Mutually polar retractions on convex cones

A. B. Németh
Faculty of Mathematics and Computer Science
Babeș Bolyai University, Str. Kogălniceanu nr. 1-3
RO-400084 Cluj-Napoca, Romania
email: nemab@math.ubbcluj.ro

Abstract
Two retractions $Q$ and $R$ on closed convex cones $M$ and respectively $N$ of a Banach space are called mutually polar if $Q + R = I$ and $QR = RQ = 0$. This note investigates the existence of a pair of mutually polar retractions for given cones $M$ and $N$. It is shown that if $\dim N = 1$ (or $\dim M = 1$) then the retractions are subadditive with respect to the order relation their cone ranges endow.

1. Introduction
If $X$ is a nonempty set, then the mapping $Q : X \to X$ is said idempotent, if $Q^2 = Q$. The aim of this note is to investigate idempotent mappings with real Banach space codomains and convex cone invariant sets.

Definition 1 Let $X$ be a Banach space. The mapping $T : X \to X$ is called retraction if:

(i) It is a continuous idempotent mapping;

(ii) It is positive homogeneous, that is, $T(tx) = tTx$ for every $x \in X$ and every $t \in \mathbb{R}_+ = [0, +\infty)$;

(iii) $T(X) = \text{rng} T$ is a non-empty, non-zero closed convex cone.

(iv) $Tx \in \text{bdr} \text{rng} T$ for any $x \in X \setminus \text{rng} T$.

Definition 2 Let $X$ be a Banach space, $0$ its zero mapping and $I$ its identity mapping. The mappings $Q, R : X \to X$ are called mutually polar retractions if

(i) $Q$ and $R$ are retractions,

(ii) $Q + R = I$,
We will say that $Q$ and $R$ are polar of each other.
(For the detailed terminology and examples see the next section.)

**Remark 1** In our note [5] the retraction $Q$ onto a cone is said proper if the pair $Q$ and $I - Q$ are mutually polar retractions.

For technical reasons, instead of a proper retraction, we will use the notion of a pair of mutually polar retractions. We use tacitly this equivalence in our proofs, by applying the results of the above cited note.

While searching conditions for the existence of asymmetric vector norms, in the recent note [5], we considered mutually polar retractions, related to the order relations their cone ranges endow.

The order theoretic restrictions seem to be severe and restrict substantially the existence of mutually polar retractions. We will see that regardless of the order theoretic restrictions, the class of mutually polar retractions is substantially larger.

The main result of this note is to construct mutually polar retractions with given cone ranges for a reasonable general class of pairs of cones, and to relate our construction to order theoretic restrictions on retractions. The notion of the transversal mutually polar retractions is introduced. If one of the cone ranges of a mutually polar pair of retractions is one dimensional, then they are transversal and both of them are subadditive with respect to the order relation their cone ranges endow.

In Section 2 we will introduce our terminology and will give important examples of mutually polar retractions from the literature.

In Sections 3-5 the transversal mutually polar retractions are defined and some of their important properties are proved.

Section 6 is concerned with some order theoretic properties of the retractions.

Subsection 6.1 contains the proof of one of the main results of our note. The proof is strongly related to the ideas and techniques in [5], some of which are reproduced there.

To keep the spirit of the note, we complete our exposition with some of the main results in [5], which are reproduced without proofs in Subsections 6.2 and 6.3.

## 2. Terminology and preliminaries

Let $X$ be a real vector space.

The nonempty set $K \subset X$ is called a convex cone if

(i) $\lambda x \in K$, for all $x \in K$ and $\lambda \in \mathbb{R}_+$ and if

(ii) $x + y \in K$, for all $x, y \in K$. 

(iii) $QR = RQ = 0$. 

We will say that $Q$ and $R$ are polar of each other.
(For the detailed terminology and examples see the next section.)
A convex cone $K$ is called pointed if $K \cap (-K) = \{0\}$.

A convex cone is called generating if $K - K = X$.

If $X$ is a Banach space, then a closed, pointed, convex cone with non-empty interior is called proper.

The relation $\leq$ defined by the pointed convex cone $K$ is given by $x \leq y$ if and only if $y - x \in K$. Particularly, we have $K = \{x \in X : 0 \leq x\}$. The relation $\leq$ is an order relation, that is, it is reflexive, transitive and antisymmetric; it is translation invariant, that is, $x + z \leq y + z$, $\forall x, y, z \in X$ with $x \leq y$; and it is scale invariant, that is, $\lambda x \leq \lambda y$, $\forall x, y \in X$ with $x \leq y$ and $\lambda \in \mathbb{R}_+$.

Conversely, for every $\leq$ scale invariant, translation invariant and antisymmetric order relation in $X$, there is a pointed convex cone $K$, defined by $K = \{x \in X : 0 \leq x\}$, such that $x \leq y$ if and only if $y - x \in K$. The cone $K$ is called the positive cone of $X$ and $(X, \leq)$ (or $(X, K)$) is called an ordered vector space; we use also the notation $\leq = \leq_K$.

