A Revised Version of a Lexicographical-based Method for Solving Fully Fuzzy Linear Programming Problems with Inequality Constraints

Boris Pérez-Cañedo a, Eduardo René Concepción-Morales b and Seyyed Ahmad Edalatpanah c

aDepartment of Mathematics, Faculty of Economics and Business Sciences, University of Cienfuegos, Cienfuegos, Cuba; bSchool of Information Systems, Metropolitan University (UMET), Quito, Ecuador; cDepartment of Applied Mathematics, Faculty of Mathematical Sciences, Ayandegan Institute of Higher Education, Tonekabon, Iran

ABSTRACT
Ezzati et al. (A new algorithm to solve fully fuzzy linear programming problems using the MOLP problem. Appl Math Model. 2015;39(12):3183–3193) introduced a lexicographic criterion for ranking triangular fuzzy numbers (TFNs), and proposed a method to solve fully fuzzy linear programming (FFLP) problems based on the lexicographic method of multi-objective optimisation; the authors assumed that fuzzy inequality constraints can be transformed into fuzzy equality constraints by introducing non-negative fuzzy slack and surplus variables. They illustrated the proposed method by means of a fully fuzzy investment problem. Bhardwaj and Kumar (A note on “A new algorithm to solve fully fuzzy linear programming problems using the MOLP problem”. Appl Math Model. 2015;39(19):5982–5985) demonstrated that introducing fuzzy slack and surplus variables is mathematically incorrect, and showed that the solution of the fuzzy investment problem is unfeasible. Towards the end of their paper, they claimed that there is no feasible solution to the fuzzy investment problem when considering Ezzati et al.’s ranking criterion. In this paper, we propose a revised version of Ezzati et al.’s method whereby the optimal solution of FFLP problems with equality and inequality constraints can be obtained. Furthermore, by using the revised method, we show that feasible solutions of the fuzzy investment problem actually exist, and therefore Bhardwaj and Kumar’s claim is false. To show the applicability of the revised method, we also consider a fully fuzzy project scheduling problem with budget constraint.

ARTICLE HISTORY
Received 5 December 2019
Revised 30 March 2020
Accepted 8 April 2020

KEYWORDS
Fully fuzzy linear programming; fuzzy inequality constraint; lexicographic ranking criterion; fully fuzzy investment problem; fully fuzzy project scheduling problem; project crashing

1. Introduction and Preliminaries
Since the foundational works on fuzzy decision-making and optimisation by Bellman and Zadeh [1], Tanaka et al. [2] and Zimmermann [3] fuzzy linear programming (FLP) has experimented a considerable development and has spread into almost every area of
decision-making. For example, methodologies of FLP and applications in fuzzy assignment, scheduling, transportation, matrix games, recommender systems and logistics network design problems with a single objective function or multiple ones are reported in [4–14]. Recent surveys by Ebrahimnejad and Verdegay [15] and Nasseri et al. [16] offer a detailed exposition of several FLP methodologies, from the very beginning of FLP to its modern approaches.

One such approach was developed by Ezzati et al. [17] for solving fully fuzzy linear programming (FFLP) problems; i.e. FLP problems in which all parameters and decision variables take on fuzzy numbers. Specifically, Ezzati et al. [17] considered FFLP problems with unrestricted triangular fuzzy parameters and non-negative triangular fuzzy decision variables. The authors found the motivation for their proposal in the shortcomings and limitations of the methods of Hosseinzadeh Lotfi et al. [18] and Kumar et al. [19]. Ezzati et al. [17] noticed that in the case of Hosseinzadeh Lotfi et al.’s [18] method, it is necessary to approximate all the problem parameters to their corresponding nearest symmetric triangular fuzzy numbers (TFNs); therefore, the obtained solutions are approximate and do not satisfy the problem constraints exactly. In [19], on the other hand, a ranking function was used to transform the FFLP problem into a crisp linear programming problem. Since by using a ranking function different fuzzy numbers may be mapped into the same real number, the information encompassed by the fuzzy numbers is lost and therefore cannot be used to guide the search for an optimal solution. To overcome these shortcomings and limitations, Ezzati et al. [17] introduced a new lexicographic criterion for ranking TFNs and transformed the FFLP problem into a three-objective crisp linear programming problem; the lexicographic method of classical multi-objective optimisation [20, chap. 5, pp. 128–130] was subsequently used to solve the problem. Recently, Ebrahimnejad [21] proposed a more efficient formulation of Ezzati et al.’s [17] method. However, Ebrahimnejad [21] approached FFLP problems having only equality constraints. We shall show shortly that Ezzati et al.’s [17] method cannot be used to solve FFLP problems with inequality constraints; considering the vast number of FFLP applications, this is a serious limitation from a practical viewpoint.

Before discussing on the shortcomings and limitations of Ezzati et al.’s [17] method, it is convenient to present some basic definitions concerning TFNs; the reader may refer to Ezzati et al.’s [17] paper for the introductory concepts on fuzzy numbers and other related definitions.

**Definition 1.1 ([17]):** A fuzzy number \( \tilde{v} = (v^l, v^c, v^u) \) is said to be a TFN if its membership function is given by:

\[
\mu_{\tilde{v}}(x) = \begin{cases} 
\frac{x - v^l}{v^c - v^l}, & v^l \leq x \leq v^c \\
\frac{v^u - x}{v^u - v^c}, & v^c \leq x \leq v^u \\
0, & \text{otherwise}
\end{cases}
\]

A TFN \( \tilde{v} = (v^l, v^c, v^u) \) is non-negative if \( v^l \geq 0 \), non-positive if \( v^u \leq 0 \), and unrestricted if \( v^c \) is an arbitrary real number.

**Definition 1.2 ([17]):** Two TFNs \( \tilde{v}_1 = (v_{1}^l, v_{1}^c, v_{1}^u) \) and \( \tilde{v}_2 = (v_{2}^l, v_{2}^c, v_{2}^u) \) are said to be equal, \( \tilde{v}_1 = \tilde{v}_2 \), if and only if \( v_{1}^l = v_{2}^l, v_{1}^c = v_{2}^c, \) and \( v_{1}^u = v_{2}^u \).
**Definition 1.3 ([17]):** Let \( \tilde{v}_1 = (v'_1, v'_1, v''_1) \) and \( \tilde{v}_2 = (v'_2, v'_2, v''_2) \) be two TFNs, addition \( \oplus \), subtraction \( \ominus \) and multiplication \( \otimes \) of \( \tilde{v}_1 \) and \( \tilde{v}_2 \) are defined as follows:

\[
\tilde{v}_1 \oplus \tilde{v}_2 = \left( v'_1 + v'_2, v'_1 + v'_2, v''_1 + v''_2 \right)
\]

\[
\tilde{v}_1 \ominus \tilde{v}_2 = \left( v'_1 - v'_2, v'_1 - v'_2, v''_1 - v''_2 \right)
\]

If \( \tilde{v}_1 \) is unrestricted and \( \tilde{v}_2 \) is non-negative, then

\[
\tilde{v}_1 \otimes \tilde{v}_2 = \begin{cases} 
\left( v'_1 v'_2, v'_1 v'_2, v''_1 v''_2 \right), & \text{if } v'_1 \geq 0 \\
\left( v'_1 v'_2, v'_1 v'_2, v''_1 v''_2 \right), & \text{if } v'_1 < 0 \text{ and } v''_1 \geq 0 \\
\left( v'_1 v'_2, v'_1 v'_2, v''_1 v''_2 \right), & \text{if } v''_1 < 0
\end{cases}
\]

Ezzati et al. [17] illustrated their method for solving FFLP problems by means of three examples. In particular, the last example is a fully fuzzy investment problem in which a corporation seeks to allocate \((25, 30, 40)\) million to its four subsidiaries. A minimal level of funding has been established for each subsidiary; these levels are \((2, 3, 5)\), \((4, 5, 6)\), \((5, 8, 9)\) and \((7, 8, 14)\) million, respectively. Each subsidiary can conduct three projects. A rate of return, as a percent of investment, has been estimated for each project, and a limited level of investment has been fixed for each project.

The mathematical model for the above decision-making situation is given by FFLP problem (1), and it must be decided how much is to be allocated to each project of each subsidiary in order to maximise the expected return of the investment. All the problem parameters and decision variables are TFNs.

\[
\text{max} (5, 7, 8) \otimes \bar{x}_{11} \oplus (3, 5, 6) \otimes \bar{x}_{12} \oplus (4, 8, 9) \\
\quad \otimes (3, 5, 7) \otimes \bar{x}_{21} \oplus (4, 7, 8) \otimes \bar{x}_{22} \\
\quad \oplus (8, 9, 10) \otimes \bar{x}_{23} \oplus (7, 10, 11) \otimes \bar{x}_{31} \\
\quad \otimes (6, 8, 10) \otimes \bar{x}_{32} \oplus (4, 7, 8) \otimes \bar{x}_{33} \\
\quad \oplus (4, 6, 8) \otimes \bar{x}_{41} \oplus (3, 5, 7) \otimes \bar{x}_{42} \oplus (7, 9, 11) \otimes \bar{x}_{43}
\]

s.t. \( \sum_{i=1}^{4} \sum_{j=1}^{3} \bar{x}_{ij} = (25, 30, 40) \),

\[
\sum_{j=1}^{3} \bar{x}_{1j} \geq (2, 3, 5), \quad \sum_{j=1}^{3} \bar{x}_{2j} \geq (4, 5, 6), \\
\sum_{j=1}^{3} \bar{x}_{3j} \geq (5, 8, 9), \quad \sum_{j=1}^{3} \bar{x}_{4j} \geq (7, 8, 14), \\
\bar{x}_{11} \leq (4, 6, 7), \bar{x}_{12} \leq (3, 5, 6), \bar{x}_{13} \leq (8, 9, 10), \\
\bar{x}_{21} \leq (5, 7, 8), \bar{x}_{22} \leq (8, 10, 11), \bar{x}_{23} \leq (3, 4, 5),
\]

(1)
\( \tilde{x}_{i,j} \), for \( i = 1, 2, 3, 4 \) and \( j = 1, 2, 3 \) are non-negative TFNs.

