A BOUNDARY CROSS THEOREM
FOR SEPARATELY HOLOMORPHIC FUNCTIONS

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Abstract. Let $D \subset \mathbb{C}^n$, $G \subset \mathbb{C}^m$ be pseudoconvex domains, let $A$ (resp. $B$) be an open subset of the boundary $\partial D$ (resp. $\partial G$) and let $X$ be the 2-fold cross $((D \cup A) \times B) \cup (A \times (B \cup G))$. Suppose in addition that the domain $D$ (resp. $G$) is locally $C^2$ smooth on $A$ (resp. $B$). We shall determine the ”envelope of holomorphy” $\hat{X}$ of $X$ in the sense that any function continuous on $X$ and separately holomorphic on $(A \times G) \cup (D \times B)$ extends to a function continuous on $\hat{X}$ and holomorphic on the interior of $\hat{X}$. A generalization of this result for an $N$-fold cross is also given.

1. Introduction and statement of the main results

In order to recall here the classical cross theorem and to state our results, we need to introduce some notation and terminology. In fact, we keep the main notation from the book by Jarnicki and the first author [6] and from the survey article by Sadullaev [16].

1.1. Plurisubharmonic measures. Let $\Omega \subset \mathbb{C}^n$ be an open set. For any function $u$ defined on $\Omega$, let

$$\hat{u}(z) := \begin{cases} u(z), & z \in \Omega, \\
\limsup_{w \in \Omega, w \to z} u(w), & z \in \partial \Omega. \end{cases}$$

For a set $A \subset \overline{\Omega}$ put

$$h_{A,\Omega} := \sup \{ u : u \in \mathcal{PSH}(\Omega), u \leq 1 \text{ on } \Omega, \hat{u} \leq 0 \text{ on } A \},$$

where $\mathcal{PSH}(\Omega)$ denotes the set of all functions plurisubharmonic on $\Omega$.

We first suppose that $\Omega$ is bounded. Then the plurisubharmonic measure of $A$ relative to $\Omega$ is given by

$$(1.1) \quad \omega(z, A, \Omega) := \overline{h_{A,\Omega}^*}(z), \quad z \in \Omega \cup A,$$

where $u^*$ denotes the upper semicontinuous regularization of a function $u$.

From now on let $\Omega$ be an arbitrary (not necessarily bounded) open set and we shall be concerned with the following two cases.

Case I: $A \subset \Omega$.

In this case, we define the plurisubharmonic measure of $A$ relative to $\Omega$ as follows.

$$\omega(\cdot, A, \Omega) := \lim_{k \to +\infty} h_{A \cap \Omega_k, \Omega_k}^*,$$

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where \((\Omega_k)_{k=1}^\infty\) is a sequence of relatively compact open sets \(\Omega_k \subset \Omega_{k+1} \in \Omega\) with \(\bigcup_{k=1}^\infty \Omega_k = \Omega\). Observe that the definition is independent of the exhausting sequence \((\Omega_k)_{k=1}^\infty\). Moreover, \(\omega(\cdot, A, \Omega) \in \mathcal{PSH}(\Omega)\).

**Case II:** \(A\) is an open subset of \(\partial \Omega\).

We suppose in addition that \(\Omega\) is *locally* \(C^2\) smooth on \(A\) (i.e. for any \(\zeta \in A\), there exist an open neighborhood \(U = U_\zeta\) of \(\zeta\) in \(\mathbb{C}^n\) and a real function \(\rho = \rho_\zeta \in C^2(U)\) such that \(\Omega \cap U = \{z \in U : \rho(z) < 0\}\) and \(d\rho(\zeta) \neq 0\).

In this case, the *plurisubharmonic measure* of \(A\) relative to \(\Omega\) is a function on \(\Omega \cup A\) given by

\[
\omega(z, A, \Omega) := \lim_{k \to +\infty} \omega(z, A_k, \Omega_k), \quad z \in \Omega \cup A,
\]

where the function \(\omega(\cdot, A_k, \Omega_k)\) is given by (1.1) and \((\Omega_k)_{k=1}^\infty\) is a sequence of relatively compact open sets \(\Omega_k \Subset \Omega\) and \((A_k)_{k=1}^\infty\) is a sequence of open subsets of \(A\) such that

(i) \(\Omega_k \subset \Omega_{k+1}\) and \(\cup_{k=1}^\infty \Omega_k = \Omega\);

(ii) \(A_k \subset A_{k+1}\) and \(A_k \subset \partial \Omega \cap \partial \Omega_k\) and \(\cup_{k=1}^\infty A_k = A\);

(iii) for any point \(\zeta \in A\) there is an open neighborhood \(V = V_\zeta\) of \(\zeta\) in \(\mathbb{C}^n\) such that \(V \cap \Omega = V \cap \Omega_k\) for some \(k\).

In Section 3 below, we shall prove that the definition is independent of the chosen exhausting sequences \((\Omega_k)_{k=1}^\infty\) and \((A_k)_{k=1}^\infty\). Moreover, \(\omega(\cdot, A, \Omega)|_\Omega \in \mathcal{PSH}(\Omega)\).

### 1.2. Cross and separate holomorphicity.

Let \(N \in \mathbb{N}, \ N \geq 2\), and let \(\emptyset \neq A_j \subset D_j \subset \mathbb{C}^{n_j}\), where \(D_j\) is a domain, \(j = 1, \ldots, N\). We define an \(N\)-fold cross \(X\), its interior \(X^o\) and a new set \(A\) as

\[
X = \mathcal{X}(A_1, \ldots , A_N; D_1, \ldots , D_N)
\]

\[
:= \bigcup_{j=1}^N A_1 \times \cdots \times A_{j-1} \times (D_j \cup A_j) \times A_{j+1} \times \cdots \times A_N \subset \mathbb{C}^{n_1 + \cdots + n_N} = \mathbb{C}^n,
\]

\[
X^o = \mathcal{X}^o(A_1, \ldots , A_N; D_1, \ldots , D_N)
\]

\[
:= \bigcup_{j=1}^N A_1 \times \cdots \times A_{j-1} \times D_j \times A_{j+1} \times \cdots \times A_N,
\]

\[
A := A_1 \times \cdots \times A_N.
\]

In particular, if \(A_j \subset D_j, \ j = 1, \ldots, N\), then we have \(X = X^o\). Moreover put

\[
\omega(z) := \sum_{j=1}^N \omega(z_j, A_j, D_j), \quad z = (z_1, \ldots , z_N) \in (D_1 \cup A_1) \times \cdots \times (D_N \cup A_N).
\]

For an \(N\)-fold cross \(X = \mathcal{X}(A_1, \ldots , A_N; D_1, \ldots , D_N)\) let

\[
\hat{X} := \{z = (z_1, \ldots , z_N) \in (D_1 \cup A_1) \times \cdots \times (D_N \cup A_N) : \omega(z) < 1\}.
\]

Then the set of all interior points of \(\hat{X}\) is given by

\[
\hat{X}^o := \{z = (z_1, \ldots , z_N) \in D_1 \times \cdots \times D_N : \omega(z) < 1\}.
\]
We say that a function \( f : X \rightarrow \mathbb{C} \) is separately holomorphic on \( X^o \) and write \( f \in \mathcal{O}_s(X^o) \), if for any \( j \in \{1, \ldots, N\} \) and \( (a', a'') \in (A_1 \times \cdots \times A_{j-1}) \times (A_{j+1} \times \cdots \times A_N) \) the function \( f(a', \cdot, a'') \mid_{D_j} \) is holomorphic on \( D_j \).

Finally, throughout the paper, the notation \( |f|_M \) denotes \( \sup_M |f| \).

1.3. **Motivations for our work.** We are now able to formulate what we will quote in the sequel as the classical cross theorem.

**Theorem 1.** (Alehyane \& Zeriahi \cite{1}) Let \( D_j \subset \mathbb{C}^{n_j} \) be a pseudoconvex domain and \( A_j \subset D_j \) a locally pluriregular subset, \( j = 1, \ldots, N \). Then for any function \( f \in \mathcal{O}_s(X) \), there is a unique function \( \hat{f} \in \mathcal{O}(\hat{X}) \) such that \( \hat{f} = f \) on \( X \).

There is a long list of papers dealing with this theorem under various assumptions. For a historical discussion, see the survey article \cite{14}.

The question naturally arises how the situation changes when the sets \( A_j \) live on the boundary \( \partial D_j \), \( j = 1, \ldots, N \).

The first results in this direction are obtained by Malgrange–Zerner \cite{17}, Komatsu \cite{10} and Drużkowski \cite{2}, but only for some special crosses. Recently, Gonchar \cite{3, 4} has proved the following remarkable more general theorem.

**Theorem 2.** Let \( D_j \subset \mathbb{C} \) be a Jordan domain and let \( \emptyset \neq A_j \) be an open set of the boundary \( \partial D_j \), \( j = 1, \ldots, N \). Then for any function \( f \in \mathcal{C}(X) \cap \mathcal{O}_s(X^o) \) there is a unique function \( \hat{f} \in \mathcal{C}(\hat{X}) \cap \mathcal{O}(\hat{X}^o) \) such that \( \hat{f} = f \) on \( X \). Moreover, if \( |f|_X < \infty \) then
\[
|\hat{f}(z)| \leq |f|_A^{1-\omega(z)}|f|_X^{\omega(z)}, \quad z \in \hat{X}.
\]

It should be observed that under the hypothesis of Theorem 2 one has \( X \subset \hat{X} \). We remark that the original formulation of Gonchar is slightly different from Theorem 2. However, his proof is still valid also in this new formulation.

The main purpose of this work is to generalize Gonchar’s theorem to higher dimensions.

1.4. **The main result.** We are now ready to state the main result.

**Main Theorem.** Let \( D_j \subset \mathbb{C}^{n_j} \) be a pseudoconvex domain and let \( \emptyset \neq A_j \) be an open set of \( \partial D_j \), \( j = 1, \ldots, N \). Suppose in addition that each domain \( D_j \) is locally \( \mathcal{C}^2 \) smooth on \( A_j \), \( j = 1, \ldots, N \). Then \( X \subset \hat{X} \) and for any function \( f \in \mathcal{C}(X) \cap \mathcal{O}_s(X^o) \), there is a unique function \( \hat{f} \in \mathcal{C}(\hat{X}) \cap \mathcal{O}(\hat{X}^o) \) such that \( \hat{f} = f \) on \( X \). Moreover, if \( |f|_X < \infty \) then
\[
|\hat{f}(z)| \leq |f|_A^{1-\omega(z)}|f|_X^{\omega(z)}, \quad z \in \hat{X}.
\]

We will give here a short outline of the proof.

The main idea is to combine Gonchar’s theorem and the classical cross theorem with the slicing method. More precisely, for each domain \( D_j \) we shall associate a family of \( \mathcal{C}^2 \) smooth planar domains which are, roughly speaking, the intersection of an open tubular neighborhood of \( A_j \) in \( D_j \cup A_j \) with the family of normal complex lines to \( A_j \) parameterized by \( A_j \). One important property of this family is that
the harmonic measures for its domains depend, in some sense, continuously on the parameter of $A_j$. Another important property is that there is a relation between the plurisubharmonic measure of $D_j$ and the harmonic measure of the domains in the above family. Applying Gonchar’s theorem and the slicing method, we shall find an extension $\hat{f}$ such that $\hat{f}$ is holomorphic on a subdomain of each domain in this family. The two important properties mentioned above, combined with a variant of the classical cross theorem, will allow us to propagate the holomorphicity from these one-dimensional subdomains to the desired envelope of holomorphy.

This paper is organized as follows.

We begin Section 2 by collecting some background of the potential theory and some classical results. Next we establish a uniform estimate for the Poisson kernels which will play an important role in the proof of the main theorem.

Based on the results of Section 2, Section 3 develops necessary estimates for the plurisubharmonic measures that will be used later in Section 5.

Section 4 provides the first step of the proof. More precisely, on the one hand we will consider the following mixed situation where there is at least one index $j$ such that the factor $A_j$ of the cross $X$ is inside $D_j$. On the other hand, we will establish some quantitative versions of the classical cross theorem.

Section 5 establishes the main theorem in the case of a 2-fold cross.

The complete proof of the main theorem will be given in Section 6 together with some concluding remarks and open questions.

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2. Auxiliary results

2.1. Harmonic measure for a planar domain. We recall some classical facts from the book of Ransford [15]. Let $D$ be a proper subdomain of $\mathbb{C} \cup \{\infty\}$ such that the boundary $\partial D$ (with respect to $\mathbb{C} \cup \{\infty\}$) is non-polar. Let $\mathcal{P}_D$ be the Poisson projection of $D$ and $A$ a Borel subset of $\partial D$. Consider the bounded function

$$1_{\partial D \setminus A}(\zeta) := \begin{cases} 1, & \zeta \in \partial D \setminus A, \\ 0, & \zeta \in A. \end{cases}$$

Then, by Theorem 4.3.3 of [15], the harmonic measure of the set $\partial D \setminus A$ (or equivalently $h_{A,D}$) is exactly the Perron solution of the generalized Dirichlet problem with boundary data $1_{\partial D \setminus A}$. In other words, one has

$$h_{A,D} = \mathcal{P}_D[1_{\partial D \setminus A}].$$

2.2. A uniqueness theorem and a Two-Constant Theorem. The following uniqueness theorem is very useful.

**Theorem 2.1.** (see [5]) Let $D \subset \mathbb{C}^n$ be a domain such that $D$ is locally $\mathcal{C}^1$ smooth on some open set $U$ of $\partial D$. If a set $E \subset D \cup U$ has a positive $(2n - 1)$-dimensional Hausdorff measure, then, for $f \in \mathcal{C}(D \cup U) \cap \mathcal{O}(D)$, $f = 0$ on $E$ implies $f \equiv 0$. 
Proof. The only nontrivial case is that \( E \subset U \). In this case by taking the intersection of \( D \) with a bundle of complex lines and applying the classical one-dimensional boundary uniqueness theorem of Privalov, one may find a set \( E' \subset D \) close to \( E \) such that \( E' \) has a positive \( 2n \)-dimensional Hausdorff measure and \( f = 0 \) on \( E' \). This completes the proof. \( \square \)

The following Two-Constant Theorem for plurisubharmonic functions will play a vital role in this paper.

Theorem 2.2. If \( u \) is a plurisubharmonic function in a bounded open set \( D \subset \mathbb{C}^n \) and \( u \leq M \) on \( D \) and \( \hat{u} \leq m \) on some subset \( A \) of \( D \), then
\[
\hat{u}(z) \leq m(1 - \omega(z, A, D)) + M \cdot \omega(z, A, D), \quad z \in \overline{D}.
\]

Proof. It follows immediately from the definition of \( \omega(\cdot, A, D) \) given in Subsection 1.1. \( \square \)

2.3. Uniform estimate for the Poisson kernels of a family of \( C^2 \) smooth domains. In what follows we fix an integer \( N \geq 2 \) and let \( \text{dist}(\cdot, \cdot) \) denote the Euclidean distance and let \( B(a, r) \) (\( a \in \mathbb{R}^N, r > 0 \)) denote the Euclidean ball of center \( a \) and radius \( r \). We say that a domain \( D \subset \mathbb{R}^N \) is \( C^2 \) smooth if \( D \) is bounded and admits a defining function \( \rho \in C^2(\mathbb{R}^N) \) such that \( d\rho(z) \neq 0 \) for all \( z \in \partial D \). Let \( P_D \) denote its Poisson kernel. We begin this subsection with the following result due to N. Kerzman (see [8] and [11]).