The mapping $R : (X, \leq) \to (X, \leq)$ is said to be isotope if $x \leq y \Rightarrow Rx \leq Ry$, and subadditive if $R(x + y) \leq Rx + Ry$, for any $x, y \in X$.

The ordered vector space $(X, \leq)$ is called lattice ordered if for every $x, y \in X$, there exists $x \lor y := \sup\{x, y\}$. In this case the positive cone $K$ is called a lattice cone. A lattice ordered vector space is called a Riesz space. Denote $x^+ = 0 \lor x$ and $x^- = 0 \lor (-x)$. Then, $x = x^+ - x^-$, $x^+$ is called the positive part of $x$ and $x^-$ is called the negative part of $x$. The absolute value of $x$ is defined by $|x| = x^+ + x^-$. The mapping $x \mapsto x^+$ is called the positive part mapping.

**Example 1** Let $(X, \leq)$ be an ordered Banach space with $K$ its closed positive cone. The following assertions are part of the classical vector lattice theory (see e.g. [3] or [7].)

1. Using the properties of the positive part operator we can see that $+$ is a proper retraction.

2. The operator $^+ : X \to K$ is obviously idempotent.

3. We have $(tx)^+ = tx^+ \forall t \in \mathbb{R}_+, \forall x \in X$.

4. Let $x^- = (-x)^+$. Then $x = x^+ - x^-$, that is, $^+ - ^- = I$.

5. $(I - ^+)^+ = 0$.

6. The mapping $^- = I - ^+$ is a proper retraction onto $-K$.

Hence, $^+$ and $^-$ are mutually polar retractions with the cone range $K$ and $-K$ respectively.

Beside the above properties, the positive part operator $^+$

6. is isotope, i.e., $x \leq_K y \Rightarrow x^+ \leq_K y^+$;

7. is subadditive, i.e., $(x + y)^+ \leq_K x^+ + y^+$, $\forall x, y \in X$ (from the definition of the supremum).
In the particular case of $X = H$, where $(H, \langle \cdot, \cdot \rangle)$ is a separable Hilbert space of scalar product $\langle \cdot, \cdot \rangle$, we will need some further notions. Let $K \subseteq H$ be a closed convex cone. Recall that 

$$K^\circ = \{ x \in H : \langle x, y \rangle \leq 0, \forall y \in K \}$$

is called the polar cone of $K$. The cone $K^\circ$ is closed and convex, and if $K$ is generating, then $K^\circ$ is pointed (this is the case for example if $K$ is latticial).

**Example 2** Let $P : H \to K$ be the projection mapping onto the closed convex cone $K$, that is, the mapping defined by 

$$Px = \text{argmin} \{ \| x - y \| : y \in K \}.$$

The projection mapping $P$ is a proper retraction. This is the consequence of the following theorem proved in [4].

**Theorem 1 (Moreau)** Let $H$ be a Hilbert space, $K, L \subset H$ two mutually polar closed convex cones in $H$. Then, the following statements are equivalent:

(i) $z = x + y, \ x \in K, \ y \in L$ and $\langle x, y \rangle = 0$,

(ii) $x = P_K(z)$ and $y = P_L(z)$.

Let us denote by $P$ the projection onto $K$. Since $K$ and $K^\circ$ are mutually polar (Farkas lemma), we have from Moreau’s theorem the identity

$$x = Px + (I - P)x \quad \text{with} \quad \langle Px, (I - P)x \rangle = 0,$$

and the important consequence, that if $x = u + v$ with $u \in K, \ v \in K^\circ$ and $\langle u, v \rangle = 0$, then we must have $u = Px$ and $v = (I - P)x$.

From (1) $Px = 0$ if and only if $x \in (I - P)H = K^\circ$ and thus

$$P(I - P) = 0, \quad (I - P)P = 0.$$

Thus $P$ is a proper retraction and hence $P$ and $I - P$ are mutually polar retractions with the cone ranges $K$ and respectively $K^\circ$.

When $(H, \langle \cdot, \cdot \rangle) = (\mathbb{R}^m, \langle \cdot, \cdot \rangle)$, the $m$-dimensional Euclidean space, the set

$$K = \{ t^1x_1 + \cdots + t^mx_m : \ t^i \in \mathbb{R}_+, \ i = 1, \ldots, m \}$$

with $x_1, \ldots, x_m$ linearly independent vectors is called a simplicial cone. A simplicial cone is closed, pointed and generating.

The simplicial cones are related to vector lattices through the following important result of Youdine ( [8]).

**Theorem 2 (Youdine)** The pair $(\mathbb{R}^m, K)$ is a vector lattice with continuous lattice operations if and only if $K$ is a simplicial cone.
(For this reason in the vector lattice theory simplicial cones sometimes are called Youdine cones as well.)

**Definition 3** The Hilbert space $H$ ordered by the order relation induced by the cone $K$ is called a Hilbert vector lattice if (i) $K$ is a lattice cone, (ii) $\|x\| = \|x\|, \forall x \in H$, (iii) $0 \leq x \leq y$ implies $\|x\| \leq \|y\|$.

