EQUIVARIANT HODGE POLYNOMIALS OF HEAVY/LIGHT MODULI SPACES

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Abstract. Let $\overline{M}_{g,m\mid n}$ denote Hassett’s moduli space of weighted pointed stable curves of genus $g$ for the heavy/light weight data $(1^m, 1/n^1)$, and let $\mathcal{M}_{g,m\mid n} \subset \overline{M}_{g,m\mid n}$ be the locus parameterizing smooth, not necessarily distinctly marked curves. We give a change-of-variables formula which computes the generating function for $(S_m \times S_n)$-equivariant Hodge–Deligne polynomials of these spaces in terms of the generating functions for $S_n$-equivariant Hodge–Deligne polynomials of $\overline{M}_{g,n}$ and $\mathcal{M}_{g,n}$.

1. Introduction

Given nonnegative integers $g$, $m$, and $n$ satisfying

$$2g - 2 + m + \min(n, 1) > 0,$$

set $\overline{M}_{g,m\mid n}$ to be Hassett’s moduli space $\overline{M}_{g,\mathcal{A}}$ of weighted pointed stable curves of genus $g$, for the weight data

$$\mathcal{A} = \left(\underbrace{1, \ldots, 1}_{m}, \underbrace{1/n, \ldots, 1/n}_{n}\right).$$

This space is a connected, smooth, and proper Deligne-Mumford stack over $\mathbb{Z}$, and is a compactification of the moduli space $\mathcal{M}_{g,m+n}$ of smooth pointed algebraic curves of genus $g$ [Has03]; this family of weight data has been called heavy/light in the literature [CHMR16, KKL21]. We also set $\mathcal{M}_{g,m\mid n} \subset \overline{M}_{g,m\mid n}$ to be the locus of smooth, not necessarily distinctly marked curves. In this paper we study the $(S_m \times S_n)$-equivariant Hodge–Deligne polynomials of $\mathcal{M}_{g,m\mid n}$ and $\overline{M}_{g,m\mid n}$. Throughout this paper we will work with the coarse moduli spaces of these stacks, as the mixed Hodge structure on the rational cohomology of a Deligne-Mumford stack coincides with that of its coarse moduli space.

If $X$ is a $d$-dimensional complex variety with an action of $S_m \times S_n$, its complex cohomology groups are $(S_m \times S_n)$-representations in the category of mixed Hodge structures. The $(S_m \times S_n)$-equivariant Hodge–Deligne polynomial of $X$ is given by the formula

$$h_{X}^{S_m \times S_n}(u, v) := \sum_{i,j,q=0}^{2d} (-1)^{i} \text{ch}_{m,n} \left( \text{Gr}^{P}_{p} \text{Gr}^{W}_{q} H^{i}_{c}(X; \mathbb{C}) \right) u^{p}v^{q} \in \Lambda^{(2)}[u, v],$$

where $\Lambda^{(2)} = \Lambda \otimes \Lambda$ is the ring of bisymmetric functions, and $\text{ch}_{m,n}(V) \in \Lambda^{(2)}$ is the Frobenius characteristic of an $(S_m \times S_n)$-representation $V$. The Hodge–Deligne polynomial has also been referred to as the $E$-polynomial and the mixed Hodge polynomial in the literature. If the mixed Hodge structure on each cohomology group is pure, as is the case for the coarse moduli...
space of $\overline{\mathcal{M}}_{g,m|n}$, the Hodge–Deligne polynomial specializes to the usual Hodge polynomial

$$
\sum_{p,q=0}^{2d} (-1)^{p+q} \text{ch}_{m,n}(H^{p,q}(X; \mathbb{C})) u^p v^q.
$$

For more details on mixed Hodge structures, see [PS08] or [CLNS18].

We assemble all of the equivariant Hodge–Deligne polynomials for heavy/light Hassett spaces with fixed genus into series with coefficients in $\Lambda^{(2)}$:

$$a_g := \sum_{m,n} h_{\mathcal{M}_{g,m|n}}^{S_m \times S_n}(u, v), \quad \overline{a}_g := \sum_{m,n} h_{\overline{\mathcal{M}}_{g,m|n}}^{S_m \times S_n}(u, v) \in \Lambda^{(2)}[[u, v]].$$

We also define

$$b_g := \sum_n h_{\mathcal{M}_{g,n}}^{S_n}(u, v), \quad \overline{b}_g := \sum_n h_{\overline{\mathcal{M}}_{g,n}}^{S_n}(u, v) \in \Lambda[[u, v]].$$

In the above, for a variety $X$ with action of $S_n$, we have set $h_X^{S_n}(u, v)$ for the $S_n$-equivariant Hodge–Deligne polynomial of $X$, defined analogously to (1.1), replacing $\text{ch}_{m,n}$ with the Frobenius characteristic $\text{ch}_n$ of an $S_n$-representation, and replacing $\Lambda^{(2)}$ with $\Lambda$.

Given $f \in \Lambda$, we set $f^{(j)} \in \Lambda^{(2)}$ for the inclusion of $f$ into the $j$th tensor factor, $j \in \{1, 2\}$. These extend to maps $\Lambda[[u, v]] \to \Lambda^{(2)}[[u, v]]$. Let $p_i \in \Lambda$ be the $i$th power sum symmetric function. The coproduct $\Lambda \to \Lambda^{(2)}$ defined by $p_i \mapsto p_i^{(1)} + p_i^{(2)}$ also extends to a map $\Delta : \Lambda[[u, v]] \to \Lambda^{(2)}[[u, v]]$. There are two plethysm operations $\circ_1, \circ_2$ defined on $\Lambda^{(2)}$, and these extend to $\Lambda^{(2)}[[u, v]]$ by

$$p_n^{(i)} \circ_i q = q^n,$n

$$p_n^{(i)} \circ_j q = p_n^{(i)} q^n,$$

for $\{i, j\} = \{1, 2\}$ and $q \in \{u, v\}$. See Section 2.1 for more details and references on symmetric functions and the Frobenius characteristic.

The main result of this paper is the following formula, which encodes the combinatorial relationship between the generating functions defined above.