Theorem 2.3. Let \( D \subset \mathbb{R}^N \) be a \( C^2 \) smooth domain that satisfies
\[
diam(D) := \sup_{x,y \in D} |x - y| \leq M \quad \text{for some finite constant } M.
\]

Then
1) there is a positive number \( r = r(D) \) such that for each \( y \in \partial D \) there are balls \( B(c_y, r) \subset D \) and \( B(\overline{c}_y, r) \subset \mathbb{R}^N \setminus \overline{D} \) that satisfy
\[
B(\overline{c}_y, r) \cap \overline{D} = \{y\}, \quad B(c_y, r) \cap (\mathbb{R}^N \setminus D) = \{y\};
\]
2) there is a finite constant \( C \) which depends only on \( N, r, \) and \( M \) such that
\[
P_D(x, y) \leq C \frac{\text{dist}(x, \partial D)}{|x - y|^N}, \quad x \in D, \ y \in \partial D.
\]

Proof. This theorem is implicitly proved in Lemmas 8.2.3–8.2.5 and Proposition 8.2.6 in the book of Krantz [11]. We only mention here that Kerzman’s idea is to compare the Green function and the Poisson kernel for \( D \) with the corresponding functions for the internally and externally tangent balls \( B(c_y, r) \) and \( B(\overline{c}_y, r) \) (and also for their complement). \( \square \)

Now we reformulate Kerzman’s theorem in order to obtain an uniform upper bound for the Poisson kernels of a family of domains which depend, in some sense, continuously on a parameter.
Corollary 2.4. Let \((D_\alpha)_{\alpha \in I}\) be a family of \(C^2\) smooth domains in \(\mathbb{R}^N\) indexed by a set \(I\). Suppose that
1) there is a finite constant \(M\) such that for all \(\alpha \in I\),
\[
\text{diam}(D_\alpha) \leq M;
\]
2) there is a finite positive number \(r\) such that for each \(\alpha \in I\), \(y \in \partial D_\alpha\), there are balls \(B(c_{y,\alpha}, r) \subset D_\alpha\) and \(B(\bar{c}_{y,\alpha}, r) \subset \mathbb{R}^N \setminus D_\alpha\) that satisfy
\[
\overline{B(\bar{c}_{y,\alpha}, r) \cap D_\alpha} = \{y\},
\]
\[
\overline{B(c_{y,\alpha}, r) \cap (\mathbb{R}^N \setminus D_\alpha)} = \{y\}.
\]
Then there exists a finite constant \(C\) such that
\[
P_{D_\alpha}(x, y) \leq C \frac{\text{dist}(x, \partial D_\alpha)}{|x - y|^N}, \quad x \in D_\alpha, \ y \in \partial D_\alpha, \ \alpha \in I.
\]

Proof. It follows immediately from Theorem 2.3. \(\square\)

We conclude this section with an example of a family of \(C^2\) smooth domains satisfying the hypothesis of Corollary 2.4.

Let \(D\) be domain in \(\mathbb{C}^n\) such that \(D\) is locally \(C^2\) smooth on an open neighborhood of a point \(P \in \partial D\). Let \(T^C_P\) (resp. \(T^R_P\)) denote the complex (resp. real) tangent hyperplane to \(\partial D\) at \(P\) and \(\pi\) (resp. \(\pi^C\)) the orthogonal projection from \(\mathbb{C}^n\) onto \(T^R_P\) (resp. \(T^C_P\)).

By an affine transformation, we may suppose without loss of generality that \(P = 0\), \(T^C_P = \{z_1 = 0\}\) and \(T^R_P = \{\text{Re } z_1 = 0\}\). Moreover, there are an open neighborhood \(U\) of the origin and a function \(\rho \in C^2(U)\) such that
\[
\rho(0) = 0, d\rho(0) = (1, 0, \ldots, 0) \quad \text{and} \quad U \cap D = \{\rho < 0\}.
\]

For any domain \(V \subset U\), any \(Q := (0, z') = (0, z_2, \ldots, z_n) \in T^C_P\), consider the planar domain
\[
V_Q := \text{env} \left( V \cap \left\{(t, z'), \ t \in \mathbb{C}\right\} \right),
\]
where env\((G)\) denotes the smallest simply connected domain containing (a given planar domain) \(G\), in other words, env\((G)\) is obtained from \(G\) by adding all its holes.

Proposition 2.5. Under the above hypothesis and notation, there are open neighborhoods \(U_1\) of \(P\) in \(T^C_P\), \(U_2\) of \(P\) in \(T^R_P\) and \(U_3\) of \(P\) in \(\mathbb{C}^n\) and a \(C^2\) smooth subdomain \(V \subset D\) such that
1) \(U_1 = U_2 \cap T^C_P\);
2) \(\partial V \cap \partial D\) is an open neighborhood of \(P\) in \(\partial D\) and in \(\partial V\) and \(\pi\) is one-to-one from \(\partial V \cap \partial D\) onto an open neighborhood of \(U_2\);
3) \((V_Q)_{Q \in U_1}\) is a family of \(C^2\) smooth planar simply connected domains which satisfies 1) and 2) in Corollary 2.4 and \(V_Q \subset D\);
4) there is a finite constant \( C \) such that for all \( Q \in U_1, z \in V_Q \cap U_3 \) and \( \zeta \in \partial V \cap \partial D \) satisfying \( \pi(\zeta) = \pi(z) \), 
\[
\text{dist}(z, \partial V_Q) \leq C \cdot \text{dist}(z, \partial D) \quad \text{and} \quad \text{dist}(z, \zeta) \leq C \cdot \text{dist}(z, \partial D); 
\]
in other words, the quantities \( \text{dist}(z, \partial V_Q), \text{dist}(z, \zeta) \) and \( \text{dist}(z, \partial D) \) are equivalent.

Proof. Since \( D \) is locally \( C^2 \) smooth on an open neighborhood of a point \( P \in \partial D \), a geometric argument (see [11, p. 325]) shows that there is an \( r > 0 \) such that the sphere \( \partial B \) is internally tangent to \( D \) at \( P \), where the ball \( B \) is given by \( B := B((-r,0,\ldots,0),r) \).

Consider the following defining function for the ball \( B \)
\[
\phi(z) := \frac{(x_1 + r)^2 + y_1^2 + |z'|^2 - r^2}{2r}, \quad z = (x_1 + iy_1, z') \in \mathbb{C}^n.
\]
A straightforward computation gives that \( |d\phi| = 1 \) on \( \partial B \). Next fix a radial function \( \psi \in C_0(\mathbb{C}^n) \) such that \( 0 \leq \psi \leq 1, \psi(z) = 1 \) for \( |z| \leq 1 \) and \( \psi(z) = 0 \) for \( |z| \geq 2 \).

Since \( d\rho(0) = d\phi(0) \), we may choose a sufficiently small \( \epsilon_0 \) such that \( 0 < \epsilon_0 < \frac{r}{4} \) and
\[
\text{(2.5)} \quad |(d\rho - d\phi)(z)| < \frac{1}{8}, \quad |z| \leq 2\epsilon_0.
\]
Now define for any \( 0 < \epsilon < \epsilon_0 \),
\[
\text{(2.6)} \quad \psi_\epsilon(z) := \psi\left(\frac{z}{\epsilon}\right), \quad \rho_\epsilon := \phi + \psi_\epsilon(\rho - \phi).
\]
Observe that \( \rho_\epsilon(z) = \rho(z) \) for \( |z| \leq \epsilon \) and \( \rho_\epsilon(z) = \phi(z) \) for \( |z| \geq 2\epsilon \). Moreover using (2.5)–(2.6) and the identities \( \rho(0) = \phi(0), d\rho(0) = d\phi(0) \), we have for \( |z| \leq 2\epsilon \),
\[
|(d\rho_\epsilon - d\phi)(z)| \leq \psi_\epsilon(z) |(d\rho - d\phi)(z)| + |d\psi_\epsilon(z)| \cdot |(\rho - \phi)(z)|
\]
\[
\leq \frac{1}{8} + C' \epsilon^2, \quad \text{where } C' \text{ is a finite constant.}
\]
Therefore there exists \( \epsilon_1 > 0 \) such that for all \( 0 < \epsilon < \epsilon_1 \),
\[
\text{(2.7)} \quad |(d\rho_\epsilon - d\phi)(z)| \leq \frac{1}{4}, \quad |z| \leq 2\epsilon.
\]
For any \( 0 < \epsilon < \min\{\epsilon_0, \epsilon_1\} \) define
\[
\text{(2.8)} \quad V' := \{z \in \mathbb{C}^n : \rho_\epsilon(z) < 0\}
\]
and let \( V \) be the connected component of \( V' \) satisfying \( P \in \partial V \). This, combined with (2.7) implies that \( |d\rho_\epsilon(z)| > \frac{1}{2} \) for \( |z| \leq 2\epsilon \). Since \( 0 \leq \psi \leq 1 \) and \( \rho_\epsilon(z) = \phi(z) \) for \( |z| \geq 2\epsilon \), we deduce from (2.6) that \( V \) is a \( C^2 \) smooth subdomain of \( D \).

Now let
\[
U_3 := B(0,\epsilon), \quad U_2 := U_3 \cap T_P^{\mathbb{R}}, \quad U_1 := U_3 \cap T_P^{\mathbb{C}}
\]
Then in virtue of (2.4)–(2.6), we see that Parts 1) and 2) are satisfied when \( \epsilon \) in (2.8) is sufficiently small.
We next turn to Part 3). Fix any $Q \in U_1$ and $z \in \partial V_Q$, then there are two cases. If $|z| \leq 2\epsilon$, then by (2.7)

$$|d_{z_1}\rho_\epsilon| \geq |d_{z_1}\phi| - |d\rho_\epsilon - d\phi| > \frac{|z_1 + r|}{r} - \frac{1}{4} > \frac{1}{4},$$

If $|z| \geq 2\epsilon$, then by (2.6)

$$|d_{z_1}\rho_\epsilon| = |d_{z_1}\phi| = \frac{|z_1 + r|}{r} > 0.$$ Thus for any $Q = (0, z') \in U_1$ the region $V \cap \{(t, z'), t \in \mathbb{C}\}$ is a $C^2$ smooth planar region contained in $D$. Since for sufficiently small $\epsilon > 0$, $\partial D \cap U_3$ is a graph over $\mathbb{T}_P$, a geometric argument shows that $V_Q$ is also a $C^2$ smooth planar simply connected region contained in $D$. We see that one may assume that $V_Q$ is a domain.

To complete Part 3) we still need to check that the family $(V_Q)_{Q \in U_1}$ satisfies 1) and 2) in Corollary 2.4. Indeed, let $\rho_Q$ be the restriction of $\rho$ on the complex line containing $V_Q$. Clearly, the Hessian $d^2\rho_Q$ depends continuously on the parameter $Q \in U_1$. This, combined with the proof of the geometric fact (see [11, p. 325]) implies the remaining assertion of Part 3).

It remains to establish Part 4). Also by [11, p. 325], when $\epsilon > 0$ in (2.8) is sufficiently small, for any $z \in U_3 \cap D$ there are a unique point $\theta \in \partial D$ and a unique point $\eta \in \partial V_Q$ such that

$$|z - \theta| = \text{dist}(z, \partial D) \quad \text{and} \quad |z - \eta| = \text{dist}(z, \partial V_Q).$$

Let $n_\theta$ (resp. $n_\eta$) be the inward unit normal vector to $\partial D$ (resp. $\partial V_Q$) at $\theta$ (resp. $\eta$). Then a geometric argument shows that the orthogonal projection of the real line containing $n_\eta$ onto $V_Q$ passes through $z$. Since $Q$ is close to $P$, the angle between two vectors $z - \eta$ and $n_\eta$ and the angle between $n_\eta$ and $n_\theta$ are sufficiently small when $\epsilon$ in (2.8) is sufficiently small. Thus the angle between two vectors $z - \eta$ and $z - \theta$ is sufficiently small. Since $|d\rho(0)| = 1$ and $\rho \in C^2(U)$, it follows that

$$|z - \theta| \leq |z - \eta| \leq C|z - \theta|$$

for some finite constant $C$, which proves that $\text{dist}(z, \partial V_Q) \leq C \cdot \text{dist}(z, \partial D)$.

The second estimate of Part 4) can be proved in exactly the same way. This completes the proof.

3. Estimates for the plurisubharmonic measures

In this section we apply the result of the previous one to establish some inequalities concerning the plurisubharmonic measures. These estimates will be crucial for the proof of the main theorem.

**Proposition 3.1.** Let $D$ be a bounded planar domain with $C^2$ smooth boundary. Then there is a constant $C$ with the following property: For any union $A$ of a finite number of open connected arcs on $\partial D$, one has

$$\omega(z, A, D) \leq C \frac{\text{dist}(z, \partial D)}{\text{dist}(z, \partial D \setminus A)^2}, \quad z \in D \cup A.$$
Proof. By Theorem 2.3 we know that there is a finite constant $C'$ such that
\[ P_D(x, y) \leq C' \frac{\text{dist}(x, \partial D)}{|x - y|^2}, \quad x \in D, \ y \in \partial D. \]
This, combined with identity (2.1), implies that
\[ \omega(z, A, D) \leq C' \cdot \int_{\partial D \setminus A} \frac{\text{dist}(z, \partial D)}{|z - \zeta|^2} d\sigma(\zeta), \]
where $d\sigma$ is the Lebesgue measure on $\partial D$. We easily see that the right side of the latter estimate is dominated by
\[ C \cdot \frac{\text{dist}(z, \partial D)}{\text{dist}(z, \partial D \setminus A)^2}. \]
Hence the proof is complete. \hfill \square

Observe that as in Theorem 2.3 and Corollary 2.4, the constant $C$ in Proposition 3.1 depends only on $\sigma(\partial D \setminus A)$, $\text{diam}(D)$ and the radius $r(D)$.

**Proposition 3.2.** Let $D \subset C^n$ be a bounded open set and let $A$ be an open set of $\partial D$ such that $D$ is locally $C^2$ smooth on $A$. Then for any set $K \in A$, there is a finite constant $C = C_K$ such that
\[ \omega(z, A, D) \leq C \cdot \text{dist}(z, K), \quad z \in D \cup A. \]
In particular, $\omega(\cdot, A, D) = 0$ on $A$.

*Proof.* Since $\omega(\cdot, A, D) \leq \omega(\cdot, B, G)$ if $B \subset A$ and $G \subset D$ and by using a compactness argument and applying Proposition 2.5, we may suppose without loss of generality that $K \in A$ is the intersection of a sufficiently small ball centered $U$ at $P$ with $A$ and $D$ is a $C^2$ smooth domain such that Proposition 2.5 is applicable in this context. Namely, keeping the notation in (2.2) and (2.3), we assume without loss of generality that $P = 0 \in C^n$ and $(V_Q)_{Q \in U_1}$ is a family of $C^2$ smooth planar simply connected domains satisfying 1) and 2) of Corollary 2.4.