The order relation \( \preceq \) is defined as follows [see 17, Definition 2.6].

Let \( \tilde{v}_1 = (v_{11}, v_{12}, v_{13}) \) and \( \tilde{v}_2 = (v_{21}, v_{22}, v_{23}) \) be two arbitrary TFNs. We say \( \tilde{v}_1 \) is relatively less than \( \tilde{v}_2 \), which is denoted by \( \tilde{v}_1 \prec \tilde{v}_2 \), if and only if:

1. \( v_{1}^c < v_{2}^c \) or,
2. \( v_{1}^c = v_{2}^c \) and \( (v_{1i} - v_{1j}) > (v_{2i} - v_{2j}) \) (equivalently, \( (v_{1i} - v_{1j}) < (v_{2i} - v_{2j}) \)) or,
3. \( v_{1}^c = v_{2}^c \), \( (v_{1i} - v_{1j}) = (v_{2i} - v_{2j}) \) and \( (v_{1i} + v_{1j}) < (v_{2i} + v_{2j}) \).

\( \tilde{v}_1 \preceq \tilde{v}_2 \) if and only if \( \tilde{v}_1 \prec \tilde{v}_2 \) or \( \tilde{v}_1 = \tilde{v}_2 \).

To solve FFLP problem (1), Ezzati et al. [17] transformed the fuzzy inequality constraints of FFLP problem (1) into fuzzy equality constraints by means of non-negative triangular fuzzy slack and surplus variables \((\tilde{s}_k, k = 1, 2, \ldots, 16)\) Thus, the authors obtained FFLP problem (2).

\[
\text{max}(5, 7, 8) \odot \tilde{x}_{11} \odot (3, 5, 6) \odot \tilde{x}_{12} \odot (4, 8, 9) \odot \tilde{x}_{13} \\
\odot (3, 5, 7) \odot \tilde{x}_{21} \odot (4, 7, 8) \odot \tilde{x}_{22} \odot (8, 9, 10) \odot \tilde{x}_{23} \\
\odot (7, 10, 11) \odot \tilde{x}_{31} \odot (6, 8, 10) \odot \tilde{x}_{32} \odot (4, 7, 8) \\
\odot \tilde{x}_{33} \odot (4, 6, 8) \odot \tilde{x}_{41} \odot (3, 5, 7) \odot \tilde{x}_{42} \odot (7, 9, 11) \odot \tilde{x}_{43}
\]

s.t. \( \sum_{i=1}^{3} \sum_{j=1}^{3} \tilde{x}_{i,j} = (25, 30, 40), \)
\[
\sum_{j=1}^{3} \tilde{x}_{1j} \odot \tilde{s}_1 = (2, 3, 5), \sum_{j=1}^{3} \tilde{x}_{2j} \odot \tilde{s}_2 = (4, 5, 6), \\
\sum_{j=1}^{3} \tilde{x}_{3j} \odot \tilde{s}_3 = (5, 8, 9), \sum_{j=1}^{3} \tilde{x}_{4j} \odot \tilde{s}_4 = (7, 8, 14), \\
\tilde{x}_{11} \odot \tilde{s}_5 = (4, 6, 7), \tilde{x}_{12} \odot \tilde{s}_6 = (3, 5, 6), \tilde{x}_{13} \odot \tilde{s}_7 = (8, 9, 10), \\
\tilde{x}_{21} \odot \tilde{s}_8 = (5, 7, 8), \tilde{x}_{22} \odot \tilde{s}_9 = (8, 10, 11), \tilde{x}_{23} \odot \tilde{s}_{10} = (3, 4, 5), \\
\tilde{x}_{31} \odot \tilde{s}_{11} = (4, 5, 7), \tilde{x}_{32} \odot \tilde{s}_{12} = (2, 3, 6), \tilde{x}_{33} \odot \tilde{s}_{13} = (4, 7, 9), \\
\tilde{x}_{41} \odot \tilde{s}_{14} = (4, 6, 7), \tilde{x}_{42} \odot \tilde{s}_{15} = (4, 5, 9), \tilde{x}_{43} \odot \tilde{s}_{16} = (2, 4, 5),
\]

\( \tilde{x}_{i,j}, \tilde{s}_k \) for \( i = 1, 2, 3, 4, j = 1, 2, 3, \) and \( k = 1, 2, \ldots, 16 \) are non-negative TFNs.

By using Ezzati et al.’s [17] method, the solution of FFLP problem (2) is given as:

\( \tilde{x}_{11} = (0, 0, 1), \tilde{x}_{12} = (0, 0, 1), \tilde{x}_{13} = (8, 9, 9), \tilde{x}_{21} = (0, 0, 1), \tilde{x}_{22} = (1, 1, 1), \tilde{x}_{23} = (3, 4, 4), \)
\( \tilde{x}_{31} = (4, 5, 5), \tilde{x}_{32} = (2, 3, 3), \tilde{x}_{33} = (0, 0, 1), \tilde{x}_{41} = (4, 4, 5), \tilde{x}_{42} = (1, 1, 5), \tilde{x}_{43} = (2, 3, 4), \)

with fuzzy objective value \( V_{Ezzati \ et \ al.} = (133, 245, 362) \).
However, Bhardwaj and Kumar [22] noticed that this solution is not satisfying all the constraints of FFLP problem (1). For example, \( \tilde{x}_{13} = (x^l_{13}, x^c_{13}, x^u_{13}) = (8, 9, 9) \) must satisfy that \( \tilde{x}_{13} \preceq (8, 9, 10) \), but it does not; note that, according to the definition of the order \( \preceq \),
\[
(x^l_{13}, x^c_{13}, x^u_{13}) \preceq (8, 9, 10) \Leftrightarrow x^c_{13} \leq 9, \text{ or if } x^c_{13} = 9 \text{ then } x^u_{13} - x^l_{13} \geq 2, \text{ or if } x^c_{13} = 9 \text{ and } x^u_{13} - x^l_{13} = 2 \text{ then } x^l_{13} + x^u_{13} \leq 18. \]
Since we have that \( x^c_{13} = 9 \) and \( x^u_{13} - x^l_{13} = 1 \), then \( \tilde{x}_{13} \not\preceq (8, 9, 10) \); hence, the solution is not feasible. Similarly, it can be shown that \( \tilde{x}_{23} \not\preceq (3, 4, 5) \), \( \tilde{x}_{31} \not\preceq (4, 5, 7) \) and \( \tilde{x}_{32} \not\preceq (2, 3, 6) \).

Towards the end of their paper, Bhardwaj and Kumar [22] claimed that it is not possible to find any feasible fuzzy solution of FFLP problem (1) according to the ranking criterion of Ezzati et al. [17]. However, the reader can easily verify that
\[
\begin{align*}
\tilde{x}_{11} &= (0, 0, 0), \\
\tilde{x}_{12} &= (0, 0, 5), \\
\tilde{x}_{13} &= (8, 9, 10), \\
\tilde{x}_{21} &= (0, 0, 0), \\
\tilde{x}_{22} &= (1, 1, 1), \\
\tilde{x}_{23} &= (3, 4, 5), \\
\tilde{x}_{31} &= (4, 5, 7), \\
\tilde{x}_{32} &= (2, 2, 2), \\
\tilde{x}_{33} &= (1, 1, 1), \\
\tilde{x}_{41} &= (4, 4, 4), \\
\tilde{x}_{42} &= (0, 0, 0), \text{ and } \\
\tilde{x}_{43} &= (2, 4, 5)
\end{align*}
\]
with fuzzy objective value \( \tilde{v} = (134, 248, 370) \) is indeed a feasible solution of FFLP problem (1); therefore, Bhardwaj and Kumar’s [22] claim on the unfeasibility of FFLP problem (1) is false.

Bhardwaj and Kumar [22] focused their discussion on two points: (1) to prove that transforming the fuzzy inequality constraints into fuzzy equality constraints by means of non-negative triangular fuzzy slack and surplus variables leads to a contradiction with Ezzati et al.’s [17] ranking criterion, and (2) to show, as we did above, the existence of this contradiction in the solution of FFLP problem (1) given in [17]. Although Bhardwaj and Kumar [22] successfully addressed both points, the authors did not provide a method for solving FFLP problems with inequality constraints considering the ranking criterion of Ezzati et al. [17]; so, until now, this issue has remained unsolved. Based on the above discussion, the present paper makes the following contributions:

- A revised version of Ezzati et al.’s [17] method for solving FFLP problems with inequality constraints that overcomes the shortcomings and limitations pointed out in [22].
- Bhardwaj and Kumar’s [22] claim on the unfeasibility of Ezzati et al.’s [17] fully fuzzy investment problem is shown to be false.
- Three numerical examples that discuss practical problems show the applicability of the revised method in real-world problems.