**Theorem A.** Let $h_n \in \Lambda$ denote the $n$th homogeneous symmetric function. For $f \in \Lambda^{(2)}[[u, v]]$ set

$$\text{Exp}^{(2)}(f) = \sum_{n>0} h_n^{(2)} \circ_2 f.$$

Then we have

$$a_g = \Delta(b_g) \circ_2 \text{Exp}^{(2)}(p_1^{(2)})$$

and

$$\overline{a}_g = \Delta(\overline{b}_g) \circ_2 \left( p_1^{(2)} - \frac{\partial b_0^{(2)}}{\partial p_1^{(2)}} \right) \circ_2 \text{Exp}^{(2)}(p_1^{(2)}).$$

A formula for the series $b_0$ has been given by Getzler [Get95]; therefore Theorem A determines $a_g$ and $\overline{a}_g$ in terms of $b_g$ and $\overline{b}_g$. Moreover, this transformation is invertible, as $\text{Exp}^{(2)}$ has a plethystic inverse $\text{Log}^{(2)}$ and $p_1^{(2)} - \partial b_0^{(2)}/\partial p_1^{(2)}$ is inverse to $p_1^{(2)} + \partial \overline{b}_0^{(2)}/\partial p_1^{(2)}$. There
is a numerical analogue of Theorem A which deals with the non-equivariant Hodge–Deligne polynomials, defined by the assignment

\[ h_X(u, v) := \sum_{i,p,q=0}^{2d} (-1)^i \dim \left( \Gr_F^{Gr_{p+q}} H^i_c(X; \mathbb{C}) \right) u^p v^q \in \mathbb{Q}[u, v]. \]

Set

\[ a_g := \sum_{m,n} h_{\mathcal{M}_{g,m|n}}(u, v) \frac{x^m y^n}{m! n!} \quad \text{and} \quad \bar{a}_g := \sum_{m,n} h_{\mathcal{M}_{g,m|n}}(u, v) \frac{x^m y^n}{m! n!} \in \mathbb{Q}[u, v, x, y], \]

and similarly put

\[ b_g := \sum_{n} h_{\mathcal{M}_{g,n}}(u, v) \frac{x^n}{n!} \quad \text{and} \quad \bar{b}_g := \sum_{n} h_{\mathcal{M}_{g,n}}(u, v) \frac{x^n}{n!} \in \mathbb{Q}[u, v, x]. \]

**Corollary B.** We have

\[ a_g = b_g |_{x \rightarrow w} \]

where \( w = x + e^y - 1 \), and

\[ \bar{a}_g = \bar{b}_g |_{x \rightarrow z}, \]

where

\[ z = x + e^y + \frac{e^{uvy} - uv \cdot e^y + uv - 1}{uv - u^2 v^2} - 1. \]

Theorem A and Corollary B allow for many explicit computations. For example, Getzler [Get98] gives a recursive formula for the calculation of \( \bar{b}_1 \), which allows us to compute \( \bar{a}_1 \); sample calculations are included at the end of the paper in Tables 1 and 2. Similarly, Chan et al. in [CFGP19] give a formula for the \( S_n \)-equivariant weight 0 compactly supported Euler characteristic of \( \mathcal{M}_{g,n} \) in arbitrary genus, so Theorem A gives a practical method to compute the \( (S_m \times S_n) \)-equivariant weight 0 compactly supported Euler characteristic of \( \mathcal{M}_{g,m|n} \). Sample computations for \( g = 2 \) have been included in Table 3.

1.1. **Context.** The heavy/light moduli space \( \overline{\mathcal{M}}_{g,m|n} \) has been studied in several algebro-geometric contexts. It is of interest in its own right, as a modular compactification of \( \mathcal{M}_{g,m+n} \) which admits a birational morphism from the Deligne–Mumford–Knudsen moduli space \( \overline{\mathcal{M}}_{g,m+n} \). It arises in the theory of stable quotients [MOP11] and in tropical geometry [CHMR16, MUW21, AL21]. As \( g, m, \) and \( n \) vary, the spaces \( \overline{\mathcal{M}}_{g,m|n} \) form the components of Losev–Manin’s extended modular operad [LM04]; when \( g = 0 \) and \( m = 2 \), the space \( \overline{\mathcal{M}}_{0,2|n} \) is a toric variety, and it coincides with the Losev–Manin moduli space of stable chains of \( \mathbb{P}^1 \)'s [LM00].

1.2. **Related work.** The first part of Corollary B follows from an observation made in [KLSY]: that in the Grothendieck ring of varieties, one has an equality

\[ [\mathcal{M}_{g,m|n}] = \sum_{k=1}^{n} S(n, k) [\mathcal{M}_{g,m+k}], \]

where \( S(n, k) \), the Stirling number of the second kind, counts the number of partitions of \( \{1, \ldots, n\} \) with \( k \) parts. It follows that the generating function \( a_g \) can be obtained from \( b_g(x + w) \) by making the substitution \( w = e^y - 1 \); this transformation is called the Stirling
Theorem A can be viewed as an application of the equivariant version of the Stirling transform.

In genus zero, the problem of computing the equivariant Hodge polynomials of $\overline{M}_{0,m|n}$ has been studied by Bergström–Minabe [BM13, BM14] and by Chaudhuri [Cha16]. Our formula gives a third approach to this problem, which applies in arbitrary genus.

Also in genus zero, the Chow groups of $\overline{M}_{0,m|n}$ have been computed by Ceyhan [Cey09]. The Chow ring has been computed by Petersen [Pet17] and by Kannan–Karp–Li [KKL21].

The techniques of this paper are based on prior work on the operad structure of moduli of stable curves and maps, by Getzler [Get95, Get98], Getzler–Kapranov [GK98], and Getzler–Pandharipande [GP06]. In particular, the main tool of the paper is a generalization of Getzler–Pandharipande’s Grothendieck ring of $S$-spaces, which encodes sequences of varieties with $S_n$-actions, to the setting of $S^2$-spaces, which allows us to keep track of $(S_m \times S_n)$-actions as $m$ and $n$ vary.

1.3. Outline of the paper. We review the necessary background on symmetric functions, the Frobenius characteristic, and Hassett spaces in Section 2. In Section 3, we define the Grothendieck ring of $S^2$-spaces and its composition operations as categorifications of plethysm. We then use these composition operations to prove Theorem A in Section 4. Sample calculations and accompanying remarks are included at the end of the paper, in Section 5.

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2. Background

Here we briefly recall some background on symmetric functions and on Hassett spaces.

2.1. Symmetric functions and the Frobenius characteristic. For a more detailed background on the ring of symmetric functions, see Macdonald [Mac95], Stanley [Sta99], or Getzler–Kapranov [GK98, §7]. The ring $\Lambda$ of symmetric functions over $\mathbb{Q}$ is defined as

$$\Lambda = \lim_{\leftarrow} \mathbb{Q}[[x_1, \ldots, x_n]]^{S_n}.$$ 

We have that

$$\Lambda = \mathbb{Q}[[p_1, p_2, \ldots]]$$

where $p_i = \sum_{k>0} x_k^i$ is the $i$th power sum symmetric function. The ring $\Lambda$ is graded by degree, where $p_i$ has degree $i$. The Schur functions $s_\lambda$ for $\lambda \vdash n$ form a basis for the homogeneous degree $n$ part of $\Lambda$. Given an $S_n$-representation $V$, the Frobenius characteristic $\text{ch}_n(V)$ is defined by

$$\text{ch}_n(V) := \frac{1}{n!} \sum_{\sigma \in S_n} \text{Tr}_V(\sigma) p_{\lambda(\sigma)},$$