Observe that it suffices to prove the proposition for the case where $z$ is sufficiently close to $K$. Now let $Q := \pi^C(z)$ and note that $z \in D_Q$. Then under this assumption, Part 4) of Proposition 2.5 gives a finite constant $C'$ such that
\[ \text{dist}(z, K) \leq \text{dist}(z, \partial V_Q) \leq C' \cdot \text{dist}(z, K). \]
Combining Propositions 2.5, 3.1 and the remark made at the end of the proof of Proposition 3.1, we see that there is a finite constant $C''$ such that
\[ \omega(z, A \cap \partial V_Q, V_Q) \leq C'' \cdot \text{dist}(z, K \cap \partial V_Q). \]
Next observe that for any $u \in PSH(D)$ with $u \leq 1$ on $D$ and $\hat{u} \leq 0$ on $A$, the following estimate holds by the very definition
\[ u(z) \leq \omega(z, A \cap \partial V_Q, V_Q). \]
This, combined with (3.1) and (3.2) implies that
\[ \omega(z, A, D) \leq C' C'' \cdot \text{dist}(z, K), \]
which completes the proof of the first desired estimate. The desired identity
\( \omega(\cdot, A, D) = 0 \) on \( A \) follows immediately from this estimate. Hence the proof is
finished. \( \square \)

The next result tells us that the definition of the plurisubharmonic measure formulated in Case II in Subsection 1.1 is well-defined.

**Proposition 3.3.** Let \( D \subset \mathbb{C}^n \) be an open set and let \( A \) be an open subset of \( \partial D \) such that \( D \) is locally \( \mathcal{C}^2 \) smooth on \( A \). Then there is a function plurisubharmonic in \( D \) which we denote by \( \omega(\cdot, A, D) \), with the following property:

Let \( (D_k)_{k=1}^{\infty} \) be a sequence of relatively compact open sets \( D_k \subset \mathbb{C}^n \) and \( (A_k)_{k=1}^{\infty} \) a sequence of open subsets of \( A \) such that

1. \( D_k \subset D_{k+1} \) and \( \cup_{k=1}^{\infty} D_k = D \);
2. \( A_k \subset A_{k+1} \) and \( A_k \subset \partial D \cap \partial D_k \) and \( \cup_{k=1}^{\infty} A_k = A \);
3. for any point \( \zeta \in A \) there is an open neighborhood \( V = V_\zeta \) of \( \zeta \) in \( \mathbb{C}^n \) such that \( V \cap D = V \cap D_k \) for some \( k \).

Then \( \omega(\cdot, A, D) = \lim_{k \to \infty} \omega(\cdot, A_k, D_k) \) on \( D \).

**Proof.** First observe that such sequences \( (D_k)_{k=1}^{\infty} \) and \( (A_k)_{k=1}^{\infty} \) always exist. For example, one may take \( D_k := D \cap B(0, k) \) and \( A_k := A \cap B(0, k) \), \( k \in \mathbb{N} \). Let \( (D_k')_{k=1}^{\infty} \) and \( (A_k')_{k=1}^{\infty} \) be another sequences which verify (i)–(iii). It is easy to see that the following limits of decreasing sequences

\[
\begin{align*}
u := \lim_{k \to \infty} \omega(\cdot, A_k, D_k) \quad \text{and} \quad \nu' := \lim_{k \to \infty} \omega(\cdot, A_k', D_k')
\end{align*}
\]

exist and define two plurisubharmonic functions in \( D \).

Fix an \( k \) and let \( \zeta \) be any point in \( A_k \) and \( K \) be any compact neighborhood of \( \zeta \) in \( A_k \). In virtue of (i)–(iii), there are a sufficiently large integer \( N \) and a bounded open neighborhood \( U \) of \( K \) in \( \mathbb{C}^n \) such that \( U \cap A \subset A_n' \) and \( U \cap D_n' = U \cap D \) for any \( n \geq N \).

Therefore, applying Proposition 3.2 to the open set \( D \cap U \), we may find a finite constant \( C \) such that

\[
\omega(z, A_n', D_n') \leq \omega(z, U \cap A, U \cap D) \leq C \cdot \text{dist}(z, K), \quad z \in U \cap D, \quad n \geq N.
\]

This implies that

\[
\hat{u}'(\zeta) := \limsup_{w \in D, w \to \zeta} u'(w) = 0.
\]

Thus \( \hat{u}' = 0 \) on \( A_k \) and therefore \( \omega(\cdot, A_k, D_k) \geq u' \). This implies that \( u \geq u' \). Similarly, one gets \( u' \geq u \). Hence \( u = u' \) and the proof is finished. \( \square \)

One should mention that in virtue of Proposition 3.3, Proposition 3.2 still holds when \( D \) is an arbitrary (not necessarily bounded) open set. An immediate consequence of Proposition 3.3 is the following result.
Proposition 3.4. Let $D \subset \mathbb{C}^n$ be an open set and let $A$ be an open set of $\partial D$ such that $D$ is locally $C^2$ smooth on $A$. Let $(A_k)_{k=1}^\infty$ be a sequence of open subsets of $\partial D$ such that $A_k ↗ A$ as $k ↗ \infty$. Then
\[
\lim_{k \to \infty} \omega(\cdot, A_k, D) = \omega(\cdot, A, D) \quad \text{on } D.
\]
Proof. It suffices to choose the sequence $(D_k)_{k=1}^\infty$ with $D_k = D$. Then the desired conclusion follows from Proposition 3.3. □

Proposition 3.5. Let $D \subset \mathbb{C}^n$ be an open set and let $A$ be an open set of $\partial D$ such that $D$ is locally $C^2$ smooth on $A$. Then for any $\delta > 0$, there is an open subset $T_\delta$ of $D$ such that
\begin{enumerate}
\item $T_\delta \subset T_{\delta'}$ for $0 < \delta_1 < \delta_2$,
\item $T_\delta \cup A$ is an open neighborhood of $A$ in $A \cup D$, 
\item $\omega(z,A,D) - \delta \leq \omega(z,T_\delta,D) \leq \omega(z,A,D)$ for $z \in D$, and 
\item $\sup_{T_\delta} \text{dist}(\cdot, A) < \delta$.
\end{enumerate}
Proof. Fix a sequence $(A_k)_{k=1}^\infty$ of open subsets of $A$ such that
\begin{enumerate}
\item $A_k \in A_{k+1} \subset A$,
\item $A_k ↗ A$ as $k ↗ \infty$, 
\item $A_k$ consists of finite open connected components.
\end{enumerate}
By Propositions 3.2 and 3.3, for any $k$ there is a finite constant $C_k > 1$ such that
\[
\omega(z,A,D) \leq C_k \text{dist}(z,A_k).
\]
For an $\delta > 0$ consider the following open subset of $D$
\[
T_\delta := \{ z \in D : C_k \text{dist}(z,A_k) < \delta \ \text{for some} \ k \in \mathbb{N} \}
\]
In virtue of (3.3)–(3.4) and (i)–(iii), Part 1) and 2) are proved. Moreover
\[
\omega(z,A,D) \leq \delta \quad \text{and} \quad \text{dist}(z,A) < \delta, \quad z \in T_\delta,
\]
which implies that $\omega(\cdot, A, D) - \delta \leq \omega(\cdot, T_\delta, D)$ on $D$.
On the other hand, by Part 2) and the definition of plurisubharmonic measures, we deduce that $\omega(\cdot, T_\delta, D) \leq \omega(\cdot, A, D)$ on $D$. Hence the proof is complete. □

The rest of this section is devoted to some applications of the previous results.

Proposition 3.6. Let $D \subset \mathbb{C}^n, G \subset \mathbb{C}^m$ be two domains and let $A$ (resp. $B$) be an open set of $\partial D$ (resp. $\partial G$) such that $D$ (resp. $G$) is locally $C^2$ smooth on $A$ (resp. $B$). Put $X := X(A,B;D,G)$ and $\hat{X}^o := \hat{X}^o(A,B;D,G)$. Then
\begin{enumerate}
\item for any finite subset $M \subset \hat{X}^o$, there are open sets $T \subset D, S \subset G$ and $0 < \epsilon < 1$ such that 
\[
M \subset \{ (z,w) \in D \times G : \omega(z,T,D) + \omega(w,S,G) < 1 - \epsilon \} \subset \hat{X}^o;
\]
\item the open set $\hat{X}^o$ is connected;
\item $X \subset \hat{X}$.
Proof. Fix an \( \epsilon > 0 \) such that
\[
\omega(z, A, D) + \omega(w, B, G) < 1 - 2\epsilon, \quad (z, w) \in M.
\]
Applying Proposition 3.5, we find two open sets \( T \subset D, S \subset G \) of the form (3.4) such that
\[
|\omega(z, A, D) - \omega(z, T, D)| < \frac{\epsilon}{2}, \quad z \in D,
\]
\[
|\omega(w, B, G) - \omega(w, S, G)| < \frac{\epsilon}{2}, \quad w \in G.
\]
Therefore,
\[
M \subset \{(z, w) \in D \times G : \omega(z, T, D) + \omega(w, S, G) < 1 - \epsilon\} \subset \hat{X}^o,
\]
which finishes Part 1).

To prove Part 2) let \((z_1, w_1)\) and \((z_2, w_2)\) be two arbitrary points in \(\hat{X}^o\). Put
\[
M := \{(z_1, w_1), (z_2, w_2)\}. \quad \text{By Part 1) there are open sets } T \subset D, S \subset G \text{ and } 0 < \epsilon < 1 \text{ such that}
\]
\[
M \subset \{(z, w) \in D \times G : \omega(z, T, D) + \omega(w, S, G) < 1 - \epsilon\} \subset \hat{X}^o.
\]
Since the set \(\{(z, w) \in D \times G : \omega(z, T, D) + \omega(w, S, G) < 1 - \epsilon\}\) is connected (see, for example, Lemma 4 in [7]), the desired conclusion of Part 2) follows.

Part 3) holds by applying Proposition 3.2 and taking into account the remark made just after Proposition 3.3. Hence the proof is complete. \(\square\)

The next result tells us that the open set \(\hat{X}^o\) is still connected in the following mixed situation.

**Proposition 3.7.** Let \(D \subset \mathbb{C}^n, G \subset \mathbb{C}^m\) be two domains and let \(A \subset D\) and \(B\) is an open subset of \(\partial G\). Assume that \(A\) is locally pluriregular and \(G\) is locally \(C^2\) smooth on \(B\). Let \(X := X(A, B; D, G)\) and \(\hat{X}^o := \hat{X}^o(A, B; D, G)\). Then
1) for any finite subset \(M \subset \hat{X}^o\), there are an open set \(S \subset G\) and a number \(0 < \epsilon < 1\) such that
\[
M \subset \{(z, w) \in D \times G : \omega(z, A, D) + \omega(w, S, G) < 1 - \epsilon\} \subset \hat{X}^o;
\]
2) the open set \(\hat{X}^o\) is connected;
3) \(X \subset \hat{X}\).

**Proof.** We proceed as in the proof of Proposition 3.6 making the obviously necessary changes. Hence, the proof is complete. \(\square\)

The last result of this section studies the level sets of plurisubharmonic measures.

**Proposition 3.8.** Let \(D \subset \mathbb{C}^n\) be an open set and let \(A\) be an open set of \(\partial D\) such that \(D\) is locally \(C^2\) smooth on \(A\). For any \(0 < \epsilon < 1\) let
\[
D_\epsilon := \{z \in D : \omega(z, A, D) < 1 - \epsilon\}.
\]
A BOUNDARY CROSS THEOREM

Then

1) 
\[
\lim_{\epsilon \to 0} \omega(\cdot, A, D_\epsilon) = \omega(\cdot, A, D) \quad \text{on } D \quad \text{and} \quad \omega(\cdot, A, D_\epsilon) = \frac{\omega(\cdot, A, D)}{1 - \epsilon} \quad \text{on } D_\epsilon;
\]

2) if \( z \in D \) verifies \( \omega(z, A, D) < 1 \), then, for any \( 0 < \epsilon < 1 - \omega(z, A, D) \), the connected component of \( D_\epsilon \) which contains \( z \) is locally \( C^2 \) smooth on a nonempty open subset of \( A \);

3) for any \( 0 < \epsilon_0 < 1 \), there is an open neighborhood \( U \) of \( A \) in \( D \cup A \) such that for all \( \epsilon \leq 1 - \epsilon_0 \) there exists exactly one connected component of \( D_\epsilon \) containing \( U \cap D \) and \( \omega(\cdot, A, D_\epsilon) < \epsilon_0 \) on \( U \).

Proof. For \( k \in \mathbb{N} \) let \( A_k := A \). It suffices to check that the sequences \( (D_{\frac{1}{k}})_k \in \mathbb{N} \) and \( (A_k)_{k=1}^\infty \) satisfy the properties (i)-(iii) of Proposition 3.3 for a sufficiently large positive integer \( N \). Observe that the only nontrivial verification is for (iii). But this follows immediately from an application of Proposition 3.2. Hence the first identity of Part 1) is proved. To verify the second one, observe by the very definition that \( \omega(\cdot, A, D_\epsilon) \geq \frac{\omega(\cdot, A, D)}{1 - \epsilon} \) on \( D_\epsilon \).

On the other hand, consider the function

\[
u(z) := \begin{cases} 
\max \{ \omega(z, A, D), (1 - \epsilon)\omega(z, A, D_\epsilon) \}, & z \in D_\epsilon, \\
\omega(z, A, D), & z \in D \setminus D_\epsilon.
\end{cases}
\]

Clearly, \( u \in \mathcal{PSH}(D) \) and \( u \leq 1 \) on \( D \). By Proposition 3.2, \( \hat{\nu} = 0 \) on \( A \). Thus \( u \leq \omega(\cdot, A, D) \) which completes the proof of the second identity of Part 1).

By Part 1), \( \omega(z, A, D_\epsilon) < 1 \) for all \( 0 < \epsilon < 1 - \omega(z, A, D) \). This proves Part 2).

For Part 3) it suffices to show that for every point \( \zeta \in A \) there is a neighborhood \( U \) of \( \zeta \) in \( D \cup A \) which possesses the required properties. By Proposition 3.2 one may find an open neighborhood \( U \) of \( \zeta \) such that for some finite constant \( C_1 \),

\[
\omega(\cdot, A, D) \leq C_1 \cdot \text{dist}(z, U \cap A) < \epsilon_0 \quad \text{on } U \cap D.
\]

This shows that \( U \cap D \subset D_{1-\epsilon_0} \). We now choose a relatively compact neighborhood \( U \) of \( \zeta \) such that \( U \Subset U \). Then applying Proposition 3.2 and shrinking \( U \), if necessary, we also have

\[
\omega(\cdot, A, D_{1-\epsilon_0}) \leq C_2 \cdot \text{dist}(z, A) < \epsilon_0 \quad \text{on } U \cap D,
\]

which completes the last part of the proposition. \( \square \)

4. A Mixed Cross Theorem and Two Quantitative Cross Theorems

The main result of this section is the following mixed cross theorem.

