |k_1 - k_2 + 2l_i - l_j - l_k|
\]
whenever \((\lambda, \chi) \in \Sigma\), \( \lambda = k_1 \omega_1 + k_2 \omega_2 \), \( \chi = l_1 \varepsilon_1 + l_2 \varepsilon_2 + l_3 \varepsilon_3 \); and \((\lambda, \chi) \in \Sigma\) whenever \( k_1 - k_2 \equiv l_1 + l_2 + l_3 \pmod{3} \) and \( n \geq 0 \).

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