where $\lambda(\sigma)$ is the cycle type of the permutation $\sigma$, and for a partition $\lambda \vdash n$ we set $p_\lambda := \prod_{i} p_{\lambda_i}$. The Frobenius characteristic $\text{ch}_n(V)$ determines the $S_n$-representation $V$: if

$$V = \bigoplus_{\lambda \vdash n} W_\lambda^{\oplus a_\lambda}$$

with $a_\lambda$ the multiplicity of $W_\lambda$ in $V$, then

$$\text{ch}_n(V) = \prod_{\lambda \vdash n} (1 + p_\lambda)^{a_\lambda}.$$
is the decomposition of $V$ into Specht modules, then

$$\text{ch}_n(V) = \sum_{\lambda \vdash n} a_{\lambda} s_{\lambda}.$$ 

We define the homogeneous symmetric functions $h_n$ by

$$h_n := \text{ch}_n(\text{Triv}_n),$$

where $\text{Triv}_n$ is the trivial $S_n$-representation of dimension one. Note that $h_n = s_n$.

There is an associative operation $\circ$, called plethysm, on $\Lambda$, characterized by the following properties:

(i) for any $g \in \Lambda$, the map $f \mapsto f \circ g$ defines an algebra homomorphism $\Lambda \to \Lambda$;
(ii) for all $n$, the map $f \mapsto p_n \circ f$ defines an algebra homomorphism $\Lambda \to \Lambda$;
(iii) $p_n \circ p_m = p_{nm}$.

Plethysm has an interpretation on the level of Frobenius characteristics: see \cite{GK98, Proposition 7.3} or \cite[Chapter 7, Appendix 2]{Sta99}.

All of these constructions generalize to $(S_m \times S_n)$-representations. As in the introduction, we set

$$\Lambda^{(2)} := \Lambda \otimes \Lambda;$$

we call $\Lambda^{(2)}$ the ring of bisymmetric functions. Given $f \in \Lambda$, we write $f^{(j)}$ for the inclusion of $f$ into the $j$th tensor factor. Then we have

$$\Lambda^{(2)} = \mathbb{Q}[p_1^{(1)}, p_1^{(2)}, p_2^{(1)}, p_2^{(2)}, \ldots].$$

Given an $(S_m \times S_n)$-representation $V$, its Frobenius characteristic is the bisymmetric function

$$\text{ch}_{m,n}(V) := \frac{1}{m! \cdot n!} \sum_{(\sigma, \tau) \in S_m \times S_n} \text{Tr}_V(\sigma, \tau)p_{\lambda(\sigma)}^{(1)}p_{\lambda(\tau)}^{(2)}.$$ 

Just as in the single variable case, the bisymmetric function $\text{ch}_{m,n}(V)$ completely determines the $(S_m \times S_n)$-representation $V$: if

$$V = \bigoplus_{\lambda \vdash m, \mu \vdash n} (W_{\lambda} \otimes W_{\mu})^{\otimes a_{\lambda\mu}},$$

then

$$\text{ch}_{m,n}(V) = \sum_{\lambda, \mu} a_{\lambda\mu} s_{\lambda}^{(1)} s_{\mu}^{(2)}.$$ 

The ring $\Lambda^{(2)}$ has two plethysm operations $\circ_1$ and $\circ_2$, characterized by:

(i) for all $g$, the map $f \mapsto f \circ_1 g$ is an algebra homomorphism $\Lambda^{(2)} \to \Lambda^{(2)}$;
(ii) for all $n$, the map $f \mapsto p_n^{(i)} \circ_1 f$ is an algebra homomorphism $\Lambda^{(2)} \to \Lambda^{(2)}$;
(iii) $p_n^{(i)} \circ_1 p_m^{(j)} = p_{nm}^{(j)}$ for any $i, j \in \{1, 2\}$; and
(iv) $p_n^{(i)} \circ_2 f = p_n^{(i)}$ if $i \neq j$;

see Chaudhuri \cite{Cha16}. The ring $\Lambda$ is a Hopf algebra, with coproduct $\Delta : \Lambda \to \Lambda^{(2)}$ defined by

$$p_i \mapsto p_i^{(1)} + p_i^{(2)}.$$
On the level of Frobenius characteristic, we have
\[(2.1) \Delta(ch_n(V)) = \sum_{k=0}^{n} ch_{k,n-k}(\text{Res}_{S_k \times S_{n-k}}^S V)\]

There is a rank homomorphism
\[(2.2) \text{rk} : \Lambda \to \mathbb{Q}[[x]],\]
determined by
\[ch_n(V) \mapsto \dim(V) \frac{x^n}{n!},\]
or equivalently, \(p_1 \mapsto 1\) and \(p_n \mapsto 0\) for \(n > 1\). This takes plethysm into composition of power series. We use the same notation for the morphism
\[(2.3) \text{rk} : \Lambda(2) \to \mathbb{Q}[[x,y]]\]
determined by
\[ch_{m,n}(V) \mapsto \dim(V) \frac{x^m y^n}{m! n!},\]
or \(p_1^{(1)} \mapsto x\), \(p_1^{(2)} \mapsto y\), and \(p_n^{(j)} \mapsto 0\) for \(n > 1\). In this case, the two plethysm operations \(\circ_1\) and \(\circ_2\) are carried into composition in \(x\) and \(y\), respectively.

### 2.2. Hassett spaces

Let \(g \geq 0\), \(n \geq 1\) be two integers, and let \(\mathcal{A} = (a_1, \ldots, a_n) \in ((0,1] \cap \mathbb{Q})^n\) be a weight datum such that \(2g - 2 + a_1 + \cdots + a_n > 0\). Let \(C\) be a curve, at worst nodal, with \(p_1, \ldots, p_n\) smooth points of \(C\). We say that \((C, p_1, \ldots, p_n)\) is \(\mathcal{A}\)-stable if

(i) the twisted canonical sheaf \(K_C + a_1p_1 + \cdots + a_n p_n\) is ample;

(ii) whenever a subset of the marked points \(p_i\) for \(i \in S \subset \{1, \ldots, n\}\) coincide, we have \(\sum_{i \in S} a_i \leq 1\).