Theorem 4.1. Let \( D \subset \mathbb{C}^n \) be a bounded pseudoconvex domain, \( G \subset \mathbb{C}^m \) a domain, \( A \subset D \), and \( B \subset \partial G \). Assume that \( A \) is a locally pluriregular relatively compact subset of \( D \) and \( A = \bigcup_{k=1}^\infty A_k \) with \( A_k \) locally pluriregular compact subsets of \( D \) and that \( B \) is an open subset of \( \partial G \) such that \( G \) is locally \( C^2 \) smooth.
on $B$. Let $X := \mathbb{X}(A, B; D, G)$, $X^o := \mathbb{X}^o(A, B; D, G)$, $\hat{X} := \mathbb{X}(A, B; D, G)$, and 
$\hat{X}^o := \mathbb{X}^o(A, B; D, G)$.

Let $\mathcal{C}_s(X)$ be the space of all functions defined on $X$ such that

(i) $f$ is locally bounded on $X$;

(ii) for any $z \in A$, $f(z, \cdot) \in \mathcal{C}(G \cup B)$.

1) Then for any function $f \in \mathcal{C}_s(X) \cap \mathcal{O}_s(X^o)$ there is a unique function $\hat{f} \in \mathcal{C}(\hat{X}) \cap \mathcal{O}(\hat{X}^o)$ such that $\hat{f} = f$ on $X$.

2) If, moreover, there is a set $B' \subset \partial G$ such that

(i) $f$ is locally bounded on $A \times (G \cup B')$,

(ii) for any $z \in A$, $f(z, \cdot) \in \mathcal{C}(G \cup B')$,

(iii) $\omega(\cdot, B, G) \in \mathcal{C}(G \cup B')$,

then $\hat{f}$ extends continuously to every point $(z, \eta) \in \overline{\hat{X}} \cap (D \times B')$.

A remark is in order. Under the hypothesis of Theorem 4.1, it follows from Part 3) of Proposition 3.7 that $X \subset \hat{X}$.

Proof. First we prove Part 1). We argue as in the proof of Theorem 3.5.1 in [6]. For the sake of completeness, we give here a sketchy proof. Fix an $f \in \mathcal{C}_s(X) \cap \mathcal{O}_s(X^o)$.

Step I: Reduction to the case where $D$ is strongly pseudoconvex, $A$ is a locally pluriregular compact subset of $D$ and $|f|$ is bounded on $X$.

One proceeds as in the first and second step in that proof. More precisely, let $(G_k)_{k=1}^\infty$ be an exhausting sequence of $G$ which verifies the properties (i)–(iii) in Proposition 3.3 (with $B$ instead of $A$). Let $B_k := B \cap \partial G_k$. Since $D$ is a domain of holomorphy, we may find an exhausting sequence $(D_k)_{k=1}^\infty$ of relatively compact, strongly pseudoconvex subdomains of $D$ with $A_k \subset D_k \nearrow D$.

By reduction assumption, for each $k$ there exists an $\hat{f}_k \in \mathcal{C}\left(\hat{X}(A_k, B_k; D_k, G_k)\right) \cap \mathcal{O}\left(\hat{X}^o(A_k, B_k; D_k, G_k)\right)$ such that $\hat{f}_k = f$ on $X(A_k, B_k; D_k, G_k)$.

By Theorem 2.1 and Proposition 3.7 and taking into account that $f_{k+1} = f_k = f$ on $X(A_k, B_k; D_k, G_k)$, one can show that $f_{k+1} = f_k = f$ on $\hat{X}(A_k, B_k; D_k, G_k)$. On the other hand, by Proposition 3.3, one gets $\hat{X}(A_k, B_k; D_k, G_k) \nearrow \hat{X}$ as $k \nearrow \infty$. Therefore, we may glue $f_k$ together to obtain a function $\hat{f} \in \mathcal{C}(\hat{X}) \cap \mathcal{O}(\hat{X}^o)$ such that $\hat{f} = f_k = f$ on $X(A_k, B_k; D_k, G_k)$. Thus $\hat{f} = f$ on $X$. The uniqueness of such an extension $\hat{f}$ follows from Theorem 2.1 and Proposition 3.7. This completes Step I.

Step II: The case where $D$ is strongly pseudoconvex, $A$ is a locally pluriregular compact subset of $D$ and $|f| \leq 1$ on $X$.

The key observation is that we are still able to apply the classical method of doubly orthogonal bases of Bergman type (see for example [12], [13] for a systematic study of this method).

Next one observes that Lemma 3.5.10 in [6] is still valid in the present context. Look at Step 3 in that proof. In the sequel, we will use the notations from [6].
Let $\mu := \mu_{A,D}$, $H_0 := L^2_h(D)$, $H_1 :=$ the closure of $H_0|_A$ in $L^2(A, \mu)$ and let $(b_k)_{k=1}^\infty$ be the basis from Lemma 3.5.10 in [6], $\nu_k := \|b_k\|_{H_0}$, $k \in \mathbb{N}$, and $\nu_k \nearrow \infty$.

For any $w \in B$, we have $f(\cdot, w) \in H_0$ and $f(\cdot, w)|_A \in H_1$. Hence
\[
(4.1) \quad f(\cdot, w) = \sum_{k=1}^\infty c_k(w) b_k,
\]
where
\[
(4.2) \quad c_k(w) = \frac{1}{\nu_k^2} \int_D f(z, w) \overline{b_k(z)} d\Lambda_{2n}(z) = \int_A f(z, w) \overline{b_k(z)} d\mu(z), \quad k \in \mathbb{N}.
\]

Taking the hypothesis $|f| \leq 1$ on $X$ and $f \in \mathcal{C}_s(X) \cap \mathcal{O}_s(X)$ into account and applying the Lebesgue’s Dominated Convergence Theorem, we see that the formula
\[
(4.3) \quad \hat{c}_k(w) := \int_A f(z, w) \overline{b_k(z)} d\mu(z), \quad w \in G \cup B, \quad k \in \mathbb{N};
\]
defines a bounded function which is holomorphic in $G$. Moreover, by (4.2)–(4.3) it follows that
\[
(4.4) \quad \lim_{w \in G, w \to \eta} \hat{c}_k(w) = \hat{c}_k(\eta) = c_k(\eta), \quad \eta \in B.
\]
Thus $\hat{c}_k \in \mathcal{C}(G \cup B) \cap \mathcal{O}(G)$.

Observe that as in [6] and using (4.2)–(4.4), we obtain the following estimates
\[
\limsup_{w \in G, w \to \eta} \frac{\log |\hat{c}_k(\eta)|}{\log \nu_k} \leq \frac{\log \sqrt{\mu(A)}}{\log \nu_k} - 1, \quad \eta \in B, \quad k \in \mathbb{N}.
\]
This shows that for any $\epsilon > 0$, there is a sufficiently large $N$ such that for all $k \geq N$,
\[
(4.5) \quad \frac{\log |\hat{c}_k|}{\log \nu_k} \leq \omega(\cdot, B, G) + \epsilon - 1 \quad \text{on } G.
\]

Take a compact $K \subset D$ and let $\alpha > \max_K h^*_A$ and $\epsilon > 0$ so small such that $\alpha + 2\epsilon < 1$. Consider the open set
\[
G_K := \{ w \in G : \omega(\cdot, B, G) < 1 - \alpha - 2\epsilon \}.
\]
By (4.5) there is a constant $C'(K)$ such that
\[
(4.6) \quad \|\hat{c}_k\| \leq C'(K) \nu_k^{\omega(\cdot, B, G) + \epsilon - 1} \leq C'(K) \nu_k^{-\alpha - \epsilon}, \quad k \geq 1.
\]
Now we wish to show that
\[
(4.7) \quad \sum_{k=1}^\infty \hat{c}_k(w) b_k(z)
\]
converges locally uniformly in $\hat{X}^\circ$. Indeed, by (4.6) and Lemma 3.5.10 in [6],
\[
\sum_{k=1}^{\infty} \|c_k\|_{G_K} \|b_k\|_K \leq \sum_{k=1}^{\infty} C'(K)\nu_k^{-\alpha-\epsilon}C(K,\alpha)\nu_k^\circ \leq C'(K)C(K,\alpha)\sum_{k=1}^{\infty} \nu_k^{-\epsilon} < \infty,
\]
which gives the normal convergence on $K \times G_K$. Moreover, by Proposition 3.2 one gets $B \subset \partial G_K$. Therefore, the previous argument also shows that the series in (4.7) converges normally on $K \times (G_K \cup B)$. Since the compact set $K \subset D$ and $\epsilon > 0$ are arbitrary, the series in (4.7) converges uniformly on compact subsets of $\hat{X}$. Let $\hat{f}$ be the sum limit. Then obviously $\hat{f} \in \mathcal{C}(\hat{X}) \cap \mathcal{O}(\hat{X}^\circ)$. Taking (4.1), (4.4) and (4.7) into account, it follows that
\[
\hat{f} = f \quad \text{on } D \times B.
\]
Consequently, an application of Theorem 2.1 gives that $\hat{f} = f$ on $X$. This completes the proof of Part 1).

We now turn to Part 2) using the proof of Part 1). Observe that by (4.3) and (i')–(ii'), $\hat{c}_k \in \mathcal{C}(G \cup B') \cap \mathcal{O}(G)$. Next fix an $\eta \in B'$ and $z \in D$. We use hypothesis (iii') in order to choose $\epsilon > 0$ and a compact neighborhood $K$ of $z$ such that $K \times (G_K \cup \{\eta\})$ is a neighborhood of $(z,\eta)$ in $\overline{X} \cap (D \times B')$. The rest of the proof follows essentially along the same lines as that of Part 1). This completes the proof of Part 2). \(\square\)

The last two results of this section give quantitative versions of the classical cross theorem (cf. Theorem 1).

**Theorem 4.2.** Let $D \subset \mathbb{C}^n$, $G \subset \mathbb{C}^m$ be bounded domains and let $A \subset D$, $B \subset G$ be locally pluriregular sets. Assume that $D$ is pseudoconvex and $A$ is of the form $A = \bigcup_{k=1}^{\infty} A_k$ with $A_k$ compact subset of $D$. Let $X := \mathbb{H}(A,B;D,G)$ and $\hat{X} := \mathbb{H}(A,B;D,G)$. Then for any $f \in \mathcal{O}_s(X)$ there is a unique function $\hat{f} \in \mathcal{O}(\hat{X})$ such that $\hat{f} = f$ on $X$. Moreover, if $|f|_X < \infty$ then
\[
(4.8) \quad |\hat{f}(z,w)| \leq |f|_{A \times B}^{1-\omega(z,A,D)-\omega(w,B,G)}|f|_{X}^{\omega(z,A,D)+\omega(w,B,G)}, \quad (z,w) \in \hat{X}.
\]

**Proof.** We proceed in two steps.

**Step 1:** Proof of the equality $|\hat{f}|_{\hat{X}} = |f|_X$.

In order to reach a contradiction suppose that there is a point $(z_0,w_0) \in \hat{X}$ such that $|\hat{f}(z_0,w_0)| > |f|_X$. Put $\alpha := \hat{f}(z_0,w_0)$ and consider the function
\[
(4.9) \quad g(z,w) := \frac{1}{f(z,w) - \alpha}, \quad (z,w) \in X.
\]
Clearly, $g \in \mathcal{O}_s(X)$. Hence by Theorem 3.5.1 in [6], there is exactly one function $\hat{g} \in \mathcal{O}(\hat{X})$ with $\hat{g} = g$ on $X$. Therefore, by (4.9) we have on $X : g(f - \alpha) \equiv 1$. Thus $\hat{g}(\hat{f} - \alpha) \equiv 1$ on $\hat{X}$. In particular,
\[
0 = \hat{g}(z_0,w_0)(\hat{f}(z_0,w_0) - \alpha) = 1,
\]
which is a contradiction. Hence the inequality $|\hat{f}|_{\hat{X}} \leq |f|_X$ is proved. The converse inequality is trivial since $X \subset \hat{X}$ (see, for example, [6]). Thus Step 1 is complete.
Step 2: Proof of inequality (4.8).

Fix now \((z_0, w_0) \in \widehat{X}\). For every \(\eta \in B\), we have
\[
|f(\zeta, \eta)| \leq |f|_{A \times B}, \quad \zeta \in A \quad \text{and} \quad |f(z, \eta)| \leq |f|_X, \quad z \in D.
\]
Therefore, Two-Constant Theorem (Theorem 2.2) implies that
\[
|f(z, \eta)| \leq |f|_{A \times B}^{1-\omega(z, A, D)} |f|_X^{\omega(z, A, D)}, \quad z \in D, \quad \eta \in B.
\]
Consider the function \(\hat{f}(z_0, \cdot) \in \mathcal{O}(G_{z_0})\), where
\[
G_{z_0} := \{w \in G : \omega(w, B, G) < 1 - \omega(z_0, A, D)\}.
\]
Observe that \(|\hat{f}(z_0, \cdot)|_{G_{z_0}} \leq |f|_X\) and \(\omega(w, B, G_{z_0}) = \frac{\omega(w, B, G)}{1-\omega(z_0, A, D)}\). Consequently, using (4.10) and applying the Two-Constant Theorem to the function \(\hat{f}(z_0, \cdot)\), (4.8) for \((z_0, w_0)\) follows. Hence the theorem is proved. \(\square\)

Theorem 4.3. Let \(D \subset \mathbb{C}^n\), \(G \subset \mathbb{C}^m\) be domains and let \(A\) (resp. \(B\)) be an open subset of the boundary \(\partial D\) (resp. \(\partial G\)). Suppose in addition that \(D\) (resp. \(G\)) is locally \(C^2\) smooth on \(A\) (resp. \(B\)) and \(D\) is pseudoconvex. Put \(X := \widehat{X}(A; B; D, G)\), \(\widehat{X} := \widehat{X}(A, B; D, G)\) and \(\widehat{X}^o := \widehat{X}(A, B; D, G)\). Then for any function \(f \in \mathcal{C}(\widehat{X}) \cap \mathcal{O}(\widehat{X}^o)\) the following inequality holds
\[
|f(z, w)| \leq |f|_{A \times B}^{1-\omega(z, A, D)-\omega(w, B, G)} |f|_X^{\omega(z, A, D)+\omega(w, B, G)}, \quad (z, w) \in \widehat{X}.
\]

Proof. Fix a point \((z_0, w_0) \in \widehat{X}\), an arbitrary number \(\epsilon > 0\) and let \(\delta > 0\).

