THE CANONICAL SYMMETRY AND HAMILTONIAN FORMALISM.
II. HAMILTONIAN OPERATORS

A. N. Leznov and A. V. Razumov
Institute for High Energy Physics, 142284 Protvino, Moscow Region, Russia

Abstract
It is shown how the canonical symmetry is used to look for the hierarchy of the Hamiltonian operators relevant to the system under consideration. It appears that only the invariance condition can be used to solve the problem.

1E-mail: razumov@mx.ihep.su
1 Introduction

It has been recently observed [1, 2] that most of nonlinear integrable equations, not only two–dimensional ones, possess a special discrete symmetry, that appears very useful in many respects. In Ref. [3] we demonstrated that this symmetry considered as a transformation of the phase space of the system is a canonical transformation. This fact was one of the reasons to call the considered transformations the canonical symmetry. In the next paper [4] we considered the behaviour of the densities of local conservation laws under the canonical symmetry for the case of the nonlinear Schrödinger equation. These densities appeared to change under the action of the canonical symmetry by the total space derivatives. We have noted in [4] that this observation does not give us a constructive method to find the local conservation laws for the system under consideration.

In the present paper we continue the investigation of the properties of the canonical symmetry from the Hamiltonian point of view. In particular, we show how the canonical symmetry can be used to look for different Hamiltonian operators relevant to the system. It is known that if two Hamiltonian operators, that can be used to write the considered evolution equations as the Hamilton equations, form a Hamiltonian pair, then we can find the whole hierarchy of the involutive conservation laws, starting from the given one [5]. It appears that the use of the canonical symmetry leads us to a set of the Hamiltonian operators, any two of which form a Hamiltonian pair. Thus, the canonical symmetry not only gives a new characterization of the local conservation laws but also helps us to find them.

In this paper we use the notations and conventions of our paper [4]. Thus, we consider the case of two independent variables \( t, x \) and \( A \) dependent variables \( u^a, a = 1, \ldots, A \). We denote by \( \mathcal{A} \) the space of the functions of the variables \( x, u^a \), and the derivatives of \( u^a \) over \( x \) up to some finite order. For a general function of such type we use the following notation

\[
 f[u] = f(x, u, (1)u, \ldots, (K)u),
\]

where for any \( k \) \((k)u\) is the set formed by the \( k \)-th derivatives \((k)u^a\) of the dependent variables \( u^a \) over \( x \). We assume also that \((0)u^a \equiv u^a\). Recall that we call the Hamilton equations evolution equations, that can be written in the form

\[
 u_t^a = J^{ab}[u] \frac{\delta h[u]}{\delta u^b},
\]

where \( J \) is a Hamiltonian operator [4, 5]. In the present paper we consider Hamiltonian operators that include inverse powers of the operator of the total derivative over \( x \). To deal with such operators we have to use a generalization of the usual binomial coefficients. This generalization is reminded of in section 2. In section 3 we give the necessary facts on integro–differential operators. Section 4 is devoted to the consideration of the invariance of a matrix integro–differential operator under the action of a differential transformation. This condition for the case of a Hamiltonian operator is equivalent to the requirement of the canonicity of the transformation. In section 5 we consider a partial case of the second order transformations. Using as examples the nonlinear Schrödinger equation and modified nonlinear Schrödinger equation we find all matrix skew-symmetric operators which satisfy the invariance condition for the corresponding canonical symmetry. It appears that they
are actually the Hamiltonian operators, forming the well known hierarchies of Hamiltonian operators for the nonlinear equations under consideration.