Condition (i) is equivalent to that for each irreducible component \(E\) in \(C\), we have
\[(2.4) 2g(E) - 2 + |(E \cap C \setminus E) \cup \text{Sing}(E)| + \sum_{i|p_i \in E} a_i > 0,\]
see [Uli15, Proposition 3.3]. Hassett shows that there exists a connected Deligne-Mumford stack \(\overline{M}_{g,A}\) of dimension \(3g - 3 + n\), smooth and proper over \(\mathbb{Z}\), which parameterizes \(\mathcal{A}\)-stable curves of genus \(g\) [Has03]. When \(\mathcal{A} = (1, \ldots, 1)\) is a sequence of \(n\) ones, Hassett stability coincides with the Deligne–Mumford–Knudsen stability, and \(\overline{M}_{g,A} = \overline{M}_{g,n}\). Each stack \(\overline{M}_{g,A}\) is equipped with a birational morphism
\[\rho_A : \overline{M}_{g,n} \to \overline{M}_{g,A}.\]

In this paper we are interested in the family of weight data
\[\mathcal{A}_{m,n} = \left(\frac{1, \ldots, 1}{m}, \frac{1/n, \ldots, 1/n}{n}\right),\]
and as in the introduction, we put \(\overline{M}_{g,m,n}\) for the resulting moduli space, called the heavy/light Hassett space. We say that a curve is \((m|n)\)-stable if it is \(\mathcal{A}_{m,n}\)-stable. We now characterize \((m|n)\)-stability in combinatorial terms.
Definition 2.5. For \((C,p_1,\ldots,p_{m+n}) \in \overline{M}_{g,m+n}\), let \(T \subset C\) be a union of irreducible components of \(C\). We say \(T\) is a rational tail if \(T\) is a connected curve of arithmetic genus zero, and \(T\) meets \(C \setminus T\) in a single point.

Definition 2.6. Let \(S \subseteq \{1,\ldots,m+n\}\). Given \((C,p_1,\ldots,p_{m+n}) \in \overline{M}_{g,m+n}\), we say a rational tail \(T \subset C\) supports \(S\) if for each \(i \in S\), we have \(p_i \in T\).

Given a rational tail \(T\), we say that an irreducible component \(E\) of \(T\) is a middle component if \(|E \cap C \setminus E'| = 2\), and we say it is terminal if \(|E \cap C \setminus E'| = 1\). The following lemma determines \((m|n)\)-stability in terms of rational tails.

Lemma 2.7. Let \((C,p_1,\ldots,p_{m+n}) \in \overline{M}_{g,m+n}\). Then \(C\) is \((m|n)\)-stable if and only if \(C\) does not have any rational tail which supports only markings with indices in \(\{m+1,\ldots,m+n\}\).

Proof. First assume \((C,p_1,\ldots,p_{m+n})\) is \((m|n)\)-stable. Then each of its irreducible components \(E\) satisfies

\[
2g(E) - 2 + |(E \cap C \setminus E) \cup \text{Sing}(E)| + \sum_{i|p_i \in E} a_i > 0.
\]

If \(T\) is a rational tail, the inequality reduces to \(\sum_{i|p_i \in E} a_i > 0\) for its middle components, and \(\sum_{i|p_i \in E} a_i > 1\) for the terminal one, so the marked points on \(T\) cannot be only subset of the last \(n\), since the sum of their weights would be at most 1.

Now assume \((C,p_1,\ldots,p_{m+n})\) is not \((m|n)\)-stable. Let \(E\) be a component of \(C\). The inequality (2.4) is satisfied if \(g(E) \geq 1\) or \(g(E) = 0\) with \(|(E \cap C \setminus E)| \geq 2\). Therefore, there must be a rational tail \(T\) consisting of a single component such that \(\sum_{i|p_i \in T} a_i \leq 1\), and this can happen only if \(T\) supports \(S\) with \(S \subseteq \{m+1,\ldots,m+n\}\). \(\square\)

3. The Grothendieck ring of \(S^2\)-spaces

A \(G\)-variety is a variety with an action of a group \(G\). An \(S\)-space is a sequence of \(S_n\)-varieties \(X_n\) for \(n \geq 0\). Getzler–Pandharipande define a Grothendieck ring of \(S\)-spaces [GP06]. We generalize this formalism to the case of \((S_m \times S_n)\)-varieties. First, we define the Grothendieck group

\[
K_0(\text{Var}, S_m \times S_n)
\]

of \((S_m \times S_n)\)-varieties. This group is constructed by first taking the free abelian group generated by isomorphism classes of \((S_m \times S_n)\)-varieties, and then imposing the relation

\[
[X] = [X \setminus Y] + [Y]
\]

whenever \(Y\) is an \((S_m \times S_n)\)-invariant subvariety of \(X\). We define an \(S^2\)-space \(\mathcal{X}\) to be a collection of varieties \(\mathcal{X}(m,n)\) together with an action of \(S_m \times S_n\) for each pair \((m,n)\) with \(m, n \geq 0\). We refer to \(\mathcal{X}(m,n)\) as the \(ary\) \((m,n)\) component of \(\mathcal{X}\). We define the Grothendieck group of \(S^2\)-spaces as the product

\[
K_0(\text{Var}, S^2) := \prod_{m,n \geq 0} K_0(\text{Var}, S_m \times S_n).
\]

We can make \(K_0(\text{Var}, S^2)\) into a ring using the \(\boxtimes\)-product on \(S^2\)-spaces:

\[
(\mathcal{X} \boxtimes \mathcal{Y})(m,n) = \prod_{i=0}^{m} \prod_{j=0}^{n} \text{Ind}_{S_i \times S_{m-i} \times S_j \times S_{n-j}}^{S_m \times S_n} \mathcal{X}(i,j) \times \mathcal{Y}(m-i,n-j).
\]
The ring $K_0(\text{Var}, S^2)$ is an algebra over the subring $K_0(\text{Var}) = K_0(\text{Var}, S_0 \times S_0)$, which is nothing but the usual Grothendieck group of varieties. For $X, Y \in K_0(\text{Var}, S^2)$ with $Y(0, 0) = \emptyset$, we can define two composition operations, $\circ_1$ and $\circ_2$, as follows:

$$
(3.1) \quad (X \circ_1 Y)(m, n) = \prod_{i=0}^{\infty} \prod_{j=0}^{m} \text{Ind}_{S^m \times S_j \times S_{n-j}}(X(i, j) \times Y_S^j(m - j))/S_i,
$$

and

$$
(3.2) \quad (X \circ_2 Y)(m, n) = \prod_{i=0}^{\infty} \prod_{j=0}^{m} \text{Ind}_{S_i \times S_{m-i} \times S_n}(X(i, j) \times Y_S^i(m - i, n))/S_j.
$$

Given an $S^2$-space $X$, we define its Hodge–Deligne series by

$$
e(X) := \sum_{m, n \geq 0} h_{X(m, n)}^{S^m \times S_n}(u, v) \in \Lambda[[u, v]].
$$

The lift of this series to the Grothendieck ring of mixed Hodge structures has been called the Serre characteristic or the Hodge-Grothendieck characteristic [Bag19].