The rest of this paper is organised as follows. In Section 2, based on a recent methodology for handling fuzzy inequality constraints [23], we make some corrections to Ezzati et al.’s [17] method that overcome the shortcomings and limitations pointed out by Bhardwaj and Kumar [22]. This new revised version of Ezzati et al.’s [17] method is illustrated in Section 3 by solving the fully fuzzy investment problem and a fully fuzzy project scheduling problem with budget constraint. In the case of the fully fuzzy investment problem, it is shown that, contrary to Bhardwaj and Kumar’s [22] claim, this problem has infinitely many optimal fuzzy solutions. A comparison with existing methods is carried out in Section 4. Concluding remarks and directions for future research are provided in Section 5.

2. Revised Version of Ezzati et al.’s Method

The general form of the FFLP problem with arbitrary triangular fuzzy parameters and non-negative triangular fuzzy decision variables can be formulated as follows:
Remark 2.1: At this point, Ezzati et al. [17] introduce non-negative triangular fuzzy slack variables to transform each fuzzy inequality constraint of FFLP problem (5) into a fuzzy equality constraint. This transformation has also been used, for example, in [7, 24]. However, as shown by Bhardwaj and Kumar [22] and Gupta et al. [25], such a transformation is not mathematically correct; therefore, we do not use it here and proceed to Step 3.

\[
\max \sum_{j=1}^{n} \tilde{c}_j \otimes \tilde{x}_j
\]
\[
\text{s.t. } \sum_{j=1}^{n} \tilde{a}_{ij} \otimes \tilde{x}_j \{ \leq, =, \geq \} \tilde{b}_i, \text{ for } i = 1, 2, \ldots, m
\]  

\(\tilde{x}_j\) is a non-negative TFN, for \(j = 1, 2, \ldots, n\)

To facilitate the following discussion, we rewrite the definition of the order relation \(\preceq\), given by Ezzati et al. [17], as in Definition 2.1.

**Definition 2.1:** Let \(\preceq_{\text{lex}}\) be the lexicographic order relation in \(\mathbb{R}^3\) and \(\tilde{v}_1 = (v_{1}^l, v_{1}^c, v_{1}^u)\) and \(\tilde{v}_2 = (v_{2}^l, v_{2}^c, v_{2}^u)\) two arbitrary TFNs. We say \(\tilde{v}_1\) is relatively less than \(\tilde{v}_2\), which is denoted by \(\tilde{v}_1 \prec \tilde{v}_2\), if and only if \((v_{1}^l, v_{1}^c - v_{1}^c, v_{1}^c + v_{1}^u) \preceq_{\text{lex}} (v_{2}^l, v_{2}^c - v_{2}^c, v_{2}^c + v_{2}^u)\). We say \(\tilde{v}_1\) is relatively less than or equal to \(\tilde{v}_2\), which is denoted by \(\tilde{v}_1 \preceq \tilde{v}_2\), if and only if \((v_{1}^l, v_{1}^c - v_{1}^c, v_{1}^c + v_{1}^u) \preceq_{\text{lex}} (v_{2}^l, v_{2}^c - v_{2}^c, v_{2}^c + v_{2}^u)\) and \(v_{1}^u = v_{2}^u\) [17].

**Remark 2.1:** \(\tilde{v}_1 = \tilde{v}_2\) if and only if \(v_{1}^l = v_{2}^l\), \(v_{1}^c - v_{1}^c = v_{2}^c - v_{2}^c\) and \(v_{1}^c + v_{1}^u = v_{2}^c + v_{2}^u\) [17].

In what follows, we will go through the steps of a method for solving FFLP problem (3) and point out the differences with the method of Ezzati et al. [17] and other approaches from the literature.

**Step 1.** Let \(\tilde{c}_j = (c_{1j}^l, c_{1j}^c, c_{1j}^u)\), \(\tilde{a}_{ij} = (a_{ij}^l, a_{ij}^c, a_{ij}^u)\), \(\tilde{b}_i = (b_{1i}^l, b_{1i}^c, b_{1i}^u)\) and \(\tilde{x}_j = (x_{1j}^l, x_{1j}^c, x_{1j}^u)\) then FFLP problem (3) can be written as:

\[
\max \sum_{j=1}^{n} (c_{1j}^l, c_{1j}^c, c_{1j}^u) \otimes (x_{1j}^l, x_{1j}^c, x_{1j}^u)
\]
\[
\text{s.t. } \sum_{j=1}^{n} (a_{ij}^l, a_{ij}^c, a_{ij}^u) \otimes (x_{1j}^l, x_{1j}^c, x_{1j}^u) \{ \leq, =, \geq \} (b_{1i}^l, b_{1i}^c, b_{1i}^u), \text{ for } i = 1, 2, \ldots, m
\]

\((x_{1j}^l, x_{1j}^c, x_{1j}^u)\) is a non-negative TFN, for \(j = 1, 2, \ldots, n\)

**Step 2.** Let \((s_{1j}^l, s_{1j}^c, s_{1j}^u) = (c_{1j}^l, c_{1j}^c, c_{1j}^u) \otimes (x_{1j}^l, x_{1j}^c, x_{1j}^u)\) and \((m_{1ij}^l, m_{1ij}^c, m_{1ij}^u) = (a_{ij}^l, a_{ij}^c, a_{ij}^u) \otimes (x_{1j}^l, x_{1j}^c, x_{1j}^u)\), then FFLP problem (4) is rewritten as follows:

\[
\max \sum_{j=1}^{n} (s_{1j}^l, s_{1j}^c, s_{1j}^u)
\]
\[
\text{s.t. } \sum_{j=1}^{n} (m_{1ij}^l, m_{1ij}^c, m_{1ij}^u) \{ \leq, =, \geq \} (b_{1i}^l, b_{1i}^c, b_{1i}^u), \text{ for } i = 1, 2, \ldots, m
\]

\((x_{1j}^l, x_{1j}^c, x_{1j}^u)\) is a non-negative TFN, for \(j = 1, 2, \ldots, n\)

**Remark 2.2:** At this point, Ezzati et al. [17] introduce non-negative triangular fuzzy slack variables to transform each fuzzy inequality constraint of FFLP problem (5) into a fuzzy equality constraint. This transformation has also been used, for example, in [7, 24]. However, as shown by Bhardwaj and Kumar [22] and Gupta et al. [25], such a transformation is not mathematically correct; therefore, we do not use it here and proceed to Step 3.
**Step 3.** Denote by $I_e$, $I_{le}$ and $I_{ge}$ the index sets of the fuzzy equality, less-than-or-equal-to and greater-than-or-equal-to constraints of FFLP problem (5). By using Definitions 1.2 and 2.1, transform FFLP problem (5) into the following lexicographic optimisation problem:

\[
\text{lex max } \left( \sum_{j=1}^{n} s_j^c, \sum_{j=1}^{n} (s_j^l - s_j^u), \sum_{j=1}^{n} (s_j^l + s_j^u) \right)
\]

s.t. \( \sum_{j=1}^{n} m_{ij}^l = b_i^l, \sum_{j=1}^{n} m_{ij}^c = b_i^c, \sum_{j=1}^{n} m_{ij}^u = b_i^u \), for \( i \in I_e \) (6)

\[
\left( \sum_{j=1}^{n} m_{ij}^c, \sum_{j=1}^{n} (m_{ij}^l - m_{ij}^u), \sum_{j=1}^{n} (m_{ij}^l + m_{ij}^u) \right) \{ \leq_{\text{lex}}, \geq_{\text{lex}} \} (b_i^c, b_i^l - b_i^u, b_i^l + b_i^u), \text{ for } i \in I_{le} \cup I_{ge}
\]

\( x_j^l \geq 0, x_j^c - x_j^l \geq 0, x_j^u - x_j^c \geq 0, \text{ for } j = 1, 2, \ldots, n \)