## 2 Binomial Coefficients

Here we define the binomial coefficients \( \binom{n}{k} \) for arbitrary integers \( n \) and \( k \), such that \( n \geq k \). Recall that for \( n, k \geq 0 \) and \( n \geq k \) we have

\[
\binom{n}{k} = \frac{n!}{k!(n-k)!}.
\]

(2.1)

Let us write the well known relation

\[
(1 + x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k.
\]

(2.2)

It is convenient to put for \( k < 0 \)

\[
\binom{n}{k} = 0.
\]

(2.3)

Hence, we can rewrite Eq. (2.2) in the form

\[
(1 + x)^n = \sum_{k=-\infty}^{n} \binom{n}{k} x^k.
\]

(2.4)

We use this formula as the definition of the binomial coefficient for \( n < 0 \). In particular, for \( n = -1 \) we have

\[
(1 + x)^{-1} = \sum_{k=-\infty}^{-1} (-1)^{-k-1} x^k,
\]

(2.5)

hence, for \( k \leq -1 \)

\[
\binom{-1}{k} = (-1)^{-k-1}.
\]

(2.6)

For any \( n < 0 \) we can write

\[
(1 + x)^n = (-1)^{-n-1} \frac{1}{(-n-1)!} \frac{d^{-n-1}}{dx^{-n-1}} (1 + x)^{-1}.
\]

(2.7)

Differentiating Eq. (2.5), we get

\[
\binom{n}{k} = (-1)^{-n-k} \binom{-k-1}{-n-1}.
\]

(2.8)

From this relation we have subsequently

\[
\binom{n}{n} = 1, \quad \binom{n}{n-1} = n, \quad \binom{n}{n-2} = \frac{n(n-1)}{2!}.
\]

(2.9)

It is clear that for any \( k > 0 \)

\[
\binom{n}{n-k} = \frac{n(n-1)\ldots(n-k+1)}{k!}.
\]

(2.10)

Note, that equality (2.10) is valid also for \( n > 0 \).
3 Integro–Differential Operators

In this section we shall give necessary facts on integro–differential operators. A more detailed exposition can be found, for example, in [6]. An integro–differential operator is a formal series of the form

\[ a = \sum_{k=-\infty}^{N(a)} a_k D^k, \] (3.1)

where \( a_k \in \mathcal{A} \). We suppose that \( a_{N(a)} \neq 0 \), and say that the number \( N(a) \) is the order of the operator \( a \).

The multiplication of integro–differential operators is defined as follows. The product of two integro–differential operators \( a \) and \( b \) is the integro–differential operator

\[ c = ab = \sum_{k=-\infty}^{N(c)} c_k D^k, \] (3.2)

where \( N(c) = N(a) + N(b) \), and

\[ c_k = (ab)_k \equiv \sum_{l=k-N(b)}^{N(a)} \sum_{m=k-N(a)}^{l} \binom{l}{m} a_l D^{l-m}(b_{k-m}) \]

\[ = \sum_{l=k-N(a)}^{N(b)} \sum_{m=k-N(a)}^{l} \binom{k-m}{k-l} a_{k-m} D^{l-m}(b_l). \] (3.3)

This relation can be formally obtained by the multiplication of the corresponding formal series and by the appropriate resummation. The direct calculation shows that thus defined product is associative.

The transposed operator \( a^T \) of the integro–differential operator \( a \) is the operator

\[ a^T = \sum_{k=-\infty}^{N(a)} (a^T)_k D^k, \] (3.4)

where

\[ (a^T)_k \equiv \sum_{l=k}^{N(a)} (-1)^l \binom{l}{k} D^{l-k}(a_l). \] (3.5)

In particular, we have

\[ D^T = -D. \] (3.6)

We shall need below the following relation

\[ (ab^T)_k = \sum_{l=k-N(a)}^{N(b)} \sum_{m=k}^{l+N(a)} (-1)^l \binom{m}{k} a_{m-l} D^{m-k}(b_l) \]

\[ = \sum_{l=k-N(b)}^{N(a)} \sum_{m=k}^{l+N(b)} (-1)^{m-l} \binom{m}{k} a_l D^{m-l}(b_{m-l}). \] (3.7)
An integro–differential operator $a$ is said to be invertible if there exists an integro–differential operator $b$ such that

$$ba = 1.$$  \hfill (3.8)

If it is the case we write $b = a^{-1}$. It is clear that $N(a^{-1}) = -N(a)$.