The composition operations (3.1) and (3.2) should be viewed as categorifications of plethysm in the following sense: if $X, Y$ are $S^2$-spaces, then

$$
e(X \circ_i Y) = e(X) \circ_i e(Y)
$$

for $i = 1, 2$. This holds because the Hodge–Deligne polynomial is a motivic invariant. See Getzler–Pandharipande [GP06, §5] for details. We will put $\varsigma_n^{(1)}$ for the $S^2$-space given by the class of $\text{Spec } \mathbb{C}$ with trivial action of $S_n \times S_0$ in arity $(n, 0)$ and $\emptyset$ for every other pair, and $\varsigma_n^{(2)}$ for the analogous space given by $\text{Spec } \mathbb{C}$ with the action of $S_0 \times S_n$ in arity $(0, n)$ and $\emptyset$ everywhere else. Note that

$$
e(\varsigma_n^{(j)}) = h_n^{(j)}. $$

We have two analogues of the exponential function $e^x - 1$, given by

$$\text{Exp}^i(X) = \sum_{n > 0} \varsigma_n^{(i)} \circ_i X$$

for $i = 1, 2$ and $X(0, 0) = \emptyset$. These have inverse operations $\text{Log}^i$, which we will not use in this paper, but which allow for the inversion of Theorem A. Finally, given an $S$-space $Z$, there are at least three natural ways to view $Z$ as an $S^2$-space. First, we define

$$\Delta Z(m, n) := \text{Res}_{S^m \times S_n}(Z(m + n)).$$

By (2.1), we have

$$
e(\Delta Z) = \Delta(e(Z)),$$

where for an $S$-space $Z$, one defines

$$e(Z) := \sum_{n \geq 0} h_{Z(n)}^{S_n}(u, v) \in \Lambda[[u, v]].$$

Next, we put

$$I_1 Z(m, n) := \begin{cases} Z(m) & \text{if } n = 0, \\ \emptyset & \text{else} \end{cases}$$
and

\[ I_2 \mathcal{Z}(m, n) := \begin{cases} 
\mathcal{Z}(n) & \text{if } m = 0, \\
\emptyset & \text{else,}
\end{cases} \]

so

\[ e(I_j \mathcal{Z}) = e(\mathcal{Z})^{(j)} \]

for \( j = 1, 2 \). The \( S \)-space \( \varsigma_n \), which contains \( \text{Spec } \mathbb{C} \) with trivial \( S_n \)-action in arity \( n \), and \( \emptyset \) elsewhere, satisfies \( I_j(\varsigma_n) = \varsigma_n^{(j)} \).

### 4. Proof of Theorem A

Our main theorem is proven using the composition operations defined above. First, for each \( g \geq 0 \), define \( S^2 \)-spaces as follows:

\[
M^\text{hl}_g(m, n) = \begin{cases} 
\mathcal{M}_{g,m|n} & \text{if } 2g - 2 + m + \min(n, 1) > 0, \\
\emptyset & \text{else},
\end{cases}
\]

\[
\overline{M}^\text{hl}_g(m, n) = \begin{cases} 
\overline{\mathcal{M}}_{g,m|n} & \text{if } 2g - 2 + m + \min(n, 1) > 0, \\
\emptyset & \text{else}.
\end{cases}
\]

We will also make use of the \( S \)-spaces

\[
\mathcal{M}_g(n) = \begin{cases} 
\mathcal{M}_{g,n} & \text{if } 2g - 2 + n > 0, \\
\emptyset & \text{else},
\end{cases}
\]

and

\[
\overline{\mathcal{M}}_g(n) = \begin{cases} 
\overline{\mathcal{M}}_{g,n} & \text{if } 2g - 2 + n > 0, \\
\emptyset & \text{else}.
\end{cases}
\]

For an \( S \)-space \( \mathcal{Z} \), we put

\[ \delta \mathcal{Z}(n) := \text{Res}_{S_n}^{S_{n+1}} \mathcal{Z}(n+1); \]

note that

\[ e(\delta \mathcal{Z}) = \frac{\partial e(\mathcal{Z})}{\partial p_1}. \]

**Proposition 4.3.** We have

\[ M^\text{hl}_g = \Delta \mathcal{M}_g \circ_2 \text{Exp}^{(2)} \left( \varsigma_1^{(2)} \right) \]

**Proof.** We set

\[ Y = \text{Exp}^{(2)} \left( \varsigma_1^{(2)} \right). \]

Note that \( \varsigma_n^{(2)} \circ_2 \varsigma_1^{(2)} = \varsigma_n^{(2)} \), so \( Y \) is nonempty in arity \((m, n)\) if and only if \( m = 0 \) and \( n \geq 1 \), in which case it is equal to \( \varsigma_n^{(2)} \). We have that

\[ Y^{SS}(m - i, n) = \emptyset \]
unless \( i = m \), so for any \( S^2 \)-space \( \mathcal{X} \), we have

\[
(\mathcal{X} \circ_2 \mathcal{Y})(m, n) = \prod_{i=0}^{m} \prod_{j=0}^{\infty} \text{Ind}_{S^i \times S_{m-i} \times S_n}(\mathcal{X}(i, j) \times \mathcal{Y}^{\delta j}(m - i, n))/S_j
\]

\[
= \prod_{j=0}^{\infty} (\mathcal{X}(m, j) \times \mathcal{Y}^{\delta j}(0, n))/S_j
\]

\[
= \prod_{j=0}^{\infty} \left( \mathcal{X}(m, j) \times \prod_{k_1 + \cdots + k_j = n} \text{Ind}_{S_{k_1} \times \cdots \times S_{k_j}} \text{Spec } \mathbb{C} \right) /S_j.
\]

Now let us return to the \( S_m \times S_n \) space \( M_{g,m|n} \). This space admits a stratification: for \( 1 \leq j \leq n \), let \( Z_{m,j} \subset M_{g,m|n} \) denote the locally closed stratum in which there are precisely \( j \) distinct marked points among the last \( n \). Then we can write

\[
Z_{m,j} \cong \left( \prod_{k_1 + \cdots + k_j = n} \text{Res}_{S_m \times S_j} M_{g,m+j} \times \text{Ind}_{S_{k_1} \times \cdots \times S_{k_j}} \text{Spec } \mathbb{C} \right) /S_j.
\]

Since

\[
M_{g}^{hl}(m, n) = \sum_{j=1}^{n} Z_{m,j},
\]

we see that

\[
M_{g}^{hl} = \Delta M_{g} \circ_2 \text{Exp}^{(2)} \left( s_1^{(2)} \right)
\]

upon summing over \( j, m, \) and \( n \) on both sides of (4.4). \( \square \)

Towards proving our theorem for the compact moduli space \( \overline{M}_{g}^{hl} \), it is useful to introduce an auxiliary moduli space.