**Step 4.** By introducing binary variables \( y_{ik} \), for \( i \in I_{le} \cup I_{ge} \) and \( k \in \{1, 2, 3\} \), transform lexicographic optimisation problem (6) into crisp mixed 0–1 lexicographic linear programming (MLLP) problem (7),

\[
\text{lex max } \left( \sum_{j=1}^{n} s_j^c, \sum_{j=1}^{n} (s_j^l - s_j^u), \sum_{j=1}^{n} (s_j^l + s_j^u) \right)
\]

s.t. \( \sum_{j=1}^{n} m_{ij}^l = b_i^l, \sum_{j=1}^{n} m_{ij}^c = b_i^c, \sum_{j=1}^{n} m_{ij}^u = b_i^u \), for \( i \in I_e \) (7a)

\[
\epsilon y_{i1} \leq b_i^c - \sum_{j=1}^{n} m_{ij}^c \leq Ly_{i1}, \text{ for } i \in I_{le} \quad (7b)
\]

\[
-Ly_{i1} + \epsilon y_{i2} \leq b_i^l - b_i^u - \sum_{j=1}^{n} (m_{ij}^l - m_{ij}^u) \leq Ly_{i2}, \text{ for } i \in I_{le} \quad (7c)
\]

\[
-L(y_{i1} + y_{i2}) + \epsilon y_{i3} \leq b_i^l + b_i^u - \sum_{j=1}^{n} (m_{ij}^l + m_{ij}^u) \leq Ly_{i3}, \text{ for } i \in I_{le} \quad (7d)
\]

\[
\epsilon y_{i1} \leq \sum_{j=1}^{n} m_{ij}^c - b_i^c \leq Ly_{i1}, \text{ for } i \in I_{ge} \quad (7e)
\]

\[
-Ly_{i1} + \epsilon y_{i2} \leq \sum_{j=1}^{n} (m_{ij}^l - m_{ij}^u) - b_i^l + b_i^u \leq Ly_{i2}, \text{ for } i \in I_{ge} \quad (7f)
\]

\[
-L(y_{i1} + y_{i2}) + \epsilon y_{i3} \leq \sum_{j=1}^{n} (m_{ij}^l + m_{ij}^u) - b_i^l - b_i^u \leq Ly_{i3}, \text{ for } i \in I_{ge} \quad (7g)
\]
\[ y_{ik} \in \{0, 1\}, \text{ for } i \in I_{le} \cup I_{ge} \text{ and } k \in \{1, 2, 3\} \quad (7h) \]
\[ x_j^l \geq 0, x_j^c - x_j^l \geq 0, x_j^u - x_j^c \geq 0, \text{ for } j = 1, 2, \ldots, n \quad (7i) \]
for positive real values of \( \epsilon \) and \( L \), sufficiently small and large, respectively.

**Remark 2.3:** Notice that the lexicographic less-than-or-equal-to (\( \leq_{\text{lex}} \)) and greater-than-or-equal-to (\( \geq_{\text{lex}} \)) constraints of problem (6) are represented, in MLLP problem (7), by constraint sets (7b)–(7d) and (7e)–(7g), respectively. Using the element-wise inequalities \( \leq \) and \( \geq \) in place of \( \leq_{\text{lex}} \) and \( \geq_{\text{lex}} \), as in e.g. [26] and [15, chap. 4, p. 299], makes the resulting optimisation problem unnecessary over-constrained and generally leads to suboptimal solutions or unfeasible problems; this last point will be illustrated in Section 3.

**Step 5.** Solve MLLP problem (7) by using the lexicographic method of classical multi-objective optimisation [20, chap. 5, pp. 128–130] to obtain an optimal solution \( x_l^*, x_c^* \) and \( x_u^* \), and put their values into \( \tilde{x}_j^* = (x_l^*, x_c^*, x_u^*) \) to obtain an optimal fuzzy solution of FFLP problem (3).

**Step 6.** Evaluate \( \sum_{j=1}^{n} \tilde{c}_j \otimes \tilde{x}_j^* \) to obtain the optimal fuzzy objective value of FFLP problem (3).

**Theorem 2.1 ([23]):** MLLP problem (7) is equivalent to FFLP problem (3).

**Proof:** The proof can be divided into two parts. Firstly, it is necessary to show that FFLP problem (3) is equivalent to lexicographic optimisation problem (6); secondly, that problem (6) is equivalent to MLLP problem (7). We omit the first part as it is easily shown by contradiction using Definition 2.1. For the second part, it is sufficient to show that the lexicographic constraints of problem (6) are equivalent to constraints (7b)–(7h) of MLLP problem (7). Thus, for the \( \leq_{\text{lex}} \)-type constraints, we analyse the following cases:

1. If \( y_{ik} = (1, *, *) \) for \( k = 1, 2, 3 \), where \( * \) means 0 or 1, then by substituting into constraint set (7b)–(7d) it follows that \( \epsilon \leq b_i^c - \sum_{j=1}^{n} m_{ij}^c \leq L \) which implies \( \sum_{j=1}^{n} m_{ij}^c < b_i^c \) hence, from Definition 2.1, the \( \leq_{\text{lex}} \)-type constraints of problem (6) are satisfied.

2. If \( y_{ik} = (0, 1, *) \), then by substituting into constraint set (7b)–(7d) it follows that \( 0 \leq b_i^c - \sum_{j=1}^{n} m_{ij}^c \leq 0 \) and \( \epsilon \leq b_i^c - b_i^l - \sum_{j=1}^{n} (m_{ij}^l - m_{ij}^u) \leq L \), which implies \( \sum_{j=1}^{n} m_{ij}^c = b_i^c \) and \( \sum_{j=1}^{n} (m_{ij}^l - m_{ij}^u) < b_i^l - b_i^u \); again, from Definition 2.1, the \( \leq_{\text{lex}} \)-type constraints of problem (6) are satisfied.

The remaining cases \( y_{ik} = (0, 0, 1) \) and \( y_{ik} = (0, 0, 0) \) can be shown to hold similarly. Conversely, in problem (6),

1. If \( \sum_{j=1}^{n} m_{ij}^c = b_i^c \), \( \sum_{j=1}^{n} (m_{ij}^l - m_{ij}^u) = b_i^l - b_i^u \) and \( \sum_{j=1}^{n} (m_{ij}^l + m_{ij}^u) = b_i^l + b_i^u \), then by substituting into (7b) we get \( \epsilon y_{i1} \leq 0 \leq L y_{i1} \), implying that \( y_{i1} = 0 \); by substituting \( y_{i1} = 0 \) and \( \sum_{j=1}^{n} (m_{ij}^l - m_{ij}^u) = b_i^l - b_i^u \) into (7c) we get \( \epsilon y_{i2} \leq 0 \leq L y_{i2} \), implying that \( y_{i2} = 0 \);
Lastly, by substituting $y_{i1} = 0$, $y_{i2} = 0$ and $\sum_{j=1}^{n} (m_{ij}^{l} + m_{ij}^{u}) = b_{ij}^{l} + b_{ij}^{u}$ into (7d) we get

$$\epsilon y_{i3} \leq 0 \leq L y_{i3},$$

implying that $y_{i3} = 0$.

(2) If $\sum_{j=1}^{n} m_{ij}^{l} = b_{ij}^{l}$, $\sum_{j=1}^{n} (m_{ij}^{l} - m_{ij}^{u}) = b_{ij}^{l} - b_{ij}^{u}$ and $b_{ij}^{l} + b_{ij}^{u} - \sum_{j=1}^{n} (m_{ij}^{l} + m_{ij}^{u}) = s_{i3} > 0$, then we get $y_{i1} = 0$, $y_{i2} = 0$ and $\epsilon y_{i3} \leq s_{i3} \leq L y_{i3}$, which holds for $y_{i3} = 1$ and positive real values of $\epsilon$ and $L$ sufficiently small and large, respectively so that $s_{i3} \in [\epsilon, L]$.

(3) If $\sum_{j=1}^{n} m_{ij}^{l} = b_{ij}^{l}$, $b_{ij}^{l} - b_{ij}^{u} - \sum_{j=1}^{n} (m_{ij}^{l} - m_{ij}^{u}) = s_{i2} > 0$ and $b_{ij}^{l} + b_{ij}^{u} - \sum_{j=1}^{n} (m_{ij}^{l} + m_{ij}^{u}) = s_{i3} \in \mathbb{R}$, then it follows that $y_{i1} = 0$, $y_{i2} = 1$ (similar to the previous case) and $-L + \epsilon y_{i3} \leq s_{i3} \leq L y_{i3}$; these inequalities are satisfied for $y_{i3} = 1$, with a sufficiently large positive value of $L$ so that $s_{i3} \in [-L + \epsilon, L]$.

The remaining cases and the proof for the $\geq_{\text{lex}}$-type constraints can be shown to hold in a similar manner.

**Remark 2.4:** Ezzati et al. [17] assumed that fuzzy inequality constraints can be transformed into equality constraints by means of non-negative triangular fuzzy slack and surplus variables. They based their results on this false assumption; therefore, their Theorem 3.1 is valid only for FFLP problems having only equality constraints. Steps 3 and 4 of the revised method and Theorem 2.1 give an appropriate way in which Ezzati et al.’s [17] ranking criterion can be used for handling fuzzy inequality constraints.

### 3. Numerical Examples

In this section, we illustrate the revised method by means of three examples. In the first example, the fully fuzzy investment problem of Ezzati et al. [17] is solved and shown to have infinitely many optimal fuzzy solutions, the second example is a fully fuzzy project scheduling problem and third one is a fully fuzzy project crashing problem. We remark that before the introduction of the revised method these problems could not be solved considering Ezzati et al.’s [17] ranking criterion; thus, they are illustrative of the advantages of the revised method. The examples in this section were solved using the linear programming modeller PuLP version 1.6.0 [27] and CBC\(^1\) version 2.9.9 [28] on a computer with an Intel\(^{\text{®}}\) Core\(^{\text{TM}}\) i3-4005U@1.70 GHz x4 and 4GB RAM running Ubuntu 18.04.4.