Suppose that an operator $a$ is invertible, then from Eq. (3.3) for $k = 0$ we get

$$(a^{-1})_{-N(a)}a_{N(a)} = 1.$$ \hfill (3.9)

Hence, the operator $a$ is invertible only if

$$a_{N(a)} \neq 0,$$ \hfill (3.10)

and in this case

$$(a^{-1})_{-N(a)} = \frac{1}{a_{N(a)}}.$$ \hfill (3.11)

From Eq. (3.3) for $k < 0$ we have

$$(a^{-1})_{k-N(a)}a_{N(a)} + \sum_{l=k-N(a)+1}^{-N(a)} \sum_{m=k-N(a)}^{l} \binom{l}{m} (a^{-1})_{l}D^{l-m}(a_{k-m}) = 0.$$ \hfill (3.12)

We can consider Eq. (3.12) as a recursive relation for the determination of the quantities $(a^{-1})_{k}$, and conclude from it that the operator $a$ is invertible if and only if Eq. (3.10) is satisfied.

The matrix integro–differential operator $A$ is a matrix with the matrix elements being integro–differential operators. The transposition operation can be naturally extended to the case of such operators. It is clear also that a matrix integro–differential operator $A$ is invertible if and only if the matrix $A_{N(A)}$ is invertible.

4 Invariance Condition

Recall [4] that the invertible differential transformation

$$\tilde{u}^{a} = \varphi^{a}[u] = \varphi^{a}(x, u, (1)u, \ldots, (K)u)$$ \hfill (4.1)

is a symmetry of Hamilton equations (1.2) if

$$\varphi'[u]J[u]\varphi'^{T}[u] = J[\varphi[u]]$$ \hfill (4.2)

and

$$h[\varphi[u]] - h[u] \in \text{Ker } \delta/\delta u.$$ \hfill (4.3)

In Eq. (4.2) $\varphi'$ denotes the Fréchet derivative of the differential transformation (4.1) [4, 4].
In papers [3, 4] we have shown that the canonical symmetry of a nonlinear system is a symmetry of the corresponding Hamilton equations. It is known that there are different ways to write integrable equations in the Hamiltonian form. In this the role of the corresponding Hamiltonians is played by different conserved quantities. We proved in [4], for the case of nonlinear Schrödinger equation, that the densities of local conservation laws change under the action of the canonical symmetry by total space derivatives. In other words, if we consider the density of a conservation law as the density of the corresponding Hamiltonian, then we see that it satisfies Eq. (4.3). It is natural to suppose that the corresponding Hamiltonian operator satisfies Eq. (4.2), which we call the invariance condition. We shall not demonstrate that directly, but consider a more general problem: what the Hamiltonian operators are that satisfy Eq. (4.2) for given differential transformation (4.1).

In this paper we consider only the case of two dependent variables $u^a$ and use the notation $q \equiv u^1$, $r \equiv u^2$. In this case the operator $J$ can be written as

$$J = \left(\begin{array}{cc} a & b \\ c & d \end{array}\right). \quad (4.4)$$

The condition $J = -J^T$ gives

$$a^T = -a, \quad d^T = -d, \quad (4.5)$$
$$c^T = -b, \quad b^T = -c, \quad (4.6)$$

thus, we can write

$$J = \left(\begin{array}{cc} a & b \\ -b^T & d \end{array}\right). \quad (4.7)$$