The cone $K$ is called self-dual, if $K = -K^\circ$. If $K$ is self-dual, then it is a generating, pointed, closed convex cone.

In all that follows we will suppose that $\mathbb{R}^m$ is endowed with a Cartesian reference system with the standard unit vectors $e_1, \ldots, e_m$.

The set
\[ \mathbb{R}^+_m = \{x = (x^1, \ldots, x^m) \in \mathbb{R}^m : x^i \geq 0, i = 1, \ldots, m\} \]
is called the nonnegative orthant of the above introduced Cartesian reference system. A direct verification shows that $\mathbb{R}^+_m$ is a self-dual cone.

Beside the terminology introduced in this section, we need some standard notions (hyperplane, core, relative interior etc.) and some classical results on cones and convex sets from functional analysis and convex geometry contained in the monographs [2], [6].

### 3. Transversal cones; transversal mutually polar retractions

**Definition 4** The pair of closed, pointed cones $M$ and $N$ in the Banach space $X$ will be called transversal, if the following conditions hold:

1. $M \cap N = \{0\}$;

2. There exist a straight line $\Delta$, called transversal line through $0$ in $X$ such that
   
   (a) $(M \setminus \{0\}) \cap \Delta \neq \emptyset$ and $(N \setminus \{0\}) \cap \Delta \neq \emptyset$;
   
   (b) $(\text{int } M \cup \text{int } N) \cap \Delta \neq \emptyset$.

**Definition 5** The mutually polar retractions $Q, R : X \rightarrow X$ are called transversal with the transversal line $\Delta$ through $\{0\}$, if for each two dimensional plane $L$ containing $\Delta$ it holds $Q : L \rightarrow L$ and $R : L \rightarrow L$.

**Proposition 1** If $Q, R : X \rightarrow X$ are mutually polar retractions and one of their cone ranges has interior points, then the cone ranges are transversal.

**Proof.**

Suppose that $\text{int } \text{rng } Q \neq \emptyset$. Take $z \in -\text{int } \text{rng } Q$. Then there exist $e \in \text{rng } Q$ and $u \in \text{rng } R$ such that $z = e + u$. Then

\[ u = z - e \in (-\text{int } \text{rng } Q - \text{rng } Q) \cap \text{rng } R \subset -\text{int } \text{rng } Q \cap \text{rng } R. \]

Hence $\Delta = \text{sp}\{u\}$ is a transversal line of $\text{rng } Q$ and $\text{rng } R$. \qed
Proposition 2 Suppose that \( Q, R : X \to X \) are mutually polar retractions with \( N = \text{rng } R \). If \( \dim N = 1 \), then they are transversal mutually polar retractions with the transversal line \( \text{sp } N \).

Proof.

Let us take \( x \in X \setminus (M \cup N) \). Denote \( Y = \text{sp}\{x, \text{sp } N\} \). We should check that \( Q, R : Y \to Y \).

Take \( x + y \) with some \( y \in \text{sp } N \). Obviously \( R(x + y) \in N \subset Y \). By definition \( x + y = R(x + y) + Q(x + y) \). Hence,

\[
Q(x + y) = x + y - R(x + y) \in x + \text{sp } N \subset Y.
\]

\[\blacksquare\]

4. Mutually polar retractions in \( \mathbb{R}^2 \)

Consider \( \mathbb{R}^2 \) endowed with a Cartesian reference system.

For a pair of transversal cones \( K_1 \) and \( K_3 \) in \( \mathbb{R}^2 \), the following two geometrical configurations are possible:

(i) Let \( I, II, III, IV \) be the quadrants of this reference system. Let \( e_1, e_2, u_1, u_2 \in \mathbb{R}^2 \) such that \( e_1 \in \text{int } I, \ e_2 \in \text{int } II, \ u_1 \in \text{int } III, \ u_2 \in \text{int } IV \).

\[
K_1 = \text{cone}\{e_1, e_2\}, \text{ and } K_3 = \text{cone}\{u_1, u_2\}.
\]

(ii) With the above notations, let \( e_1, e_2, u_1, u_2 \in \mathbb{R}^2 \) such that \( e_1 \in \text{int } I, \ e_2 \in \text{int } II, \ u = u_1 = u_2 \in III \cap IV \setminus \{0\} \).

\[
K_1 = \text{cone}\{e_1, e_2\}, \text{ and } K_3 = \text{cone}\{u\}.
\]

Theorem 3 If \( M \) and \( N \) are transversal closed pointed cones in \( \mathbb{R}^2 \), then they determine a unique pair of mutually polar retractions \( Q \) and \( R \), such that the cone range of \( Q \) and \( R \) are \( M \) and \( N \), respectively.

Proof.

