The Union-Closed Sets Conjecture for Small Families

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Abstract We prove that the Union-Closed Sets Conjecture is true for separating union-closed families \( \mathcal{A} \) with \(|\mathcal{A}| \leq 2 \left( m + \frac{m}{\log_2(m) - \log_2 \log_2(m)} \right) \) where \( m \) denotes the order of the universe \( \bigcup_{A \in \mathcal{A}} A \).

Keywords Union-closed sets · Frankl’s conjecture

1 Introduction

A family \( \mathcal{A} \) of sets is said to be union-closed if for any two member sets \( A, B \in \mathcal{A} \) their union \( A \cup B \) is also a member of \( \mathcal{A} \).

A famous conjecture is the Union-Closed Sets Conjecture which is also called Frankl’s conjecture:

Conjecture 1 For any finite union-closed family of sets containing at least one non-empty set there exists an element that is contained in at least half of the family’s member sets.

There are many papers considering this conjecture. It is known to be true if \( \mathcal{A} \) has at most 12 elements [9] or at most 50 member sets [4, 8] or if the number of member sets is large compared to the number \( m \) of elements, that is, \( |\mathcal{A}| \geq \left( \frac{2}{3} - c \right) 2^m \) for some positive constant \( c \) [2]. Nevertheless, the conjecture is still far from being proved or disproved. A good survey on the current state of this conjecture is given by Bruhn and Schaudt [1]. In January 2016 Gowers started the Polymath project FUNC on the problem [5].
In this paper we consider the case where the number of member-sets is small compared to the number of elements. But first we recall some basic definitions and results. Let $\mathcal{A}$ be a union-closed family. We call $U(\mathcal{A}) = \bigcup_{A \in \mathcal{A}} A$ the universe of $\mathcal{A}$. For an element $x \in U(\mathcal{A})$ the cardinality of $|\{A \in \mathcal{A} : x \in A\}|$ is called the frequency of $x$. Thus the Union-Closed Sets Conjecture states that there exists an element $x \in U(\mathcal{A})$ of frequency at least $\frac{1}{2} |\mathcal{A}|$.

A family $\mathcal{A}$ is called separating if for any two distinct elements $x, y \in U(\mathcal{A})$ there exists a set $A \in \mathcal{A}$ that contains exactly one of the elements $x$ and $y$. We can restrict ourselves to separating union-closed families: If there exist elements $x$ and $y$ such that each member set $A \in \mathcal{A}$ contains $x$ if and only if it contains $y$, then we can delete $x$ from each such set and obtain a new family of the same cardinality that is still union-closed. Falgas-Ravry showed that there are some sets in $\mathcal{A}$ satisfying certain conditions which help us to analyze small separating union-closed families:

**Theorem 1** (Falgas-Ravry [3]) Let $\mathcal{A}$ be a separating union-closed family and let $x_1, \ldots, x_m$ be the elements of $U(\mathcal{A})$ labeled in order of increasing frequency. Then there exist sets $X_0, \ldots, X_{m-1} \in \mathcal{A}$ such that

$$x_i \notin X_i \quad \forall i \in \{1, \ldots, m-1\}$$  \hspace{1cm} (1)

and

$$\{x_{i+1}, \ldots, x_m\} \subseteq X_i \quad \forall i \in \{0, \ldots, m-1\}.$$  \hspace{1cm} (2)

**Proof** As $\mathcal{A}$ is separating, for any $1 \leq i < j \leq m$ there exists a set $X_{ij} \in \mathcal{A}$ such that $x_i \notin X_{ij}$ and $x_j \in X_{ij}$. For all $1 \leq i \leq m-1$ let $X_i = \bigcup_{j=i+1}^{m} X_{ij}$ and set $X_0 = U(\mathcal{A})$. \hfill \Box

The previous theorem directly implies that the conjecture is satisfied for small families:

**Corollary 1** Any separating family on $m$ elements with at most $2m$ member sets satisfies the Union-Closed Sets Conjecture.