**Definition 4.5.** We set

\[
\overline{M}_{g}^{(k)} \subset \overline{M}_{g,n}
\]

to be the locus of curves which have no rational tails whose support consists of any subset of the last \( k \) markings. We define an \( S^2 \)-space \( \overline{M}_{g}^{(n)} \) by

\[
\overline{M}_{g}(m, n) := \overline{M}_{g,m+n}^{(n)}.
\]

The following proposition expresses the \( S^2 \)-space \( \Delta \overline{M}_{g} \) in terms of \( \overline{M}_{g}^{(n)} \) and the composition operation. The basic idea has appeared in the literature before, in the main theorem of [Get98], see also [Pet12].

**Proposition 4.6.** We have

\[
\Delta \overline{M}_{g} = \overline{M}_{g} \circ_2 \left( s_1^{(2)} + I_2 \delta \overline{M}_{0} \right).
\]

**Proof.** Let \( \mathcal{X} \) denote the class on the right-hand side of the claimed equality, and set

\[
\mathcal{Y} = s_1^{(2)} + I_2 \delta \overline{M}_{0} = I_2 \left( s_1 + \delta \overline{M}_{0} \right).
\]
Then $Y(m, n) = \emptyset$ unless $m = 0$. Hence, for any $j > 0$, we have that $Y^{\otimes j}(m, n) = \emptyset$ unless $m = 0$. Moreover, a point of the $S_n$-space $Y^{\otimes j}(0, n)$ corresponds to an ordered tuple of varieties

$$(X_1, \ldots, X_j)$$

such that:

1. for all $i$, $X_i$ is either $\text{Spec } \mathbb{C}$ or a pointed stable curve of arithmetic genus zero whose marked points are labelled by $\{0, \ldots, r_i\}$ for some $r_i \geq 2$;
2. there is a chosen bijection:

$$\{X_i \mid X_i = \text{Spec } \mathbb{C}\} \cup \{p \mid p \text{ is a nonzero marked point of } X_j \text{ for some } j\} \to \{1, \ldots, n\}.$$

The group $S_n$ acts on the chosen bijection, and $S_j$ acts by reordering the tuple. Now recall that

$$X(m, n) = \bigoplus_{j=0}^{\infty} (\overline{\mathcal{M}}_g(m, j) \times Y^{\otimes j}(0, n)) / S_j.$$

We can see that $X(m, n)$ is canonically isomorphic to $\text{Res}^{S_{m+n}}_{S_m \times S_n} \overline{\mathcal{M}}_{g,m+n}$: one takes the ordered tuple $(X_1, \ldots, X_j)$ represented by a point of $Y^{\otimes j}(0, n)$, and glues in the indicated order to the $j$ distinguished marked points of a pointed curve in $\overline{\mathcal{M}}_g(m, j)$. This has the effect of adding rational tails which support subsets of the final $n$ markings. The quotient by the diagonal action of $S_j$ is necessary to make this gluing procedure into an isomorphism. □

The final ingredient of the proof of Theorem A is the following formula, analogous to Proposition 4.3.

Proposition 4.7. We have

$$\overline{\mathcal{M}}_g \circ_{\text{Exp}^{(2)}} (\varsigma_1^{(2)}) = \overline{\mathcal{M}}_g^{hl}$$

Proof. The proof is essentially the same as that of Proposition 4.3: stratify $\overline{\mathcal{M}}_{g,m|n}$ by

$$\mathcal{W}_{m,j} = \{(C, p_1, \ldots, p_{n+m}) \mid \text{ there are } j \text{ distinct points among the last } m\},$$

and observe that

$$\mathcal{W}_{m,j} \cong \bigotimes_{k_1+\ldots+k_j=n \atop k_r>0 \forall r} (\overline{\mathcal{M}}_g^{(j)} \times \text{Ind}_{S_{k_1} \times \ldots \times S_{k_j}}^{S_n} \text{Spec } \mathbb{C}) / S_j,$$

by Lemma 2.7. The proof is complete upon summing over $j$. □

We can now prove the main theorem.

Proof of Theorem A. The first part of the theorem follows from taking $e(\cdot)$ on both sides of Proposition 4.3 and using both (3.3) and (3.4). From Proposition 4.6 and (3.4), we see that

$$(4.8) \quad \Delta(\overline{\varsigma}_g) = e(\overline{\mathcal{M}}_g) \circ_{\text{Exp}^{(2)}} \left( p_1^{(2)} + \frac{\partial \overline{\mathcal{B}}_0^{(2)}}{\partial p_1^{(2)}} \right),$$

as $e(\varsigma_1) = p_1$ and $e(\delta \overline{\mathcal{M}}_0) = \partial \overline{\mathcal{B}}_0 / \partial p_1$. The symmetric functions

$$p_1 + \frac{\partial \overline{\mathcal{B}}_0}{\partial p_1} \text{ and } p_1 - \frac{\partial \overline{\mathcal{B}}_0}{\partial p_1}$$

...
are plethystic inverses; this is because $b_0$ and $\overline{b}_0$ are Legendre transforms of one another, as explained in [Get95]. We thus perform the operation

$$\circ_2 \left( p_1^{(2)} - \frac{\partial b_0^{(2)}}{\partial p_1^{(2)}} \right)$$

on both sides of (4.8) to see that

$$\Delta(\overline{b}_0) \circ_2 \left( p_1^{(2)} - \frac{\partial b_0^{(2)}}{\partial p_1^{(2)}} \right) = e(\overline{M}_g^*) .$$

The theorem is now proven upon applying Proposition 4.7. \qed

To prove Corollary B, one uses the rank morphisms (2.2) and (2.3). We apply $\text{rk}$ to both sides of Theorem A, and use that

$$\text{rk} \left( \text{Exp}^{(2)} \left( p_1^{(2)} \right) \right) = e^y - 1.$$

The corollary follows from the formula

$$\text{rk} \left( p_1^{(2)} - \frac{\partial b_0^{(2)}}{\partial p_1^{(2)}} \right) = y - \sum_{n \geq 2} h_{\overline{M}_{0,n+1}}(u, v) \cdot \frac{y^n}{n!}$$

$$= y + \frac{(y + 1)uv - uvy - 1}{uv - u^2v^2},$$

due to Getzler [Get95].

5. Calculations

We conclude the paper by including three tables containing sample calculations, done with SageMath, based on Theorem A.\footnote{Sage code for these computations is available at https://sites.google.com/view/siddarthkannan/research} The first, Table 1, contains the $(S_m \times S_n)$-equivariant Hodge polynomial of $\overline{M}_{1,m|n}$ for $m + n \leq 5$. These rely on the calculation of the series $\overline{b}_1$ by Getzler [Get98]. For $n \leq 10$, the mixed Hodge structures on the cohomology groups of the moduli space $\overline{M}_{1,n}$ are polynomials in $L = H^2_c(\mathbb{A}^1; \mathbb{C})$, the mixed Hodge structure of the affine line. A consequence is that $\overline{M}_{1,n}$ has only even dimensional cohomology for $n \leq 10$, and only the diagonal Hodge numbers $\dim H^{p,p}$ are nonzero. By Theorem A, the same is true for $\overline{M}_{1,m|n}$ for $m + n \leq 10$. Therefore, Table 1 displays the equivariant Poincaré polynomial

$$h_{\overline{M}_{g,m|n}}^{S_m \times S_n}(t, t),$$

and the Hodge polynomial can be recovered by setting $t^2 = uv$.

