Subdiffusive concentration in first-passage percolation

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Abstract

We prove exponential concentration in i.i.d. first-passage percolation in \( \mathbb{Z}^d \) for all \( d \geq 2 \) and general edge-weights \((t_e)\). Precisely, under an exponential moment assumption \( (\mathbb{E}e^{\alpha t_e} < \infty \) for some \( \alpha > 0 \)) on the edge-weight distribution, we prove the inequality

\[
\mathbb{P}
\left|
T(0, x) - \mathbb{E}T(0, x)
\right| \geq \lambda \sqrt{s_1 \log s_1}
\leq ce^{-c'\lambda}, \; s_1 > 1
\]

for the point-to-point passage time \( T(0, x) \). Under a weaker assumption \( \mathbb{E}t_e^2(\log t_e)_+ < \infty \) we show a corresponding inequality for the lower-tail of the distribution of \( T(0, x) \). These results extend work of Benaïm-Rossignol [5] to general distributions.

1 Introduction

1.1 The model

Let \((t_e)_{e \in \mathcal{E}^d}\) be a collection of non-negative random variables indexed by the nearest-neighbor edges \( \mathcal{E}^d \) of \( \mathbb{Z}^d \). For \( x, y \in \mathbb{Z}^d \), define the passage time

\[
T(x, y) = \inf_{\gamma: x \to y} T(\gamma),
\]

where \( T(\gamma) = \sum_{e \in \gamma} t_e \) and the infimum is over all lattice paths \( \gamma \) from \( x \) to \( y \). \( T \) defines a pseudometric on \( \mathbb{Z}^d \) and First-Passage Percolation is the study of the asymptotic properties of \( T \). The model was introduced by Hammersley and Welsh [12] in 1965 and has recently been the object of much rigorous mathematical progress (see [6, 11, 13] for recent surveys).

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Under minimal assumptions on \((t_e)\), the passage time \(T(0, x)\) is asymptotically linear in \(x\), but the lower order behavior has resisted precise quantification. If \(d = 1\), the passage time is a sum of i.i.d. variables, so its fluctuations are diffusive, giving \(\chi = 1/2\), where \(\chi\) is the (dimension-dependent) conjectured exponent given roughly by \(\text{Var} T(0, x) \propto \|x\|^{2\chi}\). For \(d = 2\), the minimization in the definition of \(T\) is expected \([14]\) to create subdiffusive fluctuations, with a predicted value \(\chi = 1/3\). Subdiffusive behavior is expected in higher dimensions as well.

In this paper, we prove an exponential version of subdiffusive fluctuations for \(T(0, x)\) under minimal assumptions on the law of \((t_e)\). This result follows up on work done by the authors (extending the work of \([2, 5]\)) in \([8]\), in which it was shown that

\[
\text{Var} T(0, x) \leq C \frac{\|x\|_1}{\log \|x\|_1} \quad \text{for } \|x\|_1 > 1
\]

given only that the distribution of \(t_e\) has \(2 + \log\) moments. Our concentration inequalities apply to a nearly optimal class of distributions: for the upper tail inequality in \([14]\) we require that \(t_e\) has exponential moments and for the lower tail inequality \([15]\), that \(t_e\) has \(2 + \log\) moments. In contrast to existing work on subdiffusive concentration listed below, our methods do not rely on any properties of the distribution other than the tail behavior.

In 1993, Kesten \([16, \text{Eq. (1.15)}]\) gave the first exponential concentration inequality for \(T\), showing that if \(\mathbb{E} e^{\alpha t_e} < \infty\) for some \(\alpha > 0\), then one has

\[
P\left( |T(0, x) - \mathbb{E} T(0, x)| \geq \lambda \sqrt{\|x\|_1} \right) \leq c e^{-c' \lambda} \quad \text{for } \lambda \leq c'' \|x\|_1.
\]

This was improved by Talagrand \([20, \text{Eq. (8.31)}]\) to Gaussian concentration under the same moment assumption. After the influential work of Benjamini-Kalai-Schramm \([2]\) on sublinear variance, Benaïm and Rossignol proved a subdiffusive concentration inequality of the type \([14]\) for a class of distributions that they called nearly Gamma. These are continuous distributions that satisfy an entropy bound analogous to the logarithmic Sobolev inequality for the gamma distribution: for nearly gamma \(\mu\) and, for simplicity, \(f\) smooth,

\[
\text{Ent}_\mu f^2 := \int f^2(x) \log \frac{f^2(x)}{\mathbb{E}_\mu f^2} \mu(dx) \leq C \int (\sqrt{x} f'(x))^2 \mu(dx).
\]

Although the nearly Gamma class includes, for example, gamma, uniform and distributions with smooth positive density on an interval, it excludes all power law distributions, those with infinite support which decay too quickly, those with zeros on its support and all noncontinuous distributions.

The main goal of this paper is to prove subdiffusive concentration inequalities without the nearly Gamma assumption. The strategy is similar to the one introduced by Benaïm-Rossignol and used in \([8]\). However, several problems arise when attempting to implement this program for general distributions. These come in two forms: those dealing with the low moment assumption \([13]\) for the lower-tail inequality \([15]\) and those involved in the main entropy bound. Although the proof of this entropy bound follows the outline of \([8]\), through a Bernoulli encoding and analysis of greedy lattice animals, many of the estimates now require exponential concentration. In particular, one must decouple bounds for discrete derivatives similar to those in \([8]\) from quantities like \(e^{t_e T(0, x)}\) – see the terms in Theorem \([4.1]\). This is the goal of Section \([5]\) where ideas from the “entropy method” of Boucheron-Lugosi-Massart \([4]\) and estimates on lattice animals are explored.
1.2 Main result

The main assumptions are as follows: \( P \), the distribution of \((t_e)\), is a product measure with marginal \( \mu \) satisfying

\[
\mu(\{0\}) < p_c ,
\]

where \( p_c \) is the critical probability for \( d \)-dimensional bond percolation. Furthermore, we will generally assume either

\[
\mathbb{E}e^{\alpha t} < \infty \text{ for some } \alpha > 0 \tag{1.2}
\]

or

\[
\mathbb{E}t^2 (\log t) < \infty . \tag{1.3}
\]

**Theorem 1.1.** Let \( d \geq 2 \). Under (1.1) and (1.2), there exist \( c_1, c_2 > 0 \) such that for all \( x \in \mathbb{Z}^d \) with \( \|x\|_1 > 1 \),

\[
P \left( |T(0, x) - \mathbb{E}T(0, x)| \geq \frac{\|x\|_1^{1/2}}{(\log \|x\|_1)^{1/2}} \lambda \right) \leq c_1 e^{-c_2 \lambda} \text{ for } \lambda \geq 0 . \tag{1.4}
\]

Assuming (1.1) and (1.3), there exist \( c_1, c_2 > 0 \) such that for all \( x \in \mathbb{Z}^d \) with \( \|x\|_1 > 1 \),

\[
P \left( T(0, x) - \mathbb{E}T(0, x) \leq -\frac{\|x\|_1^{1/2}}{(\log \|x\|_1)^{1/2}} \lambda \right) \leq c_1 e^{-c_2 \lambda} \text{ for } \lambda \geq 0 . \tag{1.5}
\]

**Remark 1.2.** If (1.1) fails, then \( T(0, x) \) itself is sublinear in \( x \) and the model has a different character (see [15, Theorem 6.1] and [7, 21] for more details). Because \( T(0, x) \) is bounded below by the minimum of the \( 2d \) weights of edges adjacent to \( 0 \), it is necessary to assume (1.2) to obtain an upper-tail exponential concentration inequality. For the lower tail, our methods require \( 2 + \log \) moments and this is the same assumption made in [8] for a sublinear variance bound.

The strategy of the proof is to use a relation, stated in Lemma 2.2, between bounds on \( \text{Var } e^{\lambda T(0,x)} \) and exponential concentration. To obtain the required variance bound (Theorem 2.3), we follow the method of Benaim-Rossignol, applying the Falik-Samorodnisky inequality (Lemma 2.4) to the variable \( e^{\lambda F_m} \), where \( F_m \) is an averaged version of the passage time. From here, bounding the variance follows a broadly similar outline to that given in [5]: representing the passage times as a push-forward of Bernoulli sequences and the bound follows after a careful analysis of discrete derivatives. The main complications arise in giving these bounds and are dealt with using estimates on greedy lattice animals in Section 5.

1.3 Preliminary results

We will need a couple of results on the length of geodesics. By Proposition 1.3 below, condition (1.1) ensures that

\[
P(\exists \text{ a geodesic from } x \text{ to } y) = 1 \text{ for all } x, y \in \mathbb{Z}^d ,
\]

where a geodesic is a path \( \gamma \) from \( x \) to \( y \) that has \( T(\gamma) = T(x, y) \).

The fundamental estimate is from Kesten [15, Proposition 5.8].
Proposition 1.3 (Kesten). Assuming (1.1), there exist \(a, C_1 > 0\) such that for all \(n \in \mathbb{N}\),

\[
P\left( \exists \text{ self-avoiding } \gamma \text{ containing } 0 \text{ with } \#\gamma \geq n \text{ but } T(\gamma) < an \right) \leq e^{-C_1n} . \tag{1.7}
\]

As a consequence, we state a bound used in work of one of the authors and N. Kubota [9]. For this, let \(G(0, x)\) be the maximal number of edges in any self-avoiding geodesic from 0 to \(x\). An application of Borel-Cantelli to (1.7) implies

\[
\text{under (1.1), } \liminf_{\|x\|_1 \to \infty} \frac{T(0, x)}{\|x\|_1} \geq a > 0 \text{ almost surely } \tag{1.8}
\]

and so \(G(0, x)\) is finite almost surely.

Proposition 1.4. Assume (1.1) and let \(a\) be from Proposition 1.3. There exists \(C_2\) such that the variable

\[
Y_x = G(0, x)1_{\{T(0, x) < aG(0, x)\}}
\]

satisfies \(P(Y_x \geq n) \leq e^{-C_2n} \) for all \(x \in \mathbb{Z}^d\) and \(n \in \mathbb{N}\).

Proof. Defining

\[
A_m = \left\{ \exists \text{ self-avoiding } \gamma \text{ from } 0 \text{ with } \#\gamma \geq m \text{ but } T(\gamma) < a\#\gamma \right\}
\]

and summing (1.7) over \(n\), one has, for some \(C_2 > 0\),

\[
P(A_m) \leq e^{-C_2m} \text{ for all } m \in \mathbb{N} . \tag{1.9}
\]

For \(x \in \mathbb{Z}^d\), assume \(Y_x \geq n \geq 1\) and let \(\gamma\) be any self-avoiding geodesic from 0 to \(x\) with length \(G(0, x) = Y_x\). Then because \(Y_x \neq 0, G(0, x) > (1/a)T(0, x)\) and so

\[
T(\gamma) = T(0, x) < aG(0, x) = a\#\gamma ,
\]

with \(\#\gamma \geq n\). So \(A_n\) occurs and (1.9) completes the proof. \(\square\)

We can state a couple of relevant consequences of this proposition.

Corollary 1.5. Assume (1.1).

1. There exists \(C_3\) such that

\[
\mathbb{E}G(0, x)^2 \leq C_3\mathbb{E}T(0, x)^2 \text{ for all } x \in \mathbb{Z}^d . \tag{1.10}
\]

2. Under (1.2), there exists \(\alpha_1 > 0\) such that

\[
\sup_{0 \neq x \in \mathbb{Z}^d} \frac{\log \mathbb{E}e^{\alpha_1 G(0, x)}}{\|x\|_1} < \infty . \tag{1.11}
\]
Proof. Estimate

\[ \mathbb{E}G(0,x)^2 \leq a^{-2}\mathbb{E}T(0,x)^2 + \mathbb{E}Y_x^2, \]

where \( Y_x \) is from Proposition 1.4. Because \( \mathbb{E}Y_x^2 \) is bounded uniformly in \( x \), (1.8) (which gives \( \mathbb{E}T(0,x) \to \infty \) as \( \|x\|_1 \to \infty \)) shows (1.10). Assuming (1.2), for \( \beta > 0 \),

\[ \mathbb{E}e^{\beta G(0,x)} \leq \mathbb{E}e^{(\beta/a)T(0,x)} + \mathbb{E}e^{\beta Y_x}. \]

For \( \beta < C_2/2 \), the second term is bounded in \( x \). On the other hand letting \( \gamma_x \) be a deterministic path from 0 to \( x \) of length \( \|x\|_1 \), the first term is bounded by

\[ \mathbb{E}e^{(\beta/a)T(\gamma_x)} = \left( \mathbb{E}e^{(\beta/a)t}\right)^{\|x\|_1}. \] (1.12)

So we conclude for \( x \neq 0 \) and some \( C_4 \geq 1 \),

\[ \frac{\log \mathbb{E}e^{\beta G(0,x)}}{\|x\|_1} \leq \frac{\log \left( \mathbb{E}e^{(\beta/a)t} + C_4 \right)^{\|x\|_1}}{\|x\|_1} = \log \left( \mathbb{E}e^{(\beta/a)t} + C_4 \right) < \infty \]

when \( \beta < \min\{C_2/2, \alpha/2\} \), where \( \alpha \) is from (1.2).