**Example 3.1:** (Optimal solution of the fully fuzzy investment problem): Assuming $\tilde{x}_{ij} = (x_{ij}^{l}, x_{ij}^{u}, x_{ij}^{u})$ and following Steps 1–4 of the method outlined in Section 2, with $\epsilon = 10^{-4}$ and $L = 100$, FFLP problem (1) is transformed into the following MLLP problem.

$$\text{lex max} (7x_{11}^{c} + 5x_{12}^{c} + 8x_{13}^{c} + 5x_{21}^{c} + 7x_{22}^{c} + 9x_{23}^{c} + 10x_{31}^{c} + 8x_{32}^{c} + 7x_{33}^{c} + 6x_{41}^{c} + 5x_{42}^{c} + 9x_{43}^{c}, 5x_{11}^{l} - 8x_{11}^{u} + 3x_{12}^{l} - 6x_{12}^{u} + 4x_{13}^{l} - 9x_{13}^{u} + 3x_{21}^{l} - 7x_{21}^{u} + 4x_{22}^{l} - 8x_{22}^{u} + 8x_{23}^{l} - 10x_{23}^{u} + 7x_{31}^{l} - 11x_{31}^{u} + 6x_{32}^{l} - 10x_{32}^{u} + 4x_{33}^{l} - 8x_{33}^{u} + 4x_{41}^{l} - 8x_{41}^{u})$$

\(^1\) Computational Infrastructure for Operations Research (COIN-OR) Branch-and-Cut solver.
Ezzati et al. [17] lexicographic ranking criterion. The optimal fuzzy solution does satisfy all the constraints of FFLP problem (1) with respect to the other variables take on the same values as in the previous solution. As happens with classical linear programming, it can be shown that the convex combination of the two optimal fuzzy solutions is also optimal for FFLP problem (1). So, FFLP problem (1) is not classical linear programming, it can be shown that the convex combination of the two optimal fuzzy solutions is also optimal for FFLP problem (1). So, FFLP problem (1) is not

\[
\begin{align*}
3x_{42}^l - 7x_{42}^u + 7x_{43}^u - 11x_{43}^u, 5x_{11}^l + 8x_{11}^u + 3x_{12}^l + 6x_{12}^u + 4x_{13}^l \\
+ 9x_{13}^u + 3x_{21}^u + 7x_{21}^u + 4x_{22}^u + 8x_{22}^u + 10x_{23}^u + 7x_{31}^u \\
+ 11x_{31}^u + 6x_{32}^u + 10x_{32}^u + 4x_{33}^u + 8x_{33}^u + 4x_{41}^u + 8x_{41}^u \\
+ 3x_{42}^l + 7x_{42}^u + 7x_{43}^u + 11x_{43}^u)
\end{align*}
\]

s.t. \[\sum_{i=1}^{4} \sum_{j=1}^{3} x_{ij} = 25, \sum_{i=1}^{4} \sum_{j=1}^{3} x_{ij}^c = 30, \sum_{i=1}^{4} \sum_{j=1}^{3} x_{ij}^f = 40\]

\[
\epsilon y_{i1} \leq \sum_{j=1}^{3} x_{ij}^c - b_{i1} \leq Ly_{i1}, \text{ for } i = 1, 2, 3, 4
\]

\[-Ly_{i1} + \epsilon y_{i2} \leq \sum_{j=1}^{3} (x_{ij}^l - x_{ij}^u) - b_{i2} \leq Ly_{i2}, \text{ for } i = 1, 2, 3, 4
\]

\[-L(y_{i1} + y_{i2}) + \epsilon y_{i3} \leq \sum_{j=1}^{3} (x_{ij}^l + x_{ij}^u) - b_{i3} \leq Ly_{i3}, \text{ for } i = 1, 2, 3, 4
\]

\[
[ b_{ik} ] = \begin{bmatrix}
3 & -3 & 7 \\
5 & -2 & 10 \\
8 & -4 & 14 \\
8 & -7 & 21
\end{bmatrix}, \quad [ p_{ij1} ] = \begin{bmatrix}
6 & 5 & 9 \\
7 & 10 & 4 \\
5 & 3 & 7 \\
6 & 5 & 4
\end{bmatrix}, \text{ for } i = 1, 2, 3, 4 \text{ and } j = 1, 2, 3
\]

\[
[ p_{ij2} ] = \begin{bmatrix}
-3 & -3 & -2 \\
-3 & -3 & -2 \\
-3 & -4 & -5 \\
-3 & -5 & -3
\end{bmatrix}, \quad [ p_{ij3} ] = \begin{bmatrix}
11 & 9 & 18 \\
13 & 19 & 8 \\
11 & 8 & 13 \\
11 & 13 & 7
\end{bmatrix}, \text{ for } i = 1, 2, 3, 4 \text{ and } j = 1, 2, 3
\]

\[y_{ik} \in \{0, 1\}, z_{ijk} \in \{0, 1\}, \text{ for } i = 1, 2, 3, 4, j = 1, 2, 3 \text{ and } k = 1, 2, 3
\]

\[x_{ij}^l \geq 0, x_{ij}^l - x_{ij}^u \geq 0, x_{ij}^u - x_{ij}^f \geq 0, \text{ for } i = 1, 2, 3, 4, j = 1, 2, 3
\]

Steps 5 and 6 yield: \(\tilde{x}_{11} = (0, 0, 0), \tilde{x}_{12} = (0, 0, 4.9999), \tilde{x}_{13} = (7, 9, 9), \tilde{x}_{21} = (0, 0, 0), \tilde{x}_{22} = (1, 1, 1), \tilde{x}_{23} = (3, 4, 5), \tilde{x}_{31} = (4, 5, 7), \tilde{x}_{32} = (2.9999, 2.9999, 2.9999), \tilde{x}_{33} = (0.0001, 0.0001, 0.0001), \tilde{x}_{41} = (4, 4, 4), \tilde{x}_{42} = (0, 0, 0), \tilde{x}_{43} = (3, 4, 6.0001)\) with fuzzy objective value \(\tilde{v} = (138.9998, 248.9999, 374.0003)\). The reader is encouraged to verify that the obtained optimal fuzzy solution does satisfy all the constraints of FFLP problem (1) with respect to Ezzati et al.'s [17] lexicographic ranking criterion.

An alternative optimal fuzzy solution is given by \(\tilde{x}_{31} = (3, 5, 6), \tilde{x}_{43} = (4, 4, 7.0001)\) and the other variables take on the same values as in the previous solution. As happens with classical linear programming, it can be shown that the convex combination of the two optimal fuzzy solutions is also optimal for FFLP problem (1). So, FFLP problem (1) is not
an unfeasible problem as Bhardwaj and Kumar [22] claimed; it actually has infinitely many optimal fuzzy solutions.

The feasible solution of FFLP problem (1) given in the introduction of the present paper is obtained by imposing the integer condition on all variables of the above MLLP problem. Lastly, it is important to mention that if we use the element-wise inequality in place of $\leq_{lex}$, then the resulting optimisation problem becomes unfeasible; this may have motivated Bhardwaj and Kumar’s [22] claim.

Example 3.2: (Fully fuzzy project scheduling problem): A project is composed of interrelated activities, whose structure can be represented by a directed acyclic graph (see, e.g. Figure 1). The set of vertices $V$ represent events and the set of edges $A$ represent activities. Formally, a project can be represented by the triplet $(V, A, D)$, where $A \subseteq V \times V$ and $d_{ij} \in D$ is the time required for the completion of activity $(i,j) \in A$. In what follows, it is assumed that the duration of each activity is estimated by a non-negative TFN. If $\tilde{t}_j$ denotes the fuzzy time of event $j \in A$, then for each activity $(i,j) \in A$ we must have that $\tilde{t}_i \oplus d_{ij} \leq \tilde{t}_j$ to ensure that all precedence relationships hold.

In the case of the project depicted in Figure 1, we have $V = \{1, 2, \ldots, 9\}$; $A$ and $D$ are shown in Table 1.

To schedule a project, we must find the shortest time interval in which all precedence relationships are satisfied; therefore, the fully fuzzy mathematical programming model for

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure1.png}
\caption{Precedence relationship network of the fully fuzzy project scheduling problem (8).}
\end{figure}

\begin{table}[h]
\centering
\caption{Data of the project network depicted in Figure 1.}
\begin{tabular}{ll}
\hline
Activity index & Duration $\tilde{d}_{ij}$ \\
\hline
(1, 2) & (10, 12, 14) \\
(2, 3) & (11, 14, 15) \\
(2, 4) & (12, 16, 18) \\
(2, 5) & (13, 15, 17) \\
(3, 6) & (15, 17, 20) \\
(4, 6) & (10, 13, 16) \\
(4, 7) & (16, 18, 20) \\
(5, 7) & (10, 12, 15) \\
(6, 8) & (17, 19, 21) \\
(7, 8) & (11, 14, 17) \\
(8, 9) & (10, 15, 18) \\
\hline
\end{tabular}
\end{table}
Table 2. Solution of the fully fuzzy project scheduling problem (8).

| Event index | Ezzati et al.’s [17] methoda | Revised methodb |
|-------------|-----------------------------|------------------|
| 1           | (0, 0, 0)                   | (0, 0, 0)        |
| 2           | (10, 12, 14)                | (10, 12, 14)     |
| 3           | (21, 28, 32)                | (21, 26, 29)     |
| 4           | (22, 28, 32)                | (28, 28, 31)     |
| 5           | (28, 34, 38)                | (23, 27, 31)     |
| 6           | (36, 45, 52)                | (36, 43, 49)     |
| 7           | (38, 46, 53)                | (44, 46, 51)     |
| 8           | (53, 64, 74)                | (53, 62, 70)     |
| 9           | (63, 79, 92)                | (63, 77, 88)     |

aOmitting Ezzati et al.’s [17] second criterion due to its unboundedness.
bUsing Ezzati et al.’s [17] full ranking criterion.