**Proof** Consider the sets $X_0, \ldots, X_{m-1}$ constructed in Theorem 1 and observe that the most frequent element $x_m$ is contained in all these sets. As these sets are pairwise different, $x_m$ is contained in at least $m$ of all member sets of $\mathcal{A}$. \hfill \Box

In this paper we show that the Union-Closed Sets Conjecture is also satisfied for families that contain slightly more than $2m$ member sets. Considering such families is motivated by a result of Hu (see also [1]):

**Theorem 2** (Hu [6]) Suppose there is a number $c > 2$ so that any separating union-closed family $\mathcal{A}'$ with $|\mathcal{A}'| \leq c|U(\mathcal{A}')|$ satisfies the Union-Closed Sets Conjecture. Then, for every union-closed family $\mathcal{A}$, there is an element $x \in U(\mathcal{A})$ of frequency

$$|\{A \in \mathcal{A} : x \in A\}| \geq \frac{c-2}{2(c-1)} |\mathcal{A}|.$$  \hspace{1cm} (3)
Therefore, if the Union-Closed Sets Conjecture is satisfied for 'small' separating families, then for any union-closed family there exists an element that appears with a frequency at least a constant fraction of the number of member sets. In this paper we push the bound of Corollary 1 to a quantity which is slightly bigger than \(2m\), but which remains of order \((2 + o(1))m\).

**2 Frankl’s Conjecture for Small Families**

Combining and extending the idea of the proof of Theorem 1 and an argument of Knill [7] we get the main result of this paper.

**Theorem 3** The Union-Closed Sets Conjecture is true for separating union-closed families \(\mathcal{A}\) with a universe containing \(m\) elements satisfying

\[
|\mathcal{A}| \leq 2 \left( m + \frac{m}{\log_2(m) - \log_2 \log_2(m)} \right).
\]

*Proof* Let \(\mathcal{A}\) be a separating union-closed family, let the elements \(x_1, \ldots, x_m\) of \(U(\mathcal{A})\) be labeled in order of increasing frequency and set \(n = |\mathcal{A}|\). Assume that each element appears in at most \(m + d\) member sets. We compute an upper bound on the size of \(n\).

For \(i \in \{1, \ldots, m\}\) we set

\[
M_i = \bigcup_{A \in \mathcal{A} : x_i \notin A} A
\]

(4)

to be the union of all sets that do not contain \(x_i\) and we set \(M_0 = \bigcup_{A \in \mathcal{A}} A\). If the sets \(X_i, i \in \{0, \ldots, m - 1\}\), are chosen as in Theorem 1, then we have \(X_i \subseteq M_i\) for all \(i \in \{0, \ldots, m - 1\}\) and thus

\[
\{x_{i+1}, \ldots, x_m\} \subseteq M_i.
\]

(5)

Let \(\tilde{U} = \{x_i : \exists A \in \mathcal{A} \text{ such that } \max\{j : x_j \in A\} = i\}\) be the set of all \(x_i\) which are the elements with the highest index in some set \(A\).

For \(x_i \in \tilde{U}\) we set

\[
A_i = \bigcup_{A \in \mathcal{A} : \max\{j : x_j \in A\} = i} A.
\]

(6)

By definition \(x_i \in A_i\). Now consider \(j > i\). As \(x_j \notin A_i\) we have \(A_i \subseteq M_j\). Together with (5) we have

\[
x_i \in M_j \quad \forall x_i \in \tilde{U}, \quad j \in \{0, \ldots, m\}, \quad i \neq j.
\]

(7)

Observe that every non-empty member set of \(\mathcal{A}\) meets \(\tilde{U}\). Following an argument of Knill [7] let \(\tilde{U} \subseteq \tilde{U}\) be minimal such that every non-empty set of \(\mathcal{A}\) meets \(\tilde{U}\). Then for all \(x_i \in \tilde{U}\) there exists a set \(A \in \mathcal{A}\) with \(\tilde{U} \cap A = \{x_i\}\); if not, \(\tilde{U} \setminus \{x_i\}\) still contains an element of every member set of \(\mathcal{A}\) contradicting the minimality of \(\tilde{U}\). Therefore as \(\mathcal{A}\) is union-closed, for each \(B \subseteq \tilde{U}\) there exists at least a set \(P_B \in \mathcal{A}\) with \(P_B \cap \tilde{U} = B\).
We choose a set of distinct representatives $P_B$. Let $\mathcal{P} = \{P_B : B \subseteq \hat{U}\}$. The sets in $\mathcal{P}$ are distinct and each element $x_i \in \hat{U}$ is contained in at least half of the sets. Setting $k = |\hat{U}|$, we conclude that there are at most $2^k$ sets in $\mathcal{P}$ containing in total $k2^{k-1}$ elements from $\hat{U}$.