Table 2 contains the non-equivariant Hodge polynomial of $\overline{M}_{1,0|n}$ for $n \leq 11$, computed with Corollary B and Getzler’s calculation of $\overline{b}_1$. We highlight the case where all markings are light, because this is the case in which the moduli space has no curves with rational tails. It seems that this simplification greatly reduces the complexity of the moduli space: comparing with the table [Get98, p.491], one notices that $\overline{M}_{1,0|n}$ appears to have much less cohomology than $\overline{M}_{1,n}$. For example, we have $\dim H^*(\overline{M}_{1,0|10}) = 232,076$ while $\dim H^*(\overline{M}_{1,10}) = 16,275,872$. One also notes that just as in the case of $\overline{M}_{1,11}$, the space $\overline{M}_{1,0|11}$ has odd-dimensional cohomology; this is true of $\overline{M}_{1,m|n}$ whenever $m + n = 11$.\footnote{Sage code for these computations is available at https://sites.google.com/view/siddarthkannan/research}
Finally, Table 3 contains the \((S_m \times S_n)\)-equivariant compactly supported weight zero Euler characteristic of \(\mathcal{M}_{g,m|n}\) for \(m + n \leq 6\), which is equal to
\[
h_{\mathcal{M}_{g,m|n}}^{S_m \times S_n}(0, 0),
\]
the constant term of the Hodge–Deligne polynomial. We also include the numerical weight zero Euler characteristic. This table was computed using the first part of Theorem A, together with the formula of Chan et al. for \(h_{\mathcal{M}_{g,n}}^{S_n}(0, 0)\) \([\text{CFGP}19]\). We also note that this table and our techniques apply to compute the equivariant Euler characteristic
\[
\chi_{S_m \times S_n}(\Delta_{g,m|n}) := \sum_i (-1)^i \text{ch}_{m|n}(H^i(\Delta_{g,m|n}; \mathbb{Q})) \in \Lambda^{(2)}
\]
where \(\Delta_{g,m|n}\) is the tropical heavy/light Hassett space, studied in \([\text{CHMR}16, \text{CMP}^+20, \text{KLSY}, \text{KKL}21]\). Indeed, one has
\[
\chi_{S_m \times S_n}(\Delta_{g,m|n}) = s_{m}^{(1)} s_{n}^{(2)} - h_{\mathcal{M}_{g,m|n}}^{S_m \times S_n}(0, 0)
\]
when \(\Delta_{g,m|n}\) is connected, which holds when \(g \geq 1\), and when \(g = 0\) and \(m + n > 4\). See \([\text{KLSY}, \S 4]\).

We exclude the case \(n = 1\) from Tables 1 and 3, as
\[
h_{\mathcal{M}_{g,m|1}}^{S_m \times S_1}(u, v) = \left(\frac{\partial h_{\mathcal{M}_{g,m+1}}^{S_m+1}(u, v)}{\partial p_1}\right)^{(1)} \cdot s_{1}^{(2)},
\]
and the analogous formula holds for the open moduli spaces.
| $(m, n)$ | $\mathcal{M}_{1,m/n}(t, t)$ |
|---------|--------------------------|
| (0, 2)  | $(t^4 + 2t^2 + 1) s_2^{(2)}$ |
| (0, 3)  | $(t^4 + t^2) s_{2,1}^{(2)} + (t^6 + 2t^4 + 2t^2 + 1) s_3^{(2)}$ |
| (1, 2)  | $(t^4 + t^2) s_{1,1}^{(1)} s_2^{(2)} + (t^6 + 4t^4 + 4t^2 + 1) s_1^{(1)} s_2^{(2)}$ |
| (0, 4)  | $(t^6 + 2t^4 + t^2) s_{2,2}^{(2)} + (t^6 + 2t^4 + t^2) s_{3,1}^{(2)} + (t^8 + 2t^6 + 3t^4 + 2t^2 + 1) s_4^{(2)}$ |
| (1, 3)  | $(3t^6 + 6t^4 + 3t^2) s_{1,1}^{(1)} s_{2,1}^{(2)} + (t^8 + 5t^6 + 9t^4 + 5t^2 + 1) s_1^{(1)} s_3^{(2)}$ |
| (2, 2)  | $((t^6 + 2t^4 + t^2) s_{1,1}^{(1)} + (2t^6 + 4t^4 + 2t^2) s_2^{(2)}) s_{1,1}^{(2)} + ((2t^6 + 4t^4 + 2t^2) s_{1,1}^{(1)} + (t^8 + 7t^6 + 13t^4 + 7t^2 + 1) s_2^{(1)}) s_2^{(2)}$ |
| (0, 5)  | $(t^6 + t^4) s_{2,2,1}^{(2)} + (t^8 + 3t^6 + 3t^4 + t^2) s_{3,2}^{(2)} + (t^8 + 3t^6 + 3t^4 + t^2) s_{4,1}^{(2)} + (t^{10} + 2t^8 + 3t^6 + 3t^4 + 2t^2 + 1) s_5^{(2)}$ |
| (1, 4)  | $(2t^6 + 2t^4) s_{1,1}^{(1)} s_{2,1,1}^{(2)} + (2t^8 + 8t^6 + 8t^4 + 2t^2) s_1^{(1)} s_{2,2}^{(2)} + (4t^8 + 14t^6 + 14t^4 + 4t^2) s_{3,1}^{(1)} s_{3,1}^{(2)} + (t^{10} + 6t^8 + 15t^6 + 15t^4 + 6t^2 + 1) s_1^{(1)} s_4^{(2)}$ |
| (2, 3)  | $(t^6 + t^4) s_{1,1}^{(1)} + (t^6 + t^4) s_{2,1,1}^{(1)} + ((2t^8 + 9t^6 + 9t^4 + 2t^2) s_{1,1}^{(1)} + (5t^8 + 20t^6 + 20t^4 + 5t^2) s_2^{(1)}) s_{2,1}^{(2)} + ((3t^8 + 11t^6 + 11t^4 + 3t^2) s_{1,1}^{(1)} + (t^{10} + 9t^8 + 26t^6 + 26t^4 + 9t^2 + 1) s_2^{(1)}) s_{3,1}^{(2)}$ |
| (3, 2)  | $t^{10} s_{1,1}^{(1)} s_2^{(2)} + t^8 (2s_{2,1}^{(1)} + 3s_3^{(1)}) s_{1,1}^{(2)} + (5s_{2,1}^{(1)} + 10s_3^{(1)}) s_2^{(2)} + t^6 ((s_{1,1}^{(1)} + 9s_{2,1}^{(1)} + 12s_3^{(1)}) s_{1,1}^{(2)} + (s_{1,1}^{(1)} + 20s_{2,1}^{(1)} + 30s_3^{(1)}) s_2^{(2)}) + \cdots$ |