the project network depicted in Figure 1 is given by FFLP problem (8).

$$\min \tilde{t}_9$$

s.t. $\tilde{t}_i \oplus \tilde{d}_{ij} \leq \tilde{t}_j$, for $(i, j) \in A$  \hspace{1cm} (8)

$\tilde{t}_j$ is an unrestricted TFN, for $j \in \{1, 2, \ldots, 9\}$

There are several approaches in the literature for solving fuzzy project scheduling problems [see, e.g. 8,29–36]. However, not all of those studies considered fully fuzzified problems, and none has defined the fuzzy inequality constraints lexicographically. The solution of FFLP problem (8), obtained by using the revised method, is shown in Table 2. The critical path is given by the sequence of activities $(1, 2) \rightarrow (2, 3) \rightarrow (3, 6) \rightarrow (6, 8) \rightarrow (8, 9)$, since their corresponding fuzzy constraints are satisfied in strict equality. On the other hand, FFLP problem (8) is wrongly considered unbounded by Ezzati et al.’s [17] method, since, by using the lexicographic method of classical multi-objective optimisation, the second criterion is not bounded over the corresponding feasible region. However, if Ezzati et al.’s [17] second criterion is omitted, then we obtain the solution shown in Table 2.

That solution is, nevertheless, unfeasible according to their own criterion. To show this, notice that according to the precedence relationships we must have $\tilde{t}_4 \oplus \tilde{d}_{47} \leq \tilde{t}_7$, and after substituting the values of $\tilde{t}_4$, $\tilde{t}_7$ we get $\tilde{t}_{47} = (x_{47}^l, x_{47}^c, x_{47}^u) = (38, 46, 52)$ and $\tilde{t}_{47} = (x_{47}^l, x_{47}^c, x_{47}^u) = (38, 46, 53)$ for the left-hand side and right-hand side values of this fuzzy inequality constraint, respectively. According to Ezzati et al.’s [17] ranking criterion, $(x_{47}^l, x_{47}^c, x_{47}^u) \leq (38, 46, 53) \Rightarrow x_{47}^c \leq 46$, or if $x_{47}^c = 46$ then $x_{47}^u - x_{47}^l \geq 15$, or if $x_{47}^c = 46$ and $x_{47}^u - x_{47}^l = 15$ then $x_{47}^l + x_{47}^u \leq 91$; since we have that $x_{47}^c = 46$ but $x_{47}^u - x_{47}^l = 14$, then $\tilde{t}_{47} \not\leq \tilde{t}_{47}$. Similarly, it can be shown that $\tilde{t}_6 \oplus \tilde{d}_{68} \not\leq \tilde{t}_8$; therefore, the solution obtained by using Ezzati et al.’s [17] method is unfeasible. Remarkably, the solution obtained by using the revised method has lower fuzzy objective value (shorter project completion time) as compared with the fuzzy objective value obtained by using Ezzati et al.’s [17] method, according to their criterion; this value is also intuitively lower as seen from Figure 2. Clearly, the revised method allows to obtain an optimal solution of this problem.
Figure 2. Project completion times obtained by using the revised method (solid line) and Ezzati et al.’s [17] method (dashed line).

Table 3. Data of the fully fuzzy project crashing problem (9).

| Activity index | Maximum allowable crash time $\tilde{T}_{ij}$ | Crash cost $\tilde{c}_{ij}$ |
|----------------|---------------------------------------------|------------------------------|
| (1, 2)         | (3, 5, 7)                                  | (10, 13, 15)                |
| (2, 3)         | (4, 6, 8)                                  | (17.5, 20, 22.5)            |
| (2, 4)         | (4, 7, 9)                                  | (12, 15, 16.5)              |
| (2, 5)         | (2, 5, 8)                                  | (11.5, 13, 14)              |
| (3, 6)         | (5, 6, 7)                                  | (19, 21, 23)                |
| (4, 6)         | (3, 4, 5)                                  | (13, 14, 15)                |
| (4, 7)         | (6, 8, 9)                                  | (15, 16, 17)                |
| (5, 7)         | (2, 4, 7)                                  | (16.5, 18, 20.5)            |
| (6, 8)         | (4, 8, 9)                                  | (10.5, 12, 13)              |
| (7, 8)         | (3, 5, 8)                                  | (20, 23, 24.5)              |
| (8, 9)         | (4, 7, 9)                                  | (14, 17.5, 21)              |

Example 3.3: (Fully fuzzy project crashing problem): Frequently, it is necessary to complete a project in a time less than the critical path. To this aim, additional resources must be allocated to the project activities so that the duration of each activity can be modified accordingly. Suppose that, for the project depicted in Figure 1, the decision-maker has a budget estimated in $\tilde{B} = (540, 950, 1250)$, let us denote by $\tilde{t}_i$ the quantity by which the duration of activity $(i,j)$ is crashed and $\tilde{c}_{ij}$ the unitary cost of crashing activity $(i,j)$ in $\tilde{T}_{ij}$ units. It is further assumed that the duration of each activity $(i,j)$ can be reduced by at most $\tilde{T}_{ij}$ time units. The model to obtain an optimal solution of the project crashing problem is given by FFLP problem (9); all the problem parameters are TFNs as shown in Table 3.

$$\begin{align*}
\min & \quad \tilde{t}_0 \\
\text{s.t.} & \quad \tilde{t}_i \oplus \tilde{d}_{ij} \preceq \tilde{t}_j + \tilde{T}_{ij}, \text{ for } (i,j) \in A, \\
& \quad \tilde{T}_{ij} \preceq \tilde{T}_{ij}, \text{ for } (i,j) \in A, \\
& \quad \sum_{(i,j) \in A} \tilde{c}_{ij} \otimes \tilde{T}_{ij} \preceq \tilde{B} = (540, 950, 1250), \\
& \quad \tilde{t}_1 = (0, 0, 0), \\
& \quad \tilde{t}_{ij} \text{ is non-negative TFN, for } (i,j) \in A \\
& \quad \tilde{T}_{ij} \text{ is an unrestricted TFN, for } j \in \{1, 2, \ldots, 9\}
\end{align*}$$
Figure 3. Project completion times, obtained by using the revised method, before crashing (dashed line) and after crashing (solid line).

Table 4. Solution of the fully fuzzy project crashing problem (9), up to two decimal places, obtained by using the revised method.

| Activity index | Units crashed $\tilde{t}_i$ | Event index | Event time $\tilde{t}_j$ |
|---------------|--------------------------|-------------|---------------------|
| (1, 2)        | (3, 5, 7)                | 1           | (0, 0, 0)           |
| (2, 3)        | (3, 6, 7)                | 2           | (7, 7, 7)           |
| (2, 4)        | (3.99, 7, 8.99)          | 3           | (15, 15, 15)        |
| (2, 5)        | (4.99, 4.99, 4.99)       | 4           | (15, 16, 16)        |
| (3, 6)        | (5.70, 5.70, 5.70)       | 5           | (15, 17, 19)        |
| (4, 6)        | (0.70, 2.70, 2.70)       | 6           | (24.29, 26.29, 29.29) |
| (4, 7)        | (5.70, 5.70, 6.24)       | 7           | (26.56, 28.29, 31.02) |
| (5, 7)        | (0.70, 0.70, 0.70)       | 8           | (37.29, 37.29, 41.29) |
| (6, 8)        | (4, 8, 9)                | 9           | (43.29, 45.29, 50.29) |
| (7, 8)        | (1.96, 5, 6.96)          |             |                     |
| (8, 9)        | (4, 7, 9)                |             |                     |

The solution of FFLP problem (9), using the revised method, is shown in Table 4. The project completion times, before and after crashing, are depicted in Figure 3. For this particular FFLP problem, the corresponding multi-objective crisp linear programming problems obtained by using Ezzati et al.’s [17] method and by using the element-wise inequality in place of $\leq_{lex}$ are both unfeasible; hence, to the best of our knowledge, only the method given in this paper can be used to solve FFLP problem (9), assuming that the decision-maker ranks TFNs according to Ezzati et al.’s [17] criterion.

4. Some Remarks on the Selected Problems and Alternative Solution Methods

Besides Ezzati et al.’s [17] method, there are several other methods in the literature for solving FFLP problems [see, e.g. 37–45]. Among the existing methods, those using ranking functions are the most commonly used, but can lead to unsatisfactory results. For example, the existing methods [41, sec. 5.4.2] and [40, Algorithm 1] use the ranking functions $R(\tilde{v}) = (v^{l} + 2v^{c} + v^{u})/4$ and $EV(\tilde{v}) = (v^{l} + 4v^{c} + v^{u})/6$, respectively to transform an FFLP
problem with inequality constraints into a crisp one. This transformation is carried out by applying \( R(\cdot) \), \( EV(\cdot) \) or any other ranking function to the objective function and to both sides of the inequality constraints of the FFLP problem. An optimal solution for the resulting crisp problem is then considered optimal for the fuzzy one. However, the ranking functions may map different fuzzy numbers into the same real number, and therefore have little discriminative capabilities. In this sense, using ranking functions to solve FFLP problems may not yield satisfying solutions.