For the remainder of the paper we assume (1.1).

2 Setup for the proof

Instead of showing concentration for \( T(0,x) \), we use an idea from [2]: to show it for \( T(z,z+x) \), where \( z \) is a random vertex near the origin. So, given \( x \in \mathbb{Z}^d \), fix \( \zeta \) with \( 0 < \zeta < 1/4 \) and define

\[ m = \lfloor \|x\|_1^\zeta \rfloor \text{ and } F_m = \frac{1}{\#B_m} \sum_{z \in B_m} T_z, \]

where \( T_z = T(z,z+x) \) (this particular randomization was used by both [11] and [19]). For \( \lambda \in \mathbb{R} \) we define

\[ G = G_\lambda = e^{\lambda F_m}. \] (2.1)

Below are the concentration inequalities for \( F_m \) analogous to (1.4). In the next subsection, we will show why they suffice to prove Theorem 1.1.

**Theorem 2.1.** Assuming (1.2), there exist \( c_1, c_2 > 0 \) such that for all \( x \in \mathbb{Z}^d \) with \( \|x\|_1 > 1 \),

\[ \mathbb{P} \left( |F_m - \mathbb{E}F_m| \geq \frac{\|x\|_1^{1/2}}{(\log \|x\|_1)^{1/2}} \lambda \right) \leq c_1 e^{-c_2 \lambda} \text{ for } \lambda \geq 0. \]

Assuming (1.3), there exist \( c_1, c_2 > 0 \) such that for all \( x \in \mathbb{Z}^d \) with \( \|x\|_1 > 1 \),

\[ \mathbb{P} \left( F_m - \mathbb{E}F_m < -\frac{\|x\|_1^{1/2}}{(\log \|x\|_1)^{1/2}} \lambda \right) \leq c_1 e^{-c_2 \lambda} \text{ for } \lambda \geq 0. \]
This theorem is a consequence on a bound for \( \text{Var} e^{\lambda F_m} \), and this is what we focus on from Section 3 onward. The link between a variance bound and concentration is given by the following lemma from [5, Lemma 4.1] (which itself is a version of [17, Corollary 3.2]). We have split the statement from [5] into upper and lower deviations.

**Lemma 2.2.** Let \( X \) be a random variable and \( K > 0 \). Suppose that
\[
\text{Var} e^{\frac{\lambda X}{2}} \leq K \lambda^2 e^{\lambda X} < \infty \text{ for } 0 \leq \lambda < \frac{1}{2 \sqrt{K}} .
\]
Then
\[
P \left( X - \mathbb{E}X > t\sqrt{K} \right) \leq 2e^{-t} \text{ for all } t \geq 0 .
\]
If
\[
\text{Var} e^{\frac{\lambda X}{2}} \leq K \lambda^2 e^{\lambda X} < \infty \text{ for } -\frac{1}{2 \sqrt{K}} < \lambda \leq 0 ,
\]
then
\[
P \left( X - \mathbb{E}X < -t\sqrt{K} \right) \leq 2e^{-t} \text{ for all } t \geq 0 .
\]

Taking \( K = \frac{C ||x||_1 \log ||x||_1}{||x||_1^2} \) for \( ||x||_1 > 1 \) and \( X = F_m \) in the previous lemma shows that to prove Theorem 2.1 it suffices to show the following variance bound.

**Theorem 2.3.** Assuming (1.2), there exists \( C_5 > 0 \) such that
\[
\text{Var} e^{\frac{\lambda F_m}{2}} \leq K \lambda^2 e^{\lambda F_m} < \infty \text{ for } |\lambda| < \frac{1}{2 \sqrt{K}} \text{ and } ||x||_1 > 1 ,
\]
where \( K = \frac{C_5 ||x||_1 \log ||x||_1}{||x||_1^2} \). Assuming (1.3), there exists \( C_6 > 0 \) such that
\[
\text{Var} e^{\frac{\lambda F_m}{2}} \leq K \lambda^2 e^{\lambda F_m} < \infty \text{ for } -\frac{1}{2 \sqrt{K}} < \lambda \leq 0 \text{ and } ||x||_1 > 1 ,
\]
where \( K = \frac{C_6 ||x||_1 \log ||x||_1}{||x||_1^2} \).

The proof of this bound will be broken into several sections below.

### 2.1 Theorem 2.1 implies Theorem 1.1

Assume first that we have the concentration bound
\[
P \left( |F_m - \mathbb{E}F_m| \geq \frac{||x||_1^{1/2}}{(\log ||x||_1)^{1/2} \lambda} \right) \leq be^{-c\lambda} , \ \lambda \geq 0
\]
for some \( b, c > 0 \) and that (1.2) holds. We will derive from (2.4) the corresponding estimate for the passage time \( T = T(0,x) \):
\[
P \left( |T(0,x) - \mathbb{E}T(0,x)| \geq \frac{||x||_1^{1/2}}{(\log ||x||_1)^{1/2} \lambda} \right) \leq b'e^{-c\lambda} , \ \lambda \geq 0 .
\]
Write
\[ T(0, x) - \mathbb{E}T(0, x) = F_m - \mathbb{E}T + T(0, x) - F_m, \]
and note that \( \mathbb{E}F_m = \mathbb{E}T \). If both events \( \{|F_m - \mathbb{E}F_m| < \lambda/2\} \) and \( \{|T(0, x) - F_m| < \lambda/2\} \) occur, then the triangle inequality implies that we have the bound
\[ |T(0, x) - \mathbb{E}T(0, x)| < \lambda. \]
This results in the estimate
\[ \mathbb{P}(|T(0, x) - \mathbb{E}T(0, x)| \geq \lambda) \leq \mathbb{P}(|F_m - \mathbb{E}F_m| \geq \lambda/2) + \mathbb{P}(|T(0, x) - F_m| \geq \lambda/2). \] (2.6)
By subadditivity, we can write
\[ |T(0, x) - F_m| = \left| T(0, x) - \frac{1}{\#B_m} \sum_{z \in B_m} T(z, z + x) \right| \leq \frac{1}{\#B_m} \sum_{z \in B_m} |T(0, x) - T(z, z + x)| \leq \frac{1}{\#B_m} \sum_{z \in B_m} (T(0, z) + T(x, x + z)). \]
Repeating the argument for (1.12) (bounding \( T(0, z) \) by the passage time of a deterministic path), we have for \( \alpha \geq 0 \) and each \( z \in B_m \)
\[ \mathbb{E}e^{\alpha T(0, z)} \leq (\mathbb{E}e^{\alpha x}) \mathbb{E} \| x \|_1^\alpha = C(\alpha) \mathbb{E} \| x \|_1^\alpha. \]
Here \( \alpha \) is from (1.2). We now obtain a bound for the second term on the right in (2.6). Let \( M > 0 \). First, by the triangle inequality:
\[ \mathbb{P} \left( \frac{1}{\#B_m} \sum_{z \in B_m} (T(0, z) + T(x, x + z)) \geq 2M \right) \leq 2 \mathbb{P}(\max_{z \in B_m} T(0, z) \geq M). \]
The last quantity is bounded by
\[ 2(\#B_m) \cdot \mathbb{P}(T(0, z) \geq M) \leq 2(\#B_m) \cdot e^{-\alpha M C(\alpha) \mathbb{E} \| x \|_1^\alpha} \leq 2 \mathbb{E} \| x \|_1^{dK} e^{-\alpha M C(\alpha) \mathbb{E} \| x \|_1^\alpha}. \]
Choosing \( 2M = \lambda \mathbb{E} \| x \|_1^{1/2} / (\log \| x \|_1)^{1/2} \) and adjusting constants, we find the bound
\[ \mathbb{P} \left( |T(0, x) - F_m| \geq \lambda \mathbb{E} \| x \|_1^{1/2} / (\log \| x \|_1)^{1/2} \right) \leq b' e^{-c' \lambda}. \]
Combined with (2.4) in (2.6), this shows (2.5).
We now move to proving that under assumption (1.3), if we prove Theorem 2.1 then there exist \( b'', c'' > 0 \) such that
\[ \mathbb{P} \left( T(0, x) - \mathbb{E}T(0, x) < -\frac{\| x \|_1^{1/2}}{(\log \| x \|_1)^{1/2} \lambda} \right) \leq b'' e^{-c'' \lambda}, \lambda \geq 0. \]
Defining $S = \sum_{e \in B_m} t_e$, where the sum is over all edges with both endpoints in $B_m$, then

$$\mathbb{P}(S \leq 2E_S) \geq 1/2.$$ 

By the Harris-FKG inequality [4, Theorem 2.15], if we put $c_x = \frac{\|x\|_{1/2}}{(\log \|x\|_1)^{1/2}}$ and $S' = \sum_{e \in x + B_m} t_e$, then

$$\mathbb{P}(T(0, x) - ET(0, x) \leq -\lambda c_x, S \leq 2ES, S' \leq 2ES') \geq (1/4)\mathbb{P}(T(0, x) - ET(0, x) \leq -\lambda c_x).$$

This means that

$$\mathbb{P}(T(0, x) - ET(0, x) \leq -\lambda c_x) \leq 4\mathbb{P}(T(0, x) - ET(0, x) \leq -\lambda c_x, S \leq 2ES, S' \leq 2ES').$$

However the event on the right implies that for any $z \in B_m$, $T(z, z + x) \leq T(0, x) + 4ES$. Therefore

$$\mathbb{P}(T(0, x) - ET(0, x) \leq -\lambda c_x) \leq 4\mathbb{P}(F_m - EF_m \leq -\lambda c_x + 4ES).$$

Now we can bound $4ES$ by $C_7\|x\|^{d_4}_1$, so

$$\mathbb{P}(T(0, x) - ET(0, x) \leq -\lambda c_x) \leq 4\mathbb{P}(F_m - EF_m \leq -\lambda c_x + C_7\|x\|^{d_4}_1)$$

and this is bounded by

$$4\mathbb{P} \left( F_m - EF_m \leq - \left( \lambda - C_7 \frac{\|x\|^{d_4}_1}{c_x} \right) c_x \right) \leq 4c_1 \exp \left( -c_2 \left( \lambda - C_7 \frac{\|x\|^{d_4}_1}{c_x} \right) \right),$$

as long as $\lambda \geq C_7 \frac{\|x\|^{d_4}_1}{c_x}$.

To finish, we simply choose $\zeta = d/4$, so that $\|x\|^{d_4}_1/c_x \leq C_8$ for $\|x\|_1 > 1$ and some $C_8 > 0$. This implies

$$\mathbb{P}(T(0, x) - ET(0, x) \leq -\lambda c_x) \leq 4c_1 \exp \left( -c_2 \left( \lambda - C_8 \right) \right)$$

for $\lambda \geq 0$, $\|x\|_1 > 1$, giving the bound $C_9 e^{-C_{10} \lambda}$.

### 2.2 Falik-Samorodnitsky and entropy

Enumerate the edges of $\mathcal{E}^d$ as $e_1, e_2, \ldots$ and write $e^{\Lambda F_m}$ as a sum of a martingale difference sequence:

$$G - EG = \sum_{k=1}^{\infty} V_k,$$ 

where

$$V_k = \mathbb{E}[G \mid \mathcal{F}_k] - \mathbb{E}[G \mid \mathcal{F}_{k-1}].$$

We have written $\mathcal{F}_k$ for the sigma-algebra $\sigma(t_{e_1}, \ldots, t_{e_k})$, with $\mathcal{F}_0$ trivial. In particular if $F \in L^1(\Omega, \mathbb{P})$,

$$\mathbb{E}[F \mid \mathcal{F}_k] = \int F((t_e)) \prod_{i \geq k+1} \mu(dt_{e_i}).$$

(2.8)

To prove concentration for $F_m$, we bound the variance of $G$; the lower bound comes from the proof of [10, Theorem 2.2].
Lemma 2.4 (Falik-Samorodnitsky). If $\mathbb{E}G^2 < \infty$,
\[
\sum_{k=1}^{\infty} \text{Ent}(V_k^2) \geq \text{Var} G \log \left[ \frac{\text{Var} G}{\sum_{k=1}^{\infty} (\mathbb{E}|V_k|)^2} \right] .
\] (2.9)

In the above lemma, we have used $\text{Ent}$ to refer to entropy:

**Definition 2.5.** If $X$ is a non-negative random variable with $\mathbb{E}X < \infty$ then the entropy of $X$ is defined by:
\[
\text{Ent} X = \mathbb{E}X \log X - \mathbb{E}X \log \mathbb{E}X .
\]

We will need some basic results on entropy. This material is taken from [8, Section 2], though it appears in various places, including [4]. By Jensen’s inequality, $\text{Ent} X \geq 0$. There is a variational characterization of entropy [18, Section 5.2] that we will use.

