ISOSPECTRALITY OF MARGULIS-SMILGA SPACETIMES FOR IRREDUCIBLE REPRESENTATIONS OF REAL SPLIT SEMISIMPLE LIE GROUPS

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Abstract. In this article we show that, under certain conditions, equality of the Margulis-Smilga invariant spectra of two Margulis-Smilga spacetimes induce an automorphism of the ambient affine Lie group.

In particular, we show that equality of the Margulis-Smilga invariant spectra of two Margulis-Smilga spacetimes, coming from the adjoint representation of a real split simple algebraic Lie group $G$ with trivial center and Lie algebra $g$, induce an automorphism of the affine group $G \ltimes g$.

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Let $G$ be a noncompact real semisimple Lie group, let $V$ be a finite dimensional vector space and let $\mathfrak{r} : G \to \text{GL}(V)$ be a faithful irreducible representation. We denote the affine group obtained from this representation by $G \ltimes_{\mathfrak{r}} V$. In the main body of this article, when there is no confusion, we will omit the subscript $\mathfrak{r}$ from $G \ltimes_{\mathfrak{r}} V$. Moreover, for a hyperbolic group $\Gamma$, let $\rho : \Gamma \to G \ltimes_{\mathfrak{r}} V$ be a representation such that the projection of $\rho(\gamma)$ in $G$ is loxodromic for all nonidentity element $\gamma \in \Gamma$. In a series of works [Smi16a, Smi18, Smi16b] Smilga shows that if the faithful irreducible representation $\mathfrak{r}$ satisfy a few conditions (which are in particular satisfied by the adjoint representation), then for the nonabelian free group $\Gamma$ with finitely many generators, there exist $\rho : \Gamma \to G \ltimes_{\mathfrak{r}} V$ such that $\rho(\Gamma)$ act properly on $V$. In such a situation, we call the quotient space $\rho(\Gamma)\backslash V$ a Margulis-Smilga spacetime. More precisely,

**Definition 0.1.** Let $G$ be a real semisimple Lie group of noncompact type with trivial center, let $V$ be a finite dimensional vector space and let $\mathfrak{r} : G \to \text{GL}(V)$ be a faithful irreducible representation. Moreover, let $\rho : \Gamma \to G \ltimes_{\mathfrak{r}} V$ be a representation with its linear part $L_\rho : \Gamma \to G$ be such that $L_\rho(\gamma)$ is loxodromic for all nonidentity element $\gamma \in \Gamma$ and $L_\rho(\Gamma)$ is Zariski dense inside $G$. Then $\rho(\Gamma)\backslash V$ is called a Margulis-Smilga spacetime if and only if $\rho(\Gamma)$ act properly on $V$.

Interest in these spaces started from counter intuitive examples introduced by Margulis [Mar83, Mar84], to answer a question of Milnor [Mil77] regarding the Auslander Conjecture [Aus64]. The examples constructed by Margulis are examples of Margulis-Smilga spacetimes when $G \cong \text{SL}(\mathbb{R}^2)$ and $\mathfrak{r}$ is the adjoint representation. Note that the adjoint action of $\text{SL}(\mathbb{R}^2)$ can also be seen as the linear action of $\text{SO}(2,1)$ on $\mathbb{R}^3$. Later on, similar examples were constructed for the linear action of $\text{SO}(2n,2n-1)$ on $\mathbb{R}^{2n-1}$ by Abels–Margulis–Soifer [AMS02]. Recently, similar examples were constructed for adjoint representaions of any noncompact semisimple Lie group in [Smi16a] and for any general representation satisfying certain special criteria in [Smi18, Smi16b]. These criteria are technical in nature but they can be roughly translated to mean the following: the unit eigenspace of $\mathfrak{r}_g$ for every loxodromic element $g \in G$ contains a nontrivial subspace $V^0_g$ such that any two subspaces $V^0_g$ and $V^0_h$ can be canonically identified with each other via a map $\pi_{g,h} : V^0_g \to V^0_h$ with $\pi_{g,g}^{-1}$ never being the identity map (for more details please see the Main Theorem at page 4 of [Smi16b]). In the case where $G$ is split, the spaces $V^0_g$ for $g \in G$ are precisely the unit eigenspaces of $\mathfrak{r}_g$ and the criteria on $\mathfrak{r}$ boils down to the existence of nontrivial zero weight spaces and the action of the Weyl group on the zero weight space being nontrivial. A complete classification of such representations has recently been obtained, in the split case by LeFloch–Smilga [LFS18] and in the general case by Smilga [Smi20].

In the original construction of Margulis [Mar83, Mar84] a certain real valued invariant played a central role in the detection of proper affine actions. These invariants are called *Margulis invariants*. Later, similar real valued invariants were introduced in [AMS02] to detect proper affine actions.
of representations into $SO(2n, 2n - 1) \ltimes \mathbb{R}^{4n-1}$. Recently, these invariants were generalized by Smilga in [Smi16a, Smi18, Smi16b] into vector valued invariants to detect proper affine actions into $G \ltimes \mathbb{R} V$. We call these generalized vector valued invariants introduced by Smilga as Margulis-Smilga invariants. The definition of these invariants are also technical in nature but roughly they can be thought of as follows: for any $(g, X) \in G \ltimes \mathbb{R} V$ we denote the projection of $X$ onto $V^0_g$, with respect to some canonical decomposition of eigenspaces of $R_g$, by $X^0$, then the Margulis-Smilga invariant $M(g, X)$ is the class $[\pi_{g,h}(X^0) \mid h \in G]$ (for a more precise definition in the split case see Definition 5.3 and for the general case see Definition 7.19 of [Smi16b]).

The marked spectrum of Margulis-Smilga invariants of a Margulis-Smilga spacetime closely resemble the marked length spectrum of a hyperbolic surface. In fact, it is not very difficult to show that the marked Margulis-Smilga invariant spectrum of a Margulis-Smilga spacetime $\rho$ satisfy the following: $M(\rho(\gamma)) \neq 0$ for all non identity element $\gamma \in \Gamma$. Also, the additivity of Margulis-Smilga invariants play a crucial role in constructing examples of proper affine actions. In the case when $G \cong SL(\mathbb{R}^2)$ and $R$ is the adjoint representation, Drumm–Goldman [DG01] showed that the Margulis-Smilga invariant spectrum of two Margulis-Smilga spacetimes $\rho$ and $\varrho$ are same if and only if there exists an isomorphism $\sigma : SL(\mathbb{R}^2) \ltimes sl(\mathbb{R}^2) \to SL(\mathbb{R}^2) \ltimes sl(\mathbb{R}^2)$ such that $\rho = \sigma \circ \varrho$. Later, Kim [Kim05] generalized this result for examples constructed by Abels–Margulis–Soifer. In the case when $G \cong SO(2n, 2n - 1)$ and $R$ is the inclusion map into $GL(\mathbb{R}^{4n-1})$, he showed that the Margulis-Smilga invariant spectrum of two Margulis-Smilga spacetimes $\rho$ and $\varrho$ are same if and only if there exists an isomorphism $\sigma : SO(2n, 2n - 1) \ltimes \mathbb{R}^{4n-1} \to SO(2n, 2n - 1) \ltimes \mathbb{R}^{4n-1}$ such that $\rho = \sigma \circ \varrho$. Recently, the author proved [Gho19] that an infinitesimal version of Kim’s result is also true. In this article we generalize the isospectrality results from [DG01] and [Kim05] to show that for a large class of other interesting cases too the Margulis-Smilga invariant spectrum of a Margulis-Smilga spacetime do indeed determine the Margulis-Smilga spacetime.

Now we mention the criteria we use in this article more precisely and state the Theorems we prove.

**Convention 0.2.** Throughout this article unless otherwise stated we follow the following convention for the tuple $(G, V, R, \Gamma, \rho, \varrho)$:

1. $G$ denotes a real split connected semisimple algebraic Lie group with trivial center,
2. $V$ denotes a finite dimensional vector space with $\dim V > 1$,
3. $R : G \to GL(V)$ denotes a faithful irreducible algebraic representation which admits zero as a weight,
4. $G \ltimes V$ denote an affine group whose multiplication law is defined as follows: for all $g, h \in G$ and $X, Y \in V$, $(g, X)(h, Y) := (gh, X + R(g)Y),$
5. $L : G \ltimes V \to G$ denotes the map for which $L(g, X) = g$ for all $g \in G$ and $X \in V$,
6. $T : G \ltimes V \to V$ denotes the map for which $T(g, X) = X$ for all $g \in G$ and $X \in V$. 


Theorem 0.7. Let \((\mathcal{G}, \mathbf{V}, \mathbf{R}, \Gamma, \rho, \varrho)\) be as in Convention 0.2, let \(L \circ \rho = L \circ \varrho\) and let \(\mathcal{M}(\rho(\gamma)) = \mathcal{M}(\varrho(\gamma))\) for all \(\gamma \in \Gamma\). Then there exists an inner automorphism \(\sigma\) of \(G \ltimes_R \mathbf{V}\) such that \(\sigma \circ \rho = \varrho\).

Moreover, we prove two stronger results for a special class of representations \(\mathbf{R}\). Let \(\mathbf{R}\) be absolutely irreducible and let \(\mathbf{R}\) be conjugate to its dual (\(R^t\))$^{-1}$. We call representations which are conjugate to their dual as self-contragredient (for more details see Section 3.11 of \([\text{Sam90}]\)). Then by Lemma 1.3 of \([\text{Gro71}]\) the representation \(\mathbf{R}\) admits an invariant symmetric bilinear form \(B_{\mathbf{R}}\). We denote the norm coming from this bilinear form by \(\| \cdot \|_{\mathbf{R}}\) and prove the following results:

Theorem 0.4. Let \((\mathcal{G}, \mathbf{V}, \mathbf{R}, \Gamma, \rho, \varrho)\) be as in Convention 0.2, let \(\mathbf{R}\) be an absolutely irreducible self-contragredient representation, let \(L \circ \rho = L \circ \varrho\) and let \(\|\mathcal{M}(\rho(\gamma))\|_{\mathbf{R}} = \|\mathcal{M}(\varrho(\gamma))\|_{\mathbf{R}}\) for all \(\gamma \in \Gamma\). Then there exists an automorphism \(\sigma : G \ltimes_R \mathbf{V} \to G \ltimes_R \mathbf{V}\) such that \(\sigma \circ \rho = \varrho\). Moreover, \(\sigma\) is conjugation by an element \((\mathbf{A}, \mathbf{Y}) \in \mathbf{O}(B_{\mathbf{R}}) \ltimes \mathbf{V}\) such that \(\mathbf{A}\) centralizes \(R(G)\).

Theorem 0.5. Let \((\mathcal{G}, \mathbf{V}, \mathbf{R}, \Gamma, \rho, \varrho)\) be as in Convention 0.2, let \(\mathbf{R}\) an absolutely irreducible self-contragredient representation and let \(\|\mathcal{M}(\rho(\gamma))\|_{\mathbf{R}} = \|\mathcal{M}(\varrho(\gamma))\|_{\mathbf{R}}\) for all \(\gamma \in \Gamma\). Then either of the following holds:

1. both \(\rho(\Gamma)\) and \(\varrho(\Gamma)\) are Zariski dense inside some conjugates of \(G\),
2. there exists an automorphism \(\sigma : G \ltimes_R \mathbf{V} \to G \ltimes_R \mathbf{V}\) such that \(\sigma \circ \rho = \varrho\) and \(\sigma\) is conjugation by an element \((\mathbf{A}, \mathbf{Y}) \in \mathbf{O}(B_{\mathbf{R}}) \ltimes \mathbf{V}\) such that \(\mathbf{A}\) normalizes \(R(G)\).

In fact, we also prove the following characterization:

Theorem 0.6. Let \((\mathcal{G}, \mathbf{V}, \mathbf{R}, \Gamma, \rho)\) be as in Convention 0.2 and let \(\mathbf{R}\) an absolutely irreducible self-contragredient representation. Then the following are equivalent:

1. \(\|\mathcal{M}(\rho(\gamma))\|_{\mathbf{R}} = 0\) for all \(\gamma \in \Gamma\),
2. \(\mathcal{M}(\rho(\gamma)) = 0\) for all \(\gamma \in \Gamma\),
3. is conjugate to \(L \circ \rho(\Gamma) = (e, \mathbf{Y})\rho(\Gamma)(e, \mathbf{Y})^{-1}\) for some \(\mathbf{Y} \in \mathbf{V}\).

Hence we obtain the following result:

Theorem 0.7. Let \((\mathcal{G}, \mathbf{V}, \mathbf{R}, \Gamma, \rho, \varrho)\) be as in Convention 0.2 and let \(\rho\) and \(\varrho\) be two Margulis-Smilga spacetimes. Then the following holds:

1. If \(\rho\) and \(\varrho\) are conjugate via some inner automorphism of \(G \ltimes_R \mathbf{V}\), then they have the same Margulis-Smilga invariant spectrum.
2. If \(\rho, \varrho\) have the same marked Margulis-Smilga invariant spectrum and \(L \circ \rho = L \circ \varrho\), then there exists \(\sigma\), an inner isomorphism of \(G \ltimes_R \mathbf{V}\), such that \(\rho = \sigma \circ \varrho\).
3. If $\rho$, $\varrho$ have the same marked Margulis-Smilga invariant spectrum and $R$ is absolutely irreducible self-contragredient, then $R$ preserves a symmetric bilinear form $B_R$ and there exists $(A, Y) \in O(B_R) \ltimes V$ such that $\rho = (A, Y)g(A, Y)^{-1}$.