By Proposition 3.5 one may find an open set \(T_\delta \subset D\) such that
\[
\omega(z, A, D) - \delta \leq \omega(z, T_\delta, D) \leq \omega(z, A, D), \quad z \in D.
\]
By Proposition 3.3 one may find a (not necessarily pseudoconvex) bounded subdomain \(G_\delta\) of \(G\) such that \(\overline{G_\delta} \subset G \cup B\), \(G_\delta\) is locally \(C^2\) smooth on the open subset \(\partial G_\delta \cap B\) and
\[
0 \leq \omega(w_0, \partial G_\delta \cap B, G_\delta) - \omega(w_0, B, G) < \delta.
\]
Since \(f \in \mathcal{C}(\widehat{X})\), there is an open subset \(A_\delta\) of \(T_\delta\) such that \(A \cup A_\delta\) is an open neighborhood of \(A\) in \(A \cup D\) and moreover
\[
|f(z, w)| \leq |f|_X + \epsilon, \quad z \in A_\delta, \quad w \in G_\delta.
\]
It is also clear from (4.12) and the above properties of \(A_\delta\) that
\[
\omega(z, A_\delta, D) - \delta \leq \omega(z, T_\delta, D) \leq \omega(z, A_\delta, D), \quad z \in D.
\]
Let \(D_\delta\) be a strongly pseudoconvex subdomain of \(D\) such that \(D_\delta \subset D\) and
\[
0 \leq \omega(z_0, A_\delta \cap D_\delta, D_\delta) - \omega(z_0, A_\delta, D) < \delta.
\]
Since \(G_\delta\) is locally \(C^2\) smooth on the open subset \(\partial G_\delta \cap B\) of \(B\) and \(f \in \mathcal{C}(\widehat{X})\), one may find an open subset \(B_\delta\) of \(G_\delta\) such that \(B \cup B_\delta\) is an open neighborhood of \(\partial G_\delta \cap B\) in \((\partial G_\delta \cap B) \cup G_\delta\) and moreover
\[
|f(z, w)| \leq |f|_X + \epsilon, \quad z \in D_\delta, \quad w \in B_\delta.
\]
By taking the intersection of $B_\delta$ with the level open set given by Proposition 3.5 with respect to the open set $G_\delta$, one may assume that

\[(4.18) \quad \omega(w, \partial G_\delta \cap B, G_\delta) - \delta \leq \omega(w, B_\delta, G_\delta) \leq \omega(w, \partial G_\delta \cap B, G_\delta), \quad w \in G_\delta.\]

Consider following crosses

\[X_\delta := \mathbb{X}(A_\delta \cap D_\delta, B_\delta; D_\delta, G_\delta) \quad \text{and} \quad \hat{X}_\delta := \mathbb{X}(A_\delta \cap D_\delta, B_\delta; D_\delta, G_\delta).\]

By Theorem 1, there is a function $f_\delta \in \mathcal{O}(\hat{X}_\delta)$ such that $f_\delta = f$ on $X_\delta$.

If one chooses $\delta$ such that $0 < 10\delta < 1 - \omega(z_0, A, D) - \omega(w_0, B, G)$, then it follows from (4.12), (4.13), (4.15), (4.16) and (4.18) that

\[(z_0, w_0) \in \{(z, w) \in D_\delta \times H_\delta : \omega(z, A_\delta \cap D_\delta, D_\delta) + \omega(w, B_\delta, H_\delta) < 1 - 5\delta\} \subset \hat{X}_\delta,\]

where $H_\delta$ is the connected component of $G_\delta$ containing $w_0$.

In addition we recall that $f_\delta = f$ on $X_\delta$. Therefore, $f(z_0, w_0) = f_\delta(z_0, w_0)$. Consequently, applying Theorem 4.2 and taking (4.14) and (4.17) into account, we deduce that

\[|f(z_0, w_0)| \leq |f|_X + \epsilon.\]

Since $\epsilon > 0$ and $(z_0, w_0) \in \hat{X}$ are arbitrary, it follows that $|f|_\hat{X} \leq |f|$. The converse inequality is trivial as $X \subset \hat{X}$ by Part 3) of Proposition 3.6. Thus we have shown that $|f|_\hat{X} = |f|$.

Therefore, arguing as in Step 2 of Theorem 4.2 and applying the second identity of Part 1) of Proposition 3.8, inequality (4.11) follows. \qed

5. Proof of the Main theorem for $N = 2$

In this section we simplify the notation and rephrase the Main Theorem for the case $N = 2$ as follows.

**Theorem 5.1.** Let $D \subset \mathbb{C}^n$, $G \subset \mathbb{C}^m$ be pseudoconvex domains and let $A$ (resp. $B$) be an open subset of the boundary $\partial D$ (resp. $\partial G$). Suppose in addition that $D$ (resp. $G$) is locally $\mathcal{C}^2$ smooth on $A$ (resp. $B$). Put $X := \mathbb{X}(A, B; D, G)$, $X^o := \mathbb{X}^o(A, B; D, G)$, $\hat{X} := \mathbb{X}(A, B; D, G)$ and $\hat{X}^o := \mathbb{X}^o(A, B; D, G)$. Then for any function $f \in \mathcal{C}(X) \cap \mathcal{O}_n(X^o)$, there is a unique function $\hat{f} \in \mathcal{C}(\hat{X}) \cap \mathcal{O}(\hat{X}^o)$ such that $\hat{f} = f$ on $X$. Moreover,

\[(5.1) \quad |\hat{f}(z, w)| \leq |f|_{A \times B}^{1-\omega(z, A, D)-\omega(w, B, G)} |f|_X^{\omega(z, A, D)+\omega(w, B, G)}, \quad (z, w) \in \hat{X}.\]

**Proof.** We proceed by several steps. First observe that by Theorem 2.1 and Part 3) of Proposition 3.6, the function $\hat{f}$ is uniquely determined (if exists).

**Step 1: Reduction to the case where $D$ and $G$ are bounded pseudoconvex domains**

**Proof of Step 1.** Fix any sequences of bounded pseudoconvex subdomains $(D_k)_{k=1}^\infty$ (resp. $(G_k)_{k=1}^\infty$) of $D$ (resp. $G$) such that the sequences $(D_k)_{k=1}^\infty$ and $(A_k)_{k=1}^\infty$ (resp. $(G_k)_{k=1}^\infty$ and $(B_k)_{k=1}^\infty$) satisfy (i)--(iii) of Proposition 3.3, where $A_k := A \cap \partial D_k$ and $G_k := B \cap \partial G_k$. Let

\[X_k := \mathbb{X}(A_k, B_k; D_k, G_k) \subset X\]

and note that $\hat{X}_k \supset \hat{X}$ by Propositions 3.3.

Let $f \in \mathcal{C}(X) \cap \mathcal{O}_n(X^o)$ be given. Clearly, $f \in \mathcal{C}(X_k)$. Therefore, by the reduction assumption, for each $k$ there exists an $\hat{f}_k \in \mathcal{C}(\hat{X}_k) \cap \mathcal{O}(\hat{X}_k^o)$ with $\hat{f}_k = f$ on $X_k$. By
Theorem 2.1 and Proposition 3.6, \( \hat{f}_{k+1} = \hat{f}_k \) on \( \hat{X}_k \). Therefore, gluing the \( \hat{f}_k \)'s, we obtain an \( \hat{f} \in C(\hat{X}) \cap O(\hat{X}^o) \) with \( \hat{f} = f \) on \( X \). To reduce estimate (5.1) to the case where \( D \) and \( G \) are bounded pseudoconvex domains, we proceed in the same way as above. This completes Step 1. \( \square \)

From now on we assume that the hypothesis of Step 1 is fulfilled.

We introduce a new terminology. A subset \( A \) of an open subset \( A \) of \( \partial D \) is said to be a ball in \( A \) with center \( \zeta \) and radius \( r \) if \( A = B(\zeta, r) \cap A \) for a point \( \zeta \in A \) and a positive number \( r \) verifying \( 2r < \text{dist}(\zeta, \partial A) \). Moreover, for a ball \( A \) in \( A \) and a number \( 0 < \lambda \leq 2 \), \( \lambda A \) denotes the open set \( B(\zeta, \lambda r) \cap \partial D \).

**Step 2:** We keep the hypothesis of Theorem 5.1 and assume in addition that \( G \) is a Jordan planar simply connected domain. Then we shall prove the following local version of Theorem 5.1:

For any point \( P \in A \), there is a ball \( A \) in \( A \) with center \( P \) such that the following property holds: For any function \( f \in C(X) \cap O_s(X^o) \), there is a unique function \( \hat{f} \in C(\hat{X}_A) \cap O(\hat{X}^o_A) \) such that \( \hat{f} = f \) on \( X_A \), where \( X_A := X(A, B; D, G) \), \( \hat{X}_A := \hat{X}(A, B; D, G) \) and \( \hat{X}^o_A := \hat{X}^o(A, B; D, G) \).

**Proof of Step 2.** First, we apply Proposition 2.5 to the domain \( D \) which is locally \( C^2 \) smooth on an open neighborhood of \( P \) in \( \partial D \). Consequently, we may find an open neighborhood \( U \) of \( P \) satisfying (2.2) such that Proposition 2.5 is applicable there.

In the sequel the notation \( U, U_1, U_3, \pi^C, \pi, V \) and \( V_Q \) have the same meanings as in Proposition 2.5. Now we can fix a ball \( A \) of \( A \)

\[
(5.2) \quad A := A \cap B(P, r),
\]

where the radius \( r \) is sufficiently small such that \( 2A \subset A \), \( 2A \subset (U \cap D) \cap \partial V \), \( 2A \subset U_3 \) and \( \pi^C(2A) \subset U_1 \).

For any \( \delta \) small enough, by Proposition 3.5 we may find an open subset \( T_\delta \) of \( U \) such that

\[
\omega(z, A, D) - \delta \leq \omega(z, T_\delta, D) \leq \omega(z, A, D), \quad z \in D,
\]

\[
(5.3) \quad \sup_{T_\delta} \text{dist}(\cdot, A) < \delta \quad \text{and} \quad \pi^C(T_\delta) \subset U_1.
\]

A geometric argument based on Proposition 2.5 and definition (5.2) shows that one may find \( \delta_0 > 0 \) small enough such that for any \( z \in D \cup A \) with \( \text{dist}(z, 2A) < \delta_0 \), \( z \in U_3 \) and there is a unique \( Q_z \in U_1 \) such that \( z \in V_{Q_z} \). In addition by Part 4) of Proposition 2.5 we have

\[
(5.4) \quad \text{dist} \left( z, \partial V_{Q_z} \cap \frac{3}{2} A \right) \leq C_1 \cdot \text{dist}(z, A)
\]

for any \( z \in D \cup A \) with \( \text{dist}(z, A) < \delta_0 \).

On the other hand, combining Corollary 2.4 and Propositions 2.5 and 3.1, we get the following estimate

\[
(5.5) \quad \omega(z, \partial V_{Q_z} \cap 2A, V_{Q_z}) \leq C_2 \cdot \text{dist} \left( z, \partial V_{Q_z} \cap \frac{3}{2} A \right),
\]
where $C_1$, $C_2$ are finite constants independent of $z \in D \cup A$ with $\text{dist}(z, A) < \delta_0$.

For each $Q \in \pi c(2, A)$, we apply Gouchar’s Theorem (Theorem 2) to the function $f \in C(\mathbb{X}(\partial V_Q \cap 2A, B; V_Q, G)) \cap \mathcal{O}_s(\mathbb{X}^o(\partial V_Q \cap 2A, B; V_Q, G))$ in order to obtain an extension function $\tilde{f}_Q \in C\left(\mathbb{X}(\partial V_Q \cap 2A, B; V_Q, G)\right) \cap \mathcal{O}\left(\mathbb{X}^o(\partial V_Q \cap 2A, B; V_Q, G)\right)$ such that $\tilde{f}_Q = f$ on $\mathbb{X}(\partial V_Q \cap 2A, B; V_Q, G)$.

Collecting the family $(\tilde{f}_Q)_{Q \in \pi c(2, A)}$, we obtain an extension function $\tilde{f}$ defined on the following set

$$\tilde{X}_A := \{(z, w) \in (D \cup 2A) \times (G \cup B) : \exists Q \in \pi c(2, A), \exists \omega(z, \partial V_Q \cap 2A, V_Q) + \omega(w, B, G) < 1\},$$

which is not necessarily open; moreover

$$\tilde{f} = f \quad \text{on} \quad \mathbb{X}(A, B; T_0, G).$$

In virtue of (5.3)–(5.5) we obtain an $\delta_0$ small enough such that for $0 < \delta < \delta_0$,

$$\omega(z, \partial V_Q \cap 2A, V_Q) \leq C_1 C_2 \cdot \text{dist}(z, A) < C_1 C_2 \delta < 1, \quad z \in T_\delta.$$  

Therefore, by (5.6), (5.8) and Theorem 2 for $0 < \delta < \delta_1 := \min \left\{ \delta_0, \frac{1}{2C_1 C_2} \right\}$ and $z \in T_\delta$, $\tilde{f}(z, \cdot)$ is holomorphic on the open set

$$G_\delta := \{w \in G : \omega(w, B, G) < 1 - 2C_1 C_2 \delta\}.$$

We need the following

**Lemma 5.2.** For any $(\zeta_0, w_0) \in \overline{A} \times (G \cup B)$, there are an open neighborhood $U$ of $\zeta_0$ in $D \cup A$ and an open neighborhood $V$ of $w_0$ in $G \cup B$ such that $U \times V \subset \tilde{X}_A$ and $|\tilde{f}|_{U \times V} < \infty$.