Nasser et al. [38], on the other hand, have used the partial order \( \tilde{v}_1 = (v_1^l, v_1^c, v_1^u) \preceq \tilde{v}_2 = (v_2^l, v_2^c, v_2^u) \Leftrightarrow v_1^l \leq v_2^l, v_1^c \leq v_2^c, v_1^u \leq v_2^u \) to handle the fuzzy inequality constraints, and \( R(\cdot) \) with the objective function; therefore, they have used two different approaches for the inequality of TFNs. We note that, with respect to Nasser et al.’s [38] partial order, if \( \tilde{v}_1 \preceq \tilde{v}_2 \), then \( R(\tilde{v}_1) \leq R(\tilde{v}_2) \), but the converse is not true. Ebrahimnejad and Verdegay [15, chap. 4, p. 298] claimed that it is not correct to use two different approaches for the inequality of two fuzzy numbers in the same problem. Although we share their view in general, we also recognise that it can be open to discussion, since decision-makers might be interested in checking solution feasibility and comparing objective values by different criteria; in any case, the choice of these criteria must be carefully justified on an application basis. In this paper, we choose to use the same lexicographic criterion for the problem objective function and its inequality constraints because our main objective is to present a revised version of Ezzati et al.’s [17] method that overcomes its shortcomings and limitations. It should be noted that the revised method presented in Section 2 can be straightforwardly modified to account for cases in which decision-makers check feasibility and compare objective values by different lexicographic criteria.

Stanojević et al. [45], based on the work of Dong and Wan [10], used the interval expectation of trapezoidal fuzzy numbers and a partial order for intervals to transform the FFLP problem into a two-objective crisp linear programming problem. The weighted sum approach was subsequently used to transform the multi-objective problem into a single-objective one.

In the following section, we compare the revised method with the existing methods [38, 40, 41, 45].

4.1. Discussion on the Results Obtained by Using Existing Methods

The results obtained by using the existing methods [41, sec. 5.4.2] and [40, Algorithm 1] to solve FFLP problem (8) are shown in Table 5. We see from those results that most event times are crisp instead of fuzzy; hence, they do not properly represent the inherent fuzziness of project scheduling problem (8). Furthermore, it seems unreasonable for this particular problem that the most optimistic and possible values of the few fuzzy event times are zero. We therefore conclude that the existing methods [41, sec. 5.4.2] and [40, Algorithm 1] do not effectively solve FFLP problem (8).

The results obtained by using Nasseri et al.’s [38] method are shown also in Table 5. Contrary to the existing methods [41, sec. 5.4.2] and [40, Algorithm 1], Nasseri et al.’s [38] method does yield a truly fuzzy solution of FFLP problem (8); actually, the project completion time \( \tilde{t}_9 = (63, 77, 88) \) is exactly the same obtained by using the revised method; these results are however unfeasible according to Ezzati et al.’s [17] ranking criterion.
Table 5. Solution of the fully fuzzy project scheduling problem (8) with alternative methods.

| Event index | [41, sec. 5.4.2] | [40, Algorithm 1] | [38] | [45, $\alpha = 0.2, \beta = 0.5$] |
|-------------|------------------|-------------------|------|----------------------------------|
| 1           | (0, 0, 0)        | (0, 0, 0)         | (0, 0) | (0, 0, 0)                        |
| 2           | (12, 12, 12)     | (12, 12, 12)      | (10, 12, 14) | (11, 11, 15) |
| 3           | $\left(\frac{51}{2}, \frac{51}{2}, \frac{51}{2}\right)$ | $\left(\frac{77}{3}, \frac{77}{3}, \frac{77}{3}\right)$ | (21, 26, 29) | (23.5, 23.5, 31.5) |
| 4           | $\left(\frac{55}{2}, \frac{55}{2}, \frac{55}{2}\right)$ | $\left(\frac{83}{3}, \frac{83}{3}, \frac{83}{3}\right)$ | (22, 28, 32) | (17.75, 31.5, 31.5) |
| 5           | (0, 0, 108)      | (0, 0, 162)       | (23, 27, 37) | (23, 23, 51)                      |
| 6           | $\left(\frac{171}{4}, \frac{171}{4}, \frac{171}{4}\right)$ | $\left(\frac{257}{6}, \frac{257}{6}, \frac{257}{6}\right)$ | (36, 43, 49) | (39.5, 39.5, 52.5) |
| 7           | $\left(\frac{191}{4}, \frac{191}{4}, \frac{191}{4}\right)$ | $\left(\frac{287}{6}, \frac{287}{6}, \frac{287}{6}\right)$ | (38, 46, 52) | (32.75, 50.5, 50.5) |
| 8           | $\left(\frac{247}{4}, \frac{247}{4}, \frac{247}{4}\right)$ | $\left(\frac{371}{6}, \frac{371}{6}, \frac{371}{6}\right)$ | (53, 62, 70) | (49, 66, 66) |
| 9           | (0, 0, 305)      | (0, 0, 459)       | (63, 77, 88) | (70, 70, 95)                      |

Figure 4. Triangular fuzzy numbers representing the project completion times after crashing obtained by methods [38, 45] and the revised method, crash cost obtained by Stanojević et al.’s [45] method and the available budget.

Stanojević et al.’s [45] method yields the results shown in the last column of Table 5. Those results were obtained by setting parameters $\alpha = 0.2$ and $\beta = 0.5$ in Stanojević et al.’s [45] method. Parameter $\beta$ weights the objective functions and $\alpha$ is the acceptance degree for the violation of the fuzzy inequality constraints. By fixing $\beta = 0.5$ and varying $\alpha$, we observed that values of $\alpha$ over 0.25 lead to unbounded problems in this particular situation. As in the case of Nasseri et al. [38], Stanojević et al.’s [45] solution is unfeasible according to Ezzati et al.’s [17] ranking criterion.

As for the fully fuzzy project crashing problem (9), a similar conclusion, as given above, can be drawn with respect to the use of the existing methods [41, sec. 5.4.2] and [40, Algorithm 1] for solving this FFLP problem. Nasseri et al.’s [38] method, on the other hand, yields a slightly higher project completion time ($\hat{t}_9$)$_{\text{Nasseri et al.}} = (43, 45.36, 50.64)$, considering Ezzati et al.’s [17] ranking criterion, as compared with the one yielded by the revised method ($\hat{t}_9$)$_{\text{Revised method}} = (43.29, 45.29, 50.29)$. Stanojević et al.’s [45] method yields the shortest project completion time ($\hat{t}_9$)$_{\text{Stanojević et al.}} = (43.70, 43.70, 51.65)$, as seen
from Figure 4(a); however, from Figure 4(b), we see that the corresponding crash cost exceeds the available budget.

5. Conclusions

In this paper, we confirmed the results of Bhardwaj and Kumar [22] that demonstrate that the method proposed by Ezzati et al. [17] for solving FFLP problems cannot be used to find optimal solutions of FFLP problems with inequality constraints. We proposed a revised version of Ezzati et al.’s [17] method that overcomes its shortcomings and limitations. By using the revised method, we showed that Bhardwaj and Kumar’s [22] claim on the unfeasibility of the fully fuzzy investment problem of Ezzati et al. [17] is false, and this problem actually has infinitely many optimal fuzzy solutions. To further illustrate the applicability of the revised method, a fully fuzzy project scheduling problem with budget constraint was solved. The example problems could not be solved by using Ezzati et al.’s [17] original method.

Future research will focus on using the revised method to solve real-world FFLP problems, assessing its efficiency and proposing improvements to its implementation. Taking into account that most FLP problems of practical interest have inequality constraints, and the interpretation of fuzzy inequality constraints is a major concern for decision-makers, we believe that the suitability of the revised method to address diverse application problems with fuzzy inequality constraints such as recommender systems for improving health conditions [12], multi-item solid transportation problems [7] and reverse logistics network design problems [11] is worthy of investigation.

Acknowledgements

The authors would like to thank the anonymous reviewers for their helpful comments and suggestions.

Disclosure statement

No potential conflict of interest was reported by the author(s).

Funding

The authors did not receive any funding for this research.

Notes on contributors

Boris Pérez-Cañedo is currently with the Department of Mathematics, Faculty of Economics and Business Sciences, at the University of Cienfuegos, Cuba. He received his Bachelor degree in Informatics Engineering in 2010 and his MSc in Applied Mathematics in 2014, both from the University of Cienfuegos. His research interests include fuzzy optimisation, metaheuristics and inverse problems. He is currently working towards his PhD degree.

Eduardo René Concepción-Morales is currently a Professor at the Informatics Department, University of Cienfuegos, Cuba. He received his PhD in Computer Science from the University of the Basque Country, Department of Computer Science and Artificial Intelligence, Spain, in 2010. He received his BSc in Applied Mathematics in 1984 from the Baku State University, Azerbaijan, and his MSc in Applied Mathematics in 1998 from the University of Cienfuegos, Cuba. His research interests include Fuzzy
Linear Programming, Computer Vision and Pattern Recognition. He has published academic papers in peer-reviewed journals and conferences.