The operator $\varphi'$ in the case under consideration may be represented as

$$\varphi' = \left(\begin{array}{cc} \alpha & \beta \\ \gamma & \delta \end{array}\right), \quad (4.8)$$

and invariance condition (4.2) is equivalent to the relations

$$\begin{align*}
\tilde{a} &= \alpha \alpha a^T - \beta b^T \alpha^T + \alpha b^T \beta^T + \beta d^T \beta^T, \\
\tilde{b} &= \alpha \alpha \gamma^T - \beta b^T \gamma^T + \alpha b^T \delta^T + \beta d^T \delta^T, \\
-\tilde{b}^T &= \gamma \alpha a^T - \delta b^T \alpha^T + \gamma b^T \beta^T + \delta d^T \beta^T, \\
\tilde{d} &= \gamma \alpha \gamma^T - \delta b^T \gamma^T + \gamma b^T \delta^T + \delta d^T \delta^T,
\end{align*} \quad (4.9)-(4.12)$$

where $\tilde{a}[u] \equiv a[\varphi[u]]$, etc. Note that Eq. (4.11) can be obtained from Eq. (4.10) via the transposition. Thus, we have only three independent relations. In all the cases that we consider in this paper $\alpha = 0$ and $\beta, \gamma$ are simply functions. Therefore, we can rewrite system (4.9)-(4.12) in the form

$$\begin{align*}
\tilde{a} - \beta d \beta &= 0, \\
\delta d + \tilde{b}^T \beta^{-1} + \gamma b &= 0, \\
\delta \beta^{-1} \tilde{b} + \gamma b \delta^T + \gamma a \gamma - \tilde{d} &= 0.
\end{align*} \quad (4.13)-(4.15)$$
From this system we conclude, in particular, that

\[ N(b) = N(a) + N(\delta), \quad N(d) = N(a). \]  

(4.16)

Representing \( a \) and \( d \) as

\[ a = \sum_{k=-\infty}^{N(a)} a_k D^k, \quad d = \sum_{k=-\infty}^{N(d)} d_k D^k, \]  

(4.17)

and using Eq. (3.3), we get from Eq. (4.13) the following equalities

\[ \tilde{a}_k - \sum_{m=k}^{N(d)} \binom{m}{k} \beta D^{m-k} \delta = 0, \quad k \leq N(a). \]  

(4.18)

In the same way, using Eqs. (3.3) and (3.5), we write Eq. (4.14) in the component form \((k \leq N(b))\):

\[ \sum_{l=k-N(b)}^{N(\delta)} \sum_{m=k-N(b)}^{l} \binom{l}{m} \delta_l D^{l-m} (d_{k-m}) + \sum_{m=k}^{N(\delta)} (-1)^m \binom{m}{k} D^{m-k} (\beta^{-1} \tilde{b}_m) + \gamma b_k = 0. \]  

(4.19)

Using now Eq. (3.7), we get from Eq. (4.15) for \( N(a) < k \leq N(b) + N(\delta) \) the relations

\[ \sum_{l=k-N(\delta)}^{N(\delta)} \sum_{m=k-N(\delta)}^{l} \binom{l}{m} \delta_l D^{l-m} (\beta^{-1} \tilde{b}_{k-m}) + (-1)^l \binom{m + N(b)}{k} \gamma D^{m+N(b)-k} (\delta_l) b_{m+N(b)-l} = 0, \]  

(4.20)

while for \( k \leq N(a) \) we have

\[ \sum_{l=k-N(b)}^{N(\delta)} \sum_{m=k-N(b)}^{l} \binom{l}{m} \delta_l D^{l-m} (\beta^{-1} \tilde{b}_{k-m}) + (-1)^l \binom{m + N(b)}{k} \gamma D^{m+N(b)-k} (\delta_l) b_{m+N(b)-l} + \sum_{m=k}^{N(a)} \binom{m}{k} \gamma D^{m-k} (\gamma) a_m - \tilde{d}_k = 0. \]  

(4.21)

5 Invariance Condition for Second Order Transformations
5.1 General Consideration

In this section we consider invariance condition (4.13)–(4.15) for the case \( N(\delta) = 2 \), i.e. for the case when

\[
\delta = \delta_2 D^2 + \delta_1 D + \delta_0. \tag{5.1}
\]