**Proof of Lemma 5.2.** Fix a point $(\zeta_0, w_0) \in \overline{A} \times (G \cup B)$. Let

$$\epsilon := \frac{1 - \omega(w_0, B, G)}{3}$$

and choose an open neighborhood $V$ of $w_0$ in $G \cup B$ such that

$$\omega(w, B, G) < \omega(w_0, B, G) + \epsilon, \quad w \in V.$$

Moreover, choose a sufficiently small open neighborhood $U$ of $\zeta_0$ in $D \cup A$ such that by (5.4) and (5.5) we have

$$\omega(z, \partial V_Q \cap 2A, V_Q) < \epsilon, \quad z \in U.$$

Next, by Proposition 3.3, we may find a subdomain $G^\epsilon$ of $G$ such that $w_0 \in G^\epsilon$, $\overline{G^\epsilon} \subset G \cup B$ and $G^\epsilon$ is locally $C^2$ smooth on the open subset $B_\epsilon := \partial G \cap \partial G^\epsilon$ of $B$, $B_\epsilon \Subset B$ and

$$\omega(w_0, B, G) \leq \omega(w_0, B_\epsilon, G^\epsilon) < \omega(w_0, B, G) + \epsilon.$$

By shrinking $V$ if necessary, we may assume that

$$\omega(w, B_\epsilon, G^\epsilon) < \omega(w_0, B_\epsilon, G^\epsilon) + \epsilon, \quad w \in V.$$
Since $f \in C(X)$ and $2A \Subset A$, by shrinking $U$, if necessary, we may find a finite constant $M$ such that
\begin{equation}
|\tilde{f}|_{2A \times G^2} < M \quad \text{and} \quad |\tilde{f}|_{U \times B_\epsilon} < M.
\end{equation}
Consequently, for each $Q \in \pi C(2A)$ we apply Gonchar's Theorem (Theorem 2) to the function $f \in C(X(\partial V_Q \cap 2A, B_\epsilon; V_Q, G^2)) \cap O_s(X(\partial V_Q \cap 2A, B_\epsilon; V_Q, G^2))$ and obtain the inequality $|\tilde{f}| < M$ on the following set
\[
\tilde{X}_A := \{(z, w) \in (D \cup 2A) \times (G^2 \cup B_\epsilon) : \exists Q \in \pi C(2A), z \in V_Q \text{ and } \omega(z, \partial V_{Q_z}, 2A, V_{Q_z}) + \omega(w, B_\epsilon, G^2) < 1\},
\]
On the other hand, using (5.6) and (5.11)–(5.14), we see that
\begin{equation}
\tilde{X}_A := \{(z, w) \in (D \cup 2A) \times (G^2 \cup B_\epsilon) : \exists Q \in \pi C(2A), z \in V_Q \text{ and } \omega(z, \partial V_{Q_z}, 2A, V_{Q_z}) + \omega(w, B_\epsilon, G^2) < 1\},
\end{equation}
\[
\text{Hence } |\tilde{f}|_{U \times \mathcal{Y}} < M, \text{ which completes the proof of the lemma.} \quad \Box
\]
Now for any $0 < \delta < \delta_1$, we are able to apply Theorem 4.1 to the function
\[
\tilde{f} \in C_s(X(T_\delta, B; D, G_\delta)) \cap O_s(X(\partial V_{Q_z}, 2A, B_\epsilon; V_{Q_z}, G^2))
\]
and obtain a function $\hat{f}_\delta \in C(\hat{X}(T_\delta, B; D, G_\delta)) \cap O(\hat{X}(T_\delta, B; D, G_\delta))$ such that
\begin{equation}
\hat{f}_\delta = \tilde{f} \quad \text{on } X(T_\delta, B; D, G_\delta).
\end{equation}
We are now in a position to define the desired extension function $\hat{f}$. Indeed, one glues $\left(\hat{f}_\delta\right)_{0 < \delta < \delta_1}$ together to obtain $\hat{f}$ in the following way
\begin{equation}
\hat{f} := \lim_{\delta \to 0} \hat{f}_\delta \quad \text{on } \hat{X}(A, B; D, G) \setminus (A \times (G \cup B)).
\end{equation}
One now checks that the limit (5.17) exists and possesses all the required properties. This is an immediate consequence of the following

**Lemma 5.3.** For any point $(z, w) \in \hat{X}(A, B; D, G) \setminus (A \times (G \cup B))$ let $\delta_{z,w}$ be the unique positive number $\delta$ which verifies
\begin{equation}
\delta + \frac{2C_1C_2\delta}{1 - \delta} = 1 - \omega(z, A, D) - \omega(w, B, G).
\end{equation}
Then $f(z, w) = \hat{f}(z, w)$ for all $0 < \delta < \delta_{z,w}$.

**Proof of Lemma 5.3.** Fix a point $(z_0, w_0) \in \hat{X}(A, B; D, G) \setminus (A \times (G \cup B))$. Then by (5.3), (5.18) and the second identity of Part 1) of Proposition 3.8, for all $0 < \delta < \delta_{z_0, w_0}$,
\[
(z_0, w_0) \in \hat{X}(T_\delta, B; D, G_\delta).
\]
Therefore, for any $0 < \delta' < \delta < \delta_{z_0, w_0}$, $(z_0, w_0)$ is contained in the following set
\begin{equation}
\hat{X}(T_{\delta'}, B; D, G_{\delta'}) \cap \hat{X}(T_\delta, B; D, G_\delta).
\end{equation}
Let $H_\delta$ be the connected component of $G_\delta$ containing $w_0$. Let $B_\delta$ be the largest open subset of $B$ such that $H_\delta$ is locally $C^2$ smooth on $B_\delta$. By Part 2) of Proposition 3.8,
Applying Theorem 2.1, we deduce that \( \hat{\omega}(\cdot, w, B_\delta, H_\delta) = \omega(\cdot, B, G_\delta) \), \( w \in H_\delta \). On the other hand, using (5.3) and the inclusion \( G_\delta \subset G'_\delta \), we see that the set (5.19) contains the set
\[
\{(z, w) \in D \times H_\delta : \omega(z, T_\delta, D) + \omega(w, B_\delta, H_\delta) < 1 - \delta\}.
\]
By Proposition 3.7 this open set is connected. Moreover, it contains the point \((z_0, w_0)\). In addition by (5.7) and (5.16), one gets
\[
\hat{f}_\delta = \hat{f} = f \quad \text{on } T_\delta \times B_\delta.
\]
Applying Theorem 2.1, we deduce that \( \hat{f}_\delta = \hat{f} \) on the domain given by (5.20). In particular \( \hat{f}_\delta(z_0, w_0) = \hat{f}(z_0, w_0) \). This completes the proof. \( \square \)

Another consequence of Lemma 5.3 is that \( \hat{f} \in \mathcal{O}\left(\hat{\mathbb{X}}(\mathcal{X}(\mathcal{A}, B; D, G))\right) \). Now we define \( \hat{f} \) on \( \mathcal{A} \times (G \cup B) \) as follows
\[
(5.21) \quad \hat{f} := f \quad \text{on } \mathcal{A} \times (G \cup B).
\]
Thus \( \hat{f} \) is well-defined on the whole \( \hat{\mathbb{X}}(\mathcal{X}(\mathcal{A}, B; D, G)) \).

To complete Step 2, it remains to show that \( f \in \mathcal{C}(\mathbb{X}(\mathcal{A}, B; D, G)) \) and \( \hat{f} = f \) on \( \mathbb{X}(\mathcal{A}, B; D, G) \).

First we prove that \( \hat{f} \) is continuous on \( D \times B \). For this let \((z_0, \eta_0) \in D \times B\). By Proposition 3.2 there are an open neighborhood \( \mathcal{U} \) of \( z_0 \) in \( D \) and an open neighborhood \( \mathcal{V} \) of \( \zeta_0 \) in \( G \cup B \) such that
\[
\lambda := \sup_{z \in \mathcal{U}, w \in \mathcal{V}} (\omega(z, \mathcal{A}, D) + \omega(w, B, G)) < 1.
\]
Now let \( \delta > 0 \) be a positive number which verifies \( \delta + \frac{2C_1 \delta}{1 - \delta} < 1 - \lambda \). Then Lemma 5.3 implies that \( f = \hat{f}_\delta \) on \( \mathcal{U} \times \mathcal{V} \). Since by Theorem 4.1 we have known that \( \hat{f}_\delta \) is continuous on \( D \times B \), so is \( \hat{f} \) and moreover \( \hat{f} = f \) on \( D \times B \). This, combined with (5.21) implies that \( \hat{f} = f \) on \( \mathbb{X}(\mathcal{A}, B; D, G) \).

Finally, it remains to check the continuity of \( \hat{f} \) on \( \mathcal{A} \times (G \cup B) \). Fix a point \((\zeta_0, w_0) \in \mathcal{A} \times (G \cup B) \) and a number \( 0 < \epsilon < 1 \). From the hypothesis \( f \in \mathcal{C}(\mathcal{X}(\mathcal{A}, B; D, G)) \) and by Lemma 5.2 it follows that there are an open connected neighborhood \( \mathcal{U} \) of \( \zeta_0 \) in \( D \cup \mathcal{A} \), an open connected neighborhood \( \mathcal{V} \) of \( w_0 \) in \( G \cup B \) and a finite constant \( M \) such that
\[
|f(\zeta_0, w_0) - f(\zeta, w)| < \epsilon^2, \quad \zeta \in \mathcal{A} \cap \mathcal{U}, \ w \in \mathcal{V},
\]
\[
|\hat{f}|_{\mathcal{U} \times \mathcal{V}} < \frac{M}{2}.
\]
Moreover, by shrinking \( \mathcal{U} \) and \( \mathcal{V} \), if necessary, and applying Proposition 3.2, we may suppose that
\[
\sup_{\mathcal{U} \times \mathcal{V}} (\omega(z, \mathcal{A}, D) + \omega(w, B, G)) < 1.
\]
Therefore, \( \mathcal{U} \times \mathcal{V} \subset \hat{\mathbb{X}}(\mathcal{X}(\mathcal{A}, B; D, G)) \). Moreover, by Lemma 5.3 and (5.17), there is an \( \delta > 0 \) such that \( \hat{f} = \hat{f}_\delta = \hat{f} \) on the nonempty open set \( (T_\delta \cap \mathcal{U}) \times \mathcal{V} \). Thus
\[
(5.23) \quad \hat{f} = \hat{f} \quad \text{on } \mathcal{U} \times \mathcal{V}.
\]
By shrinking $\mathcal{U}$, if necessary, we may suppose that for all $z \in \mathcal{U}$, there is exactly one point $\zeta_z \in A$ such that $\pi(\zeta_z) = \pi(z)$. By Part 4) of Proposition 2.5 we have

$$z \in V_{\zeta_z}, \quad \zeta_z \in A \cap \partial V_{\zeta_z}, \quad \text{and} \quad \text{dist}(z, \zeta_z) \approx \text{dist}(z, \partial V_{\zeta_z}).$$

Therefore, we are able to apply the Two-Constant Theorem to the function $\hat{f}(\cdot, w) - \hat{f}(\zeta_z, w) \in C((\partial V_{\zeta_z} \cap \mathcal{U}) \cup (V_{\zeta_z} \cap \mathcal{U})) \cap \mathcal{O}(V_{\zeta_z} \cap \mathcal{U})$, which is, by (5.22), bounded by $M$ for any $z \in \mathcal{U}$, $w \in \mathcal{V}$.

Consequently, taking (5.22) and (5.23) into account, we deduce that

$$|\hat{f}(z, w) - \hat{f}(\zeta_z, w)| < \frac{2(1-\omega(z, \partial V_{\zeta_z} \cap \mathcal{U}, V_{\zeta_z} \cap \mathcal{U}))}{M} \omega(z, \partial V_{\zeta_z} \cap \mathcal{U}, V_{\zeta_z} \cap \mathcal{U})$$

for all $(z, w) \in \mathcal{U} \times \mathcal{V}$. Thus for $(z, w) \in (D \cup A) \times (G \cup B)$ sufficiently close to $(\zeta_0, w_0)$ we have by Proposition 3.2 and (5.22),

$$|\hat{f}(z, w) - \hat{f}(\zeta_0, w_0)| \leq |\hat{f}(z, w) - \hat{f}(\zeta_z, w)| + |\hat{f}(\zeta_z, w) - \hat{f}(\zeta_0, w_0)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$

which proves the continuity of $\hat{f}$ at $(\zeta_0, w_0)$.

Hence Step 2 is finished. \qed

**Step 3: The case where $G$ is a Jordan planar simply connected domain.**

**Proof of Step 3.** Fix a sequence $(A_k)_{k=1}^\infty$ of open subsets of $A$ such that $A_k \subseteq A_{k+1}$ and $A_k \nearrow A$ as $k \nearrow \infty$. By Proposition 3.4, we obtain $\hat{\mathcal{X}}(A_k, B; D, G) \nearrow \hat{\mathcal{X}}(A, B; D, G)$. Using a routine uniqueness argument (Theorem 2.1 and Proposition 3.6) and the gluing procedure, we are reduced to the proof that for any $k$, there is a function $\hat{f}_k \in C\left(\hat{\mathcal{X}}(A_k, B; D, G)\right) \cap \mathcal{O}\left(\hat{\mathcal{X}}^o(A_k, B; D, G)\right)$ satisfying $\hat{f}_k = f$ on $\hat{\mathcal{X}}(A_k, B; D, G)$.

Now fix an $k \in \mathbb{N}$. First we shall show that one may find a sufficiently small positive number $\delta_0$ with the following properties:

- For any $0 < \delta < \delta_0$, there is a finite number of open balls $(A_j)_{j=1}^N$ of $A$ with radius $\delta$ (depending on $\delta$) such that
  
  (i) $A_k \subseteq \bigcup_{j=1}^N A_j$;
  
  (ii) $\bigcup_{j=1}^N 2A_j \subseteq A_{k+1}$, where $2A_j$ is the ball with the same center as $A_j$ but with double radius;
  
  (iii) for each $1 \leq j \leq N$, Step 2 can apply to the open ball $2A_j$; more precisely, Step 2 provides a function $\hat{f}_j \in C\left(\hat{\mathcal{X}}_{2A_j}\right) \cap \mathcal{O}\left(\hat{\mathcal{X}}^o_{2A_j}\right)$ satisfying $\hat{f}_j = f$ on $\hat{\mathcal{X}}_{2A_j}$;
  
  (iv) for any $0 < \delta < \delta_0$ there is an open subset $T_\delta$ of $D$ such that
    
    $$\omega(z, A_k, D) - \delta \leq \omega(z, T_\delta, D) \leq \omega(z, A_k, D), \quad z \in D,$$
    
    $$\sup_{T_\delta} \text{dist}(\cdot, A_k) < r,$$
    
    for some $0 < r := r_\delta < \delta$;
  
  (v) for any $0 < \delta < \delta_0$ and $z \in T_\delta$ there is a unique nearest point $\zeta_z \in \bigcup_{j=1}^N A_j$ such that $\text{dist}(z, \zeta_z) = \text{dist}(z, \partial D)$ and for any $1 \leq j \leq N$ such that $\zeta_z \in A_j$
we have
\[ \sup_{t \in [z, \zeta_z]} \omega(t, 2A_j, D) < \delta, \]
where \([z, \zeta_z]\) denotes the real segment connecting \(z\) to \(\zeta_z\).

Indeed, using the result of Step 2 and by a compactness argument we see that one may find \(\delta_0 > 0\) sufficiently small such that the properties (i)-(iii) are fulfilled.

On the other hand, using Proposition 3.2 we see that there is an \(r := r_\delta\) sufficiently small such that
\[ (5.24) \quad \omega(z, 2A_j, D) < \delta, \]
for all \(1 \leq j \leq N\) and \(z \in D\) with \(\text{dist}(z, A_j) < r\).

By examining carefully the proof of Proposition 3.5, we may arrange that property (iv) is fulfilled with \((5.25)\) and hence the function \(\tilde{f}\) is well-defined. Indeed, in virtue of (iv)-(v) and (5.25), for any \((z, w) \in (T_\delta \cup A_k) \times G_\delta\) there is at least an \(j\) such that \(\zeta_z \in A_j\) and \((t, w) \in \tilde{X}_{2A_j}\), for \(t \in [z, \zeta_z]\).

On the other hand, suppose that there is another index \(l\) such that \(\zeta_z \in A_l\). Observe that \(\tilde{f}_j = \tilde{f}_1 = f\) on \((A_j \cap A_l) \times G\). Therefore, we may apply Theorem 2.1 and Proposition 3.6 and conclude that \(\tilde{f}_j = \tilde{f}_1\) on the connected component of \(\tilde{X}_{2A_j} \cap \tilde{X}_{2A_l}\) which is locally \(C^2\) smooth on \((A_j \cap A_l) \times G_\delta\). However we have already shown in the previous paragraph that \((t, w) \in \tilde{X}_{2A_j} \cap \tilde{X}_{2A_l}\) for \(t \in [z, \zeta_z]\) and clearly \((\zeta_z, w) \in (A_j \cap A_l) \times G\). Consequently, the above mentioned connected component contains the point \((z, w)\). Thus \(\tilde{f}_j(z, w) = \tilde{f}_1(z, w)\), and hence the function \(\tilde{f}\) is well-defined.