We note that the adjoint representations of connected real split simple algebraic Lie groups with trivial center are absolutely irreducible self-contragredient representations. Hence, along with all the known results from the literature, our result also covers in its full generality, adjoint representations of real split simple algebraic Lie groups with trivial center. Furthermore, we note that the techniques used in this article to prove Theorem 0.7 can be used to prove a more general result in the split case which might include some more representations $R$ but we do not include it in this article because the conditions on $R$ does not look natural enough in that generality.

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1. Jordan Decomposition

In this section we recall certain basic results about the structure theory of real semisimple algebraic Lie groups of noncompact type with trivial center and their Jordan decomposition. These results will be used later in the article to obtain our main result.

Let $G$ be a real semisimple algebraic Lie group of noncompact type with trivial center and let $\mathfrak{g}$ be its Lie algebra. We denote the identity element of $G$ by $e$. Let $C_\mathfrak{g}$ be the conjugation map on $G$, i.e. for any $g, h \in G$ we have $C_\mathfrak{g}(h) = ghg^{-1}$ and let $\text{Ad}_g$ be the differential of this at identity. Hence we obtain a homomorphism $\text{Ad} : G \to \text{SL}(\mathfrak{g})$. Moreover, let $\text{ad}$ be the differential of $\text{Ad}$ at the identity element. We fix a Cartan involution $\theta : \mathfrak{g} \to \mathfrak{g}$ and consider the corresponding decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ where $\mathfrak{k}$ (respectively $\mathfrak{p}$) is the eigenspace of eigenvalue 1 (respectively -1). Let $\mathfrak{a}$ be the maximal abelian subspace of $\mathfrak{p}$. We denote the space of linear forms on $\mathfrak{a}$ by $\mathfrak{a}^*$ and for all $\alpha \in \mathfrak{a}^*$ we define

$$\mathfrak{g}^\alpha := \{ X \in \mathfrak{g} | \text{ad}_H(X) = \alpha(H)X \text{ for all } H \in \mathfrak{a} \}.$$ 

We call $\alpha \in \mathfrak{a}^*$ a restricted root if and only if both $\alpha \neq 0$ and $\mathfrak{g}^\alpha \neq 0$. Let $\Sigma \subset \mathfrak{a}^*$ be the set of all restricted roots. As $\mathfrak{g}$ is finite dimensional, it follows that $\Sigma$ is finite. Moreover, we note that

$$\mathfrak{g} = \mathfrak{g}^0 \oplus \bigoplus_{\alpha \in \Sigma} \mathfrak{g}^\alpha.$$ 

We choose $\mathfrak{a}^{++}$, a connected component of $\mathfrak{a} \setminus \bigcup_{\alpha \in \Sigma} \ker(\alpha)$ and denote its closure by $\mathfrak{a}^+$. Let $K \subset G$ (respectively $A \subset G$) be the connected subgroup whose Lie algebra is $\mathfrak{k}$ (respectively $\mathfrak{a}$) and let $A^+ := \text{exp}(\mathfrak{a}^+)$. We note that $K$ is a maximal compact subgroup of $G$.

Let $B$ be the Killing form on $\mathfrak{g}$ i.e. for any $X, Y \in \mathfrak{g}$ we have $B(X, Y) := \text{tr}(\text{ad}_X \circ \text{ad}_Y)$ and denote $B(X, X)$ by $\|X\|_B^2$. We define $\Sigma^+ \subset \Sigma$ to be
the set of restricted roots which take positive values on \( a^+ \) and note that 
\[ \Sigma = \Sigma^+ \cup -\Sigma^+. \]
We consider the following nilpotent subalgebras:
\[ n^\pm := \bigoplus_{\pm \alpha \in \Sigma^+} g^\alpha. \]

Let \( K, A, N \) be the Lie subgroups of \( G \) generated respectively by \( k, a, n^+ \). Let \( g \in G \). Then
1. \( g \) is called elliptic if and only if some conjugate of \( g \) lies in \( K \),
2. \( g \) is called hyperbolic if and only if some conjugate of \( g \) lies in \( A \),
3. \( g \) is called unipotent if and only if some conjugate of \( g \) lies in \( N \).

**Theorem 1.1** (Jordan decomposition). Let \( G \) be a connected real semisimple algebraic Lie group of noncompact type with trivial center. Then for any \( g \in G \), there exist unique \( g_e, g_h, g_u \in G \) such that the following hold:
1. \( g = g_e g_h g_u \),
2. \( g_e \) is elliptic, \( g_h \) is hyperbolic and \( g_u \) is unipotent,
3. the elements \( g_e, g_h, g_u \) commute with each other.

**Definition 1.2.** Let \( G \) be a connected real semisimple algebraic Lie group of noncompact type with trivial center and let \( g \in G \). Then the Jordan projection of \( g \), denoted by \( Jd_g \), is the unique element in \( a^+ \) such that
\[ g_h \text{ is a conjugate of } \exp(Jd_g). \]

**Remark 1.3.** We note that \( Jd \) is continuous. Indeed, we use Lemmas 6.32 and 6.33 (ii) of [BQ16] and Appendix V.4 of [Whi72] to deduce it (see also [Tit71]).

**Definition 1.4.** Let \( G \) be a connected real semisimple algebraic Lie group of noncompact type with trivial center and let \( g \in G \). Then \( g \) is called loxodromic if and only if \( Jd_g \in a^{++} \).

Moreover, let \( M \) be the centralizer of \( a \) inside \( K \) and \( m \) be the Lie subalgebra of \( g \) coming from \( M \). We note that \( g = n^+ \oplus g^0 \oplus n^- \) and \( g^0 = a \oplus m \).

**Remark 1.5.** If \( G \) is split then \( m \) is trivial and \( M \) is a finite group (see Theorem 7.53 of [Kna02]).

**Proposition 1.6** (Dang, see Proposition 2.31 of [Dan19]). Let \( G \) be a connected real semisimple algebraic Lie group of noncompact type with trivial center and let \( g \in G \) be loxodromic. Then the following holds:
1. \( g_u \) is trivial,
2. for \( h_g \in G \) with \( g_h = h_g \exp(Jd_g) h_g^{-1} \), we have \( m_g := h_g^{-1} g_u h_g \in M \),
3. for \( (h_g, m_g) \in G \times M \) as above, we have \( g = h_g m_g \exp(Jd_g) h_g^{-1} \),
4. if \( (h, m) \in G \times M \) satisfy \( g = h \exp(Jd_g) h^{-1} \), then there exists a unique \( c \in MA \) such that \( h = h_g c \) and \( m = c^{-1} m_g c \).

We observe that the Jordan projection are invariant under conjugation, i.e. for all \( g, h \in G \) we have \( Jd_{gh} = Jd_g \).

2. **Weights and eigenspaces**

In this section we recall some basic results about the structure theory of finite dimensional faithful irreducible representations of real semisimple Lie
groups of noncompact type with trivial center. These results will be used later in the article to obtain our main result.

Let $G$ be a real semisimple Lie group of noncompact type with trivial center, $V$ be a finite dimensional vector space and let $R : G \to GL(V)$ be a faithful irreducible representation. Hence, we obtain a Lie algebra representation $\hat{R} : g \to gl(V)$, by taking the differential $\hat{R}$ of the representation $R$ at the identity. We recall that $a^*$ denotes the space of all linear forms on $a$ and for all $\lambda \in a^*$ we define

$$V^\lambda := \{ X \in V \mid \hat{R}_H(X) = \lambda(H) X \text{ for all } H \in a \}. $$

We call $\lambda \in a^*$ a restricted weight of the representation $R$ if and only if both $\lambda \neq 0$ and $V^\lambda \neq 0$. Let $\Omega \subset a^*$ be the set of all restricted weights. As $V$ is finite dimensional, it follows that $\Omega$ is finite. Moreover, we note that

$$V = V^0 \oplus \bigoplus_{\lambda \in \Omega} V^\lambda.$$

**Remark 2.1.** Henceforth, we will only consider representations $R$ such that $V^0$ is nontrivial and we will denote $\bigoplus_{\lambda \in \Omega} V^\lambda$ by $V^\neq 0$.

**Notation 2.2.** Henceforth we will also use the expression $R_g$ to denote $R(g)$ for any $g \in G$.

**Lemma 2.3.** Let $R : G \to GL(V)$ and $\hat{R} : g \to gl(V)$ be as above. Then for any $X \in a$ and $t \in \mathbb{R}$ we have $R(\exp(tX)) = \exp(\hat{R}(X)).$

**Proof.** We observe that both $\{R(\exp(tX))\}_{t \in \mathbb{R}}$ and $\{\exp(\hat{R}(X))\}_{t \in \mathbb{R}}$ are one parameter subgroups of $GL(V)$ passing through identity element with the same tangent vector i.e. $R(\exp(0)) = \exp(0)$ and

$$\frac{\partial}{\partial t} \bigg|_{t=0} R(\exp(tX)) = \hat{R}(X) = \frac{\partial}{\partial t} \bigg|_{t=0} \exp(\hat{R}(X)).$$

Hence for all $t \in \mathbb{R}$ we have $R(\exp(tX)) = \exp(\hat{R}(X)).$ \hfill $\Box$

**Lemma 2.4.** Let $g \in A$ and let $X_g \in a$ be such that $g = \exp(X_g)$. Then for any $X \in V^\lambda$ we have $R_g X = \exp(\lambda(X_g))X$.

In particular, we obtain that

$$V^\lambda = \{ X \in V \mid R_g(X) = \exp(\lambda(X_g))X \text{ for all } g \in A \}.$$  

**Proof.** As $A$ is abelian we note that the exponential map from $a$ to $A$ is surjective. Moreover, we observe that

$$\exp(\hat{R}(X_g))|_{t=0} X = X = \exp(t\lambda(X_g))|_{t=0} X,$$

$$\frac{\partial}{\partial t} \bigg|_{t=0} \exp(\hat{R}(X_g))X = \hat{R}(X_g)X = \lambda(X_g)X = \frac{\partial}{\partial t} \bigg|_{t=0} \exp(t\lambda(X_g))X.$$  

Hence, both $\exp(\hat{R}(X_g))X$ and $\exp(t\lambda(X_g))X$ have the same value at $t = 0$ and their derivatives at $t = 0$ are also equal. It follows that $\exp(\hat{R}(X_g))X = \exp(t\lambda(X_g))X$ for all $t \in \mathbb{R}$. We take $t = 1$ and use Lemma 2.3 to conclude that $R_g X = \exp(\lambda(X_g))X$. Therefore, we obtain

$$V^\lambda \subset \{ X \in V \mid R_g(X) = \exp(\lambda(X_g))X \text{ for all } g \in A \}.$$
Furthermore, let \( X \in \mathcal{V} \) be such that \( R_g(X) = \exp(\lambda(X_g))X \) for all \( g \in \mathcal{A} \). Hence, for any \( Y \in \mathfrak{a} \) we observe that
\[
\hat{R}(Y)X = \left. \frac{\partial}{\partial t} \right|_{t=0} \exp(\hat{R}(tY))X = \left. \frac{\partial}{\partial t} \right|_{t=0} \exp(\lambda(tY))X = \lambda(Y)X.
\]

Therefore, it follows from the definition of \( \mathcal{V}^\lambda \) that
\[
\mathcal{V}^\lambda \supset \{ X \in \mathcal{V} \mid R_g(X) = \exp(\lambda(X_g))X \text{ for all } g \in \mathcal{A} \},
\]
and we conclude our result.

**Lemma 2.5.** Let \( G \) be as in Convention 0.2. Then for any \( g \in \mathcal{M} \mathcal{A} \) we have \( R_g \mathcal{V}^\lambda = \mathcal{V}^\lambda \). Moreover, for \( \lambda = 0 \) we have
\[
\mathcal{V}^0 = \{ X \in \mathcal{V} \mid R_g(X) = X \text{ for all } g \in \mathcal{M} \mathcal{A} \}.
\]

**Proof.** As \( G \) is split we have \( \mathfrak{m} = 0 \). Hence \( \mathcal{M} \) is a discrete group. It follows that the connected component of \( \mathcal{M} \) containing identity is a singleton. Now we use Theorem 7.53 of [Kna02] and observe that \( \mathcal{M} \subset \exp(i\mathfrak{a}) \). Hence for any \( m \in \mathcal{M} \) there exists \( X_m \in \mathfrak{a} \) such that \( m = \exp(iX_m) \). It follows that for any \( X \in \mathcal{V}^\lambda \) we have
\[
(2.1) \quad R_mX = \exp(i\hat{R}(X_m))X = \exp(i\lambda(X_m))X.
\]

Also, as \( X \) and \( R_mX \) lie inside the real part we obtain that \( \exp(i\lambda(X_m)) \in \mathbb{R} \). Therefore, \( R_mX \in \mathcal{V}^\lambda \) if and only if \( X \in \mathcal{V}^\lambda \).