First, suppose that there exists a Hamiltonian operator \( J \) of form (4.7) with the matrix elements being functions. In other words, we suppose that there exists a Hamiltonian operator of the order 0, having a special form. From Eq. (4.16) it follows that in this case \( a = d = 0 \). Denote the function, playing the role of the operator \( b \), by \( g \), and consider Eqs. (4.13)–(4.15). Eq. (4.13) is satisfied by definition. In the case under consideration Eq. (4.14) takes the form

\[
\tilde{g} + \beta \gamma g = 0. \tag{5.2}
\]

Note, that in all known cases the canonical symmetry has the following property. The function \( f[u] \), satisfying the relation

\[
f[\varphi[u]] = f[u], \tag{5.3}
\]

is simply a constant. From this property it follows that if there exists a function \( g \), satisfying Eq. (5.2), then this function is unique up to a constant factor.

Eq. (4.13) in our case is equivalent to

\[
\delta \gamma g = \gamma g \delta^T. \tag{5.4}
\]

It is easy to show that the function \( g \) satisfies Eq. (5.4) if and only if it satisfies the relation

\[
D(\delta_2) \gamma g - \delta_2 D(\gamma g) - \delta_1 \gamma g = 0, \tag{5.5}
\]

that can be written as

\[
D \left( \frac{\delta_2}{\gamma g} \right) = \frac{\delta_1}{\gamma g}. \tag{5.6}
\]

Proceed now to the investigation of invariance condition (4.13)–(4.15), for the case when the order of the operator \( b \) is an arbitrary integer \( N \). Denote the corresponding operator by \( J^{(N)} \). It is a priori unknown, that this operator exists. On the other hand, if it exists then it is defined at least up to an arbitrary linear combination of the operators \( J^{(k)} \) with \( k < N \).

Let us suppose that the operator \( J^{(N)} \) exists. From Eq. (4.20) in the case \( k = N + 2 \) we get

\[
\tilde{b}_N + \beta \gamma b_N = 0. \tag{5.7}
\]

Thus for any \( N \) the function \( b_N \) is determined up to a constant factor and is proportional to the function \( g \). Thus, taking into account Eq. (5.3) we see that

\[
D(\delta_2) \gamma b_N - \delta_2 D(\gamma b_N) - \delta_1 \gamma b_N = 0. \tag{5.8}
\]

For \( k = N + 1 \) from Eq. (4.20) we get the following relation

\[
\delta_2 (\bar{b}_{N-1} - \beta \gamma b_{N-1}) + ((N + 2)D(\delta_2) - \delta_1) \beta \gamma b_N + \delta_1 \tilde{b}_N + 2 \delta_2 \beta D(\beta^{-1} \tilde{b}_N) = 0. \tag{5.9}
\]
Using Eqs. (5.4) and (5.8), we can rewrite this equality in the form

$$\delta_2 (\tilde{b}_{N-1} + \beta \gamma b_{N-1}) + N \beta \gamma D(\delta_2) b_N = 0. \tag{5.10}$$

It is convenient to introduce the functions

$$f_k = b_k/b_N. \tag{5.11}$$

For the function $f_{N-2}$ from Eq. (5.10) we have

$$\tilde{f}_{N-1} - f_{N-1} = N \frac{D(\delta_2)}{\delta_2}. \tag{5.12}$$

Hence the function $f_{N-1}$, if it exists, is defined up to an additive constant. The function $b_{N-1}$, in turn, is defined up to a constant multiplied by the function $b_N$. Recall that the function $b_N$ is proportional to the function $g$. It is clear, that the operator $J(N)$ is defined up to a linear combination of the operators $J(k)$ with $k < N$. It is the reason of the ambiguity that we have come across.