Note, that $\mathcal{P}$ might contain the sets $M_i$ for $x_i \in \hat{U}$ and one additional set $M_j$ with $\hat{U} \subseteq M_j$. But then $\{M_0, \ldots, M_m\}$ contains $m - k$ sets that are not in $\mathcal{P}$ and each of these sets contains all elements of $\hat{U}$.

Before we compute an upper bound for the number of elements in $A$ we summarize the previous observations:

- Each of the $k$ elements in $\hat{U}$ appears in at most $m + d$ member sets,
- there are exactly $k2^{k-1}$ appearances of members of $\hat{U}$ in the at most $2^k$ sets in $\mathcal{P}$,
- there are at least $m - k$ additional member sets, each containing all elements of $\hat{U}$ and
- all remaining member sets in $A$ contain at least one element of $\hat{U}$.

Thus the total number of appearances of members of $\hat{U}$ in sets in $A$ is bounded above by $k(m + d)$ and below by $k2^{k-1} + (m - k)k + (|A| - 2^k - (m - k))$. We conclude:

$$n = |A| \leq k(m + d) + (2^k - k2^{k-1}) + (m - k)(1 - k) \quad (8)$$
$$= m + kd + (2 - k)2^{k-1} + k^2 - k. \quad (9)$$

Suppose the Union-Closed Sets Conjecture does not hold for $A$, that is, $n > 2(m + d)$ or $\frac{n}{2} - m > d$. Then

$$n \leq m + k\left(\frac{n}{2} - m\right) + (2 - k)2^{k-1} + k^2 - k \quad (10)$$

or

$$n \geq \frac{(k - 1)m + (k - 2)2^{k-1} + k - k^2}{k - 2} \quad (11)$$

$$\geq 2\left(m + 2^{k-1} + \frac{m}{k - 2} - k - 3\right). \quad (12)$$

We conclude that the conjecture is true for all $n$ satisfying

$$n \leq 2\left(m + \min_{k \in \mathbb{N}}\left(2^{k-1} + \frac{m}{k - 2} - k - 3\right)\right). \quad (13)$$

The function $f_m(k) := 2^{k-1} + \frac{m}{k - 2} - k - 3$ is convex. Živković and Vučković [9] showed that the Union-Closed Sets Conjecture is satisfied for $m \leq 12$ so we can assume that $m \geq 13$. In this case the minimum of $f_m(k)$ is obtained in the interval $[5, \log_2(m)]$ and we get
\[ f_m(k) = \max \left\{ 2^{k-1}, \frac{m}{k-2} \right\} + \left( \min \left\{ 2^{k-1}, \frac{m}{k-2} \right\} - 3 - k \right) \]  \hspace{0.5cm} (14)

\[ \geq \max \left\{ 2^{k-1}, \frac{m}{k-2} \right\} \]  \hspace{0.5cm} (15)

\[ \geq \min_{k'} \left( \max \left\{ 2^{k'-1}, \frac{m}{k'-2} \right\} \right) \]  \hspace{0.5cm} (16)

\[ \geq \max_{k'} \left( \min \left\{ 2^{k'-1}, \frac{m}{k'-2} \right\} \right). \]  \hspace{0.5cm} (17)

The last inequality is due to the fact that \( 2^{k-1} \) is increasing in \( k \) while \( \frac{m}{k-2} \) is decreasing in \( k \).

Setting \( k' = \log_2(m) - \log_2 \log_2(m) + 2 \) we get

\[ \log_2 \left( \frac{m}{k'-2} \right) = \log_2(m) - \log_2 \left( \log_2(m) - \log_2 \log_2(m) \right) \]

\[ = \log_2(m) - \log_2 \log_2(m) - \log_2 \left( 1 - \frac{\log_2 \log_2(m)}{\log_2(m)} \right) \]

\[ \leq \log_2(m) - \log_2 \log_2(m) + 1 \]

\[ = \log_2(2^{k'}). \]

Inserting this result in (13) and (17) we finally obtain that the Union-Closed Sets Conjecture is true for all \( n \) satisfying

\[ n \leq 2 \left( m + \frac{m}{\log_2(m) - \log_2 \log_2(m)} \right). \]  \hspace{0.5cm} (18)

\( \square \)

**Acknowledgments** The author thanks Henning Bruhn-Fujimoto for pointing him to the Union-Closed Sets Conjecture and the two anonymous referees for their valuable comments.

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