**Proposition 2.6.** We have the formula
\[
\text{Ent} X = \sup \{ \mathbb{E}XY : \mathbb{E}e^Y \leq 1 \} .
\]

The second fact we need is a tensorization for entropy. For an edge $e$, write $\text{Ent}_e X$ for the entropy of $X$ considered only as a function of $t_e$ (with all other weights fixed).

**Proposition 2.7.** If $X$ is a non-negative measurable function of $(t_e)$ then
\[
\text{Ent} X \leq \sum_e \mathbb{E}\text{Ent}_e X .
\]

## 3 Bound on influences

To bound the sum $\sum_{k=1}^{\infty} (\mathbb{E}|V_k|)^2$ we start with a simple lemma from [8, Lemma 5.2]. For a given edge-weight configuration $(t_e)$ and edge $e$, let $(t_{e^c}, r)$ denote the configuration with value $t_f$ if $f \neq e$ and $r$ otherwise. Let $T_z(t_{e^c}, r)$ be the variable $T_z = T(z, z + x)$ in the configuration $(t_{e^c}, r)$ and define $\text{Geo}(z, z + x)$ as the set of edges in the intersection of all geodesics from $z$ to $z + x$.

**Lemma 3.1.** For $e \in \mathcal{E}^d$, the random variable
\[
D_{z,e} := \sup \{|r \geq 0 : e \text{ is in a geodesic from } z \text{ to } z + x \text{ in } (t_{e^c}, r)\} \cup \{0\}
\]
has the following properties almost surely.

1. $D_{z,e} < \infty$.
2. For $0 \leq s \leq t$,
\[
T_z(t_{e^c}, t) - T_z(t_{e^c}, s) = \min\{t - s, (D_{z,e} - s)_+\} .
\]
3. For $0 \leq s < D_{z,e}$, $e \in \text{Geo}(z, z + x)$ in $(t_{e^c}, s)$.

We need one more lemma from [8] bounding the length of geodesics. Let $\mathcal{G}$ be the set of all finite self-avoiding geodesics.
Lemma 3.2. Assuming $\mathbb{E} t_e^2 < \infty$, there exists $C_{11}$ such that for all finite $E \subset \mathcal{E}^d$,

$$\mathbb{E} \max_{\gamma \in \mathcal{G}} \#(E \cap \gamma) \leq C_{11} \text{diam } E.$$ 

With these two tools we can bound the influences in the denominator of the logarithm of (2.9). The following proof is very similar to the one of Benaim-Rossignol [5, Theorem 4.2].

Proposition 3.3. Assuming $\mathbb{E} t_e^2 < \infty$, there exists $C_{12}$ such that

$$\sum_{k=1}^\infty (\mathbb{E} |V_k|)^2 \leq C_{12}^2 \mathbb{E} ((1 + e^{\lambda t_e}) t_e)^2 \|x\|_1^{2(d(1-d)/2)} \mathbb{E} e^{2\lambda F_m} \text{ for all } x \in \mathbb{Z}^d.$$

This inequality holds for any $\lambda$ for which the left side is defined.

Under (1.2), we require $\lambda \in [0, \alpha]$ for the left side to be defined. Under (1.3), one can take $\lambda \leq 0$.

Proof. Let $F_m^{(k)}$ be the variable $F_m$ with the edge weight $t_e$ replaced by an independent copy $t'_e$. Then we can give the upper bound

$$\mathbb{E} |V_k| \leq \mathbb{E} \left| e^{\lambda F_m} - e^{\lambda F_m^{(k)}} \right| = 2\mathbb{E} \left( e^{\lambda F_m} - e^{\lambda F_m^{(k)}} \right) + 2\mathbb{E} \left( e^{\lambda F_m} - e^{\lambda F_m^{(k)}} \right) = 2\mathbb{E} \left( e^{\lambda F_m} - e^{\lambda F_m^{(k)}} \right) + 2\mathbb{E} \left( e^{\lambda F_m} - e^{\lambda F_m^{(k)}} \right).$$

We will use (3.1) when $\lambda > 0$ and (3.2) when $\lambda \leq 0$. With these restrictions, the integrands in both cases above are only nonzero when $F_m^{(k)} > F_m$. Apply the mean value theorem to get

$$\mathbb{E} |V_k| \leq \begin{cases} \lambda e^{(e^{\lambda F_m} - F_m) F_m^{(k)}} & \text{when } \lambda > 0, \\ |\lambda| e^{(e^{\lambda F_m} - F_m) F_m^{(k)}} & \text{when } \lambda \leq 0. \end{cases}$$

To combine these, when $\lambda > 0$, we use $F_m^{(k)} \leq F_m + t'_e$ to find $e^{\lambda F_m^{(k)}} \leq e^{\lambda F_m} e^{\lambda t'_e}$, so we obtain for both cases

$$\mathbb{E} |V_k| \leq |\lambda| \mathbb{E} \left[ e^{\lambda F_m} \left( (1 + e^{\lambda t'_e}) (F_m^{(k)} - F_m) \right) \right].$$

By Cauchy-Schwarz, we have the following two bounds:

$$\sum_{k=1}^\infty \mathbb{E} |V_k| \leq \lambda^2 \mathbb{E} e^{2\lambda F_m} \mathbb{E} \left( \sum_{k=1}^\infty (1 + e^{\lambda t'_e}) (F_m^{(k)} - F_m) \right)^2$$

and

$$\mathbb{E} |V_k| \leq \lambda^2 \mathbb{E} e^{2\lambda F_m} (1 + e^{\lambda t'_e})^2 (F_m^{(k)} - F_m)^2.$$

We will bound these terms using Lemma 3.1. Write

$$\mathbb{E} (1 + e^{\lambda t'_e})^2 (F_m^{(k)} - F_m)^2 = \mathbb{E} (1 + e^{\lambda t'_e})^2 \left( \frac{1}{\#B_m} \sum_{z \in B_m} (T^{(k)}_z - T_z) \right)^2.$$
We have used the assumption incorporating the factor $\lambda$ in independent copy $t$. By convexity of the function $x \mapsto (x_+)^2$, we obtain the bound
\[
\frac{1}{\#B_m} \sum_{z \in B_m} \mathbb{E}(1 + e^{\lambda t'_e})^2(T_z^{(k)} - T_z)^2_+
\]
By Lemma 3.1, $(T_z^{(k)} - T_z)_+ \leq (t'_e - t_e) + 1_{\{t_e < D_z,e_k\}}$, so
\[
\mathbb{E}(1 + e^{\lambda t'_e})^2(F_m^{(k)} - F_m)^2_+ \leq \frac{1}{\#B_m} \sum_{z \in B_m} \mathbb{E}((1 + e^{\lambda t'_e})t'_e)^2 1_{\{t_e < D_z,e_k\}}
\]
\[
\leq \mathbb{E}((1 + e^{\lambda t_e})t_e)^2 \frac{1}{\#B_m} \sum_{z \in B_m} \mathbb{P}(e_k \in Geo(z, z + x)).
\]
By translation invariance, the final probability equals $\mathbb{P}(e_k - z \in Geo(0, x))$:
\[
\mathbb{E}(1 + e^{\lambda t'_e})^2(F_m^{(k)} - F_m)^2_+ \leq \frac{\mathbb{E}((1 + e^{\lambda t_e})t_e)^2 \mathbb{E}\#\{e_k - z \in Geo(0, x) : z \in B_m\}}{\#B_m}
\]
\[
\leq C_{13} ||x||_1^{\zeta(1-d)} \mathbb{E}((1 + e^{\lambda t_e})t_e)^2.
\] (3.5)
We have used the assumption $\mathbb{E}t_e^2 < \infty$ and Lemma 3.2 to bound the expectation above. After incorporating the factor $\lambda^2 \mathbb{E}e^{2\lambda F_m}$, this is our bound for (3.4).

For (3.3), write
\[
\mathbb{E} \left( \sum_{k=1}^{\infty} (1 + e^{\lambda t'_e})(F_m^{(k)} - F_m)_+ \right)^2 \leq \frac{1}{\#B_m} \sum_{z \in B_m} \mathbb{E} \left( \sum_{k=1}^{\infty} ((1 + e^{\lambda t'_e})t'_e) 1_{\{t_e < D_z,e_k\}} \right)^2.
\] (3.6)
The expectation equals
\[
\sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \mathbb{E}(1 + e^{\lambda t'_e})(1 + e^{\lambda t'_j})t'_e(t'_j) 1_{\{t_k < D_z,e_k, t_j < D_z,e_j\}}
\]
\[
\leq \mathbb{E} \left( (1 + e^{\lambda t_e})t_e \right)^2 \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \mathbb{P}(t_k < D_z,e_k, t_j < D_z,e_j) \leq \mathbb{E} \left( (1 + e^{\lambda t_e})t_e \right)^2 \mathbb{E}\#Geo(z, z + x)^2.
\]
By (1.10) and $\mathbb{E}t_e^2 < \infty$, the last expression is bounded by $C_{14} ||x||_1^2 \mathbb{E}((1 + e^{\lambda t_e})t_e)^2$.

We can now finish the proof with this bound and (3.5):
\[
\sum_{k=1}^{\infty} (\mathbb{E}|V_k|)^2 \leq \sup_j \mathbb{E}|V_j| \sum_{k=1}^{\infty} \mathbb{E}|V_k| \leq C_{15} \lambda^2 ||x||_1^{2\zeta(1-d)} \mathbb{E}((1 + e^{\lambda t_e})t_e)^2 \mathbb{E}e^{2\lambda F_m}.
\]
4 Entropy bound

The purpose of the present section is to give an intermediate upper bound for the sum of entropy terms in the left side of (2.9). Namely we will prove the following inequality, recalling that $F$ is the distribution function of $t_e$.

**Theorem 4.1.** For some $C_{16} > 0$ independent of $\lambda$,

$$\sum_{k=1}^{\infty} \operatorname{Ent}(V^2_k) \leq \lambda^2 \frac{C_{10} C_{\lambda}}{\# B_{m}} \sum_{x \in \mathbb{Z}^d} \mathbb{E} \left[ e^{2\lambda F_m} \sum_{z \in \mathbb{Z}^d} (1 - \log F(t_e)) \right]$$

for all $x \in \mathbb{Z}^d$.

The constant $C_\lambda$ is determined as follows:

1. Assuming (1.2) and $\lambda \in [0, \alpha/2)$, $C_\lambda = \operatorname{Ent}_e(t_e e^{\lambda t_e})^2 + \mathbb{E}[t_e e^{\lambda t_e}]^2$.
2. Assuming (1.3) and $\lambda \leq 0$, $C_\lambda = (\mathbb{E} e^{2\lambda t_e})^{-1}$.

We will prove this in a couple of steps. First we use the Bernoulli encoding from [8] to give an upper bound (Lemma 4.3 below) in terms of discrete derivatives relative to Bernoulli sequences. Next we split into two cases, $\lambda \geq 0$ and $\lambda \leq 0$. The first is handled in Proposition 4.4 and the second in Proposition 4.7. These three results will prove Theorem 4.1.

4.1 Bernoulli encoding

We will now view our edge variables as the push-forward of Bernoulli sequences. Specifically, for each edge $e$, let $\Omega_e$ be a copy of $\{0,1\}^\mathbb{N}$ with the product sigma-algebra. We will construct a measurable map $T_e : \Omega_e \to \mathbb{R}$ using the distribution function $F$. To do this, we create a sequence of partitions of the support of $\mu$. Recalling $I := \inf \operatorname{supp}(\mu) = \inf \{x : F(x) > 0\}$, set

$$a_{0,j} = I \text{ and } a_{i,j} = \min \left\{ x : F(x) \geq \frac{i}{2^j} \right\} \text{ for } j \geq 1 \text{ and } 1 \leq i \leq 2^j - 1 .$$

Note that by right continuity of $F$, the minimum above is attained; that is,

$$F(a_{i,j}) \geq \frac{i}{2^j} \text{ for } j \geq 1 \text{ and } 0 \leq i \leq 2^j - 1 . \quad (4.1)$$

Let us note two properties of the sequence.