After some rather long calculations from Eq. (4.20) for the case $k = N$ we get the following equality

$$\delta_2 (\tilde{b}_{N-2} + \beta \gamma b_{N-2}) - 2 \delta_2 \beta \gamma D \left( \frac{b_{N-1}}{b_N} \right) b_N - N \delta_2 \beta \gamma D \left( \frac{\delta_1}{\delta_2} \right) b_N$$

$$+ (N - 1) \beta \gamma D(\delta_2) b_{N-1} + \frac{1}{2} N(N - 1) \beta \gamma D^2(\delta_2) b_N = 0, \tag{5.13}$$

that can be written in terms of the functions $f_k$ as

$$\tilde{f}_{N-2} - f_{N-2} = -2D(f_{N-1}) - ND \left( \frac{\delta_1}{\delta_2} \right)$$

$$+ (N - 1) \frac{D(\delta_2)}{\delta_2} f_{N-1} + \frac{1}{2} N(N - 1) \frac{D^2(\delta_2)}{\delta_2}. \tag{5.14}$$

We see that the function $b_{N-2}$ is defined ambiguously, and the reason of this ambiguity is the same as for the function $b_{N-1}$ (see above).

Consider now Eq. (4.19). For $k = N$ we get

$$d_{N-2} = ((-1)^N - 1) \gamma \frac{b_N}{\delta_2}. \tag{5.15}$$

Thus for an odd $N$ we have

$$d_{N-2} = -2 \gamma \frac{b_N}{\delta_2}, \tag{5.16}$$

and for an even $N$

$$d_{N-2} = 0. \tag{5.17}$$

Note, that we have not actually used the fact that the operator $d$ must be skew-symmetric, but Eq. (5.17) is already in agreement with it.
We get further

$$d_{N-3} = -(N-2)D\left(\frac{2\gamma b_N}{\delta_2}\right) = \frac{N-2}{2}D(d_{N-2})$$  \hspace{1cm} (5.18)

for an odd \( N \), and

$$d_{N-3} = -\frac{2\gamma b_{N-1}}{\delta_2} - \frac{N\delta_1\gamma b_N}{\delta_2^2}$$  \hspace{1cm} (5.19)

for an even \( N \). Eq. (5.18) is again in agreement with the skew–symmetricity of the operator \( d \).

At last we get

$$d_{N-4} = -\frac{2\gamma b_{N-2}}{\delta_2} + \frac{2\delta_1\gamma b_{N-1}}{\delta_2^2} + (N+1)D\left(\frac{\gamma b_{N-1}}{\delta_2}\right)$$

$$+ \frac{2\delta_0\gamma b_N}{\delta_2^2} - \frac{1}{2}(N^2 - 3N + 4)D^2\left(\frac{\gamma b_N}{\delta_2}\right)$$  \hspace{1cm} (5.20)

for an odd \( N \), and

$$d_{N-4} = \frac{N-3}{2}D(d_{N-3})$$  \hspace{1cm} (5.21)

for an even \( N \).

The functions \( a_k \) can be determined from Eq. (4.18).

Let us make here some conclusions from our consideration of the invariance condition. The ambiguity in the definition of the operator \( (N)^{\sim}J \) can be described as follows. Let for any \( k \) the operator \( (k)^{\sim}J \) be some solution of the invariance condition. A general operator, satisfying the invariance condition, and having the order \( N \), is given by the formula

$$J = \sum_{k=-\infty}^{N} \nu_k (k)^{\sim}J,$$  \hspace{1cm} (5.22)

where \( \nu_k \) are some constants.

If we have two solutions, for example \( (0)^{\sim}J \) and \( (1)^{\sim}J \), of the invariance condition, and the operator \( (0)^{\sim}J \) is invertible, then the operator

$$\left(2\right)^{\sim}J = (1)^{\sim}J (0)^{\sim}J^{-1} (1)^{\sim}J$$  \hspace{1cm} (5.23)

is of the order two and also satisfies the invariance condition. Analogously, we get the operators, having any positive order. If the operator \( (1)^{\sim}J \) is invertible, then we can construct an operator of any negative order. For example, the operator of the order \(-1\) has the form

$$(-1)^{\sim}J = (9)^{\sim}J (1)^{\sim}J^{-1} (0)^{\sim}J.$$  \hspace{1cm} (5.24)
5.2 Nonlinear Schrödinger Equation