Finally, using the definition of \( \mathcal{V}^0 \) we obtain that
\[
\mathcal{V}^0 \supset \{ X \in \mathcal{V} \mid R_g(X) = X \text{ for all } g \in \mathcal{M} \mathcal{A} \}.
\]

Also, using Equation 2.1, for \( X \in \mathcal{V}^0 \) we get \( R_mX = \exp(0)X = X \) and using Lemma 2.4 we obtain that \( R_gX = X \) for all \( g \in \mathcal{M} \mathcal{A} \). Hence,
\[
\mathcal{V}^0 \subset \{ X \in \mathcal{V} \mid R_g(X) = X \text{ for all } g \in \mathcal{M} \mathcal{A} \}
\]
and we conclude our result.

**Lemma 2.6.** Let \( G \) be as in Convention 0.2 and let \( g \in \mathcal{G} \) be a loxodromic element. Then the dimension of the unit eigenspace of \( R_g \) is atleast \( \dim \mathcal{V}^0 \).

Moreover, the set of loxodromic elements \( h \in \mathcal{G} \) such that the dimension of the unit eigenspace of \( R_h \) is exactly \( \dim \mathcal{V}^0 \), is a non-empty open dense subset of \( \mathcal{G} \).

**Proof.** We use Proposition 1.6 and Lemma 2.5 to deduce that the unit eigenspace of \( R_g \) for a loxodromic element \( g \) is atleast \( \dim \mathcal{V}^0 \).

Suppose \( Y \in \mathfrak{a} \) is such that \( \alpha(Y) \neq 0 \) for all \( \alpha \in \Sigma \) and \( \lambda(Y) \neq 0 \) for all \( \lambda \in \Omega \), then by Lemma 2.4 we get that \( R(\exp(Y))X = \exp(\lambda(Y))X \). We again use Theorem 7.53 of [Kna02] and the fact that \( G \) is split to conclude that for any \( m \in \mathcal{M} \) there exists \( X_m \in \mathfrak{a} \) such that \( m = \exp(iX_m) \) and
\[
R_mX = \exp(i\lambda(X_m))X
\]
with \( \exp(i\lambda(X_m)) \in \mathbb{R} \). Hence, \( \exp(i\lambda(X_m)) = \pm 1 \) and for any \( X \in \mathcal{V}^\lambda \) with \( \lambda \neq 0 \) we have
\[
R(m\exp(Y))X = \exp(\lambda(Y))R_mX = \pm \exp(\lambda(Y))X \neq X.
\]
Also, using Lemma 2.5 we obtain that $R_g(X) = X$ for all $g \in MA$ and for all $X \in V^0$. Therefore, our result follows by using Remark 1.3, Proposition 1.6 and observing that the set 

$$a \setminus \left( \bigcup_{\alpha \in \Sigma} \ker(\alpha) \cup \bigcup_{\lambda \in \Omega} \ker(\lambda) \right)$$

is a non-empty open dense subset of $a$ as the sets $\Sigma$ and $\Omega$ are finite. \hfill \Box

3. Characteristic Polynomial

In this section we recall the definition of the minimal polynomial and the characteristic polynomial of a linear transformation. We also prove some preliminary results which will play a central role in the proof of our main theorem.

**Definition 3.1.** Let $A \in \text{gl}(V)$ and let $I \in \text{gl}(V)$ be the diagonal matrix with all its diagonal entries equal to 1. Then the characteristic polynomial of $A$ in the indeterminate $x$ is defined by the following expression:

$$\det(xI - A).$$

**Notation 3.2.** Let $(G,V,R)$ be as in Convention 0.2 and let $g \in G$. We alternately denote $R(g)$ by $R_g$. Then $R_g \in \text{GL}(V) \subset \text{gl}(V)$. Hence $(R_e - R_g) \in \text{gl}(V)$. Also we observe that $R_e = I$. In order to simplify our notations, in this article we will denote the characteristic polynomial of $(R_e - R_g)$ in the indeterminate $x$ by $\text{CP}_g$, i.e.

$$\text{CP}_g(x) = \det(xR_e - (R_e - R_g)) = \det((x - 1)R_e + R_g).$$

**Theorem 3.3** (Cayley–Hamilton, see [Fro77]). Let $g \in G$ and let $\text{CP}_g$ be the characteristic polynomial of $(R_e - R_g)$. Then $\text{CP}_g(R_e - R_g) = 0$.

**Remark 3.4.** One can use the Cayley–Hamilton Theorem 3.3 to deduce that the characteristic polynomial of $(R_e - R_g)$ has the following expression:

$$\text{CP}_g(x) = \sum_{k=0}^{\dim V} (-1)^{\dim V - k} \text{tr}(\wedge^{\dim V - k}(R_e - R_g))x^k.$$ 

Hence the coefficients of the characteristic polynomial are also algebraic.

Let $\mathbb{R}[x]$ be set of all polynomials in the indeterminate $x$ and note that $\mathbb{R}[x]$ is a principal ideal domain i.e. any ideal is generated by a single polynomial, which is unique up to units in $\mathbb{R}[x]$. Now for $A \in \text{gl}(V)$ we consider:

$$I_A := \{p(x) \in \mathbb{R}[x] \mid p(A) = 0\}$$

and observe that $I_A$ is a proper ideal of $\mathbb{R}[x]$.

**Definition 3.5.** The minimal polynomial of $A \in \text{gl}(V)$ is the unique monic polynomial which generates $I_A$. It is the monic polynomial of least degree inside $I_A$.

**Notation 3.6.** Let $g \in G$. In order to simplify our notations, in this article we will denote the minimal polynomial of $(R_e - R_g)$ in the indeterminate $x$ by $\text{MP}_g(x)$.
Remark 3.7. We observe that by definition $\text{CP}_g(x) \in \mathbb{I}(\mathbb{R}_e - \mathbb{R}_g)$ and hence $\text{MP}_g(x)$ divides $\text{CP}_g(x)$. In fact, one can deduce that $\text{CP}_g(x)$ and $\text{MP}_g(x)$ have the same irreducible factors in $\mathbb{R}[x]$. Moreover, when $g$ is loxodromic, using Proposition 14 in Chapter 7.5.8 of [Bou03] we get that $\text{MP}_g(x)$ has no multiple factors.

Lemma 3.8. Let $g \in G$ be a loxodromic element. Then $\mathbb{R}_g$ is diagonalizable over $\mathbb{C}$.

Proof. We use Theorem 2.4.8 (ii) of [Spr09] to conclude our result. □

Proposition 3.9. Let $g \in G$ be loxodromic and let $\text{CP}_g(x)$ be the characteristic polynomial of $(\mathbb{R}_e - \mathbb{R}_g)$ with variable $x$. Then $P_g(x) := \text{CP}_g(x)/x^\dim V^0$ is a polynomial.

Moreover, let $P_g(x) = \sum_{k=0}^{\dim V^0} a_k(g)x^k$. Then the coefficient $a_k(g)$, for any $k \in \{0, 1, ..., \dim V\}$, is algebraic in $g$.

Proof. Using Lemma 2.6 we obtain that $\mathbb{R}_g$ has eigenvalue 1 with multiplicity at least $\dim V^0$. Hence, $(\mathbb{R}_e - \mathbb{R}_g)$ has at least $\dim V^0$ many 0 as eigenvalues. Moreover, by Lemma 3.8 we know that $\mathbb{R}_g$ is diagonalizable over $\mathbb{C}$. Hence $\text{CP}_g(x)$ is divisible by $x^\dim V^0$ and it follows that $P_g(x)$ is a polynomial.

As $\text{CP}_g(x)$ is the characteristic polynomial of $(\mathbb{R}_e - \mathbb{R}_g)$, we have

$$\text{CP}_g(x) = \sum_{k=0}^{\dim V^0} (-1)^{\dim V^0 - k} \text{tr}((\wedge^{\dim V^0 - k}(\mathbb{R}_e - \mathbb{R}_g))) x^k.$$  

We denote $\dim V$ by $n$ and $\dim V^0$ by $n_0$. Moreover, as $x^{n_0}$ divides $\text{CP}_g(x)$ we have $\text{tr}((\wedge^{n_0 - k}(\mathbb{R}_e - \mathbb{R}_g))) = 0$ for all $0 \leq k \leq n_0 - 1$ and hence

$$P_g(x) = \sum_{k=n_0}^{n} (-1)^{n-k} \text{tr}((\wedge^{n-k}(\mathbb{R}_e - \mathbb{R}_g))) x^k - n_0 = \sum_{k=0}^{n-n_0} (-1)^{n-n_0-k} \text{tr}((\wedge^{n-n_0-k}(\mathbb{R}_e - \mathbb{R}_g))) x^k.$$  

As $\mathbb{R}$ is algebraic, we conclude by observing that

$$a_k(g) = (-1)^{n-n_0-k} \text{tr}((\wedge^{n-n_0-k}(\mathbb{R}_e - \mathbb{R}_g)))$$

is algebraic in $g$ for all $k$. □

Lemma 3.10. Let $g \in G$ be a loxodromic element such that the dimension of the unit eigenspace of $\mathbb{R}_g$ is exactly $\dim V^0$. Then $P_g(0) \neq 0$.

Proof. We use Lemma 3.8 and the fact that $V = V^0 \oplus V^{\neq 0}$ to conclude our result. □

4. Unit Eigenspace Projection

In this section we deduce a formula for the projection operator onto eigenspace of unit eigenvalues with respect to the eigenspace decomposition of a linear operator. Moreover, for $(G,V,\mathbb{R})$ as in Convention 0.2 and $g \in G$, we relate the unit eigenspace projections of $\mathbb{R}_g$ with the projection onto the zero weight space of $\mathbb{R}$. 
Let $\pi_0$ be the projection onto the $V^0$ component with respect to the decomposition: $V = V^0 \oplus V^\neq 0$.

**Lemma 4.1.** Let $g \in G$ be a loxodromic element and let $P_g$ be as in Proposition 3.9. Then
\[
(R_e - R_g)P_g(R_e - R_g) = 0.
\]

**Proof.** Let $g_h$ be the hyperbolic part of $g$ with respect to the Jordan decomposition. Let $h \in G$ be such that $g_h = h(\exp \lambda_g)h^{-1}$. We use Proposition 1.6 to obtain that $c := h^{-1}gh \in \mathcal{M}A$. As $g$ is loxodromic, using Lemma 3.8 we obtain that $R_c$ is diagonalizable over $\mathbb{C}$. It follows that the minimal polynomial $\mathcal{M}P_c(x)$ of $(R_e - R_c)$ is a product of distinct monic linear factors and hence is divisible by $x$ but not by $x^2$ (See Remark 3.7). Also $\mathcal{M}P_c(x)$ divides $\mathcal{C}P_c(x)$ and hence $\mathcal{M}P_c(x)$ divides $xP_c(x)$. We also know that $\mathcal{M}P_c(R_e - R_c) = 0$ and it follows that $(R_e - R_c)P_c(R_e - R_c) = 0$.

As $g = hch^{-1}$ we have $R_e - R_g = R_h(R_e - R_c)R_h^{-1}$. Hence $P_g = P_c$ and
\[
P_g(R_e - R_g) = P_c(R_e - R_g) = R_hP_c(R_e - R_c)R_h^{-1}.
\]

Therefore, we conclude that
\[
(R_e - R_g)P_g(R_e - R_g) = R_h(R_e - R_c)P_c(R_e - R_c)R_h^{-1} = 0.
\]

**Proposition 4.2.** Let $c \in \mathcal{M}A$ and let $P_c$ be as in Proposition 3.9. Then for any $X \in V$:
\[
P_c(R_e - R_c)X = P_c(0)\pi_0(X).
\]

**Proof.** As $R_c$ is diagonalizable over complex numbers, $R_e - R_c$ is also diagonalizable over complex numbers. Moreover, as $R_eZ = Z$ for all $Z \in V^0$ we have $V^0 \subset \ker(R_e - R_c)$. We will prove our result in two separate cases:

1. $V^0 \neq \ker(R_e - R_c)$: In this case $P_c(x)$ is divisible by $x$. Hence $P_c(x)$ is divisible by the minimal polynomial $\mathcal{M}P_c(x)$. Moreover, $\mathcal{M}P_c(R_e - R_c)$ vanishes and we obtain that $P_c(R_e - R_c) = 0$. Also, as $P_c(x)$ is divisible by $x$ we have $P_c(0) = 0$. Therefore,
\[
P_c(R_e - R_c)X = 0 = P_c(0)\pi_0(X).
\]

2. $V^0 = \ker(R_e - R_c)$: In this case, given any $Y \in V^\neq 0$ there exists an $Y' \in V^\neq 0$ such that $(R_e - R_c)Y' = Y$. Indeed, as $R_eV^\neq 0 \subset V^\neq 0$ we obtain that
\[
(R_e - R_c) : V^\neq 0 \rightarrow V^\neq 0
\]
is a linear map with kernel $V^0 \cap V^\neq 0 = \{0\}$ and hence $(R_e - R_c)$ is invertible on $V^\neq 0$. Therefore, for any $X \in V$ there exists $Y \in V^\neq 0$ such that $(X - \pi_0(X)) = (R_e - R_c)Y$. It follows that
\[
P_c(R_e - R_c)(X - \pi_0(X)) = P_c(R_e - R_c)(R_e - R_c)Y = 0
\]
and hence for any $X \in V$ we have $P_c(R_e - R_c)X = P_c(R_e - R_c)\pi_0(X)$. Moreover, as $(R_e - R_c)\pi_0(X) = 0$, we conclude by observing that
\[
P_c(R_e - R_c)\pi_0(X) = \sum_{k=0}^{\dim(V^\neq 0)} a_k(c)(R_e - R_c)^k\pi_0(X)
= a_0(c)\pi_0(X) = P_c(0)\pi_0(X).
\]
Our result is complete. \(\square\)

**Proposition 4.3.** Let \(g \in G\) be a loxodromic element such that the dimension of the unit eigenspace of \(R_g\) is exactly \(\dim V^0\). Then the map

\[
P_g(0)^{-1}P_g(R_e - R_g) : V \to V
\]

is the projection onto the unit eigenspace of \(R_g\) with respect to the eigenspace decomposition of \(R_g\).