(a) Using the notations introduced above consider the following cones:

1. \( K_1 = \text{cone}\{e_1, e_2\} \),
2. \( K_2 = \text{cone}\{e_2, u_1\} \),
3. \( K_3 = \text{cone}\{u_1, u_2\} \),
4. \( K_4 = \text{cone}\{u_2, e_1\} \).
Observe that the transversal cones $M = K_1$ and $N = K_3$ admit such a representation (accepting the particular case $u_1 = u_2$).

Then $\text{int } K_i \cap \text{int } K_j = \emptyset$ if $i \neq j$ and $\mathbb{R}^2 = K_1 \cup K_2 \cup K_3 \cup K_4$.

Let $x \in \mathbb{R}^2$ be arbitrary. We define the following operators $Q, R : \mathbb{R}^2 \rightarrow \mathbb{R}^2$:

\[
Qx = \begin{cases} 
  x & \text{if } x \in K_1 \\
  \lambda e_1 & \text{if } x \in K_4 \text{ and } x = \lambda e_1 + \mu u_2 \\
  \lambda e_2 & \text{if } x \in K_2 \text{ and } x = \lambda e_2 + \mu u_1 \\
  0 & \text{if } x \in K_3
\end{cases}
\]

\[
Rx = \begin{cases} 
  x & \text{if } x \in K_3 \\
  \mu u_2 & \text{if } x \in K_4 \text{ and } x = \lambda e_1 + \mu u_2 \\
  \mu u_1 & \text{if } x \in K_2 \text{ and } x = \lambda e_2 + \mu u_1 \\
  0 & \text{if } x \in K_1
\end{cases}
\]

A direct verification shows that $Q$ is a proper retraction with the cone range $K_1$, $R$ is a proper retraction with cone range $K_3$, that $Q + R = I$, and $QR = RQ = 0$. Hence, $Q$ and $R$ are mutually polar retractions.

(b) Our constructions of the mutually polar proper retractions $Q$ and $R$ holds also when $u_1 = u_2 \in K_3 = \{(0, -t) : t \in \mathbb{R}_+\} \setminus \{0\}$. In both cases $K_1$ and $K_3$ are transversal cones.

(c) Suppose that $Q$ and $R$ are mutually polar retractions in $\mathbb{R}^2$. We will show that their cone ranges are transversal.

From the relation $Q + R = I$, it follows that

\[K_1 + K_3 = \mathbb{R}^2, \quad \text{with } K_1 = \text{rang } Q, \ K_3 = \text{rang } R. \quad (3)\]

This relation shows that $K_1$ and $K_3$ cannot be both of dimension one. Hence, we can assume for instance that $\text{int } K_1 \neq \emptyset$. Then, by Proposition (b) the cones are transversal.

(d) For technical reasons, for $e \in \mathbb{R}^2 \setminus \{0\}$, we denote by

\[\text{ray } e = \{te : t \in \mathbb{R}_+\}\]

the one dimensional cone engendered by $e$.

Suppose that $e_1, e_2$ and $u_1, u_2$ are vectors generating $K_1 = \text{rang } Q$ and $K_3 = \text{rang } R$, respectively.

By an appropriate choice of the the reference system, we can realize the transversal line $\Delta$ of $K_1$ and $K_3$ be the perpendicular axis of the reference system.

Hence we must have such a geometric picture as in (a) (or (b)).

(d) We will finally see that $Q$ and $R$ are exactly of the form as in (a) (or (b)).

Take an element in the interior of $K_4 = \text{cone} \{e_1, u_2\}$, say $x = e_1 + u_2$. We will see that $Qx = e_1$ and $Rx = u_2$.

Obviously $x = e_1 + u_2 = Q(e_1 + u_2) + R(e_1 + u_2) \in K_1 + K_3$, since $Q + R = I$.

By definition we must have $Qx \in \text{bdr } K_1 = \text{ray } e_1 \cup \text{ray } e_2$. $Qx \neq 0$, since $Qx = 0$ would lead to the contradiction $x \in K_3$. 


If we had \( z = Qx \in \text{ray} e_2 \), then it would follow, by the continuity of \( Q \), that the line segment \( xw \), joining \( x \) and \( w \in \text{ray} e_1 \), is mapped by \( Q \) in a continuous curve on \( \text{bdr} K_1 \) joining \( w \) and \( z \). Such a curve must contain 0, as the image by \( Q \) of an interior point \( y \) of the segment \( xw \). The equality \( Qy = 0 \) would imply that an interior point \( y \) of \( K_4 \) is in \( K_3 \), which is a contradiction.

Hence, we must have \( Qx \in \text{ray} e_1 \).

By a similar argument, \( Rx \in \text{ray} u_2 \). Since \( x \) can uniquely be represented in the positive quadrant of the reference system endowed by the vectors \( e_1 \) and \( u_2 \), we must have \( Qx = e_1 \), \( Rx = u_2 \). Exploiting the positive homogeneity of retractions, it follows for an arbitrary \( y \in K_4 \), \( y = \lambda e_1 + \mu u_2 \), \( \lambda, \mu \in \mathbb{R}_+ \) that \( Qy = \lambda e_1 \), \( Ry = \mu u_2 \). (If \( \text{ray} u_1 = \text{ray} u_2 \), then \( \text{sp} u_i \cap \text{int cone} \{ e_1, e_2 \} \neq \emptyset \) and our proof is similar.)