As a first concrete example, we consider the nonlinear Schrödinger equation. In fact to define the canonical symmetry we should consider the following complex extension of it [3]:

\[ i\dot{q} + q'' - 2\epsilon rq^2 = 0, \quad i\dot{r} - r'' + 2\epsilon qr^2 = 0, \]  
(5.25)

where \( q \) and \( r \) are arbitrary complex functions of the variables \( x \) and \( t \), \( \epsilon \) is the coupling constant. In Eq. (5.25) and below dot and prime mean the partial derivative over \( t \) and \( x \), respectively. The canonical symmetry for this system has the form [1, 2, 7]

\[ \tilde{q} = \frac{1}{\epsilon r}, \quad \tilde{r} = \epsilon r^2 q - r'' + \frac{r'^2}{r}. \]  
(5.26)

Thus, we have

\[ \beta = -\frac{1}{\epsilon r^2}, \quad \gamma = \epsilon r^2, \]  
(5.27)

\[ \delta = -D^2 + 2\frac{r'}{r} D + 2\epsilon rq - \frac{r'^2}{r^2}. \]  
(5.28)

Equations (5.25) can be written in Hamiltonian form (1.2) if we choose

\[ J = ^{(0)}J = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \]  
(5.29)

and

\[ h[q, r] = r'q' + \epsilon r^2 q^2. \]  
(5.30)

It is not difficult to show that the operator \(^{(0)}J\) satisfies the invariance condition [3, 4]. Hence, in this case the function \( g \) is a constant function.

Recall [3, 4] that for the conserved quantity \( rq \) we have

\[ \tilde{r}\tilde{q} = rq - D \left( \frac{r'}{\epsilon r} \right). \]  
(5.31)

Consider now the case \( N = 1 \). From Eq. (5.7) we get

\[ \tilde{b}_1 - b_1 = 0, \]  
(5.32)

hence, the function \( b_1 \) is a constant function. It is convenient to normalize the operator \(^{(1)}J\) by the condition

\[ b_1 = -1. \]  
(5.33)

From Eq. (5.10) we have

\[ \tilde{b}_0 - b_0 = 0. \]  
(5.34)

Let us put

\[ b_0 = 0. \]  
(5.35)
Eq. (5.13) is more nontrivial:

\[ \tilde{b}_{-1} - b_{-1} + 2D \left( \frac{r'}{r} \right) = 0. \] (5.36)

Comparing this relation with Eq. (5.31), we see that we can choose

\[ b_{-1} = 2\epsilon rq. \] (5.37)

From Eqs. (5.16), (5.18) and (5.20) we find

\[ d_{-1} = -2\epsilon r^2, \quad d_{-2} = 2\epsilon r', \quad d_{-3} = -2\epsilon rr'', \] (5.38)

while Eq. (4.18) gives

\[ a_{-1} = -2\epsilon q^2, \quad a_{-2} = 2\epsilon qq', \quad a_{-3} = -2\epsilon qq''. \] (5.39)

Taking into account the expressions obtained for \( b_k, d_k \) and \( a_k \) we can suppose that the operator \( J^{(1)} \) has the form

\[
J^{(1)} = \begin{pmatrix}
-2\epsilon qD^{-1}q & -D + 2\epsilon qD^{-1}r \\
-D + 2\epsilon rD^{-1}q & -2\epsilon rD^{-1}r
\end{pmatrix}.
\] (5.40)

A direct calculation shows that the operator, given by Eq. (5.40) satisfies the invariance condition. Thus, we get the well known second Hamiltonian operator for the nonlinear Schrödinger equation \[8, 9\] as a solution of the invariance condition.