**Proof.** We use Lemma 3.10 and observe that \(P_g(0) \neq 0\). Hence the map \(P_g(0)^{-1}P_g(R_e - R_g) : V \to V\) is a well defined linear map. Moreover, as \(g\) is loxodromic, we use Proposition 1.6 and obtain that there exists \(h \in G\) such that \(c := h^{-1}gh \in MA\). Now we use Proposition 4.2 and obtain that

\[
P_c(0)^{-1}P_c(R_e - R_c)X = \pi_0(X)
\]

for all \(X \in V\). Hence, \(P_c(0)^{-1}P_c(R_e - R_c) = \pi_0\) is a projection operator projecting onto \(V^0\). Moreover, as \(c = h^{-1}gh\), we deduce that

\[
P_g(0)^{-1}P_g(R_e - R_g) = R_h \circ \pi_0 \circ R_h^{-1}.
\]

It follows that \(P_g(0)^{-1}P_g(R_e - R_g)\) is a projection onto the space \(R_hV^0\). Therefore, we will be done once we show that \(R_hV^0\) is the unit eigenspace of \(R_g\). Finally, we observe that \(R_eX = X\) if and only if \(R_gR_hX = R_hX\) and conclude our result using Lemmas 2.5 and 2.6. \(\square\)

5. Margulis-Smilga invariant

In this section we define the Margulis-Smilga invariants corresponding to a faithful irreducible representation of a real split connected semisimple Lie group with trivial center. We also relate these invariants with the unit eigenspace projections introduced in the previous section.

Let \((G, V, R)\) be as in Convention 0.2. We consider the group \(G \ltimes V\) as follows: for any \(g, h \in G\) and \(X, Y \in V\) we have \((g, X), (h, Y) \in G \ltimes V\) and \((g, X)(h, Y) := (gh, X + R_gY)\). Moreover, we denote the affine action of \(G \ltimes V\) on \(V\) by \(Af\) i.e. for any \((g, X) \in G \ltimes V\) and \(Y \in V\) we have:

\[
Af_{(g, X)}Y := R_gY + X.
\]

Let \(L : G \ltimes V \to G\) be the map such that \(L(g, X) = g\) and let \(T : G \ltimes V \to V\) be the map such that \(T(g, X) = X\) for all \(g \in G\) and \(x \in V\). Image under \(L\) of \((g, X) \in G \ltimes V\) is called the linear part of \((g, X)\) and the image under \(T\) is called the translation part of \((g, X)\).

**Lemma 5.1.** Let \((g, X) \in G \ltimes V\) be such that \(g\) is loxodromic and \(g_h\) be its hyperbolic part with respect to the Jordan decomposition. Let \(h_1, h_2\) be such that

\[
h_1\exp(Jd_y)h_1^{-1} = g_h = h_2\exp(Jd_y)h_2^{-1}.
\]

Then \(\pi_0(R_{h_1}^{-1}X) = \pi_0(R_{h_2}^{-1}X)\).

**Proof.** We recall that by Lemma 2.5, for any \(c \in MA\) we have \(R_cV^{\neq 0} = V^{\neq 0}\) and \(R_cX = X\) for any \(X \in V^0\).

Also by Proposition 1.6 there exist some \(c \in MA\) such that \(h_2 = h_1c\). For \(i \in \{1, 2\}\) we denote the component of \(R_{h_i}^{-1}X\) inside \(V^{\neq 0}\) by \(Y_i\) and
the component of $R_{h_1}^{-1}X$ inside $V^0$ by $Z_1$. As $h_2 = h_1 c$ we deduce that

$$(Y_1 + Z_1) = R_c(Y_2 + Z_2)$$

and hence

$$R_c Y_2 - Y_1 = Z_1 - R_c Z_2 = Z_1 - Z_2.$$ We notice that $(R_c Y_2 - Y_1) \in V^{\neq 0}$, $(Z_1 - Z_2) \in V^0$ and $V^0 \cap V^{\neq 0} = \{0\}$. Therefore, $Z_1 = Z_2$ and we conclude that $\pi_0(R_{h_1}^{-1}X) = \pi_0(R_{h_2}^{-1}X)$.

**Lemma 5.2.** Let $(g, X) \in G \ltimes V$ be such that $g$ is loxodromic and $g_h$ be its hyperbolic part with respect to the Jordan decomposition. Let $h \in G$ be such that $g_h = \exp(J_d g)h^{-1}$. Then for any $Y \in V$ we have

$$\pi_0(R_{h_1}^{-1}(Af(g, X)Y - Y)) = \pi_0(R_{h_1}^{-1}X).$$

**Proof.** As $Af(g, X)Z = R_g Z + X$ and $\pi_0$ is linear, proving this lemma is equivalent to showing that $\pi_0(R_{h_1}^{-1}(R_g Z - Z)) = 0$ for all $Z \in V$. We denote $Y := R_{h_1}^{-1}Z$ and observe that

$$R_{h_1}^{-1}(R_g Z - Z) = R_{h_1}^{-1}gh Y - Y.$$ We recall from Proposition 1.6 that $l := h^{-1}gh \in MA$. Hence using Lemma 2.5 we deduce that $R_l(Y - \pi_0(Y)) \in V^{\neq 0}$ and it follows that

$$\pi_0(R_l(Y - \pi_0(Y))) = 0.$$

Also, as $l \in MA$, using Lemma 2.5 we have $R_l \pi_0(Y) = \pi_0(Y)$. Therefore, we conclude by observing that

$$\pi_0(R_l(Y - \pi_0(Y))) = \pi_0(R_l Y - R_l \pi_0(Y)) = \pi_0(R_l Y - \pi_0(Y))$$

$$= \pi_0(R_l Y) - \pi_0(\pi_0(Y)) = \pi_0(R_l Y) - \pi_0(Y)$$

$$= \pi_0(R_l Y - Y)$$

and hence $\pi_0(R_{h_1}^{-1}(R_g Z - Z)) = \pi_0(R_l Y - Y) = 0$. 

**Definition 5.3.** Let $(g, X) \in G \ltimes V$ be such that $g$ is loxodromic and $g_h$ be its hyperbolic part with respect to the Jordan decomposition. Let $h \in G$ be such that $g_h = \exp(J_d g)h^{-1}$. Then the Margulis-Smilga invariant of $(g, X)$ denoted by $M(g, X)$ is defined as follows:

$$M(g, X) := \pi_0(R_{h_1}^{-1}X).$$

**Remark 5.4.** Note that by Definition 6.2 of [Smilga16], Proposition 7.8 of [Smilga16b] and Lemma 2.5, the definition of a Margulis-Smilga invariant given here is the same as the definition of a Margulis invariant given in Definition 7.19 of [Smilga16b]. Smilga was the first to modify real valued Margulis invariants into vector valued invariants and he used these invariants in [Smilga16a, Smilga18, Smilga16b] to construct proper affine actions of Schottky groups.

**Proposition 5.5.** Let $g \in G$ be a loxodromic element and let $h \in G$ be such that $g_h = \exp(J_d g)h^{-1}$. Then for any $Y \in V$ we have

$$P_g(R_c - R_g)Y = P_g(0)R_h M(g, Y).$$
Proof. Let \( c := h^{-1}gh \). Then by Proposition 1.6 we have \( c \in \mathbb{MA} \). Now using Proposition 4.2 we obtain that \( P_c(R_e - R_e)X = P_c(0)\pi_0(X) \) for all \( X \in V \). Also, we have \( P_e(x) = P_g(x) \). Hence, we deduce that
\[
P_g(R_e - R_e)Y = R_eP_c(R_e - R_e)R_N^{-1}Y = P_c(0)R_h\pi_0(R_N^{-1}Y) = P_g(0)R_hM(g, Y)
\]
and our result follows. \( \square \)

6. Isospectrality: fixed linear part

In this section we show that the Margulis-Smilga invariant spectrum of two faithful irreducible algebraic representations with fixed linear parts of a split connected semisimple Lie group with trivial center are completely determined by the isomorphism class.

Let \((G, V, R, \Gamma, \rho, \varrho)\) be as in Convention 0.2. We denote \( L(\rho(\gamma)) \) by \( L_\rho(\gamma) \), \( T(\rho(\gamma)) \) by \( T_\rho(\gamma) \), \( M(\rho(\gamma)) \) by \( M_\rho(\gamma) \).

**Definition 6.1.** The map \( M_\rho : \Gamma \to V \) (respectively \( M_\varrho \)) is called the marked Margulis-Smilga invariant spectrum of the representation \( \rho \) (respectively \( \varrho \)).

**Proposition 6.2.** Let \((G, V, R, \Gamma, \rho)\) be as in Convention 0.2. Then either \( \rho(\Gamma) \) is Zariski dense inside \( G \ltimes V \) or \( \rho(\Gamma) \) is conjugate to \( L_\rho(\Gamma) \) under the action of some element of \( \{e\} \ltimes V \).

**Proof.** Let \( X \) be the Zariski closure of \( \rho(\Gamma) \) inside \( G \ltimes V \) and let \( \gamma \in \Gamma \). As \( \rho(\gamma)\rho(\gamma)^{-1} = \rho(\Gamma) \), we obtain that \( \rho(\gamma)\rho(\gamma)^{-1} = X \). Also, \( L \) is a homomorphism. Hence, we have \( L_\rho(\gamma)L(X)L_\rho(\gamma)^{-1} = L(X) \) for all \( \gamma \in \Gamma \). As \( X \) normalizes \( \{e\} \ltimes V \) using the Corollary to Proposition A at page 54 of [Hum75] we obtain that \( X(\{e\} \ltimes V) \) is a Zariski closed subgroup of \( G \ltimes V \). Also, as \( L_\rho(\Gamma) \) is Zariski dense inside \( G \), we obtain that \( gL(X)g^{-1} = L(X) \) for all \( g \in G \). It follows that \( L(X) \) is normal inside \( G \). Moreover, as \( L_\rho(\Gamma) \) is Zariski dense inside \( G \) and \( L_\rho(\Gamma) \subset L(X) \), we deduce that \( L(X) = G \).

Now we consider the map \( L|_X : X \to G \) and note that
\[
\ker(L|_X) = (\{e\} \ltimes V) \cap X.
\]
We will prove our result in two parts as follows:

\( \diamond \) Let \( \ker(L|_X) \) be trivial, then \( L|_X \) is an isomorphism. Hence, for all \( g \in G \) there exists \( X_g \in V \) such that \( X_g = X_g + R_gX_h \) and \( X = \{g, X_g \mid g \in G\} \). As \( G \) is connected, we use Whitehead’s Lemma (see end of section 1.3.1 in page 13 of [Rag07]) and deduce that there exists \( X \in V \) such that \( X = X - R_gX \). Therefore, we have \( T_\rho(\gamma) = X - R_{L_\rho(\gamma)}X \) for all \( \gamma \in \Gamma \). Hence \( \rho(\gamma) = (e, X)(L_\rho(\gamma), 0)(e, X)^{-1} \) for all \( \gamma \in \Gamma \).

\( \diamond \) Let \( \ker(L|_X) \) be non trivial. Then there exist \( X \in V \) with \( X \neq 0 \) such that \( (e, X) \in \ker(L|_X) \). As \( \ker(L|_X) \) is normal inside \( X \) we obtain that for any \( (h, Y) \in X \) we have \( (h, Y)(e, X)(h, Y)^{-1} \in \ker(L|_X) \). We also notice that for all \( (h, Y) \in G \ltimes V \) we have
\[
(h, Y)(e, X)(h, Y)^{-1} = (h, 0)(e, X)(h, 0)^{-1} = (e, R_hX).
\]
Moreover, as \( L(X) = G \) we deduce that \( (e, R_hX) \in \ker(L|_X) \) for all \( h \in G \). As \( R \) is irreducible, we use Lemma A.1 and obtain that \( \ker(L|_X) = (\{e\} \ltimes V) \). Furthermore, as \( L(X) = G \), we conclude that \( X = G \ltimes V \).

