Space of Reasons and Mathematical Model

Logical Understanding III

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Abstract

Inferential relations govern our concept use. In order to understand a concept it has to be located in a “space of implications” (Wilfrid Sellars). There are different kinds of conditions for statements, i.e. that the conditions represent different kinds of explanations, e.g. causal or conceptual explanations. The crucial questions is: How can the conditionality of language use be represented. The conceptual background of representation in models is discussed and in the end I propose how implications of propositional logic and conceptual determinations can be represented in a model of a neural network.

1 Introduction

1.1 Context – Commitments

In the last paper it was shown that conceptual relations exist within a space of implications. This means that statements are embedded in a logical space of conditions. There are different kinds of conditions for statements: they can represent reasons, conceptual relations, causes, or also motivational states, which explain actions. Inferring from the statements to its conditions is part of abductive reasoning. Charles Sanders Peirce is very honest about his own mistake of confusing inductive reasoning and abductive reasoning (hypothesis): “Only in almost everything I printed before the beginning of this century I more or less mixed up Hypothesis and Induction...”[12] (CP 8.227) This mix up is very common.

1.2 Problem

Abductive inferences are also often forgotten in the context of reasoning. Behind the confusion of induction and abdution lies the problematic point of mathematical representation of reasoning. Induction is easier to represent and can be based on statistical tools. Markov logic e.g. is a combination of statistics and predicate logic. Universal formulas, e.g. “All M are K.” \( \forall x(xM \rightarrow xK) \), can represent rules or conceptual relations (characteristics). But what happens if we encounter a counterexample? Only one counterexample makes them false,
but if they are taken as weighted, valid, statistical regularities, this problem might be bypassed. Therefore, if a (possible) world violates one formula, then the probability of the formula of being true goes down. If there are only few worlds violated, the formula (statement) might probably be true. The worlds are constraints on the universally quantified formulas: “A first-order KB can be seen as a set of hard constraints on the set of possible worlds: if a world violates even one formula, it has zero probability.” Domingos et al. (2016) state that the “basic idea in Markov logic is to soften these constraints: when a world violates one formula in the KB it is less probable, but not impossible. The fewer formulas a world violates, the more probable it is. Each formula has an associated weight that reflects how strong a constraint it is: the higher the weight, the greater the difference in log probability between a world that satisfies the formula and one that does not, other things being equal.”

The crucial question is here, how the similarities or differences are calculated and with regard to which concept or idea. This is a question from abductive reasoning. E.g. a plastic tree can be in size and form similar to a real oak tree and is therefore with regard to size and form more similar than a bonsai tree, but with regard to the material the bonsai tree is more similar to an oak tree. Abductive reasoning is conceptually richer and can be found often in the field of engineering and in philosophy of technology. The following table is from Christoph Hubig[7] (208):

| Inference on the case | Perceptual abduction | Conceptual abduction (middle term) |
|----------------------|----------------------|----------------------------------|
| Inference on the rule | object of perception  | characteristics as relevant for building a class |
| Inference on the best explanation | explanation of perception | conceptual rule of subsumption |
| Inference on the best strategy of explanation | strategy of perception | system of classification |
Causal abduction

Presuppositional abduction

| Inference on the case | cause | means, instrument as reliable/proven |
|----------------------|-------|-------------------------------------|
| Inference on the rule | lawlike connections | techniques as yielding results |
| Inference on the best explanation | theories | sciences, technologies |
| Inference on the best strategy of explanation | paradigms, patterns of interpretation | general principle of relation to the world, conception of technology |

How can these different kinds of inferences be represented in mathematical models? One example will be picked out to shape the kind of mathematical thinking behind causal reasoning.

2 Causal Reasoning – Explanations

In statistics one tries to identify the confounder not as a condition that messes up the data. The confounder is therefore not the real cause. The real cause has to be revealed in order to make the data “meaningful”. According to Judea Pearl, statistics lacks the idea of causality and statisticians are only interested in correlation. That is a strong statement that might not be applicable for every statistician. Pearl writes that “[d]ata are the ingredients that go into the estimand recipe. It is critical to realize that data are profoundly dumb about causal relationships. They tell us about quantities like P(L | D) or P(L | D, Z). It is the job of the estimand to tell us how to bake these statistical quantities into one expression that, based on the model assumptions, is logically equivalent to the causal query – say, P(L | do(D)).” According to him the “whole notion of estimands does not exist in traditional methods of statistical analysis. There, the estimand and the query coincide.” Now, if we e.g. would be “interested in the proportion of people among those with Lifespan L who took the Drug D, we simply write this query as P(D | L). The same quantity would be our estimand.” This should require “no causal knowledge. For this reason, some statisticians to this day find it extremely hard to understand why some knowledge lies outside the province of statistics and why data alone cannot make up for lack of scientific knowledge.”