For $j \geq 1$, $a_{0,j} \leq a_{1,j} \leq \cdots \leq a_{2^j-1,j} . \quad (4.2)$

For $i = 0, \ldots, 2^j - 1$, $x \geq a_{i,j}$ if and only if $F(x) \geq \frac{i}{2^j}$ and $x \geq a_{0,j} . \quad (4.3)$

Each $\omega \in \Omega_e$ gives us an “address” for a point in the support of $\mu$. Given $\omega = (\omega_1, \omega_2, \ldots)$ and $j \geq 1$, we associate a number $T_j(\omega)$ by

$$T_j(\omega) = a_{i(\omega,j),j}, \text{ where } i(\omega,j) = \sum_{l=1}^{j} 2^{j-l} \omega_l .$$
Lemma 4.3. Assume possibly at \( \omega \) where
\[
\circ T \text{ the map by the previous lemma,}
\]
with action
\[
\text{Call } T \text{ will be important to note that if } \omega \leq \hat{\omega} \text{ then } i(\omega, j) \leq i(\hat{\omega}, j) \text{ for all } j \geq 1.
\]
This, combined with the monotonicity statement (1.2), implies
\[
\omega \leq \hat{\omega} \Rightarrow T_j(\omega) \leq T_j(\hat{\omega}) \text{ for all } j \geq 1.
\]
(4.4)

It is well-known that one can represent Lebesgue measure on \([0, 1]\) using binary expansions and Bernoulli sequences. One way to view the encoding \( T \) in Lemma 4.2 is a composition of this representation with the right-continuous inverse of the distribution function \( F \). The function \( T \) instead uses an inverse approximated by simple functions taking dyadic values.

The following is \([8, \text{Lemma 6.1}]\).

**Lemma 4.2.** For each \( \omega \), the numbers \((T_j(\omega))\) form a non-decreasing sequence and have a limit \( T(\omega) \). This map \( T : \Omega_e \to R \cup \{\infty\} \) is measurable and has the following properties.

1. (Monotonicity) If \( \omega \leq \hat{\omega} \) then \( T(\omega) \leq T(\hat{\omega}) \).
2. (Nesting) For any \( \omega \in \Omega_e \) and \( j \geq 1 \), if \( i(\omega, j) < 2^j - 1 \) then
   \[
a_i(\omega, j, j) \leq T(\omega) \leq a_i(\omega, j, j + 1).
   \]
3. If \( \omega_k = 0 \) for some \( k \geq 1 \) then \( T(\omega) < \infty \).
4. Letting \( \pi \) be the product measure \( \prod_{i \in \mathbb{N}} \pi_i \), with each \( \pi_i \) uniform on \( \{0, 1\} \), we have
   \[
   \pi \circ T^{-1} = \mu.
   \]

By part 3, \( T \) is \( \pi \)-almost surely finite.

We now view \( G = e^{\lambda f_m} \) as a function of sequences of Bernoulli variables, as in \([8]\). So define \( \Omega_B = \prod_e \Omega_e \) with product sigma-algebra and measure \( \pi := \prod_e \pi_e \), where \( \pi_e \) is a product of the form \( \prod_{j \geq 1} \pi_{e,j} \) with \( \pi_{e,j} \) uniform on \( \{0, 1\} \). An element of \( \Omega_B \) will be denoted
\[
\omega_B = \{\omega_{e,j} : e \in \mathcal{E}^d, j \geq 1\}.
\]

Call \( T_e \) the map from the previous lemma on \( \Omega_e \) and define the product map \( T := \prod_e T_e : \Omega_B \to \Omega \) with action
\[
T(\omega_B) = (T_e(\omega_e) : e \in \mathcal{E}^d).
\]
By the previous lemma, \( \pi \circ T^{-1} = \mathbb{P} \).

In what follows, we will consider functions \( f \) on the original space \( \Omega \) as functions on \( \Omega_B \), through the map \( T \). We will suppress mention of this in the notation and, for instance, write \( f(\omega_B) \) to mean \( f \circ T(\omega_B) \). We will estimate discrete derivatives, so for a function \( f : \Omega_B \to \mathbb{R} \), set
\[
(\Delta e,j f)(\omega_B) = f(\omega_{B,e,j}^+) - f(\omega_{B,e,j}^-),
\]
where \( \omega_{B,e,j}^+ \) agrees with \( \omega_B \) except possibly at \( \omega_{e,j} \), where it is 1, and \( \omega_{B,e,j}^- \) agrees with \( \omega_B \) except possibly at \( \omega_{e,j} \), where it is 0. Then, exactly the same proof as in \([8, \text{Lemma 6.3}]\) gives

**Lemma 4.3.** Assume (1.2) and \( \lambda \in [0, \alpha/2) \) or (1.3) and \( \lambda \leq 0 \). We have the following inequality:
\[
\sum_{k=1}^{\infty} \text{Ent}(V^2_k) \leq \sum_{e,j} \mathbb{E}_\pi (\Delta e,j G)^2.
\]
4.2 Derivative bound: positive exponential

For the next derivative bounds we continue with \( G = e^{\lambda F_m} \) and set \( H \) as the derivative of \( G \); that is, \( H = \lambda e^{\lambda F_m} \).

**Theorem 4.4.** Assume \( H \geq 0 \). For some \( C_1 > 0 \), \( C_\lambda = \text{Ent}_e(t_e e^{\lambda t_e})^2 + \mathbb{E}[t_e e^{\lambda t_e}]^2 \) and all \( \lambda \in [0, \alpha/2) \),

\[
\sum_{e,j} \mathbb{E}_\pi (\Delta_{e,j} G)^2 \leq \lambda^2 \mathbb{E} \left[ \frac{C_\lambda}{\# B_m} \sum_{z \in B_m} \mathbb{E} \left[ e^{2 \lambda F_m} \sum_{e \in \text{Geo}(z, z+x)} (1 - \log F(t_e)) \right] \right] \quad \text{for all } x \in \mathbb{Z}^d .
\]

**Proof.** Write \( \mathbb{E}_e \) for expectation relative to \( \prod_{f \neq e} \pi_f \) and for any \( i \geq 1 \), let \( \pi_{e,i} \) be the measure \( \prod_{k \geq i} \pi_{e,k} \). Further, for \( j \geq 1 \) write

\[
\omega_B = (\omega_{e,e}, \omega_{e,j}, \omega_{e,j}, \omega_{e,j}) ,
\]

where \( \omega_{e,e} \) is the configuration \( \omega_B \) projected on the coordinates \( (\omega_{f,k} : f \neq e, \ k \geq 1) \), \( \omega_{e,<j} \) is \( \omega_B \) projected on the coordinates \( (\omega_{e,k} : k < j) \) and \( \omega_{e,j} \) is \( \omega_B \) projected on the coordinates \( (\omega_{e,k} : k > j) \).

Then

\[
\mathbb{E}_\pi (\Delta_{e,j} G)^2 = \mathbb{E}_e \mathbb{E}_{x_{<j}} \cdots \mathbb{E}_{x_{j-1}} \left[ \mathbb{E}_{x_{e,j}} (\Delta_{e,j} G)^2 \right] \]

\[
= \mathbb{E}_e \left[ \frac{1}{2j-1} \sum_{\sigma \in \{0,1\}^{j-1}} \left[ \mathbb{E}_{x_{e,j}} (\Delta_{e,j} G(\omega_{e,e}, \sigma, \omega_{e,j}, \omega_{e,j}))^2 \right] \right] ,
\]

and the innermost term is

\[
\mathbb{E}_{x_{e,j}} (G(\omega_{e,e}, \sigma, 1, \omega_{e,j})) - G(\omega_{e,e}, \sigma, 0, \omega_{e,j}))^2 .
\]

Applying the mean value theorem, we get an upper bound of

\[
\mathbb{E}_{x_{e,j}} (H(\omega_{e,e}, \sigma, 1, \omega_{e,j}))(F_m(\omega_{e,e}, \sigma, 1, \omega_{e,j}) - F_m(\omega_{e,e}, \sigma, 0, \omega_{e,j})))^2 .
\]

Convexity of \( x \mapsto x^2 \) gives the bound

\[
\mathbb{E}_\pi (\Delta_{e,j} G)^2 \leq \frac{1}{\# B_m} \sum_{z \in B_m} \mathbb{E}_e \frac{1}{2j-1} \]

\[
\times \left[ \sum_{\sigma \in \{0,1\}^{j-1}} \mathbb{E}_{x_{e,j}} \left[ H^2(\omega_{e,e}, \sigma, 1, \omega_{e,j}) (T_z(\omega_{e,e}, \sigma, 1, \omega_{e,j}) - T_z(\omega_{e,e}, \sigma, 0, \omega_{e,j}))^2 \right] \right] ,
\]

where we have written \( T_z = T(z, z+x) \). Because of Lemma 3.1, we can rewrite the inner summand of (4.7) as

\[
\mathbb{E}_{x_{e,j}} H^2(\omega_{e,e}, \sigma, 1, \omega_{e,j}) \min\{T_z(\sigma, 1, \omega_{e,j}) - T_z(\sigma, 0, \omega_{e,j}), (D_z/e - T_z(\sigma, 0, \omega_{e,j}))_+ \}^2 .
\]

To simplify notation in the case \( j \geq 2 \), we write the values \( a_{1,j-1}, \ldots, a_{2j-1,j-1} \) as \( a_1, \ldots, a_{2j-1} \) and for a fixed \( \sigma \in \{0,1\}^{j-1} \), \( a_\sigma \) for \( a_i((\sigma, 0, \omega_{e,j}), j-1), j-1 \) (note that this does not depend on the
configuration outside of \( \sigma \). Also we write \( a'_\sigma \) for the element of the partition that follows \( a_\sigma \) (when there is one; that is, when \( \sigma \) is not \((1, \ldots, 1)\)). Last, we abbreviate \( T_e(\sigma, c, \omega_{e,>j}) \) by \( T_{e,j}(\sigma, c) \) for \( c = 0, 1 \). With this notation, we claim the inequalities

\[
a_\sigma \leq T_{e,j}(\sigma, 0) \leq T_{e,j}(\sigma, 1) \leq a'_\sigma \text{ when } \sigma \neq (1, \ldots, 1) \text{ and } j \geq 2.
\]

The first and third inequalities follow from the nesting part of Lemma 4.2. The second holds because of the monotonicity part. Therefore we can give an upper bound for (4.8) when \( j \geq 2 \) of

\[
\begin{align*}
0 & \quad \text{if } D_{z,e} \leq a_\sigma \\
\mathbb{E}_{\pi_{e,\geq j}} \left[ \mathcal{H}^2(\omega^{e,}, \sigma, 1, \omega_{e,>j}) \times \min \{ D_{z,e} - a_\sigma, T_{e,j}(\sigma, 1) - a_\sigma \}^2 \mathbb{1}_{\{ T_{e,j}(\sigma, 0) < D_{z,e} \}} \right] & \quad \text{if } \sigma \neq (1, \ldots, 1) \text{ and } a_\sigma < D_{z,e} \leq a'_\sigma \\
\mathbb{E}_{\pi_{e,\geq j}} \mathcal{H}^2(\omega^{e,}, \sigma, 1, \omega_{e,>j})(a'_\sigma - a_\sigma)^2 & \quad \text{if } a'_\sigma \leq D_{z,e}
\end{align*}
\]

(Here and above we have strict inequality in the condition of the indicator function since when \( T_e(\sigma, 0, \omega_{e,>j}) = D_{z,e} \), (4.8) is zero.) With this, when \( j \geq 2 \), the integrand of \( \mathbb{E}_{\pi_{e,\geq j}} \) in (4.7) is no bigger than

\[
\frac{1}{2^{j-1}} \mathbb{E}_{\pi_{e,\geq j}} \mathcal{H}^2(\omega^{e,}, \sigma(D_{z,e}), 1, \omega_{e,>j}) \left[ (a_1 - a_0)^2 + \cdots + (a_s - a_{s-1})^2 \right]
\]

\[
\times \min \{ D_{z,e} - a_s, T_{e,j}(\sigma(D_{z,e}), 1) - a_s \}^2 \mathbb{1}_{\{ T_{e,j}(\sigma(D_{z,e}), 0) < D_{z,e} \}} \mathbb{1}_{\{ I < D_{z,e} \}}.
\]

(4.9)