Table 1: The $(S_m \times S_n)$-equivariant Poincare polynomials of $\overline{\mathcal{M}}_{1,m/n}$ for $m + n \leq 5$. The omitted terms are determined by Poincaré duality.
| $n$ | $h_{\mathcal{M}_{1,0|n}}(u, v)$ |
|-----|-------------------------------|
| 1   | $uv + 1$                      |
| 2   | $u^2v^2 + 2uv + 1$            |
| 3   | $u^3v^3 + 4u^2v^2 + 4uv + 1$  |
| 4   | $u^4v^4 + 7u^3v^3 + 13u^2v^2 + 7uv + 1$ |
| 5   | $u^5v^5 + 11u^4v^4 + 35u^3v^3 + 35u^2v^2 + 11uv + 1$ |
| 6   | $u^6v^6 + 16u^5v^5 + 81u^4v^4 + 140u^3v^3 + 81u^2v^2 + 16uv + 1$ |
| 7   | $u^7v^7 + 22u^6v^6 + 168u^5v^5 + 476u^4v^4 + 476u^3v^3 + 168u^2v^2 + 22uv + 1$ |
| 8   | $u^8v^8 + 29u^7v^7 + 323u^6v^6 + 1456u^5v^5 + 2458u^4v^4 + 1456u^3v^3 + 323u^2v^2 + 29uv + 1$ |
| 9   | $u^9v^9 + 37u^8v^8 + 591u^7v^7 + 4201u^6v^6 + 11901u^5v^5 + 11901u^4v^4 + 4201u^3v^3 + 591u^2v^2 + 37uv + 1$ |
| 10  | $u^{10}v^{10} + 46u^9v^9 + 1051u^8v^8 + 11850u^7v^7 + 55975u^6v^6 + 94230u^5v^5 + 55975u^4v^4 + 11850u^3v^3 + 1051u^2v^2 + 46uv + 1$ |
| 11  | $u^{11}v^{11} + 56u^{10}v^{10} + 1848u^9v^9 + 33451u^8v^8 + 258940u^7v^7 + 710512u^6v^6 - u^{11} - v^{11} + 710512u^5v^5 + 258940u^4v^4 + \ldots$ |

Table 2: The Hodge polynomial of $\mathcal{M}_{1,0|n}$ for $n \leq 11$. 
| $(m, n)$ | $h^{S_m \times S_n}_{\mathcal{M}_{2,m|n}} (0, 0)$ | $h_{\mathcal{M}_{2,m|n}} (0, 0)$ |
|----------|----------------------------------------|------------------|
| $(0, 2)$ | $-s_2^{(2)}$ | $-1$ |
| $(0, 3)$ | $-s_{2,1}^{(2)} - s_3^{(2)}$ | $-3$ |
| $(1, 2)$ | $-s_1^{(1)} s_2^{(2)}$ | $-1$ |
| $(0, 4)$ | $-s_{2,1,1}^{(2)} - s_{2,2}^{(2)} - s_{3,1}^{(2)} - s_4^{(2)}$ | $-9$ |
| $(1, 3)$ | $-s_1^{(1)} (s_{1,1,1}^{(2)} + s_{2,1}^{(2)})$ | $-3$ |
| $(2, 2)$ | $\left(s_{1,1}^{(1)} - s_2^{(1)}\right) s_{1,1}^{(2)} + \left(-s_{1,1}^{(1)} + s_2^{(1)}\right) s_2^{(2)}$ | $-2$ |
| $(0, 5)$ | $s_{2,1,1}^{(2)} - s_{2,2,1}^{(2)} - s_{3,1,1}^{(2)} - s_{3,2}^{(2)} - s_{4,1}^{(2)} - s_5^{(2)}$ | $-25$ |
| $(1, 4)$ | $-s_1^{(1)} s_{2,1,1}^{(2)}$ | $-3$ |
| $(2, 3)$ | $s_2^{(1)} (s_{1,1,1}^{(2)} + s_3^{(2)}) + \left(2s_2^{(1)} - s_{1,1}^{(1)}\right) s_{2,1}^{(2)}$ | $4$ |
| $(3, 2)$ | $\left(s_{2,1}^{(1)} + 2s_3^{(1)}\right) s_{1,1}^{(2)} + \left(s_{2,1}^{(1)} + 2s_3^{(1)}\right) s_2^{(2)}$ | $8$ |
| $(0, 6)$ | $s_{1,1,1}^{(1)} - s_{2,1,1,1}^{(2)} - s_{2,2,1}^{(2)} - s_{3,1,1,1}^{(2)} - s_{3,2,1}^{(2)} - s_{4,1,1}^{(2)} - s_{4,2}^{(2)} - s_{5,1}^{(2)} - s_6^{(2)}$ | $-71$ |
| $(1, 5)$ | $-s_1^{(1)} (2s_{1,1,1,1}^{(2)} + s_{2,1,1,1}^{(2)} + s_{2,2,1}^{(2)}$ | $-11$ |
| $(2, 4)$ | $\left(3s_{1,1}^{(1)} - s_2^{(1)}\right) s_{1,1,1,1}^{(2)} + \left(-4s_{1,1}^{(1)} + 2s_2^{(1)}\right) s_{2,1,1}^{(2)} + \left(-s_{1,1}^{(1)} + s_2^{(1)}\right) s_{2,2}^{(2)} + \left(-s_{1,1}^{(1)} + s_2^{(1)}\right) s_{3,1}^{(2)}$ | $-14$ |
| $(3, 3)$ | $\left(5s_{1,1}^{(1)} - s_2^{(1)} + s_3^{(1)}\right) s_{1,1,1}^{(2)} + \left(-2s_{1,1}^{(1)} - 4s_2^{(1)}\right) s_{2,1}^{(2)} + \left(s_{1,1}^{(1)} - s_{2,1}^{(1)} - s_3^{(1)}\right) s_3^{(2)}$ | $-32$ |
| $(4, 2)$ | $\left(-3s_{1,1,1,1}^{(1)} - 5s_{2,1,1}^{(1)} - 3s_{2,2}^{(1)} - 2s_{3,1}^{(1)}\right) s_{1,1}^{(2)} + \left(-s_{1,1,1,1}^{(1)} - 5s_{2,2}^{(1)} - 2s_{3,1}^{(2)} - 3s_4^{(1)}\right) s_2^{(2)}$ | $-50$ |

Table 3: The $(S_m \times S_n)$-equivariant and numerical weight zero compactly supported Euler characteristics of $\mathcal{M}_{2,m|n}$. 
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