In virtue of (5.26), it is also clear that
\[ \tilde{f} \in \mathcal{C} \left( (T_\delta \cup A_k) \times G_\delta \right) \cap \mathcal{O} \left( T_\delta \times G_\delta \right). \]
Let \(\tilde{f}_\delta\) be the trace of \(\tilde{f}\) on \(\tilde{X}(T_\delta, B; D, G_\delta)\). Applying Theorem 4.1 to the function \(\tilde{f}_\delta \in \mathcal{C} \left( \tilde{X}(T_\delta, B; D, G_\delta) \right) \cap \mathcal{O} \left( \tilde{X}(T_\delta, B; D, G_\delta) \right)\), we obtain an extension function
\[ \hat{f}_\delta \in \mathcal{C} \left( \tilde{X}(T_\delta, B; D, G_\delta) \right) \cap \mathcal{O} \left( \tilde{X}(T_\delta, B; D, G_\delta) \right) \]
satisfying \(\hat{f}_\delta = f\) on \(D \times B\).
Finally, one proceeds as in the end of Step 2. Observe that Lemma 5.3 is still valid in the present context. As in formula (5.17), one may glue \( \hat{f} := \lim_{\delta \to 0} \hat{f}_\delta \) to obtain an extension function \( \hat{f} \), which is holomorphic on \( \hat{X}^o \) and continuous on \( D \times B \).

Since \( \tilde{f} \in C((T_\delta \cup A_k) \times G_\delta) \) for \( 0 < \delta < \delta_0 \), Lemma 5.3 in the present context also gives that \( \hat{f} \in C(\hat{X}(A, B; D, G)) \).

Hence Step 3 is complete. □

**Step 4:** We keep the hypothesis of Theorem 5.1 and prove the following local version of this theorem:

For any point \( P \in B \), there is a ball \( B \) in \( B \) with center \( P \) such that the following property holds: For any function \( f \in C(X) \cap O_s(X^o) \), there is a unique function \( \hat{f} \in C(\hat{X}_B) \cap O(\hat{X}_B^o) \) such that \( \hat{f} = f \) on \( X_B \), where

\[
X_B := X(A, B; D, G), \quad \hat{X}_B := \hat{X}(A, B; D, G) \quad \text{and} \quad \hat{X}_B^o := \hat{X}^o(A, B; D, G).
\]

**Proof of Step 4.** We proceed using Step 3 in exactly the same way as we proved Step 2 using Theorem 2. Therefore we shall only indicate briefly the outline of the proof.

First we apply Proposition 2.5 to the domain \( G \) which is locally \( C^2 \) smooth on an open neighborhood of \( P \) in \( \partial G \). Consequently, we may find an open neighborhood \( U \) of \( P \) satisfying (2.2) such that Proposition 2.5 is applicable there. In the sequel the notation \( U, U_1, \pi^C, V \) and \( V_Q \) have the same meanings as in Proposition 2.5.

Now we can fix a ball \( B \) of \( B : \mathcal{B} := B \cap B(P, r) \), where the radius \( r \) is sufficiently small such that \( 2B \in B \) etc.

Arguing as in (5.2)–(5.3) we can choose an \( \delta_0 > 0 \) sufficiently small such that for any \( 0 < \delta < \delta_0 \) there is an open subset \( S_\delta \) of \( G \) satisfying

\[
\omega(w, B, G) - \delta \leq \omega(w, S_\delta, G) \leq \omega(w, B, G), \quad w \in G,
\]

\[
\sup_{S_\delta} \text{dist}(\cdot, B) < \delta.
\]

Arguing as in (5.4)–(5.8), there is a finite constant \( C_3 \) such that

\[
\omega(w, \partial V_Q \cap 2B, V_Q) \leq C_3 \text{dist}(w, B) \leq C_3 \delta,
\]

for \( 0 < \delta < \delta_0 \), \( w \in S_\delta \) and \( Q_w := \pi^C(w) \).

Lemma 5.2 is still valid in the present context making the obviously necessary changes in notation. There is only one important difference between Step 2 and the present step. In Step 2 we apply Gonchar’s Theorem to (5.15) but in this step we appeal to Theorem 4.3.

Lemma 5.3 is also valid in the present context making the obviously necessary changes in notation.
For each $Q \in \pi^C(2B)$, we apply the result of Step 3 to the function $f \in \mathcal{C}(X(A, \partial V_Q \cap 2B; D, V_Q)) \cap \mathcal{O}_s(X^o(A, \partial V_Q \cap 2B; D, V_Q))$ in order to obtain an extension function $\hat{f}_Q \in \mathcal{C}(\hat{X}(A, \partial V_Q \cap 2B; D, V_Q)) \cap \mathcal{O}(\hat{X}^o(A, \partial V_Q \cap 2B; D, V_Q))$ such that $\hat{f}_Q = f$ on $X(A, \partial V_Q \cap 2B; D, V_Q)$.

Gluing the family $(\hat{f}_Q)_{Q \in \pi^C(2B)}$, we obtain an extension function $\tilde{f}$ defined on

$$\{(z, w) \in D \times G : \exists Q \in \pi^C(2B), w \in V_Q \text{ and } \omega(z, A, D) + \omega(w, \partial V_Q \cap 2B, V_Q) < 1\}.$$ 

For $0 < \delta < \delta_0$ put

(5.30) $D_\delta := \{\omega(z, A, D) < 1 - 2C_3\delta\}.$

As in Step 2, taking (5.27)–(5.30) into account we see that

$$\tilde{f} \in \mathcal{C}_s(X(A, S_\delta; D_\delta, G)) \cap \mathcal{O}_s(X^o(A, S_\delta; D_\delta, G)).$$

Therefore, we are in a position to apply Theorem 4.1 and obtain an extension function

$$\hat{f}_\delta \in \mathcal{C}(\hat{X}(A, S_\delta; D_\delta, G)) \cap \mathcal{O}(\hat{X}^o(A, S_\delta; D_\delta, G)).$$

Using (5.17) we may glue $(\hat{f}_\delta)_{0 < \delta < \delta_0}$ together in order to obtain the desired extension function $\hat{f}$. The rest of the proof follows along the same lines as in Step 2 making use of Two-Constant Theorem and Lemmas 5.2 and 5.3. This finishes Step 4.

**Step 5: The general case.**

The same argument which has been used to go from Step 2 to Step 3 will enable us to go from Step 4 to Step 5. Consequently, there is an extension function $\hat{f} \in \mathcal{C}(\hat{X}) \cap \mathcal{O}_s(\hat{X}^o)$ such that $\hat{f} = f$ on $X$. It is also clear that $\hat{f}$ is uniquely determined. Finally, it remains to establish estimate (5.1). But it follows immediately from Theorem 4.3.

This completes the last step of the proof. □

6. **Proof of the Main Theorem and concluding remarks**

In order to prove the Main Theorem, we proceed by induction (I) on $N \geq 2$. Suppose the Main Theorem is true for $N - 1 \geq 2$. We have to discuss the case of an $N$-fold cross $X := X(A_1, \ldots, A_N; D_1, \ldots, D_N)$, where $D_1, \ldots, D_N$ are pseudoconvex domains and $A_1, \ldots, A_N$ are open subsets of $\partial D_1, \ldots, \partial D_N$ such that $D_j$ is locally $C^2$ smooth on $A_j$ ($1 \leq j \leq N$). Fix an $f \in \mathcal{C}(X) \cap \mathcal{O}_s(X^o)$.

We proceed again by induction (II) on the positive integer $j$ ($1 \leq j \leq N$) such that $D_{j-1}, \ldots, D_N$ are Jordan planar domains.

For $j = 1$, we are reduced to Theorem 2.

Suppose the Main Theorem is true for the case where $D_{j-1}, \ldots, D_N$ are Jordan planar domains ($j \geq 2$). We have to discuss the case where $D_j, \ldots, D_N$ are Jordan planar domains. The proof given below follows essentially the schema of that of Theorem 5.1. It is divided into three steps.
Step 1: Reduction to the case where $D_1, \ldots, D_{j-1} \text{ are bounded pseudoconvex domains.}$

Proof of Step 1. We proceed in exactly the same way as in Step 1 of Theorem 5.1. This completes Step 1.

From now on we assume that the hypothesis of Step 1 is fulfilled.

Step 2: We prove the following local version of the Main Theorem:

For any point $P \in A_1$, there is a ball $\mathcal{A}$ in $A_1$ with center $P$ such that the following property holds: For any function $f \in C(X) \cap \mathcal{O}_s(X^o)$, there is a unique function $\hat{f} \in C(\hat{X}_A) \cap \mathcal{O}(\hat{X}_A^o)$ such that $\hat{f} = f$ on $X_A$, where

$$X_A := X(A, A_2, \ldots, A_N; D_1, \ldots, D_N), \quad \hat{X}_A := \hat{X}(A, A_2, \ldots, A_N; D_1, \ldots, D_N),$$

$$\hat{X}_A^o := \hat{X}^o(A, A_2, \ldots, A_N; D_1, \ldots, D_N).$$

Proof of Step 2. As in Step 2 in the proof of Theorem 5.1 we first apply Proposition 2.5 to the domain $D_1$ which is locally $C^2$ smooth on an open neighborhood of $P$ in $\partial D_1$. Consequently, we may find an open neighborhood $U$ of $P$ satisfying (2.2) such that Proposition 2.5 is applicable there. In the sequel the notation $U, U_1, \pi, V$ and $V_0$ have the same meanings as in Proposition 2.5. Now we can fix a ball $\mathcal{A}$ of $A_1 : A := A_1 \cap B(P, r)$, where the radius $r$ is sufficiently small such that $2\mathcal{A} \subseteq A_1$ etc.

Arguing as in (5.2)–(5.3) and (5.4)–(5.8), we can choose an $\delta_0 > 0$ sufficiently small and a finite constant $C$ such that for any $0 < \delta \leq \delta_0$ there is an open subset $T^1_\delta$ of $D_1$ satisfying

$$\omega(z_1, A, D_1) - \delta \leq \omega(z_1, T^1_\delta, D_1) \leq \omega(z_1, A, D_1), \quad z_1 \in D_1,$$

(6.1)

$$\sup_{T^1_\delta} \text{dist}(\cdot, A) < \frac{\delta}{C}, \quad T^1_\delta \subset \bigcup_{Q \in \pi(2\mathcal{A})} V_Q,$$

and

$$\omega(z_1, \partial V_{Q_{z_1}} \cap 2\mathcal{A}, V_{Q_{z_1}}) \leq C \text{dist}(z_1, A) \leq \delta,$$

(6.2)

for $0 < \delta \leq \delta_0$, $z_1 \in T^1_\delta$ and $Q_{z_1} := \pi(z_1)$.

Similarly, for each $2 \leq k \leq N$ there is an open subset $T^k_\delta$ of $D_k$ satisfying

$$\omega(z_k, A_k, D_k) - \delta \leq \omega(z_k, T^k_\delta, D_k) \leq \omega(z_k, A_k, D_k), \quad z_k \in D_k.$$

(6.3)

For each $Q \in \pi(2\mathcal{A})$, we apply the reduction assumption (II) to the function

$$f \in C(X(2\mathcal{A} \cap \partial V_Q, A_2, \ldots, A_N; V_Q, D_2, \ldots, D_N))$$

$$\cap \mathcal{O}_s(X^o(2\mathcal{A} \cap \partial V_Q, A_2, \ldots, A_N; V_Q, D_2, \ldots, D_N))$$

in order to obtain an extension function

$$\hat{f}_Q \in C(\hat{X}(2\mathcal{A} \cap \partial V_Q, A_2, \ldots, A_N; V_Q, D_2, \ldots, D_N))$$

$$\cap \mathcal{O}(\hat{X}^o(2\mathcal{A} \cap \partial V_Q, A_2, \ldots, A_N; V_Q, D_2, \ldots, D_N))$$

(6.4)
such that
\[\hat{f}_Q = f\quad\text{on } \mathbb{X}(\mathcal{A} \cap \partial V_Q, A_2, \ldots, A_N; V_Q, D_2, \ldots, D_N).\]
Collecting the family \((\hat{f}_Q)_{Q \in \pi^C(2A)}\), we obtain an extension function \(\tilde{f}\) defined on
\[\{ (z_1, \ldots, z_N) \in D_1 \times \cdots \times D_N : \exists Q \in \pi^C(2A),\quad z_1 \in V_Q \quad\text{and}\quad \omega(z_1, 2\mathcal{A} \cap \partial V_Q, V_Q) + \sum_{k=2}^N \omega(z_k, A_k, D_k) < 1 \}\]
which satisfies
\[\tilde{f} = f\quad\text{on } \mathbb{X}(\mathcal{A}, A_2, \ldots, A_N; T_{\delta_0}^1, D_2, \ldots, D_N).\]

For \(0 \leq \delta < \delta_0\) put
\[D_\delta := \{ (z_2, \ldots, z_N) \in D_2 \times \cdots \times D_N : \omega(z_2, A_2, D_2) + \omega(z_2, T_\delta^3, D_3) + \cdots + \omega(z_N, T_\delta^N, D_N) < 1 - N\delta \}\]
and
\[D_\delta^k := \{ z_k \in D_k : \omega(z_k, A_k, D_k) < 1 - N\delta \}, \quad 1 \leq k \leq N.\]
Consequently, in virtue of (6.1)–(6.4) and (6.6) for any fixed \(z_1 \in T_\delta^1\) and \(0 < \delta < \delta_0\), the restriction \(\tilde{f}(z_1, \cdots)\) is holomorphic on \(D_\delta\).