This proves that \( Q \) and \( R \) are exactly the retractions constructed in (a) or (b).

\[ \square \]

**Corollary 1** Suppose that \( \mathbb{R}^2 \) is equipped with a norm. In what follows we will suppose that \( ||e_1|| = ||e_2|| = ||u_1|| = ||u_2|| = 1 \).

From the formulas defining the mappings \( Q \) and \( R \), it follows that they are continuous functions of the vectors \( e_1, e_2, u_1, u_2 \).

## 5. Mutually polar retractions on pair of transversal cones

**Proposition 3** Suppose that \( Q, R : X \to X \) are mutually polar retractions with \( M = \text{rng} Q \), \( N = \text{rng} R \). If \( \dim N = 1 \), and \( \text{int} M \neq \emptyset \), then they are transversal cones with \( \Delta = \text{sp} N \) their transversal line.

**Proof.** We have from Proposition 2 that \( \Delta \) is the transversal line of the retractions \( Q \) and \( R \). From the definition of transversal mutually polar retractions and Theorem 3, for each two-dimensional plane \( Y \) through \( \Delta \) we must have that \( M_Y = Y \cap M \) and \( N_Y = Y \cup N \) are mutually polar two dimensional cones in \( Y \). Since \( \dim N_Y = 1 \), the line \( \text{sp} N_Y \) (which in fact is exactly \( \Delta \)) must meet \( M_Y \) in its relative interior point \( y \). From convex geometric reasons, this means that \( y \in \text{core} M = \text{int} M \). This proves that \( M \) and \( N \) are transversal cones with \( \text{sp} N \) their transversal line.

\[ \square \]

**Theorem 4** Consider the transversal cones \( M \) and \( N \) in the Banach space \( X \) with the transversal line \( \Delta \). If one of the following conditions hold

\begin{enumerate}
  \item \( \text{int} M \neq \emptyset \), \( \text{int} N \neq \emptyset \), or
  \item \( \dim N = 1 \),
\end{enumerate}
then there exists a unique pair of transversal mutually polar retractions \( Q \) and \( R \) with transversal line \( \Delta \) and the cone ranges \( M \) and \( N \), respectively.

Proof.

(i) Suppose that \( M \) and \( N \) is a pair of transversal cones in \( X \) with the transversal line \( \Delta \).

We will construct a pair of mutually polar proper retractions \( Q \) and \( R \) with the cone range \( M \) and \( N \) respectively. 

Take \( x \in X \setminus (M \cup N) \) Let \( H \) be the two dimensional plane 

\[
H = \text{sp}\{x, \Delta\}.
\]

The plane \( H \) intersects the cones \( M \) and \( N \) in the pair of transversal cones \( M_H = M \cap H \) and \( N_H = N \cap H \).

If we identify \( H \) with \( \mathbb{R}^2 \) and take an appropriate reference system therein, then we can suppose that

\[
M_H = \text{cone}\{e_1, e_2\}, \quad N_H = \text{cone}\{u_1, u_2\},
\]

with appropriate unit vectors \( e_1, e_2, u_1 \) and \( u_2 \) in \( H \). Next we can construct the mutually polar proper retractions \( Q_H \) and \( R_H \) in \( H \) as it was done in the proof of Theorem 3.

Since \( \text{int} M \neq \emptyset \) and \( \text{int} N \neq \emptyset \), from convex geometric reasons, the vectors \( e_i, u_j, \|e_i\| = \|u_j\| = 1, i, j = 1, 2 \) depend continuously on \( x \) and the same thing is valid for \( Q_H \) and \( R_H \).

Define now the mappings

\[
Qx = Q_H x \text{ if } x \in H, \quad Rx = R_H x, \text{ if } x \in H.
\]

The pair of mappings \( Q \) and \( R \) is obviously a pair of mutually polar proper retractions with the cone ranges \( M \) and \( N \), respectively.

From Theorem 3 it follows that \( Q_H \) and \( R_H \) are well determined by their cone ranges and hence so are \( Q \) and \( R \).

(ii) Our constructions of the mutually polar retractions \( Q \) and \( R \) hold also when \( u_1 = u_2 \in K_3 = \{(0, -t) : t \in \mathbb{R}_+\} \setminus \{0\} \).

\[\square\]

Corollary 2 If \( Q, R : X \rightarrow X \) are mutually polar retractions in the Banach space \( X \) with the cone ranges \( M \) and respectively \( N \) with \( \text{int} M \neq \emptyset \) and \( \dim N = 1 \), then \( Q \) and \( R \) are well defined by their cone ranges.

Proof.