Here we have written \( s \) for the largest index \( i \) such that \( a_i < D_{z,e} \) and \( \sigma(D_{z,e}) \) for the configuration such that \( a_{\sigma(D_{z,e})} = a_s \). In the case \( j = 1 \), we have the similar upper bound

\[
\mathbb{E}_{\pi_e} \mathcal{H}^2(\omega^{e,}, 1, \omega_{e,>1}) \min \{ D_{z,e} - I, T_e, 1(1) - I \}^2 \mathbb{1}_{\{ T_{e,1}(0) < D_{z,e} \}} \mathbb{1}_{\{ I < D_{z,e} \}}.
\]

(4.10)

In the case \( j \geq 2 \) we condense this by writing \( \tilde{\sigma}_{j-1} \) for the vector consisting of \( j - 1 \) zeroes, and using the inequalities \( \mathbb{1}_{\{ T_{e,j}(\sigma(D_{z,e}), 0) < D_{z,e} \}} \leq \mathbb{1}_{\{ T_{e,j}(\tilde{\sigma}_{j-1}, 0) < D_{z,e} \}} \) and \( a^2 + b^2 \leq (a + b)^2 \) for non-negative \( a, b \). This produces for all \( j \geq 1 \), after writing \( L_z(\omega^{e,}, \omega_e) = L_z(\omega^{e,}, \omega_{e,\leq j}, \omega_{e,>j}) \) for \( \mathcal{H}^2(\omega^{e,}, \omega_e) \min \{ D_{z,e}, T_e(\omega_e) \}^2 \),

\[
\mathbb{E}_{\pi_e} (\Delta_{e,j} G)^2 \leq \frac{1}{\# B_m} \sum_{z \in B_m} \frac{1}{2^{j-1}} \mathbb{E}_{\pi_{e,\geq j}} \left[ L_z(\omega^{e,}, \sigma(D_{z,e}), 1, \omega_{e,>j}) I_{\{ T_{e,j}(\tilde{\sigma}_{j-1}, 0) < D_{z,e} \}} \right] I_{\{ I < D_{z,e} \}}.
\]

(4.11)

Note that since \( \lambda \geq 0, L_z(\omega^{e,}, \omega_e) \) is an increasing function of \( \omega_{e,>j} \) (with all other variables fixed), whereas \( \mathbb{1}_{\{ T_{e,j}(\tilde{\sigma}_{j-1}, 0) < D_{z,e} \}} \) is decreasing. Therefore if \( \lambda \in [0, \alpha/2] \) (where \( \alpha \) is from (1.2) - this ensures that \( L_z \) is integrable) we can apply the Harris-FKG inequality and sum over \( j \) for the upper bound

\[
\mathbb{E}_{\pi_e} \sum_{j=1}^\infty (\Delta_{e,j} G)^2 \leq \frac{1}{\# B_m} \sum_{z \in B_m} \sum_{j=1}^\infty \frac{1}{2^{j-1}} \left[ \mathbb{E}_{\pi_{e,\geq j}} L_z(\omega^{e,}, \sigma(D_{z,e}), 1, \omega_{e,>j}) \right]
\]

\[
\times \pi_{e,\geq j} (T_{e,j}(\tilde{\sigma}_{j-1}, 0) < D_{z,e}) \mathbb{1}_{\{ I < D_{z,e} \}}.
\]

(4.12)

The goal is now to give a useful bound for this sum.
Lemma 4.5. There exists $C_{18}$ independent of $e$ such that for $\lambda \in [0, \alpha/2)$,

$$
\mathbb{E}_{\pi_e} \sum_{j=1}^{\infty} (\Delta_{e,j} G)^2 \leq C_{18} \frac{1}{\#B_m} \sum_{z \in B_m} F(D^{-}_z, e)(1 - \log F(D^{-}_z, e))\mathbb{E}_{\pi_e}[L_z N],
$$

where $N = 1 + \max \left\{ \{ j \geq 1 : \omega_{e,j} = \tilde{1}_j \} \cup \{0\} \right\}$.

Proof. We consider two types of values of $j$. Note that when $D_{z,e} > I$, $F(D^{-}_z, e) > 0$ and therefore for some $j$, $F(D^{-}_z, e) \geq 2^{-j}$. So define

$$
J(D_{z,e}) = \min\{ j \geq 2 : F(D^{-}_z, e) \geq 2^{-(j-1)} \}.
$$

Note that

$$
1 - \log_2 F(D^{-}_z, e) \leq J(D_{z,e}) \leq 2 - \log_2 F(D^{-}_z, e).
$$

(4.13)

We will estimate the term $\pi_{e,>j}(T_{e,j}(0_{j-1}, 0) < D_{z,e})$ only when $j < J(D_{z,e})$. By definition, it is

$$
\left( \prod_{k \geq j} \pi_{e,k} \right) (\{ \omega_e : T_e(0, \ldots, 0, \omega_{e,j+1}, \ldots) < D_{z,e} \}) = \pi_e(\{ \omega_e : T_e(0, \ldots, 0, \omega_{e,j+1}, \ldots) < D_{z,e} \}).
$$

The event in $\Omega_e$ listed on the right depends only on $\omega_{e,k}$ for $k > j$, so it is independent (under $\pi_e$) of the state of the first $j$ coordinates. Thus the above equals

$$
2^j \pi_e(T_e(0, \ldots, 0, \omega_{e,j+1}, \ldots) < D_{z,e}, \omega_{e,1}, \ldots, \omega_{e,j} = 0) \leq 2^j \pi_e(T_e(\omega_e) < D_{z,e}) = 2^j F(D^{-}_z, e).
$$

Using this inequality for $j < J(D_{z,e})$, (4.12) gives the bound

$$
\mathbb{E}_{\pi_e} \sum_{j=1}^{\infty} (\Delta_{e,j} G)^2 \leq \frac{1}{\#B_m} \sum_{z \in B_m} \left[ \frac{2F(D^{-}_z, e)}{J(D_{z,e})} \mathbb{E}_{\pi_e} L_z(\omega_{e^c}, 1, \omega_{e,>j})1_{I < D_{z,e}} \right]
$$

(4.14)

$$
+ \frac{2F(D^{-}_z, e)}{J(D_{z,e})} \sum_{j=2}^{J(D_{z,e})-1} \left[ \mathbb{E}_{\pi_{e,j}} L_z(\omega_{e^c}, \sigma(D_{z,e}), 1, \omega_{e,>j})1_{I < D_{z,e}} \right]
$$

(4.15)

$$
+ \sum_{j=J(D_{z,e})}^{\infty} \frac{1}{2^j-1} \left[ \mathbb{E}_{\pi_{e,j}} L_z(\omega_{e^c}, \sigma(D_{z,e}), 1, \omega_{e,>j})1_{I < D_{z,e}} \right] 1_{I < D_{z,e}}.
$$

(4.16)

We first bound $\mathbb{E}_{\pi_{e,j}} L_z(\omega_{e^c}, \tilde{1}_j, \omega_{e,>j})$, which only depends on $\omega_e$ through $\omega_{e,>j}$. By independence,

$$
\mathbb{E}_{\pi_{e,j}} L_z(\omega_{e^c}, \tilde{1}_j, \omega_{e,>j}) = 2^j \mathbb{E}_{\pi_e} L_z(\omega_{e^c}, \tilde{1}_j, \omega_{e,>j})1_{\{ \omega_{e,>j} = \tilde{1}_j \}} = 2^j \mathbb{E}_{\pi_e} L_z1_{\{ \omega_{e,>j} = \tilde{1}_j \}}.
$$

(4.17)

We can then bound the term $2F(D^{-}_z, e)\mathbb{E}_{\pi_e} L_z(\omega_{e^c}, 1, \omega_{e,>1})1_{I < D_{z,e}}$ in (4.14) by

$$
\text{(4.14)} \leq 4F(D^{-}_z, e) \mathbb{E}_{\pi_e} L_z1_{I < D_{z,e}}.
$$

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For (4.16), we consider two cases. If $D_{z,e} \leq a_{1,1}$ then, setting $\rho$ to be the sequence $(1, 0, 0, \ldots)$, we have $\sigma(D_{z,e}) < \rho$. Using the fact that $\lambda \geq 0$, $L_z(\omega^{e_c}, \sigma(D_{z,e}), 1, \omega_{e,j}) \leq L_z(\omega^{e_c}, \rho)$. So we give the bound for (4.16):

$$L_z(\omega^{e_c}, \rho) \sum_{j=J(D_{z,e})}^{\infty} \frac{1}{2j-1} 1_{\{I < D_{z,e}\}} = 4L_z(\omega^{e_c}, \rho) 2^{-(J(D_{z,e})-1)} 1_{\{I < D_{z,e}\}}.$$ 

By the Markov inequality, $L_z(\omega^{e_c}, \rho) \leq \mathbb{E}_{\pi_e} L_z/\pi_e(\omega_{e,1} = 1) = 2\mathbb{E}_{\pi_e} L_z$. With this and the bound $J(D_{z,e}) \geq 1 - \log_2 F(D_{z,e}^-)$,

$$D_{z,e} \leq a_{1,1} \Rightarrow (4.16) \leq 4F(D_{z,e}^-) \mathbb{E}_{\pi_e} \left[ L_z \max\{j \geq 1 : \omega_{e,j} = \bar{I}_j\} \right] 1_{\{I < D_{z,e}\}}.$$ 

On the other hand, if $D_{z,e} > a_{1,1}$ then $F(D_{z,e}^-) \geq 1/2$ so using (4.17) and that $L_z(\omega^{e_c}, \sigma(D_{z,e}), 1, \omega_{e,j})$ is no bigger than $L_z(\omega^{e_c}, \bar{I}_j, \omega_{e,j})$,

$$D_{z,e} > a_{1,1} \Rightarrow (4.16) \leq 4F(D_{z,e}^-) \mathbb{E}_{\pi_e} \left[ L_z \max\{j \geq 1 : \omega_{e,j} = \bar{I}_j\} \right] 1_{\{I < D_{z,e}\}}.$$ 

Combining this with the case $D_{z,e} \leq a_{1,1}$,

$$(4.16) \leq 4F(D_{z,e}^-) \mathbb{E}_{\pi_e} \left[ L_z (1 + \max\{j \geq 1 : \omega_{e,j} = \bar{I}_j\}) \right] 1_{\{I < D_{z,e}\}}.$$ 

The term (4.16) is bounded by noting that when this sum is nonempty (that is, $J(D_{z,e}) > 2$), it follows that $F(D_{z,e}^-) < 1/2$ and so $D_{z,e} \leq a_{1,1}$. Using this with (4.13) we obtain the upper bound for (4.15):

$$2F(D_{z,e}^-) \sum_{j=2}^{J(D_{z,e})-1} L_z(\omega^{e_c}, \rho) 1_{\{I < D_{z,e}\}} \leq 2F(D_{z,e}^-) (1 - \log_2 F(D_{z,e}^-)) L_z(\omega^{e_c}, \rho) 1_{\{I < D_{z,e}\}}. \quad (4.18)$$

Again by the inequality $L(\omega^{e_c}, \rho) \leq 2\mathbb{E}_{\pi_e} L_z$, this becomes

$$(4.15) \leq 4F(D_{z,e}^-) (1 - \log_2 F(D_{z,e}^-)) \mathbb{E}_{\pi_e} L_z 1_{\{I < D_{z,e}\}}.$$ 

We now combine all the bounds: setting $N = 1 + \max\{j \geq 1 : \omega_{e,j} = \bar{I}_j\} \cup \{0\}$, for some $C_{19}$,

$$\mathbb{E}_{\pi_e} \sum_{j=1}^\infty (\Delta_{e,j} G)^2 \leq \frac{1}{\#B_m} \sum_{z \in B_m} C_{19} F(D_{z,e}^-) (1 - \log_2 F(D_{z,e}^-)) \mathbb{E}_{\pi_e} [L_z N] 1_{\{I < D_{z,e}\}}. \quad (4.19)$$

To bound the above terms we use Lemma 6.5).