On the other hand, for any \(a_2 \in A_2\), by the reduction assumption (I) for an \((N-1)\)-fold cross, we obtain an extension \(\hat{f}_{a_2}\) such that
\[\hat{f}_{a_2} \in \mathcal{C}\left(\tilde{\mathbb{X}}(A_1, A_3, \ldots, A_N; D_1, \ldots, D_N)\right) \cap \mathcal{O}\left(\tilde{\mathbb{X}}^o(A_1, A_3, \ldots, A_N; D_1, \ldots, D_N)\right)\]
and
\[\hat{f}_{a_2}(z_1, z_3, \ldots, z_N) = f(z_1, a_2, z_3, \ldots, z_N),\]
\[(z_1, z_3, \ldots, z_N) \in \mathbb{X}(A_1, A_3, \ldots, A_N; D_1, D_3, \ldots, D_N).\]

Observe that by (6.1)–(6.3), (6.6), and (6.9)–(6.10), for \(0 < \delta < \delta_0\) sufficiently small, the domain of definition of \(\hat{f}_{a_2}\) \((a_2 \in A_2)\) contains \(D_\delta^1 \times T_\delta^3 \times \cdots \times T_\delta^N\) and that of \(\tilde{f}\) contains \(T_\delta^1 \times A_2 \times T_\delta^3 \times \cdots \times T_\delta^N\). Next we would like to prove that for \(0 < \delta < \delta_0\) sufficiently small and \(a_2 \in A_2\),
\[\tilde{f}(z_1, a_2, z_3, \ldots, z_N) = \hat{f}_{a_2}(z_1, z_3, \ldots, z_N),\]
\[(z_1, z_3, \ldots, z_N) \in T_\delta^1 \times T_\delta^3 \times \cdots \times T_\delta^N.\]
Indeed, in virtue (6.5) and (6.11) and by applying the reduction assumption (I) to \(\hat{f}_{a_2}\) and the reduction assumption (II) to \(\hat{f}_Q\) for any \(Q \in \pi^C(2A)\) we know that
\[\tilde{f}(z_1, a_2, z_3, \ldots, z_N) = \hat{f}_Q(z_1, a_2, z_3, \ldots, z_N) = f(z_1, a_2, z_3, \ldots, z_N) = \hat{f}_{a_2}(z_1, z_3, \ldots, z_N),\]
\[z_1 \in A \cap \partial V_Q, \quad a_2 \in A_2 \quad\text{and}\quad (z_3, \ldots, z_N) \in \mathbb{X}(A_3, \ldots, A_N; D_3, \ldots, D_N).\]
This proves (6.12). Consequently, we can define a new function $\tilde{f}_\delta$ on $X(T^1_\delta, A_2 \times T^3_\delta \times \cdots \times T^N_\delta; D^1_\delta, D_\delta')$ as follows

\[
(6.13) \quad \tilde{f}_\delta := \begin{cases} 
\tilde{f}, & \text{on } T^1_\delta \times D_\delta', \\
\hat{f}_{a_2}, & \text{on } T^1_\delta \times \{a_2\} \times T^3_\delta \times \cdots \times T^N_\delta, \ a_2 \in A_2.
\end{cases}
\]

We need the following lemmas

**Lemma 6.1.** The following assertions hold:
1) $\tilde{f}_\delta$ is locally bounded on $X(T^1_\delta, A_2 \times T^3_\delta \times \cdots \times T^N_\delta; D^1_\delta, D_\delta')$;
2) $\tilde{f}_\delta$ is locally bounded on $T^1_\delta \times (D_\delta' \cup (D_\delta' \cap X(A_2, \ldots, A_N; D_2, \ldots, D_N)))$ and $\tilde{f}_\delta(z_1, \cdots) \in C(D_\delta' \cup (D_\delta' \cap X(A_2, \ldots, A_N; D_2, \ldots, D_N)))$ for any $z_1 \in T^1_\delta$.

**Proof of Lemma 6.1.** It follows along the same lines as that of Lemma 5.2. Therefore, we only indicate a crucial difference. In Lemma 5.2 we appeal to Gonchar’s Theorem but in the present lemma we apply the hypothesis of induction (II). This completes the proof. □

**Lemma 6.2.** Let $D^k$ be a bounded open set and let $A_2$ be an open set of $\partial D^k$ such that $D^k$ is locally $C^2$ smooth on $A_2$. Let $T^k \subset D_k \subset \mathbb{C}^n$, $D_k$ a domain and $T^k$ locally pluriregular, $k = 3, \ldots, N, N \geq 3$. Put

\[
D' := \left\{ z' = (z_2, \ldots, z_N) \in D_2 \times \cdots \times D_N : \omega(z_2, A_2, D_2) + \sum_{k=3}^{N} \omega(z_k, T^k, D_k) < 1 \right\}.
\]

Then

\[
\omega(z', A_2 \times T^3 \times \cdots \times T^N, D') = \omega(z_2, A_2, D_2) + \sum_{k=3}^{N} \omega(z_k, T^k, D_k).
\]

**Proof of Lemma 6.2.** We argue as in the proof of Lemma 3(b) in [7] making use of Part 1) of Proposition 3.8. □

We now come back to the proof of the Main Theorem. Applying Lemma 6.2 and Part 1) of Proposition 3.8, we see that

\[
(6.14) \quad \omega(z', A_2 \times T^3 \times \cdots \times T^N, D_\delta') = \frac{1}{1 - N\delta} \left( \omega(z_2, A_2, D_2) + \sum_{k=3}^{N} \omega(z_k, T^k_\delta, D_k) \right)
\]

for any $z' \in D_\delta'$.

To summarize what has been done so far: for any $0 < \delta < \delta_0$ sufficiently small, we obtain, by Part 1) of Lemma 6.1, a function $\tilde{f}_\delta$ defined on a mixed cross $\tilde{f}_\delta \in C_s \left( X(T^1_\delta, A_2 \times T^3_\delta \times \cdots \times T^N_\delta; D^1_\delta, D_\delta') \right) \cap O_s \left( X^\omega \left( T^1_\delta, A_2 \times T^3_\delta \times \cdots \times T^N_\delta; D^1_\delta, D_\delta' \right) \right)$. 
Applying Theorem 4.1 to \( \hat{f}_\delta \) we obtain an extension function \( \tilde{f}_\delta \) of \( \hat{f}_\delta \) such that

\[
\tilde{f}_\delta \in \mathcal{C} \left( \hat{X} \left( T^1_\delta, A_2 \times T^3_\delta \times \cdots \times T^N_\delta; D^1_\delta, D'_\delta \right) \right) \cap \mathcal{O} \left( \hat{X}_o \left( T^1_\delta, A_2 \times T^3_\delta \times \cdots \times T^N_\delta; D^1_\delta, D'_\delta \right) \right).
\]

In virtue of (6.4)–(6.9) and (6.13)–(6.15) and by Part 2) of Lemma 6.1, we can apply Part 2) of Theorem 4.1 and conclude that \( \tilde{f}_\delta \) can be continuously extended to a new function

\[
\hat{f}_\delta \in \mathcal{C}(\hat{X}_\delta),
\]

where

\[
\hat{X}_\delta := \hat{X} \left( T^1_\delta, A_2 \times T^3_\delta \times \cdots \times T^N_\delta; D^1_\delta, D'_\delta \right)
\]

\[
\cup \left( \hat{X} \left( T^1_\delta, A_2 \times T^3_\delta \times \cdots \times T^N_\delta; D^1_\delta, D'_\delta \right) \cap \left( D^1_\delta \times X (A_2, \ldots, A_N; D_2, \ldots, D_N) \right) \right).
\]

We are now in a position to define the desired extension function \( \hat{f} \). Indeed, one glues \( \left( \hat{f}_\delta \right)_{0 < \delta < \delta_0} \) together to obtain \( \hat{f} \) in the following way

\[
\hat{f} := \lim_{\delta \to 0} \hat{f}_\delta \quad \text{on} \quad \hat{X}_A \setminus (A \times X (A_2, \ldots, A_N; D_2, \ldots, D_N)).
\]

One now checks that the limit (6.18) exists and does possess all the required properties. This is an immediate consequence of the following

**Lemma 6.3.** For any point

\[
z = (z_1, \ldots, z_N) \in \hat{X}_A \setminus (A \times X (A_2, \ldots, A_N; D_2, \ldots, D_N))
\]

let \( \delta_z \) be the unique positive number \( \delta \) which verifies

\[
\delta + \frac{2N\delta}{1 - N\delta} = 1 - \omega(z_1, A, D_1) - \sum_{k=2}^{N} \omega(z_k, A_k, D_k).
\]

Then \( f(z) = \hat{f}_\delta(z) \) for all \( 0 < \delta < \delta_z \).

**Proof of Lemma 6.3.** Fix a point \( z^0 = (z_1^0, \ldots, z_N^0) \in \hat{X}_A \setminus (A \times X (A_2, \ldots, A_N; D_2, \ldots, D_N)) \). Then by (6.1)–(6.3) and (6.14)–(6.18) and the second identity of Part 1) of Proposition 3.8, for all \( 0 < \delta < \delta_0 \), we have \( z^0 \in \hat{X}_\delta \). In particular, \( \bigcup_{0 < \delta < \delta_0} \hat{X}_\delta = \hat{X}_A \setminus (A \times X (A_2, \ldots, A_N; D_2, \ldots, D_N)) \).

Therefore, for any \( 0 < \delta < \delta < \delta_0 \), \( z^0 \) is contained in the following set

\[
\hat{X}_\delta \cap \hat{X}_\delta'.
\]

Let \( H_\delta \) be the connected component of \( D'_\delta \) containing \( (z_2^0, \ldots, z_N^0) \). Let \( B_\delta \) be the largest open subset of \( B \) such that \( H_\delta \) is locally \( C^2 \) smooth on \( B_\delta \). By (6.1)–(6.3) and (6.19) and arguing as in the proof of Lemma 5.3, we see that the above intersection contains the set

\[
\left\{ z = (z_1, z') \in D^1_\delta \times H_\delta : \omega(z_1, T^1_\delta, D^1_\delta) + \omega(z', B_\delta, H_\delta) < 1 - \delta \right\}.
\]
By Proposition 3.7 this open set is connected. Moreover, it contains the point $z^0$. In addition we deduce from (6.17) that
\[ \hat{f}_\delta = \hat{f} = \hat{f} = f \quad \text{on } T^1_\delta \times B'_\delta. \]
Applying Theorem 2.1, we deduce that $\hat{f}_\delta = \hat{f}$ on the domain given by (6.20). In particular $\hat{f}_\delta(z^0) = \hat{f}(z^0)$. This completes the proof.

An immediate consequence of Lemma 6.3 is that $\hat{f} \in \mathcal{O} \left( \hat{X}_A \right)$. Now we define $\hat{f}$ on $A \times X(A_2, \ldots, A_N; D_2, \ldots, D_N)$ as follows
\[ \hat{f} := f \quad \text{on } A \times X(A_2, \ldots, A_N; D_2, \ldots, D_N). \]
Thus $\hat{f}$ is well-defined on the whole $\hat{X}_A$ and
\[ \hat{f} \in \mathcal{C}(D_1 \times X(A_2, \ldots, A_N; D_2, \ldots, D_N)). \]
To complete Step 2, it remains to show that $\hat{f} \in \mathcal{C}(\hat{X}_A)$ and $\hat{f} = f$ on $X_A$. For this purpose we do the following trick.

We replace $D_1$ by $D_j$ $(j = 2, \ldots, N)$ and proceed as above. For example, if we replace $D_1$ by $D_2$, then we obtain a new extension function $\hat{f}$ such that in virtue of (6.21)
\[ \hat{f} \in \mathcal{C}(D_2 \times X(A_1, A_3, \ldots, A_N; D_1, D_3, \ldots, D_N)). \]
Next, using identities (6.12), (6.13), (6.16) and (6.18) and applying Theorem 2, we see that the value of $\hat{f}$ and $\hat{f}$ can be uniquely determined on $T^1_\delta \times T^2_\delta \times \cdots \times T^N_\delta$ from the value of $f$ on $A \times A_2 \times \cdots \times A_N$ for any sufficiently small $\delta > 0$. Thus
\[ \hat{f} = \hat{f} \quad \text{on } T^1_\delta \times T^2_\delta \times \cdots \times T^N_\delta. \]
Hence $\hat{f} = \hat{f}$ on $\hat{X}_A$ since $\hat{X}_A$ is a domain by Proposition 3.6. Therefore, in virtue of (6.21), (6.22) and similar conclusions when $D_1$ is replaced by $D_3, \ldots, D_N$, we conclude that
\[ \hat{f} \in \mathcal{C} \left( \hat{X}_A \setminus (A \times A_2 \times \cdots \times A_N) \right). \]
Therefore, Step 2 will be finished if we can prove that $\hat{f}$ is continuous on $A \times A_2 \times \cdots \times A_N$. To do this fix a point $a = (a_1, \ldots, a_N) \in A \times A_2 \times \cdots \times A_N$ and an arbitrary number $\epsilon > 0$. Next, we apply Proposition 2.5 to each domain $D_j$ which is locally $\mathcal{C}^2$ smooth on an open neighborhood of $a_j$, $j = 1, \ldots, N$. Consequently, we may find an open neighborhood $U^j$ of $a_j$ satisfying (2.2) such that Proposition 2.5 is applicable there. In the sequel the notation $U^j$, $U^1_j$, $\pi^c_j$, $V^j_j$ and $V^q_j$ have the same meanings for $a_j$ as $U$, $U_1$, $\pi^c$, $V$ and $V_Q$ do for $P$ in Proposition 2.5.

Since $f \in \mathcal{C}(X)$, by shrinking $U^j$, if necessary, we may assume without loss of generality that
\[ |f(\zeta) - f(\eta)| < \frac{\epsilon}{2}, \quad \zeta, \eta \in X \left( A \cap U^1, A_2 \cap U^2, \ldots, A_N \cap U^N; D_1 \cap U^1, \ldots, D_N \cap U^N \right). \]
Let \( z = (z_1, \ldots, z_N) \) be an arbitrary point of \( U^1 \times \cdots \times U^N \) and put \( Q_j := \pi_{C,j}(z_j) \). Then, in virtue of the hypothesis on \( f \), we may apply Theorem 2 to
\[
f \in \mathcal{C}(X(A \cap \partial V_{Q_1}, A_2 \cap \partial V_{Q_2}, \ldots, A_N \cap \partial V_{Q_N}; V_{Q_1}^1, \ldots, V_{Q_N}^N)) \\
\cap \mathcal{O}(X^0(A \cap \partial V_{Q_1}, A_2 \cap \partial V_{Q_2}, \ldots, A_N \cap \partial V_{Q_N}; V_{Q_1}^1, \ldots, V_{Q_N}^N)).
\]
Consequently, taking into account (6.23) and the above construction of the extension function \( \hat{f} \), we deduce that
\[
|\hat{f}(z) - f(\zeta)| < \frac{\epsilon}{2}, \quad \zeta \in X(A \cap \partial V_{Q_1}, A_2 \cap \partial V_{Q_2}, \ldots, A_N \cap \partial V_{Q_N}; V_{Q_1}^1, \ldots, V_{Q_N}^N).
\]
Hence, fixing any \( \zeta \) as above and applying again (6.23), we get
\[
|\hat{f}(z) - f(a)| \leq |\hat{f}(z) - f(\zeta)| + |f(\zeta) - f(a)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} < \epsilon,
\]
which proves the continuity of \( \hat{f} \) at \( a \). Hence the remaining assertion of Step 2 is proved.

Thus the proofs for the induction (I) and (II) are complete in this second step. □

**Step 3: The general case.**

The same argument which has been used to go from Step 2 to Step 3 in the proof of Theorem 5 will enable us to go from Step 2 to Step 3 in the present context. Consequently, there is an extension function \( \hat{f} \in \mathcal{C}(\hat{X}) \cap \mathcal{O}(\hat{X}^0) \) such that \( \hat{f} = f \) on \( X \). It is also clear that \( \hat{f} \) is uniquely determined. Finally, it remains to establish estimate (1.3). We have already proved the existence and uniqueness of the Main Theorem. Using this result we argue as in the proof of Theorem 4.2 in order to obtain (1.3). This completes the last step of the proof.

Hence the Main Theorem is proved. □

Finally, we conclude this paper by some open remarks and open questions.

1. It seems to be of interest to establish the Main Theorem under weaker assumptions than the continuity of \( f \), the smoothness of \( D_j \) on \( A_j \), and the regularity of the set \( A_j \) \( j = 1, \ldots, N \), etc. We postpone this issue to an ongoing work.

2. Does the Main Theorem still hold if we only assume that \( A_j \) is of positive \((2n_j - 1)\)-Hausdorff measure, \( j = 1, \ldots, N \)?

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