By Proposition 2 \( Q \) and \( R \) are transversal retractions, and hence, from Theorem 3, it follows that they are well defined by their cone ranges.

\[\square\]
6. Mutually polar retractions with order theoretic properties

We will say that a retraction $S$ is subadditive or isotone, if it has this property with respect to the order relation defined by its cone range $\text{rng} S$. Thus, by saying that the mutually polar retractions $Q$ and $R$ are subadditive or isotone, we mean that $Q$ has the corresponding property with respect the order relation endowed by $\text{rng} Q$ and $R$ has the corresponding property with respect to the order relation endowed by $\text{rng} R$.

6.1 Subadditive one range retractions

For the sake of completeness of the proofs, in this subsection we will reproduce some results and proofs from Section 5 and Section 6 of [5].

In [5] a retraction $R : X \to X$ was called one range, if $N = \text{rng} R = R(X)$ is one dimensional.

Lemma 1 If $R : X \to X$ is a one range retraction, then it is of form $Rx = q(x)u$, where $u \in N = \text{rng} R$, and $q$ is positive homogeneous functional with $q(u) = 1$.

Proof. The representation of $Rx$ in the form $q(x)u$, with some $u \in N \setminus \{0\}$ and $q$ a non-negative positive homogeneous functional is obvious.

By definition, $R$ is idempotent, hence we must have

$$q(x)u = Rx = R(Rx) = R(q(x)u) = q(x)Ru = q(x)q(u)u,$$

whereby $q(u) = 1$. □

Lemma 2 For the one range retraction $T$, with $Tx = q(x)u$, there exists the idempotent mapping $S : X \to X$ such that $T + S = I$ and $TS = ST = 0$ if and only if

$$q(x - q(x)u) = 0, \forall x \in X. \quad (4)$$

Proof. Suppose that $S$ is an idempotent mapping with the property in the lemma. Then $S = I - T$ and

$$Sx = S(Sx) = (I-T)((I-T)x) = (I-T)(x-Tx) = (I-T)(x-q(x)u) = x-q(x)u-T(x-q(x)u) =$$

whereby we have relation (4).

Conversely, if (4) holds, then from (3), $S$ is idempotent.

Further,

$$TSx = T(I-T)x = T(x - q(x)u) = q(x - q(x)u)u,$$

and by (1) it follows that $TS = 0$.
From the idempotent property of $T$ it follows that

$$ST = (I - T)T = T - T^2 = T - T = 0.$$ 

\[\square\]

**Remark 2**

1. We observe that for three non-linear mappings $E$, $F$ and $G$ we have the distributivity "from right", that is, by definition

$$(E - F)G = EG - FG,$$

but in general the "distributivity from left" does not hold, that is, in general

$$G(E - F) \neq GE - GF.$$

2. If in Lemma 2 the range $SX$ of $S$ would be a closed convex cone, then we would be able to assert that $T$ and $S$ are mutually polar retractions.

**Definition 6** [1] The functional $q : X \to \mathbb{R}_+$ is said an asymmetric norm if the following conditions hold:

1. $q$ is positive homogeneous, i.e., $q(tx) = tq(x)$, $\forall t \in \mathbb{R}_+$, $\forall x \in X$;
2. $q$ is subadditive, i.e., $q(x + y) \leq q(x) + q(y)$, $\forall x, y \in X$;
3. If $q(x) = q(-x) = 0$ then $x = 0$.

**Example 3** If $C \subset X$ is a closed convex set with $0 \in \text{int} C$, then the functional $q : X \to \mathbb{R}_+$ is called the gauge of $C$ if

$$q(x) = \inf \{t \in \mathbb{R}, t > 0 : x \in tC\}.$$ 

The gauge is an asymmetric norm (see e.g. [1], p. 165).

**Proposition 4** If $q : X \to \mathbb{R}_+$ is an asymmetric norm satisfying the relation (4) with some $u$, $q(u) = 1$, then $Rx = q(x)u$ and $Q = I - R$ are mutually polar retractions.

**Proof.** By Lemma 2 and item 2 of Remark 2 we only have to see that $\text{rng} Q = Q(X)$ is a closed convex cone.

From $R + Q = I$, it follows that $\text{rng} Q = \ker R = \{x \in X : Rx = 0\}$. Hence, we have to show that $\ker R$ is a closed convex cone. It is closed since $R$ is continuous. Since $\ker R = \ker q$, we must see that if $q(x) = 0$, then $q(tx) = 0$ for $t \in \mathbb{R}_+$, and that $q(x) = q(y) = 0$ implies $q(x + y) = 0$. The first relation follows from the positive homogeneity of the asymmetric norm. From the subadditivity of the asymmetric norm we have $0 \leq q(x + y) \leq q(x) + q(y) = 0$. This shows that $\text{rng} Q = \ker q$ is a convex cone. 

\[\square\]
Proposition 5 (Proposition 4 [5]) Consider the space $\mathbb{R} \times X$ and denote by $(t, x)$ the sum $t + x, t \in \mathbb{R}, x \in X$.

Suppose that $g : X \to \mathbb{R}_+$ is an asymmetric norm.