**Seyyed Ahmad Edalatpanah** is an Assistant Professor of Ayandegan Institute of Higher Education of Iran. He received his PhD in Applied Mathematics from the University of Guilan, Rasht, Iran. He is also an academic member of Guilan University and Islamic Azad University of Iran. His fields of interest are numerical modelling, soft computing, and optimisation. He has published over 100 journal and conference proceedings papers in the above research areas. He serves on the editorial boards of several international journals. He is Editor-in-chief of the International Journal of Research in Industrial Engineering at www.riejournal.com.

**ORCID**

*Boris Pérez-Cañedo* [http://orcid.org/0000-0002-5623-4039]

*Eduardo René Concepción-Morales* [http://orcid.org/0000-0003-2190-1337]

*Seyyed Ahmad Edalatpanah* [http://orcid.org/0000-0001-9349-5695]

**References**

[1] Bellman RE, Zadeh LA. Decision-making in a fuzzy environment. Manage Sci. 1970;17(4):B-141–B-164.

[2] Tanaka H, Okuda T, Asai K. On fuzzy-mathematical programming. J Cybernet. 1973;3(4):37–46.

[3] Zimmermann HJ. Description and optimization of fuzzy systems. Int J Gen Syst. 1975;2(1):209–215.

[4] Kumar A, Gupta A. Methods for solving fuzzy assignment problems and fuzzy travelling salesman problems with different membership functions. Fuzzy Inf Eng. 2011;3(1):3–21.

[5] Baykasoğlu A, Subulan K, Karaslan FS. A new fuzzy linear assignment method for multiattribute decision making with an application to spare parts inventory classification. Appl Soft Comput. 2016;42:1–17.

[6] Gupta A, Kumar A, Kaur A. Mehar’s method to find exact fuzzy optimal solution of unbalanced fully fuzzy multi-objective transportation problems. Optim Lett. 2012;6(8):1737–1751.

[7] Giri PK, Maiti MK, Maiti M. Fully fuzzy fixed charge multi-item solid transportation problem. Appl Soft Comput. 2015;27:77–91.

[8] Göçken T, Baykasoğlu A. Direct solution of time-cost tradeoff problem with fuzzy decision variables. Cybern Syst. 2016;47(3):206–219.

[9] Jana J, Kumar Roy S. Solution of matrix games with generalised trapezoidal fuzzy payoffs. Fuzzy Inf Eng. 2018;10(2):213–224.

[10] Dong J-y, Wan S-p. A new trapezoidal fuzzy linear programming method considering the acceptance degree of fuzzy constraints violated. Knowl Based Syst. 2018;148:100–114.

[11] Baykasoğlu A, Subulan K. An analysis of fully fuzzy linear programming with fuzzy decision variables through logistics network design problem. Knowl Based Syst. 2015;90:165–184.

[12] Mezei J, Nikou S. Fuzzy optimization to improve mobile health and wellness recommendation systems. Knowl Based Syst. 2018;142:108–116.

[13] Nasseri SH, Zavieh H. A multi-objective method for solving fuzzy linear programming based on semi-infinite model. Fuzzy Inf Eng. 2018;10(1):91–98.

[14] Arana-Jiménez M. Fuzzy Pareto solutions in fully fuzzy multiobjective linear programming. In: Le Thi HA, Le HM, Pham Dinh T, editors. Optimization of complex systems: theory, models, algorithms and applications. Vol. 991. Cham: Springer International Publishing; 2020. p. 509–517.

[15] Ebrahimnejad A, Verdegay JL. Fuzzy sets-based methods and techniques for modern analytics. 1st ed. (Studies in Fuzziness and Soft Computing; Vol. 364). Cham: Springer International Publishing; 2018.

[16] Nasseri SH, Ebrahimnejad A, Cao BY. Fuzzy linear programming: solution techniques and applications. (Studies in Fuzziness and Soft Computing; Vol. 379). Cham: Springer International Publishing; 2019.
[17] Ezzati R, Khorram E, Enayati R. A new algorithm to solve fully fuzzy linear programming problems using the MOLP problem. Appl Math Model. 2015;39(12):3183–3193.

[18] Hosseinzadeh Lotfi F, Allahviranloo T, Alimardani Jondabeh M, et al. Solving a full fuzzy linear programming using lexicography method and fuzzy approximate solution. Appl Math Model. 2009;33:3151–3156.

[19] Kumar A, Kaur J, Singh P. A new method for solving fully fuzzy linear programming problems. Appl Math Model. 2011;35(2):817–823.

[20] Ehrrogd M. Multicriteria optimization. 2nd ed. Berlin/Heidelberg: Springer-Verlag; 2005.

[21] Ebrahimnejad A. An effective computational attempt for solving fully fuzzy linear programming using MOLP problem. J Ind Prod Eng. 2019;36(2):59–69.

[22] Bhardwaj B, Kumar A. A note on “A new algorithm to solve fully fuzzy linear programming problems using the MOLP problem”. Appl Math Model. 2015;39(19):5982–5985.

[23] Pérez-Cañedo B, Concepción-Morales ER. A method to find the unique optimal fuzzy value of fully fuzzy linear programming problems with inequality constraints having unrestricted L-R fuzzy parameters and decision variables. Expert Syst Appl. 2019;123:256–269.

[24] Gong Z, Zhao W, Liu K. A straightforward approach for solving fully fuzzy linear programming problem with Ir-type fuzzy numbers. J Oper Res Soc Jpn. 2018;61(2):172–185.

[25] Gupta G, Kaur J, Kumar A. A note on “fully fuzzy fixed charge multi-item solid transportation problem”. Appl Soft Comput. 2016;41:418–419.

[26] Das SK, Mandal T, Edalatpanah SA. A mathematical model for solving fully fuzzy linear programming problem with trapezoidal fuzzy numbers. Applied Intelligence. 2017;46(3):509–519.

[27] Mitchell S, O’Sullivan M, Dunning I. PuLP: A linear programming toolkit for python; 2011. The University of Auckland, Auckland, New Zealand; Available from: http://www.optimization-online.org/DB_FILE/2011/09/3178.pdf.

[28] Forrest J, Lougee-Heimer R. CBC user guide. In: Greenberg HJ, Smith JC, editors. Emerging theory, methods, and applications. Chapter 10. Hanover, MD: INFORMS; 2005. p. 257–277.

[29] Leu SS, Chen AT, Yang CH. A GA-based fuzzy optimal model for construction time-cost trade-off. Int J Project Manage. 2001;19(1):47–58.

[30] Liu ST. Fuzzy activity times in critical path and project crashing problems. Cybern Syst. 2003;34(2):161–172.

[31] Chen SP, Hsueh YJ. A simple approach to fuzzy critical path analysis in project networks. Appl Math Model. 2008;32(7):1289–1297.

[32] Lin FT. Fuzzy crashing problem on project management based on confidence-interval estimates. Proceedings – 8th International Conference on Intelligent Systems Design and Applications ISDA 2008. 2008;2:164–169.

[33] Zareei A, Zaerpour F, Bagherpour M, et al. A new approach for solving fuzzy critical path problem using analysis of events. Expert Syst Appl. 2011;38(1):87–93.

[34] Gökçen T. Solution of fuzzy multi-objective project crashing problem. Neural Comput Appl. 2013;23(7-8):2167–2175.

[35] Kaur P, Kumar A. Linear programming approach for solving fuzzy critical path problems with fuzzy parameters. Appl Soft Comput. 2014;21:309–319.

[36] Zhang Q, Zhou J, Wang K, et al. An effective solution approach to fuzzy programming with application to project scheduling. Int J Fuzzy Syst. 2018;20(8):2383–2398.

[37] Allahviranloo T, Lotfi FH, Kiasary MK, et al. Solving fully fuzzy linear programming problem by the ranking function. Appl Math Sci. 2008;2(1):19–32.

[38] Nasseri S, Behmanesh E, Taleshian F, et al. Fully fuzzy linear programming with inequality constraints. Int J Ind Math. 2013;5(4):309–316. Available from: http://ijim.srbiau.ac.ir/article_2162.html.

[39] Kaur J, Kumar A. Mehar’s method for solving fully fuzzy linear programming problems with L-R fuzzy parameters. Appl Math Model. 2013;37:7142–7153.

[40] Yang XP, Cao BY, Zhou XG. Solving fully fuzzy linear programming problems with flexible constraints based on a new order relation. J Intell Fuzzy Syst. 2015;29(4):1539–1550.
[41] Kaur J, Kumar A. An introduction to fuzzy linear programming problems: Theory, methods and applications. (Studies in Fuzziness and Soft Computing; Vol. 340). Cham: Springer International Publishing; 2016.

[42] Arana-Jiménez M. Nondominated solutions in a fully fuzzy linear programming problem. Math Methods Appl Sci. 2018;41(17):7421–7430.

[43] Tosarkani BM, Amin SH. A possibilistic solution to configure a battery closed-loop supply chain: multi-objective approach. Expert Syst Appl. 2018;92:12–26.

[44] Ozkok BA. Finding fuzzy optimal and approximate fuzzy optimal solution of fully fuzzy linear programming problems with trapezoidal fuzzy numbers. J Intell Fuzzy Syst. 2019;36(2):1389–1400.

[45] Stanojević B, Dzitac S, Dzitac I. Solution approach to a special class of full fuzzy linear programming problems. Procedia Comput Sci. 2019;162:260–266.