Therefore, either \( \rho(\Gamma) \) is Zariski dense inside \( G \ltimes V \) or \( \rho(\Gamma) \) is conjugate to \( L_\rho(\Gamma) \) under the action of some element of \( \{e\} \ltimes V \) and our result follows. \( \square \)
Theorem 6.3. Let \((G, V, R, \Gamma, \rho)\) be as in Convention 0.2 and \(M_{\rho}(\gamma) = 0\) for all \(\gamma \in \Gamma\). Then \(\rho(\Gamma)\) is conjugate to \(L_{\rho}(\Gamma)\) under the action of some element of \(\{e\} \ltimes V\).

Proof. As \(M_{\rho}(\gamma) = 0\) for all \(\gamma \in \Gamma\), using Proposition 5.5 we obtain that \(P_{L_{\rho}(\gamma)}(R_e - R_{\rho(\gamma)})T_{\rho}(\gamma) = 0\) for all \(\gamma \in \Gamma\). We consider the following map:

\[ \gamma : G \ltimes V \rightarrow \mathbb{R} \]

\[ (g, X) \mapsto P_{\rho}(R_e - R_{\rho(\gamma)})X \]

and observe that it is algebraic. We denote the zero set of \(\gamma\) by \(Z_{\gamma}\) i.e.

\[ Z_{\gamma} := \{(g, X) \in G \ltimes V \mid \gamma(g, X) = 0\}. \]

We choose a \(X \neq 0\) inside \(V^0\) and a loxodromic element \(g \in G\) such that the dimension of the unit eigenspace of \(R_g\) is exactly \(\dim V^0\). Moreover, let \(h \in G\) be such that \(hgh^{-1} \in MA\). Then using Lemma 3.10 and Proposition 5.5 we obtain that

\[ \gamma(g, R_h X) = P_{\rho}(R_e - R_{\rho(\gamma)})R_h X = P_{\rho}(0)R_h \pi_0(X) = P_{\rho}(0)R_h X \neq 0. \]

Hence \(Z_{\gamma} \subseteq G \ltimes V\) and it follows that \(X\), the Zariski closure of \(\rho(\Gamma)\) inside \(G \ltimes V\), is a proper subvariety of \(G \ltimes V\) i.e. \(X \subset Z_{\gamma} \subseteq G \ltimes V\). Finally, we conclude our result by using Proposition 6.2.

Theorem 6.4. Let \((G, V, R, \Gamma, \rho, g)\) be as in Convention 0.2, let \(L_{\rho} = L_{\rho}\) and let \(M_{\rho}(\gamma) = M_{\rho}(\gamma)\) for all \(\gamma \in \Gamma\). Then there exists an inner automorphism \(\sigma\) of \(G \ltimes V\) such that \(\sigma \circ \rho = \rho\).

Proof. Let \(\eta := (L_{\rho}, T_{\rho} - T_{\rho})\). We observe that for all \(\gamma \in \Gamma\) we have

\[ M_{\eta}(\gamma) = M_{\rho}(\gamma) - M_{\rho}(\gamma) = 0. \]

Therefore, using Theorem 6.3 we obtain that there exists \(Y \in V\) such that \(\eta(\gamma) = (e, Y)(L_{\rho}(\gamma), 0)(e, Y)^{-1}\) for all \(\gamma \in \Gamma\). Hence, for all \(\gamma \in \Gamma\) it follows that \(T_{\rho}(\gamma) - T_{\rho}(\gamma) = Y - R_{\rho(\gamma)}Y\) and we conclude by observing that for all \(\gamma \in \Gamma\), the following hold : \(\rho(\gamma) = (e, Y)g(\gamma)(e, Y)^{-1}\).

7. IsoSpectrality of Norm: Zero Spectrum

In this section we restrict the space of representations we are working with and only consider those representations which are absolutely irreducible and self-contragredient. We do this in order to introduce an invariant norm on the vector space. Moreover, we characterize faithful irreducible algebraic representations of a real split connected semisimple algebraic Lie group with trivial center whose normed Margulis-Smilga invariant spectrum is zero. We note that the results in this section doesn’t follow directly from results in the previous section as the invariant norm in question might not be positive definite.

Let \((G, V, R, \Gamma, \rho, g)\) be as in Convention 0.2 and also let \(R\) be an absolutely irreducible self-contragredient representation. We note that by Lemma 1.3 of [Gro71] the representation \(R\) admits an invariant symmetric bilinear form. We denote this bilinear form by \(B_R\) and the norm coming from this invariant form by \(\| \cdot \|_R\).
**Theorem 7.1.** Let \((G, V, R, \Gamma, \rho)\) be as in Convention 0.2, let \(R\) be an absolutely irreducible self-contragredient representation and \(\|M_\rho(\gamma)\|_R = 0\) for all \(\gamma \in \Gamma\). Then \(\rho(\Gamma)\) is conjugate to \(L_\rho(\Gamma)\) under the action of some element of \(\{e\} \ltimes V\).

**Proof.** As \(\|M_\rho(\gamma)\|_R = 0\) for all \(\gamma \in \Gamma\), using Proposition 5.5 we obtain that 
\[
\|P_{L_\rho(\gamma)}(R_e - R_{L_\rho(\gamma)})T_{\rho}(\gamma)\|_R = 0
\]
for all \(\gamma \in \Gamma\). We consider the following map:
\[
\mathfrak{D} : G \ltimes V \rightarrow R
\]
\[(g, X) \mapsto \|P_g(R_e - R_g)X\|^2_R\]
and observe that it is algebraic. We denote the zero set of \(\mathfrak{D}\) by \(Z_2\) i.e.
\[
Z_2 := \{(g, X) \in G \ltimes V \mid \mathfrak{D}(g, X) = 0\}.
\]
We use Lemma 2.6 and Lemma 3.10 to obtain that there exists \(c \in MA\) such that \(P_c(0) \neq 0\). As \(B_R\), the invariant form of \(R\), is orthogonal, we use Lemma 1.1 of [Gro71] to conclude that the restriction of \(B_R\) on \(V^0\) is a non-degenerate orthogonal form. Hence, \(V^0\) admits vectors which are not self-orthogonal. Let \(X \in V^0\) be such that \(\|X\|_R \neq 0\). We use Proposition 4.2 and obtain
\[
\mathfrak{D}(c, X) = \|P_c(R_e - R_e)X\|^2_R = P_c(0)^2\|\pi_0(X)\|^2_R = P_c(0)^2\|X\|^2_R \neq 0.
\]
Hence \(Z_2 \subseteq G \ltimes V\) and it follows that \(X\), the Zariski closure of \(\rho(\Gamma)\) inside \(G \ltimes V\), is a proper subvariety of \(G \ltimes V\) i.e. \(X \subset Z_2 \subseteq G \ltimes V\). Finally, we conclude our result by using Proposition 6.2. \(\square\)

**Corollary 7.2.** Let \((G, V, R, \Gamma, \rho)\) be as in Convention 0.2. Then the following are equivalent:

1. \(M_\rho(\gamma) = 0\) for all \(\gamma \in \Gamma\),
2. \(\rho(\Gamma)\) is conjugate to \(L_\rho(\Gamma)\) under the action of some element of \(\{e\} \ltimes V\).

Moreover, if \(R\) is an absolutely irreducible self-contragredient representation. Then the following are equivalent:

3. \(\|M_\rho(\gamma)\|_R = 0\) for all \(\gamma \in \Gamma\),
4. \(M_\rho(\gamma) = 0\) for all \(\gamma \in \Gamma\).

**Proof.** We use theorem 6.3 to obtain that \((1) \implies (2)\). Now we show that \((2) \implies (1)\). Let \(\rho = (e, X)L_\rho(e, X)^{-1}\) for some \(X \in V\) and let \(\gamma \in \Gamma\) and let \(h \in G\) be such that \(L_\rho(\gamma)h = h\exp(J_{\rho}(\gamma))h^{-1}\). Then \(h^{-1}L_\rho(\gamma)h =: c \in MA\).

Therefore, we deduce that
\[
M_\rho(\gamma) = \pi_0(R_h^{-1}T_{\rho}(\gamma)) = \pi_0(R_h^{-1}(X - L_\rho(\gamma)X)) = \pi_0(R_h^{-1}X - R_h^{-1}X) = 0.
\]
Now let \(R\) be an absolutely irreducible self-contragredient representation. As \(M_\rho(\gamma) = 0\) implies \(\|M_\rho(\gamma)\|_R = 0\) for any \(\gamma \in \Gamma\) we have \((4) \implies (3)\) and using Theorem 7.1 we get that \((3) \implies (2)\). Also we have \((2) \implies (1)\). Hence \((3) \implies (4)\). \(\square\)

**Corollary 7.3.** Let \((G, V, R, \Gamma, \rho)\) be as in Convention 0.2 and there exists a \(\gamma \in \Gamma\) such that \(M_\rho(\gamma) \neq 0\). Then \(\rho(\Gamma)\) is Zariski dense inside \(G \ltimes V\).

**Proof.** We use Proposition 6.2 to obtain that either \(\rho(\Gamma)\) is Zariski dense inside \(G \ltimes V\) or \(\rho(\Gamma)\) is conjugate to \(L_\rho(\Gamma)\) under the action of some element of \(\{e\} \ltimes V\). We observe that if \(\rho(\Gamma)\) is not Zariski dense inside \(G \ltimes V\), then \(\rho(\Gamma)\) is conjugate to \(L_\rho(\Gamma)\) under the action of some element of \(\{e\} \ltimes V\) and we obtain a contradiction using Corollary 7.2. \(\square\)
8. Isospectrality of norm: fixed linear part

In this section we consider faithful absolutely irreducible algebraic self-contragredient representations of a real split connected semisimple algebraic Lie group with trivial center, which are conjugate to their dual and we show that the normed Margulis-Smilga invariant spectra of two such representations with fixed linear parts are completely determined by the isomorphism class.

Let \((G, V, R, \Gamma, \rho, \varrho)\) be as in Convention 0.2 and let \(R\) be an absolutely irreducible self-contragredient representation. Hence, \(R\) preserves an invariant symmetric bilinear form, which we denote by \(B_R\). Moreover, we denote the norm associated to \(B_R\) by \(\|\cdot\|_R\).

**Lemma 8.1.** Let \((g, X), (h, Y) \in G \ltimes V\) be such that their linear parts are loxodromic and \(\|\mathcal{M}(g, X)\|_R = \|\mathcal{M}(h, Y)\|_R\). Then

\[
\|P_g(0)P_h(R_c - R_h)Y\|_R = \|P_h(0)P_g(R_c - R_g)X\|_R.
\]

**Proof.** We use Proposition 5.5 and observe that

\[
\|P_g(0)P_h(R_c - R_h)Y\|_R = \|P_g(0)P_h(0)\mathcal{M}(h, Y)\|_R
\]

\[
= \|P_g(0)P_h(0)\mathcal{M}(g, X)\|_R = \|P_h(0)P_g(R_c - R_g)X\|_R.
\]

Our result follows. \(\square\)

**Lemma 8.2.** Let \(J : G \ltimes (V \oplus V) \to \mathbb{R}\) be such that for all \((g, X, Y) \in G \ltimes (V \oplus V)\) we have

\[
J(g, X, Y) := P_g(0)^2(\|P_g(R_c - R_g)X\|_R^2 - \|P_g(R_c - R_g)Y\|_R^2),
\]

and let \(Z_2 := \{(g, X, Y) \mid J(g, X, Y) = 0\} \subset G \ltimes (V \oplus V)\). Then

\[
Z_2 \neq G \ltimes (V \oplus V).
\]

**Proof.** We use Lemma 2.6 and Lemma 3.10 to obtain that there exists \(c \in \text{MA}\) such that \(P_c(0) \neq 0\). As \(B_R\), the invariant form of \(R\), is orthogonal, we use Lemmas 1.1 and 1.3 of [Gro71] to conclude that the restriction of \(B_R\) on \(V^0\) is a non-degenerate orthogonal form. Hence, \(V^0\) admits vectors which are not self-orthogonal. Let \(X \in V^0\) be such that \(\|X\|_R \neq 0\). We choose \(Y = 0\) and using Proposition 4.2 we observe that

\[
J(c, X, 0) = P_c(0)^2(\|P_c(R_c - R_c)X\|_R^2 - \|P_c(R_c - R_c)0\|_R^2)
\]

\[
= P_c(0)^4\|X\|_R^2 \neq 0.
\]

Hence, \(Z_2\) is a proper subvariety of \((G \ltimes (V \oplus V))\), concluding our result. \(\square\)

**Remark 8.3.** We denote the projections onto the left and right coordinates of \(G \ltimes (V \oplus V)\) by \(\pi_\rho\) and \(\pi_\varrho\) respectively i.e. \(\pi_\rho, \pi_\varrho : G \ltimes (V \oplus V) \to G \ltimes V\) be such that for all \((g, X, Y) \in G \ltimes (V \oplus V)\) we have \(\pi_\varrho(g, X, Y) = (g, X)\) and \(\pi_\rho(g, X, Y) = (g, Y)\).