Define

$$q : \mathbb{R} \times X \to \mathbb{R}_+ \text{ by } q(t, x) = (t + g(x))^+.$$ 

Then $q$ is an asymmetric norm satisfying the relation

$$q((t, x) - q(t, x)(1, 0)) = 0, \quad \forall (t, x) \in \mathbb{R} \times X,$$

and hence

$$Q : \mathbb{R} \times X \to \mathbb{R} \times X, \quad Q((t, x)) = q((t, x))(1, 0)$$

is a range one subadditive proper retraction.

Proof.

The functional $q$ is obviously positively homogeneous. To prove its subadditivity, we must verify the relation

$$q((t_1, x_1) + (t_2, x_2)) = (t_1 + t_2 + g(x_1 + x_2))^+ \leq (t_1 + g(x_1))^+ + (t_2 + g(x_2))^+ = q((t_1, x_1)) + q((t_2, x_2)).$$

If all the involved sums in the round brackets are non-negative, the relation (8) follows from the subadditivity of $g$.

Suppose that $t_1 + g(x_1) \leq 0$. Then

$$t_1 + t_2 + g(x_1 + x_2) \leq -g(x_1) + t_2 + g(x_1 + x_2) \leq -g(x_1) + t_2 + g(x_1) + g(x_2) = t_2 + g(x_2)$$

and the relation (8) follows independently from the signs of the involved terms.

Suppose $q((t, x)) = q(-(t, x)) = 0$, that is, $t + g(x) = -t + g(-x) = 0$. Then $g(x) + g(-x) = 0$ and since both the terms are non-negative, they must be zero. We have first that $x = 0$ and then that $t = 0$.

Thus $q$ is an asymmetric norm. To verify relation (4), it is sufficient to consider that $q((t, x)) = (t + g(x))^+ = 1$. Then we get

$$q((t, x) - (1, 0)) = 0, \quad \text{with } q((t, x)) = 1.$$ 

But $q((t, x) - (1, 0)) = q((t - 1, x)) = (t - 1 + g(x))^+ = 0$, and the relation (4) follows.

According to Proposition 4, the operator $Q$ defined at (7) is a proper range one retraction.

$\square$

Proposition 6 Let $Q$ be the retraction constructed in Proposition 5. Then, by denoting $K = \mathbb{R}_+(1, 0)$, we have for $K^Q = (I - Q)X$ that

$$K^Q = \{(t, x) : t + g(x) \leq 0\}$$

and

$$(I - Q)((t, x)) = (t, x) - q((t, x))(1, 0)$$

is a subadditive proper retraction with the cone range $K^Q$. 
Proof.
Since $Q$ and $I - Q$ are mutually polar retractions, it follows that $K^Q = \text{rng}(I - Q) = \ker Q = \{(t, x) : t + g(x) \leq 0\}$.

Observe that $-K \subset K^Q$ since $-1 + g(0) = -1 \leq 0$, that is, $(-1, 0) \in K^Q$. We have

$$(I - Q)x + (I - Q)y - (I - Q)(x + y) = -Qx - Qy + Q(x + y) \in -K \subset K^Q,$$

which shows the subadditivity of $I - Q$.

Definition 7 The subset $B \subset K$ is called the basis of the cone $K$ if for each $x \in K \setminus \{0\}$ there exists a unique $\lambda > 0$ such that $\lambda x \in B$.

Lemma 3 If the Banach space $X$ is separable, $Q, R : X \to X$ are mutually polar retractions with $M = \text{rng} Q$, $N = \text{rng} R$, and dim $N = 1$, then $M = \text{rng} Q$ poses a basis $B$ on a hyperplane $H_0$ with $\text{sp} N \cap \text{relint} B \neq \emptyset$.

Proof. Since $M$ is a proper cone and $X$ is separable, there exists a hyperplane $H$ through $0$ such that $(M \setminus \{0\}) \cap H = \emptyset$. From geometric reasons, $H$ can be taken such that $H \cap N = \{0\}$. Since by Proposition 2 and Proposition 3 $\text{sp} N \cap \text{int} M \neq \emptyset$, by taking $u \in (N \setminus \{0\})$, it follows that $-u \in \text{int} M$. Then $B = M \cap H_0$ with $H_0 = H - u$ will be a basis of $M$ and $-u = \text{sp} N \cap \text{relint} B \neq \emptyset$.

The main result of this subsection is the

Theorem 5 Suppose that $X$ is a separable Banach space.

1. If $Q, R : X \to X$ are mutually polar retractions with $\text{int} \text{rng} M \neq \emptyset$ and dim $\text{rng} R = 1$, then they are well defined by their cone ranges and are subadditive.

2. If $M$ and $N$ are closed transversal cones in $X$ and dim $N = 1$, then there exist a unique pair of mutually polar retractions $Q$ and $R$ with $\text{rng} Q = M$, $\text{rng} R = N$. $Q$ and $R$ are subadditive.

Proof.
(a) Using the notations in Lemma 3 we will identify $H$ with a Banach space. Translate the basis $B$ along $\text{sp} N$ to the origin. Denote the translated set by $D$, and let $g$ be its gauge in $H$.