The role of this scientific knowledge has to be explained. Pearl thinks of a “ladder of causation” that leads from “correlation” (pure statistics) to “intervention”, which means to do to something and therefore to intervene in the chain of events, and the highest point is represented by “counterfactuals” that depict the intervention and are based on causal models about the world. Statistically, events or data “only” correlate, but they need to be seen through “causal models”, which are established through interventions, and serve to obtain causal
knowledge, that predicts effects. This is shall be done by the “algorithmization of counterfactuals”.[11] (9)

Pearl talks about structural models that have causal assumptions about the world. Also, in physics they are using models to represent the world, but do they use causal models, as a layperson might assume. Max Planck sees that the physical world view with its determined systems is a model and as a model it is an idealization, because normally there are measurement errors that could lead to (slightly) different results than what the model might predict. Measuring the quantities that are represented in the equations yields not always “exact” results. The physical model of the world is at least not affected by inexact measurements.[15] – Well, the role of quantum mechanics as an indetermined physical model has to be left out here.

What role does causality play in physics? And how do we usually use the term? We seem to know it from our everyday talk, when we say things like: “Peter came late, because of the heavy traffic.” So, the heavy traffic is the cause. We use causal connections to explain the behavior of persons or also the behavior of objects. Causality is a concept that is often used in philosophy, but not really in physics, even though we would expect that it plays a prominent role there.[15] We could e.g. say that a force or an event is the cause of a movement or of another event. Strangely, physicists do not use causality in that way. There is a difference according to Erhard Scheibe between the causality of states that are determined clearly and an irreversible causality of events. David Hume and Immanuel Kant examine the irreversible form of causality of events, that has at its core the asymmetry of the earlier and later. But already Isaac Newton and then Heinrich Hertz see the equations as the essence of the necessary link between states. The system is determined and can therefore be exhibited by the formation of equations which depict or represent quantities. The system is made calculable by quantification. If something is messing up the system – a confounder – it is seen as the cause that leads to a certain effect, but the cause is yet not integrated into the system to represent clearly its determination by an equation. The determination of the system by an equation is made clearer, when more an more confounders, which might have an influence on the system, are integrated in order to sustain its calculability and thus its predictability or can be disregarded as irrelevant.[15]

Models are internally determined and they are represented by inferential and differential relations[9], like the equations represent determined models. Usually, we use causality as an explanation for the behavior of persons (we produce something and are the cause or the producer of the cause (see the intervention part in Pearl)) and we use causality as an explanation in an analog manner to explain the behavior of objects, like the movements of the planets[9], but explanations in physics do not use the concept causality. Their models try to get rid of confounders or defeasors in order to have a “consistent” model.
3 Limits of Models as Means of Representation

In the debates in the philosophy of science, according to Hubig, the modeling of dispositional predicates in if-then-clauses, like the implication fails, because the implication would be true, if the antecedent is not realized or not realizable (“under-determination”) or have to be discarded, if the consequent has not been realized under the conditions of the antecedent (“over-determination”). Within these debates the approaches in the philosophy of science change to counterfactual conditionals as means of representation and search for prognostic or indicative sentences, which are equivalent to them. To avoid that the prognostic or indicative sentences turn to be counterintuitive, pragmatic conditions of the antecedent have to be added, which correspond to “conceptions of normality”. In this way, the validity of prognostic and indicative sentences can be guaranteed or warranted.\[8\]

Such conceptions of standardized cases cannot be represented by a complete list of conditions in the form of explicit rules. If that would be possible, then a set of premises, which would represent a complete list of the circumstances or conditions ($\Sigma$) as well as a rule as the major premise ($\Phi$), would always lead to an instantiation of the rule ($\varphi$) — this is a kind of deductive reasoning ($\Sigma, \Phi \Rightarrow \varphi$).\[9\] (213)