**Proposition 8.4.** Let \(\eta : \Gamma \to G \ltimes (V \oplus V)\) be a representation whose Zariski closure inside \(G \ltimes (V \oplus V)\) is a proper subvariety. Moreover, let \(\rho := \pi_\rho \circ \eta, \varrho := \pi_\varrho \circ \eta\) and both \(\rho, \varrho\) are Zariski dense inside \(G \ltimes V\). Then there exists a continuous automorphism \(\sigma : G \ltimes V \to G \ltimes V\) such that \(\sigma \circ \rho = \varrho\).
Proof. Let us denote the Zariski closure of $\eta(\Gamma)$ inside $G \ltimes (V \oplus V)$ by $X$. As $\eta(\gamma)\eta(\Gamma)\eta(\gamma)^{-1} = \eta(\Gamma)$ for all $\gamma \in \Gamma$, we obtain that $\eta(\gamma)X\eta(\gamma)^{-1} = X$. We note that both $\pi_\eta$ and $\pi_\rho$ are homomorphisms. Hence $\rho(\gamma)\pi_\rho(X)\rho(\gamma)^{-1} = \pi_\rho(X)$ and $\rho(\gamma)\pi_\sigma(X)\rho(\gamma)^{-1} = \pi_\sigma(X)$. As both $\rho(\Gamma)$ and $\rho(\Gamma)$ are Zariski dense inside $G \ltimes V$, we obtain that both $\pi_\rho(X)$ and $\pi_\sigma(X)$ are normal inside $G \ltimes V$ (using the Corollary at page 54 of [Hum75]). Moreover, as $\pi_\rho(X) \supset X$, we use Proposition A.2 and obtain that $\pi_\rho(X) = G_\rho \ltimes V$ and $\pi_\sigma(X) = G_\sigma \ltimes V$ for some normal subgroups $G_\rho, G_\sigma$ of $G$. Also, as $\rho(\Gamma)$ and $\rho(\Gamma)$ are Zariski dense inside $G \ltimes V$ we obtain that $G_\rho = G = G_\sigma$.

We denote ker$(\pi_\rho|_X)$ by $N_\rho$ and ker$(\pi_\sigma|_X)$ by $N_\sigma$. Hence,
\[ \dim N_\rho = \dim X - \dim G \ltimes V = \dim N_\sigma. \]

Moreover, we have $\dim X \leq \dim G \ltimes (V \oplus V)$ and it follows that $\dim N_\rho = \dim N_\sigma$. Therefore, $N_\rho = X \cap \{\{e\} \ltimes (\{0\} \oplus V)\} \subset \{e\} \times (\{0\} \oplus V)$ and $N_\sigma = X \cap \{\{e\} \times (\{0\} \oplus V)\} \subset \{e\} \times (\{0\} \oplus V)$. Moreover, as $N_\rho$ is normal inside $X$, we obtain that for all $(g, X, Y) \in X$,
\[ \pi_\rho(g, X, Y)^{-1} = N_\rho. \]

But for all $Z \in V$ we have $(g, X, Y)(e, 0, Z)(g, X, Y)^{-1} = (e, 0, R_g Z)$. As $R$ is irreducible, for $Z \neq 0$ the group generated by $\{\{e\}, 0, R_g Z\}$ $g \in G$ is equal to $\{e\} \times (\{0\} \oplus V)$ (for more details see Lemma A.1). Therefore, if $(e, 0, Z) \in N_\rho$ for $Z \neq 0$ then $e \times (\{0\} \oplus V) = N_\rho \subset \{e\} \times (\{0\} \oplus V)$, a contradiction. It follows that $N_\rho$ is trivial. Using similar arguments we also obtain that $N_\sigma$ is trivial. Hence both $\pi_\rho|_X$ and $\pi_\sigma|_X$ are isomorphisms. Now we conclude by observing that $\sigma := \pi_\sigma|_X \circ \pi_\rho|_X^{-1}$ is a continuous automorphism of $G \ltimes V$ and
\[ \sigma \circ \rho = \pi_\sigma|_X \circ \pi_\rho|_X^{-1} \circ \rho = \pi_\sigma|_X \circ \eta = \varrho. \]

\[ \square \]

Proposition 8.5. Let $\sigma : G \ltimes V \rightarrow G \ltimes V$ be a continuous automorphism. Then there exists $(A, Y) \in GL(V)$ such that the action of $\sigma$ is conjugation by $(A, Y)$.

Proof. We observe that $\sigma$ induces a continuous additive map $\bar{\sigma} : V \rightarrow V$. As continuous additive maps between vector spaces are linear and $\sigma$ is an isomorphism, $\bar{\sigma}$ is an invertible linear map. Hence, there exists $A \in GL(V)$ such that $\sigma(e, X) = (A, e)X$ for all $X \in V$. Moreover, for $g, h \in G$ and $Y_{g, h}$, let $\sigma(g, 0) = (g, h Y_{g, h})$. Then $Y_{g, h} = Y_{g, h} + R_{g, h} Y_{h, g}$ for all $g, h \in G$. As $G$ is connected, we use Whitehead’s Lemma (see end of section 3.1.3 in page 13 of [Rag07]) to deduce that there exists $Y \in V$ such that $Y_{g, h} = Y - R_{g, h} Y$. We also note that for all $g \in G$ we have $R_{g, h} = R_{g, h} A$. Indeed, for any $X \in V$:
\[ (g, h Y_{g, h} + A) = \sigma(e, X) = \sigma(g, 0) = \sigma(g, 0)\sigma(e, X)^{-1} = (g, h Y_{g, h} + R_{g, h} A X) \]
and it follows that $\sigma(g, X) = (A, Y)(R_{g, h} X)(A, Y)^{-1}$. 

\[ \square \]

Theorem 8.6. Let $(G, V, R, \Gamma, \rho, \varrho)$ be as in Convention 0.2, let $R$ be an absolutely irreducible self-contragredient representation, let $L_\rho = L_\varrho$ and let $\|M_\rho(\gamma)\|_R = \|M_\varrho(\gamma)\|_R$ for all $\gamma \in \Gamma$. Then there exists an automorphism $\sigma : G \ltimes V \rightarrow G \ltimes V$ such that $\sigma \circ \rho = \varrho$. Moreover, $\sigma$ is conjugation by an element $(A, Y) \in O(B_R) \ltimes V$ such that $A$ centralizes $R(G)$. 

\[ \square \]
Proof. We will prove this result in three parts.

⋄ Let $\|M_\rho(\gamma)\|_R = 0$ for all $\gamma \in \Gamma$. Hence $\|M_\sigma(\gamma)\|_R = 0$ for all $\gamma \in \Gamma$. We use Corollary 7.2 and obtain that there exists $X,Y \in V$ such that $(e,X)\rho(e,X)^{-1} = L_0 = (e,Y)\rho(e,Y)^{-1}$. Hence, $(e,X-Y)\rho(e,X-Y)^{-1} = 0$.

⋄ Let there exists $\gamma \in \Gamma$ such that $\|M_\rho(\gamma)\|_R \neq 0$. Hence $\|M_\sigma(\gamma)\|_R \neq 0$ and using Corollary 7.3 we obtain that both $\rho(\Gamma)$ and $\sigma(\Gamma)$ are Zariski dense inside $G \ltimes V$.

Let $\eta : \Gamma \to G \ltimes (V \oplus V)$ be such that for all $\gamma \in \Gamma$ we have

$$\eta(\gamma) = (L_\rho(\gamma), T_\rho(\gamma), T_\sigma(\gamma)).$$

Let $J$ and $Z_2$ be as in Lemma 8.2. We use Lemma 8.1 and obtain that $\eta(\Gamma) \subset Z_2$. Hence $X$, the Zariski closure of $\eta(\Gamma)$ inside $G \ltimes (V \oplus V)$ is a subvariety of $Z_2$. It follows that $X$ is a proper subvariety of $G \ltimes (V \oplus V)$. Now we use Proposition 8.4 and obtain that there exists a continuous automorphism $\sigma : G \ltimes V \to G \ltimes V$ such that $\sigma \circ \rho = \rho$.

⋄ We use Proposition 8.5 and obtain that there exists $(A,Y) \in G \ltimes V$ such that $\sigma(g,X) = (A,Y)(R_g,X)(A,Y)^{-1}$. Also, as $L_\rho = L_\eta$ we obtain that $A R_g = R_g A$ for all $g \in G$. Moreover, as $X \subset Z_2$, we use Lemma 4.1 and for all $(g,X) \in G \ltimes V$ we obtain that

$$P_g(0)^2 \|A P_g(R_e - R_g) X\|_R^2 = P_g(0)^2 \|A P_g(R_e - R_g) X\|_R^2.$$ 

Hence, for $c \in MA$ with $P_c(0) \neq 0$, $X \in V^0$ and $g = hch^{-1}$ we get that

$$\|A R_g X\|_R^2 = \|R_g X\|_R^2.$$ 

As $R$ is irreducible we deduce that $\|A Y\|_R^2 = \|Y\|_R^2$ for all $Y \in V$. It follows that $A \in O(B_R)$. Hence, $A$ is in the centralizer of $R(G)$ inside $O(B_R)$ and our result follows.

\[ \square \]

9. Isospectrality of norm: General case

In this section we consider faithful absolutely irreducible algebraic self-contragredient representations of a real split connected semisimple algebraic Lie group with trivial center and we show that the normed Margulis-Smilga invariant spectrum of two such representations are completely determined by the isomorphism class.

Let $(G,V,R,\Gamma,\rho,\sigma)$ be as in Convention 0.2 and let $R$ be an absolutely irreducible self-contragredient representation. Hence, $R$ preserves an invariant symmetric bilinear form, which we denote by $B_R$. Moreover, we denote the norm associated to $B_R$ by $\| \cdot \|_R$.

Remark 9.1. Let $N_r, N_l$ be two nontrivial proper normal subgroups of $G \ltimes V$ such that $\iota : (G \ltimes V)/N_r \to (G \ltimes V)/N_l$ is a continuous isomorphism. We denote the set of all $(g_r,X_{g_r},g'_r,Y_{g_r}) \in (G \ltimes V \times G \ltimes V)$ such that $(g'_r,Y_{g_r})N_l = \iota((g_r,X_{g_r})N_r)$ by $D_l$, i.e.

$$D_l := \{(g_r,X_{g_r},g'_r,Y_{g_r}) \mid (g'_r,Y_{g_r})N_l = \iota((g_r,X_{g_r})N_r)\}.$$ 

Lemma 9.2. Let $N : G \ltimes V \times G \ltimes V \to \mathbb{R}$ be such that for all $(g,X,h,Y) \in G \ltimes V \times G \ltimes V$ we have:

$$N(g,X,h,Y) := \|P_{gh}(0)P_h(R_e - R_h) Y\|_R^2 - \|P_{gh}(0)P_g(R_e - R_g) X\|_R^2,$$

and let $Z_R := \{(g,X,h,Y) \in G \ltimes V \times G \ltimes V \mid N(g,X,h,Y) = 0\}$. Then for all $\iota$ as mentioned in Remark 9.1 we have $D_l \setminus Z_R \neq \emptyset$. In particular, we have $Z_R \subseteq G \ltimes V \times G \ltimes V$.  

Proof. Let $N_r, N_l$ be any two nontrivial proper normal subgroups of $G \ltimes V$ such that $\iota : (G \ltimes V)/N_r \to (G \ltimes V)/N_l$ is a continuous isomorphism. We use Proposition A.2 and observe that $N_r = G_r \ltimes V$ and $N_l = G_l \ltimes V$, for some nontrivial proper normal subgroup $G_r, G_l$ of $G$. Now using the third isomorphism Theorem of groups we obtain that $(G \ltimes V)/N_r$ is isomorphic to $G/G_r$ and $(G \ltimes V)/N_l$ is isomorphic to $G/G_l$. Therefore, $\iota : (G \ltimes V)/N_r \to (G \ltimes V)/N_l$ gives rise to an isomorphism $\iota : G/G_r \to G/G_l$. Now we use Lemmas 2.6 and 3.10 to observe that the set $S := \{g \in G \mid P_g(0) \neq 0\}$ is an open dense subset of $G$. Moreover, as $G/G_r$ and $G/G_l$ are the quotients of $G$ by some group action, the projection maps $\pi_r : G \to G/G_r$ and $\pi_l : G \to G/G_l$ are open. Hence $\pi_r(S)$ and $\pi_l(S)$ are open dense subsets of $G/G_r$ and $G/G_l$, respectively. It follows that $\iota \circ \pi_r(S)$ and $\iota \circ \pi_l(S)$ are open dense subsets of $G/G_r$ and $G/G_l$, respectively. Let $p \in \iota \circ \pi_r(S) \cap \pi_l(S)$. Then there exists $g_r, g_l \in S$ such that $p = \pi_l(g_l') = \iota \circ \pi_r(g_r') = \iota \circ \pi_r(g_r) \circ \pi_l(S)$ i.e. $g_l'G_l = \iota(g_r, G_r)$. It follows that $P_{g_r}(0) \neq 0$, $P_{g_l}(0) \neq 0$ and $(g_r', Y)N_l = \iota((g_r, X)N_r)$ for all $X, Y \in V$.