(b) Let us consider the space $X$ represented as

$$X = \text{sp} N + H.$$

Then $x = tu + y$ with $y \in H$, $u \in N \setminus \{0\}$, $t \in \mathbb{R}$. The element $x = tu + y$ can be denoted by $x = (t, y)$. 

13
Define $T : X \to X$ as follows:

$$T(x) = (t, y) = (t + g(y)) + u.$$ 

(c) Repeating step by step the proof in Proposition 5 and Proposition 6, we conclude that $T$ is a regular range one subadditive retraction on $N$. Thus $S = I - T$ is a retraction too, with its cone range $\{(t, y) : t + g(y) \leq 0\}$.

We will show that $M = \{(t, x) : t + g(x) \leq 0\}$. (9)

Take $(t, x) \in M \setminus \{0\}$. Then there exists the unique positive $\lambda$ with $\lambda(t, x) \in B$. Thus $\lambda(t, x) = (-1, \lambda x)$. Hence, $\lambda t = -1$ and $\lambda x \in D$. Accordingly, $g(\lambda x) \leq 1$. Putting $\lambda = -\frac{1}{t}$ in the last relation, we get $g(-\frac{1}{t}x) \leq 1$ and since $t$ is negative and $g$ is positive homogeneous, $g(x) \leq -t$, that is, $t + g(x) \leq 0$. Hence, $(t, x)$ is in the second term of the equality (9).

Conversely, if $t + g(x) < 0$, then $g(-\frac{1}{t}x) \leq 1$, hence $-\frac{1}{t}x \in D$. Thus $(-1, -\frac{1}{t}x) \in B$ and $-t(-1, -\frac{1}{t}x) = (t, x) \in M$, which completes the proof of the relation (9).

(d) Thus we have constructed the mutually polar, subabitive retractions $S$ and $T$ with the cone ranges $M$ and $N$, respectively. But then from Corollary 2, we must have $Q = S$ and $R = T$. Hence, $Q$ and $R$ are both subadditive, and the proof of item 1 of the theorem is complete.

(e) The proof of item 2 of the theorem follows by using Theorem 4, Corollary 2, and item 1 of the theorem.

\[ \square \]

6.2 Mutually polar subadditive retractions with cone ranges with nonempty interiors

To be in line with our above exposition, we give here without proof the summary of results in Section 7 in [5]:

**Theorem 6** Suppose that $Q$ and $R$ are mutually polar proper retractions with the cone ranges $M$ and $N$, respectively. If $\text{int} M \neq \emptyset$ and $\text{int} N \neq \emptyset$, then the following conditions are equivalent:

1. $Q$ and $R$ are subadditive;

2. $Q$ and $R$ are isotone;

3. $M$ or $N$ are lattice cones, $N = -M$, and $Q, R = I - Q$ are the positive part operators in the vector lattices endowed by the positive cones $M$ and $N$, respectively.

If $\dim H < \infty$ then besides the above equivalent properties we have via the Theorem of Youdine the equivalent condition: 4. The cone range of $Q$ is a simplicial cone $K$ and the cone range of $R$ is the simplicial cone $-K$. 

14
6.3 Metric projection on the cone with order theoretic properties

Some results in Section 8 of [5] can be gathered as follows:

**Theorem 7** If $K$ is a nonempty generating closed cone in the separable Hilbert space, then for the metric projection $P$ onto $K$ and its polar $I - P$ we have:

1. If $P$ is subadditive, then $I - P$ is isotone;
2. If $P$ is isotone, then $I - P$ is subadditive;
3. $P$ and $I - P$ are subadditive (isotone) if and only if $(H, K)$ is a Hilbert vector lattice;
4. If $\dim H < \infty$ the last condition is equivalent with the condition that $K$ is the positive orthant of a Cartesian reference system.

**References**

[1] S. Cobzas. *Functional Analysis in Asymmetric Normed Spaces*. Birkhauser, 2013.

[2] M. G. Krein and M. A. Rutman *Lineinye operatory ostavlyayuscie invariantnym konus v prostranstve Banaha*. Uspehi Mat. Nauk 3 no 1, (23) 3-95, 1948.

[3] W.A.J. Luxemburg and A.C. Zaanen. *Riesz Spaces*. North Holland, 1971.

[4] J. J. Moreau. *Décomposition orthogonale d’un espace hilbertien selon deux cônes mutuellement polaires*. C. R. Acad. Sci., 255:238–240, 1962.

[5] A. B. Németh and S. Z. Németh. *Subadditive retractions on cones and asymmetric vector norms*. ArXiv 2005 10508, 2020.

[6] R. T. Rockafellar. *Convex Analysis*. Princeton, N.J. 1970.

[7] H. H. Schaefer. *Banach Lattices and Positive Operators*. Springer Verlag 1974.

[8] A. Youdine. *Solution des deux problèmes de la théorie des espaces semi-ordonnés*. Comptes Rendus (Doklady) de l’Académie des Sciences de l’URSS, 23:418–422, 1939.