**Lemma 4.6.** For any $y > I$, we have

$$-F(y^-) \log F(y^-) \leq - \int_{[I,y)} \log F(a) \mu(da). \quad (4.20)$$
To finish the proof of Proposition 4.7, apply this lemma in (4.19) with \( y = D_{z,e} \):
\[
\sum_{e} \sum_{j=1}^{\infty} \mathbb{E}_{\pi} (\Delta_{e,j} G)^2 \leq \frac{C_{19}}{\#B_m} \sum_{z \in B_m} \sum_{e} \mathbb{E}_{\pi} \left[ \int_{I,D_{z,e}} (1 - \log F(t_e)) \mathbb{E}_{\pi_e} (L_z N) \mu(dt_e) \right] \tag{4.21}
\]

Letting \( F_{m,e} \) be \( F_m \) evaluated at the configuration \((t_e, 0)\), we can bound \( H^2 \leq \lambda^2 e^{2\lambda F_{m,e}} e^{2\lambda t_e} \), so
\[
\mathbb{E}_{\pi_e} (L_z N) \leq \mathbb{E}_{\pi_e} (H^2 t_e^2 N) \leq \lambda^2 e^{2\lambda F_{m,e}} \mathbb{E}_{\pi_e} \left( (t_e e^{\lambda t_e})^2 N \right) \leq H^2 \mathbb{E}_{\pi_e} \left( (t_e e^{\lambda t_e})^2 N \right) .
\]

Now since \( \lambda < \alpha/2 \), \( \text{Ent}(t_e e^{\lambda t_e})^2 < \infty \), so we use Proposition 2.6 to bound the expectation by
\[
2\text{Ent}(t_e e^{\lambda t_e})^2 + 2\mathbb{E}_{\mu}(t_e e^{\lambda t_e})^2 \log \mathbb{E}_{\pi_e} e^{N/2} .
\]

Because \( N \) has geometric distribution, this is bounded by \( C_{20} C_{\lambda} \) independently of \( e \).

Returning to (4.21), note that by Lemma 3.11 if \( t_e < D_{z,e} \) then \( e \) is in \( \text{Geo}(z, z+x) \). So applying the bound on \( \mathbb{E}_{\pi_e} (L_z N) \), we obtain for some \( C_{21} \)
\[
\sum_{e,j} \mathbb{E}_{\pi} (\Delta_{e,j} G)^2 \leq \frac{C_{21} C_{\lambda}}{\#B_m} \sum_{z \in B_m} \sum_{e} \mathbb{E} \left[ (1 - \log F(t_e)) H^2 1_{\{t_e < D_{z,e}\}} \right] \\
\leq \lambda^2 \frac{C_{21} C_{\lambda}}{\#B_m} \sum_{z \in B_m} \mathbb{E} \left[ e^{2\lambda F_{m,e}} \sum_{e \in \text{Geo}(z, z+x)} (1 - \log F(t_e)) \right] .
\]

\( \square \)

### 4.3 Derivative bound: negative exponential

**Theorem 4.7.** Assume (1.3). For some \( C_{22} > 0 \), \( C_{\lambda}' = \mathbb{E}_{\mu} e^{2\lambda t_e} \) and all \( \lambda \leq 0 \),
\[
\sum_{e,j} \mathbb{E}_{\pi} (\Delta_{e,j} G)^2 \leq \lambda^2 \frac{C_{22}}{C_{\lambda}' \#B_m} \sum_{z \in B_m} \mathbb{E} \left[ e^{2\lambda F_{m,e}} \sum_{e \in \text{Geo}(z, z+x)} (1 - \log F(t_e)) \right] \quad \text{for all } x \in \mathbb{Z}^d .
\]

**Proof.** As before,
\[
\mathbb{E}_{\pi} (\Delta_{e,j} G)^2 = \mathbb{E}_{\pi e} \left[ \frac{1}{2j-1} \sum_{\sigma \in \{0,1\}^{j-1}} \left[ \mathbb{E}_{\pi_{e,j}} (\Delta_{e,j} G(\omega_{e,c}, \sigma, \omega_{e,j}, \omega_{e,j})) \right)^2 \right] , \tag{4.22}
\]
and the innermost term is
\[
\mathbb{E}_{\pi_{e,j}} (G(\omega_{e,c}, \sigma, 1, \omega_{e,j}) - G(\omega_{e,c}, \sigma, 0, \omega_{e,j}))^2 . \tag{4.23}
\]

Applying the mean value theorem, we get an upper bound of
\[
H^2 (\omega_{e,c}, \bar{\sigma}) \mathbb{E}_{\pi_{e,j}} (F_m(\omega_{e,c}, \sigma, 1, \omega_{e,j}) - F_m(\omega_{e,c}, \sigma, 0, \omega_{e,j}))^2 ,
\]

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where $\vec{0}$ is the infinite sequence $(0, 0, \ldots)$. Convexity of $x \mapsto x^2$ gives the bound

$$
\frac{1}{\# B_m} \sum_{z \in B_m} H^2(\omega_e, \vec{0}) E_{\pi_e, \geq 1} \left( \Delta_{e,j} T_z(\omega_e, \sigma, \omega_{e,j}, \omega_{e,j}) \right)^2.
$$

Therefore

$$
\sum_{j=1}^{\infty} E_{\pi_e} (\Delta_{e,j} G)^2 \leq \frac{1}{\# B_m} \sum_{z \in B_m} E_{e} \left[ H^2(\omega_e, \vec{0}) \sum_{j=1}^{\infty} \frac{1}{2j-1} \sum_{\sigma \in \{0,1\}^{j-1}} E_{\pi_e, \geq 1} \left( \Delta_{e,j} T_z(\omega_e, \sigma, \omega_{e,j}, \omega_{e,j}) \right)^2 \right]
$$

$$
= \frac{1}{\# B_m} \sum_{z \in B_m} E_{e} \left[ H^2(\omega_e, \vec{0}) \sum_{j=1}^{\infty} E_{\pi_e} (\Delta_{e,j} T_z)^2 \right].
$$

We have now isolated the term from \cite[(6.23)]{Sh}; there it is proved under (1.3) that

$$
\sum_{j=1}^{\infty} E_{\pi_e} (\Delta_{e,j} G)^2 \leq C_{23} F(D^-_{z,e})(1 - \log F(D^-_{z,e})) 1_{\{I < D_{z,e}\}}.
$$

Thus we obtain

$$
\sum_{j=1}^{\infty} E_{\pi_e} (\Delta_{e,j} G)^2 \leq \frac{C_{23}}{\# B_m} \sum_{z \in B_m} E_{e} \left[ H^2(\omega_e, \vec{0}) F(D^-_{z,e})(1 - \log F(D^-_{z,e})) 1_{\{I < D_{z,e}\}} \right]. \quad (4.24)
$$

Use the bound

$$
H^2 \geq \lambda^2 e^{2\lambda t_e} e^{2\lambda F_m(t_e, I)},
$$

which implies $H^2(\omega_e, \vec{0}) \leq \frac{E_{e} H^2}{E_{\mu} e^{2\lambda t_e}}$. Combined with Lemma \ref{lem:chebyshev}, this gives an upper bound for the right side of (4.24) when $\lambda \leq 0$:

$$
\frac{C_{23}}{E_{\mu} e^{2\lambda t_e} \# B_m} \sum_{z \in B_m} E_{e} \left[ E_{\pi_e} H^2 \left( (1 - \log F(t_e)) 1_{\{I < D_{z,e}\}} \right) \right].
$$

Since $\lambda \leq 0$, $H^2 = \lambda^2 e^{2\lambda F_m}$ is decreasing in the variable $t_e$. However $(1 - \log F(t_e)) 1_{\{I < D_{z,e}\}}$ is also decreasing in $t_e$. Therefore the Chebyshev association inequality \cite[Theorem 2.14]{H} gives an upper bound of

$$
\frac{C_{23}}{E_{\mu} e^{2\lambda t_e} \# B_m} \sum_{z \in B_m} E \left[ H^2(1 - \log F(t_e)) 1_{\{e \in Geo(z, z+x)\}} \right].
$$

Summing over edges $e$,

$$
\sum_{e,j} E_{\pi} (\Delta_{e,j} G)^2 \leq \lambda^2 \frac{C_{23}}{E_{\mu} e^{2\lambda t_e} \# B_m} \sum_{z \in B_m} E \left[ e^{2\lambda F_m} \sum_{e \in Geo(z, z+x)} (1 - \log F(t_e)) \right]. \quad \Box
$$
5 Control by lattice animals

The next step is to use the theory of greedy lattice animals to decouple and control the terms in the expectation of Theorem 4.1. Specifically we will show

**Theorem 5.1.** Assume (1.2) with $\lambda \in [0, \alpha/2)$ or (1.3) with $\lambda \leq 0$. For some $C_{24} > 0$,

$$\sum_{k=1}^{\infty} \text{Ent}(V_k^2) \leq \lambda^2 C_{24} C_\lambda \left[ \text{Ent}(e^{2\lambda F_m}) + (1 + \mathbb{E}F_m)\mathbb{E}e^{2\lambda F_m} \right]$$

for all $x \in \mathbb{Z}^d$,

where $C_\lambda$ is from Theorem 4.1.

The theorem follows from inequalities (5.4) and (5.8), which we now set out to prove. We begin by generating a new set of “lattice animal weights” from a given realization $(t_e)$; set

$$w_e := 1 - \log(F(t_e)) \quad \text{for all } e \in \mathcal{E}^d.$$ 

**Proposition 5.2.** The collection $(w_e)$ is i.i.d. with $\mathbb{E}e^{w_e/2} < \infty$.

**Proof.** If $u \in (0, 1)$ we define $F^{-1}(u) = \inf\{x : F(x) \geq u\}$, so that $F^{-1}(u) \leq x$ if and only if $u \leq F(x)$. In particular, $u \leq F(F^{-1}(u))$ for all $u$. If $U$ is uniformly distributed on $(0, 1)$ then $F^{-1}(U)$ is distributed like $t_e$, so if $r \geq 1$,

$$P(w_e \geq r) = P(F(F^{-1}(U)) \leq e^{1-r}) \leq P(U \leq e^{1-r}) = e^{1-r}.$$

This implies $\mathbb{E}e^{\lambda w_e} < \infty$ for all $\lambda < 1$. 

For a realization of $(t_e)$, consider the edge greedy lattice animal problem. For a connected subset of edges $\gamma \subseteq \mathcal{E}^d$, define $N(\gamma) = \sum_{e \in \gamma} w_e$, and define the random variable

$$N_n := \max_{\gamma: \#\gamma = n \in \gamma} N(\gamma)$$

(here the notation $0 \in \gamma$ means that 0 is an endpoint of some edge in $\gamma$).

**Proposition 5.3.** For each $\kappa > 0$, there exists $\beta > 0$ such that

$$\sup_{n > 0} \frac{\log \mathbb{E}e^{\beta N_n}}{n} \leq \kappa.$$

**Proof.** Recall

$$\{N_n > \beta n\} = \bigcup_{\gamma: \#\gamma = n} \{N(\gamma) > \beta n\},$$

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where the union is over all lattice animals of size $n$ containing the origin. Now, there exists a constant $C_{25}$ such that the number of such lattice animals is bounded by $e^{C_{25}n}$. Therefore, letting $(w_i)$ be a sequence of i.i.d. random variables distributed as $w$, 

$$
P(N_n > \beta n) \leq e^{C_{25}n} \sum_{i=1}^n w_i > \beta n \leq e^{C_{25}n} \sum_{i=1}^n w_i/2 

= e^{C_{25}n - \beta n/2} \left[ E e^{w_i/2} \right]^n.
$$

In particular, for all $\beta$ greater than some $\beta_0$,

$$
P(N_n > \beta n) \leq e^{-\beta n/4}.
$$

Now, for all $\lambda \in [0, 1/8)$ and for each $n \geq 1$,

$$
E e^{\lambda N_n} = \lambda \int_0^\infty e^{\lambda x} \mathbb{P}(N_n \geq x) \, dx \leq \beta_0 ne^{\beta_0 n} + \lambda \int_0^{\infty} e^{\lambda n/\lambda} \mathbb{P}(N_n \geq x) \, dx 

\leq \beta_0 ne^{\beta_0 n} + n \int_0^{\infty} e^{-\Xi n} \, d\Xi 

\leq e^{\beta_1 n}
$$

for some $\beta_1 < \infty$. Now, since $E e^{\lambda N_n/2} \leq \left( E e^{\lambda N_n} \right)^{1/2}$, the proof is complete.

We now consider the first-passage model on $\mathbb{Z}^d$. For any $x \in \mathbb{Z}^d$, $x \neq 0$, let

$$
Y_x := \sum_{e \in \text{Geo}(0,x)} w_e;
$$

when we need to allow the starting point to vary as well, write

$$
Y_{z,x} := \sum_{e \in \text{Geo}(z,z+x)} w_e.
$$

5.1 The case $\lambda \geq 0$

In this section, we consider the case of upper exponential concentration. For the remainder of this section, assume (1.2) and let $\lambda \in [0, \alpha/2)$.