Hubig points out that we can not describe “complete states of the world”\[8\] (here $\Sigma$), but it is at least possible to describe a set of “potential defeasors”\[1\] (107) — also conditions of the antecedent, but the ones that should be avoided — like Robert Brandom writes, who, independently of Hubig, with reference to his ideas about “material inference”, sheds light on this issue. He states that the material inferences are “in general non-monotonic”.\[1\] (106) For him, that does not mean that the “potential defeasors” should be made explicit in a complete list. The task is therefore not to transform such inferences in monotonic ones.\[1\] (107) “The potential defeasors in this way associated with each material inference endorsed in turn define (by complementation) the range of counterfactual robustness practically associated with that inference.”\[1\] (108)

That is why the “counterfactual robustness” and the “potential defeasors” are necessary in order to understand the “conceptual content of sentences” or: “counterfactually robust inferences are an essential aspect of the articulation of the conceptual contents of sentences”.\[1\] (125) Brandom concludes from this, “that in view of the non-monotonicity of material inference, the practical task of updating the rest of one’s beliefs when some of them change is tractable in principle only if those who deploy a vocabulary practically discriminate ranges of

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1 The greek letters represent names in the metalanguage.

2 It is worth to mention that “the potential defeasors are not limited to sentences that are true”, because a “non-actual state of affairs is possible”. (125)

3 And no one supposes that such probative reasoning can always be turned into dispositive reasoning by making an explicit, exhaustive list of the potential defeasors.” Or: “There need be no definite totality of possible defeasors, specifiable in advance. Even where we have some idea how to enumerate them, unless those provisos are generally left implicit, actually stating the premises so as to draw inferences from them monotonically is impossibly cumbersome in practice.”
counterfactual robustness for many of the material inferences they endorse.”

Within such a framework it is not necessary to establish a set of circumstances or conditions, which (1.) describe a ‘complete’ state of the world and that is why it is not necessary to allow (2.) in a formal way an ‘infinite’ list of possible monotonic premises. One would either take the position of an absolute-deterministic or a formal and abstract standpoint of an empty metalanguage. From the first standpoint there would be the requirement, that there are no possible conditions that could serve as defeasors, which have to be subtracted or incorporated in order to assure the result of the inference. This would mean that all conditions are explicit. The last standpoint would allow to add every true premise like in a formal and monotonic logic. – Another way to assure the result of the consequence, the application or instantiation of the rule, would be to introduce a meta-rule, which settles the application, but that would lead to an infinite regress.

4 Models – Representational Tools

Theories use models in order to represent structures. The models encompass an object-language that is normally weaker in its expressive power than the metalanguage, because the metalanguage is here the natural language, in which we talk about the models and their structures. The models are not able to do that, because they have no representation of themselves.

Hubig and Michael Weingarten introduce an important distinction in the philosophy of science. Is something a model of or a model for something? The “models of...” are external “realizations, instantiations, exemplifications”. The structure (S) of the realization (R) is revealed via induction or abduction. If R fulfills every rule of the formulas (F) of S. At least, this is possible in an axiomatic system. (35) The “models for...” are “paradigmatic abstractions, i.e. one-sided pictures of structures”. Hubig calls them “conceptualized models” or a “model-idealism”. The only thing that counts is the “correctness of the use of the symbols”.

For example, a camera can be a model of the eye. The camera is a technical device that allows to explain the function of the eye. There is a symmetry assumed between the technical model (device) and the realization or instantiation of the model (eye). The model and the representation in the model (eye) belong as instantiations to the same class of natural laws. The function of both (camera and eye) falls under the class of the laws of optics. – It is often not seen anymore that the technical device is used as a model, because we are so accustomed to their use in our daily life, but also in the sciences.

The presupposed symmetry allows to establish an equivalence, but this is done on the level of the metalanguage, but for that it has to be clear that we are using a model to represent something else as something, e.g. the mind.

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4It might be also the other way around: e.g. an “extensional metalanguage for intensional languages” (“possible world semantics for modality”).
as a computer or a logical machine. Computation and computability are also certain aspects of human reasoning and were developed as a model of human reasoning. – Neural processes construct the reality and semantic significance. The starting point would be the neural organization of us and the emergence of mind and meaning out of it, but the supposed relation of equivalence is empty as long as the differences are not reflected. This happens when e.g. a technical device is used to form a model of something else without reflecting that it is a model, i.e. a tool to represent something. If we use the model for a certain purpose, the assumed symmetry has to prove itself in the world/reality. The model is then the means to accomplish the purpose and has to yield a certain effect. In neural networks the neurons are adjusted through backpropagation in order to bring the output closer to the desired output. One model for machine learning is the brain and it is therefore the means that is slotted in between the output and the desired output. It is a real possibility – the medium, in which the adjustments are done – and not just a formal possibility of manipulating symbols, because the specific adjustments are represented by the output. It is a representation via the model.

5 Representations of the Mind: Neural Networks

5.1 The Idea of Neural Networks

The idea is not to rebuild the human brain, but to use a model of it for machine learning. How are the neurons connected and how do they make connections? Neural networks are one-sided, paradigmatic pictures of the functioning of the brain. They are of course models of the brain, but are taken here as models for machine learning. Every neuron (approximately 100 billion in one human brain) has an axon and is via dendrites connected with other neurons. If a certain voltage exceeds a threshold the neuron fires. In neural networks the neurons are adjusted through backpropagation in order to bring the output closer to the desired output. One model for machine learning is the brain and it is therefore the means that is slotted in between the output and the desired output. It is a real possibility – the medium, in which the adjustments are done – and not just a formal possibility of manipulating symbols, because the specific adjustments are represented by the output. It is a representation via the model.

Warren McCulloch and Walter Pitts proposed in 1943 a model of a neuron. They propose an equivalence between the functioning of the neurons of the brain and propositional logic (which can be also represented by logic gates in computers). McCulloch and Pitts claim that “neither of us conceives the formal equivalence to be a factual explanation”. The physical behaviors of the neurons “in no way affect the conclusions which follow from the formal treatment of the activity of nervous nets”, i.e. the behavior of the neurons does not affect the formal connections of propositional logic represented by the logical operators. Propositional logic is the normative structure, which underlies the physical activity. (101) The determination of the states of the nervous net is based on necessary connections, which are irreciprocal (causality). There cannot be a complete knowledge (or determination), because of the incompleteness as to

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5 Causality is here conceived similar to the notion of determination (which is described above), because a certain kind of consistency of the system is required.
space and indefiniteness as to time, and the “inclusion of disjunctive relations prevents complete determination” on the logical side. Such a causal determination is equivalent to the logic of propositions and disjunctions can be added to determine the system step by step. – Even though they are concerned with the logical determination, the notion of disjunctive relations is based on the calculus of propositional logic and something else than conceptual determination (see below).

5.2 Perceptron

Frank Rosenblatt introduced then in 1958 the idea of a perceptron. The model is formulated “in terms of probability theory rather than symbolic logic.” The activations of the neurons are calculated via the “algebraic sum of excitatory and inhibitory impulse intensities”, which are “equal to or greater than the threshold ($\theta$)”. Perceptrons faced also several problems and in “1969, Minsky and his colleague Seymour Papert published Perceptrons, a book detailing the shortcomings of the eponymous algorithm, with example after example of simple things it couldn’t learn.” A learning algorithm is judged by its ability to learn and if there are basic things, that it can not do, it needs to be disregarded or improved, but the question is not only about learning, also about the ability to represent things. The problematic issue, that Pedro Domingos points out, is more about the expressive richness of the representational model. And in the case of the perceptron it cannot represent the “exclusive-OR function”, but a multilayer network can do it.

5.3 Hopfield-Model

John Hopfield, a physicist, suggested in 1982 an equivalence between neural nets and a physical system of magnetism (spin glasses). Hopfield writes:

“There are classes of physical systems whose spontaneous behavior can be used as a form of general (and error-correcting) content-addressable memory. Consider the time evolution of a physical system that can be described by a set of general coordinates. A point in state space then represents the instantaneous condition of the system. This state space may be either continuous or discrete (as in the case of $N$ Ising spins).

The equations of motion of the system describe a flow in state space. Various classes of flow patterns are possible, but the systems of use for memory particularly include those that flow toward locally stable points from anywhere within regions around those points. A particle with frictional damping moving in a potential well with two minima exemplifies such a dynamics.”

Wolfgang Ertel states that in the cases of pattern recognition, which are learned by the Hopfield-model and are trained by certain examples with a certain
amount of neurons, the states are finite. Like in the physical system the energy function will reach a minimum, but if there are too many patterns learned, it can lead to a chaotic dynamic and the system or neural network is not able to recognize the patterns. In the model of Hopfield the neurons can learn to recognize patterns, but only if there are enough neurons in the model, otherwise it changes from an ordered dynamic to a chaotic dynamic. Another problematic point is that the states in the model of Hopfield are binary. He uses a step function, while the activation of the neuron is now mostly calculated via the sigmoid function to propagate through the neural network.