### 5.3 Modified Nonlinear Schrödinger Equation

We consider here the following complex extension \[1, 4, 8\] of the modified nonlinear Schrödinger equation \[10\]

\[ i\dot{q} + q'' - 2i\epsilon (rq)q' = 0, \quad i\dot{r} - r'' - 2i\epsilon (rq)r' = 0. \] (5.41)

The canonical symmetry for this system is the transformation of the form \[1, 4, 8\]

\[
\tilde{q} = \frac{1}{\epsilon r}, \quad \tilde{r} = \epsilon r^2 q + i \left( r' - \frac{r'' r'}{r'} \right).
\] (5.42)

Thus, in this case

\[
\beta = -\frac{1}{\epsilon r^2}, \quad \gamma = \epsilon r^2, \quad \delta = -i \frac{r'}{r'} D^2 + i \left( 1 + \frac{r'' r'}{r'^2} \right) D + 2\epsilon rq - i \frac{r'''}{r'}.
\] (5.43)

Equations (5.41) can be written in Hamiltonian form \[1, 2\] for the Hamiltonian operator \( J \), given by Eq. (5.21) and

\[
h[q, r] = r'q' + \frac{i\epsilon}{2} (r^2 qq' - q^2 rr').
\] (5.44)
It is not difficult to show that the operator \( J^{(0)} \) satisfies again the corresponding invariance condition [3], and the function \( g \) is again a constant function.

Write now the formulae, describing the behaviour of the densities of the lowest conservation laws under the action of the canonical symmetry:

\[
\tilde{r}\tilde{q} = rq + \frac{i}{\epsilon} D \left( \ln \frac{r}{r'} \right), \quad (5.46)
\]

\[
-i\tilde{r}\tilde{q}' = -irq' + D \left( irq + \frac{r'}{\epsilon r} \right). \quad (5.47)
\]

Proceed now to the construction of the operator \( J^{(1)} \). The relation, determining the function \( b_1 \), is again (5.32), and we can put

\[
b_1 = -1. \quad (5.48)
\]

From Eq. (5.12) we find

\[
\tilde{f}_0 - f_0 - D \left( \ln \frac{r}{r'} \right) = 0. \quad (5.49)
\]

Taking into account Eq. (5.46), we see that we can put

\[
b_0 = i\epsilon rq. \quad (5.50)
\]

Eq. (5.14) takes in our case the form

\[
\tilde{f}_{-1} - f_{-1} - \epsilon D \left( 2irq + \frac{r'}{\epsilon r} + \frac{r''}{\epsilon r'} \right) = 0. \quad (5.51)
\]

Using Eq. (5.47), we find the following expression for \( b_{-1} \):

\[
b_{-1} = i\epsilon (rq' - r'q). \quad (5.52)
\]

As in the case of the nonlinear Schrödinger equation we obtain

\[
d_{-1} = 2i\epsilon rr', \quad d_{-2} = -i\epsilon (r'^2 + rr''), \quad d_{-3} = i\epsilon (rr''' + r'r''); \quad (5.53)
\]

\[
a_{-1} = -2i\epsilon qq', \quad a_{-2} = i\epsilon (q^2 + qq''), \quad a_{-3} = -i\epsilon (qq''' + q'q''). \quad (5.54)
\]

Thus, we can try as the operator \( J^{(1)} \) the operator of form (4.4) with

\[
a = -i\epsilon (qD^{-1}q' + q'D^{-1}q), \quad (5.55)
\]

\[
b = -D + i\epsilon rq + i\epsilon (rD^{-1}q' - r'D^{-1}q), \quad (5.56)
\]

\[
d = i\epsilon (rD^{-1}r' + r'D^{-1}r). \quad (5.57)
\]

A direct check shows that this operator satisfies the invariance condition.
6 Conclusion

In this paper we have shown how the canonical symmetry can be used for the construction of Hamiltonian operators relevant to the system under consideration. Note, that we have used only the invariance condition, and discovered that the matrix skew–symmetric operators, satisfying this condition, are the required Hamiltonian operators. These operators define the Poisson brackets, satisfying Jacobi identity, and any two of them form a Hamiltonian pair. It is clear that for an arbitrary differential transformation this will not be the case. An open problem here is to formulate conditions on the transformation that provide such results. Apparently this problem is directly connected to the problem we stated in [4], that was: what are the conditions providing the involutivity of the quasi–invariants of a differential transformation?

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