Rephrase the bound from Theorem 4.1

$$
\sum_{k=1}^\infty \text{Ent}(V_k^2) \leq \lambda^2 \frac{C_{16} C_{16}}{\# B_m} \sum_{z \in B_m} E \left[ e^{2\lambda F_m Y_{z,x}} \right].
$$

(5.1)

Applying Proposition 2.6 to the expectation on the right-hand side of (5.1) and the fact that $Y_x = Y_{z,x}$ in distribution yields a bound for some $C_{26} > 0$ such that the expectation below exists:

$$
E \left[ e^{2\lambda F_m Y_{z,x}} \right] \leq C_{26}^{-1} \text{Ent}(e^{2\lambda F_m}) + C_{26}^{-1} E \left[ e^{2\lambda F_m} \right] \log E e^{C_{26} F_m}.
$$

(5.2)

We focus our efforts on a bound for the second term of (5.2).
Proposition 5.4. Assuming (1.2), there exists \( C_{26} > 0 \) such that
\[
\sup_{x \neq 0} \frac{\log \mathbb{E} e^{C_{26} x}}{\|x\|_1} < \infty.
\]

Proof. Recall that \( G(0,x) \) is the maximal number of edges in a geodesic from \( 0 \) to \( x \). We begin by writing
\[
\mathbb{E} e^{C_{26} x} = \sum_{j=1}^{\infty} \mathbb{E} e^{C_{26} x} 1\{2^{j-1}\|x\|_1 \leq G(0,x) < 2^j\|x\|_1\}
\]
\[
\leq \sum_{j=1}^{\infty} \mathbb{E} \exp(C_{26} N_{2^j\|x\|_1}) 1\{G(0,x) \geq 2^{j-1}\|x\|_1\}
\]
\[
\leq \sum_{j=1}^{\infty} \left( \mathbb{E} \left[ \exp(2C_{26} N_{2^j\|x\|_1}) \right] \right)^{1/2} \mathbb{P}[G(0,x) \geq 2^{j-1}\|x\|_1]^{1/2},
\]
where in the last step we have used Cauchy-Schwarz. By Corollary 1.5 there exist \( \alpha_1, C_{27} \in (0, \infty) \) such that
\[
\mathbb{E} e^{\alpha_1 G(0,x)} \leq e^{C_{27}\|x\|_1} \text{ for all } x \in \mathbb{Z}^d.
\]
In particular, uniformly in \( x \), for some \( C_{28} \in (0, \infty) \),
\[
\mathbb{P}(G(0,x) \geq 2^{j-1}\|x\|_1) \leq e^{C_{28}\|x\|_1} e^{-2^{j-1}\|x\|_1 / C_{28}}.
\]
Using Proposition 5.3 we see that if \( C_{26} \) is chosen sufficiently small, then uniformly in \( j \),
\[
(\mathbb{E} \left[ \exp(2C_{26} N_{2^j\|x\|_1}) \right])^{1/2} \leq e^{2^{j-2}\|x\|_1 / C_{28}}.
\]
Applying these bounds in (5.3), for \( C_{26} \) sufficiently small,
\[
\mathbb{E} e^{C_{26} x} \leq e^{C_{28}\|x\|_1} \sum_{j=1}^{\infty} e^{-2^{j-2}\|x\|_1 / C_{28}} \text{ for all } x \in \mathbb{Z}^d.
\]

So under (1.2) with \( \lambda \in [0, \alpha/2) \), we return to (5.2) and find for some \( C_{29} > 0 \),
\[
\sum_{k=1}^{\infty} \text{Ent}(V_k^2) \leq \lambda^2 \frac{C_{29} C_{\lambda}}{\# B_m} \sum_{z \in B_m} \left[ \text{Ent}(e^{2\lambda F_m}) + \mathbb{E} e^{2\lambda F_m} \right]
\]
\[
= \lambda^2 C_{29} C_{\lambda} \left[ \text{Ent}(e^{2\lambda F_m}) + \mathbb{E} e^{2\lambda F_m} \right] \text{ for all } x \in \mathbb{Z}^d.
\]

\[\square\]
5.2 The case $\lambda \leq 0$

When $\lambda \leq 0$, the problem is again to bound above the term

$$\frac{1}{\#B_m} \sum_{z \in B_m} \mathbb{E} \left[ e^{2\lambda F_m Y_{z,x}} \right].$$

(5.5)

We will break this up differently from before, now using a variant of the idea from [9]. Let $C_{30} > 0$ be arbitrary (to be fixed later, independent of $x$). Then

$$\mathbb{E} e^{2\lambda F_m Y_{z,x}} \leq C_{30} \mathbb{E} \left[ e^{2\lambda F_m T_z} \right] + \mathbb{E} \left[ e^{2\lambda F_m Y_{z,x} 1 \{Y_{z,x} > C_{30} T_z\}} \right]$$

$$\leq C_{30} \mathbb{E} \left[ e^{2\lambda F_m} \right] \mathbb{E} T_z + \mathbb{E} \left[ e^{2\lambda F_m} Z_{z,x} \right],$$

(5.6)

where we have used the Harris-FKG inequality on the first term (since $\lambda \leq 0$, $e^{2\lambda F_m}$ is a decreasing function of $(t_e)$ whereas $T_z$ is increasing) and have defined the new variable

$$Z_{z,x} := Y_{z,x} 1 \{Y_{z,x} > C_{30} T_z\}.$$

We will bound $\mathbb{P}(Z_{z,x} \geq n)$ in what follows. Analogously to the proof of Proposition 1.4, define, for $C_{31} > 0$,

$$A'_n := \{ \exists \text{ a self-avoiding } \gamma \text{ from } z \text{ to } z + x \text{ with } N(\gamma) \geq n \text{ but } T(\gamma) < C_{31} N(\gamma) \}.$$

Our first task is to control $\mathbb{P}(A'_n)$.

**Lemma 5.5.** There exist $C_{31}, C_{32} > 0$ such that $\mathbb{P}(A'_n) \leq e^{-C_{32} n}$ for all $n \geq 1$.

**Proof.** By translation invariance we can consider $z = 0$. The content of Proposition 5.3 is that there exist constants $C_{33}, C_{34} > 0$ such that

$$\mathbb{P}(\exists \text{ a self-avoiding } \gamma \text{ from } 0 \text{ with } \#\gamma = n \text{ such that } N(\gamma) > C_{33} n) \leq e^{-C_{34} n}, \quad n \geq 1.$$

Summing this over $n$ gives $C_{35} < \infty$ such that

$$\mathbb{P}(\exists \text{ a self-avoiding } \gamma \text{ from } 0 \text{ such that } \#\gamma > n \text{ and } N(\gamma) > C_{33} \#\gamma) \leq e^{-C_{35} n}. \quad (5.7)$$

Further, given $C_{36} > 0$,

$$\mathbb{P}(\exists \text{ a self-avoiding } \gamma \text{ from } 0 \text{ with } \#\gamma \leq C_{36} n \text{ but } N(\gamma) \geq n) \leq \sum_{k=1}^{C_{36}} \mathbb{P}(N_k \geq n).$$

If we choose $\beta$ in Proposition 5.3 for $\kappa = 2$, then for some $C_{37}, C_{38} > 0$ this is bounded by

$$e^{-\beta n} \sum_{k=1}^{C_{36}} \mathbb{E} e^{\beta N_k} \leq e^{-\beta n} \sum_{k=1}^{C_{36}} e^{2k} = e^{-\beta n} \frac{e^{2(C_{36}+1)} - e^2}{e^2 - 1} \leq C_{37} e^{-C_{38} n},$$

if $C_{36}$ is small enough. Combining this with (5.7),

$$\mathbb{P}(A'_n) \leq e^{-C_{37} n} + e^{-C_{35} n} + \mathbb{P}(\exists \text{ a self-avoiding } \gamma \text{ from } 0 \text{ with } \#\gamma > C_{36} n \text{ but } T(\gamma) < C_{31} C_{33} \#\gamma).$$

By (1.7), for small $C_{31}$ the last probability is bounded by $e^{-C_{30} n}$. Therefore $\mathbb{P}(A'_n) \leq e^{-C_{40} n}$. \hfill $\square$
From Lemma 5.5 we can decompose

\[ P(Z_{z,x} \geq n) \leq P(Z_{z,x} \geq n, (A'_n)^c) + P(A'_n) \]

\[ \leq P(Z_{z,x} \geq n, (A'_n)^c) + e^{-C_{32}n}. \]

Consider some outcome in \((A'_n)^c\) such that \(Z_{z,x} \geq n > 0\). For this outcome, we must have

\[ C_{30} T_z < Y_{z,x} \leq C_{31} T_z, \]

a contradiction for \(C_{30} > C_{31}^{-1}\). This implies that independent of \(x, z\) and \(n\), there exists \(C_{31}\) such that

\[ P(Z_{z,x} \geq n) \leq e^{-C_{32}n} \]

and, in particular,

\[ \sup_x \sup_{z \neq 0} E e^\delta Z_{z,x} < \infty \] for \(\delta = C_{32}/2\).

Now, to bound the second term of (5.6) we apply Proposition 2.6 using our bound on \(Z_{z,x}\). Namely, we obtain for some \(C_{41}\)

\[ E e^{2\lambda F_m} Z_{z,x} \leq \delta^{-1} \left[ \text{Ent}(e^{2\lambda F_m}) + E e^{2\lambda F_m} \log E e^{\delta Z_{z,x}} \right] \]

\[ \leq C_{41} \left[ \text{Ent}(e^{2\lambda F_m}) + E e^{2\lambda F_m} \right], \]

implying

\[ (5.5) \leq C_{42} \left[ \text{Ent}(e^{2\lambda F_m}) + (1 + ET(0,x))E e^{2\lambda F_m} \right]. \]

So we conclude that if \(\lambda \leq 0\) and we assume (1.3), then for some \(C_{43}\),

\[ \sum_{k=1}^{\infty} \text{Ent}(V^2_k) \leq \lambda^2 C_{43} C_{\lambda} \left[ \text{Ent}(e^{2\lambda F_m}) + (1 + ET(0,x))E e^{2\lambda F_m} \right] \text{ for all } x \in \mathbb{Z}^d. \quad (5.8) \]

6 Proof of Theorem 2.3

First we must complete the upper bound for \(\sum_{k=1}^{\infty} \text{Ent}(V^2_k)\). What we have shown so far is (Theorem 5.1) that under (1.2) with \(\lambda \in [0, \alpha/2)\) or (1.3) with \(\lambda \leq 0\), setting \(C_{\lambda}\) as in Theorem 4.1

\[ \sum_{k=1}^{\infty} \text{Ent}(V^2_k) \leq \lambda^2 C_{44} C_{\lambda} \left[ \text{Ent}(e^{2\lambda F_m}) + (1 + EF_m)E e^{2\lambda F_m} \right] \text{ for all } x \in \mathbb{Z}^d. \]

This is close to the bound we would like, except there is an entropy term on the right. To bound this in terms of the moment generating function, we must use some techniques from Boucheron-Lugosi-Massart, similarly to what was done in Benaïm-Rossignol (below (15) in [5 Corollary 4.3]). Because these arguments lead us a bit astray, we place them in the appendix. By Theorem A.2 under (1.2), we can transform the upper bound into, for \(C_{44} = \min\{\alpha/4, C_{45}/2\}\) and some \(C_{46} > 0\),

\[ \sum_{k=1}^{\infty} \text{Ent}(V^2_k) \leq \lambda^2 C_{44} C_{\lambda} (\|x\|_1 + EF_m) E e^{2\lambda F_m} \text{ for } x \in \mathbb{Z}^d \text{ and } 0 \leq \lambda \leq C_{44}. \]
Using $E F_m \leq C_{47} \|x\|_1$, we obtain
\[
\sum_{k=1}^{\infty} \text{Ent}(V_k^2) \leq \lambda^2 C_{48} C_\lambda \|x\|_1 E e^{2\lambda F_m} \quad \text{for } x \in \mathbb{Z}^d \text{ and } 0 \leq \lambda \leq C_{44} \text{ under (1.2)}.\]

On the other hand, when we assume (1.3), Theorem A.4 gives the upper bound (with $C_{49}$ from that theorem)
\[
\sum_{k=1}^{\infty} \text{Ent}(V_k^2) \leq \lambda^2 C_{50} C_\lambda (1 + \|x\|_1 + E F_m) E e^{2\lambda F_m} \quad \text{for } -C_{49}/2 \leq \lambda \leq 0,
\]
which again implies
\[
\sum_{k=1}^{\infty} \text{Ent}(V_k^2) \leq \lambda^2 C_{51} C_\lambda \|x\|_1 E e^{2\lambda F_m} \quad \text{for } -C_{49}/2 \leq \lambda \leq 0 \text{ under (1.3)}.
\]

If we further restrict the range of $\lambda$ we can bound $C_\lambda$ using assumptions (1.3) and (1.2) and find for some $C_{52} > 0$,
\[
\sum_{k=1}^{\infty} \text{Ent}(V_k^2) \leq \lambda^2 C_{52} \|x\|_1 E e^{2\lambda F_m}, \quad x \in \mathbb{Z}^d, \quad \lambda \in \left[-\frac{C_{54}}{2}, 0\right],
\]
where $-C_{53} \leq \lambda \leq 0$ under (1.3) and $0 \leq \lambda \leq C_{54}$ under (1.2).

We can finally place this bound back in the Falik-Samorodnitsky inequality (2.9) along with the bound on influences from Proposition 3.3. We then obtain
\[
\text{Var} e^{\lambda F_m} \leq \left[ \log \frac{\text{Var} e^{\lambda F_m}}{C_{12} \lambda^2 \|x\|_1 E e^{2\lambda F_m}} \right]^{-1} \lambda^2 C_{52} \|x\|_1 E e^{2\lambda F_m}, \quad x \in \mathbb{Z}^d.
\]

Again, this holds for $-C_{54} \leq \lambda \leq 0$ under (1.3) and $0 \leq \lambda \leq C_{54}$ under (1.2). (Here we have used that $C_{53}$ can be slightly lowered to ensure that the term $E((1 + e^{\lambda t})_{e})^2$ is bounded by a constant under either assumption.)

From (6.2) we are almost done with the proof of Theorem 2.3. For any $d \geq 2$, $\frac{2+\zeta(1-d)}{2} \leq \frac{2-\zeta}{2}$, and so for every $\lambda$ either we have
\[
\text{Var} e^{\lambda F_m} \leq C_{12} \lambda^2 \|x\|_1^{\frac{2+\zeta(1-d)}{2}} E e^{2\lambda F_m}, \quad \text{in which case we have inequalities (2.2) and (2.3) for } \|x\|_1 > 1 \text{ (with a possibly different constant, and replacing } \lambda \text{ with } 2\lambda), \text{ or the opposite inequality holds, in which case}
\]
\[
\frac{\text{Var} e^{\lambda F_m}}{C_{12} \lambda^2 \|x\|_1^{\frac{2+\zeta(1-d)}{2}} E e^{2\lambda F_m}} \geq \|x\|_1^{\frac{\zeta}{2}}.
\]

Therefore when (6.3) fails,
\[
\text{Var} e^{\lambda F_m} \leq \lambda^2 C_{52} \|x\|_1^2 E e^{2\lambda F_m},
\]
implicating (2.2) and (2.3) again. This completes the proof of Theorem 2.3.

To recap, Lemma 2.2 shows that Theorem 2.3 suffices to prove exponential concentration for $F_m$ (Theorem 2.1). Last, Theorem 1.1 follows from Theorem 2.1 by the arguments in Section 2.1.
A Preliminary entropy bounds

A.1 Log Sobolev inequality

We will use the “symmetrized log Sobolev inequality” of Boucheron-Lugosi-Massart [4, Theorem 6.15].

**Theorem A.1.** Let $X$ be a random variable and let $X'$ be an independent copy. Then for $\lambda \in \mathbb{R}$,

$$\text{Ent } e^{\lambda X} \leq \mathbb{E} \left[ e^{\lambda X} q(\lambda (X' - X)_+) \right] \tag{A.1}$$

where $q(x) = x(e^x - 1)$.

A.2 Application to $F_m$

A.2.1 Positive exponential

**Theorem A.2.** Assuming (1.2), there exist $C_{55}, C_{45} > 0$ such that

$$\text{Ent } e^{\lambda F_m} \leq C_{55} \|x\|_1 \mathbb{E} e^{\lambda F_m} \text{ for all } x \in \mathbb{Z}^d \text{ and } \lambda \in [0, C_{45}] .$$

The proof of this bound follows from (1.11) and the following proposition. Recall that $\text{Geo}(0, x)$ is the set of edges in the intersection of all geodesics from 0 to $x$.

**Proposition A.3.** Assume (1.2). Given $A > 0$, there exists $\lambda_0 = \lambda_0(A) > 0$ such that

$$\text{Ent } e^{\lambda F_m} \leq \mathbb{E} e^{\lambda F_m} \log \exp \left( \frac{\# \text{Geo}(0, x)}{A} \right) \text{ for } 0 \leq \lambda \leq \lambda_0 \text{ and } x \in \mathbb{Z}^d .$$

**Proof.** By tensorization of entropy (Proposition 2.7),

$$\text{Ent } e^{\lambda F_m} \leq \sum_{k=1}^{\infty} \mathbb{E} \text{Ent}_{e_k} e^{\lambda F_m} .$$

Introduce $F_m^{(k)}$ as the variable $F_m$ evaluated at the configuration in which $t_{e_k}$ is replaced by an independent copy $t'_{e_k}$. Then we can apply (A.1) conditionally:

$$\text{Ent } e^{\lambda F_m} \leq \sum_{k=1}^{\infty} \mathbb{E} \text{Ent}_{e_k} e^{\lambda F_m} q(\lambda (F_m^{(k)} - F_m)_+) \leq \sum_{k=1}^{\infty} \mathbb{E} \text{Ent}_{e_k} e^{\lambda F_m} q \left( \frac{1}{\# B_m} \sum_{z \in B_m} \lambda (T_z^{(k)} - T_z)_+ \right) .$$

Convexity of $q$ on $[0, \infty)$ gives the upper bound

$$\frac{1}{\# B_m} \sum_{z \in B_m} \sum_{k=1}^{\infty} \mathbb{E} \text{Ent}_{e_k} e^{\lambda F_m} q \left( \lambda (T_z^{(k)} - T_z)_+ \right) .$$

Lemma 3.1 implies that $(T_z^{(k)} - T_z)_+ \leq t'_{e_k} 1_{\{e_k \in \text{Geo}(z, z+x)\}}$ so we get the bound

$$\frac{1}{\# B_m} \sum_{z \in B_m} \sum_{k=1}^{\infty} \mathbb{E} \text{Ent}_{e_k} e^{\lambda F_m} q(\lambda t'_{e_k}) 1_{\{e_k \in \text{Geo}(z, z+x)\}} .$$
Integrate $t'_{ek}$ first and bring the sum inside the integral for

$$\frac{\mathbb{E} q(\lambda t_e)}{\# B_m} \sum_{z \in B_m} \mathbb{E} \left[ e^{\lambda F_m} \# \text{Geo}(z, z + x) \right].$$

Note that under (1.2), $\mathbb{E} q(\lambda t_e) < \infty$ for $\lambda \in [0, \alpha)$.

To deal with this product, use Proposition 2.6, along with a parameter $p > 0$, to obtain

$$\text{Ent} e^{\lambda F_m} \leq \frac{\mathbb{E} q(\lambda t_e)}{\# B_m} \sum_{z \in B_m} \left[ \text{pEnt} e^{\lambda F_m} + p \mathbb{E} e^{\lambda F_m} \log \mathbb{E} \exp \left( \frac{\# \text{Geo}(z, z + x)}{p} \right) \right].$$

By translation invariance, the expression inside the sum does not depend on $z$. Therefore if we choose $p \leq (2 \mathbb{E} q(\lambda t_e))^{-1}$ this is no bigger than

$$\frac{1}{2} \text{Ent} e^{\lambda F_m} + \frac{1}{2} \mathbb{E} e^{\lambda F_m} \log \mathbb{E} e^{\# \text{Geo}(0,x)/p}.$$

So we end with

$$\text{Ent} e^{\lambda F_m} \leq \mathbb{E} e^{\lambda F_m} \log \mathbb{E} e^{\# \text{Geo}(0,x)/p} \text{ if } 0 \leq p \leq (2 \mathbb{E} q(\lambda t_e))^{-1} \text{ and } \lambda \in [0, \alpha).$$

As $\lambda \to 0$, $\mathbb{E} q(\lambda t_e) \to 0$ so given $A > 0$ choose $\lambda_0 \in (0, \alpha)$ such that if $0 \leq \lambda \leq \lambda_0$, then $(2 \mathbb{E} q(\lambda t_e))^{-1} \geq A$. Such $\lambda_0$ completes the proof. \(\square\)

### A.2.2 Negative exponential

The bound given below is similar to the one derived in [9] for $T$ instead of $F_m$.

**Theorem A.4.** Assume $\mathbb{E} t^2_e < \infty$. There exist $C_{56}, C_{49} > 0$ such that

$$\text{Ent} e^{\lambda F_m} \leq C_{56} \lambda^2 \|x\|_1 \mathbb{E} e^{\lambda F_m} \text{ for all } x \in \mathbb{Z}^d \text{ and } \lambda \in [-C_{49}, 0].$$

**Proof.** Again we use (A.1) with tensorization:

$$\text{Ent} e^{\lambda F_m} \leq \sum_{k=1}^{\infty} \mathbb{E} \mathbb{E}_{e_k} e^{\lambda F_m} q(\lambda (F_m^{(k)} - F_m)_+) ,$$

By the inequality $q(x) \leq x^2$ for $x \leq 0$ we obtain the upper bound

$$\lambda^2 \sum_{k=1}^{\infty} \mathbb{E} \mathbb{E}_{e_k} e^{\lambda F_m} ((F_m^{(k)} - F_m)_+)^2 .$$

By convexity of $x \mapsto (x_+)^2$, this is bounded by

$$\lambda^2 \frac{1}{\# B_m} \sum_{z \in B_m} \sum_{k=1}^{\infty} \mathbb{E} \mathbb{E}_{e_k} e^{\lambda F_m} ((T_z^{(k)} - T_z)_+)^2.$$
and using Lemma 3.1 by

$$\lambda^2 \sum_{z \in \mathcal{B}_m} \sum_{k=1}^{\infty} \mathbb{EE}_{e_k} e^{\lambda F_m}(t'_{e_k} 1_{e_k \in \text{Geo}(z, z+x)})^2 .$$

Again, integrate over $t'_{e_k}$ first to get

$$\lambda^2 \mathbb{EE}_{e'} t^2 \sum_{z \in \mathcal{B}_m} \mathbb{EE}_{e} \lambda F_m \# \text{Geo}(z, z+x) . \quad (A.2)$$

To complete the proof, take $a$ from Proposition 1.4 and upper bound the expectation as

$$(1/a) \mathbb{EE} \lambda F_m T_z + \mathbb{EE} \lambda F_m \# \text{Geo}(z, z+x) 1_{a \# \text{Geo}(z, z+x) > T_z} . \quad (A.3)$$

The variable $e^{\lambda F_m}$ is decreasing as a function of the edge-weights (since $\lambda \leq 0$) whereas $T_z$ is increasing. So apply the Harris-FKG inequality to the first term for an upper bound of

$$(1/a) \mathbb{EE} \lambda F_m \mathbb{EE} \lambda F_m T_z = (1/a) \mathbb{EE} \lambda F_m \mathbb{EE} \lambda F_m(0, x) .$$

For the second term call $Y = \# \text{Geo}(z, z+x) 1_{a \# \text{Geo}(z, z+x) > T_z}$ and use Proposition 2.6. Taking $\delta = C_{57}/2$ from Proposition 1.4, we have $\mathbb{EE} \delta Y \leq \mathbb{EE} \delta Y < C_{57}$ for some $C_{57}$ and so

$$\mathbb{EE} \lambda F_m Y \leq \delta^{-1} \text{Ent } e^{\lambda F_m} + \delta^{-1} C_{57} e^{\lambda F_m} .$$

Returning to (A.2), we obtain

$$\text{Ent } e^{\lambda F_m} \leq \lambda^2 \mathbb{EE}_{e'} t^2 \left[ (1/a) \mathbb{EE} \lambda F_m \mathbb{EE} \lambda F_m T_z + \delta^{-1} \text{Ent } e^{\lambda F_m} + \delta^{-1} C_{57} e^{\lambda F_m} \right] .$$

Now restrict to $\lambda \leq 0$ such that $-(2\mathbb{EE}_{e'} t^2 \delta^{-1})^{-1/2} \leq \lambda$ to obtain

$$\frac{1}{2} \text{Ent } e^{\lambda F_m} \leq \lambda^2 \mathbb{EE}_{e'} t^2 \left[ (1/a) \mathbb{EE} \lambda F_m \mathbb{EE} \lambda F_m T_z + \delta^{-1} C_{57} e^{\lambda F_m} \right] .$$

Last, we bound $\mathbb{EE} T(0, x) \leq C_{58} ||x||_1$ to get

$$\text{Ent } e^{\lambda F_m} \leq \lambda^2 C_{59} ||x||_1 e^{\lambda F_m} \text{ for } x \in \mathbb{Z}^d, \quad -(2\mathbb{EE}_{e'} t^2 \delta^{-1})^{-1/2} \leq \lambda \leq 0 .$$

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