5.4 Forward Propagation and Backpropagation

Backpropagating is a learning procedure for neural networks. It means to go back layer by layer and to adjust the weights, but first you need to propagate through the neural network. The calculation of the output or hypothesis is the sum of the weights:

\[ h_\theta(x) = \sigma \left( \sum_{i=1}^{n} \theta_{ji} x_i \right) \]

\( \theta_{ji} \) is the weight or parameter that connects a neuron of the layer \( i \) to a neuron on another layer \( j \). The activation of the neuron is calculated via the sigmoid function:

\[ \sigma(x) = \frac{1}{1+e^{-x}} \]

And if there is now a difference between the desired output and the real output, a learning algorithm propagates back. The loss function (sometimes also called cost function) is used to adjust the weights to minimize the difference between the actual output vector and the desired output vector.

6 Neural Networks: Intuition

The goal is to build a neural network that is able to represent the logical implication of propositional logic. The compatibilities and incompatibilities can be expressed in the following way.

\( p \rightarrow q \) is incompatible with \( \Diamond (p \land \neg q) \)

\( p \rightarrow q \) is compatible with \( \Diamond (\neg p \lor q) \)

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6A study of emergent collective effects and spontaneous computation must necessarily focus on the nonlinearity of the input-output relationship. The essence of computation is nonlinear logical operations. The particle interactions that produce true collective effects in particle dynamics come from a nonlinear dependence of forces on positions of the particles. [...] Those neurons whose operation is dominantly linear merely provide a pathway of communication between nonlinear neurons. Thus, we consider a network of ‘on or off’ neurons, granting that some of the interconnections may be by way of neurons operating in the linear regime. 

7For the backpropagation algorithm see Rumelhart et al. (1986) or a textbook like Ertel (2017).

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It is therefore possible to use the negation and the disjunction to build a neural network that has as its output the same truth values like the implication in propositional logic:

| $p$ | $q$ | $p \rightarrow q$ |
|-----|-----|------------------|
| t   | t   | t                |
| t   | f   | f                |
| f   | t   | t                |
| f   | f   | t                |

The inputs are $x_1$ and $x_2$ (above $p$ and $q$). The second layer contains $a_1^{[2]}$ and $a_2^{[2]}$. And $h_\theta(x)$ is the output of the function, which is computed by the variables and the parameters or weights ($\theta_1, \theta_2, \ldots$; as a matrix: $\Theta$). For this activation function the sigmoid function is used: $\sigma(x) = \frac{1}{1+e^{-x}}$ and a bias unit has to be added $b$. The function is thus:

$$h_\theta(x) = \sigma(b + \theta_1 x_1 + \theta_2 x_2)$$

To compute the negation:

$$h_\theta(x) = \sigma(5 - 10x_1 + 0x_2)$$

To compute the second layer $a_2^{[2]}$, which should have the same value as $x_2$:

$$h_\theta(x) = \sigma(-10 + 0x_1 + 20x_2)$$

To compute the disjunction of the second layer neurons:

$$h_\theta(a) = \sigma(-5 + 10a_1^{[2]} + 10a_2^{[2]})$$

The truth values that are calculated with the sigmoid function are approximations to 1 and 0. The calculation of the activation of the neurons has the same truth values like the implication in propositional logic:

| $x_1$ | $x_2$ | $a_1^{[2]}$ | $a_2^{[2]}$ | $h_\theta(x)$ |
|-------|-------|-------------|-------------|---------------|
| 1     | 1     | 0           | 1           | 1             |
| 1     | 0     | 0           | 0           | 0             |
| 0     | 1     | 1           | 1           | 1             |
| 0     | 0     | 1           | 0           | 1             |

Incompatibility of an implication ($\Diamond(p \land \neg q)$) can be represented in a similar way. The neural networks represent possible connections or states of conceptual relations and if the input or the information changes the parameters can be adjusted.

It was mentioned that McCulloch and Pitts introduce the notion of the “inclusion of disjunctive relations”, which “prevents complete determination” of the states of the neurophysiological and logical model. I propose the introduction of conjunctions that should be included in order to have a conceptual determination. The conjunctions can be added as inputs and connected with the second layer ($a_2^{[2]}$). They are possible determinations:
\neg p \lor \Diamond (q \land r)

Or:

\[ p \implies \Diamond (q \land \neg r) \]

These conceptual relations remain, even if \( p \) is not the case. The proposed model can represent implications and determines conceptually.

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