As $\mathcal{B}_R$, the invariant form of $R$, is orthogonal, we use Lemma 1.1 of [Gro71] to conclude that the restriction of $\mathcal{B}_R$ on $V^0$ is a non-degenerate orthogonal form. Hence, $V^0$ admits vectors which are not self-orthogonal. Let $V \in V^0$ be such that $||V||_R \neq 0$. Moreover, let $h \in G$ be such that $hgh^{-1} \in MA$. We choose $Y_{g_r} = 0, X_{g_r} = R_g V$ and using Proposition 4.2 we observe that

$$N(g_r, X_{g_r}, g_l', Y_{g_l}) = N(g_r, X_{g_r}, g_l', 0)$$

$$= \|P_{g_l}(0)P_{g_l'}(R_g - R_{g_l'})0\|^2_R - \|P_{g_l}(0)P_{g_r}(R_g - R_{g_l})X_{g_r}\|^2_R$$

$$= -\|P_{g_l}(0)P_{g_r}(0)R_h \circ \pi_0 \circ R_h^{-1}(X_{g_r})\|^2_R$$

$$= -\|P_{g_l}(0)P_{g_r}(0)V\|^2_R \neq 0.$$}

Hence, the set $(G \ltimes V \times G \ltimes V) \setminus Z_R$ is non empty and in particular it contains $(g_r, X_{g_r}, g_l', Y_{g_l})$ with $(g_l', Y_{g_l}) N_l = \iota((g_r, X_{g_r}) N_r)$, concluding our result. $\square$

Remark 9.3. We denote the projections onto the left and right components of $G \ltimes V \times G \ltimes V$ by $\pi_l$ and $\pi_r$ respectively, i.e. $\pi_l, \pi_r : G \ltimes V \times G \ltimes V \to G \ltimes V$ be such that for all $(g, X, h, Y) \in G \ltimes V \times G \ltimes V$ we have $\pi_l(g, X, h, Y) = (g, X)$ and $\pi_r(g, X, h, Y) = (h, Y)$.

Proposition 9.4. Let $(G, V, R, \Gamma, \rho, g)$ be as in Convention 0.2, let $\rho(\Gamma)$ and $g(\Gamma)$ both be Zariski dense inside $G \ltimes V$ and $X$, the Zariski closure of $(\rho, \eta)(\Gamma)$ inside $G \ltimes V \times G \ltimes V$, satisfies $D_i \setminus X \neq \emptyset$ for all $i$ mentioned in Remark 9.1. Then there exists a continuous automorphism $\sigma : G \ltimes V \to G \ltimes V$ such that $\sigma \circ \rho = g$.

Proof. As $(\rho, \rho)(\gamma)(g, \rho)(\Gamma)(\rho, \gamma)(\gamma)^{-1} = (g, \rho)(\Gamma)$ for all $\gamma \in \Gamma$, we obtain that $(\rho, \rho)(\gamma)[X(\rho, \rho)(\gamma)^{-1} = X$. We recall the projection maps $\pi_l$ and $\pi_r$ from Remark 9.3 and observe that they are homomorphisms. Hence, it follows that $\rho(\gamma) \pi_l(X) \rho(\gamma)^{-1} = \pi_l(X)$ and $\rho(\gamma) \pi_r(X) \rho(\gamma)^{-1} = \pi_r(X)$ for all $\gamma \in \Gamma$. As both $\rho(\Gamma)$ and $g(\Gamma)$ are Zariski dense inside $G \ltimes V$, we deduce that both $\pi_l(X)$ and $\pi_r(X)$ are normal inside $G \ltimes V$ (using the Corollary at page 54 of [Hum75]). We observe that $\pi_l(X) \supset \rho(\Gamma), \pi_r(X) \supset \rho(\Gamma)$ and we use Proposition A.2 to obtain that $\pi_l(X) = G_l \ltimes V$ and $\pi_r(X) = G_r \ltimes V$ for some
normal subgroups $G_l, G_r$ of $G$. Moreover, both $\rho(\Gamma)$ and $\varrho(\Gamma)$ are Zariski dense inside $G \ltimes V$ and it follows that $G_l = G_r = G$.

Now we consider the following two normal subgroups of $X$: $N_l := \ker(\pi_l|_X)$ and $N_r := \ker(\pi_r|_X)$. As $(\varrho, \rho)(\Gamma) \subseteq X$ and $N_l$ is normal in $X$, for all $\gamma \in \Gamma$ we have $(\varrho(\gamma), \rho(\gamma))N_l((\varrho(\gamma), \rho(\gamma))^{-1} \subseteq N_l$. Moreover, as $N_l = \ker(\pi_l) \cap X$, we obtain that $N_l \subseteq \{o\} \times G \ltimes V$, where $o := (e, 0)$. Hence any element of $N_l$ is of the form $(o, n)$ and we obtain that

$$(\varrho(\gamma), \rho(\gamma))(o, n)(\varrho(\gamma), \rho(\gamma))^{-1} = (o, \rho(\gamma))(o, n)(o, \rho(\gamma))^{-1}$$

for all $\gamma \in \Gamma$. As $\rho(\Gamma)$ is Zariski dense inside $G \ltimes V$, we obtain that $N_l$ is normal inside $\{o\} \times G \ltimes V$. Similarly, we obtain that $N_r$ is normal inside $G \ltimes V \times \{o\}$. Moreover, as $\pi_l(X) = G \ltimes V = \pi_r(X)$, we obtain that

$$\dim(N_l) = \dim(X) - \dim(G \ltimes V) = \dim(N_r).$$

Now using Proposition A.2 we deduce that either of the following holds:

1. $N_l = \{o\} \times G \ltimes V$ and $N_r = G \ltimes V \times \{o\}$,
2. $N_l = \{o\} \times G_l \ltimes V$ and $N_r = G_r \ltimes V \times \{o\}$ for some nontrivial proper normal subgroups $G_l, G_r$ of $G$,
3. both are trivial.

We consider these three cases separately:

- If $N_l = \{o\} \times G \ltimes V$ and $N_r = G \ltimes V \times \{o\}$, then we obtain a contradiction. Indeed, we have $G \ltimes V \times G \ltimes V = N_lN_l \subseteq X \subseteq G \ltimes V \times G \ltimes V$. Then by Goursat’s lemma [Gou89] we get that the image of $X$ inside $(G \ltimes V)/N_l \times (G \ltimes V)/N_l$ is given by the graph of an isomorphism $\sigma : (G \ltimes V)/N_r \rightarrow (G \ltimes V)/N_l$.

Now we want to show that $\sigma$ is continuous. We consider the projections

$$p_r : G \ltimes V \times \{o\} \rightarrow (G \ltimes V \times \{o\})/G_r \ltimes V \times \{o\},$$

and let $\pi_l' : X/(N_lN_l) \rightarrow (G \ltimes V)/N_l$ and $\pi_r' : X/(N_lN_l) \rightarrow (G \ltimes V)/N_l$ respectively be the quotient maps induced by $p_r \circ (\pi_l|_X)$ and $p_l \circ (\pi_r|_X)$. We note that $\sigma = (\pi_l')^{-1} \circ \pi_r'$. Hence $\sigma$ is a continuous isomorphism. It follows that, for all $g, g' \in G$ and $X, X' \in V$ with $\sigma((g, X)N_l) = (g', X')N_l$ we have $(g, X, g', X') \in X$ i.e. $D_\sigma \subseteq X$. Hence, $\emptyset = D_\sigma \setminus X \neq \emptyset$, a contradiction.

- Suppose both $N_l$ and $N_r$ are trivial then by Goursat’s lemma [Gou89] we get that $X \subseteq G \ltimes V \times G \ltimes V$ is the graph of an automorphism $\sigma$ of $G \ltimes V$. We can choose $\sigma$ to be $(\pi_l|_X)^{-1} \circ (\pi_r|_X)$ and conclude by observing that it is continuous.

**Theorem 9.5.** Let $(G, V, R, \Gamma, \rho, \varrho)$ be as in Convention 0.2, let $R$ be an absolutely irreducible self-contragredient representation and let $\|M_\rho(\gamma)\|_R = \|M_\rho(\gamma)\|_R$ for all $\gamma \in \Gamma$. Then either of the following holds:

1. both $\rho(\Gamma)$ and $\varrho(\Gamma)$ are Zariski dense inside some conjugates of $G$,
2. there exists $(A, Y) \in O(R_k) \ltimes V$ such that $A$ normalizes $R(G)$ and $\rho$ is conjugate to $\varrho$ by $(A, Y)$.

**Proof.** We will prove this result in three parts.

- Let $\|M_\rho(\gamma)\|_R = 0$ for all $\gamma \in \Gamma$. It follows that $\|M_\rho(\gamma)\|_R = 0$ for all $\gamma \in \Gamma$. We use Corollary 7.2 and obtain that there exists $X, Y \in V$ such
that \( \rho = (e, X)L_\rho(e, X)^{-1} \) and \( \varrho = (e, Y)L_\varrho(e, Y)^{-1} \). It follows that both \( \rho(\Gamma) \) and \( \varrho(\Gamma) \) are Zariski dense inside some conjugates of \( G \).

\( \diamond \) Let there exist \( \gamma \in \Gamma \) such that \( \|M_\gamma(\gamma)\|_R \neq 0 \). Hence \( \|M_\varrho(\gamma)\|_R \neq 0 \) and using Corollary 7.3 we obtain that both \( \rho(\Gamma) \) and \( \varrho(\Gamma) \) are Zariski dense inside \( G \rtimes V \).

We use Lemma 8.1 and observe that \( \|R_\gamma \|_R = 0 \) for all \( \gamma \in \Gamma \). Hence \( X \), the Zariski closure of \( (\varrho, \rho)(\Gamma) \) inside \( G \rtimes V \rtimes G \), is a subvariety of \( Z_N \) and using Lemma 9.2 we obtain that \( D_\gamma \setminus X \neq \emptyset \) for all \( \gamma \) as mentioned in Remark 9.1. Now using Proposition 9.4 we deduce that there exists a continuous automorphism \( \sigma : G \rtimes V \to G \rtimes V \) such that \( \sigma \circ \rho = \varrho \).

\( \diamond \) Let \( \sigma : G \rtimes V \to G \rtimes V \) be as above. We use Proposition 8.5 and obtain that there exists \( (A, Y) \in G \rtimes V \) such that \( \sigma(g, X) = (A, Y)(g, X)(A, Y)^{-1} \).

Now we recall Notation 3.2 and Proposition 3.9 to obtain that \( P_g = P_g \). Moreover, as \( X \subset Z_N \), we use Lemma 4.1 and for all \( (g, X) \in G \rtimes V \) we obtain that
\[
P_g(0)^2\|AP_g(R_e - R_g)X\|^2_R = P_g(0)^2\|P_g(R_e - R_g)X\|^2_R.
\]

Hence, for \( c \in M_A \) with \( P_c(0) \neq 0 \), \( X \in V^0 \) and \( g = hch^{-1} \) we get that \( \|AR_hX\|^2_R = \|R_hX\|^2_R \). As \( R \) is irreducible we deduce that \( \|AY\|^2_R = \|Y\|^2_R \) for all \( Y \in V \). It follows that \( A \in O(B_R) \). Hence, \( A \) is in the normalizer of \( R(G) \) inside \( O(B_R) \) and our result follows. \( \square \)

10. Isospectrality: general case

In this section, we show that the Margulis-Smilga invariant spectra of two faithful absolutely irreducible algebraic self-contragredient representations of a connected real split semisimple algebraic Lie group with trivial center are completely determined by the isomorphism class of Margulis-Smilga spacetimes. Finally, as a corollary we prove the main result and one application.

**Theorem 10.1.** Let \((G, V, R, \Gamma, \rho, \varrho) \) be as in Convention 0.2, let \( R \) be an absolutely irreducible self-contragredient representation and let \( M_\rho(\gamma) = M_\varrho(\gamma) \) for all \( \gamma \in \Gamma \). Then either of the following holds:

1. both \( \rho(\Gamma) \) and \( \varrho(\Gamma) \) are Zariski dense inside some conjugates of \( G \),
2. there exists a continuous automorphism \( \sigma : G \rtimes V \to G \rtimes V \) such that \( \sigma \rho = \varrho \) and \( \sigma \) is conjugation by an element \((A, Y) \in O(B_R) \rtimes V \) such that \( A \) normalizes \( R(G) \).

**Proof.** As \( M_\rho(\gamma) = M_\varrho(\gamma) \) for all \( \gamma \in \Gamma \), we obtain that \( \|M_\rho(\gamma)\|_R = \|M_\varrho(\gamma)\|_R \) for all \( \gamma \in \Gamma \). Hence, using Theorem 9.5 we obtain our result. \( \square \)

**Theorem 10.2.** Let \((G, V, R, \Gamma, \rho, \varrho) \) be as in Convention 0.2 and let \( \rho \) and \( \varrho \) be two Margulis-Smilga spacetimes. Then the following holds:

1. If \( \rho \) and \( \varrho \) are conjugate via some inner automorphism of \( G \rtimes V \), then they have the same Margulis-Smilga invariant spectrum.
2. If \( \rho, \varrho \) have the same Margulis-Smilga invariant spectrum and \( L_\rho = L_\varrho \), then there exists \( \sigma \), an inner isomorphism of \( G \rtimes V \), such that \( \rho = \sigma \circ \varrho \).
3. If \( \rho, \varrho \) have the same Margulis-Smilga invariant spectrum and \( R \) is an absolutely irreducible self-contragredient representation, then there exists \((A, Y) \in O(B_R) \rtimes V \) such that \( \rho = (A, Y) \varrho (A, Y)^{-1} \).
Proof. We will prove this result in three parts.

\( \diamond \) Let \((g, Y) \in G \ltimes V\) be such that \((g, Y)\rho(g, Y)^{-1} = \varrho\) and for all \( \gamma \in \Gamma \) let \( h_\gamma \in G \) be such that \( L_\rho(\gamma) = h_\gamma \exp(3d_\rho(\gamma))h_\gamma^{-1} \). Then for all \( \gamma \in \Gamma \) we have \( Jd_\rho(\gamma) = Jd_\varrho(\gamma) \) and \( L_\varrho(\gamma) = gh_\gamma \exp(Jd_\rho(\gamma))(gh_\gamma)^{-1} \). Hence, for all \( \gamma \in \Gamma \), we deduce that

\[
M_\varrho(\gamma) = \pi_0(R_{gh_\gamma}^{-1}(g^{-1}T_\rho(\gamma)) = \pi_0(R_{gh_\gamma}^{-1}(Y + R_\gamma T_\rho(\gamma) - R_\gamma R_{L_\rho(\gamma)}R_g^{-1}Y))
= \pi_0(R_{h_\gamma}^{-1}(R_g^{-1}Y + T_\rho(\gamma) - R_{L_\rho(\gamma)}R_g^{-1}Y))
= \pi_0(R_{h_\gamma}^{-1}T_\rho(\gamma)) + \pi_0(R_{h_\gamma}^{-1}((R_e - R_{L_\rho(\gamma)})R_g^{-1}Y))
= M_\rho(\gamma) + \pi_0((R_e - \exp(Jd_\rho(\gamma)))R_g^{-1}Y) = M_\rho(\gamma).
\]

\( \diamond \) Let \( \rho, \varrho \) be two Margulis-Smilga spacetimes with \( L_\rho = L_\varrho \) and the same Margulis-Smilga invariant spectrum. We use Theorem 6.4 to obtain that there exists \( \sigma \), an inner isomorphism of \( G \ltimes V \), such that \( \rho = \sigma \circ \varrho \).

\( \diamond \) Let \( \rho, \varrho \) be two Margulis-Smilga spacetimes with the same Margulis-Smilga invariant spectrum and let \( X \) be an absolutely irreducible self-contragredient representation. Hence \( \rho \) and \( \varrho \), act properly on \( V \). It follows that for all \( \gamma \in \Gamma \setminus \{e\} \) we have \( M_\rho(\gamma) \neq 0 \) and \( M_\varrho(\gamma) \neq 0 \). Now we use Corollary 7.3 and Theorem 10.1 to obtain our result.

\( \square \)

**Theorem 10.3.** Let \( G \) be a real split connected simple algebraic Lie group with trivial center, let \( g \) be its Lie algebra with Killing form \( B \), let \( \text{Ad} : G \to GL(g) \) be the adjoint representation and let \( \rho \) and \( \varrho \) be two Margulis-Smilga spacetimes. Then the following holds:

1. If \( \rho \) and \( \varrho \) are conjugate via some inner automorphism of \( G \ltimes \text{Ad} g \), then they have the same Margulis-Smilga invariant spectrum.
2. If \( \rho, \varrho \) have the same Margulis-Smilga invariant spectrum and \( L_\rho = L_\varrho \), then there exists \( \sigma \), an inner isomorphism of \( G \ltimes \text{Ad} g \), such that \( \rho = \sigma \circ \varrho \).
3. If \( \rho, \varrho \) have the same Margulis-Smilga invariant spectrum then there exists \((A, Y) \in SO(B) \ltimes g\) such that \( \rho = (A, Y)\varrho(A, Y)^{-1} \).

**Proof.** We observe that \( g \) is finite dimensional, the Killing form \( B \) is a non-degenerate symmetric bilinear form. As \( G \) is connected and with trivial center we obtain that the adjoint representation is faithful. Also, as \( G \) is simple we obtain that \( \text{Ad} \) is irreducible. It follows that the complexification of \( \text{Ad} \) is also irreducible and hence \( \text{Ad} \) is absolutely irreducible. Moreover, by Proposition 4.4.5 of [Spr09] we obtain that \( \text{Ad} \) is algebraic. We observe that \( \alpha \) is a zero weight space of \( \text{Ad} \) and hence \( \text{Ad} \) admits zero as a weight. Also, as the Killing form is not degenerate we obtain that \( \text{Ad} \) is self-contragredient. Hence our result follows from Theorem 10.2.

\( \square \)

**Appendix A. Normal subgroups**

In this section we prove some results about the normal subgroups of affine groups of the form \( G \ltimes V \), where \( G \) is a connected real split semisimple algebraic Lie group with trivial center acting on a vector space \( V \) via a faithful irreducible algebraic representation \( \mathbb{R} : G \to GL(V) \). We expect
that these results are known in the community but we could not find an appropriate reference in the literature.

**Lemma A.1.** Let \( G \) be a connected real split semisimple Lie group, let \( V \) be a finite dimensional vector space with \( \dim V > 1 \), let \( R : G \to \text{GL}(V) \) be an irreducible algebraic representation and let \( X \in \mathbb{V} \) with \( X \neq 0 \). Then the additive group generated by \( \{ R_g X \mid g \in G \} \subset \mathbb{V} \) is \( \mathbb{V} \).

**Proof.** If possible let us assume that \( R_g X = X \) for all \( g \in G \). Then \( R(G) \) fixes the line \( \mathbb{R}X \) and \( \mathbb{R}X \not\subset \mathbb{V} \), a contradiction to the fact that the representation \( R \) is irreducible.

Hence we can assume that there exists a \( g \in G \) such that \( R_g X \neq X \). We use Lemma 2.6 and the continuity of the action of \( G \) to deduce that we can choose \( g \) such that \( g \) is loxodromic and the dimension of the unit eigenspace of \( R_g \) is exactly \( \dim \mathbb{V} \). Let \( m \in M, \ Z \in \mathfrak{a}^{++} \) and \( h \in G \) be such that \( g = hm \exp(Z)h^{-1} \) and for all \( \lambda \in \Omega \cup \{0\} \) let \( Y_\lambda \in \mathbb{V} \) be such that

\[
Y := R_h^{-1} X = Y_0 + \sum_{\lambda \in \Omega} Y_\lambda.
\]

Moreover, as \( \Omega \) is finite, we can slightly perturb \( g \) and make sure that for all \( \lambda, \nu \in \Omega \) we have \( \lambda(Z) \neq \nu(Z) \) whenever \( \lambda \neq \nu \). As \( gX \neq X \) we obtain that \( Y \neq Y_0 \) and there exists \( \mu \in \Omega \suchthat Y_\mu \neq 0 \). We observe that \( \lambda(Z) \neq 0 \) for \( \lambda \in \Omega \) and choose

\[
t_\lambda := \left( 1 + \frac{\log 2}{\lambda(Z)} \right).
\]

It follows that

\[
R_{\exp(t_\lambda Z)} Y_\lambda = 2R_{\exp(Z)} Y_\lambda.
\]

Indeed,

\[
R_{\exp(t_\lambda Z)} Y_\lambda = \exp(\lambda(t_\lambda Z))Y_\lambda = \exp(\log 2 + \lambda(Z))Y_\lambda
\]

\[
= 2 \exp(\lambda(Z))Y_\lambda = 2R_{\exp(Z)} Y_\lambda.
\]

Therefore, for all \( \lambda \in \Omega \) we have \( (2R_{\exp(Z)} - R_{\exp(t_\lambda Z)})Y_\lambda = 0 \). It follows that for all \( \lambda \in (\Omega \cup \{0\}) \setminus \{\mu\} \) and

\[
R^\mu := (R_{\exp(Z)} - R_e) \prod_{\nu \in \Omega \setminus \{\mu\}} \left( 2R_{\exp(Z)} - R_{\exp(t_\nu Z)} \right),
\]

we have \( R^\mu Y_\lambda = 0 \). Therefore, we obtain that \( R^\mu Y \in \mathbb{V}^\mu \) and \( R^\mu Y = R^\mu Y_\mu \).

Moreover, we observe that

\[
R^\mu Y_\mu = (R_{\exp(Z)} - R_e) \prod_{\nu \in \Omega \setminus \{\mu\}} \left( 2R_{\exp(Z)} - R_{\exp(t_\nu Z)} \right) Y_\mu
\]

\[
= (\exp(\mu(Z)) - 1) \prod_{\nu \in \Omega \setminus \{\mu\}} (2 \exp(\mu(Z)) - \exp(t_\nu \mu(Z))) Y_\mu.
\]

and \( (\exp(\mu(Z)) - 1) \prod_{\nu \in \Omega \setminus \{\mu\}} (2 \exp(\mu(Z)) - \exp(t_\nu \mu(Z))) \neq 0 \). Hence

\[
\{ (R_{\exp(t Z)} - R_{\exp(s Z)}) R^\mu Y \mid t, s \in \mathbb{R} \} = \mathbb{R} Y_\mu.
\]

Let \( S \) be the additive group generated by \( \{ R_g X \mid g \in G \} \subset \mathbb{V} \) and hence \( R_h^{-1} X \in S \). We observe that \( R^\mu \) is inside the additive group generated by the set \( \{ R_g \mid g \in G \} \subset \mathfrak{gl}(V) \). It follows that \( \mathbb{R} Y_\mu \subset S \). Also, we observe that the additive group generated by \( R(G)\mathbb{R} Y_\mu \) is the same as the vector space generated by \( R(G)\mathbb{R} Y_\mu \). Moreover, the vector space generated by \( R(G)\mathbb{R} Y_\mu \)
is invariant under the action of $R(G)$ and using the irreducibility of the representation $R$ we obtain that $R(G)R Y \mu$ generates $V$. Therefore, we conclude that $S = V$. \hfill \square

**Proposition A.2.** Let $G$ be a connected real split semisimple algebraic Lie group with trivial center, let $V$ be a finite dimensional vector space with $\dim V > 1$, let $R : G \to GL(V)$ be a faithful irreducible algebraic representation and let $N$ be a normal subgroup of $G \ltimes V$. Then $N$ is either of the following subgroups:

1. the trivial group,
2. $G_i \ltimes V$, where $G_i$ is a normal subgroup of $G$.

**Proof.** Let $N$ be a nontrivial normal subgroup of $G \ltimes V$. Then there exists $(g, X) \in N$ with $(g, X) \neq (e, 0)$. Moreover, for any $(h, Y) \in G \ltimes V$ we observe that $(h, Y)^{-1} = (h^{-1}, -R^{-1}_h Y)$ and hence

$$(h, Y)(g, X)(h, Y)^{-1} = (hgh^{-1}, Y + R_h X - R_{hgh^{-1}Y}).$$

It follows that $(g, X) \in N$ if and only if $(hgh^{-1}, Y + R_h X - R_{hgh^{-1}Y}) \in N$ for all $h \in G$ and $Y \in V$.

Now we consider the linear projection map $L : G \ltimes V \to G$ and observe that for all $h \in G$, $hL(N)h^{-1} \subset L(N)$. It follows that $L(N)$ is a normal subgroup of $G$. We prove our result in the following two parts:

1. $L(N)$ is trivial: As $N$ is nontrivial, in this case we see that there exists $X \neq 0$ such that $(e, X) \in N$. Hence for all $h \in G$ we have

$$(h, 0)(e, X)(h, 0)^{-1} = (e, R_h X) \in N.$$

As the representation $R$ is irreducible using Lemma A.1 we obtain that $(e, Y) \in N$ for all $Y \in V$. Therefore, we deduce that $N = \{e\} \ltimes V$.

2. $L(N)$ is a nontrivial normal subgroup of $G$: In this case also we see that there exists $X \neq 0$ such that $(e, X) \in N$. Indeed, if not then $N \cap \{(e) \ltimes V\} = \{(e, 0)\}$ and hence

$L|_N : N \to G$

is an isomorphism onto $L(N)$. It follows that for all $g \in L(N)$ there exist $X_g \in V$ such that $X_{gh} = X_g + R_g X_h$ and

$N = \{(g, X_g) \mid g \in L(N)\}.$

Since $N$ is normal inside $G \ltimes V$, for all $Y \in V$ we have

$$(e, Y)(g, X_g)(e, Y)^{-1} = (g, Y + X_g - R_g Y) \in N.$$

Hence $Y + X_g - R_g Y = X_g$ for all $Y \in V$, a contradiction. Therefore, there exists $X \neq 0$ such that $(e, X) \in N$. Hence for all $h \in G$ we have

$$(h, 0)(e, X)(h, 0)^{-1} = (e, R_h X) \in N.$$

As the representation $R$ is irreducible, using Lemma A.1 we obtain that $(e, Y) \in N$ for all $Y \in V$ and we deduce that $L(N) \ltimes V \subset N$. Also, $N \subset L^{-1}(L(N)) = L(N) \ltimes V$. It follows that $L(N) \ltimes V = N$.

Therefore, the only nontrivial normal subgroups of $G \ltimes V$ are of the form $G_i \ltimes V$ where $G_i$ is a normal subgroup of $G$